On asymptotic stability of $N$–solitons of the Gross–Pitaevskii equation

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Abstract

We consider the Cauchy problem for the Gross–Pitaevskii (GP) equation. Using the generalization of the nonlinear steepest descent method of Deift and Zhou we derive the leading order approximation to the solution of GP in the solitonic region of space–time $|x| < 2t$ for large times and provide bounds for the error which decays as $t \to \infty$ for a general class of initial data whose difference from the non vanishing background possess's a fixed number of finite moments and derivatives. Using properties of the scattering map of NLS we derive as a corollary an asymptotic stability result for initial data which are sufficiently close to the $N$-dark soliton solutions of (GP).

1 Introduction and statement of main results

We consider the Cauchy problem with finite density initial data for the defocusing nonlinear Schrödinger/Gross-Pitaevskii (NLS/GP) equation on $\mathbb{R}$:

$$i q_t + q_{xx} - 2(|q|^2 - 1)q = 0 \quad (1.1)$$

$$q(x, 0) = q_0(x), \quad \lim_{x \to \pm \infty} q_0(x) = \pm 1. \quad (1.2)$$

The NLS equation is one of the basic models of nonlinear physics, used to study Bose-Einstein condensates, water waves, plasma physics, in addition to its many applications in nonlinear optics.

Remark. The general one dimension Gross-Pitaevskii (GP) equation that appears in the modeling of Bose-Einstein condensates on a nonzero background is $i\psi_t + \psi_{xx} - 2(|\psi|^2 - 1)\psi + V(x)\psi = 0$ where $\psi$ is the wave function of a single particle and $V$ is an external potential. Equation (1.1) is just the particular case of the (GP) equation in which the particle is free, i.e., $V \equiv 0$.

Remark. The usual form of the NLS equation is $iu_t + u_{xx} + 2\sigma u|u|^2 = 0$ where $\sigma = 1$ is called the focusing and $\sigma = -1$ the defocusing NLS equation. The change of variables $q(x, t) = u(x, t)e^{-2\sigma t}$ reduces the defocusing NLS equation to (1.1). This form has the advantage that solutions of (1.1) which satisfy (1.2) are, as $x \to \infty$, asymptotically time independent as is easily checked for plane wave solutions of (1.1) – (1.2).

It is an elementary fact that solutions of the linear Schrödinger equation $iq_t + q_{xx} = 0$ disperse, i.e., $q(x, t) = O(t^{-1/2})$ as $t \to \infty$. Once nonlinear effects are included soliton solutions appear. These
special solutions do not disperse, but instead, the nonlinear effects balance the dispersive, to create solutions which persist for all time. In the more familiar case in which the initial data \( q_0(x) \) vanishes as \( |x| \to \infty \) sufficiently quickly, only the focusing NLS equation supports soliton solutions. For the finite density type of data considered in (1.1)-(1.2) the defocusing equation also possesses soliton solutions. Let \( \partial D(0,1) = \{ z \in \mathbb{C} : |z| = 1 \} \). For \( z_0 \in \partial D(0,1) \cap \mathbb{C}^+ \), write \( z_0 = z_{0R} + iz_{0I} \) with \( z_{0I} > 0 \). Define
\[
\text{sol}(x,t,z_0) := -iz_0(i z_{0R} + z_{0I} \tanh(z_{0I}(x-2z_{0R}t))).
\] (1.3)

Then \( q(x,t) = \text{sol}(x-x_0,t,z_0) \) is a traveling wave solution of (1.1) satisfying \( \lim_{x \to \infty} q(x,t) = 1 \) and \( \lim_{x \to -\infty} q(x,t) = z_0^2 \). We call these solutions 1-solitons, or simply solitons. More generally, given a collection of points \( \{ z_k \}_{k=0}^{N-1} \subset \partial D(0,1) \cap \mathbb{C}^+ \) one can construct more elaborate exact soliton solutions \( q^{(sol),N}(x,t) \), called \( N \)-solitons which do not disperse, instead for sufficiently large times they resemble the sum of \( N \) individual 1-solitons: \( q^{(sol),N}(x,t) \sim \sum_{k=0}^{N-1} \text{sol}(x-x_k,t,z_k) \). Such solutions are constructed in Appendix A.

The soliton resolution conjecture is the vaguely stated, but widely believed statement that the evolution of generic initial data for many globally well posed nonlinear dispersive equations will in the long time limit resolve into a finite train of solitons plus some dispersing radiative component. For most dispersive evolution equations this is a wide open and active area of research. The situation is somewhat better understood in the integrable setting, where the machinery of inverse scattering transform (IST) gives one much stronger control on the behavior of solutions than purely analytic techniques. Even among the integrable evolutions, most results concern initial data with sufficient decay at spatial infinity, but there has been some more recent studies concerning non vanishing initial data for [23, 7, 30, 41, 42].

As we review below, problem (1.1)-(1.2) is integrable— as discovered by Zakharov–Shabat [44]— and its solution can be characterized in terms of an IST. Briefly, the Lax-pair representation of (1.1) (c.f. (3.1)) encodes the solution of NLS as a time evolving potential in a certain spectral problem, (3.5), on the line. Under weak assumptions on \( q_0 \) the scattering map associates to \( q_0 \) a finite set of discrete spectra \( \{ z_j \}_{j=0}^{M-1} \subset \partial D(0,1) \cap \mathbb{C}^+ \), which we will refer to as the poles, to each pole is associated a single coupling constant \( c_j \in iz_j \mathbb{R}_+ \), and on the continuous spectrum a reflection coefficient \( r \) must be computed. The set \( \{ r(z); \{ z_j, c_j \}_{j=0}^{M-1} \} \) are called the scattering data associated with \( q_0 \). The miraculous fact is that the evolution of the scattering data is trivial, and an inverse scattering map can be constructed in terms of a Riemann-Hilbert problem where the spatio-temporal dependence appears only parametrically. This characterization of the inverse map is ideally suited to rigorously analysis via the Deift-Zhou steepest descent method and has been key to deriving detailed asymptotic expansions of NLS and other integrable evolutions in various asymptotic regimes [8, 10, 18, 17, 33, 34]. In particular we mention the work of Vartanian on the problem (1.1)-(1.2) under some stricter, somewhat non-generic, conditions on the initial data [11, 12, 43].

Our first result is a partial verification of the soliton resolution conjecture for (1.1) for initial data of finite density type (1.2) which also has a certain number of derivatives and moments. To state the
theorem precisely we introduce the Japanese bracket $\langle x \rangle := \sqrt{1 + |x|^2}$ and the normed spaces

\[ L^p,\varphi(\mathbb{R}) \text{ with } \|q\|_{L^p,\varphi(\mathbb{R})} := \|\langle x \rangle^\varphi q\|_{L^p(\mathbb{R})}; \]

\[ W^{k,p}(\mathbb{R}) \text{ defined with } \|q\|_{W^{k,p}(\mathbb{R})} := \sum_{j=0}^k \|\partial^j q\|_{L^p(\mathbb{R})} \text{ where } \partial^j u \text{ is the } j^{th} \text{ weak derivative of } u \]

\[ \dot{H}^k(\mathbb{R}) \text{ defined with } \|q\|_{\dot{H}^k(\mathbb{R})} := ||x|^k \hat{q}||_{L^2(\mathbb{R})} \text{ where } \hat{u} \text{ is the Fourier transform;} \]

\[ H^k(\mathbb{R}) \text{ defined with } \|q\|_{H^k(\mathbb{R})} := ||\langle x \rangle^k \hat{q}||_{L^2(\mathbb{R})}, \text{ such that } H^k(\mathbb{R}) = \dot{H}^k(\mathbb{R}) \cap L^2(\mathbb{R}) \]

\[ H^{k,s}(\mathbb{R}) \text{ defined with } \|q\|_{H^{k,s}(\mathbb{R})} := \|\langle x \rangle^s q\|_{H^k(\mathbb{R})} \]

\[ \Sigma_k := L^{2,k}(\mathbb{R}) \cap H^k(\mathbb{R}). \]

We set $K_+ = K \setminus \{0\}$ for $K = \mathbb{R}, \mathbb{C}; \mathbb{C}^+ = \{ z \in \mathbb{C} : \pm z t > 0 \};$ and $\mathbb{R}_+ = (0, \infty).$

**Theorem 1.1.** Consider an initial datum $q_0 \in \text{tanh}(x) + \Sigma_5$ which generates scattering data \( \{z_j\}_{j=0}^{N-1} \subset \partial \Omega(0,1) \cap \mathbb{C}_+, \{c_j \in i \mathbb{R}_+\}_{j=0}^{N-1} \) and an $r(z) \in H^1(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ s.t.

\[ \| \log(1 - |r|^2) \|_{L^p(\mathbb{R})} < \infty \text{ for all } p \in [1, \infty). \quad (1.4) \]

Let the the $z_j$ be ordered such that

\[ \text{Re } z_0 > \text{Re } z_2 > \cdots > \text{Re } z_{N-1}. \quad (1.5) \]

Let $\xi_0 = \frac{z_0}{2\pi t}$ and fix a $\xi_0 \in (0,1)$. Assume that $r(z)$ satisfies (1.4). Then there exist $T = T(q_0, \xi_0)$ and $C = C(q_0, \xi_0)$ s.t. the solution of $q(x,t)$ of (1.1) satisfies

\[ \left| q(t,x) - T(\infty, \xi)^{-2}q^{(s\text{o})\cdot N}(t,x) \right| \leq C t^{-1} \text{ for all } t > T \text{ and } |\xi| \leq \xi_0. \quad (1.6) \]

Here $T(\infty, \xi)$, is the complex phase, i.e. $|T(\infty, \xi)| = 1$, defined in Lemma 6.1, and $q^{(s\text{o})\cdot N}(x,t)$ is the $N$-soliton with scattering data $\{z_j \in \mathbb{C} \cap S^1\}_{j=0}^{N-1}, \{c_j \in i \mathbb{R}_+, \xi_0\}_{j=0}^{N-1}$ where $\{c_j\}$ is given by

\[ \bar{c}_j(t,x) = c_j(t,x) \exp \left( -\frac{1}{17} \int_0^\infty \log(1 - |r(s)|^2) \left( \frac{1}{s - z_j} - \frac{1}{2s} \right) ds \right). \quad (1.7) \]

Moreover, for $t > T$, the $N$-soliton solution separates in the sense that

\[ q(t,x) = \delta_+^{-1} \left[ 1 + \sum_{k=0}^{N-1} \prod_{j<k}^N \frac{z_j^2}{z_j^2} \right] \left[ \text{sol}(t,x-x_k, z_k) - 1 \right] + \mathcal{O}(t^{-1}), \quad (1.8) \]

where $\text{sol}(t,x,z)$ is the one soliton defined by (1.3), and

\[ \delta_+ = \exp \left( \frac{1}{2\pi t} \int_0^\infty \log(1 - |r(s)|^2) \frac{ds}{s} \right), \]

\[ x_k = \frac{1}{2 \Im(z_k)} \left( \log \left( \frac{\Im(z_k)}{2 \Im(z_k)} \prod_{\ell \neq k}^N \left| \frac{z_k - z_\ell}{z_k z_\ell - 1} \right|^2 \right) - \frac{\Im(z_k)}{\pi} \int_0^\infty \frac{\log(1 - |r(s)|^2)}{|s - z_k|^2} ds \right). \quad (1.9) \]

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Remark 1.2. The restriction $|x| < 2t$ in Theorem 1.1 is used only to limit the length of this paper. This is the critical regime as the soliton speeds $v_j$ in the scaling of (1.1)-(1.2) are bounded by $|v_j| < 2$. The steepest descent method of Deift and Zhou used in this paper can also be used to study the behavior of $q(x, t)$ as $t \not\to \infty$ also in the rest of space–time. We will present these results in another paper.

Remark 1.3. The two terms in (1.9) for the asymptotic phase shifts $x_j$ have clear interpretations. The first term gives the phase shift due to interactions between the solitons while the second term is a retarding factor due to the interaction of the soliton with the radiative component of the solution.

Remark 1.4. Long time asymptotic results for (1.1)-(1.2) were previously obtained in [11, 12, 43] under the assumption that $q_0(x) - \tanh(x)$ is Schwartz class, and that the reflection coefficient $r(z)$ satisfy $|r(\pm 1)| < 1$. As we review below this is a non-generic situation in that for most data $|r(\pm 1)| = 1$. Our methods remove this non-generic condition, and can handle a much wider class of initial data while also requiring less technical estimates along the way. Furthermore, we give a simpler and more direct characterization of the long time solution in terms of an $N$-soliton wave train.

In terms of scattering data, soliton solutions correspond to reflectionless potentials $q_0(x)$ for which the scattering map gives $r(z) = 0$: the scattering data of a 1–soliton is $\{0, \{z_0, c_0\}\}$ and for an $N$–soliton $\{0, \{z_k, c_k\}_{k=1}^{N-1}\}$. Observe that if we take $z_0 = i$ in (1.3) then we have the stationary solution of (1.1)-(1.2)

$$\text{sol}(x, t, i) = \tanh(x)$$

which is called the black soliton in analogy with the nonlinear optics application where $|q|^2$ represents the intensity of the light wave. When $z_0 \neq im$ the solution is a non-stationary dark soliton, becoming increasingly ‘whiter’ as $z_0 \to \pm 1$. The map which associates to $q_0$ the scattering data is not continuous at sol(0,$x-x_0,0$) (or $q^{(\text{sol})}(x-x_0,0)$).

There is a substantial body of work treating the orbital stability of the black soliton, see [28, 22, 5, 3, 27] and therein. The asymptotic stability of a dark soliton, the case $z_0 \neq i$, is discussed in [2], while the case of the black soliton $z_0 = i$ is discussed in [27]. Orbital stability of multi–solitons is considered in [3].

A corollary of Theorem 1.1 of this paper is the following asymptotic stability type result for the multi–solitons.

**Theorem 1.5.** Consider an $M$–soliton $q^{(\text{sol})}(x, t)$ satisfying both boundary conditions in (1.2), with poles $(z_j)_{j=0}^{M-1}$ and such that for a preassigned $\xi_0 \in (0,1)$ we have

$$\xi_0 > \Re z_0 > \cdots > \Re z_{M-1} > -\xi_0.$$  

Then, there exist $\varepsilon_0 > 0$ and $C > 0$ s.t. for any initial datum $q_0$ of problem (1.1)-(1.2) with

$$\epsilon := \|q_0 - q^{(\text{sol})}(x, 0)\|_{\Sigma_\delta} < \varepsilon_0$$  

(1.10)

$q_0$ has a finite number $N \geq M$ of poles $(z_j')_{j=0}^{N-1}$ with coupling constants $(c_j')_{j=0}^{N-1}$ s.t. Re $z_j' > \Re z_k'$ when $j < k$ and so that there exists an $L \in \{0, \ldots, N-1\}$ s.t. $L + M \leq N - 1$ and

$$\max_{L \leq j \leq M+L} (|z_j - z_j'| + |c_j - c_j'|) + \max_{j \geq M+L} |1 + z_j'| + \max_{j \leq L} |1 - z_j'| < C\epsilon.$$

(1.11)

Furthermore, $q_0$ has reflection coefficient $r$ satisfying $r \in H^1(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ and (1.4). Let $\xi := \frac{\varepsilon}{\pi}$. Then there exist $T(q_0) > 0$, $C(q_0) > 0$ and $x_{k+L} \in \mathbb{R}$ for $k = 0, \ldots, M-1$ s.t. for $t > T$, $|\xi| \leq \xi_0$.
Remark 1.6. Theorem 1.5 yields when $M = 1$ and $q^{(sol), 1}(x, t) = \tanh(x)$ an asymptotic stability result for the black soliton. Minor modifications in our arguments yield asymptotic stability results also for dark solitons and for $N$-solitons with boundary conditions different from (1.2).

Our proofs of Theorem 1.1 and Theorem 1.5 take advantage of the integrability of (1.1)-(1.2). Integrability allows one access to the Inverse Scattering Transform (IST) machinery. This was the approach of Gérard–Zhang [28] to establishing orbital stability of solutions near the black soliton. We recall that the IST provides a representation of a solution $q$ of (1.1)-(1.2) of its scattering data reminiscent of the Fourier representation formula $q(x, t) = \int_{R} e^{i \epsilon k^2 + i x \kappa} \hat{g}_0(k) dk$ of the linear Schrödinger equation. One can then envisage that solutions $q(x, t)$ of an integrable equation might be estimated by nonlinear analogues of the stationary phase method or other classical tools used in asymptotic analysis. The steepest descent method of Deift–Zhou does exactly this.

We would like to highlight some of the technical aspects of the manuscript. Our proof uses a more recent version of the steepest descent method, particularly the classical tools used in asymptotic analysis. The steepest descent method of Deift–Zhou does exactly this. Unfortunately, currently the IST is not well suited to explore cases when the metric used in (1.10) is as weak as in [27], where, assuming that there is a way to associate to $q_0$ scattering data, we should expect infinitely many poles concentrating near the points $\pm 1$ (for somewhat related material see [32]). This is a situation we do not consider. Instead, in the case when $\epsilon$ in (1.10) is finite, we know by [14] that there is only a finite number of poles. So we are very far from the very general set up considered in [2, 27, 3] and in some obvious respect our asymptotic stability result in the special case of solitons is much weaker than [27].

Nonetheless, in the case treated here of a solution $q(x, t)$ of (1.1) with a finite number of poles and an $r \in H^1(R)$ sufficiently regular, the steepest descent method provides more information on the asymptotic behavior of $q(x, t)$ as $t \to \infty$ than in [2, 27]. Furthermore, we treat the case of $N$-solitons for any $N$. It is also interesting to explore the same problem using completely different theoretical frameworks (we recall that [2, 27] follow the arguments introduced by Martel and Merle, see [33–34] and therein). Obviously the approach in [2, 27], not based on a direct use of the integrability
of (1.1)-(1.2), appears more amenable to extension to non integrable NLS's and so stronger than ours also in this respect. On the other hand, apart from questions on the correct formulation of the problem and some technical complications in Sect. 6.4, the arguments in the present paper are technically rather elementary. Considering that, from the viewpoint of scattering data, distinct integrable systems might not be very different the one from the other, maybe similar arguments apply to other systems. For example, it is certainly worth checking the case of the Sine–Gordon equation (see [13] for an analysis of pure radiation solutions). As for the integrability of (1.1)–(1.2) and the non robustness of this condition in real life, we remark that we expect that it might be possible to extend the analysis to non integrable systems, like in [19], although admittedly this part of the theory seems at its infancy. For a recent paper on this topic we refer to [9]. Our paper was written independently of [27], which we learned about only after finishing the mathematical part of our paper.

2 Plan of the proof

We prove Theorems 1.1 and 1.5 by applying the inverse scattering transform (IST) to the NLS equation (1.1)-(1.2).

In Sections 3 we review the integrable structure of (1.1). The Lax-pair (3.1) gives one an eigenvalue problem (3.5) in which the solution \( q(x,t) \) of NLS appears as a potential. We construct Jost solutions of (3.5), certain normalized solutions of (3.5): \( \psi_1^-(x,t,z) \) and \( \psi_2^+(x,t,z) \), \( k = 1, 2 \), holomorphic for \( \text{Im} \, z > 0 \) with derivatives in \( \text{Re} \, z \) and \( \text{Im} \, z \) extending continuously to \( \mathbb{C}^+ \) and \( \psi_1^-(x,t,z) \) and \( \psi_2^+(x,t,z) \), \( k = 1, 2 \), holomorphic for \( \text{Im} \, z > 0 \) with derivatives in \( \text{Re} \, z \) and \( \text{Im} \, z \) extending continuously to \( \mathbb{C}^- \). We enumerate several properties of these solutions under various assumptions on the smoothness and decay rate of \( q(x,t) - \tanh(x) \). Implicit to this construction is that we have global solutions of (1.1), \( q \in \tanh(x) + \Sigma_5 \), this is shown in Appendix B.

In Section 4 we describe how one constructs the scattering data from these Jost functions. The Wronskian \( \det [\psi_1^-(x,t,z), \psi_2^+(x,t,z)] \) is shown to be independent of both \( x \) and \( t \), and its zeros are precisely the discrete spectrum of (3.5) for \( z \in \mathbb{C}^+ \). These numbers each encode a single soliton component of the solution of (1.1). The total number of solutions of \( \det [\psi_1^-(x,t,z), \psi_2^+(x,t,z)] = 0 \) is finite provided \( q_0 \in \tanh(x) + L^1(\mathbb{R}) \), (c.f.[14], Appendix B). The totality of the scattering data generated by \( q_0(x) \) consist of the zeros \( (z_j^*)_{j=0}^N \) of the Wronskian, where \( N \geq 1 \) is finite, of their corresponding coupling constants \( (c_j)_{j=0}^{N-1} \), and of the reflection coefficient \( r(z) \), which we will show belongs in \( H^1(\mathbb{R}) \) and satisfies additional estimates proved in Sect. 3. In particular we show that generically we have (4.17) that \( \lim_{z \to \pm 1} r(z) = \mp 1 \). The situation in which (4.17) does not hold is simpler. One other issue that appears in Section 4 is that the map from initial data \( q_0 \) to scattering data is not continuous at the soliton solutions. In appendix C we show that even compactly supported perturbations of the single black soliton can be multisolitonic in that the perturbed Wronskian \( \det [\psi_1^-(x,t,z), \psi_2^+(x,t,z)] \) can have up to two new zeros in \( \mathbb{C}^+ \). The new zeros however are very close to \( z = \pm 1 \) corresponding to nearly white solitons. In particular we have perturbative result in Lemma 4.4.

