Cohomology of Filippov algebras and an analogue of Whitehead’s lemma

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Abstract

We show that two cohomological properties of semisimple Lie algebras also hold for Filippov (n-Lie) algebras, namely, that semisimple n-Lie algebras do not admit non-trivial central extensions and that they are rigid i.e., cannot be deformed in Gerstenhaber sense. This result is the analogue of Whitehead’s Lemma for Filippov algebras. A few comments about the n-Leibniz algebras case are made at the end.
1 Introduction and outlook

In the last years there has been an increasing interest in the applications of various generalizations of the ordinary Lie algebra structure to theoretical physics problems. In these generalizations, which we shall denote generically as $n$-ary algebras, the two entries of the standard Lie bracket are replaced by $n > 2$ entries. There are two main ways of achieving this, depending on how the Jacobi identity (JI) of the ordinary Lie algebras is looked at. The JI can be viewed as the statement that (a) a double Lie bracket gives zero when antisymmetrized with respect to its three entries or that (b) the Lie bracket is a derivation of itself. Both (a) and (b) are equivalent for ordinary Lie algebras and (a) is indeed an identity that follows from the associativity of the composition of the Lie algebra generators.

When a $n$-ary algebra is defined using the characteristic identity that extends property (a) to a $n$-ary bracket, one is led to a generalization denoted higher order Lie algebras or generalized Lie algebras (GLA) $\mathfrak{g}$ [1,2], and the characteristic identity satisfied by its multibracket is called generalized Jacobi identity (GJI). This generalization is natural for $n$ even (for $n$ odd, the r.h.s. of the GJI, rather than being zero, is a larger bracket with $(2n - 1)$ entries [3]). Similar algebras have also been discussed in [4–8]; GLAs may also be considered as a particular case (when there is no violation of the GJI [3]) of the strongly homotopy algebras of Stasheff [9–12]. When possibility (b) is used as the guiding principle, then one is led to the Filippov identity (FI) [13] and correspondingly to $n$-Lie or Filippov algebras $\mathfrak{a}$ [13] (both terms, Filippov and $n$-Lie, will be used indistinctly), for which the characteristic identity is the FI. For $n = 2$, both algebra structures coincide and determine ordinary Lie algebras $\mathfrak{g}$; when $n \geq 3$, the GJI ($n$ even) and the FI become different characteristic identities and define, respectively, generalized Lie algebras $\mathfrak{g}$ (GLA) $\mathfrak{g}$ and $n$-Lie or Filippov algebras (FA) $\mathfrak{a}$.

Filippov algebras [13–16] have recently been found useful in the search for an effective action describing the low energy dynamics of coincident M2-branes or, more specifically, in the Bagger-Lambert-Gustavsson model (BLG) [17–23]. The field theory BLG model contains scalar and fermion fields that take values in a 3-Lie algebra plus gauge fields that are valued in the adjoint representation of the Lie algebra of the automorphisms of the 3-Lie algebra; the model has $\mathcal{N}=8$ supersymmetries. The uniqueness of the euclidean 3-Lie algebra $A_4$ was in fact found in the context of the BLG model, where it follows [24,25] by assuming that the metric needed for the BLG action has to be positive definite, a condition that may be relaxed [23] (see also [26] and references therein). We shall not discuss the BLG and related models here; we shall just mention that the original BGL action was subsequently reformulated [20] without using a three-Lie algebra, and that other models for low energy multiple M2 brane dynamics have appeared (albeit with $\mathcal{N}=6$ rather than $\mathcal{N}=8$ supersymmetries) that do not use a FA structure [27] (see footnote 7). This paper will be devoted instead to some purely mathematical aspects of Filippov algebras. Other specific ternary structures, such as Jordan-Okubo triple systems and others will not be discussed here (see [28–31] and references therein).

It is well known that semisimple Lie algebras neither admit non-trivial central extensions (see e.g. [35]) nor infinitesimal deformations [32,33] so that they are rigid or stable. This is

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1The GLA were called Lie $n$-algebras in [4], where they were independently considered (see also [5]). But, rather than betting the distinction between $n$-Lie algebras ($\equiv$ FA) and and GLAs on the precise location of a single letter ($n$-Lie alg. vs. Lie $n$-alg.), we prefer our higher order Lie algebras or GLA terminology.
so because their cohomology groups \( H_0^2(\mathfrak{g}) \) and \( H_{\rho}^2(\mathfrak{g}) \) are trivial as a result of Whitehead’s Lemma (see [34]; in fact, the \( p \)-th cohomology groups \( H_{\rho}^p(\mathfrak{g}, V) \), where \( V \) is a \( \rho(\mathfrak{g}) \)-module, are zero for \( \rho \) non-trivial\(^2\) and \( p \geq 0 \). The standard proof uses that the Cartan-Killing metric for semisimple algebras can be inverted and then the inverse allows one to show that all \( p \)-cocycles are \( p \)-coboundaries. We prove in this paper that both the triviality of central extensions and the stability of semisimple Lie algebras under deformations also hold for semisimple Filippov algebras, so that these properties hold true for all \( n \geq 2 \). We shall show this by using a route that does not require introducing a generalized Cartan-Killing form for the FA. The reason is twofold. First, the Cartan-Killing bilinear form of a Lie algebra \( \mathfrak{g} \) is defined on its vector space. In contrast, its \( n \)-Lie algebra analogue [15] (see eq. (2.6)), which one might think of using as the Lie algebra Cartan metric to mimic the proof there, is not a bilinear form on \( \mathfrak{g} \) but on their fundamental objects \( \mathfrak{X} \in \wedge^{n-1} \mathfrak{g} \) (Sec. 2 below). However, the semisimplicity criterion for a \( n \)-Lie algebra, which states that the \( 2(n-1) \)-linear Killing form generalization \( k \) [15] in (2.6) is not degenerate (eq. (2.7) below) does not guarantee that \( k \) is non-degenerate as a bilinear form on \( \wedge^{n-1} \mathfrak{g} \). Secondly, all simple \( n > 2 \) Filippov algebras are known [16, 13] and they are few when compared with the plethora of the \( n = 2 \) Cartan classification of simple Lie algebras and, further, all of them have the same general structure. Specifically, the only simple real Filippov algebras are the \((n+1)\)-dimensional \( n \)-Lie algebras of type \( A_{n+1} \) (eq. (3.10)) [16, 13], which may be thought of as \( n > 2 \) generalizations of the \( n = 2 \) \( \text{so}(3) \) and \( \text{so}(1, 2) \) ordinary Lie algebras. We shall take advantage of this fact to show first the triviality of the central extensions and deformations of simple \( n \)-Lie algebras; then, using that any semisimple Filippov algebra is the direct sum of its simple ideals [16], we shall extend the result to semisimple \( n \)-algebras as well.

As already mentioned, the central extension and deformation problems for Lie algebras are formulated in terms of the second Lie algebra cohomology groups (see e.g. [35]) for the trivial and the adjoint action respectively. Non-trivial central extensions are characterized by non-trivial two-cocycles in the Lie algebra cohomology group \( H_0^2(\mathfrak{g}) = Z_0^2(\mathfrak{g})/B_0^2(\mathfrak{g}) \) for the trivial action, whereas non-trivial infinitesimal deformations require \( H_{\rho}^2 \neq 0 \) for \( \rho = \text{ad} \). In the Filippov algebras case, the generalization is not immediate, and in fact we will show that the cocycles responsible for both extensions and deformations may be considered as one-cocycles rather than two-cocycles. The characterization of the cochains and the corresponding cohomology complexes will be given in Secs. 4.1 and 5.1. It will turn out that the cohomology complex adapted to the deformation problem obtained in Sec. 4.1 is essentially equivalent to the one introduced by Gautheron [36] (see also [37–39]), who was the first to consider the full deformation cohomology complex for Nambu algebras.

Nambu algebras are, in fact, a particular case of \( n \)-Lie algebras. Their \( n \)-bracket is provided by the Jacobian determinant of \( n \) functions or Nambu bracket [40], although Nambu did not write the characteristic identity satisfied by his \((n = 3)\) bracket, which is none other than the FI. This was done in [13, 41–44], and Nambu-Poisson structures (N-P) have been much studied

\(^2\)This is no longer true for \( \rho = 0 \); for a simple compact \( \mathfrak{g} \), for instance, the fully antisymmetric structure constants of the Lie algebra always determine a non-trivial three-cocycle.

\(^3\)There is still a parallel, however, if one realizes that in both the \( n = 2 \) and the arbitrary \( n \) cases the Killing metric is a trace form, namely \( \text{Tr}(\text{ad}X \text{ad}Y) \). This relates it to the Lie algebra of inner derivations of \( n \)-Lie algebras \( (n \geq 2) \) in general.
since Nambu’s original paper [40] and Takhtajan general study [43], see [45,42,46–49,37,50–53] (ref. [37] also considers Nambu superalgebras). In fact, since the earlier considerations of p-branes as gauge theories of volume preserving diffeomorphisms [54], the infinite dimensional FAs given by Nambu brackets have reappeared in applications to brane theory [55] and, in particular, in the Nambu three-bracket realization of the mentioned BLG model as a gauge theory associated with volume preserving diffeomorphisms in a three-dimensional space; see, in particular, [56–59].

Much in the same way the Nambu-Poisson structures follow the pattern of FAs, it is also possible to introduce generalized Poisson structures (GPS) [60,1,61] (see further [52,62,63]) whose n-even generalized Poisson brackets (GPB) satisfy the GJI and correspond to the GLAs earlier mentioned. This can also be achieved in the graded case, which corresponds to graded GLAs [64]. Note, however, that besides the two properties that each Poisson generalization share respectively with the GLAs and FAs (skewsymmetry of both n-ary Poisson brackets plus the GJI (FI) for the GP (N-P) structures, respectively), the n-ary brackets of both GPS and of N-PS satisfy an additional condition, Leibniz’s rule. There has been an extensive discussion since the papers by Nambu [40] and Takhtajan [43] about the difficulties of quantizing the N-P structures. We shall not touch the point of quantizing n-ary Poisson structures here and will just refer instead to the papers above and e.g., to [55,61,65–67] and references therein.

The plan of this paper is as follows. Sec. 2 reviews some facts on FAs in a (we hope) transparent notation. In Sec. 3 we show that Kasymov’s analogue [15] of the Cartan-Killing form for a FA $\mathfrak{g}$ (eq. (2.6) below), when viewed as a bilinear form on $\wedge^{n-1}\mathfrak{g}$, is degenerate when $\mathfrak{g}$ is semisimple. This is because the fundamental objects (Sec. 2) $X \in \wedge^{n-1}\mathfrak{g}$ may involve elements of $\mathfrak{g}$ in different simple ideals (something that cannot happen in the Lie algebra case, where the fundamental objects reduce to single elements $X \in \mathfrak{g}$), although it is non-degenerate when $\mathfrak{g}$ is simple. In Sec 4.1 we derive the conditions for the existence and triviality of a central extension of a Filippov algebra, which allows us to define one-cocycle and one-coboundary conditions respectively. By extending them to higher order cochains, this leads us naturally to the expression of the corresponding cohomology complex, which is given explicitly in that section. Sec. 4.2 contains the proof of the triviality of all central extensions of semisimple Filippov algebras. In Sec. 5.1 we derive the cohomological conditions that govern the infinitesimal deformations of FAs, and subsequently we obtain from them the action of the coboundary operator and the cohomology complex for the non-trivial action, which is the relevant one for deformations of FAs; both the left and the right actions appear naturally in its definition. Sec. 5.2 contains the proof of the rigidity of the semisimple Filippov algebras. Sec. 6 is makes some observations relating the FA and $n$-Leibniz algebra cohomologies ($n$-Leibniz algebras share the derivation property of the FI with the FAs but not the total antisymmetry of their $n$-brackets). Finally, Sec. 7 presents some remarks concerning the extension of the above results to $n$-Leibniz algebras.
2 Filippov algebras: some basic definitions and properties

We present in this section some salient features of FA in a form that will be convenient for applications later.

