

\section{Introduction}

Recently some general holographic and string theory features of 2D CFTs featuring $S_N$ orbifolds have been studied [1–6]. These are based on large $N$ expansion of the CFT which translates into a semiclassical expansion of the dual gravity/string theory. In particular the growth of states and phases of the partition function have been considered as pointers to a holographic string theory interpretation. For strings in 10D and 26D Minkowski space, it was demonstrated convincingly that the corresponding $S_N$ orbifold gives the correct Virasoro amplitudes [7, 8]. In [1, 4, 5], a very interesting higher spin symmetry group was identified in the specific model of $\mathbb{T}^4$ orbifolds classifying the states of tensionless strings on $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$ through extended $W$-symmetry [9, 10]. This greatly extends the earlier holographic studies of supersymmetric $S_N (\mathbb{T}^4/K3)$ orbifold CFTs representing D1–D5 branes [11–15], where the sector of chiral primary states [16] was studied in correspondence with (super) gravity on $\text{AdS}_3 \times S^3$. Here explicit construction of all chiral primary operators and evaluation of their three-point functions was performed in [17–26]. These were shown to compare with the gravitational interactions of compactified Supergravity in $\text{AdS}_5$. A concrete implementation of the ‘stringy exclusion principle’ was also demonstrated in these
constructions [17, 20]. For the case of general orbifold CFTs recent study on the growth of states and phases of the partition function points more generally to a holographic string theory interpretation [2, 3, 5, 6, 27].

In the present paper we are concerned with general $S_N$ orbifold CFTs with the purpose of describing interactions between general primary states. We begin by considering characters, associated with general primaries which we use as characteristic invariant variables in the $S_N$ orbifold CFT. We formulate a novel procedure for evaluating characters in general $S_N$ orbifold CFT based on the $gl(\infty)_k$ algebra\(^2\). This allows one to follow some analogies with the $U(N)$ group case, and we give a representation evaluating general characters (as polynomials of a basic set). Our representation is seen to agree (in explicit comparison) with the results of Bantay [29] who has given a beautiful mathematical procedure for orbifold characters through representations of the double. The procedure that we present will have some practical advantages, in particular we will use it to study the interacting features of primaries. In parallel with our earlier work [30] involving $W_N$ minimal model CFTs and the simple model of $U(N)$ we will describe the interactions in terms of a hierarchy of Hamiltonians. The hamiltonians are constructed with the property that general characters appear as exact polynomial eigenfunctions. These will be demonstrated to exhibit certain locality properties in the emergent space–time.

The content of the paper is as follows. In section 2 we start with describing the ‘twisted’ Hilbert space of an $S_N$ orbifold CFT representing $N$ copies of some seed (free) CFT. We define the basic set of characters associated with class primaries and give a procedure for constructing all characters as generalized Schur polynomials. To demonstrate this procedure we work out explicitly the $N = 3$ case of the $S_3$ orbifold showing agreement with the expressions of Bantay. Higher cases of $N = 4, 5$ are exhibited in the appendix. In section 3 we present the Hamiltonian(s) introduced to govern the nonlinear structure of general $S_N$ orbifold characters. We motivate this construction through the case of $U(N)$ group characters. We give the geometric interpretation of interactions through the ‘stringy’ processes of joining and splitting. Through appropriate Fourier transforms we exhibit locality of these interactions. In section 4 we present an evaluation of partition functions, as an application of our character construction, making contact with previous works.

2. $S_N$ orbifold

2.1. Hilbert space

To define Hilbert space [31] of $S_N$ orbifold CFT one starts by introducing twisted boundary conditions of primary operators\(^3\) $\phi_I$ ($I = 1, 2, \cdots, N$)

$$\phi_I(\sigma + 2\pi) = \phi_{g(I)}(\sigma) \quad (I = 1, 2, \cdots, N), \quad (1)$$

where $g$ is an element of $S_N$, and can be represented by the partition of $N$

$$g = \lambda_1 \lambda_2 \cdots \lambda_N \quad \text{with} \quad \sum_{n=1}^{N} n \lambda_n = N. \quad (2)$$

\(^2\) A similar approach can be found in [28] where $gl(\infty)_k$ was used to analyze the partition function of the two dimensional free boson.

\(^3\) In this section, we consider only one primary operator for simplicity. In general, there can be more than one primary operator. For this case, we will later generalize the analysis in the next section.
Note that elements in the same conjugacy class of \( S_N \) represent the same boundary condition. Hence, the conjugacy class \([g]\) characterizes the twisted sector \( \mathcal{H}_{[g]} \) with the boundary condition \([g]\). The full Hilbert space is decomposed into the twisted sectors. In each twisted sector represented by the conjugacy class, it is sufficient\(^4\) to consider an invariant subspace under the centralizer subgroup \( C_g \) of \([g]\) given by

\[
C_{[g]} = \prod_{n=1}^{N} \left( S_{\lambda_n} \rtimes \mathbb{Z}_{\lambda_n} \right),
\]

For the given Hilbert space \( \mathcal{H} \) of the seed CFT, the Hilbert space \( \mathcal{H}_{[g]} \) of the twisted sector associated with the boundary condition \([g]\) can be written as

\[
\mathcal{H}_{[g]} = \bigotimes_{n=1}^{N} \left[ (\mathcal{H}_n \otimes \cdots \otimes \mathcal{H}_n)^{S_n} \right]^{l_n \text{ times}},
\]

where \( \mathcal{H}_n \) is the \( \mathbb{Z}_{\lambda_n} \) invariant subspace of \( \mathcal{H}^{S_n} \) with the boundary condition corresponding to one cycle of length \( n \). Moreover, \( \mathcal{M}^{S_n} \) denotes the \( S_{\lambda_n} \) invariant subspace of \( \mathcal{M} \).

This is true for any \( S_N \) orbifold CFT whether the Hilbert space \( \mathcal{H} \) of the seed CFT is a chiral sector or a full sector. Recall that we usually focus on the holomorphic part of the full Hilbert space of the two-dimensional CFTs. Thus, we will also consider the holomorphic part of the orbifold CFT. Since \( S_N \) invariant states consist of holomorphic and anti-holomorphic part, the holomorphic part is not necessarily \( S_N \) invariant. Hence, one may relax the twisted boundary condition \((1)\) by introducing phase factor depending on \( g \)

\[
\phi_I(z^2) = e^{i \theta_I} \phi_I(z) \quad (I = 1, 2, \cdots, N)
\]

as was discussed in the cyclic orbifold case [32]. The phase factor of the holomorphic part can be cancelled with that of the anti-holomorphic part. For simplicity, let us consider one cycle of length \( a \) in \( S_a \) for \( \phi_I (I = 1, 2, \cdots, a) \) i.e.

\[
[g] = a^1.
\]

A consistent phase factor in \((5)\) is an irreducible representation \( e^{2\pi i \theta} \) of \( \mathbb{Z}_a \) (\( \theta = 0, \cdots, a - 1 \)). Namely, for the given cycle \([g] = a^1\), we have the following boundary condition labeled by \( \theta \) (\( \theta = 0, 1, \cdots, a - 1 \)).

\[
\phi_I(z e^{2\pi i}) = e^{2\pi i \theta} \phi_{I+1}(z) \quad (I = 1, 2, \cdots, a - 1),
\]

\[
\phi_a(z e^{2\pi i}) = e^{2\pi i \theta} \phi_1(z).
\]

To construct the Hilbert space satisfying the boundary condition, we define a projection operator \( P_{a,\theta} \) as follows

\[
P_{a,\theta} \equiv \frac{1}{a} \sum_{j=0}^{a-1} \exp \left[ \frac{2\pi i (-\theta + L_0)}{a} j \right],
\]

where we rescaled the length of the space by \( 1/a \) (e.g. \( 2\pi a \rightarrow 2\pi \)). It is easy to see that \( P_{a,\theta} \) is indeed a projection operator\(^5\).

\(^4\) The \( S_N \) invariance of the state can be realized by averaging over the \( S_N \) elements. The summation over \( S_N \) can be decomposed into summation over the centralizer and over the conjugacy class. At the level of character (partition function), the average over the conjugacy class is trivial.

\(^5\) When considering the holomorphic part only, one can find a problem in \((10)\) for the case of a primary with non-integer conformal dimension. In this case, one may redefine \( \theta \) by absorbing the conformal dimension of the primary into \( \theta \). And, this prescription is related to taking ‘regular part’ of characters of the seed CFT in [29].
\[
P^2_{a,\theta} = P_{a,\theta}, \quad P^i_{a,\theta} = P_{a,\theta}, \quad \sum_{\theta=0}^{a-1} P_{a,\theta} = 1. \tag{10}
\]

Hence, it projects the tensor product space \( H^{\otimes a} \) onto the subspace satisfying the boundary condition (8). E.g.,

\[
e^{-\frac{2\pi i}{a} L_a} P_{a,\theta} = e^{-\frac{2\pi i}{a} L_a} P_{a,\theta}. \tag{11}
\]

Using the projection operator, one can show that the trace over the subspace satisfying the twisted boundary condition (8) becomes

\[
\text{tr} \left( P_{a,\theta} q^{L_a} P_{a,\theta} \right) = \text{tr} \left( q^{L_a} P_{a,\theta} \right) = \frac{1}{a} \sum_{\theta=0}^{a-1} e^{-\frac{2\pi i}{a} \chi \left( \frac{\tau + j}{a} \right)}. \tag{12}
\]

where \( \chi(\tau) \) is the character of the seed CFT corresponding to the primary \( \phi \). In general, an element \([g]\) in \( S_N \) can be represented as

\[
[g] = 1^{2\lambda_2} \cdots N^{\lambda_N}. \tag{13}
\]

In (13), each cycle of length \( a \) is accompanied by the phase factor which is the irreducible representation \( e^{\frac{2\pi i}{a}} \) of \( Z_a (\theta = 0 \cdots a - 1) \). Hence, one can classify the \( \lambda_a \) cycles of length \( a \) according to \( \theta \):

\[
\lambda_a = \sum_{\theta=0}^{a-1} l_{a,\theta}, \tag{14}
\]

where \( l_{a,\theta} \) is the number of cycles of length \( a \) with the phase factor \( e^{\frac{2\pi i}{a}} \). Hence, the phase factor in (5) can be represented by

\[
\delta_{[g]} \sim \text{diag} (\delta_{1,1}, \cdots \delta_{1,1}, \delta_{2,1}, \cdots \delta_{2,1}, \delta_{2,2}, \cdots \delta_{2,2}, \cdots \delta_{N,N}, \cdots \delta_{N,N}), \tag{15}
\]

where \( \delta_{a,\theta} \equiv e^{\frac{2\pi i}{a}} \). Symmetry group of the twisted sector \( H_{e^{\alpha|g}} \) is not the centralizer subgroup \( C_g \) in (3), but is given by

\[
C_{e^{\alpha|g}} = \prod_{a=1}^{N} \prod_{\theta=0}^{a-1} (S_{l_{a,\theta}} \times \mathbb{Z}_{l_{a,\theta}}^{l_{a,\theta}}). \tag{16}
\]

Recall that the holomorphic part is transformed in the irreducible representation under \( \mathbb{Z}_{l_{a,\theta}}^{l_{a,\theta}} \) action. Likewise, the holomorphic part is not necessarily invariant under \( S_{l_{a,\theta}} \) in (16). For instance, if both of the holomorphic and anti-holomorphic parts are fully anti-symmetric, the full state is invariant. Hence, we will also consider irreducible representations with respect to \( S_{l_{a,\theta}} \). Since the twisted sector \( H_{e^{\alpha|g}} \) can be decomposed according to the choice of \( l_{a,\theta} \)’s, we can write the subsector \( H^{L}_{e^{\alpha|g}} \) as

\[
H^{L}_{e^{\alpha|g}} = \bigotimes_{a=1}^{N} \left( \bigotimes_{\theta=0}^{a-1} \frac{(H_{a,\theta} \otimes \cdots \otimes H_{a,\theta})}{l_{a,\theta} \text{ times}} \right), \tag{17}
\]

where \( L \) represents the specific choice of \( l_{a,\theta} \) for all \( a, \theta \), and will be defined fully in the next section. See (38) and (40). As mentioned, note that we consider tensor products of \( H_{a,\theta} \) because we consider all possible irreducible representations with respect to \( S_{l_{a,\theta}} \). (See \( S_{l_{a,\theta}} \) invariant subspace in (4).)

The Hilbert space \( H_{a,\theta} \) can be viewed as an infinite-dimensional vector space, and one can define a natural \( gl(\infty) \) action on \( H_{a,\theta} \) [28]. One may calculate character of \( q^{\frac{1}{2}(L_0 - \frac{c}{24})} \)
which is an element of the Cartan subgroup of the Lie group \( GL(\infty)_c \) corresponding to the Lie algebra \( gl(\infty)_c \). The character of \( q^{1/2(l_{\alpha} - \frac{c}{2\pi})} \) in the fundamental representation is given by

\[
q^{1/2(l_{\alpha} - \frac{c}{2\pi})} = \text{diag}(q^{\epsilon_1}, q^{\epsilon_2}, q^{\epsilon_3}, \ldots),
\]

where \( \epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \ldots \) is the spectrum of \( H_{a,\theta} \).

