ON MEAN ERGODIC CONVERGENCE IN THE CALKIN ALGEBRAS

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Abstract. In this paper, we give a geometric characterization of mean ergodic convergence in the Calkin algebras for Banach spaces that have the bounded compact approximation property.

1. Introduction

Let $X$ be a real or complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on $X$. Suppose that $T \in B(X)$ and consider the sequence

$$M_n(T) := \frac{I + T + \ldots + T^n}{n+1}, \quad n \geq 1.$$ 

In [3], Dunford considered the norm convergence of $(M_n(T))_n$ and established the following characterizations.

**Theorem 1.1.** Suppose that $X$ is a complex Banach space and that $T \in B(X)$ satisfies $\|T^n\| \to 0$. Then the following conditions are equivalent.

1. $(M_n(T))_n$ converges in norm to an element in $B(X)$.
2. $1$ is a simple pole of the resolvent of $T$ or $1$ is in the resolvent set of $T$.
3. $(I - T)^2$ has closed range.

It was then discovered by Lin in [6] that $I - T$ having closed range is also an equivalent condition. Moreover, Lin’s argument worked also for real Banach spaces. This result was later improved by Mbekhta and Zemánek in [9] in which they showed that $(I - T)^m$ having closed range, where $m \geq 1$, are also equivalent conditions. More precisely,

**Theorem 1.2.** Let $m \geq 1$. Suppose that $X$ is a real or complex Banach space and that $T \in B(X)$ satisfies $\|T^n\| \to 0$. Then the sequence $(M_n(T))_n$ converges in norm to an element in $B(X)$ if and only if $(I - T)^m$ has closed range.

Let $K(X)$ be the closed ideal of compact operators in $B(X)$. If $T \in B(X)$ then its image in the Calkin algebra $B(X)/K(X)$ is denoted by $\dot{T}$. By Dunford’s Theorem...
or by an analogous version for Banach algebras (without condition (3)), when \( X \) is a complex Banach space and \( \|T^n\| \to 0 \), the convergence of \( (M_n(T))_n \) in the Calkin algebra is equivalent to 1 being a simple pole of the resolvent of \( \hat{T} \) or being in the resolvent set of \( \hat{T} \). But even if we are given that the limit \( \hat{P} \in B(X)/K(X) \) exists, there is no obvious geometric interpretation of \( \hat{P} \). In the context of Theorems 1.1 and 1.2, if the limit of \( (M_n(T))_n \) exists, then it is a projection onto \( \ker(I - T) \). In the context of the Calkin algebra, the limit \( \hat{P} \) is still an idempotent in \( B(X)/K(X) \); hence by making a compact perturbation, we can assume that \( P \) is an idempotent in \( B(X) \) (see Lemma 2.7 below).

A natural question to ask is: what is the range of \( P \)? Although the range of \( P \) is not unique (since \( P \) is only unique up to a compact perturbation), it can be thought of as an analog of \( \ker(I - T) \) in the Calkin algebra setting. If \( T_0 \in B(X) \) then \( \ker T_0 \) is the maximal subspace of \( X \) on which \( T_0 = 0 \). This suggests that the analog of \( \ker T_0 \) in the Calkin algebra setting is the maximal subspace of \( X \) on which \( T_0 \) is compact. But the maximal subspace does not exist unless it is the whole space \( X \). Thus, we introduce the following concept.

Let \( X \) be a Banach space and let \((P)\) be a property that a subspace \( M \) of \( X \) may or may not have. We say that a subspace \( M \subseteq X \) is an essentially maximal subspace of \( X \) satisfying \((P)\) if it has \((P)\) and if every subspace \( M_0 \supseteq M \) having property \((P)\) satisfies \( \dim M_0/M < \infty \).

Then the analog of \( \ker T_0 \) in the Calkin algebra setting is an essentially maximal subspace of \( X \) on which \( T_0 \) is compact. It turns that if such an analog for \( I - T \) exists, then it is already sufficient for the convergence of \( (M_n(T))_n \) in the Calkin algebra (at least for a large class of Banach spaces), which is the main result of this paper.

