Motions and world-line deviations in
Einstein-Maxwell theory

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Abstract

We examine the motion of charged particles in gravitational and electro-magnetic background fields. We study in particular the deviation of world lines, describing the relative acceleration between particles on different space-time trajectories. Two special cases of background fields are considered in detail: (a) pp-waves, a combination of gravitational and electro-magnetic polarized plane waves travelling in the same direction; (b) the Reissner-Nordstrøm solution. We perform a non-trivial check by computing the precession of the periastron for a charged particle in the Reissner-Nordstrøm geometry both directly by solving the geodesic equation, and using the world-line deviation equation. The results agree to the order of approximation considered.

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1 Introduction

According to the equivalence principle, the motion of structureless test particles in a gravitational background field is determined only by the space-time geometry: the worldlines are geodesics [1]. However, as only relative accelerations (tidal forces) have an invariant meaning in general relativity, the relevant information about the space-time geometry as a gravitational influence manifests itself in the rate at which worldlines diverge (or converge). The equation describing the rate of geodesic deviation in a quantitative way is well-known since at least half a century [4], and numerous applications have been developed up to the present time [3]-[6].

Things become more complicated for test particles which are not structureless, but carry e.g. non-vanishing charge or spin. In such cases the worldline of the test particle in general is no longer a geodesic, but is modified by electro-magnetic and/or spin-orbit forces [7, 8, 9]. In these cases one also can write an equation for world-line deviations, describing relative accelerations, which are now due to the combined effect of gravitational and/or electromagnetic forces. Even a generalized string-theoretical version has recently been proposed [10].

Nevertheless, in the presence of such additional forces the geometrical interpretation of the world-line deviation is not completely lost. For example, it is well known [13, 14], that the geodesic lines in the five-dimensional Kaluza-Klein theory represent the worldlines of charged massive particles, moving under the influence of the Lorentz force in a curved four-dimensional space-time. In fact, the equivalence principle holds in five dimensions as well, although in four dimensions it applies separately to classes of particles characterized by the same value of the charge-to-mass ratio \(q/m\). Recently it has been checked [15] that also the geodesic deviation equation in the five-dimensional Kaluza-Klein manifold leads to a generalized deviation equation for the world lines of charged particles, which can be obtained by a direct variation of the world line’s equation in the four-dimensional space-time. Similarly, the world lines of particles with spin can be obtained from the supersymmetric extension of the geodesic equation for simple point particles [7, 9]. In spite of these developments, examples of a thorough analysis of solutions to the world-line deviation equations, as well as discussions of their physical implications, are scarce in the literature [16].
In this article we initiate a self-contained discussion of the world-line deviation equations in Einstein-Maxwell theory. We first derive the basic form of the equation, taking the equation of motion itself as a starting point. We also show that these equations can be derived from an action principle. We then turn to discuss two examples for which the nature of the solutions to the world-line deviation equation can be explicitly exhibited. The first case describes the effects of combined non-stationary gravitational and electromagnetic fields, in the form of a gravitational pp-wave. The second case concerns the analysis of deviation of world lines in stationary spherically symmetric field generated by an electrically charged mass (Reissner-Nordstrøm solution).

In part, the choice of these particular solutions is determined by their high symmetry properties, which make it possible to arrive at exact and explicit solutions for the background field, for the world lines, as well as for the deviation itself. But another reason for investigating this type of background solutions derives from the physical aspects: according to General Relativity the electromagnetic field acts on massive charged particles in two ways, directly via the Lorentz force it exerts, and indirectly, via its contribution to the ambient space-time curvature. The combination of these effects may lead to interesting new states of motion. Indeed, the Reissner-Nordstrøm solution provides a particularly illustrative example of this situation, as the repulsive electric force on a test particle can be compensated by the attractive gravitational force.

The supersymmetric extension of the models discussed here describes the modifications in the physics due to the presence of spin. These aspects will be discussed in a separate paper [17].

2 World-line deviation equations

Before considering the more general case, let us recall the derivation of the equation for geodesic deviation relevant to simple chargeless point particles. Consider a continuous smooth vector field $u^\mu(x)$ with the property

$$u^\nu \nabla_\nu u^\mu = 0.$$  \hfill (1)
A vector field of this kind, being transported parallel to itself, represents the field of vectors tangent to a smooth set of geodesic curves \( x^\mu(\tau) \):

\[
u^\mu = \frac{dx^\mu}{d\tau},\]  

where as usual for time-like geodesics we take the parameter \( \tau \) to represent proper time. In terms of proper-time derivatives eq.(1) then becomes

\[
\frac{D^2 x^\mu}{D\tau^2} = \frac{d^2 x^\mu}{d\tau^2} + \Gamma_\lambda^\mu_\nu \frac{dx^\lambda}{d\tau} \frac{dx^\nu}{d\tau} = 0.
\] (3)

Now consider a slicing of space-time into smooth (1+1)-dimensional space-time planes formed by time-like geodesics, and in each of these planes a second smooth set of geodesic curves \( x^\mu[\lambda; \tau] \), connecting the points on the geodesics corresponding to a fixed value of the proper-time parameter \( \tau \). Thus, for each value of \( \tau \) there is a smooth curve in the plane parametrized by the real affine parameter \( \lambda \). Let \( n^\mu(x) = dx^\mu/d\lambda \) be the field of tangent vectors to these curves. We observe that

\[
u^\nu \nabla^\nu n^\mu = \frac{d^2 x^\mu}{d\tau d\lambda} + \Gamma_\nu^\kappa_\lambda \frac{dx^\nu}{d\tau} \frac{dx^\kappa}{d\lambda} = n^\nu \nabla^\nu u^\mu.
\] (4)

Combining eqs.(1) and (4) we then obtain

\[
u^\lambda \nabla^\lambda (u^\nu \nabla^\nu n^\mu) = u^\lambda \nabla^\lambda (n^\nu \nabla^\nu u^\mu) = (n^\lambda \nabla^\lambda u^\nu) \nabla^\nu u^\mu + u^\lambda n^\nu \nabla^\lambda \nabla^\nu u^\mu
\]

\[= n^\lambda \nabla^\lambda (u^\nu \nabla^\nu u^\mu) + u^\lambda n^\nu [\nabla^\lambda, \nabla^\nu] u^\mu = R_\lambda^\mu_\nu^\kappa u^\lambda u^\kappa n^\nu.
\] (5)

In terms of derivations in parameter space (proper time derivatives) this equations reads:

\[
\frac{D^2 n^\mu}{D\tau^2} = R_\lambda^\mu_\nu^\kappa u^\lambda u^\kappa n^\nu,
\] (6)

which is the standard text-book result, as can be found e.g. in [20, 21], stating that the rate of change of the velocity at which geodesics diverge, as measured along the curves \( x^\mu(\lambda) \) at fixed \( \tau \), is proportional to the geodesic separation, with the proportionality factor given by the components of the

\[\text{1In order for these curves to be well-defined, one fixes the origin of proper time on each geodesic to be located on a smooth, differentiable curve cutting all the geodesics in the slice.}\]
Riemann curvature tensor in the direction of the geodesics and the curves of equal proper time.

It is now straightforward to extend this derivation to curves representing the world line of charged particles. We start from the equation of motion for a particle of mass \( m \) and charge \( q \):

\[
\frac{D^2 x^\mu}{D\tau^2} = u^\nu \nabla_\nu u^\mu = \frac{q}{m} F^\mu_{\nu} u^\nu. \tag{7}
\]

The right-hand side of this equation represents the electro-magnetic and Lorentz force acting on the particle. Next we obtain an equation for the relative acceleration of two world lines, as measured by the change in tangent vectors \( n^\mu = dx^\mu/d\lambda \) to curves connecting points of equal proper time on smooth planes of world lines. The equation, derived by a series of steps similar to those of (5), reads:

\[
\frac{D^2 n^\mu}{D\tau^2} = R^\mu_{\lambda\nu\kappa} u^\lambda u^\kappa n^\nu + \frac{q}{m} F^\mu_{\nu} \frac{Dn^\nu}{D\tau} + \frac{q}{m} \nabla_\lambda F^\mu_{\nu} u^\nu n^\lambda. \tag{8}
\]

In practice it is often simpler to work with the non-explicitly covariant equation

\[
\ddot{n}^\mu + \left( 2\Gamma^\mu_{\nu\kappa} u^\kappa - \frac{q}{m} F^\mu_{\nu} \right) \dot{n}^\nu + \left( u^\nu u^\sigma \partial_\nu \Gamma^\mu_{\kappa\sigma} - \frac{q}{m} u^\kappa \partial_\nu F^\mu_{\nu} \right) n^\nu = 0, \tag{9}
\]

which is obtained from (8) by substitution of the explicit expressions for the covariant derivatives and the Riemann curvature tensor.

We observe, that this equation only determines the rate of divergence of world lines of particles with the same charge-to-mass ratio \( q/m \). For neutral particles equation (9) remains valid as the special case for \( q = 0 \). Eq.(8) does not apply to the relative acceleration between particles of different charge-to-mass ratio, such as a neutral and a charged particle. On the other hand, it is clear that a neutral particle can feel the presence of an electro-magnetic field through the influence of its accompanying gravitational field [11].

We also observe, that the direct influence of the electro-magnetic field on charged particles is strictly linear in the field strength \( F_{\mu\nu} \), with coupling constant \( q/m \). No terms quadratic in \( F_{\mu\nu} \) are present in this equation in the context of classical general relativity.
Observe, that eqs. (7) and (8) can be combined to show that \( u \cdot \frac{Dn}{D\tau} \) is a constant on geodesics:

\[
\frac{d}{d\tau} \left( u \cdot \frac{Dn}{D\tau} \right) = 0. \tag{10}
\]

Actually, this equation can be integrated to give the stronger condition

\[
u \cdot \frac{Dn}{D\tau} = 0. \tag{11}\]

This follows, if we consider two worldlines \( x_1^\mu(\tau) = x^\mu(\tau; \lambda) \) and \( x_2^\mu(\tau) = x^\mu(\tau; \lambda + \Delta \lambda) \approx x_1^\mu(\tau) + \Delta n^\mu(\tau) \), with

\[
\Delta n^\mu(\tau) = n^\mu(\tau) \Delta \lambda. \tag{12}\]

As both \( x_{(1,2)}(\tau) \) are solutions of the equations of motion, it follows that

\[
g_{\mu\nu}(x_1) u_1^\mu u_1^\nu = -1 = g_{\mu\nu}(x_2) u_2^\mu u_2^\nu \approx g_{\mu\nu}(x_1) u_1^\mu u_1^\nu + 2 \Delta \lambda g_{\mu\nu}(x_1) u_1^\mu \frac{Dn^\nu}{D\tau}. \tag{13}\]

As a result only three components of the rate of change of the deviation vector \( n^\mu(\tau) \) are independent.

The equation (8) for the world-line deviation of charged particles in general relativity, and per force the geodesic deviation equation (6), can be derived from a covariant action principle. The relevant lagrangean reads

\[
L = \frac{1}{2} g_{\mu\nu} \frac{Dn^\mu}{D\tau} \frac{Dn^\nu}{D\tau} + \frac{1}{2} R_{\mu\lambda\nu\sigma} u^\mu u^\lambda n^\nu n^\sigma + \frac{q}{2m} F_{\mu\nu} n^\mu \frac{Dn^\nu}{D\tau} + \frac{q}{2m} \nabla_\mu F_{\nu\lambda} u^\lambda n^\mu n^\nu. \tag{14}\]

After some manipulations involving the Bianchi identity for the electromagnetic field, the world-line deviation equation (8) is now obtained by requiring the action \( S = \int Ld\tau \) to be stationary under variations of the deviation vector \( n^\mu \):

\[
\frac{\delta S}{\delta n^\mu} = 0. \tag{15}\]

This action can serve as a starting point for a Hamiltonian formulation and for quantization.
3 Motion in PP-waves

3.1 Particle worldlines

The combined Einstein-Maxwell equations admit parallel plane-wave solutions, known as pp-waves\(^2\). The non-singular metrics of these solutions are represented by the line-element

\[
c^2d\tau^2 = -dudv - K(u, x, y) du^2 + dx^2 + dy^2,
\]

which reduces to flat Minkowski space-time when the metric uu-component vanishes: \(K(u, x, y) = 0\). As is implicit in (16), we choose to work with the light-cone co-ordinates

\[
u = ct - z, \quad v = ct + z.
\]

As the determinant of this metric is negative definite, it is manifestly invertible everywhere.

The solutions for the vector potential of the electro-magnetic field are of the form

\[A_\mu = \delta_\mu^x A_x(u) + \delta_\mu^y A_y(u).\]

The corresponding electric and magnetic fields are wave-like, propagating in the \(z\)-direction, transverse and orthogonal:

\[
\frac{E_i}{c} = -\epsilon_{ij} B_j = A'_i(u), \quad i, j = (x, y).
\]

Here the prime denotes a derivative with respect to \(u\). These fields satisfy the free Maxwell equations in a space-time with metric (16). The Einstein equations, with the energy-momentum tensor of the electro-magnetic field, reduce now to the single p.d.e. \([19, 11, 12]\)

\[
\Delta_{\text{trans}} K = \frac{16\pi\epsilon_0 G}{c^4} E_i^2.
\]

Here \(\Delta_{\text{trans}}\) is the ordinary 2-dimensional laplacian in the \(x-y\)-plane. The general solution for the metric component \(K\) is

\[
K(x, y, u) = \frac{4\pi\epsilon_0 G}{c^4} E_i^2 (x^2 + y^2) + f(u, \zeta) + \bar{f}(u, \bar{\zeta}),
\]

\(^2\)Some reviews can be found in [19, 20, 11, 12].
with \( f, \bar{f} \) conjugate holomorphic functions of the complex transverse co-ordinates

\[
\zeta = x + iy, \quad \bar{\zeta} = x - iy.
\]  

(22)

However, as constant or linear functions of \((\zeta, \bar{\zeta})\) do not give rise to a true gravitational field (the full Riemann tensor vanishes for them), we take \( f(u, \zeta) \) to be an analytic function of the type

\[
f(u, \zeta) = \sum_{n=2}^{\infty} \kappa_n(u)\zeta^n.
\]  

(23)

Because of its behaviour under rotations in the transverse plane we can identify the first term with \( n = 2 \) as free quadrupole gravitational waves. In contrast, the first term in (21), which is the special solution due to the presence of the electro-magnetic waves, is invariant under rotations in the transverse plane.

The equations of motion (7), adapted to this case, can be obtained from the world-line lagrangean

\[
L = \frac{1}{2} \left( -\dot{u}\dot{v} - K\dot{u}^2 + \dot{x}^2 + \dot{y}^2 \right) - \frac{q}{m} (A_x\dot{x} + A_y\dot{y}).
\]  

(24)

Three of the equations of motion can be written as

\[
\dot{u} = \text{constant} \equiv \gamma,
\]

\[
\ddot{x} = -\frac{\gamma^2}{2} K_x - \frac{q\gamma}{mc} E_x, \quad \ddot{y} = -\frac{\gamma^2}{2} K_y - \frac{q\gamma}{mc} E_y.
\]  

(25)

The fourth equation, for the light-cone co-ordinate \( v \), can be replaced by the conservation of the world-line hamiltonian

\[
H = -2p_v p_u + 2K p_v^2 + \frac{1}{2} \left( \Pi_x^2 + \Pi_y^2 \right),
\]  

(26)

where the (covariant) momenta are defined by

\[
p_v = -\frac{1}{2} \dot{u} = -\frac{\gamma}{2}, \quad p_u = -\frac{1}{2} \dot{v} - K\dot{u},
\]

\[
\Pi_x = p_x + \frac{q}{m} A_x = \dot{x}, \quad \Pi_y = p_y + \frac{q}{m} A_y = \dot{y}.
\]  

(27)
Here \((p_x, p_y)\) are the standard canonical momenta in the transverse plane, as used to define the Poisson brackets. It is convenient to use the first equation (25) to change the world-line parameter from proper time \(\tau\) to light-cone time \(u\). Furthermore, equation (26) can be solved for the velocity component \(\dot{v}\), or equivalently \(v' = \gamma \dot{v}\):

\[
v' = (x')^2 - K - \frac{2H}{\gamma^2},
\]

the last term being a constant of integration. In terms of the constants \(H\) and \(p_v\) the dynamical problem is now reduced to that of solving for the motion in the transverse plane. Consider the case of electro-magnetic and pure quadrupole free waves; writing \(\kappa_+ = 4 \Re \kappa_2, \kappa_- = -4 \Im \kappa_2\) the metric component is

\[
K = \frac{4\pi \epsilon_0 G}{c^4} E^2_i (u) (x^2 + y^2) + \frac{\kappa_-(u)}{2} (x^2 - y^2) + \kappa_+(u) xy \equiv \frac{1}{2} K_{ij} x^i x^j. \tag{29}
\]

This metric allows additional Killing vectors in the \(x\)-\(y\)-plane, making the dynamical problem fully solvable in terms of first integrals of motion. The additional constants of motion are given by

\[
J = J_x \Pi_x + J_y \Pi_y + 2p_v (J'_x x + J'_y y) + \Delta
\]

\[
= J_x \dot{x} - J_y \dot{y} + \Delta,
\]

where the co-efficient functions \(J_{(x,y)}(u)\) and \(\Delta\) are the solutions of the differential equations resulting from the requirement \(\frac{dJ}{d\tau} = 0\):

\[
J_x'' + \frac{1}{2} \left( \kappa_+ + \frac{8\pi \epsilon_0 G}{c^4} E^2_i \right) J_x + \frac{\kappa_-}{2} J_y = 0,
\]

\[
J_y'' + \frac{1}{2} \left( -\kappa_- + \frac{8\pi \epsilon_0 G}{c^4} E^2_i \right) J_y + \frac{\kappa_+}{2} J_x = 0, \tag{31}
\]

\[
\Delta' = \frac{q}{mc} (E_x J_x + E_y J_y).
\]

These equations can be summarized in the compact form

\[
J'_i + \frac{1}{2} K_{ij} J_j = J''_i - R_{\kappa_2 \kappa_2} J_j = 0, \quad \Delta' = \frac{q}{mc} E_i J_j = \frac{q}{m} F_{ui} J_j. \tag{32}
\]
They admit a two-parameter set of solutions, which implies the existence of
two linearly independent constants of motion fixing the values of the velocity
components \( \dot{x} \) and \( \dot{y} \).

This can be inferred from the trivial vanishing of the Wronskian \( W(f, g) = (f'g - fg') \) of any two solutions, \( J_{(1)} \) and \( J_{(2)} \), of (30), which reduces the original set of four constants of integration of the system (31) to two constants only.

### 3.2 World-line deviation

We now discuss the generalized world-line deviation equation (8) in the con-
text of the electro-magnetic and free quadrupole gravitational waves de-
scribed by the metric (16), (29) and the electro-magnetic vector potential (18).

The only non-vanishing components of the Riemann curvature tensor in
this case are

\[
R_{xuxu} = -\frac{1}{2} K_{xx} = -\frac{1}{2} \left( \kappa_+ + \frac{8\pi\varepsilon_0 G}{c^4} E_i^2 \right),
\]

\[
R_{yuyu} = -\frac{1}{2} K_{yy} = -\frac{1}{2} \left( -\kappa_+ + \frac{8\pi\varepsilon_0 G}{c^4} E_i^2 \right),
\]

\[
R_{xuyu} = R_{yuxu} = -\frac{1}{2} K_{xy} = -\frac{1}{2} \kappa_x,
\]

whilst the non-vanishing electro-magnetic field strength has two non-zero component

\[
F_{ux} = \frac{1}{c} E_x, \quad F_{uy} = \frac{1}{c} E_y.
\]

It is then straightforward to write down the explicit form of the world-line
deviation equations. For the \( u \) component of the deviation vector it reduces to

\[
\frac{D^2 n^u}{D\tau^2} = \ddot{n}^u = 0.
\]

Hence the rate of change of \( n^u \) is constant: \( \dot{n}^u = \beta = \text{constant} \), and

\[
n^u(\tau) = n^u(0) + \beta \tau.
\]
The world-line deviation in the transverse $x$-$y$-plane is described by the equations

$$\frac{D^2 n^i}{D\tau^2} = \ddot{n}^i + \frac{\gamma}{2} \frac{dK_{\delta}}{d\tau} n^\mu + \gamma K_{,i} n^\mu$$

$$= -\frac{\gamma^2}{2} K_{ij} n^j + \frac{\gamma}{2} K_{ij} \dot{x}^j n^u - \frac{q\gamma}{mc} E_i n^u - \frac{q}{mc} E_i \dot{n}^u.\quad (37)$$

This can be simplified, using $u$ as independent variable, to read

$$n_i'' = -\frac{1}{2} K_{ij} n_j - \frac{1}{2} (K_{,i} n^u)' - \frac{1}{2} K_{,i} (n^u)' + \frac{1}{2} K_{ij} x^j n^u - \frac{q}{mc\gamma} (E_i n^u)'\quad (38)$$

This is a system of coupled ordinary linear second order differential equations with field dependent coefficients. In spite of their complicated form, the equations for the transverse components $n_x$ and $n_y$ can be solved in terms of the constants of motion (30), (31). First define the linear combinations

$$N_i \equiv n_i - x'_i n^u + x_i (n^u)', \quad i = (x, y).\quad (39)$$

By substitution of the explicit form of the equations (38) we obtain

$$N_i'' = -\frac{1}{2} K_{ij} N_j.\quad (40)$$

We observe, that these equations are identical with eqs.(31), (32). Therefore they admit the solutions

$$N_x = \alpha J_x, \quad N_y = \alpha J_y,\quad (41)$$

where $\alpha$ is an arbitrary constant of proportionality; this constant serves to normalize the solutions for $n^\mu$, and can be absorbed into the parameter $\lambda$ measuring the deviation between the world lines. Together with the solution (36) for $n^u$ we now have a full solution for three of the four components $n^\mu$.

The last remaining component is the second light-cone component $n^v$.\quad (10)
which is subject to the equation
\[
\frac{1}{\gamma^2} \frac{D^2 n^v}{D\tau^2} = \left( (n^v)' + K'n^u + K_n n^i \right)' + K'(n^u)' + K_n n^i' + \frac{1}{2} K^2 n^u
\]
\[
= K_{ij} x^i' x^j n^u - K_{ij} x^i n^j - \frac{2q}{mc\gamma} E_i \left( n^i' + \frac{1}{2} K_{ij} x^j n^u \right) - \frac{2q}{mc\gamma} E_i x^i n^u.
\]

(42)

Upon substitution of the definitions (39) once more, using the deviation equations (35) and (40), and using the equations of motion (25) to eliminate the explicit dependence on the electromagnetic fields, this equation can be cast into the form
\[
\left[ (n^v)' - 2 (N_i x_i)' + \left( K - (x_i')^2 \right)' n^u - \left( K - (x_i')^2 \right) n^u' \right]' = 0.
\]

(43)

Finally equation (28) can be substituted, which together with equation (35) gives the final result
\[
[n^v + 2v(n^u)' - v'n^u - 2N_i x_i]'' = 0.
\]

(44)

It follows that the second light-cone component of the world-line deviation is given by
\[
n^v + 2v(n^u)' - v'n^u = 2\alpha J_i' x_i + A + Bu,
\]

(45)

with $A$ and $B$ constants of integration to be fixed by the initial conditions. Observe, that starting out with $n^u = (n^u)' = 0$, the right-hand side directly gives $n^v$ itself.

It is well known [20] that the $pp$-wave solution may be encoded in another frequently used co-ordinate system, in which the line element takes on the following form:
\[
ds^2 = du dv + g_{22}(u)(dx^2)^2 + g_{33}(u)(dx^3)^2,
\]

(46)

where the light-cone variables $u$ and $v$ are defined as before.

The space-time endowed with metric (11) admits Petrov’s $G_5$ group of isometries [18, 19]. The three Killing vectors, generating the Abelian subgroup of $G_5$ are defined as:
\[
\xi_{(V)}^\mu = \delta_1^\mu, \quad \xi_{(2)}^\mu = \delta_2^\mu, \quad \xi_{(3)}^\mu = \delta_3^\mu.
\]

(47)
The first Killing vector is covariantly constant, isotropic and orthogonal to second and third vectors in (17). The Lie derivative of the metric along any Killing vector is equal to zero: \( L_\xi(g_{\mu\nu}) = 0 \).

Using these symmetries, we can explicitly integrate the world-line equations, and then also obtain the explicit solutions for the deviation 4-vector in these co-ordinates. The details of these calculations can be found in ([23]).

4 Reissner-Nordstrøm fields

4.1 Worldlines and integrals of motion

The Reissner-Nordstrøm solution of the Einstein-Maxwell system of equations describes the external gravitational and electro-magnetic fields of a static and spherically symmetric distribution of mass and charge, or —in the case one takes it as a complete solution— a static charged black hole of mass \( M \) and charge \( Q \). The gravitational field is described by the line-element (in natural co-ordinates with \( c = 1 \))

\[
g_{\mu\nu}dx^\mu dx^\nu = -d\tau^2 = -B(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),
\]

with

\[
B(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \left(1 - \frac{M}{r}\right)^2 + \frac{Q^2 - M^2}{r^2}.
\]

The corresponding solution of Maxwell’s equations in this space-time is

\[
A = A_\mu dx^\mu = -\frac{Q}{4\pi r} dt, \quad F = dA = \frac{Q}{4\pi r^2} dr \wedge dt.
\]

Details of this solution of the field equations are discussed in standard text books, see e.g. [20]. In the following we assume \( M^2 > Q^2 \).

We immediately proceed to the solution of the world-line equations for a test particle of mass \( m \) and charge \( q \). As the spherical symmetry guarantees conservation of angular momentum, the particle orbits are always confined to an equatorial plane, which we choose to be the plane \( \theta = \pi/2 \). The angular momentum \( J \) is then directed along the \( z \)-axis. Denoting its magnitude per unit of mass by \( \ell = J/m \), we have

\[
\frac{d\phi}{d\tau} = \frac{\ell}{r^2}.
\]
In addition, as the metric is static outside the horizon \( r_+ = M + \sqrt{M^2 - Q^2} \), it allows a time-like Killing vector which guarantees the existence of a conserved world-line energy (per unit of mass \( m \)) \( \varepsilon \), such that

\[
\frac{dt}{d\tau} = \frac{\varepsilon - \frac{Q}{4\pi mr}}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}.
\]

(52)

Finally, the equation for the radial co-ordinate \( r \) can be integrated owing to the conservation of the world-line Hamiltonian, i.e. the conservation of the absolute four-velocity:

\[
\left( \frac{dr}{d\tau} \right)^2 = \left( \varepsilon - \frac{qQ}{4\pi mr} \right)^2 - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) \left( 1 + \frac{\ell^2}{r^2} \right).
\]

(53)

\( \ell \)From this we derive a simplified expression for the radial acceleration:

\[
\frac{d^2r}{d\tau^2} = \frac{qQ}{4\pi mr^2} \left( \varepsilon - \frac{qQ}{4\pi mr} \right) - \frac{1}{r^3} \left( Mr - Q^2 \right) + \frac{\ell^2}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right).
\]

(54)

Eq.(53) can in principle be integrated directly. However, to get directly an approximate solution to the equations of motion one can also study perturbations of special simple orbits. We follow both paths and compare the results. First, we find a solution to the equation for bound orbits in terms of an integral, from which we can compute e.g. the periastron shift, similar to the case of the Schwarzschild solution; secondly, we can solve the simple case of circular orbits, and then study the problem of generic bound orbits by considering the world-line deviation equations for the special case of world lines close to circular orbits. This is the subject of the final section of this paper.

For generic orbits, we first construct the orbital equation from the separate equations of motions for \( r(\tau) \) and \( \varphi(\tau) \), eqs.(51) and (53):

\[
\ell^2 \left( \frac{d^2 \varphi}{d\varphi r} \right)^2 = \varepsilon^2 - 1 + \frac{4\pi mM - \varepsilon qQ}{2\pi mr} - \frac{1}{r^2} \left( \ell^2 + Q^2 - \left( \frac{qQ}{4\pi m} \right)^2 \right) + \frac{2M\ell^2}{r^3} - \frac{\ell^2Q^2}{r^4}.
\]

(55)
We parametrize the solutions of this equation for $\ell \neq 0$ as quasi-Kepler orbits

$$r(\varphi) = \frac{r_0}{1 + e \cos y(\varphi)}, \quad (56)$$

with $e < 1$ for bound states. The function $y(\varphi)$, which is linear for Kepler orbits, here remains to be determined. As the extrema of the orbit (the apoastron and periastron) are reached for $y(\varphi) = (2n + 1)\pi$ (apoastron) and $y(\varphi) = 2\pi n$ (periastron) eq. (55) evaluated at these extrema give two independent equations relating $e$ and $r_0$ to the other parameters:

$$\varepsilon^2 - 1 = \frac{(4\pi m M - \varepsilon q Q)}{2\pi m r_0} + \frac{1}{r_0^2} \left( \ell^2 + Q^2 - \left( \frac{q Q}{4\pi m} \right)^2 \right) \left( 1 + e^2 \right)$$

$$- \frac{2M\ell^2}{r_0^3} \left( 1 + 3e^2 \right) + \frac{\ell^2Q^2}{r_0^4} \left( 1 + 6e^2 + e^4 \right),$$

and

$$\frac{(4\pi m M - \varepsilon q Q)}{2\pi m r_0} = \frac{2}{r_0^2} \left( \ell^2 + Q^2 - \left( \frac{q Q}{4\pi m} \right)^2 \right) - \frac{2M\ell^2}{r_0^3} \left( 3 + e^2 \right)$$

$$+ \frac{4\ell^2Q^2}{r_0^4} \left( 1 + e^2 \right).$$

With the help of these relations we find for the function $y(\varphi)$ the first-order differential equation

$$\left( \frac{dy}{d\varphi} \right)^2 = 1 + \frac{Q^2}{\ell^2} \left[ 1 - \left( \frac{q}{4\pi m} \right)^2 \right] - \frac{6M}{r_0} + \frac{Q^2}{r_0^2} \left( 6 + e^2 \right)$$

$$- \frac{2e}{r_0} \left( M - \frac{2Q^2}{r_0} \right) \cos y + \frac{e^2Q^2}{r_0^4} \cos^2 y.$$}

$$\equiv A + B \cos y + C \cos^2 y \quad (59)$$

It follows that the total change in the orbital angle $\varphi$ between two periastrons is given by

$$\Delta \varphi = \int_0^{2\pi} \frac{dy}{\sqrt{A + B \cos y + C \cos^2 y}}.$$  

$$14$$
As a result we obtain an expression for the periastron shift per one revolution:

\[ \delta \varphi = \Delta \varphi - 2\pi \approx 2\pi \left( \frac{3M}{r_0} - \frac{Q^2}{2Mr_0} \right) + \mathcal{O} \left( e^2, \frac{M^2}{r_0^2}, \frac{Q^2}{r_0^2} \right). \]  

(61)

### 4.2 World-line deviation near circular orbits

We observe that for circular orbits \( r = R = \text{constant} \), the expressions for \( dr/d\tau \), eq.(53), and \( d^2r/d\tau^2 \), eq.(54), must both vanish at all times. This produces two relations between the three dynamical quantities \( (R, \varepsilon, \ell) \), showing that the circular orbits are characterized completely by specifying either the radial co-ordinate, or the energy, or the angular momentum of the test particle. In particular, the equation for constant radial velocity gives:

\[ \left( \varepsilon - \frac{qQ}{4\pi mR} \right)^2 = \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right) \left( 1 + \frac{\ell^2}{R^2} \right). \]  

(62)

Substitution of this result into the expression for the radial acceleration, multiplying by \( R^4 \), and equating the result to zero, leads to:

\[
\left[ \frac{\ell^2}{R} - M \left( 1 + \frac{3\ell^2}{R^2} \right) + \frac{Q^2}{R} \left( 1 + \frac{2\ell^2}{R^2} \right) \right]^2
\]

\[ = \left( \frac{qQ}{4\pi m} \right)^2 \left( 1 + \frac{\ell^2}{R^2} \right) \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right). \]

(63)

Note that the right-hand side vanishes for chargeless test particles, i.e. on geodesics. In the limiting case \( Q = 0 \) we reproduce the well-known result

\[ MR^2 - \ell^2 (R - 3M) = 0 \implies R = \frac{\ell^2}{2M} \left( 1 + \sqrt{1 - \frac{12M^2}{\ell^2}} \right), \]

(64)

leading to the requirement \( R \geq 6M \) for stable circular orbits to exist 20.

Because of the spherical symmetry and the conservation of angular momentum all test-particle orbits are planar. Therefore the deviation of any bound orbit from a circular one can always be computed with reference to a circular orbit in the same plane, which may be chosen to be the equatorial plane \( \theta = \pi/2 \). For such world-line deviations the component of the deviation out of the plane is always zero: \( n^\theta(\tau) = 0 \).
On the other hand, one might also be interested in the motion of a cloud of test particles, as for example in disks of matter surrounding stars or planets, whose orbits are close but not necessarily precisely in the same plane. In such a case the deviation between the planes of the orbits, parametrized by \( n^\theta \), oscillates between positive and negative values.

In order to analyse the deviations quantitatively, we use the connection coefficients and curvature components as listed in the appendix, together with the expression (50) for the electro-magnetic field strength, to study the world-line deviation as given by eq. (8), with reference to a nearby circular orbit of radius \( r = R \). We begin with the equation for the component \( n^\theta \) out of the equatorial plane. Using the properties of the circular reference orbit:

\[
\begin{align*}
    r &= R = \text{constant}, \quad u^r = 0, \\
    \theta &= \frac{\pi}{2}, \quad u^\theta = 0,
\end{align*}
\]  

whilst \( F^\theta_{\nu} = 0 \), one finds from the deviation equation in the form (9)

\[
\ddot{n}^\theta + (u^\phi)^2 \partial_\phi \Gamma^\theta_{\phi\phi} n^\theta = \ddot{n}^\theta + \omega^2 n^\theta = 0,
\]

with

\[
\omega^2 = (u^\phi)^2 = \frac{\ell^2}{R^4}.
\]

We see that indeed either \( n^\theta = 0 \), or else \( n^\theta \) oscillates at frequency \( f = \omega/2\pi \), as the orbit is tilted with respect to the equatorial plane.

Next we solve for the deviations from the circular orbit within the plane. For the components \( (n^t, n^r, n^\phi) \) the deviation equations take the general form

\[
\ddot{n}^i + \gamma^i_{\ j} \dot{n}^j + m^i_j n^j = 0, \quad i, j = (t, r, \phi).
\]

The coefficients in these equations have the structure

\[
\gamma = \begin{pmatrix}
    0 & \gamma^t_r & 0 \\
    \gamma^t_t & 0 & \gamma^t_\phi \\
    0 & \gamma^r_\phi & 0
\end{pmatrix}, \quad
m = \begin{pmatrix}
    0 & 0 & 0 \\
    0 & m_r & 0 \\
    0 & 0 & 0
\end{pmatrix}.
\]

The precise form of the matrix elements follows from eq. (8) or (1); they will be given shortly. Now our treatment applies to orbits deviating little from a
circular orbit, which still describe bound states. Therefore \( r = R + \Delta r \), with \( \Delta r = n^r \Delta \lambda \), must remain bounded, e.g. we look for oscillating solutions of (68). From eqs. (68), (69) it then follows that with standard initial conditions (clocks and angles synchronized in the initial apastron: \( n^t(0) = n^\varphi(0) = 0, n^r(0) = \text{maximal} \), the solutions take the form

\[
\begin{align*}
n^t &= n^t_0 \sin \omega_1 \tau, \\
n^r &= n^r_0 \cos \omega_1 \tau, \\
n^\varphi &= n^\varphi_0 \sin \omega_1 \tau.
\end{align*}
\]

With this Ansatz, eq. (68) can be written more explicitly as

\[
T^i_j n^j_0 = 0,
\]

with the matrix \( T \) defined by

\[
T = \begin{pmatrix}
-\omega_1^2 & -\omega_1 \left( 2 \Gamma_{rt}^r u^t - \frac{q}{m} F_{rt}^t \right) & 0 \\
\omega_1 \left( 2 \Gamma_{tt}^r u^t - \frac{q}{m} F_{rt}^r \right) & \partial_t \Gamma_{tt}^r (u^t)^2 + \partial_r \Gamma_{\varphi\varphi}^r (u^\varphi)^2 & 2 \omega_1 \Gamma_{\varphi\varphi}^r u^\varphi \\
0 & -2 \omega_1 \Gamma_{r\varphi}^\varphi u^\varphi & -\omega_1^2
\end{pmatrix}.
\]

The solutions for the frequency \( \omega_1^2 \) follow by solving the characteristic equation \( \det T = 0 \):

\[
\omega_1^4 \left( C + \frac{q}{m} \alpha + \omega_1^2 \right) = 0,
\]

with the coefficients defined by

\[
C = (-\partial_t \Gamma_{tt}^r + 4 \Gamma_{rt}^r \Gamma_{tt}^r) (u^t)^2 + (-\partial_r \Gamma_{\varphi\varphi}^r + 4 \Gamma_{r\varphi}^r \Gamma_{\varphi\varphi}^r) (u^\varphi)^2,
\]

\[
\frac{q}{m} \alpha = \frac{q}{m} \left( F_{tr}^r - 4 \Gamma_{rt}^r F_{tr}^t \right) u^t + \frac{q^2}{m^2} F_{tr}^r F_{tr}^t.
\]

Disregarding the trivial solution \( \omega_1^2 = 0 \), which occurs with double degeneracy, the solution of (73) of interest to us is

\[
\omega_1^2 = - \left( C + \frac{q}{m} \alpha \right).
\]
We compute the values of $\alpha$ to zeroth, and $C$ to first order in $q/m$. The connection coefficients $\Gamma_{ij}^k$ have been collected in appendix (A.a). The electromagnetic field strength is given in eq.(50). Finally, eqs. (51) and (52), together with (62), imply for the velocities

$$\left(\begin{array}{c}
\epsilon^t \\
\epsilon^r
\end{array}\right) = \left(\begin{array}{c}
\frac{R^2 + \ell^2}{R^2 - 2MR + Q^2} \\
\frac{\ell^2}{MR - Q^2}
\end{array}\right) \left(1 + \frac{qQ}{4\pi mM} + \ldots\right),$$  

(76)

where we have neglected terms of higher order in the small parameters $(M/R, Q/M, q/m)$; and

$$\left(\begin{array}{c}
\omega^t \\
\omega^\phi
\end{array}\right) = \left(\begin{array}{c}
\frac{\ell^2}{R^4}
\end{array}\right).$$  

(77)

As a result of the various substitutions one obtains

$$C = -\omega^2 \left[1 - \frac{6M}{R} + \frac{Q^2}{MR} - \frac{2qQ}{4\pi mM} + \ldots\right],$$  

(78)

$$\frac{q}{m} \alpha = -\frac{2qQ}{4\pi mM} \omega^2 + \ldots$$

Observe that all terms of order $qQ/4\pi mM$ cancel. Indeed, deviations in the orbital period are not expected from the Coulomb interaction, but only from relativistic corrections to the Coulomb interaction, which are of higher order in $(M/R, Q/M, q/m)$.

The solution of the characteristic equation in this approximation thus becomes by

$$\omega_1^2 = \omega^2 \left(1 - \frac{6M}{R} + \frac{Q^2}{MR} + \ldots\right),$$  

(79)

which gives for the frequency

$$\omega_1 = \omega \left(1 - \frac{3M}{R} + \frac{Q^2}{2MR} + \ldots\right).$$  

(80)

To complete the solutions (70), eq.(71) establishes a relation between the components $n^\mu$, which for the solutions (77) becomes

$$\left(\epsilon - \frac{qQ}{4\pi mR}\right) n^t_0 - \ell n^r_0 = \frac{qQ}{4\pi m\omega_1 R^2} u^r n^r_0.$$  

(81)
This guarantees that the perturbed orbit is again a solution of the equations of motion. We can interpret this in terms of observable co-ordinate differences by writing $\Delta x^\mu = n^\mu \Delta \lambda$.

We finish with some physical observations relating to our solution (70) with frequency (80). First, note that there are different solutions $\omega_1$ for different values of the test charge $q$: there is the frequency $\omega_0$ for neutral test particles ($q = 0$), a larger one $\omega_+$ for test and central charge of equal sign, and a smaller one $\omega_-$ for test and central charge of opposite sign.

If we compare our approximate solution with the parametrized exact solution (56):

$$r(\varphi) \approx R + \Delta r \approx R - eR \cos y(\varphi),$$

(82)

we find, using the initial conditions in the apastron ($y = -\pi$), that to first order

$$e = \frac{(\Delta r)_0}{R} = \frac{n^r_0 \Delta \lambda}{R}.$$  

(83)

Thus our solution describes an approximate ellipse (Kepler orbit), with eccentricity $e$.

We can also calculate the precession of the periastron. Observe that as the orbit reaches its extremal radius, the shifts in orbital angle and time vanish: $n^r_{\text{extr}} = \pm n^r_0$ at $\omega_1 \tau = k\pi$, with $k$ integer; therefore $n^r_{\text{extr}} = n_{\text{extr}}^r = 0$. Now we can calculate the period between two periastra of the perturbed orbit:

$$T = \int_0^{2\pi/\omega_1} d\tau \frac{dt}{d\tau} = \int_0^{2\pi/\omega_1} d\tau \left( u^t + \dot{n}^t \Delta \lambda \right)$$

$$= \frac{2\pi}{\omega_1} u^t + \left[ n^r \Delta \lambda \right]_{0}^{2\pi/\omega_1} = \frac{2\pi}{\omega_1} u^t.$$ 

(84)

Here $u^t$ denotes the constant value (76) of $dt/d\tau$ for the circular reference orbit. At this time the values of the angular co-ordinate of the perturbed and circular reference orbits coincide. The angular direction of the periastron can therefore be calculated by calculating the angular co-ordinate of the circular orbit at time $T$. The angular shift of the periastron per period is then

$$\delta \varphi = \bar{\varphi}(\bar{t} = T) - \bar{\varphi}(0) - 2\pi,$$

(85)

where we have used overlines to denote the co-ordinates of the circular orbit.
Now as $u^\rho$ is conserved, we have

\[ \bar{\varphi}(T) - \bar{\varphi}(0) = \frac{\ell}{R^2} \frac{T}{u^t} = \frac{\omega T}{u^t}. \]  

(86)

Substitution of this result into eq. (85), using (80), gives

\[ \delta \varphi = 2\pi \left( \frac{\omega}{\omega_1} - 1 \right) \approx 2\pi \left( \frac{3M}{R} - \frac{Q^2}{2MR} + \ldots \right). \]  

(87)

This is in complete agreement with eq.(61).

5 Conclusion

The method of investigation of particle motions close to the exact solutions already known, can be used in other exact solutions of Einstein-Maxwell system. We believe that it is worthwhile to pursue the efforts in this direction, in particular because this method introduces the small parameter (the norm of the deviation vector) different from the small parameters used by many authors in search for approximate solutions of the theory, such as e.g. the ratio $M/R$ or $Q/R$. It is also well-suited for computer calculations.

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A Appendix: Connections and curvatures for Reissner-Nordstrøm geometry

In this appendix we collect the expressions for the components of the connections and Riemann curvature used in the main body of the paper.

a. Connections. From the line-element (48) one derives the following expressions for the connection co-efficients:

\[ \Gamma_{rt} = -\Gamma_{rt}^r = \frac{Mr - Q^2}{r (r^2 - 2Mr + Q^2)}, \]
\[ \Gamma_{tt}^r = \frac{1}{r^5} (Mr - Q^2) (r^2 - 2Mr + Q^2), \]
\[ \Gamma_{\varphi\varphi}^r = \sin^2 \theta \Gamma_{\theta\theta}^r = -\frac{\sin^2 \theta}{r} (r^2 - 2Mr + Q^2), \]
\[ \Gamma_{r\theta} = \Gamma_{r\varphi} = \frac{1}{r}, \]
\[ \Gamma_{\varphi\varphi}^\theta = \frac{\cos \theta}{\sin \theta}, \quad \Gamma_{\varphi\varphi}^\theta = -\sin \theta \cos \theta. \]

b. Curvature components. The corresponding curvature two-form components \( R_{\mu\nu} = \frac{1}{2} R_{\kappa\lambda\mu\nu} dx^\kappa \wedge dx^\lambda \) are:

\[ R_{tr} = \frac{1}{r^4} \left( 2Mr - 3Q^2 \right) dt \wedge dr, \]
\[ R_{t\theta} = -\frac{1}{r^4} \left( Mr - Q^2 \right) (r^2 - 2Mr + Q^2) dt \wedge d\theta, \]
\[ R_{t\varphi} = -\frac{1}{r^4} \left( Mr - Q^2 \right) \left( r^2 - 2Mr + Q^2 \right) \sin^2 \theta dt \wedge d\varphi. \]
and

\[ R_{r\theta} = \frac{Mr - Q^2}{r^2 - 2Mr + Q^2} dr \wedge d\theta, \]
\[ R_{r\varphi} = \frac{Mr - Q^2}{r^2 - 2Mr + Q^2} \sin^2 \theta dr \wedge d\varphi, \]
\[ R_{\theta\varphi} = -\left(2Mr - Q^2\right) \sin^2 \theta d\theta \wedge d\varphi. \] 

(90)

References

[1] A. Einstein, Ann. Physik 49 (1916), 769

[2] J.L. Synge, Relativity: the General Theory (North-Holland; Amsterdam, 1960).

[3] G.F.R. Ellis, H. van Elst, Deviation of geodesics in FLRW spacetime; gr - qc/ 9709060 (Sep. 1997)

[4] S. Mohanty, A.R. Prasanna, Geodesic deviation of photons in Einstein and higher derivative gravity; gr - qc/ 9701009 (v3, Jan. 1997)

[5] R. Steinbauer, Geodesics and geodesic deviation for impulsive gravitational waves; J. Math. Phys. vol.39, 1998; gr - qc/ 9710119 (v2, Mar. 1998)

[6] M. Kunzinger, R. Steinbauer, A rigorous solution concept for geodesic and geodesic deviation equations in impulsive gravitational waves; gr - qc/ 9806009 (June 1998)

[7] J.W. van Holten, Theory of a charged spinning particle in a gravitational and electromagnetic field; Proc. Sem. Math. Structures in Field Theories 1986-87, CWI Syllabus vol. 26 (1990), 109 ; see also : On the electrodynamics of spinning particles; Nucl. Phys. B 356 (1991), 3

[8] I.B. Kriphlovich, JETP 69 (1989), 217

[9] R.H. Rietdijk and J.W. van Holten, Spinning particles in Schwarzschild space-time; Class. Quantum Grav. 10 (1993), 575
[10] M.D. Roberts, *The string deviation equation*; (gr - qc/ 9810043; Oct. 1998)

[11] J.W. van Holten, *Gravitational waves and black holes*; Fortschr. Phys. 45 (1997), 6

[12] J.W. van Holten, *Gravitational waves and massless particle fields*, in: *Towards Quantum Gravity*, Proc. XXXVth Karpacz School of Theoretical Physics, Polanica 1999 (Springer; Berlin, 2000); gr-qc/9906026

[13] R. Kerner, Ann. Inst. H. Poincaré , 9, 147 (1968); *ibid* 24, 431 (1981)

[14] C. Wong, Nuovo Cimento, 65 A (4), 689 (1970)

[15] R. Kerner, J. Martin, S. Mignemi, J.W. van Holten, *Geodesic deviation in Kaluza-Klein theories*, Preprint LGCR

[16] A.N. Aliev, D.V. Gal’tsov, *Radiation from Relativistic Particles in Non-geodesic Motion in a Strong Gravitational Field*, GRG-Journal 13, n0 10, 387 (1981)

[17] A. Balakin, J.W. van Holten and R. Kerner, in preparation

[18] A.Z. Petrov, *Einstein Spaces* (Pergamon Press; Oxford, 1969).

[19] D. Kramer, H. Stephani, M. MacCallum, E. Herlt, *Exact Solutions of Einstein’s Field Equations* (Berlin, 1980).

[20] C.W. Misner, K.S.Thorne, J.A. Wheeler, *Gravitation* (W. Freeman; San-Francisco, 1970).

[21] S. Weinberg, *Gravitation and Cosmology* (J. Wiley; New York, 1972).

[22] A. Balakin, unpublished (to appear as a preprint LGCR).