GLOBAL BEHAVIOR OF SOLUTIONS TO CHEVRON PATTERN EQUATIONS

H. KALANTAROVA∗, V. KALANTAROV† AND O. VANTZOS‡

Abstract. Considering a system of equations modeling the chevron pattern dynamics, we show that the corresponding initial boundary value problem has a unique weak solution that continuously depends on initial data, and the semigroup generated by this problem in the phase space $X^0 := L^2(\Omega) \times L^2(\Omega)$ has a global attractor. We also provide some insight to the behavior of the system, by reducing it under special assumptions to systems of ODEs, that can in turn be studied as dynamical systems.

1. Introduction

The chevron patterns also known as the herringbone patterns in the context of the electroconvection of nematic liquid crystals, i.e. in the electromagnetically driven motion of anisotropic liquids composed of rod-like particles, that can be oriented freely in space, were first studied by Heilmeyer and Helfrich [7] and then in detail by Orsay group [6].

The typical experimental setup involves in general the containment of the nematic liquid between two parallel transparent plates, and the application of an AC voltage of varying frequency across the plates, often in conjunction with a magnetic field parallel to the plates, see Fig. 2. Depending on the characteristics (voltage, frequency, etc.) of the external driving forces, the behavior of the fluid exhibits a wealth of nonlinear dynamical phenomena [1], [2], [5], [10], [16]. A common class of such phenomena feature the self-organization of the nematic liquid into convection cells, where the flow is regular and largely local to each cell. The formation of these cells is driven on a microscopical level by the interaction of the external forces with the constituent particles of the liquid, and therefore the orientation of the particles is of great importance to the dynamics. The theoretical study of this problem focuses then mainly on macroscopic models that attempt to capture...
directly the distribution and flow pattern (such as direction and orientation of rotation) of the convection cells, coupled with a measure of the local average orientation of the fluid particles (for instance in the form of a so-called director vector field).

\[ \vec{E} \parallel \vec{H} \]

**Figure 1.** Cross-section of a nematic liquid crystal contained between two parallel transparent plates. The liquid is composed of rod-like particles, that are free to flow and orient themselves in 3d space, under the influence of external electric \( \vec{E} \) and magnetic fields \( \vec{H} \).

We are interested in particular in the case where the fluid flow takes the form of **rolls**, i.e. zones where the fluid rotates parallel to the plates; the rolls themselves are arranged in periodical configurations, alternating between clockwise and counter-clockwise rotation, see Fig. 2. The periodicity of the rolls does not hold on larger scales, which leads to interesting formations such as the titular chevron pattern, where two sequences of alternating rolls meet at an angle, see Fig. 3. The following system of equations was proposed by Rossberg et al. [12], [13] to model the evolution of such patterns:

\[
\begin{align*}
\tau \frac{\partial A}{\partial t} &= A + \Delta A - \phi^2 A - |A|^2 A - 2ic_1\phi \frac{\partial A}{\partial y} + i\beta A \frac{\partial \phi}{\partial y}, \\
\frac{\partial \phi}{\partial t} &= D_1 \frac{\partial^2 \phi}{\partial x^2} + D_2 \frac{\partial^2 \phi}{\partial y^2} - h\phi + \phi |A|^2 - c_2 \text{Im} \left( A^* \frac{\partial A}{\partial y} \right),
\end{align*}
\]

where \( \tau, D_1, D_2, c_1, c_2, h \) are non-negative constants and \( \beta \in \mathbb{R} \). The complex valued function \( A \) succinctly represents the phase (clockwise/counter-clockwise), direction and amplitude (wave vector) of the periodical patterns, whereas the orientation of the liquid crystals is
Figure 2. A typical flow pattern where the liquid self-organizes into rotating zones, called rolls, with axes of rotation parallel to the plates and rotation orientation alternating between clockwise and counterclockwise.

Figure 3. Experimental observation of chevron patterns, where periodic groups of rolls (observable as alternating light-dark zones) meet at an angle. Reprinted FIG. 2,(a) with permission from Jong-Hoon Huh, Yoshiki Hidaka, Axel G. Rossberg, and Shoichi Kai, Phys. Rev. E 61, 2769, 2000. Copyright 2000 by the American Physical Society

represented via the real valued function $\phi$, the angle of the director vector (projected in the x-y plane) with the x-axis. The parameter $\tau$ is a function of the various physical time-scales of the problem, and $D_1$ and $D_2$ are the coefficients of the anisotropic diffusion of the director.
field for the liquid crystal particles. The rest of the parameters reflect various interactions:

- the dampening parameter $h$ measures the tendency of the director field to align with the magnetic field $\vec{H}$, corresponding to $\phi = 0$,
- the parameter $\beta$ measures the interaction between the gradient of the director field and the phase of the rolls, and
- the parameters $c_1$ and $c_2$ control the torque that the director field and the wave vector of the rolls exert on each other; when $c_1 = c_2 = 1$ the interaction is isotropic, but many experimentally interesting phenomena occur in the anisotropic regime.

In the literature, one can mostly find experimental [8] and numerical [9], [15] studies on chevron patterns, and various works on the physical derivation of the model [12], [14], [13]. In our work, we show the existence, uniqueness and the continuous dependence on initial data for the weak solutions of the model. Moreover we show the existence of a global attractor of a semigroup generated by this problem. These results are valid under the assumption that the parameter $c_1$ is in the range $[0, 1)$.

