SU(3) symmetry in theory of a weakly interacting gas of spin-1 atoms with Bose-Einstein condensate

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Abstract

We study a many-body system of interacting spin-1 particles in the context of ultracold atomic gases. Its description requires eight parameters among which there are three components of magnetization and five parameters associated with quadrupole degrees of freedom. Based on the symmetry considerations, we construct a many-body interaction Hamiltonian that includes eight generators of the SU(3) group related to the above description parameters. This Hamiltonian can be generalized to spin-S systems by considering the generators of the SU(2S+1) group. We apply the Hamiltonian to study the ferromagnetic phase of a weakly interacting gas of spin-1 atoms with Bose-Einstein condensate. It is shown how the quadrupole degrees of freedom entering the Hamiltonian modify the ground state and single-particle excitation spectra in comparison with those obtained from the Hamiltonian containing the spin operators only. We discuss the issue of taking into account the local character of interaction to obtain the correct spectra of single-particle excitations.

Keywords: high spin magnets; Ultracold gases, Bose-Einstein condensate; Ferromagnetic phase; Single-particle excitations
INTRODUCTION

The magnetic properties of spin-1/2 crystalline systems are well understood and presented in the literature \[1, 2\]. If one ignores the relativistic effects associated with the interaction of the electron magnetic moments, then the description is based on the Heisenberg model Hamiltonian and depending on the sign of the exchange integral, the system exhibits the ferromagnetic or antiferromagnetic ordering. The physical nature of the exchange interaction can be explained by considering the electrostatic interaction of two electrons between themselves and protons in a hydrogen molecule or by employing only the symmetry considerations which require the Hamiltonian to be invariant with respect to spatial rotations \[1\]. The resulting Hamiltonian is expressed in terms of the Pauli matrices representing the generators of the SU(2) group. It takes into account purely quantum (exchange) effects originating from the Fermi-Dirac statistics for electrons.

The interaction in high-spin \((S > 1/2)\) crystalline systems has a more complicated character that goes beyond the usual Heisenberg model, while their phase diagram exhibits a more rich structure \[3\]. In particular, for spin-1 systems with bilinear and biquadratic exchange interactions, the exotic orderings, such as nematic \[3–6\] and semi-ordered \[6\] phases, may exist along with the traditional ferromagnetic and antiferromagnetic phases. Moreover, the non-Heisenberg structure of the spin-spin interaction affects even the traditional phases. In recent years, there have been intensive studies of unconventional orderings in magnets with \(S \geq 1\) \[7–13\].

Nowadays, the interest in high-spin systems is attracted by the studies of ultracold atomic gases providing remarkable opportunities to examine and model various effects and phenomena in quantum many-body systems in a well controlled manner. In particular, quantum gases loaded in an optical lattice represent an artificial but effective simulator of magnetic phenomena in crystalline systems \[14, 15\]. The first theoretical studies of magnetic phases and corresponding excitations in dilute Bose gases with condensate \[16–18\] were stimulated by experiments on optical trapping a condensate of \(^{23}\text{Na}\) spin-1 atoms \[19\]. A comprehensive study of the so-called spinor Bose gases, including those of spin-2 and spin-3 atoms, has been reviewed in Refs. \[20, 21\], where the interaction between the atomic degrees of freedom was taken as a bilinear form in spin operators, like in the usual Heisenberg model. However, as was mentioned, such a form of interaction energy is not sufficient to describe properly the
magnetic properties of high-spin systems.

In this paper, proceeding solely from the symmetry considerations, we propose a general recipe for obtaining the many-body Hamiltonian describing a system of interacting spin-S particles. Such systems are characterized by \((2S + 1)^2 - 1\) parameters among which there are three components of the magnetization vector and the rest can be treated as the multipole degrees of freedom. These additional parameters are induced in a many-body system by the spin of the structural constituents of matter (particles) and appear at the macroscopic level. The spin degrees of freedom of a particle generate also the SU\((2S+1)\) symmetry of the many-body Hamiltonian in the absence of external fields. As the most intriguing case with a view to physics of ultracold Bose gases, we present a detailed study of the SU\((3)\) symmetric Hamiltonian describing a system of spin-1 particles. Then we apply it to examine the ferromagnetic phase of a weakly interacting Bose gas with condensate in a magnetic field. It is shown that the constructed Hamiltonian modifies the ground state properties and single-particle excitation spectra of the system.

FORMULATION OF THE PROBLEM AND MANY-BODY INTERACTION
HAMILTONIAN OF INTERNAL DEGREES OF FREEDOM

Consider a many-body system of spin-S particles whose reduced description is performed in terms of the single-particle density matrix \(f_{\alpha\beta}(p) = \text{Tr} \rho_a^\dagger p_\beta a_\alpha\), where \(\rho\) can be either an equilibrium or non-equilibrium statistical operator, \(a_\alpha^\dagger\) and \(a_\alpha\) are the creation and annihilation operators, with index \(\alpha\) running \(2S + 1\) values. Depending on the spin value, these operators meet the following bosonic commutation (integer spin) or fermionic anticommutation (half-integer spin) relations:

\[
[a_\alpha^\dagger, a_{p'\alpha'}^\dagger]_B = \delta_{pp'} \delta_{\alpha\alpha'}, \quad [a_\alpha, a_{p'\alpha'}]_B = 0, \quad (1)
\]

\[
\{a_\alpha^\dagger, a_{p'\alpha'}^\dagger\}_F = \delta_{pp'} \delta_{\alpha\alpha'}, \quad \{a_\alpha, a_{p'\alpha'}\}_F = 0. \quad (2)
\]

Since below we study a homogeneous interacting Bose gas, we use the momentum \(p\) to specify the individual state of a particle. For the lattice models, one should consider a lattice site index instead of \(p\).

