Complete solutions of Toda equations and cyclic Higgs bundles over non-compact surfaces

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Abstract

On a Riemann surface with a holomorphic \( r \)-differential, one can naturally define a Toda equation and a cyclic Higgs bundle with a grading. A solution of the Toda equation is equivalent to a harmonic metric of the Higgs bundle for which the grading is orthogonal. Here we focus on a general non-compact Riemann surface with an \( r \)-differential which is not necessarily meromorphic at infinity. We introduce the notion of complete solution of the Toda equation, and we prove the existence and uniqueness of a complete solution by using techniques for both Toda equations and harmonic bundles. Moreover, we show some quantitative estimates of the complete solution.

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1 Introduction

1.1 Higgs bundles associated to \( r \)-differentials and harmonic metrics

Let \( X \) be any Riemann surface. We fix a line bundle \( K_X^{1/2} \) with an isomorphism \( K_X^{1/2} \otimes K_X^{1/2} \simeq K_X \). Let \( r \) be a positive integer. We set \( K_{X,r} := \bigoplus_{i=1}^r K_X^{(r+1-2i)/2} \). We define the actions of \( G_r = \{ a \in \mathbb{C} | a^r = 1 \} \) on \( K_X^{(r+1-2i)/2} \) by \( a \cdot v = a^iv \). They induce a \( G_r \)-action on \( K_{X,r} \). For any \( r \)-differential \( q \in H^0(X, K_X^\bullet) \), let \( \theta(q) \) denote the Higgs field of \( K_{X,r} \) induced by \( \theta_i = id : K_X^{(r+1-2i)/2} \rightarrow K_X^{(r+1-2(i+1))/2} \otimes K_X(i=1,\ldots,r-1) \) and \( \theta_r = q : K_X^{(r-1)/2} \rightarrow K_X^{(r-1)/2} \otimes K_X \).

A Hermitian metric \( h \) on a Higgs bundle is called harmonic if it satisfies the Hitchin self-dual equation. Geometrically, a harmonic metric gives rise to equivariant harmonic maps from the universal cover \( \tilde{X} \) to \( SL(r, \mathbb{C})/SU(r) \). Let \( \text{Harm}(q) \) denote the set of \( G_r \)-invariant harmonic metrics \( h \) of \((K_{X,r}, \theta(q))\) such that \( \det(h) = 1 \). By the \( G_r \)-invariance, the decomposition \( K_{X,r} = \bigoplus_{i=1}^r K_X^{(r+1-2i)/2} \) is orthogonal with respect to any \( h \in \text{Harm}(q) \), and hence we obtain the decomposition \( h = \bigoplus h_i |_{K_X^{(r+1-2i)/2}} \). We say that \( h \in \text{Harm}(q) \) is real if \( h_i |_{K_X^{(r+1-2i)/2}} \) and \( h_j |_{K_X^{(r-1+2i)/2}} \) are mutually dual. Let \( \text{Harm}^R(q) \) denote the subset of \( h \in \text{Harm}(q) \) which are real.

Recall that for a compact Riemann surface \( X \), the classification of harmonic metric of \((K_{X,r}, \theta(q))\) is well-known. If \( X \) is hyperbolic, there uniquely exists a harmonic metric of unit determinant by the Hitchin-Kobayashi correspondence for Higgs bundles \((17, 22)\). Moreover, as observed by Baraglia \([1]\), the harmonic metric is \( G_r \)-invariant and real. In other words, if \( X \) is compact hyperbolic, \( \text{Harm}(q) = \text{Harm}^R(q) \) consists of a unique element. If \( X \) is an elliptic curve, it is easy to see that \( \text{Harm}(q) \) consists of a unique element if \( q \neq 0 \), and that \( \text{Harm}(0) \) is empty. If \( X = \mathbb{P}^1 \), it is easy to see that there exists no non-zero holomorphic \( r \)-differential, and that \( \text{Harm}(0) \) is empty.

When \( X \) is non-compact, the uniqueness of harmonic metrics no longer holds always and the harmonic metric of unit determinant is not necessarily \( G_r \)-invariant. When \( X = \mathbb{C}^* \) and \( q = z^n dz^r \), the solution space \( \text{Toda}(q, g) \) was obtained in \([13, 14, 27, 28]\) motivated by the relation with the tt*-geometry \([6]\).

In this paper, we study the existence and uniqueness of \( G_r \)-invariant harmonic metrics on \((K_{X,r}, \theta(q))\) over a non-compact Riemann surface \( X \).

1.2 Toda equation associated to \( r \)-differentials

Let \( g \) be any Kähler metric of \( X \). It induces a \( G_r \)-invariant Hermitian metric \( h^{(0)}(g) \) of \( K_{X,r} \). For any other \( G_r \)-invariant Hermitian metric \( h \) such that \( \det(h) = 1 \), we obtain a tuple of \( \mathbb{R} \)-valued functions \( w = (w_1, \ldots, w_r) \) such that \( \sum w_i = 0 \) by the relation \( h |_{K_X^{(r+1-2i)/2}} = e^{w_i} h^{(0)}(g) |_{K_X^{(r+1-2i)/2}} \). Then, \( h \) is contained in \( \text{Harm}(q) \) if and only if the Toda equation is satisfied:

\[
\begin{align*}
\Delta_g w_1 &= -e^{-w_1+w_2} - e^{-w_2+w_3} - \cdots - e^{-w_r}k_g \\
\Delta_g w_i &= e^{-w_{i-1}+w_i} - e^{-w_{i+1}+w_i} - \cdots - e^{-w_r+w_1} - \frac{1}{2}k_g \\
\Delta_g w_r &= e^{-w_{r-1}+w_r} - e^{-w_r+w_1} - \cdots - e^{-w_2+w_1} - \frac{1}{4}k_g \\
&\quad \quad (i = 2, \ldots, r-1)
\end{align*}
\]

Here, \( \Delta_g = \frac{1}{2} \sqrt{-1} \Lambda \partial \overline{\partial} \), \( k_g = \sqrt{-1} AR(g) \) and \( |q|^2_g = q\overline{q}/g^r \). Let \( \text{Toda}(q, g) \) denote the set of solutions \( w \) of \((1)\) satisfying \( \sum w_i = 0 \). A solution \( w \in \text{Toda}(q, g) \) is called real if \( w_i + w_{r+1-i} = 0 \). Let \( \text{Toda}^R(q, g) \) denote the set of real solutions of \((1)\). As explained, there is a natural bijection \( \text{Harm}(q) \simeq \text{Toda}(q, g) \), which induces \( \text{Harm}^R(q) \simeq \text{Toda}^R(q, g) \).

Remark 1.1 If we change the sign in the system \((1)\), we obtain the classical Toda equation studied extensively in integrable system, e.g. see \([4, 5]\). Geometrically, the classical Toda equation gives rise to harmonic maps from surface to compact flag manifolds.

Remark 1.2 For general \( r \), a solution of the Toda equation gives rise to an equivariant harmonic map \( f : \tilde{X} \rightarrow SL(r, \mathbb{C})/SU(r) \) such that \( \text{tr}(\partial f \otimes i) = 0 \) except for \( \text{tr}(\partial f \otimes r) = q \). In lower rank, the Toda equation is encoded with much richer geometry. If \( r = 2 \), the Toda equation coincides with the Bochner equation for harmonic maps between surfaces, which is studied extensively in literature, e.g. see \([13, 16, 21, 27, 28, 44, 47]\).
If \( r = 3 \), the Toda equation for a real solution coincides with Wang’s equation for hyperbolic affine spheres in \( \mathbb{R}^3 \), which is studied extensively in literature, e.g., see \cite{3, 12, 18, 24, 25, 26, 29, 32}. If \( r = 4 \), the Toda equation for a real solution coincides with the Gauss-Ricci equation for maximal surfaces in \( \mathbb{H}^{2,2} \) by the work of Collier-Tholozan-Toulisse \cite{8}, and also studied in \cite{24}.

### 1.3 Existence and uniqueness of complete solutions

A solution \( w \in \text{Toda}(q,g) \) is called complete if for each \( 2 \leq i \leq r \), \( e^{-w_{i-1}+w_i} \cdot q \) is complete. In terms of harmonic metrics, it is equivalent to the condition that the metrics \( h_{K[r^{i-1}+i+1]/2} \otimes h_{K[r^{i-1}+i+1]/2} \) induce complete distances on \( X \). Note that \( K_{X}^{(r^{i-1}+2i+1)/2} \otimes (K_{X}^{(r^{i-1}+2i+1)/2} - 1) \) is naturally identified with the tangent bundle of \( X \). Note that for any Kähler metric \( g_a \) (\( a = 1, 2 \)) there exists a natural bijection \( \text{Toda}(q,g) \simeq \text{Toda}(q,g_a) \) under which complete solutions are preserved.

As in Remark 1.2, a complete solution has rich geometric interpretations. For general \( r \), the induced metric of the harmonic map \( f : X \to SL(r, \mathbb{C})/SU(r) \) arising from a complete solution is complete. But our condition of complete solution is stronger than the condition of the induced metric being complete. When \( r = 2 \), a complete solution in \( \text{Toda}(q,g) \) is equivalent to looking for an equivariant harmonic map \( f : X \to \mathbb{H}^2 \) with Hopf differential \( q \) such that the holomorphic energy density \( |\partial f|^2 \) defines a complete metric on \( X \), see Wan \cite{37}. This is our main motivation to introduce a complete solution of the Toda equation for general \( r \). When \( r = 3 \), a complete real solution in \( \text{Toda}(q,g) \) is equivalent to looking for a complete hyperbolic affine sphere in \( \mathbb{R}^3 \) with Pick differential \( q \). When \( r = 4 \), a complete real solution in \( \text{Toda}(q,g) \) gives rise to a complete maximal surface in \( \mathbb{H}^{2,2} \) which is studied recently in \cite{20, 19}.

The main result in this paper is the following existence and uniqueness theorem of a complete solution. In the case \( r = 2 \), the theorem is proven in \cite{21, 37, 38} in which case the Toda equation reduces to a scalar equation.

The existence result makes use of the techniques in rank 2 case together with the method of super-subsolution for system developed in Guest and Lin \cite{14}. The uniqueness result makes use of the Omori-Yau and Cheng-Yau maximum principles together with Simpson’s inequality for harmonic bundles.

**Theorem 1.3 (Theorem 5.12)** Suppose \( q \neq 0 \) unless \( X \) is hyperbolic. Then, there exists a unique complete solution in \( \text{Toda}(q,g) \). Moreover, it is real.

If \( X \) is parabolic or elliptic, and if \( q = 0 \), then \( \text{Toda}(q,g) \) is empty.

Let \( w^c \) denote the complete solution and \( h^c \) the corresponding harmonic metric.

### 1.4 Uniqueness and non-uniqueness of general solutions

Suppose the \( r \)-differential \( q \) has finitely many zeros, and let \( K \) be a relatively compact open subset containing all zeros of \( q \). Set \( |q|^{2/r} = (\bar{q}q)^{2/r} \). Then \( |q|^{2/r} \) naturally induces a metric on \( X \setminus K \). We study the uniqueness of solutions in \( \text{Toda}(q,g) \) depending on whether \(|q|^{2/r}\) induces a complete metric on \( X \setminus K \). In the case \( r = 2 \) or 3, this is proven by Li \cite{21} in the setting of a scalar equation for \( X = \mathbb{C} \) and \( q \) is a polynomial \( r \)-differential.

**Theorem 1.4 (Corollary 3.30)** If \(|q|^{2/r}\) induces a complete metric on \( X \setminus K \), then \( \text{Toda}(q,g) = \{ w^c \} \).

As a direct corollary of Theorem 1.3.

**Corollary 1.5** Suppose that \( X \) is the complement of a finite subset in a compact Riemann surface \( \overline{X} \). We also assume that (i) \( q \) is meromorphic on \( \overline{X} \), (ii) the pole order of \( q \) is larger than \( r \) at each point of \( \overline{X} \setminus X \). Then, there uniquely exists a solution of the associated Toda equation. Moreover, it is complete and real.

The statement of Corollary 1.5 also follows from the method of Kobayashi-Hitchin correspondence for wild harmonic bundles, as explained in the subsequent paper \cite{28}.

**Theorem 1.6 (Proposition 5.13)** If \(|q|^{2/r}\) induces an incomplete metric on \( X \setminus K \), then there exists a solution \( u \in \text{Toda}^R(q,g) \) which is not complete and satisfies there exists a constant \( c > 0 \) such that on \( X \setminus K \),

\[-r + 1 - 2l \log|q|_g - c \leq u \leq -r + 1 - 2l \log|q|_g, \quad 1 \leq l \leq n = \lfloor r/2 \rfloor.\]

In particular, the solutions in \( \text{Toda}(q,g) \) are not unique.
1.5 Properties of complete solutions

We obtain some interesting properties of the complete solution, especially, the precise bounds between the metrics $e^{w_{i+1} - w_i}g(i = 1, \cdots, r - 1)$. Such estimates were first proven in Dai and Li [9, 10] over compact hyperbolic surfaces. When we study real solutions of (1), we set $n := \lfloor r/2 \rfloor$.

Theorem 1.7 (Theorem 4.4) The unique complete solution $w^c = (w_1, w_2, \cdots, w_r)$ satisfies that $e^{w_{i+1} - w_i}g(i = 1, \cdots, r - 1)$ are mutually bounded and $|q|^{w_{i+1} - w_i}g(i = 1, \cdots, r - 1)$ are bounded. More precisely, one of the following holds:

(i) $w_k < -\frac{r + 1 - 2k}{r} \log |q|_g$, $1 \leq k \leq n$

(ii) $w_k = -\frac{r+1-2k}{r} \log |q|_g$ for $1 \leq k \leq n$, in which case $q$ has no zeros and $|q|^{\frac{2}{n}}$ defines a complete metric;

(iii) $w_k = \log \left(\frac{(k-1)!}{(r-1)!} 2^{2(r+1-2k)} \right)$ for $1 \leq k \leq n$, in which case $q \equiv 0$ and $(X, g)$ is a complete hyperbolic surface.

Let $f : \tilde{X} \to SL(r, \mathbb{C})/SU(r)$ be the equivariant harmonic map associated to the harmonic bundle $(\mathcal{K}_{X,r}(\theta(q), h^c)$. Theorem 1.7 implies that the curvature $\kappa$ of the pullback metric of $f$ satisfies $\kappa < 0$ (Corollary 4.5 in Case (i) and (iii)).

Next we see that the complete solution dominates any other real solution.

Theorem 1.8 (Proposition 4.7) Let $w^c$ be the complete solution in Toda$(q, g)$. Suppose $u$ is a real solution in Toda$(q, g)$. Then one of the following holds: (i) $w^c = u$, (ii) $w^c_i < u_i$ $(i = 1, \cdots, n)$.

When the $r$-differential $q$ is bounded with respect to a complete hyperbolic metric $g$, we obtain a unique bounded solution.

Theorem 1.9 (Corollary 5.9) Let $(X, g)$ be a complete hyperbolic surface. If an $r$-differential $q$ on $X$ is bounded with respect to $g$, then there uniquely exists a bounded solution in Toda$(q, g)$. Moreover, it is real. Conversely, if there exists a bounded solution in Toda$(q, g)$, then $q$ is bounded with respect to $g$.

In fact, the bounded solution is the complete solution.

This result is closely related to a recent work of Labourie and Toulisse [10] on the maximal surfaces in $\mathbb{H}^{2,2}$. We also generalize the existence and uniqueness of bounded solutions to general cyclic Higgs bundles in a subsequent paper [23, Section 7].

In a subsequent paper [23], we will study a classification of general solutions of (1) for isolated singularities which are poles or some type of essential singularities.

1.6 Organization of the paper

In Section 2 we review the relation between Toda equation and the harmonic bundle. In Section 3 we prove the uniqueness of complete solutions for the Toda equation. For the uniqueness, we will apply the Omori-Yau maximum principle and Cheng-Yau maximum principle. In Section 4 we show some interesting properties of the complete solution. In Section 5 we show the existence of a complete real solution for the Toda equation. For the existence, we use the method of super-subsolution.
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2 Toda equations and harmonic bundles

Let $X$ be a Riemann surface with an $r$-differential $q = q(z)dz^r$. Let $g = g_0 dz \otimes d\bar{z}$ be a Kähler metric of $X$. The Kähler form $\omega$ is given by $\omega = \sqrt{-1} g_0 dz \wedge d\bar{z}$. We also view the Kähler metric $g$ as a Hermitian metric on $K^{-1}_X$. Let $\Lambda$ denote the contraction with respect to the Kähler form $\omega$. Let $R(g)$ denote the curvature of the Chern connection of $K^{-1}_X$ with $g$. So $R(g) = \partial \bar{\partial} \log g_0 = -\partial \bar{\partial} \log g_0$. The Laplacian with respect to $g$, the Gaussian curvature of $\text{Re}(g)$, and the square norm of $q$ with respect to $g$ are respectively:

$$\Delta_g := \frac{\sqrt{-1}}{2} \Lambda \bar{\partial}g = \frac{1}{g_0} \partial_x \partial_x, \quad k_g := -\sqrt{-1} \Lambda R(g) = -\frac{2}{g_0} \partial_x \partial_x \log g_0, \quad |q|_g^2 = q(z)\bar{q}(z)/g_0.$$

We obtain the Hermitian metrics $(g^{-1})^{\otimes (r+1-2i)/2}$ on $K^{(r+1-2i)/2}_X$, and $h^{(0)}(g) = \oplus_{i=1}^r (g^{-1})^{\otimes (r+1-2i)/2}$ on $\mathbb{K}_{X,r}$. Then

$$|(dz)^{(r+1-2i)/2}|_{h^{(0)}(g)} = g_0^{-(r+1-2i)/2}.$$

For any $\mathbb{R}^r$-valued function $w = (w_1, \ldots, w_r)$, we obtain a Hermitian metric

$$h(g, w) := \bigoplus e^{w_i} (g^{-1})^{\otimes (r+1-2i)/2}$$

on $\mathbb{K}_{X,r}$.

**Proposition 2.1** $h(g, w) \in \text{Harm}(q)$ if and only if $w$ satisfies the following system of differential equations:

$$\begin{cases}
\Delta_g w_1 = |q|_g^2 e^{w_1 - w_r} - e^{w_2 - w_1} - \frac{1}{4} k_g \\
\Delta_g w_j = e^{w_j - w_{j-1}} - e^{w_{j+1} - w_j} - \frac{r+1-2i}{2} k_g \\
\Delta_g w_r = e^{w_r - w_{r-1}} - |q|_g^2 e^{w_1 - w_r} + \frac{4}{4} k_g
\end{cases} \quad (j = 2, \ldots, r - 1) \tag{2}$$

and $\sum_{i=1}^r w_i = 0$.

In particular, in the case $r = 2$, $h(g, w) \in \text{Harm}(q)$ if and only if $w = (w_1, -w_1)$ satisfies

$$\Delta_g w_1 = |q|_g^2 e^{2w_1} - e^{-2w_1} - \frac{1}{4} k_g.$$

**Proof** Let $h(g, w)_i$, denote the restriction of $h(g, w)$ to $K^{(r+1-2i)/2}_X$. We have the decomposition $h(g, w) = \bigoplus_{i=1}^r h(g, w)_i$. We obtain the decomposition of the curvature of the Chern connection:

$$R(h(g, w)) = \bigoplus_{i=1}^r R(h(g, w)_i) = \bigoplus_{i=1}^r (R(g)^{(r+1-2i)/2} + \bar{\partial} w_i).$$

Hence, we obtain

$$\sqrt{-1} \Lambda R(h(g, w)) = \bigoplus_{i=1}^r \left( -\frac{(r+1-2i)}{2} k_g - 2\Delta_g w_i \right).$$

For $P \in X$, let $v$ denote the element of $(K_X)_P$ such that $|v|_g = 1$ by viewing $g^{-1}$ a Hermitian metric on $K_X$. We may choose $v = g_0^2 dz$. Then $\sqrt{-1} \Lambda (v \otimes \bar{v}) = 2$. 

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5
Hence, the Hitchin equation

\[
\theta(v_i) = \begin{cases} 
    v_{i+1} \otimes v & (i = 1, \ldots, r-1) \\
    \beta v_1 \otimes v & (i = r)
\end{cases}
\]

We obtain

\[
\theta^1_{h(g,w)}(v_i) = \begin{cases} 
    e^{u_{i+1} - u_i} v_{i+1} \otimes \overline{v} & (i = 2, \ldots, r) \\
    \beta e^{u_1} v_r \otimes \overline{v} & (i = 1)
\end{cases}
\]

Therefore, we obtain

\[
\sqrt{-1} \Lambda[\theta, \theta^1_{h(g,w)}](v_i) = \begin{cases} 
    2(|\beta|^2 e^{u_1} - e^{u_2} v_i) v_1 & (i = 1) \\
    2(e^{u_{i+1} - u_i} - e^{u_{i+2} - u_i}) v_i & (i = 2, \ldots, r-1) \\
    2(e^{u_r - u_{i-1}} - |\beta|^2 e^{u_1} v_r) & (i = r)
\end{cases}
\]

Hence, the Hitchin equation \( R(h(g,w)) + [\theta, \theta^1_{h(g,w)}] = 0 \) if and only if (2) holds.

\section{Uniqueness of complete solutions}

In this section, we will discuss the uniqueness of solutions for the Toda system \( (1) \). Our main tools are:

- the Omori-Yau maximum principle and the Cheng-Yau maximum principle, since they work well for complete Riemann surfaces.

