Geometrical Structures of M-Theory

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Abstract

N=(2,1) heterotic string theory provides clues about hidden structure in M-theory related to string duality; in effect it geometrizes some aspects of duality. The program whereby one may deduce this hidden structure is outlined, together with the results obtained to date. Speculations are made as to the eventual shape of the theory. Talk presented at Strings '96 (Santa Barbara, July 20-25, 1996).

1 Introduction

N=(2,1) heterotic strings seem to know deeply about the duality structure of M-theory. In recent work [1, 2], it has been shown that N=(2,1) strings realize in their target space all critical string/membrane worldvolume dynamics as different background geometries. The overarching structure is 2+2 dimensional self-dual geometry, from which the string/membrane dynamics follows by (null) dimensional reduction. The conceptual framework is pictured in figure 1.

1In fact, one obtains even more than one bargained for – not only are IIA/IIB strings, 11D supermembranes, and heterotic/type I strings obtained [3], but also what appear to be bosonic strings/membranes [4], the N=(2,2) string, and the N=(2,1) string itself [5]! While I consider this fascinating, I will mostly confine myself to spacetime supersymmetric vacua in this talk.
Figure 1. The route from $N=(2,1)$ heterotic strings to M-theory.

The background fields of the $(2,1)$ string describe the embedding of a 2+2 self-dual worldvolume into a 10+2 spacetime; specifically, one finds the 2+2 worldvolume stretched over some hyperplane, described in static gauge. Rather than focusing on the particular properties of any particular null reduction, we would like to describe the general features of these 2+2 ‘M-branes’, and what we might learn about the 10+2 dimensional theory they couple to.

There is in fact a concrete procedure to attain this goal, also indicated in figure 1. The embedding fields of the M-brane (or rather, fluctuations in them) are the background fields of the $(2,1)$ string sigma-model. Thus the action governing the 2+2 target space dynamics is the one that generates the $N=(2,1)$ sigma model beta-function equations, as well as the $(2,1)$ string S-matrix. This step has recently been completed in collaboration with D. Kutasov [4]. The result is an intriguing mixture of the Dirac-Born-Infeld (DBI) action for D-branes and the super $p$-brane action: The exact classical bosonic action is of DBI form:

$$S_{2+2} = \int d^4x \sqrt{\det[\eta_{ij} + f_{ij} + \partial_i \phi^a \partial_j \phi^a]}.$$  

(1)

A preliminary analysis of the target space fermion couplings indicates a supersymmetric completion of the sort familiar from super $p$-branes:

$$\partial_i \phi^a \rightarrow \partial_i \phi^a + \bar{\psi} \gamma^a \partial_i \psi \equiv \Pi^a_i$$

$$f_{ij} \rightarrow f_{ij} + \bar{\psi} \gamma_{(i} \partial_{j)} \psi \equiv \Pi_{ij},$$  

(2)
together with some sort of Wess-Zumino terms in $S_{2+2}$. We will describe the derivation of these results in section 2.

Recall that the $p$-brane action has both global spacetime supersymmetry and $\kappa$-symmetry. If one did not know what supergravity theory couples to the eleven-dimensional supermembrane, for example, one could deduce it by asking for the most general possible supermembrane action compatible with $\kappa$-symmetry. This then is the program: find the $\kappa$-symmetric, supersymmetric, reparametrization invariant action that reduces to $S_{2+2}$ in static gauge; then look for deformations of the theory compatible with this structure.

The organization of the talk is as follows: In section 2, we sketch the derivation of the target space action (1). Section 3 discusses the symmetry properties of the 2+2 worldvolume theory: its supersymmetry, $\kappa$-symmetry, self-duality, current algebra, etc. Section 4 contains some brief remarks about the quantization of the 2+2 theory; here analogies with matrix models of noncritical string theory might prove helpful. Section 5 returns to the issue of supersymmetry, and how the (2,1) string might point towards ‘$p$-brane democracy’ at short distances. Section 6 is an attempt to discern the ultimate structure of M-theory by extrapolating all these geometrical structures to their logical conclusion.

2 The 2+2 target space action

There are two routes to the target space action of a string theory, each giving complementary information. First, the action which generates the sigma-model beta-functions contains, at a given loop order, all orders in fields with a fixed number of derivatives. Second, the generating function of the string S-matrix gives all orders in derivatives with a fixed number of fields. Both approaches yield crucial information about the (2,1) string.

