Abstract. In the context of holomorphic families of endomorphisms of $\mathbb{P}^k$, we prove that stability in the sense of [BBD18] is equivalent to a summability condition for the post-critical mass and to the convergence of a suitably defined ramification current. This allows us to both simplify the approach of [BBD18] and better relate stability to post-critical normality.

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1. Introduction and results

A holomorphic family of degree $d \geq 2$ endomorphisms on $\mathbb{P}^k$, parametrized by a complex manifold $M$, is a holomorphic map $f : M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$, of the form $(\lambda, z) \mapsto (\lambda, f_\lambda(z))$ such that, as endomorphisms on $\mathbb{P}^k$, the maps $f_\lambda$ have the same algebraic degree $d \geq 2$. In dimension $k = 1$, the dynamical stability of such families has been independently studied by Mañé-Sad-Sullivan [MSS83] and Lyubich [Lju83]. At the heart of their theory stands a characterization of global dynamical stability by a post-critical normality condition encoding the dynamical stability of the critical set. By considering a cover of the parameter space, one may always assume that the critical points are marked, that is to say that the critical set $C_f$ of $f$ is given by the graphs of $2d - 2$ holomorphic maps $c_j : M \rightarrow \mathbb{P}^1$. Under this mild assumption, this characterization can be stated as follows.

Theorem 1.1. Let $f : M \times \mathbb{P}^1 \rightarrow M \times \mathbb{P}^1$ be a holomorphic family of rational functions with marked critical points $c_j$ and let $J_\lambda$ denote the Julia set of $f_\lambda$. Then the two following assertions are equivalent:

i) the Julia sets $J_\lambda$ move holomorphically with $\lambda$,

ii) the sequences $(f^{\circ n} \circ c_j)_n$ are normal.

The fact that the Julia sets $J_\lambda$ move holomorphically with $\lambda$ amounts to say that there exists a holomorphic lamination $\mathcal{L}$ by graphs over $M$ in $M \times \mathbb{P}^1$ whose slices $\mathcal{L}_\lambda := \mathcal{L} \cap (\{\lambda\} \times \mathbb{P}^1)$ coincide with $J_\lambda$ for every $\lambda \in M$; in that case the family $f$ is said to be stable. More generally, the stability locus $\text{Stab}(f)$ of a family $f$ is the set of parameters $\lambda \in M$ admitting a neighbourhood $D$ such that the restricted family $f|_{D \times \mathbb{P}}$ is stable. An important corollary of the above theorem is the density of the stability locus in the parameter space. Another one, due to DeMarco [deM03], is that the Lyapunov exponents $L(\lambda)$ of the maximal entropy measures of $f_\lambda$ define a plurisubharmonic function on $M$ which is pluriharmonic exactly on $\text{Stab}(f)$. 

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When studying stability of families of endomorphisms of \( \mathbb{P}^{k\geq 2} \), basic one-dimensional tools are no longer efficient, and a different approach, mainly based on pluripotential and ergodic techniques, has recently been designed in [BBD18]. To describe it, we first briefly recall some basic facts and definitions. For any parameter \( \lambda \in M \), denote by \( \mu_\lambda \) the equilibrium measure of \( f_\lambda \) and by \( J_\lambda := \text{supp}(\mu_\lambda) \) its Julia set. The space \( \mathcal{O}(M, \mathbb{P}^k) \) of holomorphic maps from \( M \) to \( \mathbb{P}^k \) is endowed with the topology of local uniform convergence; which yields a complete metric space \( (\mathcal{O}(M, \mathbb{P}^k), d_{\text{loc}}) \) on which the family \( f \) induces a topological dynamical system \( (\mathcal{O}(M, \mathbb{P}^k), \mathcal{F}) : \forall \gamma \in \mathcal{O}(M, \mathbb{P}^k), \forall \lambda \in M : \mathcal{F}(\gamma)(\lambda) := (f_\lambda \circ \gamma)(\lambda). \)

The following (possibly empty) \( \mathcal{F} \)-invariant subset of \( \mathcal{O}(M, \mathbb{P}^k) \) is also of importance

\[ \mathcal{J} := \{ \gamma \in \mathcal{O}(M, \mathbb{P}^k) : \gamma(\lambda) \in J_\lambda \text{ for every } \lambda \in M \}. \]

A key idea is to replace a holomorphic motion of Julia sets by a holomorphic motion of the equilibrium measures, meaning by that the existence of a holomorphic lamination \( \mathcal{L} \) by graphs over \( M \) in \( M \times \mathbb{P}^k \) whose slices \( L_\lambda := \mathcal{L} \cap (\{\lambda\} \times \mathbb{P}^k) \) have full \( \mu_\lambda \) measure for every \( \lambda \in M \). Such a lamination can be further required to be induced by a relatively compact subset of \( \mathcal{J} \) on which \( \mathcal{F} \) induces a \( d^k \) to 1 map and, in that case, is called an equilibrium lamination. This leads to the following definition of stability.

**Definition 1.2.** A family \( f \) is said to be \( \mu \)-stable if it admits an equilibrium lamination. The \( \mu \)-stability locus of a family \( f \), denoted \( \mu\text{-Stab}(f) \), is the set of parameters \( \lambda \in M \) for which there exists a neighbourhood \( D \) in \( M \) such that the restricted family \( f|_{D \times \mathbb{P}^k} \) is \( \mu \)-stable.

Although some post-critical normality conditions played a crucial role in the work [BBD18], they were not directly related to stability. We aim here to both introduce a new notion of post-critical normality and show that it naturally implies \( \mu \)-stability. As we shall explain in section 2, this allows to simplify the overall scheme of proof of [BBD18]. To this purpose, we borrow the concept of ramification current from the work [DS03], see also [DS10], of Dinh and Sibony on the equidistribution of iterated preimages for endomorphisms on \( \mathbb{P}^k \) and polynomial-like maps, and adapt it to the setting of holomorphic families.

**Definition 1.3.** Let \( C_f \) denote the critical set of \( f \). For every integer \( n \geq 0 \), we denote by \( R_{n,f} \) the closed positive current on \( M \times \mathbb{P}^k \) defined by

\[ R_{n,f} := d^{-\kappa f^\circ n} \ast [f(C_f)] \]

where \([f(C_f)]\) denotes the current of integration on the analytic subset \( f(C_f) \). The formal ramification current \( R_f \) of \( f \) is defined by

\[ R_f := \sum_{n \geq 0} R_{n,f} = \sum_{n \geq 0} d^{-\kappa f^\circ n} \ast [f(C_f)]. \]

The convergence domain \( \Omega(R_f) \) of \( R_f \) is the subset of points in \( M \times \mathbb{P}^k \) which admit a neighbourhood \( U \) such that the series \( \sum_{n \geq 0} \|1_U R_{n,f}\| \) converges.

Let us stress that \( R_f \) induces a closed positive \((1,1)\)-current on its convergence domain \( \Omega(R_f) \).
We can now state our main result.

**Theorem 1.4.** Let \( f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k \) be a holomorphic family of endomorphisms of degree \( d \geq 2 \) on \( \mathbb{P}^k \) and \( \pi_M : M \times \mathbb{P}^k \to M \) be the canonical projection. Then:

\[
\Omega(R_f) = \pi_M^{-1}(\mu\text{-Stab}(f)) = \pi_M(\Omega(R_f)) \times \mathbb{P}^k.
\]

As we shall see in the next section, it follows from Theorem 1.4 that \( \lambda_0 \in \mu\text{-Stab}(f) \) if and only if \( \|((f^n)^*)_*(|C_f|)||_{U \times \mathbb{P}^k} = O(d^{(k-1)n}) \) for some neighbourhood \( U \) of \( \lambda_0 \) in \( M \). This last condition already appeared in [BBD18, Proposition 3.12(3)] and, since for every \( \lambda, \|((f^n)^*)_*(|C_{f_{\lambda}}|)||_{\mathbb{P}^k} \sim_n d^{(k-1)n} \), it can be interpreted as a post-critical normality statement. In dimension \( k = 1 \), one may indeed check that it is equivalent to the second assertion of Theorem 1.1 (see [BB18b, Theorem 3.9, proof of (5) \( \implies (6) \)]).

