Quantum Hall Ferromagnets:
Induced Topological term and electromagnetic interactions

Rashmi Ray\textsuperscript{1}

Laboratoire de Physique Nucléaire
Université de Montréal
Montréal, Quebec H3C 3J7, Canada

Abstract

The $\nu = 1$ quantum Hall ground state in materials like GaAs is well known to be ferromagnetic in nature. The exchange part of the Coulomb interaction provides the necessary attractive force to align the electronic spins spontaneously. The gapless Goldstone modes are the angular deviations of the magnetisation vector from its fixed ground state orientation. Furthermore, the system is known to support electrically charged spin skyrmion configurations. It has been claimed in the literature that these skyrmions are fermionic owing to an induced topological Hopf term in the effective action governing the Goldstone modes. However, objections have been raised against the method by which this term has been obtained from the microscopics of the system. In this article, we use the technique of the derivative expansion to derive, in an unambiguous manner, the effective action of the angular degrees of freedom, including the Hopf term. Furthermore, we have coupled perturbative electromagnetic fields to the microscopic fermionic system in order to study their effect on the spin excitations. We have obtained an elegant expression for the electromagnetic coupling of the angular variables describing these spin excitations.

\textsuperscript{1}E-mail address: rray@lpshpb.lps.umontreal.ca
I. Introduction

It is well known that solitons play an important role in condensed matter physics [1]. Moreover, in various models describing condensed matter systems, solitons are found to have unexpected quantum numbers associated with them. For instance, as shown in [2], the skyrmionic solutions of the purely bosonic O(3) non-linear sigma model (NLSM) can become fermionic if a topological term (the Hopf term) is added on with the appropriate coefficient to the action.

The NLSM actually provides a natural description of the low energy dynamics of the Goldstone modes in a system where some continuous global symmetry of the action is broken spontaneously by the ground state. For instance, in superfluid \(^3\)He – \(A\), the SO(3) spin rotation group is spontaneously broken down to the group SO(2) about some preferred axis [3]. Since this state is also known to be anti-ferromagnetic, one can express the dynamics of the Goldstone modes, which here are anti-ferromagnetic magnons, in terms of a NLSM. What is more interesting is that a Hopf term is naturally generated in the effective theory for the magnons [4]. The significance of these induced Hopf terms in a variety of systems has been studied in [5]. In the liquid Helium system, the induced Hopf term happens to be P and T invariant, as the anti-ferromagnetic ground state evinces these discrete symmetries. However, in suitable systems, Hopf terms that break these discrete symmetries could be induced dynamically. We shall discuss such a system in this article.

Recently, it has been noticed [6] that the \(\nu = 1\) quantum Hall state in samples like GaAs (where the effective gyromagnetic ratio \(g \ll 2\)) is ferromagnetic. Thus, the global SU(2) spin symmetry of the microscopic action governing the 2-component fermions is broken spontaneously to U(1) by the ground state. The ferromagnetic ground state also breaks P and T. The Goldstone modes are the ferromagnetic magnons. An effective action governing the dynamics of these has been obtained from the microscopics in [7]. It is further known that the skyrmions in this system are electrically charged and are favoured over the
quasielectron excitations as the charge carriers. A natural question that one should pose is whether these skyrmions are bosonic or fermionic. It has been claimed [8] that they are fermionic due to an induced Hopf term. However, objections have been raised [9] against the derivation presented therein. An induced Hopf term with the appropriate coefficient has also been obtained, but within an ansatz, in [10]. In this article, we propose to address the issue of the existence of an induced Hopf term in a direct and straightforward manner.

The basic idea that we exploit is as follows. As we know, the ground state is ferromagnetic. Thus, the magnetisation vector is globally aligned in a particular direction, say the $\hat{z}$ direction, in the laboratory frame. In the excited state, it suffers local deviations from this fixed direction. We can however transform the system to a frame such that the magnetisation is locally in the $\hat{z}$ direction. The price that we pay for this is that the fermions are now coupled minimally to pure SU(2) valued gauge potentials (with zero curvature) arising from the transformation of coordinates to the local frame. We can then integrate the fermions out to obtain an effective action for the SU(2) potentials. Since these potentials encapsule all the information of the angular deviation of the magnetisation in the excited state, this effective action is also the effective action governing the spin excitations. The functional determinant obtained upon integrating the fermions out can be expanded within a derivative expansion scheme to obtain the required low energy effective action [17].

The spin skyrmions in this system are electrically charged. Thus it would be very interesting to investigate the manner in which the angular variables (Goldstone modes) couple to electromagnetism. The magnons are themselves neutral. Hence we should not expect minimal coupling [15]. In the sequel, we investigate the form of the non-minimal coupling by subjecting the microscopic fermions to perturbing electromagnetic fields.

The article is organised as follows. In section 2, we establish our notation and discuss the microscopic fermionic theory that we choose as our point of departure. In section 3, we give the form of the effective action upon collecting all the appropriate terms arising
from the derivative expansion. Section 4 is devoted to our conclusions and to discussions of further possibilities of investigations along similar lines. Details of the computation have been relegated to the appendices.

II. Notation and Formulation

The microscopic model for the quantum Hall system in samples like GaAs may be taken to be:

\[ S = S_1 + S_2 \]  \hspace{1cm} (2.1)

with

\[ S_1 \equiv \int dt \int d\vec{x} \, \psi^\dagger (\vec{x}, t) \left[ i\frac{\partial}{\partial t} - \frac{1}{2m} (\vec{p} - \vec{A})^2 + \mu \right] \psi (\vec{x}, t) \]  \hspace{1cm} (2.2)

\[ S_2 \equiv -\frac{V_0}{2} \int dt \int d\vec{x} \, (\psi^\dagger \psi)^2 \]  \hspace{1cm} (2.3)

where \( \partial_x A^y - \partial_y A^x = -B \), \( B \) being the external magnetic field and where \( \mu \) is the chemical potential. \( \psi (\vec{x}, t) \) is a 2-component spinor satisfying the usual anticommutator

\[ \{ \psi_\alpha (\vec{x}, t), \psi_\beta^\dagger (\vec{y}, t) \} = \delta_{\alpha\beta} \delta (\vec{x} - \vec{y}). \]

Here we have taken the limit \( g \to 0 \) and set the Pauli term in the action equal to zero. In this limit, the above action has an exact spin \( SU(2) \) symmetry. If restored, a small Pauli term leads to a soft explicit breaking of this symmetry.

