CONGRUENCES FOR BROKEN $k$-DIAMOND PARTITIONS

MARIE JAMESON

Abstract. We prove two conjectures of Paule and Radu from their recent paper on broken $k$-diamond partitions.

1. Introduction and Statement of Results

In [1], Paule and Andrews constructed a class of directed graphs called broken $k$-diamonds, and they used them to define $\Delta_k(n)$ to be the number of broken $k$-diamond partitions of $n$. They noted that the generating function for $\Delta_k(n)$ is essentially a modular form. More precisely, if $k \geq 1$, then

$$\sum_{n=0}^{\infty} \Delta_k(n)q^n = q^{(k+1)/12} \frac{\eta(2z)\eta((2k+1)z)}{\eta(z)^3\eta((4k+2)z)},$$

where $\eta(z)$ is Dedekind’s eta function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad (q = e^{2\pi i z}).$$

One can show various congruences for $\Delta_k(n)$ for $n$ in certain arithmetic progressions. For example, Xiong [4] proved congruences for $\Delta_3(n)$ and $\Delta_5(n)$ which had been conjectured by Paule and Radu in [3]. In particular, he showed that

$$\prod_{n=1}^{\infty} (1 - q^n)^4(1 - q^{2n})^6 \equiv 6 \sum_{n=0}^{\infty} \Delta_3(7n+5)q^n \mod 7$$

In this note, we prove the remaining two conjectures in [3]. First, we use (1.2) to prove the following statement (which is denoted Conjecture 3.2 in [3]).

**Theorem 1.1.** For all $n \in \mathbb{N}$, we have that

$$\Delta_3(7^3n+82) \equiv \Delta_3(7^3n+229) \equiv \Delta_3(7^3n+278) \equiv \Delta_3(7^3n+327) \equiv 0 \mod 7.$$

Now, recall that the weight $k$ Eisenstein series (where $k \geq 4$ is even) are given by

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n,$$

where $B_k$ is the $k$th Bernoulli number, and $\sigma_{k-1}(n) := \sum_{d \mid n} d^{k-1}$. Also define

$$\sum_{n=0}^{\infty} c(n)q^n := E_4(2z) \prod_{n=1}^{\infty} (1 - q^n)^8(1 - q^{2n})^2.$$
The coefficients \( c(n) \) are of interest here because they are related to broken \( k \)-diamond partitions in the following way (as conjectured in [3] and proved in [4]):

\[
\tag{1.4} c(n) \equiv 8\Delta_5(11n + 6) \pmod{11}.
\]

Here we prove the last remaining conjecture of Paule and Radu (which is Conjecture 3.4 of [3]). More precisely, we have the following theorem.

**Theorem 1.2.** For every prime \( p \equiv 1 \pmod{4} \), there exists an integer \( y(p) \) such that

\[
c \left( pn + \frac{p - 1}{2} \right) + p^8 c \left( \frac{n - (p - 1)/2}{p} \right) = y(p)c(n)
\]

for all \( n \in \mathbb{N} \).

**Remark 1.** Theorem 1.2 follows from a more technical result (see Theorem 3.1 which is proved in Section 3).

**Remark 2.** As noted in [3], one can combine (1.4) with Theorem 1.2 to see that for every prime \( p \equiv 1 \pmod{4} \) and \( n \in \mathbb{N} \) we have

\[
\Delta_5 \left( (11n + 6)p - \frac{p - 1}{2} \right) + p^8 \Delta_5 \left( \frac{11n + 6}{p} + \frac{p - 1}{2p} \right) \equiv y(p)\Delta_5(11n + 6) \pmod{11}.
\]

To prove Theorems 1.1 and 1.2 we make use of the theory of modular forms. In particular, we shall make use of the \( U \)-operator, Hecke operators, the theory of twists, and a theorem of Sturm. These results are described in [2]. We shall freely assume standard definitions and notation which may be found there.

