Turán number and decomposition number of intersecting odd cycles

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Abstract

An extremal graph for a given graph $H$ is a graph on $n$ vertices with maximum number of edges that does not contain $H$ as a subgraph. Let $s, t$ be integers and let $H_{s,t}$ be a graph consisting of $s$ triangles and $t$ cycles of odd lengths at least 5 which intersect in exactly one common vertex. Erdős et al. (1995) determined the extremal graphs for $H_{s,0}$. Recently, Hou et al. (2016) determined the extremal graphs for $H_{0,t}$, where the $t$ cycles have the same odd length $q$ with $q \geq 5$. In this paper, we further determine the extremal graphs for $H_{s,t}$ with $s \geq 0$ and $t \geq 1$. Let $\phi(n, H)$ be the largest integer such that, for all graphs $G$ on $n$ vertices, the edge set $E(G)$ can be partitioned into at most $\phi(n, H)$ parts, of which every part either is a single edge or forms a graph isomorphic to $H$. Pikhurko and Sousa conjectured that $\phi(n, H) = \text{ex}(n, H)$ for $\chi(H) \geq 3$ and all sufficiently large $n$. Liu and Sousa (2015) verified the conjecture for $H_{s,0}$. In this paper, we further verify Pikhurko and Sousa’s conjecture for $H_{s,t}$ with $s \geq 0$ and $t \geq 1$.

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1 Introduction

In this paper, all graphs considered are simple and finite. For a graph $G$ and a vertex $x \in V(G)$, the neighborhood of $x$ in $G$ is denoted by $N_G(x)$. The degree of $x$, denoted by $\deg_G(x)$, is $|N_G(x)|$. Let $\delta(G), \Delta(G)$ and $\chi(G)$ denote the minimum degree, maximum degree and chromatic number of $G$, respectively. Let $e(G)$ be the number of edges of $G$. For a graph $G$ and $S, T \subset V(G)$, let $e_G(S, T)$ be the number of edges $e = xy \in E(G)$ with $x \in S$ and $y \in T$, if $S = T$, we use $e_G(S)$ instead of $e_G(S, S)$, and $e_G(u, T)$ instead of $e_G(\{u\}, T)$ for convenience, the index $G$ will be omitted if no confusion from the context. For a subset $X \subseteq V(G)$ or $X \subseteq E(G)$, let $G[X]$ be the subgraph of $G$ induced by $X$, that is $G[X] = (X, E(X))$ if $X \subseteq V(G)$, or $G[X] = (V(X), X)$ if $X \subseteq E(G)$. A matching $M$ in $G$ is a subset of $E(G)$ with $\delta(G[M]) = \Delta([M]) = 1$. The matching number of $G$, denoted by $\nu(G)$, is the maximum number of edges in a matching in $G$. A maximum cut of $G$ is a bipartition of $V(G) = V_0 \cup V_1$ such that $e_G(V_0, V_1)$ is maximized. For $x \in V(G)$ and $A \subset V(G)$, let $E_A(x) = \{e \in E(G[A]) \mid V(e) \cap N_G(x) \neq \emptyset\}$. A cycle of length $q$ is called a $q$-cycle. Given a partition of $V(G) = V_0 \cup V_1$ and $x \in V_i \ (i = 0, 1)$, $\deg_{G[V_i]}(x)$ is called the in-degree of $x$, similarly, we call $e_G(x, V_{1-i})$ the out-degree of $x$.

Given two graphs $G$ and $H$, we say that $G$ is $H$-free if $G$ does not contain an $H$ as a subgraph. The Turán number, denoted by $\text{ex}(n, H)$, is the largest number of edges of an $H$-free graph on $n$ vertices. That is, 

$$\text{ex}(n, H) = \max\{e(G) : |V(G)| = n, \ G \text{ is } H\text{-free}\}.$$ 

For positive integers $n$ and $r$ with $n \geq r$, the Turán graph, denoted by $T_{n,r}$, is the balanced complete $r$-partite graph on $n$ vertices, where each part has size $\left\lceil \frac{n}{r} \right\rceil$ or $\left\lfloor \frac{n}{r} \right\rfloor$.

Let $s, t$ be integers and let $H_{s,t}$ be a graph consisting of $s$ triangles and $t$ cycles of odd lengths at least 5 which intersect in exactly one common vertex, called the center of $H_{s,t}$. In 1995, Erdős et al. [6] determined the value of $\text{ex}(n, H_{k,0})$ and the extremal graphs for $H_{k,0}$.

**Theorem 1.1** ([6]). For $k \geq 1$ and $n \geq 50k^2$,

$$\text{ex}(n, H_{k,0}) = e(T_{n,2}) + g(k),$$

where

$$g(k) = \begin{cases} 
  k^2 - k & \text{if } k \text{ is odd}, \\
  k^2 - \frac{3}{2}k & \text{if } k \text{ is even}.
\end{cases}$$
Moreover, when $k$ is odd, the extremal graph must be a $T_{n,2}$ with two vertex disjoint copies of $K_k$ embedding in one partite set. When $k$ is even, the extremal graph must be a $T_{n,2}$ with a graph having $2k - 1$ vertices, $k^2 - \frac{3}{2}k$ edges with maximum degree $k - 1$ embedded in one partite set.

In 2003, Chen et al. [4] generalized Erdős et al.’s result to $ex(n, F_{k,r}) = e(T_{n,r} - 1) + g(k)$, where $F_{k,r}$ is a graph consisting of $k$ complete graphs of order $r (\geq 3)$ which intersect in exactly one common vertex and $g(k)$ is the same as in Theorem 1.1. The above result were further generalized by Gelbov (2011) and Liu (2013). They determined the extremal graphs for blow-ups of paths [8], cycles and a large class of trees [11]. Recently, Hou et al. (2016) generalized Erdős et al.’s result in another way, they determined the extremal graphs for a special family of $H_{0,k}$, where the $k$ odd cycles have the same length $q$ with $q \geq 5$ (denoted by $C_{k,q}$ in [10]).

**Theorem 1.2** ([10]). For an integer $k \geq 2$ and an odd integer $q \geq 5$, there exists $n_0(k,q) \in \mathbb{N}$ such that for all $n \geq n_0(k,q)$, we have

$$ex(n, C_{k,q}) = e(T_{n,2}) + (k - 1)^2,$$

and the only extremal graph is a $T_{n,2}$ with a $K_{k-1,k-1}$ embedded in one partite set.

As we have seen from Theorems 1.1 and 1.2 that the extremal graphs for $H_{k,0}$ and $H_{0,k}$ are different. A natural and interesting problem is to determine the extremal graphs for mixed graph $H_{s,t}$. In this paper, our first main result solves the problem.

Let $C_{s,t}$ be the family of all graphs $H_{s,t}$, and let $F_{n,s,t}$ be the family of graphs with each member is a Turán graph $T_{n,2}$ with a graph $H$ embedded in one partite set, where

$$H = \begin{cases} K_{s+t-1,s+t-1} & \text{if } (s,t) \neq (3,1), \\ K_{3,3} \text{ or } 3K_3 & \text{if } (s,t) = (3,1), \end{cases}$$

where $3K_3$ is the union of three disjoint triangles.

**Theorem 1.3.** For any integers $s \geq 0, t \geq 1$ and for any $H_{s,t} \in C_{s,t}$, there exists $n_1(H_{s,t}) \in \mathbb{N}$ such that for all $n \geq n_1(H_{s,t})$,

$$ex(n, H_{s,t}) = e(T_{n,2}) + (s + t - 1)^2,$$

and the only extremal graphs for $H_{s,t}$ are members of $F_{n,s,t}$.
A parameter related to Turán number is the so-called decomposition number. Given two graphs $G$ and $H$, an $H$-decomposition of $G$ is a partition of edges of $G$ such that every part is a single edge or forms a graph isomorphic to $H$. Let $\phi(G,H)$ be the smallest number of parts in an $H$-decomposition of $G$. Clearly, if $H$ is non-empty, then

$$\phi(G,H) = e(G) - p_H(G)(e(H) - 1),$$

where $p_H(G)$ is the maximum number of edge-disjoint copies of $H$ in $G$. Define

$$\phi(n,H) = \max\{\phi(G,H) : |V(G)| = n\}.$$

This function, motivated by the problem of representing graphs by set intersections, was first studied by Erdős, Goodman and Pósa [7], they proved that $\phi(n,K_3) = \text{ex}(n,K_3)$. The result was generalized to $\phi(n,K_r) = \text{ex}(n,K_r)$, for all $n \geq r \geq 3$ by Bollobás [2]. More generally, Pikhurko and Sousa [14] proposed the following conjecture.

**Conjecture 1.4** ([14]). For any graph $H$ with $\chi(H) \geq 3$, there is an integer $n_0 = n_0(H)$ such that $\phi(n,H) = \text{ex}(n,H)$ for all $n \geq n_0$.