In the case of spin-1/2 system, the density matrix \(f_{\alpha\beta}(p)\), being a square matrix of the second order, can be written as a linear combination of the Pauli matrices \(\sigma_{\alpha\beta}^i\) and unit
matrix $I$ which form a basis for the vector space of $2 \times 2$ matrices. The scalar part of the single-particle density matrix in such a decomposition defines the density of the system, whereas its vectorial part specifies three components of the magnetization vector. The latter quantity is induced by the spin of the microscopic constituents of matter. The many-body Hamiltonian of two-particle interaction includes the so-called spin-spin interaction given by

$$V = \frac{1}{2V} \sum_{p_1,\ldots,p_4} I(p_1-p_3) a_{p_1\alpha}^T a_{p_2\beta}^T S_{\alpha\gamma}^i S_{\beta\delta}^i a_{p_3\gamma} a_{p_4\delta} \delta_{p_1+p_2+p_3+p_4}, \quad S_{\alpha\beta}^i = \frac{1}{2} \sigma_{\alpha\beta}^i,$$

where $a_{p\alpha}$, $a_{p\alpha}^T$ satisfy the permutation relations given by Eqs. (2) and $I(p_1-p_3)$ denotes the exchange interaction. Here and below, the summation over the repeated indices related to internal symmetry is assumed. The above Hamiltonian commutes with the spin operator of a many-body system,

$$S^i = \sum_{p} a_{p\alpha} S_{\alpha\alpha}^i a_{p\beta}$$

and, consequently, the later represents the conserved quantity or the integral of motion. It is related to magnetization vector $M^i$ by $M^i = 2\mu_0 S^i$, where $\mu_0 = e\hbar/2mc$ is the Bohr magneton. Note that three components of a particle spin (microscopic characteristic) generate the same number of macroscopic parameters necessary to describe a many-body system of spin-1/2 particles. The Hamiltonian given by Eq. (3) can be applied to describe a gas of spin-1/2 atoms or interacting electron gas embedded in a solid state system. Note that in the lattice models, the creation and annihilation operators carry lattice site index instead of momentum $p$.

Now we address the description of a many-body system of spin-1 particles. In this case, the single-particle density matrix, being a reduced description parameter, can be written as a linear combination of the unit $3 \times 3$ matrix $I_{\alpha\beta}$ and the Gell-Mann linearly independent traceless Hermitian matrices $\lambda_{\alpha\beta}^a$ (see Appendix):

$$f_{\alpha\beta}(p) = f^0(p) I_{\alpha\beta} + f^a(p) \lambda_{\alpha\beta}^a, \quad a = 1, \ldots, 8.$$  

The scalar $f^0(p)$ and vectorial $f^a(p)$ coefficients are given by

$$f^0 = \frac{1}{3} \text{Tr} f(p), \quad f^a(p) = \frac{1}{2} \text{Tr} f(p) \lambda^a.$$  

In contrast to spin-1/2 systems, we see that three components of spin are insufficient to describe the many-body states of the system. Indeed, according to Eqs. (4), (5), the states
are specified by eight independent parameters determined by the generators $\lambda^a$ of the SU(3) group.

To clarify the physical meaning of eight parameters associated with internal symmetry, consider the realization of spin-1 operators in the vector (Cartesian) basis $|x\rangle, |y\rangle, |z\rangle$ instead of the usual canonical (irreducible) basis $|S, m\rangle$ with $S = 1$ and $m = -1, 0, 1$. These two are related by (see e.g. [6]):

$$|x\rangle = \sqrt{1/2} (|1, 1\rangle + |1, -1\rangle), \quad |y\rangle = -i |1, 0\rangle, \quad |z\rangle = i \sqrt{1/2} (|1, 1\rangle - |1, -1\rangle).$$

In the vector basis we have

$$\langle i | k \rangle = \delta_{ik}, \quad S^i |k\rangle = i \varepsilon_{ikl} |l\rangle,$$

so that $S^i$ meet the usual commutation relations for spin operators,

$$[S^i, S^k] = i \varepsilon_{ikl} S^l.$$

From Eq. (6), one finds the matrix elements for the corresponding spin operators,

$$\langle k | S^i |l\rangle \equiv (S^i)_{kl} = -i \varepsilon_{ikl},$$

whence

$$S^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S^y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S^z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (7)$$

