On the asymptotic properties of an interesting diffraction integral

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Abstract

We introduce and study a new special integral, denoted $I_{\varepsilon}^{+,-}$, depending on two complex parameters $\alpha_1$ and $\alpha_2$. It arises from the canonical problem of wave diffraction by a quarter-plane, and is heuristically constructed to capture the complex field near the tip and edges. We establish some region of analyticity of this integral in $\mathbb{C}^2$, and derive its rich asymptotic behaviour as $|\alpha_1|$ and $|\alpha_2|$ tend to infinity. We also study the decay properties of the function obtained from applying a specific double Cauchy integral operator to this integral. These results allow us to show that this integral shares all of the asymptotic properties expected from the key unknown function $G_{+,-}$ arising when the quarter-plane diffraction problem is studied via a two-complex-variables Wiener–Hopf technique (see Assier & Abrahams, arXiv:1905.03863, 2019). As a result, the integral $I_{\varepsilon}^{+,-}$ can be used to mimic the unknown function $G_{+,-}$ and to build an efficient ‘educated’ approximation to the quarter-plane problem.

1 Introduction and motivation

We propose to study the properties (asymptotic behaviour, analyticity and more) of the integral $I_{\varepsilon}^{+,-}$ that can be considered a function of two complex variables ($\alpha_1, \alpha_2$) and is defined by

$$I_{\varepsilon}^{+,-}(\alpha_1, \alpha_2) = \int_{\frac{\pi}{2}}^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{f_1 \left( \frac{\pi}{2}, \phi \right)}{-i(\alpha_1 \cos(\phi) + \alpha_2 \sin(\phi) + i\varepsilon) \nu_1^{\nu_1+3/2}} d\phi,$$

(1.1)

where the constants $\varepsilon$ and $\nu_1$ and the function $f_1 \left( \frac{\pi}{2}, \phi \right)$ will be specified later. This work is directly motivated by the conclusions discussed in a recent article by the authors, [2], which focuses on a two-complex-variable investigation of the three-dimensional problem of wave diffraction by a quarter-plane with homogeneous Dirichlet boundary conditions subject to an incident plane wave. The quarter-plane is occupying the $(x_1 > 0, x_2 > 0, x_3 = 0)$ subspace of a $(x_1, x_2, x_3)$ Cartesian space. The total physical wave field is denoted $u(x_1, x_2, x_3)$ and the incident plane wave takes the form $e^{-i(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)}$, where the constants $a_1, a_2, a_3$ depend solely on the incident direction and the wave number $k > 0$ such that $a_1^2 + a_2^2 + a_3^2 = k^2$. Note also that the time factor $e^{-i\omega t}$, where $\omega$ is the angular frequency of the wave, has been suppressed for brevity. This physical solution can be expressed in the form of an inverse Fourier transform

$$u(x, x_3) = \frac{1}{(2\pi)^2} \int_{A_1} \int_{A_2} F_{++} \left( \alpha \right) e^{-i\alpha \cdot x} e^{i\nu_{\alpha}^{x_3}} d\alpha_1 d\alpha_2,$$

(1.2)
where \( \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \), \( \mathbf{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{C}^2 \) and the contours \( \mathcal{A}_{1,2} \) defined in Figure 1 naturally lead to the definition of the upper and lower half planes \( \text{UHP}_{1,2} \) and \( \text{LHP}_{1,2} \) of the \( \alpha_{1,2} \) complex planes. It is also useful to define the domains \( \mathcal{D}_{++} = \text{UHP}_1 \times \text{UHP}_2 \) and \( \mathcal{D}_{+-} = \text{UHP}_1 \times \text{LHP}_2 \). The crucial unknown function \( F_{++}(\mathbf{\alpha}) \) is to be determined and \( K(\mathbf{\alpha}) = (k^2 - \alpha_1^2 - \alpha_2^2)^{-1/2} \) is the kernel of the problem.

![Figure 1: Description of the lower and upper-half planes UHP\(_{1,2}\) and LHP\(_{1,2}\) lying on and above and on and below the integration contours \( \mathcal{A}_{1,2} \).](image)

One of the achievements of [2] is the reduction of the complicated problem of diffraction by a quarter-plane to two equations in the two-complex-variables Fourier space. They relate the two unknown functions \( F_{++}(\mathbf{\alpha}) \) and \( G_{+-}(\mathbf{\alpha}) \), analytic on \( \mathcal{D}_{++} \) and \( \mathcal{D}_{+-} \) respectively, as follows

\[
F_{++} = F_{++}^* + \mathcal{I}[G_{+-}] \quad \text{on} \quad \mathcal{D}_{++},
\]

\[0 = G_{+-}^* + \mathcal{J}[G_{+-}] \quad \text{on} \quad \mathcal{D}_{+-}, \tag{1.4}\]

where \( \mathcal{I} \) and \( \mathcal{J} \) are explicitly-known Cauchy integral operators depending on the kernel \( K \), and the two functions \( F_{++}^*(\mathbf{\alpha}) \) and \( G_{+-}^*(\mathbf{\alpha}) \) are also known explicitly and can be written in terms of \( K \).

In this article, we will focus on the function \( G_{+-} \), since it leads to \( F_{++} \) directly via (1.3) and hence to the sought-after \( u \) via (1.2). Though unknown, this function can be expressed in terms of the wave field \( u \) by

\[
G_{+-}(\mathbf{\alpha}) = \int_{-\infty}^{0} \int_{0}^{\infty} u(\mathbf{x}, 0) e^{i\mathbf{\alpha} \cdot \mathbf{x}} dx_1 dx_2. \tag{1.5}\]

The physical field \( u \) must obey some edge and vertex conditions, which can be written as

\[
u_1 = \sqrt{\lambda_1 + 1/4} \approx 0.7967,
\]

where \( r = \sqrt{x_1^2 + x_2^2} \) is the distance to the vertex and

\[
u_1 = \sqrt{\lambda_1 + 1/4} \approx 0.7967,
\]
with \( \lambda_1 \) being the first eigenvalue of the Laplace-Beltrami operator (LBO) with Dirichlet conditions on the cut defined by \( \{ \theta = \frac{\pi}{2}, \varphi \in [0, \frac{\pi}{2}] \} \) in the usual spherical coordinates (see e.g. [3], [4]). In the Fourier space, these conditions translate into the following asymptotic behaviour for \( G_{+-} \):

\[
\begin{align*}
G_{+-}(\alpha_1, \alpha_2) & \xrightarrow{|\alpha_1| \to \infty} \mathcal{O}\left(\frac{1}{|\alpha_1|^{\nu_1+3/2}}\right), \\
G_{+-}(\alpha_1, \alpha_2) & \xrightarrow{|\alpha_2| \to \infty} \mathcal{O}\left(\frac{1}{|\alpha_1|}\right), \\
G_{+-}(\alpha_1, \alpha_2) & \xrightarrow{|\alpha_2| \to \infty} \mathcal{O}\left(\frac{1}{|\alpha_2|^{3/2}}\right),
\end{align*}
\]

for \( \alpha_1 \in \text{UHP}_1 \) and \( \alpha_2 \in \text{LHP}_2 \).

In order to derive the equations (1.3)–(1.4) rigorously, an assumption on the key function \( G_{+-} \) had to be made in [2], mainly that \( \mathcal{I}[G_{+-}](\alpha_1, \alpha_2) \) tends to zero as \( |\alpha_2| \to \infty \).