In [14], Pikhurko and Sousa also proved that $\phi(n,H) = \text{ex}(n,H) + o(n^2)$. The error term was improved to be $O(n^{2-\alpha})$ for some $\alpha > 0$ by Allen, Böttcher, and Person [11]. Sousa verified the conjecture for some families of edge-critical graphs, namely, clique-extensions of order $r \geq 4 \ (n \geq r)$ [18] and the cycles of length 5 ( $n \geq 6$) [16] and 7 ( $n \geq 10$) [17]. In [13], Özkahya and Person verified the conjecture for all edge-critical graphs with chromatic number $r \geq 3$. Here, a graph $H$ is called edge-critical, if there is an edge $e \in E(H)$, such that $\chi(H) > \chi(H-e)$. For non-edge-critical graphs, Liu and Sousa [12] verified the conjecture for $H_{k,0}$ and recently, the result was generalized to $F_{k,r}$ for all $k \geq 2$ and $r \geq 3$ by Hou et al. [9].

Our second main result verifies that Pikhurko and Sousa’s conjecture is true for $H_{s,t}$ with $s \geq 0$ and $t \geq 1$.

**Theorem 1.5.** For any integer $s \geq 0, t \geq 1$ and for any $H_{s,t} \in C_{s,t}$, there exists $n_2(H_{s,t}) \in \mathbb{N}$ such that for all $n \geq n_2(H_{s,t})$,

$$\phi(n,H_{s,t}) = \text{ex}(n,H_{s,t}).$$

Moreover the only graphs attaining $\text{ex}(n,H_{s,t})$ are members of $F_{n,s,t}$.

The remaining of the paper is arranged as follows. Section 2 gives all the technical lemmas we need. Sections 3 and 4 give the proofs of Theorems 1.3 and 1.5 respectively.
2 Lemmas

The following two lemmas due to Chavátal and Hanson [3] and Erdös et al. [6] are used to evaluate the maximum number of edges of a graph with given maximum degree and matching number.

Lemma 2.1 ([3]). For any graph $G$ with maximum degree $\Delta \geq 1$ and matching number $\nu \geq 1$, we have $e(G) \leq f(\nu, \Delta) = \nu \Delta + \left\lfloor \frac{\nu}{2} \right\rfloor \left\lfloor \frac{\Delta}{2} \right\rfloor \leq \nu(\Delta + 1)$.

Lemma 2.2 ([6]). Let $H$ be a graph with maximum degree $\Delta$ and matching number $\nu$ and let $b$ be a nonnegative integer such that $b \leq \Delta(H) - 2$. Then
\[
\sum_{x \in V(H)} \min\{\deg_H(x), b\} \leq \nu(\Delta + b).
\]

The following two stability lemmas due to Erdős [4], Simonovits [15], Özkahya and Person [13] play an important role to determine the Turán number and decomposition number of a given graph $H$.

Lemma 2.3 ([5 15]). Let $H$ be a graph with $\chi(H) = r \geq 3$ and $H \neq K_r$. Then, for every $\gamma > 0$, there exists $\delta > 0$ and $n_0 = n_0(H, \gamma) \in \mathbb{N}$ such that the following holds. If $G$ is an $H$-free graph on $n \geq n_0$ vertices with $e(G) \geq \text{ex}(n, H) - \delta n^2$, then there exists a partition of $V(G) = V_1 \cup \cdots \cup V_{r-1}$ such that $\sum_{i=1}^{r-1} e(V_i) < \gamma n^2$.

Lemma 2.4 ([13]). Let $H$ be a graph with $\chi(H) = r \geq 3$ and $H \neq K_r$. Then, for every $\gamma > 0$, there exists $\delta > 0$ and $n_0 = n_0(H, \gamma) \in \mathbb{N}$ such that the following holds. If $G$ is a graph on $n \geq n_0$ vertices with $\phi(G, H) \geq \text{ex}(n, H) - \delta n^2$, then there exists a partition of $V(G) = V_1 \cup \cdots \cup V_{r-1}$ such that $\sum_{i=1}^{r-1} e(V_i) < \gamma n^2$.

The following lemma can be found in [10].

Lemma 2.5 (Lemma 8 in [10]). Let $n_0$ be an integer and let $G$ be a graph on $n \geq n_0 + \binom{n_0}{2}$ vertices with $e(G) = e(T_{n,2}) + j$ for some integer $j \geq 0$. Then $G$ contains a subgraph $G'$ on $n' > n_0$ vertices such that $\delta(G') \geq \delta(T_{n',2})$ and $e(G') \geq e(T_{n',2}) + j + n - n'$.

The proof of the following lemma is almost the same as the proof of Lemma 6 in [9], we give the proof here for completeness.
Lemma 2.6. Let \( n_0 \) be an integer and \( H \) be a given graph with \( \chi(H) = r \geq 3 \) and \( \text{ex}(n, H) - \text{ex}(n-1, H) \geq \delta(T_{r-1,n}) \) for all \( n \geq n_0 \). Let \( G \) be a graph on \( n > n_0 + \binom{n_0}{2} \) vertices with \( \phi(G, H) = \text{ex}(n, H) + j \) for some integer \( j \geq 0 \). Then \( G \) contains a subgraph \( G' \) on \( n' > n_0 \) vertices such that \( \delta(G') \geq \delta(T_{n',r-1}) \) and \( \phi(G', H) \geq e(T_{n',r-1}) + j + n - n' \).

Proof. If \( \delta(G) \geq \delta(T_{n,r-1}) \), then \( G \) is the desired graph and we have nothing to do. So assume that \( \delta(G) < \delta(T_{n,r-1}) \). Let \( v \in V(G) \) with \( \deg_G(v) < \delta(T_{n,r-1}) \) and set \( G_1 = G - v \). Then \( \phi(G_1, H) \geq \phi(G, H) - \deg_G(v) \geq \text{ex}(n, H) + j - \delta(T_{n,r-1}) + 1 \geq \text{ex}(n-1, H) + j + 1 \), since \( \text{ex}(n, H) - \text{ex}(n-1, H) \geq \delta(T_{n,r-1}) \). We may continue this procedure until we get a graph \( G' \) on \( n - i \) vertices with \( \delta(G') \geq \left[ \frac{n^2-2}{n-1}(n-i) \right] \) for some \( i < n - n_0 \), or until \( i = n - n_0 \). But the latter case can not occur since \( G' \) is a graph on \( n_0 \) vertices with \( e(G') \geq \phi(G', H) \geq \text{ex}(n_0, H) + j + i \geq n - n_0 > \binom{n_0}{2} \), which is impossible. \( \square \)

The following observation was given in [10].

Observation 2.7 (Observation 5 in [10]). Let \( G \) be a graph with no isolated vertex. If \( \Delta(G) \leq 2 \), then
\[
\nu(G) \geq \frac{|V(G)| - \omega(G)}{2},
\]
where \( \omega(G) \) is the number of components of \( G \).

The following is a technical lemma to determine the extremal graphs for intersecting odd cycles.

Lemma 2.8. Let \( s \geq 0 \) and \( t \geq 1 \) be two integers and \( k = t + s \). Let \( G \) be a graph with no isolated vertex and \( \nu(G) \leq k - 1 \). If for all \( x \in V(G) \) with \( \deg(x) \geq s \), we have \( \deg(x) + \nu(G - N(x)) \leq k - 1 \), then \( e(G) \leq (k-1)^2 \). Moreover, equality holds if and only if \( G = K_{k-1,k-1} \) or \( G = 3K_3 \), the latter case happens only if \( s = 3 \) and \( t = 1 \).

Proof. Note that the conditions of the lemma imply that \( \Delta(G) \leq k - 1 \) and \( k \geq 2 \).

Case 1. \( \Delta(G) \leq k - 2 \).

Then \( k \geq 3 \) in this case. By Lemma 2.1 we have
\[
e(G) \leq f(k-1,k-2) = (k-1)(k-2) + \left[ \frac{k-2}{2} \right] \left[ \frac{k-1}{(k-2)/2} \right] \leq (k-1)^2,
\]
and the equality holds only if \( \nu = k - 1 \), \( \Delta = k - 2 \) and \( k = 4 \). Now assume that \( e(G) = f(3,2) = 3^2 = 9 \). Since \( \Delta(G) \leq 2 \)
and $G$ has no isolated vertex, $|V(G)| \geq e(G) = 9$ (the equality holds if and only if $G$ is 2-regular) and $\omega(G) \leq \nu(G) = 3$. By Observation 2.7,

$$3 = \nu(G) \geq \frac{|V(G)| - \omega(G)}{2} \geq \frac{|V(G)| - 3}{2}.$$ 

Hence $|V(G)| \leq 9$. Thus $|V(G)| = 9$ (and so $G$ is 2-regular) and $\omega(G) = 3$. Therefore, $G = 3K_3$. Then, for any $x \in V(G)$, $\deg(x) + \nu(G - N(x)) = 2 + 2 > 3 = k - 1$. So, by the condition of the lemma, $2 = \deg(x) < s = 4 - t \leq 3$ (since $t \geq 1$). Therefore, we must have $s = 3$ and $t = 1$.

