Perturbations to Kink-Like Topological Defects in 2D Anti de Sitter Spacetime

Orlando Alvarez\textsuperscript{a,1}, Matthew Haddad\textsuperscript{b,1}

\textsuperscript{1}Department of Physics, University of Miami, 1320 Campo Sano Ave, Coral Gables, FL 33146, USA

Abstract We examine codimension–1 topological defects whose associated worldline is geodesically embedded in AdS\textsubscript{2}. This discussion extends a previous study of exact analytical solutions to the equations of motion of topological defects in AdS\textsubscript{n} in a particular limit where the masses of the scalar and gauge field vanish. We study perturbations to second order about the zeroth order kink-like solution and find that they are stable.

1 Introduction

This is the second in an ongoing series of articles in which we study the solutions to the equations of motion of topological defects in anti de Sitter (AdS) spacetime. In our previous paper [1], we were able to show that there exist exact analytical solutions to the equations of motion for a topological defect in AdS\textsubscript{n}, in what we dubbed the “double BPS limit” owing to the similarity of our approach with that of Bogomolny, Prasad, and Sommerfield (BPS) [2,3]. There are other studies on topological defects in AdS\textsubscript{d}, such as the work of Lugo et al. [4,5] on magnetic monopoles in AdS\textsubscript{4} and more recently the work of Ivanova et al. [6] on finite-energy Yang-Mills field theory in AdS\textsubscript{4}. Kink-like defects in flat space are discussed by Coleman in his famous Erice lectures [7]. Manton and Sutcliffe give a good treatment of topological defects in general in their book [8].

We briefly explain some notational conventions. Our work concerns analytical solutions for SO(l) Higgs field theories in a static AdS\textsubscript{n} background. These admit maximally symmetric topological defects of dimension p. The world brane associated with one of these defects is then a q = p + 1 dimensional timelike totally geodesic submanifold \(\Sigma^q \approx \text{AdS}_q\) embedded in AdS\textsubscript{n}. The value of l is the number of transverse dimensions, \(n = q + l\). Our framework allows us to seek exact analytical solutions for defects categorized by pairs \((q, l)\), where \(q \geq 1\) and \(l \geq 1\). The one-dimensional kink that we study in this paper is therefore the \((1, 1)\) defect. In a forthcoming article, we generalize this to the case of \((q, 1)\) dimensional defects in AdS\textsubscript{q+1}.

We have mentioned the double BPS limit thus far without much explanation. BPS studied monopole solutions to the equations of motion for topological defects in \(\mathbb{R}^{3+1}\) in the limit that the mass of the scalar field vanishes while preserving some boundary conditions. They found that there were exact analytical solutions. This work was done in flat space, where the length scale was set by the gauge field mass. In non-flat constant curvature spaces, the radius of curvature \(\rho\) sets a third length scale. Therefore, one can take the limit in the equations of motion where the masses of both the scalar and the gauge fields vanish and obtain equations that can have exact analytical solutions with a single length scale \(\rho\). This is the double BPS limit. For a more thorough explanation of this process, we direct the reader to our previous article [1].

The double BPS equations of motion do not follow directly from an action, but rather from a limit imposed on the equations of motion for a topological defect in curved spacetime. To this end, the double BPS solution is a starting point to a perturbative solution to these original equations in which the mass of the fields are taken to be on the same order as the perturbation. We pursue such a solution in this article, and show that the linear and quadratic perturbations are stable, and uniquely determine the third and higher order corrections.
2 Deriving the Equation of Motion

2.1 The Conformal Metric

In our previous paper, we discussed the procedure for writing the action and deriving the equations of motion for a spherically-symmetric topological defect in AdS$_n$ [1]. In that discussion, we adopted an SO(l)-gauged Higgs model embedded into AdS$_n$, where $n \geq 2$. We restrict our discussion to maximally symmetric defects, so that $n = q + l$ where $q$ is the dimension of the topological defect we are studying. In this article we focus on the $(1, 1)$ defect, which is the kink-like defect that extends from AdS$_1$ out into AdS$_2$. In this case, there is only a scalar field $\phi$. The worldline associated with the defect is $\Sigma^1 \approx$ AdS$_1$. This is a one-dimensional timelike submanifold, a timelike curve, and so we simply take the metric as $ds_{\text{AdS}_2}^2 = -d\tau^2$. If $\rho$ is the radius of curvature of AdS$_2$ and $v$ as the signed distance to a point to $\Sigma^1$, we can write the metric of AdS$_2$ as