The purpose of the present article is two-fold. On the one hand, we wish to show why we think this assumption is justified and, on the other hand, we aim to suggest an efficient approximation scheme to solve the quarter-plane problem. This will be done by introducing the explicitly defined integral \( I_{+-}^\varepsilon \), and by showing that it mimics the behaviour of \( G_{+-} \). By this we mean that \( I_{+-}^\varepsilon \) should be analytic on \( \mathcal{D}_{+-} \), should have the asymptotic behaviour (1.9)–(1.11) and should satisfy \( \mathcal{I}[I_{+-}^\varepsilon] \to 0 \) as \( |\alpha_2| \to \infty \).

The rest of the paper is organised as follows: in section 2, we give the integral expression of \( I_{+-}^\varepsilon \) again and explain where it comes from; in section 3 we highlight some important properties of the first Laplace-Beltrami eigenfunction; in section 4 we prove that \( I_{+-}^\varepsilon \) does indeed have the correct asymptotic behaviour (1.9)–(1.11); and in section 5, we prove that \( \mathcal{I}[I_{+-}^\varepsilon] \) does tend to zero as \( |\alpha_2| \) tends to infinity. Finally, we discuss the implications of our findings and conclude the paper in section 6.

## 2 An interesting integral

In this section, we aim to derive and construct explicitly a function defined by an integral that satisfies all the conditions required of \( G_{+-} \). Starting from the integral representation (1.5) and using the change of variable \( x_1 = r \cos(\varphi), x_2 = r \sin(\varphi) \), for \( r \in \mathbb{R}^+ \) and \( \varphi \in [3\pi/2, 2\pi] \), we can write

\[
G_{+-}(\alpha) = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{0}^{\infty} \hat{u}(r, \varphi)e^{ir(\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi))}rdrd\varphi,
\]

where \( \hat{u}(r, \varphi) = u(r \cos(\varphi), r \sin(\varphi), 0) \). Moreover, as explained in [2] for example, using separation of variables, it can be shown that

\[
\hat{u}(r, \varphi) \xrightarrow{r \to 0} Af_1\left(\frac{\pi}{2}, \varphi\right) r^{\nu_1-1/2},
\]

where \( f_1(\theta, \varphi) \) is the eigenfunction of the LBO associated to the first eigenvalue \( \lambda_1 \) and \( A \) is a constant. For technical reasons that will become apparent later on, let us rewrite the asymptotic behaviour (2.2) in a slightly different form

\[
\hat{u}(r, \varphi) \xrightarrow{r \to 0} Af_1\left(\frac{\pi}{2}, \varphi\right) r^{\nu_1-1/2}e^{-\varepsilon r},
\]
for some $\varepsilon > 0$. Note that (2.2) and (2.3) are equivalent to leading order since $e^{-\varepsilon r} \to 1$ as $r \to 0$. Because the far-field behaviour in the Fourier space is intrinsically linked to the near-field behaviour in the physical space, we are interested in the integral $I$ obtained by replacing $\hat{u}(r, \varphi)$ by its leading order behaviour (2.3) in (2.1):

$$I = \int_{\frac{3\pi}{2}}^{2\pi} \int_{0}^{\infty} A f_1 \left( \frac{\pi}{2}, \varphi \right) r^{\nu_1-1/2} e^{i r (\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi))} e^{-\varepsilon r} r dr d\varphi.$$ 

Noting that the integral over $r$ takes the form

$$\int_{0}^{\infty} r^{\nu-1} e^{-\mu r} dr,$$ (2.4)

for $\nu = \nu_1 + 3/2$ and $\mu = -i(\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi) + i\varepsilon)$. Note (see [6] p317, 3.381.4.) that for $\text{Re}(\nu) > 0$ and $\text{Re}(\mu) > 0$, this integral is exactly equal to $\frac{\Gamma(\nu)}{\nu}$, where $\Gamma$ is the Euler Gamma function. It is clear that $\text{Re}(\nu) > 0$ since $\nu_1 \geq 0$. Moreover, we can choose $\varepsilon$ such that $\text{Re}(\mu) > 0$. In order to do so, we refer to Figure 1, to see that for all $\alpha \in \mathcal{D}_{+\varepsilon}$, we have $\text{Im}(\alpha_1) \geq -M$ and $\text{Im}(\alpha_2) \leq M$ for some $M > 0$ depending on the choice of contour $\mathcal{A}_{1,2}$. Remembering that for $\varphi \in \left[ \frac{3\pi}{2}, 2\pi \right]$ we have $0 \leq \cos(\varphi) \leq 1$ and $-1 \leq \sin(\varphi) \leq 0$, it is possible to show that upon choosing $\varepsilon$ such that $\varepsilon > 2M$, we have $\text{Re}(\mu) > 0$ for all $\alpha \in \mathcal{D}_{+\varepsilon}$, and hence $I$ can be rewritten as

$$I = A \Gamma(\nu_1 + 3/2) \int_{\frac{3\pi}{2}}^{2\pi} \frac{f_1 \left( \frac{\pi}{2}, \varphi \right)}{(-i(\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi) + i\varepsilon))^{\nu_1+3/2}} d\varphi.$$ 

Because of our choice of $\varepsilon$ the denominator of the integrand is never zero for $\alpha \in \mathcal{D}_{+\varepsilon}$; hence the integral is a ‘+−’ function, i.e. it is analytic in $\mathcal{D}_{+\varepsilon}$. This naturally leads to the definition of the special integral $I_{+\varepsilon}^{\varepsilon}(\alpha_1, \alpha_2)$ to be studied in this paper:

$$I_{+\varepsilon}^{\varepsilon}(\alpha_1, \alpha_2) = \int_{\frac{3\pi}{2}}^{2\pi} \frac{f_1 \left( \frac{\pi}{2}, \varphi \right)}{(-i(\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi) + i\varepsilon))^{\nu_1+3/2}} d\varphi.$$ (2.5)

For the purpose of the present work, the values of $k$ and $M$ (and hence $\varepsilon$) can be any strictly positive numbers. For numerical illustration of our theoretical results, we will choose $k = 3$ and $\varepsilon = 1$.

### 3 A note on the Laplace-Beltrami eigenfunction

Before deriving various properties of the newly-introduced integral $I_{+\varepsilon}^{\varepsilon}$, it is important to know the behaviour of the eigenfunction $f_1 \left( \frac{\pi}{2}, \varphi \right)$. Its important properties are summarised in the following lemma, the proof of which (linked to the physical edge conditions) is omitted here for brevity.

**Lemma 1** Let $f_1(\theta, \varphi)$ be the first eigenfunction of the LBO. Then $f_1 \left( \frac{\pi}{2}, \varphi \right)$ is equal to zero on the cut $\varphi \in \left[ 0, \frac{\pi}{2} \right]$, and is smooth and such that $0 < f_1 \left( \frac{\pi}{2}, \varphi \right) \leq 1$ for $\varphi \in \left( \frac{\pi}{2}, 2\pi \right)$. Moreover, its behaviour at the edge of the non-zero region is given by

$$f_1 \left( \frac{\pi}{2}, \varphi \right) \begin{cases} \varphi > \pi/2 & \mathcal{O} \left( \sqrt{\varphi - \pi/2} \right) \\ \varphi \to \pi/2 & \mathcal{O} \left( \sqrt{2\pi - \varphi} \right). \end{cases}$$
In particular, there exists a constant $\beta$, such that
\[ f_1 \left( \frac{\pi}{2}, 2\pi - \psi \right) \psi \to 0, \quad \beta \sqrt{\psi}. \]

In addition, it transpires that $f_1 \left( \frac{\pi}{2}, \varphi \right)$ is strictly decreasing for $\varphi \in \left[ \frac{3\pi}{2}, 2\pi \right]$.

