Tighter Quadratically Constrained Convex Reformulations for Semi-Continuous Quadratic Programming

Xiaojin Zheng
School of Economics and Management, Tongji University
Shanghai 200092, China

Zhongyi Jiang∗
School of Information Science, Changzhou University
Jiangsu 203164, China.

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1. Introduction. While modeling real-world problems, we often encounter optimization models with semi-continuous variables. We say a variable \(x \in \mathbb{R}\) is semi-continuous if \(x \in \{0\} \cup [\alpha, \beta]\) for some \(0 < \alpha \leq \beta\). Examples of optimization models with semi-continuous variables are mean-variance portfolio selection problems with real-life constraints [7], unit commitment problems in a power system [5, 8, 16], design problems [11] and many others [13].

The model with semi-continuous variables we consider in this paper has the following mixed-integer formulation

\[
\text{(P)} \quad \min \{x^T Q x + c^T x \mid (x, y) \in \mathcal{F}\},
\]

where \(Q\) is an \(n \times n\) positive semidefinite symmetric matrix, \(c \in \mathbb{R}^n\),

\[
\mathcal{F} = \{(x, y) \in \mathbb{R}^n \times \{0, 1\}^n \mid Ax + By \leq d, \alpha_i y_i \leq x_i \leq \beta_i y_i, \ i = 1, \ldots, n\},
\]

with \(A, B \in \mathbb{R}^{m \times n}, d \in \mathbb{R}^m\) and \(0 < \alpha_i \leq \beta_i, \ i = 1, \ldots, n\). It has been shown in [1] that problem (P) is in general NP-hard. Numerical results [9, 20] show that the continuous relaxation bound of problem (P) is usually very poor. Hence, there are various researches on the improved reformulation of problem (P) in the sense...
that the continuous relaxation bound of the reformulation is at least as tight as that of problem (P). To the best of our knowledge, the state-of-the-art improved reformulation of problem (P) is the perspective reformulation [13, 9]. Note that there are fractional terms in the objective function of the perspective reformulation. The reformulation is further reformulated as a second-order cone program [13, 20, 10] or as a semi-infinite mixed-integer quadratic programming [7].

An alternative way to obtain the improved reformulation is the quadratic convex reformulation (QCR) method. The method is first studied by [14]. It is then extended to unconstrained 0-1 quadratic programming (QP) by [2], to equality constrained 0-1 QP by [4], and to the general mixed-integer QP by [3]. Recently, the QCR method is recently applied in [19] to obtain an improved reformulation whose continuous relaxation bound is at most as tight as that of the perspective reformulation.

In this paper, we propose a novel class of QCCR for semi-continuous quadratic programming (SQP). According to the construction of QCCR, we only need to introduce a new quadratic constraint obtained by surrogating the quadratic valid inequalities. Compared with the n second-order cone constraints in the second-order cone reformulation of problem (P), the structure of QCCR is much more simple than that of the second-order cone reformulation. Among the class of QCCR, we refer the reformulation corresponding with the tightest continuous relaxation bound as the “best” reformulation. We show in this paper that the problem of finding the “best” reformulation can be converted to a semidefinite programming (SDP) via strong duality and lift technique, and can be found polynomially. Furthermore, the continuous relaxation bound of QCCR is at least as tight as that of the perspective reformulation. We also illustrate via Example 1 that the continuous relaxation bound of QCCR can be strictly tighter than that of the perspective reformulation.

1.1. Structure of the paper. The paper is organized as follows. In section 2, we propose QCCR for problem (P). We show in section 3 that the problem of finding the “best” reformulation can be solved via solving an SDP. We access the quality of the continuous relaxation bound of the “best” reformulation in section 4. In section 5, we conduct preliminary computational experiments to compare the performance of QCCR with the perspective reformulation for the test problems arising from portfolio selection with real-life constraints. Finally, we conclude the paper in section 6 with some concluding remarks.

Notations: Throughout the paper, the optimal value of problem (·) is denoted by v(·). We denote by \( \mathbb{R}_+^n \) the nonnegative orthant of \( \mathbb{R}^n \), \( I_n \) the identity matrix, and \( e \) the all-one vector. We denote by \( S_n \) the set of \( n \times n \) symmetric matrices, by \( S_n^+ \) the set of positive semidefinite matrices of \( S_n \) and \( M \succeq 0 \) means \( M \in S_n^+ \). For any \( a \in \mathbb{R}^n \), we denote by \( \text{diag}(a) = \text{diag}(a_1, \ldots , a_n) \) the diagonal matrix with \( a_i \) being the \( i \)th diagonal element. Finally, for any \( a \in \mathbb{R} \), we define that \( a/0 \) is equal to \( \infty \) if \( a > 0 \), 0 if \( a = 0 \) and \( -\infty \) if \( a < 0 \).

2. The QCCR for problem (P). In this section, we propose QCCR for problem (P). The construction of QCCR consists of three steps. First, construct quadratic valid inequalities for problem (P). Second, surrogate all the quadratic valid inequalities to obtain a new quadratic constraint and add the new quadratic constraint to problem (P). Third, find the “best” possible parameters in the sense that the new reformulation is convex and the continuous relaxation bound of the new reformulation is as tight as possible.
For the sake of convenience, we restate the studied models in this paper

\[
(P) \min \{x^T Q x + c^T x \mid (x, y) \in \mathcal{F}\}.
\]

Denote by \((P)\) the continuous relaxation of problem \((P)\). In the sequel, we always assume the following constraint qualification for \((P)\).

**Assumption 1.** There is a (relative) interior point in the feasible set of \((P)\) such that all the equality constraints hold and all the inequalities hold with strict inequalities.

