Global Well-Posedness and Non-linear Stability of Periodic Traveling Waves for a Schrödinger-Benjamin-Ono System

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Abstract

The objective of this paper is two-fold: firstly, we develop a local and global (in time) well-posedness theory for a system describing the motion of two fluids with different densities under capillary-gravity waves in a deep water flow (namely, a Schrödinger-Benjamin-Ono system) for low-regularity initial data in both periodic and continuous cases; secondly, a family of new periodic traveling waves for the Schrödinger-Benjamin-Ono system is given: by fixing a minimal period we obtain, via the implicit function theorem, a smooth branch of periodic solutions bifurcating a Jacobian elliptic function called dnoidal, and, moreover, we prove that all these periodic traveling waves are nonlinearly stable by perturbations with the same wavelength.

1 Introduction

In this paper we are interested in the study of the following Schrödinger-Benjamin-Ono (SBO) system

\[
\begin{align*}
    iu_t + u_{xx} &= \alpha vu, \\
    v_t + \gamma D v_x &= \beta (|u|^2)_x,
\end{align*}
\]

where $u$ is a complex-valued function, $v$ is a real-valued function, $t \in \mathbb{R}$, $x \in \mathbb{R}$ or $\mathbb{T}$, $\alpha, \beta$ and $\gamma$ are real constants such that $\alpha \neq 0$ and $\beta \neq 0$, and $D \partial_x$ is a linear differential operator representing the dispersive term. Here $D = \mathcal{H} \partial_x$ where $\mathcal{H}$ denotes the Hilbert transform defined as

\[
\mathcal{H}f(k) = -i \text{sgn}(k) \hat{f}(k),
\]
where
\[
\text{sgn}(k) = \begin{cases} 
-1, & k < 0, \\
1, & k > 0.
\end{cases}
\]
Note that from these definitions we have that $D$ is a linear positive Fourier operator with symbol $|k|$. The system (1.1) was deduced by Funakoshi and Oikawa \((21)\). It describes the motion of two fluids with different densities under capillary-gravity waves in a deep water flow. The short surface wave is usually described by a Schrödinger type equation and the long internal wave is described by some sort of wave equation accompanied with a dispersive term (which is a Benjamin-Ono type equation in this case). This system is also of interest in the sonic-Langmuir wave interaction in plasma physics \(28\), in the capillary-gravity interaction wave \(20\), \(26\), and in the general theory of water wave interaction in a nonlinear medium \(13\), \(14\). We note that the Hilbert transform considered in \(21\) for describing system (1.1) is given as $-\mathcal{H}$.

When studying an initial value problem, the first step is usually to investigate in which function space well-posedness occurs. In our case, smooth solutions of the SBO system (1.1) enjoy the following conserved quantities
\[
\begin{align*}
G(u, v) &\equiv \text{Im} \int u(x)\overline{u_x(x)} \, dx + \frac{\alpha}{2\beta} \int |v(x)|^2 \, dx, \\
E(u, v) &\equiv \int |u_x(x)|^2 \, dx + \alpha \int |v(x)||u(x)|^2 \, dx - \frac{\alpha\gamma}{2\beta} \int |D^{1/2}v(x)|^2 \, dx, \\
H(u, v) &\equiv \int |u(x)|^2 \, dx,
\end{align*}
\]
where $D^{1/2}$ is the Fourier multiplier defined as $\hat{D^{1/2}v}(k) = |k|^{1/2}\hat{v}(k)$. Therefore, the natural spaces to study well-posedness are the Sobolev $H^s$-type spaces. Moreover, due to the scaling property of the SBO system (1.1) (see \(9\) Remark 2), we are led to investigate well-posedness in the spaces $H^s \times H^{s-1/2}$, $s \in \mathbb{R}$.

In the continuous case Bekiranov, Ogawa and Ponce \(10\) proved local well-posedness for initial data in $H^s(\mathbb{R}) \times H^{s-1/2}(\mathbb{R})$ when $|\gamma| \neq 1$ and $s \geq 0$. Thus, because of the conservation laws in (1.2), the solutions extend globally in time when $s \geq 1$, in the case $\frac{\alpha\gamma}{\beta} < 0$. Recently, Pecher \(33\) has shown local well-posedness in $H^s(\mathbb{R}) \times H^{s-1/2}(\mathbb{R})$ when $|\gamma| = 1$ and $s > 0$. He also used the Fourier restriction norm method to extend the global well-posedness result when $1/3 < s < 1$, always in the case $\frac{\alpha\gamma}{\beta} < 0$. Here, we improve the global well-posedness result till $L^2(\mathbb{R}) \times H^{-\frac{1}{2}}(\mathbb{R})$ in the case $\gamma \neq 0$ and $|\gamma| \neq 1$. Indeed, we refine the bilinear estimates of Bekiranov, Ogawa and Ponce \(10\) in Bourgain spaces $X^{0,b_1} \times Y^{-\frac{1}{2},b}$ with $b$, $b_1 < \frac{1}{2}$ (see Proposition 3.11). These estimates combined with the $L^2$-conservation law allow us to show that the size of the time interval provided by the local well-posedness theory depends only on the $L^2$-norm of $u_0$. It is worth to point out that this scheme applies for other dispersive systems. In fact Colliander, Holmer and Tzirakis \(18\) already applied this method to Zakharov and Klein-Gordon-Schrödinger systems.
They also announced the above result for the SBO system (see Remark 1.5 in [18]). However they allowed us to include it in this paper since there were not planing to write it up anymore. Note that we also prove global well-posedness in $H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ when $s > 0$ in the case $\gamma \neq 0$ and $|\gamma| \neq 1$. We take the opportunity to express our gratitude to Colliander, Holmer and Tzirakis for the fruitful interaction about the Schrödinger-Benjamin-Ono system.

In the periodic setting, there does not exist, as far as we know, any result about the well-posedness of the SBO system (1.1). Bourgain [16] proved well-posedness for the cubic nonlinear Schrödinger equation (NLS) (see (1.3)) in $H^s(\mathbb{T})$ for $s \geq 0$ using the Fourier transform restriction method. Unfortunately, this method does not apply directly for the Benjamin-Ono equation. Nevertheless, using an appropriate Gauge transformation introduced by Tao [35], Molinet [31] proved well-posedness in $L^2(\mathbb{T})$. Here we apply Bourgain’s method for the SBO system to prove its local well-posedness in $H^s(\mathbb{T}) \times H^{s-\frac{1}{2}}(\mathbb{T})$ when $s \geq 1/2$ in the case $\gamma \neq 0$, $|\gamma| \neq 1$. The main tool is the new bilinear estimate stated in Proposition 3.3. Furthermore, by standard arguments based on the conservation laws, this leads to global well-posedness in the energy space $H^1(\mathbb{T}) \times H^{1/2}(\mathbb{T})$ in the case $\alpha \gamma < 0$. We also show that our results are sharp in the sense that the bilinear estimates on these Bourgain spaces fail whenever $s < 1/2$ and $|\gamma| \neq 1$ or $s \in \mathbb{R}$ and $|\gamma| = 1$. In fact, we use Dirichlet’s Theorem on rational approximation to locate certain plane waves whose nonlinear interactions behave badly in low regularity.

In the second part of this paper, we turn our attention to another important aspect of dispersive nonlinear evolution equations: the traveling-waves. These solutions imply a balance between the effects of nonlinearity and dispersion. Depending on the specific boundary conditions on the wave’s shape, these special states of motion can arise as either solitary or periodic waves. The study of this special steady waveform is essential for the explanation of many wave phenomena observed in the practice: in surface water waves propagating in a canal, in propagation of internal waves or in the interaction between long waves and short waves as in our case. In particular, some questions such as existence and stability of these traveling waves are very important in the understanding of the dynamic of the equation under investigation.

The solitary waves are in general a single crested, symmetric, localized traveling waves, with sech-profiles (see Ono [32] and Benjamin [11] for the existence of solitary waves of algebraic type or with a finite number of oscillations). The study of the nonlinear stability or instability of solitary waves has had a big development and refinement in recent years. The proofs have been simplified and sufficient conditions have been obtained to insure the stability to small localized perturbations in the waveform. Those conditions have showed to be effective in a variety of circumstances, see for example [1], [2], [3], [12], [15], [25], [36].

The situation regarding to the study of periodic traveling waves is very different. The stability and the existence of explicit formulas of these progressive wave trains have re-
ceived comparatively little attention. Recently many research papers about this issue have appeared for specify dispersive equations, such as the existence and stability of cnoidal waves for the Korteweg-de Vries equation [5] and the stability of dnoidal waves for the one-dimensional cubic nonlinear Schrödinger equation

\[ iu_t + u_{xx} + |u|^2u = 0, \] (1.3)

where \( u = u(t, x) \in \mathbb{C} \) and \( x, t \in \mathbb{R} \) (Angulo [4], see also Angulo & Linares [6] and Gallay & Hărău [22], [23]).

In this paper we are also interested in giving a stability theory of periodic traveling waves solutions for the nonlinear dispersive system SBO (1.1). The periodic traveling waves solutions considered here will be of the general form

\[
\begin{align*}
  u(x,t) &= e^{i\omega t} e^{ic(x-ct)/2} \phi(x-ct), \\
  v(x,t) &= \psi(x-ct),
\end{align*}
\] (1.4)

where \( \phi, \psi : \mathbb{R} \rightarrow \mathbb{R} \) are smooth, \( L \)-periodic functions (with a prescribed period \( L \)), \( c > 0 \), \( \omega \in \mathbb{R} \) and we will suppose that there is a \( q \in \mathbb{N} \) such that

\[ 4q\pi/c = L. \]

So, by replacing these permanent waves form into (1.1) we obtain the pseudo-differential system

\[
\begin{align*}
  \phi'' - \sigma \phi &= \alpha \psi \phi \\
  \gamma \mathcal{H} \psi' - c\psi &= \beta \phi^2 + A_{\phi,\psi}
\end{align*}
\] (1.5)

where \( \sigma = \omega - \frac{c^2}{4} \) and \( A_{\phi,\psi} \) is an integration constant which we will set equal zero in our theory. Existence of analytic solutions of system (1.5) for \( \gamma \neq 0 \) is a difficult task. In the framework of traveling waves of type solitary waves, namely, the profiles \( \phi, \psi \) satisfy \( \phi(\xi), \psi(\xi) \to 0 \) as \( |\xi| \to \infty \), it is well known the existence of solutions for (1.5) in the form

\[
\begin{align*}
  \phi_{0,s}(\xi) &= \sqrt{\frac{2c\sigma}{\alpha\beta}} \text{sech}(\sqrt{\sigma} \xi), \\
  \psi_{0,s}(\xi) &= -\frac{\beta}{c} \phi_{0,s}^2(\xi)
\end{align*}
\] (1.6)

when \( \gamma = 0, \sigma > 0 \), and \( \alpha\beta > 0 \). For \( \gamma \neq 0 \) a theory of even solutions of these permanent waves solutions has been established in [7] (see also [8]) by using the concentration-compactness method.

For \( \gamma = 0 \) and \( \sigma > 2\pi^2/L^2 \) we prove (along the lines of Angulo [4] with regard to (1.3)) the existence of a smooth curve of even periodic traveling wave solutions for (1.5) with \( \alpha = 1, \beta = 1/2 \); note that this restriction does not imply loss of generality. This construction is based on the dnoidal Jacobian elliptic function, namely,

\[
\begin{align*}
  \phi_0(\xi) &= \eta_1 \text{dn} \left( \frac{n\eta}{2\sqrt{c}} \xi; k \right) \\
  \psi_0(\xi) &= -\frac{\eta_1}{2c} \text{dn}^2 \left( \frac{n\eta}{2\sqrt{c}} \xi; k \right),
\end{align*}
\] (1.7)
where \( \eta_1 \) and \( k \) are positive smooth functions depending on the parameter \( \sigma \). We observe that the solution in (1.7) gives us in “the limit” the solitary waves solutions (1.6) when \( \eta_1 \to \sqrt{4c\sigma} \) and \( k \to 1^- \), because in this case the elliptic function \( \text{dn} \) converges, uniformly on compacts sets, to the hyperbolic function \( \text{sech} \).

In the case of our main interest, \( \gamma \neq 0 \), the existence of periodic solutions is a delicate issue. Our approach for the existence of these solutions uses the implicit function theorem together with the explicit formulas in (1.7) and a detailed study of the periodic eigenvalue problem associated to the Jacobian form of Lame’s equation

\[
\left\{ \begin{array}{l}
\frac{d^2}{dx^2} \Psi + \left[ \rho - 6k^2 \text{sn}^2(x; k) \right] \Psi = 0 \\
\Psi(0) = \Psi(2K(k)), \quad \Psi'(0) = \Psi'(2K(k))
\end{array} \right.
\tag{1.8}
\]

where \( \text{sn}(\cdot; k) \) is the Jacobi elliptic function of type snoidal and \( K = K(k) \) represents the complete elliptic integral of the first kind and defined for \( k \in (0, 1) \) as

\[
K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.
\]

So, by fixing a period \( L \), and choosing \( c \) and \( \omega \) such that \( \sigma \equiv \omega - \frac{a}{2} \) satisfies \( \sigma > \frac{2\pi}{L^2} \), we obtain a smooth branch \( \gamma \in (-\delta, \delta) \to (\phi_\gamma, \psi_\gamma) \) of periodic traveling wave solutions of (1.5) with a fundamental period \( L \) and bifurcating from \((\phi_0, \psi_0)\) in (1.7). Moreover, we obtain that for \( \gamma \) near zero \( \phi_\gamma(x) > 0 \) for all \( x \in \mathbb{R} \) and \( \psi_\gamma(x) < 0 \) for \( \gamma < 0 \) and \( x \in \mathbb{R} \).

Furthermore, concerning the non-linear stability of this branch of periodic solutions, we extend the classical approach developed by Benjamin [12], Bona [15] and Weinstein [36] to the periodic case. In particular, using the conservation laws (1.2), we prove that the solutions \((\phi_\gamma, \psi_\gamma)\) are stable in \( H^1_{\text{per}}((0, L]) \times H^\frac{1}{2}_{\text{per}}((0, L]) \) at least when \( \gamma \) is negative near zero. We use essentially the Benjamin&Bona&Weinstein’s stability ideas because it gives us an easy form of manipulating with the required spectral conditions and the positivity property of the quantity \( \frac{d}{d\sigma} \int \phi_\gamma^2(x)dx \), which are basic information in our stability analysis.

However, we do not use the abstract stability theory of Grillakis et al. in our approach basically because of the two circumstances above. We recall that Grillakis et al. theory in general requires a study of the Hessian for the function

\[
d(c, \omega) = L(e^{i\xi/2} \phi_\gamma, \psi_\gamma) = E(e^{i\xi/2} \phi_\gamma, \psi_\gamma) + cG(e^{i\xi/2} \phi_\gamma, \psi_\gamma) + \omega H(e^{i\xi/2} \phi_\gamma, \psi_\gamma)
\]

with \( \gamma = \gamma(c, \omega) \), and a specific spectrum information of the matrix linear operator \( H_{c,\omega} = L''(e^{i\xi/2} \phi_\gamma, \psi_\gamma) \). In our case, these facts do not seem to be easily obtained.

So, for \( \gamma < 0 \) we reduce the required spectral information (see formula (5.6)) to the study of the self-adjoint operator \( \mathcal{L}_\gamma \),

\[
\mathcal{L}_\gamma = -\frac{d^2}{d\xi^2} + \sigma + \alpha \psi_\gamma - 2\alpha \beta \phi_\gamma \circ K_\gamma^{-1} \circ \phi_\gamma,
\]
where $\mathcal{K}_\gamma^{-1}$ is the inverse operator of $\mathcal{K}_\gamma = -\gamma D + c$. Hence we obtain via the min-max principle that $\mathcal{L}_\gamma$ has a simple negative eigenvalue and zero is a simple eigenvalue with eigenfunction $\frac{d}{dx}\phi_\gamma$ provide that $\gamma$ is small enough.

Finally, we close this introduction with the organization of this paper: in Section 2, we introduce some notations to be used throughout the whole article; in Section 3, we prove the global well-posedness results in the periodic and continuous settings via some appropriate bilinear estimates; in Section 4, we show the existence of periodic traveling waves by the implicit function theorem; then, in Section 5, we derive the stability of these waves based on the ideas of Benjamin and Weinstein, that is, to manipulate the information from the spectral theory of certain self-adjoint operators and the positivity of some relevant quantities.

