UNIQUENESS OF TRAVELING FRONT SOLUTIONS FOR THE
LOTKA-VOLTERRA SYSTEM
IN THE WEAK COMPETITION CASE

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Abstract. In this paper, we will prove the uniqueness of traveling front solutions with critical and noncritical speeds, connecting the origin and the positive equilibrium, for the classical competitive Lotka-Volterra system with diffusion in the weak competition, which partially answers the open problem presented by Tang and Fife in [17]. In fact, once these traveling front solutions have the same wave speed and the same asymptotic behavior at $\xi = \pm \infty$, they are unique up to translation.

1. Introduction. In this paper, we consider the classical competitive Lotka-Volterra system with diffusion

$$
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= \frac{\partial u(x,t)}{\partial x^2} + u(x,t)(1 - u(x,t) - k_1 v(x,t)), \\
\frac{\partial v(x,t)}{\partial t} &= d \frac{\partial v(x,t)}{\partial x^2} + rv(x,t)(1 - v(x,t) - k_2 u(x,t)),
\end{align*}
$$

where $k_1, k_2, r, d$ are positive constants and $u(x,t), v(x,t)$ denote the population density of two competitive species that are nonnegative.

To begin with this paper, we remark that, as stated in [4], in any bounded intervals of $\mathbb{R}$, the solutions of (1) with nonnegative initial conditions exhibit the following asymptotic behavior as $t \to +\infty$:

(i) if $0 < k_1 < 1 < k_2$, then $(u(x,t), v(x,t)) \to (1,0)$ ($u$ survives);
(ii) if $0 < k_2 < 1 < k_1$, then $(u(x,t), v(x,t)) \to (0,1)$ ($v$ survives);
(iii) if $k_1, k_2 > 1$, then $(u(x,t), v(x,t)) \to (1,0)$ or $(u(x,t), v(x,t)) \to (0,1)$ depending on the initial condition (strong competition and bistability);
(iv) if $0 < k_1, k_2 < 1$, then $(u(x,t), v(x,t))$ converges to the positive equilibrium (weak competition, $u$ and $v$ coexist).

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Here, we always assume $0 < k_1, k_2 < 1$, which is the weak competition case. In this case, system (1) has four equilibria that are $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(u^*, v^*) := \left(\frac{1-k_1}{1-k_2}, \frac{1-k_2}{1-k_1}\right)$. Moreover, we also remark that $u^* + k_1 v^* = k_2 u^* + v^* = 1$, which will be used in the sequel.

For competitive reaction diffusion systems, the existence of traveling wave solutions is a significant and of particular interesting issue, and has been proved before. For example, Tang and Fife [17] proved the existence of traveling wave solutions and traveling front solutions which connect the origin and the positive equilibrium for a general Lotka-Volterra competition model

$$
\begin{aligned}
\frac{\partial u(x, t)}{\partial t} &= \frac{\partial u(x, t)}{\partial x} + u(x, t) f(u(x, t), v(x, t)), \\
\frac{\partial v(x, t)}{\partial t} &= d \frac{\partial v(x, t)}{\partial x} + v(x, t) g(u(x, t), v(x, t)),
\end{aligned}
$$

$x \in \mathbb{R}$ (2)

where the functions $f$ and $g$ satisfy the following assumptions:

(a) $f, g \in C^1$ have a positive zero $(u^*, v^*)$, that is, $f(u^*, v^*) = g(u^*, v^*) = 0$;

(b) if $0 < u < u^*, 0 < v < v^*$, then $0 < f(u, v) < f(0, 0), 0 < g(u, v) < g(0, 0)$;

(c) if $0 < u < u^*, 0 < v < v^*$, then $\partial_u f(u, v) < 0$, $\partial_v f(u, v) < 0$, $\partial_u g(u, v) < 0$, $\partial_v g(u, v) < 0$;

