KOROVKIN TYPE APPROXIMATION OF ABEL TRANSFORMS OF $q$-MEYER-KÖNIG AND ZELLER OPERATORS

DILEK SÖYLEMEZ AND MÈHMET ÜNVER

ABSTRACT. In this paper we investigate some Korovkin type approximation properties of the $q$-Meyer-König and Zeller operators and Durrmeyer variant of the $q$-Meyer-König and Zeller operators via Abel summability method which is a sequence-to-function transformation and which extends the ordinary convergence. We show that the approximation results obtained in this paper are more general than some previous results. Finally, we obtain the rate of Abel convergence for the corresponding operators.

1. Preliminaries

Korovkin type approximation theory aims to provide some simple criteria for the convergence of a sequence of positive linear operators in some senses [16]. There is a number of main motivations in the theory. One of them is obtaining some suitable conditions for the convergence of arbitrary sequence of positive linear operators acting from one certain space to another one. Next motivation is studying some particular conditions for convergence for certain sequence of positive linear operators by using known criteria (see e.g., [3]). It is also possible to introduce the summability theory whose main idea is to make a non-convergent sequence or series to converge in some general senses whenever the sequence of positive linear operators does not converge in the ordinary sense. The leading study with this motivation gives criteria for the statistical convergence of a sequence of positive linear operators over $C[a, b]$, the space of all real continuous functions defined on the interval $[a, b]$ [13]. Following that study many authors have given several approximation results via summability theory [5, 10, 34].

In 1987, Lupaş [17] introduced the first $q$-analogue of Bernstein operators and investigated its approximation and shape preserving properties. In 1997, Phillips [24] defined another $q$-generalization of Bernstein operators. Afterwards, many generalizations of positive linear operators based on $q$-integers were introduced and studied by several authors, for example, we refer the readers to [1], [2], [4], [12], [19], [20], [21], [22], [35], [36], [37], [39].

Now, let us recall some notations from $q$-analysis [25]: For any fixed real number $q > 0$, the $q$-integer $[n]$ is defined by

$$[n] := [n]_q = \sum_{k=1}^{n} q^{k-1} = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1 \end{cases}$$

2010 Mathematics Subject Classification. 40A35, 40G10, 41A36.

Key words and phrases. Meyer-König and Zeller Operators, Abel convergence, rate of convergence.
where \( n \) is a positive integer and \([0] = 0\), the \( q\)-factorial \([n]!\) of \([n]\) is given with

\[
[n]! := \begin{cases} 
\prod_{k=1}^{n} [k], & n = 1, 2, \\
1, & n = 0.
\end{cases}
\]

For integers \( n \geq r \geq 0 \), the \( q\)-binomial coefficient is defined by

\[
[n]_q^r := \frac{[n]!_q}{[r]!_q [n-r]!_q}
\]

and \( q\)-shifted factorial is defined by

\[
(t; q)_n := \begin{cases} 
1, & n = 0 \\
\prod_{j=0}^{n-1} (1 - tq^j), & n = 1, 2, \ldots.
\end{cases}
\]

Thomae \cite{31} introduced the \( q\)-integral of function \( f \) defined on the interval \([0, a]\) as follows:

\[
\int_0^a f(t) \, dq_t := a (1 - q) \sum_{n=0}^{\infty} f(aq^n) q^n, \quad 0 < q < 1.
\]

Finally, the \( q\)-beta function \cite{31} is defined by

\[
B_q (m, n) = \int_0^1 t^{m-1} (qt; q)_{n-1} \, dq_t.
\]

The original Meyer-König and Zeller operators were introduced for \( f \in C[0, 1] \) in 1960 (see \cite{18}). Later, Cheney and Sharma \cite{9} rearranged these operators as follows:

\[
M_n (f; x) = \begin{cases} 
(1 - x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \left(\frac{n+k}{k}\right) x^k, & x \in [0, 1) \\
f(1), & x = 1.
\end{cases}
\]

Trif \cite{32} defined the \( q\)-generalization of the Meyer-König and Zeller operators as

\[
M^q_n (f; x) = \begin{cases} 
\prod_{j=0}^{n} (1 - q^j x)^{\sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \left(\frac{n+k}{k}\right) x^k, & x \in [0, 1) \\
f(1), & x = 1.
\end{cases}
\]

