1. Introduction

It is well known that the Hochschild cohomology for associative algebras has good properties only for algebras which are projective modules over the ground ring. For general algebras behavior of Hochschild cohomology is more pathological, for example there is no long cohomological exact sequence corresponding to a short exact sequence for coefficients, etc. In early 60-s Shukla \[39\] developed a cohomology theory for associative algebras with nicer properties than Hochschild theory. Quillen in \[32\] indicated that the Shukla cohomology fits in his general framework of homotopical algebra. The approach of Quillen is based on simplicial methods, which are usually quite hard to deal with. The aim of this work is to give the foundation of Shukla cohomology based on chain algebras. We also give an application to the problem of strengthening additive track theories, which is based on the comparison homomorphism between Shukla and Mac Lane cohomology theories \[37\]. We believe that our approach is much simpler than one used in \[32\] or \[39\].

Let us recall that a track category is a category enriched in groupoids. A track category \( \mathcal{T} \) is called abelian if for any arrow \( f \) the group of automorphisms of \( f \) is abelian. A track theory is an abelian track category with finite lax products. If it admits strong products then it is called strong track theory. The main result of \[5\] asserts that any track theory is equivalent to a strong one. An additive track theory is a track theory which moreover possesses a lax zero object and finite lax coproducts such that the natural map from the lax coproduct to the lax product is a homotopy equivalence and the corresponding homotopy category is an additive category. An additive track theory is called very strong if it possesses a strong zero object and strong coproducts which are also strong coproducts. By the result of \[5\], any additive track theory is equivalent to one which possesses strong products and lax coproducts or strong coproducts and lax products. We show that in general it is impossible to get both strong products and coproducts. However this is possible if certain obstructions vanish. In particular this is possible if hom’s of the corresponding homotopy category are vector spaces over a field.

The contents of the sections below are as follows. In Section 2 we recall basics on Hochschild cohomology theory and especially relationship between abelian extensions which are split over ground ring, and elements of the second Hochschild cohomology. In Section 3 we introduce crossed bimodules and crossed extensions. We recall the relationship between crossed extensions which are split over the ground ring, and elements of the third Hochschild cohomology. This section also contains a new interpretation of the classical obstruction theory in terms of crossed extensions (see Theorem 3.3.1). We also discuss a different generalization of the relationship between different sort of extensions and higher cohomology. In Section 4 we define...
Shukla cohomology as a kind of derived Hochschild cohomology on the category of
chain algebras and we prove basic properties of the Shukla cohomology including
relationship with crossed bimodules. In the original paper Shukla used an explicit
cochain complex for the definition of Shukla cohomology. Unfortunately this com-
plex is very complicated to work with. Quillen instead used closed model category
structure on the category of simplicial algebras. We use the closed model category
structure on the category of chain algebras, which is developed in the Appendix.
The Section 5 is devoted to some computations of Shukla cohomology when the
ground ring is the ring of integers or \( \mathbb{Z}/p^2\mathbb{Z} \); we also consider the relationship be-
tween the Shukla cohomology over integers and over \( \mathbb{Z}/p^2\mathbb{Z} \). In this direction we
prove the following result. Let \( A \) be an algebra over \( F_p \) and let \( M \) be a bimodule
over \( A \), then the base change morphism

\[
\text{Shukla}^i(A/\mathbb{K}, M) \to \text{Shukla}^i(A/\mathbb{Z}, M), \quad \mathbb{K} = \mathbb{Z}/p^2\mathbb{Z}
\]

is always an epimorphism. It is an isomorphism in dimensions 0, 1 and 2. We also
prove that in dimension three the kernel of this map is isomorphic to \( H^3(A, M) \).
The Section 6 solves the problem of constructing a canonical cochain complex for
computing the Shukla cohomology in the important case when the ground ring is
an algebra over a field. Our cochain complex consists of tensors, unlike the one
proposed by Shukla. The Section 7 recalls basics of Mac Lane cohomology \[27\] and
relationship with Shukla cohomology. It is well known that these two theories are
isomorphic up to dimension two. It turns out that for algebras over fields they are
also isomorphic in dimension three. The section 8.1 continues the study of track
theories which was started in \[5\]. In this section we show that the straightforward
version of the strengthening result for additive track theories is not true and we
construct the corresponding obstruction. This obstruction is defined using the exact
sequence relating third Shukla and Mac Lane cohomology and is a main application
of the theory considered in the previous sections. The Appendix contains the basic
definitions on closed model categories. It contains also a proof of the fact that
chain algebras over any ground ring form a closed model category. This fact is used
in Section 6. At the end of the Appendix we introduce a closed model category
structure on the category of crossed bimodules over any ground ring.

In a forthcoming paper we introduce the notion of a \textit{strongly additive track theory}
and we will prove that any additive track category is equivalent to a strong one.
The notion of strongly additive track theory is based on theory of square rings \[6\].

The second author is indebted to Mamuka Jibladze for the idea to modify clas-
sical obstruction theory in terms of crossed bimodules.

2. Preliminaries on Hochschild Cohomology

Here we recall the basic notion on Hochschild cohomology theory and refer to
\[26\] and \[29\] for more details. In this section \( \mathbb{K} \) denotes a commutative ring with
unit, which is considered as a ground ring, except for the section 6.

2.1. Definition. Let \( R \) be a \( \mathbb{K} \)-algebra with unit and let \( M \) be a bimodule over \( R \).
Consider the module

\[
C^n(R, M) := \text{Hom}(R^\otimes n, M)
\]
(where $\otimes = \otimes_K$ and $\text{Hom} = \text{Hom}_K$). The Hochschild coboundary is the linear map $d : C^n(R, M) \to C^{n+1}(R, M)$ given by the formula

$$d(f)(r_1, ..., r_{n+1}) = r_1 f(r_2, ..., r_{n+1}) + \sum_{i=1}^n (-1)^i f(r_1, ..., r_ir_{i+1}, ..., r_{n+1}) + (-1)^{n+1} f(r_1, ..., r_n)r_{n+1}.$$ 

Here $f \in C^n(R, M)$ and $r_1, \cdots, r_{n+1} \in R$. By definition the $n$-th Hochschild cohomology group of the algebra $R$ with coefficients in the $R$-bimodule $M$ is the $n$-th homology group of the Hochschild cochain complex $C^*(R, M)$ and it is denoted by $H^*(R, M)$. Sometimes these groups are denoted by $H^*(R/K, M)$ in order to make clear that the ground ring is $K$. We are especially interested in cases $K = \mathbb{Z}, \mathbb{F}_p, \mathbb{Z}/p^2\mathbb{Z}$. It is clear that for such a $K$ one has

$$H^i(\mathbb{F}_p/K, \mathbb{F}_p) = 0, \ i \geq 1.$$ 

In the following sections we consider two modifications of Hochschild cohomology, known as Shukla and Mac Lane cohomology. It should be noted that in both theories the algebra $\mathbb{F}_p$ has nontrivial cohomology over the ground ring $K = \mathbb{Z}$ or $K = \mathbb{Z}/p^2\mathbb{Z}$.

2.2. $K$-split exact sequences. Let

$$0 \to M_1 \xrightarrow{\mu} M \xrightarrow{\sigma} M_2 \to 0$$

be an exact sequence of bimodules over $R$. It is called $K$-split if there exists a $K$-linear map $u : M_2 \to M$ such that $\sigma \circ u = \text{Id}_{M_2}$. This condition is equivalent to the following one: there is a $K$-linear map $v : M \to M_1$ such that $v \circ \mu = \text{Id}_{M_1}$.

Let $f : M \to N$ be a morphism of bimodules over $R$. It is called $K$-split, if the following exact sequences

$$0 \to \ker(f) \to M \to \im(f) \to 0,$$

and

$$0 \to \im(f) \to N \to \coker(f) \to 0$$

are $K$-split.

If $0 \to M_1 \xrightarrow{\mu} M \xrightarrow{\sigma} M_2 \to 0$ is a $K$-split exact sequence, then

$$0 \to C^*(R, M_1) \to C^*(R, M) \to C^*(R, M_2) \to 0$$

is exact in the category of cochain complexes and therefore yields the long cohomological exact sequence:

$$\cdots \to H^n(R, M_1) \to H^n(R, M) \to H^n(R, M_2) \to H^{n+1}(R, M_1) \to \cdots$$
2.3. **Induced bimodules.** The category of bimodules over $R$ is equivalent to the category of left modules over the ring $R^e := R \otimes R^{op}$, where $R^{op}$ is the opposite ring of $R$, which is isomorphic to $R$ as a $K$-module via the map $r \mapsto r^{op}$, $R \to R^{op}$, while the multiplication structure in $R^{op}$ is given by $r^{op}s^{op} = (sr)^{op}$. The multiplication map $R \otimes R^{op} \to R$ is an algebra homomorphism. We always consider $R$ as a bimodule over $R$ via this homomorphism.

If $A$ and $B$ are left $R$-modules, then $\text{Hom}(A, B)$ is a bimodule over $R$ by the following action

$$(rfs)(a) = rf(sa), \quad r, s \in R, a \in M, f \in \text{Hom}(A, B)$$

A bimodule is called induced if it is isomorphic to $\text{Hom}(R, A)$ for an $R$-module $A$. It is well-known [29] that the Hochschild cohomology vanishes in positive dimensions on induced bimodules. For a bimodule $M$ the map

$$\mu : M \to \text{Hom}(R, M)$$

given by $\mu(m)(r) = mr$ is a homomorphism of bimodules, which is also $K$-split monomorphism, hence one has a $K$-split short exact sequence

$$0 \to M \to \text{Hom}(R, M) \to N \to 0$$

where $N = \text{Coker}(\mu)$, which yields the isomorphism

$$H^{i+1}(R, M) \cong H^i(R, N), \quad i > 0$$

This shows that there is a natural isomorphism [29]

$$H^*(R, M) \cong \text{Ext}^*_{R^e, K}(R, M)$$

where subscript $K$ indicates that Ext-groups in question are defined in the framework of relative homological algebra, where the proper class consists of $K$-split exact sequences. If $R$ is projective as a $K$-module, then one can take the usual Ext-groups $\text{Ext}^*_R(R, M)$ instead of the relative Ext-groups. In particular, the Hochschild cohomology vanishes in positive dimensions on injective bimodules, provided $R$ is projective as a $K$-module.

2.4. **Hochschild cohomology in dimension 0.** For $n = 0$ one has

$$H^0(R, M) = \{ m \in M \mid rm = mr \text{ for any } r \in R \}.$$ 

In particular $H^0(R, R)$ coincides with the center $Z(R)$ of the algebra $R$.

2.5. **Hochschild cohomology in dimension 1.** For $n = 1$ a 1-cocycle is a linear map $D : R \to M$ satisfying the identity

$$D(xy) = xD(y) + D(x)y, \quad x, y \in R.$$ 

Such a map is called a derivation from $R$ to $M$ and the $K$-module of derivations is denoted by $\text{Der}(R, M)$. A derivation $D : R \to M$ is a coboundary if it has the form

$$\text{ad}_m(r) = rm - mr$$

for some fixed $m \in M$; $\text{ad}_m$ is called an inner derivation. Therefore

$$H^1(R, M) = \text{Der}(R, M)/\{\text{Inner derivations}\}.$$ 

In particular one has the exact sequence

$$0 \to H^0(R, M) \to M \xrightarrow{\text{ad}} \text{Der}(R, M) \to H^1(R, M) \to 0$$
2.6. Hochschild cohomology in dimension 2. It is clear that a 2-cocycle of \( C^*(R, M) \) is a linear map \( f : R \otimes R \to M \) satisfying
\[
xf(y, z) - f(xy, z) + f(x, yz) - f(x, y)z = 0, \quad x, y, z \in R.
\]
For any linear map \( g : R \to M \) the formula \( f(x, y) = xg(y) - f(xy) + f(x)y \) defines a cocycle, all such cocycles are called coboundaries. We let \( Z^2(R, M) \) be the collections of all 2-cocycles and coboundaries. Hence
\[
H^2(R, M) = Z^2(R, M)/B^2(R, M).
\]
We recall the relation of \( H^2(R, M) \) to abelian extensions of algebras.

An abelian extension (sometimes called also a singular extension) of associative algebras is a short exact sequence
\[
0 \longrightarrow M \longrightarrow E \overset{p}{\longrightarrow} R \longrightarrow 0
\]
where \( R \) and \( E \) are associative algebras with unit and \( p : E \to R \) is a homomorphism of algebras with unit and \( M^2 = 0 \), in other words the product in \( E \) of any two elements from \( M \) is zero. For an elements \( m \in M \) and \( r \in R \) we put \( mr := me \) and \( rm := em \). Here \( e \in E \) is an element such that \( p(e) = r \). This definition does not depend on the choice of \( e \). Therefore \( M \) has a bimodule structure over \( R \).

An abelian extension \( (E) \) is called \( K \)-split if there exists a linear map \( s : R \to E \) such that \( ps = Id_R \).

Assume we have a bimodule \( M \) over an associative algebra \( R \), then we let \( \mathcal{E}(R, M) \) be the category, whose objects are the abelian extensions \( (E) \) such that the induced \( R \)-bimodule structure on \( M \) coincides with the given one. The morphisms \( (E) \to (E') \) are commutative diagrams
\[
0 \longrightarrow M \longrightarrow E \overset{\phi}{\longrightarrow} R \longrightarrow 0
\]
where \( \phi \) is a homomorphism of algebras with unit. Moreover, we let \( \mathcal{E}_k(R, M) \) be the category, whose objects are \( K \)-split singular extensions. It is clear that the categories \( \mathcal{E}(R, M) \) and \( \mathcal{E}_k(R, M) \) are groupoids, in other words all morphisms in \( \mathcal{E}(R, M) \) and \( \mathcal{E}_k(R, M) \) are isomorphisms. We let \( \text{Extalg}(R, M) \) and \( \text{Extalg}_k(R, M) \) be the classes of connected components of these categories. Clearly \( \text{Extalg}_k(R, M) \subset \text{Extalg}(R, M) \). According to [29] there is a natural bijection
\[
H^2(R, M) \cong \text{Extalg}_k(R, M).
\]
We also recall that the map \( H^2(R, M) \to \text{Extalg}_k(R, M) \) is given as follows. Let \( f : R \otimes R \to M \) be a 2-cocycle. We let \( M \rtimes f R \) be an associative \( K \)-algebra which is \( M \oplus R \) as a \( K \)-module, while the algebra structure is given by
\[
(m, r)(n, s) = (ms + rn + f(r, s), rs).
\]
Then
\[
0 \longrightarrow M \overset{i}{\longrightarrow} M \rtimes f R \overset{p}{\longrightarrow} R \longrightarrow 0
\]
is an object of \( \mathcal{E}_k(R, M) \). Here \( i(m) = (m, 0) \) and \( p(m, r) = r \).

2.7. Cohomology of tensor algebras. [26], [29]. Let \( V \) be a \( K \)-module. For the tensor algebra \( R = T^*(V) \) one has \( H^i(R, -) = 0 \) for all \( i \geq 2 \). An algebra is called free if it is isomorphic to \( T(V) \), where \( V \) is a free \( K \)-module.
2.8. **Cup-product in Hochschild cohomology.** For any associative algebra $R$ the cohomology $H^\ast(R, R)$ is a graded commutative algebra under the cup-product, which is defined by

$$ (f \cup g)(r_1, \ldots, r_{n+m}) := f(r_1, \ldots, r_n)g(r_{n+1}, \ldots, r_{n+m}) $$

for $f \in C^n(R, R)$ and $g \in C^m(R, R)$ (see [18]). This product corresponds to the Yoneda product under the isomorphism $H^\ast(R, R) \cong \text{Ext}_{R^{\text{op}} \otimes R}^\ast(R, R)$.

3. **Crossed bimodules and Hochschild cohomology**

3.1. **Crossed bimodules.** Let us recall that a chain algebra over $\mathbb{K}$ is a graded algebra $C_* = \bigoplus_{n \geq 0} C_n$ equipped with a boundary map $\partial : C_* \to C_*$ of degree $-1$ satisfying the Leibniz identity

$$ \partial(xy) = \partial(x) + (-1)^{|x|}x\partial(y). $$

**Definition 3.1.1.** A crossed bimodule is a chain algebra which is trivial in dimensions $\geq 2$.

Thus a crossed bimodule consists of an algebra $C_0$ and a bimodule $C_1$ over $C_0$ together with a homomorphism of bimodules

$$ C_1 \xrightarrow{\partial} C_0 $$

such that

$$ \partial(c)c' = c\partial(c'), \ c, c' \in C_1, $$

Indeed, since $C_2 = 0$ the last condition is equivalent to the Leibniz identity $0 = \partial(cc') = \partial(c)c' - c\partial(c')$.

It follows that the product defined by

$$ c \ast c' := \partial(c)c' $$

where $c, c' \in C_1$ gives an associative non-unital $\mathbb{K}$-algebra structure on $C_1$ and $\partial : C_1 \to C_0$ is a homomorphism of non-unital $\mathbb{K}$-algebras. The equivalent but less economic definition goes back at least to Dedecker and Lue [11]. The notion of crossed bimodules is an associative algebra analogue of crossed modules introduced by Whitehead [11] in the group theory framework, which plays a major role in homotopy theory of spaces with nontrivial fundamental groups [9, 24].

We let $\mathbf{Xmod}$ and $\mathbf{Xmod}_R$ be the category of crossed bimodules and crossed $R$-bimodules respectively.

We have also a category $\mathbf{Bim}/\mathbf{Alg}$, whose objects are triples $(C_0, C_1, \partial)$, where $C_0$ is an associative algebra, $C_1$ is a bimodule over $C_0$ and $\partial : C_1 \to C_0$ is a homomorphism of bimodules over $C_0$. It is clear that $\mathbf{Xmod}$ is a full subcategory of $\mathbf{Bim}/\mathbf{Alg}$ and the inclusion $\mathbf{Xmod} \subset \mathbf{Bim}/\mathbf{Alg}$ has a left adjoint functor, which assigns $\partial : \tilde{C}_1 \to C_0$ to $C_1 \to C_0$. Here $\tilde{C}_1$ is the quotient of $C_1$ under the equivalence relation $x\partial(y) = \partial(x)y \sim 0, x, y \in C_1$.

We let $\mathbf{Mod}/\mathbf{Alg}$ be the category whose objects are triples $(V, C, \partial)$, where $C$ is an associative algebra, $V$ is a $\mathbb{K}$-module and $\partial : V \to C$ is a linear map. One has the forgetful functor $\mathbf{Bim}/\mathbf{Alg} \to \mathbf{Mod}/\mathbf{Alg}$, which has a left adjoint functor sending $(V, C, \partial)$ to the triple $(M, d, C)$, where $M = C \otimes V \otimes C$ and $d$ is the unique homomorphism of bimodules which extends $\partial$. As a consequence we see that the forgetful functor $\mathbf{Xmod} \to \mathbf{Mod}/\mathbf{Alg}$ also has a left adjoint. Of special interest is the case when $C$ is a free associative algebra and $V$ is a free $\mathbb{K}$-module on $X \subset V$. In this case the corresponding crossed bimodule is called **free crossed bimodule.**
3.2. Hochschild cohomology in the dimension 3 and crossed extensions.

