Arc-Consistency computes the minimal binarised domains of an STP. Use of the result in a TCSP solver, in a TCSP-based job shop scheduler, and in generalising Dijkstra’s one-to-all algorithm

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Abstract

TCSPs (Temporal Constraint Satisfaction Problems), as defined in [Dechter et al., 1991], get rid of unary constraints by binarising them after having added an "origin of the world" variable. The constraints are therefore exclusively binary; additionally, a TCSP verifies the property that it is node-consistent and arc-consistent. Path-consistency, the next higher local consistency, solves the consistency problem of a convex TCSP, referred to in [Dechter et al., 1991] as an STP (Simple Temporal Problem); more than that, the output of path-consistency applied to an $n + 1$-variable STP is a minimal and strongly $n + 1$-consistent STP. Weaker versions of path-consistency, aimed at avoiding what is referred to in [Schwalb and Dechter, 1997] as the "fragmentation problem", are used as filtering procedures in recursive backtracking algorithms for the consistency problem of a general TCSP. In this work, we look at the constraints between the "origin of the world" variable and the other variables, as the (binarised) domains of these other variables. With this in mind, we define a notion of arc-consistency for TCSPs, which we will refer to as binarised-domains Arc-Consistency, or bdArc-Consistency for short. We provide an algorithm achieving bdArc-Consistency for a TCSP, which we will refer to as bdAC3, for it is an adaptation of Mackworth’s [1977] well-known arc-consistency algorithm AC3. We show that bdArc-Consistency computes the minimal (binarised) domains of an STP. We then show how to use the result in a general TCSP solver, in a TCSP-based job shop scheduler, and in generalising the well-known Dijkstra’s one-to-all shortest paths algorithm.

1 Introduction

The first version of the definition of a Temporal Constraint Satisfaction Problem (TCSP), as given in [Dechter et al., 1991], says that the constraints are either unary or binary. Then, in the same article, the definition has been modified through the addition of a new variable referred to as the "origin of the world" variable, which is used to binarise the unary constraints. This addition of an "origin of the world" variable makes the constraints of a TCSP exclusively binary, with the consequence that a TCSP verifies the properties of node- and arc-consistencies. While the adding of an "origin of the world" variable has the advantage of transforming the solving of an STP, i.e., a TCSP whose constraints are convex, into a shortest paths problem, it remains that, because a TCSP becomes then node- and arc-consistent, it gets easy for one’s attention to immediately skip to the next higher local consistency, path-consistency, which is exactly what seems to have happened in [Dechter et al., 1991].

In this work, while we will keep the idea of having an "origin of the world" variable, and of binarising the unary constraints, making the constraints of a TCSP exclusively binary, we will look at the constraints between the "origin of the world" variable and the other variables, as the (binarised) domains of these other variables. With this in mind, we define a notion of arc-consistency for TCSPs, which we will refer to as binarised-domains Arc-Consistency, or bdArc-Consistency for short. We provide an algorithm achieving bdArc-Consistency for a TCSP, which we will refer to as bdAC3, for it is an adaptation of Mackworth’s [1977] well-known arc-consistency algorithm AC3. We show that bdArc-Consistency computes the minimal (binarised) domains of an STP. We then show how to use the result in a general TCSP solver, in a TCSP-based job shop scheduler, and in generalising the well-known Dijkstra’s one-to-all shortest paths algorithm.

The rest of paper is organised as follows. Chapter 2 is devoted to needed background. Chapter 3 defines our notion of binarised-domains Arc-Consistency (bdArc-Consistency) for TCSPs; provides an algorithm achieving bdArc-Consistency for a TCSP; shows the result that the binarised domains of a bdArc-Consistent STP are minimal; and ends with an important corollary generalising Dijkstra’s one-to-all shortest-paths algorithm [Aho et al., 1976; Papadimitriou and Steiglitz, 1982] to $\mathbb{R}$-labelled directed graphs. Chapter 4 provides a general TCSP solver using a weak version of bdArc-Consistency as the filtering procedure during the search, whose completeness is a direct conse-
quence of our minimality result; it also makes another use of our minimality result to give a backtrack-free procedure for the search for a solution of a bdArc-Consistent STP. Chapter 5 provides a TCSP-based job shop scheduler adapted from the general TCSP solver of Chapter 4. Chapter 6 summarises the work.

2 Temporal Constraint Satisfaction Problems

Temporal Constraint Satisfaction Problems, or TCSPs for short, have been proposed in [Dechter et al., 1991] as an extension of (discrete) CSPs [Mackworth, 1977; Montanari, 1974] to continuous variables.