More importantly, we note that in the above action, we have replaced the non-local repulsive Coulomb term by a local repulsive four-fermi interaction. While this has been done to simplify the analysis, it is also true that the interesting physics associated with the spontaneous symmetry breaking (SSB) is due to the short-distance part of the Coulomb interaction [11]. The long-distance part can be added as a perturbation.

The partition function of the system is written as

\[ Z = \int D\psi \int D\bar{\psi} \, e^{i(S_1 + S_2)}. \]  \hspace{1cm} (2.4)
At this point we may use the standard property of the properly normalised SU(2) generators, $t^a_{\alpha\beta} t^a_{\gamma\delta} = \frac{1}{2} \delta_{\alpha\delta} \delta_{\beta\gamma} - \frac{1}{4} \delta_{\alpha\beta} \delta_{\gamma\delta}$ to re-organise the Coulomb term as [12]

$$S_2 = V_0 \int dt \int d\vec{x} \left[ \vec{S}^2 + \frac{1}{4} \rho^2 \right]$$

(2.5)

where $\rho \equiv \bar{\psi} \psi$ is the density and $S^a \equiv \bar{\psi} t^a \psi$ is the spin density. Here, $t^a \equiv \frac{\sigma^a}{2}$, where the $\sigma^a$ are the standard Pauli matrices.

Written in this form, it is quite clear the the contact interactions in (2.5) are attractive. This in turn permits us to perform the Hubbard-Stratonovich transformation to rewrite the quartic fermionic terms as bilinears at the cost of introducing auxiliary fields.

Upon introducing the auxiliary fields $\vec{h} \equiv h \hat{n}$, where $\hat{n}$ is a unit vector and $\phi$, the partition function is written as

$$Z = \int D\phi D\vec{h} \, e^{-\frac{1}{2V_0} \int dt d\vec{x} \left[ \phi^2 + \frac{1}{4} \vec{h}^2 \right]} \int D\bar{\psi} D\psi \, e^{i \int dt d\vec{x} \left[ \bar{\psi} \left[ i \partial_t + \mu + \phi - \frac{1}{2m} (\vec{p} - \vec{A})^2 + \frac{1}{4} \hat{n} \cdot \vec{\sigma} \right] \psi \right]}.$$  

(2.6)

Assuming that the $\phi$ integral and the $h$ integral have non-zero saddle points, the gap equations yielding these saddle points are

$$\phi_0 = -\frac{iV_0}{2} \text{tr} \left\langle \vec{x}, t \right| \frac{1}{i \partial_t + \mu + \phi_0 - \frac{1}{2m} (\vec{p} - \vec{A})^2 + \frac{1}{2} \sigma_z} |\vec{x}, t\rangle$$

(2.7)

and

$$h^a_0 = -2iV_0 \text{tr} \left\langle \vec{x}, t \right| t^a \frac{1}{i \partial_t + \mu + \phi_0 - \frac{1}{2m} (\vec{p} - \vec{A})^2 + \frac{1}{2} \sigma_z} |\vec{x}, t\rangle.$$  

(2.8)

Here $\vec{h}_0 \equiv \lambda \hat{z}$.

These may be solved to yield $\phi_0 \simeq \frac{V_0}{2} \left( \frac{B}{2\pi} \right)$ and $h^a_0 \simeq V_0 \left( \frac{B}{2\pi} \right) \delta^{a,z}$. This just means that the ground state is of uniform density $\rho_0 = \frac{B}{2\pi}$ and uniform magnetisation $|\vec{h}_0| = V_0 \rho_0$, taken to be along the $\hat{z}$ direction. This is SSB. The fluctuations in $\phi$ and in $h$ are obviously massive while Goldstone’s theorem guarantees that the fluctuations in $\hat{n}$, the Goldstone modes, are gapless. Upon evaluating the $\phi$ and the $h$ integrals at the saddle points, the partition function is written as
\[
Z = \int D\tilde{n} \int D\psi D\bar{\psi} \ e^{i \int dt dx \ \bar{\psi}[i\partial_t + \mu - \frac{1}{2\pi}(\vec{p} - \vec{A})^2 + \zeta \hat{n} \cdot \sigma]\psi}
\]

(2.9)

where \(\zeta \equiv \frac{\rho_0 V_0}{2}\).

As mentioned in the introduction, we may now transform the fermionic fields unitarily such that locally they represent spin-up fermions. Namely we introduce the space-time dependent unitary matrix \(U \in SU(2)\) such that

\[
U^\dagger \hat{\sigma} \cdot \hat{n} U = \sigma_z.
\]

(2.10)

We next define the spin-up fermionic field as

\[
\chi \equiv U^\dagger \psi
\]

and

\[
\bar{\chi} \equiv \bar{\psi} U.
\]

Let us further introduce the SU(2)-valued pure gauge potentials

\[
\Omega^a_\mu t^a \equiv U^\dagger \partial_\mu U. \tag{2.11}
\]

Since these are pure gauge configurations the corresponding field strengths vanish. Thus we have the relation [4]

\[
F^a_{\mu\nu} \equiv \partial_\mu \Omega^a_\nu - \partial_\nu \Omega^a_\mu + \epsilon^{abc} \Omega^b_\mu \Omega^c_\nu = 0. \tag{2.12}
\]

In terms of this SU(2)-valued connection, the partition function is written as

\[
Z = \int D\tilde{n} \int D\chi D\bar{\chi} \ e^{i \int dt dx \ \bar{\chi}[i\partial_t + \mu + \Omega^a_\mu t^a - \frac{1}{2m}(-i\partial_i - A^i \Omega_i^a t^a)^2 + \zeta \sigma_z]\chi}.
\]

(2.13)

Integrating the fermionic fields out, we can write the partition function as a path integral over the gapless angular degrees of freedom. Namely,
\[ Z = \int D\hat{n} \ e^{iS_{\text{eff}}} \]

where

\[ S_{\text{eff}} = -i \ \text{tr} \ \ln \left[ i\partial_t + \mu + \Omega_0^a t^a + \zeta \sigma_z - \frac{1}{2m} (-i\partial_i - A^i - \Omega_i^a t^a)^2 \right] \]. \quad (2.14)\]

If we had coupled perturbative slowly varying electromagnetic fields minimally to the microscopic fermions, the effective action would be

\[ S_{\text{eff}} = -i \ \text{tr} \ \ln \left[ i\partial_t + \mu - a_0 + \Omega_0^a t^a + \zeta \sigma_z - \frac{1}{2m} (-i\partial_i - a^i - \Omega_i^a t^a)^2 \right] \], \quad (2.15)\]

where \( a^\mu \) are the slowly varying perturbing electromagnetic potentials.