- for a harmonic bundle \( (E, \overline{\nabla}_E, \theta, h) \) over a Riemann surface \( X \) with a Kähler metric \( g \), the following inequality holds away from zeros of \( \theta \):

\[
\triangle_g \log |\theta|^2_{h,g} \ge \frac{||[\theta, \theta^1]|^2}{|\theta|^2_{h,g}} + \frac{1}{2}k_g.
\]

We will apply the two maximum principles to the above inequality.

\subsection{Maximum principles}

\textbf{3.1.1 Omori-Yau Maximum principle}

\textbf{Lemma 3.1 (Omori-Yau Maximum Principle \[30, 42\])} Suppose \((M, h)\) is a complete manifold with Ricci curvature bounded from below. Then for a \( C^2 \) function \( u : M \to \mathbb{R} \) bounded from above, there exists a \( m_0 \in \mathbb{N} \) and a family of points \( \{x_m\} \in M \) such that for each \( m \ge m_0 \),

\[
u(x_m) \ge \sup u - \frac{1}{m}, \quad |\nabla_h u(x_m)| \le \frac{1}{m}, \quad \triangle_h u(x_m) \le \frac{1}{m},
\]

where \( \nabla_h, \triangle_h \) are the gradient and the Laplacian with respect to the background metric \( h \) respectively.

The following is a variant of Omori-Yau maximum principle for manifolds with boundary.

\textbf{Lemma 3.2} Let \((M, h)\) be any complete Riemannian manifold with a smooth compact boundary \( \partial M \). Assume that the Ricci curvature of \( h \) is bounded from below. Let \( u : M \to \mathbb{R} \) be a \( C^2 \)-function bounded from above. Either one of the following holds.

- \( \max_{\partial M} u = \sup_M u \).

- There exist \( m_0 \in \mathbb{Z}_{>0} \) and a family of points \( Q_m \in M \ (m \ge m_0) \) such that

\[
u(Q_m) \ge \sup_M u - \frac{1}{m}, \quad |\nabla_h u(Q_m)| \le \frac{1}{m}, \quad \triangle_h u(Q_m) \le \frac{1}{m},
\]

where \( \nabla_h \) and \( \triangle_h \) are the gradient and the Laplacian with respect to the background metric \( h \), respectively.
Proof Suppose that $\max_{\partial M} u < \sup_M u$. There exists a compact neighbourhood $K$ of $\partial M$ in $M$ with a diffeomorphism $\Psi : K \simeq \partial M \times [0,1]$ such that $\Psi(\partial M) = \partial M \times \{0\}$. We may assume that $\max_{K} u < \sup_{M} u$. There exists a complete metric $h_1$ on $M \setminus \partial M$ such that $h_{1|M \setminus K} = h_{|M \setminus K}$ and that the Ricci curvature of $h_1$ is bounded from below. By applying Omori-Yau maximum principle (Lemma 3.1), we obtain the claim of the lemma.

3.1.2 Cheng-Yau maximum principle

Lemma 3.3 (Cheng-Yau Maximum Principle [7]) Suppose $(M, h)$ is a complete manifold with Ricci curvature bounded from below. Let $u$ be a $C^2$-function defined on $M$ such that $\Delta_h u \geq f(u)$, where $f : \mathbb{R} \to \mathbb{R}$ is a function. Suppose there is a continuous positive function $g(t) : [a, \infty) \to \mathbb{R}_+$ such that

(i) $g$ is non-decreasing;
(ii) $\liminf_{t \to \infty} \frac{f(t)}{g(t)} > 0$;
(iii) $\int_a^\infty (\int_b^t g(\tau)d\tau)^{-\frac{1}{2}}dt < \infty$, for some $b \geq a$,

then the function $u$ is bounded from above. Moreover, if $f$ is lower semi-continuous, $f(\sup u) \leq 0$.

In particular, for $\alpha > 1$, positive constants $c_0, c_1, c_2$, one can check $f(t) = c_0 t^\alpha - (c_1 t + c_2)$, $g(t) = t^\alpha$ satisfy the above three conditions (i)(ii)(iii).

The following is a variant of Cheng-Yau maximum principle for manifolds with boundary.

Lemma 3.4 Let $(M, h)$ be a connected complete Riemannian manifold with compact smooth boundary $\partial M$ whose Ricci curvature is bounded from below. Let $u$ be a $C^2$-function $M \to \mathbb{R}$ such that $\Delta_h u \geq f(u)$ for a function $f$. Suppose that the restriction of $f$ to $\{-\infty < t < a\}$ is bounded from above for any $a < \infty$. Suppose that there exists a continuous positive function $g : \{a \leq t < \infty\} \to \mathbb{R}_{>0}$ such that

- $g$ is non-decreasing;
- $\liminf_{t \to \infty} \frac{f(t)}{g(t)} > 0$;
- $\int_a^\infty \left( \int_b^t g(\tau)d\tau \right)^{-1/2}dt < \infty$ for some $b > a$.

Then, $u$ is bounded from above. Moreover, if $f$ is lower semi-continuous, either one of the following holds; (i) $\max_{\partial M} u = \sup_M u$, (ii) $f(\sup_M u) \leq 0$.

Proof Let $N_1$ be a relatively compact open neighbourhood of $\partial M$ in $M$. Let $N_2$ be a relatively compact open neighbourhood of $\partial M$ in $N_1$. There exists a complete Riemannian metric $h_0$ of $M \setminus \partial M$ such that (i) Ricci curvature of $h_0$ is bounded from below, (ii) $h = h_0$ on $M \setminus N_1$. There exists a $C^2$-function $u_0 : M \to \mathbb{R}$ such that (i) $u = u_0$ on $M \setminus N_1$, (ii) $u_0|N_2$ is constant. The condition (ii) particularly implies that $\Delta_{h_0}(u_0|N_2) = 0$. Because $\Delta_{h_0}(u_0)|N_1 \setminus \partial M$ is bounded, and because $f(0)|N_1$ is bounded from above, there exists $C_0 > 0$ such that $\Delta_{h_0} u_0 \geq f(u_0) - C_0$ on $N_1$. Because $u = u_0$ and $h = h_0$ on $M \setminus N_1$, we obtain $\Delta_{h_0} u_0 \geq f(u_0) - C_0$ on $M \setminus \partial M$. By Cheng-Yau maximum principle, $u_0$ is bounded from above. Because $u_0|N_1$ is bounded from above, and because $u = u_0$ on $M \setminus N_1$, we obtain that $u$ is bounded from above.

Set $T := \sup_M u$. Assume that $T > \max_{\partial M} u$. We may assume that $T > \sup_{N_1} u$. There exists $\epsilon > 0$ such that $\sup_{N_1} u < T - 2\epsilon$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$-function such that (i) $0 \leq \chi(t) \leq 1$ for any $t$, (ii) $\chi(t) = 0$ (t $\geq T - \epsilon$), (iii) $\chi(t) = 1$ (t $\leq T - 2\epsilon$). We set $f_0(t) := f(t) - C_0 \chi(t)$. On $M \setminus N_1$, we obtain

$$\Delta_{h_0}(u_0) = \Delta_h(u) \geq f(u) \geq f(u_0) - \chi(u_0)C_0 = f_0(u_0).$$

On $N_1$, we obtain

$$\Delta_{h_0}(u_0) \geq f(u_0) - C_0 = f(u_0) - \chi(u_0)C_0 = f_0(u_0).$$

By Cheng-Yau maximum principle (Lemma 3.3), we obtain $f_0(\sup_{M \setminus \partial M} u_0) \leq 0$. Because $\sup_{M \setminus \partial M} u_0 = T = \sup_M u$, we obtain $f(\sup_M u) \leq 0$. \hfill \Box
3.2 Preliminary from linear algebra

3.2.1 Lower estimate of the commutator with the adjoint (general case)

We recall well known lemmas from [33, Page 729] for the convenience of the readers. Let \( V \) be an \( r \)-dimensional vector space with a Hermitian metric \( h \). Let \( f \) be an endomorphism of \( V \). Let \( f_h^1 \) denote the adjoint of \( f \) with respect to \( h \).

There exists a (non-canonical) orthonormal base \( v_1, \ldots, v_r \) of \( V \) for which \( f \) is lower triangular, i.e., \( f(v_j) = \sum_{i \leq j} f_{i,j} v_i \). We define the endomorphism \( f_0 \) of \( V \) by \( f_0(v_i) = f_{i,i} v_i \) (\( i = 1, \ldots, r \)). We set \( f_1 := f - f_0 \). Note that \( |f|^2_h = |f_0|^2 + |f_1|^2_h \).

**Lemma 3.5** There exists \( C_0 > 0 \) depending only on \( r \) such that \( ||[f, f_h^1]|_h| \geq C_0|f_1|^2_h \).

**Proof** We study the claim in terms of matrices. The following argument is explained in [33, Page 729]. Let \( A = (A_{i,j}) \) be an \( r \)-square matrix such that \( A_{i,j} = 0 \) (\( i < j \)). Let \( A^\dagger \) denote the adjoint matrix of \( A \), i.e., \( A_{i,j}^\dagger = (A_{j,i}) \). We decompose \( A = A_0 + A_1 \),

\[
  (A_0)_{i,j} = \begin{cases} A_{i,j} & (i = j) \\ 0 & (i \neq j) \end{cases}, \quad (A_1)_{i,j} = \begin{cases} 0 & (i = j) \\ A_{i,j} & (i \neq j). \end{cases}
\]

We obtain \( [A, A^\dagger] = [A_0, A_1^\dagger] + [A_1, A_0^\dagger] + [A_1, A_1^\dagger] \). Note that the diagonal entries of \( [A_0, A_1^\dagger] \) and \( [A_1, A_0^\dagger] \) are 0. By direct calculations, we obtain the following for any \( i = 1, \ldots, r \):

\[
  [A_1, A_1^\dagger]_{i,i} = \sum_{1 \leq j < i} |A_{i,j}|^2 - \sum_{i < j \leq r} |A_{j,i}|^2.
\]

Note that the following holds for any \( i \):

\[
  \sum_{i < j \leq r} |A_{j,i}|^2 \leq \sum_{1 \leq j < i} |A_{i,j}|^2 - \sum_{i < j \leq r} |A_{j,i}|^2 + \sum_{1 \leq j < i} |A_{i,j}|^2 \leq \left| \sum_{1 \leq j < i} |A_{i,j}|^2 - \sum_{i < j \leq r} |A_{j,i}|^2 \right| + \sum_{1 \leq j < i} \sum_{j < k \leq r} |A_{k,j}|^2 \tag{3}
\]

By an easy induction using (3), we can prove that there exist positive constants \( C_{1,k} \) (\( k = 1, \ldots, r \)) depending only on \( r \) such that the following holds:

\[
  \sum_{i \leq k} \sum_{i < j \leq r} |A_{j,i}|^2 \leq C_{1,k} \sum_{i \leq k} \left| \sum_{1 \leq j < i} |A_{i,j}|^2 - \sum_{i < j \leq r} |A_{j,i}|^2 \right|. \tag{4}
\]

The inequality in the case \( k = r \) implies the claim of Lemma 3.5.

**Lemma 3.6** Suppose that there exists \( B > 0 \) such that any eigenvalues \( \alpha \) of \( f \) satisfies \( |\alpha| < B \), and that \( |f|^2_h \geq 2rB^2 \). Then, we obtain

\[
  ||[f, f_h^1]|_h| \geq C_0 \frac{|f_1|^2_h}{2}.
\]

Here, \( C_0 \) is a constant as in Lemma 3.5.

**Proof** Because \( |f_0|^2 \leq rB^2 \), we obtain \( |f_1|^2_h \geq rB^2 \geq |f_0|^2_h \). It implies that \( |f_1|^2_h \geq \frac{1}{2}|f|^2_h \). Thus, we obtain the claim of the lemma.
3.2.2 Lower estimate of the commutator with the adjoint in the cyclic case

Let \( e_1, \ldots, e_r \) be an orthogonal base of \((V, h)\), i.e., \( h(e_i, e_j) = 0 \) \((i \neq j)\). Let \( \alpha \) be a nonzero complex number. Let \( f \) be the endomorphism of \( V \) determined by \( f(e_i) = e_{i+1} \) \((i = 1, \ldots, r-1)\) and \( f(e_r) = \alpha^r e_1 \).

**Proposition 3.7** For any \( 0 < \epsilon < 1 \), there exists \( \delta > 0 \), depending only on \( \epsilon \) and \( r \), such that the following holds:

- If there exists \( 1 \leq i \leq r-1 \) such that \( |e_{i+1}|_h \cdot |e_i|_h^{-1} \leq \epsilon |\alpha| \), then we obtain \( ||f, f^\dagger|| \geq \delta |f|_h^2 \).

**Proof** We begin with a preliminary. We set \( Z := \{ a = (a_1, \ldots, a_r) \in \mathbb{R}_{>0}^r | \prod_{i=1}^r a_i = 1 \} \). We define the function \( F : Z \rightarrow \mathbb{R}_{>0} \) by

\[
F(a) = \sum_{i=1}^{r-1} (a_{i+1} - a_i)^2 + (a_1 - a_r)^2.
\]

We also define the function \( G : Z \rightarrow \mathbb{R}_{>0} \) by

\[
G(a) = \sum_{i=1}^r a_i.
\]

**Lemma 3.8** For any \( 0 < \epsilon < 1 \), there exists \( \delta > 0 \) such that the following holds for \( a \in Z \):

- If there exists \( 1 \leq i \leq r \) such that \( a_i < \epsilon \), then \( F(a) > \delta G(a)^2 \).

**Proof** To simplify the notation, we set \( a_0 := a_r \) and \( a_{r+1} := a_1 \). We set \( c_{\text{max}} := \max \{ a_i \} \) and \( c_{\text{min}} := \min \{ a_i \} \). We obtain \( G(a)^2 \leq r^2 c_{\text{max}}^2 \). There exists \( i \) such that \( |a_{i+1} - a_i| > r^{-1}(c_{\text{max}} - c_{\text{min}}) \). Hence, we obtain

\[
F(a) \geq r^{-2}(c_{\text{max}} - c_{\text{min}})^2 = r^{-2}c_{\text{max}}^2(1 - c_{\text{min}}^{-1})^2.
\]

Because \( \prod_{i=1}^r a_i = 1 \), we obtain \( 1 \leq c_{\text{min}}c_{\text{max}}^{-1} \), i.e.,

\[
0 < c_{\text{max}}^{-1} \leq c_{\text{min}}^{-1} \leq \epsilon^{r^2}. \]

We obtain \( c_{\text{max}}^{-1}c_{\text{min}} \leq \epsilon^{r^2} \). Then, the claim of the lemma is clear.

Let us return to the proof of Proposition 3.7. Note that

\[
|f|_h^2 = \sum_{i=1}^{r-1} \frac{|e_{i+1}|_h^2}{|e_i|_h^2} e_i + \frac{|\alpha|^{2r}}{|e_r|_h^2} |e_1|_h^2.
\]

By a direct calculation, we obtain

\[
f^\dagger_h(e_{i+1}) = \frac{|e_{i+1}|_h^2}{|e_i|_h^2} e_i \quad (i = 1, \ldots, r-1), \quad f^\dagger_h(e_1) = \frac{\pi r}{|e_r|_h^2} |e_1|_h^2 e_r.
\]

Hence, we obtain

\[
(f \circ f^\dagger_h - f^\dagger_h \circ f)(e_1) = \left( \frac{|e_1|_h^2}{|e_r|_h^2} |\alpha|^{2r} - \frac{|e_2|_h^2}{|e_1|_h^2} \right) e_1,
\]

\[
(f \circ f_h^\dagger - f \circ f^\dagger_h)(e_i) = \left( \frac{|e_i|_h^2}{|e_{i-1}|_h^2} - \frac{|e_{i+1}|_h^2}{|e_i|_h^2} \right) e_i \quad (i = 2, \ldots, r-1),
\]

\[
(f \circ f^\dagger_h - f \circ f^\dagger_h)(e_r) = \left( \frac{|e_r|_h^2}{|e_{r-1}|_h^2} - \frac{|e_1|_h^2}{|e_r|_h^2} |\alpha|^{2r} \right) e_r.
\]

It implies

\[
||f, f^\dagger_h||_h^2 = \left( \frac{|e_1|_h^2}{|e_r|_h^2} |\alpha|^{2r} - \frac{|e_2|_h^2}{|e_1|_h^2} \right)^2 + \sum_{i=2}^{r-1} \left( \frac{|e_i|_h^2}{|e_{i-1}|_h^2} - \frac{|e_{i+1}|_h^2}{|e_i|_h^2} \right)^2 + \left( \frac{|e_r|_h^2}{|e_{r-1}|_h^2} - \frac{|e_1|_h^2}{|e_r|_h^2} |\alpha|^{2r} \right)^2
\]

Then, we obtain the claim of Proposition 3.7 from Lemma 3.8.
3.2.3 Commutator with the difference of metrics

Let \( h' \) be another Hermitian metric of \( V \) such that \( e_1, \ldots, e_r \) is orthogonal with respect to \( h' \). Assume that \( \prod |e_i|_h = \prod |e_i|_{h'} = 1 \). We obtain the automorphism \( s \) determined by \( h'(x, y) = h(sx, y) \) for \( x, y \in V \), which satisfies \( \det(s) = 1 \). Let \( B > 0 \) and \( C > 0 \). Assume the following conditions on \( h \) and \( h' \):

\[
\begin{align*}
|e_i|_h \cdot |e_{i+1}|^{-1}_h & \leq B \text{ for } i = 1, \ldots, r - 1 \\
|s|_h & \leq C.
\end{align*}
\]

**Lemma 3.9** There exist \( C_1 > 0 \) and \( \epsilon_1 > 0 \), depending only on \( B \) and \( C \), such that the following holds:

- If there exists \( 0 < \epsilon < \epsilon_1 \) such that \( ||f, s||^{s^{-1/2}}_h \leq \epsilon \), then \( |s - \text{id}_V|_h \leq C_1 \cdot \epsilon \) holds.

**Proof** There exist \( a_i \in \mathbb{R}_{>0} \) (\( i = 1, \ldots, r \)) such that \( s(e_i) = a_ie_i \) and that \( \prod a_i = 1 \). Note that \( a_i < C \) and \( a_i^{-1} < C^{r-1} \). For \( i = 1, \ldots, r - 1 \), we obtain

\[
[f, s]s^{-1/2}e_i = a_i^{1/2}(1 - a_i^{-1}a_{i+1})e_{i+1}.
\]

Hence, we obtain the following for any \( i = 1, \ldots, r - 1 \):

\[
|1 - a_i^{-1}a_{i+1}| \leq (|e_i|_h \cdot |e_{i+1}|^{-1}_h)a_i^{-1/2}||f, s(h)||s(h)^{-1/2}||_h \leq BC^{(r-1)/2}\epsilon.
\]

There exists \( C_2 > 0 \) such that if \( BC^{(r-1)/2}\epsilon < 1/2 \) then the following holds:

\[
-\log(a_i) + \log(a_{i+1}) = \log(a_i^{-1}a_{i+1}) \leq C_2|1 - a_i^{-1}a_{i+1}| \leq BC^{(r-1)/2}C_2\epsilon.
\]

We obtain \( |\log(a_i) - \log(a_1)| \leq iBC^{(r-1)/2}C_2\epsilon \). Because \( \prod a_i = 1 \), we obtain

\[
r|\log a_1| \leq \frac{r(r-1)}{2}BC^{(r-1)/2}C_2\epsilon.
\]

We obtain \( |\log(a_i)| \leq \frac{r+1}{2}BC^{(r-1)/2}C_2\epsilon \) for \( i = 1, \ldots, r \). It implies the claim of the lemma.

3.3 Boundedness of Higgs field with bounded spectral curve

Let \((E, \overline{\theta}, \theta, h)\) be a harmonic bundle on a Riemann surface \( X \). Let \( g \) be a Kähler metric on \( X \). Let \( Z(\theta) \) denote the zero set of \( \theta \). Recall that there is the following standard inequality on \( X \setminus Z(\theta) \) (see [33, Page 729]):

\[
\Delta_g \log |\theta|^2_{h,g} \geq \|\theta, \theta^\bot \|^2_{h,g} + \frac{1}{2}k_g.
\]  

(5)

Here, \( \Delta_g = \frac{1}{2}\sqrt{-\text{I}_g \partial \overline{\partial}} \), and \( k_g = \sqrt{-\text{I}_g R(g)} \).

**Condition 3.10** There exists \( C_0 > 0 \) such that any eigenvalue \( \alpha \) of \( \theta \) at \( P \in X \) satisfies

\[
|\alpha|_g \leq C_0.
\]

More precisely, for any holomorphic coordinate \( z \) around \( P \), we express \( \theta \) as \( \theta = f dz \) around \( P \), and then any eigenvalue \( \alpha_1 \) of \( f \) at \( P \) satisfies \( |\alpha_1| \cdot |dz|_g \leq C_0 \).

**Lemma 3.11** If Condition 3.10 is satisfied, there exist \( C_1 > 0 \) (\( i = 1, 2 \), depending only on \( C_0 \) and \( \text{rank}(E) \)), such that the following inequality holds on \( X \setminus Z(\theta) \):

\[
\Delta_g \log |\theta|^2_{h,g} \geq C_1|\theta|^2_{h,g} - C_2 + \frac{1}{2}k_g.
\]

**Proof** By Lemma 3.6 Condition 3.10 implies that there exists \( C_3 > 0 \) depending only on \( \text{rank}(E) \), such that the following holds.
• If $|\theta|_{h,g}^2 \geq 2(\text{rank} E)C_0^2$ at $P \in X$, then $||[\theta, \theta]|_{h,g}^1| \geq C_3|\theta|_{h,g}^2$ at $P$.

