Some New Identities on the \((h, q)\)-Genocchi Numbers and Polynomials with Weight \(\alpha\)

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Abstract: In this paper, we deal with \((h, q)\)-Genocchi numbers and polynomials with weight \(\alpha\). We also derive some new properties. Also, we introduce not only new but also interesting properties of \((h, q)\)-Genocchi numbers with weight \(\alpha\) by using the fermionic \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\) and the weighted \(q\)-Bernstein polynomials.

Keywords: \((h, q)\)-Genocchi numbers and polynomials with weight \(\alpha\), weighted Bernstein polynomials, fermionic \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\).

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1 Introduction and Notations

Let \(p\) be a fixed odd prime number. Throughout this paper we use the following notations. By \(\mathbb{Z}_p\) we denote the ring of \(p\)-adic rational integers, \(\mathbb{Q}\) denotes the field of rational numbers, \(\mathbb{Q}_p\) denotes the field of \(p\)-adic rational numbers, and \(\mathbb{C}_p\) denotes the completion of algebraic closure of \(\mathbb{Q}_p\).

Let \(\mathbb{N}\) be the set of natural numbers and \(\mathbb{N}^* = \mathbb{N} \cup \{0\}\). The normalized \(p\)-adic absolute value is defined by

\[
|p|_p = \frac{1}{p}.
\]

In this paper, we will assume that \(|q - 1|_p < 1\) as an indeterminate. Let \(UD(\mathbb{Z}_p)\) be the space of uniformly differentiable functions on \(\mathbb{Z}_p\). For \(f \in UD(\mathbb{Z}_p)\), the fermionic \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\) is defined by T. Kim:

\[
I_{-q}(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_{-q}(\xi) = \lim_{n \to \infty} \sum_{k=0}^{n-1} q^{k} f(\xi) (-1)^{\xi} \quad (1)
\]

(for more information, see [28], [29] and [30]).

From (1), we easily see that

\[
q I_{-q}(f_1) + I_{-q}(f) = [2]_{q} f(0) \quad (2)
\]

where \(f_1(x) := f(x+1)\) (for details, see[2-40]).

Let \(C([0,1])\) be the space of continuous functions on \([0,1]\). For \(C([0,1])\), the weighted \(q\)-Bernstein operator for \(f\) is defined by

\[
\mathcal{B}_{n,q}^{(\alpha)}(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{n,q}^{(\alpha)}(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \left[\frac{1}{1-x^{q_k}}\right] \quad (3)
\]

where \(n, k \in \mathbb{N}^*\). Here \(B_{n,q}^{(\alpha)}(x)\) are called the weighted \(q\)-Bernstein polynomials and defined by

\[
B_{n,q}^{(\alpha)}(x) = \binom{n}{k} x^{\frac{kn}{q}} (1-x)^{\frac{n-k}{q}} \quad (3)
\]

(for more information, see [3], [32], [38] and [39]).

As it is well known, the familiar Genocchi polynomials are defined by means of the following generating function:

\[
G_n(x) = e^{xt} \frac{(e^t - 1)^n}{n!} \quad (4)
\]

where \(G_n(x) := G_n(x)\), symbolically. For \(x = 0\) in (4), we have to \(G_n(0) := G_n\), which are called Genocchi numbers and given by

\[
d^{G_n} = \sum_{n=0}^{\infty} G_n x^n = \frac{2t}{e^t + 1} \quad (5)
\]

In [4], the \(q\)-Genocchi numbers are given by

\[
G_{0,q} = 0 \quad \text{and} \quad q(qG_q + 1)^n + G_{n,q} = \begin{cases} [2]_q & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}
\]
with the usual convention about replacing \((G_{a})^{n}\) by \(G_{n,a}\).

For any \(n \in \mathbb{N}^{*}\), the \((h,q)\)-Genocchi numbers are introduced by
\[
G_{0,a}^{(h)} = 0 \quad \text{and} \quad q^{h-1} \left( qG_{a}^{(h)} + 1 \right)^{n} + G_{n,a}^{(h)} = \begin{cases} [2] \quad \text{if} \ n = 1, \\ 0 \quad \text{if} \ n \neq 1 \end{cases}
\]
with the usual convention about replacing \((G_{a})^{n}\) by \(G_{n,a}\) (for details, see [5]).