2 Notation

For any positive numbers $a$ and $b$, the notation $a \lesssim b$ means that there exists a positive constant $\theta$ such that $a \leq \theta b$. Here, $\theta$ may depend only on certain parameters related to the equation (1.1) such as $\gamma$, $\alpha$, $\beta$. Also, we denote $a \sim b$ when, $a \lesssim b$ and $b \lesssim a$.

For $a \in \mathbb{R}$, we denote by $a^+$ and $a^-$ a number slightly larger and smaller than $a$, respectively.

In the sequel, we fix $\psi$ a smooth function supported on the interval $[-2, 2]$ such that $\psi(x) \equiv 1$ for all $|x| \leq 1$ and, for each $T > 0$, $\psi_T(t) := \psi(t/T)$.

Let $L > 0$, the inner product of two functions in $L^2([0, L])$ is given by

$$<f, g> = \int_0^L f(x)\bar{g}(x)dx, \quad \forall f, g \in L^2([0, L]).$$

Now let $\mathcal{P}_L'$ the set of periodic distributions of period $L$, for all $s \in \mathbb{R}$ we denote by $H^s_{\text{per}}([0, L]) = H^s_L(\mathbb{R})$ the set of all $f$ in $\mathcal{P}_L'$ such that

$$\|f\|_{H^s_L} = \left( L \sum_{n=-\infty}^{+\infty} (1 + |n|^2)^s |\hat{f}(n)|^2 \right)^{\frac{1}{2}} < \infty,$$

where $(\hat{f}(n))_{n \in \mathbb{Z}}$ denote the Fourier series of $f$ (for further information see Iorio&Iorio [27]). Sometimes we also write $H^s(\mathbb{T})$ to denote the space $H^s_{\text{per}}([0, L])$ when the period $L$ does not play a fundamental role.

Similarly, when $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R})$ the set of all $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{H^s} = \left( \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty,$$
where $\mathcal{S}'(\mathbb{R})$ is the set of tempered distributions and $\hat{f}$ is the Fourier transform of $f$.

When the function $u$ is of the two time-space variables $(t, x) \in \mathbb{R} \times \mathbb{R}$, periodic in space of period $L$, we define its Fourier transform by

$$\hat{u}(\tau, n) = \frac{1}{(2\pi)^{1/2}L} \int_{\mathbb{R} \times [0,L]} u(t, x) e^{-i(n\pi x/t+\tau t)} dt dx,$$

and similarly, when $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, we define

$$\hat{u}(\tau, \xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} u(t, x) e^{-i(x\xi+t\tau)} dt dx.$$

Next, we introduce the Bourgain spaces related to the Schrödinger-Benjamin-Ono system in the periodic case:

$$\|u\|_{X_{s,b}^{+,\text{per}}} := \left( \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \langle \tau + n^2 \rangle \langle n \rangle^{2b} \langle n \rangle^{2s} |\hat{u}(\tau, n)|^2 d\tau \right)^{1/2}, \quad (2.1)$$

$$\|u\|_{Y_{\gamma,\text{per}}^{s,b}} := \left( \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \langle \tau + \gamma |n| \rangle \langle n \rangle^{2b} \langle n \rangle^{2s} |\hat{u}(\tau, n)|^2 d\tau \right)^{1/2}, \quad (2.2)$$

and the continuous case:

$$\|u\|_{X_{s,b}} := \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \langle \tau + \xi^2 \rangle \langle \xi \rangle^{2b} \langle \xi \rangle^{2s} |\hat{u}(\tau, \xi)|^2 d\xi d\tau \right)^{1/2}, \quad (2.3)$$

$$\|u\|_{Y_{\gamma}^{s,b}} := \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \langle \tau + \gamma |\xi| \rangle \langle \xi \rangle^{2b} \langle \xi \rangle^{2s} |\hat{u}(\tau, \xi)|^2 d\xi d\tau \right)^{1/2}, \quad (2.4)$$

where $\langle x \rangle := 1 + |x|$. The relevance of these spaces are related to the fact that they are well-adapted to the linear part of the system and, after some time-localization, the coupling terms of (1.1) verifies particularly nice bilinear estimates. Consequently, it will be a standard matter to conclude our global well-posedness results (via Picard fixed point method).

### 3 Global Well-Posedness of the Schrödinger-Benjamin-Ono System

This section is devoted to the proof of our well-posedness results for (1.1) in both continuous and periodic settings.

#### 3.1 Global well-posedness on $\mathbb{R}$
The bulk of this subsection is the proof of the following theorem:

**Theorem 3.1.** Let $0 < |\gamma| \neq 1$. Then, the SBO system is globally well-posed for initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-\frac{3}{2}}(\mathbb{R})$, when $s \geq 0$.

In the rest of this section, we will denote by $U(t) := e^{it\partial_x^2}$ and $V_\gamma(t) := e^{-\gamma t \partial_x^2}$ the unitary groups associated to the linear part of (1.1). The proof of Theorem 3.1 follows the lines of [18]. Let us first state the linear estimates:

**Lemma 3.1.** Let $0 \leq b, b_1 \leq \frac{1}{2}$, $s \in \mathbb{R}$ and $0 < T \leq 1$. Then

\[ \| \psi_T U(t) u_0 \|_{X^{s, b_1}} \lesssim T^{\frac{1}{2} - b_1} \| u_0 \|_{H^s}, \]  

(3.1)

and

\[ \| \psi_T V_\gamma(t) v_0 \|_{Y_{\gamma, s-\frac{1}{2}, b}^0} \lesssim T^{\frac{1}{2} - b} \| v_0 \|_{H^{s-\frac{3}{2}}}, \]  

(3.2)

for $\gamma \in \mathbb{R}$.

**Proof of Lemma 3.1.** Estimate (3.1) is proved in [18] Lemma 2.1 (a). Next we combine Estimate (3.1) and the fact that $V_\gamma(t) = P_+ U(\gamma t) - P_- U(-\gamma t)$ (3.3) to deduce Estimate (3.2), where

\[ \hat{P}_+ f = \chi_{(0, +\infty)} \hat{f} \quad \text{and} \quad \hat{P}_- f = \chi_{(-\infty, 0)} \hat{f}. \]

\[ \square \]

**Lemma 3.2.** (i) Let $s \in \mathbb{R}$, $0 < T \leq 1$, $0 \leq c_1 \leq \frac{1}{2}$ and $b_1 \geq 0$ such that $b_1 + c_1 \leq 1$. Then

\[ \| \psi_T \int_0^t U(t - t') z(t') dt' \|_{X^{s, b_1}} \lesssim T^{1 - b_1 - c_1} \| z \|_{X^{s, -c_1}}, \]  

(3.4)

and

\[ \| \int_0^t U(t - t') z(t') dt' \|_{C([0, T]; H^s)} \lesssim T^{\frac{1}{2} - c_1} \| z \|_{X^{s, -c_1}}, \]  

(3.5)

(ii) Let $s \in \mathbb{R}$, $\gamma \in \mathbb{R}$, $0 < T \leq 1$, $0 \leq c \leq \frac{1}{2}$ and $b \geq 0$ such that $b + c \leq 1$. Then

\[ \| \psi_T \int_0^t V_\gamma(t - t') z(t') dt' \|_{Y_{\gamma, s-\frac{1}{2}, b}^0} \lesssim T^{1 - b - c} \| z \|_{Y_{\gamma, s-\frac{1}{2}, -c}}, \]  

(3.6)

and

\[ \| \int_0^t V_\gamma(t - t') z(t') dt' \|_{C([0, T]; H^{s-\frac{3}{2}})} \lesssim T^{\frac{1}{2} - c} \| z \|_{Y_{\gamma, s-\frac{1}{2}, -c}}. \]  

(3.7)
Proof of Lemma 3.2. For the proof of Estimates (3.4) and (3.5) see [18] Lemma 2.3 (a). For Estimates (3.6) and (3.7) we combine Identity (3.3) with Estimates (3.4) and (3.5). □

Once these linear estimates are established, our task is to prove the following bilinear estimates:

**Proposition 3.1.** Let $\gamma \in \mathbb{R}$ such that $|\gamma| \neq 1$ and $\gamma \neq 0$. Then, we have for any $\frac{1}{4} < b, b_1, c, c_1 < \frac{1}{2}$

$$\|uv\|_{X^{0,-c_1}} \lesssim \|u\|\|v\|_{X^{0,b_1}}, \quad \text{if} \quad b + b_1 + c_1 \geq 1,$$

(3.8)

$$\|\partial_x(uv)\|_{X^{1/2,c-\gamma}} \lesssim \|u\|_{X^{0,b_1}}\|v\|_{X^{0,b_1}}, \quad \text{if} \quad 2b_1 + c \geq 1,$$

(3.9)

where the implicit constants depend on $\gamma$.

For the proof of these bilinear estimates, we need the following standard Bourgain-Strichartz estimates:

**Proposition 3.2.** Let $\gamma \in \mathbb{R}$ such that $|\gamma| \neq 1$ and $\gamma \neq 0$. Then

$$\|u\|_{L^3_{t,x}} \lesssim \|u\|_{X^{0,1/4}},$$

(3.10)

and

$$\|u\|_{L^3_{t,x}} \lesssim \|u\|_{Y^{0,1/4+\gamma}}.$$  

(3.11)

Finally, we recall the two following technical lemmas proved in [24]:

**Lemma 3.3.** Let $f \in L^q(\mathbb{R})$, $g \in L^{q'}(\mathbb{R})$ with $1 \leq q$, $q' \leq +\infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Assume that $f$ and $g$ are nonnegative, even and nonincreasing for positive argument. Then $f \ast g$ enjoys the same property. In particular $f \ast g$ takes its maximum at zero.

**Lemma 3.4.** Let $0 \leq a_1, a_2 < \frac{1}{2}$ such that $a_1 + a_2 > \frac{1}{2}$. Then

$$\int_{\mathbb{R}} \langle y - \alpha \rangle^{-2a_1} \langle y - \beta \rangle^{-2a_2} \lesssim \langle \alpha - \beta \rangle^{1-2(a_1+a_2)}, \quad \forall \alpha, \beta \in \mathbb{R}.$$  

After these preliminaries, we are ready to show the bilinear estimates (3.8) and (3.9):

**Proof of Proposition 3.1.** Without loss of generality we can suppose that $|\gamma| < 1$ in the rest of the proof.

We first begin with the proof of Estimate (3.8). Letting $f(\tau, \xi) = \langle \xi \rangle^{-1/2} \langle (\tau + \gamma \xi) \rangle^{b} \widehat{a}(\tau, \xi)$, $g(\tau, \xi) = \langle (\tau + \xi^2)^{b_1} \widehat{b}(\tau, \xi)$ and using duality, we deduce that Estimate (3.8) is equivalent to

$$I \lesssim \|f\|_{L^2_{\tau,\xi}} \|g\|_{L^2_{\tau,\xi}} \|h\|_{L^2_{\tau,\xi}},$$

(3.12)
where
\[
I = \int_{\mathbb{R}^4} \frac{h(\tau, \xi) \langle \xi_1 \rangle^{1/2} f(\tau_1, \xi_1) g(\tau_2, \xi_2)}{\langle \sigma \rangle^{c_1} \langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^{b_1}} d\xi d\xi_1 d\tau d\tau_1,
\]
(3.13)
with \(\xi_2 = \xi - \xi_1, \tau_2 = \tau - \tau_1, \sigma = \tau + \xi^2, \sigma_1 = \tau_1 + \gamma |\xi_1||\xi_1|\) and \(\sigma_2 = \tau_2 + \xi_2^2\). The algebraic relation associated to (3.13) is given by
\[
- \sigma + \sigma_1 + \sigma_2 = -\xi^2 + \gamma |\xi_1||\xi_1| + \xi_2^2.
\]
(3.14)

We split the integration domain \(\mathbb{R}^4\) in the following regions
\[
\mathcal{A} = \{ (\tau, \tau_1, \xi, \xi_1) \in \mathbb{R}^4 : |\xi_1| \leq 1 \},
\]
\[
\mathcal{B} = \{ (\tau, \tau_1, \xi, \xi_1) \in \mathbb{R}^4 : |\xi_1| > 1 \text{ and } |\sigma_1| = \max(|\sigma|, |\sigma_1|, |\sigma_2|) \},
\]
\[
\mathcal{C} = \{ (\tau, \tau_1, \xi, \xi_1) \in \mathbb{R}^4 : |\xi_1| > 1 \text{ and } |\sigma| = \max(|\sigma|, |\sigma_1|, |\sigma_2|) \},
\]
\[
\mathcal{D} = \{ (\tau, \tau_1, \xi, \xi_1) \in \mathbb{R}^4 : |\xi_1| > 1 \text{ and } |\sigma_2| = \max(|\sigma|, |\sigma_1|, |\sigma_2|) \},
\]
and denote by \(I_A, I_B, I_C\) and \(I_D\) the restriction of \(I\) to each one of these regions.

**Estimate for \(I_A\).** In this region \(\langle \xi_1 \rangle \leq 1\), then we deduce using Plancherel’s identity and Hölder’s inequality that
\[
I_A \lesssim \int_{\mathbb{R}^2} \left( \frac{h(\tau, \xi)}{\langle \tau + \xi^2 \rangle^{c_1}} \right)^{\gamma} \left( \frac{f(\tau, \xi)}{\langle \tau + \gamma |\xi||\xi| \rangle^{b}} \right)^{\gamma} \left( \frac{g(\tau, \xi)}{\langle \tau + \xi^2 \rangle^{b_1}} \right)^{\gamma} dtdx.
\]
This implies, together with Estimates (3.10) and (3.11), that
\[
I_A \lesssim \| f \|_{L^2_{\tau,\xi}} \| g \|_{L^2_{\tau,\xi}} \| h \|_{L^2_{\tau,\xi}},
\]
(3.15)

since \(b, b_1, c_1 > \frac{1}{4}\).

**Estimate for \(I_B\).** Using the Cauchy-Schwarz inequality two times, we deduce that
\[
I_B \lesssim \left( \sup_{\xi_1} \langle \sigma_1 \rangle^{-2b} \int_{\mathbb{R}^2} \frac{|\xi_1|}{\langle \sigma \rangle^{2c_1} \langle \sigma_2 \rangle^{2b_1}} d\xi d\sigma \right)^{\frac{1}{2}} \| f \|_{L^2_{\tau,\xi}} \| g \|_{L^2_{\tau,\xi}} \| h \|_{L^2_{\tau,\xi}}.
\]
(3.16)

Remembering the algebraic relation (3.22), we have for \(\xi_1, \sigma, \sigma_1\) fixed that \(d\sigma_2 = -2\xi_1 d\xi\). Thus we obtain, by change of variables in the inner integral of the right-hand side of (3.10),
\[
\langle \sigma_1 \rangle^{-2b} \int_{\mathbb{R}^2} \frac{|\xi_1|}{\langle \sigma \rangle^{2c_1} \langle \sigma_2 \rangle^{2b_1}} d\xi d\sigma
\]
\[
\lesssim \langle \sigma_1 \rangle^{-2b} \left( \int_{|\sigma| \leq |\sigma_1|} \frac{d\sigma}{\langle \sigma \rangle^{2c_1}} \right) \left( \int_{|\sigma_2| \leq |\sigma_1|} \frac{d\sigma_2}{\langle \sigma_2 \rangle^{2b_1}} \right) \lesssim \langle \sigma_1 \rangle^{2(1-(b+b_1+c_1))} \lesssim 1,
\]
since \(b + b_1 + c_1 \geq 1\). Combining this estimate with (3.16), we have
\[
I_B \lesssim \|f\|_{L^2_{\tau,\xi}} \|g\|_{L^2_{\tau,\xi}} \|h\|_{L^2_{\tau,\xi}}. \tag{3.17}
\]

**Estimate for I_C.** By the Cauchy-Schwarz inequality (applied two times) it is sufficient to bound
\[
\langle \sigma \rangle^{-2c_1} \int_{\mathbb{R}^2} \frac{|\xi_1|}{\langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2b_1}} d\xi_1 d\sigma_1 \tag{3.18}
\]
indently of \(\xi\) and \(\sigma\) to obtain that
\[
I_C \lesssim \|f\|_{L^2_{\tau,\xi}} \|g\|_{L^2_{\tau,\xi}} \|h\|_{L^2_{\tau,\xi}}. \tag{3.19}
\]

Now following [10], we first treat the subregion
\[
|2((1 + \gamma \text{sgn}(\xi_1))\xi_1 - \xi) - \frac{1-|\gamma|}{2}| \xi_1|.
\]

When \(\xi, \sigma\) and \(\sigma_1\) are fixed, Identity (3.16) implies that
\[
d\sigma_2 = 2((1 + \gamma \text{sgn}(\xi_1))\xi_1 - \xi) d\xi_1.
\]

Hence we deduce that
\[
\langle \sigma \rangle^{-2c_1} \int_{\mathbb{R}^2} \frac{|\xi_1|}{\langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2b_1}} d\xi_1 d\sigma_1 \lesssim \langle \sigma \rangle^{-2c_1} \left( \int_{|\sigma_1| \leq |\sigma|} \frac{d\sigma_1}{\langle \sigma_1 \rangle^{2b}} \right) \left( \int_{|\sigma_2| \leq |\sigma|} \frac{d\sigma_2}{\langle \sigma_2 \rangle^{2b_1}} \right) \lesssim \langle \sigma_1 \rangle^{2(1-(b+b_1+c_1))} \lesssim 1,
\]

since \(b + b_1 + c_1 \geq 1\).

