Nature of long range order in stripe forming systems with long range repulsive interactions

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We study two dimensional stripe forming systems with competing repulsive interactions decaying as \( r^{-\alpha} \). We derive an effective Hamiltonian with a short range part and a generalized dipolar interaction which depends on the exponent \( \alpha \). An approximate map of this model to a known XY model with dipolar interactions allows us to conclude that, for \( \alpha < 2 \) long range orientational order of stripes can exist in two dimensions, and establish the universality class of the models. When \( \alpha \geq 2 \) no long-range order is possible, but a phase transition in the KT universality class is still present. These two different critical scenarios should be observed in experimentally relevant two dimensional systems like electronic liquids (\( \alpha = 1 \)) and dipolar magnetic films (\( \alpha = 3 \)). Results from Langevin simulations of Coulomb and dipolar systems give support to the theoretical results.

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Two dimensional isotropic systems in which a short range attractive interaction competes with a repulsive interaction decaying as a power law of the form \( r^{-\alpha} \) have been widely studied [1–8]. These include, as physically relevant examples, the dipolar (\( r^{-3} \)) and the Coulomb (\( r^{-1} \)) interaction as the repulsive part of the total energy of the system. Dipolar interactions competing with exchange and uniaxial anisotropy arise, e.g. in ultra-thin ferromagnetic films with perpendicular anisotropy [9–11] while long range Coulomb interactions appear in low dimensional electron systems and may be relevant to understand the low temperature phase behavior of doped Mott insulators, two dimensional quantum Hall systems and high \( T_c \) superconductors [12–15]. It is well known that under certain conditions of relative strength of interactions and external parameters these systems develop modulated stripe-like structures in two dimensions which break space rotational symmetry, similar to classical liquid-crystal systems, giving rise to smectic, nematic and hexatic phases [16–18]. This analogy, based on the 180° rotational symmetry of stripe structures and elongated liquid-crystal molecules, allowed to apply well known results for liquid-crystal systems [19, 20] to predict the qualitative, and to some extent also quantitative phase behavior of many systems with modulated order parameters. Nevertheless, when it is important to understand the true nature of the thermodynamic phases, the analogy between stripe forming systems and classical liquid-crystals should not be taken at face value. The basic units in liquid-crystals are elongated molecules. A given molecule typically interacts with its near neighbors and due to its elongated form a rotation in 180° of a single molecule does not alter the energy of the system. On the other hand, the smallest relevant scale of a stripe system is the modulation length. At this scale, a basic cell can be considered as containing a single interface and then it is a dipole of opposite densities with an average linear size equal to the modulation length. It is important to note that such dipoles will not be, in general, elementary electric or magnetic dipoles, their character will depend on the nature of the density order parameter under consideration. Having clarified this point, in principle all realistic low energy configurations of the system can be built from these dipole cells. Clearly, a 180° rotation of a dipole does change the energy of the system and then cannot be considered a local symmetry. The system is only symmetric under global rotations of 180°. Furthermore, when long range interactions are present, it is well known that the behavior of the systems may be very different from those with only short range interactions, which represent the vast majority of classical liquid-crystal systems. A study of the nature of low temperature phases of stripe forming systems should take these elements into account.

Consider a coarse grain Hamiltonian in two dimensions of the form

\[
\mathcal{H}[\phi(\vec{x})] = \frac{1}{2} \int d^2 x \left( \nabla \phi(\vec{x}) \right)^2 + \frac{1}{2} \int d^2 x' \int d^2 x \ J(\vec{x}) \phi(\vec{x}') \phi(\vec{x}) + \frac{1}{2 \beta} \int d^2 x \ V(\phi(\vec{x})),
\]

where \( \beta = 1/k_B T \) and \( V(\phi) = -\frac{\alpha}{2} \phi^2 + \frac{\delta}{4} \phi^4 \) is a local potential that could be seen as an entropic contribution and which exact form is not important to our work. The long range repulsive interaction has the form \( J(\vec{x}) = J/|\vec{x}|^\alpha \) which allows to analyze in a unified way short range (large \( \alpha \)) and long range (small \( \alpha \)) interactions. Physically relevant examples are the Coulomb interaction (\( \alpha = 1 \)) and the dipolar interaction between out-of-plane magnetic moments (\( \alpha = 3 \)). It is well known that at low temperatures this kind of systems display stripe-like patterns in the form of spatial modulations of the density \( \phi(\vec{x}) \) [1, 21–23] in a direction represented by a wave vector \( \vec{k}_0 \). Low energy excitations of the stripes can be described in terms of a displacement field \( u(\vec{x}) \) in the form \( \phi(\vec{x}) = \sum_n \phi_n \cos(nk_0 x + nk_0 u(\vec{x})) \), where \( x \) is the average...
direction of the modulation and \( k_0 \) stands for the modulus of \( \vec{k}_0 \). If \( u(\vec{x}) \) varies smoothly in space it is possible to define a local wave vector \( \vec{k}_0 + k_0 \nabla u(\vec{x}) \).