Let us begin with the sigma-model. The (2,1) heterotic sigma-model is most conveniently described in N=(2,0) superfields. These are of two types: bosonic chiral/antichiral superfields $X^i = x^i + \theta R x^i + \ldots$ and $X^i = x^i + \tilde{\theta} R x^i + \ldots$ which are complex coordinates on the 2+2 target space; and fermionic chiral/antichiral superfields $\Lambda^{\alpha} = \lambda^{\alpha} + \theta R F^{\alpha} + \ldots$ and $\Lambda^{\alpha} = \lambda^{\alpha} + \tilde{\theta} R F^{\alpha} + \ldots$ which give a fermionic description of the ‘internal space’ of the heterotic
The sigma-model action is

\[ S_{\sigma\text{-model}} = \int d^2zd^2\theta_R[K_\mu(X,\bar{X})\partial X^\mu + h_{ab}(X,\bar{X})\Lambda^a\Lambda^b], \quad (3) \]

where \( \mu = i, \bar{i} \) and \( a = \alpha, \bar{\alpha} \) collect together the complex indices. Expanding in components, one finds a conventional heterotic sigma-model, with

\[
\begin{align*}
g_{ij} &= \partial_i K_j - \partial_j K_i & g_{ij} &= 0 \\
b_{ij} &= \partial_i K_j + \partial_j K_i & b_{ij} &= 0 \\
A_i &= h^{\frac{1}{2}} \partial_i h^{-\frac{1}{2}} & A_{\bar{i}} &= (h^{-\frac{1}{2}}) \partial_{\bar{i}} h^{\frac{1}{2}}
\end{align*}
\]

i.e. the fields are written in terms of prepotentials. Note that the expression for \( A_\mu \) is simply the Yang ansatz for self-dual gauge fields. The left-moving N=1 supersymmetry is not manifest in this formalism. It will result in a relation between the background metric and gauge fields; henceforth, when we refer to the heterotic gauge field we will mean the one for the ‘internal space’, suppressing that part which is directly determined by the metric via the left supersymmetry. A fact that will be important later is that conformal (2,0) sigma-models in the critical dimension \( D = 4 \) automatically have (4,0) supersymmetry, given by a triplet of complex structures \( I^{(r)}_\mu \), \( r = 1, 2, 3 \), obeying the algebra of imaginary quaternions:

\[ I^{(r)} I^{(s)} = -\delta^{rs} + \epsilon^{rst} I^{(t)}. \]

Thus the target space is hyperKähler (with torsion).

### 2.1 \( \beta \)-functions

Let us temporarily ignore the gauge field couplings in \( S_{\sigma\text{-model}} \), equation (3). The one-loop beta-functions for the purely gravitational sector are then

\[ R_{\mu\nu}(\Gamma) = \nabla_\mu \nabla_\nu \Phi, \quad (6) \]

where \( \Gamma^\mu_{\nu\lambda} = \{^\mu_{\nu\lambda}\} - \frac{1}{2} H^\mu_{\nu\lambda} \) is the connection with torsion. This system of equations may be integrated \[5\] to find (after a holomorphic coordinate transformation)

\[
\begin{align*}
e^{-2\Phi} \det[g] &= 1 \\
\Gamma_\mu &\equiv \Gamma^\nu_{\lambda\mu} I^\lambda_\nu = 0. \quad (7)
\end{align*}
\]
The first equation determines the dilaton $\Phi$ algebraically in terms of the metric. The second equation gives the dynamics of the Kähler vector potential; it arises from the variational principle

$$S_{2+2}^{\text{grav}} = \int d^4x \sqrt{\det[g_{i\bar{j}}]}.$$  

(8)

Now we add the heterotic gauge sector. This comes from a coupling to chiral fermions $\lambda^a$ in the sigma-model, and results in the standard shifts (again to one loop)

$$\Gamma \longrightarrow \{\text{Christoffel}\} - \frac{1}{2}(H + \omega_{3}^{\text{YM}})$$

$$g_{i\bar{j}} \longrightarrow g_{i\bar{j}} + A^a_i A^a_{\bar{j}}.$$  

(9)

Note that there is no Lorentz Chern-Simons contribution to $\Gamma$, because the N=1 left-moving supersymmetry pairs four $\lambda^a$ with the four $x^\mu$ and cancels this part of the anomaly which comes from the right-moving $\chi^\mu$. Since the expression 4 for $g_{i\bar{j}}$ in terms of the Kähler vector potential $K_{\mu}$ admits an abelian gauge invariance, we may write the expansion of the metric around flat space as $g_{i\bar{j}} = \eta_{i\bar{j}} + f_{i\bar{j}}$, with $f_{i\bar{j}}$ the “field strength” of the Kähler vector potential.

