A gauge-invariant Wigner quasi-distribution function for charged particles in classical electromagnetic fields is derived in a rigorous way. Its relation to the axial gauge is discussed, as well as the relation between the kinetic and canonical momenta in the Wigner representation. Gauge-invariant quantum analogs of Hamilton-Jacobi and Boltzmann kinetic equations are formulated for arbitrary classical electromagnetic fields in terms of the ‘slashed’ derivatives and momenta, introduced for this purpose. The kinetic meaning of these slashed quantities is discussed. We introduce gauge-invariant conditional moments and use them to derive a kinetic momentum continuity equation. This equation provides us with a hydrodynamic representation for quantum transport processes and a definition of the ‘collision force’. The hydrodynamic equation is applied for the rotation part of the electron motion. The theory is illustrated by its application in three examples. These are: Wigner quasi-distribution function and equations for an electron in a magnetic field and harmonic potential; Wigner quasi-distribution function for a charged particle in periodic systems using the $kq$ representation; two Wigner quasi-distribution functions for heavy-mass polaron in an electric field.

Keywords: Gauge invariant Wigner quasi-distribution function, hydrodynamic representation, Quantum transport equations, Quantum Boltzmann equation, Quantum Hamilton-Jacobi equation, ‘Quantum collision force’.

I. INTRODUCTION

In 1932 Wigner introduced a possible method for describing a quantum system in terms of a phase space quasi-distribution function which is now known as the Wigner quasi-distribution function. It allows quantum mechanical expectation values to be written in the form of phase space integrals similarly to the corresponding expectation values in classical mechanics (see, e.g., Balazs and Jennings, 1984; Hillery et al., 1984). The Wigner quasi-distribution function is closely connected with the Weyl (1931) correspondence principle (see also Schwinger, 1960), which has been originally formulated for systems free of external electromagnetic fields.

The Wigner quasi-distribution function,

$$\rho_W(P_{\text{can}}, X) = \rho_W(\varepsilon_{\text{can}}, P_{\text{can}}, T, R),$$

is defined as an expectation value of the so-called Wigner operator, $\hat{\rho}_W(P_{\text{can}}, X)$ obtained by a coordinate transformation of a single particle density operator, $\hat{\rho}(x, x') = \hat{\rho}(t, r, t', r')$. This transformation Wigner (1932) was chosen in the form

$$\hat{\rho}_W(P_{\text{can}}, X) = \int d^4y e^{-(i/\hbar)P_{\text{can}}y} \hat{\rho}(y, X)$$

where $X = (x + x')/2$, $y = (x' - x)$ and the subscript $\text{can}$ indicates that $P$ in this expression is the variable conjugated to the coordinate $X$. Although the transformation ($\Box$) seems quite obvious, it has a highly significant drawback of producing not gauge-invariant quantities for charged particles in an electromagnetic field. The resulting conjugate variable $P_{\text{can}} = (\varepsilon_{\text{can}}, P_{\text{can}})$ does not necessarily have a meaning of the kinetic four momentum.

Here lies an important difference between the classical and quantum descriptions. In classical mechanics we deal with the well-defined coordinate and momentum of the particle. The electromagnetic field is introduced in the Hamilton function of the system simply by considering the canonical momentum

$$P_{\text{can}} = P_{\text{kin}} + eA(x)$$
instead of the kinetic momentum. Here \( A^\mu(x) = (\varphi(x), A(x)) \) is the four vector-potential of the applied electromagnetic field. While it is well known that no problems with defining gauge-invariant quantities arise in classical mechanics, the uncertainty principle does not allow one to define simultaneously a coordinate and a momentum of a quantum particle. Quantum mechanics must, therefore, deal with a two-point density operator 
\[
\hat{\rho}(x,x') = \hat{\rho}(t,r,t',r')
\]
Then a gauge transformation converts the potential \( A(x) \) differently in these two different points, making an introduction of the canonical momentum not so harmless and even ambiguous procedure. The result is that equation (1) is not the only, nor is it the best definition of the Wigner quasi-distribution function for a system in an electromagnetic field.

Nonetheless, the transformation (1) has been used many times in the past for systems in electromagnetic fields. This may lead to certain difficulties\(^1\) in calculations and in interpretation of the results (see, e.g., Mahan, 1987; Mahan, 1990; Rammer, 1986, and the discussion in Levanda and Fleurov, 1994) since terms explicitly depending on the potential may appear, e.g., in kinetic equations.

Fleurov and Kozlov (1978) (see also Al‘tshuler, 1978) applied a transformation of the form
\[
\hat{\rho}_W(P,X) = \int d^4y e^{-i\bar{\hbar}^\mu(y)(P^\mu - e\bar{\hbar}A^\mu(X))} \hat{\rho}(y,X),
\]
which produces explicitly gauge-invariant quantities and equations in the linear approximation or in a nonlinear case of a static homogeneous electric field (Tugushev and Fleurov, 1983; Serimaa et al., 1986).

The first who tried formulating a gauge-invariant Wigner quasi-distribution function for an electron in classical electromagnetic fields was Irving (1965) who proposed to employ the transformation
\[
\hat{\rho}_W(P,X) = \int d^4y \exp \left( -\frac{i}{\hbar}Py + i\frac{e}{c\hbar} \int_{-y/2}^{y/2} y^\mu A^\mu(X + sy) ds \right) \hat{\rho}(y,X)
\]
which is truly gauge-invariant. It is worthwhile to pay a particular attention to the phase factor in (3) emerging from the proper time method of Schwinger (1951). This phase factor is related to the dynamics of the charged particle in the electromagnetic field. Really, the phase factor of an electron wavefunction which transforms as
\[
\psi(x) \rightarrow \psi(x)e^{(ie/c\hbar)A(x)}
\]
under the local gauge transformation
\[
A^\mu(x) \rightarrow A^\mu(x) - \partial^\mu A(x)
\]
with an arbitrary real function \( A(x) \), compensate for the change in the phase factor in equation (3), and the Wigner operator \( \hat{\rho}_W(P,X) \) remains unchanged. This ensures the gauge invariance of the definition (3).

It is, however, quite clear that the choice of the integration path is still ambiguous and is not anyhow restricted by the condition of the gauge invariance. Another path of integration differing from the straight line would mean a differing definition of the variable \( P \). This is the phase factor
\[
\exp \left( \frac{ie}{c\hbar} \int_{X-y/2}^{X+y/2} A^\mu(z) dz^\mu \right)
\]
which would ensure the gauge invariance of the definition (3) for any integration path (Mandelstam, 1960). Vasak et al. (1987) and Elze et al. (1986) proposed removing the above ambiguity by choosing the straight integration path.

\(^1\)Correct calculations using noninvariant quantities lead, nevertheless, to correct gauge invariant physical quantities (expectation values). The above mentioned references and section IIIA contain examples of misinterpretations, approximations and assumptions that may result from attempts to provide physical reasoning to non-gauge-invariant equations, expressions and other terms used during the calculations. The discussion of the meaning of the slashed derivatives presented in subsection IIIA implies that one should be cautious when considering the Poisson brackets for a non-gauge-invariant Boltzmann kinetic equation.
They showed that for this choice the transform \( \mathcal{H} \) becomes equivalent to the transform \( \mathcal{I} \) when substituting the covariant derivative \( \nabla^X + (e/c)A^X \) for \( \partial^X \).

In spite of all that has been done, the situation remains rather confusing. The phase factor is introduced in equation (3) only as a means of solving the problem of the gauge invariance. The arguments in favor of a specific phase factor are mainly intuitive. Hence, it is not quite clear which transformation is to be used if one prefers choosing other variables in the Wigner quasi-distribution function or if the dispersion relations for the charged particles differ from quadratic. It is our aim in this paper to derive systematically the transformation (3) and to illustrate the importance and physical significance of the phase factor by several examples.

Subsection II C formally defines the variables of the Weyl correspondence principle to be the kinetic momenta and the coordinates. We show that the form of the transformation (3) is due to the minimal coupling principle used to relate the canonical and kinetic momenta. The phase factor introduced in (3) is shown to be a gauge transformation that transforms an arbitrary gauge into an axial gauge. Working with this axial gauge, the vector potential vanishes along the integration trajectory. In addition the kinetic and canonical momenta coincide and the two transforms, (1) and (3), are identical. We shall dwell on the connection between the kinetic and canonical momentum in the Wigner representation as well as on its classical limit.

Quantum kinetic processes are in many cases treated by means of the Kubo linear response method Kubo (1957). An alternative approach, based on Keldysh (1965) diagrammatic technique, enables one to go rather easily beyond the linear response approximation. The first to formulate kinetic equations, based on the real time Green’s function technique, introduced by Martin and Schwinger (1959), were Kadanoff and Baym (1962). Fleurov and Kozlov (1978) using Keldysh formulation deduced two sets of linearized quantum kinetic equations which may in principle give an exhaustive description of the dynamic and kinetic properties of a system of interacting Fermi-particles. Since then a number of papers were published on quantum kinetic equations (Mahan, 1987; Rammer and Smith, 1986; Tugushev and Fleurov, 1983; Reizer and Sergeev, 1987; Khan et al, 1987; Reggiani et al, 1987; Davies and Wilkins, 1988; Bertoncini et al, 1989; Ferry and Grubin, 1995) and references therein. A gauge invariant formulation of the quantum kinetic equations for charged particles in an arbitrary classical electromagnetic field was derived in our paper (Levanda and Fleurov, 1994). Here, Section II briefly revisits this derivation and represents the resulting quantum Hamilton-Jacobi and Boltzmann equations in a more compact form by using the ‘slashed’ derivatives. These derivatives make the kinetic meaning of various terms in the equations more transparent.

A hydrodynamic representation of the quantum theory was introduced by Madelung (1927) who transformed the Schrödinger equation, for a free particle, into particle and momentum continuity equations in the form similar to those found in hydrodynamic theories. We use our quantum kinetic equations in the same manner to obtain a hydrodynamic representation for description of dynamics of interacting charged particles in arbitrary external electromagnetic fields. The term hydrodynamic in this context may be somewhat misleading, as the reader may infer that ‘hydrodynamic’ approximations are implied. This, however, is not the case. The equations to be obtained would not assume any sort of approximations. The full set of equations for conditional moments (hydrodynamic equations) is equivalent to the original Dyson equations and, hence, they may account for all quantum effects. It is emphasized that the hydrodynamic equations in principle allow us to find the state of the system in and out of equilibrium without further assumptions (neglecting interactions these equations are equivalent to a single particle Schrödinger equation). This is not the case with other transport equations (e.g., for conventional quasi-classical Boltzmann equation). We show that collisions lead to an introduction of a ‘collision force’ in the momentum continuity equation. The collision force has the structure proposed by Bohm (1952) in his hidden variable interpretation, to modify the Schrödinger equation.

Section V presents a few examples that emphasize different aspects of the Wigner quasi-distribution function and its equations. The Schrödinger equation and its solutions for an electron in harmonic potential and magnetic induction are found, in Subsection V A from the hydrodynamic equation. This reveals the nature of the ‘velocity’ in such a system. We place a particular emphasis on the growth of the ‘velocity’ with the coordinates. The Wigner quasi-distribution function for such a system is found from the quantum kinetic equations. After integrating over the kinetic energy variable and taking one of the fields to vanish we reduce our result to expressions found in the literature for the Wigner quasi-distribution function in a magnetic induction or in a harmonic potential. The difference between the gauge-invariant and non-gauge-invariant functions is discussed.

Subsection V B formulates the Wigner quasi-distribution function for systems with discrete translational symmetries both in time and/or space (an example is an electron in a crystal). The acceleration theorem for an electron in a crystal is discussed and its relation to the gauge is pointed out. The Green’s function of the heavy mass polaron in an electric field is found from the kinetic energy continuity equation. If the electric field vanishes, the Green’s function is the same as the one found in the literature. If an electric field is present we find two solutions, one corresponding to a polaron with constant kinetic energy and the other one corresponding to a polaron with a diverging canonical energy. This serves as an example for the kinetic meaning of the slashed derivatives.
II. WIGNER QUASI-DISTRIBUTION FUNCTION FOR A CHARGED PARTICLE IN
ELECTROMAGNETIC FIELDS

An expression for a Wigner quasi-distribution function for a charged particle in an external electromagnetic field is derived. The coordinate transformation to the Wigner coordinates (3) is deduced rigorously from the Weyl correspondence principle.

