Kemeny ranking is NP-hard for 2-dimensional Euclidean preferences

Bruno Escoffier∗,1,2, Olivier Spanjaard1, and Magdaléna Tydrichová1

1Sorbonne Université, CNRS, LIP6, F-75005, Paris, France
2Institut Universitaire de France, Paris, France

Abstract

The assumption that voters’ preferences share some common structure is a standard way to circumvent NP-hardness results in social choice problems. While the Kemeny ranking problem is NP-hard in the general case, it is known to become easy if the preferences are 1-dimensional Euclidean. In this note, we prove that the Kemeny ranking problem is NP-hard for $d$-dimensional Euclidean preferences with $d \geq 2$. We note that this result also holds for the Slater ranking preferences.

1 Introduction

Aggregation rules are ubiquitous in social choice theory [1]. Given a multiset of rankings of candidates, an aggregation rule returns a consensus ranking, i.e., a ranking that fairly reflects the various preferences expressed in the input rankings. One of the most popular aggregation rule is the Kemeny rule, that returns a ranking minimizing the sum of Kendall tau distances to the input rankings [18] (the Kendall tau distance between two rankings corresponds to the number of pairwise disagreements between them).

While the computation of a consensus ranking can be performed in polynomial time in the number of votes and candidates for some voting rules (e.g., the Borda rule, single transferable vote, etc.), it is well-known that determining a consensus ranking for the Kemeny rule –problem named Kemeny ranking in the following– is NP-hard in the general case [3]. A standard way to circumvent this drawback is to assume that the preferences are structured [12]. Structured preferences include for instance single-peaked preferences [5], single-crossing preferences, 1-Euclidean preferences [15, 17], $d$-Euclidean preferences with $d \geq 2$ [4, 8], or group-separable preferences [16, 17]. The Kemeny ranking

∗Corresponding author: bruno.escoffier@lip6.fr
problem becomes polytime if the preferences are single-peaked, single-crossing or 1-Euclidean, because the majority relation between candidates is then transitive and gives rise to a consensus ranking for the Kemeny rule.

As the Kemeny ranking problem becomes easy for 1-Euclidean preferences, a natural subsequent question is whether this result also holds for $d$-Euclidean preferences with $d \geq 2$. In this note, we show that Kemeny ranking is NP-hard for 2-Euclidean preferences (and that the decision version is NP-complete), hence for $d$-Euclidean preferences with any $d \geq 2$. Moreover, we show that it is still the case under $\ell_1$ and under $\ell_\infty$ norm, i.e., if we consider preferences that can be represented in a 2-dimensional space using these norms (see Section 2 for a formal definition). This latter setting, which is a variant of usual $d$-Euclidean preferences, has been advocated by Eguia [10] (and references therein), and Peters recently showed that the problem of recognising preference profiles that are $d$-Euclidean with respect to $\ell_1$ or $\ell_\infty$ is in NP for $d \geq 1$ [21].

The contribution of the present note is summarized in the following result:

**Theorem 1** Under norms $\ell_2$, $\ell_1$ and $\ell_\infty$, Kemeny ranking on 2-dimensional Euclidean preferences is NP-hard. This is true even if a 2-dimensional representation of preferences is given in the input.

We recall that a consensus ranking for the Slater rule is one that minimizes the number of disagreements with pairwise majority comparisons [23]. While this rule is often considered as a tournament solution (the Slater set consists of the winning candidates), it also defines a consensus ranking given a preference profile [3]. We note that, as the (positive) majority margins are all equal to 2 in the preference profiles generated in the reductions that will be used to prove Theorem 1 the Kemeny rule and the Slater rule coincide for these profiles [13]. Thus, we have the following Corollary.

**Corollary 1** Under norms $\ell_2$, $\ell_1$ and $\ell_\infty$, Slater ranking on 2-dimensional Euclidean preferences is NP-hard. This is true even if a 2-dimensional representation of preferences is given in the input.

For the sake of brevity, we only mention the Kemeny rule in the remainder of the paper.

The note is organized as follows. In Section 2 we formally define the Kemeny ranking problem, as well as $d$-Euclidean preferences under $\ell_1$, $\ell_2$ and $\ell_\infty$ norms. We also recall the definition of the feedback arc set problem (FAS), from which the reduction will be established for proving Theorem 1 (by considering the special case of FAS where the graph is bipartite, which is known to remain NP-hard). Note that the proof of NP-hardness of Kemeny ranking also uses a reduction from FAS in the general case [4] (but without assuming that the graph is bipartite). Finally, Sections 3, 4 and 5 are devoted to the presentation of the proof of Theorem 1 under norms $\ell_1$, $\ell_\infty$ and $\ell_2$. 