A FA algebra $\mathfrak{G}$ is a vector space endowed with a skew-symmetric, $n$-linear bracket,

$$ (X_1, \ldots, X_n) \in \mathfrak{G} \times \cdots \times \mathfrak{G} \mapsto [X_1, \ldots, X_n] \in \mathfrak{G} $$

(2.1)

that satisfies the FI,

$$ [X_1, \ldots, X_{n-1}, [Y_1, \ldots, Y_n]] = \sum_{a=1}^{n} [Y_1, \ldots, Y_{a-1}, [X_1, \ldots, X_{n-1}, Y_a], Y_{a+1}, \ldots Y_n] , $$

(2.2)

which states that the bracket $[X_1, \ldots, X_{n-1}, ]$, where the last entry is empty, is a derivation of the FA.

Given a basis $\{X_a\}$ of the $\mathfrak{G}$ vector space, the FA is characterized by its structure constants,

$$ [X_{a_1}, \ldots, X_{a_n}] = f_{b_{a_1} \ldots a_n}^b X_b , \quad a_i = 1, \ldots, n . $$

The properties of $n$-Lie algebras have been studied, along the lines of Lie algebra theory, by Filippov [13, 44], Kasymov [14, 15], Ling [16] and others. For instance, a subspace $I \subset \mathfrak{G}$ is an ideal of $\mathfrak{G}$ if

$$ [X_1, \ldots, X_{n-1}, Z] \subset I \quad \forall X \in \mathfrak{G} , \forall Z \in I . $$

The above bracket may be rewritten in the form

$$ [X_1, \ldots, X_{n-1}, Z] := \mathcal{X} \cdot Z \equiv [\mathcal{X}, Z] \equiv \text{ad}_{\mathcal{X}} Z , $$

(2.3)

where $\mathcal{X} \in \wedge^{n-1} \mathfrak{G}$. The objects $\mathcal{X} \in \wedge^{n-1} \mathfrak{G}$ play an important rôle in the theory of FA, and accordingly we shall call them fundamental objects. In fact, the properties of FA are largely determined by them; they also determine derivations that generate an associated Lie algebra. The fundamental objects are characterized by $(n-1)$ elements $(X_1, \ldots, X_{n-1})$ of $\mathfrak{G}$ and are skewsymmetric in them. In terms of these fundamental objects $\mathcal{X}$, the FI may be rewritten as:

$$ \mathcal{X} \cdot [Y_1, \ldots, Y_n] = \sum_{a=1}^{n} [Y_1, \ldots, \mathcal{X} \cdot Y_a, \ldots, Y_n] \quad \text{or} \quad \text{ad}_{\mathcal{X}} [Y_1, \ldots, Y_n] = \sum_{a=1}^{n} [Y_1, \ldots, \text{ad}_{\mathcal{X}} Y_a, \ldots, Y_n] . $$

(2.4)

Thus, the FI just reflects that $\text{ad}_{\mathcal{X}} \equiv \mathcal{X} \cdot \equiv [X_1, \ldots, X_{n-1}, ]$ is a derivation of the FA (which may be called inner since it is determined by elements of $\mathfrak{G}$). For the particular case of an

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4The notation $\mathcal{X} \in \wedge^{n-1} \mathfrak{G}$ reflects that the fundamental object $\mathcal{X} = (X_1, \ldots, X_{n-1}) \in \mathfrak{G} \times \wedge^{n-1} \mathfrak{G}$ is antisymmetric in its arguments and does not imply that $\mathcal{X}$ is a $(n-1)$-multivector obtained by the associative wedge product of vector fields.
ordinary Lie algebra \( g, n = 2 \), \( \mathcal{X} = X \) and thus \( ad_{\mathcal{X}} \in \text{End} \mathfrak{g} \) reduces to the standard adjoint derivative \( ad_{X} \in \text{End} \mathfrak{g} \).

A FA is simple if \([\mathfrak{g}, \ldots, \mathfrak{g}] \neq \{0\} \) and has no ideals different from the trivial ones, \( \{0\} \) and \( \mathfrak{g} \). A \( n \)-Lie algebra is semisimple if it has no solvable ideals, an ideal \( I \subset \mathfrak{g} \) being solvable [13] if the following sequence of ideals,

\[
I^{(0)} := I, \quad I^{(1)} := [I^{(0)}, \ldots, I^{(0)}], \ldots, I^{(s)} := [I^{(s-1)}, \ldots, I^{(s-1)}], \ldots
\]

ends i.e., there exists an \( s \) for which \( I^{(s)} = \{0\} \). Kasymov’s generalization [15] of the Cartan criterion then states that a \( n \)-Lie algebra is semisimple if the following \( (2n - 2) \)-linear generalization of the Cartan-Killing form

\[
k(\mathcal{X}, \mathcal{Y}) = k(X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}) := Tr(ad_{\mathcal{X}} ad_{\mathcal{Y}})
\]

is non-degenerate, i.e. iff

\[
k(Z, \mathfrak{g}, \ldots, \mathfrak{g}, \mathfrak{g}, \ldots, \mathfrak{g}) = 0 \Rightarrow Z = 0
\]

where the \( 2n - 3 \) arguments besides \( Z \) are arbitrary elements of \( \mathfrak{g} \).

It is convenient to introduce a composition law for fundamental objects \( \mathcal{X} = (X_1, \ldots, X_{n-1}), \mathcal{Y} = (Y_1, \ldots, Y_{n-1}), X_i, Y_i \in \mathfrak{g}, i = 1, \ldots, (n-1) \). The composition \( \mathcal{X} \cdot \mathcal{Y} \) is given by the sum of fundamental objects

\[
\mathcal{X} \cdot \mathcal{Y} := \sum_{a=1}^{n-1} (Y_1, \ldots, Y_{a-1}, [X_1, \ldots, X_{n-1}, Y_a], Y_{a+1}, \ldots, Y_{n-1}).
\]

For a \( n = 2 \) FA or ordinary Lie algebra, \( \mathcal{X} \cdot \mathcal{Y} \) reduces to \( X \cdot Y = [X, Y] \). With the above notation, the following Lemma follows from the FI:

**Lemma (Properties of the composition of fundamental objects)**

The dot product of fundamental objects \( \mathcal{X} \) of a \( n \)-Lie algebra \( \mathfrak{g} \) satisfies the relation

\[
\mathcal{X} \cdot (\mathcal{Y} \cdot \mathcal{Z}) - \mathcal{Y} \cdot (\mathcal{X} \cdot \mathcal{Z}) = (\mathcal{X} \cdot \mathcal{Y}) \cdot \mathcal{Z} \quad \forall \mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \wedge^{n-1} \mathfrak{g}.
\]

As a result, the images \( ad_{\mathcal{X}} \) of the fundamental objects by the the adjoint map \( ad : \wedge^{n-1} \mathfrak{g} \to \text{InDer} \mathfrak{g} \) determine (inner) derivations of the FA that satisfy\(^6\)

\[
\mathcal{X} \cdot (\mathcal{Y} \cdot Z) - \mathcal{Y} \cdot (\mathcal{X} \cdot Z) = (\mathcal{X} \cdot \mathcal{Y}) \cdot Z \quad \forall \mathcal{X}, \mathcal{Y}, Z \in \mathfrak{g}, \forall \mathfrak{Z} \in \mathfrak{g}.
\]

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\(^5\)The solvability notion for Lie algebras allows for various generalizations when moving to FAs, \( n > 2 \), because the \( n \)-bracket has more than two entries. For a \( n \)-Lie algebra the notion of \( k \)-solvability was introduced by Kasymov [14] (see also [16]) by taking \( \mathfrak{g}^{(0,k)} = \mathfrak{g}, \mathfrak{g}^{(m,k)} = [\mathfrak{g}^{(m-1,k)}, \ldots, \mathfrak{g}^{(m-1,k)}, \mathfrak{g}, \ldots, \mathfrak{g}] \), where there are \( k \) entries \( \mathfrak{g}^{(m-1,k)} \) at the beginning of the \( n \)-bracket. Filippov’s solvability [13], used above, corresponds to \( k \)-solvability for \( k = n \); \( k \)-solvability is stronger and implies \( n \)-solvability for all \( k \) [16].

\(^6\)Notice that \( Z \) in eq. (2.10) may be replaced in general by \( v \in V \), \( \mathcal{X} \cdot v := \rho(\mathcal{X}) \cdot v \), where \( \rho \) is the action that makes the vector space \( V \) a \( \rho(\mathfrak{g}) \)-module.
Proof: To prove the assertion, it is sufficient to check eq. (2.9). Let us compute first 
\( \mathcal{X} \cdot (\mathcal{Y} \cdot \mathcal{Z}) \). This is given by

\[
\mathcal{X} \cdot (\mathcal{Y} \cdot \mathcal{Z}) = \sum_{i=1}^{n-1} \mathcal{X} \cdot (Z_1, \ldots, Z_{i-1}, [Y_1, \ldots, Y_{n-1}, Z_i], Z_{i+1}, \ldots, Z_{i-1}) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (Z_1, \ldots, Z_{j-1}, [X_1, \ldots, X_{n-1}, Z_j], Z_{j+1}, \ldots, Z_{i-1}, [Y_1, \ldots, Y_{n-1}, Z_i], Z_{i+1}, \ldots, Z_{n-1}) \\
+ \sum_{i=1}^{n-1} (Z_1, \ldots, Z_{i-1}, [X_1, \ldots, X_{n-1}, [Y_1, \ldots, Y_{n-1}, Z_i]], Z_{i+1}, \ldots, Z_{n-1}).
\]

The first term in the r.h.s is symmetric in \( \mathcal{X}, \mathcal{Y} \); hence,

\[
\mathcal{X} \cdot (\mathcal{Y} \cdot \mathcal{Z}) - \mathcal{Y} \cdot (\mathcal{X} \cdot \mathcal{Z}) = \sum_{j=1}^{n-1} (Z_1, \ldots, Z_{j-1}, [X_1, \ldots, X_{n-1}, [Y_1, \ldots, Y_{n-1}, Z_j]] - [Y_1, \ldots, Y_{n-1}, [X_1, \ldots, X_{n-1}, Z_j]], Z_{j+1}, \ldots, Z_{n-1}).
\]

On the other hand, using definition (2.8), we find

\[
(\mathcal{X} \cdot \mathcal{Y}) \cdot \mathcal{Z} = \sum_{j=1}^{n-1} \sum_{i=1}^{n-1} (Z_1, \ldots, Z_{j-1}, [X_1, \ldots, X_{n-1}, Y_i], \ldots, Y_{n-1}, Z_i], Z_{j+1}, \ldots, Z_{n-1}).
\]

Now, using the FI for \([X_1, \ldots, X_{n-1}, [Y_1, \ldots, Y_{n-1}, Z_j]]\), we see that the above expression reproduces (2.11).

Obviously, the above proof carries forward for the simplest case where the fundamental object \( \mathcal{Z} \) in (2.9) is replaced by a FA element \( Z \) as in (2.10). It is sufficient to note that, by the FI,

\[
ad_{\mathcal{X} \cdot \mathcal{Y}} Z = \sum_{i=1}^{n-1} [Y_1, \ldots, Y_{i-1}, [X_1, \ldots, X_{n-1}, Y_i], Y_{i+1}, \ldots, Y_{n-1}, Z] = [X_1, \ldots, X_{n-1}, [Y_1, \ldots, Y_{n-1}, Z]] - [Y_1, \ldots, Y_{n-1}, [X_1, \ldots, X_{n-1}, Z]]
\]

which completes the proof \( \square \)

Note that eq. (2.10) shows, by exchanging \( \mathcal{X} \) and \( \mathcal{Y} \), that \( ad_{\mathcal{X} \cdot \mathcal{Y}} Z = -ad_{\mathcal{Y} \cdot \mathcal{X}} Z \) on any \( Z \in \mathfrak{g} \), and hence that

\[
ad_{\mathcal{Y} \cdot \mathcal{X}} = -ad_{\mathcal{X} \cdot \mathcal{Y}} \quad \text{or, equivalently,} \quad (\mathcal{X} \cdot \mathcal{Y}) \cdot = -(\mathcal{Y} \cdot \mathcal{X}) \cdot ,
\]

where the dots in the last expression should be noted, since the composition of fundamental objects in eq. (2.8) is not commutative, \( \mathcal{X} \cdot \mathcal{Y} \neq -\mathcal{Y} \cdot \mathcal{X} \). It follows from eqs. (2.14), (2.10) that the inner derivations \( ad_{\mathcal{X}} \in \text{End} \mathfrak{g} \) of a FA constitute an ordinary Lie algebra; therefore, we have the following
Proposition Let $\mathfrak{g}$ a $n$-Lie algebra. The inner derivations $\text{ad}_X$ associated to the fundamental objects $\mathcal{X} \in \wedge^{n-1}\mathfrak{g}$ determine an ordinary Lie algebra, the Lie algebra associated with the FA.