We can write the operator-valued argument, \( \hat{O} \) of the above functional determinant as

\[ \hat{O} \equiv i\partial_t + \mu - h_0 - V \] \quad (2.16)

where \( h_0 \) is the part that can be diagonalised readily and \( V \) is the perturbation.

Here,

\[ h_0 \equiv \frac{1}{2m} (\vec{p} - \vec{A})^2 - \zeta \sigma_z. \] \quad (2.17)

We define \( \pi^i \equiv -i\partial_i - A_i^i \). Making the holomorphic and the anti-holomorphic combinations: \( \pi \equiv \pi^x - i\pi^y \) and \( \pi^\dagger \equiv \pi^x + i\pi^y \), with

\[ [\pi, \pi^\dagger] = 2B \] \quad (2.18)

we can rewrite

\[ h_0 = \frac{1}{2m} (\pi^\dagger \pi + B) - \zeta \sigma_z. \] \quad (2.19)
The spectrum of this operator is infinitely degenerate ($\frac{B}{2\pi}$ states per unit area) and this degeneracy is exposed in terms of the so-called “guiding-centre” coordinates $X \equiv x - \frac{1}{B} \pi^y$ and $Y \equiv y + \frac{1}{B} \pi^x$. We form the combinations $Z \equiv X + iY$ and $\bar{Z} \equiv X - iY$ with the commutation relation

$$[Z, \bar{Z}] = \frac{2}{B}. \quad (2.20)$$

We see that $X$ (or $Y$) commutes with $h_0$. Thus an eigenbasis for $h_0$ is chosen to be $\{|n, X, \alpha\rangle\}$ with $n = 0, 1, 2, \ldots \infty$, $-\infty \leq X \leq \infty$ and $\alpha = \pm 1$. The index $n$ denotes a Landau level (L.L.) and $\alpha$ denotes the spin (whether “up” or “down”).

$\pi$ is the lowering operator and $\pi^\dagger$ the raising operator for the L.L. index. Namely,

$$\pi|n, X, \alpha\rangle = \sqrt{2B n}|n - 1, X, \alpha\rangle \quad (2.21)$$

and

$$\pi^\dagger|n, X, \alpha\rangle = \sqrt{2B (n + 1)}|n + 1, X, \alpha\rangle. \quad (2.22)$$

Further,

$$\hat{X}|n, X, \alpha\rangle = X|n, X, \alpha\rangle \quad (2.23)$$

and

$$\sigma_z|n, X, \alpha\rangle = \alpha|n, X, \alpha\rangle. \quad (2.24)$$

Thus,

$$h_0|n, X, \alpha\rangle = [\left(n + \frac{1}{2}\right)\omega_c - \zeta \alpha]|n, X, \alpha\rangle \quad (2.25)$$

where $\omega_c \equiv \frac{B}{m}$ is the cyclotron frequency.

From (2.25), it is clear that the gap between opposite spins for $n = 0$ is given by $2\zeta$. Thus the inter-electron interaction provides an effective Zeeman gap between opposite spins. In the following, we assume that $\zeta \equiv \frac{\mu_0 V_0}{2} \ll \omega_c$.

The ferromagnetic many-body ground state is constructed out of these single particle states by filling up all the degenerate states with $n = 0, \alpha = 1$. This precise definition of
the ground state is crucial in defining the functional determinant that we have obtained upon integrating out the fermions.

From (2.15) and (2.16), we see that

\[
S_{\text{eff}} = -i \text{tr} \ln[i\partial_t + \mu - h_0 - V].
\] (2.26)

Upon expanding the logarithm, we get

\[
S_{\text{eff}} = -i \text{tr} \ln[i\partial_t + \mu - h_0] + i \text{tr} \sum_{l=1}^{\infty} \frac{1}{l} (GV)^l.
\] (2.27)

where \([i\partial_t + \mu - h_0]G = I\).

Now let us define \(\hat{p}_0\) such that \([\hat{t}, \hat{p}_0] = -i\) with \(\langle t|\hat{p}_0 = i\partial_t \langle t|\). Let us introduce the basis \(\{|\omega\rangle\}\) with \(\hat{p}_0|\omega\rangle = \omega|\omega\rangle\). Furthermore, \(\langle \omega|t\rangle = \frac{1}{\sqrt{2\pi}} e^{i\omega t}\).

We also introduce the spin projection operators \(P_+ \equiv \frac{1}{2}(I + \sigma_z)\) and \(P_- \equiv \frac{1}{2}(I - \sigma_z)\), which project onto \(\alpha = \pm 1\) respectively.

Now, \(\{|n, X, \omega\rangle\}\) is a basis that diagonalises \(p_0 + \mu - h_0\) and consequently, \(G\).

