Energy-Variation Analysis and Orbit-Complexity Quantification

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Abstract. In state-recurrence analysis, recurrences are considered either in the space that constitutes of the observed quantities, or in an embedding space that is manufactured by time-delayed variables. An alternative approach is proposed here. In the so-called energy-variation analysis, the complexity of an orbit is quantified in terms of the statistics of the constant-speed geodesic condition. Energy-variation analysis requires significantly less operations than state-recurrence analysis, in particular, for multivariate time-series, while numerical experiments demonstrate that the resulting energy-recurrence matrix encodes information that is sufficient for quantifying the complexity of orbits.

1. Introduction
In the context of nonlinear dynamics, characterizations of the asymptotic stability of systems, for instance, in terms of Lyapunov exponents [1], the Kolmogorov-Sinai entropy [2, 3, 4, 5, 6], fractal dimensions [1, 3, 7], and homotopy methods [8], are of great importance, since they yield bounds on the degree of fidelity of a particular model, in the presence of perturbations, such as numerical and measurement noise. In practice, the considered orbits are finite, and hence, only estimates of such characterizations are used [9, 10, 11], with state-recurrence analysis providing a general framework for feature extraction [12, 13]. State-recurrence analysis relies on the work of Poincaré and Birkhoff on recurrences, as introduced in ergodic theory [4], while it employs topological covers for extracting a range of statistical features [14] that regard the evolution of noisy, nonlinear, and nonstationary systems. Further, state-recurrence analysis can be performed using a single orbit, which is particularly beneficial whenever it is not feasible to obtain a second orbit, or feasible but computationally demanding, such as for instance, in the case of infinite-dimensional dynamical systems [15].

In this work, energy-variation analysis is proposed as an alternative to state-recurrence analysis. Energy-variation analysis can be linked to the theory of geodesics and analytical mechanics [16], and while it inherits the benefits of state-recurrence analysis, it requires significantly less floating-point operations. Fundamental properties of the so-called energy-recurrence matrix can be discussed for dynamical systems with Lipschitzian-type time-shift transformations [17, 18].

The rest of the paper is organized as follows. This introduction is followed by a brief section on notation and terminology, while in section 3, state-recurrence analysis is shortly presented and is used for deducing estimates of the second-order Rényi entropy. Energy-variation analysis is introduced in section 4, while numerical experiments that demonstrate how energy-variation
analysis can be used for classification, for the analysis of bifurcation diagrams, and for the analysis of electrocardiographs follow the theoretical presentation.

2. Terminology and Notation
Dynamical systems of the form \((X, d, f)\), with \((X, d)\) being a metric space and \(f : X \rightarrow X\) a continuous time-shift transformation, are considered here. In particular, the phase-space \(X\) is assumed to be a subset of \(\mathbb{R}^M\), for some integer \(M > 0\), and is endowed with the standard Euclidean metric whose values are 
\[
d(x, y) = \|x - y\| = \|(x - y)^T(x - y)\|^{1/2},\quad \text{with } x, y \in X.
\]
The points of \(X\) are called the states of the dynamical system under consideration and they are determined by the iterates of \(f\). More precisely, provided an initial state \(x_1 \in X\), a finite orbit of length \(N > 1\) is represented by an \(M \times N\) matrix
\[
X = (x_1, x_2, x_3, \ldots, x_N) = (f^0(x_1), f^1(x_1), f^2(x_1), \ldots, f^{N-1}(x_1)) \in \mathbb{R}^{M \times N},
\]
where \(f^n = f \circ f \circ \cdots \circ f\) denotes the \(n\)-times function composition of \(f\) with itself, \(f^0 = \text{id}_X\), \(f^1 = f\), and \(x_n = f^{n-1}(x_1)\) for all time instances \(n \in [N] = \{1, 2, \ldots, N\}\). Further, recall that a transformation \(f : X \rightarrow X\) is said to be Lipschitz if there exists a real constant \(L \geq 0\) such that 
\[
d(f(x), f(y)) \leq Ld(x, y), \quad \text{for all } x, y \in X,
\]
while the smallest such constant is denoted by \(\text{Lip}(f)\). In case \(\text{Lip}(f) < 1\), \(f\) is said to be a contraction and the Banach fixed point theorem states that \(f\) has a unique fixed point and that the iterates of \(f\) converge to that fixed point, provided that \((X, d)\) is complete.

