Decoherence due to the horizon after inflation

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Abstract. The fluctuations in the inflaton field at the end of inflation which seed the density perturbations are prepared in a pure quantum state. It is generally assumed that some physics causes this pure state to decohere so that it should be treated probabilistically. We show that the entanglement entropy between the universe inside our observable horizon and that outside our horizon is sufficient to do this. For the modes which are super-Hubble at the end of inflation, this entanglement entropy grows with volume inside the horizon, rather than with the horizon’s area, and is proportional to the number of e-folds since Hubble crossing.

Keywords: cosmological perturbation theory, inflation, physics of the early universe

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1. Introduction

Inflation provides an elegant description for the large scale appearance of our universe (very large, approximately homogeneous and isothermic, and spatially flat to within measurement errors) [1]–[3]. It also provides a mechanism to explain how the density perturbations which seeded structure formation (and therefore us) came to be (for a comprehensive review see [4]). This occurs because, during inflation, the quantum state of (the approximately harmonic oscillator $k$-modes of) the inflaton field become highly squeezed [5].

This squeezing, present on super-Hubble scales at the end of inflation, means that the inflaton is effectively taking on different values at widely separated points in space, which leads to an inhomogeneity in the temperature after reheating (or certain non-zero metric elements in a different gauge choice).

One puzzle in this picture is that the state of the inflaton field is a quantum state. It is not really correct to say that it is ‘ahead’ at some points and ‘behind’ in others; its quantum state contains amplitudes to be ahead and behind at different points. Usually in treating the subsequent evolution of the system one replaces this quantum state with a classical probability distribution or ensemble with the same statistical properties. This artificially introduces a large entropy into the state, since a pure quantum state is being replaced with a statistical ensemble. It is an interesting question whether this procedure is really legitimate, and if so how it is justified.

This issue has been mostly resolved by Polarski et al., who showed [6] that the difference between performing calculations using the quantum squeezed states and the classical probabilistic distribution is exponentially suppressed in the number of e-folds of inflation. However it is still an interesting question whether the statistical treatment is in some way better justified than this. Are we actually compelled to consider the primordial fluctuations probabilistically?
Burgess et al have argued that decoherence of the quantum state actually occurs due to interactions between the long wavelength modes of the inflaton fields and shorter wavelength modes [7]. Assuming that these shorter wavelength modes are not observed, the description of the long wavelength modes should be made by integrating them out, resulting in an entanglement entropy. The calculation depends on the details of the interactions between long and short wavelength modes and on the non-observation of the short wavelength modes.

We believe that it is unnecessary to consider this division of modes into long and short wavelength to show that the universe we observe will possess a large entanglement entropy and must be described statistically. This is because the post-inflationary universe has a horizon. We can only make observations of our universe within our horizon patch. But inflation sets up correlations between our causal patch and the universe ‘outside’ that patch. Indeed, the basis which diagonalizes the wavefunctional at the end of inflation is the momentum basis, which is maximally non-local. Since it is impossible to observe the universe outside our horizon, we are obliged to integrate it out. The goal of this paper is to investigate the degree of entanglement with the universe outside our horizon. In particular we will compute the entropy of entanglement.

Our method is based on that used by Srednicki [8], who considered the entanglement entropy associated with cutting a spherical region out of space, as a toy model of a black hole’s entropy. We extend his treatment to highly squeezed quantum states. Our main conclusion is that, for the highly squeezed case, the entanglement entropy grows with the volume of the visible universe, rather than with the surface of the boundary between the observed and unobserved regions, as occurs for vacuum fluctuations. The entropy per mode scales with the log of the conformal time, which is the same as the number of e-foldings of inflation. This saturates a bound placed by Kiefer et al [9] on the entropy per mode which is allowed without destroying the phase information in the fluctuations, responsible for the acoustic peaks. Technically this conclusion applies only for the very longest wavelengths; if inflation took \( n \) e-folds to stretch our observed universe out of one Hubble patch, then the volume behavior we find is valid for fluctuations with wavelength more than \( 1/n \) of the current horizon length. (This volume dependence is consistent with the entropy conjecture of Bousso [10] because it applies only to the few infrared modes which were super-Hubble at the end of inflation and so became highly squeezed. The horizon area behavior argued by Bousso is dominated by the much more numerous shorter wavelength modes.)