Let

\[
G|n, X, \omega\rangle \equiv \Gamma^{(n)}(\omega)|n, X, \omega\rangle.
\] (2.28)

Now, the Green’s function \(G\) can be related to the mean ground state density through the standard relation \(\rho(\vec{x}, t) = -i \lim_{\delta \to 0^+} G(\vec{x}, t; \vec{x}, t + \delta)\). In this case we know that the mean density gets contributions from only the \(n = 0\) single particle states. This in turn tells us that

\[
\Gamma^{(0)}(\omega)P_+ = \frac{1}{\omega + \mu - \frac{\omega_c}{2} + \zeta + i\epsilon} P_+
\] (2.29)

\[
\Gamma^{(0)}(\omega)P_- = \frac{1}{\omega + \mu - \frac{\omega_c}{2} - \zeta + i\epsilon} P_-
\] (2.30)

and for \(n \neq 0\),

\[
\Gamma^{(n)}(\omega)P_\pm = \frac{1}{\omega + \mu - (n + \frac{1}{2})\omega_c \pm \zeta + i\epsilon} P_\pm.
\] (2.31)

Henceforth, we choose \(\mu = \frac{\omega_c}{2} - \zeta\).
Having specified the pole structure of the Green’s functions, let us now focus our attention upon the perturbation $V$. We recall that the perturbations are functions of the coordinate operators $\hat{x}$ and $\hat{y}$, or alternatively of $\hat{z} \equiv \hat{x} + i\hat{y}$ and $\hat{\bar{z}} \equiv \hat{x} - i\hat{y}$. In the L.L. basis, it is convenient to write: $\hat{z} = \hat{Z} - \frac{i}{B} \hat{\pi}^\dagger$ and $\hat{z} = \hat{Z} + \frac{i}{B} \hat{\pi}$. Thus, a function of the coordinate operators may be Taylor expanded around $\hat{Z}$ and $\hat{\bar{Z}}$ as

$$f(\hat{z}, \hat{\bar{z}}) = \sum_{p,q} \frac{1}{p! q!}(-\frac{i}{B})^p (\hat{\pi}^\dagger)^p (\hat{\pi})^q \partial_Z^p \partial_{\bar{Z}}^q f(\hat{Z}, \hat{\bar{Z}})^\sharp$$

where $\sharp \cdots \sharp$ just indicates that the normal ordering with respect to $\pi$ and $\pi^\dagger$ forces the $Z$ and $\bar{Z}$ to be anti-normal ordered. We further note that $\pi, \pi^\dagger \sim \sqrt{B}$. Thus this Taylor expansion is also an expansion in inverse powers of $B$. Therefore, for a large value of $B$ ($\sim 10$ T), the higher derivative terms should become more and more marginal.

Let us now define

$$A_0 \equiv a_0 - \Omega_0^a$$

and

$$A^i \equiv a^i + \Omega_i^a.$$  \hspace{1cm} (2.33)

In terms of these, the perturbation is written as

$$V = A_0 + \frac{1}{2m} A^i A^i - \frac{B}{2m} - \frac{1}{2m} (A \pi + \pi^\dagger \bar{A})$$

(2.34)

where $A \equiv A^x + iA^y$, $\bar{A} \equiv A^x - iA^y$ and $B \equiv \partial_x A^y - \partial_y A^x$.

Using the aforementioned Taylor expansion, we may write

$$V = V^{(\frac{1}{2})} + V^{(0)} + V^{(-\frac{1}{2})} + \cdots$$

(2.35)

where the ellipses indicate terms subleading in $B$. Here,

$$V^{(\frac{1}{2})} = -\frac{1}{2m} (\sharp A^x \pi + \pi^\dagger \sharp A^x)$$

$$V^{(0)} = \sharp[A_0 - \frac{1}{2m} B + \frac{1}{2m} (A^i)^2] \sharp$$

(2.36)

(2.37)
and
\[ V^{-\frac{1}{2}} = \frac{i}{B} \left[ \nu \partial_{\nu} A_{\nu}^0 - \pi^i \nu \partial_{\nu} A^0_{\nu} \right]. \] (2.38)

III. The Effective Action for the Goldstone Modes.

In this section, we shall compute the effective action given in (2.27) to \( O(1/B) \). The first term in (2.27) does not contain the gauge fields and is not interesting for our purposes. To the required order, we need to compute
\[
S_{\text{eff}} = i \text{tr} \left[ G V^{(0)} + \frac{1}{2} G V^{(\frac{1}{2})} G V^{(\frac{1}{2})} + G V^{(\frac{1}{2})} G V^{(-\frac{1}{2})} + G V^{(\frac{1}{2})} G V^{(\frac{1}{2})} G V^{(0)} \right] \tag{3.1}
\]
where we have used the cyclic property of the trace.

At this point let us explain our modus operandi by providing some concrete examples.

The simplest term is where there is only one insertion of the perturbing potential. Namely, we focus on
\[
S^{(1)} \equiv \text{tr} \ G V. \tag{3.2}
\]

Upon introducing the basis where \( G \) is diagonal,
\[
S^{(1)} = i \text{tr} \sum_{n=0}^{\infty} \int dX \int d\omega \Gamma^{(n)}(\omega) \langle n, X, \omega | V(t) | n, X, \omega \rangle. \tag{3.3}
\]
This in turn may be written as
\[
S^{(1)} = i \text{tr} \sum_{n=0}^{\infty} \int dt \int dX \int \frac{d\omega}{2\pi} \Gamma^{(n)}(\omega) \langle n, X | V(t) | n, X \rangle. \tag{3.4}
\]
Using the fact that
\[
\int \frac{d\omega}{2\pi} e^{i\omega \delta} \Gamma^{(n)}(\omega) = iP_+ \tag{3.5}
\]
we get
\[
S^{(1)} = -\text{tr} \ P_+ \int dt \int dX \langle 0, X | V^{(0)} | 0, X \rangle. \tag{3.6}
\]

It may be readily shown that (see Appendix A)
\[
\int_{-\infty}^{\infty} dX \langle 0, X | \nu \nu f(\hat{Z}, \hat{\bar{Z}}) \nu | 0, X \rangle = \frac{B}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ f(x, y). \]
Using this in conjunction with (2.37) and (3.6), we have
\[ S^{(1)} = \int dt \int d\mathbf{x} \left[ -\rho_0 a^0 + \frac{1}{2} \rho_0 \Omega_0^2 + \frac{\omega_c}{4\pi} \left( \partial_x \Omega_y^2 - \partial_y \Omega_x^2 \right) - \frac{\omega_c}{4\pi} \text{tr} P_+ (A^i)^2 \right] \]  
(3.7) where \( \rho_0 \equiv \frac{B^2}{2\pi} \). Let us now look at the term with two insertions of \( V \).