**Notations:** The mass of a positive current \( T \) on some Borel measurable set \( W \) is denoted \( \|T\|_W \) or \( \|T\|_W \). The set of all holomorphic maps from \( U \) to \( V \) is denoted \( \mathcal{O}(U, V) \). When \( V = \mathbb{C} \), we simply note \( \mathcal{O}(U) \). The euclidean ball of radius \( r \) and centered at point \( a \) in \( \mathbb{C}^n \) is denoted \( B(a, r) \). The determinant of the jacobian matrix of an endomorphism \( f : \mathbb{P}^k \to \mathbb{P}^k \) (respectively a polynomial map \( F : \mathbb{C}^k \to \mathbb{C}^k \)) will be denoted \( \text{Jac}(f) \) (resp. \( \text{Jac}(F) \)). The cardinality of a finite set \( A \) is denoted \( |A| \).

2. **The role of theorem 1.4 in the stability theory**

From now on, \( f \) is a holomorphic family of degree \( d \geq 2 \) endomorphisms on \( \mathbb{P}^k \), parametrized by a complex manifold \( M \) of complex dimension \( m \). We recall that the equilibrium measure \( \mu_\lambda \) of \( f_\lambda \) is a Monge-Ampère mass given by \( \mu_\lambda = (dd^c g(\lambda, z) + \omega_{\mathbb{P}^k})^k \), where \( g \) is a Green function (see [DS10, Theorem 1.16]), which is continuous on \( M \times \mathbb{P}^k \). We denote by \( L(\lambda) = \int_{\mathbb{P}^k} \ln|\text{Jac} f_\lambda(z)| \ d\mu_\lambda(z) \) the sum of Lyapunov exponents of \( (f_\lambda, \mu_\lambda) \). For every integer \( n \) and every parameter \( \lambda \), we denote by \( \mathcal{R}_n(\lambda) \) the set of \( n \)-periodic repelling points of \( f_\lambda \) which belong to the Julia set of \( f_\lambda \). We now introduce some weaker notions of stability.

**Definition 2.1.** A family \( f \) is said to be weakly stable (resp. asymptotically weakly stable) if, for every compact subset \( M_0 \subset M \), there exists a sequence of holomorphic laminations \( (\mathcal{L}_n)_n \) by graphs over \( M \) in \( M_0 \times \mathbb{P}^k \) such that \( (\mathcal{L}_n)_\lambda := \mathcal{L}_n \cap (\{\lambda\} \times \mathbb{P}^k) = \mathcal{R}_n(\lambda) \) (resp. \( (\mathcal{L}_n)_\lambda \subset \mathcal{R}_n(\lambda) \) and \( |(\mathcal{L}_n)_\lambda| \sim_n |\mathcal{R}_n(\lambda)| \)) for every \( \lambda \in M_0 \).

The following result combines those of [BBD18] with Theorem 1.4. It both offers a much direct proof of some results of [BBD18] and completes them with a sharper characterization of stability in terms of post-critical normality.

**Theorem 2.2.** Let \( f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k \) be a holomorphic family of endomorphisms of degree \( d \geq 2 \) on \( \mathbb{P}^k \) parametrized by a simply connected complex manifold \( M \). Then the following assertions are equivalent:

1) \( \mathcal{L} \) is pluriharmonic on \( M \),
2) \( \|f_*^{n*}(|C_f|)||_{\mathcal{L} \times \mathbb{P}^k} = O(d^{(k-1)n}) \) for every \( U \subset M \),
3) \( \pi_M(\Omega(R_f)) = M \),
4) \( f \) is \( \mu \)-stable,
5) \( f \) is asymptotically weakly stable.
Let us stress that in [BBD18] the assertions 1) and 2) were shown to be equivalent to weak stability in the family of all degree \( d \geq 2 \) holomorphic endomorphisms of \( \mathbb{P}^k \) or for any family when \( k = 2 \), while in [Bia19], using further techniques, Bianchi proved the equivalence of assertions 1), 4), 5) and a version of 2) in the larger setting of holomorphic families of polynomial maps of large topological degree.

We now recall some fundamental tools from [BBD18] which will also be useful here. Our main thread is that the stability properties are encoded in the Lyapunov function \( L(\lambda) \), from which they can be extracted by mean of the two following formulas:

\[
\text{dd}^c L \text{-formula: } \text{dd}^c L = \pi_{M*} \left( \left( \text{dd}^c_{\lambda,z} g(\lambda, z) + \omega_{\mathbb{P}^k} \right)^k \wedge [C_f] \right).
\]

\[
\text{Approximation formula: } L(\lambda) = \lim_{n \to \infty} \sum_{z \in \mathcal{R}_{\text{e}n}(\lambda)} \ln |\text{Jac} f_\lambda(z)|.
\]

The first formula has been obtained by Bassanelli and Berteloot in [BaBe07], it generalizes similar formulas in dimension one due to Przytycki [Prz85] and Manning [Man84] for polynomials and DeMarco [deM03] for rational functions. The second one was proved by Berteloot, Dupont and Molino in [BDM08] and a simplified proof, avoiding difficulties due to the possible resonances between the Lyapunov exponents, has been given by Berteloot and Dupont in [BD19].

We shall also need the notion of \textit{acritical equilibrium web} whose existence, as shown in [BBD18, Theorem 4.1], implies that of an equilibrium lamination and, in principle, is much easier to prove. The projections \( p_\lambda : \mathcal{O}(M, \mathbb{P}^k) \to \mathbb{P}^k \) are defined by \( p_\lambda(\gamma) := \gamma(\lambda) \).

\textbf{Definition 2.3.} An \textit{equilibrium web} for \( f \) on \( M \) is a Borel probability measure on the metric space \((\mathcal{O}(M, \mathbb{P}^k), d_{\text{uc}})\) such that:

1. \( \mathcal{F}_*, \mathcal{M} = \mathcal{M} \),
2. \( \forall \lambda \in M : (p_\lambda)_* \mathcal{M} = \mu_\lambda \),
3. \( \text{supp}(\mathcal{M}) \) is compact.

Set \( \mathcal{J}_* := \{ \gamma \in \mathcal{J} : \Gamma \cap (U_{m \geq 0} f^{-m}(U_{n \geq 0} f^n(C_f))) \neq \emptyset \} \). An equilibrium web \( \mathcal{M} \) is said \textit{acritical} if \( \mathcal{M}(\mathcal{J}_*) = 0 \).

In [BBD18] an equilibrium web was also required to be supported on \( \mathcal{J}_* \), this is actually a consequence of the above simpler definition. Indeed, if \( \gamma_0 \notin \mathcal{J} \) then there exists \( \lambda_0 \in M \) for which \( \gamma_0(\lambda_0) \notin \text{supp} \mu_{\lambda_0} \), and then, taking any neighbourhood \( \mathcal{V}_0 \) of \( \gamma_0 \) in \( \mathcal{O}(M, \mathbb{P}^k) \) such that \( p_{\lambda_0}(\mathcal{V}_0) \subset \mathbb{P}^k \setminus \text{supp} \mu_{\lambda_0} \), one gets \( \mathcal{M}(\mathcal{V}_0) \leq \mathcal{M}(p_{\lambda_0}^{-1}(p_{\lambda_0}(\mathcal{V}_0))) = \mu_{\lambda_0}(p_{\lambda_0}(\mathcal{V}_0)) = 0 \).

