INVERTING THE TURÁN PROBLEM

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Abstract. Classical questions in extremal graph theory concern the asymptotics of \( \text{ex}(G, \mathcal{H}) \) where \( \mathcal{H} \) is a fixed family of graphs and \( G = G_n \) is taken from a “standard” increasing sequence of host graphs \( (G_1, G_2, \ldots) \), most often \( K_n \) or \( K_{n,n} \). Inverting the question, we can instead ask how large \( |E(G)| \) can be with respect to \( \text{ex}(G, \mathcal{H}) \). We show that the standard sequences indeed maximize \( |E(G)| \) for some choices of \( \mathcal{H} \), but not for others. Many interesting questions and previous results arise very naturally in this context, which also, unusually, gives rise to sensible extremal questions concerning multigraphs and non-uniform hypergraphs.

1. Introduction and Motivation

For a graph \( G \) and a family of graphs \( \mathcal{H} \), the extremal number of \( \mathcal{H} \) in \( G \) is defined to be

\[
\text{ex}(G, \mathcal{H}) = \max \{|E(F)| : F \subseteq G \text{ and } \mathcal{H} \not\subseteq F \text{ for any } H \in \mathcal{H} \}.
\]

When the family consists only of a single graph, \( \text{ex}(G, \{H\}) \) is used in place of \( \text{ex}(G, \mathcal{H}) \).

A typical example of this is when \( \mathcal{H} = \{C_3, C_4, C_5, \ldots\} \) is the collection of all cycles, in which case the extremal number is simply the graphic matroid rank of \( G \), an important graph parameter in its own right.

The Turán problem, one of the cornerstones of extremal graph theory concerns the behavior of \( \text{ex}(K_n, \mathcal{H}) \) for a fixed \( \mathcal{H} \) when \( n \) is large. The first result along these lines is a theorem of Mantel (see, for instance [5]) which states that \( \text{ex}(K_n, K_3) = \lfloor n^2/4 \rfloor \). Turán [16] obtained a version for \( K_t \) in place of \( K_3 \), in particular obtaining \( \text{ex}(K_n, K_t) = \left(1 - \frac{1}{t-1} + o(1)\right)\frac{n^2}{2} \) where \( o(1) \to 0 \) as \( n \to \infty \). In a similar spirit, the Erdős-Stone Theorem [9] states that if \( \chi = \chi(H) \) is the chromatic number of \( H \), then \( \text{ex}(K_n, H) = \left(1 - \frac{1}{\chi} + o(1)\right)\frac{n^2}{2} \).

The Erdős-Stone Theorem asymptotically answers the Turán problem, except when \( H \) is bipartite, in which case the bound becomes \( o(n^2) \). In this situation, known as the degenerate case, the asymptotic behavior of very few graphs is known and is an active area of research (c.f. [12]).

Most approaches in the case of a bipartite graph instead ask about \( \text{ex}(K_{n,n}, H) \), which is known as the Zarankiewicz problem [19]. This is often seen as a more natural question and provides bounds on the Turán problem as \( \frac{1}{2} \text{ex}(K_n, H) \leq \text{ex}(K_{n/2,n/2}, H) \leq \text{ex}(K_n, H) \) for bipartite \( H \). In the special case of \( H = C_4 \), the incidence graphs showing tightness for the Zarankiewicz problem were spotted a few years before polarity graphs showing tightness for the Turán problem (see [12] Section 3).

With this in mind, we set out to explore a framework in which to ask: what is the most “natural” or “best” host graph for a fixed family of graphs? This suggests optimizing a particular monotone graph parameter over all host graphs \( G \) where \( \text{ex}(G, \mathcal{H}) \) is bounded, the simplest of which is just the edge count. Thus we define the following extremal function for \( \mathcal{H} \):

\[
\mathcal{E}_k(\mathcal{H}) := \sup\{|E(G)| : \text{ex}(G, \mathcal{H}) < k \}.
\]

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In other words, for a family $H$, we would like to determine the host graph $G$ with the most edges such that any $k$ edges from $G$ contain some copy of $H \in H$. In other words, $G$ is best at “forcing” a copy of some $H \in H$. When the family consists only of a single graph, we write $\mathcal{E}_k(H)$ in place of $\mathcal{E}_k(\{H\})$. Note that it is necessary to consider the supremum here as $\mathcal{E}_k(H)$ may be infinite. In particular, $\mathcal{E}_k(K_{1,t}) = \mathcal{E}_k(tK_2) = \infty$ for $k \geq t$ as for any $s \geq t$, $\text{ex}(K_{1,s}, K_{1,t}) = t - 1 = \text{ex}(sK_2, tK_2)$, despite both host graphs having $s$ edges.

In a similar fashion to the original Turán problem, this paper considers two questions:

• What are the asymptotics of $\mathcal{E}_k(H)$?

• When $\mathcal{E}_k(H)$ can be determined precisely, which host graphs $G$ attain $|E(G)| = \mathcal{E}_k(H)$?

On the one hand, we will show that for nonbipartite $H$, this question behaves more or less as one might expect. For example, the following theorem is close in spirit to the Erdős-Stone Theorem:

**Theorem 1.1.** If $H$ is a family of graphs with $\rho = \min \{\chi(H) : H \in \mathcal{H}\} \geq 3$, then

$$\mathcal{E}_k(H) = \left(1 + \frac{1}{\rho - 2} + o(1)\right)k.$$  

This theorem will follow as a corollary of Theorem 3.5.

Recalling our motivation from the Zarankiewicz problem, we show that complete bipartite graphs are optimal hosts for at least one natural family, namely the collection $C_e := \{C_4, C_6, \ldots\}$ of even cycles:

**Theorem** (See Theorem 2.10). For $k \geq 4$, $\mathcal{E}_k(C_e) = \left\lfloor \frac{k^2}{4} \right\rfloor$, with $K_{[k/2],[k/2]}$ being the unique extremal graph for $k \geq 6$.

On the other hand, this is already a challenge for the case $H = K_{2,2}$:

**Question 1.** What is $\mathcal{E}_k(C_4)$ and what is the optimal host graph?

One peculiar feature of our question is that it is sensible even for multigraphs or nonuniform hypergraphs.

We let $\mathcal{E}_k^*(H)$ denote the maximum number of edges among host multigraphs $G$ with $\text{ex}(G, H) < k$. The parameter $\mathcal{E}_k^*(H)$ will be important in proving bounds on $\mathcal{E}_k(H)$ when $H$ is a family of simple graphs. However, we do not even know the following:

**Conjecture** (See Section 2.3). If $H$ consists only of simple graphs, then $\mathcal{E}_k(H) = \mathcal{E}_k^*(H)$.

Curiously, for non-uniform graphs $H$ without parallel edges, the above conjecture fails:

**Theorem** (See Theorems 3.10 & 3.12). Let $O_2$ be the graph with a single edge and a loop at each end. Then $\mathcal{E}_k(O_2) = \frac{\phi k}{2}$, whereas $\mathcal{E}_k^*(O_2) \sim \phi k$, where $\phi$ is the golden ratio.

In our study of $\mathcal{E}_k(H)$ and optimal host graphs, we will also show that:

1. Cliques are best at forcing cliques (Theorem 2.6),
2. Cliques are best at forcing a cycle (Theorem 2.8),
3. Complements of matchings are best at forcing $\{P_3, K_3\}$ (Theorem 2.12),
4. Cliques with pendant edges are best at forcing $P_3$ (Theorem 2.18),
5. Two disjoint cliques or a modified power of a cycle, depending on parity, are best at forcing $P_1 \cup P_2$ (Corollary 2.21 & Theorem 2.22),
(6) For uniform hypergraphs $H$, $E_k(H)$ is only infinite for sunflowers (Proposition 3.2).
(7) For 1-uniform “multigraphs” $H$, $E_k^*(H)$ is quadratic in $k$ (Theorem 3.15).

In fact, for items 1, 3 and 5 the correct behavior of $E_k(H)$ is implicit in references [3], [10] and [11], respectively, but our results will prove uniqueness of the respective host graphs.

The organization of the paper is as follows. In Section 2 we begin our study of $E_k(H)$ by obtaining the natural analogue of Turán’s theorem. We then explore $E_k(H)$ when $H$ is a family of cycles and when $H$ consists of small graphs, in some cases extending the results to $E_k^*(H)$. In Section 3 we then explore $E_k(H)$ when $H$ is a family of hypergraphs. In addition to uniform hypergraphs, Section 3 also considers our problem in the context of non-uniform hypergraphs and 1-uniform multigraphs. Finally, in Section 4 we present conjectures and future directions.

1.1. Notation. We follow standard notation from [18]. For a graph $G = (V, E)$ and $S, T \subseteq V$, we use $G[S]$ to denote the subgraph of $G$ induced by $S$ and $G[S, T]$ to denote the subgraph of $G$ with vertex set $S \cup T$ where $xy \in E(G[S, T])$ if and only if $xy \in E$ and $x \in S$ and $y \in T$. For a graph $G$ and integer $t$, we denote the graph consisting of $t$ vertex-disjoint copies of $G$ by $tG$, e.g. $tK_2$ is the matching on $t$ edges. For integers $m \leq n$, we use $[m, n] = \{m, m+1, \ldots, n\}$ and $[n] = [1, n]$. In contrast to [18], $P_t$ will denote the path on $t$ edges. Additionally, unless stated otherwise, all graphs throughout this paper will be assumed to have no isolated vertices.

2. Graphs and Multigraphs

A natural starting point with the study of Turán-type questions is to consider equivalent versions of the theorems of Turán [10] and Erdős-Stone [9]. Theorem 1.1 follows very easily from Erdős-Stone, but we can show a much broader result in the setting of hypergraphs (see Theorem 3.5), and so the proof is postponed until Section 3. As such, we begin our study of the parameter $E_k(H)$ with $H = \{K_t\}$ as per Turán, where we can also classify the extremal graphs.

In order to do so, we establish two new definitions and a lemma which will also be used in subsequent results. Although the Turán problem is uninteresting when $H$ is a multigraph, the parameter $E_k(H)$ leads to fruitful questions. To this end, if $H$ is a family of (multi)graphs, define

$$E_k^*(H) := \sup\{|E(G)| : G \text{ a multigraph and } \text{ex}(G, H) < k\}.$$ 

If $H$ consists only of simple graphs, it is easy to observe that $E_k(H) \leq E_k^*(H)$, so we can often consider the latter parameter instead. It is unclear whether $E_k(H) = E_k^*(H)$ for every family of simple graphs $H$, and we will discuss this further in Section 2.3.

Definition 2.1. If $G$ is a multigraph and $I \subseteq V(G)$, define $G' = C_I(G)$ to be the multigraph with the same number of edges obtained by contracting together the vertices in $I$. More specifically, write $V(G') := (V(G) \cup \{z\}) \setminus I$ for some new vertex $z$, and the multiset $E(G') := \{C_I(e) : e \in E(G)\}$, where

$$C_I(e) := \begin{cases} 
zz & \text{if } e \in \binom{I}{2}; \\
zx & \text{if } e = ux \text{ for some } u \in I; \\
e & \text{otherwise.}
\end{cases}$$

Here, we think of $C_I$ as a bijection between multigraph edge sets.
To apply contractions in determining $\mathcal{E}_k(H)$, we provide the following general definition and lemma.

**Definition 2.2.** If $G$ denotes the space of all finite simple graphs and $G^*$ denotes the space of all finite multigraphs, a function $f : G^* \to G$ is called a graph simplification if it preserves vertex sets and containment. That is, for every pair of graphs $G, H$, $V(f(G)) = V(G)$ and if $H \subseteq G$, then $f(H) \subseteq f(G)$.

Examples include:

1. $f(G) = G$, where $G$ is the underlying simple graph of $G$.
2. $ab \in E(f(G)) \iff a, b$ in the same connected component of $G$.
3. $ab \in E(f(G)) \iff \text{dist}_G(a, b) \leq t$ for some fixed integer $t$.

**Lemma 2.3.** Let $f$ be a multigraph simplification such that $f(H)$ is a clique for every $H \in \mathcal{H}$. By contrast, let $G$ be a graph and $I$ be an independent set in $f(G)$. If $G' = C_f(G)$, then $\text{ex}(G', \mathcal{H}) \leq \text{ex}(G, \mathcal{H})$.

Note that an independent set in $f(H)$ is not necessarily an independent set in $H$, as seen by e.g. $u \sim v$ in $f(G) \Rightarrow u, v$ are 2-connected in $G$. However, all of the scenarios in which we will use this lemma (namely, the three examples given above), $f(H) \supseteq H$ for every $H$. In particular, we will never be contracting edges in $G$ to loops, as per the first case in the definition of $C_e(I)$ above.

**Proof.** It suffices to show that if some $F \subseteq G$ contains a copy of $H \in \mathcal{H}$, then $C_f(F) \subseteq G'$ still contains a copy of some $H' \in \mathcal{H}$. In fact, more is true; namely, if $H_0 \subseteq G$ is a copy of $H$, then $C_f(H_0) \subseteq G'$ contains a copy of $H$. To see this, as $f$ is a graph simplification, $f(H) \simeq f(H_0) \subseteq f(G)$, so as $f(H)$ is a clique, $|I \cap V(H_0)| \leq 1$. In other words, $C_f(H_0)$ is a copy of $H$, possibly with extra multiedges or loops. \hfill $\square$

For a graph simplification $f$, we say that $G$ is $f$-compressed if $f(G)$ is a clique. Further, we say that $G$ is an $f$-compressed copy of $G'$ if $G$ is $f$-compressed and there is a sequence of graphs $G' = G_0, G_1, \ldots, G_t = G$ such that $G_{i+1} = C_f(G_i)$ for some independent set $f$ in $f(G_i)$. Note that if $G$ is an $f$-compressed copy of $G'$, then $|E(G)| = |E(G')|$.

With this definition, the following corollary follows immediately from Lemma 2.3.

**Corollary 2.4.** Suppose, as above, that $f$ is a multigraph simplification where $f(H)$ is a clique for every $H \in \mathcal{H}$. If $G^*$ is an $f$-compressed copy of $G$, then $\text{ex}(G^*, \mathcal{H}) \leq \text{ex}(G, \mathcal{H})$. In particular, when computing $\mathcal{E}_k(\mathcal{H})$, it suffices to consider graphs $G$ such that $f(G)$ is a clique, i.e. $\mathcal{E}_k(\mathcal{H}) = \sup\{|E(G)| : \text{ex}(G, \mathcal{H}) < k, f(G) \simeq K_{|V(G)|}\}$.

Before finding the value of $\mathcal{E}_k(K_t)$, we first must recall some properties of Turán graphs. Define $T_{t-1}(n)$ to be the balanced complete $(t - 1)$-partite graph on $n$ vertices; Turán’s Theorem states that $\text{ex}(K_n, K_t) = |E(T_{t-1}(n))|$. Additionally, define the Turán density of $K_t$ in $K_n$ by $\alpha_n(t) := \text{ex}(K_n, K_t)/(n^2)$. We will use the following observations in the subsequent proof.

