SINGULAR POLYNOMIALS AND MODULES FOR THE SYMMETRIC GROUPS

CHARLES F. DUNKL

ABSTRACT. For certain negative rational numbers $\kappa_0$, called singular values, and associated with the symmetric group $S_N$ on $N$ objects, there exist homogeneous polynomials annihilated by each Dunkl operator when the parameter $\kappa = \kappa_0$. It was shown by de Jeu, Opdam and the author (Trans. Amer. Math. Soc. 346 (1994), 237-256) that the singular values are exactly the values $-\frac{m}{n} \in \mathbb{Q}$ with $2 \leq n \leq N$, $m = 1, 2, \ldots$ and $\frac{m}{n}$ is not an integer. For each pair $(m, n)$ satisfying these conditions there is a unique irreducible $S_N$-module of singular polynomials for the singular value $-\frac{m}{n}$. The existence of these polynomials was previously established by the author (IMRN 2004, #67, 3607-3635). The uniqueness is proven in the present paper. By using Murphy’s (J. Alg. 69(1981), 287-297) results on the eigenvalues of the Murphy elements, the problem of existence of singular polynomials is first restricted to the isotype $\tau$ (where $\tau$ is a partition of $N$ corresponding to an irreducible representation of $S_N$) satisfying the condition that $n/\gcd(m, n)$ divides $\tau_i + 1$ for $1 \leq i < l$; $l$ is the length of $\tau$, that is, $\tau_l > \tau_{l+1} = 0$. Then by arguments involving the analysis of nonsymmetric Jack polynomials it is shown that the assumption $\tau_2 \geq n/\gcd(m, n)$ leads to a contradiction. This shows that the singular polynomials are exactly those already determined, and are of isotype $\tau$, where $\tau_2 = \ldots = \tau_{l-1} = (n/\gcd(m, n)) - 1 \geq \tau_l$.

1. Introduction

The symmetric group $S_N$ on $N$ letters acts on $\mathbb{R}^N$ by permutation of coordinates. The alternating polynomial, also called the discriminant, is defined by $a_N(x) = \prod_{1 \leq i < j \leq N}(x_i - x_j)$ for $x \in \mathbb{R}^N$ and is a fundamental object associated to the group. The Macdonald-Mehta-Selberg integral for $S_N$ is

$$(2\pi)^{-N/2} \int_{\mathbb{R}^N} |a_N(x)|^{2\kappa} \exp \left(-\frac{1}{2} \sum_{i=1}^{N} x_i^2\right) \, dx = \prod_{n=2}^{N} \frac{\Gamma(n\kappa + 1)}{\Gamma(n\kappa + 1)}$$

for $\kappa \geq 0$. The right hand side is a meromorphic function of $\kappa$ without zeroes and with poles at $\kappa = -\frac{m}{n}$, for $2 \leq n \leq N$, $m = 1, 2, 3, \ldots$ and $\frac{m}{n}$ is not an integer. (For an algebraic proof of the integral, see [5, Sect. 8.7].) Do these values have another connection with the symmetric group? The purpose of this paper is to show that for each pair $(m, n)$ of natural numbers with $2 \leq n \leq N$ and $\frac{m}{n}$ not an integer there is a unique irreducible $S_N$-module of homogeneous polynomials which have a
certain singularity property with respect to a commutative algebra of differential-difference operators. In a previous paper [2] the author established the existence of a space of such polynomials for each pair \((m, n)\). This paper proves the uniqueness of the polynomials and the associated modules. By use of the Murphy elements one can find a link between the singular polynomials, the partition of \(N\) which labels the module and the nonsymmetric Jack polynomials (NSJP’s). This is the family of simultaneous eigenvectors of a commuting set \(\mathcal{U}_i(\kappa) : 1 \leq i \leq N\) of operators (involving a parameter \(\kappa\)). The singular polynomials come from the specializations of certain of NSJP’s when \(\kappa\) takes the value \(-\frac{m}{n}\). The algebra generated by the \(\mathcal{U}_i(\kappa)\) is semisimple (that is, the set of NSJP’s forms a basis for all polynomials) for generic \(\kappa\), but this property may be lost for some negative rational values. A part of the development is to show how to find limits of certain expressions in the NSJP’s as \(\kappa\) approaches \(-\frac{m}{n}\).

Murphy [9] found the eigenvalues of the Murphy elements when restricted to any irreducible \(S_N\)-module. In Section 2 we use his results to find a necessary condition on a partition to allow corresponding singular polynomials and also to prove a uniqueness result. The condition is this: suppose gcd \((m, n) = 1\) and there is an \(S_N\)-module of singular polynomials corresponding to \(\kappa = -\frac{m}{n}\), and suppose the module is labeled by the partition \(\tau\) (that is, \(\tau = (\tau_1, \tau_2, \ldots)\) with \(\sum_{i \geq 1} \tau_i = N\) and \(\tau_1 \geq \tau_2 \geq \cdots \geq 0\)) then \(n|\tau_1 + 1\) for \(1 \leq i < \ell(\tau)\), where \(\ell(\tau) = \max\{j : \tau_j \geq 1\}\).

Section 3 develops the relevant results on NSJP’s. In Section 4 it is shown that the NSJP’s as \(\kappa\) approaches a generic \(\kappa\) for certain of NSJP’s when \(\kappa\) approaches \(-\frac{m}{n}\). The space of polynomials is the set of homogeneous polynomials of degree \(\alpha\) for \(\alpha \in \mathbb{N}_0^N\) (denote the sets of integers and rational numbers respectively). For \(\alpha \in \mathbb{N}_0^N\) let \(\alpha\) as a column vector, and define the monomial \(x^\alpha\) to be \(\prod_{i=1}^N x_i^{\alpha_i}\); its degree is \(|\alpha|\). The length of a composition \(\alpha\) is \(\ell(\alpha) = \max\{j : \alpha_j > 0\}\). Consider elements of \(S_N\) as permutations on \(\{1, 2, \ldots, N\}\). Then, for \(x \in \mathbb{R}^N\) and \(w \in S_N\) let \((xw)_i = x_{w(i)}\) for \(1 \leq i \leq N\) and extend this action to polynomials by \((wf)(x) = f(xw)\). This has the effect that monomials transform to monomials: \(w(x^\alpha) = x^{\omega w \alpha}\) where \((\omega \alpha)_i = \alpha_{w^{-1}(i)}\) for \(\alpha \in \mathbb{N}_0^N\). (Consider \(x\) as a row vector, \(\alpha\) as a column vector, and \(w\) as a permutation matrix, with 1’s at the \((w(j), j)\) entries.) The reflections in \(S_N\) are the transpositions interchanging \(x_i\) and \(x_j\) and are denoted by \((i, j)\) for \(i \neq j\).

In [1] the author constructed for each finite reflection group a parametrized commutative algebra of differential-difference operators. Let \(\kappa\) be a formal parameter, that is, \(\mathbb{Q}(\kappa)\) is a transcendental extension of \(\mathbb{Q}\).

**Definition 1.** The space of polynomials is \(\mathcal{P} := \text{span}_{\mathbb{Q}(\kappa)} \{x^\alpha : \alpha \in \mathbb{N}_0^N\}\) and for \(n \in \mathbb{N}_0\) the subspace of homogeneous polynomials of degree \(n\) is \(\mathcal{P}_n := \text{span}_{\mathbb{Q}(\kappa)} \{x^\alpha : \alpha \in \mathbb{N}_0^N, |\alpha| = n\}\). For \(p \in \mathcal{P}\) and \(\alpha \in \mathbb{N}_0^N\) let \(\text{coef}(p, \alpha)\) denote the coefficient of \(x^\alpha\) in \(p\) (thus \(p = \sum_\beta \text{coef}(p, \beta) x^\beta\)).
For the symmetric group $S_N$ the operators are defined as follows:

**Definition 2.** For any polynomial $f$ on $\mathbb{R}^N$ and $1 \leq i \leq N$ let

$$D_i(\kappa) f(x) = \frac{\partial}{\partial x_i} f(x) + \kappa \sum_{j \neq i} \frac{f(x) - (ij) f(x)}{x_i - x_j}.$$ 

It was shown in [4] that $D_i(\kappa) D_j(\kappa) = D_j(\kappa) D_i(\kappa)$ for $1 \leq i, j \leq N$ and each $D_i(\kappa)$ maps $P_n$ to $P_{n-1}$ for $n \geq 1$. A specific numerical parameter value $\kappa_0$ is said to be a singular value (associated with $S_N$) if there exists a nonzero polynomial $p$ such that $D_i(\kappa_0) p = 0$ for $1 \leq i \leq N$; such a $p$ is called a singular polynomial. It was shown in [4] that the singular values are the numbers $-\frac{n}{m}$ where $n = 2, \ldots, N$, $m \in \mathbb{N}$ and $\frac{n}{m} \notin \mathbb{Z}$. Earlier, Opdam [10] showed that the $S_N$-Bessel function $J(x,y)$ considered as a function of the parameter $\kappa$ has poles precisely at these numbers (for $\kappa > 0$ the Bessel function is the entire solution of the system of equations

$$\sum_{j=1}^{N} \left( D_j(x) \right)^k J(x,y) = \left( \sum_{j=1}^{N} y_j^k \right) J(x,y), 1 \leq k \leq N, J(0,y) = 1, J(xw,y) = J(x,yw) = J(x,y) \text{ for } x,y \in \mathbb{C}^N.$$ 

Because the operators $D_i(\kappa)$ preserve homogeneity and have the $S_N$-transformation property $D_i(\kappa)(i,j) = (i,j) D_j(\kappa)$, the set of singular polynomials for a specific singular value is a direct sum of irreducible $S_N$-modules of homogeneous polynomials. The set of partitions of length $\leq N$ is denoted by $N_0^{N,P}$ and consists of all $\lambda \in N_0^N$ such that $\lambda_i \geq \lambda_i+1$ for $1 \leq i \leq N-1$. When writing partitions it is customary to suppress trailing zeros and to use exponents to indicate multiplicity, for example $(5,2^3)$ is the same as $(5,2,2,2,0) \in N_0^{5,0}$. (The exponent notation is also used for compositions.)

The irreducible representations of $S_N$ are labeled by partitions $\tau$ of $N$ (that is, $\tau \in N_0^{N,P}$ and $|\tau| = N$) and we say a polynomial $f$ is of isotype $\tau$ if $f$ is an element of an irreducible $S_N$-submodule of $P_n$ corresponding to $\tau$. It was conjectured in [4] that the two-part representations $(n-1,N-n+1)$ (with $2(n-1) \geq N$) give rise to singular polynomials for the singular values $-\frac{n}{n}$ with $\gcd(m,n) < \frac{n}{N-n+1}$, and the representations $(dn-1,n-1,\ldots,n-1,\tau_1)$ for $d,n \in \mathbb{N}$ give rise to singular polynomials for the singular values $-\frac{m}{n}$ with $\gcd(m,n) = 1$ where $l = \ell(\tau)$ and $N = (dn-1)+(l-2)(n-1)+\tau_1)$. This construction is presented in [2] in terms of nonsymmetric Jack polynomials. In this paper we show that there are no other singular polynomials. By using Murphy’s techniques in his construction of the Young seminormal representations [3] we can show that the isotype $\tau$ of any irreducible module of singular polynomials for $\kappa_0 = -\frac{n}{m}$ (with $\gcd(m,n) = 1$) must satisfy $n| (\tau_i + 1)$ for $1 \leq i < \ell(\tau)$. After that most of the work is to show that the assumption $\tau_2 < n$ leads to a contradiction. Note that the condition $\tau_2 < n$ implies for three or more parts $\tau_n = n-1$ for $2 \leq i < \ell(\tau)$ (and $\tau_i(\tau) \leq n-1$) and for two parts that $\tau_2 < \frac{\tau_1+1}{\gcd(m_1,\tau_1+1)} = n$ for the singular value $-\frac{m_1}{\tau_1+1}$. These are the restrictions described above.

The notation is almost the same as that in [2] except that the parameter has been incorporated. Key parts of the proofs depend on the behavior of polynomials as $\kappa$ approaches a singular value $\kappa_0$. The commutative algebra of the operators
defining the nonsymmetric Jack polynomials is generated by
\[
U_i (\kappa) f (x) = D_i (\kappa) x_i f (x) - \kappa \sum_{j=1}^{i-1} (j, i) f (x), 1 \leq i \leq N.
\]
(this differs by the additive constant \(\kappa\) from the notation in [5 Ch.8]). The operators act in a triangular manner on monomials, as is explained below.

**Definition 3.** For \(\alpha \in \mathbb{N}_0^N\), let \(\alpha^+\) denote the unique partition such that \(\alpha^+ = w\alpha\) for some \(w \in S_N\). For \(\alpha, \beta \in \mathbb{N}_0^N\) the partial order \(\alpha \succ \beta\) (\(\alpha\) dominates \(\beta\)) means that \(\alpha \neq \beta\) and \(\sum_{i=1}^{j} \alpha_i \geq \sum_{i=1}^{j} \beta_i\) for \(1 \leq j \leq N\); \(\alpha \triangleright \beta\) means that \(|\alpha| = |\beta|\) and either \(\alpha^+ \succ \beta^+\) or \(\alpha^+ = \beta^+\) and \(\alpha \succ \beta\). The notations \(\alpha \succeq \beta\) and \(\alpha \preceq \beta\) include the case \(\alpha = \beta\).

When acting on the monomial basis of \(P_n\), the operators \(U_i (\kappa)\) have on-diagonal coefficients involving the following rank function on \(\mathbb{N}_0^N\). We denote the cardinality of a set \(E\) by \(|E|\).

**Definition 4.** For \(\alpha \in \mathbb{N}_0^N\) and \(1 \leq i \leq N\) let
\[
\begin{align*}
\tau (\alpha, i) &= \# \{ j : \alpha_j > \alpha_i \} + \# \{ j : 1 \leq j \leq i, \alpha_j = \alpha_i \}, \\
\xi_i (\alpha; \kappa) &= (N - \tau (\alpha, i)) \kappa + \alpha_i + 1.
\end{align*}
\]

Clearly for a fixed \(\alpha \in \mathbb{N}_0^N\) the values \(\{ \tau (\alpha, i) : 1 \leq i \leq N \}\) consist of all of \(\{1, \ldots, N\}\), are independent of trailing zeros (that is, if \(\alpha' \in \mathbb{N}_0^M, \alpha'_i = \alpha_i\) for \(1 \leq i \leq N\) and \(\alpha'_i = 0\) for \(N < i \leq N\) then \(\tau (\alpha, i) = \tau (\alpha', i)\) for \(1 \leq i \leq N\)), and \(\alpha \in \mathbb{N}_0^{N,P}\) if and only if \(\tau (\alpha, i) = i\) for all \(i\) (the latter property motivated the use of “1 \leq j \leq i” rather than “1 \leq j < i” in the definition). Then (see [5, p.291]) \(U_i (\kappa) x^\alpha = \xi_i (\alpha; \kappa) x^\alpha + q_{\alpha, i} (x)\) where \(q_{\alpha, i} (x)\) is a sum of terms \(\pm \kappa x^\beta\) with \(\alpha \triangleright \beta\). The nonsymmetric Jack polynomials are the simultaneous eigenvectors of \(\{U_i (\kappa) : 1 \leq i \leq N\}\) and they are well-defined for generic \(\kappa\).

