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Analytical solutions and parameter estimation of the SIR epidemic model

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10.1 Introduction

The origin of the mathematical modeling of epidemics can be traced back to XVII\textsuperscript{th} century with the work of Daniel Bernoulli, who was trained as a physician, and whose results justified the practice of inoculation against smallpox. This represented a major milestone of public health. It is interesting to note that some subsequent major steps in the epidemiology of compartmental models were achieved not by pure mathematicians, but by public health physicians or biologists, such as Sir R.A. Ross, W.H. Hamer, A.G. McKendrick, and W.O. Kermack \cite{1}. As a notable example of such progress, the SIR (Susceptible-Infected-Removed) epidemiological model was published in 1927 by Kermack and McKendrick in 1927 to study the plague and cholera epidemics in London and Bombay \cite{2}. Even to date, the SIR model remains a cornerstone of mathematical epidemiology \cite{3,4}. It should be noted that the full model derived by Kermack and McKendrick was formulated in terms of convolution integrals, while the more popular form used in the present literature is actually only its simplified case. The full SIR model included dependence on the age of infection, that is, the time since an individual becomes infected. In this sense, the full model can be used to provide a unified approach to compartmental epidemic models. It is also worth mentioning that the SIR model in discrete time is equivalent to a renewal equation with a geometric generation-interval distribution with probability parameter $\gamma \Delta t$ \cite{5}. This comes from the fact that the limiting distribution of the geometric probability distribution is the exponential one.

Outside epidemiology, the SIR model is also extensively used in modeling of online social networks, viral marketing, diffusion of ideas, spread of computer viruses, financial network contagion, etc. (see recent survey by Rodrigues and the references therein \cite{6}).

The SIR model can be extended in two directions – either by adding a final state, e.g. “deceased” individuals – $D$; or by adding one or more intermediate nonobservable populations –
e.g. “exposed” $E$ individuals. Distinct possibilities include the SEIR and SEIS models, with an exposed period between getting infected and becoming infective, and SIRS models, with temporary immunity conveyed upon recovery from the initial infection.

The analytical solution of the SIR model was formulated for the $S$-variable in a parametric form in [7]. Recently, Barlow and Weinstein have introduced numerical solutions based on asymptotic rational approximants [8]. Prodanov established the solution in terms of numerical inversion of the parametric solution by Newton iteration for the $I$-variable [9]. The present chapter focuses on some recent theoretical developments around the SIR model from the perspective of real analysis and theory of special functions.

10.2 The SIR model

The full SIR model can be simplified under the assumptions that the rate of infection is constant and there is no demographic variation. The simplified SIR model is a deterministic model formulated in terms of ordinary differential equations (ODEs) in terms of 3 populations of individuals. The $S$ population consists of all individuals susceptible to the infection of concern. The $I$ population comprises the infected individuals. These persons have the disease and can transmit it to the susceptible individuals. The model can be schematically represented in the input-output diagram presented in Fig. 10.1.

The standard assumptions behind the SIR model are:

1. A typical member of the population $S$ gets infected by a typical member of the $I$ population with certain probability per unit time.
2. Infected individuals recover at rate $\gamma I$ per unit time.
3. The $R$ population cannot become re-infected and the recovered individuals cannot transmit the disease to others.
4. There is no entry into or departure from the overall population, except possibly through death from the disease.

It is also assumed that new cases are generated through homogeneous mixing, yielding the mass-action incidence term $\beta IS$, or the standard-incidence term $\beta IS/N$ [3, Ch. 11]. Note
that the dimensionality of $\beta$ in the two cases is different. In the former case it is $[T]^{-1}[N]^{-1}$, while in the latter it is $[T]^{-1}$.

Model parameters can be interpreted as follows. A disease carrier infects on average $\beta$ individuals per day, for an average time of $1/\gamma$ days. The $\beta$ parameter is called disease transmission rate, while $\gamma$ – recovery rate. The average number of infections arising from an infected individual is then given by

$$R_0 = \frac{\beta}{\gamma},$$

called the basic reproduction number. Other useful quantities are the running reproduction number

$$R^*(t) = \frac{S(t)\beta}{N\gamma}$$

and the herd immunity threshold $p = 1 - 1/R_0$. Typical initial conditions, modeling an outbreak, are $S(0) = S_0$, $I(0) = I_0$, $R(0) = 0$ [2]. The simplified SIR model comprises a set of three ODEs:

$$\dot{S}(t) = -\beta \frac{S(t)}{N} I(t)$$

(10.1)

$$\dot{I}(t) = \beta \frac{S(t)}{N} I(t) - \gamma I(t)$$

(10.2)

$$\dot{R}(t) = \gamma I(t)$$

(10.3)

The model assumes a constant overall population $N = S + I + R$.

The temporal course of the $I$-variable has a characteristic exponential rise, turnover, and final slower exponential decline towards 0. The $S$-variable also has three phases – a slow decrease followed by rapid quasi-linear decline followed by slow exponential decline towards $S_\infty$, while the $R$-variable has a sigmoid shape. A phase diagram of the model is plotted in Fig. 10.2.

The model can be re-parametrized using normalized variables as

$$\dot{s} = -si$$

(10.4)

$$\dot{i} = si - gi, \quad g = \frac{\gamma}{\beta} = \frac{1}{R_0}$$

(10.5)

$$\dot{r} = gi$$

(10.6)

subject to normalization $s + i + r = 1$ and time rescaling $\tau = \beta t$.