In the subregion of \(C\) where
\[
|2((1 + \gamma \text{sgn}(\xi_1))\xi_1 - \xi) - \frac{1-|\gamma|}{2}| \xi_1|.
\]

we have from (3.16) that
\[
|\xi_1| \lesssim |\xi_1^2 + \gamma |\xi_1| \xi_1 - 2\xi \xi_1| \lesssim |\sigma|.
\]

Then, we obtain (by applying Lemma 3.4):
\[
\langle \sigma \rangle^{-2c_1} \int_{\mathbb{R}^2} \frac{|\xi_1|}{\langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2b_1}} d\xi_1 d\sigma_1 \lesssim \langle \sigma \rangle^{\frac{1}{2} - 2c_1} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \langle \sigma_1 \rangle^{-2b} \langle \sigma_2 \rangle^{-2b_1} d\sigma_1 \right) d\xi_1 \lesssim \langle \sigma \rangle^{\frac{1}{2} - 2c_1} \int_{\mathbb{R}} \langle \sigma + \xi_1^2 + \gamma |\xi_1| \xi_1 - 2\xi \xi_1 \rangle^{1-2(b+b_1)} d\xi_1.
\]

Performing the change of variable \(y = (\theta \xi_1 - \theta^{-1} \xi)^2\), where \(\theta = (1 + \text{sgn}(\xi_1) \gamma)^{\frac{1}{2}}\) and noticing that \(|y| \lesssim |\sigma|\) and \(dy = 2\theta |y|^{\frac{1}{2}} d\xi_1\) we deduce that
\[
\langle \sigma \rangle^{-2c_1} \int_{\mathbb{R}^2} \frac{|\xi_1|}{\langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2b_1}} d\xi_1 d\sigma_1 \lesssim \langle \sigma \rangle^{\frac{1}{2} - 2c_1} \int_{|y| \lesssim |\sigma|} \frac{dy}{|y|^{1+(2(b+b_1))^{-1}}}. \tag{3.19}
\]

Now we use Lemma 3.3 to bound the right-hand side integral by
\[
\int_{|y| \lesssim |\sigma|} |y|^{-\frac{1}{2}} (y)^{1-(2(b+b_1))} dy \lesssim \langle \sigma \rangle^{\frac{1}{2} - 2(b+b_1)}.
\]
where $[\alpha]_+ = \alpha$ if $\alpha > 0$, $[\alpha]_+ = \epsilon$ arbitrarily small if $\alpha = 0$, and $[\alpha]_+ = 0$ if $\alpha < 0$. Therefore
\[
\langle \sigma \rangle^{-2c_1} \int_{\mathbb{R}^2} \frac{|\xi_1|}{\langle \sigma \rangle^{2b} \langle \sigma_2 \rangle^{2b_1}} d\xi_1 d\sigma_1 \lesssim \langle \sigma \rangle^{\frac{1}{2} - 2c_1 + [\frac{3}{2} - 2(b + b_1)]_+}
\]
which is always bounded using the assumptions on $b$, $b_1$ and $c_1$.

Estimate for $I_D$. By the Cauchy-Schwarz method it suffices to bound
\[
\langle \sigma_2 \rangle^{-2b_1} \int_{\mathbb{R}^2} \frac{|\xi_1|}{\langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2c_1}} d\xi_1 d\sigma_1
\] (3.20)
independently of $\xi_2$ and $\sigma_2$.

We first treat the subregion $|2((1 - \gamma \text{sgn}(\xi_1))\xi_1 + \xi_2)| \geq \frac{1 - |\nu|}{2} |\xi_1|$. When $\xi_2$, $\sigma_2$ and $\sigma$ are fixed, Identity (3.16) implies that
\[
d\sigma = 2(\xi_1 + \xi_2 - \gamma \text{sgn}(\xi_1)\xi_1) d\xi_1
\]
Thus we can estimate (3.20) by
\[
\langle \sigma_2 \rangle^{-2b_1} \left( \int_{|\sigma| \leq \xi_2} \frac{d\sigma_1}{\langle \sigma \rangle^{2b}} \right) \left( \int_{|\sigma| \leq \xi_2} \frac{d\sigma}{\langle \sigma \rangle^{2c_1}} \right) \lesssim \langle \sigma \rangle^{2(1 - (b + b_1 + c_1))},
\]
which is bounded since $b + b_1 + c_1 \geq 1$.

In the subregion $|2((1 - \gamma \text{sgn}(\xi_1))\xi_1 + \xi_2)| < \frac{1 - |\nu|}{2} |\xi_1|$, using Identity (3.16), we deduce that
\[
|\xi_1|^2 \lesssim |\xi_2^2 - \gamma |\xi_1| \xi_1 + 2\xi_2 \xi_1| \lesssim |\sigma_2|,
\]
where the implicit constant depends on $\gamma$. Then, Lemma 3.4 implies that
\[
\langle \sigma_2 \rangle^{-2b_1} \int_{\mathbb{R}^2} \frac{|\xi_1|}{\langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2c_1}} d\xi_1 d\sigma_1 \lesssim \langle \sigma_2 \rangle^{\frac{1}{2} - 2b_1} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \langle \sigma_1 \rangle^{-2b} \langle \sigma \rangle^{-2c_1} d\sigma_1 \right) d\xi_1
\]
\[
\lesssim \langle \sigma_2 \rangle^{\frac{1}{2} - 2b_1} \int_{\mathbb{R}} \langle \sigma_2 + (1 - \gamma \text{sgn}(\xi_1))\xi_1^2 + 2\xi_2 \xi_1 \rangle^{1-2(b+c_1)} d\xi_1.
\]

We perform the change of variable $y = (\theta \xi_1 + \theta^{-1} \xi_2)^2$ where $\theta = (1 - \gamma \text{sgn}(\xi_1))^\frac{1}{2}$ in the last integral and we use Lemma 3.3 plus the assumptions on $b$, $b_1$ and $c_1$ to bound (3.20) by
\[
\langle \sigma_2 \rangle^{\frac{1}{2} - 2b_1} \int_{|y| \leq \xi_2} |y|^{-\frac{1}{2}} (y - \theta^{-2} \xi_2^2 + \sigma_2)^{1-2(b+c_1)} \lesssim \langle \sigma_2 \rangle^{\frac{1}{2} - 2b_1 + [\frac{3}{2} - 2(b+c_1)]_+} \lesssim 1.
\]
Therefore, we deduce that
\[
I_D \lesssim \|f\|_{L_t^2} \|g\|_{L_t^2} \|h\|_{L_t^2},
\]
which combined with (3.15), (3.17) and (3.19) implies (3.8).
The proof of Estimate (3.9) is actually identical to that of Estimate (3.8). Indeed, letting \( f(\tau, \xi) = \langle \tau + \xi^2 \rangle^{b_1} \hat{u}(\tau, \xi) \) and \( g(\tau, \xi) = \langle \tau - \xi^2 \rangle^{b_1} \hat{\nu}(\tau, \xi) \), we conclude that (3.9) is equivalent to

\[
J \lesssim \|f\|_{L^2_{t,\xi}} \|g\|_{L^2_{t,\xi}} \|h\|_{L^2_{t,\xi}},
\]

where

\[
J = \int_{\mathbb{R}^3} \frac{|\xi| \langle \xi \rangle^{\frac{1}{2}} h(\tau, \xi) f(\tau_1, \xi_1) g(\tau_2, \xi_2)}{\langle \sigma \rangle^{c} \langle \sigma_1 \rangle^{b_1} \langle \sigma_2 \rangle^{b_1}} \, d\xi d\xi_1 d\tau d\tau_1,
\]

with \( \xi_2 = \xi - \xi_1 \), \( \tau_2 = \tau - \tau_1 \), \( \sigma = \sigma + \gamma |\xi| \xi \), \( \sigma_1 = \tau_1 + \xi_1^2 \) and \( \sigma_2 = \tau_2 - \xi_2^2 \). The algebraic relation associated to this integral is given by

\[
- \sigma + \sigma_1 + \sigma_2 = -\gamma |\xi| \xi + \xi_1^2 - \xi_2^2.
\]

Then, we note that Estimate (3.21) is exactly the same as Estimate (3.12), replacing \( c \) by \( c_1 \) and \( b_1 \) by \( b \), so we have to ask \( 2b + c \geq 1 \) instead of \( b + b_1 + c_1 \geq 1 \).

We now slightly modify the bilinear estimates of Proposition 3.1.

**Corollary 3.1.** Let \( \gamma \in \mathbb{R} \) such that \( |\gamma| \neq 1 \) and \( \gamma \neq 0 \). Then, we have for any \( \frac{1}{4} < b, b_1, c, c_1 < \frac{1}{2} \) and \( s \geq 0 \).

\[
\|uv\|_{X^{s,-c_1}} \lesssim \|u\|_{Y_t^{s,\frac{1}{2}}} \|v\|_{X^{0,b_1}} + \|u\|_{X^{0,b_1}} \|v\|_{X^{s,b_1}}, \text{ if } b + b_1 + c_1 \geq 1,
\]

\[
\|\partial_x (u\hat{v})\|_{Y_t^{s,\frac{1}{2},-c}} \lesssim \|u\|_{X^{s,b_1}} \|v\|_{X^{0,b_1}} + \|u\|_{X^{0,b_1}} \|v\|_{X^{s,b_1}}, \text{ if } 2b + c \geq 1.
\]

**Proof.** For all \( s \geq 0 \), we have from the triangle inequality \( \langle \xi \rangle^s \lesssim \langle \xi_1 \rangle^s + \langle \xi - \xi_1 \rangle^s \). Thus we obtain, denoting \( (J^s \hat{\varphi})^\wedge(\xi) = \langle \xi \rangle^s \hat{\varphi}(\xi) \) and using (3.8), that

\[
\|uv\|_{X^{s,-c_1}} \lesssim \|J^s u v\|_{X^{s,-c_1}} + \|u J^s v\|_{X^{s,-c_1}}
\]

\[
\lesssim \|J^s u\|_{Y_t^{s,\frac{1}{2}}} \|v\|_{X^{0,b_1}} + \|u\|_{Y_t^{s,\frac{1}{2}}} \|J^s v\|_{X^{0,b_1}}
\]

\[
= \|u\|_{Y_t^{s,\frac{1}{2}}} \|v\|_{X^{0,b_1}} + \|u\|_{Y_t^{s,\frac{1}{2}}} \|v\|_{X^{s,b_1}}.
\]

This proves Estimate (3.23). Estimate (3.24) follows using similar arguments with (3.9) instead of (3.8).

Finally, we conclude this subsection with the proof of theorem 3.1.

**Proof of Theorem 3.1.**

**Case s = 0.** The system (1.1) is, at least formally, equivalent to the integral system

\[
\begin{cases}
 u(t) := F_T(u, v) &= \psi_T U(t) u_0 - i \alpha \psi_T \int_0^t U(t - t') u(t') v(t') \, dt', \\
 v(t) := F_T^2(u) &= \psi_T V_\gamma(t) v_0 + \beta \psi_T \int_0^t V_\gamma(t - t') \partial_x (|u(t')|^2) \, dt'.
\end{cases}
\]

(3.25)
Let \((u_0, v_0) \in L^2(\mathbb{R}) \times H^{-\frac{1}{2}}(\mathbb{R})\), we want to use a contraction argument to solve (3.25) in a product of balls

\[
X^{\alpha,\beta}(a_1) \times Y_{\gamma}^{-\frac{1}{2}, \beta}(a_2) = \{(u, v) \in X^{\alpha,\beta} \times Y_{\gamma}^{-\frac{1}{2}, \beta} : \|u\|_{X^{\alpha,\beta}} \leq a_1, \|v\|_{Y^{-\frac{1}{2}, \beta}} \leq a_2\}. \tag{3.26}
\]

Estimates (3.1), (3.2), (3.4), (3.6), (3.8) and (3.9) imply that

\[
\|F^1_T(u, v)\|_{X^{\alpha,\beta}} \lesssim T^{\frac{1}{2} - \beta} \|u_0\|_{L^2} + T^{1 - c_1 - \beta} \|u\|_{X^{\alpha,\beta}} \|v\|_{Y^{-\frac{1}{2}, \beta}},
\]

and

\[
\|F^2_T(u)\|_{X^{-\frac{1}{2}, \beta}} \lesssim T^{\frac{1}{2} - \beta} \|v_0\|_{H^{-\frac{1}{2}}} + T^{1 - c - \beta} \|u\|^2_{X^{\alpha,\beta}},
\]

for \(\frac{1}{4} < b, c_1 < \frac{1}{4}\) such that \(b + c \leq 1, b_1 + c_1 \leq 1, b_1 + c_1 \geq 1\) and \(2b_1 + c \geq 1\). In the sequel, we fix \(b = b_1 = c = c_1 = \frac{1}{3}\). Therefore we deduce taking \(a_1 \sim T^{\frac{1}{2}} \|u_0\|_{L^2}\) and \(a_2 \sim T^{\frac{1}{2}} \|v_0\|_{H^{-\frac{1}{2}}}(a_2)\) if and only if

\[
T^{\frac{1}{2}} \|v_0\|_{H^{-\frac{1}{2}}} \lesssim 1, \tag{3.27}
\]

and

\[
T^{\frac{1}{2}} \|u_0\|_{L^2} \lesssim \|v_0\|_{H^{-\frac{1}{2}}}. \tag{3.28}
\]

This leads to a solution \((u, v)\) of (1.1) in \(C([0, T]; L^2(\mathbb{R})) \times C([0, T]; H^{-\frac{1}{2}}(\mathbb{R}))\) satisfying

\[
\|u\|_{X^{0, \frac{1}{3}}} \lesssim T^{\frac{1}{2}} \|u_0\|_{L^2} \quad \text{and} \quad \|v\|_{Y^{0, \frac{1}{3}}} \lesssim T^{\frac{1}{2}} \|v_0\|_{H^{-\frac{1}{2}}}, \tag{3.29}
\]

whenever \(T\) satisfies (3.27) and (3.28). Since the \(L^2\)-norm of \(u\) is a conserved quantity by the SBO flow, we can suppose that \(\|v_0\|_{H^{-\frac{1}{2}}} \gg \|u_0\|_{L^2}\), otherwise we can repeat the above argument and extend the solution globally in time. Hence Condition (3.28) is automatically satisfied and Condition (3.27) implies that the iteration time \(T\) must be \(T \sim \|v_0\|_{H^{-\frac{1}{2}}}^{-2}\). Then we deduce from (3.27), (3.9), (3.25) and (3.29) that there exists a positive constant \(C\) such that

\[
\|v(T)\|_{H^{-\frac{1}{2}}} \leq \|v_0\|_{H^{-\frac{1}{2}}} + CT^{\frac{1}{2}} \|u_0\|^2_{L^2},
\]

so that we obtain after \(m\) iterations of time \(T\) where \(m \sim \frac{\|v_0\|_{H^{-\frac{1}{2}}}}{T^{\frac{1}{2}} \|u_0\|_{L^2}}\) that

\[
\|v(\Delta T)\|_{H^{-\frac{1}{2}}} = \|v(mT)\|_{H^{-\frac{1}{2}}} \leq 2\|v_0\|_{H^{-\frac{1}{2}}}, \quad \text{where} \quad \Delta T \sim \frac{1}{\|u_0\|^2_{L^2}}.
\]

