LYAPUNOV FUNCTIONS AND EXPO NENTIAL ERGODICITY
FOR REFLECTED BROWNIAN MOTION IN THE ORTHANT
AND COMPETING BROWNIAN PARTICLES

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We consider semimartingale reflected Brownian motion in the positive orthant. Loosely speaking, in the interior of the orthant such process behaves like a Brownian motion with a constant drift vector and a constant covariance matrix. On each of the faces, it is reflected, not necessarily normally. Recently, conditions for its positive recurrence and existence of a stationary distribution were found. We make these results more complete. More importantly, we find several new cases when this process is exponentially ergodic. In some of these new cases, we are able to find the explicit value of the exponent of ergodicity. We find a tail estimate for the stationary distribution in all these cases. Our main tools are Lyapunov functions. We apply these results to market models of competing Brownian particles. We consider symmetric as well as asymmetric collisions.

1. Introduction.

1.1. Definition of a semimartingale reflected Brownian motion. Fix $d \geq 1$, the number of dimensions. Let $\mathbb{R}_+ := [0, \infty)$ be the positive half-axis. Let $S := \mathbb{R}_+^d$ be the positive orthant in $\mathbb{R}^d$. This is the state space for the process.

Let us now define a $d$-dimensional semimartingale reflected Brownian motion (SRBM) in $S$. We shall start with describing its parameters. Let $R = (r_{ij})_{1 \leq i,j \leq d}$ be a $d \times d$-matrix with $r_{ii} > 0$ for each $i = 1, \ldots, d$. It is called a reflection matrix. Let $r_i$, $i = 1, \ldots, d$ be the $i$th column of $R$. Fix a drift vector $\mu \in \mathbb{R}^d$. Also, fix a positive definite nondegenerate symmetric $d \times d$-matrix $A = (a_{ij})_{1 \leq i,j \leq d}$, which is called a diffusion matrix.

First, let us loosely describe a $d$-dimensional SRBM in the positive orthant $S$ with reflection matrix $R$, drift vector $\mu$ and diffusion matrix $A$. We denote it by $\text{SRBM}^d(R, \mu, A)$. This is a Markov process with state space $S$ which:

(i) behaves as a $d$-dimensional Brownian motion with drift $\mu$ and diffusion matrix $A$ in the interior of $S$;

(ii) is reflected on the face $S_i = \{ x \in S \mid x_i = 0 \}$ of the boundary $\partial S$ in the direction of $r_i$, for each $i = 1, \ldots, d$.

If $r_i = e_i$, where $e_i$ is the $i$th standard basis vector in $\mathbb{R}^d$, then the reflection is called normal. Otherwise, it is called oblique.

Now let us formulate the precise definition. It is taken from the survey [74] and is stated in terms of a solution to the Skorohod problem in the orthant $S$. Assume the usual setting: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$, where $(\Omega, \mathcal{F})$ is a measurable space, and $(\mathcal{F}_t)_{t \geq 0}$ is a right-continuous filtration. Let $C(\mathbb{R}_+, \mathbb{R}^d)$ be the space of all continuous functions $\mathbb{R}_+ \to \mathbb{R}^d$. 

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Definition 1. Assume $X \in C(\mathbb{R}_+, \mathbb{R}^d)$. A solution to the Skorohod problem in the orthant $S = \mathbb{R}_+^d$ with reflection matrix $R$ is any pair of functions $(Y, Z)$, where $Y, Z \in C(\mathbb{R}_+, \mathbb{R}^d)$, which satisfy the following conditions:

(i) for every $t \geq 0$, we have: $Z(t) = X(t) + RY(t) \in S$;

(ii) for every $i = 1, \ldots, d$, we have: $Y_i$ is a nondecreasing function with

$$Y_i(0) = 0 \quad \text{and} \quad \int_0^\infty Z_i(t) dY_i(t) = 0.$$

The last equality means that $Y_i$ can increase only when $Z_i = 0$, i.e. only when $Z \in S$. Here, $Y_i$ and $Z_i$ are the $i$th components of the functions $Y$ and $Z$.

Definition 2. Let $B = (B_t, t \geq 0)$ be an $\mathbb{R}^d$-valued stochastic process on $(\Omega, \mathcal{F})$. Then a SRBM$^d(R, \mu, \mathcal{A})$ is:

(i) a family of probability measures $(\mathbb{P}_x, x \in S)$ on $(\Omega, \mathcal{F})$; and

(ii) a continuous $(\mathcal{F}_t)_{t \geq 0}$-adapted $S$-valued process $Z = (Z_t, t \geq 0)$ such that for every $x \in S$ there exists an $(\mathcal{F}_t)_{t \geq 0}$-adapted $\mathbb{R}^d$-valued stochastic process $Y = (Y_t, t \geq 0)$ for which the following properties hold:

(a) the pair of functions $(Y, Z)$ is a solution to the Skorohod problem in the orthant $S$ for $X = B$ with reflection matrix $R$;

(b) for every $x \in S$, the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions with respect to $\mathbb{P}_x$ (right-continuity and augmentation by $\mathbb{P}_x$-null sets), and $B = (B_t, t \geq 0)$ is an $((\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$-Brownian motion with drift $\mu$ and diffusion matrix $\mathcal{A}$, starting from $x$.

Remark 1. For the sake of brevity, we will just refer to $Z$ as a SRBM$^d(R, \mu, \mathcal{A})$, in which case the measures $\mathbb{P}_x$ are implicit.

Remark 2. We can normalize the process $Y$ without loss of generality so that $r_{ii} = 1, \ i = 1, \ldots, d$. See [8, Appendix B] for detailed explanation. In the sequel, we will always implicitly assume this.

1.2. Notation. For $a \in \mathbb{R}$, let $a_+ := \max(a, 0)$ and $a_- := \max(-a, 0)$. For $a, b \in \mathbb{R}$, let $a \land b = \min(a, b)$ and $a \lor b = \max(a, b)$. For any set $A$, its indicator function is denoted by $1_A$ or $1(A)$.

We think of vectors $a \in \mathbb{R}^d$ as column-vectors, i.e. $d \times 1$-matrices. For a vector or matrix $a$, we refer to its transpose as $a^T$. Let $I_k$ be the $k \times k$-identity matrix. For a vector $x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$, let $\|x\| := (x_1^2 + \ldots + x_d^2)^{1/2}$ denote the Euclidean norm. For a $d \times d$-matrix $C = (c_{ij})_{1 \leq i, j \leq d}$, let us define its spectral norm:

$$\|C\| := \sup_{\|x\| = 1} \|Cx\| = \max \left\{ \sqrt{\lambda}, \ \lambda \text{ is an eigenvalue of } C^TC \right\}.$$  

This is a matrix norm, in the sense that it has the usual norm properties and, in addition, satisfies $\|C_1C_2\| \leq \|C_1\| \|C_2\|$ for every $d \times d$-matrices $C_1$ and $C_2$. If $C$ is symmetric, then all its eigenvalues are real, and we can express its norm in the following way:

$$\|C\| = \max \{ |\lambda|, \ \lambda \text{ is an eigenvalue of } C \}.$$  

On matrix norms, see [43, Section 5.6]. For any two vectors $x = (x_1, \ldots, x_d)^T, \ y = (y_1, \ldots, y_d)^T \in \mathbb{R}^d$, let $x \cdot y = x_1y_1 + \ldots + x_dy_d$ be their dot product. Also, $x \geq y$ means that $x_i \geq y_i$ for each $i = 1, \ldots, d$; similarly for $x \leq y, x > y, x < y$. A principal submatrix of a $d \times d$-matrix $R$ is
obtained from \( R \) by deleting all rows and columns with indices in some (possibly empty) subset of \( \{1, \ldots, d\} \).

For every \( r > 0 \), let \( C(r) := \{ x \in S \mid \| x \| \leq r \} \) be the intersection of the ball of radius \( r \) centered at the origin and the orthant \( S \). For \( i, j = 1, \ldots, d, \ i \neq j \), let \( S_{ij} = S_i \cap S_j \) be the intersection of two faces of the boundary \( \partial S = \cup_{i=1}^{d} S_i \). Let \( S_0 := S_i \cup_{j \neq i} S_{ij} \) be the smooth part of the \( i \)th face \( S_i \). Let \( \text{int} \ S = \{(x_1, \ldots, x_d)^T \mid x_i > 0 \text{ for each } i = 1, \ldots, d\} \) be the interior of the orthant \( S \). Denote \( S^0 = \text{int} \ S \cup S_0 \cup \ldots \cup S_{d}^0 \). Then \( S \setminus S^0 = \cup_{i \neq j} S_{ij} \) is the non-smooth part of the boundary.

For a real-valued function \( f \) of \( d \) real variables, let \( Df \) and \( D^2 f \) denote its gradient and the second-derivative matrix, respectively. We write \( f \in C^k(A) \) if the function \( f \) is \( k \) times continuously differentiable on the set \( A \). We may have \( k = \infty \): \( f \in C^\infty(A) \) means \( f \) is infinitely differentiable on \( A \). Denote by \( \text{mes} \) the Lebesgue measure on \( \mathbb{R}^d \). We say \( f : S \to \mathbb{R} \) is nondecreasing if \( f(x) \geq f(y) \) whenever \( x \geq y \).

Consider a measurable space \( X \), a signed measure \( \nu \) on it. The total variation of the measure \( \nu \) is defined as

\[
\| \nu \|_{TV} := \sup_{|f| \leq 1} \left| \int_X f \, d\nu \right|,
\]

where the sup is taken over all measurable functions \( f : X \to \mathbb{R} \) such that \( |f(x)| \leq 1 \) for every \( x \in X \).

For a SRBM \( Z = (Z_t, t \geq 0) \), let \( P^t(x, A) = P_x(Z_t \in A) \) be the transition probability; here, \( x \in S, \ t \geq 0, \ A \subseteq S \) is a Borel set. Let \( E_x \) denote the expectation with respect to \( P_x \). For a set \( F \subseteq S \), let \( \tau_F := \inf\{ t \geq 0 \mid Z_t \in F \} \) be the first hitting time of \( F \) by the SRBM \( Z = (Z_t, t \geq 0) \).

1.3. Organization of the paper. Section 2 contains all main definitions for SRBM. It also lists various possible conditions for SRBM which we will use. Moreover, it contains some already known results which we will directly use in the sequel. Section 3 is devoted to some necessary background regarding Lyapunov functions, exponential ergodicity and stochastic ordering. Section 4 contains all main results for SRBM. Section 5 is devoted to systems of competing Brownian particles. Section 6 contains a toy example in dimension one, which outlines the methods of proofs. Section 7 is a brief historical survey. Section 8 contains comparison of our new results with the old ones. Sections 9 and 10 contain proofs of Theorem 4.1 and Theorem 4.2, respectively. Section 11 is devoted to proofs for competing Brownian particles. Section 12 contains refinement of the main results by sharpening the total variation norm. Section 13 contains proofs of all results regarding exponential tails. The Appendix contains some auxiliary definitions and lemmas.

2. Definitions and Essential Known Results. In Section 7, we give a historical survey. Also, we refer the reader to an excellent survey [74]. First, consider the question of existence of SRBM\(^d\)(\( R, \mu, A \)).

**Definition 3.** A \( d \times d \)-matrix \( R \) is called an \( S \)-matrix if there exists \( u \in \mathbb{R}^d, \ u > 0 \) such that \( Ru > 0 \). A \( d \times d \)-matrix \( R \) is called completely-\( S \) if all its principal submatrices are \( S \)-matrices.

Such matrices are well-known in operations research under the names of completely-\( Q \) or strictly semimonotone matrices.

**Assumption 1.** \( R \) is completely-\( S \).

The following proposition establishes existence of a SRBM. It is taken from [64, 68, 74].
Proposition 2.1. There exists a SRBM\(^d\(, R, \mu, A \)) iff \(R\) is completely-S. In this case, the SRBM is unique in law and defines a Feller continuous strong Markov process.

Let us now state a definition of positive recurrence. Loosely speaking, it means that the process does not go to infinity and has an invariant probability measure (stationary distribution). We feel obliged to warn an attentive reader that a definition adopted in SRBM literature is a bit different from the one used for general Markov processes. See the latter one, e.g. in the papers [24], [58]. We stick to the one from SRBM papers, since it has already become common and standard in this area.

Definition 4 ([27], [74], [8]). The SRBM\(^d\(, R, \mu, A \)) is called positive recurrent if for each closed set \(F \subset S\) with positive Lebesgue measure we have \(\mathbb{E}_x \tau_F < \infty\) for every \(x \in S\).

Definition 5. A stationary distribution for the SRBM\(^d\(, R, \mu, A \)) \(Z = (Z_t, t \geq 0)\) is a Borel probability measure \(\pi\) on \(S\) such that for every \(t \geq 0\) and for every bounded Borel-measurable function \(f : S \to \mathbb{R}\) we have:

\[
\int_S \mathbb{E}_x f(Z_t) d\pi(x) = \int_S f(x) d\pi(x).
\]

For the positive recurrence and existence of a stationary distribution, no necessary and sufficient conditions are known. There are only some partial results.

Definition 6. A nonsingular \(d \times d\)-matrix \(R\) is called inverse-positive if each element of the matrix \(R^{-1}\) is strictly positive. It is called inverse-nonnegative if each element of \(R^{-1}\) is nonnegative. It is called strictly inverse-nonnegative if it is inverse-nonnegative and, in addition, all elements on the main diagonal of \(R^{-1}\) are strictly positive.

Assumption 2. For every \(x \in S\), the solution \((Y, Z)\) to the Skorohod problem for \(X(t) = x + \mu t\) in the orthant \(S\) with reflection matrix \(R\) has the property \(Z(t) \to 0\) as \(t \to \infty\).

