The topology of equivariant Hilbert schemes

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Abstract

For $G$ a finite group acting linearly on $\mathbb{A}^2$, the equivariant Hilbert scheme $\text{Hilb}^r[\mathbb{A}^2/G]$ is a natural resolution of singularities of $\text{Sym}^r(\mathbb{A}^2/G)$. In this paper, we study the topology of $\text{Hilb}^r[\mathbb{A}^2/G]$ for abelian $G$ and how it depends on the group $G$. We prove that the topological invariants of $\text{Hilb}^r[\mathbb{A}^2/G]$ are periodic or quasipolynomial in the order of the group $G$ as $G$ varies over certain families of abelian subgroups of $\text{GL}_2$. This is done by using the Białynicki-Birula decomposition to compute topological invariants in terms of the combinatorics of a certain set of partitions.

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1 Introduction

Let $X$ be a smooth algebraic surface carrying the action of a finite group $G$. The equivariant Hilbert scheme $\text{Hilb}^r[X/G]$ (Sect. 1.1) is a generalization of the Hilbert scheme of points on $X$ that parameterizes certain $G$-equivariant subschemes. It is a natural resolution of singularities for the symmetric product $\text{Sym}^r(X/G)$ of the quotient space. In this paper, we study how the topology of these Hilbert schemes change as the group $G$ varies.

When $G$ is a cyclic group acting linearly on $X = \mathbb{A}^2$, we exhibit (Main Theorems A and B) periodicity and quasipolynomiality for the Betti numbers of $\text{Hilb}^r[\mathbb{A}^2]/G$ as the order of the group $G$ varies. The main tool is the combinatorics of balanced partitions (Sect. 1.3), and the proof is mostly combinatorial. To our knowledge, there is a priori no geometric relationship between the equivariant Hilbert schemes for the different groups we consider and it is an interesting question to find the geometric significance of these results.

1.1 Statement of main results

Let $G$ be a finite subgroup of $\text{GL}_2$. The stack quotient $[\mathbb{A}^2/G]$ of $\mathbb{A}^2$ by the action of $G$ is a smooth two dimensional orbifold with singular coarse moduli space $\mathbb{A}^2/G$. The Hilbert scheme of points $\text{Hilb}^r[\mathbb{A}^2/G]$ is a 2$r$-dimensional quasiprojective scheme parameterizing flat families of substacks of $[\mathbb{A}^2/G]$ with constant Hilbert polynomial $r$ [31, Theorem 1.5]. Equivalently, $\text{Hilb}^r[\mathbb{A}^2/G]$ is the moduli space of $G$-equivariant ideals $I \subset \mathbb{C}[x,y]$ such that

$$\mathbb{C}[x,y]/I \cong \mathbb{C}[G]^r$$

as representations [25, Proposition 2.9]. $\text{Hilb}^r[\mathbb{A}^2/G]$ is a union of irreducible components of the fixed locus $(\text{Hilb}^r/G)(\mathbb{A}^2)^G$ [9, Proposition 4.1]. In fact $\text{Hilb}^r[\mathbb{A}^2/G]$ is smooth (Sect. 2.3).

There is a Hilbert–Chow morphism

$$\text{Hilb}^r[\mathbb{A}^2/G] \rightarrow \text{Sym}^r(\mathbb{A}^2/G)$$

sending an ideal to its support in the coarse moduli space. The restriction of this morphism to the component of $\text{Hilb}^r[\mathbb{A}^2/G]$ containing the locus of $r$ distinct free $G$-orbits is a resolution of singularities $^1$. When $r = 1$, $\text{Hilb}^1[\mathbb{A}^2/G] \rightarrow \mathbb{A}^2/G$ is the minimal resolution [24, Theorem 5.1] [22, Theorem 3.1].

From now on, we restrict to $G$ abelian. In Sect. 4.1, we will reduce our analysis to when the group is cyclic. To this end, we consider $G$ cyclic of order $n$ acting on $\mathbb{A}^2$ by $(x, y) \mapsto (\zeta^a x, \zeta^b y)$ where $\zeta$ is a primitive $n$th root of unity and $\text{gcd}(a, b) = 1$. We will denote this group by $G_{a,b,n}$ and the equivariant Hilbert scheme $\text{Hilb}^r[\mathbb{A}^2/G_{a,b,n}]$ by $H^r_{a,b,n}$.

The first result of this article concerns the behavior of the compactly supported Betti numbers $b_i(H^r_{a,b,n})$.

**Main Theorem A** If $ab > 0$, then $b_i(H^r_{a,b,n}) = b_i(H^r_{a,b,n+ab})$ for all $r > 0$ and $n > rab$.

That is, the Betti numbers of $H^r_{a,b,n}$ are eventually periodic in $n$ with period $ab$. The proof of Main Theorem A uses the Białynicki-Birula decomposition to stratify $H^r_{a,b,n}$ by locally

\(^1\text{When } G \text{ is abelian, } \text{Hilb}^r[\mathbb{A}^2/G] \text{ is connected (Corollary 2.3) and so is itself a resolution. When } G \subset \text{SL}_2, \text{Hilb}^r[\mathbb{A}^2/G] \text{ is a Nakajima quiver variety [35, Theorem 2] and so is connected [30, Theorem 6.2]. The case for general } G \text{ is unknown to the authors.}
closed affine cells. Thus, the statement of Main Theorem A lifts to the Grothendieck ring of varieties $K_0(V_C)$.

**Theorem 1.1** For $ab > 0$, the class $[H^r_{a,bn}]$ in $K_0(V_C)$ is a polynomial in $\mathbb{L} = [A^1]$ whose coefficients are periodic in $n$ with period $ab$ for $n > rab$. In particular, any motivic invariants of $H^r_{a,bn}$ are eventually periodic in $n$.

When $a = b = 1$ and $n = 3$, this explains an observation of Gusein-Zade, Luengo, and Melle-Hernandez [19, pg. 601].

Our second main result examines the behavior of the topological invariants when $ab < 0$. Recall that a function $f : \mathbb{Z} \to \mathbb{Z}$ is called quasipolynomial of period $k$ if there are polynomials $p_1, \ldots, p_k$ such that $f(n) = p_i(n)$ where $n \equiv l \mod k$.

**Main Theorem B** If $ab < 0$, then the compactly supported topological Euler characteristic $\chi_c(H^r_{a,bn})$ is a quasipolynomial in $n$ with period $|ab|$ for all $r > 0$ and $n \gg 0$.

1.2 Background and motivation

Equivariant Hilbert schemes were first introduced by Ito and Nakamura [21] for finite subgroups $G \subset SL_2$. They play a central role in the Mckay correspondence (see, for example, [4,5,33]). Indeed much of the geometry of $\text{Hilb}^1[A^2/G]$ is determined in this case by the representation theory of $G$. On the other hand, very little is known about equivariant Hilbert schemes for general finite subgroups $G \subset GL_2$ (apart from the case $r = 1$, see, for example, [22,24]).

When $G = G_{a,bn}$ is cyclic, we can use the combinatorics of balanced partitions (Sect. 1.3) to access $\text{Hilb}^1[A^2/G_{a,bn}]$. Balanced partitions carry both geometric and topological information. For example, they determine an open affine cover of $\text{Hilb}^1[A^2/G_{a,bn}]$ whose coordinate rings can be written purely combinatorially from the partitions (Sect. 2). The hope is that the combinatorial bijections used in the proofs of Theorems 1.2 and 1.3 have an interpretation on the level of the equivariant Hilbert schemes themselves that will lead to a geometric explanation for the periodicity and quasipolynomiality phenomena.

1.2.1 Toric resolutions and continued fractions

The particular case of $r = 1$, $a = 1$ and $b = k > 0$ is instructive. Then, $A^2/G_{1,k,n}$ is the affine toric variety corresponding to the cone generated by $(0,1)$ and $(n,-k)$ and $\text{Hilb}^1[A^2/G_{1,k,n}]$ is the toric minimal resolution. It then follows from a result of Hirzebruch [10, Theorem 10.2.3] that the virtual Poincare polynomial of $H^1_{1,k,n}$ is of the form

$$
\nu_{H^1_{1,k,n}}(z) = l z^2 + z^k
$$

where $l$ is the length of Hirzebruch–Jung continued fraction expansion

$$
\frac{n}{k} = \left[\frac{a_1, \ldots, a_l}{1}\right] := a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{a_4 - \ddots}}}
$$

and $a_l > 1$. This is evidently periodic in $n$ with period $k$.

