Groups in which each subgroup is commensurable with a normal subgroup.

| Questa è la Versione finale referata (Post print/Accepted manuscript) della seguente pubblicazione: |
|---------------------------------------------------------------|
| Original Citation:                                             |
| Groups in which each subgroup is commensurable with a normal  |
| subgroup / Carlo, Casolo; Ulderico,                           |
| Dardano; Silvana, Rinauro. - In: JOURNAL OF ALGEBRA. - ISSN    |
| 0021-8693. - STAMPA. - 496:(2018), pp. 48-60. [10.1016/j.jalgebra.2017.11.016] |

| Availability:                                                 |
| This version is available at: 2158/1109988 since: 2018-02-06T12:19:06Z |

| Published version:                                            |
| DOI: 10.1016/j.jalgebra.2017.11.016                          |

| Terms of use:                                                 |
| Open Access                                                   |
| La pubblicazione è resa disponibile sotto le norme e i termini|
| della licenza di deposito, secondo quanto stabilito dalla    |
| Policy per l'accesso aperto dell'Università degli Studi di    |
| Firenze (https://www.sba.unifi.it/upload/policy-oa-2016-1.pdf) |

| Publisher copyright claim:                                    |

(Article begins on next page)
Groups in which each subgroup is commensurable with a normal subgroup

to the memory of Jim Wiegold

Carlo Casolo, Ulderico Dardano, Silvana Rinauro

Abstract

A group $G$ is a $cn$-group if for each subgroup $H$ of $G$ there exists a normal subgroup $N$ of $G$ such that the index $|HN : (H \cap N)|$ is finite. The class of $cn$-groups contains properly both the well-known classes of core-finite groups and of finite-by-abelian groups. In the present paper it is shown that a $cn$-group whose periodic images are locally finite is finite-by-abelian-by-finite. Then such groups are described into some details by considering automorphisms of abelian groups. Finally, it is shown that if $G$ is a locally graded group with the property that the above index is bounded independently of $H$, then $G$ is finite-by-abelian-by-finite. ¹

1 Introduction

In a celebrated paper, B.H. Neumann [9] showed that for a group $G$ the property that each subgroup $H$ has finite index in a normal subgroup of $G$ (i.e. $|H^G : H|$ is finite) is equivalent to the fact that $G$ has finite derived subgroup ($G$ is finite-by-abelian).

The class of groups with a dual property was considered in [1]. A group $G$ is said a $cf$-group (core-finite) if each subgroup $H$ contains a normal subgroup of $G$ with finite index in $H$ (i.e. $|H : H_G|$ is finite). As Tarski groups are $CF$, a complete classification of $cf$-groups seems to be much difficult. However, in [1] and [11] it has been proved that a $cf$-group $G$ whose periodic quotients are locally finite is abelian-by-finite and there exists an integer $n$ such that $|H : H_G| \leq n$ for all $H \leq G$ (say that $G$ is $BCF$, boundedly CF).

¹Key words and phrases: locally finite, core-finite, subnormal, inert, $cf$-group.

2010 Mathematics Subject Classification: Primary 20F24, Secondary 20F18, 20F50, 20E15
Moreover a locally graded BCF-group is abelian-by-finite. Furthermore, an easy example of a metabelian (and even hypercentral) group which is CF but not BCF is given. It seems to be a still open question whether every locally graded CF-group is abelian-by-finite. Recall that a group is said abelian-by-finite if has an abelian subgroup with finite index and that a group is locally finite (locally graded, resp.) if each finitely generated subgroup is finite (has a proper subgroup with finite index, resp.).

With the aim of considering the above two classes in a common framework, recall that two subgroups $H$ and $K$ of a group $G$ are said commensurable if and only if $H \cap K$ has finite index in both $H$ and $K$. This is an equivalence relation and will be denoted by $\sim$. Clearly, if $H \sim K$, then $(H \cap L) \sim (K \cap L)$ and $HM \sim KM$ for each $L \leq G$ and $M \lhd G$.

In the present paper we consider the class of CN-groups, that is groups in which each subgroup is commensurable to a normal subgroup. Into details, for a subgroup $H$ of a group $G$ define $\delta_G(H)$ to be the minimum index $|HN : (H \cap N)|$ with $N \lhd G$. Then $G$ is a CN-group if and only if $\delta_G(H)$ is finite for all $H \leq G$. Clearly both finite-by-abelian and CF groups are CN. Moreover, the class of CN-groups is both subgroup and quotient closed.

Note that if a subgroup $H$ of a group $G$ is commensurable with a normal subgroup $N$, then $S := (H \cap N)_N$ has finite index in $H$. Thus the class of CN-groups is contained in the class of SbyF-groups, that is, groups in which each subgroup $H$ is subnormal-by-finite; that is to say that $H$ contains a subnormal subgroup $S$ of $G$ such that the index $|H : S|$ is finite. It is known that locally finite SbyF-groups are (locally nilpotent)-by-finite (see [3]) and nilpotent-by-Chernikov (see [6]).

Recall also that from results in [4] it follows that for an abelian-by-finite group properties CN and CF are equivalent. However, for each prime $p$ there is a nilpotent $p$-group with property CN which is neither finite-by-abelian nor abelian-by-finite, see Proposition 2.2 below.

Our main result is the following.

**Theorem A** Let $G$ be a CN-group such that every periodic image of $G$ is locally finite. Then $G$ is finite-by-abelian-by-finite.

Here by finite-by-abelian-by-finite group we mean a group which has a subgroup which has finite index and is finite-by-abelian. The proof of Theorem A will be completed at the end of Sect. 5. Before, in Sect. 3, we study the action of a CN-group on its abelian sections, see Theorem 3.2 and Corollary 3.3. Then in Sect. 4 we consider also BCN-groups, that is, groups
for which there is $n \in \mathbb{N}$ such that $\delta_G(H) \leq n$ for all $H \leq G$. We will show the following theorem.

