F-THRESHOLDS $c'(m)$ FOR PROJECTIVE CURVES

VIJAYLAXMI TRIVEDI

Abstract. We show that if $R$ is a two dimensional standard graded ring (with the graded maximal ideal $m$) of characteristic $p > 0$ and $I \subset R$ is a graded ideal with $\ell(R/I) < \infty$ then the F-threshold $c'(m)$ can be expressed in terms of a strong HN (Harder-Narasimahan) slope of the canonical syzygy bundle on Proj $R$. Thus $c'(m)$ is a rational number.

This gives us a well defined notion, of the F-threshold $c'(m)$ in characteristic 0, in terms of a HN slope of the syzygy bundle on Proj $R$. This generalizes our earlier result (in [TrW]) where we have shown that if $I$ has homogeneous generators of the same degree, then the F-threshold $c'(m)$ is expressed in terms of the minimal strong HN slope (in char $p$) and in terms of the minimal HN slope (in char 0), respectively, of the canonical syzygy bundle on Proj $R$.

Here we also prove that, for a given pair $(R, I)$ over a field of characteristic 0, if $(m_p, I_p)$ is a reduction mod $p$ of $(m, I)$ then $c'(m_p) \neq c'(m)$ implies $c'(m_p)$ has $p$ in the denominator, for almost all $p$.

1. Introduction

Let $(R, I)$ be a standard graded pair, i.e., $R$ is a Noetherian standard graded ring over a perfect field $k$ (unless otherwise stated) of characteristic $p > 0$ and $I$ is a graded ideal of finite colength. Let $m$ be the graded maximal ideal of $R$.

If $M$ is a finitely generated graded $R$-module then (see [T2]) we have a compactly supported continuous function $f_{M, I} : [0, \infty) \to [0, \infty)$ called the HK density function for $(M, I)$. We realize this function as the limit of a uniformly convergent sequence of compactly supported functions $\{f_n(M, I) : \mathbb{R} \to [0, \infty)\}_{n \in \mathbb{N}}$, where

$$f_n(M, I)(x) = \frac{1}{q^{d-1}}\ell(M/I[,^q]M)\lfloor_{xq}, \text{ for } q = p^n.$$ 

Moreover

$$\int_0^\infty f_{M, I}(x)dx = e_{HK}(M, I),$$

where $e_{HK}(M, I)$ denotes the invariant HK multiplicity of $M$ with respect to $I$ (introduced by P. Monsky [M]).

Since the function $f_{M, I}$ is the uniformly convergent limit of the sequence $\{f_n(M, I)\}_n$, and is also ‘additive’ and ‘multiplicative’, it has proved to be a versatile tool to handle invariants attached to it.

The focus of this paper is on the another invariant, the maximum support of the function $f_{R, I}$, namely the number $\alpha(R, I) = \sup \{x \mid f_{R, I}(x) \neq 0\}$. Here we consider the standard graded pair $(R, I)$, where $R$ is a two dimensional domain.

In the case dim $R \geq 2$ this invariant relates to another well known invariant, the F-threshold $c'(m)$ of $m$ with respect to $I$:

We acknowledge support of the Department of Atomic Energy, Government of India, under project no. 12-R&D-TFR-RTI4001.
Theorem (Theorem 4.9, [TrW]). Let \((R, I)\) be a standard graded pair and \(m\) be the graded maximal ideal of \(R\). If \(R\) is strongly \(F\)-regular on the punctured spectrum (for example if \(\text{Proj} R\) is smooth) then \(\alpha(R, I) = c'(m)\).

In particular when \(R\) is a normal domain of dimension two then \(\alpha(R, I) = c'(m)\).

We recall that for a pair of ideals \(I\) and \(J\), the \(F\)-threshold of \(J\) with respect to \(I\) is defined as

\[
c'(J) = \lim_{q \to \infty} \min \left\{ r \mid J^{r+1} \subseteq I^q \right\}.
\]

This was first introduced by Mustaţă-Takagi-Watanabe in [MTW] for regular rings, and later, in a more general setting (when \(R\) is not regular), was further studied by Huneke-Mustaţă-Takagi-Watanabe in [HMTW].

In [TrW] we studied \(\alpha(R, I) = c'(m)\) in detail when \(I\) is generated by homogeneous elements of the same degree. In this paper we generalize the results, proved in [TrW] for the two dimensional case, to the case when \(I\) has a set of homogeneous generators, but not necesssarily of the same degree. The technique used in [TrW]) does not work here. We elaborate on this now.

For a given pair \((R, I)\), let \(S\) be the normalization of \(R\) in the quotient field \(Q(R)\). Then \(X = \text{Proj} S\) is a nonsingular curve with the ample line bundle \(\mathcal{O}_X(1)\). Analogous to the notion of the HK density function \(f_{R,I}\) for the pair \((R, I)\), we can have the notion of the HK density function \(f_{V,\mathcal{O}_X(1)}\) for the pair \((V, \mathcal{O}_X(1))\), where \(V\) is a vector-bundle on \(X\) and \(\mathcal{O}_X(1)\) the ample line bundle of \(X\). The function \(f_{V,\mathcal{O}_X(1)}\) has an explicit formula in terms of the strong HN data (see Notations \([2.1]\)) of \(V\). Moreover the maximum support of \(f_{V,\mathcal{O}_X(1)}\) has an explicit formula in terms of the minimum strong HN slope (denoted by \(a_{\text{min}}(V)\)) of the vector bundle \(V\).

