On discriminants of polylinear forms

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Abstract

In this paper we propose a conceptual framework for the observed properties of discriminants of polylinear forms. The connection with classical problems of linear algebra is shown. A new class of algebraic varieties (hypergrassmanians) is introduced, the particular case of which are grassman manifolds. An algorithm is given for computing the discriminants of polylinear forms of "boundary format" (hypergrassmanian analogues of plucker coordinates). The algorithm for computing the discriminants of polylinear forms of general formats is outlined.

0 Introduction

It is convenient (and instructive) to keep for certain objects of multilinear algebra the terminology from their counterparts in linear algebra (see [1]). So for the set of elements of $d$-linear form will be used the term "$d$-dimensional matrix", and for its discriminant (see definition below) we can use the term "determinant" with the notion of "minors" also making sense.

The investigation of determinants of multidimensional matrices may be useful already in the linear algebra (see Section 1), which essentially deals with sets of rectangular matrices. A good example (Section 1.1) is a 3-dimensional interpretation of the theory of kronekker pairs, which in this context obtains a straightforward generalization. It is also interesting to look for a generalization of eigenvalue theory. The theory of eigenvalues in its different versions is equivalent to the investigation of matrices of type $A + \lambda B$, where $A$ and $B$ is a pair of $n \times n$ matrices, and invariants of this pair, which can be expressed in terms of $GL(n) \times GL(n)$-action on $n \times n \times 2$ form (3-dimensional matrix). In a similar way the "multidimensional eigenvalue theory" is reduced to the invariants of 3-dimensional matrices of larger formats.

We know, that in the case of bilinear forms (2-dimensional matrices) the determinant (as one polynomial) is defined only for square matrices. So in order to specify the size of 2-dimensional matrix which has a unique expression (determinant) as a characteristic of its degeneracy, it is enough to give one number $n$ - the number of rows or columns. For polylinear forms the difference from bilinear ones we can illustrate on the case of 3 dimensions. Take a 3-dimensional matrix with elements $a_{i_1i_2i_3}$, where $1 \leq i_1 \leq n_1$, $1 \leq i_2 \leq n_2$, $1 \leq i_3 \leq n_3$. Let us fix $n_1$ and $n_2$. Then the determinant is defined for matrices with $n_3$ satisfying the following inequality (see [1]):

\[ n_1 - n_2 + 1 \leq n_3 \leq n_1 + n_2 - 1 \]
So the size ("format") of $d$-dimensional matrices which have the determinant is defined in general by $d$ parameters while in 2-dimensional case by only 1. But in $d$-dimensional case for $d > 2$ there also exists a class of matrices the size of which is described by $d-1$ parameters. These are so called “matrices of boundary format”, the size of which corresponds to the equality in (0.1). In 2-dimensional case the matrices, the degeneracy of which can not be characterized by one expression, are rectangular matrices. In higher dimensional case these are “matrices of grassman format”, those which in case $d = 3$ do not satisfy (0.1).

It this paper we give a set of properties of matrices of boundary and grassman formats, which show that they are the proper generalizations if square and rectangular matrices correspondingly. In particular, the condition for rectangular matrices of beeing of corank 1 is that the determinants of all maximal square submatrices (maximal minors) are 0. The corresponding first degeneracy condition for $d$-dimensional matrices, defined here as the condition of beeing of corank 1 (see Section 2.1), is that the determinants of all maximal submatrices of boundary format are 0. To a rectangular matrix one can put into correspondence a set of vectors - its rows or columns. Then the condition for beeing of corank 1 for rectangular matrix has a geometric interpretation as the linear dependence of these vectors (1-dimensional matrices). To a $d$-dimensional matrix with elements $a_{i_1...i_d}$ one can put into correspondence a set of "slices" in $k$-th direction, which are $(d-1)$-dimensional matrices with elements $a_{i_1...\hat{i}_k...i_d}$. Then the condition for beeing of corank 1 for $d$-dimensional matrices of grassman format can be expressed geometrically in terms of singularities of the intersection of the span of these slices with the submanifold of $(d-1)$-dimensional matrices of corank 1. The remarkable fact, making the notion of corank 1 matrices well defined, is that this singularity condition does not depend on the direction of slicing of our matrix (the number $k$ above).

Consider the problem of finding the kernel of a linear combination $S(\lambda) := \lambda_1 A_1 + \ldots + \lambda_k A_k$ of $k$ rectangular matrices of size $m \times n$. This kernel will be an $(n-m)$-dimensional subspace, i.e. an element of $G_{m,n}$. Changing $(\lambda_1, \ldots, \lambda_k)$ we get a k-parametric subset in $G_{m,n}$. In the case when $k = n - m + 1$ the image of this subset via plukker embedding of $G_{m,n}$ will be a Veronese manifold. The Veronese manifolds which can be obtained in this way are called here "proper Veronese manifolds". The condition for a proper Veronese manifold to be degenerate can be expressed in to ways:

1) as the condition that there exists such $(\lambda_1, \ldots, \lambda_k)$ that all plukker coordinates of the kernel of $S(\lambda)$ (which are $m \times m$ minors of $S(\lambda)$) are equal to 0, or as the singularity of the intersection of $\text{span}(A_1, \ldots, A_k)$ with the submanifold $M'_{mn}$ of degenerate $m \times n$ matrices

2) as the condition that the determinant of 3-dimensional matrix of size $m \times n \times (n-m+1)$ made of the elements of $A_1, \ldots, A_{n-m+1}$ is 0.

In the case of grassman format, when $k > n - m + 1$, the condition of the existence of the intersection of $\text{span}(A_1, \ldots, A_k)$ with $M'_{mn}$ is that the determinants of all $m \times n \times (n-m+1)$ submatrices of the corresponding 3-dimensional matrix are equal to 0. It happens that the similar fact holds in general $d$-dimensional case. This allows to interprete the determinants of maximal submatrices of boundary format of matrix of grassman format as multidimensional analogues of plukker coordinates and to consider the analogue of plukker map on the space of $d$-dimensional matrices of grassman format:

$$M_{n_1...n_d} \rightarrow P^n \choose m$$
of boundary format. As in 2-dimensional case here arises the fundamental problem to find the relations between the minors of multidimensional matrix, the analogues of plukker relations, i.e. to describe the image of $M_{n_1...n_d}$ as an algebraic manifold.

