Abstract. We prove a master theorem for hypergeometric functions of Karlsson–Minton type, stating that a very general multilateral $U(n)$ Karlsson–Minton type hypergeometric series may be reduced to a finite sum. This identity contains the Karlsson–Minton summation formula and many of its known generalizations as special cases, and it also implies several “Bailey-type” identities for $U(n)$ hypergeometric series, including multivariable $10W_9$ transformations of Milne and Newcomb and of Kajihara. Even in the one-variable case our identity is new, and even in this case its proof depends on the theory of multivariable hypergeometric series.

1. Introduction

At a first glance, the theory of summation and transformation formulas for hypergeometric functions (ordinary and basic or “q”) may appear as an entangled mess of complicated formulas involving many parameters. However, since many different identities may arise as special or limit cases of a single formula, it is possible to structure the “space” of all such identities by organizing them into hierarchies. For instance, a large number of the most useful identities may be understood as a “Bailey hierarchy”, originating from Bailey’s $10W_9$ transformation formula [B1], [GR, Equation (III.28)].

However, there are identities that do not fit into the Bailey hierarchy. An example is the Karlsson–Minton summation formula [Mi, Ka]

$$ r + 2F_{r+1} \left( \begin{array}{c} a, b, c_1 + m_1, \ldots, c_r + m_r \\ b + 1, c_1, \ldots, c_r \end{array} ; 1 \right) = \frac{\Gamma(b + 1) \Gamma(1 - a)}{\Gamma(1 + b - a)} \prod_{i=1}^{r} \frac{(c_i - b)_{m_i}}{(c_i)_{m_i}}, $$

where the $m_i$ are non-negative integers and $\text{Re} (a + |m|) < 1$ (Minton proved this for a negative integer and Karlsson in general). It belongs to a hierarchy of identities for hypergeometric series with integral parameter differences; cf. [C, G1, G2, S1, S2] for related results. We will refer to such series as Karlsson–Minton type hypergeometric series. (Schlosser [S1] prefers the acronym IPD type series, since the Karlsson–Minton formula may be obtained rather easily from results known much earlier; cf. [E, FW]. However, in our opinion the term is perfectly justified, since Minton seems to have been the first to call attention to this type of series.)

The purpose of this paper is to present a master identity for series of Karlsson–Minton type, Theorem 3.1. Not only does it contain a large number of results from the papers mentioned above as special cases, but it also provides a bridge between the Karlsson–Minton hierarchy and the Bailey hierarchy. To find this bridge, and even to state our theorem, it is necessary to leave the field of one-variable series and pass to multivariable series.

2000 Mathematics Subject Classification. 33D15, 33D67.
The multivariable series that arise are so called $U(n)$ or $A_n$ hypergeometric series. Series of this type were introduced by Biedenharn, Holman and Louck [HBL], motivated by the theory of $6j$-symbols of the group $SU(n)$. During the last 25 years, the theory of $U(n)$ series (and those connected to other classical groups) has been developed extensively by Gustafson, Milne and many others, and it has been applied to problems in representation theory, number theory and combinatorics.

Experts on multivariable series have argued that certain features of one-variable series are more easily understood within a multivariable framework; for instance, that so called very-well-poised series should be viewed as series in two variables $y_1$, $y_2$ with the summation indices restricted to a line $y_1 + y_2 = 0$. The present paper goes further in this direction and uses multivariable series as a tool for studying one-variable series.

To give an example, a very degenerate case of Theorem 3.1 is the following generalization of (1):

$$
\begin{align*}
\sum_{x_1,\ldots,x_r=0}^r (a |c_i-a| x_i) & \prod_{i,k=1}^r (c_i-c_k-m_k)_{x_i} (1+c_i-c_k)_{x_i} \\
\times (a |m| & (1+a+b+d)|m|) \prod_{1 \leq i < j \leq r} (c_i+1-d)_{m_i} (c_j+1-d)_{m_j} \\
\times & \prod_{i=1}^r (c_i-a)_{x_i} \prod_{i,k=1}^r (c_i-c_k-m_k)_{x_i} (1+c_i-c_k)_{x_i},
\end{align*}
$$

where $\text{Re} \ (a+|m|+b+d) < 0$. When $d = b+1$ all terms in the sum vanish except the one with $x_i \equiv 0$, which is 1, so that we recover (1); note also that the case $m_i \equiv 0$ is Gauss’ classical $2F_1$ summation. The point is that, although (2) is certainly an interesting identity from the viewpoint of one-variable series alone, the finite sum on the right is precisely a $U(n)$ hypergeometric sum, and the identity seems difficult to prove without using the multivariable theory.

We remark that Karlsson’s proof of (1) gives an alternative expression for the left-hand side of (2) as a finite sum. Explicitly, one has

$$
\begin{align*}
\sum_{x_1,\ldots,x_r=0}^r (a |c_i-a| x_i) & \prod_{i,k=1}^r (c_i-c_k-m_k)_{x_i} (1+c_i-c_k)_{x_i} \\
\times (a |m| & (1+a+b+d)|m|) \prod_{i=1}^r (c_i-a)_{x_i} \prod_{i,k=1}^r (c_i-c_k-m_k)_{x_i} (1+c_i-c_k)_{x_i},
\end{align*}
$$

This sum is of a type that is less symmetric than $U(n)$ series and probably without much independent interest.

To see how $U(n)$ series are related to Karlsson–Minton type series we need only know that the former (we stick to the classical rather than the $q$-case) are characterized by the factor

$$
\frac{\Delta(z+y)}{\Delta(z)} = \prod_{1 \leq i < j \leq n} \frac{z_i + y_i - z_j - y_j}{z_i - z_j},
$$
where the \( y_i \) are summation indices. Suppose we restrict the summation to the line where \( y_2 = \cdots = y_n = 0 \) and put \( y_1 = k \). Then

\[
\frac{\Delta(z + y)}{\Delta(z)} = \prod_{i=2}^{n} \frac{z_1 + k - z_i}{z_1 - z_i} = \prod_{i=2}^{n} \frac{(z_1 - z_i + 1)_k}{(z_1 - z_i)_k}.
\]

If we now choose the parameters \( z_i \) so that

\[
(z_1 - z_2, \ldots, z_1 - z_n) = (c_1, c_1 + 1, \ldots, c_1 + m_1 - 1, c_2, \ldots, c_2 + m_2 - 1, \ldots, c_r + m_r - 1),
\]

where the \( m_i \) are non-negative integers with \( |m| = n - 1 \), we obtain

\[
\frac{\Delta(z + y)}{\Delta(z)} = \prod_{i=1}^{r} \frac{(c_i + m_i)_k}{(c_i)_k},
\]

which is the factor characterizing Karlsson–Minton type series. So we may view such a series as the restriction of a \( U(n) \) series to a one-dimensional subspace.

To exploit this observation we must work with \( U(n) \) series for which restriction of the summation indices to lower-dimensional subspaces gives something nice. We choose as our starting point Gustafson’s \( U(n) \) Bailey sum \([Gu1]\); cf. equation (5) below. In this sum the summation indices live on a hyperplane \( y_1 + \cdots + y_n = 0 \).

We specialize the parameters so that the terms with \( y_n < 0 \) vanish. It then turns out that the sum with \( y_n \geq 1 \) is of the same type as the original sum with \( y_n \geq 0 \). The difference of these two sums gives the restriction of the original sum to the space where \( y_n = 0 \), which hence may be computed. Iterating this procedure we eventually find an identity for the Karlsson–Minton type series obtained by restriction to a one-dimensional subspace. Moreover, the previous steps in the iteration give identities reducing multivariable Karlsson–Minton series to finite sums; these are also contained in Theorem 3.1.

The details of this derivation are worked out in Section 3. In Section 4 we state a number of corollaries to Theorem 3.1. These include summation and transformation formulas for one- and multivariable Karlsson–Minton type series from \([C, G1, G2, S1, S2]\) and also a multivariable generalization of Shukla’s \( \psi_8 \) summation due to Schlosser \([S3]\). We point out some interesting cases corresponding to lower level identities: if Theorem 3.1 is on the \( 6\psi_6 \) level, Corollary 4.17 gives the \( 1\psi_1 \) version while Corollaries 4.13 and 4.14 correspond to the \( 2\phi_1 \) level; Corollary 4.13 is the \( q \)-analogue of (3). We also indicate how Theorem 3.1 is related to some “Bailey-type” results for \( U(n) \) series. When the Karlsson–Minton type series reduces to a finite sum, we may remove the condition that the \( m_i \) are non-negative integers by a polynomial argument, and recover \( U(n) \) Watson and Bailey transformations, Corollaries 4.7 and 4.8, which were recently found by Kajihara \([K]\). Another interesting case is when the Karlsson–Minton type series is one-dimensional. It then has a symmetry which is not apparent for the finite sum; this implies \( U(n) \) Sears and Bailey transformations due to Milne and Newcomb, Corollaries 4.15 and 4.16. In Section 5 we write down the analogues of Theorem 3.1 obtained using instead of Gustafson’s \( U(n) \) \( 6\psi_6 \) sum the closely related \( U(n) \) \( 5H_5 \) and \( 2H_2 \) summation formulas from \([Gu1]\).

We finally remark that it seems worthwhile to repeat the analysis of the present paper starting from summation formulas different from those used here. In fact, in
the subsequent paper [R] we apply the $C_n \psi_6$ sum from [Gu2] to obtain a reduction formula for Karlsson–Minton type hypergeometric series on the root system $C_n$.

Acknowledgements: I would like to thank Michael Schlosser for his comments on an earlier version of this paper; in particular his suggestion that I try to derive Theorem 4.6 of [S1] using the ideas of the present paper lead to the inclusion of the more general Corollary 4.19 below. I also thank Yasushi Kajihara for providing me with the manuscript of [K].

2. Notation and a single preliminary

In the rest of the paper we will work with $q$-series, and only discuss the limit case of classical hypergeometric series briefly in Section 5. The base $q$ will be a fixed complex number with $0 < |q| < 1$. We will use the standard notation of [GR], but since $q$ is fixed we suppress it from the notation. Thus we write (note that this is different from the notation used in the introduction; cf. (15) below)

$$ (a)_k = \begin{cases} \prod_{i=1}^{k} (1 - a)(1 - aq) \cdots (1 - aq^{k-1}), & k \geq 0, \\ 1 & k < 0 \end{cases} \quad (a_1, \ldots, a_m)_k = (a_1)_k \cdots (a_m)_k, $$

and analogously for infinite products $(a)_\infty = \prod_{j=0}^{\infty} (1 - aq^j)$.

