On the Second Boundary Value Problem for a Class of Modified-Hessian Equations

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In this paper a new class of modified-Hessian equations, closely related to the Optimal Transportation Equation, will be introduced and studied. In particular, the existence of globally smooth, classical solutions of these equations satisfying the second boundary value problem will be proven. This proof follows a standard method of continuity argument, which subsequently requires various a priori estimates to be made on classical solutions. These estimates are modifications of and generalize the corresponding estimates for the Optimal Transportation Equation, presented in [15]. Of particular note is the fact that the global $C^2$ estimate contained in this paper makes no use of duality with regard to the original equation.

Keywords Fully nonlinear elliptic PDE; Global regularity; Oblique boundary value problem; Optimal transportation.

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1. Introduction

In this paper the global regularity of classical solutions solving

$$
\left(\frac{S_n}{S_l}\right)^{\frac{1}{n-l}} \left[ D^2 u - D^2 c(\cdot, T_u) \right] = B(\cdot, u), \quad \text{in} \; \Omega^-,
$$

(1.1a)

associated with the second boundary value problem

$$
T_u(\Omega^-) = \Omega^+,
$$

(1.1b)
for given $c$, $B$, $T_u$ and $\Omega^\pm \subset \mathbb{R}^n$ will be studied. Specifically, $\Omega^\pm$ are smooth domains in $\mathbb{R}^n$ and

$$
\left( \frac{S_n}{S_\ell} \right) \frac{1}{[M]} := \sigma_{n,\ell}(\lambda_1, \ldots, \lambda_n),
$$

$$
:= \left( \frac{S_n(\lambda_1, \ldots, \lambda_n)}{S_\ell(\lambda_1, \ldots, \lambda_n)} \right) \frac{1}{\lambda_1},
$$

with $\lambda_1, \ldots, \lambda_n$ being the eigenvalues of the $n \times n$ symmetric matrix $M$ and $S_\ell$ denoting the $\ell$th elementary symmetric function given by

$$
S_\ell(\lambda_1, \ldots, \lambda_n) = \sum_{1 \leq i_1 < \cdots < i_\ell \leq n} \left( \prod_{j=i_1}^{i_\ell} \lambda_j \right)
$$

for any integer $\ell < n$. The structure of $T_u$ and the precise conditions on $\Omega^\pm$, $c$ and $B$ will be given in Section 2.

The structure of (1.1a) and (1.1b) is reminiscent of the Monge–Ampère type equations emerging from optimal transportation. Indeed, the conditions required and techniques used in this paper are reflective of those in [15] for the optimal transportation case, but all of the corresponding a priori estimates required to prove existence of globally smooth solutions in the current scenario need modification. In particular, (1.1a) does not have a duality property like the optimal transportation equation, which is one of the obstacles preventing results from being carried over from optimal transportation to the current scenario.

## 2. Motivation and Main Results

Before stating the main results of the paper, the mathematical motivation to study (1.1a) and (1.1b) will be presented. Specifically, the class of boundary value problems depicted in Section 1 represent the only class of problems from which the estimates in [15] can be adapted.

### 2.1. The Structure of $F$

One can regard (1.1a) as having the general form

$$
F[D^2u - A(\cdot, u, Du)] = B(\cdot, u), \quad \text{in } \Omega^-,
$$

where $A$ is a given $n \times n$ matrix-valued function and $B$ is a given scalar-valued function, defined on $\Omega^- \times \mathbb{R} \times \mathbb{R}^n$ and $\Omega^- \times \mathbb{R}$ respectively. Subsequently, the left-hand side of (2.1) can be represented as

$$
f(\lambda_1, \ldots, \lambda_n) := F[D^2u - A(\cdot, u, Du)], \quad \text{in } \Omega^-,
$$

where $f$ is a suitably-defined (detailed below), symmetric function of $\lambda_i$, which are the eigenvalues of the modified-Hessian matrix: $[D^2u - A(\cdot, u, Du)]$.

In order to start formulating a regularity theory for solutions solving (2.1), some conditions need to be placed on $f$ (in addition to being a symmetric function), as it is
denoted in (2.1). First, it is assumed that $f \in C^2(\Gamma) \cap C^0(\overline{\Gamma})$ is a symmetric function defined on an open, convex, symmetric region $\Gamma \subset \mathbb{R}^n$, $\Gamma \neq \mathbb{R}^n$ with $\Gamma + \Gamma_+ \subset \Gamma$, where $\Gamma_+$ is the positive cone in $\mathbb{R}^n$. Moreover, it is assumed that $f$ satisfies the following conditions:

\begin{equation}
 f > 0 \quad \text{in} \quad \Gamma, \quad f = 0 \quad \text{on} \quad \partial \Gamma, \tag{2.3}
\end{equation}

\begin{equation}
 f \quad \text{is concave in} \quad \Gamma, \tag{2.4}
\end{equation}

\begin{equation}
 \sum_i \frac{\partial f}{\partial x_i} = \sum_i f_i \geq \sigma_0 \quad \text{on} \quad \Gamma_{\mu_1,\mu_2}(f), \tag{2.5}
\end{equation}

and

\begin{equation}
 \sum_i f_i \lambda_i \geq \sigma_1 \quad \text{on} \quad \Gamma_{\mu_1,\mu_2}(f), \tag{2.6}
\end{equation}

where $\Gamma_{\mu_1,\mu_2} = \Gamma_{\mu_1,\mu_2}(f) := \{ \lambda \in \Gamma(f) : \mu_1 \leq f(\lambda) \leq \mu_2 \}$ for any $\mu_2 \geq \mu_1 > 0$ and $\sigma_0, \sigma_1$ are positive constants depending on $\mu_1$ and $\mu_2$.

Before moving on, some relevant remarks based on the ones made in [12] are now presented:

**Remarks 2.1.**

1. Conditions (2.3) and (2.4) together imply the degenerate ellipticity condition:

\begin{equation}
 f_i \geq 0, \quad \text{in} \quad \Gamma \quad \text{for} \quad i = 1, \ldots, n. \tag{2.7}
\end{equation}

This combined with the concavity assumption on $f$ subsequently implies that $F[D^2u - A(\cdot, u, Du)]$ is a concave function of $D^2u - A(\cdot, u, Du)$, which is required to apply the $C^{2,\alpha}$ estimates of Lieberman and Trudinger presented in [8].

2. $\Gamma(f)$ enables the definition of an *admissible solution* to be made corresponding to (2.1). That is, a solution $u \in C^2(\Omega^-)$ is admissible if

\begin{equation}
 D^2u - A(\cdot, u, Du) \in \Gamma(f), \quad \text{in} \quad \Omega^- \tag{2.8}
\end{equation}

It is clear that (2.8) combined with an assumption that $B(x, u) > 0$, ensures that (2.1) is elliptic with respect to a solution $u \in C^2(\Omega^-)$. In this context, ellipticity and *admissibility* of functions are indeed equivalent and will be used interchangeably for the remainder of this paper. Moreover, a solution to (2.1) is admissible if and only if it is a viscosity solution (see [13]). Indeed, one may substitute the term *viscosity solution* for *admissible solution* and vice-versa anywhere in this paper. It is through this equivalence that the notion of *viscosity solutions* will be understood for the rest of the forthcoming exposition.

3. Assuming (2.3) and (2.4) to be true, conditions (2.5) and (2.6) can be shown to be equivalent to other criterion. For instance, if

\begin{equation}
 f(t, \ldots, t) \to \infty, \quad \text{as} \quad t \to \infty, \tag{2.9}
\end{equation}
then from the concavity of \( f \) and the fact that \( 0 \leq \sum_i f_i \), it is readily calculated that

\[
f(t, \ldots, t) \leq f(\lambda) + \sum_i f_i \cdot (t - \lambda_i) \leq f(\lambda) + t \sum_i f_i.
\] (2.10)

From this, (2.5) follows on any \( \Gamma_{0,\mu_2}(f) \) when \( t \) is fixed to be large enough. Indeed, (2.5) and (2.9) are equivalent. Moreover, recalling (2.3) and \( 0 \in \partial \Gamma \), (2.10) indicates that

\[
\sum_i f_i \lambda_i \leq f(\lambda) \leq \mu_2,
\] (2.11)

on \( \Gamma_{0,\mu_2}(f) \) when taking \( t = 0 \). Similarly, if for any \( \mu_1 \) and \( \mu_2 \) with \( 0 < \mu_1 \leq \mu_2 \), there is a constant \( \theta = \theta(\mu_1, \mu_2) \) such that

\[
\theta + f(\lambda) \leq f(2\lambda), \quad \forall \lambda \in \Gamma_{\mu_1,\mu_2},
\] (2.12)

then the concavity of \( f \) can once again be used to directly show

\[
f(2\lambda) \leq f(\lambda) + \sum_i f_i \lambda_i.
\]

This subsequently implies (2.5) with \( \sigma_0 = \theta \). It is clear that (2.12) is satisfied with \( \theta = (2^x - 1)\mu_1 \) if \( f \) is homogeneous of degree \( x \in (0, 1] \).

(4) The main examples of functions \( f \) satisfying (2.3)-(2.6) are those corresponding to

\[
f(\lambda) = S^{1/k}_k(\lambda) = \left( \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left( \prod_{m=i_1}^{i_k} \lambda_m \right) \right)^{1/k}
\] (2.13)

and the quotients \( \sigma_{k,l} \), \( 1 \leq \cdots \leq l < k \leq \cdots \leq n \), for which it is denoted that

\[
f(\lambda) = \sigma_{k,l} := \left( \frac{S_k(\lambda)}{S_k(\lambda_0)} \right)^{1/(k-l)}.
\] (2.14)

As \( \Gamma(S_k) \subseteq \Gamma(S_l) \) for \( l \leq k \), \( \Gamma(f) = \Gamma(S_k) \) holds when \( f \) is either \( S^{1/k}_k \) or \( \sigma_{k,l} \). For these examples, the concavity of \( f \) is verified in [2, 13, 20]; and (2.6) is clear from the argument presented in Remark 2.1(3), as both (2.13) and (2.14) depict functions which are both homogeneous of degree 1. Lastly, in the case where \( f \) is defined by either (2.13) or (2.14), it is a straight-forward calculation to show that

\[
f_i > 0, \quad \text{in } \Gamma, \text{ for } i = 1, \ldots, n;
\] (2.15)

i.e., \( f \) satisfies the regular ellipticity criterion on \( \Gamma(f) \) and not just the degenerate ellipticity condition (2.7).

(5) Conditions (2.3)-(2.6) combined with (2.15) are essentially the ones used in [20] to prove the existence of smooth solutions to a class of Hessian equations closely related to (2.1), satisfying a simpler version of the natural boundary condition depicted in (1.1b).
At this stage, $S_1/k$ and $\sigma_{k,l}$ represent the two prime candidates for $f$, for which the existence of globally-smooth solutions may be able to be proven. Next, the structure from the Optimal Transportation Problem will be used to ascertain criterion on both $A(\cdot, u, Du)$ and $T_u$ that will subsequently further reduce the possible forms of $F$.

### 2.2. Conditions from Optimal Transportation

The next step in the motivation of (1.1a) and (1.1b), some conditions and notation will be recalled from the theory of Optimal Transportation; these will be essential parts to the upcoming a priori estimates. Denoting $U \supseteq \Omega \times \Omega^+$, the cost-function $c$ (again taking the label from Optimal Transportation) is assumed to be real-valued in $C^4(U)$, satisfying the following three conditions:

(A1) For any $(x, y) \in U$ and $(p, q) \in D_x c(U) \times D_y c(U)$, there exists a unique $Y = Y(x, p)$, $X = X(y, q)$, such that $c_x(x, Y) = p$, $c_y(X, y) = q$.

(A2) For any $(x, y) \in U$, 

$$\det[D^2_{xy} c] \neq 0,$$

where $D^2_{xy} c$ is the matrix whose elements at the $i$th row and $j$th column is $\frac{\partial^2 c}{\partial x_i \partial y_j}$.

(A3w) For any $(x, y) \in U$, and $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$ such that

$$c_{ij}(x, y) = \frac{\partial^2 c(x, y)}{\partial x_i \partial y_j}, \quad \text{and} \quad [c^{i,j}] = \text{the inverse matrix of } [c_{i,j}],$$

where $c_{ij}(x, y) = \frac{\partial^2 c(x, y)}{\partial x_i \partial y_j}$, and $[c^{i,j}]$ is the inverse matrix of $[c_{i,j}]$.

**Remark.** The (A3w) condition is a degenerate form of the following (A3) condition:

(A3) There exists a constant $C_0 > 0$ such that for any $(x, y) \in U$, and $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$ such that

$$c_{ij}(x, y) = \frac{\partial^2 c(x, y)}{\partial x_i \partial y_j} c^{i,k} c^{j,l} \xi_i \xi_j \eta_k \eta_l \geq C_0|\xi|^2|\eta|^2. \quad (2.16)$$

The (A3) condition was introduced in [11] as a necessary criterion for local $C^2$ regularity of the Optimal Transportation Equation. However, for the current global regularity analysis, only the weaker (A3w) is needed to carry out the necessary estimates provided a technical barrier condition is met (see below). If the barrier condition is not met, then the full (A3) condition is needed to ascertain global $C^2$ regularity. This point will be discussed further in Section 5.3.

The regularity theory presented in [11, 15] for the optimal transportation problem gives strong indications that $A(\cdot, u, Du)$ and $T_u$ need to satisfy certain criterion in order to guarantee that globally smooth classical solutions exist in the current scenario. Indeed, much of the structural constructions from optimal transportation need to be applied to $A(\cdot, u, Du)$ and $T_u$ if any headway is to be made in proving a global regularity result. Specifically, the calculations in Section 5 require that both $A(\cdot, u, Du)$ and $T_u$ satisfy conditions laid down by Optimal Transportation.
theory in order to make an obliqueness estimate on the boundary condition, which plays a fundamental role in the subsequent global $C^2$ a priori estimate. Moreover, this structure from optimal transportation will also be of key importance in the $C^0$ estimate presented in Subsection 5.2.

Given the conditions on $c$, $T_\nu$ is defined as solving

$$Du = D_c c(\cdot, T_\nu), \quad \text{in } \Omega^-,$$  \hspace{1cm} (2.17)

with $A(\cdot, u, Du)$ defined as

$$A(\cdot, u, Du) := D^2_c c(\cdot, T_\nu), \quad \text{in } \Omega^-.$$  \hspace{1cm} (2.18)

Thus, the (A3w) condition in the current scenario can be represented as

$$D^2_{\mu\nu} A_{ij}(x, u, Du) \tilde{\xi}_i \tilde{\eta}_j = D_{\mu\nu} c_{ij}(x, y) \tilde{\xi}_i \tilde{\eta}_j \geq 0.$$  \hspace{1cm} (2.19)

The structure of $T_\nu$ and $A(\cdot, u, Du)$ embodied in (2.17) and (2.18) respectively, will be assumed along with $c$ satisfying (A1), (A2), and (A3w) for the rest of the paper, unless otherwise indicated.