In the sequel, we will construct QCCR of problem \((P)\) by the three steps. For any \((x, y) \in \mathcal{F}\), it holds

\[
y_i^2 - y_i = 0, x_i - y_i = 0, x_i^2 - (\alpha_i + \beta_i)x_iy_i + \alpha_i \beta_i y_i^2 \leq 0, i = 1, \ldots, n.
\]

Let \(h(y) = (h_1(y_1), \ldots, h_n(y_n))^T\) with \(h_i(y_i) = y_i^2 - y_i\), \(g(x, y) = \langle g_1(x_1, y_1), \ldots, g_n(x_n, y_n) \rangle^T\) with \(g_i(x_i, y_i) = x_i - x_i y_i\), and \(f(x, y) = \langle f_1(x_1, y_1), \ldots, f_n(x_n, y_n) \rangle^T\) with \(f_i(x_i, y_i) = x_i^2 - (\alpha_i + \beta_i)x_i y_i + \alpha_i \beta_i y_i^2\). Consider the following mixed-integer quadratically constrained quadratic programming,

\[
(MIQCP(w)) \quad \min \tau + c^T x \\
\text{s.t.} \quad x^T Q x - \tau \leq 0, \\
\sigma (x^T Q x - \tau) + \delta T h(y) + \lambda^T g(x, y) + \mu^T f(x, y) \leq 0, \\
(x, y) \in \mathcal{F},
\]

with \(w = (\sigma, \delta, \lambda, \mu) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n\). Denote by \((MIQCP(w))\) the continuous relaxation of problem \((MIQCP(w))\).

Note that problem \((P)\) is equivalent to the following problem \((P_t)\),

\[
(P_t) \quad \min \{\tau + c^T x \mid x^T Q x - \tau \leq 0, \ (x, y) \in \mathcal{F}\}.
\]

Meanwhile, for any \(w = (\sigma, \delta, \lambda, \mu) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+^n\), constraint (3) can be viewed as a surrogate constraint of quadratic valid inequalities (1) and (2). The following theorem is at hand.

**Theorem 1.**

1. \(v(P) = v(MIQCP(w))\) for any \(w = (\sigma, \delta, \lambda, \mu) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+^n\).
2. \(v(P) \leq v(MIQCP(w))\) for any \(w = (\sigma, \delta, \lambda, \mu) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+^n\).

Theorem 1 shows that for any \(w = (\sigma, \delta, \lambda, \mu) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+^n\), problem \((MIQCP(w))\) is an improved reformulation of problem \((P)\) in the sense that the continuous relaxation of problem \((MIQCP(w))\) is at least as tight as that of problem \((P)\).

In order to apply the off-the-shelf mixed-integer convex quadratic solver to solve problem \((MIQCP(w))\), the parameter \(w\) should be chosen to make constraint (3) convex. That is, problem \((MIQCP(w))\) is a convex reformulation of problem \((P)\) for any \(w \in \Delta\), where

\[
\Delta = \left\{w \in \mathbb{R}^{n+1} \mid w = (\sigma, \delta, \lambda, \mu) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+^n\right\},
\]

\[
\begin{pmatrix}
\sigma Q + \text{diag}(\mu) & -\text{diag}(\mu)\text{diag}(\alpha + \beta)\text{diag}(\mu) \\
-\text{diag}(\mu)\text{diag}(\alpha + \beta)\text{diag}(\mu) & \text{diag}(\delta) + \text{diag}(\mu)\text{diag}(\alpha)\text{diag}(\beta)
\end{pmatrix} \geq 0.
\]

The “best” reformulation in the sense that the continuous relaxation bound is as tight as possible among \(w \in \Delta\), can be found via the following optimization problem

\[
(BQR) \quad \max \{v(MIQCP(w)) \mid w \in \Delta\}
\]
In the following section, we discuss how to solve problem (BQR) polynomially.

3. The “Best” Reformulation. In this section, we reduce the problem of finding the “best” reformulation, i.e., problem (BQR), to an SDP.

**Theorem 2.** Problem (BQR) can be reduced to the following SDP problem.

\[
\text{(CP}_1\text{)} \quad \text{max} \left\{ -\pi^T d - \phi \right\},
\]

\[
\text{s.t. } \begin{pmatrix} Q + \text{diag}(\mu) & M(\lambda, \mu) \\ M(\lambda, \mu) & \text{diag}(\delta) + \text{diag}(\phi) \text{diag}(\mu) \end{pmatrix} \begin{pmatrix} \theta(\lambda, \pi, \eta, \zeta) \\ \vartheta(\delta, \pi, \eta, \zeta) \end{pmatrix} \geq 0,
\]

\[
(\pi, \zeta, \eta, \mu) \geq 0,
\]

where \( p_i = \alpha_i + \beta_i, \ q_i = \alpha_i \beta_i, \ i = 1, \ldots, n \) and

\[
M(\lambda, \mu) = -\frac{\text{diag}(\lambda) + \text{diag}(p) \text{diag}(\mu)}{2},
\]

\[
\theta(\lambda, \pi, \eta, \zeta) = c + A^T \pi + \eta - \zeta + \lambda,
\]

\[
\vartheta(\delta, \pi, \eta, \zeta) = B^T \pi - \text{diag}(\beta) \eta + \text{diag}(\alpha) \zeta - \delta.
\]

Suppose that \( (\phi^*, \pi^*, \eta^*, \zeta^*, \delta^*, \lambda^*, \mu^*) \) is an optimal solution to (CP1), then \( w^* = (\sigma^*, \delta^*, \lambda^*, \mu^*) \) with \( \sigma^* = 1 \) is an optimal solution to problem (BQR).