Recently, Araci et al. have defined the \((h, q)\)-Genocchi numbers with weight \(\alpha\) as
\[
\frac{\tilde{G}_{n+1,a}^{(h)}(x)}{n+1} = \int_{\mathbb{Z}_{p}} q^{(h-1)\xi} [x + \xi]_{q}^{n} \alpha \ d\mu_{-q}(\xi). \quad (6)
\]

By (6), we have the following identity:
\[
\tilde{G}_{n,a}^{(h)}(x) = \sum_{k=0}^{n} \binom{n}{k} q^{\alpha k} (\tilde{G}_{a}^{(h)})^k [x]_{q}^{n-k} = q^{-\alpha} \left( q^{\alpha} \tilde{G}_{a}^{(h)} + [x]_{q} \right)^{n} \quad (7)
\]
with the usual convention about replacing \((\tilde{G}_{a}^{(h)})^{n}\) by \(\tilde{G}_{n,a}^{(h)}\) is used (for details, [5]).

In this paper, we derive some new properties \((h, q)\)-Genocchi numbers and polynomials with weight \(\alpha\) arising from the fermionic \(p\)-adic \(q\)-integral on \(\mathbb{Z}_{p}\) and weighted \(q\)-Bernstein polynomials.

2 On the \((h, q)\)-Genocchi numbers and polynomials with weight \(\alpha\)

In this section, we consider the \((h, q)\)-Genocchi numbers and polynomials with weight \(\alpha\) by using fermionic \(p\)-adic \(q\)-integral on \(\mathbb{Z}_{p}\) and the weighted \(q\)-Bernstein polynomials. We now start with the following expression.

In [5], we have the \((h, q)\)-Genocchi numbers with weight \(\alpha\) as follows: for \(\alpha \in \mathbb{N}^{*}\) and \(n, h \in \mathbb{N}\),
\[
\tilde{G}_{0,a}^{(h)} = 0 \quad \text{and} \quad q^{h} \tilde{G}_{n,a}^{(h)}(1) + \tilde{G}_{n,a}^{(h)} = \begin{cases} [2] \quad \text{if} \ n = 1, \\ 0 \quad \text{if} \ n \neq 1 \end{cases} \quad (8)
\]

By (7) and (8), we obtain the following corollary.

**Corollary 1.** For \(\alpha \in \mathbb{N}^{*}\) and \(n, h \in \mathbb{N}\), then we have
\[
\tilde{G}_{0,a}^{(h)} = 0 \quad \text{and} \quad q^{h-\alpha} \left( q^{\alpha} \tilde{G}_{a}^{(h)} + 1 \right)^{n} + \tilde{G}_{n,a}^{(h)} = \begin{cases} [2] \quad \text{if} \ n = 1, \\ 0 \quad \text{if} \ n \neq 1 \end{cases} \quad (9)
\]

By (6), we get symmetric property by the following basic applications:
\[
\tilde{G}_{n+1,a}^{(h+1)}(1-x) = \int_{\mathbb{Z}_{p}} q^{(h-1)\xi} [1-x+\xi]q^{n}_{-q} \ d\mu_{-q}^{-1}(\xi) = (-1)^{n} q^{h+\alpha-1} \int_{\mathbb{Z}_{p}} q^{(h-1)\xi} [x+\xi]q^{n}_{-q} \ d\mu_{-q}(\xi)
\]
Thus, we obtain the following theorem.

**Theorem 1.** The following identity
\[
\tilde{G}_{n+1,a}^{(h+1)}(1-x) = (-1)^{n} q^{h+\alpha-1} \tilde{G}_{n+1,a}^{(h+1)}(x) \quad (10)
\]
is true.

By using (7), (8) and (9), we compute
\[
q^{2\alpha} \tilde{G}_{n,a}^{(h)}(2) = (q^{2\alpha} \tilde{G}_{a}^{(h)} + [2]_{q^{n}})^{n} \quad (11)
\]
\[
= \sum_{l=0}^{n} \binom{n}{l} q^{al} (q^{\alpha} \tilde{G}_{a}^{(h)} + 1)^{l}
\]
\[
= nq^{2\alpha-h} \left( [2]_{q^{l}} \tilde{G}_{1,a}^{(h)} \right) - q^{\alpha-h} \sum_{l=2}^{n} \binom{n}{l} q^{al} \tilde{G}_{l,a}^{(h)}
\]
\[
= nq^{2\alpha-h}[2]_{q} + q^{2\alpha-2h} \tilde{G}_{n,a}^{(h)} \quad \text{if} \ n > 1.
\]

After the above applications, we procure the following theorem.