**Remark.** Since admissible solutions of (1.1a) and (1.1b) must be $c$-convex, it is readily verified that Loeper’s counter-example in [9] is also valid in the case of (1.1a) and (1.1b). This—combined with the role played by the (A3w) condition in the $C^2$ estimate reduction to the boundary in Subsection 5.3—indicates that the (A3w) condition is both necessary and sufficient for the existence of globally smooth solutions solving (1.1a) and (1.1b).

In the forthcoming obliqueness estimate, the local invertability of the matrix $[D^2u - D^2_c c(\cdot, T_\nu)]$ will be of central importance. As this invertability is tantamount to

$$\det [D^2u - D^2_c c(\cdot, T_\nu)] > 0, \quad \text{in } \Omega^-,$$  \hspace{1cm} (2.20)

it is required that $\Gamma(f)$ be contained in the cone defined by (2.20). Subsequently, the only functions in (2.13) or (2.14) that have this property are $\sigma_{n,l}$ for $l < n$. In fact, by Remark 2.1(4) in Subsection 2.1, $\Gamma(\sigma_{n,l})$ is equivalent to the cone defined by (2.20), when considering (2.17) and (2.18) applied to (2.1).

Along with specifying the forms of $A(\cdot, uDu)$ and $T_\nu$, conditions need to be placed on $\Omega^-$ and $\Omega^+$ in order to make the forthcoming obliqueness and boundary $C^2$ estimates. Again, the necessity of these conditions mirrors that of the Optimal Transportation Equation. Specifically, it is required that $\Omega^-$ and $\Omega^+$ both be bounded and $C^4$, with $\Omega^-$, $\Omega^+$ being uniformly $c$-convex, $c^*$-convex (respectively) relative to each other. (See [15] for details on $c$-convexity of domains.) This notion of $\Omega^-$ being uniformly $c$-convex relative to $\Omega^+$ is explicitly stated as $\partial \Omega^- \subset C^3$ with a positive constant $\delta_0$ such that

$$[D_{i\gamma}(x) - c^{i\gamma} c_{\gamma j}(x, y) y_j(x)] \tau_i \tau_j \geq \delta_0^2, \quad \forall x \in \partial \Omega^-, y \in \Omega^+.$$  \hspace{1cm} (2.21)
where $\tau$ is a unit tangent vector of $\partial \Omega^-$ at $x$ with the outer unit normal $\gamma$. If one considers $c^*(x, y) = c(y, x)$, then (2.21) subsequently yields an analogous representation for $\Omega^+$ being $c^*$-convex relative to $\Omega^-$. This notion of $\Omega^-$, $\Omega^+$ being $c^*$-convex, $c^*$-convex (respectively) relative to one another plays a key role in the forthcoming obliqueness estimate in Subsection 5.1.

2.3. Conditions on Inhomogeneity

Next, conditions on the inhomogeneity of (2.1) will be presented. These conditions mirror those stated in [20] for classes of Hessian equations satisfying a natural boundary condition. In order to ensure uniform ellipticity of (2.1), it is required that

$$B(x, z) > 0, \quad x \in \Omega^-, \quad z \in \mathbb{R}. \quad (2.22)$$

In addition, it will also be assumed for all $x \in \Omega^-$ that

$$B(x, z) \to \infty, \quad \text{as } z \to \infty, \quad (2.23)$$

which will be necessary for making the $C^0$ solution estimate in Subsection 5.2. Lastly, in order to apply the method of continuity, unique solvability of the linearized problem is needed, which subsequently requires that

$$B_z(x, z) > 0, \quad x \in \Omega^-, \quad z \in \mathbb{R}. \quad (2.24)$$

This condition may be relaxed to the following:

$$B_z(x, z) \geq 0, \quad x \in \Omega^-, \quad z \in \mathbb{R}, \quad (2.24w)$$

via the application of the Leray–Schauder theorem (see [6, Theorem 11.6]) as done in [17], at the expense of uniqueness of an admissible solution.

Remarks.

(1) If (2.22) is relaxed to merely requiring that $B(\cdot, u)$ be non-negative rather than positive, the eigenvalues of the modified-Hessian matrix may not lie in a compact subset of $\Gamma(f)$, even if $D^2u$ is bounded. Thus, it is not possible to deduce the uniform ellipticity of (2.1) in this scenario.

(2) (2.24w) is also required in the boundary $C^2$ estimate.

(3) In general, $B(x, z)$ cannot be allowed to have a dependence on $Du$ for the case where $f = \sigma_n$. In [17, Section 6], Urbas constructs an example where a solution corresponding to $\sigma_n$ with $B$ having a $Du$ dependence has its second derivatives blowing up at the boundary in the two dimensional case; this example can be readily adapted to the current class of modified-Hessian equations with $f = \sigma_n$. Indeed, Urbas later applied this example to boundary value problems similar to the ones considered here in [18]. Theoretically, in the general case, such a dependence prohibits the existence of barriers which are required for boundary gradients estimates subsequently used in the forthcoming obliqueness and boundary $C^2$
estimates. This restriction is again reflected in [20] for a class of Hessian equations for the same reason. However, it is possible to have $B$ dependent on $Du$ if the minimum eigenvalue of the second fundamental form of $\mathcal{H}^-$ is large enough for all $x \in \mathcal{H}^-$. In this scenario, it is possible to construct barriers for the subsequent boundary gradient estimates; this is remarked upon in Subsection 5.1. In addition to this, $B$ also needs to be convex in its gradient argument in order for the $C^2$ estimate in Subsection 5.3 to be applied; this will also be remarked upon further in that subsection. As $B$ having any dependence on $Du$ introduces a dependence between $B$ and $\mathcal{H}^-$, it is not a valid criterion for a general result.

2.4. A Barrier Condition

If $A$ is only assumed to satisfy the weak (A3w) condition, then a technical barrier condition is also needed to make the global $C^2$ a priori estimates in Subsection 5.3. Indeed, if $A$ is such that the strong (A3) criterion holds, then no such extraneous condition will be needed; this is discussed in Remark 5.5 at the end of Subsection 5.3.

Taking $A$ as in (2.1), it will be assumed that there exists a function $\tilde{\phi} \in C^2(\mathcal{H}^-)$ satisfying

$$[D_{ij}\tilde{\phi}(x) - D_{ik}A_{ij}(x, z, p) \cdot D_{ik}\tilde{\phi}(x)]\xi_i\xi_j \geq \tilde{\delta} |\xi|^2$$

for some positive $\tilde{\delta} > 0$ and for all $\xi \in \mathbb{R}^n$, $(x, z, p) \in U \subset \Omega^- \times \mathbb{R} \times \mathbb{R}^n$, with $\text{Proj}_{\Omega^-}(U) = \Omega^-$. This condition places a relatively minor restriction on $\mathcal{H}^-$ when $A$ is assumed to satisfy the (A3w) condition. In [11], this condition was needed in order to make the global $C^2$ estimate for the Optimal Transportation Equation; but it was stated in that paper that this condition was removable via a duality argument. For the currently considered class of modified-Hessian equations, no such duality exists; thus, (2.25) must be kept as a separate condition in order to make the following calculations go through.

2.5. The Quotient Transportation Equation

With the preceding justification regarding the form of $F$, $A(\cdot, u, Du)$ and $T_u$, along with the hypotheses placed on $B(\cdot, u)$, attention will now be focused on the following set of boundary value problems:

$$
\left( \begin{array}{c}
\frac{S_u}{S_i} \\
\end{array} \right) \mathcal{H}^2 u - D^2 c(\cdot, T_u) = B(\cdot, u), \quad \text{in } \mathcal{H}^-, \\
T_u(\mathcal{H}^-) = \Omega^+,
$$

where $T_u$ is defined by (2.17); $c$ satisfies conditions (A1), (A2), and (A3w); $B(\cdot, u)$ satisfies (2.22)-(2.24); and both $\mathcal{H}^-$ and $\Omega^+$ are bounded and $C^4$ with $\mathcal{H}^-$, $\Omega^+$ $c$-convex, $c^*$-convex (respectively) relative to each other. (2.25a) and (2.25b) will be referred to as the Quotient Transportation Equation for the rest of this paper.
3. Main Results
We now are able to state the main results of this paper.

**Theorem 3.1.** Let $c$ be a cost-function satisfying hypotheses (A1) and (A2) with two bounded $C^4$ domains $\Omega^-, \Omega^+ \subset \mathbb{R}^n$ both uniformly $c$-convex, $c^*$-convex (respectively) with respect to each other, in addition to either

- the barrier condition stated in Subsection 2.4 holding for $c$ and $\Omega^-$, with $c$ satisfying the weak (A3w) condition or
- $c$ satisfying the strong (A3) condition.

If $B$ is a strictly positive function in $C^2(\overline{\Omega^-} \times \mathbb{R})$ satisfying (2.23) and (2.24w) with $T_u$ defined by (2.17), then any elliptic solution $u \in C^3(\overline{\Omega^-})$ of the second boundary value problem (2.25a)–(1.1b) satisfies the a priori estimate

$$|D^2 u| \leq C, \quad (3.1)$$

where $C$ depends on $c, B, \Omega^-, \Omega^+$ and $\sup_{\Omega^-} |u|$.

From the theory of linear elliptic equations, higher regularity automatically follows from better regularity on $c, \Omega^-, \Omega^+$ and $B$. For example, if $c, \Omega^-, \Omega^+$ and $B$ are all $C^\infty$, then one has that $u \in C^\infty(\overline{\Omega^-})$.

**Remark.** The dependence of the estimate (3.1) on $\sup_{\Omega^-} |u|$ may be removed if $B$ is independent of $u$.

As a consequence of Theorem 3.1, the method of continuity of continuity will be applied in Section 6 to prove the existence of classical solutions of (2.25a) and (2.25b).

**Theorem 3.2.** If the hypotheses in Theorem 3.1 hold, then there exists an elliptic solution $u \in C^3(\overline{\Omega^-})$ of the second boundary value problem (1.1a), (1.1b). If, in addition, (2.24) is satisfied, then the elliptic solution is unique.

The plan for the proof of Theorem 3.2 is as follows. First, some technical results and inequalities from other works will be reviewed in Section 4. From there, various solution estimates will be proven in Section 5. The first such estimate will be the obliqueness estimate proven in Subsection 5.1. This calculation will show that the boundary condition (1.1b) with $T_u$ defined by (2.17), where the cost-function is assumed to satisfy (A1) and (A2), is strictly oblique for functions $u$ where $[DT_u]$ is non-singular. In this obliqueness estimate, the convexity assumptions on both $\Omega^-$ and $\Omega^+$, play a critical role. This particular estimates differs from the one in [15], as calculations are made without the use of a dual formulation of (2.25a) and (2.25b). Instead, the property of $T_u$ being a local diffeomorphism is exploited to make the argument go through.

Following the obliqueness estimate, $C^0$ bounds for solutions to (1.1a) are then derived in Subsection 5.2. This calculation mimics the corresponding estimate in [20] but with a new analogy of a parabolic subsolution discovered in [15]. Without this newly-discovered function, it would not be possible to prove the $C^0$ estimate using the current methods.
In Subsection 5.3, it is proven that second derivatives of solutions of (2.1) can be estimated in terms of their boundary values, if \( A(\cdot, u, Du) \) obeys the (A3w) condition (2.19). Again, this estimation deviates from the corresponding calculation in [15], as the case when \( f = \sigma_{n, i} \) requires the use of specific technical inequalities in order to bound the second derivatives in terms of boundary values. This particular argument is carried out for general, symmetric \( A(\cdot, u, Du) \) satisfying the (A3w) condition and not necessarily having the specific form dictated by (2.18). Finally, in Subsection 5.4, the boundary estimate for second derivatives is proven in a similar manner to [15]; but a key lemma from [20] is needed to make the estimation go through. This directly leads to the global second derivative bounds stated in Theorem 3.1.

Following these \textit{a priori} estimates, the method of continuity is applied in Section 6 to ascertain the first part of the result in Theorem 3.1, regarding the existence of globally smooth solutions uniquely solving (2.25a) and (2.25b). The application of the method of continuity follows the procedure in [15]; but instead of simply integrating the equation to get solution bounds, the \( C^0 \) estimate from Subsection 5.2 is used instead. From there, the Leray–Schauder fixed point theorem is then applied to prove existence of globally smooth (albeit not necessarily unique) solutions when one only has \( B(\cdot, u) \geq 0 \) in \( \Omega^- \).

The final section discusses possible directions in research under which to proceed from the current set of results.

4. Preliminary Lemmas

In this section, some technical relations and inequalities will be recalled from other works that will be subsequently used in the \textit{a priori} estimates presented in the next section.

To gain the scope of generality of these lemmas, the following notation will be assumed. Given an arbitrary \( F : \mathbb{R}^{n \times n} \to \mathbb{R}, f : \mathbb{R}^n \to \mathbb{R} \) will be defined by

\[
    f(\lambda_1, \ldots, \lambda_n) := F[M],
\]

where \( M \) is an arbitrary \( n \times n \) matrix having \( \lambda_1, \ldots, \lambda_n \) as its eigenvalues. In addition to this, derivatives on \( F \) will be denoted by

\[
    F^{ij} := \frac{\partial F}{\partial M_{ij}} \quad \text{and} \quad F^{ijkl} := \frac{\partial^2 F}{\partial M_{ij} \partial M_{kl}},
\]

with derivatives on \( f \) correspondingly written as

\[
    f_i := f_{\lambda_i} = \frac{\partial f}{\partial \lambda_i} \quad \text{and} \quad f_{ij} := f_{\lambda_i \lambda_j} = \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j}.
\]

Lastly, in the literature, the \textit{trace} of the operator \( F \) is frequently denoted by

\[
    \mathcal{F} = \mathcal{F}(F) := F^{ii} = \sum_i f_i.
\]

With this notation and that from Section 2, it is now possible to state the supporting lemmas that will be used in the forthcoming \textit{a priori} estimates.
This first lemma states a technical relation that will be used in the calculations of the $C^2$ estimate reduction to the boundary. This lemma is proven in [5]; but this proof will not be recalled here as it does not aid in the understanding of the a priori estimates in Section 5.

Lemma 4.1. For any $n \times n$ symmetric matrix $\Xi = [\Xi_{ij}]$, one has that

$$F^{ijkl}\Xi_{ij}\Xi_{kl} = \sum_{i,j} \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \Xi_{ij} + \sum_{i \neq j} \frac{f_i - f_j}{\lambda_i - \lambda_j} \Xi_{ij}^2, \quad (4.1)$$

The second term on the right-hand side is non-positive if $f$ is concave, and is interpreted as a limit if $\lambda_i = \lambda_j$.

Remark. Lemma 4.1 was also used in [12] to make a $C^2$ a priori estimate similar to the one presented in Subsection 5.3.

The following lemma is due to Urbas (see [20]); and while the statement of this lemma will be used in the boundary $C^2$ estimate of Subsection 5.4, the proof itself contains some valuable technical relations that will subsequently be used in various locations within the forthcoming a priori estimates.

