Existence of Periodic Solutions for a Class of Fourth-Order Difference Equation

Jia Wei, Xiaoling Han, and Fumei Ye

1School of Education, Lanzhou University of Arts and Science, Lanzhou 730070, China
2Department of Mathematics, Northwest Normal University, Lanzhou 730070, China

Correspondence should be addressed to Jia Wei; weijiavick@126.com

Received 29 January 2022; Revised 17 June 2022; Accepted 26 July 2022; Published 16 August 2022

Academic Editor: Richard I. Avery

1. Introduction

In recent years, the theory of nonlinear difference equations has been widely used in the study of discrete models in the fields of economics, neural networks, ecology, etc. For the general background of difference equations, in particular, there are many authors who have discussed the existence and multiplicity of periodic solutions for discrete boundary value problems by exploiting various methods, including the method of upper and lower solutions, Leray-Schauder degree, fixed point theory, critical theory, and variational methods; see Bereanu and Mawhin [1], Cabada and Dimitrov [2], Graef et al. [3, 4], and Cai et al. [5–10] and the references therein.

Let \( \mathbb{N}^n \), \( \mathbb{Z} \), and \( \mathbb{R} \) denote the sets of all positive integers, integers, and real numbers, respectively. This paper considers the following fourth-order nonlinear difference equation:

\[
\Delta^4 u(k - 2) - a(k)u^n(k) + b(k)u^\beta(k) = 0, \quad k \in \mathbb{Z},
\]

where \( a > 0, \beta > 0 \) and \( a \neq \beta \), \( a(k), b(k) \) are \( T \)-periodic functions and \( \Delta u(k) = u(k + 1) - u(k) \) is the forward difference operator.

Equation (1) can be considered as a discrete analogue of a special case of the following fourth-order nonlinear differential equation:

\[
u(t)'''' - a(t)u^n(t) + b(t)u^\beta(t) = 0, \quad t \in \mathbb{R},
\]

which has been studied in [11, 12] when \( a = 1, \beta = 3 \). In [13], Yang and Han proved the existence of a periodic solution to equation (2) when \( a = n, \beta = n + 2 \), where \( n \) is a positive integer.

When \( a(k) \equiv 0, \beta = 1 \), Peterson and Ridenhour [14] considered the disconjugacy of the following equation:

\[
\Delta^4 u(k - 2) + b(k)u(k) = 0, \quad k \in \mathbb{Z}.
\]

In 2005, Cai et al. [5] studied the fourth-order nonlinear difference equation

\[
\Delta^4 u(k - 2) + f(k, u(k)) = 0, \quad k \in \mathbb{Z}.
\]

By applying the linking theorem, they obtained some criteria for the existence and multiplicity of periodic solutions of equation (4).

In fact, there is a big difference between the continuous case and the discrete case. For example, the basic ideas of calculus are not always applicable when studying difference equations, such as the embedding theorem. Therefore, we need to consider other methods to deal with the difference...
problem. The main tool used is the continuation theorem of Mawhin (see [15]).

Motivated by the above works, the main aim of this paper is to investigate the existence of at least one positive $T$-periodic solution of (1). In order to obtain the main results of (1), we assume that the coefficient functions $a(k)$ and $b(k)$ satisfy the following condition:

$F_1$: Suppose $a(k), b(k)$ are $T$–periodic functions and $a(k)b(k) > 0$ for all $k \in Z$. Furthermore, we assume that there exist positive constants $a, A, b, B$ such that

$$a = \min_{k \in Z} |a(k)|,$$

$$A = \max_{k \in Z} |a(k)|,$$

$$b = \min_{k \in Z} |b(k)|,$$

$$B = \max_{k \in Z} |b(k)|.$$  \hspace{1cm} (5)

Let $X$ be all real $T$-periodic sequences of the form $u = \{u(k)\}_{k \in Z}$. Then, $X$ is a Banach space under the norm $\|u\| = \max_{k \in [2T+1]} |u(k)|$.

The main results in this paper are stated next: Theorems 1 and 2.

**Theorem 1.** Let $F_1$ hold, if $\alpha < \beta$ and the period $T$ satisfies

$$16 \leq T^4 \leq \frac{8}{AR^2_1 + BR^2_1},$$

where $R_1 = (A/b)^{1/(\beta-\alpha)} + \rho$ and $\rho > 0$ small enough such that $(a/b)^{1/(\beta-\alpha)} - \rho > 0$; then, equation (1) admits at least one positive $T$-periodic solution.

