Pseudoprocesses Related to Space-Fractional Higher-Order Heat-Type Equations

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In this article, we construct pseudo random walks (symmetric and asymmetric) that converge in law to compositions of pseudoprocesses stopped at stable subordinators. We find the higher-order space-fractional heat-type equations whose fundamental solutions coincide with the law of the limiting pseudoprocesses. The fractional equations involve either Riesz operators or their Feller asymmetric counterparts. The main result of this article is the derivation of pseudoprocesses whose law is governed by heat-type equations of real-valued order $\gamma > 2$. The classical pseudoprocesses are very special cases of those investigated here.

Keywords Weyl derivatives; Riesz derivatives; Feller fractional operators; Pseudoprocesses; Stable processes; Subordinators; Continuous-time random walks.

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1. Introduction

In this article, we consider pseudoprocesses related to different types of fractional higher-order heat-type equations. Our starting point is the set of higher-order equations of the form

$$\frac{\partial}{\partial t} u_m(x, t) = \kappa_m \frac{\partial^m}{\partial x^m} u_m(x, t), \quad x \in \mathbb{R}, \ t > 0, \ m \in \mathbb{N} > 2, \quad (1.1)$$

whose solutions have been investigated by many outstanding mathematicians such as Bernstein [2], Lévy [16], Pólya [24], and also, more recently, by means of the steepest descent method, by Li and Wong [17]. In (1.1) the constant $\kappa_m$ is usually chosen in the form

$$\kappa_m = \begin{cases} 
\pm 1, & m = 2n + 1, \\
(-1)^{n+1}, & m = 2n.
\end{cases} \quad (1.2)$$

In our investigations we assume throughout that $\kappa_m = (-1)^n$ when $m = 2n + 1$. Pseudoprocesses related to (1.1) have been constructed in the same way as for the Wiener process.

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by Daletsky [4], Daletsky and Fomin [5], Krylov [11], Ladokhin [14], and Miyamoto [20]. More recently pseudoprocesses related to (1.1) have been considered by Debbi [6], Lachal [12, 13], and Mazzucchi [18]. For equations of the form

\[
\frac{\partial}{\partial t} u_\gamma(x, t) = \frac{\partial^\gamma}{\partial |x|^{\gamma}} u_\gamma(x, t), \quad x \in \mathbb{R}, \ t > 0,
\]

(1.3)

where \(0 < \gamma \leq 2\), and \(\frac{\partial^\gamma}{\partial |x|^{\gamma}}\) is the Riesz operator, the fundamental solution has the form of the density of a symmetric stable process as Riesz himself has shown. For \(\gamma > 2\), Equation (1.3) was studied by Debbi (see [6, 7]) who proved the sign-varying character of the corresponding solutions.

For asymmetric fractional operators of the form

\[
F D^{\gamma, \theta} = -\left[ \frac{\sin \frac{\pi}{2} (\gamma - \theta)}{\sin \pi \gamma} \frac{\partial^\gamma}{\partial x^\gamma} + \frac{\sin \frac{\pi}{2} (\gamma + \theta)}{\sin \pi \gamma} \frac{\partial^\gamma}{\partial |x|^{\gamma}} \right]
\]

(1.4)

the equation

\[
\frac{\partial}{\partial t} u_{\gamma, \theta}(x, t) = F D^{\gamma, \theta} u_{\gamma, \theta}(x, t), \quad x \in \mathbb{R}, \ t > 0, \ 0 < \gamma \leq 2,
\]

(1.5)

was studied by Feller [8] who proved that the fundamental solution to (1.5) is the law of an asymmetric stable process of order \(\gamma\). The fractional derivatives appearing in (1.4) are the Weyl fractional derivatives defined as below in (1.22) and (1.23). The Riesz fractional derivatives appearing in (1.3) are combinations of the Weyl’s derivatives and are defined as

\[
\frac{\partial^\gamma}{\partial |x|^{\gamma}} = -\frac{1}{2} \cos \frac{\pi \gamma}{2} \left[ \frac{+\partial^\gamma}{\partial x^\gamma} + \frac{-\partial^\gamma}{\partial |x|^{\gamma}} \right].
\]

(1.6)

This article is devoted to pseudoprocesses related to fractional equations of the form (1.3) and (1.5) when \(\gamma > 2\). The fundamental solutions of these equations are sign-varying as in the case of higher-order heat-type equations (1.1) studied in the literature (compare with [6]).

Fractional equations arise, for example, in the study of thermal diffusion in fractal and porous media ([21, 25]). Other fields of application of fractional equations can be found in Debbi [6]. Higher-order equations emerge in many contexts as in trimolecular chemical reactions [9] and in the linear approximation of the Korteweg-De Vries equation (see [1]).

In our article, we study pseudo random walks (for the definitions and properties of pseudo random walks and variables, see [13]) of the form

\[
W^{\gamma, 2k\beta}(t) = \sum_{j=1}^{N(\gamma - 2k\beta)} U_{2k}^{(1)} Q_j^{\gamma, 2k\beta}
\]

(1.7)

where the r.v.’s \(Q_j^{\gamma, 2k\beta}\) are independent from the Poisson process \(N\), from the pseudo r.v.’s \(U_{2k}^{(1)}\) and from each other and have distribution for \(0 < \beta < 1, \gamma > 0, k \in \mathbb{N}\),

\[
\Pr\left\{ Q_j^{\gamma, 2k\beta} > w \right\} = \begin{cases} 1, & w < \gamma, \\ \left(\frac{\gamma}{w}\right)^{2k\beta}, & w \geq \gamma. \end{cases}
\]

(1.8)
The $U_{j}^{2k}(1)$ are independent pseudo r.v.’s with law $u_{2k}(x, 1)$ with Fourier transform

$$
\int_{-\infty}^{\infty} e^{i\xi x} u_{2k}(x, 1) dx = e^{-|\xi|^{2k}}.
$$

(1.9)

The Poisson process $N$ appearing in (1.7) is homogeneous and has rate $\lambda = \frac{1}{\Gamma(1-\beta)}$. We prove that

$$
\lim_{\gamma \to 0} W_{\gamma}(t) \overset{\text{law}}{=} U_{2k}^{2k}(H^\beta(t))
$$

(1.10)

where $U_{2k}^{2k}$ is the pseudoprocess of order $2k$ related to the heat-type equation (1.1) for $m = 2k$ and $H^\beta$ is a stable subordinator of order $\beta \in (0, 1)$, independent from $U_{2k}^{2k}$. We show that the law of (1.10) is the fundamental solution to

$$
\frac{\partial}{\partial t} v_{2k}^{2k}(x, t) = \frac{\partial^{2k\beta}}{\partial |x|^{2k\beta}} v_{2k}^{2k}(x, t), \quad x \in \mathbb{R}, \ t > 0, \ \beta \in (0, 1), \ k \in \mathbb{N}.
$$

(1.11)

In other words, we are able to construct pseudoprocesses of order $\gamma > 2$ in the form of integer-valued pseudoprocesses stopped at stable distributed times as the limit of suitable pseudo random walks. We consider also pseudo random walks of the form

$$
N(\nu^{-\beta(2k+1)}) \sum_{j=0}^{\nu} \epsilon_{j} U_{j}^{2k+1}(1) Q_{j}^{\nu, \beta(2k+1)}
$$

(1.12)

where the $Q_{j}^{\nu, \beta(2k+1)}$ have distribution (1.8) (suitably adjusted), $U_{j}^{2k+1}(1)$ is an odd-order pseudo random variable with law $u_{2k+1}(x, 1)$ and Fourier transform

$$
\int_{-\infty}^{\infty} e^{i\xi x} u_{2k+1}(x, 1) dx = e^{-i|\xi|^{2k+1}}
$$

(1.13)

and the $\epsilon_{j}$’s are random variables which take values $\pm 1$ with probability $p$ and $q$. All the variables in (1.12) are independent from each other and also independent from the Poisson process $N$ with rate $\lambda = \frac{1}{\Gamma(1-\beta)}$. In this case, we are able to show that

$$
\lim_{\gamma \to 0} W_{\gamma}^{\nu, (2k+1)\beta}(t) \overset{\text{law}}{=} U_{1}^{2k+1}(\nu H_1^\beta(pt)) - U_{2}^{2k+1}(\nu H_2^\beta(qt))
$$

(1.14)

where $H_j^\beta, \ j = 1, 2$, are independent stable subordinators independent also from the pseudoprocesses $U_1, U_2$. We prove that the law of (1.14) satisfies the higher-order fractional equation

$$
\frac{\partial}{\partial t} w_{\beta(2k+1)}(x, t) = \Re w_{\beta(2k+1)}(x, t), \quad x \in \mathbb{R}, \ t > 0,
$$

(1.15)

where

$$
\Re = -\frac{1}{\cos \frac{\beta\pi}{2}} \left[p e^{i\pi \beta k} + \frac{\partial^{\beta(2k+1)}}{\partial x^{\beta(2k+1)}} + q e^{-i\pi \beta k} - \frac{\partial^{\beta(2k+1)}}{\partial x^{\beta(2k+1)}} \right].
$$

(1.16)
The Fourier transform of the fundamental solution of (1.15) reads
\[
\hat{w}_{\beta(2k+1)}(\xi, t) = e^{-t|\xi|^{2\beta+1}(1-i \sign(\xi) (p-q) \tan \frac{\beta\pi}{2})}
\] (1.17)

We note that (1.17) corresponds to the Fourier transform of the law of (1.14) with a suitable change of the time-scale that is
\[
E \exp \left\{ i\xi \left[ U_1^{2k+1} \left( H_1^\beta \left( \frac{pl}{\cos \frac{\beta\pi}{2}} \right) \right) - U_2^{2k+1} \left( H_2^\beta \left( \frac{ql}{\cos \frac{\beta\pi}{2}} \right) \right) \right] \right\} = e^{-t|\xi|^{2\beta+1}(1-i \sign(\xi) (p-q) \tan \frac{\beta\pi}{2})}
\] (1.18)

The mean value here and below must be understood with respect to the signed measure of the pseudoprocess (see, e.g., [6]). We study also the pseudoprocesses governed by the equation
\[
\frac{\partial}{\partial t} z_{\beta(2k+1), \theta}(x,t) = F D_{\beta(2k+1), \theta} z_{\beta(2k+1), \theta}(x,t)
\] (1.19)

where \( F D_{\beta(2k+1), \theta} \) is the operator defined in (1.4) with \( \gamma \) replaced by \( \beta(2k+1) \). Also in this case we study continuous-time random walks whose limit has Fourier transform equal to
\[
E e^{i\xi Z_{\beta(2k+1), \theta}(t)} = e^{-t|\xi|^{2\beta+1} e^{\frac{\pi}{2} \sign(t)}} , \quad \beta \in (0, 1), k \geq 1, -\beta < \theta < \beta.
\] (1.20)

When we take into account pseudo random walks constructed by means of even-order pseudo random variables we arrive at limits \( Z_{2\beta k, \theta}(t), t > 0 \), with Fourier transform
\[
E e^{i\xi Z_{2\beta k, \theta}(t)} = e^{-t|\xi|^{2\beta} e^{\frac{\pi}{2} \sign(t)}}
\] (1.21)

which shows the symmetric structure of the limiting pseudoprocess.