We now show that Theorem 2.2 can be obtained from the following key lemma, whose proof will be given in the next section. Applying Theorem 2.2 to the restricted families \( f|_{\pi_{\text{Stab}} \times \mathbb{P}^k} \) then easily leads to Theorem 1.4.

\textbf{Lemma 2.4.} If \( \lambda_0 \in \pi_M(\Omega(R_f)) \), then there exists a neighbourhood \( D_0 \) of \( \lambda_0 \) in \( M \) such that the restricted family \( f|_{D_0 \times \mathbb{P}^k} \) admits an acritical equilibrium web.

\textbf{Proof of Theorem 2.2.} 1) \( \iff \) 2). This follows immediately from the following estimate which is a direct consequence of the \textit{dd}^c \text{-formula} (see [BBD18, Lemma 3.13]). There
exists a positive constant $\alpha$, only depending on $k$ and $m$, such that
\[
\|f^n|C_f\|_{u \times \mathbb{P}^k} = \alpha \|dd^c L\|_U + O(d^{(k-1)n})
\]
for every relatively compact open subset $U$ of $M$.

2) $\Rightarrow$ 3) is obvious.

3) $\Rightarrow$ 4). By Lemma 2.4, any parameter $\lambda_0 \in M$ has a neighbourhood $D_0$ such that the restricted family $f|_{D_0 \times \mathbb{P}^k}$ admits an acritical equilibrium web $\mathcal{M}$. Using Choquet's theory, one shows that $f|_{D_0 \times \mathbb{P}^k}$ also admits an acritical web which is ergodic (see [BBD18, Proposition 2.4]). It then follows from [BBD18, Theorem 4.1] that $f|_{D_0 \times \mathbb{P}^k}$ admits an equilibrium lamination; the proof of this theorem fully exploits the ergodicity of the dynamical system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$. Finally, since $M$ is simply connected, the family $f$ itself admits an equilibrium lamination $\mathcal{L}$ by an analytic continuation argument.

4) $\Rightarrow$ 5). This has been proved by Bianchi in the wider context of holomorphic families of polynomial-like maps of large topological degree (see [Bia19, Theorem 4.11]). The proof consists on a generalization to the dynamical system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$, where $\mathcal{M}$ is an ergodic acritical equilibrium web associated to the lamination $\mathcal{L}$, of the strategy developed by Briend and Duval [BrDu99] to recover the equidistribution of the repelling cycles from the properties of the equilibrium measure.

5) $\Rightarrow$ 1). For every integer $n$ we have, by assumption, a finite collection $(\gamma_{j,n})_{1 \leq j \leq N_n}$ of holomorphic maps $\gamma_{j,n} : M \to \mathbb{P}^k$ such that $\gamma_{j,n}(\lambda) \in \mathcal{R}_n(\lambda)$ for every $\lambda \in M$ and $N_n \sim d^{kn}$. The collection is invariant by the action of $\mathcal{F}$. We thus may write
\[
d^{-kn} \sum_{z \in \mathcal{R}_n(\lambda)} \ln |\text{Jac} f_\lambda(z)| = d^{-kn} \sum_{1 \leq j \leq N_n} \ln |\text{Jac} f_\lambda(\gamma_{j,n}(\lambda))| + d^{-kn} \sum_{z \in \mathcal{R}_n(\lambda)} \ln |\text{Jac} f_\lambda(z)|,
\]
where $\mathcal{R}_n(\lambda) := \mathcal{R}_n(\lambda) \setminus \cup_{1 \leq j \leq N_n} \{\gamma_{j,n}(\lambda)\}$.

The second term in the above sum is positive since the cycles involved in $\mathcal{R}_n(\lambda)$ are repelling, and it is bounded from above by $K_\lambda d^{-kn}|\mathcal{R}_n(\lambda)| \leq K_\lambda (1 - \frac{N_n}{d^{kn}})$ where $K_\lambda$ is a positive constants which depends continuously on $\lambda$. Similarly the first term is locally uniformly bounded and, moreover, defines a pluriharmonic function on $M$. By the approximation formula, the function $L$ is thus a limit in $L^1_{loc}(M)$ of pluriharmonic functions.

To end this section we recall the role played by Misiurewicz parameters in the study of stability. A parameter $\lambda_0 \in M$ is called Misiurewicz if some repelling periodic point of $f_{\lambda_0}$ belongs to the post-critical set of $f_{\lambda_0}$ and if this configuration is not stable by small perturbations (see [BBD18, Definition 1.5] for a precise statement). Note that Misiurewicz parameters are basic examples of parameters where, in some sense, the post-critical normality fails.

The basic result about Misiurewicz parameters is the following, its proof combines the $dd^c L$-formula with an asymptotic phase-parameter transfer (see [BBD18, Proposition 3.7]).

**Proposition 2.5.** If $dd^c L = 0$ on $M$ then there are no Misiurewicz parameters on $M$. 
This result has been used by several authors to construct holomorphic families of endomorphisms for which the bifurcation locus $M \setminus \mu$-Stab$(f)$ has non empty interior (see [BT17], [Duj17], [Taf21], [Bie19]), or for estimating the Hausdorff dimension of slices in the bifurcation locus (see [BB18a]). Let us also stress that, using different techniques than in [BBD18], Bianchi generalized Proposition 2.5 to the setting of holomorphic families of polynomial-like maps of large topological degree (see [Bia19, Theorem A]).

It has also been shown that Misiurewicz parameters are dense in the bifurcation locus (see [BBD18] Theorem 1.6) and thus, taking Proposition 2.5 into account, that the theorem played an important role in the approach of [BBD18] and its proof is quite involved. The proof of Theorem 2.2 presented here avoids these difficulties.

3. PROOF OF LEMMA 2.4

We set $Y := f(C_f)$. Take $a := (\lambda_0, z_0) \in \Omega(R_f)$ and let $U := D \times B$ be a neighbourhood of $a$ in $M \times \mathbb{P}^k$ such that $\sum_{n \geq 0} \|1_U R_{n,f}\| < +\infty$. Using Baire’s theorem, one sees that $z_0$ might be slightly moved in $B$ so that

$$a \notin \cup_{n \geq 0} f^{\circ n}(Y).$$

Since the problem is local, we may shrink $D$ (resp. $B$) so that it is contained in a local chart of $M$ (resp. $\mathbb{P}^k$) and therefore assume that $D \times B \subset \mathbb{C}^{m+k}$. We then denote by $\mathcal{D}$ the set of complex lines in $\mathbb{C}^{m+k}$ passing through the point $a$ and, for each $\Delta \in \mathcal{D}$ and every $\rho > 0$, we denote by $\Delta_\rho$ the euclidean disc lying on $\Delta$, centered at $a$ and of radius $\rho$. We identify $\mathcal{D}$ with $\mathbb{P}^{m+k-1}$ and endow it with a probability measure $\mathcal{L}$ induced by the Fubiny-Study metric on $\mathbb{P}^{m+k-1}$. From now on, we fix $0 < \varepsilon < \frac{1}{2}$ and set $\delta := \frac{\varepsilon}{1+\varepsilon}$.

We will construct an acritical equilibrium web for $f|_{D_0 \times \mathbb{P}^k}$ where $D_0$ is some neighbourhood of $\lambda_0$ contained in $D$. We will proceed in five steps. The first three are directly inspired by the work of Dinh and Sibony on the equidistribution of iterated preimages (see [DS03, section 3.4] or [DS10, Section 1.4]).

