Complete minimal hypersurfaces in $S^4$ with zero Gauss–Kronecker curvature

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Abstract
We investigate the structure of 3-dimensional complete minimal hypersurfaces in the unit sphere with Gauss–Kronecker curvature identically zero.

1. Introduction
In order to consider rigidity problems, Dajczer and Gromoll [5] studied the Gauss map of hypersurfaces, which in general is not invertible. The starting point is the observation that whenever the relative nullity is constant, then one has a representation of the hypersurface by the inverse of the Gauss map on the normal bundle of its image, which they called Gauss parametrization. In particular, they proved that if $g: V \to S^{n+1}$ is a minimal immersion, where $V$ is a 2-dimensional manifold, and $N$ is the unit normal bundle of $g$, then the polar map $\Psi (p, w) = w, (p, w) \in N$, defines a minimal hypersurface with relative nullity $n−2$ on the open set of its regular points. Conversely, it is shown that, locally, every minimal hypersurface with constant relative nullity $n−2$ has such a representation.

Consider a minimal immersion $g: V \to S^4$, where $V$ is a 2-dimensional manifold, and denote by $K, K_n$ its Gaussian and normal curvature, respectively. It is proved in [5] that if $g$ has nowhere vanishing normal curvature, then the polar map of $g$ is everywhere regular and provides a minimal hypersurface in $S^4$ with Gauss–Kronecker curvature identically zero. In [1] de Almeida and Brito classified the compact minimal hypersurfaces $M^3$ in $S^4$ with Gauss–Kronecker curvature identically zero under the assumption that the second fundamental form of $M^3$ nowhere vanishes. More precisely, they proved that such a hypersurface is the boundary of a tube, of radius $\pi/2$, of a minimal immersion $g: V \to S^4$ with non-vanishing second fundamental form in any direction. Later, Ramanathan [7] removed the assumption on the second fundamental form and classified the compact minimal hypersurfaces in $S^4$ with Gauss–Kronecker curvature identically zero. In particular, he proved that such hypersurfaces are produced by applying the above construction to appropriate branched minimal surfaces in $S^4$, unless they are totally geodesic.

A minimal immersion $g: V \to S^4$ is called superminimal if it satisfies the relation $(1 - K)^2 - K_n^2 = 0$. The purpose of the present paper is to consider the structure of complete minimal immersions $f: M^3 \to S^4$ with Gauss–Kronecker curvature identically zero under some assumptions on the second fundamental form. More precisely, we show the following:
THEOREM. Let $M^3$ be a 3-dimensional complete Riemannian manifold and $f: M^3 \to \mathbb{S}^4$ a minimal isometric immersion with Gauss–Kronecker curvature identically zero. If the square $S$ of the length of the second fundamental form is nowhere zero and bounded from above, then $f(M^3)$ is the image of the polar map associated with a superminimal immersion $g: V \to \mathbb{S}^4$ with positive normal curvature. Moreover, if $S$ is bounded away from zero, then $V$ is diffeomorphic to the sphere $\mathbb{S}^2$ or to the projective plane $\mathbb{RP}^2$ and $f(M^3)$ is compact.

2. Preliminaries

Consider an isometric and minimal immersion $g: V \to \mathbb{S}^4$, where $V$ is an oriented, 2-dimensional Riemannian manifold. Denote by $(v_1, v_2; v_3, v_4)$ a local adapted orthonormal frame field along $g$ such that $(v_1, v_2)$ is an oriented orthonormal frame field in the tangent bundle and $(v_3, v_4)$ is an oriented frame field in the normal bundle of $g$. Denote by $A_3, A_4$ the shape operators corresponding to the directions $v_3$ and $v_4$. The Gaussian curvature $\mathcal{K}$ of the induced metric $\langle \cdot, \cdot \rangle$ and the normal curvature $\mathcal{K}_n$ of the immersion $g$ are given by

$$\mathcal{K} = 1 + \det A_3 + \det A_4, \quad \mathcal{K}_n = -\langle [A_3, A_4] v_1, v_2 \rangle,$$

and do not depend on the chosen frames. Then, $(1 - \mathcal{K})^2 - \mathcal{K}_n^2 \geq 0$. The isometric immersion $g$ is superminimal if it satisfies in addition $(1 - \mathcal{K})^2 - \mathcal{K}_n^2 = 0$. In this case, we can choose the adapted orthonormal frame field $(v_1, v_2; v_3, v_4)$ such that

$$A_3 \sim \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \quad A_4 \sim \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}.$$

