HS-integral and Eisenstein integral mixed Cayley graphs over abelian groups

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Abstract

A mixed graph is called second kind hermitian integral (or HS-integral) if the eigenvalues of its Hermitian-adjacency matrix of second kind are integers. A mixed graph is called Eisenstein integral if the eigenvalues of its (0, 1)-adjacency matrix are Eisenstein integers. Let $\Gamma$ be an abelian group. We characterize the set $S$ for which a mixed Cayley graph $\text{Cay}(\Gamma, S)$ is HS-integral. We also show that a mixed Cayley graph is Eisenstein integral if and only if it is HS-integral.

Keywords. Hermitian adjacency matrix of second kind, mixed Cayley graph; HS-integral mixed graph; Eisenstein integral mixed graph
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1 Introduction

A mixed graph $G$ is a pair $(V(G), E(G))$, where $V(G)$ and $E(G)$ are the vertex set and the edge set of $G$, respectively. Here $E(G) \subseteq V(G) \times V(G) \setminus \{(u, u) \mid u \in V(G)\}$. If $G$ is a mixed graph, then $(u, v) \in E(G)$ need not imply that $(v, u) \in E(G)$. An edge $(u, v)$ of a mixed graph $G$ is called undirected if both $(u, v)$ and $(v, u)$ belong to $E(G)$. An edge $(u, v)$ of a mixed graph $G$ is called directed if $(u, v) \in E(G)$ but $(v, u) \notin E(G)$. A mixed graph can have both undirected and directed edges. A mixed graph $G$ is said to be a simple graph if all the edges of $G$ are undirected. A mixed graph $G$ is said to be an oriented graph if all the edges of $G$ are directed.

For a mixed graph $G$ on $n$ vertices, its $(0, 1)$-adjacency matrix and Hermitian-adjacency matrix of second kind are denoted by $A(G) = (a_{uv})_{n \times n}$ and $H(G) = (h_{uv})_{n \times n}$, respectively, where
The Hermitian-adjacency matrix of second kind was introduced by Bojan Mohar [26]. Let \( G \) be a mixed graph. By an \textit{HS-eigenvalue} of \( G \), we mean an eigenvalue of \( \mathcal{H}(G) \). By an \textit{eigenvalue} of \( G \), we mean an eigenvalue of \( \mathcal{A}(G) \). Similarly, the \textit{HS-spectrum} of \( G \), denoted \( \text{Sp}_H(G) \), is the multi-set of the HS-eigenvalues of \( G \), and the \textit{spectrum} of \( G \), denoted \( \text{Sp}(G) \), is the multi-set of the eigenvalues of \( G \). Note that the Hermitian-adjacency matrix of second kind of a mixed graph is a Hermitian matrix, and so its HS-eigenvalues are real numbers. However, if a mixed graph \( G \) contains at least one directed edge, then \( \mathcal{A}(G) \) is non-symmetric. Accordingly, the eigenvalues of \( G \) need not be real numbers. The matrix obtained by replacing \( \frac{1+i\sqrt{3}}{2} \) and \( \frac{1-i\sqrt{3}}{2} \) by \( i \) and \( -i \), respectively, in \( \mathcal{H}(G) \), is called the \textit{Hermitian adjacency} matrix of \( G \). Hermitian adjacency matrix of mixed graphs was introduced in [13, 24].

A mixed graph is called \textit{H-integral} if the eigenvalues of its Hermitian adjacency matrix are integers. A mixed graph \( G \) is said to be \textit{HS-integral} if all the HS-eigenvalues of \( G \) are integers. A mixed graph \( G \) is said to be \textit{Eisenstein integral} if all the eigenvalues of \( G \) are Eisenstein integers. Note that complex numbers of the form \( a+b\omega_3 \), where \( a, b \in \mathbb{Z}, \omega_3 = \frac{-1+i\sqrt{3}}{2} \), are called \textit{Eisenstein} integers. An HS-integral simple graph is called an \textit{integral} graph. Note that \( \mathcal{A}(G) = \mathcal{H}(G) \) for a simple graph \( G \). Therefore in case of a simple graph \( G \), the terms HS-eigenvalue, HS-spectrum and HS-integrality of \( G \) are the same with that of eigenvalue, spectrum and integrality of \( G \), respectively.

Integrality of simple graphs have been extensively studied in the past. Integral graphs were first defined by Harary and Schwenk [14] in 1974 and proposed a classification of integral graphs. See [5] for a survey on integral graphs. Watanabe and Schwenk [32, 33] proved several interesting results on integral trees in 1979. Csikvari [11] constructed integral trees with arbitrary large diameters in 2010. Further research on integral trees can be found in [8, 7, 30, 31]. In 2009, Ahmadi et al. [2] proved that only a fraction of \( 2^{-\Omega(n)} \) of the graphs on \( n \) vertices have an integral spectrum. Bussemaker et al. [9] proved that there are exactly 13 connected cubic integral graphs. Stevanović [29] studied the 4-regular integral graphs avoiding \( \pm 3 \) in the spectrum, and Lepović et al. [22] proved that there are 93 non-regular, bipartite integral graphs with maximum degree four.

Let \( S \) be a subset, not containing the identity element, of a group \( \Gamma \). The set \( S \) is said to be \textit{symmetric} (resp. \textit{skew-symmetric}) if \( S \) is closed under inverse (resp. \( a^{-1} \notin S \) for all \( a \in S \)). Define \( \overline{S} = \{ u \in S : u^{-1} \notin S \} \). Clearly, \( S \setminus \overline{S} \) is symmetric and \( \overline{S} \) is skew-symmetric. The \textit{mixed Cayley graph} \( G = \text{Cay}(\Gamma, S) \) is a mixed graph, where \( V(G) = \Gamma \) and \( E(G) = \{ (a, b) : a^{-1}b \in S, a, b \in \Gamma \} \). If \( S \) is symmetric then \( G \) is a \textit{simple Cayley graph}. If \( S \) is skew-symmetric then \( G \) is an \textit{oriented Cayley graph}. 

\[ a_{uv} = \begin{cases} 1 & \text{if } (u, v) \in E \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad h_{uv} = \begin{cases} 1 & \text{if } (u, v) \in E \text{ and } (v, u) \in E \\ \frac{1+i\sqrt{3}}{2} & \text{if } (u, v) \in E \text{ and } (v, u) \notin E \\ \frac{1-i\sqrt{3}}{2} & \text{if } (u, v) \notin E \text{ and } (v, u) \in E \\ 0 & \text{otherwise}. \end{cases} \]
In 1982, Bridge and Mena [6] introduced a characterization of integral Cayley graphs over abelian groups. Later on, same characterization was rediscovered by Wasin So [27] for cyclic groups in 2005. In 2009, Abdollahi and Vatandoost [1] proved that there are exactly seven connected cubic integral Cayley graphs. On the same year, Klotz and Sander [20] proved that if a Cayley graph Cay(Γ, S) over an abelian group Γ is integral then S belongs to the Boolean algebra B(Γ) generated by the subgroups of Γ. Moreover, they conjectured that the converse is also true, which was proved by Alperin and Peterson [3]. In 2015, Ku et al. [21] proved that normal Cayley graphs over the symmetric groups are integral. In 2017, Lu et al. [25] gave necessary and sufficient condition for the integrality of Cayley graphs over the dihedral group Dn. In particular, they completely determined all integral Cayley graphs over the dihedral group Dp for a prime p. In 2019, Cheng et al. [10] obtained several simple sufficient conditions for the integrality of Cayley graphs over the dicyclic group \( T_{4n} = \langle a, b^2 | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle \). In particular, they also completely determined all integral Cayley graphs over the dicyclic group \( T_{4p} \) for a prime p. In 2014, Godsil et al. [12] characterized integral normal Cayley graphs. Xu et al. [34] and Li [23] characterized the set S for which the mixed circulant graph Cay(\( Z_n \), S) is Gaussian integral. In 2006, So [27] introduced characterization of integral circulant graphs. In [16], the authors provide an alternative proof of the characterization obtained in [23, 34]. H-integral mixed circulant graphs, H-integral mixed Cayley graphs over abelian groups, H-integral normal Cayley graphs and HS-integral mixed circulant graphs have been characterized in [18], [17], [16] and [19], respectively.

Throughout this paper, we consider Cayley graphs over abelian groups. The paper is organized as follows. In Section 2, some preliminary concepts and results are discussed. In particular, we express the HS-eigenvalues of a mixed Cayley graph as a sum of HS-eigenvalues of a simple Cayley graph and an oriented Cayley graph. In Section 3, we obtain a sufficient condition on the connection set for the HS-integrality of an oriented Cayley graph. In Section 4, we first characterize HS-integrality of oriented Cayley graphs by proving the necessity of the condition obtained in Section 3. After that, we extend this characterization to mixed Cayley graphs. In Section 5, we prove that a mixed Cayley graph is Eisenstein integral if and only if it is HS-integral.

2 Preliminaries

A representation of a finite group \( \Gamma \) is a homomorphism \( \rho : \Gamma \rightarrow GL(V) \), where \( GL(V) \) is the group of automorphisms of a finite dimensional vector space \( V \) over the complex field \( \mathbb{C} \). The dimension of \( V \) is called the degree of \( \rho \). Two representations \( \rho_1 \) and \( \rho_2 \) of \( \Gamma \) on \( V_1 \) and \( V_2 \), respectively, are equivalent if there is an isomorphism \( T : V_1 \rightarrow V_2 \) such that \( T\rho_1(g) = \rho_2(g)T \) for all \( g \in \Gamma \).

Let \( \rho : \Gamma \rightarrow GL(V) \) be a representation. The character \( \chi_{\rho} : \Gamma \rightarrow \mathbb{C} \) of \( \rho \) is defined by setting \( \chi_{\rho}(g) = Tr(\rho(g)) \) for \( g \in \Gamma \), where \( Tr(\rho(g)) \) is the trace of the representation matrix of \( \rho(g) \). By degree
of \( \chi_\rho \) we mean the degree of \( \rho \) which is simply \( \chi_\rho(1) \). If \( W \) is a \( \rho(g) \)-invariant subspace of \( V \) for each \( g \in \Gamma \), then we say \( W \) a \( \rho(\Gamma) \)-invariant subspace of \( V \). If the only \( \rho(\Gamma) \)-invariant subspaces of \( V \) are \{0\} and \( V \), we say \( \rho \) an irreducible representation of \( \Gamma \), and the corresponding character \( \chi_\rho \) an irreducible character of \( \Gamma \).

