The commutator and centralizer description of Sylow subgroups of alternating and symmetric groups

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Abstract

Given a permutational wreath product sequence of cyclic groups of order we research a commutator width of such groups and some properties of its commutator subgroup. Commutator width of Sylow 2-subgroups of alternating group $A_{2^k}$, permutation group $S_{2^k}$ and $C_p \wr B$ were founded. The result of research was extended on subgroups $(Syl_{2}A_{2^k})'$, $p > 2$. The paper presents a construction of commutator subgroup of Sylow 2-subgroups of symmetric and alternating groups. Also minimal generic sets of Sylow 2-subgroups of $A_{2^k}$ were founded. Elements presentation of $(Syl_{2}A_{2^k})'$, $(Syl_{2}S_{2^k})'$ was investigated. We prove that the commutator width $[1]$ of an arbitrary element of a discrete wreath product of cyclic groups $C_{p_i}$, $p_i \in \mathbb{N}$ is 1.

Key words: wreath product of group; commutator width of Sylow $p$-subgroups; commutator subgroup of alternating group, centralizer subgroup, semidirect product.

1 Introduction

A form of commutators of wreath product $A \wr B$ was briefly considered in [3]. For more deep description of this form we take into account the commutator width $(cw(G))$ which was presented in work of Muranov [1]. This form of commutators of wreath product was used by us for the research of $cw(Syl_2A_{2^k})$, $cw(Syl_2S_{2^k})$ and $cw(C_p \wr B)$. As well known, the first example of a group $G$ with $cw(G) > 1$ was given by Fite [5]. We deduce an estimation for commutator width of wreath product $B \wr C_p$ of groups $C_p$ and an arbitrary group $B$ taking into the consideration a $cw(B)$ of passive group $B$. In this paper we continue a researches which was stared in [18]. The form of commutator presentation [3] was presented by us in form of wreath recursion and commutator width of it was studied.
A research of commutator-group serves to decision of inclusion problem [6] for elements of $Syl_2A_{2k}$ in its derived subgroup $(Syl_2A_{2k})'$.

2 Preliminaries

Denote by $fun(B, A)$ the direct product of isomorphic copies of $A$ indexed by elements of $B$. Thus, $fun(B, A)$ is a function $B \rightarrow A$ with the conventional multiplication and finite supports. The extension of $fun(B, A)$ by $B$ is called the discrete wreath product of $A, B$. Thus, $A \ltimes B := fun(B, A) \rtimes B$ moreover, $bfb^{-1} = f^b$, $b \in B$, $f \in fun(B, A)$. As well known that a wreath product of permutation groups is associative construction.

Let $G$ be a group acting (from the left) by permutations on a set $X$ and let $H$ be an arbitrary group. Then the (permutational) wreath product $H \wr G$, where $G$ acts on the direct power $H^X$ by the respective permutations of the direct factors. The group $C_p$ is equipped with a natural action by left shift on $X = \{1, \ldots, p\}$, $p \in \mathbb{N}$.

The multiplication rule of automorphisms $g, h$ which presented in form of wreath recursion $g = (g(1), g(2), \ldots, g(d)) \sigma_g$, $h = (h(1), h(2), \ldots, h(d)) \sigma_h$, is given by the formula:

$$g \cdot h = (g(1)h(\sigma_g(1)), g(2)h(\sigma_g(2)), \ldots, g(d)h(\sigma_g(d))) \sigma_g \sigma_h.$$

We define $\sigma$ as $(1, 2, \ldots, p)$ where $p$ is defined by context.

We consider $B \wr (C_p, X)$, where $X = \{1, \ldots, p\}$, and $B' = \{[f, g] \mid f, g \in B\}$, $p \geq 1$. If we fix some indexing $\{x_1, x_2, \ldots, x_m\}$ of set the $X$, then an element $h \in H^X$ can be written as $(h_1, \ldots, h_m)$ for $h_i \in H$.

The set $X^*$ is naturally a vertex set of a regular rooted tree, i.e. a connected graph without cycles and a designated vertex $v_0$ called the root, in which two words are connected by an edge if and only if they are of form $v$ and $vx$, where $v \in X^*$, $x \in X$. The set $X^n \subset X^*$ is called the $n$-th level of the tree $X^*$ and $X^0 = \{v_0\}$. We denote by $v_{j,i}$ the vertex of $X^j$, which has the number $i$. Note that the unique vertex $v_{k,i}$ corresponds to the unique word $v$ in alphabet $X$. For every automorphism $g \in AutX^*$ and every word $v \in X^*$ define the section (state) $g(v) \in AutX^*$ of $g$ at $v$ by the rule: $g(v)(x) = y$ for $x, y \in X^*$ if and only if $g(vx) = g(v)y$. The subtree of $X^*$ induced by the set of vertices $\cup_{i=0}^k X^i$ is denoted by $X^{[k]}$. The restriction of the action of an automorphism $g \in AutX^*$ to the subtree $X^{[l]}$ is denoted by $g_{(v)}|_{X^{[l]}}$. A restriction $g_{(v)}|_{X^{[l]}}$ is called the vertex permutation (v.p.) of $g$ in a vertex $v$. We call the endomorphism $\alpha|_{v}$ restriction of $g$ in a vertex $v$ [7]. For example, if $|X| = 2$ then we just have to distinguish active
vertices, i.e., the vertices for which $\alpha|_v$ is non-trivial. As well known if $X = \{0,1\}$ then $\text{Aut} X^{[k-1]} \simeq C_2 \wr \cdots \wr C_2$ \cite{7}.

Let us label every vertex of $X^l$, $0 \leq l < k$ by sign 0 or 1 in relation to state of v.p. in it. Let us denote state value of $\alpha$ in $v_{ki}$ as $s_{ki}(\alpha)$ we put that $s_{ki}(\alpha) = 1$ if $\alpha|_{v_{ki}}$ is non-trivial, and $s_{ki}(\alpha) = 0$ if $\alpha|_{v_{ki}}$ is trivial. Obtained by such way a vertex-labeled regular tree is an element of $\text{Aut} X^{[k]}$. All undeclared terms are from \cite{8,9}.

Let us make some notations. The commutator of two group elements $a$ and $b$, denoted $[a, b] = aba^{-1}b^{-1}$, conjugation by an element $b$ as $a^b = bab^{-1}$,

$\sigma = (1, 2, \ldots, p)$. Also $G_k \simeq Syl_2 A_{2k}$, $B_k = \ell_{i=1}^k C_2$. The structure of $G_k$ was investigated in \cite{18}. For this research we can regard $G_k$ and $B_k$ as recursively constructed i.e. $B_1 = C_2$, $B_k = B_{k-1} \wr C_2$ for $k > 1$, $G_1 = \langle e \rangle$, $G_k = \{(g_1, g_2) \pi \in B_k \mid g_1 g_2 \in G_{k-1}\}$ for $k > 1$.