For instance, the Lie algebra associated to the $A_4$ euclidean algebra (eq. (3.16) below for $n=3$ and no signs) is $\text{so}(4) = \text{so}(3) \oplus \text{so}(3)$ for the Lorentz case (one $\varepsilon = -1$), an equally simple calculation leads to $\text{so}(1,3)$.

3 On Kasymov’s analogue of the Cartan-Killing form

To prove that the cohomology groups that govern central extensions and the infinitesimal deformations of semisimple $n$-Lie algebras are trivial in analogy to the Lie algebra case, it would be convenient to have that the form $k$ defined in (2.6), viewed as a bilinear form on $\wedge^{n-1}\mathfrak{g}$

$$ k : \wedge^{n-1}\mathfrak{g} \times \wedge^{n-1}\mathfrak{g} \longrightarrow \mathbb{K}, $$

be nondegenerate i.e., that $k(\mathcal{X}, \mathcal{Y}) = 0 \ \forall \mathcal{Y} \in \wedge^{n-1}\mathfrak{g} \Rightarrow \mathcal{X} = 0$. In this way one could try repeating for FAs the proof for Lie algebras. But we see immediately that for $n \geq 3$ this is not so. Any semisimple $n$-Lie algebra is the direct sum of its simple ideals [16],

$$ \mathfrak{g} = \bigoplus_{s=1}^{k} \mathfrak{g}(s) = \mathfrak{g}(1) \oplus \cdots \oplus \mathfrak{g}(k). $$

(3.15)

As a consequence, the $n$-bracket $[\ldots, X, \ldots, Y, \ldots] = 0$ whenever $X$ and $Y$ belong to different simple ideals. Then, if one considers, for instance, $\mathcal{X} = X_1 \wedge \cdots \wedge X_{n-2} \wedge Y$, with $X_1, \ldots, X_{n-2}$ and $Y$ in different ideals, it follows that $\text{ad}_X \in \text{End} \mathfrak{g}$ is identically zero, and so is $k(\mathcal{X}, \mathcal{Y})$ for any $\mathcal{Y}$ without $\mathcal{X}$ itself being zero. In contrast, Kasymov criterion for the semisimplicity of $n$-Lie algebras [15] establishes that a FA is semisimple iff the $2(n-1)$-linear generalization $k$ is nondegenerate in the sense of eq. (2.7).

However, for simple $n$-Lie algebras the form $k$ is nondegenerate on $\wedge^{n-1}\mathfrak{g}$. To show this, we make use of the fact that a real simple $n$-Lie algebra is one of the FA algebras given by [16,13]

$$ [e_1 \ldots \hat{e}_i \ldots e_{n+1}] = (-1)^{i+1} \varepsilon_i e_i \quad \text{or} \quad [e_{i_1} \ldots e_{i_n}] = (-1)^n \sum_{i=1}^{n+1} \varepsilon_i \varepsilon_{i_1} \ldots \varepsilon_{i_n} i e_i, $$

(3.16)

where $\varepsilon_i = \pm 1$ (no sum over the $i$ of the $\varepsilon_i$ factors) just introduce signs that affect the different terms of the sum $i$ and we have used Filippov’s notation to denote the basis $\{e_i\}$.

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7Thus, the simple euclidean $A_4$ determines $SO(4)$ as the gauge group of the $A_4$-based BLG model. The fact that $SO(4)$ is not semisimple was used [20] to reformulate the BLG action with no reference to $A_4$, with matter fields taking values in the ‘bi-fundamental’ representation of the gauge group $SU(2) \otimes SU(2)$, and with the original gauge field replaced by two $SU(2)$-gauge ones. The $N=6$ model in [27], which describes the low energy limit of the dynamics of $N$ M2 branes, is constructed with scalar and fermion fields taking values in the algebra of $U(N) \otimes U(N)$, with a double set of gauge fields taking values in the adjoint, and does not use a Filippov algebra structure. Its connection with a (non fully skewsymmetric, see Sec. 6) three-bracket structure was elucidated in [21].

8Note that we might equally well have used the $\varepsilon_{i_1} \ldots \varepsilon_{i_n}$ without signs $\varepsilon_i$ in the r.h.s. by taking $\varepsilon_{i_1} \ldots \varepsilon_{i_n} = \eta^{i_1} \varepsilon_{i_1} \ldots \varepsilon_{i_n}$ where $\varepsilon_{i_1} \ldots \varepsilon_{i_n} = +1$ and $\eta$ is a $(n+1) \times (n+1)$ diagonal metric with $+1$ and $-1$ in the places indicated by the $\varepsilon_i$’s. We shall keep nevertheless the customary $\varepsilon_i$ factors above as in e.g. [13].
of $\mathcal{G}$. Now, the bilinear form $k$ on $\wedge^{n-1}\mathcal{G}$ is determined by its values on a basis, so taking $\mathcal{X} = (e_{i_1}, \ldots, e_{i_{n-1}})$ and $\mathcal{Y} = (e_{j_1}, \ldots, e_{j_{n-1}})$, and using eq. (2.3), the action of $ad_{\mathcal{X}}ad_{\mathcal{Y}}$ on a basis vector $e_j$ is found to be

$$ad_{\mathcal{X}}ad_{\mathcal{Y}} e_j = \sum_{l,s=1}^{n+1} \varepsilon_l \varepsilon_j \epsilon_{j_1 \ldots j_{n-1} l} \epsilon_{i_1 \ldots i_{n-1} l}^s e_s ,$$

(3.17)

from which we deduce that the trace of $ad_{\mathcal{X}}ad_{\mathcal{Y}}$ is given by

$$k(\mathcal{X}, \mathcal{Y}) = \sum_{l,s=1}^{n+1} \varepsilon_l \varepsilon_s \epsilon_{j_1 \ldots j_{n-1} l} \epsilon_{i_1 \ldots i_{n-1} l}^s .$$

(3.18)

The matrix appearing on the r.h.s. of (3.18), seen as a matrix $k_{(i_1 \ldots i_{n-1})(j_1 \ldots j_{n-1})}$ with indices $(i_1 \ldots i_{n-1})$ and $(j_1 \ldots j_{n-1})$ determined by the fundamental objects above, is clearly diagonal with non-zero elements on the diagonal, for given $(i_1 \ldots i_{n-1})$, the factor $\epsilon_{i_1 \ldots i_{n-1} l}^s$ fixes the remaining indices $l$ and $s$ (and $\varepsilon_l \varepsilon_s$) so that $(j_1, \ldots, j_{n-1})$ has to be a reordering of the $(i_1 \ldots i_{n-1})$ indices. For this reason the form $k$ is diagonal with non-zero elements in it and hence non-degenerate.

4 Central extensions of $n$-Lie algebras

4.1 Cohomology and central extensions of Filippov algebras

Given a Filippov algebra $\mathcal{G}$ with a $n$-bracket $[\ldots]$, we define a central extension $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ by adding a new, central, generator $\Xi$ and modifying the bracket as follows:

$$[\tilde{X}_{a_1}, \ldots, \tilde{X}_{a_n}] := f_{a_1 \ldots a_n}^b \tilde{X}_b + \alpha^1(X_1, \ldots, X_n) \Xi ,$$

(4.19)

$$[\tilde{X}_1, \ldots, \tilde{X}_{n-1}, \Xi] = 0 ,$$

where the $f_{a_1 \ldots a_n}^b$ are the structure constants of the unextended algebra $\mathcal{G}$. One may think of adding more than one central generator, but this will not be needed here for the discussion. Clearly, $\alpha^1$ has to be a $n$-linear and fully skewsymmetric map, $\alpha^1 \in \wedge^{n-1}\mathcal{G}^* \wedge \mathcal{G}^*$, where $\mathcal{G}^*$ is the dual of $\mathcal{G}$; it will be identified with a one-cochain. Since the new bracket for the $\tilde{X}_i$ has to satisfy the FI, this gives a condition on $\alpha^1$. When one of the vectors involved is $\Xi$, the FI is trivially satisfied; when $\Xi$ is absent it follows that

$$[\tilde{X}_1, \ldots, \tilde{X}_{n-1}, [\tilde{Y}_1, \ldots, \tilde{Y}_n]] = \sum_{a=1}^n [\tilde{Y}_1, \ldots, \tilde{Y}_{a-1}, [\tilde{X}_1, \ldots, \tilde{X}_{n-1}, \tilde{Y}_a], \tilde{Y}_{a+1}, \ldots, \tilde{Y}_n] .$$

(4.20)

Using (4.19) and the FI for the original Filippov algebra, this implies

$$\alpha^1(X_1, \ldots, X_{n-1}, [Y_1, \ldots, Y_n]) - \sum_{a=1}^n \alpha^1(Y_1, \ldots, Y_{a-1}, [X_1, \ldots, X_{n-1}, Y_a], Y_{a+1}, \ldots, Y_n) = 0 .$$

(4.21)
This equation (with $Y_n = Z$), written as $(\delta \alpha^1)(\mathcal{X}, Y, Z) = 0$, will provide below the condition that characterizes $\alpha^1 \in \wedge^{n-1} \mathfrak{g}^* \otimes \mathfrak{g}^*$, $\alpha^1 : \mathcal{X} \otimes Z \mapsto \alpha^1(\mathcal{X}, Z)$ as a one-cocycle (for the trivial action of $\mathfrak{g}$ on $\alpha$). It is seen now why becomes natural to call $\alpha^1$ an one-cochain (rather than a two-cochain, as it would be in the Lie algebra cohomology case) and why we made the split $\alpha \in \wedge^{n-1} \mathfrak{g} \otimes \mathfrak{g}$ explicit rather than simply writing $\alpha \in \wedge^n \mathfrak{g}$: the number of the fundamental objects in the arguments of a cochain determines its order. As we shall see shortly, an arbitrary $p$-cochain takes $p(n - 1) + 1$ arguments in $\mathfrak{g}$; a zero-cochain is an element of $\mathfrak{g}^*$.