Taking the Yang-Mills gauge group $[U(1)]^8$ relevant to spacetime supersymmetric vacua of the (2,1) string (i.e. $A^a_i = i \partial_i \phi^a$, $A^a_{\bar{i}} = -i \partial_{\bar{i}} \phi^a$), we achieve the result 4

$$S_{2+2} = \int d^4x \sqrt{\det[\eta_{i\bar{j}} + f_{i\bar{j}} + \partial_i \phi^a \partial_{\bar{j}} \phi^a]}.$$  

(10)

The action has the form of the Dirac-Born-Infeld action for D-branes, but the determinant is \textit{two-dimensional}, not four-dimensional! This is exactly what one would ask for the dimensional reduction to two dimensions that gives target space strings, where for example one may set $x^4 = x^5$. However, one wonders what the dimensional reduction to three dimensions has to do with the DBI action of the two-brane and its three-dimensional determinant. In 2, it was shown that to leading nontrivial order, $S_{2+2}$ reduced to D=2+1 agrees with the two-brane DBI action after a coordinate transformation and a field redefinition. It would be interesting to understand the precise relation.
2.2 N=(2,1) string S-matrix

So far, the expression (10) for $S_{2+2}$ is a one-loop result. For D-branes, the DBI action suffers higher-order corrections in $\alpha'$. However, analysis of the (2,1) string S-matrix shows that $S_{2+2}$ is exact.

The only nonvanishing S-matrix element of any N=2 string is the three-point function, due to the incompatibility of Regge behavior and the finite number of physical states. The three-point function of the (2,1) string has the schematic form

\[ \bullet = \quad \sim k^3 \xi^3 \]

i.e. three fields $\xi$ and three powers of momenta. The vanishing of the four-point function implies

\[ \bigotimes = 0 = \quad + \text{two more} \quad + \]

The nonlocal terms cancel among the $s$, $t$, and $u$ channel processes due to the special features of the vertices and 2+2 kinematics. The leftover local term has the scaling $k^3 \cdot \frac{1}{k^3} \cdot k^3 \sim k^4$, and must be cancelled by a contact term of the form $k^4 \xi^4$. Proceeding in this way, one finds the $n$-point function has the structure $k^n \xi^n$. But this is exactly what one finds in the expansion of $S_{2+2}$. A higher loop term in the beta-function would imply terms in the S-matrix of the form $k^m \xi^n$ for $m > n$, which is not seen. We conclude that

\[ \text{The (2,0) string also has a term } k^5 \xi^3, \text{ reflecting the presence of a Lorentz Chern-Simons term. This term invalidates (for the (2,0) string) the power-counting argument which follows.} \]
(10) is the exact bosonic target space effective action. In order to verify these general arguments, the three- and four-point functions have been computed explicitly [4] and indeed agree with $S_{2+2}$.

2.3 Fermion terms

We have conducted a preliminary analysis of the fermion (Ramond sector) couplings of the (2,1) string. These are notoriously difficult to analyze using sigma-model techniques (although the recently developed Green-Schwarz-Berkovits formalism may be helpful here [4, 7]); therefore, we have analyzed some of the three- and four-point S-matrix elements. Much of the result appears to be summarized in the shifts

$$\partial_i \phi^a \rightarrow \partial_i \phi^a + \bar{\psi} \gamma^a \partial_i \psi \equiv \Pi_i^a$$

$$f_{i \bar{j}} \rightarrow f_{i \bar{j}} + \bar{\psi} \gamma(i \partial_{\bar{j}}) \psi \equiv \Pi_{i \bar{j}}.$$ (11)

This is in fact just what one expects for a super $p$-brane. Recall the covariant super $p$-brane action

$$S_{p\text{-brane}} = \int d^{p+1}x \sqrt{\det[\Pi_\mu^a \Pi^a_\mu]} + \epsilon^{\mu_1 \ldots \mu_{p+1}} \left[ \Pi^a_{\mu_1} \ldots \Pi^a_{\mu_p} \bar{\psi} \gamma^{a_1 \ldots a_p} \partial_{\mu_{p+1}} \psi + \ldots \right],$$ (12)

where $\Pi_\mu^a = \partial_\mu \phi^a - \bar{\psi} \gamma^a \partial_\mu \psi$, i.e. $(\phi, \psi)$ are superspace coordinates, and $\Pi_\mu^a$ is a tangent vector to superspace. The first term in (12) is the superspace volume element; combining (10), (11), one sees essentially the same structure for the M-brane, modulo the issue of 3D vs. 2D determinants in the M-brane action discussed above. The second term in (12) is a superspace Wess-Zumino term, and is required for $\kappa$-symmetry to hold. It is not seen when the fermions are set to zero, and hence would not have been found in the sigma-model approach above. In the S-matrix approach, one looks for parity-violating terms to establish this structure.