Here we recall the relation between Hochschild cohomology and crossed bimodules (see Exercise E.1.5.1 of [20] or [4]).

Let $\partial : C_1 \to C_0$ be a crossed bimodule. We put $M = \ker(\partial)$ and $R = \text{coker}(\partial)$. Then the image of $\partial$ is an ideal of $C_0$. We have also $MC_1 = 0 = C_1M$ and $M$ has a well-defined bimodule structure over $R$.

Let $R$ be an associative algebra with unit and let $M$ be a bimodule over $R$. A crossed extension of $R$ by $M$ is an exact sequence

$$0 \to M \to C_1 \overset{\partial}{\to} C_0 \to R \to 0$$

where $\partial : C_1 \to C_0$ is a crossed bimodule, such that $C_0 \to R$ is a homomorphism of algebras with unit and an $R$-bimodule structure on $M$ induced from the crossed bimodule structure coincides with the prescribed one.

A crossed extension of $R$ by $M$ is $\mathbb{K}$-split, if all arrows in the exact sequence

$$0 \to M \to C_1 \overset{\partial}{\to} C_0 \to R \to 0$$

are $\mathbb{K}$-split.

For fixed $R$ and $M$ one can consider the category $\text{Crossext}(R, M)$ whose objects are crossed extensions of $R$ by $M$. Morphisms are maps between crossed modules which induce the identity on $M$ and $R$.

**Lemma 3.2.1.** Assume $(\partial)$ is a crossed extension of $R$ by $M$ and a homomorphism $f : P_0 \to C_0$ of unital $\mathbb{K}$-algebras is given. Let $P_1$ be the pull-back of the diagram

$$\begin{array}{ccc} P_0 & \longrightarrow & C_0 \\ \downarrow & \downarrow & \downarrow \\ C_1 & \longrightarrow & C_0. \end{array}$$

Then there exists a unique crossed module structure on $P_1$$P_0$ such that the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & M & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & R & \longrightarrow & 0 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & \longrightarrow & M & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

defines a morphism of crossed extensions.

**Corollary 3.2.2.** In each connected component of $\text{Crossext}(R, M)$ there is a crossed extension

$$(P) \quad 0 \to M \to P_1 \to P_0 \to R \to 0$$

with free algebra $P_0$ and for any other object $(\partial)$ in this connected component there is a morphism $(P) \to (\partial)$. Thus $(\partial)$ and $(\partial')$ are in the same component of $\text{Crossext}(R, M)$ if and only if there exists a diagram of the form $(\partial) \leftarrow (P) \to (\partial')$.

We let $\text{Crossext}_K(R, M)$ be the subcategory of $\mathbb{K}$-split crossed extensions. Morphisms are such morphisms from $\text{Crossext}(R, M)$ that all maps involved are $\mathbb{K}$-split. Let $\text{Cros}(R, M)$ and $\text{Cros}_K(R, M)$ be the set of components of the category of crossed extensions and the category of $\mathbb{K}$-split crossed extensions respectively. Then there is a canonical bijection:

$$(3) \quad H^3(R, M) \cong \text{Cros}_K(R, M)$$
(see for example Exercise E.1.5.1 of [29] or [4]). A similar isomorphism for group cohomology was proved by Loday [25], see also [30]. We recall only how to associate a 3-cocycle to a $\mathbb{K}$-split crossed extension:

$$0 \to M \to C_1 \xrightarrow{\partial} C_0 \xrightarrow{\pi} R \to 0$$

of $R$ by $M$. We put $V := \text{Im}(\partial)$ and consider $\mathbb{K}$-linear sections $p : R \to C_0$ and $q : V \to C_1$ of $\pi : C_0 \to R$ and $\partial : C_1 \to V$ respectively. Now we define $m : R \otimes R \to V$ by $m(r, s) := q(p(r)p(s) - p(rs))$. Finally we define $f : R \otimes R \otimes R \to M$ by

$$f(r, s, t) := p(r)m(s, t) - m(rs, t) + m(r, st) - m(r, s)p(t).$$

Then $(f, g, h) \in Z^3(R/\mathbb{K}, M)$ and the corresponding class in $H^3(R/\mathbb{K}, M)$ depends only on the connected component of a given crossed extension and in this way one gets the expected isomorphism (see [4]).

3.3. Obstruction theory. Now we explain a variant of the classical obstruction theory in terms of crossed extensions (compare with Sections IV.8 and IV.9 of [29]). Let

$$0 \to M \to C_1 \xrightarrow{\partial} C_0 \to R \to 0$$

be a crossed extension of $R$ by $M$. A $\partial$-extension of $C_1$ by $R$ is a commutative diagram with exact rows

$$0 \to C_1 \xrightarrow{\mu} S \xrightarrow{\xi} R \to 0$$

$$0 \to M \to C_1 \xrightarrow{\xi} C_0 \to R \to 0$$

where $S$ is a unital $\mathbb{K}$-algebra and $\xi$ is a homomorphism of unital $\mathbb{K}$-algebras. Furthermore the equalities $\mu(x)s = \mu(x\xi(s))$ and $s\mu(x) = \mu(\xi(s)x)$ hold, where $x \in C_1, s \in S$. It follows then that product in $C_1$ induced from $S$ coincides with the $*$-product: $x * y = \partial(x)y = x\partial(y)$. Moreover one has the exact sequence

$$0 \to M \xrightarrow{\mu} S \xrightarrow{\xi} C_0 \to 0.$$

It is clear that $\partial$-extensions of $C_1$ by $R$ form a groupoid, whose set of components will be denoted by $\mathcal{E}(\mathcal{K}(R, C_1))$.

Now we assume that $\partial$ is a $\mathbb{K}$-split crossed extension. A $\partial$-extension of $C_1$ by $R$ is called $\mathbb{K}$-split if $\xi$ is a $\mathbb{K}$-split epimorphism. Of course in this case $\xi$ is $\mathbb{K}$-split as well. We let $\mathcal{E}(\mathcal{K}(R, C_1))$ be the subset of $\mathcal{E}(\mathcal{K}(R, C_1))$ consisting of $\mathbb{K}$-split $\partial$-extensions.

Theorem 3.3.1. The class of a $\mathbb{K}$-split crossed extension

$$0 \to M \to C_1 \xrightarrow{\partial} C_0 \to R \to 0$$

is zero in $H^3(R, M)$ iff $\mathcal{E}(\mathcal{K}(R, C_1))$ is nonempty. If this is the case then the group $H^2(R, M)$ acts transitively and effectively on $\mathcal{E}(\mathcal{K}(R, C_1))$.

Proof. For a crossed extension $\partial$ one considers sections $p : R \to C_0$ and $q : V \to C_1, V = \text{Im}(\partial)$ as above. We may and we will assume that $p(1) = 1$. Then the class
of \( \partial \) in \( H^3 \) is given by the cocycle \( f(r, s, t) := p(r)m(s, t) - m(rs, t) + m(r, st) - m(r, s)p(t) \) where \( m(r, s) = \mu(p(r)s - p(rs)) \). Given a \( \partial \)-extension of \( C_1 \) by \( R \):

\[
\begin{array}{ccc}
0 & \xrightarrow{\mu} & C_1 \xrightarrow{\xi} S \xrightarrow{\iota} R \xrightarrow{\eta} 0 \\
0 & \xrightarrow{\text{Id}} & M & \xrightarrow{\partial} & C_1 \xrightarrow{\iota} C_0 \xrightarrow{\text{Id}} 0
\end{array}
\]

choose a \( \mathbb{K} \)-linear section \( v : C_0 \to S \) such that \( v(1) = 1 \). One puts \( u = vp : R \to S \). Then \( \zeta u = \text{Id}_R \). One defines \( n : R \otimes R \to C_1 \) by \( \mu(n(r, s)) = u(r)u(s) - u(rs) \). We claim that

\[
(4) \quad p(r)n(s, t) - n(rs, t) + n(r, st) - n(r, s)p(t) = 0
\]

Indeed,

\[
p(r)n(s, t) = u(r)n(s, t) = u(r)u(s)u(t) - u(r)u(st).
\]

Similarly \( n(s, t)p(t) = u(r)s)u(t) - u(rs)u(t) \).

Thus \( p(r)n(s, t) - n(rs, t) + n(r, st) - n(r, s)p(t) = u(r)u(s)u(t) - u(r)u(st) - u(rs)u(t) + u(r)s)u(t) + u(rs)u(t) = 0 \).

Since \( m(r, s) = g\partial n(r, s) \), it follows that \( g(r, s) = m(r, s) - n(r, s) \) lies in \( M \). Thus we obtain a well-defined linear map \( g : R \otimes R \to M \). Then it follows from the equation \( 4 \) that

\[
f(r, s, t) = rg(s, t) - g(rs, t) + g(r, st) - g(r, s)t,
\]

which shows that the class of \( \partial \) in \( H^3 \) is zero. Given any normalized 2-cocycle \( h : R \otimes R \to M \), one can define a new \( \partial \)-extension \( S_h \) of \( R \) by \( C_1 \) by putting \( S_h = C_1 \oplus R \) with the following multiplication:

\[
(x, r)(y, s) = (x \star y + p(r)y + xp(s) + n(r, s) + h(x, y), xy).
\]

This construction yields a transitive and effective action of \( H^3(R, M) \) on \( \partial \text{Ext}_{\mathbb{K}}(R, C_1) \).

Conversely, assume that the class of \( 0 \to M \to C_1 \xrightarrow{\partial} C_0 \to R \to 0 \) is zero in \( H^3(R, M) \). Thus there is a linear map \( g : R \otimes R \to M \) such that \( f(r, s, t) = rg(s, t) - g(rs, t) + g(r, st) - g(r, s)t \). One can define \( n : R \otimes R \to C_1 \) by \( n(r, s) = m(r, s) - g(r, s) \). Then \( p(r)n(s, t) - n(rs, t) + n(r, st) - n(r, s)p(t) = 0 \) and therefore \( S = R \oplus C_1 \) with the product \( (x, r)(y, s) = (x \star y + p(r)y + xp(s) + n(x, y), xy) \) defines a \( \partial \)-extension.

3.4. Abelian and crossed \( n \)-fold extensions. An abelian twofold extension of an algebra \( R \) by an \( R-R \)-bimodule \( M \) is an exact sequence

\[
0 \longrightarrow M \xrightarrow{\alpha} N \xrightarrow{\mu} S \xrightarrow{\pi} R \longrightarrow 0
\]

where \( N \) is a bimodule over \( R \) and \( \alpha \) is a bimodule homomorphism. Moreover, \( N \) is an associative algebra with unit and \( \pi \) is a homomorphism of algebras with unit, such that \( \text{Ker}(\pi) \) is a square zero ideal of \( S \). Furthermore, for any \( s \in S \) and \( n \in N \) one has

\[
\mu(n\pi(s)) = \mu(n)s, \quad \mu(\pi(s)n) = s\mu(n).
\]

We let \( \mathcal{E}^2(R, M) \) be the category of abelian twofold extensions of \( R \) by \( M \), whose connected components are denoted by \( \text{Extalg}^2(R, M) \). As usual we have also a \( \mathbb{K} \)-split variant \( \mathcal{E}^2_{\mathbb{K}}(R, M) \) of the category \( \mathcal{E}^2(R, M) \): Objects of \( \mathcal{E}^2_{\mathbb{K}}(R, M) \) are \( \mathbb{K} \)-split twofold abelian extensions (i.e. \( \alpha, \mu \) and \( \pi \) are \( \mathbb{K} \)-splits), and the morphisms
in $E^2_K(R, M)$ are $K$-splits, accordingly we let $\text{Extalg}^2_K(R, M)$ be the connected components of $E^2_K(R, M)$.

Let us note that for any abelian twofold extension

\[
0 \to M \xrightarrow{\alpha} N \xrightarrow{\mu} S \xrightarrow{\pi} R \to 0
\]

the morphism $\mu : N \to S$ is a crossed bimodule, where the action of $S$ on $N$ is given via $\pi$. It is clear that the induced $\ast$-product on $N$ is trivial. Thus one obtains the functor $\delta^2(R, M) \to \text{Crosseq}(R, M)$, which takes the subcategory $E^2_K(R, M)$ to the category $\text{Crosseq}_K(R, M)$.

**Lemma 3.4.1.** The natural map $\text{Extalg}^2_K(R, M) \to \text{Crosseq}_K(R, M)$ is a bijection and therefore

\[
\text{Extalg}^2_K(R, M) \cong H^3(R, M)
\]

**Proof.** We just construct the inverse map $\xi : H^3(R, M) \to \text{Extalg}^2_K(R, M)$.

Consider the $K$-split short exact sequence

\[
0 \to M \to \text{Hom}(R, M) \to N \to 0
\]

and the corresponding isomorphisms (1), (2)

\[
H^3(R, M) \cong H^2(R, N) \cong \text{Extalg}(R, N).
\]

Take now an element $a \in H^3(R, M)$. It corresponds under these isomorphisms to an abelian extension $0 \to N \to S \to R \to 0$. By gluing it with $0 \to M \to \text{Hom}(R, M) \to N \to 0$ one obtains an abelian twofold extension

\[
0 \to M \to \text{Hom}(R, M) \to S \to R \to 0
\]

In this way one obtains the expected map $\xi$.

It is clear now how to introduce the notion of abelian $n$-fold extension for all $n \geq 2$ and get the same sort of isomorphism in higher dimensions.

Following the earlier work of Huebschmann [20], recently Baues and Minian [4] obtained another interpretation of Hochschild cohomology in dimensions $\geq 4$. They introduced the notion of crossed $n$-fold extension and proved that $n$-fold extensions classify $(n + 1)$-dimensional Hochschild cohomology for all $n \geq 2$. For $n = 2$ this is an isomorphism [3]. Here we give a sketch how to deduce the case $n > 2$ from the case $n = 2$ and from the classical results of Yoneda [40]. This argument gives also a new proof of Lemma 3.4.1.

Let $T$ be an additive functor from the category of bimodules over $R$ to the category of $K$-modules. Objects of the category $\delta^n(T)$ are pairs $(E, x)$, where

\[
0 \to M \to E_1 \to \cdots \to E_n \to 0
\]

is a $n$-extension of $E_n$ by $M$ in the category of $R$-$R$-bimodules and $x \in T(E_n)$. Morphisms in $\delta^n(T)$ are defined in an obvious way. Let $E^n(T)$ be the set of components of the category $\delta^n(T)$. A result of Yoneda asserts that one has a natural isomorphism:

\[
E^n(T) \cong S^nT(M)
\]

where $S^nT$ is the $n$-th satellite of $T$ [10].

Comparing with the definition of abelian twofold extension we see that

\[
\text{Exalg}^2(R, M) \cong E^1(T)
\]
where $T = \text{Extalg}(R, -)$. To show how to deduce Lemma 3.4.1 from the Yoneda isomorphism, we consider the case when $K$ is a field. Since $T \cong H^3(R, -)$ the result of Yoneda yields

$$\text{Extalg}^2(R, M) \cong S^1T(M) \cong H^3(R, M).$$

This argument works also for general $K$: we have to use a straightforward generalization of the Yoneda isomorphism in the framework of relative homological algebra.

Let $n \geq 2$. A crossed $n$-fold extension of $R$ by $M$ [4] is an exact sequence

$$0 \rightarrow M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} A \xrightarrow{\pi} R \rightarrow 0$$

of $K$-modules with the following properties:

i) $(M_1, R, \partial_1)$ is a crossed bimodule with cokernel $R$;

ii) $M_i$ is a bimodule over $R$ for $1 < i \leq n - 1$ and $\partial_i$ and $f$ are maps of bimodules over $R$. Note that $\text{Ker}(\partial_1)$ is naturally a bimodule over $R$ and therefore it makes sense to require $\partial_2$ to be a map of bimodules over $R$. We let $\text{Cros}^n(R, M)$ denote the set of connected components of the category of crossed $n$-fold extensions of $R$ by $M$. Observe that

$$\text{Cros}^n(R, M) = E^{n-2}(T)$$

where $T = \text{Cros}(R, -)$.

Now, as in [4] for simplicity we assume that $K$ is a field. Theorem 4.3 of [4] claims that there is a natural isomorphism

$$\text{Cros}^n(R, M) \cong H^{n+1}(R, M)$$

For $n = 2$ this is the isomorphism (3) and for $n > 2$ it is an immediate corollary of Yoneda’s isomorphism:

$$E^{n-2}(T) = S^{n-2}T = S^{n-2}H^3(R, -)(M) = S^{n-2}S^3H^0(R, -)(M) \cong S^{n+1}H^0(R, -)(M) = H^{n+1}(R, M)$$

Here we used the isomorphism $T \cong H^3(R, -)$ and the classical fact that $H^n(R, M) = \text{Ext}_{R\otimes R^\text{op}}^n(R, M) = S^nH^0(R, -)(M)$ see [10]. For general $K$ one needs to work in the framework of relative homological algebra [29]. The corresponding class of proper exact sequences consists of $K$-split exact sequences. Then the corresponding results hold for arbitrary $K$.

As we can see the results in this section strongly depend on the vanishing of Hochschild cohomology on (relative) injective modules.

4. Shukla Cohomology

As we already saw the Hochschild cohomology in dimensions two and three classifies $K$-split abelian and crossed extensions respectively. However, there is a variant of Hochschild cohomology due to Shukla in the early 60-s which classifies all abelian and crossed extensions. We will present these results. Our approach to Shukla cohomology is based on chain algebras and especially on the possibility of extension of Hochschild cohomology to chain algebras. Actually there are two ways for such extension. First is a very naive: one replaces $\otimes$ and $\text{Hom}$ in the definition of Hochschild cohomology by the tensor product and $\text{Hom}$ of complexes to arrive at a cosimplicial cochain complex and then one takes the homology of
the total complex. However, this definition does not respect weak equivalences of chain algebras. The second definition (called derived Hochschild cohomology) is a kind of Quillen’s derivative of the Hochschild cohomology and uses the closed model category structure on the category of chain complexes introduced in the Appendix. Since the category of algebras is the full subcategory of the category of chain algebras, the derived Hochschild cohomology restricts to a cohomology theory of algebras, which is by definition the Shukla cohomology.

4.1. Hochschild cohomology for chain algebras. In this section we give a naive definition of the Hochschild cohomology for chain algebras.

Let us recall that a chain algebra is a graded algebra \( R = \bigoplus_{n \geq 0} R_n \) equipped with a differential \( d : R_n \to R_{n-1} \) satisfying the Leibniz identity:

\[
d(xy) = d(x)y + (-1)^n xd(y), \quad x \in A_n, y \in A_m.
\]

Let \( \text{DGA} \) be the category of chain algebras. A morphism of chain algebras is a weak equivalence if it induces isomorphism in homology.

