Non-closure of constraint algebra in N=1 supergravity

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Abstract

The algebra of constraints arising in the canonical quantization of N=1 supergravity in four dimensions is investigated. Using the holomorphic action, the structure functions of the algebra are given and it is shown that the algebra does not close formally for two chosen operator orderings.

1 Introduction

N=1 supergravity, the simplest supersymmetric extension of general relativity, was first set up in [1] and [2]. Being nonrenormalizable but finite up to second order in \( \hbar \) in the perturbative expansion, finiteness at all orders is unlikely for the unbounded case [3] but still under debate in presence of boundaries [4].

What makes locally supersymmetric theories interesting in the canonical approach is the fact that the commutator of two supersymmetry transformations gives a general coordinate transformation. Hence the physical states, i.e. those state functionals that are annihilated by the quantized constraints corresponding to those transformations, are easier to find, since one only has to look for solutions to the supersymmetry constraints to find states that are also invariant under general coordinate transformations.

In the framework of canonical quantization of theories with constraints [5] a crucial aspect is that the quantized constraints are required to form an algebra in order for the quantum theory to be consistent. This means that the commutator of two constraints should give an expression of the form \( \text{structure function} \times \text{constraint} \) with the constraint operator standing on the right so that the commutator of two constraints that annihilate a physical state also annihilates this state. Although the classical constraint algebra has fully been given in [6], a check of the more involved terms of the quantized algebra is necessary.

The starting point for the canonical quantization is the action of N=1 supergravity. It is chosen to work with the holomorphic action [7], the conventions being...
according to [8], as set out in the appendix. According to [8], as set out in the appendix.\[\int e^\mu e^\rho e^\sigma^\nu \left( \psi^{\times A'}_\mu e_{A'A'} D_{\nu} \psi^{A'}_\sigma \right. \\
\left. + \frac{i}{\kappa^2} e_{A'A'} e^{A'B}_\sigma \left[ \partial_{\mu} \omega^{AB}_\nu + \omega^A_{C\mu} \omega^{CB}_\nu \right] \right) \quad (1)\]

Here $\varepsilon^{\mu\nu\rho\sigma}$ is the Lévi-Civita tensor density with $-\varepsilon^{0123} = \varepsilon^{0123} = 1$. $\psi^A_\sigma$, $A = 0, 1$, are the components of a spinor-valued one-form describing the spin-3/2 degrees of freedom, and hence are Grassmann valued. The spinor $\psi^{\times A'}_\sigma$, $A' = 0', 1'$, corresponding to the complex conjugate of $\psi^A_\sigma$ in the real theory, is considered to be independent since the complex conjugate of a holomorphic function is not holomorphic. $e^{A'A'}_\mu$ are spinor-valued tetrads standing for the gravitational or spin-2 degrees of freedom and are taken to be invertible.

The equation $\delta I/\delta \omega = 0$, it is not necessary to differentiate the $\omega$'s when it comes to differentiate the action by the other fields. The variables $e$, $\psi$ and $\psi^{\times}$ have to obey reality conditions, given below, to make the theory equivalent to the real theory. The equations of motion arising from this action are known to be the same as those of the real theory after insertion of the reality conditions [10, 11]. Due to the complexification of the theory the Lorentz algebra splits into two factors, one with $\omega$ as a gauge field, the other with $\bar{\omega}$. Since the latter does not appear in the action, the two factors differ considerably [10].

The theory is symmetric under general coordinate transformations, the variation of the fields given by their Lie derivative, as well as under left-handed local Lorentz transformations

$$\delta e^A_\mu = N^A_B e^{BA'}_\mu, \quad \delta \psi^A_\mu = N^A_B \psi^{B\mu}, \quad \delta \psi^{\times A'}_\mu = 0$$

with a parameter $N^{AB} = N^{BA}$ and correspondingly right-handed Lorentz transformations

$$\delta e^{A'A'}_\mu = \bar{N}^{A'B'} e^{BA'}_\mu, \quad \delta \psi^{A'}_\mu = 0 \quad \delta \psi^{\times A'}_\mu = \bar{N}^{A'B'} \psi^{\times B'}_\mu$$

with a parameter $\bar{N}^{A'B'} = \bar{N}^{B'A'}$. Finally there is the symmetry under left-handed supersymmetry transformations