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2 Definitions and notations

2.1 Kemeny ranking

We consider a set $C$ of candidates and a set $V$ of voters. Each voter $v$ in $V$ ranks all the candidates (total ranking, no ties). The ranking of voter $v$ is denoted by $>_{v}$, where $c_i >_{v} c_j$ if $v$ prefers $c_i$ to $c_j$. The set of preferences of voters on candidates is called a (preference) profile.

Given two rankings $>_1$ and $>_2$ on candidates, let $kt(>_1,>_2)$ denote the Kendall tau distance between $>_1$ and $>_2$, i.e., the number of pairs of candidates $\{c_i, c_j\}$ such that $c_i >_1 c_j$ and $c_j >_2 c_i$, or vice versa ($c_i >_2 c_j$ and $c_j >_1 c_i$).

The Kendall tau distance $KT(>,\mathcal{P})$ between a ranking $>$ and a profile $\mathcal{P}$ is then defined as:

$$KT(>,\mathcal{P}) = \sum_{>_v \in \mathcal{P}} kt(>,>_v)$$

Now, we are able to define the Kemeny ranking problem:

Definition 1 (Kemeny ranking) In the Kemeny ranking problem, given a preference profile $\mathcal{P}$, we want to determine a ranking $>$ on the candidates that minimizes $KT(>,\mathcal{P})$.

In the decision version of Kemeny ranking, given some integer $k$, we want to determine whether there exists a ranking $>$ such that $KT(>,\mathcal{P}) \leq k$, or not. As stated in the introduction, we recall that this problem is known to be NP-complete [3].

2.2 Euclidean preferences under $\ell_1$, $\ell_2$ and $\ell_\infty$ norms

Let $d \geq 1$ be an integer. We recall that given two points $p, q \in \mathbb{R}^d$:

- $\ell_1$ is the norm associated to the distance $d_1(p, q) = \sum_{j=1}^{d} |p_j - q_j|$, where $p_j$ is the value of $p$ on the $j$-th coordinate;
- $\ell_2$ is the norm associated to the distance $d_2(p, q) = \sqrt{\sum_{j=1}^{d} (p_j - q_j)^2}$;
- $\ell_\infty$ is the norm associated to the distance $d_\infty(p, q) = \max_{j=1}^{d} |p_j - q_j|$.

We can now define $d$-Euclidean preferences under norm $\ell_p$:

Definition 2 Let $d \geq 1$ be an integer, and $p \in \{1, 2, \infty\}$. A profile $\mathcal{P}$ over the sets $V$ of voters and $C$ of candidates is $d$-Euclidean under norm $\ell_p$ if there exists a mapping $h: V \cup C \rightarrow \mathbb{R}^d$ such that for each $v \in V$ and each $\{c_i, c_j\} \subseteq C$ ($i \neq j$):

$$c_i >_{v} c_j \iff d_p(h(v), h(c_i)) < d_p(h(v), h(c_j))$$

Given the positions of the voters and candidates in $\mathbb{R}^d$, the norm considered obviously has a very strong influence on preferences, as illustrated by the following example.
Example 1 Consider a voter \( v \) and two candidates \( c_1 \) and \( c_2 \), with \( h(v) = (0, 0) \), \( h(c_1) = (4, 4) \) and \( h(c_2) = (7, 0) \), as illustrated in Figure 1. Under norm \( \ell_2 \), the preference \( c_1 > v c_2 \) holds, while on the contrary \( c_2 > v c_1 \) under norm \( \ell_1 \).

Figure 1: A voter \( v \) and two candidates \( c_1, c_2 \) in \( \mathbb{R}^2 \).

Note that 1-Euclidean preferences under norms \( \ell_1 \), \( \ell_2 \) and \( \ell_\infty \) are equivalent, as \( d_1(p, q) = d_2(p, q) = d_\infty(p, q) \) if \( d = 1 \). Furthermore, 1-Euclidean preferences are both single-peaked and single-crossing \([14]\), but do not coincide with the set of single-peaked single-crossing preferences \([4]\) as there are examples of such profiles that are not 1-Euclidean \([7]\). While there are polynomial-time algorithms for recognizing 1-Euclidean preferences \([9, 19, 11]\), recognizing \( d \)-Euclidean preferences under norm \( \ell_2 \) is NP-hard for \( d \geq 2 \) \([21]\). Consequently, in the rest of the note, we assume that the positions of candidates and voters in \( \mathbb{R}^d \) are given as part of the input in the Kemeny ranking problem.