The effective Hamiltonian (1) when expressed in terms of \( u(\vec{x}) \) has local and non-local parts \( \mathcal{H} = \mathcal{H}_l + \mathcal{H}_{nl} \) (see Supplemental Material). Expanding the local component to quadratic order in the fluctuation field \( u \), it can be written in Fourier space as \([19, 22, 24]\):

\[
\mathcal{H}_l = \mathcal{H}_{0l} + \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} (\gamma_x k_x^2 + \gamma_y k_y^4) \hat{u}(\vec{k}) \hat{u}(-\vec{k})
\]

(2)

where \( \gamma_x \) and \( \gamma_y \) are elastic coefficients which are simply related to the parameters of the original Hamiltonian and \( \mathcal{H}_{0l} \) represents the local contribution to the energy for an unperturbed stripe. It is well known that this form for the local fluctuations of the stripe pattern leads to a divergence of the mean square of the displacement field, implying the absence of long range positional order in the system. This is the standard situation in liquid-crystalline systems. We go on to consider the effect of the tail of the long range interaction in the fluctuation spectrum. The non-local component can be taken into account properly by considering the long range interaction between a pair of stripe dipoles as shown schematically in Figure 1. The interaction between a pair of dipoles is given by:

\[
\delta \mathcal{H}_{nl} = \frac{J}{2} \int_{\delta A_1} d^2x \int_{\delta A_2} d^2x' \phi(\vec{x}) \phi(\vec{x}') |\vec{x} - \vec{x}'|^{\alpha}
\]

(3)

where \( \delta A_1 \) and \( \delta A_2 \) are the corresponding areas (see Figure 1). If \( \lambda \) is the modulation length of the stripe pattern, in the limit \( |\vec{x} - \vec{x}'| \gg \lambda \) a multipolar expansion of the interaction (3) leads to (see Supplemental Material):

\[
\mathcal{H}_{nl} = \frac{1}{2} \gamma \int d^2x \int d^2x' \Omega(|\vec{x} - \vec{x}'|) \left( \frac{\partial_y u(\vec{x}) \partial_y' u(\vec{x}')}{|\vec{x} - \vec{x}'|^{\alpha+2}} - (\alpha + 2) \frac{\vec{e}(\vec{x}) \cdot (\vec{x} - \vec{x}') \vec{e}(\vec{x}') \cdot (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^{\alpha+4}} \right).
\]

(4)

In this expression \( \gamma = \alpha J P^2 \) and \( P = \frac{1}{\pi} \int_A d\vec{x} \phi(\vec{x}) \) is the modulus of the dipolar moment. The unit vectors \( \vec{e}(\vec{x}) \) give the orientation of the dipoles which point along the local wave vector of the stripe pattern and \( \Omega(x) \) is a short range cutoff. Here we have neglected fluctuations in the modulation length and accordingly the elastic coefficient \( \gamma \) is evaluated in its mean field value (see Supplemental Material for a discussion on relevant fluctuations). As we can see from the expression obtained, the long range repulsive interaction is responsible for a generalized dipolar contribution to the total energy.

Considering again small fluctuations in the direction of the wave vector \( \vec{k}_0 \), we can write \( \vec{e}(\vec{x}) \approx \frac{\vec{k}_0}{k_0} + \nabla u(\vec{x}) \), which leads (considering that \( \vec{k}_0 \) points in \( x \) direction) to:

\[
\Delta \mathcal{H}_{nl} = \frac{1}{2} \gamma \int d^2x \int d^2x' \Omega(|\vec{x} - \vec{x}'|) \left( \frac{\partial_y u(\vec{x}) \partial_y' u(\vec{x}')}{|\vec{x} - \vec{x}'|^{\alpha+2}} - (\alpha + 2) \frac{(y-y')^2 \partial_y u(\vec{x}) \partial_y' u(\vec{x}')}{|\vec{x} - \vec{x}'|^{\alpha+4}} \right).
\]

(5)

Thus, the effective Hamiltonian for the displacement field \( u(\vec{x}) \) results:

\[
\Delta \mathcal{H} = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \left( \gamma_x k_x^2 + \gamma_y k_y^4 + \gamma_{nl} k^{\alpha-2} k_y^4 \right) \hat{u}(\vec{k}) \hat{u}(-\vec{k})
\]

(6)

where \( \gamma_{nl} = \gamma \alpha (2 + \alpha)^2 C(\alpha + 4) \) and the function \( C(\alpha) = 2^{2-\alpha} \Gamma\left(\frac{2-\alpha}{2}\right) / \Gamma\left(\frac{\alpha}{2}\right) \) with \( \Gamma(x) \) being the Gamma function. From the previous considerations we are now in a position to analyze the stability of positional and orientational order of the stripe structures when the long range interactions are taken into account.

