Integrable Lattice Models and Holography

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Abstract

We study four-dimensional Chern-Simons theory on $D \times \mathbb{C}$ (where $D$ is a disk), which is understood to describe rational solutions of the Yang-Baxter equation from the work of Costello, Witten and Yamazaki. We find that the theory is dual to a boundary theory, that is a three-dimensional analogue of the two-dimensional chiral WZW model. This boundary theory gives rise to a current algebra that turns out to be an analytically-continued toroidal Lie algebra. In addition, we show how bulk correlation functions of two and three Wilson lines can be captured by boundary correlation functions of local operators in the three-dimensional WZW model. In particular, we reproduce the leading and subleading nontrivial contributions to the rational $R$-matrix purely from the boundary theory.

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1 Introduction

A relatively novel approach to the study of integrable lattice models underlaid by the Yang-Baxter equation with spectral parameters is that of four-dimensional Chern-Simons theory, first proposed by Costello [1, 2], and subsequently studied by Costello, Witten and Yamazaki [3–5] in depth.

The theory is defined by the path integral involving the classical action

\[ S = \frac{1}{2\pi\hbar} \int_{\Sigma \times C} \omega \wedge \text{Tr}(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}), \]

(1.1)

where \( \mathcal{A} \) is a complex-valued gauge field, \( \Sigma \) is a 2-manifold, and \( C \) is a Riemann surface endowed with a holomorphic one-form \( \omega = \omega(z)dz \). It is topological along \( \Sigma \) (modulo a framing anomaly), but has holomorphic dependence on \( C \), and moreover has a complex gauge group, \( G \). As shown in [3, 4], the nontrivial operators of the theory are Wilson lines, whose correlation functions realize the Yang-Baxter equation with spectral parameters, as well as the underlying Yangian algebra, quantum affine algebra, and elliptic quantum group of its rational, trigonometric and elliptic solutions, respectively. Furthermore, the boundary Yang-Baxter equation can likewise be realized by studying 4d Chern-Simons theory on an orbifold [6, 7].

For \( C = \mathbb{R} \times S^1 \), the 4d Chern-Simons action can be dimensionally reduced along \( S^1 \) to that of 3d analytically-continued Chern-Simons theory. In fact, the corresponding quantum field theories have been shown to be T-dual [8]. However, unlike ordinary 3d Chern-Simons theory, much of the work on 4d Chern-Simons relied on the path integral and Feynman diagrams alone, and no use was made of canonical quantization or holography. This was due to the infrared-free nature of 4d Chern-Simons, whereby it was straightforward to deduce a local procedure to compute the expectation values of Wilson line configurations of interest.

Nevertheless, given the importance of the 2d chiral Wess-Zumino-Witten (WZW) model dual to 3d Chern-Simons theory as a straightforward example of a holographic dual, and for describing edge modes of the nonabelian fractional quantum Hall effect, it is of interest to investigate the existence of a holographic dual of 4d Chern-Simons theory. In this work, we shall indeed derive such a dual boundary theory for 4d Chern-Simons on \( D \times C \) (where \( D \) is the disk), which turns out to be a three-dimensional analogue of the 2d chiral WZW model.

We shall focus on the boundary dual of 4d Chern-Simons with \( C = \mathbb{C} \), which is known to give rise to rational solutions of the Yang-Baxter equation [3]. These \( R \)-matrices are intertwining operators for representations of the Yangian algebra, and thus the classical integrable lattice models of concern are equivalent to Heisenberg XXX quantum spin chains. For example, one such lattice model is the rational six-vertex model, which is equivalent to the XXX_{1/2} spin chain. As we shall see, the 3d “chiral” WZW model we derive furnishes an alternative and convenient method for computing the rational \( R \)-matrices explicitly.

Given a 3d analogue of the 2d chiral WZW model, the first natural question to ask is if it admits a current algebra analogous to an affine Kac-Moody algebra. In Section 2,
we shall show that this is indeed the case, i.e., the 3d “chiral” WZW model furnishes a particular limit of an “analytically-continued” toroidal Lie algebra.

One would also like to verify that the boundary theory captures the correlation functions of the bulk theory. For instance, the bulk correlator of two straight, perpendicular Wilson lines along $\Sigma$, at points $z_1$ and $z_2$ on $\mathbb{C}$, and in representations $R_1$ and $R_2$ of the generators of $G$, realizes the $R$-matrix, i.e., it is computed to be

$$\tilde{R}_{12}(z_1 - z_2) = 1 + \frac{\hbar}{z_1 - z_2} T_{R_1}^a \otimes T_{R_2}^a + O(\hbar^2) \quad (1.2)$$

to linear order in $\hbar$, with the full expression for the $R$-matrix following from general theorems [9, 10]. It would be satisfying to obtain this result solely from the boundary theory, and indeed, this is what we do in Section 3 by evaluating a four-point function of local boundary operators.

Furthermore, via the boundary theory, we demonstrate the topological invariance along $D$ of the bulk correlator at order $\hbar$. We also explain how one may explicitly derive higher order contributions to the $R$-matrix, and demonstrate this by computing the $\hbar^2$ contribution for two crossed Wilson lines explicitly. Subsequently, we show that the boundary theory also reproduces the correlation function of a pair of parallel Wilson lines. Finally, we consider three Wilson lines, and show that their correlation function is reproduced by a six-point function in the boundary theory.

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2 3d “Chiral” WZW Model

Let us start with 4d Chern-Simons theory with complex gauge group, $G$, defined on $\Sigma \times \mathbb{C}$, where $\Sigma$ is a disk, denoted $D$, and $\mathbb{C}$ is the complex plane, $\mathbb{C}$. Its action is

$$S = \frac{1}{2\pi \hbar} \int_{D \times \mathbb{C}} dz \wedge \Tr (\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}). \quad (2.1)$$

Here, $\mathcal{A}$ can be understood to be the partial connection

$$\mathcal{A} = A_r dr + A_\varphi d\varphi + A_z d\bar{z}, \quad (2.2)$$

where $(r, \varphi)$ are polar coordinates on $D$ and $(z, \bar{z})$ are complex coordinates on $\mathbb{C}$.

Let us first vary the action to find the equations of motion. Doing so, one finds

$$\delta S = \frac{1}{2\pi \hbar} \int_{D \times \mathbb{C}} dz \wedge \Tr (\delta \mathcal{A} \wedge \mathcal{F} + d(\delta \mathcal{A} \wedge \mathcal{A})). \quad (2.3)$$
The second term of the variation is a boundary term via Stoke’s theorem. In order to ensure that we have equations of motion free from boundary corrections, we shall impose the boundary condition $A_{\bar{z}} = 0$, whereupon the boundary term vanishes.

This boundary condition is also necessary to achieve gauge invariance in the presence of boundaries. It can be shown that (2.1) is equivalent to

$$S = -\frac{1}{2\pi \hbar} \int_{D \times \mathbb{C}} z \text{Tr} (F \wedge F) + \frac{1}{2\pi \hbar} \int_{\partial D \times \mathbb{C}} z \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

where $A$ has been extended to a full connection over $D \times \mathbb{C}$, i.e., $A = A_r dr + A_\varphi d\varphi + A_z dz + A_{\bar{z}} d\bar{z}$. The boundary term on the RHS of (2.4) depends only on the components $A_\varphi$, $A_z$ and $A_{\bar{z}}$, and vanishes using the boundary conditions $A_{\bar{z}} = 0$ and $A_z = 0$. The remaining term is gauge invariant under large gauge transformations, i.e.,

$$A \rightarrow UAU^{-1} - dUU^{-1}.$$ (2.5)

However, we ought to restrict $U$ such that the boundary conditions $A_{\bar{z}} = A_z = 0$ are preserved. We shall achieve this by insisting that $U$ tends to the identity element of $G$ at the boundary.

