Topological Quantum Field via Chern-Simons Theory, part 1

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Abstract

To understand what does Chern-Simons with compact Lie group (does not like Dijkgraaf-Witten model with finite group in 3d) attach to a point, we first give a construction of Topological Quantum Field Theory (TQFT) via Chern-Simons theory in this paper. We discuss the Topological Quantum Field Theory and Chern-Simons theory via Category, then interpret the cobordism as cospan and field of space-time as span, which ultimately deduce the construction of TQFT.

Key words: TQFT, Chern-Simons theory, Span and Cospan, Categorification

1 Introduction

In this paper, we describe a construction for an Topological Quantum Field Theory via Chern-Simons theory, based on Freed’s works which view topological field as a general set of axioms [F1][F2][F3][F4] and Morton’s work which gives a construction for an Extended Topological Quantum Field Theory (ETQFT), associating to any finite gauge group [M]. We can find that when we say the functor

$$Z : nCob \rightarrow Hilb$$

is monoidal (i.e. is a Hilb-valued TQFT), it is equivalent to say that the Chern-Simons field satisfies the ”additivity law” which is the important property of a lagrangian field theory.

In the axiomatic formulation (due to Atiyah [At]), an n-dimensional topological quantum field theory is a rule Λ which to each closed oriented manifold Σ (of dimension (n − 1)) associates a vector space ΛΣ, and to each oriented n-manifold M whose boundary is Σ associates a vector in ΛΣ. The axioms amount to saying that a TQFT is a (symmetric) monoidal functor from nCob to Vectk, i.e.

$$Z : nCob \rightarrow Vect_k$$

where nCob is the geometric category whose objects are closed (n − 1)-dimensional oriented manifolds and morphisms are n-dimensional oriented manifolds as cobordisms of (n − 1)-dimensional closed oriented manifolds, and Vectk is the algebraic category whose objects are vector spaces over k and morphisms are k-linear maps between vector spaces. I will give
the concrete definition and the example of 2d-TQFT in §2 [Ab]. Mathematical interest in TQFTs stems from the observation that they produce invariants of closed manifolds [S]. Meanwhile, Physical interest in TQFTs comes mainly from the observation that TQFTs possess certain features one expects from a theory of quantum gravity and approach to quantum field theory [DW]. It serves as a model in which one can do calculations and gain experience before embarking on the quest for full-fledged theory.

The ”degroupoidification” functor gives a representation of $\text{Span}(Gpd)$ in $\text{Veck}_k$ which can views as a TQFT that be constructed via gauge theory associates to any finite gauge group. The reader may find more details on this program in a review by Baez, Hoffnung and Walker [BHW], though it is not obvious to refer. In this paper, we get a construction via Chern-Simons theory associated to any compact Lie group.

For a compact Lie group (usually restrict to connected and simply connected case), and integral cohomology class $\lambda \in H^4(BG)$, we can construct Chern-Simons theory on 3d manifold [F2][SW]. If $X$ is a closed, oriented 3d-manifold, then the action is a complex number with unit norm. This is the exponential of $2\pi i$ times the usual action, which is a real number determined modulo the integers. If $X$ has a nonempty boundary, then the action is an element of unit normal in an abstract metrized complex line which depends on the restriction of the field to $\partial X$. This line is called Chern-Simons line, and we can prove the action satisfies the ”Functoriality” ”Orientation” ”Additivity” and ”gluing”, which illustrate the important properties of a lagrangian field theory. This part will appear in §3.

To construct the TQFT, we first should construct a functor $\natural$ from $n\text{Cob}$ to $\text{Span}(FT)$ of field space category. In §4, we view cobordisms as cospan and field space as span. Then the composition of span is just the fiber product of groupoid, which will explain the ”additivity law” of the Chern-Simons action.

