CONSTRUCTIONS OF SUBSHIFTS WITH POSITIVE TOPOLOGICAL ENTROPY DIMENSION

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ABSTRACT. The notion of entropy dimension has been introduced to measure the subexponential complexity of zero entropy systems. In this work we present a general construction of a strictly ergodic subshift of topological entropy dimension \( \alpha \) for each \( \alpha \in (0,1) \). It is shown that the system satisfies some sort of regularity in the size of atoms and the first return time. Moreover, we modify the construction to obtain a variant system that is weakly mixing.

1. INTRODUCTION

In 1958, Kolmogorov introduced the notion of entropy to dynamical systems generalizing the Shannon entropy in information theory \([20]\). Entropy measures the chaoticity of a dynamical system. In the case of a \( \mathbb{Z} \)-action, past is well defined and hence we say entropy measures the (un)predictability of a system knowing the complete past. If it has entropy zero, then we say its past determines the future. Entropy is an isomorphism invariant and is a complete invariant in the class of Bernoulli actions \([10]\). Systems of positive entropy have been studied for several decades and many of their properties are well understood at least in the case of \( \mathbb{Z} \)-actions along with their applications. The entropy theory was extended to general amenable group actions \([17]\) and recently also to nonamenable group actions \([1, 2]\).

In the study of general group actions, entropy zero systems arise rather naturally. If a general group action has a noncocompact subgroup action of finite entropy, then it is easy to see that its entropy is zero. However their subgroup actions exhibit interesting dynamics with diverse properties. To investigate the properties of entropy zero actions, directional (subgroup) properties have been explored in many different directions \([3, 11]\). In general, a zero entropy \( \mathbb{Z}^2 \)-action may have mixture of positive, infinite and zero directional entropies. We note that if one of the directions has positive entropy, then it has exponential growth rate of orbits. Clearly different complexities will give rise to different behaviors of their subgroup actions, and the complexities of general group actions together with their subgroup actions should be explored to understand the dynamics of bigger group actions of entropy zero \([19]\).
Entropy zero systems make up a dense $G_\delta$ subset of all dynamical systems. However much less is known about the properties of entropy zero dynamical systems. Systems of very small complexity, like group rotations, interval exchange maps, substitution systems and Chacon transformations have been studied and they are known to have polynomial growth rate of orbits. Some of their properties have been investigated by many authors [8, 9, 12]. Recently more examples like Pascal adic transformations [15] and nilpotent group actions [14] were investigated and shown to have polynomial growth rate.

Motivated by the study of entropy zero systems of bigger group actions, we want to investigate the properties of entropy zero $\mathbb{Z}$-actions whose orbit growth rate is subexponential. Only a few systems of entropy zero are known to have intermediate growth rate which is strictly greater than polynomial, but less than exponential [10]. To analyze the complexity of these entropy zero systems, metric and topological entropy dimension were first introduced in [11] and their properties are investigated in [12] and [10]. Positive entropy systems have exponential growth rate, hence they have entropy dimension 1, while those of polynomial orbit growth rate have entropy dimension 0. In this article we present a constructive method to build a strictly ergodic subshift of topological entropy dimension $\alpha$ for any $\alpha \in (0,1)$. Roughly speaking, the number of $n$-blocks in the subshift is in the order of $\exp(n^\alpha)$.

For ergodic systems of positive entropy we have the Shannon-McMillan-Breiman theorem which is called the equipartition property [17, 21]. The Ornstein-Weiss theorem on the return time also holds for those systems [18]. However for zero entropy systems little is known about these properties except for irrational rotations [13]. Through our example, we investigate the equipartition and return time properties for the systems of subexponential growth rate. We show that it does not have the equipartition property, that is, the size of an atom of $\bigvee_{i=0}^{n-1} T^{-i} P$, where $P$ is a generator for the system, is not necessarily in the order of $\exp(-n^\alpha)$. It is not known yet whether there are systems of zero entropy with subexponential growth rate and with corresponding equipartition property.

The outline of the article is as follows. Section 2 presents the definitions of the entropy dimension and necessary terminology. In Section 3 we construct strictly ergodic systems with given entropy dimensions. In Section 4 we see that our examples do not have the equipartition property. However we will show that they exhibit some ‘regularity’ in their sizes of atoms and return times. In Section 5 we discuss some other aspects of the systems like rigidity and weakly mixing property.

2. Background

We introduce some terminology and known results. For more details on topological entropy dimension, see [7].
Let \((X, T)\) be a topological dynamical system. If \(\mathcal{U}\) and \(\mathcal{V}\) are open covers of \(X\), let \(\mathcal{U} \lor \mathcal{V} = \{ U \cap V : U \in \mathcal{U}, V \in \mathcal{V} \}\). Also denote by \(\mathcal{N}(\mathcal{U})\) the cardinality of a smallest subcover of \(\mathcal{U}\).

Given an open cover \(\mathcal{U}\) of \(X\) and \(\alpha > 0\), define
\[
D(T, \alpha, \mathcal{U}) = \limsup_{n \to \infty} \frac{\log \mathcal{N}(\bigsqcup_{i=0}^{n-1} T^{-i}\mathcal{U})}{n^\alpha}.
\]
Then the function \(D(T, \alpha, \mathcal{U})\) on the parameter \(\alpha\) has a unique critical value, say \(D(T, \mathcal{U})\), in \([0, \infty]\), i.e., \(D(T, \alpha, \mathcal{U}) = 0\) for \(\alpha > D(T, \mathcal{U})\) and \(D(T, \alpha, \mathcal{U}) = \infty\) for \(\alpha < D(T, \mathcal{U})\). The (topological) upper entropy dimension \(D(X, T)\) of \((X, T)\) is defined by the supremum of \(D(T, \mathcal{U})\) for all finite open covers \(\mathcal{U}\) of \(X\).