The commutator length of an element $g$ of the derived subgroup of a group $G$, denoted $cl_G(g)$, is the minimal $n$ such that there exist elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ in $G$ such that $g = [x_1, y_1] \ldots [x_n, y_n]$. The commutator length of the identity element is 0. The commutator width of a group $G$, denoted $cw(G)$, is the maximum of the commutator lengths of the elements of its derived subgroup $[G, G]$.

3 Main result

We are going to prove that the set of all commutators $K$ of Sylow 2-subgroup $Syl_2 A_{2k}$ of the alternating group $A_{2k}$ is the commutant of $Syl_2 A_{2k}$.

The following Lemma follows from the corollary 4.9 of the Meldrum’s book \cite{3}.

**Lemma 1.** An element of form $(r_1, \ldots, r_{p-1}, r_p) \in W’ = (B \wr C_p)’$ iff product of all $r_i$ (in any order) belongs to $B’$, where $B$ is an arbitrary group.

**Proof.** Analogously to the Corollary 4.9 of the Meldrum’s book \cite{3} we can deduce new presentation of commutators in form of wreath recursion

$$w = (r_1, r_2, \ldots, r_{p-1}, r_p),$$

where $r_i \in B$. If we multiply elements from a tuple $(r_1, \ldots, r_{p-1}, r_p)$, where $r_i =$
$$h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1}, \ h, g \in B \text{ and } a, b \in C_p,$$

then we get a product

$$x = \prod_{i=1}^{p} r_i = \prod_{i=1}^{p} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} \in B',$$  \hspace{1cm} (1)

where $x$ is a product of correspondent commutators. Therefore we can write $r_p = r_p^{-1} \cdots r_1^{-1} x$. We can rewrite element $x \in B'$ as the product $x = \prod_{j=1}^{c w(B)} [f_j, g_j]$.

Note that we impose more weak condition on product of all $r_i$ to belongs to $B'$ then in Definition 4.5. of form $P(L)$ in [3].

In more detail deducing of our representation constructing can be reported in following way. If we multiply elements having form of a tuple $(r_1, \ldots, r_{p-1}, r_p)$, where

$$r_i = h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1}, \ h, g \in B \text{ and } a, b \in C_p,$$

then in case $c w(B) = 0$ we obtain a product

$$\prod_{i=1}^{p} r_i = \prod_{i=1}^{p} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} \in B'.$$  \hspace{1cm} (2)

Note that if we rearrange elements in (1) as $h_1^{-1} g_1 h_2^{-1} g_2^{-1} \ldots h_p^{-1} g_p^{-1}$ then by reason of such permutations we obtain a product of correspondent commutators. Therefore, following equality holds true

$$\prod_{i=1}^{p} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} = \prod_{i=1}^{p} h_i^{-1} g_i^{-1} x \in B',$$  \hspace{1cm} (3)

where $x$ is a product of correspondent commutators. Therefore,

$$(r_1, \ldots, r_p) \in W' \iff r_p^{-1} \cdots r_1^{-1} x = x \in B'$$  \hspace{1cm} (4)

Thus, one of elements from coordinate of wreath recursion $(r_1, \ldots, r_{p-1}, r_p)$ depends on rest of $r_i$. This dependence contribute that the product $\prod_{j=1}^{p} r_j$ for arbitrary sequence $\{r_j\}_{j=1}^{p}$ belongs to $B'$. Thus, $r_p$ can be expressed as:

$$r_p = r_p^{-1} \cdots r_{p-1}^{-1} x.$$

Denote a $j$-th tuple, which consists of elements of a wreath recursion, by $(r_{j_1}, r_{j_2}, \ldots, r_{j_p})$. Closedness by multiplication of the set of forms $(r_1, \ldots, r_{p-1}, r_p) \in W = (B \wr C_p)'$ follows from
\[
\prod_{j=1}^{k} (r_{j1} \cdots r_{jp-1} r_{jp}) = \prod_{j=1}^{k} \prod_{i=1}^{p} r_{ji} = R_1 R_2 \cdots R_k \in B',
\]  \hspace{1cm} (5)

where \( r_{ji} \) is \( i \)-th element from tuple number \( j \), \( R_j = \prod_{i=1}^{p} r_{ji} \), \( 1 \leq j \leq k \). As it was shown above \( R_j = \prod_{i=1}^{p-1} r_{ji} \in B' \). Therefore, the product (5) of \( R_j \), \( j \in \{1, \ldots, k\} \) which is similar to the product mentioned in (3), has the property \( R_1 R_2 \cdots R_k \in B' \) too, because of \( B' \) is subgroup. Thus, we get a product of form (2) and the similar reasoning as above are applicable.

Let us prove the sufficiency condition. If the set \( K \) of elements that satisfy the condition of this theorem that all products of all \( r_i \), where every \( i \) occurs in this forms once, belong to \( B' \), then using the elements of form

\[ (r_1, e, \ldots, e, r_1^{-1}), \ldots, (e, e, \ldots, e, r_1, e, r_1^{-1}), \ldots, (e, e, \ldots, e, r_{p-1}, r_{p-1}^{-1}), (e, e, \ldots, e, r_1 r_2 \cdots r_{p-1}) \]

we can express any element of form \( (r_1, \ldots, r_{p-1}, r_p) \in W = (C_p \wr B)' \). We need to prove that in such way we can express all element from \( W \) and only elements of \( W \). The fact that all elements can be generated by elements of \( K \) follows from randomness of choice every \( r_i \), \( i < p \) and the fact that equality (1) holds so construction of \( r_p \) is determined.

**Lemma 2.** For any group \( B \) and integer \( p \geq 2 \), \( p \in \mathbb{N} \) if \( w \in (B \wr C_p)' \) then \( w \) can be represented as the following wreath recursion

\[ w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \cdots r_{p-1}^{-1} \prod_{j=1}^{k} [f_j, g_j]), \]

where \( r_1, \ldots, r_{p-1}, f_j, g_j \in B \), and \( k \leq cw(B) \).

**Proof.** According to the Lemma \[ \]we have the following wreath recursion

\[ w = (r_1, r_2, \ldots, r_{p-1}, r_p), \]

where \( r_i \in B \) and \( r_{p-1}r_{p-2} \cdots r_2 r_1 r_p = x \in B' \). Therefore we can write \( r_p = r_1^{-1} \cdots r_{p-1}^{-1} x \).