**Positional order:** From the effective Hamiltonian (6) we can see that for \( \alpha \geq 2 \) the \( \gamma_y k_y^4 \) term dominates over \( \gamma_{nl} k^{\alpha-2} k_y^4 \) in the long wavelength limit, i.e. for sufficiently short range interaction no positional order is possible. If \( \alpha < 2 \) the term \( \gamma_{nl} k^{\alpha-2} k_y^4 \) dominates over \( \gamma_y k_y^4 \), but even in this long range interacting regime it is easy to check that the average square fluctuations \( \langle u^2 \rangle = k_B T \int \frac{d^2k}{(2\pi)^2} \left( \gamma_x k_x^2 + \gamma_y k_y^4 + \gamma_{nl} k^{\alpha-2} k_y^4 \right)^{-1} \) diverge with some power of the system size for any \( \alpha > 0 \).

**Orientialional order:** It is well known that in systems with short range interactions orientational order can be weakened by the presence of topological defects \([22, 25]\). The typical situation in two dimensional systems with continuous symmetries is that only quasi-long range order is possible when interactions are of sufficiently short range \([26]\). Nevertheless it is commonly argued that even in systems with long range interactions, like Coulomb or dipolar interactions, shielding effects make the effective interactions short ranged. Here we

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FIG. 1. Schematic representation of the long range interaction between two elementary stripe dipoles.
revisit this question, considering explicitly the effects of the range of the interactions and show that, although the shielding occurs, the effective interactions are still capable of stabilizing a long-range-ordered nematic phase in two dimensions for long enough interaction range.

At low temperatures the stripe structure can be thought of as composed by a mosaic of domains of average size \( \xi_u \) corresponding to the correlation length of the displacement field \( u(\vec{x}) \). The orientation of each domain is a natural order parameter which can be described by a unit vector \( \vec{n} \). This vector represents the mean orientation of the elementary dipoles inside a domain and consequently it is defined in terms of the unit vectors \( \vec{e}(\vec{x}) \) previously defined in (4) as:

\[
\vec{n}(\vec{x}) = \frac{\int_{\Delta A_u} d^2x \, \vec{e}(\vec{x})}{\int_{\Delta A_u} d^2x \, \vec{e}(\vec{x})}, \tag{7}
\]

where \( \Delta A_u \) is the area of the domain, and it is over this area that a coarse graining process is made. Proceeding as in the analysis of positional order, we can separate the contribution to the orientational energy into two parts, a local part coming from interactions between nearby domains and a non-local one due to interactions between far apart domains. In the long wavelength limit, at the scale of the correlation length \( \xi_u \), the effective interaction between nearby domains will be of the form:

\[
\Delta H_{el} = \frac{\gamma_{el}}{2} \int d^2x (\nabla \theta)^2(\vec{x}) \tag{8}
\]

where \( \theta(\vec{x}) \) is the angle between two neighboring domains pointing along directions \( \vec{n} \) and \( \vec{n}' \). The elastic coefficient \( \gamma_{el} \) can be estimated to be \( J_\alpha^2 P^2/(4\xi_u^2) \). To continue with our analysis we realize that over length of order \( \xi_u \), deviations of the local directors \( \vec{e}(x) \) are small. This means after a coarse graining process, the interactions between far apart well polarized domains (of typical size \( \xi_u \times \xi_u \)) has the same form of Eq. (4):

\[
H_{onl} = \frac{\gamma}{2} \int d^2x \int d^2x' \Omega(||\vec{x} - \vec{x}'||) \left( \vec{n}(\vec{x}) \cdot \vec{n}(\vec{x}') \right) \left( \frac{\vec{n}(\vec{x}) \cdot (\vec{x} - \vec{x}')\vec{n}(\vec{x}) \cdot (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^2 \alpha^4 + |\vec{x} - \vec{x}'|^2 \alpha^2} \right) \tag{9}
\]

as a consequence of the principle of superposition. Then, \( H_{onl} = H_{el} + H_{onl} \) is the complete orientational effective Hamiltonian. This is one of the main results of our work (see Supplemental Material). Note that usually the effective orientational energy is taken to be composed only by the local part, corresponding to smooth variations in the mean directions of neighboring striped domains. We will see in the sequel that the presence of the second (non-local) term can potentially change the universality class of the orientational order in the system. A renormalization group study of the orientational effective Hamiltonian \( H_{onl} \) has been done before in Ref. 27 for the case \( \alpha = 1 \), which corresponds to a dipolar XY model. In that reference, the authors were able to renormalize the model and, importantly, they showed that the universal properties are not changed by the presence of the anisotropic part of the interaction. Furthermore, they showed that a whole family of models with isotropic long range interactions of the form \( \int \delta(\vec{k}) \vec{S}(\vec{k}) \vec{S}(\vec{-k}) \) behave in qualitatively the same way as the dipolar XY model as long as the range \( \sigma < 2 \). Once the mapping between these models and ours is established then the critical properties of the stripe forming systems are known. In fact, the Fourier transform of the isotropic term in Eq. (9) is proportional to \( \int \delta(\vec{k}) \vec{S}(\vec{k}) \vec{S}(\vec{-k}) \). Then, one immediately see that for \( \alpha \geq 2 \) the leading term in \( H_{onl} \) is quadratic in \( k \). In this case the low temperature physics of the system is that of the two dimensional short range XY model, i.e. there is a phase transition of the Kosterlitz-Thouless type at a critical temperature \( T_{KT} \). In a system with dipolar interactions \( \alpha = 3 \) and then we expect it to have an isotropic-nematic phase transition of the KT type, as anticipated in previous works based on analysis of fluctuations of the local part of the effective Hamiltonian [23, 28]. In this case nematic order is quasi-long-range with algebraically decaying correlations. However, when \( \alpha < 2 \) the physics changes according to the results of Ref. 27. Now, the non-local part in \( H_{onl} \) is relevant and rules the low temperature phase transition. In fact, the long range nature of the interactions in this sector are able to stabilize a nematic phase with truly long range order below a critical temperature \( T_c \). It is possible to show, in the framework of renormalization group equations, that the critical properties of the systems for \( \alpha < 2 \) show some peculiar characteristics, for example [27]:

- in the critical region, the correlation length diverges exponentially at \( T_c \), from both sides, as \( \xi_o \propto \exp \left( \frac{b}{\sqrt{T_c - T}} \right) \), reminiscent of the KT transition behavior.
- For \( T < T_c \) in the critical region, the average dipolar moment behaves as \( M \propto \xi_o(T)^{-(2-\alpha)/2} \), showing the existence of long range order when \( \alpha < 2 \).
- The orientational susceptibility diverges as \( \chi_o \propto \xi_o(T)^{\alpha} \) in the critical region.

This kind of behavior should be observable, e.g. in systems with long range Coulomb interactions for which \( \alpha = 1 \). This case maps onto the dipolar XY model analyzed in 27 and the results may be relevant to understand the phase behavior of two dimensional electron systems. In the next section we show results from computer simulations of systems with \( \alpha = 1 \) (Coulomb) and \( \alpha = 3 \) (dipolar) which give support to the different scenarios in both systems as described before.

**Simulation results:** We performed Langevin simulations of the Hamiltonian (1). The relaxational (overdamped) Langevin dynamics of the density \( \phi(\vec{x}) \) is defined in reciprocal space by:

\[
\frac{\partial \phi(\vec{k}, t)}{\partial t} = -A_{\alpha}(k)\phi(\vec{k}, t) - u[\phi^3](\vec{k}, t) + \eta(\vec{k}, t) \tag{10}
\]

where \( A_{\alpha}(k) \) is the spectrum of fluctuations, i.e. the Fourier transform of the quadratic part of the effective Hamiltonian.
For the numerical simulations we have used an implicit first-order scheme for the numerical integration of (10) in the Fourier space, a procedure that guarantees good numerical stability with time step \( dt = 0.1 \), as established in previous works [29, 30]. In the adimensional form, the periodicity of the stripes are set by the lattice constant \( dx \) of a 2d square grid with linear size \( L = MN \), so that \( k = (k_x/dx,k_y/dx) \) with \( k_i = 2\pi n_i/L \) and \( dx = \pi/M \). Within this scheme, the stripe length span \( M \) lattice sites and the linear system size is such that contains \( N \) stripes. We fixed \( M = 11 \) in order to have smooth domain walls.

After an estimation of the equilibration and correlation times from high temperature quenches, we performed slow cooling experiments and found that below \( T \simeq 0.59 \) \((T \simeq 0.45)\) the dipolar (Coulomb) systems find themselves in the low temperature phases (with orientational order) for all system sizes. Above those temperatures the configurations are in a state usually called liquid of stripes, where both positional and orientational correlation lengths are finite. So we concentrated on equilibrium simulations for \( T = 0.57 \) \((T = 0.43)\) for system sizes ranging from \((12 \times 11)^2\) up to \((66 \times 11)^2\).

The previous analysis implies that, in the limit of large system sizes, the orientational susceptibility for interactions with \( \alpha = 1 \) and \( \alpha = 3 \) should be qualitatively different for \( T < T_c \). In the Coulomb case, the second order nature of the phase transition should imply that the susceptibility must be finite when \( N \to \infty \). On the other hand, for dipolar interactions the transition should be of the KT type, implying a monotonic (logarithmic) increase of \( \chi_o \) with system size, which should diverge in the thermodynamic limit for all \( T \leq T_{KT} \). Results for the orientational susceptibility as a function of the linear system size \((N = L/11)\) from simulations are shown in Fig. 2 for the two characteristic temperatures cited above, corresponding to the low temperature phase of each model. Although computational limitations prevent us to reach very large system sizes, it is clearly observed that the susceptibility of the Coulomb system \((\alpha = 1)\) first grows with \( N \) but eventually suffers a crossover and then saturates at a fixed value for the largest sizes. On the other hand, the susceptibility in the dipolar system \((\alpha = 3)\) shows a power law increase with system size, a behavior consistent with that of a KT-like critical phase. Of course, we cannot conclude that \( \chi_o \) will not saturate at larger \( N \)'s, but the different trend observed in both systems for equivalent parameter values is a strong indication that the theoretical results are indeed correct.

