Non-linear functionals, deficient topological measures, and representation theorems on locally compact spaces

Svetlana V. Butler

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Abstract
We study non-linear functionals, including quasi-linear functionals, p-conic quasi-linear functionals, d-functionals, r-functionals, and their relationships to deficient topological measures and topological measures on locally compact spaces. We prove representation theorems and show, in particular, that there is an order-preserving, conic-linear bijection between the class of finite deficient topological measures and the class of bounded p-conic quasi-linear functionals. Our results imply known representation theorems for finite topological measures and deficient topological measures. When the space is compact we obtain four equivalent definitions of a quasi-linear functional and four equivalent definitions of functionals corresponding to deficient topological measures.

Keywords  p-Conic quasi-linear functional · r- and s-functionals · Deficient topological measure · Right and left measure · Representation theorem

Mathematics Subject Classification  28C05 · 28A25 · 46E27 · 46G99 · 28C15

1 Introduction

Mathematical interpretations of quantum physics by Mackey and Kadison (see, for example, [13–15]) led to very interesting mathematical problems, including the extension problem for probability measures in von Neumann algebras. Aarnes showed [1,2] that the extension problem may be regarded as a special case of the linearity problem for physical states, which is closely related to the existence of quasi-linear functionals. Aarnes [3] introduced quasi-linear functionals (that are not linear) on $C(X)$ for a
compact Hausdorff space $X$ and set functions, generalizing measures (initially called quasi-measures, now topological measures). He connected the two by establishing a representation theorem, thereby giving an impetus to the field that has already resulted in a substantial body of work.

In [10] Entov and Polterovich connected the theory of quasi-linear functionals with symplectic topology. They established that quasi-linear functionals can be viewed as an algebraic way of packing certain information contained in Floer theory, and in particular in spectral invariants of Hamiltonian diffeomorphisms, and proved many new results. Paper [10] has been cited over 100 times, and quasi-linear functionals and topological measures have been studied and used in many subsequent papers, as well as in monograph [16].

Deficient topological measures are natural generalizations of topological measures. They are valuable in the study of topological measures and non-linear functionals, and also interesting in their own right. Deficient topological measures were first defined and used by Rustad and Johansen [12] and later independently reintroduced by Svistula [18, 19]. Quasi-linear functionals have been generalized to other non-linear functionals in order to represent deficient topological measures on compact spaces. See [12, section 6]; see also [19], where such functionals are called r- and l-functionals.

Rustad [17] extended Aarnes’s representation theorem for quasi-linear functionals to the locally compact setting. However, other than Rustad’s paper and a couple of preprints (not all easily obtainable and/or in a final form) there are no other works devoted to quasi-linear functionals and topological measures on locally compact spaces. The author has always found the absence of this theory for locally compact spaces to be very unfortunate, as it impeded both the development of the field and its connection with other areas of mathematics. To remedy the situation, the author has written several papers extending the theory to the locally compact setting. The current paper is a key part of that series.

In this paper we study different kinds of non-linear functionals, including quasi-linear functionals, p-conic quasi-linear functionals, d-functionals, r-functionals, and their relationships to deficient topological measures and topological measures on locally compact spaces. We prove the relevant representation theorems. As with other Riesz-Type representation theorems, our results are crucial for proving further results about (deficient) topological measures and corresponding non-linear functionals. They are also essential for investigating connections with symplectic topology, operator algebras, probability theory, and other areas of mathematics. Although (deficient) topological measures lack subadditivity, their domain has no algebraic structure, and corresponding functionals are not linear, many results of functional analysis, measure theory, probability theory, etc. still hold for these generalizations of Borel measures and linear functionals. On the other hand, there are fascinating differences and new information. Further investigation into these matters will deepen our understanding of many areas of mathematics and is likely to lead to unforeseen applications.

Our results imply known results, including the representation theorem for quasi-linear functionals on compact spaces [3], the representation theorem for deficient topological measures on compact spaces [19], and the representation theorems for quasi-linear functionals on locally compact spaces [7,17]. We also obtain new consequences where $X$ is compact, including four equivalent definitions of a quasi-
linear functional and four equivalent definitions of functionals corresponding to
deficient topological measures. This paper is influenced by many works, including [3,11,12,17,19], and [7].

The paper is organized as follows. Section 2 contains necessary definitions and
background facts. In Sect. 3 we consider cones generated by a function, as well as
p-conic and n-conic quasi-linear functionals. In Sect. 4 we consider various related
non-linear functionals, including d-, r-, l-, and s-functionals. In Sect. 5 we show how
to obtain deficient topological measures from d-functionals. In our paper (unlike [12,
18,19]) deficient topological measures are considered as functions on locally compact
spaces into extended real numbers. In Sect. 6, given a finite deficient topological
measure and a bounded continuous function, we define left and right measures and
give a criterion for them to coincide. It turns out that they coincide if one starts
from a finite topological measure. We also give examples which show that left and
right measures may or may not coincide even for \{0, 1\}-valued deficient topological
measures. In Sect. 7, using right and left measures, we obtain by integration p-conic
and l-conic quasi-linear functionals, which are also r- and l-functionals. Section 8 is
devoted to representation theorems for finite deficient topological measures in terms of
bounded p-conic and l-conic quasi-linear functionals, and also r- and l-functionals. In
particular, we show that there is an order-preserving, conic-linear bijection between
the class of finite deficient topological measures and the class of bounded p-conic
quasi-linear functionals. When \(X\) is compact, we obtain four equivalent definitions of
quasi-linear functionals and four equivalent definitions of functionals corresponding
to deficient topological measures.

2 Preliminaries

In this paper \(X\) is a locally compact (Hausdorff), connected space.

By \(C(X)\) we denote the set of all real-valued continuous functions on \(X\) with the
extended uniform norm \(\|f\| = \sup\{f(x) : x \in X\}\), by \(C_b(X)\) the set of bounded
continuous functions on \(X\), by \(C_0(X)\) the set of continuous functions on \(X\) vanishing
at infinity, and by \(C_c(X)\) the set of continuous functions with compact support. By
\(C^+_0(X)\) we denote the collection of all nonnegative functions vanishing at infinity;
similarly, \(C^-_0(X)\) is the collection of all nonpositive functions from \(C_0(X)\).

When we consider maps into \([-\infty, \infty]\) we assume that any such map attains at most
one of \(\infty, -\infty\), and is not identically \(\infty\) or \(-\infty\). By \(D(\rho)\) we denote the domain of a
functional \(\rho\). For example, we may take \(D(\rho) = C^+_0(X)\) or \(C_c(X)\). If \(D(\rho) = C(X)\),
where \(X\) is compact, then \(D(\rho)\) contains constants.

We denote by \(E\) the closure of a set \(E\), and by \(\bigcup\) a union of disjoint sets. We denote
by \(1\) the constant function \(1(x) = 1\), by \(id\) the identity function \(id(x) = x\), and by
\(1_K\) the characteristic function of a set \(K\). By \(supp f\) we mean \(\{x : f(x) \neq 0\}\).

Several collections of sets are used often. They include: \(\mathcal{O}(X)\), the collection of
open subsets of \(X\); \(\mathcal{C}(X)\) the collection of closed subsets of \(X\); \(\mathcal{K}(X)\) the collection
of compact subsets of \(X\).
Definition 2.1 Let $X$ be a topological space and $\mu$ be a nonnegative set function on $\mathcal{E}$, a family of subsets of $X$ that contains $\varnothing(X) \cup \mathcal{C}(X)$. We say that

- $\mu$ is inner regular on $A \in \mathcal{E}$ if $\mu(A) = \sup\{\mu(K) : K \in \mathcal{K}(X), K \subseteq A\}$.
- $\mu$ is outer regular on $A$ if $\mu(A) = \inf\{\mu(U) : U \in \varnothing(X), A \subseteq U\}$.
- $\mu$ is inner regular (outer regular) if $\mu$ is inner regular (outer regular) on all sets $A \in \mathcal{E}$.
- $\mu$ is regular if $\mu$ is inner regular and outer regular.
- $\|\mu\| = \mu(X)$ and $\mu$ is finite if $\mu(X) < \infty$.
- $\mu$ is compact-finite if $\mu(K) < \infty$ for any $K \in \mathcal{K}(X)$.
- $\mu$ is monotone on all Borel sets, and inner regular on all open sets, i.e. $\mu$ is monotone if $A \subseteq B$ implies $\mu(A) \leq \mu(B)$.
- $\mu$ is $\tau$-smooth on compact sets if for every decreasing net $K_\alpha \downarrow K$, $K, K_\alpha \in \mathcal{K}(X)$ we have $\mu(K_\alpha) \to \mu(K)$.
- $\mu$ is $\tau$-smooth on open sets if for every increasing net $U_\alpha \nearrow U, U_\alpha, U \in \varnothing(X)$ we have $\mu(U_\alpha) \to \mu(U)$.
- $\mu$ is simple if it only assumes values 0 and 1.

Definition 2.2 A measure on $X$ is a countably additive set function on a $\sigma$-algebra of subsets of $X$ with values in $[0, \infty]$. A Borel measure on $X$ is a measure on the Borel $\sigma$-algebra on $X$. A Radon measure $m$ on $X$ is a Borel measure that is compact-finite, outer regular on all Borel sets, and inner regular on all open sets, i.e. $m(E) = \inf\{m(U) : E \subseteq U, U \text{ open }\}$ for every Borel set $E$, and $m(U) = \sup\{m(K) : K \subseteq U, K \text{ compact}\}$ for every open set $U$. For a Borel measure $m$ that is inner regular on all open sets (in particular, for a Radon measure) we define $\text{supp } m$, the support of $m$, to be the complement of the largest open set $W$ such that $m(W) = 0$.

Note that $W$ is the union of all open sets of measure 0 [8, pp. 206, 207].

Definition 2.3 A deficient topological measure on a locally compact space $X$ is a set function $\nu : \mathcal{C}(X) \cup \varnothing(X) \to [0, \infty]$ which is finitely additive on compact sets, inner compact regular on open sets, and outer regular on closed sets, i.e.:

- (DTM1) if $C \cap K = \emptyset$, $C, K \in \mathcal{K}(X)$ then $\nu(C \cup K) = \nu(C) + \nu(K)$;
- (DTM2) $\nu(U) = \sup\{\nu(C) : C \subseteq U, C \in \mathcal{K}(X)\}$ for $U \in \varnothing(X)$;
- (DTM3) $\nu(F) = \inf\{\nu(U) : F \subseteq U, U \in \varnothing(X)\}$ for $F \in \mathcal{C}(X)$.

Clearly, $\nu(F) = \infty$ for a closed set $F$ iff $\nu(U) = \infty$ for every open set $U$ containing $F$.

Remark 2.4 Note that a deficient topological measure $\nu$ is monotone on $\varnothing(X) \cup \mathcal{C}(X)$ and $\nu(\emptyset) = 0$. If $\nu$ and $\mu$ are deficient topological measures that agree on $\mathcal{K}(X)$ then $\nu = \mu$, and if $\nu \leq \mu$ on $\mathcal{K}(X)$ (or on $\varnothing(X)$) then $\nu \leq \mu$.

Definition 2.5 A topological measure on $X$ is a set function $\mu : \mathcal{C}(X) \cup \varnothing(X) \to [0, \infty]$ satisfying the following conditions:

- (TM1) if $A, B, A \cup B \in \mathcal{K}(X) \cup \varnothing(X)$ then $\mu(A \cup B) = \mu(A) + \mu(B)$;
- (TM2) $\mu(U) = \sup\{\mu(K) : K \in \mathcal{K}(X), K \subseteq U\}$ for $U \in \varnothing(X)$;
- (TM3) $\mu(F) = \inf\{\mu(U) : U \in \varnothing(X), F \subseteq U\}$ for $F \in \mathcal{C}(X)$. 

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We denote by $TM(X)$ the collection of all topological measures on $X$, by $DTM(X)$ the collection of all deficient topological measures on $X$, and by $M(X)$ the collection of all Borel measures on $X$ that are inner regular on open sets and outer regular (restricted to $\mathcal{C}(X) \cup \mathcal{O}(X)$).

**Remark 2.6** Let $X$ be locally compact. In general,

$$M(X) \subseteq T M(X) \subseteq DTM(X).$$

When $X$ is compact, there are examples of topological measures that are not measures and of deficient topological measures that are not topological measures in numerous papers, beginning with [3,12], and [18]. When $X$ is locally compact, see [4], [6, Sections 5 and 6], and [5, Section 9] for more information on proper inclusion, criteria for a deficient topological measure to belong to $M(X)$ or $TM(X)$ (in particular, to be a Radon measure or a regular Borel measure), as well as various examples.

The proof of the next result is in [6, Section 4].

**Theorem 2.7** (I) Let $X$ be compact, and $\nu$ a deficient topological measure. The following are equivalent:

(a) $\nu$ is a topological measure.
(b) $\nu(X) = \nu(C) + \nu(X \setminus C)$, $C \in \mathcal{C}(X)$.
(c) $\nu(X) \leq \nu(C) + \nu(X \setminus C)$, $C \in \mathcal{C}(X)$.

(II) Let $X$ be locally compact, and $\nu$ a deficient topological measure. The following are equivalent:

(a) $\nu$ is a topological measure.
(b) $\nu(U) = \nu(C) + \nu(U \setminus C)$, $C \in \mathcal{K}(X)$, $U \in \mathcal{O}(X)$.
(c) $\nu(U) \leq \nu(C) + \nu(U \setminus C)$, $C \in \mathcal{K}(X)$, $U \in \mathcal{O}(X)$.

Recall the following fact (see, for example, [9, Chapter XI, 6.2]):

**Lemma 2.8** Let $K \subseteq U$, $K \in \mathcal{K}(X)$, $U \in \mathcal{O}(X)$ in a locally compact space $X$. Then there exists a set $V \in \mathcal{O}(X)$ with compact closure such that $K \subseteq V \subseteq \overline{V} \subseteq U$.

The following is proved in [6, Section 3].

**Lemma 2.9** Let $X$ be a locally compact space.

(a) A deficient topological measure is $\tau$-smooth on compact sets and $\tau$-smooth on open sets. In particular, a topological measures is additive on open sets.
(b) A deficient topological measure $\mu$ is superadditive, i.e. if $\bigcup_{t \in T} A_t \subseteq A$, where $A_t$, $A \in \mathcal{O}(X) \cup \mathcal{C}(X)$, and at most one of the closed sets (if there are any) is not compact, then $\mu(A) \geq \sum_{t \in T} \mu(A_t)$.