**Theorem 2.** Let $F_1$ hold, if $\alpha > \beta$ and the period $T$ satisfies

$$16 \leq T^4 \leq \frac{8}{AQ^2_1 + BQ^2_1},$$

where $Q_1 = (B/a)^{1/(\alpha-\beta)} + \tau$ and $\tau > 0$ small enough such that $(b/a)^{1/(\alpha-\beta)} - \tau > 0$; then, equation (1) admits at least one positive $T$-periodic solution.

**Theorem 3.** Suppose $a(k)b(k) \leq 0$ and $a(k), b(k)$ are not identical to zero for all $k \in Z$; then, equation (1) has no positive solution.

This paper is organized as follows: In Section 2, we give some lemmas needed to prove the main results. Section 3 contains the proof of Theorem 1. Section 4 contains the proof of Theorem 2. Section 5 contains the proof of Theorem 3.

## 2. Preliminary Results

In this section, we introduce some notations and well-known results which will be used in the subsequent sections.

**Definition 4** (see [7], p. 12, B.1). Let $X, Y$ be real Banach spaces, $L : \text{Dom}L \subset X \rightarrow Y$ be a linear mapping. The mapping $L$ is said to be a Fredholm mapping of index zero if

(a) $\text{Im } L$ is closed in $Y$

(b) $\dim \text{Ker } L = \text{codim } \text{Im } L < +\infty$

If $L$ is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ such that

$$\text{Im } P = \text{Ker } L,$$

$$\text{Ker } Q = \text{Im } L = \text{Im } (I - Q).$$

It follows that the restriction

$$L_{\text{Dom}L \setminus \text{Ker } P} : (I - P)X \rightarrow \text{Im } L$$

has an inverse which is denoted by $K_p$.

**Definition 5** (see [7], pp. 12–13, B.1, B.2). Let $N : X \rightarrow Y$ be a continuous mapping. If $\Omega$ is a bounded open subset of $X$, $N$ is called $L$–compact on $\Omega$ if $QN(\Omega)$ is bounded and $K_p(I - Q)N : \tilde{\Omega} \rightarrow X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $J : \text{Im } Q \rightarrow \text{Ker } L$.

**Lemma 6** (Mawhin’s continuation theorem, see [7], Theorem IV.1). Let $L$ be a Fredholm mapping of index zero, $\Omega \subset X$ is an open bounded set, and let $N$ be $L$–compact on $\Omega$.

Suppose

(1) for each $\lambda \in (0, 1)$, $Lu \neq \lambda Nu$ for all $u \in \partial \Omega \cap \text{Dom } L$

(2) $Q Nu \neq 0$, for all $u \in \partial \Omega \cap \text{Ker } L$

(3) $\deg (JQN, \Omega \cap \text{Ker } L, 0) \neq 0$, where $J : \text{Im } Q \rightarrow \text{Ker } L$ is an isomorphism.

Then, the equation $Lu = Nu$ has at least one solution in $\tilde{\Omega} \cap \text{Dom } L$.

Define the operator $L : X \rightarrow X$ by setting

$$Lu = \Delta^4 u(k - 2), u \in X.$$  \hspace{1cm} (10)

Direct calculation shows that

$$\text{Ker } L = \mathbb{R} \text{ and } \text{Im } L = \left\{ v \left| \sum_{i=2}^{T_1} v(i) = 0 \right. \right\}.$$  \hspace{1cm} (11)

Indeed, if $v = \{v(k)\}_{k \in Z} \in \text{Im } L$, then there is $u =
\{u(k)\}_{k \in \mathbb{Z}} \in X\) such that
\[v(k) = \Delta^1 u(k - 2), \quad k \in \mathbb{Z}.\] \hfill (12)

Thus,
\[\sum_{i=2}^{T+1} v(i) = \sum_{i=2}^{T+1} \Delta^1 u(i - 2).\] \hfill (13)

Note that \(\{u(k)\}_{k \in \mathbb{Z}} \in X\), it follows that
\[\{\Delta^1 u(k)\}_{k \in \mathbb{Z}} \in X.\] \hfill (14)