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we write $|z| = z_1 + \cdots + z_n$ and use the corresponding capital letter to denote the product of the coordinates: $Z = z_1 \cdots z_n$.

To prove our main theorem all we need is Gustafson’s multivariable Bailey sum [Gu1, Theorem 1.15], which we write as

$$ \sum_{y_1, \ldots, y_n = -\infty}^{\infty} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{i,k=1}^{n} (a_iz_k)_{y_k} = \frac{(q/AZ, q^{-n}BZ)_{\infty}}{(q, q^{-1-n}B/A)_{\infty}} \prod_{i,k=1}^{n} \frac{(b_i/a_k, qz_k/z_i)_{\infty}}{(q/a_k z_i, b_i z_k)_{\infty}}, $$

where

$$ \frac{\Delta(zq^y)}{\Delta(z)} = \prod_{1 \leq i < j \leq n} \frac{z_i q^{y_i} - z_j q^{y_j}}{z_i - z_j}. $$

This holds for $|q^{-1-n}B/A| < 1$, as long as no denominators vanish. When $n = 2$, (5) is Bailey’s $\psi_6$ summation [GR, Equation (II.33)]. Gustafson’s proof of (5) is based on residue calculus and uses non-trivial identities for theta functions.

3. The theorem

Our main result is the following identity. We call it a reduction formula, since it reduces a very general multilateral Karlsson–Minton type series to a finite sum.

**Theorem 3.1.** Let $m_i$ be non-negative integers and $a_i, b_i, c_i, z_i$ parameters such that $|q^{1-n}B/A| < 1$ and none of the denominators in (5) vanishes. Then the following identity holds:
Having the extra parameter \( N \)
Although we do not need it, it is not hard to check that (7) is equivalent to (6).

and after further simplifications we obtain the identity

\[
\sum_{y_1,\ldots,y_n=0}^{\infty} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq k \leq n} \frac{(c_i z_k q^{m_i})_{yk}}{(c_i z_k)_{yk}} \prod_{i,k=1}^{n} (a_i z_k)^{yk} \prod_{i,k=1}^{n} (b_i z_k)^{yk} = \left( q^N - q^{|m|AZ} \right) \times \prod_{1 \leq k \leq n} \frac{(b_i / a_k, q z_k / z_i)_{yk}}{(a_i z_k, b_i z_k)_{yk}} \prod_{1 \leq k \leq n} \frac{(q^{-m_i} b_k / c_i)_{m_i}}{(q^{1-m_i} / c_i z_k)_{m_i}} \prod_{x_1,\ldots,x_p=0}^{m_1,\ldots,m_p} \frac{\Delta(c q^x)}{\Delta(c)} q^{|x|} \frac{(q^{n-N}/AZ)_{|x|}}{(q^{1-n}/AZ)_{|x|}} \prod_{1 \leq k \leq n} \frac{(c_i / a_k)_{x_i}}{(q c_i / b_k)_{x_i}} \prod_{1 \leq k \leq n} \frac{(q^{-m_i} c_i / c_k)_{x_i}}{(q^{-m_i} / c_k)_{x_i}}.
\]

The condition \(|q^{1-m-n}B/A| < 1\) ensures that the series on the left-hand side converges absolutely, so that the series manipulations occurring in the proof are justified. This can be seen exactly as in [Gu1]; we will not discuss questions of convergence any further.

**Proof.** We will prove the following seemingly more general identity:

\[
\sum_{y_1,\ldots,y_n=0}^{\infty} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq k \leq n} \frac{(c_i z_k q^{m_i})_{yk}}{(c_i z_k)_{yk}} \prod_{i,k=1}^{n} (a_i z_k)^{yk} \prod_{i,k=1}^{n} (b_i z_k)^{yk} = q^N (q^{|m|AZ})^N \times \prod_{1 \leq k \leq n} \frac{(b_i / a_k, q z_k / z_i)_{yk}}{(a_i z_k, b_i z_k)_{yk}} \prod_{1 \leq k \leq n} \frac{(q^{-m_i} b_k / c_i)_{m_i}}{(q^{1-m_i} / c_i z_k)_{m_i}} \prod_{x_1,\ldots,x_p=0}^{m_1,\ldots,m_p} \frac{\Delta(c q^x)}{\Delta(c)} q^{|x|} \frac{(q^{n-N}/AZ)_{|x|}}{(q^{1-n}/AZ)_{|x|}} \prod_{1 \leq k \leq n} \frac{(c_i / a_k)_{x_i}}{(q c_i / b_k)_{x_i}} \prod_{1 \leq k \leq n} \frac{(q^{-m_i} c_i / c_k)_{x_i}}{(q^{-m_i} / c_k)_{x_i}}.
\]

Although we do not need it, it is not hard to check that (7) is equivalent to (6).
Having the extra parameter \( N \) will slightly simplify the proof and also be useful later.

We first prove (6) for \( m_i \equiv 1 \) by induction on \( p \). The starting point \( p = 0 \) is equivalent to (3) by a change of summation variables; cf. Section 5 in [Gu1]. Then we show that the case of general \( m_i \) may be reduced to the special case \( m_i \equiv 1 \).

For the first part of the proof we assume that (7) holds with \( n \) replaced by \( n + 1 \) and with \( m_1 = \cdots = m_p = 1 \). We also specialize to the case \( b_{n+1} = q / z_{n+1} \). Then the factor \( 1 / (b_{n+1} z_{n+1}) \) on the left-hand side vanishes unless \( y_{n+1} \geq 0 \), so that the series is supported on a half-space. Next we let \( a_{n+1} \to q / z_{n+1} \), which corresponds to a removable singularity. After cancelling some factors, the first double product on the right-hand side may be written as

\[
\prod_{i,k=1}^{n} \frac{(b_i / a_k, q z_k / z_i)_{yk}}{(q / a_k z_i, b_i z_k)_{yk}} = \prod_{i,k=1}^{n} \frac{(b_i / a_k, q z_k / z_i)_{yk}}{(q / a_k z_i, b_i z_k)_{yk}} \prod_{i=1}^{n} \frac{1 - q^{-1} b_i z_{n+1}}{1 - z_{n+1} / z_i},
\]

and after further simplifications we obtain the identity
\[ S = \sum_{y_1, \ldots, y_{n+1} \in \mathbb{Z}, \ y_{n+1} \geq 0 \atop y_1 + \cdots + y_{n+1} = N} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq k \leq n+1, \ 1 \leq \ell \leq p} \frac{(c_i z_k q y_k)}{(c_i z_k y_k)} \prod_{1 \leq k \leq n+1} (a_i z_k y_k) / (b_i z_k y_k) \]

\[ = q^{N-1}(N) (-q^{p+1} \hat{A} \hat{Z})^N \left( \frac{q^{1-p-N} / \hat{A} \hat{Z}, q^{1-N-n} \hat{B} \hat{Z}}{(q, q^{p-n} B / A)^{\infty}} \right) \prod_{i,k=1}^{n} \left( \frac{b_i / a_k, q z_k / z_i}{(q / a_k z_i, b_i z_k)^{\infty}} \right) \]

\[ \times \prod_{i=1}^{n} \frac{1 - q^{-1} b_i z_{n+1}}{1 - z_{n+1} / z_i} \prod_{1 \leq k \leq n, \ 1 \leq i \leq p} \left( 1 - q^{-1} b_k / c_i \right) \left( 1 - 1 / c_i z_k \right) \sum_{x_1, \ldots, x_p = 0}^{\infty} \left( \frac{\Delta(c q^x)}{\Delta(c)} \right) q^{x} \left( \frac{(q^{N-N} / \hat{B} \hat{Z})^{x}}{(q^{p-N} / \hat{A} \hat{Z})^{x}} \right) \]

\[ \times \prod_{1 \leq k \leq n, \ 1 \leq i \leq p} \left( \frac{q^{-1} c_i / c_k}{q c_i / b_k} \right) \prod_{i,k=1}^{n} \left( \frac{q^{-1} c_i / c_k}{q c_i / c_k} \right) \), \]

where \( \hat{A} = a_1 \cdots a_n \) and similarly for \( \hat{B} \) and \( \hat{Z} \).

We now divide the sum into two parts as

\[ S = \sum_{y_{n+1} = 0} + \sum_{y_{n+1} \geq 1} = S_1 + S_2. \]

By a change of summation variables, \( S_2 \) may be reduced to a sum of the same type as \( S \). Indeed, choosing \( w_i = z_i \) for \( 1 \leq i \leq n \) and \( w_{n+1} = q z_{n+1} \), so that

\[ \frac{\Delta(w)}{\Delta(z)} = \prod_{i=1}^{n} \frac{1 - q z_{n+1} / z_i}{1 - z_{n+1} / z_i}, \]

we have

\[ S_2 = \sum_{y_1, \ldots, y_{n+1} \in \mathbb{Z}, \ y_{n+1} \geq 1 \atop y_1 + \cdots + y_{n+1} = N} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq k \leq n+1, \ 1 \leq \ell \leq p} \frac{(c_i z_k q y_k)}{(c_i z_k y_k)} \prod_{1 \leq k \leq n+1} (a_i z_k y_k) / (b_i z_k y_k) \]

\[ = q^{N-1}(N-1) (-q^{p+1} \hat{A} \hat{Z})^{N-1} \left( \frac{q^{1-p-N} / \hat{A} \hat{Z}, q^{N-n} \hat{B} \hat{Z}}{(q, q^{p-n} B / A)^{\infty}} \right) \prod_{i,k=1}^{n} \left( \frac{b_i / a_k, q z_k / z_i}{(q / a_k z_i, b_i z_k)^{\infty}} \right) \]

\[ \times \prod_{i=1}^{n} \frac{1 - a_i z_{n+1}}{1 - z_{n+1} / z_i} \prod_{1 \leq k \leq n, \ 1 \leq i \leq p} \left( 1 - q^{-1} b_k / c_i \right) \left( 1 - 1 / c_i z_k \right) \sum_{x_1, \ldots, x_p = 0}^{\infty} \left( \frac{\Delta(c q^x)}{\Delta(c)} \right) q^{x} \left( \frac{(q^{N-N} / \hat{B} \hat{Z})^{x}}{(q^{1-p-N} / \hat{A} \hat{Z})^{x}} \right) \]

\[ \times \prod_{1 \leq k \leq n, \ 1 \leq i \leq p} \left( \frac{q^{-1} c_i / c_k}{q c_i / b_k} \right) \prod_{i,k=1}^{n} \left( \frac{q^{-1} c_i / c_k}{q c_i / c_k} \right). \]
Thus, $S_1$ may be expressed as a finite sum of the form

\begin{equation}
S_1 = S - S_2 = \sum_{x_1, \ldots, x_p=0}^{1} + \sum_{x_1, \ldots, x_p=0}^{1}.
\end{equation}