Therefore, we obtain the following on $X \setminus Z(\theta)$:

$$||[\theta, \theta]|_{h,g}^1|^2 \geq C_3^2|\theta|_{h,g}^2((|\theta|_{h,g}^2 - 2C_0^2(\text{rank} E))).$$

Then, we obtain the claim of the lemma from (5).

Let $X_1 \subset X$ be a relatively compact open subset whose boundary is smooth and compact. Let $\overline{X}_1$ denote the closure of $X_1$ in $X$.

**Proposition 3.12** In addition to Condition 3.11, we assume that $g_{|\overline{X}_1}$ is complete and that the Gaussian curvature of $g_{|\overline{X}_1}$ is bounded from below. Then, $|\theta|_{h,g}$ is bounded on $\overline{X}_1$. More precisely, there exists $C_{10} > 0$, depending only on inf $\overline{X}_1^\prime k_g, C_0$ and $\text{rank}(E)$ such that $|\theta|_{h,g} \leq \max\{C_{10}, \max_{\overline{X}_1^\prime} |\theta|_{h,g}\}$ on $X_1$.

**Proof** By the assumptions and Lemma 3.11 there exists $C_{11} > 0$, depending only on inf $\overline{X}_1^\prime k_g, C_0$ and $\text{rank}(E)$ such that the following holds on $X_1 \setminus Z(\theta)$:

$$\Delta_g \log |\theta|_{h,g}^2 \geq C_1|\theta|_{h,g}^2 - C_{11}.$$

It implies the following inequality on $X_1 \setminus Z(\theta)$:

$$\Delta_g |\theta|_{h,g}^2 \geq |\theta|_{h,g}^2(C_1|\theta|_{h,g}^2 - C_{11}). \quad (6)$$

Because the both sides are continuous on $X_1$, the inequality (6) holds on $X_1$. By Lemma 3.11 $|\theta|_{h,g}$ is bounded on $X_1$. (We set $u = |\theta|_{h,g}^2$, $f(t) = C_1t^2 - C_{11}t$ and $g(t) = C_1t^2$ in Lemma 3.11.) If sup$_{X_1} |\theta|_{h,g} > \max_{\overline{X}_1^\prime} |\theta|_{h,g}$, then we obtain sup $|\theta|_{h,g}^2 \leq C_{11}^{-1}C_{11}$, where the right hand side depends only on inf $\overline{X}_1^\prime k_g, C_0$ and $\text{rank}(E)$.

We shall use the following lemma to prove that there is no solution for the Toda equation on a parabolic Riemann surface with the trivial $r$-differential.

**Lemma 3.13** Suppose that $X$ is parabolic, i.e., $X \subset \mathbb{C}, \mathbb{C}^*$ or an elliptic curve. If $\theta$ is nilpotent, then $\theta = 0$.

**Proof** There exists a complete Kähler metric $g$ of $X$ such that $k_g = 0$. Because $\theta$ is nilpotent, Lemma 3.6 implies that there exists $C > 0$ such that the following holds on $Z(\theta)$:

$$\Delta_g \log |\theta|_{h,g}^2 \geq C|\theta|_{h,g}^2.$$

As in the proof of Proposition 3.12 we obtain

$$\Delta_g |\theta|_{h,g}^2 \geq C|\theta|_{h,g}^4 \quad (7)$$

on $X \setminus Z(\theta)$. Because the both side of (7) are continuous on $X$, the inequality (7) holds on $X$. It implies that $|\theta|_{h,g}^2$ is subharmonic on $X$. By Proposition 3.12 $|\theta|_{h,g}^2$ is bounded. Because $X$ is $\mathbb{C}, \mathbb{C}^*$, or an elliptic curve, we obtain that $|\theta|_{h,g}^2$ is constant. By (7), we obtain $\theta = 0$.

### 3.4 Toda equation

Let $X$ be a Riemann surface. Let $q$ be an $r$-differential on $X$. Note that $\theta(q)$ is nowhere vanishing.

#### 3.4.1 The case where the $r$-differential is bounded

Let $g$ be a complete Kähler metric of $X$ such that $|q|_g$ is bounded.
Proposition 3.14 Assume that the Gaussian curvature of $g$ is bounded from below. Then, there exists $C_0 > 0$ depending only on $\inf k_g$, $\sup |q|_g$ and $r$ such that the following holds for any $h \in \text{Harm}(q)$:

$$|\theta(q)|_{h,g} \leq C_0.$$ 

As a result, there exists $C_1 > 0$ depending only on $\inf k_g$, $\sup |q|_g$ and $r$ such that the following holds for any $u \in \text{Toda}(q,g)$:

$$u_{i+1} - u_i < C_1 \quad (i = 1, \ldots, r - 1), \quad u_1 - u_r + \log |q|_g^2 < C_1.$$ 

**Proof** By the boundedness of $|q|_g$, there exists $C_2 > 0$ such that any eigenvalue $\alpha$ of $\theta(q)$ at $P \in X$ satisfies $|\alpha|_g \leq C_2$. We obtain the first claim by Proposition 3.12.

Recall that $\theta(q)$ is obtained as $\theta(q) = \sum_{i=1}^r \theta(q)_i$, where $\theta(q)_i : K_X^{(r+1-2i)/2} \to K_X^{(r+1-2(i+1))/2} \otimes K_X (i = 1, \ldots, r - 1)$ are induced by the identity, and $\theta(q)_r : K_X^{(r+1)/2} \to K_X^{(r-1)/2} \otimes K_X$ is induced by the multiplication of $q$. Because $K_{X,r} = \bigoplus K_X^{(r+1-2i)/2}$ is orthogonal with respect to $h \in \text{Harm}(q)$, we obtain

$$|\theta(q)|_{h,g}^2 = \sum_{i=1}^r |\theta(q)_i|^2_{h,g}.$$ 

Let $u \in \text{Toda}(q,g)$ denote the solution corresponding to $h$, i.e., $h|_{K_X^{(r+1-2i)/2}} = e^{u_i} \cdot (g^{-1} \otimes (r+1-2i)/2)$. Note that

$$|\theta(q)_i|_{h,g}^2 = e^{u_{i+1} - u_i} \quad (i = 1, \ldots, r - 1), \quad |\theta(q)_r|_{h,g}^2 = e^{u_1 - u_r} |q|_g^2.$$ 

Then, we obtain the second claim of the proposition from the first claim.

Let $h \in \text{Harm}(q)$. For $i = 1, \ldots, r - 1$, there exists the natural isomorphism:

$$\left( K_X^{(r+1-2i)/2} \right)^{-1} \otimes K_X^{(r+1-2(i+1))/2} \simeq K_X^{-1}.$$ 

Let $g(h)_i$ be the Kähler metric of $X$ obtained as $h^{-1}|_{K_X^{(r+1-2i)/2}} \otimes h|_{K_X^{(r+1-2(i+1))/2}}$. In terms of $w \in \text{Toda}(q,g)$, the metrics $g(h)_i = e^{-w_i+w_{i+1}} \cdot g(i = 1, \ldots, r - 1)$. We will show that the curvature of the metrics $g(h)_i(i = 1, \ldots, r - 1)$ are bounded from below. This fact will be useful in applying the two maximum principles.

**Lemma 3.15** For any solution $w \in \text{Toda}(q,g)$, the metrics $e^{-w_1+w_2} \cdot g, \ldots, e^{-w_{r-1}+w_r} \cdot g$ satisfy that their Gaussian curvature are bounded from below by $-4$.

**Proof** Recall the curvature formula for a Kähler metric $g = g_0 dz \otimes d\bar{z}$ is $k_g = -\frac{2}{g_0} \Delta \log g_0$, where $\Delta = \partial_z \overline{\partial}_z$. So for $r \geq 3$,

$$k_{e^{-w_1+w_2} \cdot g} = -\frac{2}{e^{-w_1+w_2} \cdot g_0} \Delta[(w_1 + w_2) + \log g_0] = -\frac{2}{e^{-w_1+w_2} \cdot g_0} \Delta g(-w_1 + w_2) - \frac{1}{2} k_g$$

$$= 2 e^{-w_1+w_2} |q|_g^2 - 2 e^{-w_1+w_2} + e^{-w_2+w_3} > -4.$$ 

$$k_{e^{-w_k+w_{k+1}} \cdot g} = -\frac{2}{e^{-w_k+w_{k+1}} \cdot g_0} \Delta(-w_k + w_{k+1} + \log g_0)$$

$$= 2 e^{-w_k+w_{k+1}} - 2 e^{-w_k+w_{k+1}} + e^{-w_k+w_{k+1}+w_{k+2}} > -4, \quad 2 \leq k \leq r - 2.$$ 

$$k_{e^{-w_{r-1}+w_r} \cdot g} = -\frac{2}{e^{-w_{r-1}+w_r} \cdot g_0} \Delta(-w_{r-1} + w_r + \log g_0)$$

$$= 2 e^{-w_{r-2}+w_{r-1}} - 2 e^{-w_{r-2}+w_{r-1}} + e^{-w_r+w_{r+1}} |q|_g^2 > -4.$$
For $r = 2$,
\[
    k_{e^{-2w}} = -\frac{2}{e^{-2w} \cdot g_0} \Delta (-2w) + \log g_0 = -\frac{2}{e^{-2w}} \Delta g (-2w) - \frac{1}{2} k_g = 4 \frac{e^{2w_1}|g|_g^2 - e^{-2w}}{e^{-2w}} \geq -4.
\]

**Corollary 3.16** Assume that there exist a solution $w \in \text{Toda}(q,g)$ and $1 \leq k \leq r - 1$ such that $|w_{k+1} - w_k|$ is bounded. Then, there exists $C > 0$ such that the following holds for any $u \in \text{Toda}(q,g)$:

\[
    u_{i+1} - u_i < C \quad (i = 1, \ldots, r-1), \quad u_1 - u_r + \log |q|^2 < C.
\]

**Proof** The Kähler metric $\tilde{g} = e^{w_{k+1} - w_k} g$ is complete since $|w_{k+1} - w_k|$ is bounded. By Lemma 3.15, the Gaussian curvature of $\tilde{g}$ is bounded from below by a constant. And $|\tilde{g}|$ is still bounded since $|w_{k+1} - w_k|$ is bounded. Hence, we obtain the claim of the corollary from Proposition 3.14.

Let $g^{(j)} (j = 1, 2)$ be complete Kähler metrics of $X$ such that $|q|_{g^{(j)}}$ are bounded.

**Proposition 3.17** Suppose that there exist solutions $w^{(j)} \in \text{Toda}(q,g^{(j)})$ and $1 \leq k(j) \leq r - 1$ such that $|w^{(j)}_{k(j)+1} - w^{(j)}_{k(j)}|$ are bounded. Then, $g^{(1)}$ and $g^{(2)}$ are mutually bounded.

**Proof** We obtain an $\mathbb{R}_{> 0}$-valued function $A$ determined by $g^{(2)} = A g^{(1)}$. We set

\[
    \tilde{w}^{(2)}_i := w^{(2)}_i - \frac{r + 1 - 2i}{2} \log A.
\]

Then, $\tilde{w}^{(2)} \in \text{Toda}(q,g^{(1)})$. By Corollary 3.16, there exists $C_1 > 0$ such that $\tilde{w}^{(2)}_{i+1} - \tilde{w}^{(2)}_i < C_1$ for $i = 1, \ldots, r-1$. In particular, we obtain

\[
    w^{(2)}_{k(2)+1} - w^{(2)}_{k(2)} + \log A = \tilde{w}^{(2)}_{k(2)+1} - \tilde{w}^{(2)}_{k(2)} < C_1.
\]

Because $|w^{(2)}_{k(2)+1} - w^{(2)}_{k(2)}|$ is bounded, there exists $C_2 > 0$ such that $\log A < C_2$. By exchanging the roles of $(g^{(1)}, w^{(1)})$ and $(g^{(2)}, w^{(2)})$, we obtain that there exists $C_3 > 0$ such that $-\log A < C_3$. Therefore, we obtain that $g^{(1)}$ and $g^{(2)}$ are mutually bounded.

**Corollary 3.18** Suppose that there exist bounded solutions $w^{(j)} \in \text{Toda}(q,g^{(j)})$. Then, $g^{(1)}$ and $g^{(2)}$ are mutually bounded.

### 3.4.2 The case where an associated Kähler metric is complete

**Lemma 3.19** There exists $C_j (j = 1, 2)$, depending only on $r$, such that the following holds on $X$ for any $i = 1, \ldots, r-1$:

\[
    \Delta g^{(h)}, \log |\theta(q)|_{h,g^{(h)}}, \geq C_j |\theta(q)|_{h,g^{(h)}}, -C_2.
\]

**Proof** Let $P$ be any point of $X$. We fix $i$. Take a holomorphic coordinate $z$ around $P$ such that $|dz|^{2}_{g^{(h)}} = 2$ at $P$. We set $e_k = (dz)^{(r+1-2k)/2} (k = 1, \ldots, r)$. We express $q$ and $\theta(q)$ as $q = \beta(dz)^r$ and $\theta(q) = f dz$ around $P$. We obtain $f(e_k) = e_{k+1}$ for $k = 1, \ldots, r-1$ and $f(e_r) = \beta e_1$. By the construction, we obtain the following at $P$:

\[
    \frac{|e_{i+1}|^2}{|e_i|^2_{g^{(h)}}} = |dz|^{2}_{g^{(h)}} = \frac{1}{2}.
\]

By Proposition 3.24 there exist $C_{10} > 0$ and $C_{11} > 0$, depending only on $r$, such that the following holds.

- If $|\beta|^{1/r} \geq C_{10}$ at $P$, then $||f_i||^2_{h} \geq C_{11}|f|^2_{h}$ at $P$.

By Lemma 3.26 there exists $C_{12} > 0$, depending only on $r$, such that the following holds.
• If \(|\beta|^{1/r} < C_1\) and \(|f|^2_h > 2rC^2_{10}\) at \(P\), then we obtain \(\|f, f^T_h\|_{h} \geq C_{12}|f|^2_h\) at \(P\).

By our choice of \(C_{12}\) the following holds.

• If \(|\beta|^{1/r} < C_1\) at \(P\), then we obtain the following at \(P\):

\[
\|f, f^T_h\|_{h}^2 \geq C_{12}^2|f|^2_h (|f|^2_h - 2rC^2_{10}).
\]

Recall that \(k_{g(h)} \geq -4\). Then, we obtain the claim of Lemma \[\text{3.19}\].

**Proposition 3.20** If there exists \(1 \leq i \leq r - 1\) such that \(g(h)_i\) is complete, then \(|\theta(q)|_{h,g(h)_i}\) is bounded. As a result, the functions \(g(h)_j/g(h)_i\) \((j = 1,\ldots, r - 1)\) and \(|q|_{g(h)_i}\) are bounded.

**Proof** We obtain the boundedness of \(|\theta(q)|_{h,g(h)_i}\) by Cheng-Yau maximum principle and Lemma \[\text{3.19}\]. Note that \(|\theta(q)|_{h,g(h)_i}^2 = g(h)_j/g(h)_i\). (See the proof of Proposition \[\text{3.14}\] for \(\theta(q)_j\).) Because

\[
\sum_{j=1}^r |\theta(q)_j|_{h,g(h)_i}^2 = |\theta(q)|_{h,g(h)_i}^2,
\]

for each \(j = 1,\cdots, r\), \(|\theta(q)_j|^2_{h,g(h)_i}\) is bounded. For \(j = 1,\cdots, r - 1\), we obtain the boundedness of \(g(h)_j/g(h)_i\).

Let \(g\) be any Kähler metric of \(X\). Let \(u \in \text{Toda}(q,g)\) be the solution corresponding to \(h\). Also we have

\[
e^{u_1-u_2}|q|^2_{g}e^{-u_3+u_4}\cdots e^{-u_{r-2}+u_{r-1}} \leq e^{-u_{r-1}+u_r}C.
\]

Thus, \(|q|^2_{g} \leq (e^{-u_{r-1}+u_r})^r \cdot C^r\) and \(|q|_{g(h)_i}\) is bounded.

**Corollary 3.21** If \(g(h)_i\) \((i = 1,\ldots, r - 1)\) are complete, then \(g(h)_i\) are mutually bounded, and \(|q|_{g(h)_i}\) are bounded. In other words, there exists a complete Kähler metric \(g\) such that \(g\) and \(g(h)_i\) are mutually bounded, and that \(|q|_{g}\) is bounded.

**Proof** By Proposition \[\text{3.20}\] \(g(h)_i\) are mutually bounded. Then, the claims of the corollary are clear.

**Corollary 3.22** Let \(g\) be a complete Kähler metric of \(X\). If there exists a bounded solution \(w \in \text{Toda}(q,g)\), then \(|q|_{g}\) is bounded.

### 3.4.3 Uniqueness of complete solutions

**Theorem 3.23** Suppose that the following holds for \(h^{(a)}\) in \(\text{Harm}(q)\) \((a = 1, 2)\).

- \(g(h^{(a)}_i)\) \((i = 1,\ldots, r - 1)\) are complete.

Then, we obtain \(h^{(1)} = h^{(2)}\).

**Proof** By Corollary \[\text{3.21}\] there exist complete Kähler metrics \(g^{(a)}\) \((a = 1, 2)\) such that (i) \(g^{(a)}\) and \(g(h^{(a)}_i)\) \((i = 1,\ldots, r - 1)\) are mutually bounded, (ii) \(|q|_{g^{(a)}}\) are bounded. By Corollary \[\text{3.18}\] we obtain that \(g^{(1)}\) and \(g^{(2)}\) are mutually bounded. We may assume \(g^{(1)} = g^{(2)} =: g\). Moreover, we may assume that the Gaussian curvature of \(g\) is bounded from below. Note that \(|q|_{g}\) is bounded.

Let \(w^{(a)} \in \text{Toda}(q,g)\) be the solutions corresponding to \(h^{(a)}\). Because \(g(h^{(a)}_i)\) \((i = 1,\ldots, r - 1)\) are mutually bounded with \(g\), we obtain that \(|w_{i+1}^{(a)} - w_{i}^{(a)}|\) \((i = 1,\ldots, r - 1)\) are bounded. Because \(\sum w_{i}^{(a)} = 0\), we obtain that \(w^{(a)}\) are bounded. It implies that \(h^{(1)}\) and \(h^{(2)}\) are mutually bounded.

We apply the Omori-Yau maximum principle to prove \(h^{(1)} = h^{(2)}\) from the mutually boundedness of \(h^{(1)}\) and \(h^{(2)}\) as follows.

Let \(s\) be the automorphism of \(\mathbb{K}_X, r\) determined by \(h^{(2)} = h^{(1)} \cdot s\). By \[\text{3.22}\], Lemma 3.1], there exists a constant \(C > 0\) such that following inequality holds on \(X\):

\[
\Delta_g Tr(s) = C|\theta(q), s|s^{-1/2}h^{(1)} \cdot g + C|\bar{T}_{\theta q, s}^s(s)|s^{-1/2}h^{(1)} \cdot g \geq C|\theta(q), s|s^{-1/2}h^{(1)} \cdot g.
\]
By Omori-Yau maximum principle, there exists a sequence $Q_{\ell} \in X \ (\ell = 1, 2, \ldots)$ such that
\[
\triangle_g \text{Tr}(s)(Q_{\ell}) \leq \ell^{-1}, \quad \text{Tr}(s)(Q_{\ell}) > \sup_{Q \in X} \text{Tr}(s)(Q) - \ell^{-1}.
\]

Hence, there exists $C_1 > 0$ such that the following holds for any $\ell$:
\[
\left|\langle s, \theta(q) \rangle_{s^{-1/2}}h_{(i),g}(Q_{\ell})\right| \leq C_1 \ell^{-1}.
\]

Let $z_{\ell}$ be a holomorphic coordinate around $Q_{\ell}$ such that $|dz_{\ell}|_{h_{(i)}}^2 = 2$. Because $g(h^{(i)})$ and $g$ are mutually bounded, there exists $B > 0$ such that $|dz_{\ell}|(r+1-2i)/2|_{h_{(i)}} \cdot |dz_{\ell}|(r+1-2(i+1))/2|_{h_{(i)}}^{-1} \leq B$ for any $\ell$. By Lemma 3.9 there exists $C_2 > 0$ and $\ell_2$ such that $\text{Tr}(s)(Q_{\ell}) \leq r(1 + C_2 \ell^{-1/2})$ for any $\ell > \ell_2$. Hence, we obtain $\sup_{Q \in X} \text{Tr}(s)(Q) \leq r$. Because $\det(s) = 1$, we obtain $\text{Tr}(s) \geq r$ for any $Q$. Therefore, $\text{Tr}(s)$ is constantly $r$, and we obtain $s = \text{id}$.

**Corollary 3.24** Suppose that $h \in \text{Harm}(q)$ satisfies the following condition.

- $g(h)_i (i = 1, \ldots, r - 1)$ are complete.

Then, $h$ is real, i.e., $h_{K_X^{r+1-2i}/2}$ and $h_{K_X^{-(r+2-2i)/2}}$ are mutually dual.

**Proof** We obtain $h^\lor \in \text{Harm}(q)$ as the dual of $h$ by using the natural identification of $K_{X,r}$ with its dual. Because $h^\lor \in \text{Harm}(q)$ also satisfies the same condition, we obtain $h = h^\lor$ by Theorem 3.23, which implies the claim of the corollary.