To calculate character of the higher representation, we consider a tensor product of \( H_{a,\theta} \), i.e., \((H_{a,\theta})^\otimes_{l_{\alpha}}\). By Schur–Weyl duality, the tensor product \((H_{a,\theta})^\otimes_{l_{\alpha}}\) can be decomposed into irreducible subspace with respect to \( S_{l_{\alpha}} \) and \( GL(\infty)_c \), i.e.

\[
(H_{a,\theta})^\otimes_{l_{\alpha}} = \sum_{R_{a,\theta}^S} S_{l_{\alpha}}^{R_{a,\theta}} \otimes V_{GL(\infty)_c}^{R_{a,\theta}},
\]

where \( S_{l_{\alpha}}^{R_{a,\theta}} \) and \( V_{GL(\infty)_c}^{R_{a,\theta}} \) is an irreducible subspace corresponding to the irreducible representation \( R_{a,\theta} \) with respect to \( S_{l_{\alpha}} \) and \( GL(\infty)_c \), respectively. Here \( R_{a,\theta} \) is a Young diagram with \( l_{\alpha,\theta} \) boxes. Hence, the twisted sector \( \mathcal{H}^{L,R}_{c^{\otimes_{l_{\alpha}}}_g} \) can be further decomposed according to \( R_{a,\theta} \) for each \( a, \theta \). We define the twisted sub-sector \( \mathcal{H}^{L,R}_{c^{\otimes_{l_{\alpha}}}_g} \) by

\[
\mathcal{H}^{L,R}_{c^{\otimes_{l_{\alpha}}}_g} = \bigotimes_{a=1}^{N} \bigotimes_{\theta=0}^{a-1} \left( S_{l_{\alpha}}^{R_{a,\theta}} \otimes V_{GL(\infty)_c}^{R_{a,\theta}} \right).
\]

where \( \mathcal{R} \) is a collection of \( R_{a,\theta} \)’s. See (43).

With this preparation we can calculate characters by tracing over the irreducible subspace in the same way as is done for \( U(N) \) characters. As an analogue of basic \( U(N) \) traces \( \phi_m = \text{tr}(U^m) \) we introduce:

\[
\sum_{v \in \mathcal{H}_{a,\theta}} \frac{1}{m!} \langle v | \otimes \cdots \otimes (q^{1/2(l_{\alpha} - \frac{c}{2\pi})} | v) \otimes \cdots \otimes | v \rangle = \text{tr}_{\mathcal{H}_{a,\theta}} \left( P_{a,\theta}^m q^{1/2(l_{\alpha} - \frac{c}{2\pi})} P_{a,\theta} \right)
\]

\[
= \frac{1}{a} \sum_{j=0}^{a-1} e^{-2 \pi i \theta j / a} \frac{m^r + j}{a} \equiv \phi_{a,m,\theta}(\tau).
\]

Then generally, as in the case of \( U(N) \), the trace over the irreducible subspace \( V_{GL(\infty)_c}^{R_{a,\theta}} \) leads to Schur-type polynomials corresponding to the irreducible representation \( R_{a,\theta} \) in terms of the basic set of variables with \( m = 1, 2, \ldots \). However, note that we have to calculate the trace of \( q^{1/2(l_{\alpha} - \frac{c}{2\pi})} \) over the irreducible subspace \( S_{l_{\alpha}}^{R_{a,\theta}} \otimes V_{GL(\infty)_c}^{R_{a,\theta}} \). Since \( q^{1/2(l_{\alpha} - \frac{c}{2\pi})} \) acts trivially on \( S_{l_{\alpha}}^{R_{a,\theta}} \), it gives additional overall factor \( d_{R} \), the dimension of the representation \( R \), to the polynomial

\[
\text{Tr}_{S_{l_{\alpha}}^{R_{a,\theta}} \otimes V_{GL(\infty)_c}^{R_{a,\theta}}} (q^{1/2(l_{\alpha} - \frac{c}{2\pi})}) = d_{R} P_{a,\theta}^{m} (R_{a,\theta}),
\]

where \( P_{a}^{m}(R) \) is the Schur polynomial (of order \( n \)) corresponding to the representation \( R \). This can be also easily obtained by using the projection operator \( \Pi_{k} \) in section 4.1. For the invariant state in the full sector, one has to combine holomorphic part with anti holomorphic part. Note that not all combinations of them are invariant unlike fully symmetric or fully anti-symmetric cases. One has to pick up the invariant subspace from the direct product of \( S_{l_{\alpha}}^{R_{a,\theta}} \) (of the holomorphic part) and \( S_{l_{\alpha}}^{R_{a,\theta}} \) (of the anti-holomorphic part). It is easy to see that \( R_{a,\theta} \) should be equal to \( \mathcal{R}_{a,\theta} \) (therefore, \( l_{\alpha,\theta} = l_{\alpha,\theta} \)). Also, by using the orthogonality of the characters, one can show that the branching coefficient \( c_{0}^{R_{a,\theta}} \) of \( S_{l_{\alpha}}^{0} \) is 1. I.e.
\[ S_{\alpha,\theta}^{R} \otimes S_{\alpha,\theta}^{R^T} = S_{\alpha,\theta}^{R} \oplus \sum_{T} C_{T}^{R} S_{\alpha,\theta}^{T}. \]  

Therefore, the contribution of invariant states associated to the representation \( R_{\alpha,\theta} \) to the full partition function is given by

\[ Z_{R_{\alpha,\theta}} = \frac{1}{(dR)^2} |dR P_{R_{\alpha,\theta}}|^2 = |P_{R_{\alpha,\theta}}|^2. \]  

To establish contact with \([29]\), we divide the (holomorphic) trace in \((22)\) by \(dR\) in advance to define the orbifold character. Recalling the decomposition of the twisted sector in \((20)\), one can conclude that the general character of the \( S_N \) orbifold is a product of Schur polynomials of the basic set. Then, one can use the standard formula of CFT partition function where the partition function is a sum of the absolute square of characters i.e.

\[ Z = \sum \chi_{R}^2. \]  

In section 4.1, we also present another way to obtain the partition function \((24)\) of the twisted sector by constructing states consisting of holomorphic and anti-holomorphic parts from the beginning.

For the orbifold of chiral sector, neither the phase factor nor the arbitrary irreducible representation \( R_{\alpha,\theta} \) of \( S_N \) is allowed because there is no anti-holomorphic part to compensate them. For this case, the only allowed collections of representations \((R_{\alpha,\theta})\) are given by

\[ R_{\alpha,\theta} = \begin{cases} \mathbb{1} & \text{for } \theta = 0 \\ 0 & \text{for } \theta \neq 0 \end{cases} \]  

for \( n = 0, 1, 2, \cdots \).

### 2.2. Characters of \( S_N \) orbifold

Denoting the (genus one) character \( \chi_{\rho}(\tau) \) corresponding to a primary operator \( \rho \) of CFT, we define a basic set of variables \( \Phi^{(\rho)}_{a,m,\theta}(\tau) \):

\[ \Phi^{(\rho)}_{a,m,\theta}(\tau) \equiv \frac{1}{a} \sum_{j=0}^{a-1} \exp \left( -\frac{2\pi i \theta}{a} \right) \chi_{\rho} \left( \frac{mt + j}{a} \right) \]  

representing discrete Fourier transformation between the following sets:

\[ \{ \Phi^{(\rho)}_{a,m,\theta}(\tau) \mid \theta = 0, 1, 2, \cdots, a - 1 \} \quad (a, m : \text{fixed}), \]  

\[ \left\{ \chi_{\rho} \left( \frac{mt + j}{a} \right) \mid j = 0, 1, 2, \cdots, a - 1 \right\} \quad (a, m : \text{fixed}). \]

From expansion of the character \( \chi_{\rho}(\tau) \)

\[ \chi_{\rho}(\tau) = q^{b_{\rho}} c^{\frac{c}{24}} \sum_{j=0}^{\infty} d_{j} q^{j}, \]  

where \( b_{\rho} \) is the conformal dimension of the primary \( \rho \) and \( c \) is the central charge of the seed CFT, the basic variable \( \Phi^{(\rho)}_{a,m,\theta}(\tau) \) takes the form

\[ \Phi^{(\rho)}_{a,m,\theta}(\tau) \]  

Henceforth, primary operator \( \Phi^{(\rho)} \) is denoted by \( \rho \) for simplicity.
Here we define $\omega(p; a, \theta)$ by

$$\omega(p; a, \theta) = \sum_{a = 0}^{\infty} \frac{\omega_{a; \theta}}{a^a} + \sum_{a = 0}^{\infty} \frac{\omega_{a; \theta}}{a^a} \equiv q^{\omega_{a; \theta} + \omega_{a; \theta}}$$

(30)

A general primary operator of $S_N$ orbifold CFT is expressed through an $N$-tuple of primaries in the seed CFT. By permuting components of the $N$-tuple, it can be written as follows

$$\mathcal{P} = \langle p_1, \cdots, p_k \rangle \in S_N$$

with $N = \sum_{j=1}^{k} N_j$.

(32)

where $p_i$ ($i = 1, \cdots, k$) are distinct primaries in $I$ which is a set of all primaries in the seed CFT. The Hilbert space associated with the primary $\mathcal{P}$ can be decomposed into subspaces for $\mathcal{P}_j \equiv \langle p_j, \cdots, p_k \rangle$ in $S_N$ orbifold

$$\mathcal{H}^{S_N} \mathcal{P} = \bigotimes_{j=1}^{k} \mathcal{H}^{S_N} \mathcal{P}_j.$$  

(33)

One can calculate the trace over each $\mathcal{H}^{S_N} \mathcal{P}_j$ subspace in the same way as in section 2.1. It follows that $S_N$ orbifold character denoted by $\Gamma_N(\mathcal{P})$ for the primary $\mathcal{P}$ is a product of $S_N$ orbifold characters $\Gamma_N(\mathcal{P}_j)$ for a primary $\mathcal{P}_j$. Namely

$$\Gamma_N(\mathcal{P}) = \prod_{i=1}^{k} \Gamma_N(\mathcal{P}_j).$$

(34)

Therefore, it is sufficient to consider the character for the $N$-tuple of identical primaries, $\langle p, p, \cdots, p \rangle$. Henceforth, we will consider only $S_N$ orbifold characters for identical primaries for each $N$.

A general $S_N$ orbifold character can be written as a product of the Schur polynomials of $\Phi_{a,\theta}(\tau)$ for each $a, \theta$ denoted by

$$P_a(\tau; a, \theta; R) \equiv P_a(\{ \Phi_{a,1;\theta}(\tau), \Phi_{a,2;\theta}(\tau), \Phi_{a,3;\theta}(\tau), \cdots \}; R),$$

(35)

where $a = 1, 2, \cdots$ and $\theta = 0, 1, 2, \cdots, a - 1$. $R$ is a Young diagram of $S_a$. It is natural to assign a degree $am$ to the variable $\Phi_{a,\theta;\theta}$. i.e.

$$\deg(\Phi_{a,\theta;\theta}) = am.$$  

(36)

Then, the degree of the Schur polynomial of $P_a(a, b; R)$ is given by

$$\deg(P_a(a, b; R)) = na.$$  

(37)

We now introduce a simple procedure to generate general characters of $S_N$ orbifold step by step (for $\mathcal{P} = \langle p, p, \cdots, p \rangle$). In [29], the $S_N$ orbifold characters were also obtained by considering a homomorphism from the fundamental group $\mathbb{Z} \oplus \mathbb{Z}$ to the permutation group $S_N$. Our procedure is based on the Hilbert space picture described in the previous subsection. It will result in a polynomial formula in terms of Schur polynomials.

(i) Consider a partition of $N$. E.g. $N = \sum_{a=1}^{N} a \lambda_a$.

(ii) For each $\lambda_a$ ($a = 1, 2, \cdots, N$), find $a$-tuples of non-negative integers such that

$$l_a = (l_{a,0}, l_{a,1}, \cdots, l_{a,a-1}) \quad \text{with} \quad |l_a| = \sum_{\theta=0}^{a-1} l_{a,\theta} = \lambda_a \quad \text{and} \quad 0 \leq l_{a,\theta}.$$  

(38)
This corresponds to the classification of the Hilbert space according to the boundary condition (phase) in section 2.1. For each \(a\) and \(\lambda_a\), the number of possible \(l_a\) is \(\binom{\lambda_a + a - 1}{a - 1}\). For convenience, we define \(\mathcal{L}\) to be \(N\)-tuple of \(l_a\):

\[
\mathcal{L} = (l_1, l_2, \ldots, l_N) = ((l_{1,0}), \ldots, (l_{a,0}, \ldots, l_{a,a-1}), \ldots, (l_{N,0}, \ldots, l_{N,N-1})).
\]

Note that \((|\mathcal{L}|, |\mathcal{L}_2|, \ldots, |\mathcal{L}_N|)\) is a partition of \(N\) by definition.

(iii) For each \(l_{a,\theta}\) (\(\theta = 0, 1, 2, \ldots, a - 1\)), choose an irreducible representation \(R_{a,\theta}\) of \(S_{l_{a,\theta}}\).