For studying the discriminants of polylinear forms there is a fundamental question about the algorithm of explicit calculation of these discriminants. In Section 3 we develop a technique which gives an algorithm of calculating the discriminants of $d$—linear forms of boundary format (“hyperplukker koordinates”). This technique happens to be basic for calculating the discriminants of $d$—linear forms of general format. In Section 4 we outline and give an example of this general algorithm.

**Definition 0.0.1** If $p(x_1, ..., x_m) = \sum_{i_1 \leq \ldots \leq i_n} c_{i_1...i_n}x_{i_1}...x_{i_n}$ is a homogeneous polynomial of degree $n$, then the set of values of coefficients $c$ is called discriminantal if the system of equations

$$\frac{\partial p(x_1, ..., x_m)}{\partial x_i} = 0, \quad i = 1, ..., m$$

has a solution $(x_1, ..., x_m) \in (\mathbb{C}^*)^n$.

In case when the discriminantal set is an algebraic submanifold of codimension 1 in the space of coefficients, it is called the discriminant of $p$, denoted by $D(p)$.

Now we will consider a particular case of this definition. Let $V_{n_1}, ..., V_{n_d}$ be a set of linear vector spaces, such that $\text{dim}(V_{n_i}) = n_i$. Let $a \in (V_{n_1} \otimes \ldots \otimes V_{n_d})^*$ be a $d$—linear form. For a set of vectors $x^{(k)} \in V_{n_k}, k = 1, ..., d$ with coordinates $x^{(k)} = (x^{(k)}_1, ..., x^{(k)}_{n_k})$ in a chosen basis the value of the form on $\otimes_{k=1}^d x^{(k)}$ is a polinomial of $d$ sets of variables $x^{(1)}, ..., x^{(d)}$ of degree $d$

$$a(x^{(1)}, ..., x^{(d)}) = p(x^{(1)}, ..., x^{(d)})$$

and the system (0.2) becomes

$$\frac{\partial a(x^{(1)}, ..., x^{(d)})}{\partial x^{(k)}_i} = 0, \quad i = 1, ..., n_k, \quad k = 1, ..., d$$

The coefficients $a_{i_1...i_d}$ of this form are the elements of $d$—dimensional rectangular $n_1 \times ... \times n_d$ matrix.

**Definition 0.0.2** The discriminant of the polinomial $a(x^{(1)}, ..., x^{(d)})$ is called the determinant of $n_1 \times ... \times n_d$ matrix $(a_{i_1...i_d})$ and denoted by $\text{det}(a)$.

**Example.** Let $a \in (U_n \otimes V_n)^*$. Then its coefficients form a usual $n \times n$ square matrix. Then $h = \sum_{1 \leq i, j \leq n} x_{ij}a_{ij}y_j$. The system (0.2) in this case becomes:

$$\sum_{j=1}^n a_{ij}y_j = 0, \quad i = 1, ..., n \quad \sum_{i=1}^n x_{ij}a_{ij} = 0, \quad j = 1, ..., n$$

Note, that this system contains the system of linear homogeneous equations as well as its conjugate. For the discriminants of polylinear forms this property will become essential.
There is a natural action of the group $GL_{n_1} \times \ldots \times GL_{n_d}$ on $\otimes_{j=1}^d V_{n_j}$ with the induced action on $(\otimes_{j=1}^d V_{n_j})^*$. Since the system (0.2) for $p = a(x^{(1)}, \ldots, x^{(d)})$ is invariant under this action we have

**Proposition 0.0.1** The determinant of $n_1 \times \ldots \times n_d$ matrix is invariant under the action of $GL_{n_1} \times \ldots \times GL_{n_d}$.

**Notation** Denote $M_{n_1 \ldots n_d} := (\otimes_{j=1}^d V_{n_j})^*$.

### 1 Problems of linear algebra and determinants of 3–dimensional matrices

#### 1.1 General setting

Let $M_{nm}$, where $n \leq m$, be a linear space of $n \times m$ matrices. Let $M'_{nm} \subset M_{nm}$ be a submanifold of matrices of rank $n - 1$. Let $A_1, \ldots, A_k \in M_{nm}$ be a set of $n \times m$ matrices. From their elements we can make an $n \times m \times k$ matrix of coefficients of 3-linear form $(a_{i_1i_2i_3})$. There is $1 - 1$ correspondence between linear subspaces $\text{span}(A_1, \ldots, A_k) \subset M_{nm}$ for different choices of $A_1, \ldots, A_k$ and the orbits of the corresponding forms $a(A_1, \ldots, A_k)$ under the action of $GL_k$ on $M_{nmk} = (V_n \otimes V_m \otimes V_k)^*$.

**Example 1.1.1** Let $A$ and $B$ be $n \times n$ matrices.

**Proposition 1.1.1** Let $\det(B) \neq 0$. Then the following statements are equivalent:

1) $\det(a_{i_1i_2i_3}(A, B)) = 0$

2) $D(\det(A + zB)) = 0$ or $D(\det(AB^{-1} - zI)) = 0$, i.e. the characteristic polynomial of $AB^{-1}$ has multiple roots

where $D(p(z))$ denotes the discriminant of polynomial $p(z)$.