We relate the function \(f_{R,I}\) with the HK density functions of specific vector bundles on \(X\) by the formula

\[
f_{R,I}(x) = f_{V_0,\mathcal{O}_X(1)}(x) - f_{M_0,\mathcal{O}_X(1)}(x),
\]

where if \(I\) has a set of homogeneous generators \(f_1, \ldots, f_s\) of degree \(d_1, \ldots, d_s\) then there is the canonical short exact sequence

\[
0 \rightarrow V_0 \rightarrow M_0 = \bigoplus_i \mathcal{O}_X(1 - d_i) \rightarrow \mathcal{O}_X(1) \rightarrow 0
\]

(see the sequence \([2.3]\) in subsection 2.2) of locally free sheaves of \(\mathcal{O}_X\)-modules. We recall

**Theorem 6.3 ([TrW])** If \(d_1 = \cdots = d_s\) then \(\alpha(R, I) = 1 - a_{\text{min}}(V_0)/d\).

The main point here was that the bundle \(M_0\) is strongly semistable and hence

\[
a_{\text{min}}(V_0) \leq \mu(V_0) < \mu(M_0) = a_{\text{min}}(M_0),
\]

where \(\mu(W) = \deg(W)/\text{rank}(W)\) denotes the slope of \(W\). This implied that

\[
\max \supp f_{M_0,\mathcal{O}_X(1)} < \max \supp f_{V_0,\mathcal{O}_X(1)} = 1 - a_{\text{min}}(V_0)/d.
\]

The above formula for \(\alpha(R, I)\), in terms of the strong HN data of \(V_0\), straightforward gave a well defined notion of \(\alpha(R, I)\) (hence of \(c'(m)\)) in characteristic 0, as (by Lemma 1.16 [T1]) \(\lim_{p_s \to \infty} a_{\text{min}}(V^s) = \mu_{\text{min}}(V)\), where \(V^s\) denotes the reduction mod \(p_s\) of the bundle \(V\).

In particular, if \((R_s, I_s)\) is the reduction mod \(p_s\) of the pair \((R, I)\) then this implied \(\lim_{p_s \to \infty} \alpha(R_s, I_s) = 1 - \mu_{\text{min}}(V_0)\).
In particular

\[ \alpha \]

Then \( \alpha \) is known that homogeneous generators of the same degree (see Theorem 4.8) and

\[ (\alpha, 0) \]

this says that \( V \) is a \( V \)-bundle of \( \mu \)-reduction bundle.

Let \( \alpha \) be a \( \mu \)-reduction bundle of \( V_0 \) (strictly speaking, of the exact sequence (1.1) of vector bundles): Consider the HN filtration

\[ 0 = M_1 \subset M_{1-1} \subset \cdots \subset M_0 \]

of \( M_0 \) (hence for any \( m \geq 0 \), \( 0 = F^{m*} M_1 \subset F^{m*} M_{1-1} \subset \cdots \subset F^{m*} M_0 \) is the HN filtration of \( F^{m*} M_0 \)). Let \( 0 \subset V_{t_1} \subset V_{t_1-1} \subset \cdots \subset V_0 \) be the induced (this need not be the HN) filtration on \( V_0 \). Then \( V_t \) is the \( \mu \)-reduction bundle of \( V_0 \) if \( t \) is the least integer such that \( \mu_{\min}(V_t) < \mu_{\min}(M_t) \). A bundle \( V_0 \) is the \( \mu \)-reduction bundle of \( V_0 \) if \( F^{m*}(V_0) \) is the \( \mu \)-reduction bundle of \( F^{m*}(V_0) \), where \( m \) is an integer (such an integer does exist) where \( F^{m*}(V_0) \) has the strong HN filtration. Moreover we show

\[ 0 \longrightarrow V_0 \longrightarrow M_0 \longrightarrow O_X(1) \longrightarrow 0 \]

is a short exact sequence of \( O_X \)-modules. We show (in Theorem 4.3)

**Theorem A.** If \((R, I)\) is a two dimensional standard graded pair with the multiplicity

\[ d = e_0(R, m) \]

and \( V_0 \) is the strong \( \mu \)-reduction bundle of \( V_0 \) then

1. \( f_{R, I}(x) = f_{V_0, O_X(1)}(x) - f_{M_0, O_X(1)}(x) = f_{V_0, O_X(1)} - f_{M_0, O_X(1)} \)

\[ \text{and} \]

2. \( \alpha(R, I) = 1 - \frac{a_{\min}(V_0)}{d}. \)

Though \( F^{m*} V_0 \) may not be one of the bundles occuring in the HN filtration of

\[ F^{m*} V_0 \]

the slope \( a_{\min}(V_0) \) is equal to one of the strong HN slopes of \( V_0 \). In particular, \( \alpha(R, I) \) is still given in terms of the strong HN data of \( V_0 \).