**Observation 2.5.** If $n \equiv n_0 \pmod{t - 1}$, then

$$|E(T_{t-1}(n))| = \binom{n}{2} - n_0 \binom{n - 1 - n_0}{t - 1} + (t - 1 - n_0) \binom{n - n_0}{t - 1} = \left(1 - \frac{1}{t - 1} \pm O\left(\frac{1}{n}\right)\right) \binom{n}{2}.$$

As such, if $(t - 1) \nmid n$, we have

$$|E(T_{t-1}(n))| = |E(T_{t-1}(n - 1))| + (n - 1) - \left\lfloor \frac{n - 1}{t - 1} \right\rfloor.$$

In particular, this implies that if $(t - 1) \nmid n$, then $\alpha_{n-1}(t) > \alpha_n(t)$. Furthermore, $\alpha_{n-1}(t) \geq \alpha_n(t)$ for all $n$, which can be seen by averaging over subgraphs.
The following proof uses an idea by Alon (see [3, Lemma 2.1]) in the context of chromatic numbers.

**Theorem 2.6.** For any integer \( t \geq 3 \),
\[
\mathcal{E}_k(K_t) = \mathcal{E}_k^*(K_t) = \left( 1 + \frac{1}{t-2} + o(1) \right) k.
\]
Moreover, for infinitely many values of \( k \), the unique extremal graph for \( \mathcal{E}_k^*(K_t) \) and \( \mathcal{E}_k(K_t) \) is a clique.

**Proof.** Lower bound. For any positive integer \( k \), let \( n \) be the largest integer for which \( k < \text{ex}(K_n, K_t) \). As \( \text{ex}(K_n, K_t) = (1 - 1/t^2) + O\left(\frac{1}{t^4}\right) \left(\frac{n}{2}\right) \), we observe that \( \text{ex}(K_n+1, K_t) - \text{ex}(K_n, K_t) = O(n) \). Thus, \( k \leq \text{ex}(K_n, K_t) + O(n) = \text{ex}(K_n, K_t) + O(\sqrt{k}) \), so we calculate
\[
\mathcal{E}_k(K_t) \geq \left( \left(\frac{n}{2}\right) + k - O(\sqrt{k}) \right) \left(\frac{n}{2}\right) = \left( 1 + \frac{1}{t-2} + o(1) \right) k.
\]

**Upper bound.** Let \( G \) be a (multi)graph with \( \text{ex}(G, K_t) < k \). Letting \( f \) be the “underlying simple graph” simplification ((1) in Definition 2.1), as \( f(K_t) = K_t \), we may suppose that \( G \) is \( f \)-compressed by Corollary 2.4. In other words, \( G \) is a clique, possibly with parallel edges. Let \( n = |V(G)| \) and write \( |E(G)| = \binom{r}{2} + \ell \) where \( 0 \leq \ell \leq r - 1 \). As \( G \) is a copy of \( K_n \), possibly with parallel edges, we know that \( r \geq n \).

Now, let \( T \) be a copy of the Turán graph \( T_{r-1}(n) \) chosen uniformly at random on \( V(G) \), and let \( H \) be the multigraph with edge set \( \{ uv \in E(G) : uv \in E(T) \} \) (so that if \( u, v \) span multiple edges in \( G \) then they either all survive the intersection with \( T \) or all do not). As any such \( H \) is \( K_r \)-free, writing \( \alpha_n = \alpha_n(t) \), we calculate
\[
\text{ex}(G, K_t) \geq \mathbb{E}|E(H)| = |E(G)| \cdot \alpha_n \geq |E(G)| \cdot \alpha_r = \left( \binom{r}{2} + \ell \right) \frac{|E(T_{r-1}(r))|}{\binom{r}{2}} = \text{ex}(K_r, K_t) + \ell \alpha_r. \quad (2.1)
\]

Thus, for any positive integer \( k \), let \( r \) be the least integer for which \( k \leq \text{ex}(K_r, K_t) + 1 \). As above, we note that \( k \geq \text{ex}(K_r, K_t) - O(\sqrt{k}) \). Equation (2.1) shows that for any multigraph \( G \), if \( |E(G)| > \binom{r}{2} \), then \( \text{ex}(G, K_t) \geq k \), so
\[
\mathcal{E}_k(K_t) \leq \binom{r}{2} \leq \frac{k + O(\sqrt{k})}{\text{ex}(K_r, K_t)} \binom{r}{2} = \left( 1 - \frac{1}{t-2} + o(1) \right) k.
\]

As \( \mathcal{E}_k(K_t) \leq \mathcal{E}_k^*(K_t) \), this establishes the asymptotics. In particular, we have shown that if \( k = \text{ex}(K_r, K_t) + 1 \) for some integer \( r \), then \( \mathcal{E}_k(K_t) = \mathcal{E}_k^*(K_t) = \binom{r}{2} \).

**Extremal graphs.** We now wish to show that for infinitely many \( k \), the only extremal graph for \( \mathcal{E}_k^*(K_t) \), and thus for \( \mathcal{E}_k(K_t) \), is a clique.

Let \( k = \text{ex}(K_r, K_t) + 1 \) where \( r > t \) and \( (t - 1) \mid r \). In this case, we know that \( \mathcal{E}_k(K_t) = \mathcal{E}_k^*(K_t) = \binom{r}{2} \) and that \( \alpha_n > \alpha_r \) for any \( n < r \). Now, let \( G \) be an \( f \)-compressed graph which is extremal for \( \mathcal{E}_k^*(K_t) \). As before, \( |V(G)| \leq r \), and as \( \alpha_n > \alpha_r \) for any \( n < r \), the only way for \( \text{ex}(G, K_t) \leq k - 1 = \text{ex}(K_r, K_t) \) is if \( |V(G)| = r \), as shown by Equation (2.1). Thus, as \( G \) has \( \binom{r}{2} \) edges, \( r \) vertices and contains \( K_r \), it must be the case that \( G \simeq K_r \).

Now, suppose \( G \) is any graph on \( \binom{r}{2} \) edges with \( \text{ex}(G, K_t) < k \). Let \( G = G_0, G_1, \ldots, G_q = G^* \) where \( G^* \) is \( f \)-compressed and \( G_{i+1} = C_{xy}(G_i) \) for some \( xy \not\in E(G_i) \). By the above argument, we know that \( G^* \simeq K_r \). Now, suppose \( G \not\simeq K_r \); so that \( q \geq 1 \). Let \( u, v \in V(G_{q-1}) \) be such that \( G^* = C_{uv}(G_{q-1}) \). For ease of notation, we will write \( N(x) = N_{G_{q-1}}(x) \) for the remainder of the proof.

As \( G_{q-1} \) can be contracted once more, \( G_{q-1} \not\simeq K_r \). Then \( |V(G_{q-1})| > r \) and as \( G^* = K_r \), we must have \( N(u) \cup N(v) = V(G_{q-1}) \setminus \{u, v\} \) and \( V(G_{q-1}) \setminus \{u, v\} \) must induce a copy of \( K_{r-1} \). Further, \( G^* \) is simple, so
it must be the case that $G_{q-1}$ is simple, moreover $N(u) \cap N(v) = \emptyset$ otherwise $G^*$ would contain a multiedge upon contracting $uv$. We check that such a graph has a $K_t$-free subgraph which is too large.

Indeed, first suppose $|N(u)| < \lfloor \frac{k}{t-1} \rfloor$. Then let $T$ be a copy of $T_{t-1}(r-1)$ contained in $V(G_{q-1}) \setminus \{u, v\}$ with parts $X_1, \ldots, X_{t-1}$ where $X_1 \supseteq N(u)$ and $|X_1| = \lfloor \frac{k}{t-1} \rfloor$. Then if $H$ is the subgraph consisting of the edges in $T$ along with the edges incident to $u$ and edges of the form $\{vx : x \notin X_{t-1}\}$, we find that $H \subseteq T_{t-1}(r+1)$ as $uv \notin E(G_{q-1})$, so $H$ is $K_t$-free. Additionally,

$$|E(H)| = |E(T_{t-1}(r-1))| + |N(u)| + |N(v)| - |X_1 \cap N(v)|$$

$$\geq |E(T_{t-1}(r-1))| + (r-1) - \left( \left\lfloor \frac{r-1}{t-1} \right\rfloor - 1 \right)$$

$$= |E(T_{t-1}(r))| + 1 = k,$$

a contradiction. Thus, we may suppose that $|N(u)|, |N(v)| \geq \lfloor \frac{k}{t-1} \rfloor$. Additionally, as $|N(u)| + |N(v)| = r-1$, we have, without loss of generality, $|N(v)| \geq \lfloor \frac{k}{t-1} \rfloor$. As such, let $T$ be a copy of $T_{t-1}(r-1)$ contained in $V(G_t) \setminus \{u, v\}$ with parts $X_1, \ldots, X_{t-1}$ where $X_1 \subseteq N(u)$ and $X_2 \subseteq N(v)$. Now, let $H$ consist of $T$ along with all edges incident to $u$ or $v$. As $uv \notin E(G_{q-1})$, $H$ is again a subgraph of $T_{t-1}(r+1)$, and so is $K_t$-free. However,

$$|E(H)| = |E(T_{t-1}(r-1))| + |N(u)| + |N(v)|$$

$$= |E(T_{t-1}(r-1))| + r - 1$$

$$= |E(T_{t-1}(r))| + \left\lfloor \frac{r-1}{t-1} \right\rfloor \geq k,$$

another contradiction. We conclude that any (multi)graph $G$ with $|E(G)| = \binom{r}{2}$ and $\ex(G, K_t) < k$ must be a copy of $K_r$. 

It is not clear what the precise value of $\mathcal{E}_k(K_t)$ and $\mathcal{E}_k^+(K_t)$ are when $k \neq \ex(K_r, K_t) + 1$ for any $r$, but we conjecture the following:

**Conjecture 2.7.** For positive integers $r_1 \geq \cdots \geq r_\ell$, let $K(r_1, \ldots, r_\ell)$ be the multigraph consisting of “nested” copies of $K_{r_i}$: that is, on vertex set $[r_1]$, we overlay a copy of $K_{r_i}$ on $[r_i]$ for every $i$ (thus, the maximum edge-weight is $\ell$, provided every $r_i \geq 2$). For every $k$, there exist positive integers $r_1 \geq \cdots \geq r_\ell$ such that $K(r_1, \ldots, r_\ell)$ is extremal for $\mathcal{E}_k^+(K_\ell)$.

### 2.1. Cycles

We begin this section with a simple result related to the graphic matroid rank of a graph.

**Theorem 2.8.** If $C := \{C_3, C_4, \ldots\}$ is the set of all cycles, then $\mathcal{E}_k(C) = \binom{k}{2}$. Furthermore, the only extremal graph for $\mathcal{E}_k(C)$ is $K_k$.

**Proof.** Note that any $k$-edge subgraph of $K_k$ contains a cycle, hence $\mathcal{E}_k(C) \geq \binom{k}{2}$. Now, suppose that $G$ is some connected graph with $|E(G)| > \binom{k}{2}$, then $E(G)$ spans at least $k + 1$ vertices. As such, $G$ has a spanning tree with at least $k$ edges, so $\ex(G, C) \geq k$. Hence, any connected $G$ with $\ex(G, C) < k$ has $|E(G)| \leq \binom{k}{2}$. Since every cycle is connected, we are done by taking $f$ to be the connectedness simplification ((2) in Definition 2.2) in Corollary 2.4.

We now wish to argue that the only extremal graph for $\mathcal{E}_k(C)$ is $K_k$. The above shows the only connected $G$ with $\binom{k}{2}$ edges and $\ex(G, C) < k$ is $K_k$. On the other hand, suppose there were some disconnected $G$ with
Theorem 2.10. For extremal function for the class of all even cycles, denoted by which will be proved in a more general context later (see Theorem 3.5). Thus, it is natural about this

If \( H \) is also \( C \)-free, so \( \Delta(G) \leq k - 1 \). Thus

If we have equality, then certainly \( |V(G)| = k \). But there cannot be any edge of multiplicity 2 or higher; otherwise, extending this to a spanning tree of \( G \) (with multi-edges) will gain a further \( k - 2 \) edges at least, yielding a \( C \)-free subgraph of \( G \) with at least \( k \) edges. Thus \( G \) was simple, and hence must be \( K_k \) by Theorem 2.8. \( \square \)

If \( \mathcal{H} \) does not contain a bipartite graph, then the asymptotic value of \( \mathcal{E}_k(\mathcal{H}) \) is determined by Theorem 1.3 which will be proved in a more general context later (see Theorem 3.5). Thus, it is natural about this extremal function for the class of all even cycles, denoted by \( C_e \).

Theorem 2.10. For \( k \geq 4 \), \( \mathcal{E}_k(C_e) = \left\lceil \frac{k^2}{4} \right\rceil \). Furthermore, the only extremal graph for \( \mathcal{E}_k(C_e) \) is the balanced complete bipartite graph on \( k \) vertices, unless \( k = 5 \).

Proof. Lower bound. Let \( G \) be the balanced complete bipartite graph on \( k \) vertices. Naturally, any \( k \) edges from \( G \) contain a cycle, which is necessarily even as \( G \) is bipartite. Hence, \( \mathcal{E}_k(C_e) \geq |E(G)| = \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{k}{2} \right\rceil = \left\lceil \frac{k^2}{4} \right\rceil \).

For the upper bound, we again look to use Corollary 2.4 and first prove the connected case.

Lemma 2.11. If \( G \) is connected with \( |E(G)| \geq \left\lceil \frac{k^2}{4} \right\rceil \), then \( \text{ex}(G, C_e) \geq k - 1 \), with equality if and only if \( G = K_{\lceil k/2 \rceil, \lfloor k/2 \rfloor} \) or, in the case of \( k = 5 \), \( G = K_4 \).

Proof. Let \( G \) be any connected graph on \( n \) vertices with \( \text{ex}(G, C_e) \leq k - 1 \). For any spanning tree \( F \) of \( G \), \( F \) contains no even cycle, so \( |E(F)| \leq k - 1 \), or in other words, \( n \leq k \). As such, set \( k = n + q \), and assume

but that \( G \) is not the complete balanced bipartite graph. Then as \( n \leq k \), we know, by the uniqueness of the Turán graph, that \( G \) contains a triangle. We will attempt to use the triangles in \( G \) to build a large \( C_e \)-free subgraph.

Say \( T \subseteq G \) is a “triangle forest” with \( t \) triangles if \( E(T) \) is a collection of \( t \) edge-disjoint triangles such that the removal of any one edge from each triangle forms a forest. In particular, the only cycles within such a \( T \) are the \( t \) triangles. So we may extend \( T \) to a spanning subgraph \( H \) (using connectivity) with no additional cycles, thus \( H \) is still \( C_e \)-free. We deduce that \( (n - 1) + t = |E(H)| \leq k - 1 \), so we must have \( t \leq q \). In
Thus, if $q = 0$, then $G$ must be the balanced complete bipartite graph on $k$ vertices. Thus, for the remainder of the proof, we shall suppose $q \geq 1$.

Now, take such a triangle forest $T$ with:

1. $|E(T)|$ (and hence $t$) as large as possible,
2. Subject to (1), if $T = T_1 \cup \cdots \cup T_t$ is a decomposition of $T$ into connected components where $|T_1| \geq \cdots \geq |T_t|$, then $(|T_1|, \ldots, |T_t|)$ is maximal in the lexicographic ordering.

