Manifestly scale-invariant regularization
and quantum effective operators

D. M. Ghilencea

a Theory Division, CERN, 1211 Geneva 23, Switzerland
b Theoretical Physics Department, National Institute of Physics
and Nuclear Engineering (IFIN-HH) Bucharest 077125, Romania

Abstract

Scale invariant theories are often used to address the hierarchy problem, however the regularization of their quantum corrections introduces a dimensionful coupling (dimensional regularization) or scale (Pauli-Villars, etc) which break this symmetry explicitly. We show how to avoid this problem and study the implications of a manifestly scale invariant regularization in (classical) scale invariant theories. We use a dilaton-dependent subtraction function $\mu(\sigma)$ which after spontaneous breaking of scale symmetry generates the usual DR subtraction scale $\mu(\langle\sigma\rangle)$. One consequence is that “evanescent” interactions generated by scale invariance of the action in $d = 4 - 2\epsilon$ (but vanishing in $d = 4$), give rise to new, finite quantum corrections. We find a (finite) correction $\Delta U(\phi, \sigma)$ to the one-loop scalar potential for $\phi$ and $\sigma$, beyond the Coleman-Weinberg term. $\Delta U$ is due to an evanescent correction $(\propto \epsilon)$ to the field-dependent masses (of the states in the loop) which multiplies the pole $(\propto 1/\epsilon)$ of the momentum integral, to give a finite quantum result. $\Delta U$ contains a non-polynomial operator $\sim \phi^4/\sigma^2$ of known coefficient and is independent of the subtraction dimensionless parameter ($z$). A more general $\mu(\phi, \sigma)$ is ruled out since, in their classical decoupling limit, the visible sector (of the higgs $\phi$) and hidden sector (dilaton $\sigma$) still interact at the quantum level, thus the subtraction function depends on the dilaton only. The method is useful in models where preserving scale symmetry at quantum level is important.

*E-mail: dumitru.ghilencea@cern.ch
1 Introduction

There has recently been a renewed interest in the scale invariance symmetry to address the hierarchy or the cosmological constant problems. Scale symmetry is not a symmetry of the real world since it requires that no dimensionful parameters be present in the Lagrangian. One can impose this symmetry on the Lagrangian at the classical level, to forbid any mass scales. At the quantum level, the anomalous breaking of scale symmetry is in general expected. This is because regularization of the loop corrections breaks this symmetry explicitly, either by introducing a dimensionful coupling as in dimensional regularization (DR) or a mass scale (Pauli-Villars, cutoff regularizations, etc). Therefore, the presence of a subtraction (or renormalization) scale $\mu$, breaks explicitly the (classical) scale invariance of the theory and ruins the symmetry one actually wants to study. In DR the scale $\mu$ relates the dimensionless couplings to the dimensionful ones, once the theory is continued analytically from $d = 4$ to $d = 4 - 2\epsilon$ dimensions. For example, the quartic coupling $(\lambda_{\phi})$ of a Higgs-like scalar field $\phi$ acquires a mass dimension, since

$$\lambda_{\phi} = \mu^{2\epsilon} \left( \lambda_{\phi}^{(r)} + \sum_{n} a_{n}/\epsilon^{n} \right)$$

where renormalized $\lambda_{\phi}^{(r)}$ is dimensionless. Thus, the DR scale $\mu$ breaks scale invariance.

To avoid this problem in theories in which scale-invariance must be preserved during regularization, we use a scheme in which the couplings become field-dependent, something familiar in string theory. Indeed, one can replace the scale $\mu$ by a function $\mu(\sigma)$, $\mu \rightarrow \mu(\sigma)$ (also recent [2]), where the field $\sigma$ is the dilaton, for example $\mu(\sigma) \propto \sigma$. Of course, $\sigma$ must subsequently acquire a non-zero (finite) vev, otherwise this relation does not make sense due to vanishing power ($\epsilon \rightarrow 0$) in eq. (1). One cannot just replace $\mu$ by the vev of the field $\sigma$, since this would simply bring back the original problem. One therefore needs a spontaneous breaking of the scale symmetry. When the (dynamical) field $\sigma$ acquires a vev, scale invariance is broken with the dilaton $\sigma$ as its Goldstone mode. This can happen in a framework which includes (conformal) gravity in which the dilaton vev is related to the Planck scale. In this paper we shall not include gravity, but assume the dilaton acquires a vev spontaneously (fixed e.g. by Planck scale physics) and search for solutions $\langle \sigma \rangle \neq 0$.

The goal of this paper is to investigate the quantum implications of a manifestly scale-invariant regularization of a theory that is classically scale invariant, using a dilaton-dependent subtraction “scale”. This is important since scale invariant theories, see e.g. [3]-[22], do not seem to be renormalizable [23, 24], in which case the regularization of the loops should preserve all initial symmetries to avoid regularization artefacts [25]. This motivated our work, relevant for theories which study scale invariance at quantum level.

This paper continues a previous study [2], with notable differences and new results.

---

1 The exact S-matrix is renormalization scale independent. But in perturbation theory we truncate the series, so there is always a residual renormalization scheme dependence, which must be be minimised.

2 We also consider a more general dependence $\mu = \mu(\phi, \sigma)$ where $\phi$ is our scalar (higgs-like) field.
outlined below. Consider a scale invariant theory of higgs-like $\phi$ and dilaton $\sigma$ (other fields may be present). In “usual” DR, quartic couplings become dimensionless by replacing $\lambda \rightarrow \mu^2 \lambda$, see eq.(1) and this changes the scalar potential $V(\phi, \sigma)$. For a field-dependent subtraction function $\mu(\sigma)$ this change is $V(\phi, \sigma) \rightarrow \tilde{V} = \mu^2(\sigma) V(\phi, \sigma)$ which is scale invariant in $d=4-2\epsilon$ (as it should). $\tilde{V}$ acquired new “evanescent” interactions\footnote{These are defined as interactions absent in $d=4$ ($\epsilon = 0$) but generated in $d=4-2\epsilon$ by scale invariance.} due to the field dependence of $\mu(\sigma)$. This step generates new, finite corrections at the quantum level.

For example, we obtain a scale-invariant one-loop potential $U(\phi, \sigma)$ which contains a finite (quantum) correction $\Delta U(\phi, \sigma)$ beyond the “usual” Coleman-Weinberg term\footnote{These masses are $\partial^2\tilde{V}/\partial \alpha \partial \beta$, $\alpha, \beta = \phi, \sigma$ and contain corrections $\propto \epsilon$ from derivatives of $\mu(\sigma)$, see later.} for the higgs $\phi$ and dilaton $\sigma$. $\Delta U$ is a new correction overlooked by previous studies\footnote{In a scale invariant setup, in the absence of gravity, one can only predict ratios of fields vev’s.} and at the technical level it arises when the “evanescent” correction ($\propto \epsilon$) to the field-dependent masses\footnote{With scale invariance broken by gravity, the dilaton couplings to matter are expected to be very small.} in the loop, multiplies the poles $1/\epsilon$ of the loop integrals, thus giving a finite correction! Note that $\Delta U$ contains non-polynomial operators like $\phi^6/\sigma^2$ of known coefficient; such new operators generate in turn polynomial effective operators, when expanded about $\langle \sigma \rangle \neq 0$ ($\langle \sigma \rangle$ can be arranged to be much larger than the electroweak vev $\langle \phi \rangle$, see later).

The subtraction function cannot also depend on the higgs field $\phi$ (as in [2]) since this would bring non-decoupling quantum effects of the visible sector $\langle \phi \rangle$ to the hidden sector $\langle \sigma \rangle$ even in their classical decoupling limit. As a result, the dilaton-only dependent subtraction function must be of the form $\mu(\sigma) = z\sigma$ where $z$ is an arbitrary dimensionless parameter. Unlike total $U(\phi, \sigma)$, $\Delta U$ is independent of the subtraction scale ($z\langle \sigma \rangle$), being finite. Of course physics must be independent of the parameter $z$, so we check that our potential respects the Callan-Symanzik equation, see [5] for a discussion.