Since \(\Delta T\) only depends on \(\|u_0\|_{L^2}\), we can repeat the above argument and extend the solution \((u, v)\) of (1.1) globally in time. Moreover, we deduce that there exists \(c > 0\) such that

\[
\|v(\bar{T})\|_{H^{-1/2}} \leq e^{c\|u_0\|^2_{L^2}} T \max\{\|u_0\|_{L^2}, \|v_0\|_{H^{-1/2}}\}, \forall \bar{T} > 0. \tag{3.30}
\]
Case $s > 0$. Let $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ and $\tilde{T} > 0$. This time we want to solve the integral system (3.25) in a space of the type

$$Z^s(a) = \{(u, v) \in X^{s,b_1} \times Y^\gamma_{\gamma - \frac{1}{2}, b} : \|u\|_{X^{s,b_1}} \leq a, \|v\|_{Y^{\gamma - \frac{1}{2}, b}} \leq a\}. \quad (3.31)$$

Using Estimates (3.1), (3.2), (3.4), (3.6), (3.23), and (3.21) with $b = b_1 = c = c_1 = \frac{1}{3}$, we have that for $0 < T \leq 1$

$$\|F^1_T(u, v)\|_{X_{\frac{s}{3}, \frac{b}{3}}} \lesssim \|u_0\|_{H^s} + T^{\frac{1}{2}} \left(\|u\|_{X^{0, \frac{b}{3}}} + \|v\|_{X^{0, \frac{b}{3}}} + \|u\|_{X^{s, \frac{b}{3}}} + \|v\|_{Y^{\gamma - \frac{1}{2}, \frac{b}{3}}}\right),$$

and

$$\|F^2_T(u)\|_{X_{s, \frac{b}{3}}} \lesssim \|v_0\|_{H^{s-\frac{1}{2}}} + T^{\frac{1}{2}} \|u\|_{X^{0, \frac{b}{3}}} + \|u\|_{X^{s, \frac{b}{3}}}. \quad (3.32)$$

Moreover we can always suppose that $T$ satisfies (3.27) and (3.28), so that Estimate (3.29) holds. We also observe from the third conservation law in (1.2) and a priori Estimate (3.30) that

$$\max\{\|u(t)\|_{L^2}, \|v(t)\|_{H^{s-\frac{1}{2}}}\} \leq C(\|u_0\|_{L^2}, \|v_0\|_{H^{s-\frac{1}{2}}}, \tilde{T}), \quad \forall 0 < t \leq \tilde{T}. \quad (3.32)$$

Therefore we deduce taking

$$a \sim \max\{\|u_0\|_{L^2}, \|v_0\|_{H^{s-\frac{1}{2}}}\} \quad \text{and} \quad T \sim C(\|u_0\|_{L^2}, \|v_0\|_{H^{s-\frac{1}{2}}}, \tilde{T})^{-2}, \quad (3.33)$$

the existence of a unique solution $(u, v)$ of (3.25) in $Z^s(a)$ satisfying the additional regularity

$$(u, v) \in C([0, T]; H^s(\mathbb{R})) \times C([0, T]; H^{s-\frac{1}{2}}(\mathbb{R})).$$

Since the time iteration $T$ in (3.33) only depends on $\|u_0\|_{L^2}, \|v_0\|_{H^{s-\frac{1}{2}}}$ and $\tilde{T}$, we can reapply the above argument a finite number of times and extend the solution $(u, v)$ on the time interval $[0, \tilde{T}]$. This completes the proof of Theorem 3.1 if one remembers that $\tilde{T} > 0$ is arbitrary. \hfill \Box

### 3.2 Global well-posedness on $\mathbb{T}$

This subsection contains sharp bilinear estimates for the coupling terms $uv$ and $\partial_x (|u|^2)$ of the SBO system in the periodic setting and the global well-posedness result in the energy space $H^1(\mathbb{T}) \times H^{1/2}(\mathbb{T})$ (which is necessary for our subsequent stability theory).

Let us state our well-posedness result:

**Theorem 3.2** (Local well-posedness in $\mathbb{T}$). Let $\gamma \in \mathbb{R}$ such that $\gamma \neq 0$, $|\gamma| \neq 1$ and $s \geq 1/2$. Then, the SBO system (1.1) is locally well-posed in $H^s(\mathbb{T}) \times H^{s-1/2}(\mathbb{T})$, i.e. for all $(u_0, v_0) \in H^s(\mathbb{T}) \times H^{s-1/2}(\mathbb{T})$, there exists $T = T(\|u_0\|_{H^s}, \|v_0\|_{H^{s-1/2}})$ and a unique solution...
of the Cauchy problem \( (1.1) \) of the form \( (\psi_T u, \psi_T v) \) such that \((u,v) \in X^{s,1/2+}_{\text{per}} \times Y^s_{\gamma,\text{per}}^{1/2+} \). Moreover, \((u,v)\) satisfies the additional regularity
\[
(u, v) \in C([0, T]; H^s(\mathbb{T})) \times C([0, T]; H^{s-1/2}(\mathbb{T}))
\] (3.1)
and the map solution \( S : (u_0, v_0) \mapsto (u, v) \) is smooth.

Using the conservation laws \((1.2)\) as in \(33\), our local existence result implies

**Theorem 3.3** (Global well-posedness in \(T\)). Let \(\alpha, \beta, \gamma \in \mathbb{R}\) such that \(\gamma \neq 0\), \(|\gamma| \neq 1\) and \(\frac{\alpha \gamma}{\beta} < 0\). Then, the SBO system \((1.1)\) is globally well-posed in \(H^s(\mathbb{T}) \times H^{s-1/2}(\mathbb{T})\), when \(s \geq 1/2\).

The fundamental technical points in the proof of Theorem 3.2 are the following bilinear estimates. The rest of the proof follows by standard arguments, as in \(19\).

**Proposition 3.3.** Let \(\gamma \in \mathbb{R}\) such that \(\gamma \neq 0\), \(|\gamma| \neq 1\) and \(s \geq 1/2\). Then
\[
\|uv\|_{X^{s,1/2+}_{\gamma,\text{per}}} \lesssim \|u\|_{Y^{s-1/2,1/2}_{\gamma,\text{per}}} \|v\|_{X^{s,1/2}_{\text{per}}},
\] (3.2)
and
\[
\|\partial_x(u\bar{v})\|_{Y^{s-1/2,-1/2+}_{\gamma,\text{per}}} \lesssim \|u\|_{X^{s,1/2}_{\text{per}}} \|v\|_{X^{s,1/2}_{\text{per}}},
\] (3.3)
where the implicit constants depend on \(\gamma\).

These estimates are sharp in the following sense

**Proposition 3.4.** Let \(\gamma \neq 0\), \(|\gamma| \neq 1\). Then
(i) The estimate (3.2) fails for any \(s < 1/2\).
(ii) The estimate (3.3) fails for any \(s < 1/2\).

**Proposition 3.5.** Let \(\gamma \in \mathbb{R}\) such that \(|\gamma| = 1\). Then
(i) The estimate (3.2) fails for any \(s \in \mathbb{R}\).
(ii) The estimate (3.3) fails for any \(s \in \mathbb{R}\).

The following Bourgain-Strichartz estimates will be used in the proof of Proposition 3.3

**Proposition 3.6.** We have
\[
\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,3/8}_\gamma},
\] (3.4)
and
\[
\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,3/8}_\gamma},
\] (3.5)
for \(u : \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{C}\) and \(\gamma \in \mathbb{R}\), \(\gamma \neq 0\).
Proof. The first estimate of (3.4) was proved by Bourgain in [16] and the second one is a simple consequence of the first one (see for example [31]).

Proof of Proposition 3.3. Fix \( s \geq 1/2 \). Without loss of generality we can suppose that \( 0 < |\gamma| < 1 \) in the rest of the proof.

In order to prove estimate (3.2), it is sufficient to prove that

\[
\|uv\|_{X_{per}^{s,3/8}} = \left\| \frac{\langle n \rangle^s}{\langle \tau + n^2 \rangle^{3/8}} (uv)^\wedge (\tau, n) \right\|_{L_t^2 L_x^4} \lesssim \|u\|_{Y_{per}^{s-1/2,1/2}} \|v\|_{X_{per}^{s,1/2}}. \tag{3.6}
\]

Letting \( f(\tau, n) = \langle n \rangle^{s-1/2} (\tau + \gamma n|n|)^{1/2} \hat{g}(\tau, n) \), \( g(\tau, n) = \langle n \rangle^s (\tau + n^2)^{1/2} \hat{\omega}(\tau, n) \) and using duality, we deduce that Estimate (3.6) is equivalent to

\[
I \lesssim \|f\|_{L_t^2 L_x^4} \|g\|_{L_t^2 L_x^4} \|h\|_{L_t^4 L_x^8}, \tag{3.7}
\]

where

\[
I := \sum_{n,n_1 \in \mathbb{Z}} \int_{\mathbb{R}^2} \frac{\langle n \rangle^s h(\tau, n) f(\tau_1, n_1)}{\langle \tau + n^2 \rangle^{3/8} \langle n_1 \rangle^{s-1/2} (\tau_1 + \gamma|n_1|)|n_1|^{1/2}} \times \frac{g(\tau_1 - \tau, n - n_1)}{\langle n - n_1 \rangle^s (\tau_1 - \tau_1 + (n - n_1)^2)^{1/2}} \, d\tau_1 d\tau.
\tag{3.8}
\]

In order to bound the integral in (3.8), we split the integration domain \( \mathbb{R}^2 \times \mathbb{Z}^2 \) in the following regions,

\[
\mathcal{M} = \{ (\tau, \tau_1, n, n_1) \in \mathbb{R}^2 \times \mathbb{Z}^2 : n_1 = 0 \text{ or } |n| \leq c(\gamma)^{-1} |n - n_1| \},
\]

\[
\mathcal{N} = \{ (\tau, \tau_1, n, n_1) \in \mathbb{R}^2 \times \mathbb{Z}^2 : n_1 \neq 0 \text{ and } |n - n_1| \leq c(\gamma)|n| \},
\]

where \( c(\gamma) \) is a positive constant depending on \( \gamma \) to be fixed later. We also denote by \( I_M \) and \( I_N \) the integral \( I \) restricted to the regions \( \mathcal{M} \) and \( \mathcal{N} \), respectively.

Estimate on the region \( \mathcal{M} \). We observe that, since \( s \geq 1/2 \), it holds \( \frac{\langle n \rangle^s}{\langle n_1 \rangle^{s-1/2} |n - n_1|^2} \lesssim 1 \) (where the implicit constant depends on \( \gamma \)) in the region \( \mathcal{M} \). Thus, we deduce that, using the Plancherel identity and the \( L_t^2 L_x^4 \)-Hölder inequality,

\[
I_M \lesssim \sum_{n,n_1 \in \mathbb{Z}} \int_{\mathbb{R}^2} \frac{h(\tau, n) f(\tau_1, n_1) g(\tau - \tau_1, n - n_1)}{\langle \tau + n^2 \rangle^{3/8} \tau_1 + \gamma|n_1|\tau_1 + (n - n_1)^2} \, d\tau_1 d\tau
\]

\[
\lesssim \int_{\mathbb{R}^2 \times \mathbb{T}} \left( \frac{h(\tau, n)}{\langle \tau + n^2 \rangle^{3/8}} \right)^\vee \left( \frac{f(\tau, n)}{\langle \tau + \gamma|n|\tau_1 \rangle^{1/2}} \right)^\vee \left( \frac{g(\tau, n)}{\langle \tau + n_1 \rangle^{1/2}} \right)^\vee \, dt dx
\]

\[
\lesssim \left\| \frac{h(\tau, n)}{\langle \tau + n^2 \rangle^{3/8}} \right\|_{L_t^2 L_x^4} \left\| \frac{f(\tau, n)}{\langle \tau + \gamma|n|\tau_1 \rangle^{1/2}} \right\|_{L_t^2 L_x^4} \left\| \frac{g(\tau, n)}{\langle \tau + n_1 \rangle^{1/2}} \right\|_{L_t^4 L_x^8}. \tag{3.9}
\]
This implies, together with Estimates (3.14) and (3.5), that
\[ I_M \lesssim \|f\|_{L^2_t L^4_x} \|g\|_{L^2_t L^4_x} \|h\|_{L^2_t L^4_x}. \] (3.10)

**Estimate on the region \( \mathcal{N} \).** The dispersive smoothing effect associated to the SBO system (1.1) can be translated by the following algebraic relation
\[ - (\tau + n^2) + (\tau_1 + \gamma n_1|n_1|) + (\tau - \tau_1 + (n - n_1)^2) = Q_\gamma(n, n_1), \] (3.11)
where
\[ Q_\gamma(n, n_1) = (n - n_1)^2 + \gamma|n_1|n_1 - n^2. \] (3.12)
We have in the region \( \mathcal{N} \), \(|n_1| \leq (1 + c(\gamma))|n|\), so that
\[ |Q_\gamma(n, n_1)| \geq (1 - |\gamma|(1 + c(\gamma))^2 - c(\gamma)^2) (1 + c(\gamma))^{-2}|n_1|^2. \]

Now, we choose \( c(\gamma) \) positive, small enough such that
\[ (1 - |\gamma|(1 + c(\gamma))^2 - c(\gamma)^2) = \frac{1 - |\gamma|}{2}, \]
which is possible since \(|\gamma| < 1\). Therefore, we divide the region \( \mathcal{N} \) in three parts accordingly to which term of the left-hand side of (3.11) is dominant:
\[ \mathcal{N}_1 = \{(\tau, \tau_1, n, n_1) \in \mathcal{N} : |\tau + n^2| \geq |\tau_1 + \gamma|n_1|n_1|, |\tau - \tau_1 + (n - n_1)^2|\}, \]
\[ \mathcal{N}_2 = \{(\tau, \tau_1, n, n_1) \in \mathcal{N} : |\tau_1 + \gamma|n_1|n_1| \geq |\tau + n^2|, |\tau - \tau_1 + (n - n_1)^2|\}, \]
\[ \mathcal{N}_3 = \{(\tau, \tau_1, n, n_1) \in \mathcal{N} : |\tau - \tau_1 + (n - n_1)^2| \geq |\tau + n^2|, |\tau_1 + \gamma|n_1|n_1|\}. \]

We denote by \( I_{\mathcal{N}_1}, I_{\mathcal{N}_2} \) and \( I_{\mathcal{N}_3} \) the restriction of the integral \( I \) to the regions \( \mathcal{N}_1, \mathcal{N}_2 \) and \( \mathcal{N}_3 \), respectively.

In the region \( \mathcal{N}_1 \), we have \( \frac{(n_1)^s}{(n_1)^s} \sim \frac{1}{(\tau + n_1)^{3/8}} \lesssim 1 \), so that we can conclude
\[ I_{\mathcal{N}_1} \lesssim \|f\|_{L^2_t L^4_x} \|g\|_{L^2_t L^4_x} \|h\|_{L^2_t L^4_x}, \] (3.13)

exactly as in (3.9). We note that \( \frac{(n_1)^s}{(n_1)^s} \times \frac{1}{(\tau + n_1)^{3/8}} \lesssim 1 \) in the region \( \mathcal{N}_2 \). Then, using the \( L^4_t L^2_x L^4_t L^2_x \)-Hölder inequality, that
\[ I_{\mathcal{N}_2} \lesssim \left\| \left( \frac{h(\tau, n)}{(\tau + n^2)^{3/8}} \right)^{\vee} \right\|_{L^4_t L^2_x} \left\| f \right\|_{L^2_t L^4_x} \left( \frac{g(\tau, n)}{(\tau + n^2)^{1/2}} \right)^{\vee} \|L^4_t L^2_x\|.
\]
Combining this with Estimate (3.14), we obtain that
\[ I_{\mathcal{N}_2} \lesssim \|f\|_{L^2_t L^4_x} \|g\|_{L^2_t L^4_x} \|h\|_{L^2_t L^4_x}, \] (3.14)
Similarly, \( \frac{\langle n \rangle^s}{(n_1)^{s-1/2}(n-n_1)^{2/3}} \times \frac{1}{(\tau - \tau_1 + (n-n_1)^2)^{1/2}} \lesssim 1 \) in \( N_3 \) so that

\[
I_{N_3} \lesssim \| h(\tau, n) \|_{L_{t,x}^4}^{3/8} \| f(\tau, n) \|_{L_{t,x}^4}^{3/8} \| g \|_{L_{t,x}^2} \| h \|_{L_{t,x}^2}.
\]

(3.15)

Then, we gather (3.10), (3.13), (3.14) and (3.15) to deduce (3.17), which concludes the proof of the estimate (3.2).