Though not an exact result, it seems that $f_1 \left( \frac{\pi}{2}, \varphi \right)$ is approximated very well by the function $g(\varphi)$ defined by
\[
g(\varphi) = \begin{cases} 
0 & \text{if } \varphi \in [0, \frac{\pi}{2}], \\
\frac{1}{\sqrt{\sin \left( \frac{4\varphi - \pi}{3} \right)}} & \text{if } \varphi \in \left[ \frac{\pi}{2}, 2\pi \right].
\end{cases}
\]

Note that $g$ trivially satisfies the conclusions of Lemma 1, with $\beta = \sqrt{\frac{2}{3}}$. In Figure 2, we compare numerical results obtained using a surface finite element method developed in [4] and the function $g$, showing excellent agreement.

![Figure 2: Comparison between a numerical approximation of $f_1 \left( \frac{\pi}{2}, \varphi \right)$ and the function $g(\varphi)$.](image)

4 Asymptotic behaviour of $I^\varepsilon_{+-}$

In this section, we show that the integral $I^\varepsilon_{+-}(\alpha_1, \alpha_2)$ shares the same rich asymptotic behaviour as $G_{+-}(\alpha_1, \alpha_2)$; that is, it should behave like (1.9)–(1.11), as $|\alpha_{1,2}| \to \infty$ within $\mathcal{D}_{+-}$. We will summarise the key results here, the detailed proofs being given in Appendix A. Since $\alpha = (\alpha_1, \alpha_2) \in \mathcal{D}_{+-}$, whenever $|\alpha_1|$ (resp. $|\alpha_2|$) tends to infinity, we can write $\alpha_1 = |\alpha_1| e^{i\phi_1}$ (resp. $\alpha_2 = |\alpha_2| e^{i\phi_2}$) for $\phi_1 \in (0, \pi)$ (resp. $\phi_2 \in (-\pi, 0)$). Note that if $\alpha_{1,2}$ are not assumed large, we cannot write them in this way, since they may lie within the indented part of the contours $\mathcal{A}_{1,2}$.

Asymptotic behaviour when both $|\alpha_1|$ and $|\alpha_2|$ tend to infinity within $\mathcal{D}_{+-}$. This is the simplest of the three different cases to be considered. We will take both $|\alpha_1|$ and $|\alpha_2| \to \infty$ within
\(D_{+\infty}\), in such a way that there exists an \(m > 0\) such that \(|\alpha_2| = m|\alpha_1|\). In this case, we can write \(\alpha_{1,2} = |\alpha_{1,2}|e^{i\phi_{1,2}}\) and we have

\[
I^\varepsilon_{+-}(\alpha_1, \alpha_2) \xrightarrow{|\alpha_2|=m|\alpha_1|} \frac{1}{|\alpha_1|^{\nu + 3/2}} I^0_+- (e^{i\phi_1}, m e^{i\phi_2}).
\]  

(4.1)

The validity of this asymptotic behaviour is illustrated in Figure 3.

![Figure 3: Numerical illustration of the asymptotic behaviour (4.1) as both \(|\alpha_1,2| \to \infty\) for \(\phi_1 = \frac{\pi}{4}\), \(\phi_2 = -\frac{\pi}{2}\), \(\varepsilon = 1\) and \(m = 2\), using \(g(\varphi)\) instead of \(f_1\left(\frac{\pi}{2}, \varphi\right)\) in the definition of \(I^\varepsilon_{+-}\).](image)

**Asymptotic behaviour when \(|\alpha_1| \to \infty\) within UHP_1 and \(\alpha_2\) is fixed in LHP_2**  
In this case, we can write \(\alpha_1 = |\alpha_1|e^{i\phi_1}\), and we have

\[
I^\varepsilon_{+-}(\alpha_1, \alpha_2) \xrightarrow{|\alpha_1| \to \infty} \frac{\Lambda_1(\alpha_2, \varepsilon)}{\alpha_1},
\]

where

\[
\Lambda_1(\alpha_2, \varepsilon) = \frac{2i f_1 \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)}{(1 + 2\nu_1)} \times \frac{1}{(\varepsilon + i\alpha_2)^{\nu_1 + 1/2}}.
\]

As can be seen in Appendix A, the proof is slightly more subtle than the previous case, and one needs to split the \(\varphi\) integral into two parts, one where \(\cos(\varphi)\) is very small, and one where it is bounded away from zero. The validity of the asymptotic behaviour (4.2) is illustrated in Figure 4.

**Asymptotic behaviour when \(|\alpha_2| \to \infty\) within LHP_2 and \(\alpha_1\) is fixed in UHP_1**  
In this case, we can write \(\alpha_2 = |\alpha_2|e^{i\phi_2}\), and we have

\[
I^\varepsilon_{+-}(\alpha_1, \alpha_2) \xrightarrow{|\alpha_2| \to \infty} \frac{\Lambda_2(\alpha_1, \varepsilon)}{3/2},
\]

where

\[
\Lambda_2(\alpha_1, \varepsilon) = \frac{\beta \sqrt{\pi} \Gamma(\nu_1)e^{-3i\frac{\pi}{4}}}{2\Gamma(\nu_1 + 3/2)} \times \frac{1}{(\varepsilon - i\alpha_1)^{\nu_1}}.
\]

Here again, as can be seen in Appendix A, the proof is subtle and requires a particular split of the \(\varphi\) integral, this time according to whether or not \(\sin(\varphi)\) is close to being zero. The validity of the asymptotic behaviour (4.3) is illustrated in Figure 5.
Remark 1 Even though the results in this section have been derived for $\phi_1 \in (0, \pi)$ and $\phi_2 \in (-\pi, 0)$, they remain valid for $\phi_1 = 0, \pi$ and $\phi_2 = -\pi, 0$; this is easily checked numerically. Hence these asymptotic results can be used when $|\alpha_{1,2}| \to \infty$ along $\mathcal{A}_{1,2}$.

Remark 2 If we let $|\alpha_2| \to \infty$ in (4.2), we obtain a quantity behaving like $O(\alpha_1^{-1}/\alpha_2^{\nu_1+1/2})$, while if we let $|\alpha_1| \to \infty$ in (4.3), we obtain a quantity behaving like $O(\alpha_1^{-\nu_2}/\alpha_2^{3/2})$, both expressions being compatible with the behaviour (4.1), for $|\alpha_1| \propto |\alpha_2|$.