**Case 2.** $\Delta(G) = k - 1$.

Choose $x \in V(G)$ such that $\deg(x) = k - 1$. Then $\nu(G - N(x)) = 0$. Hence $e(G - N(x)) = 0$, that is $V(G) \setminus N(x)$ is an independent set of $G$. Let $N(x) = \{x_1, \ldots, x_{k-1}\}$. For each $i \in [1, k-1]$, denote $d_i = \deg(x_i)$ and $\tilde{d}_i = \deg_{G \setminus N(x)}(x_i)$. Then

$$e(G) = e(G[N(x)]) + e(N(x), V(G) \setminus N(x)) = \frac{1}{2} \sum_{i=1}^{k-1} \tilde{d}_i + \sum_{i=1}^{k-1} (d_i - \tilde{d}_i)$$

$$= \sum_{i=1}^{k-1} d_i - \frac{1}{2} \sum_{i=1}^{k-1} \tilde{d}_i \leq (k - 1)^2 - \frac{1}{2} \sum_{i=1}^{k-1} \tilde{d}_i \leq (k - 1)^2,$$

and the equality holds if and only if $d_i = k - 1$ and $\tilde{d}_i = 0$ for each $i \in [1, k-1]$, that is $G$ is a bipartite graph with partite sets $N(x) = \{x_1, \ldots, x_{k-1}\}$ and $V(G) \setminus N(x)$. To show that $G = K_{k-1,k-1}$, it suffices to prove that $|V(G) \setminus N(x)| = k - 1$. If $|V(G) \setminus N(x)| > k - 1$, then there must exist a vertex $y \in (V(G) \setminus N(x)) \setminus N(x_1)$ since $\deg(x_1) = d_1 = k - 1$. Since $G$ has no isolated vertex, $y$ must be adjacent to some vertex $x_j$ with $j \neq 1$. This implies that $\nu(G - N(x_1)) \geq 1$. Hence we have $\deg(x_1) + \nu(G - N(x_1)) \geq k$, but $\deg(x_1) = k - 1 \geq s$, a contradiction to $\deg(x_1) + \nu(G - N(x_1)) \leq k - 1$.

The following lemma states that the members of $\mathcal{F}_{n,s,t}$ are actually $H_{s,t}$-free.

**Lemma 2.9.** Each member of $\mathcal{F}_{n,s,t}$ is $H_{s,t}$-free for any $H_{s,t} \in \mathcal{C}_{s,t}$.

**Proof.** Suppose to the contrary that there is a graph $G \in \mathcal{F}_{n,s,t}$ containing a copy of $H_{s,t}$. Let $k = s + t$ and let $K$ be the copy of $K_{k-1,k-1}$ (or $3K_3$ when $(s,t) = (3,1)$) embedded in one partite set of $G$. Then each odd cycle of $H_{s,t}$ must contain odd number of the edges of $K$. Let $A = E(H_{s,t}) \cap E(K)$. Then $|A| \geq k = s + t$. We
claim that the center of $H_{s,t}$ must lie in $K$. If not, then $G[A]$ contains a matching of order at least $k$ by the structure of $H_{s,t}$, a contradiction to $\nu(K) = k - 1$. Let $x \in V(K)$ be the center of $H_{s,t}$. Assume that $\deg_{G[A]}(x) = r$. Let $A_x$ be the set of edges incident with $x$ in $G[A]$. Then at most $r$ cycles of $H_{s,t}$ intersect $A_x$, that is $G[A] - A_x$ contains a matching of $K$ of order at least $k - r$. This is impossible since $\nu(K - N_{G[A]}(x)) \leq k - r - 1$. \hfill $\square$

In the remaining of the paper, for convenience, we set $\gamma = \lfloor 400(c(H_{s,t}) + 1)k \rfloor^{-2}$ and $\beta = (c(H_{s,t}) + 1)\sqrt{\gamma}$, where $c(H_{s,t})$ is the circumference of $H_{s,t}$.

**Lemma 2.10.** Let $G$ be a graph on $n$ vertices with $\delta(G) \geq \lfloor \frac{n}{2} \rfloor$ and $e(G) \geq \text{ex}(n, H_{s,t}) > e(T_{n,2})$ if $G$ is $H_{s,t}$-free, or $\phi(G, H_{s,t}) \geq \text{ex}(n, H_{s,t}) > e(T_{n,2})$, otherwise. Let $V_0 \cup V_1$ be a partition of $V(G)$ such that $e(V_0, V_1)$ is maximized and let $m = e(V_0) + e(V_1)$ and $B = \{x \in V(G) \mid \deg_{G[V_i]}(x) > \beta n, \text{ for } x \in V_i \text{ and } i = 0, 1\}$. Then for sufficiently large $n$, the following holds:

1. $m < \gamma n^2$ and $|B| < \frac{2\gamma}{\beta} n$;
2. $\frac{n}{2} - \sqrt{n} \leq |V_i| \leq \frac{n}{2} + \sqrt{n}$ for $i = 0, 1$;
3. $e(u, V_1 - i) \geq \frac{n}{4} - \frac{1}{4}$ for $u \in V_i$ ($i = 0, 1$);
4. Moreover, $e(u, V_1 - i) \geq \frac{n}{2} - \beta n - \frac{1}{2}$ for $u \in V_i \setminus B$ ($i = 0, 1$).

**Proof.** Applying Lemma 2.9 and Lemma 2.4 to $G$, respectively, with parameter $\gamma$, we have $m < \gamma n^2$ and so $|B| \leq \frac{2m}{\beta n} < \frac{2\gamma}{\beta} n$. Let $a = \max\{||V_i| - \frac{n}{2}|, i = 0, 1\}$. Note that

$$\left\lfloor \frac{n^2}{4} \right\rfloor = e(T_{n,2}) < e(G) = m + e(V_0, V_1) < \gamma n^2 + \left|V_0\right|\left|V_1\right| = \gamma n^2 + \frac{n^2}{4} - a^2.$$  

Hence we have $a^2 \leq \gamma n^2$ and so $a \leq Cn. By the choice of $V_0$ and $V_1$, for each $u \in V_i$ ($i = 0, 1$), we have $e(u, V_1 - i) \geq \deg_{G[V_i]}(u)$. Note that $\deg_{G}(u) = \deg_{G[V_i]}(u) + e(u, V_1 - i)$ and $\delta(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$, we have

$$e(u, V_1 - i) \geq \max\{\deg_{G[V_i]}(u), \left\lfloor \frac{n}{2} \right\rfloor - \deg_{G[V_i]}(u)\}.$$  

Hence, $e(u, V_1 - i) \geq \frac{1}{2}(\deg_{G[V_i]}(u) + \left\lfloor \frac{n}{2} \right\rfloor - \deg_{G[V_i]}(u)) \geq \frac{n}{4} - \frac{1}{4}$. Moreover, if $u \in V_i \setminus B$, then $e(u, V_1 - i) \geq \left\lfloor \frac{n}{2} \right\rfloor - \deg_{G[V_i]}(u) \geq \frac{n}{2} - \beta n - \frac{1}{2}$. \hfill $\square$

The following technical lemma will be used to find copies of $H_{s,t}$ from $G$ in Sections 3 and 4. Let $G$ be a graph and $V(G) = V_0 \cup V_1$ and $x \in V(G)$. We will use $G_i$ and $N_i(x)$ instead of $G[V_i]$ and $N_G(x) \cap V_i$ for $i = 0, 1$ in the remaining of this paper.
Lemma 2.11. Let $s \geq 0, t \geq 1$ and $k = s + t$. Let $G$ be a graph on $n$ vertices satisfying that

(i) $V_0 \cup V_1$ is a partition of $V(G)$ with $\max\{|V_0|, |V_1|\} \leq (\frac{1}{2} + \sqrt{\gamma})n$ and, for each $i = 0, 1$, $V_i$ has a subset $B_i$ with $E(G[B_i]) = \emptyset$ and $|B_0 \cup B_1| < \sqrt{\gamma}n$ and

$$|N_{1-i}(u)| > \begin{cases} 2n/5 & \text{if } u \in V_i \setminus B_i \\ n/9 & \text{if } u \in B_i \end{cases},$$

(ii) $V_i$ has a subset $U_i$ with $|V_i \setminus U_i| < \sqrt{\gamma}n$ for $i = 0, 1$.

If there exist a vertex $x \in B_i$ with $\deg_{G_i}(x) \geq k$, or a vertex $x \in V_i \setminus B_i$ and a matching $M_{1-i}$ of $G[N_{1-i}(x) \setminus B_{1-i}]$ with $\deg_{G_i}(x) + |M_{1-i}| \geq s$ and

$$\deg_{G_i}(x) + |M_{1-i}| + \nu(G[V_i \setminus N_i(x)]) + \nu(E_{V_{1-i} \setminus V(M_{1-i})}(x)) \geq k,$$  

then for sufficiently large $n$, there is a copy of $H_{s,t}$, say $H$, in $G$ centered at $x$, satisfying that

(1) $H$ contains exactly $k$ edges in $E(V_0) \cup E(V_1)$,

(2) $V(H) \cap U_{1-i} = \emptyset$,

(3) if $\deg_{G_i}(x) \geq k$, then $V(H) \cap U_j = \emptyset$ for $j = 0, 1$.