**Theorem B** Let $G$ be a finite-by-abelian-by-finite group.

i) if $G$ is $cn$, then the FC-center of $G$ has finite index and is finite-by-abelian;

ii) $G$ is $cn$ if and only if it is finite-by-CF.

iii) $G$ is $bcn$ if and only if it is finite-by-$bcf$.

It follows that if the group $G$ is periodic and finite-by-abelian-by-finite, then $G$ is $bcn$ if and only if it is $cn$. Then we consider non-periodic finite-by-abelian-by-finite $bcf$- and $bcn$-groups by Proposition 4.4.

The more restrictive property $bcn$ reveals fruitful when we consider the wider class of locally graded groups.

**Corollary** A locally graded $bcn$-group is finite-by-abelian-by-finite.

Our notation is mostly standard and we refer to [10].

## 2 Preliminaries

We point out a sufficient condition for a group to be $cn$ (or even $bcn$).

**Proposition 2.1** Let $G$ be a group with a normal series $G_0 \leq G_1 \leq G$, where $G_0$ and $G/G_1$ have finite order $m$ and $n$ resp. If $H \leq G$, then $H$ is commensurable with $H_1 := (H \cap G_1)G_0 \leq G_1$ and $\delta_G(H) \leq mn \cdot \delta_{G/G_0}(H_1/G_0)$.

In particular, if each subgroup of $G_1/G_0$ is commensurable with a normal subgroup of $G/G_0$, then $G$ is a $cn$-group.

Now we give examples of non trivial $cn$-groups.

**Proposition 2.2** For each prime $p$ there is a nilpotent $p$-group with property $bcn$, which is not abelian-by-finite nor finite-by-abelian.

**Proof.** Consider a sequence $P_n$ of isomorphic groups with order $p^4$ defined by $P_n := \langle x_n, y_n \mid x_n^{p^3} = y_n^p = 1, x_n^{p^2} = x_n^{1+p^2} \rangle = \langle x_n \rangle \times \langle y_n \rangle$ where clearly $P'_n = \langle x_n^{p^2} \rangle$ has order $p$. Let $P := \bigcup_{n \in \mathbb{N}} P_n$ and consider the automorphism $\gamma$ of $P$ such that $x_n^\gamma = x_n^{1+p}$ and $y_n^\gamma = y_n$, for each $n \in \mathbb{N}$. Clearly, $\gamma$ has order $p^2$, acts as the automorphism $x \mapsto x^{1+p}$ on $P/P'$ (which has exponent $p^2$) and acts trivially on $P'$ (which is elementary abelian). Finally let $N := \langle x_0^{p^2} x_n^{p^2} \mid n \in \mathbb{N} \rangle$. Then $N$ is a subgroup of $P'$ with index $p$. Thus the $p$-group
\( G := (P \rtimes \langle \gamma \rangle)/N \) is a bcn-group by Proposition 2.1 applied to the series \( P'/N \leq P/N \leq G \).

We have that \( G' \) is infinite, since for each \( n \) we have \( x_n^{p^n} = [x_n, \gamma] \in [P_n, \gamma] > P_n' \). Moreover, we have that \( gN \in Z(P/N) \) if and only if \( \forall i \ [g, P_i] \leq N \), and \( N \cap P_i = 1 \). Thus \( Z(P/N) = Z(P)/N \) where \( Z(P) = Dr_n (x_n^{p^n}) \) has infinite index in \( P \).

If, by contradiction, \( G \) is abelian-by-finite, then there is an abelian normal subgroup \( A/N \) of \( P/N \) with finite index. Then for some \( m \in \mathbb{N} \) we have \( P = AF \), where \( F = Dr_{n<m} P_n \) is a finite normal subgroup of \( P \). Therefore \( P/N \) is center-by-finite, a contradiction. \( \square \)

3 Automorphisms of abelian groups

As in [4], for the action of a group \( \Gamma \) on a group \( A \), we consider the following properties:

- **P)** \( \forall H \leq A \ H = H^\Gamma; \)
- **AP)** \( \forall H \leq A \ |H/H^\Gamma| < \infty; \)
- **BP)** \( \forall H \leq A \ |H^\Gamma/H| < \infty; \)
- **CP)** \( \forall H \leq A \ \exists K = K^\Gamma \leq A \ such \ that \ H \sim K, \ (H, K \ are \ commensurable) \).

When \( P \) holds, one says that \( \Gamma \) acts on \( A \) by means of **power automorphisms** or that \( A \) is **\( \Gamma \)-hamiltonian** ([10],[1]). Recall that if \( \gamma \) is a power automorphism of an abelian \( p \)-group \( A \), then there exists a \( p \)-adic integer \( \alpha \) such that \( a^\gamma = a^\alpha \) for all \( a \in A \) (see [10] for details). Here \( a^\alpha \) stands for \( a^n \), where \( n \in \mathbb{N} \) is congruent to \( \alpha \) modulo the order of \( a \). On the other hand, a power automorphism of a non-periodic abelian group is either the identity or the inversion map.

