Spectral Smoothing via Random Matrix Perturbations

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Abstract

We consider stochastic smoothing of spectral functions of matrices using perturbations commonly studied in random matrix theory. We show that a spectral function remains spectral when smoothed using a unitarily invariant perturbation distribution. We then derive state-of-the-art smoothing bounds for the maximum eigenvalue function using the Gaussian Orthogonal Ensemble (GOE). Smoothing the maximum eigenvalue function is important for applications in semidefinite optimization and online learning. As a direct consequence of our GOE smoothing results, we obtain an $O((N \log N)^{1/4} \sqrt{T})$ expected regret bound for the online variance minimization problem using an algorithm that performs only a single maximum eigenvector computation per time step. Here $T$ is the number of rounds and $N$ is the matrix dimension. Our algorithm and its analysis also extend to the more general online PCA problem where the learner has to output a rank $k$ subspace. The algorithm just requires computing $k$ maximum eigenvectors per step and enjoys an $O(k(N \log N)^{1/4} \sqrt{T})$ expected regret bound.

1 Introduction

In many learning problems, the parameter being learned is a matrix. A key property of (symmetric) matrices is that they, and more generally bounded operators in functional analysis, have a (real) spectrum. This opens up the possibility of using regularizers that are spectral: i.e., functions of the matrix that depend only on the spectrum. Spectral (convex) functions not only have a beautiful theory (Lewis, 2003) but also have widespread applications in machine learning problems. For instance, consider the role of trace norm in matrix completion (Candès and Recht, 2009) and the operator norm in PCA problems (Gallic, 2011, Chap. 14).

Adding a regularizer to add enough curvature to an otherwise flat convex function is equivalent to smoothing (via ‘inf-conv’ smoothing) the function’s Fenchel conjugate. When the regularizer is spectral, the inf-conv smoothing occurs via a spectral function. However, an alternative way to smooth functions is to perform stochastic smoothing (Abernethy et al., 2014; d’Aspremont and El Karoui, 2014; Duchi et al., 2011). This paper is built on a simple but fundamental observation: if the stochastic smoothing of a spectral function uses a perturbation distribution that is unitarily invariant (i.e., the density only depends on the eigenvalues, not the eigensystem) then the resulting stochastically smoothed function remains spectral (Lemma 2.1).

We build on our fundamental observation by focusing our attention on perhaps the most important non-differentiable spectral function, viz. the maximum eigenvalue function $\lambda_{\text{max}}$. Smoothings of this function have been considered in semidefinite optimization (d’Aspremont and El Karoui, 2014; Nesterov, 2007) and online learning (Garber et al., 2013). However, rigorous results are only available for deterministic soft-max smoothing (Nesterov, 2007) and for low rank stochastic smoothings (d’Aspremont and El Karoui, 2014; Garber et al., 2015). We consider stochastic smoothing using the Gaussian Orthogonal Ensemble (GOE). The GOE is a classical random ensemble considered in
random matrix theory [Mehta, 2004]. It has the remarkable property that its entries are independent and yet the ensemble is unitarily invariant. We establish strong smoothness of the GOE smoothing of $\lambda_{\text{max}}$ w.r.t. the operator norm (Theorem 3.3). Strong smoothness w.r.t. the operator norm is a stronger requirement than strong smoothness w.r.t. the standard Euclidean norm. Previous work had established the connection between the spectral gap and strong smoothness w.r.t. Frobenius norm. We show that this connection, in fact, persists (up to a constant factor) when we consider the stronger notion of strong smoothness w.r.t. the operator norm (Theorem 3.1).

Our results on stochastic smoothing of $\lambda_{\text{max}}$ help us make progress towards an important open problem in online learning. Namely, the construction of a minimax optimal online variance minimization algorithm that avoids a full eigen-decomposition at each step but only performs a maximum eigenvector computation. We give an algorithm, based on GOE smoothing, that only requires a maximum eigenvector computation per step and has expected regret $O((N \log N)^{1/4} \sqrt{T})$. The previous best algorithm of Garber et al. (2015) (which is actually quite recent) had a worse regret guarantee of $O(\sqrt{NT})$. For the more general online PCA problem, where the learner has to output a rank $k > 1$ subspace, we give an algorithm that just computes $k$ maximum eigenvectors per step and enjoys a regret of $O(k(N \log N)^{1/4} \sqrt{T})$. Our algorithm is not only faster than the best previous algorithm (to reach a desired level of average regret per time step) but is also easier to implement and analyze.

2 Stochastic Smoothing with a Unitary Invariant Ensembles

In this section, we define spectral functions and unitary invariant ensembles (UIE) and make a key observation: smoothing a spectral function via a UIE keeps the function spectral.