Moreover, we show that the notion of strong \( \mu \)-reduction and \( \mu \)-reduction bundles behaves well under reduction mod \( p \). This leads to a well defined notion of \( \alpha(R, I) \) in characteristic 0 (Lemma 4.11 and Theorem 4.12)

**Theorem B.** Let \((R, I)\) be a two dimensional standard graded pair in characteristic

\[ 0 \]

and let \( V_0 \) be the syzygy bundle on \( X \) as in the sequence (1.1). Let \( (A, X_A, V_A) \) and \( (A, R_A, I_A) \) be spreads for \((X, V_0)\) and \((R, I)\), respectively. If \( V_t \) is the \( \mu \)-reduction bundle of \( V_0 \) then, for a closed point \( s \in \text{Spec } A \), the strong \( \mu \)-reduction bundle of \( V_0 \)

\[ V_t^s \]

or \( V_{t-1}^s \) and

\[ \lim_{p \to \infty} \alpha(R_s, I_s) = 1 - \frac{\mu_{\min}(V_t)}{d}. \]

Since the notions of \( \mu \)-reduction and strong \( \mu \)-reduction ‘coincide’ in characteristic 0, this says that \( \alpha(R, I) \) is always expressed in terms of the minimum strong HN slope of the strong \( \mu \)-reduction bundle.

We have proved the following result in [TrW] (Theorem E) with the additional hypothesis that either \( \text{Proj } R \) is nonsingular, or the ideal \( I \) is generated by a set of homogeneous generators of the same degree (see Theorem 4.8). However though it is known that \( \alpha(R, I) = \alpha(S, IS) \), it is not known to us if \( c(I) = c^{IS}(mS) \).

**Theorem C.** Let \((R, I)\) be a standard graded pair where \( R \) is a two dimensional domain. Then

\[ c(I) = \alpha(R, I). \]

In particular \( c(I) = c^{IS}(mS) \), where \( S \) denotes the normalization of \( R \) in \( Q(R) \).
The following theorem is proved in [TrW] (Theorem C) when $I$ is an ideal generated by a set of homogeneous generators of the same degree (see subsection 4.2).

**Theorem D.** Let $(R, I)$ be a standard graded pair where $R$ is a two dimensional domain in characteristic 0 with notations as in Theorem B, then

1. $c^s_\infty(m) := \lim_{p_s \to \infty} c^s(I_p(m))$ exists and
2. For $p_s \gg 0$, $c^s(I_p(m)) \geq c^s_\infty(m)$.
3. If $V_0$ is semistable then
   (a) $c^s_\infty(m) = (d_1 + \cdots + d_r)/(r-1)$, where $M_0 = \oplus_{i=1}^r \mathcal{O}_X(1-d_i)$ and
   (b) for $p_s \gg 0$, $c^s(I_p(m)) = c^s_\infty(m) \iff V_0^s$ is strongly semistable.

In particular the $F$-threshold of the reduction mod $p_s$, $c^s(I_p(m))$, characterizes the strong semistability behaviour of the syzygy bundle $V_0$ under reduction mod $p_s$.

Next we analyse the case when $c^s(I_p(m)) \neq c^s_\infty(m)$. By Theorem 3.4 and Proposition 3.8 of [HY], where $R = \mathbb{Z}[X_1, \ldots, X_n]$ and $I \subseteq m = (X_1, \ldots, X_n)$, we have a formula for the log canonical threshold in terms of $F$-pure thresholds (where $fpt_m(I) = c^s_c(I)$ denotes the first jumping number of $I$):

$$lct_m(I) = \lim_{p \to \infty} fpt_m(I_p) = \lim_{p \to \infty} c^{mp}(I_p),$$

where $m_p$ and $I_p$ are reductions mod $p$ of $m$ and $I$, respectively.

K.Schweid asked the following question: Assuming $fpt_m(I_p) \neq lct_m(f)$, is the denominator of $fpt_m(I_p)$ (in its reduced form) a multiple of $p$?

In [CHSW] the authors explored the implication of the following two conditions:

1. The characteristic does not divide the denominator of the $F$-pure threshold.
2. The $F$-pure threshold and the log canonical threshold coincide. Theorem A in [CHSW] and also the example 4.5 in [MTW] imply that for an explicit (nonhomogeneous) polynomial $f$ in a polynomial ring (note that here the $F$-pure threshold $fpt_m(I_p) = c^{mp}(f)$), the above two conditions could be distinct.

On the other hand, there are examples (see [CHSW] for the references) of homogeneous polynomials $f$ of specific types where the two conditions are equivalent. In [BS] Proposition 5.4, it was shown that for a homogeneous polynomial $f$ of degree $d$ in $R = k[X_0, \ldots, X_n]$ (where $R/(f)$ is an isolated singularity), if $p \geq nd - d - n$ then either $c^{mp}(f_p) = (n+1)/d$, or the denominator of $c^{mp}(f_p)$ is a power of $p$. In other words

$$c^{mp}(f_p) \neq lct_m(f) \implies \text{the denominator of } c^{mp}(f_p) \text{ is a power of } p.$$

In this context, here we prove the following (in Section 5).

**Theorem E.** Let $(R, I)$ be a standard graded pair, where $R$ is a 2 dimensional domain over an algebraically closed field $k$ of char 0. Let $(R_s, I_s, m_s)$ denote reductions mod $p_s$ of $(R, I, m)$, where $p_s = \text{char } R_s$. Let $c^s_c(I) = \lim_{p_s \to \infty} c^s(I_p(m))$. Then for $p_s \gg 0$, $c^s(I_p(m)) \neq c^s_\infty(m) \implies c^s(I_p(m)) = a_1/p_sb_1$,

where $a_1, b_1 \in \mathbb{Z}_+$ and $\text{g.c.d.}(a_1, p_s) = 1$.