In summary, we have shown that two-dimensional stripe forming systems with isotropic competing interactions can be classified into two universality classes: for sufficiently short-range interactions a Kosterlitz-Thouless transition from an isotropic to a quasi-long-range orientational order phase takes place with the well known phenomenology of defect-mediated phase transitions; but, for sufficiently long-range repulsive interactions a second order phase transition with some unusual characteristics drives the system from the isotropic to a fully long-range orientational order phase. These results improve considerably the understanding of the nature of phase transitions in stripe forming systems and may be relevant to a wide variety of systems, particularly the strong correlated regime of two dimensional “electronic liquid-crystals” phases and modulated phases in ultrathin magnetic films with perpendicular anisotropy.

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Supplemental information

Nature of long range order in stripe forming systems with long range repulsive interactions

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Here we show the fundamental steps which lead to the effective positional and orientational Hamiltonians, and discuss the main approximations considered.

Our starting point is the following coarse grained Hamiltonian for a scalar field \( \phi(\vec{x}) \) containing competing interactions and a local potential:

\[
\mathcal{H}[\phi(\vec{x})] = \frac{1}{2} \int d^2 x \left( \nabla \phi(\vec{x}) \right)^2 + \frac{1}{2} \int d^2 x \int d^2 x' \phi(\vec{x}) J(\vec{x} - \vec{x}') \phi(\vec{x}') + \frac{1}{2\beta} \int d^2 x V(\phi(\vec{x})),
\]

(1)

where \( \beta = 1/k_B T \) and \( V(\phi) = -\frac{\alpha}{2} \phi^2 + \frac{\gamma}{4} \phi^4 \). This particular form used for the local potential is not essential for the conclusions of our study, we chose this particular form of \( V(\phi) \) for simplicity in the numerical simulations.

The non-local interaction considered by us corresponds to a kernel of the form \( J(\vec{x}) = J/|\vec{x}|^\alpha \), and as a consequence the interacting part of the original Hamiltonian can be written in Fourier space in the form:

\[
\mathcal{H}_i[\phi(\vec{x})] = \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} A(k) \hat{\phi}(\vec{k}) \hat{\phi}(-\vec{k}),
\]

(2)

where \( A(k) \) represents the fluctuation spectrum, which is isotropic and given by \( A(k) = k^2 + JC(\alpha)k^{\alpha-2} \), where \( k \) stands for the modulus of the vector \( \vec{k} \). Here we have used the well known result:

\[
\mathcal{F} \left[ \frac{1}{r^n} \right] = C(\alpha)k^{\alpha-2}
\]

(3)

where \( C(\alpha) = 2^{2-\alpha} \Gamma \left( \frac{2-\alpha}{2} \right) / \Gamma(\frac{\alpha}{2}) \) and \( \mathcal{F}[...] \) represents the Fourier transform operator (see e.g. Ref. where a detailed discussion is present, and references therein). For \( \alpha < 4 \) the function \( A(k) \) has a minimum at a finite value of wave vector \( k_m \).

It is known that in the class of systems described by the effective Hamiltonian (1) the competition between the short range attractive interaction and the long range repulsive interaction gives rise to a modulation in the order parameter in the form of stripes, at least at zero external field. We now proceed to analyze local fluctuations of a typical stripe pattern.

Locally, one can assume without loss of generality that the modulation profile extends in the "x" direction. Then, the scalar order parameter can be written in the general form \( \phi(\vec{x}) = \sum_{n>1} \phi_n \cos(nk_0 x) \), where the amplitudes \( \phi_n \) and \( k_0 \) are parameters which minimize the free energy. In particular, note that \( k_0 \) will be in general a temperature dependent parameter proportional to the inverse modulation length of the stripe solution. We can now study the energy cost of perturbing such a solution by a fluctuation \( u(\vec{x}) \) in the phase of the modulation. The fluctuation profile changes to:

\[
\phi(\vec{x}) = \sum_{n \geq 1} \phi_n \cos(nk_0 x + nk_0 u(\vec{x})).
\]

(4)

In the small fluctuations limit the Fourier transform of \( \phi \) is given by:

\[
\hat{\phi}(\vec{k}) = \sum_{n \geq 1} \phi_n \frac{(2\pi)^2}{2} \left( \delta(\vec{k} - n\vec{k}_0) + \delta(\vec{k} + n\vec{k}_0) \right) - \sum_{n \geq 1} \phi_n \frac{n k_0}{2} \left( u(\vec{k} - n\vec{k}_0) - u(\vec{k} + n\vec{k}_0) \right),
\]

(5)

where \( \vec{k}_0 = k_0 \hat{e}_x \). Substituting this form of \( \phi \) back into equation (2) and neglecting zero average terms we get for the leading contribution:

\[
\mathcal{H}_i = \frac{V}{4} \sum_{n \geq 1} A(nk_0) \phi_n^2 + \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \mathcal{B}(\vec{k}) \hat{\phi}(\vec{k}) \hat{\phi}(-\vec{k}),
\]

