Asymptotic Stability of Singular Solution for Camassa-Holm Equation

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Abstract

The aim of this paper is to study singular dynamics of solutions of Camassa-Holm equation. Based on the semigroup theory of linear operators and Banach contraction mapping principle, we prove the asymptotic stability of the explicit singular solution of Camassa-Holm equation.

Keywords

Asymptotic Stability, Camassa-Holm Equation, Explicit Solution, Semigroup Theory, Banach Contraction Mapping Principle

1. Introduction and Main Results

1.1. Introduction

Consider the well-known Camassa-Holm equation as follows (see [1]):

$$m_t + c_0 u_x + u m_x + 2m u_x = 0,$$

(1.1)

where \( (t,x) \in \mathbb{R}^+ \times \mathbb{R} \), \( u = u(t,x) \) is the velocity of fluid, \( m \) is the momentum given by

$$m = m(t,x) = u(t,x) - \alpha x u_x(t,x),$$

where \( c_0 \in \mathbb{R} \) is the critical speed and \( \alpha \in \mathbb{R} \) relates to the length scale. Thus,

$$u_x - \alpha^2 u_{xx} + c_0 u_x + 3uu_x = \alpha^2 \left( 2u_x u_{xx} + uu_{xxx} \right).$$

(1.2)

Given the initial value as \( u(0,x) = u_0(x) \) for \( x \in \mathbb{R} \).

The Camassa-Holm equation describes unidirectional propagation of surface water waves in shallow water area. For the global well-posedness and stability of solutions, we recommend that the reader refers to [2]-[9], etc. For the wave breaking analysis, we refer the reader to [6] [10]-[15], etc. When \( c_0 = 0 \) and \( \alpha = 1 \), the Camassa-Holm equation becomes to the classical Camassa-Holm eq-
uation, which admits a bi-Hamiltonian structure [1] [5]. Moreover, the explicit peakon solution and its stability have been established in [12] [16] [17] [18] [19], etc.

Since it is rare to see the explicit stable blowup solutions of Camassa-Holm equation, in this paper, we study the stability of the explicit solution of (1.2) as follows (see [20]):

\[
\mathcal{U}(t,x) = -\frac{1}{3} \left( c_0 + \frac{x}{T-t} + \frac{1}{T-t} \right),
\]

where \( T > 0 \) is a constant.

### 1.2. Main Results

Now, we state our main result of this paper.

**Theorem 1.1.** Let \( s > 2 \) be an integer and \( \delta \) is a sufficiently small constant. Then the explicit solution (1.3) of the Camassa-Holm Equation (1.2) is asymptotic stable, i.e., if the initial data \( u_0(x) \) satisfies

\[
\left\| u_0(x) + \frac{1}{3} \left( c_0 + \frac{x}{T} + \frac{1}{T} \right) \right\|_{L^2[0,T]} \leq \delta,
\]

then there is a solution \( u(t,x) \) of (1.2) satisfying

\[
\left\| u(t,x) - \mathcal{U}(t,x) \right\|_{L^2[0,T]} \leq \frac{\tilde{C}(T-t)}{\alpha^2(1 + C \ln(T-t))}, \quad (t,x) \in (0,T) \times \mathbb{R},
\]

where \( C \) and \( \tilde{C} \) are positive constants that depend on \( s \).

### 1.3. Notations

Denote \( L^2(\mathbb{R}) = L^2 \) and \( \mathcal{H}^s(\mathbb{R}) = \mathcal{H}^s \) by the Lebesgue spaces and Sobolev spaces with norms \( \left\| \cdot \right\|_{L^2} \) and \( \left\| \cdot \right\|_{\mathcal{H}^s} \), respectively. \( \ast \) denotes the convolution. \([A,B] \) stands for the commutator.

