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Modeling COVID-19 spreading dynamics and unemployment rate evolution in rural and urban counties of Alabama and New York using fractional derivative models

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ABSTRACT

The COVID-19 pandemic has been affecting the United States (U.S.) since the outbreak documented on 2/29/2020, and understanding its dynamics is critical for pandemic mitigation and economic recovery. This study proposed and applied novel time fractional derivative models (FDMs) to quantify the spatiotemporal dynamics of the COVID-19 pandemic spreading in the states of Alabama and New York, U.S., two states with quite different population compositions, urbanization, and industry structures. Model applications revealed that the pandemic evolving in the two states exhibited an overall similar time-dependent trend with subtle differences in propagation rates. Alabama may have more inter-county communications in rural areas than urban areas, while the opposite may be true for the New York State. Further analysis using the space FDM showed that the COVID-19 pandemic spread in rural/urban areas of the two states by following the tempered stable density distributions with different indexes, while the number of the state’s pandemic epicenters affected the pattern of the COVID-19 pandemic spreading in space. Finally, applications of a novel time FDM revealed that the evolution of the economy, represented by the weekly unemployment insurance claims in the two states, exhibited different spreading and recovery rates, most likely due to their different exposures and responses to the pandemic. Therefore, COVID-19 spreading dynamics exhibited strong and subtly different spatiotemporal memories in rural and urban areas in the Alabama and New York States, motivating the application of FDMs.

Introduction

The novel coronavirus disease 2019 (COVID-19) has been affecting the United States (U.S.) and challenging public health for months since the outbreak documented on 2/29/2020. The confirmed cases in the U.S. were > 18 million with 323,527 related deaths up to 12/26/2020, while the maximum daily increase of confirmed cases in December 2020 (which were 172,355 cases on 12/20/2020) was ten times larger than the previous peak in August 2020 (14,857 cases on 8/13/2020). The number of cumulated cases, therefore, had not reached its peak in December 2020. Efficient pandemic mitigation and post-pandemic economic recovery are critical, which can be benefitted from quantitative modeling of COVID-19 spreading dynamics using reliable mathematical models. This necessity motivated this study.

Various models had been developed to simulate COVID-19 spreading. For example, Sindhu et al. [32–33] simulated the distribution of COVID-19 infectious cases in Europe and China using the Gumbel Type-II distribution and other new classes of distributions. Giordano et al. [17] proposed a new model to quantify eight stages of COVID-19 infection in Italy. Prem et al. [29] introduced an age-structured SEIR model (where S, E, I, and R stand for susceptible, exposed, infected, and recovered populations, respectively) to simulate the infection evolution of COVID-19 in Wuhan, China. Wang and Yamamoto [36] developed a partial differential equation to describe the spatial transmission of COVID-19 among county clusters in Arizona, U.S. Doe et al. [15] introduced a county-level epidemiological inference recurrent network (LSTM) based cells to produce a two-week forecast by learning the confirmed COVID-19 cases of individual counties in the U.S. Hortaçsu et al. [18] built an analytical, tractable method to estimate the proportion of undocumented COVID-19 cases in...
the U.S. under the influence of the current epicenter. Li et al. [26] used the reported COVID-19 infection cases in China and integrated with a modified SEIR model to infer the fraction of undocumented infectious cases. With the rapid growth and evolution of the COVID-19 pandemic in the U.S., an updated model which can capture the spatiotemporal evolution of the pandemic is still needed. As reviewed above, the SEIR models are one of the primary quantitative tools used in epidemiology for describing the spread of infectious diseases. A key barrier hindering the efforts of modeling the COVID-19 outbreak is our limited knowledge of how to quantify the spatiotemporally transient dynamics of the COVID-19 outbreak that can be affected by many factors in an interconnected, highly evolving system whose details, however, cannot be mapped exhaustively in a numerical model. For instance, Abbasi et al. [1] identified signatures of the fractal structure on COVID-19 data and developed a new SIR-like model to characterize the spread of the pandemic. This study proposes to use parsimonious, fractional calculus-based epidemic models for modeling the transmission, infection, recovery, and death dynamics of the COVID-19 pandemic in the U.S. A fractional calculus modeling approach based on fractional derivative models (FDMs) is selected here because it can efficiently capture complex dynamics (e.g., fractal structure characterization) in heterogeneous systems without the prohibitive computational burden to map the sub-system heterogeneity[37]. Almeida [5] explored the existence, uniqueness, and non-negativity of solutions of the fractional SEIR model and performed numerical simulations under different control conditions. Ahmad et al. [2,4] analyzed the existence of the solution of a fractional modified SIR model with different compartments. Biala et al. [10] developed a time-fractional compartmental model that incorporates the contact tracing processes for COVID-19. Nisar et al. [24] discussed a Caputo fractional SIRD mathematical model of the COVID-19 disease. Zhang et al. [38] modeled the temporal evolution of the COVID-19 pandemic in China from 1/23/2020 to 3/22/2020 and evaluated the impact of different mitigation strategies on the disease. This study will extend the model proposed by Zhang et al. [38] to explore COVID-19’s spatiotemporally spreading dynamics in the U.S. Notably, this study will focus on the modeling, applications, and data analysis for the COVID-19 pandemic spreading, which are the major practical concerns at present. Rigorous mathematical analyses (e.g., existence, stability, and/or boundness of model solutions) will be conducted in the next study, using the method reviewed from the literature focusing on mathematical analysis of similar classical and fractional SEIR models (Appendix C).

Different from the previous work reviewed above, this study will evaluate COVID-19 spreading dynamics in both rural and urban areas in the U.S. The rural areas may exhibit a different response to the pandemic from the urban areas, probably due to some labor-intensive activities (i.e., meatpacking plants and farms), specific facilities, medical vulnerability because of the limited public health system and poverty, and/or the limited testing capacity in some rural counties. For example, while the daily death due to COVID-19 in large urban cities such as New York (NY) and New Orleans began to decrease in early May 2020, outbreaks emerged in rural towns (which were unscathed one month ago in April 2020) and kept the nation in a steady march of deaths and infections in May 2020. For example, on 5/05/2020, Cook County in Illinois had >2,000 new cases per day, Dallas County in Texas reported the highest single-day increase (234) in COVID-19 cases on 5/03/2020, and Dakota County in Nebraska was ranked number three in most cases per capita in the country while there were no cases in this county before 4/11/2020. The Midwestern and Southern U.S. were quickly becoming hot spots of the pandemic in June 2020, which may affect (i.e., by delaying) the recovery of the nation, and hence the dynamics of COVID-19 evolution in both rural and urban areas need to be explored.

We select the states of Alabama (AL) and New York as the two representative states in the U.S. for exploring COVID-19 evolution behavior. On one hand, Alabama represents the typical southern state with broad rural areas. The COVID-19 pandemic has spread quickly in both rural and urban counties in Alabama since the first COVID-19 case confirmed in Montgomery County on 3/13/2020. The infected population in Alabama has been increasing rapidly (290 new cases/day in the week of 4/6/2020 ~ 4/11/2020), with the average fatality rate approaching 2.5% (after the first coronavirus death in Jackson County on 3/28/2020) and the rate of unemployment reaching 2.2% in the week of 6/21/2020. The evolution of the disease and its economic impacts are highly uncertain and vary significantly in space and time, which makes it difficult for policymakers and Alabamians to formulate appropriate responses. To address this challenge and promptly guide the next efforts, it is critical to model the COVID-19 spread and assess its potential relation with economy. On the other hand, the New York State was selected because it was the epicenter of COVID-19 in the U.S., had the most COVID-19 related deaths (~36,870 deaths in a total of ~920,000 cases up to 12/26/2020) in all states, and could be compared with the state of Alabama (with the deaths of ~4685 in ~343,000 up to 12/26/2020) for a better understanding of the pandemic spreading dynamics at different states.