Now, having imposed the boundary condition $A_{\bar{z}} = 0$, the action (2.1) is equivalent to

$$\frac{1}{2\pi \hbar} \int dz \wedge dr \wedge d\varphi \wedge d\bar{z} \text{Tr} \left( 2 A_{\bar{z}} F_{r\varphi} - A_r \partial_{\bar{z}} A_\varphi + A_\varphi \partial_{\bar{z}} A_r \right),$$

(2.6)

upon integration by parts. Varying the Lagrange multiplier field, $A_{\bar{z}}$, implements the constraint $F_{r\varphi} = 0$, which is solved by

$$A_r = -\partial_r gg^{-1}, \quad A_\varphi = -\partial_\varphi gg^{-1},$$

(2.7)

where $g$ is a map $g : D \times \mathbb{C} \rightarrow G$.

Changing variables from $A_r$ and $A_\varphi$ to $g$ in the functional integral, we note that, just as in 3d Chern-Simons theory [11], no Jacobian appears when transforming the measure, i.e.,

$$\frac{1}{\text{vol } G} \int DA_r DA_\varphi \delta(F_{r\varphi}) = \frac{1}{\text{vol } G} \int Dg,$$

(2.8)

where the expression on the RHS is the relevant Haar measure, divided by the volume of the gauge group. Furthermore, substituting the solutions (2.7) into (2.6), we obtain the action

$$S(g) = \frac{1}{2\pi \hbar} \int_{S^1 \times \mathbb{C}} d\varphi \wedge dz \wedge d\bar{z} \text{Tr}(\partial_\varphi gg^{-1} \partial_{\bar{z}} gg^{-1}) + \frac{1}{6\pi \hbar} \int_{D \times \mathbb{C}} dz \wedge \text{Tr}(dgg^{-1} \wedge dgg^{-1} \wedge dgg^{-1})$$

(2.9)

which takes the form of a three-dimensional analogue of the 2d chiral WZW model. Now, a gauge transformation (2.5) amounts to $g \rightarrow Ug$ in (2.7). As a result, we may change the value of $g$ in the interior without changing its value at the boundary, so (2.9) only depends on $g$ at the boundary. This implies that we can divide out the volume of the gauge group to obtain the path integral

$$\int Dg \ e^{iS(g)},$$

(2.10)
where $g$ is now the map $g : \partial D \times \mathbb{C} \to G$.

Varying the action (2.9) gives us
\[
\delta S = -\frac{1}{\pi \hbar} \int d\varphi \wedge dz \wedge d\bar{z} \text{Tr}(g^{-1} \delta g \partial \varphi (g^{-1} \partial \bar{z} g)), \tag{2.11}
\]
whereby we obtain the classical equation of motion
\[
\partial \varphi (g^{-1} \partial \bar{z} g) = 0, \tag{2.12}
\]
which is equivalent to $\partial \bar{z} (\partial \varphi g g^{-1}) = 0$, and is solved by
\[
g(z, \bar{z}, \varphi) = A(z, \varphi) B(z, \bar{z}). \tag{2.13}
\]
The equations $\partial \varphi (g^{-1} \partial \bar{z} g) = 0$ and $\partial \bar{z} (\partial \varphi g g^{-1}) = 0$ are in fact equivalent to the current conservation equations for the symmetry of the action under the transformation
\[
g(\varphi, z, \bar{z}) \rightarrow \tilde{\Omega}(\varphi, z) g(\varphi, z, \bar{z}), \tag{2.14}
\]
where $\tilde{\Omega}$ and $\Omega$ give rise to the conserved currents $J_\varphi = -\frac{1}{\pi \hbar} \partial \varphi g g^{-1}$ and $J_{\bar{z}} = -\frac{1}{\pi \hbar} g^{-1} \partial \bar{z} g$, respectively.

### 2.1 Current Algebra via Canonical Quantization

We are now interested in computing a Poisson bracket involving the expression $J_\varphi = -\frac{1}{\pi \hbar} \partial \varphi g g^{-1}$, which we shall eventually use to obtain a quantum mechanical commutation relation in the form of a current algebra. In what follows, we shall take $\bar{z}$ to be the (complex) time direction.

Now, given an arbitrary action that is first order in the time derivative with dynamical variables $\phi^i$, i.e.,
\[
I = \int dt \omega(\phi) \frac{d\phi^i}{dt}, \tag{2.15}
\]
its variation takes the form
\[
\delta I = \int dt \omega_{ij} \delta \phi^i \frac{d\phi^j}{dt}, \tag{2.16}
\]
where $\omega_{ij} = \frac{\partial}{\partial \phi^j} \omega^i - \frac{\partial}{\partial \phi^i} \omega^j$ is the symplectic structure on the classical phase space. The Poisson bracket of any two functions $X$ and $Y$ on the phase space is then defined by
\[
[X, Y]_{PB} = \omega_{ij} \frac{\partial X}{\partial \phi^i} \frac{\partial Y}{\partial \phi^j}, \tag{2.17}
\]
where $\omega^{jk} \omega_{kl} = \delta^j_l$ \cite{12}.

For the 3d “chiral” WZW model, the variation (2.11) implies that its phase space symplectic structure is given by $\omega = 1_\mathfrak{g} \otimes \frac{(-1)}{\pi \hbar} \frac{\partial}{\partial \varphi} \otimes 1_z$, where $1_\mathfrak{g}$ acts on the Lie algebra index, $\frac{(-1)}{\pi \hbar} \frac{\partial}{\partial \varphi}$ acts on the $\varphi$ coordinate, and $1_z$ acts on the $z$ coordinate. Its inverse is therefore
\[
\omega^{-1} = 1_\mathfrak{g} \otimes \frac{(-1)}{\pi \hbar} \left( \frac{\partial}{\partial \varphi} \right)^{-1} \otimes 1_z. \tag{2.18}
\]
Let us now compute the Poisson bracket of $X = \text{Tr} A \frac{\partial g}{\partial \varphi} g^{-1}(\varphi, z)$ and $Y = \text{Tr} B \frac{\partial g}{\partial \varphi} g^{-1}(\varphi', z')$, where $A$ and $B$ are arbitrary generators of the group $G$. In the notation of (2.15), this can be done by evaluating $\delta X \delta Y = \frac{\partial X}{\partial \varphi} \frac{\partial Y}{\partial \varphi'} \delta \varphi \delta \varphi'$, and subsequently replacing $\delta \varphi \delta \varphi'$ by $\omega^{ij}$. Proceeding in this manner, we find

$$
\delta X \delta Y = \text{Tr} g^{-1}(\varphi, z) A g(\varphi, z) \frac{\partial}{\partial \varphi} (g^{-1} \delta g(\varphi, z)) : \text{Tr} g^{-1}(\varphi', z') B g(\varphi', z') \frac{\partial}{\partial \varphi'} (g^{-1} \delta g(\varphi', z')).
$$

