Exact Algorithms for 0-1 Integer Programs with Linear Equality Constraints

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Abstract. In this paper, we show \(O(1.415^n)\)-time and \(O(1.190^n)\)-space exact algorithms for 0-1 integer programs where constraints are linear equalities and coefficients are arbitrary real numbers. Our algorithms are quadratically faster than exhaustive search and almost quadratically faster than an algorithm for an inequality version of the problem by Impagliazzo, Lovett, Paturi and Schneider (arXiv:1401.5512), which motivated our work. Rather than improving the time and space complexity, we advance to a simple direction as inclusion of many NP-hard problems in terms of exact exponential algorithms. Specifically, we extend our algorithms to linear optimization problems.

1 Introduction

The Feasibility Problem: The existence of integer solutions for a certain system of equations has been discussed as one of the fundamental problems in the theory of computation. A prominent example is the Hilbert 10th problem on Diophantine equations [18].

In this paper, we study the feasibility problem of 0-1 integer programs whose constraints are only linear equalities as follows:

Problem 1 (Feasibility of 0-1 Integer Programs with Linear Equalities).

Find \(x \in \{0, 1\}^n\) which satisfies a given set of linear equalities \(Ax = b\).

We give an exact algorithm running in \(O(1.415^n)\)-time and \(O(1.190^n)\)-space, which achieves a quadratic speedup compared to exhaustive search running in \(O(2^n)\)-time. Our algorithm can store the data of all the feasible solutions in \(O(1.415^n)\)-space, even if the number of solutions is more than \(O(1.415^n)\).

As a similar problem, which achieves quadratic speedup, there is a quantum algorithm known as Grover’s algorithm for unstructured database search problems [11]. It gives a correct answer with high probability, but our algorithm do not use randomness and always gives a correct answer. Recently, probabilistic polynomial algorithms solving a system of linear equations has been discussed by Raghavendra [19] and Fliege [8]. If we eliminate the 0-1 constraints, we can give a polynomial time algorithm by the Gaussian elimination.
The Optimization Problem: Then, we extend our algorithm for the following standard optimization problem running in $O(1.415^n)$-time and $O(1.190^n)$-space:

**Problem 2 (Optimization of 0-1 Integer Programs with Linear Equalities).**

$$\min c^T x$$
$$\text{s.t. } Ax = b,$$
$$x \in \{0, 1\}^n.$$

We know that there are many sophisticated ideas (e.g., the branch-and-bound method and the cutting-plane method) improving algorithms and implementations for computing 0-1 integer programs \[10,23\]. However, we don’t know any improvements of worst-case time complexity for such a general setting in which elements of $A$ and $b$ can be arbitrary real numbers.

Exact Algorithms for NP-hard Problems: Since there are no polynomial time algorithms for NP-hard problems unless $P=NP$, many researchers have studied exact exponential time algorithms which are faster than exhaustive search for NP-hard problems \[9,15,17,22\]. Integer programs include many NP-hard problems as special cases. For instance, the subset sum problem is a special case of Problem 1 in which the number of constraints is exactly one.

Among several such problems whose exact algorithms have been studied, some problems (e.g., the subset sum problem \[12,20\]) have the same time complexity as Problem 1 and some other problems (e.g., the exact satisfiability problem \[5\] and the exact hitting set problem \[7\]) have algorithms faster than $O(1.415^n)$-time. In particular, the exact satisfiability problem, which is also a special case of Problem 1, has been intensively studied \[5,7\].

On the other hand, it seems to be difficult to improve the time complexity of our algorithms due to a similar reason of NP-hardness. In other words, if we can improve our algorithms, then we simultaneously improve the time complexity of exact algorithms for many NP-hard problems which can be reduced to Problem 1.

Circuit Lower Bounds from Moderately Exponential Algorithms: Very recently, Impagliazzo, Lovett, Paturi and Schneider \[14\] studied the feasibility problem for the inequality version of 0-1 integer programs stated as follows:

**Problem 3 (Feasibility of 0-1 Integer Programs with Linear Inequalities).**

Find $x \in \{0, 1\}^n$ which satisfies a given set of linear inequalities $Ax \geq b$.