**Example 1.1.2** Let $A$ and $B$ be $n \times (n + 1)$ matrices.

**Proposition 1.1.2** The determinant of $n \times (n + 1) \times 2$ matrix $\det(a_{i_1i_2i_3}(A, B)) \neq 0$ iff the pair $(A, B)$ is “kronekker”, i.e. by the action of $GL_n \times GL_{n+1}$ it can be reduced to the following canonical form

$$A = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

#### 1.2 Veronese manifolds

Let $k = m - n + 1$, $A_1, \ldots, A_k \in M_{nm}$. Let $S(z_1, \ldots, z_k) := z_1A_1 + \ldots + z_kA_k$ be a point of subspace $\text{span}(A_1, \ldots, A_k) \subset M_{nm}$. Let $\Delta(z_1, \ldots, z_k) := (\Delta_{i_1 \ldots i_m}(S))_{(i_1, \ldots, i_k) \in \binom{[m]}{n}} \in \mathbb{C}$.
be a vector with components - \( n \times n \) minors of \( S \), the image of \( S(z) \) via Plukker embedding. The map \( \varphi : (z_1, ..., z_k) \mapsto \Delta(z_1, ..., z_k) \) gives a \( k \)-parametric submanifold \( \mathcal{V}(A_1, ..., A_k) \) in \( \mathbb{P}^{m \choose n} \).

**Proposition 1.2.1** For \( A_1, ..., A_k \) in general position the intersection \( M'_{nm} \cap \text{span}(A_1, ..., A_k) \) is empty.

**Proposition 1.2.2** For \( A_1, ..., A_k \) in general position the manifold \( \mathcal{V}(A_1, ..., A_k) := \varphi(\mathbb{C}^k) \subset \mathbb{P}^{m \choose n} \) is a Veronese manifold.

The Veronese manifolds obtained in this way we call proper.

**Theorem 1.2.1** The following statements are equivalent:
1) the intersection \( M'_{nm} \cap \text{span}(A_1, ..., A_k) \) is not empty
2) \( \det(a_{i_1i_2i_3}(A_1, ..., A_k)) = 0 \)
3) the manifold \( \mathcal{V}(A_1, ..., A_k) \) is singular
4) \( \mathcal{V}(A_1, ..., A_k) \) belongs to a hyperplane in \( \mathbb{P}^{m \choose n} \)

Let \((x_1, ..., x_n), (y_1, ..., y_m), (z_1, ..., z_k)\) be the coordinates in \( V_n, V_m \) and \( V_k \) correspondingly. For \( a = a(A_1, ..., A_k) \in M_{nmk} = V_n \otimes V_m \otimes V_k \) the system (0.2), where \( p = a(x, y, z) \), contains a subsystem

\[
(1.5) \quad \frac{\partial a(x, y, z)}{\partial x_i} = \sum_{j=1}^{m} (z_1 A_1 + ... + z_k A_k)_{ij} y_j = S(z) y = 0, \quad i = 1, ..., n
\]

To a given \( z \in (\mathbb{C}^*)^k \) we can put into correspondence a subspace \( \text{Ker}(S(z)) \subset V_n \) of solutions of (1.5). If \( \text{rank}(S(z)) = n \) then \( \text{Ker}(S(z)) \) has dimension \( m - n \). If \( \text{rank}(S(z)) < n \) then \( \text{Ker}(S(z)) \) has dimension greater than \( m - n \).

**Proposition 1.2.3** \( \det(a(A_1, ..., A_k)) = 0 \) iff there are values of \( z \in (\mathbb{C}^*)^k \) such that the dimension of the space of solutions of corresponding system (1.5) is greater then \( m - n \) and the \( \varphi \)- image of the set of such \( z \) is exactly the set of singular points of \( \mathcal{V}(A_1, ..., A_k) \).

## 2 Hyperveronese and hypergrassmanian manifolds

### 2.1 On the rank of polylinear forms

For a given form \( a \in M_{n_1 ... n_d} \) and an integer number \( 1 \leq k \leq d \) we can put into correspondence a set of forms \( a_{i_k}^{(k)} \in M_{n_1 \otimes ... \otimes n_d} \), \( i_k = 1, ..., n_k \), such that \( (a_{i_k}^{(k)})_{i_1...i_k...i_d} = a_{i_1...i_d} \).

For a given \( k \) there is \( 1 - 1 \) correspondence between the orbits of the forms \( a \) under the action of \( GL_{n_k} \) on \( M_{n_1 ... n_d} \) and the corresponding linear subspaces \( \text{span}(a_{i_1}^{(k)}, ..., a_{n_k}^{(k)}) \subset \)
\( M_{n_1 \ldots n_d} \). We say, that the \( GL_{m_k} \)-orbit of \( a \) has an intersection with a submanifold in \( M_{n_1 \ldots n_d} \) if \( \text{span}(a^{(1)}_1, \ldots, a^{(k)}_{n_k}) \) has an intersection with this submanifold.

**Notation.** Denote by \( M'_{n_1 \ldots n_d} \) the set of \( n_1 \times n_2 \) matrices of corank 1.

For a multi-index \( n_1 \ldots n_d \) let \( \tilde{n}_k := n_1 \ldots n_{k-1} n_{k+1} \ldots n_d \). Now by induction on \( d \) we can introduce the following definition.

**Definition 2.1.1** A subset of the space \( M_{n_1 \ldots n_d} \) is called the set of forms of corank 1 (denoted by \( M'_{n_1 \ldots n_d} \)) if for \( a \in M_{n_1 \ldots n_d} \) and any "direction" \( k = 1, \ldots, d \) the intersection \( \text{span}(a^{k}_1, \ldots, a^{k}_{n_k}) \cap M'_{n_k} \) is not in general position (i.e. when a comes onto \( M'_{n_1 \ldots n_d} \) the topology of this intersection changes).

**Theorem 2.1.1** The set \( M'_{n_1 \ldots n_d} \) is the discriminant set for the system (0.3).

As it is shown in Section 2.2 \( M'_{n_1 \ldots n_d} \) is an algebraic manifold.

**Definition 2.1.2** The space \( M_{n_1 \ldots n_d} \) is called the space of inner format if for any \( a \in M_{n_1 \ldots n_d} \) and any "direction" \( k = 1, \ldots, d \) the intersection of \( GL_{m_k} \)-orbit of \( a \) with the submanifold \( M'_{n_k} \) of \( (d - 1)- \) linear forms of corank 1 is not empty.

Otherwise the format \( n_1 \ldots n_d \) is called grassmanian. The grassmanian format \( n_1 \ldots n_d \) for which \( M'_{n_1 \ldots n_d} \) has codimension 1 is called boundary.