### 1.1. List of Symbols

For the reader’s convenience we give a short list of the most important symbols and definitions appearing in the article.

- The right Weyl fractional derivative for \( m - 1 < \gamma < m, m \in \mathbb{N}, x \in \mathbb{R} \)
  \[
  \frac{+\partial^\gamma}{\partial x^\gamma} u(x,t) = \frac{1}{\Gamma(m-\gamma)} \frac{d^m}{dx^m} \int_{-\infty}^{x} \frac{u(y,t)}{(x-y)^{\gamma+1-m}} dy
  \] (1.22)

- The left Weyl fractional derivative for \( m - 1 < \gamma < m, m \in \mathbb{N}, x \in \mathbb{R}, \)
  \[
  \frac{-\partial^\gamma}{\partial x^\gamma} u(x,t) = \frac{(-1)^m}{\Gamma(m-\gamma)} \frac{d^m}{dx^m} \int_{x}^{\infty} \frac{u(y,t)}{(y-x)^{\gamma+1-m}} dy
  \] (1.23)

- The Riesz fractional derivative for \( m - 1 < \gamma < m, m \in \mathbb{N}, x \in \mathbb{R}, \)
  \[
  \frac{\partial^\gamma}{\partial |x|^\gamma} = -\frac{1}{2 \cos \frac{\gamma\pi}{2}} \left[ \frac{+\partial^\gamma}{\partial x^\gamma} + \frac{-\partial^\gamma}{\partial x^\gamma} \right]
  \] (1.24)
We introduce the operator $\mathcal{R}$, for $\beta \in (0, 1)$, $k \in \mathbb{N}$, $p, q \in [0, 1] : p + q = 1$, $x \in \mathbb{R}$,

$$\mathcal{R} = \frac{-1}{\cos \frac{\beta \pi}{2}} \left[ pe^{i\pi \beta k} \frac{\partial^{\beta(2k+1)}}{\partial x^{\beta(2k+1)}} + q e^{-i\pi \beta k} \frac{\partial^{\beta(2k+1)}}{\partial x^{\beta(2k+1)}} \right]$$  \(1.25\)

The Feller derivative for $m - 1 < \gamma < m$, $m \in \mathbb{N}$, $\theta > 0$, $x \in \mathbb{R}$,

$$FD^{\gamma, \theta} = -\left[ \frac{\sin \frac{\pi}{2} (\gamma - \theta)}{\sin \pi \gamma} \frac{\partial^\gamma}{\partial x^\gamma} + \frac{\sin \frac{\pi}{2} (\gamma + \theta)}{\sin \pi \gamma} \frac{\partial^\gamma}{\partial x^\gamma} \right]$$  \(1.26\)

$U^m(t), t > 0$ is a pseudoprocess of order $m \in \mathbb{N}$ with law $u_m(x, t), x \in \mathbb{R}, t > 0$, governed by (1.1)

$H^\beta(t)$ is a stable subordinator of order $\beta \in (0, 1)$ with probability density $h_\beta(x, t), x \geq 0, t \geq 0$.

### 2. Preliminaries and Auxiliary Results

#### 2.1. Pseudoprocesses

Pseudoprocesses $U^m(t), t > 0$, are constructed by considering the set of real functions $x : t \in [0, \infty) \rightarrow x(t)$ (sample paths) and the cylinders

$$C = \{x : a_j \leq x(t_j) \leq b_j, j = 1, \ldots, n\}.$$

The solutions $u_m(x, t)$ to equations (1.1) (sign-varying for $m > 2$) are used to construct the measure on the cylinders as follows

$$\mu_m(C) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \int_{j=1}^{n} u_m(x_j - x_{j-1}, t_j - t_{j-1}).$$

The measure $\mu_m$ is then extended to the field generated by cylinders $C$ for fixed $t_1 < \cdots < t_j \cdots < t_n$. The measure $u_m$ obtained in this manner is Markovian in the sense that

$$\mu_m(U^m(t + T) \in B|\mathcal{F}_T) = \mu_m(U^m(t) \in B|U^m(T)),$$

where $\mathcal{F}_T$ is the natural $\sigma$-algebra associated with $U^m(t), t \geq 0$, and $B$ is an arbitrary Borel set.

#### 2.2. Weyl Fractional Derivatives

In this article, we consider higher-order heat-type equations where the space derivatives are fractional in different ways. First of all, we consider equations of the form

$$\frac{\partial}{\partial t} u(x, t) = \pm \frac{\partial^\gamma}{\partial x^\gamma} u(x, t), \quad x \in \mathbb{R}, t > 0, \gamma > 0,$$  \(2.1\)

where $\pm \frac{\partial^\gamma}{\partial x^\gamma}$ are the space-fractional Weyl derivatives defined as in (1.22) and (1.23). These fractional derivatives exist for $u \in C^\infty$ such that $u$ and all its derivatives are suitably decreasing for $x \to \pm \infty$ (see, e.g., [19]). In our analysis, the following result on the Fourier transforms of Weyl derivatives is very important.
Theorem 2.1 ([26], p. 137). The Fourier transforms of (1.22) and (1.23) read

$$\int_{-\infty}^{\infty} dx \ e^{i\xi x} \frac{\partial^{\gamma}}{\partial x^{\gamma}} u(x, t) = (-i\xi)^{\gamma} \mathcal{F}(\xi, t) = |\xi|^{\gamma} e^{\frac{i\pi\gamma}{2} \text{sign}(\xi)} \mathcal{F}(\xi, t), \quad (2.2)$$

$$\int_{-\infty}^{\infty} dx \ e^{i\xi x} \frac{-\partial^{\gamma}}{\partial x^{\gamma}} u(x, t) = (i\xi)^{\gamma} \mathcal{F}(\xi, t) = |\xi|^{\gamma} e^{\frac{i\pi\gamma}{2} \text{sign}(\xi)} \mathcal{F}(\xi, t). \quad (2.3)$$

Clearly \( \mathcal{F}(\xi, t) \) is the x-Fourier transform of \( u(x, t) \).

Proof. We give a sketch of the proof of (2.2) with some details.

$$\int_{-\infty}^{\infty} dx \ e^{i\xi x} \frac{\partial^{\gamma}}{\partial x^{\gamma}} u(x, t) = \int_{-\infty}^{\infty} dx \ e^{i\xi x} \left[ \frac{1}{\Gamma(m-\gamma)} \int_{-\infty}^{x} \frac{\partial^{m}}{\partial x^{m}} \frac{u(z, t)}{(x-z)^{\gamma-m+1}} \right]
= \int_{-\infty}^{\infty} dx \ e^{i\xi x} \left[ \frac{1}{\Gamma(m-\gamma)} \int_{0}^{\infty} \frac{\partial^{m}}{\partial x^{m}} \frac{u(x-z, t)}{z^{\gamma-m+1}} \right]
= \int_{-\infty}^{\infty} dw \ e^{iw} \frac{\partial^{m}}{\partial w^{m}} u(w, t) \frac{1}{\Gamma(m-\gamma)} \int_{0}^{\infty} dz \ e^{i\xi z} z^{m-\gamma-1}
= (-i\xi)^{m} \int_{-\infty}^{\infty} e^{iw} u(w, t) \frac{1}{\Gamma(m-\gamma)} \int_{0}^{\infty} dz \ e^{i\xi z} z^{m-\gamma-1}. \quad (2.4)$$

The result

$$\frac{(-i\xi)^{m}}{\Gamma(m-\gamma)} \int_{0}^{\infty} dz \ e^{i\xi z} z^{m-\gamma-1} = |\xi|^{\gamma} e^{-\frac{i\pi\gamma}{2} \text{sign}(\xi)} \quad (2.5)$$
can be obtained for example by applying the Cauchy integral Theorem (see [26], p. 138).

