BLOWUP SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH COMPLEX COEFFICIENT

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Abstract. We construct a finite time blow up solution for the nonlinear Schrödinger equation with the power nonlinearity whose coefficient is complex number. We generalize the range of both the power and the complex coefficient for the result of Cazenave, Martel and Zhao [2]. As a bonus, we may consider the space dimension 5. We show a sequence of solutions closes to the blow up profile which is a blow up solution of ODE. We apply the Aubin-Lions lemma for the compactness argument for its convergence.

1. nonlinear Schrödinger equation with complex coefficient

We consider the following nonlinear Schrödinger equation with complex coefficient of the power nonlinearity

\[ iu_t(t, x) + \Delta u(t, x) = \lambda |u(t, x)|^\alpha u(t, x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^N, \]

where \( i = \sqrt{-1}, u_t = \partial_t u, \Delta u = \sum_{j=1}^N \partial_{x_j}^2 u, \alpha > 0, \lambda \in \mathbb{C}\setminus\{0\} \) and \( u = u(t, x) : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C} \) is a solution. There are large number of papers for the case \( \lambda \in \mathbb{R} \) which dealt with, for examples, wellposedness and behaviours of solutions. In this case, we have the following conservation laws, charge and energy respectively

\[ \frac{d}{dt} \|u(t)\|_{L^2} = 0, \]
\[ \frac{d}{dt} \left( \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \frac{\lambda}{\alpha+2} \|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \right) = 0. \]

These laws do not hold with \( \lambda \in \mathbb{C}\setminus\mathbb{R} \) in general. In the special case \( \lambda = i \), there are results in the book written by Lions [6], the technique of monotone operators and compactness argument are applied to have existence of the solutions. Cazenave, Martel and Zhao [2] investigate (1.1) with the same setting \( \lambda = i \). More general setting \( \lambda \in \mathbb{C} \) are discussed in [5] and which are sometimes called complex Ginzburg-Landau equation. We consider (1.1) under this general setting \( \lambda \in \mathbb{C} \) with some assumptions. We investigate the finite time blow-up phenomena of the solution of (1.1). There are former results. Kita [4] proved the blow-up solution which starts with small initial data in one spatial dimension, so called, small data blow-up phenomena. Cazenave, Martel and Zhao [2] proved the blow-up solution in general dimensions. We introduce more details in [2] at Remark 2 below.

If we set the initial data \( u(0, x) = f(x) \) belonging to the Sobolev space of order 1, that is \( H^1(\mathbb{R}^N) \), from the standard argument, there exists an unique time local solution of

\[ \frac{d}{dt} \|u(t)\|_{L^2} = 0, \]
\[ \frac{d}{dt} \left( \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \frac{\lambda}{\alpha+2} \|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \right) = 0. \]
There we assume the power condition

\[ 0 < \alpha \leq \frac{4}{N - 2} \]

where it means actually \( 0 < \alpha < \infty \) for \( N = 1, 2 \) and \( 0 < \alpha \leq \frac{4}{N - 2} \) for \( n \geq 3 \), and we employ this rule throughout this paper. We introduce a blow-up profile for our argument:

\[ U(t, x) = (-\alpha \text{Im}(t - |x|^k))^\frac{\lambda}{\alpha \text{Im}(\lambda)}. \]

This is the same in [2] when \( \lambda = i \). From the elemental calculation, we have

\[ \lim_{t \to 0^-} \|U(t)\|_{H^1} = \infty. \]

Now we state the main theorem.

**Theorem 1.** Let \( N = 1, 2, 3, 4, 5 \). Let \( \alpha \) and \( \lambda \) satisfy

\[ 1 < \alpha \leq \frac{4}{N - 2}; \quad (\alpha + 2)\text{Im}(\lambda) \geq \alpha|\lambda|, \]

and

\[ (\alpha + 2)\text{Im}(\lambda) > \alpha|\lambda| \quad \text{when} \quad \alpha > \frac{N}{N - 2} \quad \text{or} \quad \alpha = \frac{4}{N - 2}. \]

Then there exists a solution \( u \in C((\infty, 0), H^1(\mathbb{R}^N)) \) of (1.1) which blows up at \( t = 0 \) in the sense of the following

\[ \lim_{t \to 0^-} \|u(t)\|_{H^1} = \infty. \]

More precisely there exist positive constants \( C, \delta \) and \( \mu \) such that

\[ \|u(t) - U(t)\|_{H^1} \leq C(-t)^\mu, \quad -\delta < t < 0 \]

where \( U \) is the blow-up profile of (1.3).

**Remark 1.** The conditions (1.4) and (1.5) allow the followings

\[ 1 < \alpha < \infty, \quad (\alpha + 2)\text{Im}(\lambda) \geq \alpha|\lambda| \quad \text{for} \quad N = 1, 2. \]
\[ 1 < \alpha \leq 3, \quad (\alpha + 2)\text{Im}(\lambda) \geq \alpha|\lambda| \quad \text{or} \]
\[ 3 < \alpha \leq 4, \quad (\alpha + 2)\text{Im}(\lambda) > \alpha|\lambda| \quad \text{for} \quad N = 3. \]
\[ 1 < \alpha < \frac{4}{N - 2}, \quad (\alpha + 2)\text{Im}(\lambda) \geq \alpha|\lambda| \quad \text{or} \]
\[ \alpha = \frac{4}{N - 2}, \quad (\alpha + 2)\text{Im}(\lambda) > \alpha|\lambda| \quad \text{for} \quad N = 4, 5. \]