Before stating this theorem, we recall that a Banach space \( Z \) has the bounded compact approximation property (BCAP) if there is a uniformly bounded net \((S_\alpha)_{\alpha \in \Lambda} \) in \( K(Z) \) converging strongly to the identity operator \( I \in B(Z) \). It is always possible to choose \( \Lambda \) to be the set of all finite dimensional subspaces of \( Z \) directed by inclusion. If the net \((S_\alpha)_{\alpha \in \Lambda} \) can be chosen so that \( \sup_{\alpha \in \Lambda} \|S_\alpha\| \leq \lambda \), then we say that \( Z \) has the \( \lambda \)-BCAP. It is known that if a reflexive space has the BCAP, then the space has the 1-BCAP. For \( T \in B(X) \), the essential norm \( \|T\|_e \) is the norm of \( \hat{T} \) in \( B(X)/K(X) \).

**Theorem 1.3.** Let \( m \geq 1 \). Suppose that \( X \) is a real or complex Banach space having the bounded compact approximation property. If \( T \in B(X) \) satisfies \( \|T^n\|_e \to 0 \), then the following conditions are equivalent.

1. The sequence \((M_n(T))_n \) converges in norm to an element in \( B(X)/K(X) \).
2. There is an essentially maximal subspace of \( X \) on which \( (I - T)^m \) is compact.

The idea of the proof is to reduce Theorem 1.3 to Theorem 1.2 by constructing a Banach space \( \hat{X} \) and an embedding \( f : B(X)/K(X) \to B(\hat{X}) \) so that if \( T \in B(X) \) and if there is an essentially maximal subspace \( M \) of \( X \) on which \( T \) is compact, then \( f(T) \) has closed range, and then applying Theorem 2.2 to \( f(\hat{T}) \). The BCAP of \( X \) is used to show that \( f \) is an embedding but is not used in the construction of \( \hat{X} \) and \( f \). The construction of \( f \) is based on the Calkin representation [1, Theorem 5.5].
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2. The Calkin Representation for Banach Spaces

In this section, \( X \) is a fixed infinite dimensional Banach space. Let \( \Lambda_0 \) be the set of all finite dimensional subspaces of \( X \) directed by inclusion \( \subseteq \). Then \( \{ \{ \alpha \in \Lambda_0 : \alpha \supseteq \alpha_0 \} : \alpha_0 \in \Lambda_0 \} \) is a filter base on \( \Lambda_0 \), so it is contained in an ultrafilter \( U \) on \( \Lambda_0 \).

Let \( Y \) be an arbitrary infinite dimensional Banach space and let \( (Y^*)^U \) be the ultrapower (see e.g., [2] Chapter 8) of \( Y^* \) with respect to \( U \). (The ultrafilter \( U \) and the directed set \( \Lambda_0 \) do not depend on \( Y \).) If \( (y^*_\alpha)_{\alpha \in \Lambda_0} \) is a bounded net in \( Y^* \), then its image in \( (Y^*)^U \) is denoted by \( (y^*_{\alpha,U})_{\alpha \in \Lambda_0} \). Consider the (complemented) subspace

\[
\hat{Y} := \left\{ (y^*_\alpha)_{\alpha,U} \in (Y^*)^U : \text{w}^*\lim_{\alpha,U} y^*_\alpha = 0 \right\}
\]

of \( (Y^*)^U \). Here \( \text{w}^*\lim_{\alpha,U} y^*_\alpha \) is the \( \text{w}^* \)-limit of \( (y^*_\alpha)_{\alpha \in \Lambda_0} \) through \( U \), which exists by the Banach-Alaoglu Theorem.

Whenever \( T \in B(X,Y) \), we can define an operator \( \hat{T} \in B(\hat{Y},\hat{X}) \) by sending \( (y^*_\alpha)_{\alpha,U} \) to \( (T^*y^*_\alpha)_{\alpha,U} \). Note that if \( K \in K(X,Y) \) then \( \hat{K} = 0 \), where \( K(X,Y) \) denotes the space of all compact operators in \( B(X,Y) \).

**Theorem 2.1.** Suppose that \( X \) has the \( \lambda \)-BCAP. Then the operator \( f : B(X)/K(X) \to B(\hat{X}) \), \( T \mapsto \hat{T} \), is a norm one \( (\lambda + 1) \)-embedding into \( B(\hat{X}) \) satisfying

\[
f(\hat{I}) = I \text{ and } f(\hat{T_1T_2}) = f(\hat{T_2})f(\hat{T_1}), \quad T_1,T_2 \in B(X).
\]

**Proof.** It is easy to verify that \( f \) is a linear map, \( f(\hat{I}) = I \), and \( f(\hat{T_1T_2}) = f(\hat{T_2})f(\hat{T_1}) \) for \( T_1,T_2 \in B(X) \). If \( T \in B(X) \), then clearly \( \|f(T)\| \leq \|T\| \), and thus we also have \( \|f(\hat{T})\| \leq \|T\|_{\text{e}} \). Hence \( \|f\| \leq 1 \). It remains to show that \( f \) is a \( (\lambda + 1) \)-embedding (i.e., \( \inf_{\|T\|_{\text{e}} > 1} \|f(T)\| \geq (\lambda + 1)^{-1} \).

