Some congruences related to harmonic numbers and the terms of the second order sequences

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Abstract. In this paper, with helps of some combinatorial identities, we investigate various basic congruences involving harmonic numbers and terms of the second order sequences \( \{U_{kn}\} \) and \( \{V_{kn}\} \).

1. Introduction

The second order sequence \( \{W_n (c, d; r, s)\} \), or briefly \( \{W_n\} \), is defined for \( n > 0 \) by

\[
W_{n+1} = rW_n + sW_{n-1}
\]

in which \( W_0 = c, W_1 = d \), where \( c, d, r, s \) are arbitrary integers. As some special cases of \( \{W_n\} \), denote \( W_n (0, 1; r, 1) \), \( W_n (2, r; r, 1) \) by \( U_n \) and \( V_n \), respectively.

When \( r = 1 \), \( U_n = F_n \) (the \( n \)th Fibonacci number) and \( V_n = L_n \) (the \( n \)th Lucas number).

If \( \alpha \) and \( \beta \) are the roots of the equation \( x^2 - rx - 1 = 0 \), the Binet formulas of the sequences \( \{U_n\} \) and \( \{V_n\} \) have the forms

\[
U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,
\]

respectively.

From [2, 3], E. Kılıç and P. Stanica derived the following recurrence relations for the sequences \( \{U_{kn}\} \) and \( \{V_{kn}\} \) for \( k \geq 0, n > 0 \). It is clearly that

\[
U_{k(n+1)} = V_k U_{kn} + (-1)^{k+1} U_{k(n-1)},
\]

\[
V_{k(n+1)} = V_k V_{kn} + (-1)^{k+1} V_{k(n-1)},
\]

where the initial conditions of the sequences \( \{U_{kn}\} \) and \( \{V_{kn}\} \) are 0, \( U_k \), and 2, \( V_k \), respectively. Binet formulas of the sequences \( \{U_{kn}\} \) and \( \{V_{kn}\} \) are

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\[ U_{kn} = \frac{\alpha^{kn} - \beta^{kn}}{\alpha - \beta} \quad \text{and} \quad V_{kn} = \alpha^{kn} + \beta^{kn}, \]

respectively. From the Binet formulas, one can see that \( U_{-kn} = (-1)^{kn+1} U_{kn} \) and \( U_{2kn} = U_{kn} V_{kn} \). Harmonic numbers are those rational numbers given by

\[ H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N} = \{1, 2, \ldots \}. \]

The first few harmonic numbers are 1, \( \frac{3}{2} \), \( \frac{11}{6} \), \( \frac{25}{12} \), \ldots.

For \( m \in \mathbb{Z}^+ \), harmonic numbers of order \( m \) are those rational

\[ H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}, \quad n \in \mathbb{N}. \]

For a prime \( p \) and an integer \( a \) with \( a \not\equiv p \), we write the Fermat quotient \( q_p(a) = (a^{p-1} - 1)/p \). Let \( \mathbb{Z} \) be the set of integers. \( \mathbb{Z}_p \) denote the set of those rational numbers whose denominator is not divisible by \( p \) and is called as the set of \( p \)-adic integer numbers. For an integer \( D \), \( \sqrt{D} \in \mathbb{Z}_p \) if \( \left( \frac{D}{p} \right) = 1 \) and \( \sqrt{D} \notin \mathbb{Z}_p \) if \( \left( \frac{D}{p} \right) = -1 \) in [6]. It is clearly that \( x^2 - x - 1 \) has two simple roots in \( \mathbb{Z}_p \) if and only if \( p \equiv \pm 1 \pmod{p} \).

In [7], Z.W. Sun and L.L. Zhao established arithmetic properties of harmonic numbers. For example, for any prime \( p > 3 \),

\[ \sum_{i=1}^{p-1} \frac{H_i}{i2^i} \equiv \frac{7}{24} p B_{p-3} \pmod{p^2}, \quad \sum_{i=1}^{p-1} \frac{H_{i,2}}{i2^i} \equiv -\frac{3}{8} B_{p-3} \pmod{p}, \]

where \( B_0, B_1, B_2, \ldots \) Bernoulli numbers.

In [1], A. Granville showed the congruence

\[ q(x) \equiv -G(x) \pmod{p}, \quad p > 3, \]

where \( q(x) = \frac{x^p - (x-1)^p - 1}{p} \) and \( G(x) = \sum_{i=1}^{p-1} \frac{x^i}{i} \).

In [4], H. Pan and Z. W. Sun showed the following lemma and proposition:

**Lemma 1.1.** Let \( p > 3 \) be a prime. Then

\[ \left( \frac{x^p + (1 - x)^p - 1}{p} \right)^2 \equiv -2 \sum_{i=1}^{p-1} \frac{(1 - x)^i}{i^2} \]

\[ -2x^{2p} \sum_{i=1}^{p-1} \frac{(1 - x^{-1})^i}{i^2} \pmod{p}. \]
Proposition 1.1. Let \( r \) and \( s \) be nonzero integers. For an odd prime \( p \) such that \( p \nmid rs \),

\[
(3) \quad \left( \frac{y_p - r^p}{p} \right)^2 \equiv -2r^2 \sum_{i=1}^{p-1} \frac{\gamma^i}{r^i i^2} - 2\delta^2p \sum_{i=1}^{p-1} \frac{\gamma^{2i}}{(-s)^i i^2} \pmod{p},
\]

\[
(4) \quad \left( \frac{y_p - r^p}{p} \right)^2 \equiv -2r\gamma^p \sum_{i=1}^{p-1} \frac{\gamma^i}{r^i i^2} - 2\delta^2p \sum_{i=1}^{p-1} \frac{r^i \gamma^i}{s^i i^2} \pmod{p},
\]

where \( y_n = W_n(2, r; r, -s) \) and \( \gamma, \delta \) are the two roots of the equation \( x^2 - rx + s = 0 \).