The most important part of this article is §5, which gives us the construction of TQFT via the functor $\natural$. For Chern-Simons theory, we can get a Hilbert space from any oriented closed 2d-manifold and a linear map from the span views as the cobordism via the functor $\natural$. That is to say we get another functor $\heartsuit$

$$\heartsuit : \text{Span}(FT) \longrightarrow \text{Hilb}$$  

Then we compose the two functors to get the TQFT.

## 2 Topological Quantum Field Theories

Consider the fundamental formula of quantum field theory

$$\langle \hat{A}_2 | U | \hat{A}_1 \rangle = \int_{\Sigma_1=A_1}^{\Sigma_2=A_2} D\! A \ exp\{iS[A]\}$$

We have now developed the appropriate mathematical language to understand this equation. We thus define a topological quantum field theory (Hilb-valued TQFT) as a symmetric monoidal functor

$$Z : n\text{Cob} \longrightarrow \text{Hilb}$$

To begin, we must give some explicit information about the categories of $n\text{Cob}$ and $\text{Hilb}$ we are interested in.
2.1 The Category $nCob$

The category which quantum field theory concerns itself with is called $nCob$, the "n dimensional cobordism category" [B], and the rough definition is as follows. Objects are oriented closed (that is, compact and without boundary) $(n-1)$-manifolds $\Sigma$, and arrows $M : \sigma_1 \rightarrow \sigma_2$ are compact oriented $n$-dimensional manifold $M$ which are cobordisms from $\Sigma_1$ to $\Sigma_2$. Composition of cobordisms $M : \sigma_1 \rightarrow \sigma_2$ and $N : \sigma_2 \rightarrow \sigma_3$ is defined by gluing $M$ to $N$ along $\Sigma_2$.

Let us fill in the details of this definition. Let $M$ be an oriented $n$-manifold with boundary $\partial M$. Then one assigns an induced orientation to the connected components $\Sigma$ of $\partial M$ by the following procedure. For $x \in \Sigma$, let $(v_1, \cdots, v_{(n-1)}, v_n)$ be a positive basis for $T_xM$ chosen in such a way that $(v_1, \cdots, v_{(n-1)}) \in T_x\Sigma$. It makes sense to ask whether $v_n$ points inward or outward from $M$. If it points inward, then an orientation for $\Sigma$ is defined by specifying that $(v_1, \cdots, v_{(n-1)})$ is a positive basis for $T_xM$. If $M$ is one dimensional, then $x \in \partial M$ is defined to have positive orientation if a positive vector in $T_xM$ points into $M$, otherwise it is defined to have negative orientation.

Let $\Sigma_1$ and $\Sigma_2$ be closed oriented $(n-1)$-manifolds. A cobordism from $\Sigma_1$ to $\Sigma_2$ is a compact oriented $n$-manifold $M$ together with smooth maps

$$\Sigma_1 \rightarrow M \leftarrow \Sigma_2$$

where $i$ is an orientation preserving diffeomorphism of $\Sigma$ onto $i(\Sigma_1) \in M$, $i'$ is an orientation reversing diffeomorphism of $\Sigma'$ onto $i'(\Sigma_2) \in M$, such that $i(\Sigma_1)$ and $i'(\Sigma_2)$ (called the in- and out-boundaries respectively) are disjoint and exhaust $\partial M$. Observe that the empty set $\emptyset$ can be considered as an $(n-1)$-manifold.

For a given $n$, one can construct the (smooth) cobordism category $nCob$. Objects are closed, oriented $(n-1)$-manifolds $\Sigma$, and arrows $M : \sigma_1 \rightarrow \sigma_2$ are cobordism. In order to make this a well-defined category with identity arrows, we must quotient out diffeomorphic cobordisms. Specifically, let $M$ and $M'$ be cobordisms from $\Sigma_1$ to $\Sigma_2$. Then they are considered equivalent if there is an orientation preserving diffeomorphism $\psi : M \rightarrow M'$ making the diagram commute.

