Application of hierarchical structures based on binary matrices with the generalized arithmetic of Pascal's triangle in route building problems

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Abstract. The paper describes a method for constructing binary matrices based on Pascal's triangle. It provides the definitions of vertical and horizontal generatrices, consisting of binary elements, and the recurrent rule of Pascal's triangle, with the help of which a binary matrix is formed. The possibility of parametrization of generatrices is shown by identifying a combinatorial root word, called a generatrix pattern, in terms of word combinatorics. A comparative analysis is provided with the method of generalizing Pascal's triangle using reduction modulo a prime. Combinatorial differences are shown. Discrete mathematical objects (an integer lattice and combinatorial lattice paths) are described. The paper provides the definitions of lattice paths and describes specific types of paths – the Motzkin paths and the McMahon paths. It gives some combinatorial properties of the Motzkin paths. A method for mapping a binary matrix of the Pascal triangle type to the set of points of an integer combinatorial lattice is described. A method for constructing and predicting the qualitative characteristics of navigation routes by combining binary matrices and integer lattices is developed. Possibilities of parametrization of a binary matrix and an integer lattice, taking into account practical problems, are presented.

1. Introduction

In recent years, unmanned aerial vehicles have become increasingly important and find more applications in various spheres of societal life-sustaining activity. Autonomous cybernetic systems, which, among other factors, include UAVs, are designed and developed taking into account the specifics of the context of the planned work, as a rule, assuming difficult conditions. Difficult conditions imply a greater dynamism of the parameters of the control object, autonomy of control, delays, a variety of situations, incomplete control of external influences, the presence of interference, variability of goals, criteria, and restrictions. The technical capabilities of cyberphysical systems should be developed taking into account difficult working conditions, contain advanced methods for collecting and evaluating data, building optimal ways of making decisions.

When researching and developing complex cyberphysical systems, an important research tool is to consider them as multi-level systems, or systems with hierarchical structure [1, 2]. Hierarchical structures, in this case, are regarded as a special case of partially ordered sets. The process of step-by-step construction of solutions to multicriteria problems with hierarchical structures can often be interpreted as a trajectory on a finite lattice [3, 4], describing the corresponding partially ordered set [5]. Such problems are often encountered when modeling data from cyberphysical systems and
processing networks and circuit-free graphs [6, 7]. An important special case of a hierarchical structure can be the class of pyramidal structures called generalized Pascal pyramids [8]. The monograph [8] proposes a diagram for constructing combinatorial numbers and polynomials based on the generalized Pascal pyramid. Developing and studying new arithmetic, combinatorial and geometric properties of arithmetic triangles and pyramids, which are generalizations of Pascal's triangle, has both fundamental [10, 11] and applied mathematical value [10].

This article describes a way to reduce a hierarchical structure in the form of a binary matrix of a Pascal's triangle type to various discrete mathematical models by identifying repeated regular structural patterns and due to the presence of special combinatorial properties.

2. Basic concepts. Binary matrices of a Pascal's triangle type

Matrices consisting entirely of elements 0 and 1 are called binary and represent an important class of matrices that are successfully used in various branches of mathematics. So binary matrices define and/or represent binary relations. A graph – a well-known discrete object – can be represented as a binary matrix, for example, as an adjacency matrix or an incidence matrix.

This paper presents a method for constructing binary matrices by specifying certain sequences of elements (generatrices) and arithmetic of Pascal's triangle, as well as some of their properties.

Let us consider a method for constructing a binary matrix [3, 5, 8] of a Pascal's triangle type. This matrix is formed from horizontal and vertical generatrices, consisting of binary symbols 0, 1.

Let us represent Pascal's triangle as an infinite rectangular table:

```
  1
 1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
\ldots
```

The infinite first row and first column are generatrices here.

Using the well-known rule of Pascal's triangle (the elements not lying on the generatrix are equal to the sum of the left and top elements), we construct a binary matrix:

```
  1 1 0 1 1
 1 0 0 1 0
 0 0 0 1 1
 1 1 0 1 1
 0 1 0 0 1
```

which we will call a Pascal-type triangle with finite generatrices [1 1 0 1 1] and [1 1 0 1 0] and binary addition.