**Remark 2.** Cazenave-Martel-Zhao [2] gave the same conclusion under the assumption \( N = 1, 2, 3, 4 \) and for the power

\[ 2 \leq \alpha \leq \frac{4}{N - 2} \]

and the coefficient \( \lambda = i \) which satisfy (1.4) and (1.5). They, in fact, proved more generalized case that any number and anywhere for the blow up points. We generalize the range of \( \lambda \) and we reduce the lower bound of \( \alpha \). For our lower bound \( \alpha > 1 \), it seems difficult to reduce the number below 1 since our argument requires to estimate \( H^1 \) norm for the difference of the two functions, that are, solution of (1.1) and the blow up profile \( U \). We remark \( \alpha = 1 \) is critical and still open as well.
In the sequential paper, we will deal with the double critical point
\[ \alpha = \frac{4}{N - 2}, \quad (\alpha + 2)\text{Im}(\lambda) = \alpha|\lambda| \]
for both the time global wellposedness and the blow up problem. We will apply the results in [5] to solve the following complex Ginzburg-Landau equation for global existence time
\[ iu_t(t, x) + (1 - i\varepsilon)\Delta u(t, x) = \lambda|u(t, x)|^\alpha u(t, x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^N. \]
The solution \( u_\varepsilon \) exists globally in the negative time \( u_\varepsilon \in C((-\infty, 0]: H^1) \) for any \( \varepsilon > 0 \) and we take limit \( \varepsilon \to 0 + 0. \)

2. Preliminaries

Before going to our proof, we collect the standard estimates.

**Lemma 2.** Let \( p > 0 \) and \( n \in \mathbb{N} \cup \{0\} \). Then the estimates
\[
||z|^{p-n}z^n - |w|^{p-n}w^n| \lesssim \begin{cases} 
|z|^{p-1} + |w|^{p-1} |z - w| & \text{if } p \geq 1, \\
|z - w|^p & \text{if } 0 < p \leq 1,
\end{cases}
\]
hold for \( z, w \in \mathbb{C} \) where the implicit constant depends on \( p, n \) and is independent of \( z, w \).

We remark for this lemma that we may consider minus power \( p - n < 0 \) of the modulus, although the total power \( (p - n) + n = p \) is positive.

For the nonlinear term, we set \( g_\alpha(u) := |u|^\alpha u. \)

**Lemma 3.** Let \( N = 1, 2, \ldots \). Let \( I \) be bounded interval. Suppose a sequence \( (u_n) \subset L^\infty(I : H^1(\mathbb{R}^N)) \) and a function \( u \in L^\infty(I : L^2(\mathbb{R}^N)) \) satisfy
\[
\sup_{n \in \mathbb{N}} \|u_n\|_{L^\infty(I : H^1)} < \infty, \\
\lim_{n \to \infty} \|u_n - u\|_{L^\infty(I : L^2)} = 0.
\]
Let \( \alpha \) satisfies subcritical or critical condition
\[ 0 < \alpha \leq \frac{4}{N - 2}. \]
Then the limit and \( g_\alpha \) of it belong to the spaces
\[ u \in L^\infty(I : H^1(\mathbb{R}^N)) \quad \text{and} \quad g_\alpha(u) \in L^\infty(I : L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)) \]
respectively, and the following convergences hold
\[
\begin{align*}
&u_n \rightharpoonup u \quad \text{in} \ L^\infty(I : H^1(\mathbb{R}^N)), \\
&\Delta u_n \rightharpoonup \Delta u \quad \text{in} \ L^\infty(I : H^{-1}(\mathbb{R}^N)), \\
&g_\alpha(u_n) \rightharpoonup g_\alpha(u) \quad \text{in} \ L^\infty(I : L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)).
\end{align*}
\]
Moreover, suppose additional bounded condition
\[
\sup_{n \in \mathbb{N}} \|u_n\|_{W^{1, \infty}(I : H^{-1})} < \infty,
\]
Remark 3. We do not need to take any subsequence \((u_{n_k})\) in the conclusions. We do not require that \(u_n\) satisfy any equation likely as \((1.1)\), neither.

Proof. From \((2.2)\), we have a subsequence \((u_{n_k})\) and \(v \in L^\infty(I : H^1)\) such as

\[ u_n \rightarrow v \quad \text{weak in} \quad L^\infty(I : H^1). \]

Since limit is unique, we obtain \(u = v \in L^\infty(I : H^1)\). If there is subsequence \((u_{n_k})\) which does not converge to \(u\), then there are its subsequence \((u_{n_{k_j}})\), some test function \(\phi \in L^1(I : H^{-1})\) and \(\delta > 0\) satisfy

\[ |\langle u_{n_{k_j}} - u, \phi \rangle| > \delta \quad \text{for any} \quad j = 1, 2, \ldots. \]

This is a contradiction since this subsequence \((u_{n_{k_j}})\) is bounded, and so, it contains a subsequence \((u_{n_{k_{j_l}}})\) which does not satisfy \((2.9)\). Therefore the whole sequence converges to the limit. We obtain \((2.4)\), and so, \((2.5)\) follows as well. Next we consider \((2.6)\). Incidentally, we have the following convergence in the norm for the subcritical power \((N - 2)\alpha < 4\). By using the Gagliard-Nirenberg inequality

\[
\|g(u_n) - g(u)\|_{L^\infty L^{\frac{N}{{N + 2\alpha + 4}}}} \lesssim (\|u_n\|_{L^\infty L^{\alpha + 2}} + \|u\|_{L^\infty L^{\alpha + 2}})\|u_n - u\|_{L^\infty L^{\alpha + 2}}
\]