One can easily seen that $S^x = \lambda^7$, $S^y = -\lambda^5$, and $S^z = \lambda^2$ (see Eqs. (4v)), so that subalgebra of these matrices generates an SU(2) subgroup of SU(3) group. The remaining five Gell-Mann matrices, due to their properties given by Eq. (51), can be expressed in terms of the quadratic combinations of spin operators:

$$\lambda^1 = -\{S^x, S^y\}, \quad \lambda^3 = (S^y)^2 - (S^x)^2, \quad \lambda^4 = -\{S^x, S^z\},$$
$$\lambda^6 = -\{S^y, S^z\}, \quad \lambda^8 = \sqrt{3}(S^z)^2 - \frac{2}{\sqrt{3}} I,$$  

where $\{\ldots\}$ denotes an anticommutator and $I$ is the unit $3 \times 3$ matrix. Since the traceless quadrupole matrix $Q^{ik} \equiv S^i S^k + S^k S^i - (4/3) \delta_{ik}$ is determined by the above five independent components,

$$Q^{ik} = \frac{1}{\sqrt{3}} \begin{pmatrix} -\lambda^8 - \sqrt{3}\lambda^3 & -\lambda^1 & -\lambda^4 \\ -\lambda^1 & -\lambda^8 + \sqrt{3}\lambda^3 & -\lambda^6 \\ -\lambda^4 & -\lambda^6 & 2\lambda^8 \end{pmatrix},$$

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we call them the quadrupole operators. These operators can be considered as the components of a single vector $q^b = (-\lambda^1, -\lambda^3, -\lambda^4, -\lambda^6, \lambda^8)$. Therefore, if the microscopic constituents of a many-body system have a unit spin, then its macroscopic state is described by the eight parameters originating from the generators of the SU(3) group:

$$\Lambda^a = \sum_p a^\dagger_{p\alpha} \lambda^a_{\alpha\beta} a_{p\beta}, \quad a = 1, \ldots, 8, \quad \alpha, \beta = x, y, z$$

(9)

that can be splitted into the spin and quadrupole operators,

$$S^i = \sum_p a^\dagger_{p\alpha} S^i_{\alpha\beta} a_{p\beta} \quad (i = x, y, z), \quad Q^b = \sum_p a^\dagger_{p\alpha} q^b_{\alpha\beta} a_{p\beta} \quad (b = 1, 3, 4, 6, 8).$$

(10)

In the second quantization method the general two-body operator (or binary operator) $A^{(2)}$ can be expressed in the form (see, e.g., [22])

$$A^{(2)} = \frac{1}{4} \sum_{\alpha_1, \alpha_2} a^\dagger_{\alpha_1} a^\dagger_{\alpha_2} A_{\alpha_1\alpha_2;\alpha_3\alpha_4} a_{\alpha_3} a_{\alpha_4}, \quad A_{\alpha_1\alpha_2;\alpha_3\alpha_4} = \langle \alpha_1, \alpha_2 | A^{(2)} | \alpha_3, \alpha_4 \rangle.$$ 

Therefore, one can write the two-body Hamiltonian describing the interaction of internal degrees of freedom in a many-body system of spin-1 particles,

$$V_{\lambda} = \frac{1}{2\mathcal{V}} \sum_{p_1, \ldots, p_4} J(p_1 - p_3) a^\dagger_{p_1\alpha} a^\dagger_{p_2\beta} \lambda^a_{\alpha\gamma} \lambda^a_{\beta\gamma} a_{p_1\gamma} a_{p_4\delta} \delta_{p_1+p_2, p_3+p_4},$$

(11)

where $\mathcal{V}$ is the volume of the system. The structure of the obtained Hamiltonian can also be justified within a phenomenological quasiparticle theory, where the energy of the system is considered to be a functional of the single-particle density matrix, like in the normal Fermi-liquid theory [23, 24]. For a not dense system, one can restrict ourselves by the energy functional quadratic in the single-particle density matrix [25]. Since each density matrix represents the eight-component vector, they should enter the energy functional as a scalar product. On the other hand, it is clear, that such a functional should be obtained from the microscopic Hamiltonian given by Eq. (11) by statistical averaging and using the Bloch–De Dominicis (or Wick’s) theorem [26].

Taking into account the commutation relations for the creation and annihilation operators as well as the properties of the structure constants $f^{abc}$ (see Eqs. (1), (49)), one can show that $[V_{\lambda}, \Lambda^a] = 0$ and, consequently, $\Lambda_a$ is the integral of motion. Note that the above Hamiltonian and the corresponding macroscopic state are SU(3) symmetric. This symmetry, however, is broken if the coupling between the spin and magnetic field (the Zeeman term) is taken into account. The role of SU(3) symmetry has been discussed when studying the dynamics and relaxation in high-spin magnets [27, 29].
As we noted, ultracold quantum gases provide a powerful tool to study various effects and phenomena in quantum many-body systems. Therefore, we apply the obtained Hamiltonian to study the ferromagnetic phase of a weakly interacting Bose gas of spin-1 atoms with Bose-Einstein condensate. To this end, we employ the Bogoliubov model based on c-number treatment of creation and annihilation operators for condensate particles. Our starting point is the following Hamiltonian consisting of the kinetic energy term $H_0$ and the terms corresponding to the potential interaction $V_p$ as well as the interaction between the internal degrees of freedom $V_\lambda$:

$$H = H_0 + V_p + V_\lambda,$$

where

$$H_0 = \sum_p a_{p\alpha}^\dagger \left[ \varepsilon_p \delta_{\alpha\beta} - \hbar S_{\alpha\beta}^z \right] a_{p\beta}, \quad S_{\alpha\beta}^z \equiv \lambda_{\alpha\beta}^2,$$