After identifying equivalent cobordisms, the "cylinder" $\Sigma \times [0, 1]$ functions as the identity arrow for $\Sigma$. The cobordism category is a geometric category which captures the way $n$-manifolds glue together.

Furthermore, $nCob$ is a symmetric monoidal category with the disjoint union as its monoidal product, $\emptyset$ as its identity, and twist diffeomorphism $T_{\Sigma_1, \Sigma_2} : \Sigma_1 \sqcup \Sigma_2 \rightarrow \Sigma_2 \sqcup \Sigma_1$ satisfies the symmetric condition. We denote this symmetric monoidal category $(nCob, \sqcup, \emptyset, T)$. For more detail, you can read the book [K].

2.2 The Category $Hilb$

The category $Hilb$ is simple whose objects are Hilbert spaces over $k$, and arrows are bounded linear maps between Hilbert spaces. Obviously, $Hilb$ is a symmetric monoidal category which we usually denote it $(Hilb, \otimes, k, \sigma)$ where $\otimes$ is the common tensor product of vector space and symmetric condition is hold for $\sigma : A \otimes B \rightarrow B \otimes A$. We don’t focus much energy on this category.
2.3 The definition of TQFT

Definition 2.3.1 An $n$-dimensional topological quantum field theory (TQFT) is a rule $\Lambda$ which to each closed oriented $(n-1)$-manifold $\Sigma$ associates a vector space $\Lambda\Sigma$, and to each oriented cobordism $M : \Sigma_1 \rightarrow \Sigma_2$ associates a linear map $\Lambda M$ from $\Lambda\Sigma_1$ to $\Lambda\Sigma_2$. This rule $\Lambda$ must satisfy the following five axioms.

1. Two equivalent cobordisms must have the same image:
   
   $$M \cong M' \implies \Lambda M = \Lambda M'$$

2. The cylinder $\Sigma \times [0,1]$, thought of as a cobordism from $\Sigma$ to itself, must be sent to the identity map of $\Lambda\Sigma$.

3. Given a decomposition $M = M_1M_2$ then
   
   $$\Lambda M = (\Lambda M_1)(\Lambda M_2)$$

4. Disjoint union goes to tensor product: if $\Sigma = \Sigma_1 \sqcup \Sigma_2$ then $\Lambda \Sigma = \Lambda \Sigma_1 \otimes \Lambda \Sigma_2$. This must also hold for cobordisms: if $M : \Sigma_0 \rightarrow \Sigma_1$ is the disjoint union of $M' : \Sigma_0' \rightarrow \Sigma_1'$ and $M'' : \Sigma_0'' \rightarrow \Sigma_1''$ then $\Lambda M = \Lambda M' \otimes \Lambda M''$.

5. The empty manifold $\Sigma = \emptyset$ must be sent to the ground field $k$.

Remark: The first two axioms express that the theory is topological: the evolution depends only on the diffeomorphism class of space-time, not on metric structure. Axiom (4) reflects a standard principle of quantum mechanics: that the state space of two independent systems is the tensor product of the two state spaces.

Towards a categorical interpretation of the axioms of the topological quantum field theory [B], we say a $\text{Hilb}$-valued TQFT [L] is just a symmetric monoidal functor from $(n\text{Cob}, \sqcup, \emptyset, T)$ to $(\text{Hilb}, \otimes, k, \sigma)$, i.e.

$$Z : (n\text{Cob}, \sqcup, \emptyset, T) \rightarrow (\text{Hilb}, \otimes, k, \sigma)$$

2.4 $2d$-topological quantum field theory

From the mathematical view, the classification of $nd$–TQFT is very important. For general $n$, this question is so difficult to solve. But the classification of Extended TQFT has been complete done by Lurie [L]. In this subsection, we will outline the results of the classification of $2d$-TQFT, rather than one of Extended TQFT.