Assuming the couplings are initially tuned at the classical level to enforce a hierarchy $\langle \phi \rangle \ll \langle \sigma \rangle$, we show the quantum correction to the mass of $\phi$, due to $\Delta U$, remains small without additional tuning. With this scale-invariant regularization and spontaneous breaking of this symmetry one can address the hierarchy problem at higher loops.

In the case of a field-dependent subtraction function there is no initial subtraction scale present in the theory, so there is no dilatation anomaly. Note that it is possible that a theory be quantum scale invariant and the couplings still run with the momentum scale\footnote{In a scale invariant setup, in the absence of gravity, one can only predict ratios of fields vev’s.}. One first performs loop calculations with a field-dependent regularization. After spontaneous breaking of scale symmetry $\langle \sigma \rangle \neq 0$, the subtraction scale and all masses and vev’s of the theory are generated, proportional to $z\langle \sigma \rangle$. After regularization and renormalization one can eventually decouple the dilaton, by taking the limit of vanishing couplings for it, while keeping the masses of the theory fixed\footnote{With scale invariance broken by gravity, the dilaton couplings to matter are expected to be very small.}.

After introducing the model (Section 2) we present the scale invariant result of the one-loop potential for a general subtraction function (Section 3); this function is shown to depend on the dilaton only (Section 4); the implications for the mass of $\phi$ are addressed (Section 5) and the Callan-Symanzik equation verified (Section 6), followed by Conclusions.
2 A generic model

Consider a Lagrangian with two real scalar fields

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - V(\phi, \sigma) \]  

(2)

The potential in \( d = 4 \) scale invariant theories has the structure \( V(\phi, \sigma) = \sigma^4 W(\phi/\sigma) \) and is an homogeneous function, therefore it satisfies the relation \( \phi \partial V/\partial \phi + \sigma \partial V/\partial \sigma = 4V \). Using this and with the notation \( x = \phi/\sigma \), the extremum conditions for \( V(\phi, \sigma) = 0 \) can be written as

\[ W(x) = W'(x) = 0 \]

if we assume that \( \langle \sigma \rangle, \langle \phi \rangle \neq 0 \). One of these conditions fixes the ratio of the fields vev’s, while the second implies a relation (tuning) among the couplings. If \( x_0 = \langle \phi \rangle / \langle \sigma \rangle \) is a solution to these two conditions, then \( \langle \phi \rangle \) is proportional to \( \langle \sigma \rangle \) which means that a flat direction exists in this theory, along the line in the plane \( (\phi, \sigma) \) with \( \phi/\sigma = x_0 \). Also since \( W(x_0) = 0 \) on the ground state, then \( V(\langle \phi \rangle, \langle \sigma \rangle) = 0 \). Thus, in scale symmetric theories a vanishing vacuum energy at a given order of perturbation theory demands a tuning of the relation among couplings in that order.

An example of a scale invariant potential that we consider below is

\[ V = \frac{\lambda_\phi}{4} \phi^4 + \frac{\lambda_m}{2} \phi^2 \sigma^2 + \frac{\lambda_\sigma}{4} \sigma^4 \]  

(3)

where note that the couplings can depend on \( \phi/\sigma \), more fields be present, etc.

In the simple case the couplings are independent of \( \phi/\sigma \), minimizing this \( V \) gives

\[ \langle \phi \rangle (\lambda_\phi \langle \phi \rangle^2 + \lambda_m \langle \sigma \rangle^2) = 0, \quad \langle \sigma \rangle (\lambda_m \langle \phi \rangle^2 + \lambda_\sigma \langle \sigma \rangle^2) = 0 \]  

(4)

One can distinguish the following situations:

**Case a):** The ground state is \( \langle \sigma \rangle = 0, \langle \phi \rangle = 0 \) and both fields are massless.

**Case b):** A more interesting case that we study in this paper is that of spontaneous breaking of scale symmetry when \( \langle \sigma \rangle \neq 0 \). A solution to both equations in (4) then exists for \( \langle \sigma \rangle < \infty \) (finite), then also \( \langle \phi \rangle \neq 0 \), and a non-trivial ground state exists provided that \( \lambda_m^2 = \lambda_\phi \lambda_\sigma \) and \( \lambda_m < 0 \), which we assume to be true in the following. Then

\[ \frac{\langle \phi \rangle^2}{\langle \sigma \rangle^2} = -\frac{\lambda_m}{\lambda_\phi}, \quad \Rightarrow \quad V = \frac{\lambda_\phi}{4} \left( \phi^2 + \frac{\lambda_m}{\lambda_\phi} \sigma^2 \right)^2, \quad (\lambda_m^2 = \lambda_\sigma \lambda_\phi; \lambda_m < 0) \]  

(5)

Then a spontaneous breaking of the scale symmetry implies electroweak symmetry breaking at tree-level, with a vanishing cosmological constant; it also demands the existence of a finite (non-zero) scale \( \langle \sigma \rangle \) (unknown) in the theory. All scales are then generated by \( \langle \sigma \rangle \).

---

7 To find this relation, use that \( V(\alpha \phi, \alpha \sigma) = \alpha^4 V(\phi, \sigma) \) (homogeneous); differentiate wrt \( \alpha \) and set \( \alpha = 1 \).

8 Values \( \langle s \rangle = 0 \) and \( \langle s \rangle = \infty \), with \( s = \phi, \sigma \) are excluded, unless eq. 4 is implemented in the sense of a limit (also the couplings \( \lambda_{m, \phi, \sigma} \) can depend on the ratio of the two fields but such case requires assumptions.
Further, one shifts the fields $\phi \rightarrow \phi + \langle \phi \rangle$ and $\sigma \rightarrow \sigma + \langle \sigma \rangle$ and Taylor expands about the ground state. The mass eigenstates are $\tilde{\phi} = \phi \cos \alpha + \sigma \sin \alpha$ and $\tilde{\sigma} = -\phi \sin \alpha + \sigma \cos \alpha$ where $\tan^2 \alpha = \frac{-\lambda m}{\lambda \phi} > 0$. A flat direction exists, so one field (dilaton $\sigma$) is massless while the second field $\phi$ that would be the Higgs boson in a realistic model, has a mass

$$m^2 = 2\lambda \phi (1 - \lambda m/\lambda \phi) \langle \phi \rangle^2 = -2\lambda m (1 - \lambda m/\lambda \phi) \langle \sigma \rangle^2$$

(6)

Ultimately, scale invariance is expected to be broken by Planck physics, thus $\sigma$ will acquire a large vev, $\langle \sigma \rangle \sim M_{\text{Planck}}$. If one would like to implement a hierarchy with $m_{\tilde{\phi}} \sim \langle \phi \rangle \sim O(100 \text{ GeV}) \ll \langle \sigma \rangle$, one should tune accordingly the couplings $\lambda_{\sigma} \ll |\lambda_m| \ll \lambda_{\phi}$. Such hierarchy of couplings is possible [15, 28]. It was observed [15] that the shift symmetry of the dilaton enables the couplings $\lambda_{m,\sigma}$ to remain ultra-weak under RG evolution.

One would like to know if at the quantum level this tree-level tuning is enough or additional tuning (beyond that of $\lambda_m$) is required to maintain this hierarchy and $m_{\tilde{\phi}}$ light. Indeed, at one-loop dangerous corrections can emerge, like $m^2 = \lambda^2 \phi \langle \sigma \rangle^2$ that would require additional tuning (of $\lambda_{\phi}$) and would re-introduce the hierarchy problem.

