Superconformal Hypermultiplets in Superspace

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Abstract

We use the manifestly N=2 supersymmetric, off-shell, harmonic (or twistor) superspace approach to solve the constraints implied by four-dimensional N=2 superconformal symmetry on the N=2 non-linear sigma-model target space, known as the special hyper-Kähler geometry. Our general solution is formulated in terms of a homogeneous (of degree two) function of unconstrained (analytic) Fayet-Iliopoulos hypermultiplet superfields. We also derive the improved (N=2 superconformal) actions for the off-shell (constrained) N=2 projective hypermultiplets, and relate them (via non-conformal deformations) to the asymptotically locally-flat (ALF) $A_k$ and $D_k$ series of the gravitational instantons. The same metrics describe Kaluza-Klein monopoles in M-theory, while they also arise in the quantum moduli spaces of N=4 supersymmetric gauge field theories with $SU(2)$ gauge group and matter hypermultiplets in three spacetime dimensions. We comment on rotational isometries versus translational isometries in the context of N=2 NLSM in terms of projective hypermultiplets.

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1 Introduction

Better understanding of non-conformal supersymmetric field theories is often facilitated by the study of superconformal field theories. One may, therefore, expect, that studying the N=2 (rigid) superconformal Non-Linear Sigma-Models (NLSM) in 3+1 spacetime dimensions may shed light on the structure of the hypermultiplet low-energy effective actions in quantized N=2 supersymmetric gauge field theories, as well as provide more insights into the moduli spaces of hypermultiplets in the type-II superstring compactifications on Calabi-Yau threefolds in the limit where the supergravity decouples, since they are all governed by hyper-Kähler metrics. The N=2 superconformal hypermultiplets also appear in describing the D3-brane world-volume field theories \[1\], and in relation to the AdS/CFT correspondence \[2\]. The universal formulation of hypermultiplets is indispensable for those purposes.

The most natural formulation of supersymmetry is provided by superspace. However, as regards N=2 supersymmetry in 1+3 dimensions, the standard N=2 superspace is not suitable to describe off-shell hypermultiplets. Since N=2 supersymmetry in the four-dimensional NLSM amounts to hyper-Kähler geometry in the NLSM target space \[3\], there should be a good reason for this failure inside the hyper-Kähler geometry. As is well known, the hyper-Kähler geometry is characterized by the existence of three, covariantly constant, complex structures, \(I^{(a)}\), \(a = 1, 2, 3\), satisfying a quaternionic algebra. In the conventional superspace approach one picks up a single complex structure to be manifest. In fact, any linear combination, \(c_1 I^{(1)} + c_2 I^{(2)} + c_3 I^{(3)}\), of the complex structures with some real coefficients \(c_a\) is again a covariantly constant complex structure provided that \(c_1^2 + c_2^2 + c_3^2 = 1\). There is, therefore, the whole sphere \(S^2\) of the hidden complex structures on the top of the manifest one. To treat all the complex structures on equal footing, one should add \(S^2\) to the hyper-Kähler manifold, which essentially amounts to its twistor extension. In the context of N=2 supersymmetry, the extension of the standard N=2 superspace by the two-sphere \(S^2\) gives rise to the off-shell (model-independent) formulation of a hypermultiplet with manifest (linearly realized) N=2 supersymmetry. The twistor extension of N=2 superspace comes in two versions known as Projective Superspace (PSS) and Harmonic Superspace (HSS). The PSS appears after adding a single projective holomorphic coordinate of \(CP^1 \sim S^2\). The PSS construction naturally leads to holomorphic potentials for hyper-Kähler metrics via the so-called generalized Legendre transform in terms of projective \(O(2p)\) hypermultiplets \[4, 5\]. In the HSS construction \[6\] one uses another isomorphism \(S^2 \sim SU(2)/U(1)\), one adds harmonics belonging to the group \(SU(2)\), and one considers only equivariant (Grassmann-)analytic functions with respect to
the $U(1)$ charge (cf. the notion of a flag manifold). The HSS thus naturally leads to analytic potentials for hyper-Kähler metrics. An off-shell, manifestly $N=2$ supersymmetric formulation of the most basic Fayet-Sohnius (FS) hypermultiplet is only possible with the infinite number of the auxiliary fields. This problem is elegantly solved in HSS that provides the universal formulation of a hypermultiplet in terms of a single unconstrained $N=2$ superfield in the analytic subspace of HSS \[6\]. At the same time, the infinite number of the auxiliary fields leads to a quite obscure connection (‘bridge’) between the HSS superfields and their physical components, which highly complicates a derivation of hyper-Kähler metrics out of the $N=2$ NLSM in terms of the FS hypermultiplet superfields. The projective hypermultiplets with the finite numbers of the auxiliary fields, are more suitable for describing the $N=2$ NLSM metrics with isometries, either translational or rotational.

The combined constraints implied by hyper-Kähler geometry and conformal invariance on the target space geometry of the (1+3)-dimensional $N=2$ supersymmetric NLSM were recently investigated by de Wit, Kleijn and Vandoren \[7\] who called the $N=2$ superconformal NLSM geometry special hyper-Kähler. They also found remarkable relations between the special hyper-Kähler, quaternionic and 3-Sasakian metrics, by using the component approach with on-shell $N=2$ supersymmetry for hypermultiplets. The approach adopted in ref. \[7\] is invariant under reparametrizations, but it formulates the special hyper-Kähler geometry in terms of complicated geometrical constraints that are unavoidable in any component approach. We use the manifestly $N=2$ supersymmetric HSS approach to solve those constraints in terms of an unconstrained homogeneous (of degree two) function of hypermultiplets, similarly to the standard $N=2$ superconformally invariant action of $N=2$ vector multiplets.

Though our general HSS construction uses the infinite number of auxiliary fields, we also derive the non-trivial $N=2$ superconformal NLSM in terms of the projective hypermultiplets with the finite numbers of the auxiliary fields. As an application, we easily reproduce $A_k$ and $D_k$ series of four-dimensional gravitational instanton metrics by combining the (improved) special hyper-Kähler potentials with different moduli and adding simple non-conformal deformations to them. Those metrics naturally arise in modern gauge and string theories that supply more motivation for the alternative and more transparent derivation of the metrics from HSS.

Distant physical problems in field theory often share common hyper-Kähler moduli space. For example, the vacuum structure of the quantized $SU(n)$-based $N=4$ supersymmetric (pure) gauge field theory in 2+1 dimensions (in the Coulomb branch) is known to be equivalent to the moduli space of $n$ BPS monopoles in the classical $SU(2)$-based ($N=0$) Yang-Mills-Higgs system in 3+1 dimensions \[8\]. The four-dimensional
hyper-Kähler spaces are necessarily self-dual, while the latter naturally arise as gravitational instantons in quantum gravity, or as the moduli spaces of solutions to the (integrable) system of Nahm equations with appropriate boundary conditions [9].

M-theory provides the unifying framework for a study of those remarkable correspondences, while it also offers their explanation via dualities. For example, the M-theory compactification on the $A_{k-1}$ Asymptotically Locally Flat (ALF) self-dual space is equivalent to the background of $k$ parallel D6-branes in the type-IIA string picture [10]. Probing this background with a parallel D2-brane gives rise to the $(2+1)$-dimensional N=4 supersymmetric effective gauge field theory with $k$ matter hypermultiplets in the D2-brane world-volume, whose moduli space is given by the ALF space. In the (dual) type-IIB string picture one gets a BPS configuration of intersecting D5- and D3-branes, which gives rise (in the infra-red limit) to the same effective field theory in the D3-brane world-volume, and whose moduli space is given by the charge-two (centered) BPS monopole moduli space with $k$ singularities [11].

Though the brane technology appears to be very efficient in establishing the equivalence between the apparently different moduli spaces, it does not offer any means for a calculation of the exact moduli space metrics. The standard hyper-Kähler quotient construction of the multi-monopole moduli space metrics from the Nahm data is very complicated in practice, so that it does not allow one to establish a simple and natural way of describing the metrics. The alternative approach is provided by twistor methods whose essential ingredients are given by holomorphic vector bundles over the twistor space [12]. The ALF metrics can be obtained from the (moduli-dependent) holomorphic potentials in the twistor space by the generalized Legendre transform that was originally deduced from PSS [4]. However, even the generalized Legendre transform techniques are usually limited to the hyper-Kähler metrics having isometries, whereas a generic monopole moduli space does not have any isometries. It is, therefore, very desirable to develop a more universal approach to this problem.

