HEAT KERNEL RENORMALIZATION ON MANIFOLDS WITH BOUNDARY

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ABSTRACT. In the monograph Renormalization and Effective Field Theory, Costello made two major advances towards the mathematical formulation of quantum field theory. Firstly, he developed an inductive position space renormalization procedure for constructing effective field theories that is based on heat kernel regularization of the propagator. Secondly, he gave a rigorous formulation of quantum gauge theory within effective field theory that makes use of the BV formalism. In this work, we extend Costello’s inductive renormalization procedure to a class of manifolds with boundary. In addition, we reorganize the presentation of the preexisting material, filling in details and strengthening the results.

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1. Introduction

Effective field theory, in the context of the renormalization group, was developed by Wilson [5] [6] based on earlier work of Kadanoff [3]. There are many variations, but the basic theme involves two steps: mode elimination and rescaling [4] [1]. In this introduction, we shall focus on the intuitive idea of mode elimination and how it relates to the body of the paper.

Suppose that we have an action functional $S[\Lambda \phi]$ describing physics below an energy scale $\Lambda_H$. Then the action functional $S[\Lambda L \phi]$ describing physics at a lower energy scale should be given by “eliminating the modes” with energy between $\Lambda_L$ and $\Lambda_H$. This is described by the renormalization group equation (RGE)

$$e^{S[\Lambda L \phi]/\hbar} = \int_{\phi' \in \mathcal{E}(\Lambda_L, \Lambda_H)} e^{S[\Lambda_H (\phi + \phi')]/\hbar} D\phi'.$$

(1)

where the integral is over $\mathcal{E}(\Lambda_L, \Lambda_H)$, the space of fields with energy between $\Lambda_L$ and $\Lambda_H$. And $S[\Lambda \phi]$ is defined on the low energy fields $\phi \in \mathcal{E}[0, \Lambda]$. Equivalently, we can write

$$S[\Lambda_L \phi] = \hbar \log \int_{\phi' \in \mathcal{E}(\Lambda_L, \Lambda_H)} e^{S[\Lambda_H (\phi + \phi')]/\hbar} D\phi'.$$

(2)

In order to define the effective action $S[\Lambda \phi]$, one might be tempted to let $\Lambda \to \infty$ and write

$$S[\Lambda \phi] = \hbar \log \int_{\phi' \in \mathcal{E}(\infty, \Lambda_H)} e^{S[\Lambda_H (\phi + \phi')]/\hbar} D\phi'.$$

(3)

but this limit will not exist due to ultraviolet divergences. However, the limit should exist after an appropriate renormalization of the functional integral (3).

The focus of the first part of this thesis will be on constructing effective field theory, albeit in a slightly different formulation, which we now begin to move towards.

For the remainder of the introduction, for expository reasons, we shall work with a scalar theory on a compact manifold $M$. Let $D$ be the Laplacian on $M$, $\mathcal{E} = C^\infty(M)$, and the “modes” the eigenvalues of $D$. Assume that the action is of the form

$$S(\phi) = -\frac{1}{2} \langle \phi, D\phi \rangle + I(\phi).$$

(4)

where $\langle \phi, D\phi \rangle = \int_M \phi D\phi$ is the quadratic part of the action. Because $\phi \in \mathcal{E}[0, \Lambda]$ and $\phi' \in \mathcal{E}(\Lambda, \infty)$ are orthogonal,

$$S(\phi + \phi') = -\frac{1}{2} \langle \phi, D\phi \rangle - \frac{1}{2} \langle \phi', D\phi' \rangle + I(\phi + \phi').$$

(5)

If $S[\Lambda \phi] = -\frac{1}{2} \langle \phi, D\phi \rangle + I[\Lambda \phi]$, then the renormalization group equation simplifies to

$$e^{I[\Lambda_L \phi]/\hbar} = \int_{\phi' \in \mathcal{E}(\Lambda_L, \Lambda_H)} e^{-\frac{1}{2} \langle \phi', D\phi' \rangle / \hbar + I[\Lambda_H (\phi + \phi')]/\hbar} D\phi'.$$

(6)

or equivalently

$$I[\Lambda_L \phi] = \hbar \log \int_{\phi' \in \mathcal{E}(\Lambda_L, \Lambda_H)} e^{-\frac{1}{2} \langle \phi', D\phi' \rangle / \hbar + I[\Lambda_H (\phi + \phi')]/\hbar} D\phi'.$$

(7)
Let $P = P(\Lambda_L, \Lambda_H)$ be the inverse of the quadratic form $\langle \phi', D\phi' \rangle$ on $\mathcal{E}(\Lambda_L, \Lambda_H)$ and let $\partial_P$ be the contraction operator associated to $P$. By Wick’s theorem on the finite dimensional vector space $\mathcal{E}(\Lambda_L, \Lambda_H)$, the integral (6) is equal to the Wick contraction
\begin{equation}
V(P, I[\Lambda H]) := e^{\hbar \partial_P e^{I[\Lambda_H]/\hbar}}.
\end{equation}
and (7) is equal to the expression
\begin{equation}
W(P, I[\Lambda H]) := \hbar \log[e^{\hbar \partial_P e^{I[\Lambda_H]/\hbar}}].
\end{equation}

While the version of effective field theory with sharp energy cutoffs described above paints an intuitive physical picture, there are disadvantages to working with it, as discussed in [1]. Costello gives an alternative approach that comes from noticing the relationship between the uncut propagator and the heat kernel. Let $K_t(x, y)$ be the heat kernel for $D$. That is
\begin{equation}
\partial_t K_t(x, y) + D_x K_t(x, y) = 0
\end{equation}
and $\lim_{t \to 0^+} \int_M K_t(x, y) \phi(y) dy = \phi(x)$. Then if the integral
\begin{equation}
G(x, y) = \int_0^\infty K_t dt
\end{equation}
exists the operator it induces provides an inverse to $D$ on $\mathcal{E}_{(0, \infty)}$. That is, away from the energy zero fields.

Instead of cutting off the space of fields, we work with the entire space of fields $\mathcal{E}$ and introduce the regularized propagator
\begin{equation}
P_L^\epsilon = \int_c^L K_t dt
\end{equation}
An effective field theory now becomes a collection of length scale regularized interactions satisfying
\begin{equation}
I[L] = \hbar \log[e^{\hbar \partial_P e^{I[\epsilon]/\hbar}}].
\end{equation}
or more compactly $I[L] = W(P_L^\epsilon, I[\epsilon]).$

We naively might try to define the scale $L$ effective interaction as
\begin{equation}
I[L] = \lim_{\epsilon \to 0^+} \hbar \log \left[ \exp \left( \hbar \partial_P^\epsilon \right) \exp \left( I[\epsilon]/\hbar \right) \right]
\end{equation}
However, this limit may not exist and expression then has to be renormalized. That is, an interaction functional $I(\epsilon)$ with counterterms for $I$ is constructed such that
\begin{equation}
I[L] = \lim_{\epsilon \to 0^+} \hbar \log \left[ \exp \left( \hbar \partial_P^\epsilon \right) \exp \left( (I - I(\epsilon))/\hbar \right) \right]
\end{equation}
exists.

In Section 2 we define the spaces to which the propagator $P$ and the interaction functional belong. We define stable Feynman graphs which give a way of organizing the combinatorics of the contractions in $V(P, I)$ and $W(P, I)$. Theorem 1 expresses $V(P, I)$ as a summation over all stable graphs while Corollary 1 expresses $W(P, I)$ as a summation over connected stable graphs.

In Section 3, we state and prove several variations of Wick’s theorem. In Section 4, we calculate the 1 dimensional Gaussian integral
\begin{equation}
I_{m, \alpha}(a, b) = \int_a^b x^m e^{-\alpha x^2/2} dx
\end{equation}
in terms of $I_{0,\alpha}(a, b)$ and $J_{i,\alpha}(a, b) = x^i e^{-\alpha x^2/2} |x=b\rangle |x=a\rangle$ for $i < m$. The formula reduces to expected results on $\mathbb{R}$ and $\mathbb{R}^+$ which are recalled in 3.1 and 3.3 respectively. In 3.4, we generalize the formula for $I_{m,\alpha}(a, b)$ to one for

$$I_{m,\alpha,\beta}(a, b) = \int_a^b x^m e^{-\alpha x^2/2 + \beta x} dx. \tag{17}$$

The proof, which is analogous to the one in 3.2 is omitted. The next two sections are focused on the many variables Wick’s theorem. That is, the computation of the integral

$$\int_P x_{m_1} \ldots x_{m_k} e^{-Q(x)/2} dx \tag{18}$$

where $Q(x)$ is a nondegenerate quadratic form. In 3.5, we recall the standard statement of Wick’s theorem on $P = \mathbb{R}^n$ and give a proof by diagonalizing the quadratic form and applying the result of 3.1. This will be used to calculate the counterterms on $\mathbb{R}^n$ in 4.4. In 3.6 it is shown that the result of 3.2 is sufficient to compute (18) inductively, when $P$ is any polytope. Lastly, we show that as long as $P$ is bounded the $Q(x)$ may be degenerate and even inhomogeneous. In this case, the result of 3.4 can be applied iteratively to compute the answer. We specialize to the case relevant for the counterterms on $\mathbb{H}^n$, the upper half space with the Euclidean metric, in 4.6.

Section 4 forms the body of the paper. We begin with 4.1 where the construction of the counterterms in general is motivated by carrying out the procedure for the Feynman weight associated to a particular 1-loop graph in the $\phi^4$-theory. The renormalization procedure is based on the ability to cover $(0, \infty)^k$ and a fortiori $(\epsilon, 1)^k$ by sets defined by inequalities of the form $t_i \leq t_j R$, where $R > 1$. In the next section, the covering lemma that was proved by Costello in [1] is strengthened and proved. Much more detail about the nature of the sets in the cover is given. Other preliminary concepts needed for the renormalization procedure like local functionals and the form of their Feynman weights are then discussed.

In 4.4, we formulate Costello’s renormalization procedure on $\mathbb{R}^n$. We give explicit formulas whenever possible and fill in a few steps in the argument omitted by Costello, such as the introduction of what we call spanning tree coordinates. In 4.5, we show how to control the error and how the basic result of 4.4 can be used inductively to provide counterterms on each of sets in the cover of $(\epsilon, 1)^k$ where $k$ is the number of edges in the Feynman graph whose weight we are trying to renormalize.

In 4.6, the renormalization is adapted to $\mathbb{H}^n$, the upper half space with the Euclidean metric. The procedure does not carry over without modification since the quadratic form in the integral computing the Feynman weight is both no longer non-degenerate and no longer homogeneous. Luckily, this difficulty can be circumvented by a clever change of coordinates in the direction normal to boundary. The counterterms have a more complicated form than those on $\mathbb{R}^n$, but we argue that the inductive procedure of 4.5 can be carried out with appropriate modifications.

In 4.7, we correct what seems to be an oversight in Costello’s reasoning in [1]. On a compact manifold $M$, Costello uses the asymptotic expansion of the heat kernel $K_t(x, y) \sim e^{-d(x, y)^2/4t} \sum_i \phi_i(x, y) t^i$, but for each chart in a cover replaces $d(x, y)$ with the coordinate distance $\|x-y\|$. Thus, taking a partition of unity, the Feynman weight under consideration becomes a sum of integrals whose integrands will contain
the exponential of a quadratic form, which allows us to apply Wick’s theorem. However, it does not seem to be correct that
\[ K_t(x,y) \sim e^{-\|x-y\|^2/4t} \sum_i \phi_i(x,y)^{t^i}, \]
at least not uniformly in \( x \) and \( y \). Again, we show how this difficulty is not fatal.

While the counterterms will not simplify as they do on \( \mathbb{R}^n \), through the introduction of spanning tree coordinates, one can still bound the error. The inductive step in the construction thus remains valid.

The culmination of these results is 4.8 where we show the renormalization procedure can be carried out on a class of compact manifolds with boundary where the argument reduces that of 4.6 near the boundary and 4.7 away from the boundary.

In Section 5 we move beyond the construction of counterterms for each Feynman weight and construct the counterterms \( I^{CT}(\epsilon) \) for the entire effective interaction.

2. Feynman Diagrams

2.1. General Setup. Let \( \mathcal{E} \) be a graded object in an appropriate symmetric monoidal category, which contains a field \( \mathbb{K} \) as its monoidal unit. For toy examples one can work with the category of finite dimensional vector spaces. For quantum field theory one will need to work with an appropriate subcategory of topological vector spaces. The identifications \( (\mathcal{E} \otimes \mathcal{F})^* \cong \mathcal{E}^* \otimes \mathcal{F}^* \) and \( \text{Hom}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^* \otimes \mathcal{F} \) will be made throughout. We will not dwell on the issue any further and direct the interested reader to the appendices of [1] and [2].

Fix an element \( P \in \text{Sym}^2(\mathcal{E}) \) which will be called a propagator. We define the algebra of formal power series on \( \mathcal{E} \).

\[ \mathcal{O}(\mathcal{E}) = \prod_{n \geq 0} \text{Hom}(\otimes^n \mathcal{E}, \mathbb{K})_{S_n} = \prod_{n \geq 0} \text{Sym}^n(\mathcal{E}^*) \]

Here \( \text{Sym} \) means taking coinvariants of the \( n \)-fold tensor product with respect to the symmetric group action. An element of \( I \in \mathcal{O}(\mathcal{E})[[\hbar]] \) is of the form \( I = \sum_{i,k \geq 0} I_{i,k} \hbar^i \), where \( I_{i,k} \in \text{Sym}^k(\mathcal{E}, \mathbb{R}) \). Let

\[ \mathcal{O}(\mathcal{E})^+[[\hbar]] \subset \mathcal{O}(\mathcal{E})[[\hbar]] \]

be the functionals of the form \( I = \sum_{i,k \geq 0} I_{i,k} \hbar^i \), where \( I_{0,k} = 0 \) for \( k < 3 \) and \( I_{1,0} = 0 \). We will see the reason for this restricted class of functionals later in the section.

We are interested in combinatorial formulas for “functional integrals” of the form

\[ V(P,I) = e^{\hbar \partial_P} e^{I/\hbar} \]

and

\[ W(P,I) = \hbar \log(e^{\hbar \partial_P} e^{I/\hbar}), \]

where \( \partial_P \) denotes the contraction operator \( \frac{1}{2} \sum_i \partial_{p^{(1)}} \partial_{p^{(2)}} \) where \( P = \sum_i p^{(1)} \otimes p^{(2)} \).

Lemma 1 (Feynman Expansion).

\[ V(P,I) = \sum_{\{n_i,k\}} \sum_j C(\{n_i,k\},j) \hbar^{p(\{n_i,k\})} \partial_{\bar{P}_i} \prod_{i,k} I_{i,k}^{n_i,k} \]
In the outer summation, we sum over the collection of double sequences of nonnegative integers \( \{n_{i,k}\}_{i,k \geq 0} \) with the requirement that for all but finitely many \( i,k \), \( n_{i,k} = 0 \), and where we have defined

\[
C(\{n_{i,k}\}, j) = \frac{1}{j!} \prod_{i,k} \frac{1}{n_{i,k}!}
\]

and

\[
p(\{n_{i,k}\}, j) = \sum_{i,k} i n_{i,k} - \sum_{i,k} n_{i,k} + j.
\]

**Proof.** By the multinomial formula

\[
\exp \left( \sum_{i,k} I_{i,k} h^{i-1} \right) = \sum_{j} \frac{(\sum_{i,k} I_{i,k} h^{i-1})^j}{j!} = \sum_{j} \prod_{i,k} \frac{h^{(i-1)n_{i,k}}}{n_{i,k}!} I_{i,k}^{n_{i,k}},
\]

where the inner sum is over sequences of nonnegative numbers \( \{n_{i,k}\} \) such that \( \sum_{i,k} n_{i,k} = j \). We can reexpress this as a single sum over sequences of almost all zero nonnegative integers \( \{n_{i,k}\} \)

\[
\exp \left( \sum_{i,k} I_{i,k} h^{i-1} \right) = \sum_{\{n_{i,k}\}} \prod_{i,k} \frac{h^{(i-1)n_{i,k}}}{n_{i,k}!} I_{i,k}^{n_{i,k}}.
\]

Thus,

\[
V(P, I) = \sum_{\{n_{i,k}\}} \sum_{j} \frac{h^j}{j!} \prod_{i,k} \frac{h^{(i-1)n_{i,k}}}{n_{i,k}!} I_{i,k}^{n_{i,k}}
\]

\[
= \sum_{\{n_{i,k}\}} \sum_{j} C(\{n_{i,k}\}, j) p(\{n_{i,k}\}, j) \prod_{i,k} \frac{I_{i,k}^{n_{i,k}}}{I_{i,k}}.
\]

It remains to investigate the combinatorial structure of the expression

\[
\partial_p \prod_{i,k} I_{i,k}^{n_{i,k}}.
\]

Before doing so, we shall make a definition.