**Proof.** Note that problem (BQR) is a max-min problem due to the definition of \( v(\cdot) \). The proof of Theorem 2 can be decomposed into two parts. Firstly, by adopting the duality theory, we derive the dual (maximization) problem of problem (MIQCP(w)). By combining the two maximization problems, we will get a maximization problem (CP_N), which is a reformulation of problem (BQR). Next, we will further prove that problem (CP_N) is equivalent to problem (CP1).

Note that problem (MIQCP(w)) can be reformulated as

\[
\text{(MIQCP}_t\text{)(w)} \quad \text{min} \left\{ \tau + c^T x \mid (2), (3), \ Ax + By \leq d, \alpha_i y_i \leq x_i \leq \beta_i y_i, \ y_i^2 - y_i \leq 0, \ i = 1, \ldots, n. \right\}
\]

Thus, we can express problem (MIQCP(w)) as a maximization problem by adopting the duality theory to problem (MIQCP_t(w)). Associate the following multipliers to the constraints in (MIQCP_t(w)):

- \( \phi \geq 0 \) for constraints (2), \( s \geq 0 \) for constraint (3), and \( \pi \in \mathbb{R}^n_+ \) for \( Ax + By \leq d; \)
- \( \zeta_i \in \mathbb{R}_+ \) for \( x_i \geq \alpha_i y_i, \ \eta_i \in \mathbb{R}_+ \) for \( x_i \leq \beta_i y_i, \) and \( \xi_i \in \mathbb{R}_+ \) for \( y_i^2 - y_i \leq 0, \ i = 1, \ldots, n. \)

Let \( \zeta, \eta, \xi \) denote the column vectors formed by \( \zeta_i, \eta_i, \xi_i \ (i = 1, \ldots, n) \), respectively. Let \( d(\varpi) \) denote the lagrangian dual function of (MIQCP_t(w)) with \( \varpi \) being the dual variables introduced above. The Lagrangian dual of (MIQCP_t(w)) is

\[
\text{max} \left\{ d(\varpi) \mid (\varrho, s, \pi, \zeta, \eta, \xi) \geq 0 \right\}.
\]

On the other hand,

\[
d(\varpi) = \min_{x \geq 0, y \geq 0} \left\{ \tau + c^T x + \pi^T (Ax + By - d) + g(x^T Qx - \tau)
\right.
\]

\[
+ \sum_{i=1}^n [\eta_i (x_i - \beta_i y_i) - \zeta_i (x_i - \alpha_i y_i) + \xi_i (y_i^2 - y_i)]
\]

\[
+ s [\sigma (x^T Qx - \tau) + \delta^T h(y) + \lambda^T g(x, y) + \mu^T f(x, y)] \}
\]
\[
\begin{align*}
&= \min_{x,y,\tau} \left\{ \tau + c^T x + \pi^T (Ax + By - d) + s\sigma(x^T Qx - \tau) \\
&\quad + \sum_{i=1}^n [\eta_i(x_i - \beta_i y_i) - \zeta_i(x_i - \alpha_i y_i)] \\
&\quad + \sum_{i=1}^n \left[ s\delta_i(y_i^2 - y_i) + s\lambda_i(x_i - x_i y_i) + s\mu_i(x_i^2 - p_i x_i y_i + q_i y_i^2) \right] \right\} \\
&= -\pi^T d + \min_{\tau} (1 - s\sigma)\tau + \min_{x,y} \bar{q}(x,y)
\end{align*}
\]

where \( p_i = \alpha_i + \beta_i, q_i = \alpha_i \beta_i, i = 1, \ldots, n \), the second equality holds due to the fact that variables \( q \) and \( \xi_i \) can be absorbed by \( \sigma \) and \( \delta_i \), \( i = 1, \ldots, n \), respectively, and \( \bar{q}(x,y) = s x^T \sigma Q + \diag(\mu)x + 2s x^T M(\lambda,\mu)y + y^T N(\delta,\mu)y + \hat{\theta}(s,\lambda,\pi,\eta,\zeta)^T x + \hat{\vartheta}(s,\delta,\pi,\eta,\zeta)^T y \) with \( M(\lambda,\mu) \) being defined as (7), \( N(\delta,\mu) \), \( \hat{\theta}(s,\lambda,\pi,\eta,\zeta) \) and \( \hat{\vartheta}(s,\delta,\pi,\eta,\zeta) \) being defined as

\[
\begin{align*}
N(\delta,\mu) &= s \text{diag}(\delta) + s \text{diag}(q) \text{diag}(\mu), \quad (11) \\
\hat{\theta}(s,\lambda,\pi,\eta,\zeta) &= c + A^T \pi + \eta - \zeta + s\lambda, \quad (12) \\
\hat{\vartheta}(s,\delta,\pi,\eta,\zeta) &= B^T \pi - \text{diag}(\beta) \eta + \text{diag}(\alpha) \zeta - s\delta. \quad (13)
\end{align*}
\]