In Sect. 5 we define a Riemann–Hilbert problem (RHP) for a sectionally meromorphic function \( m(z,x,t) \) and describe how the solution of (1.1)–(1.2) can be recovered from the solution \( m(z,x,t) \) of the RHP. We initiate the long time analysis of (1.1) in Sect. 6 by using the \( \bar{\partial} \) generalization of the Deift–Zhou steepest descent procedure following the ideas in [24]. This proceeds as a series of three explicit transformations \( m(z) \mapsto m^{(1)}(z) \mapsto m^{(2)}(z) \mapsto m^{(3)}(z) \) such that the final unknown \( m^{(3)}(z) \)
is a continuous function in the complex plane with an asymptotically small \( \partial \) derivative uniformly in the complex plane. This allows one to prove the existence of \( m^{(3)} \) using functional analytic tricky and the theory of the solid Cauchy transform.

In Sect. 6.1 we introduce the first transformation, a set of conjugations and interpolations such that the new unknown \( m^{(1)} \) has no poles following the ideas in [15, 29, 1]. The second transformation is the heart of the steepest descent method, where appropriate factorizations of the jump matrices of the RHP on the real line are introduced and certain non-analytic extensions of these factorizations are used to deform the jumps onto contours in the plane on which they are asymptotically small. The main issue here is that \(|r(1)| = 1\) prevents factorizations such as in (0.23) [17], here reviewed in (6.1)–(6.3), which play a central role in the theory. This technical problem is solved in Sect. 6.2 where in particular for the extensions of the factors we prove estimates analogous to those in Proposition 2.1 in [21].

Sect. 6.3 contains the third transformation, and perhaps the most novel element of our steepest descent analysis. In Lemma 6.5 we show that if one ignores the \( \partial \)-component of \( m^{(2)} \) what remains is a trivial conjugation of the RHP corresponding to an \( N \)-soliton whose scattering data is known exactly. By using the explicit solution of the \( N \)-soliton RHP problem given in Appendix A we are able to define a transformation which removes all of the soliton components of the problem simultaneously. This global in \( x \) construction is in stark contrast with the typical method seen in long-time type calculations in which one fixes lines \( x = ct \) before doing the long time analysis, then by varying the speed \( c \) one observes solitons when \( c \) coincides with the soliton speed and only radiation otherwise. It is the authors’ belief that the method presented here is better. Both because it is conceptually simpler and because it could potentially be better suited to studying singular situations such as the semi-classical situation in which the number of poles is accumulating with some asymptotic density or perhaps even in the energy space setting where there are infinitely many poles accumulation at the edges \( z = \pm 1 \).

Finally, in Sect. 6.4 we then prove the existence of the function \( m^{(3)} \) and estimate its size in a way similar to Sect. 2.4–2.5 in [21]. Summing up the estimates yields the proof of Theorem 1.1 in Sect. 6.5.

### 3 Jost functions

In this section we state without proof the details of the forward scattering transform for defocusing NLS for step-like initial data. The results are well known and the interested reader can find pedagogical and detailed treatments in the literature, see [24, 6, 20] for example.

The integrability of (1.1) follows from its Lax pair representation

\[
\begin{align*}
  v_x &= \mathcal{L} v, \\
  iv_t &= \mathcal{B} v.
\end{align*}
\]  

(3.1a)  

(3.1b)

The \( 2 \times 2 \) matrices \( \mathcal{L} \) and \( \mathcal{B} \) are given by

\[
\begin{align*}
  \mathcal{L} &= \mathcal{L}(z; x, t) = i\sigma_3(Q - \lambda(z)) \\
  \mathcal{B} &= \mathcal{B}(z; x, t) = -2i\lambda(z)\mathcal{L} - (Q^2 - I)\sigma_3 + iQ_x
\end{align*}
\]  

(3.2a)  

(3.2b)
Lemma 3.1. Let analytic properties as functions of $z$ approach those of the columns of (3.7) as such that (3.5). We define Jost functions which is simply (3.1a) written as an eigenvalue problem. Let now the system

$$i\sigma_3 v_x + Qv - \lambda(z)v = 0$$

(3.5)

which is just a matrix formulation of (1.1).

Fix $q(x)$ such that $\lim_{x \to \pm \infty} q(x) = \pm 1$ (appropriate reformulations of what follows hold for different boundary values in $\partial D(0, 1)$). We consider the system

$$i\sigma_3 v_x + Qv - \lambda(z)v = 0$$

(3.5)

which is simply (3.1a) written as an eigenvalue problem. Let now

$$B_\pm = B_\pm(z) = I \pm \sigma_1 z^{-1}$$

(3.6)

$$X^\pm(x, z) = B_\pm(z)e^{-i\zeta(z)\sigma_3}$$

(3.7)

$$\zeta(z) = \frac{1}{2}(z - z^{-1}).$$

(3.8)

where $X^\pm$ are the solutions of the system obtained by replacing $Q(x)$ by $\pm Q_+$ with $Q_+ = \sigma_3$ in (3.5). We define Jost functions, $\psi_j^\pm$, $j = 1, 2$, to be the column vector solutions of (3.5) whose values approach those of the columns of (3.7) as $x \to \pm \infty$. The existence of such solutions, and their analytic properties as functions of $z$, is the subject of the following Lemma.

**Lemma 3.1.** Let $q(x)$ be such that $x \to \pm \infty, q(x) = \pm 1$ (appropriate reformulations of what follows hold for different boundary values in $\partial D(0, 1)$) the system (3.5) admits solutions

$$\psi_1^+(x, z) = m_1^+(x, z)e^{-i\zeta(z)x}, \quad \psi_2^+(x, z) = m_2^+(x, z)e^{i\zeta(z)x}$$

(3.9)

such that

$$\lim_{x \to \pm \infty} m_1^+(x, z) = \left(\frac{1}{\pm z^{-1}}\right), \quad \lim_{x \to \pm \infty} m_2^+(x, z) = \left(\pm z^{-1}\right)1.$$ (3.10)

Both $\psi_1^+(x, z)$ and $\psi_2^+(x, z)$ extend into solutions of (3.5) for $z \in \mathbb{C}^-$ and $\psi_1^+(x, z)$ and $\psi_2^-(x, z)$ extend into solutions of (3.5) for $z \in \mathbb{C}^+$. Furthermore for any $x_0 \in \mathbb{R}$ we have that $z \to m_1^+(x, z)$ is a continuous map from $\overline{\mathbb{C}^+}\setminus\{-1, 0, 1\}$ (with analytic restriction in $\mathbb{C}^+$) into $C^1([x_0, \infty), \mathbb{C}) \cap W^{1, \infty}(x_0, \infty), \mathbb{C}$ in the $+$ case and into $C^1((\infty, x_0), \mathbb{C}) \cap W^{1, \infty}((\infty, x_0), \mathbb{C})$ in the - case where $m_1^+(x, z)$ is the unique solution of the integral equation

$$m_1^+(x, z) = \left(\frac{1}{\pm z^{-1}}\right) + \int_\pm^x X^\pm(x, z)X^\pm(y, z)^{-1}i\sigma_3(Q(y) \mp \sigma_1)m_1^\pm(y, z)e^{i(x-y)\zeta(z)}dy.$$ (3.11)
We have that $z \to m^\pm_2(x, z)$ is a continuous map from $\overline{\mathbb{C}}^+ \setminus \{ -1, 0, 1 \}$ (whose restriction in $\mathbb{C}^\pm$ is analytic) into $C^1([x_0, \infty), \mathbb{C}) \cap W^{1, \infty}([x_0, \infty), \mathbb{C})$ in the $+$ case and $C^1((-\infty, x_0), \mathbb{C}) \cap W^{1, \infty}((-\infty, x_0), \mathbb{C})$ in the $-$ case where $m^\pm_2(\cdot, z)$ is the unique solution of the integral equation

$$m^\pm_2(x, z) = \left( \pm \frac{1}{z} \right) + \int_{\pm \infty}^x X^\pm(x, z)X^\pm(y, z)^{-1}\sigma_3(Q(y) = \sigma_1)m^\pm_2(y, z)e^{-i(x-y)\zeta(z)}dy.$$  \hspace{1cm} (3.12)

The map $q \to m^+_1(\cdot, z)$ is a locally Lipschitz continuous map

$$\tanh(x) + L^1(\mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{C}^- \setminus \{ -1, 0, 1 \}), C^1([x_0, \infty), \mathbb{C}) \cap W^{1, \infty}([x_0, \infty), \mathbb{C})).$$ \hspace{1cm} (3.13)

Similar statements to (3.13) hold for $q \to m^+_2(\cdot, z)$ and for $q \to m^-_j(\cdot, z)$ for $j = 1, 2$.

Additionally, if $q - \tanh(x) \in L^{1,n}(\mathbb{R})$ then the maps $z \to \frac{\partial}{\partial z}m^+_2(x, z)$ are continuous from $\overline{\mathbb{C}}^+ \setminus \{ -1, 0, 1 \}$ (with analytic restriction in $\mathbb{C}^\pm$) to $C^1([x_0, \infty), \mathbb{C}) \cap W^{1, \infty}([x_0, \infty), \mathbb{C})$ and the maps $q \to \frac{\partial}{\partial z}m^+_1(\cdot, z)$ are locally Lipschitz continuous maps from

$$\tanh(x) + L^1(\mathbb{R}) \to L^\infty_{\text{loc}}(\mathbb{C}^- \setminus \{ -1, 0, 1 \}), C^1([x_0, \infty), \mathbb{C}) \cap W^{1, \infty}([x_0, \infty), \mathbb{C}).$$ \hspace{1cm} (3.14)

Specifically, there exists an increasing function $F_n(t)$, independent of $q$, such that

$$|\partial_z^m[m^+_1(x, z)]| \leq F_n((1 + |x|)^n\|q - 1\|_{L^{1,n}(x, \infty)}, z \in \overline{\mathbb{C}}\setminus \{ -1, 0, 1 \}. \hspace{1cm} (3.15)$$

Furthermore, given potentials $q$ and $\bar{q}$ sufficiently close together we have for each $z \in \overline{\mathbb{C}}\setminus \{ -1, 0, 1 \},$

$$|\partial_z^m[m^+_1(x, z) - \bar{m}^+_1(x, z)]| \leq \|q - \bar{q}\|_{L^{1,n}(x, \infty)}F_n((1 + |x|)^n\|q - 1\|_{L^{1,n}(x, \infty)}).$$ \hspace{1cm} (3.16)

Similar estimates hold for the other Jost functions.

The above lemma suggests that the Jost functions exhibit singular behavior for $z$ near $-1, 0,$ or $1$. The singular behavior of the solutions at $z = 0$ plays a non-trivial and unavoidable role in the behavior of the solution. However, the following lemma makes clear, if the initial data $q$ has a finite first moment then the singularities of the Jost functions at $z = \pm 1$ are removable.

**Lemma 3.2.** Let $q - \tanh(x) \in L^{1,1}(\mathbb{R})$. Set $x^\pm = \max\{\pm x, 0\}$. Let $K$ be a compact neighborhood of $\{ -1, 1 \}$ in $\mathbb{C}^- \setminus \{0\}$. Then there exists a $C$ such that for $z \in K$ we have

$$|m^+_1(x, z) - \left( \frac{1}{z-1} \right)| \leq C(x^-)e^{C \int_{x^-(y-x)}^{x^-(y)}|q(y)|dy}\|q - 1\|_{L^{1,1}(x, \infty)}.$$ \hspace{1cm} (3.17)

The map $z \to m^+_1(x, z)$ extends as a continuous map to the points $\pm 1$ with values in $C^1([x_0, \infty), \mathbb{C}) \cap W^{1, \infty}([x_0, \infty), \mathbb{C})$ for any preassigned $x_0 \in \mathbb{R}$.

Furthermore, the map $q \to m^+_1(\cdot, z)$ is a locally Lipschitz continuous map

$$\tanh(x) + L^{1,1}(\mathbb{R}) \to L^{\infty}(\mathbb{C}^- \setminus \{0\}, C^1([x_0, \infty), \mathbb{C}) \cap W^{1, \infty}([x_0, \infty), \mathbb{C})).$$ \hspace{1cm} (3.18)

Analogous statements hold for $m^+_2(x, z)$ and for $m^-_j(x, z)$ for $j = 1, 2$.

If $q - \tanh(x) \in L^{1,n+1}(\mathbb{R})$ then the maps $z \to \partial_z^m m^+_1(x, z)$ and $q \to \partial_z^m m^+_1(x, z)$, satisfy analogous statements and we have like in (3.25)

$$|\partial_z^m m^+_1(x, z)| \leq F_n((1 + |x|)^n\|q - 1\|_{L^{1,n+1}(x, \infty)}).$$ \hspace{1cm} (3.19)
The final Lemma in this section concerns the behavior of the Jost functions as \( |z| \to \infty \). Set

\[
D_+(x) = \| q - 1 \|_{W^{2,1}(x,\infty)} (1 + \| q - 1 \|_{W^{2,1}(x,\infty)})^2 e^{\| q - 1 \|_{L^1(x,\infty)}}
\]
\[
D_-(x) = \| q + 1 \|_{W^{2,1}(-\infty,x)} (1 + \| q + 1 \|_{W^{2,1}(-\infty,x)})^2 e^{\| q + 1 \|_{L^1(-\infty,x)}}
\]  

(3.20)

**Lemma 3.3.** Suppose that \( q - \tanh(x) \in L^1(\mathbb{R}) \) and that \( q' \in W^{1,1}(\mathbb{R}) \). Then as \( z \to \infty \) with \( \text{Im } z < 0 \) we have

\[
m_+^+(x,z) = e_1 - \frac{1}{z} \left( 1 - \frac{|q(y)|^2}{q(x)} \right) + O(D_+(x)z^{-2}),
\]

(3.21)

\[
m_+^-(x,z) = e_2 - \frac{1}{z} \left( 1 - \frac{|q(y)|^2}{q(x)} \right) + O(D_-(x)z^{-2}),
\]

(3.22)

and for \( \text{Im } z > 0 \) as \( z \to \infty \) we have

\[
m_-^+(x,z) = e_1 + \frac{1}{z} \left( 1 - \frac{|q(y)|^2}{q(x)} \right) + O(D_+(x)z^{-2}),
\]

(3.23)

\[
m_-^-(x,z) = e_2 + \frac{1}{z} \left( 1 - \frac{|q(y)|^2}{q(x)} \right) + O(D_-(x)z^{-2}),
\]

(3.24)

where the constant in each \( O(D_+(x)z^{-2}) \) is independent of \( z \).

If in addition \( q - \tanh(x) \in L^{1,n}(\mathbb{R}) \) and \( q' \in W^{1,1}(\mathbb{R}) \), then there exists an increasing function \( F_n(t) \) independent of \( q \) such that as \( z \to \infty \),

\[
|\partial^n_q[m_+^+(x,z)]| \leq |z|^{-1} F_n((1 + |x|)^n \| q - 1 \|_{L^{1,n}(x,\infty)}).
\]

(3.25)

Furthermore, given a potential \( q \), then for potentials \( \tilde{q} \) sufficiently close to \( q \) we have

\[
|\partial^n_q[m_+^+(x,z) - m_+^+(x,z)]| \leq |z|^{-1} \| q - \tilde{q} \|_{L^{1,n}(x,\infty)} F_n((1 + |x|)^n \| q - 1 \|_{L^{1,n}(x,\infty)}).
\]

(3.26)

Similar estimates hold for the other Jost functions.

The previous lemma and the symmetry \([4,4]\) immediately imply the following corollary which describes the singularities of the Jost solutions at the origin.

**Corollary 3.4.** Let \( q \) be like in Lemma 3.3. Then for \( z \in \mathbb{C}^- \) as \( z \to 0 \) we have

\[
m_+^+(x,z) = \frac{1}{z} e_2 + O(1), \quad m_+^-(x,z) = -\frac{1}{z} e_1 + O(1)
\]

(3.27)

where \( |O(1)| \leq F(\| q - 1 \|_{W^{2,1}(x,\infty)}) \), and for \( z \in \mathbb{C}^+ \) as \( z \to 0 \) we have

\[
m_+^+(x,z) = -\frac{1}{z} e_2 + O(1), \quad m_+^-(x,z) = \frac{1}{z} e_1 + O(1)
\]

(3.28)

where \( |O(1)| \leq F(\| q + 1 \|_{W^{2,1}(-\infty,x)}) \), for some growing functions \( F(t) \).

4 The scattering data

We start with the following elementary lemma.
Lemma 4.1. Let \( q - \tanh(x) \in L^1(\mathbb{R}) \). Then

1. For \( z \in \mathbb{R} \setminus \{-1, 0, 1\} \) both of the matrix-valued functions
   \[
   \Psi(\pm)(x, z) = (\psi_1^\pm(x, z), \psi_2^\pm(x, z)) = (m_1^\pm(x, z), m_2^\pm(x, z)) e^{-i\kappa(z)x\sigma_3}
   \]
   are nonsingular solutions of (3.5) and
   \[
   \det \Psi_\pm(z) = 1 - z^{-2}.
   \]

2. For \( z \in \mathbb{C}^+ \setminus \{-1, 0, 1\} \) the Jost functions \( \psi_j^\pm \) satisfy the symmetries
   \[
   \begin{align*}
   \psi_1^-(x, z) &= \sigma_1 \psi_2^+(x, z), \\
   \psi_2^-(x, z) &= \sigma_1 \psi_1^+(x, z) \\
   \psi_1^+(x, z) &= -z^{-1} \psi_2^-(x, z^{-1}) \\
   \psi_2^+(x, z) &= z^{-1} \psi_1^-(x, z^{-1})
   \end{align*}
   \] (4.3)

3. Let \( \widehat{\psi}_j^\pm \) be the Jost functions of (3.5) with \( Q(x) \) replaced by \( -\overline{Q}(-x) \). Then
   \[
   \begin{align*}
   \psi_1^-(-x, z) &= i \sigma_2 \psi_2^+(-x, z), \\
   \psi_2^-(-x, z) &= -i \sigma_2 \psi_1^+(-x, z), \\
   \psi_1^+(-x, z) &= i \sigma_2 \psi_2^-(-x, z), \\
   \psi_2^+(-x, z) &= -i \sigma_2 \psi_1^-(-x, z).
   \end{align*}
   \] (4.5)

Before proving this lemma we point out an immediate corollary which we will find convenient in the sequel.

Corollary 4.2. Let \( q - \tanh(x) \in L^1(\mathbb{R}) \) then each of the Jost functions \( \psi_j^\pm(x, z) \) satisfy
   \[
   \psi_j^\pm(x, z^{-1}) = \pm z \sigma_j \psi_j^\pm(x, z)
   \] (4.6)

upon reflecting \( z \) through the unit circle in the half-plane in which the Jost function is defined.

Proof of Lemma 4.1. The matrices \( \Psi^\pm \) are solutions of (3.5), which follows from Lemma 3.1. To establish (4.1) and thus that \( \Psi^\pm \) is nonsingular, observe that \( \text{Tr}(\mathcal{L}) = 0 \), where \( \mathcal{L} \) is the matrix (3.2a) appearing in (3.1a), so that \( \det \Psi^\pm(x, z) = \det \Psi^\pm(z) \). Finally, \( \lim_{x \to \pm \infty} \det \Psi^\pm = \det B_{1 \pm} = 1 - z^{-2} \).

To prove the symmetries (4.3)-(4.5) start with \( z \in \mathbb{R} \setminus \{-1, 0, 1\} \). Then the symmetries of the Lax matrix: \( \mathcal{L}(z) = \sigma_1 \mathcal{L}(z) \sigma_1 = \mathcal{L}(z^{-1}) \) and the "free" solution: \( X^\pm(x, z) = \sigma_1 X^\pm(x, z) \sigma_1 = \pm z^{-1} X^\pm(x, z^{-1}) \sigma_1 \) imply that for \( z \in \mathbb{R} \setminus \{-1, 0, 1\} \) the Jost matrices satisfy
   \[
   \Psi^\pm(x, z) = \sigma_1 \overline{\Psi^\pm(x, z)} \sigma_1 = \pm z^{-1} \Psi(x, z^{-1}) \sigma_1.
   \]

Analytically extending each column vector solution \( \psi_j^\pm(x, z) \) off the real axis into the half plane indicated by Lemma 3.1 gives (4.3)-(4.4).