**Definition 1.** A **stable graph** is defined by

- \( V(\gamma) \): a set of vertices
- \( E(\gamma) \): a set of edges each connecting two vertices
- \( T(\gamma) \): a set of tails each connected to a single vertex

and a function \( g : V(\gamma) \rightarrow \mathbb{Z}^+ \) associating a "genus" to each vertex.

There is a natural preorder on vertices: If \( v_1 \) has \( g(v_1) = i_1 \) and valency \( k_1 \), and \( v_2 \) has \( g(v_2) = i_2 \) and valency \( k_2 \), then \( v_1 \preceq v_2 \) if \( i_1 < i_2 \) or \( i_1 = i_2 \) and \( k_1 \leq k_2 \).
2.2. **Feynman Diagram Expansion.** Begin with the expression

\[ V(P, I) = \sum_{\{\gamma, j\}} \sum_{\gamma} \left( \frac{1}{j!2^j} \prod_{i,k} \frac{1}{1!} \right) \mathcal{H}^{\{\gamma, j\}} \left( \sum_{l} \partial_{p_i^{(l)}} \partial_{P_i^{(l)}} \right)^j \prod_{i,k} I_{i,k}^{n_{i,k}}. \]

Let \( I_{i,k,1}, \ldots, I_{i,k,n} \) be the sequence of interactions for which \( n_{i,k} \neq 0 \). Recall that the propagator \( P \in \text{Sym}^2 \mathcal{E} \) with \( P = \sum_{l} P_{l}^{(1)} \otimes P_{l}^{(2)} \), and we are assuming that \( \mathcal{E} \) is ungraded. Make the substitution \( I_{i,k} = S^k I_{i,k} / k! \) where \( S^k I_{i,k} = \sum_{\sigma \in S_k} I_{i,k}^\sigma \). Then

\[
V(P, I) = \sum_{\{\gamma, j\}} \sum_{\gamma} \left( \frac{1}{j!2^j} \prod_{i,k} \frac{1}{1!} \right) \mathcal{H}^{\{\gamma, j\}} \left( \sum_{l} \partial_{p_i^{(l)}} \partial_{P_i^{(l)}} \right)^j \prod_{i,k} (S^k I_{i,k})^{n_{i,k}}.
\]

Note that

\[
\left( \sum_{l} \partial_{p_i^{(l)}} \partial_{P_i^{(l)}} \right)^j \prod_{i,k} (S^k I_{i,k})^{n_{i,k}}.
\]

will be a sum over contractions that can be parametrized by injections \( Q : H \rightarrow V \) of the set \( H = \{1^{(1)}, 1^{(2)}, \ldots, j^{(1)}, j^{(2)}\} \) into the set of inputs to the interactions \( V = \{1^{(1)}, 1^{(2)}, \ldots, 1^{(n)}\} \).

Since \( I_{i,k} \in \text{Sym}^* \mathcal{E}^* \) and \( I_{i,k}^\sigma \in \text{Sym}^* \mathcal{E}^* \), we can reorder the contractions so that the images of the index (1) elements in \( H, Q(1^{(1)}), \ldots, Q(j^{(2)}) \) are in ascending order. There are \( j! \) contractions that will be reordered to the same contraction in this way. We can also reorder so that \( Q(\alpha^{(1)}) \) comes before \( Q(\alpha^{(2)}) \). There are \( 2^j \) contractions that will be reordered to the same contraction in this way.

Injections up to these reorderings are in one-to-one correspondence with partitions of \( V \) into \( j \) subsets with two elements and 1 additional subset containing the remaining \( |V| - 2j \) elements. Let \( Q(\{n_{i,k}\}, j) \) be the collection of such partitions and for \( Q \in Q(\{n_{i,k}\}, j) \) let \( w_Q(P, I) \) denote the corresponding contraction.

Then

\[ (24) \quad V(P, I) = \sum_{\{\gamma, j\}} \sum_{Q \in Q(\{n_{i,k}\}, j)} \left( \prod_{i,k} \frac{1}{1!} \right) \mathcal{H}^{\{\gamma, j\}} w_Q(P, I) \]

Any partition \( Q \in Q(\{n_{i,k}\}, j) \) determines a stable graph \( \gamma \) in an obvious way. Consider \( Q_\gamma(\{n_{i,k}\}, j) \), the collection of partitions which determine the same stable graph \( \gamma \). Let \( G(\{n_{i,k}\}, j) = \prod_{i,k} (S_{n_{i,k}} \otimes n_{i,k}) \). Note that

\[
|G(\{n_{i,k}\}, j)| = \prod_{i,k} n_{i,k}! (k!)^{n_{i,k}}
\]

This acts on \( V \) by permuting the interactions of type \( i, k \) and their \( k \) inputs. As a consequence, it acts on \( Q(\{n_{i,k}\}, j) \). In fact, it acts transitively on \( Q_\gamma(\{n_{i,k}\}, j) \), the group of automorphisms of the stable graph \( \gamma \). By the orbit-stabilizer theorem, the number of partitions which determine the same stable graph \( \gamma \) is given by

\[ (25) \quad \frac{|G(\{n_{i,k}\}, j)|}{|\text{Aut}(\gamma)|} = \prod_{i,k} n_{i,k}! (k!)^{n_{i,k}} \frac{|\text{Aut}(\gamma)|}{|\text{Aut}(\gamma)|} \]

Therefore,
Theorem 1 (Feynman Diagram Expansion). For a stable graph \( \gamma \), we define
\[
g(\gamma) = b(\gamma) + \sum_{v \in V(\gamma)} g(v)
\]
where \( b(\gamma) \) is the first Betti number of \( \gamma \). Let \( C(\gamma) \) be the number of connected components of \( \gamma \). Then
\[
V(P, I) = \sum_{\gamma} \frac{1}{|\text{Aut}(\gamma)|} h^{g(\gamma) - C(\gamma)} w_{\gamma}(P, I)
\]
Proof. The constant \( p(\{n_{i,k}\}, j) = \sum_{i,k} in_{i,k} - \sum_{i,k} n_{i,k} + j \) has a very simple interpretation in terms of the stable graph \( \gamma \) since
\[
\sum_{v \in V(\gamma)} g(v) = \sum_{i,k} in_{i,k},
\]
\(|V(\gamma)| = \sum_{i,k} n_{i,k}\) and \(|E(\gamma)| = j\). Using the fact that
\[
b(\gamma) = |E(\gamma)| - |V(\gamma)| + C(\gamma),
\]
and the definition
\[
g(\gamma) = b(\gamma) + \sum_{v \in V(\gamma)} g(v)
\]
we have
\[
p(\{n_{i,k}\}, j) = g(\gamma) - C(\gamma).
\]
Finally define \( w_{\gamma}(P, I) \) to be \( w_{Q}(P, I) \) where \( Q \) is any partition that determines \( \gamma \). The formula now follows from (24) and (25).

Corollary 1.
\[
W(P, I) = \sum_{\gamma} \frac{1}{|\text{Aut}(\gamma)|} h^{g(\gamma) - 1} w_{\gamma}(P, I)
\]
Proof. If \( \gamma_1 \cup \cdots \cup \gamma_k \) is the disjoint union of not necessarily distinct connected stable graphs \( \gamma_1, \ldots, \gamma_k \), then it is clear that
\[
g(\gamma_1 \cup \cdots \cup \gamma_k) = g(\gamma_1) + \cdots + g(\gamma_k)
\]
\( C(\gamma_1 \cup \cdots \cup \gamma_k) = C(\gamma_1) + \cdots + C(\gamma_2) \)
and if \( \gamma = (\cup^{k_1} \gamma_1) \cup \cdots \cup (\cup^{k_n} \gamma_n) \) where \( \gamma_1, \ldots, \gamma_n \) are distinct
\[
|\text{Aut}(\gamma)| = k_1! \cdots k_n! |\text{Aut}(\gamma_1)|^{k_1} \cdots |\text{Aut}(\gamma_n)|^{k_n}
\]
Thus,
\[
\exp \left( \frac{W(P, I)}{\hbar} \right) = \exp \left( \sum_{\gamma} \frac{1}{|\text{Aut}(\gamma)|} h^{g(\gamma) - 1} w_{\gamma}(P, I) \right)
\]
\[
= \sum_{\{k_\gamma\}} \prod_{\gamma \text{ conn}} \frac{1}{|\text{Aut}(\gamma)|^{k_\gamma} k_\gamma!} h^{k_\gamma (g(\gamma) - 1)} w_{\cup_{\gamma} \gamma}(P, I)
\]
\[
= \sum_{\gamma} \frac{1}{|\text{Aut}(\gamma)|} h^{g(\gamma) - C(\gamma)} w_{\gamma}(P, I)
\]
In the second line above, for each sequence \( \{ k_\gamma \}_{\gamma \text{ conn}} \) in the outer summation, \( k_\gamma = 0 \) for all but finitely many \( \gamma \), and \( k_\gamma \) is a nonnegative integer for all \( \gamma \). □

Corollary 2. For \( I \in \mathcal{O}(\mathcal{E})^+[[\hbar]] \),
\[
W(P, I) \in \mathcal{O}(\mathcal{E})^+[[\hbar]]
\]

3. Wick’s Theorem

3.1. Wick’s Theorem on \( \mathbb{R} \). In one variable, Wick’s theorem reduces to the statement
\[
\int_{-\infty}^{\infty} x^m e^{-\alpha x^2 / 2} \, dx = \begin{cases} 
\sqrt{2\pi} \frac{(2k)!}{\alpha^{(2k+1)/2}} & \text{if } m = 2k \\
0 & \text{if } m = 2k + 1.
\end{cases}
\]
(32)
\[
= C_m \frac{1}{\alpha^{(m+1)/2}}
\]

3.2. Wick’s Theorem on \((a, b)\). There are several ways of proving the formula for \( \mathbb{R} \) which one might try to adapt. The proof by integration by parts seems the best suited and is the one we develop here.

We wish to compute the integral
\[
I_{m, \alpha}(a, b) = \int_a^b x^m e^{-\alpha x^2 / 2} \, dx
\]
for \(-\infty \leq a \leq b \leq \infty\) and to check that the result agrees with the standard formula for \( a = -\infty \) and \( b = \infty \). Let
\[
J_{m, \alpha}(a, b) = \left. x^m e^{-\alpha x^2 / 2} \right|_{x=a}^{x=b}.
\]

By integration by parts,
\[
\int_a^b x^m e^{-\alpha x^2 / 2} \, dx = \int_a^b x^{m-1} (xe^{-\alpha x^2 / 2}) \, dx
\]
\[
= \frac{m-1}{\alpha} \int_a^b x^{m-2} e^{-\alpha x^2 / 2} \, dx - \frac{x^{m-1}}{\alpha} e^{-\alpha x^2 / 2} \bigg|_a^b
\]
That is,
(34)
\[
I_{m, \alpha}(a, b) = \frac{m-1}{\alpha} I_{m-2, \alpha}(a, b) - \frac{1}{\alpha} J_{m-1, \alpha}(a, b).
\]

For \( m \) even, we can thus express \( I_{m, \alpha}(a, b) \) in terms of \( I_{0, \alpha}(a, b) \) and \( J_{l, \alpha}(a, b) \) where \( l \) ranges over odd integers less than \( m \). For \( m \) odd, since \( I_{1, \alpha}(a, b) = -(1/\alpha) J_{0, \alpha}(a, b) \), we can express \( I_{m, \alpha}(a, b) \) in terms of \( J_{l, \alpha}(a, b) \), where \( l \) ranges over even integers less than \( m \).

We can then prove a precise formula by induction:

Proposition 1.
\[
I_{m, \alpha}(a, b) = \frac{C_m}{\alpha^{m/2}} I_{0, \alpha}(a, b) - \sum_{i=0}^{[m/2]} \frac{\tilde{C}_{i,m}}{\alpha^{i+1}} J_{m-1-2i, \alpha}(a, b),
\]
where \( C_m = 0 \) when \( m \) is odd and \( C_m = (m-1)!! \) when \( m \) is even and for all \( m \)
(35)
\[
\tilde{C}_{i,m} = \frac{(m-1)!!}{(m-1-2i)!!}
\]
Proof. The even and odd base cases when \( m = 0 \) and \( m = 1 \) are clearly satisfied. Suppose the result is true for \( I_{m,\alpha}(a,b) \). Then using (34),

\[
I_{m+2,\alpha}(a,b) = \frac{m+1}{\alpha} \frac{C_m}{\alpha^{m/2}} - \frac{m+1}{\alpha} \sum_{i=0}^{\lfloor m/2 \rfloor} \frac{\tilde{C}_{i,m}}{\alpha^{i+1}} I_{m-1-2i,\alpha}(a,b)
\]

and

\[
(m+1)\tilde{C}_{i,m} = \frac{(m+1)!!}{(m+1-2(i+1))!!} = \tilde{C}_{i+1,m+2}
\]

so

\[
\frac{m+1}{\alpha} \sum_{i=0}^{\lfloor m/2 \rfloor} \frac{\tilde{C}_{i,m}}{\alpha^{i+1}} I_{m-1-2i,\alpha}(a,b)
\]

\[
= \sum_{i=0}^{\lfloor (m+2)/2 \rfloor} \frac{\tilde{C}_{i+1,m+2}}{\alpha^{i+1}} I_{m+2-1-2i,\alpha}(a,b)
\]

The induction step is now completed by employing the fact that

\[
\frac{(m+1)!!}{\alpha^{m/2}} = \frac{C_{m+2}}{\alpha^{(m+1)/2}}
\]

As \( a \to -\infty \) and \( b \to \infty \), we have \( J_{m,\alpha} \to 0 \) and \( I_{0,\alpha} \to \sqrt{2\pi/\alpha} \). Combining this with identity for the double factorial

\[
(2k-1)!! = \frac{(2k)!}{k!2^k}
\]

we recover the statement of Wick’s theorem on \( \mathbb{R} \).

3.3. Wick’s theorem on \( \mathbb{R}^+ \). Note that if \( a = 0 \) and \( b = \infty \), then \( J_{l,\alpha} = 0 \) for \( l \neq 0 \) and \( J_{0,\alpha} = -1 \). Since \( (2k)!! = 2^kk! \),

\[
\int_0^\infty x^m e^{-ax^2/2} \, dx = \begin{cases} \frac{C_m}{\alpha^{m/2}} I_{0,\alpha}(0, \infty) & m \text{ even} \\ \frac{C_m}{\alpha^{m+1}} J_{0,\alpha}(0, \infty) & m \text{ odd} \end{cases}
\]

\[
= \begin{cases} \sqrt{2\pi} \frac{(2k)!}{(2k+1)!} \frac{1}{\alpha^{k+1}} & \text{if } m = 2k \\ 2^kk! \frac{1}{\alpha^{k+1}} & \text{if } m = 2k+1. \end{cases}
\]
3.4. Generalized Wick’s Theorem on \((a, b)\). In \[4.6\] we shall encounter integrals of polynomials with respect to inhomogeneous quadratic forms. Here we establish the one dimensional result that can be used iteratively to calculate such integrals explicitly.

We wish to compute the integral
\[
I_{m, \alpha, \beta}(a, b) = \int_{a}^{b} x^{m} e^{-\alpha x^2 / 2 + \beta x} \, dx
\]
for \(-\infty \leq a \leq b \leq \infty\) and to check that the result agrees with the standard formula for \(a = -\infty\) and \(b = \infty\). Let
\[
J_{m, \alpha, \beta}(a, b) = x^{m} e^{-\alpha x^2 / 2 + \beta x} \bigg|_{x=a}^{x=b}.
\]

Firstly, \([34]\) generalizes to
\[
(39) \quad I_{m, \alpha, \beta}(a, b) = -\frac{1}{\alpha} I_{m-1, \alpha, \beta}(a, b) + \frac{\beta}{\alpha} I_{m-1, \alpha, \beta}(a, b) + \frac{m-1}{\alpha} I_{m-2, \alpha, \beta}(a, b).
\]
The following is a generalization of Proposition \[11\]

**Proposition 2.**

\[
(40) \quad I_{m, \alpha, \beta}(a, b) = -\sum_{i=0}^{m-1} \sum_{\{a_j\}} \frac{\beta^{[a^{-1}(1)]}}{\alpha^{[l(a)]+1}} \prod_{k \in a^{-1}(2)} (m - 1 + s_k - i) J_{\alpha, m-i-1}^{(a, m-i-1)}
\]
\[
+ \sum_{\{a_i\}} \frac{\beta^{[a^{-1}(1)]}}{\alpha^{[l(a)]+1}} \prod_{k \in a^{-1}(2)} (s_k - 1) I_{\alpha, b, 0}^{(a, m, 0)}
\]

where \(\{a_j\}\) ranges over finite sequences such that \(a_j \in \{1, 2\}\) for all \(j\). We use \(l(a)\) to denote the length of the sequence \(\{a_i\}\) and \(s_k = \sum_{j=1}^{k} a_j\).

We shall not give the proof which is a straightforward induction like the proof of Proposition \[11\]. However, let us just check that it reduces to the formula of Proposition \[11\] in the case that \(\beta = 0\). Since \(0^0 = 1\) and \(0^k = 0\) for \(k > 0\) the only nonzero terms in the sums will come from sequences with \(a^{-1}(1) = \emptyset\). But there is exactly one such sequence such that \(\sum a_j = m\) for \(m\) even and it has \(l(a) = m/2\) and no such sequences for \(m\) odd. It is clear that this then becomes the formula of Proposition \[11\].