Definition 1 (TCSP [Dechter et al., 1991]). A TCSP is a pair \( P = (X, C) \) consisting of (1) a finite set \( X \) of \( n \) variables, \( X_1, \ldots, X_n \), ranging over the universe of time points; and (2) a finite set \( C \) of Dechter, Meiri and Pearl’s constraints (henceforth DMP constraints) on the variables.

A DMP constraint is either unary or binary. A unary constraint has the form \( X_i \in C_i \), and a binary constraint the form \((X_j - X_i) \in C_{ij}\), where \( C_i \) and \( C_{ij} \) are subsets of the set \( \mathbb{R} \) of real numbers, and \( X_i \) and \( X_j \) are variables ranging over the universe of time points. A unary constraint \( X_i \in C_i \) may be seen as a special binary constraint if we consider an origin of the World (time 0), represented by a variable \( X_0 \); \( X_i \in C_i \) is then equivalent to \((X_i - X_0) \in C_{0i}, \) with \( C_{0i} = C_i \). Unless explicitly stated otherwise, we assume, in the rest of the paper, that the constraints of a TCSP are all binary. Furthermore, without loss of generality, we make the assumption that all constraints \((X_j - X_i) \in C_{ij}\) of a TCSP are such that \( i < j \): if this is not the case for a constraint \((X_j - X_i) \in C_{ij}\), we replace it with the equivalent constraint \((X_i - X_j) \in C_{ij}\) (see the definition of converse below).

Definition 2 (STP [Dechter et al., 1991]). An STP (Simple Temporal Problem) is a TCSP of which all the constraints are convex, i.e., of the form \((X_j - X_i) \in C_{ij}\), \( C_{ij} \) being a convex subset of \( \mathbb{R} \).

A universal constraint for TCSPs in general, and for STPs in particular, is of the form \((X_j - X_i) \in \mathbb{R}\), and is equivalent to “no knowledge” on the difference \((X_j - X_i)\). An equality constraint is of the form \((X_j - X_i) \in \{0\}\): it “forces” variables \( X_i \) and \( X_j \) to be equal.

We now consider an \( n + 1 \)-variable TCSP \( P = (X, C) \), with \( X = \{X_0, X_1, \ldots, X_n\} \), variable \( X_0 \) representing the origin of the World. Without loss of generality, we assume that \( P \) has at most one constraint per pair of variables.

Definition 3 (network representation). The network representation of \( P \) is the labelled directed graph of which the vertices are the variables of \( P \), and the edges are the pairs \((X_i, X_j)\) of variables on which a constraint \((X_j - X_i) \in C_{ij}\) is specified. The label of edge \((X_i, X_j)\) is the set \( C_{ij} \) such that \((X_j - X_i) \in C_{ij}\) is the constraint of \( P \) on the pair \((X_i, X_j)\) of variables.

Definition 4 (matrix representation). The matrix representation of \( P \) is the \((n + 1) \times (n + 1)\)-matrix, denoted by \( P \) for simplicity, defined as follows: \( P_{ii} = \{0\}, \forall i = 0 \ldots n; \ P_{ij} = C_{ij} \) and \( P_{ji} = \{a \ s.t. (-a) \in C_{ij}\}, \) for all \( i \neq j \) such that a constraint \((X_j - X_i) \in C_{ij}\) is specified on \( X_i \) and \( X_j \); \( P_{ij} = (-\infty, +\infty), \) for all other pairs \((i, j)\).

Definition 5 ((consistent) instantiation). An instantiation of \( P \) is any \( n + 1 \)-tuple \((x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}\), representing an assignment of a value to each variable. A consistent instantiation, or solution, is an instantiation satisfying all the constraints: for all \( i, j \), \((x_i - x_j) \in C_{ij}\) satisfies the constraint, if any, specified on the pair \((X_i, X_j)\).

Definition 6 (subnetwork). A \( k \)-variable subnetwork, \( k \leq n + 1 \), is any restriction of the network to \( k \) of its variables and the constraints on pairs of those \( k \) variables.

Definition 7 ((strong) \( k \)-consistency). For all \( k = 1 \ldots (n + 1) \), \( P \) is \( k \)-consistent if any solution to any \((k - 1)\)-variable subnetwork extends to any \( k \)-th variable; it is strongly \( k \)-consistent if it is \( j \)-consistent, for all \( j \leq k \).

Strong 1-, 2- and 3-consistencies correspond to node-, arc- and path-consistencies, respectively [Montanari, 1974; Mackworth, 1977]. Strong \((n + 1)\)-consistency of \( P \) facilitates the exhibition of a solution by backtrack-free search [Freuder, 1982].