$$
\text{d}s_{\text{AdS}_2}^2 = \cosh^2 \left( \frac{v}{\rho} \right) \text{d}s_{\text{AdS}_1}^2 + \text{d}v^2
$$

where $-\infty < \tau < +\infty$ and $-\infty < v < +\infty$.

The Lagrangian for the model is

$$
\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{8} (\phi^2 - \phi_0^2)^2
$$

where $\lambda$ is the scalar field coupling and $\phi_0$ is the field’s vacuum expectation value at spatial infinity (that is, $\phi \rightarrow \phi_0$).

The action is then

$$
I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{-g} \mathcal{L} = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{d}v \text{d}\tau \cosh \left( \frac{v}{\rho} \right) \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{\lambda}{8} (\phi^2 - \phi_0^2)^2 \right]
$$

$$
= -\int_{-\infty}^{\infty} \text{d}\tau \int_{-\infty}^{\infty} \text{d}v \cosh \left( \frac{v}{\rho} \right) \left[ -\frac{1}{2 \cosh^2 (v/\rho)} \left( \frac{\partial \phi}{\partial \tau} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial \tau} \right)^2 + \frac{\lambda}{8} (\phi^2 - \phi_0^2)^2 \right]
$$

(3)

For ease of computation we will scale everything by the radius of curvature in this article, so that we can put everything in terms of dimensionless variables. This amounts to the substitutions $v \rightarrow \rho v$, $\tau \rightarrow \rho \tau$ and $\phi \rightarrow \phi_0 \phi$. We also take note of the combination $\lambda \phi_0^2$, which is the flat-space mass squared, $m_\phi^2$, of the field and then define the dimensionless mass $\mu^2 = (m_\phi \rho)^2$. The action (3) gives the equation of motion that was studied in [1]

$$
- \frac{\partial^2 \phi}{\partial \tau^2} + \cosh^2 (v) \left[ \frac{\partial^2 \phi}{\partial v^2} + \tanh (v) \frac{\partial \phi}{\partial v} - \frac{1}{2} \mu^2 \phi (\phi^2 - 1) \right] = 0
$$

(4)

In this $(1 + 1)$ dimensional case, the linear perturbation analysis is better done in coordinates where the metric is conformal to the flat space metric. We make the substitution $x = \arctan (\sinh (v))$ which gives us the metric

$$
dx^2 = \frac{\rho^2}{\cos^2 x} (-d\tau^2 + dx^2).
$$

(5)

This shows that AdS$_2$ is conformally equivalent to the strip $\mathbb{R} \times [-\pi/2, \pi/2] \subset \mathbb{M}^2$. The action in these coordinates is then

$$
I = -\phi_0^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{d}\tau dx \left[ \frac{1}{2} \left( \frac{\partial \phi}{\partial \tau} \right)^2 + \left( \frac{\partial \phi}{\partial x} \right)^2 \right]
$$

$$
+ \frac{1}{8} \mu^2 \sec^2 (x) (\phi^2 - 1)^2,
$$

(6)

with equation of motion

$$
- \frac{\partial^2 \phi}{\partial \tau^2} + \frac{\partial^2 \phi}{\partial x^2} = \frac{1}{2} \mu^2 \sec^2 (x) (\phi^2 - 1).
$$

(7)

We analyze the linearization of this equation in this paper. Note that the differential operator on the left-hand side is just the two-dimensional flat space d’Alembertian.

