Chordal varieties of Veronese varieties and catalecticant matrices

Vassil Kanev

Abstract

It is proved that the chordal variety of the Veronese variety \( v_d(P^{n-1}) \) is projectively normal, arithmetically Cohen-Macaulay and its homogeneous ideal is generated by the \( 3 \times 3 \) minors of two catalecticant matrices. These results are generalized to the catalecticant varieties \( \text{Gor}_\leq(T) \) with \( t_1 = 2 \).

Introduction

The \( r \)-secant variety to the Segre variety \( \sigma(P^{m-1} \times P^{n-1}) \) is the projectivization of the determinantal variety \( M_r(m,n) \) of matrices of rank \( \leq r \). Its homogeneous ideal is generated by the \( (r+1) \times (r+1) \) minors of the generic \( m \times n \) matrix. Similarly the homogeneous ideal of the \( r \)-secant variety to the Veronese variety \( v_2(P^{n-1}) \) is generated by the \( (r+1) \times (r+1) \) minors of a generic symmetric \( n \times n \) matrix. These two statements are equivalent to the Second Fundamental Theorem of invariant theory for the groups \( GL_n \) and \( O(n) \) (see [Weyl, DP, VP]). In both cases the secant varieties are projectively normal and arithmetically Cohen-Macaulay (ACM) (see [BW, Ku]).

The \( r \)-secant varieties \( \text{Sec}_r(v_d(P^{n-1})) \) to the higher Veronese varieties are related to an old problem of the theory of invariants: representing a homogeneous form of degree \( d \) in \( n \) variables as a sum of \( r \) powers of linear forms

\[
\tag{1}
f = L_1^d + \cdots + L_r^d
\]

Finding explicitly polynomial equations that determine set-theoretically \( \text{Sec}_r(v_d(P^{n-1})) \) is equivalent to finding explicit polynomial equations on the coefficients of \( f \) which are necessary and sufficient for \( f \) to be representable in the form (1) or its degeneration. In analogy with the above examples the following questions arise. What are the generators of the homogeneous ideal of \( \text{Sec}_r(v_d(P^{n-1})) \)? Do these varieties satisfy the properties of being projectively normal and ACM?

These questions have affirmative answer in the case of \( r \)-secants to a rational normal curve (see Example 1.2 and references therein). Little is known about these problems when \( n \geq 3 \). We answer to them affirmatively in Theorem 3.3 for the chordal variety \( \text{Sec}_2(v_d(P^{n-1})) \). We prove, assuming the ground field is algebraically closed of characteristic 0, that the
homogeneous ideal of $\text{Sec}_2(v_d(\mathbb{P}^{n-1}))$ is generated by the $3 \times 3$ minors of the catalecticant matrices $\text{Cat}_F(1, d-1; n)$ and $\text{Cat}_F(2, d-2; n)$ (see Section 1 for definitions); it is projectively normal, ACM and its affine cone has rational singularities\(^1\). In Theorem 3.2 we generalize these results to some varieties $\text{Gor}_{\leq}(T)$ which parameterize forms with prescribed dimensions of the spaces of partial derivatives (see Section 1).

Section 1 contains preliminary material on catalecticant matrices and their determinantal loci. It is mainly an extract from [IK].

Section 2 is devoted to the rank $\leq r$ locus of the generic catalecticant matrix $\text{Cat}_F(1, d-1; n)$, which is equivalently the locus of homogeneous forms of degree $d$ in $n$ variables expressible in $r$ variables after a linear change of the coordinates. The results of this section are due to O. Porras [Po] (see also [FW]). We present here a simplified proof of her theorem and give some corollaries.

Section 3 contains the results about the chordal variety $\text{Sec}_2(v_d(\mathbb{P}^{n-1}))$ that we stated above.

Section 4 contains generalization of the results of Section 3 to the varieties $\text{Gor}_{\leq}(T)$ with $t_1 = 2$.

Our approach in proving these results is applying the method of G. Kempf [Ke1, Ke2] which was later developed and used many times for calculating the ideal and the syzygies of various types of determinantal varieties (see [La, JPW, We, Po, FW]).

We assume the ground field $k$ is algebraically closed of characteristic 0. We mean by variety a separated scheme of finite type over $k$, i.e. not necessarily irreducible. Unless otherwise stated by a point of a scheme we mean a closed point of the scheme.

Acknowledgments. The author is grateful to J. Weyman for bringing attention to the paper of O. Porras [Po] and providing the manuscript [FW], as well as to A. Iarrobino for useful discussions. The hospitality of Northeastern University during the author’s visit in the Fall of 1997 is gratefully acknowledged.

1 Catalecticant varieties

Let $S = k[x_1, \ldots, x_n]$ and let $S_d$ denote the space of homogeneous polynomials in $S$ of degree $d$. It is a classical problem of the theory of invariants to find conditions of a form $f \in S_d$ such that it can be represented as a sum

$$f = L_1^d + \cdots + L_r^d$$

where $L_1, \ldots, L_d$ are linear forms. Let us denote by $PS(r, d; n)$ the algebraic closure of the set of forms representable as in (2). When no confusion arises we will use the shorter notation

\(^1\)This result is related to a conjecture posed by A. Geramita in his talk at the Midwest Algebraic Geometry Conference (Notre Dame, Nov. 7-9, 1997), that the ideal of $\text{Sec}_2(v_d(\mathbb{P}^{n-1}))$ is generated by the $3 \times 3$ minors of each of the catalecticant matrices $\text{Cat}_F(i, d-i; n)$, $2 \leq i \leq d-2$. In the same talk A. Geramita communicated a result of T. Deery that these minors generate one and the same ideal for different $i$ as above when $n = 3, 4, 5$. 

2
The Veronese map \( v_d : \mathbb{P}(S_1) \to \mathbb{P}(S_d) \) is given by \( v_d(L) = L^d \). We see that \( PS(r, d; n) \) is the affine cone of the variety \( Sec_r(v_d(\mathbb{P}^{n-1})) \) of the \( r \)-secant \((r-1)\)-planes to the Veronese variety \( v_d(\mathbb{P}^{n-1}) \). So, finding explicitly polynomial equations of \( Sec_r(v_d(\mathbb{P}^{n-1})) \) is equivalent to finding polynomial equations on the coefficients of \( f \in S_d \), which are satisfied if and only if \( f \) has a representation of the form (2) or its degeneration (the later is called generalized additive decomposition [IK]). We refer the reader to the papers [DK, ER, IK, Ge] for a modern account and references on the subject.