In fact, for $p_s \gg 0$,

$$c^s(I_p(m)) \neq c^s_\infty(m) \implies c^s(I_p(m)) = c^s_\infty(m) + \frac{a}{p_sb},$$
for some \( a, b \in \mathbb{Z}_+ \) such that \( 0 < a/b \leq 4(g-1)(r-1) \), where \( r+1 = \) the minimal generators of \( I \) and \( g = \) the genus of \( \text{Proj} \ R \).

However, there exist examples (Remark 5.2) where, for all but finitely many \( p_s \), the denominators (in its reduced form) of \( c_s^{m_s}(m_s) \) is divisible by \( p_s \), but is not a power of \( p_s \).

The organisation of this paper is as follows.

In Section 2, we give a description of the HK density function \( f_{R,I} \) in terms of the HK density functions of the syzygy vector bundles. Most of the details given here are a rephrasing of the details given in [TrW].

In Section 3 we introduce the notion of \( \mu \)-reduction and strong \( \mu \)-reduction bundles, for a choice of the sequence of the type (1.1) (this is a key new idea in the paper).

Then we prove the existence of the \( \mu \)-reduction and the strong \( \mu \)-reduction bundles, and check the relevant properties, such as the HN filtration and the HK density function of \( V \), vis-a-vis the HN filtration and the HK density function of \( V_0 \), the relation between the \( \mu \)-reduction bundle of \( V_0 \) and the \( \mu \)-reduction bundle of \( F^s(V_0) \), where \( F^s \) is the \( s \)-th iterated Frobenius map.

In Section 4 we prove the equality \( c^f(m) = \alpha(R,I) \) and express this quantity in terms of the minimum strong HN slope of the strong \( \mu \)-reduction bundle of \( V_0 \). Also in characteristic 0, we realize \( c^f_{\alpha}(m) (= \alpha^\infty(R,I)) \) in terms of the minimum HN slope of the \( \mu \)-reduction bundle of \( V_0 \).

In Section 5, we use the above mentioned characterization of \( c^f_{\alpha}(m) \) and \( c^f(m) \) in terms the invariants of a vector bundle on \( \text{Proj} \ R \), to deduce Theorem E.

2. The HK density function in dimension 2

Let \( X \) be a nonsingular projective curve over an algebraically closed field \( k \).

We recall the following notations from [TrW]. For details we refer the reader to Section 5 of [TrW].

**Notations 2.1.** Let \( V \) be a vector bundle on \( X \). The slope of \( V \) is \( \mu(V) = \deg V/\text{rank} \ V \).

1. The set \( \{(\mu_1, \mu_2, \cdots, \mu_{t+1}), \{r_1, \cdots, r_{t+1}\}\} \) is called the HN data of \( V \) if \( V \) has the HN filtration

\[
0 = F_0 \subset F_1 \subset \cdots \subset F_t \subset F_{t+1} = V,
\]

with \( \mu_i = \mu(F_i/F_{i-1}) \) and \( r_i = \text{rank}(F_i/F_{i-1}) \). We call \( \mu_i \) a HN slope of \( V \) and \( r_i \) a HN rank of \( V \).

We denote the minimum HN slope of \( V \) by \( \mu_{\min}(V) = \mu(V/F_1) \).

2. If characteristic \( k = p > 0 \), then \( \{(a_1, \ldots, a_{t+1}), \{\tilde{r}_1, \ldots, \tilde{r}_{t+1}\}\} \) is called the strong HN data of \( V \), where \( m > 0 \) is an integer such that \( F^mV \) has the strong HN filtration (such an integer \( m > 0 \) exists by Theorem 2.7 of [L])

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_t \subset E_{t+1} = F^{m*}V
\]

and \( a_i = (1/p^m)\mu(E_i/E_{i-1}) \) and \( \tilde{r}_i = \text{rank}(E_{i+1}/E_i) \). We call \( a_i \) a strong HN slope of \( V \) and \( r_i \) a strong HN rank of \( V \).

We denote the minimum strong HN slope of \( V \) by \( a_{\min}(V) = (1/p^m)\mu(E_{t+1}/E_t) \).
Remark 2.2. Let $O_X(1)$ be an ample line bundle of degree $d$ on $X$. Let $E$ be a semistable vector-bundle on $X$ with $\mu(E) = \mu$ and rank$(E) = r$. Then by Serre duality

\[
\begin{align*}
  m < -\mu/d & \implies h^1(X, E(m)) = -r(\mu + dm + (g - 1)) \\
  -\mu/d \leq m \leq -\mu/d + (d - 3) & \implies h^1(X, E(m)) = C \\
  -\mu/d + (d - 3) < m & \implies h^1(X, E(m)) = 0,
\end{align*}
\]

where $|C| \leq r(g - 1)$ and $g = \text{genus}(X)$.

2.1. The HK density functions for vector bundles on curves. Let $X$ be a nonsingular projective curve over an algebraically closed field of characteristic $p > 0$. Let $O_X(1)$ be an ample line bundle of degree $d$ on $X$. Let $V$ be a vector bundle on $X$.

