Derivation of Source-Free Maxwell and Gravitational Radiation Equations by Group Theoretical Methods

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Abstract

We derive source-free Maxwell-like equations in flat spacetime for any helicity \( j \) by comparing the transformation properties of the \( 2(2j + 1) \) states that carry the manifestly covariant representations of the inhomogeneous Lorentz group with the transformation properties of the two helicity \( j \) states that carry the irreducible representations of this group. The set of constraints so derived involves a pair of curl equations and a pair of divergence equations. These reduce to the free-field Maxwell equations for \( j = 1 \) and the analogous equations coupling the gravito-electric and the gravito-magnetic fields for \( j = 2 \).

1 Introduction

The linearized field equations for the Weyl curvature tensor on flat spacetime are well known to be very similar in form to Maxwell’s equations, evoking an analogy referred to as gravitoelectromagnetism [1]. Both sets of equations can be derived with the same approach based on eliminating the gauge modes of spin 1 and spin 2 fields using Fourier analysis and group representation theory. In this way one sees the common origin of the curl and divergence operators for vector fields and symmetric tracefree second rank tensor fields and their intertwining to form the Maxwell combinations. Here we examine the source-free field equations to focus on those differential operators for arbitrary spin.

The electromagnetic field is described in two different ways:

(i) Classical Formulation. A field is introduced having appropriate transformation properties. Not every field represents a physically allowed state. The non-physical fields must be annihilated by appropriate equations (constraints).

(ii) Hilbert space (Quantum) Formulation. An arbitrary superposition of states in this space represents a physically allowed state. But that field does not have obvious transformation properties.

For the first formulation the field is required to be “manifestly covariant.” This requires there to be a certain number of field components at every space-time point, or more conveniently, for every allowed momentum vector. In the Hilbert space formulation the number of independent components is just the allowed number of spin or helicity states.

When the number of independent field components is less than the number of components required to define the “manifestly covariant” field, there are linear combinations of these components that cannot represent physically allowed
states. The function of the field equations (constraints) is to suppress these linear combinations of components that do not correspond to physical states. Maxwell equations and the gravitational radiation equations fulfill this function.

Classically, the electromagnetic field is described by six field components: $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$, or their components after Fourier transformation, $\vec{E}(k)$ and $\vec{B}(k)$, where $k$ is a 4-vector such that $k \cdot k = \vec{k} \cdot \vec{k} - k_4 k_4 = 0$, where $\vec{k}$ is a 3-momentum vector and $k_4$ is an energy. The quantum description involves arbitrary superposition of two helicity components for each momentum vector. Then we have four linear combinations of classical field components that must be suppressed for each $k$-vector and that are annihilated by Maxwell’s equations. By comparing the transformation properties of the basis vectors for the manifestly covariant but non-unitary representations of the inhomogeneous Lorentz group, with the basis vectors for its unitary irreducible but not manifestly covariant representations, we obtain a set of constraint equations. These reduce, for $j = 1$, to Maxwell’s equations and, for $j = 2$, to gravitational radiation equations, both in free space. These equations for the gravito-electric field and the gravito-magnetic field have a structure identical to the structure of Maxwell’s equations.

The constraint equations play an important role in this derivation since they preserve the physical states in the theory but annihilate the non-physical ones so that the physics of massless particles can be described properly.

2 Inhomogeneous Lorentz Group (ILG)

The group of inhomogeneous Lorentz transformations $\{\Lambda, a\}$ has two important subgroups. These are the subgroup of homogeneous Lorentz transformations $\{\Lambda, 0\}$ and the invariant subgroup of translations $\{I, a\}$. Both their representations play a role in the derivation of relativistically covariant field equations.

2.1 Translations $\{I, a\}$

All of its unitary irreducible representations are one-dimensional, and in fact

$$\Gamma^k(\{I, a\}) = e^{i k \cdot a},$$

where $k$ and $a$ are 4-vectors and $k$ parameterizes the one-dimensional representations. We may define a basis state for the one-dimensional representation $\Gamma^k$ of $\{I, a\}$ as $|k\rangle$:

$$\{I, a\} |k\rangle = e^{i k \cdot a} |k\rangle.$$  \hspace{1cm} (2)

Physically, $k$ (or $\hbar k$) has a natural interpretation as the 4-momentum of the photon.

2.2 Homogeneous Lorentz Transformation $\{\Lambda, 0\}$

We use the Minkowski metric tensor $\eta_{\mu\nu} = \text{diag}(+1, +1, +1, -1)$ for this transformation. Any element in $SO(3, 1)$ can be expressed in a $(2j + 1)(2j' + 1)$-
dimensional representation $D^{jj'}$ as follows
\[ D^{jj'}(Λ) = \exp(\vec{θ} \cdot \vec{L} + \vec{b} \cdot \vec{B}) = \exp[\{(i\vec{θ} - \vec{b}) \cdot \vec{J}^{(1)} + (i\vec{θ} + \vec{b}) \cdot \vec{J}^{(2)}\}] = D^{j}(\vec{θ} - \vec{b}) \cdot \vec{J}^{(1)} | D^{j}(\vec{θ} + \vec{b}) \cdot \vec{J}^{(2)}\],
\]
where $Λ$ is the $4 \times 4$ Lorentz transformation matrix and the operators $\vec{L}$ generate rotations and $\vec{B}$ generate boosts. The operator $\vec{J}^{(1)}$ has $2j + 1$ dimensional representations $D^{j}$ while $\vec{J}^{(2)}$ has $2j' + 1$ dimensional representations $D^{j'}$ and they are given by
\[ \vec{J}^{(1)} = \frac{1}{2}(-i\vec{L} - \vec{B}), \quad \vec{J}^{(2)} = \frac{1}{2}(-i\vec{L} + \vec{B}). \]
These operators satisfy angular momentum commutation relations.