$$\delta e^{A'A'}_\mu = -\frac{i\kappa^2}{2} \epsilon^A \psi^{\times A'}_\mu, \quad \delta \psi^A_\mu = D_\mu \epsilon^A \quad \delta \psi^{\times A'}_\mu = 0$$

with Grassmann valued parameters $\epsilon^A$. The transformation of $\psi^{\times A'}_\mu$ under right-handed supersymmetry is, since there is no $\bar{\omega}$ to give $D_\mu \epsilon^{A'}$, more complicated
However, after inserting the reality conditions into the transformations one gets
\[
\delta e^{AA'}_\mu = \frac{i\kappa^2}{2} \bar{\epsilon}^{A'} \psi_A^\mu \quad \delta \psi^A_\mu = 0 \quad \delta \bar{\psi}^{A'}_\mu = D_\mu \bar{e}^{A'}
\]
Formulating the theory in a canonical way lets one find the constraints which generate these transformations.

2 Canonical Formulation

To get the canonical formulation of supergravity spacetime is split into space and time according to \[12\] limiting the topology of spacetime to be \(\Sigma \times \mathbb{R}\), where \(\Sigma\) is a spatial hypersurface. The "time" associated with \(\mathbb{R}\) is just a parameter and to be distinguished from a - difficult to define - physical time \[10\]. An effect of this spacetime split is that the invariance under general coordinate transformations splits into one under translations of the time parameter and one under spatial diffeomorphisms.

The spinor equivalent of the outward normal vector to the hypersurface \(\Sigma\) is given by \(n^{AA'}\) with
\[
n_{AA'} n^{AA'} = 1 \quad \text{and} \quad n_{AA'} e^{AA'} = 0
\]
which is a function of the spatial components of \(e\) (see appendix). The time component can be written as
\[
e^{AA'}_0 = N n^{AA'} + N^i e^{AA'}_i
\]
where \(N\) is the Lapse and \(N^i\) the Shift functions \[12\]. Calculating the momenta from \[10\] one has to be aware of the Grassmann valuedness of the spin-3/2 variables, hence anticommute these variables to the left before performing functional differentiation on them. The momenta of the theory are
\[
\pi_{A^j} := \frac{\delta \tilde{I}}{\delta \psi^{A^j}} = -\epsilon^{ij} \psi^{A^j} e_{AA'} \quad (2)
\]
\[
p_{AA'^j} := \frac{\delta \tilde{I}}{\delta e^{AA'}} = \frac{2i}{\kappa^2} \epsilon^{ijkl} e_{BA'} \omega_{A^l} \quad (3)
\]
Due to the 1.5 order method \(\omega\) is not treated as a canonical variable hence it has no corresponding momentum. One clear advantage of working with the holomorphic action can be seen looking at \[3\] which involves \(\psi^\times\). In the real theory there is a similar expression for the momentum of \(\bar{\psi}\) \[8\], so the four variables \(\psi, \pi, \bar{\psi}\) and \(\bar{\pi}\) are not independent and give rise to second class constraints whose treatment needs the construction of Dirac brackets \[5\] whereas here one can treat \(\psi\) and \(\pi\) as independent variables.

Choosing \(e, \psi\) and \(p, \pi\) from \[2\] and \[3\] as canonical variables, the next step is to define Poisson brackets. Holomorphic Poisson brackets for holomorphic functionals \(F\) and \(G\) of the canonical variables are defined by
2 CANONICAL FORMULATION

\{F, G\} := \int d^3u \left( \frac{\delta G}{\delta p_{AA^i}(u)} \frac{\delta F}{\delta e^{AA^i}(u)} - \frac{\delta G}{\delta e^{AA^i}(u)} \frac{\delta F}{\delta p_{AA^i}(u)} \right)
- \left( \frac{\delta G}{\delta \pi_{A^i}(u)} \frac{\delta F}{\delta \psi_{A^i}(u)} + \frac{\delta G}{\delta \psi_{A^i}(u)} \frac{\delta F}{\delta \pi_{A^i}(u)} \right)

being symmetric for the fermionic derivatives and obeying the rules set up in [13].

With (2) and (3) follows

\{\pi_{B^j}(x), \psi^{A^i}(y)\} = -\epsilon_B^A \delta_i^j \delta(x, y) \tag{4}

\{p_{BB^j}, e^{AA^i}\} = -\epsilon_B^A \epsilon_B^{A'} \delta_i^j \delta(x, y) \tag{5}

which are the only nonvanishing brackets.

