HERMITIAN FUNCTIONAL REPRESENTATION OF FREE LÉVY PROCESSES

JOSE-LUIS PÉREZ G., VÍCTOR PÉREZ-ABREU, AND ALFONSO ROCHA-ARTEAGA.

Abstract. A functional representation of free Lévy processes is established via an ensemble of unitarily invariant Hermitian matrix-valued Lévy processes. This is accomplished by proving functional asymptotics of their empirical spectral processes towards the law of a free Lévy processes. This result recovers a functional version of Wigner’s theorem and introduces a functional version of Marchenko-Pastur’s theorem providing the free Poisson process as the noncommutative limit process.

Key words: Asymptotic spectral distribution, Burgers equation, free Brownian motion, free infinitely divisible distribution, Hermitian Brownian motion, Hermitian Lévy process, interacting particles system, measure-valued process, Bercovici-Pata bijection.

1. Introduction

Let \( \{B^{(n)}(t)\}_{t \geq 0} = \{(b_{jk}(t))\}_{t \geq 0} \) be the \( n \times n \) Hermitian matrix-valued Brownian motion where \( (b_{jj}(t))_{j=1}^n, (\text{Re} b_{jk}(t))_{j < k}, (\text{Im} b_{jk}(t))_{j < k} \) is a set of \( n^2 \) independent one-dimensional Brownian motions with parameter \( \frac{1}{2}(1 + \delta_{jk}) \). This matrix-valued process was first considered by Dyson [10] and the study of its eigenvalue process leads to several primary results in Random Matrix Theory (RMT), noncolliding particles, free probability, and laws of noncommutative processes.

For any fixed \( t > 0 \), \( B^{(n)}(t) \) is a Gaussian Unitary Ensemble (GUE) of random matrices with parameter \( t \) and its matrix distribution is invariant under unitary conjugation as well as infinitely divisible with respect to the classical convolution of matrix distributions, [1], [21]. Let \( \{(\lambda_1(t), \lambda_2(t), ..., \lambda_n(t))\}_{t \geq 0} \) be the \( n \)-dimensional stochastic process of the eigenvalues of \( B^{(n)} \) and consider the empirical spectral process of the re-scaled process \( B^{(n)}/\sqrt{n} \)

\[
\mu^{(n)}_t = \frac{1}{n} \sum_{j=1}^n \delta_{\lambda_j(t)}, \quad t \geq 0,
\]

where \( \lambda_j(t) = \lambda_j(t)/\sqrt{n} \) and \( \delta_x \) is the unit mass at \( x \).

First, from the fundamental work of Wigner [33] in RMT, for each fixed \( t > 0 \), \( \mu^{(n)}_t \) converges as \( n \) goes to infinity, weakly almost surely, to the semicircle (Wigner) distribution with parameter \( t \),

\[
w_t(dx) = \frac{1}{2\pi t} \sqrt{4t - x^2} \ 1_{[-2\sqrt{t}, 2\sqrt{t}]}(x)dx;
\]

see also [1], [21], [30]. The Wigner distribution \( w_t \) is infinitely divisible with respect to the free convolution and it also appears as the limiting distribution in the free central limit theorem [18], [23], [31].

In this sense, \( w_1 \) is the free counterpart of the Gaussian distribution in classical infinite divisibility, playing in free probability the role the Gaussian distribution does in classical probability. This is the starting point of the subject of free infinite divisibility, [18], [23], [31]. Moreover, the family \( \{w_t\}_{t \geq 0} \) is the law of free Brownian motion, a family of selfadjoint elements \( \{Z_t\}_{t \geq 0} \) in a noncommutative probability space that has free increments and is such that for each \( 0 \leq s \leq t \), \( Z_t - Z_s \) has the law \( w_{t-s} \); see Biane [9].
Second, keeping $n \geq 1$ fixed, in a pioneering work, Dyson \[16\] realized that the eigenvalue dynamics is described by a diffusion process with non-smooth drift satisfying the Itô Stochastic Differential Equation (SDE)

\[
(1.3) \quad d\lambda_i(t) = dW_i(t) + \sum_{j \neq i} \frac{df}{\lambda_i(t) - \lambda_j(t)}, \quad t \geq 0, 1 \leq i \leq n,
\]

where $W_1, \ldots, W_n$ are independent one-dimensional standard Brownian motions, see also \[1\], \[30\]. The stochastic process \{(\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t))\}_{t \geq 0}$ is called the Dyson Brownian motion corresponding to the GUE. This is a primary example of a system of interacting particles governed by SDE with strong interactions due to the non-smooth drift coefficient, a phenomenon associated with the process of eigenvalues of several matrix continuous-time processes; see \[11\], \[12\].

A similar functional asymptotic behavior in the case of other matrix (continuous-time) diffusions has been considered, by Chan \[15\], and Rogers and Shi \[29\] leading to free Brownian motion; and by \[14\], \[27\] to other noncommutative processes like the dilation of the free Poisson process. The latter is a functional version of the Marchenko–Pastur law, but the noncommutative limiting process is not a free Lévy process. The case of a symmetric fractional Brownian motion was considered in \[25\], obtaining the non-commutative law of the fractional Brownian motion introduced in \[24\].

The goal of this paper is to establish a functional representation of free Lévy processes by matrix Lévy processes. This generalizes the results for free Brownian motion to general free Lévy processes. While the classical works of Chang \[15\] and Rogers and Shi \[29\] deal with continuous diffusions, our matrix processes have jumps. Up to the best of our knowledge, this is the first time that convergence of measure-valued empirical processes arising from matrix processes with jumps are considered.

More specifically, the main result of this paper is summarized as follows. Let $\mathcal{D}(\mathbb{R}_+, \mathcal{M}(\mathbb{R}))$ denote the space of right continuous functions with left limits from $\mathbb{R}_+$ into $\mathcal{M}(\mathbb{R})$, endowed with the Skorohod topology, where $\mathcal{M}(\mathbb{R})$ is the space of probability measures on $\mathbb{R}$ endowed with the topology of weak convergence. For a given $n \times n$ Hermitian process \{$X_t^{(n)} : t \geq 0\}$ let, for each $t \geq 0$, $\lambda_1^{(n)}(t) \geq \lambda_2^{(n)}(t) \geq \cdots \geq \lambda_n^{(n)}(t)$ denote the ordered eigenvalues of $X_t^{(n)}$. The empirical spectral measure-valued process of \{$X_t^{(n)} : t \geq 0\}$ is defined as

\[
(1.4) \quad \mu_t^{(n)}(dx) = \frac{1}{n} \sum_{m=1}^{n} \delta_{\lambda_m^{(n)}(t)}(dx), \quad t \geq 0.
\]

**Theorem 1.** Given a free Lévy process \{\$Z_t : t \geq 0\$\}, there exists an ensemble of matrix Lévy processes \{$X_t^{(n)} : t \geq 0\}, n \geq 1$, such that:

a) For each $n \geq 1$, \{$X_t^{(n)} : t \geq 0\}$ is a unitarily invariant Lévy process in the space of $n \times n$ Hermitian matrices.

b) For each $n > 1$ and $t > 0$, the matrix distribution of $X_t^{(n)}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^{n^2}$ and $P \left(X_t^{(n)} \text{ has simple spectrum} \right) = 1$.
c) For each \( n \geq 1 \), the non-zero jumps of \( X_t^{(n)} \), \( \Delta X_t^{(n)} = X_t^{(n)} - X_{t^-}^{(n)} \), are of rank one.

d) The empirical spectral measure-valued processes \( \{ \mu_t^{(n)} : t \geq 0 \} \) converge weakly in probability to the law of \( \{ Z_t : t \geq 0 \} \) as \( n \to \infty \), in the space \( D(\mathbb{R}_+, \mathbb{P}(\mathbb{R})) \).

We should emphasize that our main contribution is the functional convergence in probability in the space \( D(\mathbb{R}_+, \mathbb{P}(\mathbb{R})) \) of the sequence of empirical spectral processes given by \( \{ Z_t : t \geq 0 \} \) to the law of a free Lévy process. In particular this implies the convergence of finite dimensional distributions which can be obtained by the results in \([3]\) and \([13]\), together with Voiculescu’s asymptotic freeness theorem found in \([32]\) for independent unitary invariant Hermitian random matrices.

As a particular case, we obtain a functional version of the Marchenko–Pastur theorem, where the asymptotic noncommutative process is the free Poisson process.

The strategy to prove Theorem 1 and the needed principal results are as follows. Section 2 contains the background on Hermitian Lévy processes and free Lévy processes. Section 3 introduces the ensembles of Hermitian Lévy processes \( \{ X_t^{(n)} : t \geq 0 \} \) of Theorem 1 via the appropriate characteristic triplets associated to the free Lévy process \( \{ Z_t : t \geq 0 \} \) and with the properties (a)–(c) in Theorem 1. Section 4 deals with the dynamics of the semimartingales of the eigenvalues of \( X_t^{(n)} \), for which we use an Itô formula due to \([20]\) and helpful asymptotics for the associated local martingales are also proved. Section 5 presents the proof of the tightness of the spectral measure-valued processes \( \{ \mu_t^{(n)} : t \geq 0 \} \) in the space \( D(\mathbb{R}_+, \mathbb{P}(\mathbb{R})) \), which, as expected, is more complex than the Brownian case due to the jumps. The key facts are that for each \( n \geq 1 \), all the jumps of the Hermitian Lévy process \( \{ X_t^{(n)} : t \geq 0 \} \) are of rank one. Finally, Theorem 2 in Section 6 characterizes the Burgers equation satisfied by the Cauchy transform of the limiting family of laws \( \{ \mu_t : t \geq 0 \} \) of the sequence of spectral measure valued processes \( \{ \mu_t^{(n)} : t \geq 0 \} \) in \( D(\mathbb{R}_+, \mathbb{P}(\mathbb{R})) \). This allows us to identify the family \( \{ \mu_t : t \geq 0 \} \) as the law of the free Lévy process \( \{ Z_t : t \geq 0 \} \), employing a result of Bercovici and Voiculescu \([5]\) for free infinitely divisible measures with unbounded support (see also \([13]\) and \([20]\)).

2. Preliminaries on Hermitian and Free Lévy Processes

2.1. Unitary invariant Hermitian Lévy processes. In this section we consider a class of Hermitian Lévy processes whose distributions are invariant under unitary conjugation.

Let \( \mathbb{M}_n = \mathbb{M}_n(\mathbb{C}) \) denote the linear space of \( n \times n \) matrices with complex entries with scalar product \( \langle A, B \rangle = \text{tr}(B^*A) \) and the Frobenius norm \( \| A \| = (\text{tr}(A^*A))^{1/2} \) where \( \text{tr} \) denotes the (non-normalized) trace. The set of Hermitian matrices in \( \mathbb{M}_n \) is denoted by \( \mathbb{H}_n \), \( \mathbb{H}_n^0 = \mathbb{H}_n \setminus \{ 0 \} \) and \( \mathbb{H}^1_n \) is the set of rank one matrices in \( \mathbb{H}_n \). Let \( S_n \) denote the unit sphere in \( \mathbb{H}_n \), let \( S(\mathbb{H}_n^1) = S_n \cap \mathbb{H}^1_n \) and let \( \mathbb{H}^+_n \) denote the set of nonnegative definite Hermitian matrices. We denote the \( n \times n \) identity matrix by \( I_n \).