Next, in order to prove Estimate (3.3), we argue as above so that it is sufficient to prove

\[
\| \partial_x (u\bar{v}) \|_{X^{s-1/2, -3/8}_\gamma} \lesssim \| u \|_{X^{s, 1/2}_\gamma} \| v \|_{X^{s, 1/2}_\gamma},
\]

which is equivalent by duality and after performing the change of variable \( f(\tau, n) = \langle n \rangle^2 (\tau + n^2)^{1/2} \bar{u}(\tau, n) \) and \( g(\tau, n) = \langle n \rangle^2 (\tau - n^2)^{1/2} \bar{v}(\tau, n) \) to

\[
J \lesssim \| f \|_{L_{t,x}^2} \| g \|_{L_{t,x}^2} \| h \|_{L_{t,x}^2},
\]

(3.17)

where

\[
J := \sum_{n, n_1 \in \mathbb{Z}} \int_{\mathbb{R}^2} \frac{|n| \langle n \rangle^{s-1/2} h(\tau, n) f(\tau_1, n_1)}{(\tau + \gamma |n| |n_1|^{3/8} (n_1)^{s} (\tau_1 + n_1^2)^{1/2}} \times \frac{g(\tau - \tau_1, n - n_1)}{(n - n_1)^{s} (\tau - \tau_1 - (n - n_1)^2)^{1/2}} \, d\tau d\tau_1.
\]

(3.18)

The algebraic relation associated to (3.18) is given by

\[
-(\tau + \gamma |n| |n_1|) + (\tau_1 + n_1^2) + (\tau - \tau_1 - (n - n_1)^2) = Q(\gamma(n, n_1),
\]

where

\[
Q(\gamma(n, n_1)) = -(n - n_1)^2 - \gamma |n| |n_1|.
\]

Therefore we can prove Estimate (3.17) using exactly the same arguments as for Estimate (3.7). \( \square \)

**Remark 3.1.** Observe that we obtain our bilinear estimates in the spaces \( X^{s, 1/2}_\gamma \) and \( Y^{s-1/2, 1/2}_\gamma \) which control the \( L_t^\infty H_x^s \) and \( L_t^\infty H_x^{s-1/2} \) norms respectively. Therefore, we do not need to use other norms as in the case of the periodic KdV equation [19].

**Remark 3.2.** Observe that the proof of Proposition 3.3 actually shows that the following bilinear estimates hold:

\[
\| uv \|_{X^{s-3/8, -3/8}_\gamma} \lesssim \| u \|_{X^{s, 3/8}_\gamma} \| v \|_{Y^{s-1/2, 1/2}_\gamma} + \| u \|_{X^{s, 1/2}_\gamma} \| v \|_{Y^{s-1/2, 3/8}_\gamma},
\]

(3.16)
\[\|\partial_x(u\tilde{w})\|_{Y^{s-1/2,-3/8}} \lesssim \|u\|_{X^{s,3/8}_{\text{per}}} \|w\|_{X^{s,1/2}_{\text{per}}} + \|u\|_{X^{s,1/2}_{\text{per}}} \|w\|_{X^{s,3/8}_{\text{per}}},\]

While we are not attempting to use this refined version of proposition 3.3 in this paper, we plan to apply these estimates combined with the I-method of Colliander, Keel, Stafillani, Takaoka and Tao to get global well-posedness results for the periodic SBO system below the energy space. Indeed, this issue will be addressed in a forthcoming paper.

In the proof of Proposition 3.4, we will use the following lemma which is a direct consequence of the Dirichlet theorem.

**Lemma 3.5.** Let \( \gamma \in \mathbb{R} \) such that \( \gamma \neq 0 \) and \( |\gamma| < 1 \) and \( Q_\gamma \) defined as in (3.12). Then, there exists a sequence of positive integers \( \{N_j\}_{j \in \mathbb{N}} \) such that

\[ N_j \to \infty \quad \text{and} \quad |Q_\gamma(N_j, N_j^0)| \leq 1, \quad \text{(3.19)} \]

where \( N_j^0 = \lfloor \frac{2N_j}{1+\gamma} \rfloor \) and \( \lfloor x \rfloor \) denotes the closest integer to \( x \).

**Theorem 3.4 (Dirichlet).** Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). Then, the inequality

\[ 0 < \left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2} \quad \text{(3.20)} \]

has infinitely many rational solutions \( \frac{p}{q} \).

**Proof of Lemma 3.5.** Fix \( \gamma \in \mathbb{R} \) such that \( |\gamma| < 1 \). Let \( N \) a positive integer, \( N \geq 2 \), \( \alpha = \frac{2}{1+\gamma} \) and \( N^0 = \lfloor \alpha N \rfloor \). Then, from the definition in (3.12), we deduce that

\[ |Q_\gamma(N, N^0)| \leq 1 \iff \left| \alpha - \frac{\lfloor \alpha N \rfloor}{N} \right| \leq \frac{1}{N^2}. \quad \text{(3.21)} \]

When \( \alpha \in \mathbb{Q} \), \( \alpha = \frac{p}{q} \), it is clear that we can find an infinity of positive integer \( N \) satisfying the right-hand side of (3.21) choosing \( N_j = jq, j \in \mathbb{N} \). When \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), this is guaranteed by the Dirichlet theorem. \( \square \)

**Proof of Proposition 3.4.** We will only show that the estimate (3.2) fails, since a counterexample for the estimate (3.3) can be constructed in a similar way. First observe that, letting \( f(\tau, n) = \langle n \rangle^{s-1/2} \langle \tau + n \rangle^{1/2} u(\tau, n) \) and \( g(\tau, n) = \langle n \rangle^{s} \langle \tau + n^2 \rangle^{1/2} \tilde{v}(\tau, n) \), the estimate (3.2) is equivalent to

\[ \|B_\gamma(f, g; s)\|_{L^2_{1,n}^q} \lesssim \|f\|_{L^2_{1,n}^p} \|g\|_{L^2_{1,n}^q}, \quad \forall \ f, \ g \in L^2_{1,n}, \quad \text{(3.22)} \]

where

\[ B_\gamma(f, g; s)(\tau, n) := \langle n \rangle^s \sum_{n_1 \in \mathbb{Z}} \int_{\mathbb{R}} \frac{f(\tau_1, n_1)}{(\tau + n^2)^{1/2}} \cdot \frac{g(\tau - \tau_1, n - n_1)}{(n - n_1)^{s}} \cdot \frac{d\tau_1}{(\tau_1 + n_1^{1/2})^{1/2}}, \quad \text{(3.23)} \]
for all $s$ and $\gamma \in \mathbb{R}$.

Fix $s < 1/2$ and $\gamma$ such that $|\gamma| \neq 1$; without loss of generality, we can suppose that $|\gamma| < 1$. Consider the sequence of integer $\{N_j\}$ obtained in Lemma 3.3 which we can always suppose to verify $N_j \gg 1$, and define

$$f_j(\tau, n) = a_n \chi_{1/2}(\tau + \gamma |n|) \quad \text{with} \quad a_n = \begin{cases} 1, & n = N_j^0, \\ 0, & \text{elsewhere,} \end{cases}$$

(3.24)

and

$$g_j(\tau, n) = b_n \chi_{1/2}(\tau + n^2) \quad \text{with} \quad b_n = \begin{cases} 1, & n = N_j - N_j^0, \\ 0, & \text{elsewhere,} \end{cases}$$

(3.25)

where $\chi_r$ is the characteristic function of the interval $[-r, r]$. Hence,

$$\|f_j\|_{L^2_{\mathbb{R}}} \sim \|g_j\|_{L^2_{\mathbb{R}}} \sim 1,$$

(3.26)

$$a_n b_{n-n_j} \neq 0 \quad \text{if and only if} \quad n_1 = N_j^0 \text{ and } n = N_j.$$

Using (3.11), we deduce that

$$\int_{\mathbb{R}} \chi_{1/2}(\tau_1 + \gamma |N_j^0|N_j^0) \chi_{1/2}(\tau - \tau_1 + (N_j - N_j^0)^2) d\tau_1 \sim \chi_1(\tau + N^2 + Q_\gamma(N_j, N_j^0)).$$

Therefore, we have from the definition in (3.23)

$$B_\gamma(f_j, g_j; s)(\tau, N_j) \gtrsim \frac{N_j^s \chi_1(\tau + N_j^2 + Q_\gamma(N_j, N_j^0))}{\langle \tau + N_j^2 \rangle^{1/2} N_j^{s-1/2} N_j^s},$$

(3.27)

where the implicit constant depends on $\gamma$. Thus, we deduce using (3.19) that

$$\|B_\gamma(f_j, g_j; s)\|_{L^2_{\mathbb{R}}} \gtrsim N_j^{1/2-s}, \quad \forall \ j \in \mathbb{N}$$

(3.28)

which combined with (3.19) and (3.26) contradicts (3.22), since $s < 1/2$. \hfill \Box

**Proof of Proposition 3.5.** Let $s \in \mathbb{R}$, we fix $\gamma = 1$. As in the proof of Proposition 3.4, we will only show that the estimate (3.2) fails, since a counterexample for the estimate (3.3) can be constructed in a similar way. In this case, (3.2) is equivalent to

$$\|B_1(f, g; s)\|_{L^2_{\mathbb{R}}} \lesssim \|f\|_{L^2_{\mathbb{R}}} \|g\|_{L^2_{\mathbb{R}}}, \quad \forall \ f, \ g \in L^2_{\tau n},$$

(3.29)

where

$$B_1(f, g; s)(\tau, n) := \sum_{\tau, n_1 \in \mathbb{Z}} \frac{(n_1)^s}{\langle \tau + n_1 \rangle^{1/2}} \int_{\mathbb{R}} \frac{f(\tau_1, n_1)}{\langle n_1 \rangle^{s-1}/2 \langle \tau_1 + |n_1| \rangle^{1/2}} \times \frac{g(\tau - \tau_1, n - n_1)}{\langle n - n_1 \rangle^s \langle \tau - \tau_1 + (n - n_1)^2 \rangle^{1/2}} d\tau_1.$$

(3.30)
Fix a positive integer $N$, such that $N \gg 1$, and define

$$f_N(\tau, n) = a_n \chi_{1/2}(\tau + |n|) \quad \text{with} \quad a_n = \begin{cases} 1, & n = N, \\ 0, & \text{elsewhere}, \end{cases}$$

and

$$g_N(\tau, n) = b_n \chi_{1/2}(\tau + n^2) \quad \text{with} \quad b_n = \begin{cases} 1, & n = 0, \\ 0, & \text{elsewhere}, \end{cases}$$

where $\chi_r$ is the characteristic function of the interval $[-r, r]$. Hence,

$$\|f_N\|_{L^2_{\tau_1}} \sim \|g_N\|_{L^2_{\tau_1}} \sim 1,$$

$$a_{n_1}b_{n-n_1} \neq 0 \quad \text{if and only if} \quad n_1 = N \text{ and } n = N,$$

and

$$\int_{\mathbb{R}} \chi_{1/2}(\tau_1 + N^2) \chi_{1/2}(\tau - \tau_1) d\tau_1 \sim \chi_1(\tau + N^2).$$

Therefore, we deduce from (3.30) that

$$\|B_1(f_N, g_N; s)\|_{L^2_{\tau_1}} \gtrsim N^{1/2}, \quad \forall \, N \gg 1,$$

which combined with (3.33) contradicts (3.29). The case $\gamma = -1$ is similar. □

## 4 Existence of Periodic Traveling-Wave Solutions

The goal of this section is to show the existence of a smooth branch of periodic traveling-wave solutions for (1.5). Initially we show a novel smooth branch of dNoidal waves solutions for (1.5) in the case $\gamma = 0$. After that, by using the implicit function theorem, we construct (in the case $\gamma \neq 0$) a smooth curve of periodic solutions bifurcating from these dNoidal waves.

### 4.1 Dnoidal Waves Solutions

We start by finding solutions for the case $\gamma = 0$ and $\sigma > 0$ in (1.5). Henceforth, without loss of generality, we will assume that $\alpha = 1$ and $\beta = \frac{1}{2}$. Hence, we need to solve the system

$$\begin{cases} \phi''_0 - \sigma \phi_0 = \psi_0 \phi_0 \\ \psi_0 = -\frac{1}{2c} \phi_0^2. \end{cases}$$

Then, by replacing the second equation of (4.1) into the first one, we obtain that $\phi_0$ satisfies

$$\phi''_0 - \sigma \phi_0 + \frac{1}{2c} \phi_0^3 = 0.$$
Equation (4.2) can be solved in a similar fashion to the method used by Angulo [4] (in the context of periodic traveling-wave solutions for the nonlinear Schrödinger equation (1.3)). For the sake of completeness, we provide here a sketch of the proof of this fact. Indeed, from (4.2), \( \phi_0 \) must satisfy the first-order equation

\[
[\phi_0']^2 = \frac{1}{4c}[-\phi_0^4 + 4c\sigma\phi_0^2 + 4cB_{\phi_0}] = \frac{1}{4c}(\eta_1^2 - \phi_0^2)(\phi_0^2 - \eta_2^2),
\]

where \( B_{\phi_0} \) is an integration constant and \(-\eta_1, \eta_1, -\eta_2, \eta_2\) are the zeros of the polynomial \( F(t) = -t^4 + 4c\sigma t^2 + 4cB_{\phi_0} \). Moreover,

\[
\begin{cases}
4c\sigma = \eta_1^2 + \eta_2^2 \\
4cB_{\phi_0} = -\eta_1^2\eta_2.
\end{cases}
\] (4.3)

We suppose, without loss of generality, that \( \eta_1 > \eta_2 > 0 \). Then \( \eta_2 \leq \phi_0 \leq \eta_1 \) and so \( \phi_0 \) will be a positive solution. Note that \(-\phi_0\) is also a solution of (4.2). Next, define \( \zeta = \phi_0/\eta_1 \) and \( k^2 = (\eta_1^2 - \eta_2^2)/\eta_1^2 \). It follows from (4.3) that

\[
[\zeta']^2 = \frac{\eta_1^2}{4c}(1 - \zeta^2)(\zeta^2 + k^2 - 1).
\]

Let us now define a new function \( \chi \) through \( \zeta^2 = 1 - k^2 \sin^2 \chi \). So we get that \( 4c(\chi')^2 = \eta_1^2(1 - k^2 \sin^2 \chi) \). Then for \( l = \frac{m}{2\sqrt{c}} \), and assuming that \( \zeta(0) = 1 \), we have

\[
\int_0^\chi d\xi = \int_0^l \frac{dt}{\sqrt{1 - k^2\sin^2 t}} = l \xi.
\]

Then from the definition of the Jacobian elliptic function \( sn(u; k) \), we have that \( \sin \chi = sn(l\xi; k) \) and hence \( \zeta(\xi) = \sqrt{1 - k^2 sn^2(l\xi; k)} \equiv dn(l\xi; k) \). Returning to the variable \( \phi_0 \), we obtain the novel \textit{dnoidal waves} solutions associated to equation (4.1),

\[
\begin{align*}
\phi_0(\xi) &\equiv \phi_0(\xi; \eta_1, \eta_2) = \eta_1 \, dn\left(\frac{m}{2\sqrt{c}} \xi; k\right) \\
\psi_0(\xi) &\equiv \psi_0(\xi; \eta_1, \eta_2) = -\frac{m}{2c} \, dn^2\left(\frac{m}{2\sqrt{c}} \xi; k\right),
\end{align*}
\] (4.4)

where

\[
0 < \eta_2 < \eta_1, \quad k^2 = \frac{\eta_1^2 - \eta_2^2}{\eta_1^2}, \quad \eta_1^2 + \eta_2^2 = 4c\sigma.
\] (4.5)

