Fredholm theory for band-dominated and related operators: a survey

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Abstract

This paper presents the Fredholm theory on $l^p$-spaces for band-dominated operators and important subclasses, such as operators in the Wiener algebra. It particularly closes several gaps in the previously known results for the case $p = \infty$ and addresses the open questions raised by Chandler-Wilde and Lindner [3]. The main tools are provided by the limit operator method and an algebraic framework for the description and adaption of Fredholmness and convergence. A comprehensive overview of this approach is given.

Keywords: Fredholm theory; Limit operator; Band-dominated operator.

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1 Introduction

During the last years a far reaching theory on band-dominated operators grew up and revealed deep results concerning their Fredholm property, their spectral properties, the applicability (and further side effects) of the finite section method, and was successfully applied to several more concrete subclasses of operators having more advanced properties. Among them one can find Toeplitz, Hankel or Jacobi operators, hence also discrete Schroedinger operators.

Besides, the wonderful concept of operator algebras arising from approximate projections and the limit operator method were developed. Today one may say that large parts of this theory are very well understood and finalized. However, there are some results which still require some additional assumptions that seem to be redundant but could not be removed completely yet. One prominent example is the need for a predual setting in case of band-dominated operators on $l^\infty$-spaces in the works of Lindner. A collection of eight open problems was stated by Chandler-Wilde and Lindner in the final chapter of [3].

The aim of the present text is to give an overview on the latest state of the art and to close several gaps in the theory on band-dominated operators. We show that a couple of results being known for band-dominated operators actually hold in a more general context, and we try to clear up some possible generalizations which have already been indicated in the literature. We particularly contribute to seven of the eight open questions stated in [3] answering four of them completely.
1.1 Band-dominated operators

Let us start introducing the basic notions and the precise description of two of the mentioned (and actually redundant) conditions (1) and (2).

**Sequence spaces**
Given \( N \in \mathbb{N} \), a Banach space \( X \), and the parameter \( 1 \leq p < \infty \) we let \( l^p = l^p(\mathbb{Z}^N, X) \) denote the space of all functions \( x : \mathbb{Z}^N \rightarrow X \) with the property

\[
\|x\|_p := \left( \sum_{i \in \mathbb{Z}^N} \|x(i)\|^p \right)^{1/p} < \infty.
\]

Provided with the norm \( \|x\|_p \), \( l^p \) becomes a Banach space. It is convenient to refer to such functions as sequences \((x_i)_{i \in \mathbb{Z}^N}\) with \( x_i = x(i) \). The above condition then simply means that the sequences shall be \( p \)-summable. Analogously, one introduces the respective Banach spaces \( l^\infty = l^\infty(\mathbb{Z}^N, X) \) of all bounded sequences \( x = (x_i)_{i \in \mathbb{Z}^N} \) of elements \( x_i \in X \) with the norm \( \| \cdot \|_\infty \) defined by

\[
\|x\|_\infty := \sup \{ \|x_i\| : i \in \mathbb{Z}^N \}
\]

and, finally, its closed subspace \( l^0 = l^0(\mathbb{Z}^N, X) \) consisting of all bounded sequences \((x_i)\) with \( \|x_i\| \rightarrow 0 \) as \( |i| \rightarrow \infty \).

Note that, choosing \( X = L^p((0,1)^N) \), \( l^p(\mathbb{Z}^N, X) \) is isometrically isomorphic to \( L^p(\mathbb{R}^N) \). Hence, all subsequent definitions and results for the “discrete” \( l^p \)-cases have their “continuous” \( L^p \)-counterparts. This is a well known and frequently utilized observation (see e.g. [12, 15, 9, 13, 17, 8, 3]), and we will not go into further details here.

**The operators**
Every sequence \( a = (a_i) \in l^\infty(\mathbb{Z}^N, \mathcal{L}(X)) \), where \( \mathcal{L}(X) \) denotes the Banach algebra of all bounded linear operators on \( X \), gives rise to a bounded linear operator \( aI \) of multiplication on each of the spaces \( l^p \) by \((x_i) \mapsto (a_i x_i)\).

Another basic family of operators in \( \mathcal{L}(X) \) is given by the shifts \( V_\alpha : l^p \rightarrow l^p, \quad (x_i) \mapsto (x_{i-\alpha}) \quad (\alpha \in \mathbb{Z}^N) \).

This is everything we need for the definition of band-dominated operators:

**Definition 1.** A finite sum of the form \( \sum_\alpha a_\alpha V_\alpha \) is called a band operator. The limits of sequences of band operators, taken w.r.t. the operator norm, are said to be band-dominated, and the set of all band-dominated operators on \( l^p \) shall be denoted by \( \mathcal{A}_p \).

Clearly, band operators form a (non-closed) algebra \( \mathcal{B} \) of bounded linear operators on each of the spaces \( l^p \), \( p \in \{0\} \cup [1, \infty) \). Thus, \( \mathcal{A}_p \) is a Banach algebra for every \( p \). Note that \( \mathcal{B} \) is independent of the choice of the parameter \( p \), whereas \( \mathcal{A}_p \) depends on \( p \).

For a nice introduction and a comprehensive discussion we refer to the work of Lindner (e.g. [8], [3]) and the book of Rabinovich, Roch and Silbermann [17].

1.2 The Wiener algebra

A prominent algebra which is somehow between \( \mathcal{B} \) and \( \mathcal{A}_p \) and which provides a remarkable Fredholm behavior is the so-called Wiener algebra \( \mathcal{W} \): For band operators \( A = \sum_\alpha a_\alpha V_\alpha \) set

\[
\|A\|_\mathcal{W} := \sum_\alpha \|a_\alpha\|_\infty
\]
and notice that \( \| \cdot \|_A \leq \| \cdot \|_W \). Now take \( W \) as the closure of \( B \) with respect to \( \| \cdot \|_W \). Equipped with \( \| \cdot \|_W \) the norm, \( W \) actually turns into a Banach algebra which is a subset of each \( A_p \), \( p \in \{0\} \cup [1, \infty] \). To see that it is even a proper subset, check that e.g. \( B(x_i) := (((i| + 1)^{-1}x_{-i}) \) defines an operator \( B \in A_p \setminus W \).

Note that \( W \) can be regarded as a natural (non-stationary) extension of the classical algebra of all (Laurent) operators with constant diagonals and the norm \( \| \cdot \|_W \), which is isomorphic to the Wiener algebra of all periodic functions with absolutely summable sequence of Fourier coefficients. For further details we refer to [8, Section 1.3.6].

In [7] Lindner proved that the Fredholm properties of the operators in the Wiener algebra are independent of the underlying space. By this he extended a series of predecessors in [6, 16, 17, 18, 13]. Actually, for the proofs in [7] two additional restrictions were still required:

Say that for \( A \in \mathcal{L}(l^\infty(\mathbb{Z}^N, X)) \) there exists a **predual setting**
if there is a Banach space \( Y \) and an operator \( B \in \mathcal{L}(l^1(\mathbb{Z}^N, Y)) \) such that such that \( Y^* = X \) and \( B^* = A \). (1)

Say that the Banach space \( X \) has the **hyperplane property**
if it is isomorphic to one of its subspaces of co-dimension one.
Equivalently, \( X \) has the hyperplane property if there is a \( B \in \mathcal{L}(X) \) of Fredholm index one. (2)

Now, we can state Lindner’s result as it appears in [7] and [3, Theorem 6.44]:

**Theorem 2.** Suppose that \( X \) is finite-dimensional or has the hyperplane property, and that \( A \in W \). Then

(a) If \( A \) is Fredholm on one of the spaces \( l^p \) with \( p \in \{0\} \cup [1, \infty) \), then \( A \) is Fredholm on all spaces \( l^p \), \( p \in \{0\} \cup [1, \infty] \).

(b) If for \( A \) considered as acting on \( l^\infty \) there exists a predual setting, then \( A \) is Fredholm on one of the spaces \( l^p \) if and only if \( A \) is Fredholm on all spaces \( l^p \), \( p \in \{0\} \cup [1, \infty] \).

If \( A \) is Fredholm on every space \( l^p \) then the index is the same on all these spaces.

The Open Problems No. 4 and 5 in [3] ask whether the existence of a predual setting or the hyperplane property are redundant, and we will answer both questions affirmatively within this text. This particularly simplifies Theorem 2 as follows:

**Theorem 3.** An operator \( A \in W \) is Fredholm on one of the spaces \( l^p \) if and only if it is Fredholm on all the spaces \( l^p \), \( p \in \{0\} \cup [1, \infty] \). In this case the index is the same on all the spaces.

Actually, the need for a predual setting played an important role not only for the treatment of operators in the Wiener algebra but it affected the whole general theory on Fredholmness and limit operators which we are going to discuss within this paper.

### 1.3 Approximate projections

One of the most fruitful investigations for the development of the theory of band-dominated operators is the concept of approximate projections \( \mathcal{P} = (P_n) \) and the substitution of the classical triple (compactness, Fredholmness, strong convergence) by (\( \mathcal{P} \)-compactness, \( \mathcal{P} \)-Fredholmness, \( \mathcal{P} \)-strong
convergence). This idea has its roots in the work of Simonenko on operators of local type [22], grew up in the work of Roch and Silbermann [19], [17] and became an indispensable tool in the Fredholm theory of band-dominated operators (see e.g. [17], [8] and numerous papers which led these monographs or followed them). Unfortunately, also in this concept there had been open questions (e.g. on the connections between $\mathcal{P}$-Fredholmness, invertibility at infinity \footnote{The definitions follow below.} and the usual Fredholm property) which caused some gaps and required additional conditions in some results and their applications like the above mentioned condition (1) on the existence of a preduall setting. Recently, these problems could be solved [21], and in what follows we give an overview on the latest state of the art.

Whenever the situation is unambiguous we abbreviate the spaces of interest $l^p(\mathbb{Z}^N, \mathcal{X})$, with $N \in \mathbb{N}$ and $\mathcal{X}$ being a Banach space, simply by $\mathcal{X}$.

As a foretaste of what is to come we announce an operator algebra $\mathcal{L}(\mathcal{X}, \mathcal{P})$ which marries up with the new triple ($\mathcal{P}$-compactness, $\mathcal{P}$-Fredholmness, $\mathcal{P}$-strong convergence) as $\mathcal{L}(\mathcal{X})$ does with the classical one, and by this it provides a self-contained “universe” which amazingly reflects and improves what we know from $\mathcal{L}(\mathcal{X})$: This set $\mathcal{L}(\mathcal{X}, \mathcal{P})$ is an inverse closed Banach subalgebra of $\mathcal{L}(\mathcal{X})$. It is closed with respect to $\mathcal{P}$-strong convergence, that is the $\mathcal{P}$-strong limit of a sequence in $\mathcal{L}(\mathcal{X}, \mathcal{P})$ always belongs to $\mathcal{L}(\mathcal{X}, \mathcal{P})$ again, and there are Banach Steinhaus type results for the $\mathcal{P}$-strong convergence. Multiplication with $\mathcal{P}$-compact operators turns $\mathcal{P}$-strongly converging sequences into norm converging sequences. The $\mathcal{P}$-compact operators form a closed two-sided ideal in $\mathcal{L}(\mathcal{X}, \mathcal{P})$, hence permit to introduce the $\mathcal{P}$-Fredholm property as “invertibility up to $\mathcal{P}$-compact operators”. $\mathcal{L}(\mathcal{X}, \mathcal{P})$ proves to be closed under passing to $\mathcal{P}$-regularizers, and the usual Fredholm property perfectly aligns in that new framework.