The entropy per mode which we compute is an inevitable result of the horizon structure of the post-inflationary universe and does not rely on any assumptions about the interactions of the inflaton field with other (short wavelength) degrees of freedom. It is therefore robust, and sufficient to explain the statistical nature of the universe we observe.

2. Squeezed states

A squeezed harmonic oscillator state is generated from the vacuum state by a squeezing operator, which is a function of two variables, the amplitude of squeezing \( r \) and the phase \( \Phi \). The squeezing operator for a momentum space mode of wave vector \( k \) of a quantum field is generically given by equation (2), below.
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During inflation, the state of a non-interacting, light scalar field can be described by the quantum state of each momentum space state. The cosmological expansion causes the momentum space states with wavelengths longer than the Hubble scale become highly squeezed, in a manner which has been worked out by Albrecht et al [5]. They find that the evolution equations for \( r_k \) and \( \Phi_k \) in terms of proper time \( \eta \) are

\[
U = S(r_k, \theta_k)R(\Theta_k),
\]

\[
S(r_k, \Phi_k) = \exp\left[\frac{r_k}{2} (a^\dagger_k a_k e^{-2\Phi} - a^\dagger_k a^\dagger_k e^{2\Phi})\right],
\]

\[
R(\Theta_k) = \exp[-i\Theta_k(a^\dagger_k a_k + a^\dagger_k a_{-k})],
\]

\[
r_k = -\sinh^{-1}\left(\frac{1}{2k|\eta|}\right),
\]

\[
\Phi_k = \left(-\frac{\pi}{4} + \frac{1}{2} \arctan \frac{1}{2k|\eta|}\right),
\]

\[
\Theta_k = -k|\eta| - \arctan \frac{1}{2k|\eta|}.
\]

Here \( U \) is the time evolution operator of the inflaton and \( R(\Theta_k) \) is an overall phase. Note that these solutions are for the exponentially expanding de Sitter stage, in the Bunch–Davies vacuum.

Evaluating the state of this \( k \)-mode in the field basis gives [11]

\[
\Psi[\phi(k, \eta), \phi(-k, \eta)] = \langle \phi(k, \eta), \phi(-k, \eta)|S|0\rangle = N_k \exp(-|\phi(k, \eta)|^2 \Omega_k)
\]

\[
\Omega_k = k \frac{1 + i\sin(2\Phi_k) \sinh(2r_k)}{\cosh(2r_k) - \cos(2\Phi_k) \sinh(2r_k)}
\]

\[
= k \left(\frac{k^2|\eta|^2 + i k|\eta|}{1 + k^2|\eta|^2}\right).
\]

Here \( \phi(k, \eta) \) is the scalar field, called \( y(k, \eta) \) in [11]. \( N_k \) is a normalizing constant; we will not keep track of such normalizations, ensuring the correct normalization at the end by scaling the density matrix to have trace 1. The fact that the wavefunction can be put in this simple, Gaussian form means that we can adapt the approach [8] used to calculate the entropy of a sphere of space to calculating the entropy of the squeezed states.

3. Integrating out position space modes

As discussed, we want to compute the entropy of entanglement of the visible part of the universe (that which lies within our horizon) when the invisible part (space outside our horizon) is integrated out. We are only interested in the behavior and entropy of relatively long wavelength modes, those responsible for the density perturbations on scales which are linear today or can be probed at a timescale when they were linear. At the end of inflation and in comoving coordinates, these modes had wavelengths much larger than
the Hubble scale and so were all highly squeezed. To describe only these long wavelength modes, we should implement some infrared regulator, which for convenience we take to be a lattice discretization of space. (We do not think that the choice of regulator is important to our results.) We will also make our system finite by imposing boundaries far outside the observable horizon. We will seek the limit where the boundaries are moved to infinity and will study the dependence on the infrared regulation. As a further simplification we consider a three-dimensional system where the visible universe is a cube rather than a sphere; again the qualitative behavior should be the same as for the spherical case. As a test case we also consider a one-dimensional system.