It is known that \(D(X, T) = D(T, \mathcal{U})\) if \(\mathcal{U}\) is a generating open cover. Similarly, the (topological) lower entropy dimension \(\underline{D}(X, T)\) of \((X, T)\) is defined by using \(\liminf\) instead of \(\limsup\). If \(\underline{D}(X, T) = D(X, T)\), we denote it by \(D(X, T)\) and call it the (topological) entropy dimension of \((X, T)\). They are invariant under topological conjugacy.

The notion of entropy dimension is simplified when a dynamical system is a subshift. A subshift (or shift space) is a closed \(\sigma\)-invariant subset of a full shift over some finite set \(A\) of symbols. For a subshift \(X\), denote by \(B_n(X)\) the set of all words of length \(n\) appearing in the points of \(X\) and \(\mathcal{B}(X) = \bigcup_{n \geq 0} B_n(X)\). For a subshift \(X\), since there is a natural 0-th coordinate clopen partition, the topological upper entropy dimension is a unique critical value of the function \(\overline{D}(T, \alpha)\), where
\[
\overline{D}(T, \alpha) = \limsup_{n \to \infty} \frac{\log |B_n(X)|}{n^\alpha},
\]
and the lower entropy dimension is defined analogously.

3. A construction of a subshift with given entropy dimension

In this section, we present a general construction of a topological dynamical system with positive topological entropy dimension. The constructed system is a subshift over the alphabet \(\{0, 1\}\). We first describe a construction of a subshift with entropy dimension \(1/2\) to make the argument readable. A system of topological entropy dimension \(\alpha\), for any \(0 < \alpha < 1\), can be constructed similarly. In what follows, to simplify the notation we often omit the floor function notation on the square roots and write \(\sqrt{N}\) instead of \([\sqrt{N}]\). The omissions will be clear from the context.

It is well known that for all sufficiently large \(n\), we have
\[
(3.1) \quad n \log n - n < \log n! < n \log n.
\]
Denote by \(\nu P_k\) the number of \(k\)-permutations of \(n\). We note that
\[
(3.2) \quad k \log n - k < \log \nu P_k \leq \log n^k = k \log n,
\]
for all sufficiently large $n$ and any $k$, $1 \leq k \leq n$. We write $a(n) \sim b(n)$ if the ratio $a(n)/b(n)$ goes to 1 as $n \to \infty$. From $(3.1)$ and $(3.2)$, we respectively obtain $\log n! \sim n \log n$ and $\log P_k \sim k \log n$ as $n \to \infty$.

Fix a large square number $l_1 \in \mathbb{N}$. Let $C_1$ be a set of binary words of length $l_1$ with cardinality $N_1 = |C_1| = 2\sqrt{n}$. By taking $l_1$ large enough, we may assume that $(3.1)$ and $(3.2)$ hold for all $n \geq N_1$.

For the induction step, suppose that a set $C_j$ of words of length $l_j$ has been constructed with cardinality $N_j = |C_j|$. Give an ordering on $C_j$ and write $C_j = \{u_{i,j}^{(j)} : 1 \leq i \leq N_j\}$. Consider a new word $u_{1,j+1}^{(j+1)}$ formed by concatenating all the words in $C_j$ in order:

$$u_{1,j+1}^{(j+1)} = u_{1}^{(j)} u_{2}^{(j)} u_{3}^{(j)} u_{4}^{(j)} u_{5}^{(j)} \ldots u_{8}^{(j)} u_{9}^{(j)} u_{10}^{(j)} \ldots u_{N_j}^{(j)}.$$

Put $P_j = \{2\} \cup \{i^2 : 2 \leq i \leq \sqrt{N_j}\}$ (We remark that the insertion of 2 in $P_j$ is only for notational convenience). The collection $C_{j+1}$ consists of those words of length $l_{j+1} = l_j \cdot N_j$ obtained by permuting the subwords $u_{i,j}^{(j)}$ of $u_{1,j+1}^{(j+1)}$ for $i \in P_j$ and leaving the others fixed. That is, a typical element in $C_{j+1}$ is of the form

$$u_{1,j+1}^{(j)} u_{x(2)}^{(j)} u_{3}^{(j)} u_{x(4)}^{(j)} u_{5}^{(j)} \ldots u_{x(8)}^{(j)} u_{x(9)}^{(j)} u_{10}^{(j)} \ldots u_{N_j}^{(j)},$$

where $\pi$ is a permutation on the set $P_j$. We call a word $u_{i,j}^{(j)}$ in $C_j$ permuted (or unstable) if $i \in P_j$ and unpermuted (or stable) otherwise.

Note that the cardinality $N_j$ of $C_j$ and the length $l_j$ of the words in $C_j$ satisfy the iterative formulas:

$$(3.3) \quad N_{j+1} = (\sqrt{N_j})! \quad \text{and} \quad l_{j+1} = l_j \cdot N_j \quad \text{for} \quad j \geq 1.$$

Since $u_{i,j}^{(j)}$ is a prefix of $u_{1,j+1}^{(j+1)}$ for each $j$, there is a unique limit point $w \in \{0, 1\}^\mathbb{N}$ of the sequence $\{u_{i,j}^{(j)}\}_{j \in \mathbb{N}}$. Let $X^+$ be the orbit closure of $w$ and $X$ the inverse limit of $X^+$. Equivalently, we may let $X$ be the set of all bi-infinite sequences over $\{0, 1\}$ each word of which is a subword of a word in $C_j$ for some $j \in \mathbb{N}$.

Since each word in $C_j$ occurs in every word in $C_{j+1}$, every word in $X$ occurs syndetically in $X$, so it follows that $X$ is minimal. Also any word $w \in C_j$, $j \in \mathbb{N}$, occurs exactly once in $C_{j+1}$ (with respect to $\sigma^j$), hence an irreducible component of $\sigma^j(X)$ is uniquely ergodic. As in Lemma [10, Lemma 1.9], we see that $X$ is uniquely ergodic. It follows that $X$ is strictly ergodic.