**Step 1:** Constructing a large set of discs on which $f^{\circ n}$ admits almost $d^{\kappa n}$ inverse branches.

**Lemma 3.1.** There exist $r > 0$ and $\mathcal{D} \subset \mathcal{D}$ such that:

i) $B(a, r) \cap Y = \emptyset$,

ii) $\mathcal{L}(\mathcal{D}) > 1 - \delta$,

iii) $\sum_{n=0}^{+\infty} \|R_{n,f} \wedge \Delta\| \leq \varepsilon \forall \Delta \in \mathcal{D}$,

iv) for every integer $n$ and every $\Delta \in \mathcal{D}$, there exists a set $\Gamma_n^\alpha(\Delta)$ of inverses branches of $f^{\circ n}$ above $\Delta$, such that $|\Gamma_n^\alpha(\Delta)| \geq (1 - \varepsilon)d^{\kappa n}$,

v) $\mathcal{F}(\Gamma_n^{\alpha+1}(\Delta)) \subset \Gamma_n^\alpha(\Delta)$ and $\gamma(\Delta) \cap (C_f \cup f(C_f)) = \emptyset$ for every integer $n$ and every $\gamma \in \Gamma_n^\alpha(\Delta)$.

**Proof:** Take $r > 0$ small enough so that the closed euclidean ball $\overline{B(a, r)}$ in $\mathbb{C}^{m+k}$ is contained in $U$. We shall use the following standard fact (see [DS10, Lemma 1.53]).

**Lemma 3.2.** Let $S$ be a positive closed $(1,1)$-current on $U$. Then there exist a family $\mathcal{D} \subset \mathcal{D}$ with $\mathcal{L}(\mathcal{D}) > 1 - \delta$ and a constant $A_\delta > 0$ which is independant of $S$, such that the measures $S \wedge \Delta_\rho$ are well-defined and of mass $\leq A_\delta||S||$ for every $\Delta$ in $\mathcal{D}$.
Applying Lemma 3.2 for $S = 1_U R_f$, we get $A_{\delta} > 0$ and a family of lines $\mathcal{D}' \subset \mathcal{D}$ satisfying ii) and such that $1_U R_f \wedge [\Delta_r]$ is well defined as a closed positive current with mass less than $A_{\delta} \cdot \|1_U R_f\|$ for any $\Delta \in \mathcal{D}'$. Removing some $\mathcal{L}$-negligible subset from $\mathcal{D}'$ allows to assume that the currents $1_U R_{n,f} \wedge [\Delta_r]$ are well defined for all $n \in \mathbb{N}$. This forces the series $\sum_{n \in \mathbb{N}} \|1_U R_{n,f} \wedge [\Delta_r]\|$ to converge for any line $\Delta \in \mathcal{D}'$. Indeed, for any $N \in \mathbb{N}$ we have:

$$\sum_{n=0}^{N} \|1_U R_{n,f} \wedge [\Delta_r]\| = \|\sum_{n=0}^{N} 1_U R_{n,f} \wedge [\Delta_r]\| \leq \|1_U R_f \wedge [\Delta_r]\| \leq A_{\delta} \cdot \|1_U R_f\| < +\infty.$$  

Let us set $\mathcal{D}''(N) := \{\Delta \in \mathcal{D}' : N(\Delta) \leq N\}$, where $N(\Delta)$ is the smallest positive integer for which $\sum_{n \geq N(\Delta)} \|1_U R_{n,f} \wedge [\Delta_r]\| \leq \varepsilon$. As the union $\mathcal{D}' = \bigcup_{N \in \mathbb{N}} \mathcal{D}''(N)$ is increasing, we may assume, after replacing $\mathcal{D}'$ by $\mathcal{D}''(N)$ for $N$ big enough, that

$$\forall \Delta \in \mathcal{D}' : \sum_{n \geq N} \|1_U R_{n,f} \wedge [\Delta_r]\| \leq \varepsilon. \tag{3.1}$$

Since $a \notin \bigcup_{n \leq N} f^{\circ n}(Y)$, we may reduce $r$ so that

$$B(a, r) \cap \bigcup_{n \leq N} f^{\circ n}(Y) = \emptyset.$$  

It follows that i) is satisfied and that $R_{n,f} \wedge [\Delta_r] = 0$ for any $n \leq N$. Combining this with (3.1) yields iii).

Let us establish iv). Set $\varepsilon_{n,\Delta} := \|1_U R_{n,f} \wedge [\Delta_r]\|$. Note that $d^{kn}\varepsilon_{n,\Delta}$ is the cardinality of the intersection of $f^{n}(Y)$ with $\Delta_r$, counting multiplicities and that, in particular, $\varepsilon_{n,\Delta} = 0$ for $n \leq N$. Given $\Delta \in \mathcal{D}'$, we shall prove by induction on $n$ that $f^{\circ n}$ admits at least $\nu_{\Delta}^{n}$ inverse branches which do not meet $Y$ on $\Delta_r$, where

$$\nu_{\Delta}^{n} \geq (1 - \sum_{i=0}^{n} \varepsilon_{i,\Delta}) d^{kn}. \tag{3.2}$$

The base case is covered by i). For the induction step, assume the hypothesis for $f^{\circ n}$. As $f$ realizes an unramified covering of degree $d^k$, $f^{n+1}$ admits at least $(1 - \sum_{i=0}^{n} \varepsilon_{i,\Delta}) d^{kn+1}$ inverse branches above $\Delta_r$. Among them, at most $d^{kn+1}\varepsilon_{n+1,\Delta}$ do intersect $Y$, which leads to the desired property for $f^{\circ(n+1)}$. Now iii) and (3.2) immediately yield the announced estimate $\nu_{\Delta}^{n} \geq (1 - \varepsilon)d^{kn}$.

The assertion v) directly follows from the above construction. \qed

**Step 2:** Estimating the number of points in $(f^{\circ n})^{-1}(a)$ which belong to a lot of inverse branches of the form $(f^{\circ n})^{-1}(\Delta_r)$ given by step 1.

**Lemma 3.3.** Let $r > 0$ be given by Lemma 3.1. Set $(f^{\circ n})^{-1}(a) =: \{a_{n}^{1}, \cdots, a_{n}^{n}\}$ and let $I_{n} : \mathcal{D}' \to \mathcal{D}\{1, 2, \cdots, l_{n}\}$ be the map which associates to any $\Delta \in \mathcal{D}'$ the subset of $s \in \{1, 2, \cdots, l_{n}\}$ such that there exist an inverse branch of $f^{\circ n}$ defined on $\Delta_r$ and passing through $a_{s}^{n}$.

Set $\mathcal{D}_{s, n} := \{\Delta \in \mathcal{D}' : s \in I_{n}(\Delta)\}$ and $\mathcal{D}_{\varepsilon, r, n} := \{1 \leq s \leq l_{n} : \mathcal{L}(\mathcal{D}_{s, n}) \geq 1 - 2\sqrt{\varepsilon}\}$, for $1 \leq s \leq l_{n}$. Then:

i) $(1 - \sqrt{\varepsilon})d^{kn} \leq |\mathcal{D}_{\varepsilon, r, n}| \leq d^{kn}$;
ii) there exists \( i_n : \mathcal{S}_{\epsilon,r,n} \rightarrow \mathcal{S}_{\epsilon,r,n-1} \) such that \( f(a^n) = a^{n-1}_{i_n(s)} \) and \( |i_n^{-1}(\{s\})| \leq d^k \), for any \( s \in \mathcal{S}_{\epsilon,r,n} \);

iii) the sequence \( (d^{-kn}|\mathcal{S}_{\epsilon,r,n}|)_n \) converges.