A similar computation when $r = 1$, $a = 1$ and $b = -k < 0$ yields the singular toric variety with supplementary cone. Then, the Poincare polynomial takes the same form where $l$ is the length of the continued fraction expansion of $\frac{n}{n-k}$. Quasipolynomiality can then be deduced from a geometric duality between the continued fractions of supplementary cones [32, Proposition 2.7]. In fact, the length of the continued fraction expansion of $\frac{n}{n-k}$ is a linear quasipolynomial in $n$ (Fig. 1).
For $r > 1$, we provide an analogue of the continued fraction expansion given by the set of balanced partitions defined below. We will see that the balanced partitions control the topology of the Hilbert scheme resolution of $\text{Sym}^r(\mathbb{A}^2/G_{a,b,n})$ the same way the continued fraction controls the topology of the minimal resolution of $\mathbb{A}^2/G_{1,k,n}$. Furthermore, Theorems 1.2 and 1.3, from which we deduce the main theorems, can be seen as a higher dimensional analogue of the geometric duality for continued fractions.

1.2.2 Further questions and speculations

Ultimately, the goal is to understand the total cohomology

$$\mathbb{H}_{a,b,n} := \bigoplus_{r \geq 0} H^r_c(H_{a,b,n}', \mathbb{Q})$$

and compute its graded character which is the generating function of the Betti numbers $b_i(H_{a,b,n}')$. When $(a, b) = (1, -1)$ so that $G_{1, -1,n} \subset SL_2$, $H_{1,-1,n}'$ is diffeomorphic to $\text{Hilb}^r(H_{1,-1,n})$ [28, Lemma 4.1.3] and the Göttsche formula [17, Theorem 0.1] computes this generating function as an infinite product. After specializing to the Euler characteristic, we can deduce the formula

$$\sum_{r \geq 0} \chi_c(H_{1,-1,n}') h^r = \left( \prod_{t \geq 1} \left( \frac{1}{1 - t^r} \right) \right)^n$$

from the cores-and-quotients bijective ([26, Example 1.1.8] pages 12-13). Related infinite product expansions have been computed in [36] for 3-dimensional abelian orbifolds and in [16] for surface orbifolds using similar combinatorial methods.

The work of Nakajima [29, 30] explains these infinite product formulas using representation theory of infinite dimensional Lie algebras. In particular, $\mathbb{H}_{1, -1,n}$ is a highest weight irreducible representation of a certain Heisenberg Lie algebra and this action intertwines two natural bases of $\mathbb{H}_{1, -1,n}$ coming from cores-and-quotients [28]. We expect a similar picture to be true for the more general equivariant Hilbert schemes $H'_{a,b,n'}$.

**Question 1.1** Does $\mathbb{H}_{a,b,n}$ carry a natural action of an infinite dimensional Lie algebra $\mathcal{H}_{a,b,n}$ that can be described combinatorially in terms of balanced partitions?

Computer computations with balanced partitions suggest the answer to Question 1.1 is yes and furthermore that $\mathcal{H}_{a,b,n}$ is generated in degrees $r$ for $rab < n$. This particular bound is interesting because it is the bound appearing in Main Theorem A. This suggests that if Question 1.1 has an affirmative answer, then there is some relationship between the Lie algebras $\mathcal{H}_{a,b,n}$ and $\mathcal{H}_{a,b,n+ab}$ and their representations on the corresponding cohomologies at least when $ab > 0$. 

![Fig. 1](image-url) The supplementary cones which correspond to the affine toric varieties $\mathbb{A}^2/G_{1,k,n}$ and $\mathbb{A}^2/G_{1,-k,n}$.
Moreover, these computations suggest that the Betti number generating function for $H^r_{a,b;n}$ is in general not an infinite product when $G_{a,b;n}$ is not in $SL_2$, but rather is a quasimodular form that can be written as a finite sum of infinite products. This is part of a general picture that generating functions for sheaf counting invariants on surfaces have modular properties (see [18] for a survey on this phenomena). Indeed the Euler characteristics and Poincaré polynomials of $H^r_{a,b;n}$ are naive Donaldson–Thomas-type invariants $^2$ and the modularity property, if true, would be an analogue of $S$-duality [34] for the quotient orbifolds $[h^2/G_{a,b;n}]$. It would then be an interesting question to consider how the structure of these generating functions interacts with the stabilization properties from Main Theorems A and B.

1.3 Balanced partitions

Main Theorems A and B are proved by expressing the invariants above in terms of counting certain colored partitions or Young diagrams. We call these balanced partitions.

A partition $\lambda$ of a natural number $m$ is a sequence of nonnegative integers $\lambda_1 \geq \ldots \lambda_l \geq 0$ such that $\lambda_1 + \ldots + \lambda_l = m$. We identify $\lambda$ with its Young diagram, which is a subset of $m$ boxes arranged as left justified rows so that the $i$th row contains $\lambda_i$ boxes. We will also index the boxes in $\lambda$ by coordinates in $\mathbb{Z}_{\geq 0}^2$, so as in the diagram below. We denote by $l(k)$ (resp $c(h)$) the number of blocks in the $k$th row (resp $h$th column) of $\lambda$.

Anticipating that the boxes $(i,j)$ correspond to monomials $x^i y^j$ having $G_{a,b;n}$-weight $ai + bj$, we define the $G_{a,b;n}$-content of a box $(i,j)$ of a partition to be $ai + bj \mod n$. This assigns an element of $G_{a,b;n}$ to each box in $\lambda$. We say $\lambda$ is an $(a,b;n)$-balanced partition if each element of $G_{a,b;n}$ is equally represented among the $G_{a,b;n}$-contents of $\lambda$.

In particular, the size of any $(a,b;n)$-balanced partition must be a multiple of $n$ (Fig. 2).

Denote the set of all $(a,b;n)$-balanced partitions of $rn$ by $B^r_{a,b;n}$. There is a function $\beta : B^r_{a,b;n} \to \mathbb{Z}_{\geq 0}$ we call the Betti statistic (Definition 3.1). We will show the following proposition using the Białyńnicki-Birula decomposition.

**Proposition 1.1** The Betti numbers of $H^r_{a,b;n}$ are given by

$$b_i(H^r_{a,b;n}) = \#\{\lambda \in B^r_{a,b;n} : 2\beta(\lambda) = i\}.$$

$^2$See, for example, [2,8] for Hilbert scheme invariants from the point of view of Donaldson–Thomas theory.
In particular, the Poincaré polynomial \( P_{H^r_{a,b,n}}(z) \) of \( H^r_{a,b,n} \) satisfies

\[
P_{H^r_{a,b,n}}(z) = \sum_{\lambda \in B^r_{a,b,n}} z^{2B(\lambda)}.
\]

The main theorems will then follow from the following combinatorial results.

**Theorem 1.2** Fix integers \( r > 0 \) and \( a, b \) with \( ab > 0 \). There is a natural bijection \( B^r_{a,b,n} \to B^r_{a,b,n+ab} \) that preserves the Betti statistic for \( n > rab \).

**Theorem 1.3** Fix integers \( r > 0 \) and \( a, b \) with \( ab < 0 \). The cardinality \( \#B^r_{a,b,n} \) is a quasipolynomial in \( n \) of period \( |ab| \) for \( n \gg 0 \).

2 The geometry of \( H^r_{a,b,n} \)

In this section, we give a systematic description of the geometry of \( H^r_{a,b,n} \). We discuss the natural torus action on \( H^r_{a,b,n} \) as well as smoothness and irreducibility.

2.1 Torus actions

The algebraic torus \( T = (\mathbb{C}^*)^2 \) acts naturally on \( \mathbb{A}^2 \) or equivalently on \( \mathbb{C}[x,y] \) by \((t_1, t_2)(x, y) = (t_1x, t_2y)\). This induces an action on \( \text{Hilb}^m(\mathbb{A}^2) \) by pulling back ideals,

\[
(t_1, t_2) \cdot I = \{ f(t_1x, t_2y) : f \in I \}.
\]

The fixed points of this action are the doubly homogeneous ideals, that is, the monomial ideals. These are in one-to-one correspondence with partitions \( \lambda \) of \( m \) by the assignment \( \lambda \mapsto I_{\lambda} = \{ x^iy^j : (i, j) \in \lambda \} \).

Define \( B_{\lambda} = \{ x^hy^k : (h, k) \in \lambda \} \). It is clear that \( B_{\lambda} \) forms a basis for \( \mathbb{C}[x,y]/I_{\lambda} \) so that \( I_{\lambda} \in \text{Hilb}^m(\mathbb{A}^2) \).

Every monomial ideal is fixed by \( G_{a,b,n} \). However, \( I_{\lambda} \in H^r_{a,b,n} \) if and only if \( \mathbb{C}[x,y]/I_{\lambda} = \mathbb{C}B_{\lambda} \) is isomorphic as a \( G_{a,b,n} \) representation to \( \mathbb{C}[G_{a,b,n}] \). The space \( \mathbb{C}B_{\lambda} \) decomposes as a direct sum of irreducible representations \( \mathbb{C}x^iy^j \) for \((i, j) \in \lambda \) each with weight \( ai + bj \mod n \). Since \( \mathbb{C}[G_{a,b,n}] \) decomposes as a direct sum of one copy of each irreducible representation, \( \mathbb{C}[G_{a,b,n}] \) must have \( r \) copies of each. Thus, each weight must appear \( r \) times in the decomposition of \( \mathbb{C}B_{\lambda} \) so we have proved the following:

**Lemma 2.1** The \((\mathbb{C}^*)^2\)-fixed points in \( H^r_{a,b,n} \) are in one-to-one correspondence with \( B^r_{a,b,n} \) the set of \((a, b; n)\)-balanced partitions of \( rn \).