Fractional derivative models for COVID-19 spreading dynamics: Methodology development and applications

Here we first simulate the temporal evolution of the COVID-19 pandemic reported in the states of New York and Alabama and their sub-areas (rural vs. urban counties) using a time FDM, and then quantify the in-state spatial spreading of the COVID-19 cases by proposing a space FDM coupled with the time FDM. Finally, we propose a new time FDM to quantify the temporal fluctuation of economy in the two states and their sub-areas that may be significantly affected by the COVID-19 pandemic.

A fractional SEIR model for quantifying the COVID-19 outbreak evolving in time

Model development: Temporal evolution of the COVID-19 pandemic

We propose the following time-fractional susceptible, exposure, infection, and recovery (F-SEIR) model to capture the COVID-19 evolution in time:

\[
\frac{dS_i}{dt} = -r(t)I_i S_i \frac{1}{N_i} - r_i(t)E_i \frac{S_i}{N_i} 
\]

\[
\frac{dE_i}{dt} = r_i(t)S_i \frac{1}{N_i} - p E_i + r_i(t)E_i \frac{S_i}{N_i} 
\]

\[
\frac{dI_i}{dt} = p E_i - \gamma(t) I_i + \omega \sum_{j=1}^{N} I_j 
\]

\[
\frac{dR_i}{dt} = \gamma(t) I_i 
\]

\[
\frac{dD_i}{dt} = \tau(t) I_i 
\]

where \(S, E, I, R, \) and \(D\) are the susceptible, exposed, infected, recovered, and dead populations, respectively (following the notation used in the traditional SEIR models mentioned above); the subscript \(i\) represents the \(i\)-th spatial node (i.e., county) with \(i = 1, 2, \ldots, n\) (sorted by distance from epidemic center) where \(n = 67\) for the Alabama State (containing 67 counties) and \(n = 58\) for the New York State (notably, Kings, Queens, and Richmond counties are included in the NY metropolitan area since they are close to the NY city); the diffusion term \(\omega \sum_{j=1}^{N} I_j\) describes population migration where \(\omega\) denotes the population migration rate between nodes and will be derived in the space FDM (2) (see 2(f)) for the specific value of \(\omega\); \(\tau\) denotes the number of susceptible people with whom the infected people come into contact daily; \(N\) is the sum of all the people in the study area; \(p\) is the rate for the exposed person to be
transformed into the infected one; \( r_1 \) is the number of healthy susceptible people that are contacted by the exposed people daily; \( \gamma \) and \( \gamma_p \) are the recovery and death rates, respectively; the operator \( d^a/dt^a \) or \( d^b/dt^b \) denotes the Caputo fractional derivative with the index \( a \) or \( b \) (\( 0 < a \leq 1 \) and \( 0 < b < 1 \) in this study); and \( N_i \) is the total population in each county and assumed to be constant during the pandemic (notably, the initial susceptible total population \( S \), denoted as \( S_{0i} \), can be much smaller than \( N_i \); hence, we set \( S_{0i} = 0.05 N_i \) for the Alabama State and \( S_{0i} = 0.06 N_i \) for the New York State in this study). It is also noteworthy that in the FDM (1), because \( N \) is constant and \( S_i = I_i + R_i < N_i \), the time rate change of a variable \( dS_i/dt \), \( dI_i/dt \), or \( dR_i/dt \) is not limited to be zero (although \( d^a S_i/dt^a = 0 \)).

Here the death toll \((1e)\) is expressed as a time-fractional evolution to capture the “memory” or delayed impact in COVID-19 related deaths. This is because there is usually an apparent delay between when a person’s case is reported and when their death would be reported if they fall victim to the coronavirus, which can sometimes be a month or more. In other words, the change of the infected cases may not be reflected immediately in the death toll, but rather taking a while (represented by a random waiting time for each case), which cannot be captured by a time-local model (such as the classical death toll equation \( dd/\alpha = \gamma_p I \)) but requires a time-nonlocal model such as \((1e)\). In addition, the growth of the infected population may also exhibit the memory impact (or delayed response to the exposed population), due, for example, to the incubation period of COVID-19 (which can be a random variable varying from 2 to 14 days).

**Model application: Temporal evolution of COVID-19 pandemic in NY and AL**

Applications show that the F-SEIR model (1) can capture the temporal evolution of COVID-19 cases in the states of New York and Alabama (Fig. 1). For comparison, the classical SEIR model (which is a simplified version of model (1) where the two indexes of the time fractional derivative are limited to be 1) is also applied and shown in Fig. 1. The best-fit parameters of model (1) for the two states are listed in Table 1. The New York State was the epicenter of COVID-19 in the U.S. during the initial spike in late March/early April 2020 (represented by the first peak in the current COVID-19 population shown in Fig. 1b), and

Table 1

| Study site | \( \alpha \) [-] | \( r(t) \) [day\(^{-1}\)] | \( r_1(t) \) [day\(^{-1}\)] | \( p \) [day\(^{-1}\)] | \( \gamma \) [day\(^{-1}\)] | \( \gamma_p \) [day\(^{-1}\)] |
|-----------|----------------|----------------|----------------|----------------|----------------|----------------|
| NY State  | Rural 0.75     | 4              | 0.3            | 0.4            | 1/14           | 0.01           | 0.002 0.08 \( \times \) 0.75 |
|           | Urban 0.75     | 4 + \exp(0.1 r - 1) + 0.8 | 0.4 + \exp(0.1 r - 1) | 1.2 + \exp(0.1 r - 1) | 0.01 + \exp(0.1 r - 1) | 0.002 + \exp(0.1 r - 1) |
|           | Total 0.75     | 1 + \exp(0.1 r - 1) | 0.4 + \exp(0.1 r - 1) | 1.2 + \exp(0.1 r - 1) | 0.01 + \exp(0.1 r - 1) | 0.002 + \exp(0.1 r - 1) |
| AL State  | Rural 0.88     | 4              | 0.2            | 0.32           | 1/14           | 0.01           | 0.002 0.08 \( \times \) 0.75 |
|           | Urban 0.88     | 4 + \exp(0.07 r - 1) + 0.95 | 1 + \exp(0.07 r - 1) | 1.05 + \exp(0.07 r - 1) | 0.01 + \exp(0.07 r - 1) | 0.002 + \exp(0.07 r - 1) |
|           | Total 0.88     | 1 + \exp(0.07 r - 1) | 1 + \exp(0.07 r - 1) | 1 + \exp(0.07 r - 1) | 0.01 + \exp(0.07 r - 1) | 0.002 + \exp(0.07 r - 1) |

Fig. 1. The reported COVID-19 cases (symbols) vs. the calculated ones (lines) using the F-SEIR model (1) for the cumulative infected population (a) and the current infected population (b) in the New York State, and the cumulative infected population (c) and the current infected population (d) in the Alabama State. The solid line denotes the best-fit solution using the F-SEIR model (1), and the dashed line denotes the simulated total population using the classical SEIR model with the best-fit parameters. Model parameters are listed in Table 1. In the legend, “Data: total” represents the total number of infections in both urban and rural areas, which can change with time.
then was well controlled (it was likely that lockdown/restrictions such as social distancing gradually brought COVID-19 infections to heel) until September 2020 when the 2nd peak began to form, albeit relatively slower than the first one. The Alabama state had a much smaller peak in early April 2020 (showing that it was less affected by the first hit than New York), followed by the first apparent peak in the end of August 2020 due to the opening of fall semesters of universities. The declining limb of the first peak was then overlapped with the 2nd peak starting in October 2020. The F-SEIR model (1) captures the subtle difference in the temporal evolution of COVID-19 pandemic in the states of New York and Alabama.

