QUARTIC JULIA SETS INCLUDING ANY TWO COPIES OF QUADRATIC JULIA SETS

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Abstract. If the Julia set of a quartic polynomial with certain conditions is neither connected nor totally disconnected, there exists a homeomorphism between the set of all components of the filled-in Julia set and some subset of the corresponding symbol space. The question is to determine the quartic polynomials exhibiting such a dynamics and describe the topology of the connected components of their filled-in Julia sets. In this paper, we answer the question, namely we show that for any two quadratic Julia sets, there exists a quartic polynomial whose Julia set includes copies of the two quadratic Julia sets.

1. Introduction. Let \( f \) be a polynomial of degree \( d \geq 2 \). The Fatou set \( F(f) \) is the set of normality in the sense of Montel for the family \( \{f^n\}_{n=1}^\infty \), where \( f^n = f \circ \cdots \circ f \) denotes the \( n \)-th iterate of \( f \). The filled-in Julia set \( K(f) \) is defined as
\[
K(f) = \left\{ z \in \mathbb{C} : \{f^n(z)\}_{n=0}^\infty \text{ is bounded} \right\}.
\]
The Julia set \( J(f) \) is the complement \( \mathbb{C} \setminus F(f) \) or the topological boundary \( \partial K(f) \) of the filled-in Julia set. The Julia set \( J(f) \) and the filled-in Julia set \( K(f) \) are compact. The escaping set \( I(f) \) is the complement \( \mathbb{C} \setminus K(f) \). These four sets \( F(f) \), \( K(f) \), \( J(f) \) and \( I(f) \) are completely invariant under \( f \). The connectivity of the Julia set \( J(f) \) is affected by the behavior of finite critical points.

Theorem 1.1 ([1, 5]). Let \( f \) be a polynomial of degree \( d \geq 2 \). If all finite critical points of \( f \) belong to \( I(f) \), then \( J(f) \) is totally disconnected and \( J(f) = K(f) \). Furthermore, \( f\mid_{J(f)} \) is topologically conjugate to the shift map \( \sigma\mid_{\Sigma_d} \). On the other hand, \( J(f) \) and \( K(f) \) are connected if and only if all finite critical points of \( f \) belong to \( K(f) \).

The symbol space \( \Sigma_\nu = \{1, 2, \ldots, \nu\}^\mathbb{N} \) is the countable product of \( \nu \) symbols. For \( s = (s_n) \) and \( t = (t_n) \) in \( \Sigma_\nu \), the metric \( \rho \) on \( \Sigma_\nu \) is defined as
\[
\rho(s, t) = \sum_{n=0}^\infty \frac{\delta(s_n, t_n)}{2^n}, \quad \text{where} \quad \delta(k, l) = \begin{cases} 1 & \text{if } k \neq l, \\ 0 & \text{if } k = l. \end{cases}
\]
Then \( (\Sigma_\nu, \rho) \) is a compact metric space. The shift map \( \sigma : \Sigma_\nu \to \Sigma_\nu \) is defined as
\[
\sigma((s_0, s_1, s_2, \ldots)) = (s_1, s_2, \ldots).
\]
The shift map \( \sigma \) is continuous with respect to the metric \( \rho \).

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If some critical orbits but not all critical orbits diverge, then the Julia set is disconnected and not generally totally disconnected. The author [3] simplified the dynamics of some quartic polynomial on its filled-in Julia set when the Julia set is disconnected and not totally disconnected.

Let \( f \) be a quartic polynomial and \( G = G_f \) the Green’s function associated with the filled-in Julia set \( K(f) \). For a polynomial \( p \) of degree \( d \), the Green’s function \( G_p \) is defined as:

\[
G_p(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+|p^n(z)|,
\]

where \( \log^+ x = \max\{ \log x, 0 \} \). The Green’s function \( G(z) \) is zero for \( z \in K(f) \) and positive for \( z \in I(f) \). By definition, the identity \( G(f(z)) = 4G(z) \) holds. The locus \( G^{-1}(x) \) with \( x > 0 \) is called an equipotential curve around the filled-in Julia set \( K(f) \). Note that \( f \) maps the equipotential curve \( G^{-1}(x) \) to the equipotential curve \( G^{-1}(4x) \) by a four-to-one fold covering map. Let \( \omega_1, \omega_2 \) and \( \omega_3 \) be different finite critical points of \( f \). We assume that \( G(\omega_1) = G(\omega_2) = 0 \) and \( G(\omega_3) > 0 \). The assumption indicates that \( \omega_1, \omega_2 \in K(f) \) and \( \omega_3 \in I(f) \).

