S-Duality and the Spectrum of Magnetic Monopoles in Heterotic String Theory

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Abstract
We discuss the predictions of $S$-duality for the monopole spectrum of four-dimensional heterotic string theory resulting from toroidal compactification. We discuss in detail the spectrum of “$H$-monopoles”, states that are magnetically charged with respect to the $U(1)$ groups arising from the dimensional reduction of the ten dimensional antisymmetric tensor field. Using an assumption concerning the correct treatment of collective coordinates in string theory we find results which are consistent with $S$-duality.

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1. Introduction

It has long been hoped that four-dimensional theories might be found which exhibit a duality that interchanges weak and strong coupling, thus extending the remarkable features of special two-dimensional theories to four dimensions. The discovery of such a duality would certainly have profound consequences.

There is circumstantial evidence for such a duality in N=4 super Yang-Mills theory spontaneously broken down to a subgroup containing a $U(1)$ factor and hence magnetic monopoles. This can be traced back to the GNO conjecture concerning a dual theory of magnetic monopoles [1] and the conjecture of Montonen and Olive [2] that there exist a dual gauge theory in which the coupling $g^2 \rightarrow 1/g^2$ and electric and magnetic charges are exchanged. Some evidence for this conjecture comes from the existence of a duality invariant formula for the mass of states in terms of their magnetic and electric charges. This formula must be exact for states which furnish a sixteen-dimensional (short) representation of the $N = 4$ supersymmetry algebra [3]. These states include the massive vector multiplet, the photon multiplet and, after quantization of the fermion zero modes about the monopole solution, the monopole supermultiplet [4].

A natural extension of the duality conjecture occurs when the effects of a non-zero theta parameter, $\theta$, are considered. The duality which acts on the gauge coupling as $g^2 \rightarrow 1/g^2$ is then naturally extended to a $SL(2, \mathbb{Z})$ group of transformations acting on $\lambda = \theta/2\pi + i/g^2$. It was noted by Sen [5] that this extension makes new non-trivial predictions which do not follow from the symmetry $g^2 \rightarrow 1/g^2$ alone. In particular, given the relatively weak assumption that a state with electric charge one and magnetic charge zero (i.e. a massive $W^+$ state) exists in the 16 of $N = 4$ supersymmetry for all $\lambda$, one deduces the existence of an infinite tower of stable states with electric and magnetic charges determined by a pair of relatively prime integers $(p, r)$, each in the 16 of $N = 4$. In a semi-classical analysis these states should arise from the existence of specific normalizable harmonic forms on the reduced moduli space of classical monopole solutions. In a recent remarkable paper one of these predictions, the existence of an unique normalizable anti-self-dual harmonic two-form on the two-monopole moduli space, was born out [6] (see also [7,8]).

There has also been recent discussion of the possibility of $SL(2, \mathbb{Z})$ duality (“S-duality”) in toroidally compactified heterotic string theory. Indications for such a duality come from many different sources. These include the mysterious non-compact symmetries of dimensionally reduced supergravities and in particular the $SL(2, R)$ symmetry.
which is characteristic of four-dimensional $N = 4$ supergravity \cite{3}, the soliton-like behavior of fundamental strings \cite{10}, a geometrical duality between strings and fivebranes \cite{11}, the existence of fivebrane solutions to string theory and the suggestion of a weak-strong coupling duality involving strings and fivebranes \cite{12,13,14}. On the basis of such ideas $S$-duality was proposed in general four-dimensional heterotic strings \cite{15}, but the clearest evidence at present comes from the study of toroidal compactifications \cite{16}. Since $N = 4$ super Yang-Mills is embedded in the four dimensional low-energy field theory limit of this compactification, the $S$-duality conjecture generalizes the $S$-duality in the $N = 4$ super Yang-Mills theory.

The precise group which should be involved in $S$-duality is not completely clear. At the classical level one finds an $SL(2, R)$ symmetry. It is believed that this should be broken to $SL(2, Z)$ by conventional field theory instantons \cite{15,17}. It is possible that other backgrounds or more stringy effects break it to an even smaller subgroup. The possibility of the full $SL(2, Z)$ acting is intimately connected with the fact that $E_8 \times E_8$ and $\text{Spin}(32/Z_2)$ are self-dual gauge groups in the sense of ref. \cite{1}. In this paper we will assume that the full $SL(2, Z)$ should act, but for the most part we will only discuss the implications of the $Z_2$ subgroup which takes weak to strong coupling at $\theta = 0$.

In attempting to extend $S$-duality from $N = 4$ super-Yang-Mills theory to string theory one finds new qualitative features. One of these features is that the coupling constant and the $\theta$ angle become dynamical fields. Thus $S$-duality acts on the combination

$$\lambda = \Psi + ie^{-\Phi}$$

as

$$\lambda \rightarrow \frac{a\lambda + b}{c\lambda + d}$$

where $\Psi$ is the axion field defined by

$$H = -e^{2\Phi^*}d\Psi,$$

$\Phi$ is the string dilaton field, and $ad - bc = 1$ with $a, b, c, d \in Z$. This, plus the fact that $\Psi$ changes by 1 in encircling a fundamental string suggests that $S$-duality should be regarded as a discrete gauge symmetry in string theory \cite{3}.