Assumption 3. The matrix \(R\) is nonsingular and \(R^{-1} \mu < 0\).

The following necessary condition for positive recurrence was proved in [20, Chapter 3].

Proposition 2.2. If SRBM\(^d\(, R, \mu, A \)) is positive recurrent, then Assumptions 1 and 3 hold true.

In the general case, there are no necessary and sufficient conditions for positive recurrence. There are only some necessary conditions and some other sufficient conditions, and there is a gap between them. However, in two dimensions \((d = 2)\), necessary and sufficient conditions are obtained in [42].

Proposition 2.3. In case \(d = 2\), Assumptions 1 and 3 are not only necessary but sufficient for positive recurrence. In this case, it has a unique stationary distribution.

The main sufficient condition is Proposition 2.4, which is the main result of the paper [27]; see also the survey [74]. Let us state it:

Proposition 2.4. Under Assumptions 1 and 2, SRBM\(^d\(, R, \mu, A \)) is positive recurrent and has a unique stationary distribution.
Recently, Chen in [13] proved some useful corollaries of this result. First, let us define a nonsingular $\mathcal{M}$-matrix. There are several equivalent definitions of nonsingular $\mathcal{M}$-matrices, see the book [7, Chapter 6, Theorem 2.3]. It is most convenient for us to choose the following one.

**Definition 7.** An inverse-positive matrix $R = (r_{ij})_{1 \leq i, j \leq d}$ is called an **nonsingular $\mathcal{M}$-matrix** if $r_{ij} \leq 0$ for $i \neq j$.

**Proposition 2.5 ([13, 8]).** (i) If $R$ is a nonsingular $\mathcal{M}$-matrix, then Assumption 3 implies Assumption 2.

(ii) Therefore, if $R$ is a nonsingular $\mathcal{M}$-matrix, then Assumption 3 is sufficient for SRBM$^d(R, \mu, A)$ to be positive recurrent and to have a unique stationary distribution.

**Remark 3.** If $R$ is an $\mathcal{M}$-matrix, then it is completely-$\mathcal{S}$. See [13] and [7]. Therefore, in this case Assumption 1 holds and SRBM$^d(R, \mu, A)$ is well defined.

**Assumption 4.** There exists $\gamma > 0$ such that for all $x \in S$, we have:

$$x^T R^{-1} x \geq \gamma ||x||^2.$$  

**Remark 4.** The most important case when Assumption 4 holds is when $R$ is inverse-positive, or strictly inverse-nonnegative. In this case, if $R^{-1} = (\rho_{ij})_{1 \leq i, j \leq d}$, then Assumption 4 holds for

$$\gamma = \min_{1 \leq i \leq d} \rho_{ii}$$

For example, if $R = I_d$, then $\gamma = 1$. This assumption may be true also in some other cases. For example, suppose that the matrix $\tilde{R} = (R^{-1} + (R^{-1})^T)/2$ is positive-definite. Then we can take

$$\gamma := \min\{\lambda \mid \lambda \text{ is an eigenvalue of } \tilde{R}\}.$$  

**Definition 8.** A $d \times d$-matrix $Q$ is called a **substochastic matrix** if all its entries are nonnegative, and the sum of each row is less than or equal to 1.

Below, we will need the notions of spectral radius and irreducibility. On this, consult the books [43], [56] and [7].

**Assumption 5.** $R = I_d - Q^T$, where $Q$ is a substochastic matrix with spectral radius strictly less than 1 and zeros on the main diagonal.

**Remark 5.** Under Assumption 5, we have:

$$R^{-1} = \sum_{k=0}^{\infty} (Q^T)^k.$$  

For the proof, see [56, p.618, p.682]. This means that $R$ is strictly inverse-nonnegative, and Assumption 4 holds with $\gamma$ given by (3). Moreover, $R$ is completely-$\mathcal{S}$, so Assumption 1 is satisfied.

**Remark 6.** How does Assumption 5 look like for $d = 2$? It means that

$$R = \begin{pmatrix} 1 & r_{12} \\ r_{21} & 1 \end{pmatrix},$$

where $r_{12}, r_{21} \in [-1, 0]$, but they cannot be simultaneously equal to $-1$.

**Assumption 6.** The matrix $R$ is symmetric.

This last assumption will be used in Theorem 4.1 below.
3. Lyapunov Functions, Exponential Ergodicity and Stochastic Ordering. In this section, assume we have a Markov process $X = (X_t)_{t \geq 0}$ on the state space $\mathcal{X}$. We denote transition probabilities by $P(X_t \in A \mid X_0 = x) = P^t(x, A)$, and transition kernels $P^t f(x) = \int_{\mathcal{X}} f(y) P^t(x, dy)$. By $P^x$, we denote the conditional probability under the condition $X_0 = x$.

3.1. Exponential Ergodicity. Now, let us introduce the concept of exponential ergodicity. This means that for any initial condition $x \in \mathcal{X}$, the distributions $P^t(x, \cdot)$ converge to $\pi$ exponentially fast: the distance between $P^t(x, \cdot)$ and $\pi$ is estimated by $ce^{-\kappa t}$. The variable $\kappa$ is called the exponent of ergodicity. Certainly, to make this definition precise, we should specify how exactly we measure the distance between two probability measures. It can be done in various ways, but the most convenient for us is the total variation distance, defined in $(19)$.

Definition 9. Assume we have a function $C : \mathcal{X} \to \mathbb{R}_+$ and a constant $\kappa > 0$ such that for every $t \geq 0$ and $x \in S$ we have:

$$\|P^t(x, \cdot) - \pi(\cdot)\|_{TV} \leq C(x)e^{-\kappa t}.$$ 

Then we say that the process $X$ is exponentially ergodic with exponent $\kappa$.

The next proposition is the main result of the paper [10].

Proposition 3.1. Under Assumptions 1 and 2, SRBM$^d(R, \mu, \mathcal{A})$ is exponentially ergodic.

3.2. Theory of Lyapunov Functions. The connection between Lyapunov functions for continuous-time Markov processes and exponential ergodicity (or, more generally, the rate of convergence to the stationary distribution) is investigated in the following papers: [24], [30], [3], [25], [33], [4], [11], [58]; see also the book [57] for analogous results in discrete time.

Let us give an explanation of what a Lyapunov function for a Markov process $X = (X_t)_{t \geq 0}$ is. For details and formal definitions, we refer the reader to the sources cited above, and to the Appendix. Define is generator as follows: if $f : \mathcal{X} \to \mathbb{R}$ is a real-valued function defined on the state space of this Markov process, then let

$$\mathcal{L}f(x) := \lim_{t \downarrow 0} \frac{P^t f(x) - f(x)}{t}, \quad x \in \mathcal{X},$$

for functions $f$ such that this limit exists. The set of such functions is denoted by $\mathcal{D}(\mathcal{L})$ and is called the domain of the generator. If the generator has a closed-form expression, and its domain is nice (=wide) enough, then it is convenient to define the process by its generator rather than its transition probability kernels.

Definition 10. A function $V : \mathcal{X} \to [1, \infty)$ is called a Lyapunov function if it belongs to the domain $\mathcal{D}(\mathcal{L})$ of the generator and there exists a constant $k > 0$ and a petite set $C \subseteq \mathcal{X}$ such that the following holds:

$$\mathcal{L}V(x) \leq -kV(x) \quad \text{for} \quad x \in S \setminus C, \quad \text{and} \quad V \text{ is bounded on } C.$$

Now, what is a petite set? Loosely speaking, it is a “small” set. It can be informally described as a compact set if the state space $\mathcal{X}$ is $\mathbb{R}^d$ (or some “nice” subset of $\mathbb{R}^d$, such as the positive orthant). For a SRBM, all compact sets are petite. See Appendix.
By Ito’s rule, if $X_t \not\in C$, then $dV(X_t) = \mathcal{L}V(X_t)dt + dM_t \leq -kV(X_t)dt + dM_t$, where $M = (M_t)_{t \geq 0}$ is a certain local martingale. So, on average, $V(X_t)$ “wants” to decrease exponentially fast when $X_t$ is outside this “small” set $C$. If $V(x)$ tends to $\infty$ as $x$ tends away from $C$ (and this is the case in our proofs), then $X_t$ “wants” to get back to $C$. In other words, this process “does not go to infinity”, which means recurrence.

It turns out that we can say much more than that: if we have (4), then it has a unique stationary distribution, and is exponentially ergodic. Let us state this result. It is taken from [59].

**Proposition 3.2.** Assume $X = (X_t)_{t \geq 0}$ is a Markov process on the state space $\mathcal{X}$ with generator $\mathcal{L}$. Assume it is irreducible (see precise definitions in [59], [30], and other articles cited above). Suppose there exists a Lyapunov function $V : \mathcal{X} \rightarrow [1, \infty)$, as in (4). Then this process is positive recurrent and has a unique stationary distribution $\pi$, and is exponentially ergodic with a certain exponent $\kappa$. Finally, we have:

$$\int_{\mathcal{X}} V(x)\pi(dx) < \infty.$$ 

In the general case, we are not able to find $\kappa$ explicitly. In particular, we cannot state that it is equal to $k$. However, for SRBM in the orthant, sometimes we can in fact make this conclusion! This is based on the fact that SRBM in the orthant is a *stochastically ordered process*. More on this in the next subsection.

3.3. *Stochastically ordered processes.* Informally, a Markov process is stochastically ordered if it has the following property: if it starts from a larger initial state, than it stays larger in law. More precisely:

**Definition 11.** Assume we have a Markov process $X = (X_t)_{t \geq 0}$ on the positive orthant $S = \mathbb{R}^d_+$. It is called *stochastically ordered* if for any $t \geq 0$ and $y \in S$ we have:

$$\mathbb{P}_{x_1}\{X_t \geq y\} \geq \mathbb{P}_{x_2}\{X_t \geq y\} \quad \text{for all } x_1, x_2 \in S \text{ such that } x_1 \geq x_2.$$ 

We shall call a set $C \subseteq S$ *tight* if for any $x \in C$ and $y \in S$ such that $y \leq x$ we have: $y \in C$. Examples: $C(r)$ for any $r > 0$; a cube or a $d$-dimensional rectangle $[0,a_1] \times [0,a_2] \times \ldots \times [0,a_d]$. In [55], the following result was proved:

**Proposition 3.3.** Consider a stochastically ordered Markov process on the positive orthant. Assume we constructed a Lyapunov function which satisfies (4), where the petite set $C$ is tight. Then the Markov process is exponentially ergodic with the exponent $\kappa = k$.

Actually, it was shown only for the case $d = 1$ and $C = \{0\}$, but this proof is readily generalized for more general cases, as in this proposition. See [55, Section 6] for RBM on the half-line and [55, Section 7] for the multidimensional case.

**Lemma 3.4.** Under Assumption 5, we have: the SRBM$^d(R, \mu, \mathcal{A})$ is stochastically ordered.

**Proof.** This follows from [51, Theorem 6(i)]). Within this proof, let us use their notation. Fix $x_1, x_2 \in S$ such that $x_2 \geq x_1$. Assume $B = (B(t), t \geq 0)$ is the $d$-dimensional Brownian motion with drift vector $\mu$ and covariance matrix $\mathcal{A}$, starting from zero. Let $X^i(t) = B(t) + x_i$, $i = 1, 2$. Assume $Z^i = (Z^i(t), t \geq 0)$ is the SRBM$^d(R, \mu, \mathcal{A})$ driven by $X^i = (X^i(t), t \geq 0)$, i.e. it is the solution to the Skorohod problem with reflection matrix $R$, with input $X^i$, $i = 1, 2$. Then the process...
\(X^2 - X^1\) is constant, and so it is nonnegative and nondecreasing. According to [51, Theorem 6(i)], this guarantees that \(Z^2(t) \geq Z^1(t)\) for all \(t \geq 0\). Thus, \(P_{x_2}(Z^2 \geq y) \geq P_{x_1}(Z^1 \geq y)\) for all \(y \in S\). The proof is complete. \(\square\)

We should not expect a SRBM to be stochastically ordered in the general case. Indeed, when some off-diagonal elements of \(R\) are positive, then it can start from a point \(x_1 \leq x_2\) but receive a push away from the origin if it hits one of the faces of the boundary, which will violate stochastic ordering.

3.4. Generator of a SRBM. Denote by \(E(S)\) the space of functions \(f : S \to \mathbb{R}\) such that \(f \in C^\infty(S)\) and
\[
\lim_{\|x\| \to \infty} \frac{\log |f(x)|}{\|x\|} < \infty.
\]
In other words, these are functions which grow not faster than exponentials at infinity. Then the generator of a SRBM \(d(R, \mu, A)\) is given by
\[
Lf(x) := \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + Df(x) \cdot \mu,
\]
with the domain \(D(L) \supseteq D_0(L)\), where
\[
D_0(L) := \{ f \in E(S) \mid Df(x) \cdot r_i = 0 \text{ for } x \in S_i, \ i = 1, \ldots, d \}.
\]
Just apply the analogue of Ito's and Tanaka’s formulas from [40] to any function \(f\) from this domain \(D_0(L)\), and take the expectation \(E_x\).

4. Main Results. Now, let us state the central results of this paper. The first result is a generalization and a refinement of [20, Corollary 3.3]. See also the article [13]. The second result is a refinement of Proposition 7.2.