To obtain the Poisson bracket, we ought to replace $(g^{-1} \delta g(\varphi, z))^a (g^{-1} \delta g(\varphi', z'))^b$ (where $a$ and $b$ are Lie algebra indices) by

$$
\delta^{ab} (\pi \hbar) \theta(\varphi - \varphi') \delta(z - z'),
$$

where $\theta(\varphi - \varphi')$ is an inverse of $\frac{\partial}{\partial \varphi}$. Therefore, $\frac{\partial}{\partial \varphi} (g^{-1} \delta g(\varphi, z))^a \cdot \frac{\partial}{\partial \varphi'} (g^{-1} \delta g(\varphi', z'))^b$ in (2.19) ought to be replaced by $\delta^{ab} \pi \hbar \delta'(\varphi - \varphi') \delta(z - z')$. Hence, we arrive at the Poisson bracket

$$
[X, Y]_{PB} = \pi \hbar \delta'(\varphi - \varphi') \delta(z - z') \text{Tr} g^{-1}(\varphi, z) A g(\varphi, z) g^{-1}(\varphi', z') B g(\varphi', z') \delta'(\varphi - \varphi') \delta(z - z') \text{Tr} AB.
$$

In the quantum theory, this Poisson bracket corresponds to the canonical commutation relation

$$
[X, Y] = -i \pi \hbar \delta'(\varphi - \varphi') \delta(z - z') \text{Tr} [A, B] \frac{\partial g}{\partial \varphi} g^{-1} - i \pi \hbar \delta'(\varphi - \varphi') \delta(z - z') \text{Tr} AB.
$$

This is in fact equivalent to the current algebra

$$
[\text{Tr} A J_\varphi(\varphi, z), \text{Tr} B J_\varphi(\varphi', z')] = i \delta(\varphi - \varphi') \delta(z - z') \text{Tr} [A, B] J_\varphi(\varphi, z) - \frac{i}{\pi \hbar} \delta'(\varphi - \varphi') \delta(z - z') \text{Tr} AB.
$$

To express this algebra in a more familiar form, we expand the currents in terms of their Fourier modes along $S^1$,

$$
J_\varphi(\varphi, z) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} J^n_\varphi(z) e^{in\varphi},
$$

and utilize the orthogonality of these modes, which leads us to

$$
[\text{Tr} A J^m_\varphi(z), \text{Tr} B J^n_\varphi(z')] = i \text{Tr} [A, B] J^{m+n}_\varphi(z) \delta(z - z') + \frac{2}{\hbar} (n \delta_{m+n,0}) \delta(z - z') \text{Tr} AB.
$$

This algebra has the form of a Kac-Moody algebra with generators having holomorphic dependence on the Riemann surface, $\mathbb{C}$. Note that there is no quantization condition on $\hbar$ here, unlike the current algebra derived from the boundary theory of ordinary 3d Chern-Simons theory.
To further understand this algebra, we shall write $z = \epsilon t + i \theta$, compactify the $\theta$ direction to be valued in $[0, 2\pi]$, and subsequently take $\epsilon \to 0$. Upon doing so, we may perform another expansion in Fourier modes, i.e.,

$$J^n(\theta) = \frac{1}{2\pi} \sum_{\hat{n} = -\infty}^{\infty} J^n_{\hat{n}, \theta} e^{i\hat{n} \theta}.$$  

(2.26)

Then, employing the orthogonality of these modes, the resulting algebra is

$$[\text{Tr} A J^n_{\hat{n}, \theta}, \text{Tr} B J^m_{\hat{m}, \theta}] = i \text{Tr} [A, B] J^{n+m}_{\hat{n}+\hat{m}, \theta} + \frac{4\pi}{\hbar} n \delta_{m+n,0} \delta_{\hat{n}+\hat{m},0} \text{Tr} AB.$$  

(2.27)

This is known as a two-toroidal Lie algebra (or a centrally-extended double loop algebra), which, in particular, arises as the current algebra of the four-dimensional WZW model [13]. Hence, the algebra (2.25) that we obtained can be understood to be an ‘analytical continuation’ of the two-toroidal Lie algebra (2.27), with one of the two ‘loop directions’ decompactified. This is not surprising, considering the fact that 4d Chern-Simons theory for $C = \mathbb{R} \times S^1$ can be understood to be 3d Chern-Simons theory for the loop group, but with the ‘loop direction’ complexified [14].

3 Wilson Lines and Boundary Local Operators

We shall now describe Wilson lines in 4d Chern-Simons theory in terms of local operators of the boundary theory. This is possible due to the flatness condition that restricts the gauge field components on $D$ to be pure gauge configurations, as shown in (2.7). To see this, note that a Wilson line along a curve $C$, starting at $t_i$ and ending at $t_f$, and in a representation $R$, satisfies

$$P e^{\int_C A} = g^{-1}_R(t_f) P e^{\int_C A'} g_R(t_i)$$  

(3.1)

where $A = g A' g^{-1} - dgg^{-1}$. Setting $A' = 0$, we find that

$$P e^{\int_C (-dgg^{-1})} = g^{-1}_R(t_f) g_R(t_i).$$  

(3.2)

If $t_f$ and $t_i$ are points on $\partial D = S^1$, this implies that a bulk Wilson line operator can be completely described in terms of a pair of local boundary operators. In fact, such boundary-anchored Wilson lines are automatically gauge invariant, since all gauge transformations are trivial at the boundary.

This suggests that correlation functions of Wilson line operators in 4d Chern-Simons theory can be captured by correlation functions of local operators in the 3d “chiral” WZW model. This includes correlators of crossed Wilson lines which compute the $R$-matrices of integrable lattice models. We shall now attempt to verify this.

In the bulk computation by Costello, Witten, and Yamazaki [3], the order $\hbar$ contribution to the $R$-matrix was found by performing perturbation theory around the trivial field configuration, $A = 0$, and computing free-field propagators between Wilson lines. In the same vein, we shall consider perturbation theory around the field configuration $g = 1$, and
we shall use a free-field propagator to compute the relevant correlation function involving operators appearing on the right hand side of (3.2), i.e.,

\[
\langle P \int_{\phi=0} e^{i \phi} \int_{\bar{\phi}=\pi} e^{i \bar{\phi}} A_{R_1}(z_1, \bar{z}_1) \otimes P \int_{\phi=\pi/2} e^{i \phi} A_{R_2}(z_2, \bar{z}_2) \rangle
\]

(3.3)

= \langle g_{R_1}^{-1}(0, z_1, \bar{z}_1) g_{R_1}(\pi, z_1, \bar{z}_1) \otimes g_{R_2}^{-1}(\pi/2, z_2, \bar{z}_2) g_{R_2}(3\pi/2, z_2, \bar{z}_2) \rangle.

The relevant operators are depicted in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Perpendicular Wilson lines on $D$.}
\end{figure}

Now, we may expand the field, $g$, as

\[ g = e^{\phi_a T^a} = 1 + \phi_a T^a + \ldots \]

The free part of the 3d WZW action is then

\[
\frac{1}{2\pi i} \int_{S^1 \times \mathbb{C}} d\phi \wedge dz \wedge d\bar{z} Tr(\partial_{\phi} g^{-1} \partial_{\bar{z}} g^{-1}) = -\frac{1}{2\pi i} \int_{S^1 \times \mathbb{C}} d\phi \wedge dz \wedge d\bar{z} \ n^a \partial_{\phi} \partial_{\bar{z}} \phi_a + \ldots ,
\]

where we have performed integration by parts after expanding the field $g$.