We can restate Theorem 3.23 and Corollary 3.24 in terms of solutions of the Toda equation.

**Corollary 3.25** Let $g$ be a Kähler metric of $X$.

- If $w^{(a)} \in \text{Toda}(q, g) \ (a = 1, 2)$ are complete, then we obtain $w^{(1)} = w^{(2)}$.
- If there exists a complete solution $w \in \text{Toda}(q, g)$, it is real, i.e., $w_i + w_{r+1-i} = 0$.

We also obtain the following result on bounded solutions.

**Corollary 3.26** Let $g$ be a complete Kähler metric of $X$.

- If $w^{(a)} \in \text{Toda}(q, g) \ (a = 1, 2)$ are bounded, then we obtain $w^{(1)} = w^{(2)}$.
- If there exists a bounded solution $w \in \text{Toda}(q, g)$, it is real, i.e., $w_i + w_{r+1-i} = 0$.

### 3.4.4 Estimates on an open subset with smooth compact boundary

Let $X_1 \subset X$ be an open subset whose boundary is smooth and compact. Let $\overline{X}_1$ denote the closure of $X_1$ in $X$.

**Proposition 3.27** Suppose that $h \in \text{Harm}(q)$ satisfies the following condition.

- $g(h)_i|_{\overline{X}_1} (i = 1, \ldots, r - 1)$ are complete.

Then, $g(h)_i|_{\overline{X}_1}$ are mutually bounded, and $|q|_g(h)_i$ are bounded on $\overline{X}_1$.

**Proof** We obtain Proposition 3.27 as a consequence of the following lemma, which we can prove by using the argument in the proof of Proposition 3.20 with Cheng-Yau maximum principle (Lemma 3.4).

**Lemma 3.28** Let $h \in \text{Harm}(q)$. If there exists $1 \leq i \leq r - 1$ such that $g(h)_i|_{\overline{X}_1}$ is complete, then the functions $|\theta(q)|_{h,g(h)_i}, |q|_{g(h)_i}$, and $g(h)_j/g(h)_i \ (j = 1, \ldots, r - 1)$ are bounded on $\overline{X}_1$.

The following proposition is a classification of complete solutions up to boundedness.
Proposition 3.29 Let $h^{(a)} \in \text{Harm}(q)$ ($a = 1, 2$). Suppose that $g(h^{(a)})|_{\overline{X}_1}$ $(i = 1, \ldots, r - 1)$ are complete. Then, $h^{(1)}$ and $h^{(2)}$ are mutually bounded on $X_1$. More precisely, for the automorphism $s$ of $\mathbb{R}_{X_r}$ determined by $h^{(2)} = h^{(1)} \cdot s$, we obtain $\sup_{\overline{X}_1} \text{Tr}(s) = \max_{\partial X_1} \text{Tr}(s)$. In particular, if $h^{(1)}|_{\partial X_1} = h^{(2)}|_{\partial X_1}$, we obtain $h^{(1)} = h^{(2)}$ on $X_1$.

Proof We obtain the following lemma from Proposition 3.12.

Lemma 3.30 Let $g$ be a Kähler metric of $X$ such that (i) $|q|_g$ is bounded on $\overline{X}_1$, (ii) $g_{|\overline{X}_1}$ is complete whose Gaussian curvature is bounded from below. Then, for any $h \in \text{Harm}(q)$, $|\theta(q)|_{h,g}$ is bounded on $\overline{X}_1$. Equivalently, for any $u \in \text{Toda}(q,g)$, $u_{i+1} - u_i$ $(i = 1, \ldots, r - 1)$ and $u_1 - u_r + \log |q|_g^2$ are bounded from above on $\overline{X}_1$.

As a consequence of Lemma 3.30 we obtain the following lemma.

Lemma 3.31 Let $g$ be a Kähler metric of $X$ such that (i) $|q|_g$ is bounded on $\overline{X}_1$, (ii) $g_{|\overline{X}_1}$ is complete. Suppose that there exist $w \in \text{Toda}(q,g)$ and $1 \leq k \leq r - 1$ such that $|w_{k+1} - w_k|$ is bounded on $\overline{X}_1$. Then, for any $u \in \text{Toda}(q,g)$, $u_{i+1} - u_i$ $(i = 1, \ldots, r - 1)$ and $u_1 - u_r + \log |q|_g^2$ are bounded from above on $\overline{X}_1$.

We obtain the following lemma by using the argument in the proof of Proposition 3.17 with Lemma 3.31.

Lemma 3.32 Let $g^{(j)}$ $(j = 1, 2)$ be Kähler metrics of $X$ such that (i) $|q|_{g^{(j)}}$ are bounded on $\overline{X}_1$, (ii) $g^{(j)}_{|\overline{X}_1}$ are complete. Suppose that there exist solutions $w^{(j)} \in \text{Toda}(q,g^{(j)})$ and $1 \leq k(j) \leq r - 1$ such that $|w^{(j)}_{k(j)+1} - w^{(j)}_{k(j)}|_{\overline{X}_1}$ are bounded. Then, $g^{(1)}_{|\overline{X}_1}$ and $g^{(2)}_{|\overline{X}_1}$ are mutually bounded. In particular, if there exist bounded solutions $w^{(j)} \in \text{Toda}(q,g^{(j)})$, then $g^{(1)}_{|\overline{X}_1}$ and $g^{(2)}_{|\overline{X}_1}$ are mutually bounded.

By Proposition 3.27 and Lemma 3.32 $h^{(1)}$ and $h^{(2)}$ are mutually bounded. If $\sup_{\overline{X}_1} \text{Tr}(s) > \max_{\partial X_1} \text{Tr}(s)$, Omori-Yau maximum principle (Lemma 3.2) implies that there exists a sequence $Q_\ell \subset X_1$ as in the proof of Theorem 3.23 and hence we obtain $\sup_{\overline{X}_1} \text{Tr}(s) = r \leq \max_{\partial X_1} \text{Tr}(s)$, which is a contradiction. Hence, we obtain $\sup_{\overline{X}_1} \text{Tr}(s) = \max_{\partial X_1} \text{Tr}(s)$. Thus, the proof of Proposition 3.29 is completed.

In terms of the solutions of the Toda equation, Proposition 3.27 and Proposition 3.29 are stated as follows.

Corollary 3.33 Let $g$ be a Kähler metric of $X$.

- Suppose that $w \in \text{Toda}(q,g)$ satisfies the following condition.
  
  $$(e^{w_{i+1} - w_i}g)_{|\overline{X}_1} \quad (i = 1, \ldots, r - 1) \text{ are complete.}$$

  Then, $(e^{w_{i+1} - w_i}g)_{|\overline{X}_1}$ are mutually bounded, and $e^{w_1 - 2w_r + w_{r-1}}|q|_g$ is bounded on $\overline{X}_1$.

- Suppose that $w^{(a)} \in \text{Toda}(q,g)$ $(a = 1, 2)$ satisfy the following condition.

  $$-(e^{w^{(a)}_{i+1} - w^{(a)}_i}g)_{|\overline{X}_1} \quad (i = 1, \ldots, r - 1) \text{ are complete.}$$

Then, $|w^{(1)}_i - w^{(2)}_i|$ are bounded on $\overline{X}_1$. More precisely, we obtain

$$\sup_{\overline{X}_1} \sum_{i=1}^r e^{w^{(1)}_i - w^{(2)}_i} = \max_{\partial X_1} \sum_{i=1}^r e^{w^{(1)}_i - w^{(2)}_i}.$$
3.4.5 The case where \(|q|^{2/r}\) is complete outside a relatively compact open subset

Let \(X_1 \subset X\) be an open subset whose boundary is smooth and compact.

**Proposition 3.34** Suppose that \(|q|^{2/r}|_\overline{X}_1\) is complete.

- Let \(g\) be any Kähler metric of \(X\) such that \(g = |q|^{2/r}\) on \(\overline{X}_1\). Then, for any \(u \in \text{Toda}(q,g)\), \(|u_i|\) \((i = 1, \ldots, r)\) are bounded on \(\overline{X}_1\).
- For any \(h \in \text{Harm}(q)\), \(g(h)_i\) \((i = 1, \ldots, r-1)\) are mutually bounded with \(|q|^{2/r}\) on \(\overline{X}_1\).
- Any \(h^{(a)}\) of \(Harm(q)\) \((a = 1, 2)\) are mutually bounded on \(\overline{X}_1\).

**Proof** Let \(g\) be a Kähler metric of \(X\) such that \(g|_{\overline{X}_1} = |q|^{2/r}\). Let \(u \in \text{Toda}(q,g)\). By Lemma 3.30 there exists \(C_1 > 0\) such that \(u_{i+1} - u_i \leq C_1\) \((i = 1, \ldots, r-1)\) and \(u_1 - u_r + \log |q|_g^2 \leq C_1\) on \(\overline{X}_1\). Because \(|\log |q|_g^2|\) is bounded on \(\overline{X}_1\), there exists \(C_2 > 0\) such that \(u_{i+1} - u_i \leq C_2\) \((i = 1, \ldots, r-1)\) and \(u_1 - u_r \leq C_2\) on \(\overline{X}_1\).

For any \(j = 2, \ldots, r\), we obtain \(u_j \leq u_1 + (j-1)C_2\) from \(u_{i+1} - u_i \leq C_2\) \((i = 1, \ldots, j-1)\) on \(\overline{X}_1\). Hence, we obtain \(u_j \leq u_1 + rC_2\) \((j = 2, \ldots, r)\) on \(\overline{X}_1\). By the cyclic symmetry, for any \(i\), we obtain \(u_j \leq u_1 + rC_2\) \((j \neq i)\) on \(\overline{X}_1\). Thus, we obtain \(|u_i - u_j| \leq rC_2\) for any \(1 \leq i, j \leq r\) on \(\overline{X}_1\). Because \(\sum_{i=1}^{r} u_i = 0\), there exists \(C_3 > 0\) such that \(|u_i| \leq C_3\) \((i = 1, \ldots, r)\) on \(\overline{X}_1\). It implies the claims of Proposition 3.34.

**Corollary 3.35** Suppose that there exists a relatively compact open subset \(N\) such that \(|q|^{2/r}\) is complete on \(X \setminus N\). Then, \(\text{Harm}(q)\) consists of at most a unique complete solution.

**Proof** It follows from Theorem 3.23 and Proposition 3.34.

**Corollary 3.36** Suppose that there exists a relatively compact open subset \(N\) such that \(|q|^{2/r}\) is complete on \(X \setminus N\). For any Kähler metric \(g\) of \(X\), \(\text{Toda}(q,g)\) consists of at most a unique complete solution.

3.4.6 Pull back

Let \(F : X_1 \rightarrow X_2\) be a holomorphic map of Riemann surfaces which is locally an isomorphism, i.e., the derivative of \(F\) is nowhere vanishing. Let \(q\) be an \(r\)-differential on \(X_2\).

Because \(F\) is locally an isomorphism, there exists a natural isomorphism \(F^* (\mathbb{K}_{X_1,r}, \theta(q)) \simeq (\mathbb{K}_{X_1,r}, \theta(F^* q))\). For any \(h \in \text{Harm}(q)\), it is well known and easy to check that the induced metric \(F^* (h)\) of \(\mathbb{K}_{X_1,r}\) is a harmonic metric of \((\mathbb{K}_{X_1,r}, \theta(F^* q))\). Moreover, \(F^* (h)\) is clearly \(G_r\)-invariant and satisfies \(\det(F^* (h)) = 1\), and hence \(F^* (h) \in \text{Harm}(F^* q)\). In this way, we obtain the map

\[
F^* : \text{Harm}(q) \rightarrow \text{Harm}(F^* q).
\]

Let \(g\) be a Kähler metric of \(X_2\). We obtain a Kähler metric \(F^* (g)\) of \(X_1\). For any \(w \in \text{Toda}(q,g)\), we obtain an \(\mathbb{R}^r\)-valued function \(F^* (w)\) on \(X_1\). Because \(F^* h(g, w) = h(F^* g, F^* w)\) as a Hermitian metric on \(\mathbb{K}_{X_1,r}\), we obtain \(F^* (w) \in \text{Toda}(F^* q, F^* g)\). Thus, we obtain \(F^* : \text{Toda}(q, g) \rightarrow \text{Toda}(F^* q, F^* g)\).

If \(X_1 = X_2\) and \(F^* (q) = q\), then \(F\) induces an automorphism on \(\text{Harm}(q)\). If moreover \(F^* g = g\), then \(F\) induces an automorphism of \(\text{Toda}(q, g)\).

**Proposition 3.37** Suppose that \(X_1 = X_2\), \(F^* (q) = q\) and \(F^* (g) = g\). Then, a complete solution \(w \in \text{Toda}(q, g)\) is preserved by \(F\), i.e., \(F^* (w) = w\).

**Proof** It follows from the uniqueness of complete solutions.

3.5 Appendix: Tame harmonic bundles on parabolic Riemann surfaces

In this appendix, after recalling the general theory of tame harmonic bundles due to Simpson [33], we explain the classification of tame harmonic bundles on parabolic Riemann surfaces (elliptic curves, \(\mathbb{C}\), or \(\mathbb{C}^*\)) as an easy consequence of Lemma 3.13. Though it is easy and more or less well known to specialists, the clear statements might be convenient somewhere.
3.5.1 Tame harmonic bundles on punctured Riemann surfaces

Let $X$ be a compact Riemann surface. Let $D$ be a finite subset in $X$. For any $P \in D$, let $(X_P, z_P)$ be a holomorphic coordinate neighbourhood around $P$ such that $z_P(P) = 0$. We set $X_P := X_P \setminus \{P\}$.

Let $(E, \overline{\nabla}_E, \theta, h)$ be a harmonic bundle on $X \setminus D$. We obtain the spectral curve $\Sigma_\theta \subset \mathcal{K}_X(D)$. The harmonic bundle $(E, \overline{\nabla}_E, \theta, h)$ is called tame [33] if the closure of $\Sigma_\theta$ in the logarithmic cotangent bundle $\mathcal{K}_X(D)$ is proper over $X$.

Let $\mathcal{E}^0$ denote the sheaf of holomorphic sections of $(E, \overline{\nabla}_E)$. We recall that $(\mathcal{E}^0, \theta)$ extends to a filtered regular Higgs bundle in [33].

For any $a \in \mathbb{R}$, $\mathcal{E}^0_{|X_P}$ extends to an $\mathcal{O}_{X_P}$-module $\mathcal{P}_a^h \mathcal{E}^0$ as follows. For any open subset $U \subset X_P$ such that $P \in U$, $\mathcal{P}_a^h \mathcal{E}^0(U)$ is the space of holomorphic sections $s$ of $\mathcal{E}^0_{|U \setminus \{P\}}$ satisfying $|s|_h = O(|z_P|^{-a-\epsilon})$ for any $\epsilon > 0$. According to [33], $\mathcal{P}_a^h (\mathcal{E}^0_{|X_P})$ is a locally free $\mathcal{O}_{X_P}$-module, $\theta_{|X_P}$ induces a logarithmic Higgs field of $\mathcal{P}_a^h (\mathcal{E}^0_{|X_P})$, i.e., $\theta_{|X_P} : \mathcal{P}_a^h (\mathcal{E}^0_{|X_P}) \to \mathcal{P}_a^h (\mathcal{E}^0_{|X_P}) \otimes K_{X_P}(P)$. There exist natural monomorphisms $\mathcal{P}_b^h (\mathcal{E}^0_{|X_P}) \to \mathcal{P}_a^h (\mathcal{E}^0_{|X_P})$ for any $b < a$. We define

$$\text{Gr}_a^b (\mathcal{E}^0)_P := \mathcal{P}_a^h (\mathcal{E}^0_{|X_P}) / \bigcup_{b < a} \mathcal{P}_b^h (\mathcal{E}^0_{|X_P}),$$

which we naturally regard as a finite dimensional $\mathbb{C}$-vector space.

For $a \in \mathbb{R}^D$, we obtain a locally free $\mathcal{O}_X$-module $\mathcal{P}_a^h (\mathcal{E}^0)$ from $\mathcal{E}^0$ and $\mathcal{P}_a^h (\mathcal{E}^0_{|X_P}) (P \in D)$. Thus, we obtain a regular filtered Higgs bundle $(\mathcal{P}_a^h \mathcal{E}^0, \theta)$ on $(X, D)$. As proved in [33], we obtain

$$\deg(\mathcal{P}_a^h \mathcal{E}^0) - \sum_{P \in D} \sum_{a_P - 1 < b \leq a} b \dim \text{Gr}_b^a (\mathcal{E}^0)_P = 0.$$

Recall the polystability of $(\mathcal{P}_a^h \mathcal{E}^0, \theta)$. Let $E_1 \subset E$ be a holomorphic subbundle such that $\theta(E_1) \subset E_1 \otimes K_X \setminus D$. Let $\mathcal{E}_1^0 \subset \mathcal{E}^0$ be the sheaf of holomorphic sections of $E_1$. Suppose the following holds.

**Condition 3.38** For any $a \in \mathbb{R}$, $\mathcal{E}_1^0_{|X_P}$ extends to a locally free $\mathcal{O}_{X_P}$-submodule $\mathcal{P}_a^h (\mathcal{E}_1^0_{|X_P}) \subset \mathcal{P}_a^h (\mathcal{E}^0_{|X_P})$ such that $\mathcal{P}_a^h (\mathcal{E}_1^0_{|X_P}) / \mathcal{P}_b^h (\mathcal{E}^0_{|X_P})$ is locally free. We define $\text{Gr}_a^b (\mathcal{E}_1^0)_P$ as in (9).

For any $a \in \mathbb{R}^D$, we obtain a locally free $\mathcal{O}_X$-module $\mathcal{P}_a^h (\mathcal{E}_1^0)$ from $\mathcal{E}_1^0$ and $\mathcal{P}_a^h (\mathcal{E}^0_{|X_P}) (P \in D)$. We define

$$\deg(\mathcal{P}_a^h \mathcal{E}_1^0) := \deg(\mathcal{P}_a^h \mathcal{E}^0) - \sum_{P \in D} \sum_{a_P - 1 < b \leq a} b \dim \text{Gr}_b^a (\mathcal{E}_1^0)_P.$$

Note that the left hand side is well defined in the sense that it is independent of $a$.

**Theorem 3.39** (33) For any Higgs subbundle $E_1 \subset E$ satisfying Condition 3.38, we obtain $\deg(\mathcal{P}_a^h \mathcal{E}_1^0) \leq 0$. If $\deg(\mathcal{P}_a^h \mathcal{E}_1^0) = 0$, then the orthogonal complement $E_2$ of $E_1$ in $E$ is a holomorphic subbundle such that $\theta(E_2) \subset E_2$. As a result, we obtain a decomposition of harmonic bundle $(E, \overline{\nabla}_E, \theta, h) = (E_1, \overline{\nabla}_{E_1}, \theta_1, h_1) \oplus (E_2, \overline{\nabla}_{E_2}, \theta_2, h_2)$, which induces the decomposition of filtered Higgs bundles $(\mathcal{P}_a^h \mathcal{E}^0, \theta) = (\mathcal{P}_a^h \mathcal{E}_1^0, \theta_1) \oplus (\mathcal{P}_a^h \mathcal{E}_2^0, \theta_2)$.

3.5.2 Harmonic bundles on elliptic curves

Let $X$ be an elliptic curve. A harmonic bundle $(E, \overline{\nabla}_E, \theta, h)$ of rank 1 on $X$ is easily described. Indeed, $(E, \overline{\nabla}_E)$ is a holomorphic line bundle, $h$ is a Hermitian metric of $E$ whose Chern connection is flat, and $\theta$ is a holomorphic one form on $X$. The following proposition is well known.

**Proposition 3.40** For any harmonic bundle $(E, \overline{\nabla}_E, \theta, h)$ of rank $r$ on $X$, there exist harmonic bundles of rank one $(E_i, \overline{\nabla}_{E_i}, \theta_{i}, h_i) (i = 1, \ldots, r)$ such that $(E, \overline{\nabla}_E, \theta, h) \simeq \bigoplus_{i=1}^r (E_i, \overline{\nabla}_{E_i}, \theta_{i}, h_{i}).$

**Proof** There exists a nowhere vanishing holomorphic one form $dz$ on $X$. Let $f$ be the endomorphism of $E$ determined by $\theta = fz$. We obtain the characteristic polynomial det$(t\text{id}_E - f) = \sum_{j=0}^r a_j t^j$. Then, $a_j$ are holomorphic functions on $X$, and hence they are constant, which implies that the eigenvalues of $f$ are constant.
Therefore, we obtain a decomposition of Higgs bundles \((E, \overline{\nabla}_E, \theta) = \bigoplus_i (E_i, \overline{\nabla}_E, \theta, a_i)\), where \(\theta_i, \alpha = \alpha dz \text{id}_{E_i}\) are nilpotent. Because \(\deg(E_i, \theta) \leq 0\) and \(\sum_{i} \deg(E_i, \theta) = 0\), we obtain \(\deg(E_i) = 0\). By Theorem 3.39 the orthogonal complement of \(E_i\) is also a Higgs bundle of \(E\), and hence it is equal to \(\bigoplus_{\beta \neq \alpha} \overline{\nabla}_E\). By an easy induction, we obtain that the decomposition \(E = \bigoplus E_i\) is orthogonal with respect to \(h\). Because we obtained a decomposition of harmonic bundles \((E, \overline{\nabla}_E, \theta, h) = \bigoplus_i (E_i, \overline{\nabla}_E, \theta, a_i, h_i)\), we may assume that \(\theta - \alpha dz \text{id}_E\) is nilpotent for some complex number \(\alpha\) from the beginning. Moreover, we may assume that \(\alpha = 0\), i.e., \(\theta\) is nilpotent. By Lemma 3.13 we obtain \(\theta = 0\). It implies that the Chern connection of \((E, \overline{\nabla}_E, h)\) is flat. Because the fundamental group of \(X\) is abelian, it is isomorphic to a direct sum of holomorphic line bundles with a flat Hermitian metric.