A. Derivation of the Wigner quasi-distribution function

Weyl’s correspondence principle suggests that an operator \( \hat{C} \), corresponding to a classical dynamical variable \( C(X, P_{\text{can}}) \) is obtained by replacing the variables \( P_{\text{can}} \) and \( X \) in the Fourier transform

\[
C(y, \varpi) = \frac{1}{(2\pi\hbar)^4} \int dP_{\text{can}} dX e^{-(i/\hbar)(yP_{\text{can}} + \varpi X)} C(P_{\text{can}}, X)
\]  

(7)

by the Hermitian operators, \( \hat{P}_{\text{can}} \) and \( \hat{X} \), representing these variables in quantum mechanics,

\[
\hat{C} = \frac{1}{(2\pi\hbar)^4} \int dyd\varpi e^{(i/\hbar)(y\hat{P}_{\text{can}} + \varpi \hat{X})} C(y, \varpi).
\]  

(8)

The expectation value of the operator (8) reads

\[
\langle \hat{C} \rangle = \int dX \left[ \rho(y', X) \hat{C} \right]_{y' \to 0}
\]  

(9)

where the angular brackets indicate an ensemble average. The second quantization allows one to present the density operator as

\[
\rho(y', X) = \Psi(\hat{X} + y'/2)\Psi(\hat{X} - y'/2).
\]  

(10)

by means of the creation \( \Psi(\hat{X}) \) and annihilation \( \Psi(\hat{X}) \) operators. Then its expectation value becomes

\[
\langle \hat{C} \rangle = \int dX' \left[ \langle \Psi(\hat{X}' + y'/2)\Psi(\hat{X}' - y'/2) \rangle \right]_{y' \to 0}
\]

\[
= \frac{1}{(2\pi\hbar)^8} \int dP_{\text{can}} dXC(P_{\text{can}}, X) \int dyd\varpi e^{-(i/\hbar)(yP_{\text{can}} + \varpi X)} \times
\]

\[
\int dX' \left[ \Psi(\hat{X}'e^{(i/\hbar)(y\hat{P}_{\text{can}} + \varpi \hat{X})}) \Psi(\hat{X}') \right]
\]  

(11)

If the variable \( P_{\text{can}} \) is chosen to be the canonical momentum, one can follow Moyal (1949) and write

\[
\langle \hat{C} \rangle = \frac{1}{(2\pi\hbar)^8} \int dP_{\text{can}} dXC(P_{\text{can}}, X) \int dyd\varpi e^{-(i/\hbar)(yP_{\text{can}} + \varpi X)} \times
\]

\[
\int dX' \left[ e^{iyP_{\text{can}}/2\hbar} e^{-i\varpi \hat{X}/\hbar} e^{iyP_{\text{can}}/2\hbar} \Psi(\hat{X}') \right]
\]

\[
= \frac{1}{(2\pi\hbar)^4} \int dP_{\text{can}} dXC(P_{\text{can}}, X) \rho_W(P_{\text{can}}, X).
\]  

(12)

where

\[
\rho_W(P_{\text{can}}, X) =
\]

\[
\langle \hat{\rho}_W(P_{\text{can}}, X) \rangle = \int dy e^{-(i/\hbar)yP_{\text{can}}} \left[ \Psi(\hat{X} + y/2)\Psi(\hat{X} - y/2) \right]
\]  

(13)

is the Wigner quasi-distribution function. Properties and usage of it were discussed in the introduction.

Equation (13) defines the Wigner quasi-distribution function, \( \rho_W(P_{\text{can}}, X) \) which has, however, a serious drawback. Although, the physical quantity \( \langle \hat{C} \rangle \) is certainly gauge-invariant, the function \( \rho_W(P_{\text{can}}, X) \), taken separately, depends on the gauge of the electromagnetic field. However, we are interested in a gauge invariant definition of the Wigner quasi-distribution function which can be derived by doing some modifications in the current derivation.
B. The need for a gauge-invariant Wigner quasi-distribution function

A question which we find necessary to address is why a gauge-invariant Wigner quasi-distribution function is really needed? In the papers referred to in the Introduction a gauge invariant formulation was proposed only as a convenient representation, meant either to help the intuition or to directly correspond to a classical distribution function, with the coordinates and the kinetic momenta as variables in the classical limit of the Wigner quasi-distribution function. We want to emphasize that this is not the whole story, a non-gauge-invariant Wigner quasi-distribution function does not always allow one to calculate expectation values of kinetic quantities. In particular we shall see that it is impossible to calculate the expectation value of the kinetic energy of a system in an electromagnetic field from a non-gauge-invariant function as phase-space integrals.

For the Weyl correspondence principal to apply, the operator \( \hat{C} \) must be a function of a linear combination of the operators \( X \) and \( P_{\text{can}} \) (Moyal, 1949). Although, \( X \) and \( P_{\text{can}} \) can be any complementary operators, in the following, we shall only consider the case where the two operators are the coordinate operator and the momentum operator (kinetic or canonical). The need for a linear combination of the operators can be seen by substituting \( C(aX + bP_{\text{can}}) \) for \( C(P_{\text{can}}, X) \), with \( a, b \) being constants, in equation (13), changing the variables to \( \xi = aX + bP_{\text{can}}, \eta = aX - bP_{\text{can}}, \gamma = \pi/2a + y/2b, \mu = \pi/2a - y/2b \).

\[
\hat{C} = \int d\mu d\gamma \exp \left[ \frac{i}{\hbar} \left( (\gamma - \eta)b\hat{P}_{\text{can}} + (\gamma + \eta)a\hat{X} \right) \right] \times \\
\int d\xi d\eta \exp \left[ -\frac{i}{\hbar} (\gamma \xi + \mu \eta) \right] C(\xi).
\]

Integrating over the new variables, \( \eta, \mu, \gamma \) and then over \( \xi \), one finds that \( \hat{C} = C(a\hat{X} + b\hat{P}_{\text{can}}) \). The correspondence, \( C = C(X, P_{\text{can}}) \) goes into \( \hat{C} = C(\hat{X}, \hat{P}_{\text{can}}) \), which exists only for linear combinations of the operators.

We conclude that the operators \( \hat{X} \) and \( \hat{P}_{\text{can}} \) cannot be chosen arbitrarily and the Wigner quasi-distribution function should be set up for two non-commuting operators chosen according to the measured quantity \( \langle \hat{C} \rangle \). In general a kinetic operator, e.g., kinetic energy \( P^2/2m \), not always can be written as a linear combination of canonical momenta and coordinates, hence, its expectation value not necessarily can be calculated with a non-gauge-invariant Wigner quasi-distribution function.

There are two cases in which equation (13) can be employed. The first is that of a system in homogeneous and static electric and magnetic fields. Then the vector potential is a linear function of the coordinates. Hence, the kinetic momentum is a linear combination of the canonical momentum and the coordinate. The second case is when one is interested in the expectation value of the current (it is proportional to the momentum). This can be expressed as the sum of the expectation values of the canonical momentum and of the vector potential, with each of these expectation values being calculated separately.

The gauge-invariant Wigner quasi-distribution function will be formulated in terms of the kinetic momentum and will provide us a possibility of calculating expectation values of kinetic operators.

C. Derivation of a gauge-invariant Wigner quasi-distribution function

The Cartesian coordinates and the canonical momenta are not the only possible basis operators (Scully and Cohen, 1987; Pimpale and Razavy, 1988). We may choose \( P \) to be the kinetic momentum then, using the equality (A.3) derived in the Appendix A, equation (11) becomes

\[
\langle \hat{C} \rangle = \frac{1}{(2\pi\hbar)^4} \int dPdXC(P, X) \frac{1}{(2\pi\hbar)^4} \int dyd\varphi \exp\left(\frac{i}{\hbar}(yP + \varphi X)\right) \times \\
\int dX' \left\langle \Psi^\dagger(X') e^{y\varphi^r} \exp\left(\frac{ie}{\hbar} \int_0^1 y^\mu A_\mu(\bar{X} + s\bar{y})ds\right) \right\rangle \Psi(X') \\
= \frac{1}{(2\pi\hbar)^4} \int dPdXC(P, X) \rho_W(P, X).
\]

where

\[
\rho_W(P, X) = \int dy e^{-\frac{i}{\hbar}yP} \times
\]
$$\exp \left( i e \int_{-1/2}^{1/2} y^\mu A_\mu (X + sy) ds \right) \left\langle \Psi^\dagger (X + y/2) \Psi (X - y/2) \right\rangle$$

Here, as opposed to equation (13), the resulting Wigner quasi-distribution function, \( \rho_W (P,X) \), incorporates a phase factor and coincides with the definition (3) postulated by Irving (1965).

It is appropriate to discuss here the connection of the straight integration path with the axial gauge (see, e.g., Rohrlich, 1965). A vector potential \( A_\lambda (x) \) in any gauge can be transformed to the axial gauge by means of the transformation

\[
A_\lambda (x) \rightarrow A'_\lambda (x; x'') = A_\lambda (x) + \partial^\lambda \left( (x'' - x)^\mu \int_{-1/2}^{1/2} A_\mu (x'' + x) \right) ds \]

\[
(x'' - x)^\mu \int_{-1/2}^{1/2} ds \left( \frac{1}{2} - s \right) F_{\lambda\mu} (x'' + x) = (x'' - x)^\mu \left( \frac{1}{6} - \frac{1}{6} (x'' - x)^\nu \right) \cdots F_{\lambda\nu} (x'').
\]

The resulting vector potential depends only on the electromagnetic tensor

\[
F_{\lambda\mu} (X) = \partial_\lambda A_\mu (X) - \partial_\mu A_\lambda (X)
\]

and on the coordinate \( x'' \) which can be chosen arbitrarily. This form of the vector potential has two important properties. In regions free of currents and charges, \( j_\lambda = (c \rho, j) \) generating the electromagnetic field,

\[
\partial^\lambda A'_\lambda (x) = (x'' - x)^\lambda \int_{-1/2}^{1/2} ds \left( \frac{1}{2} - s \right)^2 j_\lambda (x'') = 0
\]

and

\[
(x'' - x)^\lambda A'_\lambda (x; x'') = 0.
\]

Hence, this transformation converts any vector potential into a transverse vector field.

Now substituting this axial vector potential in the definition (16) and taking \( x'' - x = a (x' - x) = ay/2 \) with \( a \) being a constant, one sees that the integral in the phase factor becomes zero making the phase factor equal one. Therefore, the gauge-invariant Wigner quasi-distribution function coincides for this specific case with the gauge non-invariant definition (13).

All that has been done thus far can be simply restated as follows. The choice of the axial gauge with \( x'' = ay + x \), ensures that the vector potential indeed vanishes (see equation (13)) along the line that goes through \( x'' \) and \( x \). The displacement operator along this line, can equally be written as \( \exp (-i P_\text{can}'' (x'' - x)_\mu) \) or as \( \exp (-i P_\text{kin}'' (x'' - x)_\mu) \) and the Wigner quasi-distribution function of the canonical momentum variable coincides with the Wigner quasi-distribution function of the kinetic momentum variable. The function of the phase factor is to ‘transform’ an arbitrary gauge chosen for the vector potential into the axial gauge.

Two remarks are in order. This derivation of the gauge-invariant Wigner quasi-distribution function, can be applied to other correspondence principles as well, which in turn will lead to other quasi-distribution functions. As was shown in references (McCoy, 1932; Cohen, 1966; Summerfield and Zweifel, 1969; Cohen, 1966, 1976, 1987), these quasi-distribution functions can be obtained from the Wigner quasi-distribution function by simple transformations.

The equality derived in the Appendix A is applicable whenever the operators obey the relation
\[ \hat{P}_{\text{can}} = \hat{P}_W + A(\hat{X}), \]  

(20)

where \( \hat{P}_W \) is the desired variable in the Wigner representation. That is, as long as one chooses a variable that obey (20), the transformation (14) should be used.

In the language of classical mechanics it can be stated that the relation (20) holds if the coupling to the electromagnetic field is via the term \( A(X)\dot{X} \) in the Lagrangian \( L \). Then the canonical momentum is of the form

\[ P_{\text{can}} = \frac{\partial L(X, \dot{X}, A)}{\partial \dot{X}} = \frac{\partial L(X, \dot{X}, A)}{\partial \dot{X} \bigg|_{A=0}} + A(X) \overset{\text{def}}{=} P_W + A(X). \]  

(21)

Whenever an effective Hamiltonian is used the relation (20) is assumed to hold (see, e.g., Nenciu, 1991, and references therein). This however, is not necessarily the case. The particle may have a different dispersion relation due to its coupling to a set of degrees of freedom (e.g., polaron). Then if the Hamiltonian for such a composed quasi-particle is written, the minimal coupling (21) may be not necessarily applicable.

D. Kinetic and canonical momenta in the Wigner representation

The phase factor in the Wigner quasi-distribution function, equation (14), can be written as a Taylor series and integrated over \( s \) so that equation (16) takes the form

\[ \rho_W(P, X) = \int dy \exp \left( \frac{-i}{\hbar} y^{\mu} \left[ P_{\mu} - \frac{e}{c} j_0(y^{\nu} \partial_{\nu}^X/2) A_{\mu}(X) \right] \right) \times \]

\[ \langle \Psi(\dot{X} + y/2) \Psi(\dot{X} - y/2) \rangle \]

\[ = \exp \left( \frac{e}{c} j_0 \left( \frac{\hbar}{2} \partial_{\nu} P_{\nu} \partial_{\nu}^X \right) A_{\mu}(X) \partial^\mu \right) \times \]

\[ \int dy \exp \left( \frac{-i}{\hbar} y^{\mu} P_{\mu} \right) \langle \Psi(\dot{X} + y/2) \Psi(\dot{X} - y/2) \rangle \]

\[ = \exp \left( \frac{e}{c} \mathbf{A}_{\mu}(X) \partial^\mu \right) \rho_{\text{can}}(P, X) \]  

(22)

where \( \mathbf{A}_{\mu}(X) = j_0(\Delta/2) A_{\mu}(X), \Delta = \hbar \partial_{\nu} \partial_{\nu}^X, j_0(x) = \sin x/x \approx 1 - x^2/3 + \cdots \), and the derivatives with respect to \( X \) in the exponent act only on the vector potential.