To explore the county-level pandemic evolution, we also apply the F-SEIR model (1) to simulate the cumulative and current infected populations in the New York metropolitan area, NY, and Montgomery County, AL. Results show that the county-level pandemic evolution exhibited the same pattern as that of the state level, and the model (1) can capture the general pattern of the reported data, although the reported case numbers significantly fluctuated in time (Fig. 2). This demonstrates that the F-SEIR model (1) is applicable for sites with different spatial scales, which is consistent with the previous conclusion that the time FDM is feasible for various scales[37].

**A space FDM for modeling COVID-19 outbreak spreading in space**

**Model development: Spatial spreading of the COVID-19 pandemic**

The travel distance of people may affect the spread of infectious diseases. Several researchers have developed epidemic models with diffusion. For example, Samsuzzoha et al. [31] proposed a vaccinated diffusive compartmental epidemic model and considered a no-flux boundary \( \frac{\partial I}{\partial x} = 0 \). Ahmed et al. [3] proved the positivity of the SEIR reaction diffusion epidemic model and then developed a nonstandard finite difference scheme. Asif et al. [7–8] proposed a diffusive SEIR model for COVID-19 pandemic spreading and employed the meshless method to approximate the model. Furthermore, the SEIR models with the diffusion term were also applied by Wu et al. (2020), Zhang et al. [39], and Mammeri [21].

In order to determine the \( \omega \) in Eq (1c), we propose the following space FDM with initial/boundary conditions to model the spatial spreading of the COVID-19 pandemic in the state:

\[
\begin{align}
\frac{dI}{dt} &= D \frac{d^{\alpha-1}I}{dx^{\alpha-1}} \\
\left. \frac{d^{\alpha-1}I}{dx^{\alpha-1}} \right|_{x=\lambda} &= 0 \\
\left. \frac{d^{\alpha-1}I}{dx^{\alpha-1}} \right|_{x=L} &= 0 \\
I(x, t = 0) &= I_0(x) \\
\end{align}
\]

where \( I \) denotes the spatial distribution of the population infected by COVID-19, \( D \) is the diffusion coefficient, \( \gamma (1 < \gamma < 2) \) is the order of the space fractional derivative, \( \lambda \) is the spatial truncation parameter, \( L \) denotes the longest distance from the pandemic center, and \( I_0(x) \) denotes the initial condition. The no-flux boundary conditions defined by (2b) and (2c) keep the mass balance in the bounded domain. FDM (2) employs the tempered Riemann-Liouville (R-L) definition for the space fractional derivative, which was introduced in detail in Appendix A.2.

According to study by Barbosa et al. [9], the random travel lengths of people exhibit the pattern of the truncated Lévy flight, which is captured by the tempered fractional derivative in Eq. (2a). Here we apply the finite difference method to approximate model (2). For example, the explicit finite difference scheme for Eq. (2a) takes the form

\[
I_{i}^{t+1} = I_{i}^{t} + \Delta tDh^{-\alpha} \sum_{k=0}^{h^{-1}} b_{k}I_{i-k}^{t}
\]
\[ b_k = \begin{cases} 
\frac{g_{i-k+1} e^{-\lambda h(i-k)}}{g_i e^{-\lambda h(i-k+1)}} & 0 < k \leq i + 1, k \neq i, i + 1 < i < n - 1 \\
\frac{g_i e^{-\lambda h(i-k+1)}}{g_i e^{-\lambda h(i-k+1)}} & 0 < i = k < n \\
\frac{g_i e^{-\lambda h(i-k+1)}}{g_i e^{-\lambda h(i-k+1)} + \gamma e^{\lambda h(i-k)}} & i = k = 0 \\
\gamma e^{\lambda h(i-k)} - \sum_{i'=k}^{i-1} g_{i'-k} e^{-\lambda h(i-k)} & i = n, 0 \leq k < n - 1 \\
-g_{i-k} e^{\lambda h} & i = k = n \\
0 & k > i + 1 
\end{cases} \]  

(2f)

(value of \( g_{i-k+1} \) is defined by (A4) in Appendix A.1. Defining 
\[ \omega_{k+1} = \Delta t D h^{-1} b_k \]  

(2g)

we obtain the population migration rate in Eq. (1c).

To obtain the spatial propagation of the COVID-19 pandemic in the two states, we propose a coupled dynamic model framework, which contains the following five main steps:

Step 1: Discretize the total period (using day as the units) of the F-SEIR model (1) and the space FDM (2) to \( n \) nodes with a uniform interval (one day). Here we select the total period of 118 days (3/13/2020 ~ 7/8/2020), so that \( n = 118 \).

Step 2: Use the initial condition of the F-SEIR model (1) (which is the
infected population reported for each county in the state of New York or Alabama on 3/13/2020, Day 0) and the best-fit parameters obtained in Section 2.1.2 to calculate the number of COVID-19 infections in each county for the next day (3/14/2020, Day 1).

**Step 3:** Take the result from step 2 as the initial condition and employ the space FDM (2) to distribute the number of infected cases for each county on the next day (which is 3/14/2020, Day 1) affected by inter-county communications. The solution scheme is shown by Eq. (2e).

**Step 4:** Repeat steps 2 and 3 (where Day 1 is replaced by Day 2, 3, …) till the 20th nodes (i.e., 4/02/2020, Day 20). This allows enough time for the county-level quarantine to be effectively implemented (notably, here the 20th day was our fitting result, which was selected to improve the model reliability).

**Step 5:** Terminate step 4 and continue the step 2 loop until reaching the last day n (i.e., Day 118).

The reason for combining the fractional derivative models (1) and (2) in the framework mentioned above is because the F-SEIR model (1) can accurately simulate the temporal evolution of the total number of state-level COVID-19 cases (see section 2.1.2), and the space FDM (2) can optimize the spatial spreading or distribution of the infected cases in the counties by keeping the total number of cases constant in each state at a given time.

**Model application: Spatial spreading of the COVID-19 pandemic in NY and AL**

Fig. 3 shows the application results of the modeling framework mentioned above in quantifying the spatial spreading of the COVID-19 pandemic in all 64 counties in the New York State at three sampling cycles. The best-fit model parameters are listed in Table 2. The “distance” used in Fig. 3 is the relative distance from the New York City, the COVID-19 pandemic epicenter of the state, to each county, where the relative distance for the nearest county (which is Bronx County) is numbered as “1” and the furthest one (Niagara County) is numbered “64” (Fig. 3a, c, and e). Fig. 3b, d, and f rank the infectious cases, where the leading edge exhibits apparent truncation which can be well captured by the exponential truncation of super-diffusion in the space FDM (2).