**Definition 1.2** (Polynomial-like maps). The triple \((g, U, V)\), consisting of bounded simply connected domains \( U \) and \( V \) such that \( U \subseteq V \) and a holomorphic proper map \( g : U \to V \) of degree \( d \), is called a polynomial-like map of degree \( d \). The filled-in Julia set \( K(g) \) of a polynomial-like map \((g, U, V)\) is defined as:

\[
K(g) = \left\{ z \in U : g^n(z) \in U \text{ for all } n \geq 0 \right\}
\]

and the Julia set \( J(g) \) as \( J(g) = \partial K(g) \). In the case that \( d = 2 \), the triple \((g, U, V)\) is called a quadratic-like map.

**Theorem 1.3** (Straightening Theorem [2, 5]). Every polynomial-like map is hybrid equivalent to a polynomial of the same degree. Namely, for any polynomial-like map \((g, U, V)\) of degree \( d \geq 2 \), there exist a polynomial \( p \) of degree \( d \), a neighborhood \( W \) of \( K(g) \) in \( U \) and a quasiconformal map \( \varphi : W \to \varphi(W) \) such that:

1. \( \varphi(K(g)) = K(p) \),
2. the complex dilatation \( \mu_\varphi \) of \( \varphi \) is zero almost everywhere on \( K(g) \),
3. \( \varphi \circ g = p \circ \varphi \) on \( W \cap g^{-1}(W) \).

If \( K(g) \) is connected, \( p \) is unique up to conjugation by an affine map.

Let \( U \) be the bounded component of \( \mathbb{C} \setminus G^{-1}(G(f(\omega_3))) \). We suppose that \( U_A \) and \( U_B \) are the different bounded components of \( \mathbb{C} \setminus G^{-1}(G(\omega_3)) \) such that \( \omega_1 \in U_A \) and \( \omega_2 \in U_B \). Then \( U_A \) and \( U_B \) are proper subsets of \( U \). Moreover, \((f|_{U_A}, U_A, U)\) and \((f|_{U_B}, U_B, U)\) are quadratic-like maps. We define the \( A-B \) kneading sequence \((\alpha_n)_{n \geq 0}\) of \( \omega_1 \) as:

\[
\alpha_n = \begin{cases} 
A & \text{if } f^n(\omega_1) \in U_A, \\
B & \text{if } f^n(\omega_1) \in U_B.
\end{cases}
\]

We assume that the \( A-B \) kneading sequence of \( \omega_1 \) is \(((AAA)\ldots)\) and the \( A-B \) kneading sequence of \( \omega_2 \) is \(((BBB)\ldots)\). This implies that filled-in Julia sets \( K(f|_{U_A}) \) and \( K(f|_{U_B}) \) are connected.

Let \( \Sigma_6 = \{1, 2, 3, 4, A, B\}^\mathbb{N} \) be the symbol space on 6 symbols. We define its subset \( \Sigma \) in Theorem 1.4 as follows: \( s = (s_n) \in \Sigma \) if and only if:

1. \( s_n = A \Rightarrow s_{n+1} = A \),
2. \( s_n = B \Rightarrow s_{n+1} = B \).
Figure 1. An example of Theorem 1.4. The black region is the filled-in Julia set of $f(z) = a(z + 2)^2(3z^2 - 8z + 8)/3 - 2$, where $a = 1/3$. Finite critical points $-2$ and $1$ are superattracting fixed points. The orbit of the other critical point $0$ tends to $\infty$.

Figure 2. Equipotential curves. The red one is $G^{-1}(G(0))$. It is a topological figure-eight through the origin. The blue one is $G^{-1}(G(f(0)))$. It is a topological circle through the critical value $f(0) = 14/9$.