There is a nice set of polynomials in the ideal of \( Sec_r(v_d(\mathbb{P}^{n-1})) \) which we now introduce. Consider the polynomial ring \( R = k[y_1, \ldots, y_n] \) with homogeneous components \( R_i \) and consider the differential action of \( R \) on \( S \) defined as follows. For \( \phi \in R^d - i \), \( f \in S_d \) we let
\[
\phi \circ f = \phi(\partial_1, \ldots, \partial_n) f \in S_i
\]
Let us choose bases of \( R_j \) and \( S_j \) as follows. For \( R_j \) one takes
\[
Y^V = y_1^{v_1} \cdots y_n^{v_n}
\]
with \( |V| = v_1 + \cdots + v_n = j \) and for \( S_j \) one takes
\[
X^{(U)} = \frac{1}{u_1! \cdots u_n!} x_1^{u_1} \cdots x_n^{u_n}
\]
with \( |U| = u_1 + \cdots + u_n = j \). Fixing a form \( f \in S_d \)
\[
f = \sum_{|W|=d} a_W X^{(W)}
\]
one obtains for every \( i, 1 \leq i \leq d - 1 \) a linear map
\[
AP_f(i, d - i) : R_{d-i} \longrightarrow S_i, \quad \phi \mapsto \phi \circ f
\]
which in the bases introduced above has the following matrix
\[
Cat_f(i, d - i; n) = (a_{U,V})_{|U|=i, |V|=d-i}, \quad a_{U,V} = a_{U+V}
\]
This is the \( i \)-th catalecticant matrix of \( f \). Obviously
\[
^tCat_f(i, d - i; n) = Cat_f(d - i, i; n).
\]
Suppose \( f \) has a representation of the form (2). Then for every \( j \) with \( 1 \leq j \leq d - 1 \) the spaces of \( j \)-th partial derivatives \( R_j \circ f \) have dimensions \( \leq r \). Equivalently the catalecticant matrices of \( f \) have ranks \( \leq r \). Let
\[
f = \sum_{|W|=d} Z_W X^{(W)}
\]
be a generic form in \( S_d \), i.e. the coefficients \( Z_W \) are indeterminates over \( k \). Then the above can be restated as follows. The \( (r + 1) \times (r + 1) \) minors of the generic catalecticant matrix
\[
Cat_f(i, d - i; n) = (Z_{U+V})_{|U|=i, |V|=d-i}
\]
vanish on \( PS(r, d; n) \) (equivalently on \( Sec_r(v_d(\mathbb{P}^{n-1})) \)).
Example 1.1 Let $d = 2$. For a general quadratic form $F = \mathbf{1}XZX$ with $X = {}^t(x_1, \ldots, x_n)$, $Z = (Z_{ij})$, $\mathbf{1}Z = Z$ one has $\text{Cat}_F(1, 1; n) = Z$. From linear algebra $P_r = \text{PS}(r, 2; n) \subset S_2$ is defined set-theoretically by vanishing of the $(r + 1) \times (r + 1)$ minors of $(Z_{ij})$. In fact a stronger result is known [Ku] (see also Corollary 2.4), the ideal $I_r$ of $P_r$ is generated by the $(r + 1) \times (r + 1)$ minors of $(Z_{ij})$, the variety $P_r$ is projectively normal and arithmetically Cohen-Macaulay.

Example 1.2 Let $n = 2$. This case was much studied in the 19th century. The classical references are [GY, El]. A modern exposition can be found in [KR]. For a general binary form

$$F = Z_0 \frac{1}{d!} x_1^d + \cdots + Z_j \frac{1}{(d - j)!} j! x_1^{d-j} x_2^j + \cdots + Z_d \frac{1}{d!} x_2^d$$

the $i$-th catalecticant matrix is

$$\text{Cat}_F(i, d - i; 2) = \begin{pmatrix} Z_0 & Z_1 & \cdots & Z_{d-i} \\ Z_1 & Z_2 & \cdots & Z_{d-i+1} \\ \vdots & \vdots & \ddots & \vdots \\ Z_i & Z_{i+1} & \cdots & Z_d \end{pmatrix}$$

known also as Hankel matrix. When $P_r = \text{PS}(r, d; n)$ is a proper subset of $S_d$ (this holds iff $2r \leq d$) it is defined set-theoretically by vanishing of the $(r + 1) \times (r + 1)$ minors of any of the catalecticant matrices $\text{Cat}_F(i, d - i; 2)$ with $r \leq i \leq d - r$ (see e.g. [Ha, p. 103]). In fact the $(r + 1) \times (r + 1)$ minors of any of the matrices $\text{Cat}_F(i, d - i; 2)$ with $r \leq i \leq d - r$ generate the ideal of $P_r$, the variety $P_r$ is projectively normal and arithmetically Cohen-Macaulay (see [GP, El1, Wa]). A detailed exposition of this material is given also in [IK].

We now define two types of catalecticant subvarieties of $S_d$ defined by vanishing of certain minors of certain catalecticant matrices.

Definition 1.3 Consider the ideal $J_r = I_{r+1}(\text{Cat}_F(i, d - i; n))$ generated by the $(r + 1) \times (r + 1)$ minors of $\text{Cat}_F(i, d - i; n)$ (see (5)). We denote by $V_r(i, d - i; n)$ the closed affine subscheme of $S_d$ whose ideal is $J_r$ and denote by $V_r(i, d - i; n)$ the reduced subscheme $V_r(i, d - i; n)_{\text{red}}$.

Definition 1.4 With every form $f \in S_d$ one associates an ideal $\text{Ann}(f) \subset R$ consisting of polynomials $\phi$ such that $\phi \circ f = 0$. These polynomials are called apolar to $f$. The algebra $A_f = R/\text{Ann}(f)$ is a graded artinian Gorenstein algebra (see e.g. [El2, p. 527] or [IK]). Its Hilbert sequence is

$$H(A_f) = (1, \ldots, h_i = \text{dim}_k(A_f)_i, \ldots, 1)$$

It is symmetric with respect to $\frac{d}{2}$.

Definition 1.5 Let $T = (1, t_1, \ldots, t_i, \ldots, t_{d-1}, 1)$ be a sequence of $d + 1$ positive integers with $t_0 = t_d = 1$ which is symmetric with respect to $\frac{d}{2}$. Consider the subset of $S_d$

$$\text{Gor}(T) = \{ f \in S_d \mid H(A_f) = T \}$$
The variety $\text{Gor}(T)$ is quiasiaffine being an open subset of

$$\text{Gor}_<(T) = \bigcap_{i=1}^{d-1} V_{t_i}(i, d - i; n)$$

One puts a finer scheme structure on $\text{Gor}(T)$ by considering the closed subscheme of $S_d$

$$\text{Gor}_<(T) = \bigcap_{i=1}^{d-1} V_{t_i}(i, d - i; n)$$

with ideal $\sum_{i=1}^{d-1} I_{t_i+1}(\text{Cat}_F(i, d - i; n))$. Then $\text{Gor}(T)$ is the open subscheme of $\text{Gor}_<(T)$ associated with the open set $\text{Gor}(T)$

Recall we defined a differentiation action $R_i \times S_d \to S_{d-i}$ which is perfect pairing for $i = d$. We will need the following theorem proved in [IK, Chapter 2].

**Theorem 1.6** Let $f \in S_d$ and let $I = \bigoplus I_i \subset R$ be the graded ideal of polynomials apolar to $f$.

(i) Suppose $f \in V_r(i, d - i; n)$. Then we have for the tangent space at $f$ the equality

$$T_f V_r(i, d - i; n) = (I_i I_{d-i})^\perp$$

(ii) Suppose $f \in \text{Gor}(T)$. Then

$$T_f \text{Gor}(T) = (I^2)_d^\perp$$

The connection between $PS(r, d; n)$ and $\text{Gor}(T)$ is the following. Let us denote by $r_i = \dim_k R_i = \binom{n-1+i}{i}$. Suppose $d = 2t$ or $2t + 1$ and let $r \leq r_t$. Let $u$ be the maximal number $u \leq t$ such that $r_u \leq r$. As in Definition 1.5 consider the following sequence

$$T_r = (1, n, \ldots, r_u, r, \ldots, r, r_u, \ldots, n, 1)$$

Then the above considerations can be reformulated as $PS(r, d; n) \subset \text{Gor}_<(T_r)$ (in fact $\subset \text{Gor}(T)$ as shown e.g. in [IK]). It is proved in [IK] that if furthermore $r \leq r_{t-1}$, then $PS(r, d; n)$ is an irreducible component of $\text{Gor}_<(T_r)$ and the scheme $\text{Gor}_<(T_r)$ is generically smooth along $PS(r, d; n)$. This suggests the following two questions.

