LOCAL WELL-POSEDNESS OF THE FULL COMPRESSIBLE NAVIER-STOKES-MAXWELL SYSTEM WITH VACUUM

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ABSTRACT. In this paper, we prove the local well-posedness of strong solutions for a compressible Navier-Stokes-Maxwell system, provided the initial data satisfy a natural compatibility condition. We do not assume the positivity of initial density, it may vanish in an open subset (vacuum) of Ω.

1. Introduction. The Navier-Stokes-Maxwell system is a coupling of the compressible Navier-Stokes equations with the Maxwell equations through the Lorentz force in three dimensional space, which is a plasma physical model describing the motion of charged particles (ions and electrons) in electromagnetic field [5], [19], [25]. The derivation of the Navier-Stokes-Maxwell system from the Vlasov-Maxwell-Boltzmann system is given in the Appendix in [5].

In this paper, we consider the following one-fluid non-isentropic compressible Navier-Stokes-Maxwell system [13] in three space dimensions as follow,

\[\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
\rho (u_t + u \nabla u) + \nabla p - \mu \Delta u - (\lambda + \mu) \nabla \theta &= (E + u \times b) \times b, \\
\rho \theta_t + \nabla \cdot (\rho \theta u) + p \nabla \theta &= -\lambda (\nabla \cdot u)^2, \\
\rho \theta E - \text{rot} b &= -E - u \times b, \\
\theta_t + \text{rot} E &= 0, \\
\n\end{align*}\]

Here, \(\rho = \rho(t,x) \geq 0\) is the electron density, \(u = u(t,x) \in \Omega \subset \mathbb{R}^3\) is the electron velocity, \(\theta = \theta(t,x)\) is the absolute temperature, \(E = E(t,x) \in \mathbb{R}^3\), \(b = b(t,x) \in \mathbb{R}^3\) for \(t > 0, x \in \mathbb{R}^3\), represent electronic and magnetic fields respectively. The \(\Omega \subset \mathbb{R}^3\) be a bounded and simply connected domain with smooth boundary \(\partial \Omega\), and \(n\) be the unit outward normal vector to \(\partial \Omega\). The viscosity coefficients \(\mu\) and \(\lambda\) of the
fluid satisfy $\mu > 0$ and $\lambda + \frac{2}{3}\mu > 0$. $D(u) := \frac{1}{2}(\nabla u + \nabla u^t)$. $e = e(t, x)$ is the internal energy, the pressure $p = p(t, x)$ of the fluid is a polytropic ideal gas, which takes the form as follow for the positive constants $C_V$ and $R$,

$$e = C_V \theta, \quad p = R \rho \theta. \tag{2}$$

The global existence of smooth solutions for small data is established, and the zero dielectric constant limit is studied by Kawashima and Shizuta [17], [18] in $\mathbb{R}^2$. The zero dielectric constant limit is then proved in $T^3$ [14], [15]. The low Mach number limit in $T^3$ is considered [20]. Both the zero dielectric constant limit and the low Mach number limit in $\Omega \subset \mathbb{R}^3$ are studied in [7], [8].

We also mention some works where the model seems a little bit different though it is also called compressible Navier-Stokes-Maxwell system. For the one-fluid non-isentropic compressible Navier-Stokes-Maxwell system in $\mathbb{R}^3$, the global existence of solutions near constant steady states is established and the time-decay rates of perturbed solutions are obtained in [23] with different right terms from those in this paper. The global existence of regular solutions to the 2D Navier-Stokes-Maxwell system is proved by using energy estimates and Brezis-Gallouet inequality, and a blow-up criterion for solutions to 3D Navier-Stokes-Maxwell system is obtained in [16]. For the isentropic compressible Navier-Stokes-Maxwell system in one space dimension, existence and uniqueness of global strong solutions with large initial data and vacuum are established [12] for the initial boundary value problem when there is initial vacuum. The global existence and large time behavior of this model have been studied by Duan [5]. The global existence of spherical symmetric classical solution to the Navier-Stokes-Maxwell system is obtained with large initial data and vacuum [10]. For the bipolar compressible Navier-Stokes-Maxwell system in $\mathbb{R}^3$, under the assumption that the initial values are close to a equilibrium solutions, asymptotic behavior of global smooth solutions to the Cauchy problem is proved in [9] without decay rate. Moreover, the essential difference between the one-fluid Navier-Stokes-Maxwell system and the bipolar compressible Navier-Stokes-Maxwell system is shown via the phenomenon on the charge transport. The decay rate of the global smooth solutions is obtained [26] based on a detailed analysis to the Green’s function of the linearized system and some elaborate energy estimates. Hou and Zhu [11] show a regularity criterion for a compressible Navier-Stokes-Maxwell system.