2.3. Riesz Fractional Derivatives

By means of the Weyl fractional derivatives we arrive at the Riesz fractional derivative, for \( m-1 < \gamma < m, m \in \mathbb{N} \),

$$\frac{\partial^{\gamma}}{\partial |x|^{\gamma}} u(x, t) = -\frac{\partial^{m}}{\partial x^{m}} \left[ \int_{-\infty}^{x} \frac{u(y, t) \ dy}{(x-y)^{\gamma-m+1}} + \int_{x}^{\infty} \frac{(-1)^{m} u(y, t) \ dy}{(y-x)^{\gamma-m+1}} \right]
= -\frac{1}{2 \cos \frac{\pi\gamma}{2}} \left( +\frac{\partial^{\gamma}}{\partial x^{\gamma}} - \frac{\partial^{\gamma}}{\partial y^{\gamma}} \right) u(x, t) \quad (2.6)$$

In view of Theorem 2.1 we have that, for \( \gamma > 0, \gamma \notin \mathbb{N} \),

$$\int_{-\infty}^{\infty} dx \ e^{i\xi x} \frac{\partial^{\gamma}}{\partial |x|^{\gamma}} u(x, t) = -\frac{1}{2 \cos \frac{\pi\gamma}{2}} \left[ |\xi|^{\gamma} e^{-\frac{i\pi\gamma}{2} \text{sign}(\xi)} + |\xi|^{\gamma} e^{\frac{i\pi\gamma}{2} \text{sign}(\xi)} \right] \mathcal{F}(\xi, t)
= -|\xi|^{\gamma} \mathcal{F}(\xi, t). \quad (2.7)$$
Remark 2.2. The general fractional higher-order heat equation

\[
\frac{\partial}{\partial t} u(x, t) = \frac{\partial^\gamma}{\partial |x|^\gamma} u(x, t), \quad x \in \mathbb{R}, \ t > 0,
\]

(2.8)

has solution whose Fourier transform reads

\[
\hat{u}(\xi, t) = e^{-t|\xi|^\gamma}.
\]

(2.9)

For \(0 < \gamma < 2\), (2.9) corresponds to the characteristic function of the symmetric stable processes (this is a classical result due to M. Riesz himself).

3. From Pseudo Random Walks to Fractional Pseudoprocesses

We consider in this section continuous-time pseudo random walks with steps which are pseudo random variables, that is measurable functions endowed with signed measures, and with total mass equal to one (see [13]). In order to obtain in the limit pseudoprocesses whose signed law satisfies higher-order fractional equations we must construct sums of the form

\[
\sum_{j=0}^{N(t^\gamma t^{-\beta(2k+1)})} \epsilon_j U_j^{2k+1}(1) Q_j^{\gamma,\beta(2k+1)}, \quad \beta \in (0, 1), \ k \in \mathbb{N}, \ \gamma > 0,
\]

(3.1)

where

\[
\epsilon_j = \begin{cases} 
1, & \text{with probability } p, \\
-1, & \text{with probability } q, 
\end{cases} \quad p + q = 1.
\]

(3.2)

The r.v.’s \(Q_j^{\gamma,\beta(2k+1)}\) have probability distributions, for \(\beta \in (0, 1), \ k \in \mathbb{N},\)

\[
\Pr \{ Q_j^{\gamma,\beta(2k+1)} > w \} = \begin{cases} 
\left( \frac{\gamma}{w} \right)^{\beta(2k+1)}, & \text{for } w > \gamma \\
1, & \text{for } w < \gamma.
\end{cases}
\]

(3.3)

The Poisson process \(N(t), \ t > 0,\) appearing in (3.1) is homogeneous with rate

\[
\lambda = \frac{1}{\Gamma(1-\beta)}, \quad \beta \in (0, 1).
\]

(3.4)

The pseudo random variables (see Lachal [13]) \(U_j^{2k+1}(1)\) have law with Fourier transform

\[
\int_{-\infty}^{\infty} dx \ e^{i\xi x} u_{2k+1}(x, 1) = e^{-i\xi^{2k+1}}
\]

(3.5)

and the function \(u_{2k+1}(x, t), \ x \in \mathbb{R}, \ t > 0,\) is the fundamental solution to the odd-order heat-type equation, for \(k \in \mathbb{N},\)

\[
\begin{cases} 
\frac{\partial}{\partial t} u_{2k+1}(x, t) = (-1)^k \frac{\partial^{2k+1}}{\partial x^{2k+1}} u_{2k+1}(x, t), \\
u_{2k+1}(x, 0) = \delta(x).
\end{cases}
\]

(3.6)
There is a vast literature devoted to odd-order heat-type equations of the form (3.6), to the behavior of their solutions and to the related pseudoprocesses ([1, 12, 22, 23]).

The r.v.’s and pseudo r.v.’s appearing in (3.1) are independent and also independent from each other. We say that two pseudo r.v.’s (or pseudoprocesses) with signed density \( u_m \), \( u_m^2 \), are independent if the Fourier transform \( \mathcal{F} \) of the convolution \( u_m^1 \ast u_m^2 \) factorizes, that is,

\[
\mathcal{F}[u_m^1 \ast u_m^2](\xi) = \mathcal{F}[u_m^1](\xi) \cdot \mathcal{F}[u_m^2](\xi). \tag{3.7}
\]

We are now able to state the first theorem of this section.

**Theorem 3.1.** The following limit in distribution holds true

\[
\lim_{\gamma \to 0} \sum_{j=0}^{\mathcal{N}(t \gamma^{-\beta(2k+1)})} \epsilon_j U_j^{2k+1}(1) Q_j^{\gamma, \beta(2k+1)} \xrightarrow{\text{law}} U_1^{2k+1}(H_1^\beta(pt)) - U_2^{2k+1}(H_2^\beta(qt)), \tag{3.8}
\]

where \( H_1^\beta \) and \( H_2^\beta \) are independent positively skewed stable processes of order \( 0 < \beta < 1 \) while \( U_1^{2k+1} \) and \( U_2^{2k+1} \) are independent pseudoprocesses of order \( 2k + 1 \). All the random variables \( N(t) \), \( t > 0 \), \( \epsilon_j \), \( Q_j^{\gamma, \beta(2k+1)} \) are independent and also independent from the pseudo random variables \( U_j^{2k+1}(1) \). The Fourier transform of the limiting pseudoprocess reads

\[
\mathbb{E} e^{i\xi U_1^{2k+1}(H_1^\beta(pt))} - U_2^{2k+1}(H_2^\beta(qt)) = e^{-i|\xi|^{\beta(2k+1)}} \left( \cos \frac{\xi p}{2} - i \text{sign}(\xi)(p - q) \sin \frac{\xi q}{2} \right). \tag{3.9}
\]

**Proof.** In view of the independence of the r.v.’s and pseudo random variables appearing in (3.8) we have that

\[
\mathbb{E} e^{i\xi \sum_{j=0}^{\mathcal{N}(t \gamma^{-\beta(2k+1)})} \epsilon_j U_j^{2k+1}(1) Q_j^{\gamma, \beta(2k+1)}} = \mathbb{E} \mathbb{E} \left( e^{i\xi U_1^{2k+1}(1) Q^{\gamma, \beta(2k+1)}} \right)^{\mathcal{N}(t \gamma^{-\beta(2k+1)})} \mid \mathbb{N}(t \gamma^{-\beta(2k+1)})
\]

\[
= \mathbb{E} \exp \left\{ - \frac{\lambda t}{\gamma^{\beta(2k+1)}} \left( 1 - \mathbb{E} e^{i\xi U_1^{2k+1}(1) Q^{\gamma, \beta(2k+1)}} \right) \right\}
\]

\[
= \exp \left\{ - \frac{\lambda t}{\gamma^{\beta(2k+1)}} \left( 1 - p \mathbb{E} e^{i\xi U_1^{2k+1}(1) Q^{\gamma, \beta(2k+1)}} - q \mathbb{E} e^{-i\xi U_1^{2k+1}(1) Q^{\gamma, \beta(2k+1)}} \right) \right\}
\]

\[
= \exp \left\{ - \frac{\lambda t}{\gamma^{\beta(2k+1)}} \left( p + q - p \mathbb{E} e^{i\xi U_1^{2k+1}(1) Q^{\gamma, \beta(2k+1)}} - q \mathbb{E} e^{-i\xi U_1^{2k+1}(1) Q^{\gamma, \beta(2k+1)}} \right) \right\}
\]

\[
= \exp \left\{ - \frac{\lambda t}{\gamma^{\beta(2k+1)}} \left( p \left( 1 - \mathbb{E} e^{i\xi U_1^{2k+1}(1) Q^{\gamma, \beta(2k+1)}} \right) \right) + q \left( 1 - \mathbb{E} e^{-i\xi U_1^{2k+1}(1) Q^{\gamma, \beta(2k+1)}} \right) \right\}. \tag{3.10}
\]
We observe that

\[ p(1 - \mathbb{E} e^{i \xi U_{2k+1}(1) Q_y^{(0,2k+1)}}) + q(1 - \mathbb{E} e^{-i \xi U_{2k+1}(1) Q_y^{(0,2k+1)}}) \]

\[ = p \int_{-\infty}^{\infty} dw \left( 1 - \frac{\gamma^{(2k+1)} \beta (2k + 1)}{w^{(2k+1)+1}} e^{i \xi 2k+1 w_{2k+1}} \right) \]

\[ + q \int_{-\infty}^{\infty} dw \left( 1 - \frac{\gamma^{(2k+1)} \beta (2k + 1)}{w^{(2k+1)+1}} e^{-i \xi 2k+1 w_{2k+1}} \right) \]

and, therefore,

\[ \mathbb{E} e^{i \xi \sum_{j=0}^{N^{(0,2k+1)}} \epsilon_j U_j^{(2k+1)}(1) Q_y^{(0,2k+1)}} \]

\[ = \exp \left\{ -\frac{\lambda t}{\gamma^{(2k+1)}} \left[ p \int_{-\infty}^{\infty} dw \left( 1 - \frac{\gamma^{(2k+1)} \beta (2k + 1)}{w^{(2k+1)+1}} e^{i \xi 2k+1 w_{2k+1}} \right) \right] \right. \]

\[ + q \int_{-\infty}^{\infty} dw \left( 1 - \frac{\gamma^{(2k+1)} \beta (2k + 1)}{w^{(2k+1)+1}} e^{-i \xi 2k+1 w_{2k+1}} \right) \left\} \right. \]

\[ = \exp \left\{ -\frac{\lambda t}{\gamma^{(2k+1)}} \left[ p \left( 1 - e^{i(\xi \gamma)^{(2k+1)}} \right) - p i \xi 2k+1 (2k + 1) \int_{-\infty}^{\infty} dw \frac{\gamma^{(2k+1)} e^{i(\xi \gamma)^{(2k+1)}}}{w^{(2k+1)+1}} \right. \right. \]

\[ + q \left( 1 - e^{-i(\xi \gamma)^{(2k+1)}} + q i \xi 2k+1 (2k + 1) \int_{-\infty}^{\infty} dw \frac{\gamma^{(2k+1)} e^{-i(\xi \gamma)^{(2k+1)}}}{w^{(2k+1)+1}} \right) \} \]