The rest of this paper is organized as follows. In the following section, we prove well-posedness of the initial boundary value problem for the system (1.1)-(1.2) under homogeneous Dirichlet boundary conditions and dissipativity of the semigroup generated by this problem, provided that the coefficient of the nonlinear term $2i\phi \frac{\partial A}{\partial y}$ in (1.1) is restricted to the range $0 \leq c_1 < 1$. In section 3, we show that the semigroup generated by the problem is a compact semigroup and has a global attractor, under the same assumption $c_1 \in [0, 1)$. In the last section, we present an argument, by reducing the model to a dynamical system, that gives insight and supports the assumed condition on the parameter $c_1$, by showing that $c_1 = 1$ is a critical value for the dynamics of the system.

2. Existence, uniqueness and dissipativity

In this section, we study the system (1.1)-(1.2) in a bounded domain $\Omega \subset \mathbb{R}^2$ with sufficiently smooth boundary $\partial \Omega$ under the following initial and boundary conditions

\begin{equation}
A \bigg|_{t=0} = A_0, \quad \phi \bigg|_{t=0} = \phi_0, \quad A \bigg|_{\partial \Omega} = 0, \quad \phi \bigg|_{\partial \Omega} = 0,
\end{equation}

where $A_0, \phi_0 \in L^2(\Omega)$ are given functions.

To prove existence and uniqueness of a weak solution of the problem we need two a priori estimates. To find these estimates, we multiply
the equation (1.1) by $A^*$ and we multiply the equation (1.2) by $\phi$ and integrate them with respect to $x$ over $\Omega$. Then we take the real parts of the resulting identities and obtain the following inequalities

\[(2.2) \quad \frac{\tau}{2} \frac{d}{dt} \|A\|^2 - \|A\|^2 + \|\nabla A\|^2 + (\phi^2, |A|^2) + \|A\|_{L^4}^4 \leq 2c_1 |(\phi \partial_y A, A^*)|,\]

\[(2.3) \quad \frac{1}{2} \frac{d}{dt} \|\phi\|^2 + D_1 \|\partial_x \phi\|^2 + D_2 \|\partial_y \phi\|^2 + (h, \phi^2) \leq c_2 |(A^* \partial_y A, \phi)| + (|\phi|^2, |A|^2),\]

respectively, since $c_1, c_2$ are non-negative constants. We get

\[(2.4) \quad 2c_1 |(\phi \partial_y A, A^*)| \leq c_1 \|\partial_y A\|^2 + c_1 (|A|^2, \phi^2),\]

\[(2.5) \quad c_2 |(A^* \partial_y A, \phi)| \leq \frac{1}{2} c_2 \|\partial_y A\|^2 + \frac{1}{2} c_2 (|A|^2, \phi^2),\]

upon application of Hölder’s inequality and then Cauchy’s inequality. Employing the inequality (2.4) in (2.2) and the inequality (2.5) in (2.3), we obtain

\[(2.6) \quad \frac{\tau}{2} \frac{d}{dt} \|A\|^2 - \|A\|^2 + (1 - c_1) \|\partial_y A\|^2 + \|\partial_x A\|^2 + (1 - c_1)(\phi^2, |A|^2) + \|A\|_{L^4(\Omega)}^4 \leq 0,\]

and

\[(2.7) \quad \frac{1}{2} \frac{d}{dt} \|\phi\|^2 + D_1 \|\partial_x \phi\|^2 + D_2 \|\partial_y \phi\|^2 + (h, \phi^2) \leq \frac{1}{2} c_2 \|\partial_y A\|^2 + (1 + \frac{1}{2} c_2)(|\phi|^2, |A|^2),\]

respectively. Next, we multiply the inequality (2.7) by a parameter $\delta > 0$, whose value is to be determined later, and add to the inequality (2.6)

\[
\frac{1}{2} \frac{d}{dt} \left[ \frac{\tau}{2} \|A\|^2 + \delta \|\phi\|^2 \right] + \delta D_1 \|\partial_x \phi\|^2 + \delta D_2 \|\partial_y \phi\|^2 + \delta h \|\phi\|^2 \leq \frac{1}{2} c_2 \|\partial_y A\|^2 + (1 - c_1 - \frac{1}{2} \delta c_2) \|\partial_y A\|^2 + (1 - c_1 - \delta (1 + \frac{1}{2} c_2)) (\phi^2, |A|^2) + \|A\|_{L^4(\Omega)}^4 \leq 0.
\]

Choosing

\[
\delta = \delta_0 := \frac{2(1 - c_1)}{2 + c_2}, \quad \text{where} \quad c_1 < 1,
\]
we optimize the range of value of \( c_1 \) and the number of terms with nonnegative coefficients in the above inequality, which yields

\[
\frac{1}{2} \frac{d}{dt} \left[ \tau \|A\|^2 + \delta_0 \|\phi\|^2 \right] + \delta_0 D_1 \|\partial_x \phi\|^2 + \delta_0 D_2 \|\partial_y \phi\|^2 \\
+ \delta_0 h \|\phi\|^2 - \|A\|^2 + \|\partial_x A\|^2 + \|\partial_y A\|^2 + \|A\|^4_{L^4} \leq 0.
\]

Using the inequality \( 2\|A\|^2 \leq |\Omega| + \|A\|^4_{L^4(\Omega)} \), where \( |\Omega| \) denotes the measure of the domain \( \Omega \), on the last term in (2.8), we get

\[
\frac{1}{2} \frac{d}{dt} \left[ \tau \|A\|^2 + \delta_0 \|\phi\|^2 \right] + \|A\|^2 + \delta_0 h \|\phi\|^2 + \delta_0 \|\nabla A\|^2 \\
+ \delta_0 D_0 \|\nabla \phi\|^2 \leq |\Omega|,
\]

where \( D_0 = \min\{D_1, D_2\} \). It follows from (2.9), that

\[
\frac{d}{dt} \left[ \tau \|A\|^2 + \delta_0 \|\phi\|^2 \right] + k_0 \left[ \tau \|A\|^2 + \delta_0 \|\phi\|^2 \right] \leq \tau |\Omega|,
\]

where \( k_0 = \min\{\tau^{-1}, h\} \). Integrating the last inequality, we get

\[
\tau \|A(t)\|^2 + \delta_0 \|\phi(t)\|^2 \leq \left[ \tau \|A_0\|^2 + \delta_0 \|\phi_0\|^2 \right] e^{-k_0 t} + \frac{1}{k_0} |\Omega|.
\]

