Power series and integral forms of Lame equation in Weierstrass’s form and its asymptotic behaviors

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Abstract
I consider the power series expansion of Lame function in Weierstrass’s form and its integral forms applying three term recurrence formula [15]. I investigate asymptotic expansions of Lame function for the cases of infinite series and polynomials. I will show how the power series expansion of Lame functions in Weierstrass’s form can be converted to closed-form integrals for all cases of infinite series and polynomial. One interesting observation resulting from the calculations is the fact that a \(_2F_1\) function recurs in each of sub-integral forms: the first sub-integral form contains zero term of \(A_n'\)'s, the second one contains one term of \(A_n'\)'s, the third one contains two terms of \(A_n'\)'s, etc.

This paper is 7th out of 10 in series “Special functions and three term recurrence formula (3TRF)”. See section 7 for all the papers in the series. Previous paper in series deals with the power series expansion and the integral formalism of Lame equation in the algebraic form and its asymptotic behavior [19]. The next paper in the series describes the generating functions of Lame equation in Weierstrass’s form[21].

Nine examples of 192 local solutions of the Heun equation (Maier, 2007) are provided in the appendix. For each example, I show how to convert local solutions of Heun equation by applying 3TRF to analytic solutions of Lame equation in Weierstrass’s form.

Keywords: Lame equation, Integral form, Three term recurrence formula, Lame polynomials, Ellipsoidal harmonic function
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1. Introduction
In 1837, Gabriel Lame introduced second ordinary differential equation which has four regular singular points in the method of separation of variables applied to the Laplace equation in elliptic coordinates[3]. Various authors has called this equation as ‘Lame equation’ or ‘ellipsoidal harmonic equation’ [11].

Lame ordinary differential equation in Weierstrass’s form and Heun equation are of Fuchsian types with the three regular and one irregular singularities. Lame equation in Weierstrass’s form

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Lame equation in Weierstrass’s form is derived from Heun equation by changing all coefficients $\gamma = \delta = \epsilon = \frac{1}{2}$, $a = \rho^{-2}$, $\alpha = \frac{1}{2}(\alpha + 1)$, $\beta = -\frac{1}{2}a$, $q = -\frac{1}{2}\rho^{-2}$ and an independent variable $x = sn^2(z, \rho)$. [1, 2]

Due to its mathematical complexity there is no analytic solution in closed forms of Lame function [11, 12, 13]. Because its solution, in the algebraic form or in Weierstrass’s form, was a form of a power series that is expressed as three term recurrence relation [12, 13]. In contrast, most of well-known special functions consist of two term recursion relation (Hypergeometric, Bessel, Legendre, Kummer functions, etc).

In my previous paper [19], applying three term recurrence formula [15], I showed the power series expansion in closed forms of Lame function in the algebraic form (infinite series and polynomial) including all higher terms of $A_n$’s by applying three term recurrence formula [15]. I obtained representations in form of contour integrals of Lame function in the algebraic form and its asymptotic behavior of it and the boundary condition for $x$.

In this paper I will show the analytic solution of Lame equation in Weierstrass’s form. Its functions in Weierstrass’s form appear as we apply the method of separation of variables to Laplaces equation in an ellipsoidal coordinate system (Gabriel Lame 1837 [3]).

The Lame equation in Weierstrass’s form is defined by

$$\frac{d^2y}{d\xi^2} = (\alpha(\alpha + 1)\rho^2 \cdot sn^2(z, \rho) - h)y(z)$$

where $\rho$, $\alpha$ and $h$ are real parameters such that $0 < \rho < 1$ and $\alpha \geq -\frac{1}{2}$. If we take $sn^2(z, \rho) = \xi$ as independent variable, Lame equation becomes

$$\frac{d^2y}{d\xi^2} + \frac{1}{2} \left( \frac{1}{\xi} + \frac{1}{\xi - 1} + \frac{1}{\xi - \rho^{-2}} \right) \frac{dy}{d\xi} + \frac{-\alpha(\alpha + 1)\xi + h\rho^{-2}}{4\xi(\xi - 1)(\xi - \rho^{-2})} y(\xi) = 0$$

This is an equation of Fuchsian type with the four regular singularities: $\xi = 0, 1, \rho^{-2}, \infty$. The first three, namely $0, 1, \rho^{-2}$, have the property that the corresponding exponents are $0, \frac{1}{2}$ which is the same as the case of Lame equation in the algebraic form. In Ref. [19], Lame equation of the algebraic form is

$$\frac{d^2y}{dx^2} + \frac{1}{2} \left( \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} \right) \frac{dy}{dx} + \frac{-\alpha(\alpha + 1)x + q}{4(x-a)(x-b)(x-c)} y = 0$$

If we compare (2) with (3), all coefficients on the above are correspondent to the following way.

$$\begin{align*}
a &\rightarrow 0 \\
b &\rightarrow 1 \\
c &\rightarrow \rho^{-2} \\
q &\rightarrow h\rho^{-2} \\
x &\rightarrow \xi = sn^2(z, \rho)
\end{align*}$$

We obtain another expression of Lame function in Weierstrass’s form by using (4) in Ref. [19].

\[1^{st}\] higher terms of $A_n$’s” means at least two terms of $A_n$’s.
Lame equation in Weierstrass’s form

2. Power series

2.1. Polynomial in which makes $B_n$ term terminated

There are three types of polynomials in three-term recurrence relation of a linear ordinary differential equation: (1) polynomial which makes $B_n$ term terminated: $A_n$ term is not terminated, (2) polynomial which makes $A_n$ term terminated: $B_n$ term is not terminated, (3) polynomial which makes $A_n$ and $B_n$ terms terminated at the same time. In general Lame polynomial (or Lame spectral polynomial) is defined as type 3 polynomial where $A_n$ and $B_n$ terms terminated. Lame polynomial comes from a Lame equation that has a fixed integer value of $\alpha$, as it has a fixed value of $h$. In three-term recurrence formula, polynomial of type 3 I categorize as complete polynomial. In future papers I will derive type 3 Lame polynomial. In this paper I construct differential equation: (1) polynomial which makes

\[
y(z) = \sum_{n=0}^{\infty} y_n(z)
\]

\[
= c_0 \left\{ \sum_{l_0=0}^{n_0} \frac{(-\alpha)_{l_0}(\alpha_0 + \frac{1}{2} + \lambda)_{l_0}}{(1 + \frac{d}{2})_{l_0}(\frac{1}{2} + \frac{i}{2})_{l_0}} \eta^l
\]

\[
+ \left\{ \sum_{l_1=0}^{n_1} \frac{(2a - b - c)(i_0 + \frac{d}{2})^2 - a(\alpha_0 + \frac{1}{2})(\alpha_0 + \frac{1}{2} + \frac{1}{2}) + \frac{q}{2} (\alpha_0)_{l_0}(\alpha_0 + \frac{1}{2} + \lambda)_{l_0}}{(i_0 + \frac{1}{2} + \frac{d}{2})(i_0 + \frac{1}{2} + \frac{1}{2})} \right\} \mu
\]

\[
\times \sum_{l_2=0}^{n_2} \frac{(2a - b - c)(i_2 + \frac{d}{2})^2 - a(\alpha_2 + \frac{1}{2})(\alpha_2 + \frac{1}{2} + \frac{1}{2}) + \frac{q}{2} (\alpha_2)_{l_0}(\alpha_2 + \frac{1}{2} + \lambda)_{l_0}}{(i_2 + \frac{1}{2} + \frac{d}{2})(i_2 + \frac{1}{2} + \frac{1}{2})}
\]

\[
\times \prod_{k=1}^{n_1} \left( \sum_{l_{k-1}=0}^{n_{k-1}} \frac{(2a - b - c)(i_k + \frac{d}{2})^2 - a(\alpha_k + \frac{1}{2})(\alpha_k + \frac{1}{2} + \frac{1}{2}) + \frac{q}{2} (\alpha_k)_{l_{k-1}}(\alpha_k + \frac{1}{2} + \lambda)_{l_{k-1}}}{(i_k + \frac{1}{2} + \frac{d}{2})(i_k + \frac{1}{2} + \frac{1}{2})} \right)
\]

\[
\times \prod_{k=1}^{n_0} \left( \sum_{l_{k-1}=0}^{n_{k-1}} \frac{(2a - b - c)(i_{k-1} + \frac{d}{2})^2 - a(\alpha_{k-1} + \frac{1}{2})(\alpha_{k-1} + \frac{1}{2} + \frac{1}{2}) + \frac{q}{2} (\alpha_{k-1})_{l_{k-1}}(\alpha_{k-1} + \frac{1}{2} + \lambda)_{l_{k-1}}}{(i_{k-1} + \frac{1}{2} + \frac{d}{2})(i_{k-1} + \frac{1}{2} + \frac{1}{2})} \right) \eta^l \mu^k \right\}
\]

(5)

where

\[
\begin{align*}
    z &= x - a \\
    \eta &= \frac{-(x-a)^2}{(a-b)(a-c)} \\
    \mu &= \frac{-(x-a)}{a-b(a-c)}
\end{align*}
\]

2 If $A_n$ and $B_n$ terms are not terminated, it turns to be infinite series.

3 In this paper Pochhammer symbol $(x)_k$ is used to represent the rising factorial: $(x)_k = \frac{x(x+1)(x+2)\ldots(x+k-1)}{k!}$. 

\[ \]
and
\[
\begin{cases}
\alpha = 2(2\alpha_i + i + \lambda) \text{ or } -2(2\alpha_i + i + \lambda) - 1 \text{ where } i, \alpha_i = 0, 1, 2, \cdots \\
\alpha_i \leq \alpha_j \text{ only if } i \leq j \text{ where } i, j = 0, 1, 2, \cdots
\end{cases}
\]
(7)

Put (4) in (5)-(7). And take \(c_0 = 1\) as \(\lambda = 0\) for the first independent solution of Lame equation and \(\lambda = \frac{1}{4}\) for the second one into the new (5)-(7).

**Remark 1.** The representation in the form of power series expansion of the first kind of independent solution of Lame equation in Weierstrass’s form for the polynomial which makes \(B_n\) term terminated about \(\xi = 0\) as \(\alpha = 2(2\alpha_j + j)\) or \(-2(2\alpha_j + j) - 1\) where \(j, \alpha_j = 0, 1, 2, \cdots\) is

\[
y(\xi) = LF_{\alpha}(\rho, h, \alpha = 2(2\alpha_j + j) \text{ or } -2(2\alpha_j + j) - 1; \xi = \text{sn}^2(z, \rho), \mu = -\rho^2 \xi; \eta = -\rho^2 \xi^2)
\]

= \(\sum_{h=0}^{\infty} \frac{(-\alpha_0)_{\eta}(\alpha_0 + \frac{1}{\lambda})_{\eta}}{(\frac{1}{\lambda})_{\eta}(1)_{\eta}} \eta^h\)

+ \(\sum_{h=0}^{\infty} \frac{(-1 + \rho^2)_{\eta}^2 + \frac{\mu}{2\rho^2} (-\alpha_0)_{\eta}(\alpha_0 + \frac{1}{\lambda})_{\eta}}{(\frac{1}{\lambda})_{\eta}(1)_{\eta}} \sum_{i=0}^{\infty} (-\alpha_1)_{\eta}(\alpha_1 + \frac{1}{\lambda})_{\eta}(\frac{2}{\eta})_{\eta}(\frac{2}{\xi})_{\eta_{\xi_1}} \mu^i\)

+ \(\sum_{n=2}^{\infty} \frac{(-1 + \rho^2)_{\eta}^2 + \frac{\mu}{2\rho^2} (-\alpha_0)_{\eta}(\alpha_0 + \frac{1}{\lambda})_{\eta}}{(\xi)_{\eta}(1)_{\eta}} \sum_{i=0}^{\infty} (-\alpha_1)_{\eta}(\alpha_1 + \frac{1}{\lambda})_{\eta}(\frac{2}{\eta})_{\eta}(\frac{2}{\xi})_{\eta_{\xi_1}} \mu^i\)

\[\times \prod_{k=1}^{n-1} \left(\frac{-1 + \rho^2)(\xi_k + \frac{1}{\lambda})^2 + \frac{\mu}{2\rho^2} (-\alpha_k)_{\eta}(\alpha_k + \frac{1}{\lambda})_{\eta}(\xi_k + \frac{1}{\lambda})_{\eta_{\xi_k}} \mu^{2i}\right)\]

\[\times \sum_{i_n=0}^{a_n} (-\alpha_n)_{\eta}(\alpha_n + \frac{1}{\lambda})_{\eta_{\xi_{n-1}}} \frac{1}{(\xi_n + \frac{1}{\lambda})_{\eta_{\xi_n}}(\xi_n + \frac{1}{\lambda})_{\eta_{\xi_{n-1}}}} \mu^{i_n}\]

**Remark 2.** The representation in the form of power series expansion of the second kind of independent solution of Lame equation in Weierstrass’s form for the polynomial which makes \(B_n\) term terminated about \(\xi = 0\) as \(\alpha = 2(2\alpha_j + j - 1)\) or \(-2(2\alpha_j + j - 1)\) where \(j, \alpha_j = 0, 1, 2, \cdots\)

\[\text{If we take } \alpha = -\frac{1}{2}, \alpha = -2(2\alpha_i + i + \lambda) - 1 \text{ is not available any more in } (7). \text{ In this paper I consider } \alpha \text{ as arbitrary.}\]

\[\text{By definition, ‘Lame polynomial or Lame spectral polynomial’ means polynomial which makes } A_n \text{ and } B_n \text{ terms terminated: for any non-negative integer value of } \alpha \text{ there will be } 2\alpha + 1 \text{ values of } h \text{ for which the solution } y(\xi) \text{ reduces to a polynomial. In this paper I construct the power series expansion and integral formalism of Lame polynomial which makes } B_n \text{ term terminated: I treat the spectral parameter } h \text{ as a free variable. In my next papers I will work on the power series expansion and integral formalism of Lame polynomial which makes } A_n \text{ term terminated and Lame spectral polynomial.}\]
Lame equation in Weierstrass’s form

\[ y(\xi) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[ \frac{(\alpha_k + \frac{1}{2} + 2\rho)^n}{(\alpha_k + \frac{1}{2} + 2\rho)(\alpha_k + \frac{1}{2} + 2\rho)} \right] y^n \]

2.2. Infinite series

The general expression of the power series expansion of Lame equation in algebraic form for the infinite series in Ref.[19] is given by

\[ y(z) = \sum_{n=0}^{\infty} z^n \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[ \frac{(2a - b - c)(\alpha_k + \frac{1}{2} + 2\rho)^2 - \frac{2}{3} a(\alpha_k + 1) + \frac{2}{3}}{(\alpha_k + \frac{1}{2} + 2\rho)(\alpha_k + \frac{1}{2} + 2\rho)} \right] \]

\[ \times \left[ \frac{(\alpha_k + \frac{1}{2} + 2\rho)(\alpha_k + \frac{1}{2} + 2\rho)}{(\alpha_k + \frac{1}{2} + 2\rho)(\alpha_k + \frac{1}{2} + 2\rho)} \right] \]

\[ \times \left[ \frac{(\alpha_k + \frac{1}{2} + 2\rho)(\alpha_k + \frac{1}{2} + 2\rho)}{(\alpha_k + \frac{1}{2} + 2\rho)(\alpha_k + \frac{1}{2} + 2\rho)} \right] \]

The infinite series in [8] is

\[ y(\xi) = L\sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[ \frac{(\alpha_k + \frac{1}{2} + 2\rho)^n}{(\alpha_k + \frac{1}{2} + 2\rho)(\alpha_k + \frac{1}{2} + 2\rho)} \right] y^n \]

Put (4) in (8). And take \( c_0 = 1 \) as the first independent solution of Lame equation and \( \lambda = \frac{1}{2} \) for the second one into the new (8).
Remark 3. The representation in the form of power series expansion of the first kind of independent solution of Lame equation in Weierstrass’s form for the infinite series about $\xi = 0$ is

\[
y(\xi) = LF \left( \rho, h, \alpha; \xi = sn^2(z, \rho), \mu = -\rho^2 \xi; \eta = -\rho^2 \xi^2 \right) = \sum_{i=0}^{\infty} \left( \frac{-\frac{1}{2} h_i (\frac{1}{2} + \frac{1}{2})}{(\frac{1}{2} i_n (1)_n} \right) \eta^i + \left\{ \sum_{i=0}^{\infty} \left( \frac{1 + \rho^2 h_i^2}{2 h_i} \right) \left( \frac{-\frac{1}{2} h_i (\frac{1}{2} + \frac{1}{2})}{(\frac{1}{2} i_n (1)_n} \right) \sum_{i=0}^{\infty} \left( \frac{-\frac{1}{2} + \frac{1}{2} h_i (\frac{1}{2} + \frac{1}{2})}{(\frac{1}{2} i_n (1)_n} \right) \right\} \mu\right\}
\]

Remark 4. The representation in the form of power series expansion of the second kind of independent solution of Lame equation in Weierstrass’s form for the infinite series about $\xi = 0$ is

\[
y(\xi) = LS \left( \rho, h, \alpha; \xi = sn^2(z, \rho), \mu = -\rho^2 \xi; \eta = -\rho^2 \xi^2 \right) = \xi^2 \sum_{i=0}^{\infty} \left( \frac{-\frac{1}{2} h_i (\frac{1}{2} + \frac{1}{2})}{(\frac{1}{2} i_n (1)_n} \right) \eta^i + \left\{ \sum_{i=0}^{\infty} \left( \frac{1 + \rho^2 h_i^2}{2 h_i} \right) \left( \frac{-\frac{1}{2} h_i (\frac{1}{2} + \frac{1}{2})}{(\frac{1}{2} i_n (1)_n} \right) \sum_{i=0}^{\infty} \left( \frac{-\frac{1}{2} + \frac{1}{2} h_i (\frac{1}{2} + \frac{1}{2})}{(\frac{1}{2} i_n (1)_n} \right) \right\} \mu\right\}
\]
Lame equation in Weierstrass’s form