Furthermore, direct calculation shows that
\[\sum_{i=2}^{T+1} \Delta^1 u(i - 2) = [\Delta^1 u(1) - \Delta^1 u(0)] + [\Delta^1 u(2) - \Delta^1 u(1)] + \cdots + [\Delta^1 u(T) - \Delta^1 u(T - 1)] = \Delta^1 u(T) - \Delta^1 u(0) = \Delta^1 u(0) - \Delta^1 u(0) = 0.\] \hfill (15)

By virtue of the above facts, we have
\[\sum_{i=2}^{T+1} v(i) = \sum_{i=2}^{T+1} \Delta^1 u(i - 2) = 0.\] \hfill (16)

Since \(\dim X = T\) and \(L\) is a linear mapping, by the knowledge of linear algebra, we know that \(\dim \text{Ker} L < \dim \text{Im} L = \dim X\). It is easy to see that \(\dim \text{Ker} L = \text{codim} \text{Im} L = 1\) and \(\dim \text{Im} L = T - 1\). It follows that \(\text{Im} L\) is closed in \(X\). Therefore, the operator \(L\) is a Fredholm operator with index zero.

Let us define \(N : X \longrightarrow X\) by
\[(Nu)(k) = a(k)u^a(k) - b(k)u^b(k).\] \hfill (17)

We define \(P : X \longrightarrow \text{Ker} L\) and \(Q : X \longrightarrow X\) as follows:
\[(Pu)(k) = (Qu)(k) = \frac{1}{T} \sum_{i=2}^{T+1} u(i).\] \hfill (18)

The operators \(P\) and \(Q\) are projections. Hence,
\[\text{Im} P = \text{Ker} L, \quad \text{Ker} Q = \text{Im} L.\] \hfill (19)

It follows that \(L|_{\text{Dom} L_i \cap \text{Ker} P} : (I - P)X \longrightarrow \text{Im} L\) has an inverse which is denoted by \(K_P\).

In view of (17) and (18), for any \(u \in X\), we can see that
\[(QN u)(k) = \frac{1}{T} \sum_{i=2}^{T+1} [a(i)u^a(i) - b(i)u^b(i)].\] \hfill (20)

Since the Banach space \(X\) is finite dimensional, \(K_P\) is linear. By virtue of the relations (20) and (21), we see that \(QN (I - Q)\) is continuous on \(X\). Hence, we know that if \(\Omega\) is an open and bounded subset of \(X\), then \(QN(\bar{\Omega})\) is bounded. It follows that
\[K_P(I - Q)N : \Omega \longrightarrow X\] \hfill (22)

is compact. Therefore, the mapping \(N\) is \(L\)-compact on \(\Omega\) with any open and bounded subset \(\Omega \subset X\).

\textbf{Lemma 7} (see [16], Lemma 2.3). Let \(\{u(k)\}_{k \in \mathbb{Z}}\) be a real \(T\)-periodic sequence; then,
\[
\max_{2 \leq i, j \leq T + 1} |u(i) - u(j)| \leq \frac{T + 1}{16} \sum_{k=2}^{T+1} |\Delta^1 u(k - 2)|. \tag{23}
\]

\section*{3. Proof of Theorem 1}

\textit{Proof.} The content of Theorem 1 is as follows: let \(F_1\) hold, if \(a < \beta\) and the period \(T\) satisfies
\[16 \leq T^4 \leq \frac{8}{A R^{\alpha - 1} + B R^{\beta - 1}}, \tag{24}\]
where \(R_1 = (A/B)^{(1/\beta - \alpha)}\) and \(\rho > 0\) small enough such that \((a/B)^{(1/\beta - \alpha)} - \rho > 0\); then, equation (1) admits at least one positive \(T\)-periodic solution. Now, we prove that the conclusion holds. We assume that \(a < \beta\). From condition \(F_1\), we know that \(a(k)b(k) > 0\), which include both positive and negative cases. So we need to classify the cases where both \(a(k)\) and \(b(k)\) are positive and both negative.

\textit{Case 1.} If coefficient functions \(a(k)\) and \(b(k)\) are positive \(T\)-periodic functions, in view of \(F_1\), we have that \(0 < a \leq a(k) \leq A\) and \(0 < b \leq b(k) \leq B\).