In the three-point function, one sees a term

$$\int d^4x [f_{ij} (\bar{\psi} \gamma^i \partial_{\bar{j}} \psi) + h.c] \epsilon^{ij} \epsilon^{\bar{k} \bar{l}}$$ (13)

not accounted for by the Dirac-Born-Infeld term. If for a moment we relax the requirement of spacetime supersymmetry, a nonabelian heterotic gauge
sector of the (2,1) string reveals a Wess-Zumino coupling of the transverse degrees of freedom [8]

\[ \int_{\partial M_5 = M_4} \delta^5 x \ I \wedge tr[(h^{-1}dh)^3] , \]  

(14)

where \( I \) is the Kähler form. Thus one sees that Wess-Zumino terms are generic, and that there is the possibility of the right sort of Wess-Zumino terms appearing in the M-brane action so that it admits a \( \kappa \)-symmetry. It is important to complete the determination of the WZ terms – they are the expectation values of superspace antisymmetric tensor fields, and their evaluation will go a long way towards determining what sort of spacetime fields couple to the M-brane.

Finally, it is curious to note that a remarkable transmutation has taken place in \( S_{2+2} \) [10]. One begins with gauge fields \( g_{\mu \nu}, b_{\mu \nu}, A^a_\mu \); imposing the self-duality constraints of global (4,0) supersymmetry, the effective action is written in terms of the potentials \( K_\mu, \phi^a \) in a gauge fixed, yet still remarkably geometrical form. Similarly, the fermions start life as a gravitino field, and end up as superspace coordinates!

### 3 Symmetries, self-duality, and integrability

In this section we will discuss the properties, both known and expected, of \( S_{2+2} \). We begin with an expected symmetry, \( \kappa \)-symmetry; then consider a known one, self-duality, and its associated symmetries and conservation laws.

#### 3.1 Supersymmetry and \( \kappa \)-symmetry

The (2,1) string vertex algebra has manifest linear target space supersymmetry [4], which is the dimensional reduction from ten or twelve dimensions down to the four-dimensional M-brane world volume:

\[
\delta K_\mu \sim \bar{\epsilon} \gamma_\mu \psi \\
\delta \phi^a \sim \bar{\epsilon} \gamma^a \psi \\
\delta \psi \sim (f_{\mu \nu} \gamma^{\mu \nu} + \partial_\mu \phi^a \gamma^{\mu \nu} \gamma^a) \epsilon .
\]  

(15)

There are higher-order corrections to these transformation laws (c.f. [10] for a similar situation) due to the nonlinearity of \( S_{2+2} \). These transformations
are to be compared with the $\kappa$-symmetry and (nonlinearly realized \[11\]) supersymmetry of super $p$-branes (c.f. \[12\])

$$
\delta \psi = \eta + (1 + \gamma_{p+2}) \kappa
$$

$$
\delta \phi^a = \bar{\epsilon} \gamma^a \eta + \bar{\psi} \gamma^a (1 + \gamma_{p+2}) \kappa.
$$

(16)

In simpler situations, such as the Green-Schwarz string (c.f. \[13\]) and the supermembrane in D=4 \[14\], it is known that linear supersymmetry like (15) appears as a specific combination of $\eta$- and $\kappa$-supersymmetries when working in a physical gauge. This gives a strong indication that such a symmetry underlies a covariant formulation of the M-brane. We also see that the M-brane action reduced to 2+1 dimensions and the supermembrane action have the same spectrum of spontaneously broken translations and supersymmetries (in fact, one can define the $p$-brane action as the nonlinear lagrangian for partially broken global supersymmetry \[11\]), and therefore must be related by some combination of field redefinitions and field-dependent coordinate transformations (although it is not ruled out that something more indirect, such as a Backlund transformation, is involved).