Proof. We first establish the following estimate:

\[
(3.3) \quad \sum_{s=1}^{l_n} \mathcal{L}(\mathcal{D}_s^r) \geq (1 - \delta)(1 - \varepsilon)d^{kn}.
\]

Let \( \nu \) be the counting measure on \( \{1, 2, \ldots, l_n\} \) and set \( \tilde{\mathcal{L}} := \nu \otimes \mathcal{L} \). Consider the following subset \( Q_n := \{(s, \Delta) : \{s\} \in I_n(\Delta)\} \) of \( \{1, 2, \ldots, l_n\} \times \mathcal{D}' \). We compute \( \tilde{\mathcal{L}}(Q_n) \) with two different partitions of \( Q_n \). The first one is based on the value of \( s \in \{1, 2, \ldots, l_n\} \):

\[
(3.4) \quad \tilde{\mathcal{L}}(Q_n) = \tilde{\mathcal{L}} \left( \bigcup_{s=1}^{l_n} \mathcal{D}_s^r \right) = \sum_{s=1}^{l_n} \mathcal{L} \left( \mathcal{D}_s^r \right) = \sum_{s=1}^{l_n} \mathcal{L}(\mathcal{D}_s^r).
\]

The second one is based on the value of \( I_n(\Delta) \in \mathcal{P}\{1, 2, \ldots, l_n\} \):

\[
\tilde{\mathcal{L}}(Q_n) = \tilde{\mathcal{L}} \left( \bigcup_{\mathcal{J} \in \mathcal{P}\{1, 2, \ldots, l_n\}} \mathcal{J} \times I_n^{-1}(\{\mathcal{J}\}) \right) = \sum_{\mathcal{J} \in \mathcal{P}\{1, 2, \ldots, l_n\}} \nu(\mathcal{J}) \cdot \mathcal{L}(I_n^{-1}(\{\mathcal{J}\})).
\]

As soon as \( |\mathcal{J}| < (1 - \varepsilon)d^{kn} \), we get from Lemma 3.1 iv) that \( I_n^{-1}(\{\mathcal{J}\}) = \emptyset \). Then the last equality leads to

\[
(3.5) \quad \tilde{\mathcal{L}}(Q_n) \geq (1 - \varepsilon)d^{kn} \mathcal{L}(\mathcal{D}),
\]

and (3.3) immediately follows from (3.4), (3.5) and Lemma 3.1 ii).

We can now end the proof of the Lemma. Since \( l_n \leq d^{kn} \), in order to prove i), it is sufficient to prove the first inequality, namely that \((1 - \sqrt{\varepsilon})d^{kn} \leq |\mathcal{S}_{\epsilon,r,n}| \). One has

\[
\sum_{s=1}^{l_n} \mathcal{L}(\mathcal{D}_s^r) = \sum_{s \in \mathcal{S}_{\epsilon,r,n}} \mathcal{L}(\mathcal{D}_s^r) + \sum_{s \notin \mathcal{S}_{\epsilon,r,n}} \mathcal{L}(D_s^r)
\]

\[
\leq |\mathcal{S}_{\epsilon,r,n}| + (l_n - |\mathcal{S}_{\epsilon,r,n}|)(1 - 2\sqrt{\varepsilon})
\]

\[
\leq 2\sqrt{\varepsilon}|\mathcal{S}_{\epsilon,r,n}| + (1 - 2\sqrt{\varepsilon})d^{kn},
\]

which, combined with (3.3), gives that \( 2\sqrt{\varepsilon}|\mathcal{S}_{\epsilon,r,n}| \geq [(1 - \delta)(1 - \varepsilon) - (1 - 2\sqrt{\varepsilon})]d^{kn} \). Then, by our choice of \( \delta \), we get:

\[
|\mathcal{S}_{\epsilon,r,n}| \geq \left[ 1 - \frac{\varepsilon + \delta(1 - \varepsilon)}{2\sqrt{\varepsilon}} \right] d^{kn} = (1 - \sqrt{\varepsilon})d^{kn},
\]

which gives i).

Let us finally justify ii) and iii). We define the map \( i_n \) on \( \mathcal{S}_{\epsilon,r,n} \) by \( f(a^n) = a^{n-1}_{i_n(s)} \). It then follows from the assertion v) of Lemma 3.1 and the very definition of \( \mathcal{S}_{\epsilon,r,n} \) that \( i_n(\mathcal{S}_{\epsilon,r,n}) \subset \mathcal{S}_{\epsilon,r,n-1} \). Since \( f \) is a ramified covering of degree \( d^k \), one has \( |i_n^{-1}(\{s\})| \leq d^k \).

Since \( |\mathcal{S}_{\epsilon,r,n}| = \sum_{s \in \mathcal{S}_{\epsilon,r,n-1}} |i_n^{-1}(s)| \leq d^k|\mathcal{S}_{\epsilon,r,n-1}| \), the sequence \( (d^{-kn}|\mathcal{S}_{\epsilon,r,n}|)_n \) is positive non-increasing and thus converges. \( \square \)

Step 3: Extension of the inverse branches to a ball \( B(a,r) \).
We will prove here the existence of at least \((1 - \sqrt{\varepsilon})d^{kn}\) inverse branches of \(f^{on}\) on an open ball \(B(a, \tau r) \subset B(a, r)\) centered at \(a\) in \(\mathbb{C}^{m+k}\). To achieve this, we will combine Lemma 3.3 with the extension theorem of Sibony-Wong (see [SW80] or [DS10, Theorem 1.54]) which we recall below.

**Theorem 3.4. (Sibony-Wong)** Let \(\mathcal{D}'' \subset \mathcal{D}\) be such that \(\mathcal{L}(\mathcal{D}'') \geq c\) for a positive constant \(c\) and let \(\Sigma\) denote the intersection of \(\mathcal{D}''\) with \(B(a, r)\). Then there exists \(\tau \in (0, 1)\), independent from \(r\) and \(\mathcal{D}'\), such that any holomorphic function \(h\) on a neighbourhood of \(\Sigma\) can be extended to a holomorphic function \(\tilde{h}\) on \(B(a, \tau r)\). Moreover, the extended function \(\tilde{h}\) enjoys the following estimate: \(\sup_{B(a, \tau r)}|\tilde{h}| \leq \sup_{\Sigma} |h|\).

Our precise statement is as follows; we keep the notations introduced in Lemma 3.3.

**Lemma 3.5.** There exists \(\tau \in [0, 1]\) such that to every \(n \in \mathcal{I}_{e, r, n}\) is associated an inverse branch \(\gamma^n_s : B(a, \tau r) \to \mathbb{C}^m \times \mathbb{P}^k\) of \(f^{on}\) such that \(\gamma^n_s(a) = a^n_s\). Moreover, \(f \circ \gamma^n_s = \gamma^{n-1}_{i_n(s)}\) for every \(s \in \mathcal{I}_{e, r, n}\).

**Proof.** Fix \(n \in \mathbb{N}\) and \(s \in \mathcal{I}_{e, r, n}\). For every \(\Delta \in \mathcal{D}^{r,n} = \{\Delta \in \mathcal{D}' : s \in I_s(\Delta)\}\), we denote by \(\gamma^n_{s,\Delta}\) the inverse branch of \(f^{on}\) above \(\Delta\), such that \(\gamma^n_{s,\Delta}(a) = a^n_s\). Observe that the holomorphic maps \(\gamma^n_{s,\Delta}\) and \(\gamma^n_{s',\Delta'}\) are actually defined on neighbourhoods of \(\Delta_s\) and \(\Delta'_s\), and coincide on some neighbourhood of \(a\), for any pair of lines \(\Delta, \Delta' \in \mathcal{D}^{r,n}\). By analytic continuation, we can therefore consider the branch \(\gamma^n_s\) as defined on a neighbourhood of \(\bigcup_{\Delta \in \mathcal{D}^{r,n}} \Delta_s\).

By definition of \(\mathcal{I}_{e, r, n}\) (see Lemma 3.3), one has \(\mathcal{L}(\mathcal{I}_{e, r, n}) \geq 1 - 2\sqrt{\varepsilon}\) and we may therefore apply Theorem 3.4 with \(c = 1 - 2\sqrt{\varepsilon}\) and \(\mathcal{D}'' = \mathcal{D}^{r,n}\) to the coordinate functions of \(\gamma^n_s\). In this way, one sees that each \(\gamma^n_s\) extends to an inverse branch of \(f^{on}\) defined on \(B(a, \tau r)\).