It is obvious that the square of the length of the second fundamental form of $g$ is equal to $2(1 - \mathcal{K})$. If $g$ is superminimal, then the Gaussian curvature satisfies the following differential equation

$$\Delta \log (1 - \mathcal{K}) = 2(3\mathcal{K} - 1), \quad (2.1)$$

away from points where $\mathcal{K} = 1$, where $\Delta$ stands for the Laplacian operator on $V$ (cf. [8]). Using the Penrose twistor fibration of complex projective 3-space over $\mathbb{S}^4$, Bryant in [3] was able to construct superminimal immersions of $\mathbb{S}^2$ into $\mathbb{S}^4$, even with nowhere vanishing normal curvature.

We give now a brief exposition of a method developed by Dajczer and Gromoll in [5, section 1] (see also [7, theorem 1.1]) of constructing minimal hypersurfaces in $\mathbb{S}^4$ with Gauss–Kronecker curvature identically zero. Let $g: V \to \mathbb{S}^4$ be an isometric minimal immersion, where $V$ is a 2-dimensional Riemannian manifold, and

$$\mathcal{N} = \{(x, w) \in V \times \mathbb{R}^5 : |w| = 1, \ w \perp \mathbb{R} \cdot g + dg(T_x V)\}$$

be its unit normal bundle. Denote the projection to the first factor by $\pi: \mathcal{N} \to V$. The projection to the second factor $\Psi: \mathcal{N} \to \mathbb{S}^4$, $\Psi(x, w) = w$, is called the polar map associated with $g$. Choose an adapted orthonormal frame field $(v_1, v_2; v_3, v_4)$ on an open set $U \subset V$ such that the shape operators $A_3, A_4$ of $g$ corresponding to $v_3, v_4$ are represented by

$$A_3 \sim \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad A_4 \sim \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$$

and parametrize $\pi^{-1}(U)$ by $U \times \mathbb{S}^1$ via the map $(x, t) \to (x, \cos tv_3(x) + \sin tv_4(x))$. Then,
\(\Psi(x, t) = \cos tv_3(x) + \sin tv_4(x).\) Hence,

\[
d\Psi(\partial/\partial t) = -\sin tv_3 + \cos tv_4,
\]

\[
d\Psi(v_1) = -(a \cos t + c \sin t)dg(v_1) - (b \cos t)dg(v_2)
+ \omega_{34}(v_1)(- \sin tv_3 + \cos tv_4),
\]

\[
d\Psi(v_2) = -(b \cos t)dg(v_1) + (a \cos t + c \sin t)dg(v_2)
+ \omega_{34}(v_2)(- \sin tv_3 + \cos tv_4),
\]

where \(\omega_{34}\) is the connection form of the normal bundle of \(g.\) From the above relations, it follows that \(\Psi\) is regular at points \((x, t),\) for all \(t \in \mathbb{S}^1,\) if and only if \((a \cos t + c \sin t)^2 + (b \cos t)^2 \neq 0\) which is equivalent to \(K_\kappa \neq 0.\) Obviously, \(\xi(x, t) = g(x)\) defines a unit normal vector field along \(\Psi.\) Using the Weingarten formula one verifies that on the open set of its regular points, \(\Psi\) has principal curvatures

\[
k_1 = -k_3 = \frac{1}{\sqrt{(a \cos t + c \sin t)^2 + (b \cos t)^2}}, \quad k_2 = 0.
\]

Let \(M^3\) be a 3-dimensional, oriented Riemannian manifold and \(f: M^3 \to \mathbb{S}^4\) an isometric minimal immersion into the unit sphere. Denote by \(\xi\) a unit normal vector field along \(f\) with corresponding shape operator \(A\) and principal curvatures \(k_1 \geq k_2 \geq k_3.\) The Gauss–Kronecker curvature \(K\) and the square \(S\) of the length of the second fundamental form are given, respectively, by

\[
K = k_1k_2k_3, \quad S = k_1^2 + k_2^2 + k_3^2.
\]

