SURGICAL INVARIANTS OF FOUR-MANIFOLDS

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ABSTRACT. A new topological invariant of closed connected orientable four-dimensional
manifolds is proposed. The invariant, constructed via surgery on a special link, is a four-
dimensional counterpart of the celebrated SU(2) three-manifold invariant of Reshetikhin,
Turaev and Witten.

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The issue of topological classification of low-dimensional manifolds, especially of the dimension three and four, is one of the most challenging problems in modern mathematics. One of the most spectacular events in topology of three-dimensional manifolds took place in 1989, when a new (numerical) topological invariant $I_r(M)$ of a closed orientable three-dimensional manifold $M$, parametrized by the integer $r$ ($r \geq 2$), defined via surgery on a framed link, was discovered. The idea is due to a physicist [1], Edward Witten, an explicit construction to mathematicians, Reshetikhin and Turaev [2], further studies, to Kirby and Melvin [3], whereas a significant simplification of the method to Lickorish [4]. The invariant $I_r(M)$, known as the *Reshetikhin-Turaev-Witten* (RTW) invariant, is also frequently referred to as the *SU(2)-invariant* because the Kauffman bracket it bases upon (denoted in mathematical literature as $\langle \rangle$) formally corresponds, in Witten’s approach, to the average with respect to the connection $A$ (defined on a trivial SU(2) bundle on the three-manifold $M$) modulo gauge transformations, weighted by $\exp [ikcs(A)]$. Here $k$ is a positive integer, called the *level*, and $cs(A)$ is the Chern-Simons secondary characteristic class. Incidentally, the average is also denoted as $\langle \rangle$.

The construction of the RTW invariant makes use of the *fundamental theorem of surgery* of Lickorish and Wallace on presentation of every closed connected orientable three-manifold $M$ via surgery on a framed link in $S^3$, and the *linear skein theory* associated with the Kauffman bracket. The derivation of $I_r(M)$ is combinatorial, and amounts to showing the invariance of $I_r(M)$ with respect to the *Kirby moves*

In four dimensions, there is a celebrated theorem of Freedman [5] on classification of closed orientable *simply-connected* four-manifolds, provided by the *intersection form* $Q(M)$ (and the Kirby-Siebenman invariant $\alpha(M)$). The intersection form $Q(M)$ corresponds to, and for four-dimensional manifolds with boundary defined via surgery on a link in $S^3$ is equal to, the *linking matrix* $\ell k$. The elements of the symmetric matrix $\ell k$, the *linking numbers* (with framings on the diagonal), are the simplest numerical invariants
of a link. Therefore, one can ask the following questions. Can one use some stronger
('non-abelian') invariants of links, for example the Kauffman bracket polynomial, to
obtain some new 'non-abelian' invariants of four-dimensional manifolds? Can one ex-
tend the idea of RTW to the four-dimensional case? Can one treat simply-connected
and non-simply-connected manifolds uniquely? The answer to these questions seems to
be affirmative. Namely, we would like to propose a new invariant of closed connected
orientable four-manifolds, defined via surgery on a special link in $S^3$. Thus, we have
succeeded in finding a quantity invariant with respect to the four-dimensional version
of the 'Kirby moves'. The idea as well as the construction resembles the original one,
proposed by RTW in the three-dimensional case, whereas the four-dimensional version
of the Kirby calculus we need has been developed by César de Sá in [6].

**Construction of the invariant**

An arbitrary closed connected orientable four-manifold $\mathcal{M}$ can be obtained via
surgery in $S^3$ on a special framed link $(L, f)$ [6]. By definition, the special framed link
$L$ is a sum of two sorts of knots

$$L = \bigcup_{i=1}^{n} K_i \cup \bigcup_{i=1}^{\hat{n}} \hat{K}_i,$$

where $\{K_i\}_{i=1}^{n}$ are ordinary knots, and $\{\hat{K}_i\}_{i=1}^{\hat{n}}$ are special ones. The special knots are
trivial (with zero framing), and mutually unlinked unknots, and the whole link, when
regarded as a description of a three-manifold, represents a connected sum of copies of
$S^1 \times S^2$.