The ALF spaces asymptotically approach $\mathbb{R}^3 \times S^1/\Gamma$, where $\Gamma$ is a discrete subgroup of $SU(2)$. After sending the radius of $S^1$ to infinity, one gets the Asymptotically Locally Euclidean (ALE) metrics that asymptotically approach $R^4/\Gamma$. Kronheimer [13] found their A–D–E classification into two infinite ($A_k$ and $D_k$) and three exceptional $E_{6,7,8}$ cases, according to the intersection matrix of their two-cycles. Since the ALE spaces are naturally related to the enhanced symmetries, most notably, conformal invariance, it is natural to approach an ALF moduli space metric from its ALE limit, being guided by hyper-Kähler geometry and conformal invariance on the top.
of it. In the context of N=2 NLSM in superspace, the ALE potentials can be constructed by ‘switching on’ vacuum expectation values of the hypermultiplet scalars, whereas the associated ALF potentials are obtained by non-conformal deformations of them. To solve the hyper-Kähler constraints, we use an unconstrained hyper-Kähler potential in HSS. The N=2 superconformal symmetry implies extra constraints on the special hyper-Kähler potentials, which can be easily solved in HSS too.

Our paper is organized as follows. In sect. 2 we review some basic properties of the special hyper-Kähler geometry and its relation to N=2 superconformal symmetry. In sect. 3 we introduce rigid N=2 superconformal transformations in superspace. The N=2 superconformal rules for various types of the N=2 hypermultiplet superfields are discussed in sect. 4. In sect. 5 a simple general solution to the special hyper-Kähler geometry in N=2 HSS is given. In sect. 6 we turn to a construction of the improved (N=2 superconformal) actions of the O(2p) projective multiplets in HSS, and then relate them to the (ALE and ALF) A_k and D_k series of gravitational instantons. Sect. 7 is devoted to our conclusion. All efforts were made to make our presentation as simple as possible.

2 Special hyper-Kähler NLSM geometry

Let \((P_\mu, M_{\lambda\rho}, D, K_\nu)\) be the generators of the standard conformal extension of the Poincaré algebra in 3+1 spacetime dimensions, \(\mu, \nu, \ldots, = 0, 1, 2, 3\), where \(P_\mu\) stand for translations, \(M_{\lambda\rho}\) for Lorentz rotations, \(D\) for dilatations and \(K_\nu\) for special conformal transformations. The commutation relations of the conformal algebra are given by a contracted \(so(4,2)\) algebra. We use middle Greek letters to denote spacetime vector indices, whereas early Greek letters are reserved for the 2-component spinor indices, \(\alpha, \beta, \ldots = 1, 2\). A vector index \((\mu)\) is equivalent to a bi-spinor index \((\alpha \dot{\alpha})\).

The N=2 superconformal algebra extends the conformal algebra to a contracted \(su(2,2|2)\) superalgebra. The new generators are given by bosonic charges of the \(SU(2)_{\text{conf}} \times U(1)_{\text{ch}}\) internal symmetry, eight fermionic supersymmetry charges, \(Q^i_\alpha\) and \(\bar{Q}^i_{\dot{\alpha}}\), and eight fermionic special supersymmetry charges, \(S^i_\alpha\) and \(\bar{S}^i_{\dot{\alpha}}\), where \(i = 1, 2\).

In the context of N=2 supersymmetric NLSM in 3+1 dimensions, N=2 supersymmetry amounts to the hyper-Kähler NLSM target space \(\mathcal{M}\) of real dimension \(4k\), \(k = 1, 2, \ldots\), whose holonomy group is in \(Sp(\mathcal{M})\). Given the full N=2 superconformal invariance, its internal \(su(2)_{\text{conf}}\) part implies an \(su(2)\) isometry of the hyper-Kähler
NLSM metric $g_{mn}$, i.e. the existence of three Killing vectors $K_{(A)}^m$ obeying Killing equations,

$$K_{(A)}^{(m;n)} = 0 , \quad m = 1, 2, \ldots, 4k , \quad A = 1, 2, 3 ,$$

(2.1)

and forming an $su(2)$ algebra. This non-abelian isometry is necessarily rotational, i.e. it rotates complex structures in the hyper-Kähler NLSM target space $\mathcal{M}$ \cite{7}. The dilatational invariance of a Riemannian manifold $\mathcal{M}$ is equivalent to the existence of another (Eulerian) vector $X^m$ satisfying an equation \cite{3}

$$X^{m;n} = g_{mn} .$$

(2.2)

Its geometrical significance was clarified in ref. \cite{16}, where eq. (2.2) was shown to be equivalent to the following form of the metric:

$$g_{mn} dx^m dx^n = dr^2 + r^2 h_{ab} dx^a dx^b ,$$

(2.3)

where

$$x^n = (r, x^a) , \quad a, b, c = 1, 2, \ldots, 4k - 1 , \quad \text{and} \quad h_{ab} = h_{ab}(x^c) .$$

(2.4)

Eq. (2.3) means that $\mathcal{M}$ can be considered as a cone $C(B)$ over a base manifold $B$ of dimension $4k - 1$ \cite{16}. The vector $X^m$ in the coordinates (2.4) reads

$$X = r \frac{\partial}{\partial r} ,$$

(2.5)

so that it is associated with the dilatations $(r, x^a) \rightarrow (\lambda r, x^a)$ indeed.

Eq. (2.2) is obviously equivalent to the conformal Killing equation,

$$\mathcal{L}_X g_{mn} = X_{m;n} + X_{n;m} = 2g_{mn} ,$$

(2.6)

together with a condition $X_{[m;n]} = 0$ or, equivalently,

$$X_m = \partial_m f .$$

(2.7)

Eqs. (2.6) and (2.7) imply that the metric $g_{mn}$ admits a potential $f(x^n)$,

$$g_{mn} = \nabla_m \nabla_n f \quad \text{or} \quad g_{mn} X^m X^n = 2f .$$

(2.8)

In the context of hyper-Kähler geometry, the potential $f$ also generates the complex structures by differentiation, so that the function $f$ is sometimes called the hyper-Kähler potential \cite{7,17}. It is worth noticing here that the potential $f$ is a constrained function in any geometry (e.g., $\nabla_m \nabla_n \nabla_p f = 0$), whereas we are going to introduce the hyper-Kähler (pre-)potential as an unconstrained function (sect. 5).
Though the Euler vector $X$ is not a Killing vector, it is easy to verify that it implies the existence of a Killing vector $Y$ in the presence of a covariantly constant complex structure $I$ on $\mathcal{M}$ \[7, 16\],

$$Y^n = I^n_m X^m, \quad \mathcal{L}_X I = 0, \quad [X, Y] = 0.$$

The second equation means that the vector $X$ is holomorphic, i.e. it preserves the complex structure. The corresponding base manifold $B$ then carries the so-called Sasakian structure to be obtained by projection of the $(I, X, Y)$ structure of $\mathcal{M}$ on $B$ \[18]. Since in our case $\mathcal{M}$ is a (special) hyper-Kähler manifold, the base manifold $B$ should admit a 3-Sasakian structure because of the existence of three independent and covariantly constant complex structures on $\mathcal{M}$,

$$Y^n_{(A)} = I^n_{(A)m} X^m, \quad [X, Y_A] = 0.$$

One easily finds that the vector $X$ is tri-holomorphic, and \[16\]

$$\mathcal{L}_{Y_A} I_B = -2\varepsilon_{ABC} I_C, \quad [Y_A, Y_B] = -2\varepsilon_{ABC} Y_C.$$

This means that the Killing vectors $Y_A$ are rotational and form an $su(2)$ algebra indeed \[7\]. The complex structures are invariant under dilatations.

The base manifold $B$ associated with a special hyper-Kähler manifold $\mathcal{M}$ is called 3-Sasakian (see ref. \[19\] for a recent mathematical account of the 3-Sasakian manifolds). A 3-Sasakian manifold of real dimension $(4k - 1)$ is an Einstein space,

$$R_{ab} = (4k - 1) h_{ab},$$

while it takes the form of an $Sp(1)$ fibration over a quaternionic space \[19\]. In ref. \[7\] the special hyper-Kähler manifolds (of real dimension $4k$) were described as the local products of flat 4-dimensional space with a $(4k - 4)$-dimensional quaternionic manifold, i.e. as the manifolds of $Sp(k - 1)$ holonomy. The special hyper-Kähler manifolds can be equally defined as cones over 3-Sasakian manifolds \[16\]. Some applications of the 3-Sasakian manifolds in M-theory were discussed in refs. \[17, 20\].