Notation and Key Definitions. We use $S^N$ to denote the set of $N \times N$ symmetric real matrices, and $A$ to denote an arbitrary element of $S^N$. Let $\lambda : S^N \rightarrow \mathbb{R}^N$ be a function that outputs the eigenvalues of a matrix in decreasing order, and let $\text{diag}$ be a function that takes a vector and outputs a diagonal matrix whose $i$-th diagonal is the $i$-th coordinate of the argument. We will denote orthogonal matrices by $U, V$. Note that $U$ is orthogonal if $U^T U = I$. Since there is no difference between the transpose and conjugate transpose for real matrices, we will use ‘unitary’ and ‘orthogonal’ interchangeably. The $\ell_p$ norm $\|x\|_p$ is $(\sum_i |x_i|^p)^{1/p}$ and the Schatten $p$-norm of $A$ is the $\ell_p$ norm of $\lambda(A)$. The trace and operator norms are the Schatten 1 and $\infty$ norms respectively. The Bregman divergence $D_f(x, y)$ is defined as $f(x) - f(y) - \langle \nabla f(y), x - y \rangle$. A function $f$ is $\alpha$-strongly convex (resp. $\alpha$-strongly smooth) w.r.t. a norm $\|\cdot\|$ if $D_f(x, y) \geq \alpha / 2 \|x - y\|^2$ (resp. $D_f(x, y) \leq \alpha / 2 \|x - y\|^2$). A function $F$ on $S^N$ is spectral if it is a function of the eigenvalues, i.e., $F$ can be written as $F = f \circ \lambda$. We say that $F$ is a spectral extension of a vector function $f$, i.e., $F = f \circ \lambda$ for some $f : \mathbb{R}^N \rightarrow \mathbb{R}$. A unitary invariant ensemble (UIE) is a distribution over matrices such that for any $A$ in the support of $D$, any unitary transformation $U^T AU$ of $A$, where $U$ is unitary, is also in the support and has the same density as $A$. Finally, the stochastic smoothing of a function $g$ with a distribution $D$ is $\tilde{g}_D(A) = \mathbb{E}_{Z \sim D}[g(A + Z)]$.

2.1 UIE Smoothing of a Spectral Function is Spectral

We start by noting that the spectral extension $F$ of a convex vector function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is also convex (Baes, 2007, Theorem 41). Spectral functions are invariant to unitary transformations, i.e.,

\begin{equation}
F(\text{VAV}^T) = F(A), \text{ for any unitary } V.
\end{equation}

Furthermore, at points where $f$ is differentiable, the gradient has the same eigenvectors as $A$ (see, e.g., (Lewis, 1996, Corollary 3.14) and (Baes, 2007, Corollary 31)):

\begin{equation}
\nabla (f \circ \lambda)(A) = U^T \nabla f(\text{diag}(\lambda(A)))U,
\end{equation}

where $U$ is a unitary matrix such that $A = U \text{diag}(\lambda(A)) U^T$. It follows that

\begin{equation}
\nabla F(\text{VAV}^T) = \text{V}\nabla F(A)\text{V}^T, \text{ for any unitary matrix } V.
\end{equation}

We know give a key result of this section followed by a simple corollary.

\textbf{Lemma 2.1.} Let $D$ be a UIE, $A$ be a symmetric matrix, and $V$ be a unitary matrix. Then,

\[ \tilde{F}_D(VAV^T) = \tilde{F}_D(V) \] and \[ \nabla \tilde{F}_D(VAV^T) = V\nabla \tilde{F}_D(A)V^T. \]

\textbf{Corollary 2.2.} Convex conjugate of $\tilde{F}$ is also a spectral function.
2.2 Strong Smoothness/Strong Convexity of Spectral Functions

We have seen some remarkable properties of spectral functions: the spectral extension of a convex function is convex; the convex conjugate of a spectral function is spectral; the (sub)gradient of a spectral function shares the same eigensystem as the matrix we are evaluating the gradient. In view of these facts, it is natural to conjecture that if a function is \( \alpha \)-strongly convex (smooth) w.r.t. a vector norm, then its spectral extension is \( O(\alpha) \)-strongly convex (smooth) w.r.t. the spectral extension of the norm. As a matter of fact, the converse of this conjecture is easy to prove.

**Corollary 2.3.** If \( F = f \circ \lambda \) is a spectral function that is \( \alpha \)-strongly convex (smooth) w.r.t. a spectral norm, then \( f \) is \( \alpha \)-strongly convex (smooth) w.r.t. the corresponding vector norm.

To the best of our knowledge, the conjecture itself has neither been proved or disproved. Particular cases of the conjecture are actually known to be true. When the vector norm is Euclidean, [Baes, 2007, Theorem 41, part 3] showed that if \( f \) is \( \alpha \)-strongly convex (smooth) w.r.t. the Euclidean norm, then the spectral extension of \( f \) is \( \alpha \)-strongly convex (smooth) w.r.t. the Frobenius norm. This behavior is indeed expected because the Frobenius norm of a matrix is exactly the Euclidean norm of the vector representation of the matrix.

For the other norms, there is no general result but there are few important special cases of the conjecture. For example, the Shannon entropy is \( \frac{1}{2} \)-strongly convex w.r.t. the \( \ell_1 \)-norm on a probability simplex; it is known ([Kakade et al., 2012]) that its spectral extension, known as the von Neumann entropy, is \( \frac{1}{2} \)-strongly convex w.r.t. the trace norm on density matrices. Similarly, squared Schatten-\( p \) norms remain strongly convex w.r.t. the Schatten-\( p \) norm just like their \( \ell_p \) counterparts ([Ball et al., 1994]). Fortunately, our proof strategy in later section does not depend on this conjecture, as we directly prove strong smoothness of spectral functions without making explicit reference to the underlying vector function.

3 Stochastic Smoothing of the Maximum Eigenvalue with UIE

In this section, we consider various stochastic smoothings of an important spectral function, namely the maximum eigenvalue function \( \lambda_{\max}(A) \). This function naturally arises in semidefinite optimization, as a dual of primal programs with fixed trace ([Helmberg and Rendl, 2000]).

3.1 Local Smoothness and Spectral Gap

In the theorem below, we show that the local strong smoothness constant of \( \lambda_{\max} \) scales as the inverse spectral gap and hence cannot be finitely bounded. Therefore fast optimization methods for smooth functions such as Nesterov’s AGD ([Nesterov, 1983]) cannot be applied.