In \cite{32}, the author studied Korovkin type approximation properties, calculated the rate of convergence and also gave a result for monotonicity properties of these operators. Heping \cite{15} proved some approximation results for the operators \( M_n^q f \) using \( q\)-hypergeometric series. Another \( q\)-generalization of the classical Meyer-König and Zeller operators can be found in \cite{11}. Besides, Durrmeyer variant of the
$q$-Meyer König and Zeller operators \cite{14} is introduced for $f \in C[0,1], x \in [0,1]$, $n \in \mathbb{N}$ and $\alpha > 0, q \in (0,1]$ as follows:

\begin{equation}
D^q_n (f; x) = \begin{cases} 
\sum_{k=1}^{\infty} m_{n,k,q} (x) \int_0^1 \frac{1}{x_q(n,k)} t^{k-1} (qt;q)_{n-1} f(t) \, dt, & x \in (0,1) \\
m_{n,0,q} (x) f(0), & x = 1,
\end{cases}
\end{equation}

where

\[ m_{n,k,q} (x) = (x;q)_{n+1} \left\lfloor \frac{n+k}{k} \right\rfloor x^k. \]

The authors investigated some approximation properties with the help of well-known Korovkin’s theorem and compute the rate of convergence for these operators in terms of the second-order modulus of continuity \cite{14}.

Throughout this paper, we study with the sequence $(q_n)$ such that $0 < q_n \leq 1$ and $q_0 = 0$ and we define $M^q_0 f = D^q_0 f = 0$ for any $f \in C[0,1]$. It is well known that if the classical conditions

\begin{equation}
\lim_{n \to \infty} q^n = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} = 0
\end{equation}

hold, then for each $f \in C[0,1]$ the sequences $(M^q_n f)$ and $(D^q_n f)$ converge uniformly to $f$ over $[0,1]$ (see \cite{14,32}). Furthermore, we use the norm of the space $C[a,b]$ defined for any $f \in C[a,b]$ by

\begin{equation}
||f|| := \sup_{a \leq x \leq b} |f(x)|.
\end{equation}

In the present paper, taking into account the Abel convergence we obtain some approximation results for the $q$-Meyer-König and Zeller operators and Durrmeyer variant of $q$-Meyer-König and Zeller operators. We also study the rate of the convergence of these operators. We also show that the results obtained in this paper are stronger than some previous ones.

Let $x = (x_j)$ be a real sequence. If the series

\begin{equation}
\sum_{j=0}^{\infty} x_j y^j
\end{equation}

is convergent for any $y \in (0,1)$ and

\[ \lim_{y \to 1^-} (1 - y) \sum_{j=0}^{\infty} x_j y^j = \alpha \]

then $x$ is said to be Abel convergent to real number $\alpha$ \cite{20}. Korovkin type approximation via Abel convergence and other power series methods may be found in \cite{13,23,29,30,33,34,38}.

The fact given in the following remark helps us through the paper:

**Remark 1.** Let $(f_n)$ be a sequence in $C[0,1]$. If there exists a positive integer $n_0$ such that

\[ \lim_{y \to 1^-} (1 - y) \sum_{j=n_0}^{\infty} f_j y^j = 0 \]
then it is not difficult to see that
\[
\lim_{y \to 1^-} (1 - y) \left\| \sum_{j=0}^{\infty} f_j y^j \right\| = 0,
\]
i.e., while studying the Abel convergence finitely many terms do not make sense as with the ordinary convergence.

Before studying the announced approximation properties of the operators, we recall some well-known lemmas:

**Lemma 1.** Let \( n \geq 3 \) be a positive integer. Then the following hold for the operators (1.2):

\[\begin{align*}
M_n^q (e_0; x) &= 1 \\
M_n^q (e_1; x) &= x \\
x^2 &\leq M_n^q (e_2; x) \leq \frac{x}{|n - 1|} + x^2
\end{align*}\]

where \( e_i(x) = x^i \) for \( i = 0, 1, 2 \).