Thus, the dual problem (10) can be written as

\[
\max\{-\pi^T d - \phi\},
\]

s.t. \( (s,\pi,\eta,\zeta) \geq 0, \ 1 - s\sigma = 0, \ \bar{q}(x,y) \geq -\phi, \forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n, \)

which can be reduced to the following problem

\[
(D(w)) \quad \max\{-\pi^T d - \phi\},
\]

s.t. \( \begin{pmatrix} s\sigma Q + \text{diag}(\mu) & sM(\lambda,\mu) & \frac{\hat{\theta}(s,\lambda,\pi,\eta,\zeta)}{2} \\
M(\lambda,\mu) & N(\delta,\mu) & \frac{\hat{\vartheta}(s,\delta,\pi,\eta,\zeta)}{2} \\
\frac{\hat{\theta}(s,\lambda,\pi,\eta,\zeta)}{2} & \frac{\hat{\vartheta}(s,\delta,\pi,\eta,\zeta)}{2} & \phi \end{pmatrix} \succeq 0, \quad (14) \)

\[
1 - s\sigma = 0, \quad (15)
\]

\[
(s,\pi,\eta,\zeta) \geq 0,
\]

where \( M(\lambda,\mu), N(\delta,\mu), \hat{\theta}(s,\lambda,\pi,\eta,\zeta) \) and \( \hat{\vartheta}(s,\delta,\pi,\eta,\zeta) \) are defined in (7), (11), (12) and (13), respectively. By Assumption 1 and the conic duality theorem [18], the strong duality between \( \text{MIQCP}_t(w) \) and its dual holds. Therefore, problem (BQR) is equivalent to

\[
\max\{v(D(w)) \mid w \in \Delta\},
\]

which is the following problem \( \text{CP}_N \),

\[
\begin{align*}
\text{CP}_N \quad \max\{-\pi^T d - \phi\},
\end{align*}
\]

s.t. \( \begin{pmatrix} s\sigma Q + \text{diag}(\mu) & -\frac{\text{diag}(\lambda) + \text{diag}(p) \text{diag}(\mu)}{2} \\
-\frac{\text{diag}(\lambda) + \text{diag}(p) \text{diag}(\mu)}{2} & \text{diag}(\delta) + \text{diag}(q) \text{diag}(\mu) \end{pmatrix} \succeq 0, \quad (16) \)

\[
(14), (15), (s,\pi,\eta,\zeta,\sigma,\mu) \geq 0.
\]
Hence, it suffices to prove that problem \((CP_N)\) is equivalent to problem \((CP_l)\). Let \((s, \sigma, \phi, \pi, \eta, \zeta, \delta, \lambda, \mu)\) be an optimal solution to problem \((CP_N)\). According to (15), we have \(s \neq 0\), that is \(s > 0\). Thus, let \(\sigma = s\sigma, \delta = s\delta, \lambda = s\lambda, \mu = s\mu\). According to (15), we have \(\sigma = 1\). Together with (16) and \(s > 0\), we obtain that \(\overline{w} = (\overline{\sigma}, \overline{\delta}, \overline{\lambda}, \overline{\mu}) \in \Delta\). Based on (14) and the fact that \(\sigma = 1\), it holds that \((\phi, \pi, \eta, \zeta, \delta, \lambda, \mu)\) is feasible to problem \((CP_l)\), and thus \(v(CP_l) \geq -\pi^T d - \phi = v(CP_N)\). We now show the opposite direction. Suppose that \((\sigma^*, \pi^*, \eta^*, \zeta^*, \delta^*, \lambda^*, \mu^*)\) is an optimal solution to problem \((CP_l)\). Let \(s^* = 1, \sigma^* = 1\). It is easy to verify that \((s^*, \sigma^*, \phi^*, \pi^*, \eta^*, \zeta^*, \delta^*, \lambda^*, \mu^*)\) is a feasible solution to problem \((CP_N)\) and thus \(v(CP_N) \geq -(\pi^*)^T d - \phi^* = v(CP_l)\).

Therefore, \(v(CP_l) = v(CP_l) = v(BQR)\) and \(w^* = (\sigma^*, \delta^*, \lambda^*, \mu^*)\) with \(\sigma^* = 1\) is an optimal solution to problem \((BQR)\). \(\square\)

4. Tightness of QCCR. In this section, we discuss the tightness of QCCR. We show in Theorem 3 that the continuous relaxation bound of our “best” QCCR is at least as tight as that of the “best” perspective reformulation. We also illustrate by a small example (Example 1) that the continuous relaxation bound of our “best” QCCR can be strictly tighter than that of the “best” perspective reformulation.

As shown in [7], [13], [20], there are perspective reformulations for \((P)\). The “best” perspective reformulation can be found by solving the following problem

\[
\begin{align*}
(BPR) \quad \max \{ v(\overline{PR}(\rho)) \mid \rho \in \Omega \},
\end{align*}
\]

where problem \((\overline{PR}(\rho))\) is denoted as the continuous relaxation of problem \((PR(\rho))\),

\[
\begin{align*}
(PR(\rho)) \min \left\{ x^T [Q - \text{diag}(\rho)] x + c^T x + \sum_{i=1}^{n} \rho_i x_i^2/y_i \mid (x, y) \in F \right\}
\end{align*}
\]

and

\[
\Omega = \{ \rho \in \mathbb{R}^n_+ \mid Q - \text{diag}(\rho) \succeq 0 \}.
\]

It has been shown in Theorem 1 in [20] that problem \((BPR)\) can be reduced to an SDP problem \((SDP_1)\). Thus, the question arises: which one has a tighter continuous relaxation bound, the “best” perspective reformulation or the “best” QCCR? The question is answered in the following Theorem 3.

**Theorem 3.** \(v(MIQCP(w^*)) \geq v(\overline{PR}(\rho^*))\) with \(w^*\) and \(\rho^*\) being the optimal solution to problem \((BQR)\) and \((BPR)\), respectively. Therefore, \(v(BQR) \geq v(BPR)\).

**Proof.** Note that \(v(BQR) \geq v(BPR)\) holds obviously if \(v(MIQCP(w^*)) \geq v(\overline{PR}(\rho^*))\) holds. Thus, it suffices to prove that \(v(MIQCP(w^*)) \geq v(\overline{PR}(\rho^*))\) holds.