We recall the following theorem:

**Theorem 2.10** (Integration by parts for Lebesque–Stieltjes integrals) Let $F$ and $G$ be real-valued nondecreasing functions on $\mathbb{R}$ with corresponding Lebesque–Stieltjes integrals.
measures \( m_F, m_G \). Then for \( a < b \) we have:

\[
\int_a^b G(x^+) dm_F(x) + \int_a^b F(x^-) dm_G(x) = F(b^+) G(b^+) - F(a^-) G(a^-).
\]

**Definition 2.11** Let \( \rho \) be a functional on \( C(X) \). We say

- \( \rho \) is homogeneous if \( \rho(af) = a\rho(f) \) for any \( a \in \mathbb{R} \).
- \( \rho \) is positive-homogeneous if \( \rho(af) = a\rho(f) \) for any \( a \geq 0 \).
- \( \rho \) is conic-linear on a cone \( S \) if it is linear on conic combinations of elements of \( S \), i.e. \( \rho(af + bg) = a\rho(f) + b\rho(g) \) for any \( a, b \geq 0 \) and any \( f, g \in S \).
- \( \rho \) is positive if \( f \geq 0 \implies \rho(f) \geq 0 \).
- \( \rho \) is monotone if \( g \leq f \implies \rho(g) \leq \rho(f) \).
- \( \rho \) is orthogonally additive if \( f \cdot g = 0 \implies \rho(f + g) = \rho(f) + \rho(g) \).
- \( \rho \) is real-valued if \( \rho(f) \in \mathbb{R} \) for any \( f \).
- \( \|\rho\| = \sup\{|\rho(f)| : \|f\| \leq 1\} \) and \( \rho \) is bounded if \( \|\rho\| < \infty \).

The remaining definitions and facts related to quasi-linear functionals can be found in [7, Section 2]:

**Definition 2.12** Let \( X \) be locally compact.

(a) Let \( f \in C_b(X) \). Define \( A(f) \) to be the smallest closed subalgebra of \( C_b(X) \) containing \( f \) and \( 1 \). Hence, when \( X \) is compact, we take \( f \in C(X) \) and define \( A(f) \) to be the smallest closed subalgebra of \( C(X) \) containing \( f \) and \( 1 \). We call \( A(f) \) the singly generated subalgebra of \( C(X) \) generated by \( f \).

(b) Let \( B \) be a subalgebra of \( C_b(X) \). Define \( B(f) \) to be the smallest closed subalgebra of \( B \) containing \( f \). We call \( B(f) \) the singly generated subalgebra of \( B \) generated by \( f \).

We may take, for example, \( C_c(X), C_0(X) \) as \( B \).

**Remark 2.13** When \( X \) is compact, \( A(f) \) for \( f \in C(X) \) contains all polynomials of \( f \).

It is not hard to show that \( A(f) \) has the form:

\[
A(f) = \{ \phi \circ f : \phi \in C(f(X)) \}.
\]

When \( X \) is locally compact, \( B = C_0(X) \) and \( f \in C_0(X) \) (or \( B = C_c(X) \) and \( f \in C_c(X) \)) the singly generated subalgebra has the form:

\[
B(f) = \{ \phi \circ f : \phi(0) = 0, \ \phi \in C(f(X)) \}.
\]

**Definition 2.14** Let \( X \) be locally compact, and let \( B \) be a subalgebra of \( C(X) \) containing \( C_c(X) \). A real-valued map \( \rho \) on \( B \) is a signed quasi-linear functional on \( B \) if

\[ (Q11) \quad \rho(af) = a\rho(f) \text{ for } a \in \mathbb{R}; \]
We say that a signed quasi-linear functional \( \rho \) is compact-finite if \( |\rho(f)| < \infty \) for \( f \in C_c(X) \).

### 3 Cones generated by functions

When \( X \) is a compact or locally compact, non-compact space, there is a correspondence between topological measures and quasi-linear functionals. (See \([3,7,17]\)). When \( X \) is compact, there is also a correspondence between deficient topological measures and \( r \)- or \( l \)-functionals, see \([19]\) and \([12]\). When \( X \) is locally compact, we shall consider relations between deficient topological measures and various non-linear functionals. We shall start with functionals that are conic-linear on cones generated by functions.

**Definition 3.1** Let \( X \) be compact, \( f \in C(X) \). Define cones:

\[
A^+(f) = \{ \phi \circ f : \phi \text{ is non-decreasing, continuous} \}, \\
A^-(f) = \{ \phi \circ f : \phi \text{ is non-increasing, continuous} \}.
\]

If \( X \) is a locally compact, non-compact space, \( f \in C_0(X) \) let

\[
A^+(f) = \{ \phi \circ f : \phi \text{ is non-decreasing, continuous, } \phi(0) = 0 \}, \\
A^-(f) = \{ \phi \circ f : \phi \text{ is non-increasing, continuous, } \phi(0) = 0 \}.
\]

**Remark 3.2** Since \( f = id \circ f \), for \( f \neq 0 \) (\( f \neq const \) if \( X \) is compact) we have \( f \in A^+(f), f \notin A^-(f) \), and \( -f \in A^-(f), -f \notin A^+(f) \). Note that \( 0 \in A^+(f), A^-(f) \). Also, \( f^+ \in A^+(f), f^- \in A^-(f) \), since \( f^+ = (id \lor 0) \circ f, f^- = ((-id) \lor 0) \circ f \). Obviously, \( A^+(f), A^-(f) \subseteq B(f) \) (respectively, \( A^+(f), A^-(f) \subseteq A(f) \) if \( X \) is compact).

**Definition 3.3** We call a functional \( \rho \) on \( C_0(X) \) with values in \([−∞, ∞]\) (assuming at most one of \( ∞, −∞ \)) and \( |\rho(0)| < ∞ \) a \( p \)-conic quasi-linear functional if it is orthogonally additive and monotone on nonnegative functions and conic-linear on \( A^+(f) \) for each \( f \in C_0(X) \), i.e.

- (p1) If \( f, g = 0 \), \( f, g \geq 0 \) then \( \rho(f + g) = \rho(f) + \rho(g) \).
- (p2) If \( 0 \leq g \leq f \) then \( \rho(g) \leq \rho(f) \).
- (p3) For each \( f \), if \( g, h \in A^+(f) \), \( a, b \geq 0 \) then \( \rho(ag + bh) = a\rho(g) + b\rho(h) \).

Similarly, a functional \( \rho \) on \( C_0(X) \) with values in \([−∞, ∞]\) (assuming at most one of \( ∞, −∞ \)) and \( |\rho(0)| < ∞ \) is called an \( n \)-conic quasi-linear functional if it
is orthogonally additive and monotone on nonpositive functions and conic-linear on $A^-(f)$ for each $f \in C_0(X)$. In other words,

(n1) If $fg, f, g \leq 0$ then $\rho(f + g) = \rho(f) + \rho(g)$.

(n2) If $f \leq g \leq 0$ then $\rho(f) \leq \rho(g)$.

(n3) For each $f$, if $g, h \in A^-(f)$, $a, b \geq 0$ then $\rho(af + bh) = a\rho(g) + b\rho(h)$.

Remark 3.4 From [7, Lemma 20(iii), Section 3 and Lemma 44, Section 4] it follows that a positive quasi-linear functional is a $p$-conic quasi-linear functional and also an $n$-conic quasi-linear functional.

Remark 3.5 Given a functional $\rho$, consider also the functional $\pi$ defined by $\pi(f) = -\rho(-f)$ for every $f \in D(\rho)$. Then $\pi$ is an $n$-conic quasi-linear functional iff $\rho$ is a $p$-conic quasi-linear functional. This allows us to transfer results for $p$-conic quasi-linear functionals to $n$-conic quasi-linear functionals and vice versa.

Lemma 3.6 Let $\rho$ be a functional on a locally compact space with $|\rho(0)| < \infty$.

(I) Suppose a functional $\rho$ is conic-linear on $A^+(f)$ or on $A^-(f)$ for some $f$. Then $\rho(0) = 0$.

(II) Suppose a functional $\rho$ is conic-linear on $A^+(h)$ for each function $h \in C_0(X)$. Then $fg = 0$, $f \geq 0$, $g \leq 0$ implies $\rho(f + g) = \rho(f) + \rho(g)$.

(III) Suppose $0 \leq g(x) \leq f(x) \leq c$, $f = c$ on $\{x : g(x) > 0\}$. Then $f, g \in A^+(f + g)$. In particular, if $\rho$ is a $p$-conic quasi-linear functional then $\rho(af + bg) = a\rho(f) + b\rho(g)$ for any $a, b \geq 0$. If $\rho$ is a quasi-linear functional, then $\rho(af + bg) = a\rho(f) + b\rho(g)$ for any $a, b \in \mathbb{R}$.

(IV) Suppose $g \geq 0$, $0 \leq h \leq 1$, and $h = 1$ on $\{x : g(x) > 0\}$. If $\rho$ is a $p$-conic quasi-linear functional then $\rho(af + bh) = a\rho(g) + b\rho(h)$ for any $a, b \geq 0$. If $\rho$ is a quasi-linear functional, then $\rho(af + bh) = a\rho(g) + b\rho(h)$ for any $a, b \in \mathbb{R}$.

(V) Suppose $\rho$ is a $p$-conic quasi-linear functional, $f, g \in C_c(X)$, $f, g \geq 0$. Then $|\rho(f) - \rho(g)| \leq \|\rho\|\|f - g\|$. If $X$ is compact we also have:

(i) If $f \in C(X)$, and a functional $\rho$ is conic-linear on $A^+(f)$ (or on $A^-(f)$), then $\rho(f + c) = \rho(f) + \rho(c)$.

(ii) If $\rho$ is conic-linear on $A^+(h)$ for each function $h \in C(X)$ (respectively, conic-linear on $A^-(h)$ for each function $h \in C(X)$) and $\rho(1) \in \mathbb{R}$, then $\rho$ is monotone.

Proof (I) Since $0 \in A^+(f)$, $\rho$ is conic-linear on $A^+(f)$, and $|\rho(0)| < \infty$ we have $\rho(0) = 0$. A similar argument works for $A^-(f)$.

(II) Suppose $fg = 0$, $f \geq 0$, $g \leq 0$. Taking $h = f + g$, observe that $f = (id \lor 0) \circ h$, $g = (id \land 0) \circ h$, so $f, g \in A^+(h)$. Then $\rho(f + g) = \rho(f) + \rho(g)$.

(III) Note that $c \geq 0$ and $f = (id \land c) \circ (f + g)$, $g = (0 \lor (id - c)) \circ (f + g)$.

(IV) We may assume that $g \neq 0$. Since $0 \leq g \|h\| \leq \|g\|$, by part (III)

$$\rho(af + bh) = \rho \left( a\rho(g) + b\frac{b}{\|g\|}\rho(\|g\|h) \right) = a\rho(g) + b\frac{b}{\|g\|}\rho(\|g\|h) = a\rho(g) + b\rho(h).$$
(V) Suppose \( f, g \in C_c(X) \), \( f, g \geq 0 \). Choose \( h \in C_c(X) \) such that \( h \geq 0 \), \( h = 1 \) on \( \text{supp } f \cup \text{supp } g \). Since \( f - g \leq \| f - g \| h \), i.e. \( f \leq g + \| f - g \| h \), using Definition 3.3 and part (IV) we have: 
\[
\rho(f) \leq \rho(g) + \| f - g \| \rho(h) \leq \rho(g) + \| f - g \| \rho(h),
\]
so 
\[
|\rho(f) - \rho(g)| \leq \| f - g \| \| f - g \| \rho(h) \leq \| f - g \| \| \rho \|.
\]  
(3.1)

\[(i) \text{ Since every constant } c \in A^+(f) \text{ we immediately see that } \rho(f + c) = \rho(f) + \rho(c). \]

\[(ii) \text{ Note that } \rho(c) \in \mathbb{R} \text{ for any constant } c \geq 0. \text{ Let } g \leq f. \text{ Choose a constant } c > 0 \text{ such that } 0 \leq g + c \leq f + c. \text{ We have: } \rho(g) + \rho(c) = \rho(g + c) = \rho(f) + \rho(c), \text{ so } \rho(g) \leq \rho(f). \]

\[\square\]

**Proposition 3.7** Suppose \( X \) is locally compact and \( \rho \) is a functional on \( C_0(X) \) that is positive-homogeneous and monotone on nonnegative functions. If \( \rho \) is real-valued, then \( \| \rho \| < \infty \).

**Proof** The statement can be obtained by adapting the argument from Lemma 2.3 in [17] (see Proposition 50 in [7]). \(\square\)

## 4 d-Functionals

In this section we shall define several functionals. The domains of these functionals vary, but the most common are \( C(X), C_0(X), C_c(X), C_0^+(X) \). We shall not specify the domains in the definitions and results that hold for different domains, but shall indicate them later when we use these functionals on specific collections of functions.

**Definition 4.1** A functional \( \rho \) with values in \([-\infty, \infty]\) (assuming at most one of \( \infty, -\infty \)) and \( |\rho(0)| < \infty \) is called a d-functional if on nonnegative functions it is positive-homogeneous, monotone, and orthogonally additive, i.e. for \( f, g \in D(\rho) \)

\[(d1) \ f \geq 0, \ a > 0 \implies \rho(af) = a\rho(f), \]
\[(d2) \ 0 \leq g \leq f \implies \rho(g) \leq \rho(f), \]
\[(d3) \ f \cdot g = 0, \ f, g \geq 0 \implies \rho(f + g) = \rho(f) + \rho(g). \]

**Remark 4.2** It is easy to see that \( \rho(0) = 0 \), and so \( \rho \) is positive.

**Definition 4.3** We say that a functional \( \rho \) satisfies the c-level condition if

\[ f \leq c, \ f = c \text{ on } \text{supp } g \implies \rho(f + g) = \rho(f) + \rho(g), \]

where \( c \) is a constant, and \( f, g \in D(\rho) \).

If the domain of \( \rho \) includes constants, we say that \( \rho \) satisfies the constant condition if for any function \( g \in D(\rho) \) and any constant \( c \)

\[ \rho(g + c) = \rho(g) + \rho(c). \]  

(4.1)
Lemma 4.4 Suppose the domain of \( \rho \) includes constants.

(i) The \( c \)-level condition implies the constant condition.
(ii) Suppose \( \rho \) is a \( d \)-functional that satisfies the constant condition (4.1), and \( \rho(1) \in \mathbb{R} \). If \( f \leq c, f = c \) on \( \text{supp} \, g \), \( g \geq 0 \), then \( \rho(g - f) = \rho(g) + \rho(-f) \).
(iii) Suppose a functional \( \rho \) is conic-linear on \( A^+(h) \) for each function \( h \) (for example, \( \rho \) is a \( p \)-conic quasi-linear). If \( \rho \) satisfies the constant condition then \( \rho \) satisfies the \( c \)-level condition for all \( g \geq 0 \).