Moreover, it should be mentioned that one can carry over the theory on the stability of strongly converging approximation methods to $\mathcal{P}$-strongly converging methods, and actually this even turns out to be the more natural framework for such questions in a sense. This is not subject of the present paper, but can be found in e.g. [19, 15, 17, 8, 20, 21, 10].

This work is organized as follows: The second part is devoted to the mentioned algebraic framework which provides the tools for the study of convergence and the Fredholm properties of the elements in $\mathcal{L}(\mathcal{X}, \mathcal{P})$. In Section 3 we discuss the application of these techniques to band-dominated operators and certain subclasses. In particular Theorem 3 is proved there. Some possible extensions, generalizations and questions concerning the world beyond $\mathcal{A}_p$ and $\mathcal{L}(\mathcal{X}, \mathcal{P})$ are discussed in the final Section 4.

2 The $\mathcal{P}$-theory and the limit operator method

2.1 The approximate projection $\mathcal{P}$

Given a set $U \subset \mathbb{Z}^N$ we define $P_U$ acting as operator on $\mathcal{X} = l^p(\mathbb{Z}^N, \mathcal{X})$ by

$$x = (x_i) \mapsto (P_U x)_i := \begin{cases} x_i & \text{if } i \in U \\ 0 & \text{if } i \notin U. \end{cases}$$

Clearly, $P_U$ and $Q_U := I - P_U$ are complementary projections. The most important operators among them are the canonical projections $P_k := P_{(-k,...,k)^N}$ and $Q_k := I - P_k$ with $k \geq 0$. 
In the following we build upon the sequence

\[ \mathcal{P} := (P_1, P_2, P_3, \ldots), \]  

which has the following properties

- \( P_n \neq 0, \ P_n \neq I \) for every \( n \), the \( P_n \) are uniformly bounded and for every \( m \) there is an \( N_m \) such that \( P_n P_m = P_m P_n = P_m \) whenever \( n \geq N_m \) (actually \( N_m := m \) already does the job in the present case),
- \( \sup \| P_U \| < \infty \), the supremum over all finite subsets \( U \) of \( \mathbb{Z}^N \),
- \( \sup_n \| P_n x \| \geq \| x \| \) for every \( x \in X \).

Sequences \( \mathcal{P} = (P_n) \) of operators with the first property are referred to as approximate projections, in e.g. [17, 21]. If the second property is fulfilled, then one says that \( \mathcal{P} \) is uniform, and the third one makes \( \mathcal{P} \) to an approximate identity. Roughly speaking, \( \mathcal{P} \) then forms a nested sequence of operators which permit to explore the whole space \( X \) in an asymptotic sense. Although the subsequent theory was developed for uniform approximate identities in general, we restrict ourselves for simplicity to the particular canonical choice \( (3) \) within this paper.

In the classical theory, e.g. for \( p \in (1, \infty) \) and \( X = C^K \), one heavily exploits the observations that \( \mathcal{P} \) consists of compact projections and converges strongly to the identity, as well as the fact that the multiplication by compact operators always turns a strongly convergent sequence \( (A_n) \) into a norm convergent one \( (A_n K) \). Unfortunately, these classical arguments break down if \( \dim X = \infty \) or \( p = \infty \). Therefore, the \( \mathcal{P} \)-theory turns the table, and takes the sequence \( \mathcal{P} \) (for arbitrary \( p \) and \( X \)) as a starting point for the definition of adapted notions of \( \mathcal{P} \)-compactness and \( \mathcal{P} \)-strong convergence, which mimic the behavior that is known from the classical world, and translate it into a more general and more flexible framework.

2.2 \( \mathcal{P} \)-compact operators

A bounded linear operator \( K \) on \( X \) is called \( \mathcal{P} \)-compact if

\[ \| K P_n - K \| \to 0 \quad \text{and} \quad \| P_n K - K \| \to 0 \quad \text{as} \quad n \to \infty. \]

By \( \mathcal{K}(X, \mathcal{P}) \) we denote the set of all \( \mathcal{P} \)-compact operators on \( X \) and by \( \mathcal{L}(X, \mathcal{P}) \) the set of all bounded linear operators \( A \) for which \( AK \) and \( KA \) are \( \mathcal{P} \)-compact whenever \( K \) is \( \mathcal{P} \)-compact. One may say that \( \mathcal{L}(X, \mathcal{P}) \) collects the operators which are compatible with \( \mathcal{K}(X, \mathcal{P}) \).

**Proposition 4.** ([17, Proposition 1.1.8])

The set \( \mathcal{L}(X, \mathcal{P}) \) is a closed subalgebra of \( \mathcal{L}(X) \), it contains the identity operator \( I \), and \( \mathcal{K}(X, \mathcal{P}) \) is a proper closed ideal of \( \mathcal{L}(X, \mathcal{P}) \). An operator \( A \in \mathcal{L}(X) \) belongs to \( \mathcal{L}(X, \mathcal{P}) \) if and only if, for every \( k \in \mathbb{N} \),

\[ \| P_k A Q_n \| \to 0 \quad \text{and} \quad \| Q_n A P_k \| \to 0 \quad \text{as} \quad n \to \infty. \]  

At this point we shall rest for a moment just to shortly visualize the relations between the operator algebras which have been introduced so far. For this we borrow Figure 1 from [8, Figure 1]. Here \( \mathcal{K}(X) \) denotes the set of compact operators as usual.

**Example 5.** Also the following examples of operators which substantiate the proper inclusions in this picture are taken from [8].
Let $a \in X$ and $f \in X^*$ be non-zero elements and $A \in \mathcal{L}(l^1)$ be given by the rule

$$A : (x_i) \mapsto \left(\ldots, 0, 0, \sum_i f(x_i)a, 0, 0, \ldots\right).$$

This operator is compact, but does not belong to $\mathcal{L}(l^1, \mathcal{P})$ due to the characterization (4). The adjoint of $A$ provides the same outcome for the case $p = \infty$.

Let $X$ be the space $L^p(0, 1)$ of all $p$-Lebesgue integrable functions over the interval $(0, 1)$ and define $B : l^p(Z, X) \to X$, $(u_i) \mapsto v$ by

$$v(x) := \begin{cases} u_k(x) & \text{if } x \in \left(1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^k}\right) \text{ for one } k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

(As customary we let $L^\infty(0, 1)$ stand for the space of all Lebesgue measurable functions that are essentially bounded.) Then the linear operator $B : (u_i) \mapsto \left(\ldots, 0, 0, B(u_i), 0, 0, \ldots\right)$ acts boundedly on $l^p = l^p(Z, X)$ for every $p$, but does not belong to $\mathcal{L}(l^p, \mathcal{P})$ in any case.
2.3 $\mathcal{P}$-strong convergence

A sequence $(A_n) \subset \mathcal{L}(X)$ is said to converge $\mathcal{P}$-strongly to $A \in \mathcal{L}(X)$ if, for every $K \in \mathcal{K}(X, \mathcal{P})$, both $\| (A_n - A)K \|$ and $\| K(A_n - A) \|$ tend to 0 as $n \to \infty$. In this case we write $A_n \to A$ $\mathcal{P}$-strongly or $A = \mathcal{P}$-$\lim_n A_n$. By $\mathcal{F}(X, \mathcal{P})$ we denote the set of all bounded sequences $(A_n) \subset \mathcal{L}(X)$, which possess a $\mathcal{P}$-strong limit in $\mathcal{L}(X, \mathcal{P})$.

Notice that a simple calculation reveals that (cf. [17, Proposition 1.1.14]) a bounded sequence $(A_n) \subset \mathcal{L}(X)$ converges $\mathcal{P}$-strongly to $A \in \mathcal{L}(X)$ if and only if both $\| (A_n - A)P_m \|$ and $\| P_m (A_n - A) \|$ tend to 0 as $n \to \infty$ for every $m$.

**Proposition 6.** ([17, Corollary 1.1.16 et seq. or 21, Theorem 1.13])

- The algebra $\mathcal{L}(X, \mathcal{P})$ is closed with respect to $\mathcal{P}$-strong convergence, this means that if a sequence $(A_n) \subset \mathcal{L}(X, \mathcal{P})$ converges $\mathcal{P}$-strongly to $A$ then $A \in \mathcal{L}(X, \mathcal{P})$. Moreover, $(A_n)$ is bounded in this case, and hence belongs to $\mathcal{F}(X, \mathcal{P})$.

- The $\mathcal{P}$-strong limit of every $(A_n) \in \mathcal{F}(X, \mathcal{P})$ is uniquely determined.

- Provided with the linear operations $\alpha(A_n) + \beta(B_n) := (\alpha A_n + \beta B_n)$, the multiplication $(A_n)(B_n) := (A_n B_n)$, and the norm $\|(A_n)\| := \sup_n \|A_n\|$, $\mathcal{F}(X, \mathcal{P})$ becomes a Banach algebra with identity $1 := (I)$. The mapping $\mathcal{F}(X, \mathcal{P}) \to \mathcal{L}(X, \mathcal{P})$ which sends $(A_n)$ to its limit $A = \mathcal{P}$-$\lim_n A_n$ is a unital algebra homomorphism and

$$\|A\| \leq \liminf_{n \to \infty} \|A_n\|. \tag{5}$$

2.4 $\mathcal{P}$-Fredholm operators

As a third part for our new triple we are going to define an appropriate substitute for the Fredholm property of operators. But, as a start, let us first recall the most important facts about the classical notion from any textbook on functional analysis, or e.g. [4, 21].

**Fredholm operators** An operator $A \in \mathcal{L}(X)$ is said to be Fredholm, if its kernel and cokernel

$$\ker A := \{x \in X : Ax = 0\}, \quad \operatorname{coker} A := X/\operatorname{im} A$$

are of finite dimension, where $\operatorname{im} A := \{Ax : x \in X\}$ denotes the range of $A$. That is the case if and only if there is a $B \in \mathcal{L}(X)$ such that $AB - I$ and $BA - I$ are compact operators. Therefore it is equivalent to $A + \mathcal{K}(X)$ being invertible in the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$.

If $A$ is Fredholm then $\operatorname{ind} A := \dim \ker A - \dim \operatorname{coker} A$ is called the index of $A$. Moreover, $A + B$ is Fredholm for $B$ being compact or $B$ having sufficiently small norm. If both $A$ and $B$ are Fredholm operators then the product $AB$ is Fredholm as well and $\operatorname{ind} AB = \operatorname{ind} A + \operatorname{ind} B$. The latter is called Atkinsons theorem.

