THE EXCEEDANCES AND DESCENTS 
OF BI-INCREASING PERMUTATIONS

Astrid Reifegerste 
Institut für Mathematik, Universität Hannover 
Welfengarten 1, D-30167 Hannover, Germany 
reifegerste@math.uni-hannover.de 

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Abstract. Starting from some considerations we make about the relations between certain difference statistics and the classical permutation statistics we study permutations whose inversion number and excedance difference coincide. It turns out that these (so-called bi-increasing) permutations are just the 321-avoiding ones. The paper investigates their excedance and descent structure. In particular, we find some nice combinatorial interpretations for the distribution coefficients of the number of excedances and descents, respectively, and their difference analogues over the bi-increasing permutations in terms of parallelogram polyominoes and 2-Motzkin paths. This yields a connection between restricted permutations, parallelogram polyominoes, and lattice paths that reveals the relations between several well-known bijections given for these objects (e.g. by Delest-Viennot, Billey-Jockusch-Stanley, Françon-Viennot, and Foata-Zeilberger). As an application, we enumerate skew diagrams according to their rank and give a simple combinatorial proof for a result concerning the symmetry of the joint distribution of the number of excedances and inversions, respectively, over the symmetric group.

1 Motivation and preliminaries

Let $S_n$ be the set of all permutations of $[n] := \{1, \ldots, n\}$. We write any permutation $\pi \in S_n$ as word $\pi_1 \cdots \pi_n$ where $\pi_i$ means the integer $\pi(i)$.

In this paper, we will mainly investigate certain permutations in view of the behaviour of their excedances and descents. First we recall the definitions and fix some notations.

For $\pi \in S_n$, an excedance of $\pi$ is an integer $i \in [n - 1]$ such that $\pi_i > i$. Here the element $\pi_i$ is called an excedance letter. The set of excedances of $\pi$ is denoted by $E(\pi)$. By $\pi_e$ and $\pi_{ne}$ we denote the restrictions of $\pi$ on the excedances and non-excedances, respectively.

A descent of $\pi$ is an integer $i \in [n - 1]$ for which $\pi_i > \pi_{i+1}$. If $i$ is a descent we call $\pi_i$ a descent.
top and $\pi_{i+1}$ a descent bottom. The set of descents of $\pi$ is denoted by $D(\pi)$. We write $\pi_d$ to denote the subword consisting of all descent tops of $\pi$ (in order of appearance). The subword formed from the remaining letters we denote by $\pi_{nd}$.

A pair $(\pi_i, \pi_j)$ is called an inversion of $\pi$ if $i < j$ and $\pi_i > \pi_j$. The set containing all inversions of $\pi$ we denote by $I(\pi)$. (Clearly, inversions are defined in the same way for arbitrary words.)

Three of the four classical statistics count the number of occurrences of these patterns in a permutation, the fourth one, the so-called major index, is the sum of descents. We use the usual notation:

$$\text{exc}(\pi) = |E(\pi)|, \quad \text{des}(\pi) = |D(\pi)|, \quad \text{inv}(\pi) = |I(\pi)|, \quad \text{maj}(\pi) = \sum_{i \in D(\pi)} i.$$

We write $S^f_n(k)$ to denote the number of permutations in $S_n$ for which the statistic $f$ takes the value $k$. Analogously, the coefficients of the joint distribution of $f$ and $g$ on $S_n$ are denoted by $S^{(f,g)}_n(k,l)$. When we consider statistics over the set $B_n$ defined below then we write $B$ instead of $S$.

In [5], the authors studied some differences of permutation statistics. Here we will deal with two of these, namely the excedance difference and the descent difference of a permutation $\pi \in S_n$ defined by

$$d\text{exc}(\pi) = \sum_{i \in E(\pi)} (\pi_i - i) \quad \text{and} \quad d\text{des}(\pi) = \sum_{i \in D(\pi)} (\pi_i - \pi_{i+1}),$$

respectively.

For example, the permutation $\pi = 4\ 2\ 8\ 3\ 6\ 9\ 7\ 5\ 1\ 10 \in S_{10}$ has the excedances $1, 3, 5, 6$ and the descents $1, 3, 6, 7, 8$. Its excedance letter word and non-excedance letter word, respectively, are $\pi_e = 4\ 8\ 6\ 9$ and $\pi_{ne} = 2\ 3\ 7\ 5\ 1\ 10$. For the descent top word we obtain $\pi_d = 4\ 8\ 9\ 7\ 5$. The excedance difference equals

$$d\text{exc}(\pi) = (4 - 1) + (8 - 3) + (6 - 5) + (9 - 6) = 12,$$

the descent difference is equal to

$$d\text{des}(\pi) = (4 - 2) + (8 - 3) + (9 - 7) + (7 - 5) + (5 - 1) = 15.$$

This section is to show that the difference statistics are closely connected with the classical ones.

Foata showed, combinatorially, that the excedance number is equidistributed with the descent number over $S_n$. That is to say, $S^\text{exc}_n(k) = S^\text{des}_n(k)$, for all $n$ and $k$. The same result holds for the difference statistics. Foata’s bijection [11, Th. 10.2.3] also proves this fact, and hence even the equidistribution of the pairs $(\text{exc, dexc})$ and $(\text{des, ddes})$. For proof we review Foata’s proof.
Proposition 1.1 Both statistics dexc and ddes and bistatistics (exc, dexc) and (des, ddes) are equidistributed over $S_n$.

Proof. Given a permutation $\pi \in S_n$, decompose $\pi$ in distinct cycles

$$\pi = (c_{11}c_{12}\cdots c_{1l_1})(c_{21}c_{22}\cdots c_{2l_2})\cdots(c_{s1}c_{s2}\cdots c_{sl_s})$$

such that (i) each cycle is written with its largest element first, and (ii) the cycles are written in increasing order of their first element. Fixed points are regarded as singleton cycle. Define the map

$$\varphi : S_n \to S_n, \pi \mapsto c_{11}c_{1l_1}c_{1l_1-1}\cdots c_{12}c_{2l_2}c_{2l_2-1}\cdots c_{s1}c_{sl_s}c_{sl_s-1}\cdots c_{s2}.$$ 

By Foata, $\varphi$ is bijectiv, and we have $\text{exc}(\pi) = \text{des}(\phi(\pi))$.

But we have $\text{dexc}(\pi) = \text{ddes}(\phi(\pi))$ as well. To see this, first let $i$ be an excedance of $\pi$. Clearly, $i$ and $\pi_i$ belong to the same cycle $c$, and appear in $\phi(\pi)$ as consecutive letters, beginning with $\pi_i$. (If $\pi_i$ is the largest element of $c$ then $c = (\pi_i \cdots i)$, otherwise $c = (\pi_k \cdots i \pi_i \cdots)$ for some $k \neq i$.) If $i$ is a fixed point of $\pi$ condition (ii) causes that the letter following $i$ in $\phi(\pi)$ is greater than $i$. Finally, let $i$ be an integer satisfying $i > \pi_i$. Then the cycle $c$ containing $i$ and $\pi_i$ is written as either $c = (k \cdots i \pi_i \cdots)$ for some $k \neq i$, and $\pi_i$ occurs in $\phi(\pi)$, or $c = (i \pi_i \pi_i^2 \cdots \pi_i^{l-1})$, and $i \pi_i^{l-1} \pi_i^{l-2} \cdots \pi_i$ occurs in $\phi(\pi)$ where $\pi_i^k$ denotes $\pi(\pi_i^{k-1})$ for $k \geq 2$. (Note that $i$ is the largest cycle element in this case.) Consequently, the positive differences $\pi_i - i$ correspond exactly to the positive differences between consecutive letters in $\phi(\pi)$. \qed

Because of the definition, the relation between the difference statistics and their classical counterparts comes as no surprise; more amazing is the fact that dexc is closely connected with the number of inversions, as well. This result already appeared in [3, Th. 2].

We give another but just as simple proof. Its key object is the permutation $\hat{\pi} \in S_n$ that is obtained from $\pi$ by sorting the letters of $\pi_e$ and $\pi_{ne}$ in increasing order, respectively. (That is to mean, $\hat{\pi}_{i_k} = \pi_{\sigma(i_k)}$ and $\hat{\pi}_{j_k} = \pi_{\tau(j_k)}$ where $i_1, \ldots, i_e$ are the excedances, and $j_1, \ldots, j_{n-e}$ the remaining elements of $\pi$, and $\sigma$ and $\tau$ are permutations of $E(\pi)$ and the set of non-excedances, respectively, such that $\hat{\pi}_{i_1} < \ldots < \hat{\pi}_{i_e}$ and $\hat{\pi}_{j_1} < \ldots < \hat{\pi}_{j_{n-e}}$.) For example, let $\pi = 4 2 3 6 9 7 5 1 0 \in S_{10}$ as before, with $\pi_e = 4 8 6 9$ and $\pi_{ne} = 2 3 7 5 1 10$. (The letters of the subword $\pi_e$ are underlined.) Then we obtain the bi-sorted permutation $\hat{\pi} = 4 1 6 2 8 9 3 5 7 10$.

Theorem 1.2 For any permutation $\pi \in S_n$ we have $\text{inv}(\pi) = \text{dexc}(\pi) + \text{inv}(\pi_e) + \text{inv}(\pi_{ne})$. 

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Proof. First we show that $\text{exc}(\hat{\pi}) = \text{exc}(\pi)$ and $\text{dexc}(\hat{\pi}) = \text{dexc}(\pi)$.

Any excedance of $\pi$ is an excedance in $\hat{\pi}$ as well: let $i, j \in E(\pi)$ such that $i < j$ and $\pi_i > \pi_j$. Obviously, $i < j < \pi_j$ and $\pi_j < \pi_i$. Analogously, any non-excedance of $\pi$ is a non-excedance in $\hat{\pi}$. Thus the permutations $\pi$ and $\hat{\pi}$ have the same excedance set, and we obtain

$$\text{dexc}(\pi) = \sum_{i \in E(\pi)} (\pi_i - i) = \sum_{i \in E(\pi)} (\pi_{\sigma(i)} - i) = \sum_{i \in E(\hat{\pi})} (\hat{\pi}_i - i) = \text{dexc}(\hat{\pi}).$$

Now we determine the inversion number of $\hat{\pi}$. To this end, we utilize the reverse code $c(\pi) = (c_1, \ldots, c_n)$ of $\pi$ whose $i$th component is defined to be the number of integers $j \in [n]$ satisfying $j < i$ and $\pi_j > \pi_i$. (Clearly, $c_1 + \ldots + c_n = \text{inv}(\pi)$.)

Let $i_1, \ldots, i_E$ be the excedances of $\pi$ where $\pi_{i_1} < \pi_{i_2} < \ldots < \pi_{i_k} > \pi_{i_{k+1}}$. Then $c_{i_k} = 0$ and $c_{i_{k+1}} \geq 1$. For the reverse code $(c'_1, \ldots, c'_n)$ of the permutation obtained by exchanging $\pi_{i_k}$ and $\pi_{i_{k+1}}$ we have

\begin{align*}
    c'_{i_k} &= c_{i_{k+1}} - 1, \\
    c'_{i_{k+1}} &= 0, \\
    c'_j &= c_j \text{ for all } j \in [n] \setminus \{i_k, i_{k+1}\}.
\end{align*}

The last relation is evident for $j = 1, \ldots, i_k - 1, i_{k+1} + 1, \ldots, n$. By definition, $i_k + 1, \ldots, i_{k+1} - 1$ are non-excedances each of which satisfies $\pi_j \leq j < i_{k+1} < \pi_{i_{k+1}} < \pi_{i_k}$. Thus the inversion $(\pi_{i_k}, \pi_j)$ appearing in $\pi$ corresponds to $(\pi_{i_{k+1}}, \pi_j)$ in $\hat{\pi}$. Consequently, exchanging the excedance letters $\pi_{i_k}$ and $\pi_{i_{k+1}}$ decreases the number of inversions by 1. (Sorting $\pi_e$ completely requires $\text{inv}(\pi_e)$ transpositions.) By reasoning similar for non-excedances, we obtain

$$\text{inv}(\hat{\pi}) = \text{inv}(\pi) - \text{inv}(\pi_e) - \text{inv}(\pi_{ne}).$$

By construction, the words $\hat{\pi}_e$ and $\hat{\pi}_{ne}$ do not contain any inversion. Hence the component $\hat{c}_i$ of $\hat{\pi}$'s reverse code equals zero if $i$ is an excedance of $\hat{\pi}$ (or equivalently, of $\pi$), and is equal to the difference $i - \hat{\pi}_i$ otherwise. Thus

$$\text{inv}(\hat{\pi}) = \sum_{i=1}^n \hat{c}_i = \sum_{i \in E(\hat{\pi})} (i - \hat{\pi}_i) = \sum_{i \in E(\hat{\pi})} (\hat{\pi}_i - i) = \text{dexc}(\hat{\pi}),$$

and the assertion is proved. $\square$

Define two permutations $\pi, \sigma \in S_n$ as being equivalent if $\hat{\pi} = \hat{\sigma}$. The fundamental property that both exc and dexc are constant on the equivalence classes we will use in Section 6 to derive
results concerning the distribution of these statistics over \( S_n \) from analogous results proved for the set of class representatives.