### 3.2 Self-duality, integrability, and conservation laws

As mentioned above, the gauged U(1) of the right-moving N=2 local supersymmetry in the (2,1) string sits inside a global N=4 hyperKähler structure. The integrable nature of the theory is elucidated by considering the family $\mathbb{CP}^1 = SU(2)/U(1)$ of complex structures, thus passing to the twistor space $\mathcal{T}$, the bundle $\mathbb{CP}^1 \to \mathcal{M}_4$ whose coordinates $u \in \mathbb{CP}^1$, $z^i_u = z^i + u \epsilon^{ij} z^j \in \mathcal{M}_4$ define a natural complex structure. For instance, in self-dual Yang-Mills (SDYM), given a transition function $g \in G$ on the equator of $\mathbb{CP}^1$, one may factorize $g = f_+^{-1} f_-$, where $f_+(u, z^i_u)$ is holomorphic in the upper hemisphere and $f_-(u, z^i_u)$ is holomorphic in the lower hemisphere. Then one reconstructs the gauge field on $\mathcal{M}_4$ from this holomorphic bundle over $\mathcal{T}$ via $A = -(\bar{\partial} h) h^{-1}$, where $h = f_+(u = 0, z^i)$.

The twistor space relevant to (4,0) sigma-model geometry was recently constructed by Howe and Papadopolous \[15\]. However, the details of the twistor transform remain to be worked out. In principle, the complete classical physics of waves on the M-brane is under control, and it is an interesting question what the classical solutions look like.

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\[11\] In 4+0 signature; in 2+2 signature one has the disc $SU(1,1)/U(1)$. 

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Another interesting feature of the twistor construction for SDYM is that it naturally involves the complexification of the target space (one works over $G_{\mathbb{C}}$, with a gauge symmetry that reduces the dynamics in the end to a real slice). We have also seen a complexification of the world sheet in $S^{2+2}$ (a different sort of ‘complexification’ was investigated in [11]). These properties are intriguing given that Witten [17] invoked complexification of both worldsheet and spacetime in order to explain certain features of string behavior at high temperature and high energy. It may be that (2,1) strings provide a window into this regime.

Gursey and collaborators [18] have drawn analogies between 2D conformal symmetry as complex analyticity, and 4D self-duality as a kind of ‘quaternionic analyticity’. Indeed, many of the formulae of the ADHM construction of solutions to SDYM are most conveniently written in terms of quaternions. We will exploit this connection in section six when we speculate on the ultimate structure of M-theory.

Losev et.al. [19, 20] have pointed out the strong parallel between the twistor construction and the holomorphic factorization property of rational conformal field theory. Indeed, the SDYM equations have a symmetry under $h(x, \bar{x}) \rightarrow f_L(\bar{x})h(x, \bar{x})f_R(x)$ related to a 4D generalization of the Polyakov-Wiegmann identity

$$\Gamma[gh] = \Gamma[g] + \Gamma[h] - \frac{i}{2\pi} \int_{M_4} I \wedge \text{tr}[g^{-1}\partial g \cdot \bar{\partial} h^{-1}],$$

leading to corresponding current algebra and Ward identities. The gravitational action $\int d^4x \sqrt{\det[g_{ij}]}$ has a similar residual symmetry under $g_{ij} \rightarrow \partial_i f^k(x)g_{kj}(\bar{f}, \bar{x})\partial_j f^\ell(\bar{x})$ with $|\det \partial_i f^j| = 1$, i.e. area-preserving holomorphic diffeomorphisms; this is an analogue of conformal invariance in the present situation. We again see that M-brane geometry is an extremely natural generalization of string worldsheet geometry. These symmetries generate infinite towers of conservation laws which will help constrain the quantum theory.

If self-dual Yang-Mills generalizes rational conformal field theory to a 4D context, what about other string backgrounds? We now know of vast classes of nontrivial fixed points in 4D field theory, e.g. [21]. One ought to be able to suitably twist these to construct self-dual theories; one might obtain the analogue of Landau-Ginsburg models [22] in this way.
4 Quantization

While we have not thought deeply about the quantization of M-branes, it seems evident that a quantum theory exists, since the dynamics is essentially trivial – the theory is integrable, the S-matrix more or less vanishes, and of course N=(2,1) strings manifestly generate a perturbative quantization. However, the quantum theory of N=(2,1) strings is not necessarily a field theory. The first indication of this comes from the power-counting of N=2 string loops, which is two-dimensional (instead of four-dimensional) due to an extra factor of the Schwinger parameter \( \tau \) coming from the normalization of the N=2 string U(1) modulus. Another indication is that momenta in the internal directions (if we bosonize the \( \lambda^a \) to make the \( E_8 \) torus) flows off-shell in loops; in fact, once one of the 2+2 target coordinates is compactified on a circle, a rich spectrum of physical states arises \([4]\), whose interpretation is at the moment obscure. These are the analogues for the (2,1) string of the perturbative BPS states of the usual (1,0) heterotic string; when both momenta and winding are present on the circle, level matching permits the right-moving N=2 sector to remain in its ground state while the left-moving N=1 sector has essentially arbitrary excitation. Also, in the case where only winding is present on the circle, one gets a second copy of the target space M-brane \([1, 16]\).