**Lemma 2.** Let \( n \geq 3 \) be a positive integer. Then the following hold for the operators (1.3):

\[\begin{align*}
D_n^q (e_0; x) &= 1 \\
D_n^q (e_1; x) &= x \\
D_n^q (e_2; x) &= x^2 + \frac{[2] x (1 - x) (1 - q^n x)}{|n - 1|} - E_{n,q}(x),
\end{align*}\]

where
\[
0 \leq E_{n,q}(x) \leq \frac{x [2] [3] q^{n-1}}{|n - 1| |n - 2|} (1 - x) (1 - qx) (1 - q^n x).
\]

The following lemma can be proved easily:

**Lemma 3.** Let \( n \geq 3 \) be a positive integer. Then we have
\[
D_n^q (e_2; x) - x^2 \geq \frac{[2] x (1 - x) (1 - q^n x)}{|n - 1|} - \frac{x [2] [3] q^{n-1}}{|n - 1| |n - 2|} (1 - x) (1 - qx) (1 - q^n x) \geq 0.
\]

2. **Abel Transform of the sequence \((M_n^q)\)**

In this section, we study Korovkin type approximation of the operators \((M_n^q)\) defined with (1.2) by considering the Abel method. Unver proved the following Korovkin-type theorem via Abel method (see [33], Theorem 1).
Theorem 1. Let \((L_n)\) be a sequence of positive linear operators from \(C[a, b] \rightarrow B[a, b]\) such that \(\sum_{n=0}^{\infty} \|L_n(c_0)\| y^n < \infty\) for any \(y \in (0, 1)\). Then for any \(f \in C[a, b]\) we have

\[
\lim_{y \to 1^-} (1 - y) \left\| \sum_{n=0}^{\infty} (L_n f - f) y^n \right\| = 0
\]

if and only if

\[
(2.1) \quad \lim_{y \to 1^-} (1 - y) \left\| \sum_{n=0}^{\infty} (L_n e_i - e_i) y^n \right\| = 0, \quad i = 0, 1, 2.
\]

We are now ready to prove the following theorem:

Theorem 2. If the sequence \(\left(\frac{1}{[n-1]}\right)_{n=3}^{\infty}\) is Abel null then for each \(f \in C[0, 1]\) we have

\[
\lim_{y \to 1^-} (1 - y) \left\| \sum_{n=0}^{\infty} (M_q^n f - f) y^n \right\| = 0.
\]

Proof. From Lemma 1 we see that \(\sum_{n=0}^{\infty} \|M_q^n(c_0)\| y^n < \infty\) for any \(y \in (0, 1)\). If we consider Theorem 1 it suffices to show that (2.1) holds for \((M_q^n)\).

Now, considering Lemma 1, one can get for \(i = 0, 1\) that

\[
\lim_{y \to 1^-} (1 - y) \left\| \sum_{n=0}^{\infty} (M_q^n e_i - e_i) y^n \right\| = 0.
\]

Moreover, using (1.9), we have for \(n \geq 3\) that

\[
0 \leq M_q^n(e_2; x) - x^2 \leq \frac{x}{[n-1]}
\]

which gives

\[
0 \leq (1 - y) \left\| \sum_{n=3}^{\infty} (M_q^n e_2 - e_2) y^n \right\| \leq (1 - y) \sup_{0 \leq x \leq 1} \sum_{n=3}^{\infty} \frac{x}{[n-1]} y^n \leq (1 - y) \sum_{n=3}^{\infty} \frac{y^n}{[n-1]}
\]

Finally we have

\[
\lim_{y \to 1^-} (1 - y) \left\| \sum_{n=3}^{\infty} (M_q^n e_2 - e_2) y^n \right\| = 0.
\]

Hence, considering Remark 1 we get

\[
\lim_{y \to 1^-} (1 - y) \left\| \sum_{n=0}^{\infty} (M_q^n e_2 - e_2) y^n \right\| = 0
\]

which concludes the proof. \(\square\)
The following remark proves that the conditions of Theorem 2 are weaker than the classical conditions:

**Remark 2.** It is not difficult to see that the classical conditions (1.4) entail that the sequence \( \left( \frac{1}{|n-1|} \right)_{n=3}^{\infty} \) is Abel null. Conversely, if we define the sequence \( (q_n) \) with

\[
q_n := \begin{cases} 
0 & \text{if } n \text{ is a perfect cube} \\
1 & \text{otherwise}
\end{cases}
\]

then \((q_n)\) does not satisfy the classical conditions. Besides, we have for any \( n \geq 3 \) that

\[
\frac{1}{|n-1|} = \begin{cases} 
\frac{1}{n-1} & \text{if } n \text{ is a perfect cube} \\
1 & \text{otherwise}.
\end{cases}
\]

Now, since the sequence \( \left( \frac{1}{|n-1|} \right)_{n=3}^{\infty} \) is bounded and statistically convergent to zero it is Abel null [26, 28].