The theorem yields an estimate for the inversion number depending on the statistics \( \text{exc} \) and \( \text{dexc} \). Its proof needs the first one of the following auxiliary results.

**Lemma 1.3**

a) Let \( \pi \in S_n \) satisfy \( \text{inv}(\pi) = \text{dexc}(\pi) \), and let \( k \) be a non-excedance of \( \pi \). Then

\[
\{|j \notin E(\pi) : \pi_k \leq j < k\} = \{|i \in E(\pi) : i < \pi_k < \pi_i\}.
\]

b) Let \( \pi \in S_n \) be an arbitrary permutation, and let \( k \) be a non-excedance of \( \pi \). Then

\[
\{|j \notin E(\pi) : \pi_j \leq k < j\} = \{|i \in E(\pi) : i < k < \pi_i\}.
\]

**Proof.** a) By the theorem, both \( \pi_e \) and \( \pi_{ne} \) are increasing words. Thus the letters 1, 2, \ldots, \( \pi_k - 1 \) appear to the left of position \( k \). Consequently, \( k - \pi_k \) integers \( i < k \) satisfy \( \pi_i > \pi_k \) (here \( i \) is necessarily an excedance of \( \pi \)). Let \( s \) be the number of these integers for which \( i < \pi_k \) in addition. Hence among the integers \( \pi_k, \pi_k + 1, \ldots, k - 1 \) there are \( k - \pi_k - s \) excedances, and therefore, \( s \) non-excedances.

b) Evidently, \( k - 1 \) letters \( \pi_j \) different from \( \pi_k \) satisfy \( \pi_j \leq k \). Let \( t \) be the number of all these positions \( j \) being to the right of \( k \). Clearly, such a integer \( j \) is a non-excedance of \( \pi \). In \( [k - 1] \) there are \( k - 1 - t \) integers \( i \) for which \( \pi_i \leq k \), and hence \( t \) ones with \( \pi_i > k \). Obviously, the latter ones are necessarily excedances.

**Corollary 1.4** For any permutation \( \pi \in S_n \) we have \( \text{dexc}(\pi) \leq \text{inv}(\pi) \leq 2\text{dexc}(\pi) - \text{exc}(\pi) \).

**Proof.** The lower bound follows immediately from Theorem 1.2. To prove the upper one we first construct a permutation sequence \( \pi^{(0)}, \pi^{(1)}, \ldots \) satisfying \( \text{dexc}(\pi^{(k)}) = k \) and \( \text{inv}(\pi^{(k)}) = 2k - \text{exc}(\pi^{(k)}) \). Initialize \( \pi^{(0)} = 12 \cdots n \). Given \( \pi^{(k-1)} \), we define \( \pi^{(k)} \) as follows:

1. Let \( e \) be the smallest integer such that \( e + \pi^{(k-1)}_e < n + 1 \). If there is no such an \( e \) stop the procedure. Set \( \pi^{(k)}_e := \pi^{(k-1)}_e \).

2. Set \( \pi^{(k)}_e := a + 1, \pi^{(k)}_a := a \) if \( a \neq e \), \( \pi^{(k)}_{a+1} := e \), and \( \pi^{(k)}_i := \pi^{(k-1)}_i \) otherwise.

(For an example, see the following remark.) Note that \( \text{exc}(\pi^{(k)}) = e \); more exactly, we have \( E(\pi^{(k)}) = [e] \). By the construction, \( \text{dexc} \) increases with \( k \). The inversion number is increased by 2 if the integer \( e \) does not change with regard to the previous step. Otherwise we have \( \text{inv}(\pi^{(k)}) = \text{inv}(\pi^{(k-1)}) + 1 \). Hence \( \text{inv}(\pi^{(k)}) = 2k - e \).

On the other hand, this is also the maximum inversion number for a permutation \( \pi \in S_n \).
with \( e \) excedances and excedance difference \( k \). Why? By Theorem 1.2, it suffices to show \( \text{inv}(\pi_e) + \text{inv}(\pi_{ne}) \leq k - e \). As shown in the proof of 1.2, this assertion is equivalent to the following one. Let \( \sigma \in S_n \) be a permutation for which \( \text{exc}(\pi) = e \) and \( \text{inv}(\pi) = \text{dexc}(\pi) = k \). Then there are at most \( k - e \) inversions on the excedance letters and non-excedance letters of \( \sigma \) which preserve the set of excedances.

Let \( k < i \) be excedances of \( \sigma \). (Note that \( \sigma_k < \sigma_i \).) Then the exchange of \( \sigma_k \) and \( \sigma_i \) keeps all excedances if and only if \( i < \sigma_k \). Therefore, for any \( \pi \in S_n \) satisfying \( \hat{\pi} = \sigma \) we have

\[
\text{inv}(\pi_e) \leq \sum_{k \in \text{E}(\sigma)} |\{i \in \text{E}(\sigma) : i < \sigma_k < \sigma_i\}|.
\]

Now let \( j < k \) be non-excedances of \( \sigma \). (We have \( \sigma_j < \sigma_k \) again.) When exchanging \( \sigma_j \) and \( \sigma_k \) the integers \( j \) and \( k \) are still non-excedances if and only if \( j \geq \sigma_k \). By Lemma 1.3a, the number of non-excedances \( j \) satisfying \( \sigma_k \leq j < k \) equals the number of excedances \( i \) for which \( i < \sigma_k < \sigma_i \). Consequently, we have

\[
\text{inv}(\pi_{ne}) \leq \sum_{k \notin \text{E}(\sigma)} |\{i \in \text{E}(\sigma) : i < \sigma_k < \sigma_i\}|,
\]

for each \( \pi \in S_n \) with \( \hat{\pi} = \sigma \). Both together yield

\[
\text{inv}(\pi_e) + \text{inv}(\pi_{ne}) \leq \sum_k |\{i \in \text{E}(\sigma) : i < \sigma_k < \sigma_i\}| = \sum_{i \in \text{E}(\sigma)} (\sigma_i - i - 1) = \text{dexc}(\sigma) - \text{exc}(\sigma)
\]

which is the desired bound. \( \square \)

**Corollary 1.5** The conditions \( \text{exc}(\pi) = \text{inv}(\pi) \) and \( \text{exc}(\pi) = \text{dexc}(\pi) \) are equivalent in \( S_n \).

**Remarks 1.6**

a) For \( n = 6 \), the procedure given in the proof of 1.3 yields the sequence:

| \( k \) | \( \pi^{(k)} \) | \( \text{exc}(\pi^{(k)}) \) | \( \text{dexc}(\pi^{(k)}) \) | \( \text{inv}(\pi^{(k)}) \) |
|---|---|---|---|---|
| 0 | 1 2 3 4 5 6 | 0 | 0 | 0 |
| 1 | 2 1 3 4 5 6 | 1 | 1 | 1 |
| 2 | 3 2 1 4 5 6 | 1 | 2 | 3 |
| 3 | 4 2 3 1 5 6 | 1 | 3 | 5 |
| 4 | 5 2 3 4 1 6 | 1 | 4 | 7 |
| 5 | 6 2 3 4 5 1 | 1 | 5 | 9 |
| 6 | 6 3 2 4 5 1 | 2 | 6 | 10 |
| 7 | 6 4 3 2 5 1 | 2 | 7 | 12 |
| 8 | 6 5 3 4 2 1 | 2 | 8 | 14 |
| 9 | 6 5 4 3 2 1 | 3 | 9 | 15 |
It is easy to see that $\text{dexc}(\pi) \leq \frac{1}{4}n^2$ for each permutation $\pi \in S_n$. How the construction shows, for every integer $0 \leq k \leq \frac{1}{4}n^2$ there is a permutation $\pi \in S_n$ having excedance difference $k$.

b) Let $I = \{i_1, \ldots, i_e\}$ and $A = \{a_1, \ldots, a_e\}$ be ordered integer sets. The proof gives implicit a simple construction for a permutation in $S_n$ whose excedance set is $I$, whose excedance letter set is $A$, and whose inversion number is maximal. Let $A_{i_k}$ be the set consisting of all $a \in A$ satisfying $a > i_k$ where $k = 1, \ldots, e$. Define $\pi_{i_k}$ to be the smallest element of $A_{i_k}$, and delete it from $A_{i_1}, \ldots, A_{i_{e-1}}$. Then define $\pi_{i_{e-1}}$ to be the smallest element of $A_{i_{e-1}}$, and delete it from $A_{i_1}, \ldots, A_{i_{e-2}}$, and so on. Proceed analogously with the non-excedances and their letters. Let $B_{j_k}$ be consisted of the elements $b \in B$ satisfying $b \leq j_k$ where $B$ is the $[n]$-complement of $A$, and $j_1 < \ldots < j_{n-e}$ are the elements of $[n]$, different from $i_1, \ldots, i_e$. Define $\pi_{j_1}$ to be the largest element of $B_{j_1}$, and delete it from $B_{j_2}, \ldots, B_{j_{n-e}}$. Then define $\pi_{j_2}$ to be the largest element of $B_{j_2}$, and delete it from $B_{j_3}, \ldots, B_{j_{n-e}}$, and so on.

For instance, given $I = \{1, 3, 5, 6\}$, $A = \{4, 6, 8, 9\}$, and $n = 10$ we obtain

\[
\begin{align*}
A_6 &= \{8, 9\}, \quad A_5 = \{6, 8, 9\}, \quad A_3 = \{4, 6, 8, 9\}, \quad A_1 = \{A, 6, 8, 9\}; \\
B_2 &= \{1, 2\}, \quad B_4 = \{1, 2, 3\}, \quad B_7 = \{1, 2, 5, 7\}, \quad B_8 = \{1, 2, 3, 5, 7\}, \\
B_9 &= \{1, 2, 3, 5, 7\}, \quad B_{10} = \{A, 2, 3, 5, 7, 10\};
\end{align*}
\]

and hence $\pi = 3 2 4 6 5 1 0 \pi_{10} \in S_{10}$ having 20 inversions.

Clearly (by the proof of Theorem 1.2) $\hat{\pi} = 4 6 2 8 3 5 7 10$ has as few inversions as possible for a permutation in $S_{10}$ with excedance set $I$ and excedance letter set $A$, namely 12.

c) We have both $\text{inv}(\pi^{-1}) = \text{inv}(\pi)$ and $\text{dexc}(\pi^{-1}) = \text{dexc}(\pi)$ for all $\pi \in S_n$ where $\pi^{-1}$ denotes the inverse permutation. Furthermore, $\text{exc}(\pi^{-1}) = n - \text{exc}(\pi) - \text{fix}(\pi)$ for each $\pi \in S_n$ where fix counts the number of fixed points in a permutation. Thus we can strengthen the upper bound in [1.4] by replacing $\text{exc}(\pi)$ with $\max\{\text{exc}(\pi), n - \text{exc}(\pi) - \text{fix}(\pi)\}$.

For the investigations we have done until now, a certain kind of permutations plays an important role: those for which the inversion number and the excedance difference coincide. The following sections will deal with these, so-called bi-increasing, permutations in view of their excedances and descents. Here the idea is to construct one-to-one correspondences between bi-increasing permutations and other well-known combinatorial objects which encode the statistics as natural
parameters.
In Section 2, first several simple equivalent definitions of bi-increasing permutations are given. It will turn out that these permutations are very interesting for everyone who studies forbidden patterns in permutations; they are just the 321-avoiding ones. Moreover, their fixed points will be characterized.

Section 3 pursues the aim to determine the distribution of $\text{exc}$ and $\text{dexc}$ on the set of bi-increasing permutations. Recall that these permutations represent classes on which both statistics are constant. To this end, we establish first a one-to-one correspondence between bi-increasing permutations and step parallelogram polyominoes that transfers excedance number and excedance difference into width and area of the corresponding polyomino. By a simple transformation, the bijection can be given in terms of general parallelogram polyominoes as well. It proves that this correspondence connects the bijection between bi-increasing permutations and Dyck paths due to Billey, Jockusch, and Stanley with that one between parallelogram polyominoes and Dyck paths given by Delest and Viennot.