Obviously both **AP** and **BP** imply **CP**. Moreover, **these three properties are equivalent**, provided \( A \) is abelian and \( \Gamma \) is finitely generated, while they are **in fact different in the general case even when \( A \) and \( \Gamma \) are elementary abelian \( p \)-groups** (see [4]). On the other hand, the properties **AP** and **BP** have previously characterized in [5] and [2] resp., as we are going to recall. To shorten statements we define a further property:

- **\( \hat{P} \)** \( \Gamma \) **has P on the factors of a \( \Gamma \)-series** \( 1 \leq V \leq D \leq A \) where
  - i) \( V \) is free abelian with finite rank,
  - ii) \( D/V \) is divisible periodic with finite total rank,
  - iii) \( A/D \) is periodic and has finite \( p \)-exponent for each prime \( p \in \pi(D/V) \).
Theorem 3.1 [5],[2] Let $\Gamma$ be a group acting on an abelian group $A$. Then:

a) $\Gamma$ has $\text{AP}$ on $A$ if and only if there is a $\Gamma$-subgroup $A_1$ such that $A/A_1$ is finite and $\Gamma$ has either $\text{P}$ or $\tilde{\text{P}}$ on $A_1$.

b) $\Gamma$ has $\text{BP}$ on $A$ if and only if there is a $\Gamma$-subgroup $A_0$ such that $A_0$ is finite and $\Gamma$ has either $\text{P}$ or $\tilde{\text{P}}$ on $A/A_0$.

By next statement we give a characterisation of the property $\text{CP}$ along the same lines.

Theorem 3.2 Let $\Gamma$ be a group acting on an abelian group $A$. Then:

c) $\Gamma$ has $\text{CP}$ on $A$ if and only if there are $\Gamma$-subgroups $A_0 \leq A_1 \leq A$ such that $A_0$ and $A/A_1$ are finite and $\Gamma$ has either $\text{P}$ or $\tilde{\text{P}}$ on $A_1/A_0$.

The proof of Theorem 3.2 is at the end of this section. Here we deduce a corollary.

Corollary 3.3 For a group $\Gamma$ acting on an abelian group $A$, the following are equivalent:

a) $\Gamma$ has $\text{AP}$ on $A/A_0$ for a finite $\Gamma$-subgroup $A_0$ of $A$,

b) $\Gamma$ has $\text{BP}$ on a finite index $\Gamma$-subgroup $A_1$ of $A$,

c) $\Gamma$ has $\text{CP}$ on $A$.

Let us recall some basic facts from [4] where inertial automorphisms of abelian groups have been introduced. These are automorphisms $\gamma$ of a group $G$ such that $H^\gamma \sim H$ for all $H \leq G$. Clearly, if $\Gamma$ has $\text{CP}$ on $G$ and $\gamma \in \Gamma$, then $\gamma$ is inertial.

Proposition 3.4 Let $\Gamma$ be group acting on a locally nilpotent periodic group $A$. Then $\Gamma$ has $\text{AP}$, $\text{BP}$, $\text{CP}$ resp. on $A$ if and only if $\Gamma$ has $\text{AP}$, $\text{BP}$, $\text{CP}$ resp. on finitely many primary components of $A$ and $\text{P}$ on all the other ones.

Lemma 3.5 Let $\Gamma$ be a group acting on an abelian group $A$. If $\Gamma$ has $\text{CP}$, then:

i) $\Gamma$ has $\text{P}$ on the maximum periodic divisible subgroup of $A$.

ii) if $A$ is torsion-free, then each $\gamma \in \Gamma$ acts by conjugation on $A$ by either the identity or the inversion map.

Now we prove some lemmas. In the first one we do not require that the group $A$ is abelian.
Lemma 3.6 Let $\Gamma$ be a group acting on a group $A$. If $\Gamma$ has cp, then $\Gamma$ has BP on the subgroup $X := \{ a \in A \mid \langle a \rangle^\Gamma \text{ is finite} \}$ of $A$.

Proof. For any $H \leq X$ there is $K$ such that $H \sim K = K^\Gamma \leq A$. Then there is finite subgroup $F \leq X$ such that $H \leq KF$. Thus $H^\Gamma \leq KF^\Gamma$ and $|H^\Gamma : H| \leq |F^\Gamma| \cdot |HK : H|$ is finite. \(\square\)

Lemma 3.7 Let $\Gamma$ be a group acting on a $p$-group $A$ which is the direct product of cyclic groups. If $\Gamma$ has cp, then the following subgroup has finite index in $A$:

$$X := \{ a \in A \mid \langle a \rangle^\Gamma \text{ is finite} \}$$

Proof. Assume by contradiction that $A/X$ is infinite.

Let us see that, by elementary facts, there is a sequence $(a_n)$ of elements of $A$ such that
1) $\langle a_n | n \in \mathbb{N} \rangle = \text{Dr}_{n \in \mathbb{N}} \langle a_n \rangle$,
2) $A_I/A_I \cap X$ is infinite, for each infinite subset $I$ of $\mathbb{N}$, where $A_I := \langle a_i | i \in I \rangle$.

In fact, if $A/X$ has finite rank, it has a Prüfer subgroup $Q/X$. Let $Y$ be a countable subgroup such that $Q = YX$. By Kulikov Theorem (see [10]) $Y$ is the direct product of cyclic groups, so that we may choose elements $a_n \in Y$ such that $\langle a_n | n \in \mathbb{N} \rangle = \text{Dr}_{n \in \mathbb{N}} \langle a_n \rangle \leq Y$ and $|a_nX| < |a_{n+1}X|$. The claim holds. Similarly, if $A/X$ has infinite rank, we may consider its socle $S/X$ and consider a countable subgroup $Y$ such that $S = YX$. Then we may choose elements $a_n \in Y$ which are independent mod $X$ and generate their direct product as in (1).

We claim now that there are sequences of infinite subsets $I_n, J_n$ of $\mathbb{N}$ and $\Gamma$-subgroups $K_n \leq A$ such that for each $n \in \mathbb{N}$:
3) $I_n \cap J_n = \emptyset$ and $I_{n+1} \subseteq J_n$
4) $K_n \sim A_{I_n}$
5) $(K_1 \ldots K_i) \cap (A_{I_1} \ldots A_{I_i}) \leq (A_{I_1} \ldots A_{I_i})$, $\forall i \leq n$.