2. \(S_N\)-modules

In this section we find necessary conditions on a partition \(\tau\) of \(N\) for the existence of singular polynomials of isotype \(\tau\). Suppose \(f\) is a singular polynomial for some singular value \(\kappa_0\). We may assume \(f\) is homogeneous because the operators \(D_i (\kappa)\) are homogeneous and that \(f\) has rational coefficients \((D_i (\kappa_0)\) is a rational operator). Any translate of \(f\) by \(S_N\) is singular so \(\text{span}_{Q} \{ w f : w \in S_N \}\) is an \(S_N\)-module of singular polynomials for \(\kappa_0\). Suppose one of the irreducible components has isotype \(\tau\), for some partition \(\tau\) with \(|\tau| = N\). This decomposition is a computation over \(Q\) (from the representation theory of \(S_N\)). Henceforth we restrict our attention to this module, denoted by \(M\).

We turn to the application of Murphy’s results. For any given isotype he determined the eigenvalues and transformation properties of the eigenvectors of the commuting operators \(\{ \sum_{j=1}^{i-1} (i, j) : 2 \leq i \leq N \}\) (Jucys-Murphy elements). The results have to be read in reverse in a certain sense.

**Proposition 1.** Suppose \(f\) is a singular polynomial for \(\kappa = \kappa_0 \in Q\) and \(1 \leq i \leq N\), then \(U_i (\kappa_0) f = f + \kappa_0 \sum_{j=i+1}^{N} (i, j) f\).
Proof: We have the commutation $D_i (\kappa) (x_i f) = x_i D_i (\kappa) f + f + \kappa \sum_{j \neq i} (i, j) f$. Now set $\kappa = \kappa_0$ and note that $U_i (\kappa_0) f = D_i (\kappa_0) (x_i f) - \kappa_0 \sum_{j < i} (i, j) f = -\kappa_0 \sum_{j < i} (i, j) f$. \hfill $\square$

Denote the Murphy elements $\omega_i = \sum_{j=N-i+2}^N (N+1-i, j)$ for $2 \leq i \leq N$ and let $\omega_1 = 0$ (as a transformation); then $U_i (\kappa_0) f = f + \kappa_0 \omega_{N+1-i} f$ for $f \in M$. A standard Young tableau (SYT) of shape $\lambda$ is a one-to-one assignment of the numbers $\{1, \ldots, N\}$ to the nodes of the Ferrers diagram $\{(i, j) \in \mathbb{N}^2 : 1 \leq i \leq \ell (\tau), 1 \leq j \leq \tau_i\}$ so that the entries increase in each row and in each column. The notation $T (i, j)$ refers to the entry at row $i$, column $j$. There is an order on SYT's of given shape (for details see [3, p.288]) and the maximum SYT in this order, denoted by $T_0$, is produced by entering the numbers $1, 2, \ldots, N$ row by row (the first row is $1, \ldots, \tau_1$, the second is $\tau_1 + 1, \ldots, \tau_1 + \tau_2$ and so forth).

**Definition 5.** Let $\tau \in \mathbb{N}_0^{N,P}$ with $|\tau| = N$, and let $Y (\tau)$ denote the set of SYT's of shape $\tau$. Suppose $T \in Y (\tau)$ then let $rw (i, T), cm (i, T), \eta_i (T) := cm (i, T) - rw (i, T)$ denote the row, column and content, respectively, of the node of $T$ containing $i$, for $1 \leq i \leq N$.

(With this notation $T (rw (i, T), cm (i, T)) = i$.) Murphy constructed a basis $\{f_T : T \in Y (\tau)\}$ for the irreducible representation of isotype $\tau$ such that $\omega_i f_T = \eta_i (T) f_T$ for each $i$ and $T$ (actually, this is an isomorphic image of the construction, which is in terms of specific polynomials, of minimal degree). The eigenvalues $(\eta_1 (T), \ldots, \eta_N (T))$ determine the SYT $T$ uniquely thus there is a unique (up to scalar multiplication) basis $\{f_T : T \in Y (\tau)\}$ for $M$ with $U_i (\kappa_0) f_T = (1 + \kappa_0 \eta_{N+1-i} (T)) f_T$, for $1 \leq i \leq N$, $T \in Y (\tau)$.

The argument for uniqueness of $T$ is in [3]: one can reconstruct $T$ by adjoining boxes containing $2, 3, \ldots, N$ to $1$ by using the values $\eta_2 (T), \ldots, \eta_N (T)$; at any stage the locations at which one can adjoin a box to make a larger SYT have different contents.) We will show that $f_{T_0}$ is (a multiple of) $x^s + \sum_{\beta \leq \lambda} A_{\beta} x^\beta$ with coefficients $A_{\beta} \in \mathbb{Q}$ and $\lambda \in \mathbb{N}_0^{N,P}$ with

\begin{equation}
\lambda_{N+1-i} = -\kappa_0 \sum_{j=1}^{s-1} (\tau_j + 1), \text{ for } \sum_{j=1}^{s-1} \tau_j < i \leq \sum_{j=1}^s \tau_j.
\end{equation}

This implies that if $\kappa_0 = -\frac{m}{n}$ with $\gcd (m, n) = 1$ then $n | (\tau_j + 1)$ for $1 \leq j < \ell (\tau)$ (the maximum value for $s$ in the above formula). The proof relies on the triangularity properties of the $U_i (\kappa_0)$ with respect to the order $\triangleright$.

**Definition 6.** For each $T \in Y (\tau)$ let $C_T = \{\beta \in \mathbb{N}_0^N : \text{coef} (f_T, \beta) \neq 0\}$. Let $C$ be the set of $\alpha \in \cup_{T \in Y (\tau)} C_T$ such that $\alpha$ is $\triangleright$-maximal in some $C_T$ (that is, $\alpha, \beta \in C_T$ and $\beta \triangleright \alpha$ implies $\beta = \alpha$).

**Lemma 1.** If $\alpha$ is a $\triangleright$-maximal element of $C$ then $\alpha$ is a partition.

Proof. It suffices to show that for any $\beta \in \cup_{T \in C_T}$ there exists a partition $\lambda \in C$ with $\lambda \triangleright \beta$. Since $M = \text{span}_Q \{f_T : T \in Y (\tau)\}$ is $S_N$-invariant we see that for any $w \in S_N$ and $\beta \in C_T$ for some $T$ there exists $T_1 \in Y (\tau)$ such that $w \beta \in C_{T_1}$ (note that $w f_T (x) = f_T (x w)$ and $w (x^\beta) = x^{w \beta}$). So $\cup_{T \in C_T}$ is $S_N$-invariant, in particular
if $\beta \in \cup_T C_T$ then $\beta^+ \in \cup_T C_T$. Thus there exists $\gamma \in C$ such that $\gamma \triangleright= \beta^+$. Since $C$ is finite the maximal elements $\alpha$ satisfy $\alpha \triangleright= \alpha^+$, that is, $\alpha$ is a partition. \hfill \Box

The next step is to show that there is a unique maximal element in $C$ determined by equation 2.1.

**Lemma 2.** If $T \in Y(\tau)$ satisfies $\eta_{s+1}(T) \leq \eta_s(T) + 1$ for $1 \leq s < N$ then $T = T_0$.

**Proof.** We have to show that the condition implies $rw(s, T) \leq rw(s+1, T)$ for each $s$. Fix $s$ and let $T(i_1, j_1) = s$ and $T(i_2, j_2) = s+1$ so that $\eta_s(T) - \eta_{s+1}(T) = (j_1 - j_2) + (i_2 - i_1)$. We list the possibilities for these nodes in any SYT. If $s$ and $s+1$ are in the same row of $T$ then $i_2 = i_1$, $j_2 = j_1 + 1$ and $\eta_{s+1}(T) = \eta_s(T) + 1$. If $s$ and $s+1$ are in the same column of $T$ then $i_2 = i_1 + 1$, $j_2 = j_1$ and $\eta_{s+1}(T) = \eta_s(T) - 1$. The condition $i_1 < i_2$ and $j_1 < j_2$ is impossible or else $s < T(i_2, j_1) < s+1$. Also the condition $i_1 > i_2$ and $j_1 > j_2$ is impossible or else $s+1 < T(i_1, j_2) < s$. If $i_1 < i_2$ and $j_1 > j_2$ then $\eta_s(T) - \eta_{s+1}(T) \geq 2$. The case $i_1 > i_2$ and $j_1 < j_2$ is ruled out by hypothesis because it implies $\eta_s(T) - \eta_{s+1}(T) \leq -2$. \hfill \Box

**Theorem 1.** Suppose $\lambda$ is a $\triangleright$-maximal element of $C$ then $\lambda$ is $\triangleright$-maximal in $C_{T_0}$ and is given by equation 2.1.

**Proof.** By hypothesis $\lambda$ is a partition and is $\triangleright$-maximal in $C_T$ for some $T \in Y(\tau)$. By the triangularity property of $U(\kappa_0)$ we have that

$$\text{coef} (\langle 1 + \kappa_0 \eta_{N+1-i}(T) \rangle f_T, \lambda) = \text{coef} (U(\kappa_0) f_T, \lambda)$$

and $\xi_i (\lambda; \kappa_0) = (N - i) \kappa_0 + \lambda_i + 1$, for $1 \leq i \leq N$. This gives the equations

$$(N - i) \kappa_0 + \lambda_i + 1 = 1 + \kappa_0 \eta_{N+1-i}(T),$$

$$\lambda_{N+1-i} = \kappa_0 (\eta_i(T) + 1 - i).$$

Since $\lambda$ is a partition $\lambda_{N+1-i} \leq \lambda_{N-i}$ for $1 \leq i < N$ and thus $\eta_i(T) + 1 - i \geq \eta_{i+1}(T) + 1 - (i+1)$ (note that $\kappa_0 < 0$). By Lemma 2 $T = T_0$. By definition of $T_0$ for $1 \leq i \leq \tau_1$ we have $\eta_i(T_0) = i - 1$ thus $\lambda_{N+1-i} = 0$. In the range $\sum_{j=1}^{\tau_1} \tau_j + 1 \leq i \leq \sum_{j=1}^{\tau_1} \tau_j$ (row $s$ of $T_0$) $\eta_i(T_0) = \left( \sum_{j=1}^{\tau_1} \tau_j \right) - s$ and $\lambda_{N+1-i} = -\kappa_0 \left( \sum_{j=1}^{\tau_1-1} \tau_j + s - 1 \right) = -\kappa_0 \sum_{j=1}^{\tau_1} \tau_j + 1 \right).$ \hfill \Box

**Corollary 1.** There is a unique $\triangleright$-maximal element $\lambda$ of $C$ given by equation 2.1 and $n | (\tau_j + 1) \leq 1 \leq j < \ell(\tau)$ (where $\kappa_0 = -\frac{m}{n}$ and $\gcd(m, n) = 1$).

**Proof.** The uniqueness is now obvious. The equation $\lambda_{N+1-i} = m \sum_{j=1}^{\tau_1} \frac{\tau_j + 1}{n}$ for $s-1 \leq \sum_{j=1}^{\tau_1} \tau_j < i \leq \sum_{j=1}^{\tau_1} \tau_j$ shows inductively that $n | (\tau_j + 1)$ for $1 \leq j < \ell(\tau)$; since the maximum value of $s$ is $\ell(\tau)$.

**Corollary 2.** For any $\kappa_0 = -\frac{m}{n}$ and partition $\tau$ of $N$ there is at most one irreducible $S_N$-module, consisting of singular polynomials for the singular value $\kappa_0$, that has isotope $\tau$.\hfill \Box
Proof. Suppose there are two unequal modules $M$ and $M'$ satisfying the hypotheses. Let $\{f_T : T \in Y(\tau)\}$ and $\{f'_T : T \in Y(\tau)\}$ be the respective bases for $M$ and $M'$ produced by Murphy’s construction. Normalize the two bases so that both $f_{T_0}$ and $f'_{T_0}$ are monic in $x^\lambda$ (that is, $f_{T_0} = x^\lambda + \sum_{\beta \in \lambda} A_\beta x^\beta$ and $f'_{T_0}$ has the same form with $A_\beta$ replaced by $A'_\beta$), with $\lambda$ given by equation 2.1. Let $g_T = f_T - f'_T$ for $T \in Y(\tau)$, then span$_\mathbb{Q} \{g_T : T \in Y(\tau)\}$ consists of singular polynomials and its basis has the same transformation properties under the action of $S_N$ as the basis of $M$. By the Theorem $\text{coef}(g_{T_0}, \lambda) \neq 0$, which is a contradiction. □

The following summarizes the results of this section. The polynomial $f_{T_0}$ is renamed $g_\lambda$.

**Theorem 2.** Suppose there exist singular polynomials for $\kappa_0 = -\frac{m}{n}$ with $\gcd(m, n) = 1$ (and $2 \leq n \leq N$) of isotype $\tau$, a partition of $N$, then $n| (\tau_i + 1)$ for $1 \leq i < \ell(\tau)$ and there is a unique singular polynomial $g_\lambda = x^\lambda + \sum_{\beta \in \lambda} A_\beta x^\beta$ (with $A_\beta \in \mathbb{Q}$) of isotype $\tau$ such that $\mathcal{U}_i(\kappa_0) g_\lambda = \xi_i(\lambda; \kappa_0) g_\lambda$ for $1 \leq i \leq N$, where $\lambda$ is given by equation 2.1.

### 3. Nonsymmetric Jack polynomials

These polynomials are the simultaneous eigenvectors of the commuting set of operators $\{\mathcal{U}_i(\kappa) : 1 \leq i \leq N\}$. The existence follows from the triangular property and the fact that the correspondence (from compositions to eigenvalues) $\alpha \rightarrow (\xi_i(\alpha; \kappa))_{i=1}^N$ is one-to-one for generic $\kappa$. We use the notation from $\mathbb{P}$ (for now just the $x$-monic version is used but there will be a reference to the $p$-monic version).

**Definition 7.** For $\alpha \in \mathbb{N}_0^N$, let $\zeta_\alpha^x(\kappa)$ denote the $x$-monic simultaneous eigenvectors, that is, $\mathcal{U}_i(\kappa) \zeta_\alpha^x(\kappa) = \xi_i(\alpha; \kappa) \zeta_\alpha^x(\kappa)$ for $1 \leq i \leq N$ and $\zeta_\alpha^x(\kappa) = x^\alpha + \sum_{\beta \prec \alpha} A_\beta^x(\kappa) x^\beta$, with coefficients $A_\beta^x(\kappa) \in \mathbb{Q}$.\kappa.