**Theorem 3.1.** Let \( A \) have largest eigenvalue \( \lambda_{\max}(A) \) with multiplicity one. Let \( \delta(A) = \lambda_{\max}(A) - \lambda_2(A) > 0 \). Then the function \( \lambda_{\max}(A) \) is locally strongly smooth with strong smoothness constant at most \( \frac{2}{\delta(A)} \) w.r.t. the Schatten-\( \infty \)-norm and at most \( \frac{1}{\delta(A)} \) w.r.t. the Frobenius norm.

**Proof.** The strong smoothness constant with respect to the Frobenius norm is provided by ([d’Aspremont and El Karoui, 2014, Theorem 3.2]). An adaptation of their proof obtains the Schatten-\( \infty \) case result (See Appendix [D]).

[Nesterov, 2007, Sec. 2.2] showed that certain deterministic smoothings of \( \lambda_{\max} \), such as softmax of eigenvalues and inf-conv smoothing with the dual of Schatten-\( p \) norm, gives strong smoothness guarantees with a finite smoothness constant. The major downside of these smoothing techniques, however, is that they require computing a full eigenvalue decomposition. On the other hand, it is generally efficient to obtain a single unbiased estimate of the gradient of a stochastically smoothed function, as we show later in this section.

3.2 Smoothing Parameters

In order to compare various smoothing techniques, we use the following definition of smoothing parameters. The original definition is due to [Beck and Teboulle, 2012, Definition 2.1], and we present a slightly simplified version of it.
Definition 3.2. Let \( f \) be a closed proper convex function. A collection of functions \( \{ \hat{f}_\eta : \eta \in \mathbb{R} \} \) is said to be an \((\alpha, \beta)\)-smoothing with respect to \( \| \cdot \| \) if, for every \( \eta > 0 \),

\[
\begin{align*}
(\text{i}) & \quad \sup_{A \in \operatorname{dom}(f)} \{ f(A) - \hat{f}_\eta(A) \} + \sup_{A \in \operatorname{dom}(f)} \{ \hat{f}_\eta(A) - f(A) \} \leq \alpha \eta, \text{ and} \\
(\text{ii}) & \quad \hat{f}_\eta \text{ is } \frac{2}{\eta} \text{-strongly smooth with respect to } \| \cdot \|.
\end{align*}
\]

We say \( \alpha \) is the deviation parameter, and \( \beta \) is the (strong) smoothness parameter.

The smoothing parameters \((\alpha, \beta)\) capture the inherent tradeoff between the bias introduced \((\alpha)\) and the smoothness gained \((\beta)\). Note that they play a crucial role in Section 5 and 6, where we use the fact that a strongly smooth function gives rise to an online learning algorithm whose worst-case regret is characterized by the smoothing parameters.

### 3.3 Gaussian Orthogonal Ensemble

Gaussian Orthogonal Ensemble (GOE) is a distribution over real symmetric matrices whose upper triangular entries are i.i.d. normal random variables with mean zero and variance \(1/2\) and diagonal entries are i.i.d. standard normal (and also independent of the upper triangular entries). Alternatively, if \( Y \) is an \( N \times N \) random matrix with i.i.d. Gaussian entries, then \((Y + Y^T)/\sqrt{2}\) follows GOE. The density measure \( \mu \) of GOE on the space of real symmetric matrices can thus be written as

\[
\mu_{\text{GOE}}(Z) = C \exp \left( -\sum_{i<j} Z_{ij}^2 \right)
\]

where \( C \) is a normalizing constant. Using the fact that \((AA^T)_{ij} = \sum_{i,j} A_{ij}^2\), we can express \( \mu \) more concisely: \( \mu_{\text{GOE}}(Z) = C \exp(\|\text{tr}(AA^T)/2\|) \), which makes it clear that GOE is a UIE. In fact, GOE is the only UIE with independently and identically distributed entries, up to a multiplicative constant factor difference between the density functions of on-diagonal entries and that of off-diagonal entries (Liu [2004], Theorem 2).

In this subsection, we use \( \hat{\lambda}_{\max} \) to denote the GOE smoothing of max eigenvalue function with scaling parameter \( \eta \), i.e.,

\[
\hat{\lambda}_{\max}(A; \eta) = \mathbb{E}_{Z \sim \text{GOE}}[\lambda_{\max}(A + \eta Z)]
\]

Since \( \lambda_{\max} \) is differentiable almost everywhere, \( \nabla \hat{\lambda}_{\max}(A; \eta) = \mathbb{E}_{Z \sim \text{GOE}}[\nabla \lambda_{\max}(A + \eta Z)] \) and the max eigenvector of \((A + Z)\) times its transpose gives an unbiased estimate.

#### Folding a Symmetric Matrix

Any matrix \( A \in \mathbb{S}^N \) can be uniquely described by the elements in its upper triangle. For a random matrix distribution \( D \) over a subset of \( \mathbb{S}^N \), the upper triangle of \( Z \) determines the density. So, we can treat the matrices as if they are vectors of dimension \( N(N+1)/2 \).

