1 Introduction

There is a close relationship between Mumford’s geometric invariant theory (GIT) in (complex) algebraic geometry and the process of reduction in symplectic geometry. GIT was developed to construct quotients of algebraic varieties by reductive group actions and thus to construct and study moduli spaces [28, 29]. When a moduli space (or a compactification of a moduli space) over \( \mathbb{C} \) can be constructed as a GIT quotient of a complex projective variety by the action of a complex reductive group \( G \), then it can be identified with a symplectic reduction by a maximal compact subgroup \( K \) of \( G \) and techniques from symplectic geometry can be used to study its topology (for example [2, 16, 17, 20, 21, 22, 23]). Many moduli spaces arise as quotients of algebraic group actions, but the groups concerned are not necessarily reductive, so that classical GIT does not apply and different methods need to be used to construct the quotients (cf. e.g. [19, 25]). Nonetheless, in suitable situations GIT can be generalised to allow us to construct GIT-like quotients (and compactified quotients) for these actions [7, 8, 24]. This paper describes some ways in which such non-reductive compactified quotients can be studied using symplectic techniques closely related to the ‘symplectic implosion’ construction of Guillemin, Jeffrey and Sjamaar [15].

More precisely, suppose that \( U \) is a maximal unipotent subgroup of a complex reductive group \( G \) acting linearly (with respect to an ample line bundle \( L \)) on a complex projective variety \( X \), and suppose that the linear action of \( U \) on \( X \) extends to a linear action of \( G \). Then the ring of invariants \( \bigoplus_{k \geq 0} H^0(X, L^\otimes k)^U \) is finitely generated and the enveloping quotient \( X//U \) (in the sense of [7]) is the projective variety \( \text{Proj}(\bigoplus_{k \geq 0} H^0(X, L^\otimes k)^U) \) associated to the ring of invariants. Moreover if \( K \) is a maximal compact subgroup of \( G \) and \( X \) is given a suitable \( K \)-invariant Kähler form, then \( X//U \) can be identified with the imploded cross-section \( X_{\text{impl}} \) of \( X \) by \( K \) in the sense of the symplectic implosion construction of Guillemin, Jeffrey and Sjamaar [15]. Note that here \( U \) is the unipotent radical of a Borel subgroup of \( G \).

The aim of this paper is to generalise symplectic implosion to give a symplectic construction for GIT-like (compactified) quotients by the unipotent radical \( U \) of any parabolic subgroup \( P \) of a complex reductive group \( G \), when the action extends to an action of \( G \). Hence we obtain a ‘moment map’ description of such compactified quotients of projective varieties by unipotent radicals of parabolics which is analogous to the description of a reductive GIT quotient \( Y//G \) as a symplectic quotient \( \mu^{-1}(0)/K \) where \( K \) is a maximal compact subgroup of \( G \) and \( \mu \) is a moment map.

The layout of the paper is as follows. §2 reviews classical GIT and its relationship with symplectic geometry, while §3 reviews symplectic implosion from [15] and extends its construction to cover quotients by unipotent radicals of parabolics. §4 gives a brief description of the results of [7, 24] on non-reductive actions and the construction of compactified quotients (more details and a much more leisurely introduction to non-reductive GIT can be found in [7]) and finally relates them to symplectic implosion. A simple example when \( G = SL(2; \mathbb{C}) \) is worked out in detail at the very end of the paper in Example 4.8.
1.1 Index of notation

Notation is introduced in this paper as follows:

\[ \begin{align*}
\mu, \mathfrak{t}, K_\zeta & \quad \text{§2.1} \\
\hat{O}(X), X//G & \quad \text{§2.2} \\
X_{\text{impl}}, T, \mathfrak{t}, W, \{K_\zeta, K_{\zeta}\}, \Sigma, B, U_{\text{max}}, G/U_{\text{max}} & \quad \text{§3.1} \\
B^{\text{op}}, U_{\text{max}}^{\text{aff}}, V_{\Lambda}(\tau), v_{\zeta}, \mathcal{F}, \alpha_{\tau}, S, X_{\text{impl}} & \quad \text{§3.1} \\
U, P, L(P), K(P), S_P, R^+, R(S_P), Q(P), \mathfrak{t}(P), \mathfrak{z}(P), G/U, X//U, E(P) & \quad \text{§3.2} \\
V_{\tau}(P), V_{K(P)}, v_{\zeta}(P), v_{\zeta, \Lambda}, \pi_{K(P)}, \mathfrak{t}_+(P), K_\zeta(P), v_{\zeta}(P), X_{\text{impl}} & \quad \text{§3.2} \\
\tilde{X}_{K,K(P)}^{\text{aff}}, G/U & \quad \text{§3.3} \\
X^{ss}, X^+, X^{ss}, X^{ss}, X//U, \tilde{G} \times \tilde{U} X, \tilde{X}, X^{ss} & \quad \text{§4.1} \\
\tilde{U}, \tilde{X}//\tilde{U}, L_\epsilon = L_\epsilon^{(N)}(X), \tilde{X}, \tilde{X}//\tilde{U} & \quad \text{§4.2}
\end{align*} \]

2 Symplectic reduction and geometric invariant theory

The GIT quotient construction in complex algebraic geometry is closely related to the process of reduction in symplectic geometry.

2.1 Symplectic reduction

Suppose that a compact, connected Lie group \( K \) with Lie algebra \( \mathfrak{k} \) acts smoothly on a symplectic manifold \( X \) and preserves the symplectic form \( \omega \). Let us denote the vector field on \( X \) defined by the infinitesimal action of \( a \in \mathfrak{k} \) by \( x \mapsto a_x \). Recall that a moment map for the action of \( K \) on \( X \) is then a smooth map \( \mu : X \to \mathfrak{k}^* \) which satisfies

\[ d\mu(x)(\xi) \cdot a = \omega_x(\xi, a_x) \]

for all \( x \in X, \xi \in T_xX \) and \( a \in \mathfrak{k} \). Equivalently, if \( \mu_a : X \to \mathbb{R} \) denotes the component of \( \mu \) along \( a \in \mathfrak{k} \) defined for all \( x \in X \) by the pairing \( \mu_a(x) = \mu(x) \cdot a \) between \( \mu(x) \in \mathfrak{k}^* \) and \( a \in \mathfrak{k} \), then \( \mu_a \) is a Hamiltonian function for the vector field on \( X \) induced by \( a \). We shall assume that any moment map \( \mu : X \to \mathfrak{k}^* \) is \( K \)-equivariant with respect to the given action of \( K \) on \( X \) and the coadjoint action of \( K \) on \( \mathfrak{k}^* \). If the stabiliser \( K_\zeta \) of \( \zeta \in \mathfrak{k}^* \) acts freely on \( \mu^{-1}(\zeta) \) then \( \mu^{-1}(\zeta) \) is a submanifold of \( X \) and the symplectic form \( \omega \) induces a symplectic structure on the quotient \( \mu^{-1}(\zeta)/K_\zeta \) which is the Marsden-Weinstein reduction, or symplectic reduction, at \( \zeta \) of the action of \( K \) on \( X \). The quotient \( \mu^{-1}(\zeta)/K_\zeta \) also inherits a symplectic structure when the action of \( K_\zeta \) on \( \mu^{-1}(\zeta) \) is not free, but in this case it is likely to have singularities (although these will only be orbifold singularities if \( \zeta \) is a regular value of \( \mu \), or equivalently if \( K_\zeta \) acts on \( \mu^{-1}(\zeta) \) with finite stabilisers). The case when \( \zeta = 0 \) is of particular importance; \( \mu^{-1}(0)/K \) is often called the symplectic quotient of \( X \) by the action of \( K \).

Now let \( X \) be a nonsingular connected complex projective variety embedded in complex projective space \( \mathbb{P}^n \), and let \( G \) be a complex Lie group acting on \( X \) via a complex linear representation \( \rho : G \to GL(n+1; \mathbb{C}) \). By an appropriate choice of coordinates on \( \mathbb{P}^n \) we may assume that \( \rho \) maps a maximal compact subgroup \( K \) of \( G \) into the unitary group \( U(n+1) \). Then the Fubini-Study form \( \omega \) on \( \mathbb{P}^n \) restricts to a \( K \)-invariant Kähler form on \( X \), and there is a moment map \( \mu : X \to \mathfrak{k}^* \) defined (up to multiplication by a constant scalar factor depending on the convention chosen for the normalisation of the Fubini-Study form) by

\[ \mu(x) \cdot a = \frac{\bar{x} \cdot \rho_*(a) \cdot \bar{x}}{2\pi i |\bar{x}|^2} \]

(1)
for all $a \in \mathfrak{g}$, where $\hat{x} \in \mathbb{C}^{n+1} - \{0\}$ is a representative vector for $x \in \mathbb{P}^n$ and the representation $\rho : K \to U(n+1)$ induces $\rho_* : \mathfrak{g} \to u(n+1)$ and dually $\rho^* : u(n+1)^* \to \mathfrak{g}^*$.

In this situation there are two possible quotient constructions: the symplectic reduction $\mu^{-1}(0)/K$ in symplectic geometry and the GIT quotient $X//G$ in algebraic geometry described below. In fact these give us the same space, at least up to homeomorphism (and diffeomorphism away from the singularities).

### 2.2 Mumford’s geometric invariant theory

Let $X$ be a complex projective variety and let $G$ be a complex reductive group acting on $X$. Recall that over $\mathbb{C}$ a linear algebraic group $G$ is reductive if and only if it is the complexification of a maximal compact subgroup $K$. The simplest non-trivial example is the complexification $\mathbb{C}^+$ of the circle $S^1$, and more generally $GL(n; \mathbb{C})$ is the complexification of the unitary group $U(n)$ and thus is reductive. In contrast the additive group of complex numbers $\mathbb{C}$ has no nontrivial compact subgroups and so is not reductive; the same is true of any complex linear algebraic group $U$ which is unipotent (that is, $U$ is isomorphic to a closed subgroup of the group of strictly upper triangular matrices in $GL(n; \mathbb{C})$ for some $n$). In some sense reductive and unipotent groups sit at the opposite extremes of a spectrum, and any linear algebraic group $H$ has a unique maximal unipotent normal subgroup $U$ (its unipotent radical) such that the quotient group $H/U$ is reductive.

Geometric invariant theory needs an extra ingredient in addition to the action of $G$ on $X$, which is a linearisation of the action; that is, a line bundle $L$ over $X$ which restricts to a surjective $G$-invariant morphism $\chi : U \to O(L)^G$, and a geometric quotient is an orbit space $\phi : X \to \tilde{Y}$ which is affine and such that

(i) $\phi^{-1}(y)$ is a single $G$-orbit, and

(ii) if $W_1$ and $W_2$ are disjoint closed $G$-invariant subvarieties of $X$ then their images $\phi(W_1)$ and $\phi(W_2)$ in $Y$ are disjoint closed subvarieties of $Y$.

When $G$ acts linearly on $X$ as above there is an induced action of $G$ on the homogeneous coordinate ring

$$\hat{O}_L(X) = \bigoplus_{k \geq 0} H^0(X, L^\otimes k) \cong \mathbb{C}[x_0, \ldots, x_n]/\mathcal{I}_X$$

(2)

where $\mathcal{I}_X$ is the ideal in $\mathbb{C}[x_0, \ldots, x_n]$ generated by the homogeneous polynomials vanishing on $X$. The subring $\hat{O}_L(X)^G$ consisting of the elements of $\hat{O}_L(X)$ left invariant by $G$ is a finitely generated graded complex algebra because $G$ is reductive, and so we can define the GIT quotient $X//G$ to be the projective variety $\text{Proj}(\hat{O}_L(X)^G)$ associated to $\hat{O}_L(X)^G$ [28]. The inclusion of $\hat{O}_L(X)^G$ in $\hat{O}_L(X)$ determines a rational map $q$ from $X$ to $X//G$, but in general there will be points of $X \subseteq \mathbb{P}^n$ where every $G$-invariant polynomial vanishes and so this map will not be well-defined everywhere on $X$. Hence we define the set $X^{ss}$ of semistable points in $X$ to be the set of those $x \in X$ for which there exists some $f \in \hat{O}_L(X)^G$ not vanishing at $x$, and then the rational map $q$ restricts to a surjective $G$-invariant morphism from the open subset $X^{ss}$ of $X$ to the quotient variety $X//G$, which is a categorical quotient for the action of $G$ on $X^{ss}$. This restriction $q : X^{ss} \to X//G$ is not necessarily an orbit space: when $x$ and $y$ are semistable points of $X$ we have $q(x) = q(y)$ if and only
if the closures $\overline{O_G(x)}$ and $\overline{O_G(y)}$ of the $G$-orbits of $x$ and $y$ meet in $X^{ss}$. Topologically $X//G$ is the quotient of $X^{ss}$ by the equivalence relation $\sim$ such that if $x$ and $y$ lie in $X^{ss}$ then $x \sim y$ if and only if $\overline{O_G(x)}$ and $\overline{O_G(y)}$ meet in $X^{ss}$.

A stable point of $X$ (‘properly stable’ in the terminology of [28]) is a point $x$ of $X^{ss}$ with a $G$-invariant neighbourhood in $X^{ss}$ such that every $G$-orbit in this neighbourhood is closed in $X^{ss}$ and has dimension $\dim G$. If $U$ is any $G$-invariant open subset of the set $X^s$ of stable points of $X$, then $q(U)$ is an open subset of $X//G$ and the restriction $q|_U : U \to q(U)$ of $q$ to $U$ is an orbit space for the action of $G$ on $U$, so that it makes sense to write $U/G$ for $q(U)$; in fact $U/G$ is a geometric quotient for the action of $G$ on $U$. In particular there is a geometric quotient $X^s//G$ for the action of $G$ on $X^s$, and $X//G$ can be thought of as a compactification of $X^s//G$.

$$
\begin{array}{ccc}
X^s & \subseteq & X^{ss} \\
\downarrow \text{open} & & \downarrow \text{open} \\
X^s//G & \subseteq & X//G = X^{ss}//
\end{array}
$$

Remark 2.1. $X^s, X^{ss}$ and $X//G$ are unaltered if for any $k > 0$ the line bundle $L$ is replaced by $L^\otimes k$ with the induced action of $G$, so it is sometimes convenient to allow fractional linearisations $L^\otimes \ell/m$.

The subsets $X^{ss}$ and $X^s$ of $X$ are characterised by the following properties (see Chapter 2 of [28] or [29]).

Proposition 2.2. (Hilbert-Mumford criteria) (i) A point $x \in X$ is semistable (respectively stable) for the action of $G$ on $X$ if and only if for every $g \in G$ the point $gx$ is semistable (respectively stable) for the action of a fixed maximal (complex) torus of $G$.

(ii) A point $x \in X$ with homogeneous coordinates $[x_0 : \ldots : x_n]$ in some coordinate system on $\mathbb{P}^n$ is semistable (respectively stable) for the action of a maximal (complex) torus of $G$ acting diagonally on $\mathbb{P}^n$ with weights $\alpha_0, \ldots, \alpha_n$ if and only if the convex hull

$$\text{Conv}\{\alpha_i : x_i \neq 0\}$$

contains 0 (respectively contains 0 in its interior).

The GIT quotient $X//G$ is homeomorphic to the symplectic quotient $\mu^{-1}(0)//K$, and the subsets $X^{ss}$ and $X^s$ of $X$ can be described using the moment map $\mu$ at [11] above. More precisely [20], any $x \in X$ is semistable if and only if the closure of its $G$-orbit meets $\mu^{-1}(0)$, while $x$ is stable if and only if its $G$-orbit meets

$$\mu^{-1}(0)_{\text{reg}} = \{x \in \mu^{-1}(0) \mid d\mu(x) : T_xX \to \mathfrak{k}^* \text{ is surjective}\},$$

and the inclusions of $\mu^{-1}(0)$ into $X^{ss}$ and of $\mu^{-1}(0)_{\text{reg}}$ into $X^s$ induce homeomorphisms

$$\mu^{-1}(0)/K \to X//G$$

and

$$\mu^{-1}(0)_{\text{reg}} \to X^s//G.$$ 

Thus the moment map picks out a unique $K$-orbit in each stable $G$-orbit, and also in each equivalence class of strictly semistable $G$-orbits, where $x$ and $y$ in $X^{ss}$ are equivalent if the closures of their $G$-orbits meet in $X^{ss}$ (that is, if their images under the natural surjection $q : X^{ss} \to X//G$ agree).

Remark 2.3. It follows from the formula [11] that if we change the linearisation of the $G$-action of $X$ by multiplying by a character $\chi : G \to \mathbb{C}^*$ of $G$, then the moment map is modified by the addition of a central constant $c_\chi$ in $\mathfrak{k}^*$, which we can identify with the restriction to $\mathfrak{k}$ of the derivative of $\chi$. 

4
Example 2.4. Let $G = \text{SL}(2; \mathbb{C})$ act on $X = (\mathbb{P}^1)^4$ via Möbius transformations and let $K$ be the maximal compact subgroup $SU(2)$ of $G$. If we identify $\mathbb{P}^1$ with the unit sphere $S^2$ in $\mathbb{R}^3$ then there is a moment map
\[ \mu : X = (S^2)^4 \to \mathfrak{t}^* \cong \mathbb{R}^3 \]
given by $\mu(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3 + x_4$. Thus $\mu^{-1}(0)$ consists of configurations of 4 points on $S^2$ which are balanced in the sense that their centre of gravity lies at the origin, while $\mu^{-1}(0) \setminus \mu^{-1}(0)_{\text{reg}}$ consists of the configurations in which two points coincide at some $p \in S^2$ and the other two points coincide at the antipodal point $-p$. The open subset
\[ X^s = \{(x_1, x_2, x_3, x_4) \in (\mathbb{P}^1)^4 : x_1, x_2, x_3, x_4 \text{ distinct}\} \]
of $X = (\mathbb{P}^1)^4$ has a geometric quotient which, using the cross-ratio, can be identified with
\[ \mathbb{P}^1 - \{0, 1, \infty\} \]
and this in turn can be identified with $\mu^{-1}(0)_{\text{reg}}/K$. In addition
\[ X^{ss} = \{(x_1, x_2, x_3, x_4) \in (\mathbb{P}^1)^4 : \text{at most two of } x_1, x_2, x_3, x_4 \text{ coincide}\} \]
has a categorical quotient $X//G \cong X^{ss}/\sim \cong \mathbb{P}^1$ in which the points $0, 1, \infty$ each represent three strictly semistable $G$-orbits in $X$: one $G$-orbit consisting of configurations in which two points $x_i$ and $x_j$ coincide at some $p \in \mathbb{P}^1$ and the other two points $x_k$ and $x_m$ coincide at a distinct point $q \in \mathbb{P}^1$, a second consisting of configurations in which $x_i$ and $x_j$ coincide at some $p \in \mathbb{P}^1$ and the other two points $x_k$ and $x_m$ are distinct from each other and from $p$, and the third consisting of configurations in which $x_k$ and $x_m$ coincide at a some point $q \in \mathbb{P}^1$ while $x_i$ and $x_j$ are distinct from each other and from $q$. The first of these orbits is closed in $X^{ss}$ and lies in the closure of each of the other two orbits.