This is a general expression valid for an arbitrary electromagnetic field. We define \( \tilde{\rho}_W(P, X) \) and \( \tilde{P} \) to be the Wigner quasi-distribution function and its momentum as derived from equation (22) with the modified phase factor \( \exp \left( \frac{e}{c} \left[ \mathbf{A}_{\mu}(X) - A_{\mu}(X) \right] \partial^\mu \right) \). The equality

\[ \rho_W(P, X) = \tilde{\rho}_W \left( \tilde{P} - \frac{e}{c} A(X), X \right) \]  

(23)

then holds and one may say that the two momenta are related by the equation

\[ P_{\text{kin}}(X) = \tilde{P}(X) - \frac{e}{c} A(X). \]  

(24)

If all the derivatives of the electromagnetic field of order higher than one vanish, meaning only static and homogeneous fields are applied, the relation (24) is simplified and \( \tilde{P} \) becomes the usual canonical momentum. The kinetic momentum and the canonical momentum as defined by the Wigner quasi-distribution functions in this section are related by the equality,

\[ P_{\text{can}}|_{X=x''} = P_{\text{kin}}(X) = P_{\text{can}}(X; x'') - (e/c)A(X; x'') \]  

(25)

where \( A(X; x'') \) is in the axial gauge, (17).

The differential version of equation (25) is,

\[ \partial^X_{\mu} P_{\text{kin}}(X) = \partial^X_{\mu} P_{\text{can}}(X; x'') + \partial^{x''}_{\mu} P_{\text{can}}(X; x''). \]  

(26)
Equation (26) shows us that the variation of the canonical momentum is composed of two parts: variation of the kinetic momentum due to kinetic effects and variation of the kinetic momentum due to the change of the gauge as the reference point \(x''\) moves. In particular we may use equation (17) and write (for systems in static and homogeneous field)

\[
\frac{d}{dt}P_{\text{can}}(X;x'') = \frac{e}{2}E + \frac{e}{2c} \frac{dx''}{dt} \times H = \frac{1}{2}f_{\text{Lor}}(x''),
\]

where \(f_{\text{Lor}}(x'')\) is the Lorentz force acting at the point \(x''\).

Let us imagine, a quantum system confined to a small part of the three dimensional space such that it can be described as being at rest (equivalently we may think of a heavy mass particle). Another classical particle with a well-defined trajectory is chosen to define the location of the point \(x''\). The gauge part of the canonical momentum of the quantum system (as measured in the axial gauge by \(x''\) at the classical particle) will change in time according to the equation of motion of the classical particle (26). While all that time, the kinetic part of the kinetic momentum of the quantum system will not change since it is at rest. We shall return to this point twice, when discussing the heavy mass polaron and when discussing the ‘acceleration theorem’.

Now, suppose that the classical particle is another quantum system that interacts with the first system and exchanges momenta with it. Momentum conservation applies only to the canonical momentum. If the conservation laws are to be expressed in terms of kinetic momenta, the conservation laws should take into account the kinetic as well as the gauge part of the kinetic momenta. More on this point is discussed in Subsection IV B.

E. The classical limit of the Wigner quasi-distribution function.

The classical limit of the Wigner quasi-distribution function is discussed here. This limit has been already addressed by other authors (Heller, 1976, 1977; Berry, 1977) and here we give only the modifications due to the gauge invariance and appearance of an energy variable. A different approach was taken by Bund (1995).

Following Van Vleck (1928) we express the quasi-classical limit of a wave function for a state \(\alpha\) as

\[
\psi_{\alpha}(X) = N^{1/2}D^{1/2}e^{iS/\hbar}.
\]

_N_ is the normalization constant, \(S(X, \alpha) = S(R, t, \alpha)\) is the classical generator of the canonical transformation from the \(P, R\) system to a set of new variables \(\alpha, \beta\) and \(D = \det |\partial^2 S/\partial R \partial \alpha|\) is the functional determinant. The Wigner quasi-distribution function reads

\[
\rho_{\text{W}}(P, X|\alpha) = \int d^4 y \sqrt{\det |\partial^2 S(X + y/2, \alpha)/\partial R \partial \alpha| \det |\partial^2 S(X - y/2, \alpha)/\partial R \partial \alpha|} \times \exp \left[ -\frac{i}{\hbar} [(P - A(X))\mu y_{\mu} + S(X - y/2, \alpha) - S(X + y/2, \alpha)] \right].
\]

We neglected terms, non-singular in \(\hbar\), in the phase factor, since they vanish as \(\hbar \to 0\).

We conclude that the relation (23) is expected to hold in the quasi-classical limit as well as for quantum systems in static and homogeneous fields. In the classical limit the frequency and the wave length of the electron wave function are much smaller than any other time and space variations in the system, that is, the electromagnetic field is taken as static and homogeneous. This relation between the canonical and kinetic momenta are always valid in classical mechanics as well as for quantum mechanical operators. However, in the Wigner representation of the quantum mechanics this relation generally does not necessarily hold.

III. DERIVATION OF KINETIC EQUATIONS.

The diagrammatic technique proposed by Keldysh (1965) allows one to write the Dyson equation for a nonequilibrium quantum system in an arbitrary not quantized electromagnetic field. A semiclassical Boltzmann equation as well as its quantum versions may be obtained from this Dyson equation by making certain transformations and
approximations. It is our aim to present below quantum equations avoiding approximations as much as possible. In this sense, both the equations to be obtained and the Dyson equations are equally valid.

Following the five step procedure presented in our paper (Levanda and Fleurov, 1994) we arrive at equations which are quantum analogs of the Hamilton-Jacobi and the Boltzmann kinetic equations

\[
\frac{2}{\hbar} \left[ \hat{\mathbf{P}}^2 + \frac{\hbar^2}{2m} (\hat{\nabla}^X)^2 \right] \hat{\mathbf{G}}(P, X) = 2\sigma_x
\]

\[
+ \exp \left[ \frac{i\hbar}{2} \left( \frac{\partial}{\partial \varepsilon} \hat{\mathbf{P}}^T - \frac{\partial}{\partial \varepsilon} \hat{\mathbf{P}} - \nabla^x \frac{\partial}{\partial \mathbf{P}} - \frac{\partial}{\partial \mathbf{P}} \nabla^x \right) \right] 
\left( \sigma_x \hat{\Sigma}(P, X) \hat{\mathbf{G}}(P, X) + \hat{\mathbf{G}}(P, X) \hat{\Sigma}(P, X) \sigma_x \right)
\]

(30)

and

\[
i \left[ \hat{\mathbf{P}}^T + \frac{1}{2m} (\hat{\mathbf{P}} \cdot \nabla^X + \nabla^X \cdot \hat{\mathbf{P}}) \right] \hat{\mathbf{G}}(P, X) = \\
\exp \left[ \frac{i\hbar}{2} \left( \frac{\partial}{\partial \varepsilon} \hat{\mathbf{P}}^T - \frac{\partial}{\partial \varepsilon} \hat{\mathbf{P}} - \nabla^x \frac{\partial}{\partial \mathbf{P}} - \frac{\partial}{\partial \mathbf{P}} \nabla^x \right) \right] 
\left( \sigma_x \hat{\Sigma}(P, X) \hat{\mathbf{G}}(P, X) - \hat{\mathbf{G}}(P, X) \hat{\Sigma}(P, X) \sigma_x \right).
\]

(31)

where the following definitions are used

\[
\hat{\mathbf{P}} = \mathbf{P} + \frac{\hbar}{2} e j_1 \left( \frac{\Delta}{2} \right) \mathbf{E}(X) \cdot \nabla^p;
\]

\[
\hat{\mathbf{P}}^T = \frac{\partial}{\partial \mathbf{P}} + e j_0 \left( \frac{\Delta}{2} \right) \mathbf{E}(X) \cdot \nabla^P;
\]

\[
\hat{\nabla}^X = \nabla^X + e j_0 \left( \frac{\Delta}{2} \right) \mathbf{E}(X) \cdot \nabla^p;
\]

\[
\hat{\mathbf{G}} = \begin{pmatrix} 0 & G^a \\ G^r & G^\mathbf{K} \end{pmatrix}, \quad \hat{\Sigma} = \begin{pmatrix} \Sigma^K & \Sigma^r \\ \Sigma^r & 0 \end{pmatrix};
\]

(32)

\[
\hat{j}_0(x) = \sin(x)/x \quad \text{and} \quad \hat{j}_1(x) = \sin(x)/x^2 - \cos(x)/x.
\]

Here the derivative with respect to \( X \) in the operator \( \Delta = \hbar \partial^P \partial^X \) acts only on the electromagnetic fields \( \mathbf{E} \) and \( \mathbf{B} \). The other derivatives, in the left hand sides, with respect to \( X \) and \( P \) act on all the functions. In the right hand side, the first and the second derivatives of in the exponent act on the first or on the second terms of the pair in the sum, respectively.

This is a generalization of the so-called gradient expansion first used by Kadanoff and Baym (1962) for system without electromagnetic fields, by Tugushev and Fleurov (1983) for system in homogeneous and static electric field and by Mahan (1987) for homogeneous and static electric and magnetic fields. The form (30), (31) is more general and valid for arbitrary electromagnetic fields. An alternative representation of the gradient expansion was given by these authors (Levanda and Fleurov, 1994) in terms of the phase loop function (see also Appendix B).

These equations are equivalent to the Dyson equations for the Keldysh functions from which we have started. In fact, they are just alternative representations of the Dyson equations. Therefore they may provide an accurate and complete physical description for kinetic processes of a many particle system in an arbitrary electromagnetic field. We wrote two matrix equations which would correspond to six equations for complex functions. However, the number of really independent functions and, hence independent equations is only two (see, e.g., discussion in Fleurov and Kozlov,
1978). One can, for example, take an off-diagonal term in the matrix equation (30) in the representation (32) which would be an equation for, say, retarded Green’s function \( G^r \) determining the spectrum of the system. The second equation can be produced from one of the diagonal terms of the matrix equation (31) (in the same representation) and this is the equation for the function \( G^K \) which in the semi-classical approximation reduces to the conventional Boltzmann equation.

Both equations can be expanded in powers of the derivatives of the electromagnetic fields (\( \Delta \) expansion), without effecting their gauge invariance. This, however, does not correspond to the expansion in powers of \( \hbar \) of equation (30), (31).

The Liouville theorem in classical mechanics can be derived as a limit of equation (31). To see it we neglect the interaction term (the right hand side) and integrate with respect to the variable \( \varepsilon \) (the classical phase-space does not contain the energy variable). Equation (31) becomes

\[
\left[ \frac{\partial}{\partial T} + e j_0 \left( \frac{\Delta}{2} \right) \left[ E(X) + \frac{1}{m} P \times \frac{1}{c} B(X) \right] \cdot \nabla P + P \cdot \nabla X \right] G(P, X) = 0. \quad (33)
\]

If all the derivatives of the fields of order higher than one, vanish, equation (33) becomes classical Liouville equation. This should be compared with:

1. The conditions for the coincidence of the Ehrenfest equations of motion for a wave packet in an electric field (see, e.g., Messiah, 1962, p.220) with the classical equation of motion (if the electric field has at most a linear coordinate dependence).

2. The conditions for a quantum system to be deterministic. Moyal (1949) showed that if the Hamiltonian of a quantum system is a second-degree polynomial in \( X \) and \( P \) (a uniformly accelerated particle or a harmonic oscillator), then the quantum system is deterministic in the sense that the time evolution of the Wigner quasi-distribution function is given by the classical Liouville theorem. That is, the quantum nature of the system enters only via the initial conditions (Bartlett and Moyal, 1949; Lee and Scully, 1983).

**A. Kinetic meaning of the slashed terms**

We shall now demonstrate the relation of the slashed energy, momentum and derivatives to the operators of the kinetic energy and kinetic momentum. This will clarify the physical significance of these quantities and their difference from the non-slashed quantities. We also give two examples of the Wigner quasi-distribution function that demonstrate the difference between the two derivatives.