By taking the limit we get that

\[ \lim_{\gamma \to 0} \mathbb{E} e^{i \xi \sum_{j=0}^{N^{(0,2k+1)}} \epsilon_j U_j^{(2k+1)}(1) Q_y^{(0,2k+1)}} \]

\[ = \exp \left\{ -\lambda t (2k + 1) \left( -p i \xi 2k+1 \int_{0}^{\infty} dw \frac{e^{i(\xi \gamma)^{(2k+1)}}}{w^{(2k+1)+1}} - q i \xi 2k+1 \int_{0}^{\infty} dw \frac{e^{-i(\xi \gamma)^{(2k+1)}}}{w^{(2k+1)+1}} \right) \right\} \]

\[ = e^{-\lambda t (1-\beta) \left[ p(-i \xi 2k+1) q + q(i \xi 2k+1) \right]} \]

\[ = e^{-\lambda t (1-\beta) (p-\xi 2k+1) q + q(i \xi 2k+1)} \]

(3.13)

By setting \( \lambda = \frac{1}{\Gamma(1-\beta)} \) we obtain

\[ \lim_{\gamma \to 0} \mathbb{E} e^{i \xi \sum_{j=0}^{N^{(0,2k+1)}} \epsilon_j U_j^{(2k+1)}(1) Q_y^{(0,2k+1)}} = e^{-t \xi (\beta 2k+1)} e^{-\frac{1}{2} \frac{\pi}{\beta} \sin(\xi)} + q \frac{\pi}{\beta} \sin(\xi) \]

\[ = e^{-t(\beta 2k+1)} \left( \cos \frac{\pi}{\beta} + i \sin(\xi) \right) \]

(3.14)

Now we consider the Fourier transform of the law of the pseudoprocess

\[ V^{(2k+1)}(t) = U_1^{2k+1} (H_1^\beta (pt)) - U_2^{2k+1} (H_2^\beta (qt)) \]

which reads

\[ \mathbb{E} e^{i \xi V^{(2k+1)}(t)} = \mathbb{E} e^{i \xi U_2^{2k+1}(H_1^\beta (pt)) e^{-i \xi U_1^{2k+1}(H_2^\beta (qt))} \}

\[ = \left( \int_{0}^{\infty} ds e^{i \xi 2k+1 s} h_1^\beta (s, pt) \right) \left( \int_{0}^{\infty} dz e^{-i \xi 2k+1 z} h_2^\beta (z, qt) \right) \]
Proof. We start by evaluating the Fourier transform

\[ e^{-t p(-i)^{2k+1} \beta} e^{-t q(i)\theta} \]

\[ = e^{-t(p)^{[2(2k+1)]} e^{-i\theta} \sum_{n=1}^{\infty} q(2k+1)^n e^{i\theta} n} \]

\[ = e^{-t(p)^{[2(2k+1)]}(\cos \frac{\theta}{\beta} - i \text{sign}(\theta)(p-q) \sin \frac{\theta}{\beta})}, \]

(3.16)

and coincides with (3.14). In (3.16), by \( h^\beta_p(x, pt) \) and \( h^\beta_z(z, qt) \), we indicate the densities of the stable subordinators \( H^\beta(t) \) and \( H^\beta(t) \), respectively. \( \square \)

Remark 3.2. If \( \beta = \frac{1}{2k+1} \) the Fourier transform (3.16) becomes

\[ \mathbb{E} e^{i\xi U_j^{2k+1}(H^\beta(t))} \mathbb{E} e^{-i\xi U_j^{2k+1}(H^\beta(t))} = e^{-t|\xi| \cos \frac{\pi}{2k+1} + t i \text{sign}(\xi)(p-q) \sin \frac{\pi}{2k+1}}, \]

(3.17)

which corresponds to the characteristic function of a Cauchy r.v. with position parameter equal to \( t(p-q) \sin \frac{\pi}{2k+1} \) and scale parameter \( t \cos \frac{\beta}{2k+1} \). This slightly generalizes result 1.4 of Orsingher and D'Ovidio [23].

For even-order pseudoprocesses we have the following limit in distribution.

Theorem 3.3. If \( U^{2k}(t) \), \( t > 0 \), is an even-order pseudoprocess and \( N(t) \), \( t > 0 \), is a homogeneous Poisson process, independent from \( U^{2k}(t) \), \( t > 0 \), we have that

\[ \lim_{\gamma \to 0} \sum_{j=0}^{N(t \gamma^{-2k})} U_j^{2k}(1) Q_j^{2k} \overset{\text{law}}{=} U^{2k}(H^\beta(t)), \quad t > 0, \]

(3.18)

where \( H^\beta \) is a stable subordinator of order \( \beta \in (0, 1) \) and \( Q_j^{2k} \) are i.i.d. random variables with distribution

\[ \Pr\{Q_j^{2k} > w\} = \begin{cases} 1, & w < \gamma, \\ \left(\frac{\gamma}{w}\right)^{2k}, & w > \gamma. \end{cases} \]

(3.19)

The pseudoprocess \( U^{2k}(t) \), \( t > 0 \), has a signed density \( u_{2k}(x, t) \) solving the following equation

\[ \frac{\partial}{\partial t} u_{2k}(x, t) = (-1)^{k+1} \frac{\partial^{2k}}{\partial x^{2k}} u_{2k}(x, t), \quad x \in \mathbb{R}. \]

(3.20)

Proof. We start by evaluating the Fourier transform

\[ \mathbb{E} e^{i\xi \sum_{j=0}^{N(t \gamma^{-2k})} U_j^{2k}(1) Q_j^{2k}} \]

\[ = \mathbb{E} \left[ (e^{i\xi U^{2k}(1) Q^{2k}})^{N(t \gamma^{-2k})} \right] \mathbb{E} \left[ N(t \gamma^{-2k}) \right] \]

\[ = \exp \left\{ -\frac{\lambda t}{\gamma^{2k}} \right\} \]

\[ = \exp \left\{ -\frac{\lambda t}{\gamma^{2k}} \right\} \int_0^\infty dy (1 - e^{-|\xi|^2 y^{2k}}) \frac{2k y^{2k}}{y^{2k+1}} \]

\[ = \exp \left\{ -\frac{\lambda t}{\gamma^{2k}} \left[ 1 - e^{-|\xi|^2 y^{2k}} + |\xi|^2 \int_0^\infty dy e^{-|\xi|^2 y^{2k}} y^{2k-1-2k} \right] \right\} \]

(3.21)
By taking the limit we have that
\[
\lim_{\gamma \to 0} e^{\xi} \sum_{j=0}^{N(t;\gamma;2k)} U_j^{2k}(1) \gamma^{2k} = e^{-\lambda t|\xi|^{2k}} \int_0^\infty e^{-u^{2k}} dw
\]
which coincides with
\[
\mathbb{E} e^{\xi U_{2k}(H^\beta(t))} = \int_0^\infty e^{-\xi x^{2k}} \Pr\{H^\beta(t) \in ds\} = e^{-t|\xi|^{2k}}
\]
since \(\lambda = \frac{1}{\Gamma(1-\beta)}\).

**Remark 3.4.** For \(\beta = \frac{1}{k}\) the composition \(U_{2k}(H^\beta(t))\) has Gaussian distribution. For \(\beta = \frac{1}{2k}\) we have instead the Cauchy distribution and for \(\beta = \frac{1}{4k}\) we extract the inverse Gaussian corresponding to the distribution of the first passage time of a Brownian motion. The case \(\beta = \frac{1}{6k}\) yields the stable law with distribution
\[
f_{\frac{1}{3}}(x) = \frac{t}{x \sqrt{3x}} \text{Ai} \left( \frac{t}{\sqrt{3x}} \right)
\]
where \text{Ai} denotes the Airy function (see [23]).

In order to arrive at asymmetric higher-order fractional pseudoprocesses we construct pseudo random walks by adapting the Feller approach (used for asymmetric stable laws) to our context. This means that we combine independent pseudo random walks with suitable trigonometric weights as in (1.4).