3.5. Wick’s theorem on \(\mathbb{R}^n\). Suppose that \(A\) is an invertible symmetric \(n \times n\) matrix and consider the associated quadratic form \(Q(x) = \langle x, Ax \rangle = x^t A x\). We wish to compute the integral
\[
I_{J, A} = \int_{P} x_{m_1} \cdots x_{m_k} e^{-Q(x)/2} \, dx
\]
where \(J = \{j_1, \ldots, j_n\}\) is a multi-index such that \(x_{j_1}^{m_1} \cdots x_{j_n}^{m_k} = x_{m_1} \cdots x_{m_k}\) and \(P\) is a polytope.

**Theorem 2** (Wick’s Theorem on \(\mathbb{R}^n\)). For \(k\) even
\[
(42) \quad \int_{\mathbb{R}^n} x_{m_1} \cdots x_{m_k} e^{-Q(x)/2} \, dx = \frac{\sqrt{2\pi}}{\sqrt{\det(A)}} \sum_{\{\beta\}} \prod_{j=1}^{k/2} A_{\alpha, \beta}^{-1} \prod_{j=1}^{k/2} A_{\alpha, \beta}^{m_{j_1}(1), m_{j_2}(2)}
\]
where the sum is over partitions of the set 1, ..., k into k/2 subsets of 2 elements. Here $\beta_j^{(1)}$ and $\beta_j^{(2)}$ denote respectively the first and second elements of the j-th set in the partition.

Proof. Let $D$ denote the diagonalization of $A$ and assume that $D$ has diagonal entries $\alpha_1, \ldots, \alpha_n$. In this new basis, using the change of basis matrix $S$, we have a linear combination

$$\sum_{i_1, \ldots, i_k} S_{m_{i_1}}^{i_1} \ldots S_{m_k}^{i_k} \int_{\mathbb{R}^n} y_{i_1} \ldots y_{i_k} e^{-\alpha_1 y_1^2/2} \ldots e^{-\alpha_n y_n^2/2} dy.$$

Apply Wick's theorem on $\mathbb{R}$ separately in each variable. For each such integral, this gives

$$\frac{1}{\sqrt{\alpha_1 \ldots \alpha_n}} \prod_{i=1}^n C_{\alpha_i^{(1)}} D_{\alpha_i^{(2)}} = \frac{1}{\sqrt{\det(A)}} \prod_{i=1}^n C_{\alpha_i^{(1)}} D_{\alpha_i^{(2)}}$$

where the sum is over partitions of the set 1, ..., k into k/2 subsets of 2 elements. We then switch the order of summation so that the sum over partitions is the outer sum and then

$$\frac{(\sqrt{2\pi})^n}{\sqrt{\det(A)}} \sum_{\beta} \prod_{i_1, \ldots, i_k} S_{m_{i_1}}^{i_1} \ldots S_{m_k}^{i_k} \prod_{j=1}^{k/2} D_{i_{j(1)}^{(1)}, i_{j(2)}^{(2)}}^{-1} = \frac{(\sqrt{2\pi})^n}{\sqrt{\det(A)}} \sum_{\beta} \prod_{j=1}^{k/2} A_{i_{j(1)}^{(1)}, i_{j(2)}^{(2)}}$$

3.6. Wick's Theorem for Polytopes. The purpose of this section is to show that the result of 3.2 can be used inductively to calculate a Wick integral over a polytope in $\mathbb{R}^n$.

By the spectral theorem, $A$ can be diagonalized by an orthogonal transformation. This will produce a linear combination of integrals of the form

$$\int_P x_1^{k_1} \ldots x_n^{k_n} e^{-\alpha_1 x_1^2/2} \ldots e^{-\alpha_n x_n^2/2} dx.$$

where $P$ is some polytope.

Decompose the integral as

$$\int_{P'} \int_{a_n}^{b_n} x_1^{k_1} \ldots x_n^{k_n} e^{-\alpha_1 x_1^2/2} \ldots e^{-\alpha_n x_n^2/2} dx_n dx'.$$

where $P'$ is the projection of $P$ onto the hyperplane $x_n = 0$ and $a_n$ and $b_n$ are piecewise linear in the variables $x_1, \ldots, x_{n-1}$. The subdomains where $a_n$ is linear are the projections of the $(n-1)$-dimensional faces of $P$ onto the hyperplane $x_n = 0$. Denote an arbitrary projection of an $(n-1)$-dimensional face by $P^a$. Similarly, use $P^b$ for an arbitrary subdomain where $b_n$ is linear.

We apply Wick's theorem in one variable to $x_n$ to get a linear combination of elements of the form

$$\int_{P'} x_1^{k_1} \ldots x_{n-1}^{k_{n-1}} e^{-\alpha_1 x_1^2/2} \ldots e^{-\alpha_{n-1} x_{n-1}^2/2} J_{m-2t-1, \alpha_n} (a_n, b_n) dx'.$$
and an element of the form
\[ \int_{P} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}} e^{-\alpha_1 x_1^2/2} \cdots e^{-\alpha_{n-1} x_{n-1}^2/2} I_{0, \alpha_n}(a_n, b_n) \, dx. \]

By substituting the definition of \( J_{m-2i-1, \alpha} \), terms of the first form are equal to the summation
\[ \sum_{p_n} \int_{P'} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}} e^{-\alpha_1 x_1^2} \cdots e^{-\alpha_{n-1} x_{n-1}^2/2} b_n^{k_n-2i-1} e^{-\alpha_n b_n^2/2} \, dx' \]
\[ - \sum_{p_n} \int_{P'} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}} e^{-\alpha_1 x_1^2} \cdots e^{-\alpha_{n-1} x_{n-1}^2/2} c_n^{k_n-2i-1} e^{-\alpha_n c_n^2/2} \, dx' \]

We emphasize that \( b_n|_{P_n} \) is a linear function in the variables \( x_1, \ldots, x_{n-1} \) and similarly for \( a_n|_{P_n} \).

Let us focus our attention on any term involving \( b_n|_{P_n} = d_1 x_1 + \cdots + d_{n-1} x_{n-1} \); that is, those in the first summation. The analysis for terms involving \( a_n|_{P_n} \) in the second summation is similar.

Since \( P_n \) is a polytope, to complete the inductive step, it suffices to show that
\[ \alpha_1 x_1^2 + \cdots + \alpha_{n-1} x_{n-1}^2 + \alpha_u (d_1 x_1 + \cdots + d_{n-1} x_{n-1})^2 \]
is nondegenerate. Let \( d \) be the thought of as a column vector. Let \( c = \sqrt{\alpha_n} d \) and let \( A = \text{diag}(\alpha_1, \ldots, \alpha_{n-1}) \). Then
\[ \det(A + \alpha_n dd^t) = \det(A) \det(I + A^{-1} cc^t) \]
\[ = \det(A)(1 + c^t A^{-1} c) = \det(A)(1 + |A^{-1} c|^2) > 0 \]
which implies that the quadratic form is nondegenerate.

3.7. **Generalized Wick’s Theorem for Compact Polytopes.** In Section 4.6, the counterterms that will be introduced to renormalize the theory on \( \mathbb{H}^n \) will involve integrals of the form
\[ \int_{P_n} e^{-Q(z, u)} z^{k'} \, dz \]
where \( P_n \) is given by the inequalities
\[ 0 \leq u + z_1 \]
\[ 0 \leq u + z_2 - z_1 \]
\[ \ldots \]
\[ 0 \leq u + z_{m-1} - z_{m-2} \]
\[ 0 \leq u - z_{m-1}. \]
and \( Q(z, u) = \sum_{i,j} a_{ij} z_i z_j + \sum_i b_i u z_i \)

We reexpress the above inequalities in a form which makes it possible to calculate the integral.
\[ -u \leq z_1 \leq (m-1)u \]
\[ z_1 - u \leq z_2 \leq (m-2)u \]
\[ \ldots \]
\[ z_{m-3} - u \leq z_{m-2} \leq 2u \]
\[ z_{m-2} - u \leq z_{m-1} \leq u \]
So for $z^{K'} = z_1^{p_1} \ldots z_{m-1}^{p_{m-1}}$, we have $\int_{P_u} e^{-Q(z,u)} z^{K'} dz$ is equal to

$$
\int_{-u}^{(m-1)u} \ldots \int_{-u}^u e^{-\sum_{i,j} a_{ij} z_i z_j - \sum_i b_i u z_i} z_1^{p_1} \ldots z_{m-1}^{p_{m-1}} dz_{m-1} \ldots dz_1.
$$

Despite the quadratic form being inhomogeneous with homogeneous part not necessarily being nondegenerate, since the bounds are linear and finite we are able to inductively apply the result of 3.4.

4. Heat Kernel Counter Terms

4.1. A Motivating Example. Due to the complexity of the renormalization procedure for a general Feynman graph, it is helpful to begin with an example that can elucidate most of the structure that arises. For the sake of simplicity and concreteness, we will work in the $\phi^4$ theory, i.e. the scalar field theory theory with classical interaction

$$
I = \frac{1}{4!} \int \phi^4.
$$

Nothing needs to be done to renormalize the Feynman weights at the 0-loop level, since the limit

$$
\lim_{\epsilon \to 0^+} w_\gamma(P^L_\epsilon, I)
$$

already exists for any tree $\gamma$.

To illustrate what happens at the higher loop level, we will work with the 1-loop graph $\gamma$

```
\begin{center}
\begin{tikzpicture}
  \node [vertex] (v1) at (0,0) {};
  \node [vertex] (v2) at (1,0) {};
  \node [vertex] (v3) at (2,0) {};
  \draw (v1) -- (v2);
  \draw (v2) -- (v3);
\end{tikzpicture}
\end{center}
```

The Feynman weight $w_\gamma(P^L_\epsilon, I)$ is computed by labelling the vertices by the interaction $I$, the edges by the propagator $P^L_\epsilon$, and the tails by the input field $\phi$

```
\begin{center}
\begin{tikzpicture}
  \node [vertex] (v1) at (0,0) {$\phi$};
  \node [vertex] (v2) at (1,0) {$P^L_\epsilon$};
  \node [vertex] (v3) at (2,0) {$\phi$};
  \draw (v1) -- (v2);
  \draw (v2) -- (v3);
\end{tikzpicture}
\end{center}
```

and then contracting. As defined in [12], the regularized propagator $P^L_\epsilon$ is the integral of the heat kernel $K_t$ over the time interval $[\epsilon, L]$. The Feynman weight of $\gamma$ is therefore given by

$$
w_\gamma(P^L_\epsilon, I)[\phi] = \int_{[\epsilon, L]^2} f_{\gamma,t}(t_1, t_2)[\phi] dt_1 dt_2
$$
where

\begin{equation}
\tag{50}
f_{\gamma,t}(t_1, t_2)[\phi] = \int_{M^2} K_{t_1}(x_1, x_2)K_{t_2}(x_1, x_2)\phi(x_1)^2\phi(x_2)^2.
\end{equation}

We make the definition \( \Phi(x_1, x_2) = \phi(x_1)^2\phi(x_2)^2 \) to avoid unnecessary detail in subsequent equations.

In the case \( M = \mathbb{R}^n \), the heat kernel is given by

\begin{equation}
\tag{51}
K_t(x_1, x_2) = (4\pi t)^{-n/2}e^{-|x_1-x_2|^2/4t}.
\end{equation}

Making the substitution for \( K_t \) and the change of variables

\begin{align}
\tag{52}
w &= x_1 + x_2 \\
\tag{53}y &= x_1 - x_2
\end{align}

we have

\begin{equation}
\tag{54}
f_{\gamma,t}(t_1, t_2)[\phi] = C(t_1 t_2)^{-n/2} \int_{\mathbb{R}^n} e^{-|y|^2\left(\frac{1}{4t_1} + \frac{1}{4t_2}\right)}\Phi(w, y).
\end{equation}

for some constant \( C \).

Let \( \Phi^N(w, y) \) be the Taylor polynomial of degree \( N \) of \( \Phi(w, \cdot) \) and define \( f_{\gamma,t}^N(t_1, t_2)[\phi] \) by substituting \( \Phi^N(w, y) \) in place of \( \Phi(w, y) \) in the above formula for \( f_{\gamma,t}(t_1, t_2)[\phi] \).

Costello’s idea is roughly to define \( w_{\gamma}^N(P_L, I) \) by substituting \( f_{\gamma,t}^N(t_1, t_2)[\phi] \) for \( f_{\gamma,t}(t_1, t_2)[\phi] \) in formula \( \text{(19)} \). We would hope that by making \( N \) sufficiently large, we can sufficiently control the error \( |f_{\gamma,t}(t_1, t_2)[\phi] - f_{\gamma,t}^N(t_1, t_2)[\phi]| \) to force the limit \( \lim_{N \to \infty} |w_{\gamma}(P_L, I) - w_{\gamma}^N(P_L, I)| \) to exist.

This is the idea in spirit, but there are additional subtleties needed to ensure we can always sufficiently bound the error. Firstly, we want to give an ordering to the edges and carry out a procedure for each ordering. Our particular graph is symmetric with respect to interchange of the edges, so, for this particular graph, in fact we can assume in this section without loss of generality that \( t_1 \leq t_2 \).

Choose \( R > 2 \). If \( t_2^R \leq t_1 \), indeed we have the error bound

\begin{equation}
\tag{55}
|f_{\gamma,t}(t_1, t_2)[\phi] - f_{\gamma,t}^N(t_1, t_2)[\phi]| \leq C t_2^{-Rn} \int_{\mathbb{R}^n} e^{-|y|^2/2n} |y|^{N+1}
\end{equation}

\begin{equation}
\tag{56}
\leq C t_2^{-Rn} t_2^{\frac{1}{2}(N+1)+\frac{n}{2}}
\end{equation}

for some constant \( C \) and then by making \( N \) large enough we can ensure that \( \frac{1}{2}(N+1) + \frac{n}{2} - Rn \geq 0 \). Let \( N_1 \) be such an \( N \).

It is worth fleshing out the structure of \( f_{\gamma,t}^N(t_1, t_2)[\phi] \). Write \( \Phi^N(w, y) = \sum_{|K| \leq N} \Psi_K(w)y^K \)

so that

\begin{equation}
\tag{57}
f_{\gamma,t}(t_1, t_2)[\phi] = C(t_1 t_2)^{-n/2} \sum_{|K| \leq N} \int_{\mathbb{R}^n} e^{-|y|^2\left(\frac{1}{4t_1} + \frac{1}{4t_2}\right)}y^K \Psi_K(w).
\end{equation}

Applying Wick’s theorem to the integral over \( y \), we have

\begin{equation}
\tag{58}
f_{\gamma,t}(t_1, t_2)[\phi] = \sum_{|K| \leq N} F_K(t_1, t_2) \Psi_K(w).
\end{equation}

Note that for each \( K \), \( F_K(t_1, t_2) \) is a rational function of \( t_1 \) and \( t_2 \) and \( \int_{\mathbb{R}^n} \Psi_K(w) \) is a local functional of \( \phi \).

If \( t_1 \leq t_2^R \), however, one needs to do something different. Essentially, we begin by choosing the subgraph \( \gamma' \) of \( \gamma \) corresponding to the edge labelled by \( t_1 \).
and treating the edges outside $\gamma'$ as input tails. Let

$$f_{\gamma', \gamma, t}(t_1, t_2)[\phi] = f_{\gamma, t}(t_1, t_2)[\phi]$$

$$= C(t_1t_2)^{-n/2} \int_{(\mathbb{R}^n)^2} e^{-|y|^2/4t_1} \Psi(w, y, t_2)$$

where $\Psi(w, y, t_2) = e^{-|y|^2/4t_2} \Phi(w, y)$. Now define

$$f^{N}_{\gamma', \gamma, t}(t_1, t_2)[\phi] = C(t_1t_2)^{-n/2} \int_{(\mathbb{R}^n)^2} e^{-|y|^2/4t_1} \Psi^{N}(w, y, t_2)$$

where $\Psi^{N}(w, y, t_2)$ is the order $N$ Taylor polynomial $y$ of $\Psi(w, y, t_2)$. Then because the $(N + 1)$-st derivative in the $y$ variable of $\Psi(w, y, t_2)$ is bounded by a constant times $t_2^{-(N + 1)}$, we have the bound

$$|f_{\gamma', \gamma, t}(t_1, t_2)[\phi] - f^{N}_{\gamma', \gamma, t}(t_1, t_2)[\phi]| \leq C(t_1t_2)^{-n/2}t_2^{-(N + 1)} \Phi^{N}(t_1^{+1/2} + \frac{t_2}{t_1^{1/2}})$$

$$\leq C(t_2^{-(N + 1)} \Phi^{N}(t_1^{+1/2}))$$

$$\leq C(t_2^{-(N + 1)} \Phi^{N}(t_1^{+1/2})$$

This is why we require that $R > 2$, so that for $N$ sufficiently large $(\frac{R}{R} - 1)(N + 1) - \frac{R}{2} > 0$. Let $N_2$ be such an $N$.