For a group \( \Gamma \), we denote by IRR(\( \Gamma \)) and Irr(\( \Gamma \)) the complete set of non-equivalent irreducible representations of \( \Gamma \) and the complete set of non-equivalent irreducible characters of \( \Gamma \), respectively.

Throughout this paper, we consider \( \Gamma \) to be an abelian group of order \( n \). Let \( S \) be a subset of \( \Gamma \) with \( 0 \notin S \), where 0 is the additive identity of \( \Gamma \). Then \( \Gamma \) is isomorphic to the direct product of cyclic groups of prime power order, i.e.

\[
\Gamma \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k},
\]

where \( n = n_1 \cdots n_k \), and \( n_j \) is a power of a prime number for each \( j = 1, \ldots, k \). We consider an abelian group \( \Gamma \) as \( \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \) of order \( n = n_1 \cdots n_k \). We consider the elements \( x \in \Gamma \) as elements of the cartesian product \( \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \), i.e.

\[
x = (x_1, x_2, \ldots, x_k), \quad \text{where } x_j \in \mathbb{Z}_{n_j} \text{ for all } 1 \leq j \leq k.
\]

Addition in \( \Gamma \) is done coordinate-wise modulo \( n_j \). For a positive integer \( k \) and \( a \in \Gamma \), we denote by \( ka \) or \( a^k \) the \( k \)-fold sum of \( a \) to itself, \((-k)a = k(-a)\), \( 0a = 0 \), and inverse of \( a \) by \(-a\).

**Lemma 2.1.** [28] Let \( \mathbb{Z}_n = \{0, 1, \ldots, n-1\} \) be a cyclic group of order \( n \). Then IRR(\( \mathbb{Z}_n \)) = \{\( \phi_k : 0 \leq k \leq n - 1 \}, \) where \( \phi_k(j) = \omega_n^{jk} \) for all \( 0 \leq j, k \leq n - 1 \), and \( \omega_n = \exp\left(\frac{2\pi i}{n}\right) \).

**Lemma 2.2.** [28] Let \( \Gamma_1, \Gamma_2 \) be abelian groups of order \( m, n \), respectively. Let IRR(\( \Gamma_1 \)) = \{\( \phi_1, \ldots, \phi_m \}\), and IRR(\( \Gamma_2 \)) = \{\( \rho_1, \ldots, \rho_n \}\). Then

\[
\text{IRR}(\Gamma_1 \times \Gamma_2) = \{\psi_{kl} : 1 \leq k \leq m, 1 \leq l \leq n\},
\]

where \( \psi_{kl} : \Gamma_1 \times \Gamma_2 \to \mathbb{C}^* \) and \( \psi_{kl}(g_1, g_2) = \phi_k(g_1)\rho_l(g_2) \) for all \( g_1 \in \Gamma_1, g_2 \in \Gamma_2 \).

Consider \( \Gamma = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k} \). By Lemma 2.1 and Lemma 2.2, IRR(\( \Gamma \)) = \{\( \psi_\alpha : \alpha \in \Gamma \}, \) where

\[
\psi_\alpha(x) = \prod_{j=1}^{k} \omega_{n_j}^{\alpha_j x_j} \text{ for all } \alpha = (\alpha_1, \ldots, \alpha_k), x = (x_1, \ldots, x_k) \in \Gamma,
\]

and \( \omega_{n_j} = \exp\left(\frac{2\pi i}{n_j}\right) \). Since \( \Gamma \) is an abelian group, every irreducible representation of \( \Gamma \) is 1-dimensional and thus it can be identified with its characters. Hence IRR(\( \Gamma \)) = Irr(\( \Gamma \)). For \( x \in \Gamma \), let \( \text{ord}(x) \) denote the order of \( x \). The following lemma can be easily proved.

**Lemma 2.3.** Let \( \Gamma \) be an abelian group and Irr(\( \Gamma \)) = \{\( \psi_\alpha : \alpha \in \Gamma \}). Then the following statements are true.
(i) $\psi_\alpha(x) = \psi_x(\alpha)$ for all $x, \alpha \in \Gamma$.

(ii) $(\psi_\alpha(x))^{\text{ord}(x)} = (\psi_x(\alpha))^{\text{ord}(\alpha)} = 1$ for all $x, \alpha \in \Gamma$.

Let $f : \Gamma \to \mathbb{C}$ be a function. The Cayley color digraph of $\Gamma$ with connection function $f$, denoted by $\text{Cay}(\Gamma, f)$, is defined to be the directed graph with vertex set $\Gamma$ and arc set $\{(x, y) : x, y \in \Gamma\}$ such that each arc $(x, y)$ is colored by $f(x^{-1}y)$. The adjacency matrix of $\text{Cay}(\Gamma, f)$ is defined to be the matrix whose rows and columns are indexed by the elements of $\Gamma$, and the $(x, y)$-entry is equal to $f(x^{-1}y)$. The eigenvalues of $\text{Cay}(\Gamma, f)$ are simply the eigenvalues of its adjacency matrix.

**Theorem 2.4.** [4] Let $\Gamma$ be a finite abelian group and $\text{Irr}(\Gamma) = \{\psi_\alpha : \alpha \in \Gamma\}$. Then the spectrum of the Cayley color digraph $\text{Cay}(\Gamma, f)$ is $\{\gamma_\alpha : \alpha \in \Gamma\}$, where

$$
\gamma_\alpha = \sum_{y \in \Gamma} f(y) \psi_\alpha(y) \quad \text{for all } \alpha \in \Gamma.
$$

For a subset $S$ of an abelian group $\Gamma$, let $S^{-1} = \{s^{-1} : s \in S\}$.

**Lemma 2.5.** [4] Let $\Gamma$ be an abelian group and $\text{Irr}(\Gamma) = \{\psi_\alpha : \alpha \in \Gamma\}$. Then the HS-spectrum of the mixed Cayley graph $\text{Cay}(\Gamma, S)$ is $\{\lambda_\alpha : \alpha \in \Gamma\}$, where $\lambda_\alpha = \lambda_\alpha + \mu_\alpha$ and

$$
\lambda_\alpha = \sum_{s \in S \setminus \overline{S}} \psi_\alpha(s), \quad \mu_\alpha = \sum_{s \in S} \left(\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s)\right) \quad \text{for all } \alpha \in \Gamma.
$$

**Proof.** Define $f_S : \Gamma \to \{0, 1, \omega_6, \omega_6^5\}$ such that

$$
f_S(s) = \begin{cases} 
1 & \text{if } s \in S \setminus \overline{S} \\
\omega_6 & \text{if } s \in \overline{S} \\
\omega_6^5 & \text{if } s \in S^{-1} \\
0 & \text{otherwise.}
\end{cases}
$$

The adjacency matrix of the Cayley color digraph $\text{Cay}(\Gamma, f_S)$ agrees with the Hermitian adjacency matrix of the mixed Cayley graph $\text{Cay}(\Gamma, S)$. Thus the result follows from Theorem 2.4. \qed

Next two corollaries are special cases of Lemma 2.5.

**Corollary 2.5.1.** [20] Let $\Gamma$ be an abelian group and $\text{Irr}(\Gamma) = \{\psi_\alpha : \alpha \in \Gamma\}$. Then the spectrum of the Cayley graph $\text{Cay}(\Gamma, S)$ is $\{\lambda_\alpha : \alpha \in \Gamma\}$, where $\lambda_\alpha = \lambda_{-\alpha}$ and

$$
\lambda_\alpha = \sum_{s \in S} \psi_\alpha(s) \quad \text{for all } \alpha \in \Gamma.
$$

**Corollary 2.5.2.** Let $\Gamma$ be an abelian group and $\text{Irr}(\Gamma) = \{\psi_\alpha : \alpha \in \Gamma\}$. Then the spectrum of the oriented Cayley graph $\text{Cay}(\Gamma, S)$ is $\{\mu_\alpha : \alpha \in \Gamma\}$, where

$$
\mu_\alpha = \sum_{s \in S} \left(\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s)\right) \quad \text{for all } \alpha \in \Gamma.
$$
Let \( n \geq 2 \) be a fixed positive integer. Define \( G_n(d) = \{ k : 1 \leq k \leq n - 1, \gcd(k, n) = d \} \). It is clear that \( G_n(d) = dG_{n/d}(1) \). Alperin and Peterson [3] considered a Boolean algebra generated by a class of subgroups of a group in order to determine the integrality of Cayley graphs over abelian groups. Suppose \( \Gamma \) is a finite group, and \( \mathcal{F}_\Gamma \) is the family of all subgroups of \( \Gamma \). The Boolean algebra \( \mathbb{B}(\Gamma) \) generated by \( \mathcal{F}_\Gamma \) is the set whose elements are obtained by arbitrary finite intersections, unions, and complements of the elements in the family \( \mathcal{F}_\Gamma \). The minimal non-empty elements of this algebra are called atoms. Thus each element of \( \mathbb{B}(\Gamma) \) is the union of some atoms. Consider the equivalence relation \( \sim \) on \( \Gamma \) such that \( x \sim y \) if and only if \( y = x^k \) for some \( k \in G_m(1) \), where \( m = \ord(x) \).

**Lemma 2.6.** [3] The equivalence classes of \( \sim \) are the atoms of \( \mathbb{B}(\Gamma) \).

For \( x \in \Gamma \), let \([x]\) denote the equivalence class of \( x \) with respect to the relation \( \sim \). Also, let \( \langle x \rangle \) denote the cyclic group generated by \( x \).

**Lemma 2.7.** [3] The atoms of the Boolean algebra \( \mathbb{B}(\Gamma) \) are the sets \([x] = \{ y : \langle y \rangle = \langle x \rangle \} \).

By Lemma 2.7, each element of \( \mathbb{B}(\Gamma) \) is a union of some sets of the form \([x]=\{y:\langle y\rangle=\langle x\rangle\}\). Thus, for all \( S \in \mathbb{B}(\Gamma) \), we have \( S = [x_1] \cup \ldots \cup [x_k] \) for some \( x_1, \ldots, x_k \in \Gamma \). The next result provides a complete characterization of integral Cayley graphs over an abelian group \( \Gamma \) in terms of the atoms of \( \mathbb{B}(\Gamma) \).

