MODIFIED RINGEL-HALL ALGEBRAS, NAIVE LATTICE ALGEBRAS AND LATTICE ALGEBRAS

LIN JI

Abstract. For a given hereditary abelian category satisfying some finiteness conditions, in certain twisted cases it is shown that the modified Ringel-Hall algebra is isomorphic to the naive lattice algebra and there exists an epimorphism from the modified Ringel-Hall algebra to the lattice algebra. Furthermore, the kernel of this epimorphism is described explicitly. Finally, we show that the naive lattice algebra is invariant under the derived equivalences of hereditary abelian categories.

1. Introduction

The Ringel-Hall algebra $H(A)$ associated to an abelian category $A$ satisfying some finiteness conditions is a vector space over $\mathbb{Q}$ with basis parameterized by the isomorphism classes of objects in the category $A$. And the positive part of the quantum enveloping algebra can be realized by the Ringel-Hall algebras (see [4, 10, 17]). Since then much of the work was concentrated on the realization of the whole quantum group, see for example [11, 13, 14, 19, 18, 5].

Xiao [22] realized the whole quantum group via the reduced Drinfeld double of a Hall algebra by piecing together two Borel parts. Furthermore, Cramer [2] proved that the Drinfeld double Hall algebra is invariant under the derived equivalences. Recently, Bridgeland [1] considered the Hall algebras over the category of $\mathbb{Z}/2$-graded complexes with projective components which provide natural realizations of the whole quantum groups. Lu and Peng [9] introduced the modified Ringel-Hall algebra $\mathcal{MH}_{\mathbb{Z}/2}(A)$ of $A$ over the category of $\mathbb{Z}/2$-graded complexes, where $A$ is a hereditary abelian category which may not have enough projectives. Moreover, it is proved that the componentwise twisted modified Ringel-Hall algebra $\mathcal{MH}_{\mathbb{Z}/2, tw}(A)$ is isomorphic to the Drinfeld double Hall algebra of $\mathcal{H}_{tw}(A)$ which generalizes the construction of Bridgeland [1] and Gorsky [3]. In our previous work [8], we extended the construction of [9] to define the modified Ringel-Hall algebra $\mathcal{MH}(A)$ for the category of bounded complexes $\mathcal{C}^b(A)$ and showed that the twisted modified Ringel-Hall algebra is invariant under derived equivalences.

For a finitary hereditary abelian category $A$, Kapranov [6] introduced the lattice algebra $L(A)$ and the naive lattice algebra $N(A)$ for $A$ and proved that the lattice algebra $L(A)$ is invariant under derived equivalences of hereditary abelian categories. On the other hand, one may also have the so-called derived Hall algebra $\mathcal{DH}(A)$ of $A$ in the sense of [21, 23]. The relation between lattice algebra and derived Hall algebra has been investigated by Sheng and Xu in [20]. For the hereditary abelian category $A$ with enough projectives, Zhang [25] introduced the $m(m = 0$ or $m > 2)$-lattice algebra and proved that it is isomorphic to the Bridgeland’s Hall algebra of $m$-cyclic projective complexes of $A$. In particular, the so-called 0-lattice algebra is the naive lattice algebra.

In the present paper, we mainly study the relations among the modified Ringel-Hall algebra $\mathcal{MH}(A)$, the naive lattice algebra $N(A)$ and the lattice algebra $L(A)$ for a hereditary abelian category $A$ which may not have enough projectives but satisfies the following finiteness conditions over a finite field $k$:

1. $A$ is an essentially small $k$-linear category,

2. $A$ is finitary, i.e. $\dim_k \text{Hom}_A(M, N) < \infty$ and $\dim_k \text{Ext}^1_A(M, N) < \infty$ for any $M, N \in A$.

2010 Mathematics Subject Classification. 16W50, 18E10, 18E30.
Key words and phrases. Modified Ringel-Hall algebras, Naive lattice algebras, Lattice algebras, Hereditary abelian categories.

This work was supported partially by the National Natural Science Foundations of China (Grant No. 11701473) and Youth Talent Foundation of Fuyang Normal University (Grant No. rcxm201803).
In particular, we give the realizations of the naive lattice algebra and the lattice algebra via the modified Ringel-Hall algebra respectively. As an application, we obtain the derived invariance of the naive lattice algebras. Specifically, we have the following main results of this paper.

**Theorem 1.1** (Theorem 3.5). The naive lattice algebra $\mathcal{N}(A)$ is isomorphic to the componentwise twisted modified Ringel-Hall algebra $\mathcal{MH}_{rtw}(A)$.

**Theorem 1.2** (Theorem 4.10). Let $\mathcal{MH}_{rtw}(A)$ be the relative twisted modified Ringel-Hall algebra and $\mathcal{L}_*(A)$ be the Drinfeld dual lattice algebra which is isomorphic to $L(A)$. There is an epimorphism $\varphi: \mathcal{MH}_{rtw}(A) \rightarrow \mathcal{L}_*(A)$, given by

\[
K_{\alpha,n} \mapsto K_{\alpha}^{(-1)^n}, U_{\alpha,n} \mapsto Z_A^{(n)}
\]

for any $A \in \text{Iso}(A)$, $n \in \mathbb{Z}$ and $\alpha \in \mathcal{K}_0(A)$. Moreover, $\mathcal{L}_*(A) \cong \mathcal{MH}_{rtw}(A)/I$, where $I$ is the ideal of $\mathcal{MH}_{rtw}(A)$ generated by the set

\[
\{K_{\alpha,n+1}K_{\beta,n} - K_{\alpha - \beta,n+1}[\alpha, \beta] \in \mathcal{K}_0(A), n \in \mathbb{Z}\}.
\]

**Theorem 1.3** (Theorem 5.6). Let $B$ be a hereditary abelian $k$-category satisfying the finiteness conditions (1)-(2). If there exists a derived equivalence $F: D^b(A) \rightarrow D^b(B)$, then we have the following isomorphism of algebras

\[
\mathcal{N}(A) \cong \mathcal{N}(B).
\]

The paper is organized as follows. In Section 2, we recall the definition and the structure of the modified Ringel-Hall algebras. In Section 3, we prove Theorem 1.1. Section 4 is devoted to investigating the relation between the modified Ringel-Hall algebra and the lattice algebra, and Theorem 1.2 is proved in this section. In Section 5, we show Theorem 1.3 by the same method used in 8.

Unless otherwise specified, throughout this paper $k$ denotes a finite field with $q$ elements and put $v = \sqrt{q}$. $A$ is a hereditary abelian $k$-category satisfying the finiteness conditions (1)-(2).

**Acknowledgments.** The author is deeply indebted to Professor Liangang Peng for his beneficial discussions. The author is also grateful to Changjian Fu and Ming Lu for their inspirations.

2. Modified Ringel-Hall Algebras

2.1. Ringel-Hall Algebras. Let $\varepsilon$ be an essentially small exact category, linear over the finite field $k$. The set of the isomorphism classes of $\varepsilon$ is denoted by $\text{Iso}(\varepsilon)$, and $\hat{A}$ denotes the corresponding element in the Grothendieck group $K_0(\varepsilon)$ for an object $A \in \varepsilon$. Assume that $\varepsilon$ has finite morphism and extension spaces:

\[
|\text{Hom}_\varepsilon(A, B)| < \infty, \quad |\text{Ext}^1_\varepsilon(A, B)| < \infty, \quad \forall A, B \in \varepsilon.
\]

Given objects $A, B, C \in \varepsilon$, define $\text{Ext}^1_\varepsilon(A, C)_B \subseteq \text{Ext}^1_\varepsilon(A, C)$ to be the subset parameterizing extensions whose middle term is isomorphic to $B$. The Ringel-Hall algebra $\mathcal{H}(\varepsilon)$ is the $\mathbb{Q}$-vector space $\bigoplus_{A \in \text{Iso}(\varepsilon)} \mathbb{Q}[A]$ with basis parametrized by the isomorphism classes of objects endowed with the multiplication

\[
[A] \circ [C] = \sum_{[B] \in \text{Iso}(\varepsilon)} \frac{|\text{Ext}^1_\varepsilon(A, C)_B|}{|\text{Hom}_\varepsilon(A, C)|}[B].
\]

It is well-known that the algebra $\mathcal{H}(\varepsilon)$ is associative and unital with unit $[0]$, where $0$ is the zero object of $\varepsilon$, see 16 and also 17, 11, 5, 11.

In this section, we fix a hereditary abelian $k$-category $A$ satisfying the finiteness conditions (1)-(2)(cf. Section 4). For any objects $A, B \in A$, we define the Euler form

\[
(A, B) = \prod_{i \geq 0} |\text{Ext}^i_{\mathcal{A}}(A, B)|(-1)^i.
\]

It is easy to check that this form descends to a bilinear form on the Grothendieck group $K_0(A)$ of $A$ which we denote it by the same symbol:

\[
(\cdot, \cdot): K_0(A) \times K_0(A) \rightarrow \mathbb{Q}^\times.
\]

Denote by $(\hat{A}, \hat{B}) = (\hat{A}, \hat{B})\langle \hat{B}, \hat{A} \rangle$ the symmetrized Euler form.
Definition 2.1 ([17, 1]). (1) The twisted Ringel-Hall algebra \( \mathcal{H}_{tw}(A) \) is the \( \mathbb{Q}(v) \)-vector space with the same basis as \( \mathcal{H}(A) \), and the twisted multiplication is defined by

\[
[A] \ast [B] = \sqrt{(A, B)}[A] \circ [B]
\]

for any \([A], [B] \in \text{Iso}(A)\).