Proof For part (i) take \( f \equiv c \). (ii): Since \( c - f \geq 0 \) and \( (c - f)g = 0 \), we have:
\[ \rho(c + \rho(g - f)) = \rho(c + g - f) = \rho(c - f) + \rho(g) = \rho(c) + \rho(-f) + \rho(g), \]
so \( \rho(g - f) = \rho(g) + \rho(-f) \). (iii): Suppose \( \rho \) satisfies the constant condition (4.1). Let \( f \leq c, f = c \) on \( \text{supp} \, g \), \( g \geq 0 \). Let \( h = f - c \). Then \( h \leq 0 \) and \( gh = 0 \). By part (II) of Lemma 3.6, \( \rho(h + g) = \rho(h) + \rho(g) \). Then \( \rho(f + g) = \rho(h + c + g) = \rho(h + g) + \rho(c) = \rho(h) + \rho(g) + \rho(c) = \rho(h + c) + \rho(g) = \rho(f) + \rho(g) \).

Remark 4.5 If \( \rho \) is a monotone, positive-homogeneous functional that satisfies the constant condition then for any functions \( f, g \in D(\rho) \)
\[ |\rho(f) - \rho(g)| \leq \rho(\|f - g\|) = \|f - g\| \rho(1). \]

One can show this by noticing that \( f \leq g + \|f - g\| \) and using an argument similar to the one for part (V) of Lemma 3.6.

We modify condition (d3) in Definition 4.1 and obtain the following definition:

Definition 4.6 A functional \( \rho \) with values in \([-\infty, \infty]\) (assuming at most one of \( \infty, -\infty \)) and \( |\rho(0)| < \infty \) is called a \( c \)-functional if for \( f, g \in D(\rho) \)
(c1) \( f \geq 0, a > 0 \implies \rho(af) = a\rho(f) \);
(c2) \( 0 \leq g \leq f \implies \rho(g) \leq \rho(f) \);
(c3) \( f \cdot g = 0, f, g \geq 0 \) or \( f \geq 0, g \leq 0 \implies \rho(f + g) = \rho(f) + \rho(g) \).

Note that for a \( c \)-functional \( \rho \), \( \rho(f) = \rho(f^+) + \rho(-f^-) \).

Definition 4.7 A real-valued functional \( \rho \) is called an \( s \)-functional if
(s1) \( f \geq 0, a \in \mathbb{R} \implies \rho(af) = a\rho(f) \);
(s2) \( 0 \leq g \leq f \implies \rho(g) \leq \rho(f) \);
(s3) \( f \cdot g = 0, f, g \geq 0 \) or \( f \geq 0, g \leq 0 \implies \rho(f + g) = \rho(f) + \rho(g) \).

Lemma 4.8 Suppose \( \rho \) is an \( s \)-functional. Then
(i) \( \rho(f) = \rho(f^+) - \rho(f^-) \).
(ii) \( \rho(af) = a\rho(f) \) for any \( f \) and any \( a \in \mathbb{R} \).
(iii) If \( g \leq f \) then \( \rho(g) \leq \rho(f) \).
(iv) If \( f \cdot g = 0 \) then \( \rho(f + g) = \rho(f) + \rho(g) \).

Proof (i) holds because \( f^+ \cdot f^- = 0 \). (ii) follows from (i). (iii): If \( g \leq f \) then \( g^+ \leq f^+, f^- \leq g^- \) and \( \rho(g) = \rho(g^+) - \rho(g^-) \leq \rho(f^+) - \rho(f^-) = \rho(f) \). (iv): If \( f \cdot g = 0 \) then \( (f + g)^+ = f^+ + g^+, (f + g)^- = f^- + g^- \), and \( f^+ \cdot g^+ = 0 \), \( f^- \cdot g^- = 0 \). Then \( \rho(f + g) = \rho(f + g)^+ - \rho((f + g)^-) = \rho(f^+ + g^+) - \rho(f^- + g^-) = \rho(f^+) + \rho(g^+) - \rho(f^-) - \rho(g^-) = \rho(f) + \rho(g) \).
Lemma 4.8 shows that Definition 4.7 is equivalent to the following:

**Definition 4.9** A functional $\rho$ is called an s-functional if it is homogeneous, monotone, and orthogonally additive, i.e.

\begin{align*}
(\text{sa1}) \quad & \rho(af) = a\rho(f) \text{ for any } a \in \mathbb{R}; \\
(\text{sa2}) \quad & g \leq f \implies \rho(g) \leq \rho(f); \\
(\text{sa3}) \quad & f \cdot g = 0 \implies \rho(f + g) = \rho(f) + \rho(g).
\end{align*}

**Remark 4.10** Let $X$ be locally compact, and let $\rho$ be a quasi-linear functional. By [7, Lemma 20(q2), Section 3 and Lemma 44, Section 5], $\rho$ is an s-functional.

**Notation 4.11** Let $QI$ and $L$ denote, respectively, the families of all quasi-linear and linear functionals. By $\Phi^d, \Phi^c, \Phi^s, \Phi^+, \Phi^-$ we denote, respectively, the families of all d-functionals, c-functionals, s-functionals, p-conic quasi-linear functionals, and n-conic quasi-linear functionals.

**Remark 4.12** We have: $L \subseteq QI \subseteq \Phi^s \subseteq \Phi^c \subseteq \Phi^d$.

**Proposition 4.13** (i) $\rho$ is a s-functional if and only if it is a real-valued c-functional with a property that $\rho(-g) = -\rho(g)$ for every $g \leq 0$ in the domain of $\rho$.

(ii) If $\rho$ is a d-functional, $D(\rho)$ contains constants, $\rho(1) \in \mathbb{R}$, and the constants condition (4.1) is satisfied, then $\rho$ is monotone and positive.

**Proof** (i) Suppose $\rho$ is a c-functional with the property that $\rho(-g) = -\rho(g)$ for every $g \leq 0$ in the domain of $\rho$. We have: $\rho(f) = \rho(f^+) + \rho(-f^-) = \rho(f^+) - \rho(f^-)$ for any function $f$ in the domain of $\rho$. Then $-\rho(f) = -\rho(f)$ for any function $f$ in the domain of $\rho$, and condition (s1) of Definition 4.7 follows. Thus, $\rho$ is an s-functional.

(ii) Let $g \leq f$. It is enough to assume that $0 \leq g \leq f$, for choosing a positive constant $k$ such that $g + k = 0$ we see that $\rho(g) \leq \rho(f)$ iff $\rho(g) + \rho(k) \leq \rho(f) + \rho(k)$ iff $\rho(g + k) \leq \rho(f + k)$. But $\rho$ is a d-functional, so $\rho(g) \leq \rho(f)$ for $0 \leq g \leq f$. Since $\rho(0) = 0$, monotonicity of $\rho$ implies positivity. \hfill $\Box$

Now we will introduce the closely related r- and l- functionals.

**Definition 4.14** A functional $\rho$ with values in $[-\infty, \infty]$ (assuming at most one of $\infty, -\infty$) and $|\rho(0)| < \infty$ is called an r-functional if for $f, g \in D(\rho)$

\begin{align*}
(\text{r1}) \quad & a > 0 \implies \rho(af) = a\rho(f); \\
(\text{r2}) \quad & 0 \leq g \leq f \implies \rho(g) \leq \rho(f); \\
(\text{r3}) \quad & \text{If } f \cdot g = 0 \text{ where } f, g \geq 0 \text{ or } f \geq 0, g \leq 0 \text{ then } \rho(f + g) = \rho(f) + \rho(g).
\end{align*}

If $D(\rho)$ contains constants then we also require for $f \in D(\rho)$ and a constant $c$

$$\rho(f + c) = \rho(f) + \rho(c).$$

**Definition 4.15** A functional $\rho$ with values in $[-\infty, \infty]$ (assuming at most one of $\infty, -\infty$) and $|\rho(0)| < \infty$ is called an l-functional if

\begin{align*}
(\text{l1}) \quad & a > 0 \implies \rho(af) = a\rho(f);
\end{align*}
If \( g \leq f \leq 0 \implies \rho(g) \leq \rho(f) \);

If \( f \cdot g = 0 \) where \( f, g \leq 0 \) or \( f \geq 0, g \leq 0 \) then \( \rho(f + g) = \rho(f) + \rho(g) \).

If \( D(\rho) \) contains constants then we also require for \( f \in D(\rho) \) and a constant \( c \)
\[
\rho(f + c) = \rho(f) + \rho(c).
\]

**Notation 4.16** By \( \Phi^r, \Phi^l \) we denote, respectively, the families of all \( r \)-functionals, and \( l \)-functionals on \( X \).

**Remark 4.17** Here are a few easy observations.

(i) Suppose \( \rho, \mathcal{R} \) are \( r \)-functionals with the same domain that contain constants and \( \rho(1) \in \mathbb{R} \). Then \( \rho = \mathcal{R} \) iff \( \rho = \mathcal{R} \) on nonnegative functions. Indeed, for an arbitrary function \( f \) choose a constant \( c \geq 0 \) such that \( f + c \geq 0 \) and see that
\[
\rho(f) + \rho(c) = \rho(f + c) = \mathcal{R}(f + c) = \mathcal{R}(f) + \mathcal{R}(c) = \mathcal{R}(f) + \rho(c), \text{ i.e.}
\]
\[
\rho(f) = \mathcal{R}(f).
\]

(ii) We have: \( \Phi^c \subseteq \Phi^r \subseteq \Phi^d \) and from Definition 4.9 \( \Phi^c \subseteq \Phi^l \).

(iii) Given a functional \( \rho \), consider also the functional \( \pi \) defined by
\[
\pi(f) = -\rho(-f)
\]
for every \( f \in D(\rho) \). Then \( \pi \) is an \( l \)-functional iff \( \rho \) is an \( r \)-functional.

**Lemma 4.18** Each \( p \)-conic quasi-linear functional is an \( r \)-functional. Each \( n \)-conic quasi-linear functional is an \( l \)-functional. So \( \Phi^+ \subseteq \Phi^r \) and \( \Phi^- \subseteq \Phi^l \).

**Proof** Using part (II) and part (i) of Lemma 3.6 we see that a \( p \)-conic quasi-linear functional is an \( r \)-functional. The second statement follows from Remark 3.5 and part (iii) of Remark 4.17.

**Lemma 4.19** Let \( X \) be compact and \( \rho \) be a functional on \( C(X) \).

(I) If \( \rho \) is an \( r \)-functional with \( \rho(1) \in \mathbb{R} \) or an \( l \)-functional with \( \rho(-1) \in \mathbb{R} \) then \( \rho \) is monotone.

(II) If \( \rho \) is an \( r \)-functional with \( \rho(1) \in \mathbb{R} \) then
\[
|\rho(f) - \rho(g)| \leq \|\rho\| \|f - g\| = \rho(1) \|f - g\|.
\]

(III) If \( \rho \) is an \( r \)-functional with \( \rho(-1) \in \mathbb{R} \) then \( \rho \) satisfies the \( c \)-level condition for any \( g \geq 0 \).

(IV) If \( \rho \) is an \( r \)-functional, \( f \cdot g = 0, f \geq 0 \) then \( \rho(f + g) = \rho(g) + \rho(f) \).
Similarly, if \( \rho \) is an \( l \)-functional, \( f \cdot g = 0, f \leq 0 \) then \( \rho(f + g) = \rho(f) + \rho(g) \).

**Proof** (I) Suppose \( g \leq f \). Choose a constant \( c > 0 \) such that \( 0 \leq g + c \leq f + c \). Then \( \rho(g) + \rho(c) = \rho(g + c) \leq \rho(f + c) = \rho(f) + \rho(c) \), which gives \( \rho(g) \leq \rho(f) \). The monotonicity of an \( l \)-functional can be proved similarly.

(II) Use part (I) and Remark 4.5.

(III) Assume that \( f \leq c, f = c \) on \( \text{supp } g, g \geq 0 \). Then \( (f - c)g = 0, f - c \leq 0, \) and
\[
\rho(-c) + \rho(f + g) = \rho(f + g - c) = \rho(f - c) + \rho(g) = \rho(f) + \rho(-c) + \rho(g).
\]
Thus, \( \rho(f + g) = \rho(f) + \rho(g) \).

(IV) Let \( \rho \) be an \( r \)-functional, \( f \cdot g = 0, f \geq 0 \). Then \( (f + g^+)(-g^-) = 0, f \cdot g^+ = 0 \), and by Definition 4.14 \( \rho(f + g) = \rho(f + g^+ - g^-) = \rho(f + g^+) + \rho(-g^-) = \rho(f) + \rho(g^+) + \rho(-g^-) = \rho(f) + \rho(g) \).

\(\square\)
Remark 4.20 Part (IV) of Lemma 4.19 and part (iii) of Remark 4.17 were first observed for a compact space in [19, Propositions 17 and 18].

Definition 4.21 Let $\rho$ be a d-functional. Let $g \geq 0$. Define $\rho_g(f) = \rho(fg)$ for $f \in D(\rho)$.

The next lemma is easy to see.

Lemma 4.22 If $\rho$ is a d-, c-, r-, l-, s- functional or a linear functional, then so is $\rho_g$.

5 Deficient topological measures from d-functionals

Definition 5.1 Let $X$ be locally compact, and let $\rho$ be a d-functional with $C_c^+(X) \subseteq D(\rho) \subseteq C_b(X)$. Define a set function $\mu_{\rho} : \mathcal{O}(X) \cup \mathcal{C}(X) \to [0, \infty]$ as follows: for an open set $U \subseteq X$ let

$$\mu_{\rho}(U) = \sup \{ \rho(f) : f \in C_c(X), 0 \leq f \leq 1, \text{ supp } f \subseteq U \},$$

and for a closed set $F \subseteq X$ let

$$\mu_{\rho}(F) = \inf \{ \mu_{\rho}(U) : F \subseteq U, U \in \mathcal{O}(X) \}.$$

Note that Definition 5.1 is consistent for clopen sets.