### 2. Proof of Theorem 1.1

Let

\[
u(t,x) = v(t,x) + \mathcal{U}(t,x),
\]

be the solution of (1.2), where \( \mathcal{U}(t,x) = -\frac{1}{3} \left( c_0 + \frac{x}{T-t} + \frac{1}{T-t} \right) \) is the explicit solution. Substituting (2.1) into (1.2), we get

\[
\begin{align*}
v_t - \alpha^2 v_{ss} &+ \left( \frac{\alpha^2}{3} \left( c_0 + \frac{x}{T-t} + \frac{1}{T-t} \right) \right) v_{ss} + \frac{2\alpha^2}{3(T-t)} v_s \
&= \frac{x}{T-t} + \frac{1}{T-t} v + 3vv_s \quad (2.2)
\end{align*}
\]

with the initial condition \( v(0,x) = v_0(x) = u_0(x) + \frac{1}{3} \left( c_0 + \frac{x}{T} + \frac{1}{T} \right) \) for \( x \in \mathbb{R} \).
For the singular coefficients in (2.2), let \( v(t, x) = \psi(t, \rho) \) by \( \tau = -\ln(T - t) \) and \( \rho = \frac{x}{T - t} \), then (2.2) becomes to

\[
\psi_t + \rho \psi_x - \alpha^2 e^{	au} \left( \psi_{x} + 2\psi_{\rho} + \rho \psi_{\rho} + 3\psi_{\rho\rho} + \rho \psi_{\rho\rho} \right) + e^{\tau} \left[ \frac{\alpha^2}{3} \left( c_0 + \rho + e^\tau \right) \right] \psi_{\rho\rho} 
+ \frac{2\alpha^2}{3} e^{2\tau} \psi_{\rho\rho} - \left( \rho + e^\tau \right) \psi_{\rho} - \psi + 3\rho \psi_{\rho} = \alpha^2 e^{2\tau} \left( 2\psi_{\rho\rho} + \psi_{\rho\rho} \right).
\]

(2.3)

Let \( \kappa = e^{-\tau} \rho \) and \( \bar{\psi}(\tau, \kappa) = e^{-\tau} \psi(\tau, \rho) \). Then (2.3) becomes to

\[
\bar{\psi}_\tau - \alpha^2 \bar{\psi}_{\kappa\kappa} - \alpha^2 \bar{\psi}_{\kappa} + e^{\tau} \left[ \gamma + \frac{\alpha^2}{3} \left( c_0 + \kappa e^\tau + e^\tau \right) \right] \bar{\psi}_{\kappa\kappa} 
- (\kappa + 1) \bar{\psi}_{\kappa} + 3\bar{\psi}_{\kappa} = \alpha^2 \left( 2\bar{\psi}_{\kappa\kappa} + \bar{\psi}_{\kappa\kappa} \right). 
\]

(2.4)

Let the operator \( A = \left( 1 - \alpha^2 \partial_{\kappa\kappa} \right)^{\frac{1}{2}} \). Since \( 1 - \alpha^2 \partial_{\kappa\kappa} \) admits a fundamental solution \( \bar{\varphi}(x) = \frac{1}{2\alpha} e^\frac{|x|}{2\alpha} \), we have \( A^{-\frac{1}{2}} \bar{\psi} = \bar{\varphi}(\kappa) \ast \bar{\psi} \) for all \( \bar{\psi} \in L^2 \). Let \( w(\tau, \kappa) = \bar{\psi}(\tau, \kappa) - \alpha^2 \bar{\psi}_{\kappa\kappa} (\tau, \kappa) \), then \( \bar{\psi}(\tau, \kappa) = \bar{\varphi} \ast w \), where \( \kappa \in \mathbb{R} \). Furthermore, we have \( \left( \rho \ast w \right)_{\kappa\kappa} = \alpha^2 \left( \rho \ast w - w \right) \), \( \bar{\psi}_{\kappa} = \left( \rho \ast w \right)_{\kappa} \) and \( \bar{\psi}_{\kappa\kappa} = \alpha^2 \left( \left( \rho \ast w \right)_{\kappa} - w_{\kappa} \right) \). Then (2.3) can be rewritten as