The consistency problem of a TCSP, i.e. the problem of verifying whether it has a consistent instantiation, is NP-hard. Davis [1989] (cited in [Dechter et al., 1991]) showed that even the subclass of TCSPs in which the constraints are of the form \((X_j - X_i) \in C_{ij}\), with \( C_{ij} \) being a convex set or a union of two disjoint convex sets, is NP-hard (see also [Dechter et al., 1991], Theorem 4.1, Page 73). However, when we restrict ourselves to STPs, the consistency problem is polynomial [Dechter et al., 1991]. Moreover, in the case of STPs the classical path-consistency method [Montanari, 1974; Mackworth, 1977] leads to strong \((n + 1)\)-consistency [Dechter et al., 1991].

We now briefly describe the standard algebraic operations, well-known within the CSP community: converse, intersection and composition.

Definition 8 (converse). The converse of a DMP constraint \((X_j - X_i) \in C_{ij}\) is the DMP constraint \((X_i - X_j) \in C_{ij}\), such that \( C_{ij} = \{a \ s.t. (-a) \in C_{ij}\} \). The two constraints are equivalent: \((X_j - X_i) \in C_{ij} \iff (X_i - X_j) \in C_{ij}\).

Definition 9 (intersection). The intersection of two DMP constraints \((X_j - X_i) \in C_{ij}^1\) and \((X_j - X_i) \in C_{ij}^2\), on the same pair of variables, is the DMP constraint \((X_j - X_i) \in C_{ij}^1 \cap C_{ij}^2\), where \( C_{ij}^1 \) is the set-theoretic intersection of \( C_{ij}^1 \) and \( C_{ij}^2\).

Definition 10 (composition). The composition of two DMP constraints \((X_k - X_i) \in C_{ik} \) and \((X_j - X_k) \in C_{kj}\), written \((X_k - X_i) \in C_{ik} \odot (X_j - X_k) \in C_{kj}\), is the constraint \((X_j - X_i) \in C_{ij}\), on the extreme variables \( X_i \) and \( X_j \), such that \( C_{ij} = \{c : \exists a \in C_{ik}, \exists b \in C_{kj} \ s.t. c = a + b\} \). We will also say that the composition of the sets \( C_{ik} \) and \( C_{kj}\) is the set \( C_{ij}^2: C_{ij}^2 = C_{ij}\).

We refer to the operation \( C_{ij} := C_{ij} \cap C_{ik} \odot C_{kj}\), consisting of replacing, in a triangle \((X_i, X_k, X_j)\) of variables, the label \( C_{ij} \) on edge \((X_i, X_j)\) by its intersection with the composition \( C_{ik} \odot C_{kj}\) of the labels on the other two edges, as the path-consistency operation on triangle \((X_i, X_k, X_j)\).
Applying path consistency (henceforth PC) to a TCSP consists of repeating, until stability, the process of applying the path-consistency operation to each triangle \((X_i, X_k, X_j)\) of variables.

**Definition 11** (weak composition). The weak composition of two DMP constraints \((X_k - X_j) \in C_{ik}\) and \((X_j - X_k) \in C_{kj}\), written \((X_k - X_i) \in C_{ik} \otimes_w (X_j - X_k) \in C_{kj}\), is the composition of their convex closures; i.e., \((X_k - X_i) \in C_{ik} \otimes_w (X_j - X_k) \in C_{kj}\) is equal to \((X_k - X_i) \in cc(C_{ik}) \otimes (X_j - X_k) \in cc(C_{kj})\), where \(cc(S)\), for a set \(S\), is the convex closure of \(S\) (the smallest convex set containing \(S\)).

A distance graph is a complete labelled directed graph \(G = (V, E, w)\), where \(V\) is the set of vertices (nodes), \(E = V \times V\) is the set of edges, and \(w\) is a labelling function from \(E\) to the set \(\mathbb{R} \cup \{+\infty\}\), \(\mathbb{R}\) being the set of real numbers. A rooted distance graph is a pair \((G, S)\), where \(G\) is a distance graph and \(S\) is a node of \(G\). We now define the distance graph and the rooted distance graph of an STP.

**Definition 12** (rooted distance graph of an STP). If the TCSP \(P = (X, C)\) is an STP, its distance graph is the distancegraph \(G = (X, X \times X, w)\), with the set of vertices being the set \(X\) of variables of \(P\), and the labelling function \(w\) built as follows:

1. for all \(X_i\), \(w(X_i, X_i) = 0\)
2. for all variables \(X_i\) and \(X_j\), with \(i < j\), if \(P\) has a constraint \((X_j - X_i) \in C_{ij}\), with \(C_{ij}\) of the form \((-\infty, b)\), \([a, b]\) or \((a, b)\), then the label of edge \((X_i, X_j)\) is \(b\): \(w(X_i, X_j) = b\)
3. for all variables \(X_i\) and \(X_j\), with \(i < j\), if \(P\) has a constraint \((X_j - X_i) \in C_{ij}\), with \(C_{ij}\) of the form \((-\infty, b)\), \([a, b]\) or \((a, b)\), then the label of edge \((X_i, X_j)\) is \(b^-\): \(w(X_i, X_j) = b^-\)
4. for all variables \(X_i\) and \(X_j\), with \(i < j\), if \(P\) has a constraint \((X_j - X_i) \in C_{ij}\), with \(C_{ij}\) of the form \([a, +\infty)\), \([a, b]\) or \((a, b)\), then the label of edge \((X_j, X_i)\) is \(-a\): \(w(X_j, X_i) = -a\)
5. for all variables \(X_i\) and \(X_j\), with \(i < j\), if \(P\) has a constraint \((X_j - X_i) \in C_{ij}\), with \(C_{ij}\) of the form \((a, +\infty)\), \([a, b]\) or \((a, b)\), then the label of edge \((X_j, X_i)\) is \((-a)^-\): \(w(X_j, X_i) = (-a)^-\)
6. all other edges \((X_i, X_j)\) are such that \(w(X_i, X_j) = +\infty\)

We refer to the rooted distance graph \((G, X_0)\) as the rooted distance graph of the STP \(P\).

For example, if the STP contains a constraint \((X_j - X_i) \in [a, b]\), then, from Items 3 and 4, the two edges \((X_i, X_j)\) and \((X_j, X_i)\) are labelled, respectively, with \(b^-\) and \(-a\). This is so because the constraint is equivalent to the following conjunction of linear inequalities: \(X_j - X_i < b \land X_i - X_j \leq -a\). To have uniform linear inequalities, using \(-a\) (lower than or equal to) to write the inequality \(X_j - X_i < b\) as \(X_j - X_i \leq b^-\), the minus sign in \(b^-\) meaning that the upper bound \(b\) is not reached. Note that if a distance graph contains an edge \((X, Y)\) labelled with \(a\), this is interpreted as \(Y - X \leq a\).

In order to be able to apply shortest paths algorithms to a (rooted) distance graph, such as Dijkstra’s one-to-all and Floyd-Warshall’s all-to-all shortest paths algorithms [Aho et al., 1976; Papadimitriou and Steiglitz, 1982], the addition of real numbers (+) and their comparison ($) should be generalised to \(\mathbb{R} \cup \{+\infty\}\), where \(\mathbb{R}^+ = \{a^-\text{ such that a \in \mathbb{R}}\}\). This is done as follows, where \(a, b \in \mathbb{R}: a + b = a^- + b^- = a^- + b^- = (a + b)^-\) if \(a < b\); \(a^- < b^-\) if \(a \leq b\); \(a^- < b^-\) if \(a < b\). Of particular importance for the detection of negative-length circuits, \(0^- < 0\), meaning that \(0^-\) is a (strictly) negative length.

**Definition 13** (d-graph of an STP). The d-graph of an STP \(P = (X, C)\) is the distance graph \(G = (X, X \times X, w)\), with \(w\) defined as follows: \(w(X_i, X_j)\) is the length of the shortest path from node \(X_i\) to node \(X_j\) in the distance graph of \(P\).

The d-graph of an STP can be built from its distance graph using Floyd-Warshall’s all-to-all shortest paths algorithm [Aho et al., 1976; Papadimitriou and Steiglitz, 1982].

**Definition 14** (STP of a rooted distance graph). Let \((G, V_0)\) be a rooted distance graph, with \(G = (V, V \times V, w)\) and \(V = \{V_0, V_1, \ldots, V_n\}\). The STP of \((G, V_0)\) is the STP \(P = (X, C)\), with \(X = \{X_0, X_1, \ldots, X_n\}\), constructed as follows:

1. Initialise \(C\) to the empty set: \(C = \emptyset\)
2. for all vertices \(V_i\) and \(V_j\) of \(G\), with \(i < j\), such that at least one of the two edges \((V_i, V_j)\) and \((V_j, V_i)\) is not labelled with \(+\infty\):
   
   (a) let \(l_1 = w(V_i, V_j)\) and \(l_2 = w(V_j, V_i)\);
   
   (b) add to \(C\) the constraint \((X_j - X_i) \in M[l_1, l_2]\), where \(M[l_1, l_2]\) is as given by the table \(M\) below:

   
   | \(l_1\) | \(l_2\) |
   |---|---|
   | \(+\infty\) | \((-\infty, +\infty)\) |
   | \((-\infty, a]\) | \((-\infty, +\infty)\) |
   | \((-\infty, a)\) | \((-\infty, a]\) |
   | \((-\infty, a)\) | \((-\infty, a)\) |
   
   For instance, if the edges \((V_1, V_5)\) and \((V_5, V_1)\) are labelled, respectively, with 6 and \((-2)^-\), the constraint \((X_5 - X_1) \in \{2, 6\}\) is added to \(C\).