**Question 1.7** For which triples $(r, d, n)$ do the $(r + 1) \times (r + 1)$ minors of the catalecticant matrices $\text{Cat}_F(i, d - i; n)$, $1 \leq i \leq \frac{d}{2}$ determine set-theoretically $PS(r, d; n)$, or equivalently when $PS(r, d; n) = \text{Gor}_<(T_r)$?

**Question 1.8** Suppose $r, d, n$ is a triple for which the answer to Question 1.7 is affirmative. Is it true that the ideal of $PS(r, d; n)$ (equal to the ideal of $\text{Sec}_r(v_d(\mathbb{P}^{n-1}))$) is generated by the $(r + 1) \times (r + 1)$ minors of the catalecticant matrices $\text{Cat}_F(i, d - i; n)$ for $1 \leq i \leq \frac{d}{2}$? In other words when $PS(r, d; n) = \text{Gor}_<(T_r)$?
We have already seen in Examples 1.1, 1.2 that the answer to these questions is affirmative if \( d = 2 \) or \( n = 2 \). In the next two sections we will see it is also affirmative if \( r \leq 2 \).

From representation-theoretic point of view it is convenient to reformulate some of the notions we have so far encountered in terms of symmetric tensors. We consider a vector space \( V \) of dimension \( n \) with a tautological representation of \( GL_n \). We have

\[
k[x_1, \ldots, x_n] = S \cong Sym(V^*) = \bigoplus_{d \geq 0} S_d V^*
\]

Explicitly this isomorphism is: given a homogeneous form \( f \) of degree \( d \) there exists a unique symmetric covariant tensor \( \tilde{f} \in S_d V^* \), such that \( f(v) = \tilde{f}(v, \ldots, v) \) for every \( v \in V \). Then

\[
R \cong Sym(V) = \bigoplus_{i \geq 0} S_i V
\]

and the differential action of \( R_{d-i} \) on \( S_d \) defined in (3) equals \( \frac{d}{r} \) times the contraction action of tensors

\[
S_{d-i} V \times S_d V^* \to S_i V^*
\]

For later use we also need a coordinate free description of the ideal generated by the \( r \times r \) minors of \( Cat_F(i, d - i; n) \). Consider the linear map

\[
\mu_r : \Lambda^r(S_{d-i} V) \otimes \Lambda^r(S_i V) \to S_r(S_d V)
\]

defined as follows:

\[
\mu_r((\phi_1 \wedge \cdots \wedge \phi_r) \otimes (\psi_1 \wedge \cdots \wedge \psi_r)) = \det(\phi_i \psi_j).
\]

Consider the graded ring \( Sym(S_d V) = \bigoplus_{\ell \geq 0} S\ell(S_d V) \). Tensoring by \( Sym(S_d V) \) one extends \( \mu_r \) to a homomorphism of degree 0 of graded \( Sym(S_d V) \)-modules

\[
(6) \quad \Lambda^r(S_{d-i} V) \otimes \Lambda^r(S_i V) \otimes Sym(S_d V)(-r) \to Sym(S_d V)
\]

**Lemma 1.9** Let us identify as above \( S_d \) with \( S_d V^* \) and the coordinate ring \( k[S_d] \) with \( Sym(S_d V) \). Then

(i) The image of \( \mu_r \) is the linear subspace generated by the \( r \times r \) minors of \( Cat_F(i, d - i; n) \).

(ii) The image of the homomorphism (6) equals the ideal generated by the \( r \times r \) minors of \( Cat_F(i, d - i; n) \).

**Proof.** Let \( \{E_U : |U| = i\} \), \( \{E_W : |W| = d - i\} \) be the bases of \( S_i V \), \( S_{d-i} V \) which correspond to the bases \( \{Y_U\} \), \( \{Y^W\} \) of \( R_i \), \( R_{d-i} \) considered above. Then

\[
\mu_r((E_{W_1} \wedge \cdots \wedge E_{W_r}) \otimes (E_{U_1} \wedge \cdots \wedge E_{U_r})) = \det(E_{W_i} E_{U_j}) = \det(E_{W_i+U_j})
\]

The value of this symmetric tensor on \( f \in S_d V^* \) is equal to

\[
\det(E_{W_i+U_j}) (f, \ldots, f) = \det((E_{W_i+U_j}, f))
\]

The right-hand side is \( (\frac{1}{r})^r \) times the \( r \times r \) minor of \( Cat_F(i, d - i; n) \) corresponding to rows \( U_1, \ldots, U_r \) and columns \( W_1, \ldots, W_r \). This proves (i). Part (ii) is immediate from (i). \( \square \)
2 Porras’ theorem

The aim of this section is to give a simplified proof and some corollaries of a theorem of O. Porras [Po] about the catalecticant variety \( V_r(1, d - 1; n) \). Porras studies more generally rank varieties of tensors of arbitrary type. The paper [FW, Sections 4,5] is focused on the symmetric case and Porras’ theorem is generalized to several symmetric tensors. In [FW, Section 4] the reader can find a very clear exposition of the geometric method of calculating syzygies, which is an important ingredient of the original proof of Porras’ theorem and has many other applications (see [JPW, PW, We]).

**Lemma 2.1** Let \( 1 \leq r \leq n - 1 \).

(i) Suppose \( f \in V_r(1, d - 1; n) \). Then there is a linear change of coordinates \( x_i = \sum c_{ij}x'_j \), such that \( f \in k[x'_1, \ldots, x'_d] \).

(ii) \( V_1(1, d - 1; n) = PS(1, d; n) \)

**Proof.** (i). \( \text{Cat}_f(d - 1, 1; n) = \text{\`a}Cat_f(1, d - 1; n) \). So, if the rank of these matrices is \( \leq r \), then the kernel of the first one has dimension \( \geq n - r \). Choosing new coordinates \( y'_i = \sum c_{ij}y_j \) for \( R \), so that \( y'_{r+1}, \ldots, y'_n \) belong to this kernel we have for the dual coordinates of \( S \), \( x_i = \sum c_{ij}x'_j \), the equation \( \partial x'_j(f) = 0 \) for \( j \geq r + 1 \). This proves (i).

(ii). Immediate from (i). \( \square \)

Let \( X \) be the affine space \( S_dV^* \) with coordinate ring \( A = k[X] = \text{Sym}(S_dV) \) with graded components \( A_j = S_j(S_dV) \). The identification between homogeneous polynomials and symmetric tensors from the end of Section 1 together with Lemma 2.1 yield that \( V_r(1, d - 1; n) \) equals the variety \( X_r \) of symmetric tensors of rank \( r \) [Po, FW], i.e. tensors which belong to \( S_dQ^* \subset S_dV^* \) for some subspace \( Q^* \subset V^* \) of dimension \( r \). In terms of symmetric tensors the catalecticant matrix appears as follows. A generic tensor \( \tilde{F} \in S_dV^* \) yields by contraction the map

\[ \Psi : S_{d-1}V \rightarrow V^* \otimes A_1 \]

(cf. [Po, p. 703] and [FW, §5]). Its matrix is up to constant the catalecticant matrix \( \text{Cat}_F(1, d - 1; n) \).