This effectively reduces the problem to a quantum mechanical problem of the product of a large number of harmonic oscillators. We will find the wavefunction and from it the density matrix, which is most easily done in the momentum basis. Then we will transform to the position basis and integrate over the coordinates outside the horizon. The procedure is similar to that of Srednicki [8], except that we consider highly squeezed SHO states, rather than vacuum states.

In one dimension, the discretized box Hamiltonian for a free scalar field is

$$H = \sum_{j=1}^{N} \left[ -\frac{\hbar^2}{l^2} \phi(x_j) \phi(x_{j+1}) - 2\phi(x_j) + \phi(x_{j-1}) + \pi_j^2 \right]$$

$$= \frac{1}{2} \pi^\dagger \pi + \frac{1}{2} \phi(x)^\dagger M \phi(x),$$

(9)

where $l$ is a lattice spacing, $Nl$ is the extent of the universe, and $\phi(x_j)$ and $\pi_j$ are the $N$ generalized coordinates and conjugate momenta. Here $(\phi_{j+1} - 2\phi_j + \phi_{j-1})$ is a discrete approximation of $l^2 \nabla^2 \phi$, and we switch to matrix notation in the last expression. We take $\phi(0) = 0 = \phi(N+1)$, which enforces Dirichlet boundary conditions. This choice of boundary conditions should be immaterial if we take an appropriate large volume limit. We have tested this by using mixed Dirichlet–Neumann and antiperiodic boundary conditions.

The matrix $M$ is diagonalized by a sine transform. Explicitly, defining the matrix $U$ as

$$U_{jk} = \sqrt{\frac{2}{N+1}} \sin \left( \frac{jk\pi}{N+1} \right),$$

(10)

the matrix $M$ becomes $M = U^\dagger \omega^2 U$, with

$$\omega = \text{Diag}[\omega_j], \quad \omega_j^2 = \frac{4}{l^2} \sin^2 \left( \frac{\pi j}{2(N+1)} \right).$$

(11)

The wavefunction for the scalar field is easiest to express in this momentum basis. Defining $\phi_j(k) = U_{kj} \phi_i(x)$, then up to a normalization constant, the squeezed state in the $k$ space field basis is

$$\langle \phi | S \rangle = \prod_{k=1}^{N} \langle \phi_k | S_k \rangle$$

$$= \prod_{k=1}^{N} \exp(-\Omega_k \phi_k^2)$$

$$= \exp(-\phi(k)^\dagger \Omega \phi(k)),$$

(12)
where $\phi(k)$ is a vector of $Nk$ modes and $\Omega$ is the diagonal matrix with elements $\Omega_k$ given by equation (8) with $\omega_k$ replacing $k$.

We can express this in the position basis using the unitary transformation matrix $U$ found earlier. Defining $\mathcal{N} = U^\dagger \Omega U$, the state in position basis is

$$\langle \phi | S \rangle = \exp -\phi(x)^\dagger \mathcal{N} \phi(x),$$

(13)

giving the density matrix

$$\rho(\phi(x), \phi(x')^\dagger) = \langle \phi | S \rangle \langle S | \phi' \rangle = \exp - (\phi(x)^\dagger \mathcal{N} \phi(x) + \phi(x')^\dagger \mathcal{N}^\dagger \phi(x')).$$

(14)

With the density matrix in this form it is now possible to calculate the partial trace of the physical modes outside the visible universe using an approach similar to that of [8]. However before continuing it is important to note that the method used there must be altered slightly because $\mathcal{N}$ is not Hermitian, that is, $\Omega$ is not real.

Now we subdivide the $N$ lattice points into $m$ points making up the visible universe and $n = N - m$ modes lying outside our horizon, which we therefore want to integrate out. In the 1D case the $m$ modes are taken to be bounded on each side by $n/2$ modes, and in the 3D case the visible universe is taken to be the central modes of an $N \times N \times N$ cube.