An \( R_\ast \)-bimodule is a chain complex \( M_\ast \) of \( \mathbb{K} \)-modules, equipped with actions from both sides: \( R_\ast \otimes M_\ast \to M_\ast \) and \( M_\ast \otimes R_\ast \to M_\ast \), satisfying usual axioms. However, for our purposes we restrict ourselves to the case when \( M \) is concentrated in degree zero. In this case \( R_\ast \)-bimodule means simply a bimodule over \( H_0(R_\ast) \). In particular \( xm = 0 = mx \) as soon as \( m \in M \) and \( |x| \geq 1 \). For a chain algebra \( R_\ast \) and a \( H_0(R_\ast) \)-bimodule \( M \) we denote by \( C_\ast(R_\ast, M) \) the total complex of the following cosimplicial cochain complex. The \( n \)-th component of this cosimplicial object is the cochain complex

\[
C^n(R_\ast, M) := \text{Hom}(R_\ast^n, M).
\]

Here \( \otimes \) denotes the tensor product of chain complexes. The coface operations are given via Hochschild coboundary formula:

\[
d^0(f)(r_1, ..., r_{n+1}) = (-1)^{nk} r_1 f(r_2, ..., r_{n+1}), \quad f : R_\ast^n \to M, \quad r_1 \in R_k
\]

(actually this expression is zero provided \( k > 0 \))

\[
d^i(f)(r_1, ..., r_{n+1}) = f(r_1, ..., r_i r_{i+1}, ..., r_{n+1}), \quad 0 < i < n + 1
\]

\[
d^{n+1}(f)(r_1, ..., r_{n+1}) = f(r_1, ..., r_n r_{n+1}.
\]

The homology of \( C_\ast(R_\ast, M) \) is denoted by \( H^\ast(R_\ast, M) \).

The spectral sequences of a bicomplex in our situation have the following form:

\[
E^1_{pq} = H^p(\text{Hom}(R_\ast^p, M)) \Rightarrow H^{p+q}(R_\ast, M)
\]

\[
F^1_{pq} = H^p(|R_\ast|, M) \Rightarrow H^{p+q}(R_\ast, M)
\]

Here \(|R_\ast|\) denotes the underlying graded algebra of the chain algebra \((R_\ast, \partial)\).

**Lemma 4.1.1.** Let \( f : R_\ast \to S_\ast \) be a weak equivalence of chain algebras and let \( M \) be a bimodule over \( H_0(S) \). Then the induced homomorphism

\[
H^\ast(S_\ast, M) \to H^\ast(R_\ast, M)
\]

is an isomorphism provided \( R_\ast \) and \( S_\ast \) are projective \( \mathbb{K} \)-modules.
Thanks to the properties of closed model categories there exists a chain algebra to define the derived Hochschild cohomology as follows. Let $R$ be a quasi-free algebra, meaning that the underlying algebra structure is free, and let $M$ be an $H_0(R_*)$-bimodule. Then the Hochschild cohomology $H^n(A_*, M)$ is isomorphic to the $(n-1)$-st homology of the cochain complex $\text{Der}(|R_*|, M)$ provided $n > 0$.

**Proof.** This is a direct consequence of the spectral sequence related to the bicomplex $C^*(R_*, M)$ together with the fact that the Hochschild cohomology of a free algebra is zero in dimensions $> 1$.

We also recall the Künneth formula for Hochschild cohomology

**Lemma 4.2.2.** Let $\mathcal{K}$ be a field and let $R_*$ and $S_*$ be chain algebras. Assume that for each $n$, $R_n$ and $S_n$ are finite dimensional vector spaces. Then for any $R_*$-bimodule $M$ and $S_*$-bimodule $N$ one has the following isomorphism

$$H^n(R \otimes S, M \otimes N) \cong \bigoplus_{i+j=n} H^i(R, M) \otimes H^j(S, N)$$

**4.3. Derived Hochschild cohomology and Shukla cohomology.** In this section we use the closed model category structure on chain algebras described in the Appendix. Let us recall that weak equivalences in this model category are usual ones and a morphism of chain algebras is a fibration if it is surjective in all positive dimensions. We also need the fact that any cofibrant object is a retract of a quasi-free algebra. It follows from Lemma 4.1.1 that for any weak equivalence $f : R_* \to S_*$ of cofibrant chain algebras and any $H_0(S)$-bimodule $M$ the induced homomorphism $H^*(S_*, M) \to H^*(R_*, M)$ is an isomorphism. We can use this fact to define the derived Hochschild cohomology as follows. Let $R_*$ be a chain algebra. Thanks to the properties of closed model categories there exists a chain algebra morphism $f : R_* \to R_*$ which is a weak equivalence and $R_*$ is a cofibrant. For any $R$-bimodule $M$ the groups $H^*(R_*, M)$ do not depend on the cofibrant replacement
and they are called the derived Hochschild cohomology of \( R_* \) with coefficients in \( M \) and are denoted by \( H^*(R_*, M) \). Thus

\[
H^*(R_*, M) := H^*(R^c_*, M)
\]

This definition has expected functorial properties: for any morphism \( f : R_* \to S_* \) of chain algebras and any \( H_0(S) \)-bimodule \( M \) there is a well-defined homomorphism \( H^*(S_*, M) \to H^*(R_*, M) \) which depends only on the homotopy class of \( f \). Moreover it is an isomorphism provided \( f \) is a weak equivalence. One has also a natural homomorphism \( H^*(S_*, M) \to H^*(R_*, M) \) which is induced by the chain algebra homomorphism \( R^c_* \to R_* \). The following fact is a direct consequence of Lemma 4.1.4.

**Lemma 4.3.1.** If \( R_* \) is projective as a \( K \)-module then \( H^*(R_*, M) \to H^*(R_*, M) \) is an isomorphism.

Since the category of algebras is a full subcategory of the category of chain algebras we can consider the restriction of the derived Hochschild cohomology \( H^* \) on the category of algebras. The resulting theory is called the Shukla cohomology. Thus for any algebra \( R \) and any \( R \)-bimodule \( M \) the Shukla cohomology of an algebra \( R \) with coefficients in \( M \) is defined by

\[
\text{Shukla}^*(R, M) := H^*(R, M) \cong H^*(\text{Der}(R^c_*, M))
\]

where \( R^c_* \to R \) is a weak equivalence from a quasi-free chain algebra \( R^c_* \). The natural transformation

\[
H^n(R, M) \to \text{Shukla}^n(R, M), \quad n \geq 0
\]

is an isomorphism in dimensions \( n = 0, 1 \) and it is an isomorphism in all dimensions provided \( R \) is projective as a \( K \)-module. For example we have \( \text{Shukla}^i(A, -) = 0 \) provided \( A \) is a free algebra and \( i \geq 2 \).

The cup-product in Hochschild cohomology yields a (commutative graded) algebra structure on \( \text{Shukla}^*(A, A) \).

### 4.4. Shukla cohomology and extensions.

The following properties of Shukla cohomology are of special interests. They are non-\( K \)-split analogues of the isomorphisms (2) and (3).

**Theorem 4.4.1.** Let \( A \) be an associative algebra and let \( M \) be an \( A \)-bimodule. Then there are natural isomorphisms

\[
\text{Shukla}^2(A, M) \cong \text{Extalg}(A, M)
\]

\[
\text{Shukla}^3(A, M) \cong \text{Cros}(A, M).
\]

The first isomorphism is well known (see Theorem 4 of [39]). However we give an independent proof.

**Proof.** i) Let \( 0 \to M \to E \to A \to 0 \) (\( E \)) be a singular extension of algebras. Define the chain algebra \( E_* \) as follows:

\[
E_0 = E, \quad E_1 = M, \quad E_n = 0, \quad n \geq 2
\]

The only nontrivial boundary map is induced by the inclusion \( M \to E \). Then one has a map of chain algebras \( E_* \to A \) which is an acyclic fibration. Let \( A_* \to A \) be a weak equivalence with quasi-free \( A_* \). Since \( A_* \) is cofibrant there exists a lifting
Der 1-cocycle of is zero in Shukla group $g$ be a crossed extension. The algebra $C$ can be considered as a chain algebra as follows. In dimensions 0 and 1 it is already defined. In the dimension two one puts $E$ only on the isomorphism class of $(\cdot)$. The class of a crossed extension is non-split analogue of Theorem 3.3.1. Thus $h \in \text{Der}(A_0, M)$ and $f_1 = g_1 = \partial^*(h)$, which shows that the class $e(E)$ depends only on the isomorphism class of $(E)$. Conversely, if $f \in \text{Der}(A_0, M)$ is a 1-cocycle, then one can form an abelian extension according to the following diagram:

$$\cdots \rightarrow A_2 \rightarrow A_1 \rightarrow A_0 \rightarrow A \rightarrow 0$$

$$0 \rightarrow M \rightarrow E \rightarrow A \rightarrow 0.$$ 

In this way we obtain the isomorphism i).

ii) Let $0 \rightarrow M \rightarrow C_1 \overset{\partial}{\rightarrow} C_0 \rightarrow A \rightarrow 0$ be a crossed extension. The algebra $C_0$ acts on $M$ via the projection to $A$. Moreover

$$C_* = (\cdots \rightarrow 0 \rightarrow M \rightarrow C_1 \overset{\partial}{\rightarrow} C_0)$$

can be considered as a chain algebra as follows. In dimensions 0 and 1 it is already defined. In the dimension two one puts $C_2 = M$, and $C_i = 0$ for $i > 2$. The pairing $C_i \otimes C_j \rightarrow C_{i+j}$ is the given one if $i = 0$ or $j = 0$, while the pairing $C_1 \otimes C_1 \rightarrow C_2$ as well as all other pairings are zero. Then $C_* \rightarrow A$ is an acyclic fibration. Therefore we have a lifting $f_* : A_* \rightarrow C_*$, where $A_* \rightarrow A$ is a weak equivalence with quasi-free $A_*$. It is clear that $f_2 \in \text{Der}(A_0, M)$ is a 2-cocycle in $\text{Der}(A_0, M)$ and therefore gives rise to an element in $\text{Shukla}^3(A, M)$. Conversely, starting with a 2-cocycle $f \in \text{Der}(A_0, M)$ one can construct the corresponding crossed extension using the diagram

$$\cdots \rightarrow A_3 \rightarrow A_2 \rightarrow A_1 \rightarrow A_0 \rightarrow A \rightarrow 0$$

$$0 \rightarrow M \rightarrow C_1 \rightarrow C_0 \rightarrow A \rightarrow 0.$$ 

The following theorem is the non-$K$-split analogue of Theorem

\textbf{Theorem 4.4.2.} The class of a crossed extension

$$(\partial) \quad 0 \rightarrow M \rightarrow C_1 \overset{\partial}{\rightarrow} C_0 \overset{\pi}{\rightarrow} R \rightarrow 0$$

is zero in $\text{Shukla}^3(R, M)$ iff $\partial \text{Ext}(R, C_1)$ is nonempty. If this is the case then the group $\text{Shukla}^3(R, M)$ acts transitively and effectively on $\partial \text{Ext}(R, C_1)$.

\textbf{Proof.} It is clear that the crossed extension

$$0 \rightarrow M \overset{\text{Id}}{\rightarrow} M \overset{0}{\rightarrow} R \overset{\text{Id}}{\rightarrow} R \rightarrow 0$$

represents the zero element of $\text{Cros}(R, M)$. Assume $\partial \text{Ext}(R, C_1)$ is nonempty and let

$$0 \rightarrow C_1 \overset{\mu}{\rightarrow} S \overset{\xi}{\rightarrow} R \rightarrow 0$$

$$0 \rightarrow M \overset{\text{Id}}{\rightarrow} C_1 \overset{\partial}{\rightarrow} C_0 \overset{\pi}{\rightarrow} R \rightarrow 0$$
be an object of the category $\partial \text{Ext}(R, C_1)$. Then $d : M \oplus C_1 \to S$ is a crossed bimodule, where $d(m, c_1) = \mu(c_1)$ and the action of $S$ on $M \oplus C_1$ is given by $s(m, c_1) = (\xi(s)m, \xi(s)c_1)$ and $(m, c_1)s = (m\xi(s), c_1\xi(s))$. Then one has the following commutative diagram in $\text{Cross}(R, M)$:

\[
\begin{array}{c}
0 \rightarrow M \xrightarrow{\text{Id}} M \xrightarrow{\text{Id}} R \xrightarrow{\xi} \rightarrow 0 \\
0 \rightarrow M \xrightarrow{i_1} M \oplus C_1 \xrightarrow{\mu} S \xrightarrow{\varsigma} R \xrightarrow{\xi} \rightarrow 0 \\
0 \rightarrow M \xrightarrow{\text{Id}} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\pi} \rightarrow 0
\end{array}
\]

which shows that the class of $(\partial)$ in $\text{Shukla}^3(R, M)$ is zero. Here $p_1$ and $p_2$ are standard projections from the direct sums to summands and $i_1$ and $i_2$ are corresponding injections.

Conversely, assume the class of $(\partial)$ in $\text{Shukla}^3(R, M)$ is zero. It follows from Corollary 3.2.2 that there exists a commutative diagram of crossed extensions:

\[
\begin{array}{c}
0 \rightarrow M \xrightarrow{\text{Id}} M \xrightarrow{\text{Id}} R \xrightarrow{\xi} \rightarrow 0 \\
0 \rightarrow M \xrightarrow{i} M \oplus C_1 \xrightarrow{\mu} P_0 \xrightarrow{\varsigma} R \xrightarrow{\xi} \rightarrow 0 \\
0 \rightarrow M \xrightarrow{\text{Id}} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\pi} \rightarrow 0
\end{array}
\]

It follows that the restriction of $\mu$ to $\text{Ker}(p)$ is a monomorphism and therefore we have the following commutative diagram with exact rows:

\[
\begin{array}{c}
0 \rightarrow \text{Ker}(p) \xrightarrow{\mu} P_0 \xrightarrow{\varsigma} R \xrightarrow{\xi} \rightarrow 0 \\
0 \rightarrow M \xrightarrow{\epsilon} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\pi} \rightarrow 0
\end{array}
\]

One defines the $\mathbb{K}$-algebra $S$ via the exact sequence

\[
\begin{array}{c}
0 \rightarrow \text{Ker}(p) \xrightarrow{\epsilon, \mu} C_1 \oplus P_0 \rightarrow S \rightarrow 0.
\end{array}
\]

Here the product on $C_1 \oplus P_0$ is given by

$$(c, x)(c', x') := (c * c' + c\xi(x') + \xi(x)c', xx').$$

One easily checks that $\text{Ker}(p)$ is really an ideal of $C_1 \oplus P_0$ and therefore $S$ is well-defined. Now it is clear that

\[
\begin{array}{c}
0 \rightarrow C_1 \xrightarrow{\text{Id}} S \xrightarrow{\text{Id}} R \xrightarrow{\xi} \rightarrow 0 \\
0 \rightarrow M \xrightarrow{\text{Id}} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\pi} \rightarrow 0
\end{array}
\]

is an object of $\partial \text{Ext}(R, C_1)$ and the proof is finished.
Remark. One cannot get non-$K$-split versions of results of Section 3.4. In other words for $n > 2$ neither $\text{Extalg}^n(R, M)$ nor $\text{Cros}^n(R, M)$ are isomorphic to $\text{Shukla}^{n+1}(R, M)$ in general. This is because for such $n$ both groups $\text{Extalg}^n(R, M)$ and $\text{Cros}^n(R, M)$ vanish on injective bimodules, while Shukla cohomology does not. Indeed, if $K = \mathbb{Z}$ and $R = \mathbb{F}_p$, then any bimodule over $R$ is injective, while the computation in Section 4.4.1 shows that $\text{Shukla}^{2i}(\mathbb{F}_p/\mathbb{Z}, \mathbb{F}_p) = \mathbb{F}_p$ for all $i$. By the same reason the groups $\text{Extalg}^2(R, M)$ and $\text{Shukla}^2(R, M)$ are different.

On the other hand we have another interpretation of the higher Shukla cohomology using chain algebras. Indeed, the above argument can be easily modified to get the following extension of Theorem 4.4.1 to higher dimensions. A chain algebra $A_n$ is called of length $\leq n$ if $A_i = 0$ for all $i > n$. Let $R$ be an algebra and $M$ be a bimodule over $R$. For any $n \geq 1$ we let $\text{Cros}^n(R, M)$ be the category of triples $(A_n, \alpha, \beta)$ where $A_n$ is a chain algebra of length $\leq n$ with property $H_i(A_n) = 0$ for all $0 < i < n$. Moreover $\alpha : H_0(A_n) \to R$ is an isomorphism of algebras and $\beta : M \to H_n(A_n)$ is an isomorphism of $R$-bimodules, where the $R$-bimodule structure on $H_n(A_n)$ is induced via $\alpha^{-1}$. It is clear that for $n = 1$ the category $\text{Cros}^1(R, M)$ and $\text{Cros}^1(R, M)$ are equivalent. The argument given in the proof of part ii) of Theorem 4.4.1 shows that connected components of the category $\text{Cros}^n(R, M)$ are in one-to-one correspondence with elements of the group $\text{Shukla}^{n+1}(R, M)$. Furthermore, for a given object $X = (A_n, \alpha, \beta)$ of the category $\text{Cros}^n(R, M)$ one can define the category $\mathcal{C}(\text{Ext}(R; A_n))$ of objects $(C_n, \gamma, \eta)$, where $C_n$ is a chain algebra of length $\leq n$ with the property $H_i(C_n) = 0$ for all $i > 0$, $\gamma : H_0(C_n) \to R$ is an isomorphism of algebras and $\eta : C_n \to A_n$ is a chain algebra homomorphism such that the diagram

\[
\begin{array}{ccc}
H_0(C_n) & \xrightarrow{\gamma} & R \\
\downarrow{\eta} & & \downarrow{\text{id}} \\
H_0(A_n) & \xrightarrow{\alpha} & R
\end{array}
\]

commutes and $\eta_n : C_n \to A_n$ is an isomorphism. Then the category $\mathcal{C}(\text{Ext}(R; A_n))$ is nonempty iff the class of $X$ in $\text{Shukla}^{n+1}(R, M)$ is zero. If this is so, then the group $\text{Shukla}^{n+1}(R, M)$ acts transitively and effectively on the set of components of the category $\mathcal{C}(\text{Ext}(R; A_n))$.

Duskin in [13] introduced higher torsors to obtain an interpretation of elements of the cohomology groups in very general context. For associative algebras his approach also gives the interpretation of $H^3$ via crossed bimodules, but in higher dimensions his approach is totally different from one indicated here.