(3) $s_n = A$ and $s_{n-1} \neq A \Rightarrow s_{n-1} = 3$ or $4$,
(4) $s_n = B$ and $s_{n-1} \neq B \Rightarrow s_{n-1} = 1$ or $2$,
(5) if $s = (s_n) \in \Sigma_4 = \{1, 2, 3, 4\}^N$, then there exist subsequences $(t_n)_{n=1}^\infty$ and $(u_n)_{n=1}^\infty$ such that $t_n = 1$ or $2$ and $u_n = 3$ or $4$ for all $n \geq 1$.

The property (5) of $\Sigma$ is essential. For example, a sequence

$$\left\{ s^{(n)} = (1, 1, \ldots, 1, B, B, \ldots) \right\}_{n=1}^\infty$$

converges to $s = (1, 1, 1, \ldots)$ in $\Sigma_6$. However, it does not converge in $\Sigma$ because $s \notin \Sigma$. Each $s^{(n)}$ corresponds to a component of backward iterated images of the
Figure 3. Another example of Theorem 1.4. This is the filled-in Julia set of \( f(z) = a(z + 2)^2(3z^2 - 8z + 8)/3 - 2 \), where \( a = 1/3 + 83/2000 \). Finite critical points \(-2\) and \(1\) belong to \( K(f) \). The other one \(0\) belongs to \( I(f) \). The critical point \(-2\) is a superattracting fixed point.

Figure 4. Equipotential curves. The red one is \( G^{-1}(G(0)) \) and the blue one is \( G^{-1}(G(f(0))) \).

filled-in Julia set \( K(f|_{\mathcal{U}_B}) \). These backward components converge to a repelling fixed point on \( \partial K(f|_{\mathcal{U}_A}) \).

**Theorem 1.4 ([3, Theorem 1.5]).** Let \( f \) be a quartic polynomial. Suppose that

(a) its finite critical points \( \omega_1, \omega_2 \in K(f) \) and \( \omega_3 \in I(f) \) are all different,
(b) the Julia set \( J(f) \) is disconnected but not totally disconnected,
(c) the \( A-B \) kneading sequence of \( \omega_1 \) is \( (AAA\cdots) \) and the \( A-B \) kneading sequence of \( \omega_2 \) is \( (BBB\cdots) \).

Then there exist a subset \( \Sigma \) of the symbol space \( \Sigma_6 \) and a homeomorphism \( \Lambda : K(f) \to \Sigma \) such that \( \Lambda \circ F = \sigma \circ \Lambda \), where \( K(f) \) is the set of all components of \( K(f) \) with the Hausdorff metric and \( F : K(f) \to K(f) \) is the map defined as \( F(K) = f(K) \) for \( K \in K(f) \).
The question is to determine the quartic polynomials exhibiting the above dynamics and describe the topology of the connected components of their filled-in Julia sets. In this study, we construct the quartic polynomial satisfying the assumptions of Theorem 1.4. The answer is the following theorem.

**Theorem A.** For any two quadratic Julia sets, there exists a quartic polynomial whose Julia set includes copies of the two quadratic Julia sets. More precisely, for any \( c_1, c_2 \in \mathbb{C} \), there exist a quartic polynomial \( f \) and distinct three bounded simply connected domains \( U_1, U_2, V \) such that \( (f, U_1, V) \) and \( (f, U_2, V) \) are quadratic-like maps and they are hybrid equivalent to the quadratic polynomials \( p_{c_1} \) and \( p_{c_2} \) respectively, where \( p_c(z) = z^2 + c \).

**Corollary B.** If \( c_1 \) and \( c_2 \) belong to the Mandelbrot set, then the quartic polynomial obtained in Theorem A satisfies the assumptions of Theorem 1.4.

### 2. Construction of the desired quartic polynomials

In this section, we construct the desired quartic polynomial \( f \) and prove Theorem A. The construction uses Lemma 2.1 and Theorem 2.2 on quasiregular mappings. In Lemma 2.1, “log” denotes the principal branch of the logarithm.