$$V_p = \frac{1}{2Y} \sum_{p_1, \ldots, p_4} U(p_1 - p_3) a_{p_1\alpha}^\dagger a_{p_2\beta}^\dagger a_{p_3\alpha} a_{p_4\beta} \delta_{p_1 + p_2, p_3 + p_4},$$

and $V_\lambda$ is given by Eq. (11). Here $\varepsilon_p = p^2/2m$ is the kinetic energy of a particle, $a_{p\alpha}^\dagger$ and $a_{p\alpha}$ are the bosonic creation and annihilation operators satisfying Eqs. (1), $h = g \mu_B H$ with $g$, $\mu_B$, and $H$ being, respectively, the Landé hyperfine factor, the Bohr magneton, and external magnetic field directed along $z$-axis. Note that usually the interaction Hamiltonian is written in terms of the corresponding scattering lengths describing the low energy collisions of atoms at ultra low temperature. However, such parametrization of interaction does not take into account the local character of interaction and the divergencies appear when computing the ground state energy or chemical potential so that it is necessary to use the renormalization of the coupling constant. Moreover, as we see below, it may lead to an incomplete structure of the spectrum of single-particle excitations. Therefore, the interaction given by Eqs. (11), (14) is characterized by the corresponding functions $U(p)$ and $J(p)$.

Since the number of Bose condensed atoms is a macroscopic value proportional to the volume of the system $\mathcal{V}$, the next step, according to the Bogoliubov model, is to replace the creation and annihilation operators of condensed atoms with zero momentum by $c$-
numbers, \( a_0^\dagger \to \sqrt{V} \Psi_\alpha^* \) and \( a_0 \to \sqrt{V} \Psi_\alpha \), in all operators of relevant physical quantities, where \( \Psi_\alpha \) represents the condensate wave function. This procedure has been proved to be exact in the thermodynamic limit \([34]\). The \( c \)-number terms in the Hamiltonian and those that are quadratic in creation and annihilation operators allow to define the ground state and the corresponding spectrum of single-particle excitations (quasiparticles), while the higher order terms in creation and annihilation operators are relevant when describing the interaction effects between the quasiparticles themselves. Therefore, performing the above replacement with \( \Psi_\alpha \) being a variational parameter and neglecting the terms of the third and fourth order, one can obtain the Hamiltonian truncated up to quadratic terms in the creation and annihilation operators:

\[
H(\Psi) \simeq H^{(0)}(\Psi) + H^{(2)}(\Psi),
\]

where \( H^{(0)}(\Psi) \) is the \( c \)-number part of the truncated Hamiltonian given by

\[
\frac{1}{V} H^{(0)}(\Psi) = \frac{U(0)}{2} (\Psi^* \Psi)^2 + \frac{J(0)}{2} (\Psi^* \lambda^a \Psi)^2 - h (\Psi^* \lambda^2 \Psi).
\]

The quadratic part reads

\[
H^{(2)}(\Psi) = H_0^{(2)}(\Psi) + V_p^{(2)}(\Psi) + V_\lambda^{(2)}(\Psi),
\]

where \( H_0^{(2)}(\Psi) \) does not include the interparticle interactions,

\[
H_0^{(2)}(\Psi) = \sum_{\mathbf{p} \neq 0} \varepsilon_{\mathbf{p}} (a_\mathbf{p}^\dagger a_\mathbf{p}) - h \sum_{\mathbf{p} \neq 0} (a_\mathbf{p}^\dagger \lambda^2 a_\mathbf{p}).
\]

Two other terms describing the interaction have the form

\[
V_p^{(2)}(\Psi) = U(0) \sum_{\mathbf{p} \neq 0} (\Psi^* \Psi)(a_\mathbf{p}^\dagger a_\mathbf{p}) + \frac{1}{2} \sum_{\mathbf{p} \neq 0} U(\mathbf{p}) \left[ (\Psi a_\mathbf{p}^\dagger)(\Psi^* a_\mathbf{p}) + (\Psi a_\mathbf{p}^\dagger)(\Psi a_\mathbf{p}^\dagger) + \text{h.c.} \right]
\]

and

\[
V_\lambda^{(2)}(\Psi) = J(0) \sum_{\mathbf{p} \neq 0} (\Psi^* \lambda^a \Psi)(a_\mathbf{p}^\dagger \lambda^a a_\mathbf{p})
\]

\[ + \frac{1}{2} \sum_{\mathbf{p} \neq 0} J(\mathbf{p}) \left[ (a_\mathbf{p}^\dagger \lambda^a \Psi)(\Psi^* \lambda^a \Psi) + (a_\mathbf{p}^\dagger \lambda^a \Psi)(a_\mathbf{p}^\dagger \lambda^a \Psi) + \text{h.c.} \right],
\]

where we use the following notations \( (\Psi^* \Psi) \equiv \Psi_\alpha^* \Psi_\alpha \), \( (a_\mathbf{p}^\dagger a_\mathbf{p}) \equiv a_\mathbf{p}^\dagger a_\mathbf{p} \), \( (\Psi^* \lambda^a \Psi) \equiv \Psi_\alpha^* \lambda_{\alpha \beta}^a \Psi_\beta \), and so on, assuming matrix multiplication. Note that the replacement of creation and annihilation operators by \( c \)-numbers implies the gauge symmetry breaking and
leads to non-conservation of the total number of atoms. Therefore, the problem should be considered in the grand canonical ensemble, where the chemical potential \( \mu \), being a Lagrange multiplier, reflects the conservation of the total number of atoms \( N = \sum_p (a_p^\dagger a_p) \).