The counterterm is given by

$$w^{ct}_{\gamma}(P^L, I) = \int_{t_1 \leq t_2 \leq L} f^{N}_{\gamma, 1}(t_1, t_2)[\phi] + \int_{t_1 \leq t_2 \leq L} f^{N}_{\gamma', \gamma, 1}(t_1, t_2)[\phi]$$

plus the same two terms with $t_1$ and $t_2$ permuted. By construction, the limit

$$\lim_{\epsilon \to 0^+} [w_{\gamma}(P^L, I) - w^{ct}_{\gamma}(P^L, I)]$$

exists, as desired.

In the case of the Euclidean half space $\mathbb{H}^n$, the Dirichlet heat kernel is given by

$$K_t(x_1, x_2) = (4\pi t)^{-n/2}[e^{-|x_1 - x_2|^2/4t} - e^{-|x_1 - x_2|^2/4t}]$$

where $x_2$ is the reflection through the boundary. We will try to follow the same procedure writing

$$w_{\gamma}(P^L, I) = \int_{|x_1|^2} \int_{(\mathbb{H}^n)^2} K_{t_1}(x_1, x_2) K_{t_2}(x_1, x_2) \Phi(x_1, x_2)$$

which is equal to

$$\int_{|x_1|^2} [f_{\gamma, t, 0, 0}(t_1, t_2)[\phi] + f_{\gamma, t, 1, 0}(t_1, t_2)[\phi] + f_{\gamma, t, 0, 1}(t_1, t_2)[\phi] + f_{\gamma, t, 1, 1}(t_1, t_2)[\phi]]$$
where
\begin{equation}
(70) 
 f_{γ,i,j}(t_1, t_2)[φ] = C(-1)^{i+j}(t_1 t_2)^{-n/2} \int_{\mathbb{R}^4} e^{-d_i^2(x_1, x_2)/4t_1 - d_j^2(x_1, x_2)/4t_2} \Phi(x_1, x_2)
\end{equation}
where \(C\) is some constant and
\begin{equation}
(71) 
 d_i^2(x_1, x_2) = \begin{cases} 
 |x_1 - x_2|^2 & \text{if } i = 0 \\
 |x_1 - x_2|^2 & \text{if } i = 1
\end{cases}
\end{equation}

Introduce the coordinates, \(w = x_1 + x_2\) and \(y = x_1 - x_2\) and \(x_{1,n} = u + z\) and \(x_{2,n} = u - z\), where \(u \geq 0\) and \(z \in [-u, u]\).

When \(t_2^R \leq t_1\), take the Taylor expansion to order \(N\) at \(0\) of \(\Phi\) in \(y\) and \(z\) substitute it in the definition of \(f_{γ,i,j}(t_1, t_2)[φ]\) to get the expression
\begin{equation}
(72) 
 C(t_1 t_2)^{-n/2} \int_{\mathbb{R}^n} e^{g_{i,j}(w, t_1, t_2)} \int_{[-u, u]} e^{h_{i,j}(w, z, t_1, t_2)} \Phi^N(w, u, y, z)
\end{equation}

where \(C\) is some new constant and
\begin{equation}
(73) 
 g_{i,j}(w, t_1, t_2) = -u^2(δ_{i,1}t_1 + δ_{j,1}t_2)
\end{equation}

and
\begin{equation}
(74) 
 h_{i,j}(y, z, t_1, t_2) = -|y|^2(1/4t_1 + 1/4t_2) - z^2(δ_{i,0}t_1 + δ_{j,0}t_2)
\end{equation}

One can show that we get the same bound
\begin{equation}
(75) 
 |f_{γ,i,j}(t_1, t_2)[φ] - f_{γ,i,j}^N(t_1, t_2)[φ]| \leq Ct_2^{\frac{1}{2}(N+1) + \frac{n}{2} - Rn}
\end{equation}

for all \(i, j\). Details will be given in [4.6].

Upon examining the structure of the \(f_{γ,i,j}^N(t_1, t_2)[φ]\), we find that we no longer have a summation of local integrals, each weighted by the square root of some rational function in \(t_1\) and \(t_2\). For example,
\begin{equation}
(76) 
 f_{γ,i,0,0}^0(t_1, t_2)[φ] = C(t_1 t_2)^\frac{1}{2} \int_{\mathbb{R}^n} \int_{-u}^u \left[ e^{-z^2(1/4t_1 + 1/4t_2)} \right] \phi(w, u) d\Phi
\end{equation}

Besides the fourth equation, where in fact
\begin{equation}
(77) 
 f_{γ,i,1,1}^0(t_1, t_2)[φ] = 0,
\end{equation}
for \((i, j) \neq (1, 1)\),

\[
(82) \quad f^0_{\gamma, t, i, j}(t_1, t_2)[\phi] = \int_{\mathbb{R}^n} F(t_1, t_2, u)\phi(w, u)^4
\]
is no longer a local integral due to the \(u\) dependence of \(F(t_1, t_2, u)\). We shall investigate the structure of these “pseudo-local” integrals more carefully in 4.6.

Lastly, on the set where \(t_1 \leq t_2\), we introduce the notation \(f_{\gamma, t, i, j}(t_1, t_2)[\phi]\) for \(f_{\gamma, t, i, j}(t_1, t_2)[\phi]\) which we write as

\[
(83) \quad C(-1)^{i+j} (t_1, t_2)^{-n/2} \int_{(\mathbb{R}^n)^2} e^{-d_i(x_1, x_2)^2/4t_1} \Psi_j(x_1, x_2, t_2)
\]

where

\[
(84) \quad \Psi_j(x_1, x_2, t_2) = e^{-d_j(x_1, x_2)^2/4t_2} \Phi(x_1, x_2)
\]

and construct \(f^N_{\gamma, t, i, j}(t_1, t_2)[\phi]\) analogously to the way we did on \(\mathbb{R}^n\) by Taylor expanding \(\Psi_j\) in \(y\) and \(z\). We can bound the error similarly, but again \(f^N_{\gamma, t, i, j}(t_1, t_2)[\phi]\) will not be a sum of local integrals of \(\phi\) each multiplied by the square root of a rational function in \(t_1\) and \(t_2\). More details about bounding the error will be given in 4.6.

4.2. Covering \((0, \infty)^{|E(\gamma)|}\). In order to work with a more general Feynman graph, it becomes necessary to cover \((0, \infty)^{|E(\gamma)|}\) by sets generalizing those we used in the previous section when we had \(|E(\gamma)| = 2\).

Let \(k = |E(\gamma)|\). We denote \(\mathbf{t} = (t_1, \ldots, t_k)\). For each permutations \(\sigma \in S_k\), there is a subset

\[
(85) \quad S_\sigma = \{ \mathbf{t} \in (0, \infty)^k : t_{\sigma(1)} < \cdots < t_{\sigma(k)} \}
\]

and it is clear that

\[
(86) \quad \cup_{\sigma \in S_k} S_\sigma = (0, \infty)^k.
\]

The procedure we are about to describe is applied separately within each of the \(S_\sigma\), but we fix

\[
S_{\text{id}} = \{ \mathbf{t} \in (0, \infty)^k : t_1 < \cdots < t_k \}
\]

for notational clarity. Replace \(t_i\) with \(t_{\sigma(i)}\) to recover the case of a general permutation \(\sigma\).

We assume that \(R > 1\).

**Definition 2.** For \(j \in \{1, \ldots, k-1\}\), let

\[
(87) \quad B^j_R = \{ \mathbf{t} \in S_{\text{id}} : t_j < t_{j+1}^R \}
\]

For \(i, j \in \{1, \ldots, k\}\) with \(i < j\), define

\[
(88) \quad C^i_{R} = \{ \mathbf{t} \in S_{\text{id}} : t_j^R < t_j \}
\]

and define

\[
D^i_{R} = S_{\text{id}} \setminus C^i_{R}
\]

\[
= \{ \mathbf{t} \in S_{\text{id}} : t_j^R > t_j \}.
\]
And lastly for \( j \in \{2, \ldots, k-1\} \), define
\[
A^j_i = B^j_i \cap C^i_{j-1, j+1}
\]
\[
\{ t \in S_{id} : t_j < t^R_{j+1} \text{ and } t^R_i < t_1 \}
\]
and let \( A^1_i = B^1_i \) and \( A^k = C^1_{i, k} \).

Note that \( D^j_i \cup I \) is defined such that \( A^i \cap A^j \) is defined such that \( A^i \cap A^j \)

**Proposition 3.** For \( i_1 < i_2 < i_3 \),
\[
C^i_{i_1, i_2} \cap C^i_{i_2, i_3} \subset C^i_{i_1, i_3}
\]
and similarly
\[
D^i_{i_1, i_2} \cap D^i_{i_2, i_3} \subset D^i_{i_1, i_3}
\]
\[
\text{Proof. If } t \in C^i_{i_1, i_2} \cap C^i_{i_2, i_3} \text{, then } t^R_{i_2} < t_{i_1} \text{ and } t^R_{i_3} < t_{i_2}. \text{ This implies that } t^R_{i_3} < t_{i_1}. \]
The proof of the second inclusion is similar. \( \square \)

The following statements are trivially true:

**Proposition 4.** For \( j \in \{1, \ldots, k-1\} \), let
\[
\tilde{B}^j_i = \{ t \in S_{id} : t_\alpha < t^R_{j+1}, \text{ for } \alpha \leq j \text{ and } j+1 \leq \beta \}.
\]
For \( i, j \in \{1, \ldots, k\} \) with \( i < j \), define
\[
\tilde{C}^i_{i, j} = \{ t \in S_{id} : t_\alpha < t_{j}, \text{ for } \alpha \leq i \text{ and } j \leq \beta \}.
\]
Then \( \tilde{B}^j_i = B^j_i \) and \( \tilde{C}^i_{i, j} = C^i_{i, j} \).

**Proposition 5.** For \( j_1 \leq j_2 \), if \( C^i_{R, j_1} \supseteq C^i_{R, j_2} \).

The next two propositions are needed to prove Theorem 4.

**Proposition 6.** \( C^i_{R, j} \cap D^i_{R, m} = \emptyset \) for \( i \leq l \) and \( m \leq j \).
\[
\text{Proof. If } t^R_i < t_l \text{ and } t_l < t^R_m. \text{ Then } t_l < t_i < t^R_m < t^R_i < t_1,
\]
a contradiction. \( \square \)

**Proposition 7.** \( B^j_i \cap C^i_{R, j} = \emptyset \) for \( i \leq l < j \).
\[
\text{Proof. Since } B^j_i = D^j_{R, l+1}, \text{ we can apply the previous proposition.} \quad \square
\]

**Definition 3.** Fix a sequence \( s_0, \ldots, s_m \). We consider sequences of the form \( 1 = i_0 < i_1 < \cdots < i_m \leq k \), where \( m \leq k - 1 \). For any sequence of this form \( I \), we define the sets \( E^I_R = \bigcap_{j=0}^{m} E^I_{R, j} \) where \( E^I_{R, j} \) is defined such that
\[
E^I_{R, 0} = \begin{cases} 
B^1_R & \text{if } m = 0 \\
S_{id} & \text{otherwise}
\end{cases}
\]
and
\[
E^I_{R, j} = C^{i_{j-1}, i_j}_{R, j} \cap D^{i_{j-1}, i_j + 1}_{R, j}
\]
and for \( j = m \)

\[
E_{R,m}^I = \begin{cases} 
C_{R^m}^{i_{m-1},i_m} \cap D_{R^m}^{i_{m-1},i_{m+1}} \cap B_{R^m}^{i_m} & \text{if } i_m \neq k \\
C_{R^m}^{i_{m-1},i_m} & \text{if } i_m = k.
\end{cases}
\]

**Theorem 3.** Their closures \( \overline{E_R^I} \) form a cover of \((0, \infty)^k\).

**Proof.** If \( t_1 \leq t_2^{R^1} \), then \( t \in \overline{B_R^1} \). Thus let \( m = 0 \).

Otherwise, assume let \( i_1 \) be the largest integer such that \( t_{i_1}^{R^1} \leq t_1 = t_0 \). Then \( t \in C_{R^1}^{i_0,i_1} \). If \( i_1 = k \), let \( m = 1 \). If \( i_1 < k \), then \( t \in D_{R^1}^{i_0,i_1+1} \). If \( t_{i_1} \leq t_{i_1+1} \) then \( t \in \overline{B_R^{i_1+1}} \) and we let \( m = 1 \).

Otherwise, let \( i_2 \) be the largest integer such that \( t_{i_2}^{R^2} \leq t_{i_1} \). Then \( i_2 \in C_{R^2}^{i_1,i_2} \). If \( i_2 = k \), let \( m = 2 \). If \( i_2 < k \), then \( t \in D_{R^2}^{i_1,i_2+1} \). If \( t_{i_2} \leq t_{i_2+1} \) then \( t \in \overline{B_R^{i_2+1}} \) and we let \( m = 2 \).

And so on . . .

**Theorem 4.** The sets \( E_{R}^I \) are disjoint.

**Proof.** We prove this by induction. Consider the distinct sequences \( 1 = i_0 < i_1 < \cdots < i_m \leq k \) and \( 1 = j_0 < j_1 < \cdots < j_n \leq k \), where without loss of generality we assume that \( m \leq n \).

Suppose that \( i_l \neq j_l \), but \( i_1 = j_1, \ldots, i_{l-1} = j_{l-1} \). Then

\[
E_{R,l}^I \cap E_{R,l}^J \subseteq C_{R^l}^{i_{l-1},i_l} \cap D_{R^l}^{i_{l-1},i_{l+1}} \cap C_{R^l}^{i_{l},j_l} \cap D_{R^l}^{i_{l},j_{l+1}}
\]

\[
= \emptyset.
\]

because \( C_{R^l}^{i,j} \cap D_{R^l}^{i,m} = \emptyset \) for \( m < j \) by Proposition 3.

It is also possible that \( i_1 = j_1, \ldots, i_m = j_m \), but \( m < n \). Then

\[
E_{R,m}^I \cap E_{m+1}^J \subseteq B_{R^{m+1}}^{i_m} \cap C_{R^{m+1}}^{i_m,j_m+1}
\]

by Proposition 3.

Now specialize to a specific sequence \( s_0 = 1 \) and \( s_i = 2^{i-1} \) for \( i > 0 \).

**Theorem 5.** Consider the sequence \( 1 = i_0 < i_1 < \cdots < i_m \leq k \). Then

\[
E_{R}^I \subseteq A_{R^m}^{i_m}
\]

**Proof.** If \( m = 0 \), it is clear that \( E_{R}^I \subseteq A_{R^1} = B_{R}^1 \).

If \( m > 0 \),

\[
E_{R}^I \subseteq \begin{cases} 
C_{R^m}^{i_{m-1},i_m} \cap C_{R^m}^{i_{m-1},i_{m+1}} \cap B_{R^m}^{i_m} & \text{if } i_m < k \\
C_{R^m}^{i_{m-1},i_m} \cap C_{R^m}^{i_{m-1},i_{m+1}} \cap C_{R^m}^{i_{m-1},i_{m+1}} & \text{if } i_m = k
\end{cases}
\]

\[
\subseteq \begin{cases} 
C_{R^m}^{i_{m-1},i_m} \cap B_{R^m}^{i_m} & \text{if } i_m < k \\
C_{R^m}^{i_{m-1},i_m} & \text{if } i_m = k
\end{cases}
\]

\[
= A_{R^m}^{i_m}
\]
The construction of the counterterms in 4.5.2 will be based on a refinement of the covering \( \{ E_R^l \} \). For \( l < k \), given a sequence \( l = i_0 < \cdots < i_m \leq k \), introduce the more general sets \( E_R^l \) which are defined by applying the definition of \( E_R^l \), but replacing the set \( \{ t_1, \ldots, t_k \} \) with the set \( \{ t_{i_1}, \ldots, t_{i_k} \} \). For \( l = 1 \), we recover \( E_R^l \) in the sense in which it was defined earlier.

The following is a corollary of Theorem 3:

**Corollary 3.** Consider the collection of sequences of the form

\[
1 = i^{(1)}_0 < i^{(1)}_1 < \cdots < i^{(1)}_{m(1)} \\
i^{(1)}_{m(1)} = i^{(2)}_0 < i^{(2)}_1 < \cdots < i^{(2)}_{m(2)} \\
\cdots \\
i^{(p-1)}_{m(p-1)} = i^{(p)}_0 < i^{(p)}_1 < \cdots < i^{(p)}_{m(p)} = k.
\]

Then the sets

\[
E_R^{i^{(1)}_0} \cap E_R^{i^{(2)}_1} \cdots \cap E_R^{i^{(p)}_k}
\]

form a cover of \( S_{id} \).

4.3. Local Functionals and Feynman Weights.

4.3.1. Differential Operators. Let \( M \) be a smooth manifold, let \( E \) be a graded vector bundle and let \( E \) be the trivial line bundle. Let \( \mathcal{E} = \Gamma(E) \) and \( C^\infty(M) = \Gamma(\mathbb{R}) \).