Proof. Without loss of generality, assume that there is such a vertex $x \in V_0$. Let $N_0(x) = \{x_1, \ldots, x_{\ell}\}$ and $M_1 = \{w_1z_1, \ldots, w_mz_m\}$ be a matching of $G[N_1(x) \setminus B_1]$ with $\ell + m \geq s$ and

$$\ell + m + \nu(G[V_0 \setminus N_0(x)]) + \nu(E_{V_1 \setminus V(M_1)}(x)) \geq k.$$

Let $\{u_1v_1, \ldots, u_pv_p\}$ and $\{w'_1z'_1, \ldots, w'_qz'_q\}$ be two matchings of $G[V_0 \setminus N_0(x)]$ and $E_{V_1 \setminus V(M_1)}(x)$ respectively, such that

$$\ell + m + p + q = k,$$

and assume that $\{w'_1, \ldots, w'_q\} \subseteq N_1(x)$. In the case that $x \in B_0$ and $\deg_{G_0}(x) \geq k$, we simply set $\ell = k$ and $m = p = q = 0$. Suppose $H_{s,t}$ consists of $k = s + t$ odd cycles of lengths $q_1, q_2, \ldots, q_k$ respectively, where $3 = q_1 = \cdots = q_s < q_{s+1} \leq \cdots \leq q_k$.

Note that $xw_iz_i$ is a triangle for every edge $w_iz_i \in M_1$, $i = 1, \ldots, m$. Since $M_1$ is a matching, by using $w_1z_1, \ldots, w_{\min\{m,s\}}z_{\min\{m,s\}}$, we can easily find a copy of $H_{\min\{m,s\},0}$. Next we construct cycles of length $q_{\min\{m,s\}+1}, \ldots, q_k$ step by step such that these $k$ odd cycles form an $H_{s,t}$. In another words, we will show that at step $j$
with \( \min\{m, s\} < j \leq k \), we construct a cycle of length \( q_j \) which intersects previous constructed cycles only at \( x \).

**Case 1.** \( j \leq m \).

Since \( m > j > \min\{m, s\} \), we have \( m > s \). It yields that \( j > \min\{m, s\} = s \) and \( q_j \geq 5 \). Avoiding all vertices except \( x \) that have been previously used, we find vertices \( w_0^1, w_0^2, \ldots, w_0^{q_j-4}, w_0^{q_j-3} \) from \( U_0 \setminus B_0 \) and \( U_1 \setminus B_1 \) alternatively with \( w_0^1 \in U_0 \setminus B_0 \) such that \( P = z_j w_0^1 w_0^2 \cdots w_0^{q_j-3} x \) is a path of length \( q_j - 2 \). This is possible since each vertex \( u \in V_i \) has at least $e_G(u, V_{1-i}) - |B_{1-i}| - |V_{1-i} \setminus U_{1-i}| \geq (\frac{1}{9} - 2\sqrt{\gamma})n$ neighbors in \( U_{1-i} \setminus B_{1-i} \), and \( w_0^{q_j-4} \in U_0 \setminus B_0 \) and \( x \) have at least 

$$e_G(w_0^{q_j-4}, V_i) + e_G(x, V_i) - |V_i| - |V_i \setminus U_i| - |B_i| > \frac{2}{5}n + \frac{1}{9}n - \frac{1}{2}n - 3\sqrt{\gamma}n > \frac{n}{400}$$

common neighbors in \( U_1 \setminus B_1 \). Hence \( P \cup \{xw_jz_j\} \) is a desired \( q_j \)-cycle.

**Case 2.** \( m < j \leq \ell + m \).

If \( m < j \leq s \), then \( q_j = 3 \). Note that \( E(G[B_i]) = \emptyset \) for \( i = 0, 1 \). Then at least one of \( \{x, x_{j-m}\} \) is not in \( B_0 \). Hence the number of common neighbors of \( x \) and \( x_{j-m} \) in \( U_1 \setminus B_1 \) is at least

$$e_G(x, V_1) + e_G(x_{j-m}, V_1) - |V_1| - |V_1 \setminus U_1| - |B_1| > \frac{2}{5}n + \frac{1}{9}n - \frac{1}{2}n - 3\sqrt{\gamma}n > \frac{n}{400}$$

So we can find a triangle using \( x, x_{j-m} \) and a common neighbor of them which avoids all vertices that have been previously used.

If \( s < j \leq \ell + m \), then \( q_j \geq 5 \). For the same reason as in Case 1, we can find an alternating path \( P = xw_1^1 w_0^2 \cdots w_1^{q_j-4} w_0^{q_j-3} \) with vertices chosen from \( U_0 \setminus B_0 \) and \( U_1 \setminus B_1 \) alternatively with \( w_1^1 \in U_1 \setminus B_1 \) and a common neighbor of \( x_{j-m} \) and \( w_0^{q_j-3} \) in \( V_1 \setminus B_1 \), say \( w_1^{q_j-2} \), avoiding all vertices except \( x \) that have been previously used. Hence \( P \cup \{w_1^{q_j-2} x_{j-m} x\} \) is a desired \( q_j \)-cycle.

**Case 3.** \( \ell + m < j \leq \ell + m + p \).

Since \( j > \ell + m \geq s \) and \( p > 0 \), we have \( q_j \geq 5 \) and \( x \notin B_0 \). Since \( E(G[B_0]) = \emptyset \), at least one of \( \{u_{j-\ell-m}, v_{j-\ell-m}\} \) is not in \( B_0 \). Assume that \( u_{j-\ell-m} \notin B_0 \). Hence, with the same reason as in Case 1 and Case 2, avoiding all vertices except \( x \) that have been previously used, we first find a common neighbor of \( x \) and \( u_{j-\ell-m} \), say
For the same reason as the above, avoiding all vertices except $x$ is a desired $w$-P$_3$ with vertices chosen from $U$ constructed. Case 4. $U$ vertex in $G = P = \{v: v \notin F\}$, in $\{u_{j-\ell-m}w_1^1x\} \cup P \cup \{w_1^{q_2-2}v_{j-\ell-m}u_{j-\ell-m}\}$ is a desired $q_j$-cycle.

Case 4. $\ell + m + p < j \leq k$.

For the same reason as the above, avoiding all vertices except $x$ that have been previously used, we first find an alternating path $P = z_{j-\ell-m-p}w_1^1w_2^2 \cdots w_0^{q_j-3}$ of length $q_j - 3$ with vertices chosen from $U_0 \setminus B_0$ and $U_1 \setminus B_1$ alternatively, next a common neighbor of $w_0^{q_j-3} (\notin B_0)$ and $v_{j-\ell-m}$, say $w_1^{q_2-2}$ in $U_1 \setminus B_1$. Therefore, $\{u_{j-\ell-m}w_1^1x\} \cup P \cup \{w_1^{q_2-2}v_{j-\ell-m}u_{j-\ell-m}\}$ is a desired $q_j$-cycle.

Thus we always can find a copy of $H_{s,t}$ centered at $x$. In each step, the new constructed $q_j$-cycle uses exactly one edge in $E(V_0) \cup E(V_1)$ and at least one new vertex in $U_1$, so (1) and (2) hold. Moreover, if $\deg_{G_i}(x) \geq k$, then we choose $k$ neighbors of $x$ in $G_i$ and set $\ell = k$ and $p = m = q = 0$. Hence we find a copy of $H_{s,t}$ only using Case 2. In Case 2, if $q_j \geq 5$, then the $q_j$-cycle we found uses at least one vertex in $U_1$ and a vertex in $U_0$. So (3) holds. \hfill \Box

3 Proof of Theorem 1.3

We begin with a technical lemma, which is crucial to the proof of Theorem 1.3 and also will be used in next section.

Lemma 3.1. Given integers $s \geq 0, t \geq 1$ and $H_{s,t} \in \mathcal{C}_{s,t}$. Let $k = s + t$ and let $G$ be a graph on $n$ vertices with $V(G) = V_0 \cup V_1$ and $e(G) \geq e(T_{n,2}) + (k - 1)^2$. If $G$ satisfies (i) $|V_i| - \frac{n}{2} \leq \sqrt{7}n$ and $e(u, V_{1-i}) \geq \frac{n}{2} - c$ for each $u \in V_i$ and $i = 0, 1$, where $c$ is a constant, and (ii) for any vertex $x \in V_i$, $(i = 0, 1)$ and any maximum matching $M_{1-i}$ of $G[N_{1-i}(x)]$ with $\deg_{G_i}(x) + |M_{1-i}| \geq s$, we have

$$\deg_{G_i}(x) + |M_{1-i}| + \nu(G[V_i \setminus N_i(x)]) + \nu(E_{V_{1-i} \setminus V(M_{1-i})}(x)) \leq k - 1,$$

then for all sufficiently large $n$, $e(G) = e(T_{n,2}) + (k - 1)^2$. Moreover, if $G$ is $\mathcal{H}_{s,t}$-free, then $G \in \mathcal{F}_{n,s,t}$.
Proof. Let \( m = e(V_0) + e(V_1) \). Since \( m + e(V_0, V_1) = e(G) \geq e(T_{n,2}) + (k-1)^2 \), we have \( m \geq (k-1)^2 \), with equality holds only if \( G \) contains a balanced complete bipartite subgraph with partitions \( V_0 \) and \( V_1 \). Condition (ii) implies that

