A remark on the essential self-adjointness for
Klein-Gordon type operators

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Abstract
Here we discuss a new simplified proof of the essential self-adjointness
for formally self-adjoint differential operators of real principal type,
previously proved by Vasy (2020) and Nakamura-Taira (2021). For
simplicity, here we discuss the second order cases, i.e., Klein-Gordon
type operators only.

1 Introduction
We consider the second order operator of the form
\[ P = \sum_{j,k=1}^{n} D_j g^{jk}(x) D_k + \frac{1}{2} \sum_{j=1}^{n} (D_j u_j(x) + u_j(x) D_j) + u_0(x), \]
on \( L^2(\mathbb{R}^n) \), where \( D_j = -i \frac{\partial}{\partial x_j} \), \( j = 1, \ldots, n \), and \( n \geq 2 \). We suppose
all the coefficients are real-valued \( C^\infty \) functions. The top order coefficients
\( \{g^{jk}(x)\} \) is a Lorentzian cometric, and hence we suppose it is non-degenerate
for all \( x \in \mathbb{R}^n \). Moreover we suppose it is asymptotically flat, i.e., there is
a non-degenerate matrix \( \{g^{jk}_0\} \) such that \( g^{jk}(x) \rightarrow g^{jk}_0 \) as \( |x| \rightarrow \infty \). More
specifically, we suppose

Assumption A. For all \( j, k, g^{jk}(x), u_j(x), x \in \mathbb{R}^n \), are real-valued smooth
functions. Moreover, there exists \( 0 < \mu < 1 \) such that for any \( \alpha \in \mathbb{Z}_+^n \)
\[ |\partial^\alpha_x (g^{jk}(x) - g^{jk}_0)| \leq C_\alpha(x)|\alpha| - \mu^{|\alpha|}, \quad x \in \mathbb{R}^n, \ j, k = 1, \ldots, n, \]
\[ |\partial^\alpha_x u_j(x)| \leq C_\alpha(x)|\alpha| - \mu^{|\alpha|}, \quad x \in \mathbb{R}^n, \ j = 0, 1, \ldots, n, \]
with some \( C_\alpha > 0 \), where \( \langle x \rangle = (1 + |x|^2)^{1/2}. \)

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We also need the null-nontrapping condition. We write the principal
symbol by
\[ p_2(x, \xi) = \sum_{j,k=1}^{n} g^{jk}(x) \xi_j \xi_k. \]

We denote \( \exp(tH_{p_2}) \) be the Hamilton flow on \( \mathbb{R}^{2n} \) generated by the symbol
\( p_2(x, \xi) \), and we write
\[ (y(t, x_0, \xi_0), \eta(t, x_0, \xi_0)) = \exp(tH_{p_2})(x_0, \xi_0) \]
for \( t \in \mathbb{R}, (x_0, \xi_0) \in \mathbb{R}^{2n} \).

**Assumption B (Null non-trapping condition).** If \( (x_0, \xi_0) \in p_2^{-1}(\{0\}) \) and \( \xi_0 \neq 0 \), then \( |y(t, x_0, \xi_0)| \to \infty \) as \( |t| \to \infty \).

Now we can state our main result.

**Theorem 1.1.** Suppose Assumptions A and B. Then \( P \) is essentially self-
adjoint on \( C_0^\infty(\mathbb{R}^n) \).

The main purpose of this note is to simplify the argument of [14]. Here
we use a semiclassical analytic method, and it significantly simplify the
argument, especially concerning justifications of formal argument using the
Yosida approximation. The local regularity of the eigenfunction (see Section
4) is proved in this paper using the Hörmander’s propagation of singularities
theorem, whereas in [14] it is proved using the microlocal smoothing property
proved in [13], which also shorten the proof. The constructions of observables
which are decreasing along classical trajectories are refined here to make the
proof more transparent (Section 3).

The self-adjointness of the Klein-Gordon type operators on spacetime
does not obviously have physical importance. One of the motivation comes
from the construction of the Feynman propagator, which is important in
the construction of quantum field theory. The Feynman propagator may
be formally defined as the boundary value of the resolvent \( (P - z)^{-1} \)
on the Minkowski spacetime. Duistermaat-Hörmander [7] defines the Feynman
propagator on various spacetimes as an inverse with a certain wavefront
condition (see [9, Definition 1.1]) and showed the existence of an approximate
inverse with the wave front condition. However, the existence of the
Feynman propagator (that is, the actual inverse) had not been known since
then. In [8, 9, 10], the existence of the Feynman propagator is shown on
an asymptotically Minikowski spacetime under the null non-trapping condi-
tion. Independently, Dereziński and Siemssen considered the existence of
the Feynman propagators on more general spacetimes in [4, 5, 6]. In par-
ticular, they attempt to define the Feynman propagator as the boundary
value of the resolvent of the spacetime operator, and they conjectured the
following: The wave operator \( P \) is essentially self-adjoint on \( C_0^\infty \) and the
Feynman propagator defined in [4, 5] or [9, 10] coincides with the boundary value of the resolvent ([6, Conjecture 8.3]). For asymptotically Minkowski spacetimes, the first part of this problem is solved in [18] and [14] and the second part is proved in [17]. We address the first part in this paper.

Recently, Dang and Wrochna ([2, 3]) studied spectral geometry of scattering Lorentzian spaces. In [2], it is shown that the scalar curvature of scattering Lorentzian spaces is related to the integral kernel of a power of the outgoing resolvent. In order to justify the power $(P - i\varepsilon)^{-\alpha}$, the essential self-adjointness of $P$ is assumed in [2].

The essential self-adjointness of non-elliptic operators also attracted attention in another context. Colin de Verdière and Bihan showed that on generic compact Lorentzian surfaces the wave operator is not essentially self-adjoint on the space of smooth functions ([1]). Moreover, they conjectured that for a symmetric differential operator on a compact manifold, the completeness of the Hamiltonian flow is equivalent to the essential self-adjointness. This conjecture is solved for real principal type operators on the one-dimensional torus in [16].

This paper is organized as follows: In Section 2, we introduce notations, and prepare several basic tools. In Section 3, we show that microlocal Sobolev-type smoothness of eigenfunctions in the incoming region. The global regularity of the eigenfunction is proved in Section 4, and then Sobolev-type smoothness of eigenfunctions in the outgoing region is proved in Section 5. Finally we prove our main theorem in Section 6. Several technical lemmas and estimates are proved in Appendices.