The term "grassmanian format" is motivated by the fact that a certain (see Section 2.2) factorization of the space \( M_{n_1 \ldots n_d} \) gives an algebraic manifold which in case \( d = 2 \) is the grassman manifold \( G_{n_1,n_2} \).

**Example** Consider the case of 3-linear forms of format \( 2 \times 2 \times n \) for \( n \geq 4 \). These are fgrassmanounardy format. For a given \( a \in M_{22n} \) take its \( GL_n \)-orbit, i.e. the set of linear combinations \( S(z) := A_1 z_1 + \ldots + A_n z_n \), where \( 2 \times 2 \) matrices \( A_1, \ldots, A_n \) are \( 2 \times 2 \) slices of \( a \).

The intersection of \( \text{span}(A_1, \ldots, A_n) \) with the submanifold \( M'_{22} \) of degenerate \( 2 \times 2 \) matrices corresponds to the set of such \((z_1, \ldots, z_n)\) that \( \text{det}(S(z)) = 0 \). The expression \( \text{det}(S(z)) \) is a quadratic form on \( z \). The intersection of \( \text{span}(A_1, \ldots, A_n) \) with \( M'_{22} \) is described in terms of the rank of this quadratic form. Then the notion of the corank of our 3-linear form \( a(A_1, \ldots, A_n) \) can be formulated in terms of the rank of quadratic form \( \text{det}(S(z)) \) as follows:

**Proposition 2.1.1** The corank of \( 2 \times 2 \times n \) form \( a(A_1, \ldots, A_n) \) is equal to 1 iff the rank of quadratic form \( \text{det}(S(z)) \) is equal to 2.

### 2.2 Proper hyperveronese manifolds

For a set of integers \( n_1, \ldots, n_d \) let \( m_d := n_1 + \ldots + n_d + 1 - d \). Take an \( n_1 \times \ldots \times n_r \times n_{r+1} \times m_{r+1} \) matrix \((a_{i1} \ldots i_{r+1})\). Denote \( T_i := (a_{i1}^{(r+1)}), \quad i = 1, \ldots, n_{r+1} \) the \( n_1 \times \ldots \times n_r \times m_{r+1} \) "slices" of \( a \). For \((z_1, \ldots, z_{n_{r+1}}) \in (\mathbb{C}^*)^{n_{r+1}} \) let \( S(z_1, \ldots, z_{n_{r+1}}) := z_1 T_1 + \ldots + z_{n_{r+1}} T_{n_{r+1}} \) be points of \( \text{span}(T_1, \ldots, T_{n_{r+1}}) \). Denote by \( \Delta_{j_1 \ldots j_m}(S) \), where \((j_1, \ldots, j_m) \in \left( \begin{array}{c} [m_{r+1}] \\ m_r \end{array} \right) \)
the \( n_1 \times \ldots \times n_r \times m_r \) minors of \( S \). Then the problem of finding the intersection of \( \text{span}(T_1, \ldots, T_{n_r+1}) \cap M'_{n_1 \ldots n_r m_{r+1}} \) is the problem of solving the system:

\[
\Delta_{j_1 \ldots j_{mr}}(S) = 0, \quad (j_1, \ldots, j_{mr}) \in \binom{[m_{r+1}]}{m_r}
\]

\( (2.6) \)

**Proposition 2.2.1** The system \((6)\) has a nonzero solution \( z \in (\mathbb{C}^*)^{n_{r+1}} \) iff \( a \) belongs to a submanifold of codimension 1.

According to Definitions 2.1.1 and 2.1.2 this means that the format 
\( n_1 \times \ldots \times n_{r+1} \times m_{r+1} \)

is boundary grassmanian.

The set \( \Delta = (\Delta_{j_1 \ldots j_{mr}}) \) of minors gives us the components of a vector in \( \mathbf{P} \left( \binom{[m_{r+1}]}{m_r} \right) \). So we have a map:

\[
\phi : (C^*)^{n_{r+1}} \to \mathbf{P} \left( \binom{[m_{r+1}]}{m_r} \right)
\]

\[
(z_1, \ldots, z_{n_{r+1}}) \mapsto \Delta(z)
\]

the image of which is a manifold \( \mathcal{V} \) parametrized by \( (z_1, \ldots, z_{n_{r+1}}) \).

**Theorem 2.2.1** The following statements are equivalent:
1) \( \det(a_{i_1 \ldots i_{r+1}j}) = 0 \)
2) the manifold \( \mathcal{V} \) is singular
3) the system

\[
\Delta_{j_1 \ldots j_{mr}}(S(z)) = 0, \quad (j_1, \ldots, j_{mr}) \in \binom{[m_{r+1}]}{m_r}
\]

has solutions in \( (\mathbb{C}^*)^{n_{r+1}} \) and \( \phi \) gives a 1–1 correspondence between the solutions of the system and the singular points of \( \mathcal{V} \).

Comparing this statement with Teorem 1.2.1 we are lead to the following:

**Definition 2.2.1** For \( (a) \in M_{n_1 \ldots n_{r+1} m_{r+1}} \) the manifold \( \mathcal{V} \) is called proper hypveronese manifold.

### 2.3 Hypergrassmanians

Let \( (a_{i_1 \ldots i_{r+1}j}) \in M_{n_1 \ldots n_r m} \) where \( m > m_r = 1+n_1+\ldots+n_r-r \). The \( n_1 \times \ldots \times n_r \times m_r \) minors \( \Delta_{j_1 \ldots j_{mr}} \) of \( (a) \) are invariants of \( GL_{n_1} \times \ldots \times GL_{n_r} \) action on \( M_{n_1 \ldots n_r m} = (V_{n_1} \otimes \ldots \otimes V_{n_r} \otimes V_m)^* \).

For \( 1 \leq k \leq r \) take the set \( a_1^{(k)}, \ldots, a_{n_k}^{(k)} \) of \( n_1 \times \ldots \times n_k \times \ldots \times m \) "slices" of \( (a) \) in \( k \)-th direction (see Section 2.1).