**Theorem 2.8.** ([3], [6]) Let \( \Gamma \) be an abelian group. The Cayley graph \( \text{Cay}(\Gamma, S) \) is integral if and only if \( S \in \mathbb{B}(\Gamma) \).

Define \( \Gamma(3) \) to be the set of all \( x \in \Gamma \) satisfying \( \ord(x) \equiv 0 \pmod{3} \). For all \( x \in \Gamma(3) \) and \( r \in \{0, 1, 2\} \), define

\[
M_r(x) := \{ x^k : 1 \leq k \leq \ord(x), k \equiv r \pmod{3} \}.
\]

For all \( a \in \Gamma \) and \( S \subseteq \Gamma \), define \( a + S := \{ a + s : s \in S \} \) and \( -S := \{ -s : s \in S \} \). Note that \( -s \) denotes the inverse of \( s \), that is \( -s = s^{m-1} \), where \( m = \ord(s) \).

**Lemma 2.9.** Let \( \Gamma \) be an abelian group and \( x \in \Gamma(3) \). Then the following statements are true.

(i) \( \bigcup_{r=0}^{2} M_r(x) = \langle x \rangle \).

(ii) Both \( M_1(x) \) and \( M_2(x) \) are skew-symmetric subsets of \( \Gamma \).

(iii) \( -M_1(x) = M_2(x) \) and \( -M_2(x) = M_1(x) \).

(iv) \( a + M_1(x) = M_1(x) \) and \( a + M_2(x) = M_2(x) \) for all \( a \in M_0(x) \).

**Proof.** (i) It follows from the definitions of \( M_r(x) \) and \( \langle x \rangle \).
(ii) Let $\text{ord}(x) = m$. If $x^k \in M_1(x)$ then $-x^k = x^{m-k} \not\in M_1(x)$, as $k \equiv 1 \pmod{3}$ gives $m-k \equiv 2 \pmod{3}$. Thus $M_1(x)$ is a skew-symmetric subset of $\Gamma$. Similarly, $M_2(x)$ is also a skew-symmetric subset of $\Gamma$.

(iii) Let $\text{ord}(x) = m$. As $k \equiv 1 \pmod{3}$ if and only if $m-k \equiv 2 \pmod{3}$, and $-x^k = x^{m-k}$, we get $-M_1(x) = M_2(x)$ and $-M_2(x) = M_1(x)$.

(iv) Let $a \in M_0(x)$ and $y \in a + M_1(x)$. Then $a = x^{k_1}$ and $y = x^{k_1} + x^{k_2} = x^{k_1+k_2}$, where $k_1 \equiv 0 \pmod{3}$ and $k_2 \equiv 1 \pmod{3}$. Since $k_1 + k_2 \equiv 1 \pmod{3}$, we have $y \in M_1(x)$ implying that $a+M_1(x) \subseteq M_1(x)$. Hence $a+M_1(x) = M_1(x)$. Similarly, $a+M_2(x) = M_2(x)$ for all $a \in M_0(x)$. □

Let $m \equiv 0 \pmod{3}$. For $r \in \{1, 2\}$ and $g \in \mathbb{Z}$, define the following:

$$G_{m,3}^r(1) = \{ k : 1 \leq k \leq m-1, \gcd(k, m) = 1, k \equiv r \pmod{3} \},$$

$$D_{g,3} = \{ k : k \text{ divides } g, k \not\equiv 0 \pmod{3} \}, \text{ and}$$

$$D_{g,3}^r = \{ k : k \text{ divides } g, k \equiv r \pmod{3} \}.$$  

It is clear that $D_{g,3} = D_{g,3}^1 \cup D_{g,3}^2$. Define an equivalence relation $\approx$ on $\Gamma(3)$ such that $x \approx y$ if and only if $y = x^k$ for some $k \in G_{m,3}^1(1)$, where $m = \text{ord}(x)$. Observe that if $x, y \in \Gamma(3)$ and $x \approx y$ then $x \sim y$, but the converse need not be true. For example, consider $x = 5 \pmod{12}$, $y = 7 \pmod{12}$ in $\mathbb{Z}_{12}$. Here $x, y \in \mathbb{Z}_{12}(3)$ and $x \sim y$ but $x \not\approx y$. For $x \in \Gamma(3)$, let $\langle x \rangle$ denote the equivalence class of $x$ with respect to the relation $\approx$.

**Lemma 2.10.** Let $\Gamma$ be an abelian group, $x \in \Gamma(3)$ and $m = \text{ord}(x)$. Then the following are true.

(i) $\langle x \rangle = \{ x^k : k \in G_{m,3}^1(1) \}$.

(ii) $\langle -x \rangle = \{ x^k : k \in G_{m,3}^2(1) \}$.

(iii) $\langle x \rangle \cap \langle -x \rangle = \emptyset$.

(iv) $[x] = \langle x \rangle \cup \langle -x \rangle$.

**Proof.** (i) Let $y \in \langle x \rangle$. Then $x \approx y$, and so $\text{ord}(x) = \text{ord}(y) = m$ and there exists $k \in G_{m,3}^1(x)$ such that $y = x^k$. Thus $\langle y \rangle \subseteq \{ x^k : k \in G_{m,3}^1(1) \}$. On the other hand, let $z = x^k$ for some $k \in G_{m,3}^1(1)$. Then $\text{ord}(x) = \text{ord}(z)$ and so $x \approx z$. Thus $\{ x^k : k \in G_{m,3}^1(1) \} \subseteq \langle x \rangle$.

(ii) Note that $-x = x^{m-1}$ and $m-1 \equiv 2 \pmod{3}$. By Part (i),

$$\langle -x \rangle = \{ (-x)^k : k \in G_{m,3}^1(1) \} = \{ x^{(m-1)k} : k \in G_{m,3}^1(1) \} = \{ x^{-k} : k \in G_{m,3}^1(1) \} = \{ x^k : k \in G_{m,3}^2(1) \}.$$
(iii) Since $G_{m,3}^1(1) \cap G_{m,3}^2(1) = \emptyset$, so by Part (i) and Part (ii), $\langle x \rangle \cap \langle -x \rangle = \emptyset$ holds.

(iv) Since $[x] = \{x^k : k \in G_m(1)\}$ and $G_m(1)$ is a disjoint union of $G_{m,3}^1(1)$ and $G_{m,3}^2(1)$, by Part (i) and Part (ii), $[x] = \langle x \rangle \cup \langle -x \rangle$ holds. □

**Lemma 2.11.** Let $\Gamma$ be an abelian group, $x \in \Gamma(3)$, $m = \text{ord}(x)$ and $g = \frac{m}{3}$. Then the following are true.

(i) $M_1(x) \cup M_2(x) = \bigcup_{h \in D_{g,3}} [x^h]$.

(ii) $M_1(x) = \bigcup_{h \in D_{g,3}^1} \langle x^h \rangle \cup \bigcup_{h \in D_{g,3}^2} \langle -x^h \rangle$.

(iii) $M_2(x) = \bigcup_{h \in D_{g,3}^2} \langle -x^h \rangle \cup \bigcup_{h \in D_{g,3}^2} \langle x^h \rangle$.

**Proof.** (i) Let $x^k \in M_1(x) \cup M_2(x)$, where $k \equiv 1$ or $2 \pmod{3}$). To show that $x^k \in \bigcup_{h \in D_{g,3}} [x^h]$, it is enough to show $x^k \sim x^h$ for some $h \in D_{g,3}$. Let $h = \gcd(k, g) \in D_{g,3}$. Note that

$$\text{ord}(x^k) = \frac{m}{\gcd(m, k)} = \frac{m}{\gcd(g, k)} = \frac{m}{h} = \text{ord}(x^h).$$

Also, as $h = \gcd(k, m)$, we have $\langle x^k \rangle = \langle x^h \rangle$, and so $x^k = x^h j$ for some $j \in G_q(1)$, where $q = \text{ord}(x^h) = \frac{m}{h}$. Thus $x^k \sim x^h$ where $h = \gcd(k, g) \in D_{g,3}$. Conversely, let $z \in \bigcup_{h \in D_{g,3}} [x^h]$. Then there exists $h \in D_{g,3}$ such that $z = x^h j$ where $j \in G_q(1)$ and $q = \frac{m}{\gcd(m, h)}$. Now $h \in D_{g,3}$ and $q \equiv 0 \pmod{3}$ imply that $hj \equiv 1$ or $2 \pmod{3}$, and so $\bigcup_{h \in D_{g,3}} [x^h] \subseteq M_1(x) \cup M_2(x)$. Hence $M_1(x) \cup M_2(x) = \bigcup_{h \in D_{g,3}} [x^h]$.

(ii) Let $x^k \in M_1(x)$, where $k \equiv 1 \pmod{3}$. By Part (i), there exists $h \in D_{g,3}$ and $j \in G_q(1)$ such that $x^k = x^h j$, where $q = \frac{m}{\gcd(m, h)}$. Note that $k = hj$. If $h \equiv 1 \pmod{3}$ then $j \in G_{q,3}^1(1)$, otherwise $j \in G_{q,3}^2(1)$. Thus using parts (i) and (ii) of Lemma 2.10, if $h \equiv 1 \pmod{3}$ then $x^k \sim x^h$, otherwise $x^k \approx -x^h$. Hence $M_1(x) \subseteq \bigcup_{h \in D_{g,3}^1} \langle x^h \rangle \cup \bigcup_{h \in D_{g,3}^2} \langle -x^h \rangle$. Conversely, assume that $z \in \bigcup_{h \in D_{g,3}^1} \langle x^h \rangle \cup \bigcup_{h \in D_{g,3}^2} \langle -x^h \rangle$. This gives $z \in \langle x^h \rangle$ for an $h \in D_{g,3}^1$ or $z \in \langle -x^h \rangle$ for an $h \in D_{g,3}^2$. In the first case, by part (i) of Lemma 2.10, there exists $j \in G_{q,3}^1(1)$ with $q = \frac{m}{\gcd(m, h)}$ such that $z = x^h j$. Similarly, for the second case, by part (ii) of Lemma 2.10, there exists $j \in G_{q,3}^2(1)$ with $q = \frac{m}{\gcd(m, h)}$ such that $z = x^h j$. In both the cases, $hj \equiv 1 \pmod{3}$. Thus $z \in M_1(x)$.