**Theorem 4.1.** Under Assumptions 1, 3, 4, and 6, SRBM \(d(R, \mu, A)\) has a Lyapunov function with Lyapunov constant \(k = K - \varepsilon\), where \(\varepsilon > 0\) can be taken arbitrarily small, and \(K\) is given by
\[
K = \frac{\gamma a^2_*}{2d\|A\|\|R^{-1}\|^{2/3}},
\]
Here, \(\gamma\) is taken from (2), and
\[
a_* := \min_{1 \leq i \leq d} a_i > 0, \quad \text{where } a := (a_1, \ldots, a_d)^T := -R^{-1} \mu > 0.
\]

**Theorem 4.2.** Suppose we have two dimensions, \(d = 2\). Normalize \(R\) so that \(r_{11} = r_{22} = 1\), according to Remark 2. Under Assumptions 1 and 3, we have: SRBM \(d(R, \mu, A)\) has a Lyapunov function with Lyapunov constant \(k = K - \varepsilon\), where \(\varepsilon > 0\) can be taken arbitrarily small, and \(K\) is given by
\[
K = \frac{(1 - r_{12}r_{21})^2 [(R^{-1} \mu)^2 (1 + r_{12}^2)] \wedge [(R^{-1} \mu)^2 (1 + r_{21}^2)]}{4\|A\|(1 + r_{12}^2)(1 + r_{21}^2)(1 + r_{12}^2) \vee (1 + r_{21}^2) \vee (1 + r_{21}^2)\}.\]
Theorem 4.3. Suppose that either Assumptions 1, 3, 4, 6 hold, or \( d = 2 \) and Assumptions 1 and 3 hold. Then the SRBM\(^d\)(\( R, \mu, A \)) is positive recurrent, has a unique stationary distribution and is exponentially ergodic (but we cannot say exactly with which exponent of ergodicity).

Theorem 4.4. Suppose Assumptions 3, 5, 6 hold. Then the SRBM\(^d\)(\( R, \mu, A \)) is exponentially ergodic with the exponent of ergodicity \( \varepsilon = K - \varepsilon \), where \( \varepsilon \) can be taken arbitrarily small, and \( K \) is given by (5).

Theorem 4.5. Suppose \( d = 2 \), and Assumptions 3 and 5 hold. Then the SRBM\(^d\)(\( R, \mu, A \)) is exponentially ergodic with the exponent of ergodicity \( \varepsilon = K - \varepsilon \), where \( \varepsilon > 0 \) can be taken arbitrarily small, and \( K \) is given by (7).

Suppose we already proved Theorem 4.1 and Theorem 4.2. Then Theorem 4.3 is a trivial corollary from these two theorems and Proposition 3.2. In addition, Theorem 4.4 and Theorem 4.5 immediately follow from Theorem 4.1, Theorem 4.2, and Lemma 3.4. Our main goal is to prove Theorem 4.1 and Theorem 4.2. The first of these theorems will be proved in Section 9, and the second one in Section 10.

5. Applications to Competing Brownian Particles. Consider a system of \( N \) particles, formally written as one \( \mathbb{R}^N \)-valued process

\[
X = (X(t), t \geq 0), \quad X(t) = (X_1(t), \ldots, X_N(t))^T.
\]

For any vector \( x \in \mathbb{R}^N \), denote by \( x^{(1)}_t \geq x^{(2)}_t \geq \ldots \geq x^{(N)}_t \) its sorted components. We resolve ties in favor of the lowest index (see more on what it means in [5] and [48]). Let \( W = ((W_1(t), \ldots, W_N(t))^T, t \geq 0) \) be a standard \( N \)-dimensional Brownian motion. Consider a few models invented as market models in Stochastic Portfolio Theory in recent years.

5.1. Atlas-type models. Assume we have the following dynamics:

\[
dX_j(t) = \sum_{k=1}^{N} (g_k dt + \sigma_k dW_k(t)) 1 \left( X_i(t) = X_k(t) \right).
\]

Here, \( g_1, \ldots, g_N \) and \( \sigma_1, \ldots, \sigma_N \) are real numbers, and \( \sigma_1, \ldots, \sigma_N > 0 \). Loosely speaking, the \( k \)th largest process behaves as a Brownian motion with drift \( g_k \) and diffusion \( \sigma_k^2 \). This process was introduced in [5] and extensively studied in subsequent papers; see a historical review in Section 7. Assume that \( \Lambda^{j,k}(t) \) is the local time accumulated at the origin by the nonnegative semimartingale \( X(j) - X(k) \) up to time \( t \geq 0 \) for \( 1 \leq j < k \leq N \). For the sake of convenience, set \( \Lambda^{0,1}(t) = \Lambda^{N,N+1}(t) = 0 \). From the article [48] we get that \( \Lambda^{j,k}(t) = 0 \) for \( |j - k| \geq 2 \), and by [48, Section 4, Lemma 1] we have:

\[
dX_j(t) = g_j dt + \sigma_j dW_j(t) + \frac{1}{2} d\Lambda^{j,j+1}(t) - \frac{1}{2} d\Lambda^{j-1,j}(t).
\]

See also [6] on this topic. Consider the gap process, which is an \( \mathbb{R}^{N-1} \)-valued process

\[
Z = (Z(t), t \geq 0), \quad Z(t) = (Z_1(t), \ldots, Z_{N-1}(t))^T
\]

defined by

\[
Z_k(t) = X_k(t) - X_{k+1}(t), \quad k = 1, \ldots, N - 1.
\]

Let

\[
b_k = g_1 + \ldots + g_k - k\overline{g}, \quad k = 1, \ldots, N - 1, \quad \text{where} \quad \overline{g} = (g_1 + \ldots + g_N)/N.
\]
Theorem 5.1. This process is positive recurrent, in the sense of Definition 4, iff
\[ b_1 < 0, \ b_2 < 0, \ldots, \ b_{N-1} < 0. \]

In this case, the gap process is positive recurrent, has a unique stationary distribution and is exponentially ergodic, with the value of the exponent of ergodicity \( \kappa = K - \varepsilon \). Here, \( \varepsilon > 0 \) can be taken arbitrarily small, and
\[
K = \frac{4}{N} \left(1 - \cos \frac{\pi}{N}\right)^3 \|A\|^{-1} \min_{1 \leq k \leq N-1} b_k^2,
\]
where the \( d \times d \)-matrix \( A \) (the covariance matrix of the gap process) is given by
\[
A = \begin{pmatrix}
\sigma_1^2 + \sigma_2^2 & -\sigma_2^2 & 0 & \ldots & 0 \\
-\sigma_2^2 & \sigma_2^2 + \sigma_3^2 & -\sigma_3^2 & \ldots & 0 \\
 & & & & \\
0 & 0 & 0 & \ldots & \sigma_{N-1}^2 + \sigma_N^2
\end{pmatrix}.
\]

5.2. Asymmetric collisions. This model was introduced in a recent preprint [50]. In the previous model, the collisions between the particles were symmetric. Indeed, the same term \((1/2)d\Lambda^{k,k+1}(t)\) was pushing the particle \(X_k(t)\) up and the particle \(X_{k+1}(t)\) down. The local time term \(d\Lambda^{k,k+1}(t)\) was evenly divided between the \(k\)th and \(k+1\)st processes. This model has different behavior: when two particles collide, they are pushed apart with different speed (as if they had different masses).

Consider a continuous \(\mathbb{R}^N\)-valued semimartingale \(Y = ((Y_1(t), \ldots, Y_N(t))^T, t \geq 0)\) with values in the set \(W := \{(y_1, \ldots, y_N) \in \mathbb{R}^N \mid y_1 \geq \ldots \geq y_N\}\). Suppose it has the following dynamics:
\[
dY_k(t) = g_k dt + \sigma_k dW_k(t) + q_k^- d\Lambda^{k,k+1}(t) - q_k^+ d\Lambda^{k-1,k}(t), \quad k = 1, \ldots, N.
\]

Here, \(g_1, \ldots, g_N\) are real numbers, \(\sigma_1, \ldots, \sigma_N\) are positive real numbers, and the collision parameters \(q_1^-, \ldots, q_N^+\) are nonnegative real numbers satisfying \(q_{k+1}^- + q_k^+ = 1\). For each \(k = 1, \ldots, N-1\) the process \(\Lambda^{k,k+1}(t)\) is the local time accumulated at the origin by the nonnegative semimartingale \(Y_k - Y_{k+1}\). For convenience, we set \(\Lambda^{0,1}(t) = \Lambda^{N,N+1}(t) = 0\). The regulating role of these local times is to make sure the resulting process \(Y\) remains in the set \(W\).

Consider the gap process, which is an \(\mathbb{R}^{N-1}_+\)-valued process
\[
Z = (Z(t), t \geq 0), \quad Z(t) = (Z_1(t), \ldots, Z_{N-1}(t))^T
\]
defined by
\[
Z_k(t) = Y_k(t) - Y_{k+1}(t), \quad k = 1, \ldots, N-1.
\]
In [50], it was shown that under some condition (which, in fact, is a skew-symmetry condition, see (11), rewritten in terms of \(\sigma_k \) and \(q_k^\pm, \ k = 1, \ldots, N\)) it has product-of-exponentials stationary density. We give a necessary and sufficient condition for positive recurrence. Let \(R\) be the following \((N-1) \times (N-1)\)-matrix:
\[
R = \begin{pmatrix}
1 & -q_1^- & 0 & \ldots & 0 & 0 \\
-q_1^- & 1 & -q_2^- & \ldots & 0 & 0 \\
0 & -q_2^- & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & -q_{N-1}^-
\end{pmatrix}.
\]

Let the \(d \times d\)-matrix \(A\) be defined in (10). Let \(\mu \in \mathbb{R}^{N-1}\) be the vector \(\mu = (g_1 - g_2, \ldots, g_{N-1} - g_N)^T\).
Theorem 5.2. The gap process is positive recurrent, in the sense of Definition 4, iff \( R^{-1} \mu < 0 \). In this case, it has a unique stationary distribution, and is exponentially ergodic.

When \( R \) is symmetric, we could apply Theorem 4.4 and get explicit formulas for the exponent of ergodicity. If \( R \) is symmetric, it means that \( q_k^- = q_k^+ \), \( k = 2, \ldots, N - 1 \). Since we also have \( q_{k+1}^+ + q_k^- = 1 \), then the matrix \( R \) has the form

\[
R = \begin{pmatrix}
1 & -q & 0 & \cdots & 0 \\
-q & 1 & -(1 - q) & \cdots & 0 \\
0 & -(1 - q) & 1 & \cdots & 0 \\
0 & 0 & -q & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\]

for some \( q \in (0, 1) \). If \( q = 1/2 \), then we are back in the case of symmetric collisions.

Theorem 5.3. If, in addition to the conditions of the previous theorem, \( R \) is symmetric, then the exponent is given by \( \kappa = K - \varepsilon \), where \( \varepsilon > 0 \) can be taken arbitrarily small, and \( K \) is given by (5).

6. A Toy Example. Let us prove exponential ergodicity for the one-dimensional reflected Brownian motion. A reflection matrix becomes just a single number: 1. A drift vector becomes a drift coefficient \(-\alpha\), \( \alpha > 0 \), and the diffusion matrix becomes a diffusion coefficient \( \sigma^2 \). We can express this one-dimensional SRBM as \( |X(\cdot)| = (|X(t)|, t \geq 0) \), where \( X = (X(t), t \geq 0) \) satisfies the equation, see [29, Section 5.2]

\[
dX(t) = -\alpha \text{sign } X(t) dt + \sigma dW(t),
\]

The process \( |X(\cdot)| \) has a unique stationary distribution: exponential with rate \( 2\alpha/\sigma^2 \), see [29, Section 5.2]. Now, let us show it is exponentially ergodic. The proof is taken from [55, Section 6].

Proposition 6.1. This one-dimensional SRBM is exponentially ergodic with exponent \( \kappa = \alpha^2/2\sigma^2 \).

Proof. Assume \( \mathcal{L} \) is the generator for this process:

\[
\mathcal{L} f := \frac{\sigma^2}{2} f'' - \alpha f', \quad \text{for } f \in \mathcal{D}(\mathcal{L}) \supseteq \mathcal{D}_0(\mathcal{L}) := \{ f \in \mathcal{E}(\mathbb{R}_+) \mid f'(0) = 0 \}.
\]

Let us construct a Lyapunov function for this process. It should be a function \( V : \mathbb{R}_+ \to [1, \infty) \) which satisfies: \( V \in C^\infty(\mathbb{R}_+), V'(0) = 0 \), and

\[
\mathcal{L} V \leq -kV \quad \text{on } \left[ c, \infty \right)
\]

for some \( c, k > 0 \). Let us find a function \( V \) in the form of \( V = e^{\lambda x} \) on \( x > 1 \). Its values for \( x \in [0, 1] \) are irrelevant, as soon as it is smooth and \( V'(0) = 0 \). Then we have:

\[
\mathcal{L} V = \left( \frac{\sigma^2}{2} \lambda^2 - \alpha \lambda \right) V.
\]

Minimizing the coefficient near \( V \) over \( \lambda \), we get:

\[
\mathcal{L} V = -kV, \quad k = \frac{\alpha^2}{2\sigma^2}.
\]

Note that this process is stochastically ordered. This is a corollary of Lemma 3.4. Therefore, the exponent of ergodicity coincides with the Lyapunov constant: \( \kappa = k = \alpha^2/2\sigma^2 \). \( \square \)
The same pattern will be more or less visible for the other proofs in this paper. First, we will construct a linear function, or, more precisely, a function $W$ with the property $W(αx) = αW(x)$. Then, we will take $V = e^{λW}$, and then optimize over $λ$.