\[
w_t + \frac{1}{3} w - e^{\tau} \left[ \frac{1}{3} \left( c_0 + e^\tau \kappa + e^\tau \right) \right] w_{\kappa} - \frac{1}{3} \bar{\varphi} \ast w 
+ \left[ e^{\tau} \left[ \frac{1}{3} \left( c_0 + e^\tau \kappa + e^\tau \right) \right] \right] \left( \kappa + 1 \right) \left( \bar{\varphi} \ast w \right)_{\kappa} + 3 \left( \bar{\varphi} \ast w \right) \left( \bar{\varphi} \ast w \right)_{\kappa} 
= 2 \left( \bar{\varphi} \ast w \right) \left( \bar{\varphi} \ast w \right)_{\kappa} + \left( \rho \ast w \right) \left[ \left( \rho \ast w \right)_{\kappa} - w_{\kappa} \right]
\]

(2.5)

with the initial data

\[
w_0(\kappa) = u_0(x) - \alpha^2 u_0'(x) + \frac{1}{3} \left( \frac{x}{T - 1} + c_0 \right).
\]

(2.6)

and the boundary condition

\[
\lim_{|\tau| \to \infty} \bar{w}(\tau, \kappa) = 0, \quad \lim_{|\kappa| \to \infty} w_{\kappa}(\tau, \kappa) = 0.
\]

(2.7)

Before making a priori estimate of the solutions to problems (2.5)-(2.7). We recall the following commutator estimate.

**Lemma 2.1** ([21]). Let \( s > 0 \). Then it holds

\[
\left\| A^{-s} u \right\|_{L^1} \leq C \left( \left\| u \right\|_{L^\infty} \left\| A^{-s} v \right\|_{L^1} + \left\| A^{-s} w \right\|_{L^1} \right),
\]

(2.8)

where \( C \) is a positive constant that depends on \( s \).

Now, we derive a priori estimate of the solutions for (2.5).

**Lemma 2.2.** Let \( s > 2 \) and \( \alpha \neq 0 \). Assume that \( w \) be a solution of(2.5), then

\[
\left\| w \right\|_{L^1} \leq \frac{1}{\left\| w_0 \right\|_{L^1}} - Ct.
\]

(2.9)