Next, since \( dn \) has fundamental period \( 2K(k) \), it follows that \( \phi_0 \) in (4.4) has fundamental wavelength (i.e., period) \( T_{\phi_0} \) given by

\[
T_{\phi_0} = \frac{4\sqrt{c}}{\eta_1} \, K(k).
\]
Given $c > 0$, $\sigma > 0$, it follows from (4.5) that $0 < \eta_2 < \sqrt{2c\sigma} < \eta_1 < 2\sqrt{c\sigma}$. Moreover we can write

$$T_{\phi_0}(\eta_2) = \frac{4\sqrt{c}}{\sqrt{4c\sigma - \eta_2^2}} K(k(\eta_2)) \quad \text{with} \quad k^2(\eta_2) = \frac{4c\sigma - 2\eta_2^2}{4c\sigma - \eta_2^2}. \quad (4.6)$$

Then, using these formulas and the properties of the function $K$, we see that $T_{\phi_0} \in (\sqrt{\frac{2}{\sigma}} \pi, +\infty)$ for $\eta_2 \in (0, \sqrt{2c\sigma})$. Moreover, we will see in Theorem 4.1 below that $\eta_2 \mapsto T_{\phi_0}(\eta_2)$ is a strictly decreasing mapping and so we obtain the basic inequality

$$T_{\phi_0} > \sqrt{\frac{2}{\sigma}} \pi. \quad (4.7)$$

Two relevant solutions of (4.1) are hidden in (4.4). Namely, the constant and solitary wave solutions. Indeed, when $\eta_2 \to \sqrt{2c\sigma}$, i.e. $\eta_2 \to \eta_1$, it follows that $k \to 0^+$. Then since $d(u; 0^+) \to 1$ we obtain the constant solutions

$$\phi_0(\xi) = \sqrt{2c\sigma} \quad \text{and} \quad \psi_0(\xi) = -\sigma. \quad (4.8)$$

Next, for $\eta_2 \to 0$ we have $\eta_1 \to 4c\sigma^-$ and so $k \to 1^-$. Then since $dn(u; 1^-) \to sech(u)$ we obtain the classical solitary wave solutions

$$\phi_{0,s}(\xi) = 2\sqrt{c\sigma} sech(\sqrt{\sigma}\xi) \quad \text{and} \quad \psi_{0,s}(\xi) = -2\sigma sech^2(\sqrt{\sigma}\xi). \quad (4.9)$$

Our next theorem is the main result of this subsection and it proves that for a fixed period $L > 0$ there exists a smooth branch of dnoidal waves solutions with the same period $L$ to the system (4.1) (or equivalently to equation (4.2)). The construction of a family of dnoidal waves with a fixed period $L$ is an immediate consequence of the analysis made above. Indeed, let $L > 0$ be a fixed number. Choose $c > 0$ and $w \in \mathbb{R}$ real fixed numbers such that $\sigma \equiv \omega - c^2/4 > 2\pi^2/L^2$. Since the function $\eta_2 \in (0, \sqrt{2c\sigma}) \to T_{\phi_0}(\eta_2)$ is strictly decreasing (see Theorem 4.1 below) there is a unique $\eta_2 = \eta_2(\sigma) \in (0, \sqrt{2c\sigma})$ such that $\phi_0(\cdot; \eta_1(\sigma), \eta_2(\sigma))$ has fundamental period $T_{\phi_0}(\eta_2(\sigma)) = L$. We claim that the choice of $\eta_2(\sigma)$ depends smoothly of $\sigma$:

**Theorem 4.1.** Let $L$ and $c$ be arbitrarily fixed positive numbers. Let $\sigma_0 > 2\pi^2/L^2$ and $\eta_{2,0} = \eta_2(\sigma_0)$ be the unique number in the interval $(0, \sqrt{2c\sigma})$ such that $T_{\phi_0}(\eta_{2,0}) = L$. Then,

1. there are intervals $I(\sigma_0)$ and $B(\eta_{2,0})$ around of $\sigma_0$ and $\eta_2(\sigma_0)$ respectively, and an unique smooth function $\Lambda : I(\sigma_0) \to B(\eta_{2,0})$, such that $\Lambda(\sigma_0) = \eta_{2,0}$ and

$$\frac{4\sqrt{c}}{\sqrt{4c\sigma - \eta_2^2}} K(k(\sigma)) = L, \quad (4.10)$$

2. $\eta_2(\sigma)$ depends smoothly on $\sigma$:
where \( \sigma \in I(\sigma_0), \eta_2 = \Lambda(\sigma) \), and

\[
\frac{d^2E}{dk^2} = k^2 - \frac{4c\sigma - 2\eta_2^2}{4c\sigma - \eta_2^2} \in (0, 1). \tag{4.11}
\]

(2) Solutions \((\phi_0(\cdot; \eta_1, \eta_2), \psi_0(\cdot; \eta_1, \eta_2))\) given by \([4.4]\) and determined by \(\eta_1 = \eta_1(\sigma), \eta_2 = \eta_2(\sigma) = \Lambda(\sigma), \) with \(\eta_1^2 + \eta_2^2 = 4c\sigma\), have fundamental period \(L\) and satisfy \([4.1]\).

Moreover, the mapping

\[
\sigma \in I(\sigma_0) \to \phi_0(\cdot; \eta_1(\sigma), \eta_2(\sigma)) \in H_{\text{per}}^n([0, L])
\]

is a smooth function (for all \(n \geq 1\) integer).

(3) \(I(\sigma_0)\) can be chosen as \((\frac{2\pi^2}{L^2}, +\infty)\).

(4) The mapping \(\Lambda : I(\sigma_0) \to B(\eta_{2,0})\) is a strictly decreasing function. Therefore, from \([4.11]\), \(\sigma \to k(\sigma)\) is a strictly increasing function.

**Proof.** The key of the proof is to apply the implicit function theorem. In fact, consider the open set \(\Omega = \{ (\eta, \sigma) : \sigma > \frac{2\pi^2}{L^2}, \eta \in (0, \sqrt{2c\sigma}) \} \subseteq \mathbb{R}^2\) and define \(\Psi : \Omega \to \mathbb{R}\) by

\[
\Psi(\eta, \sigma) = \frac{4\sqrt{c}}{\sqrt{4c\sigma - \eta^2}} K(k(\eta, \sigma)) \tag{4.12}
\]

where \(k^2(\eta, \sigma) = \frac{4c\sigma - 2\eta_2^2}{4c\sigma - \eta_2^2}\). By hypotheses \(\Psi(\eta_{2,0}, \sigma_0) = L\). Next, we show \(\partial_\eta \Psi(\eta, \sigma) < 0\). In fact, it is immediate that

\[
\partial_\eta \Psi(\eta, \sigma) = \frac{\sqrt{c}}{(4c\sigma - \eta^2)^{3/2}} K(k) + \frac{4\sqrt{c}}{\sqrt{4c\sigma - \eta^2}} \frac{dK}{dk} \frac{dk}{d\eta}.
\]

Next, from

\[
\frac{dk}{d\eta} = -\frac{4c\sigma \eta}{k(4c\sigma - \eta^2)^2},
\]

and the relations (see \([17]\))

\[
\left\{ \begin{array}{l}
\frac{dE}{dk} = \frac{E - k}{k} \frac{dE}{dk} - \frac{1}{k} \frac{dK}{dk}, \\
kk^2 \frac{d^2E}{dk^2} + k^2 \frac{dE}{dk} + kE = 0,
\end{array} \right. \tag{4.13}
\]

with \(k^2 = 1 - k^2\), and \(E = E(k)\) being the complete elliptic integral of second kind defined as

\[
E(k) = \int_0^1 \sqrt{1 - k^2t^2} \, dt,
\]

we have the following formal equivalences

\[
\partial_\eta \Psi(\eta, \sigma) < 0 \iff k(4c\sigma - \eta^2) \left( E - k \frac{dE}{dk} \right) < -4c\sigma k \frac{d^2E}{dk^2}
\]

\[
\iff k(4c\sigma - \eta^2) \left( E - k \frac{dE}{dk} \right) < \left( \frac{dE}{dk} + \frac{k}{k^2} E \right) (4c\sigma - \eta^2)(2 - k^2)
\]

\[
\iff 2k^2 \frac{dE}{dk} + kE > 0 \iff \frac{dE}{dk} - k \frac{d^2E}{dk^2} > 0 \iff \frac{dE}{dk} + \frac{dK}{dk} > 0.
\]
So, since $E + K$ is a strictly increasing function, we obtain our affirmation.

Therefore, there is a unique smooth function, $\Lambda$, defined in a neighborhood $I(\sigma_0)$ of $\sigma_0$, such that $\Psi(\Lambda(\sigma), \sigma) = L$ for every $\sigma \in I(\sigma_0)$. So, we obtain $4.10$. Moreover, since $\sigma_0$ was chosen arbitrarily in $I = (\frac{2\pi^2}{L^2}, +\infty)$, it follows from the uniqueness of the function $\Lambda$ that it can be extended to $I$.

Next, we show that $\Lambda$ is a strictly decreasing function. We know that $\Psi(\Lambda(\sigma), \sigma) = L$ for every $\sigma \in I(\sigma_0)$, then

$$\frac{d}{d\sigma}\Lambda(\sigma) = -\frac{\partial \Psi}{\partial \sigma}/\frac{\partial \Psi}{\partial \eta} < 0 \iff \partial \Psi/\partial \sigma < 0.$$ 

Thus, using the relation $\eta^2 = (4c\sigma - \eta^2)(1 - k^2) \equiv (4c\sigma - \eta^2)k^2$, we obtain the following formal equivalences

$$\frac{\partial \Psi}{\partial \sigma} < 0 \iff (4c\sigma - \eta^2)K > \frac{\eta^2 dK}{k} \iff K > \frac{k^2 dK}{k}.$$ 

Then, since $\frac{dK}{dk} = (E - k'^2K)/(kk'^2)$, it follows that

$$\frac{\partial \Psi}{\partial \sigma} < 0 \iff k'^2K > E - k'^2K \iff (k^2 + k'^2)K > E \iff K > E.$$ 

This completes the proof of the Theorem.

\[\square\]

**Remark 4.1.** In the case that the polynomial $F(t) = -t^4 + 4c\sigma t^2 + 4cB\phi_0$ has a pure imaginary root and the other two roots are real we can show the existence of two smooth curves of periodic solutions for (4.2) of cnoidal type, more precisely we have

- $\omega \in (0, +\infty) \to b\cn\left(\sqrt{b^2 - \omega \xi}; k\right) \in H_{\text{per}}^1([0, L])$

- $\omega \in \left(-\frac{4\pi^2}{L^2}, 0\right) \to \sqrt{a^2 + 2\omega} \ cn\left(\sqrt{a^2 + \omega \xi}; k\right) \in H_{\text{per}}^1([0, L]),$

where $a, b, k$ are smooth functions of $\omega$.

The following result will be used in our stability theory.

**Corollary 4.1.** Let $L$ and $c$ be arbitrarily fixed positive numbers. Consider the smooth curve of dnoidal waves $\sigma \in (\frac{2\pi^2}{L^2}, \infty) \to \phi_0(\cdot; \eta_1(\sigma), \eta_2(\sigma))$ determined by Theorem 4.1. Then

$$\frac{d}{d\sigma} \int_0^L \phi_0^2(\xi) d\xi > 0.$$ 

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Proof. By (4.4), (4.10), and the formula \( \int_0^K d\eta^2(x; k) \, dx = E(k) \) (see page 194 in [17]) it follows that
\[
\int_0^L \phi_0^2(\xi) \, d\xi = 2\eta_1 \sqrt{c} \int_0^{2K(k)} d\eta^2(x; k) \, dx = \frac{16cK}{L} E(k)K(k).
\]
So, since \( k \to K(k)E(k) \) and \( \sigma \to k(\sigma) \) are strictly increasing functions we have that
\[
\frac{d}{d\sigma} \int_0^L \phi_0^2(\xi) \, d\xi = \frac{16c}{L} \frac{d}{dk} [K(k)E(k)] \frac{dk}{d\sigma} > 0.
\]
This finishes the Corollary.

4.2 Periodic Traveling Waves Solutions for Eq. (1.5)

In this subsection we show the existence of a branch of periodic traveling waves solutions of (1.5) for \( \gamma \) close to zero such that these solutions bifurcate the dnooidal waves solutions found in Theorem 4.1.

We start our analysis by studying the periodic eigenvalue problem considered on \([0, L] \),
\[
\begin{align*}
\mathcal{L}_0 \chi &\equiv (-\frac{d^2}{dx^2} + \sigma - \frac{3}{2c} \phi_0^2)\chi = \lambda \chi \\
\chi(0) &= \chi(L), \quad \chi'(0) = \chi'(L),
\end{align*}
\tag{4.14}
\]
where for \( \sigma > 2\pi^2/L^2 \), \( \phi_0 \) is given by Theorem 4.1 and satisfies (4.2).

Theorem 4.2. The linear operator \( \mathcal{L}_0 \) defined in (4.14) with domain \( H^2_{\text{per}}([0, L]) \subseteq L^2_{\text{per}}([0, L]) \), has its first three eigenvalues simple with zero being its second eigenvalue (with eigenfunction \( \frac{d}{dx} \phi_0 \)). Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues which are double and converging to infinity.

Theorem 4.2 is a consequence of the Floquet theory (Magnus&Winkler [30]). For convenience of the readers, we will give some basic results of this theory. From the classical theory of compact symmetric linear operator we have that problem (4.14) determines a countable infinity set of eigenvalues \( \{\lambda_n| n = 0, 1, 2, ... \} \) with \( \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq ... \), where double eigenvalue is counted twice and \( \lambda_n \to \infty \) as \( n \to \infty \). We shall denote by \( \chi_n \) the eigenfunction associated to the eigenvalue \( \lambda_n \). By the conditions \( \chi_n(0) = \chi_n(L) \), \( \chi_n'(0) = \chi_n'(L) \), \( \chi_n \) can be extended to the whole of \( (-\infty, \infty) \) as a continuously differentiable function with period \( L \).

We know that with the periodic eigenvalue problem (4.14) there is an associated semi-periodic eigenvalue problem in \([0, L] \), namely,
\[
\begin{align*}
\mathcal{L}_0 \xi &= \mu \xi \\
\xi(0) &= -\xi(L), \quad \xi'(0) = -\xi'(L).
\end{align*}
\tag{4.15}
\]
As in the periodic case, there is a countable infinity set of eigenvalues \( \{ \mu_n | n = 0, 1, 2, 3, \ldots \} \), with \( \mu_0 \leq \mu_1 \leq \mu_2 \leq \mu_3 \leq \mu_4 \leq \ldots \), where double eigenvalue is counted twice and \( \mu_n \to \infty \) as \( n \to \infty \). We shall denote by \( \xi_n \) the eigenfunction associated to the eigenvalue \( \mu_n \). So, we have that the equation

\[
L_0 f = \gamma f
\]

has a solution of period \( L \) if and only if \( \gamma = \lambda_n \), \( n = 0, 1, 2, \ldots \), while the only periodic solutions of period \( 2L \) are either those associated with \( \gamma = \lambda_n \), but viewed on \([0, 2L]\), or those corresponding to \( \gamma = \mu_n \), but extended in form \( \xi_n(L+x) = \xi_n(L-x) \) for \( 0 \leq x \leq L \), \( n = 0, 1, 2, \ldots \). If all solutions of \((4.16)\) are bounded we say that they are stable; otherwise we say that they are unstable. From the Oscillation Theorem of the Floquet theory (see [30]) we have that

\[
\lambda_0 < \mu_0 \leq \mu_1 \leq \lambda_1 \leq \lambda_2 \leq \mu_2 \leq \mu_3 \leq \lambda_3 \leq \lambda_4 \cdots.
\]  

(4.17)

The intervals \((\lambda_0, \mu_0), (\mu_1, \lambda_1), \ldots\), are called intervals of stability. At the endpoints of these intervals the solutions of \((4.16)\) are unstable in general. This is always true for \( \gamma = \lambda_0 \) (\( \lambda_0 \) is always simple). The intervals, \((-\infty, \lambda_0), (\mu_0, \mu_1), (\lambda_1, \lambda_2), (\mu_2, \mu_3), \ldots\), are called intervals of instability. The interval \((-\infty, \lambda_0)\) of instability will always be present. We note that the absence of an instability interval means that there is a value of \( \gamma \) for which all solutions of \((4.16)\) have either period \( L \) or semi-period \( L \), in other words, coexistence of solutions of \((3.10)\) with period \( L \) or period \( 2L \) occurs for that value of \( \gamma \).