By deriving a priori estimates (2.9) and (2.10), we have proved

\[
\|\phi(t)\| \leq M_0, \quad \|A(t)\| \leq M_0, \quad \forall t > 0,
\]

and

\[
\int_0^T \|\nabla \phi(t)\|^2 dt \leq M_T, \quad \int_0^T \|\nabla A(t)\|^2 dt \leq M_T.
\]

Here and in what follows \( M_0 \) denotes a generic constant, depending only on \( \|A_0\|, \|\phi_0\|, \) and \( |\Omega| \). \( M_T \) is a generic constant, which depends on \( T, \|A_0\|, \|\phi_0\| \) and \( |\Omega| \). The estimates (2.11) and (2.12) allow us by using the standard Galerkin method to prove the existence of a weak solution of the problem. Furthermore, employing the estimates (2.11) and (2.12) we can now prove that the solution of the problem is unique. Suppose that \([A, \phi] \) and \([\bar{A}, \bar{\phi}]\) are solutions of the system (1.1)-(1.2) corresponding to initial data \([A_0, \phi_0]\) and \([\bar{A}_0, \bar{\phi}_0]\), respectively, then

\[
[a, \Phi] := [A - \bar{A}, \phi - \bar{\phi}] \text{ is a solution of the system}
\]

\[
\tau \partial_t a = a + \Delta a - \phi^2 a - (\phi^2 - \bar{\phi}^2) \bar{A} - |A|^2 A + |\bar{A}|^2 \bar{A} \\
- 2i\alpha [\Phi \partial_y A + \bar{\phi} \partial_y a] + i\beta [a \partial_y \phi + \bar{A} \partial_y \Phi],
\]
\[ \partial_t \Phi = D_1 \partial_x^2 \Phi + D_2 \partial_y^2 \Phi - h \Phi + \Phi |A|^2 + \check{\phi} (|A|^2 - \check{A})^2 + c_2 \text{Im} \left[ a^\ast \partial_y A + \check{A}^\ast \partial_y a \right]. \]

Next we multiply the equation (2.13) by \( a^\ast \) and the equation (2.14) by \( \Phi \) in \( L^2(\Omega) \) and get

\[ \frac{1}{2} \frac{d}{dt} \left[ \tau \|a\|^2 + \|\Phi\|^2 \right] - \|a\|^2 + \|\nabla a\|^2 + (\phi^2, |a|^2) \]
\[ + D_0 \|\nabla \Phi\|^2 + h \|\Phi\|^2 \leq |((\phi + \check{\phi}) \Phi, \check{A} a^\ast)| + \int |A| \|\nabla \check{A}\| |a|^2 dx \]
\[ + \int |\check{A}|^2 |a|^2 dx + 2c_1 \int |\Phi| \|\nabla \Phi\| |a^\ast| dx + 2c_1 \int |\check{\phi}| \|\partial_y a\| |a^\ast| dx \]
\[ + \beta \int |\check{A}| \|\nabla \Phi\| |a^\ast| dx + (\Phi^2, |A|^2) + |(\check{\phi} \Phi, (|A| + |\check{A}|) |a|)| \]
\[ + c_2 |(a^\ast \partial_y A, \Phi)| + c_2 |(\check{A}^\ast \partial_y a, \Phi)| \]

where \( D_0 = \min\{D_1, D_2\} \), as in (2.9). We will estimate each term on the right hand side of the inequality (2.15), by employing (2.11), Hölder’s inequality, Cauchy’s inequality and the Ladyzhenskaya inequality

\[ \|u\|_{L^4(\Omega)} \leq 2^{1/4} \|u\|^{1/2} \|\nabla u\|^{1/2} \]

which is valid for each function \( u \in H^1_0(\Omega) \) with \( \Omega \subset \mathbb{R}^2 \). We start with

\[ |(\phi \Phi, \check{A} a^\ast)| \leq \|\phi\|_{L^4(\Omega)} \|\Phi\|_{L^4(\Omega)} \|\check{A}\|_{L^4(\Omega)} \|a\|_{L^4(\Omega)} \]
\[ \leq \frac{1}{2} \|\phi\|^2_{L^4(\Omega)} \|\Phi\|^2_{L^4(\Omega)} + \frac{1}{2} \|\check{A}\|^2_{L^4(\Omega)} \|a\|^2_{L^4(\Omega)} \]
\[ \leq \|\phi\| \|\nabla \phi\| \|\Phi\| \|\nabla \Phi\| + \|\check{A}\| \|\nabla \check{A}\| \|a\| \|\nabla a\| \]
\[ \leq \varepsilon_1 \|\nabla \Phi\|^2 + \varepsilon_1 \|\nabla a\|^2 + \frac{M_0^2}{4 \varepsilon_1} \left[ \|\nabla \phi\|^2 \|\Phi\|^2 + \|\nabla \check{A}\|^2 \|a\|^2 \right] \]

which gives us the estimate of the first term on the right hand side of (2.15)