The corresponding Gibbs statistical operator for the above truncated Hamiltonian reads

\[
w(\Psi) \simeq \exp \left[ \Omega - \beta \left( \mathcal{H}^{(0)}(\Psi) + \mathcal{H}^{(2)}(\Psi) \right) \right], \quad (21)
\]

where

\[
\begin{align*}
\mathcal{H}^{(0)}(\Psi) &= H^{(0)}(\Psi) - \mu \mathcal{V} (\Psi^* \Psi), \\
\mathcal{H}^{(2)}(\Psi) &= H^{(2)}(\Psi) - \mu \sum_{p \neq 0} (a_p^\dagger a_p). 
\end{align*}
\]

The grand thermodynamic potential \( \Omega \) as a function of reciprocal temperature \( \beta = 1/T \), chemical potential \( \mu \), and variational parameter \( \Psi_\alpha \) is found from the normalization condition \( \text{Tr} \ w = 1 \),

\[
\Omega = \beta \mathcal{H}^{(0)}(\Psi) - \ln \text{Tr} \left[ \exp (-\beta \mathcal{H}^{(2)}(\Psi)) \right],
\]

where the trace is taken in the space of occupation numbers of bosons with \( p \neq 0 \). In the standard Bogoliubov approach the relation between the condensate wave function and chemical potential is determined by the \( c \)-number part of the thermodynamic potential assuming that it represents the leading term,

\[
\omega^{(0)} = \frac{U(0)}{2} (\Psi^* \Psi)^2 + \frac{J(0)}{2} (\Psi^* \lambda^a \Psi)^2 - h (\Psi^* \lambda^2 \Psi) - \mu (\Psi^* \Psi), \quad (25)
\]

where we introduced the thermodynamic potential density \( \omega = \Omega/\beta \mathcal{V} \). The later, up to a sign, coincides with pressure \( P \), \( \omega = -P \) and it is employed when studying macroscopic dynamics of superfluid systems, both classical and relativistic \cite{35, 36}. The variation of Eq. (25) over \( \Psi_\alpha \) yields

\[
\mu \Psi_\alpha - U(0)(\Psi^* \Psi) \Psi_\alpha - J(0)(\Psi^* \lambda^a \Psi) \lambda^a_{\alpha \beta} \Psi_\beta + h \lambda^2_{\alpha \beta} \Psi_\beta = 0 \quad (26)
\]

(we do not write the complex conjugate equation). This equation ensures the minimum of the thermodynamic potential and gives a relation between the chemical potential and condensate wave function. The contribution of the quadratic terms in creation and annihilation operators to Eqs. (25), (26) is examined in Refs. \cite{37, 38} for atoms with zero spin.
FERROMAGNETIC GROUND STATE AND EXCITATIONS

Now we use the obtained equations to study the ground state properties and corresponding single-particle excitations of a weakly interacting Bose gas of spin-1 atoms. In order to introduce the condensate density $n_0$, consider the normalized state vector $\zeta_\alpha$:

$$\Psi_\alpha = \sqrt{n_0} \zeta_\alpha, \quad \zeta_\alpha^* \zeta_\alpha \equiv (\zeta^* \zeta) = 1.$$  \hspace{1cm} (27)

In the Cartesian basis, the ferromagnetic ordering is specified by the following vector [16]:

$$\zeta = \frac{1}{\sqrt{2}} (1, i, 0).$$ \hspace{1cm} (28)

As we have shown, the description of a many-body system of spin-1 constituents requires the introduction of additional parameters along with the ordinary magnetization vector. These parameters are determined by Eqs. (9), (10). In the problem under consideration, the above ferromagnetic state vector generates the ordinary magnetization along $z$-direction,

$$\langle S^i \rangle = \Psi^* S^i \Psi = n_0 \delta_{iz}, \quad S^i = (S^x \equiv \lambda^7, S^y \equiv -\lambda^5, S^z \equiv \lambda^2)$$ \hspace{1cm} (29)

and one more parameter associated with the quadrupole degrees of freedom,

$$\langle Q^b \rangle = \Psi^* q^b \Psi = \frac{n_0}{\sqrt{3}} \delta_{88}.$$ \hspace{1cm} (30)

The quadrupole tensor for the ferromagnetic state becomes

$$\langle Q^{ik} \rangle = \Psi^* Q^{ik} \Psi = n_0 \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}.$$ \hspace{1cm} (31)

Since $\langle Q^{xz} \rangle = \langle Q^{yy} \rangle$, the order parameter $\zeta$ is invariant with respect to rotations about $z$-axis, as it should be in the ferromagnetic state. Next, multiplying Eq. (26) by $\Psi_\alpha^*$ and performing the summation over $\alpha$, one can obtain the relation between the chemical potential and condensate density for the above state vector $\zeta$:

$$\mu = n_0 \left( U(0) + \frac{4}{3} J(0) \right) - h.$$ \hspace{1cm} (32)