**Theorem 2.2 (O. Porras)** Let \( 1 \leq r \leq n - 1 \).

(i) The variety \( X_r = V_r(1, d - 1; n) \) is irreducible of dimension \( \binom{r + d - 1}{d} + r(n - r) \). It is normal, Cohen-Macaulay with rational singularities.

(ii) Its ideal \( I(X_r) \) equals the ideal \( J_r \) generated by the \( (r + 1) \times (r + 1) \) minors of the catalecticant matrix \( \text{Cat}_F(1, d - 1; n) \).

(iii) The singular locus of \( X_r \) equals \( X_{r-1} = V_{r-1}(1, d - 1; n) \).

Before giving the proof of this theorem we make some comments on the original proof ([Po] and [FW]). It consists of three steps. First one constructs a canonical desingularization \( q : Z \rightarrow X_r \) and proves (i) by calculating \( R^q_* (O_Z) \). This step is a particular case of a theorem of G. Kempf [Ke2, p. 239]. Second, using induction and a representation-theoretic argument one reduces the proof of the equality \( I(X_r) = J_r \) to the case \( r = n - 1 \). The third
and most difficult step is to prove \( I(\mathcal{X}_{n-1}) = J_{n-1} \). This is done using a general theorem [FW, §4] by which one calculates the terms of the minimal resolution of \( I(\mathcal{X}_{n-1}) \). Our simplification is in the third step. We obtain easily the result using Theorem 1.6. For the sake of completeness we also include the proof of the first two steps following [FW].

**Proof of Theorem 2.2.**

**Step 1.** We let \( \mathcal{G} = \text{Grass}(n - r, \mathcal{V}) = \text{Grass}(r, \mathcal{V}^*) \), \( \mathcal{V} = \mathcal{V} \times \mathcal{G} \) and consider the two tautological sequences

\[
0 \rightarrow \mathcal{R} \rightarrow \mathcal{V} \rightarrow \mathcal{Q} \rightarrow 0
\]

\[
0 \rightarrow \mathcal{Q}^* \rightarrow \mathcal{V}^* \rightarrow \mathcal{R}^* \rightarrow 0
\]

We have \( S_d \mathcal{Q}^* \subset S_d \mathcal{V}^* = \mathcal{X} \times \mathcal{G} \). Let us denote by \( Z \) the total space of the vector bundle \( p : S_d \mathcal{Q}^* \rightarrow \mathcal{G} \). Then by Lemma 2.1 the projection onto the factor \( \mathcal{X} \) gives a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{c} & \mathcal{X} \times \mathcal{G} \\
\downarrow q' & & \downarrow q \\
\mathcal{X}_r & \xrightarrow{c} & \mathcal{X}
\end{array}
\]

with epimorphic \( q' \). Thus \( \mathcal{X}_r \) is irreducible. Furthermore the restriction of \( q' \) on every fiber of the vector bundle \( p : S_d \mathcal{Q}^* \rightarrow \mathcal{G} \) is closed embedding. Hence \( q' \) is bijective over \( \mathcal{X}_r - \mathcal{X}_{r-1} \). The later subset is nonempty, since it contains \( f = L_1^d + \cdots + L_r^d \) with sufficiently general \( L_1, \ldots, L_r \). This proves \( Z \) is a resolution of \( \mathcal{X}_r \), gives the dimension of \( \mathcal{X}_r \) and proves \( \mathcal{X}_r - \mathcal{X}_{r-1} \) is nonsingular.

The sheaves \( R^i q'_* (\mathcal{O}_Z) \) are coherent over the affine scheme \( \mathcal{X}_r \), so are determined completely by the \( \mathcal{A} \)-module \( H^i(Z, \mathcal{O}_Z) \). The projection \( p : Z \rightarrow \mathcal{G} \) is affine, thus

\[
H^i(Z, \mathcal{O}_Z) = H^i(\mathcal{G}, p_* \mathcal{O}_Z).
\]

Since \( Z \) is the total space of the vector bundle \( p : S_d \mathcal{Q}^* \rightarrow \mathcal{G} \) one obtains \( p_* \mathcal{O}_Z = \text{Sym}(S_d \mathcal{Q}) \).

One decomposes into a direct sum

\[
\text{Sym}(S_d \mathcal{Q}) = \bigoplus S_\lambda(\mathcal{Q})
\]

where \( S_\lambda \) are the Schur functors associated with certain Young diagrams \( \lambda_0 \geq \cdots \geq \lambda_r \geq 0 \). Using Bott’s theorem (see e.g. [Po, p. 687] or [We, p. 232] for a convenient formulation) one obtains

\[
H^i(\mathcal{G}, p_* \mathcal{O}_Z) = 0 \quad \text{for } i \geq 1
\]

\[
H^0(\mathcal{G}, p_* \mathcal{O}_Z) = \bigoplus_{\ell(\lambda) \leq r} S_\lambda(\mathcal{V})
\]

Here \( \ell(\lambda) \) is the number of columns in the Young diagram. Since \( S_d \mathcal{V} = H^0(\mathcal{G}, S_d \mathcal{Q}) \) and \( k[\mathcal{X}] = \text{Sym}(S_d \mathcal{V}) \) we obtain that \( R^i q'_* \mathcal{O}_Z \) is a factor of \( k[\mathcal{X}] \). This proves \( \mathcal{X}_r \) is normal with rational singularities which implies by [KKMS, p. 50] that it is Cohen-Macaulay.
Step 2. Let \( J_r \) be the ideal generated by the \((r+1) \times (r+1)\) minors of \( Cat_F(1, d-1; n) \). We have \( J_r \subset I(X_r) \) and we want to prove equality. Both ideals are \( GL_n \)-invariant graded ideals of \( k[X] \) and by Step 1 we have

\[
I(X_r) = \bigoplus_{\ell(\lambda) \geq r+1} S_\lambda(V)
\]