We have hence shown that $I^{\varepsilon}_{+-}$ and $G_{+-}$ have the same asymptotic behaviour at infinity. We mentioned in the introduction that in [2] we needed to make an assumption on the behaviour of $\mathcal{I}[G_{+-}]$ as $|\alpha_2| \to \infty$. In the following section we will precisely characterise this condition and show that $\mathcal{I}[I^{\varepsilon}_{+-}]$ does indeed satisfy it.
5 On the behaviour of $\mathcal{I}[I^\varepsilon_{+-}]$ as $|\alpha_2| \to \infty$

5.1 The integral operator $\mathcal{I}$

Before precisely defining this integral operator, we need to introduce the four-way factorisation of the kernel $K(\alpha_1, \alpha_2)$ discussed in [2]. On $A_1 \times A_2$, the kernel $K$ can be written as

$$K(\alpha) = K_{++}(\alpha)K_{+-}(\alpha)K_{-+}(\alpha)K_{--}(\alpha),$$

where $K_{++}(\alpha)$ is analytic on $D_{++}$, etc. Explicit integral expressions for these factors, which may be evaluated very rapidly, are given in [2], and the following asymptotic behaviour is valid

$$K_{\pm\pm}(\alpha_1, \alpha_2) \underset{|\alpha_2|\to\infty}{\approx} O(1/|\alpha_2|^{1/4}) \text{ for } \alpha \in D_{\pm\pm}. \quad (5.1)$$

It is useful to introduce the brackets $[\ ]_{+\circ}$ and $[\ ]_{+\circ}$ that are Cauchy integral sum-split operators in the $\alpha_1$ and $\alpha_2$ complex planes respectively, defined for a generic function $\Phi$ (analytic on $A_1 \times A_2$) by

$$[\Phi]_{+\circ}(\alpha_1, \alpha_2) = \frac{1}{2\pi i} \int_{A^b(\pi i)} \frac{\Phi(z, \alpha_2)}{(z-\alpha_1)} \, dz \quad \text{and} \quad [\Phi]_{+\circ}(\alpha_1, \alpha_2) = \frac{1}{2\pi i} \int_{A^b(\pi i)} \frac{\Phi(\alpha_1, z)}{(z-\alpha_2)} \, dz,$$

where $A^b$ is a contour that lies just below $A_1$ or $A_2$ as appropriate. This ensures that $[\Phi]_{+\circ}$ and $[\Phi]_{+\circ}$ can be freely evaluated (and are analytic) on $A_{1,2}$.

For the function $\Phi(\alpha_1, \alpha_2)$, the Cauchy integral operator $\mathcal{I}$ is defined as follows:

$$\mathcal{I}[\Phi] = \left[ \frac{K_{+-}(\alpha)}{K_{++}(\alpha)} \right]_{+\circ}\circ_{+\circ}, \quad (5.2)$$

where $K_{-\circ}(\alpha) = K_{--}(\alpha)K_{+-}(\alpha)$. As discussed for example in [2] and [5], the latter function can be written analytically from the $\alpha_1$ factorisation of $K(\alpha)$, and is

$$K_{-\circ}(\alpha_1, \alpha_2) = \frac{1}{\sqrt{\sqrt{k^2 - \alpha_2^2} - \alpha_1}}, \quad (5.3)$$

with a careful choice of branch-cut location (in UHP$_1$ when the function is seen as a function of $\alpha_1$). We also remind the reader that $\alpha_1$ is a constant depending on the incident angles.

In [2], we had to make the assumption that $\mathcal{I}[G_{+-}] \to 0$ as $|\alpha_2| \to \infty$. The aim of this section is to show that this assumption is valid for $\mathcal{I}[I^\varepsilon_{+-}]$.

5.2 A sufficient condition

Because of (5.1), it is clear that

$$\frac{K_{+-}(\alpha_1, \alpha_2)}{K_{++}(\alpha)} \underset{|\alpha_2|\to\infty}{\approx} O(1). \quad (5.4)$$

It is well-known (see e.g. Lemma B.1 of [2]) that if $\Phi(\alpha_1, \alpha_2)$ tends to zero as a power of $|\alpha_2|$ as $|\alpha_2| \to \infty$ on $A_2$, then the sum-split bracket $[\Phi]_{+\circ}$ tends to zero as $|\alpha_2| \to \infty$. Hence for $\mathcal{I}[I^\varepsilon_{+-}]$ to tend to zero as $|\alpha_2| \to \infty$, using (5.4), it is enough to show that

$$\left[ \frac{I^\varepsilon_{+-}}{K_{-\circ}} \right]_{+\circ}(\alpha_1, \alpha_2) \underset{|\alpha_2|\to\infty}{\approx} O(1/|\alpha_2|^{\gamma}), \quad (5.5)$$

for some $\gamma > 0$, which remains a non-trivial task, that will be completed in what follows.
5.3 Proof strategy

By definition of the Cauchy bracket, and the integral $I_{\pm}^{\varepsilon}$, we have

$$\left[ \frac{I_{\pm}^{\varepsilon}}{K_{-o}} \right]_{+o}(\alpha) = \int_{A^b} \Psi(z, \alpha_1, \alpha_2) I_{\pm}^{\varepsilon}(z, \alpha_2) dz,$$

where

$$\Psi(z, \alpha_1, \alpha_2) = \frac{1}{2\pi i(z - \alpha_1)K_{-o}(z, \alpha_2)}.$$

It is interesting to note that the singularities of the integrand of (5.6) in the $z$ UHP are exclusively those of $\Psi(z, \alpha_1, \alpha_2)$ since $I_{\pm}^{\varepsilon}(z, \alpha_2)$ is analytic there. Hence, in the $z$ UHP, the integrand of (5.6) has one simple pole at $z = \alpha_1$ and one branch point at $z = \sqrt{k^2 - \alpha_2^2}$, with a branch-cut going vertically upwards as depicted in Figure 6 (left).

![Singularity map of the integrand of (5.6) in the $z$ UHP](image)

**Figure 6**: Singularity map of the integrand of (5.6) in the $z$ UHP (left) and contour deformation around the pole and branch-cut (right).

It is hence possible to deform the contour from $A^b$ to a contour $\cup_{cut}$ surrounding the branch cut. Assuming\(^1\) for now that $\alpha_1 \neq \sqrt{k^2 - \alpha_2^2}$, the pole at $z = \alpha_1$ is picked up in the process and its contribution must be accounted for, see Figure 6 (right).

Noting that

$$2\pi i \text{Res}_{z=\alpha_1}(\Psi(z, \alpha_1, \alpha_2) I_{\pm}^{\varepsilon}(z, \alpha_2)) = \frac{I_{\pm}^{\varepsilon}(\alpha_1, \alpha_2)}{K_{-o}(\alpha_1, \alpha_2)},$$

we can write

$$\left[ \frac{I_{\pm}^{\varepsilon}}{K_{-o}} \right]_{+o}(\alpha) = \frac{I_{\pm}^{\varepsilon}(\alpha_1, \alpha_2)}{K_{-o}(\alpha_1, \alpha_2)} + I_{\text{cut}}(\alpha_1, \alpha_2),$$

where

$$I_{\text{cut}}(\alpha_1, \alpha_2) = \int_{\cup_{cut}} \Psi(z, \alpha_1, \alpha_2) I_{\pm}^{\varepsilon}(z, \alpha_2) dz.$$

Now using the fact that $\Psi$ changes sign across the cut, and that it is equal to zero at the branch point, we can rewrite this integral in the slightly simpler form

$$I_{\text{cut}}(\alpha_1, \alpha_2) = 2i \int_{0}^{\infty} \Psi \left( \sqrt{k^2 - \alpha_2^2} + it, \alpha_1, \alpha_2 \right) I_{\pm}^{\varepsilon} \left( \sqrt{k^2 - \alpha_2^2} + it, \alpha_2 \right) dt,$$

\(^1\)Since we are interested in the behaviour of this bracket for fixed $\alpha_1$ as $|\alpha_2| \to \infty$, we can make this assumption without loss of generality.
where $\Psi$ is only evaluated on the right-side of its cut.