The Lie algebra so(3,1) is isomorphic to the Lie algebra for the group of $2 \times 2$ matrices $SL(2; C)$. There are two isomorphisms which give rise to the two matrix representations: $D^{j0} : \vec{B} = i\vec{L} \rightarrow \vec{J}^{(j)}$ and $D^{j0j'} : \vec{B} = -i\vec{L} \rightarrow \vec{J}^{(j')}$ for any $j$. Here $\vec{J}^{(j)}$ are the $(2j + 1) \times (2j + 1)$ angular momentum matrices and $\vec{L} = i\vec{J}^{(j)}$ which will be useful later.

The representations of the connected component of the homogeneous Lorentz group $SO(3,1)$ are described by two angular momentum indices $j$ and $j'$ which describe each of the two representations $\vec{J}^{(1)}$ and $\vec{J}^{(2)}$ respectively. The most general representation of the group $SO(3,1)$ is
\[ D^{jj'}(\vec{θ} \cdot \vec{L} + \vec{b} \cdot \vec{B}) = D^{j0}[\vec{θ} \cdot \vec{L} + \vec{b} \cdot \vec{B}]D^{j0j'}[\vec{θ} \cdot \vec{L} + \vec{b} \cdot \vec{B}]. \]

The first representation $D^{j0}$ is obtained using the first isomorphism. In a similar way we obtain the second representation $D^{j0j'}$. Thus, $D^{jj'}$ becomes
\[ D^{jj'}(\vec{θ} \cdot \vec{L} + \vec{b} \cdot \vec{B}) = \exp[\{(i\vec{θ} - \vec{b}) \cdot \vec{J}^{(j)}\}] \exp[\{(i\vec{θ} + \vec{b}) \cdot \vec{J}^{(j')}\}]. \]
The action of $Λ$ on the basis $| j \; m \rangle \; j' \; m' \rangle$ of the total space $ξ(j, j')$ through the representation $D^{jj'}(Λ)$ is expressed by
\[ Λ \; | j \; m \rangle \; j' \; m' \rangle = | j \; l \rangle \; j' \; l' \rangle D_{ll',mm'}^{jj'}(Λ). \]

### 2.3 Representations of the Group $O(3,1)$

The full homogeneous Lorentz group $O(3,1)$ consists of those operations that are connected to the identity, that are in the subgroup $SO(3,1)$, together with the parity and time reversal operations $P$ and $T$ and their product $PT = TP$. The action of these operations on the coordinates are:
\[ P(x, y, z, ct) = (-x, -y, -z, +ct), \quad T(x, y, z, ct) = (+x, +y, +z, -ct). \]
The action of these discrete operations on the generators of infinitesimal transformations $\vec{L} = i\vec{J} (L_i = ε_{ijk}x^j\partial_k)$ and $\vec{B} = i\vec{K} (B_i = -x^j\partial_4 - x^4\partial_j)$ are
\[ PLP^{-1} = \vec{L}, \quad PBP^{-1} = -\vec{B}, \quad TLT^{-1} = \vec{L}, \quad TBT^{-1} = -\vec{B}. \]
The full homogeneous Lorentz group consists of four components, each similar to \( SO(3, 1) \). The full group can be written as follows:

\[
O(3, 1) = \{I, P, T, PT\} \otimes SO(3, 1).
\] (9)

The four operators \( \{I, P, T, PT\} \) form a discrete invariant subgroup of \( O(3, 1) \) and act as coset representatives for the quotient group \( O(3, 1)/SO(3, 1) \). The four components of the group \( O(3, 1) \) are connected to the group operations: \( I, P, T, PT \).

The discrete operations \( P, T \) interchange the operators \( \vec{J}(1) \) and \( \vec{J}(2) \). As a result, the matrices \( D_{jj'} = D_{j0} \otimes D_{0j'} \) can represent the connected component of the homogeneous Lorentz group \( SO(3, 1) \), but cannot represent the full homogeneous Lorentz group \( O(3, 1) \) unless \( j = j' \). If \( j \neq j' \) the matrix representation of the full group consists of the direct sum of the two matrix representations \( D_{j0} \oplus D_{j'0} \).

In the following sections we will be particularly interested in describing massless particles with helicity \( j \) (integer) in terms of the representations \( D_{j0} \oplus D_{0j} \) or \( D_{j\pm} = D_{j0} \otimes D_{0\pm} \). The former class of representations act on states (fields) with \( 2(2j+1) \) components while the latter act on states (potentials) with \( (j+1)^2 \) components. For the photon with helicity \( j = 1 \) there are six components (the three components of the electric field and the magnetic field, \( \vec{E} \) and \( \vec{B} \)), and four components (of the vector potential \( A_\mu \)), respectively. These observations are usefully summarized:

| Particle | Photon | Graviton |
|----------|--------|----------|
| \( j \)  | \( D_{j0} \oplus D_{0j} \) | \( D_{j2} \oplus D_{j20} \) |
| \( \vec{E}, \vec{B} \) | \( Q_E, Q_B \) | \( h_{\mu\nu} \) |

For the photon, description of the electromagnetic field is in terms of the fields \( \vec{E}, \vec{B} \) which carry the representation \( D_{j0} \oplus D_{0j} \) with \( j = 1 \), or in terms of the vector potential \( A_\mu \), which carries the representation \( D_{j2} \oplus D_{j20} \), also with \( j = 1 \). The fields \( \vec{E}, \vec{B} \) are tracefree dipole fields.