Proceed by induction on $n$. Choose an infinite subset $I_1$ of $\mathbb{N}$ such that $J_1 := \mathbb{N} \setminus I_1$ is infinite. By cp-property there exists $K_1 = K_{I_1}^\Gamma$ commensurable with $A_{I_1}$.

Suppose we have defined $I_j, J_j, K_j$ for $1 \leq j \leq n$ such that 3-5 holds. Since $(K_1 \ldots K_n) \sim (A_{I_1} \ldots A_{I_n})$, there is $m \in \mathbb{N}$ such that
6) $(K_1K_2 \ldots K_n) \cap A_N \leq (A_{I_1}A_{I_2} \ldots A_{I_n}) \langle a_1, \ldots, a_m \rangle$.

Let $I_{n+1}$ and $J_{n+1}$ be disjoint infinite subsets of $J_n \setminus \{1, \ldots, m\}$. By cp-property there exists $K_{n+1} = K_{I_{n+1}}^\Gamma$ commensurable with $A_{I_{n+1}}$. By the choice
of $I_{n+1}$ it follows that

7) \((K_1 \ldots K_i) \cap (A_{I_1} \ldots A_{I_{n+1}}) \leq (K_1 \ldots K_i) \cap (A_{I_1} \ldots A_{I_n}) \quad \forall i \leq n\)

and so (5) holds for $n+1$, as wished. The claim is proved.

Note that by (2) and (5) it follows that $A_{I_n}/A_{I_n} \cap X$ is infinite for each $n \in \mathbb{N}$ and that also the following property holds

8) \((K_1 K_2 \ldots K_n) \cap \bar{A} \leq (A_{I_1} A_{I_2} \ldots A_{I_n}) \quad \forall n, \text{ where } \bar{A} := \text{Dr}_{n \in \mathbb{N}} A_{I_n}.

Now for each $n \in \mathbb{N}$, choose an element $b_n \in (A_{I_n} \cap K_n) \setminus X$. Then we have $B := \langle b_n \mid n \in \mathbb{N} \rangle = \text{Dr}_n \langle b_n \rangle$, where $\langle b_n \rangle^\Gamma$ is infinite and $\langle b_n \rangle^\Gamma \leq K_n \sim A_{I_n}$, so that

9) \(\langle b_n \rangle^\Gamma \cap A_{I_n} \) is infinite for each $n$.

Since there exists $B_0 = B_0^\Gamma \sim B$, we may take
- $B_* := (B_0 \cap B)^\Gamma = (B_* \cap B)^\Gamma \leq B^\Gamma$ where $B_* \sim B$.

Now $B_*/(B_* \cap B)$ and $B/(B_* \cap B)$ are both finite and there is $n \in \mathbb{N}$ such that if $B_n := \langle b_1, \ldots, b_n \rangle$ we have
- $(B_* \cap B)^\Gamma = B_* \leq (B_* \cap B)B_n^\Gamma$ and
- $B = (B_* \cap B)B_n$.

Since $b_n \in K_n$ for each $n$, we have $B_n \leq \bar{K}_n := K_1 K_2 \ldots K_n$ and
- $B^\Gamma = (B_* \cap B)^\Gamma B_n^\Gamma \leq (B_* \cap B)B_n^\Gamma \leq (B_* \cap B)\bar{K}_n \leq B\bar{K}_n$, so that
- $B^\Gamma \cap \bar{A} \leq B\bar{K}_n \cap A = B(\bar{K}_n \cap A) \leq BA_{I_1} A_{I_2} \ldots A_{I_n}$ by (8) above.

Thus
- $\langle b_{n+1} \rangle^\Gamma \cap A_{I_{n+1}} \leq B^\Gamma \cap A_{I_{n+1}} \leq (BA_{I_1} A_{I_2} \ldots A_{I_n}) \cap A_{I_{n+1}} = \langle b_{n+1} \rangle$ is finite, a contradiction with (9). \qed

**Lemma 3.8** Let $\Gamma$ be a group acting on an abelian periodic reduced group $A$. If $\Gamma$ has $\text{cp}$, then there are $\Gamma$-subgroups $A_0 \leq A_1 \leq A$ such that $A_0$ and $A/A_1$ are finite and $\Gamma$ has $\text{p}$ on $A_1/A_0$.

**Proof.** By Proposition 3.4 it is enough to consider the case when $A$ is a $p$-group. If $A$ is the direct product of cyclic groups, by Lemma 3.7 we have that $A_1 := \{a \in A \mid \langle a \rangle^\Gamma \text{ is finite} \}$ has finite index in $A$. Further, by Lemma 3.6, $\Gamma$ has $\text{bp}$ on $A_1$. Then the statement follows from Theorem 3.1.

Let $A$ be any reduced $p$-group and $B_*$ be a basic subgroup of $A$. Then there is $B = B^\Gamma \sim B_*$. Since $A/B_*$ is divisible, then the divisible radical of $A/B$ has finite index. Thus we may assume that $A/B$ is divisible. By Kulikov Theorem (see [10]), also $B$ is a direct product of cyclic groups, therefore by the above there are $\Gamma$-subgroups $B_0 \leq B_1 \leq B$ such that $B_0$ and $B/B_1$ are
finite and \( \Gamma \) has \( p \) on \( B_1/B_0 \). We may assume \( B_0 = 1 \). Also, since \( A/B_1 \) is finite-by divisible, it is divisible-by-finite and we may assume it is divisible.

