Magnetocentrifugal launching of jets from disks around Kerr black holes

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ABSTRACT

Strong magnetic fields modify particle motion in the curved space-time of spinning black holes and change the stability conditions of circular orbits. We study conditions for magnetocentrifugal jet launching from accretion disks around black holes, whereby large scale black hole lines anchored in the disk may fling tenuous coronal gas outward. For a Schwarzschild black hole, magnetocentrifugal launching requires that the poloidal component of magnetic fields makes an angle less than $60^\circ$ to the outward direction at the disk surface, similar to the Newtonian case. For prograde rotating disks around Kerr black holes, this angle increases and becomes $90^\circ$ for footpoints anchored to the disk near the horizon of a critically spinning $a = M$ black hole. Thus, a disk around a critically spinning black hole may centrifugally launch a jet even along the rotation axis.

1. Introduction

The production of relativistic jets in (micro)quasars, Active Galactic Nuclei and Gamma Ray Bursts involves the conversion of rotational energy of the accreting matter and/or of the spinning black hole into plasma bulk motion (Blandford & Znajek 1977; Blandford & Payne 1982; Punsly & Coroniti 1990). This energy conversion is mediated by the magnetic field, which can both accelerate plasma by magnetic gradient forces and collimate it by hoop stresses, (see Meier et al. 2001 for review). It is not clear whether jet is powered by the accretion disk or by the spinning black hole (more precisely, whether the footpoints of magnetic field lines that eventually form the jet should pass throughout the disk, ergosphere or horizon of the black hole). Numerical simulations confirm, at least partially, that all these mechanisms can be operational (Meier et al. 2001; Koide 2004; Komissarov 2005; Tchekhovskoy et al. 2008).

There are three related issues in the production of jets: launching, acceleration and collimation. In this paper we address the conditions for jet launching from a thin accretion
disk rotating in the equatorial plane of a Kerr black hole. We assume that the disk is permeated by a large scale magnetic field, so that magnetic field lines are anchored in the disk and are dragged both by plasma rotation and the rotation of space time around the black hole. This assumes that the disk conductivity is very high, and that inside the disk the magnetic field is subdominant and does not affect plasma motion in the disk. In addition, we assume that the disk is surrounded by a tenuous coronal plasma and the magnetic field is sufficiently strong to affect particle motion in the corona. The dynamical effects of a large scale magnetic field on single particle motion are often approximated as a guiding wire with a particle playing the role of a bead. This simple approximation restricts particle motion across the field, neglecting various cross-field drifts.

We study the dynamical behavior of coronal particles, located close to the disk, which are locally restricted to move exclusively along a given magnetic field line. In such a case, under certain conditions discussed below, a coronal particle executing a circular motion around a black hole is in a state of unstable equilibrium, even though without a magnetic field it may be in a stable equilibrium. The condition of unstable equilibrium, we assume, constitutes a condition for jet launching from an accretion disk. This may be the most relevant condition for the generation of jets (Blandford & Payne 1982; Ogilvie & Livio 2001; Spruit 2008).

2. Bead on a wire rotating in Kerr space

2.1. Rotating Kerr metric

Assume that a footpoint of a magnetic field line is anchored in accreting matter and is rotating with angular velocity $\omega$ at $r = r_0$. Next, neglect the toroidal component of the magnetic field (see Appendix A) and assume that a field line makes an angle $\theta_l$ with the radial direction, Fig. 1.

As a mathematical problem, consider the motion of a bead on a wire sticking out of the equatorial plane of a Kerr black hole at an angle $\theta_l$ with the radial direction and confined to move in a given meridional plane. In Boyer-Lindquist coordinates (e.g. Misner et al. 1973), make a coordinate transformation to the frame rotating with the foot-point, $\phi \rightarrow \phi' - \omega t$. Restricting particle motion along the wire, $\tan \theta = r/(r - r_0) \tan \theta_l$, $d\phi = 0$ (see Fig. 1) implies

$$d\theta = \cot \theta_l \frac{r_0 dr}{(r - r_0)^2 + r^2 \cot^2 \theta_l}.$$  \hspace{1cm} (1)

In the coordinate frame rotating with the wire the non-vanishing components of the metric
Fig. 1.— Geometry of the model. A magnetic field line is anchored in the disk at the equatorial plane of a Kerr black hole at radius $r_0$ and makes an angle $\theta_l$ (in Boyer-Lindquist coordinates) with the radial direction.