There are some intriguing analogies between the (2,1) string on the one hand, and matrix models of noncritical strings \([23]\) on the other (see figure 2). Two-dimensional noncritical string theory is almost a field theory of the center-of-mass (“tachyon”) degree of freedom, but not quite. For instance, when compactified on a circle, there are both ‘momentum tachyons’ and ‘winding tachyons’; the partition function has an \( R \to 1/R \) symmetry. As we have just seen, the (2,1) string also has this sort of doubling.

| Matrix models/noncritical strings | N=2 strings |
|----------------------------------|-------------|
| tachyon field theory             | \( S_{2+2}[\Phi] \) |
| cubic collective field theory    | \( W[J] = S_{2+2} - J \cdot \Phi \) |
| matrix model Fermi fluid         | ??          |

Figure 2. Analogies between noncritical and N=2 strings.

There is a nonpolynomial field theory for the noncritical string tachyon field (the analogue of \( S_{2+2} \)). This field theory has a representation in terms of a
cubic collective field theory, which is obtained via an integral transform of
the tachyon field and encodes much of the symmetry and integrability. The
analogue for the $(2,1)$ string might be the Legendre transform to sources,
which would be cubic on-shell since all of the $n$-point amplitudes vanish for
$n \geq 4$; or perhaps the twistor transform, which linearizes the theory. For
the noncritical string, neither of these presentations is appropriate to the
full quantum theory, however, which is best written in terms of the fermi
fluid of the matrix model. Thus one wonders whether there might be such a
simpler presentation appropriate to the quantization of $(2,1)$ strings. Since
the classical theory is solved on twistor space, presumably one should look for
some sort of matrix dynamics whose semiclassical limit is naturally expressed
in terms of twistors, and whose collective excitations are $(2,1)$ strings.

Of course, an alternate interpretation is that $N=(2,1)$ string theory has
some overlap with M-theory but is not fundamental to it. One might regard
the above observations as evidence to that effect: The doubling of the spec-
trum, the BPS-like tower, the fact that the internal space is restricted to be
the $E_8$ torus, etc. The constructions of [1, 2] realize a particular special class
of states in particular M-theory backgrounds, but certainly do not generate
all of the theory. Perhaps $N=(2,1)$ strings know about M-theory because they
know about (a) spacetime supersymmetry, and (b) self-dual worldvolumes.
This would lead us to ignore all this excess baggage carried by $(2,1)$ strings,
and to attempt to quantize directly the 2+2 field theory $S_{2+2}$. I regard the
developments of [19, 20] as promising steps in this direction.

5 Twelve dimensions (i.e. 10+2)

The spacetime supersymmetry algebra of the $(2,1)$ string is generated by
world sheet charges [1, 2] which schematically take the form

$$Q_\alpha = \oint dz \Sigma_{\text{ghost}} S_\alpha ,$$

and obey the algebra

$$\{Q_\alpha, Q_\beta\} = \oint \Sigma_{gh} \lambda^a \lambda^b (\gamma^{ab})_{\alpha\beta} .$$

(19)

Here $S_\alpha$ is an O(10,2) Majorana-Weyl spinor, the spin field of the heterotic
fermions $\lambda^a$; thus $\lambda^a \lambda^b$ is the O(10,2) generator on this part of the theory.
However, the theory is not O(10,2) invariant; the BRS constraints impose the null reduction $\not\!\! Q = 0$ for some null vector $v$. In the supersymmetry algebra (19) this forces one of the polarizations $a,b$ on the RHS to point along the null vector; the remaining operator is a picture change of the momentum. Thus the physical states obey

$$\{Q_\alpha, Q_\beta\} = P^a v^b (\gamma^{ab})_{\alpha\beta} .$$

(20)

The RHS realizes a particular state in the natural 10+2 supersymmetry algebra [24]

$$\{Q_\alpha, Q_\beta\} = M^{ab}(\gamma^{ab})_{\alpha\beta} + Z_{a_1 \cdots a_6}(\gamma^{a_1 \cdots a_6})_{\alpha\beta} ,$$