Next we observe that, with $y_n+1 = 0$ and $z_{n+1} = 1/c_{p+1}$,

\[
\frac{\Delta(z q^u)}{\Delta(z)} = \prod_{1 \leq i < j \leq n} \frac{z_i q^{y_i} - z_j q^{y_j}}{z_i - z_j} \prod_{k=1}^{n} \frac{z_k q^{y_k} - c_{p+1}^{-1}}{z_k - c_{p+1}^{-1}}
\]

\[
= \prod_{1 \leq i < j \leq n} \frac{z_i q^{y_i} - z_j q^{y_j}}{z_i - z_j} \prod_{k=1}^{n} \frac{(c_{p+1} z_k q)_{y_k}}{(c_{p+1} z_k)_{y_k}}.
\]

so that

\[
S_1 = \sum_{y_1, \ldots, y_n = -\infty}^{\infty} \frac{\Delta(z q^u)}{\Delta(z)} \prod_{1 \leq k \leq n} \frac{(c_i z_k q)_{y_k}}{(c_i z_k)_{y_k}} \prod_{i,k=1}^{n} \frac{(a_i z_k)_{y_k}}{(b_i z_k)_{y_k}},
\]

a sum as in (7), still with $m_i \equiv 1$ but with $p$ replaced by $p + 1$. Writing the corresponding right-hand side of (6) as

\[
\sum_{x_1, \ldots, x_{p+1}=0}^{1} + \sum_{0 \leq x_1, \ldots, x_{p+1} \leq 1}^{1},
\]

it is straightforward to check that it agrees termwise with (8). Thus, (6) holds for $S_1$, and by induction for all $p$ as long as $m_i \equiv 1$.

To remove the condition $m_i \equiv 1$, we first note that we may assume $m_i \geq 1$, since if $m_i = 0$ all factors involving $m_i$ cancel. On the other hand, if $m_i > 1$ we may write

\[
\frac{(c_i z_k q^{m_i})_{y_k}}{(c_i z_k)_{y_k}} = \frac{(c_i z_k q)_{y_k}}{(c_i z_k)_{y_k}} \frac{(c_i z_k q^{2})_{y_k}}{(c_i z_k)_{y_k}} \cdots \frac{(c_i z_k q^{m_i})_{y_k}}{(c_i z_k q^{m_i-1})_{y_k}},
\]

which gives a reduction to the case $m_i \equiv 1$, with $c = (c_1, \ldots, c_p)$ replaced by

\begin{equation}
d = (c_1, q c_1, \ldots, q^{m_1-1} c_1, \ldots, c_p, q c_p, \ldots, q^{m_p-1} c_p).
\end{equation}

We now observe that, when $m_i \equiv 1$, the right-hand side of (7) contains the factor

\begin{equation}
\frac{\Delta(q x^i)}{\Delta(c)} \prod_{i,k=1}^{p} \frac{q^{-1} c_i / c_k}{(q c_i / c_k)_{x_i}}
\end{equation}

\[
= \prod_{k=1}^{n} \frac{(q^{-1})_{x_k}}{(q)_{x_k}} \prod_{1 \leq i < j \leq p} \frac{c_i q^{x_i} - c_j q^{x_j}}{c_i - c_j} \frac{(q^{-1} c_i / c_j)_{x_i} (q^{-1} c_j / c_i)_{x_j}}{(q c_i / c_j)_{x_i} (q c_j / c_i)_{x_j}}
\]

\[
= (-1)^{|x|} q^{-|x|} \prod_{1 \leq i < j \leq p} \frac{c_i q^{-x_i} - c_j q^{-x_j}}{c_i - c_j}.
\]
as is easily seen by considering the four cases \( x_i, x_j = 0, 1 \) separately. If \( c \) is replaced by \( d \) as above, this expression vanishes unless \( x \) is of the form

\[
(11) \quad x = (\overbrace{1, \ldots, 1, 0, \ldots, 0}^{m_1}, \overbrace{1, \ldots, 1, 0, \ldots, 0}^{m_\rho}), \quad 0 \leq y_i \leq m_i.
\]

Rewriting the sum using the \( y_i \) as summation variables and comparing with the corresponding right-hand side of (9), we need now only check that, with \( c, d \) and \( x, y \) related by (10) and (11), one has

\[
\prod_{i=1}^{[m]} \frac{1 - q^{-1} b_k / d_i}{1 - 1 / d_\iota z_k} \Delta(dq^x) \prod_{1 \leq k \leq m} \frac{(d_i / a_k)_{x_i}}{(qd_i / b_k)_{x_i}} \prod_{i,k=1}^{[m]} \frac{(q^{-1} d_i / d_k)_{x_i}}{(qd_i / d_k)_{x_i}} = \prod_{i=1}^{p} \frac{(q^{-m_i} b_k / c_i)_{m_i}}{(q^{1-m_i} / c_i z_k)_{m_i}} \frac{(c_i / a_k)_{y_i}}{(q d_i / b_k)_{y_i}} \prod_{i,k=1}^{p} \frac{(q^{-m_k} c_i / c_k)_{y_i}}{(c_i / c_k)_{y_i}}.
\]

Using the obvious identities

\[
\prod_{i=1}^{[m]} (a d_i)_{x_i} = \prod_{i=1}^{p} (a c_i)_{y_i}, \quad \prod_{i=1}^{[m]} (1 - a / d_i) = \prod_{i=1}^{p} (q^{1-m_i} a / c_i)_{m_i},
\]

and (11), we are left with verifying

\[
\prod_{1 \leq i < j \leq [m]} \frac{d_i q^{-x_i} - d_j q^{-x_j}}{d_i - d_j} = (-1)^{|y| q^{|y|}} \frac{\Delta(cq^y)}{\Delta(c)} \prod_{i,k=1}^{p} \frac{(q^{-m_k} c_i / c_k)_{y_i}}{(c_i / c_k)_{y_i}}.
\]

To see this we write the left-hand side as

\[
\prod_{\{i, j : i < j, x_i = x_j = 1\}} q^{-1} \prod_{\{i, j : x_i = 1, x_j = 0\}} \frac{d_i q^{-1} - d_j}{d_i - d_j} = q^{-\frac{|y|}{2}} \prod_{k,l=1}^{p} P_{k,l},
\]

where

\[
P_{k,l} = \prod_{m_1 + \cdots + m_{k-1} + 1 \leq i \leq m_1 + \cdots + m_{k-1} + y_k} \frac{d_i q^{-1} - d_j}{d_i - d_j} = \prod_{1 \leq l \leq y_k} \frac{c_k q^{-1} - c_l q^y}{c_k q^{-1} - c_l q^{y+j-1}}.
\]

\[
= \prod_{l=1}^{y_k} \frac{c_k q^{-1} - c_l q^{m_l}}{c_k q^{-1} - c_l q^y} q^{y-m_l} = \frac{(q^{-m_l} c_k / c_l)_{y_k}}{(q^{-y} c_k / c_l)_{y_k}}.
\]

It is now enough to show that

\[
\prod_{k,l=1}^{p} \frac{(q c_k / c_l)_{y_k}}{(q^{-y} c_k / c_l)_{y_k}} = (-1)^{|y| q^{|y|} + \binom{|y|}{2}} \prod_{1 \leq l < k \leq p} \frac{c_k q^{y_k} - c_l q^{y_l}}{c_k - c_l}.
\]
This follows from elementary identities for $q$-shifted factorials, after writing
\[
\prod_{k,l=1}^{p} \frac{(qck/c_l)_{yk}}{(q^{-y} c_k/c_l)_{yk}} = \prod_{k=1}^{p} \frac{(q)_{yk}}{(q^{-yk})_{yk}} \prod_{1 \leq k < l \leq p} \frac{(qck/c_l)_{yk}(qc_l/c_k)_{yi}}{(q^{-yk}c_k/c_l)_{yi}}.
\]

4. Corollaries

In this section we point out various interesting special cases and corollaries of Theorem 3.1. Throughout it is assumed that the $m_i$ are non-negative integers and that no denominators in the identities vanish.

4.1. Some immediate consequences. We first rewrite Theorem 3.1 in an alternative way, which hides much of its symmetry but facilitates comparison with related results in the literature. For this we replace $n$ by $n + 1$ and eliminate $y_{n+1} = -y_1 - \cdots - y_n$ from the summation. After the change of variables
\[
z_{n+1} \mapsto a^{-1}, \quad a_{n+1} \mapsto d, \quad b_{n+1} \mapsto aq/b,
a_i \mapsto c_i/z_i, \quad b_i \mapsto aq/c_i z_i, \quad 1 \leq i \leq n, \quad c_i \mapsto aq/f_i
\]
we obtain the following identity.