By the lexicographic order, we mean that $(a_1, \ldots, a_k) \succ (b_1, \ldots, b_k) \iff a_j > b_j$ for $j := \min\{i : a_i \neq b_i\}$. Such a lexicographic maximal $T$ means there is no $v \in T_i$ with 2 edges to the same triangle in $T_j$ for any $i < j$. If this were not the case and $wx, wy \in E(G)$, then let $T' := (T \cup \{wx, wy\}) \setminus \{wx, wy\}$ (see Figure 2.1). $T'$ is a triangle forest with the same number of edges as $T$, with $|T'_j| = |T_j|$ for all $j < i$ yet $|T'_i| \geq |T_i| + 2$, so $T'$ is lexicographically larger than $T$, contradicting (2).

\begin{figure}[h]
\centering
\def\svgwidth{0.8\textwidth}
\input{diagram21.pdf_tex}
\caption{Finding a lexicographically larger triangle forest in the case where some vertex in $T_i$ has two edges to the same triangle in $T_j$.}
\end{figure}

Thus, if $T_j$ consists of $t_j$ triangles for every $j$ (so that $|T_j| = 2t_j + 1$ and $t = \sum_j t_j$), then whenever $i < j$, every $v \in T_i$ has at most $t_j$ edges to $T_j$. Summing over all $v \in T_i$ gives $|E[T_i, T_j]| \leq (2t_i + 1)t_j$.

We now attempt to bound the remaining edges in $G$.Crudely, $|E(G[T_i])| \leq (\frac{|T_i|}{2}) = 2t_i^2 + t_i$ for all $i$.

**Case 1:** $|V(T)| \leq \frac{n}{2}$.

Let $G' := G \setminus \bigcup_i G[T_i]$. As $T$ is maximal, $G'$ must be triangle-free, so certainly $|E(G')| \leq \frac{n^2}{4}$. Therefore,

$$\frac{n^2}{4} + \frac{2nq + q^2 - 1}{4} \leq |E(G)| = |E(T)| + |E(G')| \leq \sum_{i=1}^t (2t_i^2 + t_i) + \frac{n^2}{4},$$

and so

$$\frac{2nq + q^2 - 1}{4} \leq t_1 \sum_{i=1}^t (2t_i + 1) = t_1|V(T)| \leq t|V(T)| \leq q \cdot \frac{n}{2}.$$ 

Thus, $q = 1$ as we supposed that $q \geq 1$, so we have equality everywhere. In particular, $t_1 = t = q = 1$, so $T$ is a single triangle, $|V(T)| = \frac{n}{2} \Rightarrow n = 6$, and $|E(G')| = \frac{n^2}{4} \Rightarrow G' = K_{3,3}$. Since $G$ is therefore a 6-vertex, edge-disjoint union of $K_{3,3}$ with a triangle, this uniquely determines $G$ as $K_6 \setminus K_3$, and this $G$ still has $\text{ex}(K_6 \setminus K_3, \mathcal{C}_e) \geq 7 = n + q = k$ (see Figure 2.2); a contradiction.

**Case 2:** $|V(T)| > \frac{n}{2}$.

In this case, for the triangle-free graph $G'' := G \setminus G[T] = G' \setminus \bigcup_{i,j} G[T_i, T_j]$, $V(T)$ spans an independent set
in $G''$, so we can apply a stronger version of the Mantel bound (reproved here for completeness): for each 
v ∈ V, \( d_{G''}(v) \leq \alpha(G'') = \alpha \). Now, if \( I \) is an independent set of size \( \alpha \), then every edge of \( G'' \) must meet \( V \setminus I \), so

\[
|E(G'')| \leq \sum_{v \in V \setminus I} d_{G''}(v) \leq |I||V \setminus I| = \alpha(n - \alpha).
\]

Of course, \( V(T) \) is an independent set in \( G'' \) by construction, so \( \alpha \geq |V(T)| = 2t + \ell \). As \( x(n - x) \) is strictly decreasing for \( x \geq n/2 \), we have \( |E(G'')| \leq (2t + \ell)(n - (2t + \ell)) \) as \( 2t + \ell > \frac{n}{2} \).

We run a similar calculation in this case:

\[
\left\lfloor \frac{k^2}{4} \right\rfloor \leq |E(G)| \leq \sum_{i=1}^{\ell} |E(G[T_i])| + \sum_{i<j} |E(T_i, T_j)| + |E(G'')| \\
\leq \sum_{i=1}^{\ell} (2t_i^2 + t_i) + \sum_{i<j} ((2t_i + 1)t_j) + (n - (2t + \ell))(2t + \ell) \\
\leq \sum_{i=1}^{\ell} (2t_i^2 + t_i) + \sum_{i<j} \left(t_it_j + \frac{t_i + t_j}{4}\right) + n(2t + \ell) - (2t + \ell)^2 \\
= t^2 + \sum_{i=1}^{\ell} t_i^2 + \left(\frac{\ell + 1}{2}\right)t + n(2t + \ell) - (2t + \ell)^2,
\]

so

\[
n^2 + 2qn + q^2 - 4n(2t + \ell) + 4(2t + \ell)^2 - 1_{k\text{ odd}} \leq 8t^2 + 2(\ell + 1)t \\
\Rightarrow (n + q - 2(2t + \ell))^2 + 4q(2t + \ell) - 1_{k\text{ odd}} \leq 8t^2 + 2(\ell + 1)t \leq 8qt + 2(2t)q.
\]

It follows \( |k - 2(2t + \ell)| \leq 1_{k\text{ odd}}. But the reverse is true whether \( k \) is even or odd, hence we again obtain all inequalities above at equality. So certainly \( t = q, \ell = 1, G[V(T)] = G[V(T_1)] \) is a clique, and \( \alpha(G'') = 2t + \ell \), so

\[
|E(G'')| = \sum_{v \notin V(T)} d_{G''}(v) = (n - (2t + \ell))(2t + \ell). \]

As such, \( G''[V(T)] \) is empty and \( d_{G''}(v) = 2t + \ell \) for every \( v \notin V(T) \), so \( G'' \) is the complete bipartite graph on \([V(T), V(T)]. Putting this together with the clique on \( V(T) \), deduce \( G \simeq K_n \setminus K_r \), where \( r = n - (2t + \ell) = n - (2q + 1) \).

We know \( k - (4q + 2) = \epsilon \in \{0, \pm 1\} \), so \( r = (k - q) - (2q + 1) = q + 1 + \epsilon \). Now, if \( r \geq q + 1 \), we can find a triangle forest \( F \) with \( q + 1 \) triangles (contradicting maximality of \( T \) as \( t \leq q \)) by taking a path on \( 2(q + 1) \) edges with \( q + 1 \) vertices in \( V(T) \) and \( q + 2 \leq 2q + 1 \) among \( V(T) \), and completing the \( q + 1 \) edge-disjoint copies of \( P_3 \) into \( K_3 \)’s using the edges from inside \( T \) (See Figure 2.3). Furthermore, if \( q \geq 2 \), then we may similarly choose \( P \) by instead taking \( q \leq q + 1 + \epsilon \) vertices of \( V(T) \) and \( q + 3 \leq 2q + 1 \) vertices of \( V(T) \). Otherwise, \( \epsilon = -1 \) and \( q = 1 \). In this case, we deduce that \( G \simeq K_4 \), which does have \( \text{ex}(K_4, C_e) = \text{ex}(K_{2,3}, C_e) = 4 \). \( \square \)
Upper bound. If $G$ is now arbitrary with $|E(G)| \geq \left\lceil \frac{k^2}{4} \right\rceil$, then forming any $I \subseteq V(G)$ with one vertex from each connected component gives $\text{ex}(G, C_e) \geq \text{ex}(C_I(G), C_e) \geq k-1$. If we have equality here, we know $C_I(G)$ is necessarily $K_{\lfloor k/2 \rfloor, \lceil k/2 \rceil}$ (or $K_4$) by the lemma, yet none of these graphs have a cut-vertex for $k \geq 4$. Hence, $G$ must have been connected in the first place, so $G$ is one of the claimed extremal graphs. \hfill \Box

Unfortunately, the above argument is very specific to simple graphs, so we have been unable to determine $\mathcal{E}^*_k(C_e)$ unless it happens to be the case that $\mathcal{E}^*_k(C_e) = \mathcal{E}_k(C_e)$.

2.2. Small Graphs. In this section, we will explore $\mathcal{E}_k(H)$ where $H$ is a collection of small graphs. At the end of this section, we also give a complete classification of the families which have $\mathcal{E}_k(H) = \infty$. Throughout this section, we will only focus on simple host graphs.

Recall that $P_t$ denotes the path on $t$ edges.

**Theorem 2.12.** For $H = \{P_3, K_3\}$, and $k \geq 3$,

$$\mathcal{E}_k(H) = \begin{cases} \binom{k+1}{2} - \frac{k^2}{4} & \text{if } k \text{ is even;} \\ \binom{k+1}{2} - \frac{k+1}{2} & \text{if } k \text{ is odd.} \end{cases}$$

Moreover, the only extremal graph for $\mathcal{E}_k(H)$ is

$$G_k := \begin{cases} K_{k+1} \setminus \left( \frac{k^2}{2} K_2 \cup P_2 \right) & \text{if } k \text{ is even;} \\ K_{k+1} \setminus \left( \frac{k+1}{2} K_2 \right) & \text{if } k \text{ is odd.} \end{cases}$$

Note that the first of these results has been previously noticed by Ferneyhough, Haas, Hanson and MacGillivray in [10, Corollary 2] using a bound on the domination number due to Vizing [17]. They also used the graphs $G_k$ to provide the lower bounds. We offer a self-contained proof that also shows these extremal graphs are in fact unique.

**Definition 2.13.** Given a graph $G$, a *star-packing* of $G$ is a subgraph of $G$ which is a union of vertex-disjoint stars.

It is quick to observe that $H \subseteq G$ is $\{P_3, K_3\}$-free if and only if $H$ is a star packing of $G$ with possible isolated vertices.
Lemma 2.14. Let $G$ be a graph on $n + t$ vertices. If every star-packing in $\overline{G}$ has at most $n - 2$ edges, then

$$2|E(G)| \geq f(n, t) := \begin{cases} 
  n + 2nt + t(t - 1) & \text{if } n \text{ is even}; \\
  n + 1 + 2nt + t(t - 1) & \text{if } n \text{ is odd}.
\end{cases}$$

Further, if equality holds, then $\overline{G} \simeq G_{n-1} \cup K_t$.

Proof. If $n \leq 3$, the statement is straightforward, so assume $n \geq 4$. We first claim that for any $i \geq 1$ and $S \subseteq V$ with $|S| = i$, then $S$ has at least $t - i + 2$ common neighbors in $V \setminus S$. If this were not the case, then there are at least $|V \setminus S| - (t - i + 1) = n - 1$ vertices in $V \setminus S$ which are not connected to some $v \in S$. Thus, we can find $n - 1$ edges in $\overline{G}$ that form vertex-disjoint stars with centers in $S$, contradicting the fact that every star packing has at most $n - 2$ edges. In particular this implies that

1. Taking $i = 1$, $\delta(G) = t + s + 1$ for some $s \geq 0$.
2. Taking $i = 2$, any two vertices have at least $t$ common neighbors.

Now, proceed by induction on $t$.

When $t = 0$, we have $\delta(G) \geq 1$ by (1), so $2|E(G)| \geq n + 1_{n \text{ odd}}$, with equality if and only if $G \simeq \frac{n - 3}{2}K_2$ when $n$ is even or $G \simeq \frac{n - 3}{2}K_2 \cup P_2$ when $n$ is odd. In either case, $\overline{G} \simeq G_{n-2}$.

Otherwise, $t \geq 1$, so $\text{diam}(G) \leq 2$ by (2). In this case, choose $v \in V$ with $d(v) = \delta(G) = t + s + 1$ for some $s \geq 0$ and define $N^2(v) := \{w \in G : \text{dist}(v, w) = 2\} = V \setminus (N(v) \cup \{v\})$. As $d(v) = t + s + 1$, we have $|N^2(v)| = n - s - 2$. In particular, $\{v\} \times N^2(v)$ is a star with $n - s - 2$ edges in $\overline{G}$. Thus, setting $G' := G[N(v)]$ it must be the case that every star packing in $\overline{G'}$ must have at most $s$ edges, otherwise we could find a star packing in $\overline{G'}$ with $n - 1$ edges.

Set $n' = s + 2$ and $t' = (s + t + 1) - n' = t - 1$. As $|V(G')| = n' + t'$, and every star packing in $\overline{G'}$ has at most $n' - 2$ edges, by induction,

$$2|E(G')| \geq f(n', t') = n' + 2n't' + t'(t' - 1) + 1_{n' \text{ odd}} = 2st - s + t^2 + t + 1_{n \text{ odd}}.$$ 

Additionally, we find that $|E(G[N(v), N^2(v)])| \geq t(n - s - 2) = n - s - 2$ and any two vertices have at least $t$ common neighbors. We thus obtain

$$2|E(G[N(v), N^2(v)])| + 2|E(G[N^2(v)])| = |E(G[N(v), N^2(v)])| + \sum_{w \in N^2(v)} d(w)$$

\begin{align*}
\geq 1_{n \text{ odd, } s \text{ even}} + t(n - s - 2) + (n - s - 2)(t + s + 1) & \quad (2.2) \\
= 1_{n \text{ odd, } s \text{ even}} + (n - s - 2)(2t + s + 1),
\end{align*}
since \((n - s - 2)(2t + s + 1)\) is odd whenever both \(n\) is odd and \(s\) is even. So we calculate

\[
2|E(G)| = 2|E(G[N(v), N^2(v)]| + 2|E(G[N^2(v)]| + 2d(v) + 2|E(G')|
\]

\[
\geq 1_{n \text{ odd, } s \text{ even}} + (n - s - 2)(2t + s + 1) + 2(t + s + 1) + f(n', t')
\]

\[
\geq 1_{n \text{ odd, } s \text{ even}} + (n - s - 2)s + ((n - s - 2)(2t + 1) + 2(t + s + 1)) + (2st - s + t^2 + t + 1_{s \text{ odd}})
\]

\[
= 1_{n \text{ odd, } s \text{ even}} - 1_{n \text{ even}} + 1_{s \text{ odd}} + (n - s - 2)s + (n + 2nt + t(t - 1) + 1_{n \text{ odd}})
\]

\[
= 1_{n \text{ odd, } s \text{ even}} - 1_{n \text{ odd}} + 1_{s \text{ odd}} + (n - s - 2)s + f(n, t)
\]

\[
\geq f(n, t).
\]

The last inequality follows from \(n - s - 2 = |N^2(v)| \geq 0\). We have now established the claimed bound.

When \(|E(G)| = f(n, t)|, we wish to show \(G \simeq G_{n-1} \cup K_t\). Certainly, all inequalities above are equalities, so \(s|N^2(v)| = 0\). If \(N^2(v) = \emptyset\), then \(\delta(G) = d(v) = |V| - 1\); hence, \(G \simeq K_{n+1}\), a contradiction as \(|E(K_{n+1})| > f(n, t)\) whenever \(n \geq 2\).

So instead \(s = 0\). Deduce \(\delta(G) = d(v) = t + 1\) by (1), and for any \(v \in V\), \(|N(v) \cap N(w)| \geq t\) by (2). As such, \(G' = G[N(v)] \simeq K_{t+1}\).

Equality in Equation (2.2), shows that all but (at most) one \(w \in N^2(v)\) satisfy both \(d(w) = t + 1\) and \(|N(w) \cap N(v)| = t\), and thus has exactly 1 edge inside \(N^2(v)\). There are at least \(|N^2(v)| - 1 = n - 3 \geq 1\) such \(w\), so fix one such \(w_1\) and let \(w_0\) be its unique neighbor in \(N^2(v)\). Then, \(w_0, w_1\) share \(t\) neighbors, which must therefore be some \(S \subseteq N(v)\).