**Theorem 2.** For \(n > 1\), we have
\[
\tilde{G}_{n,a}^{(h)}(2) = nq^{-h}[2]_{q} + q^{-2h} \tilde{G}_{n,a}^{(h)}.
\]

We need the following equality for sequel of this paper:
\[
[1-x]q^{n}_{\alpha} = \left( \frac{1-q^{-\alpha(1-x)}}{1-q^{-\alpha}} \right)^{n} = (-1)^{n} q^{\alpha} [x-1]q^{n}_{\alpha}. \quad (12)
\]

Now also, by using (12), we consider the following
\[
q^{h-1} \int_{\mathbb{Z}_{p}} q^{(h-1)\xi} [1-\xi]q^{n}_{-q} \ d\mu_{-q}(\xi)
\]
\[
= (-1)^{n} q^{h+n\alpha-1} \int_{\mathbb{Z}_{p}} q^{(h-1)\xi} [\xi-1]q^{n}_{-q} \ d\mu_{-q}(\xi)
\]
\[
= (-1)^{n} q^{h+n\alpha-1} \tilde{G}_{n+1,a}^{(h+1)}(1) \quad (13)
\]

By considering last identity and (10), we get the following theorem.

**Theorem 3.** The following identity holds true:
\[
\int_{\mathbb{Z}_{p}} q^{(h-1)(\xi+1)} [1-\xi]q^{n}_{\alpha} \ d\mu_{-q}(\xi) = \tilde{G}_{n+1,a}^{(h+1)}(2) \quad (14)
\]

From (13), we have the following
\[
\tilde{G}_{n+1,a}^{(h+1)}(1-x) = [2]_{q} + q^{h+1} \tilde{G}_{n+1,a}^{(h+1)}(x) \quad (15)
\]
Thus, we obtain the following theorem.

**Theorem 4.** The following identity
\[
\int_{\mathbb{Z}_{p}} q^{(h-1)\xi} [1-\xi]q^{n}_{\alpha} \ d\mu_{-q}(\xi) = [2]_{q} + q^{h+1} \tilde{G}_{n+1,a}^{(h+1)}(x) \quad (16)
\]
is true.
3 Some new identities on the \((h, q)\)-Genocchi numbers with weight \(\alpha\)

In this section, we introduce the new identities of the \((h, q)\)-Genocchi numbers with weight \(\alpha\), that is, we derive some interesting relations.

For \(x \in [0, 1]\), we recall the definition of weighted \(q\)-Bernstein polynomials as follows:

\[
B_{k,n}^{(\alpha)}(x | q) = \left(\begin{array}{c}
\alpha \\alpha \\alpha \\
(n)
\end{array}\right) \frac{x^k}{[1-x]^n} d \mu_q(x), \quad \text{where } n, k \in \mathbb{Z}_+.
\]

By expression (15), we have the symmetry property of weighted \(q\)-Bernstein polynomials, as follows:

\[
B_{k,n}^{(\alpha)}(x | q) = B_{n-k,n}^{(\alpha)}(1-x | \frac{1}{q}), \quad \text{(for details, see [32])}.
\]

Thus, (14), (15) and (16), we see that

\[
I_1 = \int_{\mathbb{Z}_p} q^{(h-1)x} B_{k,n}^{(\alpha)}(x | q) d \mu_q(x)
\]

\[
= \binom{n}{k} \int_{\mathbb{Z}_p} q^{(h-1)x} [x]_{q,a}^k [1-x]_{q,a}^{-n-k} d \mu_q(x)
\]

\[
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} q^{(h-1)x} [x]_{q,a}^k [1-x]_{q,a}^{-n-k} d \mu_q(x)
\]

\[
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{k}{l} (-1)^{k-l} \left[2q + q^{h+1} G_{n-l-1,q}(a) \right],
\]

\[
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{k}{l} (-1)^{k-l} \left[2q + q^{h+1} G_{n-l-1,q} \right].
\]

On the other hand, for \(n, k \in \mathbb{Z}_+\) with \(n > k\), we compute

\[
I_2 = \int_{\mathbb{Z}_p} q^{(h-1)x} B_{k,n}^{(\alpha)}(x | q) d \mu_q(x)
\]

\[
= \binom{n}{k} \int_{\mathbb{Z}_p} q^{(h-1)x} [x]_{q,a}^k [1-x]_{q,a}^{-n-k} d \mu_q(x)
\]

\[
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} q^{(h-1)x} [x]_{q,a}^k [1-x]_{q,a}^{-n-k} d \mu_q(x)
\]

\[
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \left[2q + q^{h+1} G_{l+k+1,q} \right],
\]

Equating \(I_1\) and \(I_2\), we have the following theorem.