At this stage, it is useful to note that by (5.3), we have

$$K_{-\circ}(\alpha_1, \alpha_2) \xrightarrow{\alpha_1 \text{ fixed}, |\alpha_2| \to \infty} O(|\alpha_2|^{-1/2}).$$

(5.8)

Using this and the asymptotic result (4.3), we find that the first term in the RHS of (5.7) behaves like

$$\frac{I_\varepsilon^\pm(\alpha_1, \alpha_2)}{K_{-\circ}(\alpha_1, \alpha_2)} \xrightarrow{\alpha_1 \text{ fixed}, |\alpha_2| \to \infty} O\left(\frac{1}{|\alpha_2|}\right),$$

(5.9)

which satisfies the condition (5.5). Moreover, we show in Appendix B that we also have

$$I_{\text{cut}}(\alpha_1, \alpha_2) \xrightarrow{\alpha_1 \text{ fixed}, |\alpha_2| \to \infty} O\left(\frac{1}{|\alpha_2|^\nu_1+1}\right),$$

(5.10)

which also satisfies the condition (5.5). Numerical evaluation of this integral confirms this finding, as illustrated on Figure 7.

![Figure 7](image_url)

**Figure 7:** Log log plot of the absolute value of $I_{\text{cut}}$ for $\alpha_1 = 1 + 3i$, and $\alpha_2$ tends to infinity on the contour $A$. Here, $A$ is parametrised by a real parameter $t$, hence the $A(t)$ notation. $A(t)$ here should be understood to take complex values, but be such that $A(t) \sim t$ as $t \to \infty$. The decay is compared to that of $1/t^{\nu_1+1}$.

In conclusion, using (5.7), (5.9) and (5.10), it is quite clear that as $|\alpha_2| \to \infty$ on $A_2$, we have

$$\left[\frac{I_\varepsilon^\pm}{K_{-\circ}}\right]_{+\circ}(\alpha_1, \alpha_2) = O\left(\frac{1}{|\alpha_2|}\right),$$

meaning that the condition (5.5) is fulfilled for $\gamma = 1$. We can hence conclude this section by saying that

$$\mathcal{I}[I_\varepsilon^\pm] \xrightarrow{|\alpha_2| \to \infty} 0.$$
6 Significance and perspectives

One of the assumptions we had to make in [2], was that both sides of a certain equation tended to zero as their argument went to infinity. This was so that we could apply Liouville’s theorem, which allowed us to constructively derive the two formulae (1.3) and (1.4), the first one including Radlow’s ansatz, and the other being coined the compatibility equation. A sufficient condition for this assumption to be correct is $I_{\epsilon}^{G} \rightarrow 0$ as $|\alpha_2| \rightarrow \infty$. The fact that $I_{\epsilon}^{\epsilon}$, which is, in most respects, very similar to $G_{+-}$, satisfies this hypothesis, gives us some confidence and insight as to why it should be true for $G_{+-}$ itself.

The exact form of the integral $I_{\epsilon}^{\epsilon}$ has the potential to be of great assistance to the design of a scheme to accurately approximate the key unknown function $G_{+-}$. One could for example consider an approximation of the type

$$G_{+-}^{(N)}(\alpha) = C I_{\epsilon}^{\epsilon}(\alpha) T_{+-}(\alpha) \left( 1 + \sum_{j=1}^{N} g_{+-}^{(j)}(\alpha) \right),$$

where $C$ is a constant, $T_{+-}(\alpha)$ is a bounded function (but not decaying to zero) at infinity, and the functions $g_{+-}^{(j)}(\alpha)$ are a set of simple functions (possibly simple poles at given locations, but with unknown residues) that decay to zero at infinity. We must choose $T_{+-}$ and $g_{+-}^{(j)}$ to be analytic in $D_{+-}$. Note that the aim here is similar to that of [1] say, i.e. to approximate functions analytic in a half-plane via Padé approximants.

We could for example choose the $T_{+-}$ function to take the form

$$T_{+-}(\alpha) = \frac{L \alpha_1 - \alpha_2}{\alpha_1 - \alpha_2},$$

for some unknown constant $L$, while the $g_{+-}^{(j)}$ could be chosen to take the form

$$g_{+-}^{(j)} = \frac{R^{(j)}}{(\alpha_1 - a_1^{(j)})(\alpha_2 - a_2^{(j)})},$$

for some specified $(a_1^{(j)}, a_2^{(j)}) \in D_{+-}$, and some unknown residues $R^{(j)}$.

For a given $N$, we will hence have $N + 2$ unknowns: $(C, L, R^{(1)}, \ldots, R^{(N)})$, which will be determined by ensuring that the compatibility equation is satisfied at a set of $N + 2$ collocation points. The implementation of such scheme is beyond the scope of the present work and will constitute the basis of further investigations by the authors.

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This is the simplest of the three different cases to be considered. We will consider that both $|\alpha_1|$ and $|\alpha_2| \to \infty$ within $\mathcal{D}_{+,-}$, in such a way that there exists an $m > 0$ such that $|\alpha_2| = m|\alpha_1|$. As discussed in section 4, since both $\alpha_1$ and $\alpha_2$ are large, we can write $\alpha_{1,2} = |\alpha_{1,2}| e^{i\phi_{1,2}}$ for $\phi_1 \in (0, \pi)$ and $\phi_2 \in (-\pi, 0)$. In this case, we have

$$I^\varepsilon_{+,-}(\alpha_1, \alpha_2) = \int_{2\pi}^{\pi} \frac{f_1\left(\frac{\pi}{2}, \varphi\right)}{\pi} (-i(\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi) + i\varepsilon))^{\nu_1+\frac{3}{2}} d\varphi$$

$$= \frac{1}{|\alpha_2|^{\nu_1+\frac{3}{2}}} \int_{2\pi}^{\pi} \frac{f_1\left(\frac{\pi}{2}, \varphi\right)}{\pi} (-i(e^{i\phi_1} \cos(\varphi) + me^{i\phi_2} \sin(\varphi) + i\varepsilon))^{\nu_1+\frac{3}{2}} d\varphi$$

$$\sim \frac{1}{|\alpha_1|^{\nu_1+\frac{3}{2}}} \int_{2\pi}^{\pi} \frac{f_1\left(\frac{\pi}{2}, \varphi\right)}{\pi} (-i(e^{i\phi_1} \cos(\varphi) + me^{i\phi_2} \sin(\varphi)))^{\nu_1+\frac{3}{2}} d\varphi$$

$$\sim \frac{1}{|\alpha_1|^{\nu_1+\frac{3}{2}}} I^0_{+,-}(e^{i\phi_1}, me^{i\phi_2}) \bigg|_{|\alpha_1| \to \infty} = \mathcal{O}\left(\frac{1}{|\alpha_1|^{\nu_1+\frac{3}{2}}}\right),$$

since the quantity $e^{i\phi_1} \cos(\varphi) + me^{i\phi_2} \sin(\varphi)$ can never be equal to zero and hence the last integral is well defined and independent of $|\alpha_1|$.