Suppose \((x^*, y^*)\) is optimal to problem \((\overline{PR}(\rho^*))\). Define \(\overline{w} = (\overline{\sigma}, \overline{\delta}, \overline{\lambda}, \overline{\mu}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n\) with \(\sigma = 1, \mu = 0\) and

\[
(\overline{\delta}, \overline{\lambda}) = \left( \rho_i \frac{x_{i}^2}{y_i}, 2\rho_i \frac{x_{i}^2}{y_i} \right), \quad i = 1, \ldots, n.
\]

(17)

Let \(\tau^* = \max\{(x^*)^T Q x^*, (x^*)^T Q x^* + \delta^T h(y^*) + \lambda^T g(x^*, y^*)\}\). According to (17), we have

\[
(x^*)^T Q x^* + \delta^T h(y^*) + \lambda^T g(x^*, y^*)
\]
If it holds that \( \varpi \in \Delta \) and \((x^*, y^*, \tau^*)\) is an optimal solution to problem (MIQCP(\(\varpi\))), then
\[
v(\text{MIQCP}(w^*)) \geq v(\text{MIQCP}(\varpi))
\]
\[
= c^T x^* + (x^*)^T [Q - \text{diag}(\rho^*)] x^* + \sum_{i=1}^n \rho_i^* (x_i^*)^2 / y_i^* = v(\text{PR}(\rho^*)).
\]

Thus, it suffices to prove that \( \varpi \in \Delta \) and \((x^*, y^*, \tau^*)\) is an optimal solution to problem (MIQCP(\(\varpi\))).

We first prove that \( \varpi \in \Delta \). In order to prove \( \varpi \in \Delta \), it suffices to prove the matrix \( M \geq 0 \) with the matrix \( M \) being defined as
\[
M = \begin{pmatrix}
Q & -\text{diag}(\lambda)/2 \\
-\text{diag}(\lambda)/2 & \text{diag}(\delta)
\end{pmatrix}.
\]

According to the Schur complement condition for positive semidefiniteness \([6]\), we have
\[
M \geq 0 \iff \begin{pmatrix}
Q - \text{diag}(\varpi)/4 & 0 \\
0 & \text{diag}(\delta)
\end{pmatrix} \succeq 0,
\]
where \( \varpi = (\varpi_1, \ldots, \varpi_n)^T \) with \( \varpi_i = \lambda_i^2 / \delta_i, \ i = 1, \ldots, n \). According to (17), we have \( \delta_i \geq 0, i = 1, \ldots, n \). Thus, it suffices to prove that \( \varpi \geq 1 \). Based on (17), we have \( \varpi_i = \lambda_i^2 / \delta_i = 4 \rho_i^2, \ i = 1, \ldots, n \). Thus, \( \varpi \geq 0 \iff Q - \text{diag}(\rho^*) \succeq 0 \). Since \( \rho^* \in \Omega \), \( Q - \text{diag}(\rho^*) \succeq 0 \) holds. Thus, \( \varpi \in \Delta \).

Now we switch to prove that \((x^*, y^*, \tau^*)\) is an optimal solution to problem (MIQCP(\(\varpi\))). It is easy to verify that \((x^*, y^*, \tau^*)\) is feasible to (MIQCP(\(\varpi\))). Thus, according to Karush-Kuhn-Tucker (KKT) conditions \([6]\), it suffices to prove that there exist \((\tau, \pi, \pi, \eta, \zeta, \xi) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_n^+ \times \mathbb{R}_n^+ \times \mathbb{R}_n^+ \times \mathbb{R}_n^+\) such that (19)-(24) hold.

\[
1 - \tau - \pi = 1,
\]
\[
2(\tau + \pi) Q x^* - \pi \text{diag}(\lambda) y^* + \pi \lambda + c + \eta - \zeta + A^T \pi = 0,
\]
\[
2\text{diag}(\delta) + \text{diag}(\xi) y^* - \text{diag}(\lambda) x^* - \text{diag}(\delta) = 0,
\]
\[
-\text{diag}(\beta) \eta + \text{diag}(\alpha) \zeta - \xi + B^T \pi = 0,
\]
\[
\tau^T (Ax^* + By^* - d) = 0, \eta_i (x^*_i - \beta_i y^*_i) = 0, \zeta_i (x^*_i - \alpha_i y^*_i) = 0,
\]
\[
\xi_i [(y^*_i)^2 - y_i^2] = 0, i = 1, \ldots, n,
\]
\[
\tau [(x^*)^T Q x^* - \tau^*] = 0.
\]
Based on (18), (19), (23) and (24), we can let
\[ \tau = 0, \pi = 1. \] (25)

Then (19), (23) and (24) hold. Substitute (17) and (25) into (20) and (21), we have
\[
2(Q - \text{diag}(\rho^*)) x^* - \bar{\pi}^T h(y^*) + \tilde{X}^T g(x^*, y^*) = 0.
\] (24)

Thus, it suffices to find the \((\pi, \eta, \zeta, \xi)\) such that (22), (26) and (27) hold.

As shown in [20], problem \((\text{PR}(\rho^*))\) is equivalent to the following second-order cone problem
\[
(\text{PR}_s(\rho^*)) \quad \min x^T [Q - \text{diag}(\rho^*)] x + e^T x + (\rho^*)^T \phi
\]
s.t. \(Ax + By \leq d, \alpha_i y_i = x_i \leq \beta_i y_i, y_i^2 - y_i \leq 0, \ i = 1, \ldots, n,
\]
\[ x_i^2 \leq \phi_i y_i, \ i = 1, \ldots, n. \] (28)

Since \((x^*, y^*)\) is an optimal solution to problem \((\text{PR}(\rho^*))\), we have that \((x^*, y^*, \phi^*)\)
with \(\phi_i^* = (x_i^*)^2 / y_i^*, \ i = 1, \ldots, n\) is an optimal solution to problem \((\text{PR}_s(\rho^*))\).