**Claim 1.** \( \max \{ \Delta_0, \Delta_1 \} \leq k - 1 \).

Condition (ii) also implies that \( \nu_i \leq k - 1 \) for \( i = 0, 1 \). Furthermore, we have

**Claim 2.** \( \nu_0 + \nu_1 \leq k - 1 \).

We prove it by contradiction. Suppose that \( \nu_0 + \nu_1 \geq k \). Let \( F_0 \) and \( F_1 \) be two maximum matchings of \( G_0 \) and \( G_1 \), respectively. Then \( |F_0| + |F_1| \geq k \). Let \( A_i = \cap_{v \in V(F_{1-i})} N_i(v) \). We first show that

**Claim 2.1** \( A_i \neq \emptyset \) and \( 2 \leq \nu_i \leq s - 1 \leq k - 2 \) for every \( i = 0, 1 \).

For each \( i = 0, 1 \), since \( ||V_i| - \frac{n}{2}| \leq \sqrt{\gamma}n \) and \( e(u, V_i) > \frac{n}{2} - c \) for all \( u \in V_{1-i} \). By definition of \( A_i \),

\[
|A_i| \geq 2|F_{1-i}|(\frac{n}{2} - c) - (2|F_{1-i}| - 1)|V_i| \\
> 2|F_{1-i}|(\frac{n}{2} - c) - (2|F_{1-i}| - 1)(\frac{n}{2} + \sqrt{\gamma}n) \\
> \frac{n}{2} - 2|F_{1-i}|(c + \sqrt{\gamma}n) \\
> (\frac{1}{2} - 2k\sqrt{\gamma})n - 2kc,
\]

the last inequality holds since \( |F_{1-i}| = \nu_{1-i} \leq k - 1 \). So \( A_i \neq \emptyset \) for sufficiently large \( n \), and furthermore, for any vertex \( x \in A_i \), \( |M_{1-i}| = \nu(G[N_{1-i}(x)]) = \nu_{1-i} \). It is easy to show that \( \deg_{G_i}(x) + \nu(G[V_i \setminus N_i(x)]) \geq |F_i| = \nu_i \). Hence

\[
\deg_{G_i}(x) + |M_{1-i}| + \nu(G[V_i \setminus N_i(x)]) + \nu(E_{V_{1-i}}(\setminus V(M_{1-i}))(x)) \geq \nu_i + \nu_{1-i} \geq k.
\]

Thus we must have \( \deg_{G_i}(x) + |M_{1-i}| \leq s - 1 \), otherwise we have a contradiction to condition (ii). So \( |M_{1-i}| = \nu_{1-i} \leq s - 1 = k - t - 1 \leq k - 2 \) for \( i = 0, 1 \) (since \( t \geq 1 \)). Therefore, \( s - 1 + \nu_i \geq \nu_{1-i} + \nu_i \geq k \) and thus \( \nu_i \geq k - s + 1 = t + 1 \geq 2 \).

**Claim 2.2** We have that \( e(G) < e(T_{n,2}) + (k-1)^2 \). So we get a contradiction to \( e(G) \geq e(T_{n,2}) + (k-1)^2 \).

For any \( x \in V_i \) \( (i = 0, 1) \), Condition (ii) implies that \( \deg_{G_i}(x) + |M_{1-i}| \leq k - 1 \). Hence,

\[
\deg_{G_i}(x) - |V_{1-i}| = \deg_{G_i}(x) + e_G(x, V_{1-i}) - |V_{1-i}| \\
\leq k - 1 - (|M_{1-i}| + |V_{1-i}| - e_G(x, V_{1-i})) \\
\leq k - 1 - \nu_{1-i},
\]
where the last inequality holds since any maximum matching of $G_{1-i}$ intersects $G[N_{1-i}(x)]$ at most $|M_{1-i}|$ edges and intersects $G_{1-i}-E(G[N_{1-i}(x)])$ at most $|V_{1-i}| - e_G(x, V_{1-i})$ edges. Now apply Lemma 2.2 to $G_i$ ($i = 0, 1$) with $\Delta_i \leq k - 1, \nu_i$ and $b = \Delta_i - \nu_{1-i} (\leq \Delta_i - 2$ by Claim 2.1), we get
\[
\sum_{x \in V_i} (\deg_G(x) - |V_{1-i}|) \leq \sum_{x \in V_i} \deg_{G_i}(x) \\
\leq \sum_{x \in V_i} \min\{\deg_{G_i}(x), \Delta_i - \nu_{1-i}\} \\
\leq \nu_i(2\Delta_i - \nu_{1-i}) \leq \nu_i(2k - 2 - \nu_{1-i}).
\]

Summing over $i$ for $i = 0, 1$, we have
\[
2e(G) - 2|V_0||V_1| \leq 2[(k - 1)(\nu_0 + \nu_1) - \nu_0 \nu_1] \\
= 2[(k - 1)^2 - (k - 1 - \nu_0)(k - 1 - \nu_1)] \\
< 2(k - 1)^2,
\]

the last inequality holds since $\nu_i \leq k - 2$ by Claim 2.1. Hence, $e(G) < |V_0||V_1| + (k - 1)^2 \leq e(T_{n,2}) + (k - 1)^2$. This completes the proof of Claim 2.2 and so of Claim 2.

**Claim 3.** If $\max\{\Delta_0, \Delta_1\} \leq k - 2$, then $k = 4$ and $e(G) = e(T_{n,2}) + (k - 1)^2$. Furthermore, if $G$ is $H_{s,t}$-free, then $(s, t) = (3, 1)$ and $G \in \mathcal{F}_{n,3,1}$.

By Lemma 2.1,
\[
m = e(V_0) + e(V_1) \leq f(\nu_0, \Delta_0) + f(\nu_1, \Delta_1) \\
\leq f(\nu_0 + \nu_1, k - 2) \leq f(k - 1, k - 2).
\]

If $k \neq 4$, then $m \leq f(k - 1, k - 2) = (k - 1)^2 - 1$, a contradiction to $m \geq (k - 1)^2$. So $k = s + t = 4$ and we have $m \leq f(3, 2) = (k - 1)^2 = 9$. Therefore, $m = (k - 1)^2 = 9$ and so $G$ contains a complete balanced bipartite subgraph with partite sets $V_0$ and $V_1$. Let $H$ be the subgraph consisting of nonempty components of $G_0 \cup G_1$. Then $H$ is a graph with $e(H) = 9$, $\Delta(H) = 2$ and $\nu(H) = 3$. Hence $|V(H)| \geq e(H) = 9$, the equality holds if and only if $H$ is 2-regular. By Observation 2.7 and the fact that $\nu(H) \geq \omega(H)$, we have $|V(H)| = 9$ and $\omega(H) = 3$, that is $H = 3K_3$. By Lemma 2.8, $s = 3$ and $t = 1$. Then $G$ must be a Turán graph $T_{n,2}$ with $H$ embedded into one class (that is $G \in \mathcal{F}_{n,3,1}$), otherwise we can easily find a vertex which contradicts condition (ii) of the lemma.
Claim 4. If $\max\{\Delta_0, \Delta_1\} = k - 1$, then $e(V_0) \cdot e(V_1) = 0$.

By Lemma 2.1 and Claim 2 we have

$$m \leq f(\nu_0, k - 1) + f(\nu_1, k - 1) \leq f(\nu_0 + \nu_1, k - 1) \leq f(k - 1, k - 1) \leq k(k - 1).$$

Wlog, we assume $\Delta_0 = k - 1$. Let $x \in V_0$ with $\deg_{G_0}(x) = k - 1$. We will show that $e(V_1) = 0$. If not, then $\nu_1 \geq 1$ and so $\nu_0 \leq k - 2$. Let $A_1 = \{u \in V_1 : \deg_{G_1}(u) > 0\}$. By (2), we have $A_1 \cap N_1(x) = \emptyset$. So $e(V_0, V_1) \leq |V_0||V_1| - |A_1| \leq e(T_{n,2}) - |A_1|$. Thus we have

$$e(T_{n,2}) + (k - 1)^2 \leq e(G) \leq e(T_{n,2}) - |A_1| + m.$$ Therefore, $|A_1| \leq m - (k - 1)^2$. That is $|A_1| + (k - 1)^2 \leq m$. Again by Lemma 2.1 we have

$$m \leq f(\nu_0, \Delta_0) + f(\nu_1, \Delta_1) \leq \nu_0(\Delta_0 + 1) + \nu_1(\Delta_1 + 1) \leq \nu_0(k - 1 - \nu_0)(\Delta_1 + 1) \quad \text{(since } \Delta_0 = k - 1 \text{ and } \nu_0 + \nu_1 \leq k - 1) \leq \nu_0(k - 1 - \nu_0)|A_1| \quad \text{(since } \Delta_1 + 1 \leq |A_1|) = \nu_0(k - |A_1|) + (k - 1)|A_1| \leq (k - 2)(k - |A_1|) + (k - 1)|A_1| \quad \text{(since } |A_1| \leq k - 1 \text{ and } \nu_0 \leq k - 2) = (k - 1)^2 + |A_1| - 1 \leq m - 1, \text{ a contradiction.}$$

Claim 3 and Claim 4 implies that $\max\{\Delta_0, \Delta_1\} = k - 1$ and $e(V_0) \cdot e(V_1) = 0$. Without loss of generality, assume that $e(V_1) = 0$. Then $m = e(V_0)$, $\Delta_0 = k - 1$ and $\Delta_1 = 0$. Let $A_0$ be the set of non-isolated vertices in $G_0$. Claim 2 implies that $\nu(G[A_0]) \leq k - 1$. Condition (ii) and Lemma 2.8 imply that $m = e(G[A_0]) \leq (k - 1)^2$. Thus we have $m = (k - 1)^2$ and $G$ contains a complete balanced bipartite subgraph with classes $V_0$ and $V_1$. Again by Lemma 2.8 $G[A_0] \cong K_{k-1,k-1}$. That is $G \in \mathcal{F}_{n,s,t}$. This completes the proof of Lemma 3.1.

