Elasticity index evaluation based on Le Cam divergence and kernel density estimator in PM space

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Abstract. This work focuses on the application of Preisach-Mayergoyz (PM) model applied to the evaluation of elastic properties of hysteretic material. In the first part, essential concepts are explained such as material hysteresis, probability density functions of PM space, optimization algorithm used, and phi-divergence measures applied in the PM identification process. Next, two different characterizations of PM space of hysterons based on kernel density estimators are proposed, either for one-dimensional PM space projection or for fully two-dimensional pyramid kernel. Finally, new index of elasticity is built up by means of the squared Le Cam divergence between probability density corresponding to the super-elastic distribution and the probability density found by our optimized PM identification process, respectively. This elasticity index describes ability of the material to absorb mechanical deformation, or alternatively, it gives an evidence about the certain degree of damage of the material. This proposed index of elasticity (IE) is evaluated for the case of experimental data measured on the steel dampers used for the protection against earthquakes.

1. PM space model characterization
In this paper, the Preisach-Mayergoyz (PM) model [1] of hysteresis is introduced as a tool for describing hysteretic material and its elastic structure. This approach is mainly associated with the nondestructive testing (NDT) defectoscopy technologies [2]. The PM space model is based on the idea that a given material is composed of a large number of small elastic units (particles, ferromagnetic elements, cracks, etc.) called here hysterons, Figure 1a). The Preisach’s operator

![Figure 1. (a) Hysterons can be found either in closed state assigned to +1 or open state assigned to -1 (b) Profile of earthquake damper I-section [3]](image)
\( \hat{\gamma}_{\alpha,\beta} \) of hysteron is mathematically expressed as

\[
\hat{\gamma}_{\alpha,\beta}(u(t)) = \begin{cases} 
-1, & u(t) \leq \beta, \\
1, & u(t) \geq \alpha, \\
k, & u(t) \in (\beta, \alpha),
\end{cases}
\]

whith \( \beta \leq \alpha \) and \( u(t) \in (\beta, \alpha) \) as an input signal, where

\[
k = \begin{cases} 
1, & \exists t^*: u(t^*) > \alpha \text{ and } \forall \tau \in (t^*, t), \\
-1, & \exists t^*: u(t^*) < \beta \text{ and } \forall \tau \in (t^*, t).
\end{cases}
\]

Our goal is the evaluation of elasticity of dissipative dampers, which are used for building protection against earthquakes. The principle instrument of this passive protection can be seen in Figure 1b) and the series of these energy dissipating devices (EDDs) are used as the web plastifying damper (WPD). The PM based method of fatigue evaluation of these dampers is the crucial point in health monitoring of nowadays architecture objects in endangered countries. In case of the dampers, \( \alpha \) and \( \beta \) represent closing (\( P_c \)) and opening (\( P_o \)) values of damper’s plastifying cells with respect to the form of loading (e.g. tensile force/pressure). Applying the input signal \( u(t) \), the output of hysteretic system, denoted by \( v(t) \), can be described in the continuous case as a double integral over the PM space

\[
v(t) = \int_{\beta \leq \alpha} \mu(\alpha, \beta) \hat{\gamma}_{\alpha,\beta}(u(t)) \, d\alpha \, d\beta,
\]

where \( \mu(\alpha, \beta) \) is a probability density on the Preisach triangle \( \beta \leq \alpha \), see Fig. 2.

![Figure 2. Schema of Preisach-Mayergoyz model [1]](image-url)
2. PM space density identification
The main objective of PM space modelling is the identification of corresponding probability density function of hysterons $\mu(P_c, P_o)$ in PM space only from the knowledge of the input load and the corresponding hysteresis curve of the damper under testing load. We applied the standard statistical distributions as Gaussian, Exponential, Weibull, Uniform, but also Guyer’s distributions [4, 5]. As an example, we present here only the third distributions of Guyer and Koen. PM distribution Guyer 3 is defined by

$$P_c = \max \cdot r_c^\alpha, \quad P_o = P_c \cdot (\gamma \times r_o) ^\beta, \quad \alpha, \beta \in \mathbb{R}^+, \quad \gamma \in (0, 1),$$

where ‘max’ is the maximum of input load, $P_c$ and $P_o$ are closing and opening values, $r_c$ and $r_o$ are random numbers uniformly distributed in interval $(0, 1)$, while $\alpha, \beta, \gamma$ represent free parameters of the distribution. Further, PM distribution Koen can be expressed as

$$P_c = \max \cdot r_c, \quad P_o = (P_c/\alpha)^\beta \cdot r_o, \quad \alpha, \beta \in \mathbb{R}_0^+,$$

where $\alpha, \beta$ represent free distributional parameters again. Notice that the Koen distribution uses quite different parametrization compared to Guyer 3. To identify potentially complex structure of PM spaces of a given material/damper, we apply statistical theory of distribution mixtures. It means, we seek for the best convex combination of $M$ density components $p_i$ in the form

$$p(x|\Theta) = \sum_{i=1}^{M} \lambda_i p_i(x|\theta_i), \quad \sum_{i=1}^{M} \lambda_i = 1,$$

where $p_i(x|\theta_i)$ are probability density functions with parameters $\theta_i$ and the component weights $\lambda_i > 0$ for all $i \in \{1, ..., M\}$.