where \( C \) is a positive constant depending upon \( s \).
Proof. Applying $A^t$ to both sides of (2.5) and taking the $\|\cdot\|_{\mathbb{L}^2}$-inner product with $A^t w$, we get
\[
\frac{1}{2} \frac{d}{d\tau} \|w\|_{\mathbb{L}^2}^2 + \frac{1}{3} \|w\|_{\mathbb{L}^2}^2 - \frac{1}{3} \int_{\mathbb{R}} A^t w A^t (\varphi \ast w) \, d\kappa
\]
\[
- e^{-t} \int_{\mathbb{R}} A^t w A^t \left[ \left( \frac{1}{3} (c_0 + e^t \kappa + e^t) \right) w_{\kappa} \right] \, d\kappa
\]
\[
+ \int_{\mathbb{R}} A^t w A^t \left[ \left( e^{-t} \left( \frac{1}{3} (c_0 + e^t \kappa + e^t) \right) \right) - (\kappa + 1) \right] (\varphi \ast w)_{\kappa} \, d\kappa.
\]
\[
(2.10)
\]
Next, we estimate each of terms in (2.10).
\[
- \frac{1}{3} \int_{\mathbb{R}} A^t w A^t (\varphi \ast w) \, d\kappa = - \frac{1}{3} \|w\|_{\mathbb{L}^2}^2, \quad (2.11)
\]
\[
- e^{-t} \int_{\mathbb{R}} A^t w A^t \left[ \left( \frac{1}{3} (c_0 + e^t \kappa + e^t) \right) w_{\kappa} \right] \, d\kappa
\]
\[
e^{-t} \int_{\mathbb{R}} \left[ \left( \frac{1}{3} (c_0 + e^t \kappa + e^t) \right) A^{2t} w \right] \, w_{\kappa} \, d\kappa
\]
\[
= - \frac{1}{3} \int_{\mathbb{R}} A^{2t} w \left[ \left( e^{-t} \left( \frac{1}{3} (c_0 + e^t \kappa + e^t) \right) \right) - (\kappa + 1) \right] (\varphi \ast w)_{\kappa} \, d\kappa.
\]
\[
= - \int_{\mathbb{R}} A^{2t} w \left[ \left( e^{-t} \left( \frac{1}{3} (c_0 + e^t \kappa + e^t) \right) \right) - (\kappa + 1) \right] (\varphi \ast w) \, d\kappa.
\]
\[
= - \int_{\mathbb{R}} A^{2t} w \left[ \left( e^{-t} \left( \frac{1}{3} (c_0 + e^t \kappa + e^t) \right) \right) - (\kappa + 1) \right] (\varphi \ast w) \, d\kappa.
\]
\[
= - \left( \frac{1}{3} - 1 \right) \int_{\mathbb{R}} A^{2t-1} w \, d\kappa + \frac{2}{3} \int_{\mathbb{R}} A^{2t} w (\varphi \ast w) \, d\kappa = \frac{1}{3} \|w\|_{\mathbb{L}^2}^2, \quad (2.12)
\]
\[
3 \int_{\mathbb{R}} A^t w A^t \left( (\varphi \ast w)_{\kappa} \right) \, d\kappa
\]
\[
= - \frac{3}{2} \int_{\mathbb{R}} w_{\kappa} A^{2t} \left( (\varphi \ast w)^2 \right) \, d\kappa \leq \frac{3}{2} \|w_{\kappa}\|_{\mathbb{L}^2} \|w\|_{\mathbb{L}^2}^2 \leq \frac{3}{2} \|w\|_{\mathbb{L}^2}^2, \quad (2.13)
\]
In addition, using (2.8), we have
\[
2 \left| \int_{\mathbb{R}} A^t w A^t \left( (\varphi \ast w)_{\kappa} \right) (\varphi \ast w - w) \, d\kappa \right|
\]
\[
= 2 \left| \int_{\mathbb{R}} \left[ A^t (\varphi \ast w - w) \right] (\varphi \ast w)_{\kappa} A^t \, d\kappa \right|
\]
\[
+ 2 \left| \int_{\mathbb{R}} (\varphi \ast w - w) A^t (\varphi \ast w)_{\kappa} A^t \, d\kappa \right|
\]
\[
\leq C \left( \|\varphi \ast w - w\|_{\mathbb{L}^2} \left\| A^{t-1} (\varphi \ast w)_{\kappa} \right\|_{\mathbb{L}^2} + \left\| A^t (\varphi \ast w - w) \right\|_{\mathbb{L}^2} \left\| (\varphi \ast w)_{\kappa} \right\|_{\mathbb{L}^2} \right) \|w\|_{\mathbb{L}^2}
\]
\[
+ 2 \left( \left\| (\varphi \ast w - w) \right\|_{\mathbb{L}^2} + \left\| (\varphi \ast w - w) \right\|_{\mathbb{L}^2} \right) \|w\|_{\mathbb{L}^2}^2
\]
\[
\leq C \|w\|_{\mathbb{L}^2}^3, \quad (2.15)
\]
similarly,
\[
\left| \int_k A_t w A^t \left[(\varphi \ast w)((\varphi \ast w)_k - w_k)\right] \, dx \right| \leq C \|w\|_{H^t}^3, \tag{2.16}
\]
where \(C\) is a positive constant depending upon \(s\).

Substituting (2.11)-(2.16) into (2.10), we get
\[
\frac{1}{2} \frac{d}{d\tau} \|w\|_{H^t}^3 \leq C \|w\|_{H^t}^3, \quad \text{and then}
\]
\[
-\frac{d}{d\tau} \|w\|_{H^t}^3 \leq C. \quad \text{Integrating this inequality above with respect to } \tau \text{ from 0 to } \tau, \text{ we get}
\]
\[
\|w\|_{H^t}^3 \leq \frac{1}{\|w_0\|_{H^t}^3 - Cr}. \tag{2.17}
\]

This completes the proof of Lemma 2.2. \(\square\)