(d) the eigenvalues of the matrix

$$
\begin{pmatrix}
u^* \partial_u f(u^*, v^*) & u^* \partial_v f(u^*, v^*) \\
v^* \partial_u g(u^*, v^*) & v^* \partial_v g(u^*, v^*)
\end{pmatrix}
$$

have negative real parts. As stated in [1], Perron-Frobenius theorem and the assumption (c) imply that the matrix in the assumption (d) has a real and negative eigenvalue. Hence, both eigenvalues are real. Furthermore in the papers [1, 2, 6], the authors extended the results in [17] into $N$-equations. Moreover, there exists a family of traveling planar waves for (2) on $\mathbb{R}^n$ with different types of reaction terms in [18]. In addition, when delays or nonlocal diffusion are taken into consideration in kinds of competition systems, there are a large number of papers focusing on the existence of traveling wave solutions, such as [3, 5, 10, 11, 12, 13, 14, 15, 16, 21, 22, 23, 24]. When the free boundary is considered, one can refer to [20].

It is easy to verify that the reaction terms in (1) under the case (iv) fully satisfy the assumptions (a)-(d). Therefore, from [17], for $c \geq c_{\min} := 2 \max\{1, \sqrt{rd}\}$, (1) admits a family of traveling wave solutions connecting $(0, 0)$ and $(u^*, v^*)$. More precisely, the traveling wave solution $(u(x, t), v(x, t)) = (\phi(\xi), \psi(\xi))$ $(\xi = x + ct, c \geq c_{\min})$ for (1) satisfies

$$
\begin{aligned}
\phi'' - c \phi' + (1 - \phi - k_1 \psi) \phi &= 0, \\
d \psi'' - c \psi' + r(1 - \psi - k_2 \phi) \psi &= 0,
\end{aligned}
$$

$$
\lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = (0, 0), \quad \lim_{\xi \to +\infty} (\phi(\xi), \psi(\xi)) = (u^*, v^*).
$$

Moreover, if $\phi$ and $\psi$ are monotone and bounded, then we call it traveling front solution.

Besides the existence, it is natural to ask whether the traveling wave solutions connecting $(0, 0)$ and $(u^*, v^*)$ for (1) are unique. In fact, Tang and Fife [17] proposed that the uniqueness of traveling wave solutions with the same wave speed is unknown. Latter Vuuren [18] had mentioned the uniqueness of monotone traveling plane waves connecting the origin and the positive equilibrium, while, he didn’t give
details. Thus, in this paper, we discuss the uniqueness of traveling front solutions connecting (0, 0) and \((u^*, v^*)\) for (1). We will prove if two traveling front solutions of (1) have the same wave speed and the same asymptotic behavior at \(\xi = \pm \infty\), then they are unique (up to a translation).

To prove the uniqueness, in \([7, 8, 9]\), under the case (i), by using the comparison theorem and the squeeze method, they obtained the uniqueness for the traveling front solutions connecting (0, 1) and (1, 0). Nevertheless, compared with the cooperative system, the comparison principle for (3) is invalid and even after the above transformation, the corresponding equilibria are (0, 1) and \((u^*, v^*)\), between which there is no any order relations. Thus, the squeeze method is no longer useful to prove the uniqueness in this case.

Recently, under the case (iv), for (1) with delays, Fang and Wu \([5]\) discussed the existence of traveling front solutions connecting (0, 0) and \((u^*, v^*)\) for \(c \geq c_{\min}\), and then proved the uniqueness of the traveling front solutions by converting (3) into the corresponding integrate systems with \(c > c_{\min}\). Very recently, Wang and Li \([19]\) had analyzed the asymptotic behavior of solutions to (3) with (4) at \(\xi = \pm \infty\). Based on the conclusion in \([19]\), we use the method in \([5]\) to prove the uniqueness of traveling front solutions of (1) satisfying (3) with (4) for \(c \geq c_{\min}\). Therefore our conclusion also includes the uniqueness of traveling front solutions connecting (0, 0) and \((u^*, v^*)\).

The paper is organized as follows. In Section 2, we recall some known results in \([19]\) and give the basic notations. In Section 3, we prove the main result.

2. Asymptotic behavior of traveling front solutions. Very recently, the asymptotic behavior of traveling front solutions had been obtained in \([19]\). Here we simply introduce such result and give some notations used in the sequel.