**Proof of Theorem 4.2.** From \((4.17)\) we have that \( \lambda_0 < \lambda_1 \leq \lambda_2 \). Since \( L_0 \frac{d^2}{dx^2} \phi_0 = 0 \) and \( \frac{d}{dx} \phi_0 \) has 2 zeros in \([0, L]\), it follows that 0 is either \( \lambda_1 \) or \( \lambda_2 \). We will show that \( 0 = \lambda_1 < \lambda_2 \). We consider \( \Psi(x) \equiv \chi(\gamma x) \) with \( \gamma^2 = 4c/\eta_1^2 \). Then from \((4.14)\) and from the identity \( k^2sn^2x + dn^2x = 1 \), we obtain

\[
\begin{cases}
\frac{d^2}{dx^2} \Psi + [\rho - 6k^2sn^2(x; k)] \Psi = 0 \\
\Psi(0) = \Psi(2K(k)), \quad \Psi'(0) = \Psi'(2K(k)),
\end{cases}
\]

(4.18)

where

\[
\rho = \frac{4c(\lambda - \sigma)}{\eta_1^2} + 6.
\]

(4.19)

Now, from Floquet theory, it follows that \((-\infty, \rho_0), (\rho_1', \mu_1')\) and \((\rho_1, \rho_2)\) are the instability intervals associated to this Lamé’s equation, where for \( i \geq 0 \), \( \mu_i' \) are the eigenvalues associated to the semi-periodic problem. Therefore, \( \rho_0, \rho_1, \rho_2 \) are simple eigenvalues for \((4.18)\) and the other eigenvalues \( \rho_3 \leq \rho_4 \leq \rho_5 \leq \rho_6 < \cdots \) satisfy \( \rho_3 = \rho_4, \rho_5 = \rho_6, \ldots \), i.e., they are double eigenvalues.

\footnote{Here we omit any empty interval obtained from a double eigenvalue.}
It is easy to verify that the first three eigenvalues $\rho_0, \rho_1, \rho_2$ and its corresponding eigenfunctions $\Psi_0, \Psi_1, \Psi_2$ are given by the formulas

\[
\begin{align*}
\rho_0 &= 2\left[1 + k^2 - \sqrt{1 - k^2 + k^4}\right], \quad \Psi_0(x) = 1 - (1 + k^2 - \sqrt{1 - k^2 + k^4})sn^2(x), \\
\rho_1 &= 4 + k^2, \quad \Psi_1(x) = snx cnx, \\
\rho_2 &= 2\left[1 + k^2 + \sqrt{1 - k^2 + k^4}\right], \quad \Psi_2(x) = 1 - (1 + k^2 + \sqrt{1 - k^2 + k^4})sn^2(x).
\end{align*}
\tag{4.20}
\]

Next, $\Psi_0$ has no zeros in $[0, 2K]$ and $\Psi_2$ has exactly 2 zeros in $[0, 2K]$, then $\rho_0$ is the first eigenvalue to (4.18). Since $\rho_0 < \rho_1$ for every $k^2 \in (0, 1)$, we obtain from (4.19) and (4.5) that

\[4c\lambda_0 = n_1^2(k^2 - 2 - 2\sqrt{1 - k^2 + k^4}) < 0 \iff \rho_0 < \rho_1.\]

Therefore $\lambda_0$ is the first negative eigenvalue to $L_0$ with eigenfunction $\chi_0(x) = \Psi_0(x/\gamma)$. Similarly, since $\rho_1 < \rho_2$ for every $k^2 \in (0, 1)$, we obtain from (4.19) that

\[4c\lambda_2 = n_1^2(k^2 - 2 + 2\sqrt{1 - k^2 + k^4}) > 0 \iff \rho_1 < \rho_2.\]

Hence $\lambda_2$ is the third eigenvalue to $L_0$ with eigenfunction $\chi_2(x) = \Psi_2(x/\gamma)$. Finally, since $\chi_1(x) = \Psi_1(x/\gamma) = \beta \frac{d}{dx}\phi_0(x)$ we finish the proof.

Next, we have our theorem of existence of solutions for (1.5). For $s \geq 0$, let $H_{per,e}^s([0, L])$ denote the closed subspace of all even functions in $H_{per,e}^s([0, L])$.

**Theorem 4.3.** Let $L, \alpha, \beta, c > 0$ and $\sigma > 2\pi^2/L^2$ be fixed numbers. Then there exist $\gamma_1 > 0$ and a smooth branch

\[\gamma \in (-\gamma_1, \gamma_1) \to (\phi_\gamma, \psi_\gamma) \in H_{per,e}^2([0, L]) \times H_{per,e}^1([0, L])\]

of solutions for Eq. (1.3). In particular, for $\gamma \to 0$, $(\phi_\gamma, \psi_\gamma)$ converges to $(\phi_0, \psi_0)$ uniformly for $x \in [0, L]$, where $(\phi_0, \psi_0)$ is given by Theorem 4.1 and it is defined by (4.4). Moreover, the mapping

\[\gamma \in (-\gamma_1, \gamma_1) \to \left(\frac{d}{d\sigma}\phi_\gamma, \frac{d}{d\sigma}\psi_\gamma\right)\]

is continuous.

**Proof.** Without loss of generality, we take $\alpha = 1$ and $\beta = 1/2$. Let $X_e = H_{per,e}^2([0, L]) \times H_{per,e}^1([0, L])$ and define the map

\[G : \mathbb{R} \times (0, +\infty) \times X_e \to L_{per,e}^2([0, L]) \times L_{per,e}^2([0, L])\]

by

\[G(\gamma, \lambda, \phi, \psi) = (-\phi'' + \lambda \phi + \phi \psi, -\gamma D \psi + c \psi + \frac{1}{2} \phi^2).\]

A calculation shows that the Fréchet derivative $G_{(\phi, \psi)} = \partial G(\gamma, \lambda, \phi, \psi)/\partial(\phi, \psi)$ exists and it is defined as a map from $\mathbb{R} \times (0, +\infty) \times X_e$ to $B(X_e; L_{per,e}^2([0, L]) \times L_{per,e}^2([0, L]))$ by

\[G(\phi, \psi)(\gamma, \lambda, \phi, \psi) = \begin{pmatrix}
-\frac{d^2}{dx^2} + \lambda + \psi \\
\phi \\
-\gamma D + c
\end{pmatrix}.\]
From Theorem 4.1 it follows that for $\Phi_0 = (\phi_0, \psi_0)$, $G(0, \sigma, \Phi_0) = \emptyset$. Moreover, from Theorem 4.2 we have that $G_{(\phi, \psi)}(0, \sigma, \Phi_0)$ has a kernel generated by $\Phi_0^t$. Next, since $\Phi_0 \notin X_e$, it follows that $G_{(\phi, \psi)}(0, \sigma, \Phi_0)$ is invertible. Hence, since $G$ and $G_{(\phi, \psi)}$ are smooth maps on their domains, the Implicit Function Theorem implies that there are $\gamma_1 > 0$, $\lambda_1 \in (0, \sigma)$, and a smooth curve

$$(\gamma, \lambda) \in (-\gamma_1, \gamma_1) \times (\sigma - \lambda_1, \sigma + \lambda_1) \to (\phi_{\gamma, 1}, \psi_{\gamma, 1}) \in X_e$$

such that $G(\gamma, \lambda, \phi_{\gamma, 1}, \psi_{\gamma, 1}) = 0$. Then, for $\lambda = \sigma$ we obtain a smooth branch $\gamma \in (-\gamma_1, \gamma_1) \to (\phi_{\gamma, \sigma}, \psi_{\gamma, \sigma}) \equiv (\phi_{\gamma}, \psi_\gamma)$ of solutions of Eq. (1.5) such that $\gamma \in (-\gamma_1, \gamma_1) \to (\frac{d}{d\sigma} \phi_{\gamma, \sigma}, \frac{d}{d\sigma} \psi_{\gamma, \sigma})$ is continuous. This shows the Theorem. \(\square\)

**Remark 4.2.** Since $\phi_0$ is strictly positive on $[0, L]$ and $\phi_\gamma \to \phi_0$, as $\gamma \to 0$, uniformly in $[0, L]$, we have that, for $\gamma$ near zero, $\phi_\gamma(x) > 0$ for $x \in \mathbb{R}$. Moreover, since the linear operator $-\gamma D + c$ is a strictly positive operator from $H^1_{per}([0, L])$ to $L^2_{per}([0, L])$ for $\gamma$ negative, we have that $\psi_\gamma(x) < 0$ for all $x \in \mathbb{R}$.

## 5 Stability of Periodic Traveling-Wave Solutions

We begin this section defining the type of stability of our interest. For any $c \in \mathbb{R}^+$ define the functions $\Phi(\xi) = e^{ic\xi/2}\phi(\xi)$ and $\Psi(\xi) = \psi(\xi)$, where $(\phi, \psi)$ is a solution of (1.5). Then we say that the orbit generated by $(\Phi, \Psi)$, namely,

$$\Omega(\Phi, \Psi) = \{ (e^{i\theta} \Phi(\cdot + x_0), \Psi(\cdot + x_0)) : (\theta, x_0) \in [0, 2\pi) \times \mathbb{R} \},$$

is stable in $H^1_{per}([0, L]) \times H^1_{per}([0, L])$ by the flow generated by Eq. (1.1), if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for $(u_0, v_0)$ satisfying $\|u_0 - \Phi\| < \delta$ and $\|v_0 - \Psi\| < \delta$, we have that $(u, v)$ solution of (1.1) with $(u(0), v(0)) = (u_0, v_0)$, satisfies that $(u, v) \in C(\mathbb{R}; H^1_{per}([0, L])) \times C(\mathbb{R}; H^1_{per}([0, L]))$ and

$$\inf_{x_0 \in \mathbb{R}, \theta \in [0, 2\pi]} \|e^{i\theta} u(\cdot + x_0, t) - \Phi\|_1 < \epsilon, \quad \inf_{x_0 \in \mathbb{R}} \|v(\cdot + x_0, t) - \Psi\|_2 < \epsilon, \quad (5.1)$$

for all $t \in \mathbb{R}$.

The main result to be proved in this section is that the periodic traveling waves solutions of (1.1) determined by Theorem 4.3 are stable for $\sigma > 2\pi^2/L^2$ and $\gamma$ negative close to 0.

**Theorem 5.1.** Let $L, \alpha, \beta, c > 0$ and $\sigma > 2\pi^2/L^2$ be fixed numbers. We consider the smooth curve of periodic traveling waves solutions for (1.5), $\gamma \to (\phi_{\gamma, \lambda}, \psi_{\gamma, \lambda})$, determined by Theorem 4.3. Then there exists $\gamma_0 > 0$ such that for each $\gamma \in (-\gamma_0, 0)$, the orbit generated by $(\Phi_{\gamma}(\xi), \Psi_{\gamma}(\xi))$ with

$$\Phi_{\gamma}(\xi) = e^{ic\xi/2}\phi_{\gamma}(\xi) \quad \text{and} \quad \Psi_{\gamma}(\xi) = \psi_{\gamma}(\xi),$$
is orbitally stable in $H^1_{\text{per}}([0, L]) \times H^\perp_{\text{per}}([0, L])$.

The proof of Theorem 5.1 is based on the ideas developed by Benjamin (12) and Weinstein (36) which give us an easy form of manipulating with the required spectral information and the positivity property of the quantity $\frac{d}{dx} \int \phi^2(x) dx$, which are basic in our stability theory. We do not use the abstract stability theory of Grillakis et al. basically by these circumstance. So, consider $(\phi, \psi)$ a solution of (1.5) obtained in Theorem 4.3. For $(u_0, v_0) \in H^1_{\text{per}}([0, L]) \times H^\perp_{\text{per}}([0, L])$ and $(u, v)$ the global solution to (1.1) corresponding to these initial data given by Theorem 3.3, we define for $t \geq 0$ and $\sigma > 2\pi^2/L^2$

$$\Omega_t(x_0, \theta) = \|e^{i\theta}(T_c u)'(\cdot + x_0, t) - \phi_\gamma'\|^2 + \sigma\|e^{i\theta}(T_c u)(\cdot + x_0, t) - \phi_\gamma\|^2$$

(5.2)

where we denote by $T_c$ the bounded linear operator defined by

$$(T_c u)(x, t) = e^{-ic(x-ct)/2}u(x, t).$$

Then, the deviation of the solution $u(t)$ from the orbit generated by $\Phi$ is measured by

$$\rho_t(u(\cdot), \phi_\gamma)^2 \equiv \inf_{x_0 \in [0, L], \theta \in [0, 2\pi]} \Omega_t(x_0, \theta).$$

(5.3)

Hence, from (5.3) we have that the $\inf \Omega_t(x_0, \theta)$ is attained in $(\theta, x_0) = (\theta(t), x_0(t))$.

**Proof of Theorem 5.1.** Consider the perturbation of the periodic traveling wave $(\phi_\gamma, \psi_\gamma)$

$$\begin{cases}
\xi(x, t) = e^{i\theta}(T_c u)(x + x_0, t) - \phi_\gamma(x) \\
\eta(x, t) = v(x + x_0, t) - \psi_\gamma(x).
\end{cases}$$

(5.4)

Hence, by the property of minimum of $(\theta, x_0) = (\theta(t), x_0(t))$, we obtain from (5.4) that $p(x, t) = \text{Re}(\xi(x, t))$ and $q(x, t) = \text{Im}(\xi(x, t))$ satisfy the compatibility relations

$$\begin{cases}
\int_0^L q(x, t)\phi_\gamma(x)\psi_\gamma(x) \, dx = 0 \\
\int_0^L p(x, t)(\phi_\gamma(x)\psi_\gamma(x))' \, dx = 0.
\end{cases}$$

(5.5)

Now we take the continuous functional $L$ defined on $H^1_{\text{per}}([0, L]) \times H^\perp_{\text{per}}([0, L])$ by

$$L(u, v) = E(u, v) + c G(u, v) + \omega H(u, v),$$

where $E, G, H$ are defined by (1.2). Then, from (5.4) and (1.3), we have

$$\Delta L(t) \triangleq L(u(t), v(t)) - L(\Phi_\gamma, \Psi_\gamma) = L(\Phi_\gamma + e^{icx/2}\xi, \psi_\gamma + \eta) - L(\Phi_\gamma, \psi_\gamma)$$

$$= \langle \mathcal{L}_\gamma p, p \rangle + \langle \mathcal{L}_\gamma^+ q, q \rangle + \frac{\alpha \beta}{2L} \int_0^L \left[ K_{\gamma}^{1/2}\eta + 2\beta K_{\gamma}^{-1/2}(\phi_\gamma p) + \beta K_{\gamma}^{-1/2}(p^2 + q^2) \right]^2 \, dx$$

$$- \frac{\alpha \beta}{2} \int_0^L \left[ K_{\gamma}^{1/2}(p^2 + q^2) \right]^2 + 4K_{\gamma}^{1/2}(\phi_\gamma p)K_{\gamma}^{-1/2}(p^2 + q^2) \, dx,$$

(5.6)
where, for $\gamma < 0$ we define $K^{-1}_\gamma$ as

$$
\hat{K}^{-1}_\gamma f(k) = \frac{1}{-\gamma |k| + c'} \hat{f}(k) \quad \text{for } k \in \mathbb{Z},
$$

which is the inverse operator of $K_\gamma : H^s_{\text{per}}([0, L]) \rightarrow H^{s-1}_{\text{per}}([0, L])$ defined by $K_\gamma = -\gamma D + c$. The operator $L_\gamma$ is

$$
L_\gamma = -\frac{d^2}{d\xi^2} + \sigma + \alpha \psi_\gamma - 2\alpha \beta \phi_\gamma \circ K^{-1}_\gamma \circ \phi_\gamma,
$$

with $\phi_\gamma \circ K^{-1}_\gamma \circ \phi_\gamma$ given by $[\phi_\gamma \circ K^{-1}_\gamma \circ \phi_\gamma](f) = \phi_\gamma K^{-1}_\gamma (\phi_\gamma f)$. Here $L^+_\gamma$ is defined by

$$
L^+_\gamma = -\frac{d^2}{d\xi^2} + \sigma + \alpha \psi_\gamma
$$

and $K^{-1/2}_\gamma$, $K^{1/2}_\gamma$ are the positive roots of $K_\gamma$ and $K^{-1}_\gamma$ respectively.