2.2 Local theory of Hilbert schemes

In this section, we recall facts about the local geometry of \( \text{Hilb}^m(\mathbb{A}^2) \) following Haiman’s description given in [20].

One can define a torus invariant open affine neighborhood \( U_{\lambda} \) of \( I_{\lambda} \) given by

\[
U_{\lambda} := \{ I : \mathbb{C}[x,y]/I \text{ is spanned by } B_{\lambda} \} \subset \text{Hilb}^m(\mathbb{A}^2).
\]

The coordinate functions on \( U_{\lambda} \) are given by \( c_{ij}^{ls}(I) \) for \((i, j) \in \lambda \) and \((l, s) \in \mathbb{Z}^2_{\geq 0} \) where

\[
x^iy^s = \sum_{(i,j) \in \lambda} c_{ij}^{ls}(I)x^iy^j \mod I.
\]
Multiplying (1) by $x$, we obtain
\[ x^{l+1} y^s = \sum_{(h,k) \in \lambda} c_{h,k}^{l,s} x^{h+1} y^k = \sum_{(h,k) \in \lambda} \sum_{(i,j) \in \lambda} c_{h,k}^{l,s} c_{i,j}^{h+1,k} x^i y^j. \]

Therefore, the coefficients satisfy the relations
\[ c_{i,j}^{l+1,s} = \sum_{(h,k) \in \lambda} c_{h,k}^{l,s} c_{i,j}^{h+1,k}. \]

Similarly, we obtain the relation
\[ c_{i,j}^{l,s+1} = \sum_{(h,k) \in \lambda} c_{h,k}^{l,s} c_{h,k+i,j}^{h+1,k+1} \]
by multiplying by $y$.

We will often denote the function $c_{i,j}^{l,s}$ as an arrow on the $\mathbb{Z}_{\geq 0}^2$ grid pointing from box $(l,s) \in \mathbb{Z}_{\geq 0}^2$ to box $(i,j) \in \lambda$. These functions $c_{i,j}^{l,s}$ are torus eigenfunctions with action given by
\[ (t_1, t_2) \cdot c_{i,j}^{l,s} = t_1^{l-i} t_2^{s-j} c_{i,j}^{l,s}. \]
Consequently, $G_{a,b,n}$ acts by
\[ c_{i,j}^{l,s} \mapsto \zeta^{a(l-i)+b(s-j)} c_{i,j}^{l,s}. \]

The actions commute so that $\text{Hilb}^m(A_2^{\times}) G_{a,b,n}$, and thus, $H^r_{a,b,n}$, inherits a $(\mathbb{C}^*)^2$ action.

For each box $(i,j) \in \lambda$, define two distinguished coordinate functions
\[ d_{ij} := c_{i,j}^{l(i),l(c(i))} \quad u_{ij} := c_{i,j}^{l(c(i)-1),l(j)-1}. \]
where $l(i)$ is the size of the $i$th column, $c(i)$ the size of the $i$th column of $\lambda$. We can picture $d_{ij}$ and $u_{ij}$ as southwest and northeast pointing arrows hugging the diagram. Note that each diagram has $2m$ such distinguished arrows associated with it, two for each box (Fig. 3).

Now, we can use these arrows to understand cotangent space to $I_\lambda \in \text{Hilb}^m(A_2^{\times})$ which we will denote $T^*_\lambda \text{Hilb}^m(A_2^{\times}) := m(I_\lambda)/m(I_\lambda)^2$. The set of $c_{i,j}^{l,s}$ vanishing at $I_\lambda$ are precisely the ones for $(l,s) \notin \lambda$. These generate $T^*_\lambda \text{Hilb}^m(A_2^{\times})$. The relation (2) expresses $c_{i,j}^{l+1,s}$ as $c_{i,j-1}^{l+1,s} + (\text{higher order terms})$ since $c_{i,j}^{l+1,s} \equiv 1$ and $c_{i,j}^{l,s} \equiv 0$ for $(i,j) \neq (\tilde{i},\tilde{j}) \in \lambda$. Thus,
\[ c_{i,j}^{l+1,s} = c_{i-1,j}^{l,s} \mod m(I_\lambda)^2. \]
as local parameters in $T^*_H\text{Hilb}^m(\mathbb{A}^2)$. Similarly, (3) implies that
\[ c_{l,s}^{i,j+1} = c_{l,s}^{i,j-1} \mod (I_\lambda)^2 \] (6)
in $T^*_H\text{Hilb}^m(\mathbb{A}^2)$.

If we denote $c_{l,s}^{i,j}$ as an arrow, then (5) and (6) imply that if we slide an arrow horizontally or vertically while keeping $(l,s) \in \mathbb{Z}_{\geq 0}^2 \setminus \lambda$ and $(i,j) \not\in \mathbb{Z}_{\geq 0}^2 \setminus \lambda$, then the arrow represents the same local parameter in $T^*_H\text{Hilb}^m(\mathbb{A}^2)$. Furthermore, if an arrow can be moved so that the head leaves the $\mathbb{Z}_{\geq 0}^2$ grid, then it is identically zero in $T^*_H\text{Hilb}^m(\mathbb{A}^2)$ because only positive degree monomials appear in $\mathbb{C}[x,y]$. In this way, every northwest pointing arrow vanishes in $T^*_H\text{Hilb}^m(\mathbb{A}^2)$ and any southwest or northeast pointing arrow can be moved until it either vanishes or is of the form $d_{i,j}$ or $u_{i,j}$, respectively. This proves the following:

**Proposition 2.1** [20, Proposition 2.4], [14, Theorem 2.4] The set \{
\begin{align*}
&d_{i,j}, u_{i,j} \\
&
\end{align*}
\} over \((i,j) \in \lambda\) generates the cotangent space of $I_{\lambda} \in \text{Hilb}^m(\mathbb{A}^2)$. In particular, $\text{Hilb}^m(\mathbb{A}^2)$ is smooth.

2.3 The cotangent space to $I_{\lambda} \in H^{r}_{a,b,n}$

We give a description of the weight space decomposition of the cotangent space to any monomial ideal $I_{\lambda} \in H^{r}_{a,b,n}$. This will be used later to compute the Białynicki-Birula cells.

By Proposition 2.1, $\text{Hilb}^m(\mathbb{A}^2)$ is smooth. It follows that the $G_{a,b,n}$-fixed locus is also smooth [15, Proposition 4]. In particular, the component $H^{r}_{a,b,n}$ is smooth. Moreover, since $G_{a,b,n}$ acts by scaling on $c_{l,s}^{i,j}$, then $c_{l,s}^{i,j}$ restricts to be nonzero on the fixed locus if and only if $G_{a,b,n}$ acts trivially on $c_{l,s}^{i,j}$. Thus, the functions $c_{l,s}^{i,j}$ for $a(l-i) + b(s-j) \equiv 0 \mod n$ generate the coordinate ring of $U_{H^{r}_{a,b,n}}$. These correspond to the arrows that start and end on a box with the same color. We call these arrows invariant (Fig. 4).

**Proposition 2.2** Let $\lambda \in B^{r}_{a,b,n}$. The cotangent space to $I_{\lambda} \in H^{r}_{a,b,n}$ has basis given by the set of $d_{i,j}$ and $u_{i,j}$ that are invariant.

**Proof** By the discussion above, these are the only local parameters of $\text{Hilb}^m(\mathbb{A}^2)$ that restrict to be nonzero in a neighborhood of $I_{\lambda}$ in $H^{r}_{a,b,n}$. On the other hand, $G_{a,b,n}$ acts trivially on the invariant arrows so they remain linearly independent in the cotangent space of the fixed locus.

**Corollary 2.1** Let $\lambda$ be an $(a, b, n)$-balanced partition of $m$. Then, exactly $2r$ of the arrows of the form $d_{i,j}$ or $u_{i,j}$ are invariant.
Proof The number of such arrows is the dimension of the cotangent space to \( I_a \in H^r_{a,brn} \).
On the other hand, \( H^r_{a,brn} \) is smooth and connected, so the Hilbert–Chow morphism is birational (see 2.3). The result then follows since \( \dim H^r_{a,brn} = 2r \). \( \square \)

Let \( T^*_A H^r_{a,brn} \) denote the cotangent space to the torus fixed point \( I_a \in H^r_{a,brn} \). Denote by \( V(a, b) \) for \( (a, b) \in \mathbb{Z}^2 \) the irreducible representation of \( (\mathbb{C}^*)^2 \) on which \( (t_1, t_2) \) acts by \( t_1^a t_2^b \).