4.5. Shukla cohomology via free crossed bimodules. Let $R$ be an associative algebra. We claim that there is a free crossed module $\partial : F_1 \to F_0$ with $\text{Coker}(\partial) = R$. Indeed, first we take a surjective homomorphism of rings $\pi : F_0 \to R$, where $F_0$ is a free $\mathbb{K}$-algebra. Then we choose a free $\mathbb{K}$-module $V$ together with an epimorphism $V \to \text{Ker}(\pi)$. Finally we take $\partial : F_1 \to F_0$ to be the free crossed bimodule generated by $V \to F_0$. Then $\partial$ has the expected property.
Proposition 4.5.1. Let \( R \) be an associative algebra and let \( M \) be a bimodule over \( R \). Let

\[(F) \quad 0 \to E \xrightarrow{j} F_1 \xrightarrow{\partial} F_0 \to R \to 0\]

be a crossed extension with free crossed bimodule \( \partial : F_1 \to F_0 \). Then there is an exact sequence

\[
\text{Hom}_{F_0}(F_1, M) \xrightarrow{j^*} \text{Hom}_{R^e}(E, M) \to \text{Shukla}^3(R, M) \to 0
\]

where \( j^*(h) = hj \), for \( h \in \text{Hom}_{F_0}(F_1, M) \).

Proof. The crossed extension \((F)\) defines an element \( e \in H^3(R, E) \). The homomorphism \( e_* : \text{Hom}_{R^e}(E, M) \to \text{Shukla}^3(R, M) \) sends an element \( f \in \text{Hom}_{R^e}(E, M) \) to \( f_*(e) \in \text{Shukla}^3(R, M) \). Take any crossed extension

\[
0 \to M \to C_1 \to C_0 \to R \to 0
\]

Since \( F_0 \) is a free algebra and \( \partial : F_1 \to F_0 \) is a free crossed bimodule, there exists a morphism of crossed extensions

\[
0 \to E \xrightarrow{j} F_1 \xrightarrow{\partial} F_0 \xrightarrow{\text{Id}} R \xrightarrow{\text{Id}} 0
\]

which shows that \( e_* : \text{Hom}_{R^e}(E, M) \to \text{Shukla}^3(R, M) \) is an epimorphism. We claim that \( j_*(e) = 0 \). Indeed, \( j_*(e) \) is represented by the bottom crossed extension in the following diagram:

\[
0 \to E \xrightarrow{j} F_1 \xrightarrow{\partial} F_0 \xrightarrow{\text{Id}} R \xrightarrow{\text{Id}} 0
\]

Obviously \( F_1 \to X \) has a retraction, hence the claim. Take any \( h \in \text{Hom}_{F_0}(F_1, M) \). Then we have

\[
e_* j^*(h) = (hj)_*(e) = h_* j_*(e) = 0
\]

Thus it remains to show that if \( f \in \text{Hom}_{R^e}(E, M) \) satisfies \( f_*(e) = 0 \), then \( f = hj \) for some \( h \in \text{Hom}_{F_0}(F_1, M) \). If \( f_*(e) = 0 \), then we can use Theorem 4.4.2 to obtain a diagram

\[
0 \to E \xrightarrow{j} F_1 \xrightarrow{\partial} F_0 \xrightarrow{\pi} R \xrightarrow{0} 0
\]

Since \( F_0 \) is a free \( K \)-algebra, the homomorphism \( t \) has a section \( s : F_0 \to S \). So we have \( ts = \text{Id}_{F_0} \). Since \( p = \pi t \), we obtain \( ps = \pi ts = \pi \). It follows that
\[ \text{ps} \partial = \pi \partial = 0, \] thus there exists a unique \( r : F_1 \to C \) such that \( s \partial = ir \). Then we have \( irj = s \partial j = 0 \) and therefore \( rj = 0 \). On the other hand

\[ \delta(g - r) = \delta g - \delta r = \partial - tir = \partial - ts \partial = 0. \]

Therefore there exists a unique \( h : F_1 \to M \) such that \( g = r + j'h \). Since \( j'f = gj = rj + j'hj = j'hj \) we obtain \( f = hj \) and we are done.

5. Some computations of Shukla cohomology

5.1. The case \( \mathbb{K} = \mathbb{Z} \). Let \( \mathbb{K} = \mathbb{Z} \) and \( R = \mathbb{Z}/n\mathbb{Z}, \ n \geq 2 \). Consider the exterior algebra \( \Lambda^*_x(x) \) on a generator \( x \) of degree 1 over \( \mathbb{Z} \). We put \( \partial(x) = n \). Then \( \Lambda^*_x(x) \) is a chain algebra, which is weakly equivalent to \( \mathbb{Z}/n\mathbb{Z} \). It is clear that the normalized Hochschild cochain complex of \( \Lambda^*_x(x) \) with coefficients in \( \mathbb{Z}/n\mathbb{Z} \) has a bicomplex structure, which is \( \mathbb{Z}/n\mathbb{Z} \) in bidegree \((i,i), i \geq 0\) and is zero elsewhere. Thus

\[ \text{Shukla}^*(R/\mathbb{Z}, R) = R[\xi], \]

where

\[ \xi \in \text{Shukla}^2(R/\mathbb{Z}, R) \]

has degree 2. Based on the interpretation of the second Shukla cohomology via abelian extensions (see Section 4.4) one easily sees that \( \xi \) represents the following extension:

\[ \xi = (0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0) \in \text{Extalg}_{\mathbb{Z}}(R, R) \]

This example can be generalized as follows. Let \( A \) be an algebra over \( R = \mathbb{Z}/n\mathbb{Z} \). We will assume that \( A \) is free as a module over \( \mathbb{Z}/n\mathbb{Z} \) (of course this holds automatically if \( n = p \) is a prime). A ring \( A_0 \) is called a lifting of \( A \) to \( \mathbb{Z} \) if there exists an isomorphism of rings \( A_0/nA_0 \cong A \) and additionally \( A_0 \) is free as an abelian group.

**Proposition 5.1.1.** Let \( A \) be an algebra over \( R = \mathbb{Z}/n\mathbb{Z} \), which is free as an \( R \)-module. If \( A \) has a lifting to \( \mathbb{Z} \) then

\[ \text{Shukla}^*(A/\mathbb{Z}, A) \cong H^*(A/R, A)[\xi] \]

**Proof:** Let \( A_\ast \) be a chain algebra over \( \mathbb{Z} \) defined as follows. As a graded algebra \( A_\ast \) is the tensor product \( A_\ast = \Lambda^*_x(x) \otimes A_0 \) where \( x \) has degree one. The boundary homomorphism is defined by \( \partial(x) = n \). Thus as a chain complex \( A_\ast \) looks as follows:

\[ \cdots \to 0 \to A_0 \xrightarrow{\partial} A_0 \]

in particular \( A_\ast \to A \) is a weak equivalence and the Künneth Theorem for Hochschild cohomology implies

\[ \text{Shukla}^*(A/\mathbb{Z}, A) \cong H^*(A/R, A) \otimes \text{Shukla}^*(R/\mathbb{Z}, R) \cong H^*(A/R, A)[\xi] \]

5.2. The case \( \mathbb{K} = \mathbb{Z}/p^2\mathbb{Z} \). Let \( p \) be a prime and \( \mathbb{K} = \mathbb{Z}/p^2\mathbb{Z} \) and \( R = \mathbb{Z}/p\mathbb{Z} \). Consider the commutative chain algebra

\[ \Lambda^*_{\mathbb{Z}/p^2\mathbb{Z}}(x) \otimes \Gamma^*_{\mathbb{Z}/p^2\mathbb{Z}}(y), \]

where \( x \) is of degree 1 and \( y \) is of degree 2. Here \( \Gamma^* \) denotes the divided power algebra. Now we put \( \partial(x) = p \) and \( \partial(y) = px \). One easily checks that in this way one obtains a chain algebra compatible with divided powers. Since the augmentation

\[ \Lambda^*_{\mathbb{Z}/p^2\mathbb{Z}}(x) \otimes \Gamma^*_{\mathbb{Z}/p^2\mathbb{Z}}(y) \to \mathbb{Z}/p\mathbb{Z} \]
Let Proposition 5.2.1. truncated polynomial algebras have lifting to \( \mathbb{Z} \). More generally, let \( \{x\} \) be a finite cartesian product. Here we use the fact that one has an isomorphism of algebras:

\[
\sigma_y \cdot \sigma_z = \mathbb{Z} \otimes \mathbb{Z} \]

where \( \Lambda^* \) is represented by the crossed extension of algebras:

\[
\mathbb{Z} \otimes \mathbb{Z} \]

Shukla \( R/\mathbb{K}, R \) is an isomorphism of algebras:

\[
\mathbb{Z} \otimes \mathbb{Z} \]

where \( \mathbb{Z} \) is represented by the following abelian extension of algebras

\[
\mathbb{Z} \otimes \mathbb{Z} \]

while \( \sigma_y \) is represented by the crossed extension of algebras:

\[
\mathbb{Z} \otimes \mathbb{Z} \]

More generally, let \( A \) be an algebra over \( \mathbb{Z}/p\mathbb{Z} \). A ring \( A_0 \) is called a lifting of \( A \) to \( \mathbb{Z}/p\mathbb{Z} \) if there exists an isomorphism of algebras \( A_0/pA_0 \cong A \) and additionally \( A_0 \) is free as a \( \mathbb{Z}/p\mathbb{Z} \)-module.

**Proposition 5.2.1.** Let \( A \) be an algebra over \( \mathbb{F}_p \). If \( A \) has a lifting to \( \mathbb{K} = \mathbb{Z}/p^2\mathbb{Z} \), then

\[
\text{Shukla}^* (A/\mathbb{K}, A) \cong H^* (A/\mathbb{F}_p, A) \otimes \text{Shukla}^* (\mathbb{F}_p/\mathbb{K}, \mathbb{F}_p)
\]

where

\[
\text{Shukla}^* (\mathbb{F}_2/\mathbb{K}, \mathbb{F}_2) \cong \mathbb{F}_2 [\sigma_2, \sigma_y, \sigma_y^{[2]}, \cdots \sigma_y^{[n]}, \cdots]
\]

and

\[
\text{Shukla}^* (\mathbb{F}_p/\mathbb{K}, \mathbb{F}_p) \cong \Lambda^* (\sigma_y, \cdots \sigma_y^{[n]}, \cdots) \otimes \mathbb{F}_p [\sigma_2, y, \cdots \sigma_y^{[n]} y^{[n]}, \cdots]
\]

if \( p \) odd.

**Proof.** Let \( A_* \) be a chain algebra over \( \mathbb{Z}/p^2 \) given as the tensor product of chain algebras:

\[
A_* = A_0 \otimes \Lambda^* \otimes \mathbb{Z}/p\mathbb{Z} \]

By the K"unneth theorem 4.2.2, \( A_* \rightarrow A \) is a weak equivalence and hence

\[
\text{Shukla}^* (A, A) \cong H^* (A, A) \otimes \text{Shukla}^* (\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})
\]

Let us observe that if a \( \mathbb{F}_p \)-algebra \( A \) has a lifting to \( \mathbb{Z} \) then it also has a lifting to \( \mathbb{Z}/p\mathbb{Z} \). It is clear that group algebras (or more generally monoid algebras), truncated polynomial algebras have lifting to \( \mathbb{Z} \). It is also known that any smooth commutative algebra has lifting to \( \mathbb{Z} \). It is also clear that the class of algebras having lifting to \( \mathbb{Z} \) (or \( \mathbb{Z}/p\mathbb{Z} \)) is closed under tensor product. It is also closed under finite cartesian products.
5.3. On relationship between Shukla cohomology over $\mathbb{Z}$ and $\mathbb{Z}/p^2\mathbb{Z}$ up to dimension three. In this section $\mathbb{K} = \mathbb{Z}/p^2\mathbb{Z}$ and $H^*$ denotes the Hochschild cohomology over $\mathbb{F}_p$.

Let $M$ be a bimodule over an $\mathbb{F}_p$-algebra $A$. Since $A$ is also an algebra over $\mathbb{Z}$ and $\mathbb{K} = \mathbb{Z}/p^2\mathbb{Z}$, we can consider not only the Hochschild cohomology $H^*(A, M)$, but also the Shukla cohomologies $\text{Shukla}^*(A/\mathbb{K}, M)$ and $\text{Shukla}^*(A/\mathbb{Z}, M)$. The ring homomorphisms $\mathbb{Z} \to \mathbb{K} \to \mathbb{F}_p$ yield the natural transformations

$$H^i(A, M) \to \text{Shukla}^i(A/\mathbb{K}, M)$$

and

$$b^i : \text{Shukla}^i(A/\mathbb{K}, M) \to \text{Shukla}^i(A/\mathbb{Z}, M)$$

which are obviously isomorphisms for $i = 0, 1$. For $i = 2$, the groups in question classify abelian extensions of $A$ by $M$, respectively in the category of algebras over $\mathbb{F}_p$, $\mathbb{K}$ and $\mathbb{Z}$. Let us observe that if $X \to Y \to Z$ is a short exact sequence of abelian groups and $pX = 0 = pZ$, then $p^2Y = 0$. Thus any abelian extension of $A$ by $M$ in the category of all rings lies in the category of algebras over $\mathbb{K}$. It follows that for $i = 2$, the first map $H^2(A, M) \to \text{Shukla}^2(A/\mathbb{K}, M)$ is a monomorphism, while the second homomorphism is an isomorphism:

$$b^2 : \text{Shukla}^2(A/\mathbb{K}, M) \cong \text{Shukla}^2(A/\mathbb{Z}, M)$$

In higher dimensions we have

**Lemma 5.3.1.** For all $n$ the homomorphism

$$b^n : \text{Shukla}^n(A/\mathbb{K}, M) \to \text{Shukla}^n(A/\mathbb{Z}, M)$$

is an epimorphism and it has a natural splitting.

**Proof.** We have only to consider the case $n \geq 3$. We have to construct the homomorphism $d^n : \text{Shukla}^n(A/\mathbb{Z}, M) \to \text{Shukla}^n(A/\mathbb{K}, M)$, which is a right inverse of $b^n$. We consider more carefully the case $n = 3$ and then we indicate how to modify the argument for $n > 3$. In terms of crossed extensions, $b = b^3 : \text{Shukla}^3(A/\mathbb{K}, M) \to \text{Shukla}^3(A/\mathbb{Z}, M)$ sends the class of a crossed extension

$$0 \to M \to C_1 \to C_0 \to A \to 0$$

of $\mathbb{Z}/p^2\mathbb{Z}$-algebras to the same crossed extension but now considered as algebras over $\mathbb{Z}$. Now we take any element from $\text{Shukla}^3(A/\mathbb{Z}, M)$, which is represented by the following crossed extension of $A$ by $M$ in the category of rings:

$$0 \to M \to D_1 \to D_0 \to A \to 0$$

Thanks to Lemma 3.2.1 and Corollary 3.2.2 without loss of generality one can assume that $D_0$ is free as an abelian group (this follows also from Section A.6). Thus $V := \text{Im}(\partial)$ is also free as an abelian group and $0 \to M \to D_1 \to V \to 0$ splits as a sequence of abelian groups. It follows that $0 \to M \to D/pD \to V/pV \to 0$ is exact. On the other hand $pV$ is a two-sided ideal in $D_0$ and therefore one has an exact sequence $0 \to V/pV \to D_0/pV \to A$. It follows that $D_0/pV$ is a $\mathbb{Z}/p^2\mathbb{Z}$-algebra. By gluing these data we get a crossed extension

$$0 \longrightarrow M \longrightarrow D_1/pD_1 \longrightarrow D_0/pV \longrightarrow A \longrightarrow 0$$
and therefore an element in Shukla\(^3\)(\(A/K, M\)). In this way we obtain the homomorphism \(d = d^3 : \text{Shukla}^3(A/Z, M) \to \text{Shukla}^3(A/K, M)\). The commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & M & \rightarrow & D_1 & \rightarrow & D_0 & \rightarrow & A & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & M & \rightarrow & D_1/pD_1 & \rightarrow & D_0/pV & \rightarrow & A & \rightarrow & 0
\end{array}
\]

shows that \(bd = \text{Id}\) and the case \(n = 3\) is done. Assume now \(n > 3\). According to Remark at the end of Section 4.4 we know that elements of \(\text{Shukla}^n(A/Z, M)\) are equivalence classes of chain algebras \(X_n\) of length \(\leq n - 2\) which are acyclic in all but the extreme dimensions:

\[0 \to M \to X_{n-2} \to \cdots \to X_0 \to A \to 0\]

Without loss of generality one can assume that \(X_0, \ldots, X_{n-3}\) are free as abelian groups (use Section A.4 or modify the argument in Lemma 3.2.1 and Corollary 3.2.2). By repeating the previous argument we can construct a diagram of the form

\[
\begin{array}{cccccc}
0 & \rightarrow & M & \rightarrow & X_{n-2} & \rightarrow & \cdots & \rightarrow & X_0 & \rightarrow & A & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & M & \rightarrow & X_{n-2}/pX_{n-2} & \rightarrow & \cdots & \rightarrow & X_0/pV & \rightarrow & A & \rightarrow & 0
\end{array}
\]

where \(V := \text{Ker}(X_0 \to A)\) and we are done.

Now we analyze the kernel of the homomorphism

\[b = b^3 : \text{Shukla}^3(A/K, M) \to \text{Shukla}^3(A/Z, M)\]

**Proposition 5.3.2.** Let \(A\) be an algebra over \(\mathbb{F}_p\) and let \(M\) be a bimodule over \(A\). Then one has a natural isomorphism

\[\text{Shukla}^3(A/K, M) \cong \text{Shukla}^3(A/Z, M) \oplus \mathbb{H}^0(A, M)\]

where \(\mathbb{K} = \mathbb{Z}/p^2\mathbb{Z}\).

**Proof** consists of several steps. We already defined the homomorphism \(d = d^3 : \text{Shukla}^3(A/Z) \to \text{Shukla}^3(A/K)\) with \(bd = \text{Id}\). Now we define the homomorphisms

\[e : \text{Shukla}^3(A/K) \to \mathbb{H}^0(A, M), \quad c : \mathbb{H}^0(A, M) \to \text{Shukla}^3(A/K)\]

with

\[ed = 0, \quad ec = \text{Id}, \quad bc = 0\]

and we prove that \((b, e) : \text{Shukla}^3(A/K) \to \mathbb{H}^0(A, M) \oplus \text{Shukla}^3(A/Z)\) is a monomorphism. From these assertions the result follows.