We define \(\tilde{R}_a\) by

\[
\tilde{R}_a = (R_{a,0}, R_{a,1}, \ldots, R_{a,a-1})
\]

and

\[
|R_{a,\theta}| \equiv (\text{The number of boxes in the Young diagram } R_{a,\theta}) = l_{a,\theta}.
\]

Furthermore, we also define \(\mathcal{R}\) to be a \(N\)-tuple of \(\tilde{R}_a\)’s:

\[
\mathcal{R} = (\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_N) = ((R_{1,0}), \ldots, (R_{a,0}, \ldots, R_{a,a-1}), \ldots, (R_{N,0}, \ldots, R_{N,N-1})).
\]

(iv) Finally, the \(S_N\) orbifold character \(\Gamma_N(p, \mathcal{R})\) equals

\[
\Gamma_N(p, \mathcal{R}) = \prod_{a=1}^{N} \prod_{\theta=0}^{a-1} P_{l_{a,\theta}}(a, \theta; R_{a,\theta}).
\]

From (30), one can see that \(\Gamma_N(p, \mathcal{R})\) takes the form

\[
\Gamma_N(p, \mathcal{R}) = q^{\Delta(p, \mathcal{R}) - \frac{Nc}{24}\rho_0 + \mathcal{O}(q)}
\]

for some non-negative integer \(\rho_0\). Here, \(\Delta(p, \mathcal{R})\) is defined by (see (31) for \(\omega(p; a, \theta)\))

\[
\Delta(p, \mathcal{R}) \equiv \sum_{a=1}^{N} \sum_{\theta=0}^{a-1} |R_{a,\theta}| \omega(p; a, \theta).
\]

Considering the leading term of \(\Gamma_N(p, \mathcal{R})\), one can deduce that \(\Delta\) is the conformal dimension of the primary \(P\).

Since \(P_{l_{a,\theta}}(a, \theta; R_{a,\theta})\) is a homogeneous polynomial of degree \(a l_{a,\theta}\), the character \(\Gamma_N(p, \mathcal{R})\) is a homogeneous polynomial of degree \(N\). I.e.

\[
\Delta(p, \mathcal{R}) \equiv \sum_{a=1}^{N} \sum_{\theta=0}^{a-1} |R_{a,\theta}| \omega(p; a, \theta).
\]

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\[\Delta(p, \mathcal{R}) = \sum_{a=1}^{N} \sum_{\theta=0}^{a-1} |R_{a,\theta}| \omega(p; a, \theta).\]

\[\Delta(p, \mathcal{R}) \equiv \sum_{a=1}^{N} \sum_{\theta=0}^{a-1} |R_{a,\theta}| \omega(p; a, \theta).\]

\[\Delta(p, \mathcal{R}) \equiv \sum_{a=1}^{N} \sum_{\theta=0}^{a-1} |R_{a,\theta}| \omega(p; a, \theta).\]

\[\Delta(p, \mathcal{R}) = \sum_{a=1}^{N} \sum_{\theta=0}^{a-1} |R_{a,\theta}| \omega(p; a, \theta).\]

\[\Delta(p, \mathcal{R}) \equiv \sum_{a=1}^{N} \sum_{\theta=0}^{a-1} |R_{a,\theta}| \omega(p; a, \theta).\]

\[\Delta(p, \mathcal{R}) \equiv \sum_{a=1}^{N} \sum_{\theta=0}^{a-1} |R_{a,\theta}| \omega(p; a, \theta).\]

\[\Delta(p, \mathcal{R}) \equiv \sum_{a=1}^{N} \sum_{\theta=0}^{a-1} |R_{a,\theta}| \omega(p; a, \theta).\]
\[
\text{deg}(\Gamma_N(p, R)) = \sum_{a=0}^{N-1} a \cdot I_{a0} = \sum_{a=1}^{N} a[I_a] = N. \tag{48}
\]

As mentioned, a primary in \(S_N\) orbifold is \(N\)-tuple of primaries and it can be be split into \(N_j\)-tuple of identical primaries (\(N_j \leq N\) and \(1 \leq j \leq k\) for some positive integer \(k\)). See (32). Following the above steps, one can choose \(R_j\) for each \(N_j\)-tuple of the identical primaries to define \(R_{j\cdot}\) denoting the \(k\)-tuple of \(R_j\)'s (\(j = 1, 2, \ldots, k\)), i.e. \(R_{j\cdot} = (R_{i_1}, \ldots, R_{i_k})\). Then, the \(S_N\) orbifold character for the primary \(P\) is given by

\[
\Gamma_N(P, R_{j\cdot}) = \prod_{j=1}^{k} \Gamma_N(P_j, R_j). \tag{49}
\]

In section 4, we will exploit \(\Gamma_N(P, R_{j\cdot})\) to calculate partition functions. We will clarify sub-sectors that we take into account for partition functions, and will show agreement with previous results [1–5, 12, 27, 31, 33].

2.3. Examples: \(S_3\) characters

To demonstrate the procedure and show agreement with examples given by Bantay [29] we now work out explicitly the \(N = 3\) case.

(i) For \(N = 3\), there are three partitions of 3

\[
1 + 1 + 1 = (3, 0, 0), \quad 1 + 2 = (1, 1, 0), \quad 3 = (0, 0, 1). \tag{50}
\]

(ii) For \((3, 0, 0)\), we have only one choice for \(L\)

\[
L = ((3), (0, 0), (0, 0, 0)). \tag{51}
\]

The only non-zero element of \(L\) is \(I_{1,0} = 3\) so that one can choose one of 3 irreducible representations of \(S_{3,1} = S_3\). Then, we have

\[
R = ((\square), (0, 0), (0, 0, 0)), \tag{52}
\]

\[
(\square), (0, 0), (0, 0, 0)), \tag{53}
\]

\[
(\square), (0, 0), (0, 0, 0)), \tag{54}
\]

where 0 represents the trivial representation. From each \(R\), we can obtain the following three characters.

\[
\Gamma_1(R) = P_3(\tau; 1, 0; \square) = \frac{1}{6} \{ (\Phi_{1,1,1})^3 + 3\Phi_{1,1,1}\Phi_{1,2,1} + 2\Phi_{1,3,1} \}
\]

\[
= \frac{1}{6} \{ (\chi(\tau))^3 + 3\chi(\tau)\chi(2\tau) + 2\chi(3\tau) \}. \tag{55}
\]

\[
\Gamma_3(R) = P_3(\tau; 1, 0; \square) = \frac{1}{6} \{ (\Phi_{1,1,1})^3 + 2\Phi_{1,3,1} \}
\]

\[
= \frac{1}{6} \{ (\chi(\tau))^3 - \chi(3\tau) \}. \tag{56}
\]
\[ \Gamma_3(\mathcal{R}) = P_3(\tau; 1, 0; \square) = \frac{1}{6} \left\{ (\Phi_{1;1;1})^3 - 3\Phi_{1;1;1}\Phi_{1;2;1} + 2\Phi_{1;3;1} \right\} \]
\[ = \frac{1}{6} \left\{ (\chi(\tau))^3 - 3\chi(\tau)\chi(2\tau) + 2\chi(3\tau) \right\}. \] (57)

(iii) For \((1, 1, 0)\), there are two choices for \(\mathcal{L}\)
\[ \mathcal{L} = ((1), (1, 0), (0, 0, 0)), \] (58)
\[ \mathcal{L} = ((1), (0, 1), (0, 0, 0)). \] (59)
And for each \(\mathcal{L}\), there is only one choice for \(\mathcal{R}\)
\[ \mathcal{R} = ((\square), (\square, 0), (0, 0, 0)), \] (60)
\[ \mathcal{R} = ((\square), (0, \square), (0, 0, 0)). \] (61)
and, they give the following characters, respectively
\[ \Gamma_1(\mathcal{R}) = P_1(\tau; 1, 0; \square) P_1(\tau; 2, 0; \square) = \Phi_{1;1;1}\Phi_{2;1;2} \]
\[ = \chi(\tau) \frac{1}{2} \left\{ \chi(\frac{\tau}{2}) + \chi(\frac{\tau + 1}{2}) \right\}. \] (62)
\[ \Gamma_1(\mathcal{R}) = P_1(\tau; 1, 0; \square) P_1(\tau; 2, 1; \square) = \Phi_{1;1;1}\Phi_{2;1;1} \]
\[ = \chi(\tau) \frac{1}{2} \left\{ \chi(\frac{\tau}{2}) - \chi(\frac{\tau + 1}{2}) \right\}. \] (63)

(iv) For \((0, 0, 1)\), we have three choices for \(\mathcal{L}\), and the corresponding possible \(\mathcal{R}\) is given by
\[ \mathcal{L} = ((0), (0, 0), (1, 0, 0)) \implies \mathcal{R} = ((0), (0, 0), (\square, 0, 0)), \] (64)
\[ \mathcal{L} = ((0), (0, 0), (0, 1, 0)) \implies \mathcal{R} = ((0), (0, 0), (0, \square, 0)), \] (65)
\[ \mathcal{L} = ((0), (0, 0), (0, 0, 1)) \implies \mathcal{R} = ((0), (0, 0), (0, 0, \square)). \] (66)
Hence, a character corresponding to each \(\mathcal{R}\) equals
\[ \Gamma_1(\mathcal{R}) = P_1(\tau; 3, 0; \square) = \Phi_{3;1;0} \]
\[ = \frac{1}{3} \left\{ \chi(\frac{\tau}{3}) + \chi(\frac{\tau + 1}{3}) + \chi(\frac{\tau + 2}{3}) \right\}, \] (67)
\[ \Gamma_1(\mathcal{R}) = P_1(\tau; 3, 1; \square) = \Phi_{3;1;1} \]
\[ = \frac{1}{3} \left\{ \chi(\frac{\tau}{3}) + e^{-i\pi} \chi(\frac{\tau + 1}{3}) + e^{i\pi} \chi(\frac{\tau + 2}{3}) \right\}, \] (68)
\[ \Gamma_1(\mathcal{R}) = P_1(\tau; 3, 2; \square) = \Phi_{3;1;2} \]
\[ = \frac{1}{3} \left\{ \chi(\frac{\tau}{3}) + e^{i\pi} \chi(\frac{\tau + 1}{3}) + e^{-i\pi} \chi(\frac{\tau + 2}{3}) \right\}. \] (69)
These agree with the $S_3$ orbifold characters found in [29]. We present further examples of $S_4$ and $S_5$ orbifold characters in the tables A1–A11.

3. Field theory of the $S_N$ orbifold

Our expression in section 2 gives a general $S_N$ orbifold character which is a product of the Schur polynomials of $\Phi_{n,m,\nu}'$s. Based on this representation we can now deduce an effective Hamiltonian for the characters. Considering Fock space of $\Phi_{n,m,\nu}'$s, we will establish a Hamiltonian which can be diagonalized by the orbifold characters. The motivation for this representation comes from an analogous construction known for the characters of $U(N)$ which we first summarize as the ‘toy model’.

3.1. Toy model: $U(N)$ character

We start with the Casimir $C_2$ of $U(N)$ given by the expression:

$$C_2(R) = \frac{1}{2}(R + 2\rho) = \frac{1}{2} \sum_{j=1}^{N} r_j (r_j + N - (2j - 1)), \quad (70)$$

where $R = \sum \eta \epsilon_i$ is an irreducible representation of $U(N)$ in the orthogonal basis $\{\epsilon_i| i = 1, 2, \cdots, N\}$, and $\rho$ is the Weyl vector defined to be $\sum_i (\frac{N + 1}{2} - i) \epsilon_i$. The well known Weyl character formula is given by

$$\Theta(R) = \frac{\sum_{\omega \in W} c(\omega) e^{(\lambda + \rho)} \prod_{r \in \tau} r(\omega)^{-1}}{\sum_{\omega \in W} e^{\lambda} \prod_{r \in \tau} r(\omega)} \quad (71)$$

The Hamiltonian $H$ which we are introducing is a representation of the Casimir operator so that the characters are (Schur polynomial) eigenstates:

$$H \Theta(R) = C_2(R) \Theta(R). \quad (72)$$

For this one uses the basic set of variables $\Phi_n$ given by

$$\Phi_n = \sum_{k=1}^{n} (-1)^{k+1} \Theta(R_k^{(n)}), \quad (73)$$

where $R_k^{(n)}$ is a Young diagram defined by $r_i$ which is the number of boxes in the $i$th row as follows.

$$r_i = \begin{cases} 
  n - k + 1, & i = 1 \\
  1, & i = 2, \cdots, k \\
  0, & i = k + 1, \cdots, n 
\end{cases} \quad (k = 1, 2, \cdots, n) \quad (74)$$

For example

$$\Phi_3 = \Theta(\begin{array}{lll} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}) - \Theta(\begin{array}{l} 1 \\ 0 \end{array}) + \Theta(\begin{array}{l} 0 \\ 0 \end{array}). \quad (75)$$

$$\Phi_4 = \Theta(\begin{array}{llll} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}) - \Theta(\begin{array}{ll} 1 & 1 \\ 0 & 0 \end{array}) + \Theta(\begin{array}{ll} 1 & 0 \\ 0 & 0 \end{array}) - \Theta(\begin{array}{l} 0 \\ 0 \end{array}). \quad (76)$$

One can easily check that $\Phi_n$ is a power-sum symmetric polynomial of order $n$. Th $U(N)$ character of the irreducible representation $R$ is a Schur polynomial of $\Phi_n$'s.
\begin{equation}
\Theta(R) = P_{|R|}(R; \{ \phi_i \}) = \frac{1}{n^!} \sum_{g \in S_{|R|}} \text{ch}_R(g) \prod_{i=1}^{\infty} (\phi_i)^{\lambda(g_i)},
\end{equation}

where \text{ch}_R(g) is the character of \( g \) in the irreducible representation \( R \) of the permutation group \( S_{|R|} \). Decomposing the Casimir into a leading (and subleading) terms (in \( 1/N \))

\begin{equation}
C_2(R) = \frac{N}{2} \left[ \sum_{j=1}^{N} r_j + \frac{1}{N} \sum_{j=1}^{N} r_j (r_j - (2j - 1)) \right]
\end{equation}

one can define unperturbed quadratic hamiltonian \( H_2 \) and can represent the perturbation as \( H_3 \) so that:

\begin{equation}
(H_2 + H_3) \Theta(R) = C_2(R) \Theta(R).
\end{equation}

Clearly the quadratic Hamiltonian is easily found to be

\begin{equation}
H_2 \equiv \frac{N}{2} \sum_{n=1}^{\infty} n \phi_n \frac{\partial}{\partial \phi_n}
\end{equation}

giving the desired unperturbed energy

\begin{equation}
H_2 \Theta(R) = \frac{|R|}{2} \Theta(R),
\end{equation}

where \( |R| \) is the total number of boxes in the Young diagram \( R \). I.e. \( |R| \equiv \sum_{i=1}^{N} r_i \)

There are several different ways to obtain the subleading \( 1/N \) generating operator \( H_3 \).