Case 1. \(N(w_0) = S \cup \{w_1\}\) (i.e. \(d(w_0) = t + 1\)). Then every other \(w \in N^2(v) \setminus \{w_0, w_1\}\) shares \(t\) neighbors with \(w_0\), none of which are \(w_1\), so must share \(S\).

Case 2. \(d(w_0) > t + 1\). So the equality in Equation (2.2) in fact shows \(d(w) = t + 1\) and \(|N(w) \cap N(v)| = t\) for every \(w \in N^2(v) \setminus \{w_0, w_1\}\). If some such \(w\) did not have \(S\) as its \(t\) neighbors in \(N(v)\), then since \(w_2\) shares \(t\) neighbors with both \(w_1\) and \(w_0\), it must be adjacent to both \(w_0\) and some \(w' \in N^2(v) \cap N(w_0)\) (possibly \(w_1\)). So in total, \(d(w) \geq t + 2\); a contradiction.

In either case, every vertex in \(S\) is connected to every vertex in \(G\), so \(S\) is a collection of isolated vertices in \(G\). As such, \(G \setminus S\) still has no star-packing with at least \(n - 2\) edges, while \(G' \setminus S\) is left with \(\frac{f(n, t)}{2} - \left(\begin{smallmatrix} t \\ 2 \end{smallmatrix}\right) - nt = \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]\) edges. Crudely \(\Delta(G \setminus S) \leq n - 2\), so \(\delta(G \setminus S) \geq 1\), hence \(G \setminus S \simeq \frac{n}{2}K_2\) (or \(\frac{n+1}{2}K_2 \cup P_2\) if \(n\) is odd). Adding \(S\) back shows \(G \simeq G_{n-1} \cup K_t\). \(\square\)

**Proof of Theorem 2.12 Lower bound.** As \(\Delta(G_k) = k - 1\), any single star in \(G_k\) hast at most \(k - 1\) edges. Additionally, as \(|V(G_k)| = k + 1\), any star-packing in \(G_k\) with \(i \geq 2\) stars has at most \(k + 1 - i \leq k - 1\) edges. Thus, \(\text{ex}(G_k, \{K_3, P_3\}) < k\), so \(E_k(\{K_3, P_3\}) \geq |E(G_k)| = \binom{k+1}{2} - \frac{k + 1 + 1_{k \text{ even}}}{2}\).

**Upper bound.** Let \(G\) be a graph with \(\text{ex}(G, \{K_3, P_3\}) < k\). Thus, every star-packing in \(G\) has at most \(k - 1\) edges. If \(G\) has at most \(k\) vertices, then

\[
|E(G)| \leq \binom{k}{2} < \binom{k + 1}{2} - \frac{k + 1 + 1_{k \text{ even}}}{2}.
\]
Thus, we may suppose $G$ has $k + 1 + t$ vertices for some $t \geq 0$. By Lemma 2.14 if every star packing in $G$ has at most $k - 1$ edges, then $2|E(G)| \geq f(k + 1, t)$. Thus,

$$|E(G)| \leq \left(\frac{k + 1 + t}{2}\right) - f(k + 1, t) = \left(\frac{k + 1}{2}\right) - \frac{k + 1 + 1_k \text{ even}}{2}.
$$

Further, if equality holds, then $G \cong G_k \cup K_t$, so as we do not consider graphs with isolated vertices, we must have $G \cong G_k$. As such, $G_k$ is the unique extremal graph for $E_k(\{K_3, P_3\})$. □

We now turn out attention to determining $E_k(P_3)$. We note that $H$ is $P_3$-free if and only if $H$ is the vertex-disjoint union of triangles, stars and isolated vertices. The following graphs will be important in determining $E_k(P_3)$ and classifying the extremal graphs.

**Definition 2.15.** For fixed positive integers $k, r_1, r_2, \ldots, r_s$ with $\sum_{i=1}^{s} r_i = k$, define the **pendant graph** $K_k^*(r_1, \ldots, r_s)$ as follows. Take a clique on some $k$-vertex set $\{v_1, \ldots, v_k\}$, called the core, and additional vertices $\{w_1, \ldots, w_s\}$, called the pendants. Partition $\{v_1, \ldots, v_k\} = W_1 \cup \cdots \cup W_s$ where $|W_i| = r_i$ and connect $w_i$ to the vertices in $W_i$. See Figure 2.4.

As such, the degree sequence of $K_k^*(r_1, \ldots, r_s)$ is $(k, \ldots, k, r_1, \ldots, r_s)$ and $|E(K_k^*(r_1, \ldots, r_s)| = \binom{k+1}{2}$.

![Figure 2.4. Pendant graphs.](image)

**Lemma 2.16.** Let $k \geq 4$ and let $r_1, \ldots, r_s$ be positive integers with $\sum_{i=1}^{s} r_i = k - 1$. We have $\text{ex}(K_{k-1}^*(r_1, \ldots, r_s), P_3) \geq k - 1$, where equality holds if and only if either $r_1 = 1$ for all $i$, or $3 \nmid k$ and $r_1 = k - 1$. In particular, $E_k(P_3) \geq \binom{k}{2}$.

**Proof.** Every vertex in the core of $G := K_{k-1}^*(r_1, \ldots, r_s)$ has degree $k - 1$, so $\text{ex}(G, P_3) \geq k - 1$ is immediate by taking any star centered at a core vertex of $G$.

Now, if $r_1 = k - 1$, then $G \cong K_k$, and it is well-known that $\text{ex}(K_k, P_3) = k - 1$ if $3 \nmid k$. If $(r_1, \ldots, r_s) = (1, \ldots, 1)$, then let $U$ denote the core of $G$. Now let $H \subseteq G$ be any $P_3$-free subgraph, so $H$ is a vertex-disjoint union of triangles, stars and isolated vertices. Now, no triangle $T$ in $H$ can contain a pendant vertex, so each $V(T) \subseteq U$, and every star contains at most one; hence $|V(S) \cap U| \geq |V(S)| - 1$ for each star $S$. Hence,
splitting up $H$ into components:

$$|E(H)| = \sum_{T \subseteq H} |E(T)| + \sum_{S \subseteq H} |E(S)|$$

$$= \sum_{T \subseteq H} |V(T)| + \sum_{S \subseteq H} (|V(S)| - 1)$$

$$\leq \sum_{T \subseteq H} |V(T) \cap U| + \sum_{S \subseteq H} |V(S) \cap U| \leq |U| = k - 1.$$

As such, $\text{ex}(G, P_3) = k - 1$. In particular, $\mathcal{E}_k(P_3) \geq |E(K_{k-1}^*(1, \ldots, 1))| = \binom{k}{2}$.

We now wish to show that if $G := K_{k-1}^*(r_1, \ldots, r_s)$ where $(r_1, \ldots, r_s)$ is neither $(k - 1)$ nor $(1, \ldots, 1)$, then $\text{ex}(G, P_3) \geq k$. Suppose that $r_1 \geq \cdots \geq r_s$, so $r_1, s \geq 2$. Let $w_1, w_2$ be the corresponding pendant vertices with degrees $r_1, r_2$, respectively. Let $v_1, v_2 \in U$ be adjacent to $w_1$ and let $v_3 \in U$ be adjacent to $w_2$ (so $v_1, v_2, v_3$ are distinct). Consider the graph $H \subseteq G$ which consists of the triangle $w_1, v_1, v_2$ and the largest star centered at $v_3$ which does not include $v_1, v_2$ (see Figure 2.4a). As $d(v_3) = k - 1$, $H$ is the vertex-disjoint union of a triangle and a star with $k - 3$ edges. In particular, $H$ is $P_3$-free, so $\text{ex}(G, P_3) \geq |E(H)| = k$.

Before determining $\mathcal{E}_k(P_3)$ exactly and classifying all extremal graphs, it is necessary consider a small case.

**Proposition 2.17.** $\text{ex}(G, P_3) = 2$ if and only if $G \in \{P_3, C_4\}$. Hence, $\mathcal{E}_3(P_3) = 4 = \binom{3}{2} + 1$.

**Proof.** Certainly $\text{ex}(P_3, P_3) = \text{ex}(C_4, P_3) = 2$.

If $\text{ex}(G, P_3) = 2$, then every set of 3 edges in $G$ forms a copy of $P_3$. Thus, $\Delta(G) \leq 2$, $G$ is connected and $|V(G)| \geq 4$, so $G$ is a cycle or a path. Both $P_3$ and $C_4$ contain a copy of $P_1 \cup P_2$, which is $P_3$-free, for $n \geq 5$, so we must have $|V(G)| = 4$. As such $G \in \{P_3, C_4\}$. Thus, $\mathcal{E}_3(P_3) = 4$. □

With this out of the way, we can now completely determine $\mathcal{E}_k(P_3)$. Unfortunately, there is a fair amount of case-work involved in the proof of this theorem in order to establish the base case for an induction. For this, we turn to NAUTY to do an exhaustive search subject to the parameters which we will establish in the following proof. As is mentioned in the proof, details about this case check can be found in Appendix A.

**Theorem 2.18.** For $k \geq 3$, if $G$ is a graph with $\text{ex}(G, P_3) < k$, then $|E(G)| \leq \binom{k}{2} + 1_{k=3}$. Furthermore, we have equality if and only if one of the following holds:

- $k = 3$ and $G \simeq C_4$,
- $k = 4$ and $G \simeq K_{2,3}$,
- $k \geq 4$ and $G \simeq K_{k-1}^*(1, 1, \ldots, 1)$, or
- $k \geq 4$, $3 \nmid k$ and $G \simeq K_k$.

Hence, $\mathcal{E}_k(P_3) = \binom{k}{2} + 1_{k=3}$ for $k \geq 3$.

**Proof.** We first note that $\text{ex}(K_{2,3}, P_3) = 3$ and $|E(K_{2,3})| = 6 = \binom{3}{2}$. Thus, along with Lemma 2.16 and Proposition 2.17, all lower bounds have been established. Additionally, Proposition 2.17 establishes the theorem when $k = 3$, so we will suppose $k \geq 4$ for the remainder of the proof. We also note that trivially, $\mathcal{E}_1(P_3) = 0 = \binom{1}{2}$ and $\mathcal{E}_2(P_3) = 1 = \binom{2}{2}$. □
As such, let \( G \) be a graph with \( \text{ex}(G, P_3) < k \) with \( |E(G)| \geq \binom{k}{2} \) and proceed by strong induction on \( k \). Note that \( \text{ex}(G, P_3) \geq \Delta := \Delta(G) \), so \( \Delta \leq k - 1 \).

Firstly, suppose \( G \) contains a triangle \( T = xyz \). If \( H \subseteq G[V \setminus T] =: G' \) is \( P_3 \)-free, then \( H \cup T \) is also \( P_3 \)-free, so \( \text{ex}(G', P_3) < k - 3 \). Thus, by induction, \( |E(G')| \leq \left( \frac{k-3}{2} \right) + 1_{k-3=3} \). Now, as \( \Delta \leq k - 1 \), \( x, y, z \) all have at most \( k - 3 \) neighbors outside \( T \), so

\[
|E(G)| \leq |E(G[V \setminus T])| + 3(k - 3) + 3 \leq \left( \frac{k-3}{2} \right) + 1_{k-3=3} + 3k - 6 = \left( \frac{k}{2} \right) + 1_{k-3=3},
\]

Using these facts, for \( 4 \leq k \leq 6 \), we can run an exhaustive search using NAUTY, the details of which can be read in Appendix \[A\]. Thus, we assume \( k \geq 7 \), so \( |E(G)| \leq \binom{k}{2} \). If equality holds, then all of \( x, y, z \) must have exactly \( k - 3 \) neighbors outside of \( T \) and \( G' \) must be one of the claimed extremal graphs, so \( G' \cong K^*_k(1, \ldots, 1) \), or \( G' \cong K_{k-3} \) and \( 3 \nmid k \), or \( G' \cong K_{2,3} \) and \( k - 3 = 4 \), possibly with isolated vertices.

We first consider the case where \( G' \cong K_{2,3} \), possibly with isolated vertices. In fact, we may suppose that for every triangle \( T' \subseteq G \), we have \( G[V \setminus T'] \cong K_{2,3} \), possibly with isolated vertices, or else we may proceed as in the remaining cases. Let the vertices of the \( K_{2,3} \) in \( G' \) have parts \( A, B \) where \( |A| = 2 \) and \( |B| = 3 \). We first note that each \( v \in T \) must have all remaining \( k - 3 = 4 \) edges to \( A \cup B \), or else there is a \( K_{1,5} \) centered at \( v \) which is disjoint from some copy of \( P_2 \) in \( A \cup B \), yielding \( \text{ex}(G, P_3) \geq 7 \); a contradiction. In particular \( G' \) has no isolated vertices. Additionally, all vertices in \( T \) must be connected to at least one vertex in \( A \); thus, by pigeonhole, there are two vertices in \( T \) adjacent to the same vertex of \( A \), say \( y, z \sim a \). Taking \( T' = yza \) shows that \( G[V \setminus T'] \cong K_{2,3} \). As such, \( x \) must be adjacent to every vertex in \( B \) and also adjacent to \( a \). In particular, \( xab \) is a triangle for all \( b \in B \), so \( G'' = G[V \setminus xa_1b] \cong K_{2,3} \). However, \( y \sim z \) and \( d_{G''}(y), d_{G''}(z) \geq 2 \), which is impossible.

Next, suppose that \( G' \cong K^*_k(1, \ldots, 1) \), possibly with isolated vertices, and let \( U \) denote the core of \( G' \). If \( x \) is not adjacent to some vertex of \( U \), then \( x \) has at least \( k - 3 \) neighbors outside of \( T \cup U \), denote two of these neighbors by \( a, b \). As \( |U| = k - 4 \geq 3 \), there must be some \( u \in U \) which is not adjacent to \( a, b \), so \( u \) is the center of a \( (k - 4) \)-edge star in \( G' \) which does not include \( a, b \). Thus, consider the graph \( H \subseteq G \) consisting of this star centered at \( u \) along with the star \( \{xy, xz, xa, xb\} \). \( H \) is \( P_3 \)-free, so \( \text{ex}(G, P_3) \geq |E(H)| = k \); a contradiction. Hence, by symmetry, \( x, y, z \) are adjacent to all vertices in \( U \). Thus, \( G \) is a pendant graph with core \( T \cup U \). Thus, \( G \) is determined to be \( K^*_k(1, \ldots, 1) \) by Lemma 2.16.