In a similar manner, the thermodynamic potential density determined by Eq. (25) is written as

$$\omega(0) = \frac{n_0^2}{2} \left( U(0) + \frac{4}{3} J(0) \right) - n_0 (h + \mu),$$

10
or eliminating the condensate density by using Eq. (32), one finds

$$\omega(0) = -\frac{1}{2} \frac{(\mu + h)^2}{U(0) + (4/3)J(0)}.$$  \hspace{1cm} (33)

In order for the equilibrium state to be stable, the thermodynamic potential density \(\omega(0)\) must be negative (the pressure is positive) that implies \(U(0) + (4/3)J(0) > 0\). Since the necessary stability condition for a condensed gas of spinless particles requires the predominance of repulsive forces over attractive forces, we also consider \(U(0) > 0\). Therefore, from Eq. (33), it becomes evident that the condition \(J(0) < 0\) must be satisfied for the stability of ferromagnetic state under consideration.

Having defined the ferromagnetic ground state, we now address the issue of single-particle excitations. To obtain the corresponding spectra, let us return to the Hamiltonian given by Eqs. (17)-(20), (23). Eliminating the chemical potential by using Eq. (32) and taking into account the explicit form for the ground state vector and for matrices \(\lambda\) (see Eqs. (27), (28), (46)), the quadratic Hamiltonian is reduced to

$$\mathcal{H}^{(2)}(n_0) = \mathcal{H}_1^{(2)}(n_0) + \mathcal{H}_2^{(2)}(n_0),$$  \hspace{1cm} (34)

where

$$\mathcal{H}_1^{(2)}(n_0) = \sum_{\mathbf{p} \neq 0} \left[ \varepsilon_{\mathbf{p}} + h + 2n_0J(\mathbf{p}) - 2n_0J(0) \right] a_{\mathbf{p},z}^+ a_{\mathbf{p},z},$$  \hspace{1cm} (35)

and

$$\mathcal{H}_2^{(2)}(n_0) = \sum_{\mathbf{p} \neq 0} a_{\mathbf{p}a}^+ A_{\alpha\beta}(\mathbf{p}) a_{\mathbf{p}\beta} + \frac{1}{2} \sum_{\mathbf{p} \neq 0} a_{\mathbf{p}a}^+ B_{\alpha\beta}(\mathbf{p}) a_{-\mathbf{p}b} + \frac{1}{2} \sum_{\mathbf{p} \neq 0} a_{\mathbf{p}a} B_{\alpha\beta}(\mathbf{p}) a_{-\mathbf{p}b}, \quad \alpha, \beta = x, y.$$  \hspace{1cm} (36)

Here \(A = A^\dagger\) and \(B = B^T\) are the Hermitian and symmetric matrices, respectively,

$$A_{\alpha\beta}(\mathbf{p}) = \begin{pmatrix} A(\mathbf{p}) & i\mathcal{A}(\mathbf{p}) \\ -i\mathcal{A}(\mathbf{p}) & A(\mathbf{p}) \end{pmatrix}, \quad B_{\alpha\beta}(\mathbf{p}) = \begin{pmatrix} B(\mathbf{p}) & iB(\mathbf{p}) \\ iB(\mathbf{p}) & -B(\mathbf{p}) \end{pmatrix}$$  \hspace{1cm} (37)

with the following matrix elements:

$$A(\mathbf{p}) = \varepsilon_{\mathbf{p}} + h + \frac{5}{3}n_0J(\mathbf{p}) - n_0J(0) + \frac{1}{2}n_0U(\mathbf{p}),$$

$$\mathcal{A}(\mathbf{p}) = \frac{1}{3}n_0J(\mathbf{p}) - \frac{1}{2}n_0U(\mathbf{p}) - n_0J(0) + h,$$

$$B(\mathbf{p}) = \frac{2}{3}n_0J(\mathbf{p}) + \frac{1}{2}n_0U(\mathbf{p}).$$  \hspace{1cm} (38)
The first part $\mathcal{H}_1^{(2)}(n_0)$ of the total quadratic Hamiltonian has already diagonal form with the following spectrum of single-particle excitations:

$$\omega^{(1)}_{p_2} = \varepsilon_p + h + 2n_0 \left[ J(p) - J(0) \right],$$

(39)

while $\mathcal{H}_2^{(2)}(n_0)$ should be diagonalized in creation and annihilation operators. Note that since $\mathcal{H}_2^{(2)}(n_0)$ commutes with $\mathcal{H}_1^{(2)}(n_0)$, it can be diagonalized independently. To this end, we apply the Bogoliubov canonical transformation method which allows to reduce the general Hermitian quadratic form in bosonic operators to a diagonal structure [26]. Therefore, let us introduce the unitary operator $U$ mixing up $a_{p\lambda}$ and $a_{p,-\lambda}^\dagger$:

$$Ua_{p\alpha}a_{p\alpha}^\dagger = \sum_{\lambda=x,y} \left[ u_{\alpha\lambda}(p)a_{p\lambda} + v_{\alpha\lambda}^*(p)a_{p,-\lambda}^\dagger \right],$$

$$Ua_{p\alpha}^\dagger a_{p\alpha} = \sum_{\lambda=x,y} \left[ u_{\alpha\lambda}^*(p)a_{p\lambda}^\dagger + v_{\alpha\lambda}(p)a_{p,-\lambda} \right]$$