\[
\lesssim (\|u_n\|_{L^\infty H^1} + \|u\|_{L^\infty H^1})\|u_n - u\|_{L^\infty L^2}^{\frac{1 - \frac{N\alpha}{2(\alpha + 2)}}{\frac{N\alpha}{2(\alpha + 2)}}} \rightarrow 0
\]
as \(n \rightarrow \infty\), where \(\frac{N\alpha}{2(\alpha + 2)} < 1\). For the critical power \((N - 2)\alpha = 4\), we shall prove the same convergence but in the * weak sense. We take any \(\phi \in L^1(I : C_0^\infty(\mathbb{R}^N))\). For each \(s \in I\), we calculate

\[
\langle g(u_n(s)) - g(u(s)), \phi(s) \rangle \leq \|g(u_n(s)) - g(u(s))\|_{L^{\frac{\alpha + 2}{\alpha + 6}}} \|\phi(s)\|_{L^{\frac{\alpha + 2}{\alpha + 4}}} \leq (\|u_n(s)\|_{L^{\alpha + 2}} + \|u(s)\|_{L^{\alpha + 2}})\|u_n(s) - u(s)\|_{L^2} \|\phi(s)\|_{L^{\frac{\alpha + 2}{\alpha + 4}}}
\]

\[
\lesssim (\|u_n(s)\|_{H^1} + \|u(s)\|_{H^1})\|u_n(s) - u(s)\|_{L^2}^{\frac{1 - \frac{N\alpha}{2(\alpha + 4)}}{\frac{N\alpha}{2(\alpha + 4)}}} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

The integrable majorant in \(s\) is the following

\[
|\langle g(u_n(s)) - g(u(s)), \phi(s) \rangle| \leq \|g(u_n(s)) - g(u(s))\|_{L^{\frac{\alpha + 1}{\alpha + 6}}} \|\phi(s)\|_{L^{\alpha + 2}}
\]

\[
\lesssim (\|u_n(s)\|_{L^{\alpha + 2}} + \|u(s)\|_{L^{\alpha + 2}})\|\phi(s)\|_{L^{\alpha + 2}} \lesssim \|\phi(s)\|_{L^{\alpha + 2}} \in L^1(I).
\]

We then apply Lebesgue’s dominated convergence theorem to obtain

\[
\lim_{n \rightarrow \infty} \int_I \langle g(u_n(s)) - g(u(s)), \phi(s) \rangle ds = \int_I \lim_{n \rightarrow \infty} \langle g(u_n(s)) - g(u(s)), \phi(s) \rangle ds = 0.
\]

Since \(L^1(I : C_0^\infty) \hookrightarrow L^1(I : L^{\alpha + 2})\) is dense, this implies as desired

\[ g(u_n) \rightharpoonup g(u) \quad \text{weak * in} \quad L^\infty(I : L^{\frac{\alpha + 4}{\alpha + 2}}). \]
Actually we can prove this weak * convergence for \( \{g(u_n)\} \) from \( L^\infty H^1 \) boundedness and \( L^\infty L^2 \) convergence of \( \{u_n\} \) in another way. Since the interval is finite, we have
\[
u_n \to u \in L^\infty(I; L^2) \hookrightarrow L^2(I \times \mathbb{R}^N),
\]
we then have a subsequence \( \{u_{n_k}\} \) which converges almost everywhere, i.e.
\[
u_{n_k} \to u \text{ a.e. in } I \times \mathbb{R}^N.
\]
This gives
\[
g(u_{n_k}) \to g(u) \text{ a.e. in } I \times \mathbb{R}^N.
\]
From Fubini’s theorem, we have
\[
g(u_{n_k})(s) \to g(u)(s) \text{ a.e. } x \in \mathbb{R}^N
\]
for a.e. \( s \in I \). We estimate the norm
\[
\|g(u_{n_k})(s)\|_{L^{\alpha+2}_x} \leq \|u_{n_k}(s)\|_{L^{\alpha+2}_x} \leq \|u_{n_k}(s)\|_{H^1_x}
\]
and this is uniformly bounded in \( k \) and \( s \in I \). From Lemma 1.3 in \( \[6\] \), we have
\[
g(u_{n_k})(s) \to g(u)(s) \text{ weak in } L^{\alpha+2}_x(\mathbb{R}^N)
\]
for a.e. \( s \in I \). Here we do not need to take a subsequence out. We estimate
\[
\|g(u)(s)\|_{L^{\alpha+2}_x} \leq \liminf_{k \to \infty} \|g(u_{n_k})(s)\|_{L^{\alpha+2}_x} \leq C
\]
with some constant \( C \). So, for any \( \phi \in L^1(I; L^{\alpha+2}) \) and this \( C \), we have
\[
|\langle g(u_{n_k})(s) - g(u(s)), \phi(s) \rangle| \lesssim C\|\phi(s)\|_{L^{\alpha+2}_x} \in L^1(I).
\]
From Lebesgue’s dominated convergence theorem again gives the result,
\[
g(u_{n_k}) \rightharpoonup g(u) \text{ weak * in } L^\infty(t, 0; L^{\alpha+2}_x).
\]
We may say from this argument that the whole sequence \( \{u_n\} \) converges to the same limit \( u \) which corresponds to \( \[2.10\] \). This complete the second proof for weak * convergence for \( \{g(u_n)\} \).