Finally, suppose \( 3 \nmid k \) and \( G' \cong K_{k-3} \), possibly with isolated vertices, and write \( S \subseteq V \setminus T \) for the vertex set of this \( K_{k-3} \). We notice that if \( x \) has at most one neighbor in \( S \), then there is a star centered at \( x \) with at least \( (k - 1) - 1 = k - 2 \) \( \geq 5 \) edges in \( G \) which is disjoint from \( S \). Thus, letting \( H \) consist of a \( (k - 4) \)-edge star in \( G' \) along with this star centered at \( x \) gives \( \text{ex}(G, P_3) \geq |E(H)| \geq k + 1 \); a contradiction. Thus, by symmetry, all of \( x, y, z \) each have at least two neighbors in \( S \). Now, suppose that there is some \( a \in V \setminus (T \cup S) \) that is adjacent to \( x \). If \( k \equiv 2 \) (mod 3), then \( y \) has at least two neighbors in \( S \), then we can partition \( S \cup \{y\} \) into \( (k - 3) + 1 = k - 2 \) vertex-disjoint triangles. Letting \( H \) consist of these triangles along with the star \( \{xz, xa\} \) yields a \( P_3 \)-free subgraph of \( G \) with \( k \) edges; another contradiction. Thus, suppose \( k \equiv 1 \) (mod 3). Either \( y \) and \( z \) share a common neighbor in \( S \) or they each have two distinct neighbors in \( S \). In either case, we can partition \( S \cup \{y, z\} \) into \( (k - 3) + 2 = k - 1 \) vertex-disjoint triangles, so letting \( H \) consist of these triangles along with the edge \( xa \) yields a \( P_3 \)-free subgraph of \( G \) with \( k \) edges; another contradiction. Hence, by symmetry, \( x, y, z \) have no neighbors outside of \( S \cup T \), so, in fact, \( G \cong K_k \).
After all of this, we have established the theorem if $G$ contains a triangle, so we may suppose that $G$ is triangle-free. As such, if $xy \in E(G)$, then $N(x) \cap N(y) = \emptyset$. Taking maximal stars with centers $x$ and $y$ (except for the edge $xy$), yields a $P_3$-free subgraph of $G$, so $k > \text{ex}(G, P_3) \geq (d(x) - 1) + (d(y) - 1)$, so $d(x) + d(y) \leq k + 1$ for every edge $xy$.

If there is some edge $xy$ with $d(x) + d(y) \leq k$, then setting $G' := G \setminus \{x, y\}$ has $|E(G')| \geq \left(\frac{k}{2}\right) - (k - 1) = \left(\frac{k-1}{2}\right)$. Additionally, adding the edge $xy$ to any $P_3$-free subgraph of $G'$ shows that $\text{ex}(G', P_3) \leq \text{ex}(G, P_3) - 1 < k - 1$. Thus, by the induction and the fact that $G'$ is triangle-free, we must have $k \in \{4, 5\}$ and $|E(G')| = \left(\frac{k}{2}\right)$. Again, these cases are established by an exhaustive search whose details are presented in Appendix A.

Hence, we may suppose $d(x) + d(y) = k + 1$ for every $xy \in E(G)$. Fix $x$ and suppose first that $d := d(x) \neq \frac{k+1}{2}$. Letting $C$ denote the connected component of $G$ containing $x$, we can partition $C = A \cup B$ where $A = \{u : d(u) = d\}$ and $B = \{u : d(u) = k + 1 - d\}$. As $d(u) + d(v) = k + 1$ for every $uv \in E(G)$ and $d \neq \frac{k+1}{2}$, $G[C]$ is a bipartite graph with parts $A, B$. Now, for any $u \in A$ and $v \in B$, by considering stars centered at $u$ and $v$ (except for the edge $uv$ if it exists), we find

$$k > \text{ex}(G, P_3) \geq \text{ex}(G[C], P_3) + \text{ex}(G[V \setminus C], P_3) \geq |N(u) \setminus \{v\}| + |N(v) \setminus \{u\}| = k + 1 - 2 \cdot 1_{uv \in E(G)}.$$

From this, we immediately find that $G[V \setminus C]$ is empty, and as the above holds for any $u, v$, we know that $G[C]$ is a complete bipartite graph. Further, as $C$ is a connected component of $G$ and we supposed $G$ has no isolated vertices, we have $G \cong K_{d,k+1-d}$. Thus $|E(G)| = d(k + 1 - d) \leq \left(\frac{k}{2}\right)$. However, we already know that $|E(G)| \geq \left(\frac{k}{2}\right)$ by assumption, so $d(k + 1 - d) = \left(\frac{k}{2}\right)$. As $k \geq 4$, the only way for this to happen is if $k = 4$ and $d \in \{2, 3\}$. Thus, $G \cong K_{2,3}$.

Otherwise, $G$ is $d := \frac{k+1}{2}$-regular. Fix $x \in V$ and set $G' := G - (N(x) \cup \{x\})$. Thus, it is clear that $\text{ex}(G', P_3) + d \leq \text{ex}(G, P_3) < k$, so $\text{ex}(G', P_3) < k - d = \frac{k-1}{2}$. Setting $k' := \frac{k-1}{2}$, we have that $|E(G')| \leq \left(\frac{k'}{2}\right) + 1_{k' = 3}$ by induction. Further, as $G$ is triangle-free, $N(x)$ spans no edges, so

$$\left(\frac{k'}{2}\right) + 1_{k' = 3} \geq |E(G')| = |E(G)| - d^2 \geq \left(\frac{k}{2}\right) - d^2,$$

so

$$d^2 \geq \left(\frac{k}{2}\right) - \left(\frac{k'}{2}\right) - 1_{k' = 3} = \frac{3}{8}(k^2 - 1) - 1_{k' = 3}.$$

As $k$ must be odd and $k \geq 4$, this is only possible if $k = 5$. Setting $k = 5$, all above inequalities become equalities, so we get $d = 3$ and $|E(G)| = \left(\frac{5}{2}\right)$. Thus, $G$ is a 3-regular graph on 10 edges; an impossibility. \hfill \Box

The last small graph we will consider in this paper is $P_1 \cup P_2$. Determining $E_k(P_1 \cup P_2)$ will also allow us to completely classify those families of graphs with $\tilde{E}_k(H) = \infty$, which we will do at the end of this section. As above, it will be important to have a complete classification of $(P_1 \cup P_2)$-free graphs.

**Lemma 2.19.** A graph $H$ is $(P_1 \cup P_2)$-free if and only if one of the following holds:

- $H \cong sK_2$ for some $s$,
- $H \cong K_{1,s}$ for some $s$,
- $H \subseteq K_4$.

**Proof.** Let $F$ be the line graph of $H$ (whereby $V(F) := E(H)$ and $e_1 \sim_F e_2$ if and only if $e_1$ and $e_2$ share a vertex). As $H$ is $(P_1 \cup P_2)$-free, for 3 distinct edges $e_1, e_2, e_3 \in E(H)$, then if $e_1 \not\sim_F e_2$ and $e_2 \not\sim_F e_3$, then it must be the case that $e_1 \not\sim_F e_3$. In particular, the relation $\{(x, y) \in V(F)^2 : x = y \text{ or } x \not\sim_F y\}$ is an
equivalence relation on $V(F)$, so we may color $V(F) = E(H)$ so that any color class is a matching, and any two edges of a distinct color are incident.

- Suppose some color class has $s \geq 3$ edges. Since these $s$ edges are disjoint, no other edge can be simultaneously incident to all of these, so every other color class must be empty. Thus $H \simeq sK_2$.
- Suppose some color class has 2 edges. Then all other edges must be incident to both of these, so $H \subseteq K_4$.
- Otherwise, there is 1 edge in each color, and they are all pairwise incident, so $H \simeq K_3$ or $H \simeq K_{1,s}$ for some $s$.

Conversely, all of these graphs are clearly $(P_1 \cup P_2)$-free.

With this classification, determining $\text{ex}(G, P_1 \cup P_2)$ for any graph $G$ is straightforward.

**Corollary 2.20.** For any $G$, $\text{ex}(G, P_1 \cup P_2) = \max\{\Delta(G), M(G)\} = t$, provided $t \geq 6$. Here $M(G)$ is the size of a maximum matching in $G$.

**Proof.** As any star in $G$ is $(P_1 \cup P_2)$-free, certainly $\text{ex}(G, P_1 \cup P_2) \geq \Delta(G)$. Similarly, $\text{ex}(G, P_1 \cup P_2) \geq M(G)$ as any matching in $G$ is also $(P_1 \cup P_2)$-free.

Conversely, take any subgraph $H \subseteq G$ with $t + 1 > 6$ edges, so $H \not\subseteq K_4$. By the definition of $t$, $H$ is neither a star nor a matching, so by Lemma [2.19] $H$ must contain a copy of $P_1 \cup P_2$. Therefore, $\text{ex}(G, P_1 \cup P_2) \leq t$. \qed

Using the above Corollary, we can provide lower bounds on $\mathcal{E}_k(P_1 \cup P_2)$.

**Corollary 2.21.** If $k \geq 7$, then $\mathcal{E}_k(P_1 \cup P_2) \geq \begin{cases} k^2 - \frac{3}{2}k & \text{if } k \text{ is even;} \\ k^2 - k & \text{if } k \text{ is odd.} \end{cases}$

**Proof.** First suppose $k$ is odd and consider $G := 2K_k$, so $\Delta(G) = M(G) = k - 1$. Therefore $\text{ex}(G, P_1 \cup P_2) < k$ by Corollary [2.20] as $k \geq 7$, so $\mathcal{E}_k(P_1 \cup P_2) \geq |E(G)| = 2\left(\frac{k}{2}\right)^2 = k^2 - k$.

On the other hand, if $k$ is even, start with the Cayley graph $H := \text{Cay}(Z_{2k-1}, \{-\frac{k-2}{2}, \frac{k-2}{2}\} \setminus \{0\})$; that is $V(H) = Z_{2k-1}$ and $xy \in E(H)$ if and only if $x - y \pmod{2k-1} \in \{-\frac{k-2}{2}, \frac{k-2}{2}\} \setminus \{0\}$. Now, look at all pairs of the form $\{xy : |x - y| = k/2\}$. Since $k/2$ and $2k-1$ are coprime, these pairs form a Hamilton cycle in the complete graph on $Z_{2k-1}$, so take any matching $M$ among them of size $k - 1$. Finally, consider the graph $G := (Z_{2k-1}, E(H) \cup M)$. As $M$ and $E(H)$ are disjoint, every vertex of $G$ has degree $(k - 1)$ with the exception of one vertex, which has degree $k - 2$. Also, $M(G) = k - 1$, so $\text{ex}(G, P_1 \cup P_2) < k$ again by Corollary [2.20]. Therefore,

$$\mathcal{E}_k(P_1 \cup P_2) \geq |E(G)| = \frac{1}{2} \sum_{v \in V(G)} d(v) = \frac{1}{2}(2(k-2)(k-1) + (k-2)) = k^2 - \frac{3}{2}k.$$

To yield upper bounds on $\mathcal{E}_k(P_1 \cup P_2)$, we prove a general bound on the number of edges of a graph based on its maximum degree and matching number. A similar theorem was proved by Abbot, Hanson and Sauer [1] in the context of the Erdős-Rado sunflower lemma, but we provide a full proof for completeness.

**Theorem 2.22.** For a graph $G$, $|E(G)| \leq (\Delta(G) + 1)M(G)$. Furthermore, the inequalities in Corollary [2.21] are in fact equalities.
In order to prove this, we will need the following proposition, which is an immediate consequence of the Gallai-Edmonds decomposition of a graph (c.f. [15] pp. 93–95).

**Proposition 2.23.** If \( G \) is a connected graph with the property that for every \( v \in V \), \( M(G - v) = M(G) \), then \( G \) has an odd number of vertices and \( M(G) = \frac{\lfloor \Delta(G) \rfloor}{2} \).

**Proof of Theorem 2.22.** Let \( G \) be a graph with \( M(G) \leq M \) and \( \Delta(G) \leq \Delta \). Suppose \( G \) has components \( S_1, \ldots, S_s, H_1, \ldots, H_t \), where \( S_i \) is a star of degree at most \( \Delta \). We will consider a series of reductions of \( G \) that maintain the matching and degree restrictions and not decrease the number of edges. We first claim that for each \( i \) and any \( v \in V(H_i) \), we may suppose that \( M(H_i - v) = M(H_i) \). To see this, suppose that this is not the case for some \( i \) and \( v \). In this case, let \( G' \) be the graph formed by replacing \( H_i \) with \( H_i' = H_i - v \) and adding a copy of \( K_{1, \Delta} \). As \( d(v) \leq \Delta \), we have \( \Delta(G') = \Delta \) and \( |E(G')| \geq |E(G)| \). Further, as every maximum matching in \( H_i \) used \( v \), \( M(H_i') = M(H_i) - 1 \), so as \( M(K_{1, \Delta}) = 1 \), we have \( M(G') = M(G) \leq M \). Thus, we may assume that \( M(H_i - v) = M(H_i) \) for all \( i \) and any \( v \in V(H_i) \).

As such, \( |V(H_i)| \) is odd and \( M(H_i) = \frac{|V(H_i)| - 1}{2} \) by Proposition 2.23. We now claim that we may suppose that \( |V(H_i)| \geq \Delta + 1 \) for all \( i \). If not, form \( G' \) by replacing \( H_i \) with a copy of \( \frac{|V(H_i)| - 1}{2} K_{1, \Delta} \). Clearly \( \Delta(G') = \Delta \), and \( M(G') = M(G) \) by the previous comment. Finally,

\[
|E(G')| - |E(G)| = \frac{|V(H_i)| - 1}{2} \Delta - |E(H_i)| \geq \frac{|V(H_i)| - 1}{2} |V(H_i)| - \frac{|V(H_i)|}{2} = 0,
\]

so we may suppose this property of \( G \). Additionally, as \( |V(H_i)| \) is odd, this property tells us \( |V(H_i)| \geq \Delta + 1 + 1_{\Delta \text{ odd}} \).

Now,

\[
M \geq M(G) = s + \frac{1}{2} \sum_{i=1}^{t} (|V(H_i)| - 1),
\]

so we find

\[
t \leq \left\lfloor \frac{2M}{\min \{|V(H_i)| - 1\}} \right\rfloor \leq \left\lfloor \frac{2M}{\Delta + 1_{\Delta \text{ odd}}} \right\rfloor.
\]

Rewriting the above equation as \( s + \frac{1}{2} \sum_{i=1}^{t} |V(H_i)| \leq M + t/2 \), we calculate

\[
|E(G)| = \sum_{i=1}^{s} |E(S_i)| + \sum_{i=1}^{t} |E(H_i)|
\leq s\Delta + \frac{\Delta}{2} \sum_{i=1}^{t} |V(H_i)|
\leq \Delta \left( M + \frac{t}{2} \right)
\leq \Delta \left( M + \frac{1}{2} \frac{2M}{\Delta + 1_{\Delta \text{ odd}}} \right)
\leq (\Delta + 1)M.
\]

Now, take any \( k \geq 7 \) and let \( G \) be a graph with \( \text{ex}(G, P_1 \cup P_2) < k \), so we must have \( \Delta(G), M(G) \leq k - 1 \). Now, when \( k \) is odd, immediately \( |E(G)| \leq (\Delta(G) + 1)M(G) \leq k(k-1) \); hence \( E_k(P_1 \cup P_2) = k^2 - k \).

When \( k \) is even, note that either \( \Delta \leq k - 2 \), in which case immediately \( |E(G)| \leq (k-1)^2 < k^2 - \frac{3}{2}k \), or else
\[ \Delta = k - 1, \] so by Equation (2.3),
\[ |E(G)| \leq (k - 1) \left( k - 1 + \frac{1}{2} \left( \frac{2(k - 1)}{k} \right) \right) = (k - 1) \left( k - 1 - \frac{3}{2}k + \frac{1}{2} \right). \]

Thus as \( k \) is even, we have \(|E(G)| \leq k^2 - \frac{3}{2}k\), so \( \mathcal{E}_k(P_1 \cup P_2) = k^2 - \frac{3}{2}k \) in this case. \( \square \)

We finally conclude this section with a classification of all families that have \( \mathcal{E}_k(\mathcal{H}) = \infty \).