For the last symmetry note that if \( q(x) \in \tanh(x) + L^1(\mathbb{R}) \) so is \( -\overline{q}(-x) \). Let \( \tilde{\mathcal{L}}(x) \) be the Lax matrix \( \mathcal{L}(x) \) with \( Q(x) \) replaced by \( -\overline{Q}(-x) \) and \( \tilde{\Psi}^\pm \) the Jost solution of \( (\partial_x - \tilde{\mathcal{L}}) \tilde{\Psi} = 0 \). Then (4.5) follows from observing that \( \sigma_2 \tilde{\mathcal{L}}(-x) \sigma_2 = -\mathcal{L}(x) \) and \( \sigma_2 B_{1 \pm} \sigma_2 = B_{1 \pm} \) imply that \( \psi_j^\pm(x, z) = \sigma_2 \psi_j^\pm(-x, z) \sigma_2 \) for \( z \in \mathbb{R} \setminus \{-1, 0, 1\} \) and extending the individual columns off the real axis as before.

\[\square\]
As the columns of $\Psi^+(x, z)$ and $\Psi^-(x, z)$ each form a basis of the solutions of (3.5) for $z \in \mathbb{R}\backslash\{-1, 0, 1\}$. It follows that the matrices satisfy the linear relation

$$\Psi^-(x, z) = \Psi^+(x, z)S(z), \quad S(z) = \begin{pmatrix} \frac{a(z)}{b(z)} & \frac{b(z)}{a(z)} \end{pmatrix}, \quad z \in \mathbb{R}\backslash\{-1, 0, 1\} \quad (4.7)$$

where the form of the scattering matrix $S(z)$ follows from (4.3). The scattering coefficients $a(z)$ and $b(z)$ define

$$r(z) := \frac{b(z)}{a(z)} \quad (4.8)$$

known as the reflection coefficient.

The following lemma records several important properties of the scattering coefficients $a(z)$ and $b(z)$.

**Lemma 4.3.** Let $z \in \mathbb{R}\backslash\{-1, 0, 1\}$ and $a(z), b(z), r(z)$ be the data in (4.7)-(4.8) generated by some $q \in \tanh(x) + L^1(\mathbb{R})$. Then

1. The scattering coefficients can be expressed in terms of the Jost functions as

$$a(z) = \{\psi_1^+(x, z), \psi_2^+(x, z)\} \quad b(z) = \{\psi_1^+(x, z), \psi_1^-(x, z)\} \quad (4.9)$$

where $\{\xi, \eta\} = \det[\xi, \eta]$. It follows that $a(z)$ extends analytically to $z \in \mathbb{C}^+$ while $b(z)$ and $r(z)$ are defined only for $z \in \mathbb{R}\backslash\{-1, 0, 1\}$.

2. For each $z \in \mathbb{R}\backslash\{-1, 0, 1\}$

$$|a(z)|^2 - |b(z)|^2 = 1. \quad (4.10)$$

In particular, for $z \in \mathbb{R}\backslash\{-1, 0, 1\}$ we have

$$|r(z)|^2 = 1 - |a(z)|^{-2} < 1. \quad (4.11)$$

3. The scattering data satisfy the symmetries

$$-a(z^{-1}) = a(z), \quad -b(z^{-1}) = b(z), \quad r(z^{-1}) = r(z) \quad (4.12)$$

wherever they are defined.

4. If additionally $q' \in W^{1,1}(\mathbb{R})$, then for $z \in \mathbb{C}^+$,

$$\lim_{z \to \infty}(a(z) - 1)z = i \int_{\mathbb{R}} (|q(x)|^2 - 1) \, dx, \quad (4.13)$$

$$\lim_{z \to 0}(a(z) + 1)z^{-1} = i \int_{\mathbb{R}} (|q(x)|^2 - 1) \, dx. \quad (4.14)$$

and for $z \in \mathbb{R}$

$$|b(z)| = O(|z|^{-2}) \text{ as } |z| \to \infty \text{ and}$$

$$|b(z)| = O(|z|^2) \text{ as } |z| \to 0. \quad (4.15)$$
Proof. The first property follows from applying Cramer’s rule to (4.7) and using (4.2), one then observes that Lemma 3.1 implies that the formula for \( a(z) \) is analytic for \( z \in \mathbb{C}^+ \). The second property is just the fact that \( \det S = 1 \) which follows from taking the determinant on each side of (4.7) using (4.2). The symmetry conditions follow immediately from (4.9) after using (4.3)-(4.4); for instance

\[
\frac{a(z^{-1})}{z^{-2}} = \frac{\det \left[ \psi_1(x, z^{-1}), \psi_2(x, z^{-1}) \right]}{1 - z^2} = \frac{1}{1 - z^2} \det \left[ \sigma_1, \left(-z\psi_1(x, z), z\psi_2^+(x, z)\right)\right] = -\frac{1}{1 - z^2} \det \left[ \psi_1^-(x, z), \psi_2^+(x, z)\right] = -a(z).
\]

To prove (4.13) first observe that

\[
|q(y) \pm 1|^2 - (2 \pm q(y) \pm \bar{q}(y)) = |q(y)|^2 - 1.
\]

Then inserting (3.23)-(3.24) from Lemma 3.3 into (4.9) gives

\[
(1 - z^{-2})a(z) = \det \left[ 1 + iz^{-1} \int_{\mathbb{R}} \left( |q(y)|^2 - 1 \right) dy, \quad 1 + iz^{-1} \int_{\mathbb{R}} \left( |\bar{q}(y)|^2 - 1 \right) dy \right] + O(z^2)
\]

\[
= 1 + iz^{-1} \int_{\mathbb{R}} \left( |q(y)|^2 - 1 \right) dy + O(z^2).
\]

To prove (4.14) write \( z = \zeta^{-1} \) and use (4.12) and (4.13); the formulae for \( b(z) \) in (4.15) are proved similarly.

Though Lemma 3.2 gives conditions on \( q \) which guarantee that the Jost functions \( \psi_j^\pm(x, z) \) are continuous for \( z \rightarrow \pm 1 \), the scattering coefficients \( a(z) \) and \( b(z) \) will generally have simple poles at these points due to the vanishing of the denominators in (4.9). Moreover, their residues are proportional: the symmetry (4.4) implies that \( \psi_1^+(x, \pm 1) = \pm \psi_2^+(x, \pm 1) \), which in turn gives

\[
a(z) = \frac{a_+}{z + 1} + O(1), \quad a_+ = \det[\psi_1^-(x, \pm 1), \psi_2^+(x, \pm 1)]. \quad (4.16)
\]

In this generic situation the reflection coefficient remains bounded at \( z \pm 1 \) and we have

\[
\lim_{z \rightarrow \pm 1} r(z) = \mp 1. \quad (4.17)
\]

By Lemmas 3.1 and 3.2 the maps \( q \rightarrow \{\psi_1^-(x, z), \psi_2^+(x, z)\} \) and \( q \rightarrow \{\psi_1^+(x, z), \psi_1^-(x, z)\} \) are locally Lipschitz maps

\[
\{q : q' \in W^{1,1}(\mathbb{R}) \text{ and } q \in \tanh(x) + L^{1,1}(\mathbb{R})\} \rightarrow L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}). \quad (4.18)
\]

Furthermore, they are locally Lipschitz maps

\[
\{q : q' \in W^{1,1}(\mathbb{R}) \text{ and } q \in \tanh(x) + L^{1,n+1}(\mathbb{R})\} \rightarrow W^{n,\infty}(\mathbb{R}) \cap H^n(\mathbb{R}) \text{ for } n \geq 1. \quad (4.19)
\]

We conclude this section with some results concerning the stability of the scattering data to perturbations of the initial data. In particular we are interested in initial data which is a small perturbation of an \( N \)-soliton solution.
Lemma 4.4. Given an $M$-soliton $q^{(sol),M}(x,t)$ and initial data $q_0(x)$ satisfying the hypotheses of Theorem 1.5 the number of solutions $z \in \mathbb{C}_+$ of det $[\psi_1^+(0,x,z),\psi_2^+(0,x,z)] = 0$, where the Jost functions correspond to $q_0(x)$, is at least $M$ and is finite when $q_0$ satisfies the hypotheses of Theorem 1.5 for $\varepsilon_0$ small enough. Furthermore (1.11) holds.

Proof. In Appendix B it is proved that when $q_0 - \tanh(x) \in L^{1,4}(\mathbb{R})$ then the number of zeros is finite. The other statements are elementary consequences of the theory which we review in Sections 3–4.

Now let $q^{(sol),M}(t)$ denote the function $q^{(sol),M}(x,t)$.

Lemma 4.5. Let $q \in \tanh(x) + L^{1,4}(\mathbb{R})$, $q' \in W^{1,1}(\mathbb{R})$ and set $\varepsilon_1 := ||q - q^{(sol),M}(0)||_{L^{1,4}(\mathbb{R})\cap W^{2,1}(\mathbb{R})}$. Then there exists $\varepsilon > 0$ s.t. for $\varepsilon < \varepsilon_1$ we have $r(z) \in H^1(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$.

Proof. For $\text{dist}(z, \{\pm 1\}) > \varepsilon_1 > 0$ we have

$$|\partial_z^2 r(z)| \leq C(z)^{-1}||q - q^{(sol),M}(0)||_{L^{1,4}(\mathbb{R})\cap W^{2,1}(\mathbb{R})}$$

for $j = 0, 1, 2$ by (3.26), its analogues for the other Jost functions, and by Lemma 3.2.

Let now $|z-1| < \varepsilon_1$. Then, using the $a_+$ in (4.16)

$$r(z) = \frac{b(z)}{a(z)} = \frac{\{\psi_1^+(x,z),\psi_1^-(x,z)\}}{\{\psi_1^+(x,z),\psi_2^+(x,z)\}} = -\frac{a_+ + \int^z_1 F(s)ds}{a_+ + \int^z_1 G(s)ds}$$

(4.21)

for $F(z) = \partial_z^2 \{\psi_1^+(x,z),\psi_1^-(x,z)\}$ and $G(z) = \partial_z \{\psi_1^+(x,z),\psi_2^+(x,z)\}$.

If $a_+ = 0$ then for $|z-1| < \varepsilon_1$

$$|r(z)| \leq \frac{\int^z_1 F(s)ds}{\int^z_1 G(s)ds} \leq C||q - q^{(sol),M}(0)||_{L^{1,4}(\mathbb{R})\cap W^{2,1}(\mathbb{R})}$$

by $F = O(||q - q^{(sol),M}(0)||_{L^{1,4}(\mathbb{R})\cap W^{2,1}(\mathbb{R})})$ and $G \sim 1$ by the Lipschitz continuity in Lemma 3.2.

Differentiating in (4.21) we get

$$|r^{(j)}(z)| \leq C||q - q^{(sol),M}(0)||_{L^{1,4}(\mathbb{R})\cap W^{2,1}(\mathbb{R})}$$

for $j = 1, 2$ by $F^{(j)} = O(||q - q^{(sol),M}(0)||_{L^{1,4}(\mathbb{R})\cap W^{2,1}(\mathbb{R})})$ and, for $G_0$ the $G$ associated to $q^{(sol),M}$, by $G^{(j)} - G_0^{(j)} = O(||q - q^{(sol),M}(0)||_{L^{1,4}(\mathbb{R})\cap W^{2,1}(\mathbb{R})})$.

If $a_+ \neq 0$ then for $|z-1| < \varepsilon_1$

$$r(z) = -1 + \frac{\int^z_1 (F(s) + G(s))ds}{a_+ + \int^z_1 G(s)ds} = c a_+^{-1}(z-1) + o(z-1),$$

(4.22)

where $c-2i = O(||q - q^{(sol),M}(0)||_{L^{1,4}(\mathbb{R})\cap W^{2,1}(\mathbb{R})})$. Differentiating (4.22) we obtain bounds for $r^{(j)}(z)$ with $j = 1, 2$.

For $z$ near $-1$ the discussion is similar.

Notice that for $a_{\pm} = 0$ we have shown that for a fixed $C = C(\varepsilon_0)$

$$||r||_{H^2(\mathbb{R})} < C||q - q^{(sol),M}(0)||_{L^{1,4}(\mathbb{R})\cap W^{2,1}(\mathbb{R})}$$

(4.23)
Lemma 4.6. Let $q$ satisfy the hypotheses of Lemma 4.5. Then, as in Lemma 4.5 there exists $\varepsilon > 0$ s.t. for $\varepsilon_1 < \varepsilon_1$ we have
\[
\| \log(1 - |r|^2) \|_{L^p(\mathbb{R})} < \infty \text{ for all } p \in [1, \infty).
\] (4.24)

Proof. For $I_\pm = [\pm 1 - \varepsilon_1, \pm 1 + \varepsilon_1]$ for a fixed small $\varepsilon_1 > 0$, we split
\[
\| \log(1 - |r|^2) \|_{L^p(\mathbb{R})} = \| \log(1 - |r|^2) \|_{L^p(\mathbb{R} \setminus (I_- \cup I_+))} + \sum_{\sigma = \pm} \| \log(1 - |r|^2) \|_{L^p(I_{\sigma})}.
\]
The first term in the r.h.s. is less than $C \| q - q^{(sot).N}(0) \|_{L^{1.3} \cap W^{1.1}}$ by (4.20). Turning to the case $I_\sigma = I_+$, if $a_+ = 0$ we can use (4.23) while if $a_+ \neq 0$ we have by (4.22)
\[
\| \log(1 - |r|^2) \|_{L^p(I_+)} \sim \| \log |z - 1| \|_{L^p(I_+)} < \infty.
\]

The following is elementary.

Corollary 4.7. If $\varepsilon_0 > 0$ in Theorem 1.5 is chosen small enough, then $q_0$ satisfies the conclusions of Lemmas 4.3, 4.6.

4.1 The discrete spectrum

At any zero $z = z_k \in \mathbb{C}^+$ of $a(z)$ it follows from (4.9) that the pair $\{ \psi^+_1(x, z_k), \psi^+_2(x, z_k) \}$ is linearly related; the symmetry (4.3) implies that $\{ \psi^-_2(x, \bar{z}_k), \psi^-_1(x, \bar{z}_k) \}$ are also linearly related. That is, there exists a constant $\gamma_k \in \mathbb{C}$ such that
\[
\psi^-_1(x, z_k) = \gamma_k \psi^+_2(x, z_k), \quad \psi^-_2(x, \bar{z}_k) = \overline{\gamma_k} \psi^+_1(x, \bar{z}_k).
\] (4.25)

These $\gamma_k$ are called the connection coefficients associated to the discrete spectral values $z_k$.

If $z_k \in \mathbb{C}^+$ then it follows that $\psi^-_1(x, z_k)$ and $\psi^-_2(x, \bar{z}_k)$ are $L^2(\mathbb{R})$ eigenfunctions of (3.5) with eigenvalue $\lambda(z_k)$ and $\overline{\lambda(z_k)}$ respectively. If $z_k \in \mathbb{R}$ then $\psi(x, z_k)$ is bounded but not $L^2(\mathbb{R})$ and we say that $z_k$ is an embedded eigenvalue. However, it follows from (4.10) and (4.14) that $|a(z)| \geq 1$ for $z \in \mathbb{R} \setminus \{ -1, 1 \}$, so the only possible embedded eigenvalues are $\pm 1$. Then as (3.5) is self-adjoint, the non-real zeros of $a(z)$ in $\mathbb{C}^+$ are restricted to the unit circle, i.e., $|z_k| = 1$, so that $\lambda(z_k)$ is real. The following lemma demonstrates that, unlike the case of vanishing data for focusing NLS, the discrete spectral data takes a very restricted form.

Lemma 4.8. Let $q \circ \tanh(x) \in L^{1,1}(\mathbb{R})$. Then we have what follows.

1. The zeros of $a(z)$ in $\mathbb{C}^+$ are simple.
2. At each $z_k$, a zero of $a(z)$:
   i. $\frac{\partial a}{\partial z}(z_k)$ and $\gamma_k$ are pure imaginary;
   ii. their arguments satisfy
   \[
   \text{sgn}(-i \gamma_k) = -\text{sgn} \left( -i \frac{\partial a}{\partial \lambda}(z_k) \right).
   \] (4.26)
Proof. Suppose \( z_k \) is a zero of \( a(z) \), and \( \gamma_k \) the connection coefficient in (4.25). Then as \( z_k \) lies on the unit circle we have \( z_k^{-1} = z_k \), applying (4.6) to (4.25) gives
\[
\frac{\partial a}{\partial \lambda} \bigg|_{z = z_k} = \frac{\det [\partial \psi^{-}_1, \partial \psi^{+}_2]}{1 - z^2} = \frac{\det [\partial \psi^{-}_1, \partial \psi^{+}_2]}{1 - z^2},
\]
Comparing this to (4.25) shows that \( \gamma_k = -z_k \), or \( \gamma_k \in i\mathbb{R} \).

To prove the remaining facts, note that \( q - \tanh(x) \in L^{1,1}(\mathbb{R}) \) implies \( \frac{\partial a}{\partial x} \) exists and we have from (4.9)
\[
\frac{\partial a}{\partial x} = \frac{\partial \lambda \psi^{-}_1(x, z), \psi^{+}_2(x, z)}{1 - z^2},
\]
Using (3.2a) one finds that
\[
\frac{\partial a}{\partial x} \bigg|_{z = z_k} = \frac{\det [\partial \lambda \psi^{-}_1, \partial \lambda \psi^{+}_2]}{1 - z^2} = \frac{\det [\partial \lambda \psi^{-}_1, \partial \lambda \psi^{+}_2]}{1 - z^2},
\]
where the cancellation in each equality follows from observing that \( \text{adj} \mathcal{L} = -\mathcal{L} \) Recalling that at each \( z_k \) the columns are linearly related by (4.25) and decay exponentially as \( |x| \to \infty \),
\[
\det [\partial \lambda \psi^{-}_1, \partial \lambda \psi^{+}_2](s, z_k) = -i \gamma_k \int_{-\infty}^{\infty} \text{det} [\partial \lambda \psi^{-}_1, \partial \lambda \psi^{+}_2](s, z_k) ds, \quad \det [\partial \lambda \psi^{-}_1, \partial \lambda \psi^{+}_2](s, z_k) = -i \gamma_k \int_{-\infty}^{\infty} \text{det} [\partial \lambda \psi^{-}_1, \partial \lambda \psi^{+}_2](s, z_k) ds.
\]
Then using (4.6) to write \( \psi^{+}_2(x, z_k) = z_k^{-1} \sigma_1 \psi^{+}_1(x, z_k) \) in the first column of the determinants, we have, after putting the terms together,
\[
\frac{\partial a}{\partial \lambda} \bigg|_{z = z_k} = \frac{-i \gamma_k}{2\zeta(z_k)} \int_{\mathbb{R}} |\psi^{+}_2(x, z_k)|^2 dx.
\]
Recalling that both \( \gamma_k \) and \( \zeta(z_k) \) are imaginary, (4.27) is both nonzero and imaginary. The simplicity of the zeros of \( a(z) \) and the signature restriction on \( \gamma_k \) follow immediately.

So the zeros of \( a(z) \) are simple and restricted to the circle. By Lemma 4.4 the zeros are finite in number. As \( a(z) \) is analytic in \( \mathbb{C}^+ \), and approaches unity for large \( z \), it admits an inner-outer factorization, see [24] p.50, which using (4.10) takes the form
\[
a(z) = \prod_{k=0}^{N-1} \left( \frac{z - z_k}{z - z_k} \right) \exp \left( -\frac{1}{\pi} \int_{\mathbb{R}} \frac{\log(1 - |r(s)|^2)}{s - z} ds \right),
\]
\footnote{adj \( M \) denotes the adjugate of the matrix \( M \), it satisfies \( M \text{adj} M = (\det M)I \).}
where \( \{z_k\}_{k=0}^{N-1} \) are the zeros of \( a(z) \) in \( \mathbb{C}^+ \). This trace formula implies a dependence between the discrete spectrum \( z_k \) and the reflection coefficient. Using \( a(0) := \lim_{z \to 0} a(z) = -1 \), that is \( 4.14 \), we have
\[
\prod_{k=0}^{N-1} z_k^2 = a(0) \exp \left( \frac{1}{\pi i} \int_{\mathbb{R}} \log \left( 1 - \frac{|r(s)|^2}{s} \right) ds \right) = - \exp \left( \frac{1}{\pi i} \int_{\mathbb{R}} \log \left( 1 - \frac{|r(s)|^2}{s} \right) ds \right). \tag{4.29}
\]
The more general case \( a(0) = e^{i\theta} \) is the "\( \theta \)-condition" in \( 24 \), formula (7.19) in Ch. 2.