The application of model (2) for Alabama is shown in Fig. 4. The

| Study site | α [ – ] | β [ – ] | λ [km⁻¹] | D [km²/day] |
|------------|---------|---------|-----------|-------------|
| New York State | Spreading in the first 20 days | 0.75 | 1.3 | 3.0 | 0.3 |
| Alabama State | Spreading in the first 20 days | 0.88 | 1.1 | 0.8 | 1.1 |

Table 2

The best-fit parameters of the space FDM (2) for the spatial spreading of COVID-19 cases (infected populations) between counties in the states of New York and Alabama. In the legend, α and β denote the time and space indexes, respectively, λ is the truncation parameter in space, and D is the diffusion coefficient. Here the scenario of “Spreading in the first 248 days” assumes a 20-day delay of quarantine (i.e., free spread of COVID-19 cases for 20 days before effective quarantine), while the scenario of “without spreading” represents the strict constrain for inter-county communication immediately after the COVID-19 outbreak (so that D = 0). See section 2.2 for description.

Fig. 4. The space FDM application for Alabama. The same legends as Fig. 3 are used here.
same spatial truncation as the New York State (with a different index $\alpha$) can be observed for counties in the state of Alabama. The model fit of the spatial distribution of COVID-19 cases (Fig. 4a, c, and e) is not as good as that of the New York State, while the “ranked number of infections” (Fig. 4b, d, and f) can be fitted slightly better than those of the New York State. Discrepancy of the pandemic’s spatial spreading in the two states will be further discussed in Section 3.1.2.

A time FDM for modeling economy evolution due to the COVID-19 pandemic outbreak

Model development: Unemployment rate evolution

We propose the following time FDM to quantify the unemployment rate evolution during the COVID-19 pandemic outbreak:

$$\begin{align*}
\partial^\mu S(t, S) &= \partial S \left( K \frac{\partial U(S, t)}{\partial S} \right) \\
U(S, t) &= q N \delta(S)
\end{align*}$$

where $U$ represents the unemployment rate, $\mu$ is the index of the time fractional derivative which captures the memory effect of unemployment over time, $N$ denotes the total population in the study site, $q$ is a coefficient describing the impact of the pandemic on the unemployment rate, $S$ is a scale characterizing the delayed response of unemployment to the pandemic (because unemployment does not necessarily start immediately after the pandemic, but may exhibit a potentially delayed period), $\partial U/\partial S$ represents the gradient of transform between unemployment and employment, and $K$ denotes the transform rate between unemployment and employment.

The time FDM (3) is a parsimonious model that can capture a wide range of patterns for the unemployment rate to evolve in time. Particularly, the index $\mu$ controls the late-time tailing behavior of the unemployment rate curve. The transform rate $K$ defines the propagation speed of the unemployment rate, and the impact coefficient $q$ affects the peak of the unemployment rate. We test the applicability of the new model (3) in the next two sub-sections with the unemployment data affected by the pandemic.

Application: Unemployment rate evolution in the New York State

The reported data show that the New York State’s unemployment rates may be significantly affected by the COVID-19 pandemic (Fig. 5a). The weekly unemployment claims were increased by 3%~16% (with an average increase of 9.1%) from 3/21/2020 to 6/27/2020 in the New York State. The unemployment rate was 15.9% in NY in July 2020, while this number was only 3.9% one year ago (in July 2019). The four sectors with the most job losses were: leisure and hospitality (~428,000 more job losses) compared to that in July 2019, especially the jobs in accommodation, food services, and drinking places), trade, transportation and utilities (~209,300 job losses), professional and business services (~180,100 job losses), and government (including public education and public health services) (~155,600 job losses), which can be more sensitive to the outbreak of the COVID-19 pandemic than the other types of jobs and confirm the impact of the COVID-19 pandemic on the economic decline in the state. In addition, the weekly unemployment insurance claim curve in the New York State exhibited two declines (Fig. 5a). The reason for the 1st decline of the unemployment rate in the week of 4/18/2020 is not clear, but it was consistent with the turnover of the COVID-19 pandemic in the state. On 4/18/2020, the New York State was the epicenter of the COVID-19 outbreak in the U.S. (with > 236,700 COVID-19 cases, accounting for 33% of all cases in the U.S.). However, New York continued to see a decline in the number of hospitalizations and daily deaths from COVID-19 in three days since 4/16/2020, implying that the state was past the plateau and was starting to descend. The 2nd drop of the unemployment rate in late May 2020 might be due to the reopening of businesses (starting with construction, manufacturing, retail, agriculture, forestry, and fishing) on 5/15/2020.

The time FDM (3) can quantify the weekly unemployment insurance claims of the state of New York (starting from 3/21/2020): the total number (blue line), the adjusted number due to COVID-19 (yellow line, which is the total number minus the base number at the same week in 2019), and the base case without the impact of COVID-19 (grey line, representing the base number at the same week in 2019). Each number represents the number of the unemployment insurance claims in the week, the modeling results of the unemployment insurance claims of the New York State using the time FDM (3): the total region (black line), the rural area gross (red line), and the urban area gross (blue line). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

| Study site         | $\mu$ [-] | $q$ [-] | $K$ [-] | $S$ [-] |
|--------------------|-----------|---------|---------|---------|
| New York State     | Rural     | 0.92    | 5.80 x 10^6 | 0.2168  | 5.0     |
|                    | Urban     | 0.98    | 0.2571  | 1.00    |
|                    | Total     | 0.99    | 0.2499  | 1.00    |
| Alabama State      | Rural     | 0.94    | 1.90 x 10^5 | 0.1989  | 5.0     |
|                    | Urban     | 0.99    | 0.2292  | 1.00    |
|                    | Total     | 0.98    | 0.2165  | 1.00    |

Table 3

The best-fit parameters of the time FDM (3) for the unemployment insurance claims in the states of New York State and Alabama. In the legend, $\mu$ is the index of the time fractional derivative, $q$ is the impact coefficient (describing the impact of pandemic on the unemployment rate), $K$ is the transform rate between unemployment and employment, and $S$ is a scale characterizing the delayed response of unemployment to the COVID-19 pandemic.
The seasonally adjusted unemployment rate was 7.5% in the professional and business services sector, the education and health service sector, and the manufacturing sector (the top 4 sectors), the hospitality sector (the same most infected sector as the New York State), persons in different sectors (26%, 9%, 8%, and 7% from the leisure and hospitality sector, the professional and business services and the manufacturing sectors. This trend was found between the two states, implying that unemployment can rise faster in the more developed areas, probably because the urban areas had more industries and workforces sensitive to coronavirus. It is also noteworthy that the rural areas in Alabama contributed a much larger portion of unemployment claims (see Fig. 7) in the total unemployment claims than that in the state of New York (Fig. 5). The same behavior was found for the cumulative/current infectious cases in the two states shown in Section 2.1.2. This discrepancy may imply that the COVID-19 pandemic can hinder more the recovery of Alabama economy recovery than that of New York.

Discussion

COVID-19 spreading dynamics in time and space

The New York State’s temporal evolution of the COVID-19 pandemic, including the peak population and the skewed shape, were dominated by the pandemic evolution in the state’s urban areas, while the infectious cases in the rural areas remained relatively stable and was somewhat isolated (Fig. 1a and b). This discrepancy may be because the urban areas in NY, consisting of 15 counties, have ~ 76% of the total population. Fig. 2a and b demonstrate that New York County, the main urban area in the state, contained half of the infected population and controlled the temporal spread pattern of the COVID-19 pandemic in the state. This also implies limited communications between the urban and rural areas in the New York State.