Let \( \gamma \in \Gamma \) and \( \alpha \) be a \( p \)-adic integer such that \( x^{\gamma} = x^\alpha \) for all \( x \in B_1 \). Consider the endomorphism \( \gamma - \alpha \) of \( A \) and note that \( B_1 \leq \ker(\gamma - \alpha) \). Thus \( A/\ker(\gamma - \alpha) \cong \text{im}(\gamma - \alpha) \) is both divisible and reduced, hence trivial. It follows \( \gamma = \alpha \) on the whole \( A \).

**Proof of Theorem 3.2** For the sufficiency of the condition note that for any subgroup \( H \leq A \) we have \( H \sim H \cap A_1 \) and the latter is in turn commensurable with a \( \Gamma \)-subgroup since \( \Gamma \) has BP on \( A_1 \) by Theorem 3.1.

Concerning necessity, we first prove the statement when \( A \) is periodic. Let \( A = D \times R_1 \), where \( D \) is divisible and \( R_1 \) is reduced. Then there is a \( R = R^1 \sim R_1 \). Thus \( DR \) and \( D \cap R \) are \( \Gamma \)-subgroups of \( A \) with finite index and order resp. Then we can assume \( A = D \times R \). Let \( X := \{ a \in A \mid \langle a \rangle^\Gamma \) is finite\}. Clearly \( D \leq X \), as \( \Gamma \) has \( p \) on \( D \) by Lemma 3.5. On the other hand, \( X \cap R \) has finite index in \( R \) by Lemma 3.8. It follows \( A/X \) is finite and by Lemma 3.6 and Theorem 3.1 the statement holds.

In the non-periodic case, note that if \( V_0 \) is a maximal free subgroup of \( A \) (hence \( A/V_0 \) is periodic), then there is \( V_1 = V_1^\Gamma \sim V_0 \). Let \( n := |V_1/(V_0 \cap V_1)| \). Thus by applying Lemma 3.5 we have - there is a free abelian \( \Gamma \)-subgroup \( V := V_1^n \) such that \( A/V \) is periodic and each \( \gamma \in \Gamma \) acts on \( V \) by either the identity or the inversion map.

Suppose that \( V \) has finite rank. Consider now the action of \( \Gamma \) on the periodic group \( A/V \) and apply the above. Then there is a series \( V \leq A_0 \leq A_1 \leq A \) such that \( A_0/V \) and \( A/A_1 \) are finite and \( \Gamma \) has either \( p \) or \( \bar{p} \) on \( A_1/A_0 \). Since \( A_0 \) has finite torsion subgroup \( T \) we can factor out \( T \) and assume \( A_0 = V \). Then \( \Gamma \) has either \( p \) or \( \bar{p} \) on \( A_1 \) as straightforward verification shows.

Suppose finally that \( V \) has infinite rank. Let \( V_2 \leq V \) be such that \( V/V_2 \) is divisible periodic and its \( p \)-component has infinite rank for each prime \( p \). We may assume \( V := V_2 \). By the above case when \( A \) is periodic, there is a \( \Gamma \)-series \( V \leq A_0 \leq A_1 \leq A \) such that \( A_0/V \) and \( A/A_1 \) are finite and \( \Gamma \) has \( p \) on \( A_1/A_0 \). We may factor out the torsion subgroup of \( A_0 \), as it is finite, and assume \( A_0 = V \).

Again let \( V_2 \leq V \) be such that \( V/V_2 \) is divisible periodic and its \( p \)-component has infinite rank for each prime \( p \). Let \( \gamma \in \Gamma \) and \( \alpha_p \) be a \( p \)-adic integer such that \( x^{\gamma} = x^{\alpha_p} \) for all \( x \) in the \( p \)-component of \( A_1/V \). Let \( \epsilon = \pm 1 \) be such that \( x^{\gamma} = x^\epsilon \) for all \( x \in V \). By Lemma 3.5, \( \gamma \) has \( p \) on
the maximum divisible subgroup $D_p/V_2$ of the $p$-component of $A_1/V_2$. Thus $\alpha_p = \epsilon$ on $D_p/V_2$. Therefore $x^\gamma = x^\epsilon$ for all $x \in V$ and for all $x \in A_1/V_1$. We claim that $a^{\gamma} = a^\epsilon$ for each $a \in A_1$. To see this, for any $a \in A_1$ consider $n \in \mathbb{N}$ such that $a^n \in V$. Then there is $v \in V$ such that $a^{\gamma} = (a^n)^\epsilon = (a^\epsilon v)^n = a^{n^\epsilon}v^n$. Thus $v^n = 1$. Therefore, as $V$ is torsion-free, we have $v = 1$, as wished. □

4 Abelian-by-finite CN-groups and Theorem B

Locally finite cf-groups are known to be abelian-by-finite and bcf (see [1]).

Proposition 4.1 Let $G$ be an abelian-by-finite group.

i) if $G$ is CN, then $G$ is CF;

ii) if $G$ is BCN, then $G$ is BCF.

Proof. Let $A$ be a normal abelian subgroup with finite index $r$. Then each $H \leq A$ has at most $r$ conjugates in $G$. If $\delta_G(H) \leq n < \infty$ then for each $g \in G$ we have $|H : (H \cap H^g)| \leq 2\delta_G(H) \leq 2n$ hence $|H/H_G| \leq (2n)^r$. More generally, if $H$ is any subgroup of $G$, then $|H/H_G| \leq r(2n)^r$. □

We state now a key fact about non-periodic CN-grups.

Lemma 4.2 Let $G$ be a CN-group and $A = A(G)$ its subgroup generated by all infinite cyclic normal subgroups. Then $G/A$ is periodic, $A$ is abelian and each $g \in G$ acts on $A$ by either the identity or the inversion map, hence $|G/C_G(A)| \leq 2$.

Proof. For any $x \in G$ there is $N \vartriangleleft G$ which is commensurable with $\langle x \rangle$. Then $n := |N : (N \cap \langle x \rangle)|$ is finite. Thus $N^n \leq \langle x \rangle$ where $N^n \vartriangleleft G$. Hence $G/A$ is periodic.