The study of $S$-duality in string theory is also complicated by the existence of several different types of magnetic monopoles. At generic points in the moduli space of Narain compactification, $\mathcal{M}_N$ \cite{18,19}, the low-energy gauge group is $U(1)^{28}$ with 16 of the $U(1)$
factors arising from the Cartan subalgebra of $E_8 \times E_8$, 6 arising from the off-diagonal components of the metric $g_{\mu n}$, and 6 arising from the off-diagonal components of the anti-symmetric tensor field $B_{\mu n}$ ($\mu, \nu = 0, \ldots , 3; m, n = 4, \ldots , 9$). Labelling the $U(1)$ field strengths as $F^a_{\mu \nu}$, $a = 1 \ldots 28$, the action of $S$-duality is

$$F^a_{\mu \nu} \rightarrow (c\lambda_1 + d)F^a_{\mu \nu} + c\lambda_2(ML)_{ab}\tilde{F}^b_{\mu \nu}$$

(1.4)

where $M$ is a $28 \times 28$ matrix of scalars (Narain moduli) and $L$ is the $O(6, 22)$ invariant metric. The scalars $M$, the Einstein metric and the fermions are all left invariant by $S$-duality.

The magnetic monopole solutions predicted by $S$-duality will thus include essentially all previously studied types of magnetic monopoles, that is 't Hooft-Polyakov or BPS monopoles, Kaluza-Klein monopoles [20], and $H$-monopoles [21, 22, 23]. In addition, as one moves in $\mathcal{M}_N$ (i.e. by varying the asymptotic values for the scalars $M$) the various $U(1)$ factors mix with each other so at a generic point the monopoles of the diagonal $U(1)$’s will be a linear combination of these various types. Furthermore these monopoles are only light compared to the string scale in some regions in $\mathcal{M}_N$. Thus in general gravitational and string corrections to the solutions must be included.

In recent work Sen has given a comprehensive review of the evidence for $S$-duality in toroidally compactified heterotic string theory and has initiated a program to test $S$-duality by finding a set of specific testable predictions which can be checked at weak-coupling and in the field theory limit [3]. The work of [3] verifies one of these predictions, but strictly speaking does not provide evidence for $S$-duality in string theory independent of its existence in $N = 4$ super-Yang-Mills theory. The other predictions made in [3] regard the spectrum after semi-classical quantization of $H$-monopoles. These are magnetic monopoles which carry magnetic charges under $U(1)$ groups which arise from dimensional reduction of the antisymmetric tensor field $B_{MN}$ in ten dimensions. The purpose of this paper is to study whether the predictions of $S$-duality for $H$-monopoles are also born out. As we will see, it does not seem possible to answer this question completely in the field theory limit, in spite of the fact that $H$-monopoles can be made arbitrarily light. However with one crucial assumption concerning the correct treatment of collective coordinates in string theory, at least one of these predictions is true. Since the $H$-monopole solutions have a non-trivial dependence on the compactified dimensions the duality predictions concern the full ten-dimensional low-energy field theory. Since the low-energy field theory is presumably not
a consistent quantum theory on its own, our results should be regarded as evidence for duality of the full string theory, or at least its toroidal compactification, if our assumption concerning collective coordinates is correct. Precisely what duality should mean for the full string theory is far from clear, at the end of this paper we will make some speculative remarks on this subject.

2. Bogomol’nyi States in Heterotic String Theory

Since $S$-duality involves the interchange of strong and weak coupling, to test whether it is a valid symmetry of string theory it is necessary, at least with our current techniques, to consider quantities that are exact at tree level and check their invariance under the duality. As discussed in [5], in the context of six dimensional toroidal compactifications their are several such quantities, including the mass spectrum of “Bogomol’nyi states”, states in the 16 of the N=4 supersymmetry algebra. These representations only exist for states whose mass saturates a Bogomol’nyi bound determined by their electric and magnetic charges. Thus the relation between mass and charge for these states must be an exact quantum relation [3]. Since the electric and magnetic charges are not renormalized in theories with $N = 4$ supersymmetry [24] we deduce that both the tree level charges and masses of Bogomol’nyi states are exact.

Thus to test $S$-duality we first determine the spectrum of electrically charged first-quantized string states which saturate the Bogomolnyi bound. Assuming that these Bogomol’nyi states exist at all values of the coupling, $S$-duality predicts the existence of magnetic monopoles and dyons with degeneracies equal to those of the corresponding dual states. Semi-classical reasoning at weak coupling then translates this into properties of monopole moduli spaces.

One rather surprising feature of heterotic string theory is the existence of an infinite tower of electrically charged Bogomol’nyi states of increasing mass [10]. Any state constructed as a tensor product of the right-moving (superstring) oscillator ground state and an arbitrary left-moving state satisfying the constraint

$$N_L - 1 = \frac{1}{2}(p_R^2 - p_L^2) \quad (2.1)$$

preserves half of the spacetime supersymmetry and satisfies a Bogomol’nyi bound

$$M^2 = \frac{1}{8}p_R^2 \quad (2.2)$$
where \((p_L, p_R) \in \Gamma_{22,6}\) are the electric charges of the state and \(\Gamma_{22,6}\) the even, self-dual, Lorentzian lattice determining the compactification. The partition function for these states is

\[
Z_{Bog}(q, \bar{q}) = \frac{\Theta_\Gamma(q, \bar{q})}{\eta(q)^{24}} \times (8 - 8)
\]

with \(\Theta_\Gamma\) the lattice theta function,

\[
\Theta_\Gamma(q, \bar{q}) = \sum_{(p_L, p_R) \in \Gamma_{22,6}} q^{p_L^2/2} \bar{q}^{p_R^2/2},
\]

\(\eta\) the Dedekind eta function,

\[
\eta(q) = q^{1/24} \prod_n (1 - q^n),
\]

and the factor of \((8 - 8)\) arises from the 8 bosonic and 8 fermionic right-moving ground states.