Lemma 5.2 For the set function $\mu_{\rho}$ from Definition 5.1 the following holds:

1. $\mu_{\rho}$ is nonnegative.
2. $\mu_{\rho}$ is monotone.
3. Given an open set $U$, for any compact $K \subseteq U$

$$\mu_{\rho}(U) = \sup \{ \rho(g) : 1_K \leq g \leq 1, g \in C_c(X), \text{ supp } g \subseteq U \}.$$ 

4. $\mu_{\rho}(K) = \inf \{ \rho(g) : g \in C_c(X), g \geq 1_K \}$ for $K \in \mathcal{K}(X)$.
5. $\mu_{\rho}(K) = \inf \{ \rho(g) : g \in C_c(X), 1_K \leq g \leq 1 \}$ for $K \in \mathcal{K}(X)$.
6. Given $K \in \mathcal{K}(X)$, for any open $U$ such that $K \subseteq U$

$$\mu_{\rho}(K) = \inf \{ \mu_{\rho}(V) : V \in \mathcal{O}(X), K \subseteq V \subseteq \text{cl } V \subseteq U \}.$$ 

7. $\mu_{\rho}(U) = \sup \{ \mu_{\rho}(K) : K \in \mathcal{K}(X), K \subseteq U \}$ for $U \in \mathcal{O}(X)$.
8. $\mu_{\rho}(K \cup C) = \mu_{\rho}(K) + \mu_{\rho}(C)$ for disjoint compact sets $K$ and $C$.
9. Suppose the domain of $\rho$ contains constants (for example, $X$ is compact), $\rho(1) \in \mathbb{R}$, and $\rho$ is an s-functional satisfying the constant condition (4.1). If $K \subseteq U$, $K \in \mathcal{K}(X)$, $U \in \mathcal{O}(X)$ then

$$\mu_{\rho}(U) = \mu_{\rho}(K) + \mu_{\rho}(U \setminus K).$$
Proof  For part y1, $\mu_\rho$ is nonnegative since $\rho$ is a positive functional by Remark 4.2. Part y2 is easy to see. Proofs for parts y3 - y8 follow proofs of the corresponding parts of [7, Lemma 35, Sect. 4]. We shall show part y9.

Let $K \subseteq U$, $K \in \mathcal{K}(X)$, $U \in \mathcal{O}(X)$. First we shall show that

$$\mu_\rho(U \setminus K) + \mu_\rho(K) \geq \mu_\rho(U). \quad (5.1)$$

If $\mu_\rho(K) = \infty$, the inequality (5.1) trivially holds, so we assume that $\mu_\rho(K) < \infty$. By Lemma 2.8 let $V \in \mathcal{O}(X)$ with compact closure be such that $K \subseteq V \subseteq V \subseteq U$. For $\epsilon > 0$ choose $W_1 \in \mathcal{O}(X)$ such that $K \subseteq W_1 \subseteq V$ and $\mu_\rho(W_1) < \mu_\rho(K) + \epsilon$. Also, there exists $W \in \mathcal{O}(X)$ with compact closure such that $K \subseteq W \subseteq W \subseteq W_1 \subseteq V \subseteq U$. Choose an Urysohn function $g \in C_c(X)$ such that $1_{W_1} \leq g \leq 1$, supp $g \subseteq W_1$. Then

$$\rho(g) \leq \mu_\rho(W_1) < \mu_\rho(K) + \epsilon.$$  

First assume that $\mu_\rho(U) < \infty$. By part y3 choose $f \in C_c(X)$ such that $1_V \leq f \leq 1_U$, and

$$\rho(f) > \mu_\rho(U) - \epsilon.$$  

Note that $0 \leq f - g \leq 1$, and, since $f - g = 0$ on $\overline{W}$, we have supp $(f - g) \subseteq U \setminus K$. Since $f = 1$ on supp $g$ and $\rho$ is an s-functional, by Remark 4.12 and part (ii) of Lemma 4.4 $\rho(g - f) = \rho(g) + \rho(-f) = \rho(g) - \rho(f)$, so

$$\rho(f - g) = \rho(f) - \rho(g). \quad (5.2)$$

Then we have:

$$\mu_\rho(U \setminus K) \geq \rho(f - g) = \rho(f) - \rho(g) \geq \mu_\rho(U) - \epsilon - \mu_\rho(K) - \epsilon,$$

which gives us inequality (5.1). If $\mu_\rho(U) = \infty$, use instead of $f$ functions $f_n$ with $1_V \leq f_n \leq 1_U$, $\rho(f_n) \geq n$ in the above argument to show that $\mu_\rho(U \setminus K) = \infty$. Then inequality (5.1) holds.

Now we would like to show that

$$\mu_\rho(U) \geq \mu_\rho(U \setminus K) + \mu_\rho(K). \quad (5.3)$$

By monotonicity of $\mu_\rho$ it is enough assume that $\mu_\rho(U \setminus K), \mu_\rho(K) < \infty$. Given $\epsilon > 0$, choose $g \in C_c(X)$, $0 \leq g \leq 1$ such that $C = \text{supp } g \subseteq U \setminus K$ and

$$\rho(g) > \mu_\rho(U \setminus K) - \epsilon.$$  

Note that $K \subseteq U \setminus C$. If $\mu_\rho(U \setminus C) = \infty$, then $\mu_\rho(U) = \infty$, so (5.3) holds. So assume that $\mu_\rho(U \setminus C) < \infty$. By part y3 choose $f \in C_c(X)$ such that $1_K \leq f \leq
1. \( \text{supp } f \subseteq U \setminus C \), and \( \rho(f) > \mu_\rho(U \setminus C) - \epsilon \). Then

\[
\rho(f) > \mu_\rho(U \setminus C) - \epsilon \geq \mu_\rho(K) - \epsilon.
\]

Since \( fg = 0 \), \( f, g \geq 0 \), we have \( \rho(f + g) = \rho(f) + \rho(g) \). Since \( f + g \in C_c(X) \) with \( \text{supp } (f + g) \subseteq U \), we obtain:

\[
\mu_\rho(U) \geq \rho(f + g) = \rho(f) + \rho(g) \geq \mu_\rho(K) + \mu_\rho(U \setminus K) - 2\epsilon.
\]

Therefore, \( \mu_\rho(U) \geq \mu_\rho(U \setminus K) + \mu_\rho(K) \).

\( \square \)

**Remark 5.3** If \( \rho \) is a quasi-linear functional on a locally compact space then formula (5.2) holds by part (III) of Lemma 3.6, and we again obtain inequality (5.1). Our means of obtaining inequality (5.1) resembles one from [17, Theorem 3.9].

**Theorem 5.4** Suppose \( X \) is locally compact, \( \rho \) is a d-functional with \( C_c(X) \subseteq D(\rho) \subseteq C_b(X) \), and \( \mu_\rho \) defined in Definition 5.1. Then

(i) \( \mu_\rho \) is a deficient topological measure.

(ii) If \( \rho \) is real-valued on \( C_c(X) \), then \( \mu_\rho \) is compact-finite.

(iii) If \( \rho \) is bounded, then \( \mu_\rho \) is finite.

(iv) If \( X \) is compact and \( D(\rho) = C(X) \) then \( \mu_\rho(X) = \rho(1) \).

(v) If the domain of \( \rho \) includes constants, \( \rho(1) \in \mathbb{R} \), and \( \rho \) is an s-functional satisfying constant condition (4.1) then \( \mu_\rho \) is a topological measure.

(vi) If \( \rho \) is a quasi-linear functional then \( \mu_\rho \) is a topological measure.

**Proof** (i): Note that since \( \rho \) is not identically \( \infty \), then neither is \( \mu_\rho \). By part y1 of Lemma 5.2 \( \mu_\rho \) is nonnegative. Part y8 of Lemma 5.2 gives (DTM1) of Definition 2.3. Definition 5.1 and part y7 of Lemma 5.2 give regularity conditions (DTM2) and (DTM3) of Definition 2.3. Thus, \( \mu_\rho \) is a deficient topological measure. (ii) follows from part y4 of Lemma 5.2. (iii) and (iv) are evident from Definition 5.1. (v) follows from part y9 (or just inequality (5.1)) of Lemma 5.2 and Theorem 2.7. (vi) follows from Remark 5.3 and Theorem 2.7. \( \square \)

6 **Left and right measures**

Given a deficient topological measure and a bounded continuous function we may consider four distribution functions.

**Definition 6.1** Let \( \mu \) be a finite deficient topological measure on a locally compact space \( X \). Let \( f \in C_b(X) \). Define the following nonnegative functions on \( \mathbb{R} \):

\[
L_1(t) = L_{1,\mu, f}(t) = \mu(f^{-1}((-\infty, t])),
\]

\[
L_2(t) = L_{2,\mu, f}(t) = \mu(f^{-1}((-\infty, t])),
\]

\[
R_1(t) = R_{1,\mu, f}(t) = \mu(f^{-1}((t, \infty))),
\]

\[
R_2(t) = R_{2,\mu, f}(t) = \mu(f^{-1}([t, \infty))).
\]
Remark 6.2 For particular \( \mu \) and \( f \) to simplify notations we use \( L_1, L_2, R_1, R_2 \). When we need to emphasize the dependence on \( \mu \) and \( f \), we use notations \( L_{1,\mu}, f, R_{1,\mu} \) and so on. When we need to use, say, \( L_1 \) as a function of \( f \) we denote it by \( L_{1,f} \).

Lemma 6.3 Let \( \mu \) be a finite deficient topological measure on a locally compact space \( X \), \( f \in C_b(X) \). Let nonnegative real-valued functions \( L_1, L_2, R_1, R_2 \) be as in Definition 6.1. Then

I. Functions \( L_1, L_2 \) are non-decreasing; \( R_1, R_2 \) are non-increasing. If \( f(X) \subset [a, b] \) then

\[
L_1(a) = L_2(a^-) = 0; \quad L_1(b^+) = L_2(b) = \mu(X), \\
R_1(a^-) = R_2(a^-) = R_2(a) = \mu(X); \quad R_1(b) = R_2(b^+) = 0.
\]

II. \( L_1 \) is left-continuous, \( R_1 \) is right-continuous.

III. \( L_1(t^-) = L_2(t^-) = L_1(t) \) for any \( t \). If \( L_2 \) is left-continuous at \( t \) (in particular, continuous at \( t \)) then \( L_1(t) = L_2(t) \). Similarly, \( R_2(t^+) = R_1(t^+) = R_1(t) \) for any \( t \), and if \( R_2 \) is right-continuous at \( t \) then \( R_1(t) = R_2(t) \). In particular, the set of \( t \) where \( L_1(t) \neq L_2(t) \) and the set of \( t \) where \( R_1(t) \neq R_2(t) \) are, at most, countable sets.

IV. \( L_1(t) + R_1(t) \leq \mu(X) \) for every \( t \).

V. If \( X \) is compact, the function \( L_2 \) is right-continuous, and \( R_2 \) is left-continuous. If \( X \) is locally compact, \( f \in C_0(X) \), then \( R_2(t) \) is left-continuous at \( a \) and any \( t > 0 \), and \( L_2(t) \) is right-continuous at \( b \) and any \( t < 0 \). In particular, \( R_2 \) is left-continuous at all \( t \) except, possibly, \( t \in E \) for some countable set \( E \subset (-\infty, 0] \setminus \{a\} \) and \( L_2(t) \) is right-continuous at all \( t \) except, possibly, \( t \in E \) for some countable set \( E_1 \subset [0, \infty) \setminus \{b\} \).

Proof I.: Easy to see.

II.: The sets \( U_s = f^{-1}((-\infty, s]) \) are open, \( U_s \not\supset U_t \) as \( s \to t^- \), so by Lemma 2.9 \( L_1 \) is left-continuous. The argument for \( R_1 \) is similar.

III.: Let \( s < t \). Then \( f^{-1}((-\infty, s]) \subset f^{-1}((-\infty, s)) \subset f^{-1}((-\infty, t)) \), and so \( L_1(t^-) \leq L_2(t^-) \leq L_1(t) \leq L_2(t) \). By left-continuity of \( L_1 \) we have \( L_1(t^-) = L_2(t^-) = L_1(t) \); if \( L_2 \) is left-continuous at \( t \) then \( L_1(t) = L_2(t) \). Similarly for \( R_1 \) and \( R_2 \).

IV.: The sets \( f^{-1}((-\infty, t)) \) and \( f^{-1}((t, \infty)) \) are disjoint open sets, so from superadditivity of \( \mu \) we see that \( L_1(t) + R_1(t) \leq \mu(X) \) for every \( t \).

V.: If \( X \) is compact, the sets \( C_a = f^{-1}((a, \infty]) \) are compact. From Lemma 2.9 it follows that \( R_2 \) is left-continuous. If \( X \) is locally compact and \( f \in C_0(X) \) then the sets \( K_a = f^{-1}((a, \infty)) \), \( a > 0 \) are compact. From Lemma 2.9 it follows that \( R_2 \) is left-continuous at any \( t > 0 \). The assertions about \( L_2 \) are proved similarly.

Remark 6.4 Let \( \mu \) be a finite deficient topological measure on a locally compact space \( X \). Let \( f \in C_0(X) \) with \( f(X) \subset [a, b] \). By Theorem 2.10 and part I of Lemma 6.3 the Riemann–Stieltjes integral

\[
\int_a^b id \, dL_1 = \int_a^b L_1(t)dt + L_1(b^+)b = \int_a^b L_1(t)dt + b\mu(X).
\]
Let \( l \) be the Lebesgue–Stieltjes measure associated with \( L_1 \), so \( l \) is a regular Borel measure on \( \mathbb{R} \). By part (III) of Lemma 6.3 we see that
\[
\int_{[a,b]} id\,dl = \int_a^b id\,dL_1 = -\int_a^b L_1(t)dt + b\mu(X) = -\int_a^b L_2(t)dt + b\mu(X).
\]

Let \( r \) be the Lebesgue–Stieltjes measure associated with \(-R_1\), a regular Borel measure on \( \mathbb{R} \). We have:
\[
\int_{[a,b]} id\,dr = \int_a^b id\,d(-R_1) = \int_a^b R_1(t)dt + a\mu(X) = \int_a^b R_2(t)dt + a\mu(X).
\]

Definition 6.5 We call \( l \) the left measure and \( r \) the right measure. When the right and left measures are equal, we set \( m = r = l \).

Remark 6.6 The measures \( r \) and \( l \) arise from functions \( R_1 = R_{1,\mu,f} \) and \( L_1 = L_{1,\mu,f} \). We use notations \( R_{1,f}, r_f, l_f \) when we need to emphasize the dependence of \( R_1 \) and measures \( r, l \) on the function \( f \). If we want to use measures \( r \) and \( l \) as functions of \( f \) and \( \mu \), we write \( r_{f,\mu}, l_{f,\mu} \).

When \( \mu \) is a topological measure, measure \( m \) is equal to \( \mu_f \) in [17] and \( m_f \) in [7]. See [7, Remark 28, Sect. 3].

Theorem 6.7 Let \( \mu \) be a finite deficient topological measure on a locally compact space \( X \), and let \( f \in C_b(X) \).