3. Integral Formalism

3.1. Polynomial in which makes $B_n$ term terminated

The general expression of the representation in the form of integral of Lame equation in algebraic form for the polynomial in which makes $B_n$ term terminated in Ref. [19] is given by

$$y(z) = \sum_{n=0}^{\infty} y_n(z)$$

$$= c_0 \sum_{n=0}^{\infty} \frac{(-\alpha_0)_{\alpha_0}(\alpha_0 + \frac{1}{2} + \lambda)_n}{(1 + \frac{1}{2})_n(\frac{1}{2} + \frac{1}{2})_n} \eta^n$$

$$+ \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} \left\{ \int_0^1 dr_{n-k} \frac{1}{r_{n-k}} \int_0^1 dr_{n-k} u_{n-k}^{(n-k-2+\lambda)} \right\} \times \frac{1}{2\pi i} \int dv_{n-k} \frac{1}{v_{n-k}} (1 - \frac{\leftrightarrow}{w_{n-k+1}} v_{n-k} (1 - r_{n-k}) (1 - u_{n-k}))^{-(n-k+4+\lambda)}$$

$$\times \left\{ \left( v_{n-k} - 1 \right) - \frac{1}{v_{n-k}} \frac{1}{1 - \frac{\leftrightarrow}{w_{n-k+1}} v_{n-k} (1 - r_{n-k}) (1 - u_{n-k})} \right\}^{\alpha_0}$$

$$\times \left\{ 2a - b - c \right\}^{\frac{1}{2}(n-k-1+\lambda)} \left\{ \frac{w_{n-k,0}}{\eta_{n-k,0}} \eta_{n-k,0} \eta_{n-k,0} \right\}^{\frac{1}{2}(n-k-1+\lambda)}$$

$$\times \left\{ a_{n-k-1} + \frac{1}{2} (n - k - 1 + \lambda) \right\} \left\{ \sum_{(\alpha_0)_{\alpha_0}(\alpha_0 + \frac{1}{2} + \lambda)_n}{(1 + \frac{1}{2})_n(\frac{1}{2} + \frac{1}{2})_n} \eta^n \right\}$$

(9)

where

$$\leftrightarrow_{i,j} = \left\{ \begin{array}{ll}
\frac{1}{(v_i - 1)} & 1 - \frac{\leftrightarrow_{i+1}}{w_{i+1}} v_i (1 - t_i) (1 - u_i) \\
\eta & \text{only if } i > j
\end{array} \right.$$

(10)
Lame equation in Weierstrass’s form

Put (4) in (9).

\[ y(\xi) = \sum_{n=0}^{\infty} y_n(\xi) \]

\[ = c_0 \xi \left\{ \sum_{n=0}^{\infty} \frac{(-\alpha_0)_n}{(1 + \frac{1}{2})_n (\frac{1}{2} + \frac{1}{2})_n} \eta^n \right\} \]

\[ + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \int_0^1 du_{n-k} \ t_{n-k}^{\frac{1}{4}(n-k-\frac{1}{2}+j)} \right\} \int_0^1 du_{n-k} \ u_{n-k}^{\frac{1}{4}(n-k-2+j)} \]

\[ \times \frac{1}{2\pi i} \oint d\nu_{n-k} \frac{1}{\nu_{n-k}} \left( 1 - \frac{w_{n-k+1,n} v_n}{(1 - t_{n-k})(1 - u_{n-k})} \right)^{(a-k+\frac{j}{2}+1)} \]

\[ \times \left( \frac{v_{n-k} - 1}{\nu_{n-k}} - \frac{1}{1 - \frac{w_{n-k+1,n} v_{n-k}}{1 - t_{n-k}}(1 - u_{n-k})} \right)^{\nu_{n-k}} \]

\[ \times \left( (1 + \rho^2) \frac{w_{n-k}}{w_{n-k,n} \partial_{w_{n-k,n}}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \left( \frac{w_{n-k,n} \partial_{w_{n-k,n}}}{w_{n-k,n}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \]

\[ \times \frac{1}{2\pi} \int_0^1 dv_{n-k} \left( \frac{1}{\nu_{n-k}} - \frac{1}{1 - \frac{w_{n-k+1,n} v_{n-k}}{1 - t_{n-k}}(1 - u_{n-k})} \right)^{\nu_{n-k}} \]

\[ \times \left( (1 + \rho^2) \frac{w_{n-k}}{w_{n-k,n} \partial_{w_{n-k,n}}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \left( \frac{w_{n-k,n} \partial_{w_{n-k,n}}}{w_{n-k,n}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \]

\[ \times \frac{1}{2\pi} \int_0^1 dv_{n-k} \left( \frac{1}{\nu_{n-k}} - \frac{1}{1 - \frac{w_{n-k+1,n} v_{n-k}}{1 - t_{n-k}}(1 - u_{n-k})} \right)^{\nu_{n-k}} \]

\[ \times \left( (1 + \rho^2) \frac{w_{n-k}}{w_{n-k,n} \partial_{w_{n-k,n}}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \left( \frac{w_{n-k,n} \partial_{w_{n-k,n}}}{w_{n-k,n}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \]

\[ \times \frac{1}{2\pi} \int_0^1 dv_{n-k} \left( \frac{1}{\nu_{n-k}} - \frac{1}{1 - \frac{w_{n-k+1,n} v_{n-k}}{1 - t_{n-k}}(1 - u_{n-k})} \right)^{\nu_{n-k}} \]

\[ \times \left( (1 + \rho^2) \frac{w_{n-k}}{w_{n-k,n} \partial_{w_{n-k,n}}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \left( \frac{w_{n-k,n} \partial_{w_{n-k,n}}}{w_{n-k,n}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \]

\[ \times \frac{1}{2\pi} \int_0^1 dv_{n-k} \left( \frac{1}{\nu_{n-k}} - \frac{1}{1 - \frac{w_{n-k+1,n} v_{n-k}}{1 - t_{n-k}}(1 - u_{n-k})} \right)^{\nu_{n-k}} \]

\[ \times \left( (1 + \rho^2) \frac{w_{n-k}}{w_{n-k,n} \partial_{w_{n-k,n}}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \left( \frac{w_{n-k,n} \partial_{w_{n-k,n}}}{w_{n-k,n}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \]

\[ \times \frac{1}{2\pi} \int_0^1 dv_{n-k} \left( \frac{1}{\nu_{n-k}} - \frac{1}{1 - \frac{w_{n-k+1,n} v_{n-k}}{1 - t_{n-k}}(1 - u_{n-k})} \right)^{\nu_{n-k}} \]

\[ \times \left( (1 + \rho^2) \frac{w_{n-k}}{w_{n-k,n} \partial_{w_{n-k,n}}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \left( \frac{w_{n-k,n} \partial_{w_{n-k,n}}}{w_{n-k,n}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \]

\[ \times \frac{1}{2\pi} \int_0^1 dv_{n-k} \left( \frac{1}{\nu_{n-k}} - \frac{1}{1 - \frac{w_{n-k+1,n} v_{n-k}}{1 - t_{n-k}}(1 - u_{n-k})} \right)^{\nu_{n-k}} \]

\[ \times \left( (1 + \rho^2) \frac{w_{n-k}}{w_{n-k,n} \partial_{w_{n-k,n}}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \left( \frac{w_{n-k,n} \partial_{w_{n-k,n}}}{w_{n-k,n}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \]

\[ = 2F_1 \left( 1 - \alpha_0, \alpha_0 + 1, \frac{3}{4}, \eta \right) \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \int_0^1 du_{n-k} \ t_{n-k}^{\frac{1}{4}(n-k-\frac{1}{2})} \right\} \int_0^1 du_{n-k} \ u_{n-k}^{\frac{1}{4}(n-k-2)} \]

\[ \times \frac{1}{2\pi} \int_0^1 dv_{n-k} \left( 1 - \frac{w_{n-k+1,n} v_n}{(1 - t_{n-k})(1 - u_{n-k})} \right)^{(a-k+\frac{j}{2}+1)} \]

\[ \times \left( \frac{v_{n-k} - 1}{\nu_{n-k}} - \frac{1}{1 - \frac{w_{n-k+1,n} v_{n-k}}{1 - t_{n-k}}(1 - u_{n-k})} \right)^{\nu_{n-k}} \]

\[ \times \left( (1 + \rho^2) \frac{w_{n-k}}{w_{n-k,n} \partial_{w_{n-k,n}}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \left( \frac{w_{n-k,n} \partial_{w_{n-k,n}}}{w_{n-k,n}} \right)^{\frac{1}{2} \frac{1}{4}(n-k+1+\frac{j}{2})} \]

Remark 6. The representation in the form of integral of the second kind of independent solution of Lame equation in Weierstrass’s form for the polynomial which makes \( B_{\eta} \) term terminated...
Lame equation in Weierstrass’s form

about $\xi = 0$ as $\alpha = 2(2\alpha_j + j) + 1$ or $-2(2\alpha_j + j + 1)$ where $j, \alpha_j = 0, 1, 2, \cdots$ is

$$y(\xi) = L_S\left(\rho, h, \alpha = 2(2\alpha_j + j) + 1 \text{ or } -2(2\alpha_j + j + 1); \xi = sn^2(z, \rho), \mu = -\rho^2 \xi; \eta = -\rho^2 \xi^2\right)$$

$$= \xi^2 \left\{ _2F_1 \left( -\alpha_0, \alpha_0 + \frac{3}{4}, \frac{5}{4}; \eta \right) + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \int_0^1 dt_w^{-k} t_w^{\frac{2}{n-k}-(\alpha-k-\frac{1}{2})} \int_0^1 du_n^{-k} u_n^{\frac{1}{2}(n-k-\frac{1}{2})} \right\} \right\}$$

$$\times \left\{ \frac{1}{2\pi i} \int dv_n^{-k} \frac{1}{v_n^{-k}} \left( 1 - \frac{\eta}{n-k+1, v_n^{-k}}(1 - t_n^{-k})(1 - u_n^{-k}) \right)^{(\alpha-k-\frac{1}{2})} \right\}$$

$$\times \left( \frac{\nu_n^{-k} - 1}{v_n^{-k}} \left( 1 - \frac{\eta}{n-k+1, v_n^{-k}}(1 - t_n^{-k})(1 - u_n^{-k}) \right) \right)$$

$$\times \left( - (1 + \rho^{-2}) \frac{w_n^{-k}}{w_n^{-k, n}}(\frac{\nu_n^{-k}}{w_n^{-k, n}})^{2} \frac{w_n^{-k}}{w_n^{-k, n}} + \frac{h}{2^2 \rho^2} \right) \right\}$$

$$\times 2 F_1 \left( -\alpha_0, \alpha_0 + \frac{3}{4}, \frac{5}{4}; \mu^2 \right) \right\}$$

3.2. Infinite series

The general expression of the representation in the form of integral of Lame equation in algebraic form for the infinite series in Ref. [19] is given by

$$y(z) = \sum_{n=0}^{\infty} y_n(z)$$

$$= c_0 \nu_0 \left\{ \sum_{i=0}^{\infty} \frac{(-\frac{1}{2} + \frac{1}{2} \nu_0)(\nu_0 + \frac{1}{2} + \frac{1}{2} \nu_0)}{(1 + \frac{1}{2} \nu_0)(\frac{1}{2} + \frac{1}{2} \nu_0)} \eta^b \right\}$$

$$+ \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \int_0^1 dt_w^{-k} t_w^{\frac{2}{n-k}-(\alpha-k-\frac{1}{2})} \int_0^1 du_n^{-k} u_n^{\frac{1}{2}(n-k-\frac{1}{2})} \right\}$$

$$\times \left\{ \frac{1}{2\pi i} \int dv_n^{-k} \frac{1}{v_n^{-k}} \left( \frac{\nu_n^{-k} - 1}{v_n^{-k}} \right) \left( 1 - \frac{\eta}{n-k+1, v_n^{-k}}(1 - t_n^{-k})(1 - u_n^{-k}) \right)^{\frac{1}{2}(\alpha-k-\frac{1}{2})} \right\}$$

$$\times \left( (2a - b - c) \frac{w_n^{-k}}{w_n^{-k, n}}(\frac{\nu_n^{-k}}{w_n^{-k, n}})^{2} \frac{w_n^{-k}}{w_n^{-k, n}} - \frac{a}{2^4 \nu_0(a + 1)} \right) \right\}$$

$$\times \sum_{i=0}^{\infty} \frac{(-\frac{1}{2} + \frac{1}{2} \nu_0)(\nu_0 + \frac{1}{2} + \frac{1}{2} \nu_0)}{(1 + \frac{1}{2} \nu_0)(\frac{1}{2} + \frac{1}{2} \nu_0)} \mu^b \right\}$$

(12)

Put (4) in (12). And put $c_0 = 1$ as $\lambda = 0$ for the first independent solution of Lame equation and $\lambda = \frac{1}{2}$ for the second one into the new (12).

Remark 7. The representation in the form of integral of the first kind of independent solution of
Lame equation in Weierstrass’s form

Lame equation in Weierstrass’s form for the infinite series about \( \xi = 0 \) is

\[
y(\xi) = LF \left( \rho, h, \alpha; \xi = sn^2(z, \rho), \mu = -\rho^2 \xi; \eta = -\rho^2 \xi^2 \right)
\]

\[
y(\xi) = 2F_1 \left( \frac{\alpha}{4}, \frac{\alpha}{4}; \frac{3}{4}; \eta \right) + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left( \int_{v_{n-k}}^{v_{n-k-1}} dt_{n-k} t_{n-k}^\frac{4}{n-k-2} \right) \right\} \cdot \frac{1}{2\pi i} \int dv_{n-k} \frac{1}{v_{n-k} - 1} \left( \frac{1 - \frac{1}{\rho} \sum_{k=0}^{n-1} \frac{1}{v_{n-k}} v_{n-k}(1 - t_{n-k})(1 - u_{n-k})^{\frac{4}{n-k-1}}}{1 - \frac{1}{\rho} \sum_{k=0}^{n-1} \frac{1}{v_{n-k}} v_{n-k}(1 - t_{n-k})(1 - u_{n-k})^{\frac{4}{n-k-1}}} \right)\frac{1}{2\pi i} \int dv_{n-k} \frac{1}{v_{n-k} - 1} \left( \frac{1 - \frac{1}{\rho} \sum_{k=0}^{n-1} \frac{1}{v_{n-k}} v_{n-k}(1 - t_{n-k})(1 - u_{n-k})^{\frac{4}{n-k-1}}}{1 - \frac{1}{\rho} \sum_{k=0}^{n-1} \frac{1}{v_{n-k}} v_{n-k}(1 - t_{n-k})(1 - u_{n-k})^{\frac{4}{n-k-1}}} \right)\]

**Remark 8.** The representation in the form of integral of the second kind of independent solution of Lame equation in Weierstrass’s form for the infinite series about \( \xi = 0 \) is

\[
y(\xi) = LS \left( \rho, h, \alpha; \xi = sn^2(z, \rho), \mu = -\rho^2 \xi; \eta = -\rho^2 \xi^2 \right)
\]

\[
y(\xi) = \xi^2 \left\{ 2F_1 \left( \frac{-\alpha}{4}, \frac{-\alpha}{4}; \frac{3}{4}; \eta \right) + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left( \int_{v_{n-k}}^{v_{n-k-1}} dt_{n-k} t_{n-k}^\frac{4}{n-k-2} \right) \right\} \cdot \frac{1}{2\pi i} \int dv_{n-k} \frac{1}{v_{n-k} - 1} \left( \frac{1 - \frac{1}{\rho} \sum_{k=0}^{n-1} \frac{1}{v_{n-k}} v_{n-k}(1 - t_{n-k})(1 - u_{n-k})^{\frac{4}{n-k-1}}}{1 - \frac{1}{\rho} \sum_{k=0}^{n-1} \frac{1}{v_{n-k}} v_{n-k}(1 - t_{n-k})(1 - u_{n-k})^{\frac{4}{n-k-1}}} \right)\frac{1}{2\pi i} \int dv_{n-k} \frac{1}{v_{n-k} - 1} \left( \frac{1 - \frac{1}{\rho} \sum_{k=0}^{n-1} \frac{1}{v_{n-k}} v_{n-k}(1 - t_{n-k})(1 - u_{n-k})^{\frac{4}{n-k-1}}}{1 - \frac{1}{\rho} \sum_{k=0}^{n-1} \frac{1}{v_{n-k}} v_{n-k}(1 - t_{n-k})(1 - u_{n-k})^{\frac{4}{n-k-1}}} \right)\]

4. Asymptotic behavior of the function \( y(\xi) \) and the boundary condition for \( \xi = sn^2(z, \rho) \)

4.1. Infinite series

The condition of convergence in Lame function in the algebraic form for the infinite series and its asymptotic function in Ref. [19] are

\[
\lim_{n \to \infty} y_n = \frac{1}{1 + a \left( \frac{1}{n-b}(x-a) \right)^2 + \left( \frac{2a-b-c}{n-b}(x-a) \right)} < 1
\]

\[
\left| \frac{(x-a)^2}{(a-b)(a-c)} + \frac{(2a-b-c)(x-a)}{(a-b)(a-c)} \right| < 1
\]
Lame equation in Weierstrass’s form

Put (4) in (13) and (14). And its asymptotic function and the boundary condition of \( \xi = \text{sn}^2(z, \rho) \) for the infinite series of Lame function is

\[
\lim_{n \to 1} y(\xi) = \frac{1}{1 + \rho^2 \text{sn}^4(z, \rho) - (1 + \rho^2)\text{sn}^2(z, \rho)} \tag{15}
\]

where \( \left| (1 + \rho^2)\text{sn}^2(z, \rho) - \rho^2 \text{sn}^4(z, \rho) \right| < 1 \) \( \tag{16} \)

Since \( 0 < \rho < 1 \), the boundary condition of \( \text{sn}^2(z, \rho) \) in (16) is given by

\[
0 \leq \text{sn}^2(z, \rho) < 1 \tag{17}
\]

4.2. The case of \( \rho \approx 0 \)

Let assume that \( \rho \) is approximately close to 0. But \( \rho \neq 0 \). Then \( B_n \) terms are negligible.