When \(s \in \mathcal{I}_{e, r, n}\), the map \(f \circ \gamma^n_s\) is an inverse branch of \(f^{on}\) which is defined on \(B(a, \tau r)\) and whose value at \(a\) is \(f(a^n_s) = a^{n-1}_{i_n(s)}\). It must therefore coincide with the branch \(\gamma^{n-1}_{i_n(s)}\) which, by assertion ii) of Lemma 3.3 and the above construction, does exist. \(\square\)

The following lemma is also a consequence of the Sibony-Wong extension theorem, we shall use it in the fifth (and last) step.

**Lemma 3.6.** Fix \(\rho > 0\). For \(u \in \mathcal{O}(B(a, \rho))\) we set \(\mathcal{U}_u := \{\Delta \in \mathcal{D} : 0 \notin u(\Delta_{\rho})\}\). There exists \(0 < \tau' < 1\) such that \(0 \notin u(B(a, \tau' \rho))\) as soon as \(\mathcal{L}(\mathcal{U}_u) > \frac{1}{2}\).

**Proof.** If \(u \in \mathcal{O}(B(a, \rho))\) and \(\mathcal{L}(\mathcal{U}_u) > \frac{1}{2}\) then \(u(a) \neq 0\) and thus, by Hurwitz lemma, the function \(u\) does not vanish on \(\Delta_{\rho}\) for every \(\Delta \in \mathcal{U}_u\). This shows that \(\mathcal{U}_u\) is closed and that the function \(h := \frac{1}{u}\) is holomorphic on some neighbourhood of \(\Sigma_u := \mathcal{U}_u \cap B(a, \rho)\). By Theorem 3.4, this function extends to \(\tilde{h} \in \mathcal{O}(B(a, \tau' \rho))\) where \(0 < \tau' < 1\) neither depends on \(u\) or \(\rho\). Since \(u\tilde{h} = uh = 1\) on \(\bigcup_{\Delta \in \mathcal{U}_u} \Delta_{\tau' \rho}\), which has positive Lebesgue measure, one has \(u\tilde{h} = 1\), and therefore \(u\) does not vanish on \(B(a, \tau' \rho)\). \(\square\)

**Step 4 : Construction of an equilibrium web.**

We will use the collection of inverse branches \((\gamma^n_s)_{n \geq 1, s \in \mathcal{I}_{e, r, n}}\) obtained in the former step to build an equilibrium web \(\mathcal{M}\) for the restricted family \(f|_{D_0 \times \mathbb{P}^k}\) and some neighbourhood \(D_0\) of \(\lambda_0\). The equidistribution of iterated preimages towards the measures \(\mu_{\lambda,\rho}\).
as well as the ergodicity of these measures, will play an important role here.

Let $D_0 \times B_0$ be a neighbourhood of $a$ contained in $B(a, \tau r)$, where $\tau > 0$ is given by Lemma 3.5. For every $z \in B_0$, we define a sequence of discrete measures $(m_n(z))_n$ on $O(D_0, \mathbb{P}^k)$, and their Cesàro means $(\mathcal{M}^n(z))_n$, by

$$m^n(z) := d^{-kn} \sum_{s \in \mathcal{S}, r, n} \delta_{\gamma^n_{s,z}} \quad \text{and} \quad \mathcal{M}^n(z) := \frac{1}{n} \sum_{r=1}^n m^r(z),$$

where $\gamma^n_s(\lambda, z) =: (\lambda, \gamma^n_{s,z}(\lambda))$. If $\text{vol}$ denotes the Lebesgue measure on $B_0$, we then set

$$m^n := \frac{1}{\text{vol}(B_0)} \int_{B_0} m^n(z) \, d\text{vol}(z) \quad \text{and} \quad \mathcal{M}^n := \frac{1}{\text{vol}(B_0)} \int_{B_0} \mathcal{M}^n(z) \, d\text{vol}(z) = \frac{1}{n} \sum_{r=1}^n m^r.$$

The equilibrium web $\mathcal{M}$ will be obtained as a rescaled weak limit of $(\mathcal{M}^n)_n$. In the above definitions, the averaging on $B_0$ is devoted to make $z$ avoid the exceptional set of $f_\lambda$ for Lebesgue-almost every $z$ and every fixed $\lambda$. The Cesàro means will allow us to get the $\mathcal{F}$-invariance of $\mathcal{M}$. The compactness of the support of $\mathcal{M}$ will be obtained from the following special case of a classical result of Ueda (see [Ued98, Theorem 2.1]).

**Lemma 3.7.** The family of all inverse branches of $f^n$ on $B(a, \tau r)$, when $n$ runs over $\mathbb{N}$, is a normal family in $O(B(a, \tau r), \mathbb{P}^k)$. In particular, for any $\alpha > 0$, we may reduce $r$ so that $d_{\text{pc}}(\gamma^n_s(z), \gamma^n_s(a)) \leq \alpha$ for all points $z \in B(a, \tau r)$ and all branches $\gamma^n_s$ given by Lemma 3.5.

We may now state the main result of this step.

**Lemma 3.8.** Let $D_0 \times B_0$ be any neighbourhood of $a$ contained in $B(a, \tau r)$. There exists a positive measure $\mathcal{M}$ on $O(D_0, \mathbb{P}^k)$ such that a subsequence of $(\mathcal{M}^n)_n$ converges to $\mathcal{M}$ and the probability measure $\mathcal{M} := \frac{\mathcal{M}}{||\mathcal{M}||}$ is an equilibrium web for the restricted family $f|_{D_0 \times \mathbb{P}^k}$.

**Proof.** By definition we have

$$||\mathcal{M}^n|| = \frac{1}{n} \sum_{r=1}^n ||m^r|| = \frac{1}{n} \sum_{r=1}^n \frac{1}{\text{vol}(B_0)} \int_{B_0} ||m^n(z)|| \, d\text{vol}(z) = d^{-kn} ||\mathcal{S}_{r,n}||,$$

which, by the last assertion of Lemma 3.3, yields the existence of $\alpha \in [0, 1[$ such that

$$\lim_{n} ||\mathcal{M}^n|| = 1 - \alpha.$$

Since the family of all inverse branches of $f^n$ above $B(a, \tau r)$, when $n \in \mathbb{N}$, is a normal family in $O(B(a, \tau r), \mathbb{P}^k)$, due to Lemma 3.7, the family of their restrictions to $D_0 \times \{z\}$ when $z \in B_0$ is normal as well. In other words, there exists a compact subset $\mathcal{K}$ of the space $(O(B(a, \tau r), \mathbb{P}^k), d_{\text{loc}})$ such that $\text{supp}(\mathcal{M}^n) \subset \mathcal{K}$ for all integers $n$. By Banach-Alaoglu theorem, the sequence $(\mathcal{M}^n)_n$ admits a cluster value $\mathcal{M}$ with support in $\mathcal{K}$.