### 3.5.3 Tame harmonic bundles on \(\mathbb{C}\)

For any positive integer \(r\), we set \(E(r) := \bigoplus_{i=1}^{r} O_{\mathbb{C}} e_i\). It is equipped with the trivial Higgs field \(0\). There exists a Hermitian metric \(h(r)\) determined by \(h(r)(e_i, e_j) = 1 (i = j)\) or \(h(r)(e_i, e_j) = 0 (i \neq j)\).

**Proposition 3.41** Any tame harmonic bundle of rank \(r\) on \(\mathbb{C}\) is isomorphic to \((E(r), 0, h(r))\).

**Proof** Let \((E, \overline{\nabla}_E, \theta, h)\) be a tame harmonic bundle on \(\mathbb{C}\). We obtain a locally free \(O_{\mathbb{C}^r}\text{-module } P^h_{0,0} E^0\) with the logarithmic Higgs field \(\theta\) as explained in 3.5.1. We may regard \(\theta\) as a section of \(\text{End}(P^h_{0,0} E^0) \otimes K_p(\infty)\). By considering the product \(K_p(\infty)^{\otimes m} \otimes K_p(\infty)^{\otimes \ell} \approx K_p(\infty)^{\otimes (m+\ell)}\) and the product on \(\text{End}(P^h_{0,0} E^0)\), we obtain a section \(\theta^m\) of \(\text{End}(P^h_{0,0} E^0) \otimes K_p(\infty)^{\otimes m}\). Because \(K_p(\infty) \approx O_{\mathbb{C}^r}(−1)\), we obtain \(\text{tr}(\theta^m) = 0\) for any \(m > 0\). It implies that \(\theta\) is nilpotent. By Lemma 3.13 we obtain \(\theta = 0\). It implies that the Chern connection of \((E, \overline{\nabla}_E, h)\) is flat. Because the fundamental group of \(\mathbb{C}\) is trivial, \((E, \overline{\nabla}_E, h)\) is isomorphic to \((E(r), h(r))\).

### 3.5.4 Tame harmonic bundles on \(\mathbb{C}^*\)

Let \((a, \alpha) \in \mathbb{R} \times \mathbb{C}\). On \(O_{\mathbb{C}^*}\), let \(h_{\alpha}\) be the Hermitian metric determined by \(h_{\alpha}(1, 1) = |z|^{−2\alpha}\). It is equipped with a Higgs field \(\theta_{\alpha} = \alpha dz/|z|^2\). Thus, we obtain a harmonic bundle \((a, \alpha) = (O_{\mathbb{C}^*}, \theta_{\alpha}, h_{\alpha})\). It is easy to see that \(L(a, \alpha)\) is isomorphic to \(L(a + n, \alpha)\) for any \(n \in \mathbb{Z}\).

**Proposition 3.42** For any tame harmonic bundle \((E, \overline{\nabla}_E, \theta, h)\) of rank \(r\) on \(\mathbb{C}^*\), there exist \((a_i, \alpha_i) \in \mathbb{R} \times \mathbb{C}\) \((i = 1, \ldots, r)\) such that \((E, \overline{\nabla}_E, \theta, h) \approx \bigoplus_{i=1}^{r} L(a_i, \alpha_i)\).

**Proof** Let \(f\) be the endomorphism of \(E\) determined by \(\theta = f dz/|z|^2\). We obtain the characteristic polynomial \(\det(t \text{id}_E - f) = \sum_{j=0}^{r} a_j(z)t^j\). Because \(f(P^h_{0,0} E^0) \subset P^h_{0,0} E^0\), \(a_j(z)\) are holomorphic at \(z = 0, \infty\), and hence constant. It implies that the eigenvalues of \(f\) are constant. We obtain the decomposition \((E, f) = \bigoplus_i (E_i, f_i)\), where \(f_i - \alpha_i \text{id}_{E_i}\) are nilpotent. It induces the decomposition of the regular filtered Higgs bundles \((P^h_{0,0} E^0, \theta) = \bigoplus_i (P^h_{0,0} E^0_i, \theta_i)\). Because \(\deg(P^h_{0,0} E^0_i) \leq 0\) and \(\sum \deg(P^h_{0,0} E^0_i) = 0\), we obtain \(\deg(P^h E^0_i) = 0\). The orthogonal complement \(E^0_i\) is a Higgs subbundle of \(E\). Hence, we obtain \(E_i = \bigoplus_{\beta \neq \alpha} \overline{\nabla}_E\beta\). By an easy induction, we obtain that the decomposition \(E = \bigoplus E_i\) is orthogonal with respect to \(h\), and we obtain the decomposition of harmonic bundle \((E, \overline{\nabla}_E, \theta, h) = \bigoplus_{i=1}^{r} L(a_i, 0)\) for some \(a_i \in \mathbb{R}\).

### 4 Estimates for a complete solution

In this section, we deduce some quantitative estimates of a complete solution.

In Corollary 3.2.1 we already see that for a complete solution \(h\), the metrics \(g(h)\) are mutually bounded and \(|g| g(h)\) is bounded. We will give precise bounds in Theorem 4.2.1.
According to Corollary 3.25, a complete solution to the system (1) must be real. The system (1) for a solution $w \in Td^g(q,g)$ is reduced to $(w_1, \ldots, w_n)(n = \frac{2}{g})$ satisfying

$$\begin{cases}
\Delta_g w_1 = e^{2w_1} |q|_g^2 - e^{-w_1+w_2} - \frac{r-1}{4}k_g \\
\Delta_g w_k = e^{-w_k-1+w_k} - e^{-w_k+w_{k+1}} - \frac{r+1}{4}k_g, & k = 2, \ldots, n-1 \\
\Delta_g w_n = e^{-w_{n-1}+w_n} - e^{-(2n-2+1)w_n} - \frac{r+1}{4}k_g.
\end{cases} \tag{10}$$

Before proving Theorem 4.3, we first prove the following proposition.

**Proposition 4.1** Suppose that the background metrics $g_1, \ldots, g_m$ are complete and their Gaussian curvature are bounded from below. Suppose $\sigma_1, \ldots, \sigma_m : X \to \mathbb{R}$ are $C^2$ functions satisfying the following system

$$\begin{cases}
\Delta_g \sigma_1 \geq (1+a)e^{\sigma_1} - (a+2) + e^{-\sigma_2} \\
\Delta_g \sigma_k \geq -e^{\sigma_{k-1}+\sigma_k} + 3e^{\sigma_k} - 3 + e^{-\sigma_{k+1}}, & k = 2, \ldots, m-1 \\
\Delta_g \sigma_m \geq -e^{\sigma_{m-1}+\sigma_m} + (2+b)e^{\sigma_m} - (1+c), \quad \text{where } a, b, c \text{ are positive constants.}
\end{cases} \tag{11}$$

Suppose $d_1, \ldots, d_m$ are positive constants satisfying

$$\begin{cases}
0 = (1+a)d_1 - (a+2) + d_2^1 \\
0 = -d_{k-1}d_k + 3d_k - 1 + d_{k+1}^1, & k = 2, \ldots, m-1 \\
0 = -d_{m-1}d_m + (2+b)d_m - (1+c).
\end{cases} \tag{12}$$

Then either $e^{\sigma_k} < d_k$ for $1 \leq k \leq m$ or $e^{\sigma_k} \equiv d_k$ for $1 \leq k \leq m$.

If replacing $\sigma_1$ by a nonzero non-negative $C^2$ function $\eta$ and suppose $\eta, \sigma_2, \ldots, \sigma_m : X \to \mathbb{R}$ are $C^2$ functions satisfying the following system

$$\begin{cases}
\Delta_g \eta \geq (1+a)\eta - (a+2) + e^{-\sigma_2} \\
\Delta_g \sigma_2 \geq -\eta e^{\sigma_2} + 3e^{\sigma_2} - 3 + e^{-\sigma_3} \\
\Delta_g \sigma_k \geq -e^{\sigma_{k-1}+\sigma_k} + 3e^{\sigma_k} - 3 + e^{-\sigma_{k+1}}, & k = 3, \ldots, m-1 \\
\Delta_g \sigma_m \geq -e^{\sigma_{m-1}+\sigma_m} + (2+b)e^{\sigma_m} - (1+c), \quad \text{where } a, b, c \text{ are positive constants.}
\end{cases} \tag{13}$$

then we obtain either $\eta < d_1, e^{\sigma_k} < d_k$ for $2 \leq k \leq m$ or $\eta \equiv d_1, e^{\sigma_k} \equiv d_k$ for $2 \leq k \leq m$.

In case $m = 1$, the equations (11), (12), (13) are respectively

$$\begin{align*}
\Delta_g \sigma &\geq (1+a)e^{\sigma} - (a+2), \\
0 &\geq (1+a)d - (a+2), \\
\Delta_g \eta &\geq (1+a)\eta - (a+2).
\end{align*} \tag{14}$$

**Proof** Let $M_i = \sup_X e^{\sigma_i}$ for $1 \leq i \leq m$. Define the constants $B_i, D_i(1 \leq i \leq m+1)$ as follows:

$$\begin{align*}
B_1 &= a(1 - M_1^{-1}), & B_k &= 2 - M_{k-1}^{-1}(2 \leq k \leq m), & B_{m+1} &= c - BM_m, \\
D_1 &= a(1 - d_1^{-1}), & D_k &= 2 - d_{k-1}^{-1}d_k - d_{k-1}(2 \leq k \leq m), & D_{m+1} &= c - bd_m.
\end{align*} \tag{17}$$

**Claim 4.2** (1) $d_1D_1 = D_2$; $M_1B_1 \leq B_2$ and $M_1 \leq \frac{a+2}{a+1}$.

(2) $d_kD_k = D_{k+1}$; if $M_{k-1} \leq 2$ where $k$ satisfies $2 \leq k \leq m$, then $M_kB_k \leq B_{k+1}$.

(3) $d_k - 1 = \frac{a+1}{a+2}D_{k+1}$; suppose $M_{k-1} \leq 2$ for every $2 \leq k \leq s$, then $M_s - 1 \leq \frac{a+1}{a+2}B_{s+1}$.

(4) $M_{k-1} \leq 2$ for every $2 \leq k \leq m$.

**Proof** (of Claim 4.2) For Part (1) and (2), the proof will use the Cheng-Yau maximum principle. Note that the background metric $g_1, g_2, \ldots, g_m$ are complete and their Gaussian curvature is bounded from below, the condition of the Cheng-Yau maximum principle on the background metric is satisfied. We only need to check whether the equations satisfy the condition.

(1) We apply the Cheng-Yau maximum principle to the equation for $\sigma_1$,

$$\rho_{\sigma_1} \geq (1+a)e^{\sigma_1} - (2+a) + e^{-\sigma_2} \geq (1+a)e^{\sigma_1} - (2+a) + M_2^{-1}. \tag{19}$$
and obtain
\[ 0 \geq (1 + a)M_1 - (2 + a) + M_s^{-1}. \]

Note that it is allowed in the equation that \( M_k = +\infty \) in which case we write \( M_k^{-1} = 0 \). So \( M_1 B_2 \leq B_2 \) and \( M_1 \leq \frac{a + 1}{a} \). Similarly, \( d_1 D_1 = D_3 \).

(2) (i) For some \( 2 \leq k \leq m - 1 \), suppose \( M_{k-1} \leq 2 \). The equation for \( \sigma_k \) is
\[
\triangle g_k \sigma_k \geq -e^{\sigma_k} + 3e^{\sigma_k} - 3 + e^{-\sigma_k+1}
\]
\[ \implies \triangle g_k \sigma_k \geq (3 - M_{k-1})e^{\sigma_k} + M_{k+1}^{-1} - 3, \]
where the coefficient of \( e^{\sigma_k} \) is a positive constant. We apply the Cheng-Yau maximum principle to (21) and obtain
\[ 0 \geq (3 - M_{k-1})M_k + M_{k+1}^{-1} - 3, \]
which is equivalent to \( M_{k} B_{k} \leq B_{k+1} \). Similarly, \( d_k D_k = D_{k+1} \).

(ii) Suppose \( M_{m-1} \leq 2 \), the equation for \( \sigma_m \) is
\[
\triangle g_m \sigma_m \geq -e^{\sigma_m} + (2 + b)e^{\sigma_m} - (1 + c)
\]
\[ \implies \triangle g_m \sigma_m \geq (b + 2 - M_{m-1})e^{\sigma_m} - (1 + c), \]
where the coefficient of \( e^{\sigma_m} \) is \( (2 + b - M_{m-1}) \), a positive constant. We apply the Cheng-Yau maximum principle to (23) and obtain
\[ 0 \geq (b + 2 - M_{m-1})M_m - (1 + c), \]
which is equivalent to \( M_mB_m \leq B_{m+1} \). Similarly, \( d_mD_m = D_{m+1} \).

(3) By the definitions of \( B_i 's \), we have
\[
1 - M^{-1}_s = B_s + (M_{s-1} - 1)
\]
\[ = B_s + M_{s-1}(1 - M^{-1}_{s-1})
\]
\[ = B_s + M_{s-1}B_{s-1} + \cdots + M_{s-1} \cdots M_2B_2 + M_{s-1} \cdots M_2(M_1 - 1)
\]
\[ = B_s + M_{s-1}B_{s-1} + \cdots + M_{s-1} \cdots M_2B_2 + \frac{1}{a}M_{s-1} \cdots M_1B_1. \]

Since \( M_1, M_2, \cdots, M_{s-1} \leq 2 \), by Part (1) and (2), \( M_k B_k \leq B_{k+1} \) for \( 1 \leq k \leq s \) and hence \( M_sM_{s-1} \cdots M_kB_k \leq B_{s+1} \) for \( 1 \leq k \leq s \). So
\[
1 - M^{-1}_s \leq B_s \cdot (s - 1 + \frac{1}{a}) \leq \frac{as + 1 - a}{a}M_{s+1}^{-1}B_{s+1}
\]
and thus \( M_s - 1 \leq \frac{as + 1 - a}{a}B_{s+1} \).

Similarly, we have \( d_s - 1 = \frac{as + 1 - a}{a}D_{s+1} \).

(4) From Part (1) of Claim 4.2, \( M_1 \leq \frac{as + 2}{as + 1} \leq 2 \). Suppose \( s \) be the largest integer such that \( M_{k-1} \leq 2 \) for \( 2 \leq k \leq s \). Note that \( 2 \leq s \leq m \). Applying Part (3) of Claim 4.2
\[
M_s - 1 \leq \frac{as + 1 - a}{a}B_{s+1} = \frac{as + 1 - a}{a}(2 - M_{s+1}^{-1} - M_s) \leq \frac{as + 1 - a}{a}(2 - M_s)
\]
\[ \implies M_s \leq \frac{2as - a + 2}{as + 1} < 2, \]
which contradicts with the assumption of \( s \) unless \( s = m \). Therefore, \( M_1, M_2, \cdots, M_{m-1} \leq 2 \).}

Applying Part (3)(4) of Claim 4.2
\[
M_m - 1 \leq \frac{am + 1 - a}{a}B_{m+1} = \frac{am + 1 - a}{a}(c - bM_m),
\]
\[
d_m - 1 = \frac{am + 1 - a}{a}D_{m+1} = \frac{am + 1 - a}{a}(c - bd_m),
\]
\[
0 \geq (1 + a)M_1 - (2 + a) + M_2^{-1}.
\]
implying that $M_m \leq d_m$. Let $t$ be the smallest integer such that $M_{t+1} \leq d_{t+1}$. Note that $0 \leq t \leq m - 1$.

Applying Part (3)(4) of Claim 4.2:

$$M_t - 1 \leq \frac{at + 1 - a}{a}B_{t+1} = \frac{at + 1 - a}{a}(2 - M_{t+1}^{-1} - M_t) \leq \frac{at + 1 - a}{a}(2 - d_{t+1}^{-1} - M_t),$$

$$d_t - 1 = \frac{at + 1 - a}{a}D_{t+1} = \frac{at + 1 - a}{a}(2 - d_{t+1}^{-1} - d_t),$$

implying that $M_t \leq d_t$, which contradicts with the assumption of $t$ unless $t = 0$. Thus $M_k \leq d_k$ for $1 \leq k \leq m$.

Applying $M_k \leq d_k$ for $1 \leq k \leq m$ into the equation system (11),

$$\begin{cases}
\Delta_{g_1}\sigma_1 \geq (1 + a)e^{\sigma_1} - (2 + a) + d_2^{-1} \\
\Delta_{g_2}\sigma_k \geq 3e^{\sigma_k} - 3 - d_{k-1}d_k + d_{k+1}^{-1}, \quad 2 \leq k \leq m \\
\Delta_{g_m}\sigma_m \geq be^{\sigma_m} - c - d_{m-1}d_m
\end{cases}$$

For each equation, $\log d_k$ is a solution from the definition of $d_k$’s. Applying the strong maximum principle to each equation, we obtain that for every $1 \leq k \leq m$, either $e^{\sigma_k} < d_k$ or $e^{\sigma_k} \equiv d_k$.

If $e^{\sigma_1} \equiv d_1$, we have $0 = \Delta_{g_1}\sigma_1 \geq (1 + a)e^{\sigma_1} - (2 + a) + e^{-\sigma_1}$ implying $e^{\sigma_2} \equiv d_2$.

If $e^{\sigma_k} \equiv d_k$, we have $0 = \Delta_{g_k}\sigma_k \geq -e^{\sigma_k-1} + \sigma_k + 3e^{\sigma_k} - 3 - e^{-\sigma_k+1}$.

Thus we obtain that either $e^{\sigma_k} < d_k$ for $1 \leq k \leq m$ or $e^{\sigma_k} \equiv d_k$ for $1 \leq k \leq m$.

If we replace $\sigma_1$ by $\eta$, then we apply the Cheng-Yau maximum principle and obtain $\eta$ is bounded from above and $\sup\eta((1 + a)^\eta - (a + 2) + M_2^{-1}) \leq 0$. Since $\sup\eta > 0$, then $(1 + a)^\eta - (a + 2) + M_2^{-1} \leq 0$. Let $M_1 = \sup\eta$, so we obtain $M_1B_1 = B_2$. The rest proof is identical to the above.

We don’t intend to give a formula of $d_k$’s satisfying (12) in Proposition 4.1, but for general $a, b, c$ since it will be too messy. The following two cases are enough for our later use.

**Lemma 4.3** Let $d_k, 1 \leq k \leq m$, be constants satisfying (12) in Proposition 4.1.

1. If $b = c$, then $d_k = 1$ for $1 \leq k \leq m$;
2. If $b = 1, c = 2$, then $d_k = \frac{(m-k)(m+ka+2-a)}{(m+k+1)(m+ka+k+2)}$ for $1 \leq k \leq m$. In particular, if $n = \lfloor \frac{m}{2} \rfloor$, $m = n - 1, a = 2n + 2 - r$, then $d_{n-k} = \frac{(k+1)(r-k)}{k(r-k)}$ for $1 \leq k \leq n - 1$.

**Proof** Applying Part (3)(4) of Claim 4.2:

$$d_m - 1 = \frac{am + 1 - a}{a}D_{m+1} = \frac{am + 1 - a}{a}(c - bd_m) \implies d_m = \frac{c(am + 1 - a) + a}{b(am + 1 - a) + a}$$

1. If $b = c$, then $d_m = 1$. Let $t$ be the smallest integer such that $d_{t+1} = 1$. Note that $0 \leq t \leq m - 1$. Applying Part (3) of Claim 4.2,

$$d_t - 1 = \frac{at + 1 - a}{a}D_{t+1} = \frac{at + 1 - a}{a}(2 - d_{t+1}^{-1} - d_t) = \frac{at + 1 - a}{a}(1 - d_t),$$

implying that $d_t = 1$, which contradicts with the assumption of $t$ unless $t = 0$. Thus $d_k = 1$ for $1 \leq k \leq m$.

2. If $b = 1, c = 2$, then $d_m = \frac{2m + 2 - a}{m + 1}$. Let $t$ be the smallest integer such that $0 \leq t \leq m - 1$ and

$$d_{t+1} = \frac{(m - (t + 1) + 2)(ma + (t + 1)a + 2) - a}{(m - (t + 1) + 1)(ma + (t + 1)a + 2)} = \frac{(m - t + 1)(ma + ta + 2)}{(m - t)(ma + ta + a + 2)}.$$

Applying Part (3) of Claim 4.2:

$$d_t - 1 = \frac{at + 1 - a}{a}D_{t+1} = \frac{at + 1 - a}{a}(2 - d_{t+1}^{-1} - d_t)$$

$$= \frac{at + 1 - a}{a}(2 - \frac{(m - t)(ma + ta + a + 2)}{(m - t + 1)(ma + ta + 2)} - d_t),$$

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implying that $d_t = \frac{(m-t+2)(ma+ka+2-a)}{(m-t+1)(ma+ka+2)}$, which contradicts with the assumption of $t$ unless $t = 0$. Thus $d_k = \frac{(m-k+2)(ma+ka+2-a)}{(m-k+1)(ma+ka+2)}$ for $1 \leq k \leq m$.