Impagliazzo, Lovett, Paturi and Schneider \[14\] gave an algorithm solving Problem 3 in $O(2^{(1 - \text{poly}(1/d))n})$-time where $dn$ is the number of constraints. It improves an algorithm for Problem 3 by Impagliazzo, Paturi and Schneider \[10\], which is faster than $O(2^n)$-time only when the number of inequalities is smaller than $0.136n$. These results are motivated from the challenge initiated by Williams \[21\] for proving lower bounds for certain circuit models. In this context, it is important to give only a modest improvement of the exponential factor from the $O(2^n)$-time exhaustive search.
Our Algorithms: Our algorithms are built on a simple combination of basic techniques on exact algorithms for NP-hard problems. In particular, we use a classic technique called the \(k\)-table method studied in \cite{12,20}. This method splits \(n\)-variables into the \(k\) sets of \(n/k\)-variables, and lists all possible \(2^{n/k}\)-assignments for each set. This preprocessing enables us to give algorithms which run faster than \(O(2^n)\)-time for certain problems.

On the other hand, it was unclear how we construct the \(k\)-table for exact algorithms to compute 0-1 integer programs with linear equalities. The ideas introduced in the two papers \cite{14,16} give us inspiration to overcome technical problems including analysis to bound the time complexity. In this paper, we introduce a notion of the vector equality problem, a variation of the vector domination problem studied by Impagliazzo, Paturi and Schneider \cite{16}, to construct the 2-table for our problems. In Section 3, we show two algorithms solving the vector equality problem and give analysis of its time complexity.

In Section 4, we describe how we compute the feasibility problem and the optimization problem for 0-1 integer programs with linear equalities by reducing them to the vector equality problem. A novel point here is an extension of the feasibility problem to the optimization problem by using the 2-table method. This is achieved by post-processing after solving the feasibility problem with extra storage of the objective function. There are no extra blow-up of the exponential time complexity. In Section 5, we improve the space complexity of our algorithms to \(O(1.190^n)\) following the idea of Shroeppel and Shamir \cite{20}.

2 Notation and Definition

Throughout the paper, we use the following notations. We denote \(m \times n\) constant matrices by \(A\), and \(i, j\)-th element of a matrix \(A\) by \(A_{i,j}\). We use \(b\) and \(c\) as constant vectors. \(c^T\) is the transpose of \(c\). We also use \(x\), \(u\) and \(v\) as variable vectors. We denote \(j\)-th element of a vector \(x\) by \(x_j\). The same notation applies for other constant and variable vectors.

The function \(\text{poly}(n)\) is some polynomial for \(n\). Following the convention in the theory of exact algorithms, we measure the time complexity by the function of \(n\), which is the number of variables. We assume \(m \in O(\text{poly}(n))\) since otherwise the input size is super-polynomial to \(n\).

In this paper, we will give algorithms for 0-1 integer programs with linear equality constraints by reducing them to following problem.

**Definition 1 (Vector Equality).** Two vectors \(u = (u_1, u_2, \ldots, u_m)\) and \(v = (v_1, v_2, \ldots, v_m)\) are equal (i.e., \(u = v\)) if and only if \(u_i = v_i\) for any \(i \leq i \leq m\). Given two sets of \(m\)-dimensional vectors \(U\) and \(V\), the vector equality problem is a problem to output information (e.g., a list of subsets of \(U\) and \(V\)) of all the pairs of two vectors \(u \in U\) and \(v \in V\) such that \(u = v\).

Note that elements of vectors can be real numbers. Therefore, we cannot combine a set of elements into one element unlike the case of integers whose absolute values are bounded \cite{6}.
We will use the lexicographical order to compare two $m$-dimensional vectors.

**Definition 2 (Lexicographical Order).** A vector $u = (u_1, u_2, \cdots, u_m)$ is larger than another vector $v = (v_1, v_2, \cdots, v_m)$ (i.e., $u > v$) if and only if there is an index $l$ ($0 \leq l \leq m$) such that $u_l > v_l$ and $u_i = v_i$ for any $i$ ($0 \leq i < l$).

## 3 Algorithms for the Vector Equality Problem

### 3.1 Overview and Comparison of Two Algorithms

In this section, we present two algorithms (Algorithm 1 and Algorithm 2) to solve the vector equality problem efficiently. Algorithm 1 is simple and uses a sort routine by lexicographical order between two $m$-dimensional vectors, while Algorithm 2 is recursive and uses the idea of measure-and-conquer [9].