(2) The extended twisted Ringel-Hall algebra \( \mathcal{H}_{ext}^{tw}(A) \) is defined as an extension of \( \mathcal{H}_{tw}(A) \) by adjoining symbols \( k_\alpha \) for classes \( \alpha \in K_0(A) \), and imposing relations

\[
k_\alpha \ast k_\beta = k_{\alpha + \beta}, \quad k_\alpha \ast [B] = \sqrt{(\alpha, B)}[B] \ast k_\alpha
\]

for \( \alpha, \beta \in K_0(A) \) and \([B] \in \text{Iso}(A)\). Note that \( \mathcal{H}_{ext}^{tw}(A) \) is an associative algebra over \( \mathbb{Q}(v) \) with a basis consisting of the elements \([B] \ast k_\alpha \) for \([B] \in \text{Iso}(A)\) and \( \alpha \in K_0(A)\).

2.2. Modified Ringel-Hall algebras. Let \( \mathcal{H}(C^b(A)) \) be the Ringel-Hall algebra of \( C^b(A) \), i.e. for any \( L, M \in C^b(A) \), the Hall product is defined to be the following sum:

\[
[L] \circ [M] = \sum_{[X] \in \text{Iso}(C^b(A))} \frac{|\text{Ext}^1_{C^b(A)}(L, M)_X|}{|\text{Hom}_{C^b(A)}(L, M)|} [X].
\]

Let \( \mathcal{H}(C^b(A))/I \) be the quotient algebra, where \( I \) is the ideal of \( \mathcal{H}(C^b(A)) \) generated by all differences \([L] - [K \circ M]\) if there is a short exact sequence \( K \rightarrow L \rightarrow M \) in \( C^b(A) \) with \( K \) acyclic. We also denote by \( \circ \) the induced multiplication in \( \mathcal{H}(C^b(A))/I \).

We denote by \( C^b_{ac}(A) \) the category of bounded acyclic complexes over \( A \). And set \( S \) to be the subset of \( \mathcal{H}(C^b(A))/I \) formed by all \( r[K], \) where \( r \in \mathbb{Q}^*, \) \( K \in C^b_{ac}(A) \). The modified Ringel-Hall algebra \( \mathcal{M}(A) \) is defined to be the right localization of \( \mathcal{H}(C^b(A))/I \) with respect to \( S \), i.e. \( \mathcal{M}(A) = (\mathcal{H}(C^b(A))/I)[S^{-1}] \) and we refer to [S] for details. Here we also denote by \( \circ \) the multiplication in \( (\mathcal{H}(C^b(A))/I)[S^{-1}] \).

Given an object \( A \in A \) and \( m \in \mathbb{Z} \), let \( K_{A,m} \) be the acyclic complex

\[
\cdots \rightarrow 0 \rightarrow A \xrightarrow{1_A} A \rightarrow 0 \rightarrow \cdots,
\]

where \( A \) sits in the degrees \( m - 1 \) and \( m \); \( U_{A,m} \) be the stalk complex with \( A \) concentrated in the degree \( m \). By abuse of notations, we use the same symbols to denote their isomorphism classes in the modified Ringel-Hall algebra, i.e. \( K_{A,m} := [K_{A,m}] \) and \( U_{A,m} := [U_{A,m}] \). It is well-defined that \( K_{\alpha, n} := \frac{1}{(\alpha, A_1)} K_{A_1, n} \circ K_{A_2, n}^{-1}, \) if \( \alpha = \widehat{A_1 - A_2} \in K_0(A) \) for two objects \( A_1, A_2 \in A \). A \( \mathbb{Q} \)-basis of the modified Ringel-Hall algebra \( \mathcal{M}(A) \) has been constructed in [S].

Proposition 2.2 ([S]). \( \mathcal{M}(A) \) has a basis consisting of elements

\[
K_{\alpha_{i+1-i}} \circ K_{\alpha_{i-2-i}} \circ \cdots \circ K_{\alpha_{l+1-l}} \circ U_{a_{i+1-l}, l} \circ U_{a_{i-l}, r} \circ U_{a_{i+1-l}, l} \circ U_{A, l},
\]

where \( r \geq l \), \( \alpha_i \in K_0(A) \) and \( A_j \in \text{Iso}(A) \) for any \( l \leq i \leq r - 1 \) and \( 1 \leq j \leq r \).

Let \( A, B, M \) and \( N \) be objects in \( A \), we define a rational number \( \gamma_{AB}^{MN} \). Let \( V(M, B, A, N) \) be the subset of \( \text{Hom}(M, B) \times \text{Hom}(B, A) \times \text{Hom}(A, N) \) consisting of exact sequences \( 0 \rightarrow M \rightarrow B \rightarrow A \rightarrow N \rightarrow 0 \). The set \( V(M, B, A, N) \) is finite and let \( \gamma_{AB}^{MN} := \frac{|V(M, B, A, N)|}{a_X a_Y} \), here \( a_X = |\text{Aut}_A(X)| \) denotes the cardinality of the automorphism group \( \text{Aut}_A(X) \) for any object \( X \in A \).

The modified Ringel-Hall algebra can be described by the generators and relations.

Proposition 2.3 ([S]). The modified Ringel-Hall algebra \( \mathcal{M}(A) \) is isomorphic to the associative and unital algebra generated by the set

\[
\{ U_{A,n}, K_{\alpha, n} \mid A \in \text{Iso}(A), \alpha \in K_0(A), n \in \mathbb{Z} \}
\]

and the multiplication of which denoted by \( \circ \) subjects to the relations (2.1)-(2.10) as follows.

\[
(2.1) \quad U_{A,n} \circ U_{B,n} = \sum_{C \in \text{Iso}(A)} \frac{|\text{Ext}^1_{A}(A, B)_C|}{|\text{Hom}_{A}(A, B)|} U_{C,n},
\]

\[
(2.2) \quad K_{\alpha,n} \circ U_{A,n} = \langle \alpha, A \rangle U_{A,n} \circ K_{\alpha,n},
\]

\[
(2.3) \quad K_{\alpha,n} \circ K_{\beta,n} = \frac{1}{(\alpha, \beta)} K_{\alpha+\beta,n},
\]
antipodes since we will not need them.

is the pairing defined by $(\Xi \times \Omega) \to \mathbb{Q}(v)$ satisfying the following conditions:

(1) $\phi(\xi, 1) = \epsilon_\Xi(\xi), \phi(1, \omega) = \epsilon_\Omega(\omega)$,
(2) $\phi(\xi\xi', \omega) = \phi^{\otimes 2}(\xi \otimes \xi', \Delta(\omega))$,
(3) $\phi(\xi, \omega\omega') = \phi^{\otimes 2}(\Delta(\xi), \omega \otimes \omega')$,

where $\Delta$ and $\epsilon$ denote the comultiplication and counit respectively and $\phi^{\otimes 2} : (\Xi \otimes \Xi) \times (\Omega \otimes \Omega) \to \mathbb{Q}(v)$ is the pairing defined by $(\xi \otimes \xi', \omega \otimes \omega') \mapsto \phi(\xi, \omega)\phi(\xi', \omega')$. We do not include here any condition on the antipodes since we will not need them.

**Definition 3.1** ([6]). Let $\Xi, m \in \mathbb{Z}$ be Hopf algebras and $\phi_m : \Xi_m \times \Xi_{m+1} \to \mathbb{Q}(v)$ be Hopf pairings. The naive lattice algebra $\mathcal{N} = \mathcal{N}(\{\Xi_m, \phi_m\})$ is generated by elements of all the algebras $\Xi_m$, so that inside each $\Xi_m$ the elements are multiplied according to the multiplication law there while for elements of different algebras we impose the relations

$\xi_m \xi_{m+1} = (Id \otimes \phi_m \otimes Id) (\Delta_{\Xi_{m+1}}(\xi_{m+1}) \otimes \Delta_{\Xi_m}(\xi_m))$

and

$\xi_m \xi_{m'} = \xi_m \xi_m, \quad |m - m'| \geq 2.$

For any sequence $(a^i)_{i \in \mathbb{Z}}$ of elements of a possibly non-commutative algebra $S$, almost all equal to 1, we define their ordered product to be

$$\prod_i a^i = a^p a^{p-1} \ldots a^{q+1} a^q,$$

where $p, q$ are integers such that $a^i = 1$ unless $q \leq i \leq p$.

**Lemma 3.2** ([6]). The ordered product map $\otimes_{m \in \mathbb{Z}} \Xi_m \to \mathcal{N}$ is an isomorphism of vector spaces.