The first statement that $2d$–TQFTs are commutative Frobenius algebras appears in Dijkgraaf’s thesis (89) [D], but mathematical proofs didn’t appear until the work of L.Abrams(95), S.Swain(95), T.Quinn(95), and B.Dubrovin(96). In order to give the correspondence, we first need the following “generators-relations” theory [K]:

Theorem 2.4.1 The monoidal category $2\text{Cob}$ is generated under composition and disjoint union by the following six cobordisms:
and, those cobordisms satisfy the following relations (figure:1-6).

Figure 1: identity relations

Figure 2: sewing indices

Figure 3: associativity and coassociativity

Figure 4: commutativity and cocommutativity
From the theorem and the definition of TQFT, we can construct the following beautiful correspondence:

**Theorem 2.4.2** There is a canonical equivalence of categories

\[ 2TQFT_k \simeq cFA_k \]  

where \( 2TQFT = \text{Rep}_k(2\text{Cob}) = \text{SymMonCat}(2\text{Cob}, \text{Hilb}_k) \) whose objects are the symmetric monoidal functors from \( 2\text{Cob} \) to \( \text{Hilb}_k \), and whose arrows are the monoidal natural transformation between such functors. \( cFA_k \) denote the category of commutative Frobenius algebras over \( k \) and Frobenius algebra homomorphisms.

### 2.5 1 − 2 − 3 theorem

The above description of \( nd - TQFT \) always read "\( (n-1) - n \)" theory, for example, the 0−1 theory \( Z \) assigns a vector space \( Z(M) \) to every closed 0-manifold \( M \). A zero dimensional manifold \( M \) is simply a finite set of points. An orientation of \( M \) determines a decomposition \( M = M_+ \sqcup M_- \) of \( M \) into "positively oriented" and "negatively oriented" points. In particular, there are two oriented manifolds which consist of only a single point, up to orientation-preserving diffeomorphism. let us denote these manifolds by \( P \) and \( Q \), then we obtain vector spaces \( Z(P) \) and \( Z(Q) \) where \( Z(Q) \) is dual space of \( Z(P) \). We write \( Z(P) = V \) and \( Z(Q) = \bar{V} \), for some finite-dimensional vector space \( V \). We must also specify the behavior of \( Z \) on 1-manifolds \( B \) with boundary which is diffeomorphic either to a closed interval \([0,1]\) or to a circle \( S^1 \). There are five cases to consider, depending on how we decompose \( \partial B \) into "incoming" and "outgoing" pieces.

- Suppose that \( B = [0,1] \), regarded as a cobordism from \( P \) to itself. Then \( Z(B) \) coincides with the identity map \( id : V \to V \).
• Suppose that $B = [0, 1]$, regarded as a cobordism from $Q$ to itself. Then $Z(B)$ coincides with the identity map $id : V \rightarrow V$.

• Suppose that $B = [0, 1]$, regarded as a cobordism from $P \sqcup Q$ to the empty set. Then $Z(B)$ is a linear map from $V \otimes \hat{V}$ into ground field $k$: namely, the evaluation map $(v, \lambda) \mapsto \lambda v$.

• Suppose that $B = [0, 1]$, regarded as a cobordism from the empty set to $P \sqcup Q$. Then $Z(B)$ is a linear map from $k$ to $V \otimes \hat{V}$. Under the canonical isomorphism $P \sqcup Q \cong \text{End}(V)$, this linear map is given by $x \mapsto xid_V$.

• Suppose that $B = S^1$, regarded as a cobordism from the empty set to itself. Then $Z(B)$ is a linear map from $k$ to itself, which we can identify with an element of $k$. Decomposing $S^1 \cong z \in C : |z| = 1 = S^1_+ \cup S^1_-$ where $S^1_+ = z \in C : (|z| = 1) \cap \text{Im}(z) \leq 0$ and $S^1_- = z \in C : (|z| = 1) \cap \text{Im}(z) \geq 0$ meeting in the subset $S^1_+ \cap S^1_- = \pm 1$, then we get the composition of the maps

$$k \cong Z(\phi) \rightarrow Z(\pm 1) \rightarrow Z(\phi) \cong k \quad (7)$$

Consequently, $Z(S^1)$ is given by the trace of the identity map from $V$ to itself: in other words, the dimension of $V$.