## 2. Proof of Theorem 1.1

First we consider the form \( \eta(3z)^4 \eta(6z)^6 \). By Theorems 1.64 and 1.65 in [2], we have that \( \eta(3z)^4 \eta(6z)^6 \in S_5 \left( \Gamma_0(72), \frac{1}{2} \right) \). Note from (1.2) that

\[
\eta(3z)^4 \eta(6z)^6 \equiv 6 \sum_{n=0}^{\infty} \Delta_3(7n + 5)q^{3n+2} \pmod{7}.
\]

It follows that

\[
f(z) := \eta(3z)^4 \eta(6z)^6 \mid U_7 \equiv 6 \sum_{n=0}^{\infty} \Delta_3(7^2n + 33)q^{3n+2} \pmod{7}.
\]

Here, \( U_d \) denotes Atkin’s \( U \)-operator, which is defined by

\[
\sum_{n=0}^{\infty} a(n)q^n \mid U_d = \sum_{n=0}^{\infty} a(dn)q^n
\]

for \( d \) a positive integer. By the theory of the \( U \)-operator (see Proposition 2.22 in [2]), it follows that \( f(z) \in S_5 \left( \Gamma_0(504), \frac{1}{2} \right) \). Now if we define \( b(n) \) by

\[
f(z) := \sum_{n=0}^{\infty} b(n)q^n,
\]

then our goal is to show that

\[
b(21n + 5) \equiv b(21n + 14) \equiv b(21n + 17) \equiv b(21n + 20) \equiv 0 \pmod{7}.
\]
In order to prove the desired congruence, consider the Dirichlet character \( \psi \) defined by \( \psi(d) := \left( \frac{d}{7} \right) \), we may consider the \( \psi \)-twist of \( f \), which is given by

\[
f_\psi(z) := \sum_{n=0}^{\infty} \psi(n)b(n)q^n.
\]

By Proposition 2.8 of [2], we have that \( f_\psi(z) \in S_5 \left( \Gamma_0(24696), \left( \frac{-1}{7} \right) \right) \).

Then consider

\[
f(z) - f_\psi(z) = \sum_{n=0}^{\infty} \left( 1 - \left( \frac{n}{7} \right) \right) b(n)q^n \in S_5 \left( \Gamma_0(24696), \left( \frac{-1}{\bullet} \right) \right).
\]

In fact, \( f(z) - f_\psi(z) \equiv 0 \pmod{7} \). This follows from a theorem of Sturm (see Theorem 2.58 in [2]), which states that \( f(z) - f_\psi(z) \equiv 0 \pmod{7} \) if its first 23520 coefficients are 0 \( \pmod{7} \) (which can be checked using a computer). Thus we have that

\[
\left( 1 - \left( \frac{n}{7} \right) \right) b(n) \equiv 0 \pmod{7}
\]

for all \( n \), and thus

\[
b(21n + 5) \equiv b(21n + 14) \equiv b(21n + 17) \equiv b(21n + 20) \equiv 0 \pmod{7}
\]

for all \( n \in \mathbb{N} \), as desired.

3. Proof of Theorem 1.2

3.1 Preliminaries. Let us first recall the Hecke operators and their properties. If \( f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi) \) and \( p \) is prime, the Hecke operator \( T_{p,k,\chi} \) (or simply \( T_p \) if the weight and character are known from context) is defined by

\[
f(z) \mid T_p := \sum_{n=0}^{\infty} \left( a(pn) + \chi(p)p^{k-1}a(n/p) \right) q^n,
\]

where we set \( a(n/p) = 0 \) if \( p \nmid n \). It is important to note that \( f(z) \mid T_p \in M_k(\Gamma_0(N), \chi) \).

In order to prove the final statement of Theorem 1.2, define

\[
g(z) = \sum_{n=0}^{\infty} b(n)q^n := E_4(4z)\eta(2z)^8\eta(4z)^2 \in S_9 \left( \Gamma_0(16), \left( \frac{-4}{\bullet} \right) \right)
\]

and note that \( c(n) = b(2n + 1) \). Thus we wish to show that for every prime \( p \equiv 1 \pmod{4} \) there exists an integer \( y(p) \) such that

\[
b \left( p(2n + 1) \right) + p^8b \left( \frac{2n + 1}{p} \right) = y(p)b(2n + 1)
\]

for all \( n \in \mathbb{N} \). By summing (and noting that \( b(n) = 0 \) when \( n \) is even) we see that this is equivalent to the statement that

\[
g(z) \mid T_p = y(p)g(z).
\]