4.2 Time evolution of the scattering data

Thus far we have considered only fixed potentials \( q = q(x) \). The advantage of the inverse scattering transform is that if \( q(x, t) \) evolves according to \( 1.1 \) then the evolution of the scattering data is linear and trivial as we see now.

By Theorem B.1 we have \( q(x, t) - \tanh(x) \in C^1([0, \infty), \Sigma_3) \). It can be shown that this implies that the Jost functions \( m^\pm(x, t, z) \) in Sect. 3 are differentiable in \( t \). Then by standard arguments, see for example \( 24, 18, 28 \), which we sketch now, the evolution of the scattering coefficients and discrete data are as follows:
\[
a(z, t) = a(z, 0), \quad b(z, t) = b(z, 0)e^{-4i\zeta(z)\lambda(z)t}, \quad r(z, t) = r(z, 0)e^{-4i\zeta(z)\lambda(z)t}, \quad z_k(t) = z_k(0), \quad \gamma_k(t) = \gamma_k(0)e^{-4i\zeta(z)\lambda(z)t}. \tag{4.30}\]

In particular here we sketch the first two equalities on the left. Due to \( 3.4 \) we can write equalities
\[
(i\partial_t + B)\Psi^\pm(x, t, z) = \Psi^\pm(x, t, z)C(t, z), \quad \text{with the } \Psi^\pm \text{ in } 4.1. \]
Integrating we get for \( m^\pm := \Psi^\pm e^{i\zeta_x \sigma_3} \)
\[
m^\pm(x, t_2, z) - m^\pm(x, t_1, z) + i \int_{t_1}^{t_2} B(x, t, z)m^\pm(x, t, z)dt = \int_{t_1}^{t_2} m^\pm(x, t, z)e^{-i\zeta_x \sigma_3}C(t, z)e^{i\zeta_x \sigma_3}dt.
\]
Taking the − case, we consider the limit for \( x \to -\infty \). Then the above equality implies that \( C(t, z) \) is diagonal and
\[
C(t, z) = -2i\lambda \left( 1 - \frac{\sigma_1}{z} \right)^{-1} \sigma_3(\sigma_1 + \lambda) \left( 1 - \frac{\sigma_1}{z} \right) = -2i\sigma_3\lambda\zeta,
\]
with the r.h.s obtained with elementary computations and the first equality obtained using
\[
\lim_{x \to \pm\infty} m^\pm(x, t, z) = 1 \pm \frac{\sigma_1}{z} \quad \text{and} \quad \lim_{x \to \pm\infty} B(x, t, z) = 2\lambda\sigma_3(\mp\sigma_1 - \lambda).
\]
Also in the + case \( C(t, z) = -2i\sigma_3\lambda\zeta \). Applying now \( i\partial_t + B \) to the first equality in \( 4.7 \), that is to \( \Psi^+(x, t, z) = \Psi^+(x, t, z)S(t, z) \), after elementary computations we get \( \dot{S} = 2i\lambda[\sigma_3, S] \). This yields the left column in \( 4.30 \).
5 Inverse scattering: set up of the Riemann Hilbert problem

For $z \in \mathbb{C}\setminus\mathbb{R}$, for $q(x,t)$ the solution to (1.1), and for $m_j^\pm(x,t,z)$, $j = 1, 2$, the (normalized) Jost functions we set

$$m(z;x,t) := \begin{cases} \frac{m^+_{\Re}(x,t,z)}{m^+_{\Im}(x,t,z)} & z \in \mathbb{C}_+ \\ \frac{m^-_{\Re}(x,t,z)}{m^-_{\Im}(x,t,z)} & z \in \mathbb{C}_- \end{cases}.$$ \hfill (5.1)

**Lemma 5.1.** We have

$$m(z) = \sigma_1 m(z) \sigma_1,$$ \hfill (5.2a)

$$m(z^{-1}) = zm(z) \sigma_1.$$ \hfill (5.2b)

**Proof.** Both are immediate consequences of the symmetries contained in Lemma 4.1 and Lemma 4.3.

Assume $q \in \tanh(x) + L^1(\mathbb{R})$ and $q'(x) \in W^{1,1}(\mathbb{R})$.

**Lemma 5.2.** For $\pm \Im z > 0$

$$\lim_{z \to \infty} \left( m(z;x) - I \right) = \begin{pmatrix} -i \int_x^\infty |q(y)|^2 - 1 \, dy \\ i \int_x^\infty |q(y)|^2 - 1 \, dy \end{pmatrix} \overline{q(x)},$$ \hfill (5.3)

$$\lim_{z \to 0} \left( m(z;x) - \sigma_1 \right) = \begin{pmatrix} i \int_x^\infty |q(y)|^2 - 1 \, dy \\ -i \int_x^\infty |q(y)|^2 - 1 \, dy \end{pmatrix} \overline{q(x)}.$$ \hfill (5.4)

**Proof.** The behavior at infinity follows immediately from Lemma 3.3 and 4.13. The behavior at the origin is then a consequence of Lemma 5.1.

It is an easy consequence of Lemma 3.1 Lemma 3.3 Lemma 4.3 (4.25), and (4.30) that $m(z;x,t)$ satisfies the following Riemann Hilbert problem.

**Riemann-Hilbert Problem 5.1** Find a $2 \times 2$ matrix valued function $m(z;x,t)$ such that

1. $m$ is meromorphic for $z \in \mathbb{C}\setminus\mathbb{R}$.
2. $m(z;x,t) = I + O(z^{-1})$ as $z \to \infty$.
3. $zm(z;x,t) = \sigma_1 + O(z)$ as $z \to 0$.
4. The non-tangential limits $m^\pm(z;x,t) = \lim_{\mathbb{C}_\pm \ni z \to z^\pm} m(z;x,t)$ exist for any $z \in \mathbb{R}_+$ and satisfy the jump relation $m^+_+(z;x,t) = m^-_-(z;x,t)V(z)$ where

$$V(z) := V_{\pm}(z) = \begin{pmatrix} 1 - |r(z)|^2 & -r(z)e^{-\Phi(z;x,t)} \\ r(z)e^{\Phi(z;x,t)} & 1 \end{pmatrix},$$ \hfill (5.5)

and

$$\Phi(z,x,t) = 2ix\zeta(z) - 4i\zeta(z)\lambda(z)t = ix(z - z^{-1}) - it(z^2 - z^{-2}).$$
4. $m(z; x, t)$ has simple poles at the points $Z = \mathbb{Z}^+ \cup \mathbb{Z}^-$, $\mathbb{Z}^+ = \{z_k\}_{k=0}^{N-1} \subset \{z = e^{i\theta} : 0 < \theta < \pi\}$, with residues satisfying

\[
\text{Res } m(z; x, t) = \lim_{z \to z_k} m(z; x, t) \begin{pmatrix} 0 & 0 \\ c_k(x, t) & 0 \end{pmatrix},
\]

and

\[
\text{Res } m(z; x, t) = \lim_{z \to \overline{z}_k} m(z; x, t) \begin{pmatrix} 0 & \overline{\tau}(x, t) \\ 0 & 0 \end{pmatrix},
\]

where

\[
c_k(x, t) = \frac{\gamma_k(0)}{a'(z_k)} e^{\Phi(z_k, x, t)} = c_k e^{\Phi(z_k, x, t)}, \quad c_k = \frac{\gamma_k(0)}{a'(z_k)} = \frac{4iz_k}{\int_{\mathbb{R}} |\psi_2^\tau(x, z_k)|^2 dx} = i2_k|c_k|.
\]

The potential $q(x, t)$ is found by the reconstruction formula, see Lemma 5.2

\[
q(x, t) = \lim_{z \to -\infty} z m_{21}(z; x, t).
\]

$N$-solitons are potentials corresponding to the case when $r(z) \equiv 0$.

**Lemma 5.3.** If a solution $m(z; x, t)$ of RHP 5.1 exists, it is unique if and only if it satisfies the symmetries of Lemma 5.1 additionally for such a solution $\det m(z; x, t) = 1 - z^{-2}$.

**Proof.** Suppose a solution $m(z)$ exists. It is trivial to verify using the symmetry $\overline{r(z)} = r(z^{-1})$ and the condition $\pi c_k \in i\mathbb{R}$ on the norming constants that both $\sigma_1 m(z) \sigma_1$ and $zm(z^{-1})\sigma_1$ are solutions as well. So uniqueness immediately implies symmetry.

Suppose the solution $m$ possesses the symmetries. Taking the determinant of both sides of the jump relation gives $\det m_+ = \det m_-$ for $z \in \mathbb{R}$ since $\det V \equiv 1$. It follows from this, the normalization condition and the residue conditions that $\det m$ is rational in $z$ with poles at some subset of $\mathbb{Z} \cup \{0\}$. However, the form of the residue relation (5.6) implies that at each $p \in \mathbb{Z}$ a single column has a pole whose residue is proportional to the value of the other column. It follows then that $\det m$ is regular at each point $p \in \mathbb{Z}$. As $z \to 0$ the normalization condition gives $z^2 \det m(z) \to -1$. So $\det m = 1 + \alpha z^{-1} - z^2$ for some constant $\alpha$. However, the symmetry (5.2b) implies that $\det m(z) = -z^2 \det m(z^{-1})$ so $\alpha \equiv 0$.

Uniqueness then follows from applying Liouville’s theorem to the ratio $m(\overline{m})^{-1}$ of any two solutions $m$, $\overline{m}$ of RHP 5.1 noting that at the origin we have

\[
\lim_{z \to 0} m(z)(\overline{m}(z))^{-1} = \lim_{z \to 0} (z^2 - 1)^{-1} zm(z)\sigma_2 z \overline{m}(z)\sigma_2 = -(\sigma_1\sigma_2)^2 = I.
\]

6  The long time analysis

6.1 Step 1: Interpolation and conjugation

In order to perform the long time analysis using the Deift-Zhou steepest descent method we need to perform two essential operations:

(i) interpolate the poles by trading them for localized jump matrices;
which the connection coefficient is bounded in time. Given a finite set of discrete data \( Z \)
Additionally, we define \( z \) which is nonnegative only when some \( \Phi(z) \)
correspond to poles \( \Phi(z) \) in \([15]\) and further refined in \([29]\) and \([1]\). To motivate our method for interpolating the poles we
Our method for dealing with the poles in the Riemann-Hilbert problem follows the ideas put forward
deform the jump contours, but first we must first introduce the pole interpolate which help account
These deformations are useful when they deform the factors into regions in which the corresponding
These sets index all of the discrete spectra in the upper (and lower) half-plane. Those
\( \Delta = \{ j : \text{Re } z_j > \xi \} \)
\( \nabla = \{ j : \text{Re } z_j \leq \xi \} \)
Additionally, we define
\[ j_\alpha = j_\alpha(\xi) = \begin{cases} j & \text{if } |\text{Re}(z_j) - \xi| < \rho \text{ for some } j \in \{0, \ldots, N - 1\} \\ -1 & \text{otherwise} \end{cases} \]
which is nonnegative only when some \( z_{j_\alpha} \) is near the line \( \text{Re } z = \xi \), so that \( e^{\Phi(z_{j_\alpha})} = O(1) \).
The connection coefficients \( c_j(x, t) \) for \( j \in \Delta \) are exponentially large for \( t \gg 1 \) and for the purpose
of steepest descent analysis we want our pole interpolate to “exchange” the \( e^\Phi \) in these residues for
\( e^{-\Phi} \) in the new jump matrix.
Define the function

\[
T(z, \xi) = \prod_{k \in \Delta} \left( \frac{z - z_k}{z z_k - 1} \right) \exp \left( -\frac{1}{2\pi i} \int_0^\infty \log(1 - |r(s)|^2) \left( \frac{1}{s - z} - \frac{1}{2s} \right) ds \right). \tag{6.8}
\]

**Lemma 6.1.** The function \( T(z, \xi) \) is meromorphic in \( \mathbb{C} \setminus [0, \infty) \) with simple poles at the \( z_j \) and simple zeros at the \( z_k \) such that \( \text{Re}(z_j) > \xi \), and satisfies the jump condition

\[
T_+(z, \xi) = T_-(x, \xi)(1 - |r(z)|^2), \quad z \in (0, \infty).
\tag{6.9}
\]

Additionally,

i. \( T(z, \xi) = T(z, \xi)^{-1} = T(z^{-1}, \xi) \),

ii. \( T(\infty, \xi) := \lim_{z \to \infty} T(z, \xi) = \left( \prod_{k \in \Delta} z_k \right) \exp \left( \frac{1}{4\pi i} \int_0^\infty \log(1 - |r(s)|^2) ds \right) \) and \( |T(\infty, \xi)|^2 = 1 \),

iii. \( |T(z, \xi)| = 1 \) for \( z \leq 0 \).

iv. The ratio \( \frac{a(z)}{T(z, \xi)} \) is holomorphic in \( \mathbb{C}_+ \) and there is a constant \( C(q_0) \) s.t.

\[
\left| \frac{a(z)}{T(z, \xi)} \right| \leq C(q_0) \text{ for } z \in \mathbb{C}_+ \text{ s.t. } \text{Re} z > 0.
\tag{6.10}
\]

and extends as a continuous function on \( \mathbb{R}_+ \) with \( \left| \frac{a(z)}{T(z, \xi)} \right| = 1 \) for \( z \in (0, \infty) \).

**Proof.** From (6.8) it’s obvious that \( T \) has simple zeros at each \( z_k \) and poles at each \( z_k \), \( k \in \Delta \). The jump relation (6.9) follows from the Plancherel formula. The first symmetry property follows immediately from the symmetry (4.12) of \( r(z) \). The second and third properties are simple computations.

Finally, consider the ratio \( \frac{a(z)}{T(z, \xi)} \). Using the representation (4.28) for \( a(z) \) we can write

\[
\frac{a(z)}{T(z, \xi)} = \left( \prod_{k \in \Delta} z_j \right) e^{\frac{1}{2\pi i} \int_0^\infty \log(1 - |r(s)|^2) ds} \prod_{k \in \Delta} \frac{z - z_k}{z - \overline{z}_k} \exp \left( \frac{1}{2\pi i} \int_{-\infty}^0 \frac{\log(1 - |r(s)|^2)}{s - z} ds \right). \tag{6.11}
\]

In the r.h.s. all factors before the last one have absolute value less than 1 for \( z \in \mathbb{C}_+ \) while the real part of the exponential can be bounded as follows, where for the r.h.s. we can use Lemma 4.6

\[
-\frac{\text{Im}(z)}{2\pi} \left( \int_{-\infty}^{\frac{1}{2}} + \int_{\frac{1}{2}}^0 \right) \frac{\log(1 - |r(s)|^2)}{(s - \text{Re}(z))^2 + \text{Im}(z)^2} ds
\leq \frac{4\text{Im}(z)}{1 + 4\text{Im}(z)^2} \| \log(1 - |r|^2) \|_{L^1(\mathbb{R}_-)} + 2^{-1} \| \log(1 - |r|^2) \|_{L^\infty(-2^{-1}, 0)}
\]

where we bound the 1st term of the r.h.s. using Lemma 4.6 and the 2nd using \( \| r \|_{L^\infty(-2^{-1}, 0)} < 1 \). Obviously the function in (6.11) extends in a continuous way on \( \mathbb{R}_+ \) and it is easy to conclude that all the factors in the r.h.s. of (6.11) are equal to 1 in absolute value. \( \square \)
Figure 1: The contours defining the interpolating transformation $m \rightarrow m^{(1)}$ (c.f. (6.12)). Around each poles $z_k \in \mathbb{Z}^+$ and its conjugate $\bar{z}_k \in \mathbb{Z}^-$ we insert a small disk, oriented counterclockwise in $\mathbb{C}^+$ and clockwise in $\mathbb{C}^-$, of fixed radius $\rho$ sufficiently small such that the disk intersect neither each other nor the real axis. The set $\Delta$ (resp. $\nabla$) consist of those poles to the right (resp. left) of the line $\text{Re} \ z = \xi$. If a pair $z_j, \bar{z}_j$ lies within $\rho$ of the line $\text{Re} \ z = \xi$ we leave that pair uninterpolated (left figure), otherwise all poles are interpolated (right figure); in either case, the singularity at the origin remains.

We are now ready to implement the interpolations and conjugations discussed at the beginning of this section. From $T(z, \xi)$ define the matrices

$$D(z) = T(z, \xi)^{\sigma_3} = \begin{pmatrix} T(z, \xi) & \text{0} \\ \text{0} & T(z, \xi)^{-1} \end{pmatrix}$$

$$D(\infty) = T(\infty, \xi)^{\sigma_3} = \begin{pmatrix} T(\infty, \xi) & \text{0} \\ \text{0} & T(\infty, \xi)^{-1} \end{pmatrix}$$

We then remove the poles by the following transformation which trades the poles for jumps on small contours encircling each pole

$$m^{(1)}(z) = \begin{cases} D(\infty)^{-1}m(z) \begin{pmatrix} 1 & 0 \\ -\frac{\zeta \phi_j}{z - z_j} & 1 \end{pmatrix} D(z), & |z - z_j| < \rho, \quad j \in \nabla \text{ and } |\text{Re}(z_j) - \xi| > \rho, \\ D(\infty)^{-1}m(z) \begin{pmatrix} 1 & \text{0} \\ 0 & \frac{\zeta \phi_j}{z - z_j} \end{pmatrix} D(z), & |z - z_j| < \rho, \quad j \in \Delta \text{ and } |\text{Re}(z_j) - \xi| > \rho, \\ D(\infty)^{-1}m(z) \begin{pmatrix} 1 & 0 \\ 0 & \frac{\zeta \phi_j}{z - \bar{z}_j} \end{pmatrix} D(z), & |z - \bar{z}_j| < \rho, \quad j \in \nabla \text{ and } |\text{Re}(z_j) - \xi| > \rho, \\ D(\infty)^{-1}m(z) \begin{pmatrix} 1 & 0 \\ 0 & \frac{\zeta \phi_j}{z - \bar{z}_j} \end{pmatrix} D(z), & |z - \bar{z}_j| < \rho, \quad j \in \Delta \text{ and } |\text{Re}(z_j) - \xi| > \rho, \\ D(\infty)^{-1}m(z)D(z), & \text{elsewhere}, \end{cases}$$

where

$$\Phi_j = \Phi(z_j) = -\Phi(\bar{z}_j) = -\frac{4}{\rho} \text{Im}(z_j) (\xi - \text{Re}(z_j))$$

Consider now the following contour, which is depicted in Figure 1.

$$\Sigma^{(1)} = \mathbb{R} \cup \bigcup_{j \in \nabla \cup \Delta} \{ z \in \mathbb{C} : |z - z_j| = \rho \text{ or } |z - \bar{z}_j| = \rho \}.$$
Lemma 6.2. The Riemann-Hilbert problem for $m^{(1)}(z)$ resulting from (6.12) is the RH problem 6.1 formulated below. Furthermore, $m^{(1)}(z)$ satisfies the symmetries of Lemma 5.1.

Riemann-Hilbert Problem 6.1 Find a $2 \times 2$ matrix-valued function $m^{(1)}(z; x, t)$ such that

1. $m^{(1)}(z; x, t)$ is meromorphic in $\mathbb{C} \setminus \Sigma^{(1)}$.
2. $m^{(1)}(z; x, t) = I + O\left(z^{-1}\right)$ as $z \to \infty$.
3. The non-tangential boundary values $m^{(1)}_{\pm}(z; x, t)$ exist for $z \in \Sigma^{(1)}$, and satisfy the jump relation $m_{+}(z; x, t) = m_{-}(z; x, t)V^{(1)}(z)$ where

$$V^{(1)}(z) = \begin{cases} 
\begin{pmatrix} 1 & -r(z)T(z)^{-2}e^{-\Phi} \\
0 & 1 
\end{pmatrix} \left( \begin{pmatrix} 1 & 0 \\
0 & 1 
\end{pmatrix} \right) & z \in (-\infty, 0) \\
\begin{pmatrix} -c_{j}T(z)^{2}e^{\Phi_{j}} & 0 \\
0 & 1 
\end{pmatrix} \left( \begin{pmatrix} 1 & 0 \\
1 & 1 
\end{pmatrix} \right) & |z - z_{j}| = \rho, \; j \in \nabla \\
\begin{pmatrix} 1 & -\frac{z-z_{j}}{c_{j}}T(z)^{-2}e^{-\Phi_{j}} \\
0 & 1 
\end{pmatrix} \left( \begin{pmatrix} 1 & 0 \\
0 & 1 
\end{pmatrix} \right) & |z - z_{j}| = \rho, \; j \in \Delta \\
\begin{pmatrix} 1 & \frac{z-z_{j}}{c_{j}}T(z)^{-2}e^{\Phi_{j}} \\
0 & 1 
\end{pmatrix} \left( \begin{pmatrix} 1 & 0 \\
1 & 1 
\end{pmatrix} \right) & |z - z_{j}| = \rho, \; j \in \nabla \\
\begin{pmatrix} \frac{z-z_{j}}{c_{j}}T(z)^{2}e^{-\Phi_{j}} & 0 \\
1 & 1 
\end{pmatrix} \left( \begin{pmatrix} 1 & 0 \\
0 & 1 
\end{pmatrix} \right) & |z - z_{j}| = \rho, \; j \in \Delta.
\end{cases}$$

4. If $(x, t)$ are such that there exist (at most one) $j_{0} \in \{0, \ldots, N-1\}$ such that $|\text{Re } z_{j_{0}} - \xi| \leq \rho$, $\xi = \frac{\rho}{2\pi}$, then $m^{(1)}(z; x, t)$ has simple poles at the points $z_{j_{0}}, z_{\overline{j_{0}}} \in \mathbb{Z}$ satisfying one of the following alternatives.