**Theorem 3.5.** Let \(X_j^{\gamma,(2k+1)\beta}\) and \(Y_j^{\gamma,(2k+1)\beta}\) be i.i.d. r.v.’s with distribution
\[
\Pr\{X_j^{\gamma,(2k+1)\beta} > v\} = \left\{ \begin{array}{ll} \left( \frac{v}{w} \right)^{(2k+1)\beta}, & w > \gamma \\ 1, & w < \gamma \end{array} \right.
\]
and let \(U_{2k+1}(t)\), \(t > 0\), be a pseudoprocess of odd-order \(2k+1\), \(k \in \mathbb{N}\). For \(0 < \beta < 1\) and \(-\beta < \theta < \beta\) we have that
\[
\lim_{\gamma \to 0} \left[ \left( \frac{\sin \frac{\pi}{2}(\beta - \theta)}{\sin \pi \beta} \right)^{1/(2k+1)\beta} \sum_{j=0}^{N(t;\gamma;2k+1)} X_j^{\gamma,(2k+1)\beta} U_j^{2k+1}(1) \right.
\]
\[
- \left( \frac{\sin \frac{\pi}{2}(\beta + \theta)}{\sin \pi \beta} \right)^{1/(2k+1)\beta} \sum_{j=0}^{N(t;\gamma;2k+1)} Y_j^{\gamma,(2k+1)\beta} U_j^{2k+1}(1) \right] \xrightarrow{\text{law}} Z^{\beta(2k+1),\theta}
\]
where
\[
\mathbb{E} e^{\xi Z^{\beta(2k+1),\theta}} = e^{-t|\xi|^{(2k+1)\beta}} e^\frac{\lambda \xi^2}{2}
\]
Proof. The Fourier transform of (3.26) is written as

\[
\mathbb{E}e^{i\xi}\left(\frac{\sin \frac{\pi}{\sin \beta}}{\sin \pi \beta}\right)^{\frac{1}{\lambda t(2k+1)\beta}} \sum_{j=0}^{N(\gamma - (2k+1)\beta)} X_j \gamma^{(2k+1)\beta} U_j^{2k+1}(1)
\times \mathbb{E}e^{-i\xi}\left(\frac{\sin \frac{\pi}{\sin \beta}}{\sin \pi \beta}\right)^{\frac{1}{\lambda t(2k+1)\beta}} \sum_{j=0}^{N(\gamma - (2k+1)\beta)} Y_j \gamma^{(2k+1)\beta} U_j^{2k+1}(1)
\] (3.28)

where the first member is given by

\[
\mathbb{E}e^{i\xi}\left(\frac{\sin \frac{\pi}{\sin \beta}}{\sin \pi \beta}\right)^{\frac{1}{\lambda t(2k+1)\beta}} \sum_{j=0}^{N(\gamma - (2k+1)\beta)} X_j \gamma^{(2k+1)\beta} U_j^{2k+1}(1)
= \exp\left\{-\frac{\lambda t}{\gamma^{(2k+1)\beta}} \left(1 - \mathbb{E}e^{i\xi}\left(\frac{\sin \frac{\pi}{\sin \beta}}{\sin \pi \beta}\right)^{\frac{1}{\lambda t(2k+1)\beta}} \sum_{j=0}^{N(\gamma - (2k+1)\beta)} X_j \gamma^{(2k+1)\beta} U_j^{2k+1}(1)\right)\right\}
\]

\[
= \exp\left\{-\frac{\lambda t}{\gamma^{(2k+1)\beta}} \int_{\gamma}^{\infty} \left(1 - e^{-i\xi}\left(\frac{\sin \frac{\pi}{\sin \beta}}{\sin \pi \beta}\right)^{\frac{1}{\lambda t(2k+1)\beta}} \sum_{j=0}^{N(\gamma - (2k+1)\beta)} X_j \gamma^{(2k+1)\beta} U_j^{2k+1}(1)\right) \int_{0}^{\infty} e^{-w} w^{-\beta} d w\right\}
\]

\[
\gamma \rightarrow 0 \exp\left\{-\frac{\lambda t}{\gamma^{(2k+1)\beta}} \left(\frac{\sin \frac{\pi}{\sin \beta}}{\sin \pi \beta}\right)^{\frac{1}{\lambda t(2k+1)\beta}} \Gamma(1-\beta)\right\}
= e^{-\frac{\lambda t}{\Gamma(1-\beta)}} e^{-\frac{\pi |\xi|^{2k+1}}{\sin \pi \beta}}\Gamma(1-\beta)\right\}
\] (3.29)

The second member of (3.26) becomes, by performing a similar calculation,

\[
\mathbb{E}e^{i\xi}\left(\frac{\sin \frac{\pi}{\sin \beta}}{\sin \pi \beta}\right)^{\frac{1}{\lambda t(2k+1)\beta}} \sum_{j=0}^{N(\gamma - (2k+1)\beta)} Y_j \gamma^{(2k+1)\beta} U_j^{2k+1}(1)
\]

\[
\gamma \rightarrow 0 e^{-\frac{\lambda t}{\Gamma(1-\beta)}} e^{-\frac{\pi |\xi|^{2k+1}}{\sin \pi \beta}}\Gamma(1-\beta)\right\}
\] (3.30)

and, thus, for \( \lambda = \frac{1}{\Gamma(1-\beta)} \) we obtain that

\[
\mathbb{E}e^{i\xi}\left(\frac{\sin \frac{\pi}{\sin \beta}}{\sin \pi \beta}\right)^{\frac{1}{\lambda t(2k+1)\beta}} \sum_{j=0}^{N(\gamma - (2k+1)\beta)} X_j \gamma^{(2k+1)\beta} U_j^{2k+1}(1)
\]

\[
\times \mathbb{E}e^{-i\xi}\left(\frac{\sin \frac{\pi}{\sin \beta}}{\sin \pi \beta}\right)^{\frac{1}{\lambda t(2k+1)\beta}} \sum_{j=0}^{N(\gamma - (2k+1)\beta)} Y_j \gamma^{(2k+1)\beta} U_j^{2k+1}(1)
\]

\[
\gamma \rightarrow 0 e^{-\frac{\lambda t}{\Gamma(1-\beta)}} e^{-\frac{\pi |\xi|^{2k+1}}{\sin \pi \beta}}\Gamma(1-\beta)\right\}
\] (3.31)

By considering symmetric pseudo random walks with the Feller construction we arrive in the next theorem at symmetric pseudoprocesses with time scale equal to \( \frac{\cos \frac{\pi}{\sin \beta}}{\sin \frac{\pi}{\sin \beta}} \), \( 0 < \beta < 1 \), \( -\beta < \theta < \beta \).

Theorem 3.6. Let \( X_j^{\gamma,2\beta k} \) and \( Y_j^{\gamma,2\beta k} \) be i.i.d. r.v.'s with distribution

\[
\Pr\{X_j^{\gamma,2\beta k} > w\} = \begin{cases} \left(\frac{\gamma}{w}\right)^{2\beta k}, & w > \gamma \\ 1, & w < \gamma \end{cases}
\] (3.32)
and let $U^{2k}(t)$, $t > 0$, be a pseudoprocess of order $2k$, $k \in \mathbb{N}$. If $N(t)$ is a homogeneous Poisson process, with parameter $\lambda = \frac{1}{(1-\beta)^r}$, independent from $X_j^{\gamma,2\beta k}$ and $Y_j^{\gamma,2\beta k}$ we have that

\[
\lim_{\gamma \to 0} \left[ \left( \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin \pi \beta} \right)^{\frac{1}{2}} \sum_{j=1}^{N(\gamma^{-2\beta k})} X_j^{\gamma,2\beta k} U_j^{2k}(1) \right. \\
+ \left. \left( \frac{\sin \frac{\pi}{2} (\beta + \theta)}{\sin \pi \beta} \right)^{\frac{1}{2}} \sum_{j=0}^{N(\gamma^{-2\beta k})} Y_j^{\gamma,2\beta k} U_j^{2k}(1) \right] \xrightarrow{law} Z^{2k\beta,\theta}, \quad t > 0, \quad (3.33)
\]

for $0 < \beta < 1$ and $-\beta < \theta < \beta$ and

\[
\mathbb{E} e^{i\xi Z^{2k\beta,\theta}} = e^{-t|\xi|^{2\beta} \sin \frac{\xi \theta}{\sin \pi \beta}} \tag{3.34}
\]

**Proof.** The Fourier transform of (3.33) is written as

\[
\mathbb{E} e^{i\xi Z^{2k\beta,\theta}} = e^{-t|\xi|^{2\beta} \sin \frac{\xi \theta}{\sin \pi \beta}} \tag{3.35}
\]

where the first member is given by

\[
\mathbb{E} e^{i\xi \left( \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin \pi \beta} \right)^{\frac{1}{2}} \sum_{j=0}^{N(\gamma^{-2\beta k})} X_j^{\gamma,2\beta k} U_j^{2k}(1) + i\xi \left( \frac{\sin \frac{\pi}{2} (\beta + \theta)}{\sin \pi \beta} \right)^{\frac{1}{2}} \sum_{j=0}^{N(\gamma^{-2\beta k})} Y_j^{\gamma,2\beta k} U_j^{2k}(1)}
\]

\[
= \exp \left\{ -\frac{\lambda t}{\gamma^{2k\beta}} \left[ 1 - \mathbb{E} e^{i\xi \left( \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin \pi \beta} \right)^{\frac{1}{2}} U_j^{2k}(1) X_j^{\gamma,2\beta k} \right] \right\}
\]

\[
= \exp \left\{ -\frac{\lambda t}{\gamma^{2k\beta}} \left[ \int_{\gamma}^{\infty} \left( 1 - e^{-i\xi \left( \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin \pi \beta} \right)^{\frac{1}{2}} \sum_{j=0}^{N(\gamma^{-2\beta k})} X_j^{\gamma,2\beta k}} \right)^{2k} \right] (2k\beta) \frac{\gamma^{2k\beta}}{\gamma^{2k\beta}} \ dy \right\}
\]

\[
\xrightarrow{\gamma \to 0} \left\{ -\frac{\lambda t |\xi|^{2k}}{\gamma^{2k\beta}} \Gamma(1-\beta) \left[ \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin \pi \beta} \right] \right\}
\]

\[
= -\lambda t |\xi|^{2k} \Gamma(1-\beta) \left[ \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin \pi \beta} \right] \tag{3.36}
\]

and by similar calculations the second member becomes

\[
\mathbb{E} e^{i\xi \left( \frac{\sin \frac{\pi}{2} (\beta + \theta)}{\sin \pi \beta} \right)^{\frac{1}{2}} \sum_{j=0}^{N(\gamma^{-2\beta k})} Y_j^{\gamma,2\beta k} U_j^{2k}(1) + i\xi \left( \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin \pi \beta} \right)^{\frac{1}{2}} \sum_{j=0}^{N(\gamma^{-2\beta k})} X_j^{\gamma,2\beta k} U_j^{2k}(1)}
\]

\[
\xrightarrow{\gamma \to 0} e^{-\lambda t |\xi|^{2k} \Gamma(1-\beta) \left[ \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin \pi \beta} \right]} \tag{3.37}
\]