In general, to test \(S\)-duality at a particular point in \(M_N\) we should first expand \(Z_{Bog}\) and identify terms with equal powers of \(q\) and \(\bar{q}\). The prefactor of such a term gives the degeneracy of states with \(U(1)^{28}\) electric charges given by \((p_L, p_R)\). Applying an \(SL(2, \mathbb{Z})\) transformation we predict the existence of magnetic monopole solutions with the dual magnetic charges and the same degeneracy. To the first few orders in \(q\) this gives

\[
Z_{Bog}(q, \bar{q}) = (8 - 8)q^{-1} \left(1 + 24q + (24 \cdot 25/2 + 24)q^2 + \cdots\right) \sum_{(p_L, p_R)} q^{p_L^2/2} \bar{q}^{p_R^2/2}.
\]

Thus at lowest order we have electrically charged states with \(p_L^2 - p_R^2 = 2\) with degeneracy 1 up to the 16 fold-degeneracy arising from \(N = 4\) supersymmetry and at the next level we have states with \(p_L^2 - p_R^2 = 0\) with degeneracy 24.

Before proceeding we should mention two possible obstacles to carrying out this procedure. The first is the continuous spectrum of the theory. Since the spectrum contains massless particles (graviton, photon, dilaton, etc.), massive electrically charged states are part of a continuum of states if we work in infinite volume. Thus it is not clear that the degeneracy of such states is well defined, particularly at strong coupling. We will not be able to shed much light on this question, and some aspects of this continuous spectrum will become evident in our quantization of the monopole degrees of freedom. We will assume nonetheless that the degeneracy of states can be defined and provides a test of \(S\)-duality. The second obstacle is connected with gravitational effects. Since there is an infinite tower
of Bogomol’nyi states in string theory, one might expect that for any fixed value of the
string coupling these must eventually be treated as (extremal) black holes since their mass
eventually exceeds the Planck mass. On the other hand the mass of these states is propor-
tional to the string coupling times the Planck mass, so by taking the coupling to be very
small we can always treat the low-lying states as solitons rather than as black holes since
their Compton wavelength exceeds their Schwarzschild radius by a factor of the inverse
string coupling.

In principle we would like to construct magnetic monopole solutions to string theory
as conformal field theories and identify and quantize their collective coordinates directly
in string theory. Since it is not yet known how to do this in detail, we will have to rely
on the low-energy field theory as a guide. To do this we need to work in a limit where
the low-energy field theory is a good approximation (modulo one problem to be discussed
later). We can achieve this by working at a point where all radii $R_i$ of the six-torus are
large compared to the string scale and where all Wilson line expectation values which break
$E_8 \times E_8$ to $U(1)^{16}$ are small compared to the string scale. In addition we should consider
only states with masses small compared to the string scale. For $R_i$ large this means we
should only consider states with vanishing winding number or equivalently states with a
Kaluza-Klein origin.

A simple example may be useful. The Narain moduli are the constant values of
the metric, anti-symmetric tensor field, and Cartan sub-algebra gauge fields with indices
tangent to $T^6$. We take $g_{mn} = R^2 \delta_{mn}, B_{mn} = 0$ and $A^n_I \neq 0$ with $R^{-1}, A^n_I \ll \sqrt{\alpha'}$. For
vanishing winding number the momenta $(p_L, p_R)$ are given by

\begin{align}
    p^I_L &= P^I \\
    p^m_L &= \frac{1}{2} g^{mn} (M_n + A^n_I P^I) \\
    p^m_R &= p^m_L
\end{align}

with $P^I$ an element of the $E_8 \times E_8$ root lattice and $M_n$ labels the momenta on $T^6$. Thus
the lightest states with $N_L = 0$, $p^2_L = p^2_R + 2$ and multiplicity 1 have $M_n = 0$, $P^I \neq 0$, mass
squared $M^2 = (A^n_I P^I)^2 / 32 R^2$, and are charged under the $U(1)^{16}$ coming from $E_8 \times E_8$ but

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1 Because of the presence of the dilaton in string theory this is not necessarily the case. For
example, the BPS monopole solution in low-energy string theory does not become a black hole
even for arbitrarily large Higgs expectation value, in contrast to Einstein-Yang-Mills-Higgs
theory.
are neutral under the $U(1)^{12}$ coming from the metric and antisymmetric tensor $U(1)$. Light states with $N_L = 1$, $p_L^2 = p_R^2$ and multiplicity 24 have $P^I = 0$, $M_n \neq 0$, mass squared $M^2 = (M_n)^2/32R^2$ and are charged under $U(1)^{12}$ but neutral under $U(1)^{16}$. More precisely these states carry charge under the $U(1)^6$ coming from the metric and are neutral under the $U(1)^6$ coming from the antisymmetric tensor field (since $(p_L \pm p_R)/2$ determine the metric and antisymmetric tensor $U(1)$ charges, respectively).

Under the transformations (1.2), (1.4) with $\Psi = 0$ and $a = d = 0, c = -b = 1$ these electrically charged states transform into states with magnetic charge. The states with $p_L^2 = p_R^2 + 2$ are dual to magnetic monopoles of the $U(1)^{16}$, that is BPS monopoles, while the states with $p_L^2 = p_R^2$ are dual to magnetic monopoles of the $U(1)^6$ coming from the antisymmetric tensor field, that is they transform into $H$ monopoles. The BPS monopoles are predicted to have multiplicity one in the 16 of $N = 4$ supersymmetry. In the limit considered here gravitational corrections to the monopole moduli space should be insignificant so the spectrum is in accord with $S$-duality as in [6].

