Scaling functions applied to three-body recombination of $^{133}$Cs atoms

L. Platter$^{1,2,∗}$ and J. R. Shepard$^{3,†}$

$^1$Department of Physics, The Ohio State University, Columbus, OH 43210
$^2$Department of Physics and Astronomy, Ohio University, Athens, OH 45701, USA
$^3$Department of Physics, University of Colorado, Boulder, CO 80309, USA

(Dated: May 7, 2009)

Abstract

We demonstrate the implications of Efimov physics in the recently measured recombination rate of $^{133}$Cs atoms. By employing previously calculated results for the energy dependence of the recombination rate of $^4$He atoms, we obtain three independent scaling functions that are capable of describing the recombination rates over a large energy range for identical bosons with large scattering length. We benchmark these and previously obtained scaling functions by successfully comparing their predictions with full atom-dimer phase shift calculations with artificial $^4$He potentials yielding large scattering lengths. Exploiting universality, we finally use these functions to determine the 3-body recombination rate of $^{133}$Cs atoms with large positive scattering length, compare our results to experimental data obtained by the Innsbruck group and find excellent agreement.

PACS numbers: 21.45.+v,34.50.-s,03.75.Nt

Keywords: Renormalization group, limit cycle, cold atoms

$^∗$Electronic address: lplatter@mps.ohio-state.edu
$^{†}$Electronic address: James.Shepard@Colorado.Edu
I. INTRODUCTION

In atomic physics the term universality refers to phenomena which are a result of a two-body scattering length $a$ much larger than the range $R$ of the underlying potential and do not depend on any further parameters describing the two-body interaction. The non-relativistic three-body system also exhibits universal properties if $a \gg R$, but an additional three-body parameter is needed for the theoretical description of observables. Therefore, one three-body observable can be used (e.g. the minimum of the three-body recombination rate $a_{s0}$) to predict all other low-energy observables of such systems. A particularly interesting signature of universality in the three-body system is a tower of infinitely many bound states (Efimov states) in the limit $a = \pm \infty$ with an accumulation point at the scattering threshold and a geometric spectrum:

$$E_T^{(n)} = (e^{-2\pi/s_0})^{n-n_*} \hbar^2 \kappa^2/m,$$

where $\kappa_*$ is the binding wavenumber of the branch of Efimov states labeled by $n_*$. The three-body system displays therefore discrete scaling symmetry in the universal limit with a scaling factor factor $e^{\pi/s_0}$. In the case of identical bosons, $s_0 \approx 1.00624$ and the discrete scaling factor is $e^{\pi/s_0} \approx 22.7$. These results were first derived in the 1970’s by Vitaly Efimov and were rederived in the last decade in the framework of effective field theories (EFT).

Recently, experimental evidence for Efimov physics was found by the Innsbruck group. Using a magnetic field to control the scattering length via a Feshbach resonance, they measured the recombination rate of cold $^{133}$Cs atoms and observed a resonant enhancement in the three-body recombination rate at $a \approx -850a_0$ which occurs because an Efimov state is close to the 3-atom threshold for that value of $a$. The three-body recombination rate for atoms with large scattering length at non-zero temperature has been calculated with a number of different models or based on the universality of atoms with large scattering lengths. However, a striking way to demonstrate universality is to describe observables of one system with information which has been extracted from a completely different system. In, the authors considered Efimov’s radial laws which parameterize the three-atom S-matrix in terms of six real universal functions which depend only on a dimensionless scaling variable, $x = (ma^2E/\hbar^2)^{1/2}$, and phase factors which only contain the three-body parameter. In this work, simplifying assumptions justified over a restricted
range of $x$ were made to reduce the six universal functions required to parameterize the three-body recombination rate to just a single function. This function was then extracted from microscopic calculations of the recombination rates for $^4$He atoms by Suno et al.\cite{12}. In a recent paper, Shepard\cite{13} calculated the recombination rates from atom-dimer elastic scattering phase shifts for four different $^4$He potentials (the so-called HDFB, TTY, LM2M2 and HFDB3FCII potentials) and was able to obtain two universal functions.

Here, we relax all but one of the simplifying assumptions made in\cite{11} and extract a set of three independent universal functions capable of parameterizing the three-body recombination rate over a wide range of energies. We test the performance of these universal functions using “data” generated from phase shift calculations\cite{13} employing artificial short-range $^4$He potentials. Finally, we use the new universal functions to calculate the scattering length and temperature dependent recombination rate for $^{133}$Cs atoms as measured by the Innsbruck group\cite{5} and comment on our results.