This last statement can be proven as follows. Let us assume that $e^{i\phi_1} \cos(\varphi) + me^{i\phi_2} \sin(\varphi) = 0$ for some $\varphi \in \left[\frac{3\pi}{2}, 2\pi\right]$. If $\varphi = 3\pi/2$, then we have $-me^{i\phi_2} = 0$, which is impossible since $m > 0$. If $\varphi = 2\pi$, then we have $e^{i\phi_1} = 0$, which is impossible. Hence our quantity cannot be zero at the
end points of the integration domain. We can hence assume that \( \cos(\varphi) \neq 0 \) and \( \sin(\varphi) \neq 0 \) and rewrite the equality as \( e^{i(\varphi_1 - \varphi_2)} = -m \tan(\varphi) > 0 \). Now taking the imaginary part on both sides, we get \( \sin(\varphi_1 - \varphi_2) = 0 \), implying that \( \varphi_1 = \varphi_2 + n\pi \) for some \( n \in \mathbb{Z} \). Clearly, from the restriction on \( \varphi_{1,2} \), we can only have \( n = -1, 0, 1, 2 \). For \( n = -1, 1 \), we would get \( -1 = -m \tan(\varphi) > 0 \), which is impossible. Hence we have \( n = 0 \) or \( 2 \), i.e. \( \varphi_1 = \varphi_2 \) or \( \varphi_1 = \varphi_2 + 2\pi \). Because of the restriction on \( \varphi_{1,2} \) this imposes \( \varphi_{1,2} = 0 \) or \( \varphi_1 = \pi \) and \( \varphi_2 = -\pi \), but these values are excluded according to the restriction on \( \varphi_{1,2} \). Contradiction.

### A.2 Proof of the asymptotic form (4.2)

We have

\[
I_{\epsilon}^\varphi(\alpha_1, \alpha_2) = \int_{-\pi}^{\pi} \frac{f_1 \left( \frac{\pi}{2}, \varphi \right)}{\left( -i(\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi) + i\epsilon) \right)^{\nu_1 + 3/2}} d\varphi.
\]

We can see that as \( |\alpha_1| \to \infty \), the denominator is dominated by the term involving \( \alpha_1 \cos(\varphi) \) for all \( \varphi \), except when \( \varphi \approx \frac{3\pi}{2} \), where \( \cos(\varphi) \) approaches zero. We can hence split the integral into two parts as \( I_{\epsilon}^\varphi(\alpha_1, \alpha_2) = I_{1,\epsilon}^\varphi(\alpha_1, \alpha_2) + I_{2,\epsilon}^\varphi(\alpha_1, \alpha_2) \), where

\[
I_{1,\epsilon}^\varphi(\alpha_1, \alpha_2) = \int_{-\frac{3\pi}{2} + \delta}^{\frac{3\pi}{2} + \delta} \frac{f_1 \left( \frac{\pi}{2}, \varphi \right)}{\left( -i(\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi) + i\epsilon) \right)^{\nu_1 + 3/2}} d\varphi,
\]

\[
I_{2,\epsilon}^\varphi(\alpha_1, \alpha_2) = \int_{\frac{3\pi}{2} + \delta}^{2\pi} \frac{f_1 \left( \frac{\pi}{2}, \varphi \right)}{\left( -i(\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi) + i\epsilon) \right)^{\nu_1 + 3/2}} d\varphi,
\]

for some small \( \delta = \delta(|\alpha_1|) > 0 \), chosen such that \( |\alpha_1|\delta \to \infty \) as \( |\alpha_1| \to \infty \) and \( \delta \to 0 \) as \( |\alpha_1| \to \infty \). Let us select \( \delta = 1/\sqrt{|\alpha_1|} \) as it will prove to work. Upon making the change of variable \( \psi = \varphi - \frac{3\pi}{2} \), these integrals become

\[
I_{1,\epsilon}^\varphi(\alpha_1, \alpha_2) = \int_{0}^{\delta} \frac{f_1 \left( \frac{\pi}{2}, \frac{3\pi}{2} + \psi \right)}{\left( -i(\alpha_1 \sin(\psi) - \alpha_2 \cos(\psi) + i\epsilon) \right)^{\nu_1 + 3/2}} d\psi,
\]

\[
I_{2,\epsilon}^\varphi(\alpha_1, \alpha_2) = \int_{\delta}^{\pi/2} \frac{f_1 \left( \frac{\pi}{2}, \psi + \frac{3\pi}{2} \right)}{\left( -i(\alpha_1 \sin(\psi) - \alpha_2 \cos(\psi) + i\epsilon) \right)^{\nu_1 + 3/2}} d\psi.
\]

For the first integral, we have

\[
I_{1,\epsilon}^\varphi(\alpha_1, \alpha_2) \sim \int_{0}^{\delta} \frac{f_1 \left( \frac{\pi}{2}, \frac{3\pi}{2} \right)}{\left( -i(\alpha_1 \psi - \alpha_2 + i\epsilon) \right)^{\nu_1 + 3/2}} d\psi.
\]

Since \( |\alpha_1| \to \infty \), remember that we can write \( \alpha_1 = |\alpha_1|e^{i\phi_1} \) with \( \phi_1 \in (0, \pi) \), and making the change of variable \( \theta = |\alpha_1|\psi \) in (A.3) we get

\[
I_{1,\epsilon}^\varphi(\alpha_1, \alpha_2) \sim \int_{0}^{\alpha_1\delta} \frac{f_1 \left( \frac{\pi}{2}, \frac{3\pi}{2} \right)}{\left( -i(e^{i\phi_1} - \alpha_2 + i\epsilon) \right)^{\nu_1 + 3/2}} \frac{d\theta}{|\alpha_1|}.
\]

\[
\sim \int_{\delta |\alpha_1| \to \infty} \frac{f_1 \left( \frac{\pi}{2}, \frac{3\pi}{2} \right)}{\left( -i(e^{i\phi_1} - \alpha_2 + i\epsilon) \right)^{\nu_1 + 3/2}} d\theta.
\]
The latter integral can be recast in the form $\int_0^\infty \frac{d\theta}{(A-B\theta)^\lambda}$, for $A = i\alpha_2 + \varepsilon$, $B = ie^{i\phi_1}$ and $\lambda = \nu_1 + 3/2$. Since we have Re($A$) $> 0$, $A/B \not\in \mathbb{R}$ and $\lambda > 1$, this integral can easily be shown to be equal to $\frac{A^{1-\lambda}}{B(1-\lambda)}$, and hence

$$I_1^{\varepsilon,\delta}(\alpha_1, \alpha_2) \xrightarrow{\alpha_2 \text{ fixed}} \frac{f_1\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)}{|\alpha_1|} \frac{ie^{-i\phi_1}}{(i\alpha_2 + \varepsilon)^{\nu_1+1/2}(\nu_1 + 1/2)}$$

$$= \frac{1}{\alpha_1} \times \frac{2if_1\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)}{(i\alpha_2 + \varepsilon)^{\nu_1+1/2}(2\nu_1 + 1)}.$$  

(A.4)