2.2 The Double BPS Solution

In our previous article, we noted the existence of an exact analytical solution to (4) in the limit where $\mu \rightarrow 0$. This solution is possible because of the nonzero radius of curvature of AdS$_2$ and we refer to it as the “double BPS” solution due to the similarity of this approach to that of Bogomolny, Prasad, and Sommerfield [2, 3] in the flat-space case. In the double BPS limit using the conformal coordinates, the equation of motion for a static defect is

$$
\frac{d^2 \phi_0}{dx^2} = 0,
$$

(8)

with boundary conditions $\phi_0(\pm \frac{\pi}{2}) = \pm 1$. The solution to this ODE is simply

$$
\phi_0 (x) = \frac{x}{\pi}.
$$

(9)
2.3 Introducing the Small-Mass Perturbation

We now use (9) as the starting point for a perturbative solution where the small perturbation parameter is the mass term. The perturbation expansion to all orders is discussed in Section 4. Here we only consider the first two terms of the expansion and write the field as \( \phi(\tau, x) = \phi_0(x) + \varepsilon \phi_1(\tau, x) + \varepsilon^2 \phi_2(\tau, x) + O(\varepsilon^3) \) with \( |\varepsilon| \ll 1 \). We now insert this into (7) and keep only the terms of \( O(\varepsilon^2) \):

\[
\frac{d^2 \phi_0}{dx^2} + \varepsilon \left( -\frac{\partial^2 \phi_1}{\partial \tau^2} + \frac{\partial^2 \phi_1}{\partial x^2} \right) + \varepsilon^2 \left( -\frac{\partial^2 \phi_2}{\partial \tau^2} + \frac{\partial^2 \phi_2}{\partial x^2} \right) = \frac{1}{2} \mu^2 \sec^2(x) \left[ \phi_0(\phi_0^2 - 1) + \varepsilon \phi_1(3\phi_0^2 - 1) \right.
+ \varepsilon^2 (3\phi_0 \phi_1^2 + \phi_2(3\phi_0^2 - 1)) \right] \tag{10}
\]

Notice that \( \phi_0(x) \) is a linear function, so its second derivative vanishes. We consider the differential equations for the first- and second-order perturbations:

\[
\frac{\partial^2 \phi_1}{\partial \tau^2} + \frac{\partial^2 \phi_1}{\partial x^2} = 0 \tag{11}
\]

\[
\frac{\partial^2 \phi_2}{\partial \tau^2} + \frac{\partial^2 \phi_2}{\partial x^2} = \frac{1}{2} \mu^2 \sec^2(x) \phi_0(\phi_0^2 - 1) \tag{12}
\]

Since \( \phi_0 \) satisfies the required boundary conditions, we have that perturbative expansion functions satisfy \( \phi_1(\tau, \pm \pi/2) = 0 \) and \( \phi_2(\tau, \pm \pi/2) = 0 \).

3 Solving the Equation of Motion

3.1 Solving the homogeneous equation

We next look at the one-dimensional wave equation (11) that gives the linear perturbation. The spatial Dirichlet boundary conditions tell us that the solution is a Fourier series with basis functions \( \sin mx(x + \pi/2) \) with \( m = 1, 2, 3, \ldots \). The solution to the wave equation is

\[
\phi_1(\tau, x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left[ A_{2n-1} \cos((2n-1)\tau) + B_{2n-1} \sin((2n-1)\tau) \right] \cos((2n-1)x) \cos((2n-1)x) \,
\]

\[
+ \frac{2}{\pi} \sum_{n=1}^{\infty} \left[ A_{2n} \sin(2n\tau) - B_{2n} \cos(2n\tau) \right] \sin(2nx) \tag{13}
\]

where the Fourier coefficients \( A_n \) and \( B_m \) are real numbers.

3.2 Solving the inhomogeneous equation

Equation (12) is an inhomogeneous linear equation in \( \phi_2 \). We know that to find the solution, we need the sum of a particular solution of the inhomogeneous equation and an appropriate solution of the corresponding homogeneous equation. We therefore write the solution in the form \( \phi_2(\tau, x) = \xi(\tau) + \eta(\tau, x) \). Because we have a \( \tau \)-independent source term, we take the particular solution \( \xi(x) \) to be a \( \tau \)-independent solution of the inhomogeneous equation

\[
\frac{d^2 \xi}{dx^2} = \frac{1}{2} \mu^2 \sec^2(x) \phi_0(\phi_0^2 - 1) , \tag{14}
\]

while \( \eta(\tau, x) \) is the general solution to the homogeneous equation

\[
\frac{\partial^2 \eta}{\partial \tau^2} + \frac{\partial^2 \eta}{\partial x^2} = 0 . \tag{15}
\]

As detailed in Section 4, we can take \( \eta = 0 \). Therefore we have \( \phi_2(\tau, x) = \xi(x) \).