3. State-Recurrence Analysis
State-recurrence matrices are binary matrices that are formed by thresholding distance matrices associated with orbits, and they have been introduced as an alternative graphical representation of orbits in \(\mathbb{R}^M\) [19]. More precisely, given real \(\varepsilon > 0\) and a finite sequence \(X\) of length \(N\), with \(x_n \in \mathbb{R}^M\) for all \(n \in [N]\), the recurrence matrix \(R_\varepsilon \in \{0, 1\}^{N \times N}\) is defined by
\[
R_\varepsilon(p, q) = \Theta(\varepsilon - d(x_p, x_q)) \quad \forall p, q \in [N],
\]
where \(\Theta : \mathbb{R} \rightarrow \{0, 1\}\) is the Heaviside step function whose values are \(\Theta(x) = 1\) for \(x > 0\) and \(\Theta(x) = 0\) for \(x \leq 0\). Since \(d\) is a metric, \(R_\varepsilon\) is symmetric and can be viewed as the adjacency (connectivity) matrix of an undirected graph whose vertices are labeled according to the time variable \(n\). Each block \(R_\varepsilon(1 : n, 1 : n)\), with \(n > 2\), encodes the relative amount of information that is contributed by the state \(x_n\) relative to the accumulated information from the beginning of time up to the state \(x_{n-1}\). By covering a sufficiently long orbit \(X\) with open balls of the form 
\[
B(x_n, \varepsilon) = \{y \in \mathbb{R}^M : d(x_n, y) < \varepsilon\},
\]
assigning to each ball the counting measure, and employing Birkhoff’s ergodic theorem, the instantaneous second-order Rényi entropy can be estimated by
\[
H^2(X(\cdot;1 : n)) = -\log(||R_\varepsilon(1 : n, 1 : n)||_F^2/n^2) = -\log c_n(\varepsilon),
\]
where \(\|\cdot\|_F\) is the Frobenius norm and 
\[
c_n(\varepsilon) = \|R_\varepsilon(1 : n, 1 : n)||_F^2/n^2.
\]
is here defined as the instantaneous state-recurrence rate and is commonly occurring in state-recurrence quantification analysis [13], among various other quantifiers. The recurrence rate can be used for estimating the correlation dimension \(\nu\), by employing a decreasing sequence of \(\varepsilon\)-values and the scaling law
\[
\frac{\log c_n(\varepsilon)}{\log \varepsilon} = \nu + h(\varepsilon), \quad \text{with } h(\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,
\]
provided that \(h\) vanishes sufficiently fast, as has been previously proposed and documented in the literature [9, 10, 11].
4. Energy-Variation Analysis

Energy-variation analysis can be motivated in various ways that stem from analytical mechanics, among which the theory of geodesics and the work-energy theorem are the most apparent ones [16]. More precisely, a twice-differentiable curve $x(t)$ that is constrained onto a Riemannian manifold is said to be a geodesic if $\dddot{x}(t)$ is either zero, or perpendicular to the tangent plane of that surface at $x(t)$ for all $t$. Geodesic curves have constant speed, since

\[ \frac{d}{dt} \| \dot{x}(t) \|^2 = \frac{d}{dt} (\dot{x}^\top \dot{x}) = 2 \dot{x}^\top \ddot{x} = 0. \]  

Further, let $n > 0$ be an integer, consider an ordered tuple $(x_n, x_{n+1}, x_{n+2})$ of data in $\mathbb{R}^M$, and observe that

\[ [d(x_{n+2}, x_{n+1})]^2 - [d(x_{n+1}, x_n)]^2 = (x_{n+2} - 2x_{n+1} + x_n)^\top (x_{n+2} - x_n) \]  

holds without any regularity assumptions. In view of analytical mechanics, a real factor $m/(2(\Delta t)^2) > 0$ transforms the left-hand side of equation (6) to the variation in the kinetic energy of a point-mass $m$ that is caused by the work of the exerted force on the right-hand side, in a discrete setting.