In [Dechter et al., 1991], it has been shown that applying path-consistency to an STP \(P\) is equivalent to applying Floyd-Warshall’s all-to-all shortest paths algorithm [Aho et al., 1976; Papadimitriou and Steiglitz, 1982] to the distance graph of \(P\). In other words, if \(P^r\) is the STP resulting from applying path consistency to \(P (P^r = PC(P))\), then \(P^r\) is exactly the STP of the d-graph of \(P\). Furthermore, \(P^r\) is minimal and strongly \(n + 1\)-consistent, \(n + 1\) being the number of variables.

**Theorem 1.** Let \(P\) be an \(n + 1\)-variable STP. If path-consistency applied to \(P\) does not detect an inconsistency, then the resulting STP \(P^r\) is minimal and strongly \(n + 1\)-consistent. Furthermore, \(P^r\) is the STP of the d-graph of \(P\).
3 An arc-consistency algorithm for TCSPs

Binarised-domains Arc-Consistency, or bdAC for short, is defined as follows:

**Definition 15** (binarised-domains arc-consistency). Let $P$ be a TCSP. $P$ is said to verify the binarised-domains arc-consistency, or to be bdArc-Consistent for short, if for all all $i, j \in \{1, \ldots, n\}$, $i \neq j$, the following holds: $P_{0j} \subseteq P_{0i} \otimes P_{ij}$.

The algorithm of Figure 1 achieves bdArc-Consistency of a TCSP $P$. We refer to the algorithm as bdAC3, for it is an adaptation of Mackworth’s [1977] well-known arc-consistency algorithm AC3. bdAC3 initialises a queue $Q$ to all pairs $(i, j)$ such that $i \neq 0$, $j \neq 0$, $i \neq j$ and $P$ contains a (binary) constraint on $X_i$ and $X_j$. Then it proceeds by taking in turn the pairs in $Q$ for propagation. When a pair $(k, m)$ is taken (and removed) from $Q$, bdAC3 calls the procedure REVISE which eventually updates $P_{km}$, which represents the binarised domain of $X_k$, if it is not a subset of the composition $P_{om} \otimes P_{nk}$. If REVISE successfully updates $P_{ok}$, the pairs $(i, k)$ such that $P$ has a constraint on $X_i$ and $K_k$ are added to $Q$ if they are not already there. The algorithm terminates if a binarised domain becomes empty, or if the queue $Q$ becomes empty.

**procedure** REVISE($i, j$)$\{$
1. DELETE=false
2. $temp = P_{0i} \cap P_{0j} \otimes P_{ji}$
3. if $temp \neq P_{0i}$
4. $P_{0i} = temp; P_{0i} = temp^-; DELETE$=true
5. return DELETE$\}$

**procedure** bdAC3($P$)$\{$
1. $Q = \{(i, j) : P$ has a constraint on $X_i$ and $X_j$, $i \neq j \neq 0$; Empty_domain = false$\}$
2. while ($Q \neq \emptyset$ and (not Empty_domain))$\{$
3. select and delete an arc $(k, m)$ from $Q$;
4. if REVISE($k, m$)
5. if ($P_{om} = \emptyset$) Empty_domain = true
6. else $Q = Q \cup \{(i, k) : P$ has a constraint on $X_i$ and $X_k$, $i \neq k \neq 0$; $k \neq i \neq m\}$
7. return(not Empty_domain)$\}$

Figure 1: The binarised-domains Arc-Consistency algorithm bdAC3, and the procedure REVISE it makes use of.

The range of a TCSP’s constraint $(X_j - X_i) \in C_{ij}$, as defined in [Dechter et al., 1991] (Definition 5.6, Page 80), supposes that $C_{ij}$ has a finite lower bound and a finite upper bound, and the range of the constraint is the distance separating these two bounds. The range of a TCSP is then defined as the maximum range over all its constraints. The range of a TCSP is used to determine termination and complexity of path consistency applied to the TCSP (Theorem 5.7, Page 80). This excludes the possibility for $C_{ij}$ to have an infinite lower bound or an infinite upper bound. In our case, the constraints are supposed general, and may have either or both bounds infinite. We consider therefore another, more realistic, definition of range for the determination of termination and complexity of bdAC3:

**Definition 16** (range of a TCSP). Let $P = (X, C)$ be an $n+1$-variable TCSP and $c$ a constraint of $P$ of the form $(X_j - X_i) \in C_{ij}$. The lower bound $lb(c)$ and the upper bound $ub(c)$ of $c$ are, respectively, the lower bound and the upper bound of $C_{ij}$. The set of finite bounds of $P$, $sfb(P)$, is defined as $sfb(P) = \mathbb{IR} \cap \bigcup_{c \in C} \{lb(c), ub(c)\}$. The range of $P$ is $rg(P) = n \times \max \{|x| : x \in sfb(P)\}$.