Proof of Theorem 1.3. Let $G$ be an extremal graph on $n$ vertices for $H_{s,t}$, where $n$ is large enough. Let $k = s + t$. By Lemma 2.9 we have $e(G) \geq e(T_{n,2}) + (k - 1)^2$. By Lemma 2.5, we may assume that $\delta(G) \geq \left\lceil \frac{n}{2} \right\rceil$. Let $E(V_0, V_1)$ be a maximum cut of $G$ and let $B$ be defined as in Lemma 2.10. By Lemma 2.10, we have

(a) $m = e(V_0) + e(V_1) < \gamma n^2$ and $|B| < \frac{1}{2}\sqrt{\gamma}n < \frac{1}{2}\beta n$.
\( \frac{9}{2} - \sqrt[3]{n} \leq |V_i| \leq \frac{9}{2} + \sqrt[3]{n} \) for \( i = 0, 1 \);
(c) \( e(u, V_{i-1}) \geq \frac{9}{4} - \frac{1}{4} \) for \( u \in V_i \) (\( i = 0, 1 \));
(d) Moreover, \( e(u, V_{i-1}) \geq \frac{9}{2} - \beta n - \frac{1}{2} \) for \( u \in V_i \setminus B \) (\( i = 0, 1 \)).

Let \( B_i = B \cap V_i \) for \( i = 0, 1 \). Then, for each \( v \in B_i \), \( (i = 0, 1) \), \( e_{G_i}(v, V_i \setminus B_i) \geq k \left\lceil \frac{1}{2k} \deg_{G_i}(v) \right\rceil \) since

\[
|B_i| \leq |B| \leq \frac{1}{8} \beta n \leq \frac{1}{2} \beta n - k \leq \deg_{G_i}(v) - k \left\lceil \frac{1}{2k} \deg_{G_i}(v) \right\rceil.
\]

Hence we can keep \( k \left\lceil \frac{1}{2k} \deg_{G_i}(v) \right\rceil \) edges of \( E(v, V_i \setminus B_i) \) and delete the other edges incident with \( v \) in \( G_i \) for each \( v \in B_i \). Denote the resulting graph by \( G' \).

Note that \( E(G'(B_i)) = \emptyset \) and condition (i) of Lemma 2.11 is guaranteed by (b) and (c). Since \( G' \) is \( H_{s,t} \)-free, \( G' \) has no vertex in \( B_i \) (\( i = 0, 1 \)), say \( v \), satisfying that \( \deg_{G'[V_i]}(v) \geq k \), otherwise we can find a copy of \( H_{s,t} \), by applying Lemma 2.11 to \( G' \) with \( B_i \) and \( U_i = V_i \) (\( i = 0, 1 \)). So we must have \( B_i = \emptyset \) (\( i = 0, 1 \)) since \( k - 1 < \frac{1}{2} \beta n \leq k \left\lceil \frac{1}{2k} \deg_{G_i}(v) \right\rceil = \deg_{G'[V_i]}(v) \) for every \( v \in B_i \) (\( i = 0, 1 \)) and thus \( G = G' \).

Now note that \( \max\{\Delta_0, \Delta_1\} \leq k - 1 \) and \( \delta(G) \geq \left\lfloor \frac{n}{2} \right\rfloor \). Together with (b), condition (i) of Lemma 3.1 is guaranteed. Meanwhile, by Lemma 2.11, since \( G \) is \( H_{s,t} \)-free, all vertices of \( V(G) \) do not satisfy inequality (2) (or equivalently, for any vertex \( x \in V_i \), (\( i = 0, 1 \)) and any maximum matching \( M_{1-i} \) of \( G[N_{1-i}(x)] \) with \( \deg_{G_i}(x) + |M_{1-i}| \geq s \), \( x \) satisfies inequality (2)). So condition (ii) of Lemma 3.1 is also guaranteed. Thus, apply Lemma 3.1 to \( G \), since \( G \) is \( H_{s,t} \)-free, we have \( G \in \mathcal{F}_{n,s,t} \).

**4 Proof of Theorem 1.5**

We need to show that provided with large enough integer \( n \), \( \phi(G, H_{s,t}) \leq \ex(n, H_{s,t}) \) for all graphs \( G \) on \( n \) vertices, with equality holds if and only if \( G \in \mathcal{F}_{n,s,t} \). Since \( \phi(G, H_{s,t}) = e(G) - p_{H_{s,t}}(G)(e(H_{s,t}) - 1) \), it suffices to show that

\[
p_{H_{s,t}}(G) \geq \frac{e(G) - \ex(n, H_{s,t})}{e(H_{s,t}) - 1}
\]

for all graphs \( G \) on \( n \) vertices with \( e(G) \geq \phi(G, H) \geq \ex(n, H_{s,t}) = e(T_n, 2) + (k - 1)^2 \) (by Theorem 1.3), with equality holds if and only if \( G \in \mathcal{F}_{n,s,t} \).
Let $G$ be such a graph. By Lemma 2.6, we may assume without loss of generality that $\delta(G) \geq \frac{n}{2}$. Let $E(V_0, V_1)$ be a maximum cut of $G$. Let $B$ be defined as in Lemma 2.10 and $B_i = B \cap V_i$ for $i = 0, 1$. Then, by Lemma 2.10 we have

(a) $m = e(V_0) + e(V_1) < \gamma n^2$ and $|B| < \frac{2\gamma}{B} n$;
(b) $\frac{n}{2} - \sqrt{\gamma}n \leq |V_i| \leq \frac{n}{2} + \sqrt{\gamma}n$ for $i = 0, 1$;
(c) $e(u, V_{i-1}) \geq \frac{n}{4} - \frac{1}{4}$ for $u \in V_i$ ($i = 0, 1$);
(d) Moreover, $e(u, V_{i-1}) \geq \frac{n}{2} - \beta n - \frac{1}{7}$ for $u \in V_i \setminus B_i$ ($i = 0, 1$).

Let

$$C(H_{s,t}) = \frac{2k(k-1)(2e(H_{s,t}) - k - 1)}{e(H_{s,t}) - 2k - 1}$$

be a constant that depends only on $H_{s,t}$. We divide the proof into two cases according to $m > c(H_{s,t})$ or $m \leq c(H_{s,t})$.

**Case 1.** $m > C(H_{s,t})$.

We do the same operation to $G$ as in the proof of Theorem 1.3. That is we keep $k \left\lfloor \frac{1}{2k} \deg_{G_i}(v) \right\rfloor$ edges of $G$ that connect $v$ to its neighbors in $V_i \setminus B_i$ and delete the other edges incident with $v$ in $G_i$ for each vertex $v \in B_i$, $i = 0, 1$. Denote the the resulting graph by $G^0$.

**Claim 1.** We have that $e(G^0[V_0]) + e(G^0[V_1]) \geq \frac{m}{2}$.

For each $i \in \{0, 1\}$, we can see

$$e(G^0[V_i]) = e(G^0[V_i \setminus B_i]) + \sum_{v \in B_i} \deg_{G^0[V_i]}(v)$$

$$= e(G[V_i \setminus B_i]) + \sum_{v \in B_i} k \left\lfloor \frac{1}{2k} \deg_{G_i}(v) \right\rfloor$$

$$\geq \frac{1}{2} e(G[V_i \setminus B_i]) + \frac{1}{2} \sum_{v \in B_i} \deg_{G_i}(v)$$

$$\geq \frac{1}{2} e(V_i).$$

Hence, $e(G^0[V_0]) + e(G^0[V_1]) \geq \frac{1}{2} e(V_0) + \frac{1}{2} e(V_1) \geq \frac{m}{2}$.

We will use the following algorithm to find enough edge-disjoint copies of $H_{s,t}$ in $G$. Initially, set $U_i = V_i \setminus B_i$, for $i = 0, 1$.