From the "0–1" theory, we can get the invariants via cutting and gluing the cobordism, and an element of $k$ to each closed $n$-manifold and a vector space to any closed $(n – 1)$-manifold. What we will gain to $(n – 2)$-manifold? This just concerns with Extend TQFT which is also called $0 – 1 – 2 – \cdots – n$ theory. There is a very beautiful correspondence between $0 – 1 – 2 – \cdots – n$ theories and dualizable objects [BD][L].

**Theorem** (Baez-Dolan-Lurie) Let $\mathcal{C}$ be a symmetric monoidal infinity $n$-category. Then the space of $0 – 1 – 2 – \cdots – n$ theories of framed manifolds with values in $\mathcal{C}$ is homotopy equivalent to the space of fully dualizable objects in $\mathcal{C}$.

In this paper, we don’t refer this topic. We will see what does Chern-Simons with compact Lie group attach to a point (i.e. $0 – 1 – 2 – 3$ theory) in part 2 via this theorem.

For $1 – 2 – 3$ theory, we have the following theory [BK]

**Theorem** A $1 – 2 – 3$ theory $F$ determines a modular tensor category $\mathcal{C} = F(S^1)$. Conversely, a modular tensor category $\mathcal{C}$ determines a $1 – 2 – 3$ theory $F$ with $F(S^1) = \mathcal{C}$.

I do not know a structure theorem for $2 – 3$ theories (i.e. 3d-TQFT) like the theorem 2.4.2. There is an open problem: can 3d–TQFT’s distinguish closed 3d manifold? It is a very interesting problem which I concern all the time.

### 3 Chern-Simons Theory

In this section, we will focus on the Chern-Simons theory [F2].

Let $X$ be a 3d-manifold and $G$ be a connected, simply connected, compact Lie group, and an invariant form $\langle \rangle$ on its Lie algebra $g$. Define the category $\mathcal{C}_X = \mathcal{C}_X^G$ of $G$ connections as follows. An object in $\mathcal{C}_X$ is a connection $\Theta$ on a principal $G$ bundle $P \rightarrow X$. A morphism $\Theta' \rightarrow \Theta$ is a bundle map $\psi : P' \rightarrow P$ covering the identity map on $X$ (i.e. a bundle morphism) such that $\Theta' = \psi^* \Theta$. Obviously, such a category is a groupoid. Let $A_P$ of all
connections on principal $G$ bundle $P \rightarrow X$, it is an affine subspace of $\Omega^1_P(g)$, the vector space of $g$-valued 1-forms on $P$. Then the objects of $C_X$ form a union of affine space

$$Obj(C_X) = \sqcup A_P$$

where $P$ is the collection of all principal $G$ bundles over $X$.

Before we give the theorem of Chern-Simons action, we first make the following integrating hypothesis on the bilinear form $\langle \rangle$.

**Hypothesis** Assume that the closed form $-\frac{1}{6} \langle \theta \wedge [\theta \wedge \theta] \rangle$ represents an integral class in $H^3(G; \mathbb{R})$, where $\theta$ is the Maurer-Cartan form.