(iii) The proof is similar to Part (ii). □

The **cyclotomic polynomial** $\Phi_m(x)$ is the monic polynomial whose zeros are the primitive $m^{th}$ roots of unity. That is,

$$\Phi_m(x) = \prod_{a \in G_m(1)} (x - \omega_m^a).$$
Clearly, the degree of $\Phi_m(x)$ is $\varphi(m)$, where $\varphi$ denotes the Euler $\varphi$-function. It is well known that the cyclotomic polynomial $\Phi_m(x)$ is monic and irreducible in $\mathbb{Z}[x]$. See [15] for more details on cyclotomic polynomials.

The polynomial $\Phi_m(x)$ is irreducible over $\mathbb{Q}(\omega_3)$ if and only if $[\mathbb{Q}(\omega_3, \omega_m) : \mathbb{Q}(\omega_3)] = \varphi(m)$. Also, $\mathbb{Q}(\omega_m)$ does not contain the number $\omega_3$ if and only if $m \not\equiv 0 \pmod{3}$. Thus, if $m \not\equiv 0 \pmod{3}$ then $[\mathbb{Q}(\omega_3, \omega_m) : \mathbb{Q}(\omega_m)] = 2 = [\mathbb{Q}(\omega_3), \mathbb{Q}]$, and therefore

$$[\mathbb{Q}(\omega_3, \omega_m) : \mathbb{Q}(\omega_3)] = \frac{[\mathbb{Q}(\omega_3, \omega_m) : \mathbb{Q}(\omega_m)] \times [\mathbb{Q}(\omega_m) : \mathbb{Q}]}{[\mathbb{Q}(\omega_3) : \mathbb{Q}]} = [\mathbb{Q}(\omega_m) : \mathbb{Q}] = \varphi(m).$$

Further, if $m \equiv 0 \pmod{3}$ then $\mathbb{Q}(\omega_3, \omega_m) = \mathbb{Q}(\omega_m)$, and so

$$[\mathbb{Q}(\omega_3, \omega_m) : \mathbb{Q}(\omega_3)] = \frac{[\mathbb{Q}(\omega_3, \omega_m) : \mathbb{Q}(\omega_3)]}{[\mathbb{Q}(\omega_3) : \mathbb{Q}]} = \frac{\varphi(m)}{2}.$$

Note that $\mathbb{Q}(\omega_3) = \mathbb{Q}(\omega_6) = \mathbb{Q}(i\sqrt{3})$. Therefore $\Phi_m(x)$ is irreducible over $\mathbb{Q}(\omega_3), \mathbb{Q}(\omega_6)$ or $\mathbb{Q}(i\sqrt{3})$ if and only if $m \not\equiv 0 \pmod{3}$.

Let $m \equiv 0 \pmod{3}$. Observe that $G_m(1)$ is a disjoint union of $G_{m,3}^1(1)$ and $G_{m,3}^2(1)$. Define

$$\Phi_{m,3}^1(x) = \prod_{a \in G_{m,3}^1(1)} (x - \omega_m^a) \quad \text{and} \quad \Phi_{m,3}^2(x) = \prod_{a \in G_{m,3}^2(1)} (x - \omega_m^a).$$

It is clear from the definition that $\Phi_m(x) = \Phi_{m,3}^1(x)\Phi_{m,3}^2(x)$.

**Theorem 2.12.** [19] Let $m \equiv 0 \pmod{3}$. Then $\Phi_{m,3}^1(x)$ and $\Phi_{m,3}^2(x)$ are irreducible monic polynomials in $\mathbb{Q}(\omega_3)[x]$ of degree $\frac{\varphi(m)}{2}$.

### 3 A sufficient condition for HS-integrality of oriented Cayley graphs over abelian groups

In this section, first we prove that $S = \emptyset$ is the only connection set for an HS-integral oriented Cayley graph $\text{Cay}(\Gamma, S)$ whenever $\Gamma(3) = \emptyset$. After that we obtain a sufficient condition on the set $S$ for which the oriented Cayley graph $\text{Cay}(\Gamma, S)$ is HS-integral.

**Lemma 3.1.** Let $S$ be a skew-symmetric subset of an abelian group $\Gamma$. If $\sum_{s \in S} i\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s)) = 0$ for all $j = 0, ..., n - 1$ then $S = \emptyset$.

**Proof.** Let $A_S = (a_{uv})_{n \times n}$ be the matrix whose rows and columns are indexed by the elements of $\Gamma$, where

$$a_{uv} = \begin{cases} i\sqrt{3} & \text{if } v - u \in S \\ -i\sqrt{3} & \text{if } v - u \in S^{-1} \\ 0 & \text{otherwise.} \end{cases}$$
Since $A_S$ is a circulant matrix, $\lambda_\alpha = \sum_{k \in S} i\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s))$ is an HS-eigenvalue of $A_S$ for each $\alpha \in \Gamma$. Therefore $\lambda_\alpha = 0$ for all $\alpha \in \Gamma$, which implies all the entries of $A_S$ are zero. Hence $S = \emptyset$.  

**Theorem 3.2.** Let $\Gamma$ be an abelian group and $\Gamma(3) = \emptyset$. Then the oriented Cayley graph $\text{Cay}(\Gamma, S)$ is HS-integral if and only if $S = \emptyset$

**Proof.** Let $G = \text{Cay}(\Gamma, S)$ and $Sp_H(G) = \{\mu_\alpha : \alpha \in \Gamma\}$. Assume that $\text{Cay}(\Gamma, S)$ is HS-integral and $n \not\equiv 0 \pmod{3}$. By Corollary 2.5.2,

$$\mu_\alpha = \sum_{s \in S} (\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s)) \in \mathbb{Z}, \text{ for all } \alpha \in \Gamma.$$ 

Note that, $\psi_\alpha(s)$ and $\psi_\alpha(-s)$ are $n^{th}$ roots of unity for all $\alpha \in \Gamma$, $s \in S$. Fix a primitive $n^{th}$ root $\omega$ of unity and express $\psi_\alpha(s)$ in the form $\omega^j$ for some $j \in \{0, 1, ..., n-1\}$. Thus

$$\mu_\alpha = \sum_{s \in S} (\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s)) = \sum_{j=0}^{n-1} a_j \omega^j,$$

where $a_j \in \mathbb{Q}(\omega_3)$. Since $\mu_\alpha \in \mathbb{Z}$, so $p(x) = \sum_{j=0}^{n-1} a_j x^j - \mu_\alpha \in \mathbb{Q}(\omega_3)[x]$ and $\omega$ is a root of $p(x)$. Since $n \not\equiv 0 \pmod{3}$, so $\Phi_n(x)$ is irreducible in $\mathbb{Q}(\omega_3)[x]$. Thus $p(\omega) = 0$ and $\Phi_n(\omega)$ is the monic irreducible polynomial over $\mathbb{Q}(\omega_3)$ having $\omega$ as a root. Therefore $\Phi_n(x)$ divides $p(x)$, and so $\omega^{-1} = \omega^{n-1}$ is also a root of $p(x)$. Note that, if $\psi_\alpha(s) = \omega^j$ for some $j \in \{0, 1, ..., n-1\}$ then $\psi_{-\alpha}(s) = \omega^{-j}$. We have

$$\sum_{s \in S} i\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s)) = \sum_{s \in S} [(\omega_6 - \omega_6^5)\psi_\alpha(s) + (\omega_6^5 - \omega_6)\psi_\alpha(-s)] = \sum_{j=0}^{n-1} a_j \omega^{-j} - \mu_\alpha = \mu - \alpha - \mu_\alpha = p(\omega^{-1}) = 0.$$

By Lemma 3.1, $S = \emptyset$. Conversely, if $S = \emptyset$ then all the HS-eigenvalues of $\text{Cay}(\Gamma, S)$ are zero. Thus $\text{Cay}(\Gamma, S)$ is HS-integral.  

**Lemma 3.3.** Let $\Gamma$ be an abelian group and $x \in \Gamma(3)$. Then $\sum_{s \in M_1(x)} (\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s))$ is an integer for each $\alpha \in \Gamma$.

**Proof.** Let $x \in \Gamma(3)$, $\alpha \in \Gamma$ and $\mu_\alpha = \sum_{s \in M_1(x)} (\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s))$.

**Case 1.** There exists $a \in M_0(x)$ such that $\psi_\alpha(a) \neq 1$. Then

$$\mu_\alpha = \sum_{s \in M_1(x)} (\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s)) = \sum_{s \in M_1(x)} \omega_6 \psi_\alpha(s) + \sum_{s \in M_2(x)} \omega_6^5 \psi_\alpha(s)$$

$$= \sum_{s \in \alpha + M_1(x)} \omega_6 \psi_\alpha(s) + \sum_{s \in \alpha + M_2(x)} \omega_6^5 \psi_\alpha(s)$$

$$= \psi_\alpha(a) \sum_{s \in M_1(x)} \omega_6 \psi_\alpha(s) + \psi_\alpha(a) \sum_{s \in M_2(x)} \omega_6^5 \psi_\alpha(s)$$

$$= \psi_\alpha(a) \mu_\alpha.$$
We have \((1 - \psi_\alpha(a))\mu_\alpha = 0\). Since \(\psi_\alpha(a) \neq 1\), we get \(\mu_\alpha = 0 \in \mathbb{Z}\).