Enlightened by the known results on well-posedness, we investigate the initial boundary value problem to the compressible non-isentropic Navier-Stokes-Maxwell system with vacuum in three space dimensions, where the right terms are different from those in [23]. The main difficulty in this paper is how to prove the estimate (8), which will be proved in section 2. Under the assumption that the natural compatibility conditions hold, we prove the local well-posedness of the non-isentropic compressible Navier-Stokes-Maxwell system in three space dimensions with vacuum by the compactness principle.

The natural compatibility conditions are given below: there exists $(g_1, g_2) \in L^2$ such that

$$-\mu \Delta u_0 - (\lambda + \mu) \nabla \text{div} u_0 + \nabla p(\rho_0, \theta_0) - (E_0 + u_0 \times b_0) \times b_0 = \sqrt{\rho_0} g_1,$$

$$\Delta \theta_0 + \frac{\mu}{2} |D(u_0)|^2 + \lambda (\text{div} u_0)^2 + |E_0 + u_0 \times b_0|^2 = \sqrt{\rho_0} g_2. \tag{3}$$

Before stating our main results, we first give a proposition below:
Remark 2. A similar result has been proved in [6], [3], [11] for the Navier-Stokes-Fourier and MHD system. However, we find that our method here is different from those in [6], [3], [11].

Proposition 1. (Local existence). Suppose that the initial data $\rho_0, u_0, \theta_0, E_0, b_0$ satisfy $\rho_0 \geq \delta$, $\theta_0 \geq 0$ and
\[
\partial_t^k \rho(0), \quad \partial_t^k u(0), \quad \partial_t^k \theta(0), \quad \partial_t^k E(0), \quad \partial_t^k b(0) \in H^{2-k}(\Omega), \quad k = 0, 1, 2. \tag{4}
\]
Then there exists a positive time $T_3 > 0$ such that the problem (1)-(2) has a unique solution $(\rho, u, \theta, E, b)$, satisfying that $\rho \geq \delta$, $\theta \geq 0$ in $\Omega \times (0, T_3)$, and for $k = 0, 1, 2$,
\[
\partial_t^k (\rho, u, \theta, E, b) \in C([0, T_3]; H^{2-k}), \quad \partial_t^k (u, \theta) \in L^2(0, T_3; H^{3-k}). \tag{5}
\]

Remark 1. The local existence can be proved in a similar way as in [28] if $\rho \geq \frac{1}{\delta}$. Therefore, we omit the details of the proof. (4) can be replaced by $\rho_0, u_0, \theta_0, E_0, b_0 \in H^5(\Omega)$.

Now we state our main results.

Theorem 1.1. Let $\rho_0, \theta_0 \geq 0$ in $\Omega$ and $\rho_0 \in W^{1,6}, u_0 \in H^1_0 \cap H^2, \theta_0, E_0, b_0 \in H^2$ and (3) hold true. Then the problem (1)-(2) has a unique strong solution $(\rho, u, \theta, E, b)$ satisfying
\[
0 \leq \rho \in L^\infty(0, T; W^{1,6}), \quad \rho_t \in L^\infty(0, T; L^6),
\]
\[
u, \nu \leq L^\infty(0, T; H^2) \cap L^2(0, T; W^{2,6}),
\]
\[
\sqrt{\rho} u_t, \sqrt{\rho} \theta_t \in L^\infty(0, T; L^2), \quad u_t, \theta_t \in L^2(0, T; H^1),
\]
\[
E, b \in L^\infty(0, T; H^2), \quad E_t, b_t \in L^\infty(0, T; H^1)
\]
for some $T > 0$.

Remark 2. A similar result has been proved in [6], [3], [11] for the Navier-Stokes-Fourier and MHD system. However, we find that our method here is different from that in [6], [3], [11].