Next, from the free action, one may construct the generating functional

\[
Z_0[J] = \frac{\int D\phi e^{i \int_{S^1 \times \mathbb{C}} d\phi \wedge dz \wedge d\bar{z}(\partial_{\phi} g^{-1} \partial_{\bar{z}} g^{-1} + 2\pi i h J_a \phi^a)}}{\int D\phi e^{i \int_{S^1 \times \mathbb{C}} d\phi \wedge dz \wedge d\bar{z}(-\phi^a \partial_{\phi} \partial_{\bar{z}} \phi_a)}}
\]

(3.5)

where $x = (\phi, z, \bar{z})$, $y = (\phi', z', \bar{z}')$, and $\Delta^{ab}$ is the propagator which obeys

\[ \partial_{\phi} \partial_{\bar{z}} \Delta^{ab}(x) = \delta^{ab} \delta(x), \]

(3.6)
and is given explicitly by
\[ \Delta^{ab}(x) = \delta^{ab} \frac{1}{2\pi i} \frac{1}{z} \Delta_{x} \]  
(3.7)
where
\[ \Delta_{x} = \frac{1}{2\pi} \left( \sum_{k=1}^{\infty} \frac{e^{ik\varphi}}{ik} + \sum_{k=-\infty}^{-1} \frac{e^{ik\varphi}}{ik} \right), \]
(3.8)
which satisfies \( \partial_{\varphi} \Delta_{x} = \frac{1}{2\pi} \left( \sum_{k=-\infty}^{\infty} e^{ik\varphi} \right) = \delta(\varphi) \). Note that (3.6) holds since
\[ \partial_{z} \left( \frac{1}{2\pi i} \frac{1}{z} \right) = -\delta(z, \bar{z}). \]
(3.9)
Moreover, the propagator satisfies \( \Delta^{ab}(x - y) = \Delta^{ba}(y - x) \). The two point function can be found from (3.5) to be
\[ \langle \phi^{a}(x) \phi^{b}(y) \rangle = -\pi i \hbar \Delta^{ab}(x - y). \]
(3.10)

Before we proceed, we first note that the propagator (3.7) has no dependence on the \( \bar{z} \) coordinate. This can be understood from the fact that there is a gauge redundancy in the 3d “chiral” WZW model, namely the invariance of the parametrizations (2.7) under transformations generated by \( \Omega \) in (2.14). This redundancy can be fixed by setting \( B(z, \bar{z}) = 1 \) in (2.13). As a result, the operator \( g \) has no \( \bar{z} \)-dependence, and therefore correlation functions involving it should not have \( \bar{z} \)-dependence either.

Another pertinent point to note is that the propagator is a multi-valued function, as it includes the expression on the RHS of (3.8) as a factor. Hence, to obtain a single-valued propagator, we ought to define it with a branch cut. We shall pick the branch cut to be \( B \) (3.10). Hence, the propagator satisfies \( \Delta^{ab}(x - y) = \Delta^{ba}(y - x) \). The two point function can be found from (3.5) to be
\[ \langle \phi^{a}(x) \phi^{b}(y) \rangle = -\pi i \hbar \Delta^{ab}(x - y). \]
(3.11)

We then only keep terms of quadratic or lower order in the fields, while taking self-contractions (i.e., correlators of operators with the same value of \( z \)) to be zero via regularization. Also note that 1-point functions can be shown to be zero using (3.5). Hence, we find
\[
\begin{align*}
1 + \langle \phi_{a}(0, z_1) \phi_{c}(\pi/2, z_2) \rangle & T_{R_{1}}^{a} \otimes T_{R_{2}}^{c} - \langle \phi_{a}(\pi, z_1) \phi_{c}(\pi/2, z_2) \rangle T_{R_{1}}^{a} \otimes T_{R_{2}}^{c} \\
- \langle \phi_{a}(2\pi, z_1) \phi_{c}(3\pi/2, z_2) \rangle T_{R_{1}}^{a} \otimes T_{R_{2}}^{c} + \langle \phi_{a}(\pi, z_1) \phi_{c}(3\pi/2, z_2) \rangle T_{R_{1}}^{a} \otimes T_{R_{2}}^{c} + O(\hbar^{2}) \\
= & 1 - \frac{\hbar}{2} \delta_{ac} \frac{1}{z_{1} - z_{2}} \tilde{\Delta}_{-\bar{z}} T_{R_{1}}^{a} \otimes T_{R_{2}}^{c} + \frac{\hbar}{2} \delta_{ac} \frac{1}{z_{1} - z_{2}} \tilde{\Delta}_{\bar{z}} T_{R_{1}}^{a} \otimes T_{R_{2}}^{c} \]
\end{align*}
\]
(3.12)
\[
\begin{align*}
= & 1 - \frac{\hbar}{2} \delta_{ac} \frac{1}{z_{1} - z_{2}} \tilde{\Delta}_{\bar{z}} T_{R_{1}}^{a} \otimes T_{R_{2}}^{c} + \frac{\hbar}{2} \delta_{ac} \frac{1}{z_{1} - z_{2}} \tilde{\Delta}_{-\bar{z}} T_{R_{1}}^{a} \otimes T_{R_{2}}^{c} + O(\hbar^{2}) \\
= & 1 + \frac{\hbar}{z_{1} - z_{2}} (\tilde{\Delta}_{-\bar{z}} - \tilde{\Delta}_{\bar{z}}) T_{R_{1}}^{a} \otimes T_{R_{2}}^{a} + O(\hbar^{2}) \\
= & 1 + \frac{\hbar}{z_{1} - z_{2}} T_{R_{1}}^{a} \otimes T_{R_{2}}^{a} + O(\hbar^{2}).
\end{align*}
\]
Here, we have used the fact that

\[ \tilde{\Delta}_{\frac{\pi}{2}} = \frac{1}{2\pi} \frac{\pi}{2} + \frac{1}{\pi} \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \right) = \frac{1}{2}, \]  

(3.13)

and likewise \( \tilde{\Delta}_{-\frac{\pi}{2}} = -\frac{1}{2} \). We have thus obtained, from our 3d “chiral” WZW model, the exact order \( \hbar \) correlation function for a pair of perpendicular Wilson lines that Costello, Witten and Yamazaki [3] computed via the bulk 4d Chern-Simons path integral.