From \([19]\) the characteristic equation of the linearized (3) at \((u^*, v^*)\) is
\[
\lambda^3 - \left(c + \frac{r}{d}\right) \lambda^3 + \left(\frac{c^2}{d} - u^* - \frac{r}{d} v^*\right) \lambda^2 + \left(\frac{c^2}{d^2} v^* + \frac{r}{d} u^*\right) \lambda + \frac{c}{d} (1 - k_1 k_2) u^* v^* = 0,
\]
which admits two different negative roots \(\lambda_2 < \lambda_1 < 0\). Moreover, \(\lambda_2\) satisfies
\[
\lambda_2^2 - c \lambda_2 - u^* > 0, \quad d \lambda_2^2 - c \lambda_2 - rv^* > 0.
\]
On the other side, the characteristic equation of the linearized (3) at (0, 0) is
\[
(\lambda^2 - c \lambda + 1)(\lambda^2 - \frac{c}{d} \lambda + \frac{r}{d}) = 0,
\]
which has four real roots
\[
\lambda_3 = \frac{c + \sqrt{c^2 - 4}}{2}, \quad \lambda_4 = \frac{c - \sqrt{c^2 - 4}}{2}, \quad \lambda_5 = \frac{c + \sqrt{c^2 - 4 rd}}{2d}, \quad \lambda_6 = \frac{c - \sqrt{c^2 - 4 rd}}{2d},
\]
since \(c \geq c_{\min} = 2 \max\{1, \sqrt{rd}\}\). Obviously \(\lambda_3 \geq \lambda_4 > 0, \lambda_5 \geq \lambda_6 > 0\).

The following lemma about the asymptotic behavior of monotone solutions to (3) with (4) as \(\xi \to \pm \infty\) was proved in \([19]\).

Lemma 2.1. Let \((\phi, \psi)\) be the solution to (3) with (4), then it has the following behaviors, as \(\xi \to -\infty\),
\[
\lim_{\xi \to -\infty} \frac{\phi(\xi)}{h_1 |\xi|^{\kappa_1}} = 1, \quad \lim_{\xi \to -\infty} \frac{\psi(\xi)}{h_2 |\xi|^{\kappa_2}} = 1,
\]
where \(l_1, l_2 \in \{0, 1\}, \kappa_1 \in \{\lambda_3, \lambda_4\}, \kappa_2 \in \{\lambda_5, \lambda_4, \lambda_5, \lambda_6\}\), and \(h_1, h_2\) are two positive constants. On the other hand, as \(\xi \to +\infty\),
\[
\lim_{\xi \to +\infty} \frac{u^* - \phi(\xi)}{h_3 e^{\lambda_2 \xi}} = 1, \quad \lim_{\xi \to +\infty} \frac{v^* - \psi(\xi)}{h_4 e^{\lambda_3 \xi}} = 1,
\]
where \(h_3, h_4\) also are two positive constants.

In the following, we claim that the monotone solutions \((\phi_i, \psi_i)\) (\(i = 1, 2\)) of (3) with (4) have the same asymptotic behavior at \(\xi = \pm \infty\), if the corresponding \(l_i\) and \(\kappa_i\) (\(i = 1, 2\)) in Lemma 2.1 are same. Moreover, according to the relations between \(c, 1\) and \(\sqrt{rd}\), we will prove the uniqueness in several cases:

**Case 1.** \(c > 2 \max\{1, \sqrt{rd}\}\). In this case, \(\lambda_3 > \lambda_4\) and \(\lambda_5 > \lambda_6\). Thus, \(l_1 = l_2 = 0, \kappa_1 \in \{\lambda_3, \lambda_4\}\) and \(\kappa_2 \in \{\lambda_3, \lambda_4, \lambda_5, \lambda_6\}\).

**Case 2.** \(\sqrt{rd} < 1\) and \(c = 2\). In this case, \(\lambda_3 = \lambda_4 = 1\) and \(\lambda_5 > \lambda_6\). Then \(l_1 \in \{0, 1\}, l_2 = 0, \kappa_1 = \lambda_3\) and \(\kappa_2 \in \{\lambda_3, \lambda_5, \lambda_6\}\).

