**Abstract.** The method of constructing Ricci–flat metrics with $l$–conformal Galilei symmetry is discussed.

1. Introduction

In recent years there has been extensive investigation of the nonrelativistic conformal algebras [1]–[9]. The conformal extension of the Galilei algebra is parametrized by a (half)integer parameter $l$ which gives rise to the name $l$–conformal Galilei algebra [10]. So far most of the studies focused on the construction of various dynamical realizations with a particular emphasis on the issues of the presence of higher derivative terms and functional independence of the acceleration generators in the algebra. Applications of the $l$–conformal Galilei symmetry within the general relativistic context are unknown. The goal of this note is to adjust the conventional group–theoretic construction so as to build Ricci–flat metrics with the $l$–conformal Galilei isometry group.

In Sect. 2 we consider Maurer–Cartan one–forms associated with the $l$–conformal Galilei algebra. Sect. 3 is devoted to the construction of a Ricci–flat spacetime of the ultrahyperbolic signature which enjoys the $l$–conformal Galilei isometry group. We summarize our results in Sect. 4.

2. $l$–conformal Galilei algebra and Maurer–Cartan one–forms

The $l$–conformal Galilei algebra involves the generators of time translation $H$, dilatation $D$, special conformal transformation $K$, spatial rotations $M_{ij}$ (with $i = 1, \ldots, d$), spatial translations $C_i^{(0)}$, Galilei boosts $C_i^{(1)}$ and accelerations $C_i^{(\alpha)}$ with $\alpha = 2, \ldots, 2l$. The structure relations of the algebra read

\[
\begin{align*}
[H, D] &= iH, & [H, K] &= 2iD, & [D, K] &= iK, \\
[H, C_i^{(n)}] &= iC_i^{(n-1)}, & [D, C_i^{(n)}] &= i(n - l)C_i^{(n)}, & [K, C_i^{(n)}] &= i(n - 2l)C_i^{(n+1)}, \\
[M_{ij}, C_k^{(n)}] &= -i(\delta_{ik}C_j^{(n)} - \delta_{jk}C_i^{(n)}), & [M_{ij}, M_{kl}] &= -i(\delta_{ik}M_{jl} + \delta_{jl}M_{ik} - \delta_{il}M_{jk} - \delta_{jk}M_{il}),
\end{align*}
\]

where $n = 0, \ldots, 2l$. 

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**References**

[1]–[9]

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**Further Information**

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Let us choose a subgroup \( L \) generated by \( D \) and \( M_{ij} \) and consider the coset space

\[
\tilde{G} = e^{iH} e^{iK} e^{ix_i^{(n)} C_i^{(n)}} \times L
\]  

(parametrized by the coordinates \( t, r \) and \( x_i^{(n)} \). Left multiplication by the group element \( g = e^{iaH} e^{ibK} e^{icD} e^{i\lambda_i^{(n)} C_i^{(n)}} e^{i\omega_{ij} M_{ij}} \) determines the action of the group on the coset

\[
t' = t + a + bt^2 + ct, \quad r' = r + b(1 - 2tr) - cr, \quad x_i^{(n)} = x_i^{(n)} - 2bt(n - l)x_i^{(n)} - c(n - l)x_i^{(n)} - \sigma_{ij}x_j^{(n)} + \sum_{s=0}^{2l} \sum_{m=s}^{2l} \frac{(-1)^{n-s} m!(2l - s)!}{s!(m - s)!(n - s)!(2l - n)!} r^{m-s} r^{n-s} \lambda_i^{(m)},
\]

where \( a, b, c, \lambda_i^{(n)} \) and \( \sigma_{ij} = -\sigma_{ji} \) are infinitesimal parameters corresponding to the time translations, special conformal transformations, dilatations, vector generators in the algebra, and spatial rotations, respectively.

The Maurer-Cartan one–forms \( \tilde{G}^{-1} d\tilde{G} = i(\omega_H H + \omega_K K + \omega_D D + \omega_i^{(n)} C_i^{(n)}) \)

\[
\omega_i^{(n)} = dx_i^{(n)} + 2r(n - l)x_i^{(n)} dt - (n + 1)x_i^{(n+1)} dt - (n - 2l - 1)x_i^{(n-1)} (r^2 dt + dr), \quad \omega_H = dt, \quad \omega_K = r^2 dt + dr, \quad \omega_D = -2r dt,
\]

in which

\[
x_i^{(-1)} = x_i^{(2l+1)} = 0
\]

are the building blocks for constructing a metric with the \( l \)--conformal Galilei isometry group. For what follows it proves useful to introduce a new temporal coordinate

\[
t = \frac{1}{2} \left( \tilde{t} + \frac{1}{r} \right)
\]

which yields

\[
\tilde{\omega}_i^{(n)} = dx_i^{(n)} + \left( r(n - l)x_i^{(n)} - \frac{1}{2}(n + 1)x_i^{(n+1)} \right) \left( dt - \frac{dr}{r^2} \right) - \frac{1}{2}(n - 2l - 1)x_i^{(n-1)} r^2 \left( d\tilde{t} + \frac{dr}{r^2} \right).
\]