### 3 Scale invariance of 1-loop potential and effective operators

To compute the one-loop potential, consider the DR scheme in $d = 4 - 2\epsilon$. Then the mass dimensions are $[L] = d$, $[\phi] = [\sigma] = (d - 2)/2$: the couplings $\lambda_\phi$, $\lambda_m$, $\lambda_\sigma$ are dimensionful, $[\lambda] = [V^{(4)}] = 4 - d$. To render the couplings dimensionless, one uses the DR scale $\mu$ and replaces $\lambda \rightarrow \mu^{4-d}$. The scale $\mu$ breaks the classical scale invariance. To avoid this problem and to preserve this symmetry during regularization replace $\mu$ by a field-dependent function (unknown), so $\mu^{4-d} \rightarrow \mu(\phi, \sigma)^{4-d}$. Then the actual Lagrangian is

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \tilde{V}(\phi, \sigma), \quad \tilde{V}(\phi, \sigma) \equiv \mu(\phi, \sigma)^{4-d} V(\phi, \sigma)$$

(7)

$L$ is scale invariant in $d$ dimensions, $[\tilde{V}] = d$, $[V] = 2d - 4$. Denote by $\tilde{M}^2$ the field-dependent mass matrix

$$(\tilde{M}^2)_{\alpha\beta} = \frac{\partial^2 \tilde{V}(\phi, \sigma)}{\partial \alpha \partial \beta}, \quad \alpha, \beta = \phi, \sigma.$$  

(8)

Then the one-loop potential, that manifestly respects scale invariance, is found from [10]

---

9 This function is assumed to be non-zero, finite, continuous, differentiable, and is determined later.

10 Formula (8) is derived in the usual diagrammatic approach (for $\tilde{V}$) and is valid at one-loop (even in non-renormalizable cases, if no higher derivative operators exist and kinetic terms are canonical). Beyond one-loop more vacuum “bubble” diagrams exist and then formula (9) receives corrections [29].
\[ U = \tilde{V}(\phi, \sigma) - \frac{i}{2} \int \frac{d^dp}{(2\pi)^d} \text{Tr} \ln \left[ p^2 - \tilde{M}^2(\phi, \sigma) + i\varepsilon \right] \] (9)

\[ = \tilde{V}(\phi, \sigma) - \frac{1}{2} \frac{1}{(2\pi)^d} \Gamma[-d/2] \text{Tr} \left[ \pi \tilde{M}^2(\phi, \sigma) \right]^{d/2} \] (10)

\[ = \tilde{V}(\phi, \sigma) - \frac{1}{64\pi^2} \sum_{s=\phi,\sigma} \tilde{M}^4_s \left[ \frac{2}{4-d} + \ln \kappa - \ln \tilde{M}^2_s \right], \quad \kappa \equiv 4\pi e^{3/2 - \gamma_E} \] (11)

The sum is over the eigenvalues \(\tilde{M}^2_s\) of the matrix \((\tilde{M}^2)_{\alpha\beta}\). Up to \(O[(4-d)^2]\) terms

\[(\tilde{M}^2)_{\alpha\beta} = \mu^{4-d} \left[ (M^2)_{\alpha\beta} + (4-d) \mu^{-2} N_{\alpha\beta} \right], \quad \alpha, \beta = \{\phi, \sigma\}, \] (12)

where

\[(M^2)_{\alpha\beta} = V_{\alpha\beta}, \quad N_{\alpha\beta} = \mu (\mu_\alpha V_{\beta} + \mu_\beta V_{\alpha}) + (\mu \mu_{\alpha\beta} - \mu_\alpha \mu_\beta) V, \] (13)

and \(\mu_\alpha = \partial \mu/\partial \alpha, \quad \mu_{\alpha\beta} = \partial^2 \mu/\partial \alpha \partial \beta, \quad V_\alpha = \partial V/\partial \alpha, \quad V_{\alpha\beta} = \partial^2 V/\partial \alpha \partial \beta\), are nonzero field dependent quantities. From the last two equations one finds, up to \(O[(4-d)^2]\) terms

\[\sum_{s=\phi,\sigma} \tilde{M}^4_s = \mu^2 (4-d) \left[ \text{Tr} \, M^4 + 2(4-d) \mu^{-2} \text{Tr} (M^2 N) \right], \] (14)

Then

\[ U = \mu(\phi, \sigma)^{4-d} \left\{ V - \frac{1}{64\pi^2} \left[ \sum_{s=\phi,\sigma} M^4_s \left( \frac{2}{4-d} - \ln \frac{M^2_s}{\mu^2(\phi, \sigma)} \right) + \frac{4 \text{Tr} (M^2 N)}{\mu^2(\phi, \sigma)} \right] \right\} \] (15)

The last term is due to the field dependence of \(\mu\) and its origin is in the second “evanescent” term in the rhs of eqs. (12) which cancels the pole to give a finite contribution. We adopt the usual \(\overline{MS}\) scheme here, in which case the counterterms are\(^{12}\)

\[ \delta U_{\text{ct}} = \frac{\mu(\phi, \sigma)^{4-d}}{64\pi^2} \sum_{s=\phi,\sigma} M^4_s \left( \frac{2}{4-d} + \ln \kappa - \frac{3}{2} \right), \] (16)

\(^{11}\) For any values of the fields, \(\det(M^2)_{\alpha\beta}\) is positive provided that \(\lambda_m^2 \in [3\lambda_0 \lambda_\sigma (3-2\sqrt{2}), 3\lambda_0 \lambda_\sigma (3+2\sqrt{2})]\) and that \(\lambda_0, \lambda_\sigma, \lambda_m\) have all the same sign. The eigenvalues are positive if \(\lambda_0, \lambda_\sigma, \lambda_m\) are positive. If \(\lambda_m < 0, \lambda_0, \lambda_\sigma > 0\) one eigenvalue is negative. For \(\lambda_m^2\) outside this interval, restrictions apply to the ratio \(u_0^2/\sigma^2\) for which the eigenvalues are both positive. Note that even in the Standard Model, the Goldstone mode (negative) field dependent squared mass leads to complex and infrared divergent corrections and then only the real part of the potential is included. A resummation of higher orders in \(V\) fixes this well known problem \(30,31\) (see also \(16,32\)). Here we proceed in general and do not study this issue that affects the Coleman-Weinberg term only, but refer the reader to \(30,31\).

\(^{12}\)One can use other subtraction schemes e.g. \(\delta U_{\text{ct}} = \mu^{4-d} \left[ a_1 \phi^4(1/\varepsilon + c_1) + a_2 \phi^3 \sigma^2(1/\varepsilon + c_2) + a_3 \sigma^4(1/\varepsilon + c_3) \right]\) where we denoted \(1/\varepsilon \equiv 2/(4-d) + \ln \kappa - 3/2\). The case of \(\overline{MS}\) corresponds to \(c_1 = c_2 = c_3 = 0\).
where $\sum_{s=\phi,\sigma} M_s^4 = V_{\phi\phi}^2 + V_{\sigma\sigma}^2 + 2V_{\phi\sigma}^2$. Using

$$\delta U_{\text{c.t.}} \equiv \mu^{-d} \left[ 1/4 (Z_{\lambda_{\phi}} - 1)\lambda_{\phi} \phi^4 + 1/2 (Z_{\lambda_{\sigma}} - 1)\lambda_{\sigma} \phi^2 \sigma^2 + 1/4 (\lambda_{\sigma} - 1)\lambda_{\sigma} \sigma^4 \right]$$

(17)
o one finds the renormalization coefficients

$$Z_{\lambda_{\phi}} = 1 + \frac{1}{8\pi^2 (4 - d)} (9\lambda_{\phi} + \lambda_m^2/\lambda_{\phi})$$

$$Z_{\lambda_{\sigma}} = 1 + \frac{1}{8\pi^2 (4 - d)} (3\lambda_{\sigma} + 3\lambda_{\sigma} + 4\lambda_m)$$

$$Z_{\lambda_{\sigma}} = 1 + \frac{1}{8\pi^2 (4 - d)} (9\lambda_{\sigma} + \lambda_m^2/\lambda_{\sigma})$$

(18)

These $Z$’s have expressions identical to those obtained at one-loop with $\mu$ a constant.