\[ |((\phi + \check{\phi}) \Phi, \check{A} a^\ast)| \leq 2 \varepsilon_1 \left[ \|\nabla \Phi\|^2 + \|\nabla a\|^2 \right] \]
\[ + \frac{M_0^2}{4 \varepsilon_1} \left[ \|\nabla \phi\|^2 \|\Phi\|^2 + \|\nabla \check{\phi}\|^2 \|\Phi\|^2 + 2 \|\nabla \check{A}\|^2 \|a\|^2 \right]. \]
Then we estimate the second through the fourth terms as follows

\begin{equation}
\int |A| |\tilde{A}| |a|^2 dx \leq \|A\|_{L^4} \|\tilde{A}\|_{L^4} \|a\|^2_{L^4} \leq 2\|A\|^{1/2} \|
abla A\|^{1/2} \|\tilde{A}\|^{1/2} \|
abla \tilde{A}\|^{1/2} \|a\| \|
abla a\| \leq \|A\| \|
abla A\| \|a\| + \|\tilde{A}\| \|
abla \tilde{A}\| \|a\| \|
abla a\| \leq 2\varepsilon_2 \|
abla a\|^2 + \frac{M_0^2}{4\varepsilon_2} \left[ \|
abla A\|^2 \|a\|^2 + \|
abla \tilde{A}\|^2 \|a\|^2 \right],
\end{equation}

\begin{equation}
\int |\tilde{A}|^2 |a|^2 dx \leq \|\tilde{A}\|^2_{L^4} \|a\|^2_{L^2} \leq 2\|\tilde{A}\| \|
abla \tilde{A}\| \|a\| \|
abla a\| \leq \varepsilon_3 \|
abla a\|^2 + \frac{M_0^2}{4\varepsilon_3} \|
abla \tilde{A}\|^2 \|a\|^2,
\end{equation}

\begin{equation}
2c_1 \int |\Phi| |\partial_y A| |a| dx \leq 2c_1 \|\Phi\|_{L^4} \|
abla A\| \|a\|_{L^4} \leq \sqrt{2c_1} \|\Phi\|^{1/2} \|
abla \Phi\|^{1/2} \|
abla A\|^{1/2} \|a\|^{1/2} \|
abla a\|^{1/2} \leq \sqrt{2c_1} \|\Phi\| \|
abla \Phi\| \|
abla A\| \|a\| \|
abla a\| \leq \varepsilon_4 \left[ \|
abla \Phi\|^2 + \|
abla a\|^2 \right] + \frac{c_2^2}{2\varepsilon_4} \|
abla A\|^2 \left[ \|a\|^2 + \|\Phi\|^2 \right].
\end{equation}

We can similarly estimate the other terms on the right hand side of (2.15), and properly choosing the positive parameters \(\varepsilon_i\) in these estimates, we get the following inequality

\[ \frac{d}{dt} \left[ \tau \|a\|^2 + \|\Phi\|^2 \right] \leq K_0 \mathcal{E}(t) \left[ \tau \|a\|^2 + \|\Phi\|^2 \right], \]

where

\[ \mathcal{E}(t) = \|
abla A(t)\|^2 + \|
abla \phi(t)\|^2 + \|
abla \tilde{A}(t)\|^2 + \|
abla \tilde{\phi}(t)\|^2. \]

Integrating this inequality and remembering (2.12), we obtain

\[ \tau \|A(t) - \tilde{A}(t)\|^2 + \|\phi(t) - \tilde{\phi}(t)\|^2 \leq e^{K_0 \int_0^t \mathcal{E}(\eta)d\eta} \left[ \tau \|A_0 - \tilde{A}_0\|^2 + \|\phi_0 - \tilde{\phi}_0\|^2 \right]. \]

It follows from the last inequality that the weak solution of the problem is unique, moreover it continuously depends on initial data.

So we proved the following theorem.
**Theorem 2.1.** The initial boundary value problem (1.1)-(1.2) and (2.1) with \( c_1 \in [0, 1) \), has a unique weak solution
\[ A, \phi \in C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H^1_0(\Omega)), \quad \forall T > 0, \]
such that
\[ (2.20) \quad \| \phi(t) \| \leq M_0, \quad \| A(t) \| \leq M_0, \quad \forall t > 0, \]
and
\[ (2.21) \quad \int_0^T \| \nabla \phi(t) \|^2 dt \leq M_T, \quad \int_0^T \| \nabla A(t) \|^2 dt \leq M_T, \quad \forall T > 0. \]
In other words this problem generates a continuous semigroup \( S(t), \quad t \geq 0 \), in the phase space \( X^0 := L^2(\Omega) \times L^2(\Omega) \). Moreover (2.10) implies that this semigroup is bounded dissipative in the phase space \( X^0 \).

### 3. Global Attractor

In this section, we prove the existence of a global attractor for the semigroup associated with the initial boundary value problem (1.1)-(1.2) and (2.1). In order to establish this result, we rely on the following compactness result.