For the graviton, description of the gravitational field is in terms of the fields \( Q_E, Q_B \) which carry the representation \( D_{j0} \oplus D_{0j} \) with \( j = 2 \), or in terms of the linear metric perturbation \( h_{\mu\nu} \), which is a symmetric tracefree tensor representing the gravitational potential and carries the representation \( D_{j2} \oplus D_{j20} \), also with \( j = 2 \). The fields \( Q_E, Q_B \) are tracefree quadrupole fields, each with five components.

3 Representations of the ILG

We construct two kinds of representations for the ILG. These are the manifestly covariant representations and the unitary irreducible representations [7].
3.1 Manifestly Covariant Representations

We construct manifestly covariant representation of the ILG by constructing direct products of the basis vectors

$$|k\rangle \otimes |j \quad j' \rangle = |k\rangle \otimes |j \quad j' \rangle$$

for the subgroups of the ILG.

The action of the ILG on these direct product bases is defined by the action of the two subgroups, homogeneous Lorentz transformation and translations, on the momentum basis states and field component basis states separately:

- The action of the $\{I, a\}$ on these basis states is defined by
  $$\{I, a\} |k\rangle = |k\rangle e^{ik \cdot a},$$
  $$\{I, a\} |j \quad j' \rangle = |j \quad j' \rangle \delta_{l' l} \delta_{m' m}.$$ 

- The action of $\{\Lambda, 0\}$ on these basis states is defined, from (11) and (6) respectively, by
  $$\{\Lambda, 0\} |k\rangle = |\Lambda k\rangle,$$
  $$\{\Lambda, 0\} |j \quad j' \rangle = |j \quad j' \rangle D_{j l' j' l} (\Lambda).$$ 

3.2 Unitary Irreducible Representations

We have a representation of $\{I, a\}$ that is unitary and irreducible. For the subgroup $\{I, a\}$ this reduces to a direct sum of irreducibles $\Gamma_k (\{I, a\})$. The basis states are $|k; \xi\rangle$, so that

$$\{I, a\} |k; \xi\rangle = |k; \xi\rangle e^{ik \cdot a},$$

where $\xi$ is a helicity for different states with the same 4-momentum.

The action of the subgroup $\{\Lambda, 0\}$, by using (15) and the invariance of the inner product, on the basis states $|k; \xi\rangle$ is defined by

$$\{\Lambda, 0\} |k; \xi\rangle = |\Lambda k; \xi'\rangle M_{\xi \xi'} (\Lambda),$$

where the matrix $M_{\xi \xi'} (\Lambda)$ remains to be determined.

We consider only the case of zero mass particles in the present discussion. To construct the matrix $M(\Lambda)$, we choose one particular 4-vector $k^0$ for this case

$$k \cdot k = 0, \quad k \neq 0, \quad k^0 = (0, 0, 1, 1)$$

where $T$ is the time reversal operator and the vector $k^0$ is called the little vector $[7]$. 

5
The effect of a homogeneous Lorentz transformation on the state \( |k^0; \xi \rangle \) is determined by writing \( \Lambda \) as a product of two group operations

\[
\Lambda = C_k H_{k^0},
\]

where \( H_{k^0} \) is the stability subgroup (little group [7-9]) of the little vector \( k^0 \)

\[
H_{k^0} k^0 = k^0.
\]

For our case the little group is \( ISO(2) \), and \( C_k \) is a coset representative

\[
C_k k^0 = k = \Lambda k^0.
\]

The generators of the little group \( H_{k^0} \) are defined by

\[
H_{k^0} = I_4 + \alpha G,
\]

and the generator \( G \) is given by

\[
G = \begin{pmatrix}
0 & -\theta_3 & -\theta_2 & -\theta_1 \\
-\theta_3 & 0 & -\theta_1 & \theta_2 \\
-\theta_2 & -\theta_1 & 0 & -\theta_3 \\
-\theta_1 & \theta_2 & \theta_3 & 0
\end{pmatrix} = \vec{\theta} \cdot \vec{L} + \vec{b} \cdot \vec{B}. \tag{21}
\]

From (18), an arbitrary element in this Lie subgroup \( H_{k^0} \) acting on \( k^0 \) must leave \( k^0 \) invariant, therefore

\[
G k^0 = 0. \tag{22}
\]

Consequently, the stability subalgebra is defined by: \( b_3 = 0, b_2 = \theta_1, b_1 = -\theta_2 \).

The generator \( G \) in this subalgebra becomes

\[
G = G_H = \theta_1 Y_1 + \theta_2 Y_2 + \theta_3 Y_3, \tag{23}
\]

where \( Y_1 = L_1 + B_2, Y_2 = L_2 - B_1, Y_3 = L_3 \). These operators satisfy the commutation relations for the group \( ISO(2) \) (inhomogeneous motions of the Euclidean plane \( R^2 \)).