**Corollary 2.2** The weight space decomposition of \( T^*_A H^r_{a,brn} \) as a representation of \( (\mathbb{C}^*)^2 \) is given by

\[
\bigoplus_{d_{ij} \text{ invariant}} V(l(j) - i, j - c(i) + 1) \oplus \bigoplus_{u_{ij} \text{ invariant}} V(i - l(j) + 1, c(i) - j).
\]

**Proof** \((\mathbb{C}^*)^2\) acts on \( e_{i,j} \) by \( t^i_1 t^j_2 \). Then, we get the result by Proposition 2.2 as well as the definition (4) of \( d_{ij} \) and \( u_{ij} \). \( \square \)

**Remark 2.1** In the literature, the weight space decomposition of the tangent space is often described in terms of the arm and leg of a box \((i, j) \in \lambda\). This description is equivalent because

\[
l(j) - i = \text{arm}(i, j) + 1 \quad c(i) - j = \text{leg}(i, j) + 1.
\]

### 2.4 Connectedness of \( H^r_{a,brn} \)

We explain why \( H^r_{a,brn} \) is connected. The idea is that for any ideal \( I \in H^r_{a,brn} \), picking a monomial order \( \varpi \) and taking initial degeneration to a monomial ideal \( I_0 := \text{in}_\varpi I \) give a rational curve in \( H^r_{a,brn} \) so that every ideal lies in the same connected component as a monomial ideal. Then, one must show that all the monomial ideals are connected by chains of rational curves. This is done more generally in [27] for multigraded Hilbert schemes. In this section, we will deduce connectedness from the results of [27].

Let \( R = \mathbb{C}[x, y] = \bigoplus_A R_a \) be the polynomial ring graded by some abelian group \( A \). For any function \( h : A \rightarrow \mathbb{Z}_{\geq 0} \), the **multigraded Hilbert scheme** \( \text{Hilb}^h(R) \) is the subvariety of \( \text{Hilb}(\mathbb{A}^2) \) parameterizing homogeneous ideals \( I \subset R \) such that

\[
\dim_{\mathbb{C}}(R/I)_a = h(a).
\]

That is, \( \text{Hilb}^h(R) \) is the moduli space of homogeneous ideals with Hilbert function \( h \).

The equivariant Hilbert scheme \( H_{a,brn}^r \) is a special case as follows. Let \( G \subset GL_2 \) be a finite abelian group, and let \( A \) be the dual group \( \text{Hom}(G, \mathbb{C}^*) \) of characters of \( G \). Then, the action of \( G \) on \( \mathbb{A}^2 \) induces an \( A \)-grading on \( R \) by

\[
R_a := \{ p(x) \in R : g \cdot p(x) = a(g)p(x) \text{ for all } g \in G \}.
\]

It is easy to see that an ideal is homogeneous if and only if it is \( G \)-invariant. Furthermore, each \( a \in A \) is the character of some irreducible representation of \( G \), so the condition \( R/I \cong \mathbb{C}[G]^r \) as representations of \( G \) is equivalent to \( \dim_{\mathbb{C}}(R/I)_a = r \) for each \( a \in A \).

Therefore,

\[
\text{Hilb}^h(R) = \text{Hilb}^r[\mathbb{A}^2/G]
\]

where \( R \) is \( A \)-graded by the action of \( G \) and \( h(a) = r \) for all \( a \).

Connectedness now follows from the following theorem of Maclagan and Smith:
Theorem 2.1 [27, Theorem 3.15] $\text{Hilb}^h(R)$ is rationally chain connected for any function $h : A \to \mathbb{Z}_{\geq 0}$ satisfying
\[\sum_{a \in A} h(a) < \infty.\]

Corollary 2.3 $H_{a,b}^r$ is irreducible, and the Hilbert–Chow morphism
\[H_{a,b}^r \to \text{Sym}^r(\mathbb{A}^2/G_{a,b,n})\]
is a resolution of singularities.

3 The Białynicki-Birula stratification

In this section, we will show how to reduce the problem of computing Betti numbers of $H_{a,b}^r$ to counting $(a, b; n)$-balanced partitions of $m$ with the Betti statistic (see Definition 3.1). The idea is to use the action of an algebraic torus $(\mathbb{C}^*)^2$ on $H_{a,b}^r$ and the theory of Białynicki-Birula [1] to stratify $H_{a,b}^r$ into affine cells. Then, a local analysis of the torus action at fixed points yields the appropriate statistic giving the Betti numbers.

These techniques are standard in the theory of Hilbert schemes of points (see, for example, [3,12,13,25]).

3.1 The Białynicki-Birula Decomposition Theorem

Let $S = (\mathbb{C}^*)^m$ be an algebraic torus and $X$ a smooth quasiprojective variety on which $S$ acts. Suppose the fixed point locus $X^S = \{p_1, \ldots, p_l\}$ is finite. Then for a generic one-dimensional subtorus $T \subset S$, we have $X^T = X^S$. We further assume that
\[\lim_{t \to 0} t \cdot x \in X^S \text{ exists for all } x \in X.\]

In this case, define
\[X_j := \{x : \lim_{t \to 0} t \cdot x = p_j\}.\]

Then, the $X_j$ are locally closed and $X = \bigsqcup X_j$.

The action of $T$ on $X$ induces an action of $T$ on the tangent space $T_{p_j}X$. Define $T_{p_j}^X$ to be the subspace of vectors on which $T$ acts with positive weight, and let $n_j$ be its dimension.

Theorem 3.1 (Białynicki-Birula Decomposition Theorem [1, Theorem 4.4])\(^3\) Let $T \subset S$ and $X$ as above. Then, each locally closed stratum $X_j$ is isomorphic to an affine space $\mathbb{A}^{n_j}$ so that
\[X = \bigsqcup_{j=1}^l \mathbb{A}^{n_j}.\]

Furthermore, the $i$th compactly supported Betti number $b_i = \dim H^i_c(X, \mathbb{Q})$ is given by
\[#\{j : 2n_j = i\}.

\(^3\)Białynicki-Birula originally proved this theorem for $X$ projective. The version we use here for quasiprojective $X$ is obtained by taking a torus equivariant compactification. See, for example, [2, Lemma B.2].
3.2 The stratification of $H_{a,b}^r$; $n$

We will apply the above results to the action of $S = (\mathbb{C}^*)^2$ on $H_{a,b}^r$. As we saw (Lemma 2.1), the fixed points are indexed by balanced partitions $\lambda$. We pick $T = (t^{-p}, t^{-q}) \subset S$ for generic $p \gg q > 0$ so that $(H_{a,b}^r)^T$ consists of only the monomial ideals.

**Lemma 3.1** For all $I \in H_{a,b}^r$, the limit
\[
\lim_{t \to 0, t \in T} t \cdot I = I_0
\]
exists in $H_{a,b}^r$.

**Proof** Consider the monomial partial order given by weight $(p, q)$. That is, $x^iy^j > x^iy^j'$ if and only if $lp + sq > ip + jq$. Let $f \in I$ be any polynomial with leading term $x^iy^j$ under this monomial partial order. Then for $t \in T$,
\[
t \cdot f = t^{-(pl+qs)}x^iy^j + \sum_{pi+qj < pl+qs} t^{-(pl+qs)}x^iy^j \in t \cdot I.
\]

Multiplying through by $t^{pl+qs}$ gives $x^iy^j + t$(lower order terms) $\in t \cdot I$. So in the limit as $t \to 0$, we get $x^iy^j$. The dimension $\dim \mathbb{C} [x,y]/I = rn$ is fixed, so the degree of the polynomials in a Gröbner basis of $I$ is bounded [11, Theorem 8.2]. Since we are taking $p \gg q > 0$, all polynomials of bounded degree have a unique leading term under this monomial partial order, so the limit ideal is the initial monomial ideal generated by these leading terms. The initial ideal $I_0$ is a flat limit of the family of ideals $I_t := t \cdot I$, so $I_0 \in H_{a,b}^r$ is a monomial ideal corresponding to some balanced partition. \(\square\)

Applying Theorem 3.1 gives a decomposition of $H_{a,b}^r$ indexed by balanced partitions $\lambda \in B_{a,b}^r$:
\[
H_{a,b}^r = \bigsqcup_{\lambda \in B_{a,b}^r} \mathbb{A}_n(\lambda)
\]
where $n(\lambda)$ is the dimension of the positive weight subspace $T^+_{\lambda}H_{a,b}^r \subset T_{\lambda}H_{a,b}^r$ of the tangent space at $I_\lambda$.

**Definition 3.1** Define the Betti statistic function $\beta : B_{a,b}^r \to \mathbb{Z}_{\geq 0}$ as follows:
\[
\beta(\lambda) = \#(d_{ij} \text{ invariant}) + \#(u_{ij} \text{ invariant and vertical}).
\]
That is, $\beta(\lambda)$ is the number of invariant arrows on $\lambda$ that are pointing either strictly north or weakly southwest (Fig. 5).