(40)

and transforming $\mathcal{H}_2^{(2)}(n_0)$ to the diagonal form

$$U\mathcal{H}_2^{(2)}(n_0)U^\dagger = \sum_{p \neq 0} \sum_{\lambda=x,y} \omega_{p\lambda} a_{p\lambda}^\dagger a_{p\lambda} + \varepsilon_0,$$

(41)

where $\omega_{p\lambda}$ are the spectra of single-particle excitations and $\varepsilon_0$ redefines the vacuum energy or the ground state thermodynamic potential. The creation and annihilation operators $Ua_{p\alpha}^\dagger U^\dagger$ and $Ua_{p\alpha}U^\dagger$ given by Eqs. (40) must satisfy the same bosonic commutation relations as the operators $a_{p\alpha}$ and $a_{p\alpha}^\dagger$. This requirement results in the following normalization and orthogonality conditions for functions $u_{\alpha\lambda}(p)$ and $v_{\alpha\lambda}(p)$:

$$\sum_{\lambda=x,y} \left[ u_{\alpha\lambda}(p)u_{\beta\lambda}^*(p) - v_{\alpha\lambda}^*(p)v_{\beta\lambda}(p) \right] = \delta_{\alpha\beta},$$

$$\sum_{\lambda=x,y} \left[ u_{\alpha\lambda}(p)v_{\beta\lambda}^*(p) - v_{\alpha\lambda}^*(p)u_{\beta\lambda}(p) \right] = 0.$$  

(42)

Note that $u(p)$ and $v(p)$ are constructed from the same quantities as the matrices $A_{\alpha\beta}(p)$ and $B_{\alpha\beta}(p)$ and, therefore, they can be considered as even functions of momentum. The energies of single-particle excitations (or quasiparticles) $\omega_{p\lambda}$ satisfy the following eigenvalue equations [26]:

$$\sum_{\lambda=x,y} \left[ A_{\alpha\lambda}(p)u_{\lambda\gamma}(p) + B_{\alpha\lambda}(p)v_{\lambda\gamma}(p) \right] = \omega_{p\gamma} u_{\alpha\gamma}(p),$$

$$\sum_{\lambda=x,y} \left[ A_{\alpha\lambda}^*(p)v_{\lambda\gamma}(p) + B_{\alpha\lambda}^*(p)u_{\lambda\gamma}(p) \right] = -\omega_{p\gamma} v_{\alpha\gamma}(p).$$

(43)
This system of homogeneous linear equations has non-zero solution when the corresponding determinant turns to zero. Therefore, taking into account Eqs. (37), one finds the equation for $\omega_{px}$:

$$(A^2 - 4B^2 - 2A\mathcal{A} + \mathcal{A}^2 - \omega_x^2)((A + \mathcal{A})^2 - \omega_x^2) = 0,$$

Next, taking into account Eqs. (38), one obtains two different excitation spectra,

$$\omega_{px}^{(II)} = \varepsilon_p + 2n_0(J(p) - J(0)) = \omega_{pz}^{(I)} + h$$ (44)

and

$$\omega_{px}^{(III)} = \left[\varepsilon_p^2 + 2\varepsilon_p\left(n_0U(p) + \frac{4}{3}n_0J(p)\right)\right]^{1/2}.$$ (45)

Note that $\omega_{py}$ satisfies exactly the same equation as $\omega_{px}$ and, consequently, the corresponding spectra are identical or degenerate [16]. Therefore, the ferromagnetic phase of a weakly interacting Bose gas with condensate is characterized by three types of excitations with the dispersion laws given by Eqs. (39), (44), (45) and any of $\omega_{px}$ can be related to operators $a_{py}^\dagger$, $a_{py}$ in the Hamiltonian determined by Eq. (41).

The spectrum given by Eq. (45) is independent of magnetic field and represents the gapless Bogoliubov mode modified by the interaction of internal degrees of freedom. At small momenta, it represents the phonon excitations,

$$\omega_{px}^{(II)} \approx cp, \quad c = \left[\frac{n_0}{m}U(0) + \frac{4}{3}J(0)\right]^{1/2},$$

where $c$ is a speed of sound. The requirement for the speed of sound to be real leads to the stability condition $U(0) + (4/3)J(0) > 0$ obtained above.