**Lemma 2.1 ([4, Lemma 6.2]).** Let \( k \in \mathbb{N}, 0 < R_1 < R_2 \) and \( \varphi_j(z) \) be analytic on a neighborhood of \( |z| = R_j \) such that \( \varphi_j(z) \) goes around the origin \( k \)-times \( (j = 1, 2) \). If
\[
\left| \log \left( \frac{\varphi_2(R_2e^{iy})}{R_2^k}, \frac{R_1^k}{\varphi_1(R_1e^{iy})} \right) \right| \leq \delta_0 \tag{*}
\]
and
\[
\left| \frac{d}{dz} \log \frac{\varphi_j(z)}{z^k} \right| \leq \delta_1, \quad z = R_je^{iy}, \quad j = 1, 2 \tag{†}
\]
hold for every \( y \in [0, 2\pi] \) and for some positive constants \( \delta_0 \) and \( \delta_1 \) satisfying
\[
C = 1 - \frac{1}{k} \left( \frac{\delta_0}{\log(R_2/R_1)} + \delta_1 \right) > 0, \tag{§}
\]
then there exists a quasiregular map 

\[ H : \{ z \in \mathbb{C} : R_1 \leq |z| \leq R_2 \} \to \mathbb{C} \setminus \{0\} \]

without critical points such that \( H = \varphi_j \) on \(|z| = R_j \) \((j = 1, 2)\) and satisfies

\[ K_H \leq \frac{1}{C}. \]

**Theorem 2.2** (Quasiconformal surgery [4, Theorem 3.1]). Let \( g : \mathbb{C} \to \mathbb{C} \) be a quasiregular mapping. Suppose that there are disjoint measurable sets \( E_j \subset \mathbb{C} \) \((j = 1, 2, \ldots)\) satisfying:

1. For almost every \( z \in \mathbb{C} \), the \( g \)-orbit of \( z \) passes \( E_j \) at most once for every \( j \);
2. \( g \) is \( K_j \)-quasiregular on \( E_j \);
3. \( K_\infty = \prod_{j=1}^{\infty} K_j < \infty \);
4. \( g \) is holomorphic almost everywhere outside \( \bigcup_{j=1}^{\infty} E_j \).

Then there exists a \( K_\infty \)-quasiconformal map \( \varphi : \mathbb{C} \to \mathbb{C} \) such that \( f = \varphi \circ g \circ \varphi^{-1} \) is an entire function.

**Notations.** Let \( R > r > 0 \), \( A \in \mathbb{C} \setminus \{0\} \) and \( c \in \mathbb{C} \). We use the notations below.

- \( p_c(z) = z^2 + c \)
- \( q_c(z) = p_c(Az)/A = Az^2 + c/A \)
- \( h(z) = z^4 - 2R^4z^2 + R^8 \)
- \( h_1(z) = h(z - R^2) + R^2 = z^4 - 4R^2z^3 + 4R^4z^2 + R^2 \)
- \( h_2(z) = h(z + R^2) - R^2 = z^4 + 4R^2z^3 + 4R^4z^2 - R^2 \)
- \( \Gamma = \{ z \in \mathbb{C} : |z| = r \} \)
- \( \Gamma_1 = \{ z \in \mathbb{C} : |z + R^2| = r \} \)
- \( \Gamma_2 = \{ z \in \mathbb{C} : |z - R^2| = r \} \)

The quadratic polynomial \( q_c \) is affine conjugate to \( p_c \). The quartic polynomial \( h_1 \) and \( h_2 \) are affine conjugate to \( h \). It is easy to check that critical points of \( h \) are 0 and \( \pm R^2 \). Hence, critical points of \( h_1 \) are 0, \( R^2 \) and \( 2R^2 \). Similarly, critical points of \( h_2 \) are 0, \( -R^2 \) and \( -2R^2 \). For the sake of convenience, we deal with the quartic polynomials \( h_1 \) and \( h_2 \) simultaneously as

\[ h_*(z) = z^4 + 4R^2z^3 + 4R^4z^2 \pm R^2. \]

**Lemma 2.3.** Let \( k = 2 \), \( R_1 = r \), \( R_2 = R \), \( \varphi_1 = q_c \) and \( \varphi_2 = h_*. \) If \( A = 4R^4 \), \( r = R/2 \) and \( R \) is large enough, then the inequality (*) of Lemma 2.1 holds for \( \delta_0 = (\log 2)/3 \).