Thus,
\[ S^{(2)} = i \frac{1}{2} \text{tr} \sum_{n=0}^{\infty} \int d\omega \int dX \int dt \Gamma^{(n)}(\omega) \langle n, X, \omega | V(t) G(t) V(t) | n, X, \omega \rangle \]  
(3.8)
where the trace is now over the spin indices. Introducing a resolution of the identity in the form of \( I = \int dt |t\rangle \langle t| \), we have
\[ S^{(2)} = \frac{i}{2} \text{tr} \sum_{n=0}^{\infty} \int dt \int dX \int d\omega \frac{2\pi}{\Gamma^{(n)}(\omega)} \langle n, X | V(t) G(i\partial_t) V(t) | n, X \rangle. \]  
(3.9)
Thus
\[ S^{(2)} = \frac{i}{2} \text{tr} \sum_{n=0}^{\infty} \int dt \int dX \int d\omega \frac{2\pi}{\Gamma^{(n)}(\omega)} \langle n, X | VG(\omega + i\partial_t) V | n, X \rangle. \]  
(3.10)

Expanding \( G(\omega + i\partial_t) \) around \( \omega \), we get
\[ S^{(2)} = \frac{i}{2} \text{tr} \sum_{n=0}^{\infty} \int dt \int dX \int d\omega \frac{\Gamma^{(n)}(\omega)}{2\pi} \langle n, X | V G(\omega + i\partial_t) V + \cdots | n, X \rangle \]  
(3.11)
where the ellipses indicate terms with higher time derivatives. Let us look at the term
\[ \sum_{n=0}^{\infty} \int dt \int dX \int d\omega \frac{\Gamma^{(n)}(\omega)}{2\pi} \langle n, X | V^{(0)} G(\omega) V^{(0)} | n, X \rangle \] 
for a moment. This gives rise to an expression of the form:
\[ \text{tr} \int dt \int dX \int d\omega \frac{1}{2\pi} e^{i\omega \delta} \frac{1}{\omega - i\epsilon} P_+ (0, X | V^{(0)} \frac{1}{\omega - 2\zeta + i\epsilon} P_+ V^{(0)} | 0, X \rangle. \]
Upon doing the \( \omega \) integration, we get
\[ -\frac{i}{2\zeta} \text{tr} \int dt \int dX P_+ (0, X | V^{(0)} P_- V^{(0)} | 0, X \rangle. \]
It can be readily shown however that all the terms with $\zeta$ in the denominator add up to zero. We shall therefore ignore these terms in what follows.

With this, we can compute the non-zero contributions from $S^{(2)}$ using the same techniques as were employed for $S^{(1)}$ and obtain:

$$S^{(2)} = S^{(2a)} + S^{(2b)} + S^{(2c)}$$  \hspace{1cm} (3.13)

where

$$S^{(2a)} = \int dt \int d\vec{x} \left[ \frac{\omega_c}{4\pi} \text{tr} P_+(A^i)^2 - \frac{\omega_c}{8\pi} (\partial_x \Omega^z_y - \partial_y \Omega^z_x) - \frac{\zeta}{8\pi} ((\vec{\Omega}_t)^2 - (\Omega^z_t)^2) - \frac{\zeta}{4\pi} (\vec{\Omega}_x \times \vec{\Omega}_y)^z \right]$$

$$S^{(2b)} + S^{(2c)} = \int dt \int d\vec{x} \left[ -\frac{1}{4\pi} e^{\mu\nu} a_\mu \partial_\nu a_\rho + \frac{1}{4\pi} e^{\mu\nu} a_\mu \partial_\nu \Omega^z_\rho + \frac{3}{16} \vec{\Omega}_0 \cdot (\vec{\Omega}_x \times \vec{\Omega}_y) \right]. \hspace{1cm} (3.14)$$

The details of the computation of $S^{(2a)}$ have been provided in Appendix B.

At this point we note a few things about the contributions $S^{(1)}$ and $S^{(2)}$. There is a gauge non-invariant term in $S^{(1)}$, namely $\text{tr} P_+(A^i)^2$, which cancels against an equal but opposite contribution from $S^{(2a)}$. Furthermore, the term $\partial_x \Omega^z_y - \partial_y \Omega^z_x$ also cancels between these two. Furthermore, the external magnetic field breaks the P symmetry and the ferromagnetic ground state breaks the T symmetry. Hence, one could almost anticipate the emergence of an $U(1)$ Chern-Simons (CS) term in the effective action, which is manifest in $S^{(2)}$. Furthermore, more interestingly, there is a mixing of the angular degrees of freedom with the electromagnetic degrees of freedom through a CS like term, which is gauge invariant due to the presence of the Levi-Civita tensor. This is the lowest order coupling of the angular degrees of freedom with the electromagnetic fields. There is also the Hopf term, $\vec{\Omega}_0 \cdot (\vec{\Omega}_x \times \vec{\Omega}_y)$ which has been induced in the effective action. However, owing to the fact that $\Omega^{\mu}_\mu$ is a pure gauge potential, the Hopf term receives corrections from the term in the effective action with three insertions of the perturbation.

Namely, we have to look at

$$S^{(3)} = -i \text{ tr } GV^{(1/2)}GV^{(1/2)}GV^{(0)}.$$  \hspace{1cm} (3.16)
This may be done straightforwardly and yields:

\[ S^{(3)} \simeq \int dt \int d\vec{x} \left[ -\frac{1}{16\pi} \vec{\Omega}_0 \cdot (\vec{\Omega}_x \times \vec{\Omega}_y) \right]. \] (3.17)

Thus combining the terms together, one obtains

\[
S_{\text{eff}} = \int dt \int d\vec{x} \left[ \frac{1}{2} \rho_0 \Omega_0^z - \frac{\zeta}{8\pi} ((\vec{\Omega}_x)^2 - (\Omega_i^z)^2) - \frac{\zeta}{4\pi} (\vec{\Omega}_x \times \vec{\Omega}_y)^z \\
+ \frac{1}{8\pi} \vec{\Omega}_0 \cdot (\vec{\Omega}_x \times \vec{\Omega}_y) - \rho_0 a^0 + \frac{\omega_c}{4\pi} b - \frac{1}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \\
+ \frac{1}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu \Omega_\rho^z \right]. \] (3.18)

Here, \( b \equiv \partial_x a^y - \partial_y a^x \) is the perturbing magnetic field.