(6)

where

\[
\mathcal{B}(\vec{k}) = \sum_{n \geq 1} \phi_n^2 n^2 k_0^2 \frac{(2\pi)^4}{4} \left( A(\vec{k} - n\vec{k}_0) + A(\vec{k} + n\vec{k}_0) \right).
\]

(7)
and \( V \) is the system size. In the long wavelength limit \((k \to 0)\) the dominant contribution to the spectrum function in (7) is around \( \pm nk_0 \), where \( k_0 \) satisfies:

\[
\frac{\partial}{\partial k_0} \left( \sum_{n \geq 1} \frac{1}{4} A(nk_0) \phi_n^2 \right) = 0.
\] (8)

An expansion of (7) in the long wavelength limit leads to:

\[
B(\vec{k}) = \gamma_x k_x^2 + \gamma_y k_y^4.
\] (9)

Equations (6) to (9) imply that the local effective Hamiltonian

\[
H_l = H_{0l} + \frac{1}{2} \int d^2k \left( \frac{\gamma_x k_x^2 + \gamma_y k_y^4}{(2\pi)^2} \right) \hat{u}(\vec{k}) \hat{u}(-\vec{k})
\] (10)

is of a more general nature than usually assumed. In particular, it is not restricted to a single mode approximation which should limit its validity only to a neighborhood of the critical point. This result also implies that although inclusion of higher order modes enhances the values of the elastic coefficients \( \gamma_x \) and \( \gamma_y \), this growing stiffness will not be able to drive the system to a state with long range or quasi long range positional order.

To proceed further, it is important to note that the previous expansion of \( B(\vec{k}) \) in the long wave limit actually represents small deviations in the local wave vector of the modulation \((k_0 + k_0 \nabla u(\vec{x}))\) from its optimum value. This means that regions far away from \( k_0 \) were neglected in the original spectrum \( A(k) \) in equation (2). In particular, energy contributions coming from the long range tail of the repulsive interaction were left out from the previous analysis.

We go on to the calculation of the explicit interaction between far apart domains (basic cells), considering the basic cell as a region containing a single interface with average length equal to the modulation length \( \lambda \), as depicted in Figure 1.

![FIG. 1. Schematic representation of the long range interaction between two elementary stripe dipoles.](image)

The interaction between a pair of dipolar cells is given by:

\[
\delta H_{nl} = \frac{J}{2} \int_{\delta A_1} d^2x \int_{\delta A_2} d^2x' \frac{\phi(\vec{x})\phi(\vec{x}')}{|\vec{x} - \vec{x}'|^{\alpha}}.
\] (11)

In the limit of interest, \(|\vec{x} - \vec{x}'| \gg \lambda\), the integral can be calculated to any order by a multipolar expansion. Keeping terms up to second order we find:

\[
\delta H_{nl} = \frac{J}{2} \left( \frac{Q_1 Q_2}{|\vec{x}_{21}|^\alpha} - \alpha \frac{Q_2 \vec{x}_{21} \cdot \vec{P}_1}{|\vec{x}_{21}|^{\alpha+2}} + \alpha \frac{Q_1 \vec{x}_{21} \cdot \vec{P}_2}{|\vec{x}_{21}|^{\alpha+2}} + \alpha \frac{\vec{P}_2 \cdot \vec{P}_1}{|\vec{x}_{21}|^{\alpha+2}} - \alpha(\alpha + 2) \frac{(\vec{P}_2 \cdot \vec{x}_{21})(\vec{P}_1 \cdot \vec{x}_{21})}{|\vec{x}_{21}|^{\alpha+2}} \right) \delta A_1 \delta A_2,
\] (12)
where $\vec{x}_{21}$ represents $\vec{x} - \vec{x}'$. $Q_i = (1/\delta A_i) \int d^2x \ \phi(\vec{x})$ is a generalized “charge” and $\bar{P}_i = (1/\delta A_i) \int d^2x \ \phi(\vec{x})$ is a generalized “dipole moment”. Then, up to second order, the interaction between two basic cells is composed by a term like a charge-charge interaction, two terms of type dipole-charge interaction and a generalized dipole-dipole type interaction.