Similarly to state-recurrence analysis, energy-recurrent states are considered in energy-variation analysis. Let $X = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{M \times N}$ be a finite orbit, with each row being associated with a scalar state variable and each column with a time instance, and form the energy time-series vector $e \in \mathbb{R}^{N-1}$ whose $n$-th component is $e_n = [d(x_{n+1}, x_n)]^2$, with $n \in [N-1]$. Provided real $\varepsilon > 0$, the energy-recurrence matrix is then defined as the $(N-1) \times (N-1)$ matrix $W_\varepsilon$ whose entries are

\[ W_\varepsilon(p, q) = \Theta(\varepsilon - |e_p - e_q|) \quad \forall p, q \in [N-1], \]  

with $\Theta : \mathbb{R} \to \{0, 1\}$ being the Heaviside step function, as before. Graphical examples of the resulting objects are depicted in figure 1. Before proceeding with a presentation of the basic properties of the energy-recurrence matrix, it is worth mentioning that its computational complexity is in $O(\max\{N^2, MN\})$, in contrast to the computational complexity of the state-recurrence matrix, which is in $O(MN^2)$. Hence, the number of floating-point operations that are required for computing the energy-recurrence matrix is less than the one required for computing the state-recurrence matrix whenever $M > 1$. Further, any quantifier that has been introduced in state-recurrence quantification analysis can be used for obtaining information from $W_\varepsilon$, although the interpretation of such quantifiers needs to be adjusted accordingly; for instance, the $\pm 1$-diagonals of $W_\varepsilon$ correspond to an $\varepsilon$-approximate test for the discrete geodesic condition (6), while similar conditions are tested by the $\pm k$-diagonals for all $k \in [N-2] \setminus \{1\}$.

Given the energy-recurrence matrix $W_\varepsilon$, the instantaneous second-order Rényi entropy of the energy states ($\text{EnEntropy}, W$-entropy?) can be defined by

\[ H^2_\varepsilon(X(:, 1 : n)) = -\log(\| W_\varepsilon(1 : n, 1 : n) \|_F^2 / n^2) = -\log w_n(\varepsilon), \]  

where $w_n(\varepsilon) = ||W_\varepsilon(1 : n, 1 : n)||_F^2 / n^2$ is here defined as the instantaneous energy-recurrence rate for all $n \in [N-1]$, and is examined in the numerical section of this work.

In terms of Lipschitz functions, the construction of the energy-recurrence matrix suggests the following bound. Assume that $X \in \mathbb{R}^{M \times N}$ is an orbit of a dynamical system $(X, d, f)$ whose transformation is Lipschitz with constant $\text{Lip}(f) \geq 0$. Then, an induction argument implies that

\[ d(x_{n+1}, x_n) = d(f^{n-1}(x_2), f^{n-1}(x_1)) \leq [\text{Lip}(f)]^{n-1} d(x_2, x_1) \quad \forall n \in [N-1], \]  

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Figure 1. Time-series (left column) of the form \(\cos(\pi t) + 0.5^\alpha \cos(7^\alpha \pi t)\) for \(\alpha = 1/4\) (top) and \(\alpha = 1/2\) (bottom). The associated matrices whose entries are \(|e_p - e_q|\) (middle column), and the corresponding energy-recurrence matrices \(W_\varepsilon\) (right column) for \(\varepsilon = 0.1\); and hence, the absolute energy-difference satisfies the inequalities

\[
|e_p - e_q| \leq [d(x_{p+1}, x_p)]^2 + [d(x_{q+1}, x_q)]^2 \leq (\text{Lip}(f))^{2(p-1)} + (\text{Lip}(f))^{2(q-1)}[d(x_2, x_1)]^2. \tag{10}
\]