**Theorem 2.** Let $P = (X, C)$ be a TCSP, with $|X| = n + 1$. bdAC3 applied to $P$ can be achieved in $O(n^2 R)$ relaxation steps (calls of the procedure REVISE of Figure 1) and $O(n^2 R^2)$ arithmetic operations, where $R = rg(P)$ is the range of $P$ expressed in the coarsest possible time units.

**Proof:** The worst case scenario of bdAC3 occurs when, whenever the procedure REVISE updates $P_{0i}$, the length of the set $P_{0i}$ decreases by one time unit. Furthermore, if $P_{0i}$ has a finite bound, it will be in the set $[−R, R]$. Therefore, $P_{0i}$ can be updated $O(R)$ times. The pairs $(i, k)$ that enter initially the queue $Q$ of bdAC3 are such that there is a constraint of $P$ on $X_i$ and $X_j$. A pair $(k, i)$ can reenter the queue $O(R)$ times, whenever $P_{0i}$ has been updated. Because there are $O(n^2)$ constraints, the number of relaxation steps is $O(n^2 R)$. A relaxation step consists of a call REVISE($i, j$) consisting mainly of the computation of a path consistency operation of the form $P_{0i} = P_{0i} \cap P_{0j} \otimes P_{ji}$, which needs $O(R^2)$ arithmetic operations since each of the three sets $P_{0i}$, $P_{0j}$ and $P_{ji}$ is a union of at most $R$ convex subsets. The whole algorithm therefore terminates in $O(n^2 R^3)$ time.

**Theorem 3.** Let $P$ be an STP. If $P$ is bdArc-Consistent, its (binarised) domains are minimal: for all $i \in \{1, \ldots, n\}$, for all $a \in P_{0i}$, there exists a solution $(X_0, X_1, \ldots, X_n)$ such that $i_t - i_0 = a$.

**Proof:** Let $P$ be a bdArc-Consistent STP, and $(G, X_0)$ its rooted distance graph, with $G = (X, X \times X, w)$. According to Theorem 1, showing that the binarised domains $P_{0i}$, with $i \in \{1, \ldots, n\}$, are minimal, is equivalent to showing that the labels (weights) $w(X_0, X_i)$ and $w(X_i, X_0)$ are the lengths of the shortest paths from $X_0$ to $X_i$ and from $X_i$ to $X_0$, respectively. Suppose that this is not the case; in other words, that for some $i \in \{1, \ldots, n\}$, either $w(X_0, X_i)$ is not the length of the shortest path from $X_0$ to $X_i$, or $w(X_i, X_0)$ is not the length of the shortest path from $X_i$ to $X_0$:

1. Case 1: $w(X_0, X_i)$ is not the length of the shortest path from $X_0$ to $X_i$. This would mean the existence of a path $< X_{i_0} = X_0, X_{i_1} = X_j, \ldots, X_{i_t} = X_i >$ from $X_0$ to $X_i$ through $X_j$, whose length is strictly smaller than $w(X_0, X_j) + w(X_j, X_i) + \cdots + w(X_{i_{t-1}}, X_{i_t}) < w(X_{i_0}, X_{i_t})$. But, because the STP is bdArc-Consistent, we have the
following:
\[
\begin{align*}
    w(X_0, X_{i_1}) & \leq w(X_0, X_{i_1}) + w(X_{i_1}, X_{i_2}) \\
    w(X_0, X_{i_2}) & \leq w(X_0, X_{i_2}) + w(X_{i_2}, X_{i_3}) \\
    & \vdots \\
    w(X_0, X_{i_k}) & \leq w(X_0, X_{i_{k-1}}) + w(X_{i_{k-1}}, X_{i_k})
\end{align*}
\]

from which we get:
\[
    w(X_0, X_{i_1}) + \sum_{l=2}^{k-1} w(X_{i_l}, X_{i_{l+1}}) \leq w(X_0, X_{i_1}) + w(X_{i_1}, X_{i_2}) + \cdots + w(X_{i_{k-1}}, X_{i_k}) + \sum_{l=2}^{k-1} w(X_{i_l}, X_{i_{l+1}}).
\]

This, in turn, gives:
\[
    w(X_0, X_{i_1}) + w(X_{i_1}, X_{i_2}) + \cdots + w(X_{i_{k-1}}, X_{i_k}),
\]

which clearly contradicts our supposition.