The $H$ monopoles on the other hand are predicted to have multiplicity 24 in the 16 of $N = 4$ and it is this prediction which we wish to test. There is a more precise prediction given by decomposing the 24 states under the unbroken symmetries of the theory. We can work at a point in $\mathcal{M}_N$ where the theory has a $SO(5) \times SO(3)$ symmetry with the $SO(5)$ arising as a subgroup of the global $SO(6)$ of $N = 4$ super Yang-Mills and $SO(3)$ the group of spatial rotations. Under this symmetry group the massive 16 of $N = 4$ decomposes as $16 \rightarrow (5, 1) + (1, 3) + (4, 2)$ while the 24 states at the first excited left-moving level transform as $16(1, 1) + (5, 1) + (1, 3)$. Note in particular that the tensor product includes massive spin two states. If $S$-duality is correct we should find the same representations in the spectrum of $H$-monopoles of charge one, that is we should find exactly $24 \times 16$ harmonic forms on the one $H$-monopole moduli space with the specified quantum numbers.

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2 The Wilson line $A^I_m$ leads to mixing between the $U(1)$ factors arising from the ten-dimensional $E_8 \times E_8$ and metric fields. What is meant here is that the diagonal $U(1)$ comes predominantly from $E_8 \times E_8$ for small $A^I_m$.

3 That $S$-duality relates electrically charged states with respect to the metric $U(1)^6$ to magnetically charged states with respect to the antisymmetric tensor $U(1)^6$ is due to the off-diagonal form of the matrix $L$. 
3. The Spectrum of \( H \)-Monopoles

3.1. \( H \)-Monopoles

In this section we will concentrate on the construction of the light \( H \) magnetic monopoles discussed in the previous section. Let us first review briefly the construction presented in [23]. Monopole solutions are most easily constructed out of periodic fivebranes i.e. by wrapping the fivebrane around the six torus \([21,12]\). To illustrate the construction without unnecessary complication we will start at a point in \( \mathcal{M}_N \) with \( SU(2) \times U(1)^{27} \) symmetry \( ( \text{with } SU(2) \subset E_8 \times E_8 ) \) and discuss the final breaking of \( SU(2) \) to \( U(1) \) explicitly. Let \( x^4 \equiv x^4 + 2\pi R \) be a coordinate on one \( S^1 \); \( x^i, i = 1 \cdots 3 \), the three spatial coordinates, and \( \mu, \nu, \ldots \) indices which run from 1 to 4. The solution of [23] obeys the ansatz

\[
F_{\mu\nu} = \pm \frac{1}{2} \epsilon_{\mu\nu}^{\lambda\sigma} F_{\lambda\sigma} \\
H_{\mu\nu\lambda} = \mp \epsilon_{\mu\nu\lambda}^{\sigma} \partial_{\sigma} \phi \\
g_{\mu\nu} = e^{2\phi} \delta_{\mu\nu}.
\]

To this order, \( H = dB - \frac{\alpha'}{30} \omega \) where \( \omega \) is the Yang-Mills Chern-Simons three form and hence the Bianchi identity,

\[
dH = -\frac{\alpha'}{30} \text{Tr} F \wedge F,
\]

determines the form of the dilaton via

\[
\nabla_\rho \nabla^\rho \phi = -\frac{\alpha'}{60} \text{Tr} F^2.
\]

The solution is thus completely determined by a (anti-) self-dual \( SU(2) \) connection on \( R^3 \times S^1 \). Such solutions can be constructed from an array of instantons on \( R^4 \) which is periodic in \( x^4 \). If the instantons all have the same scale size and gauge orientation then one finds the solution

\[
A_\mu = \tilde{\Sigma}_{\mu\nu} \nabla^\nu \ln f(x)
\]

with

\[
f(\vec{x}, x^4) = 1 + \sum_{k=-\infty}^{\infty} \frac{\rho^2}{r^2 + (x^4 - x_0^4 + 2\pi k R)^2} \\
= 1 + \frac{\rho^2}{2 R r} \sinh \frac{r}{R} \left( \cosh \frac{r}{R} - \cos \frac{x^4 - x_0^4}{R} \right)
\]

where \( r = |\vec{x} - \vec{x}_0| \) and \( (\vec{x}_0, x_0^4) \) give the location of the instanton and \( \tilde{\Sigma}_{\mu\nu} \) is the matrix-valued \('t Hooft tensor. For \( \rho \) finite the \( SU(2) \) gauge fields fall off as \( 1/r^3 \) at large \( r \) and are
thus dipole rather than monopole fields. On the other hand, one finds that \( H_{ij4} \sim \epsilon_{ijk} x^k / r^3 \) indicating that these solutions are \( H \)-monopoles as required by duality.

It is known that generalizations of these solutions exist at points where the \( SU(2) \) symmetry is broken to \( U(1) \), but unfortunately an explicit representation does not seem to be available. The solutions can be described following [26] in a somewhat heuristic manner which should be correct at large \( R \) where the multi-instanton moduli space degenerates into copies of the one-instanton moduli space, by continuity one would expect the same general structure at finite \( R \). Start with an infinite string of instantons located at \( x^i = x^i_0, x^4 = 2\pi R n, n \in \mathbb{Z} \), with identical scale size but with a gauge orientation which rotates by \( \omega \) between instantons with \( \omega \) in a \( U(1) \) subgroup of \( SU(2) \). The resulting vector potential will be periodic up to a gauge transformation by \( \omega \) and has a vanishing Wilson line about \( x^4 \). If one then performs an improper gauge transformation \( U = \omega^{\alpha(x^4/2\pi R)} \) one obtains a single periodic instanton with a non-trivial Wilson line \( \Omega(x^i) = P \exp \left( \int_0^{2\pi R} dx^4 A_4 \right) \) which breaks \( SU(2) \) to \( U(1) \) if \( \alpha \) is not an integer. By integrating the Bianchi identity (3.2) one concludes that the solution is indeed an \( H \)-monopole with magnetic charge proportional to the Pontryagin number.