Define that $M(t)$ as below:
\[
M(t) := 1 + \sup_{0 \leq s \leq t} \{ \| \rho(\cdot, s) \|_{W^{1,6}} + \| \rho_t(\cdot, s) \|_{L^6} + \| u(\cdot, s) \|_{H^2}
\]
\[
+ \| \sqrt{\rho} u(\cdot, s) \|_{L^2} + \| \theta(\cdot, s) \|_{H^2} + \| \sqrt{\rho} \theta(\cdot, s) \|_{L^2} + \| E(\cdot, s) \|_{H^2}
\]
\[
+ \| E_t(\cdot, s) \|_{H^1} + \| b(\cdot, s) \|_{H^2} + \| b_t(\cdot, s) \|_{H^1} \} + \| u \|_{L^2(0, T; W^{2,6})}
\]
\[
+ \| u_t \|_{L^2(0, T; H^1)} + \| \theta \|_{L^2(0, T; W^{2,6})} + \| \theta_t \|_{L^2(0, T; H^1)}.
\]

Theorem 1.2. Let $T_3$ be the maximal time of existence for the problem (1)-(2) in the sense of Proposition 1. Then for any $t \in [0, T_3)$, it holds that
\[
M(t) \leq C_0(M_0) \exp(\sqrt{C}(M)) \tag{8}
\]
for some given nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$.

2. Proof of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Let $(\rho^\delta, u^\delta, \theta^\delta, E^\delta, b^\delta)$ satisfy $\rho^\delta \geq \delta, \theta^\delta \geq 0$ and (4), it holds that
\[
\rho^\delta \rightarrow \rho_0, \quad \text{in } W^{1,6},
\]
\[
u^\delta \rightarrow \nu_0, \quad \text{in } H^2,
\]
\[
\theta^\delta \rightarrow \theta_0, \quad \text{in } H^2,
\]
\[
E^\delta \rightarrow E_0, \quad \text{in } H^2,
\]
\[
b^\delta \rightarrow b_0, \quad \text{in } H^2, \quad \text{as } \delta \rightarrow 0.
\]
By Proposition 1, the problem (1)-(2) has a unique strong solution \((\rho^\delta, u^\delta, \theta^\delta, E^\delta, b^\delta)\) satisfying (5).

It follows [1], [4], [24], from (8), that

\[ M(t) \leq C \]

The proof is based on [4]. Assume that Theorem 1.1 holds and \(T_\delta < \infty\) is the maximal life time of existence for the solution obtained in Proposition 1. Then for any \(0 \leq t \leq \min\{T_\delta, 1\}\), we have (8), where \(M_0 \leq D_0\) for \(0 < \delta \leq 1\) (we can take \(D_0 = M_0\)). In the sequence, we choose \(D > C_0(D_0)\) and next \(T_1 \leq 1\) such that

\[ C_0(D_0) \exp(\frac{1}{4}T_1^\frac{1}{4}C(D)) < D. \]

Let \(t < \min\{T_\delta, T_1\}\). By combining the inequalities (8) and the above inequality, we have that \(M(t) \neq D\). Besides, we can assume without restriction that \(D_0 \leq D\), so that \(M \leq D\). Since the function \(M(t)\) is continuous, we obtain

\[ M(t) \leq D \text{ for } t < \min\{T_\delta, T_1\} \text{ and } 0 < \delta \leq 1. \]

Then \(T_\delta > T_1\) for \(0 < \delta \leq 1\). Otherwise, by using the above uniform estimates and applying Proposition 1 repeatedly, one can extend the time interval of existence to \([0, T_1]\), which contradicts to the maximality of \(T_\delta\). Therefore, \(M(t) \leq D\) for any \(t \in [0, T_1]\) where \(T_1\) is independent of \(0 < \delta \leq 1\). Clearly, the conclusion is also true for \(T_\delta = \infty\) by applying the same argument.

Therefore, by taking \(\delta \to 0\) and the standard compactness principle, the proof of existence is complete.

The proof of uniqueness can be finished by the very similar calculations as that in [6], [3], we omit the details here.

Therefore, by taking \(\delta \to 0\) and the standard compactness principle, the proof of existence is complete.

The proof of Theorem 1.2 is completed.

Now, we turn to prove Theorem 1.2. In other words, we prove the inequality (8).