It is, in general, difficult to estimate the contact rate $\beta$, which depends on the particular disease and on social and behavioral factors [3]. On the other hand, different nonmedical interventions, such as wearing of personal protective equipment, contact restriction or quarantine,
could offer some degree of control over $\beta$. The above formulation offers the advantage that the dynamics imposed by the contact rate $\beta(t)$, viewed as a function of time, can be absorbed into the model by time rescaling
\[
\tau(t) = \int_{-\infty}^{t} \beta(u) du
\]
which for a causal step function obviously recovers $\tau = \beta t$. Note that $\tau(t)$ becomes a dimensionless variable.

### 10.3 Second-order systems equivalent to SIR

The SIR model can be formulated also as several equivalent second-order nonlinear systems for the different variables. The readers are directed to the original results in [7,10].

#### 10.3.1 A second order differential equation for the $i$-variable

Recently, Kudryashov et al. gave an equivalent second order differential equation for the $i$-variable [10].

**Proposition 10.1.** The SIR system is reducible to the nonlinear differential equation for the $i$-variable:
\[
\ddot{i} = -gi^2 - i \dot{i} + \frac{\dot{i}^2}{i} \tag{10.7}
\]
or the system

\[ \dot{I}_t = -gI^2 - I I_t + \frac{I^2}{I} \quad \text{(10.8)} \]
\[ \dot{I} = I_t \quad \text{(10.9)} \]

**Proof.** From the conservation law \( \dot{s} = -\dot{i} - i = -ig - \dot{i} \). Then from Eq. (10.4) \( s = g + \dot{i}/i \).

Differentiating Eq. (10.5) and substituting Eq. (10.4)

\[ \ddot{i} = -gi + isi + isi = -gi - i^2 s + si = -gi + i \left( g + \frac{i}{i} \right) - i^2 \left( g + i \right) = -gi^2 + \frac{i^2}{i} - i \]

The advantage of this formulation is that the phase space manifold is parametrized only by a single parameter. Its phase portrait for values of \( g = 2 \) and \( g = 12 \) is shown in Fig. 10.3 for reference. The system admits elementary solutions \( i = i_0 e^{-gt}, r = r_0 + i_0 \left( 1 - e^{-gt} \right) \) which correspond to \( s = 0 \). This can be verified by direct substitution into Eq. (10.7).

### 10.3.2 A second order differential equation for the s-variable

An equivalent second-order system for the s-variable was given in [7].

**Proposition 10.2.** The SIR system is reducible to the nonlinear differential equation for the s-variable

\[ \ddot{s} = \frac{s^2}{s} + (s - g)\dot{s} \]

or the system

\[ \dot{S}_t = \frac{S_t^2}{S} + (S - g)S_t \quad \text{(10.10)} \]
\[ \dot{S} = S_t \quad \text{(10.11)} \]

**Proof.** Differentiating Eq. (10.4) gives

\[ \frac{di}{dt} = -\ddot{s} + i \dot{s} = \frac{s^2}{s^2} - \frac{\ddot{s}}{s} \]

and substituting \( i \) from Eq. (10.4). Substitution in Eq. (10.5) gives

\[ \frac{s^2}{s^2} - \frac{\ddot{s}}{s} = g \frac{s}{s} - \dot{s} \]

from where the result follows.
The following proposition is also formulated in [7].

**Proposition 10.3.** The SIR system is reducible to the nonlinear differential equation for the \( r \)-variable

\[
\ddot{r} = g \dot{r} \left( e^{-\frac{r}{\bar{r}}} - 1 \right)
\]

or the system

\[
\begin{align*}
\dot{R}_t &= g R_t e^{-\frac{R_t}{\bar{r}}} - g R_t \\
\dot{R} &= R_t
\end{align*}
\]

**Proof.** Differentiating Eq. (10.6) yields \( \ddot{r} = g (s \dot{i} - g \dot{i}) = g s \dot{i} - g \dot{s} = -g \dot{s} - g \dot{r} \). On the other hand, by Eq. (10.16) we have \( r = -g \log s + c \). We use the fixed-point condition \( s_0 = s(0) = g \).
so that \( s = g e^{-r/g + r_0/g} \). Since the system is autonomous we can translate the origin to \(-\infty\), where \( r_0 = 0 \) from where the result follows.

10.4 Indeterminate analytical solution

This section develops the analytical solution in an implicit form. Since there is a first integral by construction the system can be reduced to any two of the three equations [3, Ch. 2]:

\[
\frac{di}{ds} = -1 + \frac{g}{s} \tag{10.14}
\]
\[
\frac{di}{dr} = \frac{s}{g} - 1 \tag{10.15}
\]
\[
\frac{dr}{ds} = -\frac{g}{s} \tag{10.16}
\]

From this formulation

\[
R_e = N \frac{S_0}{g} = \frac{S_0 \beta}{\gamma} \geq 1
\]

must hold for the infection to propagate. \( R_e \) is called the effective reproductive number, while the basic reproduction number is \( R_0 = R_e N [11] \). In order to solve the model we will consider the two equations separately. Direct integration of Eq. (10.14) gives

\[
i = -s + g \log s + c \tag{10.17}
\]

where the indeterminate constant \( c \) can be determined from the initial conditions.

10.4.1 The s-variable

The \( s \) variable can be determined by substitution in Eq. (10.4), resulting in the autonomous system

\[
\dot{s} = -s (-s + g \log s + c) \tag{10.18}
\]

which can be solved implicitly as

\[
\int \frac{ds}{s(s - g \log s - c)} = \tau \tag{10.19}
\]

The \( s \) variable can be represented explicitly in terms of the Lambert W function.

\[
s = -g W_\pm \left( -\frac{\frac{s - c}{e}}{g} \right) \tag{10.20}
\]
where the signs denote the two different real-valued branches of the function. Note, that both branches are of interest since the argument of the Lambert W function is negative.