Thus, we have that

\[
\mathbb{E} e^{i\xi \left( \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin \pi \beta} \right)^{\frac{1}{2}} \sum_{j=0}^{N(\gamma^{-2\beta k})} X_j^{\gamma,2\beta k} U_j^{2k}(1) + i\xi \left( \frac{\sin \frac{\pi}{2} (\beta + \theta)}{\sin \pi \beta} \right)^{\frac{1}{2}} \sum_{j=0}^{N(\gamma^{-2\beta k})} Y_j^{\gamma,2\beta k} U_j^{2k}(1)}
\]

\[
\xrightarrow{\gamma \to 0} e^{-\lambda t |\xi|^{2k} \Gamma(1-\beta) \left[ \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin \pi \beta} \right]} \left[ \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin \pi \beta} \right] \left[ \frac{\sin \frac{\pi}{2} (\beta + \theta)}{\sin \pi \beta} \right] \tag{3.38}
\]
4. Governing Equations

In the previous section, we obtained fractional pseudoprocesses as limit of suitable pseudo random walks. In this section, we will show that the limiting fractional pseudoprocesses obtained before have signed density satisfying space-fractional heat-type equations of higher-order with Riesz or Feller fractional derivatives. The order of fractionality of the governing equations is a positive real number and this is the major difference with respect to the pseudoprocesses considered so far in the literature. The pseudoprocesses dealt with in the literature (e.g., [5, 11–14, 18, 22, 23], are constructed by means of the fundamental solutions of higher-order heat equations of the form (1.1), with integer values of \( m > 2 \). Only in Debbi [6, 7] higher-order equations with non-integer order of space derivatives are considered.

We start by examining space fractional higher-order equations of order \( 2k\beta \), \( \beta \in (0, 1) \), \( k \in \mathbb{N} \), which interpolate equations of the form (1.1).

**Theorem 4.1.** The solution to the initial-value problem

\[
\begin{cases}
\frac{\partial}{\partial t} v_{2k}^\beta(x, t) = \frac{\partial}{\partial (|x|^{2k}\beta)} v_{2k}^\beta(x, t), & x \in \mathbb{R}, t > 0, k \in \mathbb{N}, \beta \in (0, 1) \\
v_{2k}^\beta(x, 0) = \delta(x)
\end{cases}
\]

(4.1)

can be written as

\[
v_{2k}^\beta(x, t) = \frac{1}{\pi x} \mathbb{E} \left[ \sin \left( x G_{2k}^{\beta} \left( \frac{1}{H_{\beta}(t)} \right) \right) \right] = \frac{1}{\pi x} \mathbb{E} \left[ \sin \left( x G_{2k}^{\beta} \left( \frac{1}{t} \right) \right) \right]
\]

(4.2)

and coincides with the law of the pseudoprocess

\[ V^{2k\beta}(t) = U^{2k}(H_{\beta}(t)), \quad t > 0, \]

(4.3)

where \( U^{2k} \) is related to equation (1.1) for \( m = 2k \) and \( H_{\beta} \) is a stable subordinator independent from \( U^{2k} \). \( G^\gamma(t) \) is a gamma r.v. with density

\[ g^\gamma(x, t) = \gamma x^{\gamma-1} e^{-\frac{x}{t}}, \quad x > 0, t > 0, \gamma > 0. \]

(4.4)

**Proof.** The Fourier transform of (4.1) leads to the Cauchy problem

\[
\begin{cases}
\frac{\partial}{\partial t} \hat{v}_{2k}^\beta(\xi, t) = -|\xi|^{2k\beta} \hat{v}_{2k}^\beta(\xi, t) \\
\hat{v}_{2k}^\beta(\xi, 0) = 1
\end{cases}
\]

(4.5)

whose unique solution reads

\[
\mathbb{E} e^{i\xi V^{2k\beta}(t)} = \int_{\mathbb{R}} dx e^{i\xi x} \int_0^\infty ds u_{2k}(x, s) h_\beta(s, t)
= \int_0^{\infty} ds e^{-x^2 s} h_\beta(s, t) = e^{-t|\xi|^{2k\beta}}.
\]

(4.6)
In (4.6), \(u_{2k}\) is the density of \(U_{2k}\), and \(h_\beta(x, t)\) is the probability density of the subordinator \(H^\beta\). Now we show that the Fourier transform of (4.2) coincides with (4.6). We have that

\[
\hat{v}_β^{2k}(ξ, t) = \int_\mathbb{R} dx e^{iξx} \frac{1}{πx} \mathbb{E}\left[ \sin \left( xG^{2k} \left( \frac{1}{H^\beta(t)} \right) \right) \right]
\]

\[
= \int_\mathbb{R} dx e^{iξx} \left[ \int_0^∞ \int_0^∞ \frac{\sin xy}{πx} \Pr \left\{ G^{2k} \left( \frac{1}{s} \right) \in dy \right\} \Pr \{ H^\beta(t) \in ds \} \right]
\]

\[
= \int_0^∞ \int_0^∞ \Pr \left\{ G^{2k} \left( \frac{1}{s} \right) \in dy \right\} \Pr \{ H^\beta(t) \in ds \} \left[ \int_\mathbb{R} dx e^{iξx} \frac{\sin xy}{πx} \right].
\]  (4.7)

By considering that the Heaviside function

\[
\mathcal{H}_α(z) = \begin{cases} 1, & z > α, \\ 0, & z < α \end{cases}
\]  (4.8)

can be represented as

\[
\mathcal{H}_α(z) = \frac{1}{2π} \int_\mathbb{R} dw e^{izw} \frac{e^{-iαw}}{iw} = -\frac{1}{2π} \int_\mathbb{R} dw e^{-izw} \frac{e^{iαw}}{iw},
\]  (4.9)

we obtain that formula (4.7) becomes

\[
\hat{v}_β^{2k}(ξ, t)
\]

\[
= \int_0^∞ \int_0^∞ \Pr \left\{ G^{2k} \left( \frac{1}{s} \right) \in dy \right\} \Pr \{ H^\beta(t) \in ds \} [\mathcal{H}_{-y}(ξ) - \mathcal{H}_y(ξ)]
\]

\[
= \int_0^∞ \int_0^∞ \Pr \left\{ G^{2k} \left( \frac{1}{s} \right) \in dy \right\} \Pr \{ H^\beta(t) \in ds \} [\Pi_{[0,∞)}(y) - \Pi_{[-∞,0)}(y)]
\]

\[
= \int_0^∞ \int_0^∞ dy ds (2ksy^{2k-1} e^{-ys}) \Pi_{[0,∞)}(y) [\Pi_{[-∞,0)}(y) - \Pi_{[-∞,y]}(y)] h_\beta(s, t).
\]  (4.10)

For \(ξ > 0\) (4.10) becomes

\[
\hat{v}_β^{2k}(ξ, t) = \int_0^∞ ds \left[ 1 - \int_0^ξ dy 2ksy^{2k-1} e^{-ys} \right] h_\beta(s, t)
\]

\[
= \int_0^∞ ds e^{-ξs} h_\beta(s, t) = e^{t|ξ|^{2β}},
\]  (4.11)

and for \(ξ < 0\) (4.10) is

\[
\hat{v}_β^{2k}(ξ, t) = \int_0^∞ ds \left[ \int_{-ξ}^∞ 2ksy^{2k-1} e^{-ys} \right] h_\beta(s, t)
\]

\[
= \int_0^∞ ds e^{-ξs} h_\beta(s, t) = e^{-t|ξ|^{2β}}.
\]  (4.12)
Since
\[
\Pr \left\{ G^{2k} \left( \frac{1}{H^\beta(t)} \right) \in dy \right\} / dy = 2ky^{2k-1} \int_0^\infty se^{-x^2} h_\beta(s, t) ds
\]
\[
= - \frac{\partial}{\partial y} \int_0^\infty e^{-x^2} h_\beta(s, t) ds
\]
\[
= - \frac{\partial}{\partial y} e^{-x^2 t}
\]
\[
= \Pr \left\{ G^{2k\beta} \left( \frac{1}{t} \right) \in dy \right\} / dy
\]
(4.13)
the second form of the solution (4.2) follows immediately.

For \( k \geq 1, \beta \in (0, \frac{1}{k}] \) the solutions (4.2) are densities of symmetric random variables, while for \( \beta > \frac{1}{k} \) the functions (4.2) are sign varying. Clearly for \( \beta = 1 \) we obtain the solution of even-order heat-type equations discussed in [23]. In the Theorem 4.2 below we consider the higher-order heat-type equation
\[
\frac{\partial}{\partial t} v^{2k+1}(x, t) = R v^{2k+1}(x, t)
\]
(4.14)
where the operator \( R \) is obtained as a suitable combination of the Weyl derivatives (1.22) and (1.23). The operator \( R \) is related to the pseudoprocesses dealt with in Theorem 3.1 and can be explicitly written, for \( \{ p, q \in [0, 1] : p + q = 1 \}, \{ \beta \in (0, 1), k \in \mathbb{N} : m - 1 < \beta(2k + 1) < m, m \in \mathbb{N} \} \) as
\[
R v^{2k+1}(x, t)
\]
\[
= -\frac{1}{\cos \frac{\pi \beta}{2}} \left[ p e^{i\pi \beta k} + \frac{\partial^{2k+1} e^{-i\pi \beta k}}{\partial x^{2k+1}} + q e^{i\pi \beta k} - \frac{\partial^{2k+1} e^{i\pi \beta k}}{\partial x^{2k+1}} \right] v^{2k+1}(x, t)
\]
\[
= -\frac{1}{\cos \frac{\pi \beta}{2}} \frac{\partial^m}{\Gamma(m - (2k + 1)\beta)} \frac{e^{i\pi \beta k} p}{(y - x)^{(2k+1)\beta - m + 1}} d\gamma + q e^{-i\pi \beta k} (-1)^m \int_x^\infty \frac{v^{2k+1}(y, t)}{(y - x)^{(2k+1)\beta - m + 1}} d\gamma,
\]
(4.15)
where the left and right Weyl fractional derivatives appear.