In this paper, we investigate the congruences involving harmonic numbers and terms of second order sequences \( \{U_{kn}\} \) and \( \{V_{kn}\} \). For example, for \( (\Delta_p) = 1 \),

\[
\Delta V_k^{(p-1)/2} \sum_{i=1}^{p} U_{2k(2i+1)} H_i \equiv \frac{1}{p} \left( (-1)^k \left( V_k^p V_{kp} - \Delta^{(p+1)/2} U_{kp} \right) - 4 \right) - 2q_p (2) V_{2k(p+1)} \pmod{p},
\]

and

\[
\sum_{i=1}^{p-1} \frac{V_k(p+i-1)}{V_k^i} H_{i,2} \equiv -\frac{(-1)^k}{2} \left( \frac{V_{kp} - V_k^p}{p} \right)^2 \pmod{p},
\]

where \( \Delta = V_k^2 + 4(-1)^{k+1} \), a prime number \( p > 3 \), and an integer \( k \) with \( p \nmid V_k \).

2. SOME LEMMAS

In this section, we need the following lemmas for further use.

Lemma 2.1. For \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \), we have the following sums:

\[
(5) \quad \sum_{j=1}^{n-1} x^j H_j = \frac{1}{1-x} \sum_{i=1}^{n-1} x^i - \frac{x^n}{1-x} H_{n-1},
\]

\[
(6) \quad \sum_{j=1}^{n-1} x^j H_{j,2} = \frac{1}{1-x} \sum_{i=1}^{n-1} \frac{x^i}{i^2} - \frac{x^n}{1-x} H_{n-1,2}.
\]

Proof. For the proof of (5), from the sum \( \sum_{i=0}^{n-1} x^i y^{n-i-1} = \frac{x^n - y^n}{x-y} \), we have

\[
\sum_{j=1}^{n-1} x^j H_j = \sum_{j=1}^{n-1} x^j \sum_{i=1}^{j} \frac{1}{i} = \sum_{i=1}^{n-1} \frac{1}{i} \sum_{j=i}^{n-1} x^j.
\]
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\[ \sum_{i=1}^{n} \frac{1}{i} \left( \sum_{j=0}^{n-1} x^j - \sum_{j=0}^{i-1} x^j \right) = \sum_{i=1}^{n} \frac{1}{i} \left( 1 - x^n - 1 - x^i \right) = \sum_{i=1}^{n} \frac{1}{i} \left( \frac{x^i - x^n}{1 - x} \right) = \sum_{i=1}^{n} \frac{1}{i} \left( \frac{x^i}{1 - x} \right) = \frac{1}{1 - x} \sum_{i=1}^{n-1} \frac{x^i}{i} - \frac{x^n}{1 - x} H_{n-1}, \]

as claimed. Similarly, the other result is proven. Thus, this ends the proof.

\[ \square \]

\textbf{Lemma 2.2.} For \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \), we have the following sums:

\[ \sum_{j=1}^{n-1} jx^j H_j = \frac{nx^n (x - 1) - x (x^n - 1) - x}{(x - 1)^2} H_{n-1} \]

\[ - \frac{x^n - x}{(x - 1)^2} + \frac{x}{(x - 1)^2} \sum_{i=1}^{n-1} \frac{x^i}{i}, \]

\[ \sum_{j=1}^{n-1} jx^j H_{j,2} = \frac{nx^n (x - 1) - x (x^n - 1) - x}{(x - 1)^2} H_{n-1,2} \]

\[ - \frac{1}{x - 1} \sum_{i=1}^{n-1} \frac{x^i}{i} + \frac{x}{(x - 1)^2} \sum_{i=1}^{n-1} \frac{x^i}{i^2}. \]

\textit{Proof.} For the first claim, from the sums

\[ \sum_{i=0}^{n-1} x^i y^{n-i-1} = \frac{x^n - y^n}{x - y}, \]

and

\[ \sum_{i=0}^{n-1} ix^i y^{n-i-1} = \frac{nx^n (x - y) - x (x^n - y^n)}{(x - y)^2}, \]

we write

\[ \sum_{j=1}^{n-1} jx^j H_j = \sum_{j=1}^{n-1} jx^j \sum_{i=1}^{j} \frac{1}{i} = \sum_{i=1}^{n-1} \frac{1}{i} \sum_{j=i}^{n-1} jx^j \]

\[ = \sum_{i=1}^{n-1} \frac{1}{i} \left( \sum_{j=0}^{n-1} jx^j - \sum_{j=0}^{i-1} jx^j \right) \]

\[ = \sum_{i=1}^{n-1} \frac{1}{i} \left( \frac{nx^n (x - 1) - x (x^n - 1)}{(x - 1)^2} \right) \]
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\[ \frac{ix^i (x - 1) - x (x^i - 1)}{(x - 1)^2} \]

\[ = \frac{nx^n (x - 1) - x (x^n - 1) - x H_{n-1}}{(x - 1)^2} \]

\[ - \frac{x^n - x}{(x - 1)^2} + \frac{x}{(x - 1)^2} \sum_{i=1}^{n-1} \frac{x^i}{i}, \]

as claimed. The other claim is similarly obtained. Thus, the proof is completed. \( \square \)