After adding the counterterms $\delta U_{\text{c.t}}$ we can safely take the limit $d \to 4$ in the remaining terms ($\mu \neq 0$), so the renormalized one-loop potential is

$$U(\phi, \sigma) = V(\phi, \sigma) + \frac{1}{64\pi^2} \left\{ \sum_{s=\phi,\sigma} M_s^4(\phi, \sigma) \left( \ln \frac{M_s^2(\phi, \sigma)}{\mu^2(\phi, \sigma)} - \frac{3}{2} \right) + \Delta U(\phi, \sigma) \right\}$$

$$\Delta U = -\frac{4}{\mu^2} \left\{ V \left[ (\mu \mu_{\phi\phi} - \mu_{\sigma\sigma}^2) V_{\phi\phi} + 2 (\mu \mu_{\phi\sigma} - \mu_{\phi\sigma}) V_{\phi\sigma} + (\mu \mu_{\sigma\sigma} - \mu_{\sigma\sigma}^2) V_{\sigma\sigma} \right]$$

$$+ 2\mu (\mu_{\phi\phi} V_{\phi\phi} + \mu_{\sigma\sigma} V_{\phi\sigma} + 2\mu_{\phi\sigma} V_{\phi\sigma} + \mu_{\sigma\sigma} V_{\sigma\sigma}) V_{\sigma} \right\}$$

(19)

In the above $M_s^2 (s = \phi, \sigma)$ are the eigenvalues of the matrix $V_{\alpha\beta}$, given by the roots of equation $\rho^2 - \rho (V_{\phi\phi} + V_{\sigma\sigma}) + (V_{\phi\phi} V_{\sigma\sigma} - V_{\phi\sigma}^2) = 0$ \[13\].

Eq. (19) is a scale-invariant one-loop result. It is a modified version of the Coleman-Weinberg potential (recovered if $\mu$ is a constant) and contains an additional correction ($\Delta U$). Note that $\Delta U$ is not exactly a counterterm but a finite one-loop effect induced by scale invariance. It is generated when the “evanescent” coefficient $(4 - d)$ in the field-dependent masses of eq. (12), multiplies the pole $1/(4 - d)$ of the one-loop integral \[14\]. This effect is missed in calculations that are not scale invariant such as the usual DR scheme. Note also that $\Delta U$ vanishes on the tree-level ground state.

$\Delta U$ contains non-polynomial operators. Even in the minimal case of taking $\mu \sim \sigma$, then the terms in $\Delta U$ proportional to $VV_{\sigma\sigma}$ contain a $\phi^6/\sigma^2$ term. Similar effective operators are expected to be generated in higher orders. Further, one can Taylor expand the expression \[13\]

$$M_s^2 = (1/2) [\nu \pm \sqrt{\Delta}], \quad \nu \equiv 3(\lambda_{\phi} + \lambda_m)\phi^2 + (3\lambda_{\sigma} + \lambda_m)\sigma^2.$$  

$$\Delta = (3\lambda_{\sigma} - \lambda_{\phi})^2 \phi^4 + (3\lambda_{\sigma} - \lambda_m)^2 \sigma^2 + 2\lambda_{\sigma}^2 \sigma^2 [3\lambda_{\sigma}(\lambda_{\phi} + \lambda_m) - 9\lambda_{\phi}\lambda_m + 7\lambda_m^2]$$

\[20\]

\[14\]In higher orders, a $n$-loop pole $1/(4 - d)^n$, upon multiplication by the $4 - d$ coefficient will actually generate a $1/(4 - d)^{n-1}$ pole, i.e. what we consider usually to account for $n - 1$ loop effects. Thus, the order of the singularity is not identical to the loop order in this case.
of the potential about the ground state, using $\sigma = \langle \sigma \rangle + \delta \sigma$, with $\delta \sigma$ a quantum fluctuation. When doing so, the operator $\phi^6/\sigma^2$ becomes a series of effective (polynomial) operators

$$\frac{\phi^6}{\sigma^2} = \frac{\phi^6}{\langle \sigma \rangle^2} \left( 1 - \frac{2 \delta \sigma}{\langle \sigma \rangle} + \frac{3 \delta \sigma^2}{\langle \sigma \rangle^2} + \cdots \right).$$

(21)

To proceed further, one needs the general expression of the function $\mu = \mu(\phi, \sigma)$. Let us first take $\mu = \mu(\sigma)$ only, which will be justified in the next section; in this case the only possibility is

$$\mu(\sigma) = z \sigma$$

(22)

which, as a "DR scale", requires $\langle \sigma \rangle \neq 0$, $\langle \sigma \rangle < \infty$. To be exact, we actually take $\mu(\sigma) = z \sigma^{2/(d-2)}$ \footnote{At the loop level the ground state is changed slightly, but we ignore that effect here.}, which accounts for the mass dimension of the field $\sigma$. For one-loop case only (as here) it is safe to use at this stage its limit for $d \to 4$, so $\mu = z \sigma$. Here $z$ is an arbitrary dimensionless parameter and the dependence of $U$ on $z$ is equivalent to the familiar subtraction scale dependence of $U$ in the "usual" regularization. With eq.(22), one obtains the following form of $\Delta U$, which is independent of $z$

$$\Delta U = -\frac{4}{\sigma^2} \left[ V_{\sigma \sigma} (2 \sigma V_{\sigma} - V) + 2 \sigma V_{\phi} V_{\phi \sigma} \right]$$

(23)

and only the Coleman-Weinberg term depends on $z$. With $V$ of eq.(\ref{eq:V})

$$\Delta U = \frac{\lambda_\phi \lambda_m \phi^6}{\sigma^2} - (16 \lambda_\phi \lambda_m + 6 \lambda_m^2 - 3 \lambda_\phi \lambda_\sigma) \phi^4 - (16 \lambda_m + 25 \lambda_\sigma) \lambda_m \phi^2 \sigma^2 - 21 \lambda_\phi^2 \sigma^4$$

(24)

As anticipated, notice the presence of the non-polynomial operator $\sim \phi^6/\sigma^2$. This operator is suppressed at large $\langle \sigma \rangle$ or for small mixing ($\lambda_m$) between $\phi$ and the dilaton $\sigma$. The sign of this operator is controlled by $\lambda_m$, assuming $\lambda_\phi > 0$. When $\lambda_m < 0$, the term $\lambda_m \phi^6$ destabilizes the potential for large values of $\phi$. A tuning $|\lambda_m| \ll \lambda_\phi$ can compensate to render this term of similar size to $\phi^4$ terms; also higher loop orders can generate similar effective operators that may stabilize the potential globally.

For the special case of a non-trivial classical vacuum of eq.(\ref{eq:V}), when $\lambda_m^2 = \lambda_\phi \lambda_\sigma$, eq.(24) becomes

$$\Delta U = \frac{\lambda_m}{\lambda_\phi^2 \sigma^2} \left[ \lambda_\phi \phi^2 + \lambda_m \sigma^2 \right] \left[ \lambda_\phi^2 \phi^4 - 4 \lambda_\phi (4 \lambda_\phi + \lambda_m) \phi^2 \sigma^2 - 21 \lambda_\phi^2 \sigma^4 \right]$$

(25)

This expression vanishes on the tree-level ground state \footnote{At the loop level the ground state is changed slightly, but we ignore that effect here.}, (see eq.(\ref{eq:V}); $\lambda_m < 0$, $\lambda_\phi > 0$).
In conclusion, the expression of $U$ at one-loop is manifestly scale invariant

$$U(\phi, \sigma) = V(\phi, \sigma) + \frac{1}{64\pi^2} \left[ \sum_{s=\phi,\sigma} M_s^4(\phi, \sigma) \left( \ln \frac{M_s^2(\phi, \sigma)}{z^2\sigma^2} - \frac{3}{2} \right) + \Delta U(\phi, \sigma) \right] \tag{26}$$

with $\Delta U$ as in eqs. (24) or (25) and $V$ of eq. (3); note that the “standard” Coleman-Weinberg term is modified into a scale invariant form. This is the main result of this section, valid under our assumption $\mu(\sigma) = z\sigma$. Further, $U$ can be Taylor expanded about $\langle \sigma \rangle$. With no mass scale in the theory, from minimising $U$ one can only predict ratios of vev’s, so all masses are generated by $\langle \sigma \rangle$ after spontaneous breaking of scale symmetry.