Fix a $G$ connection $\eta$ on $Q \rightarrow Y$, where $Y$ is an oriented closed 2-manifold. Let $\mathcal{C}_Q$ be the category whose objects are sections $q : Y \rightarrow Q$. For any two sections $q, q'$ there is a unique morphism $\psi : q \rightarrow q'$, where $\psi : Q \rightarrow Q$ is the gauge transformation such that $q' = \psi q$. Define the functor $\Xi_\eta : \mathcal{C}_Q \rightarrow \mathbb{L}$ by $\Xi_\eta(q) = \mathbb{C}$ for all $q$, where $\mathbb{C}$ has its standard metric, and $\Xi_\psi : q \rightarrow q'$ is a multiplication by $c_Y(q'\eta, g_\psi q)$, where $g_\psi : Q \rightarrow G$ is the map associated to $\psi$, and $c_Y$ is the cocycle

$$c_Y(a, g) = \exp(2\pi i \int_Y (\langle Ad_{g^{-1}}a \wedge \phi_g \rangle + W_Y(g))), a \in \Omega^1_Y(g), \ \ g : Y \rightarrow G$$

We obtain the metrized line $L_\eta = L_{Y, \eta}$ of invariant section.

We now give the theorem of Chern-Simons action $S_X(p, \theta) = \int_X (p_* \alpha(\Theta))$, where $p : X \rightarrow P$ be a section and $\alpha(\Theta)$ be the Chern-Simons form as the lagrangian of field $\Theta$.

**Theorem 3** Let $G$ be a connected, simply connected compact Lie group and $\langle \rangle : g \otimes g \rightarrow \mathbb{R}$ an invariant form on its Lie algebra $g$ which satisfies the integrality condition Hypothesis. Then the assignments

$$\eta \mapsto L_\eta, \ \ \eta \in \mathcal{C}_Y$$
$$\Theta \mapsto e^{2\pi i S_X(\Theta)}, \ \ \Theta \in \mathcal{C}_X$$

defines above for closed oriented 2-manifolds $Y$ and compact oriented 3-manifolds $X$ are smooth and satisfy:

- **(Functoriality)** If $\psi : Q' \rightarrow Q$ is a bundle map covering an orientation preserving diffeomorphism $\tilde{\psi} : Y'' \rightarrow Y$, and $\psi$ is a connection on $Q$, then there is an induced isometry

  $$\psi^* : L_\eta \rightarrow L_{\psi^* \eta}$$

  and these compose properly. If $\varphi : P' \rightarrow P$ is a bundle map covering an orientation preserving diffeomorphism $\tilde{\varphi} : X' \rightarrow X$, and $\Theta$ is a connection on $P$, then

  $$\partial \varphi^* e^{2\pi i S_X \Theta} = e^{2\pi i S_X' \varphi^* \Theta},$$

  where $\partial \varphi : \partial P' \rightarrow \partial P$ is the induced map over the boundary.

- **(Orientation)** There is a natural isometry

  $$L_{-Y, \eta} \cong L_{Y, \eta}, \quad e^{2\pi S_X(\Theta)} = e^{-2\pi S_X(\Theta)}$$

8
(Additivity) If \( Y = Y_1 \sqcup Y_2 \) is a disjoint union, and \( \eta_i \) are connections over \( Y_i \), then there is a natural isometry

\[
L_{\eta_1 \sqcup \eta_2} \cong L_{\eta_1} \otimes L_{\eta_2}.
\]

(15)

If \( X = X_1 \sqcup X_2 \) is a disjoint union, and \( \Theta_i \) are connections over \( X_i \), then

\[
e^{2\pi i S_{X_1} (\Theta_1 \sqcup \Theta_2)} = e^{2\pi i S_{X_1} (\Theta_1)} \otimes e^{2\pi i S_{X_2} (\Theta_2)}.
\]

(16)

(Gluing) Suppose \( Y \hookrightarrow X \) is a closed, oriented submanifold and \( X^{\text{cut}} \) is the manifold obtained by cutting \( X \) along \( Y \). Then \( \partial X^{\text{cut}} = \partial X \sqcup Y \sqcup (-Y) \). Suppose \( \Theta \) is a connection over \( X \), with \( \Theta^{\text{cut}} \) the induced connection over \( X^{\text{cut}} \), and \( \eta \) the restriction of \( \Theta \) to \( Y \). Then

\[
e^{2\pi i S_X (\Theta)} = Tr_\eta (e^{2\pi i S_{X^{\text{cut}}} (\Theta^{\text{cut}})})
\]

(17)

where \( Tr_\eta \) is the contraction

\[
Tr_\eta : L_{\partial \Theta^{\text{cut}}} \otimes L_\eta \otimes \overline{L_\eta} \rightarrow L_{\partial \Theta}
\]

(18)

We usually use the "gluing" law to compute the invariants of manifold, and the "additivity" law is crucial point to construct the TQFT via Chern-Simons theory.