Assume now that \(K = 0\) and that the second fundamental form is nowhere zero. Then the principal curvatures are \(k_1 := \lambda, k_2 = 0, k_3 = -\lambda,\) where \(\lambda\) is a smooth positive function on \(M^3.\) We can choose locally an orthonormal frame field \((e_1, e_2, e_3)\) of principal directions corresponding to \(\lambda, 0, -\lambda.\) Let \((\omega_1, \omega_2, \omega_3)\) and \(\omega_{ij}, i, j \in \{1, 2, 3\},\) be the corresponding dual and the connection forms, respectively. Throughout this paper we make the following convention on the ranges of indices

\[
1 \leq i, j, k, \ldots \leq 3
\]

and adopt the method of moving frames. The structural equations are

\[
d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,
\]

\[
d\omega_{ij} = \sum_l \omega_{il} \wedge \omega_{lj} - (1 + k_ik_j)\omega_i \wedge \omega_j.
\]

Consider the functions

\[
u := \omega_{12}(e_3), \quad v := e_2(\log \lambda),
\]

which will play a crucial role in the proof of our result. From the structural equations, and the Codazzi equations,

\[
e_i(k_j) = (k_i - k_j)\omega_{ij}(e_j), \quad i \neq j,
\]

\[
(k_1 - k_2)\omega_{12}(e_3) = (k_2 - k_3)\omega_{23}(e_1) = (k_1 - k_3)\omega_{13}(e_2),
\]
we easily get
\[
\begin{align*}
\omega_{12}(e_1) &= v, \quad \omega_{13}(e_1) = \frac{1}{2}e_3(\log \lambda), \quad \omega_{23}(e_1) = u, \\
\omega_{12}(e_2) &= 0, \quad \omega_{13}(e_2) = -\frac{1}{2}u, \quad \omega_{23}(e_2) = 0, \\
\omega_{12}(e_3) &= u, \quad \omega_{13}(e_3) = -\frac{1}{2}e_1(\log \lambda), \quad \omega_{23}(e_3) = -v
\end{align*}
\]  

and
\[
\begin{align*}
e_2(v) &= v^2 - u^2 + 1, \quad e_1(u) = e_3(v), \quad e_2(u) = 2uv, \quad e_3(u) = -e_1(v).
\end{align*}
\]

Observe that the integral curves of $e_2$ are geodesics of $M^3$ and their images under $f$ are geodesics of $S^4$. Furthermore, from the above equations we get
\[
[e_1, e_3] = -\frac{1}{2}e_3(\log \lambda)e_1 - 2ue_2 + \frac{1}{2}e_1(\log \lambda)e_3.
\]

3. **Proof of the theorem**

Our main result follows from a sequence of lemmas which are themselves of independent interest.

**LEMMA 1.** Under the notation introduced in section 2 we have:

(i) The function $u/\lambda^2$ is constant along the integral curves of $e_2$.

(ii) The functions $u$ and $v$ are harmonic.

**Proof.** (i) By making use of (2.2) and (2.3) we, immediately, obtain $e_2(u/\lambda^2) = 0$.

(ii) From the definition of Laplacian we have
\[
\begin{align*}
\Delta v &= e_1e_1(v) + e_2e_2(v) + e_3e_3(v) \\
&\quad - (\omega_{21}(e_2) + \omega_{31}(e_3))e_1(v) \\
&\quad - (\omega_{12}(e_1) + \omega_{32}(e_3))e_2(v) \\
&\quad - (\omega_{13}(e_1) + \omega_{23}(e_2))e_3(v)
\end{align*}
\]

or, taking into account (2.2),
\[
\begin{align*}
\Delta v &= e_1e_1(v) + e_2e_2(v) + e_3e_3(v) - \frac{1}{2}e_1(\log \lambda)e_1(v) \\
&\quad - 2ve_2(v) - \frac{1}{2}e_3(\log \lambda)e_3(v).
\end{align*}
\]

By virtue of (2.3), we get
\[
\begin{align*}
e_1e_1(v) &= -e_1e_3(u), \quad e_3e_3(v) = e_3e_1(u), \\
e_2e_2(v) &= 2ve_2(v) - 2ue_2(u) = 2v^3 - 6uv^2 + 2v.
\end{align*}
\]

Inserting (3.2) and (3.3) into (3.1), and using (2.4), we obtain
\[
\begin{align*}
\Delta v &= -e_1e_3(u) + e_3e_1(u) + 2v^3 - 6uv^2 + 2v \\
&\quad - \frac{1}{2}e_1(\log \lambda)e_1(v) - 2ve_2(v) - \frac{1}{2}e_3(\log \lambda)e_3(v) \\
&\quad = \frac{1}{2}e_3(\log \lambda)e_1(u) + 2ue_2(u) - \frac{1}{2}e_1(\log \lambda)e_3(u).
\end{align*}
\]
Let
\[ \frac{1}{2} e_1 (\log \lambda) e_1 (v) - 2v e_2 (v) - \frac{1}{2} e_3 (\log \lambda) e_3 (v) + 2v^3 - 6vv^2 + 2v. \]

Appealing to (2.3), we readily see that \( v \) is harmonic. In a similar way, we verify that \( \Delta u = 0 \).