**Lemma 2.3.** For \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \), we have the following sums:

\[ \sum_{k=1}^{n-1} k^2 x^k H_k = \frac{x^n (nx - x - n)^2 + x}{(x - 1)^3} H_{n-1} \]

\[ - \frac{nx^n (x - 1) - 3x^{n+1} + x + 2x^2}{(x - 1)^3} - \frac{x (x + 1)}{(x - 1)^3} \sum_{i=1}^{n-1} \frac{x^i}{i}, \quad (9) \]

and

\[ \sum_{k=1}^{n-1} k^2 x^k H_{k,2} = \frac{x^n (nx - x - n)^2 + x}{(x - 1)^3} H_{n-1,2} \]

\[ - \frac{x^n - x}{(x - 1)^2} + \frac{2x}{(x - 1)^2} \sum_{i=1}^{n-1} \frac{x^i}{i} - \frac{x (x + 1)}{(x - 1)^3} \sum_{i=1}^{n-1} \frac{x^i}{i^2}. \]

**Proof.** Considering the sums

\[ \sum_{i=0}^{n-1} i x^i y^{n-i-1} = \frac{nx^n (x - y) - x (x^n - y^n)}{(x - y)^2}, \quad \sum_{i=0}^{n-1} x^i y^{n-i-1} = \frac{x^n - y^n}{x - y} \]

and

\[ \sum_{k=0}^{n-1} k^2 x^k y^{n-k-1} = \frac{x^n (nx - ny - x)^2 + xy}{(x - y)^3} - xy^n (x + y), \]

the proof is clearly given. \( \square \)

**Lemma 2.4.** Let \( p \) be an odd prime. For \( \left( \frac{\Delta}{p} \right) = 1 \),

\[ (p-1)/2 \sum_{i=1}^{U_{4ki}} \frac{U_{4ki}}{i} \equiv \frac{(-1)^k}{p} \left( -V_k^p U_{kp} + (\sqrt{\Delta})^{p-1} V_{kp} \right) \pmod{p}, \quad (10) \]

\[ (p-1)/2 \sum_{i=1}^{V_{4ki}} \frac{V_{4ki}}{i} \equiv \frac{4}{p} - \frac{(-1)^k}{p} \left( V_k^p V_{kp} - (\sqrt{\Delta})^{p+1} U_{kp} \right) \pmod{p}, \quad (11) \]
Lemma 2.5. Let \( \Delta = V_k^2 + 4 (-1)^{k+1} \) and Legendre symbol \( \left( \frac{x}{p} \right) \).

Proof. For the proof of (11), using the Binet formula of the sequence \( \{V_{kn}\} \) and taking \( \frac{\alpha^{2k}}{\beta^{2k}}, \frac{\beta^{2k}}{\alpha^{2k}} \) instead of \( x \) in \( \sum_{i=1}^{(p-1)/2} x_i \equiv \frac{2}{p} - \frac{(\sqrt{\Delta+1})^p - (\sqrt{\Delta-1})^p}{p} \mod p \)[5], where any p-adic integer \( x \). We get

\[
\sum_{i=1}^{(p-1)/2} \frac{V_{4ki}}{i} = \sum_{i=1}^{(p-1)/2} \frac{\alpha^{4ki}}{i} + \sum_{i=1}^{(p-1)/2} \frac{\beta^{4ki}}{i}
\]

\[
= \sum_{i=1}^{(p-1)/2} \frac{1}{i} \left( \frac{\alpha^{2ki}}{\beta^{2ki}} \right) + \sum_{i=1}^{(p-1)/2} \frac{1}{i} \left( \frac{\beta^{2ki}}{\alpha^{2ki}} \right)
\]

\[
= \frac{4}{p} - \frac{V_k^p - \left( \sqrt{\Delta} \right)^p}{p\beta^{kp}} - \frac{V_k^p - \left( -\sqrt{\Delta} \right)^p}{p\alpha^{kp}}
\]

\[
= \frac{4}{p} - \left( -1 \right)^k \frac{V_k^p}{p} - \left( \sqrt{\Delta} \right)^p - \left( -1 \right)^k \beta^{kp} \frac{V_k^p}{p} - \left( -\sqrt{\Delta} \right)^p
\]

\[
= \frac{4}{p} - \left( -1 \right)^k \frac{V_k^p}{p} \left( \alpha^{kp} + \beta^{kp} \right) + \frac{\left( \sqrt{\Delta} \right)^p}{p} \left( -1 \right)^k \left( \alpha^{kp} - \beta^{kp} \right)
\]

\[
= \frac{4}{p} - \left( -1 \right)^k \left( V_k^p V_{kp} - \left( \sqrt{\Delta} \right)^{p+1} U_{kp} \right) \mod p.
\]

Similarly, using Binet formula of the sequence \( \{U_{kn}\} \), the proof of the congruence in (10) is given. \( \square \)