A random matrix \( X \) in \( \mathbb{H}_n \) is infinitely divisible if for all \( m \geq 1 \) there exist independent identically distributed random matrices \( X_1, ..., X_m \) in \( \mathbb{H}_n \) such that \( X_1 + ... + X_m \) and \( X \) have the same matrix distribution. In this case, the matrix distribution of \( X \) is characterized by the Lévy–Khintchine representation of its Fourier transform \( \mathbb{E}e^{i\theta X} = \exp(\varphi(\Theta)) \) with Laplace exponent

\[
\varphi(\Theta) = \text{itr}(\Theta \Psi_n) - \frac{1}{2} \text{tr}(\Theta A_n \Theta) + \int_{\mathbb{H}_n} \left( e^{i\theta \langle \Theta \xi \rangle} - 1 - \frac{\text{tr}(\Theta \xi)}{1 + \|\xi\|^2} \right) \nu_n(d\xi), \quad \Theta \in \mathbb{H}_n,
\]

where \( A_n : \mathbb{H}_n \to \mathbb{H}_n \) is a linear operator which is positive (i.e. \( \text{tr}(\Phi A_n \Phi) \geq 0 \) for \( \Phi \in \mathbb{H}_n \)) and symmetric (i.e. \( \text{tr}(\Theta_2 A_n \Theta_1) = \text{tr}(\Theta_1 A_n \Theta_2) \)) for \( \Theta_1, \Theta_2 \in \mathbb{H}_n \), \( \nu_n \) is a measure on \( \mathbb{H}_n \) (the Lévy measure) satisfying \( \nu_n(\{ 0 \}) = 0 \) and \( \int_{\mathbb{H}_n} (\|\xi\|^2 \wedge 1) \nu_n(d\xi) < \infty \), and \( \Psi_n \in \mathbb{H}_n \). The triplet \( (A_n, \Psi_n, \nu_n) \) is unique.

The following is straightforward.
Proposition 1. Fix $n \geq 1$ and let $X_n$ be an infinitely divisible $n \times n$ Hermitian random matrix with Lévy-Khintchine representation \[2.7\] with Lévy triplet $(A_n, \Psi_n, \nu_n)$, where

a) $\Psi_n = \gamma I_n, \gamma \in \mathbb{R}$.

b) $A_n \Theta = \sigma^2_n \Theta, \Theta \in \mathbb{H}_n$, and $\sigma^2_n \geq 0$.

c) $\nu_n(U E^*) = \nu_n(E)$ for each unitary $n \times n$ nonrandom matrix $U$ and $E \in \mathcal{B}(\mathbb{H}_n)$.

Then the distribution of $X_n$ is invariant under unitary conjugation.

Definition 1. An $n \times n$ matrix-valued process $\{X(t) : t \geq 0\}$ is a Hermitian Lévy process if for each $t > 0$, $X(t) \in \mathbb{H}_n$ and

i) $X(0) = 0$ with probability one,

ii) $X$ has independent increments: $\forall 0 \leq t_1 < \cdots < t_m, m \geq 1, X(t_m) - X(t_{m-1}), \ldots, X(t_2) - X(t_1)$ are independent random matrices,

iii) $X$ has stationary increments: $\forall 0 \leq s < t$, $X(t) - X(s)$ and $X(t - s)$ have the same matrix distribution, and

iv) for any $s \geq 0$, the increment $X(t + s) - X(s) \to 0_n$ in distribution as $t \to 0$, where $0_n$ is the $n \times n$ zero matrix.

A key feature of an $n \times n$ Hermitian Lévy process $X(t)$ with triplet given by Proposition 1 is that for each $t > 0$, the distribution of $X(t)$ is invariant under unitary conjugation. Furthermore, the nonzero jumps $\Delta X(t) = X(t) - X(t-)$ are random matrices of rank one.

Given any infinitely divisible $n \times n$ Hermitian random matrix $X$, there is a Hermitian Lévy process $\{X(t) : t \geq 0\}$ such that $X$ and $X(1)$ have the same distribution, and vice versa. In fact, $X(t)$ has the Fourier transform $\mathbb{E}[e^{i \text{tr}(\Theta X(t))}] = \exp(t \varphi(\Theta))$, where $\varphi$ is the above Laplace exponent.

Throughout this paper we will assume that $X(t)$ has an absolutely continuous distribution for each $t > 0$. In order for this condition to hold, we will ask that $X$ have a Gaussian component ($\sigma^2 \neq 0$) or that it satisfies condition D in \[20\]. Under this assumption, for each $t > 0$, $X(t)$ has a simple spectrum \[26\].

The following dynamics for the eigenvalues of a class of Hermitian Lévy processes is proved in \[26\].

Proposition 2. Let $\{X(t) : t \geq 0\}$ be an $n \times n$ Hermitian Lévy process with absolutely continuous distribution invariant under unitary conjugation, and with triplet $(\sigma^2 I_n \otimes I_n, \gamma I_n, \nu)$. Let $(\lambda_1(t), \ldots, \lambda_n(t))$ be the vector of eigenvalues of $X(t)$ where $\lambda_1(t) > \lambda_2(t) > \cdots > \lambda_n(t)$ for each $t \geq 0$. For each $m = 1, \ldots, n$, the eigenvalue $\lambda_m$ is a semimartingale and

\begin{equation}
\lambda_m(X_t) = \lambda_m(X_0) + \gamma \sum_{i=1}^{n} \int_0^t (D\lambda_m(X_{s-}))_{ii} ds + \sigma^2 \int_0^t \frac{1}{\lambda_m(X_{s-}) - \lambda_j(X_{s-})} ds + M^m_t
\end{equation}

\begin{equation}
+ \int_{[0,t] \times \mathbb{H}_n^0} \left[ \lambda_m(X_{s-} + y) - \lambda_m(X_{s-}) - \text{tr}(D\lambda_m(X_{s-})y)1_{\|y\| \leq 1} \right] \nu(dy) ds,
\end{equation}

with

\begin{equation}
M^m_t = \sigma \sum_{r=1}^{n} \sum_{l=1}^{n} \int_0^t (D\lambda_m(X_{s-}))_{rl} dB^{rl}_s + \int_{[0,t] \times \mathbb{H}_n^0} \left[ \lambda_m(X_{s-} + y) - \lambda_m(X_{s-}) \right] J_X(ds, dy),
\end{equation}

where $J_X(\cdot, \cdot)$ is the Poisson random measure of the jumps of $X$ on $[0, \infty) \times \mathbb{H}_n^0$ with intensity measure $\text{Leb} \otimes \nu$, independent of a family of independent one dimensional standard Brownian motions $B^{ij}_s$, $i, j = 1, \ldots, n$ and the compensated measure is given by

\begin{equation}
\tilde{J}_X(dt, dy) = J_X(dt, dy) - dt \nu(dy);
\end{equation}

and for each $s \geq 0$, $D\lambda_m(X_s)$ is the matrix of derivatives of $\lambda_m(X_s)$ with respect to the entries of $X_s$, given by

\begin{equation}
(D\lambda_m(X_s))_{ij} = 2\pi_{im}(s) u_{jm}(s)1_{i < j} + |u_{im}(s)|21_{i = j},
\end{equation}
where $u_{ij}(s)$, $i, j = 1, 2, \ldots, n$ are the entries of a unitary random matrix $U_s$ of eigenvectors of $X_s$.

**Remark 1.** If we take
\[
\tilde{M}_t^n := \sum_{\tau=1}^n \sum_{l=1}^n \int_0^t (D\lambda_m(X_{s-})_l)_{rt} dB_s^l,
\]
it is clear that for $m, m' = 1, \ldots, n$ its covariance process is given by
\[
\langle \tilde{M}_t^m, \tilde{M}_t^{m'} \rangle_t = t\delta_{mm'} \quad t > 0.
\]
Therefore, by Lévy’s Theorem, we can write, for $m = 1, \ldots, n$, the martingale term $M_t^m$ as
\[
M_t^m = \sigma W_t^m + \int_{(0, t) \times [0, 1]^n} [\lambda_m(X_{s-} + y) - \lambda_m(X_{s-})] J_X(ds, dy), \quad \text{for } t > 0,
\]
where $W^1, \ldots, W^n$ are independent one dimensional standard Brownian motions.

### 2.2. Free Lévy processes affiliated with $W^*$-probability spaces

We recall some facts on free Lévy processes acting on a $W^*$-probability space. For additional information on this subject, see [1], [4], [8], and [10]. A $W^*$-probability space is a pair $(\mathcal{G}, \tau)$ where $\mathcal{G}$ is a von Neumann algebra acting on a Hilbert space $H$ and $\tau$ is a normal faithful trace on $\mathcal{G}$. In the sequel, $(\mathcal{G}, \tau)$ will denote a $W^*$-probability space.

An unbounded operator $a$ in $H$ is not an element of $\mathcal{G}$. However, a selfadjoint linear operator $a$ in $H$ is affiliated with $\mathcal{G}$ if and only if $f(a) \in \mathcal{G}$ for any bounded Borel function $f : \mathbb{R} \to \mathbb{R}$. Here $f(a)$ is defined in the sense of spectral theory (the functional calculus). That is, for any selfadjoint operator $a$ affiliated with $\mathcal{G}$, there exists a unique probability measure $\mu_a$ on $\mathbb{R}$, concentrated on the spectrum of $a$, such that
\[
\tau(f(a)) = \int_{\mathbb{R}} f(s) \mu_a(ds),
\]
for every bounded Borel function $f : \mathbb{R} \to \mathbb{R}$. The measure $\mu_a$ is called the (spectral) distribution of $a$ and is denoted by $\mu_a = \mathcal{L}\{a\}$. Unless $a$ is bounded, the spectrum of $a$ is an unbounded subset of $\mathbb{R}$ and, in general, $\mu_a$ is not compactly supported.

**Definition 2.** Let $a_1, a_2, \ldots, a_r$ be selfadjoint operators affiliated with a $W^*$-probability space $(\mathcal{G}, \tau)$. It is said that $a_1, a_2, \ldots, a_r$ are freely independent with respect to $\tau$ if for any bounded Borel functions $f_1, f_2, \ldots, f_r : \mathbb{R} \to \mathbb{R}$, the bounded linear operators $f_1(a_1), f_2(a_2), \ldots, f_r(a_r)$ in $\mathcal{G}$ are freely independent with respect to $\tau$. That is,
\[
\tau\left\{[f_1(a_{i_1}) - \tau(f_1(a_{i_1}))][f_2(a_{i_2}) - \tau(f_2(a_{i_2}))] \cdots [f_r(a_{i_r}) - \tau(f_r(a_{i_r}))]\right\} = 0
\]
for any positive integer $m$ and any $i_1, i_2, \ldots, i_m$ in $\{1, 2, \ldots, r\}$ with $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{m-1} \neq i_m$.

A stochastic process affiliated with a $W^*$-probability space $(\mathcal{G}, \tau)$ is a family $\{Z_t : t \geq 0\}$ of selfadjoint operators affiliated with $\mathcal{G}$. Let us denote by $\mu_t = \mathcal{L}\{Z_t\}$ the (spectral) distribution of $Z_t$ for each $t \geq 0$. The family $\{\mu_t : t \geq 0\}$ of probability measures on $\mathbb{R}$ is called the family of spectral distributions of the process $\{Z_t : t \geq 0\}$. Moreover, for any $s \geq 0, t \geq 0$ such that $s \leq t$, the increment $Z_t - Z_s$ is again a selfadjoint operator affiliated with $\mathcal{G}$ and we denote its distribution by $\mu_{s, t} = \mathcal{L}\{Z_t - Z_s\}$.