**Lemma 2.** Let \( M^3 \) be a 3-dimensional, oriented, complete Riemannian manifold and \( f: M^3 \to S^4 \) a minimal isometric immersion with Gauss–Kronecker curvature identically zero and nowhere vanishing second fundamental form. Then the function \( u \) is nowhere zero.

**Proof.** Assume in the contrary that there exists a point \( x_0 \in M^3 \) such that \( u(x_0) = 0 \). Let \( \gamma(s), s \in \mathbb{R} \), be the maximal integral curve of \( e_2 \) emanating from the point \( x_0 \), where \( s \) is its arclength. Because of Lemma 1(i), the function \( u(s) := u(\gamma(s)) \) must be everywhere zero. Restricting the first equation of (2.3) along \( \gamma \), we obtain the differential equation 
\[ v'(s) = v^2(s) + 1, \]
where \( v(s) := v(\gamma(s)) \) is an entire function. This is a contradiction because this equation cannot admit entire solutions.

**Lemma 3.** Let \( M^3 \) be a 3-dimensional, oriented, complete Riemannian manifold and \( f: M^3 \to S^4 \) a minimal isometric immersion with Gauss–Kronecker curvature identically zero and nowhere vanishing second fundamental form. Let \( \xi: M^3 \to S^4 \) be the Gauss map; then there exists a 2-dimensional differentiable manifold \( V \), a submersion \( \pi: M^3 \to V \) and a minimal immersion \( \tilde{\xi}: V \to S^4 \), with nowhere vanishing normal curvature, such that \( \tilde{\xi} \circ \pi = \xi \).

**Proof.** Consider the quotient space \( V \) of leaves of \( e_2 \) with quotient map \( \pi: M^3 \to V \). Since \( M^3 \) is complete, the integral curves of \( e_2 \) are complete geodesics and their images through \( f \) are great circles of \( S^4 \). These facts ensure (cf. [6]) that \( V \) can be equipped with a structure of a 2-dimensional differentiable manifold which makes \( \pi \) a submersion. The Gauss map \( \xi \) induces a smooth map \( \tilde{\xi}: V \to S^4 \) so that \( \tilde{\xi} \circ \pi = \xi \). Consider, now, a smooth transversal \( S \) to the leaves of \( e_2 \) through a point \( x \in M^3 \) such that \( e_1 |_x, e_3 |_x \) span \( T_x S \). Because \( \pi \) is submersion, \((d\pi(e_1 |_x), d\pi(e_3 |_x))\) constitutes a base of \( T_{\pi(x)} V \). Note that
\[ d\tilde{\xi}(d\pi(e_1 |_x)) = -\lambda(x) df(e_1 |_x), \quad d\tilde{\xi}(d\pi(e_3 |_x)) = \lambda(x) df(e_3 |_x). \]
Thus \( \tilde{\xi} \) is an immersion and \( X_1 := d\pi(e_1 |_x), X_2 := d\pi(e_3 |_x) \) are orthonormal at \( \pi(x) \) with respect to the metric induced by \( \tilde{\xi} \). Let \( N_1, N_2 \) be an orthonormal frame in the normal bundle of \( \xi \) such that \( N_1 \circ \pi |_S = f |_S, N_2 \circ \pi |_S = df(e_2) |_S \). Using the Gauss formula and (2.2), we get
\[ dN_1(X_1) = \frac{1}{\lambda(x)} df(e_1 |_x), \quad dN_1(X_2) = \frac{1}{\lambda(x)} df(e_3 |_x), \]
\[ dN_2(X_1) = -\frac{v(x)}{\lambda(x)} df(e_1 |_x) + \frac{u(x)}{\lambda(x)} df(e_3 |_x), \]
\[ dN_2(X_2) = -\frac{u(x)}{\lambda(x)} df(e_1 |_x) - \frac{v(x)}{\lambda(x)} df(e_3 |_x). \]
Denote by \( \tilde{A}_1, \tilde{A}_2 \) the shape operators of \( \tilde{\xi} \) at \( \pi(x) \) corresponding to the directions \( N_1 \) and \( N_2 \). Taking into account the above relations, from Weingarten formulas it follows that at \( \pi(x) \) we have
\[ \tilde{A}_1 \sim \frac{1}{\lambda(x)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{A}_2 \sim \frac{1}{\lambda(x)} \begin{pmatrix} -v(x) & -u(x) \\ u(x) & v(x) \end{pmatrix}. \]
with respect to the orthonormal base \((X_1, X_2)\). So the immersion \(\tilde{\xi} : V \to S^4\) is a minimal immersion whose Gaussian curvature \(K\) and normal curvature \(K_n\) are given by