Following the terminology of Lickorish [4], for a given (fixed) integer $r, r \geq 2$, let
$\omega$ belonging to the linear skein of the annulus $S(S^1 \times I)$ be defined by

$$\omega = \sum_{m=0}^{2r-2} \Delta_m S_m(\alpha),$$
where the coefficients
\[
\Delta_m = \frac{(-1)^m (A^{2(m+1)} - A^{-2(m+1)})}{A^2 - A^{-2}},
\]

\(S_m(\alpha)\) is the \(m\)th Chebyshev polynomial in the generator \(\alpha\) of \(S(S^1 \times I)\), and \(A\) is a primitive \(8r\)th root of unity.

Introducing a \(\mathbb{Z}_2\)-gradation, we can decompose \(\omega\) as follows (compare [7])
\[
\omega = \omega^+ + \omega^-,
\]
where
\[
\omega^+ = \sum_{m=0}^{r-1} \Delta_{2m} S_{2m}(\alpha),
\]
\[
\omega^- = \sum_{m=0}^{r-2} \Delta_{2m+1} S_{2m+1}(\alpha).
\]
The gradation used is provided by the power of \(\alpha\), or equivalently by the degree of the Chebyshev polynomial \(S_m(\alpha)\), or alternatively by the ‘spin’ labeling irreducible representations of the (quantum) SU(2) group.

\textit{Notation.} We denote as \(\tilde{K}\) the result of pushing a knot \(K\) off itself (missing the rest of the link \(L\)) using the framing \(f\) of \(K\), whereas as \(K_1 \# b K_2\) a (band) connected sum of the two knots \(K_1, K_2\), where \(b\) is any band missing the rest of \(L\). \(\omega_K\) denotes \(\omega\) immersed in the plane as a regular neighbourhood of \(K\).

\textit{Proposition.} Let \(a^+, a^-\) be arbitrary complex numbers. Then we have the following ‘Kirby calculus’
\[
\langle \alpha K_1, \omega K_2 \rangle = \langle \alpha K_1 \# b K_2, \omega K_2 \rangle,
\]
\[
\langle \alpha^2 K_1, (a^+ \omega^+ + a^- \omega^-) K_2 \rangle = \langle \alpha^2 K_1 \# b K_2, (a^+ \omega^+ + a^- \omega^-) K_2 \rangle.
\]
The first equality expresses a standard property of \(\omega\) [4], whereas the second one follows from the observation that the even element \(\alpha^2\) respects the gradation in all cablings (the Fenn-Rourke version of this phenomenon is implicit in [7]).
Corollary. From the Proposition we can derive the following ‘Kirby equalities’

\[ \langle \omega_{K_1}, \omega_{K_2} \rangle = \langle \omega_{K_1 \#_1 \tilde{K}_2}, \omega_{K_2} \rangle, \]

\[ \langle \omega_{K_1^+}, \omega_{K_2^+} \rangle = \langle \omega_{K_1 \#_1 \tilde{K}_2}, \omega_{K_2^+} \rangle, \]

\[ \langle \omega_{K_1^+}, \omega_{K_2} \rangle = \langle \omega_{K_1 \#_1 \tilde{K}_2}, \omega_{K_2^+} \rangle, \]

Fortunately, the fourth lacking equality,

\[ \langle \omega_{K_1}, \omega_{K_2^+} \rangle = \langle \omega_{K_1 \#_1 \tilde{K}_2}, \omega_{K_2^+} \rangle, \]

is, in general, not true.

Henceforth, \( U \) denotes a trivial (with zero framing) unknot, \( H_1 \) and \( \tilde{H}_2 \) are two components of the special Hopf link \( \mathcal{H} \), ordinary and special respectively, \( \mathcal{H} = H_1 \cup \tilde{H}_2 \).