**Lemma 3.1.** Let $N_j$ and $l_j$ be as in the above. Then

$$\lim_{j \to \infty} \frac{\log N_j}{\sqrt{l_j}} = 0 \quad \text{and} \quad \lim_{j \to \infty} \frac{\log N_j}{l_j^\beta} = \infty \quad \text{for any} \quad \beta < 1/2.$$

**Proof.** It suffices to prove the lemma only when each $N_j$ is a square number. We use the mathematical induction together with $(3.1)$ and $(3.3)$ to see that

$$\log N_j \leq \frac{\log 2}{2^{j-1}} \sqrt{l_j}.$$
The first equality of the lemma is immediate from this. Once again we apply the mathematical induction and use (3.1) and (3.3) to obtain

$$\log N_j \geq \frac{\log 2}{2^{j-1}} \sqrt{l_j} - \frac{1}{2^{j-2}} \sqrt{l_j} - \frac{1}{2^{j-3}} \sqrt{l_j} - \cdots - \frac{\sqrt{l_j}}{\sqrt{l_{j-1}}}.$$ 

This yields the second equality of the lemma. \(\square\)

We now show that the constructed system \(X\) has topological entropy dimension \(1/2\). In the remaining part of this section we assume that each \(N_j\) is a square number to simplify the argument. Let \(l \in \mathbb{N}\). There is \(j \in \mathbb{N}\) such that \(l_j \leq l < l_{j+1} = l_j \cdot N_j\). Without loss of generality, we may assume that \(l = k \cdot l_j\) for \(1 \leq k < N_j\). Then by considering the number of the length \(l\) prefixes of the words in \(C_j\) and using (3.2), we have

$$\frac{1}{\sqrt{l}} \log |B_l(X)| \geq \frac{1}{\sqrt{kl_j}} \log (\sqrt{N_j^{P_{\sqrt{k}}}})$$

$$\geq \frac{1}{\sqrt{kl_j}} (\lfloor \sqrt{k} \rfloor \log \sqrt{N_j} - \lfloor \sqrt{k} \rfloor)$$

$$\geq \frac{1}{4\sqrt{l_j}} \log N_j - \frac{1}{\sqrt{l_j}}.$$ 

(3.4)

On the other hand, for each \(u \in B_l(X)\), \(u\) is a subword of a word \(B_1 \cdots B_{k+1}\) formed by concatenating \(k+1\) words in \(C_j\), and the maximal number of permuted positions in the \(j\)-th level is \(\lfloor \sqrt{k+1} \rfloor\). Hence we have

$$\frac{1}{\sqrt{l}} \log |B_l(X)| \leq \frac{1}{\sqrt{kl_j}} \log (l_{j+1} \cdot \sqrt{N_j^{P_{\sqrt{k+1}}}})$$

$$\leq \frac{1}{\sqrt{kl_j}} \log l_{j+1} + \frac{1}{\sqrt{kl_j}} \log \sqrt{N_j^{\lfloor \sqrt{k+1} \rfloor}}$$

$$\leq \frac{1}{\sqrt{kl_j}} \log l_j + \frac{1}{2\sqrt{l_j}} \left( \frac{1}{\sqrt{k}} + \frac{\sqrt{k+1}}{\sqrt{k}} \right) \log N_j$$

$$\leq \frac{1}{\sqrt{kl_j}} \log l_j + \frac{3}{2\sqrt{l_j}} \log N_j.$$ 

(3.5)

From (3.4), (3.5) and the first equality of Lemma 3.1 it follows that

$$\lim_{l \to \infty} \frac{1}{\sqrt{l}} \log |B_l(X)| = 0.$$ 

A similar calculation together with the second equality of Lemma 3.1 yields that

$$\lim_{l \to \infty} \frac{1}{l^\beta} \log |B_l(X)| = \infty$$

for every \(\beta < 1/2\). This proves that the system \(X\) has topological entropy dimension \(1/2\).

In order to construct a system with topological entropy dimension \(\alpha\) for given \(\alpha \in (0, 1)\), we let \(C_{j+1}\) consist of the words obtained by permuting \(C_j\)-words of
\[ u_1^{(j+1)} \text{ at positions } i^{(1/\alpha)} \text{ for } 1 < i \leq (N_j)^\alpha. \] If \( l_1 \) and \( N_1 \) are large enough, then the proof goes as in the case \( \alpha = 1/2 \).

**Theorem 3.2.** Let \( \alpha \in (0, 1) \). There is a strictly ergodic subshift with entropy dimension \( \alpha \).

For each ordering on \( C_j, j \in \mathbb{N} \), we obtain a different strictly ergodic system. The subshifts obtained in this way are in general not topologically conjugate. However, they are all measure theoretically isomorphic.

4. **Size of atoms and return time**

In this section we consider a variant of the subshift constructed in Section 3 as follows: Let \( C_1 \) be a set of binary words of length \( l_1 \). We assume that all members of \( C_1 \) start with 001, and 00 is not a subword of any element of \( C_1 \) but a prefix. The rest of the construction is the same as the one in Section 3. The word 001 serves as a marker, and the adjustment is adopted to control the return time of the words occurring in the constructed system \( X \).

We will look into the size of atoms of the iterated partition and the first return property of the subshift \( X \). It is well known that the Shannon-McMillan-Breiman theorem and the Ornstein-Weiss return time property hold for the systems of positive entropy. Also almost every irrational rotation map on a circle has the analogous properties \([13]\). Unfortunately, it is not the case for our system. However, it exhibits some ‘regularity’ in the size of atoms and the first return time.

Let \( \mu \) be the unique \( \sigma \)-invariant measure on \( X \). As usual, for \( u \in B(X) \), denote by \( \mu(u) \) the measure of the cylinder \( [u] = \{ x \in X : x_{[0,|u|]} = u \} \). Since \( X \) is uniquely ergodic, \( \mu(u) = \mu([u]) \) is equal to the limit of the relative frequency of \( u \) in \( w_{[1,n]} \), where \( w \) is the unique limit point obtained in the process of constructing \( X \).