Next we assume \( \[2.7\] \) additionally and show \( \[2.8\] \). From the same argument above, we have \( \partial_t u \in L^\infty(I : H^{-1}) \) where \( u \) is the limit in \( \[2.3\] \). Form \( \[2.4\] \), we have for any \( \phi \in C^1_0(I : H^1) \),
\[
\int_I \langle \partial_t u_n(s) - \partial_t u(s), \phi(s) \rangle_{H^{-1}, H^1} ds = -\int_I \langle u_n(s) - u(s), \partial_t \phi(s) \rangle_{H^{-1}, H^1} ds \to 0
\]
as \( n \to \infty \). This implies
\[
\[2.11\] \quad \partial_t u_n \rightharpoonup \partial_t u \text{ weak * in } L^\infty(I : H^{-1}).
\]
Combining with \( \[2.4\] \), we obtain \( \[2.8\] \) and this complete the proof. \( \square \)

We define the space
\[
\Sigma = \Sigma(\mathbb{R}^N) : = \{ f \in H^1(\mathbb{R}^N) : \| f \|_{\Sigma} < \infty \},
\]
\[
\| f \|_{\Sigma}^2 = \| f \|_{H^1}^2 + \| \cdot f \|_{L^2}^2.
\]

**Lemma 4.** Let \( N \in \mathbb{N} \). Then the embedding \( \Sigma(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \) is compact.

We introduce the Aubin-Lions lemma, see Simon \( \[7\] \).
Lemma 5. Let $X_0, X$ and $X_1$ be three Banach spaces with $X_0 \hookrightarrow X \hookrightarrow X_1$ where $X_0$ is compactly embedded in $X$, and $X$ is continuously embedded in $X_1$. For $1 \leq p, q \leq \infty$, define

$$\Phi := \begin{cases} \{ u \in L^p(0,T:X_0) \mid u_t \in L^q(0,T:X_1) \}, & \text{if } p < \infty, \\ \{ u \in C(0,T:X_0) \mid u_t \in L^q(0,T:X_1) \}, & \text{if } p = \infty. \end{cases}$$

Then

1. If $p < \infty$, the embedding $\Phi \hookrightarrow L^p(0,T:X)$ is compact.
2. If $p = \infty, q > 1$, the embedding $\Phi \hookrightarrow C(0,T:X)$ is compact.

3. Time global well-posedness

In this section we show the existence of solution from any initial data in $H^1$. We later consider the sequence of solutions $v_n, n = 1, 2, \ldots$ on each time interval $[0, T^*_n)$ where $T^*_n$ is maximal existence time. If we obtain the time global well-posedness, we have uniform existence time $T^*_n = \infty, n = 1, 2, \ldots$.

Theorem 6. Let $n \in \mathbb{N}, \lambda \in \mathbb{C}$ and

$$(3.1) \quad 0 < \alpha \leq \frac{4}{4 - 2 N}.$$  

Then for any $f \in H^1(\mathbb{R}^N)$ there exists $T^*, -T^* > 0$ and unique solution $u \in C((T^*, T^*), H^1(\mathbb{R}^N)).$

Moreover if we additionally assume $(\alpha + 2) \Im \lambda > \alpha |\lambda|$, then $T^* = -\infty$.

Proof. The standard argument, contraction mapping principle by using Strichartz estimate, gives the time local well-posedness where the maximal existence time $T^*$ and $T_*$ depend on $\|f\|_{H^1}$ for the subcritical power $\alpha < \frac{4}{4 - 2}$ and on the profile of $f$ for the critical power $\alpha = \frac{4}{4 - 2}$. In order to obtain the global solvability, we deduce a priori estimate.

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 = \Re \int u_t(t,x)\overline{u(t,x)}dx$$

$$= \Re \int (i\Delta u - i\lambda |u|^{\alpha}u)\overline{u}dx$$

$$= \Re \int i|\nabla u|^2 - i\lambda |u|^{\alpha+2}dx$$

$$= \Im \lambda \|u\|_{L^{\alpha+2}}^{\alpha+2} \geq 0$$

from the assumption $\Im \lambda > 0$. We also have

$$\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 = -\Re \int u_t \Delta \overline{u}dx$$

$$(3.2) \quad = \frac{\alpha + 2}{2} \Re (i\lambda) \int |u|^{\alpha} |\nabla u|^2 - \frac{\alpha}{2} \Re (i\lambda) \int |u|^{\alpha-2}u^2(\nabla \overline{u})^2$$

$$\geq \frac{\alpha + 2}{2} \Im (\lambda) \int |u|^{\alpha} |\nabla u|^2 - \frac{\alpha}{2} |\lambda| \int |u|^{\alpha-2}u^2(\nabla \overline{u})^2$$

$$\geq \left( \frac{(\alpha + 2) \Im (\lambda) - \alpha |\lambda|}{2} \right) \int |u|^{\alpha} |\nabla u|^2 dx \geq 0,$$
where the fourth line just used $\text{Re}[u^2(\nabla \bar{u})^2] \leq |u|^2|\nabla \bar{u}|^2$. Therefore we have $\frac{d}{dt} \|u(t)\|_{H^1} \geq 0$ and conclude global existence of $u$ on $(-\infty, 0]$ for the subcritical power. With respect to the critical power, we may apply the argument in [2] with all $\lambda$ of (1.5) besides the critical complex coefficient $(\alpha + 2)\text{Im}(\lambda) = \alpha|\lambda|$. Indeed we utilize (3.2) to have

$$\int_{T_*}^{0} \|u(t)\|_L^p dt \leq \frac{\|\nabla u(0)\|_L^2 - \|\nabla u(T_*)\|_L^2}{(\alpha + 2)\text{Im}(\lambda) - \alpha|\lambda|} < \infty$$

for the maximal existence time $T_* < 0$. The index satisfies the embedding $W^{1,r}(\mathbb{R}^N) \hookrightarrow L^{N/(N-2)}(\mathbb{R}^N)$ with Strichartz admissible $(r, \alpha + 2)$ which implies $T_* = -\infty$. □