Theorem. Let \( n \) and \( \nu \) be the dimension and nullity of the linking matrix \( \mathcal{L} \). Then

\[ \mathcal{T}_r(M) = \frac{\langle \prod_{i=1}^{n} \omega_{K_i}^+, \prod_{i=1}^{n} \omega_{K_i} \rangle}{\langle \omega_{U}^{+} \rangle^\nu \langle \omega_{H_1}^+, \omega_{\tilde{H}_2} \rangle^{(n-\nu)/2}} \]

is an invariant of (closed, connected, orientable four-manifold) \( M = M_L \), a complex number parametrized by the integer \( r \), \( r \geq 2 \), independent of the choice of the representative \( (L, f) \).

Below, we give a list of all the allowable ‘four-dimensional Kirby moves’, so-called \( \Gamma \)-moves [6]:

(a) sliding one of the special knots over another special one;

(b) sliding one of the ordinary knots over one of the special ones;

(c) sliding one of the ordinary knots over another ordinary one;

(d) introducing or deleting a special Hopf link;

(e) introducing or deleting a trivial unknot;

(f) isotoping the link picture in \( S^3 \).
Sketch of the proof. We should show that $I_r(\mathcal{M})$ is invariant with respect to all the \( \Gamma \)-moves. \( a \)-, \( b \)- and \( c \)-invariance of $I_r(\mathcal{M})$ immediately follows from the Corollary. \( d \)-invariance is a consequence of the following transformation rule of the linking matrix $\ell_k$, accompanying the introduction of a special Hopf link $\mathcal{H}$,

$$
\ell_k \longrightarrow \begin{pmatrix}
\ell_k & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
$$

Hence the corresponding shift of the dimension and nullity of $\ell_k$

$$
n \longrightarrow n + 2
$$
$$
\nu \longrightarrow \nu,
$$
compensates the (factorized out) Kauffman bracket in the numerator. Similarly, \( e \)-invariance corresponds to the transformation rule

$$
\ell_k \longrightarrow \begin{pmatrix}
\ell_k & 0 \\
0 & 0
\end{pmatrix},
$$

and consequently the shift

$$
n \longrightarrow n + 1
$$
$$
\nu \longrightarrow \nu + 1,
$$
also compensates the numerator. \( f \)-invariance directly follows from fundamental properties of the Kauffman bracket and the linking matrix $\ell_k$.

Remarks. (i) In the particular case of a simply-connected $\mathcal{M}$, $I_r(\mathcal{M})$ simplifies to the exponent of a linear combination of the signature and the Betti number; (ii) There is also a four-dimensional counterpart of the invariant of Turaev and Viro, defined via triangulations by Ooguri, Crane and Yetter [8].
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REFERENCES

1. E. Witten, Quantum Field Theory and the Jones Polynomial, Commun. Math. Phys. 121 (1989), 351–399.

2. N. Reshetikhin and V. G. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, Invent. math. 103 (1991), 547–597.

3. R. Kirby and P. Melvin, The 3-manifold invariants of Witten and Reshetikhin-Turaev for sl(2,C), Invent. math. 105 (1991), 473–545.

4. W. B. R. Lickorish, The skein method for three-manifold invariants, 1992 (unpublished).

5. M. H. Freedman and F. Luo, Selected Applications of Geometry to Low-Dimensional Topology, University Lecture Series, American Mathematical Society, Providence, 1989, Chapt. 4.

6. E. César de Sá, A link calculus for 4-manifolds, Topology of low-dimensional manifolds, Proc. Second Sussex Conf., Lecture Notes in Math., vol. 722, Springer, Berlin, 1979, 16–30.

7. C. Blanchet, Invariants on three-manifolds with spin structure, Comment. Math. Helvetici 67 (1992), 406–427.
8. H. Ooguri, *Topological Lattice Models in Four Dimensions*, Kyoto preprint RIMS-878 and e-preprint hep-th/9205090.

L. Crane and D. Yetter, *A categorical construction of 4D topological quantum field theories*, Kansas preprint and e-preprint hep-th/9301062.

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