We also can rewrite element \( x \in B' \) as product of commutators \( x = \prod_{j=1}^{k} [f_j, g_j] \) where \( k \leq cw(B) \). \[ \]
Lemma 3. For any group $B$ and integer $p \geq 2$, $p \in \mathbb{N}$ if $w \in B \wr C_p$ is defined by the following wreath recursion

$$w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots r_{p-1}^{-1} [f, g]),$$

where $r_1, \ldots, r_{p-1}, f, g \in B$ then we can represent $w$ as commutator

$$w = [(a_{1,1}, \ldots, a_{1,p})\sigma, (a_{2,1}, \ldots, a_{2,p})],$$

where

$$a_{1,i} = e \text{ for } 1 \leq i \leq p - 1,$$

$$a_{2,1} = (f^{-1}) r_1^{-1} \ldots r_{p-1}^{-1},$$

$$a_{2,i} = r_{i-1} a_{2,i-1} \text{ for } 2 \leq i \leq p,$$

$$a_{1,p} = g a_{2,p}^{-1}.$$

Proof. Let us to consider the following commutator

$$\kappa = (a_{1,1}, \ldots, a_{1,p})\sigma \cdot (a_{2,1}, \ldots, a_{2,p}) \cdot (a_{1,p}^{-1}, a_{1,1}, \ldots, a_{1,p-1})\sigma^{-1} \cdot (a_{2,1}, \ldots, a_{2,p})^{-1}$$

$$= (a_{3,1}, \ldots, a_{3,p}),$$

where

$$a_{3,i} = a_{1,i} a_{2,1+(i \mod p)} a_{1,i}^{-1} a_{2,i}^{-1}.$$

At first we compute the following

$$a_{3,i} = a_{1,i} a_{2,i+1} a_{1,i}^{-1} a_{2,i}^{-1} = a_{2,i+1} a_{2,i}^{-1} = r_i a_{2,i} a_{2,i}^{-1} = r_i, \text{ for } 1 \leq i \leq p - 1.$$
Then we make some transformation of $a_{3,p}$:

$$a_{3,p} = a_{1,p}a_{2,1}^{-1}a_{1,p}^{-1}a_{2,p}^{-1}$$

$$= (a_{2,1}a_{2,1}^{-1})a_{1,p}a_{2,1}^{-1}a_{2,p}^{-1}$$

$$= a_{2,1}[a_{2,1}^{-1}, a_{1,p}]a_{2,p}^{-1}$$

$$= a_{2,1}a_{2,p}a_{2,2}^{-1}[a_{2,1}^{-1}, a_{1,p}]a_{2,1}^{-1}$$

$$= (a_{2,1}a_{2,1}^{-1})^{-1}[(a_{2,1}^{-1})^{a_{2,1}}, a_{1,p}]^{-1}$$

$$= (a_{2,1}a_{2,1}^{-1})^{-1}[(a_{2,1}^{-1})^{a_{2,1}}, a_{1,p}]^{-1}.$$ 

We transform commutator $\kappa$ in such way that it is similar to the form of $w$. This gives us equations with unknown variables $a_{i,j}$:

\[
\begin{align*}
    a_{i,j}a_{2,i+1}a_{1,j}a_{2,i}^{-1} &= r_j, \quad \text{for } 1 \leq i \leq p - 1, \\
    (a_{2,p}a_{2,1}^{-1})^{-1} &= r_1^{-1} \ldots r_{p-1}^{-1}, \\
    (a_{2,1}a_{2,p}a_{2,1}^{-1}) &= f, \\
    a_{2,p}^{-1}a_{1,p}^{-1} &= g.
\end{align*}
\]

In order to prove required statement it is enough to find at least one solution of equations. We set the following

$$a_{1,i} = e \quad \text{for } 1 \leq i \leq p - 1.$$

Then we have

\[
\begin{align*}
    a_{2,i+1}a_{2,i}^{-1} &= r_i, \quad \text{for } 1 \leq i \leq p - 1, \\
    (a_{2,p}a_{2,1}^{-1})^{-1} &= r_1^{-1} \ldots r_{p-1}^{-1}, \\
    (a_{2,1}a_{2,p}a_{2,1}^{-1}) &= f, \\
    a_{2,p}^{-1}a_{1,p}^{-1} &= g.
\end{align*}
\]

Now we can see that the form of the commutator $\kappa$ is similar to the form of $w$.

Let us make the following notation

$$r' = r_{p-1} \ldots r_1.$$ 

We note that from the definition of $a_{2,i}$ for $2 \leq i \leq p$ it follows that

$$r_i = a_{2,i+1}a_{2,i}^{-1}, \quad \text{for } 1 \leq i \leq p - 1.$$
Therefore
\[ r' = (a_{2,p}a_{2,p-1}^{-1})(a_{2,p-1}a_{2,p-2}^{-1}) \ldots (a_{2,3}a_{2,2}^{-1})(a_{2,2}a_{2,1}^{-1}) = a_{2,p}a_{2,1}^{-1}. \]

And then
\[ (a_{2,p}a_{2,1}^{-1})^{-1} = (r')^{-1} = r_1^{-1} \ldots r_{p-1}^{-1}. \]

Finally let us to compute the following
\[ (a_{2,1}^{-1})a_{2,p}a_{2,1}^{-1} = (((f^{-1})r_1^{-1} \ldots r_{p-1}^{-1})r') = (f(r')^{-1})r' = f, \]
\[ a_{1,p}^{a_{2,p}} = (g^{a_{2,p}^{-1}})^{a_{2,p}} = g. \]

And now we conclude that
\[ a_{3,p} = r_1^{-1} \ldots r_{p-1}^{-1}[f, g]. \]

Thus, the commutator \( \kappa \) is presented exactly in the similar form as \( w \) has. \( \square \)

For future use we formulate previous lemma for the case \( p = 2 \)

**Corollary 4.** If \( B \) is any group and \( w \in B \wr C_2 \) is defined by the following wreath recursion
\[ w = (r_1, r_1^{-1}[f, g]), \]
where \( r_1, f, g \in B \), then \( w \) can be represent as commutator
\[ w = [(e, a_{1,2})\sigma, (a_{2,1}, a_{2,2})], \]
where
\[ a_{2,1} = (f^{-1})r_1^{-1}, \]
\[ a_{2,2} = r_1a_{2,1}, \]
\[ a_{1,2} = g^{a_{2,2}}. \]

**Lemma 5.** For any group \( B \) and integer \( p \geq 2 \) inequality
\[ cw(B \wr C_p) \leq \max(1, cw(B)) \]

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holds.