(a) If $j_{0} \in \nabla$,

$$\begin{align*}
\text{Res } m^{(1)}(z; x, t) & = \lim_{z \to z_{j_{0}}} m^{(1)}(z, x, t) \begin{pmatrix} 0 & 0 \\
c_{j_{0}}^{-1}T(z_{j_{0}})^{2}e^{\Phi(z_{j_{0}})} & 0 
\end{pmatrix}, \\
\text{Res } m^{(1)}(z; x, t) & = \lim_{z \to z_{\overline{j_{0}}}^{-}} m^{(1)}(z, x, t) \begin{pmatrix} 0 & 0 \\
c_{j_{0}}^{-1}T(z_{\overline{j_{0}}}^{*})^{-2}e^{-\Phi(z_{\overline{j_{0}}})} & 0 
\end{pmatrix}.
\end{align*}$$

(b) If $j_{0} \in \Delta$,

$$\begin{align*}
\text{Res } m^{(1)}(z; x, t) & = \lim_{z \to z_{j_{0}}} m^{(1)}(z, x, t) \begin{pmatrix} 0 & c_{j_{0}}^{-1}T(z_{j_{0}})^{-2}e^{-\Phi(z_{j_{0}})} \\
0 & 0 
\end{pmatrix}, \\
\text{Res } m^{(1)}(z; x, t) & = \lim_{z \to z_{\overline{j_{0}}}^{-}} m^{(1)}(z, x, t) \begin{pmatrix} 0 & c_{j_{0}}^{-1}T(z_{\overline{j_{0}}}^{*})^{-2}e^{-\Phi(z_{\overline{j_{0}}})} \\
0 & 0 
\end{pmatrix}.
\end{align*}$$

Otherwise, $m^{(1)}(z; x, t)$ is analytic in $\mathbb{C} \setminus \Sigma^{(1)}$. 

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Remark 1. The function $T(z, \xi)$ and the transformation $m \mapsto m^{(1)}$ defined by (6.12) can be thought of in two parts. In the first step the Blaschke product in $T$, swaps the columns in which the poles $z_j$, $j \in \Delta$ appear and gives new connection coefficient proportional to $c_j(x, t)^{-1}$ as desired. The triangular factors in (6.12) then interpolate the poles trading them for jumps on the disk boundaries $|z - z_j| = \rho$. In the second step, the Cauchy integral term in $T$ is responsible for removing the diagonal factor (6.3) from the jump matrix factorization $V = B T_0 B^-$ (cf. (6.11)) on the half-line $(0, \infty)$. Finally, we point out that factors $(zz_k - 1)$ in the Blaschke product instead of simply $(z - z_k)$ and the $\frac{1}{z^j}$ term in the Cauchy integral are introduced so that $T$ satisfies property $i.$ in Lemma 6.1 which is needed to preserve the symmetries in Lemma 5.1.

Proof of Lemma 6.2. The proof consists of a lengthy but elementary series of computations which we sketch only partially. First of all we start with the symmetries of Lemma 5.1. For instance, the region outside the disks in (6.12) is invariant by the transformations $z \mapsto 1/z$ and $z \mapsto z^{-1}$. We have

$$m^{(1)}(z) = T(z) \sigma_3 m(z) T(z) \sigma_3 = T(z) \sigma_3 m(z) T(z)^{-\sigma_3} = \sigma_1 m^{(1)}(z) \sigma_1$$

and

$$m^{(1)}(z^{-1}) = T(z) \sigma_3 m(z) T(z) \sigma_3 = z T(z) \sigma_3 m(z) T(z) \sigma_3 = zm^{(1)}(z) \sigma_1.$$
6.2 Step 2: opening \( \partial \) lenses

We now want to remove the jump from the real axis in such a way that the new problem takes advantage of the decay/growth of \( \exp(\Phi(z)) \) for \( z \notin \mathbb{R} \). Additionally we want to “open the lens” in such a way that the lenses are bounded away from the disks introduced previously to remove the poles from the problem.

To that end, fix an angle \( \theta_0 > 0 \) sufficiently small such that the set \( \{ z \in \mathbb{C} : |\text{Re}z| > \cos \theta_0 \} \) does not intersect any of the disks \( |z - z_k| \leq \rho \). For any \( \xi \in (0, 1) \), let

\[
\phi(\xi) = \min \left\{ \theta_0, \arccos \left( \frac{2|\xi|}{1 + |\xi|} \right) \right\},
\]

define

\[
\Omega_1 = \{ z : \arg z \in (0, \phi(\xi)) \}, \quad \Omega_2 = \{ z : \arg z \in (\pi - \phi(\xi), \pi) \},
\]
\[
\Omega_3 = \{ z : \arg z \in (-\pi, -\pi + \phi(\xi)) \}, \quad \Omega_4 = \{ z : \arg z \in (-\phi(\xi), 0) \},
\]

denote by \( \Omega \) their union and denote

\[
\Sigma_1 = e^{i\phi(\xi)} \mathbb{R}_+, \quad \Sigma_2 = e^{i(\pi - \phi(\xi))} \mathbb{R}_+,
\]
\[
\Sigma_3 = e^{-i(\pi + \phi(\xi))} \mathbb{R}_+, \quad \Sigma_4 = e^{-i\phi(\xi)} \mathbb{R}_+,
\]

the left-to-right oriented boundaries of \( \Omega \), see Figure 2.

**Lemma 6.3.** Set \( \xi := \frac{\tau}{2t} \) and let \( |\xi| < 1 \). Then for \( z = |z|e^{i\theta} \) and \( F(s) = s + s^{-1} \) we have

\[
\text{Re} [\Phi(t, x, z)] \geq \frac{t}{4}(1 - |\xi|)F(|z|)^2 |\sin 2\theta| \quad \text{for } z \in \Omega_1 \cup \Omega_3,
\]
\[
\text{Re} [\Phi(t, x, z)] \leq -\frac{t}{4}(1 - |\xi|)F(|z|)^2 |\sin 2\theta| \quad \text{for } z \in \Omega_2 \cup \Omega_4.
\]

**Proof.** We will consider only the case \( z \in \Omega_1 \). By elementary computation we have

\[
\text{Re} [\Phi(t, x, z)] = t \sin 2\theta \psi(z) \quad \text{with} \quad \psi(z) = F(|z|)^2 - \xi \sec \theta F(|z|) - 2
\]

Then, observing that \( F(|z|) \geq 2 \), we have for \( z \in \Omega_1 \),

\[
\psi(z) \geq F(|z|)^2 - \frac{1 + |\xi|}{2}F(|z|) - 2 \geq \frac{1 - |\xi|}{4}F(|z|)^2
\]

so that

\[
\text{Re} \Phi(z) \geq \frac{t}{4}(1 - |\xi|)F(|z|)^2 \sin 2\theta.
\]

\( \Box \)

The estimates suggest that we should use (modified versions of) the factorization \((6.2)\) for \( z < 0 \) and \((6.3)\) for \( z > 0 \) to open lenses. To do so, we need to define extensions of the off-diagonal entries of \( b(z) \) and \( B(z) \) off the real axis, which is the content of the following lemma.
Lemma 6.4. It is possible to define functions $R_j : \Omega_j \rightarrow \mathbb{C}$, $j = 1, 2, 3, 4$, with boundary values

\[
\begin{align*}
R_1(z) &= \frac{r(z)T_+(z)}{1 - |r(z)|^2} \quad z \in (0, \infty) \\
R_1(z) &= 0 \quad z \in \Sigma_1 \\
R_2(z) &= r(z)T(z)^2 \quad z \in (-\infty, 0) \\
R_2(z) &= 0 \quad z \in \Sigma_2 \\
R_3(z) &= r(z)T(z)^2 \quad z \in (-\infty, 0) \\
R_3(z) &= 0 \quad z \in \Sigma_3 \\
R_4(z) &= \frac{r(z)T_-(z)}{1 - |r(z)|^2} \quad z \in (0, \infty) \\
R_4(z) &= 0 \quad z \in \Sigma_4
\end{align*}
\]

such that for a fixed constant $c_1 = c_1(g_0)$, a fixed cutoff function $\varphi \in C^\infty_0(\mathbb{R}, [0, 1])$ with a small support near 1 such that for all $z \in \Omega_j$, $j = 1$ and 4, we have

\[
\left| \partial_z R_j(z) \right| \leq c_1|z|^{-1/2} + c_1|r'(|z|)| + c_1\varphi(|z|) \quad \text{and} \quad (6.16)
\]

\[
\left| \partial_z R_j(z) \right| \leq c_1|z|^{-1} \quad \text{and} \quad (6.17)
\]

while for $j = 2, 3$ we have (6.16) with $|z|$ replaced by $-|z|$ in the argument of $r'$ and without the term $c_1\varphi(|z|)$.

Moreover, if we set $R : \Omega \rightarrow \mathbb{C}$ by $R(z) \big|_{z \in \Omega_j} = R_j(z)$, the extension can be made to preserve the symmetry $R(z^{-1}) = R(z)$.

Proof. We will give the details of the proof for $R_1$. This will yield the proof also for $R_4$ since we can define $R_4(z) = R_1(z^{-1})$. Notice that the last equality yields the desired boundary values of $R_4$ because $r(s^{-1}) = \tau(s)$ by (4.12) and $T(z^{-1}) = T^{-1}(z)$ by Lemma 6.1. The definitions of $R_2$ and $R_3$ are similar, simpler and are a simpler version of Proposition 2.1 [21]. The estimates for the $\partial$-derivative for $j = 4$ are similar to the case $j = 1$ while the case $j = 2, 3$ is simpler and is a simpler version of Proposition 2.1 [21].

As observed in (4.16)-(4.17), $a(z)$ and $b(z)$ have simple poles at $z = \pm 1$, and $r(z) \rightarrow \mp 1$ as $z \rightarrow \pm 1$. This suggests that $R_1(z)$ is singular at $z = 1$. However, the singular behavior is exactly balanced by the factor $T(z)^{-2}$. From (4.6)- (4.10) we have

\[
\frac{r(z)}{1 - |r(z)|^2} T_+(z)^{-2} = \frac{b(z)}{a(z)} \left( \frac{a(z)}{T_+(z)} \right)^2 = \frac{J_0(z)}{J_n(z)} \left( \frac{a(z)}{T_+(z)} \right)^2,
\]

where we have temporarily introduced the notation

\[
J_0(z) = \det \begin{bmatrix} \psi_+^+(x, z) & \psi_+^-(x, z) \end{bmatrix}, \quad J_n(z) = \det \begin{bmatrix} \psi_-^+(x, z) & \psi_+^-(x, z) \end{bmatrix}.
\]

Recall that though the columns of the right/left normalized Jost functions, $\psi_j^\pm(x, z)$, $j = 1, 2$, depend on $x$, the determinants are independent of $x$ as $\text{Tr} \mathcal{L} = 0$. Using Lemmas 3.1 and 6.1 the denominator of the first and the second factor in the r.h.s. of (6.18) are non-zero and analytic in $\Omega_1$, with well defined nonzero limits on $\partial \Omega_1$. Notice also that in $\Omega_1$ and away from the point $z = 1$, the factors in the l.h.s. of (6.18) are well behaved.
We introduce a cutoff function \( \chi \in C^\infty_0(\mathbb{R}, [0, 1]) \) with a small support near 0 such that for \( z \) near 0 we have \( \chi(z) = 1 \). We set \( \chi_1(z) = 1 - \chi(z) \). We then rewrite the function \( R_1(z) \) in \( \mathbb{R}_+ \) as \( R_1(z) = R_{11}(z) + R_{12}(z) \) with
\[
R_1(z) = R_{11}(z) + R_{12}(z)
\]
where
\[
R_{11}(z) := \chi_1(z - 1) \frac{r(z)}{1 - |r(z)|^2} T_+(z)^{-2}, \quad R_{12}(z) := \chi(z - 1) \frac{J_b(z)}{J_a(z)} \left( \frac{a(z)}{T_+(z)} \right)^2.
\]
(6.20)

The purpose of (6.20) is to neutralize the effect of the singularity at 1 due to \( |r(1)| = 1 \). Fix a small \( \delta_0 > 0 \). Then extend the functions \( R_{11} \) and \( R_{12} \) in \( \Omega_1 \) by
\[
R_{11}(z) = \chi_1(|z| - 1) \frac{r(|z|)}{1 - |r(|z|)|^2} T(|z|)^{-2} \cos \left( \frac{\pi}{2\delta_0} \arg z \right),
\]
(6.21)
\[
R_{12}(z) = f(|z|) g(z) \cos(k \arg z) + \frac{i}{k} |z| \chi \left( \delta_0^{-1} \arg z \right) f'(|z|) g(z) \sin(k \arg z),
\]
(6.22)
where we used the following symbols:
\[
k := \frac{\pi}{2\delta_0}, \quad g(z) := J_a(z)^{-1} \left( \frac{a(z)}{T(z)} \right)^2, \quad f(z) := \chi(|z| - 1) J_b(|z|).
\]

Both extensions are similar to Prop. 2.1 [21], but (6.22) is somewhat more elaborate. We now bound the \( \overline{\partial} \) derivatives of (6.21)–(6.22). We have
\[
\overline{\partial} R_{11}(z) = \frac{r(|z|)}{1 - |r(|z|)|^2} \cos \left( \frac{\pi}{2\delta_0} \arg z \right) \overline{\partial} \chi_1(|z| - 1) T(|z|)^{-2} + \frac{\chi_1(|z| - 1)}{T(|z|)^2} \overline{\partial} \frac{r(|z|)}{1 - |r(|z|)|^2} \cos \left( \frac{\pi}{2\delta_0} \arg z \right). \quad (6.23)
\]

Then the r.h.s. satisfies (6.16) by \( 1 - |r(|z|)|^2 > c > 0 \) in the support of \( \chi_1(|z| - 1) \), so that for \( |T(z)|^{-2} \leq C e^{-\log c} \) in \( \Omega_1 \cap \text{supp} \chi_1(|z| - 1) \) for some fixed \( C \) and for some other fixed \( C \)
\[
\left| \overline{\partial} \frac{r(|z|)}{1 - |r(|z|)|^2} \right| \leq C |r'(|z|)| + C \left| \sin \left( \frac{\pi}{2\delta_0} \arg z \right) \frac{|r(|z|)|}{|z|} \right|.
\]

Notice that by \( r(0) = 0 \) we have \( |r(|z|)| \leq \sqrt{|z|} \). Notice also that the first term in the r.h.s. of (6.23) can be bounded by \( c_1 \varphi(|z|) \) for an appropriate \( \varphi \in C^\infty_0(\mathbb{R}, [0, 1]) \) with a small support near 1 and with \( \varphi = 1 \) on suppy \( \chi \).

We turn now to \( \overline{\partial} R_{12} \). For \( z = u + iv = \rho e^{i\phi} \) we have \( \overline{\partial} = \frac{1}{2} \left( \partial_u + \partial_v \right) = \frac{e^{i\phi}}{2} \left( \partial_\rho + \frac{i}{\rho} \partial_\phi \right) \). Then
\[
\overline{\partial} R_{12}(z) = \frac{e^{i\phi} g(z)}{2} \left[ f'(\rho) \cos(k \rho) \chi \left( \frac{\phi}{\delta_0} \right) - \frac{i f(\rho)}{k \rho} \sin(k \phi)
\right.
\]
\[
\left. + \frac{i}{k} (\rho f'(\rho))' \sin(k \phi) \chi \left( \frac{\phi}{\delta_0} \right) + \frac{i}{k \delta_0} f'(\rho) \sin(k \phi) \chi' \left( \frac{\phi}{\delta_0} \right) \right].
\]

Then we have \( |\overline{\partial} R_{12}(z)| \leq c_1 \varphi(|z|) \Re z \) for a \( \varphi \in C^\infty_0(\mathbb{R}, [0, 1]) \) supported near 1. Indeed in \( \text{supp} \chi(|z| - 1) \cap \Omega_1 \) we have \( J_b^j(|z|) = O(1) \) for \( j = 0, 1, 2, J_a^{-1}(z) = O(1) \) and we have (6.10). Furthermore we use \( |\sin(k \phi)| + \chi \left( \delta_0^{-1} \phi \right) = O(\phi) \).

\( \square \)
Figure 2: The unknown $m^{(2)}(z)$ defined by (6.24) has nonzero $\bar{\partial}$ derivatives in the regions $\Omega_j$, and jump discontinuities on the disk boundaries $|z - z_j| = \rho$. The dashed boundaries of $\Omega_j$ indicate that $m^{(2)}$ is continuous at these boundaries.

We use the extensions of Lemma 6.4 to define modified versions of the factorizations (6.1) which extend into the lenses $\Omega_j$. We have on the real axis

$$\hat{V}(z) = \hat{b}^{-1}(z)\hat{b}(z) = \hat{B}(z)\hat{B}^{-1}(z)$$

where

$$\hat{b}(z) = \begin{pmatrix} 1 & 0 \\ R_2(z)e^{\Phi} & 1 \end{pmatrix}, \quad \hat{b}^{-1}(z) = \begin{pmatrix} 1 & R_3(z)e^{-\Phi} \\ 0 & 1 \end{pmatrix},$$

$$\hat{B}(z) = \begin{pmatrix} 1 & 0 \\ R_4(z)e^{\Phi} & 1 \end{pmatrix}, \quad \hat{B}^{-1}(z) = \begin{pmatrix} 1 & R_1(z)e^{-\Phi} \\ 0 & 1 \end{pmatrix}.$$ 

We use these to define a new unknown

$$m^{(2)}(z) = \begin{cases} m^{(1)}(z)\hat{B}^{-1}(z) & z \in \Omega_1 \\ m^{(1)}(z)\hat{b}^{-1}(z) & z \in \Omega_2 \\ m^{(1)}(z)\hat{b}(z) & z \in \Omega_3 \\ m^{(1)}(z)\hat{B}(z) & z \in \Omega_4 \end{cases}$$

(6.24)

Let

$$\Sigma^{(2)} = \bigcup_{j \in \mathcal{V} \cup \Delta \setminus j \neq j_0(\xi)} \{z \in \mathbb{C} : |z - z_j| = \rho \text{ or } |z - \bar{z}_j| = \rho\}$$

(6.25)

be the union of the circular boundaries of each interpolation disk. It is an immediate consequence of (6.24) and Lemmas 6.2 and 6.4 that $m^{(2)}$ satisfies the following $\bar{\partial}$–Riemann-Hilbert problem.

\textbf{$\bar{\partial}$–Riemann-Hilbert Problem 6.2} Find a $2 \times 2$ matrix-valued function $m^{(2)}(z)$ such that

1. $m^{(2)}$ is continuous in $\mathbb{C} \setminus \Sigma^{(2)}$ and takes continuous boundary values $m^{(2)}_+(z)$ (respectively $m^{(2)}_-(z)$) on $\Sigma^{(2)}$ from the left (respectively right).
2. $m^{(2)}(z) = I + O(z^{-1})$ as $z \to \infty,$
   \[ zm^{(2)}(z) = \sigma_1 + O(z) \text{ as } z \to 0. \]
3. The boundary values are connected by the jump relation \( m^{(2)}_+ (z) = m^{(2)}_- (z) V^{(2)}(z) \) where

\[
V^{(2)}(z) = \begin{cases}
\frac{1}{z - z_j} e^{\Phi(z_j)} & |z - z_j| = \rho, \; j \in \nabla \\
1 - \frac{z - z_j}{\varepsilon_j} T(z)^{-2} e^{-\Phi(z_j)} & |z - z_j| = \rho, \; j \in \Delta \\
0 & |z - z_j| = |z_j|, \; j \in \nabla \\
\frac{z - z_j}{\varepsilon_j} T(z)^2 e^{-\Phi(z_j)} & |z - z_j| = |z_j|, \; j \in \Delta.
\end{cases}
\tag{6.26}
\]

4. For \( z \in \mathbb{C} \) we have

\[
\overline{\partial} m^{(2)}(z) = m^{(2)}(z) W(z)
\tag{6.27}
\]

where

\[
W(z) = \begin{cases}
0 & z \in \Omega_1 \\
0 & z \in \Omega_2 \\
0 & z \in \Omega_3 \\
0 & z \in \Omega_4 \\
0 & \text{elsewhere.}
\end{cases}
\tag{6.28}
\]

5. \( m^{(2)}(z; x, t) \) is analytic in the region \( \mathbb{C} \setminus \Omega \) if \( j_0 = 0 \). If \( (x,t) \) are such that there exists \( j_0 \in \{0, \ldots, N-1\} \) such that \( |\text{Re} \ z_j - \xi| \leq \rho, \; \xi = \frac{x}{t} \), then \( m^{(2)}(z; x, t) \) is meromorphic in \( \mathbb{C} \setminus \Omega \) with exactly two poles, which are simple, at the points \( z_j, \; \overline{z}_j \in \mathbb{Z} \) satisfying one of the following cases.