**Proposition 2.3.1** The intersection \( \text{span}(a_1^{(k)}, \ldots, a_{n_k}^{(k)}) \cap M'_{\tilde{n}_k} \) is not empty iff

\[
\Delta_{j_1 \ldots j_{mr}}(a) = 0, \quad \forall (j_1, \ldots, j_{mr}) \in \binom{[m]}{m_r}.
\]
This implies that the map
\[ \mathcal{P} : M_{n_1...n_r,m} \to \mathbf{P}\left(\frac{m}{m_r}\right) \]
\[ (a) \mapsto \Delta(a) \]
has the kernel \( M'_{n_1...n_r,m} \), induces an injection of the open stratum of the space of orbits \( M_{n_1...n_r,m}/GL_{n_1} \times ... \times GL_{n_r} \) into \( \mathbf{P}\left(\frac{[m]}{m_r}\right) \) and its image is a projective algebraic manifold \( G_{n_1...n_r,m} \).

**Definition 2.3.1** The manifold \( G_{n_1...n_r,m} \) is called *hypergrassmanian*.

So the coordinates on a hypergrassmanian (the open stratum of the factor space of the space \( M_{n_1...n_r,m} \) of forms of grassmanian format), as in the particular case (for \( r = 1 \)) of grassman manifolds, are given by the minors of boundary format ("hyperplukker coordinates").

### 3 Algorithm

Here we give an algorithm for computing the determinants of matrices of boundary format ("hyperplukker coordinates").

#### 3.1 Basic example

Let \( P_0, ..., P_d \) be a sequence of ordered sets \( P_k \), such that \( |P_k| = k + 1 \).

**Notation.** Denote by \( C \) the space of sequences \( q := (q_1, q_2, ..., q_d) \), where \( q_k \in P_k \).

For a pair of such sequences \( q', q'' \in C \) we say that \( q' \leq q'' \), if \( \exists K < d \), such that \( q'_k \leq q''_k \) in \( P_k \), for \( k > K \).

The sequence \( P_0, ..., P_d \) can be represented in the form of a diagram on Fig.1, where \( (k + 1) \)-th row represents the elements of the set \( P_k \) ordered from the left to the right. A sequence \( q \) may be represented in the form of a diagram on Fig.2, where \( q_k \) is represented by a cross in the \( (k + 1) \)-th row.

For a given \( q_{k+1} \in P_{k+1} \) there is a unique ordering-preserving injection \( f'_k : P_k \hookrightarrow P_{k+1} \), such that \( q_{k+1} \not\in f'_k(P_k) \) and vice versa.
For \( q \) represented by Fig.2 the corresponding sequence \( f^q := (f_1^q, ..., f_4^q) \) of injections may be represented in form of a diagram on Fig.3.

Then each sequence \( q \) defines a sequence \( p(q) := (p_1, ..., p_d) \), where \( p_k \in P_k \), such that \( p_{k+1} = f_k^q(p_k) \). The sequence \( p \), corresponding to the sequence \( q \) on Fig.2, may be represented in form of a diagram on Fig.4, where the path goes through the elements \( p_k \). The rule for drawing a path, corresponding to a given sequence \( q \) may be formulated as follows:

if \( q_{k+1} \in P_{k+1} \) lies (on the diagram) to the left from \( p_k \), then we have to go from \( p_k \) to the right closest to it point of \( P_{k+1} \), if \( q_{k+1} \in P_{k+1} \) lies to the right from \( p_k \), then we have to go from \( p_k \) to the left closest point of \( P_{k+1} \).

On the other hand, for a given \( q_k \) the corresponding \( f_k^q : P_k \to P_{k+1} \) defines a partition of \( P_k \) into two parts \( P_k^-(q) \) and \( P_k^+(q) \), where

\[
\begin{align*}
P_k^-(q) & := \{ p_k \in P_k | f_k^q(p_k) < q_{k+1} \} \\
P_k^+(q) & := \{ p_k \in P_k | f_k^q(p_k) > q_{k+1} \}
\end{align*}
\]

The sequence of partitions, corresponding to the sequence of ordering preserving injections \( f^q = (f_0^q, f_1^q, f_2^q, f_3^q) \) on Fig.3, may be represented in form of a diagram on Fig.5, where in the \( k \)-th row the elements of \( P_k^-(q) \) are represented by characters “1” and the elements of \( P_k^+(q) \) are represented by characters “2”.

Notation. Denote by \( j(q) \) the ordinal number of the element \( p_d(q) \in P_d \) with respect to the ordering on \( P_d \). For a given pair \( q', q'' \in C \) denote by \([i_1, ..., i_d, j](q', q'')\) the sequence of indecies \((i_1, ..., i_d, j)\) such that

\[
\begin{align*}
j = j(Q) \text{ and } i_{k+1} = \begin{cases} 1, & \text{if } p_k(q') \in P_k^-(q'') \\ 2, & \text{if } p_k(q') \in P_k^+(q'') \end{cases}
\end{align*}
\]

For example, for \( q' = q'' = q \) from Fig.2 we can write \((i_1, ..., i_4, j)\) using the diagram on Fig.6, which may be viewed as a “superposition” of Fig.4 and Fig.5, \((i_1, ..., i_4, j) = [i_1, i_2, i_3, i_4, j](q, q) = (2, 1, 2, 2, 4)\).

**Proposition 3.1.1**

\[
\prod_{q \in C} a_{[i_1, ..., i_d, j](q, q)}
\]

is a monomial of discriminant of polylinear form of dimension \( 2 \times 2 \times ... \times 2 \times (d + 1) \) \( d \) times with coefficients \((a_{i_1, ..., i_d, j})\).

This monomial will be called diagonal monomial.
3.2 General algorithm

Let \( n_1, ..., n_d \) be a sequence of positive integers and set \( n_0 = 1 \). Denote \( m_k := n_0 + n_1 + \ldots + n_k - k \). Let \( P_0, ..., P_d \) be a sequence of ordered sets, such that \(|P_k| = m_k\). Let \( T_1, ..., T_d \) be a sequence of ordered sets, such that \(|T_k| = n_k\).