A differential operator \( P : E \rightarrow \mathbb{R} \) is an \( \mathbb{R} \)-linear map \( E \rightarrow \mathcal{O}^k \) of the form

\[
I(s) = \sum_{\beta=1}^{m} \int_M D_{\beta,1}(s) \cdots D_{\beta,k}(s)
\]

for some collection of differential operators \( D_{i,j} : E \rightarrow \mathbb{R} \).

Substituting the local formula for the differential operators

\[
D_{\beta,j}(\alpha^i e_i) = (a_{\beta,j})^k_\ell \frac{\partial \alpha^k}{\partial x^\ell}
\]
we get that locally

\[ I(\alpha^t \epsilon_i) = \sum_{\beta=1}^{m} \int_U \left[ (a_{\beta,1} t_{\beta,1} \cdots t_{\beta,k} \frac{\partial \alpha_{\beta,i}}{\partial x^{\beta,j}} \cdots \frac{\partial \alpha_{\beta,k}}{\partial x^{\beta,k}} \right] \]

\[ = \int_U a_{j_1,\ldots,j_k} \frac{\partial \alpha^{j_1}}{\partial x^{j_1}} \cdots \frac{\partial \alpha^{j_k}}{\partial x^{j_k}} \]

for a collection of functions \(a_{j_1,\ldots,j_k}\) on \(U\).

4.3.3. Evaluation of \(w_\gamma(P, I)\). We shall work in the ungraded case. For notation simplicity, we shall also assume from hereon out that \(E = \mathbb{R}\), although the method remains valid for any vector bundle.

We would like to describe the form of \(w_\gamma(P^L_\epsilon, I)\) when \(I \in \mathcal{O}_{\text{loc}}(E)[[\hbar]]\) is a power series of local functionals and

\[ P^L_\epsilon = \int_\epsilon^L K_t dt \]

where \(K_t\) is the heat kernel of \(M\).

For each vertex \(v \in V(\gamma)\), we associate the functional \(I_{g(v), k(v)}\), where \(k(v)\) is the valency of the vertex \(v\). Assume that within a given chart \(U\),

\[ S^{k(v)} I_{g(v), k(v)}(\alpha^1_t \epsilon_i, \ldots, \alpha^k_t \epsilon_i) = \int_U a^{i_1} \cdots a^{i_k} \frac{\partial \alpha_1}{\partial x^{i_1}} \cdots \frac{\partial \alpha_k}{\partial x^{i_k}}. \]

where \(i_1, \ldots, i_k\) ranges over multi-indices with \(|i_1| + \cdots + |i_k| \leq \text{ord } I_{g(v), k(v)}\)

Choose an ordering on the set of half edges \(v^1, \ldots, v^{k(v)}\) incident on each vertex \(v\) and an orientation on each edge. Then \(\gamma\) determines the maps

\[ Q : T(\gamma) \to \bigcup_{v \in V} \{v^1, \ldots, v^{k(v)}\} \]

\[ Q_1 : E(\gamma) \to \bigcup_{v \in V} \{v^1, \ldots, v^{k(v)}\} \]

\[ Q_2 : E(\gamma) \to \bigcup_{v \in V} \{v^1, \ldots, v^{k(v)}\} \]

where \(Q_1\) and \(Q_2\) map an edge to its first and second half edges respectively, and \(Q\) maps a tail to itself. Also denote by \(v_1(e)\) and \(v_2(e)\) the first and second vertices of the edge \(e\). Similarly, let \(v(h)\) denote the vertex of the tail \(h\).

With these data, we can give the expression

\[ w_\gamma(P^L_\epsilon, I)[\alpha] = \int_{(e,L)} f_{\gamma, L}(t)[\alpha]. \]

where for \(M = \mathbb{R}^n\),

\[ f_{\gamma, L}(t)[\alpha] = \int_{\mathbb{R}^n(V(\gamma))} \prod_{v \in V(\gamma)} a^{i_1} \cdots a^{i_k} \prod_{e \in E(\gamma)} \frac{\partial K_t(x_{v_1(e)}, x_{v_2(e)})}{\partial x^{i_1} / \partial x^{i_1}} \cdots \frac{\partial K_t(x_{v_1(e)}, x_{v_2(e)})}{\partial x^{i_k} / \partial x^{i_k}} \]

where \(k\) is used to stand for \(k(v)\). If \(M\) is a compact manifold then choose a partition of unity subordinate to a finite cover of \(M\) (on which \(E\) is trivialized). Then \(f_{\gamma, L}(t)[\alpha]\) is a sum of integrals of the form

\[ \int_{U(V(\gamma))} a^{i_1} \cdots a^{i_k} \prod_{e \in E(\gamma)} \frac{\partial K_t(x_{v_1(e)}, x_{v_2(e)})}{\partial x^{i_1} / \partial x^{i_1}} \cdots \frac{\partial K_t(x_{v_1(e)}, x_{v_2(e)})}{\partial x^{i_k} / \partial x^{i_k}} \]

where \(a^{i_1} \cdots a^{i_k}\) is a power of \(x_{v_1(e)} \cdots x_{v_2(e)}\).
where $\chi$ is the partition of unity function for the open set $U$ in the cover and $\alpha^i$ are the coordinates of $\alpha$ in $U$.

Due to the symmetry of $P^{L}_\epsilon$ and $I_{i,k}$, the value of $w_\gamma(P^{L}_\epsilon, I)$ is independent of the choices of ordering and orientation.

4.4. **Counterterms on $\mathbb{R}^n$: Preliminaries.** For simplicity, in this section and subsequent sections, we shall only consider scalar field theories.

When working with $\mathbb{R}^n$, we shall really mean locally on a flat compact manifold. Essentially, all this means is that the input fields will be compactly-supported functions. The procedure can also be carried out on $\mathbb{R}^n$ in earnest by working with Schwartz functions instead of compactly supported functions.

On $\mathbb{R}^n$, the heat kernel has the simple form
\[
K_t(x,y) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}.
\]

(103)

4.4.1. **Derivatives of $K_t$.**

**Proposition 8.** For a multi-index $I = (i_1, \ldots, i_n)$, $\frac{\partial K_t}{\partial x^I}$ is equal to a polynomial in $x^1, \ldots, x^n, y^1, \ldots, y^n$ and $1/t$ multiplied by $K_t$. The degree in $1/t$ is $|I|$.

This follows from the statement where $n = 1$. See Lemma 2 in Section 4.9 which also gives explicit formulas for the single variable derivatives.

4.4.2. **Powers of $t$ in $w_\gamma(P^{L}_\epsilon, I)$**. Let $O(\gamma)$ be the sum of the orders of the local functionals $I_{g(v), k(v)}$ for all $v \in V(\gamma)$. As a consequence of Proposition 8, if we group the terms in $w_\gamma(P^{L}_\epsilon, I)$ by their powers of $t$, we see that

\[
w_\gamma(P, I)[\alpha] = \int_{(c, L) \in \mathcal{E}(\gamma)} f_{\gamma, I}(t)[\alpha].
\]

(104)

where $J$ is a multi-index and

\[
f_{\gamma, I}(t)[\alpha] = \sum_{-O(\gamma) \leq |J| \leq 0} t^{J-n/2} \int_{\mathbb{R}^n \setminus |V(\gamma)|} e^{-\sum_{e \in E(\gamma)} Q_e/4t} \Phi_J.
\]

(105)

This formula requires some explanation. The outer integral is over the time variables. Secondly, we have defined

\[
t^{J-n/2} = \prod_{e \in E(\gamma)} t^{j_e-n/2}.
\]

Lastly, in the exponential, $Q_e = \|x_{v_1(e)} - x_{v_2(e)}\|^2$.

The multi-index $I : E(\gamma) \to \mathbb{Z}$ and for each $I, \Phi_J$ a sum of terms of the form

\[
\prod_{v \in V(\gamma)} D_v \alpha(x_v)
\]

where for each vertex $v$,

\[
D_v \alpha = D_{v,1} \alpha \ldots D_{v,l} \alpha
\]

(106)

is a product of differential operators applied to $\alpha$. 
4.4.3. Spanning Tree Coordinates. We would like to evaluate (105). This will require a special change of coordinates. Choose a spanning tree \( T \) of \( \gamma \). For each edge in the tree we define a coordinate \( y_e = x_{v_1(e)} - x_{v_2(e)} \).

**Proposition 9.** Given a spanning tree \( T \), the coordinates \( y_e = x_{v_1(e)} - x_{v_2(e)} \) for \( e \in E(T) \) and
\[
w = x_1 + \cdots + x_{|V(\gamma)|}
\]
form a coordinate system on \( \mathbb{R}^{n|V(\gamma)|} \).

**Proof.** This is a linear transformation from \( \mathbb{R}^{n|V(\gamma)|} \) to \( \mathbb{R}^{n|V(\gamma)|} \). It has trivial kernel because if \( y_e = 0 \) for all \( e \in T(\gamma) \) then \( x_i = x_j \) for all \( i \) and \( j \). The condition that \( x_1 + \cdots + x_{|V(\gamma)|} = 0 \) then implies that \( x_i = 0 \) for all \( i \). \( \square \)

The quadratic form \( Q(x) = \sum_{e \in E(\gamma)} Q_e(x)/4t_e \) can be written in the spanning tree coordinates as \( Q(w,y) \).

**Proposition 10.** The quadratic form \( Q(w,y) \) is independent of \( w \).

**Proof.** For any edge \( e \in E(\gamma) \), let \( f^e_1, \ldots, f^e_l \) be the unique path of edges in \( T \) connecting \( v_1(e) \) and \( v_2(e) \). Then
\[
x_{v_1(e)} - x_{v_2(e)} = \sum_{i=1}^{l(e)} (x_{v_1(f^e_i)} - x_{v_2(f^e_i)}) = \sum_{i=1}^{l(e)} y_{f^e_i}.
\]

Therefore,
\[
Q(x) = \sum_{e \in E(\gamma)} \frac{Q_e(x)}{4t_e} = \sum_{e \in E(\gamma)} \frac{\left( \sum_{i=1}^{l(e)} y_{f^e_i} \right)^2}{4t_e} = Q(w,y)
\]
which clearly does not depend on \( w \). \( \square \)

Let \( A \) be the matrix of \( Q(0,y) = Q(w,y) \). Then \( A \) is an \( n(|V(\gamma)| - 1) \) by \( n(|V(\gamma)| - 1) \) matrix.

**Proposition 11.** The matrix \( B = (4 \prod_{e \in E(\gamma)} t_e)A \) has entries that are integer polynomials in \( \{t_e\}_{e \in E(\gamma)} \). Consequently, \( P_\gamma = \det B \) is an integer polynomial in \( \{t_e\}_{e \in E(\gamma)} \).

**Proposition 12.**
\[
\det A = 4^{-n(|V(\gamma)| - 1)} t^{-n(|V(\gamma)| - 1)} P_\gamma
\]
and
\[
A^{-1} = \frac{1}{P_\gamma} C,
\]
where \( C \) is a matrix with polynomial entries in \( t_e \).
Proof. To prove the second statement, use Cramer’s rule
\begin{equation}
B^{-1} = \frac{1}{\det B} \text{adj}(B) = \frac{1}{P_\gamma} \text{adj}(B)
\end{equation}
and that \( A^{-1} = (4 \prod_{e \in E(\gamma)} t_e) B^{-1} \). So, the statement follows by letting \( C = (4 \prod_{e \in E(\gamma)} t_e) \text{adj}(B) \). \( \square \)

4.4.4. Taylor Expansion of \( \Phi_J \). In (105), replace \( \Phi_J \) in \( f_{\gamma,I}(t)[\alpha] \) with its Taylor polynomial of degree \( N' \) in \( y \), \( \Phi_J^{N'}(w, y) = \sum_{|K| \leq N'} c_{J,K} y^K \), where \( N' \) is a non-negative integer to be determined. This gives
\begin{equation}
f_{\gamma,I}^{N'}(t)[\alpha] = \sum_{|K| \leq N'} t^{J-n/2} \int_{\mathbb{R}^n} e^{-\sum_{v \in V(\gamma)} Q_v(w,y)/4t_J} c_{J,K}(w) y^K dy dw
\end{equation}
(111)
\begin{equation}
= \sum_{|K| \leq N', K \text{ even}} t^{J-n/2} t_A^K(t) \int_{\mathbb{R}^n} c_{J,K}(w) dw
\end{equation}
(112)
where
\begin{equation}
t_A^K(t) = \int_{\mathbb{R}^n \setminus V(\gamma)} e^{-\langle y, Ay \rangle} y^K dy.
\end{equation}
(113)
and \( c_{J,K} \) is a function of \( w \) only.

In fact, \( t_A^K(t) \) is the square root of a rational function in \( t \) with integer coefficients. We determine the explicit form of \( t_A^K(t) \) in Theorem 9 in 4.9.

4.4.5. The structure of \( c_{J,K} \). In this section we prove that \( \Psi_{J,K}(\alpha) = \int_{\mathbb{R}^n} c_{J,K}(w) dw \) is a local functional.

Recall that \( c_{J,K}(w) = \frac{\partial \Phi_J}{\partial y^J}(0, w) \) and that
\begin{equation}
\Phi_J = \prod_{v \in V(\gamma)} D_v \alpha(x_v)
\end{equation}
(114)
where \( D_v \) is a product of differential operators. So
\begin{equation}
c_{J,K}(w) = \prod_{v \in V(\gamma)} \tilde{D}_v \alpha(w).
\end{equation}
(115)
\( \tilde{D}_v \) is a product of differential operators on \( \mathbb{R}^n_{x_v} \). Finally, we see that
\begin{equation}
\Psi_{J,K}(\alpha) = \int_{\mathbb{R}^n} \prod_{v \in V(\gamma)} \tilde{D}_v \alpha(w) dw
\end{equation}
(116)
is a local functional.

Alternatively, we can say that \( \Phi_J \) is a sum of terms of the form
\begin{equation}
f \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_v(h))}{\partial y^{|I^h|}}
\end{equation}
(117)
where \( f \) is a compactly supported function and \( I^h \) is a collection of multi-indices, one for each tail \( h \), satisfying the condition \( \sum_{h \in T(\gamma)} |I^h| \leq O(\gamma) \). This implies that \( \frac{\partial \Phi_J}{\partial y^J} \) is a sum of terms of the same form, but satisfying the condition \( \sum_{h \in T(\gamma)} |I^h| \leq O(\gamma) + N' + 1 \).
Proposition 13. The following bounds

\[ f_{\gamma,J}(t)[\alpha] = \sum_{|K| \leq N', K \text{ even}} \frac{P^K_A(t)}{Q^K_A(t)} \Psi_{J,K}(\alpha). \]

where \( Q^K_A \) is of homogeneous degree

\[ -|J| + \frac{n}{2}|E(\gamma)| + n(|V(\gamma)| - 1)(|E(\gamma)| - 1)(|K| + 1)/2. \]

and the degree of \( P^K_A \) is given within Theorem 9 in Section 4.4.

4.5. Counterterms on \( \mathbb{R}^n \): Error Bounds and Iteration.

4.5.1. Bounding the Error. Importantly, an elementary change of variables gives the following bounds

Proposition 13.

\[ \int_{\mathbb{R}^n|V(\gamma)|} e^{-\sum_{e \in E(\gamma)} Q_e/4t} |y^K| \leq Ct^{n(|V(\gamma)| - 1)} \]

for some constant \( C > 0 \).

The last inequality above is a consequence of the assumption \( t_k \leq 1 \).

Let \( k = |E(\gamma)| \). Assume that we order the edges so that \( t_1 \leq \cdots \leq t_k \) and \( t_k^R \leq t_1 \) so that \( \mathbf{t} \in A^k_R \) and that \( \mathbf{t} \in (0,1)^k \).

\[ |f_{\gamma,J}(t)[\alpha] - f_{\gamma,J}(t)[\alpha]| \leq \sum_{|K|=N'+1} t_k^{-RO(\gamma)} t^{-n/2} \int_{\mathbb{R}^n|V(\gamma)|} e^{-\sum_{e \in E(\gamma)} Q_e/4t} d_K(w)|y^K| \]

\[ \leq \sum_{|K|=N'+1} t_k^{-RO(\gamma)-|E(\gamma)|n/2} \int_{\mathbb{R}^n|V(\gamma)|} e^{-\sum_{e \in E(\gamma)} Q_e/4t} d_K(w)|y^K| \]

\[ \leq Ct^{n(|V(\gamma)| - 1)} t_k^{-RO(\gamma)-R|E(\gamma)|n/2} \sum_{|K|=N'+1} \int d_K(w) dw \]
using Proposition 13. In the formula above,
\[ d_K(w) = \sum_{J} \sup_{y} \left| \frac{\partial \Phi_J}{\partial y} K(y, w) \right| \]

But
\[ \sup_{y} \left| f \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_v(h))}{\partial x_v^h(h)} \right| \leq \sup_{y} |f| \cdot \sup_{y, w} \left| \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_v(h))}{\partial x_v^h(h)} \right| \]
\[ \leq \sup_{y} |f| \prod_{h \in T(\gamma)} \|\alpha\|_{p_h} \]

where \( p_h = |I^h| \). Note that \( \sup_{y} |f| \) is a compactly-supported function in the variable \( w \). Thus,
\[ \int d_K dw \leq \sum_{p} C_p \prod_{h \in T(\gamma)} \|\alpha\|_{p_h} \]

where the summation is over multi-indices \( p : T(\gamma) \to \mathbb{Z}^{\geq 0} \) such that \( \sum_{h \in T(\gamma)} p_h \leq O(\gamma) + N' + 1 \).