In Section 4, we develop an analogous model basing on 2-Motzkin paths to study the descent structure of bi-increasing permutations. First we will confine ourselves to such bi-increasing permutations whose exceedances coincide with the descents. Here a bijection due to Françon and Viennot yields the desired correspondence where $\text{des}$ and $\text{ddes}$ are translated into the number of up-steps and the sum of height, respectively. By a little refinement we can extend this correspondence to all bi-increasing permutations. Moreover, the relation with another bijection between permutations and 2-Motzkin paths given by Foata and Zeilberger will be revealed.

Section 5 starts with the proof of the equidistribution of the difference statistics over the set of bi-increasing permutations. From this we obtain a one-to-one correspondence between parallelogram polyominoes and 2-Motzkin paths which transfers all the natural parameters to each other. As an application, we enumerate the skew diagrams (or, parallelogram polyominoes) according to their rank.

In the last section, we deduce a result concerning the symmetry of the joint distribution of $\text{exc}$ and $\text{inv}$ over $\mathcal{S}_n$ from an analogous result for bi-increasing permutations.

2 Characterization of bi-increasing permutations

Theorem 1.2 characterizes the permutations for which inversion number and excedance difference are equally as those whose restrictions on exceedances and non-exceedances, respectively, are
increasing words. In view of this, we call the elements of
\[ B_n := \{ \pi \in S_n : \text{inv}(\pi) = \text{dexc}(\pi) \} \]
the \textit{bi-increasing permutations} of length \( n \). (This term was already used in [12]). In particular, any bi-increasing permutation is uniquely determined by its excedances and excedance letters.

In the following we give several simple equivalent definitions of \( B_n \).

**Proposition 2.1** A permutation \( \pi \in S_n \) is bi-increasing if and only if
\[
\pi_k - k = |\{ i > k : \pi_i < \pi_k \}| \quad \text{for } k \in E(\pi),
\]
\[
k - \pi_k = |\{ i < k : \pi_i > \pi_k \}| \quad \text{for } k \notin E(\pi).
\]

**Proof.** For any \( k \in [n] \), the number of integers \( i \in [n] \) with \( \pi_i < \pi_k \) equals
\[
k - 1 - |\{ i : i < k, \pi_i > \pi_k \}| + |\{ i : i > k, \pi_i < \pi_k \}|.
\]
Hence \( \pi_k = k + |\{ i : i > k, \pi_i < \pi_k \}| - |\{ i : i < k, \pi_i > \pi_k \}| \). If \( k \) is an excedance then each integer \( i < k \) with \( \pi_i > \pi_k \) is excedance as well. Analogously, every integer \( i > k \) with \( \pi_i < \pi_k \) is a non-excedance if \( k \) is such a one. In case of ordered words \( \pi \text{e} \) and \( \pi \text{ne} \), the sets \( \{ i : i < k, \pi_i > \pi_k \} \) for \( k \in E(\pi) \) and \( \{ i : i > k, \pi_i < \pi_k \} \) for \( k \notin E(\pi) \), respectively, are empty sets. \( \square \)

For \( i = 1, \ldots, n - 1 \), let \( s_i \) denote the adjacent transposition \((i, i + 1) \in S_n \). It is well-known that every permutation \( \pi \in S_n \) can be written as \( \pi = s_{i_1} s_{i_2} \cdots s_{i_k} \) where \( k = \text{inv}(\pi) \). The factorization of bi-increasing permutations is striking.

**Proposition 2.2** A permutation \( \pi \in S_n \) is bi-increasing if and only if
\[
\pi = (s_{i_1+j_1-1}s_{i_1+j_1-2}\cdots s_{i_1})(s_{i_2+j_2-1}s_{i_2+j_2-2}\cdots s_{i_2})\cdots(s_{i_e+j_e-1}s_{i_e+j_e-2}\cdots s_{i_e})
\]
for some positive integers \( i_1, \ldots, i_e \) and \( j_1, \ldots, j_e \) satisfying \( 1 \leq i_1 < i_2 < \ldots < i_e < n \) and \( 1 < i_1 + j_1 < i_2 + j_2 < \ldots < i_e + j_e \leq n \). In particular, \( i_1, \ldots, i_e \) are precisely the excedances.

**Proof.** We consider the decomposition of \( \pi \in B_n \) obtained by the following procedure:

1. Set \( k := 1 \) and \( \tau := \pi \).
2. For \( i = 1, \ldots, n - 1 \), if \( \tau_i > i \) then define \( a_k := \tau_i - 1 \) and \( \tau := s_{a_k} \tau \). Increase \( k \) by 1.
3. Write \( \pi = s_{a_1} s_{a_2} \cdots s_{a_r} \) where \( r \) denotes the value of \( k \) before stopping (2).
Let $i_1 < \ldots < i_e$ be the excedances of $\pi$. The first step yields $a_1 = \pi_{i_1} - 1$. After the run for $i = i_1$, we obtain $a = (\pi_{i_1} - 1, \pi_{i_1} - 2, \ldots, i_1)$ and $\tau = s_{i_1} \cdots s_{\pi_{i_1} - 1} \pi$ where $1, 2, \ldots, i_1$ are fixed points of $\tau$. Since $\pi_e$ is increasing applying $s_{i_1} \cdots s_{\pi_{i_1} - 1}$ to $\pi$ do not change the letters on the positions $i_2, \ldots, i_e$. Hence $\tau$ differs from $\pi$ in $\pi_{i_1} - i_1 + 1$ positions $j$, all different from $i_2, \ldots, i_e$. (We have $\tau_j = \pi_j + 1$, except for $j = i_1$.) Note that $j$ is an non-excedance of $\tau$ as well. Assuming the contrary, $j > i_1$ has to be a fixed point of $\pi$. By Proposition 2.1, then there appears no letter greater than $j$ at a position to the left of $j$. Hence $j + 1$ is not an excedance letter, and $\tau$ does not belong to the positions changed by the procedure. Consequently, $E(\tau) = E(\pi) \setminus \{i_1\}$ and $\text{dexc}(\tau) = \text{dexc}(\pi) - (\pi_{i_1} - i_1)$. Applying the procedure successively to $\pi$ yields the desired factorization.

By reasoning similar, one shows that the inversion number of a permutation $\pi \in S_n$ corresponding to the above product equals $\text{dexc}(\pi)$. □

The following characterization deals with forbidden patterns and was already given in [19, Lem. 5.6]. We say that a permutation $\pi \in S_n$ avoids the pattern $321$ if there are no integers $i < j < k$ such that $\pi_i > \pi_j > \pi_k$, i.e., every decreasing subsequence in $\pi$ is of length at most two. (It is usual to write $S_n(321)$ to denote the $321$-avoiding permutations of length $n$.)

**Proposition 2.3** A permutation $\pi \in S_n$ is bi-increasing if and only if it avoids the pattern $321$.

*Proof.* The “if” direction is evident. For the converse, first let $i < j$ be excedances. Obviously, there is an integer $k > j$ with $\pi_k \leq j$. Thus we have $\pi_i < \pi_j$ since $\pi_i \pi_j \pi_k$ is decreasing otherwise. Clearly, the inverse $\pi^{-1}$ avoids $321$ if and only if $\pi$ does it. If $i < j$ are non-excedances but no fixed points then we obtain $\pi_i < \pi_j$ applying above argument to $\pi^{-1}$. Let now $i = \pi_j$ and $j > i$ be a non-excedance but not fixed. Assume that $i > \pi_j$. Clearly, then there exists an integer $k < i$ with $\pi_k > i$, that means, a decreasing subsequence $\pi_k i \pi_j$. (In the reversed case, $i$ is non-excedance with $\pi_i < i$ and $j > i$ is fixed, we have $\pi_i < \pi_j$ anyway.) □

**Example 2.4** The permutation $2 6 1 3 7 4 5 8 10 9 \in S_{10}$ will appear as example throughout the following sections. It is bi-increasing since $\text{inv}(\pi) = \text{dexc}(\pi) = 8$, or equivalently, both $\pi_e = 2 6 7 10$ and $\pi_{ne} = 1 3 4 5 8 9$ are increasing words, or equivalently, $\pi = s_1 s_5 s_4 s_3 s_2 s_6 s_5 s_9$, or equivalently, the maximum length of a decreasing subsequence of $\pi$ equals two.

Making use of these descriptions, we obtain a simple criterion for being a fixed point in a bi-increasing permutation.

**Corollary 2.5** An integer $i$ is a fixed point of $\pi \in B_n$ if and only if $\pi_j < \pi_i$ for all $j < i$ and $\pi_j > \pi_i$ for all $j > i$. 
Proof. Let \( i \) satisfy \( \pi_i = i \) where \( \pi \in B_n \). By Proposition 2.1, there is no integer \( j < i \) with \( \pi_j > \pi_i \). An integer \( j > i \) for which \( \pi_j < \pi_i \) is a non-exceedance which contradicts the fact that \( \pi_{\text{ne}} \) is increasing. The converse immediately follows from Proposition 2.1.

In [18, Th. 7.5], the authors enumerated bi-increasing permutations according to the number of their fixed points. There are

\[
\sum_{i=0}^{n-k} (-1)^i \left( \frac{k + 1 + i}{n + 1} \right) \binom{2n - k - i}{n} \binom{k + i}{k}
\]

permutations in \( B_n \) having exactly \( k \) fixed points. In particular (see [18, Cor. 3.3]), the number of bi-increasing derangements of length \( n \) is just the \( n \)th Fine number \( F_n \) that may be defined by the formula \( F_{n-1} + 2F_n = C_n \). Here \( C_n \) denotes the \( n \)th Catalan number defined by \( \frac{1}{n+1} \binom{2n}{n} \).

**Corollary 2.6** The number of bi-increasing permutations of length \( n \) whose fixed points are exactly \( i_1 < i_2 < \ldots < i_s \) equals \( F_{i_1-1}F_{i_2-i_1-1}F_{i_3-i_2-1} \cdots F_{i_s-i_{s-1}-1}F_{n-i_s} \).

Proof. By the previous corollary, any permutation \( \pi \in B_n \) having fixed points \( i_1, \ldots, i_s \) can be represented as \( \pi = \sigma_1i_1\sigma_2i_2 \cdots \sigma_s i_s \sigma_{s+1} \) where \( \sigma_k \) is a bi-increasing derangement of length \( i_k - i_{k-1} - 1 \) (set \( i_0 := 0 \) and \( i_{s+1} := n + 1 \)).

## 3 Bi-increasing permutations and parallelogram polyominoes

The characterization 2.2 inspires the following graphical representation of a bi-increasing permutation: for every excedance \( i \) of \( \pi \in B_n \) draw a horizontal line of length \( \pi_i - i \) beginning at \( i \) such that the lines are arranged one below the other according to the appearance of \( i \) in \( \pi \).

![Graphical representation of 2 6 1 3 7 4 5 8 10 9 ∈ B_{10}](image)

Therefore the distribution of the excedance number and excedance difference on \( B_n \) answers the elementary question: in how many ways can a given number (corresponding to \( \text{exc} \)) of lines of a prescribed total length (corresponding to \( \text{dexc} \)) be arranged inside a stripe of width \( n - 1 \) such that every line begins strictly to the right of the beginning of the previous line and ends strictly to the right of the end of the previous line?

This representation immediately yields a connection between bi-increasing permutations and
another class of well-known combinatorial objects, the parallelogram polyominoes. A parallelogram polyomino is a finite connected union of cells in the plane $\mathbb{N}^2_0$ that can be described as region bounded by two non-intersecting lattice paths starting at the origin, using only steps $[0,1]$ and $[1,0]$, and ending in a common point. The most often studied parameters of parallelogram polyominoes are the perimeter which is the border length (twice path length), the width and height which are the coordinates of the point at that the paths end, and the area which is the number of cells.

For example, Figure 2 shows a parallelogram polyomino of perimeter 22, width 5, height 6, and area 13:

![Figure 2](image1.png)

Figure 2  A parallelogram polyomino

First, we will consider a special kind of parallelogram polyominoes. We call a parallelogram polyomino a step polyomino if each horizontal border segment is of length 1.

**Theorem 3.1** There is a bijection between bi-increasing permutations of length $n$ having $e$ excedances and excedance difference $k$ and step polyominoes of width $e + 1$, height $n + 1$, and area $k + n + 1$.

**Proof.** Let $\pi \in B_n$ be represented by a line arrangement as described above. Then the $i$th line (counted from left) is defined to correspond to the rows in common to the columns $i$ and $i + 1$ of the step polyomino. Let the first column begin at level 0, and let the last column end at level $n + 1$. It is easy to see that the cell number of the resulting polyomino is just the total length of all lines plus $n + 1$.

