Percolation in high dimensions is not understood

S. Fortunato\textsuperscript{1}, D. Stauffer\textsuperscript{2} and A. Coniglio\textsuperscript{3}

\textsuperscript{1} Faculty of Physics, Bielefeld University, D-33615 Bielefeld, Germany
fortunat@physik.uni-bielefeld.de

\textsuperscript{2} Institute for Theoretical Physics, Cologne University,
D-50923 Köln, Euroland
stauffer@thp.uni-koeln.de

\textsuperscript{3} Dipartimento di Fisica, Università di Napoli ‘Federico II’,
Via Cintia, I-80126 Naples, Euroland
Antonio.Coniglio@na.infn.it

Abstract: The number of spanning clusters in four to nine dimensions
does not fully follow the expected size dependence for random percolation.

Researchers were interested already long ago in percolation theory above
the upper critical dimension of six \cite{1,2}, and we followed \cite{3}. At the perco-
lation threshold \cite{4}, there is a theoretical consensus that the number $N$ of
spanning clusters stays finite with increasing lattice size below $d = 6$ di-
mensions, and increases with some power of the lattize size above six dimensions,
for hypercubic lattices of $L^d$ sites \cite{5}. Andronico et al.\cite{6}, however, have
worrying data in five dimensions showing an increase of $N$ with increasing
$L$. Thus we now check this question.

One Fortran program, available from stauffer@thp.uni-koeln.de, checks if
a cluster spans from top to bottom and uses free boundary conditions in this
and one other direction, while helical boundary conditions are used in the
remaining $d - 2$ directions. The spanning properties are known to depend on
boundary conditions and thus no quantitative agreement with \cite{6} is expected.
In three dimensions the average $N$ is about 0.4 for $L = 7$ to 101, roughly
independent of $L$ as predicted; that means there is often no spanning cluster.
Figure 1, however, shows for $d = 5$ an increase of $N$ with increasing $L = 3$
to 101. Figure 2 shows for $d = 7, 8$ and 9 an increase of the multiplicity as
$L^{1.65}, L^{2.49}$ and $L^{3.39}$, respectively. The points in Figs. 1 and 2 are averages
over mostly 1000 runs.

The other Fortran program uses free boundary conditions in all direc-
tions and it is available from fortunat@Physik.Uni-Bielefeld.DE. Its results
in Figs. 3 and 4, which refer mostly to a number of iterations between 10000 and 50000, are qualitatively similar to Figs. 1 and 2. However, one derives instead an increase of the spanning cluster multiplicity as $L^{0.97}$, $L^{1.53}$ and $L^{2.1}$ for $d = 7, 8$ and 9, respectively. We remark that this series of slopes is quite well reproduced by the simple formula $(d - 5)/2$, which is not predicted by any theory and which, if true, would hint the existence of infinite spanning clusters at threshold already in five dimensions. In fact, even the trend of the 6D data is quite well reproduced by a power law with exponent 0.51, which is amazingly close to the 1/2 that one would derive from the above mentioned formula. The 6D data points derived by the first program (Fig. 1) can be instead better described by a logarithmic law, in accord with theory: one sees an increase as $\log_2 (L/2)$. The best fit exponents derived by the two sets of data for $d = 6$ to 9 are listed in Table 1.

|       | P.B.C. | F.B.C. |
|-------|--------|--------|
| 6D    | 0 $(\log^2 (L/2))?$ | 0.51   |
| 7D    | 1.65   | 0.97   |
| 8D    | 2.49   | 1.53   |
| 9D    | 3.39   | 2.10   |

Table 1: Best fit scaling exponents of the spanning cluster multiplicity with the lattice size $L$, corresponding to the mixed boundary conditions (P.B.C.) of the first program and to the free boundaries (F.B.C.) of the second program. The latter series is well described by the formula $(d - 5)/2$.

For $d = 5$ both data sets show an analogous behaviour: the expected plateau is not reached even at the largest lattice used, $L = 101$ for periodic boundary conditions, $L = 70$ for free boundaries. Instead, the trend is quite well described in both cases by a logarithmic law.

