Superfluid stability near Landau critical velocity

Abstract. It has been speculated earlier, that a superflow exceeding the roton Landau critical velocity at zero temperature may undergo a second order phase transition into a layered state. We investigate thermodynamic stability of this state and discuss possible phase diagrams at low temperature.

1. Introduction
By Landau criterion \[1\], at zero temperature \(T = 0\) a flow with a relative velocity \(w = v_n - v_s\) lower than \(v_L = \min(\varepsilon_p/p)\) is dissipationless. Here \(v_n\) and \(v_s\) are normal and superfluid velocities, \(p\) and \(\varepsilon_p\) are excitation momentum and energy, and \(v_L\) is Landau critical velocity. For helium excitation spectrum the minimum is reached in the roton part \(v_L \approx \Delta/p_0\), where \(p_0\) and \(\Delta\) are the roton momentum and the energy gap.

It has been argued \[2\] that if the relative velocity exceeds the Landau critical velocity, then a phase transition into a layered state must take place. The new phase is characterized by a finite number of rotons occupying one state with momentum \(p_0\) parallel to the relative velocity. This phenomenon can be understood \[3\] as a Bose-Einstein condensation of rotons. The order of the phase transition would depend on the roton-roton interaction sign: attraction (repulsion) gives rise to first (second) order transition.

At finite temperature the Landau superfluidity criterion is presumably replaced \[4\] by thermodynamic inequalities (requirements of thermodynamic stability). The boundary of the stability region in the \(w\)–\(T\) plane gives the critical velocity \(w_c\) as a function of temperature.

In the present paper we discuss whether the occurrence of Bose-Einstein condensation of rotons might be compatible with stability conditions and speculate on possible phases and observable transitions at low temperature.

2. Stability conditions
The strongest thermodynamic inequality (Eq.15 in \[4\]) can be written in the form

\[
\left( \frac{\partial f}{\partial w} \right)_{T,P} > f \left( \frac{\partial f}{\partial P} \right)_{T,w},
\]

where \(f\) is the mass flux in the frame of reference of the normal component (the frame of zero \(v_n\)). Let \(\rho = \rho_n + \rho_s\) and \(P\) designate the total density and the pressure respectively. The

\[1\] Usually vortices impose a lower dissipation threshold. Separate consideration is required to justify neglecting of vortices in particular experiment.
superfluid and normal densities are defined by means of $f = \rho_s w$ and $j_0 = \rho w - f = \rho_n w$. The inequality (1) then becomes

$$\rho_s + w \left( \frac{\partial \rho_s}{\partial w} \right)_{T,P} > f w \left( \frac{\partial \rho}{\partial P} \right)_{T,w}. \tag{2}$$

In the following we will also use the relation

$$\left( \frac{\partial \rho}{\partial w} \right)_{P} = w \rho \left( \frac{\partial \rho_n}{\partial P} \right)_{w} - j_0 \left( \frac{\partial \rho}{\partial P} \right)_{w}, \tag{3}$$

which can be derived from the thermodynamic identity

$$d\mu = \frac{dP}{\rho} - \frac{j_0}{\rho} dw - S \frac{dT}{\rho}$$

assuming $T = \text{const}$ and using the standard technique for mixed derivatives:

$$\left( \frac{\partial \rho^{-1}}{\partial w} \right)_{P} = - \left( \frac{\partial j_0/\rho}{\partial P} \right)_{w}.$$

The inequality (2) was evaluated numerically in [4]. Two important simplifications were used: the interaction between the excitations (phonons and rotons) was neglected and the Boltzmann distribution for rotons was assumed. The approximation is valid if the excitation density is small, that is, $T \ll \Delta - p_0 w$ (see dashed line in Fig.1 of [3]). This requirement is violated at low temperature: in the vicinity of $T = 0$ both roton interaction and Bose statistics affect the critical velocity behaviour.

The account for Bose statistics at near-critical velocity is straightforward. The “modified” free energy in the frame of reference of the superfluid component ($\tilde{F}_0 = F_0 - w j_0$, $d\tilde{F}_0 = -S dT - j_0 dw$) is obtained from the excitation spectrum with the conventional formula

$$\tilde{F}_0 = T \int \ln \left( 1 - \exp \left( \frac{pw - \Delta}{T} - \frac{(p - p_0)^2}{2mT} \right) \right) \frac{dp}{(2\pi \hbar)^3}.$$

At low temperature the main contribution to the integral is given by the vicinity of the exponent maximum. Using quadratic expansion the free energy can be transformed into

$$\tilde{F}_0 = \frac{T^{5/2} \rho_0^2 m^{1/2}}{2^{3/2} \pi^3 \hbar^3 \Delta} \int \ln \left( 1 - \exp \left( - \frac{p_0 (v_L - w)}{T} - q^2 \right) \right) dq.$$