**Theorem 5.** The following identity holds true:

\[
\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l G_{l+k+1,q} = \begin{cases} 
2q + q^{h+1} G_{n-l-1,q} & \text{if } k = 0, \\
\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l G_{l+k+1, q} & \text{if } k \neq 0.
\end{cases}
\]

Let \(n_1, n_2, k \in \mathbb{Z}_+\) with \(n_1 + n_2 > 2k\). Then, we derive the followings:

\[
I_3 = \int_{\mathbb{Z}_p} q^{(h-1)x} B_{k,n_1}^{(\alpha)}(x | q) d \mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1} \binom{k}{l} (-1)^{n_1-l} \int_{\mathbb{Z}_p} q^{(h-1)x} [x]_{q,a}^k [1-x]_{q,a}^{-n_1-l} d \mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1} \binom{k}{l} (-1)^{n_1-l} \left[2q + q^{h+1} G_{n_1-l-1,q} \right],
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1} \binom{k}{l} (-1)^{n_1-l} \left[2q + q^{h+1} G_{n_1-l-1,q} \right].
\]

In other words, by using the binomial theorem, we can derive the following equation.

\[
I_4 = \int_{\mathbb{Z}_p} q^{(h-1)x} B_{k,n_1}^{(\alpha)}(x | q) d \mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1} \binom{k}{l} (-1)^{n_1-l} \int_{\mathbb{Z}_p} q^{(h-1)x} [x]_{q,a}^k [1-x]_{q,a}^{-n_1-l} d \mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1} \binom{k}{l} (-1)^{n_1-l} \left[2q + q^{h+1} G_{n_1-l-1,q} \right],
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1} \binom{k}{l} (-1)^{n_1-l} \left[2q + q^{h+1} G_{n_1-l-1,q} \right].
\]

Combining \(I_3\) and \(I_4\), we state the following theorem.

**Theorem 6.** For \(n_1, n_2, k \in \mathbb{Z}_+\) with \(n_1 + n_2 > 2k\), we have

\[
\sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l \int_{\mathbb{Z}_p} q^{(h-1)x} [x]_{q,a}^k [1-x]_{q,a}^{-n_1-n_2+2k-l} d \mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1} \binom{k}{l} (-1)^{n_1-l} \left[2q + q^{h+1} G_{n_1-l-1,q} \right],
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1} \binom{k}{l} (-1)^{n_1-l} \left[2q + q^{h+1} G_{n_1-l-1,q} \right].
\]

For \(x \in \mathbb{Z}_p\) and \(s \in \mathbb{N}\) with \(s \geq 2\), let \(n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+\) with \(\sum_{i=1}^{s} n_i > sk\). Then, we take the fermionic p-adic \(q\)-integral on \(\mathbb{Z}_p\) for the weighted \(q\)-Bernstein polynomials of degree \(n\) as follows:

\[
I_5 = \int_{\mathbb{Z}_p} q^{(h-1)x} \left\{ \prod_{i=1}^{s} B_{n_i,n}^{(\alpha)}(x | q) \right\} d \mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1} \binom{k}{l} (-1)^{n_1-l} \int_{\mathbb{Z}_p} q^{(h-1)x} [x]_{q,a}^k [1-x]_{q,a}^{-n_1-l} d \mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1} \binom{k}{l} (-1)^{n_1-l} \left[2q + q^{h+1} G_{n_1-l-1,q} \right],
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1} \binom{k}{l} (-1)^{n_1-l} \left[2q + q^{h+1} G_{n_1-l-1,q} \right].
\]

On the other hand, from the definition of weighted \(q\)-Bernstein polynomials and the binomial theorem, we easily get

\[
I_6 = \int_{\mathbb{Z}_p} q^{(h-1)x} \left\{ \prod_{i=1}^{s} B_{n_i,n}^{(\alpha)}(x | q) \right\} d \mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1} \binom{k}{l} (-1)^{n_1-l} \int_{\mathbb{Z}_p} q^{(h-1)x} [x]_{q,a}^k [1-x]_{q,a}^{-n_1-l} d \mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1} \binom{k}{l} (-1)^{n_1-l} \left[2q + q^{h+1} G_{n_1-l-1,q} \right],
\]