The slashed time derivative is defined as

\[
i \hbar \frac{\partial}{\partial t} G^<(P, X) =
\]

\[
i \hbar \frac{\partial}{\partial t} \int d^4y \exp \left( \frac{i}{\hbar} P y + \frac{ie}{c} \int_{-1/2}^{1/2} y'^\mu A_\mu(X + sy) ds \right) G^<(y, X)
\]

\[
= \int d^4y \exp \left( \frac{i}{\hbar} P y + \frac{ie}{c} \int_{-1/2}^{1/2} y'^\mu A_\mu(X + sy) ds \right) \cdot 
\]

\[
\left[ i \hbar \frac{\partial}{\partial T} + \frac{e}{c} A_0(X - y/2) - \frac{e}{c} A_0(X + y/2) \right] G^<(y, X)
\]

\[
= \int d^4y \exp \left( \frac{i}{\hbar} P y + \frac{ie}{c} \int_{-1/2}^{1/2} y'^\mu A_\mu(X + sy) ds \right) \cdot 
\]

\[
\left[ \left( i \hbar \frac{\partial}{\partial t} + \frac{e}{c} A_0(x) \right) - \left( -i \hbar \frac{\partial}{\partial t'} + \frac{e}{c} A_0(x') \right) \right] i(\psi^+(x')\psi(x)) \quad (34)
\]

This definition has been written for the \( G^<(x, x') = i\rho(x, x') \) component. The expressions for the other components
are similar. The definition of the slashed spatial derivative is

$$-i\hbar \nabla^X G^<(P, X) = \int d^4y \exp \left( -\frac{i}{\hbar} Py + \frac{ie}{c\hbar} \int \frac{1/2}{-1/2} y^\mu A_\mu (X + sy) ds \right).$$

$$\left[ \left( -i\hbar \nabla^x - \frac{e}{c} A(x) \right) - \left( i\hbar \nabla^{x'} - \frac{e}{c} A(x') \right) \right] i\langle \psi^+(x') \psi(x) \rangle$$

In a similar way one can introduce definitions for the slashed energy and momentum

$$2i\mathcal{P}^<(P, X) = \int d^4y \exp \left( -\frac{i}{\hbar} Py + \frac{ie}{c\hbar} \int \frac{1/2}{-1/2} y^\mu A_\mu (X + sy) ds \right).$$

$$\left[ \left( -i\hbar \nabla^x - \frac{e}{c} A(x) \right) + \left( i\hbar \nabla^{x'} - \frac{e}{c} A(x') \right) \right] i\langle \psi^+(x') \psi(x) \rangle$$

The operators in the square brackets in (36) and in (37) are the kinetic energy and kinetic momentum operators. The kinetic momentum operator that acts on the wave function of the system is gauge invariant and produces the values of the electron’s mass times a quantity which has a meaning of an average velocity.

We shall now show how the uniformity of the particle and the current density make \(\nabla^X \hat{G}(P, X)\), but not \(\nabla^X G(P, X)\), vanish. The system is not assumed to be homogeneous, e.g., it may be placed in a non-uniform magnetic induction or in a time dependent electric field. That is, we assume only that the density and the current do not depend on the coordinate:

$$\psi^*(x)\psi(x) = \rho(x) = \rho$$

and

$$\frac{e}{2m} \left[ \left( -i\hbar \nabla^x - \frac{e}{c} A(x) \right) + \left( i\hbar \nabla^{x'} - \frac{e}{c} A(x') \right) \right] i\psi^*(x')\psi(x) \bigg|_{x' \to x} = J(x) = j$$

and show that \(\nabla^X \hat{G}(P, X) = 0\).

A single particle ‘wave function’ of the system is written in the form \(\psi(x) = R(x) \exp[iS(X)/\hbar]\) and with the two aforementioned conditions for homogeneity, one finds, \(R(x) = \sqrt{\rho}\) and \(\nabla^x S(x) = m\jmath/e + A(x)\). Putting this form of the wave function into equation (36) the derivative \(\nabla^X \hat{G}(P, X)\) is readily seen to vanish. The derivative \(\nabla^X \hat{G}(P, X)\) vanishes as well, only if the electromagnetic fields vanish. Substituting the same wave function in equation (37) we find that \(\mathcal{P}^<(P, X) = m\jmath/eG^<(P, X)\) and \(\mathcal{P}^<(P, X) \neq m\jmath/eG^<(P, X)\) if the electromagnetic fields have non-vanishing gradients.

The Wigner quasi-distribution function for a charged particle in a static and homogeneous electric field in a system without interactions obey the equations

$$\left[ \varepsilon - \frac{P^2}{2m} + \frac{\hbar^2}{8m} \left( \nabla^X + eE\partial^x \right)^2 \right] \rho_W(P, X) = 0 \quad (38)$$

$$\left[ \frac{\partial}{\partial T} + eE\nabla^P + \frac{P}{m} \left( \nabla^X + eE\partial^x \right) \right] \rho_W(P, X) = 0 \quad (39)$$
If the condition is assumed that the derivatives of Wigner quasi-distribution function with respect to the coordinates, $\nabla^X \rho(P, X) = 0$, vanish then the Wigner quasi-distribution function reads

$$\rho_W(P, X) = \frac{m}{(m \hbar eE)^{2/3}} \text{Ai} \left( \frac{2m}{(m \hbar eE)^{2/3}} \left( \varepsilon - \frac{P^2}{2m} \right) \right) \frac{\hbar e}{2m} \delta \left( \varepsilon - \frac{P^2}{2m} \right)$$

(40)

which is a time and coordinate independent solution, Ai is the Airy function. The solution (40) implies that the application of an electric field accelerates the particle and, hence, causes a broadening of the energy uncertainty. The broadening is of the order of $(\hbar E)^{2/3}$ and is only due to the acceleration, no interactions are assumed. The Wigner quasi-distribution function is a spectral function that yields an electron dispersion relation. It does not produce the dynamics of the system. The dependence on the energy and the momentum can be rewritten in the form

$$\varepsilon - \frac{P^2}{2m} = \varepsilon - eE \left( \frac{P}{mT} - \frac{eE}{2m} T^2 \right) - \frac{(P - eET)^2}{2m} = \varepsilon - eE R(T) - \frac{P^2(T)}{2m},$$

where $R(T)$ and $P(T)$ are the coordinate and the momentum of a classical particle that moves in the same electric field. This representation reveals the hidden acceleration of the particle. If a steady current flows in the system the broadening takes place. We conclude that the Wigner quasi-distribution function of a system without interactions of the electric field, it remains such afterwards; (ii) The slashed derivative vanishes.

Now turning to a system with interactions we may use at this stage the simplest mean free time approximation in the quantum kinetic equations. This approximation has been used for more than a century in classical and quasi-classical systems (see, e.g., Ziman, 1960). To introduce the mean free time we first consider a solution without any field $\rho(P, X)|_E = 0$. Then the Liouville equation (33) allows as to make the ‘canonical transformation’

$$\varepsilon \rightarrow \varepsilon - eE \cdot (R - R_0),$$

$$P \rightarrow P - eET,$$

$$R \rightarrow R_0 + \frac{P}{mT} + \frac{e}{2m}ET^2.$$

in order to arrive to a solution in an electric field. The transformed Wigner quasi-distribution function after the canonical transformation has the following properties: (i) If the system has been homogeneous prior to the application of the electric field, it remains such afterwards; (ii) The slashed derivative vanishes.

The next step is to assume that after the interaction converts the variable $T$ into a parameter having the meaning of the mean free time that needs to be determined from the quantum kinetic Boltzmann equation. The mean free time may be a function of the momentum and the energy, $\tau(\varepsilon, P)$. The derivative with respect to $T$ in the kinetic equations vanishes ($T$ is now a parameter) as well as the derivatives with respect to $X$ and $\varepsilon$ (due to the canonical transformation of $\varepsilon$). In the left hand side of the Boltzmann kinetic equation (39) we are left with only one term, $\varepsilon \cdot \nabla^P \rho_W(\varepsilon, P, R)$, just like in a quasi-classical Boltzmann equation (see, e.g., Ziman 1960). This term should be equal to the collision integral in the right hand side of the equation, the electric force is equal to the collision force. In the left hand side of the Hamilton-Jacobi kinetic equation (38) we are left with $(\varepsilon - P^2/2m)\rho_W(\varepsilon, P, R)$ that is equal to the interaction which causes a broadening in the dispersion relation and vanishes for classical systems.

Let us look at the dependence of the Wigner quasi-distribution function on the mean free time. The Wigner quasi-distribution function in equilibrium is usually written in the form $\rho_W(P, X)|_{E=0} = A(P, X) n_F(\varepsilon)$ where $n_F(\varepsilon)$ is the Fermi function and $A(P, X)$ is the spectral function. In the linear approximation the collision integral does not vanish only if nonequilibrium corrections to the Fermi function are considered, the spectral function may be taken at equilibrium (Fleurov and Kozlov, 1978). Hence, the linear correction to the Wigner quasi-distribution function is taken in the form

$$\rho_W(P, X) = A(P, X) \left[ n_F(\varepsilon - eE \cdot (R - R_0)) \right]$$

$$\approx A(P, X) \left[ n_F(\varepsilon) - \left( \frac{\partial n_F}{\partial \varepsilon} \right) eE \cdot (R - R_0) \right]$$

$$= \rho_W(P, X)|_{E=0} + A(P, X) \left( \frac{\partial n_F}{\partial \varepsilon} \right) \frac{eE \cdot P}{m} \tau(\varepsilon, P, R).$$

(41)

This form of the Wigner quasi-distribution function may be inserted into the quantum kinetic equation in order to calculate the mean free time $\tau(\varepsilon, P, R)$. 

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IV. KINETIC MOMENTUM CONTINUITY EQUATION

Quantum kinetic equations provide us a possibility to calculate Green’s functions. Green’s functions, however, are not directly measurable, and are only used at certain steps of calculations. Green’s functions enable one to calculate quasi-distribution functions, and macroscopic measurable quantities are calculated as their averages over the phase space. Although deduction of various equations for the macroscopic measurable quantities from the quantum kinetic equations is possible, we shall consider here only the kinetic momentum continuity equation. The particle continuity equation and the quantum mechanical equivalent of the classical Hamilton-Jacobi equation were discussed in Levanda and Fleurov (1994).

A. Space conditional moments

Macroscopic measurable quantities can be written in terms of gauge invariant space conditional moments, \( \overline{P}_{\mu}^n(X) \) which can be defined as the averages of various powers \( P_{\mu}^n \) of the momentum at a given point, \( X \),

\[
\overline{P}_{\mu}^n(X) = \frac{1}{\rho(X)} \frac{1}{(2\pi \hbar)^4} \int d^4 P (P_{\mu})^n \rho W(P,X) = \frac{1}{\rho(X)} \left( -i \hbar \frac{\partial}{\partial y} \right)^n \left[ \exp \left( \frac{ie}{\hbar c} \int_{-1/2}^{1/2} y^\mu A_\mu(X + sy) ds \right) \right] \rho(X,y) \quad \text{y} \rightarrow 0. (42)
\]

Here

\[
\rho(X) = \frac{1}{(2\pi \hbar)^4} \int d^4 P \rho W(P,X)
\]

is the particle density. The canonical (non gauge-invariant) space conditional moments are defined as

\[
\overline{P}_{\mu}^{can,n}(X) = \frac{1}{(2\pi \hbar)^4} \int d^4 P (P_{\mu})^n \rho W(P_{can},X) = \frac{1}{\rho(X)} \left( -i \hbar \frac{\partial}{\partial y} \right)^n \rho(X,y) \quad (43)
\]

The relation between the canonical space conditional moments and gauge-invariant space conditional moments is trivial if equation (23) holds. In the general case one can also see that the relation

\[
\overline{P}(X;x'') = \overline{P}_{can}(X) - (e/c) A(X;x'') \quad (44)
\]

holds regardless of the nature of the electromagnetic field (as well as equation (24)).

This can be readily understood, since \( \overline{P}_{can}(X) \) is merely the expectation value of the current operator times the electron mass divided by the density,

\[
\overline{P}_{can}(X) = \frac{m \overline{J}_{can}(X)}{e \rho(X)} = \left. \frac{-i \hbar}{2} \left( \partial^{X'X} - \partial^{X'} \right) \left( \Psi^{\dagger}(X') \Psi(X) \right) \right|_{X' \rightarrow X} \quad (45)
\]

In a similar manner we have an expression for the gauge-invariant terms

\[
\overline{P}(X) = \frac{m \overline{J}(X)}{e \rho(X)} = \left. \frac{1}{2} \left( -i \hbar \partial^{X'} - \frac{e}{c} A(X') + i \hbar \partial^{X} + \frac{e}{c} A(X) \right) \left( \Psi^{\dagger}(X') \Psi(X) \right) \right|_{X' \rightarrow X} \quad (46)
\]
Relations between the canonical and gauge-invariant conditional moments are best seen with the help of the cummulant expansion generated by the function

\[
\log \left( \exp \left( \frac{ie}{\hbar} \int_{-1/2}^{1/2} y^\mu A_\mu(X + sy) ds \right) \frac{\rho(X, y)}{\rho(X)} \right)
= \frac{ie}{\hbar} \int_{-1/2}^{1/2} y^\mu A_\mu(X + sy) ds + \log \left( \frac{\rho(X, y)}{\rho(X)} \right).
\]

Then writing the density function in the form

\[
\rho(X, y) = \rho(X - y/2) \rho(X + y/2) \exp \left( \frac{-i}{\hbar} (S(X - y/2) - S(X + y/2)) \right)
\]

one obtains cummulants.

\[
\kappa_{2n+1} = \left( \frac{-i\hbar}{2} \right)^{2n+1} \left( \frac{\partial}{\partial X} \right)^{2n+1} S(X) + \left( -i\hbar \frac{\partial}{\partial y} \right)^{2n+1} \left[ \frac{ie}{\hbar} \int_{-1/2}^{1/2} y^\mu A_\mu(X + sy) ds \right] \quad y \to 0
\]

and

\[
\kappa_{2n} = \left( -i\hbar \right)^{2n} \left( \frac{\partial}{\partial X} \right)^{2n} \log \rho(X) + \left( -i\hbar \frac{\partial}{\partial y} \right)^{2n} \left[ \frac{ie}{\hbar} \int_{-1/2}^{1/2} y^\mu A_\mu(X + sy) ds \right] \quad y \to 0
\]

\[
n = 0, 1, 2 \ldots \text{The odd cummulants contain canonical cummulants plus cummulants of the phase factor. The asymmetry of the Wigner quasi-distribution function is, hence, represented by these odd cummulants, therefore the asymmetry of the gauge-invariant and canonical momenta are different. The even gauge-invariant and canonical cummulants are the same and, in particular, the conditional mean square deviation of the momenta, } \kappa_2 = \overline{P^2} - \overline{P}^2, \text{ is the same in both cases and the uncertainty relation holds for both the canonical and the gauge-invariant momenta.}
\]

**B. Derivation of kinetic momentum continuity equations**

Here we discuss an equation which can be called a kinetic momentum continuity equation. It is obtained from the trace of equation (31), after multiplying it by \( \mathbf{P} \) and integrating over the energy and momentum,

\[
\frac{\partial}{\partial T} \left( \rho(X) \overline{\mathbf{P}}(X) \right) + \nabla^\lambda \left[ \frac{P^\lambda P(X)}{m} \rho(X) \right] =
\]

\[
- \epsilon \mathbf{E}(X) \rho(X) - (e/mc) \left( \rho(X) \overline{\mathbf{P}}(X) \right) \times \mathbf{B}(X) +
\]

\[
\frac{1}{(2\pi \hbar)^3} \int d\mathbf{P} \cdot \mathbf{P} \ I_{coll}(\mathbf{P}, X).
\]
The kinetic momentum continuity equation (47) contains a collision integral, multiplied by \( \mathbf{P} \) and integrated over the momentum space. It describes the transfer of electron kinetic momentum from and to a four dimensional volume element, at \( X \), caused by the collision processes.