The condition of convergence in Lame function in the algebraic form for the case of \( 2a - b - c \gg 1 \) or \( 2a - b - c \ll -1 \) and its asymptotic function in Ref. [19] are

\[
\lim_{n \to 1} y(z) = \frac{1}{1 + \frac{(2a-b)(x-a)}{(a-b)(a-c)}} \quad \text{where} \quad a \neq b \text{ and } a \neq c \tag{18}
\]

\[
\left| \frac{(2a-b)(x-a)}{(a-b)(a-c)} \right| < 1 \tag{19}
\]

Put (4) in (18) and (19). And its asymptotic function and the boundary condition of \( \xi = \text{sn}^2(z, \rho) \) for \( \rho \approx 0 \) is

\[
\lim_{n \to 1} y(\xi) = \frac{1}{1 - (1 + \rho^2)\text{sn}^2(z, \rho)} \tag{20}
\]

The condition of convergence of \( \text{sn}^2(z, \rho) \) is

\[
0 \leq \text{sn}^2(z, \rho) < \frac{1}{1 + \rho^2} \tag{21}
\]

5. Application

Lame equation appears elsewhere in mathematical physics. For example, Recently, in “Droplet nucleation and domain wall motion in a bounded interval” [4], the authors investigate an extended model (a classical Ginzburg-Landau model) of noise-induced magnetization reversal. Lame equation arises in some specific boundary conditions. (see (8), (9) in Ref. [4]. In (9) its solution consists of the Jacobi eta, theta, and zeta functions according to Hermite’s solution of the Lame equation.) In “Group Theoretical Properties and Band Structure of the Lame Hamiltonian” [10], the authors represent a group theoretical analysis of the Lame equation, which is an example of a SGA band structure problem for \( su(2) \) and \( su(1, 1) \). (see (1), (10), (13), (14), (28), (29), (33), (38) in Ref. [10]) Applying three term recurrence formula [15], we can obtain the power series expansion in closed forms and asymptotic behaviors of Lame function analytically. And it might be possible to obtain specific eigenvalues for the Lame Hamiltonian. Again Lame equation is applicable to diverse areas such as theory of the stability analysis of static configurations.

The authors treat the analytic solution of Lame equation as Lame polynomial of type 3: for any non-negative integer value of \( \alpha \), there will be \( 2\alpha + 1 \) values of \( h \) (Energy) for which the solution \( y(\xi) \) reduces to a polynomial. In this paper I construct the power series expansion and integral formalism of Lame polynomial of type 1: I treat the spectral parameter \( h \) as a free variable. Mathematically, it can be one of possible analytic solutions of Lame equation. In future papers I will derive types 2 and 3 Lame polynomial.
Lame equation in Weierstrass’s form

in Josephson junctions[6], the computation of the distance-redshift relation in inhomogeneous cosmologies[7], magnetostatic problems in triaxial ellipsoids[8] and etc.

6. Conclusion

From the above all, applying three term recurrence formula[15], I show the power series expansion in closed forms of Lame function in Weierstrass’s form (infinite series and polynomial which makes $B_n$ term terminated) and its integral forms. I show that a $_2F_1$ function recurs in each of sub-integral forms of Lame function in Weierstrass’s form: the first sub-integral form contains zero term of $A'_n$, the second one contains one term of $A'_n$’s, the third one contains two terms of $A'_n$’s, etc. And I show asymptotic expansions of Lame function for infinite series and the special case as $\rho \approx 0$. Since we obtain the closed integral forms of Lame function in Weierstrass’s form, Lame function is able to be transformed to other well-known special functions analytically; hypergeometric function, Mathieu function, Lame function, confluent forms of Heun function and etc.

For type 3 Lame polynomial, various authors argue that the value of $\hbar \rho^{-2}$ can be chosen properly such that the Lame function is not an infinite series but a polynomial as $\alpha$ parameters of Lame functions is a positive integer. For type 1 Lame polynomial, since $\alpha$ is $2(2\alpha_i + i + \lambda)$ or $-2(2\alpha_i + i + \lambda) - 1$ as $i, \alpha_i \in \mathbb{N}_0$ in the analysis of the three term recurrence formula[15], Lame functions will be polynomial which makes $B_n$ term terminated; $\lambda$ is an indicial root which is 0 or $\frac{1}{2}$, then all possible $\alpha$ is $\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$.

According to Erdelyi (1940)[9], “there is no corresponding representation of simple integral formalisms of the solutions in ordinary linear differential equations with four regular singularities; Heun equation, Lame equation and Mathieu equation. It appears that the theory of integral equations connected with periodic solutions of Lame equation is not as complete as the corresponding theory of integral representations of, say, Legendre functions.” The reason, why the analytic integral forms of Lame functions can not be obtained, is that the coefficients in a power series expansions do not have two term recursion relations. We have a relation between three different coefficients. By using the three term recurrence formula[15], we are able to obtain analytic integral solution of any linear ordinary differential equation in which has three term recursion relations.

7. Series “Special functions and three term recurrence formula (3TRF)”

This paper is 7th out of 10.

1. “Approximative solution of the spin free Hamiltonian involving only scalar potential for the $q - \bar{q}$ system”[14] - In order to solve the spin-free Hamiltonian with light quark masses we are led to develop a totally new kind of special function theory in mathematics that generalize all existing theories of confluent hypergeometric types. We call it the Grand Confluent Hypergeometric Function. Our new solution produces previously unknown extra hidden quantum numbers relevant for description of supersymmetry and for generating new mass formulas.

2. “Generalization of the three-term recurrence formula and its applications”[15] - Generalize three term recurrence formula in linear differential equation. Obtain the exact solution of the three term recurrence for polynomials and infinite series.
Lame equation in Weierstrass’s form

3. “The analytic solution for the power series expansion of Heun function” [16] - Apply three term recurrence formula to the power series expansion in closed forms of Heun function (infinite series and polynomials) including all higher terms of $A_n$’s.

4. “Asymptotic behavior of Heun function and its integral formalism”, [17] - Apply three term recurrence formula, derive the integral formalism, and analyze the asymptotic behavior of Heun function (including all higher terms of $A_n$’s).

5. “The power series expansion of Mathieu function and its integral formalism”, [18] - Apply three term recurrence formula, analyze the power series expansion of Mathieu function and its integral forms.

6. “Lame equation in the algebraic form” [19] - Applying three term recurrence formula, analyze the power series expansion of Lame function in the algebraic form and its integral forms.

7. “Power series and integral forms of Lame equation in Weierstrass’s form and its asymptotic behaviors” [20] - Applying three term recurrence formula, derive the power series expansion of Lame function in Weierstrass’s form and its integral forms.

8. “The generating functions of Lame equation in Weierstrass’s form” [21] - Derive the generating functions of Lame function in Weierstrass’s form (including all higher terms of $A_n$’s). Apply integral forms of Lame functions in Weierstrass’s form.

9. “Analytic solution for grand confluent hypergeometric function” [22] - Apply three term recurrence formula, and formulate the exact analytic solution of grand confluent hypergeometric function (including all higher terms of $A_n$’s). Replacing $\mu$ and $\varepsilon\omega$ by 1 and $-q$, transforms the grand confluent hypergeometric function into Biconfluent Heun function.

10. “The integral formalism and the generating function of grand confluent hypergeometric function” [23] - Apply three term recurrence formula, and construct an integral formalism and a generating function of grand confluent hypergeometric function (including all higher terms of $A_n$’s).

Appendix. 9 local solutions of Lame equation in Weierstrass’s form by applying 3TRF out of 192 local solutions of Heun equation

192 local solutions of the Heun equation vs. 9 local solutions of the Lame equation in Weierstrass’s form by applying 3TRF solutions of Heun equation A machine-generated list of 192 (isomorphic to the Coxeter group of the Coxeter diagram $D_4$) local solutions of the Heun equation was obtained by Robert S. Maier(2007) [5]. In appendix of Ref.[17], I apply 3TRF to the power series expansions of Heun equation for infinite series and polynomial of type 1 and its integral forms of nine out of the 192 local solution of Heun function in Table 2 [5]. In this appendix, by changing all coefficients and independent variables of the previous nine examples of 192 local solutions of Heun function into the first kind of independent solutions of Heun equation by applying 3TRF [16, 17], I construct 9 local solutions of Lame equation in Weierstrass’s form for Frobenius solutions in closed form (for infinite series and polynomial of
Lame equation in Weierstrass’s form

Lame equation in Weierstrass’s form is a special case of Heun’s equation. Heun equation is a second-order linear ordinary differential equation of the form

\[
\frac{d^2y}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right) \frac{dy}{dx} + \frac{\alpha \beta x - q}{x(x-1)(x-a)} y = 0
\] (1)

With the condition \(\epsilon = \alpha + \beta - \gamma - \delta + 1\). The parameters play different roles: \(a \neq 0\) is the singularity parameter, \(\alpha, \beta, \gamma, \delta, \epsilon\) are exponent parameters, \(q\) is the accessory parameter which in many physical applications appears as a spectral parameter. Also, \(\alpha\) and \(\beta\) are identical to each other. The total number of free parameters is six. It has four regular singular points which are 0, 1, \(a\) and \(\infty\) with exponents \(\{0, 1 - \gamma\}, \{0, 1 - \delta\}, \{0, 1 - \epsilon\}\) and \(\{\alpha, \beta\}\).

As we compare (2) with (1), all coefficients on the above are correspondent to the following way.

\[
\begin{align*}
\gamma, \delta, \epsilon & \leftrightarrow \frac{1}{2} \\
\alpha & \leftrightarrow \rho^{-2} \\
\alpha & \leftrightarrow \frac{1}{2}(\alpha + 1) \\
\beta & \leftrightarrow -\frac{1}{2}\alpha \\
q & \leftrightarrow -\frac{1}{4}h\rho^{-2} \\
x & \leftrightarrow \xi = sn^2(z, \rho)
\end{align*}
\] (2)

**A. Power series**

In Ref. [16], the representation in the form of power series expansion of the first kind of independent solution of Heun equation for polynomial of type 1 about \(x = 0\) as \(\alpha = -2\alpha_j - j\)

---

\[1\] In this appendix, I treat \(h\) as a free variable and a fixed value of \(\alpha\) to construct polynomials of type 1 for all 9 local solutions of Lame equation. An independent variable \(sn^2(z, \rho)\) is denoted by \(\xi\). And I consider \(\alpha\) as arbitrary. The condition \(\alpha \geq -\frac{1}{2}\) is not necessary any more.
Lame equation in Weierstrass’s form

where \( j, \alpha_j \in \mathbb{N}_0 \) is given by

\[
y(x) = HF_{\alpha_j, \beta} \left( \alpha_j = -\frac{1}{2}(\alpha + j) \right) \eta \left( \frac{1 + a}{a} x; z = -\frac{1}{a} x^2 \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{(-\alpha_0)_n}{(1)_n} \eta \left( \frac{x}{2} \right) \left( \frac{1 + \frac{1}{2} x}{1 + \frac{3}{2} x} \right)^{1/2} \left( 1 + \frac{2}{3} x \right)^{1/2} \left( 1 + \frac{3}{4} x \right)^{1/2} \left( 1 + \frac{4}{5} x \right)^{1/2} \eta
\]

\[
+ \sum_{n=2}^{\infty} \sum_{k=0}^{n-1} \frac{(-\alpha_n)_n}{(1)_n} \eta \left( \frac{x}{2} \right) \left( \frac{1 + \frac{1}{2} x}{1 + \frac{3}{2} x} \right)^{1/2} \left( 1 + \frac{2}{3} x \right)^{1/2} \left( 1 + \frac{3}{4} x \right)^{1/2} \left( 1 + \frac{4}{5} x \right)^{1/2} \eta
\]

\[
+ \sum_{n=2}^{\infty} \sum_{k=0}^{n-1} \frac{(-\alpha_n)_n}{(1)_n} \eta \left( \frac{x}{2} \right) \left( \frac{1 + \frac{1}{2} x}{1 + \frac{3}{2} x} \right)^{1/2} \left( 1 + \frac{2}{3} x \right)^{1/2} \left( 1 + \frac{3}{4} x \right)^{1/2} \left( 1 + \frac{4}{5} x \right)^{1/2} \eta
\]

where

\[
\begin{align*}
\eta &= \frac{(1 + a)_n}{a} x \\
\alpha_j &\leq \alpha_j \quad \text{only if} \quad i \leq j \quad \text{where} \quad i, j = 0, 1, 2, \ldots
\end{align*}
\]

and

\[
\begin{align*}
\Gamma_0^S &= \frac{1}{2(1 + a)}(-2a_0 + \beta - \delta + a(\delta + \gamma - 1)) \\
\Gamma_k^S &= \frac{1}{2(1 + a)}(-2a_k + \beta - \delta + a(\delta + \gamma + k - 1)) \\
Q &= \frac{a}{2(1 + a)}
\end{align*}
\]
In Ref. [16], the representation in the form of power series expansion of the first kind of independent solution of Heun equation for infinite series about \( x = 0 \) is given by

\[
y(x) = HF_{a,\beta}\left(\eta = \frac{(1 + a)}{a}x; z = -\frac{1}{a^2}\right)
\]

\[
= \sum_{n=0}^{\infty} \frac{\left(\frac{\zeta}{\eta}\right)_{\beta+1}}{(\frac{\zeta}{\eta})_{n}} z^{n}n!
\]

\[
+ \sum_{n=0}^{\infty} \frac{\left(\frac{\zeta}{\eta}\right)_{\gamma+1}}{(\frac{\zeta}{\eta})_{n}} z^{n}n! \sum_{i=0}^{\infty} \frac{\left(\frac{\zeta}{\eta}\right)_{\delta+1}}{(\frac{\zeta}{\eta})_{i}} \left(1 + \frac{\zeta}{\eta}\right)^{i}\eta^{n} + O\left(\frac{\zeta}{\eta}\right)^{n+1} \left(1 + \frac{\zeta}{\eta}\right)^{i}\eta^{n} \right.
\]

\[
= \sum_{n=0}^{\infty} \frac{\left(\frac{\zeta}{\eta}\right)_{\beta+1}}{(\frac{\zeta}{\eta})_{n}} z^{n}n!
\]

\[
+ \sum_{n=0}^{\infty} \frac{\left(\frac{\zeta}{\eta}\right)_{\gamma+1}}{(\frac{\zeta}{\eta})_{n}} z^{n}n! \sum_{i=0}^{\infty} \frac{\left(\frac{\zeta}{\eta}\right)_{\delta+1}}{(\frac{\zeta}{\eta})_{i}} \left(1 + \frac{\zeta}{\eta}\right)^{i}\eta^{n} + O\left(\frac{\zeta}{\eta}\right)^{n+1} \left(1 + \frac{\zeta}{\eta}\right)^{i}\eta^{n} \right)
\]

where

\[
\begin{align*}
I^{(f)}_0 &= \frac{-1}{2\gamma(\gamma+1)}(\alpha + \beta - \delta + a(\delta + \gamma - 1)) \\
I^{(f)}_k &= \frac{-1}{2\gamma(\gamma+1)}(\alpha + \beta - \delta + k + a(\delta + \gamma - 1 + k)) \\
O &= \frac{a}{2\gamma(\gamma+1)}
\end{align*}
\]

A.1. \( (1 - x)^{-\gamma}HF(a, q - (\delta - 1)y)\alpha; \alpha - \delta + 1, \beta - \delta + 1, y, 2 - \delta; x \)

A.1.1. Polynomial of type 1

Replace coefficients \( q, \alpha, \beta \) and \( \delta \) by \( q - (\delta - 1)y \alpha; \alpha - \delta + 1, \beta - \delta + 1 \) and \( 2 - \delta \) into (A.1). Multiply \((1 - x)^{-\gamma}\) and (A.1) together. Put (2) into the new (A.1) with replacing \( a \) by \(-2(2\alpha_j + j + 1)\) where \( j, \alpha_j \in \mathbb{N}_0 \); apply \( a = -2(2\alpha_0 + 1) \) into sub-power series \( y_0(\xi) \), apply \( a = -2(2\alpha_0 + 1) \) into the first summation and \( a = -2(2\alpha_1 + 2) \) into second summation of sub-power series \( y_1(\xi) \), apply \( a = -2(2\alpha_0 + 1) \) into the first summation, \( a = -2(2\alpha_1 + 2) \) into the second summation and \( a = -2(2\alpha_2 + 3) \) into the third summation of sub-power series \( y_2(\xi) \), etc
Lame equation in Weierstrass’s form

in the new \( L \)

\[
(1 - \xi^2) y''(\xi) = (1 - \xi^2) H L \left( \rho^{-2}, \frac{1}{4} (h - 1) \rho^{-2} ; \alpha, \frac{3}{2}, \frac{1}{2} \right)
\]

\[
= (1 - \xi^2) \sum_{n=0}^{\infty} \frac{(-\alpha_0)_n}{(1)_n} \left( \frac{1}{2} \right)_n \eta^n + \sum_{n=0}^{\infty} \frac{1}{(i_0 + k + \frac{1}{2}) (i_0 + \frac{k}{2})} \left( \frac{-\alpha_0}_n \left( \frac{1}{2} + \frac{i}{2} \right) \right)_n \eta^n
\]

\[
\prod_{i=1}^{n} \left\{ \frac{(i_k + \frac{1}{2})(i_k + i + \frac{1}{2})}{(i_k + \frac{1}{2})(i_k + \frac{1}{2} + \frac{1}{2})} \right\}
\]

\[
\sum_{n=0}^{\infty} \frac{(-\alpha_n)_n}{(1 + \frac{1}{2}) (1 + \frac{1}{2})} \left( \frac{1}{2} + \frac{n}{2} \right)_n \left( \frac{1}{2} + \frac{n}{2} \right)_n \eta^n
\]

\[\text{where } \alpha = 2 \left( 2\alpha_j + j + \frac{1}{2} \right) \text{ or } -2 \left( 2\alpha_j + j + 1 \right)\]

\[\text{for all } 9 \text{ local solutions of Lame equation for polynomial of type 1 in this appendix, } \alpha_i \leq \alpha_j \text{ only if } i \leq j \text{ where } i, j, \alpha_i, \alpha_j \in \mathbb{N}_0.\]
A.1.2. Infinite series