The caveat here is that the generator of $L$ in the multi-dimensional case has the domain which is given by the conditions

$$Df(x) \cdot r_i = 0 \text{ for any } x \in S_i, \ i = 1, \ldots, d.$$  

It is harder to satisfy this condition than just $f'(0) = 0$, because the boundaries (=faces) $S_i$ lie outside a compact set. This is why we need to construct $W(x) = [x^T R^{-1} x]^{1/2}$ in the proof of Theorem 4.1, and to undergo a similar process for Theorem 4.2.

7. Historical Survey.

7.1. Origins of the notion of a SRBM. Initially, the concept of a SRBM in the orthant was introduced in connection with series of queues, as a heavy traffic approximation (when the traffic intensities on each station are close to 1). A lot of work has been done in this area: see the papers [37], [38], [65], [34], [75], the surveys [71], [54], the theses [63], [70], [49], [61], [20], Section 4.6 and the books [14], [32], [53].

One can also construct and deliver the theory of reflected Brownian motion in general polyhedral domains (finite intersections of half-spaces). See [69], [19], [34]. In [69] and [34], the construction is done in a different way. SRBM is defined as a solution to a certain submartingale problem. On this, see also [20], Theorem 3.3. However, we will not apply this theory; we are interested only in the SRBM in the positive orthant $S = \mathbb{R}_d^+$, defined as in Definition 2.

Processes similar to SRBM arise as limits of discrete-value processes. The paper [76] proves an invariance principle for a SRBM in the orthant. A recent preprint [62] introduces a class of sticky Brownian motions in many dimensions and shows similar results for them.

Also, note that SRBM can be defined not as a solution to the Skorohod problem, but as a solution to a certain submartingale problem, see [73].

7.2. Existence and uniqueness. The existence and uniqueness was first proved in [40] under some fairly restrictive conditions. In the paper [73], it was shown under the skew-symmetry condition:

$$2A = RD + DR,$$

where $D$ is the diagonal matrix with the same diagonal entries as $A$. This condition can also be stated for general polyhedral domains. We refer the reader to the articles [74], [36], [41], [35] and the survey [74]. The condition in Proposition 2.1 was proved in two papers: [64] (necessity) and [68] (sufficiency); see also the survey [74]. For convex polyhedral domains, a similar result by some different techniques was shown in [19]. It has the following geometric meaning: at any point of the boundary, there exists a linear combination of reflection vectors which points in the interior of the domain. (See [74, p.3].) More on Skorokhod problem can be found in [26].

7.3. Positive recurrence and existence of a stationary distribution. As mentioned above, there are no necessary and sufficient conditions for positive recurrence in the general case.

Let us also mention a result from [20], which closely resembles Theorem 4.1.

**Proposition 7.1.** If $R$ is inverse-positive and symmetric, then Assumptions 1 and 3 guarantee positive recurrence and existence and uniqueness of a stationary distribution.
We aim to generalize this result for non-symmetric $R$ (and, in addition, prove exponential ergodicity).

The paper [72] deals with RBM in the wedge $\{ (r \cos \theta, r \sin \theta) \mid r \geq 0, \ 0 \leq \theta \leq \alpha \}$, where $\alpha \in (0, 2\pi)$ is the angle of the wedge.

As mentioned in Section 2, for $d = 2$ Assumptions 1 and 3 together are not only necessary but sufficient. Let us explain this in more detail:

**Proposition 7.2.** (i) (See [42, 72].) In two dimensions ($d = 2$), $SRBM^d(R, \mu, A)$ is positive recurrent if and only if

$$\mu_1 + r_{12} \mu_2 < 0, \quad \text{and} \quad \mu_2 + r_{21} \mu_1 < 0.$$ 

(ii) (See [39, Appendix A].) These conditions are equivalent to Assumptions 1 and 3 together.

(iii) Therefore, Assumptions 1 and 3 are necessary and sufficient for positive recurrence.

For $d = 3$, Assumption 1 together with Assumption 3 is necessary but not sufficient for positive recurrence, see [8, Section 3]. However, Assumptions 1 and 2 do comprise a necessary and sufficient condition, see [8], [18], [28] (however, we would like to warn the reader that the latter paper contains some mistakes in the proof). Also, see [20, Corollary 3.3].

Let us mention some related papers which deal with general diffusions reflected in a convex polyhedral cone and explore conditions for positive recurrence: [2] (see also corrections [1]), and [9].

### 7.4. Properties of stationary distribution

There is a vast literature on this topic. However, the exact form is known only in a few cases, the most important of which is the skew-symmetry condition, see (11). If it holds, and Assumption 3 also holds, then $a := -R^{-1} \mu > 0$, and there exists a unique stationary distribution with product-of-exponentials density $p_0(x) = C \exp(-D^{-1}a \cdot x)$ for $x \in \text{int} S$. Here, $C$ is a normalizing constant. See [74], [36], [41], [35] and the survey [74]. A general theorem about the properties of the stationary distribution is given in [20, 74]. It states, e.g. that if it exists then it is unique and has density with respect to Lebesgue measure on $S$. Also, this stationary distribution satisfies the basic adjoint relationship, which is a certain weak form of an elliptic PDE. See the formula (3.2) in [74].

See also: (i) a stationary measure for RBM in the wedge for $d = 2$, [72]; (ii) sum-of-exponentials stationary density for RBM in the wedge for $d = 2$, [31], [23]; (iii) tail asymptotics of the stationary distribution $\pi$ for $d = 2$: some results in [39] and a complete solution of this problem in [21]; (iv) numerical methods of calculation of the stationary distribution, see the article [17] and the thesis [20]; (v) other papers on this topic: [16], [22].

### 7.5. Hitting non-smooth parts of the boundary

Under the skew-symmetry condition (11), SRBM does not hit non-smooth parts of the boundary, see [73]. For a RBM in the wedge $\{ (r \cos \theta, r \sin \theta) \mid r \geq 0, \ 0 \leq \theta \leq \alpha \}$, some results are in [72]. Some recent articles in Stochastic Financial Mathematics deal with a related topic: triple collisions of Brownian particles. See [45], [46], [50], and Ichiba’s thesis [44].

### 7.6. Exponential ergodicity

The paper [10] proves that under Assumptions 1 and 2, SRBM is exponentially ergodic. Another paper, [47], deals with a related topic: Atlas-type models of competing Brownian particles. See the next subsection.
7.7. Competing Brownian Particles. Recently, several models of competing Brownian particles were considered in Stochastic Financial Mathematics for market modeling. The simplest one is the Atlas model, which corresponds to (8) with \( g_N > 0, \ g_1 = \ldots = g_{N-1} = 0 \) and \( \sigma_1 = \ldots = \sigma_N = 1 \). Here, the lowest particle has positive drift and unit diffusion, and all other particles behave like standard Brownian motions. It was introduced in the book [29, Chapter 5].

The model given by (8) was introduced in the article [5]. In a subsequent article [60] and the thesis [44, Section 3], they found some results about the gap process: recurrence, existence and uniqueness of a stationary distribution. Also, they gave an explicit formula for this distribution in the cases which correspond to the skew-symmetry condition. The paper [45] treated absence of triple collisions (when three or more particles occupy the same position on the real line).

In the paper [47], they consider the case of competing Brownian particles with symmetric collisions (Atlas-type models) with \( \sigma_k = 1 \) for each \( k \). This topic is the closest to the current paper. Our aim is to extend and generalize their results. It is instructive to compare their results with ours. It turns out that their results are somewhat stronger, somewhat weaker than ours. However, we consider more general cases, and impose weaker conditions. In fact, we do not have any restrictions on the diffusion coefficients. Our techniques are very different from the ones in this paper. They rely heavily on the skew-symmetry condition (which has the form of linear growth condition in this context), and we do not use it. Instead, we construct Lyapunov functions, as described above.

Let us present this result, which is Theorem 1 in their article [47]. We changed notation from theirs to ours.

**Proposition 7.3.** Suppose we have an Atlas-type model with unit variances, as in (8). Consider the gap process \( Z \) on the state space \( S = \mathbb{R}_+^{N-1} \). Assume \( b_1, \ldots, b_{N-1} < 0 \). The process \( Z \) has a unique stationary distribution \( \pi \) (which is, by the way, the product of exponentials, see the thesis [44] and the articles [60, 48]). Take initial distribution \( \kappa \) for the process \( Z \). Assume it is absolutely continuous with respect to \( \pi \), and the density \( \rho = d\kappa/d\pi \in L^2(\mathbb{R}_+^{N-1}, \pi) \). Take any bounded function \( f : S \to \mathbb{R} \) with
\[
\int_S f \, d\pi = 0, \quad \int_S f^2 \, d\pi = \sigma^2, \quad \sup_{x,y \in S} |f(x) - f(y)| = \delta, \quad \sup_{x \in S} |f(x)| = C.
\]
For every \( t, r, \varepsilon > 0 \), the following estimate holds:
\[
P \left\{ \frac{1}{t} \int_0^t f(Z(s)) \, ds \geq r \right\} \leq ||\rho||_{L^2} \exp \left[ -\kappa t \max \left( \frac{r^2}{\delta^2}, 4\varepsilon(\varepsilon + \sigma^2) \sqrt{1 + \frac{r^2}{2\varepsilon(\varepsilon + \sigma^2)^2C^2}} - 1 \right) \right].
\]

Here,
\[
\kappa = 2 \left( 1 - \cos \frac{\pi}{N} \right) \min_{1 \leq k \leq N-1} b_k^2.
\]
This looks like exponential ergodicity. It is a bit another form of saying that the speed of convergence is exponential.

A recent paper [48] introduced hybrid Atlas models. This paper treats the same questions: recurrence, triple collisions, explicit formula for the stationary distribution, etc. The most recent preprint [50] introduced models with asymmetric collisions. It treats the same questions, but fails to give complete results about recurrence. It would be an interesting problem to perfect their results, as well as to study exponential ergodicity.
The articles [12], [66], [67] study behavior of these systems if they have many particles, that is, \( N \to \infty \). Also, it is worth to mention some related papers: [6], [46].

8. Comparison of Old and New Results.

8.1. General comparison. 1. These are the first results, to the best of our knowledge, that show exponential ergodicity with explicit exponents of ergodicity for SRBM. The paper [10] does not provide explicit formulas for these exponents.

2. We prove positive recurrence an exponential ergodicity under Assumptions 1, 3 and 4. They are easier to test than Assumption 2. A useful corollary is with Assumption 5, which covers many practical cases.

3. For the case \( d = 2 \), we prove exponential ergodicity with explicit values of the exponent in all cases when it is possible (i.e. when the process is positive recurrent).

4. We find general estimates for the tail of the stationary distribution. This is done in all cases we consider, not necessarily when the process is stochastically ordered. This problem, as mentioned above, was fully resolved in [21] for \( d = 2 \). However, to the best of our knowledge, general results are lacking for the case of the general dimension.

5. We have exponential ergodicity as a corollary for the gap process of competing Brownian particle systems. In [47], they provide a result for unit variances and symmetric collisions. We give another result which is much more general: it covers any positive variances, both symmetric and some cases of asymmetric collisions, as well as hybrid Atlas models. However, our result is much cruder in case of unit variances and symmetric collisions, compared with the known one. The comparison is done later in this section.

6. Our result for \( d = 2 \) from Theorem 4.2 seems to be stronger than the general result from Theorem 4.1. It is a bit hard to compare them in the general case, so we carry out this comparison for a symmetric and inverse-nonnegative \( R \).

7. We can compare our results with the obvious result for normal reflection and uncorrelated Brownian motions: \( R = I_d \). \( \mathcal{A} \) is diagonal. Our results are a bit less sharp.

8.2. Comparison of Theorem 4.1 and Theorem 4.2. Suppose \( d = 2 \): the case of two dimensions. Assume \( R \) is \( 2 \times 2 \) matrix which is symmetric and strictly inverse-nonnegative. Assume also it is normalized: \( r_{11} = r_{22} = 1 \). Then it has the form

\[
R = \begin{pmatrix} 1 & -q \\ -q & 1 \end{pmatrix}
\]

Therefore,

\[
R^{-1} = \frac{1}{1 - q^2} \begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix}
\]

Since \( R^{-1} \) is inverse-nonnegative, \( 1/(1 - q^2) > 0 \) and \( q/(1 - q^2) \geq 0 \). Then we have: \( 0 \leq q < 1 \). Let us calculate the norm of the matrix \( R^{-1} \). The norm of the matrix

\[
P = \begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix}
\]

is equal to \( 1 + q \), since its eigenvalues are \( 1 \pm q \). (Just solve the characteristic equation.) Then

\[
\|R\| = \frac{1}{1 - q^2}\|P\| = \frac{1 + q}{1 - q^2} = \frac{1}{1 - q}.
\]
Also,

\[ \gamma := \frac{1}{1 - q^2}. \]

Therefore, the Lyapunov constant from Theorem 4.1 is equal to \( \varepsilon = K_1 - \varepsilon \), where \( \varepsilon > 0 \) can be taken arbitrarily small, and

\[ K_1 := \frac{\gamma a^2_i}{2d \|A\| (1/(1 - q))^3} = \frac{a^2_i (1 - q)^2}{4(1 + a\|A\|)}. \]

And the Lyapunov constant from Theorem 4.2 is equal to \( \varepsilon = K_2 - \varepsilon \), where, again, \( \varepsilon > 0 \) can be arbitrarily small, and

\[ K_2 := \frac{a^2_i (1 - q^2)^2 (1 + q^2)}{4\|A\| (1 + q^2)^3} = \frac{a^2_i (1 - q^2)^2}{4\|A\| (1 + q^2)^2}. \]

Let us compare these results. One can prove without difficulty that \((1 + q^2)^2 \leq (1 + q)^3\) for \( q \in [0, 1) \), and so \( K_1 < K_2 \). Therefore, the constant from Theorem 4.2 is sharper. However, for some constant \( c_0 > 0 \) we have: \( c_0 (1 + q^2)^2 \geq (1 + q)^3 \). Therefore, it is only sharper by a constant multiple (not by a different order).