Proof. We can represent any $w \in (B \wr C_p)'$ by Lemma 1 with the following wreath recursion

$$w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots, r_{p-1}^{-1} \prod_{j=1}^{k} [f_j, g_j])$$

$$= (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots, r_{p-1}^{-1} [f_1, g_1]) \cdot \prod_{j=2}^{k} [(e, \ldots, e, f_j), (e, \ldots, e, g_j)],$$

where $r_1, \ldots, r_{p-1}, f_j, g_j \in B$, $k \leq cw(B)$. Now by the Lemma 3 we can see that $w$ can be represented as product of $\max(1, cw(B))$ commutators. \hfill \Box

**Corollary 6.** If $W = C_{p_k} \ast \ldots \ast C_{p_1}$ then for $k \geq 2$ $cw(W) = 1$.

Proof. If $W = C_{p_k} \ast \ldots \ast C_{p_1}$ then according to Lemma 5 implies that $cw(C_{p_k} \ast C_{p_{k-1}}) = 1$, because $C_{p_k} \ast C_{p_{k-1}}$ is not commutative group so $cw(W) > 0$. If $W = C_{p_k} \ast \ldots \ast C_{p_2}$ then according to the inequality $cw(C_{p_k} \ast C_{p_{k-1}} \ast C_{p_{k-2}}) \leq \max(1, cw(B))$ from Lemma 5 we obtain $cw(W) = 1$. Analogously if $W = C_{p_k} \ast \ldots \ast C_{p_2}$ and supposition of induction for $C_{p_k} \ast \ldots \ast C_{p_2}$ holds then using that permutational wreath product is associative construction we obtain from the inequality of Lemma 5 and $cw(C_{p_k} \ast \ldots \ast C_{p_2}) = 1$ that $cw(W) = 1$. \hfill \Box

**Corollary 7.** Commutator width $cw(Syl_p(S_p^k)) = 1$ for prime $p$ and $k > 1$ and commutator width $cw(Syl_p(A_p^k)) = 1$ for prime $p > 2$ and $k > 1$.

Proof. Since $Syl_p(S_p^k)$ $\simeq$ $\bigast_{i=1}^{k} C_p$ (see [11],[12], then $cw(Syl_p(S_p^k)) = 1$. As well known in case $p > 2$ $Syl_p(S_p^k)$ $\simeq$ $Syl_p(A_p^k)$ (see [20]), then $cw(Syl_p(A_p^k)) = 1$. \hfill \Box

**Definition 1.** Let us call the index of automorphism $\alpha$ on $X^l$ a number of active v.p. of $\alpha$ on $X^l$, denote it by $In_l(\alpha)$.

The following Lemma gives us a criteria when the element from the group $Syl_2 S_2^k$ belong to $(Syl_2 S_2^k)'$.

**Lemma 8.** An element $g \in B_k$ belongs to commutator subgroup $B'_k$ iff $g$ has even index on $X^l$ for all $0 \leq l < k$.

Proof. Let us prove the ampleness by induction by a number of level $l$ and index of $g$ on $X^l$. We first show that our statement for base of the induction is true. Actually, if
α, β ∈ B₀ then \((αβα^{-1})β^{-1}\) determine a trivial v.p. on \(X^0\). If α, β ∈ B₁ and β has an odd index on \(X^1\), then \((αβα^{-1})\) and β⁻¹ have the same index on \(X^1\). Consequently, in this case an index of the product \((αβα^{-1})β^{-1}\) can be 0 or 2. Case where α, β ∈ B₁ and has even index on \(X^1\), needs no proof, because the product and the sum of even numbers is an even number.

To finish the proof it suffices to assume that for \(B_{l-1}\) statement holds and prove that it holds for \(B_l\). Let α, β are an arbitrary automorphisms from \(\text{Aut}X^{[k]}\) and β has index \(x\) on \(X^l\), \(l < k\), where \(0 \leq x \leq 2^l\). A conjugation of an automorphism β by arbitrary \(α \in \text{Aut}X^{[k]}\) gives us arbitrary permutations of \(X^l\) where β has active v.p.

Thus following product \((αβα^{-1})β^{-1}\) admits all possible even indexes on \(X^l, l < k\) from 0 to 2x. In addition \([α, β]\) can has arbitrary permitted assignment (arrangement) of v.p. on \(X^1\). Let us present \(B_k\) as \(B_k = B_lB_{k-l}\), so elements α, β can be presented in form of wreath recursion \(α = (h_1, ..., h_{2^l})\pi_1, β = (f_1, ..., f_{2^l})\pi_2\), \(h_i, f_i \in B_{k-l}\), \(0 < i \leq 2^l\) and \(h_i, f_j\) corresponds to sections of automorphism in vertices of \(X^1\) of isomorphic subgroup to \(B_l\) in \(\text{Aut}X^{[k]}\). Actually, the parity of this index are formed independently of the action of \(\text{Aut}X^{[l]}\) on \(X^l\). So this index forms as a result of multiplying of elements of commutator presented as wreath recursion \((αβα^{-1})β^{-1} = (h_1, ..., h_{2^l})\pi_1, (f_1, ..., f_{2^l})\pi_2 = (h_1, ..., h_{2^l})f_{\pi_1(1)}, ..., f_{\pi_1(2^l)}\pi_1\pi_2\), where \(h_i, f_j \in B_{k-l}\), \(l < k\) and besides automorphisms corresponding to \(h_i\) are \(x\) automorphisms which has active v.p. on \(X^1\). Analogous automorphisms \(h_i\) has number of active v.p. equal to \(x\). As a result of multiplication we have automorphism with index \(2i : 0 \leq 2i \leq 2x\). Consequently, commutator \([α, β]\) has arbitrary even indexes on \(X^m\), \(m < l\) and we showed by induction that it has even index on \(X^l\).

Let us prove this Lemma by induction on level \(k\). Let us to suppose that we prove current Lemma (both sufficiency and necessity) for \(k - 1\). Then we rewrite element \(g \in B_k\) with wreath recursion

\[ g = (g_1, g_2)\sigma^i, \]

where \(i \in \{0, 1\}\).

Now we consider sufficiency.

Let \(g \in B_k\) and \(g\) has all even indexes on \(X^j\) \(0 \leq j < k\) we need to show that \(g \in B'_{k}\). According to condition of this Lemma \(g_1g_2\) has even indexes. An element \(g\) has form \(g = (g_1, g_2)\), where \(g_1, g_2 \in B_{k-1}\), and products \(g_1g_2 = h \in B'_{k-1}\) because \(h \in B_{k-1}\) and for \(B_{k-1}\) induction assumption holds. Therefore all products of form \(g_1g_2\) indicated in formula² belongs to \(B'_{k-1}\). Hence, from Lemma⁴ follows that \(g = (g_1, g_2) \in B'_{k}\).  