**Case 2.** Assume that \(\psi_\alpha(a) = 1\) for all \(a \in M_0(x)\). Then \(\psi_\alpha(s) = \psi_\alpha(x)\) for all \(s \in M_1(x)\) and \(\psi_\alpha(s) = \psi_\alpha(x^2)\) for all \(s \in M_2(x)\). Therefore

\[
\mu_\alpha = \sum_{s \in M_1(x)} (\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s)) = \sum_{s \in M_1(x)} \omega_6 \psi_\alpha(s) + \sum_{s \in M_2(x)} \omega_6^5 \psi_\alpha(s) = |M_1(x)| (\omega_6 \psi_\alpha(x) + \omega_6^5 \psi_\alpha(x^2)) = -|M_1(x)| (\omega_3 \psi_\alpha(x) + \omega_3 \psi_\alpha(x^2)).
\]

Since \(\psi_\alpha(x^3) = 1\) then \(\psi_\alpha(x) = \omega_3\) or \(\omega_3^2\). If \(\psi_\alpha(x) = \omega_3\) then \(\mu_\alpha = -2|M_1(x)|\). If \(\psi_\alpha(x) = \omega_3^2\) then \(\mu_\alpha = |M_1(x)|\). Thus in both cases, \(\mu_\alpha\) are integers for all \(\alpha \in \Gamma\).

For \(x \in \Gamma(3)\) and \(\alpha \in \Gamma\), define

\[
Z_x(\alpha) = \sum_{s \in \langle x \rangle} (\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s)).
\]

**Lemma 3.4.** Let \(\Gamma\) be an abelian group and \(x \in \Gamma(3)\). Then \(Z_x(\alpha)\) is an integer for each \(\alpha \in \Gamma\).

**Proof.** Note that there exists \(x \in \Gamma(3)\) with \(\text{ord}(x) = 3\). Apply induction on \(\text{ord}(x)\). If \(\text{ord}(x) = 3\), then \(M_1(x) = \langle x \rangle\). Hence by Lemma 3.3, \(Z_x(\alpha)\) is an integer for each \(\alpha \in \Gamma\). Assume that the statement holds for all \(x \in \Gamma(3)\) with \(\text{ord}(x) \in \{3, 6, ..., 3(g - 1)\}\). We prove it for \(\text{ord}(x) = 3g\). Lemma 2.11 implies that

\[
M_1(x) = \bigcup_{h \in D_{g,3}^1} \langle x^h \rangle \cup \bigcup_{h \in D_{g,3}^2} \langle -x^h \rangle.
\]

If \(\text{ord}(x) = 3g = m, h \in D_{g,3}^1 \cup D_{g,3}^2\), and \(h > 1\) then \(\text{ord}(x^h), \text{ord}(-x^h) \in \{3, 6, ..., 3(g - 1)\}\). By induction hypothesis, both \(Z_{x^h}(\alpha)\) and \(Z_{-x^h}(\alpha)\) are integers for all \(\alpha \in \Gamma\). Now we have

\[
\sum_{s \in M_1(x)} (\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(s)) = Z_x(\alpha) + \sum_{h \in D_{g,3}^1, h > 1} Z_{x^h}(\alpha) + \sum_{h \in D_{g,3}^2, h > 1} Z_{-x^h}(\alpha).
\]

By Lemma 3.3 and induction hypothesis,

\[
Z_x(\alpha) = \sum_{s \in M_1(x)} (\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(s)) - \sum_{h \in D_{g,3}^1, h > 1} Z_{x^h}(\alpha) - \sum_{h \in D_{g,3}^2, h > 1} Z_{-x^h}(\alpha)
\]

is an integer for each \(\alpha \in \Gamma\).

For \(\Gamma(3) \neq \emptyset\), define \(E(\Gamma)\) to be the set of all skew-symmetric subsets of \(\Gamma\) of the form \(\langle x_1 \rangle \cup \ldots \cup \langle x_k \rangle\) for some \(x_1, ..., x_k \in \Gamma(3)\). For \(\Gamma(3) = \emptyset\), define \(E(\Gamma) = \{\emptyset\}\).

**Theorem 3.5.** Let \(\Gamma\) be an abelian group. If \(S \in E(\Gamma)\) then the oriented Cayley graph \(\text{Cay}(\Gamma, S)\) is HS-integral.
Proof. Assume that $S \in \mathbb{E}(\Gamma)$. Then $S = \langle \langle x_1 \rangle \rangle \cup \ldots \cup \langle \langle x_k \rangle \rangle$ for some $x_1, \ldots, x_k \in \Gamma(3)$. We have

$$
\mu_\alpha = \sum_{s \in S} \left( \omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s) \right) = \sum_{j=1}^k Z_{x_j}(\alpha).
$$

Now by Lemma 3.4, $\mu_\alpha$ is an integer for each $\alpha \in \Gamma$. Hence the oriented Cayley graph $\text{Cay}(\Gamma, S)$ is HS-integral.

\[\square\]

4 Characterization of HS-integral mixed Cayley graphs over abelian groups

Let $\Gamma$ be an abelian group of order $n$. Define $E$ to be the matrix of size $n \times n$, whose rows and columns are indexed by elements of $\Gamma$ such that $E_{x,y} = \psi_x(y)$. Note that each row of $E$ corresponds to a character of $\Gamma$ and $EE^* = nI_n$, where $E^*$ is the conjugate transpose of $E$. Let $v_{\langle \langle x \rangle \rangle}$ be the vector in $\mathbb{Q}(\omega_3)^n$ whose coordinates are indexed by the elements of $\Gamma$, and the $a^{th}$ coordinate of $v_{\langle \langle x \rangle \rangle}$ is given by

$$
v_{\langle \langle x \rangle \rangle}(a) = \begin{cases} 
\omega_6 & \text{if } a \in \langle \langle x \rangle \rangle \\
\omega_6^5 & \text{if } a \in \langle \langle -x \rangle \rangle \\
0 & \text{otherwise.}
\end{cases}
$$

By Lemma 3.4, we have $Ev_{\langle \langle x \rangle \rangle} \in \mathbb{Z}^n$. For $z \in \mathbb{C}$, let $\overline{z}$ denote the complex conjugate of $z$ and $\Re(z)$ (resp. $\Im(z)$) denote the real part (resp. imaginary part) of $z$.

Lemma 4.1. Let $\Gamma$ be an abelian group, $v \in \mathbb{Q}(\omega_3)^n$ and $Ev \in \mathbb{Q}^n$. Let the coordinates of $v$ be indexed by elements of $\Gamma$. Then

(i) $\overline{v}_x = v_{-x}$ for all $x \in \Gamma$.

(ii) $v_x = v_y$ for all $x, y \in \Gamma(3)$ satisfying $x \approx y$.

(iii) $\Re(v_x) = \Re(v_{-x})$ and $\Im(v_x) = \Im(v_{-x}) = 0$ for all $x \in \Gamma \setminus \Gamma(3)$.

Proof. Let $E_x$ and $E_y$ denote the column vectors of $E$ indexed by $x$ and $y$, respectively, and assume that $u = Ev \in \mathbb{Q}^n$.

(i) We use the fact that $\overline{\psi_x(y)} = \psi_{-x}(y) = \psi_x(-y)$ for all $x, y \in \Gamma$. Again

$u = Ev \Rightarrow E^*u = E^*Ev = (nI_n)v \Rightarrow \frac{1}{n} E^*u = v \in \mathbb{Q}(\omega_3)^n$. 

\[12\]
Thus
\[ v_x = \frac{1}{n} (E^* u)_x = \frac{1}{n} \sum_{a \in \Gamma} E^*_{x,a} u_a = \frac{1}{n} \sum_{a \in \Gamma} \psi_a(x) u_a = \frac{1}{n} \sum_{a \in \Gamma} \psi_a(-x) u_a \]
\[ = \frac{1}{n} \sum_{a \in \Gamma} \psi_a(-x) u_a \]
\[ = \frac{1}{n} \sum_{a \in \Gamma} E^*_{-x,a} u_a = \frac{1}{n} (E^* u)_{-x} = \mathcal{V}_{-x}. \]

(ii) If \( \Gamma(3) = \emptyset \) then there is nothing to prove. Now assume that \( \Gamma(3) \neq \emptyset \). Let \( x, y \in \Gamma(3) \) and \( x \sim y \).
Then there exists \( k \in G_{m,3}^1(1) \) such that \( y = x^k \), where \( m = \text{ord}(x) \). Assume \( x \neq y \), so that \( k \geq 2 \).
Using Lemma 2.3, entries of \( E \) are \( m \)-th roots of unity. Fix a primitive \( m \)-th root of unity \( \omega \), and express each entry of \( E_x \) and \( E_y \) in the form \( \omega^j \) for some \( j \in \{0, 1, \ldots, m-1\} \). Thus
\[ n v_x = (E^* u)_x = \sum_{j=0}^{m-1} a_j \omega^j, \]
where \( a_j \in \mathbb{Q} \) for all \( j \). Thus \( \omega \) is a root of the polynomial \( p(x) = \sum_{j=0}^{m-1} a_j x^j - n v_x \in \mathbb{Q}(\omega_3)[x] \).
Therefore \( p(x) \) is a multiple of the irreducible polynomial \( \Phi_{m,3}(x) \), and so \( \omega^k \) is also a root of \( p(x) \), because of \( k \in G_{m,3}^1(1) \). As \( y = x^k \) implies that \( \psi_a(y) = \psi_a(x)^k \) for all \( a \in \Gamma \), we have
\[ (E^* u)_y = \sum_{j=0}^{m-1} a_j \omega^{kj} \]. Hence
\[ 0 = p(\omega^k) = \sum_{j=0}^{m-1} a_j \omega^{kj} - n v_x = (E^* u)_y - n v_x = n v_y - n v_x \Rightarrow v_x = v_y. \]

(iii) Let \( x \in \Gamma \setminus \Gamma(3) \) and \( r = \text{ord}(x) \neq 0 \mod 3 \). Fix a primitive \( r \)-th root of unity, and express each entry of \( E_x \) in the form \( \omega^j \) for some \( j \in \{0, 1, \ldots, r-1\} \). Thus
\[ n v_x = (E^* u)_x = \sum_{j=0}^{r-1} a_j \omega^j, \]
where \( a_j \in \mathbb{Q} \) for all \( j \). Thus \( \omega \) is a root of the polynomial \( p(x) = \sum_{j=0}^{r-1} a_j x^j - n v_x \in \mathbb{Q}(\omega_3)[x] \).
Therefore, \( p(x) \) is a multiple of the irreducible polynomial \( \Phi_r(x) \), and so \( \omega^{-1} \) is also a root of \( p(x) \). Since \( \psi_a(-x) = \psi_a(x)^{-1} \) for all \( a \in \Gamma \), therefore \( (E^* u)_{-x} = \sum_{j=0}^{r-1} a_j \omega^{-j} \). Hence
\[ 0 = p(\omega^{-1}) = \sum_{j=0}^{r-1} a_j \omega^{-j} - n v_x = (E^* u)_{-x} - n v_x = n v_{-x} - n v_x, \]
implies that \( v_x = v_{-x} \). This together with Part (i) imply that \( \Re(v_x) = \Re(v_{-x}) \), and that \( \Im(v_x) = \Im(v_{-x}) = 0 \) for all \( x \in \Gamma \setminus \Gamma(3) \).
\[ \square \]
Theorem 4.2. Let $\Gamma$ be an abelian group. The oriented Cayley graph $\text{Cay}(\Gamma, S)$ is HS-integral if and only if $S \in E(\Gamma)$.