Replace coefficients \(q, \alpha, \beta\) and \(\delta\) by \(q - (\delta - 1)\gamma a, \alpha - \delta + 1, \beta - \delta + 1\) and \(2 - \delta\) into (A.2). Multiply \((1 - x)^{1-\delta}\) and (A.2) together. Put \((2)\) into the new (A.2).

\[
(1 - \xi^2) y(\xi) = (1 - \xi^2) H(\rho^{-2}, -\frac{1}{4}(h-1)\rho^{-2}; \frac{\alpha}{2} + 1, -\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, \frac{3}{2}; \xi)
\]

\[
= (1 - \xi^2) \left( \sum_{i=0}^{\infty} \left( \sum_{N=0}^{\infty} \frac{\left( \frac{\alpha}{2} + \frac{1}{2} \right)_i \left( -\frac{\alpha}{2} + \frac{1}{2} \right)_i}{(1)_i} \right) x^i \right)
\]

\[
+ \sum_{n=2}^{\infty} \left( \sum_{i=0}^{\infty} \frac{i_0 (i_0 + \Gamma_0) + Q \left( \frac{\alpha}{2} + \frac{1}{2} \right)_i (1, -\frac{\alpha}{2} + \frac{1}{2}) \sum_{i=0}^{\infty} \left( \frac{\alpha + 1}{2} \right)_i \left( \frac{1}{2} \right)_i \left( \frac{1}{2} \right)_i \right)}{(1)_i} \right) x^n \right)
\]

\[
\times \left( \sum_{i=0}^{\infty} \frac{(i_0 + \frac{1}{2}) (i_0 + \Gamma_0) + Q \left( \frac{\alpha}{2} + \frac{1}{2} \right)_i (1, -\frac{\alpha}{2} + \frac{1}{2}) \sum_{i=0}^{\infty} \left( \frac{\alpha + 1}{2} \right)_i \left( \frac{1}{2} \right)_i \left( \frac{1}{2} \right)_i \right)}{(1)_i} \right) x^n \right)
\]

\[
\times \left( \sum_{i=0}^{\infty} \frac{(i_0 + \frac{1}{2}) (i_0 + \Gamma_0) + Q \left( \frac{\alpha}{2} + \frac{1}{2} \right)_i (1, -\frac{\alpha}{2} + \frac{1}{2}) \sum_{i=0}^{\infty} \left( \frac{\alpha + 1}{2} \right)_i \left( \frac{1}{2} \right)_i \left( \frac{1}{2} \right)_i \right)}{(1)_i} \right) x^n \right)
\]

(A.4)

On (A.3) and (A.4),

\[
\eta = (1 + \rho^2) x
\]

\[
\gamma = -\rho^2 x
\]

\[
\Gamma_0 = \frac{\alpha}{(1 + \rho^2) x}
\]

\[
\Gamma_k = \frac{\alpha}{b(1 + \rho^2) x}
\]

\[
Q = \frac{1 - b}{b(1 + \rho^2) x}
\]

A.2. \(x^{1-\gamma}(1 - x)^{1-\delta}H(a, q - (\gamma + \delta - 2)y, a - (\gamma - 1)(a + \beta - \gamma - \delta + 1); \alpha - \gamma - \delta + 2 + \beta - \gamma - \delta + 2, 2, \gamma, 2, -\delta; x)\)

A.2.1. Polynomial of type I

Replace coefficients \(q, \alpha, \beta, \gamma\) and \(\delta\) by \(q - (\gamma + \delta - 2)y, a - (\gamma - 1)(a + \beta - \gamma - \delta + 1), a - \gamma - \delta + 2, \beta - \gamma - \delta + 2, 2 - \gamma, 2 - \delta; x\)
power series \( y_2(\xi) \), etc in the new (A.1).

\[
\xi^\pm (1 - \xi)^\pm y(\xi) = \xi^\pm (1 - \xi)^\pm H\left(\rho^2, -\frac{1}{4}((h - 4)\rho^2 - 1); \frac{\alpha}{2} + \frac{3}{2} = 2\alpha_j + j + \frac{5}{2}, -\frac{\alpha}{2} + 1 = 2\alpha_j + j + \frac{5}{2}\right) \xi^\pm (1 - \xi)^\pm \]

\[
\xi^\pm (1 - \xi)^\pm \sum_{\alpha_0} \left(\frac{(-\alpha_0)_{\alpha_0} (\alpha_0 + \frac{5}{2})}{(1)^{\alpha_0} (\frac{5}{2})^{\alpha_0}} \right) \eta^{\alpha_0} + \sum_{n=2}^{\infty} \left\{ \frac{\alpha_0}{(i_0 + \frac{1}{2})(i_0 + \frac{3}{2})} \left(1\right)^{\alpha_0} (\frac{5}{2})^{\alpha_0} \right\} \eta^{\alpha_0} \prod_{k=1}^{n-1} \left(1 + (\frac{5}{2})_{i_k} (\frac{5}{2})_{i_{k+1}} \right) \right] \eta^{\alpha_n} \right)
\]

\[
\times \sum_{\alpha_n,i_{n+1}} \left(\frac{(-\alpha_n)_{\alpha_n} (\alpha_n + n + \frac{5}{2})_{i_{n+1}} (1 + (\frac{5}{2})_{i_{n+1}} (\frac{5}{2})_{i_{n+1}} \right) \eta^{\alpha_n} \prod_{k=1}^{n-1} \left(1 + (\frac{5}{2})_{i_k} (\frac{5}{2})_{i_{k+1}} \right) \eta^{\alpha_{n-1}} \left(\frac{5}{2})_{i_{n+1}} \right) \right] \eta^{\alpha_n} \right)
\]

\[(A.5)\]

where

\[\alpha = 2\left(2\alpha_j + j + 1\right) \text{ or } -2\left(2\alpha_j + j + \frac{3}{2}\right)\]

A.2.2. Infinite series

Replace coefficients \( q, \alpha, \beta, \gamma \) and \( \delta \) by \( q - (\gamma + \delta - 2)u - (\gamma - 1)(\alpha + \beta - \gamma + 1), \alpha - \gamma - \delta + 2, \beta - \gamma - \delta + 2, 2 - \gamma \) and \( 2 - \delta \) into (A.2). Multiply \( x^{1-\gamma}(1-x)^{1-\delta} \) and (A.2) together. Put (2) into
the new (A.3).

\[ \xi^2 (1 - \xi^2) y(\xi) \]

\[ = \xi^2 (1 - \xi^2) Hl(\rho^2, -\frac{1}{4}((\alpha - 4)\rho^{-2} - 1); \frac{\alpha}{2} + \frac{3}{2}, -\frac{\alpha}{2} + 1; \frac{3}{2} \xi) \]

\[ = \xi^2 (1 - \xi^2), \]

\[ \sum_{k=0}^{\infty} \left( \frac{\xi^2}{(\xi^2 + 1)} \right)_{i} (-\xi^2 + \frac{1}{2})_{i} \eta \]

+ \sum_{k=1}^{\infty} \left( \frac{\xi^2}{(\xi^2 + 1)} \right)_{i} (-\xi^2 + \frac{1}{2})_{i} \eta \]

\[ \times \prod_{k=1}^{n} \left( \frac{\xi^2}{(\xi^2 + 1)} \right)_{i} (-\xi^2 + \frac{1}{2})_{i} \eta \]

\[ \times \prod_{k=1}^{n} \left( \frac{\xi^2}{(\xi^2 + 1)} \right)_{i} (-\xi^2 + \frac{1}{2})_{i} \eta \]

\[ (A.6) \]

On (A.5) and (A.6).

\[ \eta = (1 + \rho^2) \xi \]

\[ \xi = -\rho^2 \xi^2 \]

\[ \Gamma_0 = \frac{2(\rho^2)}{1+\rho^2} \]

\[ \Gamma_1 = \frac{\xi + \frac{2\rho^2}{1+\rho^2}}{1+\rho^2} \]

\[ Q = \frac{4\xi^2}{(1+\rho^2)} \]

A.3. \( Hl(1 - a, -q + a\beta; \alpha, \beta, \delta, \gamma; 1 - x) \)

A.3.1. Polynomial of type 1

Replace coefficients \( a, q, \gamma, \delta \) and \( x \) by \( 1 - a, -q + a\beta, \delta, \gamma \) and \( 1 - x \) into (A.1). Put (2) into the new (A.1) with replacing \( a \) by \(-2(2\alpha_j + j + 1/2)\) where \( j, \alpha_j \in \mathbb{N}_0 \); apply \( a = -2(2\alpha_0 + 1/2) \) into sub-power series \( y_0(\xi) \), apply \(-2(2\alpha_0 + 1/2) \) into the first summation and \(-2(2\alpha_1 + 3/2) \) into second summation of sub-power series \( y_1(\xi) \), apply \(-2(2\alpha_0 + 1/2) \) into the first summation, \(-2(2\alpha_1 + 3/2) \) into the second summation and \(-2(2\alpha_2 + 5/2) \) into the third summation of sub-
Lame equation in Weierstrass's form

\[ y(\varsigma) = H(1 - \rho^2, \frac{1}{4}(h\rho^2 - a(\alpha + 1)); \frac{\alpha}{2} + \frac{1}{2} = 2a_j + j + \frac{1}{2}, -\alpha = 2a_j + j + \frac{1}{2}, \frac{1}{2}, 1 - \xi) \]

\[ = \sum_{n=0}^{\infty} \frac{(-\alpha_0)_{n}}{(1)_n} \left( a_0 + \frac{j}{n} \right)_{n} \varsigma^n \]

\[ + \{ \sum_{n=0}^{\infty} \frac{\rho^2 + Q_0}{(a_0 + \frac{1}{n}) (a_0 + \frac{j}{n})} \frac{(-\alpha_0)_{n}}{(1)_n} \left( a_0 + \frac{j}{n} \right)_{n} \sum_{k=0}^{n} \frac{(-\alpha_1)_{n} (a_1 + \frac{j}{n})_{n} (\frac{j}{n})_{n} \varsigma^n}{(1)_n (\frac{j}{n})_{n}} \} \eta \]

\[ + \sum_{n=2}^{\infty} \sum_{k=0}^{n} \frac{\rho^2 + Q_0}{(a_0 + \frac{j}{n})} \frac{(-\alpha_0)_{n}}{(1)_n} \left( a_0 + \frac{j}{n} \right)_{n} \times \prod_{k=1}^{n-1} \left( \frac{(i_k + \frac{j}{n})^2}{(a_k + \frac{j}{n})^2} \frac{(-\alpha_k)_{n}}{(1)_n} \left( a_k + \frac{j}{n} \right)_{n} \right) \eta^n \]

where

\[ \alpha = 2(2a_j + j) \text{ or } -2(2a_j + j + \frac{1}{2}) \]

\[ Q_0 = \frac{1}{2a_j - \alpha} (h\rho^2 - 4a_0 (a_0 + \frac{1}{j})) \]

\[ Q_k = \frac{1}{2a_j - \alpha} (h\rho^2 - 4(a_k + \frac{1}{j})(a_k + \frac{j}{n})) \]

A.3.2. Infinite series

Replace coefficients \( a, q, \gamma, \delta \) and \( x \) by \( 1 - a, -q + ab, \delta, \gamma \) and \( 1 - x \) into \( A.2 \). Put \( A.2 \) into the new \( A.2 \).

\[ y(\varsigma) = H(1 - \rho^2, \frac{1}{4}(h\rho^2 - a(\alpha + 1)); \frac{\alpha}{2} + \frac{1}{2} = 2a_j + j + \frac{1}{2}, -\alpha = 2a_j + j + \frac{1}{2}, \frac{1}{2}, 1 - \xi) \]

\[ = \sum_{n=0}^{\infty} \frac{\rho^2 + Q_0}{(a_0 + \frac{j}{n}) (a_0 + \frac{j}{n})} \frac{(-\alpha_0)_{n}}{(1)_n} \left( a_0 + \frac{j}{n} \right)_{n} \varsigma^n \]

\[ + \sum_{n=2}^{\infty} \sum_{k=0}^{n} \frac{\rho^2 + Q_0}{(a_0 + \frac{j}{n})} \frac{(-\alpha_0)_{n}}{(1)_n} \left( a_0 + \frac{j}{n} \right)_{n} \times \prod_{k=1}^{n-1} \left( \frac{(i_k + \frac{j}{n})^2}{(a_k + \frac{j}{n})^2} \frac{(-\alpha_k)_{n}}{(1)_n} \left( a_k + \frac{j}{n} \right)_{n} \right) \eta^n \]
where
\[ Q = \frac{h^p^{-2} - \alpha(\alpha + 1)}{8(2 - p^2)} \]

On (A.7) and (A.8),
\[
\begin{align*}
\zeta &= 1 - \xi \\
\eta &= \frac{1}{2 - p^2} \zeta \\
\xi &= \frac{1}{2 - p^2} \zeta
\end{align*}
\]

A.4. \((1 - x)^{1 - \delta} H_l(1 - a, -q + (\delta - 1)ya + (\alpha - \delta + 1)(\beta - \delta + 1); \alpha - \delta + 1, \beta - \delta + 1, -1 - \delta, \gamma; \ddot{1} - x)\)

A.4.1. Polynomial of type 1

Replace coefficients \(a, q, \alpha, \beta, \gamma, \delta\) and \(x\) by \(1 - a, -q + (\delta - 1)ya + (\alpha - \delta + 1)(\beta - \delta + 1), \alpha - \delta + 1, \beta - \delta + 1, 2 - \delta, \gamma\) and \(x - 1\) into (A.11). Multiply \((1 - x)^{1 - \delta}\) and (A.11) together. Put (2) into the new (A.11) with replacing \(\alpha\) by \(-2(2\alpha_j + j + 1)\) where \(j, \alpha_j \in \mathbb{N}_0\); apply \(-2(2\alpha_j + 1)\) into sub-power series \(y_0(\zeta)\), apply \(-2(2\alpha_j + 1)\) into the first summation and \(-2(2\alpha_j + 2)\) into second summation of sub-power series \(y_1(\zeta)\), apply \(-2(2\alpha_j + 1)\) into the first summation. \(-2(2\alpha_j + 3)\) into the second summation and \(-2(2\alpha_j + 3)\) into the third summation of sub-power series \(y_2(\zeta)\), etc in the new (A.11).

\[
(1 - \xi)^{\frac{1}{2}} y_l(\zeta) = (1 - \xi)^{\frac{1}{2}} H_l \left(1 - p^{-2}, \frac{1}{4} \left((1 + h)p^{-2} + (a - 1)(\alpha + 2)\right); \frac{\alpha}{2} + 1 = 2\alpha_j + j + \frac{3}{2} \right)
\]

\[
\xi = \frac{\alpha - 1 + \frac{1}{2} = 2\alpha_j + j + \frac{3}{2}, \frac{1}{2}, 1 - \xi}
\]

\[
(1 - \xi)^{\frac{1}{2}} \left\{ \sum_{k=0}^{\alpha} \left( -a_0 \right) b_k \left( a_0 + \frac{1}{2} \right) \right\}_{\xi_k} \eta
\]

\[
+ \left\{ \sum_{k=0}^{\infty} \left( \eta_0 \left( \frac{1}{2} \right) \right) \right\}_{\xi_k} \eta
\]

\[
\times \prod_{k=1}^{\infty} \left\{ \sum_{\iota=1}^{\alpha} \left( \eta_0 \left( \frac{1}{2} + \iota \right) \right) \right\}_{\xi_k} \eta
\]

\[
\times \sum_{\iota=\iota_0}^{\alpha} \left( -a_0 \right) b_k \left( a_n + n + \frac{1}{2} \right) \eta^\iota
\]

\[
\text{where}
\begin{align*}
\alpha &= 2(2\alpha_j + j + \frac{1}{2}) \quad \text{or} \quad -2(2\alpha_j + j + 1) \\
Q_0 &= \frac{\alpha^2 + \iota \eta \left( 1 + \frac{1}{2} + \iota \right) \left( \frac{1}{2} + \iota \right)}{\alpha^2 - \iota \eta \left( 1 + \frac{1}{2} + \iota \right) \left( \frac{1}{2} + \iota \right)} \\
Q_k &= \frac{\alpha^2 + \iota \eta \left( 1 + \frac{1}{2} + \iota \right) \left( \frac{1}{2} + \iota \right)}{\alpha^2 - \iota \eta \left( 1 + \frac{1}{2} + \iota \right) \left( \frac{1}{2} + \iota \right)}
\end{align*}
\]
Lame equation in Weierstrass’s form