The action of little group on the subspace of states \( |k^0; \xi \rangle \) is

\[
H_{k^0} |k^0; \xi \rangle = |H_{k^0} k^0; \xi' \rangle D_{\xi' \xi}(H_{k^0}) = |k^0; \xi' \rangle D_{\xi' \xi}(H_{k^0}). \tag{24}
\]

The original representation of the ILG is unitary and irreducible if and only if the representation \( D_{\xi' \xi}(H_{k^0}) \) of the little group is unitary and irreducible.

We construct the unitary and irreducible representation of \( ISO(2) \) following the method of little groups. Since \( ISO(2) \) has a two-dimensional translation invariant group, basis states in a unitary irreducible representation can be labeled by a vector \( \kappa = (\kappa_1, \kappa_2) \) in a 2-dimensional Euclidean space, \( \kappa \in R^2, \kappa \cdot \kappa \geq 0 \). The invariant length of \( \kappa \) parameterizes the representation.

Physically, we require \( \kappa = 0 \) [7-9]. With this, the physically allowable representation of the little group (\( \theta_1 \to 0, \theta_2 \to 0 \)) is

\[
D_{\xi' \xi}(H_{k^0}) = \exp(i \theta_1 Y_1 + \theta_2 Y_2 + \theta_3 Y_3) = \exp(i \theta_3 J_3) = \exp(i \xi \theta_3) \delta_{\xi' \xi}, \tag{25}
\]
where $J_3$ is the $(2j + 1)$-dimensional angular momentum diagonal matrix and $\xi$ is integer or half-integer ($-j \leq \xi \leq j$).

The action of an arbitrary element of the ILG on any state in this Hilbert space is

$$\{\Lambda, a\} |k; \xi\rangle = \{\Lambda, 0\} \{I, \Lambda^{-1}a\} |k; \xi\rangle.$$ 

By using the coset representative $C_k$ that permutes the 4-vector subspaces: $C_k |k^0; \xi\rangle = |k; \xi\rangle$, and \(25\) the last equation becomes

$$\{\Lambda, a\} |k; \xi\rangle = |k'; \xi\rangle e^{i\xi\Theta} e^{ia\Lambda k},$$

where

$$H_{k^0} = C_k^{-1} \Lambda C_k \rightarrow e^{i\xi\Theta}.$$ 

Thus, we have obtained the action of the homogeneous and the ILG on a state $|k, \xi\rangle$ in the Hilbert space in terms of their unitary irreducible representation by using the little group $H_{k^0}$. It will be useful to compare this quantum description with the classical one.

## 4 Transformation Properties

To compare these two kinds of representations for massless particles we compare transformation properties of their states in Table 1, where $\theta_± = \theta_1 \pm i\theta_2$ and $J_3 = J_1 \pm iJ_2$. By comparing (a) and (b) with (c) we conclude that the state $|k^0; \frac{j}{m} \frac{0}{0}\rangle$ transforms identically to $|k^0; \xi\rangle$ if $j = \xi$ and $\xi > 0$. The state $|k^0; \frac{j}{m} \frac{0}{0}\rangle$ transforms identically to $|k^0; \xi\rangle$ if $j' = -\xi$ and $\xi < 0$.

Therefore, when $j = \xi > 0$, the state $|k^0; \frac{j}{m} \frac{0}{0}\rangle$ is the unique physical state in the manifestly covariant representation. The remaining states are superfluous (non-physical states). Thus, the amplitudes $\langle k^0; \frac{j}{m} \frac{0}{0} | \psi \rangle$ of the states

$$|k^0; \frac{j}{m} \frac{0}{0}\rangle,$$

for $m \neq j$, all vanish because they are all non-physical states which are required in the manifestly covariant representation but are not present in the Hilbert space that carries the unitary irreducible representation.

A simple linear way to enforce this condition on the non-physical amplitudes is to require

$$k_3^0 \langle j^f^3 \rangle - j I_{2j+1} \langle k^0; \frac{j}{m} \frac{0}{0} | \psi \rangle = 0.$$ (27)

The matrix within the bracket $\{ \}$ is diagonal, with the elements $(j-j)k_3^0 = 0$ multiplying the physically allowed amplitude $\langle k^0; \frac{j}{m} \frac{0}{0} | \psi \rangle$. Therefore this amplitude is arbitrary. The non-zero elements $(m-j)k_3^0$ multiplying the non-physical amplitudes $\langle k^0; \frac{j}{m} \frac{0}{0} | \psi \rangle$, $m \neq j$, absent in the description of a physical state, require that these amplitudes be null. For $\xi < 0$, by the same argument, we obtain an equation similar to (27).