Now, we need to find a lower bound for $\Delta L(t)$. The first step will be to obtain a suitable lower bound of the last term on the right-hand side of (5.6). In fact, since $K^{-1/2}_\gamma$ is a bounded operator on $L^2_{\text{per}}([0, L])$, $\phi_\gamma$ is uniformly bounded, and from the continuous embedding of $H^1_{\text{per}}([0, L])$ in $L^4_{\text{per}}([0, L])$ and in $L^\infty([0, L])$, we have that

$$
-\frac{\alpha \beta}{2} \int_0^L \left[ |K^{-1/2}_\gamma (p^2 + q^2)|^2 + 4K^{-1/2}_\gamma (\phi_\gamma p) K^{-1/2}_\gamma (p^2 + q^2) \right] dx \geq -C_1 \|\xi\|^3 - C_2 \|\xi\|^4
$$

where $C_1$ and $C_2$ are positive constants.

The estimates for $\langle L_\gamma p, p \rangle$ and $\langle L^+_\gamma q, q \rangle$ will be obtained from the following theorem.

**Theorem 5.2.** Let $L, \alpha, \beta, c > 0$ and $\sigma > 2\pi^2/L^2$ be fixed numbers. Then, there exists $\gamma_2 > 0$ such that, if $\gamma \in (-\gamma_2, 0)$, the self-adjoint operators $L_\gamma$ and $L^+_\gamma$ defined in (5.7) and (5.8), respectively, with domain $H^2_{\text{per}}([0, L])$ have the following properties:

1. $L_\gamma$ has a simple negative eigenvalue $\lambda_\gamma$ with eigenfunction $\varphi_\gamma$ and $\int_0^L \langle L_\gamma \varphi_\gamma, \varphi_\gamma \rangle dx \neq 0$.
2. $L_\gamma$ has a simple eigenvalue at zero with eigenfunction $\frac{d}{dx} \phi_\gamma$.
3. There is $\eta_\gamma > 0$ such that for $\beta_\gamma \in \Sigma(L_\gamma) - \{\lambda_\gamma, 0\}$, we have that $\beta_\gamma > \eta_\gamma$.
4. $L^+_\gamma$ is a non-negative operator which has zero as its first eigenvalue with eigenfunction $\phi_\gamma$. The remainder of the spectrum is constituted by a discrete set of eigenvalues.

**Proof.** From (1.3) it follows that $L_\gamma \phi_\gamma = 2\phi_\gamma \psi_\gamma$ and so, from Remark 4.2, we have that for $\gamma < 0$, $\langle L_\gamma \phi_\gamma, \phi_\gamma \rangle = 2 \int_\mathbb{R} \phi_\gamma^2 \psi_\gamma dx < 0$. Therefore $L_\gamma$ has a negative eigenvalue.
Moreover, we have that $L_{\gamma} \frac{d}{dx} \phi_{\omega} = 0$. Next, for $f \in H^1_{\text{per}}([0, L])$ and $\|f\| = 1$, we have

$$
\langle L_{\gamma} f, f \rangle = \langle L_0 f, f \rangle - \frac{\alpha^2}{\gamma} \langle \phi_0 f, DK^{-1}_\gamma(\phi_0 f) \rangle + \alpha \int_0^L (\psi - \psi_0)^2 f^2 dx \\
+ \alpha^2 \int_0^L [\phi_0 f K^{-1}_\gamma(\phi_0 f) - \phi_\gamma f K^{-1}_\gamma(\phi_\gamma f)] dx
$$

(5.10)

where the last inequality is due to that $\gamma < 0$ and $DK^{-1}_\gamma$ is a positive operator. So, since

$$
\begin{align*}
\left| \int_0^L (\psi - \psi_0)^2 f^2 dx \right| & \leq \|\psi - \psi_0\|_{\infty} \\
\left| \int_0^L [\phi_0 f K^{-1}_\gamma(\phi_0 f) - \phi_\gamma f K^{-1}_\gamma(\phi_\gamma f)] dx \right| & \leq (\|\phi_\gamma\| + \|\phi_0\|)\|\phi_\gamma - \phi_0\|_{\infty},
\end{align*}
$$

(5.11)

we have from Theorem 4.3 that for $\gamma$ near $0^-$ and $\epsilon$ small, $\langle L_{\gamma} f, f \rangle \geq \langle L_0 f, f \rangle - \epsilon$. Hence, for $f \perp \chi_0$ and $f \perp \frac{d}{dx} \phi_0$, where $L_0 \chi_0 = \lambda_0 \chi_0$ with $\lambda_0 < 0$, we have from the spectral structure of $L_0$ (Theorem 4.2) that $\langle L_{\gamma} f, f \rangle \geq \eta_{\gamma} > 0$. Therefore, from min-max principle (33) we obtain the desired spectral structure for $L_{\gamma}$. Moreover, let $\varphi_\gamma$ be such that $L_{\gamma} \varphi_\gamma = \lambda_\gamma \varphi_\gamma$ with $\lambda_{\gamma} < 0$. Therefore, if $\phi_\gamma \perp \varphi_\gamma$, then from the spectral structure of $L_{\gamma}$ we must have that $\langle L_{\gamma} \phi_\gamma, \phi_\gamma \rangle \geq 0$. But we know that $\langle L_{\gamma} \phi_\gamma, \phi_\gamma \rangle < 0$. Hence, $\langle \phi_\gamma, \varphi_\gamma \rangle \neq 0$. Finally, since $L^+_{\gamma} \phi_\gamma = 0$ with $\phi_\gamma > 0$, it follows that zero is simple and it is the first eigenvalue. The remainder of the spectrum is discrete. \hfill $\square$

**Theorem 5.3.** Consider $\gamma < 0$ close to zero such that Theorem 5.2 is true. Then

(a) $\inf \{ \langle L_{\gamma} f, f \rangle : \|f\| = 1, \langle f, \phi_\gamma \rangle = 0, \} \equiv \beta_0 = 0$.

(b) $\inf \{ \langle L_{\gamma} f, f \rangle : \|f\| = 1, \langle f, \phi_\gamma \rangle = 0, \langle f, (\phi_\gamma \psi)\prime \rangle = 0 \} \equiv \beta > 0$.

**Proof.** Part (a). Since $L_{\gamma} \frac{d}{dx} \phi_{\omega} = 0$ and $\frac{d}{dx} \phi_{\omega}, \varphi_\gamma = 0$ then $\beta_0 \leq 0$. Next we will show that $\beta_0 \geq 0$ by using Lemma E.1 in Weinstein [37]. So, we shall show initially that the infimum is attained. Let $\{\psi_j\} \subseteq H^1_{\text{per}}([0, L])$ with $\|\psi_j\| = 1$, $\langle \psi_j, \phi_\gamma \rangle = 0$ and $\lim_{j \to \infty} \langle L_{\gamma} \psi_j, \psi_j \rangle = \beta_0$. Then there is a subsequence of $\{\psi_j\}$, which we denote again by $\{\psi_j\}$, such that $\psi_j \rightharpoonup \psi$ weakly in $H^1_{\text{per}}([0, L])$, so $\psi_j \rightharpoonup \psi$ in $L^2_{\text{per}}([0, L])$. Hence $\|\psi\| = 1$ and $\langle \psi, \phi_\gamma \rangle = 0$. Since $\|\psi\|^2 \leq \liminf \|\psi_j\|^2$ and $K^{-1}_\gamma(\phi_\gamma \psi_j) \to K^{-1}_\gamma(\phi_\gamma \psi)$ in $L^2_{\text{per}}([0, L])$, we have $\beta_0 \leq \langle L_{\gamma} \psi, \psi \rangle \leq \liminf \langle L_{\gamma} \psi_j, \psi_j \rangle = \beta_0$. Next we show that $\langle L^{-1}_{\gamma} \phi_\gamma, \phi_\gamma \rangle \leq 0$. From (1.5) and Theorem 4.3 we obtain for $\chi_\gamma = -\frac{d}{dx} \phi_{\omega}$ that $L_{\gamma} \chi_\gamma = \phi_\gamma$. Moreover, from Corollary 4.4 it follows that $\langle -\frac{d}{dx} \phi_0, \phi_0 \rangle < 0$ and so for $\gamma$ small enough $\langle -\frac{d}{dx} \phi_\gamma, \phi_\gamma \rangle < 0$. Hence from [37] we obtain that $\beta \geq 0$. This shows part (a) of the Theorem.
Part (b). From (a) we have that $\beta \geq 0$. Suppose $\beta = 0$. Then following a similar analysis to that used in part (a) above, we have that the infimum defined in (b) is attained at an admissible function $\zeta$. So, from Lagrange’s multiplier theory, there are $\lambda, \theta, \eta$ such that

$$\mathcal{L}_\gamma \zeta = \lambda \zeta + \theta \phi_\gamma + \eta (\phi_\gamma \psi_\gamma)' = 0.$$  \hspace{1cm} (5.12)

Using (5.12) and $\langle \mathcal{L}_\gamma \zeta, \zeta \rangle = 0$ we obtain that $\lambda = 0$. Taking the inner product of (5.12) with $\phi_\gamma'$, we have from $\mathcal{L}_\gamma \phi_\gamma' = 0$ that

$$0 = \eta \int_0^L \phi_\gamma' (\phi_\gamma \psi_\gamma)' dx,$$ \hspace{1cm} (5.13)

but the integral in (5.13) converges to

$$\int_0^L \phi_0' (\phi_0 \psi_0)' dx = \frac{-3\beta}{c} \int_0^L \phi_0^2 (\phi_0')^2 dx < 0$$

as $\gamma \to 0$. Then, from (5.13), we obtain $\eta = 0$ and therefore $\mathcal{L}_\gamma \zeta = \theta \phi_\gamma$. So, since $\mathcal{L}_\gamma (-\frac{d}{d\sigma} \phi_\gamma) = \phi_\gamma$, we obtain $0 = \langle \zeta, \phi_\gamma \rangle = \theta \langle \phi_\gamma, -\frac{d}{d\sigma} \phi_\gamma \rangle$. Therefore $\theta = 0$ and $\mathcal{L}_\gamma \zeta = 0$. Then $\zeta = \nu \phi_\gamma'$ for some $\nu \neq 0$, which is a contradiction. Thus $\beta > 0$ and the proof of the Theorem is completed. \hfill $\Box$

Theorem 5.4. Consider $\gamma < 0$ close to zero such that Theorem 5.2 is true. If $\mathcal{L}_\gamma^+$ is defined as in (5.8) then

$$\inf \{ \langle \mathcal{L}_\gamma^+ f, f \rangle : \| f \| = 1, \langle f, \phi_\gamma \psi_\gamma \rangle = 0 \} \equiv \mu > 0.$$ 

Proof. From Theorem 5.2 we have that $\mathcal{L}_\gamma^+$ is a non-negative operator and so $\mu \geq 0$. Suppose $\mu = 0$. Then, by following the ideas of the proof of Theorem 5.3, we have that the minimum is attained at an admissible function $g^* \neq 0$ and there is $(\lambda, \theta) \in \mathbb{R}^2$ such that

$$\mathcal{L}_\gamma^+ g^* = \lambda g^* + \theta \phi_\gamma \psi_\gamma.$$ \hspace{1cm} (5.14)

Thus, it follows that $\lambda = 0$. Now, taking the inner product of (5.14) with $\phi_\gamma$ it is deduced that $0 = \langle \mathcal{L}_\gamma^+ \phi_\gamma, g^* \rangle = \theta \int_0^L \phi_\gamma^2 \psi_\gamma dx$, and therefore $\theta = 0$. Then, since zero is a simple eigenvalue for $\mathcal{L}_\gamma^+$ it follows that $g^* = \nu \phi_\gamma$ for some $\nu \neq 0$, which is a contradiction. This completes the proof. \hfill $\Box$

Next we prove Theorem 5.1 by returning to (5.6). Our task is to estimate the terms $\langle \mathcal{L}_\gamma p, p \rangle$ and $\langle \mathcal{L}_\gamma^+ q, q \rangle$ where $p$ and $q$ satisfy (5.5). From Theorem 5.4 and the definition of $\mathcal{L}_\gamma^+$, we have that there is $C_1 > 0$ such that

$$\langle \mathcal{L}_\gamma^+ q, q \rangle \geq C_1 \| q \|_1^2.$$ \hspace{1cm} (5.15)
Now we estimate $\langle L_\gamma p, p \rangle$. Suppose without loss of generality that $\|\phi_\gamma\| = 1$. We write $p_\perp = p - p_\parallel$, where $p_\parallel = \langle p, \phi_\gamma \rangle \phi_\gamma$. Then, from (5.3) and the positivity of the operator $K_\gamma^{-1}$ it follows that $\langle p_\perp, (\phi_\gamma \psi_\gamma)' \rangle = 0$. Therefore from Theorem 5.3 it follows $\langle L_\gamma p_\perp, p_\perp \rangle \geq D\|p_\perp\|^2$. Now we suppose that $\|u_0\| = \|\phi_\gamma\| = 1$. Since $\|u(t)\|^2 = 1$ for all $t$, we have that $\langle p, \phi_\gamma \rangle = -\|\xi\|^2/2$. So, $\langle L_\gamma p_\perp, p_\perp \rangle \geq \beta_0\|p\|^2 - \beta_1\|\xi\|^4$. Since $\langle L_\gamma \phi_\gamma, \phi_\gamma \rangle < 0$ it follows that $\langle L_\gamma p_\parallel, p_\parallel \rangle \geq -\beta_3\|\xi\|^4$. Moreover, Cauchy-Schwarz inequality implies $\langle L_\gamma p_\parallel, p_\perp \rangle \geq -\beta_4\|\xi\|^4$. Therefore we conclude from the specific form of $L_\gamma$ that

$$\langle L_\gamma p, p \rangle \geq D_1\|p\|^2_{1,\sigma} - D_2\|\xi\|^3_{1,\sigma} - D_3\|\xi\|^4_{1,\sigma}, \quad (5.16)$$

with $D_i > 0$ and $\|f\|^2_{1,\sigma} = \|f'\|^2 + \sigma\|f\|^2$.

Next, by collecting the results in (5.9), (5.15) and (5.16) and substituting them in (5.6), we obtain

$$\Delta L(t) \geq d_1\|\xi\|^2_{1,\sigma} - d_2\|\xi\|^3_{1,\sigma} - d_3\|\xi\|^4_{1,\sigma}, \quad (5.17)$$

where $d_i > 0$. Therefore, from standard arguments, for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that, if $\|u_0 - \Phi_\gamma\|_{1,\sigma} < \delta(\epsilon)$ and $\|v_0 - \Psi_\gamma\|_{1/2} < \delta(\epsilon)$, then

$$\rho_\sigma(u(t), \phi_\gamma)^2 = \|\xi(t)\|^2_{1,\sigma} < \epsilon \quad (5.18)$$

for $t \in [0, \infty)$, and so we obtain the first inequality in (5.1).

Now, it follows from (5.6) and from the above analysis of $\xi$ that

$$\epsilon \geq \frac{\alpha}{2\beta} \int_{\mathbb{R}} \left[ K_\gamma^{1/2}\eta + 2\beta K_\gamma^{-1/2}(\phi_\gamma p) + \beta K_\gamma^{-1/2}(p^2 + q^2) \right]^2 dx.$$

Thus, from (5.18) and the equivalence of the norms $\|K_\gamma^{1/2}\eta\|$ and $\|\eta\|_{1/2}$, we obtain (5.1). This proves that $(\Phi_\gamma, \Psi_\gamma)$ is stable relative to small perturbation which preserves the $L^2_{\text{per}}([0, L])$ norm of $\Phi_\gamma$. The general case follows from that $\gamma \in (-\gamma_1, \gamma_1) \rightarrow (\phi_\gamma, \psi_\gamma)$ is a smooth branch of solutions for Eq. (1.3). \qed

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