**Definition 8.** For $1 \leq i \leq N$ the operators $\mathcal{B}_{ij}$ (with $j \neq i$) and the operator $\mathcal{B}_i$ (each maps $\mathcal{P}_n$ into itself, for $n \in \mathbb{N}_0$) are given by

$$
\mathcal{B}_{ij}p(x) := \frac{x_i p(x) - x_j p(x(i,j))}{x_i - x_j} - \left\{ \begin{array}{ll}
0, & i < j, \\
p(x(i,j)), & i > j,
\end{array} \right.
$$

\[ \mathcal{B}_i p := \sum_{j \neq i} \mathcal{B}_{ij} p, \text{ for } p \in \mathcal{P}. \]

In this notation $\mathcal{U}_i(\kappa) p(x) = \frac{\partial}{\partial x_i} (x_i p(x)) + \kappa \mathcal{B}_i p(x)$. There is an easily proved identity: $\mathcal{B}_{ij} + \mathcal{B}_{ji} = 1$, and this shows directly that

$$
\sum_{i=1}^N \mathcal{U}_i(\kappa) = N + \sum_{i=1}^N x_i \frac{\partial}{\partial x_i} + \kappa \frac{N(N-1)}{2}.
$$
For \( \alpha \in \mathbb{N}_0^N \) and \( i \neq j \) by direct computation we obtain:

\[
B_{ij} x^\alpha = \sum_{l=0}^{\alpha_i - \alpha_j} \left( \frac{x_j}{x_i} \right)^l x^\alpha, \text{ for } \alpha_j \leq \alpha_i, i < j,
\]

(3.1)

\[
B_{ij} x^\alpha = \sum_{l=0}^{\alpha_i - \alpha_j - 1} \left( \frac{x_j}{x_i} \right)^l x^\alpha, \text{ for } \alpha_i \leq \alpha_j, i > j,
\]

(3.2)

\[
B_{ij} x^\alpha = - \sum_{l=1}^{\alpha_j - \alpha_i - 1} \left( \frac{x_i}{x_j} \right)^l x^\alpha, \text{ for } \alpha_i < \alpha_j, i < j,
\]

(3.3)

\[
B_{ij} x^\alpha = - \sum_{l=1}^{\alpha_j - \alpha_i} \left( \frac{x_i}{x_j} \right)^l x^\alpha, \text{ for } \alpha_i < \alpha_j, i > j.
\]

(3.4)

There is another invariant subspace structure for \( \{U_\ell(\kappa)\} \) besides the \( \triangleright \)-triangular property. The purpose of the following arguments is to allow the computation of certain coefficients of \( \zeta^r_\alpha(\kappa) \) crucial in the arguments of Section 5.

**Definition 9.** For \( 1 \leq s \leq N \) and \( n \geq 1 \) let

\[
I_{s,n}^{(N)} := \{ \alpha \in \mathbb{N}_0^N : \alpha_j < n \text{ for } 1 \leq i \leq s, \alpha_j \leq n \text{ for } s + 1 \leq i \leq N \},
\]

\[
P_{s,n}^{(N)} := \text{span}_{\mathbb{Q}(\kappa)} \left\{ x^\alpha : \alpha \in I_{s,n}^{(N)} \right\}.
\]

Note that each \( I_{s,n}^{(N)} \) is finite and \( P_{s,n}^{(N)} \) is the direct sum of its homogeneous subspaces \( P_{s,n}^{(N)} \cap P_k, k \geq 0 \).

**Lemma 3.** Suppose \( 1 \leq s \leq N \) and \( n \geq 1 \), then \( U_\ell(\kappa) P_{s,n}^{(N)} \subset P_{s,n}^{(N)} \) for \( 1 \leq i \leq N \), and \( \text{span}_{\mathbb{Q}(\kappa)} \left\{ \zeta^r_\alpha(\kappa) : \alpha \in I_{s,n}^{(N)} \right\} = P_{s,n}^{(N)}. \)

**Proof.** Let \( \alpha \in I_{s,n}^{(N)} \). It suffices to show \( B_{ij} x^\alpha \in P_{s,n}^{(N)} \) for all \( i, j \). By formulae 3.1 and 3.2 this is obvious for \( 1 \leq i, j \leq s \) or \( s + 1 \leq i, j \leq N \), or \( \max(\alpha_i, \alpha_j) < n \). Only the two cases \( 1 \leq i \leq s, \alpha_j = n \) (thus \( j > s \)) and \( 1 \leq j \leq s, \alpha_i = n \) (with \( i > s \)) remain to be considered. Formulae 3.3 and 3.4 respectively show that \( B_{ij} x^\alpha \in P_{s,n}^{(N)} \). For any \( \alpha \in \mathbb{N}_0^N \) the eigenvector \( \zeta^r_\alpha(\kappa) \) is contained in the orbit of \( x^\alpha \) under the algebra generated by \( \{U_\ell(\kappa)\} \) hence \( x^\alpha \in P_{s,n}^{(N)} \) implies \( \zeta^r_\alpha(\kappa) \in I_{s,n}^{(N)} \). That the span of \( \{\zeta^r_\alpha(\kappa)\} \) is all of \( P_{s,n}^{(N)} \) follows easily (dimension argument, for example).

**Definition 10.** For a partition \( \lambda \) and an integer \( s \) with \( 1 \leq s \leq N \) define the insertion operator \( \iota(s; \lambda) : \mathbb{N}_0^N \to \mathbb{N}_0^{s+s(\lambda)} \) as follows: for \( \alpha \in \mathbb{N}_0^N \)

\[
(\iota(s; \lambda) \alpha)_i = \begin{cases} 
\alpha_i, & 1 \leq i \leq s \\
\lambda_i - s, & s < i \leq s + s(\lambda) \\
\alpha_i - s(\lambda), & s + s(\lambda) < i \leq N + s(\lambda).
\end{cases}
\]

The definition is only interesting when \( \alpha \in I_{s,n}^{(N)} \) where \( n = \lambda(\lambda) \), in which case the following rank equations hold: let \( \beta = \iota(s; \lambda) \alpha \) and \( k = \ell(\lambda) \), then \( r(\beta, i) = r(\alpha, i) + k \) for \( 1 \leq i \leq s \), \( r(\beta, i) = r(\alpha, i - k) + k \) for \( s + k < i \leq N + k \), and \( r(\beta, i) = i - s \) for \( s + 1 \leq i \leq s + k \).

**Theorem 3.** Suppose \( \lambda \) is a partition, \( 1 \leq s \leq N \), \( \alpha, \beta \in I_{s,n}^{(N)} \) where \( n = \lambda(\lambda) \), and \( \alpha \triangleright \beta \), then \( \text{coef} \left( \zeta^r_\alpha(s, \lambda) \alpha (\kappa), \iota(s, \lambda) \beta \right) = \text{coef} (\zeta^r_\alpha(\kappa), \beta) \).
It suffices to prove this for $\ell(\lambda) = 1$ because then one can insert one part of $\lambda$ at a time in nondecreasing order: explicitly let $\lambda^{(j)} = \left(\lambda_{\ell(\lambda)+1-j}, \lambda_{\ell(\lambda)+2-j}, \ldots, \lambda_{\ell(\lambda)}\right)$ for $1 \leq j \leq \ell(\lambda)$, then $\ell\left(s, \left(\lambda_{\ell(\lambda)+j}\right)\right) = \ell\left(s, \lambda^{(j)}\right)$; also if $\alpha, \beta \in I_{s,n}$, then $\ell\left(s, \lambda^{(j)}\right) \alpha \in I_{s,k}^{(N+2)}$ where $k = \lambda_{\ell(\lambda)+1-j}$ and $\alpha \triangleright \beta$ implies $\ell\left(s, \lambda^{(j)}\right) \alpha \triangleright \ell\left(s, \lambda^{(j)}\right) \beta$.

For arbitrary $M \geq 1$ let $\mathcal{P}^{(M)} = \text{span}_{\mathbb{Q}(\kappa)} \{x^\alpha : \alpha \in \mathbb{N}_0^M\}$ and let $U_i^{(M)}(\kappa)$ denote the operator $U_i(\kappa)$ for $M$ variables. For $M > N$ let $\pi_{MN}$ be the projection from $\mathcal{P}^{(M)}$ onto $\mathcal{P}^{(N)}$ defined by setting $x_{N+1} = x_{N+2} = \ldots = x_M = 0$. The coefficients of the $\xi^n$ do not depend on the number of variables (that is coef $(\xi^n, \beta)$ is independent of $N \geq \max(\ell(\alpha), \ell(\beta))$ because of the intertwining relation

$$\pi_{MN} U_i^{(M)}(\kappa) = \left( U_i^{(N)}(\kappa) + (M - N) \kappa \right) \pi_{MN},$$

for $1 \leq i \leq N < M$. Fix integers $n, s$ with $n \geq 1$ and $1 \leq s \leq N$ and define the map $\iota_{s,n} : \mathcal{P}^{(N)} \to \mathcal{P}^{(N+1)}$ by $\iota_{s,n} x^\alpha = x^\beta$ for $\alpha \in \mathbb{N}_0^N$ and $\beta = \ell\left(s, (n)\right) \alpha = (\alpha_1, \ldots, \alpha_s, n, \alpha_{s+1}, \ldots, \alpha_N)$ and extending by linearity to all polynomials. Direct computation yields the identities:

$$\begin{align*}
U_i^{(N+1)}(\kappa) \iota_{s,n} - \iota_{s,n} U_i^{(N)}(\kappa) &= \kappa B_{i,s+1} \iota_{s,n}, \text{ for } 1 \leq i \leq s, \\
U_i^{(N+1)}(\kappa) \iota_{s,n} - \iota_{s,n} U_i^{(N)}(\kappa) &= \kappa B_{i+1,s+1} \iota_{s,n}, \text{ for } s+1 \leq i \leq N.
\end{align*}$$

We show that if $\alpha \in I_{s,n}^{(N)}$ then $\iota_{s,n} \xi^n(\kappa)$ is congruent to $\xi^n(\iota_{s,n}(\alpha)) \alpha(\kappa)$ modulo the subspace $P_{s+1,n}^{(N+1)}$. To illustrate the argument, suppose there is a linear operator $\mathcal{V}$ with an invariant subspace $E$ and there is a vector $f$ and number $c$ so that $\mathcal{V} f - c f \in E$ then $f - (\mathcal{V} - c) |_E^{-1} (\mathcal{V} f - c f)$ is an eigenvector of $\mathcal{V}$ with eigenvalue $c$, provided that the restriction of $\mathcal{V} - c$ to $E$ is invertible. This can be adapted for simultaneous eigenvectors of pairwise commuting operators by extending the base field $\mathbb{Q}(\kappa)$, adjoining another formal variable (transcendental) $v$ and considering just one operator $\sum_{i=1}^N v^i U_i^{(N)}(\kappa)$ (or $\sum_{i=1}^{N+1} v^i U_i^{(N+1)}(\kappa)$, as appropriate). The eigenvalues $\sum_{i=1}^N v^i \xi^n(\alpha; \kappa)$ are simple ($\alpha \in \mathbb{N}_0^N$ and generic $\kappa$). Denote the field $\mathbb{Q}(\kappa, v)$ by $\mathbb{K}$.

**Lemma 4.** Suppose $1 \leq s \leq N$ and $n \geq 1$. If $\alpha \in I_{s,n}^{(N)}$ then $\iota_{s,n} \xi^n(\kappa) = \xi^n(\iota_{s,n}(\alpha)) \alpha(\kappa) + f_{\alpha}$ for some $f_{\alpha} \in P_{s+1,n}^{(N+1)}$.

**Proof.** First we show $B_{i,s+1} \iota_{s,n} P_{s,n}^{(N)} \subset P_{s+1,n}^{(N+1)}$ for $i \neq s + 1$. Let $\alpha \in I_{s,n}^{(N)}$ and $\beta = \ell\left(s, (n)\right) \alpha$. For $1 \leq i \leq s$ by Formula 6.23 $B_{i,s+1} x^\beta = - \sum_{l=1}^{n-\alpha_i-1} \left( \frac{x}{x_{s+1}} \right)^l x^\beta$ with the key (change from $\beta$) terms being $x_{s+1}^{\alpha_i+l} x_{s+1}^{n-l}$ where $\alpha_i + 1 \leq \alpha_{i+1} \leq n-1$ and $\alpha_{i+1} + 1 \leq n - l \leq n - 1$; if $\alpha_i = n - 1$ then $B_{i,s+1} x^\beta = 0$. Suppose $s + 2 \leq i \leq N + 1$; if $\alpha_i = n = \beta_i$, then $B_{i,s+1} x^\beta = 0$. By Formula 5.2, if $\alpha_i = n = \beta_i$ then

$$B_{i,s+1} x^\beta = - \sum_{l=1}^{n-\beta_i} \left( \frac{x_{s+1}}{x_{s+1}} \right)^l x^\beta$$

with key terms $x_{s+1}^{\beta_i+l} x_{s+1}^{n-l}$ where $\beta_i + 1 \leq \beta_i + l \leq n$ and $\beta_i \leq n - l \leq n - 1$. Thus $B_{i,s+1} x^\beta \in P_{s+1,n}^{(N+1)}$. 


Temporarily we use a superscript on the eigenvalues $\xi_i(\alpha; \kappa)$ to indicate the number of variables, then

$$
\xi^{(N+1)}_i(\beta; \kappa) = (N + 1 - r(\beta, i))\kappa + \beta_i + 1
$$

$$
= (N + 1 - (r(\alpha, i) + 1))\kappa + \alpha_i + 1 = \xi^{(N)}_i(\alpha; \kappa)
$$

for $1 \leq i \leq s$ and, similarly, $\xi^{(N+1)}_i(\beta; \kappa) = \xi^{(N)}_{i-1}(\alpha; \kappa)$ for $s + 2 \leq i \leq N + 1$. The eigenvalues $\{\xi^{(N+1)}_i(\beta; \kappa) : 1 \leq i \leq N + 1, i \neq s + 1 \}$ and the degree of homogeneity $|\beta| = |\alpha| + n$ determine $\xi^N_\beta(\kappa)$ uniquely, subject to $\text{coef} (\xi^N_\beta(\kappa), \beta) = 1$, because

$$
\sum_{i=1}^{N+1} U_i^{(N+1)}(\kappa) = N + 1 + \sum_{i=1}^{N+1} x_i \frac{\partial}{\partial x_i} + \kappa \frac{N(N+1)}{2}.
$$

Let

$$
\mathcal{V} := \left( \sum_{i=1}^{s} \sum_{i=s+2}^{N+1} v_i U_i^{(N+1)}(\kappa) \right).
$$

The polynomials $\{\xi^N_\gamma(\kappa) : \gamma \in \mathbb{N}_0^{N+1}, |\gamma| = |\alpha| + n \}$ form a basis of eigenvectors of $\mathcal{V}$ for $E := \text{span}_F \{x^\gamma : \gamma \in \mathbb{N}_0^{N+1}, |\gamma| = |\alpha| + n \}$ and each eigenvalue is simple. Let $F := \text{span}_F \{x^\gamma : \gamma \in I^{(N+1)}_{s+1,n}, |\gamma| = |\alpha| + n \}$ then $\mathcal{V}F \subseteq F$ by Lemma 3. Finally consider

$$
\mathcal{V} \iota_s, n \xi^x_\alpha(\kappa) = \sum_{i=1}^{s} v_i \left( \iota_s, n U_i^{(N)} + \kappa \mathcal{B}_{i,s+1} \iota_s, n \right) \xi^x_\alpha(\kappa) +
$$

$$
+ \sum_{i=s+2}^{N+1} v_i \left( \iota_s, n U_i^{(N)} + \kappa \mathcal{B}_{i,s+1} \iota_s, n \right) \xi^x_\alpha(\kappa)
$$

$$
= \sum_{i=1}^{N+1} v_i \xi^{(N+1)}_i(\beta; i) \iota_s, n \xi^x_\alpha(\kappa) + h_\alpha,
$$

where $h_\alpha = \kappa \left( \sum_{i=1}^{s} \sum_{i=s+2}^{N+1} v_i \mathcal{B}_{i,s+1} \iota_s, n \xi^x_\alpha(\kappa) \right)$ and $h_\alpha \in F$, since $\xi^x_\alpha(\kappa) \in F_s(\mathcal{N})$. Let $\mathcal{V}_\beta$ be the restriction of $\mathcal{V} - \sum_{i=1, i \neq s+1}^{N+1} v_i \xi^{(N+1)}_i(\beta; i)$ to the invariant subspace $F$ and let $f_\alpha = \mathcal{V}^{-1} h_\alpha$, then $\iota_s, n \xi^x_\alpha(\kappa) - f_\alpha = \xi^x_\beta(\kappa)$ because $\text{coef}(f_\alpha, \beta) = 0$ and $\text{coef}(\iota_s, n \xi^x_\alpha(\kappa), \beta) = \text{coef}(\xi^x_\alpha(\kappa), \alpha) = 1$. Since $f_\alpha = \iota_s, n \xi^x_\alpha(\kappa) - \xi^x_\beta(\kappa)$ the coefficients of $f_\alpha$ are in $\mathbb{Q}(\kappa)$.