2.3 The feedback arc set problem

In the feedback arc set problem (FAS), we are given a directed graph \( G \) and an integer \( k \), and we want to determine whether we can delete (at most) \( k \) arcs in \( G \) in such a way that the resulting graph is acyclic. This problem is NP-complete \([2]\). It is well known that it remains NP-hard in bipartite graphs, as one can replace each arc \( e_i = (u, v) \) by two arcs \((u, w_i), (w_i, v)\) where \( w_i \) is a new vertex associated to \( e_i \), and obtain an equivalent instance in a bipartite graph.

A classical way to prove hardness results for Kemeny ranking is to build from a directed graph \( G \), instance of FAS, an instance of Kemeny ranking, by converting the vertices of \( G \) into candidates and the arcs into voters. More precisely, suppose that we build an instance such that:

- There is one candidate \( c_i \) for each vertex \( v_i \);
- There are two voters \( f_{ij} \) and \( g_{ij} \) for each arc \((v_i, v_j)\);
- Both \( f_{ij} \) and \( g_{ij} \) prefer \( c_i \) to \( c_j \);

\( ^1 \)A profile is *single-peaked* if there exists an axis on the candidates such that the preferences of each voter are decreasing as one moves away from his/her most-preferred candidate along the axis, and *single-crossing* if there exists an axis on the voters such that, for each couple \((c_i, c_j)\) of candidates, the set of voters \( v \) for which \( c_i > v c_j \) is connected along the axis.
For any other pair \( \{c, d\} \) of candidates, exactly one voter among \( f_{ij} \) and \( g_{ij} \) prefers \( c \) to \( d \) (and one prefers \( d \) to \( c \)).

If preferences are unrestricted, such properties for the preferences of \( f_{ij} \) and \( g_{ij} \) can be obtained for instance by following the approach proposed by McGarvey [20]. Then it is well known that there is a FAS of size (at most) \( k \) if and only if there is a Kemeny ranking of score (at most) \( k + K \), for some \( K \) which depends only on the size of the graph, see [3]. Providing such a reduction proves that Kemeny ranking is NP-hard.

In the sequel, we produce reductions of the same type for preferences that are restricted to be 2-Euclidean (under the considered norm).

3 Proof of Theorem 1 under \( \ell_1 \)

We provide a reduction from FAS in bipartite graphs. Let \( G \) be a bipartite graph, with vertex set \( L \cup R \) and arc set \( A \) (each arc having one extremity in \( L \) and one in \( R \)). We denote by \( n \) the number of vertices, and by \( m \) the number of arcs.

We build an instance where candidates and voters lie on a square, whose sides are parallel to the axes, (see Figure 2). More precisely:

- Each vertex \( v_i \) corresponds to a candidate \( c_i \). If \( v_i \in L \) (resp. \( v_i \in R \)), \( c_i \) will be on the vertical left side (resp. right side) of the square. We will say that \( c_i \in L \) (resp. \( c_i \in R \)) if \( v_i \in L \) (resp. \( v_i \in R \))

- Each arc \((v_i, v_j)\) correspond to two voters \( f_{ij} \) and \( g_{ij} \). Point \( f_{ij} \) will be on the horizontal upper side of the square, while \( g_{ij} \) will be on the horizontal lower side of the square.

Let us consider an arc \((v_i, v_j)\), with \( v_i \in L \) and \( v_j \in R \). We call \( A_{ij} \) the point on the upper horizontal side such that \( d_1(c_i, A_{ij}) = d_1(c_j, A_{ij}) \) (note that such point indeed exists on the horizontal side of the square). Similarly, we call \( B_{ij} \) the point on the lower horizontal side such that \( d_1(c_i, B_{ij}) = d_1(c_j, B_{ij}) \).

We put two voters \( f_{ji} \) and \( g_{ji} \) which are positioned on the edges at \( \epsilon \) (to be specified) to the left of \( A_{ij} \) and \( B_{ij} \), respectively. If the arc had been \((v_j, v_i)\), then the voters \( f_{ji} \) and \( g_{ji} \) would have been at \( \epsilon \) to the right of \( A_{ij} \) and \( B_{ij} \).
Figure 2: The construction with two vertices $v_i, v_j$ and an arc $(v_i, v_j)$.

Assume that we choose the vertical positions of candidates in such a way that all $A_{ij}$ are distinct (and equivalently, all $B_{ij}$ are distinct), see below for an explicit construction. Note that as $B_{ij}$ is the symmetric of $A_{ij}$ with respect to the center of the square, the order of $A$-points on the upper side is the inverse order of $B$-points on the lower side.