#### Previous Analysis

d’Aspremont and El Karoui [2014] Lemma 3.1 showed that \( \{ \hat{\lambda}_{\max}(A; \eta) \} \) is an \((\sqrt{N}, 2\sqrt{N})\)-smoothing of \( \lambda_{\max} \) with respect to the Frobenius norm \( \| \cdot \|_F \). The dimension dependence of the smoothing parameters can be improved to \((\sqrt{N}, 1)\) by applying (Duchi et al., 2011, Lemma 9) instead. Now consider the smoothing parameters with respect to the operator norm \( \| \lambda(\cdot) \|_{\infty} \). (Duchi et al., 2011, Lemma 9) can be easily generalized to an arbitrary norm to show that

\[
\| \nabla \hat{\lambda}_{\max}(A) - \nabla \hat{\lambda}_{\max}(B) \|_1 \leq 1/\eta \| A - B \|_F
\]

(See Appendix B). Since \( \| A - B \|_F \leq \sqrt{2\log N}/\eta \| A - B \|_{\infty} \), it follows that GOE is an \((\sqrt{N}, \sqrt{N})\)-smoothing of \( \lambda_{\max} \) with respect to the norm \( \| \lambda(\cdot) \|_{\infty} \). An important thing to note is that the result of [Duchi et al. 2011] primarily concerns with vector problems; they take i.i.d. samples for each coordinate, and their proof relies on the rotation invariance of a Gaussian vector. Note that the full matrix defined by “unfolding” the \( N(N+1)/2 \) samples of i.i.d. Gaussian noise, however, is not unitary invariant. As we show shortly, the unitary invariance is key to a simpler and tighter analysis of GOE smoothing.

Theorem 3.3. GOE smoothing (Equation 4) is an \((\sqrt{N}, 2\log N)\)-smoothing of \( \lambda_{\max} \) with respect to the Schatten \( \infty \)-norm, and \((\sqrt{N}, 1)\)-smoothing of \( \lambda_{\max} \) with respect to the Schatten 1-norm.

Proof. Deviation parameter: GOE smoothing always overestimates \( \lambda_{\max} \), and the deviation is at most \( \eta \mathbb{E}_{Z \sim \text{GOE}}[\lambda_{\max}(Z)] = \eta \sqrt{N} \) because \( \lambda_{\max}(A + \eta Z) \leq \lambda_{\max}(A) + \eta \lambda_{\max}(Z) \).
Strong smoothness parameter: Due to the unitary invariance, we have that
\[ D_{\lambda_{\max}}(A, B) = D_{\lambda_{\max}}(U^T AU, \text{diag}(\lambda(B))) \] (6)
and \( \|A - B\| = \|U^T AU - \text{diag}(\lambda(B))\| \) where \( U \) is a matrix of eigenvectors of \( B \). So, it suffices to bound \( \sup_{b, \|b\| = 1} \nabla^2 \tilde{\Phi}(\text{diag}(b), \text{diag}(b)) \).
Let \( H \) be a matrix with entries \( H_{ij} = \frac{\partial \tilde{\Phi}}{\partial A_{ii} \partial A_{jj}} \). Then, \( \nabla^2 \tilde{\Phi}(\text{diag}(b), \text{diag}(b)) = b^T H b \). Because the trace of the gradient is fixed, the sum of each column and row of \( H \) must be 0 and the off-diagonals are negative. Therefore,
\[
\sup_{b, \|b\| = 1} b^T H_{ij} b \leq \sum_{i,j} |H_{ij}| = \sum_{H_{ij} > 0} H_{ij} + \sum_{H_{ij} < 0} -H_{ij} = 2 \sum_{H_{ij} > 0} H_{ij} = 2 \text{tr}(H).
\]

We first use the folding trick to represent the matrices as vectors with a reduced dimensionality, and then apply [Abernethy et al. 2014, Equation 9]:
\[ H_{ii} = \eta^{-1} \mathbb{E}_{Z \sim \text{GOE}}[\nabla_{ii} \lambda_{\max}(A + \eta Z)^2 Z_{ii}] \] (7)
Let \( v_1 = v_1(A + \eta Z) \) be the leading eigenvector of \( A + \eta Z \) (unique with probability 1). Since \( \nabla \lambda_{\max} = v_1 v_1^\top \) is a PSD matrix with trace equal to 1, we have that:
\[
\sum_{i=1}^N (v_1, i(A + \eta Z))^2 Z_{ii} \leq \max_i Z_{ii} \leq \sqrt{2 \log N}.
\]
To bound the smoothness with respect to the Schatten 1-norm, we bound \( b^T H b \) for \( b \) in the \( \ell_1 \) unit ball. The smoothness constant is simply the maximum diagonal entry of \( H \):
\[
\max_i H_{ii} = \eta^{-1} \max_i \mathbb{E}_{Z \sim \text{GOE}}[\nabla_{ii} \lambda_{\max}(A + \eta Z)^2 Z_{ii}] \leq \eta^{-1} \mathbb{E}_i |Z_{ii}| = \eta^{-1}.
\]

### 3.4 Multiple Rank-1 Perturbations
The two-step stochastic smoothing in the lemma below was proposed by d’Aspremont and El Karoui (2014). We show that this smoothing too preserves spectralness.

**Lemma 3.4.** Let \( F \) be a spectral function and \( Z^{(1)}, \ldots, Z^{(k)} \) be iid draws from a UIE. The two-step stochastic smoothing \( \mathbb{E}_{\max_{i=1,...,k}} F(A + \eta Z^{(i)}) \) is also a spectral function. For the case \( k \geq 3 \) and \( Z^{(i)} = z^{(i)}(z^{(i)})^\top \), \( z^{(i)} \) i.d. standard multivariate normal, we can extend (d’Aspremont and El Karoui, 2014, Proposition 3.7) where we now bound the smoothness constant w.r.t. the Schatten \( \infty \)-norm as opposed to the Frobenius norm. The upper bound on the smoothness constant is only doubled. We skip the proof since it is the same as theirs, except whenever they use the smoothness constant w.r.t. the Frobenius norm (d’Aspremont and El Karoui, 2014, Theorem 3.2), we invoke our Theorem [5,7] instead.