Although we have not done so, it should be possible to construct these solutions perturbatively in the strength of the \( SU(2) \) breaking. It is known (see e.g. [27]) that there is a well-defined perturbative expansion about the \( 5k \) parameter \( 't \) Hooft solutions which gives local parameters for the full \( 8k \) parameter moduli space described in the ADHM construction [28].

Having described a classical solution with the required properties we have to determine whether the quantum states have the structure required by duality. As usual the low-lying states may be studied by quantization of the bosonic and fermionic collective coordinates corresponding to zero-energy deformations of the monopole. Let us first discuss the bosonic collective coordinates. At finite \( \rho \) and \( \Omega = 1 \) from (3.4), (3.5), we see that the moduli consist of the “center of mass” coordinates \( (\vec{x}_0, x^4_0) \) and the scale size \( \rho \). In addition there are three collective coordinates which describe the \( SU(2) \) orientation of the instantons. Since the center of \( SU(2) \) acts trivially, these three collective coordinates are properly treated as coordinates on \( SU(2)/Z_2 = SO(3) = S^3/Z_2 \). With a non-trivial Wilson line

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Note that Sen’s definition of an \( H \)-monopole includes some of the other moduli and gauge fields but for the solutions we are considering the asymptotic behavior is exactly the same as the definition being used here.
the treatment of these three collective coordinates is somewhat different since one should only consider $SU(2)$ transformations which commute with the value of the Wilson line at spatial infinity. On the other hand, spatial rotations which cannot be undone by $SU(2)$ transformations commuting with the Wilson line $\Omega(\infty)$ will also then have to be included. With trivial Wilson line one can equally well think of the $SO(3)$ collective coordinates as arising from spatial rotations since rotations and $SU(2)/Z_2$ gauge transformations have equivalent actions on the instanton configuration. When the Wilson line is non-vanishing it is simplest to think of all three collective coordinates as arising from rotations. One again concludes that these collective coordinates parametrize $SO(3) = S^3/Z^2$.

The bosonic collective coordinates are coordinates on the one monopole moduli space, $\mathcal{M}^1$. The metric on $\mathcal{M}^1$ is determined as follows. If we let $Z^i$ denote the collective coordinates and $A^0_\mu(x, Z^i)$ the general monopole solution then zero modes are given up to a gauge transformation by varying the classical solution with respect to the $Z^i$:

$$\delta_i A_\mu = \partial_i A^0_\mu - D^0_\mu \epsilon_i,$$  \hspace{1cm} (3.6)

where the gauge parameters $\epsilon_i$ are determined by demanding that the zero modes be orthogonal to local gauge transformations

$$D^0_\mu \delta_i A_\mu = 0.$$  \hspace{1cm} (3.7)

The metric on $\mathcal{M}$ is

$$G_{ij} = \int d^3 x \int d x^4 Tr(\delta_i A_\mu \delta_j A_\mu).$$  \hspace{1cm} (3.8)

The gauge parameter $\epsilon_i$ provides a natural connection on $\mathcal{M}^1$ with covariant derivative

$$s_i = \partial_i + [\epsilon_i, \ ]$$  \hspace{1cm} (3.9)

and field strength

$$\phi_{ij} = [s_i, s_j].$$  \hspace{1cm} (3.10)

There is also a hyperKähler structure on $\mathcal{M}^1$ which is induced from the hyperKähler structure $J^{(m)}$ on $R^3 \times S^1$:

$$J^{(m)\ i\ j} = \int d^3 x \int d x^4 J^{(m)\ \mu\ \nu\ \mu\ \nu} \ Tr(\delta_i A^\mu \delta_k A^\nu) G^{kj}.$$  \hspace{1cm} (3.11)

In addition to the bosonic zero modes there are fermionic zero modes which are paired with the bosonic zero modes via the N=4 supersymmetry \[29\]. For each fermionic zero
mode we introduce a Grassmann odd collective coordinate. Techniques to carry out the explicit reduction of the four dimensional action to $\mathcal{M}^1$ are discussed in [31,32]. The result is an N=4 supersymmetric quantum mechanics based on the moduli space $\mathcal{M}^1$.

Without an explicit solution on $R^3 \times S^1$ with non-trivial Wilson line it is difficult to calculate the metric on $\mathcal{M}^1$ directly. However it seems possible to determine the metric by an indirect argument. First of all, note that the metric (3.8) has obvious Killing vectors which it inherits from translation symmetry in $R^3 \times S^1$ and from rotational symmetry in $R^3$. Furthermore, the three complex structures (3.11) transform as a triplet under the $SO(3)$ isometry. The translation zero modes about the solution (3.4),(3.5) are given by

$$\delta_\mu A_\nu = \partial_\mu A^0_\nu - D^0_\nu A^0_\mu = F^0_{\mu\nu} \quad (3.12)$$

and obviously satisfy (3.7). Substitution into (3.8) gives the flat metric on $R^3 \times S^1$. As the remaining zero modes are independent of the translation zero modes $\mathcal{M}^1$ must have the form

$$\mathcal{M}^1 = R^3 \times S^1 \times \tilde{\mathcal{M}}^1 \quad (3.13)$$

with $\tilde{\mathcal{M}}^1$ a four-dimensional hyperKähler manifold with a $SO(3)$ isometry which rotates the hyperKähler structure. Following the arguments of [33] this requires that $\tilde{\mathcal{M}}^1$ is either the Atiyah-Hitchin metric or the flat metric on $R^4/Z_2$. But we know that the metric (3.8) which is inherited from the gauge field kinetic term has a singularity as $\rho \to 0$ (ignoring possible string and gravitational corrections) so $\tilde{\mathcal{M}}^1$ must be $R^4/Z_2$ with the flat metric.