A.4.2. Infinite series

Replace coefficients \( a, q, \alpha, \beta, \gamma, \delta \) and \( x \) by \( 1 - a, -q + (\delta - 1)y + (\alpha - \delta + 1)(\beta - \delta + 1), \alpha - \delta + 1, \beta - \delta + 1, 2 - \delta, \gamma \) and \( 1 - x \) into (A.2). Multiply \((1 - x)^{1-q}\) and (A.2) together. Put (2) into the new (A.2).

\[
(1 - \xi) \frac{d}{d\xi} y(\xi) = (1 - \xi)^{1/2} H \left[ \frac{1}{4} (1 + h) \rho^{-2} + (\alpha - 1)(\alpha + 2) \right] \left( \frac{\alpha}{2} + 1, -\frac{\alpha}{2} + 1, 3; 2, 2; 1 - \xi \right)
\]

\[
= (1 - \xi)^{1/2} \left\{ \sum_{a=0}^{\infty} \frac{\left( \frac{\alpha}{2} + \frac{1}{2} \right)_a \left( \frac{\beta}{2} + \frac{1}{2} \right)_a \left( \frac{\gamma}{2} - \frac{1}{2} \right)_a \left( \frac{\delta}{2} - \frac{1}{2} \right)_a \left( \frac{1}{2} \right)_a \left( \frac{1}{2} \right)_a}{\left( \frac{1}{2} \right)_a} \right\} \eta
\]

\[
+ \sum_{n=2}^{\infty} \sum_{a=0}^{\infty} \frac{i_n \left( \frac{\alpha}{2} + \frac{1}{2} \right)_a \left( \frac{\beta}{2} + \frac{1}{2} \right)_a \left( \frac{\gamma}{2} - \frac{1}{2} \right)_a \left( \frac{\delta}{2} - \frac{1}{2} \right)_a \left( \frac{1}{2} \right)_a \left( \frac{1}{2} \right)_a}{\left( \frac{1}{2} \right)_a} \eta
\]

\[
\times \prod_{k=1}^{n-1} \left\{ \sum_{l=0}^{\infty} \frac{\left( \frac{\alpha}{2} + \frac{1}{2} \right)_l \left( \frac{\beta}{2} + \frac{1}{2} \right)_l \left( \frac{\gamma}{2} - \frac{1}{2} \right)_l \left( \frac{\delta}{2} - \frac{1}{2} \right)_l \left( \frac{1}{2} \right)_l \left( \frac{1}{2} \right)_l}{\left( \frac{1}{2} \right)_l} \right\} \eta^l \right\}
\]

(A.10)

where

\[
Q = \frac{(1 + h) \rho^{-2} + (\alpha - 1)(\alpha + 2)}{8(2 - \rho^{-2})}
\]

On (A.9) and (A.10),

\[
\zeta = 1 - \xi
\]

\[
\eta = \frac{2 - \rho^{-2}}{1 - \rho^{-2}} \zeta
\]

\[
\zeta = \frac{1}{1 - \rho^{-1}}
\]

On (A.9) and (A.10),

\[
\zeta = 1 - \xi
\]

\[
\eta = \frac{2 - \rho^{-2}}{1 - \rho^{-2}} \zeta
\]

\[
\zeta = \frac{1}{1 - \rho^{-1}}
\]

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Lame equation in Weierstrass’s form

A.5. \( x^{-\alpha} HI \left( \frac{1}{a} q + a \alpha(\alpha - \gamma - \delta + 1) - \beta + \delta; \frac{1}{x} \right) \)

A.5.1. Infinite series

Replace coefficients \( a, q, \beta, \gamma \) and \( x \) by \( \frac{1}{x}, \frac{\alpha q(\alpha - \gamma - \delta + 1) - \beta + \delta}{a}, \alpha - \gamma + 1, \alpha - \beta + 1 \) and \( \frac{1}{x} \) into (A.2). Multiply \( x^{-\alpha} \) and (A.2) together. Put (2) into the new (A.2).

\[
\xi^{1/(a+1)} y(\zeta) = \xi^{1/(a+1)} H I \left( \rho^2, -\frac{1}{4} \left( h - (1 + \rho^2)(\alpha + 1)^2 \right); \frac{1}{2}(\alpha + 1), \frac{1}{2}(\alpha + 2), \alpha + \frac{3}{2}, \frac{1}{2}; \xi^{-1} \right)
\]

\[
= \xi^{1/(a+1)} \left( \sum_{n=0}^{\infty} \frac{\left( \frac{n}{n+1} \right)_{\infty} \left( \frac{\frac{n}{2} + \frac{1}{2}}{n+1} \right)_{\infty} \left( \frac{\frac{n}{2} + \frac{1}{2}}{n+1} \right)_{\infty}}{(1)_{n} \left( \frac{n}{n+1} \right)_{n}} \eta \right)
\]

\[
+ \sum_{n=2}^{\infty} \left( \sum_{i=0}^{n-1} \frac{\left( \frac{n}{n+1} \right)_{\infty} \left( \frac{\frac{n}{2} + \frac{1}{2}}{n+1} \right)_{\infty} \left( \frac{\frac{n}{2} + \frac{1}{2}}{n+1} \right)_{\infty}}{(1)_{n} \left( \frac{n}{n+1} \right)_{n}} \eta \right)
\]

\[
\times \prod_{i=1}^{n-1} \left( \sum_{i=0}^{\infty} \frac{\left( \frac{n}{n+1} \right)_{\infty} \left( \frac{\frac{n}{2} + \frac{1}{2}}{n+1} \right)_{\infty} \left( \frac{\frac{n}{2} + \frac{1}{2}}{n+1} \right)_{\infty}}{(1)_{n} \left( \frac{n}{n+1} \right)_{n}} \eta \right)
\]

\[
\times \sum_{i=0}^{\infty} \left( \frac{\frac{n}{n+1} \left( \frac{n}{n+1} \right)_{\infty} \left( \frac{\frac{n}{2} + \frac{1}{2}}{n+1} \right)_{\infty} \left( \frac{\frac{n}{2} + \frac{1}{2}}{n+1} \right)_{\infty}}{(1)_{n} \left( \frac{n}{n+1} \right)_{n}} \eta \right)
\]

(A.11)

where

\[
\begin{align*}
\zeta &= \xi^{-1} \\
\eta &= (1 + \rho^{-2}) \xi \\
z &= -\rho^{-2} \xi^2 \\
Q &= -\frac{1}{8} \left( (1 + \rho^2)^{-1} - (\alpha + 1)^2 \right)
\end{align*}
\]
A.6. \[
(1 - \frac{x}{a})^{-\beta} H[1 - a, -q + \gamma \beta; -\alpha + \gamma + \delta, \beta, \gamma, \delta; \frac{(1 - a)x}{x - a})
\]

A.6.1. Infinite series

Replace coefficients \(a, q, \alpha \) and \(x \) by \(-a, -q + \gamma \beta, -\alpha + \gamma + \delta + \frac{1-a}{x-a} \) into (A.2). Multiply
\((1 - \frac{x}{a})^{-\beta} \) and (A.2) together. Put (2) into the new (A.2).

\[
(1 - \rho^2 \xi^2) \gamma(x)
\]

\[
= (1 - \rho^2 \xi^2)\eta\left[1 - \rho^{-2}; \frac{1}{4} (k^2 - a); -\frac{\alpha}{2} + 1, -\frac{\alpha}{2} - 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} - \rho^{-2}\right]
\]

\[
= (1 - \rho^2 \xi^2)\eta\sum_{n=0}^{\infty} \left(\frac{\eta}{\xi} \right)_n (\xi)^n
\]

\[
+ \sum_{n=2}^{\infty} \left[\sum_{i=1}^{\infty} (\xi \eta) \left(\frac{\eta}{\xi} \right)^i (\xi)^i \right] \prod_{i=1}^{n} \left(\xi \eta \right)^{\frac{\alpha}{2}}
\]

A.7. \[
(1 - x)^{1-\delta} \left(1 - \frac{x}{a} \right)^{-\beta+\delta-1} H[1 - a, -q + \gamma((\delta - 1)a + \beta - \delta + 1); -\alpha + \gamma + 1 \]

A.7.1. Infinite series

Replace coefficients \(a, q, \alpha, \beta, \delta \) and \(x \) by \(-a, -q + \gamma((\delta - 1)a + \beta - \delta + 1), -\alpha + \gamma + 1, \beta - \delta + 1, 2 - \delta \) and \(\frac{1-a}{x-a} \) into (A.2). Multiply \((1 - x)^{1-\delta} \left(1 - \frac{x}{a} \right)^{-\beta+\delta-1} \) and (A.2) together. Put
Lame equation in Weierstrass’s form

\[
(1 - \xi) \frac{\alpha^2}{2 \xi} \frac{(1 - \rho^2 \xi)^{r/2}}{\xi} y(\xi) = (1 - \xi) \frac{\alpha^2}{2 \xi} \frac{(1 - \rho^2 \xi)^{r/2}}{\xi} H(1 - \rho^{-2}, \frac{1}{4}((h - 1)\rho^{-2} + 1 - \alpha); \frac{\alpha}{2} + 1, -\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{1 - \rho^2 \xi}{\xi})
\]

\[
= (1 - \xi) \frac{\alpha^2}{2 \xi} \frac{(1 - \rho^2 \xi)^{r/2}}{\xi} \left\{ \sum_{i=0}^{\infty} \left( \frac{-\alpha^2 + \frac{1}{2}}{1} \right) \left( \frac{-\alpha^2 + \frac{3}{2}}{1} \right) \left( \frac{1 + \frac{1}{2}}{1} \right) \left( \frac{1 + \frac{3}{2}}{1} \right) \right\} \eta^i
\]

\[
+ \sum_{i=0}^{\infty} \left\{ \sum_{i=0}^{\infty} \left( \frac{-\alpha^2 + \frac{1}{2}}{1} \right) \left( \frac{-\alpha^2 + \frac{3}{2}}{1} \right) \left( \frac{1 + \frac{1}{2}}{1} \right) \left( \frac{1 + \frac{3}{2}}{1} \right) \right\} \eta^i
\]

\[
\times \prod_{k=1}^{n-1} \left\{ \sum_{i=0}^{\infty} \left( \frac{-\alpha^2 + \frac{1}{2}}{1} \right) \left( \frac{-\alpha^2 + \frac{3}{2}}{1} \right) \left( \frac{1 + \frac{1}{2}}{1} \right) \left( \frac{1 + \frac{3}{2}}{1} \right) \right\} \eta^i
\]

where

\[
\begin{align*}
\xi & = \frac{(1 - \rho^2 \xi)}{\rho^2}, \\
\eta & = \frac{\alpha^2}{2 \rho^2}, \\
\gamma & = \frac{1}{1 - \rho^2 \xi}, \\
\Gamma_0 & = \frac{-\alpha^2 + \frac{1}{2}}{2(\rho^2 - 1)}, \\
\Gamma_k & = \frac{1}{2} - \frac{\alpha^2 + \frac{3}{2}}{2(\rho^2 - 1)}, \\
Q & = \frac{(1 - \rho^2 \xi)^{r/2}}{\rho^2}
\end{align*}
\]
Lame equation in Weierstrass' form

A.8. $x^{-\alpha} H_l \left( \frac{a - 1}{a}, -q + \alpha (\delta a + \beta - \delta), a, \alpha - \gamma + 1, \delta, \alpha - \beta + 1; \frac{x - 1}{x} \right)$

A.8.1. Infinite series

Replace coefficients $a$, $q$, $\beta$, $\gamma$, $\delta$ and $x$ by $\frac{4}{\pi} \sqrt{\frac{-q + \alpha (\delta a + \beta - \delta)}{\alpha - \gamma + 1, \delta, \alpha - \beta + 1}}$ into (A.2). Multiply $x^{-\alpha}$ and (A.2) together. Put (A.2) into the new (A.2).

\[
\xi^{-\frac{1}{2}(\alpha+1)} \gamma(\xi) = \xi^{-\frac{1}{2}(\alpha+1)} H_l \left( 1 - \rho^2, \frac{1}{4} \left[ h + (\alpha + 1) \left( 1 - (\alpha + 1) \rho^2 \right) \right]; \frac{1}{2}(\alpha + 1), \frac{1}{2}(\alpha + 2), \frac{1}{2}, \alpha + 3; \frac{\xi - 1}{\xi} \right)
\]

\[
= \xi^{-\frac{1}{2}(\alpha+1)} \left\{ \sum_{n=0}^{\infty} \left( \frac{\frac{4}{\pi} \sqrt{\frac{-q + \alpha (\delta a + \beta - \delta)}}}{1} \right) \left( \frac{\frac{4}{\pi} \sqrt{\frac{-q + \alpha (\delta a + \beta - \delta)}}}{1} \right)^n \right\} \eta
\]

\[
= \xi^{-\frac{1}{2}(\alpha+1)} \left\{ \sum_{n=0}^{\infty} \left( \frac{\frac{4}{\pi} \sqrt{\frac{-q + \alpha (\delta a + \beta - \delta)}}}{1} \right) \left( \frac{\frac{4}{\pi} \sqrt{\frac{-q + \alpha (\delta a + \beta - \delta)}}}{1} \right)^n \right\} \eta
\]

where

\[
\begin{align*}
\xi &= \frac{\xi - 1}{\xi} \\
\eta &= \frac{\xi - 1}{\xi} \xi \\
z &= \frac{1}{(1 - \rho^2)(\alpha + 1)} \\
\Gamma_0 &= \frac{1}{2(1 - \rho^2)(\alpha + 1)} \\
\Gamma_k &= \frac{1}{2(1 - \rho^2)(\alpha + 1)} \\
\eta &= \frac{h(\alpha + 1) \xi^{\frac{1}{2}(\alpha + 1)}}{8(2 - \rho^2)}
\end{align*}
\]
A.9. Infinite series

Multiply Lame equation in Weierstrass’s form (B.1) is geometric series. The condition of convergence of (B.1) is given by

\[ \left| \frac{1}{a} x^2 + \frac{1}{a} \right| < 1 \]  

In Ref. [17], an asymptotic representation of Heun function about \( (\alpha, \beta, \gamma, \delta) \) is given by

\[ \begin{align*}
\left( \xi - \rho^{-2} \right)^{\frac{1}{2}(\alpha+1)} y(\xi) &= \frac{\xi - \rho^{-2}}{1 - \rho^{-2}} H_1 \left( \rho^{-\frac{1}{2}}, -\frac{1}{4} (h \rho^{-2} - (\alpha + 1)^2); \frac{1}{2} (\alpha + 1), -\frac{1}{2} (\alpha - 2), \frac{1}{2}, \frac{1}{2} \rho^2 (\xi - \rho^{-2}) \right) \\
&= \frac{\xi - \rho^{-2}}{1 - \rho^{-2}} \left\{ \sum_{k=0}^{\infty} \left( \frac{2k + 1}{2} \right) \xi^k \right\} \eta \\
&+ \sum_{\alpha=0}^{n-1} \sum_{k=0}^{\infty} \left( \frac{2k + 1}{2} \right) \xi^k \
\times \prod_{k=1}^{n-1} \left[ \sum_{\alpha=0}^{\infty} \left( \frac{2k + 1}{2} \right) \xi^k \right] \left( \frac{2k + 1}{2} \right) \xi^k \\
\times \prod_{\alpha=0}^{n-1} \left[ \sum_{k=0}^{\infty} \left( \frac{2k + 1}{2} \right) \xi^k \right] \left( \frac{2k + 1}{2} \right) \xi^k \right\} \eta^n \right\} (A.15)
\end{align*} \]

where

\[ \begin{align*}
\xi &= \frac{\xi - \rho^{-2}}{\rho^2 (\xi - \rho^{-2})} \\
\eta &= (1 + \rho^2) \xi \\
\zeta &= -\rho^2 \xi^2 \\
\gamma_0 &= -\frac{\alpha + 1}{2(1 + \rho^2)} \\
\gamma_k &= \frac{a}{\pi} + \frac{a + 1}{8(1 + \rho^2)}
\end{align*} \]

B. Asymptotic behavior

In Ref. [17], an asymptotic representation of Heun function about \( x = 0 \) for infinite series is given by

\[ \lim_{n \to \infty} H_1 (a, q; \alpha, \beta, \gamma, \delta; x) = \frac{1}{1 - \left( \frac{a}{Q} x \right)} \]  

(B.1) is geometric series. The condition of convergence of (B.1) is

\[ \left| \frac{1}{a} x^2 + \frac{1}{a} \right| < 1 \]
Lame equation in Weierstrass’s form

For \( a \approx -1 \), an asymptotic approximation of Heun function is given by

\[
\lim_{n \to \infty} H_l(a, q; \alpha, \beta, \gamma, \delta; x) = \frac{1 + x}{1 + \frac{1}{a}x^2}
\]  
(B.3)

The condition of convergence of (B.3) is

\[
\left| \frac{1}{a} \right| < 1
\]  
(B.4)

For \( |a| \gg 1 \), an asymptotic expansion of Heun function is given by

\[
\lim_{n \to \infty} \left| \frac{a}{n} \right| \gg 1 
H_l(a, q; \alpha, \beta, \gamma, \delta; x) = \frac{1}{1 - \frac{1}{a}x}
\]  
(B.5)

(B.5) is binomial series. The condition of convergence of (B.5) is

\[
\left| \frac{1}{a} \right| < 1
\]  
(B.6)

B.1. \((1 - x)^{1-\delta} H_l(a, q - (\delta - 1)y; \alpha - \delta + 1, \beta - \delta + 1, \gamma, 2 - \delta; x)\)

Replace coefficients \( q, \alpha, \beta \) and \( \delta \) by \( q - (\delta - 1)y, \alpha - \delta + 1, \beta - \delta + 1 \) and \( 2 - \delta \) into (B.1), (B.2), (B.5) and (B.6). Put (2) into the new (B.1), (B.2), (B.5) and (B.6).