3 Symplectic implosion and quotients by non-reductive groups

Ways in which classical GIT might be generalised to actions of non-reductive affine algebraic groups on algebraic varieties were studied in [7] (see also [24]) building on earlier work such as [9, 10, 11, 12, 36]. Every affine algebraic group $H$ has a unipotent radical $U \leq H$ such that $H/U$ is reductive, so we can concentrate on unipotent actions. It is shown in [7] that when a unipotent group $U$ acts linearly (with respect to an ample line bundle $L$) on a complex projective variety $X$, then $X$ has invariant open subsets $X^s \subseteq X^{ss}$, consisting of the ‘stable’ and ‘semistable’ points for the action, such that $X^s$ has a geometric quotient $X^s//U$ and $X^{ss}$ has a canonical ‘enveloping quotient’ $X^{ss} \to X^s//U$ which restricts to $X^s \to X^s//U$ where $X^s//U$ is an open subset of $X//U$. However, in contrast to the reductive case, the natural map from $X^{ss}$ to $X//U$ is not necessarily surjective, and indeed its image is not necessarily a subvariety of $X//U$, so this does not in general give us a categorical quotient of $X^{ss}$. Furthermore $X//U$ is in general only quasi-projective, not projective, though when the ring of invariants $\hat{\mathcal{O}}_L(X)^U = \bigoplus_{k \geq 0} H^0(X, L^\otimes k)^U$ is finitely generated as a $C$-algebra then $X//U$ is the projective variety $\text{Proj}(\hat{\mathcal{O}}_L(X)^U)$.

In order to obtain a compactification $\overline{X//U}$ of the enveloping quotient $X//U$ when the ring of invariants $\hat{\mathcal{O}}_L(X)^U$ is not finitely generated, and to understand its geometry even when $X//U = \overline{X//U}$ is itself projective, we can transfer the problem of constructing a quotient for the $U$-action to the construction of a quotient for an action of a reductive group $G$ which contains $U$ as a subgroup, by finding a reductive envelope. This is a projective completion
\[ \overline{G \times_U X} \]
of the quasi-projective variety $G \times_U X$ (which is the quotient of $G \times X$ by the free action of $U$ acting diagonally on the left on $X$ and by right multiplication on $G$), with a linear $G$-action on $\overline{G \times_U X}$ extending the induced $G$-action on $G \times_U X$, such that the $U$-invariants on $X$ lying in a suitable set (see Definition 4.3 below) extend to $G$-invariants on $\overline{G \times_U X}$. If the linearisation on $\overline{G \times_U X}$ is ample, then the classical GIT quotient
\[ \overline{G \times_U X}//G \]
is a compactification $\overline{X}/U$ of $X//U$, and hence also of its open subset $X^{s}/U$ if $X^{s} \neq \emptyset$. Moreover if $X^{s}$ and $X^{ss}$ denote the open subsets of $X$ consisting of points of $X$ which are stable and semistable for the $G$-action on $G \times U X$ under the inclusion

$$
X \hookrightarrow G \times U X \hookrightarrow G \times U \overline{X}
$$

then

$$
X^{s} \subseteq X^{s} \subseteq X^{ss} \subseteq X^{ss}.
$$

Note however that $X^{s}, X^{ss}$ and $\overline{X}/U$ depend in general on the choice of reductive envelope $G \times U X$ with its linear $G$-action, whereas $X^{s}, X^{ss}$ and $X//U$ depend only on the linear action of $U$ on $X$.

Just as GIT quotients by complex reductive groups are closely related to symplectic reduction, so quotients by suitable unipotent groups (in particular maximal unipotent subgroups of complex reductive groups) are closely related to the construction called symplectic implosion [15] which we will discuss below.

### 3.1 Symplectic implosion for a maximal unipotent subgroup

Let $(X, \omega)$ be a symplectic manifold on which a compact connected Lie group $K$ acts with a moment map $\mu : X \to \mathfrak{t}^{*}$ where $\mathfrak{t}$ is the Lie algebra of $K$. Let us choose an invariant inner product on $\mathfrak{t}$ and use it to identify $\mathfrak{t}^{*}$ with $\mathfrak{t}$. Let $T$ be a maximal torus of $K$ with Lie algebra $\mathfrak{k} \subseteq \mathfrak{t}$ and Weyl group $W = N_{K}(T)/T$, and let $\mathfrak{t}_{+}^{*} \cong \mathfrak{t}^{*}/W \cong \mathfrak{t}^{*}/Ad^{*}(K)$ be a positive Weyl chamber in $\mathfrak{t}^{*}$. The impled cross-section [15] of $X$ is then

$$
X_{\text{impl}} = \mu^{-1}(\mathfrak{t}_{+}^{*})/\sim
$$

where $x \sim y$ if and only if $\mu(x) = \mu(y) = \zeta \in \mathfrak{t}_{+}^{*}$ and $x = ky$ for some $k \in [K_{\zeta}, K_{\zeta}]$. Here $K_{\zeta}$ denotes the stabiliser $K_{\zeta} = \{ k \in K : (Ad^{*}k)\zeta = \zeta \}$ of $\zeta$ under the co-adjoint action of $K$ on $\mathfrak{t}^{*}$, and $[K_{\zeta}, K_{\zeta}]$ is its commutator subgroup. If $\Sigma$ is the set of faces of $\mathfrak{t}_{+}^{*}$ then

$$
X_{\text{impl}} = \bigcap_{\sigma \in \Sigma} \mu^{-1}(\mathfrak{t}_{+}^{*})/ [K_{\sigma}, K_{\sigma}] = \mu^{-1}((\mathfrak{t}_{+}^{*})^{\circ}) \cap \bigcap_{\sigma \in \Sigma} \mu^{-1}(\mathfrak{t}_{+}^{*})/ [K_{\sigma}, K_{\sigma}]
$$

where $K_{\sigma} = K_{\zeta}$ for any $\zeta \in \sigma$. The topology on $X_{\text{impl}}$ is the quotient topology induced from $\mu^{-1}(\mathfrak{t}_{+}^{*})$, and $X_{\text{impl}}$ also inherits a symplectic structure. More precisely, it is stratified by the locally closed subsets $\mu^{-1}(\sigma)/[K_{\sigma}, K_{\sigma}]$, each of which is the symplectic reduction by the action of $[K_{\sigma}, K_{\sigma}]$ of a locally closed symplectic submanifold

$$
X_{\sigma} = K_{\sigma} \mu^{-1}(\bigcup_{\tau \in \Sigma, \tau \supseteq \sigma} \tau)
$$

of $X$ (and locally near every point $X_{\text{impl}}$ can be identified symplectically with the product of the stratum and a suitable cone in the normal direction). The induced action of $T$ on $X_{\text{impl}}$ preserves this symplectic structure and has a moment map

$$
\mu_{X_{\text{impl}}} : X_{\text{impl}} \to \mathfrak{t}_{+}^{*} \subset \mathfrak{t}^{*}
$$

inherited from the restriction of $\mu$ to $\mu^{-1}(\mathfrak{t}_{+}^{*})$. If $\zeta \in \mathfrak{t}_{+}^{*}$ the symplectic reduction of $X_{\text{impl}}$ at $\zeta$ for this action of $T$ is the symplectic reduction of $X$ at $\zeta$ for the action of $K$:

$$
\frac{\mu_{X_{\text{impl}}}(\zeta)}{T} = \frac{\mu^{-1}(\zeta)}{T} \approx \frac{\mu^{-1}(\zeta)}{K_{\zeta}}.
$$

The universal impled cross-section is the impled cross-section

$$
(T^{*}K)_{\text{impl}} = K \times \mathfrak{t}_{+}^{*} / \sim
$$

of the cotangent bundle $T^{*}K \cong K \times \mathfrak{t}^{*}$ with respect to the $K$-action induced from the right action of $K$ on itself; it inherits an action of $K \times T$ from the left action of $K$ on itself and the right action of $T$ on $K$. Any other impled cross-section $X_{\text{impl}}$ can be constructed as the symplectic quotient of the product $X \times (T^{*}K)_{\text{impl}}$ by the diagonal action of $K$ [15] Theorem 4.9].
In fact \((T^*K)_{\text{impl}}\) is always a complex affine variety and its symplectic structure is given by a Kähler form. Indeed, let \(G = K_c\) be the complexification of \(K\) and let \(B\) be a Borel subgroup of \(G\) with \(G = KB\) and \(K \cap B = T\). If \(U_{\text{max}} \leq B\) is the unipotent radical of \(B\) (and hence a maximal unipotent subgroup of \(G\)), then \(U_{\text{max}}\) is a Grosshans subgroup of \(G\) [13]: that is, the quasi-affine variety \(G/U_{\text{max}}\) can be embedded as an open subset of an affine variety in such a way that its complement has (complex) codimension at least two. This means that the ring of invariants \(\mathcal{O}(G)^{U_{\text{max}}}\) is finitely generated (see for example [13]), and by [15] Proposition 6.8 there is a natural \(K \times T\)-equivariant identification

\[
(T^*K)_{\text{impl}} \cong \text{Spec}(\mathcal{O}(G)^{U_{\text{max}}})
\]

of the canonical affine completion \(\text{Spec}(\mathcal{O}(G)^{U_{\text{max}}})\) of \(G/U_{\text{max}}\) with \((T^*K)_{\text{impl}}\). It follows that if \(X\) is a complex projective variety on which \(G\) acts linearly with respect to a very ample line bundle \(L\), and \(\omega\) is an associated \(K\)-invariant Kähler form on \(X\), then the symplectic quotient \(X_{\text{impl}}\) of \(X \times (T^*K)_{\text{impl}}\) by \(K\) can be identified with the GIT quotient \((X \times \text{Spec}(\mathcal{O}(G)^{U_{\text{max}}})))/G\). Moreover

\[
\tilde{\mathcal{O}}_L(X)^{U_{\text{max}}} \cong (\tilde{\mathcal{O}}_L(X) \otimes \mathcal{O}(G)^{U_{\text{max}}})^G
\]

is finitely generated, and if we define the GIT quotient \(X//U_{\text{max}}\) to be the projective variety \(\text{Proj}(\tilde{\mathcal{O}}_L(X)^{U_{\text{max}}})\) associated to the ring of invariants \(\tilde{\mathcal{O}}_L(X)^{U_{\text{max}}}\) then

\[
X//U_{\text{max}} = \text{Proj}(\tilde{\mathcal{O}}_L(X)^{U_{\text{max}}}) \cong (X \times \text{Spec}(\mathcal{O}(G)^{U_{\text{max}}}))//G \cong X_{\text{impl}}.
\]

The proof in [15] §6 that \((T^*K)_{\text{impl}}\) is homeomorphic to the canonical affine completion

\[
\overline{G/U_{\text{max}}}^{\text{aff}} = \text{Spec}(\mathcal{O}(G)^{U_{\text{max}}})
\]

of \(G/U_{\text{max}}\) runs as follows. First it is possible to reduce to the case when \(K\) is semisimple and simply connected, by regarding \(K\) as the quotient by a finite central subgroup of \(Z(K) \times [K, K]\) where \(Z(K)\) is the centre of \(K\) and \([K, K]\) is the universal cover of the commutator subgroup \([K, K]\) of \(K\).

Following [15] §6, if \(K\) is a semisimple, connected and simply connected compact group let \(\Lambda = \ker(\exp|_t)\) be the exponential lattice in \(t\), and let \(\Lambda^* = \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z})\) be the weight lattice in \(t^*\), so that \(\Lambda^*_+ = \Lambda^* \cap t^*_+\) is the monoid of dominant weights. For \(\lambda \in \Lambda^*_+\) let \(V_\lambda\) be the irreducible \(G\)-module with highest weight \(\lambda\), and let

\[
\Pi = \{\varpi_1, \ldots, \varpi_r\}
\]

be the set of fundamental weights, which forms a \(\mathbb{Z}\)-basis of \(\Lambda^*\) and a minimal set of generators for \(\Lambda^*_+\). Recall that \(V_\lambda^* = V_{\lambda^*}\) is the irreducible \(G\)-module with highest weight \(\lambda^*\), where \(\iota : t^* \to t^*\) is the involution given by \(\iota \lambda = -w_0 \lambda\) and \(w_0\) denotes the element of the Weyl group \(W\) of \(G\) such that \(w_0 U_{\text{max}} w_0^{-1} = U_{\text{max}}^{\text{op}}\) is the unipotent radical of the Borel subgroup \(B^{\text{op}}\) of \(G\) which is opposite to \(B \geq U\) in the sense that \(B \cap B^{\text{op}}\) is the complexification \(T_c\) of \(T\) and \(U_{\text{max}} \cap U_{\text{max}}^{\text{op}}\) is the identity subgroup. We have an isomorphism of \(G\)-modules

\[
\mathcal{O}(G)^{U_{\text{max}}} \cong \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda^* \cong \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda
\]

where \(G\) acts on itself on the left and \(U_{\text{max}}\) acts on \(G\) on the right. Note that \(T_c\) normalises \(U_{\text{max}}\) and this isomorphism (9) becomes an isomorphism of \(G \times T_c\)-modules if we let \(T_c\) act on \(V_\lambda\) with weight \(-\lambda\) so that it acts on \(V_\lambda^*\) with weight \(\lambda\) (see [13] §12). Equivalently we have an isomorphism of \(G \times T_c\)-modules

\[
\mathcal{O}(G)^{U_{\text{max}}} \cong \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda^{(T)} \otimes V_\lambda^*
\]

where \(V_\lambda^{(T)}\) is the irreducible \(T_c\)-module with weight \(\lambda\), and by [13] Theorem 12.9 this isomorphism extends to an isomorphism of \(G \times G\)-modules

\[
\mathcal{O}(G) \cong \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda \otimes V_\lambda^*.
\]
In particular the algebra $\mathcal{O}(G)^{U_{\text{max}}}$ is generated by its finite-dimensional vector subspace

$$\bigoplus_{\varpi \in \Pi} V_{\varpi}^* \cong \bigoplus_{\varpi \in \Pi} V_{\varpi}^{(T)} \otimes V_{\varpi}^*.$$  

The inclusion of this finite-dimensional subspace into $\mathcal{O}(G)^{U_{\text{max}}}$ induces a closed $G \times T_c$-equivariant embedding of $\overline{G/U_{\text{max}}^\text{aff}} = \text{Spec}(\mathcal{O}(G)^{U_{\text{max}}})$ into the affine space

$$E = \bigoplus_{\varpi \in \Pi} V_{\varpi} \cong \bigoplus_{\varpi \in \Pi} (V_{\varpi}^{(T)})^* \otimes V_{\varpi},$$  

sending the identity coset $U_{\text{max}}$ in $G/U_{\text{max}} \subseteq \overline{G/U_{\text{max}}^\text{aff}}$ to a sum

$$\sum_{\varpi \in \Pi} v_{\varpi}$$

of highest weight vectors $v_{\varpi} \in V_{\varpi} \cong (V_{\varpi}^{(T)})^* \otimes V_{\varpi}$. Under this embedding $G/U_{\text{max}}$ is identified with $G E^{U_{\text{max}}}$ where $E^{U_{\text{max}}}$ is the subspace of $E$ consisting of vectors fixed by $U_{\text{max}}$. We give $E$ a flat Kähler structure $\omega_E$ via the unique $K \times T$-invariant Hermitian inner product on $E$ which satisfies $\|v_{\varpi}\| = 1$ for each $\varpi \in \Pi$. Then by [15] Proposition 6.8 there is a $K \times T$-equivariant map $\mathcal{F} : K \times t_c^* \rightarrow E$ defined on $t_c^*$ by

$$\mathcal{F}(1, \lambda) = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{r} \sqrt{\lambda(\alpha_j^\vee)} v_{\varpi_j},$$

where $\alpha^\vee = 2\alpha/(\alpha \cdot \alpha)$ and $S = \{\alpha_1, \ldots, \alpha_r\}$ is the set of simple roots corresponding to the fundamental weights $\{\varpi_1, \ldots, \varpi_r\}$ (so that $\varpi_i, \alpha_j^\vee = \delta_{ij}$ for $i, j \in \{1, \ldots, r\}$); moreover $\mathcal{F}$ induces a homeomorphism from $(T^*K)_{\text{impl}}$ to $\overline{G/U_{\text{max}}^\text{aff}}$ whose restriction to each stratum $\mu^{-1}(\sigma)(K_\sigma, K_\sigma)$ of $(T^*K)_{\text{impl}}$ is a symplectic isomorphism onto its image.

**Remark 3.1.** Let $M$ be any compact Kähler manifold on which the complexified torus $T_c$ acts in such a way that $T$ preserves the Kähler structure and has a moment map $\mu_T : M \rightarrow t^*$. In [1] Theorem 2 Atiyah shows

(a) that the image $\mu_T(\hat{Y})$ under the torus moment map $\mu_T$ of the closure $\hat{Y}$ in $M$ of the $T_c$-orbit $Y = T_c m$ of any $m \in M$ is a convex polytope $\mathcal{P}$ whose vertices are the images under $\mu_T$ of the connected components of $\hat{Y} \cap M^T$ where $M^T$ is the $T$-fixed point set in $M$,

(b) that the inverse image in $\hat{Y}$ of each open face of $\mathcal{P}$ consists of a single $T_c$-orbit, and

(c) that $\mu_T$ induces a homeomorphism of $\hat{Y}/T$ onto $\mathcal{P}$.

In fact Atiyah’s proof shows that if $\mathcal{Y} = \exp(it)$ is the orbit of $m \in M$ under the subgroup $\exp(it)$ of $T_c$ then $\mu_T$ restricts to a homeomorphism from $\hat{Y}$ onto $\mathcal{P}$, and the inverse image in $\hat{Y}$ of each open face of $\mathcal{P}$ consists of a single $\exp(it)$-orbit.