Proof. Assume that the oriented Cayley graph $\text{Cay}(\Gamma, S)$ is integral. If $\Gamma(3) = \emptyset$ then by Theorem 3.2, we have $S = \emptyset$, and so $S \in E(\Gamma)$. Now assume that $\Gamma(3) \neq \emptyset$. Let $v$ be the vector in $\mathbb{Q}^n(\omega_3)$ whose coordinates are indexed by the elements of $\Gamma$, and the $x^{th}$ coordinate of $v$ is given by

$$v_x = \begin{cases} 
\omega_6 & \text{if } x \in S \\
\omega_6^5 & \text{if } x \in S^{-1} \\
0 & \text{otherwise.}
\end{cases}$$

We have

$$(Ev)_a = \sum_{x \in \Gamma} E_{a,x} v_x = \sum_{x \in S} \omega_6 E_{a,x} + \sum_{x \in S^{-1}} \omega_6^5 E_{a,x} = \sum_{x \in S} (\omega_6 \psi_a(x) + \omega_6^5 \psi_a(-x)) = (E_\psi a)_a.$$

Thus $(Ev)_a$ is an HS-eigenvalue of the integral oriented Cayley graph $\text{Cay}(\Gamma, S)$ for each $a \in \Gamma$. Therefore $Ev \in \mathbb{Q}^n$, and hence all the three conditions of Lemma 4.1 hold.

By the third condition of Lemma 4.1, $v_x = 0$ for all $x \in \Gamma \setminus \Gamma(3)$, and so we must have $S \cup S^{-1} \subseteq \Gamma(3)$. Again, let $x \in S$, $y \in \Gamma(3)$ and $x \approx y$. The second condition of Lemma 4.1 gives $v_x = v_y$, which implies that $y \in S$. Thus $x \in S$ implies $\langle x \rangle \subseteq S$. Hence $S \in E(\Gamma)$. The converse part follows from Theorem 3.5.

The following example illustrates Theorem 4.2.

Example 4.1. Consider $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $S = \{(0,1),(2,0)\}$. The oriented graph $\text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, S)$ is shown in Figure 1a. We see that $\langle (0,1) \rangle = \{(0,1)\}$ and $\langle (2,0) \rangle = \{(2,0)\}$. Therefore $S \in E(\Gamma)$. Further, using Corollary 2.5.2 and Equation 1, the HS-eigenvalues of $\text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, S)$ are obtained as

$$\mu_\alpha = [\omega_6 \psi_\alpha(0,1) + \omega_6^5 \psi_\alpha(0,2)] + [\omega_6 \psi_\alpha(2,0) + \omega_6^5 \psi_\alpha(1,0)]$$

for each $\alpha \in \mathbb{Z}_3 \times \mathbb{Z}_3$, where

$$\psi_\alpha(x) = \omega_3^{\alpha_1 x_1} \omega_3^{\alpha_2 x_2}$$

for all $\alpha = (\alpha_1, \alpha_2), x = (x_1, x_2) \in \mathbb{Z}_3 \times \mathbb{Z}_3$.

It can be seen that $\mu_{(0,0)} = 2, \mu_{(0,1)} = -1, \mu_{(0,2)} = 2, \mu_{(1,0)} = 2, \mu_{(1,1)} = -1, \mu_{(1,2)} = 2, \mu_{(2,0)} = -1, \mu_{(2,1)} = -4$ and $\mu_{(2,2)} = -1$. Thus $\text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, S)$ is HS-integral.

Lemma 4.3. Let $S$ be a skew-symmetric subset of an abelian group $\Gamma$ and $t(\neq 0) \in \mathbb{Q}$. If $\sum_{s \in S} it\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s))$ is an integer for each $\alpha \in \Gamma$ then $S \in E(\Gamma)$.

Proof. Let $v$ be the vector, whose coordinates are indexed by the elements of $\Gamma$, defined by

$$v_x = \begin{cases} 
it\sqrt{3} & \text{if } x \in S \\
-it\sqrt{3} & \text{if } x \in S^{-1} \\
0 & \text{otherwise.}
\end{cases}$$
Since \( v \in \mathbb{Q}(\omega_3)^n \) and \( \alpha \)-th coordinate of \( Ev \) is \( \sum_{s \in S} it\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s)) \), we have \( Ev \in \mathbb{Q}^n \). By the third condition of Lemma 4.1, \( \exists(v_x) = 0 \), and so \( v_x = 0 \) for all \( x \in \Gamma \setminus \Gamma(3) \). Thus we must have \( S \cup S^{-1} \subseteq \Gamma(3) \). Again, let \( x \in S \), \( y \in \Gamma(3) \) and \( x \approx y \). The second condition of Lemma 4.1 gives \( v_x = v_y \), which implies that \( y \in S \). Thus \( x \in S \) implies \( \langle x \rangle \subseteq S \). Hence \( S \in \mathbb{E}(\Gamma) \). \( \square \)

**Lemma 4.4.** Let \( S \) be a skew-symmetric subset of an abelian group \( \Gamma \) and \( t(\neq 0) \in \mathbb{Q} \). If \( \sum_{s \in S} it\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s)) \) is an integer for each \( \alpha \in \Gamma \) then \( \sum_{s \in S \cup S^{-1}} \psi_\alpha(s) \) is an integer for each \( \alpha \in \Gamma \).

**Proof.** Assume that \( \sum_{s \in S} it\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s)) \) is an integer for each \( \alpha \in \Gamma \). By Lemma 4.3 we have \( S \in \mathbb{E}(\Gamma) \), and so \( S = \langle x_1 \rangle \cup \ldots \cup \langle x_k \rangle \) for some \( x_1, \ldots, x_k \in \Gamma(3) \). Therefore, using Lemma 2.10 we get \( S \cup S^{-1} = [x_1] \cup \ldots \cup [x_k] \in \mathbb{E}(\Gamma) \). Thus by Theorem 2.8, \( \text{Cay}(\Gamma \cup S^{-1}) \) is integral, that is, \( \sum_{s \in S \cup S^{-1}} \psi_\alpha(s) \) is an integer for each \( \alpha \in \Gamma \). \( \square \)

**Lemma 4.5.** Let \( \Gamma \) be an abelian group. The mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is HS-integral if and only if \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) is integral and \( \text{Cay}(\Gamma, \overline{S}) \) are HS-integral.

**Proof.** Assume that the mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is HS-integral. Let the HS-spectrum of \( \text{Cay}(\Gamma, S) \) be \( \{ \gamma_\alpha : \alpha \in \Gamma \} \), where \( \gamma_\alpha = \lambda_\alpha + \mu_\alpha \),

\[
\lambda_\alpha = \sum_{s \in S \setminus \overline{S}} \psi_\alpha(s) \quad \text{and} \quad \mu_\alpha = \sum_{s \in \overline{S}} (\omega_6 \psi_\alpha(s) + \omega_6^2 \psi_\alpha(-s)) , \quad \text{for } \alpha \in \Gamma .
\]

Note that \( \{ \lambda_\alpha : \alpha \in \Gamma \} \) is the spectrum of \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) and \( \{ \mu_\alpha : \alpha \in \Gamma \} \) is the HS-spectrum of \( \text{Cay}(\Gamma, \overline{S}) \). By assumption \( \gamma_\alpha \in \mathbb{Z} \), and so \( \gamma_\alpha - \gamma_-\alpha = \sum_{s \in \overline{S}} i\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s)) \in \mathbb{Z} \) for all \( \alpha \in \Gamma \). By Lemma 4.4,
we get \( \sum_{s \in S \cup S^{-1}} \psi_\alpha(s) \in \mathbb{Z} \) for all \( \alpha \in \Gamma \). Note that \( \mu_\alpha \) is an algebraic integer. Also

\[
\mu_\alpha = \frac{1}{2} \sum_{s \in S \cup S^{-1}} \psi_\alpha(s) + \frac{1}{2} \sum_{s \in \overline{S}} i\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s)) \in \mathbb{Q}.
\]

Hence \( \mu_\alpha \) is an integer for each \( \alpha \in \Gamma \). Thus \( \text{Cay}(\Gamma, \overline{S}) \) is HS-integral. Now we have \( \gamma_\alpha, \mu_\alpha \in \mathbb{Z} \), and so \( \lambda_\alpha = \gamma_\alpha - \mu_\alpha \in \mathbb{Z} \) for each \( \alpha \in \Gamma \). Hence \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) is integral.

Conversely, assume that \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) is integral and \( \text{Cay}(\Gamma, \overline{S}) \) is HS-integral. Then Lemma 2.5 implies that \( \text{Cay}(\Gamma, S) \) is integral. \( \square \)

**Theorem 4.6.** Let \( \Gamma \) be an abelian group. The mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is HS-integral if and only if \( S \setminus \overline{S} \in \mathbb{E}(\Gamma) \) and \( \overline{S} \in \mathbb{E}(\Gamma) \).

**Proof.** By Lemma 4.5, the mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is HS-integral if and only if \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) is integral and \( \text{Cay}(\Gamma, \overline{S}) \) is HS-integral. Note that \( S \setminus \overline{S} \) is a symmetric set and \( \overline{S} \) is a skew-symmetric set. Thus by Theorem 2.8, \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) is integral if and only if \( S \setminus \overline{S} \in \mathbb{E}(\Gamma) \). By Theorem 4.2, \( \text{Cay}(\Gamma, \overline{S}) \) is HS-integral if and only if \( \overline{S} \in \mathbb{E}(\Gamma) \). Hence the result follows. \( \square \)

The following example illustrates Theorem 4.6.