Let us now construct the cohomology complex relevant for central extensions of $\mathfrak{fa}$. Since $\mathfrak{g}$ does not act on $\alpha^1(\mathcal{X}, Z)$, it will be the $\mathfrak{fa}$ cohomology complex for the trivial action. We define arbitrary $p$-cochains as elements of $\wedge^{n-1} \mathfrak{g}^* \otimes \cdots \otimes \wedge^{n-1} \mathfrak{g}^* \otimes \mathfrak{g}^*$,

$$\alpha^p : (\mathcal{X}_1, \ldots, \mathcal{X}_p, Z) \mapsto \alpha^p(\mathcal{X}_1, \ldots, \mathcal{X}_p, Z),$$

where $\mathcal{X}_1, \ldots, \mathcal{X}_p$ are $p$ fundamental objects. Condition (4.21), which guarantees the consistency of $\alpha$ in eq. (4.19) with the FI (4.20), reads then

$$(\delta \alpha^1)(\mathcal{X}, Y, Z) = \alpha^1(\mathcal{X}, Y \otimes Z) - \alpha^1(\mathcal{X} \cdot Y, Z) - \alpha^1(Y, \mathcal{X} \cdot Z) = 0,$$

(4.23)

where $\mathcal{X} \cdot Z$ and $\mathcal{X} \cdot Y$ were defined in eqs. (2.3) and (2.8). It is now straightforward to extend (4.23) to a whole cohomology complex; $\delta \alpha^p$ will be a $p + 1$ cochain taking arguments on one more fundamental object than $\alpha^p$. This is done by means of the following

**Definition (FA cohomology complex (C*($\mathfrak{g}$), $\delta$) adapted to central extensions)**

Let $\alpha^p \in \wedge^{n-1} \mathfrak{g}^* \otimes \cdots \otimes \wedge^{n-1} \mathfrak{g}^* \otimes \mathfrak{g}^*$ be a $p$-cochain on a $\mathfrak{fa}$. The action of the coboundary operator $\delta$ on arbitrary $p$-cochains ($\alpha^p \in C^p(\mathfrak{g})$) is given by (see [61])

$$(\delta \alpha)(\bar{\mathcal{X}}_1, \ldots, \bar{\mathcal{X}}_{p+1}, Z) =$$

$$\sum_{1 \leq i < j} (-1)^i \alpha(\bar{\mathcal{X}}_1, \ldots, \bar{\mathcal{X}}_i, \ldots, \bar{\mathcal{X}}_{i \cdot}, \bar{\mathcal{X}}_j, \ldots, \bar{\mathcal{X}}_{p+1}, Z)$$

$$+ \sum_{i=1}^{p+1} (-1)^i \alpha(\bar{\mathcal{X}}_1, \ldots, \bar{\mathcal{X}}_i, \ldots, \bar{\mathcal{X}}_{p+1}, \bar{\mathcal{X}}_i \cdot Z).$$

(4.24)

The proof that $\delta^2 = 0$ is analogous to that for the Lie algebra coboundary operator if we think of $\mathcal{X} \cdot Y$ as a commutator, in which case eq. (2.39) plays the rôle of a Jacobi identity (we shall come back to this point in Sec. 6).

The $p$-th cohomology groups are given by $H_p^0(\mathfrak{g}) = Z_p^0(\mathfrak{g}) / B_p^0(\mathfrak{g})$, where $Z_p^0(\mathfrak{g})$ is the group (for the natural addition of cochains) of the $p$-cocycles, $Z_p^0(\mathfrak{g}) = \{ \alpha^p \in C^p(\mathfrak{g}) | \delta \alpha^p = 0 \}$, and $B_p^0(\mathfrak{g})$ is the subgroup of the $p$-coboundaries, $B_p^0(\mathfrak{g}) = \{ \alpha^p \in Z_p^0(\mathfrak{g}) | \alpha^p = \delta \alpha^{p-1}, \alpha^{p-1} \in C^{p-1}(\mathfrak{g}) \}$.

A central extension is actually trivial if it is possible to find new generators $\bar{X}' \in \mathfrak{g}$ from the old ones,

$$\bar{X}' = \bar{X} - \beta(X) \Xi,$$

(4.25)

where $\beta \in \mathfrak{g}^*$ is a zero-cochain, such that they remove $\Xi$ from the r.h.s. of eq. (4.19),

$$[\bar{X}'_{a_1}, \ldots, \bar{X}'_{a_n}] = f^b_{a_1 \ldots a_n} \bar{X}'_b$$

$$= f^b_{a_1 \ldots a_n} \bar{X}_b - \beta([X_{a_1}, \ldots, X_{a_n}]) \Xi.$$  

(4.26)
Comparing the last term above with the original expression in eq. (4.19), we conclude that a central trivial extension is defined by a one-cochain of the form $\alpha^1$ such that

$$\alpha^1(X_1, \ldots, X_n) = -\beta([X_1, \ldots, X_n]) .$$  \hfill (4.27)

This is tantamount to saying that the one-cocycle $\alpha^1$ is actually the one-coboundary generated by the zero-cochain $\beta$, $\alpha^1(\mathcal{X}, Z) = (\delta\beta)(\mathcal{X}, Z)$, $(\delta\beta)(X_1, \ldots, X_{n-1}, Z) = -\beta([X_1, \ldots, X_{n-1}, Z])$, as it is read from (4.24). Clearly, equivalent extensions correspond to one-cocycles that differ in a coboundary. The different central extensions are thus characterized by the elements of $H^1_0(G)$, and the trivial extension corresponds to the zero element of $H^1_0(G)$. This of course recovers the well known Lie algebra cohomology result for $n = 2$: the second cohomology group $H^2_0(g)$ for a Lie algebra $g$ becomes the first one $H^1_0$ when $g$ is viewed as a FA $G$, since for $n = 2$ the fundamental objects are single elements $\mathcal{X} = X$ of $G = g$.

### 4.2 Triviality of the central extensions of semisimple Filippov algebras

We now show that all the central extensions of a semisimple $n$-Lie algebra are trivial. To do so, we shall use the explicit form of the simple $n$-Lie algebras in eq. (3.16) [16, 13] to prove the statement in this case first. Then, the decomposition (3.15) will allow us to extend the result to all semisimple $n$-Lie algebras.

As a previous example, and since the reasonings below may be considered as a $n > 2$ generalization of the $so(3)$ and $so(1, 2)$ Lie algebras, let us consider these $n = 2$ cases first. Their Lie algebra commutators may be jointly expressed as $[X_i, X_j] = \varepsilon_k\epsilon_{ijk}X_k$, $i, j, k = 1, 2, 3$. The values $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ determine $so(3)$,

$$[X_1, X_2] = X_3 , \quad [X_2, X_3] = X_1 , \quad [X_3, X_1] = X_2,$$  \hfill (4.28)

whereas $so(1, 2)$ corresponds to, say, $\varepsilon_1 = \varepsilon_2 = 1$, $\varepsilon_3 = -1$,

$$[X_1, X_2] = -X_3 , \quad [X_2, X_3] = X_1 , \quad [X_3, X_1] = X_2 .$$  \hfill (4.29)

Since both $so(3)$ and $so(2, 1)$ are simple, they do not have non-trivial central extensions by Whitehead’s Lemma. This is easy to check directly. First, any two-cochain $\alpha^2$ is given by its (skewsymmetric) coordinates $\alpha^2_{ij} = \alpha^2(X_i, X_j)$. But this is also a two-cocycle since it satisfies the two-cocycle condition $\alpha^2([X_i, X_j], X_k) - \alpha^2([X_i, X_k], X_j) + \alpha^2([X_k, X_j], X_i) = 0$ (to check this, it suffices to note that, since the antisymmetrization over four indices is zero, $\epsilon_{[ij]kl}\alpha^2_{ijkl} = 0$, which gives $\epsilon_{ijk}\alpha^2_{ijl} - \epsilon_{jik}\alpha^2_{jil} + \epsilon_{ilk}\alpha^2_{ilk} = 0$). But then $\alpha^2$ is a two-coboundary, in fact the two-coboundary generated by the one-cochain $\beta$, $\alpha^2 = \delta\beta$, $(\delta\beta)(X_i, X_j) = -\beta([X_i, X_j]) = \varepsilon_{[ij]}\beta_l$ with $\beta_l = \beta(X_l)$ so that $\alpha^2_{ij} = -\varepsilon_{[ij]}\beta_l$. This may always be satisfied with $\beta_l = \varepsilon^1_l\theta_{ij}\alpha^2_{ij}$. Note, however, that this type of argument cannot be extended to other simple algebras, since for all others $\dim g$ will be larger than three, and the structure constants will not be given in terms of the three-dimensional skewsymmetric tensor.

With this preliminary remark, let us now move to the simple $n$-Lie algebra case.

**Lemma:** Any one-cochain of a simple $n$-Lie algebra is a one-coboundary (and thus a trivial one-cocycle).
Proof: Let $\alpha^1 \in \wedge^{n-1}\mathfrak{g}^* \wedge \mathfrak{g}^*$ be a one-cochain and $\mathfrak{g}$ simple. Given a basis $\{e_i\}_{i=1}^{n+1}$ of $\mathfrak{g}$, $\alpha^1$ is determined by its coordinates, $\alpha^1_{i_1...i_n} = \alpha^1(e_{i_1},...,e_{i_n})$. We now show that, in fact, a one-cochain on a simple $\mathfrak{g}$ is a one-coboundary i.e., that there exists a $\beta \in \mathfrak{g}^*$ such that

$$\alpha^1_{i_1...i_n} = -\beta([e_{i_1} \ldots e_{i_n}]) = -\sum_{k=1}^{n+1} \varepsilon_k e_{i_1...i_n} k \beta_k,$$  \hspace{1cm} (4.30)

where $\beta_k = \beta(e_k)$. Indeed, given $\alpha^1$, the zero-cochain $\beta$ given by

$$\beta_k = -\varepsilon_k \frac{n!}{n!} \sum_{i_1...i_n=1}^{n+1} e^{i_1...i_n} k \alpha^1_{i_1...i_n},$$ \hspace{1cm} (4.31)

has the desired property (4.30):

$$-\beta([e_{i_1} \ldots e_{i_n}]) = -\sum_{k=1}^{n+1} \varepsilon_k e_{i_1...i_n} k \beta_k$$

$$= \sum_{k=1}^{n+1} \varepsilon_k e_{i_1...i_n} k \frac{\varepsilon_k}{n!} \sum_{j_1...j_n=1}^{n+1} e^{j_1...j_n} k \alpha^1_{j_1...j_n}$$

$$= \frac{1}{n!} \sum_{j_1...j_n=1}^{n+1} \varepsilon_k e_{i_1...i_n} j_1...j_n \alpha^1_{j_1...j_n} = \alpha^1_{i_1...i_n}, \hspace{1cm} (4.32)\]$$

which proves the lemma $\Box$

Let now $\mathfrak{g}$ be a semisimple FA, and (3.15) the splitting in its simple components. First, we establish the following simple

Lemma: Let $\alpha^1 \in \wedge^{n-1}\mathfrak{g}^* \wedge \mathfrak{g}^*$ be a one-cocycle on a semisimple $n$-Lie algebra for the $n$-Lie algebra trivial action cohomology defined by eq. (4.24). Then, $\alpha(X_1,\ldots,X_{n-2},Y,Z) = 0$ if $Y$ and $Z$ belong to different ideals.

Proof: Let the different simple ideals be labelled by small gothic letters $\mathfrak{s}$, $\mathfrak{t}$, etc. Let $Z \in \mathfrak{g}(\mathfrak{s})$ and $Y \in \mathfrak{g}(\mathfrak{t})$, $\mathfrak{s} \neq \mathfrak{t}$. Since $\mathfrak{g}(\mathfrak{s})$ is simple, there exist $Z_1,\ldots,Z_n \in \mathfrak{g}(\mathfrak{s})$ such that $[Z_1,\ldots,Z_n] = Z$. We can now use this fact and the cocycle condition in eq. (4.24) to obtain

$$\alpha(X_1,\ldots,X_{n-2},Y,Z) = \alpha^1(X_1,\ldots,X_{n-2},Y,[Z_1,\ldots,Z_n])$$

$$= \sum_{k=1}^{n} \alpha^1([X_1,\ldots,X_{n-2},Y,Z_k],\ldots,Z_n) = 0 \hspace{1cm} (4.33)$$

because $[X_1,\ldots,X_{n-2},Y,Z] = 0$ since $Y \in \mathfrak{g}(\mathfrak{t})$ and $Z \in \mathfrak{g}(\mathfrak{s})$ and $\mathfrak{s} \neq \mathfrak{t} \hspace{1cm} \Box$

Using this lemma, we can now prove the main result of this section:

Theorem All central extensions of semisimple $n$-Lie algebras are trivial.