Although its theoretical time complexity is the same as the other one, Algorithm 2 has a practical merit when $m$ (the number of constraints) is large. Moreover, we can see the quadratic difference between the equality and inequality versions of integer programs by looking at the recursive algorithm compared to the algorithms for Problem [1], [14], [16].

To analyze the time complexity of Algorithm 2, we need to incorporate an idea using the weighted median, which is introduced very recently by Impagliazzo, Lovett, Paturi and Schneider [14]. Furthermore, we complete our analysis of the time complexity by setting a suitable choice of complexity measure, which is a novel point of this paper, for the search space in the 2-table method.

Algorithm 2 can be regarded as a vector version of the quicksort algorithm with the weighted median as a pivot. Actually, it is not necessary to use the weighted median in practice to select the pivot. A heuristical choice of the pivot may be faster in many cases, while there are no certificates of its worst-case time complexity. After the completion of the first draft including Algorithm 2, we noticed that it can be simplified as Algorithm 1. However, we consider Algorithm 2 is still beneficial to present due to several reasons as mentioned above.

### 3.2 A Simple Algorithm by Sorting

We describe a simple algorithm by sorting for solving the vector equality problem in Algorithm 1. Since the sort of $m$-dimensional vectors can be done in $O(mN \log N)$-time, we can also run Algorithm 1 in $O(mN \log N)$-time.

**Lemma 1.** The vector equality problem can be computed in $O(mN \log N)$-time where $|U| = |V| = N$ by Algorithm 1.

Our algorithm can enumerate all the possible solutions. It may sound strange that we can store the data of all possible solutions within $O(mN \log N)$-space, even if the number of all possible solutions is $\omega(mN \log N)$. This is just because we store the data as a collection of two sets of elements.
Algorithm 1 SortVectorEquality($U$, $V$)

Input: Two sets of $m$-dimensional vectors

Output: A list of two sets of $m$-dimensional vectors

Sort $U$ and $V$ in the ascending lexicographical order, respectively.
($u^k$ and $v^k$ denote the $k$-th vectors in $U$ and $V$, respectively)

Initialize two indices $\alpha = 1$ and $\beta = 1$ for $U$ and $V$.

while $\alpha \leq |U|$ and $\beta \leq |V|$ do
  if $u^\alpha > v^\beta$ then
    Increment $\beta$.
  else if $u^\alpha < v^\beta$ then
    Increment $\alpha$.
  else
    Set $\alpha' := \alpha$, $\beta' := \beta$ and $w := u^\alpha (: = v^\beta)$.
    while $w = u^\alpha$ do
      Increment $\alpha$.
    end while
    while $w = v^\beta$ do
      Increment $\beta$.
    end while
    Output ($\alpha'$, $\alpha$) and ($\beta'$, $\beta$) as representation of two subsets of $U$ and $V$.
  end if
end while

If the number of solutions is bounded by $O(mN \log N)$, then the time complexity to enumerate all the solutions is also $O(mN \log N)$. If the number of solutions is $\omega(mN \log N)$, then the time complexity depends on the number of possible solutions.

3.3 A Recursive Algorithm by Measure-and-Conquer

Another way to solve the vector equality problem is to use a notion of the weighted median to bound the time complexity of our algorithms in the way of measure-and-conquer.

Definition 3 (Weighted Median). The weighted median for a set of weighted numbers is a number such that both the total weight of numbers smaller than the weighted median and the total weight of numbers larger than the weighted median are at most half of the total weight of all the numbers.

Then, we consider a recursive algorithm (Algorithm 2) computing the vector equality problem. Following a linear time algorithm for the unweighted median problem [4], we can give a linear time algorithm for the weighted median problem [3], which is also indicated in [14].