**Lemma 2.5.** Let \( p > 3 \) be a prime. For an integer \( k \) with \( p \nmid V_k \) and \( \left( \frac{\Delta}{p} \right) = 1 \),

\[
\sum_{i=1}^{p-1} \frac{V_{k(i+p)}}{V_k^{i-1} i} \equiv \left( -1 \right)^k \frac{V_k^p V_{k(p-2)} - V_{2k(p-1)} - \left( -1 \right)^k V_{2k}}{p V_k^{p-1}} \mod p.
\]

Proof. Consider

\[
\sum_{i=1}^{p-1} \frac{V_{k(i+p)}}{V_k^{i-1} i} = \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p-2)}}{V_k^{i-1} i} + \sum_{i=1}^{p-1} \frac{\beta^{k(i+p-2)}}{V_k^{i-1} i}
\]

\[
= V_k \alpha^{k(p-2)} \sum_{i=1}^{p-1} \left( \frac{\alpha^k}{V_k} \right)^i \frac{1}{i} + V_k \beta^{k(p-2)} \sum_{i=1}^{p-1} \left( \frac{\beta^k}{V_k} \right)^i \frac{1}{i}.
\]

For \( \left( \frac{\Delta}{p} \right) = 1 \), taking \( \frac{\alpha^k}{V_k}, \frac{\beta^k}{V_k} \) place of \( x \) in (1), respectively, we write

\[
\sum_{i=1}^{p-1} \frac{V_{k(i+p)}}{V_k^{i-1} i}
\]
Consider that

\[ \sum_{i=1}^{p-1} \frac{V_{k(i+p)}}{V_{k}^{i-1} i^2} = \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p)}}{V_{k}^{i-1} i^2} + \sum_{i=1}^{p-1} \frac{\beta^{k(i+p)}}{V_{k}^{i-1} i^2} = V_k \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p)}}{V_{k}^{i-1} i^2} + V_k \sum_{i=1}^{p-1} \frac{\beta^{k(i+p)}}{V_{k}^{i-1} i^2}. \]

For \( \left( \frac{\Delta}{p} \right) = 1 \), by taking \( V_k \), \((-1)^k \) instead of \( r, s \) in (4), respectively, we have

\[ \sum_{i=1}^{p-1} \frac{V_{k(i+p)}}{V_{k}^{i-1} i^2} = -\frac{1}{2} \left( \frac{V_{kp} - V_{k}^{p}}{p} \right)^2 \ (mod \ p), \]

and from Fermat’s little theorem, the congruence \( \frac{1}{(p-k)^2} \equiv \frac{1}{k^2} \ (mod \ p) \) for \( k \not\equiv p \) and \( \alpha^{k} \beta^{k} = (-1)^{k} \), we get

\[ \beta^{2kp} \sum_{i=1}^{p-1} \frac{(V_{k} \alpha^{k})^{i}}{(p-k)^2} = \beta^{2kp} \sum_{i=1}^{p-1} \frac{V_{k}^{i} \alpha^{ki}}{(p-k)^2} = \beta^{2kp} \sum_{i=1}^{p-1} \frac{V_{k}^{p-i} \beta^{ki} (p-i)^2}{(p-i)^2} \]

\[ \equiv \beta^{2kp} \frac{V_{k}^{p}}{\beta^{kp}} \sum_{i=1}^{p-1} \frac{\beta^{ki} i^2}{V_{k}^{i-1} i^2} \equiv V_k \sum_{i=1}^{p-1} \frac{\beta^{k(i+p)}}{V_{k}^{i-1} i^2} \ (mod \ p). \]

By (12) and (13), we obtain the desired result. \( \square \)
3. The Results Involving the Terms of the Sequences \( \{U_{kn}\} \) and \( \{V_{kn}\} \)

In this section, we give congruences for the terms of the sequences \( \{U_{kn}\} \) and \( \{V_{kn}\} \). Now we start with our first result.

**Theorem 3.1.** Let \( p \) be an odd prime. For \( \left( \frac{\Delta}{p} \right) = 1 \),

\[
\Delta V_k \sum_{i=1}^{(p-1)/2} U_{2k(2i+1)} H_i \equiv -\frac{4}{p} + \frac{(-1)^k}{p} \left( V^p_k V_{kp} - \Delta^{(p+1)/2} U_{kp} \right) - 2q_p(2) V_{2k(p+1)} \quad \text{(mod } p),
\]

and

\[
V_k \sum_{i=1}^{(p-1)/2} V_{2k(2i+1)} H_i \equiv \frac{(-1)^k}{p} \left( V^p_k U_{kp} - \Delta^{(p-1)/2} V_{kp} \right) - 2q_p(2) U_{2k(p+1)} \quad \text{(mod } p),
\]

where the Fermat quotient \( q_p(2) = \frac{2p-1-1}{p} \).