(I) There are regular Borel measures \( r \) and \( l \) on \( \mathbb{R} \) such that \( \text{supp } r \subseteq \overline{f(X)}, \text{supp } l \subseteq \overline{f(X)}, \ell(\mathbb{R}) = \mu(X), r(\mathbb{R}) = \mu(X), \)
\[
\begin{align*}
& r((t, \infty)) = \mu(f^{-1}((t, \infty))) \text{ for all } t, \\
& r([t, \infty)) = \mu(f^{-1}([t, \infty))) \text{ for all } t \notin E,
\end{align*}
\]
where \( E \subseteq (-\infty, 0] \) is a countable set from part V of Lemma 6.3,
\[
\begin{align*}
& l((-\infty, t)) = \mu(f^{-1}((-\infty, t))) \text{ for all } t, \\
& l((-\infty, t]) = \mu(f^{-1}((-\infty, t])) \text{ for all } t \notin E_1,
\end{align*}
\]
where \( E_1 \subseteq [0, \infty) \) is a countable set from part V of Lemma 6.3.

(II) For any open or closed set \( A \subseteq \mathbb{R} \)
\[
\mu(f^{-1}(A)) \leq l(A), \quad \mu(f^{-1}(A)) \leq r(A).
\]

Proof Let \( \overline{f(X)} = [a, b] \).

(I) By Lemma 6.3 \( L_1 \) is left-continuous, so \( l((-\infty, t)) = L_1(t) = \mu(f^{-1}((-\infty, t))) \) for every \( t \). Next, using Lemma 2.9 we have \( l(\mathbb{R}) = \lim_{n \to \infty} l((-\infty, n)) = \lim_{n \to \infty} \mu(f^{-1}((-\infty, n))) = \mu(f^{-1}(\mathbb{R})) = \mu(X). \)
If $t \notin E_1$, then $L_2$ is right-continuous at $t$. Since $L_2 = L_1$ outside of a countable set, $l((−∞, t)) = \lim_{s \to t+} l((−∞, s)) = \lim_{s \to t+} L_1(s) = \lim_{s \to t+} L_2(s) = L_2(t) = \mu(f^{-1}((−∞, t))).$

Since $L_1$ is constant on $(−∞, a)$ and on $(b, ∞)$, we see that $l((−∞, a)) = l((b, ∞)) = 0$. It follows that $supp l \subseteq [a, b] = f(X)$. The statements for $r$ can be proved similarly.

(II) Let $(a, b) \subseteq \mathbb{R}$, $b \notin E$. By the superadditivity of $\mu$ and part (I) $\mu(f^{-1}((a, b)) \leq \mu(f^{-1}((a, ∞))) - \mu(f^{-1}([b, ∞))) = r((a, ∞)) - r([b, ∞)) = r((a, b))$. For $(a, b)$ with $b \in E$, choose $b_n \notin E$ such that $(a, b_n) \not\subset (a, b)$. Since by Lemma 2.9 $\mu \circ f^{-1}$ and $r$ are both $\tau$-smooth on open sets, we have $\mu(f^{-1}((a, b))) \leq r((a, b))$ for any $(a, b)$. We see that $\mu(f^{-1}(J)) \leq r(J)$ for any finite or infinite open interval $J$. Then the same inequality holds for any open set $W \subseteq \mathbb{R}$.

Now let $C \subseteq \mathbb{R}$ be closed. Then $\mu(f^{-1}(C)) \leq \inf\{\mu(f^{-1}(W) : C \subseteq W, W \in \mathcal{O}(X)) \leq \inf\{r(W) : C \subseteq W, W \in \mathcal{O}(X)) = r(C)$. The statements for the left measure $l$ can be proved in a similar way.

**Theorem 6.8** Let $\mu$ be a finite deficient topological measure on a locally compact space. Then $r = l$ iff $L_1(t) + R_1(t) = \mu(X)$ for a.e. $t$ with respect to the Lebesque measure.

**Proof** Using Remark 6.4 and part IV of Lemma 6.3 we note that

$$r = l \implies \int_a^b id dL_1 = \int_a^b id d(-R_1) \iff \int_a^b R_1 dx + a \mu(X)$$

$$= -\int_a^b L_1 dx + b \mu(X)$$

$$\iff \int_a^b (L_1 + R_1) dx = (b - a) \mu(X) \iff L_1 + R_1 = \mu(X) \text{ a.e.},$$

where a.e. is with respect to the Lebesque measure.

Conversely, let $L_1(t) + R_1(t) = \mu(X)$ for $t \notin D$, where $\lambda(D) = 0$. We may assume that $D$ contains sets $E$, $E_1$ from part V of Lemma 6.3 and all points where $L_1 \neq L_2$, $R_1 \neq R_2$. If $[a, b] \subseteq \mathbb{R}$ and $a, b \notin D$ then by part (I) of Theorem 6.7:

$$l([a, b]) = l((−∞, b)) - l((−∞, a)) = L_2(b) - L_1(a)$$

$$= \mu(X) - R_1(b) - \mu(X) + R_2(a) = r((a, ∞)) - r([b, ∞)) = r([a, b]).$$

An arbitrary interval $(a, b)$ can be written as $\bigcup_{n=1}^{\infty} [a_n, b_n]$, where intervals $[a_n, b_n]$ are ordered by inclusion, and $a_n, b_n \notin D$. It follows that measures $l = r$ on $\mathbb{R}$.

**Theorem 6.9** Let $\mu$ be a finite topological measure on a locally compact space $X$. Let $f \in C_0(X)$. Then for the right and left measures $r, l$ we have $r = l$.

**Proof** Let $f(X) \subseteq [a, b]$. By Theorem 6.8 it is enough to show that there is a countable set $D$ such that $L_1(t) + R_1(t) = \mu(X)$ for all $t \in \mathbb{R} \setminus D$. Let $D$ be the countable (by part III of Lemma 6.3) set consisting of 0 and all points where $L_1 \neq L_2$, $R_1 \neq R_2$.
If \( t > 0 \) then \( f^{-1}([t, \infty)) \) is compact, and it follows from (TM1) of Definition 2.5 that \( L_1(t) + R_2(t) = \mu(X) \). \( R_1(t) = R_2(t) \) for \( t \in (0, \infty) \setminus D \), so \( L_1(t) + R_1(t) = L_1(t) + R_2(t) = \mu(X) \). Similarly, for all \( t \in (-\infty, 0) \setminus D \) we have \( L_1(t) + R_1(t) = L_2(t) + R_1(t) = \mu(X) \). \( \square \)

**Theorem 6.10** Let \( \mu \) be a finite topological measure on a locally compact space \( X \). Consider the regular Borel measure \( m = r = l \).

(I) If \( f \in C_0(X) \), then \( m(A) = \mu(f^{-1}(A)) \) for any open set \( A \subseteq \mathbb{R} \) and any closed set \( A \subseteq \mathbb{R} \setminus \{0\} \).

(II) If \( X \) is compact, \( f \in C(X) \) then \( m(A) = \mu(f^{-1}(A)) \) for any open or closed set \( A \subseteq \mathbb{R} \).

**Proof** (I) First let \( (a, b) \in \mathbb{R}, b \neq 0 \). Note that in

\[
f^{-1}((a, \infty)) = f^{-1}((a, b)) \cup f^{-1}([b]) \cup f^{-1}([b, \infty)),
\]

all the sets are open except for the middle set on the right hand side, which is compact since \( f \in C_0(X) \). Applying \( \mu \) we obtain

\[
R_1(a) = \mu(f^{-1}((a, b))) + \mu(f^{-1}([b])) + \mu(f^{-1}([b, \infty))) \quad (6.1)
\]

Since \( f^{-1}([b, \infty)) \cup f^{-1}([b]) \subseteq f^{-1}((t, \infty)) \) for any \( t < b \), by superadditivity (see Lemma 2.9) we have: \( \mu(f^{-1}([b])) + \mu(f^{-1}((b, \infty))) \leq \mu(f^{-1}((t, \infty))) = R_1(t) \). Thus, from (6.1) we see that \( R_1(a) \leq \mu(f^{-1}((a, b))) + R_1(t) \). As \( t \to b^- \) we have: \( m((a, b)) = R_1(a) - R_1(b^-) \leq \mu(f^{-1}((a, b))) \). Together with part (II) of Theorem 6.7 we obtain \( m((a, b)) = \mu(f^{-1}((a, b))) \) for any interval \( (a, b), b \neq 0 \). An interval \( (a, 0) = \cup_{n=1}^{\infty} (a, -\frac{1}{n}) \). Since both \( m \) and \( \mu \) are \( \tau^- \)-smooth and additive on open sets (see Lemma 2.9), the result holds for any finite open interval in \( \mathbb{R} \), and then for any open set in \( \mathbb{R} \). Below in (III) we shall prove that \( \mu(f^{-1}(C)) = m(C) \) for closed sets in \( \mathbb{R} \setminus \{0\} \).

(II) The set \( f^{-1}([d, \infty)) \) is compact for every \( d \), and the argument as in part (I) shows that \( m(W) = \mu(f^{-1}(W)) \) for every open set \( W \subseteq \mathbb{R} \).

Now let \( C \subseteq \mathbb{R} \) be closed. Choose \( W \in \mathcal{O}(X) \) such that \( C \subseteq W \). Since \( f^{-1}(C) \) is compact and \( \mu \) is a topological measure, \( m(W) = \mu(f^{-1}(C)) = \mu(f^{-1}(W)) - \mu(f^{-1}(C)) = \mu(f^{-1}(W \setminus C)) = m(W \setminus C) \). Thus \( \mu(f^{-1}(C)) = m(C) \).

(III) Now we shall finish the proof of part (I). Let \( C \subseteq \mathbb{R} \setminus \{0\} \) be closed. Set \( C_1 = C \cap (0, \infty) \) and \( C_2 = C \cap (-\infty, 0) \). We have \( C = C_1 \cup C_2 \), and \( b = \inf C_1 > 0 \). Since \( f^{-1}(C_1) \subseteq f^{-1}([b, \infty)) \), the set \( f^{-1}(C_1) \) is compact. An argument similar to the one in part (III) shows that \( \mu(f^{-1}(C_1)) = m(C_1) \). Also, \( \mu(f^{-1}(C_2)) = m(C_2) \), and so by finite additivity of \( \mu \) and \( m \) on compact sets \( \mu(f^{-1}(C)) = m(C) \). \( \square \)

**Lemma 6.11** Let \( \mu \) be a finite deficient topological measure on a locally compact space \( X \).

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I. If $\mu$ is a simple deficient topological measure, then measures $r$ and $l$ are point masses, $l = \delta_a$, $r = \delta_b$, $b \leq a$, where

$$a = \inf\{t : L_1(t) = 1\} = \sup\{s : L_1(s) = 0\},$$

$$b = \inf\{t : R_1(t) = 0\} = \sup\{s : R_1(s) = 1\}.$$ 

II. If $X$ is compact and $f = c$ is a constant function, then the measure $m = \mu(X)\delta_c$, where $\delta_c$ is a point mass at $c$.

**Proof** I. Since $\mu$ is simple, the non-decreasing function $L_1$ assumes only two values, and has single discontinuity at $a = \inf\{t : L_1(t) = 1\} = \sup\{s : L_1(s) = 0\}$. Since $l(a) = L_1(a^+) - L_1(a^-) = 1$, we see that $l = \delta_a$. Similarly, $r = \delta_b$, where $b = \inf\{t : R_1(t) = 0\} = \sup\{s : R_1(s) = 1\}$.

If $a < b$ then $L_1(t) + R_1(t) = 2$ on interval $(a, b)$, which contradicts part IV of Lemma 6.3. Thus, $b \leq a$. II. We have $R_1(t) = \mu(X)$ for every $t < c$, and $R_1(t) = 0$ for every $t \geq c$. Then $m(\{c\}) = r(\{c\}) = R_1(c^-) - R_1(c^+) = \mu(X)$. □

**Example 6.12** Let $X = \mathbb{R}$, $D = [0, 1]$ and $\mu$ be a simple deficient topological measure as in [6, Example 46, Sect. 6], i.e. $\mu(A) = 1$ if $D \subseteq A$ and $\mu(A) = 0$ otherwise, where $A \in \mathcal{B}(X) \cup \mathcal{K}(X)$. Consider the following $f \in C_0(X) : f(0) = 1$, $f(t) = 0$ for $t \in (-\infty, -1] \cup [1, \infty)$, and $f$ is linear on $[-1, 0]$ and $[0, 1]$. Note that $D \subseteq f^{-1}((-\infty, t))$ iff $t > 1$, and $D \subseteq f^{-1}((t, \infty))$ iff $t < 0$. Thus, $L_1$ and $R_1$ are characteristic functions $1_{(1, \infty)}$, and $1_{(-\infty, 0)}$, respectively. For measures $l$, $r$ we have $l = \delta_1$ and $r = \delta_0$.

**Example 6.13** Let $X = \mathbb{R}$, the family $\mathcal{E} = \{D = [1, 2]\}$, $\lambda_0 = \delta_{3/2}$. Let $\mu = \lambda^+$ as in [6, Example 47, Sect. 6]. Consider the following $f \in C_0(X) : f(0) = 2$, $f(t) = 1$ for $t \in [1, 2]$, $f(t) = 0$ for $t \in (-\infty, -1] \cup [3, \infty)$, and $f$ is linear on $[-1, 0]$, $[0, 1]$, and $[2, 3]$. Note that $D \subseteq f^{-1}((\infty, t))$ iff $t > 1$, and $D \subseteq f^{-1}((t, \infty))$ iff $t < 1$. Then $L_1 = 1_{(1, \infty)}$, and $R_1 = 1_{(-\infty, 1)}$, and $l = r = \delta_1$.

**Remark 6.14** In Theorem 6.7 it is stated that $supp \ l$, $supp \ r \subseteq f(X)$. In Example 6.12 and Example 6.13 $supp \ l$ and $supp \ r$ are properly contained in $f(X)$. On the other hand, from part II of Lemma 6.11 we see that it is also possible to have $supp \ l = supp \ r = f(X)$.

**Remark 6.15** From part IV of Lemma 6.3 we know that $L_1(t) + R_1(t) \leq \mu(X)$. Although $L_1(t) + R_1(t) = \mu(X)$ a.e. when $\mu$ is a finite topological measure (see Theorems 6.8 and 6.9), for deficient topological measures we may have both situations: in Example 6.13 $L_1(t) + R_1(t) = \mu(X)$ a.e., but in Example 6.12 we have $L_1(t) + R_1(t) = 0 < \mu(X)$ for $t \in (0, 1)$.