In conclusion, we have shown that

**Theorem 6.**

(124)
\[ |f_{\gamma, t}(t)[\alpha] - f_{\gamma, t}^{N'}(t)[\alpha]| \leq \left( \sum_{p} C_p \prod_{h \in T(\gamma)} \|\alpha\|_{p_h} \right) \frac{1}{k}^{(N'+1)+\frac{n}{2}(|V(\gamma)|-1)-R|O(\gamma)+\frac{n}{2}|E(\gamma)|)} \]

where the summation is over multi-indices \( p : T(\gamma) \to \mathbb{Z}^{\geq 0} \) such that \( \sum_{h \in T(\gamma)} p_h \leq O(\gamma) + N' + 1 \).

By a similar argument,

**Proposition 14.**

(125)
\[ |T_A^K(t)| \leq C t^{\frac{1}{2}(|K|+n(|V(\gamma)|-1))} \leq C t^{n(|V(\gamma)|-1)} \]

and consequently,

(126)
\[ |f_{\gamma, t}^{N'}(t)[\alpha]| \leq \left( \sum_{p} C_p \prod_{h \in T(\gamma)} \|\alpha\|_{p_h} \right) \frac{1}{k}^{\frac{n}{2}(|V(\gamma)|-1)-R|E(\gamma)|+\frac{n}{2}-R|O(\gamma)|} \]
\[ \leq \|\alpha\|_{O(\gamma)+N'^k} \frac{1}{k}^{\frac{n}{2}(|V(\gamma)|-1)-R|E(\gamma)|+\frac{n}{2}-RO(\gamma)} \]
\[ \leq \|\alpha\|_{O(\gamma)+N'^k} \frac{1}{k}^{\frac{n}{2}(|V(\gamma)|-1)-R|E(\gamma)|+\frac{n}{2}-RO(\gamma)} \]

where the summation is over multi-indices \( p : T(\gamma) \to \mathbb{Z}^{\geq 0} \) such that \( \sum_{h \in T(\gamma)} p_h \leq O(\gamma) + N' \) and \( \|\alpha\|_{p_h} \) is the \( C^{p_h} \) norm of \( \alpha \).
4.5.2. Inductive Construction of the Counterterms. We shall use the results of Section 12 in particular Corollary 8 which gives a finite cover of \((0,1)^k\) given by sets of the form \(E_R^{(1)} \cap E_R^{(2)} \cdots \cap E_R^{(p)}\), for \(p \leq k\). The structure of the multi-indices \(I^{(1)}, \ldots, I^{(p)}\) is given in 8 and the sets \(E_R^{(k)}\) are defined in Definition 8.

Also we shall need Theorem 15 which states that for \(E_R^I\) where \(I\) is a sequence of the form 1 < \(i_1 < \cdots < \sum \leq k\) with \(m < k\), we have \(E_R^I \subseteq A_{Rm}^I\), where

\[
A_{Rm}^I = \{t_1 < t_2 < \cdots < t_k : t_i < t_{i+1}^R \text{ and } t_i < t_{i+1}^R\}.
\]

**Theorem 7.** For any sequence \(I^{(1)}, \ldots, I^{(p)}\) as in Corollary 8 for nonnegative integers \(N_1, \ldots, N_p\), we can construct \(f_{\gamma, I}^{N_1, \ldots, N_p}(t)[\alpha]\) by iterative Taylor expansion of \(f_{\gamma, I}(t)[\alpha]\) so that

\[
|f_{\gamma, I}(t)[\alpha] - f_{\gamma, I}^{N_1, \ldots, N_p}(t)[\alpha]| \leq \|\alpha\|^{T(\gamma)} \sum_i C_i t_i^{d_i},
\]

where \(l\) some positive integer, where \(d_i = d_i(N_1, \ldots, N_p)\) increases linearly in \(N_i\) for \(N_1, \ldots, N_p\) fixed and sufficiently large.

**Proof.** Fix an ordering of the edge set so that we can identify \(E(\gamma) = \{e_1, \ldots, e_k\} = \{1, \ldots, k\}\). The procedure should be carried out in

\[
S_\sigma = \{t \in (0, \infty)^k : t_{\sigma(1)} < \cdots < t_{\sigma(k)}\}.
\]

for each permutation \(\sigma \in S_k\). However, we shall work in \(S_{id}\) for notational simplicity.

For general \(p\), we would consider sets of the form \(E_R^{(1)} \cap E_R^{(2)} \cap \cdots \cap E_R^{(p)}\), for each of the \(k!\) possible orderings of the edges of \(\gamma\).

For illustrative purposes and notational simplicity we prove the main theorem only for \(p = 2\). The inductive step in the proof of the general case is similar. Let \(i^{(1)} = m^{(1)}\) and let \(R_1 = R^{m^{(1)}+1}\) and \(R_2 = R^{m^{(2)}+1}\) so that we are working within

\[
E_R^{(1)} \cap E_R^{(2)} \subseteq A_{R_1}^{(1)} \cap A_{R_2}^{(2)} = \{t \in S_{id} : t_{i^{(1)}} < t_1 \text{ and } t_{i^{(1)}+1} \leq t_{i^{(1)}+1} \text{ and } t_{i^{(1)}+1} \leq t_{i^{(1)+1}}\}
\]

where the inclusion follows from Theorem 5. Note that \(i^{(2)}_{m^{(2)}} = k\).

The collection of edges \(e_{m^{(1)}}, \ldots, e_{m^{(2)}}\) determines a subgraph of \(\gamma\), which we denote by \(\gamma'\). The remaining edges \(e_{i^{(1)+1}}, \ldots, e_k\) form the edge set of \(\gamma'/\gamma\).

A tail \(h \in T(\gamma)\) will either be in \(T(\gamma')\) or will be one of the two half edges forming an edge in \(E(\gamma)\). Let \(T(\gamma', \gamma) = T(\gamma') \cap T(\gamma)\) and \(E(\gamma', \gamma) = E(\gamma') \cup F(\gamma', \gamma)\), where \(F(\gamma', \gamma)\) is the set of all edges in \(\gamma\) for which one tail edge making up the edge is a tail in \(\gamma\). Let \(V(\gamma', \gamma)\) denote the vertices not in \(\gamma'\) that are incident on an edge in \(F(\gamma', \gamma)\).

The integral in the formula for \(f_{\gamma, I}(t)[\alpha]\) is over \(\mathbb{R}^{n(V(\gamma))}\) and we can order the integration so that we integrate first with respect to the vertices in \(V(\gamma')\). This inner integral, which is of the form

\[
\int_{\mathbb{R}^{n(V(\gamma'))}} \prod_{v \in V(\gamma')} a^{e_1, \ldots, e_k} (x_v) \prod_{e \in E(\gamma', \gamma)} \frac{\partial K_{\gamma}(x_{e_1}(e), x_{e_2}(e))}{\partial x^{T}^{(e_1)}(e)} \prod_{h \in T(\gamma', \gamma)} \frac{\partial K_{\gamma}(x_{e_1}(e))}{\partial x^{T}(h)}
\]
Let us use \( g_{\gamma',\gamma,I}(t)[\alpha] \) to denote the integral above. It is a function of \( t = \{ t_v \}_{v \in E(\gamma', \gamma)} \) and \( x_v \) for \( v \in V(\gamma', \gamma) \). Let \( g \) be a functional of order \( T(\gamma') \). Let \( f_{\gamma'/\gamma,I,R}(t)[\alpha] \) be defined like we would define \( f_{\gamma'/\gamma,I}(t) \) but with the functional \( g \) used for the distinguished vertex in \( \gamma'/\gamma \).

Use the same procedure which led to Corollary[4] and Theorem[5] We have that
\[
|g_{\gamma',\gamma,I}(t)[\alpha] - N'_{\gamma',\gamma,I}(t)[\alpha]| \leq C(\gamma',n,R_1) \left( \sum_p C_p \prod_{h \in T(\gamma')} \|\alpha\|_{p_h} \prod_{e \in F(\gamma',\gamma)} \|K_{t_e}\|_{p_e(h)} \right) t_{i(1)}^{N'_1 + C(\gamma',n,R_1)}
\]
where
\[
C(\gamma',n,R_1) = \frac{1}{2} + \left( |\{\gamma'\} - 1\right)^n - R_1 \left( O(\gamma') + |E(\gamma')|^{n/2}\right).
\]
But
\[
\|K_{t_e}\|_{p_e(h)} \leq Ct_e^{-\frac{N'_1}{2} + n/2} \leq Ct_{i(1)+1}^{-\frac{N'_1}{2} + n/2}
\]
for some constant \( C \), and thus
\[
\prod_{e \in F(\gamma',\gamma)} \|K_{t_e}\|_{p_e(h)} \leq Ct_{i(1)+1}^{-|F(\gamma',\gamma)|\frac{N'_1}{2} + n/2 - \sum_e F(\gamma',\gamma) \cdot p_e(h)}
\]
\[
\leq Ct_{i(1)+1}^{-|E(\gamma)|\frac{N'_1}{2} + O(\gamma') - N'_1 - 1}
\]
for some constant \( C \).

Thus, \( |g_{\gamma',\gamma,I}(t)[\alpha] - N'_{\gamma',\gamma,I}(t)[\alpha]| \) is less than or equal to
\[
\left( \sum_p C_p \prod_{h \in T(\gamma')} \|\alpha\|_{p_h} \right) t_{i(1)+1}^{N'_1 + R_1|E(\gamma)|\frac{N'_1}{2} - O(\gamma') - R_1 N'_1 - 1}
\]
So as long as \( R_1 > 2 \), \( |g_{\gamma',\gamma,I}(t)[\alpha] - N'_{\gamma',\gamma,I}(t)[\alpha]| \) and consequently
\[
|f_{\gamma'/\gamma,I,g_{\gamma',\gamma,I}}(t)[\alpha] - f_{\gamma'/\gamma,I,N'_{\gamma',\gamma,I}}(t)[\alpha]|
\]
will be bounded by a power of \( t_k \), which we denote \( d_k(N'_1) \), that grows linearly with \( N'_1 \).

To finish the argument, we need to show the same thing for
\[
|f_{\gamma'/\gamma,I,N'_{\gamma',\gamma,I}}(t)[\alpha] - N'_{\gamma'/\gamma,I,N'_{\gamma',\gamma,I}}(t)[\alpha]|
\]
From Proposition[14]
\[
|g_{\gamma',\gamma,I}(t)[\alpha]| \leq \left( \sum_p C_p \prod_{h \in T(\gamma')} \|\alpha\|_{p_h} \right) t_{i(1)}^{\frac{1}{2}(|V(\gamma')|-1) - R_1|E(\gamma)|\frac{1}{2} - R_1 O(\gamma') - R_1 N'_1}
\]
Let \( C_1(N'_1,\gamma',\gamma,n,R_1) \) be the power of \( t_{i(1)} \) in the inequality above.

We are able to bound \( |f_{\gamma'/\gamma,I,N'_{\gamma',\gamma,I}}(t)[\alpha] - f_{\gamma'/\gamma,I,N'_{\gamma',\gamma,I}}(t)[\alpha]| \) by
\[
C\|\alpha\|_{T(\gamma)} t_k^{N'_2 + \frac{1}{2}(|V(\gamma')|-1)} - R_1 O(\gamma') + N'_1 + R_1 C_1(N'_1,\gamma',\gamma,n,R_1)
\]
for some positive integer \( l \).
In conclusion, using the triangle inequality, we can bound
\[
|f_{\gamma',\beta}(t)[\alpha] - f_{\gamma',\beta}(t)[\alpha]| \leq C_1 t_k^{d_1(N_1')} + C_2 t_k^{d_2(N_1',N_2')}
\]
where by \(d_1(N_1')\) grows linearly in \(N_1'\) and \(d_2(N_1',N_2')\) grows linearly in \(N_2'\) for \(N_1'\) fixed.

4.6. **Counterterms on the Euclidean Upper Half Space.** The Dirichlet heat kernel on the upper half space \(\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}\) with the Euclidean metric is given by
\[
K_t(x,y) = (4\pi t)^{-n/2}[e^{-|x-y|^2/4t} - e^{-|x-y'|^2/4t}],
\]
where \(y^*\) is the reflection through the hyperplane \(y_n = 0\).

Note that \(K_t\) is solves the heat equation, for \(y \in \partial \mathbb{H}^n, K_t(x,y) = 0\), and
\[
\lim_{t \to 0^+} \int_{\mathbb{H}^n} K_t(x,y)\phi(y) dy = \phi(x)
\]
for any \(\phi \in C^\infty(\mathbb{H}^n)\).

As in Section 4.4, we form
\[
\nu_{\gamma}(P^L, I)[\alpha] = \int_{(e,L) \in E(\gamma)} f_{\gamma',\beta}(t)[\alpha]
\]
but now
\[
f_{\gamma',\beta}(t)[\alpha] = \sum_{\beta} \sum_{|J| 

\sum_{e \in E(\gamma)} Q_e^{(\beta_e)} / 4t e \Phi_{J,\beta}
\]
where \(Q_e^{(1)} = \|x_{v_1(e)} - x_{v_2(e)}\|^2\) and \(Q_e^{(-1)} = \|x_{v_1(e)} - x_{v_2(e)}^*\|^2\) and \(\beta\) ranges over all functions \(E(\gamma) \to \{-1, 1\}\). We wish to apply Wick’s theorem after taking the Taylor expansion of \(\Phi_{J,\beta}\).

4.6.1. **Coordinate System on** \(\mathbb{H}^n[\gamma]\). Using the decomposition \(\mathbb{H}^n = \mathbb{R}^{(n-1)} \times \mathbb{R}^{\geq 0}\) introduce the coordinates \(x_v = (\overline{x}_v, x_{v,n})\). Split the integral into an integral over \(\mathbb{R}^{(n-1)[\gamma]}\) followed by an integral over \((\mathbb{R}^{\geq 0})[\gamma]\). The quadratic form \(Q\) is equal to
\[
\sum_{e \in E(\gamma)} \|\overline{x}_{v_1(e)} - \overline{x}_{v_2(e)}\| / 4t e
\]
plus the part depending on the variables \(x_{v,n}\)
\[
\sum_{e \in \beta^{-1}(1)} |x_{v_1(e),n} - x_{v_2(e),n}|^2 / 4t e + \sum_{e \in \beta^{-1}(-1)} |x_{v_1(e),n} + x_{v_1(e),n}|^2 / 4t e
\]
We shall only concentrate on the part of the quadratic form depending on the variables \(x_{v,n}\) since the integral over the variables \(\overline{x}_v\) can be treated by the methods of Section 4.4.

Choose an ordering on the set of vertices and consider the basis, \(f_{[\gamma]} = e_1 + \cdots + e_n, f_1 = e_1 - e_2, \ldots, f_{[\gamma]} = -e_{[\gamma]} = e_{[\gamma] - 1} - e_{[\gamma]}, \) where \(e_1, \ldots, e_{[\gamma]}\) is the
standard basis of $\mathbb{R}^{V(\gamma)}$. This induces the coordinate system $u, z_1, \ldots, z_{|V(\gamma)|-1}$ on $\mathbb{R}^{V(\gamma)}$ related to standard coordinates by
\[
x_{1,n} = u + z_1 = u + \tilde{z}_1
\]
\[
x_{2,n} = u + z_2 - z_1 = u + \tilde{z}_2
\]
\[
\ldots
\]
\[
x_{|V(\gamma)|-1,n} = u + z_{|V(\gamma)|-1} - z_1 = u + \tilde{z}_{|V(\gamma)|-1}
\]
where in the proof of Proposition 10.

As in the previous section, choose coordinates $z_1, \ldots, z_{|V(\gamma)|-1}$ and $u$ on $(\mathbb{R}^{\geq 0})^{V(\gamma)}$. In this coordinate system, the part of the quadratic form depending on the coordinates in the normal direction becomes
\[
\sum_{e \in \beta^{-1}(1)} |\tilde{z}_{v_1(e)} - \tilde{z}_{v_2(e)}|^2/4t_e + \sum_{e \in \beta^{-1}(-1)} |2u + \tilde{z}_{v_1(e)} + \tilde{z}_{v_2(e)}|^2/4t_e.
\]

Let $P$ be the plane spanned by $f_i$ for $i$ between 1 and $|V(\gamma)| - 1$. Then for $u \geq 0$
\[
u(e_1 + \cdots + e_{|V(\gamma)|}) + P
\]
intersects $(\mathbb{R}^{\geq 0})^{V(\gamma)}$ in a bounded set (in particular a simplex) whose projection onto $P$ we denote $P_u$.