Then we choose $\epsilon$ sufficiently small so that between $f_{ij}$ and $A_{ij}$ there is no other $A$-point. Similarly, between $g_{ij}$ and $B_{ij}$ there is no other $B$-point.

Let us consider an arc $(v_i, v_j)$ with $v_i \in L$ and $v_j \in R$ (the other case being completely symmetric). Then:

- Both voters $f_{ij}$ and $g_{ij}$ prefer $c_i$ to $c_j$ (as $A_{ij}$ is equidistant from $c_i$ and $c_j$, the same for $B_{ij}$).
- For any other pair $\{c, d\}$ of candidates, exactly one voter among $f_{ij}$ and $g_{ij}$ prefers $c$ to $d$ (and one prefers $d$ to $c$). This is easy to see if both $c$ and $d$ belong to $L$, or if both belong to $R$. If $c \in L$ and $d \in R$, if for instance their corresponding $A$-point is on the right of $A_{ij}$, then their corresponding $B$-point is on the left of $B_{ij}$, thus $f_{ij}$ prefers $c$ to $d$ but $g_{ij}$ prefers $d$ to $c$.

Thus, this construction fulfills the conditions of the reduction given in Subsection 2.3, showing NP-hardness. We now give an explicit (polynomial time) construction which ensures that $A$-points are distinct, as well as $B$-points.

**Explicit construction**

We consider a square with side lengths $\Delta = 2^{n+1}$, where $n$ is the number of vertices of the graph. Let us consider that the bottom left corner of the rectangle has coordinates $(0,0)$.

We set the $y$-coordinate of candidate $c_i$ to $y_i = 2^i$. Then the $x$-coordinate $x_{ij}$ of $A_{ij}$ is such that $x_{ij} + \Delta - y_i = \Delta - x_{ij} + \Delta - y_j$, meaning that:

$$x_{ij} = \frac{\Delta + y_i + y_j}{2}.$$
Then we can verify that these \( y \)-values are such that all \( A \)-points are distinct. Indeed, for any distinct pairs \( \{i, j\} \) and \( \{k, \ell\} \) of indices, \( y_i + y_j \neq y_k + y_\ell \). To see this, if say \( \ell \) is the largest among the indices, then:

- If \( j = \ell \) (as the pairs are distinct), and \( y_i + y_j \neq y_k + y_\ell \).
- If \( i, j < \ell \), then \( y_i + y_j \leq 2(2^{\ell-1}) = 2\ell < y_\ell + y_k \).

Then all the values \( x_{ij} = \frac{y_i + y_j}{2} \) are distinct. Note that as \( y \)-values and \( \Delta \) are even integers, \( x_{ij} \) is an integer, and we can choose \( \epsilon = \frac{1}{2} \) (and multiply everything by 2 if we want integers).

As the coordinates can be encoded with a polynomial number of bits, the reduction is polynomial.

4 Proof of Theorem 1 under \( \ell_\infty \)

We use a construction which is similar to the case of \( \ell_1 \), but positioning candidates and voters on a square which is oriented as in Figure 3. The diagonal of the square has length \( 2\Delta \) with \( \Delta = 2^{n+1} \).

We position a candidate \( c_i \in L \) on the lower left side, at position \((-2^i, 2^i - \Delta)\). A candidate \( c_j \in R \) is on the upper right side, at position \((2^j, \Delta - 2^j)\).

Then we define two points \( A_{ij} \) and \( B_{ij} \), respectively on the upper left side and on the lower right side, both being equidistant (under \( \ell_\infty \)) from \( c_i \) and \( c_j \). Namely, the coordinates of \( A_{ij} \) are \((2^{i+j} - \Delta, 2^i + 2^j)\). Point \( B_{ij} \) is the symmetric of \( A_{ij} \) with respect to the center \( \hat{O} \) of the square.

As previously, if there is an arc \((v_i, v_j)\) with \( v_i \in L \) and \( v_j \in R \), we create two voters \( f_{ij} \) and \( g_{ij} \), point \( f_{ij} \) being positioned on the edge of \( A_{ij} \) at \( \epsilon \) to the bottom/left of \( A_{ij} \), and \( g_{ij} \) being positioned on the edge of \( B_{ij} \) at \( \epsilon \) to the bottom/left of \( B_{ij} \). If there is an arc \((v_j, v_i)\) with \( v_j \in L \) and \( v_i \in R \), then \( f_{ji} \) and \( g_{ji} \) are positioned on the edges at \( \epsilon \) to the right/up of \( A_{ij} \) and \( B_{ij} \).