Proof: Let $\alpha^1 \in \wedge^{n-1}\mathfrak{g}^* \wedge \mathfrak{g}^*$ be a one-cocycle in the FA cohomology for the trivial action, and let $\mathfrak{g}$ be semisimple. Then the theorem follows if $\alpha^1$ is a one-coboundary.
All $X \in \mathfrak{G}$ can be split in a unique way in the form $X = \sum_{s=1}^{k} X_{(s)}$, where $X_{(s)}$ is the component of $X$ in the simple ideal labelled by $s$. Then we have

$$\alpha^1(X_1, \ldots, X_n) = \sum_{s_1 \ldots s_n=1}^{k} \alpha^1(X_{1(s_1)}, \ldots, X_{n(s_n)}) = \sum_{s=1}^{k} \alpha^1(X_{1(s)}, \ldots, X_{n(s)}) , \quad (4.34)$$

where the last equality is due to the previous lemma. Every term in the above expression defines a cochain on the simple $n$-Lie algebra $\mathfrak{G}_{(s)}$, so they are coboundaries i.e. there exist $\beta_{(s)} \in \mathfrak{G}^*_{(s)}$ such that $\alpha^1(X_{1(s)}, \ldots, X_{n(s)}) = -\beta_{(s)}([X_{1(s)}, \ldots, X_{n(s)}])$. This means that there is a zero-cochain $\beta \in \mathfrak{G}^*$,

$$\beta(X) = \sum_{s=1}^{k} \beta_{(s)}(X_{(s)}) , \quad (4.35)$$

that generates $\alpha^1$:

$$- \beta([X_1, \ldots, X_n]) = -\beta \left( \sum_{(s_1) \ldots (s_n)=1}^{k} [X_{1(s_1)}, \ldots, X_{n(s_n)}] \right)$$

$$= -\beta \left( \sum_{(s)=1}^{k} [X_{1(s)}, \ldots, X_{n(s)}] \right) = - \sum_{s=1}^{k} \beta_{(s)}([X_{1(s)}, \ldots, X_{n(s)}])$$

$$= \sum_{s=1}^{k} \alpha^1(X_{1(s)}, \ldots, X_{n(s)}) = \alpha^1(X_1, \ldots, X_n) , \quad (4.36)$$

which concludes the proof $\square$

## 5 Infinitesimal deformations of $n$-Lie algebras

### 5.1 Cohomology complex adapted to deformations of Filippov algebras

In contrast with algebra extensions, which add one or more generators, deformations [32,33] do not increase the dimension of the algebra. The infinitesimally deformed $n$-bracket $[\ldots]_t$ may be written in terms of the original one $[\ldots]$ as follows:

$$[X_1, \ldots, X_n]_t = [X_1, \ldots, X_n] + t\alpha^1(X_1, \ldots, X_n) , \quad (5.37)$$

where now $\alpha^1$ is $\mathfrak{G}$-valued (the added term must belong to $\mathfrak{G}$), and $t$ is the parameter of the infinitesimal deformation [32,33]. The one-cocycle condition for the deformation problem appears when the deformed $n$-bracket in (5.37) is made to satisfy the FI so that the deformed bracket does define a FA at order $t$

$$[X_1, \ldots, X_{n-1}, [Y_1, \ldots, Y_n]]_t = \sum_{a=1}^{n} [Y_1, \ldots, Y_{a-1}, [X_1, \ldots, X_{n-1}, Y_a]_t, Y_{a+1}, \ldots, Y_n]_t . \quad (5.38)$$
In terms of fundamental objects (and setting $Y_n = Z$) the above condition reads

$$[\mathcal{X}, (\mathcal{Y} \cdot Z)_t] = [(\mathcal{X} \cdot \mathcal{Y})_t, Z] + [\mathcal{Y}, (\mathcal{X} \cdot Z)_t]$$  \hspace{1cm} (5.39)

(cf eq. (2.10)). Eq. (5.38) implies, keeping only terms linear in $t$,

$$[X_1, \ldots, X_{n-1}, \alpha^1(Y_1, \ldots, Y_n)] + \alpha^1(X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_n)$$

$$= \sum_{a=1}^{n} [Y_1, \ldots, Y_{a-1}, \alpha^1(X_1, \ldots, X_{n-1}, Y_a), Y_{a+1}, \ldots, Y_n]$$

$$+ \sum_{a=1}^{n} \alpha^1(Y_1, \ldots, Y_{a-1}, [X_1, \ldots, X_{n-1}, Y_a], Y_{a+1}, \ldots, Y_n).$$  \hspace{1cm} (5.40)

This expression may be read as the one-cocycle condition $\delta \alpha^1 = 0$ for the $\mathfrak{G}$-valued cochain $\alpha^1$. In terms of the fundamental objects it may be written, setting again $Y_n = Z$, as

$$\delta \alpha^1(\mathcal{X}, \mathcal{Y}, Z) = ad_x \alpha^1(\mathcal{Y}, Z) - ad_y \alpha^1(\mathcal{X}, Z) - (\alpha^1(\mathcal{X}, \mathcal{Y}), \mathcal{Y}) \cdot Z$$

$$\quad - \alpha^1(\mathcal{X} \cdot \mathcal{Y}, Z) - \alpha^1(\mathcal{Y} \cdot \mathcal{X}, Z) + \alpha^1(\mathcal{Y}, \mathcal{X} \cdot Z) = 0$$  \hspace{1cm} (5.41)

where, for instance for $n = 3$, the term $\alpha^1(\mathcal{X}, \mathcal{Y}) \cdot Y_2$ above is the fundamental object defined by

$$\alpha^1(\mathcal{X}, \mathcal{Y}) \cdot Y := (\alpha^1(\mathcal{X}, \mathcal{Y}), Y_1, Y_2) + (Y_1, \alpha^1(\mathcal{X}, \mathcal{Y}), Y_2)$$

$$= (\alpha^1(\mathcal{X}, Y_1), Y_2) + (Y_1, \alpha^1(\mathcal{X}, Y_2)) \quad [\alpha^1(\mathcal{X}, \mathcal{Y}), Y_i] = \alpha^1(\mathcal{X}, Y_i).$$  \hspace{1cm} (5.42)

The general action of the coboundary operator on an arbitrary $p$-cochain will be given in eq. (5.47) below; we notice at this stage that expression (5.41) involves both the left and right actions of $\mathfrak{G}$ on $\alpha^1$.

An infinitesimal deformation is trivial if there exists a redefinition $X'_i = X_i - t\beta(X_i)$, for some $\mathfrak{G}$-valued zero-cochain $\beta$ that removes the deformating term in eq. (5.37). If so, the first order deformed bracket in terms of the new, primed generators reads

$$[X'_1, \ldots, X'_n] = [X_1, \ldots, X_n]$$

$$\equiv [X_1, \ldots, X_n] - t\beta([X_1, \ldots, X_n]).$$  \hspace{1cm} (5.43)

But, again keeping terms up to order $t$ only, we find that the l.h.s. above gives

$$[X'_1, \ldots, X'_n] = [X_1, \ldots, X_n]$$

$$- t \sum_{a=1}^{n} [X_1, \ldots, X_{a-1}, \beta(X_a), X_{a+1}, \ldots, X_n]$$

$$= [X_1, \ldots, X_n] + t\alpha^1(X_1, \ldots, X_n)$$

$$- t \sum_{a=1}^{n} [X_1, \ldots, X_{a-1}, \beta(X_a), X_{a+1}, \ldots, X_n].$$  \hspace{1cm} (5.44)
Therefore, the infinitesimal deformation given by the $G$-valued one-cocycle $\alpha^1$ is trivial if there exists a $G$-valued zero-cochain $\beta$ such that $\alpha^1 = \delta \beta$ where

$$(\delta \beta)(X_1, \ldots, X_n) := -\beta([X_1, \ldots, X_n]) + \sum_{a=1}^n [X_1, \ldots, X_{a-1}, \beta(X_a), X_{a+1}, \ldots, X_n]. \quad (5.45)$$

This is the expression that defines the one-coboundary generated by $\beta$ in the FA cohomology induced by the deformation problem (eq. [5.47] below for $\alpha^0 = \beta$).

The expression of the one-coboundary may again be formulated in terms of the fundamental objects $\mathcal{X}$, as in the central extensions case, which will allow us to generalize the action of the coboundary operator $\delta$ on an arbitrary cochain $\alpha^p \in C^p(\mathcal{G}, \mathcal{G})$. Explicitly we find, relabelling the zero-cochain $\beta$ as $\alpha^0$,

$$(\delta \alpha^0)(\mathcal{X}, Z) = \mathcal{X} \cdot \alpha^0(Z) - \alpha^0(\mathcal{X} \cdot Z) + (\alpha^0(\mathcal{X}) \cdot \mathcal{X}) \cdot Z. \quad (5.46)$$

As for Lie algebras, it is the characteristic identity, here the FI, that is responsible for the structure of the whole cohomology complex. This leads to a generalization of the Lie algebra cohomology relative to the adjoint action, and it is given by the following

**Definition (Cohomology complex $(C_{ad}^\bullet(\mathcal{G}, \mathcal{G}), \delta)$ for deformations of FA)**

Let $\alpha^p \in C^p(\mathcal{G}, \mathcal{G})$ a $\mathcal{G}$-valued p-cochain, $\alpha^p : \wedge^{(n-1)} \mathcal{G} \otimes \cdots \otimes \wedge^{(n-1)} \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$. The action of the coboundary operator is given by

$$(\delta \alpha^p)(\mathcal{X}_1, \ldots, \mathcal{X}_p, \mathcal{X}_{p+1}, Z) =$$

\[\sum_{1 \leq j < k} (-1)^{j} \alpha^p(\mathcal{X}_1, \ldots, \mathcal{X}_j, \ldots, \mathcal{X}_{k-1}, \mathcal{X}_{k} \cdot \mathcal{X}_k, \mathcal{X}_{k+1}, \ldots, \mathcal{X}_{p+1}, Z)\]

\[+ \sum_{j=1}^{p-1} (-1)^j \alpha^p(\mathcal{X}_1, \ldots, \mathcal{X}_j, \ldots, \mathcal{X}_{p+1}, \mathcal{X}_j \cdot Z)\]

\[+ \sum_{j=1}^{p+1} (-1)^{j+1} \mathcal{X}_j \cdot \alpha^p(\mathcal{X}_1, \ldots, \mathcal{X}_j, \ldots, \mathcal{X}_{p+1}, Z)\]

\[+ (-1)^p(\alpha^p(\mathcal{X}_1, \ldots, \mathcal{X}_p, \mathcal{X}_{p+1}) \cdot \mathcal{X}_{p+1})\cdot Z\]

where, in the last term,

$$\alpha^p(\mathcal{X}_1, \ldots, \mathcal{X}_p, \mathcal{X}_{p+1}) \cdot \mathcal{Y} = \sum_{i=1}^{n-1} (Y_1, \ldots, \alpha^p(\mathcal{X}_{i_1}, \ldots, \mathcal{X}_{i_p}, Y_i), \ldots, Y_{n-1}) . \quad (5.48)$$

The above equations define $(C_{ad}^\bullet(\mathcal{G}, \mathcal{G}), \delta)$ as induced by the deformation problem. It is seen above that both left and right ‘ad’ actions enter into the definition of the coboundary operator. The above cohomological complex may be seen essentially equivalent to the one adapted to deformations introduced by Gautheron [36] for the Nambu-Poisson algebras; see also [37, 39, 38]. Since $H^p_{ad}(\mathcal{G}, \mathcal{G}) = Z^p_{ad}(\mathcal{G}, \mathcal{G})/B^p_{ad}(\mathcal{G}, \mathcal{G})$ it follows that the infinitesimal deformations of Filippov algebras are governed by $H^1_{ad}(\mathcal{G}, \mathcal{G})$.  

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It is not difficult to check that if one moves to the next order in the deformation of $\mathfrak{g}$ by adding $t^2\alpha^{(2)}(X, X_n)$ to the r.h.s. of eq. (5.37), where the superindex two in $\alpha^{(2)}$ refers to the order of the deformation, the FI imposes on the one-cochain $\alpha^{(2)}$ a condition of the form $\gamma(X, Y, Z) = (\delta\alpha^{(2)})(X, Y, Z)$, where $\gamma(X, Y, Z)$ is a two-cocycle which is expressed in terms of $\alpha^1$. Therefore, if $H^2_{ad}(\mathfrak{g}, \mathfrak{g}) \neq 0$, there is an obstruction that prevents extending the infinitesimal deformation to higher orders. As a result, we have the following

**Theorem (Deformations of FA algebras)**

Let $\mathfrak{g}$ be a FA. The first cohomology group $H^1_{ad}(\mathfrak{g}, \mathfrak{g})$ for the above complex governs the infinitesimal deformations of $\mathfrak{g}$. The triviality of this cohomology group, $H^1_{ad} = 0$, is a sufficient condition for the rigidity of a FA. The obstruction to expanding an infinitesimal deformation is given by the non-vanishing of the second cohomology group, $H^2_{ad}(\mathfrak{g}, \mathfrak{g}) \neq 0$.