Equation (47) can be rewritten in a more transparent form

\[
\frac{d\mathbf{P}(X)}{dT} = 
- e\mathbf{E}(X) - \frac{e}{c}\nabla(X) \times \mathbf{B}(X) - \frac{\hbar^2}{2m} \nabla^2 \sqrt{\rho(X)} \left\{ \sqrt{\rho(X)} \right\} + f_{\text{coll}}(X)
\]

(48)

where

\[
f_{\text{coll}}(X) = \frac{1}{\rho(X)} \frac{1}{(2\pi\hbar)^3} \int d\mathbf{P} \left( \mathbf{P} - m\mathbf{v}(X) \right) I_{\text{coll}}(\mathbf{P}, X),
\]

and the equality

\[
-\rho(X) \frac{\hbar^2}{2m} \nabla^2 \frac{\sqrt{\rho(X)}}{\sqrt{\rho(X)}} = \nabla^\lambda \sigma_{\lambda\nu} =
\]

\[
\nabla^\lambda \left[ \rho(X) \left( \overline{\mathbf{P}}(X) - \overline{\mathbf{F}}(X) \right) \right]
\]

(49)

is used. The left hand side of equation (49) is the gradient of the so-called ‘quantum potential’, whereas the term \( \sigma_{\lambda\nu} = (\hbar^2/4m\rho(X))\nabla^\lambda \nabla^\nu \ln(\rho(X)) \) is the stress tensor used in the hydrodynamic formulation of the Schrödinger equation (Madelung, 1927; Holland, 1993). The electron density continuity equation (equation (47) in Levanda and Fleurov, 1994) shows that the quantity \( e\rho(X)\overline{\mathbf{P}}(X)/m \) is the kinetic particle current density \( \overline{\mathbf{j}}_{\text{kin}}(X) \) at the point \( X \), hence, \( \nabla(X) \) is a quantity having a meaning of an average velocity of the particles. This averaging includes both thermodynamical and quantum averagings.

The collision term \( f_{\text{coll}}(X) \) appears due to possible processes of electron scattering. Bohm (1952) (his equation (31)) argued that hydrodynamic equation in the quantum theory should contain terms depending on the difference \( \mathbf{p} - m\nabla \) but did not present a physical mechanism that may lead to such a term. It is clear that the collision term \( f_{\text{coll}}(X) \) is a function just of this type. Its specific form is determined by the collision integral and can be now calculated explicitly for any particular system.

This continuity equation has been obtained from the gauge invariant quantum kinetic equations which depend explicitly only on electromagnetic fields and do not depend on their potentials. These equations are actually Dyson equations for the Keldysh Green’s functions in a differential form. In the absence of collision processes, they can be reduced to a Schrödinger equation and to a quantum generalization of the Liouville theorem (Moyal, 1949). All the physical processes described by the Schrödinger equation can be described by the quantum kinetic equations without any need of using potentials or vector-potentials of the field. The continuity equation for the velocity \( \nabla(X) \) can be looked at as a quantum generalization of the second law of Newton. Equation (48) converts into a classical Newton equation of motion with a Lorentz force, if the quantity \( \nabla(X) \) is replaced by the classical kinetic velocity and \( \hbar \to 0 \).

We mention, here, a few general things about the collision terms that emerge from their explicit expressions derived from equation (38) of Levanda and Fleurov (1994). The collision force is written in the form

\[
\rho(X)f_{\text{coll}}(X) =
-ih\nabla y \tilde{D}(X, y) \bigg|_{y \to 0} - \left\{ m\nabla(X) + \frac{e}{c}A(X) \right\} \tilde{D}(X, y) \bigg|_{y \to 0}
\]

(50)

where

\[
\tilde{D}(X, y) \bigg|_{y_0 = 0} = \int d^4x_1 \theta_{H}(X_0 - (x_1)_0) \tilde{D}(X, y| x_1) \bigg|_{y_0 = 0}
\]

and
If the Green’s functions and the mass operators obey the equalities \( \mathcal{G}^{<}(x_1, X) = -\mathcal{G}^{>}(x_1, X) \) and \( \mathcal{\Sigma}^{<}(X, x_1) = -\mathcal{\Sigma}^{>}(x_1, X) \), then \( \tilde{D}(X, y|x_1) = \tilde{D}(X, -y|x_1) \) and all the processes in the system are detailed balanced as discussed in Section 4 of Levanda and Fleurov (1994). In particular creation and annihilation of particles, \( \tilde{D}(X, y)|_{y=0} = 0 \), and the collision force, \( \nabla^y \tilde{D}(X, y)|_{y=0} = 0 \), vanish.

If transport processes occur in the system, the detailed balance no longer exists and the conservation of the number of particles or a vanishing collision force are not necessarily true. The function \( \tilde{D}(X, y|x_1)|_{y=0} = 0 \) can be regarded as describing the transfer of particles to a four dimensional volume element, at \( X \), from a four dimensional volume element, at \( x_1 \), caused by the collision processes. If the net transfer of particles to a point due to collision processes does not vanish, the momentum at that point changes by the amount equal to the mass transferred in a unit of time times the longitudinal part the velocity (the velocity without the rotation part). This is the meaning of the term on the right hand side of equation (5).

In the same way \( \hbar \nabla^y \tilde{D}(X, y|x_1)|_{y=0} \) stands for the transfer of momentum from a four dimensional volume element, at \( X \), to a four dimensional volume element, at \( x_1 \), caused by the collision processes. It is the quantum mechanical equivalent of the force applied to part of the system. This is the meaning of the term in the right hand side of equation (10). It obeys Newton third law, \( \hbar \nabla^y \tilde{D}(X, y|x_1)|_{y=0} = -\hbar \nabla^y \tilde{D}(x_1, y|X)|_{y=0} \), only for internal interactions. If for example an interaction with a phonon bath or impurities is considered this relation will not hold.

It has long been known that approximations used in transport theory can in principle lead to transport equations that are not necessarily consistent with all the macroscopic conservation laws (see, e.g., Mermin, 1970; Hu et al., 1989). This possible inconsistency was sometimes used to estimate the correctness of the results obtained (Thorber and Feynman, 1970). It is therefore desired to build the macroscopic conservation laws into the structure of the approximation used. Papers by Baym and Kadanoff (1961) and Baym G and Kadanoff (1962) (see also Green et al., 1985) found criteria for maintenance of the macroscopic conservation laws in systems with electron-electron interaction. The main motivation for including the conservation laws in the transport formalism comes from the division of the transport processes into two types. First, the particles interact via the interaction specified in the approximation, with this process generally having a small time constant. Then, the kinetic process proceeds, with a large time constant, via interactions with an unspecified bath. This part of the interaction is described by the macroscopic conservation laws. Using the hydrodynamic equations it is possible to build the macroscopic conservation laws into the structure of the approximation. That is, equation 18 holds for every \( X \). This is a new aspect of transport theory that comes out naturally from the hydrodynamic equations.

As an example, we mention the calculations of the conductivity in mesoscopic structures based on the quasi-classical Boltzmann equation. In these calculations the integration over the wave functions always precedes the evaluation of physical quantities (e.g., current density). Only after the integration, one equates the total influence of the electric field to the total influence of the impurities or of the phonons or of any other interaction. This procedure leads to macroscopic momentum conservation in the system but the microscopic momentum conservation is abandoned. In a paper to be soon published, we use the hydrodynamic equations to look for the current density in mesoscopic structures keeping the microscopic momentum conservation. We found that for elastic impurity s-scattering the current is close to being constant over the cross-section of the structure (i.e., the velocity of the electrons is higher near the edges of the structure). This result differs from the commonly used results that assume constant velocity. We deliberately used the simplest and most widely applied approximations that are well known and commonly used for decades with the quasi-classical Boltzmann equation to emphasize the role of the one particular assumption we changed. We require a microscopic balance of forces rather than a macroscopic balance of forces.

C. A single particle Schrödinger equation for a many particle system

The hydrodynamic equations were originally derived from the Schrödinger equation without electromagnetic fields by Madelung (1927) and with electromagnetic fields by Janossy and Ziegler-Naray (1964). Here we discuss an interesting possibility. One may introduce a modified Schrödinger equation for a wave function of a single particle excitation
in a multi-particle system which is equivalent to the above hydrodynamic equations. We mean that it is possible
to derive hydrodynamic equations for a system in electromagnetic fields with interaction, from the following
modified Schrödinger equation,

\[ i\hbar \frac{\partial \Psi(X)}{\partial T} = \left( \frac{-i\hbar \nabla - (e/c)A(X) + \int T dT' f_{coll}^t(X, T')} {2m} \right)^2 + e\varphi(X) + \int_X dX' f_{coll}^t(X, T) + \frac{i\hbar}{\rho(X)} \int dP I_{coll}(P, X) \Psi(X). \]  

(51)

Here \( f_{coll}^t(X) \) and \( f_{coll}^l(X) \) are the transverse and the longitudinal components of the collision term \( f_{coll}(X) \).

Now we demonstrate the derivation of equation (48) from equation (51) in two possible ways:

1. Introducing \( \rho(X) = \Psi(X)\Psi^*(X) \) and the Hamilton principal function \( S(X) = \left(-i\hbar/2\right)\ln(\Psi(X)/\Psi^*(X)) \) (that is writing \( \Psi(X) = \sqrt{\rho(x)} \exp(-iS(X)/\hbar) \)) one finds the continuity equation and the expression for the canonical
momentum

\[ \bar{\mathbf{p}}_{can}(X) = \mathbf{v} \cdot S(X) = \left( m\bar{\mathbf{v}}(X) + (e/c)A(X) + \int dT' f_{coll}^t(X, T') \right). \]  

(52)

It should be noted that the velocity is derived from the wave function by means of the equation

\[ \rho(X)\bar{\mathbf{v}}(X) = m\bar{\mathbf{J}}_{kin}(X) = \frac{1}{2} \left( -i\hbar \partial X' - \frac{e}{c} A(X') - \int dT' f_{coll}^t(X', T') \right) \nabla X' \rho(X') \left\vert_{X' \rightarrow X} \right. \]  

(53)

In the absence of the interaction terms this equation converges to the commonly used expression.

Now differentiating equation (52) with respect to time and using (51) we arrive at the kinetic momentum
continuity equation (48).