10.4.2 The i-variable

Based on Eq. (10.20), Eq. (10.5) can be reduced to the first-order autonomous system

\[ i = -gi \left( W_\pm \left( -\frac{e^{\frac{i-x}{g}}} g \right) + 1 \right) \]  

(10.21)

valid for two disjoined domains on the real line. This autonomous ODE can be solved for the rescaled time variable \( \tau \) as

\[ -\int \frac{di}{i \left( W_\pm \left( -\frac{e^{\frac{i-x}{g}}} g \right) + 1 \right)} = g \tau \]  

(10.22)

10.4.3 The r-variable

Finally, the \( r \) variable can also be conveniently expressed in terms of \( i \). For this purpose we solve the differential equation

\[ \frac{dr}{di} = \frac{g}{s-g} = \frac{-1}{1 + W_\pm \left( -\frac{e^{\frac{i-x}{g}}} g \right)} \]

Therefore,

\[ r = c_1 - g \log \left( -g W_\pm \left( -\frac{e^{\frac{i-x}{g}}} g \right) \right) = c_1 - g \log s \]

by Proposition 10.5. On the other hand,

\[ g \log \left( -g W \left( -\frac{e^{\frac{i-x}{g}}} g \right) \right) = g \left( \log \left( e^{\frac{i-x}{g}} \right) - W \left( -\frac{e^{\frac{i-x}{g}}} g \right) \right) = i - c - g W \left( -\frac{e^{\frac{i-x}{g}}} g \right) = s + i - c \]
So that
\[
r = g W \left( -\frac{e^{\frac{i-m}{g}}}{g} \right) - i + c + c_1 \tag{10.23}
\]

Since \((s_0, 0, 0)\) is a stable point it follows that
\[
c + c_1 = -g W \left( -\frac{e^{\frac{i-m}{g}}}{g} \right) \tag{10.24}
\]

Therefore,
\[
r = g W \left( -\frac{e^{\frac{i-m}{g}}}{g} \right) - g W \left( -\frac{e^{\frac{i-m}{g}}}{g} \right) - i \tag{10.25}
\]

**Proposition 10.4.** \(i(t)\) attains a global maximum \(i_m = c - g \log g - g\).

**Proof.** We use a parametrization for which \(i(0) = i_m\). Then
\[
i(\tau) = -gi \left( W_\pm \left( -\frac{e^{\frac{i-m}{g}}}{g} - 1 \right) + 1 \right)
\]

It follows that \(i(0) = 0, i(0) = i_m\) so \(i_m\) is an extremum. In the most elementary way since \(W(z)\) should be real-valued then
\[
-e^{\frac{i-m}{g} - 1} \geq -1/e \implies \frac{i - i_m}{g} \leq 0
\]

Hence, \(i \leq i_m\).

**10.5 Inverse parametric solution**

The implicit solution Eq. (10.22) can be computed as the definite integral
\[
\tau(i) = - \int_{i_{c+g \log g - g}}^{i} \frac{dy}{y \left( W_\pm \left( -\frac{e^{\frac{y-c}{g}}}{g} \right) + 1 \right)} \tag{10.26}
\]

which requires computation of the Lambert W function on every integration step. However, this does not seem to be efficient. Alternatively the solution can be computed more efficiently by change of variables by Proposition 10.6:
\[
\tau(i) = -g \int_{g W_\pm \left( -\frac{e^{\frac{i-m}{g}}}{g} \right)}^{g} \frac{dy}{y \left( g \log y - y + c \right)} \tag{10.27}
\]
On the other hand, from Eq. (10.19) it follows that

$$\tau(s) = -\int_{g}^{s} \frac{dy}{y (g \log y - y + c)}$$  \hspace{1cm} (10.28)

where the domain of $s$ is $[-gW_+ \left(-e^{-c/g}/g\right), -gW_- \left(-e^{-c/g}/g\right)]$.

The $\tau$-variable can also be expressed as function of $r$. We differentiate Eq. (10.23) by $s$. From where

$$-gt = \int \frac{dr}{ds} \int dt = \int dr \frac{1}{s (s - g \log s - c)} \bigg|_{s = e^{(c_1 - r)/g}} = \int \frac{r - r_1}{e^{r}} \frac{dr}{(r - (c_1 + c)) e^{r} + 1}
$$

where $c_1 = -gW_- \left(-e^{-c/g}/g\right) - c$ by Eq. (10.24). Therefore,

$$g\tau(r) = -\int_{q - c}^{r} \frac{e^{\frac{y}{g}} dy}{y \left(y - q\right) e^{\frac{y}{g}} + q}, \quad q = -gW_- \left(-e^{-c/g}/g\right)$$  \hspace{1cm} (10.29)

for the $c$-parametrization and the domain of $r$ is

$$[0, gW_+ \left(-e^{-c/g}/g\right) - gW_- \left(-e^{-c/g}/g\right)]$$

Therefore, the following theorem can be formulated.

**Theorem 10.1** (Inverse parametric solution). The inverse parametric solution of the SIR model is given by the integrals

$$\tau(s) = -\int_{g}^{s} \frac{dy}{y (g \log y - y + c)}$$

$$g\tau(i) = -g \int_{g}^{-gW_+ \left(-e^{\frac{i}{g}}\right)} \frac{dy}{y \left(g \log y - y + c\right)} = \int_{c+g \log g - g}^{i} \frac{dy}{y \left(W_\pm \left(-e^{\frac{y}{g}}\right) + 1\right)}$$

$$g\tau(r) = -\int_{q - c}^{r} \frac{e^{\frac{y}{g}} dy}{y \left(y - q\right) e^{\frac{y}{g}} + q}, \quad q = -gW_- \left(-e^{-c/g}/g\right)$$

The solutions are plotted in Fig. 10.4. A similar solution for $\tau(r)$ was obtained also in [12]. Interestingly, the authors derive an infinite series for $\tau(r)$ in terms of upper incomplete Gamma functions.
10.5.1 Peak value parametrization