To be more precise this means that in the decomposition of \( I(X_r) \) enter all irreducible components \( S_\lambda(V) \) of \( k[X] \) with number of columns \( \ell(\lambda) \geq r+1 \). One proves \( J_r = I(X_r) \) by descending induction in \( r \), the case \( r = n \) being trivial. Suppose one has that \( J_{r+1} = I(X_{r+1}) \). Using (7) one has to prove that every isotypical component of \( k[X] \) associated to a \( S_\lambda(V) \) with \( \ell(\lambda) = r \) is contained in \( J_r \). We claim it suffices to prove the equality \( J_r = I(X_r) \) in the case of vector spaces \( V \) of dimension \( r+1 \). Indeed, suppose \( r+1 < n \). Choose a basis \( e_1, \ldots, e_n \) of \( V \), let \( V' = \langle e_1, \ldots, e_{n-1} \rangle \), let \( GL_{n-1} \subset GL_n \) be the corresponding embedding and let \( V \rightarrow V' \) be the \( GL_{n-1} \)-invariant projection. It induces a projection

\[
k[X] = Sym(S_dV) \rightarrow Sym(S_dV') = k[X']
\]

with the property that every isotypical component \( \bigoplus_1^m S_\lambda(V) \) of \( k[X] \) with \( \ell(\lambda) \leq n-1 \) is transformed into isotypical component \( \bigoplus_1^m S_\lambda(V') \) of \( k[X'] \). For homogeneous forms in \( S_d \cong S_dV^* \) the effect of this projection is letting \( x_n = 0 \), so the images of \( J_r \) and \( I(X_r) \) are the corresponding ideals for polynomials in \( n-1 \) variables. Thus, if \( r+1 \leq n-1 \) proving the equality

\[
I_{r+1}(Cat_F(1, d-1; n-1)) = I(V_r(1, d-1; n-1))
\]

would imply that every isotypical component \( \bigoplus_1^m S_\lambda(V) \) with \( \ell(\lambda) = r+1 \) is contained in \( J_r \) which would imply from the induction hypothesis \( J_{r+1} = I(X_{r+1}) \) that \( J_r = I(X_r) \). Repeating this argument we see it suffices to prove that \( J_r = I(X_r) \) for \( n = r+1 \).

Step 3. Let \( r = n-1 \). Then the ideal \( J_{n-1} \) is generated by the maximal minors of the catalecticant matrix \( Cat_F(1, d-1; n) \) of type \( n \times N \) with \( N = {n+d-2 \choose d-1} \). The codimension of \( X_{n-1} \) equals

\[
\left( \begin{array}{c}
{n+d-1} \\
d
\end{array} \right) - \left( \begin{array}{c}
{n+d-2} \\
d
\end{array} \right) - (n-1) = N - n + 1 .
\]

This number is the codimension of the generic determinantal locus \( M_{n-1}(n, N) \), hence the catalecticant scheme \( V_r(1, d-1; n) \) with ideal \( J_r \) is Cohen-Macaulay (see e.g. [ACGH, p. 84]).

Since \( V_r(1, d-1; n) \) is an irreducible scheme, in order to prove that \( J_r \) is a prime ideal (and thus equal to \( I(X_r) \)) it suffices to verify that \( V_r(1, d-1; n) \) is generically smooth (see e.g. [Ei2, p. 457]). We prove smoothness at every \( f \in V_{n-1}(1, d-1; n) - V_{n-2}(1, d-1; n) \). Let \( I \subset \mathcal{A} \) be the ideal of polynomials apolar to \( f \). Then \( \dim_k I_1 = 1 \), \( \dim_k I_{d-1} = \dim_k R_{d-1} - n + 1 \) since \( rk Cat_1(1, d-1; n) = n - 1 \). Thus

\[
\dim_k I_1 I_{d-1} = \left( \begin{array}{c}
{n+d-2} \\
d-1
\end{array} \right) - n + 1
\]
which by Theorem 1.6 yields that \( f \) is a smooth point of the scheme \( V_{n-1}(1, d-1; n) \). This proves (ii).

It remains to prove that \( \text{Sing}(X_r) = X_{r-1} \). We have already proved above that \( X_r - X_{r-1} \) is nonsingular. Associating to \( f \in S_d \) the catalecticant matrix \( \text{Cat}_f(1, d-1; n) \) yields a linear map into the space of \( n \times N \) matrices \( \text{Cat} : S_d \to M(n, N) \). The \( (r+1) \times (r+1) \) minors are homogeneous polynomials on \( M(n, N) \) which vanish of order \( \geq 2 \) at every point of the rank \( \leq r-1 \) locus \( M_{r-1}(n, N) \) (see [Ha, pp. 184-185]). The same holds for the pull-back of these minors by the map \( \text{Cat} \), so from the equality \( J_r = I(X_r) \) we conclude that for every \( f \in X_{r-1} \) one has \( T_f X_r = S_d \). Therefore \( \text{Sing} X_r = X_{r-1} \). Theorem 2.2 is proved. \( \square \)

Remark 2.3 Porras’ theorem gives affirmative answer to the first two questions of Problem 11.6 in [Ge].

A particular case of Porras’ theorem is the following well-known theorem mentioned in the introduction.

Corollary 2.4 Let \( X \) be the affine space of symmetric \( n \times n \) matrices. Then the sublocus of matrices of rank \( \leq r \) is irreducible, normal, Cohen-Macaulay with rational singularities and its ideal is generated by the \( (r+1) \times (r+1) \) minors of a generic symmetric \( n \times n \) matrix. The singular locus of \( X_r \) equals \( X_{r-1} \).

Corollary 2.5 The Veronese variety \( v_d(\mathbb{P}^{n-1}) \) is projectively normal, arithmetically Cohen-Macaulay, its affine cone has rational singularities and its ideal is generated by the \( 2 \times 2 \) minors of the catalecticant matrix \( \text{Cat}_F(1, d-1; n) \).

Remark 2.6 The generation of the ideal of \( v_d(\mathbb{P}^{n-1}) \) by the \( 2 \times 2 \) minors was posed as a question in [Ge, pp. 101-102]. It is reported in [GPS, p. 213] that M. Pucci has shown independently this fact.

Corollary 2.7 The chordal variety \( \text{Sec}_2(v_3(\mathbb{P}^{n-1})) \) is projectively normal, arithmetically Cohen-Macaulay, its affine cone has rational singularities and its ideal is generated by the \( 3 \times 3 \) minors of the catalecticant matrix \( \text{Cat}_F(1, 2; n) \). Its singular locus equals the Veronese variety \( v_3(\mathbb{P}^{n-1}) \).

Proof. It is proved in [IK] that \( \dim_k PS(2, d; n) = 2n \) which equals the dimension of \( V_2(1, 2; n) \) according to Theorem 2.2(i). \( \square \)

Remark 2.8 This corollary gives affirmative answer to Problem 10.7 in [Ge, p. 102].

3 The chordal variety to a Veronese variety

The chordal variety to \( v_3(\mathbb{P}^{n-1}) \) was considered in the previous section. We assume in the present one \( d \geq 4 \). We saw in Section 1 that the affine cone to the chordal variety \( \text{Sec}_2(v_d(\mathbb{P}^{n-1})) \) which is \( PS(2, d; n) \) is contained in \( \text{Gor}_2(T_2) \), the latter variety being defined
by the vanishing of the $3 \times 3$ minors of the catalecticant matrices $\text{Cat}_F(i, d-i; n)$ for $1 \leq i \leq \frac{d}{2}$.

In fact a smaller set of equations suffices (see [Ge, p. 107] for the case $d = 3$).

**Lemma 3.1** A form $f \in S_d$ belongs to $\text{PS}(2, d; n) = P_2$ if and only if

$$\text{rk} \text{Cat}_f(1, d - 1; n) \leq 2 \quad \text{and} \quad \text{rk} \text{Cat}_f(2, d - 2; n) \leq 2.$$ 

The possibilities for $f \in P_2$ are: $0, L^d, L_1^d + L_2^d$ or $L_1L_2^{d-1}$ for some linear forms $L, L_1, L_2$.