**Lemma 3.5.** Let \( k \geq 3 \). Then,
\[
\tilde{\lambda}_{\max}(A; \eta) = \mathbb{E}_{Z \sim \mathcal{N}(0, I)} \left[ \max_{i=1,...,k} \lambda_{\max}(A + \eta z^{(i)}(z^{(i)})^\top) \right]
\]
is an \( O(N \log k), 2k/(k - 2) \)-smoothing of \( \lambda_{\max} \) with respect to the Schatten \( \infty \)-norm.

### 4 Online Linear Optimization via Stochastic Smoothing
**Problem Setting and Notations** Online linear optimization (OLO) with the decision set \( X \), and the reward set \( Y \) is defined to be the following iterative process: On round \( t = 1, \ldots, T \),
- the learner plays \( X_t \in X \).
- the adversary reveals \( A_t \in Y \).
- the learner receives a reward \( (X_t, A_t) = \text{tr}(X_t A_t^\top) \).
For online linear optimization on matrices, we require that $\mathcal{X}$ and $\mathcal{Y}$ are subsets of $S^N$ and $\mathcal{X}$ is convex. We use $A_{1:t}$ to denote the cumulative reward $\sum_{s=1}^{t} A_s$. For a given OLO problem, we use $\Phi_X$ to denote the support function of $\mathcal{X}$, which we call the baseline potential. We can write the regret in terms of the baseline potential function:

$$\text{Regret} = \Phi_X(A_{1:T}) - \langle X_t, A_t \rangle$$

This paper focuses on Gradient-Based Prediction Algorithm (GBPA) (Abernethy et al., 2014), which includes a broad family of online learning algorithms such as Follow the Regularized Leader and Follow the Perturbed Leader. The GBPA with a proxy potential function $\tilde{\Phi}$ plays $X_t = \nabla \tilde{\Phi}(A_{1:t-1})$ at each time $t$. Its regret can be written as:

$$\text{Regret} \leq \left[ \Phi_X(A_{1:t}) - \tilde{\Phi}(A_{1:t}) + \tilde{\Phi}(0) - \tilde{\Phi}_{t-1}(0) + \sum_{t=1}^{T} D_{\Delta}(A_{1:t}, A_{1:t-1}) \right]$$

**Follow the Perturbed Leader** Consider the following type of stochastic smoothing of $\Phi_X$:

$$\tilde{\Phi}(A; \eta, D) \overset{\text{def}}{=} E_{Z \sim D}[\Phi_X(A + \eta Z)] = E_{Z \sim D} \left[ \max_{x \in \mathcal{X}} \{ \langle x, A + \eta Z \rangle \} \right] \quad (9)$$

where $D$ is a continuous distribution over a subset of $S^N$ and $\eta > 0$ is a scaling parameter. The GOE smoothing and d’Aspremont and El Karoui (2014)’s smoothing with $k = 1$ are of this type. If the $\max$ inside the expectation has a unique maximizer almost everywhere, then it is differentiable with probability 1 and we can swap the expectation and gradient (Bertsekas, 1973, Proposition 2.2):

$$\nabla \tilde{\Phi}(A) = E_{Z \sim D} \left[ \arg \max_{x \in \mathcal{X}} \{ \langle x, A + \eta Z \rangle \} \right]. \quad (10)$$

Each $\arg \max$ evaluation is equivalent to the decision rule of FTPL (Hannan, 1957; Kalai and Vempala, 2005). GBPA with $\tilde{\Phi}$ can thus be seen as playing the expected action of FTPL. Since the learner gets a linear reward in online linear optimization, the regret of the GBPA with $\tilde{\Phi}$ is equal to the expected regret of FTPL.

**Follow the Leader of Perturbed Leaders** Consider the two-step smoothing (Lemma 3.4) of $\Phi_X$. Its gradient can be written as: $E_{Z^{(1)}, \ldots, Z^{(k)}} \nabla \tilde{\Phi}_X(A + Z^{(i^*)})$, where $i^* = \arg \max_{i=1, \ldots, k} \Phi_X(A + Z^{(i)})$. The GBPA with this smoothing plays the expected action of Follow the Leader of Perturbed Leaders algorithm, defined as follows:

1. Draw $Z^{(i)}$, $i \in \{1, \ldots, k\}$ i.i.d.
2. Define $i^* = \arg \max_{i=1, \ldots, k} \Phi_X(A + \eta Z^{(i)})$
3. Play $\arg \max_{x \in \mathcal{X}} \{ \langle x, A_{1:t} + \eta Z^{(i^*)} \rangle \}$

**Algorithm 1**: Follow the Leader of Perturbed Leaders (FTLPL) Algorithm

Implicit Spectral Regularization An interesting fact about the GBPA on a spectral function is that it is equivalent to Follow the Regularized Leader (FTRL) algorithm with a spectral regularizer. As in Abernethy et al. (2014), FTRL with $\tilde{\Phi}^\ast$ plays at time $t$ the move $X_t = \arg \max_{x \in \mathcal{X}} \{ \langle x, A_{1:t-1} \rangle - \tilde{\Phi}^\ast(x) \}$, which can be seen as the gradient of $\Phi(A) = \max_{x \in \mathcal{X}} \{ \langle x, A \rangle - \Phi^\ast(x) \}$, evaluated at $A_{1:t-1}$. In other words, if the decision set $\mathcal{X}$ is spectral, i.e., $\{ U^T X U : U \text{ is unitary and } X \in \mathcal{X} \} = \mathcal{X}$, then the baseline potential function is spectral and the expected action of FT(L)PL with UIE perturbations is equivalent to that of some instance of FTRL. In particular, if $\Phi$ is a spectral function $\alpha$-strongly smooth with respect to $\| \cdot \|$, then FTPL (or FTLPL) on $\tilde{\Phi}$ is equivalent to FTRL with a spectral regularizer that is $1/\alpha$-strongly convex with respect to $\| \cdot \|_\ast$.  