The metric components are

\begin{align*}
g_{00} &= -1 + \frac{1}{\Sigma} \left( 2Mr + \frac{4aMr^3\omega \cot^2 \theta_l}{\Delta_2} + \left( \frac{(a^2 + r^2)^2}{\Delta_2} - \frac{a^2 r^2 \cot^2 \theta_l}{\Delta_2} \right) \frac{r^2 \omega^2 \cot^2 \theta_l}{\Delta_2} \right) \\
g_{rr} &= \left( \frac{1}{\Delta} + \frac{r_0^2 \cot^2 \theta_l}{\Delta_2} \right) \Sigma \\
\Delta &= a^2 + r^2 - 2Mr \\
\Sigma &= r^2 + a^2 \cos^2 \theta \\
\Delta_2 &= (r - r_0)^2 + r^2 \cot^2 \theta_l
\end{align*}

where $a$ is the angular momentum per unit black hole mass and $M$ is the mass of the black hole. (We set $G = c = 1$, use $(-1, 1, 1, 1)$ sign convention and assume that the mass of a test particle is unity.) We assume that $\omega > 0$, while the parameter $a$ can be in the range $-M < a < M$ ($a > 0$ corresponds to prograde rotation of the wire). Below we refer to the metric (2) as a rotating Kerr metric.

Condition $g_{00} = 0$ defines two light cylinders. Let us consider the location of the light
cylinders for a particular case of radial magnetic field $\theta_l = 0, \theta = \pi/2$. In this case

$$g_{00} = -1 - (a^2 + r^2)\omega^2 + \frac{2M(1 - a\omega)^2}{r}$$

$$g_{rr} = \frac{r^2}{\Delta}. \quad (3)$$

For $\omega = 0$, the inner light cylinder coincides with the ergosphere, $r = 2M$. For small $\omega \ll 1/M$, the outer light cylinder is approximately at $r \sim 1/\omega$. The location of the light cylinders are given by

$$r_{in} = \sqrt{\frac{1}{\omega^2} - a^2} \left( \sin \left( \frac{1}{3} \arccos \left( -\frac{3\sqrt{3}(1 - a\omega)^2M\omega}{(1 - a^2\omega^2)^{3/2}} \right) \right) - \frac{1}{\sqrt{3}} \cos \left( \frac{1}{3} \arccos \left( -\frac{3\sqrt{3}(1 - a\omega)^2M\omega}{(1 - a^2\omega^2)^{3/2}} \right) \right) \right)$$

$$r_{out} = \frac{2}{\sqrt{3}} \sqrt{\frac{1}{\omega^2} - a^2} \cos \left( \frac{1}{3} \arccos \left( -\frac{3\sqrt{3}(1 - a\omega)^2M\omega}{(1 - a^2\omega^2)^{3/2}} \right) \right) \quad (4)$$

Since the determinant of the metric tensor is smaller than zero beyond the light cylinders, the approximation of a rigidly rotating wire is inapplicable in those regions.

Condition $\partial_r g_{00} = 0$ defines circular orbits

$$r_{in}^3\omega^2 = M(1 - a\omega)^2 \quad (5)$$

This is Kepler’s law in a Kerr metric.

Transformation to the rotating Kerr metric is physical only for angular velocities smaller than $\omega_{ph}$, angular velocity of a photon orbit, defined by the conditions of circular rotation with the speed of light $g_{00} = 0, \partial_r g_{00} = 0$. This gives $-4a^2M + r(-3M + r)^2 = 0$ [Bardeen et al. 1972]. For a given $a$ and $r$, the transformation to the rotating Kerr metric becomes meaningless for $\omega$ higher than the angular velocity of a photon circular orbit,

$$\omega_{ph} = \frac{1}{\pm|a| + 6M \cos \left( \frac{1}{3} \arccos \left( \mp|a|/M \right) \right)} \quad (6)$$