Equation (14) is an ordinary differential equation in \( x \). The defect boundary conditions require that \( \xi(x) \xrightarrow{x \rightarrow \pm \pi/2} 0 \). If we substitute expression (9) for \( \phi_0(x) \), the differential equation then becomes:

\[
\frac{d^2 \xi}{dx^2} = \frac{\mu^2}{\pi^3} (4x^2 - \pi^2) x \sec^2(x) \tag{16}\]

The right hand side is an odd function and the solution should be an odd function. Integrating the equation twice yields an unwieldy expression

\[
\xi(x) = \frac{\mu^2}{5\pi^4} \left[ -5i \left( \pi^2 - 12x^2 \right) \text{Li}_2 \left( -e^{2ix} \right) - 90x \text{Li}_3 \left( -e^{2ix} \right) + 60i \text{Li}_4 \left( -e^{2ix} \right) + 20x \log(1 + e^{2ix}) + 5i\pi^2 x^2 + 90x \zeta(3) + 10\pi^3 x + 10\pi^2x \log(1 + e^{2ix}) \right.
\]

\[
-5\pi^2 \log 2 - 5\pi^2 \log(\cos x) - i\pi^4 \right] ,
\]

where \( \text{Li}_n \) is the polylogarithm function of order \( n \) and \( \zeta(3) \approx 1.20206 \) is the Riemann zeta function evaluated at 3. The above may be rewritten as

\[
\phi_2(\tau, x) = \xi(x) = \frac{\mu^2}{5\pi^4} \left[ 5 \left( \pi^2 - 12x^2 \right) \text{Im} \left( \text{Li}_2 \left( -e^{2ix} \right) \right)
- 90x \text{Re} \left( \text{Li}_3 \left( -e^{2ix} \right) \right) + 60 \text{Im} \left( \text{Li}_4 \left( -e^{2ix} \right) \right)
- 5x \left( 4x^2 - \pi^2 \right) \log(2 \cos x) + 90\zeta(3) x \right] \tag{17}
\]

We plot the function \( \phi_2 \) in figure 1. In figure 2, we plot the static numerical solution to (7) and compare it to the zeroth order solution.
I

where \( q_{ij} \) is a real symmetric bilinear form. In our field theoretic model, we have that \( q = -\partial^2_x + \partial^2_t \) is the d’Alembertian operator. The extrema of the action gives the equation of motion

\[
0 = q_{ij} \phi^j \phi^i - \epsilon^2 \partial U / \partial \phi^i. \quad (20)
\]

We are interested in a power series expansion that leads to a perturbative solution

\[
\phi = \sum_{n=0}^{\infty} \epsilon^n \phi_n. \quad (21)
\]

The order by order equation of motion is summarized by the power series

\[
0 = q \phi_0 + \epsilon q \phi_1 + \epsilon^2 \left( q \phi_2 - U'(\phi_0) \right) + \epsilon^3 \left( q \phi_3 - q_1 U''(\phi_0) \right) + \epsilon^4 \left( q \phi_4 - \frac{1}{2} q^2 U^{(3)}(\phi_0) - q_2 U''(\phi_0) \right) + \epsilon^5 \left( q \phi_5 - \frac{1}{3} q^3 U^{(4)}(\phi_0) - q_2 q_3 U^{(3)}(\phi_0) - q_3 U''(\phi_0) \right) + \mathcal{O}(\epsilon^6). \quad (22)
\]