\[
K \circ \pi = 1 - \frac{1 + u^2 + v^2}{\lambda^2}, \quad K_n \circ \pi = \frac{4u^2}{\lambda^4} > 0.
\] (3.4)

This completes the proof.

We shall use in the proof of our theorem a result due to Cheng and Yau [4] that we recall in the following lemma.

**Lemma 4.** Let \(M^n\) be an \(n\)-dimensional, \(n \geq 2\), complete Riemannian manifold with Ricci curvature \(\text{Ric} \geq -(n-1)k^2\), where \(k\) is a positive constant. Suppose that \(h\) is a smooth non-negative function on \(M^n\) satisfying

\[
\Delta h \geq ch^2,
\]

where \(c\) is a positive constant and \(\Delta\) stands for the Laplacian operator. Then \(h\) vanishes identically.

**Proof of the Theorem.** After passing to the universal covering space of \(M^3\), we may suppose that \(M^3\) is simply connected. Since \(M^3\) is simply connected, it is oriented and the standard monodromy argument allows us to define a global orthonormal frame field \((e_1, e_2, e_3)\) of principal directions. Our assumptions imply that \(M^3\) has three distinct principal curvatures \(\lambda, \lambda_0, -\lambda\), where \(\lambda\) is a smooth positive function on \(M^3\). The functions \(u\) and \(v\) are well defined on entire \(M^3\). By virtue of Lemma 2, we may assume that \(u > 0\).

Let \(V\) be the quotient space of leaves of \(e_2\). According to Lemma 3, the immersion \(\tilde{\xi} : V \to S^4\) is minimal with normal curvature nowhere zero. Denote by \(\mathcal{N}\) the unit normal bundle of \(\tilde{\xi}\) in \(S^4\). Then the polar map \(\Psi : \mathcal{N} \to S^4, \Psi(x, w) = w\), is an immersion. Consider the map \(\tau : M^3 \to \mathcal{N}, \tau(x) = (\pi(x), f(x))\). Because \(\Psi \circ \tau = f\) it follows that \(\tau\) is a local isometry and \(\Psi(\mathcal{N}) \equiv f(M^3)\). Hence \(f(M^3)\) is the image of the polar map associated with \(\tilde{\xi}\).

Using (2.3) and the fact that \(u\) and \(v\) are harmonic, we obtain

\[
\frac{1}{2} \Delta((u - 1)^2 + v^2) = |\nabla u|^2 + |\nabla v|^2 \\
\geq (e_2(u))^2 + (e_2(v))^2 \\
= 4u^2v^2 + (v^2 - u^2 + 1)^2 \\
= ((u - 1)^2 + v^2)^2 + 4u((u - 1)^2 + v^2).
\]

Therefore we have

\[
\Delta((u - 1)^2 + v^2) \geq 2((u - 1)^2 + v^2)^2.
\]

In view of our assumptions, the Ricci curvature of \(M^3\) is bounded from below. Appealing to Lemma 4, we deduce that \((u - 1)^2 + v^2\) is identically zero. Consequently \(u \equiv 1\) and \(v \equiv 0\). From (3-4) it follows that

\[
K \circ \pi = 1 - \frac{2}{\lambda^2}, \quad K_n^2 \circ \pi = \frac{4}{\lambda^4}.
\]

So \((1 - K)^2 - K^2_n = 0\) and the isometric immersion \(\tilde{\xi}\) is superminimal.