Lemma 2 ([3,14]). The weighted median of $N$ numbers can be computed in $O(N \cdot N)$-time.
Algorithm 2 RecursiveVectorEquality($U, V, i, m$)

**Input:** Two sets of $m$-dimensional vectors and an index $i$ and the dimension $m$.

**Output:** A list of two sets of $m$-dimensional vectors.

if $U = \emptyset$ or $V = \emptyset$ then
    return an empty list
else if $i > m$ then
    return a singleton list of $(U, V)$
else
    1. Find the weighted median $k$ of the $i$-th coordinates of $U \cup V$ with weight $|V|$ and $|U|$ for each element in $U$ and $V$, respectively.
    2. Partition $U$ into three sets:
        (a) $U^+ = \{u \mid u_i > k\}$,
        (b) $U^\times = \{u \mid u_i = k\}$,
        (c) $U^- = \{u \mid u_i < k\}$.
    3. Partition $V$ into three sets:
        (a) $V^+ = \{v \mid v_i > k\}$,
        (b) $V^\times = \{v \mid v_i = k\}$,
        (c) $V^- = \{v \mid v_i < k\}$.
    4. Solve the following three subproblems:
        (a) $L_1 = \text{VectorEquality}(U^+, V^+, i, m)$
        (b) $L_2 = \text{VectorEquality}(U^\times, V^\times, i + 1, m)$
        (c) $L_3 = \text{VectorEquality}(U^-, V^-, i, m)$

    return the concatenation of the three lists $L_1$, $L_2$, and $L_3$
end if

We analyze the time complexity of Algorithm 2 for the vector equality problem in the following lemma.

**Lemma 3.** The vector equality problem can be computed in $O(mN \log N)$-time where $|U| = |V| = N$ by starting Algorithm 2 at RecursiveVectorEquality($U, V, 1, m$).

**Proof.** In Algorithm 2, we find the weighted median $k$ of the $i$-th coordinates of $U \cup V$ where all the elements in $U$ and $V$ have weight $|V|$ and $|U|$, respectively. Then, we partition each of $U$ and $V$ into three sets, respectively.

Two vectors $u \in U$ and $v \in V$ can be equal in at most one of the following three cases:

1. $u \in U^+$ and $v \in V^+$,
2. $u \in U^\times$ and $v \in V^\times$,
3. $u \in U^-$ and $v \in V^-$. 

We solve smaller subproblems of the vector equality problem for the three cases. In particular, we decrease the dimension $m$ to $m - 1$ in the case of (2).
The rule of the partition immediately gives the following equation:
\[ |V| \cdot (|U^+| + |U^-| + |U^-|) + |U| \cdot (|V^+| + |V^-| + |V^-|) = |V| \cdot |U| + |U| \cdot |V|. \]
Dividing it by \(|U| \cdot |V|\), we have
\[ \frac{|U^+| + |U^-| + |U^-|}{|U|} + \frac{|V^+| + |V^-| + |V^-|}{|V|} = 2. \]
For some constants \(s\) and \(t\) such that \(0 \leq s \leq 1\) and \(0 \leq t \leq 1\), we have
\[ \frac{|U^+|}{|U|} + \frac{|V^+|}{|V|} = 1 - s, \]
\[ \frac{|U^-|}{|U|} + \frac{|V^-|}{|V|} = 1 - t, \]
\[ \frac{|U^=|}{|U|} + \frac{|V^=|}{|V|} = s + t \]
because we partitioned \(U\) and \(V\) at the weighted median. Since \(\alpha + \beta \geq 2\sqrt{\alpha \beta}\) for any \(\alpha, \beta \geq 0\), we have
\[ \frac{|U^+|}{|U|} \cdot \frac{|V^+|}{|V|} \leq \frac{1}{4} \cdot (1 - s)^2, \]
\[ \frac{|U^-|}{|U|} \cdot \frac{|V^-|}{|V|} \leq \frac{1}{4} \cdot (1 - t)^2, \]
\[ \frac{|U^=|}{|U|} \cdot \frac{|V^=|}{|V|} \leq \frac{1}{4} \cdot (s + t)^2. \]
Collecting these inequalities, we have
\[ |U^+| \cdot |V^+| \cdot 2^m + |U^-| \cdot |V^-| \cdot 2^m + |U^=| \cdot |V^=| \cdot 2^{m-1} \]
\[ \leq \frac{1}{4} \cdot \left\{ (1 - s)^2 \cdot 2^m + (1 - t)^2 \cdot 2^m + (s + t)^2 \cdot 2^{m-1} \right\} \cdot |U| \cdot |V| \]
\[ = \frac{1}{4} \cdot \left\{ (1 - s)^2 + (1 - t)^2 \cdot \frac{1}{2} \cdot (s + t)^2 \right\} \cdot |U| \cdot |V| \cdot 2^m. \]
It means that the search space \(|U| \cdot |V| \cdot 2^m\) decreases by the factor of
\[ f(s, t) = \frac{1}{4} \cdot \left\{ (1 - s)^2 + (1 - t)^2 \cdot \frac{1}{2} (s + t)^2 \right\} \]
\[ = 0.5 - 0.5s - 0.5t + 0.375s^2 + 0.375t^2 + 0.25st \]
at each recursion.
We can conclude \(f(s, t) \leq \frac{1}{4}\) in the domain of \(0 \leq s \leq 1\) and \(0 \leq t \leq 1\) by the following argument. By taking the partial derivatives, we have
\[ \frac{\partial f(s, t)}{\partial s} = -0.5 + 0.75s + 0.25t, \]
\[ \frac{\partial f(s, t)}{\partial t} = -0.5 + 0.25s + 0.75t. \]
If \( \frac{\partial f(s,t)}{\partial t} > 0 \) (equivalently, \( t > 2 - 3s \)), then the function \( f(s,t) \) is monotonically increasing in the direction of \( s \). If \( \frac{\partial f(s,t)}{\partial s} < 0 \) (equivalently, \( t < 2 - 3s \)), then the function \( f(s,t) \) is monotonically decreasing in the direction of \( s \). The same thing applies for \( t \) instead of \( s \).