Lemma 4.2 ([20]). If $f = \sigma_{n,l}$ with $l \in \{1, \ldots, n - 1\}$, then there exists a positive constant $C(\epsilon)$ such that

$$\sum_i f_i \lambda_i^2 \leq (C(\epsilon) + \epsilon |\lambda|) \sum_i f_i, \quad \text{on } \Gamma_{\mu_1, \mu_2}(f), \quad (4.2)$$

for any $\epsilon > 0$ and $0 < \mu_1 \leq \mu_2$, with $C(\epsilon)$ depending only on $\mu_1, \mu_2$ and $\epsilon$.

Proof. Denoting

$$S_{k-1,i}(\lambda) := S_{k-1}(\lambda)|_{\lambda = 0} = \frac{\partial S_k(\lambda)}{\partial \lambda_i},$$

it is calculated that

$$f_i = \frac{1}{n - l} \left( \frac{S_n}{S_l} \right)^{\frac{1}{n-1}} \left( \frac{S_{n-1,i} - S_n S_{n-1,i}}{S_i^2} \right). \quad (4.3)$$

Next, it is readily observed that

$$\sum_i S_{k,i}(\lambda) = (n - k) S_k(\lambda), \quad (4.4)$$

for $k = 0, \ldots, n$. Summing (4.3) across $i$ and applying (4.4) yields

$$\sum_i f_i = \frac{1}{n - l} \left( \frac{S_n}{S_l} \right)^{\frac{1}{n-1}} \left( \frac{S_{n-1} S_l - (n - l + 1) S_n S_{n-1}}{S_i^2} \right). \quad (4.5)$$
Next, using the fact that
\[ S_k(\lambda) = S_{k-1,i}(\lambda)\lambda_i + S_{k,i}(\lambda) \quad \text{for each } i = 1, \ldots, n, \tag{4.6} \]
along with (4.4), it is deduced that
\[ \sum_i S_{k-1,i}(\lambda)\lambda_i = n \cdot S_k(\lambda) - \sum_i S_{k,i}(\lambda) = k \cdot S_k(\lambda). \tag{4.7} \]
On the other hand, (4.6) can also be used to show that
\[ \sum_i S_{k-1,i}(\lambda)\lambda_i^2 = \sum_i S_k(\lambda)\lambda_i - \sum_i S_{k,i}(\lambda)\lambda_i = S_1(\lambda)S_k(\lambda) - (k + 1)S_{k+1}(\lambda), \]
where \( S_{k+1}(\lambda) \) is defined to be zero if \( k = n \) and (4.7) has subsequently been used to produce the second equality. Using this with (4.4) and summing, it is next calculated that
\[ \sum_i f_i\lambda_i^2 = \frac{l + 1}{n - l} \left( \frac{S_n(\lambda)}{S_l(\lambda)} \right)^2 \frac{1}{n} \frac{S_{n+1}(\lambda)}{S_l(\lambda)}. \tag{4.8} \]
In the special case where \( l = n - 1 \), (4.8) reduces to
\[ \sum_i f_i\lambda_i^2 = n \left( \frac{S_n(\lambda)}{S_{n-1}(\lambda)} \right)^2 = nB^2, \]
which completes the proof for the case when \( l = n - 1 \).

To proceed further, the \textit{Newton inequality} is next recalled:
\[ \frac{S_k(\lambda)}{\binom{k}{\lambda}} \leq \frac{S_{k-1}(\lambda)}{\binom{k-1}{\lambda-1}} \leq \frac{S_{k-2}(\lambda)}{\binom{k-2}{\lambda-2}} \leq \cdots \leq \frac{S_0(\lambda)}{\binom{0}{\lambda}}, \]
which is valid for any \( 1 \leq l \leq k \leq n \) and any \( \lambda \in \Gamma_k \) (see [10, Section 2.15]). Taking \( k = n \), Newton’s inequality yields
\[ (n - l + 1)S_1(\lambda)S_{n-1}(\lambda) \leq \frac{l}{n}S_n(\lambda)S_{n-1}(\lambda); \]
and therefore, from (4.5) it is ascertained that
\[ \frac{1}{n} \left( \frac{S_n(\lambda)}{S_l(\lambda)} \right) \frac{1}{n} \frac{S_{n-1}(\lambda)}{S_{n-1}(\lambda)} \leq \sum_i f_i \leq \frac{1}{n - l} \left( \frac{S_n(\lambda)}{S_l(\lambda)} \right) \frac{1}{n} \frac{S_{n-1}(\lambda)}{S_{n-1}(\lambda)}. \tag{4.9} \]
Given (4.8), (4.9) and the fact that \( S_n(\lambda)/S_l(\lambda) \) is bounded between two positive constants, the following inequality now follows:
\[ C_0 \frac{S_{n+1}(\lambda)}{S_{n-1}(\lambda)} \leq \frac{\sum_i f_i\lambda_i^2}{\sum_i f_i} \leq C_1 \frac{S_{n+1}(\lambda)}{S_{n-1}(\lambda)}. \tag{4.10} \]
for some positive constants $C_1$ and $C_2$. The ratio $S_{n+1}(\lambda)/S_{n-1}(\lambda)$ is trivially bounded if $l = n - 2$, so the lemma holds in this case as well.

Now, the remaining cases are considered, which (in light of (4.10)) entail bounding the quantity $S_{n+1}(\lambda)/S_{n-1}(\lambda)$ from above. Without loss of generality, it is assumed that $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Since all $\lambda_i$ are positive, it is seen that

$$\prod_{i=1}^{k} \lambda_i \leq S_k(\lambda) \leq C_k \prod_{i=1}^{k} \lambda_i,$$

for $j = 1, \ldots, n$ and some positive constants $C_j$. Therefore, one now has the following:

$$\frac{S_{n+1}(\lambda)}{S_{n-1}(\lambda)} \leq C \frac{\prod_{i=1}^{n+1} \lambda_i}{\prod_{i=1}^{n-1} \lambda_i} \leq C \frac{S_1(\lambda)}{S_{n-1}(\lambda)} \lambda_{n+1} \lambda_n \leq C_2 \lambda_{n+1} \lambda_n. \quad (4.11)$$

Next, it is calculated that

$$C_0 \lambda_{n+1} \leq C_1 \prod_{i=n+1}^{n} \lambda_i \leq \frac{S_n(\lambda)}{S_{n-1}(\lambda)} \leq C_2;$$

and hence,

$$\lambda_n \leq C. \quad (4.12)$$

Now, an arbitrary $\epsilon > 0$ is chosen. If $\lambda_{n+1} \leq \epsilon \lambda_1$, then by (4.11) and (4.12)

$$\frac{S_{n+1}(\lambda)}{S_{n-1}(\lambda)} \leq C \lambda_{n+1} \leq C \epsilon \lambda_1. \quad (4.13)$$

If $\lambda_{n+1} > \epsilon \lambda_1$, two cases require consideration. First, if $\lambda_n \leq \epsilon$, then by (4.11), it is ascertained that

$$\frac{S_{n+1}(\lambda)}{S_{n-1}(\lambda)} \leq C \epsilon \lambda_{n+1} \leq C \epsilon \lambda_1, \quad (4.14)$$

while if $\lambda_n > \epsilon$, then

$$C_0 \epsilon^{n-l-1} \leq C_1 \prod_{i=n+1}^{n} \lambda_i \leq \frac{S_n(\lambda)}{S_{n-1}(\lambda)} \leq C_2.$$

Thus, one has the following relation:

$$\lambda_1 \leq \frac{C}{\epsilon^{n-l-1}}.$$

Since $S_n(\lambda)/S_{n-1}(\lambda)$ is assumed to be bounded between two positive constants, and $S_n/S_{n-1} = 0$ on $\partial \Gamma_n$, a positive lower bound is now implied:

$$C(\epsilon) \leq \lambda_n.$$
It then follows that
\[
\frac{S_{l+1}(\lambda)}{S_{n-1}(\lambda)} \leq C(\epsilon).
\] (4.15)

By applying (4.13), (4.14), and (4.15) to (4.10) and replacing \(\epsilon\) by \(\epsilon/C\) for a suitably large constant \(C\), it is finally derived that
\[
\sum f_i \lambda_i^2 \leq (C(\epsilon) + \epsilon|\lambda|) \sum f_i
\]
for any \(\epsilon > 0\) as required.

\[\square\]

**Remark.** The proof of Lemma 4.2 comes straight from [20]. As mentioned before, this particular exposition of proof has been recalled here, as it depicts key calculations and technical relations that will subsequently be of use in the forthcoming a priori estimates.

### 5. Solution Estimates

In this section, various a priori estimates to elliptic solutions of (1.1a), (1.1b) will be presented. These estimates will subsequently be used to prove (via the method of continuity) the existence of globally smooth solutions to the Quotient Transportation Equation.

#### 5.1. Obliqueness Estimate

In this section, it will be proven that the boundary condition (1.1b) implies a strict oblique boundary condition. This estimate will subsequently be used in the continuity estimate in Subsection 5.2, the boundary \(C^2\) estimate in Subsection 5.4, in addition to justifying the use of the results from [8] that yield \(C^{2,\gamma}\) estimates from the forthcoming \(C^2\) a priori bound.

To begin, a boundary condition of the form
\[
G(\cdot, u, Du) = 0, \quad \text{on } \partial \Omega^-,
\] (5.1)
for a second order partial differential equation in a domain \(\Omega^-\) is called oblique if
\[
G_p \cdot \gamma > 0
\]
for all \((x, z, p) \in \partial \Omega^- \times \mathbb{R} \times \mathbb{R}^n\), where \(\gamma\) denotes the unit outer normal to \(\partial \Omega^-\).

Next, it is assumed that \(\phi^-\) and \(\phi^+\) are \(C^2\) defining functions for \(\Omega^-\) and \(\Omega^+\) respectively; with \(\phi^-, \phi^+ < 0\) near \(\partial \Omega^-\), \(\partial \Omega^+\) respectively; \(\phi^- = 0\) on \(\partial \Omega^-\), \(\phi^+ = 0\) on \(\partial \Omega^+\); and \(\nabla \phi^-, \nabla \phi^+ \neq 0\) near \(\partial \Omega^-, \partial \Omega^+\) respectively. If \(u \in C^2(\Omega^-)\) is an elliptic solution of the second boundary value problem (2.25a) and (2.25b), then the following relations hold:
\[
\phi^+ \circ T_u = 0 \quad \text{on } \partial \Omega^-, \quad \phi^+ \circ T_u < 0 \quad \text{near } \partial \Omega^-.
\]
By tangential differentiation, it is ascertained that
\[ \phi^+_i (D_j T^i) \tau_j = 0, \]  
(5.2)

for all unit tangent vectors \( \tau \). Note that the subscript on \( T_u \) has been dropped without loss of clarity. From (5.2), it follows that
\[ \phi^+_i (D_j T^i) = \chi \gamma, \]

for some \( \chi \geq 0 \) and \( \gamma \) is again an outer, unit normal of \( \partial \Omega^- \). Consequently, one has that
\[ \phi^+_k c^{k,i} w_{ij} = \chi \gamma, \]  
(5.3)

where

\[ w_{ij} = u_{ij} - c_{ij}. \]

At this point, it is observed that \( \chi > 0 \) on \( \partial \Omega^- \) since \( |\nabla \phi^+| \neq 0 \) on \( \partial \Omega^- \) and \( \det DT \neq 0 \), which is subsequently implied by the ellipticity of \( u \) and the (A2) condition. Moreover, since \( u \) is assumed to be an elliptic solution to (1.1a), it is observed that
\[ \phi^+_k c^{k,i} = \chi u^{k,i} \gamma_k, \]  
(5.4)

where \( [w^{ij}] \) denotes the inverse matrix of \( [w_{ij}] \). Upon writing
\[ G(x, p) := \phi^+ \circ Y(x, p), \]  
(5.5)

one has from the (A1) condition that
\[ \beta_k := G_{\rho_k} (\cdot, Du) = \phi^+_k D_{\rho_k} Y^i = \phi^+_k c^{k,i} = \chi u^{k,i} \gamma_k, \]
which subsequently indicates that
\[ \beta \cdot \gamma = \chi w^{k,i} \gamma_k \gamma_j > 0 \]  
(5.6)

on \( \partial \Omega^- \). On the other hand, from (5.4), it is calculated that
\[ \phi^+_k c^{k,i} w_{ijk} \phi^+_l c^{l,j} = \chi \phi^+_k c^{k,i} \gamma_j = \chi (\beta \cdot \gamma). \]  
(5.7)

Eliminating \( \chi \) from (5.6) and (5.7), yields
\[ (\beta \cdot \gamma)^2 = (w^{k,i} \gamma_k)(w_{ijkl} c^{k} c^{l} \phi^+_k \phi^+_l). \]  
(5.8)

(5.8) is referred to as a formula of Urbas type, as it was proven in [18] for the Monge–Ampère equation with a natural boundary condition.

**Remark.** In the above calculations, the fact that \( w_{ij} \) is invertible has been used; and this is a consequence of \( u \) being an elliptic solution of (1.1a). It is this invertability
that requires the equations under consideration to have an Optimal Transportation structure combined with a structure that automatically implies that elliptic solutions are $c$-convex (see Appendix A for details).

Now, $\beta \cdot \gamma$ needs to be estimated from below. This calculation mimics the one in [15, 18] for the Monge–Ampère equation, but with a modification to avoid the use of a dual formulation to (1.1a). These calculations start with estimating the double normal derivatives of a solution to a Dirichlet problem related to (1.1a); this follows the key idea from [14]. Specifically, a point $x_0 \in \partial \Omega^-$ is fixed, where $\beta \cdot \gamma$ is minimized for an elliptic solution $u \in C^3(\Omega^-)$. From there a comparison argument is used to estimate $\gamma \cdot D(\beta \cdot \gamma)$ from above. Given that $\beta \cdot \gamma$ does not have any assumed concavity criterion in the gradient argument, the quantity itself needs to be modified so that a workable differential inequality can be derived. Thus, the following auxiliary function is defined:

$$v := \beta \cdot \gamma - \kappa(\phi^+ \circ T),$$

with a point $x_0$ on $\partial \Omega^-$ fixed, where $\beta \cdot \gamma$ is minimized for an elliptic solution $u \in C^3(\Omega^-)$ for sufficiently large $\kappa$, with the function $\phi^+$ now chosen so that

$$[D_{ij}(\phi^+ \circ T) - c^{k,j}c_{i,k}(\cdot, T) \cdot D_k(\phi^+ \circ T)]\xi_i \xi_j \geq \delta^+_0 |\xi|^2$$

(5.9)

near $\partial \Omega^-$, $\forall \xi \in \mathbb{R}^n$ and some positive constant $\delta^+_0$. Inequality (5.9) is possible via the uniform $c^*$-convexity of $\Omega^+$ with respect to $\Omega^-$ and taking $\phi^+$ to be of the form

$$\phi^+ = a(d^+)^2 - bd^+,$$

(5.10)

where

$$d^+(y_0) = \inf_{y \in \partial \Omega^+} |y - y_0|$$

(5.11)

with $a$ and $b$ taken to be positive constants, [6, 15].