From the (fermionic) Weyl representation of \( U(N) \) characters it can be deduced through bosonization \([34]\). Following a group theoretic way, one can use the fusion rules of conjugacy classes of the \( S_N \) group to derive the same form \([35]\). One has

\begin{equation}
H_3 = \sum_{n=0}^{N} \sum_{m=1}^{n-1} \frac{n}{2} \phi_m \phi_{n-m} \frac{\partial}{\partial \phi_n} + \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{nm}{2} \phi_{n+m} \frac{\partial}{\partial \phi_n} \frac{\partial}{\partial \phi_m},
\end{equation}

where the eigenvalue corresponding to \( \Theta(R) \) is

\begin{equation}
H_3 \Theta(R) = \left( \frac{1}{2} \sum_{j=1}^{N} r_j (r_j - 2j + 1) \right) \Theta(R).
\end{equation}

It is instructive to see the geometrical interpretation of the cubic interaction. For this we interpret the variable \( \phi_n \) (power-sum symmetric polynomial) as a (closed) loop with winding number \( n \). Then, the first term in the cubic interaction \( (83) \) is splitting of one loop into two loops. The second term represents joining of two loops into one loop. Moreover, the coefficients of the cubic interaction is natural because they correspond to the number of all possible ways in such a action (splitting or joining, respectively).

In sum, we have the total Hamiltonian:

\begin{equation}
H \equiv H_2 + H_3 = \frac{N}{2} \sum_{n=1}^{\infty} n \phi_n \frac{\partial}{\partial \phi_n} + \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \frac{n}{2} \phi_m \phi_{n-m} \frac{\partial}{\partial \phi_n} + \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{nm}{2} \phi_{n+m} \frac{\partial}{\partial \phi_n} \frac{\partial}{\partial \phi_m}
\end{equation}

and, its eigenvalue is the Casimir of the corresponding Young diagram

\begin{equation}
H \Theta(R) = C_2(R) \Theta(R).
\end{equation}

Above we have motivated the construction of the Hamiltonian starting from the fact that the characters are explicitly (Schur) polynomial eigenstates. For the \( S_N \) orbifold, we have
expressed, in the previous section, the characters as generalized Schur polynomials. From this we can infer the form of the Hamiltonian operator(s).

3.2. Hamiltonians for $S_N$ orbifold characters

In the case of $U(N)$ one had a sequence of commuting Hamiltonians $H_n$ representing an integrable hierarchy which are essentially given as $\text{tr}(P^m)$ in the matrix model description. We will now describe a similar Hamiltonian hierarchy for the $S_N$ orbifold theory.

We consider a Fock space consisting of all possible $F_{q,p,a,R}$ as in the $U(N)$ case. In this Fock space, one may define Hamiltonian hierarchies $H_n^{\text{orb}}$ which are diagonalized by the $S_N$ orbifold characters $\Gamma_N(\mathcal{P}, \mathcal{R})$ with some eigenvalue $E_n^{\text{orb}}(\mathcal{P}, \mathcal{R})$:

$$H_n^{\text{orb}} \Gamma_N(\mathcal{P}, \mathcal{R}) = E_n^{\text{orb}}(\mathcal{P}, \mathcal{R}) \Gamma_N(\mathcal{P}, \mathcal{R})$$

and, they commute each other

$$[H_n^{\text{orb}}, H_m^{\text{orb}}] = 0$$

so that the orbifold character simultaneously diagonalize them. By generalizing the Hamiltonians in the $U(N)$ example, we now give the Hamiltonian hierarchies, for $n = 2, 3$ describing the $S_N$ orbifold

$$H_2^{\text{orb}} \equiv \sum_{p \in \mathcal{P}, a=0}^{\infty} \sum_{\theta=0}^{a-1} \sum_{m=0}^{\infty} m \omega(p; a, \theta) \psi_{a,m,\theta} \frac{\partial}{\partial \psi_{a,m,\theta}},$$

$$H_3^{\text{orb}} \equiv \sum_{p \in \mathcal{P}, a=0}^{\infty} \sum_{\theta=0}^{a-1} \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{n} \psi_{a,m,\theta} \psi_{a,n-m,\theta} \frac{\partial}{\partial \psi_{a,m,\theta}} \frac{\partial}{\partial \psi_{a,n-m,\theta}} \right] \bigg|_{p=0}^{p=1},$$

where $\epsilon_{p,a,\theta}$ is an arbitrary constant. Moreover, the eigenvalues corresponding to $\Gamma_N(\mathcal{P}, \mathcal{R})$ are given by

$$E_2^{\text{orb}}(\mathcal{P}, \mathcal{R}) = \Delta(\mathcal{P}, \mathcal{R}) \equiv \sum_{p \in \mathcal{P}} \Delta(p, \mathcal{R}),$$

$$E_3^{\text{orb}}(\mathcal{P}, \mathcal{R}) = \sum_{p \in \mathcal{P}, a=0}^{\infty} \sum_{\theta=0}^{a-1} \sum_{j=0}^{N} \sum_{m=0}^{j} \frac{\epsilon_{p,a,\theta}}{2} (R_{a\theta})_j [(R_{a\theta})_j - 2j + 1].$$

Analogous expressions can be given for higher Hamiltonians. The proof for the above follows from two observations:

(i) The orbifold character is a product of the Schur polynomials, and each Schur polynomial $P_{\lambda}(p; a, \theta; R_{a\theta})$ is a function of $\psi_{a,m,\theta}$ for $m = 1, 2, \cdots$. $(a, \theta; R_{a\theta})$ are fixed.

(ii) The Schur polynomial simultaneously diagonalizes the quadratic and cubic Hamiltonians of the $U(N)$ case.

We emphasize that there is an ambiguity in choosing the coefficients $\omega(p; a, \theta)$ and $\epsilon_{p,a,\theta}$. Recall that the eigenvalue of the quadratic Hamiltonian is proportional to the number of boxes in Young diagram. Among the physical quantities in the permutation orbifold, it is the conformal dimension $\Delta(p, \mathcal{R})$ in (46) that is proportional to the number of the Young diagrams $\mathcal{R}$. Therefore, we chose $\omega(p; a, \theta)$ in (31) as the coefficient of the
quadratic Hamiltonian. On the other hand, unlike the case of $U(N)$ character, physical meaning of the eigenvalue $E_3^{orb}(P, R)$ is not clear. Thus, we leave the constant $c_{p,a,\theta}$ undetermined for now.

3.3. Geometrical interpretation

In section 3.1, we have seen the geometrical meaning of $U(N)$ character. That is, $\phi_m$ represents a loop with winding number $m$ and the cubic Hamiltonian splits and joins these loops. The Schur polynomial of $\phi_m$’s representing these loops is an eigenstate of both of quadratic and cubic Hamiltonians.

Since the $S_n$ orbifold character is also a product of Schur polynomials of $\Phi^{(p)}_{a,m,\theta}$, one may put a similar geometrical interpretation on $\Phi^{(p)}_{a,m,\theta}$. Namely, one may think $\Phi^{(p)}_{a,m,\theta}$ as a loop with winding number $m$ like $\phi_m$. However, we need to make an interpretation on other labels such as $a$ and $\theta$. For this we visualize $\chi_p\left(\frac{m\tau + j}{a}\right)$ as a torus associated a primary $p$ where there are two ‘winding numbers’. As mentioned, we already assigned $m$ to one of the winding numbers. We will assign $a$ to the other winding number of the torus which we will call ‘dual winding number’. In addition, $j$ is associated with how the torus is twisted. Then, (26) implies that $\Phi^{(p)}_{a,m,\theta}$ is (discrete) Fourier transformed torus with respect to ‘twist number’ $j$, $\theta$, which is conjugate to twist number $j$, is the analogue of the crystal momentum in the lattice. One way to show such a torus $\Phi^{(p)}_{a,m,\theta}$ graphically is to draw a box diagram with $m$ rows and $a$ columns where the left and the right sides are glued. At the same time, the top and the bottom sides are twisted according to $\theta$, and then are sewn together. See figure 1. For convenience, let us denote such a box diagram by $(p; a, m, \theta)$. This geometrical interpretation is consistent with the illustrations shown in [2]. From (26), one can see that the box diagrams $(p; a, m, b)$ for $\theta = 0, 1, 2, \cdots, a - 1$ are conjugate to $(a \times m)$ box diagrams twisted by $j = 0, 1, \cdots, a - 1$. See figure 1. From the Parseval’s theorem of the (discrete) Fourier transformation, one has

$$\sum_{\theta = 0}^{a - 1} |\text{box diagram } (p; a, m; b)|^2 = \sum_{j = 0}^{a - 1} (a \times m) \text{ box diagram twisted by } j^2. \quad (92)$$

After summing over all possible primaries $p \in \mathcal{I}$, we obtain one of the identities that are useful in calculation of partition functions

$$\sum_{p \in \mathcal{I}} \sum_{\theta = 0}^{a - 1} |\Phi^{(p)}_{a,m,\theta}|^2 = \frac{1}{a} \sum_{j = 0}^{a - 1} Z_{\text{CFI}}\left(\frac{m\tau + j}{a}\right). \quad (93)$$

Similar to $U(N)$ case, the cubic Hamiltonian $H_3^{orb}$ of the orbifold also splits and joins each torus $\Phi^{(p)}_{a,m,\theta}$ with respect to the winding number $m$ (along the red line in the figure 1). Graphically, the first term of $H_3^{orb}$ in (89) represents splitting of a torus $(p; a, m; \theta)$ into two tori $(p; a, m - n; \theta)$ and $(p; a, n; \theta)$. See figure 2. In addition, the second term represents joining of two tori $(p; a, m; \theta)$ and $(p; a, n; \theta)$ into a torus $(p; a, n + m; \theta)$. See figure 3. Note that the Hamiltonians of orbifold does not change the ‘dual winding number’ $\theta$ or primary $p$ in $\Phi^{(p)}_{a,m,\theta}$. Hence, tori with different $a$, $\theta$ and $p$ are decoupled.

3.4. Locality

In the case of D1–D5 brane system captured by the symmetric product orbifold of $\mathbb{T}^4$ or $K3$, the chiral primary operator was studied [17–19, 21–24]. In this case, a local interaction were shown to appear and were constructed in [20].
The interaction features locality in the emerging coordinates. Recall the $\theta$ was interpreted as the Fourier transformation of $j$ in (26). The interaction is local in $\theta$. The other quantum number $m$ features splitting and joining processes which are characteristic of matrix model interactions. By Fourier transformation $e^{im\varphi}$, one can see the locality in $\varphi$. All together we have $\Phi^{(p)}(a, \varphi, \theta)$ with locality in the three coordinates $(a, \varphi, \theta)$

$$H^\text{orb}_{3} \sim \sum_{p \in \mathbb{Z}} \sum_{a=0}^{a-1} \sum_{\theta=0}^{\varphi=\pi} \int d\varphi [\Phi^{(p)}(a, \varphi, \theta)]^2.$$ 

(94)

**Figure 1.** Pictorial Interpretation of $\chi_p\left(\frac{m\tau + j}{a}\right)$ and $\Phi^{(p)}_{a,m,\theta}$. This represents the discrete Fourier transformation between $\chi_p\left(\frac{2\tau + j}{3}\right)$ and $\Phi^{(p)}_{2,1,\theta}$ in (26). The cubic Hamiltonian $H^\text{orb}_{3}$ joins and splits tori along the horizontal line (red line).

**Figure 2.** Splitting of the torus. The coefficients correspond to the number of all the possible ways to cut torus along two red lines.
4. Application: partition functions

In this section, we apply the construction in section 2 to the evaluation and study of partition functions. First, we consider a construction $S_n$ invariant states containing both holomorphic and anti-holomorphic parts. Based on these we can calculate the partition functions, and show that the result agrees with the one that follows from the use of characters (e.g., see (24)). We also consider partition functions in special cases to make contact with earlier studies.