We recall the definition ((6.1) in [TrW]) of the HK density function of $V$ with respect to $O_X(1)$. Let $f_n(V, O_X(1)) : \mathbb{R} \rightarrow [0, \infty)$ be given by (where $q = p^n$)

\[
f_n(V, O_X(1))(x) = \frac{1}{q} h^1(X, F^{n*}V((x - 1)q))).
\]

and let

\[(2.1) f_{V, O_X(1)} : \mathbb{R} \rightarrow [0, \infty) \text{ given by } x \rightarrow \lim_{n \to \infty} f_n(V, O_X(1))(x)\]

The function $f_{V, O_X(1)}$ is well defined and continuous (though need not be compactly supported).

Remark 2.3. Later in the paper, we will use the following formula (given in terms of the strong HN data $\{\{a_1, \ldots, a_{l+1}\}; \{r_1, \ldots, r_{l+1}\}\}$ of $V$.

We choose $n_1 > 0$ such that $F^{n_1*}V$ has the strong HN filtration

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} = F^{n_1*}V,
\]

where $a_i = (1/p^{n_1})\mu(E_i/E_{i-1})$ and $r_i = \text{rank}(E_i/E_{i-1})$.

Since $a_1 > a_2 > \cdots > a_{l+1}$, we can choose $q > 0$ ($q = p^{n_1}$) such that

\[-\frac{a_1qq_1}{d} < \frac{a_1qq_1}{d} < (d - 3) < \frac{a_2qq_1}{d} < \frac{a_2qq_1}{d} < (d - 3) < \cdots < \frac{a_{l+1}qq_1}{d}.
\]

1. By Remark 2.2 (where $q = p^{n_1}$)

\[
qq_1f_{n+n_1}(V, O_X(1)) \left(\frac{m}{qq_1}\right) = h^1(X, F^{n+n_1*}V(m-qq_1)) = \sum_{i=1}^{l+1} h^1(X, F^{n_1*}(E_i/E_{i-1})(m-qq_1))
\]

If $g = \text{genus}(X)$ and $R_i = r_i \left[a_i + d\left(\frac{m}{qq_1} - 1\right) + \frac{(g-1)}{qq_1}\right]$ then we have

\[
f_{n+n_1}(V, O_X(1)) \left(\frac{m}{qq_1}\right) = \begin{cases} 
- \sum_{i=1}^{l+1} R_i & \text{for } \frac{m}{qq_1} < 1 - \frac{a_1}{d} \\
\frac{C_i}{qq_1} - \sum_{i=2}^{l+1} R_i & \text{for } 1 - \frac{a_1}{d} \leq \frac{m}{qq_1} < 1 - \frac{a_1}{d} + \frac{(d-3)}{qq_1} \\
- \sum_{k=i=1}^{l+1} R_k & \text{for } 1 - \frac{a_1}{d} + \frac{(d-3)}{qq_1} \leq \frac{m}{qq_1} < 1 - \frac{a_{i+1}}{d} \\
\frac{C_{i+1}}{qq_1} - \sum_{k=i+2}^{l+1} R_k & \text{for } 1 - \frac{a_{i+1}}{d} \leq \frac{m}{qq_1} < 1 - \frac{a_{i+1}}{d} + \frac{(d-3)}{qq_1} \\
0 & \text{for } 1 - \frac{a_{i+1}}{d} + \frac{(d-3)}{qq_1} \leq \frac{m}{qq_1}.
\end{cases}
\]
where $|C_i| \leq \text{rank}(V)(g-1)$ for all $i$ and $a_{l+1} = a_{\text{min}}(V)$.

(2) Taking limit as $n \to \infty$, we get the formula for $f_{V, \mathcal{O}_X(1)}$:

$$f_{V, \mathcal{O}_X(1)}(x) = \begin{cases} 
- \left[ \sum_{i=1}^{l+1} a_i r_i + d(x-1)r_i \right] & \text{for } x < 1 - a_1/d \\
- \left[ \sum_{k=i+1}^{l+1} a_k r_k + d(x-1)r_k \right] & \text{for } 1 - a_i/d \leq x < 1 - a_{i+1}/d.
\end{cases}$$

(3) Support $f_{V, \mathcal{O}_X(1)}$ lies in the interval $(-\infty, 1 - a_{\text{min}}(V)/d]$ and

$$\alpha(V, \mathcal{O}_X(1)) := \text{Sup} \{ x \mid f_{V, \mathcal{O}_X(1)}(x) > 0 \} = 1 - \frac{a_{\text{min}}(V)}{d}.$$ 

**Remark 2.4.** Replacing $R$ by $R \otimes_k \bar{k}$ does not change the function $f_{R, I}$ and the semistability behaviour of any vector bundle $V$ on $X = \text{Proj } R$. Therefore we can assume, without loss of generality, that the underlying field $k$ is algebraically closed.

### 2.2. The HK density functions of $f_{R, I}$ and the syzygy vector bundles

Let $(R, I)$ be a standard graded pair, where $R$ is a domain defined over a field of characteristic $p > 0$.