In Lemma 6.11 we have $b \leq a$. Examples 6.12 and 6.13 show that both situations $b < a$ and $b = a$ are possible. These examples also show that when $\mu$ is a deficient topological measure, we can have both situations for measures $l$ and $r$ induced by $\mu$ and a given function $f$: when $l = r$ and when $l \neq r$. 

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7 Functionals from deficient topological measures

When $\mu$ is a finite deficient topological measure (not a topological measure) the measures $r, l$ are not equal in general, and we consider two different integrals:

$$\int_{[a,b]} id \, dl \quad \text{and} \quad \int_{[a,b]} id \, dr.$$ 

**Definition 7.1** Let $\mu$ be a finite deficient topological measure on a locally compact space $X$, and let measures $r = r_{f,\mu}, l = l_{f,\mu}, m = m_{f,\mu}$ be as in Definition 6.5, Remarks 6.4 and 6.6. Define the following functionals on $C_b(X)$:

$$\mathcal{R}(f) = \mathcal{R}_\mu(f) = \int_\mathbb{R} id \, dl, \quad \mathcal{L}(f) = \mathcal{L}_\mu(f) = \int_\mathbb{R} id \, dl,$$

and

$$\rho(f) = \rho_\mu(f) = \int_\mathbb{R} id \, dm.$$

**Remark 7.2** By Theorem 6.7 $\text{supp } r, \text{supp } l \subseteq \overline{f(X)}$, so for any $[a, b]$ containing $f(X)$

$$\mathcal{R}(f) = \int_\mathbb{R} id \, dl = \int_{[a,b]} id \, dl, \quad \mathcal{L}(f) = \int_{[a,b]} id \, dl, \quad \rho(f) = \int_{[a,b]} id \, dm.$$

With functions $L_1, L_2, R_1, R_2$ as in Definition 6.1 by Remark 6.4 we have:

$$\mathcal{L}(f) = \int_{[a,b]} id \, dl = \int_{[a,b]} id \, dl$$

$$= -\int_a^b L_1(t) dt + b \mu(X) = -\int_a^b L_2(t) dt + b \mu(X). \quad (7.1)$$

$$\mathcal{R}(f) = \int_\mathbb{R} id \, dl = \int_{[a,b]} id \, dl$$

$$= \int_a^b R_1(t) dt + a \mu(X) = \int_a^b R_2(t) dt + a \mu(X). \quad (7.2)$$

If $f = c$ is a constant function then

$$\mathcal{R}(f) = \mathcal{L}(f) = c \mu(X). \quad (7.3)$$

If $f(X) \subseteq [0, b]$ we have:

$$\mathcal{R}(f) = \int_{[0,b]} id \, dl = \int_0^b R_1(t) dt = \int_0^b R_2(t) dt. \quad (7.4)$$
When \( r = l \) (in particular, when \( \mu \) is a topological measure) and \( f(X) \subseteq [0, b] \) we have

\[
\rho(f) = \int_{[0,b]} id \, dm = \int_0^b R(t) \, dt = \int_0^b R_2(t) \, dt.
\]

Similarly, if \( f(X) \subseteq [a, 0] \) we have:

\[
\mathcal{L}(f) = \int_{[a,0]} id \, dl = -\int_a^0 L_1(t) \, dt = -\int_a^0 L_2(t) \, dt.
\] (7.5)

When \( r = l \) (in particular, when \( \mu \) is a topological measure) and \( f(X) \subseteq [a, 0] \) we have

\[
\rho(f) = \int_{[a,0]} id \, dm = -\int_a^0 L_1(t) \, dt = -\int_a^0 L_2(t) \, dt.
\]

**Remark 7.3** Note that when \( \mu \) is a topological measure, we obtain familiar formulas. See, for example, [17, Proposition 3.7] and [7, Remark 43, Section 5]. These results were, in turn, generalizations of results first obtained by J. F. Aarnes for compact spaces in [3]. For example, when \( X \) is compact and \( \mu(X) = 1 \), formula (7.2) gives [3, formula (3.3)].

**Remark 7.4** We have the connection between \( \mathcal{R} \) and \( \mathcal{L} \) (which is the same as noted in [19, p. 739]). We use notations as in Remark 6.6. Observe that \( R_{1,f}(-t) = L_{1,-f}(t) \). Thus, \( l_{-f} = r_f \circ T^{-1} \), where \( T(t) = -t \) for \( t \in \mathbb{R} \). Then \( \int_{\mathbb{R}} id \, dl_{-f} = -\int_{\mathbb{R}} id \, dr_f \), i.e.

\[
\mathcal{L}(-f) = -\mathcal{R}(f).
\] (7.6)

We may prove results for \( \mathcal{R} \) and obtain similar results for \( \mathcal{L} \) by analogy (as we did, for example, in Theorem 6.7) or using relation (7.6).

**Definition 7.5** Let \( \mathcal{R} \) be the functional as in Definition 7.1. We call the functional \( \mathcal{R} \) a quasi-integral (with respect to a deficient topological measure \( \mu \)) and write:

\[
\int_X f \, d\mu = \mathcal{R}(f) = \mathcal{R}_\mu(f) = \int_{\mathbb{R}} id \, dr.
\]

**Remark 7.6** If \( \mu \) is a topological measure on \( X \), by Definitions 7.1 and 6.5 we obtain exactly the quasi-integral in [7, Definition 27, Section 3].

**Lemma 7.7** Let \( \mathcal{L}, \mathcal{R} \) be the two functionals introduced in Definition 7.1.

(i) \( \mathcal{R} \) is orthogonally additive on nonnegative functions, and \( \mathcal{L} \) is orthogonally additive on nonpositive functions.

(ii) \( \mathcal{R}(0) = 0 \) and \( \mathcal{L}(0) = 0 \).
(iii) $\mathcal{L}$, $\mathcal{R}$ are positive-homogeneous functionals.
(iv) $\mathcal{L}$, $\mathcal{R}$ are monotone. In particular, $\mathcal{L}$, $\mathcal{R}$ are positive.

**Proof** We use notations as indicated in Remark 6.6.

(i): Let $f, g \geq 0$ and $fg = 0$. Say, $f(X), g(X) \subseteq [0, b]$. For any $t > 0$ observe that $(f + g)^{-1}((t, \infty)) = f^{-1}((t, \infty)) \cup g^{-1}((t, \infty))$, so by additivity of a deficient topological measure on disjoint open sets we immediately obtain $R_{1,f+g}(t) = R_{1,f}(t) + R_{1,g}(t)$. Since $(f + g)(X) \subseteq [0, b]$, from (7.4) we have $\mathcal{R}(f + g) = \mathcal{R}(f) + \mathcal{R}(g)$. Thus, $\mathcal{R}$ is orthogonally additive on nonnegative functions. Then orthogonal additivity of $\mathcal{L}$ on nonpositive functions follows from (7.6).

(ii): Follows from part (i) or from (7.4) and (7.5).

(iii): If $c = 0$ then from (ii) we see that $\mathcal{R}(cf) = \mathcal{L}(cf) = 0$. Let $c > 0$. Since $L_{1,cf}(t) = L_{1,f}(t/c)$, from (7.1) we see that $\mathcal{L}(cf) = c\mathcal{L}(f)$. One can show that $\mathcal{R}$ is also positive-homogeneous in a similar way using (7.2) or using positive-homogeneity of $\mathcal{L}$ together with formula (7.6).

(iv): Suppose that $f \leq g$. Choose an interval $[a, b]$ which contains both $f(X)$ and $g(X)$. Since $L_{1,f} \geq L_{1,g}$ and $R_{1,f} \leq R_{1,g}$, from (7.1) and (7.2) we see that $\mathcal{L}(f) \leq \mathcal{L}(g)$ and $\mathcal{R}(f) \leq \mathcal{R}(g)$.

**Lemma 7.8** If $h = \phi \circ f \in A^+(f)$ and $[a, b]$ is any interval containing $f(X)$ then

$$\mathcal{R}(h) = \int_{[a,b]} \phi \, df_f.$$ 

Similarly, if $h = \phi \circ f \in A^-(f)$ then

$$\mathcal{L}(h) = \int_{[a,b]} \phi \, dl_f.$$ 

**Proof** Let $r_f$, $r_h$ be the right measures for functions $f$ and $h$ as in Theorem 6.7, $r_f$ is supported on $f(X) \subseteq [a, b]$. Since $\phi$ is nondecreasing, for any interval $(t, \infty)$ we have $\phi^{-1}((s, \infty)) = (s, \infty)$, where $s = \min\{y \in [a, b] : \phi(y) = t\}$. (This is similar to [19, Proposition 13(1)].) Then $r_h((t, \infty)) = \mu(h^{-1}((t, \infty))) = \phi^{-1}(\phi^{-1}((t, \infty))) = \phi^{-1}((s, \infty))) = r_f((s, \infty)) = (r_f \circ \phi^{-1})(t, \infty)$). Thus, $r_h = r_f \circ \phi^{-1}$ are equal as measures. Using formula (7.2) we have: $\mathcal{R}(h) = \int_{\mathbb{R}} i d r_h = \int_{[a,b]} \phi \, df_f$. The formula $\mathcal{L}(h) = \int_{[a,b]} \phi \, dl_f$ can be proved in a similar way.

**Lemma 7.9** The functional $\mathcal{R}$ is conic-linear on each cone $A^+(f)$, and the functional $\mathcal{L}$ is conic-linear on each cone $A^-(f)$.

**Proof** Suppose $h, g \in A^+(f)$, $h = \phi \circ f$, $g = \psi \circ f$, $f(X) \subseteq [a, b]$. Applying Lemma 7.8 we have:

$$\mathcal{R}(h + g) = \mathcal{R}(\phi \circ f + \psi \circ f) = \mathcal{R}((\phi + \psi) \circ f) = \int_{[a,b]} (\phi + \psi) \, df_f$$

$$= \int_{[a,b]} \phi \, df_f + \int_{[a,b]} \psi \, df_f = \mathcal{R}(h) + \mathcal{R}(g).$$

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Since by Lemma 7.7 \( \mathcal{R} \) is also positive-homogeneous, we see that \( \mathcal{R} \) is conic-linear on \( A^+(f) \) for each \( f \). The statements for \( \mathcal{L} \) can be proved similarly. \( \square \)

**Theorem 7.10**  (i) The functional \( \mathcal{R} \) is a \( p \)-conic quasi-linear functional, and the functional \( \mathcal{L} \) is an \( n \)-conic quasi-linear functional.

(ii) The functional \( \mathcal{R} \) is an \( r \)-functional, and \( \mathcal{L} \) is an \( l \)-functional.

**Proof**  (i) Follows from Lemmas 7.7 and 7.9. (ii): apply Lemma 4.18. \( \square \)

**Remark 7.11**  When \( \mu \) is a topological measure, the functional \( \rho = \rho_\mu \) is a quasi-linear functional (see [7, Theorem 30, Section 3]), so \( \rho \) is linear on each singly generated subalgebra. Theorem 7.10 gives an analog of this for the case when \( \mu \) is a deficient topological measure: if \( \mu \) is a deficient topological measure, then the functional \( \mathcal{R} \) obtained from \( \mu \) is \( p \)-conic linear, so, in particular, it is conic-linear on the cone \( A^+(f) \) for each \( f \).

The next lemma shows properties that relate \( \mathcal{R} \) and \( \mu \).

**Lemma 7.12**  Let \( \mu \) be a finite deficient topological measure, and \( \mathcal{R}, \mathcal{L} \) be functionals on \( C_0(X) \) obtained from \( \mu \) as in Definition 7.1.

\( z1. \) If \( U \in \mathcal{O}(X) \) and \( f \in C(X) \) is such that \( \text{supp} \ f \subseteq U, \ 0 \leq f \leq 1 \), then \( \mathcal{R}(f) \leq \mu(U) \).

\( z2. \) If \( C \in \mathcal{C}(X) \) and \( f \in C(X) \) is such that \( 0 \leq f \leq 1, \ f = 1 \) on \( C \), then \( \mathcal{R}(f) \geq \mu(C) \).

\( z3. \) For any \( f \in C_0(X) \)

\[
\mu(X) \cdot \inf_{x \in X} f(x) \leq \mathcal{R}(f) \leq \mu(X) \cdot \sup_{x \in X} f(x),
\]

\[
\mu(X) \cdot \inf_{x \in X} f(x) \leq \mathcal{L}(f) \leq \mu(X) \cdot \sup_{x \in X} f(x)
\]

Hence, \( \|\mathcal{R}\|, \|\mathcal{L}\| \leq \mu(X) \).

\( z4. \) \( \mu(U) = \sup\{\mathcal{R}(f) : f \in C(X), 0 \leq f \leq 1, \text{supp} \ f \subseteq U\} \) for \( U \in \mathcal{O}(X) \).

\( z5. \) If \( K \in \mathcal{K}(X) \) then

\[
\mu(K) = \inf\{\mathcal{R}(g) : g \in C(X), g \geq 1\}
\]

\[
= \inf\{\mathcal{R}(g) : g \in C(X), g \geq 1, 0 \leq g \leq 1\}
\]

\[
= \inf\{\mathcal{R}(g) : g \in C_0(X), g \geq 1\}
\]

\[
= \inf\{\mathcal{R}(g) : g \in C_0(X), g \geq 1, 0 \leq g \leq 1\}.
\]

\( z6. \) \( \|\mathcal{R}\| = \mu(X) = \|\mathcal{L}\|, \) so \( |\mathcal{R}(f)| \leq \|\mathcal{R}\| \|f\| \) and \( |\mathcal{L}(f)| \leq \|\mathcal{L}\| \|f\| \).