To do this, let \( T \in B(X) \) satisfy \( \|T\|_{\text{e}} > 1 \). Since \( X \) has the \( \lambda \)-BCAP, we can find a net of operators \( (S_\alpha)_{\alpha \in \Lambda_0} \subset K(X) \) converging strongly to \( I \) such that sup \( \|S_\alpha\| \leq \lambda \). Then \( \|T^*(I - S_\alpha)^*\| = \|(I - S_\alpha)T\| \geq \|T\|_{\text{e}} > 1, \alpha \in \Lambda_0 \). Thus, there exists \( (x^*_\alpha)_{\alpha \in \Lambda_0} \subset X^* \) such that \( \|x^*_\alpha\| = 1 \) and \( \|T^*(I - S_\alpha)^*x^*_\alpha\| > 1 \) for \( \alpha \in \Lambda_0 \).

Note that for every \( x \in X \),

\[
\limsup_{\alpha \in \Lambda_0} \|((I - S_\alpha)^*x^*_\alpha, x)\| = \limsup_{\alpha \in \Lambda_0} \|\langle x^*_\alpha, (I - S_\alpha)x \rangle\| \leq \limsup_{\alpha \in \Lambda_0} \|(I - S_\alpha)x\| = 0,
\]

and so the net \( ((I - S_\alpha)^*x^*_\alpha)_{\alpha \in \Lambda_0} \) converges in the \( \text{w}^* \)-topology to \( 0 \). By the construction of \( U \), this implies that

\[
\text{w}^*\lim_{\alpha,U} (I - S_\alpha)^*x^*_\alpha = 0.
\]

Therefore, due to the definition \( f(\hat{T}) = \hat{T} \), we obtain

\[
(1 + \lambda)\|f(\hat{T})\| \geq \|f(\hat{T})\| \lim_{\alpha,U} \|(I - S_\alpha)^*x^*_\alpha\| = \|f(\hat{T})\||(I - S_\alpha)^*x^*_\alpha\|_{\alpha,U} \|
\geq \|f(\hat{T})||(I - S_\alpha)^*x^*_\alpha\|_{\alpha,U} \|
= \lim_{\alpha,U} \|T^*(I - S_\alpha)^*x^*_\alpha\| \geq 1.
\]
It follows that \( \|f(\hat{T})\| \geq (1 + \lambda)^{-1} \) whenever \( \|T\|_e > 1 \). \( \square \)

**Remark 1.** We do not know whether Theorem 2.1 is true without the hypothesis that \( X \) has the BCAP.

**Remark 2.** The embedding in Theorem 2.1 is an isometry if the approximating net can be chosen so that \( \|I - S_\alpha\| = 1 \) for every \( \alpha \). This is the case if, for example, the space \( X \) has a 1-unconditional basis. However, we do not know whether the embedding is an isometry if \( X = L_p(0, 1) \) with \( p \neq 2 \).

If \( N \) is a subset of \( Y^* \), then we can define a subset \( N' \) of \( \hat{Y} \) by

\[
N' := \left\{ (y^*_\alpha)_{\alpha, U} \in \hat{Y} : \lim_{\alpha, U} d(y^*_\alpha, N) = 0 \right\},
\]

where

\[
d(y^*_\alpha, N) := \inf_{z^* \in N} \|y^*_\alpha - z^*\|.
\]

**Lemma 2.2.** If \( N \) is a \( w^* \)-closed subspace of \( Y^* \), then for every \( (y^*_\alpha)_{\alpha, U} \in \hat{Y} \),

\[
d((y^*_\alpha)_{\alpha, U}, N') \leq 2 \lim_{\alpha, U} d(y^*_\alpha, N).
\]

**Proof.** Let \( a = \lim_{\alpha, U} d(y^*_\alpha, N) \). Let \( \delta > 0 \). Then

\[
A := \{ \alpha \in \Lambda : d(y^*_\alpha, N) < a + \delta \} \in U.
\]