2 Preliminaries

2.1 Symbol classes and the quantization

We use the following symbol class: For $k, \ell \in \mathbb{R}$, we write

$$S^{k,\ell} = S(\langle x \rangle^\ell \langle \xi \rangle^k, g),$$

where $S(m, g)$ denotes the Hörmander’s symbol class ([11, §18.4]) with $g = dx^2/\langle x \rangle^2 + d\xi^2/\langle \xi \rangle^2$. Namely, $a \in S^{k,\ell}$ if for any $\alpha, b \in \mathbb{Z}_n^+$ there is $C_{\alpha \beta} > 0$ such that

$$|\partial^\alpha_x \partial^\beta_\xi a(x, \xi)| \leq C_{\alpha \beta} \langle x \rangle^{\ell-|\alpha|} \langle \xi \rangle^{k-|\beta|}, \quad x, \xi \in \mathbb{R}^n.$$

We consider semiclassical symbols, i.e., symbols with semiclassical parameter $h > 0$, and its semiclassical Weyl quantization: For $a = a(h; x, \xi)$,

$$\text{Op}_h(a) \phi(x) = (2\pi h)^{-n} \iint e^{i(x-y) \cdot \xi/h} a(h; \frac{x+y}{2}, \xi) \phi(y) dyd\xi.$$
and we denote $\text{Op} := \text{Op}_1$. We denote the Poisson bracket by

$$\{a, b\} = \frac{\partial a}{\partial \xi} \cdot \frac{\partial b}{\partial x} - \frac{\partial a}{\partial x} \cdot \frac{\partial b}{\partial \xi} = \frac{d}{dt} \exp(tH_a)b\bigg|_{t=0}$$

for functions $a(x, \xi)$ and $b(x, \xi)$ on $\mathbb{R}^{2n}$. We denote the weighted Sobolev space $H^{s,t}(\mathbb{R}^n)$ defined by

$$H^{s,t}(\mathbb{R}^n) = \langle x \rangle^{-t} \langle D_x \rangle^{-s} [L^2(\mathbb{R}^n)].$$

2.2 The first reduction

In order to show the essential self-adjointness of a symmetric operator $P$, it is sufficient to show $\ker(P^* - z_\pm) = \{0\}$ for some $z_\pm \in \mathbb{C}$, $\pm \text{Im}(z_\pm) > 0$. We concentrate on the case $z = z_+$ in the following. The other case is similar. Let $\psi \in \ker(P^* - z)$, then it implies

$$\psi \in L^2(\mathbb{R}^n), \quad (P - z)\psi = 0 \text{ in the distribution sense. (2.1)}$$

Our theorem is proved if (2.1) implies $\psi = 0$. The first step of the proof is remark that it follows if we know $\psi$ is sufficiently good function.

Lemma 2.1. If $\psi$ satisfies (2.1) and $\psi \in L^2(\mathbb{R}^n) \cap H^{1,-1}(\mathbb{R}^n)$ then $\psi = 0$.

We give the proof in Appendix A.

Remark 2.1. Actually, the condition is relaxed to $\psi \in L^2(\mathbb{R}^n) \cap H^{1/2,-1/2}(\mathbb{R}^n)$, and this condition is used in Vasy [18] and Nakamura-Taira [14], but this condition is sufficient for our purpose, and more elementary to prove.

Thus it suffices to show $\psi \in H^{1,-1}(\mathbb{R}^n)$ from (2.1). In fact, we shall show $\psi \in H^{N,-\gamma}(\mathbb{R}^n)$ with any $N,\gamma > 0$.

2.3 Basic commutator estimate

We use the following simple commutator estimate as the basic tool in the proof of $\psi \in H^{N,-\gamma}(\mathbb{R}^n)$. Let $B$ be an $h$-pseudodifferential operator, and suppose we have the following operator inequality:

$$i[B^*B, P] \geq \frac{c}{h} B^* \langle x \rangle^{-1} B - \tilde{B}^* \langle x \rangle^{-1} \tilde{B} - E^* E,$$  (2.2)

where $c > 0$ and $E$ is another pseudodifferential operator. Then by simple algebraic computations, we obtain

$$\frac{c}{2h} \| \langle x \rangle^{-1/2} B \phi \|^2 + 2(\text{Im}z) \| B \phi \|^2$$

$$\leq \frac{2h}{c} \| \langle x \rangle^{1/2} B(P - z) \phi \|^2 + \| \langle x \rangle^{-1/2} \tilde{B} \phi \|^2 + \| E \phi \|^2$$  (2.3)
Figure 1: The definition of $\tau(x, \xi)$

for $\varphi \in S(\mathbb{R}^n)$ and $\| \cdot \| = \| \cdot \|_{L^2(\mathbb{R}^n)}$. We will apply this argument in the following steps, with various remainder terms $\tilde{B}$ and $E$ and $\text{Im} \, z > 0$. Moreover, when $0 < \gamma < \frac{1}{2}$, $B, \tilde{B}$ and $E \in \bigcap_{m \in \mathbb{R}} \text{Op} S^{m, \gamma}$, the inequality (2.3) holds for $\varphi \in L^2(\mathbb{R}^n)$ with $(P - z)\varphi \in R^{0, \frac{1}{2} + \gamma}(\mathbb{R}^n)$. We also note that the eigenfunction $\psi$ defined in (2.1) satisfies these conditions.

These computations are standard, but we give the proof in Appendix B for the completeness.

3 Incoming estimates

3.1 Incoming observable

We use an operator of the following form: We write

$$v(\xi) = \partial_\xi p_0(\xi), \quad \hat{v}(\xi) = \frac{v(\xi)}{|v(\xi)|}, \quad \hat{x} = \frac{x}{|x|}, \quad \beta(x, \xi) = \hat{x} \cdot \hat{v}(\xi),$$

and we set $0 < \gamma \ll \mu$, e.g., $\gamma = \mu/10$, and we set $0 < \sigma_\infty < 1$, which is close to 1, e.g., $\sigma_\infty = 9/10$. Then we set

$$\tau(x, \xi) = |x|(c_0 \sqrt{1 - \beta(x, \xi)^2} - \beta(x, \xi)), \quad c_0 = \frac{\sigma_\infty}{\sqrt{1 - \sigma_\infty^2}} \quad (3.1)$$

for $(x, \xi)$ such that $\beta(x, \xi) \leq \sigma_\infty$. The function $\tau(x, \xi)$ is the length of the line segment $\{x + t\hat{v}(\xi) \mid t \geq 0\}$ inside $\{(x, \xi) \mid \beta(x, \xi) \leq \sigma_\infty\}$ (see Figure 1).
Now we set
\[ b_-(x, \xi) = \tau(x, \xi)^\gamma \zeta_-(x, \xi), \]
where we specify the cut-off functions \( \zeta_-(x, \xi) \) in the following. We note the weight function \( \tau(x, \xi)^\gamma \) is homogeneous of order \( \gamma \) with respect to \( x \), and order 0 with respect to \( \xi \). We note this weight is increasing in the incoming directions, and the cut-off functions \( \zeta_-(x, \xi) \) localize this operator in the (microlocally) incoming region. The cut-off functions \( \zeta_-(x, \xi) \) also eliminate the singularity of this weight. For \( 0 < \delta \ll 1 \) and \( 0 < \sigma' < \sigma < \sigma_\infty \), we set
\[ \zeta_-(x, \xi) = \zeta_-(\delta, \sigma, \sigma', R; x, \xi) = \zeta_1(x/R, \xi) \zeta_2(x, \xi) \zeta_3(x, \xi), \]
where each function \( \zeta_j(x, \xi) \) has a different role.

We choose a smooth functions \( \chi_1 \in C^\infty(\mathbb{R}) \) such that
\[ \chi_1(s) = \begin{cases} 1, & s \leq -1, \\ 0, & s \geq 0, \end{cases} \]
\[ 0 \leq \chi_1(s) \leq 1, \chi_1'(s) \leq 0 \text{ for } s \in \mathbb{R}, \text{ and } 0 < \chi_1(s) < 1 \]
We also set
\[ \chi_2(s) = \chi_1(s - 2) \chi_1(2 - s) \]
so that
\[ \chi_2(s) = \begin{cases} 1, & |s| \leq 1, \\ 0, & |s| \geq 2. \end{cases} \]

1. **Cut-off to outside of parabola.**
\[ \zeta_1(x, \xi) = \chi_1(\frac{1}{2}|x_\xi|^2 + 1) \]
where
\[ x_\xi = x \cdot \hat{v}(\xi), \quad \frac{1}{2}|x_\xi|^2 = x - (x \cdot \hat{v}(\xi))\hat{v}(\xi) \]
for \( x, \xi \in \mathbb{R}^n \).