**Proof of Theorem 1.1.** Now, we study the well-posedness for (2.5)-(2.7). Define the linear operator \(L\) as
\[
L[w] = -\frac{1}{3} w - \frac{1}{3} \varphi \ast w + e^{-\tau} \left[ \frac{1}{3} (c_0 + e' \kappa + e'') \right] w_k
\]
\[
- \left[ e^{-\tau} \left( \frac{1}{3} (c_0 + e' \kappa + e'') \right) - (\kappa + 1) \right] (\varphi \ast w)_k, \tag{2.18}
\]
then (2.5) becomes to
\[
w_t = L[w] + f(w), \tag{2.19}
\]
where \(f\) is the nonlinear terms:
\[
f(w) = -3(\varphi \ast w)(\varphi \ast w)_k + 2(\varphi \ast w)(\varphi \ast w - w)
\]
\[
+ (\varphi \ast w)[(\varphi \ast w)_k - w_k], \tag{2.20}
\]

**Lemma 2.3.** Let \(s > 2\). Then

- \(L[w]\in \mathbb{H}^s\) for \(\forall w\in D(L)\).
- \(L\) is a closed and densely defined linear operator in \(\mathbb{H}^s\).

**Proof.** It is a direct verification by the definition of \(L\). \(\square\)

**Lemma 2.4.** Let \(s > 2\). Then \(L\) is a dissipative operator in \(\mathbb{H}^s\), i.e.,
\[
(L[w], w)_s \leq 0.
\]

**Proof.** Using (2.11)-(2.14), a direct calculation shows that
\[
\int_0^1 (A^* L[w]) A^t w \, dx \kappa = -\frac{1}{3} \|w\|_{H^t}^2 + \frac{1}{3} \|w\|_{H^{t-1}}^2 - \frac{1}{6} \|w\|_{H^{t-1}}^2 - \frac{1}{3} \|w\|_{H^{t-1}}^2
\]
\[
= -\frac{1}{2} \|w\|_{H^t}^3 \leq 0. \tag{2.21}
\]

This completes the proof. \(\square\)

**Lemma 2.5** (Young inequality with \(\varepsilon\), see [22]). Let \(a,b > 0\) and \(\varepsilon > 0\). If \(p, q \in (1, \infty)\) satisfy \(\frac{1}{p} + \frac{1}{q} = 1\). Then
\[
ab \leq \varepsilon a^p + C(\varepsilon) b^q, \tag{2.22}
\]
where \(C(\varepsilon) = (\varepsilon p)^{\frac{q}{p}} q^{-1}\).
Lemma 2.6. Let \( s > 2 \). Then the operator \( L \) is invertible in \( \mathbb{H}^s \). Furthermore, it generates a \( C_0 \)-semigroup \( \{S(t)\}_{t \geq 0} \) in \( \mathbb{H}^s \).

Proof. Firstly, we show that the existence of \( L^{-1} \). Indeed, we need to prove \( L \) is injective and surjective. On the one hand, let \( w \in \mathcal{D}(L) \) such that \( L[w] = 0 \), then

\[
\int_{\mathcal{E}} A'[L[w]]A'wd\kappa = -\frac{1}{2}\|w\|^2_{H^s} = 0.
\] (2.23)

This combining with the boundary condition (2.7) gives that \( w = 0 \). So the operator \( L \) is injective. On the other hand, for all \( g \in \mathbb{H}^1 \), put

\[
L[w] = g.
\] (2.24)

Applying \( A' \) to (2.24) and multiplying the result by \( A'w \), and then integrating over \( \mathbb{R} \), we get

\[
\|w\|^2_{H^s} = -2\int_{\mathbb{R}} A'gA'wd\kappa.
\] (2.25)

It follows from the Young inequality with \( \varepsilon \) in Lemma 2.5 that

\[
\|w\|^2_{H^s} \leq C\|g\|^2_{H^s}.
\] (2.26)