**$\mathcal{P}$-Fredholm operators and invertibility at infinity** Now, replacing compact operators by $\mathcal{P}$-compact operators, we get two possible definitions.

**Definition 7.** An operator $A \in \mathcal{L}(X)$ is said to be invertible at infinity if there is a $B \in \mathcal{L}(X)$ such that $AB - I$ and $BA - I$ are $\mathcal{P}$-compact operators. In this case $B$ is referred to as a $\mathcal{P}$-regularizer for $A$. 
**Definition 8.** An operator $A \in \mathcal{L}(X, \mathcal{P})$ is called $\mathcal{P}$-Fredholm if the coset $A + \mathcal{K}(X, \mathcal{P})$ is invertible in the quotient algebra $\mathcal{L}(X, \mathcal{P})/\mathcal{K}(X, \mathcal{P})$.

Notice that invertibility at infinity is defined for all bounded linear operators, whereas for $\mathcal{P}$-Fredholmness we are restricted to $\mathcal{L}(X, \mathcal{P})$, since we need that $\mathcal{K}(X, \mathcal{P})$ forms a closed ideal in $\mathcal{L}(X, \mathcal{P})$. Both notions have been known and have been studied for many years. The monographs by Rabinovich, Roch and Silbermann [17] and Lindner [8] already contain a comprehensive theory and many applications of this approach. Nevertheless, it has been an open problem for a long time whether these two notions coincide for the operators $A \in \mathcal{L}(X, \mathcal{P})$. At least the inverse closedness of $\mathcal{L}(X, \mathcal{P})$ in $\mathcal{L}(X)$ has been known [17, Theorem 1.1.9], based on a proof of Simonenko [22]. An affirmative answer to the general question was given recently by the author.

**Theorem 9.** ([21, Theorem 1.16])
An operator $A \in \mathcal{L}(X, \mathcal{P})$ is $\mathcal{P}$-Fredholm if and only if it is invertible at infinity. In this case every $\mathcal{P}$-regularizer of $A$ belongs to $\mathcal{L}(X, \mathcal{P})$. Particularly, $\mathcal{L}(X, \mathcal{P})$ is inverse closed in $\mathcal{L}(X)$.

Its proof is essentially based on [21, Theorem 1.15] which was new as well, interesting on its own and, in the present context, reads as follows.

**Proposition 10.** Let $A \in \mathcal{L}(X, \mathcal{P})$. Then there is a uniform approximate projection $(R_n) \subset \mathcal{L}(X, \mathcal{P})$ with

- For every $m \in \mathbb{N}$ there is an $N_m \in \mathbb{N}$ such that for all $n \geq N_m$
  $$R_n P_m = P_m R_n = P_m \quad \text{and} \quad R_m P_n = P_n R_m = R_m.$$  
- The commutators $\|[A, R_n]|| := \|A R_n - R_n A\|$ tend to zero as $n \to \infty$.

**Fredholm operators and $\mathcal{P}$-compact projections.** There is a third and very fruitful way of characterizing Fredholm operators: One can “capture” their kernel and range via compact projections (As usual, an operator $P \in \mathcal{L}(X)$ is called projection if $P^2 = P$):

- An operator $A \in \mathcal{L}(X)$ is Fredholm if and only if there exist projections $P, P' \in \mathcal{K}(X)$ such that $\text{im } P = \ker A$ and $\ker P' = \text{im } A$.
- An operator $A \in \mathcal{L}(X)$ is not Fredholm if and only if for every $\epsilon > 0$ and every $l \in \mathbb{N}$ there exists a projection $Q \in \mathcal{K}(X)$ with rank $Q \geq l$ such that $\|A Q\| < \epsilon$ or $\|Q A\| < \epsilon$.

This characterization is closely related to the existence of a generalized inverse $B$ for an operator $A$, that is $A = ABA$ and $B = BAB$ holds and $P = I - BA$, $P' = I - AB$. \[2\]

Clearly, the question standing to reason is, whether the characterization of the Fredholm property for operators in $\mathcal{L}(X, \mathcal{P})$ might be even possible with $\mathcal{P}$-compact projections instead of compact ones. Also this hope comes true as it was recently proved. Here are the details:

**Theorem 11.** ([21, Proposition 1.27]) Let $A \in \mathcal{L}(X, \mathcal{P})$.

- $A$ is Fredholm if and only if there exist projections $P, P' \in \mathcal{K}(X, \mathcal{P})$ of finite rank such that $\text{im } P = \ker A$ and $\ker P' = \text{im } A$.

\[2\]We will see some more details in the proof of Corollary 12.
A is not Fredholm if and only if for every $\epsilon > 0$ and every $l \in \mathbb{N}$ there exists a projection $Q \in \mathcal{K}(X, \mathcal{P})$ with rank $Q \geq l$ such that $\|AQ\| < \epsilon$ or $\|QA\| < \epsilon$.

Hence we can embed the classical Fredholm property into the $\mathcal{P}$-framework: The implication $(A \in \mathcal{L}(X, \mathcal{P}) \Rightarrow A$ Fredholm) holds for every $A \in \mathcal{L}(X, \mathcal{P})$ if and only if $\dim X < \infty$. This easily follows since $\mathcal{K}(X, \mathcal{P}) \subset \mathcal{K}(X)$ if $\dim X < \infty$, whereas in case $\dim X = \infty$ the projections $P_m$ are not compact and hence all $Q_m$ are $\mathcal{P}$-Fredholm but not Fredholm. The converse implication (which is much more important since it guarantees that we can study all Fredholm operators by the tools that emerge from the $\mathcal{P}$-theory) always holds:

**Corollary 12.** Let $A \in \mathcal{L}(X, \mathcal{P})$. If $A$ is Fredholm then $A$ is $\mathcal{P}$-Fredholm and has a generalized inverse $B \in \mathcal{L}(X, \mathcal{P})$, i.e. $A = ABA$ and $B = BAB$. Moreover, $A$ is Fredholm of index zero if and only if there exists an invertible operator $C \in \mathcal{L}(X, \mathcal{P})$ and an operator $K \in \mathcal{K}(X, \mathcal{P})$ of finite rank such that $A = C + K$.

**Proof.** Let $A \in \mathcal{L}(X, \mathcal{P})$ be Fredholm and $P, P' \in \mathcal{K}(X, \mathcal{P})$ denote projections as given by the previous theorem. The compression $A : \ker P \to \ker P'$ is an isomorphism between Banach spaces, hence it has a bounded inverse $A^{(-1)}$ by the Banach Inverse Mapping Theorem. Now, the operator $B := (I - P)A^{(-1)}(I - P')$ is a bounded linear operator on $X$ with

$$AB = A(I - P)A^{(-1)}(I - P') = I - P'$$

and $BA = (I - P)A^{(-1)}(I - P')A = I - P$.

Thus, $B$ is a $\mathcal{P}$-regularizer for $A$ and Theorem 9 yields the $\mathcal{P}$-Fredholm property. Clearly,

$$ABA = (I - P')A = A - P'A = A \quad \text{and} \quad BAB = (I - P)B = B.$$

If, additionally, $\text{ind } A = 0$ then the projections $P, P'$ are of the same (finite) rank and we can choose a linear bijection $T : \text{im } P \to \text{im } P'$. We easily check that $K := P'TP$ is $\mathcal{P}$-compact of the same finite rank: for this we just note that

$$\|Q_nK\| \leq \|Q_nP'\|\|TP\| \quad \text{and} \quad \|KQ_n\| \leq \|P'T\|\|PQ_n\|$$

tend to zero as $n$ goes to infinity. Now $C := A + K$ is the desired invertible operator with the inverse $C^{-1} = (I - P)A^{(-1)}(I - P') + PT^{-1}P'$, where $T^{-1}$ denotes the inverse of $T : \text{im } P \to \text{im } P'$. Conversely, if $A$ arises from an invertible operator by a finite rank perturbation then it is Fredholm and $\text{ind } A = 0$. \[\square\]

The next observation from [21] which is a consequence of Theorem 11 clarifies the Open Problem No. 4 in [3] and therefore constitutes the essential ingredient to obviate (1), the need for a predual setting, in the whole theory for the $l^\infty$-case:

**Proposition 13.** ([21, Proposition 1.18 and Corollary 1.19])

Let $A \in \mathcal{L}(l^\infty, \mathcal{P})$. Then $A$ maps the subspace $l^0$ into $l^0$ and its compression $A|_{l^0} : l^0 \to l^0$ satisfies $\|A|_{l^0}\| = \|A\|$. Moreover, $A$ is Fredholm if and only if $A|_{l^0}$ is Fredholm, and in this case

$$\dim \ker A = \dim \ker A|_{l^0}, \quad \dim \text{coker } A = \dim \text{coker } A|_{l^0}, \quad \text{ind } A = \text{ind } A|_{l^0}.$$
2.5 The operator spectrum

The designation “invertibility at infinity” already suggests that operators with this property act on sequences \((x_i) \in X\) which are supported “far away from the origin” similar to invertible operators. This is substantiated by the fact that invertibility at infinity aka \(P\)-Fredholmness means invertibility up to some \(P\)-compact perturbations which have by definition their range of influence essentially in a neighborhood of the origin. The next definition lays the ground for the limit operator technique which brings these observations into precise statements.

**Definition 14.** Let \(A \in \mathcal{L}(X, P)\) and \((g_n) \subset \mathbb{Z}^N\) be a sequence such that \(|g_n| \to \infty\) as \(n \to \infty\). The limit

\[
A_g := \lim_{n \to \infty} V_{-g_n} AV_{g_n}
\]

of the sequence \((V_{-g_n} AV_{g_n})\) of shifted copies of \(A\), if it exists, is called the limit operator of \(A\) with respect to the sequence \(g\). The set \(\sigma_{\text{op}}(A)\) of all limit operators of \(A\) is referred to as its operator spectrum.

Further, we say that \(A\) is a rich operator if every sequence \(g \subset \mathbb{Z}^N\) of points whose absolute values tend to infinity has a subsequence \(h\) such that \(A_h\) exists. The set of all rich operators is denoted by \(\mathcal{L}^R(X, P)\).

The method of limit operators has been intensively studied during the last years and proved to be an extremely useful tool with a wide range of applications in the theory of band-dominated operators. Here we want to state only one of its highlights. More details will follow in Section 3.

**Theorem 15.** Let \(A\) be a rich band-dominated operator. Then \(A\) is \(P\)-Fredholm if and only if all limit operators of \(A\) are invertible and their inverses are uniformly bounded.