Suppose $n = \lceil \frac{r}{2} \rceil$, $m = n-1$, $a = 2n+2 - r$. If $r = 2n$, then $a = 2$ and

$$d_k = \frac{(m-k+2)(ma+ka+2-a)}{(m-k+1)(ma+ka+2)} = \frac{(n-k+1)(n+k-1)}{(n-k)(n+k)}$$

If $r = 2n+1$, then $a = 1$ and

$$d_k = \frac{(m-k+2)(ma+ka+2-a)}{(m-k+1)(ma+ka+2)} = \frac{(n-k+1)(n+k)}{(n-k)(n+k+1)}$$

In both cases, $d_{n-k} = \frac{(k+1)(r-k-1)}{k(r-k)}$.

With the preparation of Proposition 4.1 and Lemma 4.3, we can prove the following estimates for the complete solution.

**Theorem 4.4** Suppose $w^c = (w_1, \cdots, w_r) \in \text{Toda}(q, g)$ is the complete solution, then one of the following holds:

(i) $e^{2w_1} |q|_g^2 < \frac{1}{e^{-w_1+w_2}} < \frac{e^{-w_k+1}w_k}{e^{-w_k+w_k+1}} < 1$, $2 \leq k \leq n = \lceil \frac{r}{2} \rceil$.

(ii) $w_k = -\frac{r+1-2k}{r} \log |q|_g$ for $1 \leq k \leq n$, in which case $q$ has no zeros and $|q|^2$ defines a complete metric.

(iii) $w_k = \log (\frac{(k-1)!}{(r-k)!} 2^{r+1-2k})$ for $1 \leq k \leq n$, in which case $q \equiv 0$ and $(X, g)$ is a hyperbolic surface.

**Proof** If $q$ has no zeros and $|q|^2$ defines a complete metric, then the $w_k$’s in Case (ii) is a complete solution. If $q \equiv 0$, then $(X, g)$ has to be a hyperbolic surface for Toda(0) on the nonempty following Lemma 3.13. The $w_k$’s in Case (iii) is a complete solution.

We only need to show the inequalities in Case (i).

1. We first prove the right hand side inequalities of (24). Outside zeros of $q$,

$$\Delta_g(|q|_g^2e^w) \geq |q|_g^2e^w \log (|q|_g^2e^w) = |q|_g^2e^w(\Delta_g w + \frac{r}{2}g)$$

Since both sides are continuous on $X$, the above inequality holds on $X$. Then we have

$$\begin{align*}
\Delta_g e^{2w_1}|q|_g^2 &\geq e^{2w_1}|q|_g^2(3e^{2w_1}|q|_g^2 - 4e^{-w_1+w_2} + e^{-w_2+w_3}) \\
\Delta_g \log e^{-w_1+w_2} &\leq -e^{2w_1}|q|_g^2 + 3e^{-w_1+w_2} - 3e^{-w_2+w_3} + e^{-w_3+w_4} \\
\Delta_g e^{-w_2+1+w_3} &\geq -e^{2w_2}|q|_g^2 + 3e^{-w_2+w_3} - 3e^{-w_3+w_4} + e^{-w_4+w_5} = (3e^{2w_2}|q|_g^2 - 4e^{-w_2+w_3} + e^{-w_3+w_4}) \\
\Delta_g e^{-w_{n-1}+w_n} &\geq -e^{-w_{n-1}+w_n} + (2n+4-r)e^{-w_{n-1}+w_n} - (2n+3-r)e^{-2n+2-r} w_n
\end{align*}$$

Let $\eta = e^{2w_1}|q|_g^2$, $\sigma_2 = \log e^{-w_1+w_2}$, $\cdots$, $\sigma_n = \log e^{-w_{n-1}+w_n}$. The above system becomes

$$\begin{align*}
\Delta_g e^{-w_{n-1}+w_n} &\geq \eta(3\eta - 4 + e^{-\sigma_2}) \\
\Delta_g e^{-w_{n-1}+w_n} &\geq -\eta e^{\sigma_2} + 3e^{\sigma_3} - 3 + e^{-\sigma_3} \\
\Delta_g e^{-w_{n-1}+w_n} &\geq -e^{-\sigma_{k-1}+\sigma_k} + 3e^{\sigma_{k+1}} - 3 + e^{-\sigma_{k+1}} = 3 \leq k \leq n - 1 \\
\Delta_g e^{-2n+2-r} w_n &\geq -e^{-\sigma_{n-1}+\sigma_n} + (2n+4-r)e^{\sigma_n} - (2n+3-r)
\end{align*}$$

This is the system (13) where $m = n$, $a = 2$, $b = c = 2n+2 - r$. By Lemma 4.3 $d_1 = d_2 = \cdots = d_n = 1$. 23
Applying Proposition 4.1, we obtain that either $e^{\alpha k} < 1$ for $1 \leq k \leq n$ or $e^{\alpha k} \equiv 1$ for $1 \leq k \leq n$. In the latter case, we obtain that $q$ has no zeros and $2w_1 + 2\log |q|_g = -w_1 + w_2 = -w_2 + w_3 = \cdots = -(2n + 2 - r)w_n$, implying that $w_k = -\frac{r+1-2k}{r} \log |q|_g$ for $1 \leq k \leq n$. And the metric $e^{-w_1+w_2} \cdot g = |q|_g^2$ is complete.

II. Next we prove the left hand side inequalities of (24). We have

\[
\begin{align*}
\Delta_g \log e^{-(2n+2-r)w_n} & = e^{-w_n-2+w_n-1} - (2n+4-r)e^{-w_n-1+w_n} + (2n+3-r)w^{-(2n+2-r)w_n} \\
\Delta_g \log e^{-w_{k+1}+w_{k+1}} & = e^{-w_{k+2}+w_{k+1}} - 3e^{-w_{k+1}+w_{k+1}} + 3e^{-w_{k+2}+w_{k+1}+1} - e^{-w_{k+1}+w_{k+2}}, \quad 3 \leq k \leq n-1 \\
\Delta_g \log e^{w_1+w_2} & = e^{w_1+w_2} - 3e^{-w_1+w_2} + 3e^{-w_2+w_3} - e^{-w_3+w_4}
\end{align*}
\]

Let $\eta_1 = \log e^{-(2n+2-r)w_n}$, $\eta_2 = \log e^{-w_n-1+w_n}$, $\cdots$, $\eta_{n-1} = \log e^{-w_3+w_2}$. The above system becomes

\[
\begin{align*}
\Delta_{e^{-w_n-1+w_n}} \eta_1 & = (2n+3-r)e^{\eta_1} - (2n+4-r) + e^{-\eta_2} \\
\Delta_{e^{-w_{n-k}+w_{n-k+1}} \eta_k} & = -e^{\eta_k+1} - \eta_k - 3 + 3e^{\eta_k} - e^{-\eta_{k+1}}, \quad 2 \leq k \leq n-2 \\
\Delta_{e^{-w_1+w_2}} \eta_{n-1} & \geq -e^{\eta_n-2} + \eta_{n-1} + 3e^{\eta_n-1} - 3
\end{align*}
\]

This is the system (11) where $m = n-1$, $a = 2n + 2 - r$, $b = 1$, $c = 2$. By Lemma 4.3, $d_{n-k} = \frac{(k+1)(r-k-1)}{(r-k)}$ for $1 \leq k \leq n-1$.

Applying Proposition 4.1, we obtain that either $e^{\alpha k} < d_k$ for $1 \leq k \leq n$ or $e^{\alpha k} \equiv d_k$ for $1 \leq k \leq n$. In the latter case, we obtain that $q \equiv 0$ which cannot happen by assumption.

III. Finally, we prove the inequalities in (20). From (24), we have for each $1 \leq k \leq n$,

\[
(|q|_g^2 e^{2w_1}) \cdot (e^{-w_1+w_2} e^{2w_2} \cdots (e^{-w_{k-1}+w_k} e^{2w_k}) < (e^{-w_{k-1}+w_k})^{2k-1} \\
< (e^{-w_k+w_{k+1}})^{2k+1} \cdots (e^{-w_1} e^{2w_1})^{2n+2-r} e^{-(2n+2-r)w_n}.
\]

Hence $|q|_g^2 e^{2w_1} < e^{-\frac{2k+1-2k}{r} \log |q|_g}$ and thus $w_k < -\frac{r+1-2k}{r} \log |q|_g$.

Let $f: \tilde{X} \to N := SL(r, \mathbb{C})/SU(r)$ denote the associated equivariant harmonic map for the harmonic bundle $(\mathcal{K}_X, \sigma(q), h^e)$ where $h^e$ is the harmonic metric corresponding to the complete solution $\omega_e$. Here, we use the $SL(r, \mathbb{C})$-invariant Riemannian metric on $N$ induced by the Killing form on $sl(r, \mathbb{C})$.

**Corollary 4.5** Suppose $q$ has at least a zero or $|q|^{2/r}$ does not induce a complete metric on $X$. For each tangent plane of $f(\tilde{X})$, the sectional curvature $K^N_\sigma$ in $N$ satisfies $K^N_\sigma < 0$.

The curvature $\kappa$ of the pullback metric satisfies $\kappa < 0$.

**Proof** In this proof, $\theta(q)$ and $h^e$ are denoted by $\theta$ and $h$, respectively. By Theorem 4.1, $[\theta, \theta^\perp] \neq 0$. From the curvature formula of $K^N_\sigma$ (see [22] Proposition 5.3),

\[
\kappa \leq K^N_\sigma = -\frac{1}{2r} \frac{|[\theta, \theta^\perp]|_g^2}{|\theta^\perp|_g^4 - |tr(\theta^\perp)^2|_g^2}.
\]

Hence $\kappa \leq K^N_\sigma < 0$.

**Proposition 4.6** Suppose $\omega \in \text{Toda}(q, g)$ is the complete solution. Suppose $(u_1, \cdots, u_n)$ is a supersolution of the system (11) satisfying there exists a constant $M$ such that $w_i - u_i \leq M$ for $1 \leq i \leq n$.

Then either $w_i < u_i$, for $1 \leq i \leq n$ or $u_i \equiv w_i$, for $1 \leq i \leq n$.

The definition of a supersolution is in Definition 5.1.

**Proof** Let $\xi_i = w_i - u_i$, for $1 \leq i \leq n$, which are bounded from above. So $\xi_i$’s satisfy

\[
\begin{align*}
\Delta_{e^{-w_1+w_2}} \xi_1 & \geq (1 - e^{-2\xi_1}) \frac{e^{2w_1}|q|_g^2}{e^{-w_1+w_2}} - (1 - e^{\xi_1 - \xi_2}), \\
\Delta_{e^{-w_2+w_3}} \xi_2 & \geq (1 - e^{\xi_1 - \xi_2}) \frac{e^{-w_2+w_3}}{e^{-w_1+w_2}} - (1 - e^{\xi_2 - \xi_3}), \\
\vdots \\
\Delta_{e^{-(2n+2-r)w_n}} \xi_n & \geq (1 - e^{\xi_{n-1} - \xi_n}) \frac{e^{-w_{n-1}+w_n}}{e^{-(2n+2-r)w_n}} - (1 - e^{(2n+2-r)\xi_n}).
\end{align*}
\]
Denote $M_i = \sup e^{\xi_i}$ for $1 \leq i \leq n$. Since the metric $e^{-w_k + w_{k+1}} \cdot g$ is complete and the curvature is bounded from below, we can apply the Omori-Yau maximum principle to the equation of $\xi_k$. Suppose $s$ is the largest integer such that $M_s = \max M_i$.

If $s = 1$, then for every $k$, there exist a sequence of points $p_k \in X$ such that

$$\Delta_{e^{-w_1 + w_2}} \xi_1(p_k) \leq \frac{1}{k}, \quad \xi_1(p_k) \geq M_1 - \frac{1}{k}. $$

Then at point $p_k$, from the equation of $\xi_1$,

$$\frac{1}{k} \geq \Delta_{e^{-w_1 + w_2}} \xi_1(p_k) \geq (1 - e^{\xi_1(p_k) - \xi_1(p_k)}) \frac{e^{-w_1 + w_2} |q^2_g|_g}{e^{-w_1 + w_2}}(p_k) - (1 - e^{\xi_1(p_k) - \xi_1(p_k)}).
$$

As $k \to \infty$, we have $0 \geq (1 - \frac{M_1}{M_1})$, which contradicts with the definition of $s$.

If $2 \leq s \leq n - 1$, then for every $k$, there exist a sequence of points $p_k \in X$ such that

$$\Delta_{e^{-w_s + w_{s+1}}} \xi_s(p_k) \leq \frac{1}{k}, \quad \xi_s(p_k) \geq M_s - \frac{1}{k}.
$$

Then at point $p_k$, from the equation of $\xi_s$,

$$\frac{1}{k} \geq \Delta_{e^{-w_s + w_{s+1}}} \xi_s(p_k) \geq (1 - e^{\xi_{s-1}(p_k) - \xi_s(p_k)}) \frac{e^{-w_{s-1} + w_s} |q^2_g|_g}{e^{-w_s + w_{s+1}}}(p_k) - (1 - e^{\xi_s(p_k) - \xi_{s+1}(p_k)})$$

$$\geq (1 - \frac{M_{s-1}}{M_{s-1}} \delta - (1 - \frac{M_{s-1}}{M_{s-1}})).$$

As $k \to \infty$, we have $0 \geq (1 - \frac{M_{s-1}}{M_{s-1}}) \delta - (1 - \frac{M_{s-1}}{M_{s-1}})$, which cannot happen since $M_{s+1} < M_s$ and $M_{s-1} \leq M_s$.

Therefore $s = n$, then for every $k$, there exist a sequence of points $p_k \in X$ such that

$$\Delta_{e^{-(2n+2-r)w_n}} \xi_n(p_k) \leq \frac{1}{k}, \quad \xi_n \geq M_n - \frac{1}{k}.
$$

Then at point $p_k$, from the equation of $\xi_n$,

$$\frac{1}{k} \geq \Delta_{e^{-(2n+2-r)w_n}} \xi_n(p_k) \geq (1 - e^{\xi_{n-1}(p_k) - \xi_n(p_k)}) \frac{e^{-w_{n-1} + w_n} |q^2_g|_g}{e^{-(2n+2-r)w_n}}(p_k) - (1 - e^{(2n+2-r)\xi_n(p_k)})$$

$$\geq (1 - \frac{M_{n-1}}{M_{n-1}} \delta - (1 - (M_{n-1} - \frac{1}{k})^{2n+2-r}).$$

As $k \to \infty$, we have $0 \geq (1 - \frac{M_{n-1}}{M_{n-1}}) \delta - (1 - (M_{n-1} - \frac{1}{k})^{2n+2-r})$. So we have $M_n \leq 1$ and thus $M_i \leq 1$, $1 \leq i \leq n$.

Apply $M_i \leq 1$ for $1 \leq i \leq n$ to the system and obtain

$$\begin{align*}
\Delta_{e^{-w_1 + w_2}} \xi_1 & \geq (1 - e^{-2\xi_1}) \frac{e^{2w_1} |q^2_g|_g}{e^{-w_1 + w_2}}(1 - e^{\xi_1}), \\
\Delta_{e^{-w_2 + w_3}} \xi_2 & \geq (1 - e^{-\xi_2}) \frac{e^{-w_2 + w_3}}{e^{-w_1 + w_2}}(1 - e^{\xi_2}), \\
\cdots \\
\Delta_{e^{-(2n+2-r)w_n}} \xi_n & \geq (1 - e^{-\xi_n}) \frac{e^{-w_{n-1} + w_n}}{e^{-(2n+2-r)w_n}}(1 - e^{(2n+2-r)\xi_n}).
\end{align*}$$

Applying the strong maximum principle to each equation, for every $1 \leq k \leq n$, either $\xi_k < 0$ or $\xi_k \equiv 0$.

If $\xi_1 \equiv 0$, then $0 = \Delta_{e^{-w_1 + w_2}} \xi_1 \geq (1 - e^{-2\xi_1}) \frac{e^{2w_1} |q^2_g|_g}{e^{-w_1 + w_2}} - (1 - e^{\xi_1})$, implying that $\xi_2 \equiv 0$.

If $\xi_k \equiv 0$ for some $2 \leq k \leq n - 1$, then $0 = \Delta_{e^{-w_k + w_{k+1}}} \xi_k \geq (1 - e^{-\xi_k}) \frac{e^{-w_k + w_{k+1}}}{e^{-w_{k-1} + w_k}}(1 - e^{\xi_k - \xi_{k+1}}) = (1 - e^{\xi_{k-1}}) \frac{e^{-w_{k-1} + w_k}}{e^{-w_k + w_{k+1}}}(1 - e^{-\xi_{k+1}})$, implying that $\xi_{k+1} = \xi_{k+1} \equiv 0$.

If $\xi_n \equiv 0$, then $0 = \Delta_{e^{-(2n+2-r)w_n}} \xi_n \geq (1 - e^{-\xi_n}) \frac{e^{-w_{n-1} + w_n}}{e^{-(2n+2-r)w_n}}(1 - e^{(2n+2-r)\xi_n}) = (1 - e^{-\xi_{n-1}}) \frac{e^{-w_{n-1} + w_n}}{e^{-(2n+2-r)w_n}}$, implying that $\xi_{n-1} \equiv 0$.

Hence we obtain either $\xi_k \equiv 0$ for $1 \leq k \leq n$ or $\xi_k < 0$ for $1 \leq k \leq n$. 


Proposition 4.7 Suppose \( \mathbf{w} \) is the unique complete solution in Toda\((q,g)\) and \( \mathbf{u} \in \text{Toda}^R(q,g) \). Then either \( w_i < u_i \), for \( 1 \leq i \leq n = \lfloor \frac{q}{2} \rfloor \) or \( w_i \equiv u_i \), for \( 1 \leq i \leq n = \lfloor \frac{q}{2} \rfloor \).

Proof By Corollary \ref{prop:unique-complete-solution} there exists a positive constant \( M \) such that
\[
-u_i + w_{i+1} \leq -w_i + w_{i+1} + M, \quad 1 \leq i \leq n - 1.
\]
Since \( (w_n - u_n) - (w_{n+1} - u_{n+1}) = (2n + 2 - r)(w_n - u_n) \), then for any integer \( i_0 \) satisfying \( 1 \leq i_0 \leq n \),
\[
w_{i_0} - u_{i_0} = \sum_{k=i_0}^{n-1} [(w_k - u_k) - (w_{k+1} - u_{k+1})] + \frac{1}{2n + 2 - r)(2n + 2 - r)}(w_n - u_n) \\
\leq (n - i_0)M + \frac{1}{2n + 2 - r}M \leq nM.
\]
Then we obtain the claim of the proposition from Proposition \ref{prop:existence-complete-solution}.

5 Existence of a complete real solution

We will show the existence of a complete real solution for the system \((\ref{eq:system})\), equivalently, a solution to the system \((\ref{eq:system})\) if it satisfies weakly

Proposition 5.1 Assume that \( g \) is a Riemannian metric and \( u - v \in C^0(M) \cap W^{1,2}(M, g) \). Then there exists a positive constant \( M \) such that \( -u_i + w_{i+1} \leq -w_i + w_{i+1} + M, \quad 1 \leq i \leq n - 1 \). Since \( (w_n - u_n) - (w_{n+1} - u_{n+1}) = (2n + 2 - r)(w_n - u_n) \), then for any integer \( i_0 \) satisfying \( 1 \leq i_0 \leq n \),
\[
w_{i_0} - u_{i_0} = \sum_{k=i_0}^{n-1} [(w_k - u_k) - (w_{k+1} - u_{k+1})] + \frac{1}{2n + 2 - r)(2n + 2 - r)}(w_n - u_n) \\
\leq (n - i_0)M + \frac{1}{2n + 2 - r}M \leq nM.
\]
Then we obtain the claim of the proposition from Proposition \ref{prop:existence-complete-solution}.

5.1 Method of super-subsolution for a system

On a smooth manifold \( M \) equipped with a Riemann metric \( g \). For an integer \( 1 \leq k \leq n \), let \( F_k(x, y_1, \cdots, y_n) \) be a smooth function defined on \( M \times \mathbb{R}^n \). Consider the following system of equations of \( (u_1, \cdots, u_n) \)
\[
\Delta_g u_k = F_k(x, u_1, \cdots, u_k), \quad 1 \leq k \leq n.
\]
Assume that
\[
\frac{\partial F_k}{\partial y_j} \leq 0, \quad \text{for } j \neq k.
\]
In the case \( n = 1, \Delta_g u = f(x, u) \), we do not assume any condition on \( f \).

Definition 5.1 A vector function \( (g_1, \cdots, g_n) \), where \( g_k \in C^0(M) \cap W^{1,2}(M) \), is called a supersolution (subsolution) of the system \((\ref{eq:system})\) if it satisfies weakly
\[
\Delta_g g_k \leq (\geq) F_k(x, g_1, \cdots, g_n).
\]
We denote a supersolution by \( \mathbf{w}^+ \) and a subsolution by \( \mathbf{w}^- \).

The following existence theorem easily follows from Guest-Lin \cite{13} with a slight modification. Note here unlike the result in Guest-Lin, we do not need boundary control on the supersolution and subsolution. For two \( \mathbb{R}^n \)-valued functions \( \mathbf{w} \) and \( \mathbf{u} \), we define \( \mathbf{w} < \mathbf{u} \) (resp. \( \mathbf{w} \leq \mathbf{u} \)) if \( w_i < u_i \) (resp. \( w_i \leq u_i \)) for all \( 1 \leq i \leq n \).