Note that \( s > 2 \), then by the standard theory of elliptic equations (see [22]), there exists a unique weak solution \( w \in \mathbb{H}^1 \), moreover, we have \( w \in H^{s+1} \) if \( g \in \mathbb{H}^s \). Thus, the operator \( L \) is surjective. Secondly, by the Lumer-Phillips theorem (see [23]), the operator \( L \) generates a \( C_0 \)-semigroup \( \{S(t)\}_{t \geq 0} \) in \( \mathbb{H}^s \). This completes the proof. \( \square \)

As a consequence, we have

Proposition 2.7. Let \( s > 2 \). Then the Cauchy problem

\[
\begin{cases}
\frac{d}{d \tau} w = Lw, \\
w(0) = w_0
\end{cases}
\] (2.27)

with zero boundary condition exists a unique solution \( w(\tau) = S(\tau)w_0 \), where \( w_0 \) is the initial data defined in (2.6).

Using the Duhamel’s principle, the solutions of (2.19) satisfies the integral equation:

\[
w(\tau) = S(\tau)w_0 + \int_0^\tau S(\tau-s)f(w(s))ds.
\] (2.28)

To show this integral equation exists a solution, we define the solution space as

\[
B_\delta = \{w \in \mathbb{H}^s : \|w\|^2_{H^s} < \delta \ll 1\},
\] (2.29)

and the map \( T \) as

\[
Tw(\tau) = S(\tau)w_0 + \int_0^\tau S(\tau-s)f(w(s))ds.
\] (2.30)

We need to prove that \( T \) has a fixed point in the space \( B_\delta \).

Lemma 2.8 ([21]). Let \( s > 2 \). Then \( B_\delta \) is an algebra, and
\[ ||w||_{L^s} \leq C (||u||_{L^s} ||v||_{L^e} + ||u||_{L^e} ||v||_{L^s}), \tag{2.31} \]

where \( C \) is a positive constant depending upon \( s \).

**Lemma 2.9.** Let \( s > 2 \) be an integer. Assume that \( ||w_k||_{L^{s+1}} < \delta \) for some sufficiently small \( \delta > 0 \). Then \( T \) is a self-mapping on \( B_\delta \). Moreover, \( T \) is a contraction mapping.

**Proof.** By Lemma 2.8, we have
\[
||f(w)||_{L^2} \leq 3 ||(\varphi * w)(\varphi * w)_k||_{L^2} + 2 ||(\varphi * w)_k ((\varphi * w)_k - w_k)||_{L^2} + ||(\varphi * w)((\varphi * w)_k - w_k)||_{L^2} 
\leq C_1 (||\varphi * w||_{L^s} ||\varphi * w||_{L^e} + ||(\varphi * w)_k ||_{L^s} ||(\varphi * w - w)||_{L^e} + ||(\varphi * w)((\varphi * w)_k - w_k)||_{L^e}), \tag{2.32} \]

where \( C_1 \) is a positive constant.

Note that \( \mathbb{H}^r \subset L^s \) and \( w = A^2(\varphi(\kappa) * \tilde{v}) \), then using Lemma 2.2, we have
\[
||f(w)||_{L^2} \leq C_1 ||w||_{L^s} < \frac{C_1}{\delta^2 - C_1} < \delta \tag{2.33} \]

for sufficiently small \( \delta \). Thus, \( T \) is a self-mapping on \( B_\delta \).