This argument can be checked by explicit calculation for $SU(2)$ instantons on $R^4$ [34] or on $R^3 \times S^1$ with vanishing holonomy [35] and one finds the expected result.

### 3.2. Infrared Properties

Having determined the moduli space we should analyze the spectrum of states and compare with S-duality. Before doing this in this subsection we will first make a few comments concerning the physical interpretation of the result for $\tilde{\mathcal{M}}^1$. The most striking feature of $\tilde{\mathcal{M}}^1$ is that it is non-compact and hence the quantum-mechanical spectrum will be continuous. Of course the part of the moduli space coming from translations, $R^3 \times S^1$, is also non-compact, but the resulting continuous spectrum is just the usual continuum of momentum states associated with translation invariance of the underlying theory. The continuous spectrum on $\tilde{\mathcal{M}}^1$ means that there is essentially an additional fake “four-momentum” which labels the monopole configurations, at least in the moduli space.
approximation. What is the physical origin of this effect? The non-compact direction in \( \tilde{M}^1 \) is parametrized by the instanton scale size. In conventional non-abelian Yang-Mills theory the effective coupling will grow at large scale size, invalidating the semi-classical approximation. In the \( N = 4 \) theory considered here this is not the case since the beta function vanishes identically \[24\].

One may be tempted to think that the non-compactness arises from very large configurations. However this is not correct. It is known from studies of finite temperature instantons \[26\] and may be verified from the solution (3.5) that the effective size of an \( H \) monopole is of order \( \rho/(1 + \rho^2/12R^2)^{1/2} \) and is thus never much larger than the radius \( R \) of the \( S^1 \). Here the size is defined by the falloff of the action or topological charge density of the instanton. Thus a “plane wave” configuration in \( \tilde{M}^1 \) is a configuration in which the scale size \( \rho \) varies from 0 to \( \infty \), but the physical scale size is never large compared to \( R \).

It is also tempting to think that the non-compactness is caused by coupling to massless non-abelian gauge fields but this is also incorrect. The argument above shows that \( \tilde{M}^1 \) is non-compact even with \( SU(2) \) broken to \( U(1) \). Thus the origin of the additional continuum in the \( H \) monopole spectrum must be due to the coupling of the monopole to the photon and massless scalars of the \( N = 4 \) theory. In principle it seems that this phenomenon could occur in any theory with massless fields, but in most examples, including BPS monopoles in \( N = 4 \) Yang-Mills theory, it does not. This suggests that there may be some problem in the correct identification of collective coordinates in special theories with massless fields.\[5\].

One could try to address this issue by putting the system in a box with twisted boundary conditions which remove the gauge field and scalar zero modes following similar treatments in \[36\] and \[37\]. An immediate problem that arises in trying to study \( S \)-duality in this context is that neither electric charges nor magnetic charges exist on a compact space. For electric charges this follow from Gauss’s law and for \( H \) magnetic charges it follows from \( dH = \alpha' \text{Tr} F \wedge F \). Integrating this equation over \( T^4 \) shows that the instanton number, \( p_1(V) \), must be trivial where \( V \) is the \( E_8 \times E_8 \) bundle.

Although we have little definite to say about \( S \)-duality predictions on \( T^4 \), there are reasons for thinking this is a direction worth pursuing. These include connections with the enhanced symmetries of string theory found in \[38\], the results on \( N = 4 \) Yang-Mills theory on \( T^4 \) found in \[39\] and results on the instanton moduli space on \( T^4 \). For gauge group

\[5\] We thank N. Seiberg for this suggestion and for discussions on this section.
SU(2)/Z₂ it is known that the moduli space of single instantons on $T^4$ with boundary conditions that remove the gauge field zero mode is given by

$$\frac{T^4 \times \tilde{M}}{Z_2 \times Z_2}$$

(3.14)

where $\tilde{M}$ is an orbifold limit of a $K3$ manifold [40,41].

We find the connection with $K3$ intriguing in that in the infinite volume limit the identification of $\tilde{M}^1$ with $K3$ would naturally explain the factor of 24 degeneracy in the spectrum of $H$ monopoles since $K3$ has precisely 24 harmonic forms. On the other hand $K3$ is incompatible with the requirement of $SO(3)$ holonomy and thus with the required quantum numbers and we have argued that in fact $\tilde{M}^1 = R^4/Z_2$. For a smooth, four-dimensional, hyperkähler manifold it is impossible to reconcile the requirement of $SO(3)$ isometry with the requirement that $\tilde{M}^1$ have 24 harmonic forms. In the following section we will argue that a reconciliation may be possible for singular hyperkähler manifolds in string theory in that $R^4/Z_2$ clearly has $SO(3)$ isometry and also has 24 harmonic forms when the forms are counted using ideas from orbifolds.

3.3. Quantization on $\tilde{M}^1$

In the hope that the infrared problems can be resolved let us now discuss a direct treatment of the quantization in infinite volume. The first problem we encounter in thinking about quantization on $M^4$ is the breakdown of the low-energy field theory approximation. Although we can work in a region of the moduli space of Narain compactifications where $H$ monopoles are light, as $\rho \to 0$ the field strength becomes more and more concentrated at the center of the instanton and eventually is large compared to the string scale or Planck scale. It thus seems likely that gravitational and string corrections to the moduli space will become important for scale sizes small compared to $\sqrt{\alpha'}$. On the other hand, the moduli space is tightly constrained by the demands of $N = 4$ supersymmetry and rotational invariance. We can imagine four general possibilities.