For each \( x \in X \), denote by \( P_n(x) \) the \( n \)-cylinder \( [x_0 \cdots x_{n-1}] = \{ y \in X : y_i = x_i \text{ for } 0 \leq i < n \} \) of \( x \). Also denote by \( R_n(x) \) the first return time of \( x \) to the \( n \)-cylinder containing \( x \), i.e., \( R_n(x) = \inf \{ k > 0 : x_{[k,k+n]} = x_{[0,n]} \} \). Since \( X \) is minimal, we have \( R_n(x) < \infty \) for each \( x \in X \) and \( n \in \mathbb{N} \).

The first result states that the subexponential growth rate of the measure of \( P_n(x) \) can be any number between 0 and the topological entropy dimension 1/2 (Theorem 4.4). We begin with simple lemmas.

**Lemma 4.1.** Let \( x \in X \) and \( j \in \mathbb{N} \). If \( x_{[0,l_j]} \) is a word in \( C_j \), then each return time of \( x \) to \( P_{l_j}(x) \) is a multiple of \( l_j \). In particular, \( R_{l_j}(x) \) is a multiple of \( l_j \).

**Proof.** For \( j = 1 \), \( x_{[0,l_1]} \) begins with the marker 001. Since 00 does not occur in a word of \( C_1 \) except at its prefix, the word 001 cannot occur in the middle of \( CC^n \in C_1C_1 \). It follows that each return time of \( x \) to \( P_{l_1}(x) \) is a multiple of \( l_1 \).

Suppose that the claim holds for \( j - 1 \). If \( x_{[0,l_j]} \) is in \( C_j \), then it begins with \( v = u_1^{(j-1)} \in C_{j-1}. \) Let \( m \) be a return time of \( x \) to \( P_{l_j}(x) \). Then by the induction hypothesis, \( m \) is a multiple of \( l_{j-1} \). However, all the words in \( C_{j-1} \) other than
v are different from v, hence the first coordinate of v should occur in \(x_{(1, \infty)}\) at positions which are multiples of \(l_j\). It follows that \(m\) also should be a multiple of \(l_j\). □

**Lemma 4.2.** For each \(x \in X\) and \(j \in \mathbb{N}\), there is a unique decomposition of \(x\) into \(C_j\) words.

*Proof.* The existence follows from compactness argument and the uniqueness follows from the existence of the marker word 001 and the fact that \(u_1^{(j-1)}\), which is the first member of \(C_{j-1}\), is the prefix of each word of \(C_j\) for \(j \in \mathbb{N}\). □

**Lemma 4.3.** Let \(C_1, \ldots, C_p\) be unpermuted words in \(C_j\). Suppose that \(C_1 \cdots C_p\) be a subword of some element of \(C_{j+1}\). Then we have \(\mu(C_1) = \mu(C_1 \cdots C_p)\).

*Proof.* By Lemma 4.1, in \(w\) both \(C_1 \) and \(C_1 \cdots C_p\) occur at the positions which are multiples of \(l_j\). The result follows from the fact that the relative frequencies of \(C_1\) and \(C_1 \cdots C_p\) in \(w\) are the same. □

We now present the results on the growth rate of the size of atoms.

**Theorem 4.4.** Let \(X\) be the subshift with topological entropy dimension \(1/2\) constructed in this section. Then the following hold.

1. For \(\mu\)-a.e. \(x \in X\),
   \[
   \limsup_{n \to \infty} -\frac{1}{n^\beta} \log \mu(P_n(x))
   \]
   has the critical value \(1/2\).

2. There is a set \(\hat{X} \subset X\) of full measure with the property that for any \(\tau \in [0, 1/2]\), there is an increasing sequence \(\{n_j\}_{j \in \mathbb{N}}\) such that for each \(x \in \hat{X}\),
   \[
   \lim_{j \to \infty} -\frac{1}{(n_j)^\beta} \log \mu(P_{n_j}(x))
   \]
   has the critical value \(\tau\).

*Proof.* Let \(x \in X\). We first show that \(\limsup_{n \to \infty} -\frac{1}{n^\beta} \log \mu(P_n(x))\) has the critical value not greater than \(1/2\). By Lemma 4.2 there is a unique decomposition of \(x\) into a concatenation of words of \(C_j\) for each \(j \in \mathbb{N}\). We may write \(x = \cdots B_{-1}^{(j)}B_0^{(j)}B_1^{(j)}B_2^{(j)} \cdots\), where \(B_i^{(j)} \in C_j\) for each \(i \in \mathbb{Z}\) and \(B_0^{(j)}\) is the unique word of \(C_j\) containing the 0-th coordinate of \(x\). Note that \(B_0^{(j)}\) is a subword of \(B_0^{(j+1)}\) for each \(j\).

For each \(n \in \mathbb{N}\), there is a maximal \(j\) such that \(P_n(x)\) is a subword of \(B_0^{(j)} \cdots B_{k-1}^{(j)}\) with \(k \geq 3\) and \(B_1^{(j)} \cdots B_{k-2}^{(j)}\) is a subword of \(P_n(x)\). By the choice of \(j\) we also have \(k \leq 2N_j\). Then since the maximal number of permuted words
in $C_j$ occurring in $B_0^{(j)} \cdots B_{k-1}^{(j)}$ is $\sqrt{k}$, we have

$$\mu(P_n(x)) \geq \mu(B_0^{(j)}B_1^{(j)} \cdots B_{k-1}^{(j)})$$

$$= \lim_{l \to \infty} \frac{1}{l} \cdot \text{number of occurrences of } B_0^{(j)}B_1^{(j)} \cdots B_{k-1}^{(j)} \text{ in } w_{[1,l]}$$

$$\geq \frac{(\sqrt{N_j} - \sqrt{k})!}{l_{j+2}} = \frac{(\sqrt{N_j} - \sqrt{k})!}{l_{j+1}(\sqrt{N_j})!}$$

$$\geq \frac{1}{l_{j+1}}(\sqrt{N_j} \sqrt{k})^{-1} = \frac{1}{l_j}(N_j(\sqrt{N_j} \sqrt{k})^{-1} = (l_j(\sqrt{N_j} \sqrt{k+2}))^{-1}.$$

Since $n \geq (k-2)l_j$, we obtain

$$-\frac{1}{n^{1/3}} \log \mu(P_n(x)) \leq \frac{1}{((k-2)l_j)^{1/3}} \log(l_j \cdot \sqrt{N_j} \sqrt{k+2}) \sim \frac{1}{2} \frac{\sqrt{k} + 2}{((k-2)l_j)^{1/3}} \log N_j.$$  

Since the limit of the last sequence has the same critical value as the limit of $\frac{1}{l_j} \log N_j$, the limit superior in (1) has the critical value not greater than $1/2$. On the other hand, it is immediate from (2) with $\tau = \frac{1}{2}$ that the critical value of the limit superior is not less than $1/2$.