According to the KKT conditions, there exist \((\bar{\pi}, \bar{\eta}, \bar{\zeta}, \bar{\xi}) \in \mathbb{R}^n_+ \times \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n\) such that (22), (26) and (27) hold. Thus, it suffices to prove that \(\overline{\gamma} = -\overline{X}/2\) and \(\bar{\delta} = \bar{\pi}\) hold.

Due to (28), we have \(\left(\rho_i^* \bar{\gamma}_i \bar{c}_i \right) \in S^2_+, \ i = 1, \ldots, n.\) Note that
\[
\left(\rho_i^* x_i^* \bar{c}_i \right) \in S^2_+, \ i = 1, \ldots, n. \]

According to the Schur Product theorem in [15], we have \(\left(\rho_i^* \phi_i^* \bar{c}_i \right) \in S^2_+,\) and thus \(\rho_i^* \phi_i^* \bar{c}_i y_i^* - (\bar{\gamma}_i x_i^*)^2 \geq 0, \ i = 1, \ldots, n.\) By (31), we have \(\bar{c}_i y_i^* = -\rho_i^* \phi_i^* \bar{c}_i y_i^* - (\bar{\gamma}_i x_i^*)^2, \ i = 1, \ldots, n.\) Then,
\[
0 \leq -\rho_i^* \phi_i^* \rho_i^* \phi_i^* + 2\bar{\gamma}_i x_i^* - (\bar{\gamma}_i x_i^*)^2 = -\rho_i^* \phi_i^* \phi_i^* + \bar{\gamma}_i x_i^*, \ i = 1, \ldots, n.
\]
Therefore, \((\rho_i^* \phi_i^* + \bar{\gamma}_i x_i^*)^2 \leq 0,\) which implies that \(\rho_i^* \phi_i^* + \bar{\gamma}_i x_i^* = 0\) and then \(\bar{\gamma}_i = -\rho_i^* \phi_i^* / x_i^* = -\rho_i^* x_i^*/y_i^* = -\bar{\lambda}_i/2, \ i = 1, \ldots, n.\) Substituting it into (31), we have
\[ \bar{c}_i = -\bar{\gamma}_i x_i^*/y_i^* = \rho_i^* (x_i^*)^2 / (y_i^*)^2, \ i = 1, \ldots, n. \]

According to the proof of Theorem 3, following the lines after (18), we have the following corollary.
Corollary 1. For any \( \bar{\rho} \in \Omega \), let \((\bar{x}, \bar{y})\) be an optimal solution to problem \((\overline{\text{PR}}(\bar{\rho}))\). Define \( \bar{w} = (\bar{\sigma}, \bar{\delta}, \bar{\lambda}, \bar{\mu}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \) with \( \bar{\sigma} = 1 \) and \( \bar{\mu} = 0 \) and
\[
(\bar{\delta}_i, \bar{\lambda}_i) = (\bar{\rho}_i \bar{x}_2^2, 2\bar{\rho}_i \bar{x}_2), \quad i = 1, \ldots, n.
\]
Then it holds that \( \bar{w} \in \Delta \) and \( v(\text{MIQCP}(\bar{w})) = v(\overline{\text{PR}}(\bar{\rho})) \).

Corollary 1 shows that for any perspective reformulation of problem \((P)\), we can always find a MIQCP reformulation of problem \((P)\) such that the continuous relaxation bound of the MIQCP reformulation is the same as that of the perspective reformulation.

Now we switch back to Theorem 3. Theorem 3 shows that the continuous relaxation bound of our “best” QCCR is at least as tight as that of the “best” perspective reformulation. At this point, another question arises: Can the continuous relaxation bound of our “best” QCCR be strictly tighter than that of the “best” perspective reformulation? We will answer the question by the following example.

Example 1. Consider the following example
\[
\begin{align*}
(PE) \quad \min & \quad \{2x_1^2 - 18x_1x_2 + 41x_2^2 - 10x_2x_3 + 51x_3^2 + 6x_1 + 6x_2 - 8x_2 \} \\
\text{s.t.} & \quad x_1 + x_2 + x_3 = 1, \\
& \quad y_1 + y_2 + y_3 \leq 2, \\
& \quad 0 \leq x_i \leq 0.75y_i, \quad y_i \in \{0, 1\}, \quad i = 1, 2, 3.
\end{align*}
\]

Solve the above problem via Cplex, we get the optimal value of problem \((PE)\) is \( v(PE) = 6.3125 \) with \( x = (0.75, 0.25, 0)^T \) and \( y = (1, 1, 0)^T \).

According to Theorem 1 in [20], we can get the tightest lower bound provided by the perspective reformulation among \( \rho \in \Omega \) via solving an SDP. Solve the SDP problem \((SDP_1)\) in [20], we have \( v(BPR) = v(SDP_1) = 4.8236 \).

Now get the best QCCR of problem \((PE)\). Via solving the corresponding problem \((CP)\) in Theorem 2, we get the solution to problem \((BQR)\), \( \bar{\delta} = (1.45841, 2.5873, 2.5873)^T, \quad \bar{\lambda} = (-2.6707, 16.7631, 21.4871)^T \) and \( \bar{\mu} = (8.7452, 0, 0)^T \). Let \( \bar{w} = (1, \bar{\delta}, \bar{\lambda}, \bar{\mu}) \). Solve the continuous relaxation of problem \((\text{MIQCP}(\bar{w}))\), we have \( v(\text{MIQCP}(\bar{w})) = 5.9717 \). Thus, \( v(BQR) = 5.9717 > 4.8236 = v(BPR) \).