Let
\[\Omega_1 = \{u \in X| H_1 < u(k) < R_1\}, \tag{25}\]
which is an open set in \(X\), where
\[R_1 = R + \rho, \quad R = \left(\frac{A}{B}\right)^{(1/\beta - \alpha)}, \tag{26}\]
\[H_1 = H - \rho, \quad H = \left(\frac{a}{B}\right)^{(1/\beta - \alpha)}, \tag{27}\]
where \(\rho > 0\) small enough such that \((a/B)^{(1/\beta - \alpha)} - \rho > 0\). Obviously, \(H_1\) and \(R_1\) are well defined.
By
\[ a < \beta, 0 < a \leq a(k) \leq A, \]
\[ 0 < b \leq b(k) \leq B, \]
we obtain
\[ 0 < H_1 < H \leq \left( \frac{a(k)}{b(k)} \right)^{1/(\beta-a)} \leq R < R_1, \]  
uniformly for \( k \in \mathbb{Z}. \)
Furthermore, it follows from (26), (27), and (29) that
\[ a(k) - b(k)H_1^{\beta-a} \geq a - BH_1^{\beta-a} > 0, \]
\[ a(k) - b(k)R_1^{\beta-a} \leq A - bR_1^{\beta-a} < 0. \]  
Therefore,
\[ a(k) - b(k)H_1^{\beta-a} > 0, \]
\[ a(k) - b(k)R_1^{\beta-a} < 0. \]  
We prove that condition (1) of Lemma 6 holds. Let 0 < \( \lambda < 1 \) and \( u \) be such that
\[ \Delta^t u(k-2) - \lambda a(k)u^\alpha(k) + \lambda b(k)u^\beta(k) = 0. \]  
Summing from 2 to \( T + 1 \), we can see that
\[ \sum_{i=2}^{T+1} \left[ \Delta^t u(i-2) - \lambda a(i)u^\alpha(i) + \lambda b(i)u^\beta(i) \right] = 0. \]  

Firstly, we claim that for each \( \lambda \in (0, 1) \) and \( u \in \partial \Omega_1 \cap \text{Dom } L, Lu \neq \lambda Nu. \) In fact, in view of (25), if \( u \in \partial \Omega_1, \) then \( \|u\| = H_1 \) or \( \|u\| = R_1. \) We only prove the case of \( \|u\| = H_1, \) similar to the proof of \( \|u\| = R_1. \)

When \( \|u\| = H_1, \) we can get that \( \max_{2\leq j \leq T+1} |u(l) - u(j)| < H_1/2 \) or \( \max_{2\leq j \leq T+1} |u(l) - u(j)| \geq H_1/2 \) holds. If \( \max_{2\leq j \leq T+1} |u(l) - u(j)| < H_1/2, \) thus \( H_1/2 < u \leq H_1. \) According to the above facts, we have
\[ 0 = \sum_{i=2}^{T+1} \left[ \lambda a(i)u^\alpha(i) - \lambda b(i)u^\beta(i) \right] \]
\[ = \lambda \sum_{i=2}^{T+1} u^\alpha(i) \left[ a(i) - b(i)u^{\beta-a}(i) \right] \]
\[ \geq \lambda (a - BH_1^{\beta-a}) \sum_{i=2}^{T+1} u^\alpha(i) > 0. \]  
This is a contradiction.

If \( \max_{2\leq j \leq T+1} |u(l) - u(j)| \geq H_1/2, \) we see from Lemma 7 that
\[
0 = \left| \sum_{i=2}^{T+1} \left[ \Delta^t u(i-2) - \lambda a(i)u^\alpha(i) + \lambda b(i)u^\beta(i) \right] \right|
\geq \sum_{i=2}^{T+1} \Delta^t u(i-2) - \sum_{i=2}^{T+1} \lambda a(i)u^\alpha(i) + \lambda b(i)u^\beta(i)
\geq \sum_{i=2}^{T+1} \Delta^t u(i-2) - \sum_{i=2}^{T+1} a(i)u^\alpha(i) + b(i)u^\beta(i)
\geq 16 T^3 \max_{2\leq j \leq T+1} |u(l) - u(j)| - T (AH_1^{\alpha-1} + BH_1^{\beta-1}) \|u\|
\geq \frac{8}{T^3} |u| - T (AH_1^{\alpha-1} + BH_1^{\beta-1}) \|u\|
= \frac{8}{T^3} - T (AH_1^{\alpha-1} + BH_1^{\beta-1}) \|u\|
\geq 0.
\]
This is a contradiction.