Whenever \( \alpha \in A \), \( \|y^*_\alpha - z^*_\alpha\| < a + \delta \) for some \( z^*_\alpha \in N \). If we take \( z^*_\alpha = 0 \) for \( \alpha \notin A \), then, since \( \sup_{\alpha \in A} \|y^*_\alpha\| < \infty \),

\[
\sup_{\alpha \in A} \|z^*_\alpha\| = \sup_{\alpha \in A} \|z^*_\alpha\| \leq (a + \delta) + \sup_{\alpha \in A} \|y^*_\alpha\| < \infty.
\]

As a consequence, \( \left( z^*_\alpha - w^* - \lim_{\beta, U} z^*_\beta \right)_{\alpha, U} \in N' \), since \( N \) is \( w^* \)-closed. Therefore,

\[
d((y^*_\alpha)_{\alpha, U}, N') \leq \lim_{\alpha, U} d(y^*_\alpha, \left( z^*_\alpha - w^* - \lim_{\beta, U} z^*_\beta \right)_{\alpha, U})
\]

\[
= \lim_{\alpha, U} \left\| y^*_\alpha - z^*_\alpha + w^* - \lim_{\beta, U} z^*_\beta \right\|
\]

\[
\leq \lim_{\alpha, U} \left\| y^*_\alpha - z^*_\alpha \right\| + \left\| w^* - \lim_{\beta, U} z^*_\beta \right\|
\]

\[
\leq (a + \delta) + \left\| w^* - \lim_{\beta, U} (z^*_\beta - y^*_\beta) \right\|
\]

\[
\leq (a + \delta) + \lim_{\beta, U} \left\| z^*_\beta - y^*_\beta \right\| \leq 2(a + \delta).
\]

But \( \delta \) can be arbitrarily close to 0 so \( d((y^*_\alpha)_{\alpha, U}, N') \leq 2a = 2 \lim_{\alpha, U} d(y^*_\alpha, N) \). \( \square \)

**Proposition 2.3.** If \( X \) and \( Y \) are infinite dimensional Banach spaces and if \( T \in B(X, Y) \) has closed range then \( \hat{T} \in B(\hat{Y}, \hat{X}) \) also has closed range.
Hence there is an essentially maximal subspace $T$ of $\hat{\mathcal{Y}}$. Therefore by Theorem 3.3, there exists $(y_\alpha^*)_{\alpha,U} \in \hat{\mathcal{Y}}$ such that $\|\hat{T}(y_\alpha^*)_{\alpha,U}\| = \lim_{\alpha,U} \|T^*y_\alpha^*\| \geq c \lim_{\alpha,U} d(y_\alpha^*, \ker T^*) \geq \frac{c}{2} d((y_\alpha^*)_{\alpha,U}, (\ker T^*)')$.

But obviously $(\ker T^*)' \subset \ker \hat{T}$, and so

$$\|\hat{T}(y_\alpha^*)_{\alpha,U}\| \geq \frac{c}{2} d((y_\alpha^*)_{\alpha,U}, \ker \hat{T}), \quad (y_\alpha^*)_{\alpha,U} \in \hat{\mathcal{Y}}.$$ 

Hence $\hat{T}$ has closed range.

**Lemma 2.4.** Suppose that $X \subset Y$ and that $T \in B(X)$. Let $T_0 \in B(X, Y)$, $x \mapsto Tx$. Then $\hat{T}_0 \hat{Y} = \hat{T} \hat{X}$.

**Proof.** If $(y_\alpha^*)_{\alpha,U} \in \hat{\mathcal{Y}}$, then for each $\alpha \in \Lambda$, we have $T_0^* y_\alpha^* = T^* (y_\alpha^*)_{X}$, and $(y_\alpha^*)_{X,\alpha,U} \in \hat{\mathcal{X}}$. Thus $\hat{T}_0 (y_\alpha^*)_{\alpha,U} = (T_0^* y_\alpha^*)_{\alpha,U} = (T^* (y_\alpha^*)_{X})_{\alpha,U} = \hat{T} (y_\alpha^*)_{\alpha,U} \in \hat{T} \hat{X}$. Hence $\hat{T}_0 \hat{Y} \subset \hat{T} \hat{X}$.