Our purpose is to solve the constraints implied by the special hyper-Kähler geometry, in terms of an unconstrained special hyper-Kähler (pre-)potential. By using the established relation between the special hyper-Kähler geometry and the $N=2$ superconformal symmetry, the solution amounts to a formulation of the most general $N=2$ superconformally invariant NLSM. To do this in superspace, we need a HSS realization of $N=2$ superconformal transformations.
3 N=2 superconformal transformations in HSS

The N=2 superconformal transformations in the ordinary N-extended superspaces are known for a long time [14]. It is, therefore, straightforward to rewrite them to the case of N=2 supersymmetry in HSS [15]. We follow ref. [15] in this section.

In the HSS approach the standard N=2 superspace coordinates $Z = (x^\mu, \theta_\alpha^i, \bar{\theta}_{\bar{\alpha}}^i)$ are extended by bosonic harmonics (or twistors) $u^{\pm i}, i = 1, 2$, belonging to the group $SU(2)$ and satisfying the unimodularity condition

$$u^{+i}_i u_i^{-} = 1, \quad u^{+i} i = u_i^{-} . \quad (3.1)$$

The original motivation for an introduction of harmonics [6] was the desire to make manifest the hidden analyticity structure of the N=2 superspace constraints defining both N=2 vector multiplets and FS hypermultiplets, and find their manifestly N=2 supersymmetric solutions in terms of unconstrained N=2 superfields.

Instead of using an explicit parametrization of the sphere $S^2$, in HSS one deals with the equivariant functions of harmonics, which have definite $U(1)$ charges defined by

$$U(u^{\pm}) = \pm 1. \quad (3.2)$$

are similar to the (Berezin) integration rules in superspace. In particular, any harmonic integral over a $U(1)$-charged quantity vanishes. The harmonic covariant derivatives, preserving the defining equations (3.1) in the original (central) basis, are given by

$$D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} \equiv \partial^{++} , \quad D^{--} = u^{-i} \frac{\partial}{\partial u^{+i}} , \quad D^0 = u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}} . \quad (3.3)$$

They satisfy an $su(2)$ algebra and commute with the standard (flat) N=2 superspace covariant derivatives $D^i_\alpha$ and $\bar{D}^i_{\bar{\alpha}}$. The operator $D^0$ measures $U(1)$ charges.

The key feature of HSS is the existence of an analytic subspace parametrized by

$$\langle \zeta; u \rangle = \left \{ x^{\alpha \bar{\alpha}}_{\text{analytic}} = x^{\alpha \bar{\alpha}} - 4i \theta^{\alpha \bar{\alpha}} u^i_i (u^{-j} j) , \quad \theta^+_\alpha = \theta^i_i u^i_i , \quad \theta^-_{\bar{\alpha}} = \bar{\theta}^i_i u^i_{\bar{\alpha}} ; \quad u^\pm_i \right \} , \quad (3.4)$$

which is invariant under N=2 supersymmetry [3]:

$$\delta x^{\alpha \bar{\alpha}}_{\text{analytic}} = -4i \left ( \varepsilon^{\alpha \bar{\alpha}} \bar{\theta}^\dagger + \theta^\dagger \varepsilon^{\alpha \bar{\alpha}} \right ) u_i^- \equiv -4i \left ( \varepsilon^{\alpha \bar{\alpha}} \bar{\theta}^\dagger + \theta^\dagger \varepsilon^{\alpha \bar{\alpha}} \right ) , \quad \delta \theta^+_\alpha = \varepsilon^i_i u^i_i \equiv \varepsilon^+_\alpha , \quad \delta \bar{\theta}^\dagger_{\bar{\alpha}} = \varepsilon^i_i u^i_{\bar{\alpha}} \equiv \varepsilon^\dagger_{\bar{\alpha}} , \quad \delta u^\pm_i = 0 , \quad (3.5)$$
where only $\theta^+_{\alpha,\dot{\alpha}}$ are present, not $\theta^-_{\alpha,\dot{\alpha}}$.

The usual complex conjugation does not preserve analyticity. However, it does, after being combined with another (star) conjugation that only acts on the $U(1)$ indices as $(u^+_i)^* = u^-_i$ and $(u^-_i)^* = -u^+_i$. One has $u^{\pm i} = -u^{\pm i}$ and $\bar{u}^{\pm i} = u^{\pm i}$. 

Analytic off-shell N=2 superfields $\phi^{(q)}(\zeta(Z,u),u)$ of any positive (integral) $U(1)$ charge $q$ in HSS are defined by (cf. N=1 chiral superfields)

$$D^+_\alpha \phi^{(q)} = \bar{D}^+_{\dot{\alpha}} \phi^{(q)} = 0\,,$$

where $D^+_\alpha = D^+_\alpha u^+_i$ and $\bar{D}^+_{\dot{\alpha}} = \bar{D}^+_{\dot{\alpha}} u^+_i$. (3.6)

The analytic measure reads $d\zeta^{(-4)}du \equiv d^4 x^\mu_{\text{analytic}} d^2 \theta^+ d^2 \bar{\theta}^+ du$, and it has the $U(1)$ charge $(-4)$. The harmonic derivative $D^{++}$ in the analytic subspace (3.4) takes the form

$$D^{++}_{\text{analytic}} = \partial^{++} - 4i \theta^\alpha \bar{\theta}^{\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}},$$

it preserves analyticity, and it allows one to integrate by parts. Similarly, one easily finds that

$$D^0_{\text{analytic}} = u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}} + \theta^\alpha \frac{\partial}{\partial \theta^{\alpha}} + \bar{\theta}^{\dot{\alpha}} \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}. \quad (3.8)$$

In what follows we omit the explicit references to the analytic subspace, in order to simplify our notation.

The use of harmonics allows one to make manifest (i.e. linearly realised) the $SU(2)_R$ symmetry of N=2 supersymmetry algebra, in addition to manifest N=2 supersymmetry (see ref. [21] for more details). The relation to PSS, where the $SU(2)_R$ rotations take the form of projective transformations, becomes clear in a particular parametrization

$$u^+_i = (1, \xi), \quad u^-_i = \frac{-1}{1 + |\xi|^2} \begin{pmatrix} 1 \\ \xi \end{pmatrix}, \quad (3.9)$$

where $\xi$ is the projective (complex) $CP^1$ coordinate.

The translational and Lorentz transformation properties of the HSS coordinates are obvious, and we do not write them down. The transformation rules with respect to dilatations with the infinitesimal parameter $\rho$ are also rather evident, being dictated by conformal weights $w$,

$$w[x] = 1, \quad w[\theta] = w[\bar{\theta}] = \frac{1}{2}, \quad w[u] = 0. \quad (3.10)$$

The non-trivial part of N=2 superconformal transformations is given by the $U(2)_{\text{conf}}$ rotations with the parameters $l_{ij}$, special conformal transformations with the parameters $k_{\alpha\dot{\alpha}}$, and N=2 special supersymmetry with the parameters $\eta^i_{\alpha}$ and $\bar{\eta}^i_{\dot{\alpha}}$. 