**Corollary 2.24.** \( \mathcal{E}_k(\mathcal{H}) \leq k(k-1) \) for any \( \mathcal{H} \) not containing a star or a matching. In particular, \( \mathcal{E}_k(\mathcal{H}) = \infty \) if and only if \( \mathcal{H} \) contains \( K_{1,s} \) or \( sK_2 \) for some \( s \).

**Proof.** Suppose \( G \) is a graph with \( \text{ex}(G, \mathcal{H}) < k \). As \( \mathcal{H} \) does not contain a star or a matching, any star or matching in \( G \) is \( \mathcal{H} \)-free. Thus, \( \Delta(G), M(G) \leq k - 1 \), so \(|E(G)| \leq (\Delta(G) + 1)M(G) \leq k(k - 1) \). \( \square \)

### 2.3. Multigraphs

As mentioned earlier, if \( \mathcal{H} \) is a family of simple graphs, then \( \mathcal{E}_k(\mathcal{H}) \leq \mathcal{E}_k^*(\mathcal{H}) \). In fact, we conjecture the following:

**Conjecture 2.25.** If \( \mathcal{H} \) consists only of simple graphs, then \( \mathcal{E}_k(\mathcal{H}) = \mathcal{E}_k^*(\mathcal{H}) \).

This statement appears very difficult to prove in general. Indeed, in [4] and [6], a similar conjecture has been put forth specifically for \( C_o \), the family of odd cycles, i.e. when considering max cuts (or “judicious partitions”), but in a slightly stronger setting.

However, we can present the proof of a simple subcase.

**Proposition 2.26.** Let \( \mathcal{H} \) be a family of simple graphs and \( G \) be a multigraph. If each edge of \( G \) has the same multiplicity, then there exists a simple graph \( G' \) with \(|E(G')| = |E(G)|\) and \( \text{ex}(G', \mathcal{H}) \leq \text{ex}(G, \mathcal{H}) \).

**Proof.** Let \( G \) be a multigraph where each edge has multiplicity \( r \). Decompose \( G \) into simple graphs \( G_1, \ldots, G_r \) and let \( G' \) be the disjoint union of \( G_1, \ldots, G_r \), so certainly we have \(|E(G')| = |E(G)|\). Now, let \( F \subseteq G' \) be an \( \mathcal{H} \)-free subgraph on \( \text{ex}(G', \mathcal{H}) \) edges and set \( F_i = F \cap G_i \). Without loss of generality, suppose \(|E(F_1)| \geq |E(F_i)|\) for all \( i \) and form \( F' \subseteq G \) by replacing each edge of \( F_1 \) by \( r \) copies. As \( \mathcal{H} \) consisted only of simple graphs, \( F' \) is also \( \mathcal{H} \)-free, so
\[ \text{ex}(G, \mathcal{H}) \geq |E(F')| = r|E(F_1)| \geq r \cdot \frac{\text{ex}(G', \mathcal{H})}{r} = \text{ex}(G', \mathcal{H}) \]. \( \square \)

Unfortunately, when \( G \) is a multigraph where different edges have different multiplicities, it is unclear whether or not one can construct a simple graph \( G' \) with \(|E(G')| = |E(G)|\) and \( \text{ex}(G', \mathcal{H}) \leq \text{ex}(G, \mathcal{H}) \).

Notice (see Theorem 2.3.5) that if \( \mathcal{H} \) does not contain a bipartite graph, then \( \mathcal{E}_k^*(\mathcal{H}) = (1 + o(1))\mathcal{E}_k(\mathcal{H}) \). We can also provide the following bound which, unfortunately, is not very strong.

**Proposition 2.27.** If \( \mathcal{H} \) is a family of simple graphs, then \( \mathcal{E}_k^*(\mathcal{H}) \leq \mathcal{E}_{k \log k}(\mathcal{H}) \).

**Proof.** Both are infinite if \( \mathcal{H} \) contains a star or a matching, so we shall suppose that is not the case.

Let \( G \) be a multigraph with \( \text{ex}(G, \mathcal{H}) < k \). As above, decompose \( G \) into simple graphs \( G_1, \ldots, G_r \) where \( G_1 \supseteq \cdots \supseteq G_r \), and let \( G' \) be the disjoint union of these graphs, so certainly \(|E(G')| = |E(G)|\). We now argue that \( \text{ex}(G', \mathcal{H}) < k \log k \), which will establish the claim.
To do this, we first note that as \( \text{ex}(G, \mathcal{H}) < k \), we must have \( r \leq k - 1 \) as \( \mathcal{H} \) does not contain \( K_2 \). Further, consider any \( \mathcal{H} \)-free subgraph \( F \subseteq G_i \). As \( G_1 \supseteq \cdots \supseteq G_r \), and \( \mathcal{H} \) is a family of simple graphs, we can form an \( \mathcal{H} \)-free subgraph \( F' \subseteq G \) by replacing every edge of \( F \) by \( i \) copies. Thus, it must be the case that \( \text{ex}(G_i, \mathcal{H}) < \frac{k}{r} \). As such,

\[
\text{ex}(G', \mathcal{H}) \leq \sum_{i=1}^{r} \text{ex}(G_i, \mathcal{H}) < \sum_{i=1}^{r} \frac{k}{i} \leq k \log(r + 1) \leq k \log k. \]

\( \square \)

Interestingly, Conjecture [2.25] fails if we consider non-uniform hypergraphs, and in Section [3.1] we give an example of such a hypergraph.

3. Hypergraphs

We now explore the extremal function \( E_k(\mathcal{H}) \) when \( \mathcal{H} \) is a family of hypergraphs.

In light of the result on 2-uniform graphs, we begin by asking when \( E_k(\mathcal{H}) = \infty \) for a family of hypergraphs \( \mathcal{H} \) of higher uniformity. In fact, this is answered by the classical sunflower lemma due to Erdős and Rado [8].

**Definition 3.1.** Let \( H \) be any \( r \)-uniform (multi)hypergraph. \( H \) is said to be a sunflower if, for some \( S \subseteq V(H) \) called the core of \( H \), every pair of distinct edges \( e_1, e_2 \in E(H) \) has \( e_1 \cap e_2 = S \). Note that \( K_{1,s} \) and \( sK_2 \) fully describe all simple 2-uniform sunflowers (where \( |S| = 1, 0 \) respectively).

Crucially, whenever \( H \) is a sunflower, every \( F \subseteq H \) is also a sunflower.

**Proposition 3.2.** \( E_k(\mathcal{H}) = \infty \) for \( k \) sufficiently large if and only if \( \mathcal{H} \) contains a sunflower.

**Proof.** If \( \mathcal{H} \) contains a sunflower \( H \) with \( |E(H)| = k \) and core \( S \), then any sunflower \( G \) with \( s \) edges and core of size \( |S| \) has \( \text{ex}(G, H) = k - 1 \). Hence, \( E_k(\mathcal{H}) \geq s \) for every \( s \), so \( E_k(\mathcal{H}) = \infty \).

Conversely, take any family of hypergraphs \( \mathcal{H} \) without a sunflower and fix \( k \). By the Erdős-Rado sunflower lemma [8], any \( r \)-graph \( G \) with \( |E(G)| > r!(k-1)^{r+1} \) contains a sunflower \( F \) with at least \( k \) edges. Thus \( F \) contains no hypergraph in \( \mathcal{H} \), showing \( \text{ex}(G, \mathcal{H}) \geq k \). The contrapositive gives us \( E_k(\mathcal{H}) \leq r!(k-1)^{r+1} \). \( \square \)

We will also show later that, for most uniform hypergraphs, cliques are asymptotically best at forcing them. The only possible exceptions are when the hypergraphs are “degenerate”:

**Definition 3.3.** For an arbitrary \( r \)-uniform hypergraph family \( \mathcal{H} \) we denote by

\[
(\pi_n(\mathcal{H}) := \frac{\text{ex}(K_n^{(r)}, \mathcal{H})}{\binom{n}{r}})_{n \geq 1}
\]

the sequence of Turán densities and denote the limiting density \( \pi(\mathcal{H}) := \lim_{n \to \infty} \pi_n(\mathcal{H}) \).

\( \mathcal{H} \) is said to be degenerate if \( \pi(\mathcal{H}) = 0 \).

Note that \((\pi_n(\mathcal{H}))_{n \geq 1}\) is a decreasing sequence of densities for any \( \mathcal{H} \) by averaging over subgraphs, so the limit always exists. Furthermore, there is a standard classification:

**Proposition 3.4.** An \( r \)-uniform graph \( H \) is degenerate if and only if it is \( r \)-partite. That is to say, we may \( r \)-color \( V(H) \) so that each \( e \in E(H) \) has 1 vertex of each color, or equivalently, \( H \subseteq K_{t_1, \ldots, t_r}^{(r)} \) for some \( t \).

\(^1\)When \( G \) is simple, this can be lowered to \( r!(k-1)^r \).
Indeed, for $H$ nondegenerate, $\pi(H) \geq r!/r^r$ since the balanced $r$-partite hypergraph $K_{n/r}^{(r)}$ does not contain $H$, otherwise $\text{ex}(K_{n/r}^{(r)}, H) = o(n^r)$ is true by an induction on $r$, as was observed by Erdős [7]. In fact, these easily generalize to families of $r$-uniform graphs; namely $\pi(H) = 0$ if and only if $H$ contains a degenerate graph. See [14] for a survey on the hypergraph Turán problem.

**Theorem 3.5.** If $\mathcal{H}$ is a family of simple $r$-uniform hypergraphs not containing a degenerate graph, then

$$\mathcal{E}_k(\mathcal{H}), \mathcal{E}_k^*(\mathcal{H}) = \left(\frac{1}{\pi(\mathcal{H})} - o(1)\right) k.$$ 

**Proof.** With the exception of applying contractions, we proceed in a fashion similar to the proof of Theorem [2.6].

**Lower bound.** For a positive integer $k$, let $n$ be the largest integer for which $k > \pi_n(\mathcal{H})\binom{n}{r}$. As $\pi_n(\mathcal{H}) = \pi(\mathcal{H}) + o(1)$ and $\binom{n+1}{r} - \binom{n}{r} = O(n^{r-1})$, we observe that $\pi_{n+1}(\mathcal{H})\binom{n+1}{r} - \pi_n(\mathcal{H})\binom{n}{r} = o(n^r)$; thus $k \leq \pi_n(\mathcal{H})\binom{n}{r} + o(k)$. Then, as $\text{ex}(K_n^{(r)}, \mathcal{H}) = \pi_n(\mathcal{H})\binom{n}{r} < k$,

$$\mathcal{E}_k(\mathcal{H}) \geq |E(K_n^{(r)})| \geq \frac{k - o(k)}{\pi_n(\mathcal{H})\binom{n}{r}} \binom{n}{r} = \left(\frac{1}{\pi(\mathcal{H})} - o(1)\right) k.$$ 

**Upper bound.** Let $G$ be an $r$-uniform (multi)graph on $n$ vertices with $\text{ex}(G, \mathcal{H}) < k$, and let $F \subseteq K_n^{(r)}$ be an $\mathcal{H}$-free subgraph with $|E(F)| = \text{ex}(K_n^{(r)}, \mathcal{H}) = \pi_n(\mathcal{H})\binom{n}{r}$. Let $F'$ be a copy of $F$ chosen uniformly at random from $K_n^{(r)}$ and set $F^* = \{e \in E(G) : e \in E(F')\}$, where multiedges are preserved. Certainly as $F$ is $\mathcal{H}$-free and $\mathcal{H}$ consists only of simple graphs, $F^*$ is also $\mathcal{H}$-free. Therefore, as $\pi_n(\mathcal{H}) \geq \pi(\mathcal{H})$,

$$k > \mathbb{E}|E(F^*)| = \pi_n(\mathcal{H})|E(G)| \geq \pi(\mathcal{H})|E(G)|,$$

so $\mathcal{E}_k^*(\mathcal{H}) < \frac{k}{\pi(\mathcal{H})}$. 

As degenerate 2-uniform graphs are exactly bipartite graphs, Theorem 3.5 immediately implies Theorem 1.1 by noting that $\pi(\mathcal{H}) = 1 - \frac{1}{\rho(\mathcal{H}) - 1}$ by the Erdős-Stone Theorem [9].

Unfortunately, when it comes to hypergraphs, we cannot attain a tighter result when $\mathcal{H} = \{K_1^{(r)}\}$ as we could in Theorem 2.6. The main difficulty here is that when $r \geq 3$, it may not be possible to apply compressions to end up with a clique at the end. However, despite this difficulty, it should still be the case that cliques are extremal for $\mathcal{E}_k(K_1^{(r)})$.

**Conjecture 3.6.** If $k = \text{ex}(K_1^{(r)}, K_1^{(r)}) + 1$, then $\mathcal{E}_k(K_1^{(r)}) = \mathcal{E}_k^*(K_1^{(r)}) = \binom{n}{r}$ and the unique extremal graph is $K_1^{(r)}$.

To end this section, we present a general upper bound on $\mathcal{E}_k(H)$, which directly follows from the work of Friedgut and Kahn [11] who extended a result of Alon [2].

For two hypergraphs $H$ and $G$, let $N(G, H)$ denote the number of copies of $H$ contained in $G$, and let $N(m, H)$ denote the maximum value of $N(G, H)$ taken over all hypergraphs $G$, with $|E(G)| = m$. Also, for a hypergraph $H$, we say that $\phi : E(H) \rightarrow [0, 1]$ is a fractional cover of $H$ if $\sum_{e \ni v} \phi(e) \geq 1$ for every $v \in V(H)$. The fractional cover number of $H$, denoted $\rho^*(H)$, is the minimum value of $\sum_{e \in E(H)} \phi(e)$ where $\phi$ is a fractional cover of $H$.

**Theorem 3.7** (Friedgut and Kahn [11]). For any hypergraph $H$, $N(m, H) = \Theta(m\rho^*(H))$. 


Proposition 3.8. If \( \rho^* = \rho^*(H) \) and \( s = |E(H)| \), then there is a constant \( c = c(H) \) such that

\[
\mathcal{E}_k(H) \leq ck^{(s-1)/(s-\rho^*)}.
\]

Proof. Let \( G \) be a graph with \( \text{ex}(G, H) < k \) and \( |E(G)| = m \). Thus, by Theorem 3.7 there is a constant \( C = C(H) \) such that \( N(G, H) \leq N(m, H) \leq Cm^{\rho^*} \).

We proceed by a standard averaging argument. Let \( S \subseteq E(G) \) be a set of edges where each \( e \in E(G) \) is included in \( S \) independently with probability \( p \). Then let \( S' \subseteq S \) be attained by removing one edge per copy of \( H \) contained in \( S \). Thus \( S' \) is \( H \)-free, so

\[
k > \mathbb{E}[|S'|] \geq pm - p^sN(G, H) \geq pm - Cp^s m^{\rho^*} = pm \left(1 - C p^{s-1} m^{\rho^*-1}\right).
\]

Selecting \( p^{s-1} m^{\rho^*-1} = 1/(sC) \) yields

\[
k > \left(1 - \frac{1}{s}\right) \left(\frac{m^{s-\rho^*}}{sC}\right)^{1/(s-1)}.
\]

As such, there is some \( c = c(H) \) with \( m < ck^{(s-1)/(s-\rho^*)} \) \( \square \).

3.1. Non-uniform Hypergraphs. Recall \( \text{ex}(G, \mathcal{H}) = |E(G)| \) unless \( G \) contains a copy of some \( H \in \mathcal{H} \), so it makes sense to even ask about \( \mathcal{E}_k^*(\mathcal{H}) \) where \( \mathcal{H} \) is a family of non-uniform hypergraphs.