8.3. The case of uncorrelated components. Assume \( R = I_d \), and \( A \) is the diagonal matrix with entries \( \sigma_1^2, \ldots, \sigma_d^2 \). Then \( \text{SRBM}^d(R, \mu, A) \) is a collection of one-dimensional reflected Brownian motions on the positive half-line, and the exponent of ergodicity of each of them is equal to \( \kappa_i = \mu_i^2/2\sigma_i^2, i = 1, \ldots, d \), according to the toy example. The exponent of ergodicity of the whole \( d \)-dimensional process is equal to \( \min_{1 \leq i \leq d} \kappa_i \). It is easy to see that this is generally larger than the exponent from Theorem 4.1. Indeed, it is given by \( K - \varepsilon \), where \( \varepsilon > 0 \) can be taken arbitrarily small, and

\[ K = \frac{1}{2d} \min_{1 \leq i \leq d} \frac{\mu_i^2}{\max_{1 \leq i \leq d} \sigma_i^2}. \]

8.4. Comparison of Results for Atlas-type Models with Unit Diffusions. Assume we have a competing Brownian particle system with symmetric collisions and unit diffusions. Let us compare the result of Theorem 5.1 with the result from Proposition 7.3, [47, Theorem 1]. From there, we get:

\[ \kappa = 2 \left( 1 - \cos \frac{\pi}{N} \right) \min_{1 \leq k \leq N-1} b_k^2 = a_N^{(2)} \min_{1 \leq k \leq N-1} b_k^2, \quad a_N^{(2)} = 2 \left( 1 - \cos \frac{\pi}{N} \right) \sim \frac{\pi^2}{N^2}. \]

The spectral norm of the matrix \( R \) is equal to its maximal eigenvalue, that is, to \( 1 - \cos((N - 1)\pi/N) = 1 + \cos(\pi/N) \). Since \( A = 2R \) for this case, we have: \( \|A\| = 2\|R\| = 2(1 + \cos(\pi/N)) \). Then from Theorem 5.1 we get:

\[ K_1 = a_N^{(1)} \min_{1 \leq k \leq N-1} b_k^2, \quad \text{where} \quad a_N^{(1)} = \frac{4}{N} \left( 1 - \cos \frac{\pi}{N} \right)^3 \cdot \left( 1 + \cos \frac{\pi}{N} \right)^{-1}. \]

Let us compare \( a_N^{(1)} \) and \( a_N^{(2)} \) as \( N \to \infty \). We have:

\[ a_N^{(1)} \sim 4N^{-1} \left( \frac{\pi^2}{2} N^{-2} \right)^3 \cdot 2^{-1} = \frac{\pi^6}{4} N^{-7}, \quad a_N^{(2)} \sim \pi^2 N^{-2}. \]

The result from Proposition 7.3, which is [47, Theorem 1], is much better. But it is valid only for unit diffusions. The result from 5.1 is much less sharp, but more general.
8.5. Comparison for Atlas-type models of three particles with unit diffusions. Here, we have: 
\[ d = 2, \quad N = 3, \quad \text{and} \]
\[ R = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}, \quad r_{12} = r_{21} = -1/2. \]

Also, \( A = 2R \), and we have: \( \|A\| = 2\|R\| = 2(1 - \cos(2\pi/3)) = 3. \) The exponent from Theorem 5.1 is given by \( K - \varepsilon \), where \( \varepsilon > 0 \) can be made arbitrarily small, and
\[ K = \frac{3}{25}(b_1^2 \wedge b_2^2) \]

The result of Proposition 7.3, [47, Theorem 1] gives us
\[ \varkappa = 2(1 - \cos(\pi/3))(b_1^2 \wedge b_2^2) = b_1^2 \wedge b_2^2. \]

The result from Theorem 4.2 gives us \( \varkappa = K - \varepsilon \), where \( \varepsilon > 0 \) can be made arbitrarily small, and
\[ K = \frac{12}{25}(b_1^2 \wedge b_2^2). \]

Again, the result from Proposition 7.3, i.e. [47, Theorem 1] is the best. The other result differs by an absolute constant from this result.

9. Proof of Theorem 4.1. Let us define a function \( W : S \to \mathbb{R}_+ \):

\[ W(x) = [x^TR^{-1}x]^{1/2}. \]

By Assumption 4, see (2), \( W(x) \) is well defined for every \( x \in S \), and \( W(x) \geq \gamma^{1/2}\|x\| \). Let us show that \( W \in C^\infty(S \setminus \{0\}) \). Indeed, the function \( x \mapsto x^TR^{-1}x \) is infinitely differentiable on \( \mathbb{R}^d \), and the function \( u \mapsto \sqrt{u} \) is infinitely differentiable for \( u > 0 \). From (2), we know that \( x^TR^{-1}x > 0 \) for \( x \in S \setminus \{0\} \). Thus, \( W \in C^\infty(S \setminus \{0\}) \).

Also, \( x^TR^{-1}x \leq \|R^{-1}\||x||^2 \), so \( W(x) \leq \|R^{-1}\|^{1/2}\|x\| \).

Let us show that \( \|DW(x)\| \) is bounded and \( \|D^2W(x)\| \to 0 \) as \( \|x\| \to \infty \). Since \( R^{-1} \) is symmetric, we have:

\[ DW(x) = \frac{R^{-1}x}{2\sqrt{x^TR^{-1}x}} = \frac{R^{-1}x}{W(x)}. \]

Since \( \|R^{-1}x\| \leq \|R^{-1}\||x|| \), and \( W(x) \geq \gamma^{1/2}\|x\| \), we have:

\[ \|DW(x)\| \leq \frac{\|R^{-1}\|}{\gamma^{1/2}} =: C_0. \]

Calculating the second derivative matrix \( D^2W(x) \), we get: if \( F(x) := x^TR^{-1}x \), then

\[ D^2W(x) = \frac{1}{4F^{3/2}(x)} \left[ 2F(x)D^2F(x) - DF(x) \cdot (DF(x))^T \right]. \]

We have: \( DF(x) = 2R^{-1}x \), \( D^2F(x) = 2R^{-1} \), we have: \( \|DF(x)\| \leq 2\|R^{-1}\||x|| \). Also, \( \gamma\|x\|^2 \leq F(x) \leq \|R^{-1}\||x||^2 \). Therefore, we have:

\[ \|D^2W(x)\| \leq \frac{1}{4\gamma^{3/2}\|x\|^3} \left[ 4\|R^{-1}\||x||^2 + 4\|R^{-1}\|2\|x\|^2 \right] = 2\frac{\|R^{-1}\|^2}{\|x\|^2}. \]
Therefore, \( \|D^2W(x)\| \to 0 \) as \( \|x\| \to \infty \).

Now let us prove the two most important properties: for each \( i = 1, \ldots, d \),

\[
(13) \quad DW(x) \cdot r_i = 0 \quad \text{for} \quad x \in S_i \setminus \{0\}
\]

and there exists a certain constant \( \beta > 0 \) (its value will be determined later) such that

\[
(14) \quad DW(x) \cdot \mu \leq -\beta \quad \text{for} \quad x \in S \setminus \{0\}.
\]

For all \( x \in \mathbb{R}_+^d \setminus \{0\} \), we have:

\[
DW(x) = \frac{R^{-1}x}{W(x)}.
\]

Since \( r_i \) is the \( i \)th column of \( R \), we have: \( R^{-1}r_i = e_i \), where \( e_i \) is the \( i \)th standard unit vector in \( \mathbb{R}^d \). However, for \( x \in S_i \) we have: \( e_i \cdot x = x_i = 0 \). Therefore, by the symmetry of \( R^{-1} \), we get:

\[
R^{-1}x \cdot r_i = x \cdot R^{-1}r_i = x \cdot e_i = 0.
\]

Thus, \( DW(x) \cdot r_i = 0 \), and the first property is proved. Now,

\[
DW(x) \cdot \mu = \frac{(R^{-1} \mu) \cdot x}{W(x)}.
\]

Recall: \( a := -R^{-1} \mu > 0 \). Then we have: \( a_+ = \min_{1 \leq i \leq d} a_i > 0 \). Therefore, \( x \cdot a \geq a_+(x_1 + \ldots + x_n) = a_+ \|x\| \) for \( x \in S \). Also, we have: \( W(x) \leq \|R^{-1}\|^{1/2} \|x\| \). Therefore,

\[
\frac{x \cdot a}{W(x)} \geq \frac{a_+ \|x\|}{\|R^{-1}\|^{1/2} \|x\|} = \beta := \frac{a_+}{\|R^{-1}\|^{1/2}} > 0.
\]

Thus,

\[
DW(x) \cdot \mu = \frac{(R^{-1} \mu) \cdot x}{W(x)} = -\frac{x \cdot a}{W(x)} \leq -\beta.
\]

The function \( W \) is smooth in the whole orthant \( S \) except the origin. We need to make it smooth everywhere, to be able to apply the generator to this function. Consider a function \( W_0 = \varphi(W) \), where \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) is any \( C^\infty \) function such that \( \varphi(s) = 0 \) for \( s \in [0, 1/2] \) and \( \varphi(s) = s \) for \( s \geq 1 \). We do not care about the values of \( W \) close to the origin. What is important for Lyapunov functions is what is happening outside \( C(r) \) for every fixed \( r > 0 \). Then we have: \( W_0 \in C^\infty(S) \). The norm of the gradient vector \( \|DW_0(x)\| \) is bounded on \( S \). Indeed, on \( S \setminus C(1) \) we have: \( W_0 = W \), so \( DW_0 = DW \), while \( \|DW\| \) is bounded on \( S \setminus \{0\} \). The remaining set \( C(1) \) is compact, so a continuous function \( \|DW_0\| \) is bounded on this set. Therefore, \( \|DW_0(x)\| \) is bounded on \( S \). Moreover, \( D^2W_0(x) = D^2W(x) \) for \( x \in S \setminus C(1) \), so \( \|D^2W_0(x)\| \to 0 \) as \( \|x\| \to \infty \).

Let us show that

\[
(15) \quad DW_0(x) \cdot r_i = 0 \quad \text{for} \quad x \in S_i \quad \text{for each} \quad i = 1, \ldots, d.
\]

First, let \( x \neq 0 \). We have: \( DW_0(x) = \varphi'(W(x))DW(x) \). From (13), we have: \( DW(x) \cdot r_i = 0 \). Therefore, \( DW_0(x) \cdot r_i = 0 \) for \( x \in S_i \setminus \{0\} \). For \( x = 0 \), we have: since \( W_0 = 0 \) on \( C(1/2) \), then \( DW_0(0) = 0 \). Thus, we have \( DW_0(0) \cdot r_i = 0 \) too. The property (15) is proved. Also, we have:

\[
(16) \quad DW_0(x) \cdot \mu \leq -\beta \quad \text{for} \quad x \in S \setminus C(1).
\]
Indeed, on $S \setminus C(1)$ we have: $W = W_0$ and $D W = D W_0$.

The generator $\mathcal{L}$ of SRBM$^d(R, \mu, \mathcal{A})$ is defined as

$$\mathcal{L} f(x) := \lim_{t \to 0} \frac{P^t f(x) - f(x)}{t}.$$ 

Its domain $\mathcal{D}(\mathcal{L})$ contains the following subdomain:

$$\mathcal{D}_0(\mathcal{L}) = \{ f \in \mathcal{E}(S) \mid D f(x) \cdot r_i = 0 \text{ for } x \in S_i, \ i = 1, \ldots, d \}.$$

The generator itself can be calculated by the formula

$$\mathcal{L} f(x) = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + D f(x) \cdot \mu.$$ 

Therefore, $W_0 \in \mathcal{D}(\mathcal{L})$. Consider the function

$$V(x) = e^{\lambda W_0(x)}$$

for some parameter $\lambda > 0$.

Then $V \in C^\infty(S)$ and $D V(x) = \lambda V(x) D W_0(x)$. So $D V(x) \cdot r_i = 0$ for $x \in S_i$ for each $i = 1, \ldots, d$, and $V \in \mathcal{D}(\mathcal{L})$. Now, let us estimate $\mathcal{L} V(x)$. First of all, the second derivative matrix of the function $V$ is equal to

$$D^2 V(x) = (\lambda D^2 W_0(x) + \lambda^2 (D W_0(x))(D W_0(x))^T) V(x).$$

Since $\|D^2 W_0(x)\| \to 0$ as $\|x\| \to \infty$, we have: for every fixed $\varepsilon > 0$, there exists sufficiently large $r_\varepsilon > 1$ such that for $\|x\| > r_\varepsilon$ we have: $\|D^2 W_0(x)\| \leq \varepsilon$. Recall that we also have: $\|D W_0(x)\| = \|D W(x)\| \leq C_0$. By Lemma 14.1, $\|((D W_0(x))(D W_0(x))^T) = \|D W_0(x)\| \leq C_0^2$. Therefore, for $x \in S \setminus C(r_\varepsilon)$ we have:

$$\|D^2 V(x)\| \leq (\lambda \varepsilon + \lambda^2 C_0^2) V(x),$$

and $D V(x) \cdot \mu = \lambda V(x) D W_0(x) \cdot \mu \leq -\lambda \beta V(x)$.