The distribution function has a singularity if $w = v_L$, but the integral remains finite even at nonzero temperature and the normal density can be [2] formally calculated

$$\rho_n(v_L) = \frac{T^{3/2} \rho_0^4 m^{1/2} \zeta(3/2)}{2^{3/2} \pi^3 \hbar^3 \Delta^2}.$$

This state is unstable, however: $\partial \rho_n/\partial w|_{w=v_L} = \infty$ and the instability occurs at a lower velocity [3]

$$v_L - w_c \propto T^2. \tag{4}$$
3. Roton-roton interaction

Following [2], let us assume that a phase transition occurs between the ordinary superfluid and the roton BEC state. The condensate occupation number (and associated normal density $\rho_n$) is controlled by the roton-roton interaction. Positivity of the interaction constant $g$ implies a continuous (second order) transition, while a negative value may lead to a first order transition. The latter apparently violates the stability requirement: a jump in $\rho_n$ means $\partial \rho_s/\partial w = -\infty$ at the transition point.

Let the second order phase transition occur along the entire line $w = v_L(P)$ at low temperature. In the fluctuation region the essential part of the normal density expansion is

$$\rho_n = \rho_n(T) + B(w - v_L)^{1-\alpha} \overset{\text{def}}{=} \rho_n(T) + \rho'_n. \quad (5)$$

Here $\rho_n(T)$ is the normal density at the transition and $\alpha$ is some critical index.

- Suppose $\alpha > 0$ and $(\partial \rho'_n/\partial P)_P \to \infty$. Singular parts of the pressure and velocity derivatives are related:

$$\left( \frac{\partial \rho'_n}{\partial P} \right) \sim -\frac{w \rho}{dP/dv_L - j_0} \left( \frac{\partial \rho_n}{\partial w} \right)_P. \quad (6)$$

Substituting this in (3) we get

$$\left( \frac{\partial \rho_n}{\partial w} \right)_P = \left( \frac{w \rho}{dP/dv_L - j_0} - 1 \right) \left( \frac{\partial \rho_n}{\partial w} \right)_P.$$

The slope of the transition line is negative $dP/dv_L < 0$, hence $(\partial \rho_s/\partial w)_P \to -\infty$. By using (2), one can rewrite inequality (2) in the form

$$\rho_s + \left( w + f w \frac{d v_L}{dP} \right) \left( \frac{\partial \rho_n}{\partial w} \right)_P \gtrsim 0,$$

which immediately shows that the stability condition is violated at the transition point.

- Suppose $\alpha \leq 0$. Neglecting the right hand side of (2) and letting the mass density be constant $\partial \rho_n/\partial w = -\partial \rho_n/\partial w$ we get

$$\rho_s > w \left( \frac{\partial \rho_n}{\partial w} \right)_P.$$

This may leave a window for an over-critical state if $\alpha = 0$ (in Landau theory of phase transitions) and $Bv_L < \rho$, or if $\alpha < 0$ and

$$B(1-\alpha)(w - v_L)^{-\alpha}w < \rho_s. \quad (7)$$

4. Discussion

Observability of the layered state, even when inequality (7) is satisfied, requires further investigation. From (4) it follows that the stability critical velocity (if roton interaction is neglected) is lower than Landau velocity $v_L$. A possible phase diagram in this situation is drawn in Fig. 1. It is therefore impossible to gradually accelerate the fluid until the Bose-Einstein condensation sets in.

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2 Strictly speaking, the system discussed here is more complicated than a single-condensate superfluid. Further investigation is required to ascertain the applicability of this approach.

3 There are indications [2] that $\alpha$ is equal to the heat capacity index ($\sim -0.01$ in superfluid $^4$He).

4 To justify this we estimate $(\partial \rho_s/\partial P)_P f/(\partial \rho_n/\partial w)_P \lesssim \rho v_L d v_L/dP \sim 10^{-2}$.
Stronger roton-roton repulsion may extend the stability boundary to the Landau critical velocity. The corresponding phase diagram is depicted in Fig.2. In the mean field approximation the following estimate can be done:

$$\max \left( \frac{\partial j_0}{\partial \omega} \right)_{\rho} \sim \frac{p_0^2}{g}. \quad (8)$$

The Landau velocity is accessible if $\rho > \max(\partial j_0/\partial \omega)_{\rho}$, or if

$$g > \frac{p_0^2}{\rho} \sim 3 \cdot 10^{-37} \text{erg cm}^3. \quad (9)$$

In [2] the estimate $g = 2 \cdot 10^{-38} \text{erg cm}^3$ was used. This is apparently insufficient to satisfy (9). It may be possible to change experimental conditions (pressure?) so that (9) is valid. Otherwise, unless stability conditions in narrow channels are substantially different from those in bulk, experiments in porous media (as suggested in [6]) are unlikely to discover the microscopic layered state. It seems plausible to try the original idea of Iordanskii and Pitaevskii [3] to investigate the equilibrium of quasiparticle pulse with high initial momentum. Such technique seems to attract much experimental attention [7].

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5 Instead, a macroscopic large-scale modulation due to the development of thermodynamic instability will be observed.