Now we prove the second statement of the theorem. Let $\tilde{x}$ be the set of $x \in X$ such that $B_0^{(j)}(x)$ is an unpermuted word for all sufficiently large $j \in \mathbb{N}$. We claim that $\mu(\tilde{x}) = 1$. Let $L_j$ be the set of all $x \in X$ such that $B_0^{(j)}(x)$ is a permutated word in $C_j$. Then we have $\mu(L_j) = \frac{\sqrt{N_j}}{N_j}$, so $\sum_j \mu(L_j) < \infty$. Now the Borel-Cantelli lemma proves the claim.

From now on we will assume that $N_j$ is a square number since, if not, a slightly modified argument will work. Note that every word in $C_{j+1}$ contains $\sqrt{N_j}$ isolated permutated $C_j$-words and there is a chain of consecutive unpermuted $C_j$-words between two adjacent permuted $C_j$-words. Let $U_j$ be the collection of those chains, that is,

$$U_j = \{u_1^{(j)}, u_3^{(j)}, u_5^{(j)} \cdots u_8^{(j)}, u_{10}^{(j)} \cdots u_{15}^{(j)}, \ldots, u_{2^j+1}^{(j)} \cdots u_{2^j+2^j-1}^{(j)}, \ldots\}$$

Then the cardinality of $U_j$ is $\sqrt{N_j}$, and the numbers of unpermuted $C_j$-words in the chains in $U_j$ are $1, 1, 4, 6, 8, 10, \ldots, 2(\sqrt{N_j} - 1)$.

Given $x \in \tilde{x}$ and $j \in \mathbb{N}$ with $B_0^{(j)}(x)$ unpermuted, let $C_1 \cdots C_{p(x,j)}$ be the chain in $U_j$ containing $B_0^{(j)}(x)$. Also denote by $q(x,j)$ the integer with $C_{q(x,j)} = B_0^{(j)}(x)$. Note that $p(x,j)$ and $q(x,j)$ are well defined for all large $j \in \mathbb{N}$.

Fix $\frac{1}{4} < \eta < \frac{1}{2}$. For each $j \in \mathbb{N}$, let $T_j$ be the set of all $x \in \tilde{x}$ such that $p(x,j) > (N_j)^\eta$ and $q(x,j) < p(x,j) - (N_j)^{\frac{1}{4}}$. Then we have

$$\mu(T_j^C) < \frac{1}{N_j} \sum_{i \leq N_j^\eta/2} 2i + \sqrt{N_j \cdot (N_j)^{\frac{1}{4}}}.$$
for all large \( j \). Let \( \hat{X} \) be the set of all \( x \in \hat{X} \) such that \( x \in T_j \) for all large \( j \in \mathbb{N} \).

Since \( \sum_{j \in \mathbb{N}} \mu(T_j^c) < \infty \), by the Borel Cantelli Lemma we have \( \mu(\hat{X}) = 1 \).

Intuitively, for any \( x \in \hat{X} \), the word \( B_0^{(j)}(x) \) lies in the forepart of a long chain of consecutive unpermuted \( C_j \)-words in \( B_0^{(j+1)}(x) \). Note that for all \( x \in \hat{X} \), \( B_0^{(j)}(x) \cdots B^{(j)}(x) \) is a chain of unpermuted \( C_j \)-words for all large \( j \in \mathbb{N} \).

Let \( \tau \in [0, \frac{1}{2}] \) be given. Since \( \lim_{j \to \infty} \frac{1}{(l_j)^\beta} \log N_j \) has the critical value \( \beta = 1/2 \) and \( \lim_{j \to \infty} \frac{1}{(l_j)^\beta} \log N_j = 0 \) for all \( \beta > 0 \), there is a sequence \( n_j \) such that \( l_j \leq n_j \leq l_j \sqrt{N_j} \) and \( \lim_{j \to \infty} \frac{1}{(n_j)^\beta} \log N_j \) has the critical value \( \tau \). For any \( x \in \hat{X} \) and \( j \in \mathbb{N} \), by considering a \( C_j \)-word envelop of \( P_{n_j}(x) \) as in the proof of (1), we see that \( \mu(P_{n_j}(x)) \) is almost same as \( \mu(B_0^{(j)}(x) \cdots B_p^{(j)}(x)) \) for some \( p \), which in turn is the same as \( \mu(B_p^{(j)}(x)) \) by Lemma 4.3. Thus for each \( x \in \hat{X} \), the sequence \( \lim_{j \to \infty} \frac{1}{(n_j)^\beta} \log \mu(P_{n_j}(x)) \) has the same critical value as \( \lim_{j \to \infty} \frac{1}{(n_j)^\beta} \log N_j \).

This completes the proof of (2). \( \square \)

Remark 4.5. (1) Let \( \mathcal{P} \) be the 0-th coordinate partition for \( X \) and denote by \( H(\mathcal{Q}) \) the usual metric entropy for a partition \( \mathcal{Q} \) of \( X \). Then Theorem 4.4 directly implies that \( \limsup_{n \to \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} \sigma^{-i} \mathcal{P}) \) has the critical value 1/2 and that \( \liminf_{n \to \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} \sigma^{-i} \mathcal{P}) = 0 \) for all \( \beta > 0 \). However, this does not guarantee that the system has metric entropy dimension 1/2. Indeed, our system has metric entropy dimension 0. (For the definition of metric entropy dimension, see [6]).