### 3.1 Non-perpendicular Wilson Lines

We can generalize the calculation above to the case of non-perpendicular Wilson lines. As an example, we shall start with two perpendicular Wilson lines, and rotate the vertical Wilson line clockwise by an angle, \( \delta \), as shown in Figure 2.

\[ \text{Figure 2: Non-perpendicular Wilson lines on } D. \]

The four-point function we should compute is

\[ \langle g_{R_1}^{-1}(0, z_1) g_{R_1}(\pi, z_1) \otimes g_{R_2}^{-1}(\pi/2 - \delta, z_2) g_{R_2}(3\pi/2 - \delta, z_2) \rangle. \]  

(3.14)

Expanding each operator to linear order in \( \phi \) as in (3.11), we find

\[
\begin{align*}
1 &- \langle \phi_a(0, z_1) \phi_c(\pi/2 - \delta, z_2) \rangle T_{R_1}^a \otimes T_{R_2}^c - \langle \phi_a(\pi, z_1) \phi_c(\pi/2 - \delta, z_2) \rangle T_{R_1}^a \otimes T_{R_2}^c \\
&- \langle \phi_a(2\pi, z_1) \phi_c(3\pi/2 - \delta, z_2) \rangle T_{R_1}^a \otimes T_{R_2}^c + \langle \phi_a(\pi, z_1) \phi_c(3\pi/2 - \delta, z_2) \rangle T_{R_1}^a \otimes T_{R_2}^c + O(\hbar^2) \\
&= 1 - \hbar \frac{\delta_{ac}}{2} \frac{1}{z_1 - z_2} \tilde{\Delta}_{-\frac{\pi}{2} + \delta} T_{R_1}^a \otimes T_{R_2}^c + \frac{\hbar}{2} \frac{\delta_{ac}}{z_1 - z_2} \tilde{\Delta}_{\frac{\pi}{2} + \delta} T_{R_1}^a \otimes T_{R_2}^c \\
&+ \frac{\hbar}{2} \frac{\delta_{ac}}{z_1 - z_2} \frac{1}{\tilde{\Delta}_{-\frac{\pi}{2} + \delta} + 1} T_{R_1}^a \otimes T_{R_2}^c + \hbar \frac{\delta_{ac}}{z_1 - z_2} \frac{1}{\tilde{\Delta}_{\frac{\pi}{2} + \delta} + 1} T_{R_1}^a \otimes T_{R_2}^c + O(\hbar^2) \\
&= 1 + \hbar \frac{1}{z_1 - z_2} (\tilde{\Delta}_{\frac{\pi}{2} + \delta} - \tilde{\Delta}_{-\frac{\pi}{2} + \delta}) T_{R_1}^a \otimes T_{R_2}^a + O(\hbar^2) \\
&= 1 + \hbar \frac{1}{z_1 - z_2} (\tilde{\Delta}_{\frac{\pi}{2} + \delta} - \tilde{\Delta}_{-\frac{\pi}{2} + \delta}) T_{R_1}^a \otimes T_{R_2}^a + O(\hbar^2)
\end{align*}
\]  

(3.15)
Now, note that (3.8) can be rewritten as

$$\tilde{\Delta}_\varphi = \frac{\varphi}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(k\varphi)}{k}. \quad (3.16)$$

This implies that

$$\tilde{\Delta}_{\frac{\pi}{2} + \delta} = \frac{\pi}{2} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(k\frac{\pi}{2})\cos(k\delta) + \cos(k\frac{\pi}{2})\sin(k\delta)}{k}, \quad (3.17)$$

and

$$\tilde{\Delta}_{-\frac{\pi}{2} + \delta} = -\frac{\pi}{2} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(k\frac{\pi}{2})\cos(k\delta) - \cos(k\frac{\pi}{2})\sin(k\delta)}{k}. \quad (3.18)$$

We find that the sums over $k$ above can be separated into two types of sums, each having the form of a Fourier series, namely, the Fourier series for a square wave,

$$\sum_{k=1}^{\infty} \frac{\sin(k\frac{\pi}{2})\cos(kx)}{k} = \frac{\pi}{4} \text{sign}(\cos(x)), \quad (3.19)$$

and the Fourier series for a sawtooth wave,

$$\sum_{k=1}^{\infty} \frac{\cos(k\frac{\pi}{2})\sin(kx)}{k} = \frac{-x}{2} + \frac{l\pi}{2}, \quad \pi(l - \frac{1}{2}) < x < \pi(l + \frac{1}{2}), l \in \mathbb{Z}, \quad (3.20)$$

for $x \in \mathbb{R}$. However, single-valuedness of the propagators involved in the computation (3.15) requires that $\frac{-\pi}{2} < \delta < \frac{\pi}{2}$, implying

$$\sum_{k=1}^{\infty} \frac{\sin(k\frac{\pi}{2})\cos(k\delta)}{k} = \frac{\pi}{4}, \quad (3.21)$$

and

$$\sum_{k=1}^{\infty} \frac{\cos(k\frac{\pi}{2})\sin(k\delta)}{k} = -\frac{\delta}{2}. \quad (3.22)$$

From here we find that

$$\tilde{\Delta}_{\frac{\pi}{2} + \delta} = \frac{1}{2} \quad (3.23)$$

$$\tilde{\Delta}_{-\frac{\pi}{2} + \delta} = -\frac{1}{2}$$

for $\frac{-\pi}{2} < \delta < \frac{\pi}{2}$. As a result, (3.15) is in fact independent of the angle $\delta$, and agrees precisely with the result we found for perpendicular Wilson lines. Hence, we once again find agreement with the results of Costello, Witten and Yamazaki [3].

**Arbitrarily Crossed Wilson Lines**

We can generalize the preceding calculations further to more general configurations of crossed Wilson lines, for which we expect to obtain the same result as (3.12) due to the topological invariance of 4d Chern-Simons along $D$. For instance, we can consider
both Wilson lines rotated from perpendicularity, as shown in Figure 3. The corresponding four-point function is unaffected by the additional rotation, i.e., we find

$$
\langle g_R^{-1}(0 - \alpha, z_1) g_R(\pi - \alpha, z_1) \otimes g_R^{-1}(\pi/2 - \delta, z_2) g_R(3\pi/2 - \delta, z_2) \rangle = 1 + \frac{\hbar}{z_1 - z_2} (\tilde{\Delta}_{\frac{\pi}{2} - \alpha + \delta} - \tilde{\Delta}_{\frac{\pi}{2} - \alpha - \delta}) T_{R_1}^a \otimes T_{R_2 a} + O(h^2)
$$

(3.24)

(3.25)

(where $-\frac{\pi}{2} < -\alpha + \delta < \frac{\pi}{2}$ to ensure single-valued propagators) with the use of the identity (3.23).

Figure 3: Crossed Wilson lines, both rotated from perpendicularity.

A different generalization is that of perpendicular Wilson lines crossing at a point that is not the origin, $r = 0$, as shown in Figure 4. The four-point function in this case is also independent of the angles $\beta$ and $\rho$ shown in the figure, i.e., we have

$$
\langle g_R^{-1}(0 + \beta, z_1) g_R(\pi - \beta, z_1) \otimes g_R^{-1}(\pi/2 + \rho, z_2) g_R(3\pi/2 + \rho, z_2) \rangle = 1 + \frac{\hbar}{z_1 - z_2} \frac{1}{2} (\tilde{\Delta}_{\frac{\pi}{2} + \beta - \rho} - \tilde{\Delta}_{\frac{\pi}{2} + \beta + \rho} + \tilde{\Delta}_{\frac{\pi}{2} - \beta + \rho} - \tilde{\Delta}_{\frac{\pi}{2} - \beta - \rho}) T_{R_1}^a \otimes T_{R_2 a} + O(h^2)
$$

(3.25)

(3.26)

(where $-\frac{\pi}{2} < \beta + \rho < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \beta - \rho < \frac{\pi}{2}$ to ensure single-valuedness of propagators) using the identity (3.23). Note that the allowed ranges of $\beta + \rho$ and $\beta - \rho$ mean that the result is valid only when the Wilson lines are crossed.