2. The same Schrödinger equation can be derived in a different manner. The expectation value of the kinetic
energy density

\[ \bar{\varepsilon}(X) = \frac{-i}{\rho(X)} \int d^4P \bar{\varepsilon}_W(P, X) \]

satisfies the quantum mechanical equivalent of the classical Hamilton-Jacobi equation

\[ \bar{\varepsilon}(X) = -\frac{\hbar^2}{8m\rho(X)} \left( \nabla X \right)^2 \rho(X) + \frac{\bar{\mathbf{p}}^2(X)}{2m} + \bar{\varepsilon}_{int}(X) = \]

\[ = \frac{m\bar{\mathbf{v}}^2(X)}{2} + \frac{\hbar^2}{8m} \left( \nabla X \frac{\rho(X)}{\rho(X)} \right)^2 + \bar{\varepsilon}_{int}(X) \]

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\[ \frac{m}{2} \nabla^2 (X) + \frac{\hbar^2}{2m} \nabla \frac{\nabla \rho(X)}{\sqrt{\rho(X)}} + \Xi_{int}(X). \]  

(54)

The last equality is obtained by integrating by parts in order to show explicitly the role of the quantum potential in changing the kinetic energy. This step is legitimate since an observable is obtained after integrating over the spatial coordinates. Substituting (52) into equation (54) the expectation value of the total energy of the system is expressed as a functional

\[ \langle E \rangle = \int dX \rho(X) \left( \nabla \Phi(X) + \phi(X) \right) = \]

\[ \int dX \Psi^*(X) \left( \frac{-i \hbar \nabla - (e/c) A(X) + \int dT' f_{coll}(X, T')}{2m} \right)^2 + \phi(X) + \Xi_{int}(X) \right) \Psi(X), \]

(55)

where \( \phi(X) \) is the electric potential. The Schrödinger equation is obtained by varying equation (55) with respect to the wave function \( \Psi(X) \). This equation is, in fact, written for the wave function of a one particle excitation in a many particle system. It contains a complex potential \( \Xi_{int}(X) \) created by all possible scattering processes. Its imaginary part is simply \( \text{Im} \Xi_{int}(X) = (\hbar/\rho(X)) \int dP I_{coll}(P, X) \). These sort of terms are discussed, e.g., within the framework of the optical model (see, e.g., Hodgson, 1963). The real part, \( \text{Re} \Xi_{int}(X) \) is equal to \( \int dX f_{coll}(X) \) plus linked cluster expansion terms.

D. Rotation of the velocity

The phase of a single particle wave function represented by the Hamilton principle function \( S(X) \), must be single valued and satisfy certain boundary conditions. In a collision free system the boundary conditions lead to the Aharonov-Bohm effect (Aharonov and Bohm, 1959) and their controversial physical role has been discussed previously in several papers (Bohm and Hiley, 1979; Wódkiewicz, 1984; Liang and Ding, 1987; Peshkin and Tonomura, 1989). It has been argued that the hydrodynamic formulation uses field strength, rather than potentials, and, hence, it can describe the Aharonov-Bohm effect only when boundary conditions are added (e.g., Philippidis et al, 1981). Here, it will be shown that the Aharonov-Bohm effect can be derived from the hydrodynamic equations themselves without a use of potentials. The effect will be generalized to account for the role of collisions and the boundary conditions for the velocity in systems with interactions will be presented.

To find the boundary conditions for the velocity \( \nabla v(X) \), its rotation is calculated. The rotation of equation (18) and the identity

\[ \frac{dB}{dt} = \frac{\partial B}{\partial t} + \nabla \times (B \times v) + v(\nabla \cdot B) \]

(56)
yield the equation (compare to the derivation of Lorentz force from the Faraday’s law, e.g., by Jackson (1975))

\[ m \frac{d}{dT} \oint \nabla v(X) \cdot dl = \oint \left[ -eE(X) - \frac{e}{c} \nabla \Phi(X) \times B(X) + f_{coll}(X) \right] \cdot dl \]

\[ = \frac{e}{c} \frac{d}{dT} \oint B(X) \cdot dl + \oint f_{coll}(X) \cdot dl. \]

(57)

Equation (57) can be represented in the form

\[ m \oint \nabla v(X) \cdot dl - m \oint \nabla v(X) \cdot dl \bigg|_{B=0, f_{coll}=0} - \int^T dT \oint f_{coll}(X) \cdot dl = \]

\[ \frac{e}{c} \oint B(X) \cdot dl \]

(58)

where the integration in the left hand side of the equation is carried out along the curve bounding the integration surface in the right hand side.
The term $\oint \langle v(X) \rangle \cdot d\ell \Big|_{B \equiv f_{\text{coll}} \equiv 0}$ is the rotation of the velocity for the collisionless system without a magnetic induction. It can be calculated by means of the continuity equation and equation (48) which for a steady $(d\mathbf{v}/dT = 0)$, cylindrically symmetric system, coincide with the Schrödinger equation for particles with a well defined angular momentum. The Bohr-Sommerfeld quantization rule, therefore, follows

$$\oint m\mathbf{v}(X) \cdot d\ell \mid_{B \equiv f_{\text{coll}} \equiv 0} = n\hbar$$

with $n$ being an integer. A change by one quantum level in the rotation of the velocity corresponds to a change of a flux unit $\Phi_0 = c\hbar/e = c\pi\hbar/e$.

Equation (67) can be written with the help of equation (52) in the form

$$\frac{1}{\hbar} \oint \nabla \cdot S(X) = 2\pi n,$$

which is, in fact, the condition that the wave function is single valued.

Equation (68) is rewritten in the form

$$m \oint \mathbf{v}(X) \cdot d\ell = nh + \frac{e}{c} \oint \mathbf{B}(X) \cdot ds + \int^T dT \oint f_{\text{coll}}(X) \cdot d\ell.$$

It states that the sum of the angular momentum of the electron, of the magnetic induction and of the system, that applies the collision force, should be quantized. All these three sub-systems are coupled together and a continuous change in the angular momentum of one of them results in a continuous change in the angular momentum of the others. For example, if the magnetic induction is raised gradually to its finite static value, it induces an electric field which changes the velocity lines in such a way as to keep the total angular momentum constant (Lentz law) (c.f., Peshkin, 1981). Equation (60) tells us nothing about sudden changes of the total angular momentum in quanta of $\hbar$. This is done via coupling to an external system that is not included in our description (e.g., a current source that drives the solenoid). The external system that applies a torque to change the angular momentum of the electrons and the magnetic induction can change their angular momentum only in quanta of $\hbar$, hence, if it applies a small and uneven torque to the magnetic induction and to the electron, it can change the angular momentum of the electron but not the total angular momentum. The conserved angular momentum of the external system in this interaction does not mean that this is a forceless interaction. The internal distribution of angular momentum in the external system has changed.

Although the change of the velocity of the electrons is due to the induced electric field, the system does not relax to its state before the fields were applied just as a classical particle that does not relax to its initial velocity after the field has been removed. Relaxation may occur only if a dissipation mechanism is introduced into the system, via $f_{\text{coll}}(X)$. Equation (67) explicitly shows that the change in $\mathbf{v}(X)$ can be viewed at as being due to a local action of the electric field at earlier times, or as a non-local action of the magnetic field at the present time. The collision force can cause a non-zero rotation of the velocity in the same manner as it can be done by an electric field generated by a time dependent magnetic field.

V. EXAMPLES

A. Wigner quasi-distribution function for a harmonic potential in a permanent magnetic field

1. Schrödinger equation

A system with a harmonic potential $m\omega^2z^2/2$ in the $z$ direction and a magnetic induction $\mathbf{B} = (cm/e)\omega H$ in the $x,y$ plane is considered. Before the application of the magnetic induction the electron is moving with the momentum $p^0 = m\mathbf{v}^0$ in the $x,y$ plane. After its application the electron velocity changes according to equation (68) and becomes

$$\mathbf{v}(z) = p^0/m + (e/mc)\mathbf{B} \times \mathbf{z} = p^0/m + \omega H \times \mathbf{z}.$$ (61)

The force acting on the electron stems from three sources, harmonic potential, magnetic induction and the quantum potential. Putting them all in equation (68), and assuming that the system is in a steady state, we find an equation for the balance of forces in the $z$ direction,
\[ 0 = -m(\omega_c^2 + \omega_H^2)z - \mathbf{p}^0 \times \omega_H + \frac{\hbar^2}{2m} \frac{\partial^2\sqrt{\rho(z)}}{\partial z^2} \left( \frac{1}{\sqrt{\rho(z)}} \frac{\partial \sqrt{\rho(z)}}{\partial z^2} \right). \]  

(62)

After integration (here, a particular choice of the gauge is made taken, see equation 64) we obtain a time independent Schrödinger equation

\[
\begin{align*}
0 &= \frac{\hbar^2}{2m} \frac{\partial^2\sqrt{\rho_n(z)}}{\partial z^2} - \frac{m(\omega_c^2 + \omega_H^2)z^2}{2} - \mathbf{p}^0 \times \omega_H z + E_n \\
&= \frac{\hbar^2}{2m} \frac{\partial^2\sqrt{\rho_n(z)}}{\partial z^2} - \frac{m(\omega_c^2 + \omega_H^2)}{2} ((z + z^0)^2 - (z^0)^2) + E_n.
\end{align*}
\]

(63)

Here

\[ z^0 = \frac{\mathbf{p}^0 \times \omega_H}{m(\omega_H^2 + \omega_c^2)} \]

is the deflection of the center of the effective harmonic potential due to the Hall effect (it is the point where the Hall force equals the harmonic force). \( E_n \) and \( \rho_n(z) \) are the eigenenergy and the electron density for the \( n \hbar \) eigenstate.

The Hamiltonian can be written in the form

\[
\hat{H} = \frac{\mathbf{p}^2 - \left( \frac{\mathbf{p}^0 \times \omega_H}{\omega_H} \right)^2}{2m} + \frac{m\omega_H^2 z^2}{2} + \frac{m\omega_c^2}{2} \left( \frac{\mathbf{p}^0 \times \omega_H}{m\omega_H^2} \right)^2 \\
= \frac{\mathbf{p}^2 - \left( \frac{\mathbf{p}^0 \times \omega_H}{\omega_H} \right)^2}{2m} + \frac{m(\omega_H^2 + \omega_c^2)}{2} ((z + z^0)^2)
\]

(64)

If only the harmonic potential is present or if only the magnetic induction is present, the above Schrödinger equation takes a form similar to that which can be found in many text-books (see, e.g., Landau and Lifshitz, 1977).

Solutions of this equation read

\[
\sqrt{\rho_n(z)} = \frac{1}{2^{n/2}(n!)^{1/2}} \left( \frac{m\sqrt{\omega_c^2 + \omega_H^2}}{\pi \hbar} \right)^{1/4} \exp \left\{ -\frac{m}{2\hbar} \sqrt{\omega_c^2 + \omega_H^2} (z + z^0)^2 \right\} \times \]

\[
H_n \left( \frac{m}{\hbar} \sqrt{\omega_c^2 + \omega_H^2} (z + z^0) \right)
\]

where \( H_n \) are the Hermite polynomials. The energy levels are given by

\[
E = (n + 1/2)\hbar \sqrt{\omega_c^2 + \omega_H^2} + \frac{(\mathbf{p}^0 \times \omega_H)^2}{2m(\omega_c^2 + \omega_H^2)}
= (n + 1/2)\hbar \sqrt{\omega_c^2 + \omega_H^2} + \frac{m(\omega_c^2 + \omega_H^2)}{2} (z^0)^2.
\]

and, as expected, correspond to the levels of the effective harmonic potential shifted due to the Hall force.

The above derivation emphasizes a connection between the momentum continuity and the Schrödinger equations. An attention should be also drawn to the velocity profile. It is the velocity (but not the current) which grows with the coordinate \( z \). This can be seen directly from equation (61) or (more roughly) from the independence of \( \rho(z) \) from the coordinates \( x \) and \( y \). This suggests that the canonical momentum is constant, whereas the kinetic momentum equals to the vector potential that generates the magnetic induction.
2. Wigner quasi-distribution function

Gauge-invariant Wigner quasi-distribution function should satisfy the quantum Hamilton-Jacobi and Boltzmann equations for the system under consideration. The quantum Hamilton-Jacobi equation is found by substituting the electric field $E = m\omega^2 z$ and the magnetic induction $B = (cm/e)\omega_H$ into equation \((60)\).

$$\frac{1}{2\pi\hbar} \left\{ \varepsilon + \frac{\hbar^2}{6m\omega^2} (\partial p_z)^2 + \frac{\hbar^2}{4m} \left[ \nabla^2 - m\omega^2 z \partial^2 + m\omega_H \times \nabla \right]^2 \right\} \rho_W(\varepsilon, p, x)$$

Quantum Boltzmann equation \((31)\) has the form

$$\left\{ -m\omega^2 z \partial p_z + \frac{1}{2m} \left[ p - \frac{z}{\hbar} m\omega^2 \partial p_x \partial \varepsilon \right] \left[ \nabla^2 - m\omega^2 z \partial^2 + m\omega_H \times \nabla \right] \right\} \rho_W(\varepsilon, p, x)$$

$$= 0.$$ \hspace{1cm} \text{Equation (68)}

The set of functions that solve these equations are

$$\rho_W(\varepsilon, p, z|n) = \frac{m\sqrt{\omega^2 + \omega_H^2}}{\pi\hbar} \exp \left[ -\frac{m\sqrt{\omega^2 + \omega_H^2}}{\hbar} \left( z + Z_0 \right)^2 + \frac{6\Omega^2}{m\omega^2} \right]$$

$$\sqrt{\frac{\hbar^2}{6m\omega^2\Omega^2}} \cos \left( \frac{p_z}{\hbar} \sqrt{\frac{24\Omega^2}{m\omega^2}} \left( \frac{\omega^2}{2m} \right) \theta_H(\Omega) \right)$$

$$L_n \left( \frac{4}{\hbar\sqrt{\omega^2 + \omega_H^2}} \left( \frac{m\omega^2 + \omega_H^2}{2} \right) (z + Z_0)^2 + \frac{p_z^2}{2m} \right)$$ \hspace{1cm} \text{Equation (67)}

where $L_n$ are the Laguerre polynomials. $\Omega$ is defined by means of the equation

$$0 = \Omega^2 + \varepsilon - \left( \frac{1}{2} + n \right) \hbar \sqrt{\omega^2 + \omega_H^2} + \frac{m\omega^2 z^2}{2} - \frac{p_z^2}{2m} + \frac{(p_x - m\omega_H z)^2}{2m} \frac{\omega_H}{\omega^2 + \omega_H^2},$$

and

$$Z_0 = \frac{(p_x - m\omega_H z) \omega_H}{m(\omega^2 + \omega_H^2)}.$$ 

Integrating equation \((67)\) over $\varepsilon$ we find

$$\rho_W(P, z|n) = \exp \left[ -\frac{2}{\hbar\sqrt{\omega^2 + \omega_H^2}} \left( \frac{m(\omega^2 + \omega_H^2)}{2} \left( z + Z_0 \right)^2 + \frac{p_z^2}{2m} \right) \right]$$

$$\times L_n \left( \frac{4}{\hbar\sqrt{\omega^2 + \omega_H^2}} \left( \frac{m(\omega^2 + \omega_H^2)}{2} \left( z + Z_0 \right)^2 + \frac{p_z^2}{2m} \right) \right).$$ \hspace{1cm} \text{Equation (68)}

Equation \((68)\) is now compared with other existing expressions for the Wigner quasi-distribution function appearing in the literature for a harmonic oscillator ($\omega_H = 0$) or for a particle in a magnetic induction ($\omega_c = 0$).