When \( \|u\| = R_1, \) we can get that \( \max_{2\leq j \leq T+1} |u(l) - u(j)| < R_1/2 \) or \( \max_{2\leq j \leq T+1} |u(l) - u(j)| \geq R_1/2 \) holds. If \( \max_{2\leq j \leq T+1} |u(l) - u(j)| < R_1/2, \) thus \( R_1/2 < u \leq R_1; \) further, we have
\[
0 = \sum_{i=2}^{T+1} \left[ \lambda a(i)u^\alpha(i) - \lambda b(i)u^\beta(i) \right]
= \lambda \sum_{i=2}^{T+1} u^\alpha(i) \left[ a(i) - b(i)u^{\beta-a}(i) \right]
\leq \lambda (A - bR_1^{\beta-a}) \sum_{i=2}^{T+1} u^\alpha(i) < 0.
\]
This is a contradiction.
but this is a contradiction. Hence, for $u \in \partial \Omega_1$ and $\lambda \in (0, 1)$, we have
\[ \Delta^4 u(k-2) - \lambda a(k)u^a(k) + \lambda b(k)u^b(k) \neq 0. \] (38)

Therefore, we verify that condition (1) of Lemma 6 holds for $\Omega_1$.

If $u \in \partial \Omega_1 \cap \text{Ker } L$, then $u = \{H_1\}_{k \in \mathbb{Z}}$ or $u = \{R_1\}_{k \in \mathbb{Z}}$. By virtue of (31), we conclude that
\[ (QN\mu)(k) = \frac{1}{T} \sum_{i=2}^{T+1} u^a(i) a(i) - b(i)u^{a-a}(i) \neq 0. \] (39)

Hence, $QN\mu \neq 0$ for each $u \in \partial \Omega_1 \cap \text{Ker } L$.

Next let us consider $(H_1 + R_1)/2$, the arithmetic mean of $H_1$ and $R_1$. We define $G : X \times \mathbb{R} \rightarrow X$ as follows, for all $\mu \in (0, 1)$,
\[ G(u, \mu) = (1 - \mu) \left( u - \frac{H_1 + R_1}{2} \right) + \mu \frac{1}{T} \sum_{i=2}^{T+1} u^a(i) a(i) - b(i)u^{a-a}(i) \]. (40)

Clearly, we find that
\[ G(u, \mu) \neq 0, \forall u \in \partial \Omega_1 \cap \text{Ker } L. \] (41)

By using the homotopy invariance theorem, it is easy to see that
\[ \text{deg } (QN, \Omega_1 \cap \text{Ker } L, 0) = \text{deg } (G(u, 1), \Omega_1 \cap \text{Ker } L, 0) = \text{deg } (G(u, 0), \Omega_1 \cap \text{Ker } L, 0) = -1 \neq 0. \] (42)

Therefore, conditions (1)–(3) of Lemma 6 hold for $\Omega_1$.

Furthermore, according to the above reasoning, we deduce that (1) has at least one positive solution in $\Omega_1$.

Case 2. If the coefficient functions $a(k), b(k)$ are negative $T$-periodic functions, in view of $F_1$, we have that $-A \leq a(k) \leq -a < 0$ and $-B \leq b(k) \leq -b < 0$.

Let $\tilde{a}(k) = -a(k), \tilde{b}(k) = -b(k)$; then, we see that
\[ 0 < a \leq \tilde{a}(k) \leq A, \]
\[ 0 < b \leq \tilde{b}(k) \leq B. \] (43)

It is obvious that (1) is equivalent to the equation
\[ \Delta^4 u(k-2) + \tilde{a}(k)u^a(k) - \tilde{b}(k)u^b(k) = 0. \] (44)

Let $0 < \lambda < 1$ and $u$ be such that
\[ \Delta^4 u(k-2) + \lambda \tilde{a}(k)u^a(k) - \lambda \tilde{b}(k)u^b(k) = 0. \] (45)

Summing from 2 to $T + 1$, we can see that
\[ \sum_{i=2}^{T+1} \left[ \Delta^4 u(i-2) + \lambda \tilde{a}(i)u^a(i) - \lambda \tilde{b}(i)u^b(i) \right] = 0. \] (46)

Firstly, we claim that for each $\lambda \in (0, 1)$ and $u \in \partial \Omega_1 \cap \text{Dom } L, Lu \neq \lambda Nu$. In fact, in view of (25), if $u \in \partial \Omega_1$, then $|u| = H_1$ or $|u| = R_1$. We only prove the case of $|u| = H_1$, similar to the proof of $|u| = R_1$.