It is clear that $A$ is abelian. Let $g \in G$. If $\langle a \rangle \lhd G$ and $a$ has infinite order, then there is $\epsilon_a = \pm 1$ such that $a^g = a^{\epsilon_a}$. On the other hand, by Lemma 3.5, there is $\epsilon = \pm 1$ such that for each $a \in A$ there is a periodic element $t_a \in A$ such that $a^g = a^{t_a}$. It follows $a^{t_a^{-\epsilon}} = t$. Therefore $\epsilon_a = \epsilon$ is independent of $a$, as wished. □

Lemma 4.3 Let $G$ be an FC-group. If $G$ is a CN-group then $G$ is finite-by-abelian.
Proof. Let $H$ be any subgroup of $G$. We shall prove that $|H^G : H|$ is finite. Consider $A = A(G)$ as in Lemma 4.2. Then $H \cap A \triangleleft G$ and $H/A \cap H$ is periodic. Hence we may assume $H$ is periodic, that is, $H$ contained in the torsion subgroup of the fc-group $G$. Our claim follows then from Lemma 3.6. □

Proof of Theorem B Let $G$ be a cn-group and $G_0 \leq G_1 \leq G$ be a normal series such that $G_1/G_0$ is abelian and both $G_0$ and $G/G_1$ are finite. Then $G$ has cp on $G_1/G_0$. By Corollary 3.3, the group $G$ has bp on a subgroup $A_1/G_0 \leq G_1/G_0$ with finite index in $G_1/G_0$. Thus $A_1/G_0$ is contained in the fc-centre of $G/G_0$. Hence $A_1$ is contained in the fc-centre of $F$ of $G$. So that $G/F$ is finite. On the other hand, from Lemma 4.3 it follows that $F'$ is finite.

Finally, (ii) and (iii) follow from Proposition 2.1 and Proposition 4.1. □

Let us characterize bcf-groups among abelian-by-finite cf-groups.

Proposition 4.4 Let $G$ be a non-periodic group with an abelian normal subgroup $A$ with finite index. Then the following are equivalent:

i) $G$ is a bcf-group;

ii) $G$ is a cf-group and there is $B \leq A$ such that $B$ has finite exponent, $B \triangleleft G$ and each $g \in G$ acts by conjugation on $A/B$ by either the identity or the inversion map.

Proof. Let $T$ be the torsion subgroup of $A$. By Lemma 3.5, each $g \in G$ acts on $A/T$ as the automorphism $x \mapsto x^{\epsilon_g}$ where $\epsilon_g = \pm 1$. Then the equivalence of (i) and (ii) holds with $B := \langle A^g_{\epsilon_g} \mid g \in G \rangle$, by Theorem 3 of [4]. □

5 Proof of Theorem A

Our first statement in this section is a reduction to nilpotent groups.

Lemma 5.1 A soluble p-group $G$ with the property cn is nilpotent-by-finite.

Proof. By Theorem 3.2, one may refine the derived series of $G$ to a finite $G$-series $S$ such that $G$ has p on each infinite factor of $S$. Recall that a $p$-group of power automorphisms of an abelian $p$-group is finite (see [10]). Then the stability group $S \leq G$ of the series $S$, that is, the intersection of
the centralizers in $G$ of the factors of the series, has finite index in $G$. On the other hand, by a theorem of Ph.Hall, $S$ is nilpotent.

We recall now an elementary property of nilpotent groups.

\textbf{Lemma 5.2} \textit{Let $G$ be a nilpotent group with class $c$. If $G'$ has finite exponent $e$, then $G/Z(G)$ has finite exponent dividing $e^c$.}

\textbf{Proof.} Argue by induction on $c$, the statement being clear for $c = 1$. Assume $c > 1$ and that $G/Z$ has exponent dividing $e^{c-1}$, where $Z/\gamma_c(G) := Z(G/\gamma_c(G))$. Then for all $g, x \in G$ we have $[g^{e^{c-1}}, x] \in \gamma_c(G) \leq G' \cap Z(G)$. Therefore $1 = [g^{e^{c-1}}, x]^e = [g^e, x]$, and $g^e \in Z(G)$, as claimed. 

Next lemma follows easily from Lemma 6 in [8].

\textbf{Lemma 5.3} \textit{Let $G$ be a nilpotent $p$-group and $N$ a normal subgroup such that $G/N$ is an infinite elementary abelian group. If $H$ and $U$ are finite subgroup of $G$ such that $H \cap U = 1$, there exists a subgroup $V$ of $G$ such that $U \leq V$, $H \cap V = 1$ and $V/N \cap V$ is infinite.}

We deduce a technical lemma which is a tool for our purpose.

\textbf{Lemma 5.4} \textit{Let $G$ be a nilpotent $p$-group and $N$ be a normal subgroup such that $G/N$ is an infinite elementary abelian group. If $N$ contains the FC-center of $G$ and $G'$ is abelian with finite exponent, then there are subgroups $H, U$ of $G$ such that $H \cap U = 1$, with injective maps $n \mapsto h_n \in H$ and $(i, n) \mapsto u_{i,n} \in [G, h_i^{-1}h_n] \cap U$, where $i, n \in \mathbb{N}, i < n$.}

\textbf{Proof.} Let us show that for each $n \in \mathbb{N}$ there is an $(n + 1)$-uple $v_n := (h_n, u_{0,n}, u_{1,n}, \ldots, u_{n-1,n})$ of elements of $G$ such that:

1. $\{h_1, \ldots, h_n\}$ is linearly independent modulo $N$;
2. $u_{i,n} \in [G, h_i^{-1}h_n]$ \ \forall $i \in \{0, \ldots, n - 1\}$;
3. $\{u_{j,h} \mid 0 \leq j < k \leq n\}$ is $\mathbb{Z}$-independent in $G'$;
4. $H_n \cap U_n = 1$, where $H_n := \langle h_1, \ldots, h_n \rangle$ and $U_n := \langle u_{j,h} \mid 0 \leq j < k \leq n \rangle$. 