**Algorithm 1:** Begin with $G^0, U_0^0, U_1^0, B_0, B_1$, suppose that we have gotten $G^j$ and $U_0^j, U_1^j$ for some $j \geq 0$. A vertex $u \in V_i \setminus B_i$ ($i = 0, 1$) is active in $G^j$ if $e_{G^j}(u, V_{i-1}) >$
\[ \frac{n}{2} - 2\beta n - \frac{1}{8}, \text{ otherwise, is inactive. It's clear that all vertices in } V(G^0) \setminus B \text{ are active in } G^0 \text{ by (d) of Lemma 2.10.} \]

**Step 1.** If there is some \( u \in V_i \) with \( \deg_{G^j[V_i]}(u) \geq k \) (here the vertices of \( B \) are considered first), applying Lemma 2.11 to \( G^j \) with \( V_i, U_i^j, B_i \) for \( i = 0, 1 \), then we can find a copy of \( H_{s,t} \) in \( G^j \). Let \( G^{j+1} \) be the graph obtained from \( G^j \) by deleting the edges of the \( H_{s,t} \). Let \( U_i^{j+1} \) be the set of all active vertices of \( V_i \) for \( i = 0, 1 \). We stop at some iteration \( G^a \) and turn to Step 2 if there is no vertex \( u \in V_i \) with \( \deg_{G^a[V_i]}(u) \geq k \).

**Step 2.** If there is a matching of size \( k \) in \( G^j[V_i] \) for some \( i \in \{0, 1\} \), then, again applying Lemma 2.11 to \( G^j \), we can find a copy of \( H_{s,t} \) in \( G^j \). Update \( G^j \) to \( G^{j+1} \) by the same method as in Step 1. If there is no such a matching, we stop and denote the resulting graph by \( G' \).

Clearly, \( \Delta(G'[V_i]) \leq k - 1 \) and \( \nu(G'[V_i]) \leq k - 1 \) for \( i \in \{0, 1\} \). So by Lemma 2.1, we have \( e(G'[V_i]) \leq f(k - 1, k - 1) \leq k(k - 1) \) for \( i \in \{0, 1\} \). Note that in each step, the copy of \( H_{s,t} \) we found uses exactly \( k \) edges of \( E(G^0[V_0]) \cup E(G^0[V_1]) \). Thus the number of edge-disjoint copies of \( H_{s,t} \) we have found after Step 1 and Step 2 finished is equal to

\[
\begin{align*}
\frac{e(G^0[V_0]) + e(G^0[V_1]) - e(G'[V_0]) - e(G'[V_1])}{k} &\geq \frac{m/2 - 2k(k - 1)}{k} \\
&> \frac{m - (k - 1)^2}{e(H_{s,t}) - 1} = \frac{(e(T_{n,2}) + m) - (e(T_{n,2}) + (k - 1)^2)}{e(H_{s,t}) - 1} \\
&\geq \frac{e(G) - \text{ex}(n, H_{s,t})}{e(H_{s,t}) - 1},
\end{align*}
\]

the first inequality holds by Claim 1 and the second inequality holds since \( m > C(H_{s,t}) \). So to complete proof of Case 1, it suffices to show that Algorithm 1 can be successfully iterated. We prove it in the following claim.

**Claim 2.** Algorithm 1 can be successfully iterated.

Let \( G^j \) \( (j \geq 0) \) be the graph obtained at some point of the iteration. It suffices to verify that \( G^j \) satisfies the conditions of Lemma 2.11. Note that the total number of iterations is most \( \frac{n}{k} < \frac{n^2}{k} \). In each iteration, there are \( e(H_{s,t}) - k \) edges removed from \( E(V_0, V_1) \). Note that \( c_{G^0}(u, V_{1-i}) \geq \frac{n}{2} - \beta n - \frac{1}{8} \) for \( u \in V_i \setminus B_i \) \( (i = 0, 1) \) and \( c_{G^j}(u, V_{1-i}) < \frac{n}{2} - 2\beta n - \frac{1}{2} \) for each inactive vertex \( u \in V_i \) in \( G^j \). So the number of inactive vertices in \( G^j \) is at most \( \frac{c(H_{s,t}) - k}{\beta n} < \frac{(c(H_{s,t}) - 1)\gamma}{\beta n} \). (Recall that \( c(H_{s,t}) \) is the circumference of \( H_{s,t} \).) So \( |V_i \setminus U_i^j| < \frac{(c(H_{s,t}) - 1)\gamma}{\beta} n \leq \sqrt{\gamma} n \).
For vertex $u \in U_i^j$, since $u$ is active, we have $e_{G^j}(u, V_{1-i}) \geq \frac{n}{2} - 2\beta n - \frac{1}{4} > \frac{2}{5}n$. For vertex $u \in B_i$, $u$ was involved in at most $\deg_{G^0[V_i]}(u)/k$ previous iterations and lost $k$ edges from $E_{G^0}(u, V_{1-i})$ in each iteration. Therefore,

$$e_{G^j}(u, V_{1-i}) \geq e_{G^0}(u, V_{1-i}) - k \cdot \deg_{G^0}(u, V_i)/k = e_G(u, V_{1-i}) - e_{G^0}(u, V_i)$$

$$\geq e_G(u, V_{1-i}) - \frac{1}{2}e_G(u, V_i) - k \geq \frac{1}{2}e_G(u, V_{1-i}) - k \geq \frac{n}{8} - k - 1 > \frac{n}{9}.$$ 

For each $u \in V_i^j \setminus (U_i^j \cup B_i)$, $u$ must become inactive in some previous iteration, say in $G^{j'}$, with $j' \leq j$. Note that after $u$ became inactive, $u$ lost at most $\deg_{G^{j'}[V_i]}(u)$ edges from $E_{G^0}(u, V_{1-i})$. Hence, we have

$$e_{G^j}(u, V_{1-i}) \geq e_{G^{j'}}(u, V_{1-i}) - \deg_{G^{j'}[V_i]}(u) \geq e_{G^{j'}}(u, V_{1-i}) - \deg_{G^0[V_i]}(u)$$

$$> \frac{n}{2} - 2\beta n - \frac{1}{4} - \Delta(H_{s,t}) - \beta n = \frac{n}{2} - 3\beta n - \frac{1}{4} - \Delta(H_{s,t}) > \frac{2}{5}n.$$ 

Therefore, Step 1 of Algorithm 1 can be iterated successfully. As for Step 2, suppose that $G^j[V_i]$ has a matching $M_i$ of size $k$. It suffices to show $V(M_i)$ has a common neighbor in $V_{1-i}$. To see this, every vertex $u \in B_i$ has degree $k\lceil\frac{\deg_{G^0}(u)}{2k}\rceil$ in $G^0[V_i]$; hence in any iteration $G^{j'}$ of Step 1, $\deg_{G^{j'}[V_i]}(u)$ must be a multiple of $k$. So, after Step 1 was finished, $u \in B_i$ must have in-degree zero. Thus we have $V(M_i) \cap B_i = \emptyset$ and so the number of common neighbors of $V(M_i)$ in $V_{1-i}$ is at least

$$2k(\frac{n}{2} - 3\beta n - \frac{1}{4} - \Delta(H_{s,t})) - (2k - 1)|V_{1-i}| \geq \frac{n}{2} - 2k(\sqrt{\gamma} + 3\beta)n \geq \frac{24}{50}n.$$ 

**Case 2.** $m \leq C(H_{s,t}).$

Since $2m \leq 2C(H_{s,t}) \ll \beta n \leq \frac{n}{2} - \sqrt{n} \leq |V_i|$, both $G[V_0]$ and $G[V_1]$ have isolated vertices and $B = \emptyset$ by the definition of $B$ (hence in the following algorithm we always choose $B_0 = B_1 = \emptyset$ in each iteration). Since $\delta(G) \geq \lceil \frac{n}{2} \rceil$, we have $|V_i| \geq \lceil \frac{n}{2} \rceil$ for $i = 0, 1$. Since $\{V_0, V_1\}$ is a partition of $V(G)$, we have $\lceil \frac{n}{2} \rceil \leq |V_i| \leq \lceil \frac{n}{2} \rceil$ for $i = 0, 1$. That is $\{V_0, V_1\}$ is a balanced partition of $V(G)$. We will use the following algorithm to find enough many copies of $H_{s,t}$ in $G$. Initially, set $G^0 = G$, $V_0^0 = V_i$ and set $U_i^0$ to be the set of isolated vertices in $G[V_0^0]$ for $i = 0, 1$. Note that $|V_i^0 \setminus U_i^0| \leq 2m \leq 2C(H_{s,t}) < \sqrt{n}$ for sufficiently large $n$.

**Algorithm 2:** Begin with $G^0, U_i^0 (i = 0, 1)$. Suppose that we have get $G^j, V_i^j$ and $U_i^j (i = 0, 1)$ for some $j \geq 0$. Define $i^* = i^*(j) = \begin{cases} 0 & \text{if } |V_0^j| \geq |V_i^j|, \\ 1 & \text{otherwise.} \end{cases}$
Another clear fact is that is no vertex in $G^i$. Let $G$ to $G^i$, we find a copy of $H_{s,t}$, say $H$, with $V(H) \cap U_{i^*} \neq \emptyset$. Choose any vertex $x^j \in V(H) \cap U_{i^*}$. Let $G_{i^*}$ be the graph obtained from $G^j$ by deleting $x^j$ and $E(H)$. Set $V_{i^*}^j = V_{i^*} \setminus \{x^j\}$, $V_{i-1}^{j+1} = V_{i-1}^j \setminus \{x^j\}$ and $U_{i-1}^{j+1} = U_{i-1}^j \setminus \{x^j\}$. We stop Step 2 if there is no vertex satisfying inequality (1) of Lemma 2.11.