3. **Abel Transform of the operators \((D^n_q)\)**

In this section, we study Korovkin type approximation of the operators \((D^n_q)\) defined with (1.3) by considering the Abel method as well.

**Theorem 3.** If the sequence \( \left( \frac{[2]}{|n-1|} \right)_{n=3}^{\infty} \) is Abel null then for each \( f \in C[0,1] \) we have

\[
\lim_{y \to 1^-} (1 - y) \left\| \sum_{n=0}^{\infty} (D^n_q f - f) y^n \right\| = 0.
\]

**Proof.** Lemma 2 implies that \( \sum_{n=0}^{\infty} \|D^n_q(e_0)\| y^n < \infty \). From Theorem 1 it suffices to show that (2.1) holds for \((D^n_q)\). Using (1.10) and (1.11), we obtain for \( i = 0, 1 \) that

\[
\lim_{y \to 1^-} (1 - y) \left\| \sum_{n=0}^{\infty} (D^n_q e_i - e_i) y^n \right\| = 0.
\]

On the other hand from (1.12) and Lemma 3 we have for any \( n \geq 3 \) that

\[
\frac{[2]}{|n-1|} \left(1 - x\right) \left(1 - q^n x\right) - \frac{x [2]}{|n-1| |n-2|} \left(1 - x\right) \left(1 - q^n x\right)
\leq D^n_q(e_2; x) - x^2
\leq \frac{[2]}{|n-1|} \left(1 - x\right) \left(1 - q^n x\right)
\]
which implies

\[
0 \leq (1 - y) \left\| \sum_{n=3}^{\infty} (D_n^q e_2 - e_2) y^n \right\| \\
\leq (1 - y) \sup_{0 \leq x \leq 1} \sum_{n=3}^{\infty} \left( \frac{[2] x (1 - x) (1 - q^n x)}{n - 1} \right) y^n \\
\leq (1 - y) \sum_{n=3}^{\infty} \frac{[2]}{n - 1} y^n.
\]

Now from the hypothesis we obtain

\[
\lim_{y \to 1^{-}} (1 - y) \left\| \sum_{n=3}^{\infty} (D_n^q e_2 - e_2) y^n \right\| = 0.
\]

Therefore, from Remark 1 we can write

\[
\lim_{y \to 1^{-}} (1 - y) \left\| \sum_{n=0}^{\infty} (D_n^q e_2 - e_2) y^n \right\| = 0.
\]

which ends the proof. \(\square\)

Following remark shows that the condition of Theorem 3 is weaker than the classical conditions (1.4):

**Remark 3.** Note that if the classical conditions (1.4) hold then condition of Theorem 3 holds. In fact, if \(\lim_{n \to \infty} q_n^n = 1\) and \(\lim_{n \to \infty} \frac{1 + q_n}{n} = 0\) then we have

\[
\lim_{n \to \infty} \frac{[2]}{n - 1} = \lim_{n \to \infty} \frac{1 + q_n}{n - q_n^{n-1}} = 0.
\]

Therefore, it is Abel null. Conversely, consider the sequence \((q_n)\) given by

\[
q_n := \begin{cases} 
0 & \text{, } n \text{ is a prime} \\
1 & \text{, otherwise.}
\end{cases}
\]

We see that \((q_n)\) does not satisfy the conditions of classical Korovkin theorem. On the other hand, one can have for any \(n \geq 2\) that

\[
\frac{1 + q_n}{[n] - q_n^{n-1}} = \begin{cases} 
1 & \text{, } n \text{ is a prime} \\
2 & \text{, } n \text{ is not a prime}
\end{cases}
\]

Thus the sequence \(\left( \frac{1 + q_n}{[n] - q_n^{n-1}} \right)_{n=2}^{\infty}\) is Abel convergent to zero (since it is bounded and statistically null).