**Definition 3.** A free Lévy process is a stochastic process $\{Z_t : t \geq 0\}$ affiliated with the $W^*$-probability space $(\mathcal{G}, \tau)$ such that:

1. $Z_0 = 0$.
2. For any $m \geq 1$ and $0 \leq t_1 < \cdots < t_m$, the increments
   \[
   Z_{t_m} - Z_{t_{m-1}}, \ldots, Z_{t_2} - Z_{t_1}
   \]
   are freely independent random variables.
iii) For any \( s \geq 0, t \geq 0 \) the spectral distribution of \( Z_{t+s} - Z_s \) does not depend on \( s \).

iv) For any \( s \geq 0 \), the increment \( Z_{t+s} - Z_s \to 0 \) in distribution as \( t \to 0 \), that is, the spectral distributions \( \mathcal{L}\{Z_{t+s} - Z_s\} \) converge weakly to \( \delta_0 \) as \( t \to 0 \).

It is well known that the law \( v = \mathcal{L}(Z_1) \) of a free Lévy process \( \{Z_t : t \geq 0\} \) is free infinitely divisible. Moreover, it has the Lévy–Khintchine representation \( \phi_v(z) = t\phi_v(z) \) in terms of the Voiculescu transform

\[
\phi_v(z) = \eta + \int_{\mathbb{R}} \frac{1 + tz}{z - t} \rho(dt), \quad (z \in \mathbb{C}^+),
\]

with generating pair \((\eta, \rho)\), where \( \eta \in \mathbb{R} \) and \( \rho \) is a finite measure on \( \mathbb{R} \), see [4], [8], [31].

Finally, let \( ID(*) \) and \( ID(\|) \) be the set of classical and free infinitely divisible distributions on \( \mathbb{R} \), respectively. The Benaych–Georges bijection [6], denoted by \( \Lambda \), between \( ID(*) \) and \( ID(\|) \), is such that for each \( \mu \in ID(*) \) with Lévy triplet \((\sigma^2, \gamma, \nu)\), \( \Lambda(\mu) \in ID(\|) \) has generating pair

\[
\rho(dx) = \sigma^2 \delta_0(dx) + \frac{x^2}{1 + x^2} \nu(dx),
\]

\[
\eta = \gamma - \int_{\mathbb{R}} x \left( 1_{[-1,1]}(x) - \frac{1}{1 + x^2} \right) \nu(dx),
\]

see [3], [4].

Benaych–Georges [6] and Cabanal–Duvillard [13] explained the bijection \( \Lambda \) via random matrix models. Their work constitutes a generalization of the Wigner semicircle law for the GUE to more general random matrices. The distributions of these random matrices share similar properties to those of the GUE, such as having an infinitely divisible matrix distribution which is invariant under unitary conjugation (Lévy unitary ensemble).

3. THE APPROXIMATING HERMITIAN LÉVY PROCESSES

In this section we introduce the ensemble of Hermitian valued processes considered in this paper. Let \( \{Z_t : t \geq 0\} \) be a free Lévy process with generating pair \((\eta, \rho)\) and let \( \sigma^2 = \rho(\{0\}) \). In a fixed probability space \((\Omega, \mathcal{F}, \mathbb{P})\) we construct a sequence of \( n \times n \) Hermitian Lévy processes \( \{X^{(n)}_t\}_{n \geq 1} = \{X^{(n)}_t : t \geq 0\}_{n \geq 1} \), such that for each \( n \geq 1 \), the generating triplet \( \left( \frac{n^2}{2} I_n \otimes I_n, \gamma I_n, \nu_n \right) \) is given by

a)

\[
\sigma^2_n = \sigma^2 + \frac{n - 1}{n^2}.
\]

b)

\[
\gamma = \eta + \int_{|r| \leq 1} r \rho(dr) - \int_{|r| > 1} \frac{1}{r} \rho(dr).
\]

c) The Lévy measure \( \nu_n \) is as follows

\[
\nu_n(E) = \int_{S(H^n_1)} \int_{\mathbb{R}} 1_E(r \xi) n \rho^0_n(dr) \pi_n(d\xi), \quad E \in \mathcal{B}(H^n_1 \setminus \{0\}),
\]

where:

i) \( \pi_n \) is a distribution on \( S(H^n_1) \) satisfying

\[
\int_{S(H^n_1)} 1_B(\xi) \pi_n(d\xi) = \int_{S(C_n)} 1_{\phi_n^{-1}(B)}(u) \pi(du), \quad B \in \mathcal{B}(S(H^n_1)),
\]

where \( \phi_n \) denotes the transformation \( u \to uu^* \) and \( \pi \) is the Haar distribution of a random vector in \( S(C_n) \), the unit sphere of \( C_n \).
ii) $\rho_n^\alpha$ is a measure defined on $\mathbb{R}$ for each $n \geq 1$ and $\alpha \in (0, 1/4)$ by
\begin{equation}
(3.5) \quad \rho_n^\alpha(dr) = \frac{1 + r^2}{r^4} \rho(dr)1_{(1/n,n^{2\alpha}/(n^\alpha-1))}(|r|),
\end{equation}
where the above expression is understood in the limiting sense when $n = 1$. Note that
\( \int_{-\infty}^{\infty} (1 + r^2) \rho_n^\alpha(dr) < \infty. \)

**Remark 2.**

i) $X_t^{(n)}$ has an absolutely continuous distribution, for each $t > 0$, $n > 1$, with respect to the Lebesgue measure on $\mathbb{R}^{2n}$, since $\sigma_n^2$ in (3.1) is non-zero.

ii) The spectrum of $X_t^{(n)}$ is simple for each $t > 0$, $n > 1$, by the absolute continuity, see [23].

iii) $X_t^{(n)}$ has a unitary invariant distribution, for each $t > 0$, $n > 1$, since the assumptions of Proposition 4 are satisfied. This follows since the spherical measure $\pi_n$ is a multiple of the Haar distribution and $\rho_n^\alpha$ does not depend on $\xi \in \mathbb{S}(\mathbb{H}^1_\alpha)$.

iv) The sequence of Hermitian matrices $\{X^{(n)}\}_{n \geq 1}$ is a Lévy unitary ensemble.

v) The non-zero jumps of $X^{(n)}$ are of rank one, i.e., if $\Delta X_t^{(n)} = X_t^{(n)} - X_{t-}^{(n)} \neq 0$ then $\Delta X_t^{(n)} \in \mathbb{H}^1_\alpha$. This follows since the spherical measure $\pi_n$ is concentrated on $\mathbb{S}(\mathbb{H}^1_\alpha)$.

vi) For fixed $n > 1$ and $t > 0$, our proposed models differ from the random matrix models given in [6, [13] in the sense that the distribution of $X_t^{(n)}$ always satisfy i) and ii). This is essential in our proof of the functional convergence.

**Example.** When $Z$ is the free Poisson process, its generating pair is given by: $\eta = -\lambda$, and $\rho = \lambda \delta_1$, with $\lambda > 0$. Then, for each $n > 1$
\[ X_t^{(n)} = \sum_{j=1}^{N_t} u_j u_j^* + \frac{n - 1}{n^2} B_t I_n \]
where $\{u_j\}_{j \geq 1}$ is a sequence of independent uniformly distributed random vectors in $\mathbb{C}_n$, $N = \{N_t\}_{t \geq 0}$ is a Poisson process of parameter $\lambda$ independent of $\{u_j\}_{j \geq 1}$ and $B = \{B_t\}_{t \geq 0}$ is a one-dimensional standard Brownian motion independent of $N$ and $\{u_j\}_{j \geq 1}$. Then Theorem 1 gives a functional version of the Marchenko–Pastur theorem, in which the asymptotic noncommutative process is the free Poisson process. We point out that the functional version in [14] gives as an asymptotic process the dilation of the free Poisson distribution, which is not a free Lévy process.

4. The Dynamics of the Eigenvalues and the Measure Valued Processes

In this section we establish some results needed later on for the sequence of semimartingales corresponding to the eigenvalues and the spectral measure valued processes of the ensemble of the Hermitian Lévy processes $\{X_t^{(n)} : t \geq 0\}_{n \geq 1}$ defined in Section 3.

Given $A, B \in \mathbb{H}_n$, we denote by $\mathcal{D}f(A)(B)$ the Gâteaux derivative of $f$ at $A$ in the direction $B$, that is
\[ \mathcal{D}f(A)(B) = \frac{d}{dt} \bigg|_{t=0} f(A + tB). \]

Similarly, we denote by $\mathcal{D}f(A)(B^2)$ to the second Gâteaux derivative of $f$ at $A$ in the direction $B$ i.e.
\[ \mathcal{D}f(A)(B^2) = \frac{\partial^2}{\partial t^2} \bigg|_{t=0} f(A + tB). \]

We refer to Chapter V.3 in [5] and Chapter 6.6 in [19] for a proper definition of the Gâteaux derivative of a function on the set of Hermitian matrices. We first prepare the following lemma.
Lemma 1. For $s \geq 0$ and a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$,

$$
\int_{S(C_n)} \sum_{m=1}^{n} \left[ f'(\lambda_m(n)(X_{s-}^{(n)})) \text{tr}(D\lambda_m(n)(X_{s-}^{(n)})vv^*) \right] \pi(dv) = \int_{S(C_n)} \text{tr}(Df(X_{s-}^*)(vv^*)) \pi(dv).
$$

Proof. We first observe that using (2.3) and Proposition 4.2.3 in [18],

$$
\int_{S(C_n)} \sum_{m=1}^{n} \left[ f'(\lambda_m(n)(X_{s-}^{(n)})) \text{tr}(D\lambda_m(n)(X_{s-}^{(n)})vv^*) \right] \pi(dv)
= \int_{S(C_n)} \sum_{m=1}^{n} \left[ f'(\lambda_m(n)(X_{s-}^{(n)}))(\sum_{i=1}^{n}(D\lambda_m(n)(X_{s-}^{(n)}))_{ik}v_i(s-))\pi(s-) \right] \pi(dv)
= \frac{1}{n} \sum_{m=1}^{n} \left[ f'(\lambda_m(n)(X_{s-}^{(n)}))\sum_{i=1}^{n}(D\lambda_m(n)(X_{s-}^{(n)}))_{ii} \right]
= \frac{1}{n} \sum_{m=1}^{n} \left[ f'(\lambda_m(n)(X_{s-}^{(n)}))\sum_{i=1}^{n}|u_{im}(s-)|^2 \right]
= \frac{1}{n} \sum_{m=1}^{n} f'(\lambda_m(n)(X_{s-}^{(n)})�
\right)
(4.1)

Next, using identity (V.13) in [5] (see also Theorem 6.6.30(1) in [19]), we obtain

$$
\text{tr}(Df(X_{s-}^*)(vv^*)) = \text{tr}\left( U_{s-} \cdot [f^{[1]}(\Lambda_{s-}) \circ (U_{s-}^*vv^*U_{s-})]U_{s-}^* \right),
$$

where $\circ$ denotes the Schur-product, and $f^{[1]}(\Lambda_{s-})$ is the matrix with entries given by

$$(f^{[1]}(\Lambda_{s-}))_{ij} = \begin{cases} 
\frac{f(\lambda_i(n)(X_{s-}^{(n)}) - f(\lambda_j(n)(X_{s-}^{(n)})}}{\lambda_i(n)(X_{s-}^{(n)}) - \lambda_j(n)(X_{s-}^{(n)})} & \text{if } i \neq j \\
 f'(\lambda_i(n)(X_{s-}^{(n)})) & \text{if } i = j.
\end{cases}
$$

Since $U_{s-}^*v$ and $v$ have the same distribution under $\pi$

$$
\int_{S(C_n)} \text{tr}(Df(X_{s-}^*)(vv^*)) \pi(dv) = \int_{S(C_n)} \text{tr}\left( f^{[1]}(\Lambda) \circ (vv^*) \right) \pi(dv)
= \int_{S(C_n)} \sum_{m=1}^{n} f'(\lambda_m(n)(X_{s-}^{(n)}))|v_m|^2 \pi(dv)
= \frac{1}{n} \sum_{m=1}^{n} f'(\lambda_m(n)(X_{s-}^{(n)})).
$$

(4.2)

The result now follows from (4.1) and (4.2). \hfill \square

Throughout this paper we will use the notation

$$
\langle \mu, f \rangle := \int_{\mathbb{R}} f(x) \mu(dx),
$$

for any bounded measurable function $f$ and $\mu \in \text{Pr}(\mathbb{R})$. We denote the set of functions $f : \mathbb{R} \to \mathbb{R}$ that are $k$ times continuously differentiable with bounded derivatives by $C^k_b(\mathbb{R})$, for $k = 1, 2$. 