An automorphisms group of the subgroup \(C_2^{2k-1-1}\) is based on permutations of copies
of $C_2$. Orders of $\prod_{i=l}^{k-1} C_2$ and $C_2^{2^{k-1}-1}$ are equals. A homomorphism from $\prod_{i=l}^{k-1} C_2$ into $\text{Aut}(C_2^{2^{k-1}-1})$ is injective because a kernel of action $\prod_{i=l}^{k-1} C_2$ on $C_2^{2^{k-1}-1}$ is trivial, action is effective. The group $G_k$ is a proper subgroup of index 2 in the group $\prod_{i=l}^{k-1} C_2$ [14][18][20]. The following theorem can be used for proving structural property of Sylow subgroups.

**Theorem 9.** A maximal 2-subgroup of $\text{Aut}X^k$ acting by even permutations on $X^k$ has the structure of the semidirect product $G_k \cong B_{k-1} \rtimes W_{k-1}$ and is isomorphic to $\text{Syl}_2A_{2k}$. Also $G_k < B_k$.

The prove of this theorem is in [18].

An even easier Proposition, that needs no proof, is the following.

**Proposition 10.** An element $(g_1,g_2)\sigma^i$, $i \in \{0,1\}$ of wreath power $\prod_{i=l}^{k-1} C_2$ belongs to its subgroup $G_k$, iff $g_1g_2 \in G_{k-1}$.

**Proof.** This fact follows from the structure of elements of $G_k$ described Theorem [3] in and the construction of wreath recursion. Indeed, due to structure of elements of $G_k$ described in Theorem, we have action on $X^k$ by an even permutations because subgroup $W_{k-1}$, containing even number of transposition, acts on $X^k$ only by even permutation. The condition $g_1g_2 \in G_{k-1}$ is equivalent to index of $g$ on $X^{k-1}$ is even but this condition equivalent to condition that $g$ acting on $X^k$ by even permutation.

**Lemma 11.** An element $(g_1,g_2)\sigma^i \in G'_k$ iff $g_1,g_2 \in G_{k-1}$ and $g_1g_2 \in B'_{k-1}$.

**Proof.** Indeed, if $(g_1,g_2) \in G'_k$ then indexes of $g_1$ and $g_2$ on $X^{k-1}$ are even according to Lemma [18] thus, $g_1,g_2 \in G_{k-1}$. A sum of indexes of $g_1$ and $g_2$ on $X^l$, $l < k - 1$ are even according to Lemma [18] too, so index of product $g_1g_2$ on $X^l$ is even. Thus, $g_1g_2 \in B'_{k-1}$. Hence, necessity is proved.

Let us prove the sufficiency via Lemma [18]. Wise versa, if $g_1,g_2 \in G_{k-1}$ then indexes of these automorphisms on $X^{k-2}$ of subtrees $v_{11}X^{[k-1]}$ and $v_{12}X^{[k-1]}$ are even as elements from $G'_k$ have. The product $g_1g_2$ belongs to $B'_{k-1}$ by condition of this Lemma so sum of indexes of $g_1,g_2$ on any level $X^l$, $0 \leq l < k - 1$ is even. Thus, the characteristic properties of $G'_k$ described in Lemma [18] holds.

**Proposition 12.** The following inclusion $B'_{k} < G_k$ holds.

**Proof.** Indeed, $B'_k = \prod_{i=1}^{k-1} C_2 = B_{k-1}$ and as we define $G_k \cong B_{k-1} \rtimes W_{k-1}$ so $B'_k < G_k$.
Proposition 13. The group $G_k$ is normal in wreath product $\prod_{i=1}^k C_2$ i.e. $G_k \triangleleft B_k$.

Proof. The commutator of $B_k$ is $B'_k < B_{k-1}$. In other hand $B_{k-1} < G_k$ because $G_k \simeq B_{k-1} \rtimes W_{k-1}$ consequently $B'_k < G_k$. Thus, $G_k \triangleleft B_k$. □

There exists a normal embedding (normal injective monomorphism) $\varphi : G_k \to B_k$ i.e. $G_k \triangleleft B_k$. Actually, it implies from Proposition 13. Also according to [18] index $|B_k : G_k| = 2$ so $G_k$ is a normal subgroup that is a factor subgroup $B_k/C_2 \simeq G_k$.

Theorem 14. Elements of $B'_k$ have the following form $B'_k = \{[f,l] | f \in B_k, l \in G_k\} = \{[l,f] | f \in B_k, l \in G_k\}$.

Proof. It is enough to show either $B'_k = \{[f,l] | f \in B_k, l \in G_k\}$ or $B'_k = \{[l,f] | f \in B_k, l \in G_k\}$ because if $f = [g,h]$ then $f^{-1} = [h,g]$.

We prove the proposition by induction on $k$. $B'_1 = \langle e \rangle$.

We already know [2] that every element $w \in B'_k$ we can represent as

$$w = (r_1, r_1^{-1}[f,g])$$

for some $r_1, f \in B_{k-1}$ and $g \in G_{k-1}$ (by induction hypothesis). By the Corollary 4 we can represent $w$ as commutator of

$$(e, a_{1,2}) \sigma \in B_k \text{ and } (a_{2,1}, a_{2,2}) \in B_k,$$

where

$$a_{2,1} = (f^{-1})^{r_1^{-1}},$$
$$a_{2,2} = r_1 a_{2,1},$$
$$a_{1,2} = g^{a_{2,2}^{-1}}.$$ 

We note that as $g \in G_{k-1}$ then by proposition 10 we have $(e, a_{1,2}) \sigma \in G_k$. □

Directly from this Proposition follows next Corollary, that needs no proof.

Remark 1. Let us to note that Theorem 14 improve Corollary 4 for the case $p = 2$.

Proposition 15. If $g$ is an element of wreath power $\prod_{i=1}^k C_2 \simeq B_k$ then $g^2 \in B'_k$.