First, we can imagine that there are no $\alpha'$ corrections to the moduli space $\tilde{M}^1$ and that the spectrum is given by the spectrum of $N = 4$ supersymmetric quantum mechanics on $M^1$. In this case we find $16 \times 8$ states with the factor of 16 arising as usual from harmonic forms on $R^3 \times S^1$ and the factor of 8 arising from the $8 Z_2$ invariant forms on $R^4/Z^2$. This spectrum is not in agreement with that predicted by $S$-duality.
Second, it is possible that gravitational and other $\alpha'$ corrections change the moduli space at short distances. One natural guess is that the orbifold singularity at the origin is resolved as in the construction of $K3$ from the $T^4/Z_2$ orbifold, that is by replacing $R^4/Z_2$ by the Eguchi-Hanson metric. However on closer inspection this seems unlikely. We argued above that the moduli space should be hyperKähler with an $SO(3)$ isometry rotating the three complex structures. The Eguchi-Hanson metric is hyperKähler and has $SO(3)$ isometry, but the three complex structures are left invariant by the $SO(3)$ action.

A third possibility is that the correct low-energy monopole dynamics is not that of standard $N = 4$ supersymmetric quantum mechanics but rather involves correction terms coming from the supergravity fields. In this case the connection between states and harmonic forms may also be modified. An example of such a modification is discussed in [21]. This possibility deserves further thought, but it seems unlikely to lead to the required number of states.

A fourth possibility is that the moduli space is in fact $R^3 \times S^1 \times R^4/Z_2$, but that the correct treatment of collective coordinates in string theory differs from that in low-energy field theory due to the orbifold singularity in the moduli space. Since the treatment of collective coordinates in string theory is for the most part unknown, we can only speculate as to how string theory and particle theory might differ. However with that warning there is at least one plausible guess which turns out to be in agreement with the predictions of S-duality. The guess is that the treatment of collective coordinates in string theory involves promoting the collective coordinates not just to quantum mechanical variables, $A_\mu(x, Z^i) \rightarrow A_\mu(x, Z^i(t))$, but to two-dimensional string variables, $Z^i \rightarrow Z^i(\tau, \sigma)$ in order to be compatible with modular invariance. It seems unlikely that one really wants the full spectrum of a new string coordinate, so there must presumably be a truncation to just the low-lying spectrum, perhaps as in topological field theory. Thus our guess is that the low-lying spectrum of monopoles with a moduli space possessing orbifold singularities is given by the low-lying spectrum of superstring theory on the moduli space. Note that this

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6 $R^4$ has two sets of commuting hyperKähler structures given in terms of the 't Hooft symbols by $J_{\mu\nu}^a = -\eta_{\mu\nu}^a$, $J_{\mu\nu}^{\bar{a}} = -\bar{\eta}_{\mu\nu}^\bar{a}$. Under the $SO(4) = SU(2)_+ \times SU(2)_-$ isometry these transform as $(3, 1) + (1, 3)$. The $SO(3)$ of rotations we are interested in is the diagonal of $SU(2)_+$ and $SU(2)_-$ and both sets of complex structures transform as triplets. The Eguchi-Hanson metric has a self-dual $SU(2)$ isometry and an anti-self-dual hyperKähler structure which is thus invariant under the isometry. See [7] for more details.

7 This suggestion is due to E. Witten.
does not affect the treatment of the BPS monopole moduli space which is known in general to be non-singular.

Following this line of speculation we wish to calculate the low-lying spectrum of super-string theory on $R^3 \times S^1 \times R^4 / Z_2$. Since the $R^3 \times S^1$ part will be the same in string theory and particle theory, we can concentrate on the $R^4 / Z_2$ factor. The low-lying superstring spectrum on $R^4 / Z_2$ can be computed using orbifold techniques. The partition function is

$$ Z = \frac{1}{2} (Z_{1,1} + Z_{1,\theta} + Z_{\theta,1} + Z_{\theta,\theta}) $$

(3.15)

where $\theta$ is the $Z_2$ transformation and as usual $Z_{g,h}$ denotes the partition function for strings with boundary conditions twisted by $g$ in the $\sigma$ direction and by $h$ in the $\tau$ direction. The first two terms in (3.15) give the point particle spectrum projected onto invariant states. The low-lying states from this sector are thus 8-fold degenerate. The last two terms in (3.15) comes from twisted strings which close only up to a $Z_2$ transformation. They can be determined by modular invariance from $Z_{1,\theta}$. However since $Z_{1,\theta}$ is independent of momenta, it is the same on $R^4 / Z_2$ as on $T^4 / Z_2$. Thus the spectrum from the twisted sector is the same as on $T^4 / Z_2$, that is 16 fold degenerate. Thus the total degeneracy is 24, in agreement with $S$-duality. The basic point is that the degeneracy of 16 in the twisted sector is required by modular invariance whether we are on $T^4 / Z_2$ or on $R^4 / Z_2$. On $T^4 / Z_2$, which is an orbifold limit of $K_3$, the factor of 16 can be viewed as arising from the 16 fixed points of the $Z_2$ transformation. We might try to define string propagation on $R^4 / Z_2$ by taking the infinite radius limit of $T^4 / Z_2$, but this is problematic since if we take one of the fixed points at the origin, the 15 states at the other fixed points move off to infinity in this limit. On the other hand, we don’t see any obstacle to directly defining string propagation on $R^4 / Z_2$ with 16 states localized at the one fixed point of the $Z_2$ transformation on $R^4$.