5. Computation experiments. In this section, we conduct some computational experiments, to illustrate the effectiveness of our “best” QCCR for problem \((P)\). The computational results in this section show that our “best” QCCR favorably compares with the “best” perspective reformulation for the set of test problems.

5.1. Test problems. The test problem we used in our numerical test is the following mean-variance portfolio selection problem with cardinality constraint and buy-in threshold restriction.
\[
\begin{align*}
\min & \quad x^T Q x \\
\text{s.t.} & \quad e^T x \geq \rho, \quad (33) \\
& \quad e^T x = 1, \quad (34) \\
& \quad |\text{supp}(x)| \leq K, \quad (35) \\
& \quad \alpha_i \leq x_i \leq \beta_i, \quad i \in \text{supp}(x). \quad (36)
\end{align*}
\]
In the problem, $x = (x_1, x_2, \ldots, x_n)^T$ is the weight vector of the portfolio, which is the decision vector. The objective $x^T Q x$ is the variance of the portfolio, $c^T x$ is the expected return of the portfolio, and $\varrho$ is the prescribed return given by the investor. In (35), $\text{supp}(x) = \{i \mid x_i \neq 0\}$ and $K$ is an integer satisfying $0 < K \leq n$. Constraint (35) is referred as a cardinality constraint, which is used to limit the total number of assets in the optimal portfolio. In Constraint (36), $\beta \in \mathbb{R}^n$ is the upper bound vector of $x$ and $\alpha_i$ is the lower bound for $x_i$. Constraint (36) is referred as the buy-in threshold restriction, which is introduced to prevent the investors from holding some assets with a very small amount. As in [20], by introducing binary variables, we can reduce the problem to the following semi-continuous quadratic programming

\[
\begin{align*}
\text{(TP)} \quad & \min \ x^T Q x \\
\text{s.t.} \quad & e^T y \leq K, \ y \in \{0, 1\}^n, \\
& \alpha_i y_i \leq x_i \leq \beta_i y_i, \ i = 1, \ldots, n,
\end{align*}
\]

5.2. Implementation issues. The numerical tests were implemented in Matlab R2014a (64 bit) and run on a personal computer equipped with Intel Pentium CPU (3.6GHz) and 16 GB RAM, running Windows 7(64 bit). CVX 1.2 [12], which is a MATLAB-based modeling system for convex optimization, is used to model the SDP problems (SDP1) of [20] and (CP1). The SDP problems are solved by SeDuMi 1.2 within CVX. We use the software Cplex to solve our “best” QCCR and the “best” perspective reformulation in Cplex. Both reformulations are solved by the mixed-integer quadratic programming solver in Cplex 12.7. The software Cplex is used through C++, and the C++ programs were developed and compiled using Microsoft Visual Studio 2015. We use Cplex default settings except that we set the time limit to 1,800 seconds and the number of parallel threads to be single.

5.3. Numerical results. We conduct the numerical tests on 45 instances of problem (TP). The 45 instances have the following structure:

- there are 15 instances each for $n = 200, 300, 400$;
- for each $n$, the 15 instances are further divided into three subsets denoted by $n^+, n^0$ and $n^-$. Each subset has a different diagonal dominance in the covariance matrix $Q$.
- For each instance, we consider the cardinality constraint (37) with $K = 6, 8, 10, 12$. Thus, we have 180 instances of problem (TP) in total in our numerical test.

We use the same random generator as in [10] to generate the instances of test problem (TP). Details are as follows.

- $\alpha_i = 0.05$, $\beta_i = 0.5$, for $i = 1, \ldots, n$;
- $c$ and $\varrho$ are both uniformly drawn at random from the interval $(0.0002, 0.01)$;
- We use the random generator of [17] to generate matrix $Q$. The nondiagonal elements of matrix $Q$ are integers uniformly drawn at random from the interval $[-10, -5]$. The diagonal elements of matrix $Q$ are given by $Q_{ii} = \frac{\sum_{j \neq i} |Q_{ij}|}{1-S}$ with different dominance index $S$, $i = 1, \ldots, n$. For each value of $n$, we take the dominance index $S = 0.63$ for $n^+$ set, $S = -0.05$ for $n^0$ set and $S = -0.5$ for $n^-$ set.
We conduct computational comparisons for the “best” perspective reformulation and our “best” QCCR for the instances with different \( n \) and \( K \). We denote by (PR\(_{SOCP}\)) and (QCP) the “best” perspective reformulation and our “best” QCCR in this section. Table 1 records the average comparison results of five instances for each \( n \) and \( K \). The first column “\( n \)” in the table 1 denotes the problem size \( n \). The second column “\( K \)” denotes the cardinality \( K \). The column “\( T_{PR} \)” is the computational time (in seconds) for obtaining the parameter of the “best” perspective reformulation via solving the SDP problem (SDP\(_l\)) of [20]. The column “\( T_{QCP} \)” is the computational time (in seconds) for obtaining the parameter vector \( w \) for our “best” QCCR via solving SDP problem (CP\(_l\)). The columns “gap(%)” are the output parameters of Cplex, which measure the relative gap (in percentage) of the incumbent solution when Cplex is terminated. Note that the default tolerance of the relative gap in Cplex is 0.01%. The columns “time” and “nodes” are the computational time (in seconds) and the number of nodes explored by Cplex, respectively.