(2) It is not known whether Theorem 4.4 holds for general dynamical systems of entropy dimension \( \alpha > 0 \). We conjecture that the critical value of \( \limsup \) equals topological entropy dimension and that of \( \liminf \) equals metric entropy dimension.

We shall show similar results hold for the first return time. We begin with a lemma.

Lemma 4.6. Let \( X \) be a subshift and \( \mu \) be an invariant measure on \( X \). Suppose that \( \limsup_{n \to \infty} \frac{1}{n} \log \mu(P_n(x)) \) has the critical value \( c \) for \( \mu \)-a.e. \( x \in X \). Then the critical value of \( \limsup_{n \to \infty} \frac{1}{n} \log R_n(x) \) is less than or equal to \( c \) for \( \mu \)-a.e. \( x \in X \).

Proof. Let \( \epsilon > 0 \) be given. Take \( \alpha' > \alpha > \gamma > c \). Fix \( n \in \mathbb{N} \). For each \( D \in \bigvee_{i=0}^{n-1} \sigma^{-i} \mathcal{P} \), consider the set

\[
D_0 = \{ x \in D : R_n(x) > \exp(n^\alpha) \}.
\]

Then for all \( 1 \leq i, j \leq \exp(n^\alpha) \) with \( i \neq j \), the sets \( \sigma^i D_0 \) and \( \sigma^j D_0 \) do not intersect. So we have \( \mu(D_0) < \exp(-n^\alpha) \). Now, let

\[
Q_n = \{ x \in X : \mu(P_n(x)) > \exp(-n^\gamma) \text{ and } R_n(x) > \exp(n^\alpha) \}.
\]
Since the number of $D$’s in $\sqrt[n]{\prod_{i=0}^{n-1} \sigma^{-i} P}$ with $\mu(D) > \exp(-n^\gamma)$ is at most $\exp(n^\gamma)$, the measure of $Q_n$ can be estimated by
\[
\mu(Q_n) \leq e^{-n^\alpha} \cdot e^{n^\gamma} = e^{n^\gamma - n^\alpha},
\]
so we have $\sum \mu(Q_n) < \infty$. By the Borel-Cantelli lemma, it follows that for $\mu$-a.e. $x \in X$, $x \notin Q_n$ for all large $n$.

By the assumption, for $\mu$-a.e. $x \in X$ we have $\limsup_{n \to \infty} -\frac{1}{n^\gamma} \log \mu(P_n(x)) = 0$. Hence $\mu(P_n(x)) > \exp(-n^\gamma)$ for all large $n \in \mathbb{N}$. As $x \notin Q_n$ for all large $n \in \mathbb{N}$, we have $R_n(x) \leq \exp(n^\alpha)$ for such $n \in \mathbb{N}$. It follows that
\[
\limsup_{n \to \infty} \frac{\log R_n(x)}{n^{\alpha'}} = 0
\]
for $\mu$-a.e. $x \in X$. Since the last inequality holds for all $\alpha' > c$, the proof is complete.

**Theorem 4.7.** Let $X$ be as in Theorem 4.4. Then the following hold.

1. For $\mu$-a.e. $x \in X$,
\[
\limsup_{n \to \infty} \frac{1}{n^\beta} \log R_n(x)
\]
has the critical value $1/2$.

2. There is a set $X' \subset X$ of full measure with the property that for any $x \in X'$, there is an increasing sequence $\{n_j\}_{j \in \mathbb{N}}$ such that for each $x \in X'$,
\[
\lim_{j \to \infty} \frac{1}{(n_j)^{\beta}} \log R_{n_j}(x)
\]
has the critical value $\tau$.

**Proof.** By Lemma 4.6 and Theorem 4.4(1) it follows that the limsup in (1) has the critical value no greater than $1/2$. The inequality of the other direction is an immediate consequence of the second result of the theorem with $\tau = 1/2$.

For the proof of (2), let $X' \subset X$ be the same set as in the proof of Theorem 4.4. Given $\tau \in [0, 1/2]$, take $\{n_j\}_{j \in \mathbb{N}}$ as in (2) of Theorem 4.4. Let $x \in X'$. There is a chain of unpermuted $C_j$-words $B_0^{(j)} \cdots B_{k-1}^{(j)}$, $k \geq 3$, such that $P_{n_j}(x)$ is a subword of $B_0^{(j)} \cdots B_{k-1}^{(j)}$ and $P_{n_j}(x)$ contains $B_1^{(j)} \cdots B_k^{(j)}$ as a subword. Then both $B_0^{(j)} \cdots B_{k-1}^{(j)}$ and $B_1^{(j)} \cdots B_{k-2}^{(j)}$ occur exactly after the time $l_j + 1$, so $R_{n_j}(x) \leq l_j + 1$. By the uniqueness of the decomposition into $C_j$-words and by Lemma 4.4 it follows that $R_{n_j}(x) = l_j + 1 = l_j N_j$. It is easy to see that the limit of $\frac{1}{n_j} \log R_{n_j}(x)$ has the critical value $\tau$. \qed

5. Further properties and remarks

Sections 3 and 4 give a general method to construct strictly ergodic subshifts with positive entropy dimension. With slight modifications, we can construct such systems with specific topological and measure theoretic properties.
5.1. **Entropy generating sequence.** Let \( S = \{s_i\}_{i \in \mathbb{N}} \) be an increasing sequence of natural numbers. It is called an entropy generating sequence for a subshift \((X, \sigma)\) if \( \liminf_{n \to \infty} \frac{1}{n} \log N(\sqrt{l_1} \sigma^{-s_i} P) > 0 \). The upper dimension of \( S \) is defined by

\[
\overline{D}(S) = \inf \{ \tau \geq 0 : \limsup_{n \to \infty} \frac{n}{(s_n)^\tau} = 0 \}.
\]

The intuitive idea of an entropy generating sequence is to specify positions where the independence occur. It is known that the upper entropy dimension \( \overline{D}(X, \sigma) \) defined in Section 2 equals the supremum of \( \overline{D}(S) \) over all entropy generating sequences \( S \) [7, Theorem 3.10]. For a subshift, there is an entropy generating sequence attaining the supremum.