4.1. Partition functions

For the given holomorphic and anti-holomorphic Hilbert space $(\mathcal{H}_q, \mathcal{H}_{\bar{q}})$, one may consider them as infinite vector spaces, and define a natural $gl(\infty)_+$ Lie algebra acting on each vector space. In the corresponding group $GL(\infty)_+$, $q^{L_0 - \frac{c}{24}}$ and $q^{\bar{L}_0 - \frac{c}{24}}$ are elements of the Cartan subgroup

$$q^{L_0 - \frac{c}{24}} = \text{diag}(q^{\epsilon_1}, q^{\epsilon_2}, q^{\epsilon_3}, \cdots) \quad (\epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \cdots),$$

$$q^{\bar{L}_0 - \frac{c}{24}} = \text{diag}(q^{\bar{\epsilon}_1}, q^{\bar{\epsilon}_2}, q^{\bar{\epsilon}_3}, \cdots) \quad (\bar{\epsilon}_1 \leq \bar{\epsilon}_2 \leq \bar{\epsilon}_3 \leq \cdots),$$

where $\epsilon_j$ and $\bar{\epsilon}_j$ is the spectrum of the holomorphic and anti-holomorphic sector, respectively.

We now consider the holomorphic and anti-holomorphic parts together. A state in the (full) Hilbert space is denoted by

$$|m, \bar{m}\rangle \in \mathcal{H}_{a,\theta} \otimes \overline{\mathcal{H}}_{a,\theta},$$

where $m \in \{\epsilon_1, \epsilon_2, \cdots\}$ and $\bar{m} \in \{\bar{\epsilon}_1, \bar{\epsilon}_2, \cdots\}$. The trace of $q^{L_0 - \frac{c}{24}}q^{\bar{L}_0 - \frac{c}{24}}$ over the Hilbert space $\mathcal{H}_{a,\theta} \otimes \overline{\mathcal{H}}_{a,\theta}$ becomes

$$\sum_{m, \bar{m}} \langle m, \bar{m}| q^{L_0 - \frac{c}{24}}q^{\bar{L}_0 - \frac{c}{24}}|m, \bar{m}\rangle = \Phi_{a,\theta, 1}\overline{\Phi}_{a,\theta, 1}.$$  (98)

For the tensor product of the Hilbert spaces, $(\mathcal{H}_{a,\theta} \otimes \overline{\mathcal{H}}_{a,\theta})^{\otimes n}$, one can define $S_n \times S_n$ action by

$$(\sigma, \tau)|m_1(1), \bar{m}_1(1), \cdots |m_n, \bar{m}_n\rangle = |m_{\sigma(1)}, \bar{m}_{\tau(1)}), \cdots |m_{\sigma(n)}, \bar{m}_{\tau(n)}\rangle.$$  (99)
where \([\{m_j\}, \{\overline{m}_j\}]\) is a general state in \((H_{a,b} \otimes \overline{H}_{a,b})^{\otimes m}\) defined by

\[
|\{m_j\}, \{\overline{m}_j\}\rangle = |m_1, \overline{m}_1\rangle \otimes \cdots \otimes |m_m, \overline{m}_m\rangle.
\] (100)

It is convenient to define three commuting projection operators \(\Pi_R, \overline{\Pi}_R\) and \(\Upsilon_R\) which project Hilbert space onto irreducible representation \(R\).

\[
\Pi_R \equiv \frac{1}{n!} \sum_{\omega \in S_n} d_R \text{ch}_R(\sigma, 1), \quad \overline{\Pi}_R \equiv \frac{1}{n!} \sum_{\sigma \in S_n} d_R \text{ch}_R(\overline{\sigma})(1, \overline{\sigma})
\] (101)

and

\[
\Upsilon_R \equiv \frac{1}{n!} \sum_{\sigma \in S_n} d_R \text{ch}_R(\sigma, \sigma).
\] (102)

They satisfy the properties of projection operator. E.g.

\[
\Pi_R \Pi_T = \delta_{R,T} \Pi_R, \quad \Pi_R^\dagger = \Pi_R, \quad \sum_{R=1}^{n} \Pi_R = 1.
\] (103)

First of all, in order to calculate partition function of the \(S_n\) invariant subspace, we define \(S_n\) invariant state

\[
|\{m_j\}, \{\overline{m}_j\}\rangle_{\text{inv}} \equiv \Upsilon_0 |\{m_j\}, \{\overline{m}_j\}\rangle.
\] (104)

The trace over such states gives the desired result (see (115))

\[
Z^n = \sum_{|\{m_j\}, |\{\overline{m}_j\}|} \langle \{m_j\}, \{\overline{m}_j\}| q^{L_0} \overline{q}^{\overline{L}_0} |\{m_j\}, \{\overline{m}_j\}\rangle_{\text{inv}}
\]

\[
= \frac{1}{n!} \sum_{\rho \in S_n} \prod_{k=1}^{n} \Phi_{\rho_k, a, \overline{b}, k} \overline{\Phi}_{\rho_k, a, b, k} = \sum_{\rho \in S_n} \prod_{k=1}^{n} \frac{\Phi_{\rho_k, a, \overline{b}, k} \overline{\Phi}_{\rho_k, a, b, k}}{\rho_k!},
\] (105)

where for the given \(\rho \in S_n\), \(\rho_k\) is defined by \(\rho = 1^{\rho_1} 2^{\rho_2} \cdots n^{\rho_n}\).

As in section 2.1, one may consider irreducible subspace with respect to \(S_n\). By using the projection operator \(\overline{\Pi}_R\), we project the invariant states in (104) onto the irreducible representation \(R\) with respect to the anti-holomorphic part

\[
|R; \{m_j\}, \{\overline{m}_j\}\rangle \equiv \Upsilon_0 \overline{\Pi}_R |\{m_j\}, \{\overline{m}_j\}\rangle = \sum_{\rho, \sigma \in S_n} \frac{d_R}{(m!)^2} \text{ch}_R(\sigma, \rho \overline{\rho}) |\{m_j\}, \{\overline{m}_j\}\rangle.
\] (106)

Note that this state is also \(S_n\) invariant because \(\Upsilon_0\) and \(\overline{\Pi}_R\) commute with each other. Using the third identity in (103), one can further decompose the state (106) into irreducible representation of \(S_n\) with respect to the holomorphic part

\[
|R; \{m_j\}, \{\overline{m}_j\}\rangle = \frac{1}{(n!)^3} \sum_{\tau, \sigma, \rho, \overline{\rho} \in S_n} \sum_{\sigma \rho \overline{\sigma} \overline{\rho} \in S_n} d_R d_T \text{ch}_R(\sigma) \text{ch}_R(\overline{\sigma}) \text{ch}_R(\sigma \rho, \sigma \overline{\rho}) |\{m_j\}, \{\overline{m}_j\}\rangle
\]

\[
= \frac{1}{(n!)^2} \sum_{\sigma, \rho \in S_n} d_R \text{ch}_R(\sigma^{-1} \rho) |\{m_j\}, \{\overline{m}_j\}\rangle,
\] (107)

where we used the orthogonality of the characters. This implies that the non-zero contribution only comes from the irreducible representation \(R\) of the holomorphic part. Hence, one
may write \( |R; \{ m_j \}, \{ \bar{m}_j \} \rangle \) as
\[
|R; \{ m_j \}, \{ \bar{m}_j \} \rangle = \mathbb{Y}_{\rho} \mathbb{Y}_{\bar{\rho}} |\{ m_j \}, \{ \bar{m}_j \} \rangle.
\] (108)

The trace over this irreducible subspace is given by
\[
Z^n_R = \sum_{\{ m_j \}, \{ \bar{m}_j \}} \langle R; \{ m_j \}, \{ \bar{m}_j \} | q^{L_0 - L_\rho - L_{\bar{\rho}}} | R; \{ m_j \}, \{ \bar{m}_j \} \rangle
\]
\[
= \frac{1}{(n!)^2} \sum_{\sigma, \pi \in S_n} d_R \text{ch}_R(\sigma \pi^{-1}) \prod_{k=1}^n \Phi^{a_{\rho k}}_{a_{\bar{\rho} k}} \overline{\Phi^{a_{\rho k}}_{a_{\bar{\rho} k}}},
\] (109)

where \( \sigma = 1^n \cdots n^\alpha \) and \( \pi = 1^n \cdots n^\gamma \). One can easily show that the summation of \( Z^n_R \) over the irreducible representation \( R \) gives the full partition function in (105)
\[
Z^n = \sum_R Z^n_R = \sum_{\{ \sigma \}, \{ \pi \}} \prod_{k=1}^n \frac{\Phi^{a_{\rho k}}_{a_{\bar{\rho} k}} \overline{\Phi^{a_{\rho k}}_{a_{\bar{\rho} k}}}}{\sigma_k ! k^\gamma}.
\] (110)

Note that \( \{ \sigma_k \} \) and \( \{ \pi_k \} \) in (109), which are the partitions of \( n \), depend only on the conjugacy class. Hence, one can average the coefficient \( \text{ch}_R(\sigma \pi^{-1}) \) in (109) over the conjugacy class. Then, one can show that
\[
\frac{d_R}{||\sigma|| ||\pi||} \sum_{\rho \in [\sigma]} \sum_{\pi \in [\pi]} \text{ch}_R(\rho \pi^{-1}) = \frac{d_R}{n!^2} \sum_{g \in S_n} \text{ch}_R(g^{-1} \sigma g \pi^{-1})
\]
\[
= \frac{d_R}{n!^2} \sum_{g \in S_n} D^R(g^{-1}) D^R_\pi(\sigma) D^R_\rho(g) D^R_\bar{\rho}(\pi^{-1}) = \text{ch}_R(\sigma) \text{ch}_R(\pi^{-1}).
\] (111)

Therefore, one can conclude that
\[
Z^n_R = \frac{1}{(n!)^2} \sum_{\sigma, \pi \in S_n} \text{ch}_R(\sigma) \text{ch}_R(\sigma) \prod_{k=1}^n \Phi^{a_{\rho k}}_{a_{\bar{\rho} k}} \overline{\Phi^{a_{\rho k}}_{a_{\bar{\rho} k}}} = |P_\lambda(\alpha, \beta; R)|^2
\] (112)

which is the same as the result (24) from the \( S_N \) orbifold character.

4.2. Examples

In this section, we will calculate \( S_N \) orbifold partition function from the orbifold character \( \Gamma_N(\mathcal{P}, \mathcal{R}) \). One can express the partition function \( Z^N_{\text{orbifold}} \) of the \( S_N \) orbifold in terms of \( \Gamma_N(\mathcal{P}, \mathcal{R}) \) as follows
\[
Z^N_{\text{orbifold}} = \sum_{\mathcal{P}} \sum_{\mathcal{R}} |\Gamma_N(\mathcal{P}, \mathcal{R})|^2,
\] (113)

where the summations run over all possible \( N \)-tuple of primaries \( \mathcal{P} \) and over all possible Young diagrams \( \mathcal{R} \). One can define a generating function for \( Z^N_{\text{orbifold}} \):
\[
Z_{\text{orbifold}} = \sum_{N=0}^{\infty} t^N Z^N_{\text{orbifold}}
\] (114)
where $Z_0^{\text{orbifold}} = 1$ for consistency. Using an identity of the Schur polynomial

$$
\sum_R \frac{|P_\alpha(R; \{j_\lambda\})|^2}{|\lambda|^{\alpha}} = \sum_{(\lambda_j^a)^{\infty}} \prod_{j=1}^{n} \left[ \frac{1}{\lambda_j^a} \left( 1 - \lambda_j^a \right)^\lambda \right]
$$

(115)

it is not difficult to prove that

$$
Z_{\text{orbifold}} = \prod_{\rho \in \mathcal{I}} \left[ \sum_{N=0}^{\infty} \sum_{R} \sum_{\theta=0}^{a-1} \prod_{j=0}^{a-1} \frac{t^{d[R,\theta]}}{d[R,\theta]} |P_{[R,\theta]}(\rho; a, \theta, R_{\alpha,0})|^2 \right]
\exp \left[ \sum_{M=1}^{\infty} t^M T_M Z_{\text{CFT}}(\tau, \bar{\tau}) \right],
$$

(116)

where $Z_{\text{CFT}}(\tau, \bar{\tau})$ is the partition function of the seed CFT and $T_M$ is $M$th-Hecke operator of which action is defined as follows

$$
T_M Z_{\text{CFT}}(\tau, \bar{\tau}) \equiv \frac{1}{M} \sum_{a=1}^{\infty} \sum_{M=0}^{a-1} Z_{\text{CFT}} \left( \frac{M\tau + a\theta}{a^2} , \frac{M\bar{\tau} + a\bar{\theta}}{a^2} \right).
$$

(117)

This agrees with [2, 3, 27, 33].