**Proof:** Contained above □

For $n = 2$, and reverting to the notation that characterizes the order of cochains and cocycles by the number of their Lie algebra arguments, the above theorem recovers the standard result $[32,33]$ for Lie algebras: the infinitesimal deformations are characterized by $H^2_{ad}(\mathfrak{g}, \mathfrak{g})$ and the obstructions to go to higher orders are in the non-vanishing elements of third cohomology group $H^3_{ad}(\mathfrak{g}, \mathfrak{g})$.

### 5.2 Rigidity of semisimple Filippov algebras

In this section we prove that semisimple $n$-Lie algebras are also rigid. The pattern of the proof is similar to the case of the central extensions: first we present the proof for simple FAs and then extend it to the semisimple $n$-Lie algebras. In contrast with the central extensions problem, it is not true now that every one-cochain of a simple $n$-Lie algebra is a coboundary, so it is necessary to characterize the cocycles for simple $n$-Lie algebras, eq. (3.16). Taking a basis of $\mathfrak{g} \{e_i\}_{i=1}^{n+1}$, a one-cochain is defined by

$$\alpha^1(e_{i_1}, \ldots, e_{i_n}) = \sum_{j=1}^{n+1} (\alpha^1)^{ji_1 \ldots i_n} e_j ,$$

in terms of its coordinates $(\alpha^1)^{ji_1 \ldots i_n}$. It will turn out convenient to define new, dual quantities that are easier to manipulate:

$$\epsilon^{i_1 \ldots i_n} (\alpha^1)^{ji_1 \ldots i_n} , \quad (\alpha^1)^{ji_1 \ldots i_n} = \sum_{i=1}^{n+1} \epsilon_{i_1 \ldots i_n} (\alpha^1)^{ji_i} .$$

Using them we now prove the following

**Proposition** Let $\alpha^1$ be a $\mathfrak{g}$-valued one-cochain of a simple $n$-Lie algebra with coordinates $(\alpha^1)^{ji_1 \ldots i_n}$. Then, $\alpha^1$ is a one-cocycle for the above cohomology complex iff $(\alpha^1)^{ji} = (\alpha^1)^{ij}$.
Proof: First, we write the one-cocycle condition \((5.40)\) in terms of basis elements of \(\mathcal{G}\),

\[
0 = [e_{i_1}, \ldots, e_{i_{n-1}}, \alpha^1(e_{j_1}, \ldots, e_{j_{n-1}}, e_k)] \\
+ \alpha^1(e_{i_1}, \ldots, e_{i_{n-1}}; [e_{j_1}, \ldots, e_{j_{n-1}}, e_k]) \\
- \sum_{a=1}^{n-1} \alpha^1(e_{j_1}, \ldots, e_{j_{na-1}}; e_{i_1}, \ldots, e_{i_{n-1}}, e_{j_{na+1}}, \ldots, e_{j_{n-1}}, e_k) \\
- \alpha^1(e_{j_1}, \ldots, e_{j_{n-1}}; e_{i_1}, \ldots, e_{i_{n-1}}, e_{j_{n-1}}) \\
- \sum_{a=1}^{n-1} [e_{j_1}, \ldots, e_{j_{na-1}}; \alpha^1(e_{i_1}, \ldots, e_{i_{n-1}}, e_{j_{na+1}}, \ldots, e_{j_{n-1}}, e_k)] \\
- [e_{j_1}, \ldots, e_{j_{n-1}}; \alpha^1(e_{i_1}, \ldots, e_{i_{n-1}}, e_k)]. \tag{5.51}
\]

Using the commutation relations \((3.16)\) of the simple algebra and that

\[
\alpha^1(e_{i_1}, \ldots, e_{i_{n-1}}, e_k) = \sum_{l=1}^{n+1} e_l (\alpha^1)^l_{i_1 \ldots i_{n-1} k} = \sum_{r,s=1}^{n+1} e_r \varepsilon_{i_1 \ldots i_{n-1} k s} (\alpha^1)^r s \tag{5.52}
\]

by eq. \((5.50)\), eq. \((5.51)\) gives

\[
0 = \sum_{s,l=1}^{n+1} \left( \varepsilon_r \varepsilon_{i_1 \ldots i_{n-1} s l} (\alpha^1)^r s \\
+ \varepsilon_s \varepsilon_{i_1 \ldots i_{n-1} s l} (\alpha^1)^r s \\
- \sum_{a=1}^{n-1} \varepsilon_s \varepsilon_{i_1 \ldots i_{a-1} j_a s j_{a+1} \ldots j_{n-1} k l} (\alpha^1)^r s \\
- \sum_{a=1}^{n-1} \varepsilon_r \varepsilon_{j_1 \ldots j_{a-1} j_a j_{a+1} \ldots j_{n-1} k l} (\alpha^1)^r s \\
- \varepsilon_r \varepsilon_{i_1 \ldots i_{n-1} s l} (\alpha^1)^r s \right). \tag{5.53}
\]

Since this expression translates the condition that \(\delta \alpha^1(\mathcal{X}, \mathcal{Y}, e_k) = 0\), where \(\mathcal{X} \in \wedge^{n-1} \mathcal{G}\), \(\mathcal{Y} \in \wedge^{n-1} \mathcal{G}\) with \(\mathcal{X} = (e_{i_1}, \ldots, e_{i_{n-1}}), \mathcal{Y} = (e_{j_1}, \ldots, e_{j_{n-1}})\), it follows that it must be antisymmetric in the indices \(i\) and also in the indices \(j\) and, indeed, it may be checked explicitly in eq. \((5.53)\) that this is so. Using the antisymmetry in \(i_1, \ldots, i_{n-1}\) and in \(j_1, \ldots, j_{n-1}\), an equivalent expression is obtained by contracting with \(\varepsilon_{i_1 \ldots i_{n+1}}\) and \(\varepsilon_{j_1 \ldots j_{n+1}}\),

\[
\sum_{s,l=1}^{n+1} (\varepsilon_{i_1 i_{n+1}} \varepsilon_{j_1 j_{n+1}} - \varepsilon_{j_1 j_{n+1}} \varepsilon_{i_1 i_{n+1}} - \varepsilon_{i_1 i_{n+1}} \varepsilon_{j_1 j_{n+1}}) (\alpha^1)^s l = 0. \tag{5.54}
\]

Clearly if \((\alpha^1)^s l\) is symmetric then \((5.54)\) is satisfied. Conversely, contracting \((5.54)\) with \(\delta_{j_{n+1}}^k \delta_{i_{n+1}}^r\) we obtain

\[
(n - n^2) ((\alpha^1)^i_{jn} - (\alpha^1)^j_{in}) = 0, \tag{5.55}
\]

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which means that (5.54) holds if and only if \((\alpha^{1})^{ij}\) is symmetric □

Using the previous Proposition it now follows that the simple Filippov \(n\)-Lie algebras are stable in the sense of Gerstenhaber \([32, 33]\)

**Theorem** All infinitesimal deformations of a simple \(n\)-Lie algebra are trivial and therefore simple FAs are rigid.

**Proof:** We have to show that if \(\alpha^{1}\) is a one-cocycle it is a trivial one. Let the one-cochain be characterized by \((\alpha^{1})^{ij}\) as in eq. (5.50). In order to express the one-coboundary condition in terms of \((\alpha^{1})^{ij}\), we rewrite the coboundary condition \((5.45)\) in the basis \(\{e_{i}\}\),

\[
\alpha^{1}(e_{i_{1}}, \ldots, e_{i_{n}}) = \delta \beta(e_{i_{1}}, \ldots, e_{i_{n}}) = -\beta([e_{i_{1}}, \ldots, e_{i_{n}}])
\]

\[
+ \sum_{a=1}^{n} [e_{i_{1}}, \ldots, e_{i_{a-1}}, \beta(e_{i_{a}}, e_{i_{a+1}}, \ldots, e_{i_{n}})], \tag{5.56}
\]

which implies, after using \(\beta(e_{j}) := e_{i}\beta^{i}_{j}, i = 1, \ldots, n + 1\), that

\[(\alpha^{1})^{r_{i_{1}}...i_{n}} = -(-1)^{n} \varepsilon_{s} \varepsilon_{i_{1}...i_{n}}^{s} \beta^{r}_{s}
\]

\[+ (-1)^{n} \sum_{a=1}^{n} \varepsilon_{r} \varepsilon_{i_{1}...i_{a-1}i_{a+1}...i_{n}}^{r} \beta^{s}_{i_{a}}. \tag{5.57}\]

If we now contract this equation with \(\varepsilon^{i_{1}...i_{n}}\), we find that the coboundary condition may be rewritten as \((\alpha^{1})^{rk} = (\delta \beta)^{rk}\) with

\[(\delta \beta)^{rk} = -(-1)^{n}(\varepsilon_{k} \beta^{rk} + \varepsilon_{r} \beta^{kr}) + (-1)^{n} \sum_{s=1}^{n+1} \beta^{s}_{s} \varepsilon_{r} \delta^{rk}. \tag{5.58}\]

Let \(\alpha^{1}\) now be a cocycle. Then, it is generated by the zero-cochain \(\beta\) given by

\[(\beta)^{jk} = -\frac{(-1)^{n}}{2} \left[ \varepsilon_{k} (\alpha^{1})^{jk} - \frac{1}{n-1} \sum_{s=1}^{n+1} \varepsilon_{s} (\alpha^{1})^{s}_{s} \delta^{jk} \right]. \tag{5.59}\]

Indeed, inserting \((5.59)\) into the \(r.h.s\) of \((5.58)\) we get

\[(\delta \beta)^{jk} = \frac{1}{2} \left( (\alpha^{1})^{jk} - \frac{1}{n-1} \varepsilon_{k} \delta^{jk} \sum_{s=1}^{n+1} \varepsilon_{s} (\alpha^{1})^{s}_{s} \right)
\]

\[+ (\alpha^{1})^{kj} - \frac{1}{n-1} \varepsilon_{j} \delta^{jk} \sum_{s=1}^{n+1} \varepsilon_{s} (\alpha^{1})^{s}_{s} \right)
\]

\[- \frac{1}{2} \varepsilon_{j} \delta^{jk} \sum_{s=1}^{n+1} \varepsilon_{s} (\alpha^{1})^{s}_{s} - \frac{n+1}{n-1} \sum_{s=1}^{n+1} \varepsilon_{s} (\alpha^{1})^{s}_{s} \right)
\]

\[= (\alpha^{1})^{jk}. \tag{5.60}\]
since all sums in the expression cancel among themselves and, because \( \alpha^1 \) is a one-cocycle, \( \alpha^{jk} \) is symmetric by the previous Proposition. Therefore every one-cocycle of a simple \( n \)-Lie algebra is trivial and simple FAs are rigid \( \Box \)

Again, this result can be extended to semisimple \( n \)-Lie algebras. By eq. (5.13) every \( X \in \mathfrak{g} \) can be written as a sum \( X = \sum_{s=1}^{t} X_{(s)} \) of components in the different simple ideals, \( X_{(s)} \in \mathfrak{g}_{(s)} \). Then, the result of the action of \( \alpha^1 \) on \( n \) vectors \( X_1, \ldots, X_n \in \mathfrak{g} \) may be written as follows:

\[
\alpha^1(X_1, \ldots, X_n) = \alpha^1 \left( \sum_{s_1=1}^{k} X_{1(s_1)}, \ldots, \sum_{s_n=1}^{k} X_{n(s_n)} \right) = \sum_{s_1 \ldots s_n=1}^{k} \alpha^1(X_{1(s_1)}, \ldots, X_{n(s_n)}) = \sum_{s_1 \ldots s_n=1}^{k} \sum_{l=1}^{k} (\alpha^1)^{(l)}(X_{1(s_1)}, \ldots, X_{n(s_n)}) .
\]

As a result, a one-cochain for a semisimple FA \( \mathfrak{g} \) is determined once the components \( (\alpha^1)^{(l)}(X_{1(s_1)}, \ldots, X_{n(s_n)}) \), with \( (\alpha^1)^{(l)}(X_{1(s_1)}, \ldots, X_{n(s_n)}) \in \mathfrak{g}_l \) and \( X_{i(s_a)} \in \mathfrak{g}_{s_a}, a = 1, \ldots, n \) are known.