**Corollary 4.1.** When $|a^{1+n}q^{1-m}|/bCdE| < 1,$ the following identity holds:
\[
\sum_{y_1, \ldots, y_n = -\infty}^{\infty} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^{n} \frac{1 - az_k q^{y_k + |y|}}{1 - az_k} \frac{(b)_{|y|}}{(aq/d)_{|y|}} \prod_{i=1}^{n} \frac{(aqz_k/e_k)_{|y|}}{(aqz_k/f_i)_{|y|}} \prod_{1 \leq k \leq n \leq p} (aq^{1+mz_k/f_i})_{y_k} (a^{1+n}q^{1-m})_{|y|} \right)
\times \prod_{k=1}^{n} \frac{(dz_k)_{yk}}{(aqz_k/b)_{yk}} \prod_{i=1}^{n} \frac{(c_i z_k/z_i)_{yk}}{(aq z_k/c_i z_i)_{yk}} \prod_{1 \leq k \leq n \leq p} (aq z_k/e_k)_{y_k} (aq z_k/f_i)_{y_k}
\times \frac{(aq^{1-m})/dE, a^n q^{|bC|, aq/bd})}{(a^{n+1}q^{1-m}, bCdE, aq/d, q/b, \ldots)} \prod_{i=1}^{n} \frac{(aq z_k/e_k c_i z_i, q z_k/z_i)_{\infty}}{(q z_k/e_k z_i, a q z_k/c_i z_i)_{\infty}}
\times \prod_{k=1}^{m_1, \ldots, m_p} \frac{(aq / dc_k z_k, q/a z_k, a q z_k/b e_k, a q z_k)_{\infty}}{(q / c_k z_k, q/d z_k, a q z_k/b, a q z_k/e_k)_{\infty}} \prod_{1 \leq k \leq n \leq p} \frac{(q^{-m_i f_i / c_k z_k})_{m_i}}{(q^{-m_i f_i / a z_k})_{m_i}} \prod_{i=1}^{p} \frac{(q^{-m_i f_i / b})_{m_i}}{(q^{-m_i f_i / f_i})_{m_i}}
\times \sum_{x_1, \ldots, x_p = 0}^{\infty} \left( \frac{\Delta(q^x/f)}{\Delta(1/f)} \frac{(bC/a^n)_{x}}{(aq^{1-m}/dE)_{x}} \right)
\times \prod_{1 \leq k \leq n \leq p} \frac{(aq z_k/e_k f_i)_{x_k}}{(q c_k z_k/f_i)_{x_k}} \prod_{i=1}^{p} \frac{(aq / d f_i)_{x_i}}{(q b / f_i)_{x_i}} \prod_{i, k=1}^{p} \frac{(q^{-m_k f_k / f_i})_{x_i}}{(q f_k / f_i)_{x_i}}.
\]

The case $p = m_1 = 1$ of Corollary 4.1 is due to Schlosser [33, Theorem 3.4]. It gives a multivariable analogue of Shukla’s $8\psi_8$ summation formula, which is obtained if in addition $n = 1$. Note that in the inductive proof of Theorem 3.1 given above, the case $p = m_1 = 1$ is the first step beyond Gustafson’s formula. The observation that
Schlosser’s \( n \)-variable Shukla summation follows from Gustafson’s \((n + 1)\)-variable Bailey summation was in fact the starting point of the present work.

In \cite{S1} Theorem 4.2, Schlosser found a transformation formula for series like those of Corollary 4.1. This identity arises as an immediate consequence of Theorem 3.1, namely, from the observation that the sum on the right-hand side depends only on the product \( Z = z_1 \cdots z_n \). To obtain the less compact formulation in \cite{S1} one must first rewrite both series as in Corollary 4.1 and then replace \( y_i \) by \(-y_i\) on the right-hand side.

**Corollary 4.2** (Schlosser). Let \( w_i \) and \( z_i \) be parameters with \( z_1 \cdots z_n = w_1 \cdots w_n \). Then, assuming \(|q^{1-n-m}B/A| < 1\), the following identity holds:

\[
\sum_{y_1, \ldots, y_n = -\infty}^{\infty} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq k \leq n} \frac{(c_iz_kq^{m_i})_{y_k}}{(c_iz_k)_{y_k}} \prod_{i,k=1}^{n} \frac{(a_iz_k)_{y_k}}{(b_iz_k)_{y_k}} \\
= \prod_{i,k=1}^{n} \frac{(q/a_kw_i, b_iz_k, qz_k/z_i)}{(q/a_kz_i, b_iz_k, qw_k/w_i)} \prod_{1 \leq k \leq n} \frac{(c_iz_k)m_i}{(c_iz_k)_{m_i}} \\
\times \sum_{y_1, \ldots, y_n = -\infty}^{\infty} \frac{\Delta(wq^y)}{\Delta(w)} \prod_{1 \leq k \leq n} \frac{(c_iz_kq^{m_i})_{y_k}}{(c_iz_k)_{y_k}} \prod_{i,k=1}^{n} \frac{(a_iz_k)_{y_k}}{(b_iz_k)_{y_k}}
\]

Another immediate corollary of Theorem 3.1 is obtained by choosing \( BZ = q^n \), which reduces the sum on the right to one term. Yet again, we obtain an identity of Schlosser \cite{S1} Corollary 4.3.

**Corollary 4.3** (Schlosser). If \( BZ = q^n \) and \(|q^{1-|m|}/AZ| < 1\), the following identity holds:

\[
\sum_{y_1, \ldots, y_n = -\infty}^{\infty} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq k \leq n} \frac{(c_iz_kq^{m_i})_{y_k}}{(c_iz_k)_{y_k}} \prod_{i,k=1}^{n} \frac{(a_iz_k)_{y_k}}{(b_iz_k)_{y_k}} \\
= \prod_{i,k=1}^{n} \frac{(q/a_k, qz_k/z_i)}{(q/a_kz_i, b_iz_k)} \prod_{1 \leq k \leq n} \frac{(qc_i/b_k)m_i}{(c_iz_k)_{m_i}}
\]

When \( n = 2 \), this is an identity of Chu \cite{C} (equivalently, Chu’s identity is the case \( a = bc \) of Corollary 4.1 below). In \cite{S1}, Corollary 4.3 is derived from Corollary 4.2 by choosing \( w_k = q/b_k \) so that the right-hand side is reduced to the term with \( y_k \equiv 0 \). We point out that this also happens under the conditions \( b_k = qa_k \) for \( 1 \leq k \leq n - 1 \) and the choice \( w_k = 1/a_k \) for \( 1 \leq k \leq n - 1 \). Alternatively, we may in this situation use Corollary 4.3 below to sum the right-hand side of (6). After writing \( a_n = b \) and \( b_n = d \) we obtain with either of these two methods the following summation formula, which is a \( U(n) \) generalization of Chu’s identity different from Corollary 4.3.

**Corollary 4.4.** For \(|q^{-m}d/b| < 1\), one has
\[
\sum_{y_1, \ldots, y_n = 0}^{\infty} \frac{\Delta(zq^n)}{\Delta(z)} \prod_{1 \leq k \leq n} \frac{(c_i z_k q^{m_k}) y_k}{(c_i z_k) y_k} \prod_{1 \leq k \leq n} \frac{(az_k) y_k}{(qa z_k) y_k} \prod_{k = 1}^{n} \frac{(b z_k) y_k}{(d z_k) y_k} \\
\left( \frac{q/AbZ, AdZ}{(q)^{\infty}} \prod_{1 \leq k \leq n} (qa z_k/a)_{\infty} \right) \prod_{1 \leq k \leq n} (q z_k/z_i)_{\infty} \prod_{k = 1}^{n} (q a z_k/b, d/a_k)_{\infty} \\
\times \prod_{i = 1}^{p} \frac{(c_i A Z, c_i/a_1, \ldots, c_i/a_{n-1}) m_i}{(c_i z_1, \ldots, c_i z_n) m_i}.
\]

An important special case of Theorem \[3.1\] is when \(b_i z_i = q\) for \(1 \leq i \leq n - 1\). Then only the terms with \(y_i \geq 0\) for \(1 \leq i \leq n - 1\) are non-zero, so that we obtain a multivariable generalization of the unilateral \(\phi\)-series rather than the bilateral \(\psi\)-series. When exhibiting this case explicitly we prefer to start from Corollary \[4.1\], where we put \(c_1 = \cdots = c_n = a\).

**Corollary 4.5.** When \(|aq^{1-m}|/bdE| < 1\), the following identity holds:

\[
\sum_{y_1, \ldots, y_n = 0}^{\infty} \left( \frac{\Delta(zq^n)}{\Delta(z)} \prod_{k = 1}^{n} \frac{1 - az_k q^{y_k+|y|}}{1 - az_k} \right) \prod_{k = 1}^{n} \frac{(aq/d)_y}{(aqz_k/e_k)_y} \prod_{1 \leq k \leq n} \frac{(aq^{1+m} z_k / f_i y_k)}{(aqz_k / f_i y_k)} \left( \frac{aq^{1-m}}{bdE} \right)^{|y|} \\
\times \prod_{i = 1}^{p} \frac{(c_i A Z, c_i/a_1, \ldots, c_i/a_{n-1}) m_i}{(c_i z_1, \ldots, c_i z_n) m_i}.
\]

Yet another interesting case of Theorem \[3.1\] is when the sum on the right-hand side is supported on a hyperplane \(|x| = N\). In this case we prefer to start from \[7\] and assume that \(AZ = q^{-m}, \ BZ = q^n\). Then the factor

\[
\left( \frac{q^{1-m} - N}{AZ} \right)_{\infty} \left( \frac{q^n - N}{BZ} \right)_{|x|} = \left( q^{1-N}|x| \right)_{\infty} \left( q^{-N} \right)_{|x|}
\]
on the right vanishes unless \(|x| = N\), which reduces the finite sum to a \((p-1)\)-variable \(2n+6W_{2n+5}\) rather than a \(p\)-variable \(n+2\phi_{n+1}\). Moreover, the condition for convergence of the left-hand side reduces to \(|q| < 1\), which is automatically satisfied. This leads to the following identity, which may be viewed as a well-poised version of Theorem \[3.1\].

**Corollary 4.6.** If \(AZ = q^{-m}\) and \(BZ = q^n\) the following identity holds:
4.2. Kajihara’s transformations. Theorem 3.1 may be viewed as a transformation formula relating hypergeometric series of different dimension. Although such results are rare, some identities of this type were recently obtained by Kajihara (cf. [GR, K] for other transformations with this property). In fact, it is possible to obtain Kajihara’s identities from Theorem 3.1 by choosing the parameters so that the left-hand side is a finite sum and then applying a standard “polynomial argument”. Starting from the case \( b = q^{-N} \) of Corollary 4.5 we obtain in this way Proposition 6.1 of [K]. In the one-variable case \( n = p = 1 \) this is a Watson-type transformation that may be obtained by combining Equations (III.15) and (III.18) of [GR].