Let us now study the most general case. We shall show that the four-point function corresponding to any arbitrary configuration of crossed Wilson lines has the same expression. Such a configuration, as depicted in Figure 5, is determined by four angles, namely $\alpha$, $\beta$, $\gamma$, and $\rho$. The four-point function is then
\[(g_R^{-1}(0 + \beta, z_1) g_R(\pi - \beta, z_1)) \otimes g_R^{-1}(\pi/2 - \rho - \delta, z_2) g_R(3\pi/2 + \rho - \delta, z_2)\]
\[= 1 + \frac{\hbar}{z_1 - z_2} \frac{1}{2} (\tilde{\Delta}_{\pi/2 - \beta + \rho - \alpha + \delta} - \tilde{\Delta}_{\pi/2 + \beta + \rho - \alpha + \delta} + \tilde{\Delta}_{\pi/2 - \beta + \rho - \alpha + \delta} - \tilde{\Delta}_{\pi/2 - \beta - \rho - \alpha + \delta}) T^a_R \otimes T^a_{R2} + O(\hbar^2).\]

(3.26)

Here, to ensure single-valuedness of propagators, we require $\frac{-3\pi}{2} < \beta - \rho - \alpha + \delta < \frac{\pi}{2}$, $\frac{-3\pi}{2} < -\beta + \rho - \alpha + \delta < \frac{\pi}{2}$, $\frac{-3\pi}{2} < -\beta + \rho - \alpha + \delta < \frac{\pi}{2}$, $\frac{-3\pi}{2} < \beta - \rho - \alpha + \delta < \frac{\pi}{2}$. However, to ensure that we are considering only crossed Wilson lines, we require the stronger conditions $\frac{-\pi}{2} < \beta - \rho - \alpha + \delta < \frac{\pi}{2}$, $\frac{-\pi}{2} < -\beta + \rho - \alpha + \delta < \frac{\pi}{2}$, $\frac{-\pi}{2} < -\beta + \rho - \alpha + \delta < \frac{\pi}{2}$, $\frac{-\pi}{2} < -\beta + \rho - \alpha + \delta < \frac{\pi}{2}$. These conditions in turn allow us to use (3.23), whereby...
we find that \( (3.26) \) is
\[
1 + \frac{\hbar}{z_1 - z_2} T_{R_1}^a \otimes T_{R_2 a} + O(\hbar^2). \tag{3.27}
\]

We have thus shown that topological invariance along \( \Sigma \) of the bulk correlation function of two crossed Wilson lines is reflected in the dual four-point function of the boundary 3d “chiral” WZW model, at least up to order \( \hbar \).

3.2 Crossed Wilson Lines at Order \( \hbar^2 \)

Having found the expected \( O(\hbar) \) contribution to the four-point function of arbitrarily inserted crossed Wilson lines in \( (3.26) \), we may improve on this and compute higher order contributions by using (free-field) \( n \)-point functions defined from the generating functional \( (3.5) \), for even \( n \). We shall demonstrate this explicitly for the \( O(\hbar^2) \) contribution to the correlation function of perpendicular Wilson lines.

Firstly, from the generating functional \( (3.5) \), the (free-field) four-point function
\[
\langle \phi^a(w) \phi^b(x) \phi^c(y) \phi^d(z) \rangle = \frac{\hbar^2}{4} \left( \Delta^{ab}(w - x) \Delta^{cd}(y - z) + \Delta^{ac}(w - y) \Delta^{bd}(x - z) + \Delta^{ad}(w - z) \Delta^{bc}(x - y) \right) \tag{3.28}
\]
can be found. Expanding the operators in the RHS of \( (3.3) \) to quadratic order in \( \phi \) as
\[
g_{R_1}^{-1}(0, z_1) g_{R_1}(\pi, z_1) = 1 + (\phi_a(\pi, z_1) - \phi_a(0, z_1)) T_{R_1}^a + \left( -\phi_a(0, z_1) \phi_b(\pi, z_1) \right) T_{R_1}^a T_{R_1}^b + \ldots
\]
\[
g_{R_2}^{-1}(\pi/2, z_2) g_{R_2}(3\pi/2, z_2) = 1 + (\phi_a(3\pi/2, z_2) - \phi_a(\pi/2, z_2)) T_{R_2}^a + \left( -\phi_a(\pi/2, z_2) \phi_b(3\pi/2, z_2) \right) T_{R_2}^a T_{R_2}^b + \ldots.
\tag{3.29}
\]
we then find via (3.28) that, to order $\hbar^2$, (3.3) is
\[
\langle g_{R_1}^{-1}(0, z_1) g_{R_1}(\pi, z_1) \otimes g_{R_2}^{-1}(\pi/2, z_2) g_{R_2}(3\pi/2, z_2) \rangle
= 1 + \frac{h}{z_1 - z_2} T_{R_1} \otimes T_{R_2} + \frac{h^2}{(z_1 - z_2)^2} \left( T_{R_1}^b T_{R_1} \otimes T_{R_2}^c T_{R_2} + T_{R_1}^b T_{R_1} \otimes T_{R_2} T_{R_2}^a + O(h^3) \right),
\]
where we have used the notation $\tilde{\Delta}_{a\varphi} = \tilde{\Delta}_{\varphi} \delta_{ab}$ for brevity.

In a similar manner, one can compute contributions to the $R$-matrix of order $h^3$ and above. Note that these contributions are not expected to remain invariant under moves of the local boundary operators that correspond to rotations and translations of the bulk Wilson lines, due to the framing anomaly that arises in the bulk theory at order $\hbar^2$ for non-perpendicular Wilson lines [3]. This framing anomaly ought to be computable in our boundary WZW model as well, by taking into account its interaction terms when computing correlation functions of local operators.

A slightly more involved calculation shows that the result of (3.30) holds, modulo the framing anomaly, for arbitrarily inserted Wilson lines (as depicted in Figure 5), assuming the same constraints on the angles given below (3.26).

### 3.3 Parallel Wilson Lines

The OPEs of parallel Wilson lines in 4d Chern-Simons theory do not have the same singular behaviour as correlation functions of crossed Wilson lines. In this subsection, we shall consider such correlation functions of parallel Wilson lines and show how they are captured by correlation functions of boundary operators. We shall focus on the free-field limit, at order $\hbar$, and retrieve the expected behaviour in this regime.