One can also consider a partition function of a sub-sector of the orbifold CFT. The simplest sub-sector is the untwisted sector where all components of representation $R_j$ in $\mathcal{R}_P$ are 0 except for $(R_j)_{1,0}$ for all $j = 1, \cdots k$. Therefore, the partition function of untwisted sector is given by

$$
Z_{\text{untwisted}}^N = \sum_{\mathcal{P}, \mathcal{R}_P} |\Gamma_N(\mathcal{P}, \mathcal{R}_P)|^2 \quad \text{where} \quad (\mathcal{R}_j)_{a,\theta} = \begin{cases} R & \text{for} \ (a, \theta) = (1, 0), \\ 0 & \text{for} \ (a, \theta) \neq (1, 0), \end{cases}
$$

(118)

where the summation runs over the all possible Young diagrams $R$. We also construct a generating function for the untwisted partition function

$$
Z_{\text{untwisted}} = \sum_{N=0}^{\infty} t^N Z_{\text{untwisted}}^N.
$$

(119)

Using the orbifold characters, one can show that

$$
Z_{\text{untwisted}} = \prod_{\rho} \left[ \sum_{N=0}^{\infty} \sum_{|R|=N} t^{d[R]} |P_{[R]}(\rho; 1, 0, R)|^2 \right] = \exp \left[ \sum_{m=1}^{\infty} \frac{1}{m} t^m Z_{\text{CFT}}(m\tau, m\bar{\tau}) \right].
$$

(120)

When the seed partition function has the following form

$$
Z_{\text{CFT}}(\tau, \bar{\tau}) = \sum_{l,l} \rho(l, \bar{l}) q^l \bar{q}^{\bar{l}}
$$

(121)

the generating function of the untwisted partition function can be written as

$$
Z_{\text{untwisted}} = \prod_{l,l} \frac{1}{(1 - t^{i}q^l \bar{q}^{\bar{l}})}
$$

(122)

which agrees with [1, 3, 4].

---

9 A representation $R_p$ for a primary $P$ (see (32)) is a $k$-tuple of representations $R_j = ((R_j)_{1,0}, (R_j)_{2,0}, (R_j)_{2,1}, \cdots)$ for $N_f$-tuple of identical primaries of $S_{N_f}$ orbifold ($1 \leq j \leq k$).
It is also easy to calculate a generating function for the partition function of the 2-cycle twist sector. In the 2-cycle twist sector, a representation $\mathcal{R}_P = (\mathcal{R}_1, \ldots, \mathcal{R}_k)$ for a primary $\mathcal{P}$ takes the following form. For fixed $m \in \{1, 2, \ldots, k\}$,

$$(\mathcal{R}_j)_{a, \theta} = \begin{cases} R_{a, \theta} & \text{for } (a, \theta) = (1, 0) \\ \mathbf{0} & \text{for } (a, \theta) = (2, 0) \text{ or } (\mathcal{R}_j)_{a, \theta} = \begin{cases} R_{a, \theta} & \text{for } (a, \theta) = (1, 0) \\ \mathbf{0} & \text{for } (a, \theta) = (2, 1) \end{cases} \end{cases}$$

which means that only one primary corresponding to the label $m$ is twisted by one 2-cycle.

The 2-cycle twist partition function is obtained by summing absolute square of the characters over such representations

$$Z_{2\text{-cycle twist}}^N = \sum_{\mathcal{P}} \left| \Gamma_N(\mathcal{P}, \mathcal{R}_P) \right|^2$$

and the corresponding generating function is given by

$$Z_{2\text{-cycle twist}} = \sum_{N=0}^{\infty} t^N Z_{2\text{-cycle twist}}^N = \frac{t^2}{2} \left( Z_{\text{CFT}}(\frac{\tau}{2}, \frac{\tau}{2}) + Z_{\text{CFT}}(\frac{\tau + 1}{2}, \frac{\tau + 1}{2}) \right) Z_{\text{untwisted}}$$

$$= t^2 \sum_{l' \ell} \rho(l, \ell) q^{l \ell} q^{l' \ell} \prod_{l' \ell} \frac{1}{(1 - t q^{l' \ell})^{\rho(l, \ell)}}$$

which is consistent with the supersymmetric version of the 2-cycle twisted partition function [1, 12]. In the same way, one can calculate a generating function for $k$-cycle twist partition function

$$Z_{k\text{-cycle twist}} = Z_{\text{untwisted}} \frac{k-1}{k} \sum_{j=0}^{k-1} Z_{\text{CFT}}(\frac{\tau + j}{k}, \frac{\tau + j}{k})$$

$$= t^k \sum_{l' \ell} \rho(l, \ell) q^{l \ell} q^{l' \ell} \prod_{l' \ell} \frac{1}{(1 - t q^{l' \ell})^{\rho(l, \ell)}}.$$

So far, we have considered the partition function of the full sector (the holomorphic and the anti-holomorphic parts) so that we have summed the absolute square of the orbifold characters over all possible Young diagrams. We now consider orbifold of chiral sector. As mentioned in section 2.1, we cannot have the phase factor. In addition, one has to consider only fully symmetric representations, that is, the Young diagrams which have (at most) one row.

First of all, the generating function of the untwisted chiral partition function can be written as

$$Z_{\text{untwisted}} = \sum_{N=0}^{\infty} t^N \sum_{\mathcal{P}} \left| \Gamma_N(\mathcal{P}, \mathcal{R}_P) \right|^2$$

where $(\mathcal{R}_j)_{a, \theta} = \{ \mathbb{1} \cdots \mathbb{1} \}$ for $(a, \theta) = (1, 0)$ and $(\mathcal{R}_j)_{a, \theta} = 0$ for $(a, \theta) \neq (1, 0)$

$$= \prod_P \left[ \sum_{N=0}^{\infty} t^N P_N(p; 1, 0, \mathbb{1} \cdots \mathbb{1}) \right].$$

(127)
Recall that the Schur polynomial for the irreducible representation \( N \) is given by

\[
P_N(\lambda_1, \ldots, \lambda_N; \{x_m\}) = \sum_{\{\lambda_m\} \vdash N} \prod_{m=1}^{N} \frac{x_m^{\lambda_m}}{\lambda_m! m^{\lambda_m}}.
\]  

Using (128), we have

\[
Z_{\text{untwisted}} = \prod_{m=1}^{\infty} \exp \left[ \frac{i^n}{m} \sum_p \chi_p(m\tau) \right].
\]  

If the chiral partition function of the seed CFT takes the form

\[
\chi_{\text{CFT}}(\tau) = \sum_p \chi_p(\tau) = \sum_l \theta q^l
\]

the generating function of the untwisted chiral partition function can be written as

\[
Z_{\text{chiral untwisted}} = \prod_{l} \frac{1}{(1 - t q^l)^{\nu_l}}
\]

which is the non-supersymmetric version of [1, 5, 31]. The full chiral partition function of the \( S_N \) orbifold can be calculated in the similar way. For this we consider Young diagrams which are zero (the trivial representation) for non-zero \( \theta \). I.e. \( R_{a,0} = 0 \) for \( \theta \neq 0 \) and

\[
R_{a,\theta} = \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}
\]

Recalling the form of the expansion of \( q_{a,m,\theta}^{(p)} \) in (30), one can easily reproduce the result in [31]

\[
Z_{\text{chiral}} = \prod_{a=1}^{\infty} \prod_{l} \frac{1}{(1 - t q^l)^{\nu_{l,a}}}.
\]

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Appendix. Examples: genus one characters of \( S_N \) orbifolds.

| \( \mathcal{R} \) | \( S_2 \) orbifold character | \( E_2^{\text{orb}}(\mathcal{R}) \) | \( E_4^{\text{orb}}(\mathcal{R}) \) |
|-----------------|-----------------------------|-----------------|-----------------|
| ((), (0, 0))   | \( P_2(1, 0, 0) = \frac{1}{2}((\chi(\tau))^2 + \chi(2\tau)) \) | \( 2h_p \) | \( e_1,0 \) |
| ((), (0, 0))   | \( P_2(1, 0, 0) = \frac{1}{2}((\chi(\tau))^2 - \chi(2\tau)) \) | \( 2h_p \) | \( -e_1,0 \) |
| ((), (0, 0))   | \( P_2(2, 0, 0) = \frac{1}{2}((\chi(\tau))^2 + \chi(2\tau)) \) | \( \frac{1}{2}h_p + \frac{1}{16}e \) | 0 |
| ((), (0, 0))   | \( P_2(1, 1, 0) = \frac{1}{2}((\chi(\tau))^2 - \chi(2\tau)) \) | \( \frac{1}{2}h_p + \frac{1}{16}e + \frac{1}{2} \) | 0 |
### Table A2. Characters for primaries \((p, p, p)\) of \(S_3\) orbifold.

| \(R\) | \(S_3\) orbifold character | \(E_2^{\text{orb}}(R)\) | \(E_3^{\text{orb}}(R)\) |
|-------|----------------------------|----------------|----------------|
| \((□□□), \tilde{0}_2, \tilde{0}_3)\) | \(P_3(1, 0, \square \square \square) = \frac{1}{2}[(\chi(\tau))^{3} + 3\chi(\tau)\chi(2\tau) + 2\chi(3\tau)]\) | \(3h_p\) | \(3\epsilon_{1,0}\) |
| \((□□□), \tilde{0}_2, \tilde{0}_3)\) | \(P_3(1, 0, \square \square \square) = \frac{1}{2}[(\chi(\tau))^{3} - \chi(3\tau)]\) | \(3h_p\) | 0 |
| \((□□□), \tilde{0}_2, \tilde{0}_3)\) | \(P_3(1, 0, \square \square \square) = \frac{1}{2}[(\chi(\tau))^{3} - 3\chi(\tau)\chi(2\tau) + 2\chi(3\tau)]\) | \(3h_p\) | \(-3\epsilon_{1,0}\) |
| \((□, □, 0), \tilde{0}_3)\) | \(P_1(1, 0, \square)P_1(2, 0, \square) = \chi(\tau)\frac{1}{2}[\chi(\tau) + \chi\left(\frac{\tau+1}{2}\right)]\) | \(\frac{3}{2}h_p + \frac{1}{16}c\) | 0 |
| \((□, 0, □), \tilde{0}_3)\) | \(P_1(1, 0, \square)P_1(2, 1, \square) = \chi(\tau)\frac{1}{2}[\chi(\tau) - \chi\left(\frac{\tau+1}{2}\right)]\) | \(\frac{3}{2}h_p + \frac{1}{16}c + \frac{1}{2}\) | 0 |
| \((\tilde{0}_3, \tilde{0}_2, (□, 0, 0))\) | \(P_3(3, 0, \square) = \frac{1}{2}[\chi\left(\frac{\tau}{2}\right) + \chi\left(\frac{\tau+1}{2}\right)]\) | \(\frac{1}{4}h_p + \frac{1}{2}c\) | 0 |
| \((\tilde{0}_3, \tilde{0}_2, (0, □, 0))\) | \(P_1(3, 1, \square) = \frac{1}{2}[\chi\left(\frac{\tau}{2}\right) + e^{-i\pi}\chi\left(\frac{\tau+1}{2}\right) + e^{-\frac{3i\pi}{2}}\chi\left(\frac{\tau+2}{2}\right)]\) | \(\frac{1}{4}h_p + \frac{1}{2}c + \frac{1}{4}\) | 0 |
| \((\tilde{0}_3, \tilde{0}_2, (0, 0, □))\) | \(P_1(3, 2, \square) = \frac{1}{2}[\chi\left(\frac{\tau}{2}\right) + e^{-i\pi}\chi\left(\frac{\tau+1}{2}\right) + e^{-\frac{3i\pi}{2}}\chi\left(\frac{\tau+2}{2}\right)]\) | \(\frac{1}{4}h_p + \frac{1}{2}c + \frac{2}{5}\) | 0 |


Table A3. Characters for primaries \((p, p, p, p)\) of \(S_4\) orbifold.

| \(\mathcal{R}\) | \(S_4\) orbifold character | \(E_2^{orb}(\mathcal{R})\) | \(E_3^{orb}(\mathcal{R})\) |
|---|---|---|---|
| \((\square, \square, \circ, \circ)\) | \(P_1(1, 0, \square, \circ) = \frac{1}{4}[\chi(\tau)^4 + \chi(2\tau)(\chi(\tau))^2 + 8\chi(3\tau)\chi(\tau) + 3(\chi(2\tau))^2 + 6\chi(4\tau)]\) | 4\(h_\rho\) | 6\(e_{1,0}\) |
| \((\square, \circ, \circ, \circ)\) | \(P_1(1, 0, \square, \circ) = \frac{1}{4}[3(\chi(\tau))^4 + \chi(2\tau)(\chi(\tau))^2 - 3(\chi(2\tau))^2 - 6\chi(4\tau)]\) | 4\(h_\rho\) | 2\(e_{1,0}\) |
| \((\square, \circ, \circ, \circ)\) | \(P_2(1, 0, \square, \circ) = \frac{1}{4}[2(\chi(\tau))^2 - 8\chi(3\tau)\chi(\tau) + 6(\chi(2\tau))^2]\) | 4\(h_\rho\) | 0 |
| \((\circ, \circ, \circ, \circ)\) | \(P_4(1, 0, \square, \circ) = \frac{1}{4}[3(\chi(\tau))^4 - 6\chi(2\tau)(\chi(\tau))^2 - 3(\chi(2\tau))^2 + 6\chi(4\tau)]\) | 4\(h_\rho\) | −6\(e_{1,0}\) |
| \((\square, \circ, \circ, \circ)\) | \(P_2(1, 0, \square, \circ) = \frac{1}{4}[\chi(\tau)^4 + \chi(2\tau)(\chi(\tau))^2 + 8\chi(3\tau)\chi(\tau) + 3(\chi(2\tau))^2 - 6\chi(4\tau)]\) | 4\(h_\rho\) | 6\(e_{1,0}\) |
| \((\square, \square, \circ, \circ)\) | \(P_2(1, 0, \square, \circ) = \frac{1}{2}[\chi(\tau) + \chi(-\frac{1}{2})]\) | \(\frac{5}{2}h_\rho + \frac{1}{16}c\) | \(\epsilon_{1,0}\) |
| \((\square, \circ, \circ, \circ)\) | \(P_2(1, 0, \square, \circ) = \frac{1}{2}[\chi(\tau) + \chi(-\frac{1}{2})]\) | \(\frac{5}{2}h_\rho + \frac{1}{16}c + \frac{1}{2}\) | \(\epsilon_{1,0}\) |
| \((\circ, \circ, \circ, \circ)\) | \(P_2(1, 0, \square, \circ) = \frac{1}{2}[\chi(\tau) - \chi(-\frac{1}{2})]\) | \(\frac{5}{2}h_\rho + \frac{1}{16}c - \frac{1}{2}\) | −\(\epsilon_{1,0}\) |
| \((\square, \circ, \circ, \circ)\) | \(P_2(1, 0, \square, \circ) = \frac{1}{2}[\chi(\tau) - \chi(-\frac{1}{2})]\) | \(\frac{5}{2}h_\rho + \frac{1}{16}c + \frac{1}{2}\) | −\(\epsilon_{1,0}\) |
Table A4. Characters for primaries \((p, p, p)\) of \(S_4\) orbifold.