Two other spectra $\omega_{pz}^{(I)}$ and $\omega_{px}^{(II)}$ describe the excitations related to the internal degrees of freedom, or "spin-quadrupole" waves. When the applied magnetic field is zero ($h = 0$), both spectra become identical so that the system is described by two types of single-particle excitations. Note that the quadratic form determined by Eq. (36) must be positive definite. This requirement implies $J(p) - J(0) > 0$ at any momentum $p$, moreover $J(0) < 0$ for ferromagnetic ordering. It is worth stressing that in contrast to the previous studies of spin-1 Bose-Einstein condensates [16, 18, 20, 21], both spectra depends on the interaction parameter that is absolutely clear for the system of interacting atoms. This is due to the fact that we do not parameterize the interaction by the corresponding scattering lengths. Indeed, in doing so, $J(p) = J(0) = 4\pi\hbar^2a_s/m$, where $a_s$ is the corresponding scattering length and
the spectra become independent of the interaction parameters. Therefore, the description of the interaction effects in ultracold gases by the scattering length represents a sufficiently rough approximation that does not take into account the local character of interaction. The role of nonlocal interaction was recently discussed for ultracold Bose [38, 39] and Fermi [40, 41] gases.

Finally, if the interaction Hamiltonian is SU(2) symmetric so that it contains the spin operators $S^i$ only, then the ferromagnetic state of spin-1 condensate is characterized by the following spectra of single-particle excitations [17, 42]:

$$
\omega_p^{(1)} = \epsilon_p - 2J(0)n + 2\hbar,
\omega_p^{(2)} = \epsilon_p - n_0(J(p) - J(0)) + \hbar,
\omega_p^{(3)} = \left[\epsilon_p^2 + 2\epsilon_p n_0(U(p) + J(p))\right]^{1/2}
$$

which are in agreement with other studies [16, 18, 20] if the interaction is taken to be of the contact type,

$$
U(p) = U(0) = \frac{g_0 + 2g_2}{3}, \quad J(p) = J(0) = \frac{g_2 - g_0}{3},
$$

with $g_0$ and $g_2$ being related to the s-wave scattering lengths of the total spin-1 channel [20].

Therefore, the extension of the Hamiltonian to SU(3) symmetry leads to the appearance of new description parameters (along with the magnetization vector) and changes the ground state and single-particle excitation spectra of ferromagnetic Bose-Einstein condensate.

**CONCLUDING REMARKS**

We have proposed a general approach for describing a many-body system of interacting spin-1 objects. It has been shown that the microscopic spin-1 objects inevitably induce both SU(3) symmetry of a many-body system and appearance of additional description parameters along with the ordinary magnetization vector. These parameters are related to the quadrupole degrees of freedom. From the symmetry considerations, we have found the many-body SU(3) symmetric Hamiltonian describing the interaction of internal degrees of freedom. It includes eight generators of the SU(3) group, among which there are three spin and five quadrupole operators. The Hamiltonian has been applied to study the ferromagnetic state of a weakly interacting Bose gas of spin-1 atoms with Bose-Einstein condensate. It has been shown that the ground state is specified by non-zero magnetization vector and
additional quadrupole parameter. In addition, due to the quadrupole degrees of freedom entering the Hamiltonian, the single-particle excitation spectra are modified in comparison with those obtained from the usually employed Hamiltonian containing only the spin operators. It is also worth noting that the nonlocal character of interaction essentially affects the structure of the spectra so that the parametrization of interaction by the scattering length is a sufficiently rough approximation. Finally, the studied Hamiltonian can be easily generalized to spin-S systems by considering the generators $T^a$ of the SU(N) group with $N = 2S + 1$. The resulting Hamiltonian can be applied to study the magnetic properties of high-spin dilute Fermi and Bose gases.

The properties of Gell-Mann matrices

The Gell-Mann matrices, being the generators of the SU(3) group, are defined as follows:

\[
\begin{align*}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. 
\end{align*}
\]

(46)

They have the following property:

\[
\text{Sp} \lambda^a \lambda^b = 2\delta_{ab}
\]

(47)

and satisfy the commutation relations,

\[
[\lambda^a, \lambda^b] = 2i f^{abc} \lambda^c.
\]

(48)

The structure constants $f^{abc}$ of the SU(3) group, according to Eq. (47), are found to be

\[
f^{abc} = -\frac{i}{4} \text{Sp} \lambda^c [\lambda^a, \lambda^b],
\]

whence

\[
f^{abc} = -f^{bac} = f^{bca}.
\]

(49)

One can easily find the numerical values of the structure constants,

\[
f^{123} = 1, \quad f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2}, \quad f^{456} = f^{678} = \frac{\sqrt{3}}{2}.
\]

(50)
All other numerical values of $f^{abc}$ not related to the indicated above by permutation are zero. The anticommutator of the Gell-Mann matrices, as well as the commutator, is linear in $\lambda_a$:

$$\{\lambda^a, \lambda^b\} = \frac{4}{3}\delta_{ab} + 2d^{abc}\lambda^c.$$  \hspace{1cm} (51)

The coefficients $d^{abc}$, symmetric over all indices, are given by

$$d^{abc} = \frac{1}{4} \text{Sp} \lambda^c \{\lambda^a, \lambda^b\}.$$

The following their values are different from zero:

$$d^{118} = d^{228} = d^{338} = -d^{888} = \frac{1}{\sqrt{3}},$$  \hspace{1cm} (52)

$$d^{146} = d^{157} = d^{256} = d^{344} = d^{355} = -d^{247} = -d^{366} = -d^{377} = \frac{1}{2},$$  \hspace{1cm} (51)

$$d^{448} = d^{558} = d^{668} = d^{778} = -\frac{1}{2\sqrt{3}}.$$  \hspace{1cm} (52)

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