For infinite series,

\[
\lim_{n \to \infty} H_l\left(\rho^{-2}, -\frac{1}{4}(h - 1)\rho^{-2}; \frac{\alpha}{2} + 1, -\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}; \xi\right) = \frac{1}{1 - (-\rho^2\text{sn}^2(z, \rho) - (1 + \rho^2)\text{sn}^2(z, \rho))}
\]  
(B.7)

The condition of convergence of (B.7) is

\[
\left| -\rho^2\text{sn}^2(z, \rho) - (1 + \rho^2)\text{sn}^2(z, \rho) \right| < 1
\]  
(B.8)

As \( 0 < \rho < 1 \) in (B.8),

\[
0 \leq \text{sn}^2(z, \rho) < 1
\]

For \( \rho \approx 0 \),

\[
\lim_{\rho \to 0} H_l\left(\rho^{-2}, -\frac{1}{4}(h - 1)\rho^{-2}; \frac{\alpha}{2} + 1, -\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}; \xi\right) = \frac{1}{1 - (1 + \rho^2)\text{sn}^2(z, \rho)}
\]  
(B.9)

The condition of convergence of (B.9) is

\[
0 \leq \text{sn}^2(z, \rho) < \frac{1}{1 + \rho^2}
\]  
(B.10)
The Lame equation in Weierstrass’s form

\[ x^{\gamma-1}(1-x)^{\beta+1}H\ell(\alpha,q - (\gamma + \delta - 2)a - (\gamma - 1)(\alpha + \beta - \gamma - \delta + 1); \alpha - \gamma - \delta + 2, \beta - \gamma - \delta + 2, 2 - \gamma, 2 - \delta, x) \]

Replace coefficients \( q, \alpha, \beta, \gamma \) and \( \delta \) by \( q - (\gamma + \delta - 2)a - (\gamma - 1)(\alpha + \beta - \gamma - \delta + 1), \alpha - \gamma - \delta + 2, \beta - \gamma - \delta + 2, 2 - \gamma \) and \( 2 - \delta \) into (B.1), (B.2), (B.5) and (B.6). Put (\ref{eq:2}) into the new (B.1), (B.2), (B.5) and (B.6).

For infinite series,

\[
\lim_{n \to 1} H\ell\left( \rho^{-2}, -\frac{1}{4}\left((h - 4)\rho^{-2} - 1\right); \frac{\alpha}{2}, \frac{3}{2}, \frac{\alpha}{2}, \frac{3}{2}; 1; \frac{3}{2}, \frac{\xi}{2}, \frac{\xi}{2} \right) = \frac{1}{1 - (\rho^2 sn^2(z, \rho) - (1 + \rho^2)sn^2(z, \rho))} \tag{B.11}
\]

The condition of convergence of (B.11) is

\[
\left| -\rho^2 sn^4(z, \rho) - (1 + \rho^2)sn^2(z, \rho) \right| < 1 \tag{B.12}
\]

As \( 0 < \rho < 1 \) in (B.12),

\[
0 \leq sn^2(z, \rho) < 1
\]

For \( \rho \approx 0 \),

\[
\lim_{\rho \to 0} H\ell\left( \rho^{-2}, -\frac{1}{4}\left((h - 4)\rho^{-2} - 1\right); \frac{\alpha}{2}, \frac{3}{2}, \frac{\alpha}{2}, \frac{3}{2}; 1; \frac{3}{2}, \frac{\xi}{2}, \frac{\xi}{2} \right) = \frac{1}{1 - (1 + \rho^2)sn^2(z, \rho)} \tag{B.13}
\]

The condition of convergence of (B.13) is

\[
0 \leq sn^2(z, \rho) < \frac{1}{1 + \rho^2} \tag{B.14}
\]

\[ B.3. \ H\ell(1 - a, -q + a\beta; \alpha, \beta, \gamma, 1 - x) \]

Replace coefficients \( a, q, \gamma, \delta \) and \( x \) by \( 1 - a, -q + a\beta, \gamma \) and \( 1 - x \) into (B.1)–(B.6). Put (\ref{eq:2}) into the new (B.1)–(B.6).

For infinite series,

\[
\lim_{n \to 1} H\ell\left( 1 - \rho^{-2}, -\frac{1}{4}\left(h\rho^{-2} - \alpha(\alpha + 1)\right); \frac{1}{2}(\alpha + 1), -\frac{\alpha}{2}, \frac{1}{2}, \frac{1}{2}, 1 - \xi \right) = \frac{1}{1 - \left( \frac{(1 - sn^2(z, \rho))^2}{1 - \rho^2} + \frac{2 - \rho^2}{1 - \rho^2}(1 - sn^2(z, \rho)) \right)} \tag{B.15}
\]

The condition of convergence of (B.15) is

\[
\left| \frac{(1 - sn^2(z, \rho))^2}{1 - \rho^2} + \frac{2 - \rho^2}{1 - \rho^2}(1 - sn^2(z, \rho)) \right| < 1 \tag{B.16}
\]

For \( \rho \approx 2^{-\frac{1}{2}} \),

\[
\lim_{\rho \to 2^{-\frac{1}{2}}} H\ell\left( 1 - \rho^{-2}, -\frac{1}{4}\left(h\rho^{-2} - \alpha(\alpha + 1)\right); \frac{1}{2}(\alpha + 1), -\frac{\alpha}{2}, \frac{1}{2}, \frac{1}{2}, 1 - \xi \right) = \frac{2 - sn^2(z, \rho)}{1 + \frac{(1 - sn^2(z, \rho))^2}{1 - \rho^2}} \tag{B.17}
\]
Lame equation in Weierstrass’s form

The condition of convergence of (B.17) is

\[ \left| \frac{1 - sn^2(z, \rho)}{1 - \rho^{-2}} \right| < 1 \quad \text{(B.18)} \]

For \( \rho \approx 0 \),

\[ \lim_{\substack{n \to \infty \\left| z \right| > 1}} H_l \left( 1 - \rho^{-2}, -\frac{1}{4} \left( hp^{-2} - \alpha(\alpha + 1) \right); \frac{1}{2}(\alpha + 1), -\frac{\alpha}{2}, \frac{1}{2}, \frac{1}{2}, 1 - \xi \right) \]

\[ = \frac{1}{1 - \frac{2p^{-2}}{1-p^{-2}} (1 - sn^2(z, \rho))} \quad \text{(B.19)} \]

The condition of convergence of (B.19) is

\[ \left| \frac{2 - \rho^{-2}}{1 - \rho^{-2}} (1 - sn^2(z, \rho)) \right| < 1 \quad \text{(B.20)} \]

**B.4.** \((1 - x)^{-\delta} H_l(1 - a, -q + (\delta - 1)\gamma a + (\alpha - \delta + 1)(\beta - \delta + 1); \alpha - \delta + 1, \beta - \delta + 1, 1 - x)\)

Replace coefficients \( a, q, \alpha, \beta, \gamma, \delta \) and \( x \) by \( 1 - a, -q + (\delta - 1)\gamma a + (\alpha - \delta + 1)(\beta - \delta + 1), \alpha - \delta + 1, \beta - \delta + 1, 2 - \delta, \gamma \) and \( 1 - x \) into (B.1)–(B.6). Put (2) into the new (B.1)–(B.6).

For infinite series,

\[ \lim_{n \to 1} H_l \left( 1 - \rho^{-2}, -\frac{1}{4} \left( (1 + h)p^{-2} + (\alpha - 1)(\alpha + 2) \right); \frac{\alpha}{2} + 1, -\frac{\alpha}{2}, \frac{1}{2}, 2 \frac{1}{2}, 1 - \xi \right) \]

\[ = \frac{1}{1 - \left( \frac{1 - sn^2(z, \rho)^2}{1 - \rho^{-2}} + \frac{2 - \rho^{-2}}{1 - \rho^{-2}} (1 - sn^2(z, \rho)) \right)} \quad \text{(B.21)} \]

The condition of convergence of (B.21) is

\[ \left| \frac{(1 - sn^2(z, \rho)^2)}{1 - \rho^{-2}} + \frac{2 - \rho^{-2}}{1 - \rho^{-2}} (1 - sn^2(z, \rho)) \right| < 1 \quad \text{(B.22)} \]

For \( \rho \approx 2^{-\frac{1}{2}} \),

\[ \lim_{\substack{n \to \infty \\rho \approx 2^{-\frac{1}{2}}} H_l \left( 1 - \rho^{-2}, -\frac{1}{4} \left( (1 + h)p^{-2} + (\alpha - 1)(\alpha + 2) \right); \frac{\alpha}{2} + 1, -\frac{\alpha}{2}, \frac{1}{2}, 2 \frac{1}{2}, 1 - \xi \right) \]

\[ = \frac{2 - sn^2(z, \rho)}{1 + \frac{1 - sn^2(z, \rho)^2}{1 - \rho^{-2}}} \quad \text{(B.23)} \]

The condition of convergence of (B.23) is

\[ \left| \frac{(1 - sn^2(z, \rho)^2)}{1 - \rho^{-2}} \right| < 1 \quad \text{(B.24)} \]
Lame equation in Weierstrass’s form

For $\rho \approx 0$,

$$
\lim_{\rho \to 0} H_l \left( 1 - \rho^{-2}, -\frac{1}{4} \left( (1 + h)\rho^{-2} + (\alpha - 1)(\alpha + 2) \right); \frac{\alpha}{2} + 1, -\frac{\alpha}{2} + 1, 2; 2; 1 - \xi \right)
= \frac{1}{1 - 2\rho^{-2}(1 - sn^2(z, \rho))}
$$

(B.25)

The condition of convergence of (B.25) is

$$
\left| \frac{2 - \rho^{-2}}{1 - \rho^{-2}}(1 - sn^2(z, \rho)) \right| < 1
$$

(B.26)

B.5. $x^{-\alpha} H_l \left( \frac{1}{a}, \frac{q + \alpha((\alpha - \gamma - \delta + 1)\alpha - \beta + \delta)}{a}; \alpha, \alpha - \gamma + 1, \alpha - \beta + 1, \delta; \frac{1}{x} \right)$

Replace coefficients $a, q, \beta, \gamma$ and $x$ by $\frac{1}{a}, \frac{q + \alpha((\alpha - \gamma - \delta + 1)\alpha - \beta + \delta)}{a}$, $\alpha - \gamma + 1, \alpha - \beta + 1$ and $\frac{1}{x}$ into (B.1) and (B.2). Put (2) into the new (B.1) and (B.2).

For infinite series,

$$
\lim_{n \to 1} H_l \left( \rho^2, -\frac{1}{4} \left( h + (1 + \rho^2)(\alpha + 1) \right); \frac{\alpha}{2} + 1, \frac{1}{2}(\alpha + 2), \alpha + \frac{3}{2}, \frac{1}{2}; \xi^{-1} \right)
= \frac{1}{1 - (-\rho^{-2}sn^{-4}(z, \rho) + (1 + \rho^{-2})sn^{-2}(z, \rho))}
$$

(B.27)

The condition of convergence of (B.27) is

$$
\left| -\rho^{-2}sn^{-4}(z, \rho) + (1 + \rho^{-2})sn^{-2}(z, \rho) \right| < 1
$$

(B.28)

B.6. $\left( 1 - \frac{x}{a} \right)^{-\beta} H_l \left( 1 - a, -q + \gamma \beta; -\alpha + \gamma + \delta, \beta, \gamma; \frac{(1 - a)x}{x - a} \right)$

Replace coefficients $a, q, \alpha$ and $x$ by $1 - a, -q + \gamma \beta, -\alpha + \gamma + \delta$ and $\frac{(1 - a)x}{x - a}$ into (B.1)–(B.6).

Put (2) into the new (B.1)–(B.6).

For infinite series,

$$
\lim_{n \to 1} H_l \left( 1 - \rho^{-2}, -\frac{1}{4} \left( h + \rho^{-2} - \alpha \right); \frac{\alpha}{2} + 1, \frac{1}{2} - \frac{\alpha}{2} + 1, \frac{1}{2}; \frac{1 - \rho^{-2} \xi}{\xi - \rho^{-2}} \right)
= \frac{1}{1 - \left( \frac{1 - \rho^{-2}sn^4(z, \rho)}{(sn^2(z, \rho) - \rho^{-2})^2} + \frac{(2 - \rho^{-2})sn^2(z, \rho)}{sn^2(z, \rho) - \rho^{-2}} \right)}
$$

(B.29)

The condition of convergence of (B.29) is

$$
\left| \frac{(1 - \rho^{-2})sn^4(z, \rho)}{(sn^2(z, \rho) - \rho^{-2})^2} + \frac{(2 - \rho^{-2})sn^2(z, \rho)}{sn^2(z, \rho) - \rho^{-2}} \right| < 1
$$

(B.30)

For $\rho \approx 2^{-\frac{1}{2}}$,

$$
\lim_{\rho \to 2^{-\frac{1}{2}}} H_l \left( 1 - \rho^{-2}, -\frac{1}{4} \left( h + \rho^{-2} - \alpha \right); \frac{\alpha}{2} + 1, \frac{1}{2} - \frac{\alpha}{2} + 1, \frac{1}{2}; \frac{1 - \rho^{-2} \xi}{\xi - \rho^{-2}} \right) = \frac{1 + \frac{(1 - \rho^{-2})sn^2(z, \rho)}{(sn^2(z, \rho) - \rho^{-2})^2}}{1 + \frac{(1 - \rho^{-2})sn^2(z, \rho)}{(sn^2(z, \rho) - \rho^{-2})^2}}
$$

(B.31)
Lame equation in Weierstrass’s form

The condition of convergence of (B.31) is

$$\left| \frac{(1 - \rho^{-2})sn^4(z, \rho)}{(sn^2(z, \rho) - \rho^{-2})^2} \right| < 1$$  \hspace{1cm} (B.32)

For $\rho \approx 0$,

$$\lim_{n \to 1} \frac{HI(1 - \rho^{-2}, 1/4 (h \rho^{-2} - \alpha) ; -\alpha/2 + 1/2, -\alpha/2 + 1/2, -\alpha/2 + 1/2, (1 - \rho^{-2})\xi)}{1 - \frac{1}{(2 - \rho^{-2})sn^2(z, \rho)}} = \frac{1}{1 - \frac{1}{(2 - \rho^{-2})sn^2(z, \rho)}}$$ \hspace{1cm} (B.33)

The condition of convergence of (B.33) is

$$\left| \frac{(2 - \rho^{-2})sn^2(z, \rho)}{sn^2(z, \rho) - \rho^{-2}} \right| < 1$$ \hspace{1cm} (B.34)

\textbf{B.7.} $(1 - x)^{1-\delta} \left( 1 - \frac{x}{a} \right)^{-\beta+\delta+1} HI \left( 1 - a,-q + \gamma(\delta - 1)\alpha + \beta - 1 + 1, -\alpha + 1 \right.$

$$\left. - \beta - 1, 2 - \delta, (1 - a)x; \frac{(1 - a)x}{x - a} \right)$$

Replace coefficients $a, q, \alpha, \beta, \delta$ and $x$ by $1 - a, -q + \gamma(\delta - 1)\alpha + \beta - 1 + 1, -\alpha + 1$, $\beta - 1, 2 - \delta$ and $\frac{(1 - a)x}{x - a}$ into (B.1)–(B.6). Put (2) into the new (B.1)–(B.6).

For infinite series,

$$\lim_{n \to 1} \frac{HI(1 - \rho^{-2}, 1/4 (h - 1)\rho^{-2} + 1 - \alpha) ; -\alpha/2 + 1, -\alpha/2 + 1/2, -\alpha/2 + 3/2, (1 - \rho^{-2})\xi)}{1 - \frac{1}{(2 - \rho^{-2})sn^2(z, \rho)}}$$ \hspace{1cm} (B.35)

The condition of convergence of (B.35) is

$$\left| \frac{(1 - \rho^{-2})sn^4(z, \rho)}{(sn^2(z, \rho) - \rho^{-2})^2} + \frac{1}{(2 - \rho^{-2})sn^2(z, \rho)} \right| < 1$$  \hspace{1cm} (B.36)

For $\rho \approx 2^{-\frac{1}{4}}$,

$$\lim_{n \to 1} \frac{HI(1 - \rho^{-2}, 1/4 (h - 1)\rho^{-2} + 1 - \alpha) ; -\alpha/2 + 1, -\alpha/2 + 1/2, -\alpha/2 + 3/2, (1 - \rho^{-2})\xi)}{1 + \frac{(1 - \rho^{-2})sn^2(z, \rho)}{(sn^2(z, \rho) - \rho^{-2})^2}}$$ \hspace{1cm} (B.37)

The condition of convergence of (B.37) is

$$\left| \frac{(1 - \rho^{-2})sn^4(z, \rho)}{(sn^2(z, \rho) - \rho^{-2})^2} \right| < 1$$  \hspace{1cm} (B.38)

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Lame equation in Weierstrass’s form

For \( \rho \approx 0 \),

\[
\lim_{n \to 1 \rho} Hl \left( 1 - \rho^2, \frac{1}{4} \left( (h-1)\rho^2 + 1 - \alpha \right) ; -\frac{\alpha}{2} + 1, -\frac{\alpha}{2} \pm \frac{3}{2}, \frac{(1-\rho^2)\xi}{\xi - \rho^2} \right)
\]

\[
= \frac{1}{1 - \left( \frac{(2 - \rho^2)\sin(\zeta, \rho)}{\sin^2(\zeta, \rho) - \rho^2} \right)}
\]

(B.39)

The condition of convergence of (B.39) is

\[
\left| \frac{(2 - \rho^2)\sin^2(\zeta, \rho)}{\sin^2(\zeta, \rho) - \rho^2} \right| < 1
\]

(B.40)

B.8. \( x^{-a} Hl \left( \frac{a-1}{a}, -\frac{q + a(\alpha + \beta - \delta)}{a} ; \alpha - \gamma + 1, \delta, \alpha - \beta + 1, \frac{x - 1}{x} \right) \)

Replace coefficients \( a, q, \alpha, \beta, \gamma, \delta \) and \( x \) by \( a^{-1}, -\frac{q + a(\alpha + \beta - \delta)}{a}, \alpha - \gamma + 1, \delta, \alpha - \beta + 1 \) and \( \frac{x - 1}{x} \) into (B.1) and (B.2). Put (2) into the new (B.1) and (B.2).