We represent the elements of the generatrix \(x_1, x_2, ..., x_n\) as a sequence of letters that make up the word \(W = x_1, x_2, ..., x_n\). In the combinatorics on words [12], an associative operation of word concatenation (appending), which we will call the multiplication of words, is defined on the set of words over the given alphabet. The concept of a word degree is naturally introduced in relation to this process. A word that is not a degree of any other word is called primitive. If \(W = Z^n\), where \(Z\) is primitive and \(n > 1\), then \(Z\) is called the root of \(W\) [12].

By a generatrix pattern we mean a sequence of elements of the root of the word \(W\) composed of elements of the given generatrix. So, for the generatrix [1 1 0 1 0] its pattern will be the sequence [1 1 0], and the number of elements of the sequence is equal to the cardinality of the word \(W\). For the generatrix [1 1 0 0 0] there is no such primitive word \(Z\), for which for \(n > 1\), \(W = Z^n\); in this case the generatrix [1 1 0 0 0] will simultaneously be its own pattern.

Note that the pattern in the generatrix may be partially repeated at its last occurrence, for example, in the form of the “1 1 0 1 1 1” pattern, the “1 1 0” pattern fully occurs twice and once as its beginning. As a result, it is possible (if necessary) to form generatrices of even or odd lengths.
To build a binary matrix with Pascal's triangle arithmetic, it is enough to set two sequences of symbols that form the first line and the first column. When you expand such a rectangular table, the recurrent property of this matrix becomes apparent: each new element depends only on two elements that have already been calculated earlier:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 2 & 3 & 4 & 5 & \cdots \\
1 & 3 & 6 & 10 & 15 & \cdots \\
1 & 4 & 10 & 20 & 35 & \cdots \\
1 & 5 & 15 & 35 & 70 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]

Note that each element modulo any number (for example, 2) will not violate the properties of diagonal symmetry of the elements:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 0 & 1 & 0 & 1 & \cdots \\
1 & 1 & 0 & 0 & 1 & \cdots \\
1 & 0 & 0 & 1 & 0 & \cdots \\
1 & 1 & 1 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]

Building a binary matrix using Pascal's triangle arithmetic in the case of an arbitrary selection of horizontal and vertical generatrices can lead to a disturbance of diagonal symmetry in the matrix, for example:

\[
\begin{array}{cccccccc}
1 & 0 & 1 & 1 & 1 & \cdots \\
0 & 0 & 1 & 0 & 1 & \cdots \\
0 & 0 & 1 & 1 & 0 & \cdots \\
1 & 1 & 0 & 1 & 1 & \cdots \\
1 & 0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]

Thus, the method of building a binary matrix with the help of generatrices allows one to obtain much more different matrices than the method of reduction modulo \( p \).

In a 100 by 100 matrix, there are \( 2^{199} \) ways to choose horizontal and vertical generatrices and, as a result, you can build the same number of matrices.

3. Combinatorics of paths

Let us consider and summarize some of the issues related to lattice path enumeration [4]. Lattice paths are determined through the starting point, setting the coordinates of the path beginning and the sequence of points of the integer lattice of the plane with some rules that set the relationship between each pair of adjacent points. In other words, we consider a sequence of steps related to the path, and the step refers to an ordered pair of numbers showing the mutual location of the adjacent points of the path.

![Figure 1. An integer lattice](image.png)

There are specific types of paths, such as the McMahon paths:

Let \( u_0, \ldots, u_n \) be such a sequence of points from \( \mathbb{Z}^2 \), that:
1) \( u_0 = (0, j_0); \)
2) \( u_{k+1} - u_k = (1, 0) \) or \( (0, -1) \), \( 0 \leq k \leq l; \)
3) \( \text{alt}(u_k) \geq 0, 0 \leq k \leq l \), \( \text{alt}(u_k) \) is the height of the point \( u_k \).