The Wigner quasi-distribution function and an equation for it in one-dimensional harmonic oscillator has been derived many times in the past (e.g., Heller, 1976; Bartlett and Moyal, 1949; Groenewald, 1946; Takabayashi, 1954; Carruthers and Zachariasen, 1983). All these derivations were done for a non-gauge-invariant Wigner function and
without the energy variable. The commonly used equation for the non-gauge-invariant Wigner quasi-distribution function for a harmonic oscillator is

\[
\left\{ \varepsilon_{\text{can}} + \frac{\hbar^2}{8m} \left[ \frac{\partial^2}{\partial z^2} + em^2 \omega^2 \frac{\partial^2}{\partial P^2} \right] - \frac{P^2}{2m} - \frac{1}{2}em\omega^2 z^2 \right\} \rho_W(P, z | n) = 0.
\]

This equation does not converge to equation (67) integrated over \( \varepsilon \). However, the solution of (68) satisfies both equation (69) and (65) integrated over \( \varepsilon \). This situation of one solution for two different equations, becomes clear if we remember that \( \varepsilon_{\text{can}} = \frac{\hbar \omega}{2} \left( n + \frac{1}{2} \right) \) and that

\[
\int d\varepsilon \varepsilon \rho_W(\varepsilon, P, z) = \left[ \frac{7}{12} \frac{\hbar \omega}{6m} + \frac{1}{2} m\omega^2 z^2 \right] \int d\varepsilon \rho_W(\varepsilon, P, z).
\]

We conclude that although the Wigner quasi-distribution function, after integrating over \( \varepsilon \), is identical in the gauge-invariant and in the non-gauge-invariant cases, the equations they obey are different. This is due to the different expressions for the kinetic and canonical momenta. Before the integration over \( \varepsilon \), the two functions are different as their equations are. The fact that the two solutions are identical is connected with the fact that the gauge used in the non-gauge-invariant formalism assumes a vanishing vector potential, hence, integrating over \( \varepsilon \) has washed out all the traces of the gauge.

The Wigner quasi-distribution function for an ensemble of oscillators at temperature \( k_B T = 1/\beta \) can be found using the linearity of the distribution function.

\[
\rho_W(\varepsilon, P, z | \beta) = \sum_n e^{-(n+\frac{1}{2})\hbar \omega \beta} \rho_W(\varepsilon, P, z | n) =
\sum_n e^{-(n+\frac{1}{2})\hbar \omega \beta} \frac{1}{\pi \hbar} \sqrt{\frac{m(\omega_c^2 + \omega_H^2)}{\pi \hbar}} \times
\exp \left[ -\frac{m(\omega_c^2 + \omega_H^2)}{\hbar} (z + Z_0)^2 + \frac{6 \Omega^2}{m \omega_c^2} \right] \times
\frac{\hbar^2}{6m \omega_c^2 \Omega^2} \cos \left( \frac{P}{\hbar} \sqrt{\frac{24 \Omega^2}{m \omega_c^2}} \right) \times
L_n \left( \frac{4}{\hbar \sqrt{\omega_c^2 + \omega_H^2}} \left( \frac{m(\omega_c^2 + \omega_H^2)}{2} (z + Z_0)^2 + \frac{P^2}{2m} \right) \right).
\]

Integrating over \( \varepsilon \), using the equality (see, e.g., Gradshteyn and Ryzhik, 1980, equation 8.975)

\[
\frac{1}{1 - a^{g+1}} = \sum_{i=0}^{\infty} L_n(b) a^i, \quad |a| < 1
\]

and normalizing, we arrive at an expression that converges to a well known expression (see, e.g., Hillery et al, 1984) for harmonic oscillators (at \( \omega_H = 0 \)).

\[
\rho_W(\varepsilon, P, z | \beta) = \tanh \left( \frac{\hbar \sqrt{\omega_c^2 + \omega_H^2} \beta}{2} \right) \times
\exp \left[ -\frac{2}{\hbar \sqrt{\omega_c^2 + \omega_H^2}} \tan \left( \frac{\hbar \sqrt{\omega_c^2 + \omega_H^2} \beta}{2} \right) \left( \frac{m(\omega_c^2 + \omega_H^2)}{2} (z + Z_0)^2 + \frac{P^2}{2m} \right) \right].
\]

A comparison for the zero magnetic induction case can be done after substituting \( \omega_c = 0 \) in equation (68).
\[ \rho_W(P, z|n) = \exp \left[ - \frac{2}{\hbar \omega_H} \left( \frac{m \omega_H^2}{2} (z + Z_0)^2 + \frac{p_z^2}{2m} \right) \right] \]

\[ L_n \left( \frac{4}{\hbar \omega_H} \left( \frac{m \omega_H^2}{2} (z + Z_0)^2 + \frac{p_z^2}{2m} \right) \right) \]

\[ = \frac{1}{\pi \hbar} \exp \left[ - \frac{2}{\hbar \omega_H} \frac{p_z^2 + p_x^2}{2m} \right] L_n \left( \frac{4}{\hbar \omega_H} \frac{p_z^2 + p_x^2}{2m} \right) \]

(71)

where

\[ \varepsilon_{\text{can}} = \left( \frac{1}{2} + n \right) \hbar \omega_H - \frac{p_\mu^2}{2m}. \]

This is the expression used in the literature (see, e.g., book of Mahan, 1987).

### B. Wigner quasi-distribution function in a periodic system

It is well known that the concept of a quasi-momentum plays an important part in the theory of electrons in periodic potentials (see, e.g., Weinreich, 1965). If the potential is also periodic in time, then a quasi-energy is to be introduced, Zel’dovich, 1973; Zel’dovich et al., 1976. Hence, if a periodic (in space and/or in time) electromagnetic field is applied to otherwise homogeneous system, it may be reasonable to use a Wigner quasi-distribution function formulated in terms of quasi-momentum and/or quasi-energy. This may provide us a better physical insight and make calculations easier. Here we shall try to reformulate the Wigner quasi-distribution function using the so-called \( kq \)-representation, introduced by Zak (1972), assuming periodicity in space and/or time of the electromagnetic field. This may shine a new light on the ‘acceleration theorem’ related, as we shall see, to gauging rather than to electron dynamics.

A \( kq \)-representation can be constructed for any pair of operators \( \hat{A} \) and \( \hat{B} \) that satisfy the commutation relation \([\hat{A}, \hat{B}] = -i \hbar\). Then a four dimensional constant vector \( a_\mu \), which is the space-time period of our system, allows one to define the operators

\[ \hat{T}(a) = \exp(i \hat{B}^\mu a_\mu) \quad \text{and} \quad \hat{\tau}(2\pi \hbar/a) = \exp(i \hat{A}^\mu 2\pi \hbar/a_\mu). \]

(72)

Eigen-functions of these operators satisfy the conditions

\[ \hat{T}(a) \varphi_{kq}(x) = \exp(i k^\mu a_\mu) \varphi_{kq}(x) \]

(73)

\[ \hat{\tau}(2\pi \hbar/a) \varphi_{kq}(x) = \exp(i q^\mu (2\pi \hbar/a)_\mu) \varphi_{kq}(x). \]

(74)

and form a basis of the \( kq \)-representation.

The operator \( \hat{A} \) is chosen to be the coordinate operator \( \hat{x} = (\hat{t}, \hat{r}) \), and \( \hat{B} \) to be the energy-momentum operator \( \hat{p} = (\hat{\varepsilon}, \hat{p}) \). Then \( k \) is the quasi-energy-momentum and \( q \) is the quasi-four-coordinate. Both the coordinates and momenta, as well as their corresponding operators, can be written as (Zak, 1972),

\[ x = q + (m^\mu a_\mu)a = q + X \]

(75)

\[ p = k + (m^\mu (2\pi \hbar/a)_\mu)(2\pi \hbar/a) = k + K \]

and

\[ \hat{x} = \hat{q} + \hat{X} \]

(76)

\[ \hat{p} = \hat{k} + \hat{K}. \]

\( ^2 \)The ‘correct’ and ‘transport’ Green’s functions used by Mahan are actually non-gauge-invariant and gauge-invariant forms of the Green’s functions.
Here \( m^\mu \) is a vector of integers, \( K \) is the momentum conjugate to the quasi-coordinates and \( X \) is the coordinate conjugate to the quasi-energy-momenta. It is clear that at \( \hbar/a \to 0 \) the spacing between the discrete values of \( K \) vanishes and one approaches the continuous limit, \( K \to p, q \to x \). The opposite limit \( \hbar/a \to \infty \) also yields \( k \to p \) and \( X \to x \).

The eigenfunction of the operators \( \hat{T}(2\pi\hbar/a) \) and \( \hat{\Gamma}(a) \) can be now represented in the form

\[
\varphi_{kq}(x) = \left( \frac{V_0}{(2\pi\hbar)^2} \right)^{1/2} \sum_{m\lambda} \exp \left( ik^\mu a_\mu(m^\lambda a_\lambda) \right) \delta \left( x_\mu - q_\mu - (m^\lambda a_\lambda) a_\mu \right),
\]

where \( V_0 \) is the volume of the four dimensional periodic unit cell. These eigenfunctions can be used to define the creation and annihilation operators \( \alpha^\dagger(k, q) \), \( \alpha(k, q) \) in the \( kq \)-representation in terms of the creation and annihilation operators \( \varphi^\dagger(x) \), \( \varphi(x) \) in the \( x \) representation,

\[
\varphi(x) = \int \text{d}k\text{d}q \alpha(k, q) \varphi_{kq}(x).
\]

\[
\alpha(k, q) = \sum_{m\lambda} e^{-ik^\mu a_\mu(m^\lambda a_\lambda)} \varphi(q_\mu + (m^\lambda a_\lambda) a_\mu)
\]

Transforming the Wigner quasi-distribution function into the \( kq \)-representation the matrix elements

\[
\langle kq | p | k'q' \rangle = -i\hbar(\partial/\partial q)\delta(q' - q)\delta(k' - k)
\]

and

\[
\langle kq | x | k'q' \rangle = [i\hbar(\partial/\partial k) + q] \delta(q' - q)\delta(k' - k).
\]

are used. Then the equality

\[
\left\langle \Psi^\dagger(X + y/2)\Psi(X - y/2) \right\rangle = \int \text{d}k\text{d}q \left\langle \alpha^\dagger(k, q) \exp \left( \frac{i}{\hbar} y^\mu \partial_\mu \right) \alpha(k, q) \right\rangle.
\]

is obtained and equation (79) in the \( kq \)-representation now reads

\[
\rho_W(P, X) = \int \text{d}y e^{-(i/\hbar)(yP)} \int \text{d}k\text{d}q \left\langle \alpha^\dagger(k, q + y/2)\alpha(k, q - y/2) \right\rangle,
\]

as can be verified by substituting equation (79).

One thing that can be learned from equation (83) is that if all the particles in the system acquire an additional quasi-momentum \( \delta k \), it will not have any effect on the Wigner quasi-distribution function neither will it have an effect on the measurable quantities. This is follows from the cancellation of the additional phase factors in equation (79). Hence in order to cause a change in the physical system, some other factors should be changed. This statement applies also to the Bloch representation. To clarify, we give the connection between the \( kq \)-representation and the Bloch representation,

\[
\alpha(k, q) = \sum_n B_n(k)\Psi_{nk}(q)
\]

where the coefficients \( B_n(k) \) are the wave functions in the Bloch representation, and \( \Psi_{nk}(q) \) are the Bloch functions. If a change of the quasi-momentum is necessary which will also change the momentum one should introduce a function of the quasi-coordinate and the quasi-momentum, \( f(k, q) \) in the definition of \( \alpha(k, q) \). Usually the function \( \exp \int \lambda(k, q)dt \) is used with \( \lambda(k, q) \) chosen to be the dispersion relation of the electrons with the homogeneous fields set to zero. The issue of the validity of this procedure is reviewed by Nenciu (1991), and we shall not dwell on it here. All that has
been said so far, is that a change in the quasi-momentum cannot by itself lead to a kinetic change in the physical system. Hence we suggest that a change in the quasi-momentum due to external field is a gauge effect.