In the Alabama State, the urban areas had a similar trend as the rural areas in the current infected population fluctuation since 3/13/2020 (note that ~ 56% of the total population is in the 12 urban counties in Alabama) (Fig. 1c and d), implying stronger rural–urban communications than the New York State. Fig. 2c and d show that Montgomery County, the capital and one of the main urban counties of AL, contributed < 1/10 of Alabama’s infected population and exhibited apparent noise and multiple local peaks.

Although the rural and urban areas tended to play different roles in the COVID-19 pandemic evolution in the two states, the F-SEIR model (1) captures the COVID-19 dynamics evolving temporally in the two states and their rural/urban areas. The fitted rates, including the contact, exposure, recovery, and death rates, decline with time and follow the same functional forms (which will be plotted in Section 3.1.3) for the two states and sub-areas (Table 1). Hence, the COVID-19 pandemic evolution from 3/28/2020 to 11/15/2020 in the two states and their rural/urban areas, although exhibiting different magnitudes and internal contributions, followed the same physical law described by the F-SEIR model (1).

The best-fit parameters of the F-SEIR model (1) also revealed three subtle discrepancies in the pandemic’s temporal evolution in the two states. First, the best-fit time index \( \alpha \) for the New York State is smaller than that of Alabama (Table 1), indicating that the reported COVID-19
cases in the New York State might be more delayed than that for Alabama, which was mentioned by CNN (https://edition.cnn.com/2020/04/01/politics/testing-backlog-coronavirus-quest-invs/index.html).

This implies that some infected people might not be timely reported in NY, causing uncertainty in the best-fit rates \( r \) and \( r_1 \) in model (1). Second, in Alabama, the best-fit contact rate \( r \) and exposure rate \( r_1 \) for the rural areas are larger than those for the urban areas, while the opposite is true for the New York State (Table 1 and Fig. 8). This discrepancy might be because Alabama has more rural-based counties and tenser communications between agricultural counties than those of New York. Third, on the state level, the best-fit contact rate \( r \) and exposure rate \( r_1 \) (Table 1 and Fig. 8) for New York are generally smaller than those for Alabama although the initial rates of NY are larger than those of AL, which is expected considering the stricter isolation and quarantine policy in New York. However, the values of \( r \) and \( r_1 \) in our models need further validation in a future study, considering the facts that (1) the time index \( \alpha \) differs in the two states, and (2) the infectious cases in Alabama had not reached the peak (up to 11/15/2020, the last day of the dataset used by this study).

Fig. 9 shows the best-fit spatial distribution of the infected cases in rural and urban counties in (a) the New York State and (b) the Alabama State on 7/08/2020. In the two states, model (2) generally fits the COVID-19 spreading in the urban counties better than the rural counties. The reason that the simulation results in the developed areas are better may be that there are more population and/or larger sample sizes in the urban areas, limiting randomness and outliers in data analysis.

**COVID-19 spread dynamics in space**

The space FDM (2) predictions of the spatial spreading of the COVID-19 pandemic (shown by the solid line in Fig. 2) suggest that there might be a relatively lower exchange or transportation of the COVID-19 infected population between counties in the state of New York than that in Alabama. Table 2 shows that the best-fit time index \( \alpha \) of model (2) for the New York State is lower than that of Alabama, while the best-fit space index \( \beta \) and the truncation coefficient \( \lambda \) of model (2) for the New York State are higher than those for Alabama, implying that the relative proportion of people using long-distance travel in the New York State was lower than that in Alabama. The relatively low inter-county mobility in the New York State, which was captured by the three model parameters (\( \alpha \), \( \beta \), and \( \lambda \)), implied that residences in the New York State may follow stringent regulations of the travel advisory and/or use less inter-county public transportation (public transportation in New York is made up of some of the most extensive infrastructures in the U.S., including canals, urban mass transit, commuter railroads, intercity and international rail, and intercity bus, which can be managed under strict government policies) during the pandemic outbreak, which can limit the in-state spread of the COVID-19 pandemic.

In addition, the best-fit diffusion coefficient \( D \) in model (2) for the New York State is lower than that for Alabama (Table 2), also implying a relatively less fraction in county-level communications in New York than in Alabama. Notably, our assumption of one COVID-19 pandemic epicenter for Alabama (Mobile County) led to a larger \( D \) than that with multiple pandemic epicenters in the state, because the possible spatial spreading of COVID-19 cases from other major urban areas such as Montgomery and Jefferson Counties was attributed to the single epicenter (hence exaggerating the diffusion coefficient \( D \) for the latter). A future study will focus on the spatial spreading of the COVID-19 pandemic from multiple sources. The estimation of more inter-county communications in Alabama (than New York), which is consistent with the conclusion in Section 3.1.1 using the temporal evolution of the COVID-19 cases, may also be explained by the following two facts: 1) Alabama is known as a “drive” market where out-of-county or -state travelers often arrive via automobile, using, for example, interstate I-20 which is easily accessible by car or truck (which, however, can be more difficult to control or manage than public transportation tools during the pandemic); and 2) the relative fraction of the rural population in the total population for Alabama (44%) is apparently larger than that of the New York State (24%), enhancing communications between rural and
urban counties in Alabama. Notably, the best-fit $D$ is the same for the urban and rural areas in each state. The same $D$ for the rural and urban areas is likely due to the following three reasons: (1) although the metropolis tends to promote more communication than the urban areas, the strict quarantine policy in the urban area (i.e., face masks, social distancing, and COVID-19 event and gathering policies) can counterbalance such contact; 2) in NY, the urban and rural areas are affected by the single epicenter; and 3) in AL, the urban and rural population ratio is close to 1:1.

It is noteworthy that, on one hand, the spatial spreading of the COVID-19 infections from a single pandemic epicenter is apparent in the New York State, as implied by the fitting result using model (2) (Fig. 3). New York County, the main pandemic epicenter of the state of New York, was surrounded by counties with a relatively high rate of infected populations, while the counties with less infections were usually distributed far away from this epicenter. New York County is the economic, social, and transportation center of the state that intricately connects the other counties, and therefore, it has a significant impact on the other counties. On the other hand, the impact of Alabama’s COVID-19 epicenter, such as Mobile County, on the other counties in the same state is much less apparent (than that in the New York State). In Alabama, counties surrounding Mobile County, the early epicenter of the COVID-19 pandemic in Alabama, had low infection rates, implying that the epicenter at the early stage may have less impact on the surrounding counties (than New York County discussed above). This might be because Mobile is a coastal town filled with popular tourist attractions, where the main COVID-19 cases may be initially due to out-of-state travelers. After the beaches along Alabama’s gulf coast were closed on 3/19/2020, Alabama’s COVID-19 pandemic epicenter gradually shifted from Mobile County to Jefferson and Montgomery Counties, possibly because the outbreak reduced Mobile’s tourism industry and the population exchange rate in the county. Jefferson, who owns an airport, and Montgomery, the capital of Alabama, had then the more frequent county-level population communication. Therefore, the space FDM (2) can quantify the spatial evolution of the COVID-19 outbreak in the state with a single pandemic epicenter better than the state with random infection distributions and multiple or shifting pandemic epicenters. In addition, the combination of Figs. 3, 4 and 9 revealed that the spatial distribution of the pandemic cases followed a power-law function in the urban counties and the transition from power-law to exponential functions in the rural counties, which may simply be due to the expectation that a few rural areas can stand at the lower end of the pandemic outbreak.