**Case 3.** \(\sqrt{rd} > 1\) and \(c = 2\sqrt{rd}\). In this case, \(\lambda_3 > \lambda_4\) and \(\lambda_5 = \lambda_6\). Therefore \(l_1 = 0, l_2 \in \{0, 1\}, \kappa_1 \in \{\lambda_3, \lambda_4\}\) and \(\kappa_2 \in \{\lambda_3, \lambda_4, \lambda_5\}\).

**Case 4.** \(\sqrt{rd} = 1\) and \(c = 2\). In this case \(\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 1\). Hence \(l_1, l_2 \in \{0, 1\}\) and \(\kappa_1 = \kappa_2 = \lambda_3 = 1\).

In addition, the nonmonotone solutions \((\phi_i, \psi_i)\) (\(i = 1, 2\)) of (3) with (4) still have the same asymptotic behavior at \(\xi = \pm \infty\) as above, by refining the method in [19]. However with the method used to prove Lemma 3.1 in [5], the nonconstant traveling wave solution \((\phi, \psi)\) of (1) between \((0, 0)\) and \((u^*, v^*)\) is strictly increasing and connecting them. Since the following proof is based on the conditions \(0 \leq \phi(\xi) \leq u^*\) and \(0 \leq \psi(\xi) \leq v^*\), then we only consider the uniqueness of traveling front solution.

### 3. Uniqueness of traveling front solutions

In this section, we will use the method in [5] to prove the uniqueness of traveling front solutions connecting the origin and the positive equilibrium in each cases mentioned in the last paragraph in Section 2. First of all, we transform the differential system \(L[\phi, \psi] = 0\) into the corresponding integrate systems in each cases, where

\[
L[\phi, \psi] := \begin{pmatrix}
\phi'' - c\phi' + (1 - \phi - k_1 \psi)\phi \\
d\psi'' - c\psi' + r(1 - \psi - k_2 \phi)\psi
\end{pmatrix}.
\]

**Lemma 3.1.** Under case \(i\) for \(i = 1, 2, 3, 4\), the solution \((\phi, \psi)\) to \(L[\phi, \psi] = 0\) with (4) is equivalent to the solution of \(T_i[\phi, \psi] = (\phi, \psi)^T\) with (4), where

\[
T_1[\phi, \psi](\xi) := \begin{pmatrix}
\int_{\xi}^{+\infty} \frac{1}{\lambda_3 - \lambda_1} (e^{\lambda_3 (\xi - s)} - e^{\lambda_1 (\xi - s)})\phi(s) \phi(s) + k_1 \psi(s) ds \\
\int_{\xi}^{+\infty} \frac{r}{d(\lambda_6 - \lambda_5)} (e^{\lambda_6 (\xi - s)} - e^{\lambda_5 (\xi - s)})\psi(s) \psi(s) + k_2 \phi(s) ds
\end{pmatrix},
\]

if \(c > 2 \max\{1, \sqrt{rd}\}\);

\[
T_2[\phi, \psi](\xi) := \begin{pmatrix}
\int_{\xi}^{+\infty} (s - \xi) e^{\lambda_3 (\xi - s)} \phi(s) \phi(s) + k_1 \psi(s) ds \\
\int_{\xi}^{+\infty} \frac{r}{d(\lambda_6 - \lambda_5)} (e^{\lambda_6 (\xi - s)} - e^{\lambda_5 (\xi - s)})\psi(s) \psi(s) + k_2 \phi(s) ds
\end{pmatrix},
\]

if \(c = 2, \sqrt{rd} < 1\);
Since the characteristic equation of the linearized equation of $L$ formula, here we only treat case 2, since the other cases are similar. Remark that the proofs of these equivalences are based on the variation of constants formula, moreover, from (4), for $\phi, \psi$ \[ \phi, \psi \] satisfies \( L[\phi, \psi] = 0 \) at \((0, 0)\) is (7). Since $c = 2$, $\sqrt{rd} < 1$, then $\lambda_3 = \lambda_4 = 1$ and $\lambda_5 > \lambda_6$. Thus, let

\[
\begin{align*}
\phi(\xi) &= c_1(\xi)e^{\lambda_3 \xi} + c_2(\xi)e^{\lambda_4 \xi}, \\
\psi(\xi) &= c_3(\xi)e^{\lambda_5 \xi} + c_4(\xi)e^{\lambda_6 \xi},
\end{align*}
\]