4. **Rate of Abel convergence**

In this section, we compute the rate of the Abel convergence by means of the modulus of continuity. The modulus of continuity of \(\omega(f, \delta)\) is defined by

\[
\omega(f, \delta) = \sup_{x \in [0,1]} \sup_{|y| \leq \delta} |f(x) - f(y)|
\]
It is well known that, for any \( f \in C[a, b] \),
\[
\lim_{\delta \to 0^+} \omega (f, \delta) = 0
\]
and for any \( \delta > 0 \)
\[
|f(x) - f(y)| \leq \omega (f, \delta) \left( \frac{|x - y|}{\delta} + 1 \right)
\]
and for all \( c > 0 \)
\[
\omega (f, c\delta) \leq (1 + \lfloor c \rfloor) \omega (f, \delta)
\]
where \( \lfloor c \rfloor \) is the greatest integer less than or equal to \( c \). Now we are ready to give the following lemma:

**Lemma 4.** For any \( f \in C[0, 1] \) we have
\[
(1 - y) \left\| \sum_{n=3}^{\infty} (D_n^q f - f) y^n \right\| \leq 2 \omega (f, \varphi (y)),
\]
where
\[
\varphi (y) := \left\{ (1 - y) \sup_{0 \leq x \leq 1} \sum_{n=3}^{\infty} D_n^q (t - x)^2 y^n \right\}^{\frac{1}{2}}
\]
and the series in (4.3) is convergent for each \( y \in (0, 1) \).

**Proof.** By using (1.10), for any \( f \in C[0, 1] \) and any \( \delta > 0 \) we can write
\[
\left\| \sum_{n=3}^{\infty} (D_n^q (f; x) - f (x)) y^n \right\| \leq \sum_{n=3}^{\infty} D_n^q |f (t) - f (x)| \; y^n
\]
\[
\leq \sum_{n=3}^{\infty} D_n^q \left( \omega \left( f, \left\{ t - x \right\} \frac{|t - x|}{\delta} \right) \; ; x \right) y^n
\]
\[
\leq \sum_{n=3}^{\infty} D_n^q \left( 1 + \left\{ \frac{|t - x|}{\delta} \right\} \omega (f, \delta) \; ; x \right) y^n
\]
\[
\leq \omega (f, \delta) \sum_{n=3}^{\infty} D_n^q \left( 1 + \frac{(t - x)^2}{\delta^2} \right) y^n
\]
\[
\leq \omega (f, \delta) \sum_{n=3}^{\infty} D_n^q (c_0 (t) \; ; x) y^n
\]
\[
+ \frac{1}{\delta^2} \omega (f, \delta) \sum_{n=3}^{\infty} D_n^q \left( (t - x)^2 \; ; x \right) y^n
\]
\[
\leq \omega (f, \delta) \left( \frac{1}{1 - y} - y - y^2 \right) +
\]
\[
+ \frac{1}{\delta^2} \omega (f, \delta) \sum_{n=3}^{\infty} D_n^q \left( (t - x)^2 \; ; x \right) y^n.
\]
Thus, we reach to
\[(1 - y) \left| \sum_{n=3}^{\infty} (D_q^n (f; x) - f(x)) y^n \right| \leq \omega (f; \delta) + \frac{1}{\delta^2} \omega (f; \delta) \sum_{n=3}^{\infty} D_q^n \left( (t - x)^2; x \right) y^n \]

Now if we take \( \delta = \left( (1 - y) \sup_{0 \leq x \leq 1} \sum_{n=3}^{\infty} D_q^n \left( (t - x)^2; x \right) y^n \right)^{\frac{1}{2}} \), we get

\[
0 \leq (1 - y) \left| \sum_{n=3}^{\infty} (D_q^n f - f) y^n \right| \leq 2 \omega (f, \phi (y))
\]

which completes the proof. \(\square\)

**Remark 4.** If the sequence \( \left( \frac{[2]}{[n-1]} \right) n=2^{\infty} \) is Abel summable (need not to be zero) then the series in (4.3) is convergent for each \( y \in (0, 1) \).