Proof. As it was proved in Lemma 3 commutator $[\alpha, \beta]$ from $B_k$ has arbitrary even indexes on $X^m, m < k$. Let us show that elements of $B^2_k$ have the same structure.
Let $\alpha, \beta \in B_k$ an indexes of the automorphisms $\alpha^2, (\alpha\beta)^2$ on $X^l$, $l < k - 1$ are always even. In more detail the indexes of $\alpha^2, (\alpha\beta)^2$ and $\alpha^{-2}$ on $X^l$ are determined exceptionally by the parity of indexes of $\alpha$ and $\beta$ on $X^l$. Actually, the parity of this index are formed independently of the action of $\text{Aut}X^l$ on $X^l$. So this index forms as a result of multiplying of elements $\alpha \in B_k$ presented as wreath recursion $\alpha^2 = (h_1, ..., h_{2l})\pi_1 \cdot (h_1, ..., h_{2l})\pi_1 = (h_1, ..., h_{2l})(h_{\pi_1(1)}, ..., h_{\pi_1(2l)})\pi_1^2$, where $h_i, h_j \in B_{k-l}, \pi_1 \in B_l, l < k$ and besides automorphisms corresponding to $h_i$ are $x$ automorphisms which has active v.p. on $X^l$. Analogous automorphisms $h_i$ has number of active v.p. equal to $x$. As a result of multiplication we have automorphism with index $2i : 0 \leq 2i \leq 2x$.

Since $g^2$ admits only an even index on $X^l$ of $\text{Aut}X^{[k]}$, $0 < l < k$, then $g^2 \in B'_k$ according to lemma 5 about structure of a commutator subgroup.

Since as well known a group $G^2_k$ contains the subgroup $G'$ then a product $G^2G'$ contains all elements from the commutant. Therefore, we obtain that $G^2_k \simeq G'_k$.

**Proposition 16.** For arbitrary $g \in G_k$ following inclusion $g^2 \in G'_k$ holds.

**Proof.** Induction on $k$: for $G^2_1$ elements has form $((e,e)\sigma)^2 = e$ where $\sigma = (1,2)$ so statement holds. In general case when $k > 1$ elements of $G_k$ has form

$$g = (g_1, g_2)^i, g_1 \in B_{k-1}, i \in 0.1$$

then we have two possibilities

$$g^2 = (g_1^2, g_2^2) \text{ or } g^2 = (g_1g_2, g_2g_1).$$

We first show that

$$g_1^2 \in B'_{k-1}, g_2^2 \in B'_{k-1}$$

after we will prove

$$g_1g_2 \cdot g_2g_1 \in B'_{k-1},$$

actually, according to Proposition 14 $g_1^2, g_2^2 \in B'_{k-1}$ then $g_1^2g_2^2 \in B'_{k-1}$ and $g_1^2, g_2^2 \in G_{k-1}$ by Proposition 12 also $g_1^2, g_2^2 \in G_{k-1}$ by induction assumption. From Proposition 10 it follows that $g_1g_2 \in G_{k-1}$.

Note that $B'_{k-1} < B_{k-2}$. In other hand $B_{k-2} < G_{k-1}$ because $G_{k-1} \simeq B_{k-2} \times W_{k-2}$ consequently $B'_{k-1} < G_{k-1}$. Besides we have $g_1^2 \in B'_{k-1}$ hence $g_1^2 \in G_{k-1}$.

Thus, we can use Lemma 11 (about $G'_k$) from which yields $g^2 = (g_1^2, g_2^2) \in G'_k$. 

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Consider second case \( g^2 = (g_1g_2, g_2g_1) \)

\[ g_1g_2 \in G_{k-1} \text{ by proposition 10} \]

\[ g_2g_1 = g_1g_2g_2^{-1}g_1^{-1}g_2g_1 = g_1g_2[2g_2^{-1}, g_1^{-1}] \in G_{k-1} \text{ by propositions 12 and 10} \]

\[ g_1g_2 \cdot g_2g_1 = g_1g_2^2g_1 = g_1^2g_2^2[2g_2^{-2}, g_1^{-1}] \in B'_{k-1} \]

Note that \( g_1^2, g_2^2 \in B'_{k-1} \) according to Proposition 15, this is a reason why \( g_1^2g_2^2[2g_2^{-2}, g_1^{-1}] \in B'_{k-1} \). Thus, \((g_1g_2, g_1g_2) \in G'_k \) by Lemma 15.

Let \( X_1 = \{v_{k-1,1}, v_{k-1,2}, ..., v_{k-1,2^{k-2}}\} \) and \( X_2 = \{v_{k-1,2^{k-2}+1}, ..., v_{k-1,2^{k-1}}\} \).

We will call a distance structure \( \rho_l(\theta) \) of \( \theta \) a tuple of distances between its active vertices from \( X^l \). Let group \( Syl_2A_{2k} \) acts on \( X^{[k]} \).

**Lemma 17.** An element \( g \) belongs to \( G'_k \simeq Syl_2A_{2k} \) iff \( g \) is arbitrary element from \( G_k \) which has all even indexes on \( X^l \), \( l < k-1 \) of \( X^{[k]} \) and on \( X^{k-2} \) of subtrees \( v_{11}X^{[k-1]} \) and \( v_{12}X^{[k-1]} \).

**Proof.** Let us prove the amenability by induction on a number of level \( l \) and index of automorphism \( g \) on \( X^l \). Conjugation by automorphism \( \alpha \) from \( Autv_{11}X^{[k-1]} \) of automorphism \( \theta \), that has index \( x : 1 \leq x \leq 2^{k-2} \) on \( X_1 \) does not change \( x \). Also automorphism \( \theta^{-1} \) has the same number \( x \) of v. p. on \( X_{k-1} \) as \( \theta \) has. If \( \alpha \) from \( Autv_{11}X^{[k-1]} \) and \( \alpha \notin AutX^{[k]} \) then conjugation \( (\alpha\theta\alpha^{-1}) \) permutes vertices only inside \( X_1 \) (\( X_2 \)).

Thus, \( \alpha\theta\alpha^{-1} \) and \( \theta \) have the same parities of number of active v. p. on \( X_1 \) (\( X_2 \)). Hence, a product \( \alpha\theta\alpha^{-1}\theta^{-1} \) has an even number of active v. p. on \( X_1 \) (\( X_2 \)) in this case. More over a coordinate-wise sum by \( \mod 2 \) of active v. p. from \( (\alpha\theta\alpha^{-1}) \) and \( \theta^{-1} \) on \( X_1 \) (\( X_2 \)) is even and equal to \( y : 0 \leq y \leq 2x \).

If conjugation by \( \alpha \) permutes sets \( X_1 \) and \( X_2 \) then there are coordinate-wise sums of no trivial v. p. from \( \alpha\theta\alpha^{-1}\theta^{-1} \) on \( X_1 \) (analogously on \( X_2 \)) have form:

\((s_{k-1,1}(\alpha\theta\alpha^{-1}), ..., s_{k-1,2k-2}(\alpha\theta\alpha^{-1})) \oplus (s_{k-1,1}(\theta^{-1}), ..., s_{k-1,2k-2}(\theta^{-1}))\). This sum has even number of v. p. on \( X_1 \) and \( X_2 \) because \( (\alpha\theta\alpha^{-1}) \) and \( \theta^{-1} \) have a same parity of no trivial v. p. on \( X_1 \) (\( X_2 \)). Hence, \((\alpha\theta\alpha^{-1})\theta^{-1} \) has even number of v. p. on \( X_1 \) as well as on \( X_2 \).