If \(f\) is a contraction, then its iterates converge to its unique fixed point, according to the Banach fixed point theorem, while the right-hand side of (10) is strictly decreasing with respect to both \(p\) and \(q\). This scenario regards fixed, eventually fixed, and asymptotically fixed orbits, with the energy-recurrence matrix being eventually populated by ones, provided sufficiently many iterates whose number depends on \(\varepsilon\). Without additional constraints imposed on \(X\), for non-expansive time-shift transformations with unity Lipschitz constant, only boundedness of the absolute energy-variation can be concluded from (10), while if \(f\) is expansive, then

\[
|e_p - e_q| \leq 2(\text{Lip}(f))^{2(\max\{p,q\})-1}e_1, \tag{11}
\]

which results in a time-dependent lower bound for the Lipschitz constant. Recall that, whenever \(\text{Lip}(f) > 1\), the topological entropy yields a time-independent lower bound for the Lipschitz constant, \(h_{\text{top}} \leq M \log \text{Lip}(f)\) [20].

5. Numerical Experiments
5.1. Uniformly Noisy Data Set
As a first verification experiment, energy-recurrence analysis is used for a set of points in the presence of uniform noise. More precisely, consider a set of \(N = 2000\) points that are lying on the circles

\[
C_1 = \{(x(t), y(t)) \in \mathbb{R}^2 : x(t) = \cos(t), y(t) = \sin(t), t \in [0, 2\pi]\}, \tag{12}
\]

\[
C_2 = \{(x(t), y(t)) \in \mathbb{R}^2 : x(t) = 0.5 \cos(t), y(t) = 0.5 \sin(t), t \in [2\pi, 4\pi]\}, \tag{13}
\]

with equidistant \(t\) sampling. The sampled points in \(C_1 \cup C_2\) are collected into a matrix \(C \in \mathbb{R}^{M \times N}\), with \(M = 2\), and in ascending \(t\) order, while they are perturbed by pseudorandom
Figure 2. The data points $X_\alpha$ (left column), with $\alpha \in \{0.1, 0.15, 0.2\}$, the associated energy-recurrence matrices (middle column) for $\varepsilon = 0.01$, and the evolution of the energy-recurrence rate (right column).

Numbers that are drawn from the uniform distribution on the open interval $(-1, 1)$, that is, the final data set is $X_\alpha = C + \alpha P$, where $P$ is an $M \times N$ matrix that specifies the perturbation field and $\alpha \geq 0$ is a real number that determines the strength of that perturbation field. The matrix $P$ is held constant throughout the experiments, while the energy-recurrence matrices are computed for $\varepsilon$ values relative to the maximum $|e_p - e_q|$ value. In figure 2, an expected density reduction of energy-recurrent states is observed as $\alpha$ increases, and is also reflected in the energy-recurrence rate, with larger $\alpha$ values resulting in smaller final energy-recurrence rate values. The cumulative distribution of energy-recurrent states with respect to $\varepsilon$ is depicted in figure 3, where $\log w_{N-1}(\varepsilon)$ appears to depend linearly on $\log \varepsilon$ for sufficiently small $\varepsilon$ values. Here, since higher $\alpha$ values result in a consistent horizontal and vertical displacement of a morphologically similar graph,
5.2. Energy-Variation Analysis Applied to Bifurcation Diagrams