\textbf{Proof of Theorem 1.2.} For simplicity, we drop the superscript “\(\delta\)” in \(\rho^\delta, u^\delta, \theta^\delta, E^\delta\) and \(b^\delta\). The physical constants \(C_V\) and \(R\) do not cause any essential difficulties in our arguments. Therefore, we take \(C_V = R = 1\).

First, testing the first equation in (1) by \(\rho^{q-1}\), we see that

\[ \frac{1}{q} \frac{d}{dt} \int \rho^q dx = - \int \text{div} (\rho u) \rho^{q-1} dx = \int \rho u \nabla \rho^{q-1} dx = - \frac{q-1}{q} \int \rho^q \text{div} u dx, \]

which gives

\[ \frac{d}{dt} \|\rho\|_{L^q} \leq \|\text{div} u\|_{L^\infty} \|\rho\|_{L^q}, \]

and thus

\[ \|\rho\|_{L^q} \leq \|\rho_0\|_{L^q} \exp \left( \int_0^t \|\text{div} u\|_{L^\infty} ds \right) \quad (2 \leq q \leq \infty). \]

On the other hand, using the Gagliardo-Nirenberg inequality

\[ \|\nabla u\|_{L^\infty} \leq C \|\nabla u\|_{L^2}^{\frac{1}{2}} \|u\|_{W^{2,6}}^{\frac{1}{2}}, \]

we have

\[ \int_0^t \|\nabla u\|_{L^\infty} ds \leq \sqrt{t} C(M), \]

whence

\[ \|\rho\|_{L^q} \leq C_0(M_0) \exp(\sqrt{t} C(M)). \] (9)
Taking $\nabla$ to the first equation in (1), testing by $|\nabla\rho|\nabla\rho$, we find that
\[
\frac{d}{dt} \|\nabla\rho\|_{L^6} \leq C \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^6} + C \|\rho\|_{L^\infty} \|\nabla \div u\|_{L^6},
\]
which gives
\[
\|\nabla\rho\|_{L^6} \leq C \left( \|\nabla\rho_0\|_{L^6} + \int_0^t \|\rho\|_{L^\infty} \|\nabla \div u\|_{L^6} \, dx \right) \exp \left( \int_0^t \|\nabla u\|_{L^\infty} \, ds \right) 
\leq C \left( 1 + \int_0^t \|u\|_{W^{2,\infty}} \|\rho\|_{L^\infty} \right) \exp(\sqrt{t}C(M)) 
\leq C_0(M_0) \exp(\sqrt{t}C(M)).
\]

It is easy to see that
\[
\|u\|_{H^1} = \left\| u_0 + \int_0^t u_1 \, ds \right\|_{H^1} , \ \leq \|u_0\|_{H^1} + \int_0^t \|u_t\|_{H^1} \, ds \leq C_0 + \sqrt{t}C(M).
\]  
Similarly, we find that
\[
\|\theta\|_{H^1} \leq C_0 + \sqrt{t}C(M), \ |E|_{H^1} \leq C_0 + tC(M), \ |b|_{H^1} \leq C_0 + tC(M).
\]
Applying $\partial_t$ to the second equation in (1), testing by $u_t$ and using the first equation in (1), we discover that
\[
\frac{1}{2} \frac{d}{dt} \int \rho |\nabla u|^2 \, dx + \int (\mu |\nabla u|^2 + (\lambda + \mu)(\div u)^2) \, dx 
= \int \rho \div u \, dx - \int \rho_t |u|^2 \, dx - \int \rho_t u \cdot \nabla u \cdot u \, dx - \int \rho u_t \cdot \nabla u \cdot u \, dx 
+ \int ((E_t + u_t \times b + u \times b_t) \times b + (E + u \times b) \times b_t) u_t \, dx 
\leq \ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5.
\]

We bound $\ell_1, \ldots, \ell_5$ as follows.