### 4.6.2. Taylor Expansion of $\Phi_J$

For a fixed spanning tree of $\gamma$, choose spanning tree coordinates on $\mathbb{R}^{(n-1)||V(\gamma)||}$, $\bar{\nu}_e = \bar{\nu}_{v_1(e)} - \bar{\nu}_{v_2(e)}$ and $\bar{\nu} = \bar{\nu}_1 + \cdots + \bar{\nu}_{|V(\gamma)|}$ on $\mathbb{R}^{(n-1)||V(\gamma)||}$. As in the previous section, choose coordinates $z_1, \ldots, z_{|V(\gamma)|-1}$ and $u$ on $(\mathbb{R}^{\geq 0})^{V(\gamma)}$.

In these coordinates, the quadratic form $\sum_{e \in E(\gamma)} Q_e/t_e$ decomposes into a sum of three terms
\[
Q(\bar{\gamma}, t) + Q^{(\gamma)}(z, u, t) + Q^{(\beta)}(u, t),
\]
where as in the proof of Proposition 10
\[
Q(\bar{\gamma}, t) = \sum_{e \in E(\gamma)} \left\| \nu(e) \right\|^2/4t_e,
\]
and
\[
Q^{(\gamma)}(z, u, t) = \sum_{e \in \beta^{-1}(1)} |\tilde{z}_{v_1(e)} - \tilde{z}_{v_2(e)}|^2/4t_e + \sum_{e \in \beta^{-1}(-1)} |\tilde{z}_{v_1(e)} + \tilde{z}_{v_2(e)}|^2/4t_e
\]
\[
+ \sum_{e \in \beta^{-1}(-1)} u(\tilde{z}_{v_1(e)} + \tilde{z}_{v_2(e)})/t_e
\]
and
\[
Q^{(\beta)}(u, t) = \left( \sum_{e \in \beta^{-1}(-1)} t_e^{-1} \right) u^2.
\]

We shall Taylor expand $\Phi_J$ in $f_{\gamma}(t)[\alpha]$ in both $\bar{\gamma}$ and $z$ to order $N'$. For $f_{\gamma,N'}(t)[\alpha]$ we have a sum of integrals of the form
\[
\int \int \int_{\mathbb{R}^{n+1}} e^{Q(\bar{\gamma}, t) - Q^{(\gamma)}(z, u, t) - Q^{(\beta)}(u, t)} c_{\gamma, K, K', \beta} \bar{\gamma}^{K} z^{K'} dz du,
\]
over $|K| + |K'| \leq N', J$ even, $-O(\gamma) \leq |J| \leq 0$ and $\beta$ functions $E(\gamma) \to \{-1, 1\}$. Also, $c_{IJKK', \beta}$ the Taylor coefficient is a function of $w$ and $u$ only.

The integral over $T$ gives an answer like that of Theorem 9 and Corollary 4, but with the dimension $n$ replaced by $n - 1$ in all the formulas. The integral over $z$ exists because $P_u$ is a bounded set. Let

$$\phi_{K'}(u, t) = \int_{P_u} e^{-Q(z) \cdot z} dz$$

Let

$$I_{K, K'}(u) = \int_{P_u} \int_{\mathbb{R}^{(n-1)|V(\gamma)|-1}} e^{-Q(z) \cdot z} \theta^{K'} dz$$

and

$$P^K_A(t) = \frac{P^K_A(t)}{Q^K_A(t)} \phi_{K'}(u, t)$$

where $P^K_A(t)$ and $Q^K_A(t)$ are defined as in Corollary 4.

Then $f_{N',J}^N(t)[\alpha]$ becomes a sum of integrals

$$I_{K, K'}(u) = \int_{P_u} \int_{\mathbb{R}^{(n-1)|V(\gamma)|-1}} e^{-Q(z) \cdot z} \frac{P^K_A(t)}{Q^K_A(t)} \phi_{K'}(u, t)$$

Let $\Psi_{J, K, K'}(t, \alpha)$ denote the integral in the last line above and let $\psi_{K'}(u, t) = e^{-Q(\gamma) \cdot z} \phi_{K'}(u, t)$. Then $f_{N',J}^N(t)[\alpha]$ is not quite the product of a function of $t$ and a local functional because $\psi_{K'}(u, t)$ depends on $u$ and $t$.

Recall that $c_{J, K, K'}(\overline{w}, u) = \frac{\partial \Phi}{\partial z} |_{(0, \overline{w}, 0, u)}$ and that

$$\Phi_J = \prod_{v \in V(\gamma)} D_v \alpha(x_v)$$

where $D_v$ is a product of differential operators. So

$$c_{J, K, K'}(\overline{w}, u) = \prod_{v \in V(\gamma)} \hat{D}_v \alpha(\overline{w}, u).$$

where $\hat{D}_v$ is a product of differential operators. Finally, we see that $\Psi_{J, K, K'}(t, \alpha)$ is almost a local functional with $\psi_{K'}(u, t)$ as the $t$-dependent factor. For $t_1 \leq \cdots \leq t_k$, we do have control on the $t$ dependence

$$|\psi_{K'}(\sqrt{t_k} u, t)| \leq C t_k^{\frac{1}{2} |K| + |V(\gamma)| - 1}.$$

We will show how to the renormalization procedure can be adapted to this situation in the next section.

4.6.3. Bounding the Error. Assume that we have ordered the set of edges so that $t_1 \leq \cdots \leq t_k$ and $t_k^R \leq t_1$ for some $R > 1$. By Taylor’s theorem $|f_{N',J}(t)[\alpha] - f_{N',J}^N(t)[\alpha]|$ is bounded above by a sum of terms of the form

$$t_k^{-RO(\gamma)} t_{-n/2} \int_{P_u} \int_{\mathbb{R}^{(n-1)|V(\gamma)|}} e^{-\sum_{\epsilon \in E(\gamma)} Q_{\epsilon}^{4t_k} d_{K,K',\beta}(\overline{w}, u) \theta^{K'} \cdot z}$$

over multi-indices $|K| + |K'| = N' + 1$ and $\beta : E(\gamma) \to \{-1, 1\}$.
Proposition 15.

\[ \int_{P^{\sqrt{t}u}} \int_{B^{(n-1)\phi(V(\gamma)-1)}} e^{-Q(\mathbf{t})-Q(\sqrt{t}u,\mathbf{t})} \left| \mathbf{g}^K \right| \left| z' \right| d\mathbf{g} dz \leq t_k^{\frac{1}{2}(N' + n(\phi(V(\gamma)-1))}
\]

where \(|K| + |K'| = N'\).

Each term in (143) is bounded above by

\[ t_k^{-RO(\gamma)-R(\mathbf{E}(\gamma))n/2} \int_{B^{(n-1)\phi(V(\gamma)-1)}} e^{-\sum_{e \in E(\gamma)} Q_e/4t_k} d_{K,K',\beta}(\mathbf{w},u) \left| \mathbf{g}^K \right| \left| z' \right| \]

Let

\[ C_{K,K'}(u) = e^{-|\beta^{-1}(1)|u^2} \int_{B^{(n-1)\phi(V(\gamma)-1)}} |z'|^{K'} dz \int_{B^{(n-1)\phi(V(\gamma)-1)}} e^{-Q(\mathbf{w}^\gamma t)\mathbf{g}} dz. \]

For \(\beta^{-1}(1) \neq 0\), (145) is bounded by

\[ t_k^{\frac{1}{2}(N' + n(\phi(V(\gamma)-1))} t_k^{-RO(\gamma)-R(\mathbf{E}(\gamma))n/2} \left( \int d_{K,K',\beta}(\mathbf{w},\sqrt{t_k} u) C_{K,K'}(u) d\mathbf{g} du \right) \]

(148)

\[ \leq t_k^{\frac{1}{2}(N' + n(\phi(V(\gamma)-1))} t_k^{-RO(\gamma)-R(\mathbf{E}(\gamma))n/2} \left( \int e_{K,K',\beta}(w) C_{K,K'}(u) d\mathbf{g} du \right) \]

where we have used the fact that \(C_{K,K'}(u)\) is a Schwartz function in \(u\) for \(\beta^{-1}(1) \neq 0\), and we have defined

\[ e_{K,K',\beta}(w) = \sup_u d_{K,K',\beta}(\mathbf{w},u). \]

It remains to understand the integral

\[ \int e_{K,K',\beta}(\mathbf{w}) C_{K,K'}(u) d\mathbf{g} du \]

in terms of the field \(\alpha\) and its derivatives. Note that

\[ e_{K,K',\beta}(\mathbf{w}) = \sum_f \sup_{\mathbf{g},z,u} \left| \frac{\partial \Phi_f}{\partial \gamma^z} \right|, \]

and \(\Phi_f\) is a sum of terms of the form

\[ f \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_v(h))}{\partial \alpha_{I_v}} \]

where \(f\) is a compactly-supported function and \(I_h\) is a collection of multi-indices, one for each tail \(h\) satisfying the condition \(\sum_{h \in T(\gamma)} |I_h| \leq O(\gamma)\). This implies that \(\frac{\partial \Phi_f}{\partial \gamma^z}\) is a sum of terms of the same form, but satisfying the condition \(\sum_{h \in T(\gamma)} |I_h| \leq O(\gamma) + N' + 1\).
where \( p_h = |f^h| \). Note that \( \sup_{w,z,u} |f| \) is a compactly-supported function in the variable \( w \). Thus,
\[
\int e_{K,K',\beta}(w)C_{K,K'}(u) dw du \leq \sum_p C_p \prod_{h \in T(\gamma)} \|\alpha\|_{p_h}
\]
where the summation is over multi-indices \( p : T(\gamma) \to \mathbb{Z}_{\geq 0} \) such that \( \sum_{h \in T(\gamma)} p_h \leq O(\gamma) + N' + 1 \).

In conclusion, combining the results for \( \beta^{-1}(-1) = \emptyset \) and \( \beta^{-1}(-1) \neq \emptyset \)

\section*{Theorem 8.}
\begin{equation}
|f_{\gamma,I}(t) - f_{\gamma,I}^{N',I}(t)| \leq \left( \sum_p C_p \prod_{h \in T(\gamma)} \|\alpha\|_{p_h} \right) t_k^{(N'+1) + (|V(\gamma)| - 1)s - RO(\gamma) - R|E(\gamma)| + 1}
\end{equation}
where the summation is over multi-indices \( p : T(\gamma) \to \mathbb{Z}_{\geq 0} \) such that \( \sum_{h \in T(\gamma)} p_h \leq O(\gamma) + N' + 1 \).

\subsection*{4.6.4. Inductive Construction of the Counterterms.}
We will only note the differences from Section 4.5.4.

The proof of Theorem 7, which was given only in the case \( p = 2 \), involves two steps. In the first step, we show that \( |g_{\gamma',\gamma,I}(t)[\alpha] - g_{\gamma',\gamma,I}^{N',I}(t)[\alpha]| \) is bounded by \( t_k \) to a power that grows linearly in \( N_1' \). This implies that
\[
|f_{\gamma,I}(t)[\alpha] - f_{\gamma',\gamma,I}^{N',I}(t)[\alpha]|
\]
is bounded by a power of \( t_k \) that grows linearly in \( N_1' \), where we have used that directly from the definitions, we have
\[
f_{\gamma,I}(t)[\alpha] = f_{\gamma',\gamma,I}^{N',I}(t)[\alpha].
\]

In the second step, we show that
\[
|f_{\gamma',\gamma,I}^{N',I}(t)[\alpha] - f_{\gamma',\gamma,I}^{N',I}(t)[\alpha]|
\]
is bounded by a power of \( t_k \) that grows linearly in \( N_2' \) for \( N_1' \) fixed. This will require a slight modification from the procedure of Section 4.5.2. To construct \( f_{\gamma',\gamma,I}^{N',I}(t)[\alpha] \), we start with \( f_{\gamma',\gamma,I}^{N',I}(t)[\alpha] \), which is formed from the Feynman rules applied to the pointed graph \( \gamma'/\gamma' \), where we place the functional \( g_{\gamma',\gamma,I}^{N',I} \) on the distinguished vertex.

In the case of \( \mathbb{H}_n \), \( g_{\gamma',\gamma,I}^{N',I}(t) \) is no longer a sum of functions of \( t_1, \ldots, t_d(1) \) multiplied by local functionals applied to the inputs on the tails of \( \gamma' \). Now the local functional depends on \( t \) in the integrand.
When forming $f_{\gamma', I, g^\tau, t}^{N_k}(t)[\alpha]$ from $f_{\gamma', I, g^\tau, t}^{N_k}(t)[\alpha]$ only take the Taylor expansion of the factors in the integrand of $g^\tau_{\gamma', \gamma, t}$ that do not depend on $t$. That is, in the integrand of each $\Psi_{t, K, K'}$ neglect the first factor $\psi_{K'}(u, t)$ and only take the Taylor expansion of the second factor. Because $|\psi_{K'}(\sqrt{t_0^2 \mu}, t)| \leq t_0^{M(t_0)}(V(\gamma))^{-1}$, this will contribute factor of $t_0^{M(t_0)}(V(\gamma))^{-1}$ to the bound on

$$\left| f_{\gamma', I, g^\tau, t}^{N_k}(t)[\alpha] - f_{\gamma', I, g^\tau, t}^{N_k}(t)[\alpha] \right|$$

so the overall power of $t_k$ will in fact be the same as in the case of $\mathbb{R}^n$.

4.7. Counterterms on a Compact Manifold. The asymptotic formula for the scalar heat kernel $K_t(x, y) \sim (4\pi t)^{-n/2}e^{-d(x, y)^2/4t} \sum_i \phi_i(x, y)t^i$ states that there exists some sequence of smooth functions $\phi_i$ on $M \times M$ supported on a neighborhood of the diagonal such that

$$\|K_t(x, y) - (4\pi t)^{-n/2}e^{-d(x, y)^2/4t} \sum_{i=0}^N \phi_i(x, y)t^i\|_1 = O(t^{N-n/2-l}),$$

Let

$$K^N_t(x, y) = (4\pi t)^{-n/2}e^{-d(x, y)^2/4t} \sum_{i=0}^N \phi_i(x, y)t^i.$$

Beginning from (102), for each edge, we replace $K_t$ with $K^N_t$. Changing our notational conventions, we let $f^N_{\gamma, t}$ denote the result of making all $|E(\gamma)|$ of such substitutions. Assume that we’ve ordered the edges so that $t_1 \leq \cdots \leq t_k$. Then

$$|f_{\gamma, t}(t)[\alpha] - f^N_{\gamma, t}(t)[\alpha]| \leq \|\alpha\|^{T(\gamma)}_{O(\gamma)} \sum_{i=1}^k C_i t_i^{N-n/2-p_j} \prod_{j \neq i} t_j^{p_j-n/2}$$

for some nonnegative integers $p_j$ with $\sum_j p_j \leq O(\gamma)$ and some constants $C_i$. Thus

$$|f_{\gamma, t}(t)[\alpha] - f^N_{\gamma, t}(t)[\alpha]| \leq C\|\alpha\|^{T(\gamma)}_{O(\gamma)} t_k^{N-O(\gamma)-|E(\gamma)|/2}.$$

An analogous statement to Proposition 8 can be made for $K^N_t$ giving that

**Proposition 16.** For the heat kernel in (103)

$$\frac{\partial K^N_t}{\partial x^2} = P_{i,k}e^{-d(x, y)^2/4t}$$

where $P_{i,k}$ is a Laurent polynomial in $t$ of degree between $-k$ and $N$.

Therefore, we have a formula

$$f_{\gamma, t}(t)[\alpha] = \sum_{-O(\gamma) \leq |J| \leq |E(\gamma)|} t^{j-n/2} \int_{U^{|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e/4t} \Phi_J$$

where $Q_e = d^2(x_{v_1(e)}, x_{v_2(e)})$. 


4.7.1. Spanning Tree Coordinates. Choose a spanning tree $T$ and vertex $v_0$ of $\gamma$. For any vertex $v$ of $\gamma$, there is a unique path $e_1^{(i)}, \ldots, e_l^{(i)}$ from $v_0$ to $v$. We can then make the inductive definition

**Definition 5.** Spanning tree coordinates are defined by $w = x_{v_0}$ and $\{y_e\}_{e \in T}$ where $y_e$ is defined inductively so that $x_{v_2(e)}(y_{e_1}, \ldots, y_{e_l}, y_e) = \exp_{x_{v_1}(e)}(y_{e_1}, \ldots, y_{e_l})(y_e)$ where $e_1, \ldots, e_l$ is the unique path from $v_0$ to $v_1(e)$.