Let us now show that $\mathcal{M}$ is $\mathcal{F}$-invariant. To this purpose, we first establish the following estimate:

$$a_n(z) := ||m^{n-1}(z) - \mathcal{S}_s m^n(z)|| \leq \frac{|\mathcal{S}_{r,n-1}|}{d^k(n-1)} - \frac{|\mathcal{S}_{r,n}|}{d^k n} =: a_n, \forall z \in B_0.$$  

(3.6)
We observe that \( \mathcal{F}(\gamma^n_{s,z}) = \gamma^{n-1}_{i_n(s),z} \). Indeed, by Lemma 3.5 we have, for every \( \lambda \in D_0 \):

\[
\mathcal{F}(\gamma^n_{s,z})(\lambda) = f_\lambda(\gamma^n_{s}(\lambda, z)) = \Pi_{hk} \circ f \circ \gamma^n_{s}(\lambda, z) = \Pi_{hk} \circ \gamma^{n-1}_{i_n(s)}(\lambda, z) = \gamma^{n-1}_{i_n(s),z}(\lambda),
\]

where \( \Pi_{hk} : M \times \mathbb{P}^k \rightarrow \mathbb{P}^k \) is the canonical projection. Hence, by partitioning \( \mathcal{F}_{\varepsilon,r,n} \) on the values of \( i_n(s) \), where the function \( i_n \) is defined in Lemma 3.3, one gets

\[
\mathcal{F}_s m^n(z) = d^{-kn} \sum_{s \in \mathcal{F}_{\varepsilon,r,n}} \mathcal{F}_s(\delta_{s,n}) = d^{-kn} \sum_{s \in \mathcal{F}_{\varepsilon,r,n}} \delta_{s,n-1} = d^{-kn} \sum_{s \in \mathcal{F}_{\varepsilon,r,n}} |i_n^{-1}(s)|\delta_{s,n-1},
\]

and thus \( m^{n-1}(z) - \mathcal{F}_s m^n(z) = d^{-kn} \sum_{s \in \mathcal{F}_{\varepsilon,r,n-1}} (d^k - |i_n^{-1}(s)|)\delta_{s,n-1} \). Then, as \( |i_n^{-1}(s)| \leq d^k \) (see Lemma 3.3), we obtain

\[
a_n(z) \leq d^{-kn} \sum_{s \in \mathcal{F}_{\varepsilon,r,n-1}} d^k - |i_n^{-1}(s)| = d^{-k(n-1)}|\mathcal{F}_{\varepsilon,r,n-1}| - d^{-kn} \sum_{s \in \mathcal{F}_{\varepsilon,r,n-1}} |i_n^{-1}(s)|,
\]

which is the desired inequality, since \( \sum_{s \in \mathcal{F}_{\varepsilon,r,n-1}} |i_n^{-1}(s)| = |\mathcal{F}_{\varepsilon,r,n}| \).

It immediately follows from (3.6) that

\[
(3.7) \quad ||m^{r-1} - \mathcal{F}_s m^r|| \leq \frac{1}{\text{vol}(B_0)} \int_{B_0} a_r(z) d\text{vol}(z) \leq a_r.
\]

Now, since

\[
\mathcal{M}^n - \mathcal{F}_s \mathcal{M}^n = \frac{1}{n} \sum_{r=1}^n m^r - \frac{1}{n} \sum_{r=1}^n \mathcal{F}_s m^r = \frac{1}{n} \sum_{r=2}^n (m^{r-1} - \mathcal{F}_s m^r) + \frac{m^n - \mathcal{F}_s m^n}{n},
\]

one deduces from (3.7) that

\[
||\mathcal{M}^n - \mathcal{F}_s \mathcal{M}^n|| \leq \frac{1}{n} \sum_{r=2}^n ||m^{r-1} - \mathcal{F}_s m^r|| + 2 \leq \frac{1}{n} \sum_{r=2}^n a_r + \frac{2}{n}
\]

\[
\leq \frac{1}{n} \left( \frac{|\mathcal{F}_{\varepsilon,r,1}|}{d^k} - \frac{|\mathcal{F}_{\varepsilon,r,n}|}{d^{kn}} \right) + 2 \leq \frac{3}{n}.
\]

Since \( \mathcal{F}_s \) is continuous for the weak topology, this proves that \( \mathcal{F}_s \mathcal{M} = \mathcal{M} \).

Now \( \mathcal{M} := \frac{1}{\text{vol}(\mathcal{A})} \) is a compactly supported \( \mathcal{F} \)-invariant probability measure on the metric space \( (\mathcal{O}(\mathcal{A}, \mathcal{A}), \mathbb{P}^k, d_{\text{loc}}) \), and it remains to prove that \( (p_\lambda)_* (\mathcal{M}) = \mu_\lambda \) for any \( \lambda \in D_0 \). For \( (\lambda, z) \in D_0 \times B_0 \) and \( n \in \mathbb{N} \), let us set \( \mu_\lambda^n(z) := d^{-kn}(f_\lambda^m)^*\delta_z \). Then, by definition, we have \( 0 \leq (p_\lambda)_* (\mathcal{M}^n(z)) \leq \frac{1}{n} \sum_{r=1}^n \mu_\lambda^n(z) \) for every \( z \in B_0 \) and therefore:

\[
(3.8) \quad 0 \leq (p_\lambda)_* (\mathcal{M}^n) \leq \frac{1}{\text{vol}(B_0)} \int_{B_0} \frac{1}{n} \sum_{r=1}^n \mu_\lambda^n(z) d\text{vol}(z).
\]

As the exceptional set of \( f_\lambda \) is a proper pluripolar subset of \( \mathbb{P}^k \) [FS95], it follows from the equidistribution theorem [DS10] that \( (\mu_\lambda^n(z))_n \) is weakly converging to \( \mu_\lambda \) for almost every \( z \in B_0 \). Then, by Lebesgue convergence Theorem, the right hand side of (3.1) is weakly converging to \( \mu_\lambda \) and thus

\[
p_{\lambda_* \mathcal{M}} \leq \mu_\lambda, \forall \lambda \in D_0.
\]
Since \( \| p_{\lambda^*} \| = \| \mathcal{M} \| = 1 - \alpha \) where \( \alpha \in [0, 1] \) we are done if \( \alpha = 0 \). Otherwise this allows to write \( \mu_\lambda = (1 - \alpha) \frac{(p_{\lambda^*})_1}{1-\alpha} + \alpha \frac{\lambda - (p_{\lambda^*})_1}{\alpha} \) which, by the ergodicity of \( \mu_\lambda \), implies that \( \frac{(p_{\lambda^*})_1}{1-\alpha} = p_{\lambda^*}(\mathcal{M}) = \mu_\lambda \), as well. \( \square \)

**Step 5:** The equilibrium web \( \mathcal{M} \) is acritical.

We first show that we can reduce \( r \) and the neighbourhood \( D_0 \times B_0 \) of \( a \) so that \( D_0 \times B_0 \subset B(a, \tau r) \), where \( 0 < \tau < 1 \) is given by Lemma 3.6, and, moreover, for every \( p \geq 0 \) and every branch \( \gamma^n_s \) used in the definition of \( \mathcal{M} \), there exists a single holomorphic function defining \( f^{op}(Y) \) on some open set containing \( \gamma^n_s(B(a, \tau r)) \).

To do that, we first reduce \( r \) so that \( \gamma^n_s(B(a, \tau r)) \) is contained in the domain of definition of some holomorphic section of the canonical projection \( \pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k \) for any \( \gamma^n_s \). It suffices for that to use Lemma 3.7 with \( a > 0 \) small enough. Then we shrink \( D_0 \times B_0 \) so that \( D_0 \times B_0 \subset B(a, \tau' \tau r) \), and that the family \( f|_{D_0 \times \mathbb{C}^k} \) can be lifted to some holomorphic family \( F|_{D_0 \times \mathbb{C}^{k+1}} \) of non-degenerate homogeneous polynomials of \( \mathbb{C}^{k+1} \). Let \( C_F \) be the critical set of \( F \). As the maps \( F^{op} : D_0 \times \mathbb{C}^{k+1} \to D_0 \times \mathbb{C}^{k+1} \) are proper, the function \( \varphi_p(z) := \Pi_{i=1}^d \text{Jac} F(\rho_i(z)) \), where \( \{\rho_1(z), \ldots, \rho_d(z)\} \) is the set of preimages of \( z \) counted with multiplicities, is a holomorphic defining function for the analytic hypersurface \( F^{op}(C_F) \) on \( D_0 \times \mathbb{C}^{k+1} \). It follows that for any open set \( U \) in \( \mathbb{P}^k \) on which \( \pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k \) admits a holomorphic section, each analytic set \( f^{op}(C_F) \) is defined by a single holomorphic function on \( D_0 \times U \).