\( z7. \) If \( f, g \in C(X), f, g \geq 0, \text{supp} \ f, \text{supp} \ g \subseteq K \) where \( K \) is compact, then

\[
|\mathcal{R}(f) - \mathcal{R}(g)| \leq \|f - g\| \mu(K).
\]

If \( X \) is compact we also have:

(i) If \( c \) is a constant then \( \mathcal{R}(c) = c\mu(X) \) and, hence, \( \mathcal{R}(f + c) = \mathcal{R}(f) + c\mu(X) \).
(ii) For any functions \( f, g \in D(\rho) \)
\[
|\mathcal{R}(f) - \mathcal{R}(g)| \leq \mathcal{R}(1)\|f - g\| = \mu(X)\|f - g\|.
\]

Proof z1. Since \( R_1(t) \leq \mu(U) \) for \( t \in [0, 1] \), the statement follows from (7.4).
z2. Since \( R_2(t) \geq \mu(f^{-1}([1, \infty))) \geq \mu(C) \) for \( t \in [0, 1] \), the statement follows from (7.4).
z3. Let \( a = \inf_{x \in X} f(x) \), \( b = \sup_{x \in X} f(x) \). It is enough to assume \( a \notin E \) (see part (I) of Theorem 6.7), for otherwise we may take \( a_n \nrightarrow a, a_n \notin E \). Then
\[
a\mu(X) = a\mu(f^{-1}([a, \infty))) = ar([a, \infty)) = ar([a, b])
= \int_{[a,b]} id\,dr = \mathcal{R}(f) \leq br([a, \infty)) = b\mu(X).
\]
Because of formula (7.6), the statement for \( L(f) \) also holds.
z4. By part z1, \( \sup\{\mathcal{R}(f) : f \in C_c(X), 0 \leq f \leq 1, \text{supp } f \subseteq U \} \leq \mu(U) \). For \( \epsilon > 0 \) choose \( K \in \mathcal{K}(X) \) such that \( \mu(U) - \mu(K) < \epsilon \). Pick \( f \in C_c(X) \) such that \( 0 \leq f \leq 1 \), \( \text{supp } f \subseteq U \), \( f \equiv 1 \) on \( K \). Then by part z2 \( \mathcal{R}(f) \geq \mu(K) \geq \mu(U) - \epsilon \).

It follows that \( \sup\{\mathcal{R}(f) : f \in C_c(X), 0 \leq f \leq 1, \text{supp } f \subseteq U \} = \mu(U) \).
z5. We shall show the first equality; the rest are similar. Take \( U \in \mathcal{O}(X) \) containing \( K \).
Taking an Urysohn function \( f \in C_c(X) \) such that \( 1_K \leq f \leq 1_U \) and using part z1 we see that \( \inf\{\mathcal{R}(g) : g \in C_c(X), g \geq 1_K \} \leq \mathcal{R}(f) \leq \mu(U) \). Taking the infimum over all open sets containing \( K \) we have \( \inf\{\mathcal{R}(g) : g \in C_c(X), g \geq 1_K \} \leq \mu(K) \).

To prove the opposite inequality, take any \( g \in C_c(X) \) such that \( g \geq 1_K \). Let \( 0 < \delta < 1 \).
Let \( U = \{x : g(x) > 1 - \delta\} \). Then \( U \) is open and \( K \subseteq U \). Consider function \( h = \inf\{g, 1 - \delta\} \in C_c(X) \). Since \( 0 \leq h \leq g \) we have \( \mathcal{R}(h) \leq \mathcal{R}(g) \). Because
\[
\frac{h}{1-\delta} = 1 \text{ on } U, \text{ for any function } f \in C_c(X), 0 \leq f \leq 1, \text{supp } f \subseteq U \text{ we have }
\]
\[
f \leq \frac{h}{1-\delta} \text{ and so by parts (iii) and (iv) of Lemma 7.7 }
\]
\[
\mathcal{R}(f) \leq \mathcal{R}\left(\frac{h}{1-\delta}\right) = \frac{\mathcal{R}(h)}{1-\delta} \leq \frac{\mathcal{R}(g)}{1-\delta}.
\]
Then
\[
\mu(K) \leq \mu(U) = \sup\{\mathcal{R}(f) : f \in C_c(X), 0 \leq f \leq 1_U, \text{supp } f \subseteq U \} \leq \frac{\mathcal{R}(g)}{1-\delta}.
\]
Thus, for any \( g \in C_c(X) \) such that \( g \geq 1_K \) and any \( 0 < \delta < 1 \)
\[
(1-\delta)\mu(K) \leq \mathcal{R}(g).
\]
Therefore, \( \mu(K) \leq \inf\{\mathcal{R}(g) : g \in C_b(X), g \geq 1_K\} \).
z6. By part z4 we see that \( \mu(X) \leq \|\mathcal{R}\| \). Using also part z3 we have \( \|\mathcal{R}\| = \mu(X) \), and by formula (7.6) also \( \|L\| = \mu(X) \).
z7. It is enough to consider $K = \text{supp } f \cup \text{supp } g$. For any function $h \in C_c(X)$ such that $h \geq 0$, $h = 1$ on $K$ as in the proof of formula (3.1) we have: $|\mathcal{R}(f) - \mathcal{R}(g)| \leq \|f - g\| \mathcal{R}(h)$. Taking the infimum over functions $h$, by part z5 we obtain the assertion. Suppose $X$ is compact.

(i): By formula (7.3), $\mathcal{R}(c) = c\mu(X)$, and the rest of the statement follows from Theorem 7.10. (ii): Since $\mathcal{R}$ is monotone, an r-functional, and $\mu(X) = \|\mathcal{R}\| = \mathcal{R}(1)$ the statement follows from Remark 4.5.

Remark 7.13 The proof of part z5 is similar to the one for [7, Lemma 35(p4), Sect. 4], which, in turn, follows a proof given by D. Grubb in a lecture.

Remark 7.14 Part z7 of Lemma 7.12 means that on $C_c(X)$ the functional $\mathcal{R}$ is continuous with respect to topology of uniform convergence on compact sets.

Definition 7.15 Let $f$ be a continuous bounded function on a locally compact space $X$. Consider the $\sigma$-algebra of subsets of $X$

$$\Sigma_f = \{f^{-1}(A) : A \in \mathcal{B}(\mathbb{R})\},$$

where $\mathcal{B}(\mathbb{R})$ are the Borel subsets of $\mathbb{R}$. Let $r$, $l$, $m$ be measures on $\mathbb{R}$ from Theorem 6.7. On $\Sigma_f$ define measure $n_r$, $n_l$, $n$ by setting for each $E \in \Sigma_f$

$$n_r(E) = r(f(E)), \quad n_l(E) = l(f(E)), \quad n(E) = m(f(E)).$$

Remark 7.16 Definition 7.15 leads to another way to represent functionals $\mathcal{R}$ and $\mathcal{L}$. Let $\Sigma_f$ and measures $n_r$, $n_l$ be as in Definition 7.15. Then for any set $A \in \mathcal{B}(\mathbb{R})$ we have: $n_r(f^{-1}(A)) = r(f(f^{-1}(A))) = r(A), \quad n_l(f^{-1}(A)) = l(A)$, i.e.

$$n_r \circ f^{-1} = r, \quad n_l \circ f^{-1} = l \quad \text{and} \quad n \circ f^{-1} = m \text{ on } \mathcal{B}(\mathbb{R}).$$

(7.7)

By formula (7.2)

$$\mathcal{R}(f) = \int_{\mathbb{R}} id \, dr = \int_{\mathbb{R}} id \, d(n_r \circ f^{-1}) = \int_X f \, dn_r.$$ 

Similarly,

$$\mathcal{L}(f) = \int_X f \, dn_l.$$ 

Lemma 7.17 Let $\mu$ be a finite deficient topological measure on a locally compact space $X$, $f \in C_0(X)$, and $n_r$, $n_l$ be measures defined in Definition 7.15.

(i) $n_r(f^{-1}((t, \infty))) = \mu(f^{-1}((t, \infty)))$ for all $t$,

$$n_r(f^{-1}((t, \infty))) = \mu(f^{-1}((t, \infty))) \text{ for all } t \notin E,$$
where $E \subseteq (-\infty, 0]$ is a countable set from part V of Lemma 6.3;
\[ n_l(f^{-1}((-\infty, t])) = \mu(f^{-1}((-\infty, t])) \text{ for all } t, \]
\[ n_l(f^{-1}((-\infty, t])) = \mu(f^{-1}((-\infty, t])) \text{ for all } t \notin E_1, \]

where $E_1 \subseteq [0, \infty)$ is a countable set from part V of Lemma 6.3.

(ii) For any open or closed set $A \subseteq \mathbb{R}$
\[ \mu(f^{-1}(A)) \leq n_r(f^{-1}(A)), \mu(f^{-1}(A)) \leq n_l(f^{-1}(A)). \]

(iii) If $\mu$ is a finite topological measure and $f \in C_0(X)$, then also $n(f^{-1}(A)) = \mu(f^{-1}(A))$ for any open or closed set $A \subseteq \mathbb{R}\backslash\{0\}$.

(iv) If $\mu$ is a finite topological measure and $X$ is compact, then $n(f^{-1}(A)) = \mu(f^{-1}(A))$ for any open or closed set $A \subseteq \mathbb{R}$.

Proof Follows from (7.7), Theorems 6.7, and 6.10.  \[ \square \]

Remark 7.18 Definition 7.15 and Remark 7.16 were first observed for the case of the compact space $X$ by Svistula, see [19, (3.4)].

8 Representation Theorems for deficient topological measures

Theorem 8.1 (Representation theorem) Let $\mu$ be a finite deficient topological measure on a locally compact space $X$.

(i) Then there exists a unique $p$-conic quasi-linear functional $\mathcal{R}$ on $C^+_0(X)$ of finite norm such that $\mu = \mu_{\mathcal{R}}$ and $\|\mathcal{R}\| = \mu(X)$. In fact, $\mathcal{R} = \mathcal{R}_\mu$.

(ii) If $X$ is compact, the unique functional $\mathcal{R}$ can be taken to be a real-valued $r$-functional on $C(X)$.

Proof (i): Let $\mu$ be a finite deficient topological measure on a locally compact space $X$, and let $\mathcal{R} = \mathcal{R}_{\mu}$ be a functional on $C^+_0(X)$ obtained from $\mu$ as in Definition 7.1. Note that $\mathcal{R}$ is a $d$-functional by Theorem 7.10 and Remark 4.17. Then $\mu_{\mathcal{R}}$ defined as in Definition 5.1 from $\mathcal{R}$ is a deficient topological measure by Theorem 5.4. From part y4 of Lemma 5.2 and part z5 of Lemma 7.12 we see that $\mu(K) = \mu_{\mathcal{R}}(K)$ for every compact $K$. Thus, $\mu = \mu_{\mathcal{R}}$. By part z6 of Lemma 7.12 $\mu(X) = \|\mathcal{R}\|$.

Now we shall show the uniqueness. Let $\eta$ be another $p$-conic quasi-linear functional on $C^+_0(X)$ of finite norm such that $\mu_{\eta} = \mu_{\mathcal{R}}$. Then $\|\eta\| \leq a \mu(X)$ for some $a \geq 1$.

Let $f \in C^+_0(X)$. Both $\rho$ and $\eta$ are positive-homogeneous, so we may assume that $f(X) = [0, 1]$. For $n \in \mathbb{N}$ let $t_i = i/n$, $i = 0, \ldots, n$. For $i = 1, \ldots, n$ consider functions $\phi_i$ defined as follows:
\[ \phi_i(t) = \begin{cases} 0 & \text{if } t \leq t_{i-1} \\ t - t_{i-1} & \text{if } t_{i-1} < t < t_i \\ \frac{1}{n} & \text{if } t \geq t_i. \end{cases} \]
With functions $f_i = \phi_i \circ f$, we have $f = \sum_{i=1}^{n} f_i$. Since each $\phi_i$ is non-decreasing, and $\phi_i(0) = 0$, each $f_i \in A^+(f)$. Since $\mathcal{R}$ and $\eta$ are both $p$-conic quasi-linear functionals, we have

$$
\mathcal{R}(f) = \mathcal{R} \left( \sum_{i=1}^{n} f_i \right) = \sum_{i=1}^{n} \mathcal{R}(f_i), \quad \eta(f) = \sum_{i=1}^{n} \eta(f_i). \tag{8.1}
$$

By part 1 of Lemma 7.12, $\mathcal{R}(n f_1) \leq \mu(X)$, so $0 \leq \mathcal{R}(f_1) \leq \frac{1}{n} \mu(X) \leq \frac{a}{n} \mu(X)$. We have $0 \leq \eta(n f_1) \leq \|\eta\| \leq a \mu(X)$, i.e. $0 \leq \eta(f_1) \leq \frac{a}{n} \mu(X)$. Thus,

$$
|\mathcal{R}(f_1) - \eta(f_1)| \leq \frac{a \mu(X)}{n}. \tag{8.2}
$$

For each $i = 2, \ldots, n$ let $K_i = f^{-1}([t_i, \infty))$, $V_i = f^{-1}((t_i-1, \infty))$. Choose an open set $U_i \subseteq V_i$ such that $\mu(U_i) - \mu(K_i) < \frac{1}{n}$ and then pick an Urysohn function $g_i \in C_c(X)$ such that $0 \leq g_i \leq \frac{1}{n}$, $g_i = \frac{1}{n}$ on $K_i$ and $\text{supp } g_i \subseteq U_i \subseteq V_i$. Since $n g_i = 1$ on $K$ and $\mu = \mu_\mathcal{R} = \mu_\eta$, by part y4 of Lemma 5.2 and Definition 5.1, $\mu(K_i) \leq \mathcal{R}(n g_i)$, $\eta(n g_i) \leq \mu(U_i)$, and so $|\mathcal{R}(n g_i) - \eta(n g_i)| \leq \mu(U_i) - \mu(K_i) \leq \frac{1}{n}$. Then

$$
|\mathcal{R}(g_i) - \eta(g_i)| \leq \frac{1}{n^2}. \tag{8.3}
$$

Let $g = \sum_{i=2}^{n} g_i$, $h = \sum_{i=2}^{n} f_i$. Since $\text{supp } (g_3 + \cdots + g_n) \subseteq K_2$ and $g_2 = 1$ on $K_2$, by part (III) of Lemma 3.6, $\rho(g_2 + g_3 + \cdots + g_n) = \rho(g_2) + \rho(g_3 + \cdots + g_n)$. Similarly for $\eta$. By induction

$$
\mathcal{R}(g) = \sum_{i=2}^{n} \mathcal{R}(g_i), \quad \eta(g) = \sum_{i=2}^{n} \eta(g_i). \tag{8.4}
$$

Then

$$
|\mathcal{R}(g) - \eta(g)| \leq \sum_{i=2}^{n} |\mathcal{R}(g_i) - \eta(g_i)| \leq \frac{n-1}{n^2} < \frac{1}{n}. \tag{8.5}
$$

Note that $\|g - h\| \leq \frac{1}{n}$, so by part (V) of Lemma 3.6

$$
|\mathcal{R}(g) - \mathcal{R}(h)| \leq \frac{\mu(X)}{n}, \quad |\eta(g) - \eta(h)| \leq \frac{a \mu(X)}{n}. \tag{8.6}
$$
Using (8.2), (8.6), and (8.5) we obtain:

\[
|R(f) - \eta(f)| \leq |R(f_1) - \eta(f_1)| + |R(h) - \eta(h)|
\leq |R(f_1) - \eta(f_1)| + |R(h) - R(g)| + |R(g) - \eta(g)| + |\eta(g) - \eta(h)|
\leq \frac{a\mu(X)}{n} + \frac{\mu(X)}{n} + \frac{1}{n} + \frac{a\mu(X)}{n} \leq \frac{1}{n}(3a\mu(X) + 1).
\]

Thus, \( R = \eta \).