Before coming to the constraints, it is useful to discuss the reality conditions. The reality conditions are given by

\begin{align*}
R_{1AA'}^i & := e^{AA'}_i \\
R_{2AA'^k} & := i \epsilon^{ijk} \psi_A \psi^{A'}_j \\
R_{3AA'^j} & := p_{AA'^j} + \frac{i}{\kappa^2} \epsilon^{ijk} \partial_i e_{AA'^k} - \frac{1}{2} \epsilon^{ijk} \psi^{A'}_k \psi_{Ak}
\end{align*}

The first two conditions state the reality of \(e\) and the fact that \(\psi^x\) is the complex conjugate of \(\psi\) in the real theory. The third reality condition arises from claiming that \(p + p^x\) should be real, \(p^x\) being a holomorphic function corresponding to \(\bar{\rho}\) after insertion of the first two reality conditions. However, \(p\) itself is not required to be real [12]. The resulting second class constraints \(\text{Im}(R_i) \approx 0, \approx 0\) meaning ”weakly zero” [12], cause no problems as the Dirac brackets that follow from them are equal to the holomorphic Poisson brackets [10, 11]. Hence for each nonholomorphic field \(F\) a holomorphic field \(F^x\) can be found, being equal to \(F\) modulo the reality conditions, and can be used instead, since

\(\{G, \bar{F}\}_s = \{G, F^x\}_s = \{G, F^x\}\)

where \(\{ \ldots \}_s\) are the Dirac brackets with respect to \(\text{Im}(R_i) \approx 0\). \(R_{1AA'}^i\) and the projections \(R_{2BB'}^{(s)}(= e^{AA'} e_{BB'})\), \(\ldots\) denoting symmetrization in the indices, form a set of 18 commuting reality conditions, meaning that there is a real configuration space, described by those variables whose reality is enforced by these 18 conditions.

In the quantized theory the reality conditions will become exact operator identities that restrict the possible scalar product of physical states.

The constraints arise as follows. A primary constraint follows from (3)

\[ \mathcal{J}_{A'B'}^x := -e^{A'j} p_{AB'}^j \approx 0 \tag{6} \]

The variation of the canonical variables as given by \(\delta \chi = \{\chi, \int d^3y \bar{N}^x A'B' \mathcal{J}_{A'B'}^x\}\) with the parameter \(\bar{N}^x A'B' = \bar{N}^x B'A'\) are

\[ \delta e^{D'S'} = \bar{N}_{A'B'} e^{D'A'} \delta p_{DD'}^S = -\bar{N}_{D'A'} p_{DA'}^S \delta \psi^S = 0 \quad \delta \pi_{D'}^S = 0 \tag{7} \]

thus identifying \(\mathcal{J}^x\) as the generator of right-handed Lorentz transformations. To find the generator of left-handed Lorentz transformations, which in the real theory is the complex conjugate of \(\mathcal{J}^x\), one takes the complex conjugate of (3), uses the torsion equation of the real theory to get a holomorphic function \(\bar{\omega}\) and finally uses the reality conditions to replace the remaining nonholomorphic variables by holomorphic ones. This leads to

\[ \bar{\rho}_{AA'}^j = p_{AA'}^j + \frac{2i}{\kappa^2} \epsilon^{ijk} \partial_i e_{AA'}^k - \epsilon^{ijk} \bar{\psi}_{A'i} \psi_{Ak} \tag{8} \]
With this, one finds the generator of left-handed Lorentz transformations.

\[ J_{AB} = -e^{A'}_{(Ak} p_{B)A'}^k + \frac{i}{\kappa^2} \partial_l (\varepsilon^{ikl} e_{(AA')k} e^{A'B)l}) - \psi_{(Bl} \pi_{A)}^l \]  

(9)

The secondary constraints are given by

\[ S_A := \frac{\delta L}{\delta \psi^A_0} = D_i \pi^i_A \approx 0 \]  

(10)

\[ S^A \times_{A'} := -\frac{\delta L}{\delta \psi^A \times_{A'}_0} = -\varepsilon^{ijk} e_{AA'k} D_{ij} \psi_{A'}^k \approx 0 \]  

(11)

\[ H_{AA'} := \frac{\delta L}{\delta e_{AA'}_0} = 2i \kappa^2 \varepsilon^{ijk} e_{BA'k} [\partial_i \omega_{A'j} + \omega_{AC} \omega_{CB} j] - \varepsilon^{ijk} \psi_{A'}^i D_{ij} \psi_{Ak} \approx 0 \]  

(12)

where \( S_A \) is the generator of left-handed, \( S^A \times_{A'} \) that of right-handed supersymmetry transformations and \( H_{AA'} \) the combined generator of time translations (Wheeler-deWitt generator) and of spatial diffeomorphisms plus Lorentz and supersymmetry transformations [11].