We can apply this to the compactification $M = \mathbb{P}(\mathbb{C} \oplus E)$ of the affine space $E$. The moment map $\mu_T^E : E \rightarrow t^*$ for the $T$-action on $E$ with its chosen flat Kähler structure is given (up to multiplication by a positive constant) by

$$\sum_{\varpi \in \Pi} u_{\varpi} \mapsto \sum_{\varpi \in \Pi} \|u_{\varpi}\|^2_{\varpi}$$

when $u_{\varpi} \in V_{\varpi}$ for $\varpi \in \Pi$, while the moment map $\mu_T^{\mathbb{P}(\mathbb{C} \oplus E)} : \mathbb{P}(\mathbb{C} \oplus E) \rightarrow t^*$ for the $T$-action on $\mathbb{P}(\mathbb{C} \oplus E)$ with the induced Fubini-Study Kähler structure is given (up to multiplication by a positive constant) by

$$[z : \sum_{\varpi \in \Pi} u_{\varpi}] \mapsto \sum_{\varpi \in \Pi} \|u_{\varpi}\|^2_{\varpi} / |z|^2 + \sum_{\varpi \in \Pi} \|u_{\varpi}\|^2.$$
when $z \in \mathbb{C}$ and $u_\varpi \in V_\varpi$ for $\varpi \in \Pi$ are not all zero. Comparing these two moment maps on $E$ (regarded as an open subset of $\mathbb{P}(\mathbb{C} \oplus E)$ in the usual way) we see that the image under $\mu^\delta_\mu E$ of the closure $\tilde{\mathcal{Y}}$ in $E$ of the exp(t)-orbit $\mathcal{Y}$ in $E$ of the vector $\sum_{\varpi \in \Pi} v_\varpi$ corresponding to the identity coset $U_{\text{max}}$ in $G/U_{\text{max}}$ is the cone in $t^*$ spanned by the half-lines $\mathbb{R}_+ \varpi$ for $\varpi \in \Pi$, which is of course the positive Weyl chamber $t^*_+$. We also find that the restriction
\[
\mu^\delta_\mu E|_{\tilde{\mathcal{Y}}} : \tilde{\mathcal{Y}} \to t^*_+
\] is a homeomorphism, and it is easy to check that the map $\mathcal{F} : t^*_+ \to E$ of \cite{[12]} Proposition 6.8 defined at \cite{[12]} above can be identified with the composition of the inverse $(\mu^\delta_\mu E)^{-1} : t^*_+ \to \tilde{\mathcal{Y}}$ of \cite{[13]} and the inclusion of $\tilde{\mathcal{Y}}$ in $E$. From this it can be deduced that its $K \times T$-equivariant extension $\mathcal{F} : K \times t^*_+ \to E$ induces a bijection from $(T^*K)_{\text{impl}}$ onto the closure $G/U_{\text{max}}^{\text{aff}}$ of $G(\sum_{\varpi \in \Pi} v_\varpi) \equiv G/U_{\text{max}}$ in $E$ using

(i) the Iwasawa decomposition
\[
G = K \exp(t) U_{\text{max}}
\] of $G$ which tells us that $G/U_{\text{max}}^{\text{aff}} = K \tilde{\mathcal{Y}} = \mathcal{F}(K \times t^*_+)$, and

(ii) Lemma 6.2 of \cite{[15]}, which shows that for each face $\sigma$ of $t^*_+$ the stabiliser in $K$ of
\[
\sum_{\varpi \in \sigma} v_\varpi
\] is $[K_{\sigma}, K_{\sigma}]$.

Guillemin, Jeffrey and Sjamaar also construct a $K \times T$-equivariant desingularisation $(T^*K)_{\text{impl}}$ for the universal imploded cross-section $(T^*K)_{\text{impl}} \cong G/U_{\text{max}}^{\text{aff}}$ and a partial desingularisation $\sim X_{\text{impl}}$ for $X_{\text{impl}}$. In \cite{[15]} §7 they show that if the action of $K$ on $X$ has principal face the interior $(t^*_+)^0$ of $t^*_+$ (where the principal face is the minimal open face $\sigma$ of $t^*_+$ such that $\mu(X) \cap t^*_+$ is contained in $\sigma$), then $\sim X_{\text{impl}}$ can be identified with the symplectic quotient of $X \times (T^*K)_{\text{impl}}$ by the induced action of $K$ (and they observe without proof that the same is true for any principal face). Moreover $(T^*K)_{\text{impl}}$ can be identified as a Hamiltonian $K$-manifold with the homogeneous complex vector bundle
\[
\sim G/U_{\text{max}}^{\text{aff}} = G \times_B E^{U_{\text{max}}}
\] over the flag manifold $G/B$, where the restriction to $G \times E^{U_{\text{max}}}$ of the multiplication map $G \times E \to E$ induces a birational $G$-equivariant morphism
\[
p_{U_{\text{max}}} : \sim G/U_{\text{max}}^{\text{aff}} \to \sim G/U_{\text{max}}^{\text{aff}} = (T^*K)_{\text{impl}} \subseteq E.
\] Note that the fixed point set $E^{U_{\text{max}}}$ of $U_{\text{max}}$ in $E$ is the closure in $E$ of the $T$-orbit of $\sum_{\varpi \in \Pi} v_\varpi$. If $\lambda_0 \in t^*$ is regular dominant and $\epsilon > 0$ is sufficiently close to 0, and if $\omega_0$ is the Kähler form on $G/B$ given by regarding $G/B$ as the coadjoint $K$-orbit through $\epsilon \lambda_0$, then $p_{U_{\text{max}}}^{\epsilon} \omega_E + q^* \omega_0$ is a Kähler form on $\sim G/U_{\text{max}}^{\text{aff}}$ where $q : G \times_B E \to G/B$ is the projection.

It is also shown in \cite{[15]} §7 that the partial desingularisation $\sim X_{\text{impl}}$ can alternatively be obtained from $X_{\text{impl}}$ via a symplectic cut with respect to the $T$-action and the polyhedral cone $\epsilon \lambda_0 + \tau$ where $\tau$ is the principal face of $X$ and $\lambda_0 \in \tau$ and $\epsilon > 0$ is sufficiently close to 0; that is, $\sim X_{\text{impl}}$ is the symplectic reduction at $\epsilon \lambda_0$ for the diagonal $T$-action on the product of $X_{\text{impl}}$ and the symplectic toric manifold associated to the polyhedron $-\tau$ (see \cite{[20]} \cite{[27]}).

### 3.2 Symplectic implosion for the unipotent radical of a parabolic subgroup

Now suppose that $U$ is the unipotent radical of a parabolic subgroup $P$ of the complex reductive group $G$. Recall (see e.g. \cite{[1]} \cite{[2]}) that a parabolic subgroup of $G$ is a closed subgroup which contains some Borel
subgroup, and its unipotent radical is its unique maximal normal unipotent subgroup; thus by replacing $P$ with a suitable conjugate in $G$ if necessary, we can assume that $P$ contains the Borel subgroup $B$ of $G$ and $U \leq U_{\text{max}}$. Then $P = U L^{(P)} \cong U \times L^{(P)}$, where the Levi subgroup $L^{(P)}$ of $P$ contains the complex maximal torus $T_c$ of $G$, and we can assume in addition that $L^{(P)}$ is the complexification of its intersection $K^{(P)} = L^{(P)} \cap K = P \cap K$

with $K$. For some subset $S_P$ of the set $S$ of simple roots, $P$ is the unique parabolic subgroup of $G$ which contains $B$ such that the root space $g_{-\alpha}$ for $\alpha \in S$ is contained in the Lie algebra of $P$ if and only if $\alpha \in S_P$. The Lie algebra of $L^{(P)}$ is generated by the root spaces $g_\alpha$ and $g_{-\alpha}$ for $\alpha \in S_P$ together with the Lie algebra $t_c = t \otimes \mathbb{C}$ of the complexification $T_c$ of $T$. In addition the Lie algebra of $U$ is

where $R^+$ is the set of positive roots for $G$, while the Lie algebra of $P$ is

where $R(S_P)$ is the union of $R^+$ with the set of all roots which can be written as sums of negatives of the simple roots in $S_P$. If we identify $S$ with the set of vertices of the Dynkin diagram of $K$ then the Dynkin diagram of the semisimple part $Q^{(P)} = [K^{(P)}, K^{(P)}]$ of $K^{(P)}$ is the subdiagram given by leaving out the vertices which do not belong to $S_P$. We can decompose $\mathfrak{t}^{(P)} = \text{Lie} K^{(P)}$ and $t$ as

where $[\mathfrak{t}^{(P)}, \mathfrak{t}^{(P)}]$ is the Lie algebra of $Q^{(P)} = [K^{(P)}, K^{(P)}]$, while $\mathfrak{t}^{(P)}$ is the Lie algebra of the maximal torus $T^{(P)} = T \cap [K^{(P)}, K^{(P)}]$ of $Q^{(P)}$, and $\mathfrak{j}^{(P)}$ is the Lie algebra of the centre $Z(K^{(P)})$ of $K^{(P)}$. As before let $B^{\text{op}} = T, U_{\text{max}}^{\text{op}}$ be the Borel subgroup of $G$, with unipotent radical $U_{\text{max}}^{\text{op}}$, which is opposite to $B$ in the sense that $B \cap B^{\text{op}} = T_c$ and $U_{\text{max}} \cap U_{\text{max}}^{\text{op}} = \{1\}$, and let $\iota : \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ be the involution given by $\iota \lambda = -w_0 \lambda$ where $w_0$ denotes the element of the Weyl group $W$ of $G$ such that $w_0 U_{\text{max}} w_0^{-1} = U_{\text{max}}^{\text{op}}$.

By [13] Theorem 2.2 $U$ is a Grosshans subgroup of $G$, and so, just as in the case when $U = U_{\text{max}}$, the ring of invariants $O(G)^U$ is finitely generated and $G/U$ has a canonical affine completion

such that the complement of $G/U$ in $\overline{G/U}^{\text{aff}}$ has codimension two.

**Remark 3.2.** When $U = U_{\text{max}}$ the Iwasawa decomposition

enables us to identify $G/U_{\text{max}}$ with $K \exp(itU_{\text{max}})$. More generally we have an analogous decomposition

which enables us to identify $G/U$ with $K \exp(it^{(P)})U$.

Let $X$ be a complex projective variety on which $G$ acts linearly with respect to a very ample line bundle $L$, and let $\omega$ be an associated $K$-invariant Kähler form on $X$. Then it follows by the Borel transfer theorem (see e.g. [5] Lemma 4.1) that

is finitely generated, and the associated projective variety

$$X//U = \text{Proj}(\hat{O}_L(X)^U)$$

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is isomorphic to the GIT quotient $(\overline{G/U^\text{aff}} \times X)/G$. Just as in the case when $U = U_{\max}$, if we have a suitable $K$-invariant Kähler form on $G/U^\text{aff}$, then we will be able to identify $X/U$ with a symplectic quotient of $\overline{G/U^\text{aff}} \times X$ by $K$, and obtain a symplectic description of $X/U$ analogous to symplectic implosion, with $G/U^\text{aff}$ playing the rôle of the universal impled cross-section $(T^*K)^\text{impl}$. As is observed in [15] §6, the easiest case is when $K$ is semisimple and simply connected (for example when $K = SU(r + 1)$); for general compact connected $K$ one can reduce to this case by considering the product $\overline{K}/\overline{\mathcal{Y}}$, where $\mathcal{Y}$ is a finite central subgroup of $\overline{K}$.

Therefore, as in the previous subsection, let $K$ be a semisimple, connected and simply connected compact group, let $\Lambda = \ker(\exp|_1)$ be the exponential lattice in $t$, and let $\Lambda^* = \hom_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ be the weight lattice in $t^*$, so that $\Lambda^*_+ = \Lambda^* \cap t^*_+$ is the monoid of dominant weights. For $\lambda \in \Lambda^*_+$ let $V_\lambda$ be the irreducible $G$-module with highest weight $\lambda$, and let $\Pi = \{\varpi_1, \ldots, \varpi_r\}$ be the set of fundamental weights, forming a $\mathbb{Z}$-basis of $\Lambda^*$ and a minimal set of generators for $\Lambda^*_+$. Recall that we have an isomorphism of $G \times G$-modules

$$\mathcal{O}(G) \cong \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda \otimes V_\lambda^* \cong \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda \otimes V_{\lambda^*}$$

which restricts to an isomorphism of $G \times T_e$-modules

$$\mathcal{O}(G)^{U_{\text{max}}} \cong \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda^{(T)} \otimes V_\lambda^* \cong \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda^*$$

which is generated as an algebra by its finite-dimensional vector subspace

$$E^* = \bigoplus_{\varpi \in \Pi} V_{\varpi}^*$$

giving us a closed $G \times T_e$-equivariant embedding of $\overline{G/U_{\max}} = \text{Spec}(\mathcal{O}(G)^{U_{\max}})$ into the affine space $E$ equipped with a flat Kähler structure. We have seen how Guillemin, Jeffrey and Sjamaar identify $(T^*K)^\text{impl}$ with $\overline{G/U^\text{aff}}_{\max}$ equipped with the Kähler structure obtained from this embedding in $E$. To extend their construction to $\overline{G/U^\text{aff}}_{\max}$ when $U$ is the unipotent radical of a parabolic subgroup $P \geq B$ as above, we first observe from the proof of [14] Theorem 2.2 that $\mathcal{O}(G)^U$ is generated by any finite-dimensional $L^{(P)}$-invariant (or equivalently $K^{(P)}$-invariant) vector subspace of

$$\mathcal{O}(G) \cong \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda \otimes V_\lambda^* \cong \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda \otimes V_{\lambda^*}$$

which contains

$$E^* = \bigoplus_{\varpi \in \Pi} V_{\varpi}^* \cong \bigoplus_{\varpi \in \Pi} V_{\varpi}^{(T)} \otimes V_{\varpi}^*.$$  

Here as above $V_{\varpi}^{(T)}$ is the irreducible $T_e$-equivariant vector subspace of $V_\varpi$ while $K^{(P)} = K \cap L^{(P)} = K \cap P$ is a maximal compact subgroup of the Levi subgroup $L^{(P)} = K \cap P$ of $P$, and $K^{(P)}$ acts on $\mathcal{O}(G)$ via left multiplication on $G$.

Let $E^{(P)}$ be the dual of the smallest $K^{(P)}$-invariant subspace $(E^{(P)})^*$ of $\mathcal{O}(G)$ containing $E^*$; then $(E^{(P)})^*$ is fixed pointwise by $U$ since $K^{(P)}$ normalises $U$ and $U$ is a subgroup of $U_{\max}$ which fixes $E$ pointwise. The inclusion of $(E^{(P)})^*$ in $\mathcal{O}(G)^U \subseteq \mathcal{O}(G)$ induces a closed $L^{(P)} \times G$-equivariant embedding of $\overline{G/U^\text{aff}} = \text{Spec}(\mathcal{O}(G)^U)$ into the affine space $E^{(P)}$, whose projection to $E$ induces the embedding of $\overline{G/U_{\max}}$ described in the previous subsection.

$(E^{(P)})^*$ decomposes under the action of $K \times K^{(P)}$ as a direct sum of irreducible $K \times K^{(P)}$-modules

$$(E^{(P)})^* = \bigoplus_{\varpi \in \Pi} (V_{\varpi}^{(P)})^*$$
where \((V^{(P)})^*\) is the smallest \(K \times K^{(P)}\)-invariant subspace of \(\mathcal{O}(G)\) containing \(V^{*}_{\varpi}\). As in [3, §12] we have

\[
(V^{(P)})^* \cong V^{K^{(P)}} \otimes V^{*}_{\varpi}
\]

where \(V^{K^{(P)}}\) is the irreducible \(K^{(P)}\)-module with highest weight \(\varpi\), so

\[
E^{(P)} = \bigoplus_{\varpi \in \Pi} V^{(P)}_{\varpi} = \bigoplus_{\varpi \in \Pi} (V^{K^{(P)}})^{*}_{\varpi} \otimes V^{*}_{\varpi}.
\]

Moreover, if \(v^{(P)}_{\varpi}\) is the vector in \(V^{(P)}_{\varpi} \cong (V^{K^{(P)}})^*_{\varpi} \otimes V^{*}_{\varpi}\) representing the inclusion of \(V^{K^{(P)}}_{\varpi}\) in \(V^{*}_{\varpi}\) then the embedding of \(G/U \subseteq \overline{G/U^{\text{aff}}}\) in \(E^{(P)}\) induced by the inclusion of \((E^{(P)})^*\) in \(\mathcal{O}(G)^{U}\) takes the identity coset \(U\) to \(\sum_{\varpi \in \Pi} v^{(P)}_{\varpi}\). Let

\[
V^{K^{(P)}}_{\varpi} = \bigoplus_{\lambda \in \Lambda^{*}_{\varpi}} V^{K^{(P)}_{\varpi}}_{\lambda}
\]

be the decomposition of \(V^{K^{(P)}}_{\varpi}\) into weight spaces with weights \(\lambda \in t^*\) under the action of the maximal torus \(T\) of \(K^{(P)}\). Then \(V^{(P)}_{\varpi}\) decomposes as a \(K \times T\)-module into a sum of irreducible \(K \times T\)-modules

\[
V^{(P)}_{\varpi} \cong \bigoplus_{\lambda \in \Lambda^{*}_{\varpi}} V^{(P)}_{\varpi} \otimes (V^{K^{(P)}_{\varpi}})^{*}_{\lambda}
\]

and \(v^{(P)}_{\varpi} = \sum_{\lambda} v^{(P)}_{\varpi,\lambda}\) where \(v^{(P)}_{\varpi,\lambda} \in V^{(P)}_{\varpi} \otimes (V^{K^{(P)}_{\varpi}})^{*}_{\lambda}\) represents the inclusion of \(V^{K^{(P)}_{\varpi}}_{\lambda}\) in \(V^{*}_{\varpi}\). In particular \(v^{(P)}_{\varpi,\lambda}\) is a highest weight vector for the action of \(K \times K^{(P)}\) on \(V^{(P)}_{\varpi}\).