2. **Cut-off to the incoming area**
\[ \zeta_2(x, \xi) = \chi_1((\beta(x, \xi) - \sigma)/(\sigma - \sigma')) \]
so that
\[ \zeta_2(x, \xi) = \begin{cases} 1, & \text{if } \beta(x, \xi) \leq \sigma', \\ 0, & \text{if } \beta(x, \xi) \geq \sigma. \end{cases} \]

3. **Cut-off in the momentum variable**
\[ \zeta_3(x, \xi) = \chi_2(\frac{|\xi|^2 - 1}{\lambda(x, \xi)}), \quad \lambda(x, \xi) = 2\delta - \delta \langle \tau(x, \xi) \rangle^{-\nu}, \]
with \( 0 < \nu < \mu \).
We note $\tau(x, \xi)$ is decreasing along the free trajectory $x + t\hat{v}(\xi)$ inside the cone $\Gamma_-(\delta, \sigma, R)$, and hence $\lambda(x, \xi)$ is also decreasing there. Thus $\zeta_3(x, \xi)$ has support size decreasing along the trajectory, and $\lambda(x, \xi) = \delta$ at the boundary: $(x, \xi) : \beta(x, \xi) = \sigma$. Note also $\lambda(x, \xi) \to 2\delta$ as $t \to -\infty$ along the free trajectory.

The cut-off function $\zeta_-(x, \xi)$ satisfies the following properties: We denote

$$\Gamma_-(\delta, \sigma, R) = \left\{(x, \xi) \mid 1 - \delta \leq |\xi|^2 \leq 1 + \delta, \beta(x, \xi) \leq \sigma, |x| \geq R\right\}$$

where $\delta > 0$, $\sigma \in [-1, 1]$ and $R > 0$.

**Lemma 3.1.** For $0 < \delta \ll 1$ and $0 < \sigma' < \sigma < 1$, there are $R_0$ and $C_0 > 1$ such that for $R \geq R_0$ there exists $\zeta_-(\cdot, \cdot) = \zeta_-(\delta, \sigma, \sigma', R; \cdot, \cdot) \in S^{0,0}$ such that

$$\text{supp}[\zeta_-] \subset \Gamma_-(4\delta, \sigma, R), \quad \zeta_-(x, \xi) = 1 \text{ on } \Gamma_-(\delta, \sigma, C_0R),$$

and

$$\{p_2, \zeta_-\}(x, \xi) \leq 0$$

for all $(x, \xi) \in \mathbb{R}^{2n}$.

**Remark 3.1.** The product $\zeta_-$ belongs to a good symbol class $S^{0,0}$, although $\zeta_1 \notin S^{0,0}$. This is mainly because the projection of $\text{supp}\nabla \zeta_1 \cap \text{supp}\zeta_2$ into the $x$-space is compact. See Figure 2.

The proof is elementary, and we give it in Appendix C.

We need to compute an upper bound of $\{p_2, b_-\}$ by a nonpositive function. By straightforward computations, we can show
Lemma 3.2. (i) \( \{ p_2, \tau \} + |v(\xi)| \in S^{1-\mu} \) away from \( \{ x = 0 \} \cup \{ \xi = 0 \} \).

(ii) Under the above assumptions, for sufficiently large \( R > 0 \), there is \( c_1 > 0 \) such that

\[
\{ p_2, b_- \} \leq -c_1(x)^{-1}b_-.
\]

Remark 3.2. We note \( b_- \in S^{0,\gamma} \). In fact \( \zeta_-(x, \xi) \) is compactly supported in \( \xi \), and hence the decay with respect to \( \xi \) is irrelevant. Since \( p_2 \in S^{2,0} \), we have \( \{ p_2, b_- \} \in S^{1-1+\gamma} \).

Proof. (i) We recall

\[
\tau(x, \xi) = c_0 \sqrt{|x|^2 - (x \cdot \hat{v}(\xi))^2} - x \cdot \hat{v}(\xi) = c_0|x_x^*| - x \cdot \hat{v}(\xi),
\]

and hence

\[
\partial_x \tau(x, \xi) = c_0 x_x^\perp - \hat{v}(\xi),
\]

and in particular,

\[
-\hat{v}(\xi) \cdot \partial_x \tau(x, \xi) = 1.
\]

(3.2)

Hence, recalling \( \partial_x p_2 - v(\xi) \in S^{1-\mu} \), we have

\[
\{ p_2, \tau \} = \partial_x \tau \cdot \partial_p p_2 - \partial_x \tau \cdot \partial_x p_2 = -|v(\xi)| + S^{1-\mu}
\]

away from \( \{ x = 0 \} \cup \{ \xi = 0 \} \).

(ii) We look at

\[
\{ p_2, b_- \} = \{ p_2, \tau(x, \xi) \gamma \} \zeta_- + \tau(x, \xi) \gamma \{ p_2, \zeta_- \} = \gamma \tau(x, \xi) \gamma^{-1} \{ p_2, \tau(x, \xi) \} \zeta_- + \gamma \tau(x, \xi) \gamma \{ p_2, \zeta_- \} \leq \gamma \tau(x, \xi) \gamma^{-1} \{ p_2, \tau(x, \xi) \} \zeta_-.
\]

Thus we learn

\[
\{ p_2, b_- \} \leq -\gamma \tau(x, \xi) \gamma^{-1}(|v(\xi)| + O(|x|^{-\mu})|\xi|) \zeta_- \leq -c_1(x)^{-1} |v(\xi)| \tau(x, \xi) \gamma \zeta_- \leq -c_1(x)^{-1} b_-.
\]

since \( c|x| \leq \tau(x, \xi) \leq C|x| \) with some constants \( 0 < c < C < \infty \), and \( 1 - 4\delta \leq \gamma \gamma \) on \( \text{supp } \zeta_- \).

\( \square \)

3.2 Incoming regularities

We denote \( P_2 = \text{Op}(p_2) = h^{-2} \text{Op}_h(p_2) \). The sharp Gårding inequality and Lemma 3.2 imply, if we set \( B_0 = \text{Op}_h(b_-) \), then

\[
i |B_0^2 - P_2| \geq \frac{2c_0}{h} B_0(x)^{-1} B_0 + \tilde{E}_0, \quad \tilde{E}_0 = \text{Op}_h(\tilde{r}_0),
\]

where \( \tilde{r}_0 \in S^{-N-2+2\gamma} \) with any \( N \geq 0 \), and \( \tilde{r}_0 \) is supported in \( \text{supp } [b_0] \) modulo \( h^{-1} S^{-\infty,-\infty} \) terms.
Now we consider the total Hamilton operator $P$, and we denote $P = P_2 + Q$ and $Q = \text{Op}_h(q)$. By simple computations, we learn

$$q(h; x, \xi) = h^{-1} \sum_{j=1}^{n} u_j(x)\xi_j + u_0(x) + \frac{1}{4} \sum_{j,k=1}^{n} \partial_{x_j} \partial_{x_k} g^{jk}(x),$$

and in particular $q \in h^{-1}S^{1,-\mu}$. This implies the symbol of $-i[B_0^2, Q]$ is in $S^{0,-1-\mu+2\gamma}$.