**Proof.** For the proof of (14), by the Binet formula of the sequence \( \{U_{kn}\} \), we have

\[
\Delta V_k \sum_{i=1}^{(p-1)/2} U_{2k(2i+1)} H_i = V_k \sqrt{\Delta} \sum_{i=1}^{(p-1)/2} \alpha^{2k(2i+1)} H_i - V_k \sqrt{\Delta} \sum_{i=1}^{(p-1)/2} \beta^{2k(2i+1)} H_i.
\]

Writing \((p+1)/2\) place of \( n \) and \( \alpha^{4k}, \beta^{4k} \) place of \( x \) in (5), respectively, we write

\[
\left( 1 - \alpha^{4k} \right) \sum_{i=1}^{(p-1)/2} \alpha^{4ki} H_i = \sum_{i=1}^{(p-1)/2} \frac{\alpha^{4ki}}{i} - \alpha^{2k(p+1)} H_{(p-1)/2},
\]

\[
\left( 1 - \beta^{4k} \right) \sum_{i=1}^{(p-1)/2} \beta^{4ki} H_i = \sum_{i=1}^{(p-1)/2} \frac{\beta^{4ki}}{i} - \beta^{2k(p+1)} H_{(p-1)/2}.
\]

Since \( \alpha^{2k} = \beta^{2k} \alpha^{4k} \) and \( \beta^{2k} = \alpha^{2k} \beta^{4k} \), we can rewrite

\[
- V_k \sqrt{\Delta} \sum_{i=1}^{(p-1)/2} \alpha^{4ki+2k} H_i = \sum_{i=1}^{(p-1)/2} \frac{\alpha^{4ki}}{i} - \alpha^{2k(p+1)} H_{(p-1)/2},
\]

\[
V_k \sqrt{\Delta} \sum_{i=1}^{(p-1)/2} \beta^{4ki+2k} H_i = \sum_{i=1}^{(p-1)/2} \frac{\beta^{4ki}}{i} - \beta^{2k(p+1)} H_{(p-1)/2}.
\]
By (16) and (17), we get
\[
\Delta V_k \sum_{i=1}^{(p-1)/2} U_{2k(2i+1)} H_i
\]
\[
= - \sum_{i=1}^{(p-1)/2} \frac{\alpha^{4ki}}{i} + \alpha^{2k(p+1)} H_{(p-1)/2} - \sum_{i=1}^{(p-1)/2} \frac{\beta^{4ki}}{i} + \beta^{2k(p+1)} H_{(p-1)/2}
\]
\[
= - \sum_{i=1}^{(p-1)/2} \frac{V_{4ki}}{i} + V_{2k(p+1)} H_{(p-1)/2},
\]
which, by (11) and the congruence
\[
H_{(p-1)/2} \equiv -2q_p(2) \pmod{p},
\]
equivalents
\[
\frac{1}{p} \left( (-1)^k \left( V_k^{p(p-1)/2} U_p - (\sqrt{\Delta})^{p+1} U_{kp} \right) - 4 \right) - 2q_p(2) V_{2k(p+1)} \pmod{p}.
\]
Similarly, using the Binet formula of the sequence \( \{V_k\} \), (16), (17), (10) and the congruence \( H_{(p-1)/2} \equiv -2q_p(2) \pmod{p} \), the other claim is obtained.

For example, by taking \( k = 1 \) in Theorem 3.1, for \( \left( \frac{r^2+4}{p} \right) = 1 \),
\[
r \left( r^2 + 4 \right) \sum_{i=1}^{(p-1)/2} U_{4i+2} H_i
\]
\[
\equiv -\frac{1}{p} \left( r^p V_p - (r^2 + 4)^{p+1} U_p + 4 \right) - 2q_p(2) V_{2p+2} \pmod{p},
\]
and
\[
r \sum_{i=1}^{(p-1)/2} V_{4i+2} H_i
\]
\[
\equiv -\frac{1}{p} \left( r^p U_p - (r^2 + 4)^{p-1} V_p \right) - 2q_p(2) U_{2p+2} \pmod{p}.
\]

**Theorem 3.2.** Let \( p \) be an odd prime. For \( \left( \frac{\Delta}{p} \right) = 1 \),
\[
\Delta V_k^2 \sum_{i=1}^{(p-1)/2} i U_{4ki} H_i \equiv (2U_{2k(p+1)} - V_k V_{2kp}) q_p(2) - U_{2k(p-1)}
\]
\[
\quad - \frac{(-1)^k}{p} \left( V_k^{p U_p - \Delta(p-1)/2 V_{kp}} \right) \pmod{p},
\]
and
\[
\Delta V_k^2 \sum_{i=1}^{(p-1)/2} i V_{4ki} H_i \equiv (2V_{2k(p+1)} - \Delta V_k U_{2kp}) q_p(2) - V_{2k(p-1)} + 2
\]
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\[ (19) \quad \frac{1}{p} \left( 4 - (-1)^k \left( V_k^p V_{kp} - \Delta^{(p+1)/2} U_{kp} \right) \right) \pmod{p}, \]

where \( q_p(2) \) as before.