Using Lemma 4 and Remark 1 the following theorem which gives the rate of the Abel convergence for \( (D_q^n) \) can be proved:

**Theorem 4.** Let \( \phi \) be defined as Lemma 4 If \( \omega (f, \phi (y)) = o(\mu(y)) \) as \( y \to 1^- \) then we have

\[
(1 - y) \left| \sum_{n=1}^{\infty} (D_q^n f - f) y^n \right| = o(\mu(y)) \text{ as } y \to 1.
\]

The rate of Abel convergence for \( (M_q^n) \) defined with (1.2) can be proved by using the similar idea for \( (D_q^n) \):

**Theorem 5.** If \( \omega (f, \phi (y)) = o(\mu(y)) \) as \( y \to 1^- \). then we have

\[
(1 - y) \left| \sum_{n=1}^{\infty} (M_q^n f - f) y^n \right| = o(\mu(y)) \text{ as } y \to 1^-
\]

where

\[
\phi (y) := \left\{ (1 - y) \sup_{0 \leq x \leq 1} \sum_{n=3}^{\infty} M_q^n (t - x)^2 y^n \right\}^{\frac{1}{2}}.
\]

**References**

[1] Acu, A. M. Barbosu, D. and Sofonea, D. F. A Note on A \( q \)-Analogue of Stancu-Kantorovich Operators, Miskolc Mathematical Notes, 16 (1) (2015), 3-15.

[2] Agratini O. and Doğru, O. Weighted approximation by \( q \)-Szász-King type operators, Taiwanese J. of Math., 14 (4) (2010), 1283-1296.

[3] Altomare F. and Campiti, M. Korovkin-type approximaton theory and its applications, Walter de Gruyter, Berlin-New York, 1994.

[4] Aral, A. and Gupta, V., R.P. Agarwal, Applications of \( q \)-Calculus in Operator Theory, Springer, New York, 2013

[5] Atlıhan O. G. and Orhan, C. Matrix summability and positive linear operators, Positivity, 11 (3) (2007), 387-398.

[6] Atlıhan O. G. and Unver, M. Abel transforms of convolution operators. Georgian Math. J. 22 (3) (2015), 323-329.