### Time-dependent behavior of the model parameters

Compared to the constant parameters in the time FDM model fitted for COVID-19 evolution in China which adopted a nationwide quarantine policy for the COVID-19 pandemic (e.g., $r$ and $r_1$, see Zhang et al. [38] for details), the exposure/contact rates fitted by the reported COVID-19 cases for the states of New York and Alabama declined in time, following the trend of sigmoid functions (Table 1). According to the inter-county tourist flow chart built by Sun et al. [34], the U.S. population of inter-county communication began to dramatically decline since 3/13/2020, until converging to its minimum by the end of March 2020. Their finding supports our assumption of the time-dependent contact coefficient depicted in Fig. 8.

To catch the 2nd outbreak of the COVID-19 pandemic (starting from September 2020), we calibrated the model parameters (listed in Table 4) by trial and error. It is difficult to explain precisely the cause for the 2nd outbreak, which caused the time-dependent model parameters, although the return to school for U.S. students and the season change might be the contributing factors.

### Impact of COVID-19 spread dynamics on state economy

#### New York State vs. Alabama State

The best-fit impact coefficient $q$ in the new time FDM (3) for the New York state is smaller than that of Alabama, showing a lower unemployment proportion in the total population of New York. According to the statistical analysis by Futurelearn [16], the New York City was ranked 17th in the world’s 100 major cities in its efficiency in managing the post-COVID economy, due especially to its excellence in government policies (i.e., government effectiveness, and immigration rates and openness) and gender equality. These active policies and factors may limit the overall impact of the COVID-19 pandemic on the state economy. In addition, the relatively high disposable income, high cost of living for New Yorkers, and the urbanization may motivate people to search for jobs in a short time, accelerating the recovery from unemployment.

The best-fit transform rate between unemployment and employment ($K$) of the New York State, however, was larger than that of Alabama. The larger $K$ in the New York State implied a faster increase in the unemployment rate, or a faster spreading of job losses among major industry sectors in the state of New York than the state of Alabama. This result, however, is no surprise, since New York’s economy is more exposed to coronavirus (ranked the 5th most exposed state in the U.S. by WalletHub [35] than Alabama (ranked the 26th most exposed state) because of its highly affected industries and workforce. The faster increase of the unemployment population, however, did not overturn the capability of the New York City in managing the post-COVID economy; indeed, the reported data show that it took less time (~0.5 month less) for the New York State’s unemployment claims to drop ~ 50% from the peak than the Alabama State (Figs. 5 and 7). Therefore, on one hand, the higher exposure to COVID-19 made the New York State’s unemployment spread faster; on the other hand, the New York State’s excellent government policies, gender equality, larger urbanization areas, and resources for businesses to cope better with the crisis helped the New York State economy to better recover from the COVID-19 pandemic than Alabama.

Compared with the New York State, the industries and workforce in Alabama were much less affected by coronavirus (ranked 6th in the least affected in all states in the U.S.), but the relatively poor resources for businesses to cope with the crisis and the relatively larger rural areas and population percentage made Alabama suffer more from the crisis. Therefore, the COVID-19 pandemic might cause more delay in economic recovery in Alabama than the state of New York.

### Table 4

| Study site | $t_2$ [date] | $a_2$ [-] | $S_0$ [-] | $r(t)$ [-] |
|------------|--------------|-----------|-----------|-----------|
| New York State | Rural | 9/29/2020 | 1 | 1.039$^{+0.02}$ | 1.003$^{-0.02}$ |
|             | Urban | 8/30/2020 | 1 | 1.039$^{+0.02}$ | 1.015$^{-0.02}$ |
|             | Total | 9/19/2020 | 1 | 1.039$^{+0.02}$ | 1.006$^{-0.02}$ |
|             | New York Metro | 9/19/2020 | 1 | 1.039$^{+0.02}$ | 1.006$^{-0.02}$ |
| Alabama State | Rural | 8/30/2020 | 1 | 1.025$^{+0.01}$ | 1 |
|             | Urban | 8/30/2020 | 0.98 | 1.022$^{+0.01}$ | 1 |
|             | Total | 8/30/2020 | 0.96 | 1.034$^{+0.01}$ | 1 |
|             | Montgomery | 7/21/2020 | 1 | 1.067$^{+0.01}$ | 1.046$^{-0.01}$ |
Heavy tailed process: Infectious vs. Unemployment

The F-SEIR model (1) is an effective model for describing the evolution of infectious diseases, since the evolution of infected cases tended to be delayed for various reasons (such as the ban of travel, the adjustment of government policy, and the improvement of public health capacity). For the unemployment rate evolution, the new time FDM (3) however cannot apparently improve the quantification of the standard model. The unemployment rate exhibited a rapid decline after reaching its peak (Figs. 5 and 7), probably due to the government employment incentive policy; e.g., the state government released a plan to restart the U.S. on 4/16/20, including three phases to return to work to ease the recession which may result in the unemployment rate falling much faster than the daily infected population. Meanwhile, returning to work for areas with a high infection rate may increase the probability of infection, delay the pandemic mitigation, and strengthen the heavy tailed phenomenon of the infectious evolution.

Conclusion

This study applied the spatiotemporally fractional derivative models to quantify the spatiotemporal spreading of the COVID-19 pandemic and the temporal evolution of unemployment claims affected by coronavirus in the states of Alabama and New York, as well as their rural and urban areas. Model applications revealed the following three conclusions.

First, applications of the F-SEIR model (1) showed that the temporal evolution of the COVID-19 pandemic followed the same physical law described by the time FDM, including the memory in time and the temporally declining rates, in the states of Alabama and New York, two states quite different in population compositions, urbanization, and industry structures, while subtle discrepancies in the pandemic evolution in the two states did exist. Alabama might have more inter-county communications in the rural areas than that in the urban areas, and the best-fit contact and exposure rates during the COVID-19 pandemic evolution for the rural areas are also larger than those for the urban areas; while the opposite is true for the New York State. Hence, the pandemic evolution in the state of New York’s urban counties (such as New York County) dominated the state’s temporal evolution of the pandemic, including the peak population and the skewed shape. In Alabama, both the rural and urban counties contributed to the state’s pandemic evolution characteristics.

Second, applications of the space FDM (2) showed that the COVID-19 outbreak’s spatial spreading in rural and urban areas follow the tempered stable density in space, whose detailed properties may be affected by the number of pandemic epicenters in the state. The space FDM (2) fitted better the infected population in the urban areas than those in the rural areas (likely due to their larger population and sample size), and it also worked better for the state with a single epicenter (i.e., the New York State) than that with multiple epicenters or a shifting epicenter (i.e., Alabama).

Third, applications of the new time FDM (3) showed that the evolution of economy represented by the weekly unemployment insurance claims in the two states differed because of their different capabilities in coping with the COVID-19 pandemic. On one hand, the higher exposure to the COVID-19 pandemic made the New York State unemployment spread faster. On the other hand, the excellent government policies, gender equality, larger urbanization areas, and resources for businesses to cope better with the crisis helped the New York State to probably recover relatively faster than Alabama.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Definitions of the fractional derivatives

This appendix introduces the definition for the fractional derivatives used in the main text. There are three types of commonly used fractional derivatives, which are the Grünwald-Letnikov (G-L), Riemann-Liouville (R-L), and Caputo fractional derivatives.