4 More general $\mu(\phi, \sigma)$ and implications

The subtraction function could in principle be more general and could depend on $\phi$ too, $\mu = \mu(\phi, \sigma)$. In this section we show that such dependence is not physical and conclude that $\mu$ must be a function of the dilaton only. First, consider the following example

$$\mu(\phi, \sigma) = z (\xi_\phi \phi^2 + \xi_\sigma \sigma^2)^{1/2} \tag{27}$$

This was used in earlier similar studies [2, 3] where scale invariant models had a non-minimal coupling to gravity, with this expression to fix the Planck scale upon spontaneous breaking of scale symmetry. With this $\mu$ and $V$ of eq. (3) one finds that $\Delta U$ contains leading power terms $\phi^8$ and $\sigma^8$ as shown in

$$\Delta U = - (\xi_\phi \phi^2 + \xi_\sigma \sigma^2)^{-2} \left[ (21 \lambda_\phi \xi_\phi + \lambda_m \xi_\sigma) \xi_\phi \lambda_\phi \phi^8 + (21 \lambda_\sigma \xi_\sigma + \lambda_m \xi_\phi) \xi_\sigma \lambda_\sigma \sigma^8 + \cdots \right] \tag{28}$$

The dots stand for remaining $\phi^6\sigma^2$, $\phi^4\sigma^4$ and $\phi^2\sigma^6$ terms which we do not display since their coefficients are too long. The coefficients of $\phi^8$, $\sigma^8$ are positive irrespective of the values of $\xi_{\phi,\sigma}$, if $\lambda_m^2 \geq 21^2 \lambda_\phi \lambda_\sigma$, (with $\lambda_{\phi,\sigma} > 0$). This condition is not respected on the ground state of $V$ (with $\lambda_m^2 = \lambda_\phi \lambda_\sigma$). We thus encounter terms unbounded from below, that otherwise vanish on the tree level ground state. A small fluctuation about the critical point can then destabilize the potential.

It is intriguing that even if the classical $V$ contains no interaction terms between “visible” ($\phi$) and “hidden” ($\sigma$) sectors, i.e. $\lambda_m = 0$, such terms are still generated by quantum corrections, for $\mu(\phi, \sigma)$ of eq. (27). Indeed, one has

\[16\]The non-minimal coupling is $\mathcal{L}_G = -\frac{1}{2} (\xi_\phi \phi^2 + \xi_\sigma \sigma^2) R$, and is added in some models to generate the Planck mass from $\mu(\sigma, \phi\ldots)$, in (spontaneously broken) scale invariant theories [2, 3]. The relative signs of $\xi_\phi$, $\xi_\sigma$ are important to ensure a positive Newton constant (for a review see [33]). When going to the Einstein frame, this coupling generates a suppression of the tree level potential by a factor $1/(\xi_\phi \phi^2 + \xi_\sigma \sigma^2)^2$, while in the case discussed in the text (where no such coupling is included), such suppression is shown to be generated in $\Delta U$ at 1-loop, see later.
\[ \Delta U \bigg|_{\lambda_m=0} = -3 \left[ \xi \phi \xi \sigma \left( \lambda_\phi (9 \lambda_\phi + \lambda \phi) \phi^6 \sigma^2 + \lambda_\sigma (\lambda_\phi + 9 \lambda_\sigma) \phi^2 \sigma^6 \right) \right. \\
+ 7 \left( \lambda_\phi^2 \xi \phi \phi^8 + \lambda_\sigma^2 \xi \sigma \sigma^8 \right) - \left( \xi \phi^2 + \xi \sigma^2 \right) \lambda_\phi \lambda_\sigma \phi^4 \sigma^4 \left( \xi \phi \phi^2 + \xi \sigma \sigma^2 \right)^{-2} \] (29)

This simplifies further if also \( \lambda_\sigma = \lambda_\phi^2 / \lambda_\phi \to 0 \), but the term \( \propto \xi \phi \xi \sigma \lambda_\phi^2 \phi^6 \sigma^2 \) does not vanish. Such term ultimately arise from the expression of the \( \mu \)-dependent factor in \( \tilde{V} \), via terms like \( \mu \phi \tilde{V}_\phi \) and \( (\mu \phi \phi - \mu_0^2) \tilde{V}_\phi \) in eq. (19). The two sectors still “communicate” at the quantum level, due to scale invariance even if they are classically decoupled! This concerning effect is only removed for vanishing \( \xi \phi \) or \( \xi \sigma \), which means \( \mu \propto \sigma \). \(^{17}\)

More generally, consider
\[ \mu(\phi, \sigma) = z \sigma e^{g(\phi/\sigma)} \] (30)

Here \( g \) is some arbitrary function of the ratio \( \phi/\sigma \). In this case, in the classical “decoupling” limit \( \lambda_m \to 0 \), also with \( \lambda_\sigma = \lambda_\phi^2 / \lambda_\phi \to 0 \), there are non-vanishing quantum interactions terms
\[ \Delta U \bigg|_{\lambda_m=0} = -3 \lambda_\phi^2 \left[ \frac{\phi^3}{\sigma} g'(\phi/\sigma) + \frac{\phi^6}{\sigma^2} g''(\phi/\sigma) \right] \] (31)

Again, the two sectors still communicate at the quantum level only. To avoid such concerning behaviour, we must take \( g=0 \) (or constant \(^{18}\)). Therefore, the subtraction function is independent of \( \phi \) and thus \( \mu(\sigma) = z \sigma \). This result is the minimal scenario used in the previous section and justifies our choice in eq. (22) and our result in eq. (26). We conclude that it is the dilaton alone that generates the subtraction scale after spontaneous breaking of scale symmetry.

5 The mass spectrum

Let us minimise the one-loop potential \( U \). We restrict the analysis to the simpler case of a hierarchy of the couplings considered in \(^{2} \text{[15]} \). We take
\[ \lambda_\sigma \ll |\lambda_m| \ll \lambda_\phi. \] (32)

To enforce this hierarchy, introduce \( \lambda_m = \tilde{\lambda}_m \varepsilon \) and \( \lambda_\sigma = \tilde{\lambda}_\sigma \varepsilon^2 \), where \( \varepsilon \approx 1/M_{\text{Planck}}^2 \ll 1 \), and \( \lambda_\phi, \tilde{\lambda}_m \) and \( \tilde{\lambda}_\sigma \) are now of similar magnitude. One then expands \( U \) up to \( \mathcal{O}(\varepsilon^3) \sim \mathcal{O}(\lambda_m^3) \)

\(^{17}\) up to a relabeling, see the symmetry \( \phi \leftrightarrow \sigma \), at which stage one decides which field denotes the dilaton.