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5 Online Variance Minimization

In the online variance minimization problem (Warmuth and Kuzmin, 2006), at each round, adversary plays a covariance matrix $A$ and learner plays a vector $w$. The variance of the random loss with covariance $A$ is $\text{Var}(w \cdot \ell) = w^T A w$. We consider the case where the learner’s decision set is the unit Euclidean sphere. The baseline potential function is the maximum eigenvalue norm of matrix:

$$\Phi(A) = \max_{w: ||w||_2 = 1} w^T A w = \max_{X: \text{tr}(X) = 1, X \in S_N^+} \text{tr}(X A) = ||\lambda(A)||_{\infty}$$

where the maximum over $w$ is achieved at the leading eigenvector of $A$. In this section, we use the smoothness results from Section 2 and prove the regret of the GBPA on various stochastic smoothings of $\lambda_{\text{max}}$. The same regret bounds hold in expectation for FT(L)PL algorithms. We first state a key lemma that will greatly simplify our regret analysis.

**Lemma 5.1.** (Abernethy et al., 2014, Corollary 4) Let $\Phi_X$ be the baseline potential for an online linear optimization problem. Suppose $\{\tilde{\Phi}_\eta\}$ is a $(\alpha, \beta)$-smoothing of $\Phi_X$ with respect to $\| \cdot \|$. Then, the GBPA with $\tilde{\Phi}$ has regret at most

$$\text{Regret} \leq \eta \alpha + \frac{\beta}{2\eta} \sum_{t=1}^T ||A_t||^2 \leq \sqrt{\frac{\alpha \beta}{2} \sum_{t=1}^T ||A_t||^2}.$$ 

**Theorem 5.2.** The GBPA with low-rank stochastic smoothing as in Equation 8 (and the corresponding FTPL presented in Algorithm 1) has regret at most

$$\sum_{t=1}^T \eta^{-1} 2(\lambda_{\text{max}}(A_t))^2 + \eta N = O(\sqrt{NT}).$$

*Proof.* Apply Lemma 5.1 with smoothness parameters (Lemma 3.3) w.r.t. Schatten-$\infty$ norm. □

The strong smoothness guarantee of Lemma 5.1 holds for $k > 3$. Recently, Garber et al. (2015) showed that $k = 1$ case still has $O(\sqrt{NT})$ regret bound. Although $k = 1$ gives an unbounded strong smoothness constant of $1/(\lambda_1(A + zz^T) - \lambda_2)$, the gap is big with high probability. If the rare event of small eigengap occurs, they use a worst-case constant bound on the Bregman divergence itself: $D_{\lambda_{\text{max}}}(A_{1:t}, A_{1:t-1}) \leq ||\lambda(A_t)||_{\infty}$ that follows from the Lipschitzness of $\lambda_{\text{max}}$.

**Theorem 5.3.** The GBPA with GOE perturbation (and the corresponding FTPL algorithm) has (expected) regret at most

$$\sum_{t=1}^T \sqrt{\log N (\lambda_{\text{max}}(A_t))^2} + \eta \sqrt{N}$$

where $\eta > 0$ is the scaling parameter.

*Proof.* Apply Lemma 5.1 with smoothness parameters (Theorem 5.3) w.r.t. Schatten-$\infty$ norm. □

If $\lambda_{\text{max}}(A_t) \leq 1$, then the above theorem implies that FTPL with GOE has $O(\sqrt{T(N \log N)^{1/4}})$ expected regret. Since FTPL computes one leading eigenpair per iteration ($O(N^2)$ time), it takes $O((\sqrt{N \log N}) (N^2) /\epsilon^2)$ time to converge to $\epsilon$ average regret. The convergence rate is faster than the previously known $O(N^3/\epsilon^2)$ (Garber et al., 2015) with rank-1 smoothing and $O(N^3 \log N / \epsilon^2)$ with von Neumann entropy regularization (Warmuth and Kuzmin, 2006).

6 Online PCA

We consider the online PCA problem (Warmuth and Kuzmin, 2008) where the learner finds the best $k$-rank subspace of a covariance matrix online. We show that FTPL converges faster to $\epsilon$-regret than Exponentiated Gradient (Nie et al., 2013) does for this problem.