**First step.** The homomorphism \(e : \text{Shukla}^3(A/K, M) \to \mathbb{H}^0(A, M)\). Let

\[(\partial) \quad 0 \to M \to C_1 \xrightarrow{\partial} C_0 \xrightarrow{\pi} A \to 0\]

be a crossed extension, where \(C_0\) and \(C_1\) are \(\mathbb{K}\)-algebras. Since \(A\) is an algebra over \(\mathbb{F}_p\), one has \(\pi(p1) = 0\), where \(1 \in C_0\) is the unit of \(C_0\). Therefore one can write \(p1 = \partial([P])\) for a suitable \([P]\) in \(C_1\). Now we put:

\[e(\partial) = p[P] \in M\]

it is easy to check that \(e\) is a well-defined homomorphism. Let us observe that \(e(\partial) = 0\) if \(pC_1 = 0\). It follows that \(ed = 0\).
Second step. The canonical class $\langle \sigma \rangle_A \in \text{Shukla}^3(A/\mathbb{K}, A)$. Let $X$ be an abelian group. We let $\mathbb{Z}[X]$ be the free abelian group generated by $X$ modulo the relation $[0] = 0$. Here $[x]$ denotes an element of $\mathbb{Z}[X]$ corresponding to $x \in X$. Then we have a canonical epimorphism $\eta : \mathbb{Z}[X] \to X$, $\eta([x]) = 0$ which gives rise to the canonical free resolution of $X$:

$$0 \to R(X) \to \mathbb{Z}[X] \to X \to 0$$

For any $x, y \in X$ we put

$$(x, y) := [x] + [y] - [x + y] \in R(X)$$

We now assume that $pX = 0$, that is $X$ is a vector space over $\mathbb{F}_p$. By applying the functor $(-) \otimes \mathbb{Z}/p^2\mathbb{Z}$ to the canonical free resolution we obtain the following exact sequence

$$(\sigma)_X : 0 \to X \overset{i}{\to} R(X)/p^2R(X) \overset{\sigma}{\to} \mathbb{Z}/p^2\mathbb{Z}[X] \overset{\eta}{\to} X \to 0$$

Here we used the well-known isomorphism $V \cong \text{Tor}_1(V, \mathbb{Z}/p^2\mathbb{Z})$ for any $\mathbb{F}_p$-vector space $V$ considered as an abelian group (the $\text{Tor}$ and $\otimes$ are taken of course over $\mathbb{Z}$ and not over $\mathbb{K} = \mathbb{Z}/p^2\mathbb{Z}$). The homomorphism $i$ has the following form

$$i(x) = \sum_{j=0}^{p-1} p \langle jx, x \rangle \mod(p^2R(X))$$

Let us turn back to our situation. We can take $X = A$. The multiplicative structure on $A$ can be extended linearly to $\mathbb{Z}[A]$ to get an associative algebra structure on it. Then not only $\eta$ is a ring homomorphism, but the exact sequence $(\sigma)_A$ is a crossed extension and therefore we obtain an element

$$\langle \sigma \rangle_A = (0 \to A \to R(A)/p^2R(A) \overset{\sigma}{\to} \mathbb{Z}/p^2\mathbb{Z}[A] \to A \to 0) \in \text{Shukla}^3(A/\mathbb{K}, A)$$

It is clear that $A \to (\sigma)_A$ is a functor from $\mathbb{F}_p$-algebras to the category of crossed extensions of $\mathbb{Z}/p^2\mathbb{Z}$-algebras. Since

$$\sigma \left( \sum_{j=0}^{j=p-1} \langle j, 1 \rangle \right) = p[1]$$

one has

$$c(\langle \sigma \rangle_A) = 1 \in H^0(A, A) \subset A.$$  

On the other hand $p^2R(A)$ is an ideal of $\mathbb{Z}[A]$. Thus we have a commutative diagram

$$\begin{array}{ccc}
0 & \rightarrow & R(A)/p^2R(A) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathbb{Z}[X]/p^2R(A)
\end{array}$$

$$\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathbb{Z}/p^2\mathbb{Z}[A]
\end{array}$$

\begin{array}{ccc}
\downarrow & & \downarrow \\
A & \rightarrow & A
\end{array}$$

It follows from Theorem 13.2.2 that the class $\langle \sigma \rangle_A$ has the following important property:

$$0 = b(\langle \sigma \rangle_A) \in \text{Shukla}^3(A/\mathbb{Z}, A).$$

Third step. The homomorphism $c : H^0(A, M) \to \text{Shukla}^3(A/\mathbb{K}, M)$. Using the class $\langle \sigma \rangle_A$ we now define the homomorphism $c : H^0(A, M) \to \text{Shukla}^3(A/\mathbb{K}, M)$ by

$$c(m) = f^*_m(\langle \sigma \rangle_A).$$
Here \( m \in \text{H}^0(A, M) \) and \( f_m : A \to M \) is the unique bimodule homomorphism with \( f_m(1) = m \) and \( f_m' : \text{Shukla}^3(A/K, A) \to \text{Shukla}^3(A/K, M) \) is the induced homomorphism in cohomology. Since \( e \) and \( b \) are natural transformations of functors it follows that for any \( m \in M \) we have \( ec(m) = ef_m^*((\sigma)_A) = f_m^*e((\sigma)_A) = f_m(1) = m \) and \( bc(m) = bf_m^*((\sigma)_A) = f_m^*b((\sigma)_A) = 0 \). Thus \( ec = \text{Id} \) and \( bc = 0 \).

**Fourth step.** It remains to show that

\[
(b, e) : \text{Shukla}^3(A/K) \to \text{H}^0(A, M) \oplus \text{Shukla}^3(A/Z)
\]

is a monomorphism. Let

\[
0 \to M \to C_1 \to C_0 \to A \to 0
\]

be a crossed extension of \( \mathbb{Z}/p^2\mathbb{Z} \)-algebras which lies in \( \text{Ker}(b, e) \). Since it goes to zero in \( \text{Shukla}^3(A/Z, M) \) one has the following diagram:

\[
\begin{array}{cccccc}
0 & \to & C_1 & \mu & \to & S & \to & R & \to & 0 \\
\downarrow{\text{Id}} & & \downarrow{\xi} & & \downarrow{\text{Id}} & & \\
0 & \to & M & \to & C_1 & \to & C_0 & \to & R & \to & 0
\end{array}
\]

where \( S \) is a ring. Since \( \xi \) is a homomorphisms of algebras with unit we have \([P] = p1_S\), where \( 1_S \) is the unit of \( S \). Therefore \( e(\partial) = p^21_0\), because \( \partial \) goes also to zero under the map \( e \). It follows that \( S \) is an algebra over \( \mathbb{Z}/p^2\mathbb{Z} \). Theorem 4.4.2 shows that the class of \( 0 \to M \to C_1 \to C_0 \to A \to 0 \) in \( \text{Shukla}^3(A/K, M) \) is zero and the proof is finished.

6. A bicomplex computing Shukla cohomology

6.1. **Construction of a bicomplex.** In this section following [48] we construct a canonical bicomplex which computes the Shukla cohomology in the special case, when the ground ring \( K \) is an algebra over a field \( k \). In this section, contrary to other parts of the paper the tensor product \( \otimes \) denotes \( \otimes_k \) and not \( \otimes_K \). The same is for \( \text{Hom} \).

Let \( R \) be a \( K \)-algebra and let \( M \) be a bimodule over \( R \), where \( K \) is a commutative algebra over a field \( k \). Thus \( R \) is also an algebra over \( k \). We let \( C^*(R, M) \) be the Hochschild cochain complex of \( R \) considered as an algebra over \( k \). Similarly, we let \( C^*(R/K, M) \) be the Hochschild cochain complex of \( R \) considered as an algebra over \( K \). Accordingly \( H^*(R, M) \) and \( H^*(R/K, M) \) denotes the Hochschild cohomology of \( R \) with coefficients in \( M \) over \( k \) and \( K \) respectively.

We let \( K^{**}(K, R, M) \) be the following bicosimpicial vector space:

\[
K^{pq}(K, R, M) = \text{Hom}(K^p \otimes q \otimes R^q, M)
\]

The \( q \)-th horizontal cosimplicial vector space structure comes from the identification

\[
K^{*q}(K, R, M) = C^*(K^q, C^q(R, M)),
\]

where \( C^q(R, M) = \text{Hom}(R^q, M) \) is considered as a bimodule over \( K^q \) via

\[
((a_1, \ldots, a_q)f(b_1, \ldots, b_q))(r_1, \ldots, r_q) := a_1 \cdots a_qf(b_1r_1, \ldots, b_qr_q).
\]
Here $f \in \text{Hom}(R^q, M)$ and $a_i, b_j \in K, r_k \in R$. The $p$-th vertical cosimplicial vector space structure comes from the identification

$$K^{p*}(K, R, M) = C^*(K^{\otimes p} \otimes R, M)$$

where $M$ is considered as a bimodule over $K^{\otimes p} \otimes R$ via

$$(a_1 \otimes \cdots \otimes a_p \otimes r)m(b_1 \otimes \cdots \otimes b_p \otimes s) := (a_1 \cdots a_pr)(b_1 \cdots b_ps).$$

We allow ourselves to denote the corresponding bicomplex by $K^{**}(K, R, M)$ as well. Thus $K^{**}(K, R, M)$ looks as follows:

![Diagram](attachment:image.png)

Therefore for $f : K^\otimes pq \otimes R^\otimes q \to M$ the corresponding linear maps

$$d(f) : K^\otimes(p+1)q \otimes R^\otimes q \to M$$

and

$$\delta(f) : K^\otimes p(q+1) \otimes R^\otimes (q+1) \to M$$

are given by

$$df(a_{01}, \ldots, a_{0q}, a_{11}, \ldots, a_{1q}, \ldots, a_{pq}, r_1, \ldots, r_q) =$$

$$a_{01} \cdots a_{0q}f(a_{11}, \ldots, a_{1q}, \ldots, a_{pq}, r_1, \ldots, r_q) +$$

$$\sum_{0 \leq i < p} (-1)^{i+1}f(a_{01}, \ldots, a_{0q}, \ldots, a_{0i}a_{i+1,1}, \ldots, a_{iq}a_{i+1,q}, \ldots, a_{pq}, r_1, \ldots, r_q) +$$

$$(-1)^{p+1}f(a_{01}, \ldots, a_{0q}, \ldots, a_{p-1,1}, \ldots, a_{p-1,q}, a_{p1}r_1, \ldots, a_{pq}r_q).$$

and

$$\delta(f)(a_{10}, \ldots, a_{1q}, \ldots, a_{p0}, \ldots, a_{pq}, r_0, \ldots, r_q) =$$

$$(-1)^p a_{10} \cdots a_{p0}r_0f(a_{11}, \ldots, a_{1q}, a_{p1}, \ldots, a_{pq}, r_1, \ldots, r_q) +$$

$$\sum_{0 \leq i < q} (-1)^{i+p+1}f(a_{10}, \ldots, a_{1i}a_{i+1,1}, \ldots, a_{pi}a_{pi+1}, \ldots, a_{pq}, r_0, \ldots, r_ir_{i+1}, \ldots, r_q) +$$

$$(-1)^{q+p+1}f(a_{10}, \ldots, a_{1q-1}, \ldots, a_{pq}, \ldots, a_{p,q-1}, r_0, \ldots, r_{q-1}, a_{1q} \cdots a_{pq}r_q.$$
We let $H^*(\mathbb{K}, R, M)$ be the homology of the bicomplex $K^{**}(\mathbb{K}, R, M)$. We also consider the following subbicomplex $\tilde{K}^{**}(\mathbb{K}, R, M)$ of $K^{**}(\mathbb{K}, R, M)$:

\[
\begin{array}{ccccccccc}
M & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \cdots \\
\downarrow \delta & & & & & & \\
\text{Hom}(R, M) & \rightarrow & \text{Hom}(\mathbb{K} \otimes R, M) & \rightarrow & \text{Hom}(\mathbb{K} \otimes \mathbb{K} \otimes R, M) & \rightarrow & \cdots \\
\downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
\text{Hom}(R \otimes R, M) & \rightarrow & \text{Hom}(\mathbb{K} \otimes^2 \otimes R \otimes^2, M) & \rightarrow & \cdots \\
\downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
\vdots & & \vdots & & \vdots & & \\
\end{array}
\]

It is clear that $H^*(\mathbb{K}, R, M) \cong H^*(\tilde{K}^{**}(\mathbb{K}, R, M))$.

6.2. The homomorphism $\alpha$. It follows from the definition that

\[\text{Ker}(d : K^{*0} \rightarrow K^{*1}) \cong C^*(R/\mathbb{K}, M).\]

Therefore one has the canonical homomorphism

\[\alpha^n : H^n(R/\mathbb{K}, M) \rightarrow H^n(\mathbb{K}, R, M), \ n \geq 0.\]

**Theorem 6.2.1.** i) The homomorphisms $\alpha^0$ and $\alpha^1$ are isomorphisms. The homomorphism $\alpha^2$ is a monomorphism.

ii) If $R$ is projective over $\mathbb{K}$, then $\alpha^n : H^n(R/\mathbb{K}, M) \rightarrow H^n(\mathbb{K}, R, M)$ is an isomorphism for all $n \geq 0$.

iii) The groups $H^*(\mathbb{K}, R, M)$ are canonically isomorphic to Shukla$^*(R/\mathbb{K}, M)$.

**Proof:** i) is an immediate consequence of the definition of the bicomplex $\tilde{K}^{**}(\mathbb{K}, R, M)$. ii) The bicomplex gives rise to the following spectral sequence:

\[E_1^{pq} = H^q(\mathbb{K}^{\otimes p}, C^p(R, M)) \Rightarrow H^{p+q}(\mathbb{K}, R, M).\]

Let us recall that if $X$ and $Y$ are left modules over an associative algebra $S$, then $\text{Ext}_S^q(X, Y) \cong H^q(S, \text{Hom}(X, Y))$, where $\text{Hom}(X, Y)$ is considered as a bimodule over $S$ via $(sft)(x) = sf(tx)$. Here $x \in X$, $s, t \in S$ and $f : X \rightarrow Y$ is a lineal map. Having this isomorphism in mind, we can rewrite $E_1^{pq} \cong \text{Ext}_{\mathbb{K}^{\otimes q}}^q(R^{\otimes p}, M)$. By our assumptions $R^{\otimes p}$ is projective over $\mathbb{K}^{\otimes p}$. Therefore the spectral sequence degenerates and we get $H^*(\mathbb{K}, R, M) \cong H^*(C(R/\mathbb{K}, M)) = H^*(R/\mathbb{K}, M)$. Here we used the obvious isomorphism

\[\text{Hom}_{\mathbb{K}^{\otimes q}}(R \otimes R \otimes \cdots \otimes R, M) = \text{Hom}_{\mathbb{K}}(R \otimes_{\mathbb{K}} R \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} R, M).\]

iii) We let $\hat{K}^*(\mathbb{K}, R, M)$ denote the total cochain complex associated to the bicomplex $\tilde{K}^{**}(\mathbb{K}, R, M)$. Then this construction has an obvious extension to the category of chain $\mathbb{K}$-algebras. Unlike Lemma 4.1.1, for any weak equivalence $R_* \rightarrow S_*$ of chain $\mathbb{K}$-algebras the induced map $\hat{K}^*(\mathbb{K}, S_*, M) \rightarrow \hat{K}^*(\mathbb{K}, R_*, M)$ is a weak equivalence. This is because the definition of $\hat{K}^*(\mathbb{K}, R, M)$ involves the tensor products and hom’s over the field $k$ and not over $\mathbb{K}$. Furthermore, by ii) $H^*(\mathbb{K}, R_*, M)$ is isomorphic to the Hochschild cohomology, provided $R_*$ is degreewise projective over
there exists a linear map \( R \otimes M \) and \( g \) are linear maps and the equalities

\[ h_R : R \otimes R \to M \text{ is given by } (a, r) := (am + g(a, r), ar). \]

Similarly, we have \( H^3(\mathbb{K}, R, M) = Z^3(\mathbb{K}, R, M)/B^3(\mathbb{K}, R, M) \), where \( Z^3(\mathbb{K}, R, M) \) consists of triples \((f, g, h)\) such that \( f : R \otimes R \otimes R \to M \) and \( g : \mathbb{K} \otimes R \otimes R \to M \) are linear maps and the relations

\[
\begin{align*}
    rf(s, t) - f(r, st) & = 0 \\
    rf(s, t) - f(r, st) & = 0
\end{align*}
\]

hold. Here \( a, b \in \mathbb{K} \) and \( r, s, t \in R \). Moreover, \((f, g)\) belongs to \( B^2(A, R, M) \) iff there exists a linear map \( h : R \to M \) such that \( f(r, s) = rh(s) - h(rs) + h(r)s \) and \( g(a, r) = ah(r) - h(ar) \). Starting with \((f, g)\) in \( Z^2(\mathbb{K}, R, M) \) we construct an abelian extension of \( R \) by \( M \) by putting \( S = M \oplus R \) as a vector space. A \( \mathbb{K} \)-module structure on \( S \) is given by \( (m, r)(n, s) = (ms + rn + f(r, s), rs) \). Conversely, given an abelian extension

\[ 0 \to M \to S \to R \to 0 \]

we choose a \( k \)-linear section \( h : R \to S \) and then we put \( f(r, s) := h(r)h(s) - h(rs) \) and \( g(a, r) := ah(r) - h(ar) \). One easily checks that \((f, g)\) in \( Z^2(\mathbb{K}, R, M) \) and one gets i). Similarly, we have \( H^3(\mathbb{K}, R, M) = Z^3(\mathbb{K}, R, M)/B^3(\mathbb{K}, R, M) \). Here \( Z^3(\mathbb{K}, R, M) \) consists of triples \((f, g, h)\) such that \( f : R \otimes R \otimes R \to M \), \( g : \mathbb{K} \otimes R \otimes R \to M \) and \( h : \mathbb{K} \otimes \mathbb{K} \otimes R \to M \) are linear maps and the following hold:

\[
\begin{align*}
    r_1f(r_2, r_3, r_4) - f(r_1, r_2, r_3)_4 & = 0 \\
    abcf(r, s, t) - f(ar, bs, ct) & = 0 \\
    abg(c, d, x, y) & = 0 \\
    ah(b, c, x) & = 0
\end{align*}
\]

Moreover, \((f, g, h)\) belongs to \( B^3(\mathbb{K}, R, M) \) iff there exist linear maps \( m : R \otimes R \to M \) and \( n : \mathbb{K} \otimes R \to M \) such that

\[
\begin{align*}
    f(r, s, t) & = rm(s, t) - m(rs, t) + m(r, st) - m(r, s)t \\\n    g(a, b, r, s) & = am(b, r) - m(ab, rs) - am(b, s) + m(ab, rs) - m(a, x)bs \\\n    h(a, b, r) & = an(b, r) - n(ab, r) + n(a, br)
\end{align*}
\]

Let \( 0 \to M \to C_1 \xrightarrow{\partial} C_0 \xrightarrow{\varphi} R \to 0 \)
be a crossed extension. We put \( V := \text{Im}(\partial) \) and consider \( k \)-linear sections \( p : R \rightarrow C_0 \) and \( q : V \rightarrow C_1 \) of \( \pi : C_0 \rightarrow R \) and \( \partial : C_1 \rightarrow V \) respectively. Now we define \( m : R \otimes R \rightarrow V \) and \( n : A \otimes R \rightarrow V \) by \( m(r, s) := q(p(r)p(s) - p(rs)) \) and \( n(a, r) := q(ap(r) - p(ar)) \). Finally we define \( f : R \otimes R \otimes R \rightarrow M, g : \mathbb{K}^3 \otimes R \otimes R \rightarrow M \) and \( h : \mathbb{K} \otimes \mathbb{K} \otimes R \rightarrow M \) by

\[
\begin{align*}
  f(r, s, t) &:= p(r)m(s, t) - m(rs, t) + m(r, st) - m(r, s)p(t) \\
  g(a, b, r, s) &:= p(as)n(b, s) - n(ab, rs) + bn(a, x)p(y) - abn(r, s) + m(ax, by) \\
  h(a, b, r) &:= an(b, r) - n(ab, r) + n(a, bx).
\end{align*}
\]

Then \( (f, g, h) \in \mathbb{Z}^3(A, R, M) \) and the corresponding class in \( H^3(A, R, M) \) depends only on the connected component of a given crossed extension. Thus we obtain a well-defined map \( \text{Cros}(A, R, M) \rightarrow H^3(\mathbb{K}, R, M) \) and a standard argument (see [1]) shows that it is an isomorphism.