\(^{18}\) We disregard a second solution for which the rhs of eq. (31) vanishes, since it is not continuous in \( \phi = 0. \)
\[ U = \frac{\lambda}{4} \phi^4 + \frac{\lambda_m}{2} \phi^2 \sigma^2 + \frac{\lambda}{4} \sigma^4 + \frac{1}{64\pi^2} \left\{ M_1^4 \left[ \ln \frac{M_1^2}{z^2 \sigma^2} - \frac{3}{2} \right] + M_2^4 \left[ \ln \frac{M_2^2}{z^2 \sigma^2} - \frac{3}{2} \right] \right\} + \lambda_m \phi \sigma \left\{ 16 \lambda_m \lambda_m + 6 \lambda_m^2 - 3 \lambda_m \lambda \right\} + \lambda \sigma^4 + \frac{1}{64} \pi^2 \{M_4^1 \left[ \ln \frac{M_1^2}{z^2 \sigma^2} - \frac{3}{2} \right] + M_4^2 \left[ \ln \frac{M_2^2}{z^2 \sigma^2} - \frac{3}{2} \right] \} + O(\lambda_m^4) \] (33)

One can minimise \( U \) and find the solution for \( \langle \phi \rangle / \langle \sigma \rangle \) that satisfies \( U_{\phi} = U_{\sigma} = 0 \); \( U \) being manifestly scale invariant, these conditions ensure a flat direction exists and also that vacuum energy vanishes in this order. To the lowest order in \( \varepsilon \), one finds

\[
\frac{(\phi)^2}{\langle \sigma \rangle^2} = -\frac{\lambda_m}{\lambda} \left[ 1 - \frac{6\lambda_\phi}{64\pi^2} \left( 4 \ln 3 \lambda_\phi - 17/3 \right) \right] + O(\lambda_m^2) \] (34)

This brings a correction to the tree level case, eq.(5); here \( \lambda_m < 0, \lambda_\phi > 0 \) and \( \lambda_m^2 = \lambda_\phi \lambda_\sigma \). To obtain eq.(34) we fixed the subtraction parameter \( z \) under the log term in \( U \) to

\[ z = \langle \phi \rangle / \langle \sigma \rangle, \quad \text{then} \quad \mu(\langle \sigma \rangle) = \langle \phi \rangle \] (35)

on the ground state. This value for \( \mu \) is the standard choice for the subtraction scale, made to minimize the Coleman-Weinberg log-term dependence on it. As mentioned, \( \Delta U \) itself is scheme-independent (being independent of \( z \)).

The potential in (33) is scale invariant, the dilaton remains massless at one-loop while the higgs-like scalar \( \phi \) has a mass

\[ m_\phi^2 = [U_{\phi\phi} + U_{\sigma\sigma}]_{\text{min}} \] (36)

Let us consider only the contribution \( \delta m_\phi^2 \) from \( \Delta U \) alone to the mass of \( \phi \). The interest is to examine if potentially “dangerous” corrections of the type \( \lambda_\phi^2 (\langle \sigma \rangle)^2 \), etc, can emerge from the new contribution \( \Delta U \). These would require an additional tuning (of \( \lambda_\phi \)) beyond that of \( \lambda_m \) done at the tree level, in order to keep \( \phi \) light compared to \( \langle \sigma \rangle \sim M_{\text{Planck}} \). In general, one has

\[
\delta m_\phi^2 \equiv \frac{1}{64\pi^2} \left( \Delta U_{\phi\phi} + \Delta U_{\sigma\sigma} \right)_{\text{min}} = -\frac{(\langle \sigma \rangle)^2}{32\pi^2} \left[ 4\lambda_m^2(4 + 13\rho) + 18\lambda_\sigma(7\lambda_\sigma - \lambda_\phi\rho) + \lambda_m \left[ 25\lambda_\sigma(1+\rho) - 3\lambda_\phi\rho(-32 + 5\rho + \rho^2) \right] \right] \] (37)

where \( \rho = (\langle \phi \rangle^2 / \langle \sigma \rangle^2) \). This mass correction contains terms proportional to \( \lambda_m \) or \( \lambda_\sigma = \lambda_m^2 / \lambda_\phi \ll \lambda_m \) but not to \( \lambda_\phi \) alone. Therefore no extra tuning is needed beyond that at classical level of eqs.(5, 6), in order to maintain \( \delta m_\phi^2 \) and \( m_\phi^2 \sim (\langle \phi \rangle)^2 \sim \lambda_m (\langle \sigma \rangle)^2 \) close to the electroweak scale. It is possible that this nice behaviour survives to higher or all orders, as a result of the manifest scale invariance and spontaneous breaking of this symmetry.
This suggests that the hierarchy problem could be solved with only one initial (classical) tuning of $\lambda_m$ (no tuning of higgs self-coupling $\lambda_\phi$).

6 Further remarks

The method we used to generate dynamically the subtraction scale of the DR scheme as the dilaton vev deserves further study.

First, note that the potential $U$ must respect the Callan-Symanzik equation i.e. it must be independent of the choice of the dimensionless parameter $z$ and thus of the subtraction scale $z \langle \sigma \rangle$ after spontaneous scale symmetry breaking [5]. In our one-loop approximation this demands that

$$\frac{dU}{d \ln z} = \left( \frac{\partial U}{\partial \ln z} + \beta_{\lambda_j} \frac{\partial}{\partial \lambda_j} \right) U = O(\lambda^3) \quad (38)$$

where $U$ is that of eq.(26) and $\Delta U$ of (24) and the Coleman-Weinberg term is the only one that depends explicitly on $z$. To check if condition (38) is respected, we need the one-loop beta functions of the theory; these are obtained from the condition that the “bare” couplings of the Lagrangian are independent of subtraction scale $z \langle \sigma \rangle$, where $z$ is arbitrary: $d(\lambda_j Z_{\lambda_j})/d \ln z = 0$, where $j = \phi, m, \sigma$ (fixed) and $Z_{\lambda_j}$ are given in eq.(17). One finds

$$\beta_{\lambda_\phi} = \frac{d\lambda_\phi}{d \ln z} = \frac{1}{8\pi^2} \left( 9\lambda_\phi^2 + \lambda_m^2 \right)$$

$$\beta_{\lambda_m} = \frac{d\lambda_m}{d \ln z} = \frac{1}{8\pi^2} \left( 3\lambda_\phi + 4\lambda_m + 3\lambda_\sigma \right) \lambda_m$$

$$\beta_{\lambda_\sigma} = \frac{d\lambda_\sigma}{d \ln z} = \frac{1}{8\pi^2} \left( \lambda_m^2 + 9\lambda_\sigma^2 \right)$$

which are the same as in the case the theory was regularized with $\mu =$-constant. Using these beta functions one easily checks that eq.(38) is respected. This shows that the change of parameter $z$ is “moved” into the running couplings of the potential and physics is indeed independent of $z$: $U(\lambda_j(z), z) = U(\lambda_j(z_0), z_0)$, where $j = \phi, m, \sigma$ and $z, z_0$ are different subtraction parameters (ultimately corresponding to subtraction scales $z \langle \sigma \rangle$, $z_0 \langle \sigma \rangle$).

Regarding renormalizability of scale invariant models, previous studies [23] identified at three-loop order a UV counterterm to the original Lagrangian $L$, of the form

$$\frac{1}{(16\pi^2)^3} \frac{1}{(4 - d)^2} \left( \xi_\sigma \frac{\phi_\sigma}{\sigma^2} \right)^2 (\Box \phi^2)^2 \quad (40)$$

19To be exact, in $d = 4 - 2\epsilon$, one actually imposes $d((\sigma z)^{2\epsilon} \lambda_j Z_{\lambda_j})/d \ln z = 0$, giving that beta functions are shifted from those above, $\beta_{\lambda_j} = -2\epsilon\lambda_j + (\ldots)$ where $(\ldots)$ denotes the rhs in each of eqs.(39).

20This is expected since we only found new finite terms, but no new counterterms.

21 Thus there is no dilatation anomaly, yet the couplings still run as usual [5, 6].
In [23], \( \mu \sim (\xi_\phi \phi^2 + \xi_\sigma \sigma^2)^{1/2} \), just like in eq. (27). This UV divergence was due to a new vertex generated by the Taylor expansion of \( \mu(\phi, \sigma) \) wrt \( \phi \); this vertex is ultimately due to new interactions that \( \mu(\phi, \sigma) \) itself brought in \( \hat{V} \) but absent in initial \( V! \) Given this counterterm, the theory is then non-renormalizable and non-local. The same conclusion is expected for any subtraction function that depends on additional fields other than dilaton.