By using the coset operator $C_k$: $C_k |k^0; \xi\rangle = |k; \xi\rangle$ and $C_k |k^0; \frac{j}{m} \frac{0}{0}\rangle = |k| \langle \frac{j}{m} \frac{j'}{m'} | D_{12,mm'}^{j'}(C_k)$ we can express the constraint equation (27) for any $k = (k_1, k_2, k_3, k_4)$, $k_1^2 + k_2^2 + k_3^2 = k_4^2$. The condition on the amplitudes in
Table 1: Manifestly covariant versus unitary irreducible representations.

| Classical | Quantum |
|-----------|---------|
| **Manifest. Covariant. Rep.** | **Unit. Irreducible. Rep.** |
| basis state: $|k⟩\left\{ \begin{array}{cc} j \end{array} _m, j' \end{array} _{m'}, k = \Lambda k^0 \right.$ | basis state: $|k; \xi⟩, k = \Lambda k^0$ |
| $\{H_{k^0}, 0\}|k^0⟩\left\{ \begin{array}{cc} j \end{array} _m, j' \end{array} _{m'} = |k^0⟩|j, j'⟩D^{j'}_{m, mm'}(H_{k^0})$ | $\{H_{k^0}, 0\}|k^0, \xi⟩ = |k^0, \xi⟩e^{i\xi \Theta}$ (c) |
| for $j' = 0 \; \vec{B} = i\vec{L}$ | $D^{j0}(H_{k^0}) = \exp[i\theta_3 J_3^{(j)} + i\theta_- J_-^{(j)}]$ |
| $\{H_{k^0}, 0\}|k^0⟩\left\{ \begin{array}{cc} j \end{array} _m, 0 \right.$ $\rightarrow$ $|k^0⟩\left\{ \begin{array}{cc} j \end{array} _m, 0 \right.$ $e^{i\xi \Theta}$ (a) | $e^{i\xi \Theta} = \exp(\Theta Y_3 + \theta_1 Y_1 + \theta_2 Y_2)$ |
| $D^{j'}(H_{k^0}) = \exp[i\theta_3 J_3^{(j')} + i\theta_+ J_+^{(j')}$ | for $j = 0 \; \vec{B} = -i\vec{L}$ |
| if $j' = -m' = -\xi, \xi < 0$ | $\{H_{k^0}, 0\}|k^0⟩\left\{ \begin{array}{cc} 0 \end{array} _{0, j'} \right.$ $\rightarrow$ $|k^0⟩\left\{ \begin{array}{cc} 0 \end{array} _{0, j'} \right.$ $e^{i\xi \Theta}$ (b) |
| $\langle \vec{J}^{(j)} \cdot \vec{K} - j k_4 I_{2j+1} \rangle |k; \left\{ \begin{array}{cc} j \end{array} _m, 0 \right.$ $|\psi⟩ = 0, \quad (28)$ |

The subspace $|k⟩$ is related to condition in the subspace $|k^0⟩$ by a similarity transformation. Considering $C_k$ as the product of a boost $B_z$ in the $z$ direction followed by a rotation $R(\vec{k})$, the constraint equation for any $k$ and $\xi > 0$ becomes

$$\langle \vec{J}^{(j)} \cdot \vec{K} - j k_4 I_{2j+1} \rangle |k; \left\{ \begin{array}{cc} j \end{array} _m, 0 \right.$ $|\psi⟩ = 0, \quad (28)$$

where $\vec{J}^{(j)}$ are the three $(2j + 1) \times (2j + 1)$ angular momentum matrices. For any $\xi < 0$, we obtain a similar constraint equation.

## 5 The Constraint Equation

The constraint equation is conveniently expressed in the coordinate rather than the momentum representation by inverting the original Fourier transform that brought us from the coordinate to the momentum representation. Since $e^{ik \cdot x} = e^{ik \cdot x'} = e^{i(k \cdot \vec{x} - k_4 ct)}$, we can replace $\vec{k} \rightarrow \vec{k} \cdot \nabla$ and $k_4 \rightarrow -\frac{1}{c} \frac{\partial}{\partial t}$. The Fourier inversion is explicitly

$$\langle k| x \rangle \left\{ \begin{array}{cc} \vec{J}^{(j)} \cdot \frac{\vec{x}}{i} + j \frac{1}{i} \frac{\partial}{\partial (ct)} I_{2j+1} \right\} \langle x| k \rangle \langle k; \left\{ \begin{array}{cc} j \end{array} _m, 0 \right.$ $|\psi⟩ = 0, \quad (29)$$
We define complex spherical tensor fields

\[ \psi_{jm} = \langle x | k \rangle \langle k | m \rangle | \psi \rangle = T_{E}^{(j)}(x) + iT_{B}^{(j)}(x), \]

(30)

where \(-j \leq m \leq j; x = (\vec{x}, t)\) and \(T_{E}^{(j)}\) and \(T_{B}^{(j)}\) are a spherical tensor and pseudo-tensor respectively, of rank \(j\) with \(2j + 1\) components each. Thus, in coordinate space, the constraint equation becomes

\[
\left\{ \frac{1}{i} \vec{J}^{(j)} \cdot \nabla + \frac{1}{i} \frac{\partial}{\partial(ct)} I_{2j+1} \right\} \left( T_{E}^{(j)} + iT_{B}^{(j)} \right) = 0. \tag{31}
\]

We define the curl operator in a new way [10]: \(\text{curl} = \frac{1}{i} \vec{J}^{(j)} \cdot \nabla\), which is equivalent to the standard definition of the curl operator in three dimensions. With this definition we obtain

\[
\left\{ \text{curl} - \frac{1}{c} \frac{\partial}{\partial t} I_{2j+1} \right\} \left( T_{E}^{(j)} + iT_{B}^{(j)} \right) = 0. \tag{32}
\]

Finally, we take the real and imaginary parts of this equation and obtain:

\[
\text{curl}T_{E}^{(j)} + \frac{1}{c} \frac{\partial}{\partial t} T_{B}^{(j)} = 0
\]

\[
\text{curl}T_{B}^{(j)} - \frac{1}{c} \frac{\partial}{\partial t} T_{E}^{(j)} = 0. \tag{33}
\]

Thus we conclude that the zero mass particles with helicity \(j\) can be described, at least in free space, by a pair of real coupled, interacting, oscillating spherical tensor fields with \((2j + 1)\) components each.