**Remark 3.1** Note from the definition (4) of $u_{ij}$, it is vertical if and only if $i = l(j) - 1$.

**Proposition 3.1** For any $\lambda \in B_{a,b}^r$, we have $\beta(\lambda) = \dim T^+_{\lambda}H_{a,b}^r$.

**Proof** Corollary 2.2 gives us the weight space decomposition of the cotangent space $T^+_{\lambda}H_{a,b}^r$. The tangent space $T_{\lambda}H_{a,b}^r$ is the dual space and so has weight space decomposition
\[
\bigoplus_{d_{ij} \text{ invariant}} V(-(l(j) - i), -(j - c(i) + 1)) \oplus \bigoplus_{u_{ij} \text{ invariant}} V(-(i - l(j) + 1), -(c(i) - j)).
\]
Considering the subtorus \( T = (t^{-p}, t^{-q}) \), we see the weight spaces for this subtorus are generated by the invariant \( d_{i,j} \) with weight \( p(l(j) - i) + q(j - c(i) + 1) \) and invariant \( u_{i,j} \) with weight \( l(i - r(j) + 1) + q(c(i) - j) \). The \( d_{i,j} = c_{r(j) - l(j) - 1} \) arrows point southwest and so satisfy \( l(j) > i \). Since \( p \gg q > 0 \), this means the weight \( p(l(j) - i) + q(j - c(i) + 1) > 0 \).

On the other hand, a \( u_{i,j} = c_{r(i) - c(i) - 1} \) vector points northeast. If it points strictly northeast, then \( c(i) > j \) but \( l(j) - 1 < i \) and so the weight \( p(i - l(j) + 1) + q(c(i) - j) < 0 \). If it points strictly north, then \( l(j) - 1 = i \) and \( c(i) > j \) so that \( p(i - l(j) + 1) + q(c(i) - j) = q(c(i) - j) > 0 \). Therefore, the positive weight vectors are exactly counted by the Betti statistic. \( \square \)

This proves Proposition 1.1 which we repeat here for convenience:

**Proposition 1.1** The Betti numbers of \( H^r_{a,b,n} \) are given by

\[
b_i(H^r_{a,b,n}) = \# \{ \lambda \in H^r_{a,b,n} : 2\beta(\lambda) = i \}.
\]

In particular, the Poincare polynomial \( P_{H^r_{a,b,n}}(z) \) of \( H^r_{a,b,n} \) satisfies

\[
P_{H^r_{a,b,n}}(z) = \sum_{\lambda \in B^r_{a,b,n}} z^{2\beta(\lambda)}
\]

and the topological Euler characteristic is given by \( \chi(H^r_{a,b,n}) = \# B^r_{a,b,n} \).

This reduces Main Theorems A and B to the combinatorial statements in Theorems 1.2 and 1.3. The proofs of these will be given in Sect. 4.

### 3.3 Grothendieck ring of varieties

In this section, we will discuss the Grothendieck ring of varieties. Due to the Białynicki-Birula decomposition, any statements about Betti numbers (for example, Main Theorem A) lift to the Grothendieck ring of varieties.

Recall the **Grothendieck ring of varieties** \( K_0(V_C) \) is the ring generated by isomorphism classes \([X]\) of varieties \( X/C \) under the cut-and-paste relations:

\[
[X] = [U] + [X \setminus U] \quad U \subset X \text{ open}.
\]

The ring structure is given by \([X][Y] = [X \times Y]\) with unit \([pt] = 1\). We denote by \( \mathbb{L} = [\mathbb{A}^1] \in K_0(V_C) \). Then, \([\mathbb{A}^n] = \mathbb{L}^n\).

If \( X = \bigsqcup X_i \) where \( X_i \subset X \) are a finite collection of locally closed subvarieties, then

\[
[X] = \sum_i [X_i].
\]

Thus, the Białynicki-Birula decomposition induces a decomposition of the class in \( K_0(V_C) \). We get the following:
Proposition 3.2  The class of $H_{a,b;n}^r$ in $K_0(V_C)$ is given by

\[ [H_{a,b;n}^r] = \sum_{\lambda \in B_{a,b;n}^r} \lambda^{(\lambda)}. \]

The ring $K_0(V_C)$ is universal with respect to ring valued invariants of varieties satisfying cut-and-paste and splitting as a product for $X \times Y$. These include compactly supported Euler characteristic, virtual Poincare polynomials, and virtual mixed Hodge polynomials. These are often called motivic invariants.

Proposition 3.2 allows us to compute all motivic invariants of $H_{a,b;n}^r$ in terms of the Betti statistic on the set $B_{a,b;n}^r$ of balanced partitions. Then, we apply Theorems 1.2 and 1.3 proven below to obtain Theorem 1.1.

4 Proofs of the theorems

In Sect. 3, we showed how the main theorems follow from Theorems 1.2 and 1.3. In this section, we will give combinatorial proofs of these results after making an initial reduction.

4.1 The Chevalley–Shephard–Todd Theorem

Here, we reduce to the case where both $a$ and $b$ are coprime to $n$ using the Chevalley–Shephard–Todd theorem.

Let $G$ be a finite group acting linearly and faithfully on $A_k$. We say that an element $\gamma \in G$ is a pseudoreflection if it fixes a hyperplane in $A_k$. We recall the following classical theorem:

Theorem 4.1  (Chevalley–Shephard–Todd [6, §5 Thm 4]) The following are equivalent:

(a) $G$ is generated by pseudoreflections,
(b) $A_k/G$ is smooth,
(c) $A_k/G \cong A_k$,
(d) the natural map $A_k \to A_k/G$ is flat.

Let $Y_{r,G} \subset \operatorname{Hilb}^r[A_k/G]$ denote the irreducible component containing the locus of $r$ distinct free $G$-orbits in $A_k$.

Corollary 4.1  The restriction $h_1 : Y_{1,G} \to A_k/G$ of the Hilbert–Chow morphism to $Y_{1,G}$ is an isomorphism if and only if any of the equivalent conditions of the Chevalley–Shephard–Todd theorem hold.

Proof  Suppose the conditions of the theorem hold so that $A_k \to A_k/G$ is flat. Then, this is a flat family of $G$-orbits in $A_k$ and so induces a map

\[ A_k/G \to \operatorname{Hilb}^1[A_k/G] \]

which is a section to $h$. This is an isomorphism on a dense open subset of $Y_{1,G}$ with inverse given by $h_1$ and so is an isomorphism everywhere.

For the converse, suppose $h_1 : Y_{1,G} \to A_k/G$ is an isomorphism. We have a commutative diagram
where $\mathcal{U}_1$ is the universal family over $Y_{1,G}$ and $\mathcal{U}_1 \to \mathbb{A}^k$ is $G$-equivariant.

We claim that $\mathcal{U}_1$ is the pullback of $\mathbb{A}^k$ along $h$. Indeed let $f : T \to Y_{1,G}$ and $g : T \to \mathbb{A}^k$ be maps commuting with the projections to $\mathbb{A}^k/G$. These maps induce a diagram

$$
\begin{array}{ccc}
Z & \longrightarrow & T \times \mathbb{A}^k \\
\downarrow \pi & & \downarrow s \\
T & & \end{array}
$$

where $\pi$ is a flat family of $G$-equivariant subschemes of $\mathbb{A}^k$ over $T$ and $s$ is a section of the projection. The fact that $f$ and $g$ commute with the maps to $\mathbb{A}^k/G$ and that $h$ is an isomorphism implies that $s$ is in fact a section of $\pi$. Composing the section $s : T \to Z$ with the map $Z \to \mathcal{U}_1$ gives the required universal map $T \to \mathcal{U}_1$ exhibiting $\mathcal{U}_1$ as the pullback.

It follows that $\mathcal{U}_1 \to \mathbb{A}^k$ is an isomorphism. Therefore, $\mathbb{A}^k \to \mathbb{A}^k/G$ is flat, so the equivalent conditions of the theorem hold. \(\square\)

The above results allow us to reduce to the case where our group has no pseudoreflections. Let $H \subset G$ be the subgroup generated by pseudoreflections. First note that if $\gamma \in H$ fixes the hyperplane $\mathcal{H} \subset \mathbb{A}^k$, then $g\gamma g^{-1}$ fixes $g\mathcal{H}$ for any $g \in G$ so that $H$ is a normal subgroup. Denote $\tilde{G} := G/H$.

**Proposition 4.1** In the situation above, there is a natural morphism

$$
\text{Hilb}^r[\mathbb{A}^k/G] \to \text{Hilb}^r[\mathbb{A}^k/\tilde{G}]
$$

which induces an isomorphism $Y_{r,G} \cong Y_{r,\tilde{G}}$.