By the work of Green [4] and Xiao [22], the comultiplication $\Delta$ and counit $\epsilon$ of twisted extended Ringel-Hall algebra $\mathcal{H}_{tw}(A)$ are given by

$$\Delta : \mathcal{H}_{tw}(A) \to \mathcal{H}_{tw}(A) \otimes \mathcal{H}_{tw}(A), \quad \epsilon : \mathcal{H}_{tw}(A) \to \mathbb{Q}(v),$$

$$\Delta([A] * k_\alpha) = \sum_{[B] \in \text{Iso}(A)} \sqrt{(B, C)} \frac{|\text{Ext}_A^1(B, C)| \cdot a_A}{|\text{Hom}_A(B, C)|} \cdot [A] * |k_{C+\alpha} | \otimes |C| * k_\alpha,$$

$$\epsilon([A] k_\alpha) = \delta_{[A], 0}$$

for any $[A] \in \text{Iso}(A)$ and $\alpha \in K_0(A)$. Then $(\mathcal{H}_{tw}(A), *, [0], \Delta, \epsilon)$ is a topological bialgebra defined over $\mathbb{Q}(v)$. Here topological means that everything should be considered in the completed space.
It is well-known that there exists a non-degenerate symmetric bilinear pairing
\[ \phi : \mathcal{H}_w^e(A) \otimes \mathcal{H}_w^e(A) \to \mathbb{Q}(v) \]
defined by
\[ \phi([M] \ast K_\alpha, [N] \ast K_\beta) = (\alpha, \beta) \delta_{[M],[N]} q_M, \]
which is a Hopf pairing. Xiao [22] showed that the extended twisted Ringel-Hall algebra has an antipode
and it is a Hopf algebra. The naive lattice algebra \( \mathcal{N}(A) \) for the hereditary category \( A \) is the naive lattice
algebra for \( \Xi_m = \mathcal{H}_w^e(A) \) and \( \phi_m = \phi \) for \( m \in \mathbb{Z} \).

According to the construction, the naive lattice algebra \( \mathcal{N}(A) \) can also be described by the generators
and relations. By Lemma 3.2 and the proof of Proposition 1.5.3 of [6], one can easily get the following proposition.

**Proposition 3.3.** The naive lattice algebra \( \mathcal{N}(A) \) is generated by the symbols \( Y^{(n)}_A \) and \( K^{(n)}_\alpha \) for all \( A \in \text{Iso}(A), n \in \mathbb{Z} \) and \( \alpha \in K_0(A) \). And the following relations (3.1)-(3.10) are the defining relations of \( \mathcal{N}(A) \).

\begin{align*}
(3.1) \quad Y_A^{(n)} Y_B^{(n)} &= \sum_{C \in \text{Iso}(A)} \sqrt{\langle A, B \rangle} |\text{Ext}^1_B(A, B)| \gamma_C^{(n)} Y_C, \\
(3.2) \quad K^{(n)}_\alpha Y_A^{(n)} &= \sqrt{(\alpha, A)} Y_A^{(n)} K^{(n)}_\alpha, \\
(3.3) \quad K^{(n)}_\alpha K^{(n)}_\beta &= K^{(n)}_{\alpha + \beta}, \\
(3.4) \quad Y_A^{(n)} K^{(n+1)}_\alpha &= \sqrt{(\alpha, A)} K^{(n+1)}_\alpha Y_A^{(n)}, \\
(3.5) \quad K^{(n)}_\alpha Y_A^{(n+1)} &= Y_A^{(n+1)} K^{(n)}_\alpha, \\
(3.6) \quad K^{(n)}_\alpha K^{(n+1)}_\beta &= \sqrt{(\alpha, \beta)} K^{(n+1)}_\beta K^{(n)}_\alpha, \\
(3.7) \quad Y_B^{(n)} Y_A^{(n+1)} &= \sum_{M, N \in \text{Iso}(A)} \gamma_{AB}^{MN} \gamma_{AB}^{MN} \sqrt{\langle M - N, M - B \rangle} Y_B^{(n+1)} Y_A^{(n)} K^{(n+1)}_{B-M}.
\end{align*}

If \( |m - n| \geq 2 \), then
\begin{align*}
(3.8) \quad Y_A^{(m)} Y_B^{(n)} &= Y_B^{(n)} Y_A^{(m)}, \\
(3.9) \quad K^{(m)}_\alpha Y_B^{(n)} &= Y_B^{(n)} K^{(m)}_\alpha, \\
(3.10) \quad K^{(m)}_\alpha K^{(n)}_\beta &= K^{(n)}_\beta K^{(m)}_\alpha.
\end{align*}

### 3.2. Componentwise Twisted Modified Ringel-Hall Algebras.

In the following, we define the componentwise Euler form on \( \text{Iso}(C^b(A)) \)
\[ \langle -, - \rangle_{\text{cw}} : \text{Iso}(C^b(A)) \times \text{Iso}(C^b(A)) \to \mathbb{Q}(v), \]
by setting \( \langle [M], [N] \rangle_{\text{cw}} = \prod_{i \in \mathbb{Z}} (M_i, N_i) \) for \( [M], [N] \in \text{Iso}(C^b(A)) \). This form descends to a bilinear form
\[ \langle -, - \rangle_{\text{cw}} : K_0(C^b(A)) \times K_0(C^b(A)) \to \mathbb{Q}(v). \]

The multiplication in the componentwise twisted modified Ringel-Hall algebra \( \mathcal{MH}_{\text{cw}}(A) \) is defined by
\[ [M_1] \ast [M_2] := ([M_1] \ast [M_2])_{\text{cw}} [M_1] \circ [M_2], \]
for any \( [M_1], [M_2] \in \text{Iso}(C^b(A)) \).

So it is not hard to obtain that the componentwise twisted modified Ringel-Hall algebra \( \mathcal{MH}_{\text{cw}}(A) \) is also generated by the set
\[ \{ U_{A,n}, K_{\alpha, n} | A \in \text{Iso}(A), \alpha \in K_0(A), n \in \mathbb{Z} \}, \]
but subject to the relations (3.11)-(3.20) in the following proposition.
Proposition 3.4. For any \( A, B \in \text{Iso}(A) \), \( \alpha, \beta \in K_0(A) \), and \( m, n \in \mathbb{Z} \), we have

\[
\begin{align*}
U_{A,n} \ast U_{B,n} &= \sum_{C \in \text{Iso}(A)} \sqrt{\langle \hat{A}, \hat{B} \rangle} |\text{Ext}^1_A(A, B)_{\text{C}}| U_{C,n}, \\
K_{\alpha,n} \ast U_{A,n} &= \sqrt{\langle \alpha, \hat{A} \rangle} U_{A,n} \ast K_{\alpha,n}, \\
K_{\alpha,n} \ast K_{\beta,n} &= K_{\alpha + \beta,n}, \\
U_{A,n} \ast K_{\alpha,n+1} &= \sqrt{\langle \alpha, \hat{A} \rangle} K_{\alpha,n+1} \ast U_{A,n}, \\
K_{\alpha,n} \ast U_{A,n+1} &= U_{A,n+1} \ast K_{\alpha,n}, \\
K_{\alpha,n} \ast K_{\beta,n+1} &= \sqrt{\langle \alpha, \beta \rangle} K_{\beta,n+1} \ast K_{\alpha,n}, \\
U_{B,n} \ast U_{A,n+1} &= \sum_{M, N \in \text{Iso}(A)} \gamma_{MN}^{AB} \frac{a_{A,n} a_{B,n}}{a_{M,n}} \sqrt{\langle \hat{M} - \hat{N}, \hat{M} - \hat{B} \rangle} U_{N,n+1} \ast U_{M,n} \ast K_{\hat{B} - \hat{M}, n+1}, \\
\end{align*}
\]

if \( |m - n| \geq 2 \), then

\[
\begin{align*}
U_{A,m} \ast U_{B,n} &= U_{B,n} \ast U_{A,m}, \\
K_{\alpha,m} \ast U_{B,n} &= U_{B,n} \ast K_{\alpha,m}, \\
K_{\alpha,m} \ast K_{\beta,n} &= K_{\beta,n} \ast K_{\alpha,m},
\end{align*}
\]

in \( \mathcal{MH}_{ctw}(A) \), where \( K_{\alpha,n} = K_{A_1,n} \ast K_{-A_2,n}^{-1} \) if \( \alpha = A_1 - A_2 \in K_0(A) \).