Based on this observation, and by a standard semiclassical analysis argument, we can show the following:

We set $0 < \delta_0 < \delta_1 < \cdots < \delta_\infty \ll 1$, $0 < \sigma_0 < \sigma_1 < \cdots < \sigma_\infty < 1$, $0 < \sigma'_0 < \sigma'_1 < \cdots < \sigma'_\infty < 1$, $\sigma'_j < \sigma_j$ for $j = 0, 1, \ldots,$ and we choose $R_0 > R_1 > \cdots > R_\infty > 0$ so that

$$\zeta_j(x, \xi) = \zeta_-(\delta_j, \sigma_j, \sigma'_j, R_j; x, \xi)$$

satisfies the conditions of Lemma 3.1 for all $j$. We may suppose the constant $c_0$ in Lemma 3.1 is independent of $j$, i.e., $\{p_2, b_j\} \leq -c_0\langle x \rangle^{-1}b_j$ for all $j$.

We then set $b_j(x, \xi) = \tau(x, \xi)^7\zeta_j(x, \xi)$ and

$$B_j = \text{Op}_h(b_j) \in \bigcap_{k \in \mathbb{R}} \text{Op}_h(S^{k,\gamma}).$$

**Lemma 3.3.** For each $j = 0, 1, 2 \ldots$, there is a positive constant $\alpha_j > 0$ such that

$$i[B_j^2, P] \geq \frac{c_0}{h} B_j \langle x \rangle^{-1} B_j - \alpha_j B_{j+1} \langle x \rangle^{-1} B_{j+1} - E_j^* E_j$$

where $\|E_j\| = O(h^{\infty})$ as $h \to 0$.

**Proof.** We note that the principal symbol of $ih[B_j^2, P_2] - 2c_j B_j \langle x \rangle^{-1} B_j$ is

$$-\{p_2, b_j^2\} - 2c_j \langle x \rangle^{-1} b_j^2 = 2b_j(-\{p_2, b_j\} - c_j \langle x \rangle^{-1} b_j),$$

and by Lemma 3.2, we learn it is non-negative with suitable choice of $c_j > 0$. We also note the symbol is in $S^{0,-1+2\gamma}$. Then, by the sharp Gårding inequality, there is $a_0 \in S^{0,-2+2\gamma}$ such that

$$ih[B_j^2, P_2] - 2c_j B_j \langle x \rangle^{-1} B_j \geq -h\text{Op}_h(a_0),$$

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and we may assume supp[$a_0$] $\subset$ supp[$b_j$] modulo $O(h^\infty)$ smoothing terms. On the other hand, the symbol $a_1$ of $-i[B_j^2, Q]$ satisfies $a_1 \in S^{0,-1-\mu+2\gamma}$, and also supp[$a_1$] $\subset$ supp[$b_j$] modulo $O(h^\infty)$ smoothing terms. By the construction of $b_{j+1}$, we have $(a_0 + a_1)b_{j+1}^2 \in S^{0,-1}$. Using these observation and the construction of the parametrix, we can construct $\tilde{a}$ modulo $O(h^\infty)$ smoothing terms. Hence we have

$$\langle \varphi, Op_h(a_0 + a_1)\varphi \rangle = \langle (x)^{-1/2}B_{j+1}\varphi, Op_h(\tilde{a})((x)^{-1/2}B_{j+1})\varphi \rangle + O(h^\infty)\|\varphi\|^2$$

$$\leq \alpha_j\|\langle x \rangle^{-1/2}B_{j+1}\psi\|^2 + O(h^\infty)\|\psi\|^2$$

where $\alpha_j > 0$. Combining these, we conclude

$$i[B_j^2, P] \geq \frac{2c_j}{\hbar}B_j(x)^{-1}B_j - \alpha_jB_{j+1}(x)^{-1}B_{j+1} + O(h^\infty),$$

which completes the proof. \qed

Then, by (2.3) with $\psi$, $(P - z)\psi = 0$, we learn

$$\frac{c}{2\hbar}\|\langle x \rangle^{-1/2}B_j\psi\|^2 + 2(\text{Im}z)\|B_j\psi\|^2 \leq \alpha_j\|\langle x \rangle^{-1/2}B_{j+1}\psi\|^2 + \|E_j\psi\|^2$$

for each $j$, and in particular, for any $N \in \mathbb{N}$,

$$\|\langle x \rangle^{-1/2}B_j\psi\|^2 \leq h(2\alpha_j/c)\|\langle x \rangle^{-1/2}B_{j+1}\psi\|^2 + M_jh^{2N}\|\psi\|^2, \quad (3.3)$$

with some $C, M_j > 0$. At first, setting $j = 2N$ in (3.3), we learn $\|\langle x \rangle^{-1/2}B_{2N}\psi\| = O(\sqrt{h})$ since $\langle x \rangle^{-1/2}B_{2N+1}$ is bounded in $L^2(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^n)$. Then we use this and (3.3) with $j = 2N - 1$, we learn $\|\langle x \rangle^{-1/2}B_{2N-1}\psi\| = O(h)$. Iterating this procedure $2N$ times, and we arrive at

$$\|\langle x \rangle^{-1/2}B_1\psi\| = O(h^N)$$

for arbitrary $N \in \mathbb{N}$. Since $\text{Im}z > 0$, this then implies

$$\|B_0\psi\| = O(h^N) \quad (3.4)$$

with any $N$.

We set $\chi_3(\xi) = \chi_1(1 - |\xi|)$. Let $\sigma_0, \sigma_0'$ and $R_0$ as in Lemma 3.3, and we denote

$$\zeta_-(x, \xi) = \zeta_0^0(\sigma_0, \sigma_0'; R_0; x, \xi) = \zeta_1(x/R_0, \xi)\zeta_2(\sigma_0, \sigma_0'; x, \xi)\chi_3(\xi),$$

i.e., the cut-off function $\zeta_-(x, \xi)$ without the cut-off in the momentum variable, but with a cut-off $\chi_3(\xi)$ to eliminate singularities at $\xi = 0$. We also denote

$$b_0^0(x, \xi) = \tau(x, \xi)^\gamma\zeta_-(x, \xi).$$

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ζ^0 and b^0 are homogeneous of order 0 in \( \xi \), except for \( \chi_3(\xi) \). We set

\[
\tilde{\Gamma}_-(\sigma, R) = \{(x, \xi) \mid \beta(x, \xi) \leq \sigma, |x| \geq R\} \subset T^*\mathbb{R}^n,
\]

Then we have the following lemma.

**Lemma 3.4.** Suppose \( \psi \in L^2(\mathbb{R}^n) \) and \( (P - z)\psi = 0 \) with \( \text{Im} z > 0 \). Then we have the following lemma.

1) \( \text{Op}(b^0)\psi \in H^\infty(\mathbb{R}^n) = \bigcap_{j=0}^{\infty} H^j(\mathbb{R}^n) \).

2) \( (x)^{-\gamma}\text{Op}(\zeta^0)\psi \in H^\infty(\mathbb{R}^n) \).