As has been shown explicitly in Appendix C, we can express the unitary matrix \( U \in SU(2) \) in terms of the Euler angles, \( \theta, \phi, \chi \). Namely, we can write

\[ U = e^{-i\frac{\phi}{2}\sigma_z} e^{-i\frac{\theta}{2}\sigma_y} e^{-i\frac{\chi}{2}\sigma_z}. \] (3.19)

Again, as \( U \sigma_z U^\dagger = \vec{\sigma} \cdot \hat{n} \), the unit vector \( \hat{n} \) is given in terms of the Euler angles as

\[ \hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \] (3.20)

Also,

\[ (\vec{\Omega}_x \times \vec{\Omega}_y)^z = \frac{1}{2} \epsilon^{ij} \hat{n} \cdot (\partial_i \hat{n} \times \partial_j \hat{n}). \] (3.21)

and

\[ (\vec{\Omega}_i)^2 - (\Omega_i^z)^2 = (\partial_i \hat{n})^2. \] (3.22)

With these, we can write the effective action as

\[
S_{\text{eff}} = \int dt \int d\vec{x} \left[ \frac{B}{4\pi} \cos \theta \partial_t \phi - \frac{\zeta}{8\pi} (\partial_t \hat{n})^2 - \zeta \rho_p + \frac{1}{48\pi} \epsilon^{\mu\nu\rho} \vec{\Omega}_\mu \cdot (\vec{\Omega}_\nu \times \vec{\Omega}_\rho) \\
+ \frac{1}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu \Omega_\rho^z - \rho_0 a^0 + \frac{\omega_c}{4\pi} b - \frac{1}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \right]. \] (3.23)

Here, \( \rho_p \equiv \frac{1}{8\pi} \epsilon^{ij} \hat{n} \cdot (\partial_i \hat{n} \times \partial_j \hat{n}) \) is a topological density (the Pontryagin index density).

This means that \( \int d\vec{x} \rho_p = \text{integer} \).
Let us look at the various terms in the effective action. The first term, with a single time derivative is a so called Wess-Zumino term [13]. It is actually a Berry phase [7] and is ubiquitous in the path integral representation of spin systems. The interesting point is that in this derivation, the Berry phase has emerged through a derivative expansion [14]. In principle higher order corrections to the Berry phase with a higher number of time derivatives could be straightforwardly computed. The second term is the standard kinetic energy term of the NLSM. Together, the first two terms tell us that the dispersion relation of the Goldstone bosons are that of ferromagnetic magnons. If we set the perturbing electromagnetic field to zero momentarily, we see that the fourth term, the Hopf term, has the appropriate coefficient to make the solitons of the NLSM fermionic. When the electromagnetic field is turned on, there is a mixing between the angular degrees of freedom and the electromagnetic potentials. This is given by the fifth term in (3.23), which can be rewritten in terms of the Euler angles as $-\frac{1}{4\pi} \sin \theta e^{\mu \nu \rho} \partial_\nu \theta \partial_\rho \phi$. This provides a rather elegant expression for the electromagnetic coupling of the angles parametrising the coset SU(2)/U(1). It is also gratifying to note that the angle $\chi$ has dropped out of the expression, as it should, since only two angles are needed to describe the coset. An interesting point to note is that one would not get this term by naively covariantising the term $\rho_p a_0$ which arises naturally in this context, to $j^\mu_p a_\mu$, where

$$j^\mu_p \equiv \frac{1}{8\pi} \epsilon^{\alpha \beta \gamma} \hat{n} \cdot (\partial_\beta \hat{n} \times \partial_\gamma \hat{n})$$

is a conserved topological current, and $\rho_p = j^0_p$. It is well known that skyrmions that exist as topological excitations in the system are characterised by their winding number $N_p \equiv \int d\vec{x} \rho_p$, the same winding number also gives the electrical charge of the skyrmion ($e = 1$). Thus it is natural that the response of the system to an electrostatic potential $a^0$ should be given by a term $\int dt \int d\vec{x} \rho_p a^0$ in the effective action. In (3.23), the electromagnetic interaction (the fifth term) can be written as

$$S_{em}^{eff} = -\int dt \int d\vec{x} \left[ \rho_p a^0 + \frac{1}{4\pi} \left\{ a^x (\partial_y \Omega_z^0 - \partial_0 \Omega_y^z) + a^y (\partial_0 \Omega_x^z - \partial_x \Omega_0^z) \right\} \right]. \tag{3.24}$$
The first term is as expected. The second term, within parentheses does not meet with our naive expectations. In terms of the perturbing electromagnetic fields, this term can be expressed as

\[ S_{\text{em}}^{\text{eff}} = -\frac{1}{4\pi} \left[ \Omega_0^z b + \Omega_x^z e^y - \Omega_x^z e^x \right] \]  

(3.25)

where

\[ b \equiv \partial_x a^y - \partial_y a^x \]

and

\[ e^i \equiv -(\partial_0 a^i + \partial_i a^0). \]

What is quite remarkable is that this second term does not depend on the details of the interaction \( V_0 \). The purely electromagnetic terms are familiar from previous studies concerning the electromagnetic effective action of spinless quantum Hall fermions [16,17].

From the effective action, we can readily compute the mean electromagnetic currents in the spin-textured (excited) state.

Thus,

\[ \langle j_0 \rangle = \rho_0 + \rho_p - \frac{1}{2\pi} b \]

\[ \langle j_x \rangle = \frac{1}{2\pi} e^y + \frac{1}{4\pi} (\partial_y \Omega_0^z - \partial_0 \Omega_x^z) \]

\[ \langle j_y \rangle = -\frac{1}{2\pi} e^x + \frac{1}{4\pi} (\partial_0 \Omega_x^z - \partial_x \Omega_0^z). \]  

(3.26)

This shows explicitly that the density in the excited state changes from its ground state value by an amount which is a topological index density.

Given the effective action, one can also compute the mean magnetisation in the excited state. It is given by

\[ M^a(\vec{x}, t) \equiv \langle \bar{\chi} t^a \chi \rangle = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \Omega_0^a}. \]  

(3.27)

For instance the \( z \) component of the magnetisation changes from its value of \( \frac{1}{2} \rho_0 \) in the ground state. It is given by

\[ M^z(\vec{x}, t) = \frac{1}{2} \rho_0 + \frac{1}{2} \rho_p - \frac{1}{4\pi} b \]  

(3.28)
which is precisely equal to half the value of the mean density in the excited state.