The upper sign corresponds to prograde rotation. Particular values are $a = 0$, $r_{ph} = 3M$, $\omega_{ph} = 1/(3\sqrt{3}M)$ for Schwarzschild black hole, $a = M$, $r_{ph} = M$, $\omega_{ph} = 1/(2M)$ for prograde and $a = -M$, $r_{ph} = 4M$, $\omega_{ph} = 1/(7M)$ for retrograde photon orbits. For $1/(7M) < \omega < 1/(2M)$ this requires sufficiently high $a$, satisfying $\omega < 1/ (a + 2\sqrt{2}M \left( 1 + \cos \left( \frac{2}{3} \arccos \left( -\frac{a}{M} \right) \right) \right))^{3/2}$ (for $a = 0$ this requires $\omega < 1/(3\sqrt{3})$). For $\omega \rightarrow 1/(2M)$ both light cylinders merge on the black hole horizon at $a = M = r$. For higher $\omega$, the transformation to the rotating frame becomes meaningless everywhere (Fig. 2).
Fig. 2.— The locations of light cylinders for different $\omega$ as function of $a$. The light cylinders always lie outside the horizon (dotted curve). For a given $\omega$, the inner and outer light cylinders merge at the photon circular orbit (dashed line). The physically meaningful region lies in between the light cylinders, to the right of the light cylinders curve. For $\omega \to 0$, the inner light cylinder coincides with the ergosphere $r = 2M$. As $\omega$ increases, the outer light cylinder moves to smaller $r$; the radial location of the inner light cylinder is a complicated function of $\omega$. For $\omega > 1/(7M)$ there is a region for sufficiently small $a$, for which the transformation to the rotating frame is unphysical. For high $\omega \to 1/(2M)$ and high $a > M/\sqrt{2}$, the inner light cylinder moves inside the ergosphere of a Kerr black hole. For $\omega > 1/(2M)$, the transformation to rotating frame is unphysical everywhere.

2.2. Motion of particle in rotating Kerr metric.

Since metric (3) is time independent, the Hamilton - Jacobi equation,

$$g^{\mu\nu} \partial_\mu S \partial_\nu S = g^{00} (\partial_t S)^2 + g^{rr} (\partial_r S)^2 = -1$$

(7)
has a solution in the form $S = -E_0 t + S_r (r)$ with

$$S_r = \int dr \sqrt{-(E_0^2 + g_{00}) \frac{g_{rr}}{g_{00}}}$$

(8)

Differentiating with respect to $E_0$ gives the equation of motion

$$\left(\partial_t r\right)^2 = -(1 + g_{00}/E_0^2) \frac{g_{00}}{g_{rr}}$$

(9)

Transforming to proper time $dt/d\tau = E_0/(-g_{00})$ gives

$$\left(\partial_\tau r\right)^2 = -\frac{E_0^2 + g_{00}}{g_{00}g_{rr}} \equiv V$$

(10)

This defines the effective potential $V$. The extremum of the potential $\partial_r V = 0$ defines the angular velocity of a circular orbit. Direct calculations give, naturally, condition (5).

2.2.1. Reversal of centrifugal acceleration at photon circular orbit

The above relation simply in case of a radial magnetic field, $\theta_l = 0$. In this limit

$$V = -\Delta \frac{E_0^2}{r^2} \left(1 - \frac{2M}{r} (1 - a\omega)^2 - \omega^2 (r^2 + a^2)\right) = 0 \text{ (for circular orbits)}$$

$$\left.\frac{\partial^2 r}{\partial \tau^2}\right|_{E_0 = \sqrt{-g_{00}}} = \frac{1}{2} \partial_r V = -\frac{Mr - a^2}{r^2} + \left(\frac{a + (r^2 + a^2)\omega}{r^2}\right) \frac{a(M - r) + \omega (r^2(r - 3M) + a^2(r + M))}{r^2}$$

(11)

For a Schwarzschild black hole, $a = 0$, Eq. (11) shows the reversal of centrifugal acceleration at the photon circular orbit $r = 3M$ (Abramowicz & Prasanna 1990):

$$\frac{\partial^2 r}{\partial \tau^2} = -\frac{M}{r^2} + (r - 3M)\omega^2$$

(12)

For a Kerr black hole, no clear separation can be made between the effects of the wire rotation and the rotation of space-time, so that the notion of a centrifugal force becomes ill-defined (see Abramowicz & Prasanna 1990; de Felice 1991; Abramowicz & Lasota 1997; Iyer & Prasanna 1993, for discussion).