3) \( \text{WF}(\psi) \cap \tilde{\Gamma}_-(\sigma_0', R_0) = \emptyset \), where \( \text{WF}(\cdot) \) denotes the wave front set.

**Proof.** The statement i) follows from (3.4) and the standard semiclassical characterization of the smoothness, or equivalently, the Besov space argument. The statement ii) follows from i), since \( \tau(x, \xi)^\gamma \leq (x)^\gamma \). The last statement iii) follows as well as i) from (3.4) and the semiclassical characterization of the wave front set since \( \zeta^0(x, \xi) \) does not vanish on \( \tilde{\Gamma}_-(\sigma_0', R_0) \).

4 Overall smoothness: Propagation of singularities theorem

Now we use the nontrapping assumption and the celebrated propagation of singularities theorem of Hörmander to show that \( \psi \) is smooth everywhere.

By the nontrapping condition, for any \((x_0, \xi_0) \in \tilde{\mathcal{P}}^{-1}(\{0\})\) with \( \xi_0 \neq 0 \), there exists \( \xi_- \neq 0 \) such that \( \eta(t; x_0, \xi_0) \to \xi_- \) as \( t \to -\infty \), and also \( y(t; x_0, \xi_0)/t \to v(\xi_-) \) (see [14, (B.1)]), and hence we have

\[
\dot{y}(t; x_0, \xi_0) \cdot \dot{v}(\eta(t; x_0, \xi_0)) \to -1 \quad \text{as} \quad t \to -\infty.
\]

In particular, the trajectory enters \( \tilde{\Gamma}_-(\sigma_0', R_0) \) for \( t \ll 0 \). Then by the result of the last step and the propagation of singularities theorem, we learn \((x_0, \xi_0)\) is not in the wave front set of \( \psi \). Thus we have the following lemma from Lemma 3.4 and the propagation of singularities theorem ([11] Theorem 23.2.9):

**Lemma 4.1.** Suppose \( \psi \in L^2(\mathbb{R}^n) \) and \( (P - z)\psi = 0 \) with \( \text{Im} z > 0 \). Then \( \psi \in C^\infty(\mathbb{R}^n) \).

5 Outgoing estimates

5.1 Outgoing observable

In this section, we use the symbols \( \sigma, \sigma', \sigma_\infty \), etc., as in Section 3, but here we assign different values. Let \( \sigma_\infty \in (-1, 0) \), close to \(-1 \), e.g., \( \sigma_\infty = -9/10 \).
Then we set
\[
\tau(x, \xi) = |x| (\beta(x, \xi) - c_0 \sqrt{1 - \beta(x, \xi)^2}), \quad c_0 = \frac{\sigma_\infty}{\sqrt{1 - \sigma_\infty^2}} \tag{5.1}
\]
for \((x, \xi)\) such that \(\beta(x, \xi) \geq \sigma_\infty\).

Let \(\sigma_\infty < \sigma < \sigma' < 0\), \(0 < \delta \ll 1\) and \(R > 0\). For the outgoing cut-off we can use the following construction:
\[
\zeta_+(\delta, \sigma, \sigma', R; x, \xi) = \tilde{\zeta}_1(x/R, \xi)\tilde{\zeta}_2(x, \xi)\tilde{\zeta}_3(x, \xi),
\]
where
\[
\tilde{\zeta}_1(x, \xi) = \chi_1(-x_\xi - \frac{1}{2}|x_\xi|^2 + 1),
\]
\[
\tilde{\zeta}_2(x, \xi) = \chi_1((\sigma - \beta(x, \xi))/(\sigma' - \sigma)),
\]
\[
\tilde{\zeta}_3(x, \xi) = \chi_2\left(\frac{|\xi|^2 - 1}{\lambda_+(x, \xi)}\right), \quad \lambda_+(x, \xi) = \delta_0 + \delta_0 \langle \tau(x, \xi) \rangle^{-\nu}.
\]

We note that here \(\tau(x, \xi)\) is defined by (5.1) and increasing along the free classical trajectory \(x + tv(\xi)\). Thus \(\lambda_+(x, \xi)\) is a decreasing function on the support of \(\zeta_+\).

Analogously to the incoming case, we set
\[
\Gamma_+(\delta, \sigma, R) = \{(x, \xi) \mid 1 - \delta \leq |\xi|^2 \leq 1 + \delta, \beta(x, \xi) \geq \sigma, |x| \geq R\}
\]
where \(\delta > 0\), \(\sigma \in [-1, 1]\) and \(R > 0\). Then we have

**Lemma 5.1.** For \(0 < \delta \ll 1\) and \(-\sigma_\infty < \sigma < \sigma' < 0\), there is \(R_0\) and \(C_0 > 0\) such that for \(R \geq R_0\) there exists \(\zeta_+(\cdot, \cdot) = \zeta_+(\delta, \sigma, \sigma', R; \cdot, \cdot) \in S^{0,0}\) such that
\[
\text{supp}[\zeta_+] \subset \Gamma_+(4\delta, \sigma, R), \quad \zeta_+(x, \xi) = 1 \text{ on } \Gamma_+(\delta, \sigma', C_0 R),
\]
and
\[
\{p_2, \zeta_+(x, \xi)\} \leq \rho(x, \xi), \quad (x, \xi) \in \mathbb{R}^{2n}, \tag{5.2}
\]
where \(\rho = \rho(\delta, \sigma, \sigma', R ; \cdot, \cdot) \in S^{0,-1+\gamma}\) such that
\[
\text{supp}[\rho] \subset \{(x, \xi) \mid 1 - 4\delta \leq |\xi|^2 \leq 1 + 4\delta, |x| \leq C_0 R \text{ or } \sigma \leq \beta(x, \xi) \leq \sigma'\}.
\]

The proof is essentially the same as Lemma 3.1, and we sketch the proof in Appendix C.

We set
\[
0 < \delta_0 < \delta_1 < \cdots < \delta_\infty \ll 1, \quad 0 > \sigma_0 > \sigma_1 > \cdots > \sigma_\infty
\]
\[
0 > \sigma'_0 > \sigma'_1 > \cdots > \sigma'_\infty > \sigma_\infty, \quad \sigma_j < \sigma'_j \quad \text{for } j = 0, 1, \ldots,
\]
and we choose
\[
R_0 > R_1 > \cdots > R_\infty > 0
\]
Then, as well as Lemma 3.3, we similarly have

\[ b_j^+(x, \xi) = \tau(x, \xi)\zeta_j^+(x, \xi) \quad \text{and} \quad B_j^+ = \text{Op}_h(b_j^+). \]

We denote \( \rho(x, \xi) \) in Lemma 5.1 with the constants \( \delta_j, \sigma_j, \sigma_j', R_j \) by \( \rho_j(x, \xi) \).