4 Span and Cospan

To construct the TQFT via Chern-simons theory, we first need to construct a functor \( \natural \) from \( nCob \) to \( \text{Span}(FT) \) of field space category. Let us give the definition of Span and Cospan.

4.1 The category \( \text{Span}(FT) \)

**Definition 4.1** Given any category \( \mathcal{C} \), a span \((S, s, t)\) between objects \( X_1, X_2 \in \mathcal{C} \) is a diagram in \( \mathcal{C} \) of the form

\[
X_1 \leftarrow S \rightarrow X_2
\]

**Definition 4.2** Given two spans \((S, s, t)\) and \((S', s', t')\) between \( X_1 \) and \( X_2 \), a morphism of spans is a morphism \( g : S \rightarrow S' \) making the diagram commutes.

Composition of spans \( S \) from \( X_1 \) to \( X_2 \) and \( S' \) from \( X_2 \) to \( X_3 \) is given by pullback: that is, an object \( R \) with maps \( f_1 \) and \( f_2 \) making the diagram which satisfies the "universal property" commutes.

**Definition 4.3** A cospan in \( \mathcal{C} \) is a span in \( \mathcal{C}^{\text{op}} \), morphisms of cospan are morphisms of span in \( \mathcal{C}^{\text{op}} \), and composition of cospans is given by pullback in \( \mathcal{C}^{\text{op}} \)-that is, by pushout in \( \mathcal{C} \).

From the definitions of "Cobordism" and "Cospan", obviously, a cobordism from \( \Sigma_1 \) to \( \Sigma_2 \) can be viewed as a cospan from \( \Sigma_1 \) to \( \Sigma_2 \).

As we describe above, for a manifold \( X \) with \( \partial X = Y_0 \sqcup Y_1 \) (\( Y_0 \) and \( Y_1 \) may be \( \emptyset \)), We have a groupoid \( \mathcal{C}_X \) which is the field space of Chern-Simons theory. We can construct \( \mathcal{C}_{Y_0} \) and \( \mathcal{C}_{Y_1} \) by restriction \( t_0 : \mathcal{C}_X \rightarrow \mathcal{C}_{Y_0} \) and \( t_1 : \mathcal{C}_X \rightarrow \mathcal{C}_{Y_1} \). Obviously, \( (\mathcal{C}_X, t_0, t_1) \) is a span between \( \mathcal{C}_{Y_0} \) and \( \mathcal{C}_{Y_1} \). Then we obtain the category \( \text{Span}(FT) \) which is a monoidal category, composition as its monoidal product.
4.2 Functor from $3\text{Cob}$ to $\text{Span}(\text{FT})$

From the above consideration, we have the following property.

**Property 4.2** There is a monoidal functor

$$
\sharp : 3\text{Cob} \rightarrow \text{Span}(\text{FT}) \quad (19)
$$

$$
Y \mapsto C_Y
$$

$$
Y_0 \rightarrow X \leftarrow Y_1 \mapsto C_{Y_0} \leftarrow C_X \rightarrow C_{Y_1} \quad (20)
$$

where $Y$ is an object of $3\text{Cob}$.