**Corollary 3.** Suppose $\alpha, \gamma \in I^{(N)}_{s,n}$ then

$$
\text{coef} \left( \xi^x_{\iota(s,(n))\alpha}(\kappa), \iota(s,(n)) \gamma \right) = \text{coef} \left( \xi^x_\alpha(\kappa), \gamma \right).
$$

**Proof.** By definition $\text{coef} (\xi^x_\alpha(\kappa), \gamma) = \text{coef} (\iota_s, n \xi^x_\alpha(\kappa), \iota(s,(n)) \gamma)$. Also $\iota(s,(n)) \gamma)_{s+1} = n$ and thus $\text{coef} (f, \iota(s,(n)) \gamma) = 0$ for any $f \in F_{s+1,n}$.

This completes the proof of Theorem 3.

The poles of the coefficients of $\xi^x_\alpha(\kappa)$ play a key role in the analysis of singular polynomials. Knop and Sahi [8] found an algorithm for the evaluation of the coefficients. It uses the idea of extending the definition of Ferrers diagrams to compositions and associating a hook-length to each node in the diagram. The Ferrers diagram of a composition $\alpha \in \mathbb{N}_0^{N}$ is the set $\{(i, j) : 1 \leq i \leq \ell(\alpha), 0 \leq j \leq \alpha_i\}$. 


For each node \((i, j)\) with \(1 \leq j \leq \alpha_i\) there are two special subsets of the Ferrers diagram, the arm \(\{(i, l): j < l \leq \alpha_i\}\) and the leg \(\{(l, j): l > i, j \leq \alpha_l \leq \alpha_i\} \cup \{(l, j - 1): l < i, j - 1 \leq \alpha_l < \alpha_i\}\). The node itself, the arm and the leg make up the hook. The definition of hooks for compositions is from [8, p.15]. The cardinality of the leg is called the leg-length, formalized by the following:

**Definition 11.** For \(\alpha \in \mathbb{N}_0^N\), \(1 \leq i \leq \ell(\alpha)\) and \(1 \leq j \leq \alpha_i\) the leg-length is

\[
L(\alpha; i, j) := \#\{l : l > i, j \leq \alpha_l \leq \alpha_i\} + \#\{l : l < i, j \leq \alpha_l + 1 \leq \alpha_i\}.
\]

For \(t \in \mathbb{Q}(\kappa)\) the hook-length and the hook-length product for \(\alpha\) are given by

\[
h(\alpha, t; i, j) = (\alpha_i - j + t + \kappa L(\alpha; i, j))^{\frac{\ell(\alpha)}{\alpha_i}}
\]

\[h(\alpha, t) = \prod_{i=1}^{\ell(\alpha)} \prod_{j=1}^{\alpha_i} h(\alpha, t; i, j),\]

Note that the indices \(\{i : \alpha_i = 0\}\) are omitted in the product \(h(\alpha, t)\). In [2] and [3] we used the notation

\[E_\varepsilon(\alpha) = \prod\left\{1 + \frac{\varepsilon \kappa}{\kappa (r(\alpha, i) - r(\alpha, j)) + \alpha_j - \alpha_i} : i < j, \alpha_i < \alpha_j\right\}, \varepsilon = \pm.\]

The denominator also equals \(\xi_j(\alpha; \kappa) - \xi_i(\alpha; \kappa)\). The relation to \(h(\alpha, t)\) (for the values \(t = 1, \kappa + 1\) which are of concern here) is the following:

**Lemma 5.** For \(\alpha \in \mathbb{N}_0^N\), \(h(\alpha, \kappa + 1) = h(\alpha^+, \kappa + 1) E_+ (\alpha)\) and \(h(\alpha, 1) = \frac{h(\alpha^+, 1)}{E_- (\alpha)}\).

**Proof.** We use induction on adjacent transpositions. The statements are true for \(\alpha = \alpha^+\). Fix \(\alpha^+\) and suppose \(\alpha_i > \alpha_{i+1}\) for some \(i\). Let \(\sigma = (i, i + 1)\). Consider the ratio \(\frac{h(\sigma \alpha, t)}{h(\alpha, t)}\). The only node whose hook-length changes (in the sense of interchanging rows \(i\) and \(i + 1\) of the Ferrers diagram) is \((i, \alpha_{i+1} + 1)\). Explicitly \(h(\sigma \alpha, t; s, j) = h(\alpha, t; s, j)\) for \(s \neq i, i + 1\) and \(1 \leq j \leq \alpha_s\), \(h(\sigma \alpha, t; i, j) = h(\alpha, t; i + 1, j)\) for \(1 \leq j \leq \alpha_{i+1}\) and \(h(\sigma \alpha, t; i + 1, j) = h(\alpha, t; i, j)\) for \(1 \leq j \leq \alpha_i\) except for \(j = \alpha_{i+1} + 1\). Thus \(\frac{h(\sigma \alpha, t)}{h(\alpha, t)} = \frac{h(\sigma \alpha, t; i + 1, \alpha_{i+1} + 1)}{h(\alpha, t; i, \alpha_{i+1} + 1)}\). Note that

\[L(\sigma \alpha; i + 1, \alpha_{i+1} + 1) = L(\alpha; i, \alpha_{i+1} + 1) + 1\]

(the node \((i, \alpha_{i+1})\) is adjoined to the leg). Let

\[E_1 = \{s : s \leq i, \alpha_s \geq \alpha_i\} \cup \{s : s > i, \alpha_s > \alpha_i\},\]

\[E_2 = \{s : s \leq i + 1, \alpha_s \geq \alpha_{i+1}\} \cup \{s : s > i + 1, \alpha_s > \alpha_{i+1}\},\]

thus by definition \(r(\alpha, i) = \#E_1\) and \(r(\alpha, i + 1) = \#E_2\). Now \(E_1 \subset E_2\) thus \(r(\alpha, i + 1) - r(\alpha, i) = \#(E_2 \setminus E_1)\) and \(E_2 \setminus E_1 = \{s : s < i, \alpha_s \geq \alpha_{i+1}\} \cup \{i\} \cup \{s : s > i + 1, \alpha_i \geq \alpha_s > \alpha_{i+1}\}\). This shows that \(\#(E_2 \setminus E_1) = 1 + L(\alpha; i, \alpha_{i+1} + 1)\), and

\[h(\alpha, t; i, \alpha_{i+1} + 1) = \kappa (r(\alpha, i + 1) - r(\alpha, i)) + t + \alpha_i - \alpha_{i+1} - 1,\]

\[h(\sigma \alpha, t; i + 1, \alpha_{i+1} + 1) = \kappa (r(\alpha, i + 1) - r(\alpha, i)) + t + \alpha_i - \alpha_{i+1} - 1.\]
Lemma 6. Suppose $\kappa$ is the unique

Thus

the latter equation is proven in Theorem 8.5.8, from [3] p.302, and

Thus $h(\alpha, 1; i, \alpha_{i+1} + 1)$ have the same transformation properties under adjacent transpositions and hence are equal. Similarly $h(\alpha, 1) = \frac{h(\alpha, 1; 1, \alpha_{i+1} + 1)}{E_+(\alpha)}$. \qed

Knop and Sahi [3] Theorem 5.1 showed that $h(\alpha, \kappa + 1) \zeta_\alpha^\kappa(\kappa)$ has all coefficients in $\mathbb{N}_0[\kappa]$ for each $\alpha \in \mathbb{N}_0^N$. When $\kappa$ takes on a negative rational number $\kappa_0$ it may happen that two different compositions have the same eigenvalues $(\xi_i(\alpha; \kappa_0))_{i=1}^N$ so one can not claim the existence of a basis of simultaneous eigenvectors of $\{U_i(\kappa_0) : 1 \leq i \leq N\}$. We recall the following from [2].

Definition 12. Let $\alpha, \beta \in \mathbb{N}_0^N$ and let $m, n \in \mathbb{N}$ with $\text{gcd}(m, n) = 1$ then say $(\alpha, \beta)$ is a $(-\frac{m}{n})$-critical pair (for $\alpha$) if $\alpha \triangleright \beta$ and $(n\kappa + m)$ divides $(r(\beta, i) - r(\alpha, i))\kappa + \alpha_i - \beta_i$ (in $\mathbb{Q}[\kappa]$) for $1 \leq i \leq N$. The definition implies $\xi_i(\alpha; -\frac{m}{n}) = \xi_i(\beta; -\frac{m}{n})$ for $1 \leq i \leq N$. We can deduce the existence of simple poles at $\kappa = -\frac{m}{n}$ in a certain coefficient.

Lemma 6. Suppose $\alpha, \beta \in \mathbb{N}_0^N$, $h(\alpha, \kappa + 1)$ has a simple zero at $\kappa_0 \in \mathbb{Q}$ and $(\alpha, \beta)$ is the unique $\kappa_0$-critical pair for $\alpha$, then $\text{cof}(\zeta_\alpha^\kappa(\kappa), \beta)$ has a simple pole at $\kappa_0$.

Proof. Since $\text{cof}(\zeta_\alpha^\kappa(\kappa), \beta)$ is independent of the number of variables $N$ provided $N \geq \max(\ell(\alpha), \ell(\beta))$ we may assume $N = \ell(\alpha) + |\alpha|$. Let $\gamma = (0^{\ell(\alpha)}, 1^{\ell(\beta)}) \in \mathbb{N}_0^N$ then by [3] $\text{cof}(\zeta_\alpha^\kappa(\kappa), \gamma) = (|\alpha|)!k^{\ell(\alpha)}|/h(\alpha, \kappa, 1)$. Let $f = \lim_{\kappa \to \kappa_0} (\kappa - \kappa_0) \zeta_\alpha^\kappa(\kappa)$ which exists as a polynomial over $\mathbb{Q}$ by hypothesis and is not zero because $\lim_{\kappa \to \kappa_0} (\kappa - \kappa_0) \text{cof}(\zeta_\alpha^\kappa(\kappa), \gamma) \neq 0$. The polynomial $f$ is a simultaneous eigenvector for $\{U_i(\kappa_0) : 1 \leq i \leq N\}$ because $U_i(\kappa_0) f = \lim_{\kappa \to \kappa_0} (\kappa - \kappa_0)U_i(\kappa) \zeta_\alpha^\kappa(\kappa) = \xi_i(\alpha; \kappa_0) f$. Let $\gamma$ be a $\triangleright$-maximal element of $\{\delta \in \mathbb{N}_0^N : \text{cof}(f, \delta) \neq 0\}$. By $\triangleright$-triangularity $\xi_i(\alpha; \kappa_0) = \xi_i(\delta; \kappa_0)$ for each $i$, thus $\delta = \alpha$ or $\delta = \beta$ by definition of critical pairs. It is impossible for $\delta = \alpha$ since $\text{cof}(f, \alpha) = \lim_{\kappa \to \kappa_0} (\kappa - \kappa_0) = 0$ hence $\delta = \beta$. So $\text{cof}(f, \beta) = \lim_{\kappa \to \kappa_0} (\kappa - \kappa_0) \text{cof}(\zeta_\alpha^\kappa(\kappa), \beta) \neq 0$. \qed

In the next sections the Lemma will be combined with Theorem 3.

Example 1. The conceptual proof of the Lemma may be the only reasonably effective method. For example in the next section we need the conclusion of Lemma 7 for $\text{cof}(\zeta_{(5, 6)}^N(\kappa), (2, 0, 3, 3, 3))$, which arises for $N = 5, \tau = (3, 2), \kappa_0 = -\frac{3}{2}$.
There is a combinatorial formula for the coefficients of $\zeta_\alpha^\beta (\kappa)$ due to Knop and Sahi $[8]$, which requires a sum over $3! \times 1721$ configurations for this example (the factorial comes from permuting the indices $(3, 4, 5)$). By direct (computer algebra) calculations this coefficient equals
\[
\frac{30c_4^3(1 + \kappa)^2(62\kappa^3 + 135\kappa^2 + 78\kappa + 40)}{(2\kappa + 3)(2\kappa + 5)(\kappa + 2)^2(\kappa + 3)(\kappa + 4)(\kappa + 5)}.
\]
The expression suggests that there is no practical closed form.

We address the problem of the relationship of a simultaneous eigenvector of $\{U_i (\kappa_0) : 1 \leq i \leq N\}$ to the nonsymmetric Jack polynomials; namely, how can such a polynomial be expressed as a limit as $\kappa \to \kappa_0$? For a given $\alpha \in \mathbb{N}_0^N$ and $\kappa_0 = -\frac{m}{n}$ let $C (\alpha, \kappa_0) = \{\beta : (\alpha, \beta)$ is a $\kappa_0$-critical pair$\}$. In the proof we again use the field $\mathbb{K} = \mathbb{Q} (\kappa, v)$ and the operator $\sum_{i=1}^N v^i U_i (\kappa)$; otherwise to each $\gamma \in E$ one has to associate some $i$ for which $\xi_i (\alpha; \kappa_0) \neq \xi_i (\gamma; \kappa_0)$. The expressions we consider are all rational in $\kappa$ (now with values in $\mathbb{Q} (v)$) so having no pole at $\kappa_0$ is equivalent to being analytic in a neighborhood of $\kappa_0$.

**Theorem 4.** Suppose for some $\alpha \in \mathbb{N}_0^N$ and $\kappa_0 = -\frac{m}{n}$ that there exists a simultaneous eigenvector $g_\alpha = x^\alpha + \sum_{\beta < \alpha} A_{\beta} x^\beta$ of $\{U_i (\kappa_0) : 1 \leq i \leq N\}$, with the coefficients $A_{\beta} \in \mathbb{Q}$, then there are coefficients $B_{\beta} (\kappa) \in \mathbb{K}$ defined for $\beta \in C (\alpha, \kappa_0)$ such that the polynomial $g_\alpha (\kappa) = \zeta_\alpha^\beta (\kappa) + \sum B_{\beta} (\kappa) \zeta_\beta^\gamma (\kappa) : \beta \in C (\alpha, \kappa_0)$ has no pole at $\kappa_0$ and $\lim_{\kappa \to \kappa_0} g_\alpha (\kappa) = g_\alpha$.