Proof. For the proof of (19), using the Binet formula of the sequence \( \{V_{kn}\} \), we have

\[ \Delta V_k^{(p-1)/2} \sum_{i=1}^{(p-1)/2} iV_{4ki} H_i = \Delta V_k^{(p-1)/2} \sum_{i=1}^{(p-1)/2} i\alpha^{4ki} H_i + \Delta V_k^{(p-1)/2} \sum_{i=1}^{(p-1)/2} i\beta^{4ki} H_i. \]

Putting \( (p + 1)/2 \) instead of \( n \) and \( \alpha^{4k}, \beta^{4k} \) instead of \( x \) in (7), respectively, we write

\[ \left( \alpha^{4k} - 1 \right)^2 \sum_{i=1}^{(p-1)/2} i\alpha^{4k(i-1)} H_i = \left( \frac{p + 1}{2} \alpha^{2k(p-1)} \left( \alpha^{4k} - 1 \right) - \alpha^{2k(p+1)} \right) H_{(p-1)/2} \]

\[ - \left( \alpha^{2k(p-1)} - 1 \right) + \sum_{i=1}^{(p-1)/2} \alpha^{4ki} \]

and

\[ \left( \beta^{4k} - 1 \right)^2 \sum_{i=1}^{(p-1)/2} i\beta^{4k(i-1)} H_i = \left( \frac{p + 1}{2} \beta^{2k(p-1)} \left( \beta^{4k} - 1 \right) - \beta^{2k(p+1)} \right) H_{(p-1)/2} \]

\[ - \left( \beta^{2k(p-1)} - 1 \right) + \sum_{i=1}^{(p-1)/2} \beta^{4ki} \]

From the equalities \( \alpha^{2k} = \beta^{2k} \alpha^{4k} \) and \( \beta^{2k} = \alpha^{2k} \beta^{4k} \), we have

\[ \Delta V_k^{(p-1)/2} \sum_{i=1}^{(p-1)/2} i\alpha^{4ki} H_i = \left( \frac{p + 1}{2} \sqrt{\Delta V_k \alpha^{2kp} - \alpha^{2k(p+1)}} \right) H_{(p-1)/2} \]

(20)

\[ - \left( \alpha^{2k(p-1)} - 1 \right) + \sum_{i=1}^{(p-1)/2} \frac{\alpha^{4ki}}{i}, \]

\[ \Delta V_k^{(p-1)/2} \sum_{i=1}^{(p-1)/2} i\beta^{4ki} H_i = \left( -\frac{p + 1}{2} \sqrt{\Delta V_k \beta^{2kp} - \beta^{2k(p+1)}} \right) H_{(p-1)/2} \]

(21)

\[ - \left( \beta^{2k(p-1)} - 1 \right) + \sum_{i=1}^{(p-1)/2} \frac{\beta^{4ki}}{i}. \]
Using the Binet formulas of the sequences \( \{U_{kn}\} \) and \( \{V_{kn}\} \), by (20) and (21), we rewrite

\[
\Delta V_k^2 \sum_{i=1}^{(p-1)/2} iV_{4ki}H_i = \left( \frac{p+1}{2} \sqrt{\Delta} V_k \alpha^{2kp} - \alpha^{2k(p+1)} \right) H_{(p-1)/2} - \left( \alpha^{2k(p-1)} - 1 \right)
+ \left( -\frac{p+1}{2} \sqrt{\Delta} V_k \beta^{2kp} - \beta^{2k(p+1)} \right) H_{(p-1)/2} - \left( \beta^{2k(p-1)} - 1 \right)
+ \sum_{i=1}^{(p-1)/2} \alpha^{4ki} \frac{i}{i} + \sum_{i=1}^{(p-1)/2} \frac{\beta^{4ki}}{i}
= \left( \frac{p+1}{2} \Delta V_k U_{2kp} - V_{2k(p+1)} \right) H_{(p-1)/2} - V_{2k(p-1)} + 2 + \sum_{i=1}^{(p-1)/2} \frac{V_{4ki}}{i}.
\]

From (11) and the congruence \( H_{(p-1)/2} \equiv -2q_p(2) \pmod{p} \), we have

\[
\Delta V_k^2 \sum_{i=1}^{(p-1)/2} iV_{4ki}H_i \equiv (2V_{2k(p+1)} - \Delta V_k (p + 1) U_{2kp}) q_p(2) - V_{2k(p-1)} + 2 + \frac{4}{p} - \frac{(-1)^k}{p} \left( V_k^p V_{kp} - \left( \sqrt{\Delta} \right)^{p+1} U_{kp} \right) \pmod{p},
\]

as claimed. Similarly, the other congruence is given. Thus, the proof is completed. \(\square\)

For example, when \( k = r = 1 \) in Theorem 3.2, we have the congruences as follows: For \( \left( \frac{5}{p} \right) = 1 \),

\[
5 \sum_{i=1}^{(p-1)/2} iF_{4i}H_i \equiv (2F_{2p+2} - L_{2p}) q_p(2) - F_{2p-2}
+ \frac{F_p - 5(p-1)/2 L_p}{p} \pmod{p},
\]

and

\[
5 \sum_{i=1}^{(p-1)/2} iL_{4i}H_i \equiv (2L_{2p+2} - 5F_{2p}) q_p(2) - L_{2p-2} + 2
+ \frac{L_p - 5(p+1)/2 F_p + 4}{p} \pmod{p}.
\]
Theorem 3.3. Let $p$ be an odd prime. For $\left(\frac{\Delta}{p}\right) = 1$,

$$
\Delta^2 V_k^{(p-1)/2} \sum_{i=1}^{(p-1)/2} i^2 U_{4ki} H_i
\equiv -V_{2k} \left( 3 + \frac{4}{p} - \frac{(-1)^k}{p} \left( V_k^p V_{kp} - (\sqrt{\Delta})^{p+1} U_{kp} \right) \right)
+ V_{2kp} \left( 3 - q_p (2) \left( \frac{V_k}{2} + 3 \right) \right) - \frac{\Delta}{2} U_{2k(p-1)} U_{2k} \pmod{p},
$$

and

$$
\Delta V_k^{(p-1)/2} \sum_{i=1}^{(p-1)/2} i^2 V_{4ki} H_i
\equiv U_{2kp} \left( 3 - q_p (2) \left( \frac{V_k}{2} + 3 \right) \right) - U_{2k} \left( \frac{1}{2} V_{2k(p-1)} + 1 \right)
+ \frac{(-1)^k}{p} V_{2k} \left( V_k^p U_{kp} - (\sqrt{\Delta})^{p-1} V_{kp} \right) \pmod{p}.
$$