**Remark 3.3.** The embedding of \(G/U \subseteq \overline{G/U^{\text{aff}}}\) in \(E^{(P)}\) induced by the inclusion of \((E^{(P)})^*\) in \(\mathcal{O}(G)^{U}\) takes the identity coset to \(\sum_{\varpi \in \Pi} v^{(P)}_{\varpi}\). From the decomposition \(G = K \exp(\mathfrak{t}(P))U\) (see Remark 3.2 above) and the compactness of \(K\) it follows that the closure \(\overline{G/U^{\text{aff}}}\) of the \(G\)-orbit of \(\sum_{\varpi \in \Pi} v^{(P)}_{\varpi}\) in \(E^{(P)}\) is given by the \(K\)-sweep

\[
\overline{G/U^{\text{aff}}} = \text{Ker}(\exp(\mathfrak{t}(P)) \sum_{\varpi \in \Pi} v^{(P)}_{\varpi})
\]

of the closure in \(E^{(P)}\) of the \(\exp(\mathfrak{t}(P))\)-orbit of \(\sum_{\varpi \in \Pi} v^{(P)}_{\varpi}\). Similarly the closure in \(E^{(P)}\) (or equivalently in the linear subspace \(\bigoplus_{\varpi \in \Pi} (V^{K^{(P)}_{\varpi}})^* \otimes V^{K^{(P)}_{\varpi}}\) of \(E^{(P)}\)) of the \(L^{(P)}\)-orbit of \(\sum_{\varpi \in \Pi} v^{(P)}_{\varpi}\) (which is a free orbit since \(U \cap L^{(P)} = \{1\}\)) is given by \(K^{(P)}(\exp(\mathfrak{t}(P)) \sum_{\varpi \in \Pi} v^{(P)}_{\varpi})\). Note also that \(\mathfrak{t}(P) = \bigcup_{k \in K^{(P)}} \text{Ad}(k)\mathfrak{t}\) and so

\[
\exp(\mathfrak{t}(P)) = \bigcup_{k \in K^{(P)}} k \exp(\mathfrak{t}) k^{-1}.
\]

Let \(S_{P} = \{\alpha_{1}, \ldots, \alpha_{r(P)}\} \subseteq S = \{\alpha_{1}, \ldots, \alpha_{r}\}\) be the set of simple roots for the root system of \((K^{(P)}, T)\) with corresponding positive Weyl chamber

\[
\mathfrak{t}_{+,P} = \{\zeta \in t^* : \zeta \cdot \alpha \geq 0\text{ for all }\alpha \in S_{P}\} = \mathfrak{t}_{+}^{(P)} \oplus \mathfrak{z}_{(P)}^{*}
\]

where \(\mathfrak{z}_{(P)}^{*}\) is the Lie algebra of the centre \(Z(K^{(P)}) \leq T\) of \(K^{(P)}\) and \(\mathfrak{t}_{+}^{(P)}\) is the positive Weyl chamber for the semisimple part

\[
Q^{(P)} = [K^{(P)}, K^{(P)}]
\]

of \(K^{(P)}\) with respect to the maximal torus \(T^{(P)} = T \cap [K^{(P)}, K^{(P)}]\) of \(Q^{(P)}\) and simple roots given by restricting \(S_{P}\) to \(T^{(P)}\). If \(\varpi \cdot \alpha = 0\) for all \(\alpha \in S_{P}\) (or equivalently if \(\varpi = \varpi_{j}\) for \(j > r(P)\)) then \(\varpi \in \mathfrak{z}_{(P)}^{*}\) and \(V^{K^{(P)}}_{\varpi}\) is one-dimensional; in this situation \(Q^{(P)}\) acts trivially on \(V^{K^{(P)}}_{\varpi}\) and we have \(V^{K^{(P)}}_{\varpi} = V^{K^{(P)}_{\varpi}}\) with \(V^{K^{(P)}_{\varpi}} = 0\) if \(\lambda \neq \varpi\) and \(v^{K^{(P)}_{\varpi}} = v^{K^{(P)}_{\varpi,\lambda}}\) while \(v^{K^{(P)}_{\varpi,\lambda}} = 0\) if \(\lambda \neq \varpi\). On the other hand if \(j \leq r(P)\) then \(\varpi = \varpi_{j}\) restricts to a fundamental weight for \(Q^{(P)}\) and \(V^{K^{(P)}}_{\varpi} = V^{Q^{(P)}}_{\varpi}\) is the irreducible \(Q^{(P)}\)-module with highest weight \(\varpi|_{Q^{(P)}}\) on which \(Z(K^{(P)})\) acts as scalar multiplication by \(\varpi|_{Z(K^{(P)})}\).
There is a unique $K \times K^{(P)}$-invariant Hermitian inner product on $E^{(P)} = \bigoplus_{\varpi \in \Pi} V^{(P)}_{\varpi}$ satisfying $\|v^{(P)}_{\varpi}\| = 1$ for each $\varpi \in \Pi$, which is obtained from $K$-invariant Hermitian inner products on the irreducible $K$-modules $V_{\varpi}$ and their restrictions to $K^{(P)}$-invariant Hermitian inner products on the irreducible $K^{(P)}$-modules $V^{(P)}_{\varpi}$. This gives $E^{(P)}$ a flat Kähler structure which is $K \times K^{(P)}$-invariant.

**Remark 3.4.** Recall that

$$E^{(P)} = \bigoplus_{\varpi \in \Pi} V^{(P)}_{\varpi} = \bigoplus_{\varpi \in \Pi} (V^{K^{(P)}}_{\varpi})^* \otimes V_{\varpi}$$

where $V^{K^{(P)}}_{\varpi} \subseteq V_{\varpi}$, and the embedding of $G/U \subseteq \overline{G/U}^{\text{aff}}$ in $E^{(P)}$ induced by the inclusion of $(E^{(P)})^*$ in $\mathcal{O}(G)^U$ takes the identity coset $U$ to $\sum_{\varpi \in \Pi} v^{(P)}_{\varpi}$ where $v^{(P)}_{\varpi} \in V^{(P)}_{\varpi} \cong (V^{K^{(P)}}_{\varpi})^* \otimes V_{\varpi}$ represents the inclusion of $V^{K^{(P)}}_{\varpi}$ in $V_{\varpi}$. Thus

$$\sum_{\varpi \in \Pi} v^{(P)}_{\varpi} \in \bigoplus_{\varpi \in \Pi} (V^{K^{(P)}}_{\varpi})^* \otimes V^{K^{(P)}}_{\varpi} \subseteq E^{(P)}$$

where $\bigoplus_{\varpi \in \Pi} (V^{K^{(P)}}_{\varpi})^* \otimes V^{K^{(P)}}_{\varpi}$ is invariant under the action of the subgroup $K^{(P)} \times K^{(P)}$ of $K \times K^{(P)}$ on $E^{(P)}$, and indeed is invariant under the action of $L^{(P)} \times L^{(P)}$. If we identify $(V^{K^{(P)}}_{\varpi})^* \otimes V^{K^{(P)}}_{\varpi}$ with $\text{End}(V^{K^{(P)}}_{\varpi})$ equipped with the Hermitian structure

$$\langle A, B \rangle = \text{Trace}(AB^*)$$

in the standard way, then $v^{(P)}_{\varpi}$ is identified with the identity map in $\text{End}(V^{K^{(P)}}_{\varpi})$. If $V$ is any Hermitian vector space then the moment map for the action of the product of unitary groups $U(V) \times U(V)$ on $End(V)$ by left and right multiplication is given (up to a nonzero real scalar) by

$$A \mapsto (iAA^*, iA^*A)$$

(cf. [30] §3.3). Thus the moment map for the action of $K^{(P)} \times K^{(P)}$ on

$$\bigoplus_{\varpi \in \Pi} (V^{K^{(P)}}_{\varpi})^* \otimes V^{K^{(P)}}_{\varpi} \cong \bigoplus_{\varpi \in \Pi} \text{End}(V^{K^{(P)}}_{\varpi})$$

is given (up to multiplication by a nonzero real scalar) by

$$\sum_{\varpi \in \Pi} A_{\varpi} \mapsto (\pi^{K^{(P)}}(\sum_{\varpi \in \Pi} iA_{\varpi}A^*_{\varpi}), \pi^{K^{(P)}}(\sum_{\varpi \in \Pi} iA^*_{\varpi}A_{\varpi}))$$

(24)

where $\pi^{K^{(P)}}: u(\bigoplus_{\varpi \in \Pi} V^{K^{(P)}}_{\varpi})^* \rightarrow (\mathfrak{k}^{(P)})^*$ is the projection induced by the inclusion of $K^{(P)}$ as a subgroup of the unitary group $U(\bigoplus_{\varpi \in \Pi} V^{K^{(P)}}_{\varpi})$. In particular if $g$ belongs to the complexification $L^{(P)}$ of $K^{(P)}$ and $g_{\varpi}: V^{K^{(P)}}_{\varpi} \rightarrow V^{K^{(P)}}_{\varpi}$ is the action of $g$ on $V^{K^{(P)}}_{\varpi}$, then

$$g \sum_{\varpi \in \Pi} v^{(P)}_{\varpi} = \sum_{\varpi \in \Pi} g_{\varpi}$$

and the moment map for the left $K^{(P)}$-action sends this to

$$\pi^{K^{(P)}}(\sum_{\varpi \in \Pi} ig_{\varpi}g^*_{\varpi}) \in \mathfrak{k}^{(P)}.$$

Using the decomposition $\mathfrak{k}^{(P)} = [\mathfrak{k}^{(P)}, \mathfrak{k}^{(P)}] \oplus \mathfrak{z}^{(P)}$ we can decompose $\pi^{K^{(P)}}: u(\bigoplus_{\varpi \in \Pi} V^{K^{(P)}}_{\varpi})^* \rightarrow (\mathfrak{k}^{(P)})^*$ as

$$\pi^{K^{(P)}} = \pi^{[K^{(P)}, K^{(P)}]} \oplus \pi^{\mathfrak{z}^{(P)}}: u(\bigoplus_{\varpi \in \Pi} V^{K^{(P)}}_{\varpi})^* \rightarrow [\mathfrak{k}^{(P)}, \mathfrak{k}^{(P)}]^* \oplus \mathfrak{z}^{(P)}^*.$$

(25)

If $g = yz$ with $y \in [L^{(P)}, L^{(P)}] = Q^{(P)}_c$ and $z \in Z(L^{(P)}) = Z(K^{(P)})_c$, then the $K^{(P)}$-moment map above sends $g \sum_{\varpi \in \Pi} v^{(P)}_{\varpi}$ to

$$\pi^{[K^{(P)}, K^{(P)}]}(\sum_{1 \leq j \leq r^{(P)}} iy^{(P)}_{\varpi j} y^{(P)*}_{\varpi j}) + \pi^{\mathfrak{z}^{(P)}}(\sum_{1 \leq j \leq r^{(P)}} iz^{(P)}_{\varpi j} z^{(P)*}_{\varpi j}) \in [\mathfrak{k}^{(P)}, \mathfrak{k}^{(P)}]^* \oplus \mathfrak{z}^{(P)}^*. $$
It follows by the arguments of [30] §3 (in particular Proposition 3.10) that the $T_c^{(P)}$-orbit of $\sum_{\varpi \in \Pi} v^{(P)}_{\varpi}$ is mapped diffeomorphically onto $t^{(P)}$ by the moment map

$$y \sum_{\varpi \in \Pi} v^{(P)}_{\varpi} \mapsto \pi^{(P)}(\sum_{1 \leq j \leq r(P)} iy_{\varpi_j} y^{*}_{\varpi_j})$$

(26)

for the action of $T^{(P)}$ on $E^{(P)}$, since its image in the projective space $\mathbb{P}(E^{(P)})$ is mapped diffeomorphically by the associated moment map onto the convex hull of the set $\{w \varpi : \varpi \in \Pi, w \in W^{(P)}\}$ where $W^{(P)}$ is the Weyl group of $Q^{(P)} = [K^{(P)}, K^{(P)}]$ (cf. Remark 3.3).

Now consider the moment map $\mu^{E^{(P)}}_T$ for the restriction to $T$ of the $K^{(P)}$-action on $E^{(P)}$. This is given (up to multiplication by a positive constant) by

$$\sum_{\varpi, \lambda} u_{\varpi, \lambda} \mapsto \sum_{\varpi, \lambda} |u_{\varpi, \lambda}|^2 \lambda$$

when $u_{\varpi, \lambda} \in V^{K^{(P)}}_{\varpi, \lambda}$ for $\varpi \in \Pi$ and $\lambda \in \Lambda^*_+ \subseteq \Lambda^*$. The embedding of $G/U \subseteq G/U^{aff}$ in $E^{(P)}$ induced by the inclusion of $(E^{(P)})^*$ in $O(G)^*$ takes the coset of $t \in T_c$ to

$$\sum_{\varpi, \lambda} \lambda(t)^{-1} v^{(P)}_{\varpi, \lambda},$$

so the value taken by this moment map on the coset $tU$ of $t = t_1 t_2 \in T_c$ where $t_1 \in T_c^{(P)}$ and $t_2 \in Z(L^{(P)}) = Z(K^{(P)})_c$ is given by

$$\sum_{\varpi, \lambda} |\lambda(t)|^{-2} |v^{(P)}_{\varpi, \lambda}|^2 \lambda = \sum_{j=1}^{v^{(P)}} |\varpi_j(t_2)|^{-2} \sum_{\lambda} |\lambda(t_1)|^{-2} |v^{(P)}_{\varpi_j, \lambda}|^2 \lambda + \sum_{j=r(P)+1}^{r(P)} |\varpi_j(t_2)|^{-2} |v^{(P)}_{\varpi_j, \lambda}|^2 \lambda + \sum_{j=1}^{r(P)} |\varpi_j(t_2)|^{-2} \sum_{\lambda} |\lambda(t_1)|^{-2} |v^{(P)}_{\varpi_j, \lambda}|^2 \lambda |_{T^{(P)}}$$

(27)

where the $j$th sum over $\lambda$ runs over all the weights of the irreducible $K^{(P)}$-module $V^{K^{(P)}}_{\varpi_j, \lambda}$ with highest weight $\varpi_j$. When we decompose $t^*$ as $t^{(P)*} \oplus j^{(P)*}$ this has component $\sum_{j=1}^{r(P)} |\varpi_j(t_2)|^{-2} |v^{(P)}_{\varpi_j, \lambda}|^2 \lambda |_{Z(K^{(P)})}$ in $j^{(P)*}$ and $\sum_{j=1}^{r(P)} |\varpi_j(t_2)|^{-2} \sum_{\lambda} |\lambda(t_1)|^{-2} |v^{(P)}_{\varpi_j, \lambda}|^2 \lambda |_{T^{(P)}}$ in $t^{(P)*}$.

**Definition 3.5.** Let $t_{(P)^*}^+$ be the cone

$$t_{(P)^*}^+ = \bigcup_{w \in W^{(P)}} \text{Ad}^*(w)t_{(P)^*}^+$$

in $t^*$, where $W^{(P)}$ is the Weyl group of $Q^{(P)} = [K^{(P)}, K^{(P)}]$ (which is a subgroup of the Weyl group $W$ of $K$).

**Lemma 3.6.** The restriction to the closure $\exp(it) \sum_{\varpi \in \Pi} v^{(P)}_{\varpi}$ of the $\exp(it)$-orbit in $E^{(P)}$ of $\sum_{\varpi \in \Pi} v^{(P)}_{\varpi}$ of the moment map $\mu^{E^{(P)}}_T$ for the action of $T$ on $E^{(P)}$ is a homeomorphism onto the cone $t_{(P)^*}^+$ in $t^*$. Its inverse provides a continuous injection

$$\mathcal{F}^{(P)} : t_{(P)^*}^+ \to \overline{G/U^{aff}} \subseteq E^{(P)}$$

(28)

such that $\mu^{E^{(P)}}_T \circ \mathcal{F}^{(P)}$ is the identity on $t_{(P)^*}^+$. Moreover $\exp(it) \sum_{\varpi \in \Pi} v^{(P)}_{\varpi}$ is the union of finitely many $\exp(it)$-orbits, each of the form

$$\mathcal{F}^{(P)}(\sigma) = \exp(it) \sum_{\varpi \in \Pi, \lambda \in \Lambda^* \cap \sigma} v^{(P)}_{\varpi, \lambda}$$

where $\sigma$ is an open face of $t_{(P)^*}^+$. 

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Proof. This follows by applying the results of \( \Xi \) to the compactification \( \mathbb{P}(\mathbb{C} \oplus E^{(P)}) \) of the affine space \( E^{(P)} \), as in Remark 3.1 and observing that the convex hull of the weights \( \lambda \) of the \( T \)-action on the \( K^{(P)} \)-module \( V^{(P)}_{\mathfrak{m}} \) is the convex hull of \( \{ w \varpi : w \in W^{(P)} \} \), and thus the convex hull of the half-lines \( \mathbb{R}_+ \lambda \) for \( \lambda \in \Lambda_{\mathfrak{m}} \) with \( \varpi \in \Xi \) is the cone \( t^{(P)}_+ \).

Lemma 3.7. (cf. [12] Lemma 3.12) The image of the closure \( T_c \sum_{\varpi \in \Xi} v^{(P)}_{\varpi} \) of the \( T_c \)-orbit in \( E^{(P)} \) of \( \sum_{\varpi \in \Xi} v^{(P)}_{\varpi} \) under the \( K^{(P)} \)-moment map \( \mu^{E^{(P)}} : E^{(P)} \to (t^{(P)}_+)^* \) is contained in \( t \).

Proof: The orthogonal complement to \( t \) in \( t^{(P)}_+ \) is \( [t^{(P)}_+, t] \), and if \( \zeta \in t \) and \( \xi \in t^{(P)}_+ \) and \( t \in T_c \) then by Remark 3.4

\[
\mu^{E^{(P)}}(t \sum_{\varpi \in \Xi} v^{(P)}_{\varpi}) : \xi, \zeta = \sum_{\varpi \in \Xi} \text{Trace}(i[\xi, \zeta] t_{\varpi} t^{*}_{\varpi}) = \sum_{\varpi \in \Xi} \text{Trace}(i[\xi, t_{\varpi} t^{*}_{\varpi}]) = 0
\]
since \( [\zeta, t_{\varpi} t^{*}_{\varpi}] = 0 \).

Corollary 3.8. The restriction of the \( K^{(P)} \)-moment map \( \mu^{E^{(P)}} : E^{(P)} \to (t^{(P)}_+)^* \) to the closure

\[
\exp(\mathfrak{t}^{(P)}_+) \sum_{\varpi \in \Xi} v^{(P)}_{\varpi}
\]
of the \( \exp(\mathfrak{t}^{(P)}_+) \)-orbit in \( E^{(P)} \) of \( \sum_{\varpi \in \Xi} v^{(P)}_{\varpi} \) is a homeomorphism from \( \exp(\mathfrak{t}^{(P)}_+) \sum_{\varpi \in \Xi} v^{(P)}_{\varpi} \) onto the closed subset

\[
t^{(P)}_+ = \text{Ad}^* (K^{(P)}) t^{(P)}_+
\]
of \( t^{(P)}_+ \). Moreover \( \exp(\mathfrak{t}^{(P)}_+) \sum_{\varpi \in \Xi} v^{(P)}_{\varpi} \) is the union of finitely many \( \exp(\mathfrak{t}^{(P)}_+) \)-orbits which correspond under this homeomorphism to the open faces of \( t^{(P)}_+ \).