By Lemma 14.2, we have:

$$\left| \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij} \frac{\partial^2 V(x)}{\partial x_i \partial x_j} \right| \leq d\|\mathcal{A}\|\|D^2 V(x)\| \leq d\|\mathcal{A}\| \left( \lambda \varepsilon + \lambda^2 C_0^2 \right).$$

Now we can estimate $\mathcal{L} V(x)$: for $x \in S \setminus C(r_\delta)$, we have:

$$(17) \quad \mathcal{L} V(x) \leq \frac{1}{2}\|\mathcal{A}\| \left( \lambda \varepsilon + \lambda^2 C_0^2 \right) - \lambda \beta V(x) = \left[ \frac{1}{2} d C_0^2 \|\mathcal{A}\| \lambda^2 + \left( \frac{1}{2} \varepsilon d\|\mathcal{A}\| - \beta \right) \lambda \right] V(x).$$

Let us minimize the expression in brackets. Its minimal value is obtained when $\lambda = \lambda_\varepsilon$, and it is equal to $-K_\varepsilon$, where

$$\lambda_\varepsilon = \beta - \frac{1}{2} \varepsilon d\|\mathcal{A}\|, \quad K_\varepsilon = \frac{1}{2 d C_0^2 \|\mathcal{A}\|} \left( \frac{1}{2} \varepsilon d\|\mathcal{A}\| - \beta \right)^2.$$ 

As $\varepsilon \to 0$, we have: $K_\varepsilon \to K$, where

$$K := \frac{\beta^2}{2 d C_0^2 \|\mathcal{A}\|}.$$
So we proved that for every $\delta > 0$, there exists $r_\delta > 1$ such that on the set $S \setminus C(r_\delta)$ we have:

$$\mathcal{L}V \leq -(K - \delta)V.$$ 

Note that $V \in C^\infty(S)$, so $\mathcal{L}V(x)$ is a continuous function, and the set $C(r_\delta)$ is compact (therefore, petite - see Appendix). Therefore, $\mathcal{L}V(x)$ is bounded from above on $C(r_\delta)$. We can state this result as follows: any positive number strictly less than

$$K = \frac{\beta^2}{2dC_0^2||A||}, \quad \text{where} \quad \beta := \frac{a_*}{||R^{-1}||^{1/2}}, \quad C_0 := \frac{||R^{-1}||}{\gamma^{1/2}},$$

can serve as a Lyapunov constant. Let us write $K$ more explicitly: plugging in $\beta$ and $C_0$, we get the expression (5). \(\square\)

10. Proof of Theorem 4.2. Now consider the case $d = 2$. We know the following necessary and sufficient conditions for positive recurrence: Assumptions 1 and 3. Let us show that under these assumptions, the SRBM is exponentially ergodic. We follow the same steps as in the previous proof. However, the first step, construction of the function $W$, takes more effort. We introduce a curve in the first quadrant that intersects each ray coming from the origin only once. Then we set $W$ to be equal to 1 on this curve and to be homogeneous: $W(\xi x) = \xi W(x)$ for $x \in \mathbb{R}^2_+$, $\xi \geq 0$. After this, we repeat all the steps.

First of all, rewrite the conditions for positive recurrence in angular form. Let $\alpha \in (0, \pi)$ be the angle such that

$r_2 = \langle r_{12}, 1 >\rangle = ||r_2|| < \cos \alpha, \sin \alpha >; \quad \alpha = \arctg r_{12}.$

Let $\beta \in (\pi/2, 3\pi/2)$ be the angle such that

$-r_1 = \langle -1, -r_{21} >\rangle = ||r_1|| < \cos \beta, \sin \beta >, \quad \beta = \arctg r_{21} + \pi.$

Let $\gamma \in [-\pi/2, 3\pi/2)$ be the angle such that

$$-\mu = ||\mu|| < \cos \gamma, \sin \gamma > .$$

Now, rewrite Assumptions 1 and 3 in terms of these angles. We have:

$$R = \begin{pmatrix} -||r_1|| \cos \beta & ||r_2|| \cos \alpha \\ -||r_1|| \sin \beta & ||r_2|| \sin \alpha \end{pmatrix}, \quad R^{-1} = \frac{1}{1 - r_{12}r_{21}} \begin{pmatrix} ||r_2|| \sin \alpha & -||r_2|| \cos \alpha \\ ||r_1|| \sin \beta & -||r_1|| \cos \beta \end{pmatrix},$$

Therefore, we have

$$-R^{-1} \mu = \frac{||\mu||}{||r_1|| ||r_2|| \sin(\beta - \alpha)} < ||r_2|| \sin(\alpha - \gamma), \quad ||r_1|| \sin(\beta - \gamma) > .$$

We have: $r_{12}r_{21} < 1$, from [39], so det $R = 1 - r_{12}r_{21} > 0$. Therefore, $\sin(\beta - \alpha) > 0$. Since $0 < \alpha < \pi$ and $\pi/2 < \beta < 3\pi/2$, it follows that $\alpha < \beta < \alpha + \pi$. Also, $-R^{-1} \mu > 0$, so $\sin(\alpha - \gamma) > 0$ and $\sin(\beta - \gamma) > 0$. Therefore, $\gamma < \alpha$, but $\gamma > \beta - \pi$. Thus, we have:

$$\beta - \pi < \gamma < \alpha < \beta.$$ 

These are Assumptions 1 and 3 rewritten in terms of angles $\alpha, \beta, \gamma$.

For any nonzero vector $x = \langle x_1, x_2 >\rangle \in \mathbb{R}^2$, its direction $\theta$ is a unique angle from $[0, 2\pi)$ such that

$$x_1 = ||x|| \cos \theta, \quad x_2 = ||x|| \sin \theta.$$
**Lemma 10.1.** There exists a smooth $C^\infty$ curve $C$ in $\mathbb{R}^2_+ \setminus \{0\}$ such that:

(i) it intersects each ray $x := \{\langle x, \xi \rangle \mid \xi > 0\}$, $x \in \mathbb{R}^2_+ \setminus \{0\}$ only once, at a certain point $N(x)$;

(ii) the tangent vector is parallel to $r_i$ at the intersection with $S_i = \{x \in S \mid x_i = 0\}$, $i = 1, 2$;

(iii) if $\psi(\theta)$ is the direction of the tangent vector at $N(\theta) = N((\cos \theta, \sin \theta))$, $0 \leq \theta \leq \pi/2$, then $\psi(0) = \alpha$, $\psi(\pi/2) = \beta$, and $\psi$ is nondecreasing;

(iv) the ratio of the maximal and the minimal distances from the origin is equal to

$$\frac{\max_{y \in C}\|y\|}{\min_{y \in C}\|y\|} = K + \varepsilon$$

where $\varepsilon > 0$ can be made arbitrarily small by the choice of $C$, and

$$K = (1 + (r_{12}^2)_{1/2}) \vee (1 + (r_{21}^2)_{1/2}).$$

The proof is postponed until the end of this section. Having done this, we define the function $W : \mathbb{R}^2_+ \to \mathbb{R}^+$ by two properties:

(i) $W(\xi x) = \xi W(x)$ for $x \in \mathbb{R}^2_+$ and $\xi \geq 0$;

(ii) $W(x) = 1$ for $x \in C$.

Then we have: $DW(\xi x) = DW(x)$, $x \in \mathbb{R}^2_+$, $\xi > 0$. In addition, $\|D^2W(x)\| \to 0$ as $\|x\| \to \infty$.

The function $W$ is continuous on $\mathbb{R}^2_+$ and is $C^\infty(\mathbb{R}^2_+ \setminus \{0\})$. Assume $n(\theta)$ is the unit normal vector at the point $N(\theta)$. Then $n(\theta) = \langle \sin \psi(\theta), -\cos \psi(\theta) \rangle$. Then

$$DW(x) = \frac{n(\theta)}{\|x\|}.$$ 

Therefore,

$$-DW(x) \cdot \mu = \frac{\|\mu\|}{\|x\|} \sin(\psi(\theta) - \gamma).$$

Since $\beta - \pi < \gamma < \alpha < \beta$, and $\psi(0) = \alpha$, $\psi(\pi/2) = \beta$, and $\psi$ is always between $\alpha$ and $\beta$, we have:

$$\sin(\psi(\theta) - \gamma) \geq \sin(\alpha - \gamma) \land \sin(\beta - \gamma).$$

Therefore,

$$-DW(x) \cdot \mu \geq \frac{\|\mu\|(\sin(\alpha - \gamma) \land \sin(\beta - \gamma))}{\max_{y \in C}\|y\|} =: c_0 > 0.$$ 

From (10), we have:

$$\|\mu\| \sin(\alpha - \gamma) = (1 - r_{12}r_{21})(-R^{-1}_1\mu_1)\|r_2\|^{-1}, \text{ and } \|\mu\| \sin(\beta - \gamma) = (1 - r_{12}r_{21})(-R^{-1}_2\mu_2)\|r_1\|^{-1},$$

and, therefore,

$$(1 - r_{12}r_{21})(-R^{-1}_1\mu_1)\|r_1\| \land (-R^{-1}_2\mu_2)\|r_2\| =: c_0 > 0.$$ 

Also, $DW(x) \cdot n_i = 0$ for $x \in S_i$, $i = 1, 2$, from the property (ii) of this curve. Now, repeat the corresponding steps of the last proof. Take $W_0 = \varphi(W)$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a function described in the proof of Theorem 4.1, and take $V = e^{\lambda W_0}$ for some $\lambda > 0$. Then $V \in \mathcal{D}(L)$ (the domain of the generator), and we have:

$$DV = \lambda WD, \quad D^2V = \left[\lambda^2(DW)(DW)^T + \lambda D^2W\right]V$$
Then for $S \setminus C(1)$. Then we get:

$$
\mathcal{L}V = \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j} + DV \cdot \mu.
$$

As before, we have: for $x \in S \setminus C(1)$,

$$
\left| \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j} \right| \leq 2 \|A\| \|D^2V\| \leq 2 \|A\| \left( \lambda^2 \|DW\|^2 + \lambda \|D^2W\| \right) V,
$$

so

$$
\mathcal{L}V \leq \|A\| \left( \lambda^2 \|DW\|^2 + \lambda \|D^2W\| \right) V - c_0 \lambda V.
$$

Similarly to the last proof, we have: $\|D^2W(x)\| \to 0$ as $\|x\| \to \infty$, and for every $\delta > 0$ there exists large enough $r_\delta > 1$ such that for $x \in S$ with $\|x\| \geq r_\delta$ we have: $\|D^2W(x)\| < \delta$. Also, for every $x \in S \setminus \{0\}$, we have:

$$
\|DW(x)\| = \frac{1}{\|N(x)\|} \leq \frac{1}{\min_{x \in C} \|x\|}
$$

Then for $x \in S \setminus C(r_\delta)$ we have:

$$
\mathcal{L}V(x) \leq \lambda(\|A\| \delta - c_0) V + \|A\| \lambda^2 \frac{1}{(\min_{x \in C} \|x\|)^2} V.
$$

By minimizing the coefficient near $V$ over $\lambda$, we get:

$$
\mathcal{L}V(x) \leq -kV(x), \quad k = \frac{\left( c_0 - \delta \|A\| \right)^2 (\min_{y \in C} \|y\|)^2}{4 \|A\|}.
$$

As $\delta \to 0$, this tends to

$$
k_0 := \frac{\left( c_0 - \delta \|A\| \right)^2 (\min_{y \in C} \|y\|)^2}{4 \|A\|} = \left( \frac{\min_{y \in C} \|y\|}{\max_{y \in C} \|y\|} \right)^2 \frac{\left( (-R^{-1} \mu)^2 \|r_1\|^2 \right) \wedge \left( (-R^{-1} \mu)^2 \|r_2\|^2 \right)}{4 \|A\| \|r_1\|^2 \|r_2\|^2}.
$$

Rewrite this as

$$
k_0 = \frac{\left( (-R^{-1} \mu)^2 \|r_1\|^2 \right) \wedge \left( (-R^{-1} \mu)^2 \|r_2\|^2 \right)}{4 \|A\| \|r_1\|^2 \|r_2\|^2 (K + \varepsilon)^2}.
$$

It suffices to note that $\mathcal{L}V$ is bounded on $C(r_\delta)$ for each $\delta > 0$. Therefore, $V$ is a Lyapunov function, and exponential ergodicity holds. Note that

$$
\|r_1\|^2 = 1 + r^2_{21}, \quad \|r_2\|^2 = 1 + r^2_{12}, \quad K = (1 + (r_{12})^2) / 2 + (1 + (r_{21})^2) / 2.
$$

Let $\varepsilon \to 0$ and get: $\varepsilon_0 \to K$, where $K$ is given by (7). \(\square\)

10.1 \textbf{Proof of Lemma 10.1.} For $\theta \in \mathbb{R}$, let $P(\theta) = (\cos \theta, \sin \theta)$ be the point on the unit circle. Let $P_1 = (0, 1)$ and $P_2 = (1, 0)$. Let $l_i$ be the line passing through $P_i$ and parallel to $r_i$, where $i = 1, 2$. We will consider several cases, in each of them (except the last one) the proof will go as follows: we will construct a piecewise-smooth curve $C$ which satisfies all requirements, except smoothness (it will be piecewise-smooth, not smooth). For a continuous curve $C$ in $\mathbb{R}^2 \setminus \{0\}$, let

$$
\Delta(C) = \max_{y \in C} \|y\| / \min_{y \in C} \|y\|.
$$
1. Assume $\alpha < \pi/2$ and $\beta > \pi$, which means $r_{12} > 0$ and $r_{21} > 0$. Fix small angles $\theta_1, \theta_2 > 0$. Draw a tangent line to the unit circle at the point $P(\pi/2 - \theta_2)$. Assume $Q_1$ is the point of intersection of this tangent line and $l_1$. Draw a tangent line to the unit circle at the point $P(\theta_1)$. Assume $Q_2$ is the point of intersection of this tangent line and $l_2$. Consider the curve which consist of the following parts: the segment $P_2Q_2$, the segment $Q_2P(\theta_1)$, the arc $P(\theta_1)P(\theta_2)$ of the unit circle, the line segment $P(\theta_2)Q_1$ and the segment $Q_1P_1$. Then $\min_{y \in C} |y| = 1$, and $\max_{y \in C} |y| \to 1$ as $\theta_1, \theta_2 \to 0$. Therefore, $\Delta(C) \to 1$, and $\Delta = ((r_{12})^2 + 1) \cup ((r_{21})^2 + 1)$.