**Proof.** By the triangularity property $U_i (\kappa_0) g_\alpha = \xi_i (\alpha; \kappa_0) g_\alpha$. For generic $\kappa$ there are coefficients $B_{\gamma} (\kappa)$ defined for all $\gamma \prec \alpha$ so that $g_\alpha = \zeta_{\alpha}^\gamma (\kappa) + \sum_{\gamma < \alpha} B_{\gamma} (\kappa) \zeta_{\gamma}^\gamma (\kappa)$ (because the nonsymmetric Jack polynomials form a basis and the change-of-basis matrix is unimodular and triangular). Let $E = \{\gamma : \gamma \prec \alpha, \gamma \notin C (\alpha, \kappa_0)\}$. Apply the operator
\[
\mathcal{V} (\kappa) := \prod_{\gamma \in E} \frac{\sum_{i=1}^N v^i (U_i (\kappa) - \xi_i (\gamma; \kappa))}{\sum_{i=1}^N v^i (\xi_i (\alpha; \kappa) - \xi_i (\gamma; \kappa))}
\]
to both sides of the equation for $g_\alpha$, thus annihilating all $\zeta_{\gamma}^\gamma (\kappa)$ with $\gamma \in E$. The right hand side becomes
\[
\mathcal{V} (\kappa) g_\alpha = \zeta_{\alpha}^\gamma (\kappa) + \sum_{\beta \in C (\alpha, \kappa_0)} B_{\beta} (\kappa) \left( \prod_{\gamma \in E} \frac{\sum_{i=1}^N v^i (\xi_i (\beta; \kappa) - \xi_i (\gamma; \kappa))}{\sum_{i=1}^N v^i (\xi_i (\alpha; \kappa) - \xi_i (\gamma; \kappa))} \right) \zeta_{\beta}^\gamma (\kappa)
\]
with the last equation implicitly defining the coefficients $B_{\beta} (\kappa)$. We use the operators $\mathcal{B}$ from Definition $[8]$. To evaluate $\mathcal{V} (\kappa) g_\alpha$ directly we consider
\[
(U_i (\kappa) - \xi_i (\gamma; \kappa)) g_\alpha - (\xi_i (\alpha; \kappa) - \xi_i (\gamma; \kappa)) g_\alpha
\]
\[
- (U_i (\kappa_0) + (\kappa - \kappa_0) B_i - \xi_i (\alpha; \kappa)) g_\alpha
\]
\[
= (\xi_i (\alpha; \kappa_0) - \xi_i (\alpha; \kappa) + (\kappa - \kappa_0) B_i) g_\alpha
\]
\[
= (\kappa - \kappa_0) (r (\alpha, i) - N + B_i) g_\alpha.
\]
Thus for each \( \gamma \in E \) we have
\[
\sum_{i=1}^{N} v^i (U_i (\kappa) - \xi_i (\gamma; \kappa)) g_\alpha = g_\alpha + (\kappa - \kappa_0) \sum_{i=1}^{N} v^i (r (\alpha, i) - N + B_i) g_\alpha.
\]
The latter term has no pole at \( \kappa_0 \), since \((\alpha, \gamma)\) is not a \( \kappa_0 \)-critical pair. Apply this computation repeatedly to obtain \( V (\kappa) g_\alpha = g_\alpha + (\kappa - \kappa_0) p (\kappa) \), where \( p \) is polynomial in \( x \), rational in \( \kappa \) and has no pole at \( \kappa_0 \). Hence set \( g_\alpha = V (\kappa) g_\alpha \), then \( \lim_{\kappa \to \kappa_0} g_\alpha (\kappa) = g_\alpha \), and this completes the proof. \( \Box \)

If we apply this result to the hypothetical singular polynomial described in Theorem 2 that is \( g_\lambda = x^{\lambda_1} + \sum_{\beta \subset \lambda} A_\beta x^\beta \), we obtain
\[
V (\kappa) g_\lambda = \zeta^x (\kappa) + \sum \{ B_\beta (\kappa) \zeta^x (\kappa) : \beta \in C (\lambda, \kappa_0) \},
\]
which has no pole at \( \kappa_0 \). More importantly, since \( D_\lambda \) is polynomial in \( \kappa \), the relation \( \lim_{\kappa \to \kappa_0} D_i (\kappa) V (\kappa) g_\lambda = D_i (\kappa_0) g_\lambda = 0 \) holds. This is a key ingredient in the proof that \( \tau_2 < n \), because we can now apply the known formulae for \( D_i (\kappa) \zeta^x (\kappa) \).

The basic step is the formula for \( D_{R(\alpha)} (\kappa) \zeta^x (\kappa) \) for \( \alpha \in \mathbb{N}^N_0 \). The computation involves a cyclic shift. For \( 1 \leq i \leq N \) let \( \varepsilon (i) \in \mathbb{N}^N_0 \) denote the standard basis element, that is, \( \varepsilon (i)_{j} = \delta_{ij} \).

**Definition 13.** For \( 1 < k \leq N \) let \( \theta_k = (1, 2) (2, 3) \ldots (k - 1, k) \in S_N \), (thus, \( \theta_k \alpha = (\alpha_k, \alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots) \) for \( \alpha \in \mathbb{N}^N_0 \). If \( \alpha \in \mathbb{N}^N_0 \) satisfies \( \ell (\alpha) = k \) for \( 1 < k \leq N \) set \( \tilde{\alpha} = \theta_k (\alpha - \varepsilon (k)) = (\alpha_k - 1, \alpha_1, \ldots, \alpha_{k-1}, 0, \ldots) \).

In 2 the formula for \( D_k (\kappa) \) is stated for the p-basis \{\( \zeta_\alpha (\kappa) \).\} To use the result here it suffices to invoke the transformation formula \( \zeta_\alpha (\kappa) = \frac{h_0 (\kappa, \alpha + 1)}{h_0 (\alpha, 1)} \zeta_\alpha (\kappa) \) for \( \alpha \in \mathbb{N}^N_0 \). The ratio does not have to be computed explicitly since only the values of \( \frac{h_0 (\alpha, 1)}{h_0 (\kappa, \alpha + 1)} \) for \( t = 1, \kappa + 1 \) are needed.

**Lemma 7.** Let \( \alpha \in \mathbb{N}^N_0 \) and suppose \( \ell (\alpha) = k \) then
\[
\frac{h_0 (\alpha, t)}{h_0 (\kappa, t)} = (k - r (\alpha, k)) \kappa + t + \alpha_k - 1.
\]

**Proof.** Heuristically the Ferrers diagram for \( \tilde{\alpha} \) is produced from that of \( \alpha \) by deleting the node at \((k, 1)\) and moving the remainder of row \( k \) to the top (row zero); then every node still has the same hook-length and the required ratio is \( h (a, t, k, 1) \). Explicitly, \( h (\tilde{\alpha}, t, i, j) = h (\alpha, t, i - 1, j) \) for \( 2 \leq i \leq k, 1 \leq j \leq \alpha_{i-1} \) and \( h (\tilde{\alpha}, t, 1, j - 1) = h (\alpha, t, k, j) \) for \( 2 \leq j \leq \alpha_k \) because
\[
L (\alpha, k, j) = \# \{ l : l < k, j \leq \alpha_l + 1 \leq \alpha_k \}
= \# \{ l : 1 \leq l, j - 1 \leq \alpha_l - 1 \leq \alpha_k - 1 \} = L (\tilde{\alpha}; 1, j - 1).
\]
Also
\[
L (\alpha, k, 1) = \# \{ l : l < k, 1 \leq \alpha_l + 1 \leq \alpha_k \} = \# \{ l : 1 \leq l, 0 \leq \alpha_l < \alpha_k \}
= k - \# \{ l : l \leq k, \alpha_l \geq \alpha_k \} = k - r (\alpha, k),
\]
thus \( h (\alpha, t, 1, k) = \kappa (k - r (\alpha, k)) + t + \alpha_k - 1 \). \( \Box \)

**Proposition 2.** Let \( \alpha \in \mathbb{N}^N_0 \) and suppose \( \ell (\alpha) = k \) then
\[
D_k (\kappa) \zeta^x (\kappa) = \frac{(k - r (\alpha, k)) \kappa + \alpha_k}{(k + 1 - r (\alpha, k)) \kappa + \alpha_k} ((N + 1 - r (\alpha, k)) \kappa + \alpha_k) \theta_k^{-1} \zeta^x (\kappa).
\]
Proof. In [2] Theorem 3.5] it was shown that
\[ D_k (\kappa) \zeta_\alpha (\kappa) = ((N + 1 - r (\alpha, k)) \kappa + \alpha_k) \theta_k^{-1} \zeta_\alpha (\kappa). \]
To modify this equation to hold for the \( x \)-monic polynomials multiply the right hand side by
\[ \frac{h (\alpha, 1) h (\bar{\alpha}, \kappa + 1)}{h (\bar{\alpha}, 1) h (\alpha, \kappa + 1)} = \frac{(k - r (\alpha, k)) \kappa + \alpha_k}{(k + 1 - r (\alpha, k)) \kappa + \alpha_k}. \]

The last topic for the section is the action of \( D_i (\kappa) \) with respect to the order \( \triangleright \). In the lemma the operator is modified to be degree-preserving to simplify the statement.

Lemma 8. Suppose \( \alpha \in \mathbb{N}_0^N \) and \( 1 \leq i \leq N \), if \( \mathrm{coef} (x_i D_i (\kappa) x^\beta, \alpha) \neq 0 \) then \( \beta = \alpha \) or \( \beta^+ \triangleright \alpha^+ \) or \( \beta = (i, j) \alpha \) with \( \alpha_i > \alpha_j, 1 \leq j \leq N \).

Proof. By direct computation for \( \beta \in \mathbb{N}_0^N \) we have
\[ x_i D_i (\kappa) x^\beta = \beta^\alpha x^\beta + \kappa \sum_{\beta_j \leq \beta} \frac{\beta_j - \beta_i - 1}{\beta_j > \beta} \sum_{l=0}^{\beta_i - \beta_j - 1} \left( \frac{x_j}{x_i} \right)^l x^\beta - \kappa \sum_{\beta_j \geq \beta} \sum_{l=1}^{\beta_i - \beta_j} \left( \frac{x_i}{x_j} \right)^l x^\beta. \]
The term \( x^\alpha \) appears in the sum if (i) \( \alpha = \beta \), (ii) (with coefficient \( \kappa \)) for some \( j \), \( \beta_i > \beta_j \) and \( \alpha_i = \beta_i - l, \alpha_j = \beta_j + l \) with \( 0 \leq l \leq \beta_i - \beta_j - 1 \), (iii) (with coefficient \( -\kappa \)) for some \( j \), \( \beta_i < \beta_j \) and \( \alpha_i = \beta_i + l, \alpha_j = \beta_j - l \) with \( 1 \leq l \leq \beta_j - \beta_i \) for (ii) and (iii) \( \alpha_k = \beta_k \) for \( k \neq i, j \). In case (ii) \( \alpha = \beta \) for \( l = 0 \) and \( \beta^+ \triangleright \alpha^+ \) for \( 1 \leq l \leq \beta_i - \beta_j - 1 \) by [5] Lemma 8.2.3. In case (iii) \( \alpha = (i, j) \beta \) for \( l = \beta_j - \beta_i = \alpha_i - \alpha_j > 0 \) and \( \beta^+ \triangleright \alpha^+ \) for \( 1 \leq l \leq \beta_i - \beta_j - 1 \) as before.

The Lemma will be used in analyzing the effect of \( D_{\ell (\alpha)} (\kappa) \) on \( q_\alpha (\kappa) \), the polynomial defined in Theorem 4. The aim will be to show it suffices to consider \( D_{\ell (\alpha)} (\kappa) \zeta_\alpha (\kappa) \). We point out that for any given partition \( \tau \) with \( \tau_2 \geq n \) there may be several reasons why there can be no singular polynomial of isotype \( \tau \), notably there may be no eigenfunction of \( \{ U_i (\kappa_0) \mid 1 \leq i \leq N \} \) with the respective eigenvalues \( \{ \xi_i (\lambda; \kappa_0) \mid 1 \leq i \leq N \} \), where \( \lambda \) is as specified in Theorem 2. Our proof singles out one aspect, a certain nonvanishing coefficient of \( D_{\ell (\lambda)} (\kappa_0) g_\lambda \) which applies to all cases.

4. The two-part case

In this section we consider the simplest case where \( \tau = (\tau_1, \tau_2), \kappa_0 = -\frac{m}{n} \) with \( \gcd (m, n) = 1 \) and \( \tau_2 = n \). By Corollary 4 \( \tau_1 = dn - 1 \) for some \( d \geq 2 \) (since \( \tau_1 \geq \tau_2 \)). We will show that there is no singular polynomial for \( \kappa_0 \) of isotype \( \tau \). By Theorem 2 if there exist singular polynomials for \( \kappa_0 \) of isotype \( \tau \) then there exists \( g_\lambda = x^\lambda + \sum_{\beta < \lambda} A_\beta x^\beta \) with \( \lambda = ((md)^n) \), \( D_i (\kappa_0) g_\lambda = 0 \) and \( U_i (\kappa_0) g_\lambda = \xi_i (\lambda; \kappa_0) g_\lambda \) for \( 1 \leq i \leq N \). In fact, \( g_\lambda = \lim_{\kappa \to \kappa_0} \zeta_\lambda (\kappa) \). This follows from Theorem 4 because there is no \( \kappa_0 \) critical pair \( (\lambda, \beta) \) with \( \ell (\beta) \leq N \). (The background for this is detailed in [2]; briefly \( \mathrm{coef} (\zeta_\beta, \beta) \) is independent of the number \( M \) of variables provided \( \max (\ell (\alpha), \ell (\beta)) \leq M \) (see equation 4.3); also if \( \ell (\alpha) \leq N < \ell (\alpha) + |\alpha| \) then not every factor of \( h (\alpha, \kappa + 1) \) need appear as a pole of \( \zeta_\lambda \).) We start by computing \( h (\lambda, \kappa + 1) \) and showing there is a unique \( \beta \) so that \( (\lambda, \beta) \) is a \( \kappa_0 \) critical pair for \( \lambda \) and \( \ell (\beta) = N + 1 \).

For the rectangular diagram \( \lambda = ((md)^n) \) it is clear that \( L (\lambda, i, j) = n-i \) for \( 1 \leq i \leq n, 1 \leq j \leq md \) so that \( h (\lambda, \kappa + 1) = \prod_{i=1}^{n} \prod_{j=1}^{md} ((n-i+1) \kappa + md + 1 - j) = \)
\[ \prod_{k=1}^{\infty} \prod_{j=1}^{\infty} (i \kappa + j). \] Since \( \gcd(m, n) = 1 \) the multiplicity of \((n \kappa + m)\) in \( h(\lambda, \kappa + 1) \) is one, occurring as \( h(\lambda, \kappa + 1; 1, m\kappa - m + 1) \). The algorithm of \([3]\) yields \( \beta = (0^n, m^\infty) \) for a \( \kappa_0 \)-critical pair \((\lambda, \beta)\). Note \( \ell(\beta) = n + nd = \tau_2 + (\tau_1 + 1) = N + 1 \). Also recall the easily proved rule: for any critical pair \((\alpha, \gamma)\) it always holds that if \( i > \ell(\alpha) \) and \( \gamma_i = 0 \) then \( \gamma_j = 0 \) for all \( j > i \); since \( r(\alpha, i) = i = r(\gamma, i) \).

**Proposition 3.** For \( \lambda = (|md)^n_+\); \( \kappa_0 = -\frac{m}{n} \) with \( \gcd(m, n) = 1 \) let \( \beta = (0^n, m^\infty) \) then \((\lambda, \beta)\) is the unique \( \kappa_0 \)-critical pair for \( \lambda \).