3. Stability of circular orbits in magnetic field

Circular orbits are defined by $\partial_r r = 0 \ (V = 0, \ E_0 = \sqrt{-g_{00}})$ and $\partial^2_\tau r = 0 \ (V' = 0$, condition (5)). The stability of an orbit depends on the second derivative of the potential
V, Eq. (10).

\[ \kappa^2 = -\frac{1}{2} V''_{rr} \bigg|_{E_0=\sqrt{-g_{00}}, r=r_0} = \Delta \left( 3a^2 - 4a\sqrt{M}r + r^2(1 - 3\cot^2 \theta_l) \right) \left( r^4(r - 3M) + 2a\sqrt{M}r^{7/2} \right)(\Delta + r^2\cot^2 \theta_l) M, \]  
(13)

where we replaced \( r_0 \) by \( r \). Stability requires the epicyclic frequency to be real, \( \kappa^2 > 0 \). We identify parameters for which \( \kappa^2 < 0 \) as a necessary condition for jet launching.

The sign of \( \kappa^2 \) in Eq. (13) determines whether a particle confined to move along the magnetic field anchored at radius \( r \) to the disk is in a state of stable equilibrium \( (\kappa^2 > 0) \). In Eq. (13), the denominator changes sign at the photon circular orbit \( r^4(r - 3M) + 2a\sqrt{M}r^{7/2} = 0 \), Eq. (6). The physically realizable case corresponds to radii larger than the radius of the photon circular orbit. The numerator in Eq. (13) changes sign at

\[ \cot \theta_l = \sqrt{\frac{1}{3} - \frac{4a\sqrt{M}}{3r^{3/2}} + \frac{a^2}{r^2}} \]  
(14)

When the magnetic field lines make an angle with the radial direction less than the one given by Eq. (14), a particle tied to the field line in the equatorial plane is in a state of unstable equilibrium and will be flung away from the disk.

An analysis of Eq. (14) reveals, first, that the magnetic field cannot stabilize particle motion for an arbitrary (especially radial) direction of the magnetic field (angle \( \theta_l \) is never zero). In the Schwarzschild metric, \( \cot \theta_l = \sqrt{1/3}, \theta_l = \pi/3 \), which is exactly the same result as in Newtonian mechanics (Blandford & Payne 1982). For prograde rotation, \( a > 0 \), the instability region is larger, \( \theta_l > \pi/3 \), so that the flow is more unstable in the Kerr metric if compared with the Newtonian and Schwarzschild cases. Or, equivalently, a flow my be launched over a wider range of angles.

Physically meaningful cases correspond to footpoint motion along stable circular orbits in a Kerr space-time. For prograde rotation around a critically spinning Kerr black hole, the outer and inner light cylinders, photon circular orbit and horizon all coincide at \( r = M \) in Boyer-Lindquist coordinates; in proper radial coordinates they remain distinct (Bardeen et al. 1972). According to Eq. (14), in the limit \( a \rightarrow M \), the angle \( \theta_l \rightarrow \pi/2 \), so that the motion along all field lines, including those aligned with the axis of rotation, becomes unstable. Thus, in the case of a critically spinning Kerr black hole, a jet may be launched by magnetocentrifugal mechanism across the rotation axis of a black hole.

For any \( a \neq M \), the condition \( \theta_l = \pi/2 \) is satisfied only inside the horizon, so that for \( a \neq M \) there is always a set of magnetic field lines around the direction of hole’s rotation where a particle is in a state of stable equilibrium. A jet won’t be launched along those field lines.
Fig. 3.— The maximum unstable angle $\theta_l$ of the magnetic field inclination to the plane of the disk evaluated at the innermost stable orbit. For angles less that $\theta_l$, a particle on a circular orbit is in an unstable equilibrium and will be flung away by centrifugal forces. For a Schwarzschild black hole, $a = 0$, $\theta_l = 60^\circ$, similar to the Newtonian case. For $a \to M$, a particle can be accelerated along the rotation axis. For counter rotating disk around critically spinning black hole the critical angle decreases to $\cot \theta_l = 4\sqrt{2}/9$, $\theta_l = 57.85^\circ$