V. Conclusions

In this article, we have derived some new results and have rederived some well-known ones on the subject of quantum Hall ferromagnets.

Previously, as in [7], the \( \nu = 1 \) system has been studied, but exclusively within the L.L.L. projection approximation. Mixing with higher L.L., due to the Coulomb interaction, was mainly disregarded, except in [10], where some effects of the higher L.L. mixing were considered. However, since any interaction that is non-diagonal in the L.L. basis will cause mixing of the levels, in particular electromagnetic interactions, it is important to have a calculational technique that avoids explicit L.L.L. projection. In this article, we have exploited precisely one such method. The fermionic operators have not been projected onto the L.L.L. whilst the information regarding the ground state of the system (namely that only the L.L.L. single particle states are filled), has been coded into the pole structure of the Green’s function. In this method, the effects due to higher L.L. mixing appear naturally through subleading terms in the derivative expansion scheme that we have adopted.

In this manner, we have shown how a non-linear sigma model, describing the spin excitations of the electrons, emerges simply in terms of the angular variables describing the deviation of the magnetisation vector from its ferromagnetic ground state orientation. We have shown further that a Hopf term also emerges in terms of these same angular variables, with a coefficient that is appropriate for turning the skyrmionic excitations in the system into fermions. The Hopf term had also been derived in [8] but the manner of the derivation there has met with criticism (see [9] for details). The same reservations, however, should not exist against our rather straightforward derivative expansion technique.

Apart from the very economical rederivation of known results listed above, we have also investigated the obviously non-minimal electromagnetic coupling of the spin excitations in the system [15]. To the leading order, we get a gauge invariant “Chern-Simons like” term
which couples the angular variables describing the spin excitations, to externally applied electromagnetic fields. This term clearly shows how the electromagnetic currents in the system are affected by the spin excitations. This, we believe is a new result.

Within our approach, the U(1) valued electromagnetic potentials are treated on the same footing as the SU(2) valued potentials describing the spin dynamics. Since P and T are violated in the system, the emergence of “Chern-Simons like’ terms is only to be expected. In fact, three such terms are obtained: A pure electromagnetic CS term, a Hopf term purely in terms of the angular degrees of freedom and the term described in the previous paragraph which mixes the two.

If we set all angular excitations to zero, thereby freezing the spin degree of freedom, we obtain the effective electromagnetic interaction of planar polarised electrons, as has been described, for instance, in [16,17]. On the other hand, upon setting the perturbative electromagnetic fields to zero, we obtain the magnon effective action of [7]. Thus our work also provides an unifying treatment of these two different systems.

As has been discussed in [16, 17] and in references therein, the gapless chiral edge excitations are the lowest-lying excitations in the spin-polarised Hall systems. In the Hall systems addressed by the present article, the gapless spin waves are the lowest-lying excitations. Realistically such a system would have a boundary and we could expect excitations at the boundary. The relationship of these excitations to the magnons could bear investigating. Furthermore, it would be interesting to shed light on the spin-textured edge states beginning from the microscopics. Work in this direction is currently in progress.

Acknowledgements

I wish to acknowledge J. Soto for early discussions on the subject and for a fruitful collaboration on a previous article on a similar topic. T.H. Hansson must be thanked for suggesting that the spin collective modes might be extracted through a unitary transformation. I am indebted to R. Mackenzie and M. Paranjape for sharing their insights with
me and B. Sakita and V.P. Nair for their encouragement. The work is partially supported by the N.S.E.R.C of Canada and the F.C.A.R of Quebec.
Appendix A

In this appendix, we shall discuss the transformation of the integral over the guiding centre coordinate \( X \), with the L.L. index \( n = 0 \) to an integral over the spatial coordinates \( x \) and \( y \).

An expression that we repeatedly encounter in section III is

\[
\int dX \langle 0, X|\hat{\Psi}(\hat{Z}, \hat{\bar{Z}})|\hat{\Psi}|0, X\rangle. \tag{A.1}
\]

In view of the fact that the normal ordered products of \( \hat{\pi} \) and \( \hat{\pi}^\dagger \) give zero matrix elements in the L.L.L., we can rewrite (A.1) as

\[
\int d\hat{x} \langle 0, \hat{x}|\hat{\Psi}(\hat{x}, \hat{\bar{y}})|\hat{\Psi}|0, \hat{x}\rangle. \tag{A.2}
\]

Inserting the identity in the form of \( I = \int d\vec{x} |\vec{x}\rangle\langle \vec{x}| \) into (A.2), we get

\[
\int d\vec{x} f(x, y) \int dX |\langle \vec{x}|0, X\rangle|^2. \tag{A.3}
\]

Now in the Landau gauge \( \vec{A} \equiv (0, -B x) \), the L.L.L. wavefunction is

\[
\langle \vec{x}|0, X\rangle = \left(\frac{B}{\pi}\right)^{\frac{3}{4}} \left(\frac{B}{2\pi}\right)^{\frac{1}{2}} e^{-iBXy} e^{-\frac{B}{2}(x-X)^2}. \tag{A.4}
\]

Using (A.4) in (A.3), we get

\[
\int dX \langle 0, X|\hat{\Psi}(\hat{Z}, \hat{\bar{Z}})|\hat{\Psi}|0, X\rangle = \frac{B}{2\pi} \int d\vec{x} f(\vec{x}). \tag{A.5}
\]

Appendix B

In this appendix, we shall, as an example, work out the contribution \( S^{(2a)} \) to the effective action, in reasonable detail.

From (3.12),

\[
S^{(2a)} = \frac{i}{2} \text{tr} \sum_{n=0}^{\infty} \int dt \int d\vec{x} \int \frac{d\omega}{2\pi} \Gamma^{(n)}(\omega) \langle n, X|V^{(\frac{1}{2})}G(\omega)V^{(\frac{1}{2})}|n, X\rangle. \tag{B.1}
\]
Using the explicit form of \( V^{(\frac{1}{2})} \) given in (2.36), we note that only \( n = 0 \) and \( n = 1 \) from the infinite sum over \( n \) will contribute, as \( V^{(\frac{1}{2})} \) can change the L.L. index by at most one and the integral over \( \omega \) is non zero if and only if \( \Gamma^{(0)}(\omega) \) is involved.