Proof. Since the restriction of the componentwise twisted multiplication on \( \text{Iso}(A) \) coincides with the twisted multiplication of \( \mathcal{H}_{ctw}^A(A) \), it is trivial to check the relations (3.11)-(3.13). And it is not hard to check the relations (3.14)-(3.16) by the definition of the componentwise Euler form. For (3.17), we have

\[
\begin{align*}
U_{B,n} \ast U_{A,n+1} &= U_{B,n} \circ U_{A,n+1} \\
&= \sum_{M, N \in \text{Iso}(A)} \gamma_{MN}^{AB} \frac{a_{A,n} a_{B,n}}{a_{M,n}} \sqrt{\langle \hat{B} - \hat{M}, \hat{M} \rangle} U_{N,n+1} \circ U_{M,n} \\
&= \sum_{M, N \in \text{Iso}(A)} \gamma_{MN}^{AB} \frac{a_{A,n} a_{B,n}}{a_{M,n}} \sqrt{\langle \hat{B} - \hat{M}, \hat{M} \rangle} \sqrt{\langle \hat{B} - \hat{M}, \hat{M} \rangle} \sqrt{\langle \hat{B} - \hat{M}, \hat{N} \rangle} \sqrt{\langle \hat{B} - \hat{M}, \hat{N} \rangle} \sqrt{\langle \hat{B} - \hat{M}, \hat{N} \rangle} \\
&= \sum_{M, N \in \text{Iso}(A)} \gamma_{MN}^{AB} \frac{a_{A,n} a_{B,n}}{a_{M,n}} \sqrt{\langle \hat{M} - \hat{N}, \hat{M} - \hat{B} \rangle} U_{N,n+1} \ast U_{M,n} \ast K_{\hat{B} - \hat{M}, n+1},
\end{align*}
\]

which completes the proof.

Theorem 3.5. The naive lattice algebra \( \mathcal{N}(A) \) is isomorphic to the componentwise twisted modified Ringel-Hall algebra \( \mathcal{MH}_{ctw}(A) \).

Proof. Following Proposition 3.3 and Proposition 3.4, there is an isomorphism of algebras

\[
\Theta : \mathcal{MH}_{ctw}(A) \to \mathcal{N}(A),
\]

by setting

\[
U_{A,n} \mapsto Y_{(n)}^A \text{ and } K_{\alpha,n} \mapsto K_{(n)}^\alpha.
\]
4. Modified Ringel-Hall Algebras and Lattice Algebras

4.1. Derived Hall algebras. Let $\mathcal{T}$ be a $k$-additive triangulated category with translation $[1]$ satisfying

(i) $\dim_k \text{Hom}_{\mathcal{T}}(X, Y) < \infty$ for any two objects $X$ and $Y$;
(ii) $\text{End}_{\mathcal{T}}(X)$ is local for any indecomposable object $X \in \mathcal{T}$;
(iii) $\mathcal{T}$ is (left) locally finite; that is, $\sum_{i \geq 0} \dim_k \text{Hom}_{\mathcal{T}}(X[i], Y) < \infty$ for any $X$ and $Y$ in $\mathcal{T}$.

The derived Hall algebra $\mathcal{DH}(\mathcal{T})$ of the triangulated category $\mathcal{T}$ is the $\mathbb{Q}$-space with the basis $\{[X] | X \in \mathcal{T}\}$ and the multiplication is defined by

$$[X][Y] = \sum_{[L] \in \text{Iso}(\mathcal{T})} \frac{|\text{Ext}_{\mathcal{T}}^1(X, Y)_L|}{\prod_{i \geq 0} |\text{Hom}_{\mathcal{T}}(X[i], Y)[(-1)^i]L|},$$

where $\text{Ext}_{\mathcal{T}}^1(X, Y)_L$ is defined to be $\text{Hom}_{\mathcal{T}}(X, Y[1])_L$ which denotes the subset of $\text{Hom}(X, Y[1])$ consisting of morphisms $l : X \rightarrow Y[1]$ whose cone $\text{Cone}(l)$ is isomorphic to $L[1]$. Here we use the structure coefficient given by M. Kontsevich and Y. Soibelman in [7], which is different from the one introduced by B. Toën [21] and Xiao-Xu [23], however, it is proved in [24] that both derived Hall algebras with these two different structure coefficients are isomorphic.

In particular, for the hereditary abelian category $A$ satisfying the finiteness conditions (1) and (2) in Section 1, it is easy to describe the derived Hall algebra of $A$ by generators

$$\{Z^n_A | A \in \text{Iso}(A), n \in \mathbb{Z}\},$$

and relations in terms of $A$, where $Z^n_A$ is the stalk complex with the non-zero component $A$ sitting in the degree $n$.

**Proposition 4.1.** $\mathcal{DH}(A)$ is an associative and unital $\mathbb{Q}$-algebra generated by the set

$$\{Z^n_A | A \in \text{Iso}(A), n \in \mathbb{Z}\},$$

and subject to the following relations:

\begin{align*}
(4.1) \quad Z^n_A Z^n_B &= \sum_{C \in \text{Iso}(A)} |\text{Ext}_{A}(A, B)_C| Z^n_C, \\
(4.2) \quad Z^n_B Z^{n+1}_A &= \sum_{M, N \in \text{Iso}(A)} \gamma^M_{AB} a_{M} a_{N} 1_{\langle N, M \rangle} Z^{n+1}_N Z^n_M, \\
(4.3) \quad Z^n_B Z^n_A &= \langle \hat{A}, \hat{B} \rangle^{(-1)^{m-n}} Z^n_Y Z^n_X \text{ for } m > n + 1.
\end{align*}

For any $[M], [N] \in \text{Iso}(D^b(A))$, define the Euler form

$$\langle [M], [N] \rangle_t = \sqrt{\prod_{i \in \mathbb{Z}} \text{Hom}_{D^b(A)}(M, N[i])[(-1)^i]},$$

also it can descend to the Grothendieck group $K_0(D^b(A))$. Moreover it coincides with the Euler form of $K_0(A)$ over the stalk complexes. And then the multiplication of the twisted derived Hall algebra $\mathcal{DH}_{tw}(A)$ is given by

$$[M] \Delta [N] = \langle \hat{M}, \hat{N} \rangle_t [M][N],$$

for any $[M]$, $[N] \in \text{Iso}(D^b(A))$.

**Definition 4.2** ([20]). The extended twisted derived Hall algebra $\mathcal{DH}^e_{tw}(A)$ is the associative algebra generated by all the elements

$$Z^n_A, K_\alpha,$$
for all $A \in \text{Iso}(A), n \in \mathbb{Z}$ and $\alpha \in K_0(A)$, and with the following defining relations.

\begin{align*}
(4.4) \quad K_\alpha K_\beta &= K_{\alpha + \beta}, & K_\alpha Z_A^{[n]} &= \sqrt{\langle \hat{\alpha}, \alpha \rangle^{-1}} Z_A^{[n]} K_\alpha, \\
(4.5) \quad Z_A^{[n]} Z_B^{[n]} &= \sum_{C \in \text{Iso}(A)} \sqrt{\langle \hat{\alpha}, \beta \rangle} |\text{Ext}_A^1(A, B)_C| Z_C^{[n]}, \\
(4.6) \quad Z_B^{[n]} Z_A^{[n+1]} &= \sum_{M, N \in \text{Iso}(A)} \gamma_{AB}^{MN} a_{AB} a_{MN} \frac{1}{\sqrt{\langle \hat{B}, \alpha \rangle \langle \hat{\alpha}, \beta \rangle}} Z_A^{[n+1]} Z_B^{[n]}, \\
(4.7) \quad Z_B^{[n]} Z_A^{[m]} &= \sqrt{\langle \hat{\alpha}, \beta \rangle^{-(1)^{n-m}}} Z_A^{[n]} Z_B^{[m]} \quad \text{for } m > n + 1.
\end{align*}

4.2. Lattice algebras. For any objects $A, B, C \in A$, we use the symbol $g_{AB}^C$ to denote the number of subobject $B'$ of $C$ such that $B' \cong B$ and $C/B' \cong A$. Then one has the Riedtmann-Peng formula (see [15, [22])

$$g_{AB}^C = \frac{|\text{Ext}_A^1(A, B)_C|}{|\text{Hom}_A(A, B)|} a_{AB} a_{CB}.$$  

Definition 4.3 ([6]). The lattice algebra $L(A)$ is generated by the elements

$$X_A^{(n)}, K_\alpha,$$

for all $A \in \text{Iso}(A), n \in \mathbb{Z}$ and $\alpha \in K_0(A)$, subject to the relations

\begin{align*}
(4.8) \quad K_\alpha K_\beta &= K_{\alpha + \beta}, & K_\alpha X_A^{(n)} &= \sqrt{\langle \hat{\alpha}, \alpha \rangle^{-1}} X_A^{(n)} K_\alpha, \\
(4.9) \quad X_A^{(n)} X_B^{(n)} &= \sum_{C \in \text{Iso}(A)} \sqrt{\langle \hat{\alpha}, \beta \rangle} g_{AB}^C X_C^{(n)}, \\
(4.10) \quad X_B^{(n)} X_A^{(n+1)} &= \sum_{M, N \in \text{Iso}(A)} \gamma_{AB}^{MN} a_{AB} \frac{1}{\sqrt{\langle \hat{B}, \alpha \rangle \langle \hat{\alpha}, \beta \rangle}} X_A^{(n+1)} X_B^{(n)} K_B^{-(1)^{n+1}}, \\
(4.11) \quad X_B^{(n)} X_A^{(m)} &= \sqrt{\langle \hat{\alpha}, \beta \rangle^{-(1)^{n-m}}} X_A^{(n)} X_B^{(m)} \quad \text{for } |m - n| \geq 2.
\end{align*}

By the result of Sheng and Xu [20], we know that the lattice algebra coincides with the extended twisted derived Hall algebra $\mathcal{D}H_{\text{tw}}(A)$ in Definition 1.22.