2. Assume $r_{12} \leq -1$, which is equivalent to $\alpha \geq 3\pi/4$. Let $l_2$ intersect the $y$-axis at the point $R$, and let $R'$ lie a bit lower than $R$ on the $y$-axis. Let $l_1'$ be the line passing through $R'$ parallel to $r_1$. Let $Q = l_1' \cap l_2$, and let $C'$ consist of two segments: $P_2Q$ and $QR'$. Then $\Delta(C) = (\sin \alpha)^{-1} + \varepsilon$, where $\varepsilon > 0$ can be made arbitrarily small by choosing $R'$. It suffices to note that $(\sin \alpha)^{-1} = ((r_{12})^2 + 1)^{1/2} = ((r_{21})^2 + 1) \cup ((r_{21})^2 + 1)$.

3. The case $r_{21} \leq -1$, i.e. $\beta \leq 3\pi/4$, is considered analogously. Note that these two cases cannot occur simultaneously, since $\alpha < \beta$.

4. Let $3\pi/4 > \alpha > \pi/2$ and $\beta > \pi$, which corresponds to $-1 < r_{12} < 0$, $r_{21} > 0$. Let $Q$ be the second point of intersection of $l_2$ with the unit circle (the first point is $P_2$ itself). Let $\theta > 0$ be small, let $R$ be the intersection of the tangent line to the unit circle at the point $P(\pi/2 - \theta)$ with $l_2$. Consider a curve $C$ consisting of the segment $P_2Q$, the arc $QP(\pi/2 - \theta)$ of the unit circle, and the segment $RP_1$.

5. The case $r_{12} > 0$ and $-1 < r_{21} < 0$ is treated similarly.

6. If $-1 < r_{12} < 0$ and $-1 < r_{21} < 0$, then let $Q_i$ be the second intersection of $P_i$ with the unit circle (the first one is $P_i$), $i = 1, 2$. If $Q_2$ has smaller $y$-coordinate than $Q_1$, then consider the following curve $C$: the segment $P_2Q_2$, the arc $Q_2Q_1$ of the unit circle, and the segment $Q_1P_1$. Otherwise, let $Q$ be the intersection of $l_1$ and $l_2$, and let $C$ consist of two segments $P_2Q$ and $QP_1$. Then again it is straightforward to show that $\Delta(C) \leq ((r_{12})^2 + 1) \cup ((r_{21})^2 + 1)$.

7. The cases $r_{12} = 0$, $-1 < r_{21} < 0$ and $r_{12} = 0$, $r_{21} > 0$ are treated similarly to the cases 4 and 1, respectively. The difference is that now, starting from $P_2$, we do not construct two line segments and then an arc of the unit circle; we immediately proceed to an arc of the unit circle.

8. The cases $r_{21} = 0$, $-1 < r_{12} < 0$ and $r_{21} = 0$, $r_{12} > 0$ are analogous to the cases in 7.

9. Finally, when $r_{12} = r_{21} = 0$, this is just a unit circle, or, more precisely, its arc in the first quadrant. This is the easiest case. □

11. Competing Brownian Particles.

11.1. Proof of Theorem 5.2. The process $Z$ is nothing else but a SRBM$^{N-1}(R, \mu, A)$, where

$$A = \begin{pmatrix}
\sigma_1^2 + \sigma_2^2 & -\sigma_2^2 & 0 & \cdots & 0 \\
-\sigma_2^2 & \sigma_2^2 + \sigma_3^2 & -\sigma_3^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \sigma_{N-1}^2 + \sigma_N^2
\end{pmatrix}$$

Indeed, since $q_k^- + q_{k+1}^+ = 1$, we have:

\[dZ_k(t) = (g_k - g_{k+1})dt + \sigma_k dW_k(t) - \sigma_{k+1} dW_{k+1}(t)\]

\[+ q_k^- d\Lambda^{k,k+1}(t) - q_k^+ d\Lambda^{k-1,k}(t) + q_{k+1}^- d\Lambda^{k,k+1}(t) - q_{k+1}^- d\Lambda^{k+1,k+2}(t) = \]

\[= \mu_k dt + \sigma_k dW_k(t) - \sigma_{k+1} dW_{k+1}(t) + d\Lambda^{k,k+1}(t) - q_k^+ d\Lambda^{k-1,k}(t) - q_{k+1}^- d\Lambda^{k+1,k+2}(t)\]
Let $B_k(t) = \sigma_k W_k(t) - \sigma_{k+1} W_{k+1}(t)$. The vector $B(t) = (B_1(t), \ldots, B_{N-1}(t))^T$ has the form $B(t) = SW(t)$, where $S$ is the following $(N-1) \times N$-matrix:

$$S = \begin{pmatrix}
\sigma_1 & -\sigma_2 & 0 & \ldots & 0 \\
0 & \sigma_2 & -\sigma_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & -\sigma_N
\end{pmatrix}$$

Therefore, the process $B = (B(t), t \geq 0)$ is an $\mathbb{R}^{N-1}$-dimensional Brownian motion with diffusion matrix $SS^T = \mathcal{A}$. This proves that the gap process $Z$ is a SRBM$^d(R, \mu, \mathcal{A})$ for the given $R, \mu, \mathcal{A}$ and the dimension $d = N - 1$.

Let us show that $R$ is an inverse-positive matrix. On this topic, see the book [7]. Since the off-diagonal entries of $R$ are nonpositive, it suffices to prove that $R$ is inverse-positive. Let $Q = I_{N-1} - R$. Note that $Q$ is a nonnegative irreducible matrix, all its column sums are less than or equal to 1, and the column sum for the first column strictly less than 1. Therefore, its spectral radius is strictly less than 1. The proof is in [56, p.682]; see also [56, Exercise 8.3.7(b)]. Therefore, $R = I_{N-1} - Q$ is inverse-positive, see [7, Chapter 6, Lemma 2.1].

Since, in addition, $r_{ij} \leq 0$ for $i \neq j$, we have: $R$ is an $\mathcal{M}$-matrix. This SRBM satisfies Assumption 3: $R^{-1} \mu < 0$. By Proposition 2.5, Assumption 3 implies Assumption 2. So this SRBM satisfies Assumptions 1 and 2. By Proposition 2.4, the process is positive recurrent and has a unique stationary distribution. By Proposition 3.1, the process is exponentially ergodic. The proof is complete. □

Proof of Theorem 5.3. Immediate corollary of Theorem 4.1. □

11.2. Proof of Theorem 5.1. This is just a particular case of Theorem 5.3: $q_k^+ = 1/2$ for $k = 1, \ldots, N$. Here,

$$R = \begin{pmatrix}
1 & -1/2 & 0 & \ldots & 0 \\
-1/2 & 1 & -1/2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}$$

Recall that

$b_1 = g_1 - \overline{g}, \ b_2 = g_1 + g_2 - 2\overline{g}, \ldots, \ b_{N-1} = g_1 + \ldots + g_{N-1} - (N-1)\overline{g}$.

Let us express $g_1, g_2, \ldots, g_{N-1} - g_N$ in terms of $b_1, \ldots, b_{N-1}$. We have:

$g_1 - \overline{g} = b_1, \ g_2 - \overline{g} = b_2 - b_1, \ g_3 - \overline{g} = b_3 - b_2, \ldots, g_N - \overline{g} = -b_{N-1}$.

Therefore,

$g_1 - g_2 = 2b_1 - b_2, \ g_2 - g_3 = -b_1 + 2b_2 - b_3, \ldots, g_{N-1} - g_N = 2b_{N-1} - b_{N-2}$.

We can write it in vector form: $\mu = (g_1 - g_2, g_2 - g_3, \ldots, g_{N-1} - g_N)^T = 2Rb$, where $b = (b_1, \ldots, b_{N-1})^T$. Therefore, $a = R^{-1} \mu = 2b$. Thus, $b < 0$ if and only if $R^{-1} \mu < 0$. So the statement about positive recurrence and existence and uniqueness of a stationary distribution, which is proved in [60], [44] and [48], follows directly from the general statement of Theorem 5.3.

Now let us find the exponent of ergodicity. The general result from Theorem 4.1 gives us $K_1 - \varepsilon$, where $\varepsilon > 0$ can be made arbitrarily small, and $K_1$ is given by (5). Let us find $K_1$. We have: $R$ is
given by (11.2), while \( \mu = (g_1 - g_2, \ldots, g_{N-1} - g_N)^T \). Let us find the norm of \( R^{-1} \). The eigenvalues of \( R \) are given by (see, e.g. [52])

\[
\lambda_k = 1 - \cos \frac{k\pi}{N}, \quad k = 1, \ldots, N - 1.
\]

The eigenvalues of \( R^{-1} \) are \( \lambda_k^{-1} \), \( k = 1, \ldots, N - 1 \). The matrix \( R^{-1} \) is symmetric, so its norm is equal to the absolute value of its maximal eigenvalue. Therefore,

\[
\|R^{-1}\| = \lambda_1^{-1} = \left(1 - \cos \frac{\pi}{N}\right)^{-1}.
\]

Finally, \( R^{-1}\mu = 2b \), and

\[
\min_{1 \leq k \leq N-1} (R^{-1}\mu)^2_k = 4 \min_{1 \leq k \leq N-1} b_k^2.
\]

Finally, let us find \( \min_{1 \leq k \leq N-1} (R^{-1})_{kk} \). Let us compute the diagonal elements of \( R^{-1} \). By Corollary 4.2 from [15], we have:

\[
(R^{-1})_{ii} = \frac{1}{-1/2} \frac{U_{i-1}(-1)U_{N-1-i}(-1)}{U_{N-1}(-1)},
\]

where \( U_k(x) \) is the \( k \)th Chebyshev polynomial. See the definitions in [15]. Its value at \(-1\) is given by \((-1)^k(k + 1)\). Therefore,

\[
(R^{-1})_{ii} = -2\frac{(-1)^{N-2}(i)(N - i)}{(-1)^n(N - 1N)} = 2\frac{i(N - i)}{N}.
\]

The minimal element among those on the main diagonal corresponds to \( i = 1 \), and is equal to \( 2(N - 1)/N \). To summarize, we are in conditions of Theorem 5.3, where

\[
d = N - 1, \quad a_s = 2 \min_{1 \leq k \leq N-1} b_k, \quad \gamma = \frac{2(N - 1)}{N}, \quad \|R^{-1}\| = \left(1 - \cos \frac{\pi}{N}\right)^{-1}.
\]

Plugging in all these results into (5), we get the desired formula. \( \square \)

12. Refinement of the Mode of Convergence. Actually, we can strengthen the statement about exponential ergodicity a bit. Let us introduce another distance between two probability measures.

Consider a measurable space \( \mathcal{X} \), a signed measure \( \nu \) on it, and a measurable function \( g : \mathcal{X} \to \mathbb{R}_+ \). The \( g \)-norm of the measure \( \nu \) is defined as

\[
\|\nu\|_g := \sup_{|f| \leq g} \left| \int \mathcal{X} f d\nu \right|,
\]

where the sup is taken over all measurable functions \( f : X \to \mathbb{R} \) such that \( |f(x)| \leq g(x) \) for every \( x \in X \). For \( g = 1 \), this is the total variation norm. Let \( X = (X_t)_{t \geq 0} \) be a Markov process with transition kernel \( P^t \), as in Section 3. We shall introduce the following definition, see e.g. [57].

**Definition 12.** Assume we have a function \( V : S \to [1, \infty) \) and constants \( C_0, \kappa > 0 \) such that for every \( t \geq 0 \) and \( x \in S \) we have:

\[
\|P^t(x, \cdot) - \pi(\cdot)\|_V \leq C_0 V(x) e^{-\kappa t}.
\]

Then we say that SRBM\(_d\)(\( R, \mu, \mathcal{A} \)) is \( V \)-uniformly ergodic with exponent \( \kappa \) and function \( V \).
The following result was proved in the paper [55]. This is a ramification of Proposition 3.3.

**Proposition 12.1.** Consider an $S$-valued Markov process $X = (X_t)_{t \geq 0}$. Assume it is stochastically ordered. Assume also that there exists a Lyapunov function $V$ corresponding to the Lyapunov constant $k$. Finally, let $V$ be nondecreasing. Then we have $V$-uniform ergodicity with $\varkappa = k$ and $C_0 = 2(1 + b/k)$.

This allows us to take test functions which are not necessarily bounded, but instead growing not faster than exponentials $e^{\lambda ||x||}$ for some $\lambda$, because in our cases $V$ grows as an exponential function.