11
Then the statement is true for $H := \bigcup_{n \in \mathbb{N}} H_n$ and $U := \bigcup_{n \in \mathbb{N}} U_n$. 

Let $h_0 := 1$ and choose $h_1 \in G \setminus N$. Since $N \geq F$, the $\mathbb{F}_2$-center of $G$, we have that $\gamma_1$ has an infinite numbers of conjugates in $G$, hence $[G, h_1]$ is infinite and residually finite. Thus we may choose $u_{0,1} \in [G, h_1]$ such that $\langle u_{0,1} \rangle \cap \langle h_1 \rangle = 1$.

Assume then that we have defined $v_i$ for $i \leq n$, that is, we have elements $h_0, \ldots, h_n, u_{j,k}$, with $0 \leq j < k \leq n$ such that conditions 1-4 hold. To define an adequate $v_{n+1}$, note that by Lemma 5.3 we have that there exists $V_n \leq G$ such that $H_n \leq V_n$, $U_n \cap V_n = 1$ and $V_nN/N$ is infinite. Then choose $i)$ $h_{n+1} \in V_n \setminus NU_nH_n$.

Note that $h_{n+1} \not\in FH_n \leq NH_n$, so that $\{h_1, \ldots, h_{n+1}\}$ is independent mod $F$. In particular $\forall i \in \{0, \ldots, n\}$, $h_i^{-1}h_{n+1} \not\in F$, hence also $[G, h_i^{-1}h_{n+1}]$ is infinite.

Since $G'$ is residually finite, we may recursively choose $u_{0,n+1}, \ldots, u_{n,n+1}$ such that $\forall i \in \{0, \ldots, n\}$

$$i) \quad u_{n+1} \in [G, h_i^{-1}h_n]$$

$$iii) \quad \langle u_{i,n+1} \rangle \cap U_n\langle u_{h_{n+1}} \mid 0 < h < i \rangle H_{n+1} = 1$$

Then properties 1-3 hold for $v_{n+1}$. Finally suppose there are $h \in H_n$, $u \in U_n$, $s, t_0, \ldots, t_n \in \mathbb{Z}$ such that

$$iii) \quad a = hh_{n+1}^s = uu_{0,n+1}^{t_0} \cdots u_{n,n+1}^{t_n} \in H_{n_1} \cap U_{n+1}.$$ 

Then from (iii) it follows $u_{0,n+1}^{t_n} = \ldots = u_{0,n+1}^{t_1} = 1$. Hence $a = hh_{n+1}^s = u \in V_n \cap U_n = 1$ and 4 holds. 

\begin{lemma}
Let $G$ be a nilpotent $p$-group. If $G$ is CN, then $G'$ has finite exponent.
\end{lemma}

\begin{proof}
If, by contradiction, $G'$ has infinite exponent, then the same happens to the abelian group $G'/\gamma_3(G)$ and there is $N$ such that $G' \geq N \geq \gamma_3(G)$ and $G'/N$ is a Prüfer group. We may assume $N = 1$, that is, $G'$ itself is a Prüfer group and $G' \leq Z(G)$. Let us show that for any $H \leq G$ we have $|H^G : H| < \infty$, hence $G'$ is finite, a contradiction. In fact we have that, by CN-property there is $K < G$ such that $K \sim H$. Thus $H$ has finite index in $HK$ and we can also assume $H = HK$, that is, $H/H_G$ is finite. Thus, we can assume $H_G = 1$ and $H \cap G' = 1$, that is, $H$ is finite with order $p^n$ and $HG'$ is an abelian Chernikov group. It follows that $H$ is contained in the $n$-th socle $S$ of $HG' < G$, where $S$ is finite and normal in $G$, as wished.
\end{proof}

\begin{lemma}
Let $G$ be a nilpotent $p$-group. If $G$ is CN, then $G$ is finite-by-abelian-by-finite.
\end{lemma}

12
Proof. Assume, by contradiction, $G$ is a counterexample. Then both $G'$ and $G/Z(G)$ are infinite. However, they have finite exponent by Lemmas 5.5 and 5.2. Moreover, even the $\text{fc}$-center $F$ of $G$ has infinite index by Lemma 4.3. On the other hand, $G/F$ has finite exponent, since $F \geq Z(G)$.

Then $N := FGpG'$ has infinite index in $G$, otherwise the abelian group $G/FG'$ has finite rank and finite exponent, hence it is finite. This implies that the nilpotent group $G/F$ is finite, a contradiction.

If $G'$ is abelian we are in a condition to apply Lemma 5.4 and get infinite elements and subgroups $h_n \in H$, $u_{i,n} \in U$ as in that statement. By $\text{cn}$-property there is $K$ such that $H \sim K \trianglelefteq G$. So that the set $\{h_n(H \cap K) / n \in \mathbb{N}\}$ is finite. Hence there is $i \in \mathbb{N}$ and an infinite set $I \subseteq \mathbb{N}$ such that for each $n \in I$ we have $h_i^{-1}h_n \in H \cap K$ and $u_{i,n} \in U \cap [G, H \cap K] \leq U \cap K$. Therefore $U \cap K$ is infinite, in contradiction with $U \cap K \sim U \cap H = 1$.