$\square$
Proposition 3. Let \( \mu^{(n)}_t : t \geq 0 \) be the spectral measure-valued process \( \{X^{(n)}_t : t \geq 0\} \) of \( \{X_t^{(n)} : t \geq 0\} \).

Then, for each \( n \geq 1, t \geq 0, \) and \( f \in C_c^0(\mathbb{R}) \)
\[
\langle \mu^{(n)}_t, f \rangle = \langle \mu^{(n)}_0, f \rangle + M^{n,f}_t + \frac{\sigma^2}{2} \int_0^t \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu^*_n(dx)\mu^*_n(dy)ds + \gamma \int_0^t f'(x)\mu^*_n(dx)ds
\]
\[+ \int_0^t \int_{S(\mathbb{C}_n)} \int_{\mathbb{R}} \left[ f(X^{(n)}_{s^*} + rv^*) - f(X^{(n)}_{s^*}) - DF(X^{(n)}_{s^*})(rv^*)1_{\{r \leq 1\}} \right] \rho^*_n(dr)\pi_n(dy)ds. \tag{4.3}\]

Proof. For each \( n \geq 1, t \geq 0 \) let \( \lambda^{(n)}_1(t) > \cdots > \lambda^{(n)}_n(t) \) denote the eigenvalues of \( X^{(n)}_t \). Then, by Proposition 2
\[
\lambda^{(n)}_m(X^{(n)}_t) = \lambda_m(X^{(n)}_0) + \gamma \sum_{i=1}^n \int_0^t \langle D\lambda^{(n)}_m(X^{(n)}_{s^-}) \rangle_{ii} ds + \frac{\sigma^2}{n} \int_0^t \sum_{j \neq m} \frac{1}{(\lambda^{(n)}_m - \lambda^{(n)}_j)(s^-)} ds + M^{n,m}_t
\]
\[+ \int_0^t \int_{S(\mathbb{C}_n)} \int_{\mathbb{R}} \left[ \lambda^{(n)}_m(X^{(n)}_{s^*} + ry) - \lambda^{(n)}_m(X^{(n)}_{s^-}) - \text{tr}(D\lambda^{(n)}_m(X^{(n)}_{s^-}))ry 1_{\{|r| \leq 1\}} \right] \rho^*_n(dr)\pi_n(dy)ds, \tag{4.4}\]

where the process \( (M^{n,m}_t)_{t \geq 0} \) is a martingale.

From (4.4) and an application of Itô’s formula we obtain that
\[
\langle \mu^{(n)}_t, f \rangle = \langle \mu^{(n)}_0, f \rangle + M^{n,f}_t + \frac{\sigma^2}{n^2} \sum_{m=1}^n \int_0^t f'(\lambda^{(n)}_m(X^{(n)}_{s^-})) \sum_{j \neq m} \frac{1}{(\lambda^{(n)}_m - \lambda^{(n)}_j)(s^-)} ds
\]
\[+ \frac{\gamma}{n} \sum_{m=1}^n \sum_{i=1}^n \int_0^t f'(\lambda^{(n)}_m(X^{(n)}_{s^-})) (D\lambda^{(n)}_m(X^{(n)}_{s^-}))_{ii} ds + \frac{1}{2n} \sum_{m=1}^n \int_0^t f''(\lambda^{(n)}_m(X^{(n)}_{s^-})) d\langle M^{n,m}, M^{n,m} \rangle_s
\]
\[+ \sum_{m=1}^n \int_0^t f'(\lambda^{(n)}_m(X^{(n)}_{s^-})) \int_{S(\mathbb{C}_n)} \int_{\mathbb{R}} \left[ \lambda^{(n)}_m(X^{(n)}_{s^*} + ry) - \lambda^{(n)}_m(X^{(n)}_{s^-}) - \text{tr}(D\lambda^{(n)}_m(X^{(n)}_{s^-}))ry 1_{\{|r| \leq 1\}} \right] \rho^*_n(dr)\pi_n(dy)ds
\]
\[
+ \sum_{m=1}^n \int_0^t \int_{S(\mathbb{C}_n)} \int_{\mathbb{R}} \left[ f(\lambda^{(n)}_m(X^{(n)}_{s^*} + ry)) - f(\lambda^{(n)}_m(X^{(n)}_{s^-})) - f'(\lambda^{(n)}_m(X^{(n)}_{s^-}))(\Delta \lambda^{(n)}_m(X^{(n)})) \right] \rho^*_n(dr)\pi_n(dy)ds, \tag{4.5}\]

where \( (M^{n,f}_t)_{t \geq 0} \) is a local martingale, given by
\[M^{n,f}_t = \frac{\sigma^2}{n^{3/2}} \sum_{m=1}^n \int_0^t f'(\lambda^{(n)}_m(X^{(n)}_{s^-})) dW^m_s + \frac{\sigma^2}{2n} \sum_{m=1}^n \int_0^t f''(\lambda^{(n)}_m(X^{(n)}_{s^-})) d\langle M^{n,m}, M^{n,m} \rangle_s, \tag{4.6}\]

and \( W^m, m = 1, \ldots, n \) are independent one-dimensional Brownian motions.

Following Remark 1 it is easy to check that
\[
\frac{1}{2n} \sum_{m=1}^n \int_0^t f''(\lambda^{(n)}_m(X^{(n)}_{s^-})) d\langle M^{n,m}, M^{n,m} \rangle_s = \frac{\sigma^2}{2n^2} \sum_{m=1}^n \int_0^t f''(\lambda^{(n)}_m(X^{(n)}_{s^-})) ds,
\]
and therefore
\[
\frac{\sigma^2}{n^2} \sum_{m=1}^n \int_0^t f'(\lambda^{(n)}_m(X^{(n)}_{s^-})) \sum_{j \neq m} \frac{1}{(\lambda^{(n)}_m - \lambda^{(n)}_j)(s^-)} ds + \frac{1}{2n} \sum_{m=1}^n \int_0^t f''(\lambda^{(n)}_m(X^{(n)}_{s^-})) d\langle M^{n,m}, M^{n,m} \rangle_s
\]
\[= \frac{\sigma^2}{2} \int_0^t \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu^*_n(dx)\mu^*_n(dy)ds. \]
The drift term in (4.5) is expressed, using (2.3), as
\[
\sum_{m=1}^{n} \sum_{i=1}^{n} \int_{0}^{t} f'(\lambda_m^{(n)}(X_{s,-}^{(n)})) (D\lambda_m^{(n)}(s-))_{ii} ds = \sum_{m=1}^{n} \sum_{i=1}^{n} \int_{0}^{t} f'(\lambda_m^{(n)}(X_{s,-}^{(n)})) |\|\|_{im} u_{im}^{(n)} ds
\]
\[
= \sum_{m=1}^{n} \int_{0}^{t} f'(\lambda_m^{(n)}(X_{s,-}^{(n)})) ds = \gamma \int_{0}^{t} f'(x) \mu_s^{(n)}(dx) ds.
\]
The last two terms in (4.5) can be written, using (3.3) and (3.4), as
\[
\sum_{i=1}^{n} \sum_{m=1}^{n} \int_{S(C_{n})} \int_{\mathbb{R}} f(\lambda_m^{(n)}(X_{s,-}^{(n)} + rvv^*)) - f(\lambda_m^{(n)}(X_{s,-}^{(n)}))
\]
\[
- f'(\lambda_m^{(n)}(X_{s,-}^{(n)})) \text{tr}(D\lambda_m^{(n)}(X_{s,-}^{(n)})rvv^*)1_{\{|r| \leq 1\}} \rho_n^{(n)}(dr) \pi(dx) ds.
\]
Thus (4.5) is expressed as
\[
\langle \mu_t^{(n)}, f \rangle = \langle \mu_0^{(n)}, f \rangle + M_t^{n,f} + \frac{\sigma^2}{2} \int_{0}^{t} \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu_s^{(n)}(dx) \mu_s^{(n)}(dy) ds + \gamma \int_{0}^{t} \int_{\mathbb{R}} f'(x) \mu_s^{(n)}(dx) ds
\]
\[
+ \int_{0}^{t} \int_{S(C_{n})} \int_{\mathbb{R}} \sum_{m=1}^{n} \left[ f(\lambda_m^{(n)}(X_{s,-}^{(n)} + rvv^*)) - f(\lambda_m^{(n)}(X_{s,-}^{(n)}))
\]
\[
- f'(\lambda_m^{(n)}(X_{s,-}^{(n)})) \text{tr}(D\lambda_m^{(n)}(X_{s,-}^{(n)})rvv^*)1_{\{|r| \leq 1\}} \rho_n^{(n)}(dr) \pi(dx) ds.
\]
Using the fact that the last term in (4.7) is integrable, we obtain (4.3) by an application of Fubini’s theorem together with Lemma 1.

In the rest of this section we will study the convergence of the martingale term, to this end we need the following auxiliary result.