Equating \(I_5\) and \(I_6\), we discover the following theorem.
Theorem 7. For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $\sum_{l=1}^s n_l > sk$. Then, we have
\[
\sum_{l=0}^{n_1-1} \binom{n_1-1}{l+sk+1} (-1)^l \frac{G_{l+sk+1}}{l+sk+1} \cdot \frac{[2]_q + q^{h+1}}{m_1 + m_2 + \cdots + m_s + 1}
\]
\[
= \begin{cases} 
\sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+k+1} \frac{[2]_q + q^{h+1}}{m_1 + m_2 + \cdots + m_s + 1} & \text{if } k = 0, \\
\sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+k+1} \frac{[2]_q + q^{h+1}}{m_1 + m_2 + \cdots + m_s + 1} & \text{if } k \neq 0.
\end{cases}
\]

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References

[1] M. Acikgoz and S. Araci, A study on the integral of the product of several type Bernstein polynomials, IST Transaction of Applied Mathematics-Modelling and Simulation, 1, 10–14 (2010).

[2] Y. Simsek and M. Acikgoz, A New generating function of $q$-Bernstein type polynomials and their interpolation function, Abstract and Applied Analysis, Article ID 769095, 12 pages.

[3] S. Araci, D. Erdal and J. J. Seo, A study on the fermionic $p$-adic $q$-integral representation on $\mathbb{Z}_p$ associated with weighted $q$-Bernstein and $q$-Genocchi Polynomials, Abstract and Applied Analysis, 2011, Article ID 649248, 10 pages.

[4] S. Araci, D. Erdal and D.-J. Kang, Some New Properties on the $q$-Genocchi numbers and Polynomials associated with $q$-Bernstein polynomials, Honam Mathematical J., 33, 261-270 (2011).

[5] S. Araci, J. J. Seo and D. Erdal, New construction weighted $(h, q)$-Genocchi numbers and polynomials related to Zeta type function, Discrete Dynamics in Nature and Society, 2011, Article ID 487490, 7 pages.

[6] S. Araci, M. Acikgoz and J. J. Seo, A study on the weighted $q$-Genocchi numbers and polynomials with Their Interpolation Function, Honam Mathematical J., 34 , 11-18 (2012).

[7] S. Araci, M. Acikgoz, F. Qi, H. Jolany, A note on the modified $q$-Genocchi numbers and polynomials with weight $\alpha$ and $\beta$ and their interpolation function at negative integers, Accepted in Fasc. Math. Journal.

[8] S. Araci, M. Acikgoz and Feng Qi, On the $q$-Genocchi numbers and polynomials with weight zero and their applications, http://arxiv.org/abs/1202.2643

[9] S. Araci, M. Acikgoz, K. H. Park and H. Jolany, On the unification of two families of multiple twisted type polynomials by using $p$-adic $q$-integral on $\mathbb{Z}_p$ at $q = -1$, Bulletin of the Malaysian Mathematical Sciences Society (accepted for publication)

[10] S. Araci, M. Acikgoz and K. H. Park, A note on the $q$-analogue of Kim’s $p$-adic log gamma type functions associated with $q$-extension of Genocchi and Euler numbers with weight $\alpha$, accepted in Bulletin of the Korean Mathematical Society.

[11] S. Araci, M. Acikgoz, and J. J. Seo, Explicit formulas involving $q$-Euler numbers and polynomials, Abstract and Applied Analysis, 2012, Article ID 298531, 11 pages.

[12] H. Jolany, S. Araci, M. Acikgoz and J. J. Seo., A note on the generalized $q$-Genocchi measure with weight $\alpha$, Bol. Soc. Parana. Math., 31, 17-27 (2013).

[13] H. Ozden and Y. Simsek, A new extension of $q$-Euler numbers and polynomials related to their interpolation functions, Applied Mathematics Letters, 21, 934–939 2008.

[14] I. N. Cangul, V. Kurt, H. Ozden, and Y. Simsek, On the higher-order $w$-$q$-genocchi numbers, Advanced Studies in Contemporary Mathematics, 19, 39–57 (2009).

[15] I. N. Cangul, V. Kurt, Y. Simsek, H. K. Pak, and S.-H. Rim, An invariant $p$-adic $q$-integral associated with $q$-Euler numbers and polynomials, Journal of Nonlinear Mathematical Physics, 14, 8–14 (2007).

[16] T. Kim, On the $q$-extension of Euler and Genocchi numbers, J. Math. Anal. Appl., 326, 1458-1465 (2007).