The upper terminal of integration can be determined by the requirement for the real-valuedness of \( i \). This value of \( i \) is denoted as \( i_m \); that is

\[
W_\pm \left(-\frac{\frac{i_m - c}{g}}{e^{\frac{i_m - c}{g}}}\right) = -1
\]

Therefore,

\[
i_m = c + g \log g - g \quad (10.30)
\]

The peak-value parametrization is supported by Proposition 10.4.
Chapter 10

If we consider formally the phase space \((z \times y = -gz\left(W_{\pm}\left(-e^{\frac{z-im}{g}}\right) + 1\right))\) the following argument allows for the correct branch identification. For \(i \to -\infty\) \(W_+\left(-e^{\frac{z-im}{g}}\right) \to -\infty\) so \(y < 0\); while \(W_+\left(-e^{\frac{z-im}{g}}\right) \to 0^+\) so \(y > 0\). Therefore, if we move the origin as \(t(0) = i_m\) then conveniently

\[
-\int_{i_0}^{i} \frac{dz}{z\left(W_+\left(-e^{\frac{z-im}{g}}\right) + 1\right)} = g\tau, \quad \tau > 0 \tag{10.31}
\]

\[
-\int_{i_0}^{i} \frac{dz}{z\left(W_-\left(-e^{\frac{z-im}{g}}\right) + 1\right)} = g\tau, \quad \tau \leq 0 \tag{10.32}
\]

Furthermore, the recovered population under this parametrization is

\[
r = gW_{\pm}\left(-e^{\frac{z-im}{g}}\right) - gW_-\left(-e^{\frac{z-im}{g}}\right) - i \tag{10.33}
\]

under the same choice of origin.

10.5.2 Initial value parametrization

As customarily accepted, the SIR model can be recast as an initial value problem. In this case, the indeterminate constant \(c\) can be eliminated using the initial condition

\[
i_0 = -s_0 + g \log s_0 + c
\]

Therefore,

\[
i = i_0 + s_0 - s + g \log s/s_0 = 1 - s + g \log \frac{s}{1 - i_0} - i \tag{10.34}
\]

For this case, the following autonomous differential equation can be formulated:

\[
\dot{t} = -g t \left(W_{\pm}\left(-\frac{1 - i_0}{g} e^{\frac{i+1}{g}}\right) + 1\right) \tag{10.35}
\]

This can be solved implicitly by separation of variables as

\[
-\int_{i_0}^{i} \frac{dz}{z\left(W_-\left(-\frac{1-i_0}{g} e^{\frac{i+1}{g}}\right) + 1\right)} = g\tau, \quad \tau \leq \tau_m \tag{10.36}
\]
- \int_{i_0}^{i} \frac{dz}{z \left( \frac{1 - i_0 g e^{-\frac{z}{g}}}{W_+} + 1 \right)} = g\tau, \quad \tau > \tau_m \tag{10.37}

It is noteworthy that the time to the peak of infections \( \tau_m \) can be calculated as [9]:

\[ \tau_m = \frac{\log g/s_0}{\int_0^d \frac{du}{s_0 e^u - gu - (s_0 + i_0)} } \]

The result follows by considering the autonomous system (10.14) and fixing the upper terminal of integration \( s = g \). However, by Proposition 10.7 this definite integral can be evaluated only numerically.

### 10.6 Analysis of the incidence variable

The incidence \( i \)-function of the SIR model appears to be an interesting object of study on its own. It was demonstrated that the function is nonelementary in the sense of the definition given below [9].

**Definition 10.1.** An elementary function is defined as a function built from a finite number of combinations and compositions of algebraic, exponential, and logarithm functions under algebraic operations (+, −, ., /)

Allowing for the underlying field to be complex numbers – \( \mathbb{C} \), trigonometric functions become elementary as well.

**Definition 10.2** (Liouvillian function). We say that \( f(x) \) is a Liouvillian function if it lies in some Liouvillian extension of \((C(x), \cdot)\) for some constant field \( C \).

As a first point we establish the nonelementary character of the integral in Eq. (10.22). The necessary introduction to the theory of differential fields is given in Appendix 10.B. From the work of Liouville it is known that a function of the form \( F(x) = f(x) e^{q(x)} \), where \( f, q \) are elementary functions, has an elementary antiderivative of the form [13]

\[ \int F(x) dx = \int f e^q dx = h e^q \]

for some elementary function \( h(x) \) [14]. Therefore, differentiating we obtain

\[ f e^q = h' e^q + h q' e^q \]

so that if \( e^q \neq 0 \) \( h' + h q' = f \) holds. The claim can be strengthened to demand that \( h \) be algebraic for algebraic \( f \) and \( q \) (see Theorem 10.1).
Theorem 10.2. The integrals

\[ I_{\pm}(\xi) = \int \frac{d\xi}{\xi \left( W_{\pm} \left( -\frac{\xi - c}{g} \right) + 1 \right)} \]

are not Liouvillian.