**Remark.** (5.10) represents only one particular example of $\phi^+$ that may be chosen that satisfy (5.11). Indeed, there are several alternative barrier functions listed for convex domains in [6, Section 14.2], (5.10) is a generalized construction of a barrier, in that only derivatives up to second order of a barrier work into the subsequent boundary gradient estimates. As a bound on the gradient is only needed at its maximum point on the boundary, only a neighborhood of such a point is considered. In this neighborhood, a diffeomorphic mapping can be applied to straighten the boundary, to see that (5.10) indeed represents a second-order Taylor expansion about the extremal point $x_0$, along the inner-normal of the boundary (see [3] for examples of such calculations).

For clarity in the forthcoming calculations, the following definition is made:

$$H(x, p) := G_p(x, p) \cdot \gamma(x) - \kappa G(x, p),$$

(5.12)
where \( G(x, p) \) is defined by (5.5); that is,
\[
v(x) = H(x, Du(x)).
\]
Calculating, it is seen that
\[
\begin{align*}
D_1v &= D_1H + (D_{p_1}H)u_{k_1} \\
D_{ij}v &= D_{ij}H + (D_{p_{i_1}}H)u_{j_{k_1}} + (D_{p_{j_1}}H)u_{i_{k_1}} + (D_{p_{k_1}}H)u_{i_{j_1}} + (D_{p_{k_1}}H)u_{i_{j_1}},
\end{align*}
\]
Next, taking \( F := \sigma_{n,t} \), equation (1.1a) in the general form (2.1) is differentiated to ascertain that
\[
F^{ij}[D_ku_{ij} - D_kA_{ij} - (D_{p_i}A_{ij})u_{ik}] = B_k + B_zD_ku.
\]
Introducing the linearized operator \( \mathcal{L} \):
\[
\mathcal{L}v = F^{ij}[D_{ij}v - (D_{p_i}A_{ij})D_kv];
\]
and using (5.13) and (5.14) with some simple estimation, it is calculated that
\[
\mathcal{L}v = F^{ij}[D_kH + 2(D_{p_i}H)u_{j_{k_1}} + (D_{p_i}H)u_{j_{k_1}}u_{j_{k_1}} - (D_{p_i}A_{ij})D_kH]
+ (D_{p_i}H)(B_{n_i} + B_zD_ku + F^{ij}D_kA_{ij})
\leq F^{ij}D_kH + F^{ij}u_{j_{k_1}}[D_{p_i}H + \delta_{k_1}] + F^{ij}[(D_{p_i}H)(D_{j_{p_i}}H)\delta_{k_1} - (D_{p_i}A_{ij})D_kH]
+ (D_{p_i}H)(B_{n_i} + B_zD_ku + F^{ij}D_kA_{ij})
\leq F^{ij}u_{j_{k_1}}[D_{p_i}H + \delta_{k_1}] + C(F^{ii} + 1).
\]
In the above estimation, the gradient bound on the solution \( u \) (which is implied by (2.17); the boundedness of \( \Omega^+ \), \( \Omega^- \); and the continuity of \( c \)) has been used. Now, with an elementary calculation and condition (A1) applied to (5.5), one now has the following:
\[
D_{p,p_j}G = D_{p_i}(\phi^+_k \phi^+_l \phi^+_p) = \phi^+_k \phi^+_l \phi^+_p - \phi^+_k \phi^+_l \phi^+_p
= \phi^+_k \phi^+_l \phi^+_p - \phi^+_k \phi^+_l \phi^+_p,
\]
Utilizing the criterion for \( \phi^+_p \) in (5.9), it is subsequently calculated that
\[
[D_{p,p_j}G(x, Du)]_{\xi_i \xi_j} \geq \delta^+_k \sum_{i} |c^{i,j} \xi_j|^2 \geq \delta^+_i \xi_i \xi_j
\]
for a further positive constant \( \delta^+_k \). Thus, by choosing \( \kappa \) sufficiently large, one has
\[
[D_{p,p_j}H(x, Du)]_{\xi_i \xi_j} \leq -\frac{1}{2}\kappa |\xi|^2
\]
holding true near \( \partial \Omega^- \). Substituting this into 5.15 yields
\[
\mathcal{L}v \leq -\frac{1}{4}\kappa \delta^+_i F^{ij}u_{i_\beta}u_{i_\beta} + C(F^{ii} + 1)
\]
where \( C \) is a constant depending on \( c, B, \Omega^-, \Omega^+ \) and \( \kappa \).
Before proceeding, a technicality in the above argument must be addressed. The calculations that lead to the inequality in (5.16) depend on the explicit structure of \( \phi^+ \). Indeed, unless \( \phi^+ \) extends to all of \( \Omega^+ \) such that (5.9) holds for all \( T \in \Omega^+ \), there is no control near \( \partial \Omega^- \) to validate (5.16) and thus (5.17). To get around this, one can simply modify the definition of \( G \) in (5.12) by a function satisfying (5.16) in all of \( \Omega^- \) and agreeing with (5.5) near \( \partial \Omega^- \). One example of such a \( G \) (suggested in [15]) is given by:

\[
G(x, p) := \rho_k \ast (\max\{\phi^+ \circ Y(x, p), C_0(|p|^2 - C_1^2)\}),
\]

where \( C_0 \) and \( C_1 \) are positive constants with \( C_0 \) sufficiently small and \( C_1 > \max |Du| \) and with \( \epsilon \) being sufficiently small with \( \rho_k \) being a standard mollification.

A suitable barrier is now provided by the uniform \( c \)-convexity of \( \Omega^- \) which implies analogously to the case of \( \Omega^+ \) above, that there exists a defining function \( \phi^- \) for \( \Omega^- \) satisfying

\[
[D_j \phi^- - c^{ij} c_{ij}(T) D_k \phi^-] \xi_j \geq \delta_0 |\xi|^2,
\]

near \( \partial \Omega^- \). Specifically, \( \phi^- \) is defined in a similar manner to \( \phi^+ \) in (5.10):

\[
\phi^- := a(d^-)^2 - bd^-.
\]

for some positive constants \( a \) and \( b \). Given the elementary fact that \( D(d^-) = -\gamma \), the definition above can be combined with (2.21) and the definition of \( \mathcal{L} \) in (5.14) to yield

\[
\mathcal{L} \phi^- \geq F_{ii} \delta_0 (b - 2ad^-) + F_{ij} \frac{2a}{(b - 2ad^-)^2} D_i \phi^- D_j \phi^- , \tag{5.20}
\]

in a sufficiently small neighborhood of \( \partial \Omega^- \); that is,

\[
\mathcal{L} \phi^- (x_0) \geq F_{ii} \delta_0 b + F_{ij} 2a \gamma_i \gamma_j , \tag{5.21}
\]

Now the fact that \( F_{ij} \) is bounded from below on \( \Gamma_0, \mu \), is recalled from (2.5) and Remark 2.1(3). With this, (5.17) and (5.21) indicate that one can pick \( a \) and \( b \) in (5.19) such that

\[
\mathcal{L} \phi^- \geq \mathcal{L} v ,
\]

in a small enough, fixed neighborhood of \( \partial \Omega^- \). From this, it is inferred by the standard barrier argument (see [6, Chapter 14]) that

\[
\gamma \cdot Dv(x_0) \leq C ,
\]

where again \( C \) is a constant depending on \( c, \Omega^-, \Omega^+ \) and \( B \). From (5.18) and since \( x_0 \) is a minimum point of \( v \) on \( \partial \Omega^- \), it can be written that

\[
Dv(x_0) = \tau \gamma(x_0) \tag{5.22}
\]
where $\tau \leq C$. Next, it is calculated that
\[
D_i(\beta \cdot \gamma) = D_i[\phi_i^+ c^{i,j}] = \phi_i^+ D_i(T) c^{i,j} \gamma_j + \phi_i^+ c^{i,j} D_i \gamma_j = \phi_i^+ c^{i,j} (\phi_i^+ c^{i,j} + c^{i,j} D_i T).
\]

Multiplying by $\phi_i^+ c^{i,j}$ and summing over $i$, one subsequently has the inequality
\[
\phi_i^+ c^{i,j} D_j(\beta \cdot \gamma) = \phi_i^+ c^{i,j} (D_j \gamma_j - c^{i,j} c_{ij,j}) + (\phi_i^+ c^{i,j} c_{kl} c^{k,l}) D_j T.
\]

By virtue of the uniform $c$-convexity of $\Omega^-$, the $c^*$-convexity of $\Omega^+$ and (5.4). From (5.12) and (5.22), it is derived that at $x_0$ that the following relation holds:
\[
-w_{kl} c^{i,k} c^{j,l} \phi_i^+ \phi_j^+ \leq C(\beta \cdot \gamma) - \tau_0,
\]

for positive constants $C$ and $\tau_0$. Hence if $\beta \cdot \gamma \leq \tau_0/2C$, one has the lower bound
\[
w_{kl} c^{i,k} c^{j,l} \phi_i^+ \phi_j^+ \geq \frac{\tau_0}{2k}.
\]

At this point in the argument, a Legendre transform (or in the case of Optimal Transportation a $c$-transform) would usually be invoked to derive a dual equation to (1.1a), with the intention to bound $w^{i,j} \gamma_i \gamma_j$ at $x_0$ from below. In the current situation, such a dual formalism cannot be explicitly constructed; indeed, the transformed Quotient Transportation Equation would not have the same form as (2.25a) and (2.25b). Instead of doing this, (5.22) is manipulated another way via the fact that $T_u$ is a local diffeomorphism (in fact, it’s a global diffeomorphism).

At $x_0$, (5.22) indicates that
\[
w^{i,j} \gamma_i D_j v = \tau w^{i,j} \gamma_i \gamma_j \leq C w^{i,j} \gamma_i \gamma_j.
\]

Using (5.3) and following the same vein as the calculation done in (5.23), one calculates that
\[
w^{i,j} \gamma_i D_j(\beta \cdot \gamma) = \chi^{-1} \phi_i^+ c^{i,j} D_i(\phi_i^+ c^{i,j}) = \chi^{-1} \phi_i^+ c^{i,j} c^{i,j} (D_i \gamma_j - c^{i,j} c_{ij,j}) + \chi^{-1} \phi_i^+ c^{i,j} c_{kl} c^{k,l} (\phi_i^+ c^{i,j} c_{kl} c^{k,l} c_{kl} c^{k,l} c_{kl} c_{kl} c_{kl} c_{kl} c_{kl}) \geq \frac{\delta_0}{2} \sum_j |\phi_i^+ c^{i,j} c_{ij,j}|^2 + \frac{\delta_0}{2} \sum_j |\gamma_i c^{i,j} c_{ij,j}|^2 \geq \frac{\delta_0}{2} \sum_j |\gamma_i c^{i,j} c_{ij,j}|^2.
\]
where \( \delta^- \) and \( \delta^+ \) are the constants associated with the uniform \( c \)-convexity of \( \Omega^- \) and the \( c^* \)-convexity of \( \Omega^+ \), respectively.

Combining (5.25) with (5.26) and using the definition for \( v \) along with (5.7), it is ascertained that

\[
\frac{\partial^i}{\partial x_i} \frac{\partial^j}{\partial x_j} D_j v \geq \delta^+ \sum_l |\gamma_k e^{i,j,l}|^2 - \chi^{-1} \kappa \phi \phi^*_j c^{i,j,l} w_{kl}
\]

at \( x_0 \). Setting \( \tau^+_0 := \delta^+_0 \sum_k |\gamma_k e^{i,j,k}|^2 \) at \( x_0 \), the above reduces to

\[
\frac{\partial^i}{\partial x_i} \frac{\partial^j}{\partial x_j} \geq \frac{\tau^+_0}{C} - \frac{\kappa}{C} (\beta \cdot \gamma).
\]

Thus, if \( (\beta \cdot \gamma) \leq \frac{\tau^+_0}{2C} \), then it follows that

\[
\frac{\partial^i}{\partial x_i} \frac{\partial^j}{\partial x_j} \geq \frac{\tau^+_0}{2C}.
\]  

(5.27)

**Remark.** In order to estimate \( \phi^*_i \phi^*_j c^{i,j,k} c^{i,j,l} w_{kl} \), only the positivity of the second term of the right-hand side of (5.23) (due to \( c^* \)-convexity) was used. That is, the estimate for \( \phi^*_i \phi^*_j c^{i,j,k} c^{i,j,l} w_{kl} \) only depends on the value of \( \delta^+_0 \) and not-necessarily uniform \( c^* \)-convexity of \( \Omega^+ \). On the other hand, the estimate for \( \frac{\partial^i}{\partial x_i} \frac{\partial^j}{\partial x_j} \) depends on \( \delta^-_0 \) and the not-necessarily uniform \( c \)-convexity of \( \Omega^- \).

Combining (5.8), (5.24), and (5.27), the uniform obliqueness estimate is thus derived:

\[
G_{\rho \cdot \gamma} > \delta,
\]  

(5.28)

for some positive constant \( \delta \) which depends on \( \Omega^- \), \( \Omega^+ \), \( c \) and \( B \). This estimate is now stated formally in the following result.

**Theorem 5.1.** Let \( c \in C^3(\mathbb{R} \times \mathbb{R}^n) \) be a cost-function satisfying conditions (A1) and (A2) with respect to bounded \( C^3 \) domains \( \Omega^-, \Omega^+ \in \mathbb{R}^n \), which are respectively, uniformly \( c \)-convex and \( c^* \)-convex with respect to each other. In this case, if \( B(x, z) \) is a positive, bounded function in \( C^3(\Theta^- \times \mathbb{R}^n) \) and \( T_\rho \) a mapping satisfying (2.17), then any elliptic solution \( u \in C^3(\Theta^-) \) of the second boundary value problem (2.25a) and (2.25b) satisfies the obliqueness estimate (5.28).

**Remark 5.2.**

(1) The key difference in this obliqueness proof is that a dual formulation to (1.1a) is not used anywhere in the estimate. Instead, the fact that \( T_\rho \) is a local diffeomorphism is exploited to make the estimate (5.24) and (5.27) go through.

(2) As mentioned earlier, the restriction that \( B \) be independent of the gradient in the general case is unattractive. However, a similar criterion was required by Urbas in [20] in the case where equations involved the operator \( \sigma_{n,l} \) applied to
Hessian matrices. The reason for this restriction is due to (5.20) holding for general barriers and the fact that $F^u$ can remain bounded even as $|D^2u| \to \infty$. This observation is clear, recalling the relation (4.5) derived in Section 4 and applying MacLauren’s Inequality.