| \( n \) | \( K \) | \( T_{PR} \) | \( T_{QCP} \) | \( (PR_{SOCP}) \) | \( (QCP) \) |
|---|---|---|---|---|---|
| 200\(^{0}\) | 6 | 18.94 | 114.81 | 3.02 | 1800.00 | 8905 | 0.02 | 727.71 | 533846 |
| 200\(^{0}\) | 8 | 18.35 | 138.23 | 3.15 | 1800.00 | 7726 | 0.91 | 1753.91 | 1777822 |
| 200\(^{+}\) | 10 | 18.74 | 110.77 | 3.49 | 1800.00 | 6052 | 1.67 | 1800.00 | 2084138 |
| 200\(^{+}\) | 12 | 18.35 | 124.77 | 3.52 | 1800.00 | 6100 | 1.86 | 1800.00 | 1941049 |
| 200\(^{0}\) | 6 | 20.40 | 124.90 | 34.75 | 1800.01 | 11038 | 27.09 | 1800.01 | 2183238 |
| 200\(^{0}\) | 8 | 17.60 | 126.74 | 33.74 | 1800.01 | 10520 | 28.67 | 1800.01 | 2020237 |
| 200\(^{0}\) | 10 | 16.68 | 116.01 | 34.17 | 1800.00 | 6104 | 27.61 | 1800.00 | 2612416 |
| 200\(^{0}\) | 12 | 16.31 | 126.23 | 33.17 | 1800.00 | 8758 | 28.65 | 1800.01 | 2293060 |
| 200\(^{0}\) | 6 | 18.78 | 129.74 | 58.70 | 1800.01 | 12628 | 50.00 | 1800.01 | 1990354 |
| 200\(^{0}\) | 8 | 19.52 | 125.89 | 58.91 | 1800.00 | 11850 | 53.12 | 1800.01 | 2428332 |
| 200\(^{0}\) | 10 | 19.00 | 125.75 | 58.75 | 1800.01 | 10990 | 55.25 | 1800.01 | 1911470 |
| 200\(^{0}\) | 12 | 17.80 | 128.07 | 58.80 | 1800.00 | 11449 | 55.41 | 1800.01 | 1456779 |
| 300\(^{0}\) | 6 | 48.31 | 314.17 | 3.01 | 1447.96 | 4793 | 1.97 | 1800.01 | 1357610 |
| 300\(^{0}\) | 8 | 49.16 | 298.87 | 3.37 | 1445.44 | 3134 | 1.88 | 1441.80 | 1249562 |
| 300\(^{0}\) | 10 | 48.54 | 320.01 | 3.37 | 1454.77 | 2186 | 2.04 | 1800.01 | 1024344 |
| 300\(^{0}\) | 12 | 48.83 | 340.17 | 3.32 | 1502.79 | 2077 | 2.37 | 1800.01 | 811791 |
| 300\(^{0}\) | 6 | 46.55 | 298.79 | 40.90 | 1800.00 | 3321 | 32.88 | 1800.01 | 2435679 |
| 300\(^{0}\) | 8 | 43.08 | 295.82 | 40.66 | 1800.01 | 3452 | 34.04 | 1800.00 | 2132799 |
| 300\(^{0}\) | 10 | 39.67 | 301.40 | 40.49 | 1800.00 | 3146 | 34.83 | 1800.00 | 1855355 |
| 300\(^{0}\) | 12 | 42.47 | 279.30 | 40.24 | 1800.00 | 3648 | 35.28 | 1800.00 | 1701796 |

Table 1. Comparison results of perspective reformulation and QCCR for Problem (TP)
For the test problems, most of the reformulations can not be solved to the optimality when Cplex terminated except for some problems of set 200+ and 300+. From the table, we find that the relative gap tends to increase as the dominance index decreases for the test problems. That is, the relative gap of problem set n− is the largest, while the relative gap of problem set n+ is the smallest. We can also find from Table 1 that, the average computational time used to solve the large SDP problem (CP_l) is much longer than to solve the SDP problem (SDP_l) of [20]. From the table, the time of solving problem (CP_l) is about 6 times longer than that of solving problem (SDP_l). This may be caused by the dimension of the linear inequality matrix of problem (CP_l) is about 2 times larger than that of problem (SDP_l). On the other hand, it can be noted from the table that the time cost for solving the SDP is often negligible since the computational time of the branch-and-bound algorithm is much larger than the time spent for solving large SDP. Furthermore, the average relative gap of our “best” QCCR is less than that of the “best” perspective reformulation when Cplex terminated. These observations support that the quality of the lower bound generated by the continuous relaxation of reformulation has a significant positive impact on the efficiency of branch-and-bound procedure.

6. Concluding remarks. We have proposed QCCR for problem (P). Compared with the perspective reformulation of problem (P), the structure of QCCR is much more simple than that of the second-order cone reformulation of perspective reformulation. Furthermore, the continuous relaxation bound of our new reformulation is at least as tight as that of the perspective reformulation. We also illustrated by Example 1 that the continuous relaxation bound of our “best” QCCR can be strictly tighter than that of the “best” perspective reformulation. For the problem of finding the “best” reformulation among QCCR in the sense that the continuous relaxation bound is as tight as possible, we have shown in the paper that the problem could be solved via solving an SDP. We finally test our reformulation on a class of portfolio selection problems with real-life constraints. The computational results showed that our reformulation favorably compared with the perspective reformulation for the test problems. We expect that the framework of this paper can shed some light on deriving good lower bounds that contribute to the performance of searching algorithms for SQP.

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E-mail address: xjzheng@tongji.edu.cn
E-mail address: jiangzy@fudan.edu.cn