### 7. Applications to Mac Lane cohomology

In this section we are working with rings. So our ground ring is the ring of integers \( \mathbb{K} = \mathbb{Z} \).

#### 7.1. Eilenberg-MacLane \( Q \)-construction and Mac Lane cohomology

The definition of the Mac Lane cohomology \( Z \) of a ring \( R \) with coefficients in an \( R \)-bimodule \( M \) is based on the work of Eilenberg and Mac Lane on Eilenberg-Mac Lane spaces \( \text{13} \). Namely, for any abelian group \( A \) Eilenberg and Mac Lane constructed a chain complex \( Q_*(A) \) whose homology is the stable homology of Eilenberg-Mac Lane spaces

\[
H_q(Q_*(A)) \cong H_{n+q}(K(A, n)), \quad n > q.
\]

In low dimensions \( Q_*(A) \) is defined as follows \( \text{13} \), \( 27 \), \( 28 \), \( 26 \). The group \( Q_0(A) = \mathbb{Z}[A] \) is the free abelian group generated by elements \( [a] \), \( a \in A \) modulo the relation \([0] = 0\). The group \( Q_1(A) \) is the free abelian group generated by pairs \([a, b], a, b \in A \) modulo the relations \([a, 0] = 0 = [0, a], a \in A \), while the group \( Q_2(A) \) is the free abelian group generated by 4-tuples \([a, b, c, d] \) modulo the relations

\[
[a, b, 0, 0] = [0, 0, c, d] = [a, 0, c, 0] = [0, b, 0, d] = [a, 0, 0, d] = 0
\]

in general \( Q_n(A) \) is generated by \( 2^n \)-tuples modulo some relations \( \text{13} \), \( 27 \), \( 23 \). The boundary map is given by

\[
d[a, b] = [a] + [b] - [a + b] \\
d[a, b, c, d] = [a, b] + [c, d] - [a + c, b + d] - [a, c] - [b, d] + [a + b, c + d].
\]

For any \( a \in A \), the element \( \gamma(a) := [0, a, 0, 0] \in Q_2(A) \) is a two-dimensional cycle and \( \gamma \) yields an isomorphism (see \( \text{13} \), \( 27 \))

\[
\gamma : A/2A \cong H_2(Q_*(A)).
\]

Moreover for any abelian groups \( A \) and \( B \) there is a natural pairing

\[
Q_*(A) \otimes Q_*(B) \rightarrow Q_*(A \otimes B)
\]

(see for example \( 27 \), \( 23 \) or \( 26 \)). For any ring \( R \), this pairing allows us to put a chain algebra structure on \( Q_*(R) \). For example, in very low dimensions we have

\[
[x][y] = [xy], \quad [x][y, z] = [xy, xz], \quad [x, y][z] = [xz, yz], \\
[x][y, z, u, v] = [xy, xz, xu, xv], \quad [x, y, z, t][u] = [xu, yu, zu, tu]
\]
By definition the Mac Lane cohomology $HML^*(R, M)$ is defined as the Hochschild cohomology of $Q_*(R)$ with coefficients in $M$. One can also introduce the dual objects – Mac Lane homology. It was proved in [37] that Mac Lane homology is isomorphic to the topological Hochschild homology of Bökstedt [7]. It is also isomorphic to the stable $K$-theory thanks to a result of Dundas and McCarthy [12].

7.2. Relation with Shukla cohomology in low dimensions. Since $H_0(Q_*(R)) \cong R$ we have a natural augmentation $\epsilon : Q_*(R) \to R$. Since $Q_*(R)$ is free as an abelian group the chain algebra $V_*(R) = ( \cdots \to 0 \to \text{Ker}(\epsilon) \to Q_0(R) \to R) \to 0$ is $\mathbb{Z}$-free and $V_*(R) \to R$ is a weak equivalence. Hence $V_*(R)$ can be used to compute the Shukla cohomology. Thus the morphism of chain algebras

$\cdots \to Q_2(R) \to Q_1(R) \to Q_0(R) \to R \to 0$

$\downarrow \quad \downarrow \quad \downarrow$ $\quad \downarrow \quad \downarrow$

$0 \to \text{Ker}(\epsilon) \to Q_0(R) \to R \to 0$

yields the natural transformation

$$\text{Shukla}^j(R/\mathbb{Z}, M) \to HML^j(R, M)$$

which is an isomorphism in dimensions 0, 1 and 2. Thus $HML^2(R, M)$ classifies singular extensions of $R$ by $M$ in the category of rings, see also [27]. According to Theorem 9 of [28] in the dimension 3 one has the following exact sequence (see also Theorem 7.3.1)

$$0 \to \text{Shukla}^3(R/\mathbb{Z}, M) \to HML^3(R, M) \to H^0(R, 2M)$$

The connecting map $HML^3(R, M) \to H^0(R, 2M)$ is defined via $\gamma$ (see [28]).

**Proposition 7.2.1.** Let $R$ be an algebra over $\mathbb{F}_p$ and $M$ be an $R$-bimodule. Then the natural map

$$\text{Shukla}^3(R/\mathbb{Z}, M) \to HML^3(R, M)$$

is an isomorphism.

**Proof.** If $p \neq 2$ then this is an immediate consequence of the exact sequence (7), because $2M = 0$. So we have to consider only the case $p = 2$. For any $\mathbb{F}_2$-algebra $R$ we have the canonical homomorphism $\mathbb{F}_2 \to R$, which yields the following commutative diagram

$$0 \to \text{Shukla}^3(R/\mathbb{Z}, M) \to HML^3(R, M) \to H^0(R, M)$$

$$0 \to \text{Shukla}^3(\mathbb{F}_2/\mathbb{Z}, M) \to HML^3(\mathbb{F}_2, M) \to H^0(\mathbb{F}_2, M) = M$$

It is well known that $HML^3(\mathbb{F}_2, M) = 0$ see for example [16] or [7]. Since the last vertical arrow is a monomorphism we are done.

Based on Proposition 7.2.1 and Proposition 6.3.2 we obtain the following
Corollary 7.2.2. Let $A$ be an algebra over $\mathbb{F}_p$ and let $M$ be an $A$-bimodule. Then one has a split exact sequence

$$0 \rightarrow H^0(A,M) \rightarrow \text{Shukla}^3(A/\mathbb{K}, M) \rightarrow \text{HML}^3(A,M) \rightarrow 0$$

where $\mathbb{K} = \mathbb{Z}/p^2\mathbb{Z}$.

Remark. The homomorphism $\text{Shukla}^3(R/\mathbb{Z}, M) \rightarrow \text{HML}^3(R,M)$ in general is not an isomorphism. For example, if $R = \mathbb{Z}$, then $\text{Shukla}'(\mathbb{Z}, -) = 0$ for all $i \geq 1$, thanks to Lemma 7.3. On the other hand $\text{HML}^*(\mathbb{Z}, -)$ is quite nontrivial (see [7], [17]) and in particular $\text{HML}^3(\mathbb{Z}, \mathbb{F}_2) = \mathbb{F}_2$. More about $\text{HML}^*(\mathbb{Z}, -)$ see at the end of Section 7.3.

7.3. Relation with Shukla cohomology in higher dimensions. The relationship between Shukla cohomology $\text{Shukla}^*(A/\mathbb{Z}, M)$ and Mac Lane cohomology $\text{HML}^*(A,M)$ in higher dimensions is more complicated. Let us first consider the crucial case $A = \mathbb{Z}/p^k\mathbb{Z}$. We already saw $\text{Shukla}'(A/\mathbb{Z}, M), A = \mathbb{Z}/p^k\mathbb{Z}$, is $M$ if $i$ is even and is zero otherwise. Unlike the Shukla cohomology, the behavior of $\text{HML}^*(A,M)$ depends on whether $k = 1$ or $k > 1$. If $k = 1$, then similarly to Shukla cohomology the group $\text{HML}^1(A,M)$ is $M$ if $i$ is even and is zero otherwise. However the natural map

$$\text{Shukla}^i(\mathbb{F}_p/\mathbb{Z}, M) \rightarrow \text{HML}^i(\mathbb{F}_p, M)$$

is an isomorphism only for $i = 0, \cdots, 2p-1$, and it is zero for $i > 2p-2$. This follows from the fact that $\text{Shukla}^*(\mathbb{F}_p, \mathbb{F}_p)$ is a polynomial algebra on the generator $x$ of dimension two and $\text{HML}^*(\mathbb{F}_p, \mathbb{F}_p)$ is a divided power algebra on the same generator $x$ [10]. If $k > 1$, then situation with Mac Lane cohomology is more complicated. A computation made in [36] shows that

$$\text{HML}^{2n}(\mathbb{Z}/p^k\mathbb{Z}, \mathbb{F}_p) = (\mathbb{F}_p)^t, \quad \text{HML}^{2n-1}(\mathbb{Z}/p^k\mathbb{Z}, \mathbb{F}_p) = (\mathbb{F}_p)^s,$$

where $t = 1 + [\frac{n}{p}]$ and $s = [\frac{n+1}{p}]$. The full computation of $\text{HML}^i(\mathbb{Z}/p^k\mathbb{Z}, -)$ was obtained by Brun [9].

The relationship between Mac Lane cohomology and Shukla cohomology for general rings in all dimensions is given by the following theorem proved in [37] (see also [35]).

Theorem 7.3.1. Let $R$ be a ring. Then for any $R$-bimodule $M$ there is a spectral sequence

$$E_2^{pq}(\mathbb{K}) = \text{Shukla}^p(R/\mathbb{K}, \text{HML}^q(\mathbb{K}, M)) \Longrightarrow \text{HML}^{p+q}(R,M)$$

which is natural in $R$ and $M$. The spectral sequence in low dimensions gives rise to the exact sequence:

$$0 \rightarrow \text{Shukla}^3(R/\mathbb{Z}, M) \rightarrow \text{HML}^3(R,M) \rightarrow H^0(R,M) \rightarrow \text{Shukla}^4(R/\mathbb{Z}, M) \rightarrow \text{HML}^4(R,M).$$

For the proof of the first part we refer to [37] and [35]. The second part first was proved in [22]. It is an immediate consequence of the existence of the spectral sequence together with the following computation due to Bökstedt [7] (see also [10] and [17]).

$$\text{HML}^{2n}(\mathbb{Z}, M) = M/nM, \quad \text{HML}^{2n-1}(\mathbb{Z}, M) = nM, \quad n > 0.$$

$$\text{HML}^{2n}(\mathbb{F}_p, M) = M, \quad \text{HML}^{2n-1}(\mathbb{F}_p, M) = 0.$$
7.4. Mac Lane cohomology and cohomology of small categories. In this section we recall the relationship between Mac Lane cohomology and cohomology of small categories \[23\]. We assume that the reader is familiar with definition of cohomology of small categories with coefficients in a natural system \[2, 5\]. Let us recall that any bifunctor gives rise to a natural system, and therefore we can talk about the cohomology of small categories with coefficients in a bifunctor.

For a ring \( R \) we let \( R\text{-mod} \) be the category of finitely generated free \( R \)-modules. Actually we will assume that objects of \( R\text{-mod} \) are natural numbers and morphisms from \( n \) to \( m \) are the same as \( R \)-linear maps \( R^n \to R^m \), or \( m \times n \)-matrices over \( R \).

Let \( M \) be a bimodule over \( R \). There is a bifunctor \( \text{Hom}(-, M \otimes_R -) : R\text{-mod}^{\text{op}} \times R\text{-mod} \to \text{Ab} \) given by

\[
\text{Hom}(-, M \otimes_R -)(X, Y) = \text{Hom}_R(X, M \otimes_R Y).
\]

Therefore one can consider the cohomology \( H^\bullet(R\text{-mod}, \text{Hom}(-, M \otimes_R -)) \) of the category \( R\text{-mod} \) with coefficients \( \text{Hom}(-, M \otimes_R -) \) in the sense of Baues and Wirshing \[2\] (see also \[5\]). A result of \[23\] asserts that one has an isomorphism:

\[
H^\bullet_{\text{ML}}(R, M) \cong H^\bullet(R\text{-mod}, \text{Hom}(-, M \otimes_R -)).
\]

Comparing this isomorphism with the natural homomorphism \( \text{Shukla}^\bullet(R, M) \to H^\bullet(R\text{-mod}, \text{Hom}(-, M \otimes_R -)) \), \( \bullet \geq 0 \).

Now we recall the description of this homomorphism in terms of extensions for \( i = 2 \). Let

\[
0 \to M \xrightarrow{i} S \xrightarrow{p} R \to 0
\]

be an abelian extension of rings. Then

\[
0 \to \text{Hom}(-, M \otimes_R -) \to S\text{-mod} \xrightarrow{p} R\text{-mod} \to 0
\]

is a linear extension of categories \[2 5\], where the functor \( p_* \) is given by \( p_*(A) = A \otimes_S R \), \( A \in S\text{-mod} \) (having in mind the identification of \( R\text{-mod} \) as the category of natural numbers and matrices, the functor \( p_* \) is the identity on objects and is given by applying \( p \) on matrices). Let us recall that for fixed \( R \) and \( M \) the equivalence classes of abelian extensions of \( R \) by \( M \) form a group \( \text{Ext}_S(R, M) \), which is isomorphic to the second Shukla cohomology of \( R \) with coefficients in \( M \) (see Theorem \[14.1\]), while linear extensions are classified using the second cohomology of small categories \[2, 5\], thus we obtain the homomorphism

\[
\text{Shukla}^2(R, M) \to H^2(\text{-mod}(R), \text{Hom}(-, M \otimes_R -))
\]

which is an isomorphism according to isomorphisms \[5\] and \[6\]. One easily shows that any biadditive bifunctor \( D \) on \( R\text{-mod} \) is of the form \( D = \text{Hom}(-, M \otimes_R -) \), where \( M = D(R, R) \). Thus one can conclude that any extension of the category \( R\text{-mod} \) by a biadditive bifunctor is also of the form \( S\text{-mod} \), for some ring \( S \). In particular it is an additive category, more generally any linear extension of an additive category by biadditive functor is an additive category. This fact is an immediate consequence of Lemma 5.1.2 of \[5\].
8. Applications to strengthening of additive track theories

8.1. Additive and very strongly additive track theories. Let us recall that a track category $\mathcal{T}$ is a category enriched in groupoids $\mathcal{G}$. Thus $\mathcal{T}$ consists of objects and for each pair of objects $X, Y$ of $\mathcal{T}$ there is given the Hom-groupoid $[X,Y]$, whose objects are termed maps, while 2-arrows — tracks. To any track category $\mathcal{T}$ there is an associated category $\mathcal{T}_\approx$ with the same objects as $\mathcal{T}$, while for objects $A$ and $B$ of $\text{Ob}(\mathcal{T})$ the set of morphisms $[A,B]$ in $\mathcal{T}_\approx$ is the set of connected components of the groupoid $[X,Y]$.

A track category is abelian if for any 1-arrow $f : X \to Y$, the group $\text{Aut}(f)$ of tracks from $f$ to itself is abelian. Any abelian track category defines a natural system $D = D_\approx$ on $\mathcal{T}_\approx$ and a canonical class $\text{Ch}(\mathcal{T}) \in H^3(\mathcal{T}_\approx, D)$ — see section 2.3 of [5]. Conversely for any category $C$, any natural system $D$ on $C$ and any element $a \in H^3(C, D)$ there exists an abelian track category $\mathcal{T} = \mathcal{T}_{C,D,a}$ unique up to equivalence such that $\mathcal{T}_\approx = C$ and $\text{Ch}(\mathcal{T}) = a$ (see [5]). In fact for a given natural system $D$ on a category $C$ there is a category $\text{Trext}(C, D)$ whose objects are abelian track categories $\mathcal{T}$ with $\mathcal{T}_\approx = C$ and $D_\approx = D$ and the set of connected components of $\text{Trext}(C, D)$ is isomorphic to the third dimensional cohomology $H^3(C, D)$:

\begin{equation}
\pi_0(\text{Trext}(C, D)) \cong H^3(C, D)
\end{equation}

A lax coproduct $A \vee B$ in a track category $\mathcal{T}$ is an object $A \vee B$ equipped with maps $i_1 : A \to A \vee B$, $i_2 : B \to A \vee B$ such that the induced functor

$$(i_1^\ast, i_2^\ast) : [A \vee B, X] \to [A, X] \times [B, X]$$

is an equivalence of groupoids for all objects $X \in \mathcal{T}$. The coproduct is strong if the functor $(i_1^\ast, i_2^\ast)$ is an isomorphism of groupoids. By duality we have also the notion of lax product and strong product.

A lax zero object in a track category $\mathcal{T}$ is an object 0 such that the categories $[0, X]$ and $[X, 0]$ are equivalent to the trivial groupoid for all $X \in \mathcal{T}$. Let us recall that a trivial groupoid has only one object and one arrow. A strong zero object in a track category $\mathcal{T}$ is an object 0 such that the categories $[0, X]$ and $[X, 0]$ are trivial groupoids.

A theory is a category possessing finite products. A track theory (resp. strong track theory) is a track category $\mathcal{T}$ possessing finite lax products (resp. strong products) $\mathcal{G}$. If $\mathcal{T}$ is a track theory, then $\mathcal{T}_\approx$ is a theory. In this case the corresponding natural system on $\mathcal{T}_\approx$ is a so called cartesian natural system, meaning that it is compatible with finite product in an appropriate sense $\mathcal{G}$. Conversely, if $\mathcal{T}$ is a track category, with property that $\mathcal{T}_\approx$ is a theory and corresponding natural system is a cartesian natural system then $\mathcal{T}$ is a track theory $\mathcal{G}$.

Morphisms of track theories are enriched functors which are compatible with lax products. An equivalence of track theories is a track theory morphism which is a weak equivalence $\mathcal{G}$ and two track theories are called equivalent if they are made so by the smallest equivalence relation generated by these. Two track theories $\mathcal{T}$ and $\mathcal{T}'$ are equivalent iff there is an equivalence of categories $\mathcal{T}_\approx \cong \mathcal{T}'_\approx$ and after identification of these categories one should have $D_\approx = D_\approx'$ and $\text{Ch}(\mathcal{T}) = \text{Ch}(\mathcal{T}')$.