**Corollary 4.7 (Kajihara).** The following identity holds:

\[
\sum_{y_1, \ldots, y_n \geq 0} \left( \frac{\Delta(zq^m)}{\Delta(z)} \prod_{k=1}^{n} \frac{1 - az_k q^{m_k + |y|}}{1 - az_k} \frac{(q^{-N})_{|y|}}{(aq/d)_{|y|}} \prod_{i=1}^{p} \frac{f_i}{(aq g_i)_{|y|}} \right)
\times \prod_{1 \leq k \leq n} \frac{1}{(aq^{1+N}z_k)_{y_k}} \prod_{1 \leq i \leq p} \frac{(aq z_k)_{y_k}}{(aq z_k/f_i)_{y_k}} \frac{(g_i z_k)_{y_k}}{(aq z_k/f_i)_{y_k}} \frac{(a z_k)_{y_k}}{(aq z_k)_{y_k}} \frac{1}{(aq/dz_k)_{y_k}} \prod_{1 \leq k \leq n} \frac{(aq z_k)_{y_k}}{(aq z_k)_{y_k}} \prod_{i=1}^{p} \frac{f_i}{(aq g_i)_{|y|}}
\times \prod_{x_1, \ldots, x_p \geq 0} \left( \frac{\Delta(q^x f_i)}{\Delta(1/f_i)} \frac{(q^{-N})_{|x|}}{(aq^{1+N} f_i)_{|x|}} \prod_{1 \leq k \leq n} \frac{(aq z_k)_{x_i}}{(aq z_k/f_i)_{x_i}} \prod_{i=1}^{p} \frac{(aq/d f_i)_{x_i}}{(aq/d f_i)_{x_i}} \prod_{i, k=1}^{p} \frac{(aq g_k f_i)_{x_i}}{(aq g_k f_i)_{x_i}} \right).
\]

Proof. When \( g_i = aq^{1+m_i}/f_i \), this is the case \( b = q^{-N} \) of Corollary 4.5. Initially we have this only when \( |q^m| > |aq^{1+N}/dE| \). However, since both sides of the identity depend rationally on \( d \), it extends to all non-negative integers \( m_i \). Thus, if we multiply the identity with \( (aq/g_1, \ldots, aq/g_p)_N \) both sides will be polynomials in the variables \( 1/g_i \) that agree at an infinite number of points and are thus identical. \( \square \)

If we repeat the same procedure starting from Corollary 4.6, we recover Kajihara’s beautiful Bailey-type transformation formula [K, Proposition 6.2]. Namely, in the case \( b_i = q/z_i \) the left-hand side of Corollary 4.6 becomes a finite sum. We may
then, as in the proof of Corollary 4.7, remove the condition that the $m_i$ are non-negative integers and obtain the following identity. When $n = p = 2$ it is a $10W_9$ transformation which may be obtained by iterating $[GR$ Equation (III.28)]$. From the proof given above it is apparent that Corollary 4.8 is actually the special case $d = aq^N$, $EFG = a^p q^n$ of Corollary 4.7, but since this is not noted in [K] we list it as a separate corollary.

Corollary 4.8 (Kajihara). For $W = ABZ$, the following identity holds:

$$ \sum_{y_1, \ldots, y_n \geq 0 \atop y_1 + \cdots + y_n = N} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{i,k=1}^n \frac{(a_i z_k)_{y_k}}{(q z_k / z_i)_{y_k}} \prod_{1 \leq k \leq n} \frac{(b_i z_k)_{y_k}}{(w_i z_k)_{y_k}} $$

$$ = \sum_{x_1, \ldots, x_p \geq 0 \atop x_1 + \cdots + x_p = N} \frac{\Delta(wq^x)}{\Delta(w)} \prod_{i,k=1}^p \frac{(w_i/b_k)_{x_i}}{(q w_i/w_k)_{x_i}} \prod_{1 \leq k \leq n} \frac{(w_i/a_k)_{x_i}}{(w_i z_k)_{x_i}}. $$

4.3. Low values of $n$. The right-hand side of (3) is a $p$-variable analogue of a balanced $n+2\phi_{n+1}$-series. In view of the classical $3\phi_2$ summation and $4\phi_3$ transformation formulas, one would expect the cases $n = 1$ and $n = 2$ to be of special interest. Indeed, when $n = 1$ the left-hand side of Theorem 3.1 reduces to 1, so that the multivariable $3\phi_2$ on the right-hand side can be summed. After replacing $p$ by $n$ and relabelling the parameters, we recover the following multivariable $q$-Saalschütz summation formula due to Milne [M3, Theorem 4.1]. The resulting new proof of Milne’s identity is not very natural, but it gives a first illustration of how Theorem 3.1 forms a bridge between different types of identities for hypergeometric functions.

Corollary 4.9 (Milne). If $q^{1-|m|}ab = cd$, the following identity holds:

$$ \sum_{x_1, \ldots, x_n = 0}^{m_1, \ldots, m_n} \frac{\Delta(zq^x)}{\Delta(z)} q^{x[z]} \prod_{i=1}^n \frac{(a_i z_i)_x}{(c_i)_x} \frac{(b_i z_i)_x}{(d_i)_x} \frac{(z_i)_x}{(z_k)_x} = \frac{(d/b)_{|m|}}{(d/ab)_{|m|}} \prod_{i=1}^n \frac{(dz_i/a)_{m_i}}{(dz_i)_{m_i}}. $$

(12)

If we put $n = 1$ in Corollary 4.6, or equivalently multiply (12) with $(d/ab)_{|m|}$ and then let $a = q^{-N}$, $c = q^{1-N}$, we obtain

$$ \sum_{x_1, \ldots, x_p = 0 \atop x_1 + \cdots + x_p = N}^{m_1, \ldots, m_p} \frac{\Delta(zq^x)}{\Delta(z)} \prod_{i=1}^p \frac{(q^{m_i} z_i)_{x_i}}{(dz_i)_{x_i}} \prod_{i,k=1}^p \frac{(q^{-m_i} z_i / z_k)_{x_i}}{(q z_i / z_k)_{x_i}} = \frac{(q^{-|m|})_N}{(q)_N} \prod_{i=1}^p \frac{(q^{m_i} z_i)_N}{(dz_i)_N}. $$

By a polynomial argument (cf. the proof of Corollary 4.7), this is equivalent to a multivariable $q$-Dougall summation of Milne [M2, Theorem 6.17] (stated in more transparent notation in [MN, Theorem A.5]). The observation that Milne’s $q$-Dougall sum follows easily from his $q$-Saalschütz sum appears to be new.
Corollary 4.10 (Milne). The following identity holds:

\[
\sum_{x_1, \ldots, x_n \geq 0 \atop x_1 + \cdots + x_n = N} \frac{\Delta(zq^x)}{\Delta(z)} \prod_{i=1}^{n} \frac{(d_{zi}/E)_{x_i}}{(d_{zi})_{x_i}} \prod_{i, k=1}^{n} \frac{(e_k z_i/z_k)_{x_i}}{(q z_i/z_k)_{x_i}} = \frac{(E)_N}{(q)_N} \prod_{i=1}^{n} \frac{(d_{zi}/e_i)_{N}}{(d_{zi})_{N}}.
\]

Next we turn to the case \( n = 2 \) of Theorem 3.1, when the left-hand side is a one-variable very-well-poised \( p+6\psi_p + 6 \) series, and the sum on the right a \( p \)-variable terminating balanced \( _4\psi_3 \). We write it out explicitly by letting \( n = 1 \) and (without loss of generality) \( z_1 = 1 \) in Corollary 4.1.

Corollary 4.11. For \( |a^2q^{1-|m|}/bcd| < 1 \), the following identity holds:

\[
\sum_{y=-\infty}^{\infty} \frac{1 - a q^{2y}}{1 - a} \frac{(b, c, d, e)_y}{(aq/b, aq/c, aq/d, aq/e)_y} \prod_{i=1}^{p} \frac{(f_i, aq^{1+m_i}/f_i)_y}{(q^{-m_i} f_i, aq/f_i)_y} \frac{(a^2q^{1-|m|})^y}{(bcd)} = \frac{(aq, q/a, q/bc, aq/bd, aq/bc, aq/ce, aq^{1-|m|}/de)_\infty}{(q/b, q/c, q/d, q/e, aq/b, aq/c, aq/d, aq/e, a^2q^{1-|m|}/bcd)_\infty} \times \prod_{i=1}^{p} \frac{(q^{-m_i} f_i, q^{-m_i} f_i/c)_{m_i}}{(q^{-m_i} f_i, q^{-m_i} f_i/a)_{m_i}} \sum_{x_1, \ldots, x_p = 0}^{m_1, \ldots, m_p} \frac{\Delta(q^{x}/f)}{\Delta(1/f)} \frac{(bc/a)|x|}{(aq^{1-|m|}/d)|x|} \times \prod_{i=1}^{p} \frac{(aq/df_i, aq/e f_i)_{x_i}}{(q/b f_i, q/c f_i)_{x_i}} \prod_{i, k=1}^{p} \frac{(q^{-m_k} f_k/f_i)_{x_i}}{(q f_k/f_i)_{x_i}}.
\]

In our opinion, Corollary 4.11 is an interesting result both from the viewpoint of one- and multivariable \( q \)-series. It is interesting both that the bilateral series on the left may be reduced to a finite sum and that the multivariable \( _4\psi_3 \) on the right may be written as a single series. Corollary 4.11 may be compared with Theorem 1.7 of [MI], which also reduces the left-hand side to a finite sum but in a less symmetric way, similar to (3).

If we put \( e = a \) in Corollary 4.11 or \( n = 1 \) in Corollary 4.3, we obtain the following unilateral identity.