**Lemma 3.1.** The semigroup \( S(t) : X^0 \to X^0, \quad t \geq 0 \), is a compact semigroup.

**Proof.** Multiplying (1.1) in \( L^2 \) by \( |A|^2 A^* \), taking \( 2 \text{Re} \) parts of the obtained relation and using the inequality
\[ (3.1) \quad -2 \text{Re} \int_\Omega |A|^2 A^* \Delta A dx = 4 \int_\Omega |\nabla A|^2 |A|^2 dx + 2 \text{Re} \int_\Omega (\nabla A)^2 (A^*)^2 \]
\[ \geq 2 \int_\Omega |\nabla A|^2 |A|^2 dx \]
we obtain
\[ \frac{\tau}{2} \frac{d}{dt} \int_\Omega |A|^4 dx - 2 \int_\Omega |A|^4 dx + 2(|\nabla A|^2, |A|^2) + 2(\phi^2, |A|^4) + 2 \int_\Omega |A|^6 dx \]
\[ \leq 4c_1 \int_\Omega |\phi| |\partial_y A||A|^3 dx \leq 2c_1 \int_\Omega \phi^2 |A|^4 dx + 2c_1 \int_\Omega |\nabla A|^2 |A|^2 dx. \]
Combining the above inequality with the inequality \( 2 \int_\Omega |A|^4 dx \leq \int_\Omega |A|^6 dx + \frac{32}{7^2} |\Omega| \) gives us
\[ (3.2) \quad \frac{\tau}{2} \frac{d}{dt} \int_\Omega |A|^4 dx + (2 - 2c_1)(|\nabla A|^2, |A|^2) + (2 - 2c_1)(\phi^2, |A|^4) \]
\[ + \int_\Omega |A|^6 dx \leq \frac{32}{2^7} |\Omega|. \]
Next we multiply (1.2) by $\partial_t \phi$ and after a simple calculation obtain
\[
\frac{d}{dt}[D_1\|\partial_x \phi\|^2 + D_2\|\partial_y \phi\|^2 + h\|\phi\|^2] \leq (\phi^2, |A|^4) + c_2^2(|\nabla A|^2, |A|^2).
\]
Now we multiply last inequality by $\delta_1 := \min\left\{\frac{1-c_1}{c_2^2}, 1-c_1\right\}$ and add to (3.2)
\[
\frac{d}{dt}[\tau_2 \int_{\Omega} |A|^4 dx + \delta_1 D_1\|\partial_x \phi\|^2 + \delta_1 D_2\|\partial_y \phi\|^2 + \delta_1 h\|\phi\|^2] \\
+ (2-2c_1-\delta_1 c_2^2)(|\nabla A|^2, |A|^2) + (2-2c_1-\delta_1)(\phi^2, |A|^4) \\
+ \int_{\Omega} |A|^6 dx \leq \frac{32}{27}|\Omega|.
\]
Adding (2.9) and the last inequality gives
\[
\frac{d}{dt} E_a(t) + E_b(t) \leq \frac{59}{27}|\Omega|,
\]
where
\[
E_a(t) := \frac{\tau}{2} \int_{\Omega} |A|^4 dx + \delta_1 D_1\|\partial_x \phi\|^2 + \delta_1 D_2\|\partial_y \phi\|^2 + \delta_1 h\|\phi\|^2 \\
+ \frac{\tau}{2} |A|^2 + \frac{\delta_0}{2} \|\phi\|^2,
\]
and
\[
E_b(t) := |A|^2 + \delta_0 h\|\phi\|^2 + \delta_0 |\nabla A|^2 + \delta_0 D_0\|\nabla \phi\|^2 \\
+ (2-2c_1-\delta_1 c_2^2)(|\nabla A|^2, |A|^2) + (2-2c_1-\delta_1)(\phi^2, |A|^4) + \int_{\Omega} |A|^6 dx.
\]
The inequality (3.4) implies that
\[
\frac{d}{dt}(tE_a(t)) - E_a(t) + tE_b(t) \leq t \frac{59}{27}|\Omega|.
\]
Integrating this inequality and utilizing the estimates (2.21) we obtain
\[
||\nabla \phi(t)||, \int_0^T t||A(t)||^6_{L^6(\Omega)} dt, \int_0^T t||A|^2, |\nabla A|^2|| dt \leq M_0(T), \forall t \in (0, T),
\]
where $M_0(T)$ depends only on $T, ||A_0||, ||\phi_0||, |\Omega|$. 
To get an estimate similar to (3.5) for $A$, we multiply (1.1) this time by $-\Delta A^*$, take $2Re$ parts and get

(3.6) $\frac{d}{dt} \|\nabla A\|^2 - 2\|\nabla A\|^2 + 2\|\Delta A\|^2 + 2(|A|^2, |\nabla A|^2)$

$\leq 2[\phi A^*, \Delta A^*] + 4c_1(\|\phi\|\|\nabla A\|, |\Delta A^*|) + 2\beta(\|A\|\|\partial_y \phi\|, |\Delta A^*|)$

$\leq 3\varepsilon\|\Delta A\|^2 + \frac{1}{\varepsilon}(\|\phi\|^4, |A|^2) + \frac{4c_1^2}{\varepsilon}(\phi^2, |\nabla A|^2) + \frac{\beta^2}{\varepsilon}(|A|^2, |\partial_y \phi|^2)$.

Employing the H"older inequality and the Young's inequality, we estimate the second term on the right hand side of (3.6) as follows

(3.7) $\frac{1}{\varepsilon}(\|\phi\|^4, |A|^2) \leq \frac{1}{\varepsilon}\|\phi\|^6_{L^6(\Omega)} \|A\|^2_{L^6(\Omega)} \leq \|A\|^6_{L^6(\Omega)} + \frac{1}{4\varepsilon^2}\|\phi\|^6_{L^6(\Omega)}.$

Furthermore, we use the following version of the Gagliardo-Nirenberg interpolation inequalities (see [3])

(3.8) $\|u\|_{L^4(\Omega)} \leq \beta_0\|u\|^{\frac{2}{3}}\|\Delta u\|^{\frac{1}{3}}, \|\nabla u\|_{L^4(\Omega)} \leq \beta_0\|u\|^{\frac{4}{3}}\|\Delta u\|^{\frac{1}{3}}$

and obtain the following estimates of the third and the fourth terms

(3.9) $\frac{4c_1^2}{\varepsilon}(\phi^2, |\nabla A|^2) \leq \frac{4c_1^2}{\varepsilon}\|\phi\|^2_{L^4(\Omega)} \|\nabla A\|^2_{L^4(\Omega)}$