That is, we need only show that \( g(z) \) is an eigenform of the Hecke operator \( T_p \) for all \( p \equiv 1 \pmod{4} \).
To see this, we let $F$ be the weight 2 Eisenstein series given by
\[ F(z) := \frac{\eta(4z)^8}{\eta(2z)^4} = \sum_{n=0}^{\infty} \sigma_1(2n + 1)q^{2n+1} \in M_2(\Gamma_0(4)), \]
let $\theta_0(z)$ be the theta-function given by
\[ \theta_0(z) := \sum_{n=-\infty}^{\infty} q^{n^2}, \]
and let $h(z)$ be the normalized cusp form
\[ h(z) := \eta(4z)^6 = \sum_{n=1}^{\infty} a(n)q^n = q - 6q^5 + 9q^9 + \cdots \in S_6(\Gamma_0(16), \left(\frac{-4}{\bullet}\right)). \]
Then $h(z)$ is a modular form with complex multiplication (see Section 1.2.2 of [2]), and for prime $p$ we have
\[ a(p) = \begin{cases} 2x^2 - 2y^2 & p = x^2 + y^2 \text{ with } x, y \in \mathbb{Z} \text{ and } x \text{ odd} \\ 0 & p \equiv 2, 3 \pmod{4}. \end{cases} \]
Then we may define $f_1, f_2, f \in S_9(\Gamma_0(16), \left(\frac{-4}{\bullet}\right))$ by
\[ f_1(z) = \sum_{n=0}^{\infty} d_1(n)q^n := E_4(4z)F(z) \left[ 4\theta_0^4(4z) - \theta_0^6(2z) + 4\theta_0^4(2z)\theta_0^2(4z) - 6\theta_0^2(2z)\theta_0^4(4z) \right] \]
\[ f_2(z) = \sum_{n=0}^{\infty} d_2(n)q^n := E_4(4z)F(2z)h(z) \]
\[ f(z) = \sum_{n=0}^{\infty} d(n)q^n := f_1(z) + 8i\sqrt{3}f_2(z). \]
We prove the following theorem involving these forms.

**Theorem 3.1.** The forms $f(z)$ and $\overline{f(z)}$ are eigenforms of the Hecke operator $T_p$ for all primes $p$. Furthermore we have that
\[ \mathbb{T}_g = \langle f, \overline{f} \rangle, \]
where $\mathbb{T}_g$ is the subspace of $S_9(\Gamma_0(16), \left(\frac{-4}{\bullet}\right))$ spanned by $g$ together with $g \mid T_p$ for all primes $p$.

**Proof.** First note that $f$ and $\overline{f}$ are eigenforms of the Hecke operator $T_p$ for all primes $p$. To see this, note that there is a basis of Hecke eigenforms of the space $S_9(\Gamma_0(16), \left(\frac{-4}{\bullet}\right))$. Also, both $f$ and $\overline{f}$ are eigenforms of $T_5$ with eigenvalue 258, one can compute that this eigenspace $\ker(T_5 - 258)$ is 2-dimensional. Finally, both $f$ and $\overline{f}$ are eigenforms of the Hecke operator $T_7$, and they have different eigenvalues.

Now, note that
\[ g = \left(\frac{1}{2} + \frac{i}{2\sqrt{3}}\right)f + \left(\frac{1}{2} - \frac{i}{2\sqrt{3}}\right)\overline{f} \]
and thus $T_g$ is a two-dimensional subspace of $\langle f, \overline{f} \rangle$. Thus $T_g = \langle f, \overline{f} \rangle$, as desired. 

3.2. **Proof of Theorem 1.2.** Suppose $p$ is a prime with $p \equiv 1 \pmod{4}$. Then we need only check that $f$ and $\overline{f}$ are eigenforms of $T_p$ with the same eigenvalue. Since these eigenvalues are the coefficients of $q^p$ in the expansions of $f$ and $\overline{f}$, we need only show that
\[ d(p) = \overline{d(p)}, \]
i.e., $d(p) \in \mathbb{R}$.

Now, note that $d_2(p) = 0$ since the coefficients of $f_2$ are only supported on indices that are congruent to 3 mod 4 by the descriptions of $E_4(4z), F(2z)$, and $h(z)$ given above. Thus $d(p) = \overline{d_1(p)} \in \mathbb{R}$, as desired.

**References**

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Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322

E-mail address: mjames7@emory.edu