II. THREE-BODY RECOMBINATION

Three-body recombination is a process in which three atoms collide to form a diatomic molecule (dimer). If the scattering length is positive and large compared to the range of the interaction, we have to differentiate between deep and shallow dimers. Shallow dimers have an approximate binding energy of $E_{\text{shallow}} \simeq \hbar^2/(ma^2) \ll \hbar^2/(mR^2)$. The binding energy of deep dimers cannot be expressed in terms of the effective range parameters and $E_{\text{deep}} \gtrsim \hbar^2/(mR^2)$. If the underlying interaction supports deep bound states, recombination processes can occur for either sign of $a$. In a cold thermal gas of atoms, recombination processes lead to a change in the number density of atoms $n_A$

$$\frac{d}{dt} n_A = -L_3 n_A^3,$$

where $L_3$ denotes the loss rate constant. The recombination coefficient, to which $L_3$ is proportional, can be decomposed into

$$K_3(E) = K_{\text{shallow}}(E) + K_{\text{deep}}(E),$$

and the recombination rate into the shallow dimer can be further decomposed into contributions from the channels in which the total orbital angular momentum of the three
atoms has a definite quantum number $J$ according to

$$K_{\text{shallow}}(E) = \sum_{J=0}^{\infty} K^{(J)}(E). \quad (4)$$

For now, let us consider recombination via the shallow dimer only. If the collision energy $E$ is small compared to the natural energy scale $\hbar^2/(mR^2)$, the recombination rate $K_{\text{shallow}}(E)$ is a universal function of the collision energy $E$, scattering length $a$ and three-body parameter $a_{s0}$. The universal function depends on the dimensionless scaling variable defined as

$$x = (ma^2E/\hbar^2)^{1/2}. \quad (5)$$

For $J > 0$ the recombination rate does not depend on the three-body parameter $a_{s0}$ and the implications of universality are therefore particularly simple, namely

$$K^{(J)} = f_J(x)\hbar a^4/m. \quad (6)$$

However, $K^{(0)}$ depends log-periodically on $a_{s0}$ (this is the signature of Efimov physics!) and is related to the S-matrix for elastic atom-dimer scattering through

$$K^{(0)}(E) = \frac{k}{x^4}(1 - |S_{AD,AD}|^2), \quad (7)$$

Efimov’s radial law then gives the dependence on complex universal functions and the three-body parameter $a_{s0}$ which defines the scattering length for which the recombination rate has a minimum as

$$S_{AD,AD} = s_{22}(x) + \frac{s_{12}^2(x)e^{2is_{0}\ln(a/a_{s0})}}{1 - s_{11}(x)e^{2is_{0}\ln(a/a_{s0})}}. \quad (8)$$

The functions $s_{11}$ and $s_{12}$ are known at threshold

$$s_{11}(0) = -e^{-2\pi s_{0}},$$
$$s_{12}(0) = \sqrt{1 - e^{-4\pi s_{0}}e^{i\delta_{\infty}}},$$
$$s_{22}(0) = e^{2i\delta_{\infty}}e^{-2\pi s_{0}}, \quad (9)$$

with $\delta_{\infty} = 1.737$. It follows that $|s_{11}(0)| \simeq 0.002$. The first simplifying assumptions being made in [11] was that this function remains small (i.e.; $\ll 1$) for all $x$ and can be ignored. Then the energy dependent recombination rate can be written as

$$K^{(0)}(E) = \frac{144\sqrt{3}\pi^2}{x^4} \left[ 1 - (r_{22}^2 - r_{12}^4 + 2r_{22}r_{12}^2\cos[\Phi + 2s_{0}\ln(a/a_{s0})]) \right] \frac{\hbar a^4}{m}, \quad (10)$$
where we have set \( s_{ij} = r_{ij} \exp(i\phi_{ij}) \) and \( \Phi = \phi_{22} - 2\phi_{12} \). Under the assumption that \( s_{11} \) can be neglected the recombination rate depends therefore on the three real-valued function \( r_{12}(x), r_{22}(x) \) and \( \Phi(x) \). It is worth noting that the expression in Eq. (10) is symmetric under exchange of \( r_{12} \) and \( r_{22} \). However, the threshold conditions in Eq. (9) can be used to attribute the correct fit solutions to the universal function.