When $|u| = H_1$, we can get that
\[ \max_{2 \leq j \leq T+1} |u(l) - u(j)| < \frac{H_1}{2} \] (47)

or
\[ \max_{2 \leq j \leq T+1} |u(l) - u(j)| \geq \frac{H_1}{2} \] (48)

holds.

If $\max_{2 \leq j \leq T+1} |u(l) - u(j)| < H_1/2$, thus
\[ \frac{H_1}{2} < u \leq H_1. \] (49)

Further, we have
\[ 0 = \sum_{i=2}^{T+1} \left[ \lambda \tilde{a}(i)u^a(i) - \lambda \tilde{b}(i)u^b(i) \right] = \lambda \sum_{i=2}^{T+1} \left[ \tilde{a}(i) - \tilde{b}(i) \right] u^{a-a}(i) \geq \lambda \left( a - BH_1^{a-a} \right) \sum_{i=2}^{T+1} u^a(i) > 0. \] (50)

This is a contradiction.

If $\max_{2 \leq j \leq T+1} |u(l) - u(j)| \geq H_1/2$, we see from Lemma 7 that
\[ 0 \geq \sum_{i=2}^{T+1} \left[ \Delta^4 u(i-2) - \lambda \tilde{a}(i)u^a(i) + \lambda \tilde{b}(i)u^b(i) \right]. \]

\[ \geq \sum_{i=2}^{T+1} \left[ \Delta^4 u(i-2) - \lambda \tilde{a}(i)u^a(i) + \lambda \tilde{b}(i)u^b(i) \right] \]
\[ \geq \frac{16}{T^3} \max_{2 \leq j \leq T+1} |u(l) - u(j)| - T \left( A H_1^{a-a} + B H_1^{b-b} \right) \|u\| \]
\[ \geq \frac{8}{T^3} \|u\| - T \left( A H_1^{a-a} + B H_1^{b-b} \right) \|u\| \]
\[ = \left[ \frac{8}{T^3} - T \left( A H_1^{a-a} + B H_1^{b-b} \right) \right] \|u\| \geq 0, \] (51)
but this is a contradiction. Hence, for all \( u \in \partial \Omega_1 \) and \( \lambda \in (0, 1) \), we have

\[
\Delta^4 u(k - 2) + \lambda \hat{a}(k) u^\alpha(k) - \lambda \hat{b}(k) u^\beta(k) \neq 0.
\] (52)

Therefore, we verify that condition (1) of Lemma 6 holds for \( \Omega_1 \).

The remaining proof is similar to the proof of Case 1, and so we omit it. Furthermore, according to the above reasoning, we deduce that (44) has at least one positive solution in \( \Omega_1 \).

4. Proof of Theorem 2

Proof. The content of Theorem 2 is as follows: let \( F_1 \) hold, if \( \alpha > \beta \) and the period \( T \) satisfies

\[
16 < T^4 \leq \frac{8}{AQ_1^{-1} + BQ_1^{-1}},
\] (53)

where \( Q_1 = (B/a)^{1/(\alpha-\beta)} + \tau \) and \( \tau > 0 \) small enough such that \( (B/A)^{1/(\alpha-\beta)} - \tau > 0 \); then, equation (1) admits at least one positive \( T \)-periodic solution. Now, we prove that the conclusion holds. Similarly, in the case of \( \alpha > \beta \), we need to discuss the case where the coefficient functions \( a(k) \) and \( b(k) \) are both positive and negative, respectively.

Case 1. If coefficient functions \( a(k) \) and \( b(k) \) are positive \( T \)-periodic functions, we have that \( 0 < a \leq a(k) \leq A \) and \( 0 < b \leq b(k) \leq B \).

Let

\[
\Omega_2 = \{u \in X|P_1 < u(k) < Q_1\},
\] (54)

which is an open set in \( X \), where

\[
Q_1 = Q + \tau, Q = \left(\frac{B}{a}\right)^{1/(\alpha-\beta)},
\] (55)

\[
P_1 = P - \tau, P = \left(\frac{b}{A}\right)^{1/(\alpha-\beta)}.
\] (56)

where \( \tau > 0 \) small enough such that \( (B/A)^{1/(\alpha-\beta)} - \tau > 0 \). Obviously, \( P_1 \) and \( Q_1 \) are well defined.

By \( \alpha > \beta \), \( 0 < a \leq a(k) \leq A \), and \( 0 < b \leq b(k) \leq B \), we obtain

\[
0 < P_1 < P = \left(\frac{b}{A}\right)^{1/(\alpha-\beta)} \leq Q < Q_1,
\] (57)

uniformly for \( k \in Z \).