The main result of [5] is the so called strengthening theorem, which asserts that any abelian track theory is equivalent to a strong one.
An additive track theory is a track category $\mathcal{T}$ such that $\mathcal{T}_\omega$ is an additive category and the corresponding natural system is a biadditive bifunctor. We are going now to give an equivalent definition, but let us before that discuss the definition of an additive category. Let $\mathcal{A}$ be a category with zero object $0$ which possesses also finite coproducts and finite products. For objects $A$ and $B$ we have canonical inclusions $i_1 = (\text{Id}, 0) : A \to A \times B$ and $i_2 = (0, \text{Id}) : B \to A \times B$ and therefore also the canonical morphism $\kappa : A \vee B \to A \times B$. The category $\mathcal{A}$ is called semi-additive if the canonical morphism $\kappa : A \vee B \to A \times B$ is an isomorphism for all $A$ and $B$. If $\mathcal{A}$ is a semi-additive category and $f, g : A \to B$ are morphisms in $\mathcal{A}$, we let $f + g : A \to B$ be the following composite:

$$
\xymatrix{ A \ar[r]^\Delta & A \times A \\
\ar[r]^{(f,g)} & B \times B \\
\ar[r]^(0.3){\kappa^{-1}} & B \vee B \\
\ar[r]^\nabla & B
}
$$

where $\Delta = (\text{Id}, \text{Id})$ is the diagonal and $\nabla$ is the codiagonal. Thus in a semi-additive category hom’s are commutative monoids and the composition law is biadditive. If these monoids are abelian groups then a semi-additive category is called additive. This happens iff the identity morphism $\text{Id}_A$ admits the additive inverse $-\text{Id}_A$, for each object $A$.

Now we pass to the 2-world. Let $\mathcal{T}$ be a track theory with lax zero object. Then for any objects $A$ and $B$ of $\mathcal{T}$, there is a map $i_1 : A \to A \times B$ and tracks $p_{1i_1} \Rightarrow \text{Id}_A, p_{2i_1} \Rightarrow 0$. Similar meaning has $i_2 : B \to A \times B$. A semi-additive track category is an additive track theory with strong zero object, such that for any two objects $A$ and $B$ the lax product $A \times B$ is also lax coproduct via $i_1 : A \to A \times B$ and $i_2 : A \to A \times B$. It is clear that the homotopy category $\mathcal{T}_\omega$ of a semi-additive track theory is a semi-additive category.

One can prove that a track category $\mathcal{T}$ is an additive track theory iff it is a semi-additive track category and additionally the semi-additive category $\mathcal{T}_\omega$ is an additive category.

An additive track category is called very strong if it admits strong zero object $0$, strong finite products and for any two objects $A$ and $B$ the strong product $A \times B$ is also the strong coproduct by $i_1 : A \to A \times B$ and $i_2 : A \to A \times B$.

As we said a strengthening theorem of [5] asserts that any track theory is equivalent to a strong one. In particular, any additive track category is equivalent to one which possesses strong products. Since the dual of an additive track category is still a track category, we see that it is also equivalent to one which possesses strong coproducts. Can we always get strong products and coproducts simultaneously? In other words, is every additive track category $\mathcal{T}$ equivalent to a very strong one? We will see that the answer is negative in general, but positive provided the corresponding homotopy category $\mathcal{T}_\omega$ (which is an additive category in general) is $\mathbb{F}_2$-linear, or $2$ is invertible in $\mathcal{T}_\omega$ (meaning that all $\text{Hom}$’s are modules over $\mathbb{Z}[\frac{1}{2}]$). More precisely the following is true:

**Theorem 8.1.1.** Let $\mathcal{T}$ be a small additive track theory with the homotopy category $\mathcal{C} = \mathcal{T}_\omega$ and a canonical bifunctor $D = D_{\mathcal{T}}$. Let $2D$ be the two-torsion part of $D$. Then there is a well-defined element $o(\mathcal{T}) \in H^0(\mathcal{C}, 2D)$, which is nontrivial in general and such that $o(\mathcal{T}) = 0$ iff $\mathcal{T}$ is equivalent to a very strongly additive track theory. The class $o(\mathcal{T})$ is zero provided $\text{hom}$’s of the additive category $\mathcal{C}$ are modules either over $\mathbb{Z}[\frac{1}{2}]$) or over $\mathbb{F}_2$. 

The reader should compare Theorem 8.1.1 with the exact sequence (11) and Proposition 7.2.1. The similarity of these results is not accidental. Indeed, let us give a quick proof of the Theorem 8.1.1 in the key case when $C = R$-mod is the category of finitely generated free modules over a ring $R$.

The proof of Theorem 8.1.1 in the general case is a repetition of the proof given below in the special case, except that one has to use ringoids instead of rings and we leave it as an exercise to the interested reader.

**Proof of Theorem 8.1.1**  The case $C = R$-mod. For any biadditive bifunctor $D$ on $R$-mod one has an isomorphism $D \cong \text{Hom}(-, M \otimes_R -)$, where $M = D(R, R)$. Here we used the notations of Section 7.4. By Proposition 8.2.1 Shukla$^3(R, M)$ classifies all very strong additive track categories (up to equivalence) $\mathcal{T}$ with $\mathcal{T}_0 = R$-mod and $D(R, R) = M$, where $D$ is the canonical bifunctor associated with $M$. On the other hand the isomorphism (10) and the isomorphism (8) show that $\text{Ch}: \mathcal{T} \rightarrow \mathcal{T}_0$ is an isomorphism (up to equivalence) with $\mathcal{T}_0 = R$-mod and $D(R, R) = M$. Let $\mathcal{T}$ be an additive track category, then up to isomorphism (8) one can assume that $\text{Ch}(\mathcal{T}) \in \text{HML}^3(R, M)$. Thanks to the exact sequence (12) we can take $o(\mathcal{T})$ to be the image of $\text{Ch}(\mathcal{T})$ in $\text{H}^0(R, \mod M)$. Now Theorem 8.1.1 is a consequence of the exact sequence (12) and Proposition 7.2.1.

The example $R = \mathbb{Z}$ and $M = \mathbb{R}_2$ shows that the map $\text{HML}^3(R, M) \rightarrow \text{H}^0(R, M)$ is not trivial in general. It follows that the function $o$ is not trivial in general.

**Remarks.** 1) The following example introduces a well-known example from topology of a track category $\mathcal{T}$, which represents the generator of $\text{HML}^3(\mathbb{Z}, \mathbb{F}_2) = \mathbb{F}_2$. Following 8 we consider the track category $\text{Top}^*$ of compactly generated Hausdorff spaces with basepoint $\ast$. Maps in $\text{Top}^*$ are pointed maps. A track $\alpha : f \Rightarrow g$ between pointed maps $f, g : A \rightarrow B$ is a homotopy class of a homotopy relative to $A \times \partial I$. Now we take $\mathcal{T}_k$ to be the full subcategory of $\text{Top}^*$ consisting of finite one-point unions of spheres $S^k$, $k \geq 2$. Then $\mathcal{T}_k$ is an abelian track category and $(\mathcal{T}_k)_0$ is equivalent to $\mathbb{Z}$-mod. For $k \geq 3$ the corresponding bifunctor is $\text{Hom}(-, \mathbb{F}_2 \otimes -)$ and therefore $\mathcal{T}_k$ is an additive track theory, whose class in $\text{H}^1(\mathbb{Z}$-mod, $\text{Hom}(-, \mathbb{F}_2 \otimes -)) \cong \text{HML}^3(\mathbb{Z}, \mathbb{F}_2) = \mathbb{F}_2$ is nontrivial.

2) One can describe the function $o$ in Theorem 8.1.1 as follows. Let $\mathcal{T}$ be an additive track theory. Let $\vee$ denote the weak coproduct in $\mathcal{T}$ and let 0 be the weak zero object. For objects $X, Y$ one has therefore “inclusions” $i_1 : X \rightarrow X \vee Y$ and $i_2 : Y \rightarrow X \vee Y$. Since $X \vee Y$ is also a weak product of $X$ and $Y$ in $\mathcal{T}$ it follows that one has also projection maps $p_1 : X \vee Y \rightarrow X$ and $p_2 : X \vee Y \rightarrow Y$. For each $X$ we choose maps $i_X : X \rightarrow X \vee X$ and $t_X : X \vee Y \rightarrow Y \vee X$ in such a way that classes of $i_X$ and $t_X$ in $\mathcal{T}_0$ are the codiagonal and twisting maps in the additive category $\mathcal{T}_0$. It follows that there is a unique track

$$\alpha_X : i_X \Rightarrow t \circ i_X$$

such that $p_i(\alpha_X) = 0$ for $i = 1, 2$. Now, let $(1, 1) : X \vee X \rightarrow X$ be a map which lifts the codiagonal map in $\mathcal{T}_0$. Then $(1, 1)_\ast \alpha_X$ is a track $\text{Id}_X \rightarrow \text{Id}_X$ and therefore it differs from the trivial track by an element $o(X) \in D(X, X)$. One can prove that the assignment $X \mapsto o(X)$ is an expected one.

3) Corollary 7.2.1 shows that if $\mathcal{T}$ is an additive track theory such that $\mathcal{T}_0$ is an $\mathbb{F}_p$-linear category then $\mathcal{T}$ is equivalent to a strong additive track theory with $\mathbb{Z}/p^2\mathbb{Z}$-linear hom’s. This fact for a special track theory arising in the theory of
the “secondary Steenrod algebra” was proved by the first author by completely different methods and was a starting point of this work.

4) Based on quadratic categories and square rings we in the forthcoming paper we introduce the notion of strongly additive track theories and we prove that any additive track category is equivalent to strong one.

8.2. Crossed bimodules and very strongly additive track theories. Let us recall that for a category \( C \) and a bifunctor \( D \) there is a category \( \text{Trext}(C,D) \) such that \( \pi_0(\text{Trext}(C,D)) \cong H^2(C,D) \). The objects of \( \text{Trext}(C,D) \) are abelian track categories \( \mathcal{F} \) with \( \mathcal{T}_\mathcal{F} \cong C \) and \( D_{\mathcal{F}} = D \). If additionally \( C \) is an additive category and \( D \) is a biaadditive bifunctor, then any such \( \mathcal{F} \) is an additive track theory. We let \( \text{Strext}(C,D) \) be the full subcategory of \( \text{Trext}(C,D) \) whose objects are very strongly additive track theories.

**Proposition 8.2.1.** Let \( R \) be a ring and let \( M \) be a bimodule over \( R \). There is a functor

\[
\text{Croxt}(R,M) \to \text{Strext}(R\text{-mod}, \mathcal{Hom}(-, M \otimes_R -))
\]

which is an equivalence of categories.

**Proof.** Let \( \partial : C_1 \to C_0 \) be a crossed bimodule. We let \( \mathcal{F} = \mathcal{F}(\partial) \) be the following track category. The objects of \( \mathcal{F} \) are the same as the objects of \( R\text{-mod} \), i.e., natural numbers. For any natural numbers \( n \) and \( m \) the maps from \( n \) to \( m \) (which is the same as objects of the groupoid \( \mathcal{F}(n,m) \)) are \( m \times n \)-matrices with coefficients in \( C_0 \). For \( f, g \in \text{Mat}_{m \times n}(S) \) the set of tracks \( f \to g \) (which is the same as the set of morphisms from \( f \) to \( g \) in the groupoid \( \mathcal{F}(n,m) \)) is given by

\[
\text{Hom}_{\mathcal{F}(n,m)}(f,g) = \{ h \in \text{Mat}_{m \times n}(C_1) \mid \partial(h) = f - g \}.
\]

The composition of 1-arrows is given by the usual multiplication of matrices, while the composition of tracks is given by the addition of matrices. One easily checks that in this way one really obtains a very strongly additive track theory \( \mathcal{F}(\partial) \). It is clear that \( \mathcal{T}_{\mathcal{F}} = R\text{-mod} \), where \( R = \text{Coker}(\partial) \) and the bifunctor associated to \( \mathcal{F} \) is \( D = \mathcal{Hom}(-, M \otimes_R -) \). Thus we obtain a functor

\[
\text{Croxt}(A,M) \to \text{Strext}(R\text{-mod}, \mathcal{Hom}(-, M \otimes_R -)).
\]

Now we construct the functor in the opposite direction. Let \( \mathcal{F} \) be an object of \( \text{Strext}(R\text{-mod}, \mathcal{Hom}(-, M \otimes_R -)) \). Let \( \mathcal{T}_0 \) be the category with the same objects as \( \mathcal{F} \) and with maps (i.e., 1-arrows) of \( \mathcal{F} \) as morphisms. Since \( \mathcal{F} \) is a strongly additive track theory, we see that \( \mathcal{T}_0 \) is an additive category and therefore it is equivalent to \( S\text{-mod} \), where \( S = \text{End}_{\mathcal{T}_0}(1) \). The restriction of the quotient functor \( \mathcal{F} \to \mathcal{T}_0 \) yields the homomorphism of rings \( S \to R \). One defines \( X \) to be the set of pairs \( (h,x) \), where \( x \in \text{Hom}_{\mathcal{T}_0}(1,1) \) and \( h : x \Rightarrow 0 \) is a track in the groupoid \( \mathcal{F}(1,1) \). Moreover we put \( \partial = \partial_{\mathcal{F}}(h,x) = x \). Then \( X \) carries a structure of a bimodule over \( S \), and

\[
0 \to M \to X \xrightarrow{\partial} S \to R \to 0
\]

is a crossed extension. Then \( \mathcal{F} \mapsto \partial_{\mathcal{F}} \) yields the functor

\[
\text{Strext}(R\text{-mod}, \mathcal{Hom}(-, M \otimes_R -)) \to \text{Croxt}(A,M).
\]

One easily checks that these two functors yield the expected equivalence of categories.
Appendix A. Closed model category structure on chain algebras and crossed bimodules

A.1. Closed model categories. We recall the definition of a closed model category introduced by Quillen [31]. We refer the reader to [14] for the basic facts on the closed model category theory. Let \( C \) be a category. A morphism \( f \) is a retract of a morphism \( g \) if there exists a commutative diagram of the form

\[
\begin{array}{ccc}
A & \rightarrow & C \\
\downarrow^f & & \downarrow^g \\
B & \rightarrow & D \\
\end{array}
\]

in which the horizontal composites are identities. Let \( i : A \rightarrow B \) and \( p : X \rightarrow Y \) be morphisms in \( C \). Then \( i \) has left lifting property with respect to \( p \) and \( p \) has right lifting property with respect to \( i \), if for every commutative diagram

\[
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow^g & & \downarrow^f \\
B & \rightarrow & Y \\
\end{array}
\]

there exists a commutative diagram

\[
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow^i & & \downarrow^f \\
B & \rightarrow & Y \\
\end{array}
\]

Then \( h \) is called a lifting.

Definition A.1.1. A closed model category consists of a category \( C \) together with three distinguished classes of morphisms called respectively weak equivalences, cofibrations and fibrations, so that the following 5 axioms hold.

CM 1. \( C \) has all finite limits and colimits. All 3 classes form a subcategory.
CM 2. If \( f \) and \( g \) are composable arrows in \( C \) and two of the three morphisms \( f, g, gf \) are weak equivalences, then so is the third.
CM 3. A retract of a fibration (resp. cofibration, weak equivalence) is still a fibration (resp. cofibration, weak equivalence).
CM 4. Fibrations have the right lifting property with respect to acyclic cofibrations and cofibrations have left lifting property with respect to acyclic fibrations. Here a map is called an acyclic fibration (resp. acyclic cofibration) if it is both a fibration (resp. cofibration) and a weak equivalence.
CM 5. Any arrow \( f : A \rightarrow B \) has factorizations \( f = pi \) and \( f = qj \), where \( i \) and \( j \) are cofibrations, \( p \) and \( q \) are fibrations and \( p \) and \( j \) are weak equivalences too.

Here is more language corresponding to closed model categories. An object \( X \) is called cofibrant if \( \emptyset \rightarrow X \) is a cofibration. An object \( Y \) is called fibrant if \( Y \rightarrow * \) is a fibration. Here \( \emptyset \) and \( * \) are respectively initial and terminal objects in \( C \). For any object \( X \) there are weak equivalences \( X \rightarrow X^f \) and \( X \rightarrow X^c \) with fibrant \( X^f \) and cofibrant \( X^c \). This is an easy consequence of CM 5. Any such \( X^c \) (resp. \( X^f \)
is called a cofibrant replacement (resp. fibrant replacement). It follows from the axioms that a map \( i \) is a cofibration iff it has the left lifting property with respect to acyclic fibrations. Moreover \( i \) is an acyclic cofibration iff it has the left lifting property with respect to fibrations. Therefore fibrations and weak equivalences completely determine cofibrations. The dual properties hold for fibrations.

Let \( \mathcal{C} \) be a closed model category. We let \( \mathcal{W} \) be the class of all weak equivalences. Then there exists a category \( \mathcal{H}o := \mathcal{C}[\mathcal{W}^{-1}] \) together with a functor \( \mathcal{C} \to \mathcal{H}o \) which takes all morphisms from \( \mathcal{W} \) to isomorphisms and which is universal with respect to this property. Clearly the category \( \mathcal{H}o \) is determined uniquely up to equivalence of categories. It has the following description: objects of \( \mathcal{H}o \) are the same as those of \( \mathcal{C} \), while morphisms are given by

\[
\text{Hom}_{\mathcal{H}o}(X, Y) := \text{Hom}_{\mathcal{C}}(X^e, Y^f)/\sim,
\]

where \( \sim \) is an appropriate homotopy relation, which is defined as follows. Let \( f, g : A \to B \) be two maps. Then \( f \sim g \) if there exists a map \( h : IA \to B \) such that \( f = h \circ i_1 \) and \( g = h \circ i_2 \). Here \( IA \) and the maps \( i_1, i_2 : A \to IA \) satisfy the following conditions: the canonical map \((\text{id}, \text{id}) : A \coprod A \to A \) is a composite \( A \coprod A \xrightarrow{A} IA \to A \), where the first map is a cofibration and the second one is an acyclic fibration. It turns out that this relation is an equivalence relation on \( \text{Hom}_{\mathcal{C}}(A, B) \) if \( A \) is cofibrant and \( B \) is fibrant. Moreover it is compatible with the composition law in \( \mathcal{C} \) and the category \( \mathcal{H}o \) is well defined.