$\leq \frac{c_04c_1^2}{\varepsilon}\|\phi\|^2_{L^4(\Omega)} \|A\|^2_{L^4(\Omega)} \|\Delta A\|^2$

$\leq \varepsilon\|\Delta A\|^2 + b_\varepsilon\|\phi\|^4_{L^4(\Omega)} \|A\|^2$

$\leq \varepsilon\|\Delta A\|^2 + 4b_\varepsilon\|\phi\|^4 \|\nabla \phi\|^4 \|A\|^2,$

(3.10) $\frac{\beta^2}{\varepsilon}(\|A|^2, |\partial_y \phi|^2) \leq \frac{\beta^2}{\varepsilon}\|A\|^2_{L^4(\Omega)} \|\nabla \phi\|^2_{L^4(\Omega)}$

$\leq C_\varepsilon\|\Delta \phi\|^2 \|\phi\|^2 \|\Delta A\|^2 \|A\|^2$

$\leq \varepsilon\|\Delta A\|^2 \|A\|^6 + c_\varepsilon\|\Delta \phi\|^2 \|\phi\|^2.$

Utilizing the inequalities (3.7)-(3.10) in (3.6) we obtain

$$\frac{d}{dt} \|\nabla A\|^2 + (2 - 4\varepsilon - \varepsilon\|A\|^6)\|\Delta A\|^2 \leq 2\|\nabla A\|^2 + \|A\|^6_{L^6(\Omega)} + a_\varepsilon\|\phi\|^6_{L^6(\Omega)}$$

$$+ 4b_\varepsilon\|\phi\|^4 \|A\|^2 \|\nabla \phi\|^4 + c_\varepsilon\|\phi\|^2 \|\Delta \phi\|^2.$$
Choosing $\varepsilon > 0$ small enough and using (2.20) we infer from the last inequality that

$$
(3.11) \quad \tau \frac{d}{dt} \|\nabla A\|^2 + \|\Delta A\|^2 \\
\leq 2\|\nabla A\|^2 + 2\|A\|_{L^p(\Omega)}^p + a_\varepsilon \|\phi\|^p_{L^p(\Omega)} + 4b_\varepsilon M_0^6 \|\nabla \phi\|^4 + c_\varepsilon M_0^2 \|\Delta \phi\|^2.
$$

Now, we multiply (1.2) by $-\mathcal{L} \phi := -D_1 \frac{\partial^2 \phi}{\partial x^2} - D_2 \frac{\partial^2 \phi}{\partial y^2}$ in $L^2(\Omega)$ and using the following inequality (see e.g. [11])

$$
\nu_0 \|\Delta u\|^2 \leq \|\mathcal{L} u\|^2 \leq \nu_1 \|\Delta u\|^2
$$

we obtain

$$
\frac{1}{2} \frac{d}{dt} \left[ D_1 \|\partial_x \phi\|^2 + D_2 \|\partial_y \phi\|^2 \right] + \nu_0 \|\Delta \phi\|^2 + hD_1 \|\partial_x \phi\|^2 + hD_2 \|\partial_y \phi\|^2 \\
\leq (|\phi| A, |\mathcal{L} \phi|) + c_2(|A| \|\partial_y A\|, |\mathcal{L} \phi|) \\
\leq \frac{\nu_0}{4} \|\Delta \phi\|^2 + \frac{\nu_1}{\nu_0} (\phi^2, |A|^4) + \frac{\nu_0}{4} \|\Delta \phi\|^2 + \frac{\nu_1 c_2^2}{\nu_0} (|A|^2, |\nabla A|^2).
$$

Multiplying the last inequality by $\eta := \min \left\{ \frac{\nu_0 (1-c_1)}{\nu_1 c_2^2}, \frac{\nu_0 (1-c_1)}{\nu_1} \right\}$ and adding to (3.2):

$$
(3.12) \quad \frac{1}{2} \frac{d}{dt} \left[ \tau \|\nabla A\|_{L^4(\Omega)} + \eta D_1 \|\partial_x \phi\|^2 + \eta D_2 \|\partial_y \phi\|^2 \right] \\
+ \frac{1}{2} \eta \nu_0 \|\Delta \phi\|^2 + \eta d_3 \|\nabla \phi\|^2 + (1-c_1) (\phi^2, |A|^4) \\
+ \eta d_0 \|\nabla A\|^2 + \eta |A|^{6-p} + \|A\|_{L^p(\Omega)}^p \leq \frac{32}{27} |\Omega|,
$$

where $d_0 = \min \{hD_1, hD_2\}$.

Finally, we multiply (3.10) by $\eta_0 := \min \{1, \frac{\nu_0}{4c_2M_0^6} \}$ and add to (3.12)

$$
(3.13) \quad \frac{1}{2} \frac{d}{dt} E_\nabla(t) + \eta d_0 \|\nabla \phi\|^2 + \eta_0 \|\Delta A\|^2 \leq F_\nabla(t),
$$

where

$$
E_\nabla(t) := \tau \|\nabla A\|_{L^4(\Omega)} + \eta D_1 \|\partial_x \phi\|^2 + \eta D_2 \|\partial_y \phi\|^2 + 2\eta_0 \tau \|\nabla A\|^2,
$$

$$
F_\nabla(t) := 2\eta_0 \|\nabla A\|^2 + a_\varepsilon \eta_0 \|\phi\|^p_{L^p(\Omega)} + 4b_\varepsilon \eta_0 M_0^6 \|\nabla \phi\|^4 + \frac{32}{27} |\Omega|.
$$