To express the canonical Hamiltonian density and hence the secondary constraints in terms of the canonical variables, it is necessary to invert (2) and (3). This inversion takes place on the hypersurface in phase space given by the vanishing of the primary constraint \( J^\times \), the surface on which the canonical Hamiltonian is defined, and leads to

\[ \psi_{\times B'}^i = \pi^i_A D_{ij}^{AB'} \]  

(13)

\[ \omega_{AB}^i = -\frac{i\kappa^2}{2} p_A^B D_{ij}^{AA'} \]  

(14)

with

\[ D_{jk}^{AA'} := -\frac{2i}{\sqrt{\hbar}} e_{AB'}^k e_{BB'}^j n^{RA'} \]

\[ = \varepsilon_{jkp} e^{AA'}_{Ap} + \frac{i}{\sqrt{\hbar}} \hbar_{jk} n^{AA'}, \]  

(15)

because of

\[ D_{rj}^{CE'} \varepsilon^{lm} e_{CC'm} = \epsilon_{C'}^{E'} \delta^l_j \]  

(16)

On this hypersurface, the part of the rhs of (13) that is antisymmetric in \( A \) and \( B \) vanishes, yielding an expression for \( \omega \) with the correct number of degrees of freedom. \( J \) and \( J^\times \) are multiplied by Lagrangian multipliers \( \omega_{AB}^0 \) and \( \bar{\omega}^{A'B'}_0 \) and added to the canonical Hamiltonian density to give the total Hamiltonian density

\[ H_T = -e^{A'A'}_0 H_{AA'} - \omega_{AB}^0 J_{AB} - \psi_{A}^0 S_A - S^A \times_{A'}^0 - \bar{\omega}^{A'B'}_0 \bar{J}_{A'B'} \]

which is the typical picture in reparametrization-invariant theories: The total Hamiltonian vanishes weakly. The secondary constraints can now all be given as functions of the canonical variables

\[ S_A = \partial_l \pi^i_A + \frac{i\kappa^2}{2} p_{AA'}^i D_{ij}^{BA'} \pi_{B}^i \]  

(17)
\begin{align}
S_A^\lambda &= -\varepsilon^{ijk} e_{AA'} \partial_j \psi_{A_k} - \frac{i\kappa^2}{2} p_{BA'} \partial_{ij} \psi_{B_k} \\
H_{AA'} &= \partial_\mu p_{AA'}^\lambda - \varepsilon^{ijk} p_{AC'}^\mu D_{j\mu}^\nu \partial_{ij} \epsilon_{BA'k} + \frac{i\kappa^2}{2} p_{AC'}^\mu D_{il}^\nu \epsilon_{CA'i} \\
&+ \frac{i\kappa^2}{2} \varepsilon^{ijk} p_{AC'}^m D^B_{A'i} D_{jm}^C \psi_{Ck} \pi_B^l \\
&+ \varepsilon^{ijk} D^B_{A'i} \partial_j (\psi_{Ak}) \pi_B^l
\end{align}

Now that the canonical variables and constraints are known it is possible to proceed to quantize the theory.

3 Constraint Algebra

To quantize the theory canonically, one has to find operators corresponding to the canonical variables fulfilling the following quantization prescription for even variables $E$ and odd variables $O$

\[ [\hat{E}_1, \hat{E}_2] = i\hbar \{E_1, E_2\} \quad [\hat{O}, \hat{E}] = i\hbar \{O, E\} \quad [\hat{O}_1, \hat{O}_2]_+ = i\hbar \{O_1, O_2\} \]

where $[\ , \ ]_+$ stands for the anticommutator. It is not necessary to consider a specific representation of the operators that correspond to the canonical variables because for the algebra of constraints one only needs the commutation relations of those operators that are given by $[\ ]$ and $\{\}$ multiplied by $i\hbar$. A representation giving the correct form of the Lorentz generators is given in $[14, 11]$.