**Proof.** By Lemma 2.1 the first condition gives that after a change of coordinates $f \in k[x_1', x_2']_d$. The second one holds if and only if there is a form $\phi \in k[y_1', y_2']_s$ such that $\phi \circ f = 0$. For every $s \leq \frac{d}{2}$ the variety $V_s(s, d - s; 2)$ is irreducible. Indeed, one constructs its smooth resolution as follows. One considers

$$\tilde{Y} \subset \mathbb{P}(k[y_1', y_2']_s \times k[x_1', x_2']_d)$$

consisting of pairs $([\phi], f)$ with $\phi \circ f = 0$. Then the first and second projection yield a diagram

$$\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\pi_2} & V_s(s, d - s; 2) \\
\downarrow \pi_1 & & \\
\mathbb{P}(k[y_1', y_2']_s) & & \\
\end{array}$$

such that $\pi_1$ is a vector bundle and $\pi_2$ is a birational morphism. The varieties $V_s(s, d - s; 2)$ have open dense subsets consisting of forms $f = L_1^d + \cdots + L_s^d$ with nonproportional $L_i \in k[x_1', x_2']_1$, these forms being apolar to $\phi \in k[y_1', y_2']_s$ with $s$ distinct roots (see e.g. [GY, El, IK]). In our particular case $s = 2$ we obtain $f \in \text{PS}(2, d; n)$ and if $f \neq 0$ one has the following cases

(a) $f = L^d$ if $\text{rk} \text{Cat}_f(1, d - 1; n) = 1$

and if $\text{rk} \text{Cat}_f(1, d - 1; n) = 2$

(b) $f = L_1^d + L_2^d$ if $\phi$ has 2 distinct roots;

(c) $f = L_1L_2^{d-1}$ if $\phi$ has a double root.

This proves the lemma. □

Let us make the relative analog of the above. We consider the diagram as in the proof of Theorem 2.2

$$\begin{array}{ccc}
Z & \xrightarrow{p} & G \\
\downarrow q & & \\
X_2 & \subset & X \\
\end{array}$$
and the map of vector bundles over $Z$

\[ \varphi : S_{d-2}(p^*Q) \to S_2(p^*Q^*) \]

which is defined on the fiber over $f \in S_d(Q^*) = Z$ as contraction with $f$. If we let $A = S_{d-2}(p^*Q), E = S_2(p^*Q^*)$ with ranks $a = d - 1, e = 3$ respectively we are in the situation considered in [La, p. 216]. Let us denote by $Y \subset Z$ the corank 1 determinantal subscheme of the map $\varphi$, defined as the closed subscheme of $Z$ with ideal sheaf $J_Y$ generated by the $3 \times 3$ minors of $\varphi$, i.e. $J_Y$ is the image of

\[ \varphi^2 : \Lambda^3 A \otimes \Lambda^3 E^* \to \mathcal{O}_Z \]

**Theorem 3.2** The scheme $Y$ is integral, normal, Cohen-Macaulay with rational singularities. The resolution of $\mathcal{O}_Y$ is given by the Eagon-Northcott complex.

\[ 0 \to \Lambda^a A \otimes S_{a-e} E^* \otimes \Lambda^e E^* \to \cdots \to \Lambda^j A \otimes S_{j-e} E^* \otimes \Lambda^e E^* \to \cdots \to \Lambda^e A \otimes \Lambda^e E^* \to \mathcal{O}_Z \to \mathcal{O}_Y \to 0 \]

where $d - 1 = a \geq j \geq e = 3$, $A = p^*S_{d-2}Q, E^* = p^*S_2Q$.

**Proof.** The Grassman bundle $G_1(E)$ of quotients of rank 1 considered in [La, p. 216] is by duality equal to

\[ \text{Grass}(1, E^*) = \mathbb{P}(p^*S_2Q) = \mathbb{P}(S_2Q) \times_G Z \]

The relative (over $G$) analog of (8), (9) is a variety

\[ \tilde{Y} \subset \mathbb{P}(S_2Q) \times_G Z \]

with two projections $\pi_1, \pi = \pi_2$

\[ \begin{array}{c}
\tilde{Y} \\
\downarrow \pi_1
\end{array} \xymatrix{
\pi \ar[r] & Y \\
\mathbb{P}(S_2Q) \ar[r] & G
} \]

(13)

The projection $\pi_1$ makes $\tilde{Y}$ a vector bundle over $\mathbb{P}(S_2Q)$, so it is smooth. The second one $\pi_2 = \pi$ maps $\tilde{Y}$ birationally onto $Y$, so all condition required in Proposition 2.4 and Theorem 2.6 of [La] are satisfied. The arguments in [La, p. 217] and [Ke1, pp. 181-182] yield $R^p\pi_*\mathcal{O}_{\tilde{Y}} = 0$ for $p \geq 1$, $\pi_*\mathcal{O}_{\tilde{Y}} = \mathcal{O}_Y$ and easy calculation of the terms $\mathcal{E}_{1-j,i}$ of the spectral sequence considered in [La, p. 217] yield the Eagon-Northcott complex. The remaining statements of the theorem follow from [KKMS, p. 50].
Let $P_2 = PS(2, d; n)$. We obtain from (10) the following diagram

\[
\begin{array}{ccc}
Y & \subset & Z \\
\downarrow q & & \downarrow p \\
P_2 & \subset & X_2 & \subset & X
\end{array}
\]  

(14)

**Theorem 3.3** Let $P_2 = PS(2, d; n) \subset S_d$.

(i) The variety $P_2$ is normal, Cohen-Macaulay with rational singularities.

(ii) The ideal $I(P_2) \subset k[S_d]$ is generated by the $3 \times 3$ minors of the catalecticant matrices $Cat_F(1, d - 1; n)$ and $Cat_F(2, d - 2; n)$.

(iii) The singular locus of $P_2$ equals $P_1 = PS(1, d; n)$.

**Proof.** The case $d = 3$ is proved in Corollary 2.7. So we may assume $d \geq 4$. The map $q': Z \to X_2$ is birational over $X_2 - X_1$ and $X_1 = V_1(1, d - 1; n)$ is properly contained in $P_2$ (Lemma 2.1(ii)). Thus $q : Y \to P_2$ is birational. By [Ke2, Theorem 5] in order to verify (i) it suffices to prove that $R^i q_* \mathcal{O}_Y = 0$ for $i \geq 1$ and $q_* \mathcal{O}_Y = P_2$. Since $P_2$ is a closed affine subvariety of $X$ it is equivalent to prove that $H^i(Y, \mathcal{O}_Y) = 0$ for $i \geq 1$ and $H^0(Y, \mathcal{O}_Y)$ is a factor of $k[X] = \text{Sym}(S_d V)$. The map $p : Z \to \mathbb{G}$ is affine, so pushing forward the Eagon-Northcott resolution of $\mathcal{O}_Y$ one obtains a resolution for $p_* \mathcal{O}_Y$:

\[
\cdots \to \Lambda^j(S_{d-2} \mathcal{Q}) \otimes S_{j-3}(S_2 \mathcal{Q}) \otimes \Lambda^{d-1}(S_2 \mathcal{Q}) \otimes \text{Sym}(S_d \mathcal{Q})(-j) \to \\
\cdots \to \Lambda^3(S_{d-2} \mathcal{Q}) \otimes \Lambda^3(S_2 \mathcal{Q}) \otimes \text{Sym}(S_d \mathcal{Q})(-3) \to \text{Sym}(S_d \mathcal{Q}) \to p_* \mathcal{O}_Y \to 0
\]

where $d - 1 \geq j \geq 3$. With the grading $\text{Sym}(S_d \mathcal{Q})(-j)$ the differentials of the complex are of degree 0. We need a lemma.