\[
\ell_1 = \int (\rho \theta + \rho \theta_t) \div u \, dx \leq \left( \|\rho\|_{L^2} \|\theta\|_{L^\infty} + \|\sqrt{\rho}\|_{L^\infty} \|\sqrt{\rho} \theta_t\|_{L^2} \right) \|\nabla u\|_{L^2} 
\leq C(M) \|\nabla u_t\|_{L^2};
\]

\[
\ell_2 = \int \div (\rho u) |u_t|^2 \, dx = -\int \rho \nabla |u_t|^2 \, dx \leq \|\sqrt{\rho}\|_{L^\infty} \|u\|_{L^\infty} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u_t\|_{L^2} 
\leq C(M) \|\nabla u_t\|_{L^2};
\]

\[
\ell_3 \leq \|\rho_t\|_{L^2} \|u\|_{L^6} \|\nabla u\|_{L^6} \|u_t\|_{L^6} \leq C(M) \|u_t\|_{L^6} \leq C(M) \|u_t\|_{L^6} \leq C(M) \|\nabla u_t\|_{L^2};
\]

\[
\ell_4 \leq \|\sqrt{\rho}\|_{L^\infty} \|\sqrt{\rho} u_t\|_{L^2} \|u_t\|_{L^6} \|\nabla u\|_{L^3} \leq C(M) \|u_t\|_{L^6} \leq C(M) \|\nabla u_t\|_{L^2};
\]

\[
\ell_5 = \int [ (E_t + u \times b_t) \times b + (E + u \times b) \times b_t ] u_t \, dx - \int |u_t \times b|^2 \, dx 
\leq \left( \|E_t\|_{L^2} + \|u\|_{L^\infty} \|b_t\|_{L^2} \right) \|b\|_{L^\infty} \|u_t\|_{L^2} + \left( \|E\|_{L^\infty} + \|u\|_{L^\infty} \|b\|_{L^\infty} \right) \|b_t\|_{L^2} \|u_t\|_{L^2} 
\leq C(M) \|u_t\|_{L^2} \leq C(M) \|\nabla u_t\|_{L^2}.
\]

Inserting the above estimates into (13) and integrating it over $(0, t)$, we have
\[
\int_0^t \int |\nabla u_t|^2 \, dx \, ds \leq C_0 + \sqrt{t}C(M).
\]  

(14)
By the $H^2$-theory of elliptic system, using the first equation and second one in (1), (9), (10), (11) and (12), (14), we derive

$$
\|u\|_{H^2} \leq C\|\mu \Delta u + (\lambda + \mu) \nabla \text{div } u\|_{L^2} = C\|\rho u_t + \rho u \cdot \nabla u + \nabla \rho - (E + u \times b) \times b\|_{L^2} \\
\leq C\|\sqrt{\rho}\|_{L^\infty} \|\sqrt{\rho} u_t\|_{L^2} + C\|\rho\|_{L^\infty}\|u\|_{L^\infty}\|\nabla u\|_{L^3} \\
+ C\|\theta\|_{L^6}\|\nabla \rho\|_{L^3} + C\|\rho\|_{L^\infty}\|\nabla \theta\|_{L^2} + C\|E\|_{L^6}\|b\|_{L^3} + C\|u\|_{L^6}\|b\|_{L^6}^2,
$$

which gives

$$
\|u\|_{H^2} \leq C_0(M_0) \exp(\sqrt{t}C(M)).
$$

Here we used the Gagliardo-Nirenberg inequality

$$
\|\nabla u\|_{L^2}^2 \leq C\|\nabla u\|_{L^6}\|u\|_{H^2}.
$$

It follows from the first equation in (1), (9), (10) and (15) that

$$
\|\rho_t\|_{L^6} = \|u \nabla \rho + \rho \text{div } u\|_{L^6} \\
\leq \|u\|_{L^\infty}\|\nabla \rho\|_{L^6} + \|\rho\|_{L^\infty}\|\text{div } u\|_{L^6} \\
\leq C_0(M_0) \exp(\sqrt{t}C(M)).
$$

Using the fourth equation and the seventh one in (1), we observe that

$$
\text{rot } b \times n = 0 \text{ on } \partial \Omega \times (0, \infty).
$$

Using the fifth equation and the seventh one in (1), and (16), we infer that

$$
\text{rot }^2 E \times n = 0 \text{ on } \partial \Omega \times (0, \infty).
$$

Taking \(\text{rot }^2\) to the fourth equation and the fifth equation in (1), testing the results by \(\text{rot }^2 E\) and \(\text{rot }^2 b\), respectively, summing up the results and using (17), we get

$$
\frac{1}{2} \frac{d}{dt} \int (|\text{rot }^2 E|^2 + |\text{rot }^2 b|^2) dx + \int |\text{rot }^2 E|^2 dx \\
= - \int \text{rot }^2 (u \times b) \text{rot }^2 E dx \leq C\|u\|_{H^2}\|b\|_{H^2}\|E\|_{H^2} \leq C(M).
$$