(3) This estimate can be derived in special cases where $B$ has a $Du$ dependence provided that the second fundamental form of $\partial \Omega^-$ has a large enough minimum eigenvalue. Specifically, it is required that

$$2F^u [G_{i,p_i,p_i}^\gamma_k - \kappa G_i^1] + (G_{p_i,p_i}^\gamma_k - \kappa G_{p_i})B_{p_i} \leq -2F^u G_{p_i,p_i} D_i \gamma_k. \quad (5.29)$$

This can be realized by examining (5.15) and noting the appearance of a $H_{p_i}B_{p_i}u_{rs}$ term when $B$ has a $Du$ dependence. If (5.29) holds, this new term can be removed immediately from the calculation; and thus, it would not affect the subsequent barrier construction. As (5.29) implies a dependence between $B$ and $\Omega^-$, this situation cannot be assumed for the general case.

### 5.2. $C^0$ Estimate

In [11], $C^0$ solution estimates are naturally handled as the Optimal Transportation problem corresponds to a Monge–Ampère equation, which subsequently correlates to the determinant of a Jacobian corresponding to a measure-preserving coordinate transform. Thus, simply integrating the equation will lead directly to uniform integral bounds that, in turn, directly lead to $C^0$ estimates of the solution. In the case of the Quotient Transportation Equation, there are no such simplification given that $\sigma_{n,l}$ has no direct correspondence to a Jacobian of a non-singular coordinate transform. Instead, the estimate set forth by Urbas in [20] will be adapted to the current situation, with the use of a special auxiliary function first introduced in [15].

#### 5.2.1. Supremum Solution Bound.

First, an upper-bound for an elliptic solution $u$ of (2.25a) and (2.25b) will be proven. The argument for this part of the estimate is the same as the one [20] and needs a trivial modification for the current scenario. Recalling (2.10), the concavity of $f$ implies

$$f(\hat{\lambda}) \leq f(1, \ldots, 1) + \sum f_i(1, \ldots, 1)(\hat{\lambda}_i - 1);$$

and thus,

$$F[u] \leq C_1 + \sigma \Delta \sigma(u - c). \quad (5.30)$$

where $\sigma = \sum_i f_i(1, \ldots, 1)$. Integrating (5.30) and applying the divergence theorem, it is readily calculated that

$$\int_{\partial \Omega^-} B(x, u) dx \leq C_1 \text{Vol}(\Omega^-) + \sigma \int_{\partial \Omega^-} \Delta \sigma(u - c) dx \leq C_2 \text{Vol}(\Omega^-) + \sigma \int_{\partial \Omega^-} (\gamma \cdot Du) dx < C_3,$$

where $\gamma$ is the unit outer-normal vectorfield relative to $\partial \Omega^-$. The last inequality comes from the gradient solution bound implied by (2.17); the continuity of $c$; and
the boundedness of both $\Omega^-$ and $\Omega^+$. From (5.31) and (2.23), it is clear that $u$ is bounded from above somewhere in $\Omega^-$; thus, by the aforementioned bound on $Du$, it is ascertained that

$$\sup_{\Omega^-} u < C,$$

where $C$ depends on $B$, $c$, $\Omega^-$, and $\Omega^+$.

5.2.2. Infimum Solution Bound. Before proceeding to the infimum estimate, the following definition is needed.

**Definition 5.3.** A $c$-convex function $\phi$ on an arbitrary domain $U$ is said to be **uniformly $c$-convex** on $U$, if $\phi$ satisfies the following inequality

$$[\phi_{ij} - c_{ij}(\cdot, T_u)] > 0, \quad \text{in } U, \quad (5.32)$$

where $T_u$ is defined by (2.17).

**Remark.** It is clear that in the case where $f := \sigma_{a,i}$, (5.32) corresponds to the ellipticity criterion (2.8).

To get the lower bound for $u$, a specific type of auxiliary function $u_0(x)$ is considered, which is uniformly $c$-convex. This auxiliary function was first used as the basis for an alternate proof of Lemma 5.1 in [15]. Let $y_0$ be a point in $\Omega^+$ and $u_0$ be the $c^*$-transform of the function

$$\psi(y) = -\sqrt{r^2 - |y - y_0|^2},$$

given by

$$u_0(x) = \sup_{y \in B_r(y_0)} \{c(x, y) - \psi(y)\}, \quad (5.33)$$

for sufficiently small $r > 0$. $u_0$ is a locally uniformly $c$-convex function defined in some ball $B_R(0)$, with $R \to \infty$ as $r \to 0$; and the image of its $c$-normal mapping satisfies

$$T_{u_0}(\Omega^-) \subset B_r(y_0),$$

where $T_{u_0}$ is a diffeomorphism between $\Omega^-$ and $T_{u_0}(\Omega^-)$. The reader is referred to [15, 16] for the derivation and discussion of these properties of $u_0$. With the above construction, it is clear that for given $x_0 \in \Omega^-$ and $y_0 \in \Omega^+$, $r > 0$ can be fixed small enough so that one has $u_0$ is uniformly $c$-convex on $\Omega^- \Subset B_R(0)$ and

$$T_{u_0}(\Omega^-) \subset B_r(y_0) \subset \Omega^+. \quad (5.34)$$
Since, \( B_R(0) \subseteq \Omega^- \) (by the smoothness of both \( u_0 \) and \( c \) combined with (5.32)), it is clear that

\[
\left( \frac{S_n}{S_t} \right) \left[ D^2 u_0 - D^2 c(\cdot, T_\alpha) \right] \geq C(r), \quad \text{in } \Omega^-,
\]

for some positive constant \( C(r) \). Next, it is supposed that

\[
B(\cdot, u) < C(r), \quad \text{in } \Omega^-.
\]

From this, one has that

\[
\left( \frac{S_n}{S_t} \right) \left[ D^2 u_0 - D^2 c(\cdot, T_\alpha) \right] > B(\cdot, u) = \left( \frac{S_n}{S_t} \right) \left[ D^2 u - D^2 c(\cdot, T_\alpha) \right],
\]

so \( u_0 - u \) obtains its maximum on \( \partial \Omega^- \); this maximal point is denoted \( x_0 \). With that, it is calculated that

\[
D^- (u_0 - u)(x_0) \geq 0 \quad \text{and} \quad \delta^- (u_0 - u)(x_0) = 0,
\]

where \( \gamma^- \) is the unit normal vector of \( \Omega^- \) at \( x_0 \) and \( \delta^- \) denotes the tangential part of the gradient; that is, \( \delta^- := D - \gamma (\gamma^- \cdot D) \). By a suitable rotation of coordinate axes, it is assumed that \( \gamma^- = e_1 \) at \( x_0 \); so one now has the following:

\[
D_1 u_0(x_0) \geq D_1 u(x_0),
\]

\[
D_x u_0(x_0) = D_x u(x_0), \quad \text{for } x = 2, \ldots, n.
\]

Using the definition of \( T_\alpha \), explicitly stated in (2.17), the above is seen to be equivalent to

\[
c_1(x_0, T_\alpha(x_0)) \geq c_1(x_0, T_\alpha(x_0)),
\]

\[
c_x(x_0, T_\alpha(x_0)) = c_x(x_0, T_\alpha(x_0)), \quad \text{for } x = 2, \ldots, n.
\]

From here, it is recalled that \( [DT_\alpha] > 0 \), as \( u \) is an elliptic solution to (2.25a) and (2.25b) with \( c \) satisfying condition (A2). Using this fact and the obliqueness estimate (5.28), it is seen that the pull-back of \( \gamma^+ \), the unit normal vector \( \Omega^+ \) at \( T_\alpha(x_0) \), satisfies

\[
0 < \gamma_k^{+} T_k^l \gamma_l^- = \gamma_k^{+} w_{ij} e^{i+j} \gamma_i^-.
\]

Since \( w_{ij} \) is positive definite, as \( u \) is an elliptic solution of (2.25a) and (2.25b) with \( [DT_\alpha] > 0 \), (5.38) along with \( e^{i+j} \) being diagonalizable, as it is symmetric in its indices, implies that

\[
c_{1,1}(x_0, T_\alpha(x_0)) > 0,
\]

with our coordinates again labeled such that \( \gamma^- = e_1 \). Moreover, (5.38) implies that \( p_0 = T_\alpha(x_0) \) must lie in the set \( \partial \Omega^+ := \{ p \in \partial \Omega^+ : \gamma^+_i(p) \geq 0 \} \). Thus, the \( \gamma^- \)-gradient
of $c$, at $(x_0, T_u(x_0))$ has a non-trivial component in the normal direction of $\partial \Omega^-$ at $x_0$. This combined with (5.37) subsequently implies that

\[(T_m(x_0) - T_u(x_0)) \cdot \gamma^- > 0.\]  

(5.40)

However, by taking $c^*(x, y) = c(x, y)$ in (2.21) and pulling back by the mappings $c(x, \cdot)$, it is seen that $\Omega^-$ being $c^*$-convex relative to $\Omega^+$ implies that the image of $c(x, \Omega^+)$ is uniformly convex with respect to $x \in \Omega^-$ [15]. Subsequently, combining this observation with (5.37) and (5.40), thus implies that $T_m(x_0)$ lies outside of $\Omega^+$, which contradicts (5.34). Therefore, (5.35) is false, and so it must be that $B(x, u) > \sigma(r)$ somewhere in $\Omega^-$. Thus, the bound on the gradient of $u$ implies a lower bound on the solution:

\[C < \inf_{\Omega^-} u.\]

With this, the following result is thus proven:

**Theorem 5.4.** Let $c \in C^3(\mathbb{R} \times \mathbb{R}^n)$ be a cost-function satisfying conditions (A1) and (A2) with respect to bounded $C^3$ domains $\Omega^-, \Omega^+ \in \mathbb{R}^n$, which are respectively, uniformly $c$-convex, $c^*$-convex with respect to each other. In this case, if $B(x, z)$ is a positive function in $C^1(\overline{\Omega^-} \times \mathbb{R}^n)$ satisfying (2.23) and $T_u$ a mapping satisfying (2.17), then any elliptic solution $u \in C^3(\overline{\Omega^-})$ of the second boundary value problem (2.25a) and (2.25b) satisfies the solution bound:

\[\sup_{\Omega^-} |u| < C,\]

where $C$ depends on $c$, $B$, $\Omega^-$ and $\Omega^+$. **Remarks.**

1. In the infimum estimate the obliqueness estimate was used, which subsequently utilized the fact that $\sum_i f_i$ is bounded away from zero on $\Gamma_{0,\mu_2}$. It is important to note that the obliqueness estimate does not depend on $\mu_1$. Obviously, if this were the case then the infimum estimate above would fall victim to circular logical argument.

2. Applying (5.41) to the condition (2.23), indicates that $B$ is bounded from above and away from zero. Thus, the forthcoming calculations pertaining to (2.25a) and (2.25b) will be carried out on $\Gamma_{\mu_1,\mu_2}(\sigma_{n,1})$, where $0 < \mu_1 \leq \mu_2$ without loss of generality.

**5.3. Global Second Derivative Estimates**

In this section, global bounds for second derivatives of elliptic solutions of equation (1.1a) will be shown to be estimated in terms of their boundary values. The following arguments mimic those presented in [11, 15], which are subsequent modification of the arguments presented in [6, Section 17.6].
For this estimate, it suffices to consider the more general form of the Quotient Transportation Equation depicted in (2.1), under the assumption that the matrix valued function \( A \in C^4(\Omega^- \times \mathbb{R} \times \mathbb{R}^n) \) satisfies condition (A3w); that is
\[
D_{p, q, p, q}^2 A_{ij}(x, z, p) \xi_i \xi_j (\eta_1) \geq 0,
\]
for all \((x, z, p) \in \Omega^- \times \mathbb{R} \times \mathbb{R}^n, \xi, \eta \in \mathbb{R}^n \) and \( \xi \perp \eta \). It is also assumed that is \( A \) is symmetric, hence diagonalizable. Lastly, the barrier condition from Subsection 2.4 is also assumed to hold for \( A \). This condition is recalled here for convenience: it is assumed that there exists a function \( \phi \in C^2(\Omega^-) \) satisfying
\[
[D_{ij} \tilde{\phi}(x) - D_{p, q} A_{ij}(x, z, p) \cdot D_{k} \tilde{\phi}(x)] \xi_i \xi_j \geq \tilde{\delta} |\xi|^2,
\]
for some positive \( \tilde{\delta} > 0 \) and for all \( \xi \in \mathbb{R}^n \), \((x, z, p) \in U \subset \Omega^- \times \mathbb{R} \times \mathbb{R}^n \), with \( \text{proj}_{\Omega^-(U)} = \Omega^- \). Remark 5.5, at the end of this section, will discuss how this barrier condition can be removed if \( A \) satisfies the strong (A3) criterion.

To begin, \( u \in C^4(\Omega^-) \) is assumed to be an elliptic solution of (2.1) with \((x, u(x), Du(x)) \in U \) for \( x \in \Omega^- \) and \( \xi \in S^2 \). The auxiliary function \( v \) is now defined as
\[
v := \log(w_{ij} \xi_i \xi_j) + \kappa \tilde{\phi},
\]
where \( w_{ij} := u_{ij} - A_{ij} \). Differentiation (2.1) yields
\[
F^{ij}[D_{ij} u_{ij} - D_{ij} A_{ij} - (D_{ij} A_{ij}) D_{ij} u - (D_{p, q} A_{ij}) u_{ij} ] = B_z + B_\xi D_\xi u.
\]

Another differentiation subsequently produces
\[
F^{ij, kl} D_{ij} w_{ij} D_{ij} u_{kl} + F^{ij}[D_{ij} u_{ij} - D_{ij} A_{ij} - (D_{ij} A_{ij}) (D_{ij} u)^2 - (D_{p, q} A_{ij}) u_{ij} ]\]
\[
+ (D_{ij} A_{ij}) (D_{ij} u) - 2(D_{ij} A_{ij}) D_{ij} u - 2(D_{ij} A_{ij}) u_{ij} ]
\]
\[
= D_{\xi} B + (D_{\xi} B) (D_{\xi} u)^2 + 2(D_{\xi} B) D_{\xi} u + (D_{\xi} B) u_{\xi}.
\]