**Example 3.2** Let $\pi = 26137458109 \in B_{10}$ as before. The corresponding step polyomino is:

![Figure 3](image2.png)

Figure 3  Step polyomino associated with 26137458109
For a more formal description, we code a step polyomino by two integer sequences that contain the lengths of the vertical border segments. Let $\alpha = (l_1 - 1, l_2, l_3, \ldots, l_w)$ and $\beta = (k_1, k_2, \ldots, k_{w-1}, k_w - 1)$ where $l_i$ and $k_i$ denote the lengths of the ith vertical border segment of the upper and lower path, respectively. We will identify a step polyomino with the pair $(\alpha, \beta)$ and vice versa.

For instance, Figure 3 shows the step polyomino $((1, 4, 1, 3, 1), (1, 1, 3, 4, 1))$.

Given a step polyomino of width $w$ and height $n + 1$, obviously, $\alpha$ and $\beta$ are compositions of $n$ into $w$ positive parts. On the other hand, it is easy to say when a pair of compositions describes a step polyomino.

**Proposition 3.3** Let $C_{n,w}$ be the set of compositions of $n$ into $w$ positive parts. Any pair $(\alpha, \beta) \in C_{n,w}^2$ corresponds to a step polyomino of width $w$ and height $n + 1$ if and only if $\alpha \geq \beta$ (dominance order), i.e., $\alpha_1 + \ldots + \alpha_i \geq \beta_1 + \ldots + \beta_i$ for all $i \geq 1$.

**Proof.** If the dominance condition fails for some integer $i$ then the border paths intersect at the point $(i, \alpha_1 + \ldots + \alpha_i + 1)$. $\blacksquare$

To determine the distribution of $\text{exc}$ on bi-increasing permutations first we give a bijection between step polyominoes and objects whose numbers is well-known.

**Theorem 3.4** There is a bijection between step polyominoes of width $w$ and height $n + 1$ and Young diagrams which fit into the shape $(n - 1, n - 2, \ldots, 1)$ and have $w - 1$ corners.

**Proof.** Let $(\alpha, \beta) \in C_{n,w}^2$. Define $\lambda = (n - \alpha_1, n - \alpha_1 - \alpha_2, \ldots, n - \alpha_1 - \ldots - \alpha_{w-1})$ and $\mu = (\beta_1 + \ldots + \beta_{w-1}, \ldots, \beta_1 + \beta_2, 1)$. Hence $\lambda$ and $\mu$ are partitions into distinct parts, all at most $n - 1$. (The Young diagrams of $\lambda$ and $\mu$ appear when we consider the polyomino as being contained in an $w \times (n + 1)$-rectangle.)

There is exactly one partition $\Lambda$ such that

(i) the distinct parts of $\Lambda$ are just the parts of $\lambda$, and

(ii) the distinct parts of the conjugate to $\Lambda$ are just the parts of $\mu$.

Given $\lambda$ and $\mu$, we construct $\Lambda$ as follows. Beginning with the diagram of $\lambda$, add $\mu_1 - (w - 1)$ squares to each of the first $\lambda_{w-1}$ columns, then add $\mu_2 - (w - 2)$ squares to each of the next $\lambda_{w-2} - \lambda_{w-1}$ columns, then add $\mu_3 - (w - 3)$ squares to each of the next $\lambda_{w-3} - \lambda_{w-2}$ columns, and so on. Since $\mu_{w-i} \geq i$ this procedure is always possible. By the construction, the squares $(\mu_{w-i}, \lambda_i)$ where $i = 1, \ldots, w - 1$ are just the corners of the Young diagram of $\Lambda$. (Here the first coordinate denotes the rows, and the second one denotes the columns of the diagram, beginning at the left-hand upper corner.) Thus the partition $\Lambda$ is uniquely determined, and it satisfies (i)
and (ii).

In addition, Proposition 3.3 yields 

$$\mu_{w-i} + \lambda_i = \beta_1 + \ldots + \beta_i + n - \alpha_1 - \ldots - \alpha_i \leq n \text{ for each } i,$$

that is, $\Lambda$’s diagram fits into $(n-1, n-2, \ldots, 1)$.

\[\square\]

**Example 3.5** For the step polyomino $((1, 4, 1, 3, 1), (1, 1, 3, 4, 1))$ we obtain the partitions $\lambda = (9, 5, 4, 1)$ and $\mu = (9, 5, 2, 1)$, and hence $\Lambda = 9 5 4^2 1^4$.

\[\text{Figure 4} \quad \text{One-to-one correspondence between step polyomino and restricted Young diagram}\]

**Remark 3.6** Connecting the bijections given in [3.2] and [3.4] yields a one-to-one correspondence between bi-increasing permutations and restricted Young diagrams.

Let $(\alpha, \beta) \in C^2_{n,w}$ be the step polyomino associated with $\pi \in B_n$. Then the partial sums 

$$\beta_1, \beta_1+\beta_2, \ldots, \beta_1+\ldots+\beta_{w-1} \text{ and } \alpha_1+1, \alpha_1+\alpha_2+1, \ldots, \alpha_1+\ldots+\alpha_{w-1}+1$$

are exactly the excedances and excedance letters of $\pi$, respectively. Therefore the bi-increasing permutation $\pi$ corresponds to the Young diagram having corners $(i_k, n+1-\pi_i)$ where $i_k$ denotes the excedances of $\pi$.

The restricted Young diagrams are an item in Stanley’s list of combinatorial objects counted by Catalan numbers (see [20, Ex. 6.19]).

Consider the diagram as being contained in an $n \times n$-rectangle. Then the lattice path going along the diagram boundary from the lower left-hand rectangle corner to the upper right-hand one never falls below the line $x = y$. Paths of this kind were counted in [22]. There are

$$N_{n,w} = \frac{1}{n} \binom{n}{w} \binom{n}{w-1}$$

paths with $w$ horizontal segments, or equivalently, Young diagrams with $w-1$ corners fitting into $(n-1, n-2, \ldots, 1)$. The integers $N_{n,w}$ are called Narayana numbers.

**Corollary 3.7** The statistics $\text{exc}$ is Narayana distributed over $B_n$; we have

$$B_n^{\text{exc}}(e) = \frac{1}{n} \binom{n}{e} \binom{n}{e+1}$$

for all $n$ and $e$.  

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Making use of the several bijections established above, we obtain the following combinatorial interpretations of Catalan numbers and Narayana numbers, respectively.

**Corollary 3.8** The Catalan number \( C_n \) (Narayana number \( N_{n,w} \)) counts the number of

a) arrangements of \((w-1)\) lines inside a stripe of width \(n-1\), as described above,

b) step polyominoes of height \(n+1\) (and width \(w\)),

c) pairs \((\alpha, \beta)\) of compositions of \(n\) with the same number \((w)\) of parts such that \(\alpha \geq \beta\).

**Remark 3.9** Certainly, the bijection given in [3.4] is not the most obvious one proving Corollary 3.7. In the following we give an elementary transformation that takes step polyominoes to general parallelogram polyominoes, and proves [3.7] as well. However, the one-to-one correspondence between bi-increasing (or 321-avoiding) permutations and restricted Young diagrams yields a simple bijection between those permutations and 132-avoiding ones which we have described in [17].

Let \((\alpha, \beta) \in C_{n,w}^2\) be a step polyomino of width \(w\) and height \(n+1\). Define the components of the sequences

\[
\gamma = (\alpha_1, \alpha_2 - 1, \alpha_3 - 1, \ldots, \alpha_w - 1) \quad \text{and} \quad \delta = (\beta_1 - 1, \beta_2 - 1, \ldots, \beta_{w-1} - 1, \beta_w)
\]

to be the vertical segment lengths of the lattice paths bounding a parallelogram polyomino having perimeter \(\sum_{i=1}^{w} (\gamma_i + \delta_i) + 2w = 2n + 2\). (The border paths are generated by adjoining alternately \(\gamma_i\) and \(\delta_i\), respectively, north steps and one east step.) Since \(\alpha \geq \beta\) the path described by \(\delta\) never rises above the path encoded by \(\gamma\). Note that \(\gamma\) and \(\delta\) are compositions of \(n+1-w\) into non-negative parts. Hence the resulting polyomino is of width \(w\) and height \(n+1-w\).

For example, the step polyomino \(((1,4,1,3,1),(1,1,3,4,1))\) is transformed into the parallelogram polyomino \(((1,3,0,2,0),(0,0,2,3,1))\):

![Figure 5](image.png)  
**Figure 5** Transformation of a step polyomino into a general parallelogram polyomino
It is known (for instance, see [6]) that the number of parallelogram polyominoes having width \( w \) and perimeter \( 2n + 2 \) equals the Narayana number \( N_{n,w} \).

Obviously, the area of \((\gamma,\delta)\) is equal to the area of \((\alpha,\beta)\) minus \( w \). Thus parallelogram polyominoes encode \text{exc} and \text{dexc} by their width and area. More exactly we have

**Theorem 3.10** Let \( \pi \in B_n \) be a bi-increasing permutation with \( \text{exc}(\pi) = e \) and \( \text{dexc}(\pi) = k \). Then \( \pi \) corresponds in a one-to-one fashion to a parallelogram polyomino of perimeter \( 2n + 2 \), width \( e + 1 \), and area \( n + k - e \).

In [2], the authors gave a bijection between 321-avoiding permutations of length \( n \) and parallelogram polyominoes of perimeter \( 2n + 2 \). Their idea is as follows.

West described in [23] a method to construct the set \( B_n \) recursively, by means of generating trees. A basic term in this context is that of active sites of a permutation. Given \( \pi \in B_n \), an integer \( i \in [n+1] \) is called an active site of \( \pi \) if the permutation \( \pi^{(i)} := \pi_1\pi_2\cdots\pi_{i-1} n+1 \pi_i\pi_{i+1}\cdots\pi_n \) avoids 321 as well. Barcucci et al. studied the consequences for the number of active sites and the inversion number when inserting \( n+1 \) into the site \( i \).

On the side of parallelogram polyominoes, all polyominoes of perimeter \( 2(n+1) \) can be constructed from those having perimeter \( 2n \) by adding a cell onto the last column, or adding a column to the right of the last column such that both end at the same level. Here [2] considered the changes of width, height, cell number of the last column, and number of cells having an adjacent cell on its right under these operations.

The comparison yields a correspondence between the following parameters:

| bi-increasing permutations | parallelogram polyominoes |
|---------------------------|---------------------------|
| length \( n \)            | perimeter \( 2n + 2 \)    |
| number of active sites minus one | number of cells belonging to the last column |
| inversion number          | number of cells with adjacent cell on its right |

It is easy to see that our bijection carries out these transformations.

**Proposition 3.11** The bijection given in 3.10 translates the permutation statistics into the polyomino parameters, as described above.

**Proof.** Let \( \pi \in B_n \) and \( (\gamma,\delta) \) its corresponding parallelogram polyomino. It is evident from the structure of bi-increasing permutations that \( \pi^{(i)} \in B_{n+1} \) if (and only if) \( i > i_e \) where \( i_e \) denotes the greatest excedance of \( \pi \). By construction, the last column of \( (\gamma,\delta) \) contains precisely \( n - i_e \) cells. Clearly, the number of cells having an adjacent cell on its right equals area minus height of
(γ, δ). (Note that the half-perimeter is just the sum of height and width.) Since \(\text{inv}(\pi) = \text{dexc}(\pi)\), the bijection 3.10 translates the inversion number as desired.

In detail, \(\pi^{(n+1)}\) corresponds to the parallelogram polyomino obtained from \((\gamma, \delta)\) by adding an addition cell on top of the last column. (The permutations \(\pi\) and \(\pi^{(n+1)}\) have the same excedances and excedance letters.) For \(i = i_e + 1, \ldots, n\), a new excedance with difference \(n+1 - i\) is arisen in \(\pi^{(i)}\). (All the other excedances are preserved with their letters.) Thus \(\pi^{(i)}\)'s parallelogram polyomino develops from \((\gamma, \delta)\) by adding an additional column to the right of the last column, consisting of \(n+1 - i\) cells, each with an adjacent cell on its left.

\[ \square \]

**Example 3.12** Let \(\pi = 4 \ 1 \ 2 \ 5 \ 3 \ 6 \in \mathcal{B}_6\), with maximal excedance 4. Thus \(\pi^{(5)}, \pi^{(6)}, \text{ and } \pi^{(7)}\) are bi-increasing permutations corresponding to the following parallelogram polyominoes:

\[
\begin{align*}
\pi &= 4 \ 1 \ 2 \ 5 \ 3 \ 6 \\
\pi^{(5)} &= 4 \ 1 \ 2 \ 5 \ 7 \ 3 \ 6 \\
\pi^{(6)} &= 4 \ 1 \ 2 \ 5 \ 3 \ 7 \ 6 \\
\pi^{(7)} &= 4 \ 1 \ 2 \ 5 \ 3 \ 6 \ 7
\end{align*}
\]

**Figure 6** Permutations obtained from 4 1 2 5 3 6 by West’s method with their corresponding parallelogram polyominoes

Using a result of [23], the correspondence between the maximum excedance and the number of active sites in bi-increasing permutations yields the following enumerative statement.