The static zeroth order solution \( \phi_0(\tau, x) = 2x/\pi, \) see (9), already satisfies the spatial boundary conditions \( \phi(\tau, \pm \pi/2) = \pm 1. \) This means that all the higher order terms satisfy vanishing Dirichlet spatial boundary conditions \( \phi_n(\tau, \pm \pi/2) = 0 \) for \( n \geq 1. \) An important consequence is that if we write the \( n \)-th order perturbative equation of motion as \( q \phi_n = s_n \) for \( n \geq 2, \) where \( s_n \) is a source term, then \( s_0(x) = 0 \) at the spatial boundaries \( x = \pm \pi/2. \) With vanishing boundary conditions, the wave equation, \( q \psi = 0, \) has non-trivial solutions (13). This means that the solution to \( q \phi_n = s_n \) is not unique because of the non-trivial kernel of \( q. \) A Greens’ function \( G \) satisfies \( qG = I, \) and the general solution to the equation \( q \psi = s \) may be written\(^1\) as \( \psi = \psi_0 + G s \) where \( q \psi_0 = 0. \) The uniqueness issue for solutions of \( q \phi_n = s_n \) may be resolved by an appropriate choice of a cokernel for the operator \( q. \) This is equivalent to choosing a particular Greens’ function\(^2,\) i.e., a right partial inverse for \( q. \) Well-known choices for the d’Alembertian operator in Minkowski space are: the advanced Greens’ function, the retarded Greens’ function, the Feynman Greens’ function, etc. Once we have made a choice of cokernel, we can uniquely specify the solution to \( q \phi_n = s_n \) as \( \phi_n = G s_n. \) The kernel of \( q \) only appears\(^3\)

\[
q_{ij} \phi^j \phi^i - \epsilon^2 U(\phi).
\]

\[
I(\phi) = \frac{1}{2} \]

Putting all of this together, we can now write down the full perturbative solution to the equation of motion to second order in \( \epsilon: \)

\[
\phi(\tau, x) = \frac{2}{\pi} x + \epsilon \phi_1(x, \tau) + \epsilon^2 \phi_2(x) + \mathcal{O}(\epsilon^3). \quad (18)
\]

where \( \phi_1(\tau, x) \) is (13), \( \phi_2(x) \) is (17). Note that all of the dependency on the mass to this order is contained inside the \( \phi_2(x) \) term. Here, we are able to absorb the solution to the homogeneous wave equation (15) into the first-order term through an appropriate choice of Greens’ function. For more detail, we refer the reader to Section 4.

3.3 Perturbative solution to the equation of motion

4 General perturbation expansion of the equation of motion

Here we discuss general properties of our action and the equation of motion. Assume \( \phi \in \mathbb{R}^n, \) then our action is schematically of the form

\[
I(\phi) = \frac{1}{2} \]

The \( U \) terms all vanish at \( x = \pm \pi/2, \) this means that the source is in the image of the operator \( q. \) This is necessary for the construction of a right inverse.

\( ^1 \) The difference of two choices of Greens’ function is solution to the wave equation.

\( ^2 \) The difference of two choices of Greens’ function is solution to the wave equation.

\( ^3 \) The difference of two choices of Greens’ function is solution to the wave equation.
in the first term $\phi_1$ which can be freely specified. Once $\phi_1$ is
given then all the higher order corrections $\phi_n$, for $n \geq 3$, are
uniquely determined by using (22). Said differently, once a
free massless wave oscillating in the background of the zeroth
order kink is specified then all higher order corrections
are completely determined. Note that the second order cor-
rection $\phi_2$ is independent of the choice of traveling wave $\phi_1,
see eq. (17). Summarizing, we have constructed a perturba-
tive solution to the equation of motion.

Remark 1 In quantum mechanical time-independent pertur-
bation theory, where you have a positive definite inner pro-
duct in the Hilbert space, the perturbations are taken to be or-
thogonal to the zeroth order solution. The argument we gave
above is the generalization of this notion to our situation.