[7] Braha N. L. Some properties of Baskakov-Schurer-Szász operators via power summability methods, Quaestiones Mathematicae, doi.org/10.2989/16073666.2018.1523248, (2018).
[8] Braha N. L. Some properties of new modified Szász-Mirakyan operators in polynomial weight spaces via power summability methods, Bull. Math. Anal. Appl. 10 (3) (2018), 53-65.
[9] Cheney E. W. and Sharma, A. Bernstein power series, Canad. J. Math., 16 (1964), 241-253.
[10] Doğru, O. Direct and Inverse Theorems on Statistical Aproximations by Positive Linear Operators, Rocky Mountain J. Math. 38 (6) (2008), 1959-1973.
[11] Doğru O. and Duman, O. Statistical approximation of Meyer-König and Zeller operators based on $q$-integers. Publ. Math. Debrecen, 68 (2006) (1-2), 199-214.
[12] Doğru O. and Gupta, V. Korovkin-type approximation properties of bivariate $q$-Meyer-König and Zeller operators. Calcolo, 43 (2006), 51-63.
[13] Gadžiev A. D. and Orhan, C. Some approximation theorems via statistical convergence, Rocky Mountain J. Math. 32 (2002), 129-138.
[14] Govil, N. K. and Gupta, V. Convergence of $q$-Meyer-König-Zeller-Durrmeyer operators, Advanced Studies in Contemporary Mathematics, 19 (1) (2009), 97-108.
[15] Heping, W. Properties of convergence for the $q$-Meyer-König-Zeller operators. J. Math. Anal. Appl. 335 (2)(2007), 1360-1373.
[16] Korovkin, P. P. Convergence of linear positive operators in the spaces of continuous functions (Russian), Doklady Akad. Nauk. SSSR (N.S)
[17] Lupas, A. A $q$-analogue of the Bernstein operators. Seminar on Numerical and Statistical Calculus 9, University of Cluj-Napoca, Cluj-Napoca, 1987.
[18] Meyer-König, W. Zeller, K. Bernsteinsche Potenzreihen, Studia Math., 19(1960), 89-94.
[19] Mishra, V. N., Khatri, K. and Mishra, L. N. Statistical approximation by Kantorovich-type discrete $q$-Betaoperators. Advances in Difference Equations, 2013 (1), 345.
[20] Mishra, V. N., Khatri, K. and Narayan Mishra, L. Some approximation properties of Baskakov-Beta-Stancu type operators. Journal of Calculus of Variations, 2013.
[21] Mursaleen, M. and Ansari, K. J. Approximation of $q$-Stancu-Beta operators which preserve $x^2$. Bulletin of the Malaysian Mathematical Sciences Society, 2015.
[22] Ostrovskaya, S. $q$-Bernstein polynomials and their iterates. Journal of Approximation Theory, vol. 123 (2)(2003), 232-255.
[23] Özgüç, İ. $L_p$-approximation via Abel convergence. Bull. Belg. Math. Soc. Simon Stevin 22 (2) (2015), 271-279.
[24] Phillips, G. M. Bernstein polynomials based on the $q$-integers, Ann. Numer. Math. 4 (1997) 511-518.
[25] Phillips, G. M. Interpolation and approximation by polynomials. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 14. Springer-Verlag, New York, 2003.
[26] Powell, R. E. and Shah, S. M. Summability Theory and Its Applications, Prentice-Hall of India, New Delhi, 1988.
[27] Sakaoglu, İ and Unver, M. Statistical approximation for multivariable integrable functions, Miskolc Math. Notes 13(2)(2012), 485-491.
[28] Schoenberg, I. J. The integrability of certain functions and related summability methods. The American Mathematical Monthly, 66 (5) (1959), 361-775.
[29] Taş, E. and T. Yurdakadim and Ö. G. Atlıhan, Korovkin type approximation theorems in weighted spaces via power series method, Oper. Matr. 12 (2) (2018), 529-535.
[30] Taş, E. and T. Yurdakadim, Approximation to derivatives of functions by linear operators acting on weighted spaces by power series method. Computational analysis, (2016) , Springer Proc. Math. Stat., 155, Springer, 363-372.
[31] Thomae, J. Beitrage zur Theorie der durch die Heinesche Reihe, J. Reine. Angew. Math. 70 (1869), 258-281.
[32] Trif, T. Meyer-König and Zeller operators based on the $q$-integers, Rev. Anal. Numer. Theor. Approx., 29(2000), 221-229.
[33] Ünver, M. Abel transforms of positive linear operators. In: ICNAAM 2013. AIP Conference Proceedings, 1558 (2013), 1148-1151.
[34] Ünver, M. Abel transforms of positive linear operators on weighted spaces. Bulletin of the Belgian Mathematical Society-Simon Stevin, 21(5) (2014), 813-822.
[35] Süleymanzade-Özden, D. Başcanbaz-Tunc, G. and Aral, A. Approximation by Complex $q$-Baskakov Operators. Oradea Fascicola Mathematica. 21(1) (2014), 167-181.
[36] Süleymanzade, D. On $q$-Bleimann, Butzer and Hahn-Type Operators, Abstract and Applied Analysis Volume 2015, Article ID 480925.
[37] Söylemez, D. Modified $q$-Baskakov Operators, Communications Faculty Of Science University of Ankara Series A1 Mathematics and Statistics Volume 65 (1) (2016), 1-9.

[38] Söylemez, D. and Unver, M. Korovkin type theorems for Cheney–Sharma Operators via summability methods, Res. Math., 73(3)(2017), 1601-1612.

[39] Srivastava, H. M., Mursaleen, M., Alotaibi, A., Nasiruzzaman, Md. and Al-Abied, A. A. H., Some approximation results involving the $q$-Szász–Mirakjan-Kantrovich type operators via Dunkl’s generalization, Math. Meth. App. Sci., 40 (15) (2017), 5437-5452.

E-mail address: dsoylemez@ankara.edu.tr

Ankara University, Elmadağ Vocational School, Department of Computer Programming, 06780 Ankara, Turkey, Ankara University, Faculty of Science, Department of Mathematics, 06100 Ankara TURKEY

E-mail address: munver@ankara.edu.tr