This theorem has been the engine for the development of the limit operator method, in a sense, and it has a long history from which we particularly mention the pioneering paper [6] of Lange and Rabinovich. The proof of the if part is based on a construction of a \(P\)-regularizer, which has its roots in an idea of Simonenko [22] and can be found in [16] and [8], for example. The only if part was discussed in [16] and [17], Theorem 2.2.1 (for \(1 < p < \infty\)), in [8] (all \(p\) and with the additional assumption (1) on the existence of a predual setting in the case \(p = \infty\)), and in [3], Theorem 6.28 (all \(p\)). Our present approach and the new results of Theorems 9 and 11 provide this implication already on the much more general level \(\mathcal{L}(X, P)\):

**Theorem 16.** Let \(A \in \mathcal{L}(X, P)\) be \(P\)-Fredholm. Then all limit operators of \(A\) are invertible and their inverses are uniformly bounded. Moreover, the operator spectrum of every \(P\)-regularizer \(B\) equals

\[
\sigma_{\text{op}}(B) = (\sigma_{\text{op}}(A))^{-1} := \{A_g^{-1} : A_g \in \sigma_{\text{op}}(A)\}.
\]

**Proof.** Let \(B\) be a \(P\)-regularizer for \(A\) and let \(g = (g_n)\) be such that \(A_g\) exists. It is quite obvious from the definition that the operator spectrum of \(P\)-compact operators \(K\) is trivial: \(\sigma_{\text{op}}(K) = \{0\}\). Thus, besides \(V_{-g_n} AV_{g_n} \to A_g\), we also have \(V_{-g_n}(AB - I)V_{g_n} \to 0\) and \(V_{-g_n}(BA - I)V_{g_n} \to 0\) \(P\)-strongly. Then, for every \(T \in \mathcal{K}(X, P)\),

\[
\|T\| = \|V_{-g_n}IV_{g_n}T\| \leq \|V_{-g_n}BV_{g_n}\|\|V_{-g_n}AV_{g_n}T\| + \|V_{-g_n}(I - BA)V_{g_n}T\|.
\]

Consequently (for \(n \to \infty\) and with a constant \(D > 0\) independent of \(g\) and \(T\))

\[
\|T\| \leq D\|A_gT\| \quad \text{for all} \quad T \in \mathcal{K}(X, P).
\]
The dual estimate $\|T\| \leq D\|TA_g\|$ for all $T \in \mathcal{K}(X, P)$ follows analogously. Due to Theorem 11, $A_g$ must be Fredholm with trivial kernel and cokernel, hence invertible, and Theorem 9 yields that $A_g^{-1}$ belongs to $\mathcal{L}(X, P)$. Moreover,

$$V_{-g_n}BV_{g_n} - A_g^{-1} = V_{-g_n}BV_{g_n}(A_g - V_{-g_n}AV_{g_n})A_g^{-1} + V_{-g_n}(BA - I)V_{g_n}A_g^{-1},$$

hence $\|(V_{-g_n}BV_{g_n} - A_g^{-1})T\| \to 0$ as $n \to \infty$ for every $T \in \mathcal{K}(X, P)$. Analogously we find $\|T(V_{-g_n}BV_{g_n} - A_g^{-1})\| \to 0$ and deduce that $A_g^{-1} \in \sigma_{op}(B)$. Thus the inclusion “$\supset$” in (6) is proved.

Interchanging the roles of $A$ and $B$, we can apply the above result to the $\mathcal{P}$-Fredholm operator $B \in \mathcal{L}(X, P)$ and its $\mathcal{P}$-regularizer $A$, find that every operator $B_g \in \sigma_{op}(B)$ yields a limit operator $A_g = B_g^{-1}$ of $A$, and obtain the inclusion “$\subset$” in (6).

Finally, by Eq. (5) in Proposition 6, the operators in $\sigma_{op}(B)$ are uniformly bounded, which provides the uniform boundedness of the inverses $A_g^{-1}$.

We point out the following nice properties of the set $\mathcal{L}^b(X, P)$ of rich operators which generalize [17, Proposition 1.2.8].

**Corollary 17.** We have

- The set $\mathcal{L}^b(X, P)$ forms a closed subalgebra of $\mathcal{L}(X, P)$ and contains $\mathcal{K}(X, P)$ as a closed two-sided ideal.
- Every $\mathcal{P}$-regularizer of a rich $\mathcal{P}$-Fredholm operator is rich. Thus, $\mathcal{L}^b(X, P) / \mathcal{K}(X, P)$ is inverse closed in $\mathcal{L}(X, P) / \mathcal{K}(X, P)$ and $\mathcal{L}^b(X, P)$ is inverse closed in both $\mathcal{L}(X, P)$ and $\mathcal{L}(X)$.
- For every $A \in \mathcal{L}^b(X, P)$, $X = lp, p \in \{0\} \cup [1, \infty)$, we have $A^* \in \mathcal{L}^b(X^*, P)$ and

$$\sigma_{op}(A^*) = (\sigma_{op}(A))^* := \{ A_g^* : A_g \in \sigma_{op}(A) \}.$$  

(7)

**Proof.** The first part only requires some straightforward calculations, and the second part is an immediate consequence of the previous proof: Every sequence $h$ has a subsequence $g$ such that $A_g$, and hence also $B_g = A_g^{-1}$, exists.

Now, let $A \in \mathcal{L}^b(X, P)$. Then, by Proposition 4, $A^* \in \mathcal{L}(X^*, P)$ and $A_g \in \sigma_{op}(A)$ always yields $(A_g)^* = (A^*)_g \in \sigma_{op}(A^*)$. In particular, $A^* \in \mathcal{L}^b(X^*, P)$. Conversely, if $(A^*)_h \in \sigma_{op}(A^*)$ then, due to the richness of $A$, there is a subsequence $g$ of $h$ such that $A_g$ exists, and then necessarily $(A_g)^* = (A^*)_g = (A^*)_h$.

Notice that the converse implication of Theorem 16, which is true for rich band-dominated operators by Theorem 15, does not hold in the general case:

**Example 18.** Consider the space $l^2(Z, \mathbb{C})$, the symbol function

$$a : T \to \mathbb{C}, \quad e^{it} \mapsto -t/\pi + 1, \quad t \in [0, 2\pi),$$

and the Toeplitz plus Hankel operator $A = I - i\chi_H(a)\chi_+ I$. From [5, 2.4.2 (3) and Example 2.2] we get that that this operator is not Fredholm, hence not $\mathcal{P}$-Fredholm. However, it is obviously rich and its operator spectrum is the singleton $\{I\}$. 

11
On the "big question" The "big question" in the limit operator business as it is stated in [8, Section 3.9], and also as the Open Problem No. 8 in [3], is the following:

Is the operator spectrum of a rich operator automatically uniformly invertible if it is elementwise invertible? (8)

For this we say that the operator spectrum of an operator $A$ is elementwise invertible if all limit operators of $A$ are invertible, and uniformly invertible means that additionally the inverses are uniformly bounded. Clearly, it would be a great improvement and simplification of Theorem 15 if this question could be answered affirmatively. Currently only the following partial answer is known:

**Theorem 19.** Let $p \in \{0, 1, \infty\}$ and $A \in \mathcal{L}^p(l^p, P)$. If $\sigma_{op}(A)$ is elementwise invertible then it is uniformly invertible.

**Proof.** Actually, this has been studied and proved for rich band-dominated operators so far. A comprehensive survey as well as the respective proof are given in [8, Section 3.9] and [3, Theorem 6.28].

Let $p = \infty$. Now, having Proposition 10 available and plugging in the operators $(R_n)$ which asymptotically commute with $A \in \mathcal{L}(l^p, P)$, every step of the proof in [8] works without taking any properties of band-dominated operators into account. Besides that replacement there are no further modifications of that proof needed, so we omit to repeat its details again. However, it should be emphasized that this affirmative answer is, in fact, not caused by the particular advantages of band-dominated operators, but only by the structure of $\mathcal{L}^\infty(l^\infty, P)$ on these particular spaces $l^\infty$.

If $A \in \mathcal{L}^1(l^1, P)$ has elementwise invertible operator spectrum then, by Corollary 17, its adjoint $A^* \in \mathcal{L}^\infty(l^\infty, P)$ has elementwise, hence even uniformly, invertible operator spectrum. Since $\|A_g^{-1}\| = \|(A_g^{-1})^*\|$ for every $A_g \in \sigma_{op}(A)$, we get the claim for $p = 1$. The case $l^0$ is treated analogously, using its duality to $l^1$. 

For the cases $p \in (1, \infty)$ this main problem is still open in general, and could be answered affirmatively only for some special classes of operators, such as those in the Wiener algebra or for band-dominated operators with slowly oscillating coefficients (see [17, 8] and below).

## 2.6 A first roundup

We recall the following collection of conditions from [3, (5.14)]:

(a) $A$ is invertible

(b) $A$ is Fredholm

(c) $A$ is $P$-Fredholm

(d) $\sigma_{op}(A)$ is uniformly invertible

(e) $\sigma_{op}(A)$ is elementwise invertible

(f) All limit operators of $A$ are injective.

We have seen that for $A \in \mathcal{L}(X, P)$ all implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f)$ hold. Concerning the converse implications we have the following: $(c) \Rightarrow (b)$ holds in case of compact $P = (P_n)$, $(d) \Rightarrow (c)$ is the striking advantage of rich band-dominated operators, which we will study in more detail within the next section. The implication $(e) \Rightarrow (d)$ is the big question, and
particularly true for rich operators in the extremal cases \( p \in \{0, 1, \infty\} \), or for all rich operators in the Wiener algebra. The latter will be subject of Section 3.2. Finally, also \((f) \Rightarrow (e)\) holds true in a surprisingly wide range of cases. We will focus on that in Section 3.3.

3 Fredholm theory of band-dominated operators

3.1 General band-dominated operators

Let us start this section with mentioning again that all band-dominated operators belong to the algebra \( \mathcal{L}(X, \mathcal{P}) \). This is obvious for shifts and operators of multiplication and then easily follows for all \( A \in \mathcal{A}_p \) since \( \mathcal{L}(X, \mathcal{P}) \) is a Banach algebra.

Therefore, the results presented in the previous section suggest to study the Fredholm property of band-dominated operators with the help of limit operators. Actually, in the beginning most of the above results had been studied for band-dominated operators only. Later on, with the wisdom of hindsight, it turned out that large parts of the theory are not specific for \( \mathcal{A}_p \) but hold in \( \mathcal{L}(X, \mathcal{P}) \). Of course, one striking advantage of rich band-dominated operators is already outlined in Theorem 15, the equivalence of \( \mathcal{P} \)-Fredholmness and the uniform invertibility of the operator spectrum. Another remarkable speciality in the case \( \dim X < \infty \) is the fact that every band-dominated operator turns out to be rich [17, Corollary 2.1.17]. The following collection of results shall underline that \( \mathcal{A}_p \) is self-contained and closed with respect to the terms of the \( \mathcal{P} \)-theory.

**Theorem 20.** ([17, Prop 2.1.7. et seq.] or [8, Propositions 1.46, 2.9, 2.11, 3.6])

For \( \mathcal{A}_p \) and \( A \in \mathcal{A}_p \) we have the following.