Using the equations (17) to (19) as the quantum constraints with the given operator ordering and employing

\[ [p_{AA'}^\mu(x), D_{jk}^{BB'}(y)] = i\hbar \varepsilon^{rsi} D^B_{A'jr} D^B_{A'Ask} \delta(x, y) \]

(20)

(which follows from (15)) one gets the well-known results

\[ [S_A^\lambda(x), S_B^\lambda(y)]_+ = 0 \quad [S_A^\lambda(x), S_B(y)] = 0 \quad [S_A(x), S_B^\lambda(y)]_+ = -\frac{\hbar \kappa^2}{2} H_{AA'} \delta(x, y) \]

To allow for partial integration in these calculations the constraints have been contracted with Grassmann valued transformation parameters and integrated over $x$ and $y$. Also, the partial derivative of the square of the delta function is taken to be zero. It is assumed that one can find regularized operators for the theory that fulfill this requirement. The calculation of $[S_A^\lambda, \mathcal{H}_{BB'}]$ can be performed in the same straightforward manner using (23) and (1) yielding

\[ [S_A^\lambda(x), \mathcal{H}_{BB'}(y)] = \]

\[ i\hbar \kappa^2 \varepsilon^{lmn} \epsilon_{A'B'} (\partial_m \psi_{Bn} + \frac{i\kappa^2}{2} p_{BD} \partial^{B'} \psi_{E_{mn}} ) \times \]

\[ \frac{1}{\sqrt{\hbar}} n^{CG'} e^{GG'} \mathcal{J}_{CG} \delta(x, y) \]

\[ = \frac{i\hbar \kappa^2}{\sqrt{\hbar}} \varepsilon_{A'B'} D_m \psi_{Bn} \frac{1}{\sqrt{\hbar}} n^{CG'} e^{GG'} \mathcal{J}_{CG} \delta(x, y) \]

where in the last line the correspondence between (10) and (17) with the chosen operator ordering was used to define the ordering of an operator version of $\omega$. Since the result is a constraint times a structure function appearing on the left hand side, this commutator shows no sign of non-closure of the algebra of constraints.
Working out \([S_A, H_{BB'}]\) by the same methods, making use of (24) and
\[
D^{CC'}_{sj} \: p_{CC'}^j = -\frac{2i}{\sqrt{h}} n^{GC'} \: e^{G'}_{cs} \: J_{G/C'}^x
\]
leads to
\[
[S_A(x), H_{BB'}(y)] = \frac{i\hbar^2}{\sqrt{h}} \varepsilon^{lmn} \varepsilon_{AB} n^{GC'} \: e^{G'}_{gn} \: J_{G/C'}^x \times
\]
\[
\left( \partial_m \psi_{BB'}^{x} + \frac{ik^2}{2} D^{CC'}_{mn} p_{CB'}^i \psi_{C/l}^{x} - \varepsilon_{sqr} D^{0l}_{qm} \partial_s \epsilon_{DB'r} \psi_{C/l}^{x} + \frac{ik^2}{2} \varepsilon^{spq} D^{ED'}_{sm} \psi_{Eq} \psi_{B'r}^{x} \psi_{C/l}^{x} \right)
\]
(21)
where the terms in brackets can be interpreted as \(D_m \psi_{BB'}^{x}\) when choosing a holomorphic \(\bar{\omega}\) to have the above operator ordering. Note that this ordering differs from that of \(\omega\) since \(S_A\) (10) is used in its left-ordered form (17). Commuting \(J^x\) through to the right in (21) using (7) gives rise to the divergent expression
\[
[S_A(x), H_{BB'}(y)]_{\text{divergent}} =
- \frac{\hbar k^2}{\sqrt{h}} \delta(0) \varepsilon^{lmn} \varepsilon_{AB} n^{DD'} \: e_{DB'n} \: D_m \psi_{BB'}^{x} \: \delta(x,y)
\]
Writing \(D_m \psi_{BB'}^{x}\) as \((D_m D^F_D q_l) \pi_{Fq}\) plus \(D^F_D q_l D_m \pi_{Fq}\) by properly introducing \(\omega\) and using (16) one gets one term involving \(S_D\) and one involving \(D_m e^{BB'} D_n\). However, since in the latter expression \(\bar{\omega}\) appears right-ordered with respect to \(p\) whereas \(\omega\) is left-ordered, it leads to
\[
\varepsilon^{lmn} D_m \bar{e}^{BB'} D_n = -\frac{ik^2}{2} \varepsilon^{lmn} (\psi_{Dm} \bar{\psi}_{n}^{x} - D^{BB'}_{Dmn} \delta(0))
\]
Introducing this into the above equation and using (22) finally yields
\[
[S_A(x), H_{BB'}(y)]_{\text{divergent}} =
\]
\[
= \frac{\hbar k^2}{2h} n^{DB'} S D \: \delta(0) \: \delta(x,y) + \frac{\hbar k^4}{4h} e^{F}_{B^k} \pi_{F} \: \delta(0)^2 \: \delta(x,y)
\]
The first term does not lead to difficulties, since it involves a constraint sitting on the right-hand side. The second term, however, clearly leads to non-closure of the algebra of these operators.