(21)

where the bosonic charges are evaluated as $M^{ab} = P^{[a} v^{b]}$, $Z_{a_1 \cdots a_6} = 0$. This algebra is to be contrasted with 10+1 supersymmetry

$$\{Q_\alpha, Q_\beta\} = P^a(\gamma^a)_{\alpha\beta} + M^{ab}(\gamma^{ab})_{\alpha\beta} + Z_{a_1 \cdots a_5}(\gamma^{a_1 \cdots a_5})_{\alpha\beta} .$$

(22)

Comparing this algebra with (21), one sees that the 11 $P^a$ and 55 $M^{ab}$ of 10+1 dimensions have been “unified” into the 66 $M^{ab}$ of 10+2 dimensions (the 11D five-form and 12D self-dual six-form both have 462 components). Since $P^a$ couples to the metric $e_{\mu a}$ while $M^{ab}$ couples to the antisymmetric tensor gauge field $A_{\mu ab}$, an unbroken realization of this symmetry would unify all electric charges. Such a unification might explain why scalars coming from the antisymmetric tensor and the metric form multiplets of a larger symmetry (U-duality) in lower dimensions (including exceptional groups; see below). Actually, the unbroken algebra (21) would imply a complete ‘p-brane democracy’ [25], since the algebra (21) is OSp(1|32); the commutator of the $M^{ab}$ closes on the $Z_{a_1 \cdots a_6}$. Clearly such a structure is incompatible with flat spacetime, and is not seen in the N=2 string. Yet it is tempting to ask whether O(10,2) is related to the conformal group, spontaneously broken to O(9,1). Then the Planck scale would be dynamically generated (and perhaps related to the N=(2,1) string tension?), and the high energy behavior governed by scale-invariant physics and the unbroken realization of (21). Indeed, quantum cohomology provides a clue that the distinction between different degree forms disappears at short distances (e.g. K3 moduli space in string theory is O(20,4), not just the O(19,3) that acts on $H^2(\mathbb{Z})$; the extra generators mix zero-, two-, and four-forms for small volume K3’s).
6 12+4 and beyond

The question arises whether there is any limit to adding hidden dimensions to a theory. I think the N=(2,1) string reveals sufficient additional structure to provide an answer to this question. The quest for unification has seen a progression from particle world lines (\(\mathbb{R}\) analyticity) of quantum field theory, to string world sheets (\(\mathbb{C}\) analyticity) in order to incorporate gravity, and now M-brane worldvolumes (\(\mathbb{H}\) analyticity) in order to realize duality. Clearly there is room for one further step, to octonionic (\(\mathbb{O}\)) analyticity. Just as supersymmetry picks out transverse dimensions \(\nu = 1, 2, 4, 8\) corresponding to the division algebras \(\mathbb{K}_\nu = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\) – we are now exploring symmetries of longitudinal directions. In fact, there is a ‘magic square’ of orthogonal algebras \(^{26}\) (based on \(H_2(\mathbb{K}_\nu) = 2 \times 2\) Hermitean matrices over \(\mathbb{K}_\nu\))

| Transverse \(\mathbb{R}\) | \(\mathbb{C}\) | \(\mathbb{H}\) | \(\mathbb{O}\) |
|--------------------------|-----------|-----------|-----------|
| \(\mathbb{R}\)           | O(2)     | O(3)     | O(5)     | O(9)     |
| \(\mathbb{C}\)           | O(2,1)   | O(3,1)   | O(5,1)   | O(9,1)   |
| \(\mathbb{H}\)           | O(3,2)   | O(4,2)   | O(6,2)   | O(10,2)  |
| \(\mathbb{O}\)           | O(5,4)   | O(6,4)   | O(8,4)   | O(12,4)  |

These groups are constructed in the same manner as the usual Tits-Freudenthal magic square, which uniformly constructs the exceptional groups using \(H_3(\mathbb{K}_\nu) = 3 \times 3\) Hermitean matrices over \(\mathbb{K}_\nu\):

| Longitudinal \(\mathbb{R}\) | \(\mathbb{C}\) | \(\mathbb{H}\) | \(\mathbb{O}\) |
|---------------------------|-----------|-----------|-----------|
| \(\mathbb{R}\)           | SO(3)    | SU(3)    | Sq(3)    | F_4       |
| \(\mathbb{C}\)           | SL(3,\mathbb{R}) | SL(3,\mathbb{C}) | SL(3,\mathbb{H}) | E_6       |
| \(\mathbb{H}\)           | Sp(6,\mathbb{R}) | SU(3,3)   | Sp(6,\mathbb{H}) | E_7       |
| \(\mathbb{O}\)           | F_4      | E_6      | E_7      | E_8       |

Given the intimate association of the exceptional groups to octonions, it is quite possible that octonionic worldvolume structures could provide an explanation for the appearance of exceptional groups in U-duality.