[17] T. Kim, On the multiple $q$-Genocchi and Euler numbers, Russian J. Math. Phys., 15, 481-486 (2008).

[18] T. Kim, A Note on the $q$-Genocchi Numbers and Polynomials, Journal of Inequalities and Applications, 2007, Article ID 71452, 8 pages (2007).

[19] T. Kim, A Note on $q$-Bernstein Polynomials, Russ. J. Math. Phys., 18, 41-50 (2011).

[20] T. Kim, $q$-Volkenborn integration, Russ. J. Math. Phys., 9, 288-299 (2002).

[21] T. Kim, $q$-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients, Russ. J. Math. Phys., 15, 51-57 (2008).

[22] T. Kim, J. Choi, Y. H. Kim and C. S. Ryoo, On the fermionic $p$-adic integral representation of Bernstein polynomials associated with Euler numbers and polynomials, J. Inequal. Appl. (2010), Art ID 864247, 12pp.

[23] T. Kim, J. Choi and Y. H. Kim, Some identities on the $q$-Bernstein polynomials, $q$-Stirling numbers and $q$-Bernoulli numbers, Adv. Stud. Contemp. Math., 20, 335-341 (2010).

[24] T. Kim, An invariant $p$-adic $q$-integrals on $\mathbb{Z}_p$, Applied Mathematics Letters, 21, 105-108 (2008).

[25] T. Kim, J. Choi and Y. H. Kim, q-Bernstein Polynomials Associated with q-Stirling Numbers and Carlitz’s q-Bernoulli Numbers, Abstract and Applied Analysis, Article ID 150975, 11 pages.

[26] T. Kim, A note on the $q$-Genocchi numbers and polynomials, Journal of Inequalities and Applications, Article ID 71452, 8 pages.

[27] T. Kim, J. Choi, Y. H. Kim, and L. C. Jang, On the fermionic $p$-adic analogue of $q$-Bernstein polynomials and related integrals, Discrete Dynamics in Nature and Society, Article ID 179430, 9 pages.

[28] T. Kim, $q$-Euler numbers and polynomials associated with $p$-adic $q$-integrals, J. Nonlinear Math. Phys., 14, 15–27 (2007).

[29] T. Kim, New approach to $q$-Euler polynomials of higher order, Russ. J. Math. Phys., 17, 218–225 (2010).
[30] T. Kim, Some identities on the $q$-Euler polynomials of higher order and $q$-Stirling numbers by the fermionic $p$-adic integral on $\mathbb{Z}_p$, *Russ. J. Math. Phys.*, **16**, 484–491 (2009).

[31] T. Kim, On the weighted $q$-Bernoulli numbers and polynomials, *Advanced Studies in Contemporary Mathematics*, **21**, 207–215 (2011).

[32] T. Kim, A. Bayad and Y. H. Kim, A study on the $p$-adic $q$-integrals representation on $\mathbb{Z}_p$ associated with the weighted $q$-Bernstein and $q$-Bernoulli polynomials, *Journal of Inequalities and Applications*, Article ID 513821, 8 pages.

[33] T. Kim, On a $q$-analogue of the $p$-adic log gamma functions and related integrals, *J. Number Theory*, **76**, 320–329 (1999).

[34] T. Kim, A new approach to $p$-adic $q$-L-functions, *Adv. Stud. Contemp. Math.* (Kyungshang), **12**, 61–72 (2006).

[35] N. Koblitz, *p*-adic Analysis: A short course on recent work, *London Math. Soc. Lecture Note Ser.*, **46**, (1980).

[36] A. M. Robert, *A course in $p$-adic Analysis*, *Springer-Verlag*, New York, (2000).

[37] C. S. Ryoo, A note on the weighted $q$-Euler numbers and polynomials, *Adv. Stud. Contemp. Math.*, **21**, 47-54 (2011).

[38] H. Y. Lee, N. S. Jung and C. S. Ryoo, Some identities of the Twisted $q$-Genocchi numbers and polynomials with weight $\alpha$ and $q$-Bernstein polynomials with weight $\alpha$, *Abstract and Applied Analysis*, **2011**, (2011), Article ID 123483, 9 pages.

[39] N. S. Jung, H. Y. Lee and C. S. Ryoo, Some relations between twisted $(h,q)$-Euler numbers with weight $\alpha$ and $q$-Bernstein polynomials with weight $\alpha$, *Discrete Dynamics in Nature and Society*, **2011**, (2011), Article ID 176296, 11 pages.