**Lemma 2.** Let $A \in \mathbb{H}_n$.
(i) For any $f \in C_b^1(\mathbb{R})$,
\[
\int_{S(C_{n})} |\text{tr}(Df(A)(vv^*))|^2 \pi(dx) \leq \|f''\|_\infty^2.
\]
(ii) For any $f \in C_b^2(\mathbb{R})$,
\[
\int_{S(C_{n})} |\text{tr}(D^2f(A)((vv^*)^2))| \pi(dx) \leq 2\|f''\|_\infty,
\]
and
\[
|\text{tr}(D^2f(A)((n^{-1}I_n)^2))| \leq 2\|f''\|_\infty.
\]

**Proof.** (i) Let $A = UAU^*$, where $\Lambda$ is a diagonal matrix, then by identity (V.13) in [5] (see also Theorem 6.6.30(1) in [19]) together with the fact that $U^*U = I$
\[
\text{tr}(Df(A)(vv^*)) = \text{tr}\left(U\left[f^{[1]}(\Lambda) \circ (U^*vv^*U)\right] U^*\right) = \text{tr}\left(\left[f^{[1]}(\Lambda) \circ (U^*vv^*U)\right]\right),
\]
where $\circ$ denotes the Schur product. Therefore using that $U^*v$ and $v$ have the same distribution under $\pi$
\[
\int_{S(C_{n})} |\text{tr}(Df(A)(vv^*))|^2 \pi(dx) = \int_{S(C_{n})} \left[\text{tr}\left(\left[f^{[1]}(\Lambda) \circ (vv^*)\right]\right)^2\right] \pi(dx) \leq \|f''\|_\infty^2.
\]
(ii) To obtain the second identity we use Exercise V.3.9 in [5] (see also Theorem 6.6.30(2) in [19]) to note that for a matrix $B \in \mathbb{H}_n$

\begin{equation}
(\text{4.9}) \quad \mathcal{D}^2 f(\Lambda)(B^2) = 2 \sum_{i,j,k} f^{[2]}(\lambda_i, \lambda_j, \lambda_k) P_i B P_j B P_k,
\end{equation}

where $P_i$ are the projections onto the coordinate axes, $f^{[2]}(\lambda_i, \lambda_j, \lambda_k) = (f^{[1]}(\lambda_i, \lambda_j) - f^{[1]}(\lambda_i, \lambda_k))/\lambda_j - \lambda_k$ if $\lambda_i, \lambda_j, \lambda_k$ are distinct, and defined by continuity otherwise. By a straightforward computation we have $(P_i B P_j B P_k)_{lm} = (B)_{ij}(B)_{jk} 1_{\{i=m=k\}}$ for $l, m = 1, \ldots, n$, this implies that

\begin{equation}
(\text{4.10}) \quad \text{tr}(P_i B P_j B P_k) = (B)_{ij}(B)_{ji} 1_{\{i=j\}}.
\end{equation}

Therefore, using that $U^* v$ and $v$ have the same distribution under $\pi$

\[
\int_{\mathbb{S}(\mathbb{C}_n)} |\text{tr}(\mathcal{D}^2 f(A) ((vv*)^2))| |\pi(\mathbf{dv})| = \int_{\mathbb{S}(\mathbb{C}_n)} |\text{tr}(U [\mathcal{D}^2 f(\Lambda) ((U^* vv^* U)^2)] U^*)| |\pi(\mathbf{dv})|
\]

\[
= 2 \int_{\mathbb{S}(\mathbb{C}_n)} \left| \sum_{i,j} f^{[2]}(\lambda_i, \lambda_j, \lambda_i)(vv^*)_{ij}(vv^*)_{ji} \right| \pi(\mathbf{dv})
\]

\[
\leq 2 \|f''\|_{\infty} \int_{\mathbb{S}(\mathbb{C}_n)} \sum_{i,j} |v_i|^2 |v_j|^2 \pi(\mathbf{dv}) = 2 \|f''\|_{\infty}.
\]

For the remaining identity we note that using (4.9) together with (4.10) we obtain

\[
|\text{tr}(\mathcal{D}^2 f(A) ((n^{-1}\mathbf{1}_n)^2))| \leq \frac{2}{n} \sum_{i} |f^{[2]}(\lambda_i, \lambda_j, \lambda_i)| \leq 2 \|f''\|_{\infty}.
\]

\[
\text{Lemma 3.} \quad \text{For any } f \in C^2_b(\mathbb{R}), \text{ the martingale } (M_t^{n,f})_{t \geq 0} \text{ in (4.6) satisfies}
\]

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} |M_t^{n,f}| = 0 \quad \text{in probability},
\]

for any $T > 0$.

**Proof.** We show the convergence of each term in (4.6). Let $\varepsilon > 0$. By Doob’s inequality applied to the first term on the right hand side of (4.6)

\[
P \left( \sup_{0 \leq t \leq T} \left| \frac{\sigma_n}{n^{3/2}} \int_0^t \sum_{m=1}^n f'(\lambda_m(n)(X_{s-}^m)) dW_s^m \right| > \varepsilon \right) \leq \frac{1}{\varepsilon^2} \frac{\sigma_n^2}{n^3} \mathbb{E} \left[ \left( \int_0^T \sum_{m=1}^n f'(\lambda_m(n)(X_{s-}^m)) dW_s^m \right)^2 \right]
\]

\[
\leq \frac{1}{\varepsilon^2} \frac{\sigma_n^2}{n^3} \mathbb{E} \left[ \int_0^T \sum_{m=1}^n (f'(\lambda_m(n)(X_{s-}^m)))^2 ds \right] \leq \frac{1}{\varepsilon^2} \frac{\sigma^2 + 1}{n^2} \|f''\|_{\infty}^2 T,
\]

which converges to 0 as $n \to \infty$.

Let us consider $n > 1$, then by an application of Doob’s inequality in the second term on the right hand side of (4.6) together with the mean value theorem and Lemma (2)(i),
\[ \mathbb{P} \left( \sup_{0 \leq t \leq T} \frac{1}{n} \int_0^t \int_{S(H^1_n)} \int_{\mathbb{R}} \text{tr} \left[ f(X^{(n)}_{s^-} + ry) - f(X^{(n)}_{s^-}) \right] J_X(ds, dr, dy) \right) > \varepsilon \]

\[ \leq \frac{1}{n^2} \frac{1}{\varepsilon^2} \mathbb{E} \left[ \left( \int_0^T \int_{S(C_n)} \int_{\mathbb{R}} \text{tr} \left[ f(X^{(n)}_{s^-} + ry) - f(X^{(n)}_{s^-}) \right] J_X(ds, dr, dy) \right)^2 \right] \]

\[ = \frac{1}{n} \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \int_{S(C_n)} \text{tr} \left( \left[ f(X^{(n)}_{s^-} + rvv^*) - f(X^{(n)}_{s^-}) \right]^2 \pi(dv) \rho_n^\alpha(dr)ds \right) \]

\[ \leq \| f' \|^2 \frac{1}{n} \frac{1}{\varepsilon^2} T \int_{\mathbb{R}} r^2 \rho_n^\alpha(dr) \leq \left( \frac{1}{n} + \frac{\alpha}{n(n^\alpha - 1)^2} \right) C(f, T), \]

for some constant \( C(f, T) > 0 \). Hence \( \left( \frac{1}{n} + \frac{\alpha}{n(n^\alpha - 1)^2} \right) C(f, T) \to 0 \) as \( n \to \infty \). This concludes the proof. \( \square \)

5. Tightness

Let \( \{(\mu_t^{(n)})_{t \geq 0} : n \geq 1\} \) be the family of measure valued-processes of the Hermitian Lévy process ensemble \( (X^{(n)})_{n \geq 1} \) introduced in Section 3. In this section we prove that this family is tight in the space \( \mathcal{D}(\mathbb{R}_+, \mathbb{P}(\mathbb{R})) \). We denote the set of functions \( f : \mathbb{R} \to \mathbb{R} \) that are twice differentiable with compact support by \( C^2_0(\mathbb{R}) \).

The keys to the proof are the eigenvalue semimartingale estimates of Section 4, the fact that for each \( n \geq 1 \) all the jumps of the Hermitian Lévy process \( \{X^{(n)}_t : t \geq 0\} \) are of rank one and Lemma 2.

**Proposition 4.** The family of measures \( \{(\mu_t^{(n)})_{t \geq 0} : n \geq 1\} \) is tight in the space \( \mathcal{D}(\mathbb{R}_+, \mathbb{P}(\mathbb{R})) \).

**Proof.** First we will start by establishing that the family of measure valued-processes \( \{(\mu_t^{(n)})_{t \geq 0} : n \geq 1\} \) is tight in the space \( \mathcal{D}(\mathbb{R}_+, \mathbb{P}(\mathbb{R})) \), when \( \mathbb{P}(\mathbb{R}) \) is endowed with the topology of the vague convergence. To this end, we use the Aldous–Rebolledo Criterion (see [2], [17], [28]) to prove that for each \( f \in C^2_0(\mathbb{R}) \) the sequence of real processes \( \{(\mu_t^{(n)}, f)_{t \geq 0} : n \geq 1\} \) is tight. We split the proof of the tightness of the semimartingale \( (\mu^{(n)}, f) \) into two steps: the first is on the bounded variation part and the second on the martingale part.

For any \( f \in C^2_0(\mathbb{R}) \) we have from [14,3] that

\[ \langle \mu_t^{(n)}, f \rangle - \langle \mu_s^{(n)}, f \rangle = \frac{\sigma^2}{2} \int_s^t \int_{\mathbb{R}^2} f'(x) f'(y) \mu^{(n)}(dx) \mu^{(n)}(dy) dx dy + \gamma \int_s^t \int_{\mathbb{R}} f'(x) \mu^{(n)}(dx) dx \]

\[ + M_{n,f}^{t,s} - M_{n,f}^{s,s} + \int_s^t \int_{S(C_n)} \int_{\mathbb{R}} \text{tr} \left[ f(X^{(n)}_{u^-} + rvv^*) - f(X^{(n)}_{u^-}) - \mathcal{D} f(X^{(n)}_{u-})(rvv^*) 1_{\{ |v| \leq 1 \}} \right] \rho_n^\alpha(dr) \pi(dv) du. \]

Let us denote by \( V^{n,f} \) the bounded variation part of the semimartingale \( (\mu^{(n)}, f) \). Then, for each \( 0 \leq s \leq t \),

\[ V_t^{n,f} - V_s^{n,f} = \frac{\sigma^2}{2} \int_s^t \int_{\mathbb{R}^2} f'(x) f'(y) \mu^{(n)}(dx) \mu^{(n)}(dy) dx dy + \gamma \int_s^t \int_{\mathbb{R}} f'(x) \mu^{(n)}(dx) dx \]

\[ + \int_s^t \int_{S(C_n)} \int_{\mathbb{R}} \text{tr} \left[ f(X^{(n)}_{u^-} + rvv^*) - f(X^{(n)}_{u^-}) - \mathcal{D} f(X^{(n)}_{u-})(rvv^*) 1_{\{ |v| \leq 1 \}} \right] \rho_n^\alpha(dr) \pi(dv) du. \]