(9)

where $c_i(\xi)$, $i = 1, 2, 3, 4$, are determined later. Then by the variation of constants formula, $c_i(\xi)$, $i = 1, 2, 3, 4$, satisfy

\[
\begin{align*}
\lambda_3 c_1'(\xi)e^{\lambda_3 \xi} + c_2'(\xi)e^{\lambda_4 \xi} + \lambda_3 c_2'(\xi)e^{\lambda_4 \xi} &= \phi(\xi)(\phi(\xi) + k_1 \psi(\xi)), \\
\lambda_5 c_3'(\xi)e^{\lambda_5 \xi} + c_4'(\xi)e^{\lambda_6 \xi} &= 0,
\end{align*}
\]

(10)

D\lambda_3 c_1'(\xi)e^{\lambda_3 \xi} + D\lambda_5 c_3'(\xi)e^{\lambda_5 \xi} = 0.

Hence

\[
\begin{align*}
c_1'(\xi) &= -\xi e^{-\lambda_3 \xi} \phi(\xi)(\phi(\xi) + k_1 \psi(\xi)), \\
c_2'(\xi) &= e^{-\lambda_3 \xi} \phi(\xi)(\phi(\xi) + k_1 \psi(\xi)), \\
c_3'(\xi) &= \frac{r}{\lambda_5 - \lambda_6} e^{-\lambda_5 \xi} \psi(\xi)(\psi(\xi) + k_2 \phi(\xi)), \\
c_4'(\xi) &= \frac{r}{\lambda_6 - \lambda_5} e^{-\lambda_6 \xi} \psi(\xi)(\psi(\xi) + k_2 \phi(\xi)).
\end{align*}
\]

Moreover, from (4), for $i = 1, 2, 3, 4$, \( \lim_{\xi \to +\infty} c_i(\xi) = 0 \). Thus by integrating (10) over \((\xi, +\infty)\), we have

\[
\begin{align*}
c_1(\xi) &= \int_{\xi}^{+\infty} \text{se}^{-\lambda_3 \xi} \phi(s)(\phi(s) + k_1 \psi(s))ds, \\
c_2(\xi) &= -\int_{\xi}^{+\infty} e^{-\lambda_3 \xi} \phi(s)(\phi(s) + k_1 \psi(s))ds, \\
c_3(\xi) &= -\int_{\xi}^{+\infty} \frac{r}{\lambda_5 - \lambda_6} e^{-\lambda_5 \xi} \psi(s)(\psi(s) + k_2 \phi(s))ds, \\
c_4(\xi) &= -\int_{\xi}^{+\infty} \frac{r}{\lambda_6 - \lambda_5} e^{-\lambda_6 \xi} \psi(s)(\psi(s) + k_2 \phi(s))ds.
\end{align*}
\]

(11)

Then by substituting (11) into (9), $T_2[\phi, \psi] = (\phi, \psi)^T$ holds. On the other hand, it is easy to verify that if $(\phi, \psi)$ satisfies $T_2[\phi, \psi] = (\phi, \psi)^T$, then it also satisfies $L[\phi, \psi] = 0$. Moreover by L’Hôpital’s rule, $(\phi, \psi)$ satisfies (4).

Now, after these preparations, we are in a position to show and prove the main result in this paper.
Proof. First we consider case 1. To begin the proof, we introduce the following two functions
\[
P(\xi) := |\phi_1(\xi) - \phi_2(\xi + \xi_0)|e^{-\lambda_2 \xi}, \quad Q(\xi) := |\psi_1(\xi) - \psi_2(\xi + \xi_0)|e^{-\lambda_2 \xi},
\]
where \(\xi_0 = \frac{1}{\lambda_2} \ln \frac{\lambda_2 - \lambda_3}{1 - u^*} \). Since \(\lambda_2 < 0\), \(\phi\) and \(\psi\) are bounded, obviously, \(P(-\infty) = Q(-\infty) = 0\).