Proof: We have already observed that the restriction of the \( T \)-moment map \( \mu^{E^{(P)}}_T : E^{(P)} \to t^* \) to the closure

\[
\exp(\mathfrak{t}^+) \sum_{\varpi \in \Xi} v^{(P)}_{\varpi}
\]
of the \( \exp(\mathfrak{t}^+) \)-orbit of the image \( \sum_{\varpi \in \Xi} v^{(P)}_{\varpi} \) of the identity coset \( U \) under the embedding of \( G/U \) in \( E^{(P)} \) is a homeomorphism from this closure onto the cone \( t^*_+ \). Since \( \mu^{E^{(P)}}_T \) is the projection of \( \mu^{E^{(P)}} \) onto \( t^* \), it follows immediately from Lemma 3.7 above that the restriction of \( \mu^{E^{(P)}} : E^{(P)} \to (t^{(P)}_+)^* \) to this closure \( \exp(\mathfrak{t}^+) \sum_{\varpi \in \Xi} v^{(P)}_{\varpi} \) is a homeomorphism onto the cone \( t^+_+ \) when \( t^* \) is identified with \( t \subseteq t^{(P)}_+ \) via the restriction of the fixed invariant inner product on \( t \). Replacing the maximal torus \( T \) with \( kTK^{-1} \) for any \( k \in K^{(P)} \) it follows that the restriction of \( \mu^{E^{(P)}} : E^{(P)} \to (t^{(P)}_+)^* \) to the closure \( k \exp(\mathfrak{t}^+) k^{-1} \sum_{\varpi \in \Xi} v^{(P)}_{\varpi} \) of the \( \exp(\mathfrak{t}^+) \)-orbit of the image \( \sum_{\varpi \in \Xi} v^{(P)}_{\varpi} \) of the identity coset \( U \) under the embedding of \( G/U \) in \( E^{(P)} \) is a homeomorphism onto the cone \( \text{Ad}^* (k) t^*_+ \). Putting these homeomorphisms together for \( k \in K^{(P)} \) we get a homeomorphism \( \mathcal{M} \) from

\[
Z = \{(kN^{(P)}_T, x) \in K^{(P)}/N^{(P)}_T \times E^{(P)} : x \in k \exp(\mathfrak{t}^+) k^{-1} \sum_{\varpi \in \Xi} v^{(P)}_{\varpi} \}
\]
where \( N^{(P)}_T \) is the normaliser of \( T \) in \( K^{(P)} \), to

\[
K^{(P)} \times N^{(P)}_T t^*_+ = \{(kN^{(P)}_T, \xi) \in K^{(P)}/N^{(P)}_T \times t^*_+ \mid \xi \in \text{Ad}^* (k) t^*_+ \}
\]
which fits into a diagram

\[
\begin{array}{ccc}
Z & \rightarrow & K^{(P)} \times N^{(P)}_T t^*_+ \\
\downarrow & & \downarrow \beta \\
\exp(\mathfrak{t}^{(P)}_+) \sum_{\varpi \in \Xi} v^{(P)}_{\varpi} & \rightarrow & \bigcup_{k \in K^{(P)}} \text{Ad}^* (k) t^*_+ = t^{(P)}_+
\end{array}
\]
where the first horizontal map is the homeomorphism $\mathcal{M}$ and the second is $\mu^{E(P)}$. Since the image of $\alpha$ is dense and $K^{(P)}$ is compact, it follows that $\alpha$ is surjective. Moreover $\beta$ is surjective, and $\beta(kN_T^{(P)}, \xi) = \beta(k'N_T^{(P)}, \xi')$ if and only if $Ad^*(k'k^{-1})\xi$ lies in an open face $\sigma$ of $t_+$ such that $k'k^{-1} \in K_\sigma^{(P)}$, in which case $\alpha(M^{-1}(kN_T^{(P)}, \xi)) = \alpha(M^{-1}(kN_T^{(P)}, \xi))$. Thus

$$
\mu^{E(P)} : \exp(i\mathcal{H}(P)) \sum_{w \in \Pi} \tilde{v}_w^{(P)} \mapsto v_+^{(P)}
$$

is a continuous bijection, which is a homeomorphism since $K$ is compact and $\mathcal{M}$ is a homeomorphism.

The inverse of $\mu^{E(P)} : \exp(i\mathcal{H}(P)) \sum_{w \in \Pi} \tilde{v}_w^{(P)} \mapsto v_+^{(P)}$ gives us a continuous $K^{(P)}$-equivariant map

$$
\mathcal{F}^{(P)} : \chi(P)^* \mapsto G/U \subseteq E^{(P)}
$$

extending (28) such that $\mu^{E(P)} \circ \mathcal{F}^{(P)}$ is the identity on $\chi(P)^*$. This in turn extends to a continuous $K \times K^{(P)}$-equivariant map

$$
\mathcal{F}^{(P)} : K \times \chi(P)^* \mapsto G/U \subseteq E^{(P)}
$$

which is surjective since $G/U = K(\exp(i\mathcal{H}(P)) \sum_{w \in \Pi} \tilde{v}_w^{(P)})$ by Remark 3.3.

**Definition 3.9.** If $\xi \in \chi(P)^*$ is $Ad^*(K^{(P)}) t_+^{(P)^*}$, let $\zeta = Ad^*(k)\xi$ with $k \in K^{(P)}$ and $\xi \in t_+$, and let $\sigma_0$ be the open face of $t_+^{(P)^*}$ containing $\xi$. Let $\sigma_0(P)$ be the open face of $t_+^{(P)^*}$, and $\overline{\sigma_0(P)} = \{ \zeta \in t^* : \zeta \cdot c = 0 \text{ for all } c \in R_{\sigma_0} \setminus R^{(P)} \}$

where $R$ and $R^{(P)}$ are the sets of roots of $K$ and $K^{(P)}$, and $R_{\sigma_0} = \{ \alpha \in R : \zeta \cdot c = 0 \text{ for all } \zeta \in \sigma_0 \}$, so that $\sigma_0(P)$ is an open subset of the open face containing $\sigma_0$ of the cone $t_+^{(P)^*}$. Finally let $K_\xi(P) = kK_\xi k^{-1}$ where $K_\xi(P) = K_{\sigma_0}(P)$ is the stabiliser under the adjoint action of $K$ of any element of $\sigma_0(P)$.

Note that $K_\xi(P) \leq K_\xi$ for any $\xi \in \chi(P)^*$.

**Lemma 3.10.** (cf. [15] Lemma 6.2)

Let $\sigma$ be an open face of $t_+^{(P)^*}$ and let

$$
\nu_\sigma^{(P)} = \sum_{w \in \Pi, \lambda \in Ad^*(W(P)) \varpi \cap \sigma} \tilde{v}_w^{(P)}
$$

If $\xi \in \sigma$ then the stabiliser of $\nu_\sigma^{(P)}$ in $K$ is $[K_\xi(P), K_\xi(P)]$.

**Proof:** Recall that $t_+^{(P)^*} = \bigcup_{w \in W^{(P)}} Ad^*(w) t_+$, so there is an element $w_0$ of the Weyl group $W^{(P)}$ of $Q^{(P)} = [K^{(P)}, K^{(P)}]$ such that $Ad^*(w_0)\zeta \in t_+^{(P)}$ and $Ad^*(w_0)\sigma$ contains an open face $\sigma_0$ of $t_+^{(P)}$ such that $w_0$ contains an open subset of $\sigma$. First assume that $\xi = Ad^*(w_0)\zeta$ lies in $\sigma_0$. Then if $w \in \Pi$ and $w \in W^{(P)}$ we have $Ad^*(w) \varpi \in \sigma$ if and only if $Ad^*(w) \varpi \in \varpi$ if and only if $\varpi \in \varpi$ if and only if $\varpi$ is in the linear subspace of $t_+$ spanned by $\Pi \cap \sigma = \Pi \cap \sigma_0$, and since $\xi \in \sigma_0$ this happens if and only if $K_\xi \leq wK_{w^{-1}}$, so that

$$
\bigcap_{w \in \Pi, \varpi \varpi \in W^{(P)} \varpi \cap \sigma} wK_{w^{-1}} = K_\xi.
$$

As in the proof of [15] Lemma 6.2 we find that if $w \in W^{(P)}$ the stabiliser in $G = K_c$ of $[\nu_\sigma^{(P)}] = \mathbb{P}(\mathbb{C}[V_{K^{(P)}}]^*) \otimes V_\sigma$ is $wP_{w^{-1}}$, where $P_{w^{-1}}$ is the parabolic subgroup of $G$ associated to $\varpi$, and thus the stabiliser in $K$ of $\nu_\sigma^{(P)}$ is the conjugate by $w_0$ of

$$
\{ k \in \bigcap_{w \in \Pi, \varpi \varpi \in W^{(P)} \varpi \cap \sigma} wK_{w^{-1}} : \lambda(g) = 1 \text{ for all } \lambda \in \Lambda^* \cap \bar{\sigma} \}
$$

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\( \{ k \in K_\xi \mid \lambda(g) = 1 \text{ for all } \lambda \in \Lambda^* \cap \sigma \} = \{ K_\xi, K_\xi \} = [K_\sigma, K_\sigma], \) (31)

In general if \( \xi = \text{Ad}^*(w_0) \xi \) lies in \( t_+^* \cap \text{Ad}^*(w_0) \sigma \) then there is a unique open face \( \sigma_0 \) of \( t_+^* \) containing \( \xi \). Let \( \sigma_0(P) \) be as in Definition 3.9 then \( \overline{\sigma} \cap t_+^* = \overline{\sigma_0(P)} \), and so by the previous paragraph the stabiliser of \( w_0^{(P)} \) in \( K \) is

\[
 w_0[K_{\sigma_0(P)}, K_{\sigma_0(P)}]w_0^{-1} = [K_\xi(P), K_\xi(P)].
\]

\[ \square \]

Thus we extend the definition of the imploded cross-section \( X_{\text{impl}} \) to a \( K^{(P)} \)-imploded cross-section \( X_{\text{impl}}^{K,K^{(P)}} \) as follows.

**Definition 3.11.** Let \((X, \omega)\) be a symplectic manifold on which \( K \) acts with a moment map \( \mu : X \to \mathfrak{t}^* \). As before let

\[
t_+^{(P)*} = \text{Ad}^*(K^{(P)})(t_+^{(P)}) = \text{Ad}^*(Q^{(P)})(t_+^{(P)}) \subseteq \mathfrak{t}^{(P)*}
\]

be the sweep of \( t_+^{(P)} \) under the co-adjoint action of \( K^{(P)} \) on \( \mathfrak{t}^{(P)} \), and let \( \Sigma^{(P)} \) be the set of open faces of \( t_+^{(P)*} \).

If \( \zeta \in \mathfrak{t}^{(P)*} \) let \( K_\xi(P) \) be defined as in Definition 3.9. The \( K^{(P)} \)-imploded cross-section of \( X \) is

\[
 X_{\text{impl}}^{K,K^{(P)}} = \mu^{-1}(t_+^{(P)*})/\approx_{K^{(P)}}
\]

where \( x \approx_{K^{(P)}} y \) if and only if \( \mu(x) = \mu(y) = \zeta \in t_+^{(P)*} \) and \( x = \kappa y \) for some \( \kappa \in [K_\xi(P), K_\xi(P)] \).

The universal \( K^{(P)} \)-imploded cross-section is the \( K^{(P)} \)-imploded cross-section

\[
 (T^*K)_{\text{impl}}^{K,K^{(P)}} = K \times t_+^{(P)*}/\approx_{K^{(P)}}
\]

for the cotangent bundle \( T^*K \cong K \times \mathfrak{t}^* \) with respect to the \( K \)-action induced from the right action of \( K \) on itself.

**Theorem 3.12.** The map \( \mathcal{F}^{(P)} : K \times t_+^{(P)*} \to \overline{G/U}^{(P)} \) of (30) induces a \( K \times K^{(P)} \)-equivariant homeomorphism

\[
 (T^*K)_{\text{impl}}^{K,K^{(P)}} = K \times t_+^{(P)*}/\approx_{K^{(P)}} \to \overline{G/U}^{(P)} \subseteq E^{(P)}.
\]

Moreover under this identification of \( K \times t_+^{(P)*}/\approx_{K^{(P)}} \) with \( \overline{G/U}^{(P)} \subseteq E^{(P)} \), the moment map for the action of \( K \times K^{(P)} \) on \( E^{(P)} \) is induced by the map \( (K \times t_+^{(P)*})/\approx_{K^{(P)}} \to t^* \times t^{(P)*} \) given by

\[
 (k, \zeta) \mapsto (\text{Ad}^*(k)(\zeta), \zeta).
\]

**Proof:** By Lemma 3.10 \( \mathcal{F}^{(P)} \) induces a continuous map \( (T^*K)_{\text{impl}}^{K,K^{(P)}} \to \overline{G/U}^{(P)} \subseteq E^{(P)} \), which is surjective since \( \overline{G/U}^{(P)} = K(\exp(\mathfrak{i} \mathfrak{t}^{(P)})) \sum_{\mathfrak{w} \in \Pi} \mathfrak{w}^{(P)} \) by Remark 3.3. The map \( (K \times t_+^{(P)*})/\approx_{K^{(P)}} \to t^* \times t^{(P)*} \) given by

\[
 (k, \zeta) \mapsto (\text{Ad}^*(k)(\zeta), \zeta))
\]

is the composition of \( \mathcal{F}^{(P)} : K \times t_+^{(P)*} \to \overline{G/U}^{(P)} \) with the restriction to \( \overline{G/U}^{(P)} \subseteq E^{(P)} \) of the moment map \( \mu^{E^{(P)}} \) for the action of \( K \times K^{(P)} \) on \( E^{(P)} \). Moreover \( \mathcal{F}^{(P)} \) is continuous and surjective and restricts to a homeomorphism from \( \exp(\mathfrak{i} \mathfrak{t}^{(P)})) \sum_{\mathfrak{w} \in \Pi} \mathfrak{w}^{(P)} \) to \( t_+^{(P)*} \) by Corollary 3.8. If \( \mathcal{F}^{(P)}(k_1, \zeta_1) = \mathcal{F}^{(P)}(k_2, \zeta_2) \) then it follows by applying \( \text{Ad}^{E^{(P)}} \) that \( (\text{Ad}^*(k_1)(\zeta_1), \zeta_1) = (\text{Ad}^*(k_2)(\zeta_2), \zeta_2) \) and therefore \( \zeta_1 = \zeta_2 \) and \( k_1^{-1} k_2 \in K_\zeta = K_{\zeta} \). Thus

\[
 \mathcal{F}^{(P)}(1, \zeta_1) = (k_1^{-1}, 1) \mathcal{F}^{(P)}(k_1, \zeta_1) = (k_1^{-1}, 1) \mathcal{F}^{(P)}(k_2, \zeta_2) = (k_1^{-1} k_2, 1) \mathcal{F}^{(P)}(1, \zeta_2) = (k_1^{-1} k_2, 1) \mathcal{F}^{(P)}(1, \zeta_1).
\]

Since \( \zeta_1 = \zeta_2 \in t_+^{(P)*} = \text{Ad}^*(K^{(P)})(t_+^{(P)*}) \) we can write \( \zeta_1 = \zeta_2 = \text{Ad}^*(k_0)(\zeta) \) where \( \zeta \in t_+^{(P)*} \) and \( k_0 \in K^{(P)} \), so

\[
 \mathcal{F}^{(P)}(1, \zeta) = (1, k_0^{-1}) \mathcal{F}^{(P)}(1, \zeta_1) = (k_1^{-1} k_0, k_0^{-1}) \mathcal{F}^{(P)}(1, \zeta_1) = (k_1^{-1} k_2, 1) \mathcal{F}^{(P)}(1, \zeta).
\]

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By Lemma 3.6 $\mathcal{F}^{(P)}(1, \zeta)$ lies in the exp(it)-orbit of $\sum_{\pi \in \Pi, \lambda \in \Lambda_{\pi}^+} v^{(P)}_{\pi, \lambda}$ where $\sigma$ is the open face of $t^*_+(P)$ containing $\zeta$. Hence by Lemma 3.10 $k_1^{-1}k_2 \in [K_\zeta(P), K_\zeta(P)]$, and thus $\mathcal{F}^{(P)}$ induces a continuous bijection $(T^* K)^{\text{K}}_{\text{impl}} \to \overline{G/U}^{\text{aff}} \subseteq E^{(P)}$. Since $K$ is compact and so the map $(K \times t^*_+(P)) \approx (P) \to t^* \times t(P)^* \approx \overline{G/U}^{\text{aff}}$ given by $(k, \zeta) \mapsto (Ad^*(k)(\zeta), \zeta)$ is proper, this continuous bijection is a homeomorphism.

Remark 3.13. If $K^{(P)} = T$ and $\zeta \in t^*_+(P)$, then $K_\zeta(P) = K_\zeta$, and so $X_{\text{impl}}^{\text{K}}$ is the standard imploded cross-section $X_{\text{impl}}^{\text{K}}$ of $[15]$. On the other hand if $K^{(P)} = K$ then $K_\zeta(P)$ is conjugate to $T$ and $[K_\zeta(P), K_\zeta(P)]$ is trivial for all $\zeta \in t^*_+(P)$, so $X_{\text{impl}}^{\text{K}} = T^* K$.

Of course $\overline{G/U}^{\text{aff}}$ inherits a $K \times K^{(P)}$-invariant Kähler structure as a complex subvariety of $E^{(P)}$. The subvariety $G/U^{\text{aff}}$ (which is in general singular) is stratified by the (finitely many) $G$-orbits in $G/U^{\text{aff}}$, and the $K \times K^{(P)}$-invariant Kähler structure on $E^{(P)}$ restricts to a $K \times K^{(P)}$-invariant symplectic structure on each stratum, which gives $G/U^{\text{aff}}$ a stratified symplectic structure. Under the homeomorphism $(T^* K)^{\text{K}}_{\text{impl}} \to \overline{G/U}^{\text{aff}}$ of Theorem 3.12 these strata correspond to the locally closed subsets

$$K \times K^{(P)} \cong K^{(P)} \times K^{(P)} \cong K^{(P)} \times K^{(P)} \cong K^{(P)} \times K^{(P)} \cong K^{(P)} \times K^{(P)}$$

of $(T^* K)^{\text{K}}_{\text{impl}}$ where $\sigma \in \Sigma$ runs over the open faces of $t^*_+$. So the homeomorphism $(T^* K)^{\text{K}}_{\text{impl}} \to \overline{G/U}^{\text{aff}}$ of Theorem 3.12 induces a stratified $K \times K^{(P)}$-invariant symplectic structure on the universal $K^{(P)}$-imploded cross-section $(T^* K)^{\text{K}}_{\text{impl}}$. As in [15] the induced symplectic structure on

$$K^{(P)} \times K^{(P)} \cong K^{(P)} \times K^{(P)}$$

can be described directly, and can be expressed in terms of the symplectic reduction by the action of the subgroup $[K_\sigma(P), K_\sigma(P)]$ of $K$ on a locally closed symplectic submanifold of $T^* K$ (cf. [15] §2).

Using this symplectic structure on $(T^* K)^{\text{K}}_{\text{impl}}$ we obtain the following corollary.

Corollary 3.14. Let $K$ act on a symplectic manifold $X$ with moment map $\mu : X \to t^*$. Then the symplectic quotient of $\overline{G/U}^{\text{aff}} \times X = (T^* K)^{\text{K}}_{\text{impl}} \times X$ by the diagonal action of $K$ can be identified via $\mathcal{F}^{(P)}$ with $X^{K^{(P)}}_{\text{impl}}$.

Remark 3.15. In particular if $X$ is a projective variety with a linear action of the complexification $G$ of $K$, then $X^{K^{(P)}}_{\text{impl}}$ can be identified with the GIT quotient of $\overline{G/U}^{\text{aff}} \times X$ by the diagonal action of $G$.