Proposition 5.2 (Method of super-subsolution) On a non-compact manifold \( M \) equipped with a Riemannian metric \( g \), consider the system \((\ref{eq:system})\) which satisfies Assumption \((\ref{assumption})\). Suppose \( \mathbf{w}^- \) and \( \mathbf{w}^+ \) are a subsolution and a supersolution respectively satisfying \( \mathbf{w}^- \leq \mathbf{w}^+ \). Then there exists a smooth solution \( \mathbf{w} \) satisfying \( \mathbf{w}^- \leq \mathbf{w} \leq \mathbf{w}^+ \).

Proof Let \( \{M_i\}_{i=1}^{\infty} \) be a sequence of compact submanifolds of \( M \) with smooth boundary satisfying \( M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots \) and \( \bigcup_{i=1}^{\infty} M_i = M \). Let \( \Phi_i = (\phi_{1,i}, \cdots, \phi_{n,i}) \) be a smooth vector function over \( M_i \) satisfying \( \mathbf{w}^- \leq \Phi_i \leq \mathbf{w}^+ \). The existence of \( \Phi_i \) is assured by \( \mathbf{w}^- < \mathbf{w}^+ \). Denote \( \Phi^- = (q_1, \cdots, q_n) \) and \( \Phi^+ = (g_1, \cdots, g_n) \).

Step 1: We first prove the existence of a solution \( \mathbf{w}_i \) on \( M_i \) satisfying \( \mathbf{w}^- \leq \mathbf{w}_i \leq \mathbf{w}^+ \).
On $M_i$, set $\mathbf{w}_i^{(0)} = (w_{i,1}^{(0)}, \ldots, w_{i,n}^{(0)}) := (g_1, \ldots, g_n)$.

On $M_i$, given a smooth vector function $\mathbf{w}_i^{(l)} = (w_{i,1}^{(l)}, \ldots, w_{i,n}^{(l)})$ (e.g. see [35] Theorem 1.6 in Chapter 14) satisfying the following linear elliptic Dirichlet problem: for each $1 \leq k \leq n$,

$$
\triangle_g w_{k,i}^{(l+1)} - d_{k,i} \cdot w_{k,i}^{(l+1)} = G_k(x, \mathbf{w}_i^{(l)}), \quad \text{in } M_i
$$

$$
\partial_{\nu_g} w_{k,i}^{(l+1)} = \phi_{k,i}, \quad \text{on } \partial M_i,
$$

where $d_{k,i} := \max \left\{ \frac{\partial G_k}{\partial y_j}(x, y_1, \ldots, y_n) \right\}$ is a nonnegative constant and

$$
G_k(x, y_1, \ldots, y_n) := F_k(x, y_1, \ldots, y_n) - d_{k,i} \cdot y_k.
$$

Since (1) $\frac{\partial G_k}{\partial y_j} = \frac{\partial F_k}{\partial y_j} - d_{k,i} \leq 0$ if $j \leq i \leq j$ for all $j$ and (2) $\frac{\partial G_k}{\partial y_j} \geq \frac{\partial F_k}{\partial y_j} \leq 0$ for $j \neq k$, we obtain $G_k(x, y_1, \ldots, y_n)$ is decreasing in each $y_i$ if for all $j$, $j \leq i \leq j$.

Clearly, $\mathbf{w}_- \leq \mathbf{w}_i^{(0)} = \mathbf{w}_+$, then $G_k(\mathbf{w}_-) \geq G_k(\mathbf{w}_i^{(0)}) = G_k(\mathbf{w}_+)$. Since on $\partial M_i$, $q_k \leq w_{k,i}^{(1)} = \phi_{k,i} \leq g_k$, by the weak maximum principle (e.g. see [35] Theorem 8.1), we obtain $q_k \leq w_{k,i}^{(1)} \leq g_k$ in $M_i$. Thus $\mathbf{w}_- \leq \mathbf{w}_i^{(1)} \leq \mathbf{w}_+$. Assuming that $\mathbf{w}_- \leq \mathbf{w}_1^{(l)} \leq \mathbf{w}_{i-1}^{(l)} \leq \mathbf{w}_+$, then $G_k(\mathbf{w}_-) \geq G_k(\mathbf{w}_1^{(l)}) \geq G_k(\mathbf{w}_{i-1}^{(l)}) \geq G_k(\mathbf{w}_+)$. Since on $\partial M_i$, $q_k \leq w_{k,i}^{(1)} = w_{k,i}^{(l)} = \phi_{k,i} \leq g_k$, by the maximum principle for weak super(-sub)solutions (e.g. see [35] Theorem 8.1), we conclude $q_k \leq w_{k,i}^{(1)} \leq w_{k,i}^{(l)} \leq g_k$ in $M_i$. Thus $\mathbf{w}_- \leq \mathbf{w}_1^{(l)} \leq \mathbf{w}_{i-1}^{(l)} \leq \mathbf{w}_+$.

On $M_{i+3}$, $\mathbf{w}_i^{(l)}$ is uniformly bounded independent of $l$ and $G_k$ is smooth. In the following context, the constant $C$ varies in different places and does not depend on $l$. Using the interior $L^p$-estimates for linear elliptic equations ($p > \dim M$) (e.g. see [35] Theorem 9.11)

$$
\left\| w_{k,i+3}^{(l)} \right\|_{W^{2,p}(\Omega_{i+3})} \leq C \left( \left\| w_{i+3}^{(l)} \right\|_{L^p(\Omega_{i+3})} + \left\| \nabla_g w_{k,i+3}^{(l)} \right\|_{L^p(\Omega_{i+3})} \right) \leq C.
$$

We conclude the sequence $\{ \mathbf{w}_i^{(l)} \}_{l=1}^{\infty}$ is bounded in $W^{2,p}(\Omega_{i+3})$ and hence is bounded in $C^{1,0}(\Omega_{i+3})$ by the Sobolev embedding theorem (e.g. see [35] Corollary 7.11). Note that $\left\| G_k(x, \mathbf{w}_i^{(l-1)}) \right\|_{C^{0}(\Omega_{i+3})} \leq C \cdot \left\| w_i^{(l-1)} \right\|_{C^{0}(\Omega_{i+3})}$, by the Schauder interior estimates (e.g. see [35] Theorem 6.2)

$$
\left\| w_{k,i+3}^{(l)} \right\|_{C^{2,0}(\Omega_{i+3})} \leq C \left( \left\| w_{k,i+3}^{(l)} \right\|_{C^{0}(\Omega_{i+3})} + \left\| \nabla_g w_{k,i+3}^{(l)} \right\|_{C^{0}(\Omega_{i+3})} \right) \leq C.
$$

Using the Arzela-Ascoli theorem, the sequence $\{ \mathbf{w}_i^{(l)} \}_{l=1}^{\infty}$ converges to $\mathbf{w}_{i+3}$ in $C^2(\Omega_i)$. Therefore, the limit $\mathbf{w}_{i+3}$ is a $C^2$ solution of the system (30) over $M_i$. By the elliptic regularity and the bootstrap argument, we have that $\mathbf{w}_{i+3}$ is $C^\infty$. Moreover, it is clear that $\mathbf{w}_- \leq \mathbf{w}_{i+3} \leq \mathbf{w}_+$. Denote $\mathbf{w}_{i+3}$ by $\mathbf{w}_i$, we finish the proof of step 1.

Step 2: We prove the existence of solution over $M$. On $M_{i+3}$, the solutions $\mathbf{w}_{i+3}$ are uniformly bounded independent of $l$ and $F_k$ is smooth. In the following context, the constant $C$ varies in different places and does not depend on $l$. By the boundedness of $\mathbf{w}_{i+3}$ on $M_{i+3}$, using the $L^p$-estimates for linear elliptic equations ($p > \dim M$)

$$
\left\| w_{i+3} \right\|_{W^{2,p}(\Omega_{i+3})} \leq C \left( \left\| \mathbf{w}_{i+3} \right\|_{L^p(\Omega_{i+3})} + \left\| \nabla_g \mathbf{w}_{i+3} \right\|_{L^p(\Omega_{i+3})} \right) \leq C,
$$

we conclude the sequence $\{ \mathbf{w}_{i+3} \}_{l=1}^{\infty}$ is bounded in $W^{2,p}(\Omega_{i+3})$ and hence is bounded in $C^{1,0}(\Omega_{i+3})$ by the Sobolev embedding theorem. Note that $\left\| F_k(x, \mathbf{w}_i^{(l+1)}) \right\|_{C^{0}(\Omega_{i+3})} \leq C \cdot \left\| \mathbf{w}_{i+3} \right\|_{C^{0}(\Omega_{i+3})}$. By the Schauder interior estimates (e.g. see [35] Theorem 6.2)

$$
\left\| \mathbf{w}_{i+3} \right\|_{C^{2,0}(\Omega_{i+3})} \leq C \left( \left\| \mathbf{w}_{i+3} \right\|_{C^{0}(\Omega_{i+3})} + \left\| \nabla_g \mathbf{w}_{i+3} \right\|_{C^{0}(\Omega_{i+3})} \right) \leq C.
$$

Using the Arzela-Ascoli theorem and a diagonal trick, we can find a subsequence $\{ \mathbf{w}_{i+3} \}_{l=1}^{\infty}$ which converges to $\mathbf{w}$ in $C^{2,0}(\Omega_i)$, so that $\mathbf{w}$ is a $C^2$ solution of the system. It follows from the elliptic regularity and the bootstrap argument that $\mathbf{w}$ is indeed a $C^\infty$ solution of the system over $M$.  

\[\blacksquare\]
Remark 5.3 We may compare the method of super-subsolution developed in Proposition 5.2 with Theorem 9 in Wan [37] for a scalar equation $Δu = f(x, u)$. The difference here is that (1) we do not impose the condition $\frac{∂f}{∂u} ≥ 0$; (2) the non-compact manifold $(M, g)$ is not necessarily complete.

Lemma 5.4 (1) Suppose $(ψ_1, \ldots, ψ_n)$ and $(η_1, \ldots, η_n)$ are two supersolutions. Define $g_k = min\{ψ_k, η_k\}$, $k = 1, \ldots, n$, then $(g_1, \ldots, g_n)$ is again a supersolution.

Lemma 5.5 (1) Suppose $(ξ_1, \ldots, ξ_n)$ and $(ϕ_1, \ldots, ϕ_n)$ are two subsolutions. Define $q_k = max\{ξ_k, ϕ_k\}$, $k = 1, \ldots, n$, then $(q_1, \ldots, q_n)$ is again a subsolution.

Proof (1) Suppose $(ψ_1, \ldots, ψ_n)$ and $(η_1, \ldots, η_n)$ are two supersolutions. Since $\frac{∂f_k}{∂u_j} ≤ 0$, $j ≠ k$ and $g_k ≤ ψ_k, η_k$, then for $g_k$ we have

\[ Δgψ_k ≤ F_k(x, ψ_1, ψ_2, \ldots, ψ_n) ≤ F_k(x, g_1, \ldots, g_k-1, ψ_k, g_k+1, \ldots, g_n), \]

\[ Δgη_k ≤ F_k(x, η_1, η_2, \ldots, η_n) ≤ F_k(x, g_1, \ldots, g_k-1, η_k, g_k+1, \ldots, g_n). \]

Since $g_k = min\{ψ_k, η_k\}$ and using the fact that the minimum of two supersolutions is still a supersolution (see [11] for example), one gets

\[ Δg g_k ≤ F_k(x, g_1, \ldots, g_n). \]

Hence $(g_1, \ldots, g_n)$ is a supersolution.

(2) Suppose $(ξ_1, \ldots, ξ_n)$ and $(ϕ_1, \ldots, ϕ_n)$ are two subsolutions. Then $(-ξ_1, \ldots, -ξ_n)$ and $(-ϕ_1, \ldots, -ϕ_n)$ are two supersolutions to the system

\[ Δg u_k = G_k(u_1, \ldots, u_n) := -F_k(x, -u_1, \ldots, -u_n). \]

Clearly $G_k$ satisfies the Assumption (31). By Part (1), $(-q_1, \ldots, -q_n)$ is a supersolution to the system (33) and thus it is a subsolution to the system (30).

5.2 Existence of a complete real solution

In this subsection, we will show the existence of a complete real solution to the system (1) for a Riemann surface $X$ and a holomorphic $r$-differential $q$. The plan of the proof goes as follows.

1. We first show the existence in Proposition 5.8 when $X$ is the unit disk $D$ with an extra condition that $q(z)|(1 - |z|^2)^r$ is bounded. With this extra condition, it is easy to construct a supersolution and a subsolution and we can apply the method of super-subsolution.

2. Next, we remove the condition that $q(z)|(1 - |z|^2)^r$ is bounded and show the existence in Proposition 5.7 for any $q$ on the unit disk $D$. We start with an exhaustion of the unit disk by a sequence of disks. Since on each smaller disk, the boundedness condition on $q$ is automatically satisfied and the existence follows from Step 1. Then the main difficulty is to show the sequence of solutions converges in $C_0^{loc}(D)$.

3. We then show the existence in Proposition 5.10 when $X$ is a hyperbolic Riemann surface, that is, it has a universal cover as the unit disk $D$. By lifting $(q, g)$ to the universal cover $D$, we obtain a complete solution for the lifted pair from Step 2. The solution will descend to a solution on $X$ because of the uniqueness property of a complete solution.

4. We show the existence in Proposition 5.11 when $X$ is the complex plane $C$. This is again done by constructing a supersolution and a subsolution. Unlike Step 1, the supersolution and subsolution are not easy to be constructed directly. We cover the complex plane with a disk and the complement of a smaller disk. On each piece, we obtain a complete solution by Step 3 since both of them are hyperbolic Riemann surfaces. The supersolution and subsolution arise from a gluing of these two solutions.

5. Using a similar method as Step 3, we then show the existence in Proposition 5.12 when $X$ is a parabolic Riemann surface, that is, it has a universal cover as the complex plane $C$.

In this way, we finish proving the existence of a complete real solution for any Riemann surface $X$ and a holomorphic $r$-differential $q$ over $X$. In the end of the subsection, we also prove the existence of a real solution in Proposition 5.13 if $q$ has finitely many zeros which is asymptotic to the behavior of $|q|$. This will lead to a discussion of the non-uniqueness of solutions if we remove the completeness condition.
When $X = \mathbb{C}$ or any subdomain of $\mathbb{C}$, by choosing the background Kähler metric as $dz \otimes d\bar{z}$, the system \[(10)\] is just looking for a solution $(\tilde{w}_1, \ldots, \tilde{w}_n)$ satisfying
\[
\begin{align*}
\Delta \tilde{w}_1 &= e^{2\tilde{w}_1}|q|^2 - e^{-\tilde{w}_1} + \tilde{w}_2 \\
\Delta \tilde{w}_2 &= e^{\tilde{w}_1} + \tilde{w}_2 - e^{-\tilde{w}_2} + \tilde{w}_3 \\
& \quad \vdots \\
\Delta \tilde{w}_n &= e^{-\tilde{w}_{n-1} + \tilde{w}_n} - e^{-(2n+2-r)\tilde{w}_n},
\end{align*}
\] (34)
where $\Delta = \partial_z \partial_{\bar{z}}$. We denote $(\tilde{w}_1, \ldots, \tilde{w}_n)$ by $\tilde{w}$. From now on, if we use $\tilde{w}$ instead of $w$, that means we are dealing with the background metric $dz \otimes d\bar{z}$.

Consider a domain $\Omega$ equipped with a hyperbolic Kähler metric $g = g_0 dz \otimes d\bar{z}$, that is, $\Delta \log g_0 = \frac{1}{2}g_0$ for $\Delta = \partial_z \partial_{\bar{z}}$. If $q = 0$, the system \[(33)\] admits a unique complete solution $(\tilde{w}_1, \ldots, \tilde{w}_n)$ where
\[
\tilde{w}_l = \log\left(\frac{(l-1)!}{(r-l)!}\left(\frac{g_0}{4}\right)^{-\frac{r+1-2l}{2}}\right), \quad 1 \leq l \leq n.
\]
We denote the solution by $\tilde{w}_{\text{base}}$. In particular, on the unit disk $\mathbb{D}$, $g_0 = \frac{1}{1-|z|^2}$. Each entry of $\tilde{w}_{\text{base}}$ goes to $-\infty$ as $z \to \partial\mathbb{D}$. Clearly, $\tilde{w}_{\text{base}}$ is a supersolution for general $q$.

On a general domain $\Omega$, if $q \not= 0$, define the vector function
\[
\tilde{w}_q = \left(-\frac{r-1}{r}(\log |q|), -\frac{r-3}{r}(\log |q|), \cdots, -\frac{r+1-2n}{r}\log |q|\right),
\]
which is a solution to the system \[(34)\] except at zeros of $q$. Moreover, each entry of $\tilde{w}_q$ goes to $+\infty$ at each zero of $q$.

**Remark 5.5** (i) For a domain $\Omega \subset \mathbb{C}$, we may equip two natural Kähler metrics: $dz \otimes d\bar{z}$ and $g_0 dz \otimes d\bar{z}$. For example, on a disk $\mathbb{D}$, $g_0 = \frac{1}{1-|z|^2}$. Let $w = (w_1, \ldots, w_n)$ where $e^{w_l} = e^{\tilde{w}_l} \cdot g_0^{\frac{r+1-2l}{2}} (1 \leq l \leq n)$. Then $\tilde{w}$ is a solution to \[(10)\] on $(\Omega, dz \otimes d\bar{z})$ if and only if $w$ is a solution to \[(10)\] on $(\Omega, g_0 dz \otimes d\bar{z})$.

(ii) Suppose $\Omega$ is a hyperbolic domain. Denote $w_{\text{base}}$ where $w_l = \log\left(\frac{l-1}{l-r}(\log |q|)^{-\frac{r+1-2l}{2}}\right)$ for $1 \leq l \leq n$, which is a solution to \[(10)\] for $q = 0$ case. Then $w_{\text{base}}$ is a solution to \[(10)\] which corresponds to the solution $\tilde{w}_{\text{base}}$ to \[(34)\] for $q = 0$ case.

(iii) Define $w_q$ where $w_l = -\frac{r+1-2l}{2}(\log |q|)$ for $1 \leq l \leq n$, which is a solution to \[(10)\] except at zeros of $q$. Then $w_q$ is a solution to \[(10)\] which corresponds to the solution $\tilde{w}_q$ to \[(34)\] except at zeros of $q$.

**Proposition 5.6** On the unit disk $\mathbb{D}$, suppose $q(z)$ is a holomorphic $r$-differential on $\mathbb{D}$ satisfying $|q(z)|(1-|z|^2)^r$ is bounded, then there exists a smooth solution $\tilde{w} = (\tilde{w}_1, \cdots, \tilde{w}_n)$ of the system \[(34)\] satisfying
\[
\begin{align*}
\tilde{w}_{\text{base}} - c \leq \tilde{w} \leq \tilde{w}_{\text{base}}(\leq 0),
\end{align*}
\]
where $c$ is a positive constant only depending on $\sup_{\mathbb{D}} |q(z)|(1-|z|^2)^r$ and $r$, and $u - c := (u_1 - c, \cdots, u_n - c)$. Moreover, $\tilde{w}$ is a complete solution satisfying
\[
\begin{align*}
e^{-\tilde{w}_l + \tilde{w}_{l+1}} &\geq l(r-l)\frac{1}{(1-|z|^2)^2} \quad (1 \leq l \leq n-1), \quad e^{-(2n+2-r)\tilde{w}_n} \geq \frac{1}{(n-r-1)^2}, \\
e^{-\tilde{w}_l} &\geq \frac{(r-l)!}{(l-1)!}\left(\frac{1}{1-|z|^2}\right)^{(r+1-2l)/2} \quad (1 \leq l \leq n).
\end{align*}
\] (35) (36)

**Proof** Let $g_0 = \frac{1}{1-|z|^2}$, which satisfies $\Delta \log g_0 = \frac{1}{2}g_0$ for $\Delta = \partial_z \partial_{\bar{z}}$. Then $g = g_0 dz \otimes d\bar{z}$ is the unique complete hyperbolic Kähler metric on $\mathbb{D}$. By assumption $|q||g| = |q(z)|(1-|z|^2)^r$ is bounded.

Clearly, $\tilde{w}_+ = \tilde{w}_{\text{base}}$ is a supersolution. Next we construct a subsolution. Define an $n$-tuple of negative constants $(a_1, \cdots, a_n)$ satisfying
\[
a_{k+1} - a_k = \log(kE)(1 \leq k \leq n-1), \quad a_n = -\log(nE),
\]
29
where $E$ is a positive constant to be determined later. Define $\xi_k = a_k + \log(\frac{M}{k})^{-\frac{r-1-2j}{2}}$. Let $M = \sup_{\mathbb{D}} |q|^2$. For $E$ large enough, $(\xi_1, \cdots, \xi_n)$ satisfies

$$
\begin{align*}
-\frac{r-1}{2} \triangle \log g_0 &= \Delta \xi_1 \geq M \cdot e^{2 \xi_1} - e^{\xi_2 - \xi_1} \geq (e^{2a_1}|q|^2 - e^{a_2 - a_1}) \cdot \frac{M}{k} \\
-\frac{r-2}{2} \triangle \log g_0 &= \Delta \xi_2 \geq e^{\xi_2 - \xi_1} - e^{\xi_3 - \xi_2} = (e^{a_2 - a_1} - e^{a_3 - a_2}) \cdot \frac{M}{k} \\
&\vdots \\
-\frac{r-1-2n}{2} \triangle \log g_0 &= \Delta \xi_n \geq e^{\xi_n - \xi_{n-1}} - e^{-(2n+2-r)(2n+2-r)} \xi_n = (e^{a_n - a_{n-1}} - e^{-(2n+2-r)a_n}) \cdot \frac{M}{k}.
\end{align*}
$$

Thus $\bar{w}_- = (\xi_1, \cdots, \xi_n)$ is a subsolution.