**Proposition.** If \( \alpha^1 \) is a one-cocycle of a semisimple \( n \)-Lie algebra, then

\[
(\alpha^1)^{(l)}(X_{1(s_1)}, \ldots, X_{n(s_n)}) = 0
\]

when there are at least three indices among the simple ideals \( \mathfrak{t}, \mathfrak{s}_1, \ldots, \mathfrak{s}_n \) that are different \( i.e. \), when the above expression involves components in at least three different ideals.

**Proof:** Without loss of generality we can choose \( \mathfrak{s}_n \neq \mathfrak{t} \). Since the ideal \( \mathfrak{g}_{s_n} \) is simple, there exist \( Y_{1(s_n)}, \ldots, Y_{n(s_n)} \in \mathfrak{g}_{s_n} \) such that

\[
X_{n(s_n)} = [Y_{1(s_n)}, \ldots, Y_{n(s_n)}] .
\]

Then, the projection on the simple ideal \( \mathfrak{g}_l \) of the cocycle condition (5.40) tells us that

\[
(\alpha^1)^{(l)}(X_{1(s_1)}, \ldots, X_{n(s_n)}) = (\alpha^1)^{(l)}(X_{1(s_1)}, \ldots, X_{n-1(s_{n-1})}, [Y_{1(s_n)}, \ldots, Y_{n(s_n)}]) = -[X_{1(s_1)}, \ldots, X_{n-1(s_{n-1})}, (\alpha^1)^{(l)}(Y_{1(s_n)}, \ldots, Y_{n(s_n)})] + \sum_{a=1}^{n} (\alpha^1)^{(l)}(Y_{1(s_n)}, \ldots, Y_{a-1(s_n)}, X_{1(s_1)}, \ldots, X_{n-1(s_{n-1})}, Y_{a(s_n)}), Y_{a+1(s_n)}, \ldots, Y_{n(s_n)}) + \sum_{a=1}^{n} (\alpha^1)^{(l)}(Y_{1(s_n)}, \ldots, Y_{a-1(s_n)}, [X_{1(s_1)}, \ldots, X_{n-1(s_{n-1})}, Y_{a(s_n)}], Y_{a+1(s_n)}, \ldots, Y_{n(s_n)}) = 0 .
\]

The first and third term in the last equality in (5.63) vanish because we are assuming that at least three different ideals are involved and the only possibility for a non-vanishing result is that \( \mathfrak{t} = \mathfrak{s}_1 = \cdots = \mathfrak{s}_{n-1} \) and \( \mathfrak{s}_1 = \cdots = \mathfrak{s}_{n-1} = \mathfrak{s}_n \) respectively. The second term vanishes because \( \mathfrak{s}_n \neq \mathfrak{t} \) \( \Box \)
The previous Proposition allows us to simplify (5.61) for the one-cocycle \( \alpha^1 \) to the case where one or two different ideals are involved. This means that \( \alpha^1(X_1, \ldots, X_n) \) gives rise to the following terms:

\[
\alpha^1(X_1, \ldots, X_n) = \sum_{s=1}^{k} (\alpha^1)^{s}(X_1(s), \ldots, X_{n(s)})
\]

\[
+ \sum_{s \neq t}^{k} (\alpha^1)^{t}(X_1(s), \ldots, X_{n(s)})
\]

\[
+ \sum_{s \neq t}^{k} \sum_{a=1}^{n} (\alpha^1)^{s}(X_1(s), \ldots, X_{a-1(s)}, X_{a(t)}, X_{a+1(s)}, \ldots, X_{n(s)}) .
\]  

(5.64)

**Proposition.** Let \( X_{n(t)} = [Y_{1(t)}, \ldots, Y_{n(t)}] \). If \( t \neq s \), then

\[
(\alpha^1)^{s}(X_1(s), \ldots, X_{n-1(s)}, X_{n(t)}) = -[X_1(s), \ldots, X_{n-1(s)}], (\alpha^1)^{s}(Y_{1(t)}, \ldots, Y_{n(t)})] .
\]  

(5.65)

**Proof:** First, we notice that all the terms in the last line of (5.64) have the structure of the l.h.s. of (5.65), and therefore eq. (5.65) will apply to all of them. To prove now this relation, we again make use of eq. (5.63), particularized to the case where the a priori different ideals labelled \( t, s_1, \ldots, s_n \) are such that the first \( n \) are equal to \( s \), say, and the \((n+1)\)-th one \( s_n \) is different, say \( t \). Then,

\[
(\alpha^1)^{s}(X_1(s), \ldots, X_{n-1(s)}, X_{n(t)}) =
\]

\[
= (\alpha^1)^{s}(X_1(s), \ldots, X_{n-1(s)}, [Y_{1(t)}, \ldots, Y_{n(t)}])
\]

\[
= - [X_1(s), \ldots, X_{n-1(s)}], (\alpha^1)^{s}(Y_{1(t)}, \ldots, Y_{n(t)})]
\]

\[
+ \sum_{a=1}^{n} [Y_{1(t)}, \ldots, Y_{a-1(t)}], (\alpha^1)^{s}(X_1(s), \ldots, X_{n-1(s)}, Y_{a(t)}, Y_{a+1(t)}, \ldots, Y_{n(t)})
\]

\[
+ \sum_{a=1}^{n} (\alpha^1)^{s}(Y_{1(t)}, \ldots, Y_{a-1(t)}, [X_1(s), \ldots, X_{n-1(s)}, Y_{a(t)}], Y_{a+1(t)}, \ldots, Y_{n(t)}) ,
\]  

(5.66)

and we see that the terms in the last two lines vanish, which proves eq. (5.65) \( \square \)

Using the two previous propositions, we now prove the following

**Theorem.** Let \( (\alpha^1) \) be a one-cocycle for the deformation cohomology of a semisimple \( n \)-Lie algebra \( \mathfrak{g} \), eq. (5.47). Then there exists a zero-cochain \( \beta \) such that

\[
\alpha^1(X_1, \ldots, X_n) = (\delta \beta)(X_1, \ldots, X_n)
\]

\[
= -\beta([X_1, \ldots, X_n]) + \sum_{a=1}^{n} [X_1, \ldots, X_{a-1}, \beta(X_a), X_{a+1}, \ldots X_n] .
\]  

(5.67)

Therefore, any semisimple \( FA \) is rigid.
Indeed, we have, from the definition of $\delta_\beta$ and we know that then $\alpha$ is a cocycle, hence the notation $\beta^{(s)}$. But since $\mathcal{G}_s$ is simple there exists a $\mathcal{G}_s$-valued zero-cochain that takes arguments in $\mathcal{G}_s$, $\beta^{(s)}$, such that eq. (5.43) reads

$$(\alpha^1)^{(s)}(X_{1(s)}, \ldots, X_{n(s)}) = -\beta^{(s)}([X_{1(s)}, \ldots, X_{n(s)}])$$

$$+ \sum_{a=1}^{n} [X_{1(s)}, \ldots, X_{a-1(s)}, \beta^{(s)}(X_{a(s)}), X_{a+1(s)}, \ldots X_{n(s)}].$$

(5.68)

Consider now the $\mathcal{G}_t$-valued one-cochain with arguments in $\mathcal{G}_s$ ($t \neq s$) given by $(\alpha^1)^{(t)}(X_{1(s)}, \ldots, X_{n(s)})$ (eq. (5.64), second line). This part also satisfies the one-cocycle condition since $\alpha^1$ is a cocycle, but the one cocycle condition (5.40) reduces to the terms

$$(\alpha^1)^{(t)}(X_{1(s)}, \ldots, X_{n-1(s)}, [Y_{1(s)}, \ldots, Y_{n(s)}])$$

$$= \sum_{a=1}^{n} (\alpha^1)^{(t)}(Y_{1(s)}, \ldots, Y_{a-1(s)}, [X_{1(s)}, \ldots, X_{n-1(s)}, Y_{a(s)}], Y_{a+1(s)}, \ldots Y_{n(s)}).$$

(5.69)

This is the one-cocycle condition that we considered already for the trivial action (eq. (4.21)), and we know that then $\alpha^1$ may be generated by a zero-cochain $\beta$; namely

$$(\alpha^1)^{(t)}(X_{1(s)}, \ldots, X_{n(s)}) = (\delta \beta^{(t)})(X_{1(s)}, \ldots, X_{n(s)}) = -\beta^{(t)}([X_{1(s)}, \ldots, X_{n(s)}]),$$

(5.70)

where the $\mathcal{G}_t$-valued zero-cochain takes arguments in $\mathcal{G}_s$, hence the notation $\beta^{(t)}$. Using these zero-cochains, we check that $\alpha^1 = \delta \beta$ with

$$\beta(X) := \sum_{s, t=1}^{k} \beta^{(t)}(X_{s(t)}).$$

(5.71)

Indeed, we have, from the definition of $\delta \beta$ (eq. (5.45)),

$$\delta \beta(X_1, \ldots, X_n) = -\beta([X_1, \ldots, X_n])$$

$$+ \sum_{a=1}^{n} [X_1, \ldots, X_{a-1}, \beta(X_a), X_{a+1}, \ldots X_n]$$

$$= -\sum_{s, t=1}^{k} \beta^{(t)}([X_1, \ldots, X_n]^{(s)})$$

$$+ \sum_{a=1}^{n} \sum_{s, t=1}^{k} [X_1, \ldots, X_{a-1}, \beta^{(t)}(X_{a(s)}), X_{a+1}, \ldots X_n]$$

$$= -\sum_{s, t=1}^{k} \beta^{(t)}([X_{1(s)}, \ldots, X_{n(s)}])$$

$$+ \sum_{a=1}^{n} \sum_{s, t=1}^{k} [X_{1(t)}, \ldots, X_{a-1(t)}, \beta^{(t)}(X_{a(s)}), X_{a+1(t)}, \ldots X_{n(t)}].$$

(5.72)
The proof is now completed by checking that the coboundary (5.71) generates the one-cocycle $\alpha^1$ in (5.64) since, as argued there, all other cases give zero. This is achieved by means of the Proposition expressed by eq. (5.65). Using eq. (5.65) in the last terms of (5.64), eq. (5.68) in the first term of (5.64) and eq. (5.70) both in the second term plus in the terms which have now appeared from the r.h.s of (5.65) we obtain that the one-cocycle $\alpha^1$ gives rise to

$$
\alpha^1(X_1, \ldots, X_n) = - \sum_{s=1}^{k} \beta^{(s)}([X_{1(s)}, \ldots, X_{n(s)}])
$$

$$
+ \sum_{s=1}^{k} \sum_{a=1}^{n} [X_{1(s)}, \ldots, X_{a-1(s)}, \beta^{(s)}(X_{a(s)}), X_{a+1(s)}, \ldots X_{n(s)}]
$$

$$
- \sum_{s \neq t}^{k} \beta^{(t)}([X_{1(s)}, \ldots, X_{n(s)}])
$$

$$
+ \sum_{s \neq t}^{k} \sum_{a=1}^{n} [X_{1(s)}, \ldots, X_{a-1(s)}, \beta^{(s)}(Y_{a1(t)}, \ldots, Y_{an(t)}), X_{a+1(s)}, \ldots X_{n(s)}]
$$

$$
= - \sum_{s, t=1}^{k} \beta^{(t)}([X_{1(s)}, \ldots, X_{n(s)}])
$$

$$
+ \sum_{s, t=1}^{k} \sum_{a=1}^{n} [X_{1(s)}, \ldots, X_{a-1(s)}, \beta^{(s)}(Y_{a(t)}), X_{a+1(s)}, \ldots X_{n(s)}],
$$