At the innermost stable orbit (Bardeen et al. 1972), the angle $\theta_l$ is, generally, close to $60^\circ$, except in a narrow region of parameter $a \sim M$ for prograde rotation, where $\theta_l$ approaches $90^\circ$, see Fig. 3. For a retrograde disk rotating around a critically spinning black hole, the maximum launching angle decreases to $\cot \theta_l = 4\sqrt{2}/9$, $\theta_l = 57.8^\circ$ at the last retrograde stable orbit corresponding to $r = 9M$. In Fig. 4 we plot the values of the maximum unstable angle $\theta_l$ of the magnetic field inclination to the plane of the disk as function of radius various values of black hole spin $a$. 
Fig. 4.— Values of the maximum unstable angle $\theta_l$ of the magnetic field inclination to the plane of the disk as function of radius for various values of black hole spin $a$. Dashed line is the values of $\theta_l$ at the innermost stable orbit, see Fig. 3. Dotted line corresponds to Schwarzschild black hole, $a = 0$, $\theta_l = 60^\circ$

4. Force-free magnetospheres

Conditions for stability of particle motion at the base of the jet derived in $\S 3$ should be applied to the magnetic structures of black hole magnetospheres to determine if a particular magnetic field solution is consistent with the jet launching conditions at the disk surface. Analytical models of black hole magnetospheres are usually limited to the force-free approximation, when matter inertia and pressure forces are neglected. The equation governing force-free magnetic configurations in the Kerr metric was derived by Blandford & Znajek (1977) (see also Thorne et al. (1986)). No analytical solution for the magnetic field structure of fast spinning black holes, $a \sim M$, is known. (The vacuum Wald solution (Wald 1974) has large parallel electric fields; this limits its astrophysical applicability. In addition, the Wald field is orthogonal to the equatorial plane.)
Analytical solutions of force-free structures for Schwarzschild and slowly spinning Kerr black hole magnetospheres \((a \to 0)\) are generally limited to the case of zero poloidal current. In this case \((a = 0)\) the governing Grad-Shafranov equation \((\text{Shafranov 1966})\) for the poloidal flux function \(P\) becomes \(\text{(Blandford & Znajek 1977; Thorne et al. 1986)}\)
\[
[r^2 \partial_r ((1 - 2M/r) \partial_r P) + (1 - \mu^2) \partial_\mu^2 P] = 0 \tag{15}
\]
where \(\mu = \cos \theta\). Magnetic fields measured by a locally stationary observer are then given by \(\text{(Thorne et al. 1986)}\)
\[
B_\varphi = \frac{\partial_\theta P}{r^2 \sin \theta} \\
B_\theta = \sqrt{1 - 2M/r} \frac{\partial_r P}{r \sin \theta} \tag{16}
\]
(Note, that a form of the flux function \(r(\theta)\) as measured by a set of locally stationary observer is given by \(\text{dr/d}\theta = B_\theta/B_\varphi = \sqrt{1 - 2M/r} (\partial_r P/\partial_\theta P)\), not just \(P = \text{constant}\) as in the flat space).

There is a number of known solutions of Eq. \((15)\). Those connecting to the accretion disk (as opposed exclusively to the black hole) are (i) a Schwarzschild black hole in the constant magnetic field \(P \propto r^2 \sin^2 \theta\); (ii) the non-separable parabolic solution of \(\text{Blandford & Znajek (1977)}\) \(P \propto (r - 2M)(1 - \mu) - 2M(1 + \mu) \ln(1 + \mu)\); (iii) the separable Schwarzschild paraboloid \(P \propto (r + 2M \ln(r - 2M))(1 - \mu)\) (and corresponding higher order multipoles \(\text{Ghosh 2000}\)). These are all linear solutions, so that any combination is a possible solution as well.