Choosing the right-ordered version \(S^R_A\)
\[
S^R_A = \partial_i \pi^A_i + \frac{ik^2}{2} D^{BB'}_{ij} \: p_{AB^j} \: \pi^B_i
\]
leads, via \([S^R_A(x), S^R_A(y)] = -\frac{\hbar k^2}{2} H^R_{AA'} \: \delta(x,y)\) to the expression
\[
H^R_{AA'} = \partial_i \varepsilon_{AA'}^l - \varepsilon^{ijk} D^{BC'}_{jl} \: p_{AC'}^i \: \partial_i \epsilon_{BA^j}^k + \frac{ik^2}{2} D^{CC'}_{ij} \: p_{AC'}^i \: \varepsilon^{B^j}_{CA^i}
\]
\[
+ \frac{ik^2}{2} \varepsilon^{ijk} D^{B}_{AA^j} D^{CC'}_{jm} \: p_{AC'}^m \: \psi_{ck} \: \pi^B l + \varepsilon^{ijk} D^{B}_{AA^j} \: \partial_j (\psi_{Ak}) \: \pi^B l
\]
If one calculates the commutator \([S^R_A, H^R_{BB'}]\), one arrives at
\[
[S^R_A(x), H^R_{BB'}(y)] = \frac{i\hbar k^2}{\sqrt{h}} \varepsilon_{A'B'} \: \varepsilon^{lmn} n^{C^G_i} \: e^{GG'} \times
\]
\[
\left( \partial_m \psi_{BB} \right) J_{CG} + \frac{ik^2}{2} D^{ED'}_{mq} \: \psi_{En} \: J_{CG} \: p_{BD'}^{q} \right)
\]
Commuting $p$ with $\mathcal{J}$ leads to divergent terms of the form

$$[S_A^x(x), H_{BB'}(y)]_{\text{divergent}} =$$

$$= \frac{2i}{\sqrt{\hbar}} \delta(0) \epsilon^{lmn} n_{BG'} e^{EG'}_l n^{FD'} e^{C'}_m \psi_{En} J^x_{D'C'}$$

$$+ i\hbar \delta(0) n_{BG'} \psi_{En} (p^{EG'} n + \frac{2i}{\kappa^2} \epsilon^{lmn} \epsilon^{C'GE} \delta^{ijq} \partial_j \epsilon^{E_D'l})$$

Again, divergent terms without a constraint in the right-hand position arise so that the algebra of the right-handed operators does not close either. Apart from the closure issue, these simple and straightforward calculations also gave the structure functions of the classical algebra.

4 Discussion and Acknowledgements

The above calculations show that using the holomorphic formulation of N=1 supergravity one can see that there is no formal closure of the constraint algebra for the two orderings chosen. This means that - in the sense of Dirac [5] - the canonical quantization has failed since it leads to inconsistencies. A different viewpoint would be to take e.g. $[S_A, H_{BB'}]$ on as a new constraint. However, it would still be necessary to verify the closure of the entire algebra. In any case it substantially reduces the set of physical states. Whether this still remains a meaningful theory is a topic for further investigation as well as the question whether the non-closure holds for all possible operator orderings.

It has to be kept in mind that physically meaningful results concerning the algebra can only be derived using regularized operators, since in the formal calculations delta function identities are used [16, 17]. The methods and results presented here hence are paving the way for a more involved regulated calculation. They also serve as a further demonstration showing the usefulness of the holomorphic formulation of supergravity: The expressions for the constraints and hence the calculation of the algebra become relatively simple as compared to the real theory [8]. Also, giving the structure functions of the quantum algebra explicitly, those of the classical algebra, as given in [6], are found as well.