Proof. We use \( i_m \) parametrization. Let \( c = i_m + g - g \log g \). The proof proceeds by change of variables – first \( \xi = g \log y - yg + g + i_m \); followed by \( z = \log ((g \log y + i_m + g)/g) \).

\[ I = \int \frac{d\xi}{\xi \left( W_{\pm} \left( -e \frac{\xi - im}{g} \right) + 1 \right)} = \int \frac{y - 1}{y (g \log(y) - gy + i_m + g) \left( W (-ye^{-y}) + 1 \right)} dy \]

\[ = -\int \frac{dy}{y(g \log(y) - gy + i_m + g)} = \frac{1}{g} \int \frac{e^{z + i\frac{im}{g} + 1}}{e^{z} - e^{z + i\frac{im}{g} + 1}} dz \]

since \( W_{\pm} (-ye^{-y}) = -y \). The last integral has the form

\[ \int \frac{Ae^z}{e^z - Ae^z} dz \]

which allows for the application of the Liouville theorem in the form of Corollary 10.1. We can identify

\[ \int f e^z dx = h e^z, \quad f(z) = \frac{A}{e^z - Ae^z}, \quad A = e^{i\frac{im}{g} + 1} \]

so that

\[ \frac{A}{e^z - Ae^z} = h'(z) + h(z) \]

for some unknown algebraic \( h(z) \). Since the left-hand side of the equation is transcendental in \( z \) so is the right-hand side, which is a contradiction. Therefore, the integrand has no elementary antiderivative.

The result of the last paragraph leaves the question about the form of \( i(\tau) \) somehow wanting. Here we demonstrate a functional equation for the \( i \) variable exhibiting its non-Liouville character.
Ritt established that if an elementary function has an elementary inverse it is a sequential composition of algebraic and transcendental functions [15]. Prelle and Singer proved in Corollary 3 that if the autonomous system \( y' = f(y) \) has an elementary first integral then

\[
g(y) = \int \frac{dy}{f(y)}
\]

is also elementary [16]. This presents a direct way of proving that the incidence function is nonelementary by virtue of Theorem 10.2 but leaves open the question about its non-Liouvillian character. In order to clarify the form of \( i(\tau) \) we make use of the integral \( I(\xi) \).

On the first place, the following identity holds for its kernel:

\[
\frac{1}{\xi \left( W_\pm \left( -\frac{e^{-\xi/c}}{g} \right) + 1 \right)} = \frac{1}{\xi} - \frac{W_\pm \left( -\frac{e^{-\xi/c}}{g} \right)}{\xi \left( W_\pm \left( -\frac{e^{-\xi/c}}{g} \right) + 1 \right)}
\]

Therefore,

\[
I_\pm(\xi) = \log \xi - \int \frac{W_\pm \left( -\frac{e^{-\xi/c}}{g} \right) d\xi}{\xi \left( W_\pm \left( -\frac{e^{-\xi/c}}{g} \right) + 1 \right)} = \log \xi + \int \frac{dz \left( \frac{\xi}{z} - 1 \right) W_\pm \left( -\frac{e^{g \log z - z + c}}{g} \right)}{(g \log z - z + c) \left( W_\pm \left( -\frac{e^{g \log z - z + c}}{g} \right) + 1 \right)} \bigg|_{z = -W_\pm \left( -\frac{e^{-\xi/c}}{g} \right)} = \log \xi + \int \frac{dz}{g \log z - z + c} \bigg|_{z = -W_\pm \left( -\frac{e^{-\xi/c}}{g} \right)} = \log \xi + L \left( -W_\pm \left( -\frac{e^{-\xi/c}}{g} \right) \right)
\]

where we have used the defining identity for the Lambert W function and defined the auxiliary function up to a constant as the indefinite integral

\[
L(z) = \int \frac{dz}{g \log z - z + c} \tag{10.38}
\]

which is a Liouvillian extension nonelementary integral by Proposition 10.8. Exponentiating and using the notation \( L(z) \) it can be seen that

\[
\exp(I_\pm(\xi)) = \xi e^{L \left( -W_\pm \left( -\frac{e^{-\xi/c}}{g} \right) \right)} = e^{-g \tau}
\]
However, $\xi = i(\tau)$; therefore

$$i(\tau) = e^{-g\tau - L\left(-W_{\pm}\left(-\frac{i(\tau) - c}{g}\right)\right)}$$  \hspace{1cm} (10.39)$$

or

$$i(\tau) = e^{-g\tau - L\left(-\Omega_{\pm}\left(\frac{i(\tau) - c}{g} - \log g\right)\right)}, \quad \Omega_{\pm}(z) = \Omega(z \pm j\pi)$$  \hspace{1cm} (10.40)$$

from which we can infer the non-Liouvillean character of $i(\tau)$ from the form of the equation. This can be done as follows. For any given $\tau$ the RHS is a composition of a Liouvillian function with a non-Liouvillean one, i.e. the Wright $\Omega$ function being not a Liouvillian one [17]. It can be seen that the only scenario, where $i(\tau)$ is elementary is when the second term, containing the $L$ function vanishes. This is exactly the scenario, when $s_0 = 0$ as indicated by Eq. (10.7), for in this case $i = -gi$ and $i$ is exponential in $\tau$. To remove the ambiguity in the definition of $L(z)$ we observe that $z = s/g$. Therefore,

$$L(z) := \int_1^z \frac{du}{g \log u/g - u + g + i} - \log i_m$$

so that $L(1) = -\log i_m$ ensuring the identity at $\tau = 0$. Furthermore, it can be claimed that

**Theorem 10.3.** The incidence function $i(t)$, defined by the differential equation (10.21), is not Liouvillian.

### 10.7 Asymptotic analysis of the SIR model

In their seminal publication, Kermack and McKendrick develop asymptotics for $r$ under the assumption that $r/g$ is small. Recently, Barlow and Weinstein [8] derived Padé approximation scheme for the $s$-variable. The approximant for the $s$-variable by construction, matches the correct $t \to \infty$ behavior and whose expansion about $t = 0$ is exact to $n^{th}$-order.

In the previous publication [9], a double exponential form of the asymptotic for the $i$-variable was given, however without performing comprehensive study.