5.1 Symmetries of the Spherical Tensor Fields

For integer \(j\), the fields \(T_{E}^{(j)}\) and \(T_{B}^{(j)}\) are tracefree rank-\(j\) tensor and pseudo-tensor fields, respectively. They transform under \(D^{(j)}(SO(3))\) of the proper rotation group \(SO(3)\), and each has \((2j + 1)\) components. Their transformation properties under the discrete group operations \(P\) and \(T\) are summarized in this table:

| Discrete Operation | \(T_{E}^{(j)}\) | \(T_{B}^{(j)}\) |
|-------------------|----------------|----------------|
| \(P\)             | \((-1)^j\)     | \((-j)^j\)     |
| \(T\)             | \((-1)^j\)     | \((-j)^j\)     |

Under these transformation properties, the curl equations are invariant under the subgroup of discrete operations: \(\{I, P, T, PT\}\).

6 The Source Free Equations

Eq. (33), or (32), describes the source-free Maxwell equations for \(j = 1\) and the gravitational radiation equations for \(j = 2\) in the weak field limit. We exhibit these equations below.
6.1 Maxwell’s Equations

In Cartesian coordinates the equation (32) reduces, for \( j = 1 \), to

\[
\begin{pmatrix}
-\frac{i}{c} \frac{\partial}{\partial t} & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\
-\frac{\partial}{\partial z} & -\frac{i}{c} \frac{\partial}{\partial t} & -\frac{\partial}{\partial x} \\
-\frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & -\frac{i}{c} \frac{\partial}{\partial t}
\end{pmatrix}
\begin{pmatrix}
E_x + iB_x \\
E_y + iB_y \\
E_z + iB_z
\end{pmatrix} = 0.
\] (34)

These three equations are expressed as a vector equation by

\[-\frac{i}{c} \frac{\partial}{\partial t} (\vec{E} + i\vec{B}) + \nabla \times (\vec{E} + i\vec{B}) = 0.\] (35)

The real and imaginary parts of these equations are equal to zero, so we obtain

\[\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0\]
\[\nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = 0.\] (36)

These are the two Maxwell “curl” equations.

The two Maxwell “div” equations are also present. Eq. (28) that forces the non-physical components of the state \( |1 0 \rangle_{0^0} \) to vanish says that, in the special frame with light vector \( k^0 = (0, 0, 1, 1) \), the only nonvanishing component is the component with \( m = j = 1 \). The coordinates of this component are \(-(x + iy)\) and the spatial part of the \( k \) vector is \((0, 0, 1)\). In this frame \( x \) and \( y \) coordinates are arbitrary. In general, this states that \( \vec{k} \cdot (\vec{E} + i\vec{B}) = 0 \). With the substitution \( \vec{k} \rightarrow \frac{1}{i} \nabla \), the real and imaginary parts of this complex equation reduce to

\[\nabla \cdot \vec{B} = 0,\]
\[\nabla \cdot \vec{E} = 0.\] (37)

Thus, we obtain the four Maxwell’s equations for the electromagnetic field (in vacuum). The four equations (36) and (37) are necessary to obtain the source-free wave equation for these fields: \((\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2})\vec{E} = 0\) and \((\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2})\vec{B} = 0\).

6.2 Gravitational Radiation Equations

For \( j = 2 \), equations (33) describe two interacting tracefree quadrupole fields \( Q_E \) and \( Q_B \) (tensors of rank two). Specifically, the equations are

\[\text{curl}Q_E + \frac{1}{c} \frac{\partial Q_B}{\partial t} = 0,\]
\[\text{curl}Q_B - \frac{1}{c} \frac{\partial Q_E}{\partial t} = 0.\] (38)
In Cartesian coordinates these coupled interaction equations, for \( Q_E = F \) and \( Q_B = G \), become

\[
\frac{1}{2} \left( \begin{array}{ccccc}
0 & 0 & 0 & \partial x & \partial y \\
\partial y & 0 & \partial z & 0 & 0 \\
\partial x & \partial z & 0 & \partial y & \partial x \\
\partial y & 0 & \partial x & \partial x & \partial y \\
\partial z & \partial y & \partial x & 0 & \partial z \\
\end{array} \right) \begin{pmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
F_5 \\
\end{pmatrix} + \frac{1}{c} \frac{\partial}{\partial t} \begin{pmatrix}
G_1 \\
G_2 \\
G_3 \\
G_4 \\
G_5 \\
\end{pmatrix} = 0,
\]