It is sometimes stated, that the dynamics of an electron in a lattice can be described by the so called acceleration theorem which is generally written in the form

$$\frac{d\mathbf{k}}{dt} = -e\mathbf{E} - \left(\frac{e}{c}\right)v_{av} \times \mathbf{H}$$

where \( \mathbf{k} \) is the quasi-momentum of the particle and \( v_{av} \) is the average velocity of the electron. It is usually stated that one may find a basis of eigenfunctions in such a way that it will make the average velocity of the electron, after a sufficient long time, to be \( v_{av} = \nabla^k \lambda(\mathbf{k}) \), where \( \lambda(\mathbf{k}) \) is the dispersion relation (Adams and Argyres, 1956; Wilson, 1965).

The acceleration theorem is regarded as a quasi-classical approximation that can be used only if the external fields are weak and if the movement of the Bloch electron is restricted to a small part of the Brillouin zone. The existing derivations of the acceleration theorem (Wilson, 1965; Landau and Lifshitz, 1984; Kittel, 1963; Kohn, 1959) are very complicated, controversial and are not rigorous. Hence, their validity is not well defined. An exception is the paper of Zak (1972) who has derived the acceleration theorem for an electric field in a rigorous way.

It is interesting to note a similarity between the acceleration theorem and equation (27). Here \( v_{av} \) corresponds to the velocity of the observer \( dz''/dt'' \) and the canonical momentum to the quasi-momentum. To make it more clear we remark that \( \overline{P}_{can}(X, z'') \) is equal to the expectation value of the momentum \( \hat{p} = \hat{k} + \hat{K} \). Remembering that the expectation value of \( \hat{K} \) can take only discrete values, we realize that there are short periods of time \( \Delta t'' \), when the change of the expectation value of the momentum is equal to the change of the expectation value of \( \hat{k} \). At these periods of time the acceleration theorem and equation (27) are identical.

C. Canonical energy change of a heavy mass polaron in electric field

The problem of a heavy mass polaron (van Haeringen, W, 1965) is solved with the help of the quantum mechanical equivalent of the classical Hamilton-Jacobi equation. We shall solve here the problem of a heavy mass polaron in a constant and homogeneous electric field. This example is also interesting in and of itself, since this problem has never been solved by means of a Dyson equation, (see discussion in the book of Mahan (1990), p.509-510).

We are concerned with a homogeneous system with a single electron in a conduction band and assume an electron-phonon interaction with optical phonons. For that kind of system \( G^c = 0 \) and only one Green’s function should be evaluated (in the general case there are two independent Green’s functions) and we shall take it to be \( G^c \). This function satisfies the quantum Hamilton-Jacobi equation (30) that for our system takes the form

$$\frac{1}{2\pi \hbar} \left[ \varepsilon - \varepsilon_p + \frac{k^2}{8m} \left( \nabla^X + e\mathbf{E}\partial\varepsilon \right)^2 \right] G^c(\mathbf{p}, \varepsilon, \mathbf{X}) = 1 + \Sigma^c(\mathbf{p}, \varepsilon, \mathbf{X})G^c(\mathbf{p}, \varepsilon, \mathbf{X})$$

with \( \varepsilon_p = \mathbf{p}^2/(2m) \). The right hand side of the equation takes this simple form since the polaron has a heavy mass and all the derivatives of the Green’s function with respect to the momenta are assumed to vanish. Hence, the phase loop function (that represents kinetic effects) vanishes. The self-energy term has the general form

$$\Sigma^c(\mathbf{p}, \varepsilon) = \int d\omega dq G^c(\mathbf{p}, \varepsilon - \omega - \varepsilon_q) D^c(\mathbf{q}, \omega) \Gamma(\mathbf{p}, \varepsilon, \mathbf{q}, \omega)$$

with \( D^c(\mathbf{q}, \omega) \) is the phonon Green’s function and the vertex part is given by the Ward identity (Engelsberg and Schrieffer, 1963)

$$\Gamma(\mathbf{p}, \varepsilon, \mathbf{q}, \omega) = \frac{[G^c(\mathbf{p}, \varepsilon - \omega - \varepsilon_q)]^{-1} - [G^c(\mathbf{p}, \varepsilon)]^{-1}}{\omega + \varepsilon_q}$$

The heavy mass polaron is discussed only for the sake of a demonstration. We therefore do not provide an introduction to the problem of polarons and assume that the reader is familiar with the problem. The background can be found in text-books (see, e.g., Mahan, 1990).
Fourier transforming equation (86), one obtains
\[
\left[ -i\hbar \frac{d}{dt} - \varepsilon_p + \frac{1}{8m} \left( -i\hbar \nabla X + eEt \right)^2 - \Phi(t) \right] G_c(p, t) = \hbar \delta(t). \tag{89}
\]

where
\[
\Phi(t) = \int_0^t dT d\mathbf{q} D_c(q, T) e^{i\varepsilon qT}. \tag{90}
\]

The solution of this equation is
\[
G_c(p, t, X) = \theta(t) \exp \left[ \frac{i}{\hbar} \left( \varepsilon_p + eEX \right) t + \int_0^t dT \Phi(T) \right]. \tag{91}
\]

As can be seen from this equation the energy of the polaron is composed of three parts corresponding to its kinetic, potential and phonon interaction energies. It is instructive to search for a solution of equation (86) that does not depend on \(X\)
\[
\frac{1}{2\pi \hbar} \left[ \varepsilon - \varepsilon_p + \frac{\hbar^2}{8m} (eE \partial^\mu)^2 \right] G_c(p, \varepsilon) = 1 + \Sigma^c(p, \varepsilon) G_c(p, \varepsilon). \tag{92}
\]

It reads
\[
G_c(p, t) = \theta(t) \exp \left[ \frac{i}{\hbar} \left( \varepsilon_p t - \frac{(eE)^2}{24m} t^3 + \int_0^t dT \Phi(T) \right) \right]. \tag{93}
\]

Going back to equation (27) we see that the change of the canonical momentum of the particle at rest in a static and homogeneous electric field is \(eEt\). The change of the canonical energy due to the electric field is
\[
\varepsilon_{can}(t) = \int_0^t dt_1 \frac{e}{2m} EP(t_1) = \frac{1}{m} \left( \frac{e}{2} E \right)^2 \int_0^t dt_1 t_1 = \frac{1}{m} \left( \frac{e}{2} E \right)^2 t^2 \frac{t^2}{2}. \tag{94}
\]

Integrating over time \(t\) we arrive at the phase factor in equation (90) which can be now written in the form
\[
G_c(p, t) = \theta(t) \exp \left[ \frac{i}{\hbar} \left( \int_0^t \left( \varepsilon_p - \varepsilon_{can}(T) + \Phi(T) \right) dT \right) \right]. \tag{95}
\]

It does not contain the potential energy term. There is a new term instead which corresponds to the energy that a free particle would acquire if it were moving freely in the electric field.

VI. SUMMARY

In this paper we formulate a gauge-invariant formalism for transport processes starting from the definition of the gauge-invariant Wigner quasi-distribution function, we also derive quantum transport equations and their hydrodynamic representation. Throughout the paper we emphasized the kinetic vs canonical meaning of the quantities discussed. We believe that this formulation will both simplify studying transport phenomena and reveal new aspects that are not easily shown with other available formalisms (e.g., quasi-classical Boltzmann equation or Kubo correlators).

APPENDIX A

The equality
\[
\exp \left( \frac{i}{\hbar} \hat{p}_{can} \hat{q}_\mu \right) \exp \left( \frac{i}{\hbar} \hat{p}_{\text{kin}} \hat{q}_\mu \right) =
\]
\[
\exp \left( i e c \frac{1}{\hbar} \int_0^1 \hat{y}^\mu A_\mu(\hat{X} + s\hat{y})ds \right)
\]  
(A.1)

in which the canonical and kinetic momenta are connected by the equation

\[
\hat{P}_{\text{kin}} = \hat{P}_{\text{can}} - \frac{e}{c} A(\hat{X}) = -\frac{i\hbar}{e} \frac{\partial}{\partial \hat{X}} - \frac{e}{c} A(\hat{X}).
\]  
(A.2)

is derived.

First, the operator

\[
U(s) = \exp \left( -\frac{i}{\hbar} s \hat{P}_\mu^\text{can} \hat{y}_\mu \right) \exp \left( \frac{i}{\hbar} s \hat{P}_\mu^\text{kin} \hat{y}_\mu \right)
\]  
(A.3)

is defined. It satisfies the following two conditions

\[
U(s)|_{s=0} = 1
\]  
(A.4)

and

\[
\frac{\partial U(s)}{\partial s} = \frac{ie}{c\hbar} \hat{y}^\mu A_\mu(\hat{X} + s\hat{y})U(s)
\]  
(A.5)

where

\[
A(\hat{X} + s\hat{y}) = \exp \left( -\frac{i}{\hbar} s \hat{P}_\mu^\text{can} \hat{y}_\mu \right) A(\hat{X}) \exp \left( \frac{i}{\hbar} s \hat{P}_\mu^\text{kin} \hat{y}_\mu \right).
\]  
(A.6)

Then integrating equation (A.5) the equality (A.1) is obtained.

**APPENDIX B**

The phase loop function was introduced in our previous paper (Levanda and Fleurov, 1994). Here we consider its shape for systems in a static and homogeneous field. The phase loop function is another representation of the slashed derivatives appearing in equations (30) and (31) and discussed in detail in main body of the paper.

The phase loop function corresponding to an electron loop with three vertices is defined by the equation

\[
M(\pi | 3, F(X)) = M((\pi_i) | F(X)) = 
\int d^4x_1 d^4x_2 d^4x_3 \exp \left( \frac{i}{\hbar} \sum_{i=1}^3 \pi_i^\mu x_i^\mu \right) \cdot \exp \left( \frac{ie}{c\hbar} \Phi (\{x_i\}) \right)
\]  
(B.1)

where

\[
\Phi (\{x_k\}) = \sum_{i=1}^3 y_k^\mu \int_{-1/2}^{1/2} A_\mu (X_k + s_k y_k)ds_k,
\]  
(B.2)

\[
y_k = x_{k+1} - x_k, \quad X_k = (x_{k+1} + x_k)/2, \quad \text{and cyclic conditions } x_{3+i} = x_i \text{ being assumed. Such a phase loop function depends on three four-dimensional variables } \{\pi_i^\mu\}_{i=1}^3, \text{ on the electromagnetic field tensor } F_{\lambda\mu}(X) \text{ and is explicitly gauge invariant. In the absence of electromagnetic fields the phase loop function takes its trivial form}
\]

\[
M(\pi | 3, F(X)) = \prod_{\mu,i} \delta(\pi_i^\mu).
\]  
(B.3)

Choosing \(x'' = x_1\) in equation (B.1) we find that
This is just an expression for the flux through a triangle that has its three vertices at the points \( x_1, x_2, x_3 \),

\[
\Phi \left( \{ x_k \} \right) = \int_{S\{x_1,x_2,x_3\}} F^{\lambda \mu} dx_\mu \wedge dx_\lambda,
\]

where \( S\{x_1,x_2,x_3\} \) is the surface covering this triangle.

The integration in the definition of the phase loop function is now straightforward and results in

\[
M(\pi \mid 3, F_{\mu \lambda}) = \delta \left( \sum_{i=1}^{3} \pi_i / \hbar \right) \exp \left[ \left( -ie\hbar/4c \right) F_{\mu \lambda} \partial^{\pi_2 \mu} \partial^{\pi_3 \lambda} \right] \prod_{i=2}^{3} \delta(\pi_i / \hbar).
\]

The latter equation (B.6) can be easily generalized to a \( N \) vertex electron loop in a homogeneous static field. The polygon with \( N \) vertices can be subdivided into \( N-2 \) triangles and the phase loop function for such \( N \) vertex electron loop is

\[
M(\pi \mid N, F_{\mu \lambda}) = \delta \left( \sum_{i=1}^{N} \pi_i / \hbar \right) \exp \left[ \left( -ie\hbar/4c \right) F_{\mu \lambda} \sum_{k=2}^{N-1} \partial^{\pi_k \mu} \partial^{\pi_{k+1} \lambda} \right] \prod_{i=2}^{N} \delta(\pi_i / \hbar).
\]

Comparing this equation with equation (B.3), one realizes that the momentum or energy are no longer conserved at each vertex. In this representation, the phase loop function shows in a simple way, how the field interferes with the four-momentum conservation. Derivatives of the delta function mean that derivatives of the Green’s functions and of the vertex functions appear in analytical expressions for the diagrams. These derivatives stand for the electron recoil due to the field (compare this to the recoil expansion of van Hearingen, 1965). Equation (B.7) converges to equation (B.3) as \( F_{\mu \lambda} \) approaches zero.

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