**Lemma 3.4** Let $\mathcal{F}_j = \bigoplus_n (\mathcal{F}_j)_n$ be the $j$-th graded sheaf of the resolution of $p_* \mathcal{O}_Y$. Then $H^i(\mathbb{G}, (\mathcal{F}_j)_n) = 0$ for $i \geq 1$.

**Proof.** This is immediate from Bott’s theorem since each sheaf $(\mathcal{F}_j)_n$ decomposes as direct sum of $S_\lambda(\mathcal{Q})$ for some Young diagrams $\lambda_0 \geq \lambda_1 \cdots \geq \lambda_r \geq 0$. \hfill \Box

*Proof continued.* Splitting the complex $\mathcal{F}_\bullet$ into a sequence of short exact sequences we deduce from the lemma using induction that the higher cohomology groups of each syzygy sheaf is 0, hence

\[
H^i(Y, \mathcal{O}_Y) = H^i(\mathbb{G}, p_* \mathcal{O}_Y) = 0 \text{ for } i \geq 1
\]

and there is an exact sequence

\[
0 \to \Gamma(\mathcal{F}_{d-3}) \to \cdots \to \Gamma(\mathcal{F}_1) \to \Gamma(p_* \mathcal{O}_Z) \to \Gamma(p_* \mathcal{O}_Y) \to 0
\]

(16)

One obtains that $H^0(Y, \mathcal{O}_Y)$ is a factor of $H^0(Z, \mathcal{O}_Z)$ which equals $H^0(X_2, \mathcal{O}_{X_2})$ by Theorem 2.2. This proves (i).
Borel-Weyl’s theorem yields the following diagram of maps

\[
\Lambda^3(S_{d-2}V) \otimes \Lambda^3(S_2V) \otimes \text{Sym}(S_dV)(-3) \xrightarrow{\mu} \text{Sym}(S_dV)
\]

\[
\Gamma(\Lambda^3(S_{d-2}Q) \otimes \Lambda^3(S_2Q) \otimes \text{Sym}(S_dQ)(-3)) \xrightarrow{d_1} \Gamma(\text{Sym}(S_dQ))
\]

The map \(\mu\) is the one defined in (6), the map \(d_1\) is a differential of the complex (16) and its image is \(\Gamma(p_*J_Y)\). The vertical maps are epimorphisms induced from the isomorphism \(V \xrightarrow{\sim} \Gamma(G, Q)\). The diagram is commutative as one easily sees by restricting the sections of the bottom line to an arbitrary element of the Grassmanian \(G\). This implies by Lemma 1.9 that \(\Gamma(p_*O_Y)\) is generated by the images of the \(3 \times 3\) minors of \(\text{Cat}_F(2, d-2; n)\). Now, \(\Gamma(G, p_*J_Y) = \Gamma(Z, J_Y)\) and the isomorphism \(q^* : \Gamma(X_2, O_{X_2}) \xrightarrow{\sim} \Gamma(Z, O_Z)\) transforms \(\Gamma(X_2, J_{P_2})\) onto \(\Gamma(Z, J_Y)\). We conclude that the images of the \(3 \times 3\) minors of \(\text{Cat}_F(2, d-2; n)\) in \(k[X_2]\) generate the ideal \(I(P_2) \subset k[X] = k[S_d]\) (Theorem 2.2). This proves (ii).

For the proof of (iii) we refer to Theorem 4.3(iii).

\[\square\]

4 The varieties \(\text{Gor}_{\leq}(T)\)

The varieties \(PS(2, d; n)\) studied in Section 3 are particular cases of the varieties \(\text{Gor}_{\leq}(T)\) with \(t_1 = 2\) (see Definition 1.5).

**Lemma 4.1** Suppose \(f \in V_2(1, d-1; n) - V_1(1, d-1; n)\). Then there is a number \(s \geq 2\) with \(2s \leq d + 2\) such that the Hilbert sequence of \(A_f = R/\text{Ann}(f)\) is

\[
H(A_f) = T_{2,s} = (1, 2, \ldots, \frac{s-1}{s}, s, \ldots, \frac{d-s+1}{s}, s-1, \ldots, 2, 1)
\]

**Proof.** By Lemma 2.1 after a change of variables \(f\) becomes a binary form and the above Hilbert sequence is one of the possible Hilbert sequences for binary forms (see e.g. [IK]). \[\square\]

We see that the only possible Hilbert sequences \(T\) with \(t_1 = 2\) are \(T_{2,s}\) defined above. The forms \(f\) which have \(H(A_f) = T_{2,s}\) can be explicitly described as follows. Let \(z_1, z_2\) be a basis of the space of linear forms that vanish on \(\text{Ker} \text{Cat}_f(d-1, 1; n)\). Then there exist linear forms \(L_i = \alpha_i z_1 + \beta_i z_2, i = 1, \ldots, m\) not proportional to each other, and polynomials \(G_i(z_1, z_2)\) not divisible by \(L_i\), of degrees \(d_i - 1\) such that

\[
f = G_1 L_1^{d-d_1+1} + \ldots + G_m L_m^{d-d_m+1}
\]

and \(s = \sum d_i\). Furthermore this representation is unique up to multiplication of the \(L_i\)'s by nonzero constants if \(2s \leq d + 1\) (see [GX] or [IK]).
Lemma 4.2 Let $s \geq 2$, $2s \leq d + 2$. A form $f \in S_d$ belongs to $\text{Gor}_{\leq}(T_{2,s})$ if and only if

$$\text{rk} \text{Cat}_f(1,d-1;n) \leq 2, \quad \text{rk} \text{Cat}_f(s,d-s;n) \leq s$$

The variety $\text{Gor}_{\leq}(T_{2,s})$ is irreducible.

Proof. The proof is left to the reader. The same argument as that of Lemma 2.1 works using (8) and (9).

Notice that for the maximal possible values of $s$, namely $s = \frac{d+1}{2}$ if $d$ is odd and $s = \frac{d+2}{2}$ if $d$ is even, the rank condition $\text{rk} \text{Cat}_f(s,d-s;n) \leq s$ is fulfilled automatically, so in these cases $\text{Gor}_{\leq}(T_{2,s}) = V_2(1,d-1;n)$, the variety studied in Section 2. The following theorem generalizes Theorem 3.3.

Theorem 4.3 Let $s \geq 2$, $2s \leq d$. Consider $\text{Gor}_{\leq}(T_{2,s})$ (see (17) and Definition 1.5). Then

(i) The variety $\text{Gor}_{\leq}(T_{2,s})$ is normal, Cohen-Macaulay with rational singularities.

(ii) The ideal of $\text{Gor}_{\leq}(T_{2,s})$ is generated by the $3 \times 3$ minors of $\text{Cat}_F(1,d-1;n)$ and the $(s+1) \times (s+1)$ minors of $\text{Cat}_F(s,d-s;n)$.

(iii) If $s \geq 3$ the singular locus of $\text{Gor}_{\leq}(T_{2,s})$ equals $\text{Gor}_{\leq}(T_{2,s-1})$. The singular locus of $\text{Gor}_{\leq}(T_{2,2}) = PS(2,d;n)$ equals $PS(1,d;n)$.