**Corollary 3.13** There are \(\binom{n-1+k}{k} - \binom{n-1+k}{k-1}\) bi-increasing permutations of length \(n\) whose greatest excedance equals \(k\).

**Proof.** Clearly, the number of active sites of \(\pi \in \mathcal{B}_n\) having greatest excedance \(k\) is equal to \(n+1-k\). By [23, Th. 2.12] (applied on reverse permutations), we obtain the given number. \(\square\)

In [3], the authors also determined the generating function of \(\mathcal{B}_n\) according to length and inversion number. This yields the distribution coefficient \(B_n^{\text{dexc}}(k)\) we have looked for. (As shown above, \(B_n^{\text{dexc}}(k)\) is the number of parallelogram polyominoes of perimeter \(2n+2\) for which area minus height equals \(k\).)

**Theorem 3.14** (2, Th. 3.4) The number of bi-increasing permutations having length \(n\) and excedance difference \(k\) is the coefficient of \(x^n q^k\) in the quotient

\[
\frac{J_1(x, q)}{J_0(x, q)}
\]
where the function $J_r$ is defined by

$$J_r(x, q) = \sum_{n \geq 0} \frac{(-1)^n x^n r q^{\frac{1}{2} n(n+2r+1)}}{(x)^n (q)^n}$$

with $(a)_n := (1-a)(1-aq)(1-aq^2) \cdots (1-aq^{n-1})$.

Because the enumeration of polyominoes is a topic on its own we leave it here at the combinatorial interpretation for the distribution coefficients $B_n^{(\text{exc, dexc})}(e, k)$. We will determine these integers in a forthcoming paper.

A nice property of the joint distribution of $\text{exc}$ and $\text{dexc}$ on the set of bi-increasing permutations can be proved already now.

**Corollary 3.15** $B_n^{(\text{exc, dexc})}(e, k) = B_n^{(\text{exc, dexc})}(n - 1 - e, n - 1 - 2e + k)$ for all $e$ and $k$.

**Proof.** We give an elementary involution on the parallelogram polyominoes which yields the desired symmetry. Given a parallelogram polyomino, we reverse each of the border paths and replace a north step $N$ by an east step $E$ and vice versa.

For example, $((1, 3, 0, 2, 0), (0, 0, 2, 3, 1))$ corresponds to $((2, 0, 2, 0, 1), (1, 0, 0, 1, 0, 3))$ in this way:

(The upper border path NENNNEENEE of the left-hand polyomino is transformed into NNEEN-NEEENE, the upper border path of the right-hand polyomino.)

Obviously, perimeter and area are not changed by this map. The width (resp. height) of a polyomino equals the height (resp. width) of the corresponding one. Theorem 3.10 yields the assertion. \qed

321-avoiding permutations have been studied in a manifold way. In particular, several authors gave one-to-one correspondences to lattice paths (for example, see [4] and [14]). Before we do this in the next section as well, we complete this section with a note on the close connection between two well-known bijections dealing with Dyck paths. A **Dyck path** is a lattice path in the plane $\mathbb{N}_0^2$ from the origin to $(2n, 0)$ consisting only of up-steps $[1, 1]$ and down-steps $[1, -1]$.

Any point of the Dyck path connecting an up-step with a following down-step we call a **peak**, any point connecting a down-step with a following up-step we call a **valley** of the path.

In [4, p. 361], Billey, Jockusch, and Stanley established the bijection $\Phi_{BJS}$ between 321-avoiding
permutations of length $n$ and Dyck paths of length $2n$.

In [7, Sect. 4], Delest and Viennot gave the bijection $\Phi_{DV}$ between parallelogram polyominoes of perimeter $2n + 2$ and Dyck paths having length $2n$.

It is not difficult to see that the bijection from 3.10 connects the both ones.

**Proposition 3.16** Let $\pi \in B_n$ be a bi-increasing permutation and $(\gamma, \delta)$ its corresponding parallelogram polyomino. Then we have $\Phi_{BJS}(\pi) = \Phi_{DV}((\gamma, \delta))$.

**Proof.** Let $i_1 < \ldots < i_e$ be the exceedances of $\pi \in B_n$. The bijection $\Phi_{BJS}$ takes $\pi$ to the Dyck path defined as follows. For $k = 1, \ldots, e$, let $a_k := \pi_{i_k} - 1$, and $a_0 := 0$, $a_{e+1} := n$. Furthermore, for $k = 1, \ldots, e$, let $b_k := i_k$, and $b_0 := 0$, $b_{e+1} := n$. Beginning at the origin adjoin alternately $a_k - a_{k-1}$ up-steps and $b_k - b_{k-1}$ down-steps where $k = 1, \ldots, e + 1$. Note that $a_k - a_{k-1} = \alpha_k$ and $b_k - b_{k-1} = \beta_k$ for $k = 1, \ldots, e + 1$. Here $(\alpha, \beta)$ is the step polyomino associated with the permutation $\pi$ by Theorem 3.1.

By the construction, the heights (the ordinates) of the peaks are $a_k - b_{k-1}$ with $k = 1, \ldots, e + 1$. The valleys are of height $a_k - b_k$ where $k = 1, \ldots, e$.

On the other hand, the number of cells belonging to the $k$th column of $(\gamma, \delta)$ is just

$$\gamma_1 + \ldots + \gamma_k - (\delta_1 + \ldots + \delta_{k-1}) = \alpha_1 + \ldots + \alpha_k - (\beta_1 + \ldots + \beta_{k-1}) = a_k - b_{k-1}.$$ 

For the number of cells adjacent to the columns $k$ and $k + 1$ of $(\gamma, \delta)$ we obtain

$$\gamma_1 + \ldots + \gamma_k - (\delta_1 + \ldots + \delta_k) = \alpha_1 + \ldots + \alpha_k + 1 - (\beta_1 + \ldots + \beta_k) = a_k - b_k + 1.$$ 

The correspondence between column sizes and peak heights and between the numbers of rows in common of adjacent columns and valley heights, respectively, is precisely the description of $\Phi_{DV}$.  

By the way, the Young diagram constructed in the proof of Theorem 3.4 we also find again: the Dyck path describes just the boundary of the diagram when considered as being contained in a $n \times n$-rectangle.

**Example 3.17** We consider $\pi = 2 \ 6 \ 1 \ 3 \ 7 \ 4 \ 5 \ 8 \ 10 \ 9 \in B_{10}$ again. Billey-Jockusch-Stanley's bijection takes $\pi$ to the Dyck path

![Diagram](image)

**Figure 7** Dyck path $\Phi_{BJS}(2 \ 6 \ 1 \ 3 \ 7 \ 4 \ 5 \ 8 \ 10 \ 9)$ resp. $\Phi_{DV}((1, 3, 0, 2, 0), (0, 0, 2, 3, 1))$
which is exactly the path obtained from the parallelogram polyomino corresponding to \( \pi \) (see the right-hand polyomino in Figure 5) by Delest-Viennot’s bijection. Note that \( \text{exc}(\pi) \) is the number of valleys and \( \text{dexc}(\pi) \) equals the sum of the heights of the valleys, each increased by 1.

**Remark 3.18** Corollary 2.5 says, if an integer \( i \) is a fixed point of \( \pi \in B_n \) then there is no line in \( \pi \)’s graphical representation which starts or ends at level \( i \), or runs through the point \( i \). Consequently, the corresponding step polyomino has two consecutive rows consisting of one cell. The transformation into general parallelogram polyominos deletes one of these cells. Therefore, the number of fixed points of a bi-increasing permutation equals the number of singleton rows of the associated parallelogram polyomino.

Hence there are \( F_n \) parallelogram polyominos of perimeter \( 2n + 2 \) whose all rows are of length at most two (see also \( [9] \)’s list of objects counted by Fine numbers).

## 4 Bi-increasing Permutations and 2-Motzkin Paths

From the nature of bi-increasing permutations follows: an integer \( i \) is a descent if and only if \( i \) is an excedance but \( i + 1 \) is none. In particular, there are no consecutive descents.

Consequently, the excedance number and descent number have not the same distribution over \( B_n \). We have \( \text{des}(\pi) \leq \frac{n}{2} \) for all \( \pi \in B_n \).

In this section we will develop a correspondence between bi-increasing permutations and certain lattice paths that makes it possible to determine the distribution of the descent number and descent difference on \( B_n \) by means of path enumeration.

First we consider only a subset of \( B_n \), namely those permutations with identical excedance number and descent number. Note that the condition \( \text{exc}(\pi) = \text{des}(\pi) \) for \( \pi \in B_n \) is equivalent to \( E(\pi) = D(\pi) \). Permutations satisfying this can easily be characterized.

**Proposition 4.1** Let \( \pi \in S_n \) be a permutation. Then the words \( \pi_d \) and \( \pi_{nd} \) are increasing if and only if \( \pi \in B_n \) and \( \text{exc}(\pi) = \text{des}(\pi) \).

**Proof.** Let \( \pi \in S_n \) be a permutation for which both the descent top word \( \pi_d \) and the word \( \pi_{nd} \) consisting of the remaining letters are increasing. Clearly, \( \pi \) avoids the pattern 321. As mentioned at the very beginning, hence every descent is an excedance of \( \pi \). But these are all excedances: assume that \( i, i + 1, \ldots, i + k \in E(\pi) \) and \( i - 1, i + k + 1 \notin E(\pi) \) for some \( k > 0 \). Obviously, there exists an integer \( j > i \) (even \( j > i + k \)) satisfying \( \pi_j \leq i \). Then \( j \) can not be
a non-descent since \( i \) is such a one and \( \pi_j < \pi_i \). On the other hand, \( j \) is not a descent as well since \( i + k \) is a descent and \( \pi_j < \pi_{i+k} \). The converse is evident (\( E(\pi) \) equals \( D(\pi) \)). \( \square \)

**Remark 4.2** In [12, Th. 2] a natural expression for Denert’s permutation statistic \( \text{den} \) was given: 
\[
\text{den}(\pi) = \text{inv}(\pi_e) + \text{inv}(\pi_n) + i_1 + \ldots + i_e
\]
where \( i_1, \ldots, i_e \) denote the excedances of \( \pi \in S_n \). In case of bi-increasing permutations, \( \text{den} \) equals the sum of excedances.

Since every descent of a bi-increasing permutation is an excedance as well, the conditions \( \text{den}(\pi) = \text{maj}(\pi) \) and \( \text{exc}(\pi) = \text{des}(\pi) \) are equivalent in \( B_n \).

Calculating the number of permutations \( \pi \in B_n \) satisfying \( \text{exc}(\pi) = \text{des}(\pi) \) for some small values \( n \) yields the first terms of another well-known number sequence, the Motzkin numbers. The **Motzkin numbers** may be defined by
\[
M_0 = 1, \quad M_n = M_{n-1} + \sum_{i=0}^{n-2} M_i M_{n-2-i} \quad \text{for } n \geq 1.
\]

One of their numerous combinatorial interpretations is the following one: \( M_n \) counts the number of lattice paths from the origin to \((n,0)\), with steps \([1,1]\) (called up-steps), \([1,-1]\) (called down-steps), and \([1,0]\), never going below the \( x \)-axis. Such paths are called **Motzkin paths**.

A modification are the **2-Motzkin paths** which distinguish two kinds of \([1,0]\) steps: solid steps and broken steps. We denote the set of 2-Motzkin paths of length \( n \) by \( \mathcal{M}_n \). By a simple substitution, a 2-Motzkin path of length \( n \) can be transformed into a Dyck path of length \( 2n + 2 \).

Replace a step \( u \) by two up-steps, a step \( d \) by two down steps, a step \( s \) by an up-down combination, a step \( b \) by a down-up combination and adjoin an additional up-step at the beginning and an additional down-step at the end. In particular, this bijection shows \( |\mathcal{M}_n| = C_{n+1} \).

Together with a weight function, the set \( \mathcal{M}_n \) corresponds to the symmetric group \( S_n \). Several authors ([13], [12], [3]) gave bijections regarding this whose connections was studied in [3].