\begin{equation}
(5.1)
\end{equation}

Let \( \delta > 0 \) and \( \theta \in [0, \delta] \). Let \( T' > 0 \) and let \( (\tau_n)_{n \geq 1} \) be a sequence of stopping times such that \( 0 \leq \tau_n < T' \).
Next we estimate each term in the right hand of (5.1). By the mean value theorem we have for any \( x, y \in \mathbb{R} \)
\[
|f'(x) - f'(y)| \leq \|f''\|_\infty |x - y|.
\]
Hence, for the first term in (5.1) we obtain
\[
\sigma_n \int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu_u^{(n)}(dx)\mu_u^{(n)}(dy) \, du \leq (\sigma^2 + 1)^{1/2}\|f''\|_\infty \delta.
\]
For the second term in the right hand of (5.1),
\[
\int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu_u^{(n)}(dx)\mu_u^{(n)}(dy) \, du \leq \|f''\|_\infty \delta.
\]
For the jump part in (5.1), we can apply Fubini’s theorem to obtain
\[
\int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{S}(\mathbb{C}_n)} \int_{\mathbb{R}} \text{tr} \left[ f(X_{u_n}^{(n)} + r vv^*) - f(X_{u_n}^{(n)}) - \mathcal{D}f(X_{u_n}^{(n)})(r vv^*)1_{(|r|\leq 1)} \right] \rho_n^\alpha(dr)\pi(dv) \, du
\]
\[
= \int_{\tau_n}^{\tau_n+\theta} \int_{|r|\geq 1} \int_{\mathbb{S}(\mathbb{C}_n)} \text{tr} \left[ f(X_{u_n}^{(n)} + r vv^*) - f(X_{u_n}^{(n)}) \right] \pi(dv)\rho_n^\alpha(dr) \, du
\]
\[
+ \int_{\tau_n}^{\tau_n+\theta} \int_{|r|\leq 1} \int_{\mathbb{S}(\mathbb{C}_n)} \text{tr} \left[ f(X_{u_n}^{(n)} + r vv^*) - f(X_{u_n}^{(n)}) - \mathcal{D}f(X_{u_n}^{(n)})(r vv^*)1_{(|r|\leq 1)} \right] \pi(dv)\rho_n^\alpha(dr) \, du.
\]
For the first term in the right hand side of (5.4) we use Lemma III.5 in [13] and the fact that \( f \) has compact support to obtain
\[
\int_{\tau_n}^{\tau_n+\theta} \int_{|r|\geq 1} \int_{\mathbb{S}(\mathbb{C}_n)} \text{tr} \left[ f(X_{u_n}^{(n)} + r vv^*) - f(X_{u_n}^{(n)}) \right] \pi(dv)\rho_n^\alpha(dr) \, du
\]
\[
\leq \|f\|_1^\prime \int_{\tau_n}^{\tau_n+\theta} \int_{|r|\geq 1} \rho_n^\alpha(dr) \, du \leq C_1(f)\delta,
\]
where \( C_1(f) > 0 \) is a constant and \( \|g\|_1^\prime = \sup_{x_1 \leq y_1 \leq x_2 \leq \cdots \leq y_n} \sum_{i=1}^n |g(y_i) - g(x_i)| \) for any \( g \in C^2(\mathbb{R}) \).

For the second term in the right hand of (5.4) we use Lemma 2(ii) and Taylor’s theorem to obtain
\[
\int_{\tau_n}^{\tau_n+\theta} \int_{|r|\leq 1} \int_{\mathbb{S}(\mathbb{C}_n)} \text{tr} \left[ f(X_{u_n}^{(n)} + r vv^*) - f(X_{u_n}^{(n)}) - \mathcal{D}f(X_{u_n}^{(n)})(r vv^*)1_{(|r|\leq 1)} \right] \pi(dv)\rho_n^\alpha(dr) \, du
\]
\[
= \int_{\tau_n}^{\tau_n+\theta} \int_{|r|\leq 1} \int_{\mathbb{S}(\mathbb{C}_n)} \int_0^1 r^2(1 - \xi) \left[ \text{tr} \left( \mathcal{D}^2 f(X_{u_n}^{(n)} + r\xi vv^*) (vv^*)^2 \right) \right] d\xi \pi(dv)\rho_n^\alpha(dr) \, du
\]
\[
\leq \|f\|_1^\prime \int_{\tau_n}^{\tau_n+\theta} \int_{|r|\leq 1} r^2 \rho_n^\alpha(dr) \, du \leq C_2(f)\delta,
\]
where \( C_2(f) > 0 \) is a constant.

From (5.2), (5.3), (5.5) and (5.6) we conclude that there exists a constant \( K_1 > 0 \), which does not depend on \( n \geq 1 \), such that
\[
\sup_{n \geq 1} \sup_{\theta \in [0,\delta]} \mathbb{E} \left[ \left| V_{\tau_n+\theta}^{n,f} - V_{\tau_n}^{n,f} \right| \right] < K_1\delta.
\]
The next step is to prove the tightness of the laws of the martingale part of the semimartingale \(\langle \mu_t^{(n)}, f \rangle\). Recall that the quadratic variation of the martingale \(M^{n,f}\) is given by

\[
\langle M^{n,f}, M^{n,f} \rangle_t = \frac{\sigma_n^2}{n^3} \int_0^t \sum_{m=1}^n (f'(\lambda_m^{(n)}(X^{(n)}_{s-})))^2 ds \\
+ \frac{1}{n} \int_0^t \int_{\mathcal{S}(\mathcal{C}_n)} \int_{\mathbb{R}} \left( \text{tr} \left[ f(X^{(n)}_{s-} + rvv^*) - f(X^{(n)}_{s-}) \right] \right)^2 \rho_n^\alpha(dr)\pi(dv) ds.
\]

For the first term, note that

\[
\frac{\sigma_n^2}{n^3} \mathbb{E} \left[ \int_{\tau_n}^{\tau_n + \theta} \sum_{m=1}^n (f'(\lambda_m^{(n)}(X^{(n)}_{s-})))^2 ds \right] \leq \frac{(\sigma + 1)^2}{n^2} \|f'\|^2 \delta.
\]

For the second term, by the proof of Lemma 3 together with Lemma III.5 in [13], one obtains for \(n \geq 1\)

\[
\frac{1}{n} \mathbb{E} \left[ \int_{\tau_n}^{\tau_n + \theta} \int_{\mathcal{S}(\mathcal{C}_n)} \int_{\mathbb{R}} \left( \text{tr} \left[ f(X^{(n)}_{s-} + rvv^*) - f(X^{(n)}_{s-}) \right] \right)^2 \rho_n^\alpha(dr)\pi(dv) ds \right]
\leq \frac{\delta}{n} \left( \|f\|^2 \int_{|r| \leq 1} (1 + r^2)\rho(dr) + (\|f\|^4)^2 \int_{|r| > 1} \frac{1 + r^2}{r^2}\rho(dr) \right) \leq \delta C_3(f),
\]

for some constant \(C_3(f) > 0\). From (5.8) and (5.9) there exists a constant \(K_2 > 0\) independent of \(n \geq 1\) such that

\[
\sup_{n \geq 1} \sup_{\theta \in [0,\delta]} \mathbb{E} \left[ \langle M^{n,f}, M^{n,f} \rangle_{\tau_n + \theta} - \langle M^{n,f}, M^{n,f} \rangle_{\tau_n} \right] < \delta K_2.
\]

Let us fix \(T > 0\). Then proceeding as in the first part of the proof, it can be seen that there exists a constant \(K_1(T) > 0\) depending on \(T\) such that

\[
\sup_{n \geq 1} \mathbb{E} \left[ \left( \sup_{t \in [0,T]} \langle M^{n,f}, M^{n,f} \rangle_t \right)^2 \right] < K_1(T).
\]

On the other hand, from the proof of Lemma 3 there exists a constant \(K_2(T) > 0\), that depends on \(T\), such that

\[
\sup_{n \geq 1} \mathbb{E} \left[ \left( \sup_{t \in [0,T]} \langle \mu_t^{(n)}, f \rangle \right)^2 \right] < K_2(T).
\]

Therefore there exists a constant \(K(T) > 0\) depending on \(T\) such that

\[
\sup_{n \geq 1} \mathbb{E} \left[ \left( \sup_{t \in [0,T]} \langle \mu_t^{(n)}, f \rangle \right)^2 \right] < K(T).
\]

Now, from (5.7), (5.10) and (5.11), we can use the Aldous–Rebolledo criterion (see [2], [17], [28]) to conclude that the sequence of real processes \(\{(\mu_t^{(n)}, f)\}_{t \geq 0} : n \geq 1\) is tight, and that consequently the sequence of processes \(\{\mu_t^{(n)} : n \geq 1\} \) is tight in the space \(\mathcal{D}(\mathbb{R}_+, \text{Pr}(\mathbb{R}))\) with \(\text{Pr}(\mathbb{R})\) endowed with the topology of vague convergence.

It remains to extend the above result to the case when \(\text{Pr}(\mathbb{R})\) is endowed with the topology of weak convergence. Note that taking \(f = 1\), the sequence of real-valued processes \(\{(\mu_t^{(n)}, f)\}_{t \geq 0} : n \geq 1\) is tight. On the other hand note that Lemma 3 implies that for any convergent subsequence of
\{(\mu_i^{(n)})_{t\geq 0} : n \geq 1\}, the limit is strongly continuous. Therefore by an application of the Méléard–Roelly criterion (see [22]), it follows that the sequence \{(\mu_i^{(n)})_{t\geq 0} : n \geq 1\} is tight in the space \(D(\mathbb{R}_+, \text{Pr}(\mathbb{R}))\).

\section{Characterization of the weak limit of the measure valued-processes}

Let \(z \in \mathbb{C} \setminus \mathbb{R}\) and \(f_z(x) = (z - x)^{-1}\) for \(x \in \mathbb{R}\). For any continuous function \((\nu_t)_{t\geq 0} \in C(\mathbb{R}_+, \text{Pr}(\mathbb{R}))\), the Cauchy–Stieltjes transform of \(\nu_t\) is defined as

\[\psi_\nu(t, z) := \int_{\mathbb{R}} f_z(x) \nu_t(dx)\]

We identify the weak limit of the sequence \(\{(\mu_i^{(n)})_{t\geq 0} : n \geq 1\}\) to be the family \((\mu_t)_{t\geq 0}\) such that its Cauchy–Stieltjes transform satisfies the Burgers equation that characterizes the Cauchy–Stieltjes transform of the law of the free Lévy process \(\{Z_t : t \geq 0\}\), appearing in the proof of Theorem 5.10 in Bercovici and Voiculescu [8] (see also [13] proof of Theorem III.2 and [20] (Theorem 4.5 and Appendix A.1)). This is shown in the following result.

\textbf{Theorem 2.} Assume that \(\mu_0^{(n)}\) converges weakly to \(\delta_0\). Then the family of measure-valued processes \(\{(\mu_i^{(n)})_{t\geq 0} : n \geq 1\}\) converges weakly in \(D(\mathbb{R}_+, \text{Pr}(\mathbb{R}))\) to a unique continuous probability-measure-valued function \((\mu_t)_{t\geq 0}\), satisfying for each \(t \geq 0\),

\begin{equation}
\frac{\partial}{\partial t} \psi_\mu(t, z) = -\sigma^2 \psi_\mu(t, z) \frac{\partial}{\partial z} \psi_\mu(t, z) - \eta \frac{\partial}{\partial z} \psi_\mu(t, z) - \frac{\partial}{\partial z} \psi_\mu(t, z) \int_{\mathbb{R} \setminus \{0\}} \frac{\psi_\mu(t, z) + r}{1 - r \psi_\mu(t, z)} \rho(dr).
\end{equation}

The following auxiliary lemma can be obtained from the proof of Lemma III.6 in [13].

\textbf{Lemma 4.} Let \(X = (X_1, \ldots, X_n), Y = (Y_1, \ldots, Y_n)\) be random vectors and let \(V = (V_1, \ldots, V_n)\) be an independent random vector Haar distributed on \(S(\mathbb{C}_n)\). Then for any \(\varepsilon > 0\) and any bounded function \(f : \mathbb{R} \rightarrow \mathbb{C}\) there exists a constant \(C(f) > 0\) not dependent on \(X\) such that

\[\mathbb{P} \left( \left\| \sum_{i=1}^{n} |V_i|^2 f(X_i) - \frac{1}{n} \sum_{i=1}^{n} f(X_i) \right\| \geq \varepsilon \right) \leq C(f) \exp \left( -\frac{(n-1)}{8\|f\|_{\infty}^2} \varepsilon^2 \right).
\]

The following lemma will be useful for identifying the limit law of the sequence of empirical measures \(\{(\mu^{(n)})_{t\geq 0} : n \geq 1\}\).