**Proof.** For \( y \in [0, 2\pi] \), we obtain that

\[ \left| \log \left( \varphi_2 \left( \frac{R_2 e^{iy}}{R_1} \right) \right) \right| = \left| \log \left( \frac{h_*(R e^{iy})}{R^2} \right) \right| \]

\[ \leq \left| \log \left( \frac{R^4 e^{iy} \pm 4R^5 e^{2iy} + 4R^6 e^{3iy} \pm R^2}{A r^2 e^{2iy} + r^2} \right) \right| \]
Similarly, the inequality
\[ \left| z \frac{d}{dz} \log \frac{\varphi_2(z)}{z^k} \right| = \left| z \frac{d}{dz} \log \frac{h^*(z)}{z^2} \right| = \left| \frac{2z^2 + 4R^2z + 4R^4 \pm R^2z^{-2}}{z^2 + 4R^2z + 4R^4 \pm R^2z^{-2}} \right| \leq \frac{2R^2e^{i2y} + 4R^3e^{iy} + 2e^{-i2y}}{R^2e^{i2y} + 4R^3e^{iy} + 4R^4 \pm e^{-i2y}} \leq \delta_1 \]
holds if \( R \) is large enough.

\begin{lemma}
Let \( k = 2 \), \( R_1 = r \), \( R_2 = R \), \( \varphi_1 = q_c \) and \( \varphi_2 = h^* \). If \( A = 4R^4 \), \( r = R/2 \) and \( R \) is large enough, then the inequality (\( \dagger \)) of Lemma 2.1 holds for \( \delta_1 = 2/3 \).
\end{lemma}

\begin{proof}
If \( R \) is sufficiently large, we obtain that
\[ \left| z \frac{d}{dz} \log \frac{\varphi_1(z)}{z^k} \right| = \left| z \frac{d}{dz} \log \frac{q_c(z)}{z^2} \right| = \left| \frac{-2c}{A} \right| \frac{2|c|}{Ar^2e^{i2y} + \frac{c}{A}} \leq \frac{|c|}{2R^4} \leq \delta_1. \]
Similarly, the inequality
\[ \left| z \frac{d}{dz} \log \frac{\varphi_2(z)}{z^k} \right| = \left| z \frac{d}{dz} \log \frac{h^*(z)}{z^2} \right| = \left| \frac{2z^2 + 4R^2z + 4R^4 \pm R^2z^{-2}}{z^2 + 4R^2z + 4R^4 \pm R^2z^{-2}} \right| \leq \frac{2R^2e^{i2y} + 4R^3e^{iy} + 2e^{-i2y}}{R^2e^{i2y} + 4R^3e^{iy} + 4R^4 \pm e^{-i2y}} \leq \delta_1 \]
holds if \( R \) is large enough.
\end{proof}

\begin{lemma}
Let \( \delta_0 = (\log 2)/3 \), \( \delta_1 = 2/3 \), \( k = 2 \), \( R_1 = r \) and \( R_2 = R \). If \( r = R/2 \), then the inequality (\( \diamond \)) of Lemma 2.1 holds and \( C = 1/2 \).
\end{lemma}

\begin{proof}
\[ C = 1 - \frac{1}{k} \left( \frac{\delta_0}{\log (R_2/R_1)} + \delta_1 \right) = 1 - \frac{1}{2} \left( \frac{\log 2}{3} + \frac{2}{3} \right) = \frac{1}{2} > 0. \]
\end{proof}

\begin{lemma}
If \( |z| > 1 + \sqrt{1 + R^4} \), then \( |h(z)| > 2|z| \) and \( h^n(z) \to \infty \) as \( n \to \infty \).
\end{lemma}

\begin{proof}
For \( x \geq 0 \), the inequality \( x^2 - R^4 > 2x \) is equivalent to \( x > 1 + \sqrt{1 + R^4} \). Hence, if \( |z| > 1 + \sqrt{1 + R^4} \), the inequality
\[ |h(z)| = |z^2 - R^4|^2 \geq (|z|^2 - R^4)^2 > \left( 2|z| \right)^2 > 2|z| \]
holds. Therefore, we obtain that
\[ |h^n(z)| > 2^n |z| \to \infty. \]
\end{proof}
Lemma 2.7. Let \( r = R/2 \). If \( R \) is large enough, then the orbit of any point in \( \Gamma_1 \cup \Gamma_2 \) tends to infinity or

\[
h^n(re^{iy} + R^2) \to \infty.
\]