- The algebra \( \mathcal{A}_p \) contains \( \mathcal{K}(X, \mathcal{P}) \) as a closed two-sided ideal.
- The limit operators of band-dominated operators always belong to \( \mathcal{A}_p \) again, i.e. \( \sigma_{\text{op}}(A) \subset \mathcal{A}_p \).
- If \( A \) is invertible then \( A^{-1} \in \mathcal{A}_p \). Thus \( \mathcal{A}_p \) is inverse closed in \( \mathcal{L}(X) \) and \( \mathcal{L}(X, \mathcal{P}) \).
- If \( A \) is \( \mathcal{P} \)-Fredholm then every \( \mathcal{P} \)-regularizer of \( A \) is band-dominated as well.

The following observations which additionally address the usual Fredholm property are probably new.

**Theorem 21.** Let \( A \in \mathcal{A}_p \).

- If \( A \) is Fredholm then there exists a regularizer which belongs to \( \mathcal{A}_p \). More precisely, there is even a generalized inverse \( B \in \mathcal{A}_p \) for \( A \), which means that \( ABA = A \), \( BAB = B \) holds and \( I - AB \), \( I - BA \) are both compact and \( \mathcal{P} \)-compact projections.
- If \( A \) is invertible, \( \mathcal{P} \)-Fredholm or Fredholm and \( B \) is an inverse, \( \mathcal{P} \)-regularizer or Fredholm regularizer in \( \mathcal{A}_p \) for \( A \), respectively, then \( \sigma_{\text{op}}(B) = (\sigma_{\text{op}}(A))^{-1} \). Moreover, \( A \) is rich if and only if \( B \) is rich.

**Proof.** The first assertion is a consequence of Corollary 12 together with Theorem 20. The second assertion follows from Theorem 16 and Corollary 17. \( \square \)
The case \( N = 1 \) and the Fredholm index

We now suppose that \( N = 1 \) and our aim is to state index formulas for operators in terms of their limit operators.

We already know by Corollary 12 that a Fredholm operator \( A \in \mathcal{A}_p \) has the \( \mathcal{P} \)-Fredholm property and hence, due to Theorem 16, its limit operators are invertible. Rabinovich, Roch and Roe [14] proposed to consider the following compressions of the limit operators: Let \( P := P_N \) and set \( Q := I - P \). For a given operator \( A \in \mathcal{A}_p \) define

\[
A_+ := PAP + Q \quad \text{and} \quad A_- := QAQ + P
\]

and further introduce the plus-(and minus-)index of \( A \) as \( \text{ind}^\pm A := \text{ind} A_\pm \). Finally let \( \sigma_\pm(A) \) denote the set of all limit operators \( A_g \) of \( A \) w.r.t. sequences \( g \) tending to \( \pm\infty \), respectively. Then the main observations of [14] for Fredholm operators \( A \in \mathcal{A}_p \) in the case \( \dim X < \infty \) and \( p = 2 \) have been the formulas

\[
\text{ind} A = \text{ind}_+ A + \text{ind}_- A \quad \text{and} \quad \text{ind}_\pm A = \text{ind}_\pm A_g
\]

for arbitrary limit operators \( A_g \in \sigma_\pm(A) \), respectively. This result was generalized to the case \( \dim X < \infty \) and \( p \in (1, \infty) \) in [18], and later on in [13] to Banach spaces \( X \) with a certain additional property, the symmetric approximation property, and for a particular class of operators of the form \( A = I + K \) with rich \( K \in \mathcal{A}_p \) having compact matrix entries. The most general version of such an index formula, without any restrictions on \( X \) and for arbitrary band-dominated operators, has been proved in [21, Theorem 3.7] and, in the present notation, reads as follows:

**Theorem 22.** Let \( A \in \mathcal{A}_p \) be Fredholm. Further, let \((u_n), (l_n) \subset \mathbb{Z} \) be sequences tending monotonically to \( +\infty \) or \( -\infty \), respectively, such that \( A_u \in \sigma_+(A) \) and \( A_l \in \sigma_-(A) \) exist. Finally set \( L_n := P_{\{l_n, \ldots, u_n\}} \). If \( (A_u)_+ \) and \( (A_l)_- \) are Fredholm operators then there is a number \( n_0 \in \mathbb{N} \) such that for \( n \geq n_0 \) the operators \( L_nAL_n \in \mathcal{L}(\text{im} L_n) \) are Fredholm as well and

\[
\text{ind} A = \text{ind} L_nAL_n + \text{ind}_+ (A_u) + \text{ind}_- (A_l).
\]

(9)

Notice that in all cases which have been studied before (i.e. for \( \dim X < \infty \) or the operators with compact entries of [13]), the Fredholm property of \( A \) automatically implies the Fredholm property of \((A_u)_+ \) and \((A_l)_-\), and the compressions \( L_nAL_n \) always have index zero. This again leads to the simplified formula

\[
\text{ind} A = \text{ind}_+ (A_u) + \text{ind}_- (A_l).
\]

(10)

It should also be mentioned that Theorem 22 even holds for the more general quasi-banded operators which will be discussed in our final Section 4.

3.2 Operators in the Wiener algebra

Recall the Wiener algebra \( \mathcal{W} \) which is defined as the closure of the algebra \( \mathcal{B} \) of all band operators \( A = \sum_\alpha a_\alpha V_\alpha \) with respect to the norm

\[
\|A\|_{\mathcal{W}} := \sum_\alpha \|a_\alpha\|_{\infty}.
\]

Before we come to the proof of Theorem 3 let us shortly summarize to what extend the previous observations for \( \mathcal{A}_p \) specify to \( \mathcal{W} \). From [17, Theorem 2.5.2] we know that, for every \( p \in \{0\} \cup [1, \infty) \),
\( \mathcal{W} \) is inverse closed in \( \mathcal{L}(l^p) \), hence also in \( \mathcal{L}(l^p, \mathcal{P}) \) and \( \mathcal{A}_p \). Moreover, by [17, Propositions 2.5.1 and 2.5.6], the property of the operators \( A \in \mathcal{W} \) to be rich does not depend on \( p \), the operator spectrum of rich operators in the Wiener algebra is independent from \( p \) as well, and it is always a subset of \( \mathcal{W} \). This leads to the great advantage of this class that one can extend the affirmative answer to the big question, via interpolation, to every \( p \) (cf. [6], [17, Theorem 2.5.7] and [3, Theorem 6.40]):

**Theorem 23.** Let \( A \in \mathcal{W} \) be rich. Then the following are equivalent

1. \( A \) is \( \mathcal{P} \)-Fredholm on one of the spaces \( l^p \).
2. \( A \) is \( \mathcal{P} \)-Fredholm on all the spaces \( l^p \).
3. All limit operators of \( A \) are invertible on one of the spaces \( l^p \).
4. All limit operators of \( A \) are invertible on all the spaces \( l^p \) and

\[
\sup_{p \in \{0\} \cup [1, \infty]} \sup_{A_g \in \sigma_{op}(A)} \| A_g^{-1} \|_{\mathcal{L}(l^p)} < \infty. \tag{11}
\]

**Proof.** The implications \( 4. \Rightarrow 2. \Rightarrow 1. \Rightarrow 3. \) are clear by Theorems 15 and 16. So let \( 3. \) hold true. Since the operator spectrum of \( A \in \mathcal{W} \) does not depend on \( p \) and since \( \mathcal{W} \) is inverse closed on every \( l^p \), we see that all limit operators of \( A \) are invertible on all the spaces \( l^p \). For \( p \in \{1, \infty\} \) there exists a uniform bound by Theorem 19. Finally, the Riesz-Thorin Interpolation Theorem and Proposition 13 provide

\[
\| A_g^{-1} \|_{l^p} \leq \max \{ \| A_g^{-1} \|_1, \| A_g^{-1} \|_\infty \}
\]

for all \( p \in \{0\} \cup [1, \infty] \) and all \( A_g \in \sigma_{op}(A) \). Thus (11) holds.

Let us now again turn our attention to the (classical) Fredholm property and the proof of Theorem 3. In fact, our arguments follow in large parts those of the original ones [7], just profit from the new and improved results of Section 2, and use an alternative construction of Fredholm operators of prescribed index that does not require the hyperplane property of \( X \). We start with an auxiliary lemma.

**Lemma 24.** For every \( \kappa \in \mathbb{Z} \) there exists an operator \( S_{\kappa} \in \mathcal{W} \) which is Fredholm on \( l^p = l^p(\mathbb{Z}^N, X) \) of index \( \kappa \) for every \( p \in \{0\} \cup [1, \infty] \).

**Proof.** Choose a projection \( \tilde{R} \in \mathcal{L}(X) \) of rank 1 (see e.g. [11, B.4.9]) and define a projection \( R : l^p \rightarrow l^p, (x_i) \mapsto (\tilde{R}x_i) \). Moreover introduce the projections \( \hat{P} := P_{\mathbb{N} \times \{0\}^{N-1}} \) and \( \hat{Q} := I - \hat{P} \). Now set

\[
S_{|\kappa|} := \hat{Q} + \hat{P}(I - R + V_{|\kappa|,0,...,0})\hat{P} \quad \text{and} \quad S_{-|\kappa|} := \hat{Q} + \hat{P}(I - R + V_{|\kappa|,0,...,0})\hat{P}
\]

and easily check that \( S_{|\kappa|}S_{-|\kappa|} = I \), whereas the spaces \( \text{ker } S_{|\kappa|} = \im R_{\mathbb{N} \times \{0\}^{N-1}} \) of the dimension (resp. codimension) \( |\kappa| \), independent from the choice of \( p \). This easily shows that \( S_{|\kappa|} \) and \( S_{-|\kappa|} \) are one-sided invertible Fredholm operators in \( \mathcal{W} \) of the index \( |\kappa| \) and \(-|\kappa|\), respectively. To convey a better understanding of how these operators
act, we mention that for \( N = 1 \) the matrix representation of \( S_{[\kappa]} \) is given by

\[
[S_{[\kappa]}] = \begin{pmatrix}
\vdots & & & & & I & & \\
& & & & & I & & \\
& & & & & I - \hat{R} & & 0 \\
& & & & & 0 & & I - \hat{R} \\
& & & & & 0 & & 0 \\
& & & & & \vdots & & \vdots
\end{pmatrix},
\]

This particularly shows that the sequence spaces \( \ell^p \) have the hyperplane property, independently from the geometry of \( X \). The proof of Theorem 3 is now straightforward:

**Proof.** Let \( A \in W \) be Fredholm on one of the spaces \( \ell^p \) and \( \kappa := \text{ind} A \). Then, with \( S_{[\kappa]} \) as in Lemma 24, \( AS_{[\kappa]} \) is Fredholm of index zero, and Corollary 12 provides an operator \( K \in \mathcal{K}(\ell^p, \mathcal{P}) \) of finite rank \( d \) such that \( AS_{[\kappa]} - K \) is invertible on \( \ell^p \). From the definitions, we easily derive that the operators \( P_m K P_m \) belong to \( W \cap \mathcal{K}(\ell^p, \mathcal{P}) \), have finite rank less than or equal to \( d \) and converge to \( K \) in the operator norm on \( \ell^p \) as \( m \to \infty \). Since the set of invertible operators is open, we get that for a sufficiently large \( m \) also \( B := AS_{[\kappa]} - P_m K P_m \) is invertible and, moreover, belongs to \( W \). Due to the inverse closedness of \( W \), its inverse \( B^{-1} \) is in the Wiener algebra as well. Thus, we have

\[
I = BB^{-1} = AS_{[\kappa]}B^{-1} - P_m K P_m B^{-1} = AC - T
\]

with \( C := S_{[\kappa]}B^{-1} \in W \) and \( T := P_m K P_m B^{-1} \in W \cap \mathcal{K}(\ell^p, \mathcal{P}) \) where \( \text{rank } T \leq d \). Since \( P_m K P_m \) considered as operator on \( \ell^r \), \( r \in \{0\} \cup [1, \infty) \), vanishes on \( \text{im } Q_m \) and has a range being a finite dimensional subspace of \( \text{im } P_m \) we see that \( T \) is an operator of finite rank \( d \) on \( \ell^r \) as well. Proceeding in a completely symmetric way one also arrives at an equation \( I = C' A - T' \) with \( C' \in W \) and \( T' \in W \cap \mathcal{K}(\ell^r, \mathcal{P}) \) of finite rank. Since finite rank operators are compact, we see that \( A + \mathcal{K}(\ell^r) \) is invertible in the Calkin algebra \( \mathcal{L}(\ell^r)/\mathcal{K}(\ell^r) \), hence \( A \) is Fredholm on \( \ell^r \). From the equalities

\[
0 = \text{ind}(I) = \text{ind}(AC - T) = \text{ind}(AS_{[\kappa]}B^{-1}) = \text{ind } A + \text{ind } S_{[\kappa]}
\]

we derive the desired relation \( \text{ind } A = \kappa \) on \( \ell^r \). Actually,

\[
C = IC = (C' A - T') C = C' (AC - T) + C' T - T' C = C' + (C' T - T' C),
\]

hence the difference \( C - C' \) is of finite rank, and we can even conclude that both operators \( C, C' \) are Fredholm regularizers for \( A \) and belong to \( W \).

Actually, this proof does not only provide that Fredholmness of operators in the Wiener algebra and their indices are independent of the underlying space, but it also yields regularizers in \( W \). Therefore we can specify Theorem 21 as follows.

**Corollary 25.** Let \( A \in W \) be Fredholm. Then there exists an operator \( B \in W \) which is a Fredholm regularizer for \( A \) on every space \( \ell^p \). In this case the operator spectrum \( \sigma_{\text{op}}(B) \) of \( B \) coincides with \( \{A_g^{-1} : A_g \in \sigma_{\text{op}}(A)\} \). Moreover, \( A \) is rich if and only if \( B \) is rich.
3.3 Collective compactness and Favard’s condition

Chandler-Wilde and Lindner [2] have studied another more concrete class of band-dominated operators on $l^\infty(\mathbb{Z}, X)$ for which the Fredholm criteria become much simpler in the sense that already the injectivity of all of their limit operators is sufficient for the Fredholmness of such operators.

**Definition 26.** We say that an operator $A$ on $l^\infty(\mathbb{Z}, X)$ is subject to Favard’s condition if every limit operator of $A$ is injective.

Let us first introduce the precise framework.

**Definition 27.** Let $\mathcal{UM}$ denote the set of all operators $K \in A_\infty$ with the property that the operators $k_{ij} := P_0 V_{i,j}^2 V_{j} P_0$ (i, j $\in \mathbb{Z}$) form a collectively compact set, that is
\[
\{k_{ij} x : i, j \in \mathbb{Z}, x \in l^\infty(\mathbb{Z}, X), \|x\| \leq 1\} \text{ is relatively compact.}
\]

If one interprets $K$ as an infinite matrix which acts on the sequence space $l^\infty(\mathbb{Z}, X)$ then the operators $k_{ij}$ can be regarded as its matrix entries. \(^3\)

Note that the set $\mathcal{UM}$ is a Banach space, and its elements are locally compact operators in the sense of [13]. Moreover, $\mathcal{UM}$ is a Banach subalgebra and a left-sided ideal of $A_\infty$. In case $\dim X < \infty$ it holds that $\mathcal{UM} = A_\infty$. These properties are proved among others in [2, Section 2] or [3, Section 6.3]. Here comes the striking advantage of this class as it appears in [2, Theorem 3.1] and [3, Theorem 6.31].

**Theorem 28.** Let $A = I - K$ with $K \in \mathcal{UM}$ be rich and all limit operators be injective. Then all limit operators of $A$ are invertible.

**Corollary 29.** Let $A = I - K$ with $K \in \mathcal{UM}$ be rich. Then the following are equivalent.

1. All limit operators of $A$ are injective (Favard’s condition).
2. The operator spectrum of $A$ is uniformly invertible.
3. $A$ is $\mathcal{P}$-Fredholm.
4. $A$ is Fredholm.
5. There exists a rich $B \in A_\infty$ s.t. $ABA = A$, $BAB = B$ holds and $I - BA$, $I - AB$ are compact projections onto the kernel or parallel to the range of $A$, respectively.

We want to emphasize again that this covers all operators $A \in A_\infty$ in case $\dim X < \infty$.

**Proof.** The implication 1. $\Rightarrow$ 2. is Theorem 28 together with Theorem 19. 2. $\Rightarrow$ 1. is obvious and 2. $\Leftrightarrow$ 3. follows from Theorem 15. Furthermore, 4. $\Rightarrow$ 5. holds by Theorem 21. 5. $\Rightarrow$ 4. is evident, and Corollary 12 provides the implication 4. $\Rightarrow$ 3. Therefore it remains to prove 3. $\Rightarrow$ 4.: Let $B$ be a $\mathcal{P}$-regularizer and let $X$ stand for $l^\infty(\mathbb{Z}, X)$. Then $(I - K)B - I = AB - I =: T \in \mathcal{K}(X, \mathcal{P})$ hence $B = I + KB + T$. Thus
\[
A(I + KB) = (I - K)(I + KB) = I - K(I - B + KB) = I + KT
\]
and $A(I + KB)(I - KT) = I - KTKT$. Since $T \in \mathcal{K}(X, \mathcal{P})$ it holds that $\|TP_m K P_m T - TKT\| \to 0$ as $m \to \infty$. Furthermore, $P_m K P_m$ is compact, hence $TKT$ and even $KTKT$ are compact, which shows that $A + \mathcal{K}(X)$ has a right inverse in $\mathcal{L}(X)/\mathcal{K}(X)$. Analogously we find a left inverse for $A + \mathcal{K}(X)$, and we conclude that $A$ is Fredholm.

\(^3\)A rigorous description and justification of this perspective can be found in [17] or [8].
This Corollary already appeared in [3] as Corollary 6.32 in large parts, but under the additional assumption (1) on the existence of a predual setting, because at that time Corollary 12 and hence the implication 4. \( \Rightarrow 3. \) were not available.

The Open Problem No. 6 in [3] conjectures that Theorem 28 may also hold in case \( l^\infty(Z^N, X), N > 1. \) Unfortunately, the following example demonstrates that this is wrong.

**Example 30.** Let \( X \) be a Banach space and \( \hat{R} \in \mathcal{L}(X) \) be a rank-one-projection. Define projections \( R \) and \( M \) on \( l^\infty(Z^2, X) \) by the rules \( R(x_i) := (\hat{R}x_i) \) and \( M(x_i) := (y_i) \) with

\[
y_i := \begin{cases} x_i & \text{if } i = (k, l) \in Z^2 \text{ with either } l \leq 0 \text{ or } k \geq l^2 \\ 0 & \text{otherwise.} \end{cases}
\]

Furthermore, set \( K = (I - V(1,0))MR \) and \( A = I - K. \) Clearly, \( R, M, K \) and \( A \) are rich band operators and \( K \in \mathcal{UM}. \)

We easily check that the operator spectrum of \( M \) consists of the zero operator, the identity and all shifted copies \( V_\alpha M_k V_\alpha \) \( (\alpha \in Z^2) \) of the operators \( M_k \) \( (k = 1, 2) \) which are given by \( M_k(x_i) := (y^\alpha_k) \) with

\[
y^1_k := \begin{cases} x_i & \text{if } i = (k, l) \text{ with } l \leq 0 \\ 0 & \text{otherwise,} \end{cases}
y^2_k := \begin{cases} x_i & \text{if } i = (k, l) \text{ with } l < 0 \text{ or } k \geq l = 0 \\ 0 & \text{otherwise.} \end{cases}
\]

This yields that \( \sigma_{op}(A) \) consists of the identity, the operator \( (I - R) + V(1,0)R \) and all shifted copies of the operators \( I - (I - V(1,0))M_k R \ (k = 1, 2). \) With the representation

\[
I - (I - V(1,0))M_k R = (I - R) + R((I - M_k) + V(1,0)M_k) R
\]

we finally check that all limit operators are injective, but \( I - (I - V(1,0))M_2 R \) is not invertible.

We finish this section with generalized versions of [2, Proposition 4.1, and Corollary 4.3] which combine the specialities of both, the algebras \( W \) and \( \mathcal{UM}, \) in order to extend the previous observations to all spaces \( l^p. \) Our present result is homogenous w.r.t. all \( p \in \{0\} \cup [1, \infty] \) and there are not any restrictions on the Banach space \( X \) remaining, in particular for the index formula.

**Corollary 31.** Let \( A = I - K \) with \( K \in \mathcal{UM} \cap W \) be rich. The following are equivalent.

**FC.** All limit operators of \( A \) are injective on \( l^\infty \) (Favard’s condition).

1. All limit operators of \( A \) are invertible on one of the spaces \( l^p. \)
2. \( A \) is \( \mathcal{P} \)-Fredholm on one of the spaces \( l^p. \)
3. \( A \) is Fredholm on one of the spaces \( l^p. \)
4. There exists a Fredholm regularizer \( B \in W \) for \( A \) on one of the spaces \( l^p. \)

1’.-4’. The same as in 1.-4. but with one replaced by all.
If $A$ is Fredholm then its index is the same on all the spaces $l^p$ and is given by the index formula (10), $\text{ind}(A) = \text{ind}_+ (A) + \text{ind}_- (A)$ with arbitrary $A_u \in \sigma_+ (A)$ and $A_l \in \sigma_- (A)$. Furthermore, the regularizers and the operator spectra of such Fredholm operators $A$ are independent from $p$ and fulfill (11) as well as (6).

One may ask, if $l^\infty$ in the first statement $FC$ could even be replaced by $l^p$, but already the simple band operator $A = I - V_1$ dashes this hope, hence the first condition in the $l^\infty$-case is really stronger than in the $l^p$-case. Indeed, $\ker A = \{(c) : c \in \mathbb{C}\} \subset l^\infty$, and $\sigma_{\text{op}} A = \{A\}$, since $A$ is shift invariant, hence all limit operators of $A$, regarded as operator on $l^p$ with $p < \infty$, are injective, but, regarded as operator on $l^\infty$, that is obviously not true.