It follows from Corollary 3.14 that if $(X, \omega)$ is any symplectic manifold on which $K$ acts with moment map $\mu : X \to t^*$ then $X^{K^{(P)}}_{\text{impl}}$ inherits a stratified $K \times K^{(P)}$-invariant symplectic structure

$$X^{K^{(P)}}_{\text{impl}} = \bigsqcup_{\sigma \in \Sigma} \mu^{-1}(\sigma) \cong K^{(P)} \cong K^{(P)}$$

$$= \bigsqcup_{\sigma \in \Sigma} \mu^{-1}(\sigma) \cong K^{(P)} \cong K^{(P)}$$

(33)

with strata indexed by the set $\Sigma$ of open faces of $t^*_+$, which are locally closed symplectic submanifolds of $X^{K^{(P)}}_{\text{impl}}$. The induced action of $K^{(P)}$ on $X^{K^{(P)}}_{\text{impl}}$ preserves this symplectic structure and has a moment map

$$\mu_{X^{K^{(P)}}_{\text{impl}}} : X^{K^{(P)}}_{\text{impl}} \to t^*_+ \subseteq t^{(P)*}$$
inherited from the restriction of \( \mu \) to \( \mu^{-1}(t^{(P)}_+) \).

**Remark 3.16.** In order to identify \( G/U \) with \( (T^* K)^{K,K(p)}_{\text{impl}} \) we made the assumption that \( K \) is semisimple and simply connected. However the construction of \( X^{K,K(p)}_{\text{impl}} \) makes sense whenever \( K \) is a compact connected Lie group with a Hamiltonian action on the symplectic manifold \( X \), and as in \cite{15} we can identify \( G/U \) with \( (T^* K)^{K,K(p)}_{\text{impl}} \) in this more general situation by expressing \( K \) as the quotient of the product of its centre \( Z(K) \) and the universal cover of \([K,K]\) by a finite central subgroup. We then get an identification of \( X^{K,K(p)}_{\text{impl}} \) with the symplectic quotient of \( G/U \times X \) by \( K \) in the general case.

### 3.3 Wonderful compactifications, symplectic cuts and partial desingularisations

Recently Paradan \cite{30} has introduced a generalisation of the technique of symplectic cutting (originally due to Lerman \cite{26}) which is valid for a (not necessarily abelian) compact connected group \( K \) and is motivated by the wonderful compactifications of De Concini and Procesi. He defines a \( K \)-adapted polytope in \( t^* \) to be a \( W \)-invariant Delzant polytope \( P \) in \( t^* \) whose vertices are regular elements of the weight lattice \( \Lambda^* \). If \( \{\lambda_1,\ldots,\lambda_n\} \) are the dominant weights lying in the union of all the closed one-dimensional faces of \( P \), then there is a \( G \times G \)-equivariant embedding of \( G = K_c \) into

\[
P(\bigoplus_{i=1}^N V^*_\lambda \otimes V_{\lambda_i})
\]

associating to \( g \in G \) its representation on \( \bigoplus_{i=1}^N V_{\lambda_i} \). The closure \( \mathcal{X}(P,K) \) of the image of \( G \) in this projective space is smooth and has moment map

\[
\mu_{K\times K}^{P} : \mathcal{X}(P,K) \rightarrow t^* \times t^*
\]

whose image is

\[
\mu_{K\times K}^{P}(\mathcal{X}(P,K)) = \{(Ad^*(k_1)\xi, -Ad^*(k_2)\xi) : \xi \in P, k_1, k_2 \in K\}.
\]

The symplectic cut \( X(P,K) \) defined by Paradan of a symplectic manifold \( X \) under a Hamiltonian \( K \)-action with respect to such a \( K \)-adapted polytope \( P \) is given by the symplectic quotient of \( \mathcal{X}(P,K) \times X \) by \( K \), so that if \( X \) is a complex projective variety with a linear \( K \)-action then \( X(P,K) \) is the GIT quotient

\[
X(P,K) = (\mathcal{X}(P,K) \times X)/G
\]

where \( G = K_c \). Then \( X(P,K) \) inherits a Hamiltonian \( K \)-action with moment map \( \mu_{X}^{P}(P,K) : X(P,K) \rightarrow t^* \)

whose image is

\[
\mu_{X}^{P}(P,K)(X(P,K)) = \mu(X) \cap Ad^*(K)(P).
\]

Moreover if \( U_{P} = Ad^*(K)(P^o) \) where \( P^o \) is the interior of \( P \) then \( \left(\mu_{X}^{P}(P,K)\right)^{-1}(U_{P}) \) is an open dense subset of \( X(P,K) \) which is \( K \)-equivariantly diffeomorphic to the open subset \( \mu^{-1}(U_{P}) \) of \( X \). This diffeomorphism is a quasi-symplectomorphism in the sense that there is a homotopy of symplectic forms taking the symplectic form on \( \left(\mu_{X}^{P}(P,K)\right)^{-1}(U_{P}) \) to the pullback of the symplectic form on \( \mu^{-1}(U_{P}) \).

Recall from \cite{15} §7 that if \( P_c \) is the polyhedral cone \(-c\lambda_0 + t^*_+ \) where \( \lambda_0 \) is a generic element of \( \mu(X) \cap t^*_+ \) and \( 0 < c \ll 1 \), then the imploded cross-section \( X_{\text{impl}} = X_{\text{Kimp} \text{TT}} \) has a partial desingularisation

\[
\widetilde{X}_{\text{impl}}^c = (X_{\text{impl}})(P_c,T)
\]

which is the symplectic reduction of \( \mathcal{X}(-c\tau_c,T) \times X_{\text{impl}} \) at \( c\lambda_0 \). Similarly, just as in \cite{15}, if \( P \geq B \) is a parabolic subgroup of \( G = K_c \) with maximal compact subgroup \( K^{(P)} = K \cap P \) and unipotent radical \( U \), then we can construct a \( K \times K^{(P)} \)-equivariant desingularisation \( (T^* K)^{K,K(p)}_{\text{impl}} \) for the universal imploded cross-section
\[(T^*K)^{K,K}_{\text{impl}} \cong G/U^{\text{aff}}\] and a partial desingularisation \(X_{\text{impl}}^{K,K(P)}\) for \(X^{K,K(P)}_{\text{impl}}\), which can be identified with the symplectic quotient of \(X \times (T^*K)^{K,K}_{\text{impl}}\) by the induced action of \(K\). Moreover \((T^*K)^{K,K(P)}_{\text{impl}}\) can be identified as a Hamiltonian \(K\)-manifold with

\[
\widetilde{G/U} = G \times P (L(P) \sum_{\pi \in \Pi} v^{(P)}_{\pi}) = K \times K^{(P)} (L(P) \sum_{\pi \in \Pi} v^{(P)}_{\pi})
\]

where \(L(P) \sum_{\pi \in \Pi} v^{(P)}_{\pi}\) is the closure in \(E(P)\) (or equivalently in the linear subspace \(\bigoplus_{\pi \in \Pi} (V^{K^{(P)}}_{\pi})^* \otimes V^{K^{(P)}}_{\pi}\) of \(E(P)\)) of the \(L(P)\)-orbit (or equivalently the \(P\)-orbit) of \(\sum_{\pi \in \Pi} v^{(P)}_{\pi}\), and the restriction to \(G \times L(P) \sum_{\pi \in \Pi} v^{(P)}_{\pi}\) of the multiplication map \(G \times E(P) \rightarrow E(P)\) induces a birational \(G\)-equivariant morphism

\[
p_U : \widetilde{G/U} \rightarrow \widetilde{G/U} = (T^*K)^{K,K}_{\text{impl}} \subseteq E(P).
\]

It follows from Theorem 3.5 of [30] that \(L(P) \sum_{\pi \in \Pi} v^{(P)}_{\pi}\) is a nonsingular subvariety of \(\bigoplus_{\pi \in \Pi} (V^{K^{(P)}}_{\pi})^* \otimes V^{K^{(P)}}_{\pi} \subseteq \mathbb{P}(\bigoplus_{\pi \in \Pi} (V^{K^{(P)}}_{\pi})^* \otimes V^{K^{(P)}}_{\pi})\).

If \(\lambda_0 \in \mu(X) \cap t_+ \cap I^{(P)}_{+}\) is generic and \(\epsilon > 0\) is sufficiently close to 0, and if \(\omega_\epsilon\) is the Kähler form on \(G/P\) given by regarding \(G/P\) as the coadjoint \(K\)-orbit through \(\epsilon \lambda_0\), then \(p_U^* \omega_{E(P)} + q_U^* \omega_\epsilon\) is a Kähler form on \(\widetilde{G/U}\) where \(q_U : G \times P E(P) \rightarrow G/P\) is the projection.

The partial desingularisation \(X_{\text{impl}}^{K,K(P)}\) can alternatively be obtained from \(X^{K,K(P)}_{\text{impl}}\) via a symplectic cut following Paradan [30]. Let \(W^{(P)}\) be the Weyl group of the compact subgroup \(K^{(P)}\) of \(K\); then we have an identification

\[
X_{\text{impl}}^{K,K(P)} = (X^{K,K(P)}_{\text{impl}})/(P^+_+,K^{(P)})
\]

where the cut is with respect to the \(K^{(P)}\)-action and the polyhedral cone \(P^+_+ = -(\epsilon \lambda_0 + t^{(P)}_{+})\). If we wish we can cut with respect to a suitable \(W^{(P)}\)-invariant Delzant polytope \(P_\epsilon\) in this cone which is large enough that its complement does not meet the compact subset \(\mu(X)\), but then the identification [30] is not quite symplectic according to Paradan’s construction; as in Remark 3.1 we have to distinguish between the flat Kähler metric on

\[
\bigoplus_{\pi \in \Pi} (V^{K^{(P)}}_{\pi})^* \otimes V^{K^{(P)}}_{\pi} \subseteq E(P)
\]

and the Fubini-Study metric on

\[
\bigoplus_{\pi \in \Pi} (V^{K^{(P)}}_{\pi})^* \otimes V^{K^{(P)}}_{\pi} \subseteq \mathbb{P}(\bigoplus_{\pi \in \Pi} (V^{K^{(P)}}_{\pi})^* \otimes V^{K^{(P)}}_{\pi}) \subseteq \mathbb{P}(\mathbb{C} \otimes E(P)).
\]

### 4 Non-reductive geometric invariant theory

The last section discussed a generalisation of symplectic implosion which is closely related to a GIT-like quotient construction for a linear action of the unipotent radical \(U\) of a parabolic subgroup \(P\) of a complex reductive group \(G\) on a complex variety \(X\). This section will recall from [7] a version of GIT for non-reductive group actions and then relate it to symplectic implosion.

#### 4.1 Background

Let \(H\) be an affine algebraic group, with unipotent radical \(U\) (that is, \(U\) is the unique maximal normal unipotent subgroup of \(H\)), acting linearly on a complex projective variety \(X\) with respect to an ample line
bundle $L$. If we wish to generalise Mumford’s GIT to this non-reductive situation, the first problem to be faced is that the ring of invariants

$$\hat{O}_L(X)^H = \bigoplus_{k \geq 0} H^0(X, L^\otimes k)^H$$

is not necessarily finitely generated as a graded complex algebra, so that $\text{Proj}(\hat{O}_L(X)^H)$ is not well-defined as a projective variety. Note, however, that in the case considered in §3 when the unipotent radical $U$ of a parabolic subgroup of a reductive group $G$ acts linearly on $X$ and the linear action extends to $G$, then the ring of invariants is finitely generated. Even when $\hat{O}_L(X)^H$ is not finitely generated $\text{Proj}(\hat{O}_L(X)^H)$ does make sense as a scheme, and the inclusion of $O_L(X)^H$ in $\hat{O}_L(X)$ gives us a rational map of schemes $q$ from $X$ to $\text{Proj}(\hat{O}_L(X)^H)$, whose image in $\text{Proj}(\hat{O}_L(X)^H)$ is constructible (that is, a finite union of locally closed subschemes).

We will only consider the case when $H = U$ is unipotent, since $H/U$ is always reductive and classical GIT allows us to deal with quotients by reductive groups. A more leisurely introduction to non-reductive GIT and details and proofs of the results quoted below can be found in [7].

**Definition 4.1.** (See [7].) Let $I = \bigcup_{m>0} H^0(X, L^\otimes m)^U$ and for $f \in I$ let $X_f$ be the $U$-invariant affine open subset of $X$ where $f$ does not vanish, with $O(X_f)$ its coordinate ring. The (finitely generated) *semistable set* of $X$ is

$$X^{ss} = X^{ss,f,g} = \bigcup_{f \in I^{ss}} X_f$$

where $I^{ss}$ consists of $f \in I$ such that $O(X_f)^U$ is finitely generated. The set of (locally trivial) *stable* points is

$$X^s = X^{lts} = \bigcup_{f \in I^{lts}} X_f$$

where $I^{lts}$ is the set of $f \in I$ such that $O(X_f)^U$ is finitely generated, and $q : X_f \longrightarrow \text{Spec}(O(X_f)^U)$ is a locally trivial geometric quotient. The set of *naively semistable* points of $X$ is the domain of definition

$$X^{nss} = \bigcup_{f \in I} X_f$$

of the rational map $q$, and the set of *naively stable* points of $X$ is

$$X^{ns} = \bigcup_{f \in I^{ns}} X_f$$

where $I^{ns}$ consists of those $f \in I$ such that $O(X_f)^U$ is finitely generated, and $q : X_f \longrightarrow \text{Spec}(O(X_f)^U)$ is a geometric quotient.

The *enveloped quotient* of $X^{ss}$ is $q : X^{ss} \longrightarrow q(X^{ss})$, where $q(X^{ss})$ is a dense constructible subset (but not necessarily a subvariety) of the *enveloping quotient*

$$X/UI = \bigcup_{f \in I^{ns}, f \neq 0} \text{Spec}(O(X_f)^U)$$

of $X^{ss}$.

**Lemma 4.2.** ([7] 4.2.9 and 4.2.10). The enveloping quotient $X/UI$ is a quasi-projective variety, and if $\hat{O}_L(X)^U$ is finitely generated then it is the projective variety $\text{Proj}(\hat{O}_L(X)^U)$.

Let $G$ be a complex reductive group with $U$ as a closed subgroup, and let $G \times_U X$ denote the quotient of $G \times X$ by the free action of $U$ defined by $h(g, x) = (gh^{-1}, hx)$, which is a quasi-projective variety by [31] Theorem 4.19. There is an induced $G$-action on $G \times_U X$ given by left multiplication of $G$ on itself. If the action of $U$ on $X$ extends to an action of $G$ there is an isomorphism of $G$-varieties

$$G \times_U X \cong (G/U) \times X$$

(36)
given by \([g, x] \mapsto (gH, gx)\). When \(U\) acts linearly on \(X\) with respect to a very ample line bundle \(L\) inducing an embedding of \(X\) in \(\mathbb{P}^n\), and \(G\) is a subgroup of \(SL(n + 1; \mathbb{C})\), then there is a very ample \(G\)-linearisation (which we will also denote by \(L\)) on \(G \times_U X\) via the embedding

\[G \times_U X \hookrightarrow G \times_U \mathbb{P}^n \cong (G/U) \times \mathbb{P}^n,\]

and using the trivial bundle on the variety \(G/U\) which is quasi-affine by [13] Corollary 2.8. For large enough \(m\) we can choose a \(G\)-equivariant embedding of \(G \times_U X\) in \(\mathbb{C}^m\) with a linear \(G\)-action to get a \(G\)-equivariant embedding of \(G \times_U X\) in \(\mathbb{P}^m \times \mathbb{P}^n \subseteq \mathbb{P}^{m+n+m+n}\) and the \(G\)-invariants on \(G \times_U X\) are given by

\[
\bigoplus_{m \geq 0} H^0(G \times_U X, L^\otimes m)^G \cong \bigoplus_{m \geq 0} H^0(X, L^\otimes m)^U = \hat{O}_L(X)^U.
\]

(37)

**Definition 4.3.** ([7] §5.2). A finite separating set of invariants for the linear action of \(U\) on \(X\) is a collection of invariant sections \(\{f_1, \ldots, f_n\}\) of positive tensor powers of \(L\) such that, if \(x, y\) are any two points of \(X\) then \(f(x) = f(y)\) for all invariant sections \(f\) of \(L^\otimes k\) and all \(k > 0\) if and only if

\[f_i(x) = f_i(y) \quad \forall i = 1, \ldots, n.\]

If \(G\) is any reductive group containing \(U\), a finite separating set \(S\) of invariant sections of positive tensor powers of \(L\) is a finite fully separating set of invariants for the linear \(U\)-action on \(X\) if

(i) for every \(x \in X^s\) there exists \(f \in S\) with associated \(G\)-invariant \(F\) over \(G \times_U X\) (under the isomorphism (37)) such that \(x \in (G \times_U X)_F\) and \((G \times_U X)_F\) is affine; and

(ii) for every \(x \in X^{ss}\) there exists \(f \in S\) such that \(x \in X_f\) and \(S\) is a generating set for \(O(X_f)^U\).

By [7] Remark 5.2.3 this definition is in fact independent of the choice of \(G\).

A \(G\)-equivariant projective completion \(G \times_U \bar{X}\) of \(G \times_U X\), together with a \(G\)-linearisation with respect to a line bundle \(L\) which restricts to the given \(U\)-linearisation on \(X\), is a reductive envelope of the linear \(U\)-action on \(X\) if every \(U\)-invariant \(f\) in some finite fully separating set of invariants \(S\) for the \(U\)-action on \(X\) extends to a \(G\)-invariant section of a tensor power of \(L\) over \(G \times_U X\). If \(L\) is ample on \((G \times_U X)\) it is an ample reductive envelope.

There always exists an ample reductive envelope for any linear \(U\)-action on a projective variety \(X\), at least if we replace the line bundle \(L\) with a suitable positive tensor power of itself (see [7] Proposition 5.2.8).