Integrating the above estimate over \((0, t)\), we have

$$
\|\text{rot }^2 E(\cdot, t)\|_{L^2} + \|\text{rot }^2 b(\cdot, t)\|_{L^2} \leq C_0 + tC(M).
$$

(18)

Taking \(\nabla \text{div}\) to the fourth equation in (1) and testing by \(\nabla \text{div}\), we have

$$
\frac{1}{2} \frac{d}{dt} \int |\nabla \text{div } E|^2 dx + \int |\nabla \text{div } E|^2 dx = - \int \nabla \text{div } (u \times b) \nabla \text{div } E dx \\
\leq C\|u\|_{H^2}\|b\|_{H^2}\|E\|_{H^2} \leq C(M),
$$

which yields

$$
\|\nabla \text{div } E(\cdot, t)\|_{L^2} \leq C_0 + tC(M).
$$

(19)

Taking \(\partial_t \text{rot}\) to the fourth equation and the fifth equation in (1), testing by \(\partial_t \text{rot } E\) and \(\partial_t \text{rot } b\), respectively, summing up the results and using (16), we obtain

$$
\frac{1}{2} \frac{d}{dt} \int (|\text{rot } E_t|^2 + |\text{rot } b_t|^2) dx + \int |\text{rot } E_t|^2 dx
$$

...
which leads to
\[ \forall w \in (1), \text{ we obtain} \]
\[ \text{which implies} \]
\[ \| \text{rot} E_t (\cdot, t) \|_{L^2} + \| \text{rot} b_t (\cdot, t) \|_{L^2} \leq C_0 + \sqrt{C} M. \]  

(20)

Applying \( \partial_t \text{div} \) to the fourth equation in (1) and testing by \( \partial_t \text{div} E \), we have
\[ \frac{1}{2} \frac{d}{dt} \int (\text{div} E_t)^2 dx + \int (\text{div} E_t)^2 dx \]
\[ = - \int \text{div} (u_t \times b + u \times b_t) \text{div} E_t dx \leq C(M)(\| \nabla u_t \|_{L^2} + 1), \]

which leads to
\[ \| \text{div} E_t (\cdot, t) \|_{L^2} \leq C_0 + \sqrt{C} M. \]  

(21)

Now we recall the following Poincaré inequality [22]:
\[ \| w \|_{L^2} \leq C(\| \text{div} w \|_{L^2} + \| \text{rot} w \|_{L^2}), \]

(22)

for any \( w \in H^1(\Omega) \) with \( w \cdot n = 0 \) or \( w \times n = 0 \) on \( \partial\Omega \).

We will also use the following two inequalities [2], [7]:
\[ \| w \|_{H^s(\Omega)} \leq C(\| \text{div} w \|_{H^{s-1}(\Omega)} + \| \text{rot} w \|_{H^{s-1}(\Omega)} + \| w \cdot n \|_{H^{s-\frac{1}{2}}(\partial\Omega)} + \| w \|_{H^{s-\frac{1}{2}}(\partial\Omega)}), \]

(23)

\[ \| w \|_{H^s(\Omega)} \leq C(\| \text{div} w \|_{H^{s-1}(\Omega)} + \| \text{rot} w \|_{H^{s-1}(\Omega)} + \| w \times n \|_{H^{s-\frac{1}{2}}(\partial\Omega)} + \| w \|_{H^{s-\frac{1}{2}}(\partial\Omega)}), \]

(24)

for any \( w \in H^s(\Omega) \) and \( s \geq 1 \).