With the hindsight of the functional equation (10.39) a different scheme is more appropriate. On the first place, the functional equation immediately reveals the long-time asymptotic behavior of $i$.

$$\lim_{t \to \infty} i(\tau) = \lim_{\tau \to \infty} e^{-g\tau - L\left(-\Omega_{\pm}\left(\frac{i(\tau) - c}{g} - \log g\right)\right)} = 0$$
since \( i(\tau) \) is monotonously decreasing for \( \tau_m > 0 \). Therefore, it follows that

\[
i(\tau) \approx e^{-g\tau - \int_{\frac{\tau_m}{\alpha}} \log g + \frac{\tau_m}{\alpha} - \int_{\frac{\tau_m}{\alpha}} \log g}
\]

for large \( \tau \).

Since the solution is Lipschitz everywhere in \( \mathbb{R} \) one can make use of the Banach Fixed Point Theorem. Notably, one can use the nonlinear approximation scheme of Daftardar-Gejji-Jafari for solving the equivalent integral equations [18]. If we treat the equations of the SIR model as independent we can formally solve Eq. (10.4) as

\[
s(\tau) = s_0 e^{-\int_i(t) d\tau}
\]

On the other hand, Eq. (10.5) can be transformed formally as

\[
i' + gi = si \iff e^{-g\tau} d (e^{g\tau} i) = si d\tau \iff \frac{d (e^{g\tau} i)}{e^{g\tau} i} = s d\tau
\]

Therefore, formally,

\[
i(\tau) = k_2 e^{-g\tau} + \int s(\tau) d\tau
\]

Starting from the 0th order approximation is \( i^{(0)} \approx i_0 \), it follows that

\[
s^{(0)} \approx s_0 e^{-i_0 \tau}
\]

This approximation is valid around the fixed point \( s \approx g \), which we can take as an initial condition \( s_0 = g \) since then \( di/ds = 0 \). However, this corresponds to the peak-value parametrization so \( i_0 = i_m \), \( i'(0) = 0 \). From this, the 1st order approximation for the \( i \)-variable becomes

\[
i^{(1)} = i_m e^{\frac{g}{i_m} (1-e^{-im\tau})-gt}
\]

A second iteration of the loop results in a nonelementary \( \gamma \) integral. Let

\[
J := \int e^{\frac{g}{i_m} (1-e^{-im\tau})-g\tau} d\tau
\]

Then, by change of variables

\[
J = -\int \frac{g}{i_m} \frac{1}{e^{\frac{g}{i_m} - \frac{g}{i_m} e^{-im\tau}}} dy \Bigg|_{y=e^{-im\tau}} = \frac{\Gamma \left( \frac{g}{i_m}, \frac{g}{i_m} e^{-im\tau} \right)}{\left( \frac{g}{i_m} \right)^{\frac{g}{i_m}}}
\]
where $\Gamma$ is the upper incomplete Euler’s gamma function. Therefore,

$$ r^{(1)} = g J = k_3 \frac{\Gamma \left( \frac{g}{i_m}, \frac{g e^{-i_m \tau}}{i_m} \right)}{\left( \frac{g}{i_m} \right)^{i_m}} $$

Matching the long-term limit results in the equation

$$ r^{(1)} = g \left( W_+ \left( -e^{-\frac{i_m}{\pi} - 1} \right) - W_- \left( -e^{-\frac{i_m}{\pi} - 1} \right) \right) \frac{\Gamma \left( \frac{g}{i_m}, \frac{g e^{-i_m \tau}}{i_m} \right)}{\Gamma \left( \frac{g}{i_m} \right)} $$

which can be used also for fitting purposes. Then

$$ s^{(2)} = k_2 e^{-J} = k_2 \exp \left( - \frac{\Gamma \left( \frac{g}{i_m}, \frac{g e^{-i_m \tau}}{i_m} \right)}{\left( \frac{g}{i_m} \right)^{i_m}} \right) $$

Matching the initial condition gives

$$ s^{(2)} = g \exp \left( \left( \Gamma(a, a) - \Gamma \left( a, ae^{-i_m \tau} \right) \right) \left( \frac{e^a}{a} \right)^a \right), \quad a = \frac{g}{i_m} $$

Finally, following the same procedure for the i-variable we obtain

$$ i^{(2)} = i_m \exp \left( g e^q \int_0^\tau e^{-\Gamma(a,ae^{-i_m \tau})(\xi)^a} d\xi - g \tau \right), \quad q = \Gamma(a, a) \left( \frac{e^a}{a} \right)^a $$

where $a = \frac{g}{i_m}$. The asymptotics of the i-variable are shown in Fig. 10.5.

### 10.8 Numerical approximation

The i-function can be efficiently approximated by the Newton’s method [9]. The Newton iteration scheme is given as follows for the c-parametrization:

$$ i_{n+1} = i_n + g i_n \left( W_+ \left( -\frac{e^{-\frac{i_n}{\pi} - c}}{g} \right) + 1 \right) \left( t - \int_0^t g W_+ \left( -\frac{e^{-\frac{i_n}{\pi}}}{g} \right) \frac{dy}{y (g \log y - y + c)} \right) $$

(see Theorem 10.1). This form has the advantage of requiring only 1 Lambert W function evaluation per iteration.

A point of attention here is the choice of the initial value for the iteration scheme. According to the presented analysis, a suitable elementary initializing function is $i^{(1)}(t)$ given by Eq. (10.41).
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Figure 10.5: Asymptotic solutions $i(\tau)$. Asymptotic solutions compared to parametric plots of $(\tau(i), i)$ parametrized by $i_m = 3.7283$ and $g = 2.0$. Legends: LB denotes the left branch involving $W_-(x)$, RB denotes the right branch involving $W_+(x)$, Eq. (10.27). A denotes the asymptotic solution. A – the asymptotic is computed from Eq. (10.41); B – the asymptotic is computed from Eq. (10.44). Plots were produced using the \textit{quad_qags} Maxima numerical integration command.

| g    | $R_0$ | T [days] | $i_m$     | fit        |
|------|-------|----------|-----------|------------|
| 0.5493 | 1.8206 | 6.4849   | 296.2197  | asymptotic |
| 0.4710 | 2.1233 | 6.3609   | 303.9163  | sir        |

Table 10.1: Incidence parameters.