A.2. Cofibrantly generated model categories. Suppose \( \mathcal{C} \) is a category with all colimits. Let \( I \) be a class of maps in \( \mathcal{C} \). Following [19] we call a morphism \( I \)-injective (resp. \( I \)-projective) if it has the right (resp. left) lifting property with respect to every morphism in \( I \). The class of \( I \)-injective and \( I \)-projective morphisms are denoted \( I \)-inj and \( I \)-proj respectively. A morphism is called an \( I \)-cofibration (resp. \( I \)-fibration) if it has the left (resp. right) lifting property with respect to every morphism in \( I \)-inj (resp. \( I \)-proj). The class of \( I \)-cofibrations and \( I \)-fibrations are denoted \( I \)-cof and \( I \)-fib respectively. Assume now \( I \) is a set of morphisms. A morphism \( f : A \to B \) is called a relative \( I \)-cell complex if there is an ordinal \( \lambda \) and a \( \lambda \)-sequence

\[
X_0 \to X_1 \to \cdots \to X_\beta \to \cdots,
\]

\( \beta \leq \lambda \), with \( A = X_0 \) and \( B = \text{colim}X_\beta \) such that for all \( \beta \) with \( \beta + 1 < \lambda \) there is a pushout diagram

\[
\begin{array}{ccc}
C_\beta & \xrightarrow{g_\beta} & D_\beta \\
\downarrow & & \downarrow \\
X_\beta & \xrightarrow{} & X_{\beta+1}
\end{array}
\]

such that \( g_\beta \in I \). The class of relative \( I \)-cell complexes is denoted \( I \)-cell. An object \( A \) is called an \( I \)-cell complex if \( 0 \to A \) is a relative \( I \)-cell complex.

We will say that an object \( A \) is small relative to a class of morphisms \( I \) if there exists a cardinal \( \kappa \) such that for each \( \kappa \)-filtered ordinal \( \lambda \) and a \( \lambda \)-sequence \( X_0 \to X_1 \to \cdots \to X_\beta \to \cdots \) one has

\[
\text{colim} \text{Hom}_{\mathcal{C}}(A, X_\beta) \cong \text{Hom}_{\mathcal{C}}(A, \text{colim}X_\beta).
\]
If \( A \) is small with respect of \( \mathcal{C} \) then \( A \) is called small. The following result is well-known (see for example Theorem 2.1.19 of [19]).

**Proposition A.2.1.** Suppose \( \mathcal{C} \) is a category with all colimits and limits. Suppose \( W \) is a subcategory of \( \mathcal{C} \) and \( I \) and \( J \) are two sets of morphisms of \( \mathcal{C} \) such that the following conditions hold:

(i) The subcategory \( W \) is closed under retracts and satisfies the CM2 axiom.

(ii) The domains of \( I \) (resp. \( J \)) are small relative to \( I \)-cell (resp. \( J \)-cell).

(iii) \( J \)-cell \( \subset W \cap I \)-cof.

(iv) \( J \)-inj \( = W \cap J \)-inj.

Then there is a close model category structure on \( \mathcal{C} \) with \( W \) as the subcategory of weak equivalences, \( I \)-cof as the class of cofibrations, \( J \)-inj as the class of fibrations. Moreover \( J \)-inj is the class of acyclic fibrations and \( J \)-cof is the class of acyclic cofibrations.

The closed model categories obtained in this way are called cofibrantly generated model categories.

**A.3. Chain algebras.** We fix a commutative ring \( \mathbb{K} \) and all algebras in what follows in this section are \( \mathbb{K} \)-algebras. Let us recall that a chain algebra is a graded algebra \( A = \bigoplus_{n \geq 0} A_n \) equipped with a differential \( d : A_n \to A_{n-1} \) satisfying the Leibniz identity:

\[
d(xy) = d(x)y + (-1)^{n}xd(y), \quad x \in A_n, y \in A_m.
\]

Let \( \text{DGA} \) be the category of chain algebras.

**Theorem A.3.1.** Define a morphism of chain algebras to be

(i) a weak equivalence if it induces isomorphism in homology

(ii) a fibration if it is a surjection in positive dimensions

(iii) a cofibration if it has the left lifting property with respect to all maps which are fibrations and weak equivalences

Then with these choices \( \text{DGA} \) is a cofibrantly generated closed model category.

To prove the theorem, we first introduce two classes of chain algebras. They play the role of discs and spheres. For \( n \geq 1 \) we let \( D(n) \) be the following chain algebra. As graded algebra it is freely generated by elements \( x \) and \( dx \) of degree \( n \) and \( n-1 \) respectively. The boundary map assigns \( dx \) to \( x \). For \( n = 0 \) we let \( D(0) \) be the algebra freely generated by an element \( x \) of degree 0 (of course \( d(x) = 0 \) in this case). Moreover we define \( S(n) \) to be the trivial algebra \( \mathbb{K} \) if \( n = -1 \) and the algebra freely generated by an element \( y \) of degree \( n \) with zero boundary \( d(y) = 0 \) provided \( n \geq 0 \). Then for all \( n \geq 0 \) we have a canonical homomorphism \( S(n-1) \to D(n) \) which takes the generator \( y \) to \( dx \). We let \( \coprod \) denote the coproduct in \( \text{DGA} \). One has the following isomorphism of chain complexes

\[
A_n \coprod D(n) \cong A_* \oplus (A_* \otimes C_* \otimes A_*) \oplus (A_* \otimes C_* \otimes A_* \otimes C_* \otimes A_*) \oplus \cdots .
\]

Here \( C_* \) is a chain complex, which is zero in all dimensions except for dimensions \( n \) and \( n-1 \), where it is \( \mathbb{K} \) and the unique nontrivial boundary map is the identity. Therefore the inclusion \( A_* \to A_* \coprod D(n) \) is a weak equivalence, provided \( n > 0 \).

One observes that for any chain algebra \( A_* \), one has the isomorphisms

\[
\text{Hom}_{\text{DGA}}(D(n), A_*) \cong A_n,
\]

\( 11 \)
\[ \text{Hom}_{\text{DGA}}(S(n), A_\ast) \cong \ker(d : A_n \to A_{n-1}). \]

We let \( W \) be the subcategory of all weak equivalences in \( \text{DGA} \). Moreover we put
\[
J := \{\mathbb{K} \to D(n)\}_{n \geq 1},
\]
\[
I := J \cup \{S(n-1) \to D(n)\}_{n \geq 0}.
\]

Then the conditions i) and ii) of Proposition A.2.1 hold, because \( D(n) \) and \( S(n) \) are small thanks to isomorphisms 12 and 13. We will show that all conditions of Proposition A.2.1 hold as well. To this end we need some preparations.

Since \( \mathbb{K} \) is the initial object in \( \text{DGA} \) a morphism \( f : X_\ast \to Y_\ast \) is in \( J \)-inj iff for any diagram
\[
\begin{array}{ccc}
X_\ast & \xrightarrow{f} & Y_\ast \\
\downarrow & & \downarrow \\
D(n) & \xrightarrow{g} & D(n)
\end{array}
\]
there exists a morphism \( h : D(n) \to X \) such that \( f = gh \). Now the isomorphism \((12)\) gives that \( f : X_\ast \to Y_\ast \) is in \( J \)-inj iff \( f_n \) is surjective for all \( n > 0 \). Thus we proved the following

**Lemma A.3.2.** A map \( f : X_\ast \to Y_\ast \) is a \( J \)-inj iff it is fibration.

**Lemma A.3.3.** Let \( f : X_\ast \to Y_\ast \) be a morphism in \( \text{DGA} \). Then the following conditions are equivalent:

(i) \( f \) is \( I \)-injective

(ii) \( f \) is fibration and weak equivalence

(iii) \( f_n \) is surjective for all \( n \geq 0 \) and \( \ker f \) is acyclic, that is \( H_*(\ker f) = 0 \).

**Proof.** Lemma A.3.2 and the homology exact sequence show that iii) \( \implies \) ii). Thanks to the isomorphism \((13)\) a morphism \( f \) lies in \( I \)-inj iff it is a fibration with the following property: for all \( x \in X_{n-1} \) and \( y \in Y_n \) with \( dx = 0 \) and \( fx = dy \) there exists \( z \in X_n \) such that \( dz = x \) and \( fz = y \). If the last condition holds, then \( f_0 \) is surjective and \( \ker f \) is acyclic. Thus by Lemma A.3.2 we have i) \( \implies \) iii). Conversely, assume iii) holds. Suppose \( x \in X_{n-1} \) and \( y \in Y_n \) are given with \( dx = 0 \) and \( fx = dy \). Then there is \( u \in X_n \) such that \( fu = y \). Since \( d(x - du) = 0 \) and \( d(x - du) = 0 \) it follows that \( x - du = dv \) for some \( v \in X_n \) and therefore \( x = d(u + v) \) which shows that iii) \( \implies \) i). To show ii) \( \implies \) iii) it suffices to show that \( X_0 \to Y_0 \) is surjection. But this follows from the commutative diagram

\[
\begin{array}{cccccc}
\cdots & \to & X_1 & \to & X_0 & \to & H_0(X_\ast) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \cong & & \\
\cdots & \to & Y_1 & \to & Y_0 & \to & H_0(Y_\ast) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \\
0 & & 0 & & & & & &
\end{array}
\]

\[\square\]

**Corollary A.3.4.** We have \( I \)-inj = \( W \cap J \)-inj.
Recall that a morphism \( f : X_* \to Y_* \) belongs to \( I\text{-}cof \) if it has the left lifting property with respect to all maps from \( I\text{-}inj \). Thanks to Lemma [A.3.3] this happens if \( f \) is a cofibration.

A chain algebra \( A_* \) is called \( \mathbb{K}\text{-}projective if each \( A_n \) is projective as a \( \mathbb{K}\text{-}module. A chain algebra \( A_* \) is called \emph{quasi-free} if its underlying algebra is free. Let us recall that a graded algebra is free if it is isomorphic to the tensor algebra \( T(V) \) of a graded \( \mathbb{K}\text{-}module \( V_* \), which is free as a \( \mathbb{K}\text{-}module. A map of chain algebras \( f : A_* \to B_* \) is called quasi-free if \( B_* \) as a graded algebra is a coproduct \( A_* \coprod X_* \) where \( X_* \) is a free algebra.

**Lemma A.3.5.** Quasi-free maps are cofibrations.

**Proof.** Let

\[
\begin{array}{ccc}
  A_* & \xrightarrow{g} & X_* \\
  \downarrow{i} & & \downarrow{p} \\
  B_* & \xrightarrow{h} & Y_*
\end{array}
\]

be a commutative diagram of chain algebras, in which \( i \) is quasi-free and \( p \) is an acyclic fibration. We have to prove that there is a chain map \( f : B_* \to X_* \) such that \( g = fi \) and \( h = pf \). By assumptions we have an isomorphism of algebras \( B_* \cong A_* \coprod C_* \), where \( C_* \) is a free algebra. Let \( E \) be the set of free generators of \( C_* \). Then \( E \) is the union of subsets \( E_n \) of degree \( n \) elements, \( n \geq 0 \). In order to define \( f \) one needs to specify elements \( f(x) \) for \( x \in E_n, n \geq 0 \) with two properties

a) \( \partial f(x) = f(\partial x), \)

b) \( pf(x) = h(x). \)

We will work by induction on \( n \). First consider the case \( n = 0 \). Since \( p \) is surjective, there exists \( f_0(x) \in X_0 \) such that \( pf(x) = h(x) \). Consider now the case \( n > 0 \). Suppose for all \( m < n \) we already defined \( f(x) \) for all \( x \in E_m \) such that a) and b) holds for all \( x \in E_j, 1 \leq j < n \). Take now \( x \in E_n \). Since \( p \) is surjective we can choose an element \( \bar{f}(x) \in X_n \) such that \( pf(x) = g(x) \). Since \( \partial x \) lies in the subalgebra generated by \( A_* \) and \( E_j, j < n \), the element \( f(\partial x) \) is already defined.

Set \( z = \partial f(x) - f(\partial x) \). Then \( \partial(z) = 0 \) and \( p(z) = 0 \). Therefore \( z = \partial(u) \) for some element \( u \in \text{Ker}(p) \). Now we put \( f(x) = \bar{f}(x) - u \). It is clear that \( f(x) \) satisfies properties a) and b). Thus induction step is finished and hence the lemma.

**Corollary A.3.6.** The canonical maps \( k \to D(n) \), \( k \to S(n) \) and \( S(n-1) \to D(n) \) are cofibrations.

**Proof of Theorem A.3.1** As was mentioned already the conditions i) and ii) of Proposition [A.2.1] hold. By Corollary [A.3.4] the condition iv) holds as well. Thus we have to show that \( J\text{-}cell \subset W \cap I\text{-}cof \). We have \( J\text{-}cell \subset I\text{-}cof \) because \( J \subset I \). Since the domain of all maps from \( J \) is \( \mathbb{K} \), which is an initial object, we see that all pushouts in the definition of a relative \( J\text{-}cell \) complex are coproducts with \( D(n) \) for some \( n > 0 \). It follows that all such morphisms are weak equivalences and quasi-free maps and the result follows from Lemma [A.3.5].

Let us note that a similar theorem for cochain algebras was proved by Jardine [21]. Moreover our argument is merely a variant of the one given there (compare also with [3]).
A.4. **Truncated chain algebras.** Let us fix a natural number $m \geq 1$. We let $\text{DGA}_m$ be the full subcategory of $\text{DGA}$ which consists of objects $X_*$ such that $X_i = 0$ for all $i > m$.

For any chain complex $(X_*, d)$ we let $\tau_{\leq m}(X_*)$ be the following chain complex:

$$(\tau_{\leq m}(X_*))_i = X_i, \text{ if } i < m$$

$$(\tau_{\leq m}(X_*))_m = X_m/d(X_{m+1})$$

$$(\tau_{\leq m}(X_*))_i = 0, \text{ if } i > m$$

The quotient map $X_* \to \tau_{\leq m}(X_*)$ is a chain map. Moreover $H_i(\tau_{\leq m}(X_*)) \cong H_i(X_*)$ if $i \leq m$ and $H_i(\tau_{\leq m}(X_*)) = 0$ provided $i > m$. It is also clear that, if $X_*$ is a chain algebra, then there is a unique chain algebra structure on $\tau_{\leq m}(X_*)$ such that the quotient map $X_* \to \tau_{\leq m}(X_*)$ is a chain algebra homomorphism. Thus

$$\tau_{\leq m} : \text{DGA} \to \text{DGA}_m$$

is a well-defined functor, which is the left adjoint to the inclusion functor $\text{DGA}_m \subset \text{DGA}$.

**Theorem A.4.1.** Define a map in $\text{DGA}_m$ to be a weak equivalence (resp. fibration) if it is a weak equivalence (resp. fibration) in $\text{DGA}$. Define a map in $\text{DGA}_m$ to be a cofibration if it has left lifting property with respect to all acyclic fibrations. Then this defines a cofibrantly generated model structure on $\text{DGA}_m$.

**Proof.** We introduce two classes of morphisms in $\text{DGA}_m$:

$$J_m := \{ K \to \tau_{\leq m}D(n) \}_{n \geq 1},$$

$$I_m := J_m \bigcup \{ \tau_{\leq m}S(n-1) \to \tau_{\leq m}D(n) \}_{n \geq 0}.$$ 

We have to show that all assertions of Proposition A.2.1 hold. Conditions i) and ii) are clear. Formal argument with adjoint functors shows that a morphism $f : X_* \to Y_*$ in $\text{DGA}_m$ considered as a morphism of $\text{DGA}$ lies in $J_{\text{inj}}$ (resp. $I_{\text{inj}}$) iff it is in $J_{m-\text{inj}}$ (resp. $I_{m-\text{inj}}$). Therefore $f$ is a fibration (resp. acyclic fibration) iff it is in $J_{m-\text{inj}}$ (resp. $I_{m-\text{inj}}$) and the condition iv) holds. We also have $J_{m-\text{cell}} \subset I_{m-\text{cell}}$ because $J_m \subset I_m$. Thus it remains to show that $J_{m-\text{cell}} \subset W$. Comparing the definitions we see that any morphism from $J_{m-\text{cell}}$ can be written as $\tau_{\leq m}(g)$, where $g \in J_{\text{cell}}$. In particular $g \in W$. Since $\tau_{\leq m}$ preserves weak equivalences we are done.

A.5. **A closed model category structure on crossed bimodules.** Of the special interest is the case, when $m = 1$. In this case Theorem A.4.1 gives the closed model category structure on the category $\text{Xmod}$ of crossed bimodules. A map of crossed bimodules

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\partial} & C_0 \\
\downarrow f & & \downarrow g \\
C_1' & \xrightarrow{\partial'} & C_0'
\end{array}
$$

is a fibration if $f$ is a surjective homomorphism. Moreover, $(f, g) : \partial \to \partial'$ is a weak equivalence if induced maps $\text{Ker}(\partial) \to \text{Ker}(\partial')$, $\text{Coker}(\partial) \to \text{Coker}(\partial')$ are isomorphisms. It follows that if $(f, g)$ is an acyclic fibration, then $g$ is a surjection:
and the induced map $\text{Ker}(f) \rightarrow \text{Ker}(g)$ is an isomorphism, in other words one has the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccc}
0 & 0 &  &  &  &  &  \\
\downarrow & \downarrow &  &  &  &  &  \\
\text{Ker}(f) & \cong & \text{Ker}(g) &  &  &  &  \\
\downarrow & \downarrow &  &  &  &  &  \\
0 & 0 & \rightarrow & \text{Ker}(\partial) & \rightarrow & C_1 & \rightarrow & C_0 & \rightarrow & \text{Coker}(\partial) & \rightarrow & 0 \\
\downarrow & \downarrow & \cong & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cong & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & \text{Ker}(\partial') & \rightarrow & C_1' & \rightarrow & C_0' & \rightarrow & \text{Coker}(\partial') & \rightarrow & 0 \\
\downarrow & \downarrow &  &  &  &  &  &  &  &  &  &  \\
0 & 0 &  &  &  &  &  &  &  &  &  &  \\
\end{array}
$$

Thus we proved the following

**Lemma A.5.1.** If $(f, g) : \partial \rightarrow \partial'$ is an acyclic fibration in $\mathbb{X}_\text{mod}$, then $g$ is surjective and

$$
\begin{array}{ccccccc}
C_1 & \rightarrow & C_0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\text{Coker}(\partial) & \rightarrow & \text{Coker}(\partial') & \rightarrow & 0 \\
\end{array}
$$

is a pullback diagram.

**Lemma A.5.2.** A crossed bimodule $\delta : R_1 \rightarrow R_0$ is a cofibrant objects in $\mathbb{X}_\text{mod}$ provided $R_0$ is a free algebra.

**Proof.** Let $(f, g) : \partial \rightarrow \partial'$ be an acyclic fibration of crossed bimodules and let $(a', b') : \delta \rightarrow \delta'$ be a morphism of crossed bimodules. We have to lift it to a morphism $(a, b) : \delta \rightarrow \partial$. Since $g$ is a surjective homomorphism of $\mathbb{K}$-algebras and $R_0$ is a free $\mathbb{K}$-algebra, we can lift $b'$ to a homomorphism $b : R_0 \rightarrow C_0$ of $\mathbb{K}$-algebras. Then we have the following commutative diagram

$$
\begin{array}{ccccccc}
R_1 & \rightarrow & R_0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
C_1 & \rightarrow & C_0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\text{Coker}(\partial) & \rightarrow & \text{Coker}(\partial') & \rightarrow & 0 \\
\end{array}
$$

and we have the unique homomorphism $a : R \rightarrow C_1$ which fits in the diagram. It is now clear that $(a, b)$ is an expected lifting.

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