Proof. The proof of (i) and (ii) is essentially the same as that of Theorem 3.3 and we leave the details to the reader. We only indicate what changes one needs to do. Instead of (11) one considers the contraction map

$$\varphi : S_{d-s}(p^*Q) \to S_s(p^*Q^*)$$

and the rank-$s$ determinantal subscheme $Y \subset Z$ whose ideal sheaf is generated by the $(s+1) \times (s+1)$ minors of $\varphi$, i.e. by the image of

$$\varphi^* : \Lambda^s(S_{d-1}A) \otimes \Lambda^s(S_{d-1}E^*) \to \mathcal{O}_Z$$

where $A = S_{d-s}(p^*Q)$ has rank $a = d - s + 1$ and $E = S_s(p^*Q^*)$ has rank $e = s + 1$. The same statements for $Y$ as in Theorem 3.2 hold with the Eagon-Northcott complex defined by $A, E, a$ and $e$ as above.

Next, one replaces $P_2$ by $\text{Gor}_{\leq}(T_{2,s})$ in the diagram (14) and pushes forward by $p : Z \to \mathbb{G}$ the Eagon-Northcott complex. Then exactly by the same arguments as in the proof of Theorem 3.3 one verifies (i) and (ii).

For the proof of (iii) one observes that the open subset $U \subset \text{Gor}_{\leq}(T_{2,s})$ consisting of $f$ with $\text{rk} \text{Cat}_f(2,d-2;n) = s$ is nonsingular. This follows from the fact that the birational projection $Y \to \text{Gor}_{\leq}(T_{2,s})$ is biregular over $U$ and the preimage of $U$ is nonsingular in $Y$ as evident from the rank-$s$ analog of the diagram (13). For the proof that $\text{Gor}_{\leq}(T_{2,s-1})$ (resp. $PS(1,d;n)$ for $s = 2$) belongs to the singular locus of $\text{Gor}_{\leq}(T_{2,s})$ one uses the same argument as that of Theorem 2.2(iii) applied to the map

$$\text{Cat} : X_2 \to M \left( \begin{pmatrix} n+1 \\ 2 \end{pmatrix}, \begin{pmatrix} n+d-3 \\ d-2 \end{pmatrix} \right)$$

given by the catalecticant matrix $\text{Cat}_F(2,d-2;n)$.

□
References

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of Algebraic Curves. Vol. I*, Grundlehren Math. Wiss. Bd. 267, Springer-Verlag, Berlin, Heidelberg and New York, 1985.

[BW] W. Bruns and U. Vetter, *Determinantal rings*, Lecture Notes in Math., vol. 1327, Springer-Verlag, Berlin and New York, 1988.

[DP] C. De Concini and C. Procesi, *A characteristic free approach to invariant theory*, Adv. Math. 21 (1976), 330–354.

[DK] I. Dolgachev and V. Kanev, *Polar covariants of plane cubics and quartics*, Adv. Math. 98 (1993), 216–301.

[ER] R. Ehrenborg and G.-C. Rota, *Apolarity and canonical forms*, European J. Combin. 14 (1993), 157–182.

[Ei1] D. Eisenbud, *Linear sections of determinantal varieties*, Amer. J. Math. 110 (1988), 541–575.

[Ei2] D. Eisenbud, *Commutative algebra*, Graduate Texts in Math., vol. 150, Springer-Verlag, Berlin and New York, 1995.

[El] E. B. Elliot, *An introduction to the algebra of quantics*, 2nd ed., Oxford Univ. Press, London and New York, 1913.

[FW] T. Fukui and J. Weyman, *The classification of Thom-Boardman strata which are Cohen-Macaulay along determinantal strata*, preprint, 1997.

[Ge] A. V. Geramita, *Inverse systems of fat points: Waring’s problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals*, Queen’s Papers in Pure and Appl. Math. 102 (1996), 1–131.

[GPS] A. V. Geramita, M. Pucci and Y. S. Shin, *Smooth points of Gor(T)*, J. Pure Appl. Algebra 122 (1997), 209–241.

[GY] J. H. Grace and A. Young, *The algebra of invariants*, Cambridge Univ. Press 1903; reprint Chelsea, New York.

[GP] L. Gruson and C. Peskine, *Courbes de l’espace projectif: variétés de secantes*, Enumerative Geometry and Classical Algebraic Geometry (P. Le Barz and Y. Hevier eds.), Progress in Math. vol. 24, Birkhäuser, Boston, 1982, pp. 1–31.

[Ha] J. Harris, *Algebraic geometry*, Graduate Texts in Math., vol. 133, Springer-Verlag, Berlin and New York, 1992.
[IK] A. Iarrobino and V. Kanev, *The length of a homogeneous form, determinantal loci of catalecticants and Gorenstein algebras*, book, preprint, 1996, (latest version in preparation).

[JPW] T. Józefiak, P. Pragacz and J. Weymann, *Resolutions of determinantal varieties and tensor complex associated with symmetric and antisymmetric matrices*, Astérisque 87/88 (1981), 109–189.

[Ke1] G. Kempf, *On the geometry of a theorem of Riemann*, Ann. of Math. (2) 98 (1973), 178–185.

[Ke2] G. Kempf, *On the collapsing of homogeneous bundles*, Invent. Math. 37 (1976), 229–239.

[KKMS] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, *Toroidal embeddings I*, Lecture Notes in Math., vol. 339, Springer-Verlag, Berlin and New York, 1973.

[KR] J. Kung and G.-C. Rota, *The invariant theory of binary forms*, Bull. Amer. Math. Soc. (N.S.) 10 (1983), 27–85.

[Ku] R. E. Kutz, *Cohen-Macaulay rings and ideal theory of invariants of algebraic groups*, Trans. Amer. Math. Soc. 194 (1974), 115–129.

[La] A. Lascoux, *Syzygies des variétés déterminantales*, Adv. Math. 30 (1978), 202–237.

[Po] O. Porras, *Rank varieties and their desingularizations*, J. Algebra 186 (1996), 677–723.

[PW] P. Pragacz and J. Weyman, *On the construction of resolutions of determinantal ideals: a survey*, Lecture Notes in Math., vol. 1220, Springer-Verlag, Berlin and New York, 1986, pp. 73–92.

[VP] E. B. Vinberg and V. L. Popov, *Invariant theory*, (Russian), Algebraic geometry - 4, Itogi Nauki i Tekhniki: Sovr. Probl. Mat., vol. 55, VINITI, Moscow, 1989, pp. 137–314; English transl. in Encyclopaedia Math. Sci., Springer-Verlag.

[Wa] J. Watanabe, *Hankel matrices and Hankel ideals*, Queen’s Papers in Pure Appl. Math. 102 (1996), 351–363; and Proc. School Sci. Tokai Univ. 32 (1997), 11–21.

[Weyl] H. Weyl, *The classical groups*, 2nd ed., Princeton Univ. Press, Princeton, 1946.

[We] J. Weyman, *On the equations of conjugacy classes of nilpotent matrices*, Invent. Math. 98 (1989), 229–245.

Institute of Mathematics, Bulgarian Academy of Sciences,
Acad. G. Bonchev Str. bl. 8, Sofia, Bulgaria 1113

E-mail address: kanev@math.acad.bg