Corollary 4.12. For \( |aq^{1-|m|}/bcd| < 1 \), the following identity holds:

\[
\sum_{y=0}^{\infty} \frac{1 - a q^{2y}}{1 - a} \frac{(a, b, c, d)_y}{(aq/b, aq/c, aq/d)_y} \prod_{i=1}^{p} \frac{(f_i, aq^{1+m_i}/f_i)_y}{(q^{-m_i} f_i, aq/f_i)_y} \frac{(aq^{1-|m|})^y}{(bcd)} = \frac{(aq, aq/bc, aq/bd, aq/ce)_\infty}{(aq/b, aq/c, aq/d, aq^{1-|m|}/bcd)_\infty} \prod_{i=1}^{p} \frac{(q^{-m_i} f_i/b, q^{-m_i} f_i/c)_{m_i}}{(q^{-m_i} f_i, q^{-m_i} f_i/a)_{m_i}} \frac{(q^{1-|m|}/d)_{|m|}}{(q^{-1-|m|}/c)_{|m|}} \times \sum_{x_1, \ldots, x_p = 0}^{m_1, \ldots, m_p} \frac{\Delta(q^{x}/f)}{\Delta(1/f)} \frac{(bc/a)|x|}{(q^{1-|m|}/d)|x|} \prod_{i=1}^{p} \frac{(aq/df_i, q f_i)_{x_i}}{(q/b f_i, q c f_i)_{x_i}} \prod_{i, k=1}^{p} \frac{(q^{-m_k} f_k/f_i)_{x_i}}{(q f_k/f_i)_{x_i}}.
\]

The case \( p = 1 \) of Corollary 4.12 gives a version of Watson’s transformation formula, relating a non-terminating \( _4W_7 \) and a terminating \( _4\psi_3 \), which may be obtained by combining equations (III.15) and (III.20) from [GR]. The case \( a = bc \) gives an
identity of Gasper [G2]. Another interesting case arises if we let \(a, b\) and \(f_i\) tend to 0 in such a way that \(aq/b\) and \(aq/f_i\) are fixed. After relabelling the parameters we obtain the following identity. When \(d = bq\), it reduces to Gasper’s [G1], [GR, Equation (II.26)] \(q\)-analogue of the Karlsson–Minton summation formula.

**Corollary 4.13.** For \(|dq^{-|m|}/ab| < 1\), the following identity holds:

\[
\sum_{y=0}^{\infty} \frac{(a,b)_y}{(q,d)_y} \prod_{i=1}^{p} \frac{(c_i q^{m_i})_y}{(c_i)_y} \left( \frac{dq^{-|m|}}{ab} \right)^y = \frac{(d/a,b/d)_\infty}{(d,q^{-|m|}d/ab)_\infty} \prod_{i=1}^{p} \frac{(q c_i/d)_{m_i}}{(c_i)_{m_i}} \left( \frac{d}{q} \right)^{|m|} \frac{q^{1-|m|}/a}{|m|} \prod_{i=1}^{p} \frac{(c_i/a)_{x_i}}{(q c_i/d)_{x_i}} \prod_{i,k=1}^{p} \frac{(q^{-m_k} c_i/c_k)_{x_i}}{}.
\]

If we put \(c = a\) in Corollary 4.11 and then let \(a, b\) and \(f_i\) tend to 0 in such a way that \(aq/b\) and \(aq/f_i\) are fixed, we obtain the following identity.

**Corollary 4.14.** For \(|dq^{-|m|}/ab| < 1\), the following identity holds:

\[
\sum_{y=0}^{\infty} \frac{(a,b)_y}{(q,d)_y} \prod_{i=1}^{p} \frac{(c_i q^{m_i})_y}{(c_i)_y} \left( \frac{dq^{-|m|}}{ab} \right)^y = \frac{(d/a,b/d)_\infty}{(d,q^{-|m|}d/ab)_\infty} \prod_{i=1}^{p} (q^{-m_i} d/c_i)_{m_i} \times \sum_{x_1,\ldots,x_p=0}^{m_1,\ldots,m_p} \frac{\Delta(c q^{x})}{\Delta(c)} \left( \frac{q^{x}}{b} \right)^{|x|} \prod_{i=1}^{p} \frac{(c_i/a, c_i/b)_{x_i}}{(c_i, q c_i/d)_{x_i}} \prod_{i,k=1}^{p} \frac{(q^{-m_k} c_i/c_k)_{x_i}}{(q c_i/c_k)_{x_i}}.
\]

The equality of the left-hand sides of Corollaries 4.13 and 4.14 implies a transformation formula between the sums on the right. More generally, we may start from the observation that the left-hand side of Corollary 4.11 is invariant under interchanging \(b\) and \(d\). This symmetry is peculiar to the case \(n = 2\) of Theorem 3.1. Replacing \(p\) by \(n\) and relabelling the parameters we obtain the following multivariable analogue of Sears’ transformation formula. As is explained below, it is a degenerate case of a multivariable \(_{10}W_9\) transformation due to Milne and Newcomb. If we let \(a\) and \(d\) tend to 0 in Corollary 4.13 with \(a/d\) fixed we recover the \(_3\phi_2\) transformation that connects Corollaries 4.13 and 4.14.

**Corollary 4.15.** If \(q^{1-|m|} abc = def\), the following identity holds:

\[
\sum_{x_1,\ldots,x_n=0}^{m_1,\ldots,m_n} \frac{\Delta(z q^{x})}{\Delta(z)} q^{x|z|} \prod_{i=1}^{n} \frac{(b z_i, c z_i)_{x_i}}{(d z_i, f z_i)_{x_i}} \prod_{i,k=1}^{n} \frac{(q^{-m_k} z_i/z_k)_{x_i}}{} = \frac{(f/b)_{|m|}}{(q^{1-|m|} b/d)_{|m|}} \prod_{i=1}^{n} \frac{(q^{1-|m|} b z_i/d)_{m_i}}{}.
\]

\[
\times \sum_{x_1,\ldots,x_n=0}^{m_1,\ldots,m_n} \frac{\Delta(z q^{x})}{\Delta(z)} q^{x|z|} \prod_{i=1}^{n} \frac{(e/c)_{|x|}}{(q^{1-|m|} b f)_{|x|}} \prod_{i,k=1}^{n} \frac{(e z_i, q^{1-|m|} b z_i/d)_{x_i}}{} \prod_{i,k=1}^{n} \frac{(q^{-m_k} z_i/z_k)_{x_i}}{}.
\]
Finally we consider the “well-poised” specialization of Corollary 4.15 (similar to Corollaries 4.6, 4.8 and 4.10 above). If we multiply (13) with \((q^{-m}/d)_i\) and then let \(a = q^{-N}, \ d = q^{1-N}, \ c = eq^L, \ f = bq^{L-m}\) with \(N\) and \(L\) non-negative integers, we obtain

\[
\sum_{x_1, \ldots, x_n=0}^{m_1, \ldots, m_n} \frac{\Delta(zq^x)}{\Delta(z)} q^{|x|} \prod_{i=1}^{n} \frac{1-az_iq^{x_i+|x|}}{1-az_i} \frac{(b_{z_i}, c_{z_i}, d_{z_i})_{x_i}}{(aqz_i/e, aqz_i/f, aqz_i/g)_{x_i}} \times \prod_{i,k=1}^{n} \frac{(q^{-m_k}z_i/z_k)_{x_i}}{(qz_i/z_k)_{x_i}} = \frac{(q)_L(q^{-m})_N}{(q)_N(q^{-m})_L} \prod_{i=1}^{n} \frac{(f_i, aq^{1+m}/f_i)_{y}}{(aq/e, q^{1+m}/e, aq^{1+L}/e, q^{L+1})_{y}} \prod_{i=1}^{n} \frac{(q^{-m}, f_i, aq/f_i)_{y}}{q^{y}}.
\]

(14)

This corresponds to writing the series

\[
\sum_{y=-\infty}^{\infty} \frac{1-aq^{2y}}{1-a} \frac{(e, aq^{-N}/e, q^{N+L-m}/e, aq^{-L}/e)_{y}}{(aq/e, q^{N+L}/e, aq^{1+m}/e, q^{L+1})_{y}} \prod_{i=1}^{n} \frac{(f_i, aq^{1+m}/f_i)_{y}}{(aq/f_i)_{y}} q^{y}
\]

(with \(a = q^{N+L-m}bc, \ f_i = q/z_i\) as a finite sum in two ways using Corollary 4.11.)

Equation (14) is a multivariable \(10W_9\) transformation closely related to those of Milne and Newcomb [MN]. We indicate how to recover [MN, Theorem 3.1] from (14). One should then replace \(n\) by \(n+1\) in (14) and eliminate \(x_{n+1}\) from the summations. By a polynomial argument, as in the proof of Corollary 4.7, one may first remove the condition \(m_{n+1} \in \mathbb{N}\) and then the conditions \(N \in \mathbb{N}\) and \(L \in \mathbb{N}\). After relabelling the parameters, one obtains the following identity.

**Corollary 4.16** (Milne and Newcomb). Assuming that \(bcdefg = a^3q^{2|m|}\) and writing \(\lambda = qa^{2/bcf}\), one has the identity

\[
\sum_{x_1, \ldots, x_n=0}^{m_1, \ldots, m_n} \frac{\Delta(zq^x)}{\Delta(z)} q^{|x|} \prod_{i=1}^{n} \frac{1-az_iq^{x_i+|x|}}{1-az_i} \frac{(b_{z_i}, c_{z_i}, d_{z_i})_{x_i}}{(aqz_i/e, aqz_i/f, aqz_i/g)_{x_i}} \times \prod_{i,k=1}^{n} \frac{(q^{-m_k}z_i/z_k)_{x_i}}{(qz_i/z_k)_{x_i}} = \left(\frac{\lambda}{\lambda}\right)^{|m|} \frac{(\lambda/q, \lambda q/d)^{|m|}}{(aq/c, aq/d)^{|m|}} \prod_{i=1}^{n} \frac{(aqz_i, \lambda qz_i/g)_{m_i}}{(aqz_i, aqz_i/g)_{m_i}} \times \sum_{x_1, \ldots, x_n=0}^{m_1, \ldots, m_n} \frac{\Delta(zq^x)}{\Delta(z)} q^{|x|} \prod_{i=1}^{n} \frac{1-\lambda z_iq^{x_i+|x|}}{1-\lambda z_i} \frac{(aqz_i/e, cz_i, dz_i)_{x_i}}{(aqz_i/e, cz_i, dz_i)_{x_i}} \times \prod_{i,k=1}^{n} \frac{(aqz_i/e, cz_i, dz_i)_{x_i}}{(aqz_i/e, cz_i, dz_i)_{x_i}} \frac{(\lambda z_i)_{|x|}}{(\lambda z_i)_{|x|}}.
\]