Suppose now that \(0 < \inf S \leq \sup S < \infty\). At first we will show that \(V\) is complete with respect to the metric \(\langle, \rangle\) induced by \(\tilde{\xi}\). Arguing indirectly, assume that \(V\) is not complete.
Then, there exists a divergent curve \(a: [0, \infty) \to V\) with finite length. Consider a unit vector field \(\eta\) along \(a\), which is normal to \(\tilde{\xi}\) and parallel with respect to the normal connection of \(\tilde{\xi}\). The curve \(c: [0, \infty) \to \mathcal{N}, \ c(t) := (a(t), \eta(t))\), is divergent. Because the metric induced on \(\mathcal{N}\) by \(\Psi\) is complete, \(c\) has infinite length. Moreover, we have

\[
\Psi(c(t)) = \eta(t),
\]

\[
d\Psi(c'(t)) = \frac{\nabla \eta}{dt} = -\tilde{A}_\eta(a'(t)),
\]

where \(\nabla\) stands for the Levi–Civita connection on \(\mathbb{R}^5\) and \(\tilde{A}_\eta\) is the shape operator of \(\tilde{\xi}\) associated with \(\eta\). Because \(\inf S > 0\), we easily see that the length \(\|\tilde{A}_\eta\|\) of \(\tilde{A}_\eta\) satisfies

\[
\|\tilde{A}_\eta\| < \sqrt{2(1 - \mathcal{K})} = 2\sqrt{\frac{2}{S}} \leq 2\sqrt{\frac{2}{\inf S}}.
\]

Then, using the Cauchy–Schwarz inequality, we have

\[
\int_0^\infty |c'(t)|dt = \int_0^\infty |\tilde{A}_\eta(a'(t))|dt \\
\quad \leq \int_0^\infty \|\tilde{A}_\eta\|(a'(t))|dt \\
\quad \leq 2\sqrt{\frac{2}{\inf S}} \int_0^\infty |a'(t)|dt < \infty,
\]

which leads to a contradiction. Hence \(V\) must be complete. Now endow \(V\) with the conformal metric

\[
\langle \cdot, \cdot \rangle = (1 - \mathcal{K})^{\frac{1}{2}} \langle \cdot, \cdot \rangle.
\]

Note that \(\langle \cdot, \cdot \rangle\) is complete, since \(1 - \mathcal{K} \geq 4/\sup S\). The Gaussian curvature \(\tilde{\mathcal{K}}\) of the new metric is given by

\[
\tilde{\mathcal{K}} = \frac{\mathcal{K}}{(1 - \mathcal{K})^{\frac{1}{2}}} - \frac{\Delta \log (1 - \mathcal{K})^{\frac{1}{2}}}{2(1 - \mathcal{K})^{\frac{3}{2}}},
\]

where, here, \(\Delta\) stands for the Laplacian with respect to the metric \(\langle \cdot, \cdot \rangle\). Bearing in mind the equation (2.1), we find

\[
\tilde{\mathcal{K}} = \frac{1}{3(1 - \mathcal{K})^{\frac{1}{2}}} \geq \frac{1}{3} \left(\frac{\inf S}{4}\right)^{\frac{1}{2}}.
\]

Therefore, \(\tilde{\mathcal{K}}\) is bounded away from zero and thus, by Myers’ theorem, \(V\) is compact and \(f(M^3)\) is compact. Furthermore, by a result due to Asperti [2, theorem 1], \(V\) is diffeomorphic to \(S^2\) or to \(\mathbb{R}P^2\) and this completes the proof.

**Remark.** It is worth noticing that the quotient space \(V\) in the Theorem may be non-orientable, although \(M^3\) is orientable. To illustrate this, consider the Veronese surface

\[
g: S^2_{1/3} \to \mathbb{S}^4,
\]

\[
g(x, y, z) = \left(\frac{xy}{\sqrt{3}}, \frac{xz}{\sqrt{3}}, \frac{yz}{\sqrt{3}}, \frac{x^2 - y^2}{2\sqrt{3}}, \frac{x^2 + y^2 - 2z^2}{6}\right),
\]

where \(S^2_{1/3}\) denotes the two-sphere of curvature 1/3. This induces an isometric embedding \(\tilde{g}: \mathbb{R}P^2 \to \mathbb{S}^4\). Then, the unit normal bundle \(\mathcal{N}\) of \(\tilde{g}\) is compact and the polar map \(\Psi: \mathcal{N} \to \mathbb{S}^4\)
provides a minimal isoparametric hypersurface with principal curvatures $\sqrt{3}, 0, -\sqrt{3}$, the so-called Cartan hypersurface. Since the immersion $\Psi$ admits $\tilde{\gamma}$ as global normal vector field, the manifold $\mathcal{N}$ is orientable.

Concluding, we raise the following question: does there exist any complete minimal hypersurface $f: M^3 \to S^4$ with $K = 0$ and $S > 0$ whose Gauss image is not superminimal? Of course, if such an example exists, then its $S$ must be unbounded.

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