In solving $q\psi = s$, we note that we can write two Fourier
expansions

$$
\psi(\tau, x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{m=1}^{\infty} \tilde{\psi}_m(\omega) e^{-i\omega x} \sin \left[ m \left( x + \frac{\pi}{2} \right) \right],
$$

$$
s(\tau, x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{m=1}^{\infty} \tilde{s}_m(\omega) e^{-i\omega x} \sin \left[ m \left( x + \frac{\pi}{2} \right) \right].
$$

Our choice of solution to the equation $q\psi = s$ is given by

$$
\psi(\tau, x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{m=1}^{\infty} \frac{1}{\alpha^2 - m^2} \tilde{s}_m(\omega) e^{-i\omega x} \sin \left[ m \left( x + \frac{\pi}{2} \right) \right]
\times \sin \left[ m \left( x + \frac{\pi}{2} \right) \right]
$$

(23)

The choice of Greens’ function is determined by the poles
and the integration contour in the $\alpha$ complex plane. For our
purposes, it is not necessary to specify the choice of Greens’
function explicitly.

4.1 Classical Energy

The conserved energy $E(\phi)$ is given by the expression

$$
\rho \frac{E(\phi)}{\phi_0^2} = \int_{-\pi/2}^{\pi/2} \left( \frac{1}{2} (\partial_x \phi)^2 + \frac{1}{2} (\partial_t \phi)^2 + \varepsilon^2 U(\phi) \right) dx,
$$

(24)

where $U(\phi) = \frac{1}{8} \mu^2 \sec^2(x)(\phi^2 - 1)^2$. The factor of the radius
of curvature $\rho$ appears because the coordinates $(\tau, x)$ are dimensionless.
Inserting perturbation expansion (21) into the
expression for $E(\phi)$, using the fact that because of the
spatial boundary conditions we have that $\int_{-\pi/2}^{\pi/2} \partial_x \phi_0 dx = 0$
for $n \geq 1$, and substituting the $O(\varepsilon^3)$ equation of motion
found in (22), we conclude that

$$
\rho \frac{E(\phi)}{\phi_0^2} = \frac{2}{\pi}
+ \varepsilon^2 \int_{-\pi/2}^{\pi/2} \left( U(\phi_0) + \frac{1}{2} (\partial_x \phi_1)^2 + \frac{1}{2} (\partial_t \phi_1)^2 \right) dx
$$

$$
+ \varepsilon^4 \int_{-\pi/2}^{\pi/2} \left( \frac{1}{2} \mu^2 (\phi_0)^2 + \frac{1}{2} U''(\phi_0) \phi_1^2 \right.
$$

$$
+ (\partial_x \phi_1)(\partial_x \phi_1) + (\partial_t \phi_1)(\partial_t \phi_1) \right) dx
$$

$$
+ \varepsilon^5 \int_{-\pi/2}^{\pi/2} \left( \frac{1}{6} U^{(3)}(\phi_0) \phi_1^3 + U''(\phi_0) \phi_2 \phi_1 \right.
$$

$$
+ (\partial_x \phi_1)(\partial_x \phi_2) + (\partial_t \phi_1)(\partial_t \phi_2) \right) dx + O\left( \varepsilon^6 \right),
$$

(25)

where $\phi_0(\tau, x) = 2\chi_1$. Note that here $U'(\phi) = \partial U/\partial \phi$.
You can verify the perturbation expansion of $\phi$ to $O(\varepsilon^n)$
determines perturbation expansion of $E(\phi)$ to $O(\varepsilon^{n+1})$.

We can substitute into the above expression the explicit
formulas for $\phi_0$ and $\phi_2$ obtained for our choice of $U$. This
leads to some integrals that can either be computed explicitly
or numerically. The result is

$$
\rho \frac{E(\phi)}{\phi_0^2} = \frac{2}{\pi} + \frac{12 \zeta(3)}{\pi^3} \varepsilon^2 \mu^2
$$

$$
+ \varepsilon^2 \int_{-\pi/2}^{\pi/2} \left( \frac{1}{2} (\partial_x \phi_1)^2 + \frac{1}{2} (\partial_t \phi_1)^2 \right) dx
$$

$$
- 0.339506 \varepsilon^4 \mu^4 + \varepsilon^4 \int_{-\pi/2}^{\pi/2} \left( \frac{1}{2} U''(\phi_0) \phi_1^2 \right.
$$

$$
+ (\partial_x \phi_1)(\partial_x \phi_1) + (\partial_t \phi_1)(\partial_t \phi_1) \right) dx
$$

$$
+ \varepsilon^5 \int_{-\pi/2}^{\pi/2} \left( \frac{1}{6} U^{(3)}(\phi_0) \phi_1^3 + U''(\phi_0) \phi_2 \phi_1 \right.
$$

$$
+ (\partial_x \phi_1)(\partial_x \phi_2) + (\partial_t \phi_1)(\partial_t \phi_2) \right) dx + O\left( \varepsilon^6 \right),
$$

(25)

where $12 \zeta(3)/\pi^3 \approx 0.465218$. We reiterate that once $\phi_1$ is
specified, everything in the energy formula (25) is known in
principle.