\textbf{Lemma 5.} For \(z \in \mathbb{C} \setminus \mathbb{R}\) and \(t > 0\),

\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^t \int_{S(\mathbb{C}_n)} \int_{\mathbb{R}} \text{tr} \left[ f_z(X^{(n)}_{u^+} + rvv^*) - f_z(X^{(n)}_{u^-}) - Df_z(X^{(n)}_{u^-})(rvv^*)1_{\{|r| \leq 1\}} \right] 
- \text{tr} \left[ f_z \left( X^{(n)}_{u^-} + \frac{r}{n} I_n \right) - f_z(X^{(n)}_{u^-}) - Df_z(X^{(n)}_{u^-})(\frac{r}{n} I_n)1_{\{|r| \leq 1\}} \right] \rho_n^\ast(dr) \pi(dv)du \right] = 0.
\]

\textbf{Proof.} (i) First we will obtain some estimations needed for the rest of the proof. For \(r \in \mathbb{R}\) and \(\xi > 0\) let us denote by \(\{\lambda_i^{(n)}(u, v, r, \xi)\}_{i=1}^{n}\) to the family of eigenvalues of the matrix \(X^{(n)}_{u^+} + \frac{r}{n} I_n + r \xi (vv^* - \frac{1}{n} I_n)\). Additionally, we define the diagonal matrix \(\Lambda(u, v, r, \xi) := \sum_{i=1}^{n} \lambda_i^{(n)}(u, v, r, \xi) P_i\), where \(P_i\) are the projections onto the coordinate axes of \(\mathbb{H}_n\).
Hence, using Lemma 4 we obtain that for every \( r \in \mathbb{R} \)

\[
\int_{S(C_n)} \left| \text{tr} \left[ f_z(X_{u-}^{(n)} + r vv^*) - f_z \left( X_{u-}^{(n)} + \frac{r}{n} I_n \right) \right] \right| \pi(dv)
\]

\[
\leq \int_{S(C_n)} \int_0^1 |r| \left| \text{tr} \left[ D f_z \left( X_{u-}^{(n)} + \frac{r}{n} I_n + r \xi \left( vv^* - \frac{1}{n} I_n \right) \right) \right] \right| d\xi \pi(dv)
\]

\[
= \int_{S(C_n)} \int_0^1 |r| \left| \text{tr} \left[ D f_z (\Lambda(u, v, r, \xi)) \circ \left( vv^* - \frac{1}{n} I_n \right) \right] \right| d\xi \pi(dv)
\]

\[
= \int_{S(C_n)} \int_0^1 |r| \sum_{i=1}^n |v_i|^2 f'_z(\lambda_i^{(n)}(u, v, r, \xi)) - \frac{1}{n} \sum_{i=1}^n f'_z(\lambda_i^{(n)}(u, v, r, \xi)) \right| d\xi \pi(dv),
\]

where in order to obtain the first equality we applied \( (4.3) \) together with the fact that under \( \pi, vv^* \) is invariant under unitary conjugations.

Therefore, for every \( \varepsilon > 0 \)

\[
\mathbb{E} \left[ \int_{S(C_n)} \left| \text{tr} \left[ f_z(X_{u-}^{(n)} + r vv^*) - f_z \left( X_{u-}^{(n)} + \frac{r}{n} I_n \right) \right] \right| \pi(dv) \right]
\]

\[
\leq \int_{S(C_n)} \int_0^1 2|r| \| f_z \|_\infty \mathbb{P} \left( \sum_{i=1}^n |v_i|^2 f'_z(\lambda_i^{(n)}(u, v, r, \xi)) - \frac{1}{n} \sum_{i=1}^n f'_z(\lambda_i^{(n)}(u, v, r, \xi)) \geq \varepsilon \right) d\xi \pi(dv)
\]

\[
+ \varepsilon \int_{S(C_n)} \int_0^1 |r| d\xi \pi(dv).
\]

Hence, using Lemma 4 we obtain that for every \( r \in \mathbb{R} \), and \( \varepsilon > 0 \)

\[
\mathbb{E} \left[ \int_{S(C_n)} \left| \text{tr} \left[ f_z(X_{u-}^{(n)} + r vv^*) - f_z \left( X_{u-}^{(n)} + \frac{r}{n} I_n \right) \right] \right| \pi(dv) \right]
\]

\[
\leq |r| \left[ \varepsilon + 2\| f'_z \|_\infty C(f_z) \exp \left( \frac{(n-1)\varepsilon^2}{8\| f_z \|^2_{\infty}} \right) \right].
\]

(6.2)

Finally, using \( (4.3) \) together with Lemma 4

\[
\mathbb{E} \left[ \int_{S(C_n)} \left| \text{tr} [D f_z(X_{u-}^{(n)})(vv^* - \frac{1}{n} I_n)] \right| \pi(dv) \right] \leq \mathbb{E} \left[ \int_{S(C_n)} \left| \sum_{i=1}^n f'_z(\lambda_i^{(n)}(X_{u-}^{(n)})) \left( |v_i|^2 - \frac{1}{n} \right) \right| \pi(dv) \right]
\]

\[
\leq \varepsilon + 2\| f'_z \|_\infty \int_{S(C_n)} \mathbb{P} \left( \sum_{i=1}^n f'_z(\lambda_i^{(n)}(X_{u-}^{(n)})) \left( |v_i|^2 - \frac{1}{n} \right) \geq \varepsilon \right) \pi(dv)
\]

\[
\leq \varepsilon + 2\| f'_z \|_\infty C(f_z) \exp \left( \frac{(n-1)}{8\| f_z \|^2_{\infty}} \varepsilon^2 \right).
\]

(6.3)
(ii) Let us consider $\beta \in (1/4, 1/2)$. Then using (6.2) with $\varepsilon = n^{-\beta}$ and the fact that $\alpha \in (0, 1/4)$ we obtain

$$
\mathbb{E} \left[ \int_0^t \int_{S(C_n)} \int_{|r| \geq 1} \tau \left[ f_x(X_{u-}^{(n)} + rvv^*) - f_x(X_{u-}^{(n)}) \right] \rho_n^\alpha(dr) \pi(dv) du \right.
$$

$$
- \int_0^t \int_{S(C_n)} \int_{|r| \geq 1} \tau \left[ f_x \left( X_{u-}^{(n)} + \frac{r}{n} I_n \right) - f_x(X_{u-}^{(n)}) \right] \rho_n^\alpha(dr) \pi(dv) du \right]
$$

$$
\leq t \int_{|r| \geq 1} \left( \frac{1 + r^2}{r^2} \right) 1_{(1/n, n^{2\alpha/(n^{\alpha-1})})}(|r|) \rho(dr) \left[ \frac{1}{n^{\beta}} + 2 \| f_x^\prime \|_\infty C(f_x) \exp \left( -\frac{(n-1)}{8 \| f_x^\prime \|_\infty^2 n^{2\beta}} \right) \right]
$$

\begin{align}
(6.4) & \\
\leq t \int_{|r| \geq 1} \left( \frac{1 + r^2}{r^2} \right) \rho(dr) \left[ \frac{n^{2\alpha}}{n^{\beta}(n^{\alpha-1})} + 2 \| f_x^\prime \|_\infty C(f_x) \frac{n^{2\alpha}}{(n^{\alpha-1})} \exp \left( -\frac{(n-1)}{8 \| f_x^\prime \|_\infty^2 n^{2\beta}} \right) \right] \frac{n^{\alpha}}{n^{\alpha}} \to 0.
\end{align}

(iii) Using (6.2) together with (6.3) we obtain for each $r \in \mathbb{R}$

$$
\limsup_{n \to \infty} \mathbb{E} \left[ \int_{S(C_n)} \tau \left[ f_x(X_{u-}^{(n)} + rvv^*) - f_x(X_{u-}^{(n)}) - Df_x(X_{u-}^{(n)})(rvv^*) \right]
$$

$$
- \tau \left[ f_x \left( X_{u-}^{(n)} + \frac{r}{n} I_n \right) - f_x(X_{u-}^{(n)}) - Df_x(X_{u-}^{(n)}) \left( \frac{r}{n} I_n \right) \right] \pi(dv) \right]
$$

$$
\leq \limsup_{n \to \infty} \left\{ \mathbb{E} \left[ \int_{S(C_n)} \tau \left[ f_x(X_{u-}^{(n)} + rvv^*) - f_x(X_{u-}^{(n)}) \right] \pi(dv) \right] + \mathbb{E} \left[ \int_{S(C_n)} \tau \left[ Df_x(X_{u-}^{(n)}) (rvv^* - \frac{r}{n} I_n) \right] \pi(dv) \right] \right\}
$$

\begin{align}
(6.5) & \\
& \leq \limsup_{n \to \infty} 2|r| \left[ \varepsilon + 2 \| f_x^\prime \|_\infty C(f_x) \exp \left( -\frac{(n-1)}{8 \| f_x^\prime \|_\infty^2 \varepsilon^2} \right) \right] \leq 2|r| \varepsilon.
\end{align}

So taking $\varepsilon \downarrow 0$ in (6.5) we obtain

$$
\lim_{n \to \infty} \mathbb{E} \left[ \int_{S(C_n)} \tau \left[ f_x(X_{u-}^{(n)} + rvv^*) - f_x(X_{u-}^{(n)}) - Df_x(X_{u-}^{(n)})(rvv^*) \right] \pi(dv) \right]
$$

$$
- \tau \left[ f_x \left( X_{u-}^{(n)} + \frac{r}{n} I_n \right) - f_x(X_{u-}^{(n)}) - Df_x(X_{u-}^{(n)}) \left( \frac{r}{n} I_n \right) \right] \pi(dv) \right] = 0.
$$

(6.6)

We also note that by using Taylor’s theorem and Lemma (ii)

$$
\int_{S(C_n)} \tau \left[ f_x(X_{u-}^{(n)} + rvv^*) - f_x(X_{u-}^{(n)}) - Df_x(X_{u-}^{(n)})(rvv^*) \right] \pi(dv)
$$

$$
- \tau \left[ f_x \left( X_{u-}^{(n)} + \frac{r}{n} I_n \right) - f_x(X_{u-}^{(n)}) - Df_x(X_{u-}^{(n)}) \left( \frac{r}{n} I_n \right) \right] \pi(dv) \right]
$$

$$
\leq \int_{S(C_n)} \int_0^1 r^2(1-\xi) \left[ \tau \left[ Df_x(X_{u-}^{(n)} + r\xi vv^*) ((vv^*)^2) - Df_x \left( X_{u-}^{(n)} + \frac{r\xi}{n} I_n \right) \left( \frac{1}{n} I_n \right)^2 \right] \right] d\xi\pi(dv)
$$

\begin{align}
(6.7) & \\
& \leq 4 \| f_x'' \|_\infty r^2 \int_0^1 (1-\xi) d\xi = 2r^2 \| f_x'' \|_\infty.
\end{align}
Hence by (6.7) we can apply dominated convergence and using (6.6) we obtain

$$
\lim_{n \to \infty} E \left[ \int_0^t \int_{S(C_n)} \int_{|r| \leq 1} \left[ \begin{array}{c}
\operatorname{tr} \left( f_z(X_{u-}^{(n)} + rvv^*) - f_z(X_{u-}^{(n)}) - D f_z(X_{u-}^{(n)})(rvv^*) \right) \\
- \operatorname{tr} \left[ f_z \left( X_{u-}^{(n)} + \frac{r}{n}I_n \right) - f_z(X_{u-}^{(n)}) - D f_z(X_{u-}^{(n)})(\frac{r}{n}I_n) \right] \right] \right] \\
\leq \int_0^t \int_{|r| \leq 1} \lim_{n \to \infty} E \left[ \int_{S(C_n)} \operatorname{tr} \left[ f_z(X_{u-}^{(n)} + rvv^*) - f_z(X_{u-}^{(n)}) - D f_z(X_{u-}^{(n)})(rvv^*) \right] \right] \pi(dv) \left( \frac{1 + r^2}{r^2} \right) \rho(dr)du = 0.
$$

(6.8)

The result now follows from (6.4) and (6.8). \( \square \)

**Proof of Theorem 2.** Following the discussion previous to Theorem 2, it is enough to prove the convergence to (6.1) for a suitable subsequence. From Proposition 4, the family \( \{(\mu_t^{(n)})_{t \geq 0} : n \geq 1\} \) is relatively compact. Hence, there exists a subsequence \( \{n_k\}_{k \geq 1} \) such that \( \{(\mu_t^{(n_k)})_{t \geq 0} : k \geq 1\} \) converges weakly to some \( \mu_t \) in \( D(\mathbb{R}_+, \mathbb{P}(\mathbb{R})) \).