The reader is encouraged to compare these results with the observations of Section 2.6. We also note again that, in the case $\dim X < \infty$, $\mathcal{UM} = \mathcal{A}_\infty$ and every band-dominated operator is rich, thus this corollary applies to all $A \in \mathcal{W}$ (cf. also [3, Section 6.5]).

**Proof.** We additionally introduce the conditions

5. $A$ is $\mathcal{P}$-Fredholm on $l^\infty$.

6. $A$ is Fredholm on $l^\infty$.

$FC. \iff 5. \iff 6.$ is Corollary 29. 5. $\iff 1.' \iff 1. \iff 2.' \iff 2.$ and Equation (11) are provided by Theorem 23. Equation (6) is proved in Theorem 16. 6. $\iff 3.' \iff 3.$ as well as the independence of the index follow from Theorem 3. Furthermore, Corollary 25 yields $3.' \Rightarrow 4.'$ whereas $4.' \Rightarrow 4. \Rightarrow 3.$ are evident.

So, let $A = I - K$ be Fredholm. The operator $K \in \mathcal{W}$ is the norm limit of a sequence $(K^{(m)})$ of operators $K^{(m)} := \sum_{\alpha=1}^{m} a_{\alpha} V_{\alpha}$, where every entry of every diagonal $a_{\alpha}$ is a compact operator on $X$. Consequently, every limit operator $K_g$ of $K$ is the norm limit of the respective limit operators $K_g^{(m)}$. The latter are still finite sums of the form $\sum_{\alpha=1}^{m} b_{\alpha}^{(m)} V_{\alpha}$ having compact entries. $PK_g^{(m)} Q$ (analogously $QK_g^{(m)} P$) are

$$PK_g^{(m)} Q = P \sum_{\alpha=1}^{m} b_{\alpha}^{(m)} V_{\alpha} Q = P \sum_{\alpha=1}^{m} \sum_{\beta \in \mathbb{Z}} b_{\alpha}^{(m)}(\beta) P_{(\beta)} V_{\alpha} Q = P \sum_{\alpha=1}^{m} \sum_{\beta=1}^{\infty} b_{\alpha}^{(m)}(\beta) P_{(\beta)} V_{\alpha} Q,$$

i.e. finite sums of compact operators. Therefore $PK_g Q$ and $QK_g P$ are compact, hence $PA_g P + QA_g Q$ are compactly perturbed copies of the invertible operators $A_g$. So we see that all restricted limit operators $(A_u)_+$ and $(A_l)_-$ of $A$ as they appear in Theorem 22 are Fredholm, and we obtain the index formula (9) which even simplifies to (10) since $L_n K L_n$ is compact due to the above reasons. $\square$

### 4 Extensions and Generalizations

**One-sided definitions** In [3] it is also discussed whether one can weaken and replace the definitions of $\mathcal{K}(X, \mathcal{P})$, $\mathcal{L}(X, \mathcal{P})$ and $\mathcal{P}$-strong convergence by one sided analogues (see the Open Problems No. 1 to 3 there). More precisely, one may define ([3, Lemma 3.3, Corollary 3.5])

$$SN(X) := \{K \in \mathcal{L}(X) : \|K Q_n\| = \|K P_n - K\| \to 0 \text{ as } n \to \infty\}$$

$$S(X) := \{A \in \mathcal{L}(X) : KA \in SN(X) \text{ for every } K \in SN(X)\}$$

$$= \{A \in \mathcal{L}(X) : \|P_m A Q_n\| \to 0 \text{ as } n \to \infty \text{ for every } m\}$$

19
and say that \((A_n) \subset \mathcal{L}(X)\) \(s\)-converges to \(A\) (or has the \(s\)-limit \(A\)) if

\[
\|K(A_n - A)\| \to 0 \text{ for every } K \in SN(X).
\]

Let us consider the following example which illustrates that weakening the definition of \(K(X, \mathcal{P})\) necessitates also the modification of \(\mathcal{L}(X, \mathcal{P})\) and the notion of convergence:

**Example 32.** Let \(\psi : L^p(0, 1) \to L^p(\mathbb{R})\) be an isometric Banach space isomorphism. As an example one may take the following construction: The mapping

\[
\phi : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}), \quad (\phi f)(t) := e^{-t/p}f(e^t)
\]

is an isometry, and its inverse mapping is \((\phi^{-1}g)(s) = s^{-1/p}g(-\log s), s > 0.\) The same holds true for the restriction \(\phi : L^p(0, 1) \to L^p(\mathbb{R}_+)\). Thus \(\psi := \phi \circ \phi : L^p(0, 1) \to L^p(\mathbb{R})\) does the job. Now, let \(X := l^p(\mathbb{Z}, L^p(0, 1))\) and define \(B \in \mathcal{L}(X)\) by

\[
(B(x_n))_k := \chi_{(0,1)} V_{-k} \psi x_0, \quad k \in \mathbb{Z}.
\]

Clearly \(B\) is an isometric isomorphism between \(im P_0\) and \(X, B\) belongs to \(SN(X)\), but \(\|Q_n B P_0\|\) does not tend to zero as \(n \to \infty\). Thus, we see that \(SN(X)\) is not a subset of \(\mathcal{L}(X, \mathcal{P})\), hence would not serve as an ideal there. Moreover, \(\|(A_n - A)B\| \to 0\) would always imply \(\|A_n - A\| \to 0\), so a two-sided definition of convergence based on \(SN(X)\) instead of \(K(X, \mathcal{P})\) would be nothing but the usual norm convergence.

The motivation for such modified definitions is obvious: One can now try to develop an analogous theory for the larger family \(S(X) \supset \mathcal{L}(X, \mathcal{P})\). To be more concrete, one may ask which of the results of Section 2 translate to this more general setting and, in particular, whether there is an analogous Fredholm and limit operator theory available.

An initial dawn of hope arises with the observation that Propositions 4 and 6 are valid in the present setting as well:

**Proposition 33.** Let \(X\) be a Banach space with uniform approximate identity \(\mathcal{P} = (P_n)\).

- The set \(S(X)\) is a closed subalgebra of \(\mathcal{L}(X)\), it contains the identity operator \(I\), and \(SN(X)\) is a proper closed ideal of \(S(X)\).
- The algebra \(S(X)\) is closed with respect to \(s\)-convergence, this means that if \((A_n) \subset S(X)\) \(s\)-converges to \(A\) then \(A \in S(X)\).
- \((A_n) \subset \mathcal{L}(X)\) has the \(s\)-limit \(A\) iff it is bounded and \(\|P_m(A_n - A)\| \to 0\) for every \(m\).
- Let \(FS(X)\) denote the collection of all sequences \((A_n) \subset \mathcal{L}(X)\) which possess a \(s\)-limit in \(S(X)\). Then the \(s\)-limit of every \((A_n) \in FS(X)\) is uniquely determined.
- Provided with pointwise operations and the supremum norm, \(FS(X)\) becomes a Banach algebra with identity \(I := (I)\). The mapping \(FS(X) \to S(X)\) which sends \((A_n)\) to its \(s\)-limit \(A\) is a unital algebra homomorphism and

\[
\|A\| \leq B_P \liminf_{n \to \infty} \|A_n\| \quad \text{where} \quad B_P := \limsup_{n \to \infty} \|P_n\|.
\]
Proof. The first assertion is [3, Lemmata 3.32, 3.3 and Corollary 3.5]. For the second assertion let $K \in SN(X)$. Then $KA_n \in SN(X)$ and $\|KA_n - KA\| \to 0$. Since $SN(X)$ is closed, this implies that $KA$ belongs to $SN(X)$, thus $A \in S(X)$. The if part of the third assertion easily follows from the estimate $\|K(A_n - A)\| \leq \|KQ_m\||(A_n - A)| + \|K\||P_m(A_n - A)||$. The boundedness in the only if part can be proved as [3, Lemma 4.2] and the rest is trivial since $P \subset SN(X)$. The fourth assertion is quite obvious: If $\|P_m(A_n - A)\|$ and $\|P_m(A_n - B)\|$ tend to zero as $n$ goes to infinity for every fixed $m$, then $P_m(A - B) = 0$ for every $m$, hence $A - B = 0$ since $P$ is an approximate identity.

The proof of $FS(X)$ being a normed algebra is again straightforward, and we only note that if $(A_n), (B_n)$ s-converge to $A, B \in S(X)$, resp., then they are bounded and

$$\|K(A_nB_n - AB)\| \leq \|K(A_n - A)B_n\| + \|KA(B_n - B)\| \to 0$$

for every $K \in SN(X)$, as $n \to \infty$, since $A \in S(X)$ implies $KA \in SN(X)$. For the estimate (12) we can replace $(A_n)$ by one of its subsequences which realize the $\liminf$, cancel arbitrarily but finitely many entries at the beginning of this subsequence and get from the third assertion that $\|P_mA\| \leq B_P \liminf \|A_n\| + \epsilon$ for every $\epsilon > 0$ and every $m$. Then we conclude (12) since $P$ is an approximate identity. Finally, let $((C^m_n))_m$ be a Cauchy sequence of sequences $(C^m_n) \in FS(X)$, where $C^m$ shall denote the s-limit of $(C^m_n)$, respectively. For every $n$, $(C^m_n)$ converges in $L(X)$ to an element $C_n$, and the sequence $(C_n)$ is uniformly bounded. Furthermore, (12) yields that $(C^m_n)$ is a Cauchy sequence with a limit $C \in S(X)$. Now one easily checks that $(C_n)$ s-converges to $C$, thus $FS(X)$ is complete.

Notice that in cases of $(P_n)$ being a sequence of compact operators and such that their adjoints $P_n^*$ converge strongly to the identity we do not get anything new for Fredholm theory, since then $SN(X) = K(X)$ and $S(X) = L(X)$. This particularly involves the cases $X = l^p(Z, X)$ with $p \in \{0\} \cup (1, \infty)$ and $dim X < \infty$.

On inverse closedness of $S(X)$ and generalized Fredholmness The Open Problem No. 3 in [3] asks whether $S(X)$ is inverse closed. In the above mentioned cases where $S(X) = L(X)$ this is obviously true. Another partial answer in [3, Section 3.2] extends that to the cases of compact $(P_n)$ on Banach spaces $X$ which are complete w.r.t. a certain topology. By this means, one gets an affirmative answer for all cases $X = l^p(Z, X)$ with $p \in \{0\} \cup [1, \infty]$ and $dim X < \infty$. The most popular application with $dim X = \infty$ is $X = l^p(Z, L^p(0, 1)) \cong L^p(R)$. Unfortunately, already in this natural situation the picture changes, as the following examples demonstrate.