Moreover, in terms of alternative generators

$$Z_A^{(n)} = X_A^{(n)} a_{AB},$$

for all $A \in \text{Iso}(A), n \in \mathbb{Z}$ and $\alpha \in K_0(A)$, one can easily get that the lattice algebra $L(A)$ is isomorphic to the algebra $L_*(A)$ which is described by the following proposition and is called the Drinfeld dual lattice algebra of $A$.

Proposition 4.4. The lattice algebra $L(A)$ is isomorphic to the algebra $L_*(A)$ generated by the symbols $Z_A^{(n)}$ and $K_\alpha$, for all $A \in \text{Iso}(A), n \in \mathbb{Z}$ and $\alpha \in K_0(A)$, with defining relations as follows.

\begin{align*}
(4.12) \quad K_\alpha K_\beta &= K_{\alpha + \beta}, & K_\alpha Z_A^{(n)} &= \sqrt{\langle \hat{\alpha}, \alpha \rangle^{-1}} Z_A^{(n)} K_\alpha, \\
(4.13) \quad Z_A^{(n)} Z_B^{(n)} &= \sum_{C \in \text{Iso}(A)} \sqrt{\langle \hat{\alpha}, \beta \rangle} |\text{Ext}_A^1(A, B)_C| Z_C^{(n)}, \\
(4.14) \quad Z_B^{(n)} Z_A^{(n+1)} &= \sum_{M, N \in \text{Iso}(A)} \gamma_{AB}^{MN} a_{AB} \frac{1}{\sqrt{\langle \hat{B}, \alpha \rangle \langle \hat{\alpha}, \beta \rangle}} Z_B^{(n+1)} Z_A^{(n)} K_B^{-(1)^{n+1}}, \\
(4.15) \quad Z_B^{(n)} Z_A^{(m)} &= \sqrt{\langle \hat{\alpha}, \beta \rangle^{-(1)^{n-m}}} Z_A^{(n)} Z_B^{(m)} \quad \text{for } |m - n| \geq 2.
\end{align*}

By Proposition 3.3.2 of [6], one can obtain a $\mathbb{Q}$-basis of $L_*(A)$ as follows.

Proposition 4.5. The elements $K_\alpha Z_{A^i}^{(r)} Z_{A^{r-i}}^{(r-i)} \cdots Z_{A^{l}}^{(l)}$ forms a basis of $L_*(A)$, for all $A^i \in \text{Iso}(A)(r, l \in \mathbb{Z}, l \leq i \leq r)$ and $\alpha \in K_0(A)$. 

4.3. Relative twisted modified Ringel-Hall algebras. In [S], we define the twisted modified Ringel-Hall algebra $\mathcal{MH}_{tw}(A)$ by the Euler form for $\text{Iso}(C^b(A))$, i.e.

$$[M] \ast [N] = ([M], [N])[M] \circ [N]$$

for any $[M], [N] \in \text{Iso}(C^b(A))$, where $\langle [M], [N] \rangle = \prod_{p=0}^{\infty} \text{Ext}^p_{C^b(A)}(M, N)((-1)^p)$. In particular, the quantum torus of acyclic complexes is commutative with all the elements of $\mathcal{MH}_{tw}(A)$ and we have the following result.

**Proposition 4.6** ([S]). $\mathcal{MH}_{tw}(A)$ is generated by the set

$$\{U_{A,n}, K_{\alpha,n} | A \in \text{Iso}(A), \alpha \in K_0(A), n \in \mathbb{Z} \}$$

with defining relations as follows.

\begin{align*}
(4.16) & \quad U_{A,n} \ast U_{B,n} = \sum_{C \in \text{Iso}(A)} \langle \hat{A}, \hat{B} \rangle \frac{\text{Ext}^1_{A}(A, B)_{C}}{\text{Hom}_{A}(A, B)} U_{C,n}, \\
(4.17) & \quad K_{\alpha,n} \ast U_{A,n} = U_{A,n} \ast K_{\alpha,n}, \\
(4.18) & \quad K_{\alpha,n} \ast K_{\beta,n} = K_{\alpha + \beta,n}, \\
(4.19) & \quad U_{A,n} \ast K_{\alpha,n+1} = K_{\alpha,n+1} \ast U_{A,n}, \\
(4.20) & \quad K_{\alpha,n} \ast U_{A,n+1} = U_{A,n+1} \ast K_{\alpha,n}, \\
(4.21) & \quad K_{\alpha,n} \ast K_{\beta,n+1} = K_{\beta,n+1} \ast K_{\alpha,n}, \\
(4.22) & \quad U_{B,n} \ast U_{A,n+1} = \sum_{M, N \in \text{Iso}(A)} \frac{\gamma_{MN}^{MN}}{\frac{\hat{A} \ast \hat{B}}{\hat{B} \ast \hat{B}}} U_{N,n+1} \ast U_{M,n} \ast K_{B-\hat{M},n+1},
\end{align*}

and if $m > n + 1$, then

\begin{align*}
(4.23) & \quad U_{B,n} \ast U_{A,m} = \langle \hat{B}, \hat{A} \rangle^{-(m-n)} U_{A,m} \ast U_{B,n}, \\
(4.24) & \quad U_{B,m} \ast K_{\alpha,n} = K_{\alpha,n} \ast U_{B,m}, \\
(4.25) & \quad K_{\beta,n} \ast K_{\alpha,m} = K_{\alpha,m} \ast K_{\beta,n},
\end{align*}

for any $A, B \in \text{Iso}(A)$, $\alpha, \beta \in K_0(A)$ and $m, n \in \mathbb{Z}$. And $K_{\alpha,n} = K_{A_1,n} \ast K_{A_2,n}^{-1}$, if $\alpha = \hat{A}_1 - \hat{A}_2 \in K_0(A)$.

**Definition 4.7.** For any $[M], [N] \in \text{Iso}(C^b(A))$, define the relative Euler form for $\text{Iso}(C^b(A))$ by setting

$$\langle [M], [N] \rangle_r = \prod_{i,j \in \mathbb{Z}} \langle \hat{M}^i, \hat{N}^j \rangle^{(-1)^{m-n}(m-n+1)} ,$$

and it is clear that it can descend to $K_0(C^b(A))$.

**Proposition 4.8.** For any $A, B \in \text{Iso}(A)$, $\alpha, \beta \in K_0(A)$ and $m, n \in \mathbb{Z}$, we have

\begin{align*}
(4.26) & \quad \langle U_{A,n}, U_{B,m} \rangle_r = \sqrt{\langle \hat{A}, \hat{B} \rangle^{-(m-n+1)(m-n+1)}}, \\
(4.27) & \quad \langle K_{\alpha,n}, U_{B,m} \rangle_r = \sqrt{\langle \alpha, \hat{B} \rangle^{-(m-n)}}, \\
(4.28) & \quad \langle U_{B,m}, K_{\alpha,n} \rangle_r = \sqrt{\langle \hat{B}, \alpha \rangle^{-(m-n+1)}}, \\
(4.29) & \quad \langle K_{\alpha,n}, K_{\beta,m} \rangle_r = 1.
\end{align*}

**Proof.** The identity (4.26) is induced by the Definition 4.7.

For the identity (4.27), if $\alpha = \hat{A} \in K_0(A)$ for an object $A \in A$, then

$$\langle K_{\hat{A},n}, U_{B,m} \rangle_r = \langle U_{A,n}, U_{B,m} \rangle_r \langle U_{A,n-1}, U_{B,m} \rangle_r = \sqrt{\langle \hat{A}, \hat{B} \rangle^{-(m-n)}}.$$

If $\alpha = \hat{A}_1 - \hat{A}_2 \in K_0(A)$ for objects $A_1, A_2 \in A$, then $K_{\alpha,n} = K_{A_1,n}^{-1} \ast K_{A_2,n}^{-1}$. So

$$\langle K_{\alpha,n}, U_{B,m} \rangle_r = \langle K_{A_1,n}, U_{B,m} \rangle_r \langle K_{A_2,n}^{-1}, U_{B,m} \rangle_r = \sqrt{\langle \alpha, \hat{B} \rangle^{-(m-n)}}.$$

Similarly, one can get the identity (4.29).
For the identity \((4.20)\), it suffices to prove
\[
\langle K_{\hat{A}, n}, K_{B, m} \rangle_r = 1,
\]
which is deduced from the identities \(\langle K_{\hat{A}, n}, K_{B, m} \rangle_r = \langle K_{\hat{A}, n}, U_{B, m} \rangle_r \langle K_{\hat{A}, n}, U_{B, m-1} \rangle_r\) and \(\mathcal{M}_{r,tw}(A)\).

Let \(\mathcal{M}_{r,tw}(A)\) be the relative twisted modified Ringel-Hall algebra, with the relative twisted multiplication defined by
\[
[M] \circ [N] = \langle [M], [N] \rangle_r [M] \ast [N],
\]
i.e. \([M] \circ [N] = ([M], [N])_r ([M], [N]) \circ [N] \) for any \([M], [N] \in \text{Iso}(C^b(A))\). Similarly, it is easy to get the following description of the relative twisted modified Ringel-Hall algebra by the generators and relations.