Then, as well as Lemma 3.3, we similarly have

\[ i[(B_j^+)^2, P] \geq \frac{c_0}{\hbar} B_j^+ \langle x \rangle^{-1} B_j^+ - \alpha_j B_j^+ \langle x \rangle^{-1} B_{j+1}^+ - S_j - E_j^* E_j \]

where \( S_j \) is an \( h \)-pseudodifferential operator in \( \text{Op}(h^{-1}S^{0,-1+\gamma}) \) with the principal symbol \( h^{-1}\rho_j = h^{-1}\rho(\delta_j, \sigma_j, \sigma_j', R_j; \cdot, \cdot) \) and it has the same support as \( \rho_j \), and \( \|E_j\| = O(h^\infty) \). Thus, we again learn, for any \( N \in \mathbb{N} \),

\[ \frac{c}{2h\hbar} \|\langle x \rangle^{-1/2}B_j^+\psi\|^2 + 2(\text{Im} z)\|B_j^+\psi\|^2 \leq \alpha_j\|\langle x \rangle^{-\gamma}B_{j+1}^+\psi\|^2 + \langle \psi, S_j\psi \rangle + \|E_j\psi\|^2 \]

for each \( j \). We note that the support of \( \rho_j \) is contained in \( \Gamma_j(4\delta_j, \sigma_j, \sigma_j', R) \) outside a compact set in \( x \)-space, and hence by Lemmas 3.4 and 4.1, we learn \( \langle \psi, S_j\psi \rangle = O(h^N) \) with any \( N \) as \( h \to 0 \). In particular, we have

\[ \|\langle x \rangle^{-1/2}B_j^+\psi\|^2 \leq \text{Ch}_j\|\langle x \rangle^{-\gamma}B_{j+1}^+\psi\|^2 + C_j h^{2N}\|\psi\|^2. \quad (5.3) \]

Now using the same iteration step as in the incoming case, we have:

\[ \|B_j^+\psi\| = O(h^N) \quad (5.4) \]

for arbitrary \( N \in \mathbb{N} \). Now we have the regularity theorem in the outgoing case, as well as the incoming case.

Let \( \sigma_0, \sigma_0', R_0 \) as above. Analogously to the incoming case, we denote

\[ \zeta_0^+(x, \xi) = \zeta_0^+(\sigma_0, \sigma_0', R_0; x, \xi) = \zeta_1(x/R_0, \xi)\zeta_2(\sigma_0, \sigma_0'; x, \xi)\chi_3(\xi), \]

\[ b_0^+(x, \xi) = \tau(x, \xi)\zeta_0^+(x, \xi), \]

\[ \Gamma_0(\sigma, R_0) = \{ (x, \xi) \mid \beta(\sigma, \xi) \geq \sigma, |x| \geq R \} \subset T^*\mathbb{R}^n. \]

**Lemma 5.2.** Suppose \( \psi \in L^2(\mathbb{R}^n) \) and \( (P - z)\psi = 0 \) with \( \text{Im} z > 0 \). Then

i) \( \text{Op}(b_0^+\psi) \in H^\infty(\mathbb{R}^n) \).

ii) \( \langle x \rangle^{-\gamma}\text{Op}(\zeta_0^+)\psi \in H^\infty(\mathbb{R}^n) \).

### 6 Proof of Theorem 1

Combining results of Lemmas 3.4, 4.1 and 5.2, we learn \( \psi \in H^{-N,\gamma}(\mathbb{R}^n) \) with any \( N \in \mathbb{N} \) and sufficiently small \( \gamma > 0 \), provided \( (P - z)\psi = 0 \) and \( \psi \in L^2(\mathbb{R}^n) \), \( \text{Im} z > 0 \). Then, by Lemma 2.1, we conclude \( \psi = 0 \) and hence \( \text{Ker}(P^* - z) = \{0\} \). Similarly we can prove \( \text{Ker}(P^* - z) = \{0\} \) when \( \text{Im} z < 0 \). We note we have used the *incoming* nontrapping condition in the above argument, but we use the *outgoing* nontrapping condition when \( \text{Im} z < 0 \). \( \square \)
Appendix A  Proof of Lemma 2.1

Suppose $\psi$ satisfies the conditions of Lemma 2.1. At first we note that if $\varphi \in H^1(\mathbb{R}^n)$ and $P_\varphi \in L^2(\mathbb{R}^n)$, then by the definition of the distributional derivative, we learn
\[
\langle \varphi, P\varphi \rangle = \sum_{j,k=1}^n \int g^{jk}(x) D_j \varphi(x) D_k \varphi(x) dx + \operatorname{Re} \left( \sum_{j=1}^n \int u_j \varphi(x) D_j \varphi(x) dx \right) + \int |u_0| |\varphi(x)|^2 dx \in \mathbb{R}.
\]

We choose a smooth function $\chi \in C_0^\infty(\mathbb{R}^n; [0,1])$ such that $\chi(x,\xi) = 1$ for $|x| \leq 1$. We set $X_R \varphi(x) = \chi(x/R) \varphi(x)$ for $R > 0$ and $\varphi \in L^2(\mathbb{R}^n)$. Then $X_R \psi \in H^1(\mathbb{R}^n)$ for each $R > 0$, and hence we learn
\[
\operatorname{Im} \langle X_R \psi, (P - z) X_R \psi \rangle = -\operatorname{Im} z \|X_R \psi\|^2.
\]

On the other hand, we have
\[
\langle X_R \psi, (P - z) X_R \psi \rangle = \langle X_R \psi, X_R (P - z) \psi \rangle + \langle X_R \psi, [P, X_R] \psi \rangle = \langle X_R \psi, [P, X_R] \psi \rangle.
\]

It is easy to observe that $[P, X_R]$ is a first order differential operator with the coefficients uniformly bounded by $C|x|^{-1}$, and converges to 0 pointwise as $R \to \infty$. Thus $[P, X_R] \psi$ is bounded by an $L^2$ function, and then by the dominated convergence theorem, we have $\langle X_R \psi, [P, X_R] \psi \rangle \to 0$ as $R \to \infty$. Now we conclude
\[
-\operatorname{Im} z \|\psi\|^2 = \lim_{R \to \infty} (\operatorname{Im} z \|X_R \psi\|^2) = \lim_{R \to \infty} \langle X_R \psi, [P, X_R] \psi \rangle = 0,
\]
and thus $\psi = 0$.

Appendix B  Proof of the basic commutator estimate

In this appendix, we prove a basic inequality used in Subsection 2.3. More precisely, we show
\[
i[B^* B, P] \geq \frac{c}{\hbar} B^* \langle x \rangle^{-1} B - \tilde{B}^* \langle x \rangle^{-1} \tilde{B} - E^* E,
\]
implies
\[
\frac{c}{2\hbar} \|\langle x \rangle^{-1/2} B \varphi\|^2 + 2(\operatorname{Im} z) \|B \varphi\|^2 \\
\leq \frac{2\hbar}{c} \|\langle x \rangle^{1/2} B(P - z) \varphi\|^2 + \|\langle x \rangle^{-1/2} \tilde{B} \varphi\|^2 + \|E \varphi\|^2,
\]

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where \( \| \cdot \| = \| \cdot \|_{L^2(\mathbb{R}^n)} \).