By virtue of (55) and (56), we obtain

\[
a(k)P_1^{\alpha-\beta} - b(k) \leq AP_1^{\alpha-\beta} - b < 0,
\] (58)

\[
a(k)Q_1^{\alpha-\beta} - b(k) \geq aQ_1^{\alpha-\beta} - B > 0.
\]

Therefore,

\[
a(k)P_1^{\alpha-\beta} - b(k) < 0,
\] (59)

\[
a(k)Q_1^{\alpha-\beta} - b(k) > 0,
\]

uniformly for \( k \in Z \).

The remaining proof is similar to the proof of Theorem 1, and so we omit it. Furthermore, we conclude that (1) has at least one positive \( T \)-periodic solution in \( \Omega_2 \).

Case 2. If the coefficient functions \( a(k), b(k) \) are negative \( T \)-periodic functions, we have that \( -A \leq a(k) \leq -a < 0 \) and \( -B \leq b(k) \leq -b < 0 \).

Let \( \tilde{a}(k) = -a(k), \tilde{b}(k) = -b(k) \). Then, we can see that

\[
0 < a \leq \tilde{a}(k) \leq A,
\] (60)

\[
0 < b \leq \tilde{b}(k) \leq B.
\]

It is obvious that

\[
-\tilde{a}(k)P_1^{\alpha-\beta} + \tilde{b}(k) > 0,
\] (61)

\[
-\tilde{a}(k)Q_1^{\alpha-\beta} + \tilde{b}(k) < 0,
\]

uniformly for \( k \in Z \).

The remaining proof is similar to the proof of Theorem 1, and so we omit it. Furthermore, we conclude that (1) has at least one positive \( T \)-periodic solution in \( \Omega_2 \).

5. Proof of Theorem 3

Proof. The content of Theorem 3 is as follows: suppose \( a(k) \) \( b(k) \leq 0 \) and \( a(k), b(k) \) are not identical to zero for all \( k \in Z \); then, equation (1) has no positive solution.

Summing equation (1) from 2 to \( T + 1 \), we obtain that

\[
\sum_{i=2}^{T+1} \left[ \Delta^4 u(i) - a(i)u^\alpha(i) + b(i)u^\beta(i) \right] = 0.
\] (62)

In view of

\[
\text{Dom} \ L = \{u|u \in X, \Delta u(k + T) = \Delta u(k), \Delta^3 u(k + T) = \Delta^3 u(k)\}.
\] (63)

Hence,

\[
\sum_{i=2}^{T+1} \left[ -a(i)u^\alpha(i) + b(i)u^\beta(i) \right] = 0.
\] (64)

If \( a(k) > 0 \) and \( b(k) \leq 0 \), it follows from (64) that (1) does not have any positive solution. Other cases are similar. \( \square \)
6. Example

Example 1. The difference equation

\[ \Delta^2 u(k - 2) - \left( \frac{1}{100} \sin \left( \frac{2\pi k}{T} \right) + \frac{1}{50} \right) u(k) + \left( \frac{1}{200} \cos \left( \frac{2\pi k}{T} \right) + \frac{1}{100} \right) u^3(k) = 0, \]

is one of the form (1), where \( a = 1/100, A = 3/100, b = 1/100, B = 3/200, \alpha = 1, \) and \( \beta = 3. \) Let \( \rho = \sqrt{6}/6, \) we obtain

\[ 16 < T^4 \leq \frac{3200}{31 + 6\sqrt{2}}. \]

Therefore, we can prove that (65) has at least one positive \( T \)-periodic solution in \( \Omega_1, \)

\[ \Omega_1 = \left\{ u \in X \left| \frac{\sqrt{2}}{6} < u(k) < \frac{6\sqrt{3} + \sqrt{6}}{6} \right. \right\}. \]

Example 2. The difference equation

\[ \Delta^2 u(k - 2) + \left( \frac{1}{5000} \cos \left( \frac{2\pi k}{T} \right) + \frac{1}{2500} \right) u^3(k) - \left( \frac{1}{200} \sin \left( \frac{2\pi k}{T} \right) + \frac{1}{500} \right) u(k) = 0, \]

is one of the form (1), where \( -a = -1/5000, -A = -3/5000, -b = -3/2000, -B = -1/400, \alpha = 3, \) and \( \beta = 1. \) Let \( \tau = \sqrt{10}/4, \) we obtain

\[ 16 < T^4 \leq \frac{64000}{83 + 12\sqrt{5}}. \]