**Theorem 4.2.** The solution to the problem
\[
\begin{align*}
\frac{\partial}{\partial t} v^{2k+1}(x, t) &= R v^{2k+1}(x, t), \quad x \in \mathbb{R}, t > 0, \beta \in (0, 1), k \in \mathbb{N}, \\
v^{2k+1}(x, 0) &= \delta(x),
\end{align*}
\]
(4.16)
is given by the signed law of the pseudoprocess
\[
\widetilde{\mathcal{V}}^{2k+1}(t) = U^{2k+1}_1 \left( H^{\beta} \left( \frac{pt}{\cos \frac{\beta \pi}{2}} \right) \right) - U^{2k+1}_2 \left( H^{\beta} \left( \frac{qt}{\cos \frac{\beta \pi}{2}} \right) \right),
\]
(4.17)
where $U_{1}^{2k+1}$, $U_{2}^{2k+1}$ are independent odd-order pseudoprocesses and $H_{1}^β$, $H_{2}^β$, are independent stable subordinators.

**Proof.** The Fourier transform of (4.15) is written as

$$
\mathcal{F}\left[\Re u_{2k+1}^\beta(x,t)\right](\xi) = \mathcal{F}\left[-\frac{1}{\cos \frac{\beta \pi}{2}} \left[ p e^{i\pi \beta k} \right] \right] \left[ \hat{v}_{2k+1}^\beta(x,t) \right] (\xi)
$$

and, therefore, we have that

$$
\hat{v}_{2k+1}^\beta(\xi, t) = e^{-t|\xi|^\beta(2k+1)(1-i(p-q)\tan \frac{\beta \pi}{2})}
$$

(4.19)

In view of (3.16) we get

$$
\mathbb{E}e^{i\xi \hat{u}^{(2k+1)\beta}(t)} = e^{-t|\xi|^\beta(2k+1)(1-i(p-q)\tan \frac{\beta \pi}{2})}
$$

(4.20)

and this confirms that the solution to (4.16) is given by the law of the pseudoprocess (4.17).

**Remark 4.3.** For $p = q = \frac{1}{2}$ the fractional operator (4.15) coincides with the Riesz fractional derivative since the Fourier transforms of both operators coincide. This can be ascertained from (4.18).

We now pass to the derivation of the governing equation of the fractional pseudoprocesses studied in Theorem 3.5. We first recall the definition of the Feller space-fractional derivative which is

$$
F D^{\beta,\theta} u(x) = - \left[ \frac{\sin \frac{\pi}{2}(\beta - \theta)}{\sin(\pi \beta)} \frac{d}{dx} + \frac{\sin \frac{\pi}{2}(\beta + \theta)}{\sin(\pi \beta)} \frac{d}{dx} \right] u(x).
$$

(4.21)

We recall that

$$
\mathcal{F}[F D^{\beta,\theta} u(x)](\xi) = -|\xi|^\beta e^{i\pi \theta \frac{\xi}{2}} \text{sign}(\xi) \hat{u}(\xi),
$$

(4.22)
as can be shown by means of the following calculation valid for \( \beta > 0 \)

\[
\int_{\mathbb{R}} dxe^{i\xi x} F D^{\beta, \theta} u(x) = -\left[ \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin (\pi \beta)} (i \xi)^\beta + \frac{\sin \frac{\pi}{2} (\beta + \theta)}{\sin (\pi \beta)} (i \xi)^\beta \right] \hat{u}(\xi)
\]

\[
= -\frac{|\xi|^\beta}{2t \sin \pi \beta} \left[ \left( e^{\frac{i\pi}{2} \beta} e^{-\frac{i\pi}{2} \theta} - e^{-\frac{i\pi}{2} \beta} e^{\frac{i\pi}{2} \theta} \right) e^{-\frac{i\pi}{2} \beta \text{sign}(\xi)} \right]
\]

\[
+ \left( e^{\frac{i\pi}{2} \beta} e^{\frac{i\pi}{2} \theta} - e^{-\frac{i\pi}{2} \beta} e^{-\frac{i\pi}{2} \theta} \right) e^{\frac{i\pi}{2} \beta \text{sign}(\xi)} \right] \hat{u}(\xi)
\]

\[
= \begin{cases} 
-\xi^\beta e^{i\pi \theta} \hat{u}(\xi), & \xi > 0, \\
-(\xi)^\beta e^{-i\pi \theta} \hat{u}(\xi), & \xi < 0 
\end{cases}
\]

\[
= -|\xi|^\beta e^{i\pi \text{sign}(\xi)} \hat{u}(\xi)
\]

(4.23)

where we used the results of Theorem 2.1. The explicit form of the Fourier transform of the solution to

\[
\frac{\partial}{\partial t} u(x, t) = F D^{\beta, \theta} u(x, t), \quad u(x, 0) = \delta(x), \quad x \in \mathbb{R}, t > 0,
\]

is written as

\[
\hat{u}(\xi, t) = e^{-|\xi|^\beta t e^{i\pi \text{sign}(\xi)}}
\]

(4.25)

and for \( \beta \in (0, 2], 4m - 1 < \theta < 4m + 1, m \in \mathbb{N} \), represents the characteristic function of a stable r.v. The last condition on \( \theta \) is due to the fact that

\[
|\hat{u}(\xi, t)| \leq 1 \text{ if and only if } \cos \frac{\theta \pi}{2} \in (0, 1].
\]

(4.26)

Condition (4.26) must be assumed also for \( \beta > 2 \) where (4.25) however fails to be the characteristic function of a genuine r.v.. For \( \theta = \beta < 1 \) (4.25) becomes totally negatively skewed. By interchanging \( \sin(\beta - \theta) \frac{\pi}{2} \) with \( \sin(\beta + \theta) \frac{\pi}{2} \) we obtain instead

\[
\hat{u}(\xi, t) = e^{-|\xi|^\beta t e^{-i\pi \text{sign}(\xi)}}
\]

(4.27)

which is totally positively skewed for \( \theta = \beta < 1 \).

We are now ready to prove the following Theorem.

**Theorem 4.4.** Let \( Z^{(2k+1), \theta}(t) \), \( t > 0 \), be the limiting fractional pseudoprocess studied in Theorem 3.5. The signed density of \( Z^{(2k+1), \theta}(t) \) is the solution to

\[
\begin{cases} 
\frac{\partial}{\partial t} z^{(2k+1), \theta}(x, t) = F D^{\beta, \theta} z^{(2k+1), \theta}(x, t) \\
z^{(2k+1), \theta}(x, 0) = \delta(x)
\end{cases}
\]

(4.28)

and coincides with the signed distribution of the composition for \( \beta \in (0, 1), -\beta < \theta < \beta, \)

\[
Z^{(2k+1), \theta}(t) = U^{2k+1}_1 \left( H^\beta \left( \frac{\sin \frac{\pi}{2} (\beta + \theta)}{\sin \pi \beta} t \right) \right) - U^{2k+1}_2 \left( H^\beta \left( \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin \pi \beta} t \right) \right)
\]

(4.29)
where $H_j^\beta$, $j = 1, 2$ are independent stable r.v.’s and the independent pseudoprocesses $U_j^{2k+1}$, $j = 1, 2$, are related to the odd-order heat-type equation

$$\frac{\partial}{\partial t} u_{2k+1}(x, t) = (-1)^k \frac{\partial^{2k+1}}{\partial x^{2k+1}} u_{2k+1}(x, t).$$

(4.30)

The positivity of the time scales in (4.29) implies that $-\beta < \theta < \beta$.

**Proof.** By profiting from the result (4.22) we note that the Fourier transform of (4.28) is written as

$$\begin{align*}
\frac{\partial}{\partial t} \omega_2^{(2k+1), \theta}(\xi, t) &= -|\xi|^{\beta(2k+1)} e^{i \frac{\pi}{2} \text{sign}(\xi)} \omega_2^{(2k+1), \theta}(\xi, t) \\
\omega_2^{(2k+1), \theta}(\xi, 0) &= 1.
\end{align*}$$

(4.31)

which is satisfied by the Fourier transform

$$\omega_2^{(2k+1), \theta}(\xi, t) = e^{-t|\xi|^{\beta(2k+1)} e^{i \frac{\pi}{2} \text{sign}(\xi)}}.$$ 

(4.32)

We now prove that the Fourier transform of (4.29) coincides with (4.32). In view of the independence of the r.v.’s and pseudo r.v.’s involved we write that

\begin{align*}
\mathbb{E} e^{i \xi \omega_2^{(2k+1), \theta}(t)} &= \mathbb{E} e^{i \xi \omega_2^{(2k+1)}(H_2^\beta(x, t)) - U_2^{2k+1}(H_2^\beta(x, t))} \\
&= \left[ \int_R \int_0^\infty ds u_{2k+1}(s, \alpha) h_\beta(\alpha) \left( s, \frac{\sin \frac{\pi}{2} (\beta + \theta)}{\sin \pi \beta} t \right) \right] \\
&\times \left[ \int_R \int_0^\infty ds u_{2k+1}(s, \alpha) h_\beta(\alpha) \left( s, \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin \pi \beta} t \right) \right] \\
&= \left[ \int_0^\infty e^{-i \xi \omega_2^{(2k+1)}} h_\beta \left( s, \frac{\sin \frac{\pi}{2} (\beta + \theta)}{\sin \pi \beta} t \right) ds \right] \left[ \int_0^\infty e^{i \xi \omega_2^{(2k+1)}} h_\beta \left( s, \frac{\sin \frac{\pi}{2} (\beta - \theta)}{\sin \pi \beta} t \right) ds \right] \\
&= e^{-t \sin \frac{\pi}{2} (\beta + \theta)} e^{-t \sin \frac{\pi}{2} (\beta - \theta)} \\
&= e^{-t i \xi \omega_2^{(2k+1)} \sin \frac{\pi}{2} (\beta + \theta)} e^{-t i \xi \omega_2^{(2k+1)} \sin \frac{\pi}{2} (\beta - \theta)} \\
&= e^{-t i \xi \omega_2^{(2k+1)} \left( e^{i \frac{\pi}{2} \text{sign}(\xi)} - e^{-i \frac{\pi}{2} \text{sign}(\xi)} \right)} \\
&= e^{-t |\xi|^{\beta(2k+1)} e^{i \frac{\pi}{2} \text{sign}(\xi)}} \\
&= e^{-t |\xi|^{\beta(2k+1)} e^{i \frac{\pi}{2} \text{sign}(\xi)}}.
\end{align*}

(4.33)

which coincides with (4.32). □

**Remark 4.5.** We note that different combinations of $\beta$ and $2k+1$ produce pseudoprocesses whose signed density measure has Fourier transform coinciding with that of the solution to (4.28).