**Proof.** The check is straightforward from the above details.

5 TQFT via Chern-Simons Theory

Now we begin to construct the TQFT. We need another monoidal functor

$$
\heartsuit : (\text{Span}(\text{FT})) \rightarrow \text{Hilb} \quad (21)
$$

5.1 The monoidal functor $\heartsuit$

For any closed 2-manifold $Y$ associated to $C_Y$, an object of $\text{Span}(\text{FT})$, we can get a metrized line $L_\eta$ to each $\eta \in C_Y$. Assume there exist measures $\mu_X$, $\mu_Y$ on the spaces $C_X$, $C_Y$, then define the Hilbert space $H_Y = L^2(C_Y)$. For the morphism $C_{Y_0} \leftarrow C_X \rightarrow C_{Y_1}$ of the monoidal category $\text{Span}(\text{FT})$, we define the linear map as the push-pull-with-kernel $e^{iS_X}$:

$$
t_* \circ e^{iS_X} \circ s^* : H_{Y_0} \rightarrow H_{Y_1} \quad (22)
$$

In short, we get the monoidal functor as follow.

**Theorem 5.1** There is a monoidal functor $\heartsuit$ from the category $\text{Span}(\text{FT})$ to $\text{Hilb}$ assigning

$$
C_Y \mapsto H_Y \quad (23)
$$

$$
C_{Y_0} \leftarrow C_X \rightarrow C_{Y_1} \mapsto t_* \circ e^{iS_X} \circ s^* : H_{Y_0} \rightarrow H_{Y_1} \quad (24)
$$

where $t : C_X \rightarrow C_{Y_0}$ and $s : C_X \rightarrow C_{Y_1}$.

**Proof.** From the Chern-Simons theory, we can easily to get $C_Y \mapsto H_Y$, and the above constructions just interpret that $\heartsuit$ is a functor. What we need to check is that this functor is a monoidal, i.e.

$$
t'_* \circ \exp(iS_{X'}) \circ s'^* \circ t_* \circ \exp(iS_X) \circ s^* = t'_* \circ r'_* \circ \exp(iS_{X'_{oX}}) \circ r^* \circ s^* \quad (25)
$$

where $t' : C_{X'} \rightarrow C_{Y_2}$, $s' : C_{X'} \rightarrow C_{Y_1}$, $r : C_{X'_{oX}} \rightarrow C_X$, and $r' : C_{X'_{oX}} \rightarrow C_{X'}$

and

$$
\heartsuit(C_{Y_0\cup Y_1}) = H_{Y_0} \otimes H_{Y_1} \quad (26)
$$

(25) is hold, since the pushforward $t_*$ and $t'_*$ are integrations and $e^{2\pi i S_{X\cup X'}(\Theta_1 \cup \Theta_2)} = e^{2\pi i S_X(\Theta)} \otimes e^{2\pi i S_{X'}(\Theta_2)}$.

And (26) is hold, since $L_{\eta_1 \cup \eta_2} \cong L_{\eta_1} \otimes L_{\eta_2}$.

The proof is complete.

From the above proof, we can conclude that the functor $\heartsuit$ is equivalent to the ”additivity” law of Chern-Simons theory.
5.2 Construction of TQFT

Using the above two monoidal functors, we can construct our 3d-TQFT.

**Theorem 5.2** The functor

\[ Z = \bigtriangledown \circ \sharp \]

is a 3d-TQFT.

**Proof.** Since both \( \bigtriangledown \) and \( \sharp \) are monoidal functors, so is \( Z \). Then \( Z \) is a 3d-TQFT from the definition of nd-TQFT. The proof is complete.

6 Conclusion

For abstract TQFT, how to construct a physical one is very interesting problem. In this paper, we construct a TQFT via Chern-Simons theory, which provides invariants of 3d-manifolds, though it appeared in Freed’s paper as axioms. In fact, ideas of categorification has been in physical theory [BL], which also can be viewed as our main philosophy: how "high-algebraic" ideas from category theory can illuminate questions in string theory, quantum field theory and geometry.

In the next part, we will see how the higher structure function in the "Extended TQFT", which ultimately concern with the Chern-Simons theory attach to point.

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