As far as the comparison with theory is concerned, neither of the sets of exponents of Table 1 agrees with existing predictions. Moreover, they do not agree either with the following plausible argument:
Let us assume that above the upper critical dimensionality the linear dimension of the system $L$ does not scale asymptotically with the correlation length $\xi$, instead it scales with a “thermodynamical” length $\xi_T$. This length diverges as the critical point is approached with an exponent $\nu_T = 3/d$ for percolation and $\nu_T = 2/d$ for Ising \[3, 7\] models.

What is the meaning of this length $\xi_T$? We believe \[2, 5, 6\] that the number of incipient infinite clusters $N_1$ in a region of linear dimension $\xi$ scales as

$$N_1 \propto \xi^{d-6} \quad (d > 6).$$

The average distance $\xi_1$ between the ”centers” of these clusters is given by $(\xi/\xi_1)^d \propto \xi^{d-6}$. Consequently $\xi_1 \propto \xi^{6/d} \propto \xi_T$. So $\xi_T$ is the average distance between the ”centers” of the spanning clusters in a region of linear dimension $\xi$.

How many spanning clusters are there in a region of linear dimension $\xi_T$? If the clusters did not interpenetrate one would find only one cluster. However, since the clusters do interpenetrate there are many more, depending strongly on the boundary conditions. As first approximation we can assume that there are $N_1 \propto \xi^{d-6}$ spanning clusters.

Using the relation $\xi_T \propto \xi^{6/d}$, we obtain $N_1 \propto \xi_T^{d(d-6)/6}$. Since $\xi_T$ scales as $L$, we get the result that the number of spanning clusters $N_1$ scales as

$$N_1 \propto L^{d(d-6)/6}$$

which gives the exponents 1.17 ($d = 7$), 2.67 ($d = 8$) and 4.5 ($d = 9$). From Table 1 we see that if, on the one hand, the predictions for $d = 7, 8$ can be taken as possible interpolations of the two numerical values we found, the results in nine dimensions (3.39, 2.10) seem to rule out this possibility, being both sensibly smaller than the predicted value (4.5).

Of course, one can always say that the simulated lattice sizes were too small, but nevertheless the discrepancies are worrying.

**Acknowledgements:** This paper was partly written up while DS was at Ecole de Physique et Chimie Industrielles, Lab. PMMH, in Paris; he thanks D. Tiggemann for help. AC would like to acknowledge partial support from MIUR-PRIN 2002 and MIUR-FIRB 2002. SF acknowledges the financial support of the TMR network ERBFMRX-CT-970122 and the DFG Forschergruppe FOR 339/1-2.
References

[1] A. Aharony, Y. Gefen and A. Kapitulnik, J. Phys. A 17, L197 (1984)

[2] A. Coniglio in *Finely Divided Matter* [Proc. Les Houches Winter Conf.],
    N. Boccara and M. Daoud Eds. Springer New York (1985)

[3] A. Aharony and D. Stauffer, Physica A 215, 242 (1995)

[4] P. Grassberger, Phys. Rev. E 67, 036101 (2003)

[5] L. de Arcangelis, J. Phys. A 20, 3057 (1987)

[6] G. Andronico, A. Coniglio, S. Fortunato, hep-lat/0208009 at www.arXiv.org = Nuclear Physics B Proc. Suppl. 119, 876 (2003)

[7] E. Luijten, K. Binder and H. W. J. Blöte, Eur. Phys. J. B 9, 289 (1999)
Random percolation: $d=4(\pm), 5(x), 6(*)$.

Figure 1: Average number $N$ of spanning cluster versus linear lattice dimension $L$ in four, five and six dimensions. (Horizontal axis is logarithmic.)

Random percolation: $d=7(\pm), 8(x), 9(*)$.

Figure 2: As Fig. 1 but in seven, eight and nine dimensions. (Both axes are logarithmic.) The slight curvature suggests lower asymptotic slopes.
Figure 3: As Fig. 1 but with free boundary conditions in all $d$ directions.

Figure 4: As Fig. 2 but with free boundary conditions in all $d$ directions.