As also discussed in Ref. [11], the effects of deep dimers can easily be incorporated through one additional parameter \( \eta^* \) by making the substitution

\[
\ln a^* \rightarrow \ln a^* - i\eta^*/s_0 \quad (11)
\]

in, e.g. Eq. (10). Employing unitarity the resulting effect on the recombination into shallow dimers can be written as [17]

\[
K^{(0)}_{\text{shallow}}(E) = \frac{144\sqrt{3\pi^2}}{x^4}\left(1 - |s_{22}(x) + s_{12}(x) e^{2i\eta^*} - 2\eta^*|s_{12}(x)|^2 \right) \frac{\hbar a^4}{m}. \tag{12}
\]

Note that in deriving this expression we assumed again that \( s_{11} \approx 0 \). In the same manner one can derive an expression for the recombination rate into deep dimers

\[
K_{\text{deep}}(E) = \frac{144\sqrt{3\pi^2}}{x^4}(1 - e^{-4\eta^*})(1 - |s_{12}(x)|^2) \frac{\hbar a^4}{m}. \tag{13}
\]

III. ALTERNATIVE PARAMETERIZATIONS

Starting with S-matrix element for 3-atom to dimer-atom scattering, it was shown in [11] that under the assumption \( s_{11} = 0 \) the recombination rate can be written as

\[
K^{(0)}(E) = C_{\text{max}} \left| \sin[s_0 \ln\left(\frac{a}{a^*}\right)](1 + h_1(x) + ih_3(x)) \\
+ \cos[s_0 \ln\left(\frac{a}{a^*}\right)](h_2(x) + ih_4(x)) \right|^2 \frac{\hbar a^4}{m}, \tag{14}
\]

where \( C_{\text{max}} \approx 67.1 \) and the \( h_i \) are real-valued functions of \( x \). Additionally, it was assumed that the imaginary part of the above amplitude can be neglected

\[
K^{(0)}(E) = C_{\text{max}} \sin[s_0 \ln(a/a^*)](1 + h_1(x)) + \cos[s_0 \ln(a/a^*)]h_2(x) |^2 \frac{\hbar a^4}{m}. \tag{15}
\]

This is well justified by direct calculations of the \( J = 0 \) recombination rates for \(^4\)He atoms which display pronounced minima at approximately \( E_{\text{breakup}} \approx 20 \text{ mK} \) [12] and which can
be explained by this assumption. Then the functions $h_3$ and $h_4$ can be set to 0 in Eq. (10). The resulting expressions were employed in [13] to extract $h_1$ and $h_2$ for $x < 1.1$. Although $h_1$ and $h_2$ were determined by fitting to values of $K^{(0)}(E)$ calculated using just two of the four atom-atom potentials considered, they were found to accurately account for the results for all 4 potentials as expected from universality. We have recalculated the $h$-functions using the results for the three-body recombination obtained using the LM2M2 and HFDB3FCII potentials and have fitted a polynomial to our results over the energy range $0 < x < 1.2$

$$h_1(x) = -0.0234437x + 0.0550298x^2 - 1.03776x^3 + 1.18985x^4 - 0.471592x^5,$$

$$h_2(x) = 0.0338266x - 0.233836x^2 + 0.182564x^3 - 0.0895055x^4 + 0.0461793x^5. \quad (16)$$

The functions are displayed in Fig. [1]

The effect of deep dimers on the recombination rate into the shallow dimer can easily be incorporated by making the substitution $\ln a_{s0} \rightarrow \ln a_{s0} - i \eta_s/s_0$ in Eq. (13)

$$K^{(0)}(E) = C_{\max} \left[ \cosh^2 \eta_s \left( \sin[s_0 \ln(a/a_{s0})](1 + h_1(x)) + \cos[s_0 \ln(a/a_{s0})]h_2(x) \right)^2 
+ \sinh^2 \eta_s \left( \cos[s_0 \ln(a/a_{s0})](1 + h_1(x)) - \sin[s_0 \ln(a/a_{s0})]h_2(x) \right)^2 \right] \frac{\hbar a^4}{m}. \quad (17)$$

To take the effects of the recombination rate into deep dimers into account it was assumed in [11] that $K_{\text{deep}}(E)$ is a function varying slowly with energy and that it can therefore be approximated with

$$K_{\text{deep}} = \frac{C}{4} (1 - e^{-4\eta_s}) \frac{\hbar a^4}{m}. \quad (18)$$
FIG. 2: The exact recombination rates and the corresponding results obtained with scaling (solid lines) and universal functions (dashed lines) of the HFDB (circles) and TTY (triangles) potentials.

IV. EXTRACTION OF THE UNIVERSAL FUNCTIONS

By fitting Eq. (10) to the recombination rates of all four $^4$He potentials, we were able to determine the functions $r_{12}(x)$, $r_{22}(x)$ and $\Phi(x)$. Our results are smooth functions for $x > 0.2$ and the radial functions approach the known threshold values from Eq. (9) for decreasing $x$. For $x < 0.2$, we are not able to find a reliable fit which is indicated by the rapid variation of the function $\Phi$ in Fig. 3 in this region.