The angle \(\theta_b\) that magnetic field lines make with the equatorial plane is
\[
\tan \theta_b = \frac{-B_\theta}{B_\varphi} = \sqrt{1 - 2M/r} \left. \frac{r \partial_r P}{\partial_\theta P} \right|_{\theta = \pi/2} \tag{17}
\]
The constant magnetic field solution (i) has a magnetic field orthogonal to the disk at all points. For the parabolic solution of \(\text{Blandford & Znajek (1977)}\) (ii), the angle that the magnetic field lines make with the equatorial plane decreases as the footpoint approaches the horizon, \(\tan \theta_b = \sqrt{1 - 2M/r}\); for large radii, \(r \gg M\), field lines stick out at \(45^\circ\) to the plane of the disk. Thus, for the parabolic solution of \(\text{Blandford & Znajek (1977)}\), the particles confined to move along field lines are always in an unstable equilibrium, so that the jet can be launched from any point on the disk. For the separable Schwarzschild paraboloid solution (iii), the angle that magnetic field lines make with the equatorial plane

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\(^1\)One may also find spheromak-type solutions for linear dependence of the poloidal current on the poloidal magnetic flux, but they do not seem to correspond to any physically interesting case.
is $\tan \theta_l = \left(\sqrt{1 - 2M/r} \left(1 + (2M/r) \ln(r - 2M)\right)\right)^{-1}$. It reaches $\theta_l = \pi/2$ somewhere close to the horizon (at the root of $1 + (2M/r) \ln(r - 2M) = 0$) and then again at the horizon $r = 2M$, see Fig. (5). Thus, the force-free magnetic field structures of Schwarzschild black holes allow for a variety of field configurations with different inclinations of magnetic field lines at the disk surface. We may only speculate that Kerr holes have a magnetic field structure with various magnetic field inclinations at the disk surface as well.
5. Conclusion

In this paper we discussed the conditions for the centrifugal launching of jets from a thin accretion disk around spinning black holes. Typically, it is believed that “for acceleration by the centrifugal mechanism to be effective, the field lines have to be inclined outward” (Spruit 2008). This leads to a collimation problem: a flow should first accelerate by magnetocentrifugal forces outwards in cylindrical radius and later, beyond the Alfvén surface, it needs to be collimated along the axial direction by hoop stresses of bent back magnetic field lines (Blandford & Payne 1982). For relativistic motion, such collimation becomes very ineffective (Eichler 1993).

We found that Schwarzschild black holes conform to these expectations: conditions for jet launching (defined as an instability of a particle on a circular orbit confined to move along a magnetic field line) remain the same as for Newtonian disks: a field should be inclined to the disk surface at an angle $\theta_l < 60^\circ$ (when viewed in Boyer-Lindquist coordinates) at any radius in order to launch a jet. On the other hand, in the case of a critically spinning Kerr black hole, the centrifugal acceleration along magnetic field lines anchored in the disk may occur virtually along the rotation axis of the black hole. Qualitatively, this removes the need for a flow to expand sideways to get accelerated.

The present result has implications for the transport of angular momentum by large scale magnetic fields. Since the particle orbits in the inner regions of accretion disks around spinning black holes are unstable to large scale magnetic fields with a wide range of angles at the disk surface, this make the transport angular momentum away from the disk more efficient than in the case of Schwarzschild black holes.

The centrifugal acceleration mechanism (a fling) isn’t the only way to magnetically launch a jet. Alternatively, a jet may be launched by stresses of rotating twisted magnetic field (a spring), (see, e.g. Tchekhovskoy et al. 2008). In a “spring”-type launching the disk or the central black hole act as a Faraday disk, which launches an outflow with the terminal speed equal, approximately, to the Alfvén velocity in the corona. If the coronal plasma is strongly magnetized, the terminal velocity may be relativistic.

It is tempting to relate the possibility of centrifugal acceleration along the rotation axis in critically spinning black holes to the results of numerical simulations of jet launching. It is well established that the efficiency of jet production is a strong function of the black hole spin (Meier et al. 2001); for a rapidly spinning black hole the efficiency of converting the accreting energy into jet power reaches the radiation efficiency of thin disks (De Villiers et al. 2003, 2005). We speculate that this is because jet launching for critically spinning black hole is more efficient due to effect discussed in this paper.
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A. Effects of toroidal component of magnetic field

It is expected that for relativistic flow with rotation rates nearing the speed of light, the toroidal component of the magnetic field is not negligible. In this Appendix we show that inclusion of the toroidal component does not change the stability conditions described above. First note, that in the case of Keplerian (non-relativistic) rotation, the azimuthal component of the wire direction does not change the stability condition, as can be trivially shown (both centrifugal force and the force of gravitational attraction do not have \( \phi \)-component, and thus their projections on any oblique direction become equal for the same parameters, as in the case of no toroidal component of the wire).