According to \(u^* + k_1 v^* = k_2 u^* + v^* = 1\), \(\phi_i \leq u^*, \psi_i \leq v^*, i = 1, 2\), and Lemma 3.1, it is easy to see that
\[
P(\xi) \leq \int_{\xi}^{+\infty} f_1(\xi - s)[(1 + u^*)P(s) + k_1 u^*Q(s)]e^{-\lambda_2(\xi-s)}ds,
\]
\[
Q(\xi) \leq \int_{\xi}^{+\infty} f_2(\xi - s)[(1 + v^*)Q(s) + k_2 v^*P(s)]e^{-\lambda_2(\xi-s)}ds,
\]
where \(f_1(\xi) = \frac{1}{\lambda_4 - \lambda_5}(e^{\lambda_4 \xi} - e^{\lambda_5 \xi})\) and \(f_2(\xi) = \frac{r}{\lambda_6 - \lambda_5}(e^{\lambda_6 \xi} - e^{\lambda_5 \xi})\). The above inequalities imply
\[
P(\xi) \leq \int_{-\infty}^{0} f_1(\omega)[(1 + u^*)P(\xi - \omega) + k_1 u^*Q(\xi - \omega)]e^{-\lambda_2 \omega}d\omega,
\]
\[
Q(\xi) \leq \int_{-\infty}^{0} f_2(\omega)[(1 + v^*)Q(\xi - \omega) + k_2 v^*P(\xi - \omega)]e^{-\lambda_2 \omega}d\omega. \quad (12)
\]

Define
\[
F_1(\lambda_2, \omega) = \begin{cases} 0, & \omega \geq 0, \\ f_1(\omega)e^{-\lambda_2 \omega}, & \omega < 0, \end{cases} \quad F_2(\lambda_2, \omega) = \begin{cases} 0, & \omega \geq 0, \\ f_2(\omega)e^{-\lambda_2 \omega}, & \omega < 0, \end{cases}
\]
and
\[
F(\lambda_2, \omega) = \begin{pmatrix} (1 + u^*)F_1(\lambda_2, \omega) & k_1 u^*F_1(\lambda_2, \omega) \\ k_2 v^*F_2(\lambda_2, \omega) & (1 + v^*)F_2(\lambda_2, \omega) \end{pmatrix},
\]
where the items in the matrix \(F(\lambda_2, \omega)\) are nonnegative followed from the nonnegativeness of \(F_i(\lambda_2, \omega), i = 1, 2\). Hence, from (12), we have
\[
\int_{\mathbb{R}} F(\lambda_2, \omega) \begin{pmatrix} P(\xi - \omega) \\ Q(\xi - \omega) \end{pmatrix} d\omega - \begin{pmatrix} P(\xi) \\ Q(\xi) \end{pmatrix} \geq 0. \quad (13)
\]

Now we compute the integration \(\int_{\mathbb{R}} F(\lambda_2, \omega)d\omega\). Remark that \(\lambda_3 + \lambda_4 = c\) and \(\lambda_3 \cdot \lambda_4 = 1\), \(\lambda_5 + \lambda_6 = \frac{\pi}{2}\) and \(\lambda_5 \cdot \lambda_6 = \frac{\pi}{2}\), then
\[
\int_{-\infty}^{0} f_1(\omega)e^{-\lambda_2 \omega}d\omega = \int_{-\infty}^{0} \frac{1}{\lambda_4 - \lambda_3}(e^{\lambda_4 \omega} - e^{\lambda_3 \omega})e^{-\lambda_2 \omega}d\omega = \frac{1}{\lambda_4^2 - c\lambda_2 + r},
\]
\[
\int_{-\infty}^{0} f_2(\omega)e^{-\lambda_2 \omega}d\omega = \int_{-\infty}^{0} \frac{r}{d(\lambda_6 - \lambda_5)}(e^{\lambda_6 \omega} - e^{\lambda_5 \omega})e^{-\lambda_2 \omega}d\omega = \frac{r}{d\lambda_2^2 - c\lambda_2 + r}.
\]
Thus,
\[
\int_{\mathbb{R}} F(\lambda_2, \omega)d\omega - I = -A(\lambda_2), \quad (14)
\]
where