For infinite series,

\[
\lim_{n \to 1 \rho} Hl \left( 1 - \rho^2, \frac{1}{4} \left[ h + (a + 1) \left( 1 - (a + 1)\rho^2 \right) \right] ; \frac{1}{2}(\alpha + 1), \frac{1}{2}(\alpha + 2), \frac{1}{2}, \frac{3}{2}, \frac{\xi - 1}{\xi} \right)
\]

\[
= \frac{1}{1 - \left( \frac{(2 - \rho^2)\sin^2(\zeta, \rho)}{\sin^2(\zeta, \rho) - \rho^2} \right)}
\]

(B.41)

The condition of convergence of (B.41) is

\[
\left| \frac{1}{1 - \rho^2 \left( \frac{\sin^2(\zeta, \rho) - 1}{\sin^2(\zeta, \rho)} \right)^2 + \frac{2 - \rho^2}{1 - \rho^2} \left( \frac{\sin^2(\zeta, \rho) - 1}{\sin^2(\zeta, \rho)} \right)} \right| < 1
\]

(B.42)

B.9. \( \left( \frac{x - a}{1 - a} \right)^{-a} Hl \left( a, q - (\beta - \delta)a, \alpha, -\beta + \gamma + \delta, \delta, \gamma, \frac{a(\xi - 1)}{x - a} \right) \)

Replace coefficients \( q, \beta, \gamma, \delta \) and \( x \) by \( q - (\beta - \delta)a, -\beta + \gamma + \delta, \delta, \gamma \) and \( \frac{a(\xi - 1)}{x - a} \) into (B.1), (B.2), (B.5) and (B.6). Put (2) into the new (B.1), (B.2), (B.5) and (B.6).

For infinite series,

\[
\lim_{n \to 1 \rho} Hl \left( \rho^2, \frac{1}{4} \left( h\rho^2 - (a + 1)\rho^2 \right) ; \frac{1}{2}(\alpha + 1), \frac{1}{2}(\alpha + 2), \frac{1}{2}, \frac{1}{2}, \frac{\xi - 1}{\rho^2(\xi - \rho^2)} \right)
\]

\[
= \frac{1}{1 - \left( \frac{(\sin^2(\zeta, \rho) - 1)^2}{\rho^2(\sin^2(\zeta, \rho) - \rho^2)^2} + \frac{(1 + \rho^2)(\sin^2(\zeta, \rho) - 1)}{\rho^2(\sin^2(\zeta, \rho) - \rho^2)} \right)}
\]

(B.43)

The condition of convergence of (B.43) is

\[
\left| \frac{(\sin^2(\zeta, \rho) - 1)^2}{\rho^2(\sin^2(\zeta, \rho) - \rho^2)^2} + \frac{(1 + \rho^2)(\sin^2(\zeta, \rho) - 1)}{\rho^2(\sin^2(\zeta, \rho) - \rho^2)} \right| < 1
\]

(B.44)
Lame equation in Weierstrass’s form

For $\rho \approx 0$,

$$\lim_{\rho \to 0} H\left(\rho^{-2} - \frac{1}{4}(\rho^{-2} - (\alpha + 1)^2) ; \frac{1}{2}(\alpha + 1), -\frac{1}{2}(\alpha - 2), \frac{1}{2}, \frac{1}{2}, \frac{\xi - 1}{\rho^2(\xi - \rho^{-2})}\right)$$

$$\approx \frac{1}{1 - \frac{(1+\rho^2)(sn(z, \rho) - 1)}{\rho^2(sn(z, \rho) - \rho^{-2})}} \tag{B.45}$$

The condition of convergence of (B.45) is

$$\frac{|(1 + \rho^2)(sn(z, \rho) - 1)|}{\rho^2(sn(z, \rho) - \rho^{-2})} < 1 \tag{B.46}$$

C. Integral representation

In Ref. [17], the integral representation of Heun equation about $x = 0$ of the first kind for polynomial of type 1 as $\alpha = -2a_j - j$ where $j, a_j = 0, 1, 2, \ldots$ is given by

$$y(x) = H_{\alpha, \beta} \left(\alpha_j = -\frac{1}{2}(\alpha + j) \big| \big| \eta \right) \eta = \frac{(1 + \eta)}{a} x ; z = \frac{1}{a} x^2$$

$$= \binom{a_0}{\beta/2, 1/2} \frac{1}{1 + \frac{1}{2} - \eta} \sum_{n=1}^{\infty} \left\{ \left( \prod_{k=0}^{n-1} \left( \int_{0}^{1} dt_{n-k} t_{n-k}^{(a-k-2)} \right) \right) \sum_{k=0}^{n} \left( \int_{0}^{1} du_{n-k} u_{n-k}^{(a-k-1)} \right) \right\}$$

$$\times \left( \frac{1}{2\pi i} \int dv_{n-k} \left( 1 - \frac{1}{v_{n-k}} \right)^{\frac{1}{2n}} \left( 1 - \frac{w_{n-k+1,n} v_{n-k}(1 - t_{n-k})(1 - u_{n-k})}{w_{n-k,n} v_{n-k}} \right)^{\frac{1}{2}(\alpha + \beta)} \right)$$

$$\times \left( \frac{w_{n-k,n}}{w_{n-k,0}} \left( - \frac{w_{n-k,n} \partial w_{n-k,n}}{w_{n-k,n}} + \sum_{k=0}^{n} \left( \partial w_{n-k,n} \right) \right) + Q \right)$$

$$\times \left( \frac{w_{n-k,n}}{w_{n-k,0}} \left( - \frac{w_{n-k,n} \partial w_{n-k,n}}{w_{n-k,n}} + \sum_{k=0}^{n} \left( \partial w_{n-k,n} \right) \right) + Q \right) \tag{C.1}$$

where

$$\left\{ \begin{array}{l}
z = \frac{1}{a} x^2 \\
\eta = \frac{1}{a} x \\
\alpha_i \leq \alpha_j \text{ only if } i \leq j \text{ where } i, j = 0, 1, 2, \ldots \end{array} \right.$$

and

$$\left\{ \begin{array}{l}
\Omega_{n-k}^{(S)} = \frac{1}{2\pi i} \left( - 2\alpha_{n-k-1} + \beta - \delta + a(\delta + n - k - 2) \right) \\
Q = \frac{1}{2\pi i} \end{array} \right.$$

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Lame equation in Weierstrass’s form

In Ref. [17], the integral representation of Heun equation about \( x = 0 \) of the first kind for infinite series is given by

\[
y(x) = H_{\alpha, \beta}(\eta) = \left(1 + \frac{a}{x}\right) e^{\frac{\eta}{x}} z = -\frac{1}{a} x^2
\]

\[
y = 2F_1\left(\alpha, \beta, \frac{1}{2}, \frac{1}{2}; z\right) + \sum_{n=1}^{\infty} \left\{ \int_0^1 dt_{n-k} u_{n-k}^{\frac{1}{2}((n-k)-2)} \right\}
\]

\[
\times \frac{1}{2\pi i} \oint d\nu_{n-k} \left(1 - \frac{1}{\nu_{n-k}}\right) \nu_{n-k}^{\frac{1}{2}((n-k)+\alpha)} \left(1 - \nu_{n-k+1} \nu_{n-k}(1 - u_{n-k})(1 - u_{n-k}^\beta)\right)
\]

\[
\times \left\{ w_{-n-k, n}^{\frac{1}{2}((n-k)-1)} w_{n-k, \nu_{n-k}}^{\frac{1}{2}((n-k)-1)} \left( \frac{w_{n-k, \nu_{n-k}} \partial w_{n-k, \nu_{n-k}} + \Omega_{n-k-1}^{(f)} + Q}\right)\right\}
\]

\[
\times 2F_1\left(\alpha, \beta, \frac{1}{2}, \frac{1}{2}; w_{n-k, \nu_{n-k}}\right)\right\} \eta^n
\]

(C.2)

where

\[
\begin{align*}
\Omega_{n-k-1}^{(f)} &= \frac{1}{\xi} \left(\alpha + \beta - \delta + n - k - 1 + a(\delta + \gamma + n - k - 2)\right) \\
Q &= \frac{1}{\xi(1 + a)}
\end{align*}
\]

On (C.1) and (C.2).

\[
\omega_{i,j} = \left\{ \frac{v_j}{(\nu_{j-1} - \nu_j)(1 - \nu_{j+1})\nu_j(1 - u_i)} \right\} \text{ if only } i > j
\]

(C.1.1) Polynomial of type 1

Replace coefficients \( q, \alpha, \beta \) and \( \delta \) by \( q - (\delta - 1)\gamma \alpha, \alpha - \delta + 1, \beta - \delta + 1, \gamma, 2 - \delta; \alpha \)

(C.1.1.1) Multiply \((1 - \chi)^{1-\delta} H(\alpha, q - (\delta - 1)\gamma \alpha; \alpha - \delta + 1, \beta - \delta + 1, \gamma, 2 - \delta; \chi)\)

(C.3)
where

$$\alpha = 2 \left( 2\alpha_j + j + \frac{1}{2} \right) \text{ or } -2 \left( 2\alpha_j + j + 1 \right)$$

C.1.2. Infinite series

Replace coefficients $q, \alpha, \beta$ and $\delta$ by $q - (\delta - 1)\gamma a, \alpha - \delta + 1, \beta - \delta + 1$ and $2 - \delta$ into (C.2). Multiply $(1 - x)^{1-\delta}$ and (C.2) together. Put (2) into the new (C.2).

\[
(1 - \xi)^{\delta} y(\xi) = (1 - \xi)^{\delta} H_1 \left( \rho^{-2}, -\frac{1}{4}(h - 1)\rho^{-2}; \frac{\alpha}{2} + 1, \frac{\alpha}{2} + \frac{1}{2}, -\frac{1}{2}, \frac{3}{2} \xi \right)
\]

\[
= (1 - \xi)^{\delta} \left\{ 2F_1 \left( \frac{\alpha}{4} + \frac{1}{2}, -\frac{\alpha}{4} + \frac{1}{4}, \frac{3}{4}; \frac{\xi}{\gamma} \right) + \sum_{n=1}^{\infty} \left\{ \left( \int_0^{\infty} dt_n \right) \int_0^1 dt_n \right\} \sum_{k=0}^{n-1} \int_0^1 dt_n \right\} \left( 1 - \frac{1}{\xi} \right)^{\delta(n-k+\frac{3}{2})}
\]

\[
\times \omega_n \left( \frac{w_{n-k,1}}{w_{n-k,1,n}} \right)^{\Omega_{n-k-1}} \left( \frac{w_{n-k,1,n}}{w_{n-k,1,n}} \right)^{\Omega_{n-k-1}} \left( \frac{w_{n-k,1,n}}{w_{n-k,1,n}} \right)^{\Omega_{n-k-1}}
\]

\[
\times 2F_1 \left( \frac{\alpha}{4} + \frac{1}{2}, -\frac{\alpha}{4} + \frac{1}{4}, \frac{3}{4}; \frac{\xi}{\gamma} \right) \left( \xi \right)^{\eta}
\]  \hspace{1cm} \text{(C.4)}

On (C.3) and (C.4),

\[
\begin{align*}
\eta &= (1 + \rho^2) \xi \\
\zeta &= -\rho^2 \xi^2 \\
\Omega_{n-k-1} &= \frac{1}{2} (n - k - 1 + \rho^2)
\end{align*}
\]

C.2. \(x^{1-\gamma}(1 - x)^{1-\delta} H_1(a, q - (\gamma + \delta - 2)a - (\gamma - 1)(\alpha + \beta - \gamma - \delta + 1); \alpha - \gamma - \delta + 2 \quad \beta - \gamma - \delta + 2, 2 - \gamma, 2 - \delta; x)\)

C.2.1. Polynomial of type 1

Replace coefficients $q, \alpha, \beta, \gamma$ and $\delta$ by $q - (\gamma + \delta - 2)a - (\gamma - 1)(\alpha + \beta - \gamma - \delta + 1), \alpha - \gamma - \delta + 2, \beta - \gamma - \delta + 2, 2 - \gamma, 2 - \delta$ into (C.1). Multiply $x^{1-\gamma}(1 - x)^{1-\delta}$ and (C.1) together. Put (2) into the new (C.1) with replacing $\alpha$ by $-2(2\alpha_j + j + 3/2)$ where $j, \alpha_j \in \mathbb{N}_0$; apply $a = -2(2\alpha_0 + 3/2)$ into sub-integral $y_0(\xi)$, apply $-2(2\alpha_0 + 3/2)$ into the first summation and $-2(2\alpha_1 + 5/2)$ into second summation of sub-integral $y_1(\xi)$, apply $-2(2\alpha_1 + 3/2)$ into the first summation, $-2(2\alpha_1 + 5/2)$ into the second summation and $-2(2\alpha_2 + 7/2)$ into the third summation of sub-integral $y_2(\xi)$, etc.
in the new (C.1).

\[
\xi^\alpha (1 - \xi) y(\xi) = \xi^\alpha (1 - \xi) H\left(\rho^2, -\frac{1}{4}; (h - 4)\rho^2 - 1\right); \quad \frac{\alpha + 3}{2} = 2\alpha x + j + \frac{5}{2}, \quad -\frac{\alpha}{2} + 1 = 2\alpha x + j + \frac{5}{2}, \quad \frac{3}{2} \frac{3}{2} \xi
\]

\[
\xi^\alpha (1 - \xi) H\left(\rho^2, -\frac{1}{4}; (h - 4)\rho^2 - 1\right) + \sum_{n=1}^{\infty} \left\{ \int_{0}^{1} dt_{n-k} t_{n-k}^\alpha \int_{0}^{1} \frac{du_{n-k}}{u_{n-k}} \left( 1 - \sum_{n=1}^{\infty} \frac{w_{n-k,0} \partial_{w_{n-k,0}} + \Omega_{n-k}}{w_{n-k}} \right) \right\}
\]

(C.5)

where

\[
\alpha = 2 \left(2\alpha x + j + 1\right) \quad \text{or} \quad -2 \left(2\alpha x + j + \frac{3}{2}\right)
\]

**C.2.2. Infinite series**

Replace coefficients \(q, \alpha, \beta, \gamma\) and \(\delta\) by \(q-(\gamma+\delta-2)\alpha-(\gamma-1)(\alpha+\beta-\gamma+\delta+1), \alpha-\gamma-\delta+2, \beta-\gamma-\delta+2, 2-\gamma\) and \(2-\delta\) into (C.2). Multiply \(x^{\gamma-1}(1-x)^{\delta-\gamma}\) and (C.2) together. Put (C.2) into the new (C.2).

\[
\xi^\alpha (1 - \xi) y(\xi) = \xi^\alpha (1 - \xi) H\left(\rho^2, -\frac{1}{4}; (h - 4)\rho^2 - 1\right) + \sum_{n=1}^{\infty} \left\{ \int_{0}^{1} dt_{n-k} t_{n-k}^\alpha \int_{0}^{1} \frac{du_{n-k}}{u_{n-k}} \right\}
\]

\[
\times \frac{1}{2\pi i} \int_{\gamma - \delta + 2}^{\gamma + \delta - 2} \frac{1}{\xi - \rho^2} \left( 1 - \frac{1}{V_{n-k}} \right) - \frac{1}{V_{n-k}} + \frac{1}{w_{n-k}} \left( 1 - \sum_{n=1}^{\infty} \frac{w_{n-k,0} \partial_{w_{n-k,0}} + \Omega_{n-k}}{w_{n-k}} \right) \right\}
\]

\[
\times 2 F_1 \left( \frac{\alpha + 3}{2} - \frac{\alpha}{2} + 1 \frac{3}{2} \xi \right) \right\}
\]

(C.6)

On (C.5) and (C.6),

\[
\begin{align*}
\eta &= (1 + \rho^2) \xi \\
z &= -\rho^2 \xi^2 \\
\Omega_{n-k-1} &= \frac{4}{\rho^2} \left( n - k + \frac{1}{\rho^2} \right) \\
Q &= \frac{4 - \rho^2 \xi^2}{8(1 + \rho^2)}
\end{align*}
\]

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Lame equation in Weierstrass’s form

C.3. \( H(1 - a, -q + a \beta; \alpha, \beta, \delta, \gamma; 1 - x) \)

C.3.1. Polynomial of type I

Replace coefficients \( a, q, \gamma, \delta \) and \( x \) by \( 1 - a, -q + a \beta, \delta, \gamma \) and \( 1 - x \) into (C.1). Put (2) into the new (C.1) with replacing \( \alpha \) by \(-2(2\alpha_j + j + 1/2)\) where \( j, \alpha_j \in \mathbb{N}_0 \); apply \( \alpha = -2(2\alpha_0 + 1/2) \) to sub-integral \( y_0(\varsigma) \), apply \(-2(2\alpha_0 + 1/2) \) into the first summation and \(-2(2\alpha_1 + 3/2) \) into second summation of sub-integral \( y_1(\varsigma) \), apply \(-2(2\alpha_0 + 1/2) \) into the first summation, \(-2(2\alpha_1 + 3/2) \) into the second summation and \(-2(2\alpha_2 + 5/2) \) into the third summation of sub-integral \( y_2(\varsigma) \), etc in the new (C.1).