The choice of the coordinates of candidates ensure that all the \( A \)-points and \( B \)-points are distinct (for the same reason as in the proof for \( \ell_1 \) norm), and integral, so we can choose \( \epsilon = 1/2 \) to ensure that there is no \( A \)-points between \( f_{ij} \) and \( A_{ij} \) (neither \( B \)-points between \( g_{ij} \) and \( B_{ij} \)).
5 Proof of Theorem 1 under $\ell_2$

We make the reduction from FAS (in general graphs, not in bipartite graphs), and will position now candidates and voters on a circle, centered at point $O$ of coordinates $(0,0)$. More precisely (see Figure 4):

- Each vertex $v_i$ corresponds to a candidate $c_i$ positioned on the circle.

- Let us call $D_{ij}$ the line of equidistant points (under $\ell_2$) between $c_i$ and $c_j$, and $A_{ij}$ and $B_{ij}$ the two points of $D_{ij}$ on the circle. Each arc $(v_i,v_j)$ correspond to two voters $f_{ij}$ and $g_{ij}$, both positioned on the circle. Point $f_{ij}$ is on the same side of $D_{ij}$ as $c_i$, with an angle $\epsilon$ between $A_{ij}$ and $f_{ij}$. Similarly, $g_{ij}$ is on the same side of $D_{ij}$ as $c_i$, with an angle $\epsilon$ between $B_{ij}$ and $g_{ij}$.
As previously, suppose that we choose the positions of candidates in such a way that all the points $A_{ij}$ and $B_{ij}$ are distinct.

Then we choose $\epsilon$ sufficiently small such that each $A$-point or $B$-point lies neither between $f_{ij}$ and $A_{ij}$, nor between $g_{ij}$ and $B_{ij}$.

Let us consider an arc $(v_i, v_j)$. Then:

- Both voters $f_{ij}$ and $g_{ij}$ prefer $c_i$ to $c_j$ (as $A_{ij}$ is equidistant from $c_i$ and $c_j$, the same for $B_{ij}$).

- For any other pair $\{c, d\}$ of candidates, exactly one voter among $f_{ij}$ and $g_{ij}$ prefers $c$ to $d$ (and one prefers $d$ to $c$). This follows from the fact that all $D$-lines intersect in $O$, meaning that $f_{ij}$ and $g_{ij}$ cannot be on the same side of the $D$-line corresponding to $\{c, d\}$.

**Explicit construction**

Let us call $\Theta_i$ the angle (polar coordinate, in radian) of $c_i$ (i.e., the angle between the horizontal axis and $\overrightarrow{OC_i}$). Then we shall choose $\Theta_i$ in such a way that all the points $A_{ij}$ and $B_{ij}$ are distinct. This appears as soon as $(\Theta_i + \Theta_j)$ are distinct, as the angle of the line $D_{ij}$ is $\frac{\Theta_i + \Theta_j}{2}$.

Let us fix $\Theta_i = \frac{2^i}{2} = 2^{i-n}$. By the same reasoning as in the proof of the $\ell_1$ norm, all $(\Theta_i + \Theta_j)$ are distinct (note that $0 \leq \Theta_i \leq \Pi/2$ so $(\Theta_i + \Theta_j)$ are indeed distinct modulo $2\Pi$). We can fix $\epsilon = 1/2^{n+1}$, to fulfill the property for $f_{ij}$ and $g_{ij}$.

We note that the actual preference profile can be easily built from this embedding of points in the 2-dimensional space. Indeed, if $i > j$, voter $f_{ij}$ has angle $\frac{2^i+2^j+1/2}{2^n}$, and he prefers $c_k$ to $c_j$ iff $|2^k-a_{ij}| < |2^l-a_{ij}|$, where $a_{ij} = 2^i+2^j+1/2$ (if $i < j$ it is the same with $a_{ij} = 2^i+2^j-1/2$). Voter $g_{ij}$ has the reverse preference on all pairs but $\{c_i, c_j\}$. Thus, the reduction is polynomial.

6 Conclusion

In this note, we have addressed an open problem, namely the computational complexity of the Kemeny ranking problem when the input preferences are $d$-Euclidean with $d \geq 2$. We have proved that the problem remains NP-hard, contrary to the case of 1-Euclidean preferences. A direct consequence is that computing an optimal consensus ranking for the Slater rule is also NP-hard for $d$-Euclidean preferences with $d \geq 2$, because the majority margins are all equal to 2 in the preference profiles generated in the reductions.

Natural research directions to pursue would be to investigate the impact of 2-Euclidean preferences on the complexity of other NP-hard social choice problems.

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