If we let \(a, \ d, \ f, \ g \to 0\) in Corollary 4.16 in such a way that \(a/d, a/f\) and \(a/g\) are fixed, we recover Corollary 4.15.
4.4. **Generalized $1\psi_1$ series.** Next we consider identities that may be obtained from Theorem 3.1 by multiplying both sides of (7) with some function $f(N)$ and then summing over $N$. In general, this gives

$$
\sum_{y_1, \ldots, y_n = -\infty}^{\infty} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq k \leq n}^{\infty} \frac{(c_i z_k q^m y_k)}{(c_i z_k y_k)} \prod_{i,k=1}^{n} (a_i z_k y_k) f(|y|)
$$

$$
= \frac{(q^{1-m}/AZ, q^{1-n}BZ)_{\infty}}{(q, q^{1-m-n}B/A)_{\infty}} \prod_{i,k=1}^{n} \frac{(b_i/a_k, bq z_k/z_i)_{\infty}}{(q/a_k z_i, b z_k)_{\infty}} \prod_{1 \leq k \leq n}^{\infty} \frac{(q^{-m_i} b_k/c_i)_{m_i}}{(q^{1-m_i}/c_i z_k)_{m_i}}
$$

$$
\times \sum_{x_1, \ldots, x_p = 0}^{m_1, \ldots, m_p} \left( \frac{\Delta(q^{y-x})}{\Delta(c)} \frac{(q^n/BZ)_{|x|}}{(q^{1-m}/AZ)_{|x|}} \prod_{1 \leq k \leq n}^{\infty} \frac{(c_i/a_k)_{x_i}}{(q c_i/b_k)_{x_i}} \prod_{i,k=1}^{p} \frac{(q^{-m_k} c_i/c_k)_{x_i}}{(q c_i/c_k)_{x_i}} \right)
$$

which is valid whenever the series involved are absolutely convergent. This is of course most interesting when the sum in $N$ simplifies. As an example, when $f(N) = t^N$ an application of Ramanujan’s $1\psi_1$ sum [GR, Equation (II.29)] yields the following identity, which reduces to Gustafson’s multivariable $1\psi_1$ sum [Gu1, Theorem 1.17] when $p = 0$.

**Corollary 4.17.** For $|q^{1-m}| - n B/A| < |t| < 1$, the following identity holds:

$$
\sum_{y_1, \ldots, y_n = -\infty}^{\infty} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq k \leq n}^{\infty} \frac{(c_i z_k q^m y_k)}{(c_i z_k y_k)} \prod_{i,k=1}^{n} \frac{(a_i z_k y_k)}{(b_i z_k y_k)} t^{|y|}
$$

$$
= \frac{(AZ t, q/AZ t)_{\infty}}{(t, q^{1-n}B/A t)_{\infty}} \frac{1}{(q^n A t/B)_{|x|}} \prod_{i,k=1}^{n} \frac{(b_i/a_k, q z_k/z_i)_{\infty}}{(q/a_k z_i, b z_k)_{\infty}} \prod_{1 \leq k \leq n}^{\infty} \frac{(q c_i/b_k)_{m_i}}{(c_i z_k)_{m_i}}
$$

$$
\times \sum_{x_1, \ldots, x_p = 0}^{m_1, \ldots, m_p} \frac{\Delta(q^{y-x})}{\Delta(c)} \frac{(q^n A t/B)_{|x|}}{(q^n t/B)_{|x|}} \prod_{1 \leq k \leq n}^{\infty} \frac{(c_i/a_k)_{x_i}}{(q c_i/b_k)_{x_i}} \prod_{i,k=1}^{p} \frac{(q^{-m_k} c_i/c_k)_{x_i}}{(q c_i/c_k)_{x_i}}.
$$

It may be interesting to note that if we choose $f(N) = t^N \prod_{i=1}^{r} (q/d_i q^i)_{N}/(d_i)_N$ above, then the sum in $N$ is reduced to a finite sum by the case $n = 1$ of Corollary 4.17. It follows that the more general series

$$
\sum_{y_1, \ldots, y_n = -\infty}^{\infty} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{i=1}^{r} \frac{(d_i q^i)_{|y|}}{(d_i)_{|y|}} \prod_{1 \leq k \leq n}^{\infty} \frac{(c_i z_k q^m y_k)}{(c_i z_k y_k)} \prod_{i,k=1}^{n} \frac{(a_i z_k y_k)}{(b_i z_k y_k)} t^{|y|}
$$

may be reduced to a finite sum. The resulting identity is awkward and we do not write it out explicitly. However, the special case when $p = 0$ is much nicer and we state it as Corollary 4.18. Note that we have proved this identity by first applying
the case $p = 0$ of Theorem [3.1] that is, Gustafson’s identity (3), next Ramanujan’s
$1\psi_1$ sum and finally the case $n = 1$ of Theorem [3.1], which as we have seen is due to
Milne (Corollary [4.9]). Thus we have actually derived Corollary 4.18 by combining
previously known results only.

**Corollary 4.18.** For $|q^{1-|l|-n}B/A| < |t| < 1$, the following identity holds:

$$
\sum_{y_1,\ldots,y_n=-\infty}^{\infty} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{i=1}^{r} \frac{(d_iq^l)_{|y|}}{(d_i)_{|y|}} \prod_{i,k=1}^{n} \frac{(a_i z_k)_{y_k} t^{|y|}}{(b_i z_k)_{y_k}}
= \frac{(AZt, q/AZt)_{\infty}}{(AZtu, q/AZtu)_{\infty}} \prod_{i=1}^{n} \frac{(b_i/a_k, q z_k/z_i)_{\infty}}{(q/a_k z_i, b_i z_k)_{\infty}} \prod_{i=1}^{r} \frac{(q^n d_i/BZ)_{l_i}}{(d_i)_{l_i}}
\times \sum_{x_1,\ldots,x_r=0}^{l_1,\ldots,l_r} \frac{\Delta(qx^l)}{\Delta(d)} \left(\frac{q^n|y| A_t}{B} \right) \prod_{i=1}^{r} \frac{(d_i/AZ)_{x_i}}{(q^n d_i/BZ)_{x_i}} \prod_{i=1}^{r} \frac{(q^{-k} d_i/d_k)_{x_i}}{(q d_i/d_k)_{x_i}}.
$$

The sum on the right-hand side of Corollary [4.18] depends effectively on considerably fewer parameters than the one on the left. We may exploit this to find a
transformation formula for the left-hand side, similarly as in Corollary [4.2].

**Corollary 4.19.** Let $l_i$, be non-negative integers and $a, b, z \in \mathbb{C}^n$, $\bar{a}, \bar{b}, \bar{z} \in \mathbb{C}^m$, $t, u \in \mathbb{C}$ be parameters with $\tilde{A}Z = uAZ$, $\tilde{B}Z = q^{m-n}uBZ$ and $|q^{1-|l|-n}B/A| < |t| < 1$. Then the following transformation formula holds:

$$
\sum_{y_1,\ldots,y_n=-\infty}^{\infty} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{i=1}^{r} \frac{(d_iq^l)_{|y|}}{(d_i)_{|y|}} \prod_{i,k=1}^{n} \frac{(a_i z_k)_{y_k} t^{|y|}}{(b_i z_k)_{y_k}}
= \frac{(AZt, q/AZt)_{\infty}}{(AZtu, q/AZtu)_{\infty}} \prod_{i=1}^{n} \frac{(b_i/a_k, q z_k/z_i)_{\infty}}{(q/a_k z_i, b_i z_k)_{\infty}} \prod_{i=1}^{r} \frac{(q/a_k)_{l_i}}{(d_i)_{l_i}}
\times \sum_{y_1,\ldots,y_m=-\infty}^{\infty} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{i=1}^{r} \frac{(ud_iq^l)_{|y|}}{(ud_i)_{|y|}} \prod_{i,k=1}^{m} \frac{(a_i z_k)_{y_k} t^{|y|}}{(b_i z_k)_{y_k}}.
$$

In the special case when $m = n$ and $\bar{a}_j b_j = a_j \bar{b}_j$, this is equivalent to Theorem 4.6
in [4.1]. Another interesting case is $m = 1$, when the right-hand side reduces to a
one-variable $r+1^r+1$, or an $r+1^r+1$ if we choose $\bar{b}_1 z_1 = q$, $u = q^n/BZ$.

5. The Case $q = 1$

In this section we consider the case of classical hypergeometric series ($q = 1$). Thus
we will (instead of (4)) use the standard notation

$$
(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} a(a+1) \cdots (a+k-1), & k \geq 0, \\ \frac{1}{(a-1)(a-2) \cdots (a-k)}, & k < 0. \end{cases}
$$
In [Gu1], Gustafson proved the identities

\begin{align*}
\sum_{y_1,...,y_n=-\infty}^{\infty} \frac{\Delta(z+y)}{\Delta(z)} \prod_{i,k=1}^{n} \frac{(a_i + z_k)y_k}{(b_i + z_k)y_k} \\
= \frac{\Gamma(1 + |b| - |a| - n)}{\Gamma(1 - |a| - |z|) \Gamma(1 + |b| + |z| - n)} \prod_{i,k=1}^{n} \frac{\Gamma(1 - a_k - z_i) \Gamma(b_i + z_k)}{\Gamma(b_i - a_k) \Gamma(1 + z_k - z_i)}
\end{align*}

(16)

and

\begin{align*}
\sum_{y_1,...,y_n=-\infty}^{\infty} \frac{\Delta(z+y)}{\Delta(z)} \prod_{1\leq k\leq n \atop 1\leq i \leq n+1} \frac{(a_i + z_k)y_k}{(b_i + z_k)y_k} \\
= \frac{\Gamma(|b| - |a| - n)}{\prod_{1 \leq i,k \leq n+1} \Gamma(b_i - a_k) \prod_{1 \leq i \leq n+1} \Gamma(1 + z_k - z_i)} \prod_{1 \leq i \leq n+1} \Gamma(1 - a_i - z_k) \Gamma(b_i + z_k),
\end{align*}

(17)

which reduce to Dougall’s $_5H_5$ and $_2H_2$ summation formulas, when $n = 2$ and $n = 1$, respectively. The identity (16) may, at least formally, be obtained from (17) by rescaling the parameters and letting $q$ tend to 1.

Both these identities may be used to prove Karlsson–Minton type identities, exactly as in the proof of Theorem 3.1. Starting with (16) gives the following identity, which may also be obtained as the formal limit of Theorem 3.1 as $q \to 1$.