**Definition 4.4.** Let \(X\) be a projective variety with a linear \(U\)-action and a reductive envelope \(G \times_U \bar{X}\). Let \(i : X \hookrightarrow G \times_U X\) and \(j : G \times_U X \hookrightarrow G \times_U \bar{X}\) be the inclusions, and \(G \times_U \bar{X}^s\) and \(G \times_U \bar{X}^{ss}\) be the stable and semistable sets for the linear \(G\)-action on \(G \times_U \bar{X}\). Then the set of completely stable points of \(X\) with respect to the reductive envelope is

\[X^s = (j \circ i)^{-1}(G \times_U \bar{X}^s)\]

and the set of completely semistable points is

\[X^{ss} = (j \circ i)^{-1}(G \times_U \bar{X}^{ss}),\]

**Theorem 4.5.** ([7] 5.3.1). Let \(X\) be a normal projective variety with a linear \(U\)-action, for \(U\) a connected unipotent group, and let \((G \times_U X, L)\) be any ample reductive envelope. Then there is a diagram

\[
\begin{array}{cccccccc}
X^s & \subseteq & X^n & \subseteq & X^{ns} & \subseteq & X^{ss} & \subseteq & X^\mathcal{F} = X^{nss} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^s/U & \subseteq & X^n/U & \subseteq & X^{ns}/U & \subseteq & X^\mathcal{F}/U & \subseteq & G \times_U X^\mathcal{F}/U
\end{array}
\]

where all the inclusions are open and the first three vertical maps provide quasi-projective geometric quotients of the stable sets \(X^s\), \(X^n\) and \(X^{ns}\) by the action of \(U\). The fourth vertical map is the enveloping quotient \(q : X^{ss} \to X^\mathcal{F}/U\) defined in Definition 4.3 and \(X^\mathcal{F}/U\) is an open subvariety of the projective variety \(G \times_U X^\mathcal{F}/G\).

Note however that, even when \(\hat{O}_L(X)^U\) is finitely generated so that

\[X^\mathcal{F}/U = \text{Proj}(\hat{O}_L(X)^U) = G \times_U X^\mathcal{F}/G,\]

the maps \(q : X^{ss} \to X^\mathcal{F}/U\) and \(X^{ss} \to G \times_U X^\mathcal{F}/G\) are not necessarily surjective, and their images are in general only constructible subsets and not subvarieties.
4.2 Some examples of reductive envelopes

Now let us assume that $U = (\mathbb{C}^+)^r$ where $\mathbb{C}^+$ is the additive group of complex numbers and $r$ is any positive integer.

Remark 4.6. Each affine algebraic group $H$ over $\mathbb{C}$ has a unipotent radical $U$, which is the unique maximal normal unipotent subgroup of $H$ and has a reductive quotient group $R = H/U$ (see e.g. [4, 24] for more details). Given a linear action of $H$ on a projective variety $X$ with respect to an ample line bundle $L$, we can hope to quotient first by the action of $U$, and then by the induced action of the reductive group $H/U$, provided that the unipotent quotient (or compactified quotient) is sufficiently canonical to inherit an induced linear action of $H/U$. For example, if the algebra of invariants $\hat{O}_L(X)^U$ is finitely generated then the enveloping quotient $X//U = \text{Proj}(\hat{O}_L(X)^U)$ is a projective variety with an induced linear action of $H/U$ on an induced ample line bundle on $X//U$, and then classical GIT allows us to construct $X/H = \text{Proj}(\hat{O}_L(X)^H)$ as a GIT quotient $(X//U)//(H/U)$ of $X//U$ by the reductive group $H/U$; even when $\hat{O}_L(X)^U$ is not finitely generated, the same is true for $\text{Proj}(\hat{O}_L(X)^U_m)$ where $m$ is a sufficiently large positive integer and $\hat{O}_L(X)^U_m$ is the subalgebra of $\hat{O}_L(X)^U$ generated by invariant sections of $L^\otimes j$ for $1 \leq j \leq m$. Moreover the unipotent radical $U$ has canonical sequences of normal subgroups such that each successive subquotient is isomorphic to $(\mathbb{C}^+)^r$ for some $r$ (for example by taking the ascending or descending central series of $U$), so we can hope to quotient successively by unipotent groups of the form $(\mathbb{C}^+)^r$, and then finally by the reductive group $R$. Therefore the case when $U \cong (\mathbb{C}^+)^r$ for some $r$ is less special than it might appear at first sight.

Note that when $U = (\mathbb{C}^+)^r$ we have $\text{Aut}(U) \cong \text{GL}(r; \mathbb{C})$; let

$$\hat{U} = \mathbb{C}^* \ltimes U$$

be the semidirect product where $\mathbb{C}^*$ is the centre of $\text{Aut}(U)$. The centre of $\hat{U}$ is finite and meets $U$ in the trivial subgroup, so $U$ is isomorphic to a closed subgroup of the reductive group $G = SL(\mathbb{C} \oplus u)$ via the inclusion

$$U \hookrightarrow \hat{U} \rightarrow \text{Aut}(\hat{U}) \rightarrow \text{GL}(\text{Lie}\hat{U}) = \text{GL}(\mathbb{C} \oplus u)$$

where $u$ is the Lie algebra of $U$ and $\hat{U}$ is identified with its group of inner automorphisms. Then $U$ is the unipotent radical of a parabolic subgroup $P$ of $G = SL(r + 1; \mathbb{C})$, where $P$ is the stabiliser of the $r$-dimensional linear subspace $u$ of $\mathbb{C} \oplus u$, so we are in the situation of §3.2 above. The parabolic $P = U \ltimes \text{GL}(r; \mathbb{C})$ in $G = SL(r + 1; \mathbb{C})$ has Levi subgroup $\text{GL}(r; \mathbb{C})$ embedded in $SL(r + 1; \mathbb{C})$ as

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & \det g^{-1} \end{pmatrix}.$$ 

Note that

$$G/U \cong \{ \alpha \in (\mathbb{C}^r)^* \otimes \mathbb{C}^{r+1} | \alpha : \mathbb{C}^r \rightarrow \mathbb{C}^{r+1} \text{ is injective} \}$$

with the natural $G$-action $ga = g \circ \alpha$. Since the injective linear maps from $\mathbb{C}^r$ to $\mathbb{C}^{r+1}$ form an open subset in the affine space $(\mathbb{C}^r)^* \otimes \mathbb{C}^{r+1}$ whose complement has codimension two, we see directly in this case that $U = (\mathbb{C}^+)^r$ is a Grosshans subgroup of $G = SL(r + 1; \mathbb{C})$ and hence that

$$\mathcal{O}(G)^U \cong \mathcal{O}(G/U) \cong \mathcal{O}((\mathbb{C}^r)^* \otimes \mathbb{C}^{r+1})$$

is finitely generated [13] with

$$G/U^{\text{aff}} = \text{Spec}\mathcal{O}(G)^U = (\mathbb{C}^r)^* \otimes \mathbb{C}^{r+1}.$$ 

Now suppose that the linear action of $U = (\mathbb{C}^+)^r$ on $X$ extends to a linear action of $G = SL(r + 1; \mathbb{C})$, giving us an identification of $G$-spaces

$$G \times_U X \cong (G/U) \times X$$

as at (36) via $[g, x] \mapsto (gH, gx)$. Then (as in the Borel transfer theorem [4, Lemma 4.1])

$$\hat{O}_L(X)^U \cong \hat{O}_L(G \times_U X)^G \cong [\mathcal{O}(G/U) \otimes \hat{O}_L(X)]^G$$

(38)
is finitely generated \[14\] and we have a reductive envelope
\[
G \times_U X = \mathbb{P}(\mathbb{C} \oplus ((\mathbb{C}^\ast)^* \otimes \mathbb{C}^{r+1})) \times X
\]
with
\[
\overline{G \times_U X} / G \cong X / U = \text{Proj}(\hat{\mathcal{O}}_L(X)^U)
\]
where we choose for our linearisation on \(G \times_U X\) the line bundle
\[
L^{(N)} = \mathcal{O}(\mathbb{C} \oplus ((\mathbb{C}^\ast)^* \otimes \mathbb{C}^{r+1}))(N) \otimes L
\]
with \(N > 0\) sufficiently large (see \[24\] \S 4.1). This reductive envelope is ample and so satisfies Theorem \ref{4.5} in addition by \[21\] \S 4.1(6) we have
\[
X^s = X^s \text{ and } X^{ss} = X^{ss}.
\]
Thus we have a diagram
\[
\begin{array}{ccc}
X^s & \subseteq & X^{ss} \\
\downarrow & & \downarrow \\
X^s / U & \subseteq & X^{ss} / U = \overline{G \times_U X} / G
\end{array}
\]
but the enveloping quotient map \(q : X^{ss} \rightarrow X^{ss} / U = \overline{G \times_U X} / G\) is not necessarily surjective, so in contrast to the reductive situation we cannot describe \(X^{ss} / U\) topologically as the quotient of \(X^{ss}\) by an equivalence relation.

In order to describe \(X^{ss} / U\) topologically (and geometrically) it is useful to consider the linear action of the Levi subgroup \(GL(r; \mathbb{C}) \leq P\) on the closure \(\overline{P \times_U X} = \mathbb{P}(\mathbb{C} \oplus ((\mathbb{C}^\ast)^* \otimes \mathbb{C}^{r+1})) \times X\) of \(P \times_U X \cong L^{(P)} \times X\) in \(G \times_U X = \mathbb{P}(\mathbb{C} \oplus ((\mathbb{C}^\ast)^* \otimes \mathbb{C}^{r+1})) \times X\). We have
\[
G \times_U X \cong G \times P \left( P \times_U X \right)
\]
where \(P / U \cong GL(r; \mathbb{C})\) and \(G / P \cong P^r\) is projective, so \(G \times P \left( P \times_U X \right)\) is a projective completion of \(G \times_U X\).

The induced linearisation of the action of \(G\) on \(G \times P \left( P \times_U X \right)\) is not ample: if we regard \(G \times P \left( P \times_U X \right)\) as a subvariety in the obvious way of
\[
G \times P \left( P \times_U X \right) = G \times P \left( \mathbb{P}(\mathbb{C} \oplus ((\mathbb{C}^\ast)^* \otimes \mathbb{C}^{r+1})) \times X \right) \cong (G / P) \times \mathbb{P}(\mathbb{C} \oplus ((\mathbb{C}^\ast)^* \otimes \mathbb{C}^{r+1})) \times X
\]
\[
\cong \mathbb{P}^r \times \mathbb{P}(\mathbb{C} \oplus ((\mathbb{C}^\ast)^* \otimes \mathbb{C}^{r+1})) \times X
\]
then the birational morphism
\[
G \times P \left( P \times_U X \right) \rightarrow \overline{G \times_U X} \cong \mathbb{P}(\mathbb{C} \oplus ((\mathbb{C}^\ast)^* \otimes \mathbb{C}^{r+1})) \times X
\]
given by \([g, y] \mapsto gy\) extends to the projection
\[
\mathbb{P}^r \times \mathbb{P}(\mathbb{C} \oplus ((\mathbb{C}^\ast)^* \otimes \mathbb{C}^{r+1})) \times X \rightarrow \mathbb{P}(\mathbb{C} \oplus ((\mathbb{C}^\ast)^* \otimes \mathbb{C}^{r+1})) \times X
\]
and the induced line bundle is the restriction to \(G \times P \left( P \times_U X \right)\) of \(\mathcal{O}(\mathbb{C} \oplus ((\mathbb{C}^\ast)^* \otimes \mathbb{C}^{r+1}))(N) \otimes L\). However, if \(\epsilon \in \mathbb{Q} \cap (0, \infty)\), the tensor product \(\hat{L}_\epsilon = L^{(N)}\) of this line bundle with the pullback via the morphism
\[
G \times P \left( P \times_U X \right) \rightarrow G / P \cong \mathbb{P}^r,
\]
of the fractional line bundle \(\mathcal{O}_{P^r}(\epsilon)\) provides an ample fractional linearisation for the action of \(G\) on \(G \times P \left( P \times_U X \right)\) when, with \(\epsilon\) is sufficiently small, an induced surjective birational morphism
\[
\overline{X^{ss}} / G = \mathbb{P}(\mathbb{C} \oplus ((\mathbb{C}^\ast)^* \otimes \mathbb{C}^{r+1})) \times X / G = X / U
\]
which is an isomorphism over
\[
(G \times_U X^\xi) / G \cong X^\xi / U = X^s / U.
\]
This line bundle \(\hat{L}_\epsilon\) can be thought of as the bundle \(G \times P \left( \mathcal{O}(\mathbb{C} \oplus ((\mathbb{C}^\ast)^* \otimes \mathbb{C}^{r+1}))(N) \otimes L \right)\) on \(G \times P \left( P \times_U X \right)\), where now the \(P\)-action on \(\mathcal{O}(\mathbb{C} \oplus ((\mathbb{C}^\ast)^* \otimes \mathbb{C}^{r+1}))(N) \otimes L\) is no longer the restriction of the \(G\)-action on the line
bundle $O_{\mathbb{P}(\mathbb{C}^r \oplus \mathbb{C}^{r+1})}(N) \otimes L$ but has been twisted by $\epsilon$ times the character of $P$ which restricts to the determinant on $GL(r; \mathbb{C})$.

Since $GL(r; \mathbb{C}) = P/U$ has a central one-parameter subgroup $\mathbb{C}^*$ we can modify the linearisation of any linear actions of $P$ and $GL(r; \mathbb{C})$ by multiplying by $\epsilon$ times the standard character $\det$ of $GL(r; \mathbb{C})$ for any $\epsilon \in \mathbb{Q}$. By the Hilbert-Mumford criteria (Proposition 2.2 above) we have

$$P \times_U X^{ss, P, \epsilon} \subseteq P \times_U X^{ss, GL(r; \mathbb{C}), \epsilon} \subseteq P \times_U X^{ss, SL(r; \mathbb{C})} \tag{41}$$

where $P \times_U X^{ss, GL(r; \mathbb{C}), \epsilon}$ and $P \times_U X^{ss, SL(r; \mathbb{C})}$ (independent of $\epsilon$) denote the $GL(r; \mathbb{C})$-semistable and $SL(r; \mathbb{C})$-semistable sets of $P \times U X$ after twisting the linearisation by $\epsilon$ times the character $\det$ of $GL(r; \mathbb{C})$; this character is of course trivial on $SL(r; \mathbb{C})$. It turns out (see [24] §4.1(11)) that if $\epsilon$ is chosen appropriately (close to $-N/2$ where $N$ is as in the choice of linearisation above) then

$$P \times_U X^{ss, GL(r; \mathbb{C}), \epsilon} = (P(\mathbb{C} \oplus ((\mathbb{C}^r)^* \otimes \mathbb{C}^r)) \times X)^{ss, GL(r; \mathbb{C}), \epsilon} = GL(r; \mathbb{C}) \times X \tag{42}$$

and so quotienting we get

$$\frac{P \times U X}{U_{\sim \epsilon^{-N/2}}} GL(r; \mathbb{C}) \cong X. \tag{43}$$

Therefore

$$\mathcal{X} = \text{def} \ P \times_U X/\!//SL(r; \mathbb{C}) = (P(\mathbb{C} \oplus ((\mathbb{C}^r)^* \otimes \mathbb{C}^r)) \times X)/\!//SL(r; \mathbb{C}) \tag{44}$$

is a projective variety with a linear action of $\mathbb{C}^* = GL(r; \mathbb{C})/SL(r; \mathbb{C})$ which we can twist by $\epsilon$ times the standard character of $\mathbb{C}^*$, such that when $\epsilon = -N/2$ we get

$$\mathcal{X}/\!//\mathbb{C}^* \cong X \tag{45}$$

while for $\epsilon > 0$ sufficiently small we have a surjection from an open subset $(\mathcal{X}/\!//\epsilon \mathbb{C}^*)^{ss}$ of $\mathcal{X}/\!//\epsilon \mathbb{C}^*$ onto $\tilde{X}/\!//\tilde{U}$, and hence onto $X/\!//U$ (see [24] Proposition 4.6). More precisely let $(\mathcal{X}/\!//\epsilon \mathbb{C}^*)^s$ be the open subset $P \times_U X^{s, P, \epsilon}/GL(r; \mathbb{C})$ of

$$P \times_U X^{s, GL(r; \mathbb{C}), \epsilon}/GL(r; \mathbb{C}) = (P \times_U X^{s, GL(r; \mathbb{C}), \epsilon})/SL(r; \mathbb{C})/\mathbb{C}^* = \mathcal{X}^{s, \epsilon}/\mathbb{C}^* \subseteq \mathcal{X}/\!//\epsilon \mathbb{C}^*$$

and let $\mathcal{X}^{s, \epsilon} = \pi^{-1}((\mathcal{X}/\!//\epsilon \mathbb{C}^*)^{ss})$ and $\mathcal{X}^{s, \epsilon} = \pi^{-1}((\mathcal{X}/\!//\epsilon \mathbb{C}^*)^s)$ where $\pi : \mathcal{X}^{s, \epsilon} \to \mathcal{X}/\!//\epsilon \mathbb{C}^*$ is the quotient map, so that

$$(\mathcal{X}/\!//\epsilon \mathbb{C}^*)^s = \mathcal{X}^{s, \epsilon}/\mathbb{C}^*. \tag{46}$$

In this construction we can replace the compactification $P(\mathbb{C} \oplus ((\mathbb{C}^r)^* \otimes \mathbb{C}^r))$ of $GL(r; \mathbb{C})$ by its wonderful compactification $P(\mathbb{C} \oplus ((\mathbb{C}^r)^* \otimes \mathbb{C}^r))$ given by blowing up $P(\mathbb{C} \oplus ((\mathbb{C}^r)^* \otimes \mathbb{C}^r)) = \{[z : (z_{ij})_{i,j=1}]\}$ along the (proper transforms of the) subvarieties defined by

$$z = 0 \text{ and rank}(z_{ij}) \leq \ell$$

for $\ell = 0, 1, \ldots, r$ and by

$$\text{rank}(z_{ij}) \leq \ell$$

for $\ell = 0, 1, \ldots, r - 1$ [18]. The action of $SL(r; \mathbb{C})$ on $P(\mathbb{C} \oplus ((\mathbb{C}^r)^* \otimes \mathbb{C}^r))$, linearised with respect to a small perturbation of the pullback of $O_{P(\mathbb{C} \oplus ((\mathbb{C}^r)^* \otimes \mathbb{C}^r))}(1)$, satisfies

$$P(\mathbb{C} \oplus ((\mathbb{C}^r)^* \otimes \mathbb{C}^r))^{ss} = P(\mathbb{C} \oplus ((\mathbb{C}^r)^* \otimes \mathbb{C}^r))^s \text{ and } P(\mathbb{C} \oplus ((\mathbb{C}^r)^* \otimes \mathbb{C}^r))/SL(r; \mathbb{C}) \cong P1.$$
then the properties of $X$ given above are satisfied by $\tilde{X}$, and if $X$ is nonsingular then

$$\tilde{X}//U = \text{def } G \times_{P} (\mathbb{P}(\mathbb{C} \oplus ((\mathbb{C}^*)^r \otimes \mathbb{C}^{r+1})) / X) / G$$

is a partial desingularisation of $X//U$ and a compactification of $X^*/U$. Indeed it is shown in [24] Proposition 4.6 (combined with [24] Remark 4.8) that if $\epsilon > 0$ is sufficiently small then the natural rational map from $\tilde{X}//\mathbb{C}^*$ to $\tilde{X}//U$ restricts to surjective morphisms

$$(\tilde{X}//\mathbb{C}^*)^\epsilon \to \tilde{X}//U \to X//U$$

and

$$(\tilde{X}//\mathbb{C}^*)^\epsilon \to X^*/U.$$ Using the theory of variation of GIT [6, 33, 35], we can relate the quotient $\tilde{X}//\mathbb{C}^*$ to $\tilde{X}//\mathbb{N}/2\mathbb{C}^* \simeq X$ via a sequence of flips which occur as walls are crossed between the linearisations corresponding to $\epsilon$ and to $-N/2$. Thus we have a diagram

\[
\begin{array}{ccc}
(\tilde{X}//\mathbb{C}^*)^\epsilon & \subseteq & (\tilde{X}//\mathbb{C}^*)^\epsilon \subseteq \tilde{X}//\mathbb{C}^* \quad \longrightarrow \quad X = \tilde{X}//\mathbb{N}/2\mathbb{C}^* \\
X^*/U & \subseteq & \tilde{X}//U \\
X^*/U & \subseteq & X//U
\end{array}
\] (48)

where the vertical maps are all surjective, and the inclusions are all open.