To show \( T \) is a contraction mapping, we choose \( w, \tilde{w} \in B_\delta \), by Lemma 2.8 and a direct calculation show that
\[
||f(w) - f(\tilde{w})||_{L^2} 
= ||-3(\varphi * w)((\varphi * w)_k + 2(\varphi * w)_k (\varphi * w) - 2(\varphi * w)_k (\varphi * w) + 3(\varphi * \tilde{w})(\varphi * \tilde{w}) - 2(\varphi * \tilde{w})(\varphi * \tilde{w}) + 2(\varphi * \tilde{w})(\varphi * \tilde{w}) - (\varphi * \tilde{w})(\varphi * \tilde{w}) - \varphi * \tilde{w} - (\varphi * \tilde{w})||_{L^2} 
\leq 3 ||(\varphi * \tilde{w})(\varphi * \tilde{w})||_{L^2} + (\varphi * \tilde{w})(\varphi * \tilde{w})||_{L^2} + 2 ||(\varphi * \tilde{w})(\varphi * \tilde{w})||_{L^2} + (\varphi * \tilde{w})(\varphi * \tilde{w})||_{L^2} + 3 ||(\varphi * \tilde{w})(\varphi * \tilde{w})||_{L^2} + (\varphi * \tilde{w})(\varphi * \tilde{w})||_{L^2} + ||(\varphi * \tilde{w})(\varphi * \tilde{w})||_{L^2} + (\varphi * \tilde{w})(\varphi * \tilde{w})||_{L^2} + ||(\varphi * \tilde{w})(\varphi * \tilde{w})||_{L^2} + (\varphi * \tilde{w})(\varphi * \tilde{w})||_{L^2}
\leq C_2 ||w - \tilde{w}||_{L^2}. \tag{2.34} \]

Thus,
\[
||T(w)(\tau) - T(\tilde{w})(\tau)||_{L^2} \leq C_2 ||w - \tilde{w}||_{L^2}. \tag{2.35} \]

Since \( \delta > 0 \) is sufficiently small, \( T \) is a contraction mapping. \( \square \)

Thus, we have the following existence results.

**Proposition 2.10.** Let \( s > 2 \) be a fixed integer and \( \delta > 0 \) is a sufficiently small constant. Then

- if \( ||w||_{L^{s+1}} < \delta \), there exists a unique solution \( w \in B_\delta \) to (2.5) with the ini-
tial data (2.6) and the boundary condition (2.7).

- there exists a global solution $\psi(\tau, \rho) \in H^s$ to (2.3) with the initial data (2.6) and the boundary condition (2.7). Moreover, if the initial data $\psi_0$ satisfies $\|\psi_0\|_{H^{s+1}} < \delta$, then

$$\|\psi\|_{H^s} \leq \frac{\tilde{C}}{\alpha^2 e^{\epsilon} (1-C \tau)}.$$  \hspace{1cm} (2.36)

Here $C$ and $\tilde{C}$ are two positive constants that depend on $s$.

**Proof.** By Lemma 2.9 and the Banach fixed point theorem, the map $T$ has a fixed point in $B_\delta$, which is a solution of Equation (2.5). Thus, there exists a global solution of (2.3) as

$$\psi(\tau, \rho) = \epsilon^\gamma \psi (\tau, e^{-\gamma} \rho) = \epsilon^\gamma \left( (\varphi \ast w)(\tau, e^{-\gamma} \rho) \right).$$  \hspace{1cm} (2.37)

Furthermore, we have

$$v_{pp} = \psi_{pp} = (\varphi \ast w)_{xx} e^{-\gamma} = \alpha^2 e^{-\gamma} (\varphi \ast w - w).$$  \hspace{1cm} (2.38)

Thus, by Lemma 2.2, we get

$$\|\psi_{pp}\|_{H^{s-2}} \leq \alpha^2 e^{-\gamma} \|\varphi \ast w - w\|_{H^{s-2}} \leq \tilde{C} \alpha^2 e^{-\gamma} \|\psi\|_{H^{s-2}} \leq \frac{\tilde{C}}{\alpha^2 e^{\epsilon} (1-C \tau)},$$  \hspace{1cm} (2.39)

where we have used $\delta < 1$ in the last inequality. This completes the proof. $\square$

As a consequence, we obtain that the global well-posedness of the initial value problem (2.2). This implies that the asymptotic stability of the explicit singular solution (1.3) for the Camassa-Holm Equation (1.2). Hence, we complete the proof of Theorem 1.1.

### 3. Conclusion

In this paper, the Semigroup theory of linear operators has been used to study the asymptotic stability of the explicit blowup solution of Camassa-Holm equation. This result shows that the explicit solution is a meaningful physical solution. However, this explicit solution does not depend on the wavelength (i.e., it does not depend on $\alpha$). Thus, further studies are needed to construct the explicit solutions that depend on $\alpha$, and then prove their stability.

### Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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