Recall the function $V$ constructed in the proofs of our theorems. When is it nondecreasing? We can always take the smoothing function $\varphi$ to be nondecreasing. Therefore, it suffices to ensure that the function $W(x) = [x^T R^{-1} x]^{1/2}$ from Theorem 4.1 is nondecreasing, and that the function $W(x)$ from Theorem 4.2 is nondecreasing. The former function is nondecreasing if $R$ is inverse-nonnegative, for example, if Assumption 5 holds. The latter function is nondecreasing if the curve $C$ has the following property: the angle $\psi$ must be between $\pi/2$ and $\pi$, which, in turn, is equivalent to $r_{12}, r_{21} \leq 0$. It is easy to check that this is also equivalent for $R$ to be inverse-nonnegative. This condition is automatically fulfilled if Assumption 5 is true. Let us state this in the form of corollary.

**Corollary 12.2.** If Assumptions 3, 5 and 6 hold true, then the SRBM$^d(R, \mu, A)$ is $V$-exponential ergodic with exponent $\varkappa$ given by Theorem 4.1 and with $V$ constructed in the proofs of Theorem 4.4.

**Corollary 12.3.** If $d = 2$ and Assumptions 3 and 5 hold true, then the SRBM$^d(R, \mu, A)$ is $V$-uniformly ergodic with exponent $\varkappa$ given by Theorem 4.5.

13. **Subexponential Tails.** We can also show that the stationary distribution $\pi$ has a subexponential tail, in the sense that its moment generating function is finite for some positive argument.

**Theorem 13.1.** Under Assumptions 1, 3, 4, and 6, for every $\lambda < \Lambda$, where

$$\Lambda := \frac{2 \min_{1 \leq i \leq d} (R^{-1}_i \mu) i \gamma^{3/2}}{d \|R^{-1}\|^{5/2} \|A\|},$$

we have:

$$\int_S e^{\lambda ||x||} \pi(dx) < \infty.$$

**Proof.** Return to the proof of Theorem 4.1. From (17), we have:

$$\mathcal{L} V(x) \leq -k(\lambda)V(x), \text{ for all } x \in S \setminus C(r_\varepsilon).$$

Here,

$$k(\lambda) := \frac{1}{2} d c^2(R) \|A\| \lambda^2 + \left(\beta - \frac{1}{2} \varepsilon \|d\| \|A\|\right) \lambda.$$

We can take any

$$\lambda < \frac{\beta - \frac{1}{2} \varepsilon \|d\| \|A\|}{\frac{1}{2} d c^2(R) \|A\|},$$

and have $k(\lambda) > 0$. Since $\varepsilon > 0$ can be taken arbitrarily small, we have: for any $\lambda < \lambda_0 := (2\beta)/(d c^2(R) \|A\|)$, there exist constants $\varepsilon > 0$, $r_\varepsilon \geq 1$, $k > 0$, such that for $V := e^{\lambda W_0}$ we have:

$$\mathcal{L} V(x) \leq -k V(x) \text{ for } x \in S \setminus C(r_\varepsilon).$$
By Proposition 3.2, we have: for any $\lambda < \lambda_0$, 
\[
\int_S e^{V(x)} \pi(dx) < \infty.
\]
However, $W(x) \geq \gamma \|x\|$, and for $x \in S \setminus C(1)$ we have: $V(x) = e^{\lambda W(x)} \geq e^{\lambda \gamma \|x\|}$, and so for any $\lambda < \Lambda := \gamma \lambda_0 = \frac{2 \beta \gamma}{dC^2(R)\|A\|}$ we have $\int_S e^{\lambda \|x\|} d\pi(x) < \infty$.

It suffices to plug in $\beta, \gamma, C(R)$ and finish the proof. \hfill \Box

**Theorem 13.2.** Under Assumptions 1 and 3, in case of two dimensions, $d = 2$, we have: for every $\lambda < \Lambda$, where
\[
\Lambda := \left(1 - r_{12}^2r_{21}\right) \left(\frac{-R^{-1}\mu_1\|r_1\|}{\|A\|(1 + r_{12}^2)^{1/2}(1 + r_{21}^2)^{1/2}(1 + (r_{12})^2) \vee (1 + (r_{21})^2)}\right) \wedge \left(\frac{-R^{-1}\mu_2\|r_2\|}{\|r_1\|\|r_2\|\|A\|k^2}\right),
\]
we have: $\int_S e^{\lambda \|x\|} \pi(dx) < \infty$.

**Proof.** Follows the same lines as the previous proof, except that
\[
W(x) \geq c \|x\|, \quad c := \left(\max_{x \in C} \|x\|\right)^{-1}.
\]

Therefore, we can take
\[
\Lambda := \left(\min_{x \in C} \|x\|\right)^2 \frac{c_0}{2\|A\|} \left(\max_{x \in C} \|x\|\right)^{-1} = \left(1 - r_{12}^2r_{21}\right) \left(\frac{-R^{-1}\mu_1\|r_1\|}{\|A\|(1 + r_{12}^2)^{1/2}(1 + r_{21}^2)^{1/2}(1 + (r_{12})^2) \vee (1 + (r_{21})^2)}\right) \wedge \left(\frac{-R^{-1}\mu_2\|r_2\|}{\|r_1\|\|r_2\|\|A\|k^2}\right),
\]
and then use Lemma 10.1. \hfill \Box

It is instructive to compare the tail estimates with explicit formulas for the stationary distribution under the skew-symmetry condition. For example, consider the competing Brownian particle system with unit diffusions and drifts $g_1, \ldots, g_N$ such that $b_1, \ldots, b_{N-1} < 0$. The stationary distribution is known in this case: it is the product of exponentials with rates $2b_i$, $i = 1, \ldots, N - 1$. Therefore,
\[
\int_S e^{\lambda \|x\|} d\pi(x) < \infty, \quad \text{for} \ \lambda < 2 \min_{1 \leq i \leq N-1} |b_i|.
\]

What does the estimate from Theorem 13.1 give us? Here, $R$ is given by (11.2), and $A = 2R$. So $\|R^{-1}\| = (1 - \cos(\pi/N))^{-1}$, and $\|A\| = 2(1 + \cos(\pi/N))$. Also, $a_* = 2 \min_{1 \leq k \leq N-1} |b_k|$, since $a = R^{-1}\mu = 2b$. Recall that $\gamma = 2(N - 1)/N$. Thus, we have:
\[
\Lambda := \frac{2 \min_{1 \leq i \leq N-1} |b_i| \left(\frac{2(N-1)}{N}\right)^{3/2}}{(N-1)(1 - \cos(\pi/N))^{-5/2}(1 + \cos(\pi/N))} = \frac{c_N}{\min_{1 \leq i \leq N-1} |b_i|},
\]

where
\[
c_N \sim \frac{\pi^5}{4} N^{-6}.
\]

We see that this estimate is very inexact.

14. Appendix.
14.1. Technical lemmas.

**Lemma 14.1.** For \( a \in \mathbb{R}^d \), we have: \( \| a a^T \| = \| a \|^2 \).

**Proof.** If \( a = 0 \), there is nothing to prove. Let \( a \neq 0 \). Take \( x \in \mathbb{R}^d \), \( x \neq 0 \); then

\[
\frac{\| a a^T x \|}{\| x \|} = |a \cdot x| \frac{\| a \|}{\| x \|} \leq \| a \| \cdot \| x \| \frac{\| a \|}{\| x \|} = \| a \|^2.
\]

Taking the supremum over \( x \in \mathbb{R}^d \setminus \{0\} \), we get: \( \| a a^T \| \leq \| a \|^2 \). This estimate is exact: take \( x = a \). Then

\[
\frac{\| a a^T x \|}{\| x \|} = \frac{\| a a^T a \|}{\| a \|} = |a \cdot a| \frac{\| a \|}{\| a \|} = \| a \|^2.
\]

\( \square \)

**Lemma 14.2.** For \( d \times d \)-matrices \( B_1 = (b'_{ij})_{1 \leq i,j \leq d} \), \( B_2 = (b''_{ij})_{1 \leq i,j \leq d} \), we have:

\[
\left| \sum_{i=1}^{d} \sum_{j=1}^{d} b'_{ij} b''_{ij} \right| \leq d \| B_1 \| \| B_2 \|.
\]

**Proof.** By Cauchy-Schwartz inequality, we have:

\[
\left| \sum_{i=1}^{d} \sum_{j=1}^{d} b'_{ij} b''_{ij} \right| \leq \left( \sum_{i=1}^{d} \sum_{j=1}^{d} (b'_{ij})^2 \right)^{1/2} \left( \sum_{i=1}^{d} \sum_{j=1}^{d} (b''_{ij})^2 \right)^{1/2}.
\]

Now, it suffices to show that for every \( d \times d \)-matrix \( B = (b_{ij})_{1 \leq i,j \leq d} \), we have:

\[
\left( \sum_{i=1}^{d} \sum_{j=1}^{d} b^2_{ij} \right)^{1/2} \leq \sqrt{d} \| B \|.
\]

Rewrite the left-hand side of the last inequality as \( \text{tr}(BB^T) \). Note that \( \| B \|^2 \) is the maximal eigenvalue of the matrix \( BB^T \). There are \( d \) eigenvalues (not all of them distinct), and their sum is \( \text{tr}(BB^T) \). Therefore, \( d \| B \|^2 \geq \text{tr}(BB^T) \). \( \square \)

14.2. Ergodic theory of Markov processes. We must introduce some definitions from the ergodic theory of continuous-time Markov processes. On this, see the articles \([59], [58], [25], [24]\) and the book \([57]\), which extensively covers ergodic theory for a similar setting of discrete-time Markov processes.

**Definition 13.** Assume \( q \) is a probability measure on \( \mathbb{R}_+ \). Let

\[
K_q(x, A) := \int_0^\infty P^t(x, A)q(dt), \quad \text{and} \quad K_qf(x) := \int_0^\infty P^t f(x)q(dt)
\]

for \( x \in S \), for Borel measurable sets \( A \subseteq S \) and Borel measurable functions \( f : S \to \mathbb{R} \) which are positive or bounded. Then \( K_q \) is called a weighted kernel (with weight \( q \)).
DEFINITION 14. A nonempty Borel set $C \subseteq S$ is called petite if there exist a distribution $q$ on $\mathbb{R}_+$ and a nontrivial measure $\nu_q$ on $S$ such that for $x \in C$ and Borel sets $A \subseteq S$ we have: $K_q(x, A) \geq \nu_q(A)$. If this distribution is concentrated at a single point, then this set is called small.

DEFINITION 15. The process $Z$ is called $\psi$-irreducible for some $\sigma$-finite Borel measure $\psi$ on $S$ if

$$\psi(A) > 0 \Rightarrow \mathbb{E}_x \int_0^\infty 1_{\{Z_t \in A\}} dt > 0.$$  

If it is $\psi$-irreducible, it is called aperiodic if for some small set $C$ there exists $T$ such that for all $t \geq T$ and $x \in C$ we have: $P^t(x, C) > 0$.

**Lemma 14.3.** For $Z = \text{SRBM}^d(R, \mu, A)$, we have:

(i) every compact subset of $S$ is petite;

(ii) $Z$ is mes-irreducible and aperiodic.

**Proof.** First, let us show that for every $x \in \text{int} S$ and every Borel subset $C \subseteq \text{int} S$ with $\text{mes}(C) > 0$ we have: $P^t(x, C) > 0$ for every $t > 0$. Recall that $B = (B_t)_{t \geq 0}$ is the $d$-dimensional Brownian motion with drift $\mu$ and diffusion matrix $A$ which served as a basis for constructing $Z = \text{SRBM}^d(R, \mu, A)$ in Definition 2. Note that $Z_t = B_t$ until $\tau_{\partial S}$, the hitting moment of the boundary of the orthant. We have:

$$\mathbb{P}_x \{ \forall s \in [0, t] \ Z_s \in \text{int} S, \ Z_t \in C \} = \mathbb{P}_x \{ \forall s \in [0, t] \ B_s \in \text{int} S, \ B_t \in C \} > 0.$$  

This is also true if $C \subseteq S$, i.e. if $C$ does not necessarily lie in the interior, and may intersect the boundary of $S$. Indeed, since $\text{mes}(\partial S) = 0$, then for $C' := C \setminus \partial S$ we have: $\text{mes}(C') > 0$, and $P^t(x, C) \geq P^t(x, C') > 0$. Also, it is true for $x \in \partial S$. Thus, for every $x \in S$, every $t > 0$ and every Borel $C \subseteq S$ with $\text{mes}(C) > 0$ we have: $P^t(x, C) > 0$.

Let us now show that every compact set $C \subseteq \text{int} S$ is small. Fix any $t > 0$. Take the following Borel measure on int $S$:

$$\nu_t(x, A) := \mathbb{P}_x \{ \forall s \in [0, t] \ B_s \in \text{int} S, \ B_t \in A \} = \mathbb{P}_x \{ \forall s \in [0, t] \ Z_s \in \text{int} S, \ Z_t \in A \}.$$  

It has density with respect to the Lebesgue measure mes on int $S$. Denote this density by $p_t(x, y)$. It is strictly positive for $x \in \text{int} S$, and continuous in $x$. Therefore, $q_t(y) := \inf_{x \in C} p_t(x, y) > 0$, and let $\nu_t := q_t(y) dy$. This proves that $C$ is small. Therefore, it is petite.

Let us now show mes-irreducibility: if $\text{mes}(C) > 0$, then

$$\mathbb{E}_x \int_0^\infty 1_{\{Z_t \in C\}} dt = \int_0^\infty P^t(x, C) dt > 0.$$  

Let us show aperiodicity: take any compact set $C \subseteq \text{int} S$ with $\text{mes}(C) > 0$ (which is small) and get: $P^t(x, C) > 0$ for $t > 0$ and $x \in \text{int} S$, i.e. all $x \in C$.

Since this process is mes-irreducible, by [57, Proposition 6.2.8] we have: all compact subsets of $S$ are petite. \hfill $\Box$

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