In Kerr metric, we can repeat the previous derivation allowing for finite displacement in the azimuthal direction, along a direction given by angle \( \phi_l \) with respect to the radial direction. For a given radial displacement \( dr \) the change in the azimuthal angle is

\[
\frac{d\phi}{dr} = \cot \theta \frac{r_0 dr}{\sin \theta (r - r_0)^2 + r^2 \cot^2 \theta} \tan \phi_l
\]

(A1)

The non-zero components of the metric tensor are \( g_{00}, g_{rr}, g_{rt} \) (not given here explicitly). It is important that in the rotating Kerr coordinates, inclined with respect to radial direction, the \( g_{00} \) component of the metric tensor remains the same, Eq. (2).

Solution of the Hamilton-Jacobi equations are now

\[
S = -E_0 t - \int dr \frac{E_0 g_{rt} \pm \sqrt{-(E_0^2 + g_{00})(g_{00}g_{rr} - g_{rt}^2)}}{g_{00}}
\]

(A2)

Equation of motion is

\[
\left( \frac{dr}{dt} \right)^2 = \left( \frac{g_{rt}}{g_{00}} \pm \frac{E_0}{g_{00}} \sqrt{g_{00}g_{rr} - g_{rt}^2} \right)^{-2}
\]

(A3)
Transformation to proper time

\[
\frac{d\tau}{dt} = -\frac{g_0\sqrt{g_{00}g_{rr} - g_{rt}^2}}{\sqrt{-(E_0^2 + g_{00})g_{rt} - E_0\sqrt{g_{00}g_{rr} - g_{rt}^2}}}
\]

(A4)

gives the effective potential

\[
\left(\frac{dr}{d\tau}\right)^2 = -\frac{E_0^2 + g_{00}}{g_{00}g_{rr} - g_{rt}^2} \equiv V
\]

(A5)

Conditions of circular orbit give \(V = 0\) \((E_0 = -\sqrt{-g_{00}}), V_r' \propto g_{00,r} = 0\). Thus, similar to the case of radial wire, the epicyclic frequency of an oblique wire is proportional to \(g_{00,rr}\). It changes sign at the same vertical angle \(\theta_l\) as in the case \(\phi_l = 0\). Eq. (14).

### B. Pseudo-potentials

General relativistic effects are often modeled using pseudo-Newtonian potentials. For Schwarzschild black holes such a potential was first proposed by \cite{PaczynskyWiita1980}, \(\Phi_{PW} = M/(r - 2M)\). For the Paczyński-Wiita potential and centrifugal potential \((1/2)\omega^2 r^2\), the angular velocity corresponding to circular orbits is \(\omega = \sqrt{M/(r_0(r_0 - 2M)^2)}\) and the critical angle

\[
\tan^2 \theta_{PW} = 3 + \frac{4M}{r - 2M}
\]

(B1)

This value is close to the exact value in Schwarzschild geometry. Even at the last stable orbit \((r = 6M), \tan^2 \theta_{PW} = 4\) it differs from 60° by only 3.43° degrees. Thus, in the case of Schwarzschild black hole the Paczyński-Wiita potential reproduces reasonably well the launching conditions at the disk surface.

In principle, general relativistic effects should also be taken into account in calculation of centrifugal acceleration. Using relation \(\text{[12]}\) for the centrifugal force, the pseudo-centrifugal potential becomes \((1/2)\omega^2(r - 3M)^2\); the angular velocity corresponding to circular orbits is then \(\omega = \sqrt{M/((r_0 - 3M)(r_0 - 2M)^2)}\) and the critical angle

\[
\tan^2 \theta_{PW} = 3 + \frac{4M}{r - 2M} + \frac{3M}{r - 3M}
\]

(B2)

At the last stable orbit (assuming it is at \(r = 6M\)) \(\tan^2 \theta_{PW} = 5\), again reasonably close to the exact value \(\tan^2 \theta = 3\).