**Proposition 4.9.** The relative twisted modified Ringel-Hall algebra \(\mathcal{M}_{r,tw}(A)\) is isomorphic to the associative and unital algebra generated by the set
\[
\{ U_{A, n}, K_{\alpha, n} | A \in \text{Iso}(A), \alpha \in K_0(A), n \in \mathbb{Z} \}
\]
and subject to the following relations \((4.30)-(4.34)\).

\[
\begin{align*}
(4.30) \quad K_{\alpha, n} \circ K_{\beta, m} &= K_{\beta, m} \circ K_{\alpha, n}, \quad K_{\alpha, n} \circ K_{\beta, n} = K_{\alpha + \beta, n} \\
(4.31) \quad K_{\alpha, m} \circ U_{A, n} &= \sqrt{\langle \hat{A}, B \rangle (-1)^{n-m}} U_{A, n} \circ K_{\alpha, m}, \\
(4.32) \quad U_{A, n} \circ U_{B, n} &= \sum_{C \in \text{Iso}(A)} \sqrt{\langle \hat{A}, \hat{B} \rangle} \left| \dfrac{\text{Ext}^1_{A}(A, B)_{C}}{\text{Hom}_{A}(A, B)} \right| U_{C, n}, \\
(4.33) \quad U_{B, n} \circ U_{A, n+1} &= \sum_{M, N \in \text{Iso}(A)} \gamma_{MN}^{1} \hat{A}_{N} \hat{C}_{M} \sqrt{\langle \hat{M}, \hat{N} \rangle \hat{M} \hat{N}} \\
&= U_{N, n+1} \circ U_{M, n} \circ K_{B-\hat{M}, n+1}, \\
(4.34) \quad U_{B, n} \circ U_{A, m} &= \sqrt{\langle \hat{A}, \hat{B} \rangle (-1)^{n-m}(n-m+1)} U_{A, m} \circ U_{B, n}, \quad m - n \geq 2,
\end{align*}
\]

for any \(A, B \in \text{Iso}(A)\), \(\alpha, \beta \in K_0(A)\) and \(n \in \mathbb{Z}\). And \(K_{\alpha, n} = K_{\hat{A}, n} \circ K_{\hat{A}, n}^{-1}\), if \(\alpha = \hat{A}_1 - \hat{A}_2 \in K_0(A)\).

**Proof.** The relation \((4.30)\) is a consequence of the commutativity of quantum torus of acyclic complexes in the twisted modified Ringel-Hall algebra and the identity \((4.20)\). The relation \((4.32)\) is a direct consequence of Definition 4.7.

For the relation \((4.31)\), we have
\[
K_{\alpha, m} \circ U_{A, n} = \langle K_{\alpha, m}, U_{A, n} \rangle_r K_{\alpha, m} * U_{A, n}
\]
\[
= \sqrt{\langle \hat{A}, B \rangle (-1)^{n-m}} U_{A, n} * K_{\alpha, m}
\]
\[
= \sqrt{\langle \hat{A}, B \rangle (-1)^{n-m}} \dfrac{1}{\langle U_{A, n}, K_{\alpha, m} \rangle_r} U_{A, n} \circ K_{\alpha, m}
\]
\[
= \sqrt{\langle \hat{A}, B \rangle (-1)^{n-m}} U_{A, n} \circ K_{\alpha, m}
\]

for any \(m, n \in \mathbb{Z}\).
And for the relation \(4.33\), we have

\[
U_{B,n} \circ U_{A,n+1} = \langle U_{B,n}, U_{A,n+1} \rangle \cdot U_{B,n} * U_{A,n+1}
\]

\[
= \sum_{M,N \in \text{Iso}(A)} \gamma_{AB}^{MN} \frac{a_{AB} a_{MN}}{a_{M} a_{N}} U_{N,n+1} * U_{M,n} * K_{B-M,n+1}
\]

\[
= \sum_{M,N \in \text{Iso}(A)} \gamma_{AB}^{MN} \frac{a_{AB} a_{MN}}{a_{M} a_{N}} (U_{N,n+1}, U_{M,n}) (U_{N,n}, K_{B-M,n+1}) \langle U_{N,n+1}, K_{B-M,n+1} \rangle
\]

\[
= \sum_{M,N \in \text{Iso}(A)} \gamma_{AB}^{MN} \frac{a_{AB} a_{MN}}{a_{M} a_{N}} \sqrt{(\tilde{M} - \tilde{N}, \tilde{M} - \tilde{B}) U_{N,n+1} \circ U_{M,n} \circ K_{B-M,n+1}}.
\]

For any \(m \geq n + 2\), we have

\[
U_{B,n} \circ U_{A,m} = \langle U_{B,n}, U_{A,m} \rangle \cdot U_{B,n} * U_{A,m}
\]

\[
= \sqrt{(\tilde{B}, \tilde{A}) (-1)^{m-n+1} (m-n+1) (\tilde{B}, \tilde{A}) (-1)^{m-n}} U_{A,m} * U_{B,n}
\]

\[
= \sqrt{(\tilde{B}, \tilde{A}) (-1)^{m-n+1} (m-n+1) (\tilde{B}, \tilde{A}) (-1)^{m-n}} U_{A,m} \circ U_{B,n}
\]

\[
= \sqrt{(\tilde{A}, \tilde{B}) (-1)^{n-m} (n-m+1)} U_{A,m} \circ U_{B,n}.
\]

**Theorem 4.10.** (1) There is an epimorphism \(\varphi : \text{MH}_{rtw}(A) \to \mathcal{L}_n(A)\), given by

\[
K_{\alpha,n} \mapsto K_{\alpha}^{(-1)^n}, U_{A,n} \mapsto Z_A^{(n)}
\]

for any \(A \in \text{Iso}(A)\), \(n \in \mathbb{Z}\) and \(\alpha \in K_0(A)\).

(2) \(\mathcal{L}_n(A) \cong \text{MH}_{rtw}(A)/I\), where \(I\) is the ideal of \(\text{MH}_{rtw}(A)\) generated by the set

\[
\{K_{\alpha,n+1} K_{\beta,n} - K_{\alpha-n,\beta,n+1} | \alpha, \beta \in K_0(A), n \in \mathbb{Z}\}.
\]

**Proof.** (1) We just need to prove that \(\varphi\) is a homomorphism. According to Proposition 4.9 and Proposition 4.10, it suffices to prove that the relations \(4.30\) and \(4.31\) are satisfied.

For any \(\alpha, \beta \in K_0(A)\) and \(n, m \in \mathbb{Z}\), we have

\[
\varphi(K_{\alpha,n}) \varphi(K_{\beta,m}) = K_{\alpha}^{(-1)^n} K_{\beta}^{(-1)^m}
\]

\[
= K_{\alpha}^{(-1)^m} K_{\beta}^{(-1)^n}
\]

\[
= \varphi(K_{\beta,m}) \varphi(K_{\alpha,n}).
\]

In particular, for \(m = n\) we have

\[
\varphi(K_{\alpha,n}) \varphi(K_{\beta,n}) = K_{\alpha}^{(-1)^n} K_{\beta}^{(-1)^n}
\]

\[
= K_{\alpha + \beta}^{(-1)^n}
\]

\[
= \varphi(K_{\alpha + \beta,n}).
\]

On the other hand, for any \(A \in \text{Iso}(A)\), we have

\[
\varphi(K_{\alpha,m}) \varphi(U_{A,n}) = K_{\alpha}^{(-1)^m} Z_A^{(n)}
\]

\[
= \sqrt{(\tilde{A}) (-1)^n Z_A^{(n)} K_{\alpha}^{(-1)^m}}
\]

\[
= \sqrt{(\tilde{A}) (-1)^{n-m} Z_A^{(n)} K_{\alpha}^{(-1)^m}}
\]

\[
= \varphi(U_{A,n}) \varphi(K_{\alpha,m}).
\]
(2) It is equivalent to prove that $\text{Ker}(\varphi) = I$, and it is obviously that $I \subseteq \text{Ker}(\varphi)$. One can easily see that the image of the basis described in Proposition 2.2 is the generators of the lattice algebra $L_s(A)$.