**Proof.** Suppose \( \gamma \in \mathbb{N}_0^M \) for some \( M \geq N \), and \( \gamma \) satisfies the conditions \( \lambda \geq \gamma \) and \( R_1 : (r(\gamma, i) - i) m = (\lambda_i - \gamma_i) n \) for \( 1 \leq i \leq M \) (as usual, define \( \lambda_i = 0 \) for any \( i > \ell(\lambda) \), the equation is a restatement of \((r(\gamma, i) - i) \kappa_0 + (\lambda_i - \gamma_i) = 0 \)). We must show \( \gamma = \lambda \) or \( \gamma = \beta \). Since \( \gcd(m, n) = 1 \) there exists \( \eta \in \mathbb{N}_0^M \) so that \( \gamma = m \eta \) (componentwise; note \( r(\gamma, i) = r(\eta, i) \) for each \( i \)). By condition \( R_{n+1} \) we have \((r(\eta, n + 1) - n - 1) = -n \eta_{n+1} \) so that \( \eta_{n+1} = 1 - \frac{1}{n}(r(\eta, n + 1) - 1) \leq 1 \) and thus \( \eta_{n+1} = 1 \) or \( \eta_{n+1} = 0 \).

If \( \eta_{n+1} = 1 \) then \( r(\eta, n + 1) = 1 \), which implies \( \eta_i = 0 \) for \( 1 \leq i \leq n \) and \( \eta_i \leq 1 \) for \( i > n + 1 \). Since \( |\eta| = \frac{1}{m} |\gamma| = \frac{1}{m} |\lambda| = nd \) we see that \( \eta^\infty = (1^\infty) \) and in fact \( \eta_i = 1 \) for \( n + 1 \leq i \leq n (d + 1) \), since \( \eta_j = 0 \) and \( \eta_{n+1} = 1 \) is impossible for \( j > n \). Thus \( r(1, \eta) = nd + 1 \) and condition \( R_1 \) becomes \((nd + 1 - 1) m = (md - 0) n \). So \( \gamma = \beta \); the other conditions \( R_i \) are verified similarly.

If \( \eta_{n+1} = 0 \) then \( r(\eta, n + 1) = n + 1 \) and \( \ell(\eta) = \ell(\gamma) = n \). But the conditions \( \lambda_1 = \lambda_1 \) for \( 1 \leq i \leq n = \ell(\lambda) \), \( \lambda \geq \gamma \) and \( \ell(\gamma) = \ell(\lambda) \) together imply \( \gamma = \lambda \). \( \square \)

**Corollary 4.** The polynomial \( \zeta_{\alpha}^\gamma(\kappa) \) in \( N = nd + n - 1 \) variables has no pole at \( \kappa_0 \). The (hypothetical) singular polynomial \( g_{\lambda} = \lim_{\kappa \to \kappa_0} \zeta_{\alpha}^\gamma(\kappa) \).

**Proof.** By Theorem \([4]\) \( g_{\lambda} = \lim_{\kappa \to \kappa_0} \zeta_{\alpha}^\gamma(\kappa) \), since there is no \( \gamma \in \mathbb{N}_0^N \) so that \((\lambda, \gamma)\) is \( \kappa_0 \)-critical. By \([2]\) Theorem 4.8 \( \zeta_{\alpha}^\gamma(\kappa) \) in \( nd + n - 1 \) variables has no pole at \( \kappa_0 \) (since \( N = nd + n - 1 = \ell(\beta) \)). \( \square \)

This implies \( D_n(\kappa_0) g_{\lambda} = \lim_{\kappa \to \kappa_0} D_n(\kappa) \zeta_{\alpha}^\gamma(\kappa) \). In the notation of Section 3 (noting \( N + 1 - r(\lambda, n) = (nd + n - 1) + 1 - n = nd \))

\[ D_n(\kappa) \zeta_{\alpha}^\gamma(\kappa) = \frac{md}{\kappa + md} d(n\kappa + m) \theta_n^{-1} \zeta_{\alpha}^\gamma(\kappa), \]

where \( \tilde{\lambda} = (md - 1, (md)^{n-1}) \). We will show that the coefficient of \( x^\gamma \) in the equation does not converge to zero as \( \kappa \to \kappa_0 \), where \( \gamma = \theta_n (m - 1, 0^{n-1}, m^{d-1}) = (0^{n-1}, m - 1, m^{d-1}) \) and \( \ell(\gamma) = N \). The following is similar to Proposition \([3]\)

**Proposition 4.** For \( \tilde{\lambda} = (md - 1, (md)^{n-1}) \), \( \kappa_0 = -\frac{m}{n} \) with \( \gcd(m, n) = 1 \) let \( \beta = (m - 1, 0^{n-1}, m^{d-1}) \) then \( (\tilde{\lambda}, \tilde{\beta}) \) is the unique \( \kappa_0 \)-critical pair for \( \tilde{\lambda} \).

**Proof.** Suppose \( \gamma \in \mathbb{N}_0^M \) for some \( M \geq N \), and \( \gamma \) satisfies the conditions \( \tilde{\lambda} \geq \gamma \) and \( R_i : (r(\gamma, i) - r(\tilde{\lambda}, i)) m = (\lambda_i - \gamma_i) n \) for \( 1 \leq i \leq M \). The conditions \( R_i \)
specialize to:

\[
(r(\gamma,1) - n)m = (md - 1 - \gamma_1)n, \text{ for } i = 1,
\]
\[
(r(\gamma,i) - i + 1)m = (md - \gamma_i)n, \text{ for } 2 \leq i \leq n,
\]
\[
(r(\gamma,i) - i + 1)m = -n\gamma_i, \text{ for } i > n.
\]

Thus \( \gamma_1 \equiv m - 1 \mod m \) and \( \gamma_i \equiv 0 \mod m \) for \( i \geq 2 \). Consider the condition \( R_{n+1} : (r(n,n+1) - n - 1)m = -n\gamma_{n+1}, \) equivalent to \( \gamma_{n+1}/m = 1 - (r(\gamma,n+1) - 1)/n \leq 1 \). Thus \( \gamma_{n+1} = m \) or \( \gamma_{n+1} = 0 \). If \( \gamma_{n+1} = m \) then \( r(\gamma,n+1) = 1 \), implying that \( \gamma_i < m \) for \( 1 \leq i \leq n \). The congruence conditions imply \( \gamma_1 = m - 1 \) and \( \gamma_i = 0 \) for \( 2 \leq i \leq n \). Arguing similarly to Proposition 4 we see that \( \gamma_i = m \) or \( \gamma_i = 0 \) for \( i \geq n + 1 \); the condition \( nmd - 1 \) shows that \( \gamma = (m - 1, 0, \ldots, 0) = (m - 1, 0, \ldots, 0) \). To verify \( R_1 \) note \( r(\gamma,1) = nd \) so \( nd - n)m = ((md - 1) - (m - 1))n = mn(d - 1) \) is satisfied. For \( 2 \leq i \leq n \) we have \( r(\gamma,i) = nd + i - 1 \) and \( \gamma_i = 0 \) so the condition \( R_i \), namely, \( (nd + i - 1 - i + 1)m = mdn \) is satisfied.

If \( \gamma_{n+1} = 0 \) then \( \ell(\gamma) = n \); for this particular \( \bar{\lambda} \) the conditions \( \left(\frac{\bar{\lambda}}{}\right)^+ \geq \gamma^+ \) and \( \ell(\gamma) = n \) together imply \( \left(\frac{\bar{\lambda}}{}\right)^+ = \gamma^+ \); then \( \bar{\lambda} \geq \gamma \) implies \( \bar{\lambda} = \gamma \). \( \square \)

By Lemma 4 \( h(\bar{\lambda}, \kappa + 1) = h(\lambda, \kappa + 1) \) and \( (\kappa + md) \) has multiplicity one in \( h(\bar{\lambda}, \kappa + 1) \). Next we will show \( \text{coef} (\zeta_\lambda (\kappa), \beta) \) has a simple pole at \( \kappa_0 \), where \( \beta \) is defined in the Proposition.

For \( w \in S_N \) and \( \alpha \in \mathbb{N}_0^N \) since \( w(x^\alpha) = x^{w\alpha} \) the transformation property \( \text{coef} (p, \alpha) = \text{coef} (wp, w\alpha) \) holds for any polynomial \( p \).

**Theorem 5.** Suppose \( \kappa_0 = -\frac{m}{n} \) with \( \gcd(m, n) = 1 \) and \( \tau = (dn - 1, n) \) with \( d \geq 2, n \geq 2 \) so that \( N = (d + 1)n - 1 \), then there are no singular polynomials for \( \kappa_0 \) of isotype \( \tau \).

**Proof.** By Corollary 4 if there is a singular polynomial of isotype \( \tau \) for the singular value \( \kappa_0 \) then \( g_\kappa = \lim_{\kappa \to \kappa_0} \zeta_\lambda(\kappa) \) is singular, where \( \lambda = ((md)^\kappa) \). By Proposition 2 \( D_\kappa(g_\kappa) = \frac{md}{\kappa + md} \). Next we will show \( \text{coef} (\zeta^x(\kappa), \beta) \) has a simple pole at \( \kappa_0 = -\frac{m}{n} \) and \( f(\kappa_0) \neq 0 \). Note \( \theta_\kappa^{-1}\beta = (0^{n-1}, m - 1, m^{n-1}) \). Thus

\[
\text{coef} (D_\kappa(g_\kappa), \theta_\kappa^{-1}\beta) = \frac{md^2(nk + m)f(\kappa)}{(\kappa + md)(nk + m)} = \frac{md^2f(\kappa)}{\kappa + md}
\]

and

\[
\text{coef} (D_\kappa(g_\kappa))_{\lambda, \theta_\kappa^{-1}\beta} = \lim_{\kappa \to \kappa_0} \frac{md^2f(\kappa)}{\kappa + md} \neq 0,
\]

and so \( g_\lambda \) is not singular for \( \kappa_0 \), a contradiction. \( \square \)

This argument will serve as the key ingredient for the general case \( \tau \).
5. The general case

In this section we consider the singular value $\kappa_0 = -\frac{m}{n}$ with $\gcd(m,n) = 1$ and $2 \leq n \leq N$ for the isotype $\tau$, where $\tau$ has two or more parts and $\tau_2 > n$. Let $l = \ell(\tau) \geq 2$. We assume there exists a singular polynomial for $\kappa_0$ of isotype $\tau$ and will eventually arrive at a contradiction. By Corollary 4 there are integers $d_1, d_2, \ldots, d_{l-1}$ so that $\tau_i = d_i n - 1$ for $1 \leq i < l$. Because $\tau$ is a partition it follows that $d_1 \geq d_2 \geq \cdots \geq d_{l-1}$. By hypothesis $\tau_1 \geq \tau_2 > n$ so that $d_1 \geq 2$, and $d_2 \geq 2$ if $l \geq 3$. By Theorem 2 there is a corresponding partition $\lambda$ and a singular polynomial $g_\lambda = x^\lambda + \sum_{|\beta| \leq \lambda} A_\beta x^\beta$ with $U_i(\kappa_0) g_\lambda = \zeta_i (\lambda; \kappa_0) g_\lambda$ for $1 \leq i \leq N$. The computations are expressed in terms of (with $1 \leq i \leq l - 1$):

$$
t_i := \sum_{j=1}^{i-1} d_j = \sum_{j=1}^{i-1} \frac{\tau_j + 1}{n},
$$

$$
p_i := \sum_{j=i+1-l}^{l} \tau_j,
$$

$$
\lambda := ((mt_1)^{\tau_1}, (mt_2)^{\tau_2-1}, \ldots, (mt_{l-1})^{\tau_2}, 0^\tau_1),
$$

$$
\gamma := ((mt_1)^{\tau_1}, (mt_2)^{\tau_2-1}, \ldots, (mt_{l-1})^{\tau_2-n}, 0^{n-1}, m-1, m^{\tau_1}),
$$

$$
\alpha := ((mt_1)^{\tau_1}, (mt_2)^{\tau_2-1}, \ldots, (mt_{l-1})^{\tau_2-n}, 0^{n-1}, m, m^{\tau_1}).
$$

Also let $p_0 = 0, p_1 = N$. Note that $p_{i-1} = \ell(\lambda) = N - \tau_1$, $t_{l-1} = d_1 \geq 2$ and $|\gamma| + 1 = |\alpha| = |\lambda|$ because $(\tau_1 + 1) m = nd_1 m = nm t_{l-1}$. By Theorem 4 there exists a polynomial $q_\lambda (\kappa) = \zeta_\lambda^\tau (\kappa) + \sum_{\beta \trianglelefteq \lambda} B_{\beta} (\kappa) \zeta_\beta^\nu (\kappa) : \beta \in C (\lambda, \kappa_0)$ which has no pole at $\kappa_0$ and $\lim_{\kappa \to \kappa_0} q_\lambda (\kappa) = g_\lambda$; the coefficients $B_{\beta} (\kappa) \in \mathbb{Q} (\kappa, v)$ and $C (\lambda, \kappa_0)$ is the set of $\beta$ such that $(\lambda, \beta)$ is a $\kappa_0$-critical pair.

We will show $\lim_{\kappa \to \kappa_0} D_{\ell(\lambda)} (\kappa) q_\lambda (\kappa) \neq 0$ by showing $\lim_{\kappa \to \kappa_0} \text{cof} (D_{\ell(\lambda)} (\kappa) q_\lambda (\kappa), \gamma) \neq 0$. The proof has two parts: firstly we show that $\text{cof} (D_{\ell(\lambda)} (\kappa) q_\lambda (\kappa), \gamma) = \text{cof} (D_{\ell(\lambda)} (\kappa) \zeta_\lambda^\tau (\kappa), \gamma)$ and secondly we use the Insertion Theorem 3 and the result from the previous section.

Lemma 9. Suppose $\delta \in \mathbb{N}_0^N, \lambda \triangleright= \delta$ and $\text{cof} (D_{\ell(\lambda)} (\kappa) x^\delta, \gamma) \neq 0$ then $\delta \triangleright= \alpha$.

Proof. By construction $\text{cof} (D_{\ell(\lambda)} (\kappa) x^\delta, \gamma) = \text{cof} (x_{\ell(\lambda)} D_{\ell(\lambda)} (\kappa) x^\delta, \alpha)$. By Lemma 3 $\delta = \alpha$ or $\delta^+ > \alpha^+$ or $\delta = (\ell(\lambda), j) \alpha$ with $\alpha_{\ell(\lambda)} > \alpha_j$; but in the latter case $j < \ell(\lambda)$ (in fact $\ell(\lambda) - n \leq j < \ell(\lambda)$ and $\alpha_j = 0$) so that $\delta \triangleright \alpha$. Thus $\delta \triangleright \alpha$. □

To complete the first part of the argument we need only show that there is no $\kappa_0$-critical pair $(\lambda, \beta)$ with $\beta \triangleright \alpha$.

Theorem 6. Suppose $\beta \in \mathbb{N}_0^M$ (with some $M \geq N$), $\lambda \triangleright \beta \triangleright \alpha$ and $\beta$ satisfies the rank equation $(r(\beta, i) - i) m = (\lambda_i - \beta_i) n$ for $i \geq 1$ then $\beta = \lambda$.