4. Estimates of $U$

We estimate the profile $U$ in (1.3). This function satisfies the following ODE

$$iU_i = \lambda |U|^\alpha U, \quad t \neq 0, x \in \mathbb{R}^N.$$  

We collect the estimates on $U$ which are from [2] or slight modifications. We have for $1 \leq p \leq \infty$ and any $t < 0$,

$$\|U(t)\|_L^p \lesssim (-t)^{-\frac{1}{\alpha} + \frac{N}{2p}},$$

$$\|\nabla U(t)\|_L^p \lesssim (-t)^{-\frac{1}{\alpha} + \frac{1}{2} + \frac{N}{4}},$$

$$\|\Delta U(t)\|_L^2 \lesssim (-t)^{-\frac{1}{\alpha} + \frac{3}{2} + \frac{N}{2}},$$

$$\|\nabla \Delta U(t)\|_L^2 \lesssim (-t)^{-\frac{1}{\alpha} + \frac{3}{2} + \frac{N}{2}},$$

$$\|1 + |\cdot||\Delta U(t)\|_L^2 \lesssim (1 + (-t)^{\frac{1}{2}})\|\Delta U(t)\|_L^2.$$  

Proof. These estimates (4.2) – (4.5) follows by the calculation in [2]. The estimate (4.6) is new and follow by the scaling argument as

$$\|1 + |\cdot||\Delta U(t)\|_L^2 \lesssim (1 + (-t)^{\frac{1}{2}})\|\Delta U(t)\|_L^2$$

and use (4.4) for small $t < 0$. □

5. Difference between the solution and the profile

Since the profile $U$ blows up at $t = 0$, in order to obtain the blow up phenomena it suffice to estimate the difference between the solution of (1.1) and the profile which converges to 0 as $t \to 0 - 0$. We actually discuss the approximate solutions with the initial data $U(T_n)$ at the initial time $T_n = -\frac{1}{n}$ for each $n = 1, 2, \ldots$. As we saw in section 3 that (1.1) is time globally wellposed, there is no need to worry about the degeneration of the existence times for the sequence of these solutions. We consider the Cauchy problem of (1.1) with the initial data $U$ defined above at the initial time $T_n = -\frac{1}{n}$,

$$u_t = i\Delta u - \lambda |u|^\alpha u, \quad -\infty < t \leq T_n, \quad x \in \mathbb{R}^N,$n

$$u(T_n) = U(T_n), \quad x \in \mathbb{R}^N.$$  

Theorem 6 gives the unique existence of the global solution of (5.1) for each $n$ $u \in C((\infty, T_n], H^1)$.

We define and estimate the following

$$\varepsilon(t, x) := u(t, x) - U(t, x).$$
Although it seems better to denote \(u_n\) and \(\varepsilon_n\) for \(u\) and \(\varepsilon\) respectively in each \(n\), we abbreviate them when there is no confusion. Then \(\varepsilon\) satisfies
\[
\partial_t \varepsilon = i\Delta \varepsilon - i\lambda|u|^\alpha u + i\lambda|U|^\alpha U + i\Delta U, \quad -\infty < t \leq T_n, \ x \in \mathbb{R}^N,
\]
\[
\varepsilon(T_n) = 0, \quad x \in \mathbb{R}^N.
\]

**Proposition 7.** There are positive constants \(C_1, C_2, \mu, \gamma\) and \(\delta\) such that for any \(n\)
\[
\|\varepsilon_n(t)\|_{H^1} \leq C_1(T_n - t)^\mu, \quad T_n - \delta \leq t \leq T_n,
\]
\[
|||x|\varepsilon_n(t)||_{L^2} \leq C_2(T_n - t)^\gamma, \quad T_n - \delta \leq t \leq T_n.
\]

**Proof.** We start with the estimate on \(L^2\) norm.
\[
\frac{1}{2} \frac{d}{dt} \|\varepsilon(t)\|^2_{L^2} = \text{Re} \int (i\Delta \varepsilon - i\lambda|u|^\alpha u - |U|^\alpha U + i\Delta U)\overline{\varepsilon} dx
\]
\[
= \text{Re} \int (-i\lambda|u|^\alpha u - |U|^\alpha U + i\Delta U)\overline{\varepsilon} dx =: I + \text{Re} \int i\Delta U\overline{\varepsilon} dx.
\]
We estimate \(I\) that is the first term plus second term. We apply the mean value theorem for the two variables function \(F(z) = F(z, \bar{z}) = |z|^\alpha z\),
\[
|u|^\alpha u - |U|^\alpha U
\]
\[
= \int_0^1 F_z(U + \theta(u - U))(u - U) + F_\bar{z}(U + \theta(u - U))(\overline{u - U}) d\theta
\]
where \(F_z = \frac{\alpha + 2}{2}|z|^\alpha, F_\bar{z} = \frac{\alpha - 2}{2}|z|^\alpha \bar{z}^2\). We estimate
\[
-I = \text{Re} \int \int_0^1 i\lambda \frac{\alpha + 2}{2}|U + \theta(u - U)|^\alpha|u - U|^2 d\theta dx
\]
\[
+ \text{Re} \int \int_0^1 i\lambda \frac{\alpha - 2}{2}|U + \theta(u - U)|^{\alpha - 2}(U + \theta(u - U))^2(\overline{u - U})^2 d\theta dx
\]
\[
\leq -\text{Im}(\lambda) \int \int_0^1 \frac{\alpha + 2}{2}|U + \theta(u - U)|^\alpha|u - U|^2 d\theta dx
\]
\[
+ |\lambda| \int \int_0^1 \frac{\alpha - 2}{2}|U + \theta(u - U)|^{\alpha - 2}d\theta dx \leq 0
\]
where we used (1.4) at the last inequality. We therefore obtain
\[
\frac{1}{2} \frac{d}{dt} \|\varepsilon(t)\|^2_{L^2} \geq \text{Re} \int i\Delta U\overline{\varepsilon} dx \geq -\|\Delta U\|_{L^2}\|\varepsilon\|_{L^2}.
\]
where we used \(\text{Re} \int i\Delta \varepsilon dx = 0\) and \(I_2 \geq 0\) in (5.4). We may write
\[
\frac{d}{dt} \|\varepsilon(t)\|_{L^2} \geq -\|\Delta U\|_{L^2} \geq -C(-t)^{-\frac{1}{\alpha} - \frac{4 - N}{2k}}.
\]
We integrate this on the interval \((t, T_n)\) and apply (2.1) to have
\[
\|\varepsilon(t)\|_{L^2} \leq C((-t)^{\mu_1} - (T_n)^{\mu_1}) \leq C(T_n - t)^{\mu_1}
\]
where
\[
0 < \mu_1 = 1 - \frac{1}{\alpha} - \frac{4 - N}{2k} < 1
\]
for sufficiently large $k$. We next estimate $\dot{H}^1$ norm.