As long as $E$ is large enough, we have $\bar{w}_- < \bar{w}_+$. Applying Proposition 5.2, there is a $C^\infty$ solution $\bar{w}$ satisfying $\bar{w}_- \leq \bar{w} \leq \bar{w}_+$.

For $1 \leq k \leq n-1$, $e^{-\bar{w}_k + \bar{w}_{k+1}}|dz|^2 \geq e^{-a_{k+1}} \cdot \frac{k(r-1)}{k} |dz|^2$ which is a complete metric; $e^{-(2n+2-r)\bar{w}_n} |dz|^2 \geq \frac{(r-n)!}{(n+1)!} |dz|^2$ which is a complete metric. Hence $\bar{w}$ is a complete solution and by Proposition 4.4 it follows that the solution $\bar{w}$ satisfies

$$
\frac{(k-1)(r-k+1)}{k(r-k)} < e^{-\bar{w}_k + \bar{w}_{k+1}}(2 \leq k \leq n-1), \quad \frac{(n-1)(r-n+1)}{n(r-n)} < e^{-\bar{w}_n + \bar{w}_{n+1}}.
$$

Since $e^{-(2n+2-r)\bar{w}_n} \geq \frac{(r-n)!}{(n+1)!} |dz|^2 \geq n(r-n) \frac{1}{(1-|z|^2)^2}$, together with the above inequalities, we obtain the required estimates in (55)-(56).

Now we show the existence of a complete solution without the condition that $|q(z)|(1 - |z|^2)^r$ is bounded. We will use the method developed in Wan [77] by using the exhaustion of the unit disk for a scalar equation. Some steps there are simplified.

**Proposition 5.7** Let $q$ be a holomorphic $r$-differential on the disk $\mathbb{D}$, then the system (77) admits a complete $C^\infty$-solution $\bar{w} = (\bar{w}_1, \cdots, \bar{w}_n)$. Moreover, the solution satisfies Inequality (55)-(56).

**Proof** For any positive integer $k$, let $D_k = \{ z \in \mathbb{D}, |z| < R_k \}$, where $R_k = \frac{e^k}{e^k+1} < 1$. Then we have $D_1 \subset \cdots \subset D_k \subset D_{k+1} \subset \cdots \subset \mathbb{D}$, and $\mathbb{D} = \cup_{k=1}^\infty D_k$. On $D_k$, it has the Poincaré metric $\frac{4R_k^2}{(1-|z|^2)^2} dz \otimes d\bar{z}$.

On each $D_k$, $|q|$ is bounded and hence $|q(R_k^2 - |z|^2)|^r$ is bounded. By Proposition 5.6 for each $k$, there is a complete solution $\bar{w}(k)$ to the system (54) on $D_k$.

Since $\bar{w}(k)$ restricts to a bounded solution on $D_k$ and $\bar{w}(k)$ goes to $-\infty$ as approaching to $\partial D_k$, by Proposition 4.4 we have $\bar{w}(k) \geq \bar{w}(k)$ on $D_k$. Since $\bar{w}_{base}$ restricts to a supersolution on each $D_k$ and $\bar{w}(k)$ goes to $-\infty$ as approaching to $\partial D_k$, by Proposition 4.11 the complete solution $\bar{w}(k)$ satisfies $\bar{w}(k) \leq \bar{w}_{base}$. We conclude the sequence $\{\bar{w}(k)\}_k$ is monotone increasing and is bounded from above. Hence the sequence $\bar{w}(k)$ converges pointwise to a vector function $\bar{w}$.

On $\overline{D}_{k+2}$, the sequence $\{\bar{w}^{(k+1)+3}\}_{j=1}^\infty$ is uniformly bounded independent of $j$. In the following context, the constant $C = C(k)$ varies in different places and does not depend on $j$. Using the interior $L^p$-estimates for linear elliptic equations ($p > 2$) (e.g. see [36] Theorem 9.11)

$$
||\bar{w}^{(k+1)+3}||_{W^{2,p}(\overline{D}_{k+2})} \leq C(||\bar{w}^{(k+1)+3}||_{L^p(\overline{D}_{k+2})} + ||\Delta \bar{w}^{(k+1)+3}||_{L^p(\overline{D}_{k+2})}) \leq C,$
$$
we conclude the sequence $\{\bar{w}^{(k+1)+3}\}_j$ is bounded in $W^{2,p}(\overline{D}_{k+2})$ and is also bounded in $C^{1,\alpha}(\overline{D}_{k+2})$ by the Sobolev embedding theorem (e.g. see [36] Corollary 7.11). Since $|F_k(x, \bar{w}^{(k+1)+3})|_{C^{1,\alpha}(\overline{D}_{k+2})} \leq C \cdot ||\bar{w}^{(k+1)+3}||_{C^{1,\alpha}(\overline{D}_{k+2})}$, by the Schauder interior estimates (e.g. see [36] Theorem 6.2),

$$
||\bar{w}^{(k+1)+3}||_{C^{2,\alpha}(\overline{D}_{k})} \leq C(||\bar{w}^{(k+1)+3}||_{C^{1,\alpha}(\overline{D}_{k+1})} + ||\Delta \bar{w}^{(k+1)+3}||_{C^{1,\alpha}(\overline{D}_{k+1})}) \leq C = C(k).
$$

Using the Arzela-Ascoli theorem and a diagonal trick, we can find a subsequence $\{\bar{w}^{(k'+j)+3}\}_{j'=1}^\infty$ which converges to $\bar{w}$ in $C^{2,\alpha}_{loc}(\mathbb{D})$, so that $\bar{w}$ is a $C^2$ solution of the system. By the elliptic regularity and the bootstrap argument, we have that $\bar{w}$ is indeed a $C^\infty$ solution of the system.
Finally, by Proposition 5.6 for each $k$, we have
\[
e^{-w^{(k)}_j + w^{(k)}_{i+1}} \geq l(r-l) \frac{R_k^2}{(R_k^2 - |z|^2)^2} \quad (1 \leq l \leq n-1),
\]
\[
e^{-(2n+2-r)\bar{w}^{(k)}_n} \geq n(r-n) \frac{R_k^2}{(R_k^2 - |z|^2)^2}.
\]
It follows that
\[
e^{-\bar{w}_i + \bar{w}_{i+1}} \geq l(r-l) \frac{1}{(1 - |z|^2)^2} \quad (1 \leq l \leq n-1),
\]
\[
e^{-(2n+2-r)\bar{w}_n} \geq n(r-n) \frac{1}{(1 - |z|^2)^2}.
\]
Thus the solution $\bar{w}$ is complete. The estimates follow from the above equalities.

**Proposition 5.8** Let $(X, g)$ be a complete hyperbolic surface. Let $q$ be a holomorphic $r$-differential on $X$, then the system $[\text{(10)}]$ admits a complete solution $w$.

Moreover,
\[
e^{-w_l + w_{l+1}} \geq \frac{l(r-l)}{4} \quad (1 \leq l \leq n-1),
\]
\[
w_l \leq \log \left( \frac{(l-1)!}{(r-l)!} 2^{(r+1-2l)} \right) \quad (1 \leq l \leq n).
\]

In particular, if $q$ is bounded with respect to $g$, then the system $[\text{(10)}]$ admits a bounded solution $w$.

**Proof** Let $X$ be covered by $\mathbb{D}$ under the map $p : \mathbb{D} \to X$, with the covering transformation group of $X$ be $\Gamma < \text{Aut}(\mathbb{D}) = PSL(2, \mathbb{R})$, i.e. $X = \mathbb{D}/\Gamma$. Lift $q,g$ to $\hat{q}, \hat{g}$ on $\mathbb{D}$, which are invariant under $\Gamma$. By Proposition 5.7, there exists a complete solution $\hat{w} \in \text{Toda}(\hat{q}, d\hat{z} \otimes d\hat{\bar{z}})$, equivalently, there exists a complete solution $\hat{w} \in \text{Toda}(\hat{q}, \hat{g})$. By Proposition 5.31 $\gamma^*(\hat{w}) = \hat{w}$, for $\gamma \in \Gamma$. Hence $\hat{w}$ descends to a solution $w$ on $\mathbb{D}/\Gamma = X$ of the system which is still complete. The estimates directly follow from (35)-(36).

If $q$ is bounded with respect to $g$, by Proposition 5.6, there exists a bounded solution $\hat{w} \in \text{Toda}(\hat{q}, \hat{g})$ which is complete. By Proposition 5.31 $\gamma^*(\hat{w}) = \hat{w}$, for $\gamma \in \Gamma$. Hence $\hat{w}$ descends to a solution $w$ on $\mathbb{D}/\Gamma = X$ of the system which is still bounded.

**Corollary 5.9** Let $(X, g)$ be a complete hyperbolic surface. If an r-differential $q$ on $X$ is bounded with respect to $g$, there uniquely exists a bounded solution in $\text{Toda}(q,g)$. Moreover, it is real. Conversely, if there exists a bounded solution in $\text{Toda}(q,g)$, then $q$ is bounded with respect to $g$.

In fact, the bounded solution is the complete solution.

**Proof** If an $r$-differential $q$ on $X$ is bounded with respect to $g$, by Proposition 5.8 and Corollary 3.26, there exists a unique bounded solution. Conversely, if there exists a bounded solution in $\text{Toda}(q,g)$, by Corollary 3.22, $q$ is bounded with respect to $g$.

Obviously, the bounded solution is complete.

Now we move to the case of complex plane. We will use the method developed in Au-Wan [38] to construct an appropriate supersolution and subsolution.

**Proposition 5.10** Let $q$ be a nonzero holomorphic $r$-differential on $\mathbb{C}$, then the system $[\text{(34)}]$ admits a complete solution.

**Proof** We will use the method of super-subsolution for system to show the existence by constructing a supersolution and a subsolution in the weak sense.

Without loss of generality, we assume $q(0) \neq 0$. Choose $R_1$ such that the zeros of $q$ are outside the ball $B_{R_1}$. Choose $R_2 < R_1$. Let $\bar{u}$ be the unique complete solution to $[\text{(31)}]$ on $B_{R_1}$ and $\bar{v}$ be the unique complete solution to $[\text{(34)}]$ in the complement of the closed disk $\overline{B}_{R_2}$ as described in Proposition 5.8.

We first construct a subsolution $\bar{w}_-$ as follows: on $\{|z| < R_2\}$, let $\bar{w}_- = \bar{u}$;
on $\{|z| \leq R_1\}$, let $\bar{w}_- = \max\{\bar{u}, \bar{v}\}$;
on \(\{|z| > R_1\}\), let \(\tilde{w}_- = \tilde{v}\).

By Proposition 5.8 \(e^{-\frac{1}{2}\frac{i}{2}\log |z|^2}\) dominates the hyperbolic metric on \(B_{R_1}\) up to a constant and hence blows up at boundary of \(B_{R_1}\). Then at the neighborhood of \(\partial B_{R_1}\), \(\tilde{w}_- = \tilde{v}\). Similarly, by Proposition 5.8 \(e^{-\frac{1}{2}\frac{i}{2}\log |z|^2}\) dominates the hyperbolic metric on the complement of \(B_{R_2}\) up to a constant and hence blows up at boundary of \(B_{R_2}\). Then at the neighborhood of \(\partial B_{R_2}\), \(\tilde{w}_- = \tilde{u}\). Hence \(\tilde{w}_-\) is continuous. Since both \(\tilde{u}\) and \(\tilde{v}\) are solutions on the annulus \(\{R_2 \leq |z| \leq R_1\}\), by Proposition 5.4 the maximum of \(\tilde{u}\) and \(\tilde{v}\) is a subsolution in the weak sense. Therefore \(\tilde{w}_- \in C^0(\mathbb{C}) \cap W^{1,2}_{loc}(\mathbb{C})\) is a subsolution.

Secondly, we construct a supersolution \(\tilde{w}^+\). Choose \(R_1' < R_2'\) such that \(R_2 < R_2' < R_1' < R_1\). For a constant \(c\) and a solution \(\tilde{v}\) on a domain \(U \subset \mathbb{C}\), \(\tilde{v} + c = (\tilde{v}_1 + c, \tilde{v}_2 + c, \cdots, \tilde{v}_n + c)\) is a supersolution (subsolution) on \(U\) for \(c \geq 0\) \((c \leq 0)\). In fact, for \(c \geq 0\) \((c \leq 0)\),

\[
\begin{align*}
\Delta(\tilde{v}_1 + c) &= \Delta\tilde{v}_1 \leq (\geq)e^{(\tilde{v}_1 + c)|q|}^2 - e^{-(\tilde{v}_1 + c) + (\tilde{v}_2 + c)} + \cdots \\
\Delta(\tilde{v}_2 + c) &= \Delta\tilde{v}_2 = e^{-(\tilde{v}_1 + c) + (\tilde{v}_2 + c)} - e^{(\tilde{v}_2 + c) + \cdots} \\
\cdots
\end{align*}
\]

On the annulus \(\{R_2' < |z| \leq R_1'\}\), \(\tilde{v}\) has a minimum and \(-\frac{r+1-2l}{r}\log |q|\) has a maximum since all the zeros of \(q\) are outside \(B_{R_1}\). Then on the annulus \(\{R_2' \leq |z| \leq R_1'\}\), there exists a constant \(c\) such that

\[-\frac{r+1-2l}{r}\log |q| \leq \tilde{v}_l + c, \quad 1 \leq l \leq n.
\]

We construct the supersolution \(\tilde{w}^+\) as follows:

- on \(|z| < R_1\), let \(\tilde{w}^+ = \tilde{w}_q\);
- on \(|z| \geq R_2\), let \(\tilde{w}^+ = \min\{\tilde{w}_q, \tilde{v} + c\}\). By the choice of \(c\), \(\tilde{w}^+ = \tilde{w}_q\) in the neighborhood of \(\partial B_{R_2'}\). So \(\tilde{w}^+\) is continuous. Since \(\tilde{w}_q, \tilde{v} + c\) are both supersolutions on \(\{|z| \geq R_2\}\), by Proposition 5.4 the minimum of \(\tilde{w}_q\) and \(\tilde{v} + c\) is a supersolution in the weak sense. Therefore \(\tilde{w}^+ \in C^0(\mathbb{C}) \cap W^{1,2}_{loc}(\mathbb{C})\) is a supersolution.

By Theorem 4.4 the solution \(\tilde{u}\) satisfies \(\tilde{u} < \tilde{w}_q\), since \(|q|^{\frac{r}{2}}\) does not define a complete metric on \(B_{R_1}\). It is then clear that \(\tilde{w}_- < \tilde{w}^+\). Applying Proposition 5.2 there is a \(C^\infty\) solution \(\tilde{w}\) satisfying \(\tilde{w}_- \leq \tilde{w} \leq \tilde{w}^+\). Outside \(B_{R_1}\), \(\tilde{v} = \tilde{w}_- \leq \tilde{w} \leq \tilde{w}^+ = \tilde{v} + c\). Since \(\tilde{v}\) is a complete solution outside \(B_{R_1}\), the solution \(\tilde{w}\) is complete on \(\mathbb{C}\).

**Proposition 5.11** Let \(X\) be a parabolic Riemann surface with an Euclidean metric \(g\) and a holomorphic \(r\)-differential \(q\). If \(q \neq 0\), then the system (10) admits a unique complete solution. If \(q = 0\), then the system (10) admits no solution, that is, \(\text{Toda}(q, g) = \emptyset\).

**Proof** If \(q = 0\), the statement follows directly from Lemma 3.13.

Suppose now \(q \neq 0\). Let \(X\) be covered by \(\mathbb{C}\) under the map \(p: \mathbb{C} \rightarrow X\), with the covering transformation group of \(X\) as \(\Gamma < \text{Aut}(\mathbb{C}) = \{z \mapsto az + b, a \in \mathbb{C}^*, b \in \mathbb{C}\}, i.e. X = \mathbb{C}/\Gamma\). Lift \(q, g\) to \(\hat{q}, \hat{g}\) on \(\mathbb{C}\) which are invariant under \(\Gamma\). By Proposition 5.10 there exists a unique complete solution \(w \in \text{Toda}(\hat{q}, \hat{g})\). By Proposition 3.17 \(\gamma^*(\tilde{w}) = \tilde{w}\).

Hence \(\tilde{w}\) descends to a solution \(w\) on \(\mathbb{C}/\Gamma = X\) which is still complete.

Combining Theorem 3.16 Proposition 5.8 and Proposition 5.11 we obtain

**Theorem 5.12** Let \(X\) be any non-compact Riemann surface with any Kähler metric \(g\) and a holomorphic \(r\)-differential \(q\). We assume that \(q \neq 0\) if \(X\) is parabolic. Then, there exists a unique complete solution \(w \in \text{Toda}(q, g)\). Moreover, it is real.

In the case \(q\) has finitely many zeros, we can construct a solution whose asymptotic behavior is purely controlled by \(q\) using the method of super-subsolution again. This solution is not necessarily complete. For example, when \(X = \mathbb{C}, q = ze^z\), the solution is clearly not complete.

**Proposition 5.13** If \(q\) has finitely many zeros, then there exists a smooth solution \(w\) of the system (10) satisfying outside a relatively compact open set containing all zeros of \(q\),

\[w_q - c \leq w \leq w_q.
\]
Proof. Denote by $Z$ the set of zeros of $q$. In the case $Z$ is empty, $w_0$ is a solution as desired.

In the case $Z$ is nonempty, we will use the method of super-subsolution for the system to show the existence by constructing a supersolution and a subsolution.

For each point $P \in Z$, choose a neighbourhood $N_P$ of $P$ such that (i) $N_P$ is embedded into $\mathbb{C}$, which induces a coordinate $z_P$ on $N_P$, (ii) $\{|z_P| < 1\}$ is relatively compact in $N_P$, (iii) $N_P \cap N_Q = \emptyset$ for $P, Q \in Z$.

Let $\Omega_{P,2} = \{|z_P| < 1\}$. Choose neighbourhoods $\Omega_{P,0}$ and $\Omega_{P,1}$ of $P$ such that $\Omega_{P,0} \cap \Omega_{P,1} \cap \Omega_{P,2}$. Let $\Omega_i = \cup_{P \in Z} \Omega_{P,i}(i = 0, 1, 2)$. Since $Z$ is finite, $\Omega_i(i = 0, 1, 2)$ are relatively compact. By Proposition 5.8, the system admits a complete solution $u_P = (u_{P1}, \cdots, u_{Pn})$ on $\Omega_{P,2}$. Denote by $u$ the solution on $\Omega_2$ whose restriction to $\Omega_{P,2}$ is $u_P$.

The relatively compact subset $\Omega_1 \setminus \Omega_0$ does not contain any zeros of $q$, then $-\frac{r+1-2l}{r} \log |q| (l = 1, \cdots, n)$ is bounded from above. Therefore, there exists a constant $c > 0$ such that on $\Omega_1 \setminus \Omega_0$,

$$u_i \geq -\frac{r+1-2l}{r} \log |q| - c, \quad \forall 1 \leq l \leq n. \quad (39)$$

We construct a supersolution as follows: on $\Omega_1$, let $w_+ = \min\{u + c, w_q\}$; outside $\Omega_1$, let $w_+ = w_q$. We must check the continuity of $w_+$ on $\partial \Omega_1$ and at zeros of $q$. Because of the choice of $c$, $u + c \geq w_q$ in $\Omega_1 \setminus \Omega_0$, so $w_+ = w_q$ in the neighborhood of $\partial \Omega_1$. Since $u$ is continuous on $\Omega_1$ and $w_q$ goes to $+\infty$ at zeros of $q$, the vector function $\min\{u + c, w_q\} = u + c$ is continuous at zeros of $q$. Since $u$ is a solution in $\Omega_1$, $u + c$ is a supersolution. By Proposition 5.3 on $\Omega_1$, $w_+$ is still a supersolution. Hence on $X$, $w_+$ is a supersolution.

We construct a subsolution as follows: on $\Omega_0$, let $w_- = u$; on the annulus $\Omega_2 \setminus \Omega_0$, let $w_- = \max\{u, w_q - c\}$; outside $\Omega_2$, let $w_- = w_q - c$. We must check the continuity of $w_-$ on $\partial \Omega_0$ and $\partial \Omega_2$. From the choice of $c$, $u \geq w_q - c$ in the neighborhood of $\partial \Omega_0$, so $w_- = u$ in the neighborhood of $\partial \Omega_0$. By Proposition 5.8, $-\frac{r+1-2l}{r} \log |q| (k = 1, \cdots, n)$ and $\log(1 - |z|^2)$ are mutually bounded on $\Omega_{P,2}$. Hence $u_{P,k}(z) \to -\infty$ as $z \to \partial \Omega_{P,2}$. So $w_- = w_q - c$ in the neighborhood of $\partial \Omega_2$. Since $w_q$ is a solution on $X \setminus Z$, $w_q - c$ is a subsolution. By Proposition 5.4 on $\Omega_2$, $w_-$ is still a subsolution. Hence on $X$, $w_-$ is a subsolution.

By Theorem 4.4, $u < w_q$ holds on $\Omega_2$, since $Z$ is nonempty. It is then clear that $w_- < w_+$. Applying Proposition 5.2 there is a $C^\infty$ solution $w$ satisfying $w_- \leq w \leq w_+$. The inequalities of $w$ are clear.

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