Let $S = \oplus_m S_m$ be the integral closure of $R$ in its quotient field. Then the inclusion map $\pi : R \hookrightarrow S$ is a graded finite map of degree 0, where $S$ is a normal domain and $Q(R) = Q(S)$. The additivity of the HK density function (Proposition 2.14 of [T2]) implies that

$$f_{R, I}(x) = f_{S, I}(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{1}{q} \ell \left( \frac{S}{I^{[n]}S} \right)_{[xq]}.$$ 

Since $R$ is a standard graded ring over $k$, the canonical embedding $Y = \text{Proj } R \hookrightarrow \mathbb{P}^n_k$ gives the very ample line bundle $\mathcal{O}_Y(1)$ on $Y$. Let $X = \text{Proj } S$ with the canonical map $\pi : X \to Y$ and let $\mathcal{O}_X(1) = \pi^* \mathcal{O}_Y(1)$ be the ample line bundle on $X$.

Note that $X$ is a nonsingular projective curve. For a choice of homogeneous generators $h_1, \ldots, h_\mu$ of $I$ of degrees $d_1, \ldots, d_\mu$, we have the canonical (locally split) exact sequence of locally free sheaves of $\mathcal{O}_X$-modules

$$0 \to V_0 \to M_0 = \oplus_{i=1}^\mu \mathcal{O}_X(1 - d_i) \to \mathcal{O}_X(1) \to 0,$$

where the map $\mathcal{O}_X(1 - d_i) \to \mathcal{O}_X(1)$ is given by the multiplication by the element $h_i$.

Since, for $q = p^k \gg 0$,

$$f_n \left( \frac{m + q}{q} \right) = \frac{1}{q} \ell \left( \frac{S}{I^{[n]}S} \right)_{m+q} = \frac{1}{q} \left[ h^1(X, (F^n V_0)(m)) - h^1(X, (F^n M_0)(m)) \right]$$

we have

$$f_{R, I}(x) = f_{V_0, \mathcal{O}_X(1)}(x) - f_{M_0, \mathcal{O}_X(1)}(x), \text{ for } x \geq 1.$$ 

If $a_{\text{min}}(V_0) < a_{\text{min}}(M_0)$ then by (2.2) $\alpha(R, I) = 1 - a_{\text{min}}(V_0)/d$. This holds true when $d_1 = \cdots = d_s$, as $M_0$ is strongly semistable and therefore $\mu(M_0) = a_{\text{min}}(M_0)$ and $a_{\text{min}}(V_0) \leq \mu(V) < \mu(M_0)$.

However, it may happen that $a_{\text{min}}(V_0) = a_{\text{min}}(M_0)$ and the HK density functions for $V_0$ and $M_0$ may coincide in a neighbourhood of their common maximum support point (see Remark 4.1).

In the next section we introduce the notion of $\mu$-reduction and the strong $\mu$-reduction for a short exact sequence of type (2.3). Using the strong $\mu$-reduction bundle $V_{t_0}$ ($\subset V_0$) we replace the short exact sequence (2.3) by another sequence

$$0 \to V_{t_0} \to M_{t_0} \to \mathcal{O}_X(1) \to 0$$
such that

1. \( a_{\text{min}}(V_0) < a_{\text{min}}(M_0) \)
2. \( f_{V_0,\mathcal{O}_X} - f_{M_0,\mathcal{O}_X} = f_{V_0,\mathcal{O}_X} - f_{M_0,\mathcal{O}_X} \)
3. \( a_{\text{min}}(V_i) < a_{\text{min}}(M_i) \)

In particular, we express \( \alpha(R, I) \) in terms of one of the strong HN slopes of the syzygy vector bundle \( V_0 \). In characteristic 0, using the \( \mu \)-reduction bundle \( V_i \) (whose minimum HN slope occurs in the HN data of \( V_0 \)) we are able to express \( \alpha(\infty)(R, I) \) (the maximum support point of the HK density function in characteristic 0) in terms of one of the HN slopes of \( V_0 \).

Using this formula for \( \alpha(R, I) \), in terms of the strong HN data of a single vector bundle, and Remark 2.3 (1), we are able to prove the equality \( c^I(m) = \alpha(R, I) \). This enables us to study various properties (Theorem D and Theorem E) of the F-thresholds \( c^I(m) \) and \( \alpha^I(m) \) for two dimensional standard graded pair \( (R, I) \).

**3. \( \mu \)-reduction and strong \( \mu \)-reduction bundles**

Let \( X \) denote a nonsingular projective curve with an ample line bundle \( \mathcal{O}_X(1) \) of degree \( d \) over a field \( k \) of arbitrary characteristic and let

\[
(3.1) \quad 0 \rightarrow V_0 \xrightarrow{f_0} M_0 = \oplus_{i=1}^\mu \mathcal{O}_X(1 - d_i) \rightarrow \mathcal{L} = \mathcal{O}_X(1) \rightarrow 0
\]

be a short exact sequence of sheaves of \( \mathcal{O}_X \)-modules, where \( d_1 \leq d_2 \cdots \leq d_\mu \) are positive integers and where the map \( \mathcal{O}_X(1 - d_i) \rightarrow \mathcal{O}_X(1) \) is a multiplication map given by \( h_i \in H^0(X, \mathcal{O}_X(d_i)) \).