**Notation.** Denote by \( \mathcal{C}_k := \{Q_k \subset P_k : |Q_k| = m_k - m_{k-1}\} \) and by \( \mathcal{C} \) the space of sequences \( Q := (Q_1, ..., Q_d) \), where \( Q_k \in \mathcal{C}_k \). For a pair \( S', S'' \subset P_k \), where \( S' = (s'_1, ..., s'_{n'_k}) \), \( S'' = (s''_1, ..., s''_{n''_k}) \) and the elements of \( S' \) and \( S'' \) are written in the order induced by the ordering on \( P_k \), we say that \( S' \leq S'' \) if \( \exists \epsilon < n_k \) such that \( s'_\epsilon \leq s''_\epsilon \) for \( n > \epsilon \). This gives the ordering of \( \mathcal{C}_k \). For a pair of sequences \( Q', Q'' \in \mathcal{C} \) we say that \( Q' \preceq Q'' \) if \( \exists K < d \), such that \( Q'_k \preceq Q''_k \), for \( k > K \). For a given \( Q_{k+1} \in \mathcal{C}_{k+1} \) there exists a unique order preserving injection \( f_k^Q : P_k \mapsto P_{k+1} \setminus Q_{k+1} \). Then each sequence \( Q \in \mathcal{C} \) defines a sequence \( p(Q) := (p_0, p_1, ..., p_d) \), where \( p_k \in P_k \), such that \( p_{k+1} = f_k^Q(p_k) \) which is called \( Q \)-path. An example of a \( Q \)-path is shown on Fig.4.

On the other hand, since \( P_k \) are ordered, then a given \( Q_k \) defines a partition \( R_k = (R_k^1, ..., R_k^n) \) of \( P_k \setminus Q_k \) into \( n_k \) subsets \( R_k^i \). Then any injection \( \phi : P_{k-1} \mapsto P_k \setminus Q_k \) induce a map \( g_k : P_{k-1} \mapsto T_k \) as follows:

\[
(3.7) \quad \text{if } p \in \phi^{-1}(R_k^i) \text{ then } g_k(p) = i \in T_k
\]

where we write number \( i \) instead of the \( i \)-th element of \( T_k \).

**Definition 3.2.1** For a given \( Q_k \in \mathcal{C}_k \) such a map will be called \( Q \)-admissible and the sequence \( g^Q = (g_1, ..., g_d) \) of \( Q \)-admissible maps is called a \( Q \)-diagram. If in a \( Q \)-diagram all \( g_k \) are induced by the order preserving injections \( f_k^Q \), then this \( Q \)-diagram is called initial.

An example of the initial \( Q \)-diagram is shown on Fig.5. Now let us take a pair \( Q', Q'' \in \mathcal{C} \). Then for \( Q' \) we take the \( Q' \)-path \( p(Q') := (p_0, p_1, ..., p_d) \) and for \( Q'' \) we choose a \( Q'' \)-diagram \( g^{Q''} = (g_1, ..., g_d) \) from the set of \( Q'' \)-admissible diagrams. This gives us a sequence of indecies:

\[
(3.8) \quad I(Q', g^{Q''}) = (i_1, ..., i_d), \quad \text{where } i_k := g^{Q''}(p_{k-1}^Q).
\]

**Definition 3.2.2** The pair \( (p^{Q'}, g^{Q''}) \) is called \( Q' \)-path over a \( Q'' \)-diagram.

An example of this construction is shown on Fig.6. **Notation.** Denote by \( \mathcal{C}^{(k)} \) the space of subsequences \( Q^{(k)} := (Q_{k+1}, ..., Q_d) \).

**Definition 3.2.3** For a given \( Q^{(k)} \in \mathcal{C}^{(k)} \) two sequences \( Q', Q'' \in \mathcal{C} \) will be called \( Q^{(k)} \)-conjugate if \( Q'_m = Q''_m = Q^{(k)}_m \) for \( m > k \). For a given \( Q^{(k)} \in \mathcal{C}^{(k)} \) and \( p := (p_1, ..., p_d) \) two sequences \( Q', Q'' \in \mathcal{C} \) will be called \( (Q^{(k)}, p) \)-conjugate, if \( Q'_m = Q''_m = Q^{(k)}_m \) for \( m > k \), and \( p(Q') = p(Q'') = p_k \in P_k \).

**Notation.** Denote by \( \mathcal{D}(Q^{(k)}) \) the set of all \( Q^{(k)} \)-conjugate sequences and by \( \mathcal{D}(Q^{(k)}, p_k) \) the set of all \( (Q^{(k)}, p_k) \)-conjugate sequences in \( \mathcal{C} \). Set by definition \( \mathcal{D}(Q^{(d)}) := \mathcal{C} \).
Proposition 3.2.1 For a given $k \leq d$

$$|\mathcal{D}(Q^{(k)}, p_k)| = \frac{(m_k - 1)!}{(n_1 - 1)!...(n_k - 1)!}$$

for any $p_k \in P_k$.

On each $\mathcal{D}(Q^{(k)}, p_k)$ there is an ordering induced by the ordering on $\mathcal{C}$.

**Notation.** For a given $k$ denote by $\mathcal{L}_k$ the set of ordinal numbers, enumerating the elements of each $\mathcal{D}(Q^{(k)}, p_k)$.

**Remark:** Proposition 2.1 implies, that $|\mathcal{L}_0| = 1$ and $|\mathcal{L}_1| = 1$.

Then for a given $Q \in \mathcal{C}$ we have a sequence of integer numbers $L(Q) := (l_1, ..., l_d)$, where $l_k(Q)$ is the ordinal number of $Q$ in $\mathcal{D}(Q^{(k)}, p_k)$.

**Notation.** For a given $Q^{(k)}$ and $l \in \mathcal{L}_{k-1}$ denote by $\mathcal{E}(Q^{(k)}, l)$ the set of sequences $Q$, such that $l_{k-1}(Q) = l$. Denote by $S\mathcal{C}_k$ the group of permutations of the elements of $\mathcal{C}_k$.