2. Case 2: \(w(X_i, X_0)\) is not the length of the shortest path from \(X_i\) to \(X_0\). We show in a similar way that the supposition leads to a contradiction. \(\blacksquare\)

As an immediate consequence of Theorem 3, the following corollary generalising Dijkstra’s one-to-all shortest paths algorithm to \(\mathbb{R}\)-labelled directed graphs.

**Corollary 1.** Let \((G, V)\) be a rooted distance graph. bdArc-Consistency can be used to compute the shortest paths from \(V\) to all vertices of \(G\).

**Proof.** Proceed as follows: linearly transform the input rooted distance graph \((G, V)\) to its STP \(P\); apply bdArc-Consistency to \(P\); linearly transform the STP resulting from bdArc-Consistency to its rooted distance graph. \(\blacksquare\)

4. **wbDAC3 as the filtering procedure of a TCSP solver**

We define a refinement of a TCSP \(P\) to be any TCSP \(P'\) on the same set of variables such that for all constraint \((X_j - X_i) \in P_{ij}\) of \(P'\), the corresponding constraint \((X_j - X_i) \in P_{ij}\) of \(P\) verifies \(P'_{ij} \subseteq P_{ij}\). A constraint \((X_j - X_i) \in P_{ij}\) such that \(P'_{ij} \subseteq P_{ij}\) is called sublabel of \((X_j - X_i) \in P_{ij}\). A refinement is convex if it is an STP [Dechter et al., 1991]. The solver can now be described as follows (see Figure 2). As the filtering procedure during the search, it uses a weak version of the bdArc-Consistency algorithm bdAC3, which we refer to as wbdAC3, and consists of replacing composition by weak composition in the REVISE procedure of Figure 1, the aim being to avoid the “fragmentation problem” [Schwabl and Dechter, 1997]. If \(P\) is the input TCSP, the recursive procedure \(consistent(P)\) is called, which works as follows. The filtering procedure wdbAC3 is applied (the very first application of the filtering, at the root of the search space, consists of the preprocessing step). If the wdbAC3 filtering detects an inconsistency (line 1), by reducing a binarised domain to the empty set, a failure (dead end) is reached and the procedure returns false. Otherwise, if \(P\) has no disjunctive edge (line 11) then the result of the wdbAC3 filtering is a (bdArc-Consistent therefore consistent) STP; and the procedure returns true. If the result of the wdbAC3 filtering is not an STP then there are still disjunctive edges (line 3). A disjunctive edge is selected (lines 4 and 5) and instantiated with one of its sublabels (line 7), and the recursive call \(consistent(P')\) is made (line 9), where \(P'\) is the result of the instantiation of the selected disjunctive edge with the chosen sublabel (line 8). If \(consistent(P')\) returns true, \(consistent(P)\) returns true: this means here that a consistent refinement of the original TCSP has already been found, and that, because we are only interested in the consistency problem of the original TCSP, the remaining part of the search space will not be explored, and that all that is needed is, for the current node, to return the information to the parent node, which in turn returns it to its parent node, and so on, until it gets to the root of the search space, which will then make the whole procedure terminate with success. If now \(consistent(P')\) returns false, the next sublabel, if any, of the edge being instantiated is chosen, and the recursive procedure \(consistent\) is called again. If all sublabels have been already chosen (line 10) then \(consistent(P)\) returns false, which means that: it terminates with a negative answer to the consistency problem of original TCSP, if the current edge was the very first to have been instantiated (in other words, if we are at the root of the search space); it backtracks to the next most recently instantiated edge (chronological backtracking), and reiterates the process, otherwise. Completeness of the solver is guaranteed by completeness of bdArc-Consistency for STPs (Theorem 3). Furthermore, the solver can be adapted so that, when \(P\) is consistent, it returns a bdArc-Consistent refinement \(P'\), which is an STP. From \(P'\), a solution can be computed using the polynomial backtrack-free procedure \(backtrack_free(P)\) returning a true/false answer to the consistency problem of \(P\); and a polynomial backtrack-free procedure \(backtrack_free(P)\) returning a singleton-binarised-domains bdArc-Consistent refinement of a bdArc-Consistent STP \(P\).