10.9 Cast study I: application to influenza A

The influenza data are tabulated in [4]. The task here is to fit the influenza incidence in a boarding school. The fitting equation is given by

$$I_t \sim N \cdot i(t - T|g, i_m)$$

where $I_t$ is the observed incidence. The fitting functionality was implemented MATLAB®. Quadratures were estimated by the default MATLAB integration algorithms. The parametric fitting was conducted using least-squares constrained optimization algorithm. A least square fit obtained $R^2 = 53.1163$ for the asymptotic fit and $R^2 = 62.9415$ for the full model. Fitted parameters are presented in Table 10.1. The plots are presented in Fig. 10.6. The fitting produced peak estimate of 296.2197 for the asymptotic equation and 303.9163 for the full model in excellent agreement with the raw data (see Table 10.1).
10.10 Cast study II: application to COVID-19

The second case study uses COVID datasets were downloaded from the European Centre for Disease Prevention and Control (ECDC) website: https://opendata.ecdc.europa.eu/covid19/casedistribution/csv. The data were imported in the SQLite database, filtered by country and transferred to MATLAB for parametric fitting using native routines. Quadratures were estimated by the default MATLAB integration algorithms. Estimated parameter values were stored in the same database. The processing is described in [9]. For numerical stability reasons the time variable was rescaled by a factor of 10. The estimate represented in Fig. 10.8 has been published in [9]. The asymptotic approach is exemplified with data from Italy (Fig. 10.7).

10.11 Discussion and conclusions

The present chapter demonstrates that the SIR model could be solved completely. Recent developments have exhibited the solutions of the model in terms of special functions. As anticipated in previous literature, these functions are nonelementary and in particular the incidence $i$-function is non-Liouvillian. On the other hand, the asymptotic solutions of the model could be readily established in terms of iterated exponents or incomplete gamma functions.

In conclusion, the SIR model has proven to be valuable in modeling epidemic outbreaks. In many situations it has shown excellent agreement with the historical data [3,4] as well as with the data coming from the present COVID-19 pandemics [9,8,10]. On the other hand, it
Figure 10.7: Parametric fitting of the first and second waves in Italy.
A, B – case fatality fitting; C, D – incidence fitting; ‘asym’ refers to parametric fit using the asymptotic formula Eq. (10.41), ‘sir’ refers to fitting the i-variable computed by numerical inversion using the parameters estimated by Eq. (10.41). Note the pronounced weekly variation of the reported numbers.

should be remembered that any model is only a cartoon of reality and it is always important to challenge model assumptions with the available facts on the ground. A distinction should be always made between the mathematical consistency or even beauty of a certain model and its empirical applicability. In particular, predictions based on empirically false models may be harmful, and it is essential to distinguish between assumptions that simplify but do not alter the predicted effects substantially, and wrong assumptions, which are not supported by the underlying ecology or social dynamics.
Appendix 10.A The Lambert W function and related integrals

The Lambert W function can be defined implicitly by the equation

\[ W(z) e^{W(z)} = z, \quad z \in \mathbb{C} \]

We observe that by Lemma 10.1 \( W(z) \) is transcendental. Furthermore, the Lambert function obeys the differential equation for \( x \neq -1 \)

\[ W'(x) = \frac{e^{-W(x)}}{1 + W(x)} \]

The W function is nonelementary and in particular it is non-Liouvillian [17]. Its indefinite integral is:

\[ \int W(x) \, dx = xW(x) + \frac{x}{W(x)} - x \]

The Lambert W is a multivalued function. Properties of the W function are given in [19].

**Proposition 10.5.**

\[ \int \frac{dy}{1 + W \left( -\frac{y - c}{g} \right)} = g \log \left( -gW \left( \frac{\frac{y - c}{g}}{g} \right) \right) + C \]
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Proof. We differentiate

\[ g \left( \log W \left( -\frac{e^{\frac{y-c}{g}}}{g} \right) \right)' = -\frac{e^{\frac{y-c}{g}}}{gW \left( -\frac{e^{\frac{y-c}{g}}}{g} \right) \left( 1 + W \left( -\frac{e^{\frac{y-c}{g}}}{g} \right) \right)} = \frac{1}{1 + W \left( -ge^{\frac{y-c}{g}} \right)} \]

Proposition 10.6.

\[
\int_{\log g-g+c}^{i} \frac{d\xi}{\xi \left( W \left( -\frac{\xi-c}{g} \right) + 1 \right)} = g \int_{g}^{-gW_{\pm} \left( -\frac{i+c}{g} \right)} \frac{dy}{y \left( g \log y - y + c \right)}
\]

Proof. We use the change of variables \( \xi - c = g \log y - y \) and then simplification by the defining identity of the Lambert W function.

\[
\int_{\log g-g+c}^{i} \frac{d\xi}{\xi \left( W \left( -\frac{\xi-c}{g} \right) + 1 \right)} = \int_{A}^{B} \frac{\frac{g}{y} - 1}{\left( g \log y - y + c \right) \left( W \left( -\frac{g \log y - y + c - \frac{g}{y}}{g} \right) + 1 \right)} dy = \int_{A}^{B} \frac{g - y}{y \left( g \log y - y + c + cy \right) + gy \log y - y^2 + cy} dy = g \int_{A}^{B} \frac{dy}{y \left( g \log y - y + c \right)}
\]

where

\[ g \log A - A + c = g \log g - g + c, \quad g \log B - B + c = i \]

Therefore, \( A = g \) and \( B = -gW \left( -\frac{i+c}{g} \right) \).