We will use the first one, due to Françon and Viennot, in the notation of [5] to enumerate the bi-increasing permutations for which excedance number and descent number are equally.

Given a permutation \( \pi \in S_n \) we separate \( \pi \) into its **descent blocks** by putting in a dash between the letters \( \pi_i \) and \( \pi_{i+1} \) whenever \( i \) is a non-descent. For example, the permutation \( \pi = 2 \; 7 \; 4 \; 3 \; 1 \; 6 \; 5 \; 8 \; 10 \; 9 \in S_{10} \) has the descent block decomposition \( 2 \; -7 \; 4 \; 3 \; 1 \; -6 \; 5 \; -8 \; -10 \; 9 \).

Clearly, if \( \pi \) is bi-increasing (or 321-avoiding) then the maximum length of a descent block is two.
Theorem 4.3 Let $\pi \in B_n$ satisfy $\text{exc}(\pi) = \text{des}(\pi) = k$. Then $\pi$ corresponds in an one-to-one fashion to a Motzkin path of length $n$ with $k$ up-steps.

**Proof.** Given a permutation $\pi \in S_n$, Françon-Viennot’s bijection defines the 2-Motzkin path $c = c_1 \cdots c_n$ (which corresponds together with its weight to $\pi$) as follows:

$$
c_{\pi_i} = \begin{cases} 
s & \text{if } \pi_{i-1} < \pi_i < \pi_{i+1}, \\
b & \text{if } \pi_{i-1} > \pi_i > \pi_{i+1}, \\
d & \text{if } \pi_{i-1} < \pi_i > \pi_{i+1}, \\
u & \text{if } \pi_{i-1} > \pi_i < \pi_{i+1} \end{cases}
$$

where $\pi_0 := 0$ and $\pi_{n+1} := n + 1$. Using the descent block decomposition, this means: if the letter $i$ is the first (last) one in a descent block of length at least 2 then the $i$th step is a down-step (up-step). If $i$ is a letter lain strictly inside a block then the $i$th step is broken. If $i$ belongs to a singleton block then the $i$th step is solid. (For example, the permutation $2 7 4 3 1 6 5 8 10 9$ is associated with the path $u_s b u d s u d$.)

Let $\pi \in B_n$ with $\text{exc}(\pi) = \text{des}(\pi) = k$. Obviously, $c$ has no broken step. Since the words $\pi_d$ and $\pi_{nd}$ are increasing now the map which takes any bi-increasing permutation to a Motzkin path is a bijection. Any descent of $\pi$ corresponds to a down-step. Hence the descent number of $\pi$ equals the number of down-steps (or, equivalently, up-steps) in $c$.

Françon-Viennot’s map also transforms the descent difference into a natural statistic of the 2-Motzkin path.

Proposition 4.4 Let $\pi \in S_n$, and let $c$ be the 2-Motzkin path to which $\pi$ is taken by the map described above. Denote by $h_i$ the height of the $i$th step of $c$ defined to be the ordinate of its starting point. Then we have $\text{d des}(\pi) = h_1 + \ldots + h_n$.

**Proof.** It is evident that $\text{d des}(\pi)$ is just the difference of the sum of such letters which are the first ones and the sum of those which are the last ones in a block of length at least 2 obtained by the descent block decomposition of $\pi$. Each of the first mentioned letters corresponds to a down-step while each of the last mentioned ones corresponds to an up-step of $c$. It is easy to see that

$$
\sum_{c_i = d} i - \sum_{c_i = u} i = h_1 + \ldots + h_n.
$$

Example 4.5 For $\pi = 3 1 7 2 4 5 6 8 10 9 \in B_{10}$ we have $\text{exc}(\pi) = \text{des}(\pi) = 3$. Following the proof, we obtain $u u d s s s d s u d$ as corresponding Motzkin path. Conversely, given the path
Figure 8  Motzkin path corresponding to $31724568109$

we can retrieve $\pi$ as follows: first form blocks consisting of the indices of the $k$th down-step and the $k$th up-step. We obtain $31-72-109$. Then insert the remaining numbers as singleton blocks such that the blocks are increasing ordered by their last letter.

A refinement of Motzkin path enumeration according to the number of up-steps was done in [10]. Using this result, Theorem 4.3 yields

**Corollary 4.6** The number of bi-increasing permutations $\pi \in \mathcal{S}_B$ satisfying $\text{exc}(\pi) = \text{des}(\pi)$ equals $M_n$. In particular, there are $\binom{n}{2k}C_k$ such permutations with $k$ descents.

**Remarks 4.7**

a) In [1], the authors constructed classes of permutations of length $n$ which avoid certain patterns. The enumeration of these permutations yields an integer sequence whose first term is the $n$th Motzkin number and whose limit is the $n$th Catalan number.

The first term counts the number of permutations $\pi \in \mathcal{S}_n$ which avoid the patterns 321 and 3142. The latter one means that any subsequence of type 231 in $\pi$ must be contained in a subsequence of type 3142.

These are exactly the permutations considered above. Let $i < j < k$ such that $\pi_k < \pi_i < \pi_j$ where $\pi \in \mathcal{B}_n$ with $\text{exc}(\pi) = \text{des}(\pi)$. Clearly, $i$ and $j$ are excedances (or, equivalently, descents) but $k$ is none. Since $\pi$ avoids 321 there is a non-excedance $l$ for which $i < l < j$ and $\pi_l < \pi_k$. (The second condition follows from Proposition 4.1.)

Note that the integer sequences $(a_k(n))$ whose $k$th term is defined as number of bi-increasing permutations $\pi \in \mathcal{B}_n$ for which $\text{exc}(\pi) - \text{des}(\pi) \leq k-1$ are of a similar behaviour as the sequences in [1]. We have $a_1(n) = M_n$, $a_{n-2}(n) = C_n - 1$, and $a_k(n) = C_n$ for all $k \geq n - 1$.

b) The one-to-one correspondence 4.3 yields a simple proof for an observation made by Deutsch ([8]). For $n > 1$ there are as many Motzkin paths of length $n$ with no horizontal steps on the $x$-axis as Motzkin paths of length $n - 1$ with at least one horizontal step on the $x$-axis.

By the construction, the Motzkin path corresponding to $\pi \in \mathcal{B}_n$ connects the lattice points $(i - 1, 0)$ and $(i, 0)$ by a solid step if and only if $i$ is a fixed point of $\pi$. (See also Corollary 2.5.)
Let $\sigma \in B_{n-1}$ satisfy $\text{exc}(\sigma) = \text{des}(\sigma)$ with minimal fixed point $i$. Define the permutation $\pi$ to be obtained from $\sigma$ by inserting the letter $n$ between $\sigma_{i-1}$ and $\sigma_i$ and sorting the descents such that $\pi_d$ is increasing. For example, for $\sigma = 2136457$ we obtain $\pi = 2\underline{1}638\underline{4}57$ (the underlined subword is just $\pi_d$). Obviously, $\pi \in B_n$. Any fixed point forms a singleton descent block. By inserting $n$ before $\sigma_i$, a new descent arises. Note that sorting $\pi_d$ preserves all descents and non-descents since $\sigma_d$ and $\sigma_{nd}$ are increasing. Moreover, $\pi$ is a derangement: if $\sigma_j = j + 1$ for some $j > i$ then the letter $\sigma_j$ is moved to the left while sorting since $j$ is a descent (equally, excedance).

In particular, we have $\text{exc}(\pi) = \text{exc}(\sigma) + 1$. Consequently, we can refine Deutsch’s statement regarding the number of up-steps.

By a little additional convention we can extend the correspondence \[4.3\] to a bijection which takes any bi-increasing permutation to a lattice path, with preserving all parameters as above.

**Theorem 4.8** There is a bijection between bi-increasing permutations $\pi \in B_n$ with $\text{des}(\pi) = d$ and $\text{ddes}(\pi) = k$ and 2-Motzkin paths of length $n$ having $d$ up-steps and no broken step of height 0 such that the sum of heights of all steps equals $k$.

**Proof.** The only difference from the situation studied above is that the letter forming a descent block of length one may be an excedance letter now. We encode this information by the path as follows. With the convention $\pi_0 := 0$ and $\pi_{n+1} := n + 1$ we set

$$c_{\pi_i} = \begin{cases} 
\text{s} & \text{if } \pi_{i-1} < \pi_i < \pi_{i+1} \text{ and } i \text{ is not an excedance}, \\
\text{b} & \text{if } \pi_{i-1} < \pi_i < \pi_{i+1} \text{ and } i \text{ is an excedance}, \\
\text{d} & \text{if } \pi_{i-1} < \pi_i > \pi_{i+1}, \\
\text{u} & \text{if } \pi_{i-1} > \pi_i < \pi_{i+1}.
\end{cases}$$

By Remark \[4.7b\], the horizontal steps at level zero corresponds to fixed points which are clearly non-excedances. The transformation of the permutation statistics follows from Theorem \[4.3\] and from the proof of Proposition \[4.4\]. In particular, the number of broken steps equals $\text{exc}(\pi) - \text{des}(\pi)$.

This yields another combinatorial interpretation for the Catalan numbers. The second statement results from \[13, \text{Cor. 3.3}\] (Fine numbers count bi-increasing derangements). The generalization in part c) follows from Theorem 7.5 in the same paper. Clearly, $\sum_{k=0}^n m_{n,k} = C_n$. Since $|M_n| = C_{n+1}$, we have in addition $\sum_{k=0}^n 2^k m_{n,k} = C_{n+1}$. 

\[24\]
Corollary 4.9

a) The number of 2-Motzkin paths of length \( n \) having no broken steps on the \( x \)-axis is the \( n \)th Catalan number \( C_n \).

b) The number of 2-Motzkin paths of length \( n \) having no horizontal steps on the \( x \)-axis is the \( n \)th Fine number \( F_n \).

c) The number of 2-Motzkin paths of length \( n \) having no broken steps but \( k \) solid steps on the \( x \)-axis equals

\[
m_{n,k} = \sum_{i=0}^{n-k} (-1)^i \binom{k+1+i}{n+1} \binom{2n-k-i}{n} \binom{k+i}{k}.
\]

Example 4.10

The running example of the previous section \( \pi = 2\, 6\, 1\, 3\, 7\, 4\, 5\, 8\, 10\, 9 \in \mathcal{B}_{10} \) is taken to the 2-Motzkin path

![Figure 9](image-url)

having \( \text{des}(\pi) = 3 \) up-steps and height sum \( \text{ddes}(\pi) = 9 \).

Let \( \mathcal{M}_n^* \) be the set of all 2-Motzkin paths of length \( n \) whose broken steps are all of positive height. The enumeration of bi-increasing permutations of length \( n \) according to their descent number is equivalent to the enumeration of paths in \( \mathcal{M}_n^* \) with regard to the number of the up-steps. The explicit determination of these numbers remains open.

In the previous section we exhibit a one-to-one correspondence between bi-increasing permutations and parallelogram polyominoes that transfers \( \text{exc} \) and \( \text{dexc} \) to natural polyomino statistics. The bijection between the symmetric group and weighted 2-Motzkin paths due to Foata and Zeilberger (see [12]) yields a correspondence between \( \mathcal{B}_n \) and \( \mathcal{M}_n^* \) that expresses the excedance-based statistics as path parameters, too.

Given a permutation \( \pi \in \mathcal{S}_n \), Foata-Zeilberger’s map defines the 2-Motzkin path \( c = c_1 \cdots c_n \) by

\[
c_i = \begin{cases} 
  b & \text{if } i \text{ is both an excedance and an excedance letter,} \\
  s & \text{if } i \text{ is both a non-excedance and a non-excedance letter,} \\
  u & \text{if } i \text{ is both an excedance and a non-excedance letter,} \\
  d & \text{if } i \text{ is both a non-excedance and an excedance letter.} 
\end{cases}
\]

By the construction, no broken step can be of height 0; hence \( c(\pi) \in \mathcal{M}_n^* \) for any \( \pi \in \mathcal{S}_n \). The reduction on bi-increasing permutations yields a bijection between \( \mathcal{B}_n \) and \( \mathcal{M}_n^* \). From
the indices of up-steps and broken steps we obtain the excendances, from the indices of down-steps and broken steps we obtain the excendance letters. For a bi-increasing permutation, these informations are enough to determine the permutation completely.

Obviously, the excendance number of \( \pi \in S_n \) equals the number of \( u \) and \( b \) in the corresponding path. Moreover, [\text{proof of Th. 10}] showed that \( d\text{exc}(\pi) \) is just the sum of heights in terms of path.