The total contribution of this interaction energy in the continuum limit is:

$$
\mathcal{H}_{nl} = \frac{J}{2} \int d^2x \int d^2x' \left( \frac{Q(\vec{x}) Q(\vec{x}')}{|\vec{x}_{21}|^\alpha} - \alpha \frac{Q(\vec{x}) \vec{P}(\vec{x}_2) \cdot \vec{P}(\vec{x}_1)}{|\vec{x}_{21}|^{\alpha+2}} + \alpha \frac{Q(\vec{x}) \vec{P}(\vec{x}_2) \cdot \vec{P}(\vec{x}_1)}{|\vec{x}_{21}|^{\alpha+2}} \right) \Omega(|\vec{x}_{21}|),
$$

(13)

where $\Omega(|\vec{x}_{21}|)$ is a short distance cutoff. Now, translational invariance can be invoked to conclude that $(Q(\vec{x}_2) \bar{P}(\vec{x}_1)) = (Q(\vec{x}_1) \bar{P}(\vec{x}_2))$, where $\langle \cdots \rangle$ means a thermal average. This implies that the “dipole-charge” interactions do not contribute to the free energy. In other words, charge and dipolar degrees of freedom are decoupled, and because of this the “charge-charge” interactions are irrelevant for the orientational order. Thus, the effective Hamiltonian with the degrees of freedom responsible for orientational order reduces to:

$$
\mathcal{H}_{nl} = \frac{J}{2} \int d^2x \int d^2x' P(\vec{x}) P(\vec{x}') \Omega(|\vec{x} - \vec{x}'|) \left( \frac{\vec{e}(\vec{x}) \cdot \vec{e}(\vec{x}')}{|\vec{x} - \vec{x}'|^{\alpha+2}} - (\alpha + 2) \frac{\vec{e}(\vec{x}) \cdot (\vec{x} - \vec{x}') \vec{e}(\vec{x}') \cdot (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^{\alpha+4}} \right),
$$

(14)

which is equation (4) in the letter, where $\vec{e}(\vec{x}) = \bar{P}(\vec{x})/P(\vec{x})$ represents the unit vector pointing in the direction of the dipolar moment of a given cell. The directors can be written in terms of the displacement fields $u(\vec{x})$ as $\vec{e}(\vec{x}) \approx \frac{\vec{u}_1}{k_0} + \nabla u(\vec{x})$ and then, to the leading order in $u(\vec{x})$ Eq. (14) can be written as:

$$
\mathcal{H}_{nl} = \Delta \mathcal{H}_{nl} + \frac{J}{2} \alpha \int d^2x \int d^2x' P(\vec{x}) P(\vec{x}') \Omega(|\vec{x} - \vec{x}'|) \left( \frac{\partial_{\vec{u}} u(\vec{x}) \partial_{\vec{u}'} u(\vec{x}')}{|\vec{x} - \vec{x}'|^{\alpha+2}} - (\alpha + 2) \frac{(y - y')^2 \partial_{\vec{u}} u(\vec{x}) \partial_{\vec{u}'} u(\vec{x}')}{|\vec{x} - \vec{x}'|^{\alpha+4}} \right).
$$

Up to this point, the dipolar moments in the orientational effective Hamiltonian can fluctuate both in orientation and in magnitude. It is worth to note that in this kind of systems a disordered stripe structure or “liquid of stripes” develops at energies or temperatures larger than the establishment of orientational or translational order. After phase separation has been settled in the form of a liquid of stripes, the stiffness of the stripes in the longitudinal direction greatly exceeds the stiffness in the transversal direction. Actually, this is a reflection of the existence of Goldstone soft modes in the transversal direction, evident in the isotropic nature of the spectrum of fluctuations at wave vector $k_m$ in the stripe liquid state, which plays a major role in the kind of order at low temperature in these systems. The possible order-disorder transition and phase separation in these stripe systems are in different energy scales. It is around the phase separation transition point that fluctuations in the width of the stripes (magnitude of generalized dipolar moments) are important. At lower temperatures, when the liquid stripe structure has already been settled, longitudinal fluctuations are very small. Then, in order to study the nature of orientational order, it seems justified to consider $|\bar{P}(\vec{x})|$ as being constant. In this way we arrive at the final form of the orientational effective Hamiltonian (eq. (5) in the letter):

$$
\Delta \mathcal{H}_{nl} = \gamma \int d^2x \int d^2x' \Omega(|\vec{x} - \vec{x}'|) \left( \frac{\partial_{\vec{u}} u(\vec{x}) \partial_{\vec{u}'} u(\vec{x}')}{|\vec{x} - \vec{x}'|^{\alpha+2}} - (\alpha + 2) \frac{(y - y')^2 \partial_{\vec{u}} u(\vec{x}) \partial_{\vec{u}'} u(\vec{x}')}{|\vec{x} - \vec{x}'|^{\alpha+4}} \right)
$$

(15)

where $\gamma = \alpha J P^2$. It is instructive to analyse the non-local effective Hamiltonian in Fourier space. It is known that

$$
\mathcal{F} \left[ \frac{1}{r^\alpha} \right] = C(\alpha) k^{\alpha - 2},
$$

(16)

where $C(\alpha) = 2^{2-\alpha} \Gamma(\frac{2-\alpha}{2})/\Gamma(\alpha)$ and $k = \sqrt{k_x^2 + k_y^2}$. Then it is straightforward to conclude that $\mathcal{F} \left[ \frac{1}{r^{\alpha+2}} \right] = C(\alpha + 2) k^{\alpha}$, and

$$
\mathcal{F} \left[ \frac{y^2}{r^{\alpha+4}} \right] = -C(\alpha + 4) \frac{\partial^2 (k^{2+\alpha})}{\partial k_y^2} = -C(\alpha + 4)(2 + \alpha) k^{\alpha} - C(\alpha + 4)(2 + \alpha) k^{\alpha - 2} k_y^2.
$$