Proof. Using the Binet formulas of the sequences $\{U_{kn}\}, \{V_{kn}\}$, by (9), (10) and the congruence $H_{(p-1)/2} \equiv -2q_p (2) \pmod{p}$, we obtained the desired result. \qed

Now, we will give the congruences with harmonic numbers of order 2, $H_{n,2}$.

Theorem 3.4. Let $p > 3$ be a prime. For $\left(\frac{\Delta}{p}\right) = 1$,

$$
\sum_{i=1}^{p-1} \frac{V_{k(i+p-1)}}{V_k^i} H_{i,2} \equiv -\frac{(-1)^k}{2} \left( \frac{V_{kp} - V_k^p}{p} \right)^2 \pmod{p}.
$$

Proof. From Binet formula of the sequence $\{V_{kn}\}$, we consider

$$
(-1)^k \sum_{i=1}^{p-1} \frac{V_{k(i+p)-1}}{V_k^i} H_{i,2}
= (-1)^k \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p)}}{V_k^i} H_{i,2} + \sum_{i=1}^{p-1} \frac{\beta^{k(i+p)}}{V_k^i} H_{i,2}
= \frac{(-1)^k}{\alpha^k V_k} \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p)}}{V_k^{i-1}} H_{i,2} + \frac{(-1)^k}{\beta^k V_k} \sum_{i=1}^{p-1} \frac{\beta^{k(i+p)}}{V_k^{i-1}} H_{i,2}
= \frac{\beta^k}{V_k} \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p)}}{V_k^{i-1}} H_{i,2} + \frac{\alpha^k}{V_k} \sum_{i=1}^{p-1} \frac{\beta^{k(i+p)}}{V_k^{i-1}} H_{i,2}.
$$
By taking \( p \) instead of \( n \) and \( \frac{\alpha^k}{V_k}, \frac{\beta^k}{V_k} \) instead of \( x \) in (6), respectively, we have

\[
(22) \left( \frac{V_k - \alpha^k}{V_k} \right) \sum_{i=1}^{p-1} \left( \frac{\alpha^k}{V_k} \right)^{i+p} H_{i,2} = \sum_{i=1}^{p-1} \left( \frac{\alpha^k}{V_k} \right)^{i+p} \frac{1}{i^2} - \left( \frac{\alpha^k}{V_k} \right)^{2p} H_{p-1,2},
\]

\[
(23) \left( \frac{V_k - \beta^k}{V_k} \right) \sum_{i=1}^{p-1} \left( \frac{\beta^k}{V_k} \right)^{i+p} H_{i,2} = \sum_{i=1}^{p-1} \left( \frac{\beta^k}{V_k} \right)^{i+p} \frac{1}{i^2} - \left( \frac{\beta^k}{V_k} \right)^{2p} H_{p-1,2}.
\]

From (22), (23) and the congruence \( H_{p-1,2} \equiv 0 \pmod{p} \), we get

\[
(-1)^k \sum_{i=1}^{p-1} \frac{V_{k(i+p-1)}}{V_k} H_{i,2} \equiv \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p)}}{V_k^{i-1}i^2} + \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p)}}{V_k^{i-1}i^2} \pmod{p}.
\]

Using Binet formula of the sequence \( \{V_k\} \) and Lemma 2.6, we have

\[
(-1)^k \sum_{i=1}^{p-1} \frac{V_{k(i+p-1)}}{V_k^i} H_{i,2} \equiv \sum_{i=1}^{p-1} \frac{V_{k(i+p)}}{V_k^{i-1}i^2} \equiv -\frac{1}{2} \left( \frac{V_{kp} - V_k^p}{p} \right)^2 \pmod{p}.
\]

which settles the proof. \( \square \)

**Theorem 3.5.** Let \( p > 3 \) be a prime. For an integer \( k \) with \( p \nmid V_k \) and \( \left( \frac{\Delta}{p} \right) = 1 \),

\[
\sum_{i=1}^{p-1} i \frac{V_{k(i+p-3)}}{V_k^i} H_{i,2} \equiv (-1)^k \frac{V_k^p V_{k(p-2)} - V_{2k(p-1)} - (-1)^k V_{2k} - 1}{2} \left( \frac{V_{kp} - V_k^p}{p} \right)^2 \pmod{p}.
\]