3. Jaya numerical optimization technique
The optimization problem consists of minimizing distance measure between the calculated hysteresis curve and the observed hysteresis curve in each iteration step. We applied classical $L^2$-distance or Hellinger and Le Cam $\phi$-divergences, which are more robust against measurement errors. We focus here on the Le Cam divergence

$$LC^2(p, q) = \int \frac{(p - q)^2}{p + q} \, d\mu,$$

where $p$ and $q$ are probability densities with respect to a $\sigma$-finite measure $\mu$ on $\mathbb{R}$. $LC$ is a special case of the general information-theoretic measure called $\phi$-divergence between $p$ and $q$,

$$D_{\phi}(p, q) = \int q \phi(p/q) \, d\mu,$$

for convex function $\phi(t) \in [0, \infty)$, $\phi(1) = 0$, strictly convex at $t = 1$ (details in [6, 7]).

The identification process of PM density is carried out by choosing a novel numerical minimization meta-heuristic algorithm called Jaya presented by Rao [8] in 2016. It is based on Jaya operator applied in each iteration to the argument $x = (x_1, x_2, ..., x_D)$ of the minimized functional $f(x)$. We present the pseudo-code of this Jaya algorithm below. Performance of the Jaya algorithm was tested many times and it surpassed the well-known Simulated Annealing algorithm in most cases of our PM density identification task.
Algorithm 1: Pseudo-code for Jaya numerical minimization algorithm (random based)

1. Generate initial population, evaluate $f(x)$
2. Choose the best ($x_{best}$) and the worst ($x_{worst}$) solution in the population
3. while Stopping rule do
   4. for $i = 1$ to $NP$ do // $NP$ number of populations
      5. for $j = 1$ to $D$ do // $D$ dimension of vector $x$
         6. $u_{i,j} = x_{i,j} + rand_1,j \cdot (x_{best,j} - |x_{i,j}|) - rand_2,j \cdot (x_{worst,j} - |x_{i,j}|)$ // Jaya op.
      7. end
   8. Evaluate $f(x)$ in $u_i$
   9. if $f(u_i) \leq f(x_i)$ then
      10. $x_i = u_i$
   11. else
      12. $x_i = x_i$
   13. end
   14. end
   15. Actualize $x_{best}$ and $x_{worst}$
16. end

4. Elasticity index of hysteretic dampers

We have at disposal the signals obtained from 6 vibration tests of the web plastifying dampers measured at the University of Granada by prof. A. Gallego research group. The goal is to propose and evaluate a new index of damage/elasticity [3]. Our new procedure is the following. First, the numerical PM space density automatic identification for all 6 damper cycles is proceed, then the PM points are generated. Second, these points from PM triangle are projected onto the right leg of PM space, followed by the one-dimensional nonparametric kernel estimator evaluated for optimally chosen bandwidth under Gaussian kernel. The scheme of this evaluation process is shown in Figure 3. Alternatively, we have used fully two-dimensional kernel estimator on Preisach triangle for the specific triangle binning leading to pyramid kernel. These designs of 1D-projection and 2D-triangle binning is based on the knowledge of PM space of ideally elastic non-hysteretic material when all the PM points are located at the diagonal of PM space and these hysterons are moving to the bottom leg of the PM triangle as damage level increases.

The final PM space Jaya-Le Cam identifications for the 3 first cycles of earthquake EDD damper under increasing damage are shown in Figure 4 for 1000 hysterons (plastifying cells).
Consequently, for all 6 damper loading cycles, the indexes of elasticity/damage $IE$ were computed as appropriately scaled and normalized Le Cam divergence between the identified PM density kernel estimate against referential ideally elastic PM density, i.e.

$$IE = \frac{(LC \cdot 100)^{1 + \frac{LC}{100}}}{IE_{max}} \in [0, 1]$$

where $IE_{max}$ is the maximal value of $IE$ for absolutely nonelastic PM space. These relative $IE$ indexes for successive 6 testing cycles of earthquake damper are:

| Test # | #1    | #2    | #3    | #4    | #5    | #6    |
|--------|-------|-------|-------|-------|-------|-------|
| $IE$   | 0.538 | 0.872 | 0.899 | 0.908 | 0.91  | 0.915 |

The greatest increase of the proposed PM elasticity/damage index occurred during the first cyclic loading, afterwards growing gradually up to 0.915 referring to significant damage of this earthquake damper and developing disruption of its elasticity (plasticity).

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