\[ A(\lambda_2) = \begin{pmatrix}
\lambda_2^2 - c\lambda_2 - u^* & -k_1 u^* \\
\frac{\lambda_2^2 - c\lambda_2 + 1}{d\lambda_2^2 - c\lambda_2 + r} & \frac{\lambda_2^2 - c\lambda_2 + 1}{d\lambda_2^2 - c\lambda_2 + r}
\end{pmatrix}. \]

Next from Lemma 2.1, as \( \xi \to +\infty \), we see \( |u^* - \phi_1(\xi)|e^{-\lambda_2\xi} \) and \( |u^* - \phi_2(\xi + \xi_0)|e^{-\lambda_2\xi} \) are converging. Therefore, \( P(\xi) \) is also converging when \( \xi \) converges to \( \infty \). Similar \( Q(+\infty) \) exists. Now, we begin to prove that \( P(+\infty) = Q(+\infty) = 0 \). Otherwise, firstly suppose \( P(+\infty), Q(+\infty) > 0 \), then, for all \( \eta_1, \eta_2 > 0 \),

\[
0 \leq \lim_{\xi \to +\infty} \left( \eta_1 P^{-1}(\xi) \begin{pmatrix} 0 \\ \eta_2 Q^{-1}(\xi) \end{pmatrix} \left( \int_{\mathbb{R}} F(\lambda_2, \omega) \begin{pmatrix} P(\xi - \omega) \\ Q(\xi - \omega) \end{pmatrix} d\omega - \left( \begin{pmatrix} P(\xi) \\ Q(\xi) \end{pmatrix} \right) \right) \right)
= -A(\lambda_2) \left( \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right).
\]

From the definition of \( \lambda_2, \lambda_2^2 - c\lambda_2 + 1, d\lambda_2^2 - c\lambda_2 + r > 0 \). Thus, combined (6), we obtain

\[
\frac{\lambda_2^2 - c\lambda_2 - u^*}{\lambda_2^2 - c\lambda_2 + 1}, \quad \frac{d\lambda_2^2 - c\lambda_2 - ru^*}{d\lambda_2^2 - c\lambda_2 + r} > 0, \quad \frac{-k_1 u^*}{\lambda_2^2 - c\lambda_2 + 1}, \quad \frac{-k_2 v^* r}{d\lambda_2^2 - c\lambda_2 + 4} < 0.
\]

Hence by choosing large \( \eta_1 \) and small \( \eta_2 \) such that \( -A(\lambda_2)(\eta_1, \eta_2)^T \) has different signs, which is a contradiction. Secondly, if one of \( P(+\infty) \) or \( Q(+\infty) \) equals to zero and the other is positive. For example \( P(+\infty) = 0 \) and \( Q(+\infty) > 0 \) implying \( \phi_1(+\infty) = \phi_2(+\infty). \) Then by substituting \( \phi_1(\xi) - \phi_2(\xi - \xi_0) \) into the first equation of (3) and taking \( \xi \to +\infty \), we easily obtain \( \psi_1(+\infty) = \psi_2(+\infty) \), which leads to \( Q(+\infty) = 0 \). Therefore, \( P(+\infty) = Q(+\infty) = 0 \).