The mass of the zeroth order kink is $2\phi_0^2/\pi \rho$. There are
two distinct $O(\varepsilon^3)$ corrections to the energy. The first is due
to the presence of the potential energy term, and the second
is the energy of the free wave that moves in the static back-
ground of the zeroth order kink. Note that the $\varepsilon$ and $\varepsilon^3$ terms
in the expansion vanish.

5 Conclusion

Starting with the double BPS solution (9), and adding a small
perturbation in the mass term to the equation of motion (7),
we have obtained the first-order and the second order perturbative corrections to the solutions of the equations of motion beginning with the zeroth order kink solution. This perturbative solution is an oscillating kink-like (codimension 1) topological defect in AdS$_2$. In addition, in section 4, we laid out the method for building this perturbative expansion at each order and we see that all corrections of $O(e^n)$ for $n \geq 3$ are determined by the choice of linear perturbation wave solution $\phi_1$. From (18), we can see that all of the $\tau$ dependence in the solution originates from $\phi_1(\tau,x)$. All of the normal modes in the linearized term have frequencies that are non-zero real and therefore these modes are stable. You can think of $\phi_2(x)$ as a zero frequency correction to the kink.

Appendix A: Killing Vectors and the Linearized Equation of Motion

In this section we review the role played by the Killing vectors for the metric on AdS$_2$ on the solutions to the equation of motion for the (1,1) defect. Let $\Phi = (\Phi^1, \Phi^2, \ldots)$ be a real scalar field in a potential $U$. In general, the equations of motion for this field will be of the form

$$\Box \Phi^I - \frac{\partial U}{\partial \Phi^I} = 0 \quad (A.1)$$

where $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the appropriate second-order differential d’Alembertian operator for AdS$_2$.

Now suppose that $K$ is a Killing vector for the metric $g_{\mu\nu}$. By definition, Lie differentiation in the direction of $K$ leaves the metric unchanged, therefore, it commutes with the geometric d’Alembertian: $\xi_K \Box = \Box \xi_K$. If we take the Lie derivative along $K$ of (A.1), we obtain:

$$0 = \xi_K \Box \Phi^I - \xi_K \frac{\partial U}{\partial \Phi^I} = \Box \xi_K \Phi^I - \frac{\partial^2 U}{\partial \Phi^I \partial \Phi^J} \xi_K \Phi^J = \Box \left( K^\mu \frac{\partial \Phi^I}{\partial x^\mu} \right) - \frac{\partial^2 U}{\partial \Phi^I \partial \Phi^J} \left( K^\mu \frac{\partial \Phi^J}{\partial x^\mu} \right) \quad (A.2)$$

Next let $\Psi^I = K^\mu \frac{\partial \Phi^I}{\partial x^\mu}$, then we see that

$$\Box \Psi^I - \frac{\partial^2 U}{\partial \Phi^I \partial \Phi^J} \partial^J \Psi^I = 0. \quad (A.3)$$

For each Killing vector $K$, we potentially obtain a non-trivial solution $\Psi^I = K^\mu \partial^\mu \Phi$ to the linearization of (A.1).

In flat space, some of these linearized solutions would correspond to zero-frequency modes in the oscillating part of the solution, because the translational Killing vectors do not depend on the timelike coordinate. In our case of the (1,1) defect in AdS$_2$, we will not find non-trivial zero-frequency modes. Two of the Killing vectors are no longer time independent, however, the modes generated are non-zero frequency solutions of the linearized equation of motion.