It follows from (12), (18), (19), (20), (21), (22), (23) and (24) that
\[ \| E (\cdot, t) \|_{H^2} + \| b (\cdot, t) \|_{H^2} + \| E_t (\cdot, t) \|_{H^1} + \| b_t (\cdot, t) \|_{H^1} \]
\[ \leq C_0(M_0) \exp(\sqrt{C} M). \]

(25)

Applying \( \partial_t \) to the third equation in (1), testing by \( \theta_t \) and using the first equation in (1), we obtain
\[ \frac{1}{2} \frac{d}{dt} \int \rho \theta_t^2 dx + \int |\nabla \theta_t|^2 dx \]
\[ = - \int \rho_t \theta_t^2 dx - \int \rho u_t \cdot \nabla \theta \cdot \theta_t dx - \int \rho u_t \cdot \nabla \theta \cdot \theta_t dx - \int \rho_t \text{div} u dx \]
\[ - \int p \theta_t \text{div} u dx + \int \theta_t \partial_t |2 \mu |D(u)|^2 + \lambda (\text{div} u)^2| dx + \int \theta_t \partial_t |E + u \times b|^2 dx \]
\[ =: \sum_{i=6}^{12} \ell_i. \]  

(26)
We bound $\ell_6, \cdots, \ell_{12}$ as follows.

\[
\ell_6 = \int \theta_i^2 \div (\rho u) dx = - \int \rho u \nabla \theta_i^2 dx \\
\leq \|\sqrt{\rho} \|_{L^\infty} \|\sqrt{\theta}_i \|_{L^2} \|\nabla \theta_i \|_{L^2} \leq C(M) \|\nabla \theta_i \|_{L^2};
\]

\[
\ell_7 \leq \|\rho_t \|_{L^6} \|u \|_{L^6} \|\nabla \theta \|_{L^3} \|\theta_i \|_{L^3} \leq C(M) \|\theta_i \|_{L^3};
\]

\[
\ell_8 \leq \|\sqrt{\rho} \|_{L^\infty} \|\sqrt{\rho u} \|_{L^2} \|\nabla \theta \|_{L^2} \|\theta_i \|_{L^3} \leq C(M) \|\theta_i \|_{L^3};
\]

\[
\ell_9 = - \int (\rho_\theta + \rho \theta_t) \div u dx \\
\leq \|\rho_t \|_{L^2} \|\theta \|_{L^\infty} \|\theta_i \|_{L^3} \|\nabla u \|_{L^6} + \|\sqrt{\rho} \|_{L^\infty} \|\sqrt{\theta}_i \|_{L^2} \|\theta_i \|_{L^3} \|\div u \|_{L^6} \\
\leq C(M) \|\theta_i \|_{L^3};
\]

\[
\ell_{10} \leq \|\sqrt{\rho} \|_{L^\infty} \|\sqrt{\theta}_i \|_{L^2} \|\theta \|_{L^\infty} \|\div u \|_{L^2} \leq C(M) \|\nabla u \|_{L^2};
\]

\[
\ell_{11} \leq C \|\nabla u \|_{L^6} \|\nabla u_t \|_{L^2} \|\theta_i \|_{L^3} \leq C \|\nabla u \|_{L^6} \|\nabla u_t \|_{L^2}^2 + \epsilon \|\theta_i \|_{L^3}^2 \\
\leq \|\nabla u \|_{L^6}^2 \|\nabla u_t \|_{L^2}^2 + C\epsilon \|\theta_i \|_{L^2}^2 + \epsilon \|\nabla \theta_i \|_{L^2}^2 \\
\leq \|\nabla u \|_{L^6}^2 \|\nabla u_t \|_{L^2}^2 + C\epsilon \|\nabla \theta_i \|_{L^2}^2 + C(M) \\
\leq \|\nabla u \|_{L^6}^2 \|\nabla u_t \|_{L^2}^2 + \frac{1}{8} \|\nabla \theta_i \|_{L^2}^2 + C(M)
\]

by taking $\epsilon$ small enough.

Here we have used the Poincaré inequality [21]:

\[
\left\| w - \frac{\int \rho wdx}{\int \rho dx} \right\|_{L^2} \leq C \|\nabla w\|_{L^2},
\]

\[
\ell_{12} = 2 \int \theta_i^2 (E_t + u_t \times b + u \times b_t)(E + u \times b) dx \\
= 2 \int \theta_i^2 (E_t + u \times b_t)(E + u \times b) dx + 2 \int \theta_i (u_t \times b)(E + u \times b) dx \\
\leq 2 \|\theta_i \|_{L^2} \|E_t \|_{L^2} + \|u \|_{L^\infty} \|b_t \|_{L^2} \|E + u \times b\|_{L^\infty} \\
+ 2 \|\theta_i \|_{L^2} \|u_t \|_{L^5} \|b \|_{L^\infty} \|E + u \times b\|_{L^3} \|u \|_{L^6} \\
\leq C(M) \|\theta_i \|_{L^2} + 2 \|b \|_{L^6} \|E + u \times b\|_{L^3} \|\theta_i \|_{L^3} \|u \|_{L^6} \\
\leq \|b \|_{L^6}^2 \|E + u \times b\|_{L^2}^2 \|\nabla u_t \|_{L^2}^2 + \frac{1}{8} \|\nabla \theta_i \|_{L^2}^2 + C(M). 
\]