The proof of Theorem 10.2 establishes also the character of two related integrals:

Proposition 10.7. The integral

\[ I = \int \frac{dy}{y + c - e^y} \]

is not elementary.
Proof. By change of variables $y = e^x$

$$I = \int \frac{dy}{y (\log y - y + c)} = -\int \frac{dx}{e^x - x - c}$$

\[\square\]

**Proposition 10.8.** $L(z)$ is a nonelementary integral.

Proof. This can be established by an argument using change of variables as

$$L(z) = \int \frac{dz}{g \log z - z + c} = \int \frac{e^u du}{-e^u + gu + c} \bigg|_{z=e^u}$$

further applying the same reasoning as in the proof of Theorem 10.2.

\[\square\]

**Appendix 10.B Differential fields**

**Definition 10.3.** Denote by $\mathbb{C}(x, c_i, \theta_i)$ the complex-valued ring, generated by the finite set of rational functions $\{\theta_i\}_i^n$ and constants $\{c_i\}_i^n$.

**Definition 10.4.** An element $\theta$ is called algebraic if $P(x, \theta) = 0$ for some polynomial

$$P(x, t) = t^m + a_{m-1}t^{m-1} + \ldots + a_0,$$

where $a_i$ can be also rational functions of $x$, or else it is called transcendental.

**Lemma 10.1** (Composition lemma). Denote by $a$ and $t$ the algebraic or transcendental elementary functions, respectively. The following composition rules hold

$$a \circ a = a, \quad t \circ a = t, \quad a \circ t = t$$

Proof. The case $a \circ a$ when $a(x)$ is a polynomial is trivial. Suppose that $a$ and $b$ are both algebraic:

$$P(x, a) = 0, \quad Q(x, b) = 0$$

Without loss of generality suppose that $a_i$ are polynomial. Formally, $b = \tilde{f}_k(x)$ for any branch $k$ with $\tilde{f}$ algebraic since it is a root of a polynomial, where the bar denotes the inverse function in order to avoid confusion with exponentiation. Therefore,

$$a \circ b = b^m + a_{m-1}b^{m-1} + \ldots + a_0 = \tilde{f}_k^m(x) + a_{m-1}\tilde{f}_k^{m-1}(x) + \ldots + a_0$$

is algebraic since it is computed by a finite sequence of algebraic operations.
Suppose that $t = \exp(x)$. Then $\exp(a)$ is not algebraic, hence it is transcendental. $P(x, e^x) = e^{xm} + a_{m-1}e^{m-x} + \ldots + a_0$ is exponential.

Suppose that $t = \log(x)$. Then $\log(a)$ is not algebraic, hence it is transcendental. $P(x, \log(x)) = \log^m x + a_{m-1} \log^{m-1} x + \ldots + a_0$ is not algebraic, hence it is transcendental.

In what follows is assumed that the differential field is of characteristic zero and has an algebraically closed field of constants. An element $y$ of a differential field is said to be an exponential of an element $A$ if $y' = Ay$, an exponential of an integral of an element $A$ if $y' = Ay$; logarithm of an element $A$ if $y' = A'/A$, and an integral of an element $A$ if $y' = A$.

The next definition is due to [17].

**Definition 10.5.** Let $(k, d/dx)$ be a differential field of characteristic 0. A differential extension $(K, d/dx)$ of $k$ is called Liouvillian over $k$ if there are $\theta_1, \ldots, \theta_n \in K$, such that $K = C(x, \theta_1, \ldots, \theta_n)$ and for all $i$, at least one of the following

1. $\theta_i$ is algebraic over $k(\theta_1, \ldots, \theta_{n-1})$
2. $\theta_i' \in k(\theta_1, \ldots, \theta_{n-1})$
3. $\theta_i'/\theta_i \in k(\theta_1, \ldots, \theta_{n-1})$

holds. The constant subfield $C(K)$ of $K$ is defined to be the set of $c$ in $K$, such that $c' = 0$.

The next theorem is due to [14].

**Theorem 10.4.** If $K$ is an elementary field, then it is closed under differentiation.

An elementary integrability theorem due to Conrad [14].

**Theorem 10.5 (Rational Liouville criterion).** For $f, g \in C(x)$ with $f$ and $g$ nonconstant the function $f(x)e^{g(x)}$ can be integrated in elementary terms if and only if there exists a rational function $h \in C(x)$ such that $h' + g' h = f$.

The last result can be extended to algebraic functions as follows.

**Corollary 10.1 (Algebraic Liouville criterion).** For $f(x)$, $g(x)$ algebraic and nonconstant, the function $f(x)e^{g(x)}$ can be integrated in elementary terms if and only if there exists an algebraic function $h(x)$, for which $h' + g' h = f$. 
Proof. Suppose that $f$ and $g$ are arbitrary elementary algebraic functions. Denote the primitive of $f$ by capital $F$ in the juxtaposition $f \div F$. The integral can be integrated by parts

$$ I = \int f(x)e^{g(x)}\,dx = \int e^{g(x)}dF = F(x)e^{g(x)} - \int F(x)\left(e^{g(x)}\right)\,dx $$

Therefore,

$$ \int \left(f(x) + F(x)g'(x)\right)e^{g(x)}\,dx = F(x)e^{g(x)} $$

We observe that $g'(x)$ is elementary by Theorem 10.4. The L.H.S. has the form $fe^{g}$ and since $f(x) + F(x)g'(x)$ is elementary we can identify

$$ h \equiv F, \quad f_1 \equiv f + Fg' = h' + hg' $$

so that $(h' + hg')e^{g} = (he^{g})'$ and the claim follows. $\square$

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