Conversely, if $(x_\alpha^*)_{\alpha,U} \in \hat{\mathcal{X}}$ then we can extend each $x_\alpha^*$ to an element $y_\alpha^* \in Y^*$ such that $\|y_\alpha^*\| = \|x_\alpha^*\|$. Thus we have $(y_\alpha^* - w^{*\text{-lim}} y_\beta^*)_{\alpha,U} \in \hat{\mathcal{Y}}$. Note that

$$T_0^*\left(w^{*\text{-lim}} y_\beta^*\right) = w^{*\text{-lim}} T_0^* y_\beta^* = w^{*\text{-lim}} T^* x_\beta^* = T^*\left(w^{*\text{-lim}} x_\beta^*\right) = 0.$$ 

This implies that

$$\hat{T}(x_\alpha^*)_{\alpha,U} = (T^* x_\alpha^*)_{\alpha,U} = (T_0^* y_\alpha^*)_{\alpha,U} = (T_0^* (y_\alpha^* - w^{*\text{-lim}} y_\beta^*))_{\alpha,U} = \hat{T}_0 (y_\alpha^* - w^{*\text{-lim}} y_\beta^*)_{\alpha,U} \in \hat{T}_0 \hat{Y}.$$ 

Therefore $\hat{T} \hat{X} \subset \hat{T}_0 \hat{Y}$.

**Proposition 2.5.** Suppose that $T \in B(X)$ and that there exists an essentially maximal subspace $M$ of $X$ on which $T$ is compact. Then $\hat{T}$ has closed range.

**Proof.** Without loss of generality, we may assume that $X$ is a subspace of $Y = \ell_\infty(J)$ for some set $J$. Define $T_0 \in B(X, \ell_\infty(J))$, $x \mapsto Tx$. Then by assumption, there is an essentially maximal subspace $M$ of $X$ on which $T_0$ is compact. By Theorem 3.3, there exists $K \in K(X, \ell_\infty(J))$ such that $K_{|M} = T_0_{|M}$.

We now show that $T_0 - K \in B(X, \ell_\infty(J))$ has closed range. Since $M \subset \ker (T_0 - K)$ and $M$ is an essentially maximal subspace of $X$ on which $T_0 - K$ is compact, $\ker (T_0 - K)$ is an essentially maximal subspace of $X$ on which $T_0 - K$ is compact.

Let $\pi$ be the quotient map from $X$ onto $X/\ker (T_0 - K)$. Define the (one-to-one) operator $R : X/\ker (T_0 - K) \to \ell_\infty(J)$, $\pi x \mapsto (T_0 - K)x$. If $R$ does not have closed range, then by Proposition 2.4, $R$ is compact on an infinite dimensional subspace $V$ of $X/\ker (T_0 - K)$. Hence, $T_0 - K$ is compact on $\pi^{-1}V$ and so by the essential maximality of $\ker (T_0 - K)$, we have $\dim \pi^{-1}V/\ker (T_0 - K) < \infty$. Thus, $V = \pi^{-1}V/\ker (T_0 - K)$ is finite dimensional, which contradicts the definition of $V$. 
Therefore, $R$ has closed range and so $T_0 - K$ also has closed range. By Proposition 2.3, $T_0 - K$ has closed range. But $K = 0$ so $T_0$ has closed range and by Lemma 2.4 $T$ has closed range. □

**Lemma 2.6.** Suppose that $P \in B(X)$ and that $\hat{P}$ is an idempotent in $B(X)/K(X)$. Then $P$ is the sum of an idempotent in $B(X)$ and a compact operator on $X$.

**Proof.** We first treat the case where the scalar field is $\mathbb{C}$. From Fredholm theory (see e.g. [3] Chapters XI and XVII], we know that since $\sigma(\hat{P}) \subset \{0, 1\}$, the only possible cluster points of $\sigma(P)$ are 0 and 1. Thus, there exists $0 < r < 1$ such that $\{z \in \mathbb{C} : |z - 1| = r\} \cap \sigma(P) = \emptyset$. Then $\hat{P} = \frac{1}{2\pi i} \oint_{|z - 1| = r} (zI - \hat{P})^{-1} dz$ and so $P - \frac{1}{2\pi i} \oint_{|z - 1| = r} (zI - P)^{-1} dz$ is an idempotent in $B(X)$ (see e.g. [10] Theorem 2.7). This completes the proof in the complex case.