(a) If \( j_0 \in \nabla \),

\[
\text{Res} \ m^{(2)}(z; x, t) = \lim_{z \to z_{j_0}} m^{(2)}(z; x, t) \begin{pmatrix} 0 & 0 \\ C_{j_0} e^{\Phi(z_{j_0})} & 0 \end{pmatrix}, \quad \text{with} \; C_{j_0} = e_{j_0} T(z_{j_0})^2,
\tag{6.29}
\]

(b) If \( j_0 \in \Delta \),

\[
\text{Res} \ m^{(2)}(z; x, t) = \lim_{z \to \overline{z}_{j_0}} m^{(2)}(z; x, t) \begin{pmatrix} 0 & 0 \\ \overline{C}_{j_0} e^{-\Phi(z_{j_0})} & 0 \end{pmatrix}, \quad \text{with} \; C_{j_0} := c_{j_0}^{-1} T'(z_{j_0})^{-2}.
\tag{6.30}
\]
6.3 Step 3: Removing the poles, the asymptotic N-soliton solution.

Our next step is to remove the Riemann-Hilbert component of the solution, so that all that remains is a new unknown with nonzero \( \partial \)-derivatives in \( \Omega \), and is otherwise bounded and approaching identity for \( |z| \to \infty \). Once this is complete, the remaining problem is analyzed using the “small-norm” theory for the solid Cauchy operator. This is done in the following section, Section 6.4.

Lemma 6.5. Let \( m^{(\text{sol})} \) denote the solution of the Riemann-Hilbert problem which results from simply ignoring the \( \partial \)-component of RHP 6.2, that is, let

\[
m^{(\text{sol})}(z) \text{ solves } \partial \text{-RHP 6.2 with } W \equiv 0.
\]

For any admissible scattering data \( \{r(z), \{z_j, c_j\}_{j=0}^{N-1}\} \) in RHP 6.2, the solution \( m^{(\text{sol})} \) of this modified problem exists, and is equivalent, by an explicit transformation, to a reflectionless solution of the original Riemann Hilbert problem, RHP 5.1, with the modified scattering data \( \{r(z) \equiv 0, \{z_j, c_j\}_{j=0}^{N-1}\} \) where

\[
\tilde{c}_j(x,t) = c_j(x,t) \exp \left( -\frac{1}{i\pi} \int_0^\infty \log(1 - |s|)^2 \left( \frac{1}{s - z_j} - \frac{1}{2s} \right) ds \right). \tag{6.31}
\]

Proof. With \( W \equiv 0 \), the \( \partial \)-RHP for \( m^{(\text{sol})} \) reduces to a Riemann Hilbert problem for a sectionally meromorphic function with jump discontinuities on the set of circles used to interpolate the poles of the original poles of RHP 5.1 except, when \( \tilde{z}_0(\xi) \neq 0 \), the simple poles at \( \tilde{z}_0 \) and \( \frac{\pi}{\rho} \) which are left uninterpolated. The following transformation, contracts each of circular jumps to simple poles at each \( z_k \) or \( \pi_k \) in \( Z \), and reverses the triangularity effected by (6.8) and (6.12):

\[
\tilde{m}(z) = \prod_{k \in \Delta} \left( \frac{1}{z_k} \right)^{\sigma_3} m^{(\text{sol})}(z) F(z) \prod_{k \in \Delta} \left( \frac{z - z_k}{zz_k - 1} \right)^{-\sigma_3}, \tag{6.32}
\]

where

\[
F(z) = \begin{cases}
\begin{pmatrix} 1 & 0 \\ \frac{c_j}{z - z_j} T(z)^2 e^{\Phi_j} & 1 \end{pmatrix} & |z - z_j| = \rho, \ j \in \nabla \\
\begin{pmatrix} 1 & 0 \\ \frac{z - z_j}{c_j} T(z)^{-2} e^{-\Phi_j} & 1 \end{pmatrix} & |z - z_j| = \rho, \ j \in \Delta \\
\begin{pmatrix} \frac{c_j}{z - z_j} T(z)^{-2} e^{\Phi_j} & 0 \\ 0 & 1 \end{pmatrix} & |z - z_j| = \rho, \ j \in \nabla \\
\begin{pmatrix} \frac{z - z_j}{c_j} T(z)^2 e^{-\Phi_j} & 0 \\ 0 & 1 \end{pmatrix} & |z - z_j| = \rho, \ j \in \Delta,
\end{cases}
\]

Clearly, the transformation to \( \tilde{m} \) preserves the normalization conditions at the origin and infinity. Comparing (6.32) to (6.26) it is immediate to see that the new unknown \( \tilde{m} \) has no jumps. From (6.8), RHP 6.2 and (6.32) it follows that \( \tilde{m}(z) \) has simple poles at each of the points in \( \mathcal{Z} \), the discrete spectrum of the original Riemann Hilbert problem, RHP 5.1. A straightforward calculation shows that the residues have the same algebraic as (5.6), but with (5.7) replaced by (6.31). Thus, \( \tilde{m}(z) \) is precisely the solution of RHP 5.1 with scattering data \( \{r(z) \equiv 0, \{z_k, \tilde{c}_k\}_{k=1}^{N}\} \). The symmetry \( r(s^{-1}) = \overline{r(s)} \), \( s \in \mathbb{R} \), implies that the argument of the exponential in (6.31) is purely real so that the perturbed connection coefficients maintain the reality condition \( \tilde{c}_j = iz_j \overline{c}_j \). Thus, \( m^{(\text{sol})} \) is the solution of RHP 5.1 corresponding to an N-soliton, reflectionless, potential \( \tilde{q}(x,t) \) which
generates the same discrete spectrum $Z$ as our initial data, but whose connection coefficients (6.31) are perturbations of those for the original initial data by an amount related to the reflection coefficient of the initial data. The solution of this discrete RHP is a rational function of $z$, whose exact solution always exists and can be obtained as described in Appendix A.

As described in the proof above and Appendix A, the RHP for $m^{(s)}$ can be solved exactly in closed form, but we will instead give the solution using the small norm theory of Riemann-Hilbert problems as this more naturally leads to the asymptotic form of the solution for $t \gg 1$.

The Riemann Hilbert problem for $m^{(s)}$ is ideally set up for asymptotic analysis. The jump matrix $V^{(2)}(z)$ satisfies

$$\|V^{(2)} - I\|_{L^p(\Sigma(z))} \leq K_p \sup_{z \in \Sigma(z)} e^{-C|z| |x-2tzn|} \leq K_p e^{-Cp^2t}, \quad 1 \leq p \leq \infty, \quad (6.33)$$

for some constant $K_p \geq 0$ independent of $(x,t)$. This implies that the jump matrices do not meaningfully, contribute to the asymptotic behavior of the solution. Instead, the dominant contribution to the solution comes from the simple poles of $m^{(s)}$, those at $z = 0$ and, if the critical line $\text{Re } z = \xi$ is passing through the neighborhood of one of the discrete spectra $z_j \in Z$ of the original problem RHP 5.1 those at $z_j$ and $\tau_{j0}$. Indeed, the following lemma describes this further simplification of $m^{(s)}$ explicitly.

**Lemma 6.6.** Let $\xi = \frac{\pi}{4\tau}$ and let $j_0 = j_0(\xi) \in \{-1, 0, 1, \ldots, N-1\}$, be defined by (6.7). Suppose

$$m^{(s)}_{j_0}(z) \text{ solves RHP 6.2 with } W(z) \equiv 0 \text{ and } V^{(2)} \equiv I. \quad (6.34)$$

Then, for any $(x,t)$ such that $|x/t| < 1$ and $t \gg 1$, uniformly for $z \in \mathbb{C}$ we have

$$\left| m^{(s)}(z) \right| = \left| m^{(s)}_{j_0}(z) \right| \left[ I + O \left( e^{-2\rho^2t} \right) \right],$$

and, in particular, for large $z$ we have

$$m^{(s)}(z) = m^{(s)}_{j_0}(z) \left[ I + \frac{Ce^{-2\rho^2t}}{z} + O \left( z^{-2} \right) \right]. \quad (6.35)$$

Moreover, the unique solution $m^{(s)}_{j_0}(z)$ to the above Riemann Hilbert problem, (6.34), is as follows:

i. if $j_0(\xi) = -1$, then all the $z_j$ are away from the critical line and

$$m^{(s)}_{j_0}(z) = I + \frac{\sigma_1}{z}; \quad (6.36a)$$

ii. if $j_0(\xi) \in \nabla$, then

$$m^{(s)}_{j_0}(z) = I + \frac{\sigma_1}{z} + \frac{\alpha_0^\nabla(x,t)}{z-z_{j_0}} + \frac{\beta_0^\nabla(x,t)}{z+\tau_{j_0}}; \quad (6.36b)$$

$$\alpha_0^\nabla(x,t) = -z_{j_0}/\tau_{j_0}(x,t), \quad \beta_0^\nabla(x,t) = \frac{2i \text{ Im}(z_{j_0}) z_{j_0} e^{-2\varphi_{j_0}}}{1 + e^{-2\varphi_{j_0}}},$$
iii. if \( j_0(\xi) \in \Delta \), then

\[
m^{(\text{sol})}_{j_0}(z) = I + \frac{\sigma_1}{z} + \left( \begin{array}{c}
\frac{\alpha_0^{\Delta}(x,t)}{z - z_0} \\
\frac{\beta_0^{\Delta}(x,t)}{z - z_0}
\end{array} \right)
\]

(6.36c)

\[
\alpha_0^{\Delta}(x,t) = -\frac{1}{z - z_0} \nabla \beta_0^{\Delta}(x,t), \quad \beta_0^{\Delta}(x,t) = -\frac{2i \text{Im}(z_{j_0}) \sigma_{j_0} e^{2\varphi_{j_0}}}{1 + e^{2\varphi_{j_0}}}.
\]

In cases ii. and iii. the real phase \( \varphi_{j_0} \) is given by

\[
x_{j_0} = \frac{1}{2 \text{Im}(z_{j_0})} \left( \log \left( \frac{|c_{j_0}|}{2 \text{Im}(z_{j_0})} \prod_{k \neq j_0} \left| \frac{z_{j_0} - z_{j_0} - z - z_{j_0}}{z_{j_0} - z_{j_0} - z - z_{j_0}} \right| \right) - \frac{\text{Im}(z_{j_0})}{\pi} \int_0^\infty \log(1 - |r(s)|^2) \, ds \right).
\]

(6.36d)

**Proof.** We begin by proving that (6.36) solves (6.34). The assumption that \( V \equiv I \) and \( W \equiv 0 \) implies that \( m^{(\text{sol})}_{j_0}(z) \) is meromorphic with simple poles at \( z = 0 \) and, if \( j_0 \neq -1 \), at both \( z_{j_0} \) and \( z_{j_0} \). If \( j_0 = -1 \), then (6.36a) is an immediate consequence of the condition 2 in RHP 6.2 and Liouville’s theorem. For \( j \neq 0 \), observe that \( C_0 := c_{j_0} T(z_{j_0})^2 \) satisfies \( C_0 = i z_{j_0}|C_0| \) since \( c_{j_0} = i z_{j_0}|c_{j_0}| \) and \( T(z) \in \mathbb{R} \) for \( |z| = 1 \), which follows from claim i. in Lemma 6.1. For \( j_0 \in \nabla \), this means that the RHP for \( m^{(\text{sol})}_{j_0}(z) \), is equivalent to the reflectionless, i.e., \( r = 0 \), version of RHP 5.1 with poles at the origin and at the points \( z_{j_0} \) and \( z_{j_0} \) with associated connection coefficient \( C_0 \). Then the symmetries (5.2a)-(5.2b) inherited by \( m^{(\text{sol})}_{j_0} \) and (6.2b) imply that \( \alpha_0^{\Sigma} = -z_{j_0} \beta_0^{\Sigma} \) and

\[
m^{(\text{sol})}_{j_0}(z) = I + \frac{\sigma_1}{z} + \left( \begin{array}{c}
\frac{\beta_0^{\Sigma}}{z - z_0} \\
\frac{\alpha_0^{\Sigma}}{z - z_0}
\end{array} \right)
\]

the residue conditions (6.29) then yield four linearly dependant equations for the single unknown \( \beta_0^{\Sigma} \), each equivalent to \( \beta_0^{\Sigma} = C_0 (1 - z_{j_0} |\beta_0^{\Sigma}| (z_{j_0} - z_{j_0})^{-1}) \), which gives (6.36b) upon setting \( \frac{|C_0|}{\text{Im}(z_{j_0})} \).

For \( j \in \Delta \), the computation is similar, but the new pole conditions (6.30) exchanges the columns in which the two poles occur; we have \( \alpha_0^{\Delta} = -z_{j_0} \beta_0^{\Delta} \) and

\[
m^{(\text{sol})}_{j_0}(z) = I + \frac{\sigma_1}{z} + \left( \begin{array}{c}
\frac{\beta_0^{\Delta}}{z - z_0} \\
\frac{\alpha_0^{\Delta}}{z - z_0}
\end{array} \right)
\]

(6.30)

Then residue relation (6.30) leads to one linearly independent equation which can be solved trivially yielding the second line of (6.36c). Now we show that \( m^{(\text{sol})}_{j_0}(z) \) gives the leading order behavior to \( m^{(\text{sol})}(z) \) for \( t \gg 1 \). The ratio \( m^{(\text{err})}(z) = m^{(\text{sol})}(z) \left( m^{(\text{sol})}_{j_0}(z) \right)^{-1} \) has no poles (the computation proving this is identical to (6.40)-(6.41) below) and its jump matrix \( V^{(\text{err})}(z) = \left( m^{(\text{sol})}_{j_0}(z) \right)^{-1} \) satisfies the same estimate as in (6.33) since \( m^{(\text{sol})}_{j_0}(z) - I - \frac{\sigma_1}{z} = O \left( e^{-2\rho t^2} \right) \) for \( z \in \Sigma^{(2)} \).
It then follows from the small norm theory for Riemann Hilbert problems that

\[ m^{(\text{sol})}(z) = m_{j_0}^{(\text{sol})}(z) \left[ I + \frac{1}{2\pi i} \int_{\Sigma(2)} \frac{(I + \mu(s))(V^{(\text{err})})(s) - I}{s - z} \, ds \right] \]

where \( \mu \in L^2(\Sigma(2)) \) is the unique solution of \( (1 - C_{V^{(\text{err})}})\mu = C_{V^{(\text{err})}}I \), where \( C_{V^{(\text{err})}} : L^2(\Sigma(2)) \to L^2(\Sigma(2)) \) is the Cauchy projection operator

\[ C_{V^{(\text{err})}}[f](z) = C_-[f(V^{(\text{err})}) - I] = \lim_{z' \to z} \int_{\Sigma(2)} \frac{f(s)(V^{(\text{err})})(s) - I}{s - z'} \, ds \]

where the limit is understood (possibly in the \( L^2 \) sense) to be taken non-tangentially from the minus (right) side of the oriented contour \( \Sigma(2) \). Existence and uniqueness of \( \mu \) follows from the boundedness of the Cauchy projection operator \( C_- \), which immediately implies

\[ \|C_{V^{(\text{err})}}\|_{L^2(\Sigma(2)) \to L^2(\Sigma(2))} = \mathcal{O}\left(e^{-2\rho^2t}\right). \]

The last two equations in the lemma follow immediately. \( \square \)

**Remark 6.7.** The different formulae for \( m_j^{(\text{sol})}(z) \) for \( j \in \nabla \) or \( j \in \Delta \) in Lemma 6.6 is an artifact of the conjugation by \( T(z) \) in (6.12) which transforms exponentially growing pole residues into decaying residues. As is shown below, near the line \( x = 2t \Re(z_j) \) the dominant contribution to \( m(z) \) the solution of the original Riemann Hilbert problem is of the form

\[ q_j^{(\text{sol})}(t, x) \equiv T(\infty, \xi)^{-2} \lim_{z \to \infty} z(m_j^{(\text{sol})})_{21}(z, t, x) \]

\[ = \begin{cases} 
    e^{i\theta_+} (1 + \beta_j^\nabla(x, t)) & x < 2t \Re(z_j) \\
    z_j^2 e^{i\theta_+} (1 + \beta_j^\Delta(x, t)) & x > 2t \Re(z_j), 
\end{cases} \]

(6.37)

where \( \theta_+ \) is a real constant, and \( \beta_j^\nabla \) and \( \beta_j^\Delta \) are given by (6.36b) and (6.36c) respectively and the extra factor of \( z_j^2 \) for \( x > 2t \Re(z_j) \) accounts for the additional factor in \( T(\infty, \xi) \) for \( j \in \Delta \). However, since \( \bar{z}_j = z_j^{-1} \), it’s a simple algebraic exercise to show that the two formulae are identical, so that either formula gives

\[ q_j^{(\text{sol})}(t, x) = e^{i\theta_+} \frac{1 + z_j^2 e^{-2\rho_j}}{1 + e^{-2\rho_j}} = e^{i\theta_+} \text{sol}(t, x - x_j, z_j) \]

where \( \text{sol}(t, x, z) \) defined by (1.3) is the formula for the dark 1-soliton.

We now complete the original goal of this section by using \( m^{(\text{sol})} \) to reduce \( m^{(2)} \) to a pure \( \overline{\partial} \)-problem which will be analyzed in the next section.

**Lemma 6.8.** Define the function

\[ m^{(3)}(z) = m^{(2)}(z) \left(m^{(\text{sol})}(z)\right)^{-1}. \]

(6.38)

Then, \( m^{(3)} \) satisfies the following \( \overline{\partial} \)-problem.

**\( \overline{\partial} \) Problem 6.1** Find a \( 2 \times 2 \) matrix-valued function \( m^{(3)}(z) \) such that

1. \( m^{(3)}(z) \) is continuous in \( \mathbb{C} \), and analytic in \( \mathbb{C} \setminus \Omega \).
2. \( m^{(3)}(z) = I + \mathcal{O}\left(z^{-1}\right) \) as \( z \to \infty \).

3. For \( z \in \mathbb{C} \) we have

\[
\bar{\partial}m^{(3)}(z) = m^{(3)}(z)W^{(3)}(z)
\]

where \( W^{(3)} := m^{(sol)}(z)W(z)\left(m^{(sol)}(z)\right)^{-1} \), with \( W(z) \) defined by (6.27), is supported in \( \Omega \).