Thus,

\[
S^{(2a)} = \frac{i}{8m^2} \text{tr} \int dt \int d\vec{x} \int \frac{d\omega}{2\pi} \left[ \Gamma^{(0)}(\omega)\langle 0, X|\pi AG(\omega)\bar{A}\pi|0, X\rangle + \Gamma^{(1)}(\omega)\langle 1, X|\pi^{\dagger}\bar{A}G(\omega)A\pi|1, X\rangle \right].
\]

We have already mentioned in the main body of the article that we shall drop all terms with \( \zeta \) appearing in the denominator as they add up to zero. Thus, using \( \pi, \pi^{\dagger} \) on the single-particle states, using the result (A.5) from Appendix A and invoking the cyclicity of the trace, we get,

\[
S^{(2a)} = \frac{i\omega^2}{4\pi} \int dt \int d\vec{x} \text{tr} \int d\omega \Gamma^{(0)}(\omega)A\Gamma^{(1)}(\omega)\bar{A}.
\]

We know that only \( \Gamma^{(0)}(\omega)P_{+} \) has the pole structure to give a non zero value for the \( \omega \) integral. Hence,

\[
S^{(2a)} = \frac{i\omega^2}{4\pi} \int dt \int d\vec{x} \text{tr} \int d\omega \frac{1}{2\pi} \frac{1}{\omega - i\epsilon} P_{+}A \left[ \frac{1}{\omega - \omega_{c} + i\epsilon} P_{+} + \frac{1}{\omega - \omega_{c} + 2\zeta + i\epsilon} P_{-} \right] \bar{A}.
\]

Now, as \( \zeta \ll \omega_{c} \), we can expand in powers of \( \frac{\zeta}{\omega_{c}} \) and write

\[
S^{(2a)} \approx \frac{\omega_{c}}{4\pi} \int dt \int d\vec{x} \left[ \text{tr} \ P_{+}A\bar{A} - \frac{2\zeta}{\omega_{c}} \text{tr} \ P_{+}A\bar{A} + \frac{2\zeta}{\omega_{c}} \text{tr} \ P_{+}AP_{+}\bar{A} \right].
\]

Normally, we would have dropped the second term in (B.5) as it is subdominant with respect to the first. In this case, however, the first term cancels out against in \( S^{(1)} \). Now, upon using the definitions of \( A \) and \( \bar{A} \) from (2.34) and the trace relations for the generators of SU(2), we obtain (3.14).

**Appendix C**

As stated in (3.19), the unitary matrix \( U \in SU(2) \) can be expressed in terms of the Euler angles \( \theta, \phi, \chi \). Thus

\[
U = e^{-i\frac{\phi}{2}\sigma_{z}} e^{-i\frac{\theta}{2}\sigma_{y}} e^{-i\frac{\chi}{2}\sigma_{z}}.
\]
Upon using the definition $U^\dagger i \partial_\mu U \equiv \Omega_\mu^a \overline{z}_a$, we can write

\[
\begin{align*}
\Omega^x_\mu &= -\sin \theta \cos \chi \partial_\mu \phi + \sin \chi \partial_\mu \theta \\
\Omega^y_\mu &= \sin \theta \sin \chi \partial_\mu \phi + \cos \chi \partial_\mu \theta \\
\Omega^z_\mu &= \cos \theta \partial_\mu \phi + \partial_\mu \chi.
\end{align*}
\] (C.2)

From (C.2), it may be checked explicitly that

\[
\partial_\mu \Omega_\nu^a - \partial_\nu \Omega_\mu^a + \epsilon^{abc} \Omega_\mu^b \Omega_\nu^c = 0.
\] (C.3)

Again, from the relation $U\sigma_z U^\dagger = \vec{\sigma} \cdot \hat{n}$, we obtain

\[
\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).
\] (C.4)

Using (C.2) and (C.4), we get

\[
\epsilon^{zab} \Omega_x^a \Omega_y^b = \frac{1}{2} \epsilon^{ij} \hat{n} \cdot (\partial_i \hat{n} \times \partial_j \hat{n}).
\] (C.5)

Using the same equations, it is also simple to verify that

\[
(\Omega_i^a)^2 - (\Omega_i^z)^2 = (\partial_i \hat{n})^2.
\] (C.6)

References

[1] R. Rajaraman, *Solitons and Instantons*, (North Holland, Amsterdam, 1982)
[2] F. Wilczek and A. Zee, Phys. Rev. Lett **51**, 2250, (1983)
[3] A.M.J. Schakel, cond-mat/9805152
[4] G.E. Volovik & V.M. Yakovenko, J. Phys.: Condens. Matter **1**, 5263, (1989)
[5] Z. Hlousek et. al., Phys. Rev. **D 41**, 3773, (1990)
[6] S.L. Sondhi et. al., Phys. Rev. **B 47**, 16419, (1993)
[7] K. Moon et. al., Phys. Rev. B 51, 5138, (1995); W. Apel & Yu.A. Bychkov, Phys. Rev. Lett. 78, 2188, (1997); R. Ray & J. Soto, cond-mat/9708067

[8] W. Apel & Yu.A. Bychkov, Phys. Rev. Lett. 78, 2188, (1997)

[9] G.E. Volovik & V.M. Yakovenko, Phys. Rev. Lett. 79, 3791, (1997)

[10] R. Ray & J. Soto, cond-mat/9708067

[11] K. Moon et. al., Phys. Rev. B 51, 5138, (1995); R. Ray & J. Soto, cond-mat/9708067

[12] K. Moon et. al., Phys. Rev. B 51, 5138, (1995)

[13] E. Fradkin, Field Theories of Condensed Matter Systems, (Addison-Wesley Publishing Company)

[14] R. Jackiw, Int. Jour. Mod. Phys. A3, 285, (1988); P. de Sousa Gerbert, Ann. Phys. 189, 155, (1989); D. Düsedau, Phys. Lett. B 205, 312, (1988)

[15] J.M. Roman & J. Soto, cond-mat/9709298

[16] R. Ray & B. Sakita, Ann. Phys. 230, 131, (1994)

[17] R. Ray & J. Soto, Phys. Rev. B 54, 10709, (1995)