Therefore, we can verify that it is maximized at two edges \((s,t) = (0,0), (1,1)\) as \( f(s,t) = 0.5 \) and minimized at the middle point \((s,t) = (0.5,0.5)\) as \( f(s,t) = 0.25 \). Moreover, maximal points except the two edges are only two points \((s,t) = (0,1), (1,0)\) as \( f(s,t) = 0.375 \).

The recursions occur at most \( \log_2(|U| \cdot |V| \cdot 2^m) \in O(m \log N) \) depth. At each depth \( d \) of the recursion, we need to solve at most \( 3^d < N \) subproblems of the vector equality problem, but the total number of elements is at most \( 2N \). Therefore, we can solve the weighted median in linear time \( O(|U| + |V|) = O(N) \) as a whole at each depth of the recursion.

As a consequence, we conclude that the total time complexity of Algorithm 2 is \( O(mN \log N) \). \( \square \)

## 4 Exact Algorithms for 0-1 Integer Programs

In this section, we give an exact algorithm for solving the feasibility and optimization problem of 0-1 integer programs with linear equality constraints by reducing it to the vector equality problem described in the previous section.

**Theorem 1.** The feasibility and optimization problem of 0-1 integer programs with linear equalities (Problem 7 and Problem 2) can be computed in \( O(m \cdot 2^{n/2} \text{poly}(n)) \)-time.

**Proof.** We solve the feasibility problem of \( Ax = b \) by reducing it to the vector equality problem. First, we partition the set of variables \( X = \{x_1, \ldots, x_n\} \) into two disjoint subsets \( X_1 \) and \( X_2 \). Here, we assume the number of variables \( n \) is even without loss of generality.

Let \( \varphi(x_j) \) be assignments of \( x_j \). Then, we define vectors \( u \) and \( v \) by

\[
\begin{align*}
u_i &= \sum_{x_j \in X_1} A_{ij} \cdot \varphi(x_j), \\
v_i &= b_i - \sum_{x_j \in X_2} A_{ij} \cdot \varphi(x_j).
\end{align*}
\]

for each assignment of \( X_1 \) and \( X_2 \). Let \( U \) and \( V \) be two sets of \( 2^{n/2} \) such vectors \( u \) and \( v \), respectively.

Taking into account the linearity of the objective function \( c^T x \) of Problem 2, we can extend the algorithm for the feasibility problem to one for the optimization problem. For this purpose, we additionally calculate weight

\[
\begin{align*}
w(u) &= \sum_{x_j \in X_1} c_j \cdot \varphi(x_j), \\
w(v) &= \sum_{x_j \in X_2} c_j \cdot \varphi(x_j)
\end{align*}
\]

for each of \( u \in U \) and \( v \in V \), respectively.
From the construction of $U$ and $V$, there is a 0,1-vector $x \in \{0, 1\}^n$ satisfying $Ax = b$ if and only if there is a pair of two vectors $u \in U$ and $v \in V$ satisfying $u_i = v_i$ for all $i \ (1 \leq i \leq m)$.