$$
\frac{d}{dt} \| \nabla \varepsilon(t) \|_{L^2}^2 = \text{Re} \int \nabla (i \Delta \varepsilon - i \lambda(|u|^\alpha u - |U|^\alpha U) + i \Delta U) \nabla \varepsilon dx
$$

$$
= \text{Re} \int (-i \lambda) \left[ \frac{\alpha + 2}{2} (|u|^\alpha \nabla u - |U|^\alpha \nabla U) + \frac{\alpha}{2} (|u|^{\alpha - 2} u^2 \nabla \bar{u} - |U|^{\alpha - 2} U^2 \nabla \bar{U}) \right] \nabla \varepsilon + i \nabla \Delta U \nabla \varepsilon dx
$$

$$
= \text{Re} \int (-i \lambda) \left[ \frac{\alpha + 2}{2} (|u|^\alpha - |U|^\alpha) \nabla U + \frac{\alpha + 2}{2} |u|^\alpha \nabla \varepsilon + \frac{\alpha}{2} (|u|^{\alpha - 2} u^2 - |U|^{\alpha - 2} U^2) \nabla \bar{U} + \frac{\alpha}{2} |u|^{\alpha - 2} \nabla \bar{U} \right] \nabla \varepsilon
$$

$$
+ i \nabla \Delta U \nabla \varepsilon dx.
$$

We estimate the sum of second and fourth terms which is the worst if we consider the modulus of it in the sense of decay as $t \to 0 - 0$. However it can be estimated since we just consider the real part of it likely as

$$
\text{Re} \int (-i \lambda) \left[ \frac{\alpha + 2}{2} |u|^\alpha |\nabla \varepsilon|^2 + \frac{\alpha}{2} |u|^{\alpha - 2} u^2 (\nabla \varepsilon)^2 \right] dx
$$

$$
\geq \left( \text{Im}(\lambda) \frac{\alpha + 2}{2} - |\lambda| \frac{\alpha}{2} \right) \int |u|^\alpha |\nabla \varepsilon|^2 dx.
$$

Next we estimate the first term plus the third term where the coefficient $\lambda$ is estimated by its modulus. We use Lemma 2 for $\alpha \geq 1$ to have

$$
||u|^\alpha - |U|^\alpha| + ||u|^{\alpha - 2} u^2 - |U|^{\alpha - 2} U^2| \lesssim (|u|^{\alpha - 1} + |U|^{\alpha - 1}) |u - U|
$$

$$
\lesssim (|u - U|^{\alpha - 1} + |U|^{\alpha - 1}) |u - U| = |\varepsilon|^\alpha + |U|^{\alpha - 1} |\varepsilon|
$$

where we used $|u|^{\alpha - 1} \lesssim |u - U|^{\alpha - 1} + |U|^{\alpha - 1}$ at the beginning of the second line. In the long run, we have

$$
-\frac{d}{dt} \| \nabla \varepsilon(t) \|_{L^2}^2 \lesssim \int (|\varepsilon|^\alpha + |U|^{\alpha - 1} |\varepsilon|) |\nabla U| |\nabla \varepsilon| + |\nabla \Delta U| |\nabla \varepsilon| dx.
$$

We estimate the first term in (5.7). We separate it into the following two cases.

$$
1 < \alpha \leq \frac{N}{N - 2},
$$

$$
2 \leq \alpha \leq \frac{4}{N - 2}.
$$

In the former case (5.8), we estimate

$$
\int |\varepsilon|^\alpha |\nabla U| |\nabla \varepsilon| dx \leq ||\varepsilon||_{L^r}^\alpha ||\nabla U||_{L^\infty} ||\nabla \varepsilon||_{L^2}
$$

$$
\lesssim ||\varepsilon||_{L^2}^{\alpha - \frac{\alpha}{r} (\alpha - 1)} ||\nabla U||_{L^\infty} ||\nabla \varepsilon||_{L^2}^{1 + \frac{\alpha}{r} (\alpha - 1)}
$$

$$
\lesssim (-t)^{\frac{\alpha}{r}} ||\nabla \varepsilon||_{L^2}^{1 + \frac{\alpha}{r} (\alpha - 1)}
$$