**Proof.** It follows directly from (6.38) that \( m^{(3)} \) has no jumps on the disk boundaries \( |z - z_j| = \rho \) nor \( |z - \bar{z}_j| = \rho \) since \( m^{(sol)} \) has exactly the same jumps as \( m^{(2)} \) on these contours. The normalization condition and \( \bar{\partial} \) derivative of \( m^{(3)} \) follow immediately from the properties of \( m^{(2)} \) and \( m^{(sol)} \). It remains to show that the ratio also has no isolated singularities. At the origin we have

\[
(m^{(sol)}(z))^{-1} = (1 - z^{-2})^{-1}\sigma_2 m^{(sol)}(z)\sigma_2
\]

so \( m^{(3)}(z) \) is regular at the origin. If \( m^{(2)} \) has poles at \( z_{j0} \) and \( \bar{z}_{j0} \) on the unit circle then from the form of the residue relation we have in a neighborhood of \( z_{j0} \), local expansions

\[
m^{(2)}(z) = \begin{bmatrix} m_{12}^{(2)}(z_{j0}) \\ m_{22}^{(2)}(z_{j0}) \end{bmatrix} \begin{bmatrix} 0 \\ \frac{c_{j0}}{z - z_{j0}} \end{bmatrix} + \begin{bmatrix} *_{11} & 0 \\ *_{21} & 0 \end{bmatrix} + \mathcal{O}(z - z_{j0})
\]

\[
m^{(sol)}(z)^{-1} = \frac{z_{j0}^2}{z_{j0}^2 - 1} \left[ \begin{bmatrix} 1 \\ -c_{j0} \end{bmatrix} \begin{bmatrix} m_{22}^{(sol)}(z_{j0}) \\ -m_{12}^{(sol)}(z_{j0}) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mathcal{O}(z - z_{j0}) \right]
\]

where \(*_{jk} \) and \( \$_{jk} \) are constants. Taking the product gives

\[
m^{(3)}(z) = \frac{z_{j0}^2}{z_{j0}^2 - 1} \left[ \begin{bmatrix} *_{11} \\ *_{21} \end{bmatrix} \begin{bmatrix} m_{22}^{(sol)}(z_{j0}) \\ -m_{12}^{(sol)}(z_{j0}) \end{bmatrix} + \begin{bmatrix} m_{12}^{(2)}(z_{j0}) \\ m_{22}^{(2)}(z_{j0}) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mathcal{O}(1) \right]
\]

which shows that \( m^{(3)}(z) \) is bounded locally and the pole is removable. A similar argument shows that the pole at \( \bar{z}_{j0} \) is removable. Finally, because \( \det m^{(sol)}(z) = (1 - z^{-2}) \) we must check that the ratio is bounded at \( z = \pm 1 \). This follow from observing that the symmetries \( m(z) = \sigma_1 m(\bar{z})\sigma_1 = z^{-1}m(z^{-1})\sigma_1 \) applied to the local expansion of \( m^{(2)} \) and \( m^{(sol)} \) imply that

\[
m^{(2)}(z) = \left( \frac{c}{\pm\tau} \pm\epsilon \right) + \mathcal{O}(z - 1) \quad m^{(sol)}(z)^{-1} = \frac{\pm 1}{2(z + 1)} \left( \begin{bmatrix} \tau & \mp\gamma \end{bmatrix} \right) + \mathcal{O}(1)
\]

for some constants \( c \) and \( \gamma \). Taking the product it’s immediately clear the principal part of \( m^{(3)}(z) \) vanishes at \( z = \pm 1 \). \( \square \)

In Sect. 6.4 we will prove the following lemma.

**Lemma 6.9.** There exist constants \( T \) and \( c \) such that for \( t \geq T \) we have

\[
m^{(3)}(z) = 1 + \frac{m^{(3)}_1}{z} + o(z^{-1}) \quad \text{with} \quad |m^{(3)}_1| \leq c|z^{-1}|
\]
6.4 Step 4: Solution of the $\bar{\partial}$ problem 6.1 and asymptotics as $t \not\to \infty$

Lemma 6.10. Consider the following operator $J$:

$$JH(z) := \frac{1}{\pi} \int_{\Sigma} \frac{H(\xi)W^{(3)}(\xi)}{\xi - z} dA(\xi).$$  \hspace{1cm} (6.43)

Then we have $J : L^\infty(\Sigma) \to L^\infty(\Sigma) \cap C^0(\Sigma)$ and for any fixed $\xi_0 \in (0, 1)$ there exists a $C = C(q_0, \xi_0)$ s.t.

$$\|J\|_{L^\infty(\Sigma) \to L^\infty(\Sigma)} \leq Ct^{-\frac{1}{2}} \text{ for all } t > 1 \text{ and for } \left|\frac{x}{2t}\right| \leq \xi_0. \hspace{1cm} (6.44)$$

Proof. To prove (6.44) we follow the argument in Prop. 2.2 [21]. It is not restrictive to consider only the proof of $\|JH\|_{L^\infty(\Sigma)} \leq C t^{-\frac{1}{2}} \|H\|_{L^\infty(\Sigma)}$ for $H \in L^\infty(\Omega_1)$. Recall that the definition of $W^{(3)}(z) := m^{(sol)}(z)W(z) (m^{(sol)}(z))^{-1}$. So, using Lemma 5.3 for the 1st inequality and Lemma 6.6 for the 2nd, there exists a fixed constant $C_1$ s.t. pointwise

$$|W^{(3)}(z)| \leq |m^{(sol)}(z)|^2(1 - z^{-2})^{-1}|W(z)| \leq C_1(1 - z^{-2})^{-1}|W(z)|$$

This implies that

$$|JH(z)| \leq \|H\|_{L^\infty(\Sigma)} \int_{\Omega_1} \frac{\overline{\partial}R(\xi)e^{-\text{Re} \Phi(\xi)}}{|\xi - z|} dA(\xi). \hspace{1cm} (6.45)$$

To simplify notation we will normalize the problem and suppose $\theta_0 = \pi/4$ so that $\Omega_1$ is the sector defined by $\arg(z) \in [0, \pi/4]$. Now we insert in the integral the function $\chi_{[0,1]}(|\xi|) + \chi_{[1,2]}(|\xi|) + \chi_{[2,\infty]}(|\xi|)$. We prove first the following, where the 1st inequality is obvious:

$$\int_{\Omega_1} \frac{\overline{\partial}R(\xi)e^{-\text{Re} \Phi(\xi)}}{|\xi - z|} dA(\xi) \leq \int_{\Omega_1} \frac{\overline{\partial}R(\xi)e^{-\text{Re} \Phi(\xi)}}{|\xi - z|} dA(\xi) \leq C t^{-\frac{1}{2}}. \hspace{1cm} (6.46)$$

Set $\xi = u + iv, z = z_R + iz_I, 1/q + 1/p = 1$ with $p > 2$. Lemma 6.3. The hypothesis that there is a constant $\xi_0 \in (0, 1)$ s.t. $|\xi| \leq \xi_0$ is crucial in order to have $|\text{Re} \Phi(\xi)| \geq c|uv|$ for a fixed $c = c(\xi_0) > 0$. Notice also that (6.45) contains an extra singularity with respect to Proposition 2.2 in [21]. It is to offset this that our extensions of $R(z)$ in Lemma 6.4 in particular formula (6.22), where somewhat more elaborate than in [21].

To prove the 2nd inequality in (6.46) we replace $|\overline{\partial}R|$ by the 3 terms in the r.h.s. of (6.16). When replacing $|\overline{\partial}R(\xi)|$ with the term $|\xi|^{-\frac{3}{2}}$ we have

$$\int_0^\infty dv e^{-c_2 - \frac{1}{2}tv} \int_v^\infty \frac{\chi_{[1,\infty]}(|\xi|)}{|\xi - z|} du \leq \int_0^\infty dv e^{-c_2 - \frac{1}{2}tv} \frac{1}{\|\xi - z\|^{rac{1}{2}} \|L^p(\Sigma, \Sigma)||\xi - z|^{-1}\|L^p(\Sigma, \Sigma)}$$

$$\leq c e^t \int_0^\infty dv e^{-c_2 - \frac{1}{2}tv v^{-\frac{1}{2}}|v - z_I|^{-\frac{1}{2}}} \leq 4c e^t \int_0^\infty dv e^{-c_2 - \frac{1}{2}tv v^{-\frac{1}{2}}} \leq Ct^{-\frac{1}{2}} \hspace{1cm} (6.47)$$

while replacing $|\overline{\partial}R(\xi)|$ with $f(|\xi|) = r'(|\xi|), \varphi(|\xi|)$ we get

$$\int_0^\infty dv e^{-c_2 - \frac{1}{2}tv} \int_v^\infty \frac{\chi_{[1,\infty]}(|\xi|) f(|\xi|)}{|\xi - z|} du \leq c\|f\|_{L^2} \int_0^\infty dv e^{-c_2 - \frac{1}{2}tv |v - z_I|^{-\frac{1}{2}}} \leq Ct^{-\frac{1}{2}} \|f\|_{L^2}. \hspace{1cm} (6.48)$$
We have used
\[ \int_{\Omega} |\bar{\partial}R(\zeta)e^{-Re\Phi(\zeta)}\chi_{[1,2]}(|\zeta|)| dA(\zeta) \leq c_1 \int_{\Omega} \frac{e^{-Re\Phi(\zeta)}\chi_{[1,2]}(|\zeta|)}{|\zeta - z|} dA(\zeta) \leq C t^{-\frac{1}{2}}.\] (6.50)

This follows immediately from (6.48) and (6.48) for \(|z| = \chi_{[1,2]}(|z|)\). From (6.46) and (6.50) we conclude that for some \(C(q_0, \xi_0)\)
\[ \int_{\Omega} |\bar{\partial}R(\zeta)e^{-Re\Phi(\zeta)}\chi_{[1,\infty]}(|\zeta|)| dA(\zeta) \leq C(q_0, \xi_0)t^{-\frac{1}{2}}.\] (6.51)

We change notation writing \(z^{-1}\) for \(z\) and we now show
\[ \int_{\Omega} \left| \left| \frac{\partial R}{\zeta} \left( \frac{w}{w - z} \right) \right| \right|_{L^2} dA(w) \leq C t^{-\frac{1}{2}}.\] (6.52)

For \(w = 1/\zeta\), by \(dA(w) = |\zeta|^{-4} dA(\zeta)\) and \(\Phi(t, x, \zeta^{-1}) = -\Phi(t, x, \zeta)\), (6.52) becomes
\[ \int_{\Omega} \left| \left| \frac{\partial R}{\zeta} \right| \right|_{L^2} \left| \frac{|\zeta|^{-4} dA(\zeta)}{\zeta^{-1} - z^{-1}} \chi_{[\infty, |\zeta|]}(\zeta) \right| dA(\zeta) = \int_{\Omega} \left| \left| \frac{\partial R}{\zeta} \right| \right|_{L^2} \left| \frac{\chi_{[1,\infty]}(\zeta)}{|\zeta - z| |\zeta - 1|} \right| dA(\zeta)\] (6.53)

If \(|z| \leq 10\) we are back to (6.51). If \(|z| \geq 10\) we can bound the r.h.s. of (6.53) by
\[ 10 \int_{|\zeta| \geq 10} \left| \left| \frac{\partial R}{\zeta} \right| \right|_{L^2} \chi_{\Omega}(\zeta) dA(\zeta) + 10 \int_{|\zeta| \leq 10} \left| \left| \frac{\partial R}{\zeta} \right| \right|_{L^2} \chi_{\Omega}(\zeta) dA(\zeta)\]
and we are back to the previous case. So we have proved (6.44).

Lemma 6.10 implies \(m^{(3)} = 1 + Jm^{(3)}\). Indeed, since \(\frac{1}{\pi} \frac{1}{z - w} \frac{d\phi}{w} = \phi\) for any test function \(\phi \in C_0^\infty(\mathbb{C}, \mathbb{C})\), we can write
\[ \int_{\mathbb{C}} m^{(3)}(w)W^{(3)}(w)\phi(w) dA(w) = \int_{\mathbb{C}} m^{(3)}(w)W^{(3)}(w) \left| \frac{1}{\pi} \int_{\mathbb{C}} \frac{d\phi(z)}{z - w} dA(z) \right| dA(w) = - \int_{\mathbb{C}} Jm^{(3)}(z) dA(z)\]
where we exploit the fact, proved in the course of Lemma 6.10, that \(m^{(3)}(w)W^{(3)}(w)\bar{d\phi}(z) \in L^1(\mathbb{C}^2)\), so that we can exchange order of integration. Since Lemma 6.10 implies that \(Jm^{(3)}(z)\) is a continuous function in \(z\) uniformly bounded in \(\mathbb{C}\), we conclude that \(\delta (m^{(3)} - Jm^{(3)}) = 0\) in distributional sense. By elliptic regularity \(m^{(3)} - Jm^{(3)}\) is smooth, see p.179 [38], and so it is holomorphic in \(\mathbb{C}\). Finally, by 2. in the RH problem 6.1 we get \(m^{(3)} = 1 + Jm^{(3)}\).
Proof of Lemma 6.9. By the above discussion can write

\[ m_1^{(3)} = -\frac{1}{\pi} \int_{\mathcal{C}} m^{(3)}(z)W^{(3)}(z)dA(z). \]  

(6.54)

We need to bound integrals like

\[ \int_{\Omega_1} |e^{-\Phi}\partial R(z)||z - 1|^{-1}\chi_{[1,\infty]}(|z|)dA(z) \leq C(q_0, \xi_0)t^{-1} \]  

(6.55)

We first replace \( \chi_{[1,\infty]}(|z|) \) with \( \chi_{[2,\infty]}(|z|) \). The we can ignore the factor \( |z - 1|^{-1} \leq 1 \) and get the upper bound (recall \( q < 2 \))

\[ \int_{\Omega_1} |e^{-\Phi}\partial R(z)|\chi_{[2,\infty]}(|z|)dA(z) \leq \int_{\Omega_1} e^{-\text{Re}(\Phi(z))}(|z|^{-\frac{1}{2}} + \sum_{f=\sigma', \varphi} |f(|z|)|\chi_{[1,\infty]}(|z|))dA(z) \]

\[ \leq C_1 \int_{\Omega_1} e^{-c^{-\frac{1}{2}} + \sum_{f=\sigma', \varphi} |f(|z|)|\chi_{[1,\infty]}(|z|))dA(z) \]

(6.56)

By the above inequality for \( f = \chi_{[1,2]} \) and by (6.17) we have

\[ \int_{\Omega_1} |e^{-\Phi}\partial R(z)|\chi_{[1,\infty]}(|z|)dA(z) \leq c_1 \int_{\Omega_1} e^{-\text{Re}(\Phi)|z|}\chi_{[1,\infty]}(|z|))dA(z) \leq Ct^{-1}. \]  

(6.57)

By the change of variables \( z = \zeta^{-1} \) we bound

\[ \int_{\Omega_4} |e^{-\Phi}\partial R(\zeta)|\zeta - 1|^{-1}\chi_{[0,1]}(|\zeta|))dA(\zeta) = \int_{\Omega_4} |e^{\Phi}\partial R(\zeta)|\zeta - 1|^{-1}\chi_{[1,\infty]}(|\zeta|)|\zeta|^{-1}dA(\zeta). \]

The latter is bounded by the r.h.s. of (6.55).

\[ \square \]

6.5 Proofs of Theorems 1.1 and 1.5

Proof of Theorem 1.1. For \( z \) large and in \( \mathbb{C}\backslash\overline{\Omega} \) we have \( m^{(1)}(z) = m^{(2)}(z) \) and so by (6.38)

\[ m(z) = T(\infty, \xi)^{\sigma_3}m^{(2)}(z)(T(z, \xi)^{-\sigma_3} = T(\infty, \xi)^{\sigma_3}m^{(sol)}(z)(m^{(3)}(z))^{-1}T(\infty, \xi)^{-\sigma_3} \]

\[ \times \left( I - z^{-1}\left( \sum_{k \in \Delta} 2i\text{Im}(z_k) - \frac{1}{2\pi i} \int_{0}^{\infty} \log(1 - |r(s)|^2)^{-\sigma_3} + o(z^{-1}) \right) \right), \]

by the asymptotic expansion of \( T(z, \xi) \). Then, since the first two terms of the factor in the last line are diagonal, by \( m^{(3)}(z) = I + z^{-1}O(t^{-1} + o(z^{-1})) \), by (6.35) and by (6.36a)–(6.36c) we obtain for \( |x - 2\text{Re}(z_j)| \leq pt \):

\[ q(x, t) = \lim_{z \to \infty} zm_{21}(z) = -T(\infty, \xi)^{-2}i\text{Im}(z_{j_0}) + \text{Re}(z_{j_0}) + \text{Im}(z_{j_0})\tanh \varphi_{j_0}) + O(t^{-1}). \]  

(6.58)
For \(|x - 2\text{Re}(z_{j_0})| \leq \rho |t| and \(j_0 \in \Delta\) we have instead
\[
q(x,t) = \lim_{z \to x} zm_{21}(z) = -T(\infty, \xi)^{-2} i\pi j_{j_0} (i \text{Re}(z_{j_0}) + \text{Im}(z_{j_0}) \tanh \varphi_{j_0}) + \mathcal{O}\left(t^{-1}\right) .
\] (6.59)

In (6.58), the main term can be written as
\[
\delta_{+}^{-1} \prod_{k < j_0} z_k^2 \text{sol}(t, x - x_{j_0}, z_{j_0}) \quad \text{where} \quad \delta_+ := e^{\frac{i}{2}} \int_{-1}^{1} \frac{\log(1 - |s|^2)}{s} ds
\] (6.60)

using the formula for \(T(\infty, \xi)\), the obvious fact that \(\Delta = \Delta \setminus \{j_0\}\) for \(j_0 \in \nabla\) and by (1.3), which implies \(\Delta \setminus \{j_0\}\) = \(\{k : k < j_0\}\). (6.60) represents the main term also in (6.59). By \(\lim_{x \to \infty} \text{sol}(t, x_j, z_j) = 1\) and \(\lim_{x \to -\infty} \text{sol}(t, x_j, z_j) = z_j^2\) it is elementary to see that (6.60) differs from the r.h.s. of (1.8) by \(\mathcal{O}\left(t^{-1}\right)\). We obtain similarly (1.8) also when \(j_0 = 0\), that is when \(|x - 2\text{Re}(z_{j_0})| > \rho|t|, where we have
\[
q(x,t) = \lim_{z \to x} zm_{21}(z) = -T(\infty, \xi)^{-2} + \mathcal{O}\left(t^{-1}\right) = \delta_{+}^{-1} \prod_{k \leq \text{sup} \Delta} z_k^2 + \mathcal{O}\left(t^{-1}\right).
\] (6.61)

It is elementary to see that (6.61) differs from the r.h.s. of (1.8) by \(\mathcal{O}\left(t^{-1}\right)\). Finally notice that for \(q^{(sol),N}(t, x)\) the \(N\)-soliton in Lemma 6.5 our analysis proves (1.6) since formulas (6.58), (6.59) and (6.61) hold also for \(q^{(sol),N}(t, x)\).

\[\square\]

**Proof of Theorem 1.5** Given \(q_0\) close to the \(M\)-soliton \(q^{(sol),M}(x, 0)\) we obtain the information on the poles and coupling constants in (1.11) by the Lipschitz continuity of maps such \(f_{\alpha}\) in Lemma 3.1 and \(\pm \) in Lemma 3.2. Furthermore we can apply to \(q_0\) Lemmas 4.5, 4.6. Hence we can apply to \(q_0\) Theorem 1.1 obtaining (1.8). By elementary computations (1.8) yields (1.12).

\[\square\]

## A N-solitons

Consider \(N\) points \(z_j = e^{i\theta_j}\), labeled such that \(0 < \theta_0 < \cdots < \theta_{N - 1} < \pi\) and set
\[
a(z) = \prod_{k = 0}^{N - 1} \frac{z - z_k}{z_k}.
\] (A.1)

Notice that
\[
\prod_{k = 0}^{N - 1} z_k^2 = a(0).
\] (A.2)

Consider also corresponding coupling constants \(c_j\) with \(c_j = i z_j |c_j|\) and let \(c_j(x, t) = c_j e^{\Phi(z_k, x, t)}\) like in (5.7). Then consider the unique (by the proof of Lemma 5.3) solution of the corresponding RH problem 5.1 (with \(r(z) \equiv 0\)) satisfying the symmetries of Lemma 5.1. It is a meromorphic function approaching identity as \(z \to \infty\) with \(2N + 1\) simple poles \(m(z, x, t)\) with a partial fraction expansion of the form
\[
m(z, x, t) = I + \frac{\sigma_1}{z} + \sum_{k = 0}^{N - 1} \frac{1}{z - z_k} \begin{pmatrix} \alpha_k(x, t) & 0 \\ \beta_k(x, t) & 0 \end{pmatrix} + \sum_{k = 1}^{N} \frac{1}{z - z_k} \begin{pmatrix} 0 & \tilde{\beta}_k(x, t) \\ 0 & \tilde{\alpha}_k(x, t) \end{pmatrix}.
\] (A.3)
Assuming for a moment that \( m(z, x, t) \) exists we will consider the \( N \)-soliton
\[
q^{(\text{sol}), N}(x, t) := \lim_{z \to \infty} z m_{21}^{(\text{sol})}(z; x, t) = 1 + \sum_{k=1}^{N-1} \beta_k(x, t). 
\] (A.4)

Before discussing the boundary values of \( q^{(\text{sol}), N}(x, t) \) and proving Lemma B.2 we study the existence of \( m(z, x, t) \). By (5.2a) we have
\[
\tilde{\alpha}_k(x, t) = \alpha_k(x, t), \quad \tilde{\beta}_k(x, t) = \beta_k(x, t) 
\] (A.5)
and by (5.2b) the additional symmetry
\[
\alpha_k(x, t) = -z_k \tilde{\beta}_k(x, t). 
\] (A.6)