(39)

\[
\frac{1}{2} \left( \begin{array}{ccccc}
0 & 0 & 0 & \partial x & \partial y \\
\partial y & 0 & \partial z & 0 & 0 \\
\partial x & \partial z & 0 & \partial y & \partial x \\
\partial y & 0 & \partial x & \partial x & \partial y \\
\partial z & \partial y & \partial x & 0 & \partial z \\
\end{array} \right) \begin{pmatrix}
G_1 \\
G_2 \\
G_3 \\
G_4 \\
G_5 \\
\end{pmatrix} - \frac{1}{c} \frac{\partial}{\partial t} \begin{pmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
F_5 \\
\end{pmatrix} = 0.
\]

There is another pair of equations for gravitational radiation, in complete analogy with the “divergence” equations of electromagnetism. In the frame with \( \mathbf{k}^0 = (0, 0, 1, 1) \) the state \( | \frac{\mathbf{2}}{2} 0 \rangle \) vanishes unless \( m = 2 \). This state has Cartesian coordinates \( (x + iy)^2 = (x^2 - y^2) + 2ixy \). All other components in Cartesian coordinates: \( yz, zx, \) and \( (2z^2 - x^2 - y^2) \) are forced to vanish. In an arbitrary coordinate system, where the spatial component of the \( \mathbf{k} \) vector is \( \mathbf{k} \), this condition is

\[
k^i S_{ij} = 0
\]

and this becomes, after the standard Fourier transform,

\[
\frac{\partial}{\partial x^i} S_{ij} = 0.
\]

Here \( S_{ij} \) is a tracefree second order symmetric tensor on \( R^3 \) obeying

\[
S_{ij} = S_{ji}
\]

and

\[
S_{ii} = 0.
\]

The tracefree tensor \( S_{ij} \) and the five-components of \( Q_E = F \) and \( Q_B = G \) are related as follows:

For \( F \)

\[
F_1 = F_{xy}, \quad F_2 = F_{yz}, \quad F_3 = F_{zx}, \quad F_4 = F_{\frac{1}{\sqrt{3}}(x^2 - y^2)}, \quad F_5 = F_{\frac{1}{2\sqrt{3}}(2z^2 - x^2 - y^2)},
\]

\[
S_{ij} = \begin{pmatrix}
F_{11} & F_{12} & F_{13} \\
F_{21} & F_{22} & F_{23} \\
F_{31} & F_{32} & F_{33} \\
\end{pmatrix} = \begin{pmatrix}
F_4 & -\frac{1}{\sqrt{3}} F_5 & F_1 \\
-\frac{1}{\sqrt{3}} F_5 & F_3 & -\frac{1}{\sqrt{3}} F_5 \\
F_1 & -\frac{1}{\sqrt{3}} F_5 & F_2 \\
\end{pmatrix}.
\]

(40)

For \( G \), the relation with \( S_{ij} \) is similar to (40). Therefore the divergence equations for gravitational radiation, for the fields \( F \) and \( G \) in the representation given above, are

\[
\frac{\partial}{\partial x^i} F_{ij} = 0,
\]

\[
\frac{\partial}{\partial x^i} G_{ij} = 0,
\]

(41)

where \( i, j = 1, 2, 3 \).

The decomposition of Weyl tensor into electric and magnetic tensors \( E_{ab} \) and \( H_{ab} \) respectively has a long history [11-13]. In terms of these tensors the gravitational radiation equations (no sources) is described [1] by

\[
(\nabla \times E)_{ab} + \frac{1}{c} \frac{\partial H_{ab}}{\partial t} = 0,
\]

(42)

\[
(\nabla \times H)_{ab} - \frac{1}{c} \frac{\partial E_{ab}}{\partial t} = 0,
\]

where \( i, j = 1, 2, 3 \).
where \([14, 15]\)

\[ E_{ab} = C_{acbd}u^c u^d, \quad H_{ab} = \frac{1}{2} \eta_{acde}C_{bf}^{ae} u^c u^f, \]

with \(u^a\) a four-velocity and \(C_{acbd}\) is the Weyl Tensor \([16]\). In four dimensions and in terms of the Riemann Tensor \(R_{abcd}\), the Ricci tensor \(R_{ac}\) and the scalar curvature \(R\), the Weyl tensor can be written as follows

\[ C_{abcd} = R_{abcd} + \frac{1}{2} (g_{ad} R_{cb} + g_{bc} R_{da} - g_{ac} R_{db} - g_{bd} R_{ca}) + \frac{1}{6} (g_{ac} g_{db} - g_{ad} g_{cb}) R. \]

Both tensors \(E_{ab}\) and \(H_{ab}\) are spatial symmetric tracefree rank-two tensors (irreducible tensors) \([1, 14, 15]\). Each of them has 5 independent components: \(E_i\) and \(H_i\), \(i = 1, 2, 3, 4, 5\) respectively, which satisfy the transformation properties of an irreducible rank-two tensor under the discrete group operations \(P\) and \(T\).

By using the transformation properties table of section 5.1 for \(j = 2\), we have

| Discrete Operation | \(E_i\) | \(H_i\) |
|--------------------|--------|--------|
| \(P\)              | +      | -      |
| \(T\)              | -      | +      |

Thus, the Weyl tensor splits irreducibly and covariantly into these two symmetric tracefree tensors.