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The N=2 superconformal extension of the spacetime conformal transformations,
\[ \delta x^{\dot{a}\dot{a}} = \rho x^{\dot{a}\dot{a}} + k_{\beta \beta} x^{\dot{a} \dot{a}} x^{\beta \beta} , \]  
(3.11)
is dictated by the requirement of preserving the unimodularity and analyticity conditions in eqs. (3.1) and (3.6), respectively. As regards the non-trivial part of the N=2 superconformal transformation laws, one finds \[ \delta x^{\alpha \alpha} = -4i\lambda^ij u_{\dot{i}}^- u_j^- \theta^{\alpha+} \bar{\theta}^{\dot{\alpha}+} + k_{\beta \beta} x^{\alpha \alpha} x^{\beta \beta} + 4i \left( x^{\alpha \beta} \theta^{\alpha+} \bar{\eta}^-_{\beta} - x^{\dot{\alpha} \dot{\beta}} \theta^{\dot{\alpha}+} \bar{\eta}^+_{\dot{\beta}} \right) , \]  
(3.12)
\[ \delta \theta^{\alpha+} = \lambda^i u_{\dot{i}}^+ u_j^+ \theta^{\alpha+} + k_{\beta \beta} x^{\alpha \beta} \bar{\theta}^{\dot{\alpha}+} - 2i(\theta^{\beta+} \bar{\eta}^-_{\beta} + x^{\alpha \beta} \bar{\eta}^+_{\beta}) , \]  
(3.13)
\[ \delta \bar{\theta}^{\dot{\alpha}+} = -\left( \bar{\delta} \theta^{\alpha+} \right) , \]  
(3.14)
\[ \delta u_{\dot{i}}^+ = \left[ \lambda^{kj} u_{\dot{k}}^+ u_j^+ + 4i k_{\alpha \alpha} \theta^{\alpha+} \bar{\theta}^{\dot{\alpha}+} + 4i \left( \theta^{\alpha+} \bar{\eta}^+_{\alpha} + \bar{\eta}^-_{\alpha} \theta^{\alpha+} \right) \right] u_{\dot{i}}^- , \]  
(3.15)
\[ \delta u_{\dot{i}}^- = 0 . \]  
(3.16)
Since the building blocks of any invariant action in HSS are given by the measure, analytic superfields and HSS covariant derivatives, only their transformation properties under rigid N=2 superconformal transformations are going to be relevant for our purposes. It follows from eq. (3.12) that \[ \text{Ber} \frac{\partial (\zeta', u')}{\partial (\zeta, u)} = 1 - 2\Lambda , \]  
(3.17)
where the HSS superfield parameter
\[ \Lambda = -\left( \rho + k_{\alpha \alpha} x^{\alpha \alpha} \right) + \left( \lambda^i + 4i \theta^{\alpha+} \bar{\eta}^-_{\alpha} + 4i \bar{\eta}^+_{\alpha} \bar{\theta}^{\dot{\alpha}+} \right) u_{\dot{i}}^+ u_{\dot{i}}^- \]  
(3.18)
has been introduced. Similarly, one easily finds that
\[ (D^{++})' = D^{++} - (D^{++}\Lambda)D^0 \text{ and } (D^0)' = D^0 . \]  
(3.19)
The ‘truly’ N=2 superconformal (infinitesimal) component parameters can thus be nicely encoded into the single scalar superfield \[ \Lambda \]  
subject to the HSS constraint
\[ (D^{++})^2 \Lambda = 0 , \]  
(3.20)
and the reality condition
\[ \star(\Lambda^{++}) = \Lambda^{++} , \text{ where } \Lambda^{++} \equiv D^{++}\Lambda . \]  
(3.21)
The transformations rules of the harmonics in eq. (3.12),
\[ \delta u_{\dot{i}}^+ = \Lambda^{++} u_{\dot{i}}^- , \ \delta u_{\dot{i}}^- = 0 , \]  
(3.22)together with eqs. (3.13), (3.15), (3.16) and (3.17) represent the very simple and convenient description of rigid N=2 conformal supersymmetry in HSS.
4 Superconformal hypermultiplet superfields

The $O(2p)$ projective (or generalized tensor) multiplets in the standard N=2 super-
space are described by the N=2 superfields $L^{i_1 \cdots i_{2p}}$ that are totally symmetric with
respect to their $SU(2)_R$ indices, being subject to the constraints [5]

$$D^{(k} L^{i_1 \cdots i_{2p})} = D_{\alpha}^{(k} L^{i_1 \cdots i_{2p})} = 0, \quad (4.1)$$

and the reality condition

$$\mathcal{T}_{i_1 \cdots i_{2p}} \equiv (L^{i_1 \cdots i_{2p}})^* = \varepsilon_{i_1 j_1} \cdots \varepsilon_{i_{2p} j_{2p}} L^{j_1 \cdots j_{2p}}. \quad (4.2)$$

The N=2 projective multiplets are all irreducible off-shell representations of N=2 supersymmetry with superspin $Y = 0$ and superisospin $I = p - 1$ as long as $p \geq 1$. The list of their off-shell, $8(2p-1)$ bosonic and $8(2p-1)$ fermionic, field components is most conveniently represented in terms of the $SU(2)$ Young tableaux [5]:

where the boxes with dots and stars denote the N=2 superspace covariant derivatives, $D^i_{\alpha}$ and $\bar{D}^{i \dot{\alpha}}_{\alpha}$, respectively. It follows from matching the numbers of the bosonic and
fermionic degrees of freedom in eq. (4.3) that the vector $V_{\alpha\alpha}$ in the N=2 tensor multiplet ($p = 1$) is conserved, $\partial^{\alpha} V_{\alpha\alpha} = 0$. The vector $V_{\alpha\alpha}^{i_1\cdots i_{2p}}$ of any projective N=2 multiplet with $p > 1$ is an unconstrained (general) vector field.

The PSS naturally comes out of the efforts to construct an N=2 supersymmetric self-interaction of the projective multiplets in N=2 superspace [4, 5]. Let’s introduce a function $G(L_A; \xi, \eta)$, whose arguments are given by some number ($A = 1, 2, \ldots, k$) of the $O(2p)$ projective superfields (of any type) and two extra complex coordinates, $\xi$ and $\eta$. Let’s also impose four linear differential equations on this function,

$\nabla_{\alpha} G \equiv (D_{\alpha}^{1} + \xi D_{\alpha}^{2}) G = 0 \ , \quad \Delta_{\alpha} G \equiv (D_{\alpha}^{1} \cdot + \eta D_{\alpha}^{2}) G = 0 . \quad (4.4)$

It follows from the defining constraints (4.1) that a general solution to eq. (4.4) reads

$G = G(Q_A(\xi); \xi) , \quad \eta = \xi , \quad Q_{(2p)}(\xi) \equiv \xi_{i_1} \cdots \xi_{i_{2p}} L^{i_1\cdots i_{2p}} , \quad \xi_i \equiv (1, \xi) , \quad (4.5)$

in terms of an arbitrary function $G(Q_A; \xi)$. Since the function $G$ does not depend upon a half of the Grassmann coordinates of N=2 superspace by construction, its integration over the rest of the coordinates is invariant under N=2 supersymmetry. This leads to the following universal N=2 supersymmetric action for the projective multiplets in PSS [4, 5]:

$S[L_A] = \int d^4 x \frac{1}{2\pi i} \oint_C d\xi (1 + \xi^2)^{-4} \tilde{\nabla}^2 \tilde{\Delta}^2 G(Q_A, \xi) + \text{h.c.} , \quad (4.6)$

where the new derivatives,

$\tilde{\nabla}_{\alpha} \equiv \xi D_{\alpha}^{1} - D_{\alpha}^{2} , \quad \tilde{\Delta}_{\alpha} \equiv \xi D_{\alpha}^{1} \cdot - D_{\alpha}^{2} \cdot , \quad (4.7)$

in the directions orthogonal to the ‘vanishing’ directions of eq. (4.4) have been introduced. The integration contour $C$ in the complex $\xi$-plane is supposed to make the action (4.6) non-trivial. The factor $(1 + \xi^2)^{-4}$ in the action (4.6) was introduced to simplify the transformation properties of the integrand under $SU(2)_R$ [21]. The projective variable $\xi \in CP^1$ has the rational transformation law,

$\xi' = \frac{\bar{a} \xi - \bar{b}}{a + b \xi} , \quad (4.8)$

whose complex parameters $(a, b)$ are constrained by the condition $|a|^2 + |b|^2 = 1$. In general, the action (4.6) is neither conformally nor $SU(2)_R$ invariant.

After being expanded in components, the action (4.6) depends upon the bosonic Lagrange multipliers given by the vector fields ($V$) and the scalars ($C$), that can be removed via their algebraic equations of motion or by a duality transformation.
This procedure is known as the generalized Legendre transform \[4\] that leads to a hyper-Kähler metric in the bosonic NLSM part of the theory (4.6).

The constraints (4.1) and (4.2) take the simple form in HSS,

\[ D^{++}L^{++}=0, \quad \overline{L}^{++}\star L^{++}=L^{++}, \tag{4.9} \]

where (cf. eq. (4.5))

\[ L^{++}=u^+_i\cdots u^+_p L^{i_1\cdots i_2p}. \tag{4.10} \]

Requiring the invariance of the constraints (4.9) under the N=2 superconformal transformations (3.15) and (3.18) gives rise to the covariant transformation laws for the projective superfields,

\[ \delta L^{++}=w\Lambda L^{++}, \quad \text{with} \quad w=2p. \tag{4.11} \]

The choice of \(2p=1\) in eq. (4.1) defines the most basic FS hypermultiplet (with vanishing central charge), whose physical components comprise only scalars and spinors. It is not difficult to verify that the constraints (4.1) in this case imply free equations of motion, \(\Box L^i=0\). An off-shell FS hypermultiplet is naturally described in HSS by an unconstrained complex analytic superfield \(q^+\) of \(U(1)\) charge (+1) \[6\]. Its free HSS action reads \[6\]

\[ S[q]_{\text{free}}=-\int d\zeta (-4)du^* plus_t q^+D^{++}q^+. \tag{4.12} \]

This action is invariant under the N=2 superconformal transformations provided that \(D^{++}q^+\) transforms covariantly like \(q^+\) itself, which implies that \(q^+\) is of conformal weight one \[13\],

\[ \delta q^+=\Lambda q^+. \tag{4.13} \]

The free HSS equations of motion, \(D^{++}q^+=0\), imply \(q^+=L^i(Z)u^+_i\) together with the on-shell FS hypermultiplet supperspace constraints, \(D^\alpha (i L^j)=\bar{D}^\alpha (i L^j)=0\).