Recalling that a gradient bound for an elliptic solution \( u \) exists (implied by the boundary condition (1.1a) and (2.17)), one can write the above as
\[
F^{ij, kl} D_{ij} w_{ij} D_{ij} u_{kl} + F^{ij}[D_{ij} u_{ij} - (D_{p, q} A_{ij}) u_{ij} ] - 2(D_{ij} A_{ij}) D_{ij} u_{ij}
\]
\[
\geq -C[1 + (1 + w_{ij}) F^{ij}] \geq -C[1 + w_{ij}] F^{ij},
\]
where the last inequality comes from the fact that \( 0 < \sigma \leq F^{ij} \) on \( \Gamma_{0, \alpha} \). To further reduce this, the (A3w) condition is now used. Specifically, a point \( x_0 \in \Omega^- \) is fixed and coordinates are rotated such that \( [w_{ij}] \) (and thus \( F^{ij} \)) is diagonal. With such a rotation, it can also be enforced that the eigenvalues of \([w_{ij}]\) are ordered as \( 0 < \lambda_1 \leq \cdots \leq \lambda_1 \). Estimating, one sees that
\[
F^{ij} D_{p, q} A_{ij} D_{\xi} u D_{\xi} u = F^{ij} A_{ij, k} (w_{ij} + A_{ij}) (w_{ij} + A_{ij})
\]
\[
\geq F^{ij} A_{ij, k} w_{ij} + C(1 + w_{ij}) F^{ij}
\]
\[ \geq \sum_{k \text{ or } l=r} f_i A_{r,k} (\lambda_k \xi_k) (\lambda_i \xi_i) - C(1 + w_{ij}) F^{ii} \]
\[ \geq -C(F^{ii} + w_{ij} F^{ii} + |w| f_i \lambda_i) \]
\[ \geq -C(F^{ii} + |w| + |w| F^{ii}), \quad (5.44) \]

where \( |w| = \sum_i w_{ii} \); and the fact that \( f_i \xi_i \) is bounded on \( \Gamma_{p_1,p_2} \) (as shown in (2.11)) has been used to ascertain the last inequality. Using this, (5.43) can now be rewritten as
\[ F^{ij}[D_j u_{\xi \xi} - (D_p A_{ij}) D_k u_{\xi \xi}] \geq -F^{ij,kl} D_{\xi} w_{ij} D_\xi w_{kl} - C(F^{ii} + w_{ii} + w_{ij} F^{ii}). \quad (5.45) \]

(5.45) is used to guide the definition of the linear operator that will be used to analyze \( v \):
\[ \mathcal{L} u_{\xi \xi} := F^{ij}[D_j u_{\xi \xi} - (D_p A_{ij}) D_k u_{\xi \xi}]. \]

From (5.44), it is calculated that (recalling \( w_{ij} = u_{ij} - A_{ij} \))
\[ \mathcal{L} w_{\xi \xi} \geq -F^{ij,kl} D_{\xi} w_{ij} D_\xi w_{kl} - C(F^{ii} + w_{ii} + w_{ij} F^{ii}), \]
for a further constant \( C \). Next, the auxiliary function is differentiated to get
\[ D_i v = \frac{D_i w_{\xi \xi}}{w_{\xi \xi}} + \kappa D_i \tilde{\phi} \]
\[ D_{ij} v = \frac{D_{ij} w_{\xi \xi}}{w_{\xi \xi}} - \frac{D_i w_{\xi \xi} D_j w_{\xi \xi}}{w_{\xi \xi}^2} + \kappa D_{ij} \tilde{\phi}. \quad (5.46) \]

Given (5.42), it is clear that \( \mathcal{L} \tilde{\phi} \geq F^{ii} \tilde{\phi} \). Using this fact along with (5.46), the following inequality emerges:
\[ \mathcal{L} v = \frac{\mathcal{L} w_{\xi \xi}}{w_{\xi \xi}} - F^{ij} \frac{D_i w_{\xi \xi} D_j w_{\xi \xi}}{w_{\xi \xi}^2} + \kappa \mathcal{L} \tilde{\phi} \]
\[ \geq -\frac{1}{w_{\xi \xi}} F^{ij,kl} D_{\xi} w_{ij} D_\xi w_{kl} - F^{ij} \frac{D_i w_{\xi \xi} D_j w_{\xi \xi}}{w_{\xi \xi}^2} - \frac{C}{w_{\xi \xi}} (F^{ii} + w_{ii} + w_{ij} F^{ii}) + \kappa \tilde{\phi} F^{ii}. \quad (5.47) \]

Next, it is supposed that \( v \) takes its maximum point at \( x_0 \in \Omega^- \) in a direction corresponding to the vector \( \xi \). Coordinates are then relabeled so that \( \xi = e_1 \).

To proceed, the first two terms on the final line of (5.47) need to be estimated at this maximum point. Specifically, it is claimed that
\[ -\frac{1}{w_{11}} F^{ij,kl} D_i w_{ij} D_k w_{kl} - F^{ij} \frac{D_i w_{11} D_j w_{11}}{w_{11}^2} \geq 0; \quad (5.48) \]
that is,
\[ -w_{11} F^{ij,kl} D_i w_{ij} D_k w_{kl} \geq F^{ij} D_i w_{11} D_j w_{11}. \quad (5.49) \]
To show (5.48), it is first claimed that the following inequality holds:

\[-w_{11}F^{ij,k1}D_iw_{j}D_kw_{i1} \geq -\lambda_1 \left[ \frac{\partial^2 f}{\partial \lambda_1^2} |D_1w_{11}|^2 + 2 \sum_{r=2}^{n} \frac{f_1 - f_r}{\lambda_1 - \lambda_r} |D_rw_{11}|^2 \right]. \tag{5.50}\]

This relation will be fully justified at the end of the proof. With (5.50), one is able to bound the left-hand side of (5.49) from below. Using this, it is observed that (5.49) is true if

\[-\lambda_1 \left\{ \frac{\partial^2 f}{\partial \lambda_1^2} |D_1w_{11}|^2 + 2 \sum_{r=2}^{n} \frac{f_1 - f_r}{\lambda_1 - \lambda_r} |D_rw_{11}|^2 \right\} \geq \sum_{r=1}^{n} f_r |D_rw_{11}|^2\]

holds, where coordinates have been rotated so that $F^{ij}$ is diagonal. By breaking this inequality up term-by-term and dividing each resulting inequality by $|D_rw_{11}|^2$, one sees that if

\[-\lambda_1 \frac{\partial^2 f}{\partial \lambda_1^2} \geq f_1, \tag{5.51}\]

\[-2\lambda_1 \frac{f_1 - f_r}{\lambda_1 - \lambda_r} \geq f_r, \quad \text{for } r \in \{2, \ldots, n\} \tag{5.52}\]

are shown to be true, then (5.50) and hence (5.47) will be proven. For the following calculations, coordinates are relabeled so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, keeping in mind that $\lambda_n \geq \mu_1 > 0$ for $\lambda \in \Gamma_{\mu_1, \mu_2}$. (5.51) will first be verified. To begin, (4.3) is differentiated to ascertain

\[\frac{\partial^2 f}{\partial \lambda_1^2} = \left( \frac{1}{n-1} - 1 \right) \left( \frac{S_n}{S_i} \right)^{n-2} \left( \frac{S_i S_n \nabla^2_i - S_n \nabla_i S_i}{S_i^2} \right)^2 + \left( \frac{2}{n-1} \right) \left( \frac{S_n}{S_i} \right)^{n-1} \left( \frac{S_i^2 S_n - S_n S_i n \nabla^2_i - S_n n \nabla^2_i S_i}{S_i^3} \right).\]

Using this with (4.3) along with (4.4), it is calculated that

\[-\lambda_1 \frac{\partial^2 f}{\partial \lambda_1^2} = \frac{2\lambda_1 S_{l_{-1,1}}}{S_i} - \left( \frac{1}{n-l} - 1 \right) \left( \frac{S_i S_{n-l,1} - S_n S_{l_{-1,1}}}{S_n S_i} \right) \lambda_1 \]

\[= \frac{2\lambda_1 S_{l_{-1,1}}}{S_i} - \left( \frac{1}{n-l} - 1 \right) \left( \frac{S_{l_{-1,1}}}{S_i} \right) \]

\[= \frac{2\lambda_1 S_{l_{-1,1}} + S_{l_{-1,1}}}{S_i} - \frac{1}{n-l} \left( \frac{S_{l_{-1,1}}}{S_i} \right) \]

\[= 1 + \frac{1}{S_i} \left( \lambda_1 S_{l_{-1,1}} - \frac{S_{l_{-1,1}}}{n-l} \right). \tag{5.53}\]

Next, the observation is made that

\[S_{l_{-1,1}} \leq (n-l)\lambda_2 S_{l_{-1,1}}, \quad l \in \{1, \ldots, n-1\},\]
with equality holding when all \( \lambda_i \) equal and \( l = n - 1 \). Thus, it is seen from (5.53) that

\[
-\frac{\lambda_1}{f_1} \frac{\partial^2 f}{\partial \lambda_1^2} \geq 1 + \frac{1}{S_l} \left( \lambda_1 S_{l-1,1} - \frac{(n-l)\lambda_2 S_{l-1,1}}{n-l} \right) = 1 + \frac{S_{l-1,1}}{S_l} (\lambda_1 - \lambda_2) \geq 1,
\]

which directly implies (5.51) as \( f_i > 0 \) on \( \Gamma_{\mu_1, \mu_2} \).

Next, (5.52) is proven for arbitrary \( r \in [2, \ldots, n] \). Again, using (4.3) and (4.4), one finds the following:

\[
\frac{\lambda_1 f_1}{\lambda_j f_j} = \frac{\lambda_1 (S_{n-1,1} S_l - S_n S_{l-1,1})}{\lambda_j (S_{n-1,j} S_l - S_n S_{l-1,j})} = \frac{(S_n - S_{n,1}) S_l - (S_l - S_{l,1}) S_n}{(S_n - S_{n,j}) S_l - (S_l - S_{l,j}) S_n} = \frac{S_{l,1} S_n - S_{n,1} S_l}{S_{l,j} S_n - S_{n,j} S_l} = \frac{\lambda_2 S_{l-1,1,j} + S_{l,1,j}}{\lambda_1 S_{l-1,1,j} + S_{l,1,j}},
\]

where the fact that \( S_{n,i} = 0 \) for any \( i \) has been used to gain the fourth equality. From the current selection of coordinates, it is clear that the following inequalities hold:

\[
2\lambda_j^2 \leq \lambda_1^2 + \lambda_2 \lambda_j, \quad 2\lambda_j \leq \lambda_1 + \lambda_2.
\]

This combined with (5.54) yields

\[
\left( \frac{2\lambda_j}{\lambda_1 + \lambda_j} \right) \frac{\lambda_1 f_1}{\lambda_j f_j} \leq \frac{2\lambda_j^2 S_{l-1,1,j} + 2\lambda_j S_{l,1,j}}{(\lambda_1^2 + \lambda_2 \lambda_j) S_{l-1,1,j} + (\lambda_1 + \lambda_2) S_{l,1,j}} \leq 1,
\]

which subsequently implies (5.52) via an elementary algebraic manipulation, using the fact that \( f_i > 0 \) on \( \Gamma_{\mu_1, \mu_2} \).

In light of the above calculation, the claim in (5.48) has been proven. With this, (5.47) is further reduced to find that

\[
\mathcal{L} v \geq -\frac{C}{w_{11}} (F^{ii} + w_{ii} + w_{i} F^{ii}) + \kappa F^{ii} \\
\geq -C \left[ 1 + \frac{F^{ii}}{w_{11}} + F^{ii} \right] + \kappa F^{ii} \\
\geq -C \frac{F^{ii}}{w_{11}} + (\kappa - C) F^{ii},
\]

where the fact from Remark 2.1(2) that \( F^{ii} > \sigma > 0 \) on \( \Gamma_{\mu, \mu_2} \) for any \( \mu_2 \) has been used to gain the last inequality. On the other hand, since \( x_0 \) is a maximum of \( v \),
one has that
\[ \mathcal{L} v(x_0) = F^{ij} [D_{ij} v(x_0) - (D_{jk} A_{ij}) D_{k} v(x_0)] \]
\[ = F^{ij} D_{ij} v(x_0) \leq 0. \]

This, combined with (5.55) yields
\[ \frac{C}{\kappa - C} \geq w_{11}(x_0), \quad (5.56) \]
provided \( \kappa > C \), where \( C \) depends on \( \sup |u|, \sup |D u|, A, \sup |B|, \Omega^- \) and \( \Omega^+ \).

With this, all that is left to be done to prove (5.56) is to demonstrate that (5.50) holds in the case where \( f = \sigma_{m,l} \). To do this, it is first calculated that when \( f = \sigma_{m,l} \), one has that
\[ f_{ij} = \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} = \left( \frac{1}{n-l} - 1 \right) \left( \frac{1}{n-l} \right) \left( \frac{S_n}{S_l} \right)^{\frac{n-2}{2}} \left( \frac{S_{n-1,l}}{S_l} - \frac{S_n S_{1-l,i}}{S_l^2} \right) \]
\[ \cdot \left( \frac{S_{n-1,i}}{S_l} - \frac{S_{n,1-i}}{S_l^2} \right) \]
\[ + \frac{1}{n-l} \left( \frac{S_n}{S_l} \right)^{\frac{n-1}{2}} \]
\[ \times \left( \frac{S_{n-2,i}}{S_l} - \frac{S_{n-1,1-i}}{S_l^2} - \frac{S_{n-1,i} S_{1-1,i}}{S_l^2} - \frac{S_{n} S_{1-1,i} S_{1-1,j}}{S_l^2} + 2 \frac{S_n S_{1-l,i} S_{1-1,j}}{S_l^2} \right). \]

Next, applying the ellipticity criterion of \( f_i > 0 \) from ((2.15)) to (4.3) one sees that
\[ S_n S_{l-1,i} < S_{n-1,i} S_l. \quad (5.58) \]

From (5.58) it is clear that the first term on the right-hand side of (5.57) is negative. To analyze the second term on the right-hand side of (5.57), it is first calculated that
\[ \frac{S_{n-2,ij}}{S_l} - \frac{S_n S_{1-2,ij}}{S_l} = \frac{S_n}{\lambda_i \lambda_j S_l} - \frac{S_n S_{1-2,ij}}{S_l^2} \]
\[ = \frac{S_n}{\lambda_i \lambda_j S_l} \left( 1 - \frac{\lambda_i \lambda_j S_{1-2,ij}}{S_l} \right) \]
\[ = \frac{S_n}{\lambda_i \lambda_j S_l} \left( 1 - \frac{S_i - S_{1,ij}}{S_l} \right) = \frac{S_n S_{1,ij}}{\lambda_i \lambda_j S_l^2}. \]

This enables one to write the quantity inside the parentheses contained in the second term on the right-hand side of (5.57) as
\[ \frac{S_n S_{1,ij}}{\lambda_i \lambda_j S_l^2} \left( S_n S_{1,ij} - S_{1,ij} - S_{l,i} S_{l,j} + 2 S_{1,i} S_{l,j} \right). \]
Using the observation that

\[ S_i = S_{i,i} + S_{i,j} - S_{i,j}, \quad (5.61) \]

on (5.60) enables one to conclude that quantity inside the parentheses contained in the second term on the right-hand side of (5.57) is equal to

\[ \frac{S_{n-2;i}}{S_i^n} [S_{i,i}(S_{i,i} + S_{i,j}) - (S_{i,i}^2 + S_{i,j}^2)] \leq 0, \quad (5.62) \]

where the inequality is immediately obvious. Thus, it has been shown that

\[ \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \leq 0; \quad (5.63) \]

i.e., \( \sigma_{n,t} \) is indeed concave.

Defining \( M_{ij} = D_i w_{ij} D_j w_{ij} \) where \( w \) corresponds to the \( w \) in Section 5.3 and rotating coordinates so that \( M_{ij} \) is diagonal with \( w_{11} \) corresponding to the maximum of \( D^2 w \) at \( x_0 \), it is clear that the second derivative terms on the right-hand side of (5.50) are all individually non-negative.