Then $S\mathcal{C}_k$ acts on $\mathcal{E}(Q^{(k)}, l)$ as follows: for $\sigma \in S\mathcal{C}_k$ and a given $Q = (Q_1, ..., Q_k, Q^{(k)}) \in \mathcal{E}(Q^{(k)}, l)$,

$$\sigma Q = (Q_1, ..., \sigma Q_k, Q^{(k)}).$$

If to each $\mathcal{E}(Q^{(k)}, l_{k-1})$ we put into correspondence a group $S(Q^{(k)}, l_{k-1}) \cong S\mathcal{C}_k$ with the action discribed above, then on the whole $\mathcal{C}$ we have the action of the group

$$\Sigma := \prod_{k=0}^{d-1} \prod_{Q^{(k+1)}} \prod_{l_k} S(Q^{(k+1)}, l_k)$$

Each $Q \in \mathcal{C}$ defines the following subgroup $\Sigma_Q$ of $\Sigma$:

$$\Sigma_Q = S\mathcal{C}_1(Q) \times ... \times S\mathcal{C}_d(Q)$$

where $S\mathcal{C}_k(Q) = S(Q^{(k)}, l_{k-1}(Q))$. $\Sigma_Q$ acts on $Q$ componentwise: for $\tau = (\tau_1, ..., \tau_d) \in \Sigma_Q$

$$\tau Q = (\tau_1 Q_1, ..., \tau_d Q_d).$$

The role of the group $\Sigma$ is analogous to the role of symmetric group in calculation of the determinant of $n \times n$ matrix.

**Proposition 3.2.2** If $Q \in \mathcal{C}$, $\sigma \in \Sigma$ and $\sigma_Q \in \Sigma_Q \subset \Sigma$ is the $\Sigma_Q$-component of $\sigma$ then

$$\sigma Q = \sigma_Q Q.$$

For $\sigma \in \Sigma$ denote

$$\text{sign}(\sigma) := \prod_{k=0}^{d-1} \prod_{Q^{(k+1)}} \prod_{l_k} \text{sign}(\sigma(Q^{(k+1)}, l_k))$$

where $\text{sign}(\sigma(Q^{(k+1)}, l_k))$ is the signature of permutation $\sigma(Q^{(k+1)}, l_k) \in S(Q^{(k+1)}, l_k)$.  

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Let us put into correspondence to each $Q^{(k)} \in C^{(k)}$ and $l_k \in \mathcal{L}_k$ a set $G(Q^{(k)}, l_k)$ of $Q_{k+1}$-admissible functions $g_{k+1}$ and take their direct product:

$$
\Gamma := \prod_{k=0}^{d-1} \prod_{Q^{(k)}} G(Q^{(k)}, l_k)
$$

Denote by $j(Q)$ the ordinal number of the element $p_d(Q) \in P_d$ with respect to the ordering on $P_d$. For given $Q \in C$, $\sigma \in \Sigma$ and $\gamma \in \Gamma$ denote by $[i_1, ..., i_d](Q, \sigma, \gamma)$ the sequence of indices $(i_1, ..., i_d, j)$, where

$$
j = j(Q), \quad i_k = g_k(p_{k-1}(Q)) \quad \text{and} \quad g_k = \gamma(Q^{(k)}, l_k(Q)) \in G((\sigma Q^{(k)}), l_k(Q)).
$$

**Theorem 3.2.1** If $\Omega = (a_{i_1, ..., i_d})$ is a polylinear form of dimension $n_1 \times ... \times n_d \times m_d$, then its discriminant

$$
D_\Omega = \sum_{\sigma \in \Sigma} \text{sign}(\sigma) \prod_{Q \in C} a_{[i_1, ..., i_d, j](Q, \sigma Q, \gamma)}
$$

**Example.** Let $d = 1, n_1 = n$. Then $m_1 = n, C = \mathcal{C}_1 = \{Q_1 \subset P_1 : |Q| = n - 1\}$ and each $Q$ is defined by the value of $j(Q)$. Since $|\mathcal{L}_0| = 1$, then for each $Q = Q_1$ the set $G(Q^{(0)})$ consists of only one element $g^Q$, such that the ordinal number of $g^Q(p_0)$ in $T_1$ is equal to $j(Q)$, and $\Gamma = \prod_{Q^{(0)}} G(Q^{(0)})$ consists of only one element $\gamma = \prod_{Q \in C} g^Q$. Also $\Sigma = S_{\mathcal{C}_1} \cong S_n$. Then for $\Omega = (a_{ij})_{1 \leq i, j \leq n}$

$$
D_\Omega = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{Q \in C} a_{[i, j](Q, \sigma Q, \gamma)} = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{j=1}^{n} a_{\sigma_{i_1}, j}
$$

is the determinant of the square matrix $(a_{ij})_{1 \leq i, j \leq n}$.

## 4 Closed determinant

**Definition 4.0.4** For $n_1 \times ... \times n_d$ $d-$dimensional matrix $(a_{i_1, ..., i_d})$ the product of all its minors (including the determinant) is called the **closed determinant** (denoted by $\text{Det}(a)$)

The term "closed" comes from the fact that as an algebraic manifold $\text{Det}(a)$ corresponds to the projectively dual to the closure of the $(C^*)^{n_1+...+n_d}$ orbit of $(1, 1) \otimes ... \otimes (1, 1) \in V_{n_1} \otimes ... \otimes V_{n_d}$.