We now carry out this computation explicitly, using (5) as our metric. Killing’s equations are given by

$$\xi_K g_{ab} = K^c \partial_c g_{ab} + g_{c(a} \partial_{b)c} K^c + g_{ab} \partial_c K^c = 0 \quad (A.4)$$

This leads to the following system of differential equations:

$$\begin{align*}
\frac{\partial K^\tau}{\partial \tau} &= -\tan(\lambda) K^x \\
\frac{\partial K^x}{\partial \tau} &= \frac{\partial K^\tau}{\partial x} \\
\frac{\partial K^x}{\partial x} &= -\tan(\lambda) K^x
\end{align*} \quad (A.5)$$

The solutions to these equations are three Killing vector fields:

$$K_1^\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad K_2^\mu = \begin{pmatrix} -\sin(\lambda) \sin(x) \\ \cos(\lambda) \cos(x) \end{pmatrix}, \quad K_3^\mu = \begin{pmatrix} \cos(\lambda) \sin(x) \\ \sin(\lambda) \cos(x) \end{pmatrix} \quad (A.6)$$

The first vector is just $K_1 = \partial_\tau$ and corresponds to time translations. The second corresponds to space translations in flat space, which we can see from looking at the limit of $\rho \to \infty$ (i.e., $\tau/\rho \to 0$ and $x/\rho \to 0$). In this limit, $K_2 \to a_2$. Additionally, we see that $K_3 \to \lambda \delta_z + \tau \delta_x$, which is the generator of boosts in the $\tau x$-plane in flat space.

We now apply the first part of this discussion: if we differentiate a known solution to our equation of motion in these directions, we should still have a solution. For our known solution, we take (9). Since this is time-independent, $K_1 \phi(x) = 0$ leads to a trivial solution to the equation of motion (7). Differentiating using the other Killing vectors produces

$$K_2 \phi(x) = \frac{2}{\pi} \cos(\lambda) \cos(x), \quad K_3 \phi(x) = \frac{2}{\pi} \sin(\lambda) \cos(x) \quad (A.7)$$

Looking back at the oscillating part of our perturbative solution (13), we can see that the $n = 1$ mode in the first summand corresponds to the term

$$\frac{2}{\pi} \left( A_1 \cos(\lambda) + B_1 \sin(\lambda) \right) \cos(x) = A_1 K_2 \phi(x) + B_1 K_3 \phi(x) \quad (A.8)$$

As we can see, the functions in (A.7) correspond to modes of frequency 1 in the perturbative solution given by (13).

References

1. O. Alvarez, M. Haddad, Journal of High Energy Physics 2018(3), 12 (2018). DOI 10.1007/JHEP03(2018)012. URL https://doi.org/10.1007/JHEP03(2018)012
2. E.B. Bogomolny, Soviet Journal of Nuclear Physics 24, 449 (1976)
3. M.K. Prasad, C.M. Sommerfield, Physical Review Letters 35(12), 760 (1975). DOI 10.1103/PhysRevLett.35.760. URL https://doi.org/10.1103/PhysRevLett.35.760
4. A.R. Lugo, F.A. Schaposnik, Physics Letters B 467(1-2), 43 (1999). DOI 10.1016/S0370-2693(99)01178-8. URL https://doi.org/10.1016/S0370-2693(99)01178-8

5. A.R. Lugo, E.F. Moreno, F.A. Schaposnik, Physics Letters B 473(1-2), 35 (2000). DOI 10.1016/S0370-2693(99)01481-1. URL https://doi.org/10.1016/S0370-2693(99)01481-1

6. T.A. Ivanova, O. Lechtenfeld, A.D. Popov, Journal of High Energy Physics 2017(11), 17 (2017). DOI 10.1007/jhep11(2017)017. URL https://doi.org/10.1007/JHEP11(2017)017

7. S. Coleman, *Aspects of Symmetry: Selected Erice Lectures* (Cambridge University Press, Cambridge, 1985). DOI 10.1017/CBO9780511565045

8. N. Manton, P. Sutcliffe, *Topological Solitons*. Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2004). DOI 10.1017/cbo9780511617034. URL https://www.cambridge.org/core/books/topological-solitons/0A9670253EB1C8254BDAACA4EE30C3AA3