**Step 1.** If there exists some vertex $x^j \in V_{i^*}^j$ with $\deg_{G[V_{i^*}^j]}(x^j) \geq k$, then, apply Lemma 2.11 to $G^j$, we find a copy of $H_{s,t}$, say $H$, with $V(H) \cap U_{i^*} \neq \emptyset$. Choose any vertex $u^j \in V(H) \cap U_{i^*}$. Let $G_{i^*}^{j+1}$ be the graph obtained from $G^j$ by deleting $u^j$ and $E(H)$. Set $V_{i^*}^{j+1} = V_{i^*}^j \setminus \{u^j\}$. Since in each previous iteration $u$ lost at most $\Delta(H_{s,t}) + 1 = 2k + 1$ neighbors in $V_{i-1}$, we have

$$e_{G^j}(u, V_{i-1}^j) \geq e_{G_0}(u, V_{i-1}) - (2k + 1)b \geq \left\lfloor \frac{n}{2} \right\rfloor - m - (2k + 1)b \geq \left\lfloor \frac{n}{2} \right\rfloor - 4C(H_{s,t}).$$

An important point is that $|V_{i^*}^j \setminus U_{i^*}^j| \leq 2m \leq 2C(H_{s,t}) < \sqrt{n - j}$ for $i \in \{0, 1\}$ and $j \in \{1, 2, \ldots, b\}$. So Algorithm 2 can be successfully iterated.

**Step 2.** If there exists some vertex $x^j \in V_{i^*}^j$ satisfying inequality (1) of Lemma 2.11 then, apply Lemma 2.11 to $G^j$, we find a copy of $H_{s,t}$, say $H$, with $x^j \in V(H) \cap U_{i^*} \neq \emptyset$. Set $w^j = x^j$ if $i = i^*$, otherwise choose any $w^j \in V(H) \cap U_{i^*}$. Let $G_{i^*}^{j+1}$ be the graph obtained from $G^j$ by deleting $w^j$ and $E(H)$. Set $V_{i^*}^{j+1} = V_{i^*}^j \setminus \{w^j\}$, $V_{i-1}^{j+1} = V_{i-1}^j \setminus \{w^j\}$ and $U_{i-1}^{j+1} = U_{i-1}^j \setminus \{w^j\}$. We stop Step 2 if there is no vertex satisfying inequality (1) in $G^b$ for some integer $b \geq a$.

**Remark.** Clearly, $b \leq m/k < C(H_{s,t})/k$ can be bounded by a constant since in each iteration we find a copy of $H_{s,t}$ intersecting $E_G(V_0) \cup E_G(V_1)$ exactly $k$ edges and $C(H_{s,t})$ is a constant. So we always assume that $|V(G^j)| = n - j$ is sufficiently large and $\ex(n - j, H_{s,t}) = e(T_{n-j}) + (k - 1)^2$ for each $j = 0, 1, \ldots, b$ (by Theorem 1.3). Also note that in each iteration we delete one vertex from the bigger part, so $\{V_0^j, V_1^j\}$ is balanced for all $j \in \{0, 1, \ldots, b\}$ since $\{V_0^0, V_1^0\}$ is balanced.

**Claim 3** For any $j \in \{0, 1, \ldots, b\}$, $e_{G^j}(u, V_{i-1}^j) > \left\lfloor \frac{n}{2} \right\rfloor - 4C(H_{s,t})$ for each $u \in V_{i^*}^j$, $i = 0, 1$. In particular, Algorithm 2 can be successfully iterated.

Let $u \in V_{i^*}^j$. Since in each previous iteration $u$ lost at most $\Delta(H_{s,t}) + 1 = 2k + 1$ neighbors in $V_{i-1}$, we have

$$e_{G^j}(u, V_{i-1}^j) \geq e_{G_0}(u, V_{i-1}) - (2k + 1)b \geq \left\lfloor \frac{n}{2} \right\rfloor - m - (2k + 1)b \geq \left\lfloor \frac{n}{2} \right\rfloor - 4C(H_{s,t}).$$

Another clear fact is that $|V_{i^*}^j \setminus U_{i^*}^j| \leq 2m \leq 2C(H_{s,t}) < \sqrt{n - j}$ for $i \in \{0, 1\}$ and $j \in \{1, 2, \ldots, b\}$. So Algorithm 2 can be successfully iterated.

**Claim 4**

$$p_{H_{s,t}}(G) \geq \frac{e(G) - \ex(n, H_{s,t})}{e(H_{s,t}) - 1},$$

with equality holds if and only if $G \in F_{n, s, t}$.

We prove $p_{H_{s,t}}(G^j) \geq \frac{e(G^j) - \ex(n, H_{s,t})}{e(H_{s,t}) - 1}$ inductively for $j = b, b - 1, \ldots, 0$. Now since there is no vertex in $G^b$ satisfying inequality (1) of Lemma 2.11, we must have $e(G^b) \leq \left\lfloor \frac{n}{2} \right\rfloor - 4C(H_{s,t})$.
\(e(n - b, H_{s,t})\), otherwise, \(e(G^b) > e(n - b, H_{s,t})\), then by Lemma 3.1 we have \(e(G^b) = e(n - b, H_{s,t})\), a contradiction. So we have

\[
p_{H_{s,t}}(G^b) \geq 0 \geq \frac{e(G^b) - e(n - b, H_{s,t})}{e(H_{s,t}) - 1},
\]

with equality holds if and only if \(G^b \in F_{n-b,s,t}\).

Now suppose that \(p_{H_{s,t}}(G^{j+1}) \geq \frac{e(G^{j+1}) - e(n - j - 1, H_{s,t})}{e(H_{s,t}) - 1}\)

holds for some \(j + 1 \in [1, b]\). We show that the above inequality holds for \(j\). By Algorithm 2, suppose that \(G^{j+1}\) is obtained from \(G^j\) by deleting \(u^j\) and \(E(H)\), where \(H\) is a copy of \(H_{s,t}\). Clearly, we have

\[
e(G^{j+1}) = e(G^j) - e(H) = \deg_{G^j}(u^j) + \deg_{H}(u^j).
\]

If \(u^j \in V(H) \cap U^j_{i^*}\), then \(\deg_{G^j}(u^j) = \left\lfloor \frac{n-j}{2} \right\rfloor\) by definition of \(U\) and \(i^*\). Note that \(\deg_{H}(u^j) \geq \delta(H_{s,t}) = 2\), and \(\{V^j_0, V^j_1\}\) is balanced. Hence we have

\[
p_{H_{s,t}}(G^j) \geq p_{H_{s,t}}(G^{j+1}) + 1
\]

\[
\geq \frac{e(G^{j+1}) - e(n - j - 1, H_{s,t})}{e(H_{s,t}) - 1} + 1
\]

\[
\geq \frac{(e(G^j) - e(H) - \left\lfloor \frac{n-j}{2} \right\rfloor + 2) - (e(n - j, H_{s,t}) - \left\lfloor \frac{n-j}{2} \right\rfloor)}{e(H_{s,t}) - 1} + 1
\]

\[
= \frac{e(G^j) - e(n - j, H_{s,t}) + 1}{e(H_{s,t}) - 1}
\]

\[
> \frac{e(G^j) - e(n - j, H_{s,t})}{e(H_{s,t}) - 1}.
\]

If \(u^j \notin V(H) \cap U^j_{i^*}\), then the case only happens in Step 2 when \(u^j \in V^j_{i^*}\) is the center of \(H\). So \(\deg_{H}(u^j) = 2k\). Meanwhile, since it happens in Step 2, \(u^j\) has in-degree
at most \( k - 1 \). So we have that \( \deg_{G^j}(u^j) \leq k - 1 + \left\lfloor \frac{n-j}{2} \right\rfloor \). Thus

\[
p_{H,s,t}(G^j) \geq p_{H,s,t}(G^{j+1}) + 1 \geq \frac{e(G^{j+1}) - \text{ex}(n - j - 1, H_{s,t})}{e(H_{s,t}) - 1} + 1 \geq \frac{e(G^j) - e(H) - (k - 1 + \left\lfloor \frac{n-j}{2} \right\rfloor) + 2k - (\text{ex}(n - j, H_{s,t}) + \left\lfloor \frac{n-j}{2} \right\rfloor)}{e(H_{s,t}) - 1} + 1 \]

\[
= \frac{e(G^j) - \text{ex}(n - j, H_{s,t}) + k}{e(H_{s,t}) - 1} > \frac{e(G^j) - \text{ex}(n - j, H_{s,t})}{e(H_{s,t}) - 1}
\]

Therefore, we can inductively conclude that \( p_{H,s,t}(G^j) \geq \frac{e(G^j) - \text{ex}(n - j, H_{s,t})}{e(H_{s,t}) - 1} \) for all \( j \in [0, b] \), with equality holds if and only if \( b = 0 \) and \( G = G^0 \in \mathcal{F}_{n,s,t} \). The proof is now completed. \( \square \)

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