Throughout this section, for a graph \( G \), we will use \( E_i(G) := \{e \in E(G) : |e| = i\} \).

Proposition 3.9. If \( H \) is a non-uniform hypergraph, then \( \mathcal{E}_k^*(H) \leq 2(k - 1) \). Additionally, if \( \mathcal{H} \) is a finite family of non-uniform hypergraphs, then \( \mathcal{E}_k^*(\mathcal{H}) \) is always finite.

Proof. As \( H \) is non-uniform, there is some \( r \neq s \) with \( E_r(H), E_s(H) \neq \emptyset \). Now, let \( G \) be any hypergraph with \( \text{ex}(G, H) < k \). As any \( F \subseteq G \) with \( E_r(F) = \emptyset \) or \( E_s(F) = \emptyset \) is trivially \( H \)-free, we know that \( |E_r(G)| < k \) and \( |E(G) \setminus E_r(G)| < k \); therefore, \( |E(G)| \leq 2(k - 1) \).

Similarly, for a finite family of non-uniform graphs \( \mathcal{H} \), let \( U = \{i \in \mathbb{Z} : E_i(H) \neq \emptyset \) for some \( H \in \mathcal{H}\} \). Let \( G \) be a hypergraph with \( \text{ex}(G, \mathcal{H}) < k \); certainly we may suppose that the edges in \( G \) are only of the sizes in \( U \). Thus, by the same argument as above, \( |E_i(G)| < k \) for all \( i \in U \) as each \( H \in \mathcal{H} \) is non-uniform, so \( |E(G)| \leq |U|(k - 1) \), which is finite as \( \mathcal{H} \) consisted only of finitely many graphs. \( \square \)

We quickly remark that \( \mathcal{E}_k^*(\mathcal{H}) \) is not necessarily finite when \( \mathcal{H} \) is not of finite size. Namely, for positive integers \( r, t \), let \( H_{r,t} \) be the non-uniform hypergraph consisting of two disjoint edges \( e, s \) with \( |e| = r, |s| = t \). Then \( \mathcal{H} = \{H_{r,t} : 1 \leq r < t\} \) has \( \mathcal{E}_k^*(\mathcal{H}) = \infty \) when \( k \geq 2 \), as is realized by taking a host graph with disjoint edges \( e_1, \ldots, e_s \) where \( |e_i| = i \).

We now turn our attention to a non-uniform hypergraph which yields a surprising answer to \( \mathcal{E}_k^*(H) \); namely \( \mathcal{E}_k^*(H) \sim xk \) where \( x \) is an irrational number. The \( r \)-necklace, denoted \( \mathcal{O}_r \), is the hypergraph with vertex set \( \{x_1, \ldots, x_r\} \) and edge set \( \{\{x_1, \ldots, x_r\}, \{x_1\}, \ldots, \{x_r\}\} \). That is, \( \mathcal{O}_r \) is the hypergraph consisting of a single \( r \)-edge with a loop at each vertex.

Theorem 3.10. For \( r \geq 2 \), \( \mathcal{E}_k^*(\mathcal{O}_r) = (\alpha_r - o(1))k \) where \( \alpha_r \) is the unique positive solution to \( X^r + X = 1 \).

Proof. Upper bound. Let \( G \) be any hypergraph with \( \text{ex}(G, \mathcal{O}_r) < k \); certainly we may assume \( G \) contains only contains edges of uniformities 1 and \( r \). Now, let \( V' \subseteq V(G) \) be formed by including each vertex in \( V' \)
with probability $\alpha_r$, and form $G' \subseteq G$ by taking any loops on a vertex of $V'$ along with any $r$-uniform edge which is not completely contained in $V'$. By construction, $G'$ is $O_r$-free, so

$$k > E|E(G')| = \alpha_r|E_1(G)| + (1 - \alpha_r)|E_r(G)| = \alpha_r(\|E_1(G)\| + |E_r(G)|) = \alpha_r|E(G)|.$$ 

Therefore, $|E(G)| < k/\alpha_r$, so the same is true of $E_k^*(O_r)$.

**Lower bound.** We will show $E_k^*(O_r) \geq k/\alpha_r - O(k^{\pi/r})$.

Fix a large $k$, and construct the multigraph $G$ on $n = \Theta(k^{\pi/r})$ vertices with:

- $t := \left\lfloor \frac{k}{\binom{n}{r}} \cdot \frac{1}{\alpha_r + r\alpha_r^r} \right\rfloor$ parallel hyperedges spanning every $r$-set of vertices, and
- $s := \left\lfloor \frac{k}{n} \cdot \frac{r\alpha_r^{r-1} - \frac{r^2}{n-r}}{\alpha_r + r\alpha_r^r} \right\rfloor$ loops at each vertex $^4$

This way, $G$ has $|E(G)| = k\left(\frac{1}{\alpha_r + r\alpha_r^r} + \frac{r\alpha_r^{r-1} - \frac{r^2}{n-r}}{\alpha_r + r\alpha_r^r}\right) - O(k^{\pi/r}) = k/\alpha_r - O(k^{\pi/r})$.

Now, take any $O_r$-free subgraph $H \subseteq G$. We will show $|E(H)| < k$.

Let $L \subseteq V(H)$ be the vertices of $H$ with at least one loop. Write $\beta n := |L|$, then certainly $H$ has at most $\beta ns$ loops in total. If $\beta n < r$, then we calculate

$$|E(H)| \leq \beta ns + t\left(\frac{n}{r}\right) \leq r\left(\frac{k}{n} \cdot \frac{r\alpha_r^{r-1} - \frac{r^2}{n-r}}{\alpha_r + r\alpha_r^r}\right) + \frac{k}{\alpha_r + r\alpha_r^r} = \frac{k}{\alpha_r + r\alpha_r^r} + \Theta(k^{\pi/r}) < k$$

for $k$ sufficiently large as $\alpha_r + r\alpha_r^r = 1 + (r-1)\alpha_r^r > 1$, so suppose $\beta n \geq r$. In this case, we note the inequality

$$\frac{\binom{\beta n}{r}}{\binom{n}{r}} = \frac{\beta n \beta n - 1 \cdots \beta n - r + 1}{n \cdot n \cdots n - r + 1} = \beta \left(\beta - (1 - \beta) \frac{1}{n-1}\right) \left(\beta - (1 - \beta) \frac{2}{n-2}\right) \cdots \left(\beta - (1 - \beta) \frac{r-1}{n-r+1}\right) \geq \beta^r - \beta^{r-1}(1 - \beta) \sum_{i=1}^{r-1} \frac{i}{n-i} \geq \beta^r - \beta^{r-1}(1 - \beta) \frac{r^2}{n-r}. $$

We also note that $\beta \neq \alpha_r$ as $\beta$ is rational and $\alpha_r$ is irrational; therefore, by the mean value theorem, there is some $\theta$ strictly between $\alpha_r$ and $\beta$ such that $(\alpha_r - \beta)r\theta^{r-1} = \alpha_r^r - \beta^r$. Thus,

$$\beta r\alpha_r^{r-1} + \alpha_r + \alpha_r^r - \beta^r = \alpha_r + r(\alpha_r - \beta)(\theta^{r-1} - \alpha_r^{r-1}) + r\alpha_r^r < \alpha_r + r\alpha_r^r. $$

$^4$The constants $1/(\alpha_r + r\alpha_r^r)$ and $r\alpha_r^{r-1}/(\alpha_r + r\alpha_r^r)$ may be found by solving the natural linear program, but this is not necessary for the proof.
as either $\alpha < \theta < \beta$ or $\beta < \theta < \alpha$. Now, as $H$ is $O_r$-free, there are no $r$-edges spanned by $L$, so, noting that $\beta^{r-1}(1 - \beta) \leq \beta$, we have:

$$\frac{|E(H)|}{k} \leq \frac{\beta ns + t \left( \binom{n}{2} - (\beta \alpha)^r \right)}{k} \leq \frac{\beta r \alpha^{r-1} - \frac{r^2}{n-r} + (1 - \beta r + \beta^{r-1}(1 - \beta) \frac{r^2}{n-r})}{\alpha_r + r \alpha_r} \leq \frac{\beta r \alpha^{r-1} + (1 - \beta r)}{\alpha_r + r \alpha_r} = \frac{\beta r \alpha^{r-1} + \alpha_r + \alpha_r - \beta r}{\alpha_r + r \alpha_r} = 1.$$ 

In the case of $r = 2$, where $O_2$ is a 2-uniform edge with a loop at both ends, we attain an interesting corollary.

**Corollary 3.11.** $\mathcal{E}_k^*(O_2) = (\phi - o(1))k$ where $\phi = 1.618 \ldots$ is the golden ratio.

We now show that for a simple, non-uniform hypergraph $H$, it can be the case that $\mathcal{E}_k(H)$ and $\mathcal{E}_k^*(H)$ differ. This is perhaps surprising as we believe it should be the case that $\mathcal{E}_k(H) = \mathcal{E}_k^*(H)$ if $H$ is a simple $r$-uniform graph as mentioned in Conjecture 2.25.

**Theorem 3.12.** If $G$ is a graph with 1-uniform edges and 2-uniform edges, where each vertex has at most one loop (but any 2-uniform edges can have higher multiplicity), then $\text{ex}(G, O_2) \geq \frac{2}{3}|E(G)|$.

**Proof.** Let $G$ be a graph with only 2-uniform edges and loops where each vertex has at most one loop. As before, let $E_i(G)$ denote the set of $i$-uniform edges, so $E(G) = E_1(G) \cup E_2(G)$. We begin by claiming that we may suppose that every vertex of $G$ has a loop. If some $v \in V(G)$ does not have a loop, then either $v$ is isolated, in which case we may simply delete $v$, or $v$ is incident to some $e \in E_2(G)$. Let $G'$ be formed by deleting $e$ and adding a loop around $v$. Certainly $\text{ex}(G', O_2) \leq \text{ex}(G, O_2)$ as the edge $e \in E_2(G)$ cannot be used in any copy of $O_2$. After this reduction, we know that $|E(G)| = |E_1(G)| + |E_2(G)| = |V(G)| + |E_2(G)|$. Additionally, we may suppose that every vertex is incident to some $e \in E_2(G)$. To see this, suppose $v \in V(G)$ is not incident to any edge in $E_2(G)$; pick any $e \in E_2(G)$ and form $G'$ by removing the loop from $v$ and adding an additional copy of the edge $e$. As the loop around $v$ cannot be used in any copy of $O_2$ in $G$, we see that $\text{ex}(G', O_2) \leq \text{ex}(G, O_2)$.

We now prove the statement by induction on $|V(G)|$.

For the base case, suppose that $E_2(G)$ is bipartite with partite sets $A, B$ where $|A| \geq |B|$. In this case, if we take every edge in $E_2(G)$ and every loop around a vertex in $A$, we end up with an $O_2$-free graph as no two loops are joined by an edge. Now, as $G$ has no vertices not incident to a 2-edge, we must have $|E_2(G)| \geq |B|$, so as $|A| \geq |B|$, we have $\text{ex}(G, O_2) \geq |E_2(G)| + |A| \geq \frac{2}{3}|E(G)|$.

Now suppose that $E_2(G)$ is not bipartite and let $C \subseteq G$ be an induced copy of $C_{2t+1}$ for some $t$, possibly with some multiedges. Set $G' := G \setminus C$.

Now, for a fixed set of vertices $S \subseteq V(C)$, we may form $H_S \subseteq G$ by collecting together the following edges:
Recalling that by induction, $t = 3$, suppose we choose probability $2/3$. 1-Uniform Graphs. This context is that it also settles the question for multi-stars. For positive integers $d_1, \ldots, d_t$, we quickly note that a 1-uniform graph $G$ is a star on $t$ vertices if and only if there is an injection $f : [n] \to [t]$ such that for every $i \in [n]$, $d_i = x_{f(i)}$. We quickly note that a 1-uniform graph $H$ is a sunflower if and only if it is of the form $H = (1, 1, \ldots, 1)$ or $H = (r)$ for some $r$. All 2-edges in the cycle $C$ itself (there are at least $2t + 1$ of these), The 2-edges from $C \setminus S$ to $V \setminus C$, All loops in $S$, $E(H')$ for some extremal $O_2$-free $H' \subseteq G'$. Provided $S$ contains no two adjacent vertices in $C$, $H_S$ is $O_2$-free.

So, suppose we choose $S \subseteq V(C)$ by picking an independent set of size $\lceil \frac{2t+1}{3} \rceil$ uniformly at random with probability $\frac{2t+1}{3} - \frac{\lceil \frac{2t+1}{3} \rceil}{3}$, otherwise an independent set of size $\lfloor \frac{2t+1}{3} \rfloor$ uniformly at random. Since $\frac{2t+1}{3} \leq t = o(C_{2t+1})$, this is a nontrivial probability space. Furthermore, the event $\{v \in S\}$ occurs with probability $\frac{1}{3}$ for each $v \in C$.

Recalling that by induction, $|E(H')| = ex(G', O_2) \geq \frac{2}{3}|E(G')|$, we have in total

$$\mathbb{E}[|E(H_S)|] = |E(C)| + \frac{2}{3}|E(G[C, V \setminus C])| + \frac{1}{3}(2t + 1) + \frac{2}{3}|E(G')|$$

$$\geq \frac{2}{3}|E(C)| + \frac{2}{3}|E(G[C, V \setminus C])| + \frac{2}{3}(2t + 1) + \frac{2}{3}|E(G')| = \frac{2}{3}|E(G)|.$$

So some such $S$ yields an $O_2$-free $H_S$ with at least this many edges, as desired.

Thus, we have the following corollary which shows that Conjecture $2.25$ can fail for non-uniform graphs.

**Corollary 3.13.** $E_k(O_2) < \frac{2}{3}k$ whereas $E_k^*(O_2) = (\phi - o(1))k \approx (1.618 - o(1))k$.

**3.2. 1-Uniform Graphs.** A 1-uniform graph on $n$ vertices is equivalent to its degree sequence $(d_1, \ldots, d_n)$ where $d_i$ is the number of loops at vertex $i$. For 1-uniform graphs $H = (d_1, \ldots, d_n)$ and $G = (x_1, \ldots, x_t)$, $H \subseteq G$ if and only if there is an injection $f : [n] \to [t]$ such that for every $i \in [n]$, $d_i = x_{f(i)}$. We quickly note that a 1-uniform graph $H$ is a sunflower if and only if it is of the form $H = (1, 1, \ldots, 1)$ or $H = (r)$ for some $r$.

Although the Turán problem for 1-uniform graphs is quite uninteresting as every simple 1-uniform graph is a sunflower, determining $E_k^*(H)$ requires some more thought. One reason for caring about 1-uniform graphs in this context is that it also settles the question for multi-stars. For positive integers $d_1, \ldots, d_t$, the multi-star $S_{d_1, \ldots, d_t}$ is a star on $t + 1$ vertices whose edges have multiplicities $d_1, \ldots, d_t$.

**Observation 3.14.** For positive integers $d_1, \ldots, d_t$, if $H = (d_1, \ldots, d_t)$, then $E_k^*(S_{d_1, \ldots, d_t}) = E_k^*(H)$.