Proof. The rank equation and definition of $\alpha$ imply $m|\beta_1$ and $m|\alpha_1$ for each $i$; since the definition of $\triangleright$ implies that $\mu \triangleright \nu$ if and only if $\mu m \triangleright \nu m$ for arbitrary compositions $\mu, \nu$ (where $(m\mu)_i := m \mu_i$) we will assume that $m = 1$ in the rest of the proof. Since $\beta$ is trapped between $\lambda = (t_1^{\tau_1}, t_2^{\tau_2-1}, \ldots, t_{l-1}^{\tau_2}, 0^{\tau_1})$ and $\alpha = (t_1^{\tau_1}, t_2^{\tau_2-1}, \ldots, t_{l-1}^{\tau_2-n}, 0^{n-1}, 1^{\tau_1+1})$ we deduce that $\beta^+$ agrees with $\lambda$ in the first $N -$
\(\tau_1 - n\) entries, that is, \((\beta^+)_i = \lambda_i = \alpha_i\) for \(1 \leq i \leq N - \tau_1 - n = p_{l_1} - n\). None of the entries of \(\beta\) equal to some \(t_j\) can “move to the left” (lower index). For \(j, k\) with \(1 \leq j \leq k \leq l - 1\) suppose that the first appearance (least index) of \(t_k\) in \(\beta\) is at an index \(i\) with \(\lambda_i = t_j\), that is, \(p_{j-1} + 1 \leq i \leq p_j\), then \(r(\beta, i) = p_{k-1} + 1\) and the rank equation implies

\[
i = r(\beta, i) - n(\lambda_i - \beta_i) = p_{k-1} + 1 - n(t_j - t_k)
\]

\[
= p_{k-1} + 1 - n \sum_{s = l - k + 1}^{l - j} d_s = p_{k-1} + 1 - \sum_{s = l - k + 1}^{l - j} (\tau_s + 1).
\]

Furthermore

\[
0 \leq i - (p_{j-1} + 1) = p_{k-1} - p_{j-1} - \sum_{s = l - k + 1}^{l - j} (\tau_s + 1)
\]

\[
= \sum_{s = l + 2 - k}^{l + 1 - j} \tau_s - \sum_{s = l - k + 1}^{l - j} \tau_s - (k - j) = \tau_{l+1-j} - \tau_{l-k+1} - (k - j)
\]

\[
\leq j - k \leq 0.
\]

The inequality \(\tau_{l+1-j} - \tau_{l-k+1} \leq 0\) holds because \(\tau\) is a partition and \(l - k + 1 \leq l - j + 1\) by hypothesis. The chain of inequalities shows that \(j = k\) and \(i = p_{k-1} + 1\) (the possibility \(i > p_{k-1} + 1\) has not yet been ruled out).

The key to the argument is the value of \(\beta_{\ell(\lambda)+1}\) (recall \(\ell(\lambda) = p_{l-1}\)). The case \(\beta_{p_{l-1}+1} = t_j\) is impossible for \(1 \leq j \leq l - 1\); indeed suppose \(\beta_{p_{l-1}+1} = t_j\) then \(r(\beta, p_{l-1}+1) \geq p_{j-1} + 1\) and the rank equation is \(p_{l-1} + 1 = r(\beta, p_{l-1} + 1) + nt_j\) (note \(\lambda_{p_{l-1}+1} = 0\)) thus

\[
0 \leq (p_{l-1} + 1 - nt_j) - (p_{j-1} + 1)
\]

\[
= \tau_{l+1-j} - \tau_l - (l - j) \leq j - l < 0,
\]

which is a contradiction (the calculation is similar to the previous one, replacing \(k\) by \(l\) and \(t_k\) by \(t_1\)). The condition \(\beta^+ \leq \lambda\) now implies that \(\beta_{p_{l-1}+1} < t_{l-1}\) and

\[
\# \{ j : \beta_j > \beta_{p_{l-1}+1} \} \geq \ell(\lambda) - n = p_{l-1} - n\text{, hence } r(\beta, p_{l-1} + 1) \geq p_{l-1} - n + 1\text{. The rank equation is } -n\beta_{p_{l-1}+1} = r(\beta, p_{l-1} + 1) - (p_{l-1} + 1) \geq -n \text{ and so } \beta_{p_{l-1}+1} \leq 1.
\]

Suppose \(\beta_{p_{l-1}+1} = 0\) then \(r(\beta, p_{l-1} + 1) = (p_{l-1} + 1)\) which implies \(\beta_i = 0\) for \(i > p_{l-1} + 1\) and \(\beta_i > 0\) for \(i \leq p_{l-1}\). Since \(\beta^+\) differs from \(\lambda\) in at most the last \(n\) entries and \(\lambda_i = t_{l-1}\) for \(\ell(\lambda) - n < i \leq \ell(\lambda)\) it follows that \(\beta^+ = \lambda\) (note if \((t^n_{l-1}) \succeq \mu\) where \(\mu\) is a partition and \(\ell(\mu) = n\) then \(\mu = (t^n_{l-1})\)). Since the entries of \(\beta\) cannot move to the left, \(\beta = \lambda\); in detail, argue inductively that the only possible value for \(\beta_i\) when \(1 \leq i \leq p_1 = 1\), then the only possible value when \(p_1 + 1 \leq i \leq p_2 = 2\), and so on. (If \(l = 2\) then this argument is not needed.)

Suppose \(\beta_{p_{l-1}+1} = 1\). In this part replace the bound \(\beta \geq \lambda\) by \(\beta \geq \alpha' := (t^0_1, t^0_2, \ldots, t^0_{l-1-n}, 0, 1^{\tau+1})\), a weaker restriction since \(\alpha \geq \alpha'\) (note that \((\lambda, \alpha')\) is a \((-\frac{1}{n})\)-critical pair). We will show that \(\beta = \alpha'\), which contradicts the assumption \(\ell(\beta) = N\). Recall \(t_j > t_{l-1} \geq 2\) for \(1 \leq j < l - 1\), by the hypothesis \(\tau_1 \geq \tau_2 > n\).

The rank equation yields \(r(\beta, p_{l-1} + 1) = (p_{l-1} + 1) - n\beta_{p_{l-1}+1} = p_{l-1} + 1 - n\). For \(i < p_{l-1} + 1\) this implies \(\beta_i = t_j\) for some \(j\) or \(\beta_i < 1\), that is, \(\beta_i = 0\); for \(i > p_{l-1} + 1\) the rank implies \(\beta_i = t_j\) for some \(j\) or \(\beta_i < 1\). This forces the values of \(\beta\) other than \((t^0_1, t^0_2, \ldots, t^0_{l-1-n})\) to be 0 or 1, that is, \((\beta^+)_{l-1} \leq 1\) for \(i > p_{l-1} - n\). The condition \(|\lambda| = |\beta|\) shows that \(\# \{ j : \beta_j = 1 \} = nt_{l-1} = \tau_1 + 1\). Since \(\beta_1 = 1\) is ruled out for
$i \leq p_{l-1}$ it follows that $\beta_i = 1$ for $p_{l-1} + 1 \leq i \leq p_{l-1} + \tau_1 + 1 = N + 1$; indeed suppose the $j^{th}$ occurrence of 1 in $\beta$ is at index $i$, that is, $r(\beta, i) = p_{l-1} + j - n$, $\beta_i = 1$ and $i > p_{l-1}$ then the rank equation implies $r(\beta, i) = (p_{l-1} + j - n) - i = -n\beta_i = -n$ thus $i = p_{l-1} + j$, for $1 \leq j \leq \tau_1 + 1$. The $n$ remaining values of $\beta_i$ (for $1 \leq i \leq N + 1$) are zero, and so $\beta^+ = (\alpha^+)^\tau$. The condition $\beta \succeq \alpha'$ implies $\beta \succeq \alpha'$ (by definition) which shows $\beta_i = \alpha_i'$ for $1 \leq i \leq p_{l-1} - n$ (if $p_{l-1} + 1 \leq i \leq p_j$ and $j < l - 1$ then $\beta_i = t_j$ and if $p_{l-1} + 1 \leq i \leq p_{l-1} - n$ then $\beta_i = t_{l-1}$). Thus $\beta_i = 0$ for $p_{l-1} - n < i \leq p_{l-1}$ and $\beta = \alpha'$. The proof is finished since $\ell(\alpha') = N + 1$.

\begin{corollary}
\text{coef} \left(D_{\ell(\lambda)}(\kappa)\right) q_\lambda(\kappa), \gamma) = \text{coef} \left(D_{\ell(\lambda)}(\kappa)\right) \xi^\tau(\kappa), \gamma) .
\end{corollary}

\begin{proof}
Suppose $\beta \in C(\lambda, \kappa_0)$. If $\text{coef} \left(D_{\ell(\lambda)}(\kappa)\right) \xi^\tau(\kappa), \gamma) \neq 0$ then by Lemma 9 there exists $\delta \in \mathbb{N}_N^\ast$ such that $\beta \succeq \delta \succeq \alpha$, which contradicts the Theorem. Hence $\text{coef} \left(D_{\ell(\lambda)}(\kappa)\right) \xi^\tau(\kappa), \gamma) = 0$ for each $\beta \in C(\lambda, \kappa_0)$.
\end{proof}

\begin{example}
In the context of the Theorem there may well be compositions $\beta$ other than $\alpha'$ for which $(\lambda, \beta)$ is a $\kappa_0$-critical pair. Suppose $N = 10$, $\tau = (3, 3, 3, 1)$ and $\kappa_0 = -\frac{1}{4}$, then $\lambda = (6, 4^3)$ and $\alpha' = (6, 4^3, 0, 0, 1^4)$; the multiplicity of $(2\kappa + 1)$ in $h(\lambda, \kappa + 1)$ is 3. Among other compositions $\beta$ with $(\lambda, \beta)$ being $(-\frac{1}{3})$-critical are $(6, 1^3, 2^3, 3)$ and $(6, 0^3, 2, 4^2, 1^4, 4)$; the latter is a permutation of $\alpha'$. For another example take $N = 14$, $\tau = (8, 6)$ and $\kappa_0 = -\frac{1}{4}$, then $\lambda = (3^6)$ and $\alpha' = (3^3, 0^3, 1^9)$; the multiplicity of $(3\kappa + 1)$ in $h(\lambda, \kappa + 1)$ is 2 and both $(1^6, 2^6)$ and $(1^3, 0^3, 2^6, 1^3)$ form $(-\frac{1}{3})$-critical pairs with $\lambda$. The algorithm of [3] was used to produce the $\beta$’s.

Let $k = \ell(\lambda) = N - \tau_1$ and from Definition 13 let $\theta_k = (1, 2) \ldots (k - 1, k) \in S_N$, a cyclic shift and let

$$\bar{\lambda} = \left(\left(mt_{l-1} - 1, (mt_1)^{\tau_1}, (mt_2)^{\tau_1 - 1}, \ldots, (mt_{l-1})^{\tau_1 - n}, 0^\tau\right),\right)$$

$$\bar{\alpha} = \left(m - 1, (mt_1)^{\tau_1}, (mt_2)^{\tau_1 - 1}, \ldots, (mt_{l-1})^{\tau_1 - n}, 0^{n - 1}, m^{\tau_1}\right),$$

so that $\bar{\alpha} = \theta_k \gamma$. By Proposition 2

$$D_{\ell(\lambda)}(\kappa) \xi^\tau(\kappa) = \frac{mt_{l-1}}{\kappa + mt_{l-1}} \cdot \left(\left((N + 1 - k) \kappa + mt_{l-1}\right) \theta_k^{-1} \xi^\tau(\kappa)\right)$$

\begin{align*}
\text{and} (N + 1 - k) \kappa + mt_{l-1} = (\tau_1 + 1) \kappa + md_1 = (nk + m) d_1 \quad (\text{recall } \tau_1 + 1 = nd_1).
\end{align*}

Thus

$$\text{coef} \left(D_{\ell(\lambda)}(\kappa)\right) \xi^\tau(\kappa), \gamma) = \frac{md_1^2}{\kappa + md_1} (nk + m) \text{coef} \left(\theta_k^{-1} \xi^\tau(\kappa), \gamma)\right)$$

$$= \frac{md_1^2}{\kappa + md_1} (nk + m) \text{coef} \left(\xi^\tau(\kappa), \theta_k \gamma\right).$$

We finish the argument by using the Insertion Theorem \[3\] Let

$$\mu = \left((mt_1)^{\tau_1}, (mt_2)^{\tau_1 - 1}, \ldots, (mt_{l-1})^{\tau_1 - n}\right),$$

$$\nu = \left(mt_{l-1} - 1, (mt_{l-1})^{n - 1}, 0^{\tau_1}\right),$$

$$\sigma = \left(m - 1, 0^{n - 1}, m^{\tau_1}\right),$$

then $\lambda = \iota(1, \mu) \nu$ and $\bar{\alpha} = \iota(1, \mu) \sigma$. 
Lemma 10. \( \lim_{\kappa \to \kappa_0} \text{coef} (D_{\ell}(\kappa) \zeta^\kappa_\lambda (\kappa), \gamma) \neq 0. \)

Proof. Let \( p = mt_{l-1} = md_1, \) and \( M = n + \tau_1 \) then \( \nu, \sigma \in f^{(M)}_{l,p}. \) By Theorem 3 \( \text{coef} (\zeta^\nu_\lambda (\kappa), \alpha) = \text{coef} (\zeta^\sigma_\lambda (\kappa), \sigma) \) and by Proposition 4 and Lemma 6 \( \text{coef} (\zeta^\nu_\lambda (\kappa), \sigma) \) has a simple pole at \( \kappa = \kappa_0 \) (this is the same argument used in the previous section). Thus there exists \( f(\kappa) \in \mathbb{Q}(\kappa) \) so that \( \text{coef} (\zeta^\nu_\lambda (\kappa), \sigma) = \frac{f(\kappa)}{n^\nu + \nu} \) and \( f(\kappa_0) \neq 0. \) To conclude,

\[
\text{coef} (D_{\ell}(\kappa) \zeta^\kappa_\lambda (\kappa), \gamma) = \frac{md^2}{\kappa + md_1} (n\kappa + m) \text{coef} (\zeta^\kappa_\lambda (\kappa), \alpha)
= \frac{md^2}{\kappa + md_1} f(\kappa)
\]

which has a nonzero limit at \( \kappa_0. \) \( \square \)

Theorem 7. Suppose \( \kappa_0 = -\frac{m}{n} \) with \( \gcd (m,n) = 1 \) and \( \tau \) is a partition of \( N \) such that \( n(\tau_1 + 1) \) for \( 1 \leq i < \ell (\tau). \) If \( \tau_2 \geq n \) then there are no singular polynomials for \( \kappa_0 \) of isotype \( \tau. \)

Proof. For \( \tau_2 > n \) if there is a singular polynomial of isotype \( \tau \) for the singular value then \( \lim_{\kappa \to \kappa_0} D_{\ell}(\kappa) q_\lambda(\kappa) = 0 \) for the polynomial \( q_\lambda(\kappa) \) described above. But \( \lim_{\kappa \to \kappa_0} \text{coef} (D_{\ell}(\kappa) q_\lambda(\kappa), \gamma) = \lim_{\kappa \to \kappa_0} \text{coef} (D_{\ell}(\kappa) \zeta^\kappa_\lambda (\kappa), \gamma) \neq 0, \) and so these singular polynomials do not exist. The case \( \tau_2 = n, \ell (\tau) = 2 \) was done in the previous section. \( \square \)