**Example 4.2.** Consider \( \Gamma = \mathbb{Z}_4 \times \mathbb{Z}_3 \) and \( S = \{(0, 1), (1, 0), (2, 0)\} \). The mixed graph \( \text{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_3, S) \) is shown in Figure 1b. Here \( \overline{S} = \{(0, 1)\} = \langle(0, 1)\rangle \in \mathbb{E}(\Gamma) \) and \( S \setminus \overline{S} = \{(1, 0), (2, 0)\} = \{(1, 0)\} \in \mathbb{E}(\Gamma) \). Further, using Lemma 2.5 and Equation 1, the HS-eigenvalues of \( \text{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_3, S) \) are obtained as

\[
\gamma_\alpha = [\psi_\alpha(1, 0) + \psi_\alpha(2, 0)] + [\omega_6^1 \psi_\alpha(0, 1) + \omega_6^2 \psi_\alpha(0, 2)] \text{ for each } \alpha \in \mathbb{Z}_4 \times \mathbb{Z}_3.
\]

One can see that \( \gamma(0, 1) = \gamma(1, 0) = \gamma(1, 2) = \gamma(2, 0) = 0 \), \( \gamma(0, 0) = \gamma(0, 2) = 3 \) and \( \gamma(1, 1) = \gamma(2, 1) = -3 \). Thus \( \text{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_3, S) \) is HS-integral.

## 5 Characterization of Eisenstein integral mixed Cayley graphs over abelian groups

Let \( \Gamma \) be a finite abelian group of order \( n \). For an \( S \subseteq \Gamma \) with \( 0 \notin S \), consider the function \( \alpha : \Gamma \to \{0, 1\} \) defined by

\[
\alpha(s) = \begin{cases} 
1 & \text{if } s \in S \\
0 & \text{otherwise}, 
\end{cases}
\]

in Theorem 2.4. We see that \( \sum_{s \in S} \psi_\alpha(s) \) is an eigenvalue of the mixed Cayley graph \( \text{Cay}(\Gamma, S) \) for all \( \alpha \in \Gamma \). For \( x, y, \alpha \in \Gamma(3) \) and \( \alpha \in \Gamma \), define

\[
C_x(\alpha) = \sum_{s \in [x]} \psi_\alpha(s) \quad \text{and} \quad T_y(\alpha) = \sum_{s \in \langle y \rangle} i\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s)).
\]
Note that $C_x(\alpha)$ is an eigenvalue of the mixed Cayley graph $\text{Cay}(\Gamma, [x])$ for each $\alpha \in \Gamma$.

Lemma 5.1. Let $x \in \Gamma(3)$. Then $T_x(\alpha)$ is an integer for each $\alpha \in \Gamma$.

Proof. We have

$$Z_x(\alpha) = \sum_{s \in \langle \langle x \rangle \rangle} (\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s)) = \frac{1}{2} \sum_{s \in [x]} \psi_\alpha(s) + \frac{i \sqrt{3}}{2} \sum_{s \in \{x\}} (\psi_\alpha(s) - \psi_\alpha(-s))$$

$$= \frac{C_x(\alpha)}{2} + \frac{T_x(\alpha)}{2}.$$ By Lemma 3.4, $T_x(\alpha) = 2Z_x(\alpha) - C_x(\alpha)$ is an integer for each $\alpha \in \Gamma$. 

Lemma 5.2. Let $\Gamma$ be a finite abelian group and the order of $x \in \Gamma(3)$ be $3m$, with $m \not\equiv 0 \pmod{3}$. Then

$$x^m[x^3] = \begin{cases} \langle x \rangle & \text{if } m \equiv 1 \pmod{3} \\ \langle -x \rangle & \text{if } m \equiv 2 \pmod{3} \end{cases}.$$ 

Proof. Assume that $m \equiv 1 \pmod{3}$. Let $y \in x^m[x^3]$. Then $y = x^{m+3r}$ for some $r \in G_m(1)$. We have $\gcd(r, m) = 1$, which implies that $\gcd(m + 3r, 3m) = 1$ and $m + 3r \equiv 1 \pmod{3}$. Thus $x^m[x^3] \subseteq \langle x \rangle$. Since size of both $x^m[x^3]$ and $\langle x \rangle$ are same, so $x^m[x^3] = \langle x \rangle$. Similarly, if $m \equiv 2 \pmod{3}$ then we have $x^m[x^3] = \langle -x \rangle$. 

Lemma 5.3. Let $\Gamma$ be a finite abelian group and the order of $x \in \Gamma(3)$ be $3m$, with $m \not\equiv 0 \pmod{3}$. Then

$$T_x(\alpha) = \begin{cases} \pm 3C_x(\alpha) & \text{if } \psi_\alpha(x^m) \neq 1 \\ 0 & \text{otherwise.} \end{cases}$$ for all $\alpha \in \Gamma$. 

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Proof. We have

\[ T_x(\alpha) = i\sqrt{3} \sum_{s \in \langle x \rangle} (\psi_\alpha(s) - \psi_\alpha(-s)) \]

\[ = \begin{cases} 
  i\sqrt{3} \sum_{s \in \langle x \rangle} (\psi_\alpha(s) - \psi_\alpha(-s)) & \text{if } m \equiv 1 \pmod{3} \\
  -i\sqrt{3} \sum_{s \in \langle -x \rangle} (\psi_\alpha(s) - \psi_\alpha(-s)) & \text{if } m \equiv 2 \pmod{3} 
\end{cases} \]

\[ = \begin{cases} 
  i\sqrt{3} \sum_{s \in x^m \langle x \rangle} (\psi_\alpha(s) - \psi_\alpha(-s)) & \text{if } m \equiv 1 \pmod{3} \\
  -i\sqrt{3} \sum_{s \in x^m \langle x \rangle} (\psi_\alpha(s) - \psi_\alpha(-s)) & \text{if } m \equiv 2 \pmod{3} 
\end{cases} \]

\[ = \begin{cases} 
  i\sqrt{3} \sum_{s \in x^m \langle x \rangle} (\psi_\alpha(x^m)s - \psi_\alpha(-x^m)s\psi_\alpha(-s)) & \text{if } m \equiv 1 \pmod{3} \\
  -i\sqrt{3} \sum_{s \in x^m \langle x \rangle} (\psi_\alpha(x^m)s - \psi_\alpha(-x^m)s\psi_\alpha(-s)) & \text{if } m \equiv 2 \pmod{3} 
\end{cases} \]

\[ = \begin{cases} 
  -2\sqrt{3}\Im(\psi_\alpha(x^m)) \sum_{s \in x^m \langle x \rangle} \psi_\alpha(s) & \text{if } m \equiv 1 \pmod{3} \\
  2\sqrt{3}\Im(\psi_\alpha(x^m)) \sum_{s \in x^m \langle x \rangle} \psi_\alpha(s) & \text{if } m \equiv 2 \pmod{3} 
\end{cases} \]

\[ = \pm 2\sqrt{3}\Im(\psi_\alpha(x^m))C_{x^3}(\alpha). \]

Since \( \psi_\alpha(x^m) \) is a 3-rd root of unity, \( \Im(\psi_\alpha(x^m)) = 0 \) or \( \pm \frac{\sqrt{3}}{2} \). Thus

\[ T_x(\alpha) = \begin{cases} 
  \pm 3C_{x^3}(\alpha) & \text{if } \psi_\alpha(x^m) \neq 1 \\
  0 & \text{otherwise.} 
\end{cases} \]

Lemma 5.4. Let \( \Gamma \) be a finite abelian group and the order of \( x \in \Gamma(3) \) be \( k = 3^t m \), with \( m \neq 0 \pmod{3} \) and \( t \geq 2 \). Then

\[ T_x(\alpha) = \begin{cases} 
  3\sqrt{3}i \sum_{r \in G'_{3^t-1-m,3}^{(1)}} (\psi_\alpha(x^r) - \psi_\alpha(-x^r)) & \text{if } \psi_\alpha(x^{\frac{3}{2}}) = 1 \\
  0 & \text{otherwise.} 
\end{cases} \]
Proof. Note that $G_{k,3}^1(1) = G_{k,3}^1(1) \cup \left( \frac{x}{3} + G_{k,3}^1(1) \right) \cup \left( \frac{2x}{3} + G_{k,3}^1(1) \right)$. Therefore

$$T_x(\alpha) = i\sqrt{3} \sum_{s \in \langle x \rangle} (\psi_\alpha(s) - \psi_\alpha(-s))$$

$$= i\sqrt{3} \sum_{r \in G_{k,3}^1(1)} (\psi_\alpha(x^r) - \psi_\alpha(-x^r))$$

$$= i\sqrt{3} \left[ \sum_{r \in G_{k,3}^1(1)} (\psi_\alpha(x^r) - \psi_\alpha(-x^r)) + \sum_{r \in G_{k,3}^1(1)} (\psi_\alpha(x^r)\psi_\alpha(x^r) - \psi_\alpha(x^r)\psi_\alpha(-x^r)) \right]$$

$$+ \sum_{r \in G_{k,3}^1(1)} (\psi_\alpha(x^{\frac{r}{3}})\psi_\alpha(x^r) - \psi_\alpha(x^{\frac{r}{3}})\psi_\alpha(-x^r))$$

$$= i\sqrt{3} \left[ \sum_{r \in G_{k,3}^1(1)} (\psi_\alpha(x^r) - \psi_\alpha(-x^r)) + \psi_\alpha(x^{\frac{r}{3}}) \sum_{r \in G_{k,3}^1(1)} (\psi_\alpha(x^r) - \psi_\alpha(-x^r)) \right]$$

$$= i\sqrt{3} \left( 1 + \psi_\alpha(x^{\frac{r}{3}}) + \psi_\alpha(x^{\frac{r}{3}}) \right) \sum_{r \in G_{k,3}^1(1)} (\psi_\alpha(x^r) - \psi_\alpha(-x^r))$$

$$= \begin{cases} 3\sqrt{3} \sum_{r \in G_{k,3}^1(1)} (\psi_\alpha(x^r) - \psi_\alpha(-x^r)) & \text{if } \psi_\alpha(x^{\frac{r}{3}}) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 5.5. Let $\Gamma$ be a finite abelian group and $x \in \Gamma(3)$. Then $\frac{T_x(\alpha)}{3}$ is an integer for each $\alpha \in \Gamma$.