By using the matrix form for the irreducible tensors \(E_{ab}\) and \(H_{ab}\) and the definition \([14]\) of the curl operator \(((\nabla \times T)_{ab})\) in \([12]\) we obtain two matrix equations. Each of them gives rise to 5 independent equations. The 5 equations from the first one are

\[-\frac{1}{2} \partial_x E_3 + \frac{1}{2} \partial_y E_2 + \partial_z E_4 + \frac{1}{c} \frac{\partial H_1}{\partial t} = 0,\]

\[-\frac{1}{2} \partial_x (E_4 + \sqrt{3} E_5) - \frac{1}{2} \partial_y E_1 + \frac{1}{2} \partial_z E_3 + \frac{1}{c} \frac{\partial H_2}{\partial t} = 0,\]

\[\frac{1}{2} \partial_x E_1 - \frac{1}{2} \partial_y (E_4 - \sqrt{3} E_5) - \frac{1}{2} \partial_z E_2 + \frac{1}{c} \frac{\partial H_3}{\partial t} = 0,\]

\[\frac{1}{2} \partial_x E_2 + \frac{1}{2} \partial_y E_3 - \partial_z E_1 + \frac{1}{c} \frac{\partial H_4}{\partial t} = 0,\]

\[\frac{\sqrt{3}}{2} \partial_x E_2 - \frac{\sqrt{3}}{2} \partial_y E_3 + \frac{1}{c} \frac{\partial H_5}{\partial t} = 0,\]

(43)

and the 5 equations from the second one are

\[-\frac{1}{2} \partial_x H_3 + \frac{1}{2} \partial_y H_2 + \partial_z H_4 - \frac{1}{c} \frac{\partial E_1}{\partial t} = 0,\]

\[-\frac{1}{2} \partial_x (H_4 + \sqrt{3} H_5) - \frac{1}{2} \partial_y H_1 + \frac{1}{2} \partial_z H_3 - \frac{1}{c} \frac{\partial E_2}{\partial t} = 0,\]

\[\frac{1}{2} \partial_x H_1 - \frac{1}{2} \partial_y (H_4 - \sqrt{3} H_5) - \frac{1}{2} \partial_z H_2 - \frac{1}{c} \frac{\partial E_3}{\partial t} = 0,\]

\[\frac{\sqrt{3}}{2} \partial_x H_2 - \frac{\sqrt{3}}{2} \partial_y H_3 - \frac{1}{c} \frac{\partial E_4}{\partial t} = 0,\]

(44)
\[
\frac{1}{2} \partial_x H_2 + \frac{1}{2} \partial_y H_3 - \partial_z H_1 - \frac{1}{c} \frac{\partial E_4}{\partial t} = 0, \\
\frac{\sqrt{3}}{2} \partial_x H_2 - \frac{\sqrt{3}}{2} \partial_y H_3 - \frac{1}{c} \frac{\partial E_5}{\partial t} = 0.
\] (44)

We have 10 equations in total. By comparing this set of equations with the 10 equations of (39) that come from the constraint equation for \( j = 2 \), we find the components of the two oscillating interacting fields \( F \) and \( G \) in terms of the components of the electric and magnetic Weyl tensor \( E \) and \( H \). The relations are
\[
F_i = E_i, \quad G_i = H_i, \quad (45)
\]
and they satisfy the transformation properties rules under \( P \) and \( T \) for \( F \) and \( G \) (table of section 5.1).

7 Conclusion

In the quantum description of the electromagnetic field, photons are the fundamental building blocks. Photons are described by a 4-vector \( k \) that obeys \( k \cdot k = 0 \) in free space, and a helicity index indicating a projection of an angular momentum \( \pm 1 \) (in units of \( \hbar \)) along the direction of propagation of the photon. Every physical state is described by a superposition of the photon basis states, and every superposition describes a possible physical state. In this description of the electromagnetic field no equations are necessary.

The classical description of the electromagnetic field proceeds along somewhat different lines. A multicomponent field \((\vec{E}, \vec{B})\) is introduced at each point in space-time. The components of the field (tensor) transform in a very elegant way under homogeneous Lorentz transformations. If the field is Fourier transformed from the coordinate to the momentum representation, then each 4-momentum has six components associated with it. Since the quantum description has only two independent components associated with each 4-momentum, there are four dimensions worth of linear combinations of the classical field components that do not describe physically allowed states, for each 4-momentum. Some mechanism must be derived for annihilating these non-physical superpositions. This mechanism is the set of equations discovered by Maxwell for the electromagnetic field in the absence of sources. Similar equations hold for gravitational radiation in Minkowski spacetime. In this sense, these equations are an expression of our ignorance.

Group theory, by pointing to the appropriate Hilbert space for the electromagnetic field, allows us to relate physical states to arbitrary superpositions of basis states. Since no superpositions are forbidden, no equations are necessary.

We have derived constraint equations on the \( 2(2j + 1) \) components of a manifestly covariant field transforming under the representation \( D^{j0+0j} \) of the ILG. There are two constraint equations:
\[ \text{curl}(T_E^{(j)} + iT_B^{(j)}) - \frac{i}{c^2} \partial_t (T_E^{(j)} + iT_B^{(j)}) = 0, \]
\[ \text{div}(T_E^{(j)} + iT_B^{(j)}) = 0, \] (46)

For \( j = 1 \) these are the electric and magnetic fields. For \( j = 2 \) these are
the second order tracefree symmetric gravito-electric and gravito-magnetic field
tensors that are projected from the Weyl tensor.

It is remarkable that gravitational radiation in flat space-time can be
described in essentially the same way as Maxwell’s classical equations.

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