Inserting the above estimates into (26) and integrating it over $(0, t)$, we arrive at

\[
\int \rho \theta_i^2 dx + \int_0^t \|\nabla \theta_i \|_{L^2}^2 dx ds \leq C_0 \exp(\sqrt{t} C(M)).
\] (27)

By the $H^2$-theory of Poisson equation, using the first equation and the third one in (1), (9), (27), (15), (12) and (27), we have

\[
\|\theta\|_{H^2} \leq C \|\Delta \theta\|_{L^2} \\
= C \|\rho u + \rho \cdot \nabla \theta + \rho \div u - 2\mu|D(u)|^2 - \lambda (\div u)^2 - (E + u \times b)^2\|_{L^2} \\
\leq C \|\sqrt{\rho} \|_{L^\infty} \|\sqrt{\theta}_i \|_{L^2}^2 + C \|\rho\|_{L^\infty} \|u\|_{L^\infty} \|\nabla \theta\|_{L^2}^2
\]
+ C\|\rho\|_{L^\infty} \|\theta\|_{L^6} \|\text{div } u\|_{L^3} + C\|\nabla u\|_{L^4}^2 + C\|E + u \times b\|_{L^4}^2 \\
leq C_0(M_0) \exp(\sqrt{t}C(M)).

By the $W^{2,6}$-theory of elliptic system, using the first and second equation in (1), (9), (10), (15), (25), (14) and (27), we have

\[ \|u\|_{W^{2,6}} \leq C\|\mu \Delta u + (\lambda + \mu)\nabla \text{div } u\|_{L^6} \]
\[ = C\|\rho u_t + \mu u \cdot \nabla u + \nabla p - (E + u \times b) \times b\|_{L^6} \]
\[ \leq C\|\rho\|_{L^\infty} \|u_t\|_{L^6} + C\|\rho\|_{L^\infty} \|u\|_{L^\infty} \|\nabla u\|_{L^6} + C\|\rho\|_{L^\infty} \|\nabla \theta\|_{L^6} + C\|\theta\|_{L^\infty} \|\nabla \rho\|_{L^6} + C\|E\|_{L^6} + \|\|u\|_{L^\infty} \|b\|_{L^6}\|b\|_{L^\infty}, \]

which gives

\[ \|u\|_{L^2(0,t;W^{2,6})} \leq C_0(M_0) \exp(\sqrt{t}C(M)). \]

Similarly, we conclude that

\[ \|\theta\|_{W^{2,6}} \leq C\|\Delta \theta\|_{L^6} \]
\[ = C\|\rho \theta_t + \mu u \cdot \nabla \theta + p \text{div } u - 2\mu |D(u)|^2 - \lambda (\text{div } u)^2 - |E + u \times b|^2\|_{L^6} \]
\[ \leq C\|\rho\|_{L^\infty} \|\theta_t\|_{L^6} + C\|\rho\|_{L^\infty} \|\nabla \theta\|_{L^6} + C\|\rho\|_{L^\infty} \|\theta\|_{L^\infty} \|\text{div } u\|_{L^6} \]
\[ + C\|\nabla u\|_{L^1}^2 + C\|E + u \times b\|_{L^6} \|E + u \times b\|_{L^6} \]
\[ \leq C_0(M_0) \exp(\sqrt{t}C(M))(1 + \|\nabla \theta_t\|_{L^2}^2) + C\|\nabla u\|_{L^1}^2 \]
\[ \leq C_0(M_0) \exp(\sqrt{t}C(M))(1 + \|\nabla \theta_t\|_{L^2}^2) + C\|u\|_{L^\infty} \|u\|_{W^{2,6}}, \]

which leads to

\[ \|\theta\|_{L^2(0,t;W^{2,6})} \leq C_0(M_0) \exp(\sqrt{t}C(M)). \]

This completes the proof. \(\square\)

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