It is worth noticing that the \(SU(2)_{\text{conf.}}\) transformations, which are part of the N=2 superconformal symmetry, are different from the \(SU(2)_{\text{R}}\) transformations, with the latter being defined by their natural action on the Latin indices, \(i,j=1,2\), as \(\delta_{SU(2)_R}u^\pm_i=\lambda^i_j u^\pm_j\), etc. For example, as regards the free off-shell theory (4.12), one finds \[13\]

\[ \delta_{SU(2)_R}q^+=\delta_{SU(2)_{\text{conf.}}}q^++\lambda^{ij}_i u^-_i u^-_j D^{++}q^+, \tag{4.14} \]

so that the \(SU(2)_{\text{conf.}}\) and \(SU(2)_{\text{R}}\) transformations coincide only if \(D^{++}q^+=0\), i.e. on-shell.
5 General N=2 superconformal NLSM actions

We are now in a position to discuss the N=2 superconformal hypermultiplet actions in HSS. We use the pseudo-real $Sp(1)$ notation for a single FS hypermultiplet superfield,

$$q^+_a = \left( q^+_1, q_2^+ \right), \quad a = 1, 2, \quad q^+_a = \epsilon^{ab} q^+_b, \quad (5.1)$$

and further generalize it to the case of several FS hypermultiplets, $q^+_a \to q^{+A}$ and $q^+_A = \Omega_{AB} q^+_B$, with a constant (antisymmetric) $Sp(k)$-invariant metric $\Omega_{AB}$, $A, B = 1, \ldots, 2k$.

First, we recall that the most general N=2 supersymmetric NLSM can be most naturally formulated in HSS, in terms of the FS hypermultiplet superfields, as

$$S_{NLSM}[q] = -\frac{1}{\kappa^2} \int d\zeta (-4) du \left[ \frac{1}{2} q^+_A D^{++} q^{+A} + \mathcal{K}^{(+4)}(q^{+A}, u^{\pm}) \right], \quad (5.2)$$

where the real analytic function $\mathcal{K}^{(+4)} = \mathcal{K}^{(+4)}$ of $U(1)$ charge $(+4)$ is called a hyper-Kähler (pre-)potential. By manifest N=2 supersymmetry of the NLSM action (5.2), the NLSM metric must be hyper-Kähler for any choice of $\mathcal{K}^{(+4)}$. Unfortunately, an explicit general relation between a hyper-Kähler potential and the corresponding hyper-Kähler metric is not available (see, however, refs. [22, 23] for the explicit hyper-Kähler potentials of the (ALE) multi-Eguchi-Hanson, (ALF) multi-Taub-NUT and Atiyah-Hitchin metrics, and their derivation from the NLSM (5.2) in terms of FS hypermultiplet superfields, in HSS).

Eq. (5.2) formally solves the hyper-Kähler constraints on the NLSM metric in terms of an arbitrary function $\mathcal{K}^{(+4)}$. It is, therefore, quite natural to impose extra N=2 superconformal invariance on this function, in order to determine a general solution to the special hyper-Kähler geometry, since the free part (4.12) of the action (5.2) is automatically N=2 superconformally invariant. In general, the invariance of the HSS action (5.1) merely implies the invariance of the HSS Lagrangian up to a total derivative, because of the identity

$$\int d\zeta (-4) du D^{++} X^{++} = 0. \quad (5.3)$$

However, in the case of unconstrained FS analytic superfields $q^+$, the hyper-Kähler potential should be invariant too. Eqs. (3.13), (3.18) and (4.13) now imply

$$\Lambda \left[ \frac{\partial \mathcal{K}^{(+4)}}{\partial q^{+A}} q^{+A} - 2 \mathcal{K}^{(+4)} \right] + \Lambda^{++} \frac{\partial \mathcal{K}^{(+4)}}{\partial u^{+}_i} u^{-}_i = 0. \quad (5.4)$$

\[3\text{We now choose our HSS superfields to be dimensionless, by the use of the dimensionful coupling constant $\kappa$ in front of their actions.} \]
Equation (5.4) is equivalent to two constraints,

$$\frac{\partial K^{(+4)}}{\partial q^{A+}} q^{A+} = 2K^{(+4)} \quad \text{and} \quad \frac{\partial K^{(+4)}}{\partial u^-} = 0 \ .$$

(5.5)

This means that the special hyper-Kähler potential of the N=2 superconformally invariant NLSM, in terms of the analytic FS superfields $q^{A+}$ in HSS, is given by a homogeneous (of degree two) function $K^{(+4)}(q^{A+}, u^-)$ of $q^{A+}$. There is no restriction on the dependence of $K^{(+4)}$ upon $u^-$, while it should be independent upon $u^+_i$. This represents one of our main new results in this paper.

Our simple description of the N=2 superconformal hypermultiplet actions in terms of the off-shell N=2 superfields is to be compared to the well-known description of the (abelian) N=2 vector multiplet actions in the standard N=2 (chiral) superspace [24],

$$S[W] = \int d^4xd^4\theta \mathcal{F}(W_A) + \text{h.c.} \ ,$$

(5.6)

in terms of the N=2 restricted chiral superfields $W_A$ representing the N=2 abelian gauge field strengths. The N=2 superconformal invariance of the action (5.6) implies that $\mathcal{F}(W_A)$ is a homogeneous (of degree two) function of $W_A$ [25]. The special Kähler geometry, associated with the scalar NLSM part of the action (5.6), and the special hyper-Kähler geometry (sect. 2) are, however, very different at the level of components, as well as in the geometrical terms.

A non-trivial special hyper-Kähler potential exists even in the case of a single FS hypermultiplet, e.g.,

$$K^{(+4)}(q^+, u^-) = C \left[ \frac{q^+}{q^++u^-} \right]^2 ,$$

(5.7)

where $C$ is a real constant. Eq. (5.7) is not invariant with respect to $SU(2)_R$ because of its explicit dependence upon harmonics, whereas it is invariant under the $SU(2)_{\text{conf.}}$ part of the N=2 superconformal symmetry by construction. The corresponding special hyper-Kähler metric interpolates between the Eguchi-Hanson (ALE) metric described by a hyper-Kähler potential

$$K^{(+4)}_{\text{EH}} = \left[ \frac{\xi^{++}}{q^+u^- - q^+u^-} \right]^2 ,$$

(5.8)

in the limit $(q^+ + q^+) \rightarrow \xi^{++} = \xi_{ij} u^i u^j$ with constant $\xi_{ij}$, and the Taub-NUT (ALF) metric described by a hyper-Kähler potential (in the limit $q^+u^- - q^+u^- = \text{const.}$)

$$K^{(+4)}_{\text{Taub–NUT}} = \text{Const.} \left( q^+ \right)^2 .$$

(5.9)
6 N=2 superconformal projective multiplets

We now turn to a HSS construction of the N=2 superconformal (improved) actions for the $O(2p)$ projective multiplets introduced in sect. 4. Unlike the FS hypermultiplets described by unconstrained (analytic) superfields in HSS, the projective multiplets are described by the constrained (off-shell) analytic superfields that give rise to the finite numbers of the auxiliary fields. It is, therefore, straightforward to deduce the component hyper-Kähler metrics out of their HSS actions. The N=2 supersymmetric selfinteraction (4.6) of the projective multiplets ($p < \infty$) in PSS is known to be merely a subclass of the most general N=2 NLSM described by eq. (5.2) in HSS [26], while the projective superfields are of higher conformal weight than FS superfields — see eqs. (4.11) and (4.13). We should, therefore, expect severe constraints on the N=2 superconformal actions in terms of the projective multiplets. This is known to be the case for the standard N=2 tensor multiplet ($p = 1$) indeed, whose N=2 improved action was constructed many years ago, first in components [27], then in N=1 superspace [28] and N=2 PSS [4], and finally in HSS [29]. We begin with a simple derivation of the improved N=2 tensor multiplet action in HSS, and then discuss its N=2 superconformal generalizations and non-conformal deformations.