**Proof** We construct this isomorphism explicitly. Let $\mathcal{U}_r \to \text{Hilb}^r[\mathbb{A}^k/G]$ be the universal family. It comes equipped with a $G$-equivariant map $\mathcal{U}_r \to \mathbb{A}^k$. The fiberwise quotient $\mathcal{U}_r/H \to \text{Hilb}^r[\mathbb{A}^k/G]$ is a flat family of $\tilde{G}$-equivariant schemes of length $r|G|/|H| = r|\tilde{G}|$.

To see flatness, note that $\mathcal{O}_{\mathcal{U}_r/H} = \mathcal{O}_{\mathcal{U}_r}^H$ is a direct summand of the flat module $\mathcal{O}_{\mathcal{U}_r}$ since we are in characteristic 0.

The natural map

$$
\mathcal{U}_r/H \to \mathbb{A}^k/H \cong \mathbb{A}^k
$$

induces a map $\mathcal{U}_r/H \to \text{Hilb}^r[\mathbb{A}^k/G] \times \mathbb{A}^k$. We need to check that this is an embedding or equivalently that the $G$-equivariant morphism of $\mathcal{O}_{\text{Hilb}^r[\mathbb{A}^k/G]}$-algebras

$$
\mathbb{C}[\mathbb{A}^n]^H \otimes \mathcal{O}_{\text{Hilb}^r[\mathbb{A}^k/G]} \to \mathcal{O}_{\mathcal{U}_r}^H
$$

is surjective. We can check surjectivity on fibers; over the point $[J] \in \text{Hilb}^r[\mathbb{A}^k/G]$ corresponding to some $G$-invariant ideal, this map is just $\mathbb{C}[\mathbb{A}^n]^H \to (\mathbb{C}[\mathbb{A}^n]/J)^H = \mathbb{C}[\mathbb{A}^n]^H/(J \cap \mathbb{C}[\mathbb{A}^n]^H)$ which is surjective.
Moreover, the regular representation is preserved by taking invariants, $\mathbb{C}[G]^H \cong \mathbb{C}[\tilde{G}]$. Thus, $\mathcal{U}_r/H \to \text{Hilb}'[A^k/G]$ is a flat family of $\tilde{G}$-equivariant subschemes of $A^k$ of length $r|\tilde{G}|$ carrying $r$ copies of the regular representation and so induces a morphism

$$\varphi: \text{Hilb}'[A^k/G] \to \text{Hilb}'[A^k/\tilde{G}].$$

To construct an inverse over $Y_{r,\tilde{G}}$, take the universal family

$$\varrho: V_r \to A^k \times_{A^k/H} A^k \to Y_{r,\tilde{G}}.$$ 

Since $H$ is generated by pseudoreflections, pulling back the quotient map $A^k \to A^k/H \cong A^k$ along $\varrho$ gives a flat family $Z_r$:

$$Z_r = V_r \times_{A^k} A^k \to A^k \to Y_{r,\tilde{G}}.$$ 

A general fiber of $Z_r \to Y_{r,\tilde{G}}$ consists of $r$ distinct free $G$ orbits, so by flatness every fiber carries $r$ copies of the regular representation of $G$. Furthermore, closed embeddings are stable under base change, so $Z_r \to Y_{r,\tilde{G}}$ is a flat family of subschemes of $A^k$ inducing a morphism

$$\psi: Y_{r,\tilde{G}} \to Y_{r,G} \subset \text{Hilb}'[A^k/G].$$

Since $\psi$ and $\varphi$ are inverses on the dense open subset parameterizing $r$ distinct free orbits, they give an isomorphism everywhere. \hfill \Box

**Remark 4.1** Note that in the above proof, there is always a morphism $\text{Hilb}'[A^k/G] \to \text{Hilb}'[A^k/\tilde{G}]$ for any normal subgroup $H \subset G$. The fact that $H$ is generated by pseudoreflections is only used in constructing the inverse over $Y_{r,\tilde{G}}$.

Proposition 4.1 justifies our restriction to considering only the cyclic subgroups $G_{a,b,n} \subset GL_2$. Indeed if $G \subset GL_2$ is any abelian subgroup with no pseudoreflections, then it must be cyclic [7, Satz 2.9]. By Corollary 2.3 $\text{Hilb}'[A^2/G] = Y_{r,G}$ for $G$ abelian. Consequently, every equivariant Hilbert scheme for an abelian group action on $A^2$ is isomorphic to $\text{Hilb}'[A^2/G_{a,b,n}]$ for some $a$, $b$ and $n$.

**Corollary 4.2** Suppose that $a$ is not coprime to $n$ so that $a = da'$ and $n = dn'$ with $d = \gcd(a, n)$. Then, $H_{a,b,n}^r \cong H_{a',bn'}^r$.

**Proof** The generator of $G_{a,b,n}$ satisfies

$$\begin{pmatrix} \zeta_n^a & 0 \\ 0 & \zeta_n^b \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & \zeta_n^{bn'} \end{pmatrix}.$$
This is a nontrivial pseudoreflection generating a cyclic subgroup $H \subset G_{a,b;n}$ of order $d$. The quotient $G_{a,b;n}/H$ is a cyclic group of order $n'$ acting by weights $a'$ and $b$, so we get the required isomorphism.

In light of Corollary 4.2, it suffices to consider only the case when $n$ is coprime to $a$ and $b$. Indeed if $n = dn'$ and $a = da'$, then sending $n$ to $n + |ab|$ is equivalent by the corollary to sending $n'$ to $n' + |a'b|$ and so is compatible with periodicity and quasipolynomiality statements with period $|ab|$. We will only need this reduction in the proof of Theorem 1.3.

4.2 Proof of Theorem 1.2

Recall we are going to prove the following:

**Theorem 1.2** Fix integers $r > 0$ and $a, b$ with $ab > 0$. There is a natural bijection $B'_{a,bn} \rightarrow B'_{a,bn+ab}$ that preserves the Betti statistic for $n > rab$.

Since the involution $(a, b) \mapsto (-a, -b)$ does not change the family of cyclic groups $|G_{a,b;n}|$, we assume without loss of generality that $a, b > 0$. For any $(a, b; n)$-balanced partition $\lambda$ and $0 \leq k \leq n - 1$, we denote by

$$S_k := \{(i, j) \in \lambda \mid ai + bj = k \mod n\}$$

the set of boxes in $\lambda$ labeled by $k$ (mod $n$). We also denote by

$$D_k := \{(i, j) \in \mathbb{Z}_{\geq 0}^2 \mid ai + bj = k\}$$

the $k$th diagonal.

First we need the following lemmas:

**Lemma 4.1** Suppose $\lambda$ is an $(a, b; n)$-balanced partition and $n > rab$. Let $k$ be an integer with $rab \leq k \leq n - 1$ satisfying $k = rab + au + bv$ (*) for some nonnegative integers $u$ and $v$. Then, the set $S_k$ can be split into two disjoint sets $A_k$ and $B_k$ which satisfy the properties:

1. If $(i, j) \in A_k$, then either $i < b$ or $(i - b, j + a) \in A_k$.
2. If $(i, j) \in B_k$, then either $j < a$ or $(i + b, j - a) \in B_k$.

**Proof** First notice that for $rab \leq k \leq n - 1$ satisfying (*), the number of $(i, j) \in \mathbb{Z}_{\geq 0}^2$ such that $ai + bj = k$ is at least $r+1$; therefore, there exist $(i_0(k), j_0(k))$ such that $ai_0(k) + bj_0(k) = k$ and $(i_0(k), j_0(k)) \notin \lambda$. This means that the entries labeled $k$ (mod $n$) split into two sets, $A_k$ and $B_k$ defined by

$$A_k := \{(i, j) \in \lambda \mid ai + bj = k \mod n, i \leq i_0(k)\}$$

$$B_k := \{(i, j) \in \lambda \mid ai + bj = k \mod n, j \leq j_0(k)\}$$

Note that these sets must be disjoint because if $(i, j) \in \lambda$ satisfies $i < i_0(k)$ and $j < j_0(k)$, then $ai + bj < k < n$ so $(i, j) \notin S_k$.

We define a map $\varphi_k : S_k \rightarrow S_{k-\text{ab}}$ by $(i, j) \in A_k$ maps to the entry $(i, j - a)$ and $(i, j) \in B_k$ maps to the entry $(i - b, j)$. We first claim this gives an injective map. The map $\varphi_k$ is clearly injective on each set $A_k$ or $B_k$ individually, so suppose there is $(i, j) \in A_k$ and $(i', j') \in B_k$ such that $\varphi_k(i, j) = \varphi_k(i', j')$. Then, $(i, j - a) = (i' - b, j')$ so $(i, j)$ and $(i', j')$ are on the corners of an $b \times a$ rectangle. The label $k$ (mod $n$) appears in such a rectangle at most
twice, namely at \((i, j)\) and \((i', j')\). This contradicts the fact that there is a box \((i_0, j_0)\) \(\not\in \lambda\) labeled by \(k\) with \(i \leq i_0\) and \(j \leq j_0\).