6. Concluding remarks

Together with the results of [2, Theorem 2.7] we have a complete description of singular polynomials for the group \( S_N. \) For each pair \( (m_0, n_0) \in \mathbb{N}^2 \) with \( 2 \leq n_0 \leq N \) and \( \frac{m_0}{n_0} \notin \mathbb{N}, \) let \( d = \gcd (m_0, n_0), m = \frac{m_0}{d}, n = \frac{n_0}{d}, \) then there is a unique irreducible \( S_N \)-module of singular polynomials for the singular value \( \kappa_0 = -\frac{m}{n} \) of isotype \( \tau, \) where

\[
\begin{align*}
l &= \left\lfloor \frac{N-n_0+1}{n-1} \right\rfloor + 1 \\
\tau &= \left( n_0 - 1, (n-1)^{l-2}, n_l \right).
\end{align*}
\]

The number \( l = \ell (\tau) \) is the solution of the inequality \( 1 \leq \tau_1 = (N-n_0+1) - (l-2)(n-1) \leq n-1 \) \( \lfloor r \rfloor \) denotes the smallest integer \( \geq r). \) Then the index for the corresponding singular polynomial is given by:

\[
\lambda = \begin{cases} 
(m_0^{l-2}, 0^{n_0-1}), & l = 2 \\
(m_0 + (l-2) m)^\tau, (m_0 + (l-3) m)^{n-1}, \ldots, m_0^{n-1}, 0^{n_0-1}), & l \geq 3
\end{cases}
\]

Note that \( l = 2 \) is equivalent to \( N-n_0+1 < n \) or \( d < \frac{m_0}{N-n_0+1}, \) and \( \tau_2 = N-n_0+1. \)

For \( l \geq 3 \) the computation for \( \lambda \) uses the notation of the previous section with \( t_i = d + l - i - 1, nt_i = m_0 + (l-1 - i) m \) for \( 1 \leq i \leq l-1. \) The \( S_N \)-module of singular polynomials is span\( \{ w\zeta^\alpha_\lambda (\kappa_0) : w \in S_N \} \) and the basis corresponding to Murphy’s construction is exactly the set of \( \zeta^\alpha_\lambda (\kappa_0) \) such that \( \alpha \) is a reverse lattice permutation of \( \lambda. \) There are no other singular polynomials.
The relation of modules of singular polynomials to monodromy representations of the Hecke algebra was discussed in [4 Sect.6]. The parameter is \( q = e^{-2\pi i \kappa}; \) the existence of singular polynomials of isotype \( \tau \) shows that the monodromy representation corresponding to \( \tau \) contains the trivial representation. There is a general result on the connection between monodromy (called the KZ-functor) and the dual Specht modules in [7 Sect. 6.2].

Recall the definition of the rational Cherednik algebra (see [7 Sect. 3] and [6]). We consider the image \( A(\kappa) \) under the faithful representation on \( \mathcal{P}; \) indeed \( A(\kappa) \) is the \( \mathbb{Q}(\kappa) \)-algebra generated by \( \{ D_i(\kappa) : 1 \leq i \leq N \} \cup \{ x_i : 1 \leq i \leq N \} \cup S_N \) (where \( x_i \) denotes the multiplication map \( p(x) \mapsto x_i p(x) \) and \( w \in S_N \) acts by \( p(x) \mapsto p(xw) \) for \( p \in \mathcal{P} \)). In the sequel, when \( \kappa \) is specialized to a rational \( \kappa_0 \), we use \( \mathcal{P} \) to denote the polynomials with rational coefficients (that is, \( \text{span}_\mathbb{Q} \{ x^\alpha : \alpha \in \mathbb{N}^N \} \)). Here are some basic results about \( A(\kappa) \)-submodules of \( \mathcal{P} \).

**Proposition 5.** Suppose \( M \) is a nontrivial proper \( A(\kappa_0) \)-submodule of \( \mathcal{P} \) for some \( \kappa_0 \in \mathbb{Q} \), then \( M \) is the direct sum of its homogeneous components \( M_n := M \cap \mathcal{P}_n \) for \( n \in \mathbb{N}_0 \), the nonzero component \( M_{n_0} \) of least degree \( (M_j = \{ 0 \} \) for \( j < n_0 \) is an \( S_N \)-module of singular polynomials and \( \kappa_0 \) is a singular value.

**Proof.** The identity \( \sum_{i=1}^N x_i D_i(\kappa) = \sum_{i=1}^N x_i \frac{\partial}{\partial x_i} + \kappa \sum_{1 \leq i < j \leq N} (1 - (i, j)) \) implies that the Euler operator \( \sum_{i=1}^N x_i \frac{\partial}{\partial x_i} \in A(\kappa) \). Hence \( M = \sum_{n=0}^\infty (M \cap \mathcal{P}_n) \). There exists \( n_0 > 0 \) such that \( M_{n_0} \neq \{ 0 \} \) and \( M_j = \{ 0 \} \) for \( 0 \leq j < n_0 \) or else \( M_0 \neq \{ 0 \} \), \( 1 \in M \) and so \( M = \mathcal{P} \). Then \( D_i(\kappa_0) p = 0 \) for any \( p \in M_{n_0} \) and \( 1 \leq i \leq N \). \( \square \)

Say that the degree of an \( A(\kappa_0) \)-submodule \( M \) is the least degree of nonzero homogeneous components of \( M \), that is, \( \min \{ j : M_j \neq \{ 0 \} \} \). There is a symmetric bilinear form on \( \mathcal{P} \) defined by

\[
\langle p, q \rangle_\kappa = p(D_1(\kappa), \ldots, D_N(\kappa)) q(x) |_{x=0}.
\]

The radical was defined in [4 Sect. 4] to be

\[
\text{Rad}(\kappa) := \{ p \in \mathcal{P} : \langle p, q \rangle_\kappa = 0 \text{ for all } q \in \mathcal{P} \}
\]

and was shown to be an \( A(\kappa) \)-submodule. For \( \kappa_0 \in \mathbb{Q} \), \( \text{Rad}(\kappa_0) \neq \{ 0 \} \) exactly when \( \kappa_0 \) is a singular value.

**Proposition 6.** For any singular value \( \kappa_0 \) the radical \( \text{Rad}(\kappa_0) \) is the largest proper \( A(\kappa_0) \)-submodule of \( \mathcal{P} \).

**Proof.** Suppose \( M \) is a nontrivial proper \( A(\kappa_0) \)-submodule. Suppose \( p \in M_n = M \cap \mathcal{P}_n \) for some \( n > 0 \) and \( p \neq 0 \). Then for any \( q \in \mathcal{P}_n \) we have \( \langle p, q \rangle_{\kappa_0} = \langle q, p \rangle_{\kappa_0} = q(D_1(\kappa), \ldots, D_N(\kappa)) p(x) |_{x=0} \). Hence \( p \in \text{Rad}(\kappa_0) \) and \( M \subset \text{Rad}(\kappa_0) \). \( \square \)

Our complete description of irreducible \( S_N \)-modules of singular polynomials leads to some explicit results about \( A(\kappa_0) \)-submodules. We use the notation from equations (6.1) and (6.2).

**Definition 14.** For any pair \( (m_0, n_0) \in \mathbb{N} \times \mathbb{N} \) with \( 2 \leq n_0 \leq N \) and \( \frac{m_0}{n_0} \notin \mathbb{N} \) let \( M(m_0, n_0) = \{ \sum_{i=1}^{n_0} p_i \xi_{w_i \lambda}(\kappa_0) : p_i \in \mathcal{P} \} \), where \( n_\tau \) is the degree of the representation \( \tau \) and \( \{ w_i \lambda : 1 \leq i \leq n_\tau \} \) is the set of reverse lattice permutations of \( \lambda \) (that is, \( \{ \xi_{w_i \lambda}(\kappa_0) : 1 \leq i \leq n_\tau \} \) is a basis for the singular polynomials corresponding to the pair \( (m_0, n_0) \)).
The following is from [2 Sect. 6]:

**Proposition 7.** \( M(m_0, n_0) \) is a proper \( A(\kappa_0) \)-submodule, and its degree is

\[
\frac{1}{2} (l - 2)(n - 1)(2d + l - 3) + \tau_l (d + l - 2)
\]

(where \( d = \gcd(m_0, n_0) \) \( n = n_0/d \) and \( m = m_0/d \)).

**Proof.** Clearly \( M(m_0, n_0) \) is closed under multiplication by \( P \) and the action of \( S_N \). Suppose \( f = pg \) where \( p \in P \) and \( g \in \text{span} \{ \zeta_{w,\lambda}(\kappa_0) : 1 \leq i \leq n_\tau \} \) (that is, \( g \) is singular). By the product rule,

\[
D_1(\kappa_0) f = p D_1(\kappa_0) g + g \left( \frac{\partial}{\partial x_i} p \right) + \kappa_0 \sum_{j \neq i} ((i, j) g) \frac{p(x) - p(x_{i,j})}{x_i - x_j}
\]

for \( 1 \leq i \leq N \). But \( D_1(\kappa_0) g = 0 \) and \( \frac{p(x) - p(x_{i,j})}{x_i - x_j} \) is a polynomial, thus \( D_1(\kappa_0) f \in M(m_0, n_0) \). The degree is \( |\lambda| \) (as in equation 6.3).

An equivalent formula for \( \frac{|\lambda|}{mn} \) is \((N - nd + 1)(d + l - 2) - \frac{1}{2}(n - 1)(l - 1)(l - 2)\). For a given pair \((m, n)\) with \( \gcd(m, n) = 1 \) (and \( 2 \leq n \leq N \)) there are the \( A(\kappa_0) \)-submodules \( \{ M(dm, dn) : 1 \leq d \leq \left\lfloor \frac{N}{m} \right\rfloor \} \) (\( \lfloor r \rfloor \) denotes the largest integer \( \leq r \)). The degree of \( M(dm, dn) \) decreases as \( d \) increases. This is obvious because the nonzero part of the index \( \lambda \) for \( ((d + 1)m, (d + 1)n) \) is a substring of the index \( \lambda' \) for \( M(dm, dn) \); for example take \( N = 10, \kappa_0 = -\frac{1}{3} \) then the values of \( \lambda \) (from equation 6.3) are \((4^2, 3^2, 2^2, 1^2, 0^2), (4, 3^2, 2^2, 0^2), (3^2, 0^2)\) for \( d = 1, 2, 3 \) respectively. For direct computation, let \( l, \tau, \lambda \) and \( l', \tau', \lambda' \) denote the expressions defined in equations 6.1 and 6.2 for \((m_0, n_0)\) equal to \((dm, dn)\) and \(((d + 1)m, (d + 1)n)\) respectively. If \( \tau_l = 1 \) then \( l' = l - 2 \) and \( \tau_{l-2} = n - 1 \); if \( 2 \leq \tau_l \leq n - 1 \) then \( l' = l - 1 \) and \( \tau_{l-1} = \tau_l - 1 \). For both cases \(|\lambda| - |\lambda'| = m(dm + l - 2)\). Thus for any given degree of homogeneity there is at most one irreducible \( S_N \)-module of singular polynomials of that degree (for \( \kappa_0 = -\frac{m}{n} \)). The singular polynomials of least degree correspond to \((mq, nq)\) where \( q = \left\lfloor \frac{N}{m} \right\rfloor \).

**Proposition 8.** Suppose \( \gcd(m, n) = 1 \) and \( N = nq + r \) with \( 0 \leq r \leq n - 1 \) (so \( q = \left\lfloor \frac{N}{m} \right\rfloor \)) and \( k \) denotes the degree of \( \text{Rad}(-\frac{m}{n}) \). If \( r < n - 1 \) then \( k = mq(r + 1) \), \( \text{Rad}(-\frac{m}{n}) \cap P_k \) is of isotype \((nq - 1, N - nq + 1)\), and equals span \( \{ w \zeta_{w,\lambda}(-\frac{m}{n}) : w \in S_N \} \) where \( \lambda = \left( (mq)^r + 1, 0^{nq - 1} \right) \). If \( r = n - 1 \) then \( k = m(nq + 1) \) and \( \text{Rad}(-\frac{m}{n}) \cap P_k \) is of isotype \((nq - 1, n - 1, 1)\) with corresponding \( \lambda = \left( (m + 1), (mq)^{n - 1}, 0^{nq - 1} \right) \).

As well as the maximum \( A(\kappa_0) \)-submodule there is a minimum one, namely, \( M(m, n) \).

**Proposition 9.** Suppose \( \gcd(m, n) = 1 \) and \( 2 \leq n \leq N \) then \( M(m, n) \) is contained in each nontrivial \( A(-\frac{m}{n}) \)-submodule.

**Proof.** Let \( M \) be a proper nontrivial \( A(-\frac{m}{n}) \)-submodule. Let \( s_0 \) be the degree of \( M \). By Proposition \( M_{s_0} \) is an \( S_N \)-module of singular polynomials, thus \( s_0 = |\lambda| \) for some \( \lambda \) given by Equation 6.2. Suppose \( \lambda \) corresponds to the pair \((dm, dn)\) with \( 1 \leq d \leq \left\lfloor \frac{N}{m} \right\rfloor \), then \( M(dm, dn) \subset M \). The intersection of any two nontrivial \( A(-\frac{m}{n}) \)-submodules \( M_1 \) and \( M_2 \) is a nontrivial \( A(-\frac{m}{n}) \)-submodule (if \( f \in M_1 \) and \( g \in M_2 \) then \( fg \in M_1 \cap M_2 \)). Thus \( M(dm, dn) \cap M(m, n) \) is a nontrivial submodule
of $M(m,n)$ which must equal $M(m,n)$ because the degree of $M(m,n)$ is the maximum for the degrees of $M(dm,dn)$. Hence $M(m,n) \subset M(dm,dn) \subset M$. \hfill \Box

One could speculate that $\{ M(dm,dn) : 1 \leq d \leq \left\lfloor \frac{N}{n} \right\rfloor \}$ is the collection of all non-trivial proper $A(-\frac{m}{n})$-submodules and that they are nested, that is, $M(dm,dn) \subset M((d+1)m,(d+1)n)$. This would be a characterization of $\text{Rad}(-\frac{m}{n})$.

References

[1] C. Dunkl, Differential-difference operators associated to reflection groups: *Trans. Amer. Math. Soc.* 311 (1989), 167-183.
[2] C. Dunkl, Singular polynomials for the symmetric groups, *Int. Math. Research Notices* 2004 (2004), #67, 3607-3635.
[3] C. Dunkl, Hook-lengths and pairs of compositions, preprint arXiv:math.CO/0410466, Oct. 2004.
[4] C. Dunkl, M. de Jeu, and E. Opdam, Singular polynomials for finite reflection groups, *Trans. Amer. Math. Soc.* 346 (1994), 237-256.
[5] C. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*, Encycl. of Math. and its Applications 81, Cambridge University Press, Cambridge, 2001.
[6] P. Etingof and V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space and deformed Harish-Chandra isomorphism, *Invent. Math.* 147 (2002), 243-348.
[7] V. Ginzburg, N. Guay, E. Opdam and R. Rouquier, On the category $\mathcal{O}$ for rational Cherednik algebras, *Invent. Math.* 154 (2003), 617-651.
[8] F. Knop and S. Sahi, A recursion and a combinatorial formula for Jack polynomials, *Invent. Math.* 128 (1997), 9-22.
[9] G. Murphy, A new construction of Young’s seminormal representation of the symmetric groups, *J. Algebra* 69 (1981), 287-297.
[10] E. Opdam, Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group, *Compos. Math.* 85 (1993), 333-373.

Department of Mathematics, University of Virginia, P.O.Box 400137 Charlottesville VA 22904-4137

E-mail address: cfdSz@virginia.edu

URL: http://www.people.virginia.edu/~cfdSz/