To display the qualities of our fit we compare the exact recombination rates obtained with the TTY and HDFB potentials to the rates calculated with the newly obtained universal functions. These results for these functions are displayed in Fig. 2. This figure contains also the recombination rate obtained with the $h$-functions. While the new set of universal functions seem to provide slightly better results for the HFDB potential at larger energies, the $h$-functions perform equally well for these potentials at lower energies.

To test our new parameterizations we have generated three artificial potentials (which we call I, II and III) characterized by different three-body parameters $a_{*0}$ (with $a/a_{*0} = 1.384, 1.188$ and $1.780$, respectively) but having approximately the same ratio of $R/a$ as the real $^4$He potentials used in this work. We have calculated the recombination rates for these potentials and use these results to benchmark our universal functions. Our results are displayed in Fig. 4. We find that the new set of functions is capable of describing the recom-
FIG. 3: The universal functions $r_{12}, r_{22}$ and $\Phi$ as function of $x$. 
FIG. 4: The exact recombination rates and the corresponding results obtained with the $h$-functions (solid lines) and the universal functions (dashed lines) for the potentials I (circles), II (squares) and III (triangles) potentials.

Combination rates of these potentials over a relatively large range of $x$. Again we benchmark also the rates obtained with $h_1$ and $h_2$ and find that this set of scaling functions describes the exact results better at $x < 1$ than the scaling functions $r_{12}(x)$, $r_{22}(x)$ and $\Phi(x)$. This is surprising at first sight since one certainly expects to obtain a better description of the recombination rate with three instead of two functions. We speculate that the functional form in Eq. (10) results in stronger constrains on the universal functions than Eq. (14) does on the $h$ functions. All the potentials, however, contain finite range effects which are not accounted for in Eq. (8). It is therefore very likely that better fits – using the same approximation – can be obtained from recombination rates calculated in the exact zero-range limit.

V. RESULTS FOR CESIUM

In the previous section we found that we can obtain a very good overall description of the recombination rate of systems with a large scattering length if we employ the functions $h_1$ and $h_2$ for energies smaller than $E_{\text{shallow}}$ and the universal function $r_{12}(x)$, $r_{22}(x)$ and $\Phi(x)$ for energies larger than $E_{\text{shallow}}$. Using these functions at energies close to the minimum in the recombination guarantees a more appropriate treatment of the effect of deep dimers on
FIG. 5: The 3-body recombination length $\rho_3$ for $^{133}$Cs for $a_{*0} = 210a_0$ and three different values of the parameter $\eta_\ast:0$ (solid line), 0.01 (dashed line), and 0.06 (dotted lines) plotted together with the experimental results of the Innsbruck experiment (triangles).

The recombination rate, which are expected to have the largest effect in this region.

The form of the functions $f_J(x)$ in Eq. (6) and therefore the contribution to the recombination from channels with higher total angular momentum $J$ has been previously analyzed in [11, 13], we thus take these channels into account by using appropriate parameterizations for the functions $f_J(x)$. $^{133}$Cs atoms can recombine into deep and shallow dimers. As mentioned above, a deep dimer is so strongly bound that it cannot be described within the EFT for short-range interactions as the binding energy is larger than $\hbar^2/2mR^3$. We account for such processes by letting $\ln a_{*0} \rightarrow \ln a_{*0} - i\eta_\ast/s_0$ as also discussed above. We then calculate the temperature dependent recombination rate by calculating

$$\alpha(T) = \frac{\int_0^\infty dE E^2 e^{-E/(k_B T)} K_3(E)}{6 \int_0^\infty dE E^2 e^{-E/(k_B T)}}.$$  

The weight factor $E^2$ comes from using hyperspherical variables for the Jacobi momenta.

In Fig. 5 we display our results for the recombination length $\rho_3 = \left(\frac{mK_3}{\sqrt{3}\hbar}\right)^{1/4}$ of $^{133}$Cs atoms. It can be seen that the results agree very well with the experimental results obtained by the Innsbruck group at $T = 200$ nK.
VI. SUMMARY

In this paper we have used the results from different $^4$He atom-atom potentials to extract and to test the predictive power of universal functions. In doing so, we have relaxed all but one simplifying assumptions which was made in previous work [11, 13]. We have determined a third universal scaling function which allows for a description of the three-body recombination rate of systems with large scattering length over a greater range of breakup energies.