**Notations 3.1.** For the sequence (3.1), we denote the HN filtration of \( M_0 \) by

\[
0 \subset M_{l_1-1} \subset \cdots \subset M_0
\]

and let

\[
V_{l_1-1} \subseteq \cdots \subseteq V_1 \subseteq V_0
\]

denote the induced (need not be the HN) filtration on \( V_0 \), where \( V_i = M_i \cap V_0 \). For every \( 0 \leq i \leq l_1 - 1 \), let \( f_i : V_i \rightarrow M_i \) be the canonical inclusion map.

Here \( M_i \) can explicitly be given as follows: Let \( \{d_1, \ldots, d_\mu\} = \{\tilde{d}_1, \ldots, \tilde{d}_l\} \), with

\[
\tilde{d}_1 > \tilde{d}_l \ldots > \tilde{d}_{l_1-1} > \cdots > \tilde{d}_1.
\]

Then \( M_{l_1-i} = \oplus \mathcal{O}_X(1 - \tilde{d}_1) \oplus \cdots \oplus \mathcal{O}_X(1 - \tilde{d}_i) \). In particular \( \mu_{\text{max}}(M_0) = (1 - \tilde{d}_1)d \) and \( \mu_{\text{min}}(M_0) = (1 - \tilde{d}_l)d \), where \( d \) is the degree of \( \mathcal{L} \).

It is easy to check that the bundle \( V_{l_1-1} = 0 \) iff \( M_{l_1-1} \) is a line bundle.

**3.1. The \( \mu \)-reduction bundle.**

**Definition 3.2.** The bundle \( V_i \) is the \( \mu \)-reduction bundle of \( V_0 \) (of sequence (3.1)) if \( t < l_1 \) such that \( V_i \neq 0 \) and

1. \( \mu_{\text{min}}(V_i) = \mu_{\text{min}}(M_i) \) for \( i < t \)
2. \( \mu_{\text{min}}(V_i) < \mu_{\text{min}}(M_i) \).

Strictly speaking we should be refering to \( V_i \) as the \( \mu \)-reduction bundle of the sequence (3.1) as the notion depends on the sequence (3.1) too. Since in the paper there would not be any ambiguity about the associated sequence, we will refer the bundle \( V_i \) as the \( \mu \)-reduction bundle of \( V_0 \).

Next we prove relevant properties of the filtration \( \{V_i\}_i \) and then prove the existence of the \( \mu \)-reduction bundle of \( V_0 \).

We would repeatedly use the following two obvious properties of the sequence (3.1).

1. The induced map \( M_{l_1-1} \rightarrow \mathcal{L} \) is nonzero and
2. \( \mu_{\text{max}}(M_0) = \mu(M_{l_1-1}) < \mu(\mathcal{L}) \).
Remark 3.3. The following are well known facts (can also refer to Remark 5.5 in [TrW]).

(1) If $0 \to V' \to V \to V'' \to 0$ is a short exact sequence of nonzero vector bundles on $X$, then
(2) either $\mu(V') \leq \mu(V) \leq \mu(V'')$ or $\mu(V') \geq \mu(V) \geq \mu(V'')$.
(3) For a nonzero map of bundles $E \to W$, where $W$ is semistable, $\mu_{\min}(E) \leq \mu(W)$. In particular, if $0 \to V' \to V \to V/V' \to 0$ is an exact sequence of nonzero bundles such that $V/V'$ is semistable and $W \subseteq V$ is a nonzero bundle such that $\mu_{\min}(W) > \mu(V/V')$ then $W \subseteq V'$.
(4) For a nonzero bundle $V$ on $X$, we have $\mu_{\min}(F_{m*}(V)) \leq p^m \mu_{\min}(V)$, for any $m \geq 1$.

Lemma 3.4. (1) The sequence $V_{l_{t-1}} \subset \cdots \subset V_1 \subset V_0$ is a sequence of distinct subbundles and
(2) $\mu_{\min}(V_j) \leq \mu_{\min}(M_j)$, for $0 \leq j \leq l_1 - 1$ and same holds for $j = l_1 - 1$ if the bundle $V_{l_{t-1}}$ is nonzero.
(3) If $i < l_1 - 1$ such that $\mu_{\min}(V_j) = \mu_{\min}(M_j)$, for $0 \leq j \leq i$, then the canonical sequence

$$0 \to V_{i+1} \xrightarrow{f_{i+1}} M_{i+1} \to \mathcal{L} \to 0$$

is a short exact sequence and $V_j/M_{j+1} \simeq M_j/M_{j+1}$, for $0 \leq j \leq i$.

Proof. For $0 \leq i \leq l_1 - 1$, the induced map $M_i \to \mathcal{L}$ is nonzero and factors through the injective map $M_i/f_i(V_i) \to \mathcal{L}$. This implies coker $f_i \neq 0$, for every $0 \leq i \leq l_1 - 1$.

(1) If $V_i = V_{i+1}$, for some $i < l_1 - 1$ then we have $M_i/M_{i+1} \simeq$ coker $f_i/coker f_{i+1}$, where coker $f_i/coker f_{i+1}$ is a subquotient (but not a subsheaf) of $\mathcal{L}$, and hence a torsion sheaf of $\mathcal{O}_X$-modules, on the other hand $M_i/M_{i+1}$ is a nonzero locally free sheaf. Hence coker $f_i/coker f_{i+1} = 0$.