Figure 2: A look-ahead recursive procedure \(consistent(P)\) returning a true/false answer to the consistency problem of \(P\); and a polynomial backtrack-free procedure \(backtrack_free(P)\) returning a singleton-binarised-domains bdArc-Consistent refinement of a bdArc-Consistent STP \(P\).
5  wbdAC3 as the filtering procedure of a TCSP-based job shop scheduler

A scheduling TCSP is a TCSP \( P = (X, C) \), with \( X = \{X_0, X_1, \ldots, X_n\} \), \( X_0 \) being the “origin of the world” variable standing for a global release date, \( n \) being the number of (non-preemptive) tasks. For \( i \in \{1, \ldots, n\} \), variable \( X_i \) stands for the starting date of task \( i \); the duration of task \( i \) is \( d_i \) (the tasks have fixed durations, known in advance); the release and due dates of task \( i \), if any, are \( r_{d_i} \) and \( d_{d_i} \). A conjunctive (or precedence) constraint between tasks \( i \) and \( j \) has the form \((X_j - X_i) \in [d_i, +\infty)\). A disjunctive constraint between tasks \( i \) and \( j \) has the form \((X_j - X_i) \in (-\infty, -d_j] \cup [d_i, +\infty)\). A release (respectively, due) date constraint has the form \((X_j - X_i) \in [r_{d_i}, +\infty)\) (respectively, \((X_i - X_0) \in [0, d_{d_i} - d_i]\)). Finally, \( X_0 \) standing for a global release date, we add the \( n \) constraints \((X_i - X_0) \in [0, +\infty), i \in \{1, \ldots, n\}\). We define the duration \( d_{\text{dur}}(s) \) of a solution \( s \) as \( X_0 = t_0, X_1 = t_1, \ldots, X_n = t_n \) of \( P \) as \( d_{\text{dur}}(s) = \max_{i=1}^n (t_i + d_i) - t_0; \) the latency period \( d_{\text{lat}}(s) \) of \( s \) as the time between stimulus and response, the stimulus being given at time \( t_0 \), and the response obtained at the effective beginning of the very first task: \( d_{\text{lat}}(s) = \min_{i=1}^n t_i - t_0 \). The optimum of \( P \), \( t^\text{opt}(P) \), is defined as \( t^\text{opt}(P) = \min_{s \in \text{sol}(P)} d_{\text{dur}}(s), \) with \( \text{sol}(P) \) being the set of solutions of \( P \). The scheduler we propose (see below), which minimises the makespan and initialises the optimum \( z \) to +\infty, is an adaptation of the TCSP solver of Figure 2. The lower bound of the optimum of \( P \), \( \text{OLB}(P) \), is defined as \( \text{OLB}(P) = \max_{i=1}^n (a_i + d_i) \), where \( a_i \) is the lower bound of the binarised domain \( P_{0i} \). Note that \( t^\text{opt}(P) \leq \text{OLB}(P) \), which is used by the scheduler to backtrack whenever \( z < \text{OLB}(P) \) (line 1). Furthermore, whenever the wbdAC3 filtering leads to a (bdArc-Consistent) STP (line 10), the optimum is updated in a way one can justify as follows. Because the binarised domains \( P_{0i} \) of a bdArc-Consistent STP \( P \) are minimal (Theorem 3), a direct consequence of results in [Dechter et al., 1991] (Corollaries 3.2 and 3.4, Pages 69 and 70) is that the solution realising the optimum of \( P \) is given by the lower bounds of these binarised domains, and the optimum itself is \( \text{OLB}(P) \):

procedure optimum(P){
1. if (not wbdAC3(P) or \( z \leq \text{OLB}(P) \)) return
2. else {
3. if \( P \) has disjunctive edges{
4. select a disjunctive edge \((X_i, X_j)\);
5. let \( P' \) be the label of \((X_i, X_j)\);
6. for \( k = 1 \) to 2{
7. instantiate edge \((X_i, X_j)\) with \( P' \);
8. let \( P'' \) be the resulting refinement;
9. optimum\( (P'') \)}
10. else update\( (z); z = \text{OLB}(P) \))
}

6  Summary and future work

The importance of Arc-Consistency in binary discrete CSPs is well-known [Montanari, 1974; Mackworth, 1977; Mohr and Henderson, 1986; Bessière, 1994]. In particular, it is used in solution search algorithms as the filtering procedure. The binarisation of the unary constraints of a TCSP [Dechter et al., 1991], through the addition of an “origin of the world” variable, making a TCSP node- and arc-consistent, and its constraints exclusively binary, made the attention skip to the next higher local consistency, path consistency, and to its use as the filtering procedure in general TCSP solvers. With this in mind, we defined a notion of arc-consistency for TCSPs, binarised-domains Arc-Consistency, and provided, and studied the worst-case computational behaviour of an algorithm achieving it, which we showed leads to the minimal binarised domains when the input TCSP is convex. We then showed how to use the main result in a general TCSP solver and in a TCSP-based job shop scheduler.

An important future work that has always retained our attention, whose importance grows with the presented work, is to contribute to the addition of tools to Prolog libraries such as \( \text{CLP}(\mathbb{Q}, \mathbb{R}) \); tools such as a TCSP solver and a TCSP-based job shop scheduler using bdArc-Consistency, or its weak version wbdArc-Consistency, as the filtering procedure during the search.

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