**Theorem 3.15.** For every 1-uniform hypergraph $H = (d_1, d_2, \ldots, d_t)$ with $d_1 \geq \cdots \geq d_t \geq 1$ where $d_1, t \geq 2$, there exists a constant $c_H$ such that $E_k^*(H) = (c_H + o(1))k^2$. Additionally, $c_H$ can be determined in polynomial time and satisfies $\frac{1}{(t-1)(d_1-1)} \leq c_H \leq \frac{1}{(t-1)(d_1-1)}$.
Proof. Let $H = (d_1, \ldots, d_t)$ where $d_1 \geq \cdots \geq d_t \geq 1$ and $d_1, t \geq 2$.

We note that $F \subseteq G$ with $F = (f_1, \ldots, f_n)$ can be assumed to have $f_1 \geq \cdots \geq f_n$. Thus, it is clear that $F$ is $H$-free if and only if there is some $t' \in [t]$ such that $f_{t'} < d_{t'}$. Thus, for $t' \in [t]$, let $G_{t'} = (x_1', \ldots, x_{t'}')$ where $x_i' = x_i$ for $i < t'$ and $x_i' = \min\{x_i, d_{t'} - 1\}$ for all $i \geq t'$. By the earlier note, $G_{t'}$ is $H$-free for every $t' \in [t]$, and further, any $F \subseteq G$ that is $H$-free must be contained in some $G_{t'}$.

Thus, we may suppose

$$
\text{ex}(G, H) = \max_{t' \in [t]} |E(G_{t'})|.
$$

As $H \neq (1, 1, \ldots, 1)$, we know that if $\text{ex}(G, H) < k$, then $n \leq k - 1$. Thus, we may formulate the following non-linear integer program for $\mathcal{E}_k^{*}(H)$:

$$
\mathcal{E}_k^{*}(H) = \max \sum_{i=1}^{k-1} x_i
\text{ s.t. } \sum_{i=1}^{t'+1} x_i + \sum_{i=t'+1}^{k-1} \min\{x_i, d_{t'} - 1\} \leq k - 1 \quad \text{for all } t' \in [t]
$$

$$
x_i \in \mathbb{Z}_{\geq 0} \quad \text{for all } i \in [k-1].
$$

Fix a feasible $G$. Note that, since $t \geq 2$, taking $t' = 2$ shows $x_1 \leq x_1 + \sum_{i=2}^{k-1} \min\{x_i, d_2 - 1\} \leq k - 1$.

Now, define $j := \max\{i : x_i \geq d_1\}$. Then $\sum_{i=j}^{k-1} x_i \leq (k-1)(d_1 - 1) \leq d_1 k$. Furthermore, if the largest $j$ vertices $(x_1, \ldots, x_j)$ differ in degree by $\geq 2$, then certainly $x_i \geq x_{i+1} + 1 \geq \cdots \geq x_{t-1} + 1 \geq x_t + 2$ for some $i < \ell \leq j$.

Then forming $G'$ by replacing $x_i, x_\ell$ with $x_i - 1, x_\ell + 1$ respectively (noting the degree sequence is still decreasing) is still feasible, for otherwise the first condition is violated for some $t' \in [t]$. This would mean

$$
\min\{x_i - 1, d_{t'} - 1\} + \min\{x_\ell + 1, d_{t'} - 1\} > \min\{x_i, d_{t'} - 1\} + \min\{x_\ell, d_{t'} - 1\},
$$

as only $x_i$ and $x_\ell$ changed in value when forming $G'$. Thus $\min\{x_\ell + 1, d_{t'} - 1\} = x_\ell + 1$ so $x_\ell \leq d_{t'} - 2 < d_1$; a contradiction.

Thus, we may suppose $G = (x_1, \ldots, x_n)$ where $|x_i - x_\ell| \leq 1$ for all $i, \ell \leq j$. From this, define $G^{(1)} := (x_j, x_j, \ldots, x_j, 0, \ldots, 0)$, which is also feasible and has

$$
\sum_{i=j}^{j} (x_i - x_j) + \sum_{i=j+1}^{k-1} x_i \leq j + d_1 k = O(k).
$$

As such, we have $f^{(1)}(H) \leq \mathcal{E}_k^{*}(H) \leq f^{(1)}(H) + O(k)$ where

$$
f^{(1)}_k(H) = \max \sum_{i=1}^{j} jx_i
\text{ s.t. } (t' - 1)x + \sum_{i=t'}^{j} \min\{x_i, d_{t'} - 1\} \leq k - 1 \quad \text{for all } t' \in [t]
$$

$$
x, j \in \mathbb{Z}_{\geq 0}, x \geq d_1,
$$

where the lower bound follows from the fact that for a feasible pair $(x, j)$, the 1-graph $(x, x, \ldots, x, 0, \ldots, 0)$ satisfies the original program.

To simplify further, note that $x > d_{t'} - 1$ for any $t' \in [t]$ as $x \geq d_1$, so we know that $\min\{x, d_{t'} - 1\} = d_{t'} - 1$. Further, if a feasible $(x, j)$ has $j < t$, then the objective is $xj < (k-1)t = O(k)$. Whether or not the optimum is among such $(x, j)$, this shows we decrease the objective by at most $O(k)$ upon imposing the restriction
Several further questions follow naturally from our line of inquiry. For example:

We now relax the integrality of $x, j$ to attain

$$f_k^{(3)}(H) = \max_{x, j \geq 0} jx$$

satisfies $f_k^{(3)}(H) \leq f_k^{(2)}(H) \leq f_k^{(3)}(H) + O(k)$.

We now relax the integrality of $x, j$ to attain

$$f_k^{(4)}(H) = \max_{x, j \geq 0} jx$$

which is independent of $k$ and depends only on the 1-graph $H$. As $\mathcal{E}_k(H) = f_k^{(4)}(H) + O(k)$, we finally attain $\mathcal{E}_k^*(H) = (c_H + o(1))k^2$.

Now, although the program for $c_H$ is not linear, it is clearly solvable in polynomial time. Further, notice that for $(x, j) = (\frac{1}{2(t-1)}, \frac{1}{2(d_t-1)})$, we have

$$(t' - 1)x + (d_{t'} - 1)j \leq \frac{1}{2} + \frac{1}{2} = 1,$$

for all $t' \in [t]$, so $c_H \geq \frac{1}{4(t-1)(d_t-1)}$. Additionally, only considering the constraints $(1 - 1)x + (d_t - 1)j \leq 1$ and $(t - 1)x + (d_t - 1)j \leq 1$, we find that $x \leq \frac{1}{d_t}$ and $j \leq \frac{1}{d_t}$, so $c_H \leq \frac{1}{(t-1)(d_t-1)}$. \hfill \Box

4. Conclusion and Further Directions

In our study of the extremal function $\mathcal{E}_k(H)$, the largest open question is whether or not $\mathcal{E}_k(H) = \mathcal{E}_k^*(H)$ when $H$ is a family of simple, $r$-uniform graphs (see Conjecture [2.25]). Also very natural is the question of the behavior of $\mathcal{E}_k(C_4)$ (also discussed in the Introduction). Note for example that $\Omega(k^{4/3}) \leq \mathcal{E}_k(C_4) \leq O(k^{3/2})$ where the upper bound follows from Proposition [3.8] and the lower bound follows from the fact that $\text{ex}(K_n, C_4) = \Theta(n^{3/2})$.

Several further questions follow naturally from our line of inquiry. For example:
Question 2. What are the exact asymptotics of $\mathcal{E}_k(P_t)$?

We note that, for a fixed $t$, $\mathcal{E}_k(P_t) = \Theta(k^2)$ where the upper bound follows from Corollary 2.24 or Proposition 3.8 and the lower bound follows from the fact that $\text{ex}(K_n, P_t) = \frac{k-1}{2}n$. More specifically, what are the extremal graphs for $\mathcal{E}_k(P_t)$? Corollary 2.4 implies that there are extremal graphs for $\mathcal{E}_k(P_t)$ with diameter at most $t$. Gyárfás, Rousseau and Schelp [13] prove that if $n$ is sufficiently large compared to $t$, then

$$\text{ex}(K_{n,n}, P_t) = \begin{cases} \frac{t-1}{2}(2n-t+1) & \text{for } t \text{ odd}; \\ \frac{t}{2}(2n-t+2) & \text{for } t \text{ even.} \end{cases}$$

This implies that for all $t \geq 5$ and $n$ sufficiently large, $\text{ex}(K_{\sqrt{2}n}, P_t) < \text{ex}(K_{n,n}, P_t)$, despite having the same number of edges, so most likely, the extremal graphs for $\mathcal{E}_k(P_t)$ look more similar to cliques, as we showed was the case with $P_3$. However, $\text{ex}(K_{\sqrt{2}n}, P_4) \approx \frac{3}{2^{2/3}}n > 2n \approx \text{ex}(K_{n,n}, P_4)$, so it may very likely be the case that the extremal graphs for $\mathcal{E}_k(P_4)$ are bipartite. As there is this discrepancy, it would be very interesting to just determine the extremal graphs for $\mathcal{E}_k(P_3)$ and why $P_4$ may behave differently from $P_t$ for all other $t$.

Next, in regard to necklaces (see Theorem 3.10), we found that there is a multigraph $G$ on $(\phi - o(1))k$ edges with $\text{ex}(G, O_2) < k$, but whenever $G'$ is a multigraph with $\text{ex}(G', O_2) < k$ where each vertex has at most one loop, then $|E(G')| \leq \frac{3}{2}k < \phi k$. As such, it seems natural to ask about how $\mathcal{E}_k^*(H)$ changes if $H$ is a non-uniform graph and the edges of different uniformities are weighted differently to reflect the fact that there are more possible edges of uniformity 2 in the host graph than are of uniformity 1: one could more generally define $\text{ex}(G, H) := \max\{(\sum_{e \in F} w(|e|) : F \subseteq G, F \text{ } H\text{-free})$, where $w$ is an arbitrary weighting of the uniformities.

Question 3. How do $\mathcal{E}_k(H)$ and $\mathcal{E}_k^*(H)$ vary with the weight $w$ for non-uniform graphs?

In fact, since the main obstacle to forcing non-uniform graphs appears to be the edges having irreconcilable “types,” which suggests asking equivalent questions in the uniform case by artificially enforcing distinct edge-types on graphs that are already uniform.

Question 4. Suppose $H$ is a graph consisting of both red and blue edges. How many edges can a red-blue colored graph $G$ have such that any $k$-edge subgraph contains a copy of $H$ (with the correct colors)?

Finally, recall that we originally defined $\mathcal{E}_k$ by deciding that a host graph being “best at forcing” meant optimizing specifically the its edge count, but one could just as easily ask this for any other monotone graph parameter $P$. That is, we could study $\mathcal{E}_{P,k}(\mathcal{H}) := \sup\{P(G) : \text{ex}(G, H) < k\}$. One particularly interesting example may be when $P = \chi$, the chromatic number. In this case, $\mathcal{E}_{\chi,k}(K_{1,t})$ and $\mathcal{E}_{\chi,k}(tK_2)$ are not trivial.

Question 5. If $\mathcal{H}$ is a family of (multi)(hyer)graphs, what is $\mathcal{E}_{\chi,k}(\mathcal{H})$?

To this end, we quickly note that as any graph $G$ has $|E(G)| \geq (\chi(G))^2$, Theorem 1.1 implies that if $\mathcal{H}$ is a family of simple, 2-uniform graphs with $\rho(\mathcal{H}) = \rho \geq 3$, then $\mathcal{E}_{\chi,k}(\mathcal{H}) = \sqrt{(2 + \frac{2}{\rho - 2} + o(1))k}$; so again, it is most interesting to focus on families of bipartite graphs.

Acknowledgements

We would like to thank Wesley Pegden for suggesting this problem and for helpful discussion. We would also like to thank Linhua Feng for spotting the relevance of [1].
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Appendix A. Omitted details of Theorem 2.18

We present the details behind the case check in Theorem 2.18. To do this case check, we employ NAUTY after making reductions. The first necessary reduction is that we reduce our search by only considering connected graphs. Suppose that the graphs claimed in the theorem are the only connected extremal graphs for $\mathcal{E}_k(P_3)$, then for $k \geq 4$, let $G$ be a graph with $\text{ex}(G, P_3) < k - 1$ and $|E(G)| \geq \binom{k}{2}$. Let $I$ consist of one vertex from each connected component of $G$, so $\text{ex}(C_I(G), P_3) \leq \text{ex}(G, P_3)$ by Corollary 2.3. As such, $C_I(G) \in \{K_{2,3}, K_{k-1}^*(1, \ldots, 1), K_k\}$. If $G$ was not connected, then it must be that $C_I(G) \simeq K_{k-1}^*(1, \ldots, 1)$ as $K_k$ and $K_{2,3}$ do not have cut vertices. Let $G'$ be a copy of $K_{k-1}^*(1, \ldots, 1)$ with a single pendant vertex removed, then it must be the case that $G \simeq G' \cup P_1$. However, it is quick to verify that $\text{ex}(G' \cup P_1, P_3) \geq k$ by taking this isolated edge along with a star with $k - 1$ edges centered at a vertex of the core of $G'$; a contradiction.

Thus, if we can find all connected graphs $G$ with $\text{ex}(G, P_3) < k - 1$ and $|E(G)| \geq \binom{k}{2}$, we will have established the theorem. We now outline the parameters of the search, which were deduced in the proof of the theorem. If $G$ has a triangle:

- $k = 4$: $|E(G)| = \binom{4}{2}$, $\Delta(G) \leq 3$.
- $k = 5$: $|E(G)| = \binom{5}{2}$, $\Delta(G) \leq 4$.
- $k = 6$: $|E(G)| \in \{\binom{6}{2}, \binom{6}{2} + 1\}$, $\Delta(G) \leq 5$. 
If $G$ is triangle-free:

- $k = 4$: $|E(G)| = \binom{4}{2}$, $\Delta(G) \leq 3$.
- $k = 5$: $|E(G)| = \binom{5}{2}$, $\Delta(G) \leq 4$.

As the conditions for $k \in \{4, 5\}$ are the same whether or not $G$ has a triangle, it suffices to search over all connected graphs satisfying the indicated conditions.

For the case of $k = 6$, it is necessary to provide one additional reduction in order to reduce the number of graphs which must be considered. Recall that in the proof of the theorem, we have a triangle $T = xyz$ and $G' = G[V \setminus T]$ has $\text{ex}(G', P_3) < k - 3$. In the case of $k = 6$, this means that $\text{ex}(G', P_3) \leq 2$. Further, as $|E(G)| \geq \binom{6}{2}$ and $\Delta(G) \leq 5$, we see that $|E(G')| \geq 3$, so $G'$ is either a $P_3$ or $C_4$, possibly with isolated vertices; let $H$ be this copy of $P_3$ or $C_4$ in $G'$. Now, if any of $x, y, z$, say $x$, has two neighbors outside of $T \cup H$, say $a, b$, then the $2P_2$ in $H$ along with the star $\{xy, xz, xa, xb\}$ forms a $P_3$-free subgraph of $G$ with 6 edges; a contradiction. Thus, each of $x, y, z$ has at most one neighbor outside of $T \cup H$. Therefore, $|V(G)| \leq |V(H)| + |T| + 3 = 10$. Further, in order to have $|E(G)| \geq \binom{6}{2}$, it is necessary for one of $x, y, z$ to have degree 5; thus, $\Delta(G) = 5$. As such, when considering $k = 6$, it is enough to restrict to graphs with at most 10 vertices.

See [https://github.com/chrisorcox/Inverse_Turan](https://github.com/chrisorcox/Inverse_Turan) for the SAGE worksheet containing this search.