Using (3.2), the correlation function of the operators we are interested in (depicted in Figure 6) is
\[
\langle P e^{i \varphi z_1 / 2} A_{R_1}(z_1, \bar{z}_1) \otimes P e^{i \varphi \bar{z}_1 / 2} A_{R_2}(z_2, \bar{z}_2) \rangle = \langle g_{R_1}^{-1}(0, z_1, \bar{z}_1) g_{R_1}(3\pi/2, z_1, \bar{z}_1) \otimes g_{R_2}^{-1}(\pi/2, z_2, \bar{z}_2) g_{R_2}(\pi, z_2, \bar{z}_2) \rangle,
\]
(note the difference from (3.3) in ordering of the boundary operators when $z_1 = z_2$ and $R_1 = R_2$).
Expanding each operator to linear order in $\phi$ and keeping only terms of quadratic or lower order in the fields (as in the discussion below (3.11)), we have

\[
1 + \langle \phi_a(2\pi, z_1)\phi_c(\pi/2, z_2) \rangle T^a_{R_1} \otimes T^c_{R_2} - \langle \phi_a(2\pi, z_1)\phi_c(\pi, z_2) \rangle T^a_{R_1} \otimes T^c_{R_2}
\]

\[
- \langle \phi_a(3\pi/2, z_1)\phi_c(\pi/2, z_2) \rangle T^a_{R_1} \otimes T^c_{R_2} + \langle \phi_a(3\pi/2, z_1)\phi_c(\pi, z_2) \rangle T^a_{R_1} \otimes T^c_{R_2} + O(h^2)
\]

\[
= 1 - \frac{h}{2} \delta_{ac} \Delta_{z_1 - z_2} T^a_{R_1} \otimes T^c_{R_2} + \frac{h}{2} \delta_{ac} \Delta_{\phi} T^a_{R_1} \otimes T^c_{R_2}
\]

\[
+ \frac{h}{2} \delta_{ac} \Delta_{z_1 - z_2} T^a_{R_1} \otimes T^c_{R_2} + \frac{h}{2} \delta_{ac} \Delta_{\phi} T^a_{R_1} \otimes T^c_{R_2} + O(h^2)
\]

\[
= 1 + O(h^2),
\]

which is non-singular for $z_1 = z_2$ at order $h$, as expected. Here, we have used the previously derived fact that $\Delta_\phi = \frac{1}{2}$, as well as $\Delta_{z} = \frac{1}{2}$, which follows from (3.17), (3.19) and (3.20), and

\[
\tilde{\Delta}_\pi = \frac{1}{2\pi} \left( \sum_{k=1}^{\infty} \frac{(-1)^k}{ik} + \pi + \sum_{k=-\infty}^{-1} \frac{(-1)^k}{ik} \right)
\]

\[
= \frac{1}{2\pi} (\ln 2 + \pi - \ln 2)
\]

\[
= \frac{1}{2}
\]

(3.33)

Also, note that we must define the propagator here with a different branch cut from that of previous sections, namely, from $r = 0$ to $(r = R, \varphi = 0)$. This effectively restricts $\varphi$ in (3.8) to take values in $(0, 2\pi)$.  

Figure 6: Parallel Wilson lines on $D$. 

Expanding each operator to linear order in $\phi$ and keeping only terms of quadratic or lower order in the fields (as in the discussion below (3.11)), we have
3.4 Three Wilson Lines

We next consider correlation functions of three Wilson lines, all crossing each other but otherwise inserted arbitrarily (c.f. Figure 7), which corresponds to the following boundary correlator:

\[
(g_{R_1}^{-1}(0 + \beta - \rho)g_{R_1}(\pi - \beta - \rho) \otimes g_{R_2}^{-1}(\frac{\pi}{2} - \delta - \alpha)g_{R_2}(\frac{3\pi}{2} + \delta - \alpha) \otimes g_{R_3}^{-1}(\pi - \gamma - \zeta)g_{R_3}(0 + \gamma - \zeta)),
\]

where the dependence on \( C \) has been suppressed for brevity. Expanding each operator in

![Figure 7: Three Wilson lines.](image)

(3.34) to linear order in \( \phi \) we find

\[
1 + \left( \langle \phi_a(0 + \beta + \rho, z_1) \phi_c(\pi/2 - \delta - \alpha, z_2) \rangle - \langle \phi_a(\pi - \beta - \rho, z_1) \phi_c(\pi/2 - \delta - \alpha, z_2) \rangle \right)
- \langle \phi_a(2\pi + \beta - \rho, z_1) \phi_c(3\pi/2 + \delta - \alpha, z_2) \rangle + \langle \phi_a(\pi - \beta - \rho, z_1) \phi_c(3\pi/2 + \delta - \alpha, z_2) \rangle
+ \left( \langle \phi_a(0 + \beta - \rho, z_1) \phi_c(\pi/2 - \gamma' - \zeta', z_3) \rangle - \langle \phi_a(2\pi + \beta - \rho, z_1) \phi_c(3\pi/2 + \gamma' - \zeta', z_3) \rangle \right)
- \langle \phi_a(\pi - \beta - \rho, z_1) \phi_c(\pi/2 - \gamma' - \zeta', z_3) \rangle + \langle \phi_a(\pi - \beta - \rho, z_1) \phi_c(3\pi/2 + \gamma' - \zeta', z_3) \rangle
+ \left( \langle \phi_a(\pi/2 - \delta - \alpha, z_2) \phi_c(\pi - \gamma - \zeta, z_3) \rangle - \langle \phi_a(\pi/2 - \delta - \alpha, z_2) \phi_c(0 + \gamma - \zeta, z_3) \rangle \right)
- \langle \phi_a(3\pi/2 + \delta - \alpha, z_2) \phi_c(\pi - \gamma - \zeta, z_3) \rangle + \langle \phi_a(3\pi/2 + \delta - \alpha, z_2) \phi_c(2\pi + \gamma - \zeta, z_3) \rangle
\]

\[
\otimes T_{R_1}^a \otimes T_{R_2}^a \otimes 1 + \frac{h}{z_1 - z_2} \left( \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) T_{R_1}^a \otimes T_{R_2}^a \otimes 1
\]

\[
= 1 + \frac{h}{z_1 - z_2} \left( \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) T_{R_1}^a \otimes T_{R_2}^a \otimes 1 + \frac{h}{z_1 - z_3} \left( \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) T_{R_1}^a \otimes T_{R_3}^a
\]

\[
- \frac{h}{z_3 - z_2} \left( \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) 1 \otimes T_{R_2}^a \otimes T_{R_3}^a + O(h^2)
\]

\[
= 1 + \frac{h}{z_1 - z_2} T_{R_1}^a \otimes T_{R_2}^a \otimes 1 + \frac{h}{z_1 - z_3} T_{R_1}^a \otimes 1 \otimes T_{R_3}^a + \frac{h}{z_2 - z_3} 1 \otimes T_{R_2}^a \otimes T_{R_3}^a + O(h^2),
\]
where various constraints on the angles are necessary for single-valuedness of propagators and to ensure that the Wilson lines are all crossed. Once again, there is agreement with the bulk 4d Chern-Simons computation.