| \(\mathcal{R}\) | \(S_4\) orbifold character | \(E_{2,0}^{orb}(\mathcal{R})\) | \(E_{3,1}^{orb}(\mathcal{R})\) |
|-----------------|-----------------------------|-----------------|-----------------|
| \((\Box, \Box, (\Box, 0, 0, \Box))\) | \(P_1(1, 0 \Box) R_1(3, 0, \Box) = \chi(\tau) \frac{1}{2} \left\{ \chi\left(\frac{5}{2}\right) + \chi\left(\frac{5}{2} + 1\right) + \chi\left(\frac{5}{2} - 1\right) \right\} \) | \(\frac{1}{2} h_p + \frac{1}{2} c\) | 0 |
| \((\Box, \Box, (0, \Box, 0, \Box))\) | \(P_1(1, 0 \Box) R_1(3, 1, \Box) = \chi(\tau) \frac{1}{2} \left\{ \chi\left(\frac{5}{2}\right) + e^{2\pi \chi\left(\frac{5}{2} + 1\right)} + e^{2\pi \chi\left(\frac{5}{2} - 1\right)} \right\} \) | \(\frac{1}{2} h_p + \frac{1}{2} c + \frac{1}{2}\) | 0 |
| \((\Box, \Box, (0, 0, \Box, \Box))\) | \(P_1(1, 0 \Box) R_1(3, 2, \Box) = \chi(\tau) \frac{1}{2} \left\{ \chi\left(\frac{5}{2}\right) + e^{2\pi \chi\left(\frac{5}{2} + 1\right)} + e^{2\pi \chi\left(\frac{5}{2} - 1\right)} \right\} \) | \(\frac{1}{2} h_p + \frac{1}{2} c + \frac{1}{2}\) | 0 |
| \((\Box, \Box, (\Box, 0, 0, \Box))\) | \(P_2(2, 0, \Box) = \chi\left(\frac{5}{2}\right) + \chi\left(\frac{5}{2} + 1\right) + \chi\left(\frac{5}{2} - 1\right) \) | \(h_p + \frac{1}{2} c\) | \(\epsilon_{2,0}\) |
| \((\Box, (\Box, 0, 0, \Box))\) | \(P_2(2, 1, \Box) = \chi\left(\frac{5}{2}\right) + \chi\left(\frac{5}{2} + 1\right) + \chi\left(\frac{5}{2} - 1\right) \) | \(h_p + \frac{1}{2} c + 1\) | \(\epsilon_{2,1}\) |
| \((\Box, (0, \Box, 0, \Box))\) | \(P_2(2, 1, \Box) = \chi\left(\frac{5}{2}\right) + \chi\left(\frac{5}{2} + 1\right) + \chi\left(\frac{5}{2} - 1\right) \) | \(h_p + \frac{1}{2} c + 1\) | \(-\epsilon_{2,1}\) |
| \((\Box, (\Box, \Box, 0, \Box))\) | \(P_3(2, 0, \Box) = \chi\left(\frac{5}{2}\right) + \chi\left(\frac{5}{2} + 1\right) + \chi\left(\frac{5}{2} - 1\right) \) | \(h_p + \frac{1}{2} c + 2\) | 0 |
| \((\Box, \Box, (\Box, 0, 0, \Box))\) | \(P_3(2, 1, \Box) = \chi\left(\frac{5}{2}\right) + \chi\left(\frac{5}{2} + 1\right) + \chi\left(\frac{5}{2} - 1\right) \) | \(h_p + \frac{1}{2} c + 2\) | 0 |
| \((\Box, (0, \Box, 0, \Box))\) | \(P_3(2, 1, \Box) = \chi\left(\frac{5}{2}\right) + \chi\left(\frac{5}{2} + 1\right) + \chi\left(\frac{5}{2} - 1\right) \) | \(h_p + \frac{1}{2} c + 2\) | 0 |
| \((\Box, (\Box, \Box, 0, \Box))\) | \(P_3(2, 1, \Box) = \chi\left(\frac{5}{2}\right) + \chi\left(\frac{5}{2} + 1\right) + \chi\left(\frac{5}{2} - 1\right) \) | \(h_p + \frac{1}{2} c + 2\) | 0 |
Table A5. Characters for primaries $\langle p, p, p, p \rangle$ of $S_5$ orbifold.

| $R$ | $S_5$ orbifold character | $E_{1\text{th}}^\text{orb}(R)$ | $E_{3\text{th}}^\text{orb}(R)$ |
|-----|--------------------------|---------------------------------|--------------------------|
| (4,3,2,1,0,0) | $P_S(1, 0, \begin{array}{c} \emptyset \end{array}) = \frac{1}{5!} (\chi(\tau))^5 + 10\chi(2\tau)(\chi(\tau))^3 + 20\chi(3\tau)(\chi(\tau))^2 + 30\chi(4\tau)(\chi(\tau)) + 24\chi(5\tau) + 15\chi(2\tau)(\chi(\tau) + 20\chi(2\tau)(\chi(3\tau))]$ | $5h_p$ | $10\epsilon_{1,0}$ |
| (4,3,2,1,0,0) | $P_S(1, 0, \begin{array}{c} \emptyset \end{array}) = \frac{1}{5!} (4(\chi(\tau))^5 + 20\chi(2\tau)(\chi(\tau))^3 + 20\chi(3\tau)(\chi(\tau))^2 - 24\chi(5\tau) - 20\chi(2\tau)(\chi(3\tau)]$ | $5h_p$ | $5\epsilon_{1,0}$ |
| (4,3,2,1,0,0) | $P_S(1, 0, \begin{array}{c} \emptyset \end{array}) = \frac{1}{5!} (4(\chi(\tau))^5 + 24\chi(5\tau) - 30(\chi(2\tau))^2(\chi(\tau)]$ | $5h_p$ | $2\epsilon_{1,0}$ |
| (4,3,2,1,0,0) | $P_S(1, 0, \begin{array}{c} \emptyset \end{array}) = \frac{1}{5!} (4(\chi(\tau))^5 + 24\chi(5\tau) - 30(\chi(2\tau))^2(\chi(\tau)]$ | $5h_p$ | $0$ |
| (4,3,2,1,0,0) | $P_S(1, 0, \begin{array}{c} \emptyset \end{array}) = \frac{1}{5!} (5(\chi(\tau))^5 - 10\chi(2\tau)(\chi(\tau))^3 - 20\chi(3\tau)(\chi(\tau))^2 + 30\chi(4\tau)(\chi(\tau)) + 15\chi(2\tau)(\chi(\tau) - 20\chi(2\tau)(\chi(3\tau)]$ | $5h_p$ | $-2\epsilon_{1,0}$ |
| (4,3,2,1,0,0) | $P_S(1, 0, \begin{array}{c} \emptyset \end{array}) = \frac{1}{5!} (4(\chi(\tau))^5 - 20\chi(2\tau)(\chi(\tau))^3 + 20\chi(3\tau)(\chi(\tau))^2 - 24\chi(5\tau) + 20\chi(2\tau)(\chi(3\tau)]$ | $5h_p$ | $-5\epsilon_{1,0}$ |
| (4,3,2,1,0,0) | $P_S(1, 0, \begin{array}{c} \emptyset \end{array}) = \frac{1}{5!} (4(\chi(\tau))^5 - 10\chi(2\tau)(\chi(\tau))^3 + 20\chi(3\tau)(\chi(\tau))^2 - 30\chi(4\tau)(\chi(\tau)) + 24\chi(5\tau) + 15\chi(2\tau)(\chi(\tau) - 20\chi(2\tau)(\chi(3\tau)]$ | $5h_p$ | $-10\epsilon_{1,0}$ |
Table A6. Characters for primaries \langle p, p, p, p \rangle of $S_5$ orbifold.

| $\mathcal{R}$ | $S_5$ orbifold character | $E_2^{\text{orb}}(\mathcal{R})$ | $E_3^{\text{orb}}(\mathcal{R})$ |
|---------------|---------------------------|-------------------------------|-------------------------------|
| \begin{align*} \langle \square, (0, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5 \rangle \\ \langle \square, (0, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5 \rangle \end{align*} | $P_1(1, 0, \square)P_1(2, 0, \square) = \frac{1}{6}[\chi(r)^3 + 3\chi(r)\chi(2r) + 2\chi(3r)]$ | $\frac{7}{2}h_p + \frac{1}{16}e$ | $3e_{1,0}$ |
| \begin{align*} \langle \square, (0, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5 \rangle \end{align*} | $\times \frac{1}{3}\left[\chi\left(\frac{r}{2}\right) + \chi\left(\frac{r}{2} + \frac{1}{2}\right)\right]$ | | |
| \begin{align*} \langle \square, (0, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5 \rangle \end{align*} | $P_1(1, 0, \square)P_1(2, 1, \square) = \frac{1}{6}[\chi(r)^3 + 3\chi(r)\chi(2r) + 2\chi(3r)]$ | $\frac{7}{2}h_p + \frac{1}{16}e + \frac{1}{2}$ | $3e_{1,0}$ |
| \begin{align*} \langle \square, (0, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5 \rangle \end{align*} | $\times \frac{1}{3}\left[\chi\left(\frac{r}{2}\right) - \chi\left(\frac{r}{2} + \frac{1}{2}\right)\right]$ | | |
| \begin{align*} \langle \square, (0, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5 \rangle \end{align*} | $P_1(1, 0, \square)P_1(2, 1, \square) = \frac{1}{6}[\chi(r)^3 - \chi(3r)\chi(2r) + 2\chi(3r)]$ | $\frac{7}{2}h_p + \frac{1}{16}e + \frac{1}{2}$ | $0$ |
| \begin{align*} \langle \square, (0, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5 \rangle \end{align*} | $\times \frac{1}{3}\left[\chi\left(\frac{r}{2}\right) - \chi\left(\frac{r}{2} + \frac{1}{2}\right)\right]$ | | |
| \begin{align*} \langle \square, (0, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5 \rangle \end{align*} | $P_1(1, 0, \square)P_1(2, 0, \square) = \frac{1}{6}[\chi(r)^3 - 3\chi(r)\chi(2r) + 2\chi(3r)]$ | $\frac{7}{2}h_p + \frac{1}{16}e$ | $-3e_{1,0}$ |
| \begin{align*} \langle \square, (0, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5 \rangle \end{align*} | $\times \frac{1}{3}\left[\chi\left(\frac{r}{2}\right) + \chi\left(\frac{r}{2} + \frac{1}{2}\right)\right]$ | | |
| \begin{align*} \langle \square, (0, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5 \rangle \end{align*} | $P_1(1, 0, \square)P_1(2, 1, \square) = \frac{1}{6}[\chi(r)^3 - 3\chi(r)\chi(2r) + 2\chi(3r)]$ | $\frac{7}{2}h_p + \frac{1}{16}e + \frac{1}{2}$ | $-3e_{1,0}$ |
| \begin{align*} \langle \square, (0, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5 \rangle \end{align*} | $\times \frac{1}{3}\left[\chi\left(\frac{r}{2}\right) - \chi\left(\frac{r}{2} + \frac{1}{2}\right)\right]$ | | |
Table A7. Characters for primaries $\langle p, p, p, p \rangle$ of $S_5$ orbifold.