At first, we prove (B.2) for \( \varphi \in S(\mathbb{R}^n) \). If \( \varphi \in S(\mathbb{R}^n) \), we have

\[
i\langle \varphi, [B^* B, P] \varphi \rangle = i\langle \varphi, (B^* BP - PB^* B) \varphi \rangle
= i(\langle B \varphi, B(P - z) \varphi \rangle - \langle B(P - z) \varphi, B \varphi \rangle)
= -2\text{Im}(\langle B \varphi, B(P - z) \varphi \rangle) - 2(\text{Im} z)\|B \varphi\|^2
\leq 2\|\langle \rangle^{-1/2} B \varphi\| \|\langle \rangle^{1/2} B(P - z) \varphi\| - 2(\text{Im} z)\|B \varphi\|^2.
\]

On the other hand, we have

\[
c\frac{\langle \varphi, B \langle x \rangle^{-1} B \varphi \rangle - \langle \varphi, \tilde{B} \langle x \rangle^{-1} \tilde{B} \varphi \rangle - \langle \varphi, E^* E \varphi \rangle}{h}
= c\frac{\|\langle \rangle^{-1/2} B \varphi\|^2 - \|\langle \rangle^{-1/2} \tilde{B} \varphi\|^2 - \|E \varphi\|^2}{h}.
\]

Combining them with our assumption (B.1), we learn

\[
c\frac{\langle \varphi, B \langle x \rangle^{-1} B \varphi \rangle - \langle \varphi, \tilde{B} \langle x \rangle^{-1} \tilde{B} \varphi \rangle - \langle \varphi, E^* E \varphi \rangle}{h}
\leq 2\|\langle \rangle^{-1/2} B \varphi\| \|\langle \rangle^{1/2} B(P - z) \varphi\| + \|\langle \rangle^{-1/2} \tilde{B} \varphi\|^2 + \|E \varphi\|^2.
\]

Now we use the elementary bound:

\[
\|\langle \rangle^{-1/2} B \varphi\| \|\langle \rangle^{1/2} B(P - z) \varphi\| \leq \frac{c}{4h} \|\langle \rangle^{-1/2} B \varphi\|^2 + \frac{h}{c} \|\langle \rangle^{1/2} B(P - z) \varphi\|^2
\]

in the right hand side, and we obtain (B.2) for \( \varphi \in S(\mathbb{R}^n) \).

In applications, we use (B.2) for \( \varphi \in L^2(\mathbb{R}^n) \) such that \( (P - z) \varphi \in H^{0,1/2+\gamma} \), and we need to show the inequality extends to such functions. Since \( B, \tilde{B}, E \in \bigcap_{m \in \mathbb{R}} \text{Op} S^{m,\gamma} \), it is easy to observe that (B.2) is extended to \( \varphi \in \cap_{\ell \in \mathbb{R}} H^{0,\ell} \).

Now let \( A \) be one of the operators \( \langle x \rangle^{-1/2} B, B, \langle x \rangle^{1/2} B(P - z), \langle x \rangle^{-1/2} \tilde{B} \) and \( E \). Let \( X_R \) be the operator used in the last Appendix. Then \([A, X_R]\) is a pseudodifferential operator with the symbol which is bounded in \( S^{0,-1/2+\gamma} \) and supported in supp\(\nabla X_R\) \( \subset \{|x| \geq R\} \). These imply

\[
\|[A, X_R]\|_{L^2 \rightarrow L^2} \leq CR^{-1/2+\gamma} \rightarrow 0, \quad R \rightarrow \infty
\]

by the \( L^2 \)-boundedness theorem for pseudodifferential operators. Using this, and since \( X_R \varphi \in \bigcap_{\ell \in \mathbb{R}} H^{0,\ell}(\mathbb{R}^n) \) if \( \varphi \in L^2(\mathbb{R}^n) \), we have

\[
\frac{c}{2h} \|\langle x \rangle^{-1/2} B \varphi\|^2 + 2(\text{Im} z)\|B \varphi\|^2
= \lim_{R \rightarrow \infty} \left( \frac{c}{2h} \|X_R \langle x \rangle^{-1/2} B \varphi\|^2 + 2(\text{Im} z)\|X_R B \varphi\|^2 \right)
\leq \lim_{R \rightarrow \infty} \left( \frac{2h}{c} \|X_R \langle x \rangle^{1/2} B(P - z) \varphi\|^2 + \|X_R \langle x \rangle^{-1/2} \tilde{B} \varphi\|^2 + \|X_R E \varphi\|^2 \right)
= \frac{2h}{c} \|\langle x \rangle^{1/2} B(P - z) \varphi\|^2 + \|\langle x \rangle^{-1/2} \tilde{B} \varphi\|^2 + \|E \varphi\|^2,
\]

provided \( \varphi \in L^2(\mathbb{R}^n) \) and \( (P - z) \varphi \in H^{0,1/2+\gamma} \). □
Appendix C  Proof of Lemmas 3.1 and 5.1

Proof of Lemma 3.1. It suffices to prove that for each \( j = 2, 3 \),

\[
\{ p_2, \zeta_1(x/R, \xi) \} \leq 0, \quad \{ p_2, \zeta_j(x, \xi) \} \leq 0
\]
on supp\([\zeta_1(x/R, \cdot)\zeta_2\zeta_3]\) for sufficiently large \( R \).

Throughout this proof, we denote \( \delta = \sigma - \sigma' > 0 \) for simplicity. At first, we consider the estimate for \( \zeta_1(x/R, \xi) \). We note

\[
supp[\zeta_1] \subset \{ (x, \xi) \mid x_\xi^\| \leq -1 + \frac{1}{2}|x_\xi^\perp|^2 \} \subset \{ (x, \xi) \mid |x| \geq 1 \},
\]
and

\[
supp[\partial_{(x, \xi)}\zeta_1] \subset \{ (x, \xi) \mid -2 + \frac{1}{2}|x_\xi^\perp|^2 \leq x_\xi^\| \leq -1 + \frac{1}{2}|x_\xi^\perp|^2 \}.
\]

Moreover,

\[
\hat{v}(\xi) \cdot \partial_x \zeta_1(x, \xi) = \chi'_1(x_\xi^\perp - \frac{1}{2}|x_\xi^\perp|^2 + 1) \leq 0.
\]

Since \( \partial_\xi p_2 = v(\xi) + O(|\xi|\hat{x}^{-\mu}) \), we have

\[
\partial_\xi p_2 \cdot \partial_x \zeta_1(x/R, \xi) = R^{-1}\chi'_1((x/R)_\xi^\perp - \frac{1}{2}|(x/R)_\xi^\perp|^2 + 1) \cdot (1 + O(R^{-\mu}))|\xi|.
\]

Similarly, since \( \partial_x p_2 = O(|\xi|^2\hat{x}^{-1-\mu}) \) and \( \zeta_1(x, \xi) \) is homogeneous in \( \xi \), we learn

\[
\partial_x p_2 \cdot \partial_x \zeta_1(x/R, \xi) = \chi'_1((x/R)_\xi^\perp - \frac{1}{2}|(x/R)_\xi^\perp|^2 + 1) \cdot O(R^{-1-\mu})|\xi|.
\]

These imply

\[
\{ p_2, \zeta_1(x/R, \xi) \} = R^{-1}\chi'_1((x/R)_\xi^\perp - \frac{1}{2}|(x/R)_\xi^\perp|^2 + 1) \cdot (1 + O(R^{-\mu}))|\xi| \leq 0
\]
on supp\([\zeta_1(x/R, \xi)\zeta_2(x, \xi)\zeta_3(x, \xi)]\) for sufficiently large \( R \).