### 3. \( l \)--conformal Galilei symmetry in the general relativistic context

In order to construct a metric which holds invariant under the action of the \( l \)--conformal Galilei group, we first note that the one–forms \( \tilde{\omega}_i^{(n)} \) are invariant under the time translation and \( \lambda_i^{(n)} \) transformations, while with respect to the dilatations, special conformal transformations and rotations they transform homogeneously

\[
\tilde{\omega}_i^{(n)} = (1 - c(n - l))\omega_i^{(n)}, \quad \tilde{\omega}_i^{(n)} = \left( 1 - b(n - l) \left( \tilde{t} + \frac{1}{r} \right) \right) \omega_i^{(n)},
\]

\[
\tilde{\omega}_i^{(n)} = (\delta_{ij} - \sigma_{ij})\omega_j^{(n)}. \quad \omega_i^{(n)} = (\delta_{ij} - \sigma_{ij})\omega_j^{(n)}.
\]

As a result, the quadratic form

\[
\tilde{s}^2 = \left( r^2 d\tilde{t}^2 - \frac{dr^2}{r^2} \right) + S_{n,m} \tilde{\omega}_i^{(n)} \tilde{\omega}_i^{(m)}, \quad S_{n,m} = S_{m,n} = k_n \delta_{n+m,2l},
\]

where

\[
S_{n,m} = \sum_{s=0}^{2l} \sum_{m=s}^{2l} \frac{(-1)^{n-s} m!(2l - s)!}{s!(m - s)!(n - s)!(2l - n)!} r^{m-s} r^{n-s} \lambda_i^{(m)},
\]

are the non–compact generalizations of the linearized conformal Galilei Galilei group.

The quadratic form \( \tilde{s}^2 \) is invariant under the action of the \( l \)--conformal Galilei group.

\[
\tilde{s}^2 = \left( r^2 d\tilde{t}^2 - \frac{dr^2}{r^2} \right) + S_{n,m} \tilde{\omega}_i^{(n)} \tilde{\omega}_i^{(m)}, \quad S_{n,m} = S_{m,n} = k_n \delta_{n+m,2l},
\]
with $k_n$ to be fixed below, has the $l$–conformal Galilei isometry group.

In order to promote (9) to a Ricci–flat metric, we minimally extend it by an extra coordinate $y$
\[ ds^2 = \alpha(y) \left( r^2 dr^2 - \frac{dr^2}{r^2} \right) + S_{n,m} \omega_1^{(n)} \omega_1^{(m)} + \epsilon dy^2, \]
where $\alpha(y)$ is a function to be fixed below and $\epsilon = \pm 1$. It is assumed that $y$ remains intact under the $l$–conformal Galilei transformations so that the metric maintains the symmetry of its predecessor (9). The vacuum Einstein equations then fix $\alpha(y)$
\[ \alpha(y) = c_1 y^2, \quad c_1 = \epsilon + \frac{l(l+1)(2l+1)de}{6} - \frac{de}{4} \sum_{p=0}^{2l-1} \sum_{q=1}^{2l} (p+1)(q-2l-1)\tilde{S}^{p+1,q-1}S_{p,q}, \]
and impose the recurrence relations on the form of the components $S_{n,m}$
\[ n^2 S_{n-1,2l-n+1} + (2l-n)^2 S_{n+1,2l-n-1} - (n+1)^2 \tilde{S}^{n+1,2l-n-1} + (n-2l-1)^2 \tilde{S}^{n-1,2l-n+1} \right) (S_{n,2l-n})^2 = 0, \]
where $n = 0, \ldots, 2l$. In Eq. (12) $\tilde{S}^{n,m} = \tilde{S}^{m,n} = \frac{1}{S_{n,m}} = \frac{1}{S_{n,m}^{2l+1}}$ (no sum in $n$) stands for the inverse of $S_{n,m}$ and it is assumed
\[ S_{2l+1,n} = S_{-1,n} = \tilde{S}^{2l+1,n} = \tilde{S}^{-1,n} = 0. \]
It is straightforward to verify that Eqs. (12) algebraically relate all the components $S_{n,m}$ to $S_{0,2l}$, the latter being unspecified.

4. Conclusion
Let us discuss the status of the metrics constructed above. Given the spatial dimension $d$ in which the original $l$–conformal Galilei algebra (1) is realized and the value of the (half)integer parameter $l$, the $AdS_2$–spacetime in (10) is extended by $(2l+1)d$ extra dimensions parametrized by the coordinates $x_i^{(0)}, \ldots, x_i^{(2l)}$, $i = 1, \ldots, d$. For half–integer $l$, these are split into $(2l+1)d$ spatial and $(2l+1)d$ temporal dimensions, while for integer $l$ one reveals $ld$ temporal and $(l+1)d$ spatial dimensions or vice versa depending on which sign is chosen for the components $S_{n,m}$ linked to $S_{0,2l}$. Depending on whether $\epsilon = 1$ or $\epsilon = -1$ is chosen in the last term in (10), the remaining coordinate $y$ brings about one more temporal or spatial dimension. The resulting Ricci–flat spacetime is thus $[(2l+1)d+3]$–dimensional and of the ultrahyperbolic signature.

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