6.1 Sparse Online PCA

At each time $t$, the learner chooses a projection matrix $X_t$ of rank $k$. Then, adversary reveals the next unit vector $a_t$ and the learner suffers a loss

$$||a_t - X_t a_t||_2^2 - 1 = \text{tr}((I - X_t) a_t a_t^\top) - 1 = \text{tr}(a_t a_t^\top) - \text{tr}(X_t a_t a_t^\top) - 1 = -\text{tr}(X_t a_t a_t^\top).$$
Or equivalently, the learner gains a linear reward $\text{tr}(X_tA_t)$ where $A_t = a_t a_t^T$. This problem is called sparse online PCA because the reward matrix $A_t$ is rank-1. In order to cast the online PCA as OLO, we allow the learner to choose an action from the convex hull of the set of rank $k$ projection matrices, called the $k$-fantope defined as $\{X : 0 \preceq X \preceq I \text{ and } \text{tr}(X) = k\}$ (Dattorro, 2008). The baseline potential for this OLO is

$$
\Phi_{\text{OPCA}}(A) = \max_{X \in k\text{-fantope}} \langle X, A \rangle = \text{sum of top } k \text{ eigenvalues}
$$

whose gradient exists whenever all the top $k$ eigenvalues have multiplicity 1:

$$
\nabla \Phi_{\text{OPCA}}(A) = \arg \max_{X \in k\text{-fantope}} \langle X, A \rangle = \sum_{i=1}^{k} v_k v_k^T
$$

where $v_k$ is the $k$-th leading eigenvector of $A$.

**Theorem 6.1.** For the sparse online PCA, the GBPA on the GOE smoothing of $\Phi_{\text{OPCA}}$ (and the corresponding FTPL algorithm) has (expected) regret of the order $O(\sqrt{kT} \sqrt{N})$.

**Proof.** Note that adversary’s reward set $\{aa^T : \|a\|_2 = 1\}$ is a subset of $\{A : \text{tr}(A) = 1, 0 \preceq A\}$. By Lemma 5.1, it suffices to show that the GOE smoothing is an $(\sqrt{N}, 1)$-smoothing of $\Phi_{\text{OPCA}}$ with respect to the Schatten 1-norm. For the deviation parameter, note that

$$
\mathbb{E}[\Phi_{\text{OPCA}}(Z) \leq k \mathbb{E}[\|\lambda_{\text{max}}(Z)\|] = O(k \sqrt{N})
$$

(11)

The strong smoothness parameter of 1 can be directly obtained from Theorem 3.3.

MEG has a regret bound of $\sqrt{2kT \log \frac{N}{k}}$. Combined with $O(N^3)$ per-iteration time complexity for SVD, it takes $O(2kN^3 \log \frac{N}{k}/\epsilon^2)$ convergence rate to $\epsilon$-regret. On the other hand, FTPL requires one leading eigenpair computation which requires $O(N^2)$ time, and thus it takes $O(\sqrt{N^2} \frac{2}{\epsilon^2})$ time to converge to the average regret of $\epsilon$, which is faster than MEG for any choice of $k$.

Another advantage of FTPL is the easier implementation; MEG outputs an element in the $k$-fantope, and one needs to decompose it into a convex combination of rank $k$ projection matrices and randomly sample one in order to play a valid move. FTPL, on the other hand, plays a rank $k$ matrix $\nabla \Phi_{\text{OPCA}}(A + Z)$ every round. Furthermore, since the eigenvalues of $Z$ repel each other, we know that $A + Z$ has no eigenvalue of multiplicity $> 1$ with probability one, and with high probability it has a large eigengap which improves the convergence rate for most commonly used iterative methods for computing leading eigenpairs.

The differential privacy literature also offers a very interesting interpretation of Gaussian smoothing in this setting. In fact, the notion of privacy has a very close connection to strong smoothness, and the technique of adding Gaussian perturbations, referred to as Gaussian mechanism, has been well studied. In particular, a recent FTPL analysis by Dwork et al. (2014, Theorem 9) via differential privacy obtains a near-optimal regret bound of $O(\sqrt{\frac{N}{k}} \log T \sqrt{N})$.

### 6.2 Dense Online PCA

In the dense online PCA problem, adversary plays a full-rank covariance matrix with $\lambda_{\text{max}} \leq 1$.

**Theorem 6.2.** For the dense online PCA, the GBPA on the GOE smoothing (and the corresponding FTPL algorithm) of $\Phi_{\text{OPCA}}$ has (expected) regret of the order $O(\sqrt{kN} \log N)^{1/4} \sqrt{T}$.

**Proof.** Since adversary is constrained in the Schatten $\infty$-norm, by Lemma 5.1, it suffices to show that the GOE smoothing is an $(\sqrt{N}, 1)$-smoothing of $\Phi_{\text{OPCA}}$ with respect to the Schatten $\infty$-norm. The deviation parameter is the same as in the sparse online PCA problem (Equation 11). The strong smoothness parameter can be extrapolated from Theorem 3.3, from Equation 7, we use the fact that the gradient of $\Phi_{\text{OPCA}}$ has trace equal to $k$.

Similarly to the sparse online PCA, although MEG has a better regret bound of $k \sqrt{2T \log \frac{N}{k}}$, it takes $O(k^2 N^3 \log \frac{N}{k^2}/\epsilon^2)$ time to achieve $\epsilon$-regret, while FTPL takes $O(k^2 N^2 \frac{2}{\epsilon^2} \log N/\epsilon^2)$ time.
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A  Proof of Lemma 2.1

Proof. For simplicity, we omit the subscript $D$. Since $Z$ and $V Z V^T$ have the same density,
\[
\tilde{F}(V AV^T) \overset{\text{def}}{=} \mathbb{E}_Z[F(V AV^T + Z)] = \mathbb{E}_Z[F(V AV^T + V Z V^T)] = \mathbb{E}_Z[F(V (A + Z)V^T)] = \mathbb{E}[F(A + Z)] = \tilde{F}(A).
\]

Similarly,
\[
\nabla \tilde{F}(V AV^T) = \mathbb{E}[\nabla \Phi(V AV^T + Z)] = \mathbb{E}[\nabla \Phi(V AV^T + V Z V^T)] = \mathbb{E}[\nabla \Phi(V (A + Z)V^T)] = V \mathbb{E}[\nabla \Phi(A + Z)] V^T = V \nabla \tilde{F}(A)V^T.
\]

Note that in the first step, we were able to swap expectation and gradient because $F$ is convex (Bertsekas 1973).