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5 Appendix

Throughout the text spinor-valued tetrads $e^{AA'}_\mu$ are used to describe the gravitational degrees of freedom. Spinor indices take the values 0 and 1 or, respectively, 0’ and 1’. The indices $\mu, \nu, \rho \ldots$ are spacetime indices taking values from 0 to 3. The relations between spinor-valued and normal (complex) tetrads are given by

$$e^{AA'}_\mu := e^\alpha_\mu \sigma_\alpha^{AA'}$$

and

$$e^\alpha_\mu = -e^{AA'}_\mu \sigma^\alpha_{AA'},$$

where $\alpha, \beta$ are flat indices running from 0 to 3. Flat indices are pulled up and down with the Minkowskian metric $\eta_{\alpha\beta} = \eta^{\alpha\beta} = \text{diag}(-1,1,1,1)$. The $\sigma^\alpha_{AA'}$ are the Infeld van der Waerden symbols, defined by

$$\sigma^\alpha_{AA'} := \sigma_\alpha$$
\[ \sigma_0 \text{ being } -1/\sqrt{2} \text{ times the unit } 2 \times 2 \text{ matrix whereas the other } \sigma \text{’s are } 1/\sqrt{2} \times 2 \text{ matrix whereas the other } \sigma \text{’s are } 1/\sqrt{2} \text{ times the Pauli matrices. The outward normal spinor on a hypersurface described by the spatial components of the tetrad, } e^{AA'}_i, i = 1, 2, 3, \text{ is defined by} \]

\[ n_{AA'} n^{AA'} = 1 \quad \text{and} \quad n_{AA'} e^{AA'}_i = 0 \]

and is a function of the spatial components \( e^{AA'}_i \) alone, given by

\[ n^{AA'} = \frac{i}{3\sqrt{h}} \varepsilon^{ijk} e^{AB'}_i e^{BB'}_j e^{BA'}_k \]

Using the relations for \( n^{AA'} \) and the properties of the Infeld van der Waerden symbols \([15]\) one gets the following useful identities

\[ e_{AA'}^i e^{AB'}_j = \frac{1}{2} h_{ij} e_{A'B'} - i \sqrt{h} \varepsilon_{ijk} n_{AA'} e^{AB'k} \]
\[ e_{AA'}^i e^{BA'}_j = \frac{1}{2} h_{ij} e_{A'B'} + i \sqrt{h} \varepsilon_{ijk} n_{AA'} e^{BA'k} \]
\[ n^{AD'} e_{BD'}^i = -n_{BD'} e^{AD'}_i \]

Spinor indices are contracted by the means of \( \epsilon_{AB} \) od \( e^{AB} \) according to

\[ \xi^A = \xi^B \epsilon_{AB} \quad \xi_A = \xi^B \epsilon_{BA} \]

\( \epsilon_{AB} \) is antisymmetric in its indices and obeys

\[ \epsilon_{AB} \epsilon_{BC} = \epsilon^A_C = -\delta_C^A \quad \epsilon_{AB} \epsilon_{CB} = \epsilon_C^A = \delta_C^A \]

Analogous relations hold for \( \epsilon_{AB'} \). From the definition of \( n^{AA'} \) it follows that

\[ n_{AA'} n^{AB'} = \frac{1}{2} \epsilon_{A'B'} \]

The one-component spinors used, like \( \psi^A_\mu \), are taken to be Grassmann valued which directly leads to the identity

\[ \psi^{A'}_{[\mu} \psi^{A'}_{\nu]} = 0 \quad (22) \]

From \([15]\) one gets the useful formulas

\[ \varepsilon^{nmp} D^{H}_{A'tn} D^{A'}_{Cmi} = -\frac{2i}{\sqrt{h}} n^{H}_{D'r} e^{D'}_{C'i} \delta_{pr} \quad (23) \]
\[ \varepsilon^{irs} D^{DA'}_{jk} D^{C}_{A'kr} = -\varepsilon_{jkp} h^{ps} \epsilon^{CD} - \frac{2i}{\sqrt{h}} h_{kj} n_{A'C} e^{DA's} \quad (24) \]

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