There are a few indications of 12+4 structure in the (2,1) string. First, the construction of the (2,1) string worldsheet as the target space of the (2,1) string turns out to have N=8 global right-moving supersymmetry, not

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\(^{4}\)The ideas presented here were initiated in discussions with P.G.O. Freund.
just $N=4$. Second, Moore [27] has suggested that maximal symmetry in string theory arises when treating left- and right-movers independently, via Narain compactification of all dimensions. Applying this reasoning to the (2,1) string leads to the Narain group $O(12,4)$ from 10+2 left-movers, and 2+2 right-movers. This symmetry becomes apparent in the regime where all radii are of order the string scale of the underlying (2,1) string, lending further support to the idea of symmetry restoration at short distances.

Finally, one will need additional degrees of freedom to describe the M-brane in a reparametrization invariant way. The reparametrization invariant 11D supermembrane action utilizes 10+1 scalars (8 transverse and 2+1 longitudinal). Dualizing a transverse scalar on the worldvolume yields 9+1 scalars (7 transverse and 2+1 longitudinal) and the 2+1 components of a vector [28]. This parametrization is the one generated by the $N=(2,1)$ string, albeit in static gauge where the 2+1 longitudinal scalars have been eliminated by gauge-fixing. A covariant description of the M-brane before reparametrization gauge fixing and null reduction would require 10+2 scalars (8 transverse and 2+2 longitudinal) as well as the 2+2 components of a vector. If present, octonionic symmetry would have to relate the vector and the scalars; however this is just what is called for if one is to realize the $O(10,1)$ symmetry of 11D supergravity.

Additional spacelike coordinates may be required to match expected hidden structure of the IIB theory [29, 30], which seems to require 11+1 signature. Recall that our construction realizes both the IIB string and the IIA two-brane within the moduli space of (2,1) strings [1, 2]. How can we get the IIB string, with its supercharges both 16’s of the same chirality in $O(9,1)$, when the supersymmetry charges (18) form a single 32 of $O(10,2)$? The answer lies in the fact that we are in static gauge, where only half of the supersymmetry is linearly realized; it turns out that, for both the supermembrane and the IIB string, the unbroken supercharges in static gauge are compatible with the null reduction of an $O(10,2)$ spinor. However, a covariant formulation would have to accommodate both IIA and IIB superalgebras; $O(12,4)$ spinors have plenty of room (the fermionic partners of the 12+4 bosons would fill a 128 of $O(12,4)$). Related to this is the fact that the eleventh dimension of IIA supergravity comes in our construction from the field space of the vector; the ten spatial dimensions of 11D supergravity are not the same as the ten spatial dimensions of 10+2, one of which is always eliminated in the null reduction.
7 Conclusions

The construction of M-branes from N=(2,1) heterotic strings encodes key aspects of duality already in the classical theory, indicating that we are approaching the underlying degrees of freedom of the theory. In particular, more of the symmetry is manifest. The worldvolume structure is a ‘complexification’ of string worldsheets whereby 4D self-duality (H analyticity) supplants 2D conformal invariance (C analyticity) as the world-volume symmetry principle. There are a few indications that an extension to octonionic analytic structures might be possible. I expect that M-branes will prove to be as useful a probe of structure in M-theory as D-branes and p-branes have. We do not know how to quantize the latter objects either, yet they have been immensely helpful in our understanding of geometry, duality, and nonperturbative physics. It is premature to say whether and how M-branes are the ‘fundamental’ objects of M-theory. Even so, their many magical properties warrant further investigation.

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Note added: At the conference, C. Hull informed me that he had obtained the gravitational part of $S_{2+2}$; I. Bars [31] presented a framework unifying both IIA and IIB superalgebras, starting from 11+2 dimensions. He also found the realization of the 10+2 superalgebra in equation (29). L. Susskind presented work relating M-theory to SU(N) matrix mechanics along the lines of [32, 33], giving arguments that the standard 11D supermembrane is obtained as a collective excitation in a particular semiclassical limit. G. Chapline pointed out his work relating fermi fluids and self-dual gravity [34], and other work related to [20]. After the conference, work of Jevicki has
appeared relating self-dual gravity to large N matrix mechanics; and Nishino and Sezgin also have investigated the realization of 10+2 supersymmetry.

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