The inequality (3.13) implies that

$$
\frac{d}{dt} [t E_\nabla(t)] - E_\nabla(t) + t\eta d_0 \|\nabla \phi\|^2 + t\eta_0 \|\Delta A\|^2 \leq tF_\nabla(t).
$$
We integrate the last inequality over the interval $(0,t)$, then use the estimates (2.20), (2.21), (3.5) and obtain the desired estimates

\[(3.14) \quad \|\nabla \phi(t)\| \leq M_T, \quad \|\nabla A(t)\| \leq M_T, \quad \forall t \in (0,T).\]

Thus the semigroup $S(t) : X^0 \to X^0$, $t \geq 0$ generated by the problem (1.1)-(2.1) is a compact semigroup. \hfill \Box

Thanks to (2.10) this semigroup is also bounded dissipative. Therefore the following theorem holds true (See, e.g. [17]).

**Theorem 3.2.** If $c_1 \in [0,1)$, then the semigroup $S(t) : X^0 \to X^0$, $t \geq 0$, generated by the problem (1.1)-(2.1) possesses a global attractor $A$, i.e. a compact, invariant set that attracts uniformly each bounded set of the phase space $X^0$.

4. **Basic Dynamics**

The purpose of this section is to provide some insight to the behaviour of the system of PDEs (1.1)-(1.2), by reducing it under special assumptions to systems of ODEs, that can in turn be studied as dynamical systems. In particular, we are interested in examining a) the possibility of pattern formation, and b) the special role that Theorem 3.2 gives to the critical value $c_1 = 1$.

4.1. **Polar Form.** Starting again from the equations (1.1)-(1.2), we consider the change of variables $A = \rho e^{i\psi}$ with $\rho = \rho(x,y,t)$ and $\psi = \psi(x,y,t)$. As a preliminary, we note that

\[(4.1) \quad \frac{\partial A}{\partial t} = e^{i\psi} \frac{\partial \rho}{\partial t} + i\rho e^{i\psi} \frac{\partial \psi}{\partial t},\]

(and similar for the derivatives w.r.t. $x$ and $y$), and

\[(4.2) \quad \Delta A = e^{i\psi} (\Delta \rho - \rho |\nabla \psi|^2) + i\rho e^{i\psi} \left( \Delta \psi + \frac{2\nabla \rho \cdot \nabla \psi}{\rho} \right)\]

Also

\[-2ic_1 \phi \frac{\partial A}{\partial y} + i \frac{\partial \phi}{\partial y} A = -2ic_1 \phi \left( e^{i\psi} \frac{\partial \rho}{\partial y} + i\rho e^{i\psi} \frac{\partial \psi}{\partial y} \right) + i\beta \frac{\partial \phi}{\partial y} \rho e^{i\psi} \]

\[e^{i\psi} \left( 2c_1 \rho \frac{\partial \psi}{\partial y} \right) + i\rho e^{i\psi} \left( -2c_1 \rho^{-1} \frac{\partial \rho}{\partial y} + \beta \frac{\partial \phi}{\partial y} \right)\]

and

\[A^* \frac{\partial A}{\partial y} = \rho e^{-i\psi} \left( e^{i\psi} \frac{\partial \rho}{\partial y} + i\rho e^{i\psi} \frac{\partial \psi}{\partial y} \right) = \frac{\partial \rho}{\partial y} + i\rho^2 \frac{\partial \phi}{\partial y}\]
The system then can be written, in terms of the polar variables, as

\[ \tau \frac{\partial \rho}{\partial t} = \Delta \rho - \rho |\nabla \psi|^2 + \rho + 2c_1 \rho \phi \frac{\partial \psi}{\partial y} - \phi^2 \rho - \rho^3 \]  \hspace{1cm} (4.3)  

\[ \tau \frac{\partial \psi}{\partial t} = \Delta \psi + \frac{2\nabla \rho \cdot \nabla \psi}{\rho} - 2c_1 \phi \rho^{-1} \frac{\partial \rho}{\partial y} + \beta \frac{\partial \phi}{\partial y} \]  \hspace{1cm} (4.4)  

\[ \frac{\partial \phi}{\partial t} = \text{div}(D \nabla \phi) - h \phi + \phi \rho^2 - c_2 \rho^2 \frac{\partial \psi}{\partial y} \]  \hspace{1cm} (4.5)  

4.2. Spatially Uniform Dynamics. We study first the case where all the variables are constant in space, i.e. \( \rho = \rho(t), \psi = \psi(t) \) and \( \phi = \phi(t) \). Equivalently, this can be thought of as the result of performing the rescaling \( x \to x/\varepsilon, y \to y/\varepsilon \) and dropping the higher order terms w.r.t. \( \varepsilon \). In any case, we end up with the following reduced system:

\[ \tau \frac{\partial \rho}{\partial t} = \rho(1 - \phi^2 - \rho^2) \]  \hspace{1cm} (4.6)  

\[ \tau \frac{\partial \psi}{\partial t} = 0 \]  \hspace{1cm} (4.7)  

\[ \frac{\partial \phi}{\partial t} = \phi(\rho^2 - h) \]  \hspace{1cm} (4.8)  

It follows immediately that the phase \( \psi \) of \( A \) is decoupled from the rest of the system, and can be ignored. The dynamics of the reduced system (4.6)&(4.8) depend on the value of the dampening parameter parameter \( h \):
(1) Fig. 4 (left), $h = 0$: There are two saddle points at $(\rho, \phi) = (\pm 1, 0)$. The line segment $\rho = 0$, $\phi \in (-1, 1)$ consists of degenerate unstable critical points, whereas the rest of the $\phi$ axis is made of stable degenerate critical points.