On dimensional reasons, the most general N=2 supersymmetric action of a single N=2 tensor multiplet superfield $L^{++}$, subject to the constraints (4.9), reads in HSS as [26]

$$S[L] = \frac{1}{\kappa^2} \int d\zeta (-4) du L^{(++)}(L^{++}; u^\pm) .$$

(6.1)

A free bilinear action in $L^{++}$ is obviously not N=2 superconformally invariant, so that it has to be improved in some non-trivial way. A power series in terms of $L^{++}$, as the naive Ansatz for the HSS Lagrangian $L^{(++)}$, also does not work here, because of the need to balance the conformal weights defined by eqs. (3.13) and (4.11). A resolution of this problem was suggested in ref. [27], where it was noticed that the improved Lagrangian must be topologically non-trivial, i.e. it should contain a Dirac-like string of singularities parametrized by an arbitrary $SU(2)$ triplet of constants $c^{ij}$,

$$c^{ij} = c^{ji} , \quad (c^{ij}) = \varepsilon_{ik} \varepsilon_{jl} c^{kl} , \quad c^2 = \frac{1}{2} c_{ij} c^{ij} .$$

(6.2)

This essentially amounts to extracting a ‘fake’ vacuum expectation value out of the N=2 tensor superfield $L^{ij}$,

$$L^{ij} = c^{ij} + l^{ij} , \quad \text{or, equivalently,} \quad L^{++} = c^{++} + l^{++} .$$

(6.3)

The N=2 superconformal invariance of the action makes it to be independent upon the constants $c^{ij}$ because of the $SU(2)_{\text{conf}}$ symmetry. Since the normalization of $c^2$
can always be changed by dilatations, we temporarily set $c^2 = 1$ for simplicity of our calculations in what follows. The definitions

$$c^{++} = c^{ij}u^+_iu^+_j, \quad c^{+-} = c^{ij}u^+_iu^-_j, \quad c^{-+} = c^{ij}u^-_iu^-_j,$$

(6.4)

imply the identities [29]

$$D^{++}c^{-+} = 2c^{+-}, \quad D^{++}c^{+-} = c^{++}, \quad D^{++}c^{++} = 0,$$

(6.5)

and

$$c^{++}c^{-+} - (c^{+-})^2 = c^2 = 1,$$

(6.6)

with the latter being the corollary of the completeness relation for harmonics,

$$u^+_iu^-_j - u^-_ju^+_i = \delta^i_j.$$

(6.7)

Equations (6.2), (6.3) and (6.5) imply that $l^{++}$ also satisfies the initial off-shell constraints (4.9),

$$D^{++}l^{++} = 0, \quad \frac{l^*}{l^{++}} = l^{++}.$$

(6.8)

The natural Ansatz for the improved N=2 tensor multiplet action in HSS is given by [29]

$$S[L]_{impr.} = \frac{1}{\kappa^2} \int d\zeta (-4) du (l^{++})^2 f(y), \quad y \equiv l^{++}c^{-+},$$

(6.9)

where the function $f(y)$ is at our disposal. Since the action in question is supposed to improve the naive (quadratic) one, the function $f$ should obey the boundary condition

$$f(0) = 1.$$  

(6.10)

A more general HSS Ansatz for the HSS Lagrangian may include a term $(c^{++})^2 g(y, c)$ in eq. (6.9), with yet another function $g(y, c)$ to be discussed below.

The identities (6.5) and (6.6) together with the constraint (6.8) further imply that

$$2y(D^{++})^2 y - (D^{++}y)^2 = 4(l^{++})^2 \quad \text{and} \quad (D^{++})^3 y = 0.$$

(6.11)

The N=2 superconformal transformation laws of the new variables $l^{++}$ and $y$ follow from their definitions in eqs. (6.3) and (6.9), by the use of eqs. (3.18) and (4.11) with $p = 1$ and $w = 2$. We find

$$\delta l^{++} = 2\Lambda l^{++} + 2D^{++} \left( \Lambda c^{++} - \Lambda^{++}c^{-+} \right),$$

(6.12)
and
\[ \delta y = 2\Lambda y + 2 \left[ \Lambda c^{++} - \Lambda^{++}c^{--} \right] c^{--} \]
\[ = 2\Lambda y + 2D^{++} \left[ \Lambda c^{--} - \Lambda^{++}(c^{--})^2 \right] + 4c^{--} \left[ \Lambda^{++}c^{--} - \Lambda c^{++} \right] . \]  

(6.13)

Varying the action (6.9) by the use of eqs. (6.12) and (6.13), integrating by parts via eq. (5.3), and using the identities (6.11) yield
\[ \delta \int d\zeta (4) d\xi (t^{++})^2 f(y) = - \int d\zeta (4) d\xi \Lambda(D^{++}y)^2 \left[ y(y+1)f'' + \frac{1}{2}(7y+6)f' + \frac{3}{2}f \right] \]
\[ + \int d\zeta (4) d\xi \Lambda^{++}(D^{++}y) \left[ y^2 f'' + y(6-y)f' - yf \right] = 0 . \]  

(6.14)

Note that the second line of eq. (6.14) is also a total derivative in the case of a single N=2 tensor multiplet. The N=2 superconformal invariance of the action (6.9) thus amounts to the second-order ordinary differential equation on the function \( f(z) \),
\[ z(1-z)f'' + \frac{1}{2}(6-7z)f' - \frac{3}{2}f = 0 , \qquad z = -y . \]  

(6.15)

This is the very particular hyper-geometric equation, whose well-known general form depending upon three parameters \( \alpha, \beta, \gamma \) is given by
\[ z(1-z)F'' + [\gamma - (\alpha + \beta + 1)z] F' - \alpha \beta F = 0 . \]  

(6.16)

Hence, a regular solution to our problem (6.15) with the boundary condition (6.10) is given by the hyper-geometric function \( F(\alpha, \beta, \gamma; z) \) with \( \alpha = 1, \beta = \frac{3}{2} \) and \( \gamma = 3 \). It appears to be an elementary function
\[ f(z) = F(1, \frac{3}{2}, 3; z) = \left[ \frac{1 + \sqrt{1-z}}{2} \right]^{-2} , \]  

(6.17)
in full agreement with ref. [29], where the recursive methods were used. As was demonstrated in ref. [29], integration over the Grassmann and harmonic coordinates of HSS in the action defined by eqs. (6.9) and (6.17),
\[ S[L]_{\text{impr.}} = \frac{4}{\kappa^2} \int d\zeta (4) d\xi \left[ \frac{L^{++} - c^{++}}{1 + \sqrt{1 + (L^{++} - c^{++})c^{--}/c^2}} \right]^2 , \]  

(6.18)
results in the improved component action of ref. [27]. The equivalent PSS action (4.6) has a holomorphic potential [4]
\[ G(Q(\xi), \xi) = (Q(\xi) - c(\xi)) [\ln(Q(\xi) - c(\xi)) - 1] , \]  

(6.19)
where \( Q = \xi_i \xi_j L^{ij} \) and \( c(\xi) = \xi_i \xi_j e^{ij} \) with \( \xi_i = (1, \xi) \), while the contour \( C \) in the complex \( \xi \)-plane encircles the roots of a quadratic equation \( (Q - c)(\xi)) = 0 \). Like the
HSS action (6.9), the equivalent PSS action (4.6) with the potential (6.19) does not depend upon the constants $c^{ij}$.

It is not difficult to verify that another Ansatz for the N=2 tensor multiplet HSS Lagrangian of the form $(c^{++})^2 g(y, c^{+-})$ gives rise to another N=2 superconformal invariant,

$$\int d\zeta^{(-4)} du (c^{++})^2 L^{++} c^{-} .$$

(6.20)

However, it vanishes after integration over harmonics and the anticommuting N=2 superspace coordinates, because of the conservation law $\partial_{\alpha\dot{\alpha}} V^{\alpha\dot{\alpha}} = 0$ for the vector component of the N=2 tensor multiplet in eq. (4.3).

The N=2 superconformally invariant action (6.18) leads to a flat NLSM metric (in disguise) after the generalized Legendre transform [27]. The improved N=2 tensor multiplet action can, nevertheless, serve as the key building block for a construction of non-trivial four-dimensional hyper-Kähler metrics. For example, the $A_k$ series of the ALE gravitational instanton metrics arise when one sums the improved N=2 tensor multiplet Lagrangians with different moduli $c^{ij}_a$,

$$S[L]_{\text{ALE-} A_k} = \frac{4}{k^2} \int d\zeta^{(-4)} du \sum_{a=1}^{k+1} \left[ \frac{L^{++} - c^{++}_a}{1 + \sqrt{1 + (L^{++} - c^{++}_a)c^--_a/c^2}} \right]^2 .$$

(6.21)

The associated PSS potential reads

$$G(Q(\xi), \xi) = \sum_{a=1}^{k+1} (Q(\xi) - c_a(\xi)) \left[ \ln(Q(\xi) - c_a(\xi)) - 1 \right] ,$$

(6.22)

while its generalized Legendre transform is known to lead to the $A_k$ ALE metrics indeed [30]. Another simple non-conformal deformation is given by the naive (bilinear) action of the N=2 tensor multiplet,

$$S[L]_{\text{naive}} = m \int d\zeta^{(-4)} du (L^{++})^2 .$$

(6.23)

After being added to the action (6.21), it leads to the $A_k$ series of the ALF (multi-Taub-NUT) metrics with a Taub-NUT mass parameter $m$ (see e.g., ref. [21]). As is clear from eq. (6.22), the moduli of the $A_k$ metrics are naturally described by fixed real sections $c_a(\xi)$ of the $O(2)$ holomorphic bundle, while one of them can be arbitrarily chosen. In the context of M-theory/type-IIA superstring compactification (sect. 1), the $A_k$ moduli describe the D6-brane positions in the transverse space.