The following fact will be crucial. Recall that \( \mathcal{D} \) is the set of one-dimensional lines \( \Delta \subset \mathbb{C}^{m+k} \) passing through \( a \).

**Lemma 3.9.** If \( B(a, \rho) \subset \Omega(R_f) \) then the series \( \sum_{(\geq 0)} \int_{\mathcal{D}} \| 1_B(a, \rho) R_{l, f} \wedge [\Delta] \| \) is converging.

**Proof.** Recall the Crofton formula \( \int_{\mathcal{D}} |\Delta| = (dd^c \log |z - a|)^{m+k-1} \) (see for example [Dem, Corollary III.7.11]). In particular, since the p.s.h function \( \log |z - a| \) is locally bounded on \( \mathbb{C}^{m+k} \setminus \{a\} \), the currents \( u \int_{\mathcal{D}} |\Delta| \) and \( dd^c u \wedge \int_{\mathcal{D}} |\Delta| \) are well defined for any p.s.h function \( u \) (see [Dem, Proposition III.4.1]). Taking for \( u \) any local potential of \( R_{l, f} \) and exploiting the fact that \( u \) is bounded in a neighborhood of \( a \), one may check that \( R_{l, f} \wedge [\Delta] \) is \( \mathcal{L} \)-integrable, and that \( \int_{\mathcal{D}} R_{l, f} \wedge [\Delta] = R_{l, f} \wedge \int_{\mathcal{D}} [\Delta] \). We then have for any \( N \in \mathbb{N} \):

\[
\sum_{l=0}^{N} \int_{\mathcal{D}} 1_B(a, \rho) R_{l, f} \wedge [\Delta] = \sum_{l=0}^{N} \left( \int_{\mathcal{D}} 1_B(a, \rho) R_{l, f} \wedge [\Delta] \right) = \sum_{l=0}^{N} 1_B(a, \rho) R_{l, f} \wedge \int_{\mathcal{D}} [\Delta] = \sum_{l=0}^{N} \left( \int_{\mathcal{D}} 1_B(a, \rho) R_{l, f} \wedge [\Delta] \right).
\]

The assertion follows. \( \square \)

We have to show that \( \mathcal{M}(\mathcal{J}_s) = 0 \) where \( \mathcal{J}_s \) is the set of \( \gamma \in \mathcal{J} \) whose graphs \( \Gamma_\gamma \) do meet the grand orbit of \( C_f \), or equivalently \( Y \), by \( f \) (see definition 2.3). The following lemma reduces the problem to estimating the mass of some open subsets of \( (\mathcal{J}, d_{\text{loc}}) \).

For any integer \( p \), we define

\[
Y_p := \{ \gamma \in \mathcal{J} : \Gamma_\gamma \cap f^{op}(Y) \neq \emptyset \text{ and } \Gamma_\gamma \nsubseteq f^{op}(Y) \}.
\]
Lemma 3.10. If \( \mathcal{M}(Y_p) = 0 \) for all \( p \), then \( \mathcal{M}(\mathcal{I}_s) = 0 \).

Proof. Note that \( \mathcal{I}_s = \bigcup_{n \in \mathbb{N}} \mathcal{I}_s^n \), where \( \mathcal{I}_s^n := \{ \gamma \in \mathcal{I} : \Gamma \gamma \cap (f_{\text{on}})^{-1}(\bigcup_{n \in \mathbb{N}} f_{\text{op}}(Y)) \neq \emptyset \} \) and that \( \mathcal{I}_s^0 = \bigcup_{n \in \mathbb{N}} \mathcal{Y}_p \) where \( \mathcal{Y}_p := \{ \gamma \in \mathcal{I} : \Gamma \gamma \cap f_{\text{op}}(Y) \neq \emptyset \} \). As the measure \( \mathcal{M} \) is \( \mathcal{I} \)-invariant, the conclusion follows immediately from the inclusion \( \mathcal{I}_s^n \subset (\mathcal{F}_{\text{on}})^{-1}(\mathcal{I}_s^0) \) and the fact that \( \mathcal{M}(\mathcal{Y}_p \setminus Y_p) = 0 \) (see [BBD18, end of proof of Corollary 1.7]). \[ \square \]

Let \( p \geq 0 \) be a fixed integer. It follows from Hurwitz lemma that \( Y_p \) is an open subset of \((\mathcal{O}(D_0, \mathbb{P}^k), d_{\text{luc}})\). We will show that \( \lim_n \mathcal{M}^n(Y_p) = 0 \).

By the very definition of \( \mathcal{M}^n \) and since \( D_0 \times B_0 \subset B(a, \tau^p \tau r) \), we have:

\[
\mathcal{M}^n(Y_p) \leq d^{-kn} \left| \{ s \in \mathcal{I}_{\varepsilon, r, n} : \gamma_s^n(B(a, \tau^p \tau r)) \cap f_{\text{op}}(Y) \neq \emptyset \} \right|.
\]

Let \( \varphi_{p, n} \) be a holomorphic function defining \( f_{\text{op}}(Y) \) on \( \gamma_s^n(B(a, \tau)) \) and set

\[
\{ s_1, s_2, \ldots, s_N_n \} := \{ s \in \mathcal{I}_{\varepsilon, r, n} : 0 \in \varphi_{p, n} \circ \gamma_s^n(B(a, \tau^p \tau r)) \}.
\]

Then the estimate (3.9) can be written as

\[
\mathcal{M}^n(Y_p) \leq d^{-kn} N_n.
\]

For every \( 1 \leq k \leq N_n \), let us set

\[
E_k := \{ \Delta \in \mathcal{D} : 0 \in \varphi_{p, n} \circ \gamma_s^n(\Delta_{\tau r}) \}.
\]

Then \( \sum_{k=1}^{N_n} \mathcal{I}_{E_k}(\Delta) \) is the number of inverse branches of \( f_{\text{on}} \) whose restrictions to \( \Delta_{\tau r} \) meet \( f_{\text{op}}(Y) \) and thus, for a generic \( \Delta \in \mathcal{D} \), we have

\[
\sum_{k=1}^{N_n} \mathcal{I}_{E_k}(\Delta) \leq |f_{\text{on}}(n+p)(Y) \cap \Delta_{\tau r}| \leq d^{kn+p} \| R_{n+p, f} \wedge [\Delta_{\tau r}] \|.
\]

On the other hand, since \( \varphi_{p, n} \circ \gamma_s^n \) vanishes on \( B(a, \tau^p \tau r) \) for every \( 1 \leq k \leq N_n \), it follows from Lemma 3.6 that \( \mathcal{L}(E_k) > \frac{1}{2} \) which yields:

\[
\frac{N_n}{2} \leq \sum_{k=1}^{N_n} \mathcal{L}(E_k) = \int_{\mathcal{D}} \sum_{k=1}^{N_n} \mathcal{I}_{E_k}(\Delta).
\]

By combining (3.10), (3.11), and (3.12), we obtain that

\[
\frac{1}{2} \mathcal{M}^n(Y_p) \leq d^{kn} \int_{\mathcal{D}} \| R_{n+p, f} \wedge [\Delta_{\tau r}] \|
\]

which, according to Lemma 3.9, implies that \( \lim_n \mathcal{M}^n(Y_p) = 0 \). Since \( \mathcal{M}^n \) is compactly supported in \((\mathcal{O}(D_0, \mathbb{P}^k), d_{\text{luc}})\) and is a weak limit of the sequence \( (\mathcal{M}^n) \) (see Lemma 3.8), it follows that \( \mathcal{M}(Y_p) = 0 \). By Lemma 3.10, the equilibrium web \( \mathcal{M} \) is acritical. \[ \square \]

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