Proof. Let $x \in \Gamma(3)$ and order of $x$ be $k = 3^t m$ with $m \not\equiv 0 \pmod{3}$ and $t \geq 1$. If $t = 1$ then by Lemma 5.3, $\frac{T_x(\alpha)}{3}$ is an integer for each $\alpha \in \Gamma$. Assume that $t \geq 2$. If $\psi_\alpha(x^{\frac{r}{3}}) \neq 1$ then by Lemma 5.4, $\frac{T_x(\alpha)}{3}$ is an integer for each $\alpha \in \Gamma$. If $\psi_\alpha(x^{\frac{r}{3}}) = 1$ then by Lemma 5.1 and Lemma 5.4, $i\sqrt{3} \sum_{r \in G_{k,3}^1(1)} (\psi_\alpha(x^r) - \psi_\alpha(-x^r))$ is a rational algebraic integer, and hence an integer for each $\alpha \in \Gamma$.

Lemma 5.6. Let $\Gamma$ be a finite abelian group and $x \in \Gamma(3)$. Then $C_x(\alpha)$ and $\frac{T_x(\alpha)}{3}$ are integers of the same parity for each $\alpha \in \Gamma$.

Proof. Let $x \in \Gamma(3)$ and $\alpha \in \Gamma$. By Lemma 3.4, $T_x(\alpha) + C_x(\alpha) = 2Z_x(\alpha)$ is an even integer, therefore $T_x(\alpha)$ and $C_x(\alpha)$ are integers of the same parity. By Lemma 5.5, $\frac{T_x(\alpha)}{3}$ is an integer. Hence $C_x(\alpha)$ and $\frac{T_x(\alpha)}{3}$ are integers of the same parity.

Let $S$ be a subset of $\Gamma$. For each $\alpha \in \Gamma$, define

$$f_\alpha(S) = \sum_{s \in S \setminus S} \psi_\alpha(s) \quad \text{and} \quad g_\alpha(S) = \sum_{s \in S} (\omega \psi_\alpha(s) + \overline{\omega} \psi_\alpha(-s)),$$
where \( \omega = \frac{1}{2} - \frac{i\sqrt{3}}{6} \). It is clear that \( f_\alpha(S) \) and \( g_\alpha(S) \) are real numbers. We have

\[
\sum_{s \in S} \psi_\alpha(s) = f_\alpha(S) + g_\alpha(S) + \left( \frac{-1}{2} + \frac{i\sqrt{3}}{2} \right) (g_\alpha(S) - g_{-\alpha}(S)).
\]

Note that \( f_\alpha(S) = f_{-\alpha}(S) \) for each \( \alpha \in \Gamma \). Therefore if \( f_\alpha(S) + g_\alpha(S) \) is an integer for each \( \alpha \in \Gamma \), then \( g_\alpha(S) - g_{-\alpha}(S) = [f_\alpha(S) + g_\alpha(S)] - [f_{-\alpha}(S) + g_{-\alpha}(S)] \) is also an integer for each \( \alpha \in \Gamma \). Hence, the mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is Eisenstein integral if and only if \( f_\alpha(S) + g_\alpha(S) \) is an integer for each \( \alpha \in \Gamma \).

**Lemma 5.7.** Let \( S \) be a subset of a finite abelian group \( \Gamma \) with \( 0 \notin S \). Then the mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is Eisenstein integral if and only if \( 2f_\alpha(S) \) and \( 2g_\alpha(S) \) are integers of the same parity for each \( \alpha \in \Gamma \).

**Proof.** Suppose the mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is Eisenstein integral and \( \alpha \in \Gamma \). Then \( f_\alpha(S) + g_\alpha(S) \) and \( g_\alpha(S) - g_{-\alpha}(S) = \sum_{s \in S} \frac{1}{2} \left[ i\sqrt{3} (\psi_\alpha(s) - \psi_{-\alpha}(s)) \right] \) are integers. By Lemma 4.4, \( \sum_{s \in S} \psi_\alpha(s) \in \mathbb{Z} \). Since

\[
2g_\alpha(S) = \sum_{s \in S} \psi_\alpha(s) - \sum_{s \in S} \frac{i\sqrt{3}}{3} (\psi_\alpha(s) - \psi_{-\alpha}(s)),
\]

we find that \( 2g_\alpha(S) \) is an integer. Therefore, \( 2f_\alpha(S) = 2(f_\alpha(S) + g_\alpha(S)) - 2g_\alpha(S) \) is also integer of the same parity with \( g_\alpha(S) \).

Conversely, assume that \( 2f_\alpha(S) \) and \( 2g_\alpha(S) \) are integers of the same parity for each \( \alpha \in \Gamma \). Then \( f_\alpha(S) + g_\alpha(S) \) is an integer for each \( \alpha \in \Gamma \). Hence the mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is Eisenstein integral.

**Lemma 5.8.** Let \( S \) be a subset of finite abelian group \( \Gamma \) with \( 0 \notin S \). Then the mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is Eisenstein integral if and only if \( f_\alpha(S) \) and \( g_\alpha(S) \) are integers for each \( \alpha \in \Gamma \).

**Proof.** By Lemma 5.7, it is enough to show that \( 2f_\alpha(S) \) and \( 2g_\alpha(S) \) are integers of the same parity if and only if \( f_\alpha(S) \) and \( g_\alpha(S) \) are integers. If \( f_\alpha(S) \) and \( g_\alpha(S) \) are integers, then clearly \( 2f_\alpha(S) \) and \( 2g_\alpha(S) \) are even integers. Conversely, assume that \( 2f_\alpha(S) \) and \( 2g_\alpha(S) \) are integers of the same parity. Since \( f_\alpha(S) \) is an algebraic integer, the integrality of \( 2f_\alpha(S) \) implies that \( f_\alpha(S) \) is an integer. Thus \( 2f_\alpha(S) \) is even, and so by the assumption \( 2g_\alpha(S) \) is also even. Hence \( g_\alpha(S) \) is an integer.

**Theorem 5.9.** Let \( S \) be a subset of a finite abelian group \( \Gamma \) with \( 0 \notin S \). Then the mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is Eisenstein integral if and only if \( \text{Cay}(\Gamma, S) \) is HS-integral.

**Proof.** By Lemma 5.8, it is enough to show that \( f_\alpha(S) \) and \( g_\alpha(S) \) are integers for each \( \alpha \in \Gamma \) if and only if \( \text{Cay}(\Gamma, S) \) is HS-integral. Note that \( f_\alpha(S) \) is an eigenvalue of the Cayley graph \( \text{Cay}(\Gamma, S \setminus \mathcal{S}) \). By Theorem 2.8, \( f_\alpha(S) \) is an integer for each \( \alpha \in \Gamma \) if and only if \( S \setminus \mathcal{S} \in \mathcal{B}(\Gamma) \).
Assume that \( f_\alpha(S) \) and \( g_\alpha(S) \) are integers for each \( \alpha \in \Gamma \). Then \( \sum_{s \in S} -\frac{i\sqrt{3}}{6}(\psi_\alpha(s) - \psi_\alpha(-s)) = g_\alpha(S) - g_{-\alpha}(S) \) is also an integer for each \( \alpha \in \Gamma \). Using Theorem 2.8 and Lemma 4.3, we see that \( S \setminus S \) and \( S \) satisfy the conditions of Theorem 4.6. Hence \( \text{Cay}(\Gamma, S) \) is HS-integral.

Conversely, assume that \( \text{Cay}(\Gamma, S) \) is HS-integral. Then \( \text{Cay}(\Gamma, S \setminus S) \) is integral, and hence \( f_\alpha(S) \) is an integer for each \( \alpha \in \Gamma \). By Theorem 4.6, we have \( S \in \mathcal{E}(\Gamma) \), and so \( S = \langle x_1 \rangle \cup \ldots \cup \langle x_k \rangle \) for some \( x_1, \ldots, x_k \in \Gamma(3) \). Then

\[
g_\alpha(S) = \frac{1}{2} \sum_{s \in S \cup S^{-1}} \psi_\alpha(s) - \frac{1}{6} \sum_{s \in S} i\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s))
\]

\[
= \frac{1}{2} \sum_{j=1}^{k} \sum_{s \in \langle x_j \rangle} \psi_\alpha(s) - \frac{1}{6} \sum_{j=1}^{k} \sum_{s \in \langle x_j \rangle} i\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s))
\]

\[
= \frac{1}{2} \sum_{j=1}^{k} C_{x_j}(\alpha) - \frac{1}{3} \sum_{j=1}^{k} T_{x_j}(\alpha)
\]

\[
= \frac{1}{2} \sum_{j=1}^{k} \left( C_{x_j}(\alpha) - \frac{1}{3} T_{x_j}(\alpha) \right).
\]

By Lemma 5.6, \( C_{x_j}(\alpha) - \frac{1}{3} T_{x_j}(\alpha) \) is an even integer for each \( j \in \{1, \ldots, k\} \). Hence \( g_\alpha(S) \) is an integer for each \( \alpha \in \Gamma \).

The following example illustrates Theorem 5.9.

**Example 5.1.** Consider the HS-integral graph \( \text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, S) \) of Example 4.2. By Theorem 5.9, the graph \( \text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, S) \) is Eisenstein integral. Indeed, the eigenvalues of \( \text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, S) \) are obtained as

\[\gamma_\alpha = \psi_\alpha(0,1) + \psi_\alpha(1,0) + \psi_\alpha(2,0).\]

We have \( \gamma(0,0) = 3 \), \( \gamma(0,1) = 2 + \omega_3 \), \( \gamma(0,2) = 1 - \omega_3 \), \( \gamma(1,0) = 0 \), \( \gamma(1,1) = -1 + \omega_3 \), \( \gamma(1,2) = -2 - \omega_3 \), \( \gamma(2,0) = 0 \), \( \gamma(2,1) = -1 + \omega_3 \) and \( \gamma(2,2) = -2 - \omega_3 \). Thus \( \gamma_\alpha \) is an Eisenstein integer for each \( \alpha \in \mathbb{Z}_3 \times \mathbb{Z}_3 \). 

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