### 5. Some Remarks

We give various forms for the density $u^\gamma(x, t)$ of symmetric pseudoprocesses of arbitrary order $\gamma > 0$. For integer values of $\gamma = 2n$ or $\gamma = 2n + 1$ the analysis of the structure of these densities is presented in Orsingher and D’Ovidio [23]. We give here an analytical
representation of \( v^\gamma(x, t) \) for non-integer values of \( \gamma \), which is an alternative to (4.2), as a power series and in integral form (involving the Mittag-Leffler functions). Furthermore, in Figure 1 we give some curves for special values of \( \gamma \). We also give the distribution of the sojourn time of compositions of pseudoprocesses with stable subordinators (totally positively skewed stable r.v.'s).

**Remark 5.1.** For \( \gamma > 1 \) the inverse of the Fourier transform
\[
\hat{v}^\gamma(\xi, t) = e^{-t|\xi|^\gamma}
\] (5.1)
can also be written as
\[
v^\gamma(x, t) = \frac{1}{\pi} \int_0^\infty \cos(\xi x)e^{-t|\xi|^\gamma} d\xi
\]
\[
= \frac{1}{\pi \gamma} \sum_{k=0}^\infty (-1)^k x^{2k} \frac{\Gamma \left( \frac{2k+1}{\gamma} \right)}{(2k)!} \frac{1}{t^{\frac{2k+1}{\gamma}}}
\]
\[
= \frac{1}{\pi \gamma} \sum_{k=0}^\infty (-1)^k x^{2k} \frac{B \left( \frac{2k+1}{\gamma}, (2k+1) \left( 1 - \frac{1}{\gamma} \right) \right)}{t^{\frac{2k+1}{\gamma}} \Gamma \left( (2k+1) \left( 1 - \frac{1}{\gamma} \right) \right)}
\]
\[
= \frac{1}{\pi \gamma} \sum_{k=0}^\infty (-1)^k x^{2k} \frac{1}{t^{\frac{2k+1}{\gamma}} \Gamma \left( (2k+1) \left( 1 - \frac{1}{\gamma} \right) \right)} \int_0^1 dy y^{\frac{2k+1}{\gamma}-1} (1 - y)^{(2k+1)\left( 1 - \frac{1}{\gamma} \right)-1}
\]
\[
= \frac{1}{\pi \gamma} \int_0^1 dy \sum_{k=0}^\infty \frac{(-1)^k \left( xy^{\frac{1}{\gamma}} (1 - y)^{1 - \frac{1}{\gamma}} \right)^{2k} y^{\frac{1}{\gamma}-1} (1 - y)^{-\frac{1}{\gamma}}}{t^{\frac{2k+1}{\gamma}} \Gamma \left( (2k+1) \left( 1 - \frac{1}{\gamma} \right) \right)}
\]
\begin{equation}
\frac{t^{-\frac{1}{\gamma}}}{\pi^\gamma} \int_0^1 dy E_2(1 - \frac{1}{\gamma}, 1 - \frac{1}{\gamma}) \left( - \left( xy^{\frac{1}{\gamma}} (1 - y)^{1 - \frac{1}{\gamma}} \right)^2 t^{-\frac{1}{\gamma}} \right) y^{\frac{1}{\gamma} - 1} (1 - y)^{-\frac{1}{\gamma}} = \frac{w^{1 - \frac{1}{\gamma}}}{\pi^\gamma} \int_0^\infty dw E_2(1 - \frac{1}{\gamma}, 1 - \frac{1}{\gamma}) \left( -x^2 \left( \frac{w^{\frac{1}{\gamma}}}{1 + w} \right)^2 t^{-\frac{1}{\gamma}} \right) \frac{w^{\frac{1}{\gamma}}}{1 + w} \tag{5.2}
\end{equation}

and for \( \gamma < 2 \) coincides with the characteristic function of symmetric stable processes. The two-parameter Mittag-Leffler functions appearing in (5.2) are defined as

\begin{equation}
E_{\nu, \mu}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(j \nu + \mu)}, \quad \nu, \mu > 0, \quad x \in \mathbb{R}. \tag{5.3}
\end{equation}

Formula (5.2) is an alternative to the probabilistic representation (4.2) for \( \gamma = 2k \beta \). For \( 1 < \gamma < 2 \) it represents the density of a symmetric stable r.v.

**Remark 5.2.** We note that

\begin{equation}
v^{\nu}(0, t) = \frac{t^{-\frac{1}{\gamma}}}{\pi} \Gamma \left( 1 + \frac{1}{\gamma} \right) \tag{5.4}
\end{equation}

as can be inferred from (5.2). In the neighbourhood of \( x = 0 \) the density \( v^{\nu}(x, t) \) can be written as

\begin{equation}
v^{\nu}(x, t) \approx \frac{1}{\pi^\gamma} \left( \frac{1}{t^{\frac{1}{\gamma}} \Gamma \left( \frac{1}{\gamma} \right)} - \frac{x^2 \Gamma \left( \frac{3}{2} \right)}{2 t^{\frac{3}{2}}} \right) = v^{\nu}(0, t) \left( 1 - x^2 \frac{C_{\gamma}}{2t^{\frac{3}{2}}} \right) \tag{5.5}
\end{equation}

where

\begin{equation}
C_{\gamma} = \frac{\Gamma \left( \frac{1}{\gamma} + \frac{1}{3} \right) \Gamma \left( \frac{1}{\gamma} + \frac{2}{3} \right)}{\Gamma \left( \frac{2}{3} \right)} \frac{3^{\frac{3}{2} - \frac{1}{\gamma}}}{2\pi}. \tag{5.6}
\end{equation}

In the above calculation the triplication formula of the Gamma function (see [15] p. 14) has been applied

\begin{equation}
\Gamma(z) \Gamma \left( z + \frac{1}{3} \right) \Gamma \left( z + \frac{2}{3} \right) = \frac{2\pi}{3^{3z - \frac{1}{2}}} \Gamma(3z). \tag{5.7}
\end{equation}

**Remark 5.3.** For even-order pseudoprocesses \( U^{2k}(t), \ t > 0 \), the distribution of the sojourn time

\begin{equation}
\Gamma_t(U^{2k}) = \int_0^t \mathbb{1}_{(0, \infty)}(U^{2k}(s)) ds \tag{5.8}
\end{equation}
follows the arcsine law for all \( k \geq 1 \) (see Krylov [11]). Therefore, the distribution of the sojourn time of \( U^{2k}(H^\beta(t)), t > 0, \beta \in (0, 1) \), reads

\[
\Pr\{\Gamma_1(U^{2k}(H^\beta)) \in dx\} = \int_0^\infty \Pr\{\Gamma_3(U^{2k}) \in dx\} \Pr\{H^\beta(t) \in ds\}
\]

\[
= \frac{dx}{\pi} \int_x^\infty \frac{1}{\sqrt{x(s-x)}} \Pr\{H^\beta(t) \in ds\}.
\]

(5.9)

In the odd-order case, the distribution of the sojourn time

\[
\Gamma_1(U^{2k+1}) = \int_0^t \mathbb{I}_{[0,\infty)}(U^{2k+1}(s))ds
\]

is written as (see [12])

\[
\Pr\{\Gamma_1(U^{2k+1}) \in dx\} = dx \frac{\sin \frac{\pi}{2k+1}}{\pi} s^{-\frac{1}{2k+1}} (t-x)^{-\frac{2k}{2k+1}} \mathbb{I}_{[0,t]}(x)
\]

(5.11)

and, thus, we get

\[
\Pr\{\Gamma_1(U^{2k+1}(H^\beta)) \in dx\} = \frac{dx}{\pi} \int_x^\infty \frac{1}{2k+1 \sqrt{x(s-x)^{2k}}} \Pr\{H^\beta(t) \in ds\}.
\]

(5.12)

For \( \beta = \frac{1}{2} \) the integral (5.9) can be evaluated explicitly

\[
\Pr\{\Gamma_1(U^{2k}(H^{1/2})) \in dx\} = \frac{dx}{\pi} \int_x^\infty \frac{1}{\sqrt{x(s-x)}} \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds
\]

\[
= \frac{dx}{\pi} t \int_0^1 e^{-\frac{t^2}{2x}} \sqrt{\frac{1}{1-xy}} dy
\]

\[
= \frac{dx}{\pi} t \int_0^1 e^{-\frac{t^2}{2x}} \sqrt{\frac{1}{1-w}} dw
\]

\[
= \frac{dx}{\pi} t \sum_{k=0}^\infty \left(-\frac{t^2}{2x}\right)^k \frac{1}{k!} \int_0^1 w^k (1-w)^{-\frac{3}{2}} dw
\]

\[
= \frac{dx}{\pi} t \frac{E_{1,\frac{3}{2}} \left(-\frac{t^2}{2x}\right)}{x}.
\]

(5.13)

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