Since \(\lambda\) is a balanced partition, the number of boxes with each label has the same cardinality and so the injective map \(\varphi_k\) must in fact be bijective. Suppose for the sake of contradiction that the first condition in the lemma is violated, so we have \((i, j) \in A_k\) but \(i \geq b\) and \((i - b, j + a) \not\in A_k\). Then, the entry \((i - b, j)\) would be in \(\lambda\), labeled \(k - ab\) but \(\varphi_k^{-1}(i - b, j) \not\in \lambda\), which gives a contradiction. We can argue similarly for the entries in \(B_k\) (Fig. 6).

\[
\square
\]

**Lemma 4.2** The decomposition \(S_k = A_k \cup B_k\) above does not depend on a choice of \((i_0(k), j_0(k))\) \(\not\in \lambda\) on the \(k\)th diagonal. In particular, the decomposition is the unique one for which the map \(\varphi_k\) is a bijection.

**Proof** Suppose that there is an \((i_1(k), j_1(k)) \in D_k\) with \((i_1(k), j_1(k)) \not\in \lambda\) and a decomposition of \(S_k\) as

\[
A'_k := \{(i, j) \in \lambda \mid ai + bj = k \mod n, i \leq i_1(k)\} \\
B'_k := \{(i, j) \in \lambda \mid ai + bj = k \mod n, j \leq j_1(k)\}
\]

that is different than the one induced by \((i_0(k), j_0(k))\). Then, there must be some \((i', j'), (i'', j'') \in \lambda \cap D_k\) such that \(i_0 \leq i' - b < i' \leq i'' < i'' + b \leq i_1\) with \((i' - b, j' + a), (i'' + b, j' - a) \not\in \lambda\). That is \((i', j'), (i'', j'') \in B_k\) but \((i', j'), (i'', j'') \in A'_k\). In this case, we can compute

\[
\varphi_k(i', j') = (i' - b, j') \in S_{k-ab} \\
\varphi_k(i'', j'') = (i'' - b, j'') \in S_{k-ab}
\]

On the other hand, \((i'', j'' - a) \in S_{k-ab}\) and \(\varphi_k^{-1}(i'', j'' - a) = (i'' + b, j'' - a) \not\in \lambda\) contradicting that \(\varphi_k\) is a bijection.

\[
\square
\]

**Corollary 4.3** Let \((i_0, j_0) \in D_{rab}\) such that \((i_0, j_0) \not\in \lambda\). Define

\[
A = \{(i, j) \mid i < i_0, j \geq j_0\} \\
B = \{(i, j) \mid i \geq i_0, j < j_0\}
\]

Then for any \(k\) satisfying the conditions of Lemma 4.1, \(A_k = A \cap S_k\) and \(B_k = B \cap S_k\) and the maps \(\varphi_k\) extend to an map \(\varphi : A \cup B \rightarrow \mathbb{Z}^2_{\geq 0}\) that is surjective onto \(S_{k-ab}\) for all such \(k\).
Proof First notice that $A$ and $B$ are disjoint. Since $rab \leq k < n$, it follows that $S_k \subset A \cup B$.

The boxes $(i,j)$ with $i \geq i_0$ and $j \geq j_0$ are not in $\lambda$. On the other hand, for every such $k$ there exists $(i',j') \in D_k$ with $i' \geq i_0$ and $j' \geq j_0$. Consequently for $(i,j) \in \lambda$, if $i < i'$, then $i < i_0$ and similarly if $j < j'$, then $j < j_0$ so that $A_k \subset A$ and $B_k \subset B$. The first result follows. Finally, we can extend the maps $\psi_k$ by defining

$$\psi(i,j) = \begin{cases} (i, j-a) & \text{if } (i,j) \in A \\ (i-b, j) & \text{if } (i,j) \in B \end{cases}$$

Proof of Theorem 1.2 We construct a bijection $B^r_{a,b;n} \rightarrow B^r_{a,b;n+ab}$. Let $\lambda \in B^r_{a,b;n}$. We will add boxes to the columns and rows of $\lambda$ as follows:

1. If $(i,j) \in A_k$ and $n - b \leq k \leq n - 1$, then increase the length of column $j$ by $a$ boxes;
2. If $(i,j) \in B_k$ and $n - a \leq k \leq n - 1$, then increase the length of row $i$ by $b$ boxes.

We can check that the process terminates in a partition since Lemma 4.1 guarantees that if $i > i'$, column $i'$ had at least as many boxes added as column $i$, and similarly for rows. We call the resulting partition $\psi(\lambda)$. Note that $\lambda \subset \psi(\lambda)$ as a subset of $\mathbb{Z}_{\geq 0}^2$.

We need to check that $\psi(\lambda) \in B^r_{a,b;n+ab}$. We can interpret the algorithm above as inserting $a$ boxes directly below each $(i,j) \in A_k$ with $n - b \leq k \leq n - 1$ and inserting $b$ boxes directly to the right of each $(i,j) \in B_k$ with $n - a \leq k \leq n - 1$. It is clear that these new boxes are labeled with $(a,b;n+ab)$-weight in the range $n \leq k \leq n + ab - 1$ and that the boxes of $\lambda$ are in bijection with the boxes of $\psi(\lambda)$ labeled with $(a,b;n+ab)$-weight in the range $0 \leq k \leq n - 1$. Thus, it suffices to check that we have inserted $r$ boxes of each weight $n \leq k \leq n + ab - 1$, and let

$$R_k = \{(i,j) \in \psi(\lambda) | ai + bj = k \pmod{n + ab}\}.$$ 

Then, $R_k \subset A \cup B$ and the restriction $\varphi : R_k \rightarrow S_{k-ab}$ is a bijection. Consequently, $\#R_k = \#S_{k-ab} = r$ since $\lambda$ is balanced and $\psi(\lambda) \in B^r_{a,b;n+ab}$.

To check that $\psi$ is a bijection, we start with $\mu \in B^r_{a,b;n+ab}$ and produce a $\lambda \in B^r_{a,b;n}$ with $\psi(\lambda) = \mu$. Indeed we obtain $\lambda$ by deleting all boxes of $\mu$ labeled by $k \pmod{n + ab}$ with $n \leq k \leq n + ab - 1$. Then, the boxes of $\lambda$ labeled with $(a,b;n)$-weight $k$ correspond to the boxes of $\mu$ labeled with $(a,b;n+ab)$-weight $k$ with $0 \leq k \leq n - 1$ and it is clear that we can recover $\mu$ by inserting back in the boxes with larger $(a,b;n+ab)$-weight, i.e., $\psi(\lambda) = \mu$.

Lastly, we need to check that the Betti statistic is preserved. First note that $\psi$ sends invariant arrows to invariant arrows without changing the direction (though possibly changing the slope) of the arrow. This is because $\psi$ induces a bijection on the boxes with labels $0 \leq k \leq n - 1$, so stretching the arrow by applying $\psi$ does not affect whether it is invariant. On the other hand, we can shrink any invariant arrow of $\psi(\lambda)$ onto an invariant arrow of $\lambda$ by moving the head and tail by the number of boxes deleted from $\psi(\lambda)$ to obtain $\lambda$, i.e., by applying $\psi$ to the head and tail of the arrow.

For clarity, we will illustrate the above proof in the following example where $(a,b) = (2,3)$, $r = 2$ and $n = 13$. Consider the $(2,3;13)$-balanced partition below:

| 0 | 2 | 4 | 6 | 8 | 10 | 12 | 13 | 3 | 5 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 |
| 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 25 |
We have labeled the boxes by the weight $k \pmod{n}$. The box labeled by $a \bullet$ is the box in the diagonal $D_{12}$ that is not contained in $\lambda$ which we use to decompose $\lambda$ into the sets $A$ and $B$. The boxes $(i,j) \in A$ labeled with $n - b \leq k \leq n - 1$ and the boxes $(i,j) \in B$ labeled with $n - a \leq k \leq n - 1$ are shaded in.

Applying $\psi$ gives us the following $(2, 3; 19)$-balanced partition.

The new boxes that are inserted by $\psi$ are shaded in and these are exactly the boxes with labels $n \leq k \leq n + ab - 1$.

### 4.3 Proof of Theorem 1.3

Recall we are going to prove the following:

**Theorem 1.3** Fix integers $r > 0$ and $a, b$ with $ab < 0$. The cardinality $\#B'_{a,b;n}$ is a quasipolynomial in $n$ of period $|ab|$ for $n \gg 0$.

The idea of the proof is to identify $(a, b; n)$-balanced partitions with a special subset of $(1, -1; n)$-balanced ones which we then analyze in Proposition 4.3 using a version of the cores-and-quotients bijection (see [23, Section 2.7]).

By Corollary 4.2, it suffices to prove Theorem 1.3 for $a$ and $b$ both coprime to $n$. Without loss of generality, we assume that $a$ is positive and $b$ is negative. Let $P_m$ be the set of partitions of $m$. Consider the map $f : P_m \rightarrow P_{|rabn|}$ where for each $\lambda \in P_m$, $f(\lambda)$ is the partition of $|rabn|$ obtained by replacing each box of $\lambda$ by an $a \times |b|$ rectangle.