Finally we show that \( P(\xi) = Q(\xi) = 0 \) by contradiction. Assume that \( \xi_1^* \) and \( \xi_2^* \) are the points such that \( P \) and \( Q \) attain their maximum positive values respectively, and let the positive constants \( P^* \) and \( Q^* \) denoting this value for short. Thus from (13) and (14), \( A(\lambda_2)(P^*, Q^*)^T \leq 0 \) holds. Due to (15) and \( |A(\lambda_2)| = 0 \), which is derived from (5), we know that

\[
A(\lambda_2) \begin{pmatrix} P^* \\ Q^* \end{pmatrix} = 0.
\]

Hence, at \( \xi_1^* \) and \( \xi_2^* \), on the one hand, (14) and (16) yield that

\[
\int_{\mathbb{R}} F(\lambda_2, \omega)d\omega \begin{pmatrix} P^* \\ Q^* \end{pmatrix} = \begin{pmatrix} P^* \\ Q^* \end{pmatrix}.
\]

On the other hand, from (13) and by remarking the positiveness of all the items in \( F(\lambda_2, \omega) \), we have

\[
\begin{pmatrix} P^* \\ Q^* \end{pmatrix} \leq \int_{\mathbb{R}} F(\lambda_2, \omega) \begin{pmatrix} P(\xi_1^* - \omega) \\ Q(\xi_2^* - \omega) \end{pmatrix} d\omega \leq \int_{\mathbb{R}} F(\lambda_2, \omega)d\omega \begin{pmatrix} P^* \\ Q^* \end{pmatrix}.
\]

Therefore, from (17) and (18), we have

\[
\int_{\mathbb{R}} F(\lambda_2, \omega) \begin{pmatrix} P(\xi_1^* - \omega) \\ Q(\xi_2^* - \omega) \end{pmatrix} d\omega = \int_{\mathbb{R}} F(\lambda_2, \omega)d\omega \begin{pmatrix} P^* \\ Q^* \end{pmatrix}.
\]
which implies, by noting the definition of $F(\lambda_2, \omega)$,
\[
\int_{\mathbb{R}} (1 + u^*) F_1(\lambda_2, \omega)(P^* - P(\xi_i^* - \omega)) + k_1 u^* F_1(\lambda_2, \omega)(Q^* - Q(\xi_i^* - \omega)) d\omega = 0,
\]
\[
\int_{\mathbb{R}} k_2 v^* F_2(\lambda_2, \omega)(P^* - P(\xi_i^* - \omega)) + (1 + v^*) F_2(\lambda_2, \omega)(Q^* - Q(\xi_i^* - \omega)) d\omega = 0.
\]

Then by noting $P^* \geq P(\xi_i^* - \omega)$, $Q^* \geq Q(\xi_i^* - \omega)$ and the nonnegativeness of $F_i(\lambda_2, \omega)$, $i = 1, 2$, we have $P(\xi_i^* - \omega) = P^*$ and $Q(\xi_i^* - \omega) = Q^*$ for all $\omega \in (-\infty, 0]$. Thus, $P^* = Q^* = 0$ follows from $P(\pm \infty) = Q(\pm \infty) = 0$, which is a contradiction.

Now, we consider case 2. Similarly, from $u^* + k_1 v^* = k_2 u^* + v^* = 1$, $\phi_i \leq u^*$, $\psi_i \leq v^*$, $i = 1, 2$, and Lemma 3.1, it is easy to see that
\[
P(\xi) \leq \int_{-\infty}^{0} f_1(\omega)[(1 + u^*) P(s - \omega) + k_1 u^* Q(s - \omega)] e^{-\lambda_2 \omega} d\omega,
\]
\[
Q(\xi) \leq \int_{-\infty}^{0} f_2(\omega)[(1 + v^*) Q(s - \omega) + k_2 v^* P(s - \omega)] e^{-\lambda_2 \omega} d\omega,
\]
where $f_1(\omega) = -\omega e^{\lambda_3 \omega}$ and $f_2(\xi) = \frac{1}{\lambda_4 - \lambda_2}(e^{\lambda_5 \xi} - e^{\lambda_6 \xi})$. Since $c = 2 = 2\lambda_3$, then
\[
\int_{-\infty}^{0} f_1(\omega) e^{-\lambda_2 \omega} d\omega = \int_{-\infty}^{0} -\omega e^{\lambda_3 \omega} e^{-\lambda_2 \omega} d\omega = \frac{1}{\lambda_2^2 - c\lambda_2 + 1}.
\]
The rest of the proof is same. The proofs for case 3 and case 4 are also similar and we omit the details here. Up to now we have finished the proof of the theorem. \qed

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