If $X$ is a real Banach space, then let $X_C$ and $P_C$ be the complexifications (see [4] page 266) of $X$ and $P$, respectively. Thus, $P_C$ is an idempotent in $B(X_C)/K(X_C)$. Since the only possible cluster points of $\sigma(P_C)$ are 0 and 1, there exists a closed rectangle $R$ in the complex plane symmetric with respect to the real axis such that 1 is in the interior of $R$, 0 is in the exterior of $R$, and $\sigma(P_C)$ is disjoint from the boundary $\partial R$ of $R$. By [4] Lemma 3.4, the idempotent $\frac{1}{2\pi i} \oint_{\partial R} (zI - P_C)^{-1} dz$ in $B(X_C)$ is induced by an idempotent $P_0$ in $B(X)$. Since $P_C - \frac{1}{2\pi i} \oint_{\partial R} (zI - P_C)^{-1} dz \in K(X_C)$, we see that $P - P_0 \in K(X)$.

**Proof of Theorem 1.3.** “(1)⇒(2)”: Let $\hat{P} := \lim_{n \to \infty} \hat{T} + \hat{T}^2 + \ldots + \hat{T}^n$. Since $\lim_{n \to \infty} \frac{\|\hat{T}^n\|}{n} = 0$,

\[
(\hat{T} - \hat{P}) = \lim_{n \to \infty} (\hat{T} - \hat{T}^2 + \ldots + \hat{T}^n) = \lim_{n \to \infty} \frac{\hat{T} - \hat{P}^2}{n + 1} = 0.
\]

Thus $\hat{T}P = \hat{P}$, and so

\[
\hat{P}^2 = \lim_{n \to \infty} \frac{\hat{P} + \hat{P}' + \ldots + \hat{P}'^n}{n + 1} = \lim_{n \to \infty} \frac{(n + 1)\hat{P}}{n + 1} = \hat{P}.
\]

Hence $\hat{P}$ is an idempotent in $B(X)/K(X)$. By Lemma 2.6 there exists an idempotent $P_0$ in $B(X)$ such that $P - P_0 \in K(X)$. Replacing $P$ with $P_0$, we can assume without loss of generality that $P$ is an idempotent in $B(X)$. Equation (2.1) also implies that $(I - T)P \in K(X)$, which means that $I - T$ is compact on $PX$. Hence $(I - T)^m$ is compact on $PX$.

We now show that $PX$ is an essentially maximal subspace of $X$ on which $(I - T)^m$ is compact. Suppose that $(I - T)^m$ is compact on a subspace $M_0$ of $X$ containing $PX$. Let

\[
f_n(z) := \frac{n + (n - 1)z + (n - 2)z^2 + \ldots + z^{n - 1}}{n + 1}, \quad z \in \mathbb{C}, \quad n \geq 1.
\]

Note that $\hat{T} - \frac{\hat{T} + \hat{T}^2 + \ldots + \hat{T}^n}{n + 1} = (\hat{T} - \hat{P})f_n(\hat{T})$. Therefore,

\[
\hat{T} - \hat{P} = (\hat{T} - \hat{P})^m = \lim_{n \to \infty} f_n(\hat{T})^m(\hat{T} - \hat{P})^m,
\]

and so

\[
\lim_{n \to \infty} \| (I - P) - (f_n(T)^m(I - T)^m + K_n) \| = 0,
\]
for some $K_1, K_2, \ldots \in K(X)$.

Since $(I - T)^m$ is compact on $M_0$, the operator $f_n(T)^m(I - T)^m$ is compact on $M_0$ and so is $f_n(T)^m(I - T)^m + K_n$ on $M_0$. Thus $(I - P)|_{M_0}$ is the norm limit of a sequence in $K(M_0, X)$, and so $I - P$ is compact on $M_0$. Since $PX \subseteq M_0$, we have that $(I - P)M_0 \subseteq M_0$. Therefore, $(I - P)(I - P)M_0 = I((I - P)M_0)$ is compact, and so $(I - P)M_0$ is finite dimensional. In other words, $\dim M_0/PX < \infty$.

“(2)⇒(1)”: By Proposition 2.5, $(I - T)^m = (I - \hat{T})^m$ has closed range. Since by assumption $\lim_{n \to \infty} \|T^n\|_e = 0$, $\lim_{n \to \infty} \|\hat{T}^n\| = \lim_{n \to \infty} \frac{\|T^n\|}{n} = 0$. By Mbekhta-Zemánek’s Theorem 1.2, the sequence $(M_n(\hat{T}))_n$ converges in norm to an element in $B(\hat{X})$. By Theorem 2.1 the result follows.

\[ \square \]

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