For the second integral, we can use the fact that $|\alpha_1 \sin(\psi)| > |\alpha_1 \sin(\delta)| \to \infty$ to simplify the denominator and obtain

$$I_2^{\varepsilon,\delta}(\alpha_1, \alpha_2) \xrightarrow{|\alpha_1| \to \infty} \int_0^\pi \frac{f_1\left(\frac{\pi}{2}, \psi + \frac{3\pi}{2}\right)}{(-i(\alpha_1 \sin(\psi)))^{|\nu_1|+3/2}} d\psi$$

$$= \frac{1}{|\alpha_1|^{|\nu_1|+3/2}} \int_0^\pi \frac{f_1\left(\frac{\pi}{2}, \psi + \frac{3\pi}{2}\right)}{(-i\alpha_1 \sin(\psi))^{|\nu_1|+3/2}} d\psi.$$ 

Hence, since by Lemma 1, $0 \leq f_1\left(\frac{\pi}{2}, \psi + \frac{3\pi}{2}\right) \leq f_1\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and because $|\sin(\delta)| \leq |\sin(\psi)|$, we have

$$|I_2^{\varepsilon,\delta}(\alpha_1, \alpha_2)| \xrightarrow{|\alpha_1| \to \infty} \frac{1}{|\alpha_1|^{|\nu_1|+3/2}} \int_0^\pi \frac{f_1\left(\frac{\pi}{2}, \psi + \frac{3\pi}{2}\right)}{|-i\alpha_1 \sin(\psi)|^{|\nu_1|+3/2}} d\psi$$

$$\leq \frac{\pi\frac{f_1\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)}{|\sin(\delta)| |\alpha_1|^{|\nu_1|+3/2}}}{|\sin(\delta)| |\alpha_1|^{|\nu_1|+3/2}} \sim \frac{\pi\frac{f_1\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)}{|\alpha_1|^{|\nu_1|+3/2}}}{|\alpha_1|^{|\nu_1|+3/2}} \delta \to 0 = O\left(\frac{1}{|\alpha_1|^{|\nu_1|+1/2}}\right) = o\left(\frac{1}{|\alpha_1|}\right)$$

since $\frac{\nu_1}{2} + \frac{3}{4} \approx 1.15 > 1$. Hence, overall, $I_2^{\varepsilon,\delta}$ can be neglected to leading order and, using (A.4), we obtain

$$I_+^{\varepsilon,\delta}(\alpha_1, \alpha_2) \xrightarrow{|\alpha_1| \to \infty} \frac{\Lambda_1(\alpha_2, \varepsilon)}{\alpha_1}, \quad \text{where} \quad \Lambda_1(\alpha_2, \varepsilon) = \frac{2if_1\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)}{(\varepsilon + i\alpha_2)^{|\nu_1|+1/2}(1+2\nu_1)},$$

as required.

### A.3 Proof of the asymptotic form (4.3)

We have

$$I_+^{\varepsilon,\delta}(\alpha_1, \alpha_2) = \int_{\frac{3\pi}{2}}^{2\pi} \frac{f_1\left(\frac{\pi}{2}, \varphi\right)}{(-i(\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi) + i\varepsilon))^{\nu_1+3/2}} d\varphi,$$

and we can see that as $|\alpha_2| \to \infty$, the denominator is dominated by the term involving $\alpha_2 \sin(\varphi)$ for all $\varphi$, except when $\varphi \sim 2\pi$, in which region $\sin(\varphi)$ approaches zero. We should hence split the integral into two parts as $I_+^{\varepsilon,\delta}(\alpha_1, \alpha_2) = I_3^{\varepsilon,\delta}(\alpha_1, \alpha_2) + I_4^{\varepsilon,\delta}(\alpha_1, \alpha_2)$, where

$$I_3^{\varepsilon,\delta}(\alpha_1, \alpha_2) = \int_{2\pi-\delta}^{2\pi} \frac{f_1\left(\frac{\pi}{2}, \varphi\right)}{(-i(\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi) + i\varepsilon))^{\nu_1+3/2}} d\varphi,$$  

(A.5)

$$I_4^{\varepsilon,\delta}(\alpha_1, \alpha_2) = \int_{\frac{3\pi}{2}}^{2\pi-\delta} \frac{f_1\left(\frac{\pi}{2}, \varphi\right)}{(-i(\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi) + i\varepsilon))^{\nu_1+3/2}} d\varphi.$$  

(A.6)
for some small $\delta = \delta(\alpha_2) > 0$, chosen such that $|\alpha_2|\delta \to \infty$ as $|\alpha_2| \to \infty$ and $\delta \to 0$ as $|\alpha_2| \to \infty$. For specificity, let us select $\delta = 1/|\alpha_2|^{1/4}$. Upon making the change of variable $\psi = 2\pi - \varphi$, these integrals become

\[
I_3^{\varepsilon, \delta}(\alpha_1, \alpha_2) = \int_0^\delta \frac{f_1(\frac{\pi}{2}, 2\pi - \psi)}{(-i(\alpha_1 \cos(\psi) - \alpha_2 \sin(\psi) + i\varepsilon))^{\nu_1 + 3/2}} d\psi,
\]

\[
I_4^{\varepsilon, \delta}(\alpha_1, \alpha_2) = \int_\delta^{\frac{\pi}{2}} \frac{f_1(\frac{\pi}{2}, 2\pi - \psi)}{(-i(\alpha_1 \cos(\psi) - \alpha_2 \sin(\psi) + i\varepsilon))^{\nu_1 + 3/2}} d\psi.
\]

For the first integral, using Lemma 1, we have

\[
I_3^{\varepsilon, \delta}(\alpha_1, \alpha_2) \approx \begin{cases} 
\int_0^{\delta} \frac{\beta \sqrt{\psi}}{(-i(\alpha_1 - \alpha_2 \psi + i\varepsilon))^{\nu_1 + 3/2}} d\psi & \text{if } \alpha_2 = |\alpha_2| e^{i\phi_2} \text{ and } \theta = |\alpha_2| \psi \\
\int_0^{\alpha_2 \delta} \frac{\beta \sqrt{\theta}}{|\alpha_2|} \frac{1}{\nu_1 + 3/2} d\theta & \text{if } |\alpha_2| \delta \to \infty
\end{cases}
\]

The latter integral can be recast in the form $\int_0^\infty \frac{\sqrt{\theta}}{(A-B\theta)^3} d\theta$, with $A = (-i\alpha_1 + \varepsilon)$, $B = -ie^{i\phi_2}$ and $\lambda = \nu_1 + 3/2$. Since we have $\text{Re}(A) > 0$, $A/B \notin \mathbb{R}$ and $\lambda > 3/2$, this integral can be shown to be equal to $\frac{4^{\lambda} \gamma(\lambda - 3/2)}{(A-B\theta)^{3/2}}$ (see [6] p285, 3.194.3.), and hence

\[
I_3^{\varepsilon, \delta}(\alpha_1, \alpha_2) \approx \begin{cases} 
\frac{\beta}{|\alpha_2|^{3/2}} \frac{(-i\alpha_1 + \varepsilon)^{-\nu_1 - 3/2} \sqrt{\pi} \Gamma(\nu_1)}{2 \left(\frac{i\varepsilon}{-i\alpha_1 + \varepsilon}\right)^{3/2} \Gamma(\nu_1 + 3/2)} & \text{if } |\alpha_2| \to \infty \\
1 & \text{if } |\alpha_2| \to \infty
\end{cases}
\]

For the second integral, since $|\alpha_2 \sin(\psi)| > |\alpha_2 \sin(\delta)| \to \infty$, we have

\[
I_4^{\varepsilon, \delta}(\alpha_1, \alpha_2) \approx \frac{1}{(i\alpha_2)^{\nu_1 + 3/2}} \int_0^{\alpha_2 \delta} \frac{f_1(\pi, 2\pi - \psi)}{\sin(\psi))^{\nu_1 + 3/2}} d\psi
\]

Hence, using Lemma 1, we have

\[
|I_4^{\varepsilon, \delta}(\alpha_1, \alpha_2)| < \frac{f_1(\frac{\pi}{2}, 3\pi/2) \pi}{|\alpha_2|^{\nu_1 + 3/2}} \delta = \frac{\pi f_1(\pi, 3\pi/2)}{2 |\alpha_2|^{\nu_1 + 3/2}} = \mathcal{O}\left(\frac{1}{|\alpha_2|^{3/2}}\right),
\]

since $\frac{3\pi}{4} + \frac{9}{8} \approx 1.72 > \frac{3}{2}$.