It is intricate to describe an entropy generating sequence for a general system constructed in Section 3. However, imposing some additional conditions in the process of the construction, we can obtain a system of which an entropy generating sequence is rather obvious. Suppose that the \( j \)-th word set \( C_j \) is constructed by the induction. Recall that \( C_j \) consists of \( N_j \) words of length \( t_j \) and that \( N_{j+1} = (\sqrt{N_j})! \) and \( l_{j+1} = l_j N_j \). Any word \( u \) in \( C_j \) can be written uniquely as \( u = u_1 u_2 \cdots u_{N_j} \) where \( u_k \)'s are words in \( C_j \), and distinguished by the sequence \( \{u_k\}_{k \in P_{j-1}} \), where \( P_{j-1} = \{2\} \cup \{i^2 : 2 \leq i \leq \sqrt{N_{j-1}}\} \). We consider the subset

\[
\mathcal{D}_j = \{u \in C_j : \{u_k\}_{k \in Q_{j-1}} \text{ are distinct}\}
\]

where \( Q_{j-1} = \{2\} \cup \{i^2 : 2 \leq i \leq \sqrt{N_{j-1}}/2\} \). The cardinality of \( \mathcal{D}_j \) is \( \sqrt{N_{j-1}} P \sqrt{N_{j-1}/2} \). Now we order the elements of \( C_j \) so that the permuted positions \( k \in P_j \) are occupied by elements in \( \mathcal{D}_j \): if

\[
u_1^{(j+1)} = u_1^{(j)} u_2^{(j)} u_3^{(j)} u_4^{(j)} u_5^{(j)} \cdots u_{N_j}^{(j)}
\]

is the word formed by concatenating all the words of \( C_j \) in order, then \( u_k^{(j)} \) is in \( \mathcal{D}_j \) for all \( k \in P_j \). Put \( S_1 = [0, l_1] \),

\[
S_j = \bigcup_{k \in Q_j} (k \cdot l_{j-1} + S_{j-1}) \quad (\text{for } j > 1)
\]

and \( S = \bigcup_{j \in \mathbb{N}} S_j \). Note that

\[
|S_j| = \frac{N_j}{2^{j-1}} \prod_{i=1}^{j-1} \sqrt{N_i}.
\]

By a straightforward calculation it can be seen that \( \liminf_{j \to \infty} \frac{1}{|S_j|} \log N_j > 0 \), and we have \( \overline{D}(S) = 1/2 \).

5.2. **Weak mixing.** The subshift constructed in Section 3 is not topologically weak mixing (hence not metrically weak mixing), since for any \( u \in C_j \), the set of \( k \)'s with \( \sigma^k \{u\} \cap \{u\} = \emptyset \) has positive density.
By inserting a spacer symbol at each step, it is possible to construct a metrically and topologically mixing subshift. Let \( b \) be a symbol other than 0 and 1. Given \( C_j \) with \( j \in \mathbb{N} \), let \( u_1^{(j+1)} \) be the concatenation of \( C_j \) words with the spacers in front of each permuted words:

\[
u_1^{(j+1)} = u_1^{(j)} u_2^{(j)} u_3^{(j)} b_{\pi(j)} u_5^{(j)} \cdots u_8^{(j)} b_{\pi(9)} u_{10}^{(j)} \cdots u_{N_j}^{(j)},
\]

and the collection \( C_{j+1} \) is obtained by permuting the subwords \( u_i^{(j)} \) for \( i > 1 \) and leaving the others fixed. Hence a typical word in \( C_{j+1} \) is of the form

\[
u_1^{(j)} u_2^{(j)} u_3^{(j)} b_{\pi(j)} u_5^{(j)} \cdots u_8^{(j)} b_{\pi(9)} u_{10}^{(j)} \cdots u_{N_j}^{(j)},
\]

where \( \pi \) is a permutation on the set \( \{i^2 : 1 < i \leq \sqrt{N_j}\} \). Since \( b \) occurs rarely, it does not affect much on the calculations in the previous sections: Our new system \( X \) is strictly ergodic, has entropy dimension 1/2, and satisfies Theorem 4.4 and Theorem 4.7. A similar argument in §5.2 applies to \( X \), hence \( X \) is also metrically rigid.

As \( X \) is strictly ergodic, metric weak mixing implies topological weak mixing. The existence of a spacer symbol forces \( X \) to be weakly mixing by an approximation argument. Suppose that \( f : X \to \mathbb{C} \) is a measurable eigenfunction with eigenvalue \( \lambda \). We may assume that \( |f| = 1 \). Let \( \epsilon > 0 \) be small enough. By letting \( \mathcal{A}_j = \{\sigma^i([u]) : u \in C_j \text{ is unpermuted and } 0 \leq i < l_j\} \), it is easy to see that the Borel \( \sigma \)-algebra of \( X \) is generated by \( \mathcal{A}_j \)'s, \( j \in \mathbb{N} \). Note that \( \mathcal{A}_j \) does not cover \( X \), but \( \mu(\bigcup \mathcal{A}_j) > (1 - \frac{1}{l_j})(1 - \frac{1}{\sqrt{N_j}}) \).

There is a sequence of simple functions \( f_j \) converging to \( f \) and for each \( j \in \mathbb{N} \) and \( f_j \) is constant on each element of \( \mathcal{A}_j \). Then there is \( j \) with \( \mu\{x : |f(x) - f_j(x)| < \epsilon\} > 1 - \epsilon \) and \( \mu(\bigcup \mathcal{A}_j) > 1 - \epsilon \). Let \( F = \{x : |f(x) - f_j(x)| < \epsilon\} \).

Now there is \( B = \sigma^i([u]) \in \mathcal{A}_j \) such that \( \mu(B \cap F) > (1 - 2\epsilon)\mu(B) \). If not, then we have

\[
1 - 2\epsilon < \mu(F) - \epsilon < \mu(\bigcup \mathcal{A}_j \cap F) < (1 - 2\epsilon)\mu(\bigcup \mathcal{A}_j) < 1 - 2\epsilon,
\]

which is a contradiction. Without loss of generality we may assume \( i = 0 \).