A.1. Grünwald-Letnikov definition

The G-L definition has a clear relation between the integer and fractional derivatives and plays an important role in numerical simulations of the FDMs. Considering the classical n-th order integer derivative

\[ f^{(n)}(x) = \lim_{h \to 0} \sum_{i=0}^{\infty} h^{-n} \left( -1 \right)^i \binom{n}{i} f(x - ih) \]  

(A1)

where

\[ \binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i!} \]  

(A2)

is the binomial coefficient where \( i \) is an integer, and \( h \) denotes the grid space interval.

We then rewrite Eq. (A1) as

\[ \Delta^{\alpha} f(x) = \sum_{i=0}^{\infty} \left( -1 \right)^i \binom{\alpha}{i} f(x - ih) \]  

(A3)

where

\[ \binom{\alpha}{i} = \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \]  

(A4)
We use \( g^\alpha_i = (-1)^i \binom{\alpha}{i} \) to represent the Grünwald weight, where \( \alpha \) represents the fractional index and \( \Gamma(\cdot) \) denotes the Gamma function. The Grünwald weight follows the power law asymptote \( g^\alpha_i \sim -\alpha/\Gamma(1-\alpha)i^{\alpha-1} \) [22], as value of \( g^\alpha_i \) decreasing as a power law function \( i^{\alpha-1} \). This property may be valid for COVID-19 pandemic spreading since the probability density function (PDF) for persons in this study moving \( i \) steps outside the counties follow a power law distribution. Eq. (A3) is the commonly used finite difference formula for the G-L definition. Now we can obtain the limit form of the fractional derivative with the G-L definition:

\[
\left[\frac{d^\alpha}{dx^\alpha}f(x)\right]_{\xi} = \lim_{h \to 0} \frac{\Delta^\alpha f(x)}{h^\alpha} = \lim_{h \to 0} \frac{1}{(\alpha-1)!} \sum_{i=0}^{n} g^\alpha_i h^\alpha f(x - ih).
\]

Moreover, a shifted G-L definition was proposed by Meerschaert & Tadjeran [23]:

\[
\left[\frac{d^\alpha}{dx^\alpha}f(x)\right]_{\xi} = \lim_{h \to 0} \sum_{i=0}^{n} h^\alpha g_i f(x - (i-j)h)
\]

where \( j \) denotes the shift (such as one). This modified formula is widely used in the numerical solution of FDMs to overcome the problem of numerical instability.

### A.2. Riemann-Liouville (R-L) definition

The R-L definition takes the form

\[
\frac{R_L^\alpha}{\xi} f(x) = \frac{d^\alpha}{dx^\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^nf}{dx^n} \int_{x}^{\infty} (x-\xi)^{n-\alpha-1} d\xi
\]

where the index \( \alpha > 0 \), \( n \) is the smallest integer greater than \( \alpha \) (i.e., \( n-1 < \alpha < n \)), and \( \Gamma(\cdot) \) denotes the Gamma function. The spatial nonlocal operator (A7) describes people migrating from each node (i.e., county) in the bounded domains denoted as \( 0 < \xi < x \).

Based on (A7), the tempered fractional R-L derivative was proposed [12]:

\[
\frac{\tau_{\lambda}^\alpha}{\xi} f(x) = \frac{d^{\alpha\lambda}}{dx^{\alpha\lambda}} f(x) = \frac{\Gamma(-\alpha)}{\Gamma(n-\alpha)} \frac{d^nf}{dx^n} \int_{x}^{\infty} \xi^{-\alpha} (x-\xi)^{n-\alpha-1} d\xi
\]

where \( \lambda \) denotes the truncation parameter which transits the PDF of the travel distance from a power-law function to an exponential function. Note that the definition of (A8) satisfies the generalized central limit theory (i.e., the second and higher-order moments are finite) [23].

In this study, FDM (2) employs the tempered R-L fractional derivative defined by (A8), which can describe the long-distance migration of COVID-19 when taking \( 1 < \alpha < 2 \) and \( n = 2 \); meanwhile, the tempering factor in Eq. (A8) removes unrealistic, infinite jumps/travels made by any person.

### A.3. Caputo definition

The R-L definition (A7) plays an important role in the development of pure theory and applications for FDMs. However, there are limitations of the R-L definition: the R-L definition leads to the initial condition containing values of the fractional derivatives, which violates the utilization of physically interpretable initial values [28]. The following Caputo definition was proposed by Caputo [13], which makes an excellent tradeoff between practical needs and the well-established theory

\[
\frac{C^\alpha}{\xi} f(x) = \frac{d^\alpha}{dx^\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{x}^{\infty} \frac{d^nf}{d\xi^n} (x-\xi)^{n-\alpha-1} d\xi
\]

where \( 0 < \alpha < 1 \) and \( \eta = 1 \) in this study. Notable, by taking the derivative outside the integral, we can get the R-L definition. The Caputo fractional derivative was widely used in the temporal differential equations since it captures the real-world interpretable initial condition. The following formula shows the Laplace transform of R-L and Caputo type of fractional derivatives:

\[
L\left[\frac{R_L^\alpha}{\xi} f(t)\right] = s^{-\alpha} \hat{f}(s) - \sum_{i=0}^{n-1} \frac{R_L^{\alpha,i-1}}{\xi} f(t)\big|_{t=0^+}, (n-1 < \alpha < n)\]

\[
L\left[\frac{C^\alpha}{\xi} f(t)\right] = s^{-\alpha} \hat{f}(s) - \sum_{i=0}^{n-1} f(0)s^{\alpha-i}, (n-1 < \alpha < n)
\]

where \( s \) denotes variable of Laplace transform, and \( \hat{f}(s) \) represents the image function of \( f(t) \) in the Laplace transform space. For the Caputo definition, we obtain the image function associated with the interpretable initial condition (i.e., the initial value with an integer order derivative), while we cannot find the suitable initial condition in real world for the R-L fractional derivative.

In this study, the Caputo definition was employed in model (1) which has a constant initial value and characterizes the memory effect (of COVID-19) underlying the convolution of the fractional derivatives.
Appendix B. FDM parameter sensitivity analysis and property description

To explore the spatiotemporal dynamics of COVID-19 spread described by the FDM, here we conduct further numerical experiments and analyze parameter sensitivity. The following FDM is considered:

\[
\frac{dS_i}{dt} = -0.1 I_i \frac{S_i}{N_i} - 0.1 E_i \frac{S_i}{N_i} \tag{B1}
\]

\[
\frac{dE_i}{dt} = 0.1 I_i \frac{S_i}{N_i} + \frac{1}{14} E_i + 0.1 E_i \frac{S_i}{N_i} \tag{B2}
\]

\[
\frac{d^\alpha I_i}{dt^\alpha} = \frac{1}{14} E_i - 0.02 I_i + \omega_0 \sum_{j=0}^{s^2} I_j \tag{B3}
\]

\[
\frac{dR_i}{dt} = 0.02 I_i \tag{B4}
\]

\[
\frac{d^\beta D_i}{dt^\beta} = 0.001 r^\gamma - a I_i \tag{B5}
\]

where \(\omega_0\) follows the form of Eq. (2g) (i.e., a spatial FDM), which includes diffusion coefficient \(D\), time step \(t = 1\), grid space \(h = 1\) and the space fractional order \(\gamma\). Here we assume \(x = 1 \sim 67\) and \(t = 1 \sim 117\) day. The parameters \(D\), \(a\), \(\beta\) and \(\gamma\) can have a wide range of variations (i.e., they are not constrained by the values fitted in the main text). Note that the model solution at each node needs to be integrated to obtain the number of infections over the whole area. Next, we illustrate the physical properties of the FDM (B1)~(B5) using numerical examples.