**Example 34.** Consider $X = l^\infty(Z, L^\infty(0, 1))$. For $k \in \mathbb{N}$ define

$$B_k : L^\infty \left( \frac{1}{k+1}, \frac{1}{k} \right) \to L^\infty(0, 1), \quad B_k f(x) = f \left( \frac{1}{k+1} + x \left( \frac{1}{k} - \frac{1}{k+1} \right) \right)$$

and the respective extensions $C_k : L^\infty(0, 1) \to L^\infty(0, 1)$, $C_k = B_k \chi_{\left( \frac{1}{k+1}, \frac{1}{k} \right)} I$. Figuratively speaking, these operators single out a certain part of a given function and stretch it. Now, let the operator $A$ on $X$ be given by

$$(Ax)_n = \begin{cases} x_n & : n < 0 \\ C_{\frac{n}{2}+1} x_0 & : n \geq 0 \text{ even} \\ x_{\frac{n}{2}+1} & : n \geq 0 \text{ odd.} \end{cases}$$
J\text{morphisms, define }Q\text{ but it may not belong to an invertible coset in }X.\text{etry on recall the operators }\mathcal{I}\text{ operator with multiplication: The operator }P\text{ inverse may be outside }\mathcal{I}\text{. Moreover this yields that }\mathcal{I}\text{ and }\mathcal{I}\text{, assume that }\mathcal{I}\text{, and this, passing to a weaker definition of convergence. However, as a start we point out that }\mathcal{I}\text{setting without further restrictions and assumptions.}

\begin{equation}
\begin{pmatrix}
\ddots & & \\
I & & \\
& C_1 & \\
0 & I & \\
& C_2 & 0 & 0 \\
& 0 & 0 & I & 0 \\
& C_3 & 0 & 0 & 0 & 0 \\
& 0 & 0 & 0 & I & 0 \\
& \vdots & \vdots & & & & \\
\end{pmatrix}
= \begin{pmatrix}
\ddots & & \\
J_2 & \cdots & \\
& J_1 B_{-2} & \\
& J_2 & J_1 B_{-1} & \\
& J_2 & \cdots & \\
& B_0 & J_1 B_1 & J_2 & \\
& J_2 & \cdots & J_1 B_2 & J_2 \\
& J_2 & \cdots & \cdots & J_1 B_3 & J_2 \\
& \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix},
\end{equation}

Figure 2: Matrix representations of the Examples 34 and 35.

Its matrix representation is shown in Figure 2. Clearly, this operator belongs to \(S(X)\) and it is invertible where its inverse can be obtained by reflecting the matrix w.r.t. the main diagonal and replacing \(C_n\) by the inverse of \(B_n\). This inverse does not belong to \(S(X)\).

**Example 35.** Let \(\theta_1 : L^p(0,1) \to L^p(0,1/2)\) and \(\theta_2 : L^p(0,1) \to L^p(1/2,1)\) be isometric isomorphisms, define \(J_3 := \text{diag}(\ldots, \theta_1, \theta_1, I, \theta_1, \theta_1, \ldots)\) and \(J_2 := \text{diag}(\ldots, \theta_2, \theta_2, 0, \theta_2, \theta_2, \ldots)\), and recall the operators \(\psi\) and \(B\) from Example 32. Then \(A := J_1 B + J_2\) is an invertible isometry on \(X := L^p(\mathbb{Z}, L^p(0,1))\) which belongs to \(S(X)\), but \(A^{-1} \notin S(X)\). With the definition \(B_k := \chi(0,1) V_{-k} \psi : L^p(0,1) \to L^p(0,1)\) its matrix representation is shown in Figure 2.

Notice that one cannot expect a comparable Fredholm theory as well. Clearly, every invertible operator \(A \in S(X)\) is regularizable w.r.t. \(SN(X)\) in the spirit of Definition 7 (invertible at infinity), but it may not belong to an invertible coset in \(S(X)/SN(X)\) (as in Definition 8) in general since the inverse may be outside \(S(X)\). Actually, the “invertibility at infinity” would not even be compatible with multiplication: The operator \(A\) from Example 34 is invertible, hence “invertible at infinity”, and so is \(Q_1\), but \(AQ_1\) is not: Assume that there is a \(SN(X)\)-regularizer \(B\) for \(AQ_1\) and let \(R\) denote the projection

\[
(x_n) \mapsto (y_n), \quad y_n := \begin{cases} 
  x_n & : n \geq 0 \text{ even} \\
  0 & : \text{otherwise}. 
\end{cases}
\]

Then \(R \in S(X)\) and, due to the equality \(R = R(I - AQ_1 B)\), it would even belong to \(SN(X)\), a contradiction. Moreover this yields that \(A\) has no regularizer w.r.t. \(SN(X)\) in \(S(X)\) at all: To see this, assume that \(C \in S(X)\) is a regularizer for \(A\), then \(I - AQ_1 C = I - AC + AP_1 C \in SN(X)\) and \(I - C AQ_1 = I - CA + CAP_1 \in SN(X)\), that means \(C\) is a regularizer for \(AQ_1\), again a contradiction. Of course, a similar observation can easily be made for the operators on \(l^p\)-spaces from Example 35 as well.

Thus, we see that it is hardly possible to achieve a comparable Fredholm theory in the \(S(X)\)-setting without further restrictions and assumptions.

**On limit operators** Given an operator \(A\) one has to expect a larger operator spectrum after passing to a weaker definition of convergence. However, as a start we point out that \(s\)-convergence
cannot provide any additional benefit as long as one is only interested in the limit operators of band-dominated operators. For this, we denote by \( \sigma_{op}^P(A) \), \( \sigma_{op}^s(A) \) the operator spectra of \( A \) w.r.t. \( P \)-strong convergence or \( s \)-convergence, respectively.

**Proposition 36.** For every \( A \in \mathcal{A}_p \) we have \( \sigma_{op}^P(A) = \sigma_{op}^s(A) \).

**Proof.** The inclusion \( \subset \) is obvious. Assume that \( A_g \in \sigma_{op}^s(A) \setminus \sigma_{op}^P(A) \). Then there is a subsequence \( h \) of \( g \) such that \( \| (V_{-h_n}AV_{h_n} - A_g)P_m \| \to c > 0 \). Since \( A \) and \( A_g \) are band-dominated there exist \( k, m \) and \( n_0 \) such that \( \| Q_k(V_{-h_n}AV_{h_n} - A_g)P_m \| < c/2 \) for all \( n \geq n_0 \). To see this just approximate \( A \) an \( A_g \) by band operators. This yields \( \| P_k(V_{-h_n}AV_{h_n} - A_g) \| \not\to 0 \), a contradiction. \( \square \)

So, let us now look at operators outside \( \mathcal{A}_p \). When the \( P \)-strong convergence is replaced by \( s \)-convergence, the operator spectrum of invertible operators becomes larger and, unfortunately, may contain non-invertible limit operators, as the next example shows. Thus, Theorem 16, the backbone of the limit operator method, does not remain valid for \( s \)-convergence.

**Example 37.** Consider \( X = L^p(\mathbb{Z}, \mathbb{C}) \) with the canonical projections \( (P_n) \), let \( I_m \) denote the \( m \times m \) identity matrix and \( C_m \) the \( m \times m \) circulant matrix

\[
C_m := \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & 1
\end{pmatrix}.
\]

Also set \( B := \begin{pmatrix}
\iddots & 1 \\
1 & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & \ddots \\
\end{pmatrix} \).

Now define \( A \) by the infinite block diagonal matrix \( \text{diag} \{ Q, I_1, C_1, I_2, C_2, I_3, C_3, \ldots \} \). Then \( A \) belongs to \( \mathcal{L}(X,P) \), is invertible, but has the non-invertible limit operator \( B \in \sigma_{op}^s(A) \).

**Quasi-banded operators** Although the above examples do not condemn the hope for a reasonable Fredholm or limit operator theory w.r.t. \( s \)-convergence in total, they at least show that important results are not available in this more general \( S(X) \)-setting. However, there is still a wide field of possible generalizations between the pleasant and well understood class \( \mathcal{A}_p \) of band-dominated operators and the whole algebra \( \mathcal{L}(X,P) \), equipped with \( P \)-strong convergence. One should draw attention e.g. to the following class of operators:

**Definition 38.** An operator \( A \in \mathcal{L}(X) \) is said to be quasi-banded if

\[
\lim_{m \to \infty} \sup_{n > 0} \| Q_{n+m}AP_n \| = \lim_{m \to \infty} \sup_{n > 0} \| P_nAQ_{n+m} \| = 0. \tag{13}
\]

As in [10] one can verify that the set \( \mathcal{Q}_p \) of all quasi-banded operators is a Banach algebra, \( \mathcal{A}_p \subset Q_p \subset \mathcal{L}(X,P) \) and the inclusions are proper in general. In particular, the flip operator is quasi-banded, and so are also e.g. Laurent operators with quasi-continuous or slowly oscillating symbols. Moreover, \( Q_p \) is inverse closed and closed under passing to \( P \)-regularizers.

In the case \( N = 1 \), a shift invariant operator \( A \) belongs to \( Q_p \), if and only if \( PAQ \) and \( QAP \) are \( P \)-compact. \( \tag{14} \)
Moreover, Theorem 22 and, in particular, the index formula (9) also hold for operators in $Q_p$.

The most striking argument for the consideration of these operators is the observation that the well known and intensively studied results on the applicability of the finite section method in terms of limit operators extend from $A_p$ to $Q_p$. This is subject of [10] for the case $l^p(Z, L^p(0,1))$, but the arguments there also work for more general situations $l^p(Z, X)$, and suggest analogues for the spaces $l^p(Z^N, X)$.

5 Conclusions

Within this paper we have presented an overview of the recent state of the art in Fredholm theory for band-dominated and related operators which is based on the beautiful and most complete $\mathcal{P}$-approach. It turned out that replacing the classical functional analytic approach and the notions of compactness, Fredholmness and strong convergence by the respective $\mathcal{P}$-triple provides a Banach algebraic framework which perfectly fits to the problems one is interested in. It permits to treat the whole scale of spaces $l^p(\mathbb{Z}^N, X)$ (and even more) in a completely homogeneous way. In fact, there is no need for reflexivity, for a predual setting (1), for the hyperplane property of $X$ (2), or for any other restrictions on $X$, e.g. on its dimension. We have particularly answered the Open Problems No. 4 and 5 of [3]. Moreover, the conjectures No. 7 (Sufficiency of Favard’s condition for $N > 1$) and 3 (inverse closedness of $S(X)$) turned out to be wrong. We have also seen that the important partial answers to the big question (Problem No. 8) actually hold for all rich operators in the framework $L(l^p, \mathcal{P})$, $p \in \{0, 1, \infty\}$, and are not specific for band-dominated operators.

Concerning the Open Problems No. 1 and 2 we could make the following observations: The passage from the classical approach to $L(X, \mathcal{P})$ turned out to be a great step. The algebraic and topological structures and relations remained essentially the same, this $\mathcal{P}$-framework is consistent and self-contained, and one could achieve a much higher flexibility and generality. Also the proofs became more transparent. The possible next step towards $S(X)$ and s-convergence is less promising since several important basic results which are necessary for a Fredholm theory and the limit operator method cannot be extended in general, and will not provide a comparable toolbox.

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