Hence, overall, $I_4^{\varepsilon, \delta}$ can be neglected to leading order and, using (A.8), we obtain

\[
I_+^{\varepsilon}(\alpha_1, \alpha_2) \approx \frac{\alpha_1^{\nu_1 + 3/2} \Lambda_2(\alpha_1, \varepsilon)}{\alpha_2^{3/2}}, \text{ where } \Lambda_2(\alpha_1, \varepsilon) = \frac{\beta e^{-3\pi/4} \sqrt{\pi} \Gamma(\nu_1)}{2(\varepsilon - i\alpha_1)^{\nu_1} \Gamma(\nu_1 + 3/2)},
\]

as required.
B  Asymptotic behaviour of $I_{\text{cut}}$

Since we are only interested in the behaviour of $I_{\text{cut}}$ when $|\alpha_2| \to \infty$ and $\alpha_2 \in \mathcal{A}_2$, we can consider $\alpha_2$ to be real here. Let us assume that $\alpha_2 > 0$ and $\alpha_2 \to \infty$; the $\alpha_2$ negative case can be dealt with in a similar fashion. Using the definitions of the functions $K_{-\alpha}(\alpha_1, \alpha_2)$ and $\sqrt{k^2 - \alpha_2^2}$ given in [2], one can show that, on the right shore of the cut, $1/K_{-\alpha}(\sqrt{k^2 - \alpha_2^2} + it, \alpha_2) = \sqrt{i} e^{\frac{3\pi}{4}}$ and that, as $\alpha_2 \to \infty$, $\sqrt{k^2 - \alpha_2^2} \sim i\alpha_2$, we obtain

$$I_{\text{cut}}(\alpha_1, \alpha_2) \approx 2i \int_0^\infty \Psi \left(\sqrt{k^2 - \alpha_2^2} + it, \alpha_1, \alpha_2\right) I_{+\alpha}^\infty \left(\sqrt{k^2 - \alpha_2^2} + it, \alpha_2\right) dt$$

Now make the substitution $t = \alpha_2 u$ to get

$$I_{\text{cut}}(\alpha_1, \alpha_2) \approx \frac{e^{\frac{3\pi}{4}}}{\pi} \int_0^\infty \frac{\sqrt{\alpha_2 u}}{(i\alpha_2 + i\alpha_2 u - \alpha_1)} \frac{f_1(\frac{\pi}{2}, \varphi)}{(\alpha_2 e^{-i\varphi} + \alpha_2 u \cos(\varphi) + \varepsilon)^{\nu_1+\frac{3}{2}}} du$$

Now the denominators are never zero, even when neglecting the small terms involving $1/\alpha_2$, so we have

$$I_{\text{cut}}(\alpha_1, \alpha_2) \approx \frac{e^{\frac{3\pi}{4}}}{\pi} \frac{1}{\alpha_2^{\nu_1+1}} \int_0^\infty \frac{\sqrt{u}}{(i(u + 1) - \alpha_1)} \frac{f_1(\frac{\pi}{2}, \varphi)}{(e^{-i\varphi} + u \cos(\varphi) + \varepsilon)^{\nu_1+\frac{3}{2}}} du.$$  \hspace{1cm} (B.1)

We need to check that the double integral actually exists. There are no convergence problems as $u \to 0$ and no singularity of the integrand for $u \in \mathbb{R}^+$. So, we just need to ensure that the integral converges at $\infty$. For this purpose, it is enough to study the behaviour of $J(u)$ as $u \to \infty$.

The only possible issue occurs when $\cos(\varphi) \approx 0$; to resolve this, consider a small fixed constant $\delta > 0$, and write

$$J(u) = \int_{\frac{3\pi}{2} + \delta}^{\frac{3\pi}{2}} \frac{f_1(\frac{\pi}{2}, \varphi)}{(e^{-i\varphi} + u \cos(\varphi))^{\nu_1+\frac{3}{2}}} d\varphi + \int_{\frac{3\pi}{2} + \delta}^{2\pi} \frac{f_1(\frac{\pi}{2}, \varphi)}{(e^{-i\varphi} + u \cos(\varphi))^{\nu_1+\frac{3}{2}}} d\varphi$$

Let us start by studying $J_2(u)$:

$$|J_2(u)| = \left| \int_{\frac{3\pi}{2} + \delta}^{2\pi} \frac{f_1(\frac{\pi}{2}, \varphi)}{(e^{-i\varphi} + u \cos(\varphi))^{\nu_1+\frac{3}{2}}} d\varphi \right| \approx \left| \int_{\frac{3\pi}{2} + \delta}^{2\pi} \frac{f_1(\frac{\pi}{2}, \varphi)}{(u \cos(\varphi))^{\nu_1+\frac{3}{2}}} d\varphi \right| < \frac{1}{u^{\nu_1+\frac{3}{2}}} \frac{f_1(\frac{\pi}{2}, \frac{3\pi}{2}) \frac{\pi}{2}}{\delta^{\nu_1+\frac{3}{2}}} = O\left(\frac{1}{u^{\nu_1+\frac{3}{2}}} \right).$$
Now consider the slightly more subtle case of $J_1$, and using Lemma 1, we obtain

$$J_1(u) = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2} + \delta} f_1 \left( \frac{\pi}{2}, \varphi \right) \frac{d\varphi}{(e^{-i\varphi} + u \cos(\varphi))^{\nu_1 + 3/2}}$$

$$\approx \int_{\frac{3\pi}{2}}^{\frac{3\pi}{2} + \delta} f_1 \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \frac{d\varphi}{(i + uz)^{\nu_1 + 3/2}}$$

$$\approx \frac{1}{u} \int_{0}^{\delta u} \frac{dx}{(i + x)^{\nu_1 + 3/2}}$$

$$\approx \frac{1}{u} \int_{0}^{\delta u} \frac{dx}{(i + x)^{\nu_1 + 3/2}} = \mathcal{O} \left( \frac{1}{u} \right).$$

Hence, since $\nu_1 + 3/2 > 1$, $J_2(u)$ can be neglected to leading order, and, overall $J(u) = \mathcal{O}(1/u)$. This ensures that the double integral in (B.1) converges. Since it is independent from $\alpha_2$, it is quite clear that

$$I_{\text{cut}} = \mathcal{O} \left( \frac{1}{|\alpha_2|^{\nu_1 + 1}} \right),$$

as claimed in section 5.3, and as confirmed numerically.