**B.1. Memory effect in pandemic evolution captured by the temporal fractional derivative**

The first experiment explores the memory effect of the index \(\alpha\) on the temporal and spatial spreading of the COVID-19 pandemic. We set the initial condition as \(I^0 = 10 \exp(-0.1x)\), \(E^0 = 100 \exp(-0.1x)\), \(R^0 = 0\), \(D^0 = 0\), \(N = 10,000 \times 67\) (representing 67 nodes in total, each node/county containing 10,000 people), \(S^0 = 500\), and the migration rate \(\omega = 0\). Model results show that the time fractional order \(\alpha\) is negatively correlated with the growth rate of the number of infections: with the decrease of \(\alpha\), the growth rate of the infected cases is slower and approached to its asymptote earlier (Fig. B1(a)). For example, the infected cases calculated by the FDM with \(\alpha = 0.4\) exhibit a curve flattened quickly during the first 10 days and then reach the number of 400 at the last simulation day (the 117th day), while the infected-case curve for the classical SEIR model has a quick change of slopes during the first 20 ~ 40 days and approaches 1,400 cases at the 117th day. The index \(\alpha\) may capture the influence of public healthcare facilities and pandemic-related travel restriction policies on COVID-19 spreading. In addition, the index \(\alpha\) can characterize the distribution of the infected populations over the whole study area. Fig. B1(b) shows that the nodes/counties with high numbers of COVID-19 cases (i.e., locations 1 ~ 10) are more affected by \(\alpha\) than the other places, which may because the pandemic mitigation policy and medical care have a stronger impact in alleviating pandemic spreading in COVID-19 hot spots.

**B.2. Nonlocality in pandemic spreading captured by the spatial fractional derivative**

The second experiment explores the impact of “nonlocality” of the spatial fractional-derivative order on epidemic spreading. Here the concept of “nonlocality” can be interpreted as the probability of long-distance transport (of pandemic). Three model parameters controlling this “nonlocality”, including the spreading time, the spatial fractional order \(\gamma\), and the truncation parameter \(\lambda\), are selected for sensitivity analysis. To distinguish the distribution of COVID-19 cases with and without spreading, we set a “point source” of infections by defining \(F^0 = 1\) at the first node and \(F^0 = 0\) at the other nodes (and \(E^0 = 10^7\)). The other initial conditions are the same as those used in Appendix B.1. The results illustrate that the lower index \(\gamma\) leads to a heavier leading tail of the spatial distribution of the infected cases (Fig. B2(a)), implying that a
smaller $\gamma$ captures a higher probability of long-range spreading of the pandemic. When the truncation parameter $\lambda$ increases, the heavy-tailed pandemic distribution is tempered Fig. B2(b), representing the decrease of extremely long-distance spreading from the pandemic center.

B.3. Effect of the spreading time and diffusivity

The third example evaluates the influence of the spreading time and diffusivity (i.e., the diffusion coefficient $D$) on the spatial distribution of the infectious cases. If the infected people are only located at the pandemic center initially, the other places are not infected without people migration (i.e., when $D = 0$, see the curve “Without spreading” shown in Fig. B2(c)), meaning that the human mobility plays an important role in the distribution/spreading of the infectious cases. Furthermore, considering the pandemic mitigation policy that constrain travel, the pandemic free-spreading time is limited. The numerical results reveal that the 20-day spreading scenario exhibits a heavier leading-tail in the infected-case curve than that of a 10-day spreading scenario. This discrepancy is expected, since a longer spreading time would transfer more infected people from the pandemic center to its surrounding nodes (Fig. B2(c)). In addition, the variation of the diffusion coefficient $D$ does not significantly change the pattern of the spatial distribution of infected cases, since the other two parameters ($\gamma$ and $\lambda$) dominate the non-local travel process.

Appendix C. Mathematical analysis of the fractional SEIR model: Literature review

This section reviews mathematical analysis for the classical and fractional SEIR models conducted by previous studies. Various studies had been focused on the mathematical solution of the SEIR models. For example, Annas et al. [6] analyzed the global stability of the classical SEIR model for COVID-19 spread in Indonesia (under the free-disease and endemic equilibriums). Nuugulu et al. [25] analyzed the existence of uniqueness of solution, positivity of solution, equilibrium points, reproductive number, and stability of the fractional SEIR model for COVID-19 in Namibia (see their Eq. (9.1), which is similar to our model (1) except for (1e)), by referring to the work of Lin [19] who theoretically investigated the existence, uniqueness and boundedness of the non-negative solution of the fractional SEIR model (see the model (1) proposed by Lin [19], which is a generalized version of our fractional SEIR model (1)). In addition, various generalized fractional SEIR models were analyzed mathematically by researchers. For instance, Demirci and Unal [14] introduced a fractional SEIR model with a density dependent death rate (see their Eq. (2.1)) and proved the existence of non-negative solutions and stability of the solutions. Ahmad et al. [2,4] proved the boundedness of the fractional SEIR model solution (see their model (25)), as well as its non-negativity and stability. Rezapour et al. [30] analyzed the fractional SEIR model with different fractional-derivative definitions (see their model (1), including the Caputo–Fabrizio derivative and the Mittag-Leffler kernel definition. Moreover, Lu et al. [20] employed a general fractional SEIHDR model with inter-city network spreading and conducted the nonnegativity, boundedness and stability analysis.

In our study, the nonlocal spatial diffusive process was also considered. The diffusive model (i.e., Eq. (2)) could be regarded as a discrete version of the system of nonlinear fractional diffusion–reaction equations, which generally take the form according to Baeumer et al. [11]:

$$\frac{d^\alpha u(x,t)}{dt^\alpha} = D \frac{d^\gamma u(x,t)}{dx^\gamma} + f(u(x,t))$$

(C1)

where $\gamma$ denotes the order of the space fractional derivative; and $u(x,t)$, $\alpha$, $D$, and $f(u(x,t))$ represent the one-dimension vectors:

$$u(x,t) = (S(x,t), E(x,t), I(x,t), R(x,t))^T,$$

(C2)
\[ \alpha = (1, 1, \alpha_1), \]
\[ D = (0, 0, D, 0) \]
\[ f(u, (i) = (f_1, f_2, f_3, f_4) \]

in which \( D \) denotes the spatial diffusivity coefficient between the counties, \( \alpha \) denotes the temporal fractional order; and \( f_i \) \((i = 1, 2, 3, 4)\) take the form of the right-hand side of Eq. (1a-1d), respectively, representing each of the reaction terms.

Thus, the fractional SEIR model (1) proposed in our study shares the generality of the differential equation system constructed by Baeumer et al. [11] and Lu et al. [20]. The existence and uniqueness of the global mild solution could be proved by Theorem 2.1 in Baeumer et al. [11], and the boundedness and nonnegativity could be obtained by applying the Theorem 3.1 in Lu et al. [20]. We leave this prove for the next study, since this current study focused on the model application and COVID-19 dynamics.

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