Having established the improved action of a single N=2 tensor multiplet ($p = 1$), it is natural to look for N=2 superconformal actions in terms of several N=2 tensor
multiplets or the \textit{higher} $O(2p)$ projective multiplets as well (a geometrical motivation of the latter is discussed at the end of this section).

A generalization of the \textit{Ansatz} (6.9) to the case of several N=2 tensor multiplets ($p = 1$) is given by

\begin{equation}
S[L_a] = \frac{1}{\kappa^2} \int \! d\zeta^{(-4)} du \sum_{a,b=1}^{q} l_a^{++} l_b^{++} f_{ab}(\{y\}) , \quad a = 1, 2, \ldots, q ,
\end{equation}

where $f_{ab}(\{y\})$ is the symmetric matrix of $q(q+1)/2$ functions depending upon $q$ variables, $\{y\} = (l_1^{++} c_--., l_q^{++} c--)$, with \textit{the same} $c--$. Requiring the action (6.24) be invariant under the N=2 superconformal transformations gives rise to a system of $q(q+1)/2$ second-order ordinary differential equations,

\begin{equation}
\sum_{a=1}^{q} \left[ (q + \sum_{b=1}^{q} y_b) y_a + 2(1 + y_a) \right] f_{a(c,d)} + \sum_{a=1}^{q} (1 - \frac{3}{2} y_a) f_{cd,a} - \frac{3}{2} f_{cd} = 0 ,
\end{equation}

and extra consistency condition on the vector

\begin{equation}
V_a \equiv \sum_{b,c=1}^{q} y_b (2 - y_c) f_{ab,c} - \sum_{b=1}^{q} f_{ab} y_b
\end{equation}

to be a total derivative, i.e. $\partial_a V_b - \partial_b V_a = 0$. This gives rise to the additional equations

\begin{equation}
\sum_{a,b=1}^{q} y_a (2 - y_b) f_{a[c,d]} - 2 \sum_{a=1}^{q} f_{a[c,d]} y_a = 0
\end{equation}

that make the full set of $q^2$ equations to be overdetermined. We are unaware of any non-trivial solutions to these equations, except of the one given by a non-interacting sum of the improved actions for each N=2 tensor multiplet.

We now turn to a single $O(4)$ projective multiplet $L^{+++} \equiv L^{(4)}$ satisfying the constraints (4.9), and construct its improved action in HSS. Instead of writing down a new \textit{Ansatz}, it is much simpler to use the already established improved action (6.18) for the $O(2)$ projective (tensor) multiplet, and then take into account the known transformation property (4.11) of $L^{(4)}$ with $p = 2$ and $w = 4$. The latter implies that $L^{(4)}$ transforms as $(L^{++})^2$ under the N=2 superconformal transformations. By using the obvious identities,

\begin{equation}
L^{++} c-- = \sqrt{(L^{++})^2(c--)^2}
\end{equation}

and

\begin{equation}
(L^{++} - c^{++})^2 = (L^{++})^2 - 4(D^{++})^2 \sqrt{(L^{++})^2(c--)^2 + (c^{++})^2},
\end{equation}

\footnote{We denote differentiations by commas, like in general relativity.}
we can simply substitute \((L^+)^2\) by \(L^{(+4)}\) in eq. (6.18). It yields the N=2 superconformally invariant (improved) action of an \(O(4)\) projective multiplet in the form
\[
S[L^{(+4)}]_{\text{impr.}} = \frac{4}{\kappa^2} \int d\zeta \zeta (-4) du \left[ 1 + \frac{1}{c^2} \sqrt{L^{(+4)}(c-)^2 - \frac{c^{++}c^{--}}{c^2}} \right]^2 \times \left\{ L^{(+4)} \left( 1 - \frac{8c^{++}c^{--}}{L^{(+4)}(c-)^2} \right) + (c^{++})^2 \right\}.
\]

This action is one of our main new results in this paper. The associated PSS potential is obtained from eq. (6.19) after a substitution \(Q^{(2)} \rightarrow \sqrt{Q^{(4)}}\), i.e.
\[
G(Q^{(4)}(\xi), \xi) = \left( \sqrt{Q^{(4)}(\xi)} - c(\xi) \right) \left[ \ln \left( \sqrt{Q^{(4)}(\xi)} - c(\xi) \right) - 1 \right],
\]
while the integration contour \(C_2\) in the complex \(\xi\)-plane is now given by two circles around the branch cuts of \(\sqrt{Q^{(4)}}\) [30, 31].

It is worth noticing that the HSS superfield \(\sqrt{L^{(+4)}(c-)^2}\) transforms covariantly under the N=2 superconformal transformations with conformal weight \(w = 2\),
\[
\delta \sqrt{L^{(+4)}(c-)^2} = 2\Lambda \sqrt{L^{(+4)}(c-)^2}.
\]

This implies the existence of another non-trivial N=2 superconformal invariant that originates from eq. (6.20) and has the form
\[
S[L^{(+4)}, c]_{\text{ext.}} = \int d\zeta \zeta (-4) du (c^{++})^2 \sqrt{L^{(+4)}(c-)^2}.
\]
The holomorphic PSS potential associated with the HSS Lagrangian (6.33) is obvious,
\[
G(Q^{(4)}(\xi))_{\text{ext.}} = \sqrt{Q^{(4)}(\xi)}.
\]

We are now in a position to make use of the new improved action (6.30) of a single \(O(4)\) projective multiplet. The generalized Legendre transform in application to eq. (6.31) is supposed to yield a free NLSM metric (in disguise), similarly to the improved \(O(2)\) projective multiplet action in eqs. (6.18) and (6.19). However, a sum of the improved actions (6.30) and (6.33) with different moduli \(c_a^d\),
\[
S[L^{(+4)}]_{\text{ALE} - D_k} = \sum_{a=1}^k \left( S[L^{(+4)}, c_a]_{\text{impr.}} + S[L^{(+4)}, -c_a]_{\text{impr.}} \right) + S[L^{(+4)}, c_0]_{\text{ext.}},
\]
gives rise (after the generalized Legendre transform) to the N=2 NLSM whose non-trivial metric can be identified with the ALE \(D_k\) metric [1, 31]. The \(Z_2\) symmetric combination of the improved terms in eq. (6.35) is necessary to produce the dihedral
group $D_k$ out of the cyclic $C_k$ group. Similarly, the ALF series of $D_k$ metrics are obtained after adding to eq. (6.35) a non-conformal deformation (cf. eq. (6.23)),

$$m \int d\zeta (-4) du L^{(+4)}.$$  \hspace{1cm} (6.36)

In the context of M-theory/type-IIA compactification (sect. 1), the parameters $\{c_{ij}^a\}$ in the action (6.35) describe the positions of D6-branes and an orientifold in the transverse directions. In the context of the related three-dimensional N=4 supersymmetric gauge field theory with $k$ matter hypermultiplets in the probe D2-brane world-volume, the moduli $\{c_{ij}^a\}$ parametrize the quantum moduli space (= ALF $D_k$ with $k$ singularities) of the low-energy effective field theory, being related to the positions of monopoles in the type-IIA picture (sect. 1).

In particular, the N=2 NLSM with the famous (regular and complete) Atiyah-Hitchin (AH) metric \cite{12} is obtained by the non-conformal deformation (6.36) of the N=2 superconformal action (6.33). The corresponding PSS data,

$$\frac{1}{2\pi i} \oint G(Q_{(4)}(\xi), \xi) = \frac{m}{2\pi i} \oint_{C_0} \frac{Q_{(4)}(\xi)}{\xi} + \oint_{C_2} \sqrt{Q_{(4)}(\xi)},$$  \hspace{1cm} (6.37)

with the contour $C_0$ encircling the origin in the clock-wise direction, just describes the AH metric \cite{30}. The AH metric also appears in the quantum moduli space of the three-dimensional N=4 supersymmetric pure gauge $SU(2)$-based quantum field theory \cite{32}, and in the hypermultiplet low-energy effective action (NLSM) of a (magnetically charged) hypermultiplet in the Higgs branch \cite{33}. From the AH prospective, the ALF $D_k$ metrics, associated with the HSS potential given by a sum of eqs. (6.35) and (6.36), can be equally interpreted as the deformations of the AH metric. Some of those metrics were discovered by Dancer \cite{34}, by using the hyper-Kähler quotient construction. Their derivation via the generalized Legendre transform is due to Chalmers \cite{35} who also noticed the significance of the same metrics for the monopole moduli spaces in the completely broken $SU(3)$ gauge theory investigated by Houghton \cite{36}.