**Proof.** From Binet formula of the sequence \( \{V_k\} \) and \( \alpha^{2k} \beta^{2k} = 1 \), we have

\[
\sum_{i=1}^{p-1} i \frac{V_{k(i+p-3)}}{V_k^i} H_{i,2} = \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p-3)}}{V_k^i} H_{i,2} + \sum_{i=1}^{p-1} \frac{\beta^{k(i+p-3)}}{V_k^i} H_{i,2}
\]

\[
= \frac{1}{\alpha^{2k} V_k^2} \sum_{i=1}^{p-1} i \frac{\alpha^{k(p+i-1)}}{V_k^{i-2}} H_{i,2} + \frac{1}{\beta^{2k} V_k} \sum_{i=1}^{p-1} i \frac{\beta^{k(i+p-1)}}{V_k^{i-2}} H_{i,2}
\]

\[
= \frac{\beta^{2k}}{V_k^2} \sum_{i=1}^{p-1} i \frac{\alpha^{k(i+p-1)}}{V_k^{i-2}} H_{i,2} + \frac{\alpha^{2k}}{V_k} \sum_{i=1}^{p-1} i \frac{\beta^{k(i+p-1)}}{V_k^{i-2}} H_{i,2}.
\]
If we take \( p \) instead of \( n \) and \( \frac{\alpha^k}{V_k}, \frac{\beta^k}{V_k} \) instead of \( x \) in (8), respectively, we get

\[
\frac{\beta^{2k}}{V^2_k} \sum_{i=1}^{p-1} i \frac{\alpha^{k(i+p-1)}}{V^{i-2}_k} H_{i,2} = V_k^2 \alpha^{k(p-1)} \left( p \left( \frac{\alpha^k}{V_k} \right)^p \left( \frac{\alpha^k}{V_k} - 1 \right) - \left( \frac{\alpha^k}{V_k} \right)^{p+1} \right) H_{p-1,2}
\]

\[(24)\]

\[+ \frac{\beta^k}{\alpha^k} \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p)}}{V^{i-1}_k} + \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p)}}{V^{i-1}_k i^2},\]

and

\[
\frac{\alpha^{2k}}{V^2_k} \sum_{i=1}^{p-1} i \frac{\beta^{k(i+p-1)}}{V^{i-2}_k} H_{i,2} = V_k^2 \beta^{k(p-1)} \left( p \left( \frac{\beta^k}{V_k} \right)^p \left( \frac{\beta^k}{V_k} - 1 \right) - \left( \frac{\beta^k}{V_k} \right)^{p+1} \right) H_{p-1,2}
\]

\[(25)\]

\[+ \frac{\alpha^k}{\beta^k} \sum_{i=1}^{p-1} \frac{\beta^{k(i+p)}}{V^{i-1}_k} + \sum_{i=1}^{p-1} \frac{\beta^{k(i+p)}}{V^{i-1}_k i^2}.\]

From (24), (25) and the congruence \( H_{p-1,2} \equiv 0 \pmod{p} \), we have

\[
\sum_{i=1}^{p-1} \frac{V_k(i+p-3)}{V^i_k} H_{i,2} \equiv \frac{\beta^k}{\alpha^k} \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p)}}{V^{i-1}_k} + \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p)}}{V^{i-1}_k i^2} + \frac{\alpha^k}{\beta^k} \sum_{i=1}^{p-1} \frac{\beta^{k(i+p)}}{V^{i-1}_k} + \sum_{i=1}^{p-1} \frac{\beta^{k(i+p)}}{V^{i-1}_k i^2} ( \pmod{p}).
\]

By \( \alpha^k \beta^k = (-1)^k \), we rewrite

\[
\sum_{i=1}^{p-1} \frac{V_k(i+p-3)}{V^i_k} H_{i,2} \equiv (-1)^k \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p-2)}}{V^{i-1}_k} + \sum_{i=1}^{p-1} \frac{\alpha^{k(i+p)}}{V^{i-1}_k i^2} + (-1)^k \sum_{i=1}^{p-1} \frac{\beta^{k(i+p-2)}}{V^{i-1}_k} + \sum_{i=1}^{p-1} \frac{\beta^{k(i+p)}}{V^{i-1}_k i^2}
\]

\[= (-1)^k \sum_{i=1}^{p-1} \frac{V_k(i+p-2)}{V^{i-1}_k} + \sum_{i=1}^{p-1} \frac{V_k(i+p)}{V^{i-1}_k i^2} ( \pmod{p}),\]

which, by Lemma 2.5 and Lemma 2.6, equivalents

\[(-1)^k \frac{V_k p V_k(p-2) - V_{2k(p-1)} - (-1)^k V_{2k} - \frac{1}{2} \left( \frac{V_k p - V_k^p}{p} \right)^2}{p V_k^{p-1}} ( \pmod{p}).\]

\[\square\]
As a result of Theorem 3.5, by taking 1 instead of $k$, we have the following corollary:

**Corollary 3.1.** Let $p > 3$ be a prime. For $p \nmid r$, and $\left(\frac{A}{p}\right) = 1$,

$$
\sum_{i=1}^{p-1} \frac{V_{i+p-3}}{r^i} H_{i,2} \equiv -\frac{r^pV_{p-2} - V_{2p-2} + r^2 + 2}{p^{p-1}} - \frac{1}{2} \left(\frac{V_p - r^p}{p}\right)^2 \pmod{p}.
$$

For example, when $r = 1$ in Corollary 3.1, we have the congruence as follows:

$$
\sum_{i=1}^{p-1} iL_{i+p-3} H_{i,2} \equiv -\frac{L_{p-2} - L_{2p-2} + 3}{p} - \frac{1}{2} \left(\frac{L_p - 1}{p}\right)^2 \pmod{p}.
$$

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