We claim that if
\[ K_{\alpha_{r-1},r} \circ \cdots \circ K_{\alpha_l,l+1} \circ U_{A^r,r} \circ \cdots \circ U_{A^l,l} - K_{\beta_{r-1},r} \circ \cdots \circ K_{\beta_l,l+1} \circ U_{A^r,r} \circ \cdots \circ U_{A^l,l} \in \text{Ker}(\varphi) \]
for any two elements
\[ K_{\alpha_{r-1},r} \circ \cdots \circ K_{\alpha_l,l+1} \circ U_{A^r,r} \circ \cdots \circ U_{A^l,l} \]
and
\[ K_{\beta_{r-1},r} \circ \cdots \circ K_{\beta_l,l+1} \circ U_{A^r,r} \circ \cdots \circ U_{A^l,l} \]
in the basis of $\mathcal{M}_{\text{Herm}}(A)$, then
\[ K_{\alpha_{r-1},r} \circ \cdots \circ K_{\alpha_l,l+1} \circ U_{A^r,r} \circ \cdots \circ U_{A^l,l} - K_{\beta_{r-1},r} \circ \cdots \circ K_{\beta_l,l+1} \circ U_{A^r,r} \circ \cdots \circ U_{A^l,l} \in I. \]
It is clear that they must satisfy the following condition
\[ (-1)^{r-1} + \cdots + (-1)^{l+1} \alpha_l = (-1)^{r-1} + \cdots + (-1)^{l+1} \beta_l. \]
And it is induced by the following identity
\[ K_{\alpha_{r-1},r} \circ \cdots \circ K_{\alpha_l,l+1} = K_{\beta_{r-1},r} \circ \cdots \circ K_{\beta_l,l+1} \]
that
\[ K_{\alpha_{r-1},r}K_{\alpha_{r-2},r-1} \cdots K_{\alpha_l,l+1} = K_{\beta_{r-1},r}K_{\beta_{r-2},r-2} \cdots K_{\beta_l,l+1} \]
and $K_{\alpha_{r-1},r}K_{\alpha_{r-2},r-1} \cdots K_{\alpha_l,l+1} - K_{\beta_{r-1},r}K_{\beta_{r-2},r-2} \cdots K_{\beta_l,l+1} \in \text{Ker}(\varphi)$.

Thus one can get that
\[ K_{\alpha_{r-1},r} \circ \cdots \circ K_{\alpha_l,l+1} - K_{\beta_{r-1},r} \circ \cdots \circ K_{\beta_l,l+1} \in I \]
by induction on $r - l$. Hence
\[ K_{\alpha_{r-1},r} \circ \cdots \circ K_{\alpha_l,l+1} \circ U_{A^r,r} \circ \cdots \circ U_{A^l,l} = K_{\beta_{r-1},r} \circ \cdots \circ K_{\beta_l,l+1} \circ U_{A^r,r} \circ \cdots \circ U_{A^l,l} \in I. \]
For any
\[ x = \sum_{(\alpha_{r-1}, \ldots, \alpha_l, A^r, \ldots, A^l)} a_{\alpha_{r-1}, \ldots, \alpha_l, A^r, \ldots, A^l} K_{\alpha_{r-1},r} \circ \cdots \circ K_{\alpha_l,l+1} \circ U_{A^r,r} \circ \cdots \circ U_{A^l,l} \in \text{Ker}(\varphi), \]
it is clear that
\[ x = \sum_{(A^r, \ldots, A^l)} \left( \sum_{(\alpha_{r-1}, \ldots, \alpha_l)} a_{\alpha_{r-1}, \ldots, \alpha_l, A^r, \ldots, A^l} K_{\alpha_{r-1},r} \circ \cdots \circ K_{\alpha_l,l+1} \right) \circ U_{A^r,r} \circ \cdots \circ U_{A^l,l}. \]
And it follows from the basis of $L_s(A)$ given in Proposition 4.3 and the definition of $\varphi$ that for any $A^r, \cdots, A^l \in \text{Iso}(A)$ we can get that
\[ \sum_{(\alpha_{r-1}, \ldots, \alpha_l)} a_{\alpha_{r-1}, \ldots, \alpha_l, A^r, \ldots, A^l} K_{\alpha_{r-1},r} \circ \cdots \circ K_{\alpha_l,l+1} \in \text{Ker}(\varphi). \]
In addition we have
\[ \sum_{(\alpha_{r-1}, \ldots, \alpha_l)} a_{\alpha_{r-1}, \ldots, \alpha_l, A^r, \ldots, A^l} K_{\alpha_{r-1},r} \circ \cdots \circ K_{\alpha_l,l+1} = \sum_{(\alpha_{r-1}, \ldots, \alpha_l)} a_{\alpha_{r-1}, \ldots, \alpha_l, A^r, \ldots, A^l} \left( K_{\alpha_{r-1},r} \circ \cdots \circ K_{\alpha_l,l+1} - K_{\alpha_{r-1},r} \circ \cdots \circ K_{\alpha_l,l+1} \right) + \sum_{(\alpha_{r-1}, \ldots, \alpha_l)} a_{\alpha_{r-1}, \ldots, \alpha_l, A^r, \ldots, A^l} K_{\alpha_{r-1},r} \circ \cdots \circ K_{\alpha_l,l+1} \circ U_{A^r,r} \circ \cdots \circ U_{A^l,l} \circ U_{A^r,r} \circ \cdots \circ U_{A^l,l} \in I, \]
so by above claim and using induction on the number of $K_{\alpha_l}$, one can obtain that
\[ \sum_{(\alpha_{r-1}, \ldots, \alpha_l)} a_{\alpha_{r-1}, \ldots, \alpha_l, A^r, \ldots, A^l} K_{\alpha_{r-1},r} \circ \cdots \circ K_{\alpha_l,l+1} \in I, \]
which implies $\text{Ker}(\varphi) \subseteq I$.

\[ \text{Remark 4.11. It is inferred from Theorem 3.3 and Theorem 4.17 that in some twisted case the lattice algebra is in fact the quotient algebra of the naive lattice algebra.} \]

5. The invariance of naive lattice algebras

In this section, we apply Theorem 5.3 to prove that the naive lattice algebra is invariant under derived equivalences. In particular, it suffices to show that the componentwise twisted modified Ringel-Hall algebra is invariant under derived equivalences.

We begin with the following result concerning derived Hall algebras.

**Proposition 5.1.** For any object \([A] = [\bigoplus A^{-i}[i]]\) in $\text{Iso}(D^b(A))$, we have

\[
[A] = \sqrt{\prod_{i<j} \langle \hat{A}^i, \hat{A}^j \rangle (-1)^{i-j} \prod_n Z_A^{[n]}}
\]

in $\mathcal{D} \text{H}_{ctw}(A)$, which is called the normal form of \([A]\) and in the following we simply denote it by $Z(A)$.

**Proof.** For any object \([A] = [\bigoplus A^{-i}[i]]\) of $\text{Iso}(D^b(A))$. Assume that $A$ is of the following form

\[
\cdots \rightarrow 0 \rightarrow A^l \rightarrow \cdots \rightarrow A^r \rightarrow 0 \rightarrow \cdots,
\]

where $A^l$ is the leftmost nonzero component and $A^r$ is the rightmost nonzero component, then the width of $A$ is defined to be $r - l + 1$. If $A$ is zero, then the width of $A$ is defined to be $0$. For simplicity set $l = 0$, and inductively we have

\[
\bigoplus_{i=0}^r Z_A^{[i]} A^i = \prod_{p \geq 0} |\text{Hom}_{D^b(A)}(A^r[p], \bigoplus_{i=0}^{r-1} A^i[r-i])|^{(-1)^p} \sqrt{\prod_{0 \leq i < r} \langle \hat{A}^r, \hat{A}^i \rangle (-1)^{r-i-1} Z_A^{[i]} \bigoplus \bigoplus_{i=0}^{r-1} Z_A^{[i]} (\bigoplus_{i=0}^{r-1} Z_A^{[i]})}.
\]

It is clear that there is an automorphism $T$ of $\mathcal{D} \text{H}_{ctw}(A)$ by setting $T(Z_A^{[n]}) = Z_A^{[n+1]}$.

**Remark 5.2.** Let $A$ and $B$ be two hereditary abelian categories. If there exists a derived equivalence $F : D^b(A) \rightarrow D^b(B)$, then $\mathcal{D} \text{H}_{ctw}(A)$ is isomorphic to $\mathcal{D} \text{H}_{ctw}(B)$ by setting

\[
Z_A^{[n]} \mapsto T^n(Z(F(A))).
\]

By applying the method in [8], it is not hard to see that

**Theorem 5.3.** There is an embedding $\iota : \mathcal{D} \text{H}_{ctw}(A) \rightarrow \mathcal{M} \text{H}_{ctw}(A)$, defined by

\[
Z_A^{[0]} \mapsto U_{A,0},
\]

\[
Z_A^{[n]} \mapsto \frac{1}{\sqrt{\langle \hat{A}, \hat{A} \rangle^n}} U_{A,n} * K_{-\hat{A},-n} * K_{-\hat{A},n-1} * \cdots * K_{-\hat{A},2} * K_{-\hat{A},1},
\]

and

\[
Z_A^{[-n]} \mapsto \sqrt{\langle \hat{A}, \hat{A} \rangle^n} U_{A,-n} * K_{-\hat{A},-n+1} * K_{-\hat{A},-n+2} * \cdots * K_{-\hat{A},1} * K_{-\hat{A},0} \text{ for any } n > 0.
\]

Following [27], we define the completely extended twisted derived Hall algebra $\mathcal{D} \text{H}_{ctw}^{ce}(A)$ of $\mathcal{D} \text{H}_{ctw}(A)$ and show that the above embedding $\iota$ can be extended to an isomorphism between $\mathcal{D} \text{H}_{ctw}^a(A)$ and $\mathcal{M} \text{H}_{ctw}(A)$. 
Definition 5.4. The completely extended twisted derived Hall algebra $\mathcal{DH}_{\text{ctw}}(A)$ is the associative and unital algebra generated by the set

$$\{Z_A^n, K_A^n | A \in \text{Iso}(A), \alpha \in K_0(A) \text{ and } n \in \mathbb{Z}\},$$

and subject to the relations (5.1)- (5.3) and (5.5)-(5.7).