where we apply Hölder inequality and Gagliard-Nirenberg inequality with

$$
1 = \frac{\alpha}{r} + \frac{1}{\infty} + \frac{1}{2}.
$$
and
\[
\frac{1}{r} = \frac{1}{2} - \frac{\theta}{N}, \quad \alpha \theta = \frac{N}{2}(\alpha - 1)
\]
respectively. These satisfy
\[
0 < \theta = \frac{\alpha - 1}{\alpha} \frac{N}{2} \leq 1
\]
which is provided by (5.8). We estimate the power of \( t \) in (5.10). From (4.3) and (5.5) we have
\[
\mu_2 = \left(1 - \frac{\alpha}{2} - \frac{4 - N}{2k}\right)\left(\frac{\alpha}{2}(\alpha - 1) - \frac{1}{\alpha} \frac{1}{k}\right).
\]
We require that this is strictly greater than \(-1\). For simplicity we take \( k = \infty \) and then \( \mu_2 > -1 \) gives
\[
(\alpha - 1)\left(\alpha + 1 - \frac{N}{2}(\alpha - 1)\right) > 0
\]
and so
\[
(5.11) \quad 1 < \alpha < \frac{N + 2}{N - 2}
\]
which is provided by (1.4). In the latter case (5.9), we follow the arguments in [2].
\[
\int |\varepsilon|^\alpha \nabla U|\nabla \varepsilon|dx \lesssim \int |\varepsilon|(|U|^{\alpha - 1} + |U + \varepsilon|^{\alpha - 1})|\nabla U| |\nabla \varepsilon|dx.
\]
The first term of this is the same with the second term in (5.7), and we estimate later. We estimate the second term of this by Cauchy Schwarz and with any \( \delta > 0 \)
\[
\int |\varepsilon||U + \varepsilon|^{\alpha - 1}|\nabla U||\nabla \varepsilon|dx
\]
\[
\leq \left(\int |U + \varepsilon|^{\alpha}|\nabla \varepsilon|^2dx\right)^\frac{1}{2} \left(\int |\varepsilon|^2|U + \varepsilon|^{\alpha - 2}|\nabla U|^2dx\right)^\frac{1}{2}
\]
\[
\leq \delta \int |U + \varepsilon|^{\alpha}|\nabla \varepsilon|^2dx + \frac{1}{4\delta} \int |\varepsilon|^2|U + \varepsilon|^{\alpha - 2}|\nabla U|^2dx.
\]
We absorb the first term of this into (5.6). We estimate the second term
\[
(5.12) \quad \int |\varepsilon|^2|U + \varepsilon|^{\alpha - 2}|\nabla U|^2dx \lesssim \|\varepsilon\|^2_{L^2}\|U\|^{\alpha - 2}_{L^\infty}\|\nabla U\|^2_{L^\infty} + \|\varepsilon\|_{L^\alpha}\|\nabla U\|^2_{L^\infty}
\]
where we used \( \alpha \geq 2 \). The Gagliardo-Nirenberg inequality also uses the condition \( \alpha \geq 2 \)
\[
\|\varepsilon\|^2_{L^\alpha} \lesssim \|\varepsilon\|^{\frac{2N-\alpha(N-2)}{2}}_{L^2}\|\nabla \varepsilon\|_{L^2}^{\frac{N}{(\alpha - 2)}}.
\]
Therefore the right hand side of (5.12) is bounded by \( C(-t)^{\mu_3} \) with some \( C > 0 \) and \( \mu_3 > -1 \). We also follow the estimates in [2] for the second and third terms in (5.7). We use Hölder inequality and (4.2), (4.3) and (4.5) to have
\[
\int |U|^{\alpha - 1}|\varepsilon||\nabla U||\nabla \varepsilon|dx \lesssim \|U\|_{L^{\infty}}^{\alpha - 1}\|\varepsilon\|_{L^2}\|\nabla U\|_{L^\infty}\|\nabla \varepsilon\|_{L^2}
\]
\[
\lesssim (-t)^{-\frac{\alpha - 1}{\alpha} + \left(1 - \frac{1}{\alpha} - \frac{4 - N}{2k}\right)\frac{1}{\alpha} + \frac{1}{2}}\|\nabla \varepsilon\|_{L^2}
\]
\[
= (-t)^{-\frac{1}{\alpha} - \frac{\alpha N}{2k}}\|\nabla \varepsilon\|_{L^2}
\]
and
\[ \int |\nabla \Delta U| \nabla \varepsilon | d\textbf{x} \leq \| \nabla \Delta U \|_{L^2} \| \nabla \varepsilon \|_{L^2} \]
\[ \lesssim (-t)^{-\frac{1}{\alpha} - \frac{6-N}{2k}} \| \nabla \varepsilon \|_{L^2}. \]
We set \( \mu_4 = -\frac{1}{\alpha} - \frac{6-N}{2k} \) and \( \mu_5 = \min\{\mu_2, \mu_3, \mu_4\} \), we saw \( \mu_5 > -1 \) for sufficiently large \( k \).
We then estimate (5.7) as
\[ -\frac{d}{dt} \| \nabla \varepsilon(t) \|_{L^2}^2 \lesssim (-t)^{\mu_5}(1 + \| \nabla \varepsilon \|_{L^2}^{1+\frac{N}{2}(\alpha-1)}). \]
This and \( \varepsilon(T_n) = 0 \) give the value \( \delta > 0 \) such that
(5.13)
\[ -\frac{d}{dt} \| \nabla \varepsilon(t) \|_{L^2}^2 \lesssim (-t)^{\mu_5}, \quad T_n - \delta < t < T_n \]
uniformly with respect to \( n \). Integrate this and we have (5.2). Next we show (5.3). We set \( a > 0 \) and estimate
\[ -\frac{1}{2} \frac{d}{dt} \| e^{-a|\cdot|^2} \|_{L^2}^2 \leq 2 \| \nabla \varepsilon \|_{L^2}^2 (1 - 2a|\cdot|^2) e^{-2a|\cdot|^2} \| \cdot \|_{L^2} + \| \cdot \|_{L^2} \| \Delta U \|_{L^2} e^{-2a|\cdot|^2} \| \cdot \|_{L^2} \]
\[ \leq 2 \| \nabla \varepsilon \|_{L^2}^2 \| e^{-a|\cdot|^2} \|_{L^2} + \| \cdot \|_{L^2} \| \Delta U \|_{L^2} ^2 \| e^{-a|\cdot|^2} \|_{L^2} \| \cdot \|_{L^2}. \]
We then have from (5.13) and (4.6)
\[ -\frac{d}{dt} \| e^{-a|\cdot|^2} \|_{L^2} \lesssim \| \nabla \varepsilon \|_{L^2} + \| \cdot \|_{L^2} \| \Delta U \|_{L^2} \]
\[ \lesssim (-t)^{-\frac{1}{\alpha} - \gamma}, \quad T_n - \delta < t < T_n. \]
Integrate this to have
\[ \| e^{-a|\cdot|^2} \|_{L^2} \lesssim (T_n - t)^{1-\frac{1}{\alpha} - \gamma}, \quad T_n - \delta < t < T_n. \]
We obtain the result (5.3) by using Fatou’s lemma as \( a \to 0 + 0 \).