(ii): Now let \( X \) be compact. We shall show that the proof for part (i) still applies, although the reasoning for some estimates is different. We define the functional \( R = R_\mu \) on \( C(X) \). It is an \( r \)-functional by Theorem 7.10. Then \( |R(1)| = \|R\| = \mu(X) < \infty \), and \( |R(-1)| \leq \|R\| < \infty \). By monotonicity, \( R \) is real-valued. Let \( \eta \) be another real-valued (hence, bounded by Proposition 3.7) \( r \)-functional on \( C(X) \) such that \( \mu = \mu_R = \mu_\eta \). To show the uniqueness, by Remark 4.17 it is enough to show that \( \rho(f) = \eta(f) \) for \( f \geq 0 \). For the functions \( f_i \) from the proof for part (i) note the following: \( \text{supp} (f_2 + \ldots + f_n) \subseteq f^{-1}([t_1, \infty)) \), \( f_1 = \frac{1}{n} \) on \( f^{-1}([t_1, \infty)) \), thus by part (III) of Lemma 4.19 we may apply the c-level condition to obtain \( R(f_1 + f_2 + \ldots + f_n) = R(f_1) + R(f_2 + \ldots + f_n) \). Then by induction we may show that formula (8.1) holds for \( R \) and, similarly, for \( \eta \). In the same manner, by part (III) of Lemma 4.19 and induction we show that formula (8.4) holds. Note that (8.2), (8.3), and (8.5) hold as in the proof of part (i). Estimations (8.6) are valid by part (II) of Lemma 4.19. Now as in the end of the proof for part (i), we show that \( R = \eta \). 

\[\blacksquare\]

**Remark 8.2** Our inequality (8.3) is inspired by a similar estimate in the proof of [19, Theorem 9].

**Definition 8.3** Let \( L, QI, \Phi^s, \Phi^+, \Phi^-, \Phi^r, \Phi^l \) represent subfamilies of bounded functionals from \( L, QI, \Phi^s, \Phi^+, \Phi^-, \Phi^r, \Phi^l \) respectively. We may indicate in parenthesis the domain of functionals. For example, \( \Phi^+ = \Phi^+(C(X)) \) is the collection of all bounded \( p \)-conic quasi-linear functionals on \( C(X) \).

**Definition 8.4** Let \( DTM(X), TM(X), M(X) \) represent, respectively, subfamilies of finite set functions from \( DTM(X), TM(X), M(X) \).

**Corollary 8.5** Let \( X \) be locally compact.

(i) There is a bijection \( \Gamma : \Phi^+(C^+_0(X)) \rightarrow DTM(X) \) given by \( \Gamma(R) = \mu_R \). The inverse bijection \( \Pi = \Gamma^{-1} \) is given by \( \Pi : DTM(X) \rightarrow \Phi^+(C^+_0(X)) \) where \( \Pi(\mu) = R_\mu \). Here \( \mu_R \) and \( R_\mu \) are according to Definitions 5.1 and 7.1.

(ii) There is a bijection \( \Delta : DTM(X) \rightarrow \Phi^-(C^-_0(X)) \) given by \( \Delta(\mu) = L_\mu \), where \( \mu \) is according to Definition 7.1.

(iii) If \( X \) is compact, there is a bijection between \( DTM(X) \) and \( \Phi^r(C(X)) \) given by \( \mu \mapsto R_\mu \), and a bijection between \( DTM(X) \) and \( \Phi^l(C(X)) \).

**Proof** (i) follows from Theorem 8.1. (ii): By Remark 3.5 there is a bijection between \( \Phi^+(C^+_0(X)) \) and \( \Phi^-(C^-_0(X)) \), so we obtain bijection \( \Delta \). By formula (7.6) \( -R_\mu(-f) = L_\mu(f) \). (ii): By Theorem 8.1 there is a bijection between \( DTM(X) \) and \( \Phi^r(C(X)) \) given by \( \mu \mapsto R_\mu \). By Remark 4.17 there is a bijection between \( \Phi^r(C(X)) \) and \( \Phi^l(C(X)) \). 

\[\blacksquare\]
Corollary 8.6 Suppose \( \rho \) is a finite c-functional on \( C_0(X) \), and \( \rho = \mathcal{R}_\mu \) on \( C_0^+(X) \), where \( \mathcal{R}_\mu \) is a functional on \( C_0(X) \) for some finite deficient topological measure \( \mu \) as in Definition 7.1. Then \( \rho \) is an s-functional iff \( \mathcal{L}_\mu(g) = \rho(g) \) for all \( g \in C_0^+(X) \), where \( \mathcal{L}_\mu \) is a functional on \( C_0(X) \) as in Definition 7.1.

Proof If \( \rho(g) = \mathcal{L}_\mu(g) \) for all \( g \in C_0^+(X) \) then \( \rho(g) = \mathcal{L}_\mu(g) = -\mathcal{R}_\mu(-g) = -\rho(-g) \), i.e., \( -\rho(g) = \rho(g) \). Since \( \mu \) is finite, \( \rho \) is real-valued on \( C_0^+(X) \), so \( \rho \) is real-valued. Then \( \rho \) is an s-functional by part (i) of Proposition 4.13. The other direction can be proved similarly. \( \square \)

Theorem 8.7 Let \( X \) be locally compact. Let \( \Phi^+ = \Phi^+(C_0^+(X)) \). Consider the map \( \Pi : \mathcal{DTM}(X) \rightarrow \Phi^+ \) where \( \Pi(\mu) = \mathcal{R}_\mu \) and \( \mathcal{R}_\mu \) is the functional according to Definition 7.1. Then the map \( \Pi \) has the following properties:

(I) \( \Pi \) is conic-linear, i.e. \( \Pi(c\mu + dv) = c\Pi(\mu) + d\Pi(v) \), \( c, d \geq 0 \).

(II) \( \mu \leq v \) if and only if \( \Pi(\mu) \leq \Pi(v) \) (i.e., \( \mathcal{R}_\mu(f) \leq \mathcal{R}_v(f) \) for all \( f \in C_0^+(X) \)).

(III) \( \mu \in \mathcal{TM}(X) \) iff \( \rho \) is a quasi-linear functional on \( C_0(X) \), and \( \mu \in \mathcal{M}(X) \) iff \( \rho \) is a linear functional on \( C_0(X) \), where \( \rho(f) = \Pi(\mu)(f^+) - \Pi(\mu)(f^-) \).

(IV) \( \|\mathcal{R}_\mu\| = \mu(X) \).

(V) The map \( \Pi : \mathcal{DTM}(X) \rightarrow \Phi^+ \) is a conic-linear order-preserving bijection such that \( \|\mathcal{R}_\mu\| = \mu(X) \).

Proof (I): Let \( \lambda = c\mu + dv \), \( c, d \geq 0 \). Take any \( f \in C_0^+(X) \). For function \( R_{1,\lambda,f} \) in Definition 6.1 we see that \( R_{1,\lambda,f} = cR_{1,\mu,f} + dR_{1,v,f} \). From formula (7.4) \( \mathcal{R}_\lambda(f) = c\mathcal{R}_\mu(f) + d\mathcal{R}_v(f) \), and the statement follows.

(II): Let \( \mu \leq v \). Take any \( f \in C_0^+(X) \). Using Definition 6.1 we see that \( R_{1,\mu,f} \leq R_{1,v,f} \). Then by formula (7.4) we have \( \mathcal{R}_\mu(f) \leq \mathcal{R}_v(f) \). Thus, \( \mathcal{R}_\mu \leq \mathcal{R}_v \). Now assume that \( \mathcal{R}_\mu \leq \mathcal{R}_v \). From part 5 of Lemma 7.12 we see that \( \mu(K) \leq v(K) \) for any compact \( K \). By Remark 2.4 \( \mu \leq v \).

(III): Suppose \( \rho(f) = \mathcal{R}_\mu(f^+) - \mathcal{R}_\mu(f^-) \) is a quasi-linear functional on \( C_0(X) \). By part (i) of Corollary 8.5 and part (vi) of Theorem 5.4 \( \mu \) is a finite topological measure. Now suppose that \( \mu \) is a finite topological measure. Let \( m_f \) be the measure from Theorem 6.10 for \( f \), where \( f \in C_0(X) \). Consider functional \( \rho(f) = \int_R id \ dm_f \) on \( C_0(X) \). For \( \phi \in C(\overline{f(X)}) \) and any open set \( W \subseteq \mathbb{R} \) we have

\[
m_{\phi \circ f}(W) = \mu((\phi \circ f)^{-1}(W)) = \mu(f^{-1}(\phi^{-1}(W))) = m_f(\phi^{-1}(W)) = (m_f \circ \phi^{-1})(W),
\]

thus, \( m_{\phi \circ f} \) and \( m_f \circ \phi^{-1} \) are equal as measures on \( \mathbb{R} \), and for \( \phi \circ f \in A(f) \) we obtain \( \rho(\phi \circ f) = \int_R id \ dm_{\phi \circ f} = \int_R id \ dm_f \circ \phi^{-1} = \int_R \phi \ dm_f \). For \( \phi \circ f, \psi \circ f \in A(f) \) we have:

\[
\rho(\phi \circ f + \psi \circ f) = \int_R (\phi + \psi) \ dm_f = \int_R \phi \ dm_f + \int_R \psi \ dm_f = \rho(\phi \circ f) + \rho(\psi \circ f).
\]
For any constant $c$ we see that $\rho(cf) = \rho((c\text{id}) \circ f) = \int_R \! c \text{id} \, dm \, f = c \rho(f)$. Thus, $\rho$ is a quasi-linear functional on $C_0(X)$. Since $f^+ = (0 \vee \text{id}) \circ f \in A(f)$, and $f^- \in A(f)$, we know that $\rho(f) = \rho(f^+) - \rho(f^-)$. It is clear that $\rho(g) = R_\mu(g)$ for any $g \in C_0^+$, so $\rho(f) = \rho(f^+) - \rho(f^-) = R_\mu(f^+) - R_\mu(f^-) = \Pi(\mu)(f^+) - \Pi(\mu)(f^-)$.

If $\mu \in M(X)$ then $\rho(f) = \int id \, dm \, f = \int f \, d\mu$ is the usual integral, and the last assertion is a well known fact. 

(IV): This is part $z6$ of Lemma 7.12.

(V): Clear from part (i) of Corollary 8.5 and parts (I), (II) and (IV).

\begin{proof}
(i) By Remark 4.12 we need to show that $\Phi^s \subseteq QI$, so let $\rho \in \Phi^s$. Then $\rho \in \Phi^r$ by Remark 4.17. By Corollary 8.5 there is a unique deficient topological measure $\mu$ such that $\rho(f) = \rho_\mu(f)$ for all $f \in C(X)$. By part (v) of Theorem 5.4, $\mu$ is a topological measure. Then $\rho$ is a quasi-linear functional (see [3, Theorem 4.1] or [7, Theorem 42, Sect. 4]).

(ii) If $\rho$ is an $r$-functional, by Corollary 8.5 $\rho = R_\mu$ for a unique deficient topological measure $\mu$. By Theorem 7.10 $R_\mu$ is a $p$-conic quasi-linear functional. Thus, $\Phi^r \subseteq \Phi^+$. The other inclusion is given by Lemma 4.18. We can prove that $\Phi^- = \Phi^l$ in a similar way, using Corollary 8.5 and Lemma 4.18.
\end{proof}

\begin{remark}
From Theorem 8.1, part (iii) of Corollary 8.5, and Theorem 8.8, it follows that when $X$ is compact, in Theorem 8.7 we may take $\Phi^+$ to be $\Phi^+(C(X)) = \Phi^r(C(X))$.

\begin{theorem}
(I) Let $X$ be locally compact. For functionals on $C_0(X)$

\[ L \subseteq QI \subseteq \Phi^+ \cap \Phi^- \subseteq \Phi^r \cap \Phi^l, \]

\[ L \subseteq QI \subseteq \Phi^+ \cap \Phi^- \subseteq \Phi^r \cap \Phi^l = \Phi^s. \]

In general,

\[ L \not\subseteq QI \not\subseteq \Phi^+. \]

(II) Let $X$ be compact. Then for functionals on $C(X)$ we have:

\[ L \subseteq QI = \Phi^+ \cap \Phi^- = \Phi^r \cap \Phi^l = \Phi^s. \]

In general,

\[ L \not\subseteq QI. \]
\end{theorem}
Proof (I): The inclusion $QI \subseteq \Phi^+ \cap \Phi^-$ is given by Remark 3.4. The inclusion $\Phi^+ \cap \Phi^- \subseteq \Phi^r \cap \Phi^l$ follows from Lemma 4.18. By Remark 4.17 $\Phi^r \subseteq \Phi^c$, so from Corollary 8.6 we see that $\Phi^r \cap \Phi^l \subseteq \Phi^s$. By part (ii) of Remark 4.17 we have: $\Phi^s \subseteq \Phi^r \cap \Phi^l$. The proper inclusion $L \subsetneq QI$ follows from the existence of quasi-linear functionals that are not linear, or existence of topological measures that are not measures. The proper inclusion $QI \subsetneq \Phi^+$ follows from part (III) of Theorem 8.7 and the existence of deficient topological measures that are not topological measures. See Remark 2.6. For an example of a quasi-linear but not linear functional on a locally compact space see [7, Example 55, Sect. 5].

(II): Use the previous part and Theorem 8.8.

Remark 8.11 Let $X$ be compact. From Corollary 8.5 and Theorem 8.8 we see that the functionals corresponding to finite deficient topological measures can be described in four ways: as p-conic quasi-linear functionals, as r-functionals, as n-conic quasi-linear functionals, and as l-functionals.

From Theorem 8.10 we see that the functionals corresponding to finite topological measures can be described in four ways: as quasi-linear functionals; as s-functionals; as functionals that are both p-conic quasi-linear and n-conic quasi-linear; and as functionals that are both r- and l-functionals.

Remark 8.12 Theorem 8.10 answers positively the question posed in [19, Remark 7], of whether for a compact space $\Phi^s = \Phi^r \cap \Phi^l$. Note that by part (I) of Lemma 4.19 our definition of r- and l-functionals in the compact case coincide with those in [19, Definition 6]. By Definition 4.9 and Theorem 8.10 our definition of an s-functional coincides with the one in [19, Definition 6], where it was first introduced.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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