Next, the quotient terms in (5.50) are analyzed:

\[
\begin{align*}
&\frac{f_i - f_j}{\lambda_i - \lambda_j} = \frac{1}{\lambda_i - \lambda_j} \frac{1}{n - i} \left( \frac{S_n}{S_i} \right)^{\frac{1}{n} - 1} \left( \frac{S_{n-1;i} S_i - S_{n-1,j} S_j - S_n S_{ij} - S_n S_{i-1,j} + S_n S_{i-1,i}}{S_i^2} \right) \\
&= \frac{1}{\lambda_i - \lambda_j} \frac{1}{n - i} \left( \frac{S_n}{S_i} \right)^{\frac{1}{n} - 1} \frac{S_n}{\lambda_i \lambda_j S_i^2} [S_i (\lambda_i - \lambda_j) + \lambda_i \lambda_j (S_{i-1;i} - S_{i-1,j})] \\
&= \frac{1}{\lambda_i - \lambda_j} \frac{1}{n - i} \left( \frac{S_n}{S_i} \right)^{\frac{1}{n} - 1} \frac{S_n}{\lambda_i \lambda_j S_i^2} [S_i (\lambda_i - \lambda_j) + \lambda_i (S_{i-1;i} - S_{i-1,j}) - \lambda_j (S_i - S_{i,j})] \\
&= \frac{1}{\lambda_i - \lambda_j} \frac{1}{n - i} \left( \frac{S_n}{S_i} \right)^{\frac{1}{n} - 1} \frac{S_n}{\lambda_i \lambda_j S_i^2} [\lambda_i S_{i-1;i} - \lambda_j S_{i-1,j}] \quad (5.64)
\end{align*}
\]

It is clear that \( \lambda_i - \lambda_j \) will have the opposite sign of \( \lambda_i S_{i-1;i} - \lambda_j S_{i-1,j} \). Thus, it is seen that the quotient terms in (5.50) are individually positive, which is a stronger statement compared to the case when \( f \) is just concave. Indeed, it is the specific structure of \( \sigma_{n,t} \) that enables one to discard the terms that result in (5.50). With this, (5.50) has been shown to hold in the case where \( f = \sigma_{n,t} \), hence proving (5.56).

It is recalled that (5.56) has been shown under the assumption that \( v \) obtains its maximum at an interior point of \( \Omega^+ \). If \( v \) does not take a maximum at an interior point, then it must take a maximum on \( \partial \Omega^+ \). With this observation, the reduction of the \( C^2 \) estimate to the boundary is now complete.

**Remark 5.5.** In the case where \( A \) satisfies the strong (A3) condition, the above estimate is much simpler, and does not require the barrier condition (5.42). Specifically, one only needs to consider the simpler auxiliary function \( v := w_{ii} \) in
this case, with the definition of $\mathcal{L}$ remaining unaltered. Upon noting that the (A3) condition directly implies that
\[
F^{ij}D_{p_r}A_{ij}D_kuD_{l}u \geq CF^i|D^2u|^2,
\]
for some positive constant $C$, it is a straight-forward calculation that shows
\[
\mathcal{L}v \geq F^{ii}(C_1|D^2u|^2 - C_2(|D^2u| + 1)).
\]
From this, the second derivative estimate immediately falls out as $\mathcal{L}v \leq 0$.

With this remark, the following has now been proven:

**Theorem 5.6.** Let $u$ be an elliptic solution to (2.1) in $\Omega^-$, with $x, u(x), Du(x) \in U$ for all $x \in \Omega^-$ and $U$ bounded. If (5.42) is true, with $B(x, u)$ a positive, bounded function in $C^2(\bar{\Omega}^- \times \mathbb{R})$, with either

- $A$ satisfying the weak (A3w) condition plus $A$ and $\Omega^-$ satisfying the barrier condition stated in Subsection 2.4 or
- $A$ satisfying the strong (A3) condition;

then one has the following estimate:
\[
\sup_{\Omega^-}|D^2u| \leq C \left( 1 + \sup_{\bar{\Omega}^-}|D^2u| \right), \tag{5.65}
\]
where $C$ depends on $\sup|u|, \sup|Du|, A, \sup|B|, \Omega^- \text{ and } \Omega^+.$

From these calculations, it is seen that the (A3w) criterion for $A$ on the set $U$ is indeed sufficient to make the argument go through. Also, it is clear that the calculation holds for the Quotient Transportation Equation (1.1a); thus, the calculations of this subsection provide a crucial step to the proof of Theorem 3.1.

**Remark.** It is possible to generalize Theorem 5.6 to the case where $B(x, z, p) > 0$ with
\[
B_{p_jp_l}(x, z, p)\xi_k\xi_l > 0, \tag{5.66}
\]
for all $(x, z, p) \in U$. To do this generalization, (5.42) needs to be strengthened. Specifically, it is required that $A$ and $B$ be such that there exists a $\tilde{\phi}$ satisfying
\[
[D_{ij}\tilde{\phi}(x) - D_{p_j}(A_{ij}(x, z, p) + B(x, z, p))) \cdot D_k\tilde{\phi}(x)]\xi_i\xi_j \geq \tilde{\delta}|\xi|^2,
\]
for all $(x, z, p) \in U$. With that, the calculations proceed precisely as above with the linear operator $\mathcal{L}$ now being defined as
\[
\mathcal{L}u := F^{ij}[D_{ij}u - (D_{p_j}A_{ij})D_ku] - (D_{p_j}B)D_ku.
\]
This combined with Remark 5.2(3), indicates that the only way for $B$ to have a dependence on the gradient of $u$ and still allow for the existence of globally smooth solutions to the Quotient Transportation Equation, is if $B$ satisfies (5.66) with $\Omega^-$ of sufficiently high normal curvature so that (5.29) holds.
5.4. Boundary Estimates for Second Derivatives

In this subsection, the \( C^2 \) \textit{a priori} bound is completed by proving second derivative bounds for solutions of the Quotient Transportation Equation on \( \partial \Omega^- \). This treatment is similar to the one presented in [8, 11, 19], but requires some modification to accommodate the particular situation where \( f = \sigma_{n \lambda} \). Specifically, Lemma 4.2 is required in order to derive differential inequalities from which barriers can be constructed. A key point to the following argument is that for oblique boundary conditions of the form (5.1), where the function \( G \) is uniformly convex in the gradient, the twice tangential differentiation of (5.1) yields quadratic terms in second derivatives which compensate for the deviation of \( \beta = G_p \) from the geometric normal.

First, non-tangential second derivatives are treated. Taking \( \Psi \in C^2(\Omega^- \times \mathbb{R}) \), the following definition is made:

\[
v := \Psi(x, Du).
\]

Defining the linear operator \( \mathcal{L} \) by (5.14) and calculating as was done for (5.15), one observes that

\[
|\mathcal{L}v| \leq C_1 F^{ij} \delta_{ij} u_i u_j + C_2 (F^{ii} + 1),
\]

\[
\leq C \left(1 + \sum_{i=1}^n f_i (\lambda_i^2 + 1)\right).
\]

(5.67)

In the last inequality coordinates have been rotated so that \( F^{ij} \) is diagonal, without loss of generality. Using Lemma 4.2 and defining

\[
M := \sup_{\Omega^-} |D^2_x (u - c)|,
\]

(5.67) is further reduced to

\[
|\mathcal{L}v| \leq (C(\epsilon) + \epsilon MF^{ii}) + C(1 + F^{ii}),
\]

\[
\leq C(C(\epsilon) + \epsilon MF^{ii})
\]

\[
\leq (C(\epsilon) + \epsilon M)F^{ii},
\]

where the fact that \( F^{ii} \) is bounded away from zero has been used along with a rescaling of \( \epsilon \) by a factor of \( C \), again without a loss of generality.

Using the construction of \( \phi^+ \) from Subsection 5.4, \( \Psi \) is now set in the following manner:

\[
\Psi(x, Du) = G(x, Du) = \phi^+ \circ T_u(x, Du).
\]

(5.68)

(5.19) can be used to construct both an upper and lower barriers (using different choices of \( a \) and \( b \)), as (5.68) indicates \( v = 0 \) on \( \partial \Omega^- \). By using the same barrier argument from the obliqueness estimate, the following boundary estimate is derived:

\[
|DG| \leq (C(\epsilon) + \epsilon M), \quad \text{on} \ \partial \Omega^-.
\]
for any $\epsilon > 0$. This, in turn, implies that

$$|D(\beta \cdot Du)| \leq C(\epsilon) + \epsilon M, \quad \text{on } \partial \Omega^-,$$

where it is recalled that $\beta := G_{p_k}$. With this, $u_{\beta\beta}$ is thus estimated on the boundary, which is equivalent to

$$w_{\beta\beta} \leq C(\epsilon) + \epsilon M, \quad \text{on } \partial \Omega^-.$$  \hspace{1cm} (5.69)

**Remark.** By the strict obliqueness estimate (5.28), it is clear that

$$w_{\gamma\gamma} \leq Cw_{\beta\beta},$$  \hspace{1cm} (5.70)

where $\gamma$ is the outer unit normal to $\partial \Omega^-.$

To proceed, an explicit representation of any given vector $\xi$ is written in terms of a tangential component, $\tau(\xi)$, and $\beta$:

$$\xi = \tau(\xi) + \frac{\xi \cdot \gamma}{\beta \cdot \gamma} \beta,$$  \hspace{1cm} (5.71)

where

$$\tau(\xi) = \xi - (\xi \cdot \gamma)\gamma - \frac{\xi \cdot \gamma}{\beta \cdot \gamma} \beta^T,$$

and

$$\beta^T = \beta - (\beta \cdot \gamma)\gamma.$$

From this, it is calculated that

$$|\tau(\xi)|^2 = 1 - 2\frac{\beta^T \cdot \xi}{\beta \cdot \gamma} (\xi \cdot \gamma) - \left(1 - \frac{|\beta^T|^2}{(\beta \cdot \gamma)^2}\right)(\xi \cdot \gamma)^2$$

$$\leq 1 - 2\frac{\beta^T \cdot \xi}{\beta \cdot \gamma} (\xi \cdot \gamma) + C(\xi \cdot \gamma)^2.$$  \hspace{1cm} (5.72)

It is now assumed that the maximal tangential second derivative of $w$ over $\partial \Omega^-$ occurs at a point at $x_0 \in \partial \Omega^-$ in the direction which is taken to be $e_1$. Denoting $\tau = \tau(e_1)$ and utilizing (5.71), it is calculated that

$$w_{11} = w_{\tau\tau} + \frac{2\gamma_1}{\beta \cdot \gamma} w_{\beta\beta} + \frac{\gamma_1^2}{(\beta \cdot \gamma)^2} w_{\beta\beta}.$$ \hspace{1cm} (5.72)

Next, the boundary condition

$$G(x, Du) = 0, \quad \text{on } \partial \Omega^-,$$

is differentiated in the tangential direction to find that

$$0 = G_{x_1} + \tau G_{p_k} u_{ik} = G_{x_1} + u_{\beta i};$$  \hspace{1cm} (5.73)
that is,

\[ w_{\beta t} \leq -G_{x_t} + C \leq C. \]

Subsequently, this combined with (5.72) and (5.69) yields

\[ w_{11} \leq |x|^2 w_{11}(x_0) + C \frac{\bar{x} \cdot \bar{\gamma}}{\beta \cdot \gamma} + (C(\epsilon) + \epsilon M)\gamma_1^2 \]

\[ \leq \left( 1 - 2 \frac{\beta^T \cdot \bar{x}}{\beta \cdot \gamma} (\bar{x} \cdot \bar{\gamma}) + C(\bar{x} \cdot \bar{\gamma})^2 \right) w_{11}(x_0) + C \frac{\bar{x} \cdot \bar{\gamma}}{\beta \cdot \gamma} + (C(\epsilon) + \epsilon M)\gamma_1^2. \]  

(5.74)

Now the term \( D_{\beta} w_{11}(x_0) \) will be estimated. To do this, (2.1) is differentiated twice to find that

\[ F^{ij}D_{ij} w_{11} = -F^{ijkl}D_{ijkl} w_{11} + B_{11} + 2(B_{z11} + B_{11} + B_{zz} + D_{11} u)^2 \geq C, \]

where the concavity of \( F \) has been used along with the assumed condition of \( B_z \geq 0 \). Using this and the barrier construction in [6, Corollary 14.5], with a linear operator defined by

\[ \mathcal{P} u := F^{ij}D_{ij} u, \]

one obtains

\[ -D_{\beta} w_{11}(x_0) \leq C. \]  

(5.75)

Next, (5.73) is differentiated again in the tangential direction to get

\[ G_{p_k p_l} u_{k l} u_{t t} + G_{p_k u_{k t}} + G_{x_k} + 2G_{x_k p_k} u_{k t} = 0. \]

This combined with (5.75) and (5.16) indicates that at \( x_0 \) one has that

\[ C_0 u_{11}^2(x_0) - C_1 u_{11}(x_0) \leq C, \]

for positive constants \( C_0 \), \( C_1 \) and \( C_2 \). That is,

\[ w_{11}(x_0) \leq C. \]

Using this relation in (5.74) thus yields

\[ w_{11} \leq C(\epsilon) + \epsilon M, \text{ on } \partial\Omega^-, \]

for new constant \( C(\epsilon) \) and a rescaling of \( \epsilon \).

Combining this last relation with (5.69) and (5.70) allows one to bound the second derivatives of \( u \) on the boundary of \( \Omega^- \) in the following manner:

\[ \sup_{\partial\Omega^-} |D^2 u| \leq C(\epsilon) + \epsilon M. \]
Now utilizing (5.65) to eliminate $M$ from the above inequality finally yields

$$\sup_{\Omega^-} |D^2u| \leq \frac{C(\epsilon)}{1 - \epsilon C},$$

for a new $C$, $C(\epsilon)$ and $\epsilon$ scaling. Fixing $0 < \epsilon < \frac{1}{2}$, the following has been proven:

**Theorem 5.7.** Let $c$ be a cost-function satisfying hypotheses (A1), (A2), (A3w) with respect to bounded $C^2$ domains $\Omega^-$, $\Omega^+ \in \mathbb{R}^n$ which are respectively uniformly $c$-convex, $c^*$-convex with respect to each other. Let $B$ be a strictly positive function in $C^2(\Omega^- \times \mathbb{R} \times \mathbb{R}^n)$ satisfying (2.23) and (2.24w). Then any elliptic solution $u \in C^1(\Omega^-)$ of the second boundary value problem (1.1a), (1.1b) satisfies the a priori estimate

$$\sup_{\Omega^-} |D^2u| \leq C,$$  \hspace{1cm} (5.76)

where $C$ depends on $c$, $B$, $\Omega^-$, $\Omega^+$ and $\sup_{\Omega^-} |u|$.