Finally we can ask if this assumption is also compatible with the quantum numbers required by $S$-duality. Recall that this requires that the 24 states arising from $\tilde{M}^1$ transform as $16(1,1) + (5,1) + (1,3)$ under the $SO(5) \times SO(3)$ unbroken symmetry. For a charge one instanton it is known that all fermion zero modes can be obtained by supersymmetry transformations. Thus, before modding out $R^4$ by $Z_2$ the fermion zero modes transform as $(4, 2)$ as do the supersymmetry transformations. Quantization of the zero modes gives the representations $(5, 1) + (1, 3) + (4, 2)$ with the states $(5, 1) + (1, 3)$ created from the vacuum by the action of an even number of fermion creation operators (even cohomology) and the states $(4, 2)$ created by the action of an odd number of fermion creation operators.
(odd cohomology). The $Z_2$ acts as $-1$ on the fermion zero modes, so projecting onto $Z_2$ invariant states we get $(5, 1) + (1, 3)$. In addition there are 16 massless states arising in the “twisted sector”. The massless states arise in a sector with anti-periodic fermions and anti-periodic bosons and are thus singlets of the fermion current algebra. We thus presume that these states are singlets under $SO(5) \times SO(3)$. The tensor product with the 16 states coming from $R^3 \times S^1$ gives the spectrum required by $S$-duality, including massive spin two magnetic monopoles.

4. Conclusions

It is clear that we have not presented unambiguous new evidence for $S$-duality in string theory. However, thinking about the structures that would be required by $S$-duality has led to some specific suggestions about the structure of string theory. In particular, the question of the treatment of orbifold singularities in the moduli space of solutions to string theory may give an example where the treatment of collective coordinates in string theory differs in a qualitative way from the treatment in low-energy field theory. It also seems that the moduli spaces encountered in treating various kinds of magnetic monopoles are quite rigid. It may well be that there is a non-renormalization theorem for monopole moduli spaces which states that they receive no higher order corrections in $\alpha'$. This would be consistent with our conjecture regarding $R^4/Z_2$.

If our guess as to the treatment of orbifold singularities is correct then we would have new evidence for the existence of $S$-duality in heterotic string theory. Furthermore, our results would provide evidence for this duality not just in $N = 4$ Yang-Mills theory but also in low-energy string theory on $T^6$. The $H$-monopoles we have described depend in general on the compactified dimensions and cannot be viewed purely as four-dimensional solutions. We should also reiterate that we have probed only the first of many predictions that follow from $S$-duality in string theory. In light of the conjecture made in [6] it will be particularly interesting to investigate whether similar features appear in the multi-$H$-monopole moduli space and to extend these ideas to the infinite tower of monopoles required by $S$-duality. These excitations include states of arbitrarily high angular momentum and they are intrinsically stringy, that is they do not have a Kaluza-Klein origin.

In $N = 4$ Yang-Mills theory the monopole multiplet contains massive spin one monopoles. As the only consistent formulation of fundamental spin one objects that we
know of is spontaneously broken gauge theory, it seems likely that there is a dual formulation in which the monopoles become gauge bosons. Similarly, the existence of massive spin two monopoles in toroidal compactifications of low-energy string theory suggest the existence of a dual formulation involving extended objects such as strings. Note that the appearance of massive spin two \( H \)-monopoles does not depend on our assumptions concerning collective coordinates in string theory. Standard \( N = 4 \) supersymmetric quantum mechanics on \( R^4/Z_2 \) gives 8 of the 24 states required by \( S \)-duality and these states when tensored with the states arising from \( R^3 \times S^1 \) include spin two monopoles.

Although we lack any hard evidence as yet, it would be extremely interesting if there exists an extension of \( S \)-duality beyond toroidal compactifications. Given the suggestion of [5] that it should be thought of as a discrete gauge symmetry makes this seem quite likely, although the action is likely to be much less trivial in other backgrounds. Given that the monopoles we have discussed are really extended fivebranes wrapped around some internal dimensions, it is natural to speculate that the dual theory should be a quantum theory of heterotic fivebranes. Fivebranes occur as the natural objects which couple to the dual formulation of \( D = 1 \) supergravity [11], and explicit fivebranes solutions of low-energy string theory exist and have properties suggesting such a dual formulation [12]. In addition, the string states which are dual to the \( H \)-monopoles discussed above have certain characteristics of solitons [10] that suggest that they might arise as soliton solutions in a dual theory of fivebranes [42].

However there are severe difficulties in constructing a quantum theory of fivebranes and it is widely believed that such a theory does not exist, at least not with a spectrum which resembles that of string theory. The fivebrane solutions of [12] are based on Yang-Mills instantons. The results here suggest to us that \( S \)-duality in string theory may be more subtle than a duality involving fivebranes as extended objects. It seems instead that the moduli space of instantons encodes certain spacetime structures in somewhat the same way that Riemann surfaces and various structures on them encode spacetime properties in string theory. In studying the single \( H \) monopole moduli space we found a moduli space which was locally \( M_{ss} \times M_{bs} \) with \( M_{ss} \) the underlying four-manifold \( R^3 \times S^1 \) and \( M_{bs} = R^4/Z_2 \). Modulo our assumption about the counting of harmonic forms in string theory, \( M_{ss} \) reflects the structure of the right-moving superstring ground state in its cohomology in that the even cohomology and odd cohomology are both 8 dimensional while \( M_{bs} \) reflects the structure of the left-moving string first excited state in having no odd cohomology and 24 dimensional even cohomology. Perhaps we should be looking for a
reformulation of string theory in which the primary objects are moduli spaces of self-dual connections on four manifolds rather than moduli spaces of Riemann surfaces.

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