\[
y(\varsigma) = H(1 - \rho^{-2}, 1/4 (hp^{-2} - \alpha(\alpha + 1)); \alpha - 1/2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 - \varsigma) \]

\[
= zF(\alpha, \alpha_0 + 1, 3/4, z) + \sum_{\alpha=0}^{\infty} \left\{ \left\{ \int_{0}^{1} dt_{n-k} t_{n-k}^{(\alpha-2)} \right\} \left\{ \int_{0}^{1} du_{n-k} u_{n-k}^{(\alpha-k-2)} \right\} \right\}
\]

\[
\times \frac{1}{2\pi i} \int dv_{n-k} \frac{1}{V_{n-k}} \left( 1 - \frac{1}{V_{n-k}} \right) \left( 1 - \sum_{\alpha=0}^{\infty} \left\{ \left\{ \int_{0}^{1} dt_{n-k} t_{n-k}^{(\alpha-2)} \right\} \left\{ \int_{0}^{1} du_{n-k} u_{n-k}^{(\alpha-k-2)} \right\} \right\} \right) \left( \frac{1}{V_{n-k}} + \frac{1}{2} (n-k-1) + Q_{n-k-1} \right) \}
\]

\[
x_2 F(1 - \rho^{-2}, 1/4 (hp^{-2} - \alpha(\alpha + 1)); 1/2, 1/2, 1/2, 1 - \varsigma) \]

where

\[
Q_{n-k-1} = \frac{1}{8(2 - \rho^{-2})} \left( hp^{-2} - \alpha(\alpha + 1) \right)
\]

C.3.2. Infinite series

Replace coefficients \( a, q, \gamma, \delta \) and \( x \) by \( 1 - a, -q + a \beta, \delta, \gamma \) and \( 1 - x \) into (C.2). Put (2) into the new (C.2).

\[
y(\varsigma) = H(1 - \rho^{-2}, 1/4 (hp^{-2} - \alpha(\alpha + 1)); \alpha - 1/2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 - \varsigma) \]

\[
= zF(\alpha, \alpha_0 + 1, 3/4, z) + \sum_{\alpha=0}^{\infty} \left\{ \left\{ \int_{0}^{1} dt_{n-k} t_{n-k}^{(\alpha-2)} \right\} \left\{ \int_{0}^{1} du_{n-k} u_{n-k}^{(\alpha-k-2)} \right\} \right\}
\]

\[
\times \frac{1}{2\pi i} \int dv_{n-k} \frac{1}{V_{n-k}} \left( 1 - \frac{1}{V_{n-k}} \right) \left( 1 - \sum_{\alpha=0}^{\infty} \left\{ \left\{ \int_{0}^{1} dt_{n-k} t_{n-k}^{(\alpha-2)} \right\} \left\{ \int_{0}^{1} du_{n-k} u_{n-k}^{(\alpha-k-2)} \right\} \right\} \right) \left( \frac{1}{V_{n-k}} + \frac{1}{2} (n-k-1) + Q_{n-k-1} \right) \}
\]

\[
x_2 F(\alpha, \alpha_0 + 1, 3/4, z) \]

where

\[
Q = \frac{1}{8(2 - \rho^{-2})} \left( hp^{-2} - \alpha(\alpha + 1) \right)
\]
Lame equation in Weierstrass’s form

On (C.7) and (C.8),

\[
\begin{align*}
\zeta &= 1 - \xi \\
\eta &= \frac{2 - \rho + \zeta}{1 - \rho}, \quad \zeta = \frac{1}{1 - \rho^2}
\end{align*}
\]

C.4. \((1 - x)^{\beta - \delta} H_l(1 - a, -q + (\delta - 1)\gamma a + (\alpha - \delta + 1)(\beta - \delta + 1); \alpha - \delta + 1, \beta - \delta + 1, 2 - \delta, \gamma; 1 - x)\)

C.4.1. Polynomial of type 1

Replace coefficients \(a, q, \alpha, \beta, \gamma, \delta\) and \(x\) by \(1 - a, -q + (\delta - 1)\gamma a + (\alpha - \delta + 1)(\beta - \delta + 1), \alpha - \delta + 1, \beta - \delta + 1, 2 - \delta, \gamma\) and \(1 - x\) into (C.1). Multiply \((1 - x)^{\beta - \delta}\) and (C.1) together. Put (2) into the new (C.1) with replacing \(\alpha\) by \(-2(2\alpha_j + j + 1)\) where \(j, \alpha_j \in \mathbb{N}_0\); apply \(\alpha = -2(2\alpha_0 + 1)\) into sub-integral \(y_0(\zeta)\), apply \(-2(2\alpha_1 + 1)\) into the first summation and \(-2(2\alpha_1 + 2)\) into second summation of sub-integral \(y_1(\zeta)\), apply \(-2(2\alpha_0 + 1)\) into the first summation, \(-2(2\alpha_1 + 2)\) into the second summation and \(-2(2\alpha_2 + 3)\) into the third summation of sub-integral \(y_2(\zeta)\), etc in the new (C.1).

\[
(1 - \xi)^2 y(\zeta)
= (1 - \xi)^2 H_l(1 - \rho - \frac{1}{4}((1 + h)\rho^2 + (\alpha - 1)(\alpha + 2)); \frac{\alpha}{2} + 1 = 2 \alpha_j + j + \frac{3}{2}; 1 - \xi)
\]

\[
\begin{align*}
&= (1 - \xi)^2 \left\{ \frac{1}{2} F_1 \left( -\alpha_0, \alpha_0 + \frac{3}{4}, \frac{5}{4}, \frac{1}{2}; 1 - \xi \right) + \sum_{n=1}^{\infty} \left\{ \frac{1}{2} F_1 \left( -\alpha_{n-1}, \alpha_{n-1} + \frac{3}{4}, \frac{5}{4}, \frac{1}{2}; 1 - \xi \right) \right\} 
\right.
\end{align*}
\]

where

\[
\begin{align*}
Q_{n-k-1} &= \frac{1}{\alpha_{n-k} - \rho} \left( (h + 1)\rho^2 + 4 \left( \alpha_{n-k-1} + \frac{1}{2} \right) \left( \alpha_{n-k-1} + \frac{1}{2} \right) \right) - 2 \left( 2 \alpha_j + j + 1 \right)
\end{align*}
\]

C.4.2. Infinite series

Replace coefficients \(a, q, \alpha, \beta, \gamma, \delta\) and \(x\) by \(1 - a, -q + (\delta - 1)\gamma a + (\alpha - \delta + 1)(\beta - \delta + 1), \alpha - \delta + 1, \beta - \delta + 1, 2 - \delta, \gamma\) and \(1 - x\) into (C.2). Multiply \((1 - x)^{\beta - \delta}\) and (C.2) together. Put (2)
Lame equation in Weierstrass’s form

\[(1 - \xi)^2 y(\xi) = (1 - \xi)^2 H \left(1 - \rho^2, -\frac{1}{4} \left((1 + h)\rho^2 + \left(\alpha - 1\right)(\alpha + 2)\right), \frac{\alpha}{2} + 1, -\frac{\alpha}{2} + \frac{3}{2}; \frac{1}{2} \right)\]

\[= (1 - \xi)^2 \left\{2F_1 \left(\frac{\alpha}{4} + \frac{1}{2}, \frac{\alpha}{4} + \frac{5}{4}; \frac{\alpha}{4} + \frac{3}{4} \right) + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left( \int_0^1 dt_{n-k} t_{n-k}^\frac{1}{4}(n-k-2) \right) \int_0^1 dt_{n-k} u_{n-k}^\frac{1}{4}(n-k-\frac{1}{2}) \right\} \right\}
\]

where

\[Q = -\frac{1}{2(2 - \rho^2)} \left((h + 1)\rho^2 + (\alpha - 1)(\alpha + 2)\right)\]

On (C.9) and (C.10),

\[
\begin{align*}
\delta &= 1 - \xi \\
\eta &= \frac{2 - \rho^2}{1 - \rho^2} \xi \\
\zeta &= \frac{2 - \rho^2}{1 - \rho^2} \gamma
\end{align*}
\]

C.5. \(x^{-\alpha} H \left(\frac{1}{a} \left(q + \alpha [(\alpha - \gamma - \delta + 1)\alpha - \beta + \delta]\right) \right) ; \alpha, \alpha - \gamma + 1, \alpha - \beta + 1, \delta; 1/x\)

C.5.1. Infinite series

Replace coefficients \(a, q, \beta, \gamma\) and \(x\) by \(\frac{1}{a} \left[q + \alpha [(\alpha - \gamma - \delta + 1)\alpha - \beta + \delta]\right] \) into (C.2). Multiply \(x^{-\alpha}\) and (C.2) together. Put (C.2) into the new (C.2).
where

\[
\begin{align*}
\zeta &= \xi^{-1} \\
\eta &= (1 + \rho^{-2})\xi \\
\sigma &= -\rho^{-2}\zeta^2 \\
Q &= -\frac{1}{\xi}(h(1 + \rho^2)^{-1} - (\alpha + 1)^2)
\end{align*}
\]

C.6. \(\left(1 - \frac{\chi}{a}\right)^{-\beta} HI\left(1 - a, -q + \gamma\beta; -\alpha + \gamma + \delta, \beta, \gamma, \delta; \frac{(1 - a)x}{x - a}\right)\)

C.6.1. Infinite series

Replace coefficients \(a, q, \alpha\) and \(x\) by \(1 - a, -q + \gamma\beta, -\alpha + \gamma + \delta\) and \(\frac{1 - \alpha}{x - a}\) into (C.2). Multiply \(\left(1 - \frac{\chi}{a}\right)^{-\beta}\) and \((\text{C.2})\) together. Put (2) into the new \((\text{C.2})\).

\[
\begin{align*}
(1 - \rho^2) \xi^2 (\gamma(\xi)) &= (1 - \rho^2) \xi^2 HI(1 - \rho^{-2}, 1, 2; (1 - \rho^2) \xi) \\
&= (1 - \rho^2) \xi^2 \left\{ 2F1 \left( \begin{array}{c}
-\frac{\alpha}{4} + \frac{1}{4}, -\frac{\alpha}{2} + \frac{3}{4}; 2 \end{array} ; \frac{1}{\xi - \rho^2} \right) + \sum_{n=1}^{\infty} \left\{ \int_{0}^{1} dt_{n-k} \left( t_{n-k} \right) \frac{1}{v_{n-k}^{(n-k-1)}} \left( 1 - w_{n-k+1} v_{n-k} (1 - u_{n-k}) \right) \right\} \right\} \\
&\times 2F1 \left( \begin{array}{c}
\frac{\alpha}{4} + \frac{1}{4}, -\frac{\alpha}{2} + \frac{3}{4}; \frac{1}{v_{n-k}^{(n-k-1)}}, \frac{w_{n-k+1}}{w_{n-k}} \end{array} ; \frac{\Omega_{n-k-1}}{\Omega_{n-k}} \right) \right\} \right\}
\end{align*}
\]

where

\[
\begin{align*}
\zeta &= \xi^{-1} \\
\eta &= (1 + \rho^{-2})\xi \\
\sigma &= -\rho^{-2}\zeta^2 \\
Q &= \frac{h}{(1 + \rho^{-2})} (\chi - \rho)^{-\beta} + \frac{1}{2} (n - k - 1)
\end{align*}
\]

C.7. \(\left(1 - x\right)^{1-\delta} \left(1 - \frac{\chi}{a}\right)^{-\beta+1} HI\left(1 - a, -q + \gamma[(\delta - 1)a + \beta - \delta + 1]; -\alpha + \gamma + 1\right)\)

C.7.1. Infinite series

Replace coefficients \(a, q, \alpha, \beta, \delta\) and \(x\) by \(1 - a, -q + \gamma[(\delta - 1)a + \beta - \delta + 1], -\alpha + \gamma + 1, \beta - \delta + 1, 2 - \delta\) and \(\frac{1 - \alpha}{x - a}\) into (C.2). Multiply \(\left(1 - x\right)^{1-\delta} \left(1 - \frac{\chi}{a}\right)^{-\beta+1}\) and (C.2) together. Put
Lame equation in Weierstrass’s form 

\[ (1 - \xi \ell)^2 (1 - \rho^2 \xi)^{\ell x + 1} y(\xi) = (1 - \xi \ell)^2 (1 - \rho^2 \xi)^{\ell x + 1} H \left( 1 - \rho^2, 1, 4 ; 1 \right) \left( 1 + (a + 1) \left( (x + 1) + (x + 1) \right) \right) \]

where

\[ \xi = \frac{1 - \rho^2 \xi}{\xi} \]

\[ \eta = \frac{-\rho^2 \xi}{\xi} \]

\[ z = \frac{1 - \rho^2 \xi}{\xi} \]

\[ \Omega_{n-k-1} = \frac{1}{2} (n - k - \frac{a + 1}{2}) \]

\[ Q = \frac{1}{2} \left( n - k - \frac{a + 1}{2} \right) \]

C.8. \( x^{\alpha} H \left( \frac{a - 1}{a}, \frac{-q + \alpha (\delta a + \beta - \delta)}{a}, \alpha, \alpha - 1 + \delta \right) \]

C.8.1. Infinite series

Replace coefficients \( a, q, \beta, \gamma, \delta \) and \( x \) by \( \frac{a - 1}{a}, \frac{-q + \alpha (\delta a + \beta - \delta)}{a}, \alpha - 1 + \delta \), \( \alpha - \beta + 1 \) and \( \frac{a - 1}{a} \) into (C.2). Multiply (C.8) and (C.2) together. Put (C.8) into the new (C.2).
C.9.1. Infinite series

Multiply.

\[ \text{Lame equation in Weierstrass’s form} \]

\[
\begin{aligned}
\psi &= \frac{\xi^{1}}{\xi^{2}} \\
\eta &= \frac{2\rho^{2}}{1 - \rho^{2}} \epsilon \\
\zeta &= \frac{1}{1 - \rho^{2}} \epsilon^{2} \\
\Omega_{n-k-1} &= \frac{1}{2} (n - k + \frac{a(1 - \rho^{2})}{\rho}) \\
Q &= \frac{h + (\alpha + 1)(\alpha + 1 \rho^{2})}{n(1 + \rho^{2})}
\end{aligned}
\]

C.9. \( \left( \frac{x - a}{1 - a} \right)^{-\alpha} \) \( \text{H}(a, q - (\beta - \delta) \alpha; \alpha, -\beta + \gamma + \delta, \gamma; \frac{a(x - 1)}{x - a}) \)

C.9.1. Infinite series

Replace coefficients \( q, \beta, \gamma, \delta \) and \( x \) by \( q - (\beta - \delta) \alpha, -\beta + \gamma + \delta, \gamma \), \( \frac{a(x - 1)}{x - a} \) into (C.2). Multiply \( \left( \frac{x - a}{1 - a} \right)^{-\alpha} \) and (C.2) together. Put (2) into the new (C.2).

\[
\begin{aligned}
(\xi - \rho^{2})^{-\frac{1}{2}(\alpha + 1)} \\
\eta(\zeta)
\end{aligned}
\]

\[
= \left( \frac{\xi - \rho^{2}}{1 - \rho^{2}} \right)^{-\frac{1}{2}(\alpha + 1)} H \left( \rho^{-2}, -\frac{1}{4}(hp - (\alpha + 1)^{2}); \frac{1}{2}(\alpha + 1), -\frac{1}{2}(\alpha - 2), \frac{1}{2}, \frac{1}{2}, 1, \rho^{2}(\xi - \rho^{2}) \right)
\]

\[
= \left( \frac{\xi - \rho^{2}}{1 - \rho^{2}} \right)^{-\frac{1}{2}(\alpha + 1)} \quad \left\{ \begin{array}{l}
\frac{\alpha}{4} + \frac{\alpha}{4} + \frac{3}{2} + \frac{3}{4} \zeta
\end{array} \right\} + \sum_{\nu = 1}^{\infty} \left\{ \sum_{k = 0}^{n-1} \int_{0}^{1} dt_{n-k} \frac{1}{2(n-k-2)} \int_{0}^{1} dt_{n-k} u_{n-k} \right\} \left( \frac{1}{2(n + 1) + 4} \right)
\]

\[
\times \left( \frac{1}{2n} \int_{n-k}^{n-1} \frac{1}{v_{n-k}} \left( 1 - \frac{1}{v_{n-k}} \right)^{\frac{1}{2}(n-k-1)} \left( 1 - \frac{1}{v_{n-k}} \right)^{\frac{3}{2}(n-k-1)} \right)
\]

\[
\times \frac{1}{2n} \int_{n-k}^{n-1} \frac{1}{v_{n-k}} \left( 1 - \frac{1}{v_{n-k}} \right)^{\frac{1}{2}(n-k-1)} \left( 1 - \frac{1}{v_{n-k}} \right)^{\frac{3}{2}(n-k-1)} \right)
\]

\[
\times 2F_{1} \left( \frac{\alpha}{4} + \frac{\alpha}{4} + \frac{3}{2} + \frac{3}{4} \zeta \right)
\]

(C.15)

where

\[
\begin{aligned}
\psi &= \frac{\xi^{1}}{\xi^{2}} \\
\eta &= (1 + \rho^{2}) \epsilon \\
\zeta &= -\rho^{2} \epsilon^{2} \\
\Omega_{n-k-1} &= \frac{1}{2} (n - k + \frac{a(1 + \rho^{2})}{\rho}) \\
Q &= \frac{h^{2} + (\alpha + 1)(\alpha + 1 \rho^{2})}{n(1 + \rho^{2})}
\end{aligned}
\]

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Lame equation in Weierstrass’s form

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