Consider the dynamical system $(X, d, f)$, with $X = [0,1]$ and $f : X \rightarrow X$ being the logic map, that is, $f(x) = rx(1-x)$ with $r \in [0,4]$. A bifurcation diagram for this dynamical system is generated for 3000 equidistant $r$ parameter values in the range $[2.9, 4]$. For each such $r$ value, an orbit of length 20000 that starts at $x_1 = 0.1$ is computed, while only the last 1000 iterates are considered and arranged as the columns of a matrix $B \in \mathbb{R}^{M \times N}$, where $M = 1000$ and $N = 3000$. Here, the goal is to quantify the information that is encoded by the energy-variation matrix of $B$, that is, to examine the correlation of the energy of all generated orbit segments with respect to the parameter $r$. To do so, introduce the vector $p(\varepsilon) \in \mathbb{R}^{N-1}$ whose components count the recurrences of the energy of each orbit segment, that is, $p_n(\varepsilon) = \alpha 1^T W_{\varepsilon}(;n)$, with $1^T = (1, 1, \ldots, 1) \in \mathbb{R}^{N-1}$ and $\alpha$ being a normalization constant, here determined by the condition max$(p(\varepsilon)) = 1$. In figure 4, the bifurcation diagram of $(X, d, f)$ is shown and the vector $p(\varepsilon)$, for $\varepsilon \in \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$, is superimposed. Observe that the chaotic regime is characterized by $p_n(\varepsilon) \not\equiv 0$, while the period-doubling points, such as the point $r = 3$, also feature vanishing $p_n(\varepsilon)$ values, provided sufficiently small $\varepsilon$ values, since the associated orbits converge slowly to the corresponding periodic orbit. Further, the period-two orbit segment at $r \approx 3.3335445$ is $\varepsilon$-persistently discriminated from the rest period-two orbits, while a similar feature is also apparent at $r \approx 3.4930977$, in the period-four regime. In terms of sequential computational complexity, the time to compute the energy-recurrence matrix is 0.036 s, while the state-recurrence matrix requires 0.185 s.

5.3. Sinus Tachycardia Electrocardiograph

In this experiment, the orbit regards 9 s of a simulated electrocardiograph (ECG) for sinus tachycardia that has been obtained with a recently introduced differential model [21]. The considered ECG signal $X \in \mathbb{R}^{M \times N}$, with $M = 1$ and $N = 9379$, is depicted in figure 5, while the associated energy-recurrence matrix for $\varepsilon = 10^{-3}$ is shown in figure 6. As a first discriminating

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**Figure 3.** The energy-recurrence rate $w_{N-1}(\varepsilon)$ as a function of $\varepsilon$, in logarithmic scales, for the data sets depicted in figure 2, with $\alpha \in \{0.1, 0.15, 0.2\}$.
Figure 4. The vector $p(\varepsilon)$ used for characterizing the long-term orbit segments of the logistic map superimposed on its bifurcation diagram for $\varepsilon \in \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$.

Figure 5. A simulated electrocardiograph for sinus tachycardia.

signature, the energy-recurrence rate is $w_{N-1}(\varepsilon) \cong 0.194$, in contrast to the energy-recurrence rate of a normal rhythm, which is expected to be inversely proportional to the number of cardiac cycles that take place within the same time interval, that is, roughly less than 0.1. The cumulative distribution of energy-recurrent states with respect to $\varepsilon$ is depicted in figure 7 (left). On the right side of the same figure, the slopes $\alpha(\varepsilon)$ of the linear polynomials that are computed recursively by incorporating one additional data point $(\varepsilon, w_{N-1}(\varepsilon))$ at a time suggest that a linear scaling law, similar to (4), is well-satisfied up to $\varepsilon \cong 0.075$, where $\alpha(\varepsilon) \cong 0.24$. 
Figure 6. The energy-recurrence matrix (left) of the ECG depicted in figure 5 and a detail (right) of the shaded typical sinus tachycardia pattern.

Figure 7. The energy-recurrence rate $w_{N-1}(\varepsilon)$ as a function of $\varepsilon$ (left), in logarithmic scales, for the ECG depicted in figure 6, and the recursively evaluated slopes of the linear fits (right).

6. Conclusions
The framework of energy-variation analysis has been introduced, while natural extensions of existing complexity quantifiers have been proposed for characterizing the resulting energy-recurrence matrix. The long-term goals are to introduce novel complexity quantifiers that are based on the energy-recurrence matrix, to interpret these quantifiers in terms of the statistical and topological properties of dynamical systems, and to develop algorithms that will exploit the local behavior of these quantifiers for better understanding various aspects of evolution and for improving our modeling and simulation capacity of practical and realistic problems. In this direction, energy-variation analysis turned out to be an effective feature extraction methodology, while its reduced computational cost enables the analysis of large data sets.

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