Inserting (A.3) into (5.6) and using (A.5)-(A.6) we arrive at the reduced linear system:
\[
(I - C_{tx} Z) \cdot \beta_{tx} = C_{tx} \cdot 1 
\] (A.7)
where \( \beta_{tx}, 1 \in \mathbb{C}^N \) and \( C_{tx}, Z \in M(\mathbb{C}, N) \) are given by
\[
\beta_{tx} = \{ \beta_0(x, t), \ldots, \beta_{N-1}(x, t) \}^T, \quad 1 = \{1, \ldots, 1\}^T 
\]
\[
C_{tx} = \text{diag}(c_0(x, t), \ldots, c_N(x, t)) 
\]
\[
\{ Z_{jk} \}_{j,k=0}^{N-1} = \frac{\overline{z_j}}{z_j - z_k}. 
\] (A.8)

For general \( C_{tx} \) the matrix \( I - C_{tx} Z \) need not be invertible. However, under the reality condition \( c_j(x, t) = i z_j |c_j(x, t)| \), the system can be expressed in the more symmetric form
\[
(I + Y_{tx}) \cdot \tilde{\beta}_{tx} = b_{tx} 
\] (A.9)
where
\[
\tilde{\beta}_{tx} := \{ |c_0(x, t)|^{-1/2} \beta_1, \ldots, |c_N(x, t)|^{-1/2} \beta_{N-1} \}^T 
\]
\[
b_{tx} := \{ i |c_0(x, t)|^{1/2} z_1, i |c_2(x, t)|^{1/2} z_2, \ldots, i |c_{N-1}(x, t)|^{1/2} z_N \}^T. 
\]

Letting \( y_j = -iz_j \) ( Im \( z_j > 0 \) \( \Rightarrow \) Re \( y_j > 0 \) ) we have
\[
(Y_{tx})_{jk} = \frac{|c_j(x, t)|^{1/2} |c_k(x, t)|^{1/2}}{y_j + y_k} = |c_j(x, t)|^{1/2} |c_k(x, t)|^{1/2} \int_0^\infty e^{-(\overline{y_j} + y_k)s} ds. 
\]

Invertibility of the system then follows from the observation that \( Y_{tx} \) is positive definite:
\[
w^\dagger Y_{tx} w = \int_0^\infty \left( \sum_{j,k=0}^{N-1} |c_j(x, t)| c_k(x, t) |^{1/2} e^{-(\overline{y_j} + y_k)s} \overline{\pi_j w_k} \right) ds 
\]
\[
= \int_0^\infty \left( \sum_{k=0}^{N-1} |c_k(x, t)|^{1/2} e^{-y_k s} w_k \right)^2 ds \geq 0. 
\]

Using (A.4) and Cramer’s rule, the solution of the NLS corresponding to the given discrete scattering data is given by
\[
q^{(\text{sol}), N}(x, t) = 1 - \frac{\text{det}(I - (C_{tx} Z)_1)}{\text{det}(I - C_{tx} Z)} 
\] (A.10)
where \((C_{tx}Z)_1\) is the \((N + 1) \times (N + 1)\) matrix
\[
(C_{tx}Z)_1 := \begin{pmatrix}
C_{tx}Z & \vdots \\
1 & \ddots & 1
\end{pmatrix}.
\]

\[\text{(A.11)}\]

## B Global existence of solution of the NLS equation

Here we establish the global existence of solutions for \((1.1)\) with initial data \(q_0 \in \tanh(x) + \Sigma_5\) and show that the \(N\)-soliton solutions \(q^{(\text{sol})}, N(x, t)\) constructed in \(A\) belong to this class of data.

**Theorem B.1.** Consider the initial value problem \((1.1)\) with \(q_0 - \tanh(x) \in \Sigma_5\). Then \((1.1)\) admits a unique global solution \(q(t)\) s.t. \(q(t) - \tanh(x) \in C^0([0, \infty), H^5(\mathbb{R})) \cap C^1([0, \infty), H^3(\mathbb{R}))\). Furthermore we have \(q(x, t) - \tanh(x) \in C^0([0, \infty), \Sigma_5) \cap C^1([0, \infty), \Sigma_5)\).

**Proof.** By Gallo \cite{25} there is a unique global solution \(q(x, t)\) of \((1.1)\) s.t. the function \(v(x, t) := q(x, t) - \tanh(x)\) is in \(C^0([0, \infty), H^1(\mathbb{R}))\). Furthermore since \(v(x, 0) \in X^5(\mathbb{R}) \subset X^1(\mathbb{R})\), by \cite{26,25} we also have \(v(x, t) \in C^0([0, \infty), X^1(\mathbb{R}))\), where \(X^k(\mathbb{R}) := L^\infty(\mathbb{R}) \cap \bigcap_{j=1}^{k} H^j(\mathbb{R})\). In \cite{4} it is proven that \(v(x, t) \in C^0([0, \infty), X^5(\mathbb{R}))\). All these facts together imply \(v(x, t) \in C^0([0, \infty), H^5(\mathbb{R})) \cap C^1([0, \infty), H^3(\mathbb{R}))\).

The fact that \(v(x, t) \in C^0([0, \infty), \Sigma_5)\) con now be proved by standard arguments by multiplying the equation of \(v\) by \(x^5e^{-\varepsilon x^2}\) and, taking the limit \(\varepsilon \searrow 0\), by showing that \(x^5v(x, t) \in L^\infty([0, T], L^2(\mathbb{R}))\) for any \(T\). Indeed, \(v(x, t)\) solves (for \(v_R = \text{Re} \ v\))
\[
i v + v_{xx} - 2((|v|^2 + 2v_R \tanh(x))(v + \tanh(x)) - \text{sech}^2(x)v) = 0. \tag{B.1}
\]

Multiplying the equation by \(x^2 e^{-\varepsilon x^2}v\) for \(1 \leq j \leq 5\), taking the imaginary part and integrating in \(x\) on \(\mathbb{R}\) we obtain, for \(\{\partial_x^j, x^2 e^{-\varepsilon x^2}\} v = (x^j e^{-\varepsilon x^2})'v + 2(x^j e^{-\varepsilon x^2})'v_x\),
\[
\frac{d}{dt} \|x^j e^{-\varepsilon x^2} v\|_{L^2} \leq C(\|\partial_x^j, x^j e^{-\varepsilon x^2}\| v\|_{L^2} + \|x^j e^{-\varepsilon x^2} v\|_{L^2}) \tag{B.2}
\]

We have \(\|x^j e^{-\varepsilon x^2}\|_{L^2} \leq \|v\|_{\Sigma_{j-1}}\), where we assume the r.h.s. bounded by induction.

We have for fixed constants
\[
\|x^j e^{-\varepsilon x^2} v_x\|_{L^2} \leq C' \|x^{j-1} e^{-\varepsilon x^2} v_x\|_{L^2} \leq C(\|x^j e^{-\varepsilon x^2} v\|_{L^2} + \|\partial_x^j v\|_{L^2})). \tag{B.3}
\]

The 2nd inequality follows by the identity for \(f\) real, see \cite{35} p.1069,
\[
\int x^{2j-2} e^{-2\varepsilon x^2} (f_x)^2 = 2^{-1} \int f(x^{2j-2} e^{-2\varepsilon x^2})'' f_x^2 dx + \int x^{2j-2} e^{-2\varepsilon x^2} f f_{xx} dx.
\]

Then, by Gronwall inequality, \((B.2)-(B.3)\) imply that \(\|x^j e^{-\varepsilon x^2} v(\cdot, t)\|_{L^2(\mathbb{R})} \leq C_T\) for \(t \in [0, T]\) and all \(j = 1, \ldots, 5\). By Fatou's lemma we conclude \(v(x, t) \in L^\infty([0, T], \Sigma_5)\) for all \(T \geq 0\). But then by dominated convergence \(x^j e^{-\varepsilon x^2} v \to x^j v\) in \(L^\infty([0, T], L^2(\mathbb{R}))\) and since \(x^j e^{-\varepsilon x^2} v \in C^0([0, T], L^2(\mathbb{R}))\), we have also \(x^j v \in C^0([0, T], L^2(\mathbb{R}))\) for all \(j \leq 5\). So we conclude that \(v(x, t) \in C^0([0, T], \Sigma_5)\). From \((B.1)\) we have also \(v(x, t) \in \mathcal{C}^1([0, T], \Sigma_5)\).

\[\square\]
Finally, proceeding as in Sect. 6.5 as in (6.58) and using (A.2) we have

\[ a \text{ approaches } 0 \text{ exponentially fast as } x \to \infty. \]

Recall Lemma 6.6, that for fixed \( t \) we conclude from (A.10)–(A.11) that for a fixed \( a \) (where in the set up of Lemma B.2 we have \( \Delta = \{ c > 0 \} \)). Since for \( |x| \to \infty \) we have \( \Phi(z_j, x, t) = -2\pi \Im[z_j](1 + o(1)) \) it is elementary to conclude that \( q^{(sol), N}(x, t) - 1 \) with all its derivatives approaches 0 exponentially fast as \( x \to \infty \). We assume now

\[ \lim_{x \to -\infty} q^{(sol), N}(x, t) = a(0) \text{ for any fixed } t \geq 0 \] (B.4)

(where in the set up of Lemma B.2 we have \( a(0) = -1 \)). Then for any fixed \( t \) it is elementary to conclude from (A.10)–(A.11) that for a fixed \( c > 0 \) and for \( x \ll -1 \)

\[
\det(I - C_{t_x} Z) = (-1)^N \prod_{j=0}^{N-1} c_j(x, t) \det(Z) (1 + \mathcal{O}(e^{cx}))
\]

\[
\det(I - (CZ)_1) = (-1)^{N+1} \prod_{j=0}^{N-1} c_j(x, t) \det\left( \begin{pmatrix} Z & 1 \\ 1 \cdots 1 & 0 \end{pmatrix} \right) (1 + \mathcal{O}(e^{cx})).
\]

This implies that \( q^{(sol), N}(x, t) - a(0) \) with all its derivatives approaches 0 exponentially fast as \( x \to -\infty \). To see (B.4) we associate to our \( m(z, x, t) \) the function \( m^{(1)}(z, x, t) \) in (6.12). Notice that \( m^{(1)}(z) \) solves a Riemann–Hilbert problem in \( \Sigma^{(2)} \) since the jump matrix in \( \mathbb{R} \) is the identity. In other words, here \( m^{(1)}(z) = m^{(sol)}(z) \). Now, since as \( x \to -\infty \) we have \( \xi \to -\infty \), in this case \( \Delta = \{ 0, ..., N-1 \} \) and \( j_0(\xi) = -1 \), see (6.6)–(6.7). It is also easy to see, following the proof of Lemma 6.6, that for fixed \( t \) for a fixed \( c > 0 \) and all \( x \ll -1 \) we have

\[ m^{(sol)}(z) = I + z^{-1} \sigma_1 + z^{-1} \mathcal{O}\left(e^{-c|z|}\right) + o(z^{-1}). \]

Finally, proceeding as in Sect. 6.5 as in (6.58) and using (A.2) we have

\[
q^{(sol), N}(x, t) = \lim_{z \to -\infty} zm_{21}(z) = T(\infty, \xi)^{-2} \lim_{z \to -\infty} zm_{21}^{(sol)}(z)
\]

\[ = a(0)(1 + \mathcal{O}\left(e^{-c|z|}\right)) \to a(0) \text{ as } x \to -\infty. \]

\[ \square \]

C  Singularity of \( a(z) \) in \( z = \pm 1 \) for generic \( q_0 \)

We check here that for \( q_0 = \tanh(x) + \varepsilon f \) with \( f = f_R + if_I \) with \( f_A \in C^\infty_c(\mathbb{R}, \mathbb{R}) \) for \( A = R, I \) generic, then the function \( a(z) \) blows up at \( z = \pm 1 \). Denote by \( \psi_{\pm}^{\varepsilon}(x, z) \) the functions in (3.9). They extend to \( (z, x) \in \mathbb{C}_+ \times \mathbb{R} \) and they are smooth. Recalling (4.9) we denote

\[ a(\varepsilon, z) = \frac{W(\varepsilon, z)}{1 - z^{-2}} \text{ where } W(\varepsilon, z) := \{ \psi_{1\varepsilon}^{\pm}(x, z), \psi_{2\varepsilon}^{\pm}(x, z) \}. \]

Recall \( a(0, z) = \frac{z - 1}{z + 1} \). This yields \( W(0, z) = \frac{(z - 1)(z^2 - 1)}{(z + 1)z} \). We have the following fact.
Lemma C.1. We have

\[ W(\varepsilon, z) = -2i(z - 1) - 2\varepsilon C(\pm, f) + F_\pm(z, \varepsilon) \]

\[ C(\pm, f) := \int_{\mathbb{R}} \frac{(e^{-4y} - 1)f_r(y) + 2e^{-2y}f_{\pm}(y)}{(1 + e^{-2y})^2} \, dy \]  \hspace{1cm} (C.2)

where, for \(|z - 1| < c_f\) and \(|\varepsilon| < c_f\) for a sufficiently small constant \(c_f > 0\), the function \(F_\pm(z, \varepsilon)\) is analytic in \(z\) and for a fixed constant \(C_f\)

\[ |F_\pm(z, \varepsilon)| \leq C_f(|z - 1|^2 + \varepsilon^2). \]  \hspace{1cm} (C.3)

For generic \(f \in C_c^\infty(\mathbb{R}, \mathbb{C})\) we have \(C(\pm, f) \neq 0\). Then replacing \(F_\pm(z, \varepsilon)\) with 0 we obtain a function with a zero in

\[ \tilde{z}_\pm(\varepsilon) = \pm(1 + iC(\pm, f)) \]

and by Rouché Theorem we have that \(W(\varepsilon, z)\) has for \(\varepsilon\) small a zero

\[ z_\pm(\varepsilon) = \pm(1 + iC(\pm, f)) + O(\varepsilon^2). \]

If \(\pm\varepsilon C(\pm, f) > 0\) this yields a new zero in \(C_+\) of \(a(\varepsilon, z)\) near \(\pm 1\) and a corresponding almost white soliton. If \(\pm\varepsilon C(\pm, f) < 0\) this is a new zero of the analytic continuation of \(a(\varepsilon, z)\) below \(\mathbb{R}\), does not yield a new soliton but nonetheless makes \(a(\varepsilon, z)\) singular at \(\pm 1\). All four cases can occur.

Proof of Lemma C.1: For \((\psi_{10}^+(x, z))_j\) the \(j\)-th component of \(\psi_{10}^+(x, z)\) for \(j = 1, 2\), we set

\[ \Delta Q(x) = \begin{pmatrix} 0 & f(x) \\ f(x) & 0 \end{pmatrix}, \quad U(x, y, z) = [\psi_{10}^-(x, z), \psi_{20}^+(x, z)][\psi_{10}^-(y, z), \psi_{20}^+(y, z)]^{-1}, \]

with \([\psi_{10}^-, \psi_{20}^+\] the matrix with first column \(\psi_{10}^-\) and second column \(\psi_{20}^+\) and with the last the inverse of one such matrix. \(U(x, y, z)\) is well defined for any \(z \neq 0, 1\) in \(C\). We have \((\partial_z - \mathcal{L}(z; x))U(x, y, z) = 0\) and \(U(y, y, z) = 1\), i.e. \(U(x, y, z)\) is the fundamental solution of equation (3.3a) with \(Q\) defined using \(\tanh(x)\). Let \(f \in C_c^\infty((-M, M), \mathbb{C})\). Notice then that for \(\psi_{1e}^+\) and \(\psi_{2e}^+\) Jost functions associated to \(\eta_0\), we have \(\psi_{1e}^+(x, z) = \psi_{2e}^+(x, z)\) for \(x > M\) and \(\psi_{1e}^+(x, z) = \psi_{10}^+(x, z)\) for \(x < -M\). Then for \(x > M\) we have for any preassigned \(x_0 < -M\)

\[ \psi_{1e}^+(x, z) = \psi_{20}^+(x, z) \]

\[ \psi_{1e}^-(x, z) = U(x, x_0, z)\psi_{1e}^-([x_0, x]) + \int_{x_0}^x U(x, y, z)\sigma_3\Delta Q(y)\psi_{1e}^-(y, z) \, dy \]

\[ = \psi_{10}^-(x, z) + \int_{x_0}^x U(x, y, z)\sigma_3\Delta Q(y)\psi_{1e}^-(y, z) \, dy. \]  \hspace{1cm} (C.4)

Picking \(x > M\) and substituting (C.4) we can write

\[ W(\varepsilon, z) = \{\psi_{1e}^-(x, z), \psi_{2e}^+(x, z)\} = \{\psi_{1e}^-(x, z), \psi_{20}^+(x, z)\} \]

\[ = W(0, z) + \varepsilon \int_{\mathbb{R}} I'(y, z) \, dy \]

where \(I'(y, z) := \{U(x, y, z)\sigma_3\Delta Q(y)\psi_{1e}^-(y, z), \psi_{20}^+(x, z)\}\). Notice that we have

\[ I'(y, z) = \{[\psi_{10}^-(x, z), \psi_{20}^+(x, z)]F(y, z), \psi_{20}^+(x, z)\} \]

\[ = \{F_1(y, z)\psi_{10}^-(x, z) + F_2(y, z)\psi_{20}^+(x, z), \psi_{20}^+(x, z)\} = F_1(y, z)W(0, z), \]

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for $F(y, z)$ the $2$ components column vector

\[
\begin{pmatrix} F_1(y, z) \\ F_2(y, z) \end{pmatrix} = \left[ \psi_{10}(y, z), \psi_{20}^+(y, z) \right]^{-1} \sigma_3 \Delta Q(y) \psi_{10}(y, z)
\]

\[
= \frac{1}{W(0, z)} \begin{pmatrix} (\psi_{20}^+(y, z))_2 & -(\psi_{20}^+(y, z))_1 \\ -((\psi_{10}(y, z))_2 & (\psi_{10}(y, z))_1 \end{pmatrix} \begin{pmatrix} \tilde{f}(y)(\psi_{10}^-(y, z))_2 \\ -f(y)(\psi_{10}^+(y, z))_1 \end{pmatrix},
\]

$(\psi_{20}^+(y, z))_j$ the $j$–th component of $\psi_{20}^+(y, z)$ and similar notation for the other Jost functions. So

\[
I'(y, z) = \tilde{f}(y)(\psi_{20}^+(y, z))_2(\psi_{10}^-(y, z))_2 + f(y)(\psi_{20}^+(y, z))_1(\psi_{10}^-(y, z))_1.
\]

Furthermore by Lemma 3.2 in particular (3.18), for a fixed $C$ and when $z$ is in a preassigned compact subset of $\mathbb{C}\setminus\{0\}$, we have

\[
\|\psi_{10}^-(y, z) - \psi_{10}^+(y, z)\|_{L^\infty(-\infty, M)} = \|\psi_{10}^-(y, z) - \psi_{10}^+(y, z)\|_{L^\infty(-M, M)} < C\varepsilon.
\]

This yields

\[
W(\varepsilon, z) = W(0, z) + i\varepsilon \int_{\mathbb{R}} I(y, \pm 1)dy + \tilde{F}_\pm(z, \varepsilon)
\]

\[
I(y, z) = \tilde{f}(y)(\psi_{20}^+(y, z))_2(\psi_{10}^-(y, z))_2 + f(y)(\psi_{20}^+(y, z))_1(\psi_{10}^-(y, z))_1
\]

\[
\tilde{F}_\pm(z, \varepsilon) = i\varepsilon \int_{\mathbb{R}} [I(y, z) - I(y, \pm 1)]dy + O(\varepsilon^2)
\]

where $\tilde{F}_\pm(z, \varepsilon)$ has the properties claimed in the statement for $F_\pm(z, \varepsilon)$.

We have

\[
\psi_{20}^+(y, \pm 1) = i\psi_{10}(y, \pm 1),
\]

\[
\psi_{10}^-(y, -1) = \frac{1}{1 + e^{-2y}} \begin{pmatrix} i + e^{-2y} \\ -i + e^{-2y} \end{pmatrix} \quad \text{and} \quad \psi_{10}^+(y, 1) = \frac{1}{1 + e^{-2y}} \begin{pmatrix} -i + e^{-2y} \\ -i - e^{-2y} \end{pmatrix}.
\]

(C.5) can be derived in an elementary fashion by first substituting $z_{j_0} = i$ and $\varphi_{j_0} = x$ in formula (6.36) for $x > 0$ and formula (6.36c) for $x < 0$. This yields the formula for the matrix $m(x, z)$ in (3.1). To obtain the Jost functions one then multiplies by $a(z) = \frac{z - 1}{z + 1}$ the $1$st (resp. $2$nd) column of $m(x, z)$ if $\text{Im}(z) > 0$ (resp. $\text{Im}(z) < 0$), uses formulas (3.9) and exploits $\zeta(\pm 1) = 0$ for the function in (3.8) getting (C.5) with simple computations. After other elementary computations we get the formulas for $C(\pm, \tilde{f})$ in (C.2).

\[\square\]

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