B  Proof of Corollary 2.2

Proof. This directly follows from (Baes 2007, Theorem 30), which showed that the convex conjugate of the spectral extension of $f$ is a spectral extension of $f^*$.

C  Proof of Corollary 2.3

Proof. Note that by (2.1) $\text{diag}(\nabla f(y)) = \nabla F(\text{diag}(y))$. Suppose $\Phi$ is $\alpha$-strongly convex with respect to $\| \cdot \|_{\text{spe}}$, the spectral norm on matrices generated by a vector norm $\| \cdot \|$. Consider the Bregman divergence of $f$:
\[
f(x) - f(y) - \langle \nabla f(y), x - y \rangle = F(\text{diag}(x)) - F(\text{diag}(y)) - \langle \nabla F(\text{diag}(y)), \text{diag}(x) - \text{diag}(y) \rangle \\
\geq \frac{\alpha}{2} \| \text{diag}(x) - \text{diag}(y) \|_{\text{spe}} = \frac{\alpha}{2} \| x - y \|.
\]

The strong smoothness case follows the exact same argument except that the direction of inequality is flipped.

D  Proof of Theorem 3.1

Proof. Under the assumptions of the theorem, we have the following classical formula for the 2nd derivative (see (d’Aspremont and El Karoui 2014, Eq. (32))):
\[
\nabla^2 \lambda_{\text{max}}(A)[H, H] = \sum_{i=2}^{n} \frac{1}{\lambda_{\text{max}}(A) - \lambda_i(A)} (v_1(A)^\top H v_i(A))^2.
\]

Above, $(\lambda_i(A), v_i(A))$, $i \geq 1$ are the eigenpairs of $A$ with eigenvalues sorted in decreasing order. Note that the local smoothness constant, w.r.t. $\| \cdot \|_{\text{op}}$, of $\lambda_{\text{max}}(A)$ can be bounded as
\[
\sup_{\|H\|_{\text{op}} \leq 1} \nabla^2 \lambda_{\text{max}}(A)[H, H] = \sup_{\|H\|_{\text{op}} \leq 1} \frac{1}{2} \sum_{i=2}^{n} \frac{1}{\lambda_{\text{max}}(A) - \lambda_i(A)} (v_1(A)^\top H v_i(A))^2 \\
\leq \sup_{\|H\|_{\text{op}} \leq 1} \frac{1}{2} \sum_{i=2}^{n} \frac{1}{\lambda_{\text{max}}(A) - \lambda_2(A)} (v_1(A)^\top H v_i(A))^2 \\
\leq \frac{2}{\lambda_{\text{max}}(A) - \lambda_2(A)} \sup_{\|H\|_{\text{op}} \leq 1} \frac{1}{2} \sum_{i=1}^{n} (v_1(A)^\top H v_i(A))^2 \\
= \frac{2}{\lambda_{\text{max}}(A) - \lambda_2(A)} \frac{1}{\|H^\top v_1(A)\|_2^2} \\
= \frac{2}{\lambda_{\text{max}}(A) - \lambda_2(A)} \frac{1}{\delta(A)}.
\]
E  Proof of Equation 5

Duchi et al. (2011, Lemma 11) shows that the dual norm $\| \cdot \|_*$ on the lefthand side of (Duchi et al., 2011, Equation 39) can be any norm such that $f$ is $L_0$-Lipschitz with respect to $\| \cdot \|$. The rest of the proof for (Duchi et al., 2011, Lemma 9) does not depend on the choice of norm. Since $\lambda_{\text{max}}$ is 1-Lipschitz with respect to $\| \cdot \|_\infty$, we have

$$
\| \lambda(\nabla \tilde{\lambda}_{\text{max}}(A) - \nabla \tilde{\lambda}_{\text{max}}(B))\|_1 \leq 1/\eta \| A - B \|_F \leq \sqrt{N/\eta} \| A - B \|_\infty.
$$

F  Proof of Lemma 3.4

Proof. Due to the rotational invariance of $Z$,

$$
\tilde{F}(U^T A U) = \mathbb{E} \max_{i=1,\ldots,k} F(U^T A U + \eta Z^{(i)}) = \mathbb{E} \max_{i=1,\ldots,k} F(U^T A U + \eta U^T Z^{(i)} U) = \mathbb{E} \max_{i=1,\ldots,k} F(A + \eta Z^{(i)}) = \tilde{F}(A).
$$

G  Proof of Lemma 3.5

The strong smoothness parameter is proven in the main text, and we prove the deviation parameter here. Because perturbations $z z^T$ have a unique positive eigenvalue, $\tilde{\lambda}_{\text{max}}$ overestimates $\lambda_{\text{max}}$. The gap is at most $\mathbb{E} \max_{i=1,\ldots,k} \lambda_{\text{max}}(z^{(i)} z^{(i)\top})$ which is the maximum of $k$ i.i.d. chi-square random variables with degree of freedom $N$ (or equivalently, Gamma distribution with shape parameter $N/2$ and rate parameter $N^{-1}$). Standard extreme value theory results (Embrechts et al., 1997, p.156) give an $O(\log k)$ bound.