Proof. Since

\[
|h(z)| = |z|^4 \left| 1 - \frac{R^4}{z^2} \right|^2 \geq (R^2 - r)^4 \left| \frac{re^{2iy} + 2Rre^{iy}}{r^2e^{2iy} + 2Rre^{iy} + r^4} \right|^2
\]

\[
= \left( R^2 - \frac{R}{2} \right)^4 \left| \frac{R^2e^{2iy} \mp R^3e^{iy}}{R^4e^{2iy} \mp R^3e^{iy} + r^4} \right|^2 = R^4 \left( R - \frac{1}{2} \right)^4 \left| \frac{1}{4R^2e^{2iy} \mp 1} \frac{1}{r} e^{iy} + 1 \right|^2 > R^2 \left( R - \frac{1}{2} \right)^4 \frac{1}{2} > 1 + \sqrt{1 + R^4}
\]

holds if \( R \) is large enough. By Lemma 2.6, we obtain the result. \( \square \)

If \( R \) is large enough, the critical orbit \( h^n(0) = h^{n+1}(\pm R^2) \) tends to infinity. Therefore, \( G(0) \) and \( G(\pm R^2) \) are positive, where \( G = G_h \) is the Green’s function associated with the filled-in Julia set \( K(h) \). Since the preimages of the critical value \( R^3 = h(0) = 0 \) and \( \pm \sqrt{2}R^2 \), then the equipotential curve \( \Phi = G^{-1}(G(0)) \) is a figure-eight through \( 0 \) and \( \pm \sqrt{2}R^2 \). It has symmetry with respect to the origin because \( h \) is even. Moreover, since critical points \( \pm R^2 \) are the preimages of the critical point \( 0 \), then the equipotential curve \( G^{-1}(G(R^2)) = G^{-1}(G(-R^2)) \) has two components \( \Theta_1 \) and \( \Theta_2 \), which are the congruent quatrefoils centered at \( -R^2 \) and \( R^2 \) respectively. The quatrefoils \( \Theta_1 \) and \( \Theta_2 \) are surrounded by the figure-eight \( \Phi \). Since the critical orbit \( h^n(0) = h^{n+1}(\pm R^2) \) tends to infinity for sufficiently large \( R \), the Julia set \( J(h) \) is totally disconnected and it is surrounded by \( \Theta_1 \cup \Theta_2 \). By Lemma 2.7, it is also surrounded by \( \Gamma_1 \cup \Gamma_2 \).

The strategy of the proof of Theorem A.

1. Take \( R \) sufficiently large and cut off the Julia set \( J(h) \) by the two circles \( \Gamma_1 \) and \( \Gamma_2 \).
2. Paste two copies of the quadratic Julia sets \( J(p_{c_1}) \) and \( J(p_{c_2}) \) in the interior of the circles \( \Gamma_1 \) and \( \Gamma_2 \) respectively.
3. In order to construct a quasiregular map \( g \), interpolate \( h \) and two quadratic polynomials which are hybrid equivalent to \( p_{c_1} \) and \( p_{c_2} \) respectively.
4. To obtain the desired quartic polynomial \( f \), employ the quasiconformal surgery for \( g \).

Proof of Theorem A. Let \( r = R/2 \) and \( A = 4R^4 \). Since Lemma 2.3, Lemma 2.4 and Lemma 2.5 hold for sufficiently large \( R \), by Lemma 2.1, there exist quasiregular maps

\[
H_j : \{ z \in \mathbb{C} : R/2 \leq |z| \leq R \} \to \mathbb{C} \setminus \{0\} \quad (j = 1, 2)
\]

without critical points such that

\[
H_j = g_{c_j} \text{ on } |z| = R/2 \quad \text{and} \quad H_j = h_j \text{ on } |z| = R
\]

and satisfies

\[
K_{H_j} \leq \frac{1}{C} = 2
\]
for \( j = 1 \) and \( 2 \). We take \( R \) sufficiently large such that \( \Gamma \) surrounds the filled-in Julia sets \( K(q_{c_j}) \) and \( K(q_{c_2}) \). We define a quasiregular map \( g : \mathbb{C} \to \mathbb{C} \) as follows:

\[
g(z) = \begin{cases} 
q_{c_1}(z + R^2) - R^2 & \text{on } |z + R^2| \leq R/2, \\
H_1(z + R^2) - R^2 & \text{on } R/2 < |z + R^2| \leq R, \\
q_{c_2}(z - R^2) + R^2 & \text{on } |z - R^2| \leq R/2, \\
H_2(z - R^2) + R^2 & \text{on } R/2 < |z - R^2| \leq R, \\
h(z) & \text{otherwise.}
\end{cases}
\]