(2) This follows as $0 \to V_i/V_{i+1} \to M_i/M_{i+1}$ implies

$$\mu_{\min}(V_i) \leq \mu(V_i/V_{i+1}) \leq \mu(M_i/M_{i+1}) = \mu_{\min}(M_i).$$

(3) Note that coker $f_0 = \mathcal{L}$. It is enough to prove that if there is $l_1 - 1 > j \geq 0$ such that $\mu_{\min}(V_j) = \mu_{\min}(M_j)$ and coker $f_j = \mathcal{L}$ then coker $f_{j+1} = \mathcal{L}$ and $V_j/V_{j+1} \simeq M_j/M_{j+1}$.

Consider the short exact sequence

$$0 \to V_j/V_{j+1} \to M_j/M_{j+1} \to \mathcal{L}/\text{coker } f_{j+1} \to 0,$$

Now $\mathcal{L}/\text{coker } f_{j+1}$ is a torsion sheaf. Also $\mu(V_j/V_{j+1}) = \mu(M_j/M_{j+1})$ (as argued in (2)). Therefore

$$\text{rank } \frac{V_j}{V_{j+1}} = \text{rank } \frac{M_j}{M_{j+1}} \implies \text{deg } \frac{V_j}{V_{j+1}} = \text{deg } \frac{M_j}{M_{j+1}}.$$

Hence deg $(\mathcal{L}/\text{coker } f_{j+1}) = \ell(\mathcal{L}/\text{coker } f_{j+1}) = 0$ which implies coker $f_{j+1} = \mathcal{L}$ and hence $V_j/V_{j+1} \simeq M_j/M_{j+1}$.

Proposition 3.5. The bundle $V_0$ has $\mu$-reduction bundle $V_t$, for some $t < l_1$.

Proof. If the bundle $V_{l_{t-1}} = 0$ then $V_0$ has $\mu$-reduction bundle for some $t < l_1 - 1$, otherwise, by Lemma 3.4 (3), we have $M_{l_{t-1}} \simeq \mathcal{L}$.

Hence we can assume that $V_{l_{t-1}} \neq 0$. 
Suppose $\mu_{\min}(V_t) = \mu_{\min}(M_t)$, for every $0 \leq i \leq l_1 - 1$. Then, by Lemma 3.4 (3), the sequence $0 \longrightarrow V_{t-1} \longrightarrow M_{t-1} \longrightarrow L \longrightarrow 0$ is exact. Now, as $M_{t-1}$ is semistable, we have

$$\mu_{\min}(V_{t-1}) \leq \mu(V_{t-1}) \leq \mu(M_{t-1}) = \mu_{\min}(M_{t-1}).$$

But then we have the equality $\mu(V_{t-1}) = \mu(M_{t-1}) = \mu(L)$. Hence there is $t' < l_1$ such that $\mu_{\min}(V_{t'}) < \mu_{\min}(M_{t'})$. The smallest number $t < l_1$ such that $\mu_{\min}(V_t) < \mu_{\min}(M_t)$ gives the $\mu$-reduction bundle $V_t$ of $V_0$. \hfill $\square$

Though the bundle $V_t$ may not occur in the HN filtration of $V_0$, we can relate the HN filtration of $V_t$ and the HN filtration of $V_0$.

**Lemma 3.6.** Let $V_t$ be the $\mu$-reduction bundle of $V_0$, where $t \geq 1$. Then the HN filtration of $V_0$ is

$$\cdots \subset W_{t+1} \subset W_t \subset V_{t-1} \subset V_{t-2} \cdots \subset V_1 \subset V_0.$$

Moreover

1. $W_t \subseteq V_t \subseteq V_{t-1}$ and
2. the HN filtration of $V_t$ is
   (a) either $\cdots \subset W_{t+1} \subset W_t = V_t$ (equivalently $\mu_{\min}(V_t) > \mu_{\min}(V_{t-1})$),
   (b) or $\cdots \subset W_{t+1} \subset W_t \subset V_t$, (equivalently $\mu_{\min}(V_t) = \mu_{\min}(V_{t-1})$).

In both the cases $\mu_{\min}(V_{t-1}) = \mu(V_{t-1}/W_{t}) = \mu(V_{t-1}/V_{t})$.

**Proof.** By Lemma 3.4 (3), we have $V_t/V_{t+1} \simeq M_t/M_{t+1}$, for all $0 \leq i < t$. Let the HN filtration of $V_{t-1}$ be $\cdots \subset W_{t+1} \subset W_t \subset V_{t-1}$. Then

$$\mu\left(\frac{V_{t-1}}{W_t}\right) = \mu_{\min}(V_{t-1}) = \mu_{\min}(M_{t-1}) = \mu\left(\frac{V_{i-2}}{V_{i-1}}\right) > \mu\left(\frac{V_{i-1}}{V_i}\right) \geq \cdots \geq \mu\left(\frac{V_0}{V_1}\right).$$

Hence, by the uniqueness property of the HN filtration, the HN filtration of $V_0$ has to be the filtration

$$\cdots \subset W_{t+1} \subset W_t \subset V_{t-1} \subset V_{t-2} \cdots \subset V_1 \subset V_0$$

and $V_{t-1}/V_t$ is semistable. Moreover, by Remark 3.3 the inequality $\mu_{\min}