To the end of this section, we would like to comment on the geometrical significance of an $O(4)$ projective multiplet versus an $O(2)$ projective (tensor) multiplet, in the context of N=2 supersymmetric NLSM with four-dimensional (self-dual or hyper-Kähler) target spaces. The PSS construction of N=2 NLSM takes the universal form (4.6) for all projective multiplets, it generically breaks the $SU(2)_R$ automorphism symmetry of N=2 supersymmetry algebra, but it leaves a $U(1)$ symmetry. The latter implies an $SO(2)$ isometry in the target space of the associated N=2 NLSM. The nature of this isometry is, however, dependent upon whether one uses an $O(2)$ or $O(4)$ projective multiplet in the PSS action (4.6). Any such action in terms of a single $O(2)$
tensor multiplet necessarily leads to a translational (tri-holomorphic) isometry that arises after trading the conserved vector component of the $O(2)$ projective multiplet for a scalar by the generalized Legendre transform.

Quite generally, the existence of an isometry amounts to the existence of a Killing vector $K^m$, $m = 1, 2, 3, 4$, obeying eq. (2.1) or, equivalently, the existence of the coordinate system $(x^a, \tau)$, $a = 1, 2, 3$, where the metric components are independent upon one of the coordinates ($\tau$),

$$ds^2 = H^{-1}(d\tau + C_adx^a) + H\gamma_{ab}dx^adx^b. \quad (6.38)$$

A translational isometry implies extra condition on the Killing vector \[37\],

$$K^{[m;n]} = \ast K^{[m;n]}, \quad (6.39)$$

where the star denotes the four-dimensional dual tensor. Equation (6.39) gives rise to the existence of the coordinate system (6.38) where, in addition, we have the ‘monopole equation’ \[4, 37\]

$$\nabla H = \pm \nabla \times \vec{C}, \quad \text{and} \quad \gamma_{ab}(x) = \delta_{ab}. \quad (6.40)$$

Self-duality then amounts to a linear Laplace equation on the potential $H(x^a)$,

$$\Delta H = 0, \quad (6.41)$$

whose localized solutions,

$$H(x) = \lambda + \sum_{s=1}^{k} \frac{m}{|\vec{x} - \vec{x}_s|}, \quad (6.42)$$

just describe the $A_k$ series of self-dual metrics (ALE multi-Eguchi-Hanson metrics in the case of $\lambda = 0$ and ALF multi-Taub-NUT metrics in the case of $\lambda = 1$). It is now not very surprising that those metrics arise from the N=2 superspace NLSM in terms of an $O(2)$ tensor multiplet only.

However, if one wants to construct the hyper-Kähler metrics possessing merely a rotational isometry, the conventional way towards their derivation (in components) is much more involved. Equation (6.38) still holds, but no eqs. (6.39), (6.40) and (6.41) are available. Nevertheless, a single real scalar potential for those metrics also exists in the so-called Toda frame defined by the conditions \[37\]

$$H = \partial_3 \Psi, \quad C_1 = \pm \partial_2 \Psi, \quad C_2 = \pm \partial_1 \Psi, \quad C_3 = 0, \quad (6.43a)$$

and

$$\gamma_{11} = \gamma_{22} = e^\Psi, \quad \gamma_{33} = 1. \quad (6.43b)$$
In the Toda frame self-duality amounts to the *non-linear* 3d Toda equation \[ (\partial_1^2 + \partial_2^2) \Psi + \partial_3^2 e^\Psi = 0 , \] which is very hard to solve. By the use of an $O(4)$ projective multiplet having no conserved vector components, in the $N=2$ superspace construction of 4d hyper-Kähler metrics, we just deal with the self-dual metrics having merely a rotational isometry. The basic geometrical difference between the $N=2$ superspace NLSM actions in terms of $O(2)$ or $O(4)$ projective multiplets thus amounts to the nature of their abelian isometry: it is tri-holomorphic in the $O(2)$ case, whereas it is not triholomorphic in the $O(4)$ case. The $N=2$ NLSM in terms of higher $O(2p)$ projective multiplets are similar to that with $p = 2$. It is worth mentioning in this context that a self-dual metric with rotational isometry gives rise to a solution to the 3d Toda equation (6.44).

7 Conclusion

By the use of harmonic superspace describing hypermultiplets with manifest $N=2$ supersymmetry, we arrived at a general solution to the $N=2$ non-linear sigma-models with special hyper-Kähler geometry. Our solution is parametrized by a single (of degree two) homogeneous function, which is quite similar to the well-known $N=2$ superconformal description of $N=2$ vector multiplets in the standard $N=2$ superspace.

We also constructed the improved ($N=2$ superconformal) actions of the $O(2)$ and $O(4)$ projective multiplets, which lead to flat four-dimensional metrics in disguise. However, after being added together with different moduli, those $N=2$ superconformal actions naturally lead to the $A_k$ and $D_k$ series of the highly non-trivial self-dual metrics in the target space of the associated $N=2$ NLSM. It gives us the very natural way of derivation and classification of those self-dual metrics. In particular, the ALE metric potentials in superspace can be interpreted as the interpolating potentials between different improved (flat) potentials in terms of the $O(2)$ projective multiplet in the $A_k$ case and in terms of the $O(4)$ projective multiplet in the $D_k$ case, whereas the ALF potentials can be understood as non-conformal deformations of the ALE ones. It would be interesting to find the self-dual metrics, associated with the exceptional (simply-laced) $E_{6,7,8}$ Lie groups, in the superspace approach.

The (improved) $N=2$ superconformally invariant actions (6.18) and (6.30) of the $O(2)$ and $O(4)$ projective multiplets, respectively, are not easily generalizable to the

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\[5\] The 3d Toda equation arises from the standard (2d) $SU(N)$-based Toda system in the large-$N$ limit \[\text{[38]}.\]
case of higher $O(2p)$ projective multiplets with $p > 2$. An $O(2p)$ projective multiplet is described in HSS by the $L^{(2p+)}$ superfield satisfying the off-shell constraints (4.9). It can be used to introduce the HSS superfield

$$\left[ L^{(2p+)}(c^-)^p \right]^{1/p}$$

that covariantly transforms under the rigid N=2 superconformal transformations with conformal weight $w = 2$. It is, however, unclear to us how to define a covariant HSS superfield of $U(1)$ charge (+4) and conformal weight $w = 4$, in terms of the $L^{(2p+)}$ superfield, in order to substitute $(L^+)^2$ in eq. (6.18). This obstruction may be related to the existence of only two ($A_k$ and $D_k$) regular series of gravitational instantons. It is, however, possible to define N=2 superconformal generalizations of eqs. (6.20) and (6.33) to the case of higher projective multiplets,

$$S[L^{(2p+)}], c]_{\text{ext.}} = \int d\zeta (-4) du (c^+)^2 \left[ L^{(2p+)}(c^-)^p \right]^{1/p} .$$

(7.2)

It would be interesting to study non-conformal deformations of eq. (7.2).

Our final remark is devoted to local N=2 superconformal symmetry that implies coupling N=2 (rigidly) superconformal NLSM to N=2 (conformal) supergravity. It leads to the deformation of a given special hyper-Kähler NLSM metric $g_{mn}$ to a quaternionic metric $G_{mn}$ \[39\]. This deformation in four dimensions preserves self-duality of the Weyl tensor of the metric $g_{mn}$, but it also turns it into an Einstein metric with negative scalar curvature \[40\]. The associated quaternionic metric reads

$$G_{mn} = \frac{1}{f} g_{mn} - \frac{1}{f^2} \left( \frac{1}{2} X_m X_n + 2 Y_m^A Y_n^A \right)$$

(7.3)

in terms of the Euler vector $X_m$, the potential $f$ defined by eq. (2.7), and the Killing vectors $Y_m^A$ defined by eq. (2.9). Therefore, an explicit derivation of the quaternionic metric, associated with a given special hyper-Kähler metric, seems to be possible without going into details of the N=2 NLSM coupling to N=2 supergravity.

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