This combined with Theorem 5.6 proves the main result Theorem 3.1. Once the second derivatives are bound, (1.1a) is effectively uniformly elliptic. This combined with the obliqueness estimate yields global $C^{2,\alpha}$ estimates for solution of (2.25a) and (2.25b), from the theory of oblique boundary value problems for uniformly elliptic equations presented in [7]. Moreover, by the theory of linear elliptic equations with oblique boundary conditions (see [6]) and the assumed smoothness of data, one also has $C^{3,\alpha}(\Omega^-)$ bounds for elliptic solutions, for any $\alpha < 1$.

### 6. Method of Continuity

To complete the proof of Theorem 3.2, the standard method of continuity for nonlinear oblique boundary value problems is used, which is presented in [6, Section 17.9] and subsequently applied in [18, 20]. This procedure was modified in [15] in order to be applied to the Optimal Transportation Equation; and it is this method that is recalled here with trivial modification. Specifically, two procedures for applying the method of continuity were used in [15]. The first method discussed there utilized foliations of both $\Omega^-$ and $\Omega^+$ to construct a family of suitable boundary value problems; this procedure will not be used here. The second method proved the existence of a function approximately satisfying the associated boundary condition, which subsequently allowed for the method of continuity to be applied without a domain variation construction. This is the procedure that will be recalled here (for the sake of completeness) with only trivial modification, as the Quotient Transportation Equation is constructed off the archetype of Optimal Transportation (see Subsection 2.2).

To start off, a key lemma from [15] is recalled without proof:

**Lemma 6.1 (15).** Let the domains $\Omega^-$ and $\Omega^+$ and cost-function $c$ satisfy the hypothesis of Theorem 3.1. Then for any $\epsilon > 0$, there exists a uniformly $c^*$-convex approximating domain, $\Omega^+_\epsilon$, lying within distance $\epsilon$ of $\Omega^+$, and satisfying the corresponding condition (2.21) for fixed $\delta^0$, together with a function $u_0 \in C^4(\Omega)$ satisfying the ellipticity condition (2.8), for $f = \sigma_{n,l}$ and the boundary condition (1.1b) for $\Omega^+_\epsilon$. 
Remark. As $\Gamma(S_e) = \Gamma(\sigma_{\nu})$, the above lemma carries over trivially to the case of the Quotient Transportation Equation.

With this in mind and making the denotation

$$F[u] := \left(\frac{S_u}{S_t}\right)^{\frac{1}{m}} [D^2 u - D^2 c(\cdot, T_u)],$$

(6.1)

the following family of boundary value problems are now defined:

$$F[u_t] = tB(\cdot, u_t) + (1 - t)e^{\alpha_t - 
\partial \Omega^-} F[u_0], \quad \Omega^-$$

$$T_u(\Omega^-) = \Omega^+,$$

(6.3)

for $t \in [0, 1]$ with $u_0$ taken as the function indicated in Lemma 6.1. It is clear that $u_0$ is the unique elliptic solution of (6.2)–(6.3) at $t = 0$. In this family of equations, $\Omega^+_t$ is such that $\Omega^+_0 = \Omega^+_t$ (corresponding to $u_0$ as depicted in Lemma 6.1) and $\Omega^+_t = \Omega^+$. Given the assumed smoothness of $c$ and the definition of $T_u$ depicted in (2.17), an $\varepsilon > 0$ can be chosen in Lemma 6.1 small enough to guarantee that $\Omega^-_t$ and $\Omega^+_t$ are uniformly $c$-convex and $c^*$-convex (respectively) relative to one another with corresponding uniform convexity constants independent of $t$, as $\Omega^-$ and $\Omega^+$ are assumed to be uniformly $c$-convex and $c^*$-convex (respectively) relative to one another.

From Subsection 5.1, it is understood that the boundary condition (6.3) is equivalent to the oblique condition

$$G_t(\cdot, Du) := \phi^+_t(T_u(\cdot)) = 0, \quad \partial \Omega^-_t,$$

where $\phi^+_t$ are defined for $\Omega^+_t$ analogous to the construction of $\phi^+$ for $\Omega^+$ in Subsection 5.1. From the observations in the previous paragraph, it is clear that the family of boundary value problems correlate to uniformly oblique boundary value problems with a uniform constant of obliqueness independent of $t$.

To adapt the method of continuity from [6, Section 17.9], an $\alpha \in (0, 1)$ is fixed with $\Sigma$ set to denote the subset of $[0, 1]$ for which $t \in \Sigma$ implies the problem (6.2)–(6.3) is solvable for an elliptic solution $u_t \in C^{2,\alpha}(\Omega^-_t)$, with $T_u$ invertible. It is clear that the boundary condition (6.3) implies a uniform bound for $Du_t$ with respect to $t$. Upon noting that the inhomogeneity of (6.2) is uniformly bounded in $t$ and satisfies (2.22)–(2.24), uniform estimates in $C^{2,1}(\Omega^\pm)$ immediately follow, as all the solution estimates of Section 5 are clearly independent of $t \in [0, 1]$. By compactness, it is then inferred that $\Sigma$ is closed via the Heine–Borel Theorem. To show $\Sigma$ is open, the implicit function theorem is used along with the linear theory of oblique boundary value problems, as in [6, Chapter 17]. As $\Sigma$ is open, closed and non-trivial, it is known that $\Sigma = [0, 1]$; that is, there exists a unique elliptic solution $u \in C^1(\Omega^-)$ of the boundary value problem

$$F[u] = B(\cdot, u), \quad \Omega^-$$

$$T_u(\Omega^-) = \Omega^+.$$ 

(6.5)

Thus, the part of Theorem 3.2 corresponding to unique solutions is proven.
Remark. (2.24) guarantees uniqueness of the linearized boundary value problem via a straight-forward application of the Hopf boundary point lemma (see [6, Lemma 3.4]). Without the (2.24) condition, one cannot apply the method of continuity directly, as solutions will not be unique in general in this scenario. This will be discussed further in the next subsection.

6.1. Application of the Leray–Schauder Fixed Point Theorem

To conclude this section, the Leray–Schauder fixed point theorem (see [6, Theorem 11.6]) will be used to relax the monotonicity criterion on the inhomogeneity required by the method of continuity. The Leray–Schauder theorem has been used to similar ends in [7, 17, 20].

Using the notation in (6.1), \( u_0 \) is defined to be the unique admissible solution of

\[
F[u_0] = e^{u_0}, \quad \text{in } \Omega^-,
\]

\[
T_{u_0}(\Omega^-) = \Omega^+.
\]

Theorem 3.2 indicates that such a unique \( u_0 \) exists. Moreover, by elliptic regularity theory, \( u_0 \in C^\infty(\Omega^-) \). For \( t \in [0, 1] \) and \( \psi \in C^3(\Omega^-) \), the following family of problems are considered:

\[
F[u_t] = t(\tilde{B}(\cdot, u_0 + \psi) + e^{u_t} - u_0 - 1) + (1 - t)e^{u_t}, \quad \text{in } \Omega^- \quad (6.6)
\]

\[
T_{u_t}(\Omega^-) = \Omega^+, \quad (6.7)
\]

where \( \tilde{B} \) is only assumed to satisfy (2.22), (2.23) and (2.24w). By Theorem 3.2 and elliptic regularity theory, for each \( \psi \) and \( t \) (6.6)–(6.7) has a unique admissible solution \( u_t \in C^{1,\alpha}(\Omega^-) \) for any \( \alpha < 1 \). Consequently, the map \( \mathcal{F} : C^3(\Omega^-) \times [0, 1] \to C^3(\Omega^-) \) defined by \( \mathcal{F}(\psi, t) = u_t - u_0 \) is continuous and compact; and \( \mathcal{F}(\cdot, 0) = 0 \) for all \( \psi \in C^3(\Omega^-) \). If it can also be shown for \( t \in [0, 1] \), that for all the fixed points of \( \mathcal{F}(\cdot, t) \)—that is, for any admissible solution \( u_t \) of

\[
F[u_t] = t\tilde{B}(\cdot, u_t) + (1 - t)e^{u_t}, \quad \text{in } \Omega^- \quad (6.8)
\]

\[
T_{u_t}(\Omega^-) = \Omega^+ \quad (6.9)
\]

— the estimate

\[
\|u_t\|_{C^3(\Omega^-)} \leq C \quad (6.10)
\]

is satisfied with \( C \) independent of \( t \), then by the Leray–Schauder fixed point theorem, \( \mathcal{F}(\cdot, 1) \) has a fixed point. Subsequently, this is equivalent to

\[
F[u] = \tilde{B}(\cdot, u), \quad \text{in } \Omega^-, \quad (6.11)
\]

\[
T_u(\Omega^-) = \Omega^+, \quad (6.12)
\]

having an admissible solution \( u \) belonging to \( C^3(\Omega^-) \). As the solution estimates from Section 5 clearly apply to (6.8)–(6.9) independent of \( t \), results from [7] and linear theory for oblique elliptic boundary value problems can be applied to deduce (6.10).
Subsequently, the monotonicity criterion on $B$ may be relaxed to $B_t \geq 0$ relative to the previous existence criterion, thus finishing the proof of Theorem 3.2.

Remarks.

(1) As mentioned before, the admissible solution to (6.11)–(6.12) may not be unique as $B_t > 0$ is needed for the application of the Hopf Boundary point lemma.

(2) It is recalled that the condition of $B_t \geq 0$ is used in the boundary $C^2$ estimate in Subsection 5.4, and thus, is still a required criterion, as estimates on solution of (6.8)–(6.9) need to hold independent of $t$.

7. Conclusions

While (2.25a) and (2.25b) differ significantly from the Monge–Ampère equations coming from Optimal Transportation, the theory regarding their regularity is very similar. Specifically, the (A3w) represents both a necessary and sufficient condition for higher regularity of (1.1a) and (1.1b). The calculations in Subsection 5.3 indicate that the (A3w) condition is sufficient for Theorem 3.2. Given that the ellipticity criterion for (1.1a) and (1.1b) corresponds to solutions being $c$-convex, the counter-example that Loeper presented in [9] still applies in the current situation; and thus, (A3w) is also a necessary criterion for our higher regularity theory.

Appendix A. $c$-Convexity

By using cost-functions, one can naturally generalize the classical notions of convexity. This generalization is a powerful tool whose related notions and definitions will be used throughout the rest of this thesis.

The following definitions have become standard in the literature surrounding Optimal Transportation (see [1, 4, 11, 21] for more detailed expositions on $c$-convexity).

Definition A.1. A $c$-support function of $\phi$ at $x_0$ is a function of the form $c(x, y_0) + a$, where $y_0 \in \Omega^+$ and $a = a(x_0, y_0)$ is a constant, such that

$$
\phi(x_0) = c(x_0, y_0) + a,$$

$$
\phi(x) \leq c(x, y_0) + a, \quad \forall x \in \Omega^-.
$$

(A.1)

Using the above definition, one may also define the notion of $c^*$-support functions in a similar manner by switching $x$ and $y$, $\Omega^-$ and $\Omega^+$. Now, the notion of support functions can be used to generalize the notion of convexity in the context of cost-functions.

Definition A.2. An upper semi-continuous function $\phi$ defined on $\Omega^-$ is $c$-concave if for any point $x_0 \in \Omega^-$, there exists a $c$-support function at $x_0$. Similarly, an upper semi-continuous function $\psi$ defined on $\Omega^+$ is $c^*$-concave if for any point $y_0 \in \Omega^+$, there exists a $c^*$-support function at $y_0$.

From these definitions, one can also easily ascertain the notion of a $c$-convex function by simply switching the direction of the inequality shown in (A.1).
The definition for a $c^*$-convex function is also thus obtained by replacing $c$-support functions with $c^*$-support functions in Definition A.2 and reversing the inequality in (A.1).

Before proceeding onto other definitions, some properties of $c$-concave functions can be readily observed through facts in classical analysis. As it is assumed that the cost-function $c$ is smooth, any $c$-concave function $\phi$ is semi-concave; that is, there exists a constant $C$ such that $\phi(x) - C|x|^2$ is concave. It is a classical result that now shows $\phi$ is twice differentiable almost everywhere. In addition to this, it is readily demonstrated (via a Perron process) that if $(\phi_k)$ is a sequence of $c$-concave functions and $\phi_k \to \phi$, then $\phi$ is $c$-concave. Clearly, there are analogous results for $c$-convex and $c^*$-convex/concave functions.

Remark. It is important to note that the above definitions correspond with the classical notions of concavity in the special case where $c(x, y) = x \cdot y$. In this situation, a $c$-support function is simply a support hyperplane at a particular point. This special case provides a good basis of intuition for more general cost-functions.

In addition to using generalized definitions for concavity of functions, one also needs to consider analogous generalizations of concavity with respect to sets. In particular, such notions will be of key importance in the construction of barriers for boundary gradient estimates, in addition to being necessary for making the statements of current regularity theorems in Optimal Transportation. With that, the notion of $c$-segments (which generalize the notion of line-segments in classical convex analysis) is next defined:

Definition A.3. A $c$-segment in $\Omega^-$ with respect to a point $y$ is a solution set $\{x\}$ to $D_x c(x, y) = z$ for $z$ on a line segment in $\mathbb{R}^n$. A $c^*$-segment in $\Omega^+$ with respect to a point $x$ is a solution set $\{y\}$ to $D_y c(x, y) = z$ for $z$ on a line segment in $\mathbb{R}^n$.

By conditions (A1) and (A2), it is clear that a $c$-segment is a smooth curve; and for any two points $x_0, x_1 \in \mathbb{R}^n$ and any $y \in \mathbb{R}^n$, there exists an unique $c$-segment connecting $x_0$ and $x_1$ relative to $y$.

Definition A.4. A set $E^-$ is $c$-convex relative to a set $E^+$ if for any two points $x_0, x_1 \in E^-$ and any $y \in E^+$, the $c$-segment relative to $y$ connecting $x_0$ and $x_1$ lie in $E^+$. Analogously, it is said that $E^+$ is $c^*$-convex relative to $E^-$ if for any two points $y_0, y_1 \in E^+$ and any $x \in E^-$, the $c^*$-segment relative to $x$ connecting $y_0$ and $y_1$ lies in $E^+$.

Remarks.

(1) In general the notion of $c$-convexity, with respect to sets, is stronger than the classical notion of convexity. This is most readily observed when taking a ball for the set in question. Indeed, a ball may not be $c$-convex (relative to another given domain) at an arbitrary location. However, a sufficiently small ball will be $c$-convex if $c$ is $C^3$ smooth. In light of this example, it is insightful to consider an alternate statement of Definition A.4:

$E^-$ is $c$-convex with respect to $E^+$ if for each $y \in E^+$, the image $D_y c(E^-, y)$ is convex in $\mathbb{R}^n$. 


(2) One may reconcile the above alternative definition with (2.21) by pulling back that equation by the mappings $c(y, y)$ to see that this immediately implies that the image domains of $c(y, \Omega^+, y)$ are uniformly convex with respect to $y \in \Omega^+$ [15].

(3) An analogous statement may be made for a set that is $c^*$-convex relative to another set by considering $c^*(x, y) = c(y, x)$.

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