We check the assumptions of Theorem 2.2. Let

\[
E_1 = \{ z \in \mathbb{C} : R/2 \leq |z + R^2| \leq R \} \quad \text{and} \quad E_2 = \{ z \in \mathbb{C} : R/2 \leq |z - R^2| \leq R \}.
\]

If \( R \) is large enough, then

\[
E_1 \cup E_2 \subset \{ z \in \mathbb{C} : |z| < R^3 \}
\]

For \( z = re^{iy} + R^2 \in \Gamma_1 \cup \Gamma_2 \),

\[
|g(z)| = |q_{c_j}(z) + R^2| = \left| 4R^4 (z \pm R^2)^2 + \frac{c_j}{4R^4} \mp R^2 \right| \\
\geq \left| 4R^4 \cdot \frac{R^2}{4} \cdot e^{2iy} + \frac{c_j}{4R^4} \mp R^2 \right| \geq R^6 - \frac{|c_j|}{4R^4} - R^2 > R^3 > 1 + \sqrt{1 + R^4}
\]

if \( R \) is large enough. It indicates that

\[
g(E_1 \cup E_2) \subset \{ z \in \mathbb{C} : |z| > R^3 \}
\]

and the orbit of any point in \( E_1 \cup E_2 \) under \( g \) tends to infinity, which implies that the assumption (1) of Theorem 2.2 holds. The other assumptions (2), (3) and (4) of Theorem 2.2 obviously hold. Therefore, there exists a 4-quasiconformal map \( \varphi : \mathbb{C} \to \mathbb{C} \) such that \( f = \varphi \circ g \circ \varphi^{-1} \) is an entire function of degree 4. Hence, \( f \) is a quartic polynomial. We normalize \( \varphi \) as \( \varphi(0) = 0 \) and \( \varphi(1) = 1 \). Then the finite critical points of \( f \) are 0 and \( \varphi(\pm R^2) \).

Let \( U_1 \) and \( U_2 \) be the bounded components of \( \mathbb{C} \setminus G^{-1}(G(f(0))) \) such that \( \varphi(-R^2) \in U_1 \) and \( \varphi(R^2) \in U_2 \), where \( G = G_f \) is the Green’s function associated with the filled-in Julia set \( K(f) \). Let \( V \) be the bounded component of \( \mathbb{C} \setminus G^{-1}(G(f(0))) \). Then \( (f|_{U_1}, U_1, V) \) and \( (f|_{U_2}, U_2, V) \) become quadratic-like maps. By the straightening theorem, the quadratic-like map \( (f|_{U_j}, U_j, V) \) is hybrid equivalent to the quadratic polynomial \( q_{c_j} \), which is affine conjugate to the quadratic polynomial \( p_{c_j} \) for \( j = 1 \) and 2.
Proof of Corollary B. Let $\omega_1 = \varphi(-R^2), \omega_2 = \varphi(R^2)$ and $\omega_3 = 0$. If $c_1$ and $c_2$ belong to the Mandelbrot set, the filled-in Julia sets $K(f|U_1)$ and $K(f|U_2)$ are connected. Moreover, $\omega_1 \in K(f|U_1) \subset K(f)$ and $\omega_2 \in K(f|U_2) \subset K(f)$, which imply that the $A$-$B$ kneading sequence of $\omega_1$ is $(AAA\cdots)$ and the $A$-$B$ kneading sequence of $\omega_2$ is $(BBB\cdots)$. Since $\omega_1, \omega_2 \in K(f)$ and $f^n(0) = f^n(\omega_3) \to \infty$ as $n \to \infty$, the Julia set $J(f)$ is disconnected but not totally disconnected. Therefore, the quartic polynomial $f$ obtained in Theorem A satisfies the assumptions of Theorem 1.4.

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