If $\lambda$ is the partition with rows $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m > 0$ and length $l$, then $f(\lambda)$ is the partition with rows

\[
(a\lambda_1, \ldots, a\lambda_1, \ldots, a\lambda_m, \ldots, a\lambda_m). 
\]

That is, $f(\lambda)$ is a partition whose row lengths are multiples of $a$ and such that each row repeats a multiple of $|b|$ times.

**Proposition 4.2** If $a$ and $b$ are both prime to $n$, then the function $f$ restricts to a bijection between $(a, b; n)$-balanced partitions of $rn$ and $(1, -1; n)$-balanced partitions of $|rabn|$ of the form $(\ast)$.

**Proof** It is clear that the map $f$ is injective and that it surjects onto the set of partitions satisfying $(\ast)$ so that $f$ is a bijection between $P_m$ and the subset of $P_{|rabn|}$ satisfying $(\ast)$. To prove the claim, it suffices to show that $\lambda$ is $(a, b; n)$-balanced if and only if $f(\lambda)$ is $(1, -1; n)$-balanced.
We will use the generating function for the $G_{a,b;n}$-weights of the boxes of a partition. For any partition $\mu$, define

$$w_{a,b;n}^\mu(q) := \sum_{(ij) \in \mu} q^{ai+bj} \mod n \in \mathbb{Q}[q]/(q^n - 1).$$

Now, $\mu$ is $(a, b; n)$-balanced if and only if

$$w_{a,b;n}^\mu(q) = r(q^{n-1} + q^{n-2} + \ldots + 1) \mod q^n - 1$$

if and only if $(q - 1)w_{a,b;n}^\mu(q) = 0 \mod q^n - 1$.

Let $w_{a,b;n}^\lambda(q) = \sum_{l=1}^{rn} q^{w_l}$ (mod $q^n - 1$) where $l$ runs through the $rn$ boxes of $\lambda$. Under the map $f$, each box gets replaced with an $a \times |b|$ rectangle. Such a rectangle has $(1, -1; n)$-weight generating function given by

$$\left(\frac{q^a - 1}{q - 1}\right) \left(\frac{q^b - 1}{q - 1}\right) \mod (q^n - 1)$$

However, if the $k$th box of $\lambda$ has $(a, b; n)$-weight $w_k$, the corresponding rectangle in $f(\lambda)$ has a box with $(1, -1; n)$-weight $w_k - b + 1$ in the bottom right corner, so the above generating function must be shifted by $q^{-b+1}$. Putting this together gives

$$w_{1,-1;n}^{f(\lambda)}(q) = q^{-b+1} w_{a,b;n}^\lambda(q) \left(\frac{q^a - 1}{q - 1}\right) \left(\frac{q^b - 1}{q - 1}\right) \mod (q^n - 1).$$

for the $(1, -1; n)$-weight generating function of $f(\lambda)$.

By assumption, $a$ and $b$ are coprime to $n$, so $\frac{q^a - 1}{q - 1}$ and $\frac{q^b - 1}{q - 1}$ are units in $\mathbb{Q}[q]/(q^n - 1)$. It follows that $(q - 1)w_{1,-1;n}^{f(\lambda)}(q) = 0 \mod q^n - 1$ if and only if $(q - 1)w_{a,b;n}^\lambda(q) = 0 \mod q^n - 1$. Therefore, $\lambda$ is $(a, b; n)$-balanced if and only if $f(\lambda)$ is $(1, -1; n)$-balanced. □

**Proposition 4.3** Suppose $a, b$ are coprime to $n$ with $a > 0$ and $b < 0$. For any $r$, the number of $(a, b; n)$-balanced partitions of $rn$, $\#B_{a,b;n}^r$ is a quasipolynomial in $n$ with period $\lcm(ab) | n$.

**Proof** By Proposition 4.2, it suffices to count $(1, -1; n)$-balanced partitions of $|raban|$ satisfying $(\ast)$.

Let $\lambda \in B_{1,-1;n}^{raban}$ be a $(1, -1; n)$-balanced partition, and suppose the parts are given by $(\lambda_1, \lambda_2, \ldots, \lambda_m)$. We can construct a bi-infinite $\{0, 1\}$ word associated with $\lambda$ as follows

$$\ldots, 0, 0, 1, 1, \ldots, 1, 0, 1, \ldots, 1, 0, 0, 1, 1, \ldots, 1, 0, 1, 1, 1, \ldots \uparrow_{\lambda_m} \downarrow_{\lambda_m-\lambda_{m-1}} \uparrow_{\lambda_1-\lambda_2}$$

Notice that this word is only defined up to a $\mathbb{Z}$ shift, but there is a unique way to index it as a sequence $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ such that

$$\#\{a_i = 1, i < 0\} = \#\{a_i = 0, i \geq 0\}.$$

In this language, our condition $(\ast)$ is equivalent to saying that consecutive strings of 0’s have length divisible by $|b|$ and consecutive strings of 1’s have length divisible by $a$.

Using this sequence, we can associate with each such $\lambda$ an $n$-tuple of (possibly empty) partitions $(\lambda^{(0)}, \ldots, \lambda^{(n-1)})$ defined by taking $\lambda^{(i)}$ to be the partition corresponding to the sequence $\ldots, a_{n+i}, a_{n+i}, a_{i}, a_{i+1}, a_{i+2}, \ldots$. This tuple is known as the $n$-quotient of $\lambda$. 
All \((1, -1; n)\)-balanced partitions have the same multiset of contents mod \(n\); therefore, they all have a trivial \(n\)-core (see [23, Theorem 2.7.41]). Since the size of \(\lambda\) is \(|\lambda| = |r\lambda n|\), we conclude that the total size of the quotient is
\[
|\lambda^{(0)}| + |\lambda^{(2)}| + \cdots + |\lambda^{(n-1)}| = |r\lambda|
\]

We now define a set \(S_{a,b,r}(n, k)\) consisting of all tuples
\[
\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n-1)}
\]
satisfying (**) for which (i) no \(|ab|\) consecutive terms of \(\lambda^{(i)}\)'s are the empty partition and (ii) there exist exactly \(k\) indices, \(i\), such that the tuple
\[
\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(i)}, \emptyset, \ldots, \emptyset, \lambda^{(i+1)}, \ldots, \lambda^{(n-1)}
\]

\(|ab|\) copies

corresponds to a \((1, -1, n + |ab|)\)-balanced partition of \(r|ab|(n + |ab|)\) the form \((\ast)\).

The total number of boxes of a tuple \((\lambda^{(0)}, \ldots, \lambda^{(n)})\) satisfying (**) is always \(|r\lambda|\), so when \(n\) is really large we are guaranteed to find arbitrarily large consecutive substrings of empty partitions. Therefore, no such sequences satisfy condition (i) in the definition of \(S_{a,b,r}(n, k)\) so \(S_{a,b,r}(n, k) = \emptyset\) for \(n \gg 0\). In conclusion, the set
\[
\bigsqcup_{n,k} S_{a,b,r}(n, k)
\]
is finite.

Each \((1, -1; n')\)-balanced partition of the form \((\ast)\) that corresponds to a tuple \((\lambda^{(1)}, \ldots, \lambda^{(n')})\) with a consecutive substring of \(|ab|\) many empty partitions can be contracted to a \((1, -1, n' - |ab|)\)-balanced partition by simply removing that substring of empty partitions from the \(n'\)-quotient and reading the result as a \(n' - |ab|\) quotient. This preserves the property of the partition having the form \((\ast)\); therefore, by repeating this process we always obtain one of the elements of \(S_{a,b,r}(n, k)\) for some \(n, k\). Conversely, we see that every partition in \(B_{a,b,r}^{n'}\) is uniquely constructed by starting with an element of some \(S_{a,b,r}(n, k)\) and inserting multiple consecutive strings \(|ab|\)-many empty partitions.

Let us denote by \(F_{a,b,r}(x)\) the generating function \(\sum_{n \geq 0} |B_{a,b,r}^{n'}| x^n\). The number of elements of \(B_{a,b,r}^{n + \ell|ab|}\) that contract to a given element in \(S_{a,b,r}(n, k)\) is given by
\[
\binom{\ell + k - 1}{k - 1}
\]
so we can write
\[
F_{a,b,r}(x) = \sum_{n,k} |S_{a,b,r}(n, k)| \sum_{\ell \geq 0} \binom{\ell + k - 1}{k - 1} x^{\ell + \ell|ab|} = \sum_{n,k} |S_{a,b,r}(n, k)| \frac{x^n}{(1-x|ab|)^k}
\]
from which it is straightforward to conclude that the size of \(B_{a,b,r}^{n'}\) is quasipolynomial in \(n\) with period \(|ab|\) for large enough \(n\).

\[\square\]

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