We can solve the vector equality problem for $U$ and $V$ in $O(mN \log N)$-time by using algorithms in Section 3. After the algorithms terminates, we can get a list of submatrices which contains information of all the possible solutions.

$$(U^1, V^1), (U^2, V^2), \ldots, (U^k, V^k), \ldots, (U^l, V^l)$$

There are at most $N = 2^{n/2}$ submatrices. From the construction of the algorithms, each row and column of submatrices has no intersection.

Let $U^k \times V^k \ (U^k \subseteq U \text{ and } V^k \subseteq V)$ be one of such submatrices. Then we would like to solve the following optimization problem for each $k$.

$$\min w(u) + w(v)$$

s.t. $u \in U^k$ and $v \in V^k$.

From the linearity of $c^T$, $w(u)$ and $w(v)$ are independent. Therefore, the above minimization problem is solvable separately for $u$ and $v$. Hence, $O(|U^k| + |V^k|)$-time is sufficient to optimize.

We solve the same problem for each submatrices and take the minimum of all the problems. The total time complexity is $O(m \cdot 2^{n/2}\text{poly}(n))$. \(\square\)

5 Improved Space Complexity

Shroeppel and Shamir [20] studied the $k$-table method, which is a generalization of the 2-table method, and showed an $O(2^{n/4})$-space exact algorithm for the subset sum problem (a special case of the Problem 1) by using the 4-table method. Following the idea of Shroeppel and Shamir [20] using the priority queue, we can reduce the space complexity of our algorithms from $O(2^{n/2})$ to $O(2^{n/4})$ as in the following theorem.

**Theorem 2.** The feasibility and optimization problem of 0-1 integer programs with linear equality constraints (Problem 1 and Problem 2) can be computed in $O(m \cdot 2^{n/2}\text{poly}(n))$-time and $O(m \cdot 2^{n/4}\text{poly}(n))$-space.

**Proof.** We partition the set of variables $X = \{x_1, \cdots, x_n\}$ into four disjoint subsets $X_1$, $X_2$, $X_3$ and $X_4$. Here, we assume the number of variables $n$ can be divided by 4 without loss of generality.

Let $\varphi(x_j)$ be an assignment of $x_j$. Then, we define vectors $u$, $v$, $s$ and $t$ by

$$u_i = \sum_{x_j \in X_1} A_{ij} \cdot \varphi(x_j), \quad v_i = \sum_{x_j \in X_2} A_{ij} \cdot \varphi(x_j),$$

$$s_i = - \sum_{x_j \in X_3} A_{ij} \cdot \varphi(x_j), \quad t_i = b_i - \sum_{x_j \in X_4} A_{ij} \cdot \varphi(x_j).$$
for each assignment of $X_1$, $X_2$, $X_3$ and $X_4$. Let $U$, $V$, $S$ and $T$ be four sets of $2^{n/4}$ such vectors $u$, $v$, $s$ and $t$, respectively.

We additionally calculate weight

$$w(u) = \sum_{x_j \in X_1} c_j \cdot \varphi(x_j), \quad w(v) = \sum_{x_j \in X_2} c_j \cdot \varphi(x_j),$$

$$w(s) = \sum_{x_j \in X_3} c_j \cdot \varphi(x_j), \quad w(t) = \sum_{x_j \in X_4} c_j \cdot \varphi(x_j).$$

for each of $u \in U$, $v \in V$, $s \in S$ and $t \in T$, respectively. For each vector, we can store data of its corresponding assignment and weight within $O(n)$-space.

From the construction of the four sets, there is a 0,1-vector $x \in \{0,1\}^n$ satisfying $Ax = b$ if and only if a quartet of four vectors $u \in U$, $v \in V$, $s \in S$ and $t \in T$ such that $u + v = s + t$. We can search such quartets by Algorithm 3 in $O(mN^2 \log N)$-time and $O(mN)$-space where $|U| = |V| = |S| = |T| = N$. In the algorithm, we use priority queues in which we can push and pop any element in the logarithmic time to the number of elements.