However, we showed that \( \mu = z \sigma \) (Section 4), so the above three-loop counterterm is absent because we have \( \xi_\phi = 0 \). Despite this, the standard expectation is that higher loop orders still generate higher dimensional counterterms and the theory is non-renormalizable, due to the presence in \( U \) of the non-polynomial term \( \phi^6/\sigma^2 \) (one can still explore the possibility that in a scale symmetry-preserving calculation, all poles in quantum corrected \( L \) be those that renormalize its initial couplings and fields only (i.e. renormalizability), without other UV counterterms. This problem deserves careful investigation and is beyond the goal of this paper).

As a result of a manifestly scale-invariant regularization, the \((\text{mass})^2\) of \( \phi \) contains: quadratic contributions \( \lambda_m(\sigma)^2 \) and corrections suppressed by \( 1/(\sigma)^2 \), in addition to logarithmic terms \( \ln(\sigma) \sim \ln \mu \) present in the “usual” DR scheme. Our method to generate the subtraction scale via spontaneous breaking in a dilaton-modified DR can also be implemented in other regularizations. Also note that the role of \( \mu \sim \sigma \) as a finite, non-zero “DR scale” means that only non-zero, finite \( \langle \sigma \rangle \) is allowed. In fixing its actual numerical value, Planck scale physics (gravity) is expected to play a role.

Although we do not explore them here, our results can have interesting applications to phenomenology, such as model building beyond Standard Model (SM) [13, 14, 15]. For reference only, we provide below the one-loop potential in the scale invariant version of the SM extended by the dilaton. With the usual Coleman-Weinberg (CW) part \( \delta U_{\text{CW}} \) the one-loop scalar potential in the SM is (with \( M_0^2, M_n^2 \) as in eq. (23))

\[
U = \frac{\lambda_\phi}{4} \phi^4 + \frac{\lambda_m}{2} \phi^2 \sigma^2 + \frac{\lambda_\sigma}{4} \sigma^4 + \delta U_{\text{CW}}
\]

\[
+ \frac{1}{64 \pi^2} \left\{ \lambda_\phi \lambda_m \frac{\phi^6}{\sigma^2} - (16 \lambda_\phi \lambda_m + 6 \lambda_n^2 - 3 \lambda_\phi \lambda_\sigma) \phi^4 - (16 \lambda_m + 25 \lambda_\sigma) \lambda_m \phi^2 \sigma^2 - 21 \lambda_\sigma^2 \phi^4 \right.
\]

\[
+ M_n^2 \ln \frac{M_n^2}{z \sigma^2} \left\{ \left[ (9 \lambda_\phi^2 + \lambda_m^2) \phi^4 + 2 \lambda_m (3 \lambda_\phi + 4 \lambda_m + 3 \lambda_\sigma) \phi^2 \sigma^2 + (\lambda_m^2 + 9 \lambda_\sigma^2) \sigma^4 \right] \right\}
\]

(41)

The last two lines give the new correction \( \Delta U \), while in all CW terms one uses \( \mu = z \sigma \) with \( z = \langle \phi \rangle / \langle \sigma \rangle \). This equation can be used as the starting point in phenomenological studies.

---

22 This is just the SM with no classical mass term for the higgs in the Lagrangian.
23 \( \delta U_{\text{CW}} = \sum_i N_i M_i^4 \ln M_i^2/\mu^2 (\phi, \sigma) - C_i \), \( i = (G, S, W, Z, t) \) for Goldstone bosons, real scalars, gauge bosons, top, respectively, with \( (N_G, N_S, N_W, N_Z, N_t) = (3, 1, 6, 3, -12) \). \( C_i = 3/2 \) for fermions or scalars and 5/6 for gauge bosons. \( M_G^2 = \lambda_\phi \phi^2 + \lambda_m \sigma^2, M_S^2 = \frac{1}{2} g^2 \phi^2, M_W^2 = \frac{1}{2} (g^2 + g'^2) \phi^2, M_Z^2 = \frac{1}{2} y_t^2 \phi^2. \)

The potential \( U \) of Higgs-dilaton: \( V = \lambda_\phi |H|^4 + \lambda_m |H|^2 \sigma^2 + (\lambda_\sigma/4) \sigma^4 \) with \( H = (0, \phi)/\sqrt{2} \) (unitary gauge).
7 Conclusions

Scale invariant theories are often considered to address the hierarchy problem of the Standard Model. However, the regularization of their quantum corrections breaks explicitly the scale symmetry that one wants to study. This is because all regularizations introduce a dimensionful parameter e.g. the couplings in DR, the UV scale in other regularizations, etc. One can avoid this problem by using a manifestly scale invariant regularization in which the Goldstone mode of this symmetry (dilaton) plays a central role. We used a dilaton-dependent subtraction function \( \mu = \mu(\sigma) \) that replaces the ordinary subtraction scale. We applied this procedure to the DR scheme, to obtain a scale invariant one-loop scalar potential \( U(\phi, \sigma) \) for the (higgs-like) scalar \( \phi \) and dilaton \( \sigma \). After spontaneous breaking of scale invariance when \( \langle \sigma \rangle \neq 0 \), all mass scales of the theory, including the “usual” subtraction scale, are generated from this single vev.

The scale invariance of the action in \( d = 4 - 2\epsilon \) and the usual rescaling \( \lambda \rightarrow \mu^{2\epsilon} \lambda \) that ensures dimensionless quartic couplings, change the potential \( V(\phi, \sigma) \rightarrow \mu(\sigma)^{2\epsilon} V(\phi, \sigma) \) which now contains new (“evanescent”) interactions due to the field dependence of \( \mu \). At the quantum level, these interactions generate new, finite corrections.

We found a new, (finite) one-loop correction \( \Delta U \) to the potential, overlooked by previous studies, that is present beyond the usual Coleman-Weinberg term which is also modified into a scale invariant form. For the minimal case \( \mu(\sigma) = z\sigma \), it was shown that \( \Delta U \) also contains a non-polynomial operator \( \propto \phi^6/\sigma^2 \) with a known, finite coupling. After spontaneous breaking of scale invariance, this operator generates a series of (polynomial) terms suppressed by powers of \( \langle \sigma \rangle \neq 0 \). At higher loop orders, more such operators are expected.

Technically, \( \Delta U \) is generated from an evanescent correction \( (\propto \epsilon) \) to the field-dependent masses of the states “running” in the loop correction to the potential, which cancels the pole \( (\propto 1/\epsilon) \) of the momentum integral, to give rise to a finite \( \Delta U \). And since it is finite, \( \Delta U \) was found to be independent of the subtraction (dimensionless) parameter \( (z) \). Of course physics must be independent of the parameter \( z \) and of the subtraction scale \( \mu(\langle \sigma \rangle) = z \langle \sigma \rangle \) after spontaneous breaking of scale symmetry. We showed this by verifying that the one-loop potential \( U(\phi, \sigma) \) respects the Callan-Symanzik equation.

Further, the correction from \( \Delta U \) to the mass of the higgs-like scalar \( \phi \) remains under control (small) without additional tuning beyond that done at the tree-level to enforce the hierarchy \( \langle \phi \rangle \ll \langle \sigma \rangle \). It is possible that this behaviour survives in higher orders, in a manifest scale invariant calculation. This could provide a solution to the hierarchy problem beyond the one-loop order discussed here.

More general subtraction functions that depend on both \( \sigma \) and \( \phi \) were ruled out since in this case there are quantum operators that force the visible \( (\phi) \) and hidden \( (\sigma) \) sectors to interact in \( d = 4 \) even in their classical decoupling limit! Avoiding this behaviour dictates that the subtraction scale is generated by the dilaton only \( (\mu \sim \sigma) \), as considered above.

This study and the scale-invariant regularization are of interest to theories that study scale invariance at the quantum level.
Acknowledgements: The author thanks Hyun Min Lee, E. Dudas and G. G. Ross for discussions on this topic. This work was supported by a grant of the Romanian National Authority for Scientific Research (CNCS-UEFISCDI) under project number PN-II-ID-PCE-2011-3-0607.