Next, we deal with the estimate for \( \zeta_2 \). We recall \( \zeta_2 \) is homogenous in \( (x, \xi) \), and we note

\[
\partial_x \zeta_2(x, \xi) = (\partial_x \beta(x, \xi)) \chi'_1((\beta(x, \xi) - \sigma)/\delta)/\delta
\]

\[
= |x|^{-1}(\hat{v}(\xi) - \hat{x}\beta(x, \xi)) \cdot \chi'_1((\beta(x, \xi) - \sigma)/\delta)/\delta,
\]
and in particular

\[
\hat{v}(\xi) \cdot \partial_x \zeta_2(x, \xi) = |x|^{-1}(1 - \beta(x, \xi)^2) \chi'_1((\beta(x, \xi) - \sigma)/\delta)/\delta
\]

\[
\leq \delta^{-1}|x|^{-1}(1 - \sigma^2) \chi'_1((\beta(x, \xi) - \sigma)/\delta) \leq 0.
\]

Similarly to the argument for \( \zeta_1 \), if \( |x| \geq R \) then we have

\[
\partial_\xi p_2 \cdot \partial_x \zeta_2 = \delta^{-1}|x|^{-1}(1 - \sigma^2) \chi'_1((\beta(x, \xi) - \sigma)/\delta) \cdot (1 + O(R^{-\mu}))|\xi|.
\]
and
\[ \partial_x p_2 \cdot \partial_x \zeta_2(x, \xi) = \chi'_1((\beta(x, \xi) - \sigma)/\delta) \cdot O(|x|^{-1-\mu} |\xi|). \]

Combining these, we have
\[ \{p_2, \zeta_2\} = \delta^{-1} |x|^{-1} (1 - \sigma^2) \chi'_1((\beta(x, \xi) - \sigma)/\delta) \cdot (1 + O(R^{-\mu})) |\xi| \leq 0 \]
on \text{supp}[\zeta_1(x/R, \xi)] \zeta_2(x, \xi)] \zeta_3(x, \xi)]

Finally, we consider the estimate for $\zeta_3$. We now note $\tau(x, \xi)$ is the length of the line segment \{ $x + t\hat{v}(\xi)$ \mid $t \geq 0$ \} inside \{ $(x, \xi)$ \mid $\beta(x, \xi) \leq \sigma_\infty$ \}. We recall
\[ \tau(x, \xi) = c_0 \sqrt{|x|^2 - (x \cdot \hat{v}(\xi))^2} - x \cdot \hat{v}(\xi) = c_0 |x_x^{1/2} | - x \cdot \hat{v}(\xi), \]
and hence
\[ \partial_x \tau(x, \xi) = c_0 x_x^{1/2} - \hat{v}(\xi), \]
and in particular,
\[ -v(\xi) \cdot \partial_x \tau(x, \xi) = |v(\xi)|. \] (C.1)

We also note
\[ c_1 |x| \leq \tau(x, \xi) \leq C_1 |x| \quad \text{for} \quad (x, \xi) \in \text{supp} [\zeta_2] \]
with some $0 < c_1 < C_1$. We also note
\[ \partial_x \zeta_3(x, \xi) = \chi'_2 \left( \frac{|\xi|^2 - 1}{\lambda(x, \xi)} \right) \left( \frac{2\xi}{\lambda(x, \xi)} - (|\xi|^2 - 1) \frac{\partial_x \lambda(x, \xi)}{\lambda(x, \xi)^2} \right) \]
where $(\cdots)$ is smooth and uniformly bounded on the support of $\chi'_2(\cdots)$. On the other hand,
\[ \partial_x \zeta_3(x, \xi) = -\chi'_2 \left( \frac{|\xi|^2 - 1}{\lambda(x, \xi)} \right) \frac{\partial_x \lambda(x, \xi)}{\lambda(x, \xi)^2} \]
\[ = -\chi'_2 \left( \frac{|\xi|^2 - 1}{\lambda(x, \xi)} \right) \frac{|\xi|^2 - 1}{\lambda(x, \xi)^2} \frac{\nu \delta_0 \tau(x, \xi) \partial_x \tau(x, \xi)}{(\tau(x, \xi))^{2+\nu}} \]
and in particular, by (C.1), we have
\[ v(\xi) \cdot \partial_x \zeta_3(x, \xi) = \chi'_2 \left( \frac{|\xi|^2 - 1}{\lambda(x, \xi)} \right) \frac{|\xi|^2 - 1}{\lambda(x, \xi)^2} \frac{\nu \delta_0 \tau(x, \xi)|v(\xi)|}{(\tau(x, \xi))^{2+\nu}}. \]
This also implies
\[ \partial_x p_2(x, \xi) \cdot \partial_x \zeta_3(x, \xi) = \chi'_2 \left( \frac{|\xi|^2 - 1}{\lambda(x, \xi)} \right) \frac{|\xi|^2 - 1}{\lambda(x, \xi)^2} \frac{\nu \delta_0 \tau(x, \xi)|v(\xi)|}{(\tau(x, \xi))^{2+\nu}} (1 + O(|x|^{-\mu})). \]
Noting $\chi'_2(t) t \leq 0$ for $t \in \mathbb{R}$ and $|t| \geq 1$ on supp[$\chi'_2(t)$], we learn that
\[ \partial_x p_2(x, \xi) \cdot \partial_x \zeta_3(x, \xi) \leq -c_2 \chi'_2 \left( \frac{|\xi|^2 - 1}{\lambda(x, \xi)} \right) |\xi|^{-1-\nu} \]
with some $c_2 > 0$ on $\text{supp}[\zeta_2]$. On the other hand, using $\partial_x p_2(x, \xi) = O(|\xi|^2 \langle x \rangle^{-1-\mu})$ again, we have

$$\partial_x p_2(x, \xi) \cdot \partial_\xi \zeta_3(x, \xi) = \lambda'_2 \left( \frac{|\xi|^2-1}{\lambda(x, \xi)} \right) \times O(|\xi|^3 \langle x \rangle^{-1-\mu}).$$

Since $|\xi|$ is bounded on $\text{supp} \zeta_3$, these imply

$$\{p_2, \zeta_3\} \leq -c_2 |\lambda'_2 \left( \frac{|\xi|^2-1}{\lambda(x, \xi)} \right) | \langle x \rangle^{-1-\mu} \leq 0$$

on $\text{supp}[\zeta_1(x/R, \xi) \zeta_2(x, \xi) \zeta_3(x, \xi)]$ with sufficiently large $R$. □

**Proof of Lemma 5.1.** At first, we note if we set

$$\rho(x, \xi) = \{p_2, \tilde{\zeta}_1(x, \xi) \tilde{\zeta}_2(x, \xi)\},$$

then $\rho$ satisfies the properties of the lemma, and it suffices to show $\{p_2, \tilde{\zeta}_4\}$ is nonpositive on the support to prove the inequality (5.2). The computation is almost identical to the one in the proof of Lemma 3.1 above, but we remark necessary changes. Even though the definition of $\lambda_+(x, \xi)$ is different from $\lambda(x, \xi)$, we have the same derivative formula:

$$\partial_x \lambda_+(x, \xi) = -\nu \delta_0 \frac{\tau(x, \xi) \partial_x \tau(x, \xi)}{\tau(x, \xi)},$$

and we have the same bound eventually:

$$\partial_\xi p_2(x, \xi) \cdot \partial_x \tilde{\zeta}_3(x, \xi) \leq -c_2 |\lambda'_2 \left( \frac{|\xi|^2-1}{\lambda(x, \xi)} \right) | \langle x \rangle^{-1-\mu}.$$  

The rest of the computation is carried out without changes to conclude $\tilde{\zeta}_1 \tilde{\zeta}_2 \{p_2, \tilde{\zeta}_3\} \leq 0$ with sufficiently large $R$. □

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