For the general case, proceed by induction on the nilpotency class $c > 1$ of $G$ and assume that the statement is true for $G/Z(G)$ and even that this is finite-by-abelian. Then there is a subgroup $L \leq G$ such that $G/L$ is abelian and $L/Z(G)$ is finite. Thus $L'$ is finite and, by the above, $G/L'$ is finite-by-abelian-by-finite, a contradiction. □

Proof of Theorem A. Recall from the Introduction that all subgroups of $G$ are subnormal-by-finite. Thus, by above quoted results in [6] and [3] resp., we may assume that $G$ is locally nilpotent and soluble.

Assume first $G$ is periodic. Then, by Lemma 3.4, only finitely many primary components are non-abelian. Thus we may assume $G$ is a $p$-group and apply Lemma 5.1 and Lemma 5.6. It follows that $G$ is finite-by-abelian-by-finite.

To treat the general case, consider $A = A(G)$ as in Lemma 4.2. We may assume $A$ is central in $G$. Let $V$ be a torsion-free subgroup of $A$ such that $A/V$ is periodic. Then $G/V$ is locally finite and we may apply the above. Thus there is a series $V \leq F \leq G_1 \leq G$ such that $G$ acts trivially on $V$, $G_1/G_0$ is abelian, while $G_0/V$ and $G/G_1$ are finite. Then we can assume $G = G_1$ and note that the stabilizer $S$ of the series has now finite index. Since $S$ is nilpotent (by Ph.Hall Theorem) we can assume that $G = S$ is nilpotent. If $T$ is the torsion subgroup of $G$, then $VT/T$ is contained in the center of $G/T$. Since all factor of the upper central series of $G/T$ are torsion-free we have $G/T$ is abelian. Thus $G' \leq T \cap G_0$ is finite. □

Proof of Corollary. If the statement is false, by Theorem A we may assume there is a counterexample $G$ periodic and not locally finite. Also we may
assume $G$ is finitely generated and infinite. Let $R$ be the locally finite radical of $G$. By Theorem A again, $R$ is finite-by-abelian-by-finite. By Theorem B(i), there is a finite subgroup $G_0 \triangleleft G$ such that $R/G_0$ is abelian-by-finite. We may assume $G_0 = 1$, so that $R$ is abelian-by-finite.

We claim that $\bar{G} := G/R$ has finite exponent at most $(n + 1)!$ where $n$ is such that $n \geq \delta_G(H)$ for each $H \leq G$. In fact, for each $x \in \bar{G}$, there is $\bar{N} \triangleleft \bar{G}$ such that $|\bar{N} : (\bar{N} \cap \langle x \rangle)| \leq n$. Thus $\bar{N}^{n!} \leq \langle x \rangle$ and $\bar{N}^{n!} \triangleleft G$. Hence $\bar{N}^{n!} = 1$ and $x^{n!} = 1$.

By the positive answer (for all exponents) to the Restricted Burnside Problem, there is a positive integer $k$ such that every finite image of $\bar{G}$ has order at most $k$. Since $\bar{G}$ is finitely generated, this means that the finite residual $\bar{K}$ of $\bar{G}$ has finite index and is finitely generated as well. Since also $\bar{G}$ is locally graded (see [7]), we have $\bar{K} = 1$ and $\bar{G}$ is finite. Therefore $G$ is abelian-by-finite, a contradiction. $\square$

References

[1] J. T. Buckley, J.C. Lennox, B. H. Neumann, H. Smith, J. Wiegold, Groups with all subgroups normal-by-finite. J. Austral. Math. Soc. Ser. A 59 (1995), no. 3, 384-398.

[2] C. Casolo, Groups with finite conjugacy classes of subnormal subgroups, Rend. Sem. Mat. Univ. Padova 81 (1989), 107-149.

[3] C. Casolo, Groups in which all subgroups are subnormal-by-finite, Advances in Group Theory and Applications 1 (2016), 3345 DOI: 10.4399/97888548908173

[4] U. Dardano, S. Rinauro, Inertial automorphisms of an abelian group, Rend. Sem. Mat. Univ. Padova 127 (2012), 213-233. doi:10.4171/RSMUP/127-11

[5] S. Franciosi, F. de Giovanni, M.L. Newell, Groups whose subnormal subgroups are normal-by-finite, Comm. Alg. 23(14) (1995), 5483-5497.

[6] H. Heineken, Groups with neighbourhood conditions for certain lattices. Note di Matematica, 1 (1996), 131143.

[7] P. Longobardi, M. Maj, H. Smith. A note on locally graded groups. Rend. Sem. Mat. Univ. Padova, 94 (1995), 275-277.

[8] W. Möhres, Torsionsgruppen, deren Untergruppen alle subnormal sind. Geom. Dedicata, 31 (1989), 237244.
[9] B. H. Neumann, Groups with finite classes of conjugate subgroups, *Math. Z.* **63** (1955), 76-96.

[10] D.J.S. Robinson, *A course in the theory of groups*, Graduate Texts in Mathematics, 80, Springer-Verlag, New York, 1996.

[11] H. Smith, J. Wiegold, Locally graded groups with all subgroups normal-by-finite, *J. Austral. Math. Soc. Ser. A* **60** (1996), no. 2, 222-227.

Carlo Casolo, Dipartimento di Matematica U. Dini, Università di Firenze, Viale Morgagni 67A, I-50134 Firenze, Italy. email: casolo@math.unifi.it

Ulderico Dardano, Dipartimento di Matematica e Applicazioni “R.Caccioppoli”, Università di Napoli “Federico II”, Via Cintia - Monte S. Angelo, I-80126 Napoli, Italy. email: dardano@unina.it

Silvana Rinauro, Dipartimento di Matematica, Informatica ed Economia, Università della Basilicata, Via dell’Ateneo Lucano 10 - Contrada Macchia Romana, I-85100 Potenza, Italy. email: silvana.rinauro@unibas.it