6. CONSTRUCTION OF BLOW UP SOLUTION

Proof. We construct the solution for Theorem (1.1) from the approximate solution \( u_n \) and the difference function \( \varepsilon_n \) above. We set for \( t > 0 \),
\[ v_n(t) = u_n(T_n - t), \quad V_n(t) = U(T_n - t), \quad \eta_n(t) = \varepsilon_n(T_n - t). \]
In the following, we consider \( 0 < t < \delta \) since we had \( T_n - \delta \leq T_n - t \leq T_n \) in (5.2). We also remember \( T_n = -\frac{1}{n} < 0 \). Then we have \( v_n = V_n + \eta_n \) and
(6.1)
\[ \partial_t \eta_n = -i \Delta \eta_n - |v_n|^{\alpha} v_n + |V_n|^{\alpha} V_n - i \Delta V_n \]
\[ = -i \Delta \eta_n - |V_n + \eta_n|^{\alpha} (V_n + \eta_n) + |V_n|^{\alpha} V_n - i \Delta V_n \]
on $0 < t < \delta$. It holds by (5.2) and (5.3)

$$\|\eta_n(t)\|_{H^1} + \| |\eta_n(t)|_{L^2} \leq C t^\mu, \quad 0 < t < \delta.$$  

We estimate

$$\|V_n(t)\|_{L^p} \lesssim (t - T_n)^{-\frac{1}{\alpha + \frac{1}{N}}} = \left( t + \frac{1}{n} \right)^{-\frac{1}{\alpha + \frac{1}{N}}} \lesssim t^{-\frac{1}{\alpha + \frac{1}{N}}}$$

for sufficiently large $k$ and any $n$. From the embeddings $H^1(\mathbb{R}^N) \hookrightarrow L^{\alpha + 2}(\mathbb{R}^N)$ and $L^{\frac{\alpha + 2}{\alpha + 1}}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N)$,

$$\|V_n + \eta_n|^\alpha(V_n + \eta_n)\|_{H^{-1}} \lesssim \|V_n + \eta_n|^\alpha(V_n + \eta_n)\|_{L^{\frac{\alpha + 2}{\alpha + 1}}} \lesssim \|V_n\|^{\alpha + 1}_{L^{\alpha + 2}} + \|\eta_n\|^{\alpha + 1}_{H^1} \lesssim t^{-\kappa},$$

and therefore

$$\|\partial_t \eta_n\|_{H^{-1}} \lesssim \|\eta_n\|_{H^1} + \|V_n\|_{L^{\alpha + 2}} + \|\eta_n\|_{H^1} + \|\Delta V_n\|_{L^2}$$

$$\lesssim t^{-\kappa}$$

with some $\kappa > 0$ for any $0 < t \leq \delta$ and any $n$. Given any $\tau \in (0, \delta)$, if we restrict the interval $(\tau, \delta)$, the sequence $\{\eta_n\}$ is bounded in $L^\infty(\tau, \delta : \Sigma) \cap W^{1, \infty}(\tau, \delta : H^{-1})$. Now we apply the Aubin-Lions lemma with

$$\Sigma \hookrightarrow L^2 \hookrightarrow H^{-1}$$

to conclude that there exists a subsequence which is still written by $\eta_n$ and the limits $\eta \in L^\infty(\tau, \delta, L^2)$ such that

$$\|\eta_n - \eta\|_{L^\infty(\tau, \delta, L^2)} \to 0.$$  

We apply the diagonal argument. For sufficiently large $m$, we set $\sigma_k = \frac{1}{k}, k = m, m + 1, m + 2, \ldots$ such as $0 < \sigma_k < \delta$. We obtain (6.3) for each $\tau = \sigma_k$. We take the subsequence $\{\eta_{n_k}\}$ diagonally to have the limit $\eta$ which belongs to $L^\infty_{\text{loc}}(0, \delta : L^2)$ and satisfies

$$\|\eta_{n_k} - \eta\|_{L^\infty(\tau, \delta, L^2)} \to 0$$

for any $0 < \tau < \delta$. From this and the boundedness (6.2), we utilize Lemma 3 with $I = (\tau, \delta)$ to obtain four kinds of convergence (2.4), (2.5), (2.6) and (2.8) on the same $I = (\tau, \delta)$. Therefore the limit $\eta$ satisfies the equation which corresponds to (6.1),

$$\partial_t \eta = -i\Delta \eta - |U + \eta|^\alpha(U + \eta) + |V|^\alpha U - i\Delta U.$$  

The function $u = U + \eta$ satisfies (1.1) and for any $-\delta < t < 0$,

$$\|u(t) - U(t)\|_{H^1} = \|\eta(t)\|_{H^1} \leq \liminf_{k \to \infty} \|\eta_{n_k}(-t)\|_{H^1} \leq C(-t)^\mu.$$

\[\square\]

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