**Example 4.11** The permutation \( \pi = 26137458109 \in B_{10} \) is transformed into the 2-Motzkin path

![2-Motzkin path corresponding to 26137458109 by Foata-Zeilberger](image)

whose heights amount to \( d\text{exc}(\pi) = 8 \) and which contains \( \text{exc}(\pi) = 4 \) steps \( u \) or \( b \).

Using both, the correspondence due to Foata-Zeilberger and the correspondence from Theorem 4.8, shows trivially the equidistribution of \( d\text{exc} \) and \( d\text{des} \) on the set \( B_n \). In the next section we will give a simple combinatorial proof in terms of bi-increasing permutations for this fact.

## 5 Parallelogram polyominoes and 2-Motzkin paths

In contrast to the excendance number and descent number, the difference statistics are also equidistributed over the set of bi-increasing permutations. But the bijection \( \phi \) from Proposition 1.1 does not furnish proof because \( B_n \) is not closed under \( \phi \).

**Theorem 5.1** We have \( B_n^{d\text{exc}}(k) = B_n^{d\text{des}}(k) \) for all \( n \) and \( k \).

**Proof.** We give a bijection \( \psi : B_n \to B_n \) which takes a bi-increasing permutation \( \pi \) to a bi-increasing permutation \( \sigma \) whose excendence letters equal those of \( \pi \).

Given \( \pi \in B_n \), let \( i_1, \ldots, i_s \) be the excendances which are descents in addition, and let \( j_1, \ldots, j_t \) be the remaining excendances of \( \pi \). (As mentioned above, \( \text{des}(\pi) = s \).) The permutation \( \sigma \) is defined as the bi-increasing one whose excendence set is \( \{\pi_{i_1+1}, \ldots, \pi_{i_s+1}, \pi_{j_1}, \ldots, \pi_{j_t}\} \) and whose excendence letters are \( \pi_{i_1}, \ldots, \pi_{i_s}, \pi_{j_1}, \ldots, \pi_{j_t} \). Note that \( \sigma \) is well defined. To see this, let \( j \) be the first one of a sequence of consecutive excendances whose last term, say \( i \), is a descent in addition. Then \( \pi_j > \pi_{i+1} \) since there is at least one integer \( k > j \) with \( \pi_k < \pi_j \).

Obviously, the map \( \psi \) is bijective, and we have

\[
d\text{des}(\pi) = \sum_{k=1}^{s} (\pi_{i_k} - \pi_{i_k+1}) = \left( \sum_{k=1}^{s} \pi_{i_k} + \sum_{k=1}^{t} \pi_{j_k} \right) - \left( \sum_{k=1}^{s} \pi_{i_k+1} + \sum_{k=1}^{t} \pi_{j_k} \right) = d\text{exc}(\sigma).
\]
Example 5.2 Let \( \pi = \underline{7} 6 \underline{1} 3 4 \underline{5} 8 \underline{10} 9 \in B_{10} \) again. (The letters that are as well excedance letter as descent top are underlined; the letters belonging to only-excedances are overlined.) We obtain the permutation \( \psi(\pi) = 2 6 1 7 3 4 5 8 10 9 \).

Remarks 5.3

a) The bijection \( \psi \) preserves both the excedance number as the number of fixed points. The latter one is an immediate consequence of Corollary 2.3.

b) Let \( c \in M_n^* \) be the 2-Motzkin path to which \( \pi \in B_n \) is taken by the bijection given in Theorem 4.8. It is not difficult to see that \( c \) is precisely the path obtained from \( \psi(\pi) \in B_n \) by Foata-Zeilberger’s bijection.

The \( q \)-series whose coefficient of \( x^n q^k \) is just \( B_n^\text{ddes}(k) \) is given in Theorem 3.14. From the combinatorial interpretations of the distribution coefficients \( B_n^\text{dexc}(k) \) and \( B_n^\text{ddes}(k) \), respectively, we obtain

**Corollary 5.4** There are as many parallelogram polyominoes of perimeter \( 2n + 2 \) whose area minus height equals \( k \) as 2-Motzkin paths in \( M_n^* \) whose height sum equals \( k \).

Translated into the languages of polyominoes and lattice paths the bijection \( \psi \) can be read as follows. Let \( (\gamma, \delta) \) be a parallelogram polyomino of perimeter \( 2n + 2 \) and width \( w \). (Hence its height equals \( n + 1 - w \).) Define the integer sets \( A = \{\gamma_1 + \ldots + \gamma_i + i : i = 1, 2, \ldots, w - 1\} \) and \( B = \{\delta_1 + \ldots + \delta_i + i : i = 1, 2, \ldots, w - 1\} \), and let \( C \) be their intersection. The 2-Motzkin path \( c \in M_n^* \) associated with \( (\gamma, \delta) \) we obtain by the convention

\[
c_i = \begin{cases} 
d & \text{if } i \in A \setminus C, 
u & \text{if } i \in B \setminus C, 
b & \text{if } i \in C, 
s & \text{if } i \notin A \cup B. \end{cases}
\]

The following correspondences under this bijection are clear from the theorems 3.10 and 4.8 and Remark 5.3a:
parallelogram polyominos 2-Motzkin paths
perimeter $2n + 2$ length $n$
width $1 + \text{number of u’s} + \text{number of b’s}$
(height) $(\text{number of u’s} + \text{number of s’s})$
area minus height sum of heights
number of rows consisting of one cell number of solid steps at level 0

Example 5.5 Let $\gamma = (1, 3, 0, 2, 0)$ and $\beta = (0, 0, 1, 4, 1)$. Note that $(\gamma, \delta)$ is just the parallelogram polyomino corresponding to $26173458109 \in B_10$, the permutation $\psi(\pi)$ from the previous example. We have $A = \{2, 6, 7, 10\}$, $B = \{1, 2, 4, 9\}$, and $C = \{2\}$. Thus we obtain the following path:

![Figure 11](image)

Figure 11 Parallelogram polyomino and corresponding 2-Motzkin path

(Recall that this is exactly the 2-Motzkin path associated with the permutation $\pi \in B_{10}$ from Example 5.2.)

Parallelogram polyominones can be interpreted as connected skew diagrams. For two partitions $\lambda$ and $\mu$ with $\mu \subseteq \lambda$ (i.e., $\mu_i \leq \lambda_i$ for all $i$), the skew diagram $\lambda/\mu$ is defined to be the set theoretical difference of the Young diagrams associated with $\lambda$ and $\mu$, respectively. The pair $(\gamma, \delta)$ describes the skew diagram $1^{\delta_1}2^{\delta_2} \cdots w^{\delta_w}/1^{\gamma_1}2^{\gamma_2} \cdots (w-1)^{\gamma_w}$.

In [15], the rank of a skew diagram was introduced. For a skew diagram $\lambda/\mu$, its rank is defined as difference of outside diagonal lengths and inside diagonal lengths and is denoted by $\text{rank}(\lambda/\mu)$. For example, the skew diagram $3^24^35/1^33^2$ (or $((1,3,0,2,0),(0,0,2,3,1))$ in the previous notation)

![Figure 12](image)

Figure 12 A skew diagram and its rank
has rank 2. (All squares belonging to an outside diagonal are marked by +, the inside diagonal squares by −.)

If \( \mu = \emptyset \), i.e., if \( \lambda/\mu \) is a partition, then \( \lambda/\mu \) has only one outside diagonal (the main diagonal) and no inside diagonals. Hence in this case, \( \text{rank}(\lambda/\mu) \) is just the Durfee rank defined for partitions. Stanley gave in [21] several equivalent definitions of \( \text{rank}(\lambda/\mu) \).

**Proposition 5.6** Let \((\gamma, \delta)\) be a parallelogram polyomino and \(c \in \mathcal{M}^*_n\) its corresponding 2-Motzkin path. Then the rank of \((\gamma, \delta)\) equals

\[
1 + \text{number of dd} + \text{number of db} + \text{number of sd} + \text{number of sb}
\]

appearing in \(c\).

**Proof.** We use the rank definition dealing with the reduced code of skew diagrams. Given a skew diagram \(\lambda/\mu\) of perimeter \(2n+2\), we mark every vertical boundary part by 0 and every horizontal boundary part by 1. Reading these numbers while moving north and east along the lower boundary (starting from the left-hand edge) we obtain a binary sequence \(a(\lambda/\mu) = a_1a_2 \cdots a_{n+1}\). In a similar way, when we read the labels as we move north and east along the upper boundary we obtain a binary sequence \(b(\lambda/\mu) = b_1b_2 \cdots b_{n+1}\). The two-line array

\[
cd(\lambda/\mu) := \begin{array}{cccc} a_1 & a_2 & \cdots & a_{n+1} \\ b_1 & b_2 & \cdots & b_{n+1} \end{array}
\]

we call the reduced code of \(\lambda/\mu\). (Clearly, \(a_1 = b_{n+1} = 1\) and \(a_{n+1} = b_1 = 0\).) The concept should be clear from the example:

![Figure 13 Reduced code of a skew diagram](image)

By [21], Prop. 2.2, \(\text{rank}(\lambda/\mu)\) equals the number of columns \(\begin{array}{c} 0 \\ 1 \end{array}\) (or, equivalently, \(\begin{array}{c} 1 \\ 0 \end{array}\)) of \(cd(\lambda/\mu)\). (The figure shows a skew diagram of rank 2.)

Consider now the reduced code of \((\gamma, \delta)\). In the binary sequence \(b\) the 1’s appear at the positions \(\gamma_1 + 1, \gamma_1 + \gamma_2 + 2, \ldots, \gamma_1 + \ldots + \gamma_{w-1} + w - 1, \gamma_1 + \ldots + \gamma_w + w = n + 1\). These integers, excepting \(n + 1\), are exactly the elements of the set \(A\) appearing in the definition of \(c\). The sequence \(a\) contains a 1 at each of the positions \(1, \delta_1 + 2, \delta_1 + \delta_2 + 3, \ldots, \delta_1 + \ldots + \delta_{w-1} + w\). Apart from 1, decreasing these integers by 1 yields the elements of \(B\). Thus the \(i\)th column of the code equals
if and only if \( i \in A \) and \( i - 1 \notin B \) or \( i = n + 1 \). The first condition means the occurrence of \( dd, sd, db \) and \( sb \) in the path \( c \).

\[ \square \]

**Remark 5.7** Because of the symmetry, the step combinations \( uu, us, bu \) and \( bs \) (corresponding to code columns \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)) can be counted as well. Note that the reverse 2-Motzkin path \( c_n \cdots c_1 \) corresponds to the parallelogram polyomino obtained by rotating \( (\gamma, \delta) \) 180°.

We will apply the bijection to enumerate partitions of prescribed perimeter according their rank. (Here the perimeter of a partition is just twice the sum of its largest part and the number of its parts.)

**Lemma 5.8** Any 2-Motzkin path \( c \in M_n^* \) corresponds to a partition if and only if there exists an integer \( k \in \mathbb{Z} \) such that \( c_1, \ldots, c_k \in \{u, s\} \) and \( c_{k+1}, \ldots, c_n \in \{d, b\} \).