Then \( u_0, \ldots, u_n \) is called a path with no levels with the beginning of \( u_0 \) and the end of \( u_n \) and is designated as \( u = < u_0, \ldots, u_n >. \)

Let \( M_{i,j} \) be a set of all the paths of \( u \), for which \( \text{alt}(u_0) = i, \text{alt}(u_n) = j \) and \( i \geq \text{alt}(p) \geq j \) for \( \forall p \in u. \)

The set of \( M_{i,0} \) is a set of the McMahon paths. The McMahon paths are the shortest paths.

Integer lattice and its paths have different combinatory properties, for example, for example, for the Motzkin paths characterized by the equality \( \text{alt}(u_0) = \text{alt}(u_l) = 0 \) the following is true:

1. The number of the Motzkin paths, which have no levels, is related to the Catalan numbers \( C_n \) by equality (2.1):
   \[ m_{2n,0} = C_n \] (2.1)

2. The number of the Motzkin paths is related to the Motzkin numbers \( M_n \) by ratio (2.2):
   \[ M_n = \sum_{k=0}^{n} m_{n,k} \] (2.2)

where \( m_{n,k} = \begin{cases} \binom{n}{k} & n \equiv 0 (\text{mod} 2) \\ \binom{n}{k-1} & n \equiv 1 (\text{mod} 2) \end{cases} \)

4. **Navigation routes**

Let us carry out the mapping of the binary matrix to the set of points of the integer lattice, so that each element of the matrix \( a_{ij} \) corresponds to a point of the lattice \( a_{ij} \) with the coordinate \( (i, j) \):

![Figure 2. An integer lattice and a binary matrix](image1)

It is easy to see that each possible path will correspond to a certain sequence of elements of the binary matrix of a Pascal’s triangle type:

![Figure 3. A lattice combinatorial path](image2)

During the work of the UAV, there is a possibility of difficult conditions, which mean the presence of interference in the communication system and/or impaired visibility due to smoke, unfavorable meteorological conditions, etc. To minimize the negative impact on the drone activity, it is necessary to develop and subsequently apply methods of intelligent support for the operation of the UAV based on limited and/or incomplete data.

Consider a binary matrix describing the localization of fires in the area, formed using a satellite image.
where each element of the binary matrix represents an \((i, j)\) part of the area and takes on the value 1 if there is an ignition source on it and 0 if there is no ignition source. During the drone operation in a zone with difficult conditions, it is necessary to take into account the variability and incompleteness of control of external influences, therefore, previously available information may become unusable.

In this case, the developed tool for forecasting the evolution of external influences becomes useful to control the uncertainty and variable situations. Indeed, using a series of satellite images and binary matrices based on them and machine learning methods, it is possible to find certain patterns of the allocation of ignition sources and compose a prediction function \(F: X \rightarrow \{0,1\}\) for each calculated path vertex on an integer lattice, corresponding to the movement of the unmanned aerial vehicle. In this case, the binary matrix, which initially describes the fire allocation map, is transformed at time \(t\) in the UAV system into a binary matrix with a recurrent property and hierarchical structures, on the basis of which the forecast and analysis of the efficiency of the UAV movement is built (figure 3).

To take into account a wider variability of difficult conditions, it is possible to use hierarchical structures with a base of the numeral system \(k > 2\) and use ternary and \((0, 1, \ldots n)\) -matrices with a recurrent property. So, the use of a numeral system with a larger base will allow taking into account the quantitative attributes of the difficult conditions when building UAV.

5. Conclusion

Integer lattice characteristic serves as a generalization for any possible geographical area and many possible routes and directions of cyberphysical systems. Thus, by setting the initial (starting) and end point of the drone, you can build an integer lattice corresponding to the area between the two coordinates. The dimension of such a lattice will be parametric.

Auxiliary information for optimal decision-making on navigation routes are hierarchical structures in the form of binary matrices of a Pascal’s triangle type. A large number of variations of horizontal and vertical generatrices, as well as the setting of arbitrary recurrent rule of finding both a descendant element and a parent element allows one to describe a sufficient number of possible distributions of fractal processes. Based on the combinatorial properties of lattice paths and binary matrices of a Pascal's triangle type, it is possible to create a mathematical model that serves to make optimal decisions to build navigation routes.

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