**Theorem 5.1.** For $\text{Re} \ (|b| - |a|) > n + |m| - 1$, the following identity holds:

\begin{align*}
\sum_{y_1,...,y_n=-\infty}^{\infty} \frac{\Delta(z+y)}{\Delta(z)} \prod_{1\leq k\leq n \atop 1\leq i \leq p} \frac{(c_i + z_k + m_i)y_k}{(c_i + z_k)y_k} \prod_{i,k=1}^{n} \frac{(a_i + z_k)y_k}{(b_i + z_k)y_k} \\
= \frac{\Gamma(1 + |b| - |a| - |m| - n)}{\Gamma(1 - |a| - |z| - |m|) \Gamma(1 + |b| + |z| - n)} \prod_{i,k=1}^{n} \frac{\Gamma(1 - a_k - z_i) \Gamma(b_i + z_k)}{\Gamma(b_i - a_k) \Gamma(1 + z_k - z_i)} \\
\times \prod_{1 \leq k \leq n \atop 1 \leq i \leq p} \frac{(1 + c_i - b_k)m}{(c_i + z_k)m_i} \sum_{x_1,...,x_p=0}^{m_1,...,m_p} \frac{\Delta(c + x)}{\Delta(c)} \frac{(n - |b| - |z|)|_{[x]}^{|m|_{[x]}}}{(1 - |a| - |z| - |m|)|_{[x]}} \\
\times \prod_{1 \leq k \leq n \atop 1 \leq i \leq p} \frac{(c_i - a_k)x_i}{(1 + c_i - b_k)x_i} \prod_{i,k=1}^{p} \frac{(c_i - c_k - m_k)x_i}{(1 + c_i - c_k)x_i}.
\end{align*}

All the corollaries in Section 4 up to and including Corollary 4.16 have classical versions. In particular, the classical limit of Corollary 4.13 is (2). The classical limit
of Corollary \[ \text{Cor. 4.14} \] is
\[
\begin{align*}
&\sum_{0}^{r+2} \frac{F_{r+1}}{F_{r+1}} \left( a, b, c_1 + m_1, \ldots, c_r + m_r ; \right) \\
&= (-1)^{|m|} \frac{\Gamma(d)\Gamma(d - a - b - |m|)}{\Gamma(d - a)\Gamma(d - b)} \prod_{i=1}^{r} (1 + c_i - d)_{m_i} \\
&\times \sum_{x_1, \ldots, x_r = 0}^{m_1, \ldots, m_r} \frac{\Delta(c + x)}{\Delta(c)} \prod_{i=1}^{r} \frac{(c_i - a)_{x_i}(c_i - b)_{x_i}}{(c_i)_{x_i}(1 + c_i - d)_{x_i}} \prod_{i,k=1}^{r} \frac{(c_i - c_k - m_k)_{x_i}}{(1 + c_i - c_k)_{x_i}}.
\end{align*}
\]

Using the multivariable \(2H_2\) sum \([17]\) one may prove the following identity. The proof is very similar to that of Theorem \[ \text{3.1} \] but slightly easier, so we do not give the details.

**Theorem 5.2.** For \( \text{Re} \left( |b| - |a| \right) > n + |m| \), the following identity holds:
\[
\begin{align*}
&\sum_{y_1, \ldots, y_n = -\infty}^{0} \frac{\Delta(z + y)}{\Delta(z)} \prod_{1 \leq k \leq n}^{1 \leq i \leq p} \frac{(c_i + z_k + m_i)_{y_k}}{(c_i + z_k)_{y_k}} \prod_{1 \leq k \leq n}^{1 \leq i \leq n+1} \frac{(a_i + z_k)_{y_k}}{(b_i + z_k)_{y_k}} \\
&= (-1)^{|m|} \prod_{i,k=1}^{n+1} \frac{\Gamma(|b| - |a| - |m| - n)}{\Gamma(b_i - a_k)\Gamma(1 + z_k - z_i)} \\
&\times \prod_{1 \leq k \leq n}^{1 \leq i \leq n+1} \frac{\Gamma(1 - a_i - z_k)}{\Gamma(b_i + z_k)} \prod_{i=1}^{p} \frac{\Gamma(k=1) \Gamma(1 + c_i - b_k)_{m_i}}{\Gamma(k=1) (c_i + z_k)_{m_i}} \\
&\times \sum_{x_1, \ldots, x_p = 0}^{m_1, \ldots, m_p} \frac{\Delta(c + x)}{\Delta(c)} \prod_{1 \leq k \leq n+1}^{1 \leq i \leq p} \frac{(c_i - a_k)_{x_i}}{(1 + c_i - b_k)_{x_i}} \prod_{i,k=1}^{p} \frac{(c_i - c_k - m_k)_{x_i}}{(1 + c_i - c_k)_{x_i}}.
\end{align*}
\]

When \( n = 1 \), this may be written
\[
\begin{align*}
&\sum_{y = -\infty}^{0} \frac{(a)_{y} (b)_{y}}{(d)_{y} (e)_{y}} \prod_{i=1}^{p} \frac{(c_i + m_i)_{y}}{(c_i)_{y}} = (-1)^{|m|} \prod_{i=1}^{p} \frac{(1 + c_i - d)_{m_i}(1 + c_i - e)_{m_i}}{(c_i)_{m_i}} \\
&\times \frac{\Gamma(1 + e - a - b - |m| - 1)\Gamma(1 - a)\Gamma(1 - b)\Gamma(d)\Gamma(e)}{\Gamma(d - a)\Gamma(d - b)\Gamma(e - a)\Gamma(e - b)} \\
&\times \sum_{x_1, \ldots, x_p = 0}^{m_1, \ldots, m_p} \frac{\Delta(c + x)}{\Delta(c)} \prod_{i=1}^{p} \frac{(c_i - a)_{x_i}(c_i - b)_{x_i}}{(1 + c_i - d)_{x_i}(1 + c_i - e)_{x_i}} \prod_{i,k=1}^{p} \frac{(c_i - c_k - m_k)_{x_i}}{(1 + c_i - c_k)_{x_i}}.
\end{align*}
\]
When \( p = m_1 = 1 \) this is a \(3H_3\) summation formula of Bailey \([B2]\), and when \( e = 1 \) we recover \([18]\).

**References**

[B1] W. N. Bailey, *An identity involving Heine’s basic hypergeometric series*, J. London Math. Soc. 4 (1929), 254–257.
[B2] W. N. Bailey, *On the sum of a particular bilateral hypergeometric series 3H_3*, Quart. J. Math. Oxford (2) 10 (1959), 92–94.
[C] W. Chu, *Partial-fraction expansions and well-poised bilateral series*, Acta Sci. Math. (Szeged) 64 (1998), 495–513.
[FW] J. L. Fields and J. Wimp, *Expansions of hypergeometric functions in hypergeometric functions*, Math. Comp. 15 (1961), 390–395.

[F] C. Fox, *The expression of hypergeometric series in terms of similar series*, Proc. London Math. Soc. (2) 26 (1927), 201–210.

[G1] G. Gasper, *Summation formulas for basic hypergeometric series*, SIAM J. Math. Anal. 15 (1984), 196–200.

[G2] G. Gasper, *Elementary derivations of summation and transformation formulas for q-series*, Special Functions, q-Series and Related Topics, 55–70, Fields Inst. Commun. 14, Providence, RI, 1997.

[GR] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.

[GK] I. M. Gessel and C. Krattenthaler, *Cylindrical partitions*, Trans. Amer. Math. Soc. 349 (1997), 429–479.

[Gu1] R. A. Gustafson, *Multilateral summation theorems for ordinary and basic hypergeometric series in U(n)*, SIAM J. Math. Anal. 18 (1987), 1576–1596.

[Gu2] R. A. Gustafson, *The Macdonald identities for affine root systems of classical type and hypergeometric series very-well-poised on semisimple Lie algebras*, Ramanujan International Symposium on Analysis (Pune, 1987), 185–224, Macmillan of India, New Delhi, 1989.

[HBL] W. J. Holman, L. C. Biedenharn and J. D. Louck, *On hypergeometric series well-poised in SU(n)*, SIAM J. Math. Anal. 7 (1976), 529–541.

[K] Y. Kajihara, *Euler transformation formulas for multiple basic hypergeometric series of type A and some applications*, Adv. Math., to appear.

[Ka] P. W. Karlsson, *Hypergeometric functions with integral parameter differences*, J. Math. Phys. 12 (1971), 270–271.

[Kr] C. Krattenthaler, *Proof of a summation formula for an A\(_n\) basic hypergeometric series conjectured by Warnaar*, Contemp. Math. 291 (2001), 153–161.

[M1] S. C. Milne, *A multiple series transformation of the very well poised 2\(_{k+4}\)Ψ\(_{2k+4}\)*, Pacific J. Math. 91 (1980), 419–430.

[M2] S. C. Milne, *Multiple q-series and U(n) generalizations of Ramanujan’s 1Ψ1 sum*, Ramanujan revisited (Urbana-Champaign, Ill., 1987), 473–524, Academic Press, Boston, MA, 1988.

[M3] S. C. Milne, *Balanced \(3\phi2\) summation theorems for U(n) basic hypergeometric series*, Adv. Math. 131 (1997), 93–187.

[MN] S. C. Milne and J. W. Newcomb, U(n) very-well-poised \(10\phi9\) transformations, J. Comput. Appl. Math. 68 (1996), 239–285.

[Mi] B. Minton, *Generalized hypergeometric function of unit argument*, J. Math. Phys. 11 (1970), 1375–1376.

[R] H. Rosengren, *Karlsson–Minton type hypergeometric functions on the root system C\(_n\)*, in preparation.

[S1] M. Schlosser, *Multilateral transformations of q-series with quotients of parameters that are nonnegative integral powers of q*, Contemp. Math. 291 (2001), 203–227.

[S2] M. Schlosser, *Elementary derivations of identities for bilateral basic hypergeometric series*, Selecta Math., to appear.

[S3] M. Schlosser, *A multidimensional generalization of Shukla’s \(8\psi8\) summation*, Constr. Approx., to appear.

[Sh] H. S. Shukla, *A note on the sums of certain bilateral hypergeometric series*, Proc. Cambridge Philos. Soc. 55 (1959), 262–266.

Department of Mathematics, Chalmers University of Technology and Göteborg University, SE-412 96 Göteborg, Sweden

E-mail address: hjalmar@math.chalmers.se