(2) Fig. 4 (right), $0 < h < 1$: There are three saddle points at $(\rho, \phi) = (\pm 1, 0)$ and $(0, 0)$, and four spiral sinks on the unit circle at $(\pm \sqrt{h}, \pm \sqrt{1-h})$. Each quadrant converges to the corresponding sink.

(3) Fig. 5 (left), $h = 1$: There is a saddle point at $(\rho, \phi) = (0, 0)$ and two degenerate stable critical points at $(\pm 1, 0)$. Points in each half-space $\phi < 0$ and $\phi > 0$ converge to the corresponding critical point.

(4) Fig. 5 (right), $h > 1$: Similar behavior to the critical case $h = 1$.

Based on these cases, we expect the possibility of pattern formation in the case when $0 < h < 1$, where there are two\(^1\) distinct non-trivial critical points $(\rho, \phi) = (\sqrt{h}, \pm \sqrt{1-h})$.

4.3. Reduced Dynamics in the presence of a Phase Gradient. To reintroduce the phase variable $\psi$ into the dynamics, we assume as before that $\rho = \rho(x/\epsilon, y/\epsilon, t)$ and $\phi = \phi(x/\epsilon, y/\epsilon, t)$, but $\psi = \psi(x/\epsilon, y, t/\epsilon)$, i.e. the variation of $\psi$ in the $y$ direction is significant.

\(^1\)Bearing in mind that $\rho$ is in fact the modulus $|A|$ of the complex number $A$, we are effectively only interested in the positive half-plane $\rho \geq 0$. 

**Figure 5.** Phase diagrams for the spatially uniform case with critical dampening parameter $h = 1$ and $h > 1$. 

![Phase diagrams for the spatially uniform case with critical dampening parameter h = 1 and h > 1.](image-url)
This leads to the reduced system
\[
\begin{align*}
\tau \frac{\partial \rho}{\partial t} &= \rho \left(-|\frac{\partial \psi}{\partial y}|^2 + 1 + 2c_1 \phi \frac{\partial \psi}{\partial y} - \phi^2 - \rho^2 \right) \\
\frac{\partial^2 \psi}{\partial y^2} &= 0 \\
\frac{\partial \phi}{\partial t} &= \phi(\rho^2 - h) - c_2 \rho \frac{\partial \psi}{\partial y}
\end{align*}
\]
Because of the second equation the derivative \( \frac{\partial \psi}{\partial y} \) is constant. Making the change of variables \( \chi = \frac{\partial \psi}{\partial y} = \text{const} \), we end up with the following system of ODEs:
\[
\begin{align*}
(4.9) \quad \tau \frac{d\rho}{dt} &= \rho \left[(1 - \rho^2) - (\phi - c_1 \chi)^2 - (1 - c_1^2)\chi^2 \right] \\
(4.10) \quad \frac{d\phi}{dt} &= -h \phi - \rho^2(\phi - c_2 \chi)
\end{align*}
\]
The locus of the critical points of this dynamical system, for a given phase gradient \( \chi \), is the set
\[
C_\chi = \{(\rho, \phi) | \phi = \frac{c_2 \chi \rho^2}{\rho^2 - h}\}\cap\{(\rho, \phi) | \rho^2 + (\phi - c_1 \chi)^2 = 1 + (c_1^2 - 1)\chi^2 \vee \rho = 0\}
\]
The rational curve \( \phi = \frac{c_2 \chi \rho^2}{\rho^2 - h} \) has three branches, with asymptotes at \( \rho = \pm \sqrt{h} \) and \( \phi = c_2 \chi \), whereas the second set is the union of a circle with center \((0, c_1 \chi)\) and radius \( \sqrt{1 + (c_1^2 - 1)\chi^2} \) and the line \( \rho = 0 \). The set always includes the point \((\rho, \phi) = (0, 0)\) and, depending on the values of the parameters \(c_1\) and \(c_2\) and the phase gradient \( \chi \), up to 3 more critical points in the positive half-plane \( \rho \geq 0 \), where the branches of the rational curve intersect the circle.

The qualitative difference in the dynamics, between the subcritical \( 0 < c_1 < 1 \) and supercritical \( c_1 > 1 \) cases, appears to be due to the location and size of the aforementioned circle on the phase diagram (see Fig. 6 and Fig. 7) and the induced presence of absence of nontrivial critical points. More specifically, for small values of the phase gradient, \( |\frac{\partial \psi}{\partial t}| \ll 1 \), the circle is centered near the origin and has radius approximately 1, yielding dynamics similar to the ones presented in the previous section. As the phase gradient \( \chi \) increases in magnitude, in the case \( 0 < c_1 < 1 \) the radius \( \sqrt{1 + (c_1^2 - 1)\chi^2} \) is decreasing until there are no other critical points but the origin (Fig. 6), to which all orbits are attracted. On the other hand, when \( c_1 > 1 \) the radius of the circle increases with higher values of \( |\frac{\partial \psi}{\partial t}| \) and there is always at least one more critical point apart from the origin (Fig. 7).
Figure 6. Phase diagram in the case of $0 < c_1 < 1$ (and $0 < h < 1$), and for different values of the phase gradient $\frac{\partial \psi}{\partial y}$.

Figure 7. Phase diagram in the case of $c_1 > 1$ (and $0 < h < 1$), and for different values of the phase gradient $\frac{\partial \psi}{\partial y}$.

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E-mail address: *kalantarova@campus.technion.ac.il*

†Department of Mathematics, Koç University, Istanbul, Turkey

‡Azerbaijan State Oil and Industry University, Baku, Azerbaijan

‡Lightricks Ltd., Jerusalem, Israel