An automorphism \( \theta \) from \( G_k \) was arbitrary so number of active v. p. \( x \) on \( X_1 \) is arbitrary \( 0 \leq x \leq 2^l \). And \( \alpha \) is arbitrary from \( AutX^{[k-1]} \) so vertices can be permuted in such way that the commutator \([\alpha, \theta] \) has arbitrary even number \( y \) of active v. p. on \( X_1 \), \( 0 \leq y \leq 2x \).

A conjugation of an automorphism \( \theta \) having index \( x, 1 \leq x \leq 2^l \) on \( X^l \) by different \( \alpha \in AutX^{[k]} \) gives us all tuples of active v. p. with the same \( \rho_l(\theta) \) that \( \theta \) has on \( X^l \), by
which $\text{Aut}X^{|k|}$ acts on $X^l$. Let supposition of induction for element $g$ with index $2k - 2$ on $X^l$ holds so $g = (\alpha \theta \alpha^{-1})\theta^{-1}$, where $In_l(\theta) = x$. To make a induction step we complete $\theta$ by such active vertex $v_{l,x}$ too it has suitable distance structure for $g = (\alpha \theta \alpha^{-1})\theta^{-1}$, also if $g$ has rather different distance structure $d_l(g)$ from $d_l(\theta)$ then have to change $\theta$. In case when we complete $\theta$ by $v_{l,x}$ it has too satisfy a condition $(\alpha \theta \alpha^{-1})(v_{x+1}) = v_{l,y}$, where $v_{l,y}$ is a new active vertex of $g$ on $X^l$. Note that $v(x + 1)$ always can be chosen such that acts in such way $\alpha(v(x + 1)) = v(2k + 2)$ because action of $\alpha$ is 1-transitive. Second vertex arise when we multiply $(\alpha \theta \alpha^{-1})$ on $\theta^{-1}$. Hence $In_l(\alpha \theta \alpha^{-1}) = 2k + 2$ and coordinates of new vertices $v_{2k+1}, v_{2k+2}$ are arbitrary from 1 to $2^l$.

So multiplication $(\alpha \theta \alpha^{-1})\theta$ generates a commutator having index $y$ equal to coordinate-wise sum by $\text{mod}2$ of no trivial v.p. from vectors $(s_{11}(\alpha \theta \alpha^{-1}), s_{12}(\alpha \theta \alpha^{-1}), ..., s_{12}(\alpha \theta \alpha^{-1})) \oplus (s_{11}(\theta), s_{12}(\theta), ..., s_{12}(\theta))$ on $X^l$. A indexes parities of $\alpha \theta \alpha^{-1}$ and $\theta^{-1}$ are same so their sum by $\text{mod}2$ are even. Choosing $\theta$ we can choose an arbitrary index $x \in \theta$ also we can choose arbitrary $\alpha$ to make a permutation of active v.p. on $X^l$. Thus, we obtain an element with arbitrary even index on $X^l$ and arbitrary location of active v.p. on $X^l$.

Check that property of number parity of v.p. on $X_1$ and on $X_2$ is closed with respect to conjugation. We know that numbers of active v. p. on $X_1$ as well as on $X_2$ have the same parities. So action by conjugation only can permutes it, hence, we again get the same structure of element. Conjugation by automorphism $\alpha$ from $\text{Aut}v_{11}X^{|k-1|}$ automorphism $\theta$, that has odd number of active v. p. on $X_1$ does not change its parity. Choosing the $\theta$ we can choose arbitrary index $x \in \theta$ on $X^{|k-1|}$ and number of active v.p. on $X_1$ and $X_2$ also we can choose arbitrary $\alpha$ to make a permutation active v.p. on $X_1$ and $X_2$. Thus, we can generate all possible elements from a commutant. Also this result follows from Lemmas [11] and [8].

Let us check that the set of all commutators $K$ from $Syl_2A_2k$ is closed with respect to multiplication of commutators. Let $\kappa_1, \kappa_2 \in K$ then $\kappa_1 \kappa_2$ has an even index on $X^l$, $l < k-1$ because coordinate-wise sum $(s_{11}(\kappa_1), ..., s_{k-1,2l}(\kappa_1)) \oplus (s_{l,\kappa_1(1)}(\kappa_2), ..., s_{l,\kappa_1(2l)}(\kappa_2))$. of two $2^l$-tuples of v.p. with an even number of no trivial coordinate has even number of such coordinate. Note that conjugation of $\kappa$ can permute sets $X_1$ and $X_2$ so parities of $x_1$ and $X_2$ coincide. It is obviously index of $\alpha \kappa \alpha^{-1}$ is even as well as index of $\kappa$.

Check that a set $K$ is a set closed with respect to conjugation.

Let $\kappa \in K$, then $\alpha \kappa \alpha^{-1}$ also belongs to $K$, it is so because conjugation does not change index of an automorphism on a level. Conjugation only permutes vertices on level because elements of $\text{Aut}X^{|l-1|}$ acts on vertices of $X^l$. But as it was proved above elements of $K$ have all possible indexes on $X^l$, so as a result of conjugation $\alpha \kappa \alpha^{-1}$ we obtain an element from $K$.

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Check that the set of commutators is closed with respect to multiplication of commutators. Let $\kappa_1, \kappa_2$ be an arbitrary commutators of $G_k$. The parity of the number of vertex permutations on $X^l$ in the product $\kappa_1 \kappa_2$ is determined exceptionally by the parity of the numbers of active v.p. on $X^l$ in $\kappa_1$ and $\kappa_2$ (independently from the action of v.p. from the higher levels). Thus $\kappa_1 \kappa_2$ has an even index on $X^l$.

Hence, normal closure of the set $K$ coincides with $K$.

Lemma 18. An element $g = (g_1, g_2)\sigma^i$ of $G_k$, $i \in \{0, 1\}$ belongs to $G'_k$ iff $g$ has even index on $X^l$ for all $l < k - 1$ and elements $g_1, g_2$ have even indexes on $X^{k-1}$, that is equally matched to $g_1, g_2 \in G_{k-1}$.

Proof. The proof implies from Lemma 18 and Lemma 11.

Using this structural property of $(Syl_2 A_{2k})'$ we deduce a following result.

Theorem 19. Commutator subgroup $G'_k$ coincides with set of all commutators, put it differently $G'_k = \{[f_1, f_2] \mid f_1 \in G_k, f_2 \in G_k\}$.