References

[1] F. Englert, C. Truffin and R. Gastmans, “Conformal Invariance in Quantum Gravity,” Nucl. Phys. B 117 (1976) 407. S. Deser, “Scale invariance and gravitational coupling,” Annals Phys. 59 (1970) 248.

[2] M. Shaposhnikov and D. Zenhausern, “Quantum scale invariance, cosmological constant and hierarchy problem,” Phys. Lett. B 671 (2009) 162 [arXiv:0809.3406 [hep-th]].

[3] C. Wetterich, “Cosmology and the Fate of Dilatation Symmetry,” Nucl. Phys. B 302 (1988) 668.

[4] M. Shaposhnikov and D. Zenhausern, “Scale invariance, unimodular gravity and dark energy,” Phys. Lett. B 671 (2009) 187 [arXiv:0809.3395 [hep-th]].

[5] C. Tamarit, “Running couplings with a vanishing scale anomaly,” JHEP 1312 (2013) 098 [arXiv:1309.0913 [hep-th]]. See also [7].

[6] R. Armillis, A. Monin and M. Shaposhnikov, “Spontaneously Broken Conformal Symmetry: Dealing with the Trace Anomaly,” JHEP 1310 (2013) 030 doi:10.1007/JHEP10(2013)030 [arXiv:1302.5619 [hep-th]].

[7] R. Foot, A. Kobakhidze, K. L. McDonald and R. R. Volkas, “Poincaré protection for a natural electroweak scale,” Phys. Rev. D 89 (2014) 11, 115018 [arXiv:1310.0223 [hep-ph]].

[8] R. Foot, A. Kobakhidze, K. L. McDonald and R. R. Volkas, “A Solution to the hierarchy problem from an almost decoupled hidden sector within a classically scale invariant theory,” Phys. Rev. D 77 (2008) 035006 [arXiv:0709.2750 [hep-ph]].

[9] C. T. Hill, “Is the Higgs Boson Associated with Coleman-Weinberg Dynamical Symmetry Breaking?,” Phys. Rev. D 89 (2014) 7, 073003 [arXiv:1401.4185 [hep-ph]].

[10] B. Grinstein and P. Uttayarat, “A Very Light Dilaton,” JHEP 1107 (2011) 038 [arXiv:1105.2370 [hep-ph]].

[11] W. D. Goldberger, B. Grinstein and W. Skiba, “Distinguishing the Higgs boson from the dilaton at the Large Hadron Collider,” Phys. Rev. Lett. 100 (2008) 111802 [arXiv:0708.1463 [hep-ph]].
[12] R. H. Boels and W. Wormsbecher, “Spontaneously broken conformal invariance in observables,” [arXiv:1507.08162 [hep-th]].

[13] A. Farzinnia, H. J. He and J. Ren, “Natural Electroweak Symmetry Breaking from Scale Invariant Higgs Mechanism,” Phys. Lett. B 727 (2013) 141 [arXiv:1308.0295 [hep-ph]].

[14] E. Gabrielli, M. Heikinheimo, K. Kannike, A. Racioppi, M. Raidal and C. Spethmann, “Towards Completing the Standard Model: Vacuum Stability, EWSB and Dark Matter,” Phys. Rev. D 89 (2014) 1, 015017 [arXiv:1309.6632 [hep-ph]].

[15] K. Allison, C. T. Hill and G. G. Ross, “Ultra-weak sector, Higgs boson mass, and the dilaton,” Phys. Lett. B 738 (2014) 191 [arXiv:1404.6268 [hep-ph]].

[16] K. Endo and Y. Sumino, “A Scale-invariant Higgs Sector and Structure of the Vacuum,” JHEP 1505 (2015) 030 [arXiv:1503.02819 [hep-ph]].

[17] S. Iso, N. Okada and Y. Orikasa, “Classically conformal $B^- L$ extended Standard Model,” Phys. Lett. B 676 (2009) 81 [arXiv:0902.4050 [hep-ph]].

[18] S. Iso and Y. Orikasa, “TeV Scale B-L model with a flat Higgs potential at the Planck scale - in view of the hierarchy problem -,” PTEP 2013 (2013) 023B08 [arXiv:1210.2848 [hep-ph]].

[19] E. J. Chun, S. Jung, H. M. Lee, “Radiative generation of the Higgs potential,” Phys. Lett. B 725 (2013) 158 [Phys. Lett. B 730 (2014) 357] [arXiv:1304.5815 [hep-ph]].

[20] V. V. Khoze, “Inflation and Dark Matter in the Higgs Portal of Classically Scale Invariant Standard Model,” JHEP 1311 (2013) 215 [arXiv:1308.6338 [hep-ph]].

[21] V. V. Khoze, C. McCabe and G. Ro, “Higgs vacuum stability from the dark matter portal,” JHEP 1408 (2014) 026 [arXiv:1403.4953 [hep-ph]].

[22] S. Abel and A. Mariotti, “Novel Higgs Potentials from Gauge Mediation of Exact Scale Breaking,” Phys. Rev. D 89 (2014) 12, 125018 [arXiv:1312.5335 [hep-ph]].

[23] M. E. Shaposhnikov and F. V. Tkachov, “Quantum scale-invariant models as effective field theories,” [arXiv:0905.4857 [hep-th]].

[24] F. Gretsch and A. Monin, “Perturbative conformal symmetry and dilaton,” Phys. Rev. D 92 (2015) no.4, 045036 [arXiv:1308.3863 [hep-th]].

[25] W. A. Bardeen, “On naturalness in the standard model,” FERMILAB-CONF-95-391-T, C95-08-27-3.

[26] S. R. Coleman and E. J. Weinberg, “Radiative Corrections as the Origin of Spontaneous Symmetry Breaking,” Phys. Rev. D 7 (1973) 1888.
[27] E. Gildener and S. Weinberg, “Symmetry Breaking and Scalar Bosons,” Phys. Rev. D 13 (1976) 3333.

[28] A. Kobakhidze, “Quantum relaxation of the Higgs mass,” Eur. Phys. J. C 75 (2015) no.8, 384 [arXiv:1506.04840 [hep-ph]].

[29] L. J. Boya and J. Casahorran, “Two loop effective potential in $\lambda \phi^4$ in two-dimensions,” Nuovo Cim. A 100 (1988) 907. C. Ford, I. Jack and D. R. T. Jones, “The Standard model effective potential at two loops,” Nucl. Phys. B 387 (1992) 373 Erratum: [Nucl. Phys. B 504 (1997) 551] doi:10.1016/0550-3213(92)90165-8 [hep-ph/0111190].

[30] J. Elias-Miro, J. R. Espinosa and T. Konstandin, “Taming Infrared Divergences in the Effective Potential,” JHEP 1408 (2014) 034 [arXiv:1406.2652 [hep-ph]].

[31] S. P. Martin, “Taming the Goldstone contributions to the effective potential,” Phys. Rev. D 90 (2014) 1, 016013 [arXiv:1406.2355 [hep-ph]].

[32] E. J. Weinberg and A. Q. Wu, “Understanding Complex Perturbative Effective Potentials,” Phys. Rev. D 36 (1987) 2474. See also S. G. Matinyan and B. Muller, “Quantum fluctuations and dynamical chaos,” Phys. Rev. Lett. 78 (1997) 2515. S. G. Matinyan and B. Muller, “Quantum fluctuations and dynamical chaos: An Effective potential approach,” Found. Phys. 27 (1997) 1237 [hep-th/9610233]. K. E. Cahill, “An Effective potential that is real,” Phys. Rev. D 52 (1995) 4704 [hep-ph/9301294].

[33] I. Oda, “Conformal Higgs Gravity,” Adv. Stud. Theor. Phys. 9 (2015) 595 [arXiv:1505.06760 [gr-qc]].