**Example.** $2 \times 2 \times 2$ matrix. Let $(a_{i_1 i_2 i_3})_{i_1, i_2, i_3 = 1, 2}$ be a $2 \times 2 \times 2$ matrix. Then its closed determinant

$$
\text{Det}(a) = a_{111}a_{112}a_{121}a_{122}a_{211}a_{212}a_{221}a_{222} \times
\times (a_{111}a_{122}-a_{121}a_{112})(a_{211}a_{222}-a_{221}a_{212})(a_{111}a_{222}-a_{122}a_{212})(a_{121}a_{222}-a_{221}a_{112})(a_{111}a_{221}-a_{221}a_{112})(a_{112}a_{222}-a_{222}a_{111})
$$

$$
\times (a_{111}a_{222}^2+a_{112}a_{221}^2+a_{121}a_{212}^2+a_{211}a_{122}^2+a_{221}a_{112}^2-2a_{111}a_{122}a_{222}-2a_{111}a_{212}a_{222}-2a_{111}a_{221}a_{222}-2a_{121}a_{222}a_{112}
$$

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Proposition 4.0.3 The degree of the closed determinant of \( d \)-dimensional matrix of format \( n_1 \times \ldots \times n_d \) is equal to the degree of the (ordinary) determinant of \((d+1)\)-dimensional matrix of boundary format \( n_1 \times \ldots \times n_d \times (1 + n_1 + \ldots + n_d - d) \).

Let us take for each initial \( Q \)-diagram the \( Q \)-path on it (see Definition 3.2.1). The corresponding set of indecies will be denoted by \( I(Q) \).

Theorem 4.0.2 Let \( (a_{i_1,...i_d}) \) be a \( d \)-dimensional matrix. The monomial

\[
\prod_{Q \in C} a_{I(Q)}
\]  

is a monomial of the closed determinant \( \text{Det}(a) \).

Definition 4.0.5 The monomial \( \prod_{Q \in C} a_{I(Q)} \) is called diagonal.

There is an algorithm given in terms of paths over \( Q \)-diagrams (of which the evidencies of existence are Proposition 4.1 and Theorem 4.1) of computing closed determinants of polyninear forms, which will be published separately. Here we give an example of this procedure.

Example. 2 \( \times \) 2 \( \times \) 2 matrix. As a tool for our calculation let us draw the set of initial \( Q \)-diagrams with corresponding \( Q \)-paths on them

![Fig.7](image-url)
The diagrams are grouped into four rows with three pairs in each row. Let us enumerate the diagrams by triples of numbers \((q_1, q_2, q_3)\), where \(q_3 = 1, 2, 3, 4\) is the row number, \(q_2 = 1, 2, 3\) is the number of the group and \(q_1 = 1, 2\) is the number inside the group. For a diagram with a number \((q_1, q_2, q_3)\) its rows represent the functions \(g_k^{q_1, q_2, q_3} : P_0 \to 1, 2, g_k^{q_1, q_2, q_3} : P_1 \to 1, 2, g_k^{q_1, q_2, q_3} : P_2 \to 1, 2\) (where \(P_0, P_1, P_2\) are ordered sets from Section 3.2 such that \(|P_0| = 1, |P_1| = 2, |P_2| = 3\)), so to say “function \(g_k^{q_1, q_2, q_3}\)” is the same as to say “the \(k\)-th row of the diagram \((q_1, q_2, q_3)\)” and vice versa. Each triple \((q_1, q_2, q_3)\) is just a sequence \(Q\) of \(1\)-element subsets of \(P_1, ..., P_3\) (see Section 3.2), so here we can say “\((q_1, q_2, q_3)\)-diagram” instead of “\(Q\)-diagram for \(Q = (q_1, q_2, q_3)\)”.

The set of \(Q\)-paths on Fig.7 gives us the diagonal monomial

\[
d_{111}a_{112}a_{121}a_{122}a_{211}a_{212}a_{221}a_{222}a_{111}a_{122}a_{211}a_{212}a_{121}a_{222}a_{111}a_{221}a_{112}a_{222} \times
\]

\[
a_{111}^2a_{222}^2
\]

of \(Det(2 \times 2 \times 2)\). The rest of monomials is obtained as \(Q\)-paths over diagrams obtained from initial ones by permutations of the following type:

i) Permutations of odd type. These are permutations of rows between different diagrams. The generators are:

1) for a given \((q_2, q_3)\) and \(\tau \in S_2\) the action of \(\tau\) on \(g_1^{\bullet, q_2, q_3}\) (first rows of diagrams \((1, q_2, q_3)\) and \((2, q_2, q_3)\)) is the following

\[
\tau(g_1^{q_1, q_2, q_3}) = g_1^{\tau(q_1), q_2, q_3}
\]

(4.10)

2) for a given \((q_3)\) and \(\tau \in S_3\) the action of \(\tau\) on \(g_1^{\bullet, q_3}\) is the following

\[
\tau(g_2^{q_2, q_3}) = g_2^{\tau(q_2), q_3}
\]

(4.11)

and permutation \(\tau\) for \(q_3 = 2\) has to be chosen the same as for \(q_3 = 3\). In other words we have to permute the corresponding second rows of diagrams with \(q_3 = 2\) and \(q_3 = 3\) synchronously, so the permutations of second rows are the elements of the group \(S_3 \times S_3 \times S_3\).

We assign to an odd type permutation the sign which correspond to its parity.

ii) Permutations of even type. These are permutations of elements in the rows (elements of \(P_k\)). The generators are:

for a given \((q_3)\) permute the elements in second rows (elements of sets \(P_1\)) of \((\bullet, 2, q_3)\)-diagrams (the \(\bullet\) means that this permutation does not depend on the value of \(q_1\)) so that these permutations for diagrams with \(q_3 = 2\) and \(q_3 = 3\) coincide (synchronization condition).

We assign to even type permutations positive sign.

The permutations of different (odd and even) types commute. For a given permutation the sign of the monomial computed from \(Q\)-paths over permutation\((Q)\)-diagrams equals to the sign of odd type component of the permutation. As a prelude to the general algorithm we can remark that ”synchronization” of permutations on the space of \(Q\)-diagrams takes place for the sets of permutations which have the same domain of values of \(g_3 : P_2 \to T_3\).

As soon as we can compute the closed determinant for a form of a given format, its determinant is computed as the quotient of the closed determinant and the product of all its minors, which are the determinants of submatrices of smaller formats.
References

[1] I.Gelfand, M.Kapranov, A.Zelevinsky “Hyperdeterminants” //Adv.in Math, 1993.