We have tested the quality of our parameterizations with artificial finite range potentials which are appreciably different from the original Helium potentials but which display universal effects in three-body sector. We have found that our three real universal functions can describe the recombination of these artificial potentials reasonably well which gives further evidence that the assumptions made in [11] were well justified. We also found, however, that the previously calculated scaling functions $h_1$ and $h_2$ give an overall better description of the recombination rate for energies $E < E_{\text{shallow}}$. The scaling functions $h_1$ and $h_2$ which can be represented analytically with a simple polynomial fit given in Eq. (16) are therefore a useful tool to test recombination rate calculations for systems with large scattering length.

Finally, we have used both sets of universal functions together to compute the recombination length for $^{133}$Cs atoms for different values of the parameter $\eta_*$ which approximately accounts for the effect of deep dimer states and have compared our results with experimental data obtained by the Innsbruck group [5].

Although our results show very good agreement with the data, sensitivity to $\eta_*$ is insufficient to permit a precise determination of this parameter. Overall, we consider our results to be an excellent example of how few-body systems with large scattering length exhibit universal features. The low-energy properties of $^4$He atoms allow us to compute accurately the low-energy properties of a gas of a completely different element, $^{133}$Cs, which at first glance has little in common with $^4$He. Nevertheless, we point out that the results cannot be thought of as complete treatment of the problem at hand. For example, not only did we make the assumption that $s_{11}$ does not contribute significantly to the recombination coefficients, we also extracted the functions from data sets obtained with finite range potentials. Although the impact of range corrections is known to be small for realistic Helium atom-atom potentials as $R/a \sim 0.1$, it needs to be pointed out that range corrections are expected to be sizable.
for large enough energies. To obtain all universal functions $s_{ij}$ relevant to the recombination rate, a calculation in the limit $R \to 0$ seems therefore to be necessary $^1$. Furthermore, it is already understood how to include range corrections systematically in the framework of effective field theory $^{14, 15, 16}$. Indeed, this approach has already been used to calculate range corrections to the recombination rate into a shallow dimer $^{18, 19}$. Thus, further effort should be devoted to include these effects in the calculation of the energy-dependent recombination rate.

Acknowledgments

We are thankful to Eric Braaten and Daniel Phillips for useful discussions and comments on the manuscript. This work was supported in part by the Department of Energy under grant DE-FG02-93ER40756, by the National Science Foundation under Grant No. PHY–0354916.

[1] V. Efimov, Phys. Lett. 33B, 563 (1970).
[2] V. N. Efimov, Sov. J. Nucl. Phys. 12, 589 (1971).
[3] P. F. Bedaque, H.-W. Hammer and U. van Kolck, Phys. Rev. Lett. 82, 463 (1999).
[4] E. Braaten and H.-W. Hammer, Phys. Rept. 428, 259 (2006).
[5] T. Kraemer, M. Mark, P. Waldburger, J.G. Danzl, C. Chin, B. Engeser, A.D. Lange, K. Pilch, A. Jaakkola, H.-C. Nägerl, and R. Grimm, Nature 440, 315 (2006).
[6] J.P. D’Incao, H. Suno, and B. D. Esry, Phys. Rev. Lett. 93, 123201 (2004).
[7] M.D. Lee, T. Koehler and P.S. Julienne, Phys. Rev. A 76, 012720 (2007).
[8] S. Jonsell, Europhys. Lett. 76, 8 (2006).
[9] M.T. Yamashita, T. Frederico, and L. Tomio, Phys. Lett. A 363, 468 (2007).
[10] P. Massignan and H.T.C. Stoof, Phys. Rev. A 78, 030701 (2008).
[11] E. Braaten, D. Kang and L. Platter, Phys. Rev. A 75, 052714 (2007).
[12] H. Suno, B. D. Esry, C. H. Greene, and J. P. Burke, Phys. Rev. A 65, 042725 (2002).
[13] J. R. Shepard, Phys. Rev. A 75, 062713 (2007).

$^1$ This has been done $^{17}$ in which appeared after the first submission of this paper
[14] P. F. Bedaque, G. Rupak, H. W. Griesshammer and H. W. Hammer, Nucl. Phys. A 714, 589 (2003).
[15] H.-W. Hammer and T. Mehen, Phys. Lett. B 516, 353 (2001).
[16] L. Platter and D. R. Phillips, Few Body Syst. 40, 35 (2006).
[17] E. Braaten, H. W. Hammer, D. Kang and L. Platter, Phys. Rev. A 78, 043605 (2008).
[18] H.-W. Hammer, T. A. Lahde and L. Platter, Phys. Rev. A 75, 032715 (2007).
[19] L. Platter, C. Ji and D. R. Phillips, Phys. Rev. A 79, 022702 (2009).