| $\mathcal{R}$ | $S_5$ orbifold character | $E_{2\text{ orb}}^\text{phys}(\mathcal{R})$ | $E_{3\text{ orb}}^\text{phys}(\mathcal{R})$ |
|--------------|--------------------------|---------------------------------|---------------------------------|
| $((\square, \bar{0}, \square, 0), \bar{0}, \bar{0})$ | $P_2(1, 0, 0) P_3(3, 0, \square)$ | $\frac{7}{3} \eta \theta^p + \frac{1}{3} \eta^c$ | $\epsilon_{1,0}$ |
| | | | $\times \frac{1}{2} [\tau (\chi (\frac{1}{5}) + \chi (\frac{2}{5}) + \chi (\frac{3}{5}) + \chi (\frac{4}{5}) + \chi (\frac{5}{5}) ]$ | |
| $((\square, \bar{0}, \square, 0), \bar{0}, \bar{0}, \bar{0})$ | $P_2(1, 0, 0) P_3(3, 1, \square)$ | $\frac{7}{3} \eta \theta^p + \frac{1}{3} \eta^c + \frac{1}{3}$ | $\epsilon_{1,0}$ |
| | | | $\times \frac{1}{2} [\tau (\chi (\frac{1}{5}) + e^{-i \frac{2}{5} \pi} \chi (\frac{2}{5}) + e^{-i \frac{2}{5} \pi} \chi (\frac{3}{5}) + e^{-i \frac{2}{5} \pi} \chi (\frac{4}{5}) + \chi (\frac{5}{5}) ]$ | |
| $((\square, \bar{0}, 0, \square), \bar{0}, \bar{0}, \bar{0})$ | $P_2(1, 0, 0) P_3(3, 2, \square)$ | $\frac{7}{3} \eta \theta^p + \frac{1}{3} \eta^c + \frac{1}{3}$ | $\epsilon_{1,0}$ |
| | | | $\times \frac{1}{2} [\tau (\chi (\frac{1}{5}) + e^{-i \frac{2}{5} \pi} \chi (\frac{2}{5}) + e^{-i \frac{2}{5} \pi} \chi (\frac{3}{5}) + e^{-i \frac{2}{5} \pi} \chi (\frac{4}{5}) + \chi (\frac{5}{5}) ]$ | |
| $((\square, \bar{0}, 0, 0), \bar{0}, \bar{0})$ | $P_2(1, 0, 0) P_3(3, 2, 0)$ | $\frac{7}{3} \eta \theta^p + \frac{1}{3} \eta^c + \frac{1}{3}$ | $\epsilon_{1,0}$ |
| | | | $\times \frac{1}{2} [\tau (\chi (\frac{1}{5}) + e^{-i \frac{2}{5} \pi} \chi (\frac{2}{5}) + e^{-i \frac{2}{5} \pi} \chi (\frac{3}{5}) + e^{-i \frac{2}{5} \pi} \chi (\frac{4}{5}) + \chi (\frac{5}{5}) ]$ | |
Table A8. Characters for primaries \( (p, p, p, p) \) of \( S_5 \) orbifold.

| \( R \) | \( S_5 \) orbifold character | \( E_{12}^{\text{orb}}(R) \) | \( E_{12}^{\text{orb}}(R) \) |
|------|-----------------|-----------------|-----------------|
| \((\Box, \tilde{0}_2, \tilde{0}_3, (\Box, 0, 0, 0), \tilde{0}_3)\) | \( P_{1}(1, 0, \Box) P_{1}(4, 0, \Box) = \chi(\tau)^{1} \left[ \chi\left(\frac{1}{4}\right) + \chi\left(\frac{1}{4}\right) + \chi\left(\frac{1}{4}\right) + \chi\left(\frac{1}{4}\right) \right] \) | \( \frac{1}{4} h_p + \frac{3}{32} \epsilon \) | 0 |
| \((\Box, \tilde{0}_2, \tilde{0}_3, (\Box, 0, 0, 0), \tilde{0}_3)\) | \( P_{1}(1, 0, \Box) P_{1}(4, 1, \Box) = \chi(\tau)^{1} \left[ \chi\left(\frac{1}{4}\right) - i\chi\left(\frac{3}{4}\right) - \chi\left(\frac{3}{4}\right) + i\chi\left(\frac{3}{4}\right) \right] \) | \( \frac{1}{4} h_p + \frac{3}{32} \epsilon + \frac{1}{4} \) | 0 |
| \((\Box, \tilde{0}_2, \tilde{0}_3, (\Box, 0, 0, 0), \tilde{0}_3)\) | \( P_{1}(1, 0, \Box) P_{1}(4, 2, \Box) = \chi(\tau)^{1} \left[ \chi\left(\frac{1}{4}\right) - \chi\left(\frac{3}{4}\right) + \chi\left(\frac{3}{4}\right) - \chi\left(\frac{3}{4}\right) \right] \) | \( \frac{1}{4} h_p + \frac{3}{32} \epsilon + \frac{3}{4} \) | 0 |
| \((\Box, \tilde{0}_2, \tilde{0}_3, (\Box, 0, 0, 0), \tilde{0}_3)\) | \( P_{1}(1, 0, \Box) P_{1}(4, 3, \Box) = \chi(\tau)^{1} \left[ \chi\left(\frac{1}{4}\right) + i\chi\left(\frac{3}{4}\right) - \chi\left(\frac{3}{4}\right) - i\chi\left(\frac{3}{4}\right) \right] \) | \( \frac{1}{4} h_p + \frac{3}{32} \epsilon + \frac{3}{4} \) | 0 |
Table A9. Characters for primaries \(\langle p, p, p, p, p\rangle\) of \(S_5\) orbifold. \(\rho \equiv e^{2\pi i/5}\).

| \(\mathcal{R}\) | \(S_5\) orbifold character | \(E_{2}^{\text{orb}}(\mathcal{R})\) | \(E_{3}^{\text{orb}}(\mathcal{R})\) |
|----------------|-----------------------------|----------------|----------------|
| \((\bar{0}_1, \bar{0}_2, \bar{0}_3, \bar{0}_4, (\square, 0, 0, 0, 0))\) | \(P_{5}(5, 0, \square) = \frac{1}{5} \left[ \chi \left( t \right) + \chi \left( t^2 \right) + \chi \left( t^3 \right) + \chi \left( t^4 \right) \right] \) | \(\frac{1}{5} \delta_{p} + \frac{1}{5} c\) | 0 |
| \((\bar{0}_1, \bar{0}_2, \bar{0}_3, \bar{0}_4, (0, \square, 0, 0, 0))\) | \(P_{5}(5, 1, \square) = \frac{1}{5} \left[ \chi \left( t \right) + \rho \chi \left( t^{1/5} \right) + \rho^2 \chi \left( t^{2/5} \right) + \rho^3 \chi \left( t^{3/5} \right) + \rho^4 \chi \left( t^{4/5} \right) \right] \) | \(\frac{1}{5} \delta_{p} + \frac{1}{5} c + \frac{1}{5} \) | 0 |
| \((\bar{0}_1, \bar{0}_2, \bar{0}_3, \bar{0}_4, (0, 0, \square, 0, 0))\) | \(P_{5}(5, 2, \square) = \frac{1}{5} \left[ \chi \left( t \right) + \rho^3 \chi \left( t^{1/5} \right) + \rho^4 \chi \left( t^{2/5} \right) + \rho^5 \chi \left( t^{3/5} \right) + \rho^2 \chi \left( t^{4/5} \right) \right] \) | \(\frac{1}{5} \delta_{p} + \frac{1}{5} c + \frac{1}{5} \) | 0 |
| \((\bar{0}_1, \bar{0}_2, \bar{0}_3, \bar{0}_4, (0, 0, 0, \square, 0))\) | \(P_{5}(5, 3, \square) = \frac{1}{5} \left[ \chi \left( t \right) + \rho^4 \chi \left( t^{1/5} \right) + \rho^5 \chi \left( t^{2/5} \right) + \rho^2 \chi \left( t^{3/5} \right) + \rho^3 \chi \left( t^{4/5} \right) \right] \) | \(\frac{1}{5} \delta_{p} + \frac{1}{5} c + \frac{1}{5} \) | 0 |
| \((\bar{0}_1, \bar{0}_2, \bar{0}_3, \bar{0}_4, (0, 0, 0, 0, \square))\) | \(P_{5}(5, 4, \square) = \frac{1}{5} \left[ \chi \left( t \right) + \rho^5 \chi \left( t^{1/5} \right) + \rho^2 \chi \left( t^{2/5} \right) + \rho^3 \chi \left( t^{3/5} \right) + \rho^4 \chi \left( t^{4/5} \right) \right] \) | \(\frac{1}{5} \delta_{p} + \frac{1}{5} c + \frac{1}{5} \) | 0 |
### Table A10. Characters for primaries \((p, p, p, p, p)\) of \(S_5\) orbifold.

| \(R\) | \(S_5\) orbifold character | \(E_2^\text{orb}(R)\) | \(E_3^\text{orb}(R)\) |
|---|---|---|---|
| \(|\square, (\square, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5\)| \(P_1(1, 0, \square)P_2(2, 0, \square)\) = \(2h_p + \frac{1}{8}c\) | \(\epsilon_{2,0}\) |
| \(|\square, (\square, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5\)| \(P_1(1, 0, \square)P_2(2, 0, \square)\) = \(2h_p + \frac{1}{8}c\) | \(\epsilon_{2,0}\) |
| \(|\square, (\square, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5\)| \(P_1(1, 0, \square)P_2(2, 1, \square)\) = \(2h_p + \frac{1}{8}c + 1\) | \(\epsilon_{2,1}\) |
| \(|\square, (\square, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5\)| \(P_1(1, 0, \square)P_2(2, 1, \square)\) = \(2h_p + \frac{1}{8}c + 1\) | \(-\epsilon_{2,1}\) |
| \(|\square, (\square, 0), \bar{0}_3, \bar{0}_4, \bar{0}_5\)| \(P_1(1, 0, \square)P_2(2, 0, \square)\) = \(2h_p + \frac{1}{8}c + \frac{1}{2}\) | 0 |
Table A11. Characters for primaries \((p, p, p, p, p)\) of \(S_5\) orbifold.

| \(\mathcal{R}\) | \(S_5\) orbifold character | \(E_{2}^{\text{orb}}(\mathcal{R})\) | \(E_{3}^{\text{orb}}(\mathcal{R})\) |
|-----------------|-------------------------------|---------------------|---------------------|
| \((\bar{0}_1, (\Box, 0), (\Box, 0, 0), \bar{0}_4, \bar{0}_3)\) | \(P_{1}(2, 0, \Box)P_{1}(3, 0, \Box)\) | \(\frac{1}{2}\left[\chi\left(\frac{x}{2}\right) + \chi\left(\frac{x+1}{2}\right)\right] \times \frac{1}{2}\left[\chi\left(\frac{x}{2}\right) + \chi\left(\frac{x+1}{2}\right) + \chi\left(\frac{x+2}{4}\right)\right]\) | \(\frac{2}{6}b_0 + \frac{25}{144}c\) | 0 |
| \((\bar{0}_1, (\Box, 0), (\Box, 0, 0), \bar{0}_4, \bar{0}_3)\) | \(P_{1}(2, 0, \Box)P_{1}(3, 1, \Box)\) | \(\frac{1}{2}\left[\chi\left(\frac{x}{2}\right) + \chi\left(\frac{x+1}{2}\right)\right] \times \frac{1}{2}\left[\chi\left(\frac{x}{2}\right) + e^{-\frac{2\pi}{3}}\chi\left(\frac{x+1}{2}\right) + e^{2\pi i}\chi\left(\frac{x+2}{2}\right)\right]\) | \(\frac{2}{6}b_0 + \frac{25}{144}c + \frac{1}{4}\) | 0 |
| \((\bar{0}_1, (\Box, 0), (\Box, 0, 0), \bar{0}_4, \bar{0}_3)\) | \(P_{1}(2, 0, \Box)P_{1}(3, 2, \Box)\) | \(\frac{1}{2}\left[\chi\left(\frac{x}{2}\right) + \chi\left(\frac{x+1}{2}\right)\right] \times \frac{1}{2}\left[\chi\left(\frac{x}{2}\right) + e^{2\pi i}\chi\left(\frac{x+1}{2}\right) + e^{-\frac{2\pi}{3}}\chi\left(\frac{x+2}{2}\right)\right]\) | \(\frac{2}{6}b_0 + \frac{25}{144}c + 0\) | 0 |
| \((\bar{0}_1, (\Box, 0), (\Box, 0, 0), \bar{0}_4, \bar{0}_3)\) | \(P_{1}(2, 1, \Box)P_{1}(3, 0, \Box)\) | \(\frac{1}{2}\left[\chi\left(\frac{x}{2}\right) - \chi\left(\frac{x+1}{2}\right)\right] \times \frac{1}{2}\left[\chi\left(\frac{x}{2}\right) + \chi\left(\frac{x+1}{2}\right) + \chi\left(\frac{x+2}{2}\right)\right]\) | \(\frac{2}{6}b_0 + \frac{25}{144}c + 0\) | 0 |
| \((\bar{0}_1, (\Box, 0), (\Box, 0, 0), \bar{0}_4, \bar{0}_3)\) | \(P_{1}(2, 1, \Box)P_{1}(3, 1, \Box)\) | \(\frac{1}{2}\left[\chi\left(\frac{x}{2}\right) - \chi\left(\frac{x+1}{2}\right)\right] \times \frac{1}{2}\left[\chi\left(\frac{x}{2}\right) - e^{-\frac{2\pi}{3}}\chi\left(\frac{x+1}{2}\right) + e^{2\pi i}\chi\left(\frac{x+2}{2}\right)\right]\) | \(\frac{2}{6}b_0 + \frac{25}{144}c + 0\) | 0 |
| \((\bar{0}_1, (\Box, 0), (\Box, 0, 0), \bar{0}_4, \bar{0}_3)\) | \(P_{1}(2, 1, \Box)P_{1}(3, 2, \Box)\) | \(\frac{1}{2}\left[\chi\left(\frac{x}{2}\right) - \chi\left(\frac{x+1}{2}\right)\right] \times \frac{1}{2}\left[\chi\left(\frac{x}{2}\right) - e^{2\pi i}\chi\left(\frac{x+1}{2}\right) + e^{-\frac{2\pi}{3}}\chi\left(\frac{x+2}{2}\right)\right]\) | \(\frac{2}{6}b_0 + \frac{25}{144}c + 0\) | 0 |
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