The reason for introducing these coordinates is that for all $e \in T$,

$$d(x_{v_1(e)}, x_{v_2(e)}) = \|y_e\|.$$  

Note that the spanning tree $T$ is the union of maximal paths originating at $v_0$. Let us denote the $i$-th such path by $e_1^{(i)}, \ldots, e_l^{(i)}$. Let $V_1^{(i)}$ be a neighborhood of the zero section in $TM$ such that $\Phi_1^{(i)} : V_1^{(i)} \to U^{(i)}_1 \subseteq M \times M$ given by $(x_{v_0}, y_{e_1^{(i)}}) \mapsto (x_{v_0}, \exp_{x_{v_0}}(y_{e_1^{(i)}}))$ is a diffeomorphism. Inductively, given $\Phi_{j-1}^{(i)} : V_{j-1}^{(i)} \to U_{j-1}^{(i)} \subseteq M^j$ let $\exp_j^{(i)} = p_j \circ \Phi_{j-1}^{(i)}$, where $p_j$ is the projection onto the last factor. Now let $V_j^{(i)}$ be a neighborhood of the zero section in $(\exp_j^{(i)})^* TM$ such that the map $\Phi_{j}^{(i)} : V_{j}^{(i)} \to U_{j}^{(i)} \subseteq M^{j+1}$ given by

$$\left(x_{v_0}, y_{e_1^{(i)}}, \ldots, y_{e_j^{(i)}}\right) \mapsto \left(\Phi_{j-1}^{(i)}(x_{v_0}, y_{e_1^{(i)}}, \ldots, y_{e_{j-1}^{(i)}}), \exp_{\exp_{j-1}^{(i)}}^{-1}(x_{v_0}, y_{e_{j-1}^{(i)}})\right)(y_{e_j})$$

is a diffeomorphism.

We then take the fiber product over $M$ of the maps $\Phi_{i}^{(i)}$ which produces the desired diffeomorphism

$$V_{1}^{(i)} \times_M \cdots \times_M V_{m}^{(q)} \to M^{[V(\gamma)]}.$$  

4.7.2. Taylor Expanding $\Phi_J$ and Bounding the Error. Taking the Taylor expansion of $\Phi$ with respect to $\{y_e\}_{e \in E(\gamma)}$.

\[ f^N_{s,J}(t)[\alpha] = \sum_{\text{\scriptsize $|K| \leq N'$}} \int_{U^{(i)}_1} e^{-\sum_{e \in E(\gamma)} Q_e / 4t} c_{J,K}(w) y^K dy dw \]

\[ = \sum_{|K| \leq N', K \text{ even} \atop -O(\gamma) \leq |J| \leq N'|E(\gamma)|} \int_U c_{J,K}(w) \mathcal{I}_K^J(w,t) dw \]

\[ = \sum_{|K| \leq N', K \text{ even} \atop -O(\gamma) \leq |J| \leq N'|E(\gamma)|} \int_U c_{J,K}(w) \mathcal{I}_K^J(w,t) dw \]

This differs from the case of $\mathbb{R}^n$ where $\mathcal{I}_K^J(w,t)$ does not depend on $w$. Note that $\int_U c_{J,K}(w) \mathcal{I}_K^J(w,t) dw$ is not a local functional due to the factor of $\mathcal{I}_K^J(w,t)$, which depends on $t$. We will say below why the procedure to construct the counterterms still works.

We have the bound
Proposition 17.

\[
(156) \quad \int_{x \in \Gamma(\gamma)} e^{-\sum_{\gamma \in E(\gamma)} Q_\gamma / 4t} |y|^{\kappa} dy \leq C t_k^{\beta/2} \bigl( |\Gamma(\gamma)|^{-1} \bigr)
\]

Proof. This follows from the fact that

\[
e^{-\sum_{\gamma \in E(\gamma)} Q_\gamma / 4t} \leq e^{-\sum_{\gamma \in E(T)} Q_\gamma / 4t} = e^{-\sum_{\gamma \in E(T)} \|\gamma\|^2 / 4t}
\]

\[
\square
\]

Using Proposition 17, the bound on

\[
[f_{x,t}^{N,N'}(t)[\alpha] - f_{x,t}^{N,N'}(t)[\alpha]]
\]

can be established as in the proof of Theorem [6].

As in the case of $\mathbb{H}^n$, in the proof of Theorem [7] we make the following modification. When forming $f_{N_2,N_2'}^{N_1,N_1'}$, only take the Taylor expansion of the factors in the integrand of $g_{\gamma,\gamma',I}^{N_1,N_1'}$ that do not depend on $t$. That is, in the integrand of each term $\Psi_{J,K}$, neglect the factor $\mathcal{I}_A^K(w,t)$ and only take the Taylor expansion of the other factor.

Because $|\Psi_{J,K}(w,t)| \leq t_k^{\beta/2} \bigl( |\Gamma(\gamma)|^{-1} \bigr)$, this will contribute factor of $t_k^{\beta/2} \bigl( |\Gamma(\gamma)|^{-1} \bigr)$ to the bound on

\[
|f_{N_2,N_2'}^{N_1,N_1'}(t)[\alpha] - f_{N_2,N_1,N_1'}^{N_1}(t)[\alpha]|
\]

so the overall power of $t_k$ will be the same as in the case of $\mathbb{R}^n$.

4.8. Counterterms on a Compact Manifold with Boundary. The renormalization procedure can also be carried out in the case of compact Riemannian manifolds with boundary $M$, such that there exists a neighborhood $W$ of $\partial M$ that is isometric to a product $\partial M \times [0, \epsilon]$.

When we have such a manifold with boundary $M$, the double of $M$ will be denoted by $M'$. The Dirichlet heat kernel on $M$ is

\[
(157) \quad K_\alpha(x,y) = K'_\alpha(x,y) - K'_\alpha(x,y^*)
\]

where $K'_\alpha(x,y)$ is the heat kernel on $M'$.

The existence of an asymptotic expansion of $K'_\alpha(x,y)$ implies that

\[
(158) \quad K_\alpha(x,y) \sim e^{-d(x,y)^2 / 4t} \sum_i \phi_i(x,y) t^i + e^{-d(x,y^*)^2 / 4t} \sum_i \psi_i(x,y) t^i
\]

where $\psi_i(x,y) = -\phi_i(x,y^*)$.

This can be used to define $f_{x,t}^{N}$ and to show that $|f_{x,t}^{N}(t)[\alpha] - f_{x,t}^{N}(t)[\alpha]|$ is bounded by a power of $t_k$ and increases linearly with $N$.

Choose a finite cover $U_1, \ldots, U_m$ of $\partial M$ by coordinate neighborhoods. This induces a finite cover $U_1 \times [0, \epsilon], \ldots, U_m \times [0, \epsilon]$ of $W \cong \partial M \times [0, \epsilon]$ by coordinate neighborhoods.

Then choose a finite cover $V_1, \ldots, V_m'$ of the complement of $M \times [0, \epsilon]$.
On the open sets $U_i$ that intersect the boundary, since $M$ is a product, by the Pythagorean theorem the square distance is

$$d_M^2(x, y) = d_M^2(x, \overline{y}) + |x_n - y_n|^2$$

Therefore, on these open sets we can apply the analysis of 4.6 for the direction normal to the boundary and the analysis of 4.7 to the $\partial M$ direction. On open sets $V_i$ whose closures do not intersect the boundary,

$$K_t(x, y) \sim e^{-d(x, y)^2/4t} \sum \phi_i(x, y)t^i$$

so we can apply the analysis of 4.7.

4.9. Appendix.

**Lemma 2.** For the heat kernel in (103)

$$\frac{\partial K_t}{\partial x_i} = P_{i,k}K_t$$

where $P_{i,k}$ is polynomial in $x_i$ and $y_i$ and $1/t$. The degree of $P_{i,k}$ in $1/t$ is $k$.

**Proof.** We would like to find an explicit expression for $P_{i,k}$.

Note that

$$\frac{\partial K_t}{\partial x_i} = \frac{x_i - y_i}{2t}K_t.$$

For each sequence of the form $s_1, \ldots, s_{k'}$ where $s_j \geq 1$ for all $j$ and $s_1 + \cdots + s_{k'} = k$, consider the functions

$$F_{s_1, \ldots, s_{k'}}(t, x_i, y_i) = \begin{cases} \partial^{s_1}_{x_i} \left( \frac{x_i - y_i}{2t} \right)^{s_2} \cdots \partial^{s_{k'}-1}_{x_i} \left( \frac{x_i - y_i}{2t} \right)^{s_{k'}} \cdots & k' \text{ even} \\ \partial^{s_1}_{x_i} \left( \frac{x_i - y_i}{2t} \right)^{s_2} \cdots \partial^{s_{k'}-1}_{x_i} \left( \frac{x_i - y_i}{2t} \right)^{s_{k'}} \cdots & k' \text{ odd} \end{cases}$$

We argue by induction that

$$\frac{\partial^k K_t}{\partial x_i^k} = \sum_{s_1 + \cdots + s_{k'} = k \atop s_j \geq 1 \text{ for all } j} F_{s_1, \ldots, s_{k'}}K_t.$$

For $k = 1$, this is clearly true. Suppose it is true for some $k \geq 1$, then

$$\frac{\partial^{k+1} K_t}{\partial x_i^{k+1}} = \frac{\partial}{\partial x_i} \sum_{s_1 + \cdots + s_{k'} = k \atop s_j \geq 1 \text{ for all } j} F_{s_1, \ldots, s_{k'}}K_t$$

$$= \sum_{s_1 + \cdots + s_{k'} = k \atop s_j \geq 1 \text{ for all } j} \partial_x F_{s_1, \ldots, s_{k'}}K_t$$

$$+ \sum_{s_1 + \cdots + s_{k'} = k \atop s_j \geq 1 \text{ for all } j} F_{s_1, \ldots, s_{k'}} \left( \frac{x_i - y_i}{2t} \right) K_t$$

$$= \sum_{s_1 + \cdots + s_{k'} = k+1 \atop s_j \geq 1 \text{ for all } j} F_{s_1, \ldots, s_{k'}}K_t.$$
In fact, we can be more precise. That is, for \( k' \) even
\[
F_{s_1, \ldots, s_{k'}} = \frac{s_{k'}!}{(s_{k'} - s_{k'-1})! \cdots (s_{k'} - s_{k'} - 1 + \cdots + s_2)!} \frac{(x_1 - y_1)/2t)^{s_{k'} + s_{k'} - 2 + \cdots + s_2}}{(s_{k'} - s_{k'} - 1 + \cdots - s_1)!} \frac{(x_1 - y_1)^{s_{k'} - 1 + s_{k'} - 3 + \cdots + s_1}}{(s_{k'} - s_{k'} - 1 + \cdots + s_2)}
\]
as long as \( s_{k'} - s_{k'-1} + \cdots + s_{2i} - s_{2i-1} \geq 0 \) for all \( i \geq 1 \) such that \( 2i \leq k' \). Otherwise \( F_{s_1, \ldots, s_{k'}} = 0 \). If \( k' \) is odd then \( F_{s_1, \ldots, s_{k'}} = (\frac{x_1 - y_1}{2t})^{s_1} F_{s_2, \ldots, s_{k'}} \).

The leading term of \( P_{i,k} \) in \( \frac{1}{t} \) is \( F_k = \left( \frac{x_1 - y_1}{2t} \right)^{s_1} \).

**Theorem 9.**
\[
\mathcal{T}_A^K(t) = \frac{1}{P_t^{(K+1)/2}} P_A^K.
\]
where \( P_A^K \) is a homogeneous polynomial in \( t \) of degree \( R(\gamma, n, K) = C_1(\gamma, n) + |K| C_2(\gamma, n) \) for constants \( C_1(\gamma, n) \) and \( C_2(\gamma, n) \) which are defined in the body of the proof of the theorem.

**Proof.** Writing \( y^K = y_{m_1} \cdots y_{m_{|K|}} \), we have
\[
\mathcal{T}_A^K = \int_{\mathbb{R}^{n(\gamma) - 1}} e^{-\langle y, Ay \rangle} y_{m_1} \cdots y_{m_{|K|}} \, dy
\]
\[
= \frac{(\sqrt{\pi})^{n(\gamma) - 1}}{\sqrt{\det A}} \sum_\beta \prod_{i=1}^{|K|/2} (A^{-1})_{\beta_i}^{m_{\beta_i}}
\]
\[
= \frac{(\sqrt{\pi})^{n(\gamma) - 1}}{P_t^{1/2}} 2^{n(\gamma) - 1} t^{n(\gamma) - 1/2} \frac{1}{P_t^{|K|/2}} \sum_Q \prod_{i=1}^{|K|/2} C_{Q_i^{(1)}, Q_i^{(2)}},
\]
where we have used Proposition [12] and Wick’s Theorem on \( \mathbb{R}^n \). Let
\[
P_A^K = (\sqrt{\pi})^{n(\gamma) - 1} 2^{n(\gamma) - 1} t^{n(\gamma) - 1/2} P_t^{1/2} \sum_Q \prod_{i=1}^{|K|/2} C_{Q_i^{(1)}, Q_i^{(2)}}.
\]
Recall the definition of \( C \) which is \( \left( 4 \prod_{e \in E(\gamma)} t_e \right) \text{adj}(B) \). But \( B \) is \( n(|V(\gamma)| - 1) \) by \( n(|V(\gamma)| - 1) \) and its entries are homogeneous of degree \( |E(\gamma)| - 1 \) in \( \{t_e \} \in E(\gamma) \). So \( \text{adj}(B) \) has entries of degree \( (|E(\gamma)| - 1)(n(|V(\gamma)| - 1) - 1) \). Therefore \( C \) has entries of degree
\[
|E(\gamma)| - 1) n(|V(\gamma)| - 1) - 1) + |E(\gamma)| = (|E(\gamma)| - 1)n(|V(\gamma)| - 1) + 1
\]
This implies that \( P_A^K \) is of homogeneous degree
\[
R_\gamma(n, K) = n|E(\gamma)|(|V(\gamma)| - 1)/2 + \frac{|K|}{2} (|E(\gamma)| - 1)n(|V(\gamma)| - 1) + 1
\]
With the definition of \( P_A^K \) in hand, the theorem is now evident. \( \square \)

5. Constructions of an Effective Field Theory from a Local Functional

In this section, we show that an effective action can be constructed from a local functional \( I \in \mathcal{O}(\mathcal{E})[[h]] \) using a procedure that is based on Theorem [7].
Do the following for each sequence \( I^{(1)}, \ldots, I^{(p)} \) as in Corollary \(^3\). Let \( N'_1 \) be the smallest nonnegative integer such that \( d_1(N'_1) \geq 0 \). Let \( N'_2 \) be the smallest nonnegative integer such that \( d_2(N'_1, N'_2) \geq 0 \) and so on. Then by Theorem \(^7\),

\[
|f_{\gamma, I}(t)[\alpha] - f_{\gamma, I, N'_1 \ldots N'_p}(t)[\alpha]| \leq C
\]

for some constant \( C \). Let \( \mathcal{E}^{(1)} \ldots I^{(p)} = \bigcap_{\varepsilon} \mathcal{E}^{(1)} \ldots I^{(p)} \).

Let

\[
w_{\gamma}^{CT}(P^1_\varepsilon, I)[\alpha] = \sum_k \sum_{p=1}^{k} \sum_{I^{(1)}, \ldots, I^{(p)}} \int_{\mathcal{E}^{(1)} \ldots I^{(p)}} f_{\gamma, I, N'_1 \ldots N'_p}(t)[\alpha] \, dt
\]

We can integrate this formula on \((\varepsilon, 1)^k \cap \mathcal{E}^{(1)} \ldots I^{(p)}\). This gives

\[
|w_{\gamma}(P^1_\varepsilon, I)[\alpha] - w_{\gamma}^{CT}(P^1_\varepsilon, I)[\alpha]| \leq C(1 - \varepsilon^k).
\]

In particular, by Lebesgue’s dominated convergence theorem, we can let \( \varepsilon \to 0^+ \).

Thus, the limit as \( \varepsilon \to 0^+ \) of \( w_{\gamma}(P^1_\varepsilon, I)[\alpha] - w_{\gamma}^{CT}(P^1_\varepsilon, I)[\alpha] \) exists as well. We shall call this the renormalized Feynman weight.

The counterterms for the effective action are defined by

\[
I^{CT}_{i,k}(\varepsilon) = W^{CT}_{i,k}(P^1_\varepsilon, I - \sum_{(i', k') \prec (i, k)} I^{CT}_{i', k'}(\varepsilon)),
\]

where

\[
W^{CT}_{i,k}(P^1_\varepsilon, I) = \sum_{\gamma \text{ conn}}_{g(\gamma) = i, T(\gamma) = k} \frac{1}{|\text{Aut}(\gamma)|} \mu(\gamma, I) \left( P^1_\varepsilon, I - \sum_{(i', k') \prec (i, k)} I^{CT}_{i', k'}(\varepsilon) \right).
\]

Then the effective action is defined by

\[
I(L) = \lim_{\varepsilon \to 0^+} W(P^1_\varepsilon, I - I^{CT}(\varepsilon)).
\]

This is well-defined because for all \( i, k \),

\[
I_{i,k}[L] = \lim_{\varepsilon \to 0^+} W_{i,k}(P^1_\varepsilon, I - I^{CT}(\varepsilon))
\]

\[
= \lim_{\varepsilon \to 0^+} \left[ W_{i,k}(P^1_\varepsilon, I - \sum_{(i', k') \prec (i, k)} I^{CT}_{i', k'}(\varepsilon)) - I^{CT}_{i,k}(\varepsilon) \right].
\]

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