We can compute the minimum objective value and the corresponding assignment of the original 0-1 integer programs by Algorithm 3 where the inputs are given as four sets of $m$-dimensional vectors with their assignments $\varphi$ and weights $w$. The return value of $\infty$ means that the problem is infeasible. If it is feasible, we can retrieve the corresponding assignment $\varphi$ of variables from the values of $\text{SOL}$ in Algorithm 3.

Corollary 1. The feasibility and optimization problems of 0-1 integer programs with linear equality constraints (Problem 1 and Problem 2) can be computed in $O(1.415^n)$-time and $O(1.190^n)$-space.

6 Conclusions

In this paper, we have presented $O(1.415^n)$-time and $O(1.190^n)$-space exact algorithms for 0-1 integer programs with linear equality constraints. We can apply our algorithms to the optimization problem as well as the feasibility problem. We can also extend our algorithms to integer programs where their variables are constrained by any finite set of integers. There are several recent progress on the subset sum problem such as time-space tradeoff results [1] and improved algorithms for a certain important class of the subset sum problem [2,3]. It would be interesting to investigate in these directions with connection to our results concerned with 0-1 integer programs.

Our computational experiments show that our algorithms can solve 0-1 integer programs with around 60 variables which are generated in a random way. Some of well-known IP solvers cannot solve these instances because they do not have any favorable structures to cut down the search space. By connecting our algorithms to existing techniques for 0-1 integer programs (e.g., the branch-and-bound method), we hope that our algorithms will be useful from the practical point of view as well as theoretical analysis.
Algorithm 3 VectorSumEquality($U, V, S, T$)

Sort $U$, $V$, $S$ and $T$ in the ascending lexicographical order, respectively. ($u^k$, $v^k$, $s^k$, and $t^k$ denote the $k$-th vectors in $U$, $V$, $S$ and $T$, respectively)

Set $\text{MIN} := \infty$ and initialize two priority queues $Q_1$ and $Q_2$ as empty sets.

for $k = 1$ to $|V|$ do
  Push $(u^k, v^1)$ to the priority queue $Q_1$.
end for

for $k = 1$ to $|T|$ do
  Push $(s^k, t^1)$ to the priority queue $Q_2$.
end for

while Both of $Q_1$ and $Q_2$ are not empty do
  Take the top elements $(u^\alpha, v^\beta)$ and $(s^\gamma, t^\delta)$ from $Q_1$ and $Q_2$, respectively.
  if $u^\alpha + v^\beta < s^\gamma + t^\delta$ then
    Pop $(u^\alpha, v^\beta)$.
    if $\beta + 1 \leq |V|$ then
      Push $(u^\alpha, v^{\beta+1})$.
    end if
  else if $u^\alpha + v^\beta > s^\gamma + t^\delta$ then
    Pop $(s^\gamma, t^\delta)$.
    if $\delta + 1 \leq |T|$ then
      Push $(s^\gamma, t^{\delta+1})$.
    end if
  else
    $w := u^\alpha + v^\beta$ ($= s^\gamma + t^\delta$);
    $\text{MIN}_1 := \infty$; $\text{MIN}_2 := \infty$;
    while $Q_1 \neq \emptyset$ and $w = u^{\alpha'} + v^{\beta'}$ where $(u^{\alpha'}, v^{\beta'})$ is the top element of $Q_1$ do
      Pop $(u^{\alpha'}, v^{\beta'})$.
      if $\beta' + 1 \leq |V|$ then
        Push $(u^{\alpha'}, v^{\beta'+1})$.
      end if
    end while
    while $Q_2 \neq \emptyset$ and $w = s^{\gamma'} + t^{\delta'}$ where $(s^{\gamma'}, t^{\delta'})$ is the top element of $Q_2$ do
      Pop $(s^{\gamma'}, t^{\delta'})$.
      if $\delta' + 1 \leq |T|$ then
        Push $(s^{\gamma'}, t^{\delta'+1})$.
      end if
    end while
  end else
end while

if $\text{MIN} > \text{MIN}_1 + \text{MIN}_2$ then
  $\text{MIN} := \text{MIN}_1 + \text{MIN}_2$; $\text{SOL} := (\text{SOL}_1, \text{SOL}_2)$;
end if

Return $\text{MIN}$
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