(17)

With these results the energy (15) can be written in Fourier space as:

$$
\Delta \mathcal{H}_{nl} = \frac{\gamma}{2} \int \frac{d^2k}{(2\pi)^2} \left[ C(\alpha + 2) k^{\alpha} + C(\alpha + 4)(2 + \alpha)^2 k^{\alpha} + C(\alpha + 4)(2 + \alpha)^2 \alpha k^{\alpha - 2} k_y^2 \right] \delta(k) \delta(-k).
$$

(18)
Noting that \( C(\alpha + 2) + C(\alpha + 4)(2 + \alpha)^2 \equiv 0 \), the previous result simplifies to:

\[
\Delta H_{nl} = \frac{\gamma}{2} \int \frac{d^2k}{(2\pi)^2} k^{\alpha-2} k_y^4 \hat{u}(\vec{k}) \hat{u}(-\vec{k})
\]

(19)

where \( l = C(\alpha + 4)(2 + \alpha)^2 \alpha \). Finally, summing the local contribution of eq. (10) with the non local one of eq. (19) we arrive at the effective Hamiltonian for the positional degrees of freedom (eq. (6) of the letter):

\[
\Delta H = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} (\gamma_x k_x^2 + \gamma_y k_y^4 + \gamma_{nl} k^{\alpha-2} k_y^4) \hat{u}(\vec{k}) \hat{u}(-\vec{k})
\]

(20)

with \( \gamma_{nl} = \gamma \alpha (2 + \alpha)^2 C(\alpha + 4) \).

This is the relevant effective Hamiltonian which determines the nature of positional order in the stripe system. The next step is to obtain an effective Hamiltonian for the orientational degrees of freedom. Following the previous method of analysis, one can argue that the effective orientational Hamiltonian will be composed by two contributions. The local one, that have been obtained in previous works, can be seen as the result of a coarse graining process over lengths of the order of the positional correlation length \( \xi_u \). Beyond this lengthscale the director field of the stripes will show deviations which can be modelled, in the small deviation regime, by an effective orientational local Hamiltonian of the form:

\[
\Delta H_{ol} = \frac{\gamma_{ol}}{2} \int d^2x (\nabla \theta)^2(\vec{x})
\]

(21)

where \( \theta(\vec{x}) \) is the angle between two neighboring domains pointing along directions \( \vec{n} \) and \( \vec{n}' \). Note that these stripe domains have linear size of order \( \xi_u \), much bigger than the size of elementary dipoles considered previously for the study of positional order. The elastic coefficient \( \gamma_{ol} \) can be estimated to be \( J \alpha^2 P^2/(4 \xi_u^{\alpha-2}) \). On the other hand, the interaction between domains separated by distances much larger than \( \xi_u \) can be taken into account following formally the same steps as done for the non local interaction between positional degrees of freedom. The effective interaction between domains will take the same form as (14):

\[
\Delta H_{ond} = \frac{\gamma}{2} \int d^2x \int d^2x' \Omega(|\vec{x} - \vec{x}'|) \left( \frac{\hat{n}(\vec{x}) \cdot \hat{n}(\vec{x}')}{|\vec{x} - \vec{x}'|^{\alpha+2}} - (\alpha + 2) \frac{\hat{n}(\vec{x}) \cdot (\vec{x} - \vec{x}') \hat{n}(\vec{x}') \cdot (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^{\alpha+4}} \right).
\]

(22)

where \( \gamma = \alpha J P^2 \). Thus, \( \Delta H_o = \Delta H_{ol} + \Delta H_{ond} \) is the complete orientational effective Hamiltonian. To study the stability of an ordered orientational phase against small deviations of the local directors \( \hat{n}(\vec{x}) \) a spin wave analysis can be done. In the spin wave limit the orientational Hamiltonian reduces to:

\[
\Delta H_o = \frac{\gamma_{ol}}{2} \int d^2x (\nabla \theta)^2(\vec{x}) + \frac{\gamma}{2} \int d^2x \int d^2x' \Omega(|\vec{x} - \vec{x}'|) \left( \frac{\theta(\vec{x}) \theta(\vec{x}')}{|\vec{x} - \vec{x}'|^{2+\alpha}} - (2 + \alpha) \frac{\Delta y^2 \theta(\vec{x}) \theta(\vec{x}')}{|\vec{x} - \vec{x}'|^{4+\alpha}} \right).
\]

(23)

From Eq. (19) the Fourier transform is easily found to be:

\[
H_o = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} (\gamma_{ol} k_x^2 + \gamma_{nl} k^{\alpha-2} k_y^2) \hat{\theta}(\vec{k}) \hat{\theta}(-\vec{k})
\]

(24)

with \( \gamma_{nl} = \gamma \alpha (2 + \alpha)^2 C(\alpha + 4) \).