For each \( w \in C_{j+2} \), we define

\[
\mathcal{I}_w = \{0 \leq k < l_{j+2} : \sigma^k([w]) \subset B\}.
\]

Then it is easy to see that \( |\mathcal{I}_w| = N_{j+1} \). An integer \( k \in \mathcal{I}_w \) is called good if \( \mu(\sigma^k([w]) \cap F) > (1 - \sqrt{2\epsilon})\mu([w]) \) and bad otherwise. We will show that there exists some \( v \in C_{j+2} \) such that \( \mathcal{I}_v \) has at least \( (1 - \sqrt{2\epsilon})N_{j+1} \) good members.
Recall that there occur \(\left(\frac{\epsilon}{2}\right)\).

\[\text{Proof.} \supseteq \text{Claim.}\]

Let \(\mathcal{I}_v = \{k_1 < k_2 < \cdots < k_{N_j+1}\}\) be ordered. Note that \(k_i - k_{i-1}\) is \(l_{j+1} + 1\) if \(i\) is a square number bigger than 1, and \(l_{j+1}\) otherwise.

\[\text{Claim.} \] There are good members \(k_{i_1}, k_{i_2}, k_{i_3}, k_{i_4} \in \mathcal{I}_v\) and a positive integer \(c\) such that \(c \cdot l_{j+1} = k_{i_2} - k_{i_1} - 1 = k_{i_4} - k_{i_3}\).

\[\text{Proof.} \] Recall that there occur \((j + 2)\)-step spacers between \(l^2\)-th and \((l^2 + 1)\)-st \(C_{j+1}\) subblocks of \(v\). Otherwise the number of bad members of \(\mathcal{I}_v\) are greater than \(\alpha/2 > \sqrt{2\epsilon}\), a contradiction. Hence we have \(k_{i_3} - k_{i_1} = c \cdot l_{j+1} + 1\) for some integer \(c\). As there are at most \(\sqrt{\alpha N_{j+1}}\) spacers, \(c \cdot N_{j+1}\).

Suppose that there does not exist good members \(k_{i_1}\) and \(k_{i_3}\) such that \(c \cdot l_{j+1} = k_{i_4} - k_{i_2}\). Then at least one of \(k_i\) and \(k_{i+c}\) is bad if \((l - 1)^2 \leq i, i + c < l^2\) for any \(l \geq 2\). Therefore, the number of bad members of \(\mathcal{I}_v\) is bigger than

\[
\sum_{l \leq \sqrt{N_{j+1}}} \left\lfloor \frac{l^2 - (l - 1)^2}{2c} \right\rfloor \cdot c
\]

\[
\geq \sum_{l \leq \sqrt{N_{j+1}}} \left( l - \frac{1}{2} - c \right)
\]

\[
\geq \frac{N_{j+1}}{2} - c\sqrt{N_{j+1}}
\]

\[
\geq \frac{1 - 8\sqrt{\alpha}}{2}N_{j+1} > \sqrt{2\epsilon}N_{j+1}
\]
for all small $\epsilon > 0$, which is again a contradiction. Hence there are good members $k_{i_3}, k_{i_4} \in I_\nu$ such that $c \cdot I_{j+1} = k_{i_4} - k_{i_3}$. \hfill \Box

As $k_{i_1}$ and $k_{i_2}$ are good members of $I_\nu$, the set
\[(\sigma^{k_{i_1}}([u]) \cap F) \cap \sigma^{-(k_{i_2} - k_{i_1})}(\sigma^{k_{i_2}}([v]) \cap F)\]
is nonempty. Take a point $y$ in the set above. Since $f$ is an eigenfunction with eigenvalue $\lambda$, we have $|f(y) - f \circ \sigma^{k_{i_2} - k_{i_1}}(y)| = |f(y) - f \circ \sigma^{c_{I_{j+1}}+1}(y)| = |f(y)| \cdot |1 - \lambda^{c_{I_{j+1}}+1}| = |1 - \lambda^{c_{I_{j+1}}+1}|$. Also, since $y \in (B \cap F) \cap \sigma^{-(k_{i_2} - k_{i_1})}(B \cap F)$ and $f_j$ is constant on $B$ we have
\[|f(y) - f \circ \sigma^{k_{i_2} - k_{i_1}}(y)| \leq |f_j(y) - f_j \circ \sigma^{k_{i_2} - k_{i_1}}(y)| + 2 \epsilon = 2 \epsilon.\]
Similarly by using $k_{i_3}$ and $k_{i_4}$ we have
\[|1 - \lambda^{c_{I_{j+1}}}| < 2 \epsilon,\]
and it follows that $|1 - \lambda| = |\lambda^{c_{I_{j+1}}} - (1 - \lambda)| < 4 \epsilon$. This is true for every $\epsilon > 0$, hence we have $\lambda = 1$. Therefore $(X, \sigma)$ is metrically weakly mixing.

5.3. Metric rigidity. Let $X$ be any subshift constructed in this paper. We show that $X$ is metrically rigid with a rigidity sequence $\{I_n\}_{n \in \mathbb{N}}$, i.e., $\lim_{n \to \infty} \mu(\sigma^{I_n} A \triangle A) = 0$ for each measurable set $A$. To see this, first note that this holds for each set $A$ of the form $A = \sigma^k([u])$ with an unpermuted word $u \in C_j$ and $k \in \mathbb{N}$. For a measurable set $A$ and $\epsilon > 0$, there is $j \in \mathbb{N}$ and finitely many $I_i \in A_j$ such that $\mu(A \triangle \bigcup I_i) < \epsilon$. Since each $I_i$ satisfies $\mu(\sigma^{I_n} I_i \triangle I_i) \to 0$, by an approximation we also have $\mu(\sigma^{I_n} A \triangle A) \to 0$ as $n \to \infty$.

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