We should therefore write the matrices $U$ and $\Omega$ in block form as follows:

$$U = \begin{pmatrix} U_1 & U_3 \\ U_4 & U_2 \end{pmatrix},$$

(15)

$$\Omega = \begin{pmatrix} \Omega_n & 0 \\ 0 & \Omega_m \end{pmatrix}.$$

(16)

From here it is straightforward to see that

$$\phi^\dagger \mathcal{N} \phi = [y^\dagger \ x^\dagger] \begin{bmatrix} A_R + iA_c & C_R + iC_c \\ C_R^\dagger + iC_c^\dagger & B_R + iB_c \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix},$$

(17)

where

$$A_R = U_1^\dagger \text{Re}[\Omega_n]U_1 + U_3^\dagger \text{Re}[\Omega_m]U_4$$

(18)

$$A_c = U_1^\dagger \text{Im}[\Omega_n]U_1 + U_3^\dagger \text{Im}[\Omega_m]U_4$$

(19)

$$C_R = U_1^\dagger \text{Re}[\Omega_n]U_4 + U_3^\dagger \text{Re}[\Omega_m]U_2$$

(20)

$$C_c = U_1^\dagger \text{Im}[\Omega_n]U_4 + U_3^\dagger \text{Im}[\Omega_m]U_2$$

(21)

$$B_R = U_1^\dagger \text{Re}[\Omega_n]U_3 + U_2^\dagger \text{Re}[\Omega_m]U_2$$

(22)

$$B_c = U_3^\dagger \text{Im}[\Omega_n]U_3 + U_2^\dagger \text{Im}[\Omega_m]U_2$$

(23)

$$B = B_R + iB_c.$$  

(24)

Note that $A_{R,c}, B_{R,c}$ are all Hermitian matrices and we have split the $N$ modes into a $y$ vector of $n$ modes and an $x$ vector of $m$ modes. We will be tracing over the $y$ modes.
The partial trace of a density matrix can be found by integrating over the appropriate modes; that is,

$$
\rho_{\text{proj}}(x; x') = \int_{-\infty}^{\infty} dy \rho(x, y; x', y).
$$

Replacing \( y' \) with \( y \), the density matrix to be integrated is

$$
\rho = \exp \left[ - 2y^\dagger A_R y + y^\dagger [C_R (x + x') + i C_c (x - x')] + [(x + x')^\dagger C_R^\dagger + i(x - x')^\dagger C_c^\dagger]y 
+ x^\dagger B x + x'^\dagger B^\dagger x' \right].
$$

The Hermitian matrix \( A_R \) can be written as a unitary transformation of a diagonal matrix,

$$
A_R \equiv V^\dagger \lambda V.
$$

After completing the square for \( y \) and carrying out the Gaussian \( y \) integration, the density matrix becomes

$$
\rho_{\text{proj}} = \exp \left[ - \left( x^\dagger (K - J - i(G + G^\dagger)) + B) x + x'^\dagger (K - J + i(G + G^\dagger)) + B^\dagger) x' 
- x^\dagger (K + J + i(G - G^\dagger)) x' - x'^\dagger (K + J - i(G - G^\dagger)) x \right) 
\right]
\equiv \exp \left[ - \left( x^\dagger \Gamma_1 x + x'^\dagger \Gamma_2 x' - x^\dagger \delta_1 x' - x'^\dagger \delta_2 x \right) \right],
$$

where

$$
G \equiv C_c^\dagger V^\dagger \frac{\lambda^{-1}}{2} V C_R
$$

$$
K \equiv [V C_c + (V C_c)^*]^\dagger \frac{\lambda^{-1}}{8} [V C_c + (V C_c)^*]
$$

$$
J \equiv C_R^\dagger V^\dagger \frac{\lambda^{-1}}{2} V C_R
$$

$$
\Gamma \equiv K - J - i(G + G^\dagger) + B
$$

$$
\delta_1 \equiv K + J + i(G - G^\dagger)
$$

$$
\delta_2 \equiv K + J - i(G - G^\dagger).
$$

In the case where \( \Omega \) is purely real, this reduces to the result of [8], a result recently derived in the black hole context [12].