Proof. For the case $k = 1$ we have $G'_1 = \langle e \rangle$. So, further we consider case $k \geq 2$. In order to prove this Theorem we fix arbitrary element $w \in G'_k$ and then we represent this element as commutator of elements from $G_k$.

We already know by Lemma 14 that every element $w \in G'_k$ we can represent as follow

$$w = (r_1, r_1^{-1}x),$$

where $r_1 \in G_{k-1}$ and $x \in B'_{k-1}$. By proposition 14 we have $x = [f, g]$ for some $f \in B_{k-1}$ and $g \in G_{k-1}$. Therefore

$$w = (r_1, r_1^{-1}[f, g]).$$

By the Corollary 4 we can represent $w$ as commutator of

$$(e, a_{1,2})\sigma \in B_k \text{ and } (a_{2,1}, a_{2,2}) \in B_k,$$

where

$$a_{2,1} = (f^{-1})r_1^{-1},$$

$$a_{2,2} = r_1 a_{2,1},$$

$$a_{1,2} = g^{a_{2,2}}.$$
It is only left to show that \((e, a_{1,2})\sigma, (a_{2,1}, a_{2,2}) \in G_k\).

In order to use Proposition 10 we note that

\[
a_{1,2} = g^{a_{2,2}} \in G_{k-1} \text{ by Proposition } 13
\]

\[
a_{2,1}a_{2,2} = a_{2,1}r_1a_{2,1} = r_1[a_{2,1}, a_{2,2}]^2 \in G_{k-1} \text{ by Proposition } 12 \text{ and Proposition } 15
\]

So we have \((e, a_{1,2})\sigma \in G_k\) and \((a_{2,1}, a_{2,2}) \in G_k\). \(\square\)

**Corollary 20.** Commutator width of the group \(Syl_2A_{2^k}\) equal to 1 for \(k \geq 2\).

**Theorem 21.** The centralizer of \(Syl_2S_{2^k}\) \(\bigotimes [12]\) with \(k_i > 2\), in \(Syl_2S_n\) is isomorphic to \(Syl_2S_{2^k} \otimes Syl_2S_{2^{k_i}} \times Z(Syl_2S_{2^{k_i}})\).

**Proof.** Actually, the action of active group \(A \simeq \bigotimes_{i=1}^{k_i-1} C_2\) of \(Syl_2S_{2^k}\) on \(X^{k-1}\) is transitive since the orbit of \(A\) on \(X^{k-1}\) is one. Then \(Z(Syl_2S_{2^k}) \simeq C_2\) results by formula from Corollary 4.4 \([9]\).

For any no trivial automorphism \(\alpha\) from \(AutX^{[k_i]} \simeq Syl_2S_{2^k}\), there exists a vertex \(v_{jm}\), where v.p. from \(\alpha\) is active. Thus, there exists graph path \(r\) connecting the root \(v_0\) with a vertex of \(X^{k_i}\) and pathing through the \(v_{jm}\). We can choose vertex \(v_{li}\) on \(r\) such that \(l \neq j\). So there exists \(\beta \in AutX^{[k_i]}\) that has active v.p. in \(v_{li}\). Then we have \(\alpha \beta \neq \beta \alpha\). The center of \(AutX^{[k_i]}\) is isomorphic to \(C_2\). As a result we have \(C_{AutS_n}(Syl_2S_{2^k}) \simeq Syl_2S_{[k_i]} \otimes Syl_2S_{2^{k_i}} \times Z(Syl_2S_{2^{k_i}})\). \(\square\)

Let us present new operation \(\boxtimes\) (similar to that in \([9]\)) as an even subdirect product of \(Syl_2S_{2^k}\), \(n = 2^{k_0} + 2^{k_1} + \ldots + 2^{k_m}\), \(0 \leq k_0 < k_1 < \ldots < k_m\).

**Theorem 22.** The centralizer of \(Syl_2A_{2^k}\) with \(k_i > 2\), in \(Syl_2A_n\) is isomorphic to \(Syl_2S_{2^k} \boxtimes \ldots \boxtimes Syl_2S_{2^{k_i-1}} \boxtimes Syl_2S_{2^{k_{i+1}}} \boxtimes \ldots \boxtimes Syl_2S_{2^{k_m}} \boxtimes Z(Syl_2S_{2^{k_i}})\).

**Proof.** We consider \(G_k\) as a normal subgroup of \(\bigotimes_{i=1}^{k_i} C_2\). Actually, the action of subgroup \(A = B_{k_0} \cap \ldots \cap B_{k_i-1}\) of \(G_{k_i} \simeq Syl_2A_{2^k}\) on \(X^{k-1}\) is transitive since the orbit of \(A\) on \(X^{k-1}\) is one.

There exists a vertex \(v_{jm}\) for any no trivial automorphism \(\alpha\) from \(G_{k_i} \simeq Syl_2A_{2^k}\), where v.p. from \(\alpha\) is active. Thus there exists graph path \(r\) connecting the root \(v_0\) with a vertex of \(X^{k_i}\) and pathing through the \(v_{jm}\). We can choose vertex \(v_{li}\) on \(r\) such that \(l \neq j\). So there exists \(\beta \in AutX^{[k_i]}\) that has active v.p. in \(v_{li}\). Then we have \(\alpha \beta \neq \beta \alpha\). Since \(Syl_2A_{2^k} \simeq AutX^{[k]}\), consequently the center of \(Syl_2A_{2^k}\) is isomorphic to \(C_2\). As a result we have \(C_{AutA_n}(Syl_2S_{2^{k_i}}) \simeq Syl_2S_{2^{k_0}} \boxtimes \ldots \boxtimes Syl_2S_{2^{k_{i-1}}} \boxtimes Syl_2S_{2^{k_{i+1}}} \boxtimes \ldots \boxtimes Syl_2S_{2^{k_m}} \boxtimes Z(A_{2^{k_i}})\). \(\square\)
Also we note that derived length of \( Syl_2 A_k^2 \) is not always equal to \( k \) as it was said in Lemma 3 of \cite{20} because in case \( A_{2k} \) if \( k = 2 \) its \( Syl_2 A_4 \simeq K_4 \) but \( K_4 \) is abelian group so its derived length is 1.

4 Conclusion

The commutator width of Sylow 2-subgroups of alternating group \( A_{2k} \), permutation group \( S_{2k} \) and Sylow \( p \)-subgroups of \( Syl_2 A_k^p \) (\( Syl_2 S_k^p \)) is equal to 1. Commutator width of permutational wreath product \( B \wr C_n \), were \( B \) is arbitrary group, was researched.

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