First we note that using Lemma III.7 in [13] together with (4.2)

$$
\int_0^t \int_{S(C_{n_k})} \int_{\mathbb{R}} \operatorname{tr} \left[ f_z \left( X_{u-}^{(n_k)} + \frac{r}{n_k}I_{n_k} \right) - f_z(X_{u-}^{(n_k)}) - D f_z(X_{u-}^{(n_k)})(\frac{r}{n_k}I_{n_k}) \right] \left( 1_{|r| \leq 1} \right) \rho_{n_k}^\alpha(dr) \pi(dv)du
$$

(6.9)

Next, by (4.7) together with (6.9) we obtain

$$
\psi_{\mu^{(n_k)}}(t, z) - \psi_{\mu^{(n_k)}}(0, z) + \sigma_{n_k^2} \int_0^t \psi_{\mu^{(n_k)}}(s, z) \frac{\partial}{\partial z} \psi_{\mu^{(n_k)}}(s, z) ds + \gamma \int_0^t \frac{\partial}{\partial z} \psi_{\mu^{(n_k)}}(s, z) ds
$$

(6.10) \quad - \int_0^t \int_{\mathbb{R}} \left[ -r \frac{\partial}{\partial z} \psi_{\mu^{(n_k)}}(s, z) \frac{1}{1 - r \psi_{\mu^{(n_k)}}(s, z)} + \frac{\partial}{\partial z} \psi_{\mu^{(n_k)}}(s, z) \right] \left( 1_{|r| \leq 1} \right) \rho_{n_k}^\alpha(dr)ds = \Psi_{n_k}(t, z),

where

$$
\Psi_{n_k}(t, z) = M_t^{\mu^{(n_k)}} f_z + \int_0^t \int_{S(C_{n_k})} \int_{\mathbb{R}} \left[ \operatorname{tr} \left[ f_z(X_{u-}^{(n_k)} + rvv^*) - f_z(X_{u-}^{(n_k)}) - D f_z(X_{u-}^{(n_k)})(rvv^*) \right] \\
- \operatorname{tr} \left[ f_z \left( X_{u-}^{(n_k)} + \frac{r}{n_k}I_{n_k} \right) - f_z(X_{u-}^{(n_k)}) - D f_z(X_{u-}^{(n_k)})(\frac{r}{n_k}I_{n_k}) \right] \right] \rho_{n_k}^\alpha(dr) \pi(dv)du.
$$

(6.11)

By Lemmas 3 and 5 we obtain

$$
\lim_{k \to \infty} \Psi_{n_k}(t, z) = 0, \quad \text{in probability.}
$$

Hence from (6.10), (6.11) and the continuous mapping theorem we have that the Cauchy–Stieltjes transform of any weak limit \( \mu_t \) of the subsequence \( \{(\mu_t^{(n_k)})_{t \geq 0} : k \geq 1\} \) should satisfy the
following partial differential equation
\[
\frac{\partial}{\partial t} \psi_\mu(t, z) = -\sigma^2 \psi_\mu(t, z) \frac{\partial}{\partial z} \psi_\mu(t, z) - \gamma \frac{\partial}{\partial z} \psi_\mu(t, z) - \frac{\partial}{\partial z} \psi_\mu(t, z) \int_{0 < |r| \leq 1} \frac{\psi_\mu(t, z)}{1 - r \psi_\mu(t, z)} (1 + r^2) \rho(dr) \\
- \frac{\partial}{\partial z} \psi_\mu(s, z) \int_{|r| > 1} \frac{r}{1 - r \psi_\mu(s, z)} \left(1 + \frac{r^2}{r^2}\right) \rho(dr) \\
= -\sigma^2 \psi_\mu(t, z) \frac{\partial}{\partial z} \psi_\mu(t, z) - \gamma \frac{\partial}{\partial z} \psi_\mu(t, z) - \frac{\partial}{\partial z} \psi_\mu(t, z) \left(\int_{0 < |r| \leq 1} \frac{\psi_\mu(t, z)}{1 - r \psi_\mu(t, z)} (1 + r^2) \rho(dr) + \int_{|r| \leq 1} r \rho(dr)\right) \\
- \frac{\partial}{\partial z} \psi_\mu(s, z) \left(\int_{|r| > 1} \frac{r}{1 - r \psi_\mu(s, z)} \left(1 + \frac{r^2}{r^2}\right) \rho(dr) - \int_{|r| > 1} \frac{1}{r} \rho(dr)\right) \\
= -\sigma^2 \psi_\mu(t, z) \frac{\partial}{\partial z} \psi_\mu(t, z) - \gamma \frac{\partial}{\partial z} \psi_\mu(t, z) - \frac{\partial}{\partial z} \psi_\mu(t, z) \int_{\mathbb{R} \setminus \{0\}} \frac{\psi_\mu(t, z) + r}{1 - r \psi_\mu(t, z)} \rho(dr).
\]

\[
\square
\]

References

[1] Anderson, G. W., Guionnet, A., Zeitouni, O. An Introduction to Random Matrices. Cambridge University Press, (2009).
[2] Aldous, D. Stopping times and tightness. Ann. Probab. 6, 335–340, (1978).
[3] Barndorff-Nielsen, O. E., and Thorbjørnsen, S. A connection between free and classical infinite divisibility. Inf. Dim. Anal. Quantum Probab. 7, 573–590, (2004).
[4] Barndorff-Nielsen, O. E., and Thorbjørnsen, S. Classical and free infinite divisibility and Lévy processes. In: Quantum Independent Increment Processes II. M. Schürmann and U. Franz (eds). Lecture Notes in Mathematics 1866, Springer-Verlag, Berlin, (2006).
[5] Bhatia, R. Matrix Analysis. Graduate Text in Mathematics 169. Springer-Verlag, Berlin, (1997).
[6] Benaych-Georges, F. Classical and free infinitely divisible distributions and random matrices. Ann. Probab., 33, 1134–1170, (2005).
[7] Bercovici, H., and Voiculescu, D. Free convolution of measures with unbounded supports. Indiana Univ. Math. J. 42, 733–773, (1993).
[8] Biane, P. Free Brownian motion, free stochastic calculus and random matrices. Free Probability Theory (Waterloo, ON, Canada, 1995). Fields Inst. Commun. 12, Amer. Math. Soc., RI, pp. 1–19, (1997).
[9] Biane, P. Processes with free increments. Math. Z. 227, 143–174, (1998).
[10] Bru, M. F. Diffusions of perturbed principal component analysis. J. Multivariate Anal. 29, 127–136, (1989).
[11] Bru, M. F., Wishart processes. J. Theor. Prob. 4, 1, 725–751, (1991).
[12] Cabanal-Duvillard, T. A matrix representation of the Bercovici–Pata bijection. Electron. J. Probab. 10, 632–661, (2005).
[13] Cabanal-Duvillard, T., and Guionnet, A. Large deviations upper bounds for the laws of matrix-valued processes and non-communicative entropies. Ann. Probab. 29, 1205–1261, (2001).
[14] Chan, T. The Wigner semicircle law and eigenvalues of matrix-valued diffusions. Probab. Theory Relat. Fields 93, 249–272, (1992).
[15] Dyson, F. J. A Brownian-motion model for the eigenvalues of a random matrix. J. Math. Phys. 3, 1191–1198, (1962).
[16] Etheridge, A. An Introduction to Superprocesses. American Mathematical Society, Providence, RI, (2000).
[17] Hiai, F., and Petz, D. The Semicircle Law, Free Random Variables and Entropy. Volume 77 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, (2000).
[18] Horn, R. A., and Johnson, C. R. Topics in Matrix Analysis. Cambridge University Press, (1991).
[19] Hotta, I., and Schleinberger, S., Limits of radial multiple SLE and a Burgers–Loewner differential equation. ArXiv:1904.02556v2 [math.PR], (2020).
[20] Mehta, M. L. Random Matrices, 3rd edition. Elsevier, (2004).
[21] Méléard, S., and Roelly, S. Sur les convergences étroite ou vague de processus à valeurs mesures. C. R. Acad. Sci. Paris Sér. I Math. 317, 785–788, (1993).
Nica, A., and Speicher, R. Lectures on the Combinatorics of Free Probability. London Mathematical Society Lecture Notes Series 335, Cambridge University Press, (2006).

Nourdin, I., and Taqqu, M. Central and non-central limit theorems in a free probability setting. *J. Theoret. Probab.* 27, 220–248, (2014).

Pardo, J.C., Perez, J.-L., Perez-Abreu, V. A random matrix approximation to free fractional Brownian motion. *J. Theoret. Probab.* 29, 1581–1598, (2016).

Pérez-Abreu, V., and Rocha-Arteaga, A. On the process of the eigenvalues of Hermitian Lévy processes. In: The Fascination of Probability, Statistics and their Applications: Festschrift in Honour of Ole E. Barndorff-Nielsen, M. Podolskij, R. Stelzer, S. Thorbjørnsen, A. Veraart, eds., Springer. pp. 231-249, (2016).

Pérez-Abreu, V., and Tudor, C. On the Traces of Laguerre Processes. *Electron. J. Probab.* 14, 2241–2263, (2009).

Roelly-Coppoletta, S. A criterion of convergence of measure-valued processes: Application to measure branching processes. *Stoch. Stoch. Rep.* 17, 43–65, (1986).

Rogers L.C.G., and Shi, Z. Interacting Brownian particles and the Wigner law. *Probab. Theory Relat. Fields* 95, 555–570, (1993).

Tao, T. *Topics in Random Matrix Theory*. Amer. Math. Soc., (2012).

Voiculescu, D., Dykema, K. J., Nica, A. *Free Random Variables*. CRM Monographs Series, Vol. 1, pp. 1–18, (1992).

Voiculescu, D. "Lectures on free probability theory". *Lectures on Probability and Statistics: Ecole d’Ete de Probabilités de Saint-Flour XXVIII*. Lecture Notes in Mathematics. Springer. Vol. 1738, pp. 283-349, (2000).

Wigner, E. Characteristic vectors of bordered random matrices with infinite dimensions. *Ann. Math.* 62, 548–564, (1955).