We may further compute the correlation function (3.34) to order $\hbar^2$ by expanding the operators in (3.34) to quadratic order in $\phi$. Doing so, we find the following expression (modulo the framing anomaly):

$$1 + \frac{\hbar}{z_1 - z_2} T_{R_1}^a \otimes T_{R_2} \otimes 1 + \frac{\hbar}{z_1 - z_3} T_{R_1}^a \otimes 1 \otimes T_{R_3} + \frac{\hbar}{z_2 - z_3} 1 \otimes T_{R_1}^a \otimes T_{R_3}$$

$$+ \frac{\hbar^2}{4(z_1 - z_2)^2} (T_{R_1}^a T_{R_1}^b \otimes T_{R_2} \otimes 1 + T_{R_1}^b T_{R_1}^a \otimes T_{R_2} \otimes 1)$$

$$+ \frac{\hbar^2}{4(z_1 - z_3)^2} (T_{R_1}^a T_{R_1}^b \otimes 1 \otimes T_{R_3} + T_{R_1}^b T_{R_1}^a \otimes 1 \otimes T_{R_3})$$

$$+ \frac{\hbar^2}{4(z_2 - z_3)^2} (1 \otimes T_{R_2} T_{R_2}^b \otimes T_{R_3} + 1 \otimes T_{R_1} T_{R_1}^b \otimes T_{R_3}) + O(\hbar^3).$$

This result agrees with the bulk 4d Chern-Simons computation. To see this, let us consider the configurations in Figure 8. From the bulk theory, we know that the equivalence of these two configurations gives rise to the Yang-Baxter equation

$$\tilde{R}_{12} \tilde{R}_{13} \tilde{R}_{23} = \tilde{R}_{23} \tilde{R}_{13} \tilde{R}_{12},$$

where

$$\tilde{R}_{12} = 1 + \frac{\hbar}{z_1 - z_2} T_{R_1}^a \otimes T_{R_2} \otimes 1$$

$$+ \frac{\hbar^2}{4(z_1 - z_2)^2} (T_{R_1}^a T_{R_1}^b \otimes T_{R_2} \otimes 1 + T_{R_1}^b T_{R_1}^a \otimes T_{R_2} \otimes 1) + O(\hbar^3),$$

$$\tilde{R}_{13} = 1 + \frac{\hbar}{z_1 - z_3} T_{R_1}^a \otimes 1 \otimes T_{R_3}$$

$$+ \frac{\hbar^2}{4(z_1 - z_3)^2} (T_{R_1}^a T_{R_1}^b \otimes 1 \otimes T_{R_3} + T_{R_1}^b T_{R_1}^a \otimes 1 \otimes T_{R_3}) + O(\hbar^3),$$

$$\tilde{R}_{23} = 1 + \frac{\hbar}{z_2 - z_3} 1 \otimes T_{R_2}^a \otimes T_{R_3}$$

$$+ \frac{\hbar^2}{4(z_2 - z_3)^2} (1 \otimes T_{R_2} T_{R_2}^b \otimes T_{R_3} + 1 \otimes T_{R_2} T_{R_2}^b \otimes T_{R_3}) + O(\hbar^3).$$

We shall now make use of the identity

$$\left[ T_{R_1}^a, T_{R_2}^b \right] \otimes T_{R_3} + T_{R_1}^a \otimes \left[ T_{R_2}^a, T_{R_2}^b \right] \otimes T_{R_3} + T_{R_1}^a \otimes T_{R_2}^b \otimes \left[ T_{R_2}^a, T_{R_3} \right] = 0,$$

(3.39)
Figure 8: The Yang-Baxter equation is realized by moving a Wilson line across the intersection of two other Wilson lines.

which follows since the classical $r$-matrix, $r_{ij} = \frac{T_R^a \otimes T_R^{a_i}}{z_i - z_j}$ ($i, j = 1, 2, 3$, where $j > i$), obeys the classical Yang-Baxter equation. Using (3.39), the last three terms at order $\hbar^2$ of (3.36) can be shown to be

$$\frac{\hbar^2}{(z_1 - z_2)(z_1 - z_3)} T_{R_1}^a T_{R_2}^b \otimes T_{R_3a} \otimes T_{R_3b} + \frac{\hbar^2}{(z_1 - z_2)(z_2 - z_3)} T_{R_1}^a \otimes T_{R_2a} T_{R_3b} \otimes T_{R_3}^b$$

$$+ \frac{\hbar^2}{(z_1 - z_3)(z_2 - z_3)} T_{R_1}^a \otimes T_{R_2}^b \otimes T_{R_3a} T_{R_3b},$$

(3.40)

whereby (3.36) agrees with the LHS of (3.37). Alternatively, we can use (3.39) such that the last three terms at order $\hbar^2$ of (3.36) are

$$\frac{\hbar^2}{(z_1 - z_2)(z_1 - z_3)} T_{R_1}^a T_{R_2}^b \otimes T_{R_3a} \otimes T_{R_3b} + \frac{\hbar^2}{(z_1 - z_2)(z_2 - z_3)} T_{R_1}^a \otimes T_{R_2}^b T_{R_3a} \otimes T_{R_3b}$$

$$+ \frac{\hbar^2}{(z_1 - z_3)(z_2 - z_3)} T_{R_1}^a \otimes T_{R_2}^b \otimes T_{R_3b} T_{R_3a},$$

(3.41)

whereby (3.36) agrees with the RHS of (3.37). Thus, the boundary six-point function (3.36) is in agreement with the bulk correlation function of three Wilson lines up to order $\hbar^2$, modulo the framing anomaly. We expect that this will hold at higher orders of $\hbar$ as well.

References

[1] K. Costello, Integrable Lattice Models From Four-Dimensional Field Theories, Proc. Symp. Pure Math 88 (2014) 3-24, [arXiv:1709.09993]

[2] K. Costello, Supersymmetric Gauge Theory and the Yangian, ArXiV High-Energy Physics-Theory e-prints (March, 2013) [arXiv:1303.2632]

[3] K. Costello, E. Witten, M. Yamazaki, Gauge Theory and Integrability, I, Notices of the International Congress of Chinese Mathematicians 6 (1) (2018) 46-119 [arXiv:1709.09993]
[4] K. Costello, E. Witten, M. Yamazaki, *Gauge Theory and Integrability, II*, Notices of the International Congress of Chinese Mathematicians 6 (1) (2018) 120-146 [arXiv:1802.01579]

[5] K. Costello, M. Yamazaki, *Gauge Theory and Integrability, III*, ArXiV High-Energy Physics-Theory e-prints (August, 2019) [arXiv:1908.02289]

[6] R. Bittleston, D. Skinner *Gauge Theory and Boundary Integrability*, Journal of High Energy Physics, 05 (2019) 195 [arXiv:1903.03601]

[7] R. Bittleston, D. Skinner *Gauge Theory and Boundary Integrability II: Elliptic and Trigonometric Case*, ArXiV High-Energy Physics-Theory e-prints (December, 2019) [arXiv:1912.13441]

[8] M. Yamazaki, *New T-duality for Chern-Simons Theory*, ArXiV High-Energy Physics-Theory e-prints (April, 2019) [arXiv:1904.04976]

[9] V. G. Drinfeld, *Quantum Groups*, in Proceedings of the International Congress of Mathematicians, Berkeley, 1986, American Mathematical Society, 1987.

[10] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, 1994.

[11] S. Elitzur, G. Moore, A. Schwimmer, N. Seiberg, *Remarks on the canonical quantization of the Chern-Simons-Witten theory*, Nuclear Physics B, 326 (1) (1989) 108-134

[12] E. Witten, *Nonabelian bosonization in two dimensions*, Communications in Mathematical Physics, 92 (4) (1984) 455-472

[13] T. Inami, H. Kanno, T. Ueno, C.S. Xiong, *Two-toroidal Lie algebra as current algebra of the four-dimensional Kahler WZW model*, Physics Letters B, 399 (1-2) (1997) 97-104

[14] E. Witten, *Integrable Lattice Models From Gauge Theory*, ArXiV High-Energy Physics-Theory e-prints (November, 2016) [arXiv:1611.00592]