**Proof.** Let \( (\gamma, \delta) \) be a parallelogram polyomino of perimeter \( 2n + 2 \) and width \( w \). (Recall that \( \gamma_1 + \ldots + \gamma_w = \delta_1 + \ldots + \delta_w = n + 1 - w \).) Then \( (\gamma, \delta) \) describes a partition if and only if \( \gamma_2 = \ldots = \gamma_w = 0 \). Thus the elements of the set \( A \) appearing in the bijection definition are exactly the integers \( n + 2 - w, n + 3 - w, \ldots, n \). Consequently, \( c_i \in \{d, b\} \) if \( i \geq n + 2 - w \) and \( c_i \in \{u, s\} \) otherwise. \[ \square \]

**Lemma 5.9** For all non-negative integers \( a < b \leq c \) we have

\[ \sum_{k=0}^{c-b} \binom{a+k}{a} \binom{c-a-1-k}{b-a-1} = \binom{c}{b}. \]

**Proof.** This special case of the Chu-Vandermonde identity can be proved very easily. Draw a line of \( c \) dots and circle \( b \) dots. Then count the number of uncircled dots to the left of the \((a+1)\)st circled dot. \[ \square \]

**Theorem 5.10** There are \( \binom{n}{2r-1} \) partitions of perimeter \( 2n + 2 \) and rank \( r \).

**Proof.** Let \( \lambda \) be a partition of perimeter \( 2n + 2 \) and rank \( r \), and \( c_1 \cdots c_n \) the corresponding 2-Motzkin path. By Lemma 5.8 and Remark 5.7, the rank is one plus the number of occurrences of \( uu \) and \( us \) in \( c_1 \cdots c_k \) where \( k \) is the maximum integer for which \( c_i \in \{u, s\} \). Clearly, there are \( \binom{k-1}{r-1} \) different words \( c_1 \cdots c_k \) over \( \{u, s\} \) containing \( r - 1 \) times \( uu \) or \( us \). Obviously, \( c_{k+1} \cdots c_n \) has \( r - 1 \) letters \( d \) (as counterparts to \( r - 1 \) \( u \)'s in the first part of \( c \)), and only \( b \) otherwise. There are \( \binom{n-k}{r-1} \) such words. Consequently, the desired number equals

\[ \sum_{k=r}^{n+1-r} \binom{k-1}{r-1} \binom{n-k}{r-1} = \sum_{k=0}^{n+1-2r} \binom{k+r-1}{r-1} \binom{n-k-r}{r-1} = \binom{n}{2r-1}. \]
(For the second identity use Lemma 5.9 \((a = r - 1, \ b = 2r - 1, \ c = n)\) \(\square\))

To prove the generalization we utilize an encoding principle introduced in [9, Sect. 5]. Given a parallelogram polyomino of perimeter \(2n + 2\), the corresponding 2-Motzkin path \(c \in M_{n+1}\) is defined as follows. Beginning at the origin read the upper border path and lower border path, respectively, step by step. Reading the \(i\)th steps, set

\[
 c_i = \begin{cases} 
 u & \text{if upper path goes north, lower path goes east}, \\
 d & \text{if upper path goes east, lower path goes north}, \\
 b & \text{if both paths go east}, \\
 s & \text{if both paths go north}.
\end{cases}
\]

For instance, the parallelogram polyomino

is associated with the 2-Motzkin path \(\text{ubussbdssbd}\). Note that every 2-Motzkin path arising in this way begins with an up-step and ends with a down-step. Moreover, all steps are of height at least one, excepting the first one. Thus by deleting the very first and very last step of \(c\) we obtain a correspondence between parallelogram polyominos of perimeter \(2n + 2\) and 2-Motzkin path of length \(n - 1\) which is one-to-one.

**Theorem 5.11** The number of all connected skew diagrams of perimeter \(2n + 2\) and rank \(r\) equals to

\[
2^{n+1-2r} \left( \frac{n - 1}{2r - 2} \right) C_{r-1}.
\]

**Proof.** Let \(\lambda/\mu\) be a connected skew diagram of perimeter \(2n + 2\) and \(c\) its corresponding 2-Motzkin path of length \(n + 1\) obtained as described above. By construction, the \(i\)th column of the reduced code of \(\lambda/\mu\) equals \(0\) if and only if the \(i\)th step of the lower border path goes north, and the \(i\)th step of the upper border path goes east. Hence each such column corresponds to a down-step of \(c\). Consequently, Deutsch-Shapiro’s bijection takes a skew diagram of perimeter \(2n + 2\) and rank \(r\) to a 2-Motzkin path of length \(n - 1\) having \(r - 1\) down-steps. (The last down-step of \(c\) has been deleted.) By [10], the number of Motzkin paths of length \(n - 1\) with \(r - 1\) down-steps equals \(\left( \frac{n - 1}{2r - 2} \right) C_{r-1}\). Thus the number given in the statement counts the 2-Motzkin paths with these parameters. \(\square\)

**Corollary 5.12** There are as many 2-Motzkin paths in \(M_{n-1}\) with \(k\) down-steps as 2-Motzkin paths in \(M_n^*\) with \(k\) occurrences of double steps \(dd, \ db, \ sd, \ and \ sb\).
Remark 5.13 The statement of Lemma 5.8 holds analogously for the bijection of Deutsch and Shapiro as well.

6 An application to the distributions over the symmetric group

In the first section we introduce the surjection which takes any permutation \( \pi \in S_n \) to the bi-increasing permutation \( \hat{\pi} \in B_n \). The map induces a partition of \( S_n \) into \( C_n \) disjoint classes, each represented by a bi-increasing permutation. All the elements of a class have the same excedance number and excedance difference. This section deals first with the question: for \( \pi \in B_n \), how many permutations belong to the class \([\pi]\)?

As discussed in the proof of Corollary 1.4, every member of \([\pi]\) arises from \( \pi \) by applying some transpositions to \( \pi \) which preserve all excedance and their letters. Let \( T_\pi \) denote the set of pairs \((i,j)\) such that either \( i < j \) are excedances and \( j < \pi_i \) or \( i < j \) are non-excedances and \( i \geq \pi_j \). Clearly, exchanging \( \pi_i \) and \( \pi_j \) has no influence on \( \text{exc} \) and \( \text{dexc} \) if and only if \((i,j) \in T_\pi \). In the proof of 1.4 it was shown that \(|T_\pi| = \text{dexc}(\pi) - \text{exc}(\pi)\).

Note that if \((i,j)\) and \((i,k)\) belong to \( T_\pi \) then \((j,k)\) does it as well, provided that \( j < k \). Every permutation of \([\pi]\) can be constructed from \( \pi \) by applying a sequence of transpositions \((i_1,j_1), \ldots, (i_s,j_s) \in T_\pi \) to the positions of \( \pi \) where \( i_1 < i_2 < \ldots < i_s \) for some \( s \geq 0 \). Consequently, we have

\[
|[\pi]| = \prod_{i=1}^{n-1} \left( |\{j : (i,j) \in T_\pi\}| + 1 \right).
\]

For example, for \( \pi = 26137458109 \in B_{10} \) we have \( T_\pi = \{(2,5), (3,4), (4,6), (6,7)\} \), and hence \(|[\pi]| = 16\). In detail, the permutations

\[
\begin{align*}
26137458109 & 26137548109 & 26347158109 & 27316548109 \\
27136458109 & 27316458109 & 26317548109 & 27146538109 \\
26317458109 & 27146358109 & 26147538109 & 26347518109 \\
26147358109 & 27136548109 & 27346158109 & 27346518109
\end{align*}
\]

are precisely the elements of \([\pi]\).

Surprisingly, the number \(|[\pi]|\) can immediately read off from the polyomino connected with \( \pi \in B_n \) by the correspondence given in Section 3.

Theorem 6.1 Let \((\alpha, \beta) \in C_{n,w} \times C_{n,w} \) be the step polyomino corresponding to \( \pi \in B_n \). Denote by \( R_1, \ldots, R_{n+1} \) the rows of \((\alpha, \beta)\), and let \( a_i \) be the number of columns in common to \( R_i \) and \( R_{i+1} \), for \( 1 \leq i \leq n \). Then \(|[\pi]| = a_1a_2 \cdots a_n|\).
Proof. First we show that $a_i = |\{j \in [w-1] : \beta_1 + \ldots + \beta_j < i \leq \alpha_1 + \ldots + \alpha_j\}| + 1$.

By the bijection given in [3.1], the $j$th column begins at level $\beta_1 + \ldots + \beta_{j-1}$ where $\beta_0 := 0$ and ends at level $\alpha_1 + \ldots + \alpha_j + 1$. Hence both $R_i$ and $R_{i+1}$ contain squares of the $j$th column if $\beta_1 + \ldots + \beta_{j-1} < i < \alpha_1 + \ldots + \alpha_j + 1$. Thus, $a_i$ is the number of $j \in [w]$ satisfying either $\beta_1 + \ldots + \beta_j < i \leq \alpha_1 + \ldots + \alpha_j$ or $\beta_1 + \ldots + \beta_{j-1} < i \leq \beta_1 + \ldots + \beta_j$. Clearly, for each $i$ there exists exactly one integer $j$ for which the second condition holds. (Since $\beta_1 + \ldots + \beta_w = n = \alpha_1 + \ldots + \alpha_w$, we have $j \in [w-1]$.)

The transpositions $(i, \cdot) \in T_\pi$ have already been counted in the proof of Corollary [4.4]. Let $b_i$ denote this number, increased by 1. For any excedance $i$, we obtain $b_i = |\{j \in E(\pi) : i < j < \pi_i\}| + 1 = |\{j \in E(\pi) : i \leq j < \pi_i\}|$. It is easy to see that

$$\prod_{i \in E(\pi)} |\{j \in E(\pi) : i \leq j < \pi_i\}| = \prod_{i \in E(\pi)} |\{j \in E(\pi) : j \leq i < \pi_j\}|.$$  

On the other hand, if $i$ is a non-excedance then $b_i = |\{j \notin E(\pi) : \pi_j \leq i < j\}| + 1$, and by Lemma [1.3]b, we have $b_i = \{|\{j \in E(\pi) : j < i < \pi_j\}| + 1$. By Remark [3.6], the partial sums $\beta_1, \beta_1 + \beta_2, \ldots, \beta_1 + \ldots + \beta_{w-1}$ are just the excedances of $\pi$, and $\alpha_1 + 1, \alpha_1 + \alpha_2 + 1, \ldots, \alpha_1 + \ldots + \alpha_{w-1} + 1$ their letters. Consequently, we obtain $b_i = |\{j \in [w-1] : \beta_1 + \ldots + \beta_j < i \leq \alpha_1 + \ldots + \alpha_j + 1\}| + 1$ for any non-excedance $i$, and

$$\prod_{i \in E(\pi)} b_i = \prod_{i \in E(\pi)} (|\{j \in [w-1] : \beta_1 + \ldots + \beta_j < i \leq \alpha_1 + \ldots + \alpha_j + 1\}| + 1) = \prod_{i \in E(\pi)} a_i.$$  

\[\square\]

Corollary 6.2 Let $(\gamma, \delta)$ be the parallelogram polyomino corresponding to $\pi \in B_n$. Denote by $a_1, \ldots, a_n$ the diagonal lengths of $(\gamma, \delta)$. Then $|\pi| = a_1 a_2 \cdots a_n$.

Proof. Figuratively, the polyomino transformation from Section 3 works as follows. Given a step polyomino, move the squares contained in the $k$th column toward the bottom, each by $k-1$ units. Then remove the top squares of all columns. Clearly, common borders of adjacent rows correspond to the diagonals of the resulting polyomino. \[\square\]

Example 6.3 For $\pi = 26137458109 \in B_{10}$, we have $|\pi| = 16$ (see above). Alternatively, this can be seen from the corresponding step polyomino and parallelogram polyomino, respectively:
The product of the lengths drawn bold in the left-hand polyomino equals just $|\pi|$, just as the product of the diagonal lengths for the right-hand polyomino.

Corollary 3.15 said that there are as many bi-increasing permutations of length $n$ with $e$ excedances and excedance difference $k$ as those having $n - 1 - e$ excedances and excedance difference $n - 1 - 2e + k$. The same result holds when we consider arbitrary permutations.

**Corollary 6.4** \[ S_n^{(\text{exc}, \text{dexc})}(e, k) = S_n^{(\text{exc}, \text{dexc})}(n - 1 - e, n - 1 - 2e + k) \text{ for all } e \text{ and } k. \]

**Proof.** Corollary 3.15 has been proved by an involution in terms of polyominoes. Given $\pi \in B_n$ with $\text{exc}(\pi) = e$ and $\text{dexc}(\pi) = k$, the permutation $\sigma$ having the desired parameters corresponds to the parallelogram polyomino obtained from the one associating to $\pi$ by reflection. Clearly, this operation does not change the diagonal lengths. Hence we have $|\pi| = |\sigma|$. Since $\text{exc}$ and $\text{dexc}$ are invariant on the classes this yields the proof. 

The proof shows even more: not only the products of the numbers $a_1, \ldots, a_n$ counting possible letter exchanges are equally for $\pi$ and $\sigma$ but also the numbers themself, up to order. (If $a_i$ denotes the number of pairs $(i, \cdot)$ in $T_\pi$, and $a'_i$ denotes the number of pairs $(i, \cdot)$ in $T_\sigma$ then we have $a_i = a'_{n+1-i}$ for all $i$.) Consequently, we have

\[ |\{\tau \in S_n : \hat{\tau} = \pi, \ \text{inv}(\tau) - \text{inv}(\pi) = k\}| = |\{\tau \in S_n : \hat{\tau} = \sigma, \ \text{inv}(\tau) - \text{inv}(\sigma) = k\}| \]

for all $k$. (Recall that $\text{inv}$ and $\text{dexc}$ are identical statistics over $B_n$.) We obtain the following result that was proved analytically in \[5, \text{Cor. 12}].

**Corollary 6.5** \[ S_n^{(\text{exc}, \text{inv})}(e, k) = S_n^{(\text{exc}, \text{inv})}(n - 1 - e, n - 1 - 2e + k) \text{ for all } e \text{ and } k. \]
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