Proof. Firstly, we can extend the embedding $\iota$ in Theorem 5.3 to a morphism $\tilde{\iota}$ between $\mathcal{DH}_{\text{ctw}}(A)$ and $\mathcal{MH}_{\text{ctw}}(A)$ by setting

$$\tilde{\iota}(Z_A^n) = \iota(Z_A^n) \text{ and } \tilde{\iota}(K_A^n) = K_{\alpha,n}.$$

Similarly, one can also check that $\tilde{\iota}$ is a homomorphism of algebras. To prove it is an isomorphism, we construct the inverse homomorphism $\eta : \mathcal{MH}_{\text{ctw}}(A) \to \mathcal{DH}_{\text{ctw}}(A)$

given by

$$K_{\alpha,n} \mapsto K_A^n,$$

and for $n > 0$,

$$U_{A,0} \mapsto Z_A^0,$$

$$U_{A,n} \mapsto \sqrt{(A,A)^n}Z_A^n \triangleq K_A^{[n]} \triangleq K_A^{[n]} \triangleq K_A^{[n]} \triangleq K_A^{[n]}.$$
Proof. By Theorem \[5.3\] and Corollary \[5.6\] it suffices to prove that $\mathcal{DH}_{tw}^{ce}(A)$ is isomorphic to $\mathcal{DH}_{tw}^{ce}(B)$.

First of all, there is an isomorphism of Grothendieck group

$$K_0(A) \to K_0(D^b(A)) \to K_0(D^b(B)) \to K_0(B)$$

induced by $F$. By abuse of notation this isomorphism is still denoted by $F$ and it preserves the bilinear form, i.e. for any $\alpha, \beta \in K_0(A)$ we have

$$\langle \alpha, \beta \rangle_A = (F(\alpha), F(\beta))_B,$$

where $(-, -)_A$ and $(-, -)_B$ denote the Euler form of $K_0(A)$ and $K_0(B)$ respectively.

Then the induced map $F_* : \mathcal{DH}_{tw}^{ce}(A) \to \mathcal{DH}_{tw}^{ce}(B)$ is given by

$$Z^n_A[n] \mapsto T^n(Z(F(A)))$$

and $K^n_\alpha \mapsto K^n_{F(\alpha)}$, where $Z(F(A))$ is defined in Proposition \[5.3\]. It remains to verify that it is a homomorphism of algebras. However, by Remark \[5.2\] one can get that $F_*$ preserve the relations \[5.3\]-\[5.4\]. And it is easy to check that the relations \[5.2\], \[5.5\] and \[5.7\] are preserved.

Following \[9\] and \[2\], let $A_i$ be the full subcategory of $A$ with objects $\{A \in A | F(A) \subset B[i]\}$. For objects $A_i \in A_i, A_j \in A_j$, we have $\text{Hom}(A_i, A_j) = 0$ but for $j = i$ or $j = i + 1$, $\text{Ext}^1_A(A_i, A_j) = 0$ but for $j = i$ or $j = i - 1$. Thus

$$Z^n_{A_i \oplus A_j} = \sqrt{(A_i, A_j)}Z^n_{A_i} \Delta Z^n_{A_j},$$

for $A_i \in A_i, A_j \in A_j$ and $i < j$. Since any object $A \in A$ can be decomposed to a direct sum $\bigoplus_{i \in \mathbb{Z}} A_i$ with $A_i \in A_i$, and then for any $n \in \mathbb{Z}$, $Z^n_A$ can be written in the form of

$$Z^n_A = \prod_{i < j \in S} Z^n_{A_i} \prod_{i \in S} Z^n_{A_i},$$

where the indices in the product $\prod_{i < j} Z^n_{A_i}$ are in increasing order. Therefore we only need to check the relations \[5.3\], \[5.5\], \[5.4\] and \[5.6\] are preserved in the case that $A \in A_i$ and $\alpha \in K_0(A)$ for any $i \in \mathbb{Z}$.

Assume that $A \in A_{-1}$ and $F(A) = B[-i]$ for some object $B \in B$, then $Z(F(A)) = Z^n_B$. In the following we give the proof for the relation \[5.3\] and omit the others. We separate the proof into the following three cases.

Case (1): $n = 0$.

If $i = -1$,

$$F_*(K_0^{[0]} \triangle F_*(Z_0^{[0]})) = K_0^{[0]} \triangle Z_0^{[-1]}$$

$$= \frac{1}{\sqrt{(F(\alpha), B)}}Z_0^{[-1]} \triangle K_0^{[0]}$$

$$= \sqrt{(F(\alpha), F(A)[-1])}Z_0^{[-1]} \triangle K_0^{[0]}$$

$$= \sqrt{(\alpha, \hat{A})F_*(Z_0^{[0]}) \triangle F_*(K_0^{[0]}).}$$

If $i = 0$,

$$F_*(K_0^{[0]} \triangle F_*(Z_0^{[0]})) = K_0^{[0]} \triangle Z_0^{[0]}$$

$$= \sqrt{(F(\alpha), B)Z_0^{[0]} \triangle K_0^{[0]}}$$

$$= \sqrt{(\alpha, \hat{A})F_*(Z_0^{[0]}) \triangle F_*(K_0^{[0]}).}.$$
If $i = 1$,
\[
F_*(K_α^{[0]}) \triangle F_*(Z_A^{[0]}) = K_{F(α)}^{[0]} \triangle Z_B^{[1]}
\]
\[
= \frac{1}{\sqrt{(F(α), B)}} Z_B^{[1]} \triangle K_{F(α)}^{[0]}
\]
\[
= \sqrt{(α, \hat{A})F_*(Z_A^{[0]}) \triangle F_*(K_α^{[0]}))}.
\]

If $i > 1$ or $i < -1$,
\[
F_*(K_α^{[i]}) \triangle F_*(Z_A^{[i]}) = K_{F(α)}^{[0]} \triangle Z_B^{[i]}
\]
\[
= \frac{1}{\sqrt{(F(α), B)}(-1)^i} Z_B^{[i]} \triangle K_{F(α)}^{[0]}
\]
\[
= \sqrt{(α, \hat{A})F_*(Z_A^{[i]}) \triangle F_*(K_α^{[i]}))}.
\]

Case (2): $n = 1$.
If $i = -1$,
\[
F_*(K_α^{[1]}) \triangle F_*(Z_A^{[1]}) = K_{F(α)}^{[1]} \triangle Z_B^{[0]}
\]
\[
= \frac{1}{\sqrt{(F(α), B)}} Z_B^{[0]} \triangle K_{F(α)}^{[1]}
\]
\[
= \sqrt{(α, \hat{A})F_*(Z_A^{[1]}) \triangle F_*(K_α^{[1]}))}.
\]

If $i = 0$,
\[
F_*(K_α^{[1]}) \triangle F_*(Z_A^{[1]}) = K_{F(α)}^{[1]} \triangle Z_B^{[1]}
\]
\[
= \sqrt{(F(α), B)Z_B^{[1]} \triangle K_{F(α)}^{[1]}}
\]
\[
= \sqrt{(α, \hat{A})F_*(Z_A^{[1]}) \triangle F_*(K_α^{[1]}))}.
\]

If $i = 1$,
\[
F_*(K_α^{[1]}) \triangle F_*(Z_A^{[1]}) = K_{F(α)}^{[1]} \triangle Z_B^{[2]}
\]
\[
= \frac{1}{\sqrt{(F(α), B)}} Z_B^{[2]} \triangle K_{F(α)}^{[1]}
\]
\[
= \sqrt{(α, \hat{A})F_*(Z_A^{[1]}) \triangle F_*(K_α^{[1]}))}.
\]

If $i > 1$ or $i < -1$,
\[
F_*(K_α^{[1]}) \triangle F_*(Z_A^{[1]}) = K_{F(α)}^{[1]} \triangle Z_B^{[i+1]}
\]
\[
= \frac{1}{\sqrt{(F(α), B)(-1)^{i+1}} Z_B^{[i+1]} \triangle K_{F(α)}^{[0]}
\]
\[
= \sqrt{(α, \hat{A})F_*(Z_A^{[1]}) \triangle F_*(K_α^{[1]}))}.
\]

Case (3): $n \neq 0, 1$.
\[
F_*(K_α^{[n]}) \triangle F_*(Z_A^{[n]}) = K_{F(α)}^{[n]} \triangle Z_B^{[n+1]}
\]
\[
= Z_B^{[n+1]} \triangle K_{F(α)}^{[n]}
\]
\[
= F_*(Z_A^{[n]}) \triangle F_*(K_α^{[n]}).
\]

This completes the proof.
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Department of Mathematics, Sichuan University, Chengdu 610064, P.R.CHINA and Department of Mathematics and Statistics, Fuyang Normal University, Fuyang 236037, P.R.CHINA

E-mail address: jlin@fync.edu.cn