(5.73)

since, again, $[Y_{a1(t)}, \ldots, Y_{an(t)}] = X_{a(t)}$, which allows us to join the two double sums in the second and fourth lines above into a single one. This reproduces eq. (5.72) so that $\alpha^1 = \delta \beta$, which completes the proof $\square$

6 An observation on $n$-Leibniz algebras, FAs, and cohomology

Consider the case of ordinary or $n = 2$ Leibniz algebras $\mathcal{L}$ [68–71]. These algebras share with the Lie algebras a form of the JI identity, but not the anticommutativity of the two-bracket. As a result, their characteristic identity is the Leibniz identity, which retains the aspect (b) of the JI for Lie algebras mentioned in the Introduction. Specifically, a Leibniz algebra is a vector space $\mathcal{L}$ endowed with a bilinear operation $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ that satisfies the Leibniz identity

$$
[X, [Y, Z]] = [[X, Y], Z] + [Y, [[X, Z]]] \quad \forall X, Y, Z \in \mathcal{L},
$$

(6.74)

Actually, this defines a left Leibniz algebra. There is a right counterpart when the equation above is modified to correspond to a right derivation: the right Leibniz identity reads $[[X, Y], Z] = [[[X, Z], Y] + [X, [Y, Z]]$ and accordingly defines a right Leibniz algebra. Obviously, when the bracket is anticommutative, the Leibniz algebra $\mathcal{L}$ becomes a Lie algebra $g$ and both the left and right Leibniz identities become one and the same Lie algebra JI.

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Similarly, $n$-Leibniz algebras $\mathcal{L}$ [37, 72] (see also [38]) share with the FAs the derivation property expressed by the $n$-Leibniz identity, which follows the pattern of the FI of eq. (2.2), where now the brackets are $n$-Leibniz brackets and thus not fully antisymmetric. Thus, their characteristic identity has still the structure of eq. (2.9) which, to stress the similarity with the left character of the Leibniz identity above we shall write in the form

$$\mathcal{X} \cdot (Y \cdot Z) = (\mathcal{X} \cdot Y) \cdot Z + Y \cdot (\mathcal{X} \cdot Z) \quad \forall \mathcal{X}, Y, Z \in \otimes^{n-1} \mathcal{L},$$

where the $n$-bracket involved in the definition of the composition of fundamental objects (eq. (2.8)) is now the $n$-Leibniz bracket in $\mathcal{L}$. It is clear that the above defines a (left) $n$-Leibniz algebra. FAs $\mathfrak{G}$ may be viewed as as a particular case of $n$-Leibniz algebras $\mathcal{L}$, namely those with a fully antisymmetric $n$-bracket and fundamental objects in $\wedge^{n-1} \mathfrak{G}$ rather than in $\otimes^{n-1} \mathcal{L}$.

When we were discussing the various FA cohomology complexes, the essential ingredients in their definition were the FI and the specific action of $\mathfrak{G}$ on the cochains; in particular, the skewsymmetry of the FA $n$-bracket was not needed for the nilpotency of $\delta$ in, say, eq. (4.24). As a result, this $\delta$ also defines a coboundary operator for the cohomology of a $n$-Leibniz algebra $\mathcal{L}$, in which the $p$-cochains are now elements $\alpha \in (\otimes (n-1) \mathcal{L}^*) \otimes \times (\otimes (n-1) \mathcal{L}^*) \otimes \mathcal{L}^* = \otimes^{p(n-1)+1} \mathcal{L}^*$. The key ingredient that guarantees the nilpotency of the coboundary operator is still the (left) identity that follows from eq. (2.2), which implies eqs. (2.9), (2.10), etc. The essential difference between the $n$-Lie algebra and $n$-Leibniz algebra cohomology complexes for the trivial representation, unimportant for the nilpotency of $\delta$, is that $\mathcal{X} \in \wedge^{n-1} \mathfrak{G}$ for a $n$-Lie algebra and $\mathcal{X} \in \otimes^{n-1} \mathcal{L}$ for a Leibniz algebra, and the fact the brackets that appear in the composition of fundamental objects in eq. (2.8) are, respectively, $n$-Lie and $n$-Leibniz brackets.

Thus, with the appropriate changes in the definition of the $p$-cochain spaces $C^p$, the coboundary operator in eq. (4.24) defines the corresponding cohomologies for $n$-Lie $\mathfrak{G}$ and $n$-Leibniz $\mathcal{L}$ algebras adapted to the central extension problem, which corresponds to the trivial action. Similar considerations apply to the $n$-Leibniz algebra cohomology adapted to the deformation problem already discussed for the FAs (eqs. (5.47), (5.48)), but we shall not consider this further here. For an ordinary Leibniz algebra $\mathcal{L}$, for instance, generalizing to the case where the action is given through a representation [68, 71] $\rho$ on $\mathcal{A}$ and reverting (since $n = 2$) to the notation where $p$ indicates the number of algebra elements on which $\alpha^p$ takes arguments, $\alpha^p \in C^p (\mathcal{L}, \mathcal{A}) = \text{Hom}(\otimes^p \mathcal{L}, \mathcal{A})$, eq. (5.47) leads to

$$(\delta \alpha^p)(X_1, \ldots, X_p, X_{p+1}) =$$

$$\sum_{1 \leq j < k} (-1)^j \alpha^p(X_1, \ldots, \hat{X}_j, \ldots, X_{k-1}, [X_j, X_k], X_{k+1}, \ldots X_{p+1})$$

$$+ \sum_{j=1}^{p} (-1)^{j+1} \rho(X_j) \cdot \alpha^p(X_1, \ldots, \hat{X}_j, \ldots, X_{p+1})$$

$$+ (-1)^{p+1} \alpha^p(X_1, \ldots, X_p) \cdot \rho(X_{p+1}) ,$$

which coincides with the coboundary operator for the Leibniz algebra cohomology complex $(C^\bullet (\mathcal{L}, \mathcal{A}), \delta)$ [69, 68, 72] (there given for right Leibniz algebras). We note that if $\rho$ is a

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9This is the case of the three algebras used in [22] in the context of the BLG model.
symmetric representation \([68,71]\), the \(\rho(X_{p+1})\) in the last term above may be moved to the left and the resulting contribution may then be added as one more term to the third line by enlarging the range of the sum. The resulting expression has then the same form of the expression that gives the action of the Lie algebra cohomology coboundary operator on \(p\)-cochains valued on a \(\rho(\mathfrak{g})\)-module (see, e.g. \([35]\)).

Thus, the FA cohomologies defined by the coboundary operators \((4.24)\) that define the FA cohomology complexes (and the corresponding homology) constitute simply the translation of the \(n\)-Leibniz algebra \(\mathcal{L}\) cohomology complexes to the \(n\)-Lie algebra \(\mathfrak{g}\) case. In fact, the previous discussion shows that since Leibniz algebras largely underly the structural cohomological properties of the FAs, the FA cohomology complexes could also have been found from those for the \(n\)-Leibniz algebras by demanding full skewsymmetry for the \(n\)-Leibniz bracket to become the \(n\)-bracket of a FA, and by modifying accordingly the definition of the cochains to account for the skewsymmetry of the fundamental objects of a FA.

We conclude this section with a remark. Let \(\mathcal{L}\) be an \(n\)-Leibniz algebra. The composition \(\mathcal{X} \cdot \mathcal{Y}\) of fundamental objects, rewritten as \(\mathcal{[X,Y]}\), has the properties of a (non-antisymmetric) Leibniz algebra commutator. Indeed, eq. \((2.9)\) also holds for \(\mathcal{L}\), and with the notation \(\mathcal{X} \cdot \mathcal{Y} \equiv \mathcal{[X,Y]}\) it takes the form

\[
\mathcal{[X},[\mathcal{Y},\mathcal{Z}]] = [[\mathcal{X}, \mathcal{Y}], \mathcal{Z}] + [\mathcal{Y}, [\mathcal{X}, \mathcal{Z}]] , \quad \mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \otimes^{n-1}\mathcal{L} ,
\]

(6.77)

where \(\mathcal{[X},\mathcal{Y]} \neq -[\mathcal{Y},\mathcal{X}]\) is a non-antisymmetric two-bracket. Comparing with eq. \((6.74)\), we see that \((6.77)\) defines an ordinary (left) Leibniz algebra where the two entries in \([ , ]\) are fundamental objects \(\mathcal{X} \in \otimes^{n-1}\mathcal{L}\). Hence, given an \(n\)-Leibniz algebra \(\mathcal{L}\), the linear space of the fundamental objects endowed with the dot operation \((2.8)\) which defines a non-antisymmetric two-bracket \(\mathcal{[X,Y]} \equiv \mathcal{X} \cdot \mathcal{Y}\), becomes a (here left) Leibniz algebra \([37]\), the ordinary Leibniz algebra \(\mathcal{L}\) associated with a \(n\)-Leibniz algebra \(\mathcal{L}\).

When the fundamental objects are those of a FA, \(\mathcal{X}, \mathcal{Y} \in \wedge^{n-1}\mathfrak{g}\), and the \(n\)-bracket involved in the definition of \(\mathcal{X} \cdot \mathcal{Y}\) is therefore the fully antisymmetric bracket of a \(n\)-Lie algebra, the resulting \(\mathcal{L}\) is the Leibniz algebra associated with the Filippov algebra \(\mathfrak{g}\). For \(n=2\), the Leibniz algebra \(\mathcal{L}\) associated with the \(n=2\) FA \(\mathfrak{g}\) is in fact an ordinary Lie algebra \(\mathfrak{g}\) since the FA bracket is skewsymmetric and the FA itself is an ordinary Lie algebra.

## 7 \ Final comments

We have proved the analogue of Whitehead’s lemma for \(n\)-Lie algebras: semisimple Filippov algebras cannot be centrally extended in a non-trivial way and furthermore they are rigid because \(H^1(\mathfrak{g}, \mathfrak{g}) = 0\). Actually, Witehead’s lemma for ordinary Lie algebras is more general since it states that the Lie algebra cohomology groups for a non-trivial action \((\rho \neq 0)\) are trivial, \(H_p^\rho(\mathfrak{g}, V) = 0 \forall p \geq 0\), when \(\mathfrak{g}\) is semisimple. The analogous proof for Filippov algebras \(\mathfrak{g}\), using our procedure, would require proving first the triviality of the corresponding higher order cohomology groups for simple \(n\)-Lie algebras. This is much more involved, but the fact that the analogue \([15]\) of the Cartan-Killing form for a simple \(\mathfrak{g}\) is non-degenerate when considered as a bilinear form on \(\wedge^{n-1}\mathfrak{g}\), could help in the analysis of all these higher order cohomology groups, which we conjecture to be also trivial.
Another extension of our results would be to consider the case of Leibniz's $n$-algebras $\mathcal{L}$ mentioned in Sec. 6, since their cohomological structure, being based on the $n$-Leibniz identity, is similar. However, since the proof of the triviality of the relevant FA cohomology groups relies heavily on the skewsymmetry of the structure constants of the simple FAs, one would not expect e.g. their rigidity property to extend automatically to the corresponding Leibniz algebra case. In fact, it has been shown [73] that the simple euclidean three-Lie algebra $A_4$ may be infinitesimally deformed as a three-Leibniz algebra of a specific type. This is not surprising at the light of the above discussion, and indeed this increase of possibilities for deformations has been observed already for ordinary Lie algebras when they are treated as Leibniz algebras [74]; in other words, the Leibniz algebra deformations of a Lie algebra are richer. However, the mentioned Leibniz deformability of the simple $A_4$ euclidean algebra may not extend to the other simple FAs (the case $n=3$ is special), which may be rigid under this type of $n$-Leibniz algebra deformations [75].

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