**Remark 4.7.** The construction of a reductive envelope described here is only valid if the action of $U = (\mathbb{C}^+)^r$ on $X$ extends to an action of $G = SL(\mathbb{C} \oplus u)$ (which is a rather special situation when the ring of invariants $O_L(X)^U$ is always finitely generated). Moreover at least a priori this construction may depend on the choice of the extension of the $U$-action to a $G$-action, although $G \times_U X//G = X//U = \text{Proj}(O_L(X)^U)$ depends only on the linearisation of the $U$-action on $X$. However it is shown in [24] that we can associate to a linear $U$-action on $X$ a family of projective varieties $Y_m$ (one for every sufficiently large positive integer $m$), each of which contains $X$ and has an action of $G = SL(\mathbb{C} \oplus u)$ and a $G$-linearisation on an ample line bundle $L_{Y_m}$, which restricts to the given linearisation of the $U$-action on $X$ and is such that every $U$-invariant in a finite fully separating set of $U$-invariants on $X$ extends to a $U$-invariant on $Y_m$. Then we can embed $X$ in the $G$-variety

$$\mathbb{P}(\mathbb{C} \oplus ((\mathbb{C}^*)^r \otimes \mathbb{C}^{r+1})) \times Y_m$$

as $\{i\} \times X$ where $i \in (\mathbb{C}^*)^r \otimes \mathbb{C}^{r+1} \subseteq \mathbb{P}(\mathbb{C} \oplus ((\mathbb{C}^*)^r \otimes \mathbb{C}^{r+1}))$ is the standard embedding of $\mathbb{C}^r$ in $\mathbb{C}^{r+1}$. The closure of $GX \simeq G \times_U X$ in $\mathbb{P}(\mathbb{C} \oplus ((\mathbb{C}^*)^r \otimes \mathbb{C}^{r+1})) \times Y_m$ will provide us with a reductive envelope $G \times_U X$ (which is however not necessarily ample), and we can study the closures of the images of $X^*/U$ in $Y_m//U = \overline{Y_m//U}$ and its partial desingularisation $Y_m//U$ constructed as above.

### 4.3 Symplectic implosion for $U = (\mathbb{C}^+)^r \leq SL(r+1; \mathbb{C})$ actions

Let $X$ be a complex projective variety on which the complexification $G = SL(r+1; \mathbb{C})$ of $K = SU(r+1)$ acts linearly with respect to a very ample line bundle $L$, and let $U = (\mathbb{C}^+)^r$ be the unipotent radical of the parabolic $P = GL(r; \mathbb{C})U$ as in the previous subsection. As before let $T$ be the maximal torus of $K$ consisting of the diagonal matrices in $K$, and let $B$ be the upper triangular Borel subgroup of $G$. In the notation of §3.2 we have $L^{(P)} = GL(r; \mathbb{C})$ and $K^{(P)} = U(r)$. We can identify the Lie algebra $\mathfrak{t}^{(P)} = u(r)$ of $K^{(P)}$ with the product $\mathfrak{t}^{(P)} = \mathfrak{l}^{(P)} \oplus \mathfrak{g}^{(P)}$ of the Lie algebras of its semisimple part $Q^{(P)} = [K^{(P)}, K^{(P)}] = SU(r)$ and its centre $Z(K^{(P)}) \simeq S^1$. If we identify $\mathfrak{t}^*$ with

$$\{\zeta = (\zeta_1, \ldots, \zeta_{r+1}) \in \mathbb{R}^{r+1} : \zeta_1 + \cdots + \zeta_{r+1} = 0\}$$

in the usual way so that

$$\mathfrak{t}^*_+ = \{\zeta = (\zeta_1, \ldots, \zeta_{r+1}) \in \mathfrak{t}^* : \zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_{r+1}\},$$

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then
\[ t^r_{(P)_+} = \{ \zeta = (\zeta_1, \ldots, \zeta_{r+1}) \in t^r : \zeta_j \geq \zeta_{r+1} \text{ for } j = 1, \ldots, r \} \]
and
\[ \mathfrak{t}^{(P)*} = \{ \zeta = (\zeta_1, \ldots, \zeta_{r+1}) \in t^r : \zeta_1 = \cdots = \zeta_r \}. \]
Moreover \( \mathfrak{t}^{(P)*}_+ \) can be identified with the set of skew-Hermitian matrices in \( \mathfrak{su}(r+1)^* \) of the form
\[ \zeta = \begin{pmatrix} \xi & 0 \\ 0 & i\lambda_{r+1} \end{pmatrix} \]
where \( \xi \) is a skew-Hermitian \( r \times r \)-matrix with all its eigenvalues of the form \( i\lambda \) with \( \lambda \in \mathbb{R} \) and \( \lambda \geq \lambda_{r+1} \).
If all the eigenvalues \( i\lambda \) of \( \xi \) satisfy \( \lambda > \lambda_{r+1} \) then \( K_\xi(P) \) is conjugate to \( T \) and \( [K_\xi(P), K_\xi(P)] \) is trivial.
In general \( \text{Ad}^*(K^{(P)}) \zeta \) contains a matrix of the form
\[ \begin{pmatrix} \xi & 0 \\ 0 & i\lambda_{r+1} I_j \end{pmatrix} \]
for some \( j \in \{0, 1, \ldots, r\} \), where \( \xi \) is a skew-Hermitian \( (r-j) \times (r-j) \)-matrix with all its eigenvalues of the form \( i\lambda \) with \( \lambda \in \mathbb{R} \) and \( \lambda > \lambda_{r+1} \), and \( I_j \) is the \( j \times j \)-identity matrix. Then \( K_\xi(P) \) is conjugate in \( K^j(P) = U(r) \) to the product of a torus and the unitary group \( U(j) \) embedded in \( K = SU(r+1) \) as
\[ A \mapsto \begin{pmatrix} I_{r-j} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \det A^{-1} \end{pmatrix}, \]
and the face \( \sigma \) of \( \mathfrak{t}^{(P)*}_+ \) to which \( \zeta \) belongs is determined by \( j \) and the partition \( \pi \in \Pi_{r-j} \) given by the eigenvalues \( i\lambda \) of \( \xi \) with \( \lambda > \lambda_{r+1} \). Thus \([K_\xi(P), K_\xi(P)] \cong SU(j)\) and the universal \( K^{(P)} \)-imploded cross-section is
\[ (T^*K^j)^{(P)}_{\text{impl}} = \bigsqcup_{j=0}^{r} (K \times \mathfrak{t}^{(P)*}_{+,j,\pi}) / \approx_{K^{(P)}} = (K \times \mathfrak{t}^{(P)*}_+) \circ \bigsqcup_{j=1}^{r} \bigsqcup_{\pi \in \Pi_j} (K \times \mathfrak{t}^{(P)*}_+) / \approx_{K^{(P)}} = (K \times \mathfrak{t}^{(P)*}_+) \circ \bigsqcup_{j=1}^{r} \bigsqcup_{\pi \in \Pi_j} U(r) \times_{U(\pi_1) \times \cdots \times U(\pi_r) \times U(j)} \bigg( (K \times \mathfrak{t}^{(P)*}_{+,\pi}) / SU(j) \bigg). \]
Here \( \mathfrak{t}^{(P)*}_{+,j,\pi} \) consists of all \( \zeta \in \mathfrak{su}(r+1)^* \) of the form \( (51) \) with \( \xi \) a skew-Hermitian \( r \times r \)-matrix with all its eigenvalues of the form \( i\lambda \) with \( \lambda \in \mathbb{R} \) and \( \lambda \geq \lambda_{r+1} \) and exactly \( j \) of its eigenvalues equal to \( i\lambda_{r+1} \), and \( \mathfrak{t}^{(P)*}_{+,j,\pi} \) consists of all \( \zeta \in \mathfrak{t}^{(P)*}_+ \) of the form \( (51) \) such that the partition of \( r-j \) determined by the eigenvalues of \( \xi \) of the form \( i\lambda \) with \( \lambda > \lambda_{r+1} \) is \( \pi \). Moreover if \((k_1, \zeta_1)\) and \((k_2, \zeta_2)\) lie in \( K \times \mathfrak{t}^{(P)*}_+ \) then \((k_1, \zeta_1) \approx_{K^{(P)}} (k_2, \zeta_2)\) if and only if there is some \( \kappa \in K^{(P)} \) such that
\[ \zeta_1 = \zeta_2 = \kappa \begin{pmatrix} \xi & 0 \\ 0 & i\lambda_{r+1} I_j \end{pmatrix} \kappa^{-1} \]
and \( \kappa^{-1} k_1 k_2^{-1} \kappa \in [K_\xi(P), K_\xi(P)] \cong SU(j) \). Thus \((T^*K^j)^{(P)}_{\text{impl}} \) is isomorphic to \( G/U^{\text{aff}} = (C^r)^* \otimes C^{r+1} \) via
\[ (k, \zeta) \mapsto k \circ F(\zeta) \]
where if \( \zeta \) is at \( (51) \) then \( F(\zeta) : C^r \to C^r \subseteq C^{r+1} \) is the linear map represented by the unique \( r \times r \)-Hermitian positive definite matrix \( \alpha \) satisfying \( i\alpha^* \alpha = \xi - i\lambda_{r+1} I_r \).
Let \( \omega \) be a \( K \)-invariant Kähler form on \( X \), given in some choice of coordinates by the Fubini-Study form on the projective space into which the very ample line bundle \( L \) embeds \( X \). Then we know that
\[ \hat{O}_L(X)^U \cong (\hat{O}_L(X) \otimes O(G)^U)\]
is finitely generated, and the associated projective variety

\[ X//U = \text{Proj}(\hat{\mathcal{O}}_L(X)^U) \]

is isomorphic to the GIT quotient \((\mathcal{G}/U)_{\text{aff}} \times X)//G\), which as in §3.2 can be identified with a symplectic quotient of \(\mathcal{G}/U)_{\text{aff}} \times X\) by \(K\), and thus with the \(K(P)\)-imploded cross-section

\[ X^{K.K(P)}_{\text{impl}} = \mu^{-1}(t^{(P)*})/ \approx_{K(P)} \]

of \(X\), where \(x \approx_{K(P)} y\) if and only if \(\mu(x) = \mu(y) = \zeta \in t^{(P)*}_+\) and \(x = \kappa y\) for some \(\kappa \in [K_\zeta(P), K_\zeta(P)]\). Equivalently

\[ X^{K.K(P)}_{\text{impl}} = \mu^{-1}((t^{(P)*})^o) \sqcup \bigcup_{j=1}^r \mu^{-1}(t^{(P)*}_+) / \approx_{K(P)} \]

\[ \approx \mu^{-1}((t^{(P)*})^o) \sqcup \bigcup_{j=1}^r \bigcup_{(\pi, \ldots, \pi) \in \Pi_j} U(r) \times U(\pi_1) \times \cdots U(\pi_{\ell}) \times U(j) \left( \mu^{-1}(t^{(P)*}_+ \cap t^{*}_+) / SU(j) \right) \]  

(54)

since \([K_\zeta(P), K_\zeta(P)] \cong SU(j)\) if \(\zeta \in t^{(P)*}_+\).

The desingularisation \((T^*K)^{K.K(P)}_{\text{impl}} \) of \((T^*K)^{K.K(P)}_{\text{impl}} \) is given by

\[ (T^*K)^{K.K(P)}_{\text{impl}} = (K \times t^{(P)*}_+)/ \approx_{K(P)} \]

where \(t^{(P)*}_+ = \text{Ad}^*(K(P))(x \omega + t^{(P)*}_+)\) for \(0 < \epsilon \ll 1\) and \(x = \text{diag}(1, 1, \ldots, 1, -r) \in t^{(P)*}_+ \cap j^{(P)*}\), and if \((k_1, \zeta_1)\) and \((k_2, \zeta_2)\) lie in \(K \times t^{(P)*}_+\) then \((k_1, \zeta_1) \approx_{K(P)} (k_2, \zeta_2)\) if and only if there is some \(\kappa \in K(P) \cong U(r)\) such that

\[ \zeta_1 = \zeta_2 = \kappa \begin{pmatrix} \xi & 0 \\ 0 & i\lambda r+1 I_j \end{pmatrix} \kappa^{-1} \]

and \(\kappa^{-1} k_1 k_2^{-1}\kappa\) lies in the maximal torus \(T_j\) of \([K_\zeta(P), K_\zeta(P)] \cong SU(j)\) which is its intersection with \(T\).

The partial desingularisation \(X^{K.K(P)}_{\text{impl}}\) of \(X^{K.K(P)}_{\text{impl}}\) is the symplectic quotient of \((T^*K)^{K.K(P)}_{\text{impl}} \times X\) by the diagonal action of \(K\); as a stratified symplectic space, it is given by

\[ X^{K.K(P)}_{\text{impl}} = \mu^{-1}((t^{(P)*}_+)^o) \sqcup \bigcup_{j=1}^r \bigcup_{(\pi, \ldots, \pi) \in \Pi_j} U(r) \times U(\pi_1) \times \cdots U(\pi_{\ell}) \times U(j) \left( \mu^{-1}(x \omega + t^{(P)*}_+ \cap t^{*}_+)/ SU(j) \right) \]

and it can also be identified with the partial desingularisation \(X//U\) described in §4.2.

**Example 4.8.** Let \(U = \mathbb{C}^+\) act linearly on a projective space \(\mathbb{P}^n\), and suppose that coordinates have been chosen so that the natural generator of \(\text{Lie}(\mathbb{C}^+) = \mathbb{C}\) has Jordan normal form with blocks of sizes \(k_1 + 1, \ldots, k_s + 1\) where \(\sum_{j=1}^s (k_j + 1) = n + 1\). The \(\mathbb{C}^+\) action extends to an action of \(G = SL(2; \mathbb{C})\) by identifying \(\mathbb{C}^+\) with the group of upper triangular matrices

\[ \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C} \right\} \leq SL(2; \mathbb{C}) \]

and \(\mathbb{C}^{n+1}\) with \(\bigoplus_{j=1}^s \text{Sym}^{k_j}(\mathbb{C}^2)\) where \(\text{Sym}^{k_j}(\mathbb{C}^2)\) is the \(k_j\)th symmetric power of the standard representation \(\mathbb{C}^2\) of \(G = SL(2; \mathbb{C})\). We have

\[ G/\mathbb{C}^+ \cong \mathbb{C}^2 \setminus \{0\} \subseteq \mathbb{C}^2 \subseteq \mathbb{P}^2 = \mathcal{G}/\mathbb{C}^+ \]

and thus \(\mathbb{P}^n//\mathbb{C}^+\) is the GIT quotient \(\text{Proj}(\mathbb{C}[x_0, \ldots, x_n]^{\mathbb{C}^+}) \cong (\mathbb{P}^2 \times \mathbb{P}^n)//G\) with respect to the linearisation \(\mathcal{O}_{\mathbb{P}^2}(N) \otimes \mathcal{O}_{\mathbb{P}^n}(1)\) on \(\mathbb{P}^2 \times \mathbb{P}^n\) for \(N\) a sufficiently large positive integer. Since \((\mathbb{P}^2)^{ss,G} = \mathbb{C}^2\) and \(N\) is large we have

\[ (\mathbb{P}^2 \times \mathbb{P}^n)^{ss,G} \subseteq \mathbb{C}^2 \times \mathbb{P}^n = (G \times \mathbb{C}^+ \mathbb{P}^n) \sqcup \left( \{0\} \times \mathbb{P}^n \right) \]

\[ \subseteq (G \times \mathbb{C}^+ \mathbb{P}^n) \sqcup \left( \{0\} \times \mathbb{P}^n \right) \]
and if semistability implies stability then

\[ \mathbb{P}^n / \mathbb{C}^+ = (\mathbb{P}^n)^{s,U} / \mathbb{C}^+ \sqcup \{(0) \times \mathbb{P}^n\} // SL(2; \mathbb{C}). \]

In this example the parabolic subgroup \( P \) of \( G = SL(2; \mathbb{C}) \) is its standard (upper triangular) Borel subgroup with \( B/\mathbb{C}^+ \rightarrow \mathbb{P}^1 \) and

\[ B \times_{\mathbb{C}^+} \mathbb{P}^n = \mathbb{P}^1 \times \mathbb{P}^n, \]

while \( G \times_B (B/\mathbb{C}^+) = G \times_B \mathbb{P}^1 \) is the blow-up of \( \mathbb{P}^2 \) at the origin \( 0 \in \mathbb{C}^2 \subseteq \mathbb{P}^2 \). Similarly \( G \times_B (B \times_{\mathbb{C}^+} \mathbb{P}^n) \) is the blow-up of \( \mathbb{P}^2 \times \mathbb{P}^n \) along \( \{0\} \times \mathbb{P}^n \), and its quotient \( X/U \) is the blow-up of \( \mathbb{P}^n / \mathbb{C}^+ \) along its ‘boundary’

\[ \mathbb{P}^n // SL(2; \mathbb{C}) \cong \{(0) \times \mathbb{P}^n\} // SL(2; \mathbb{C}) \subset (\mathbb{P}^2 \times \mathbb{P}^n) // SL(2; \mathbb{C}) = \mathbb{P}^n // \mathbb{C}^+. \]

From the point of view of symplectic geometry we have

\[ \mathbb{P}^n // \mathbb{C}^+ \cong (\mathbb{P}^n)_{\text{impl}} = \mu^{-1}(\{(t^*_+)\circ\cup \mu^{-1}(0) \text{SU}(2) = \mu^{-1}(0, \infty) \cup \mu^{-1}(0) \text{SU}(2) \]

where \( t^*_+ \) is identified with \((0, \infty)\) in the usual way, and

\[ \mathbb{P}^n // \mathbb{C}^+ \cong (\mathbb{P}^n)_{\text{impl}} = \mu^{-1}(\epsilon, \infty) \cup \mu^{-1}(\epsilon) \text{S^1} \]

for \( 0 < \epsilon << 1 \).

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