Therefore, we can prove that (68) has at least one positive \( T \)-periodic solution in \( \Omega_2, \)

\[ \Omega_2 = \left\{ u \in X \left| \frac{\sqrt{10}}{4} < u(k) < \frac{11\sqrt{10}}{4} \right. \right\}. \]

Data Availability

Data sharing does not apply to this article as no data set was generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Acknowledgments

This work was supported by the Gansu University Innovation Fund Project (grant numbers 2021A-175 and 2019A-158), the Natural Science Foundation of Gansu Province (grant number 20JR10RA086), and the National Natural Science Foundation of China (grant number 12161079).

References

[1] C. Bereanu and J. Mawhin, “Boundary value problems for second order nonlinear difference equations with discrete \( \phi \)-Laplacian and singular \( \phi \),” Journal of Difference Equations and Applications, vol. 14, no. 10-11, pp. 1099–1118, 2008.

[2] A. Cabada and N. Dimitrov, “Multiplicity results for nonlinear periodic fourth order difference equations with parameter dependence and singularities,” Journal of Mathematical Analysis and Applications, vol. 371, no. 2, pp. 518–533, 2010.

[3] J. R. Graef, L. Kong, and M. Wang, “Existence of multiple solutions to a discrete fourth order periodic boundary value problem,” Discrete and Continuous Dynamical Systems, vol. 2013, pp. 291–299, 2013.

[4] J. R. Graef, L. Kong, and X. Liu, “Existence of solutions to a discrete fourth order periodic boundary value problem,” Journal of Difference Equations and Applications, vol. 22, no. 8, pp. 1167–1183, 2016.

[5] X. Cai, J. Yu, and Z. Guo, “Existence of periodic solutions for fourth-order difference equations,” Computers & Mathematics with Applications, vol. 50, no. 1–2, pp. 49–55, 2005.

[6] X. Liu, Y. Zhang, H. Shi, and X. Deng, “Periodic solutions for fourth-order nonlinear functional difference equations,” Mathematical Methods in the Applied Sciences, vol. 38, no. 1, pp. 1–10, 2015.

[7] R. Ma and C. Gao, “Bifurcation of positive solutions of a nonlinear discrete fourth-order boundary value problem,” Zeitschrift für Angewandte Mathematik und Physik, vol. 64, no. 3, pp. 493–506, 2013.

[8] H. Shi, X. Liu, Y. Zhang, and X. Deng, “Existence of periodic solutions of fourth-order nonlinear difference equations,” Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, vol. 108, no. 2, pp. 811–825, 2014.

[9] G. Wang and S. Cheng, “Periodic solutions of a neutral difference system,” Boletim da Sociedade Paranaense de Matemática, vol. 22, no. 2, pp. 117–126, 2009.

[10] S. Wang and Y. Long, “Multiple solutions of fourth-order functional difference equation with periodic boundary conditions,” Applied Mathematics Letters, vol. 104, no. 7, article 106292, 2020.

[11] C. Bereanu, “Periodic solutions of some fourth-order nonlinear differential equations,” Nonlinear Analysis, vol. 71, no. 1–2, pp. 53–57, 2009.

[12] S. Tersian and J. Chaparova, “Periodic and homoclinic solutions of extended Fisher-Kolmogorov equations,” Journal of Mathematical Analysis and Applications, vol. 260, no. 2, pp. 490–506, 2001.

[13] H. Yang and X. Han, “Existence and multiplicity of positive periodic solutions for fourth-order nonlinear differential equations,” Electronic Journal of Differential Equations, vol. 119, pp. 1–14, 2019.

[14] A. Peterson and J. Ridenhour, “The (2,2)-disconjugacy of a fourth order difference equation,” Journal of Difference Equations and Applications, vol. 1, no. 1, pp. 87–93, 1995.
[15] R. Gaines and J. Mawhin, “Coincidence degree and nonlinear differential equations,” in Lecture Notes in Math, vol. 586, Springer-Verlag, Berlin, New York, 1997.

[16] X. Han, X. Ma, and G. Dai, “Solutions to fourth-order random differential equations with periodic boundary conditions,” Electronic Journal of Differential Equations, vol. 235, 2012.