4. Entropy, eigenvalues, and eigenvectors

4.1. Diagonalization of \( \rho \) into normal modes

We now have the density matrix for the visible part of the universe. The question is, can this be represented by a stochastic distribution, or is its quantum nature still observable? To determine this we calculate the amount of decoherence of the density matrix by computing its entropy. To do that we need to find the eigenvalues and if possible the eigenvectors of the density matrix.
Since $\delta_1$ and $\delta_2$ are both Hermitian, we can further simplify the mixed term in the density matrix:

$$x^\dagger \delta_1 x' + x^\dagger \delta_2 x = z^\dagger \lambda_{\delta_1} z' + z^\dagger Q \delta_2 Q^\dagger z$$
$$= z^\dagger \lambda_{\delta_1} z + z^\dagger Q \delta_2 Q^\dagger z = x^\dagger (\delta_1 + \delta_2) x$$
$$= 2x^\dagger (K + J) x,$$  \hfill (34)

where $Q^\dagger \delta_1 Q = \lambda_{\delta_1}$ is diagonal, and $x = Q z$.

Now we split $\Gamma$ into its Hermitian and anti-Hermitian parts:

$$\Gamma = A + iD,$$  \hfill (35)
$$A = K - J + B_R,$$  \hfill (36)
$$D = -G - G^\dagger + B_z.$$  \hfill (37)

This gives us a density matrix of the form:

$$\rho_{\text{proj}} = \exp - \left[ (x^\dagger Ax + x^\dagger A x') + i(x^\dagger D x - x^\dagger D x') - 2x^\dagger (A - (B_r - 2J)) x' \right].$$  \hfill (38)

We know that $A$ is positive definite since the density matrix is normalizable. Diagonalizing $A = W A W^\dagger$ and dividing by $\sqrt{\lambda}$ gives:

$$\rho_{\text{proj}} = \exp - \left[ (u^\dagger u + u'^\dagger u') + i(u^\dagger D' u - u'^\dagger D' u') - 2u^\dagger (1 - \Lambda^{-1/2} W^\dagger (B_R - 2J) W \Lambda^{-1/2}) u' \right],$$  \hfill (39)

where $x = \Lambda^{1/2} W u$. Diagonalizing $\Lambda^{-1/2} W^\dagger (B_R - 2J) W \Lambda^{-1/2}$ gives:

$$\rho_{\text{proj}} = \left( \prod_{j=1}^m \sqrt{\epsilon_j \pi} \exp - \left[ (v_j^2 + v_j'^2) - 2v_j (1 - \epsilon_j) v_j' \right] \right) \exp (\pi i (D'' v - v'^\dagger D'' v'))$$
$$\equiv \left( \prod_{j=1}^m \rho_j(v_j; v_j') \right) \exp (\pi i (D'' v - v'^\dagger D'' v')),$$  \hfill (40)

where $v$ is in the new diagonal basis, and $\epsilon_j$ are the eigenvalues of $\Lambda^{-1/2} W^\dagger (B_R - 2J) W \Lambda^{-1/2}$. The second equation defines $\rho_j(v_j; v_j') \equiv \sqrt{(\epsilon_j/\pi)} \exp (\pi i (v_j^2 + v_j'^2) - 2v_j (1 - \epsilon_j) v_j')$.

### 4.2. Proof of unimportance of complex phase

Besides the $D''$ term, this density matrix in equation (40) is in the same form as found in [8]. We now show that the $D''$ term does not change the eigenvalues and only modifies the eigenvectors in a trivial way.

Write the eigenfunctions of $\rho(v_i; v'_i)$ as $f_{n(i);i}$, satisfying

$$\int_{-\infty}^{\infty} \rho_i(v_i; v'_i) f_{n(i);i}(v'_i) dv'_i = p_{n(i);i} f_{n(i);i}(v_i),$$  \hfill (41)

with $p_{n(i);i}$ the associated eigenvalue. Then the eigenvectors and eigenvalues of our density...
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The density matrix are a $D$ dependent phase factor times products of these, namely

$$g_{n(1),n(2),\ldots,n(m)}(v) = e^{-i\nu^1 D'' v} \prod_{j=1}^m f_{n(j),j}(v_j),$$

$$p_{n(1),n(2),\ldots,n(m)} = \prod_{j=1}^m p_{n(j),j}.$$  \hspace{1cm} (42)

This is easy to check:

$$\int_{-\infty}^{\infty} dv' \rho_{proj}(v,v') g(v') = e^{-i\nu^1 D'' v} \prod_{j=1}^m f_{n(j),j}(v_j)$$

$$= \exp(-i\nu^1 D'' v) \int_{-\infty}^{\infty} dv' \left( \prod_{j=1}^m \rho_j(v_j;v'_j) f_{n(j),j}(v'_j) \right)$$

$$= \exp(-i\nu^1 D'' v) \left( \prod_{j=1}^m p_{n(j),j} f_{n(j),j}(v_j) \right)$$

$$= p_{n(1),n(2),\ldots,n(m)} g(v).$$ \hspace{1cm} (44)

Therefore the complex phase does not actually affect the eigenvalues of the density matrix, and only adds an overall phase to the eigenvectors. However this complex phase will be important when taking further partial traces.

4.3. Entropy per mode

It was shown in \cite{8} that the density matrix $\rho_j(v_j;v'_j)$ found in equation (40) has eigenvectors and eigenvalues of

$$f_n(v_j) = \frac{1}{\sqrt{\alpha n!2^n \sqrt{\pi}}} H_n \left( \alpha^{-1/2} v_j \right) \exp \left( -\frac{\alpha}{2} v_j^2 \right),$$

$$p_n = (1-\xi)\xi^n,$$ \hspace{1cm} (45)

$$\alpha = \sqrt{2\epsilon - \epsilon^2},$$ \hspace{1cm} (46)

$$\xi = \frac{1-\epsilon}{1+\alpha}. \hspace{1cm} (47)$$

In other words, up to the phase factor which we just showed is irrelevant, there exists a basis in which the density matrix behaves like the product of density matrices for individual harmonic oscillators, with Poisson distributed occupancies of states which are the same as the conventional harmonic oscillator number states.

The entropy is the sum of the entropies for the normal mode density matrices. The entropy of the Poisson distribution described by $p_n = (1-\xi)\xi^n$ is

$$s_\xi = -\log(1-\xi) - \frac{\xi}{1-\xi} \log(\xi).$$ \hspace{1cm} (49)
As we will see, we will be interested in the limit where $\epsilon_j \ll 1$, in which case $\xi \simeq 1 - \frac{1}{2} \log(2\epsilon_j)$.

4.4. Large squeezing limit

For large squeezing, $\Omega_k \simeq k(k^2\eta^2 + ik|\eta|)$, which is small and dominated by the imaginary part. The overall scale of $\Omega_k$ is immaterial; multiplying all $\Omega_k$ by a common factor scales out in determining the $\epsilon_j$ and therefore the entropy. However, the $k$ dependence and the relative size of the real and imaginary parts is relevant. Since $k\eta$ is exponentially small in the number of e-foldings, we should study the scaling of the eigenvalues $\epsilon_j$ in the small $k|\eta|$ limit.

Ultimately the entropy depends on the sizes of the eigenvalues $\epsilon_j$ of the matrix $\Lambda^{-1/2}W^\dagger(B_R - 2J)W\Lambda^{-1/2}$ in equation (40). Therefore we should study the scaling behavior in $\eta$ of the pieces of this matrix. It is convenient to multiply $\Omega_k$ by $l/\eta$, with $l$ the lattice spacing (shortest wavelength under consideration), so

$$\Omega_k(\eta) \rightarrow k \left((lk)^2 \left(\frac{|\eta|}{l}\right) + i(lk)\right).$$

In the small $|\eta|/l$ limit, provided that the eigenvalues of $K$ are all non-zero, one may neglect $B_R - J$ in equation (36). In this case the $\eta$ dependence scales simply; the matrix $\Lambda^{-1/2}W^\dagger(B_R - 2J)W\Lambda^{-1/2}$, and therefore the $\epsilon_j$, scale as $|\eta|/l^2$. Defining $\epsilon(1)_j$ as the set of eigenvalues arising by setting $|\eta|/l = 1$ and setting $A = K$ in equation (36), the approximate values of the $\epsilon_j$ are

$$\epsilon_j = \frac{\eta^2}{l^2}\epsilon(1)_j.$$  

Therefore the entropy is approximately

$$S \simeq \sum_j \ln \frac{l}{|\eta|} + 1 - \frac{1}{2}\ln 2 - \frac{1}{2}\ln \epsilon(1)_j.$$  

This demonstrates that, for $\eta/l$ parametrically small (modes which were well outside the horizon at the end of inflation), the entropy is extensive in the number of modes and scales as the number of e-folds by which the mode was outside the horizon (the log of the level of squeezing).

Note however that this conclusion relies on the $\epsilon(1)_j$ not being exponentially large. Some of these eigenvalues could potentially be exponentially large if there is another parametrically large number in the game, such as the number $m$ of points inside the horizon. Therefore our conclusion that the entropy scales as the volume and as $-\ln |k\eta|$ is only robust if we consider fluctuations on very long scales, $m < -\ln |\eta/l|$.

We have been unable to find a closed-form solution for the eigenvalues $\epsilon(1)_j$. Instead we have performed numerical experiments to check how the residual entropy in equation (53) scales with volume. Recall that the choice of the number of points inside the visible universe corresponds to a selection of what scales of perturbations to consider, while we should seek the limit that the volume outside the visible universe becomes large.
We studied the entropy in both the one-dimensional and three-dimensional cases. The matrix manipulations rapidly become numerically unstable and were only able to achieve a large value of $m$ for the one-dimensional case. For small $m$ (an analysis only considering modes of near-horizon wavelength), we find as expected that all eigenvalues $\epsilon(1)_j$ are $O(1)$ and therefore the entropy grows with the number of modes and scales with the number of e-folds. When we increase $m$, we observe that the mean value $(1/m)\sum_{j=1...m} \ln \epsilon(1)_j$ grows roughly linearly with $m$; the largest eigenvector $v_j$ is one which avoids the boundary. This suggests (though we have not been able to show it analytically) that the volume behavior we obtain will break down when one considers wavelengths shorter than the current horizon by more than the number of e-foldings of inflation.

Return to the longest wavelength modes, where we have shown that the entropy per mode is $\approx n_{\text{efolds}}$. This is exactly what Kiefer et al [9] predicted to be the upper bound on the entropy of the modes. They argue that if the entropy were any higher, then the experimentally observed acoustic oscillations of the CMB fluctuations would not be seen. Therefore, so far as the modes which can actually be observed are concerned, the decoherence arising from entanglement with the (unobservable) region outside our horizon is sufficient to force a probabilistic description of the density fluctuations, regardless of whether or not there are additional mechanisms for decoherence.

5. Conclusion

The post-inflationary causal structure of the universe has a horizon, meaning that we can only observe physics in some finite domain of the universe. However, inflation induces a large entanglement between the IR field modes in this region and those outside our horizon. We have calculated the entanglement entropy for the IR modes responsible for density perturbation seeds. The entropy scales with the number of modes in the visible universe, that is, with the volume of the universe times the cube of the $k$-vector at which density perturbations are measured. The entropy per mode obeys the simple rule $s \approx r$, where $r$ is the squeezing factor (the number of e-foldings of inflation during which the relevant mode was outside the Hubble radius, of order 60). This far exceeds the cosmic variance limit on the amount of information available in the density perturbations. In fact, it saturates the upper bound predicted by Kiefer et al [9]. This result holds for the longest wavelength modes, those with $H_0 \lambda > 1/r$, with $H_0$ the current Hubble constant (roughly the inverse of the horizon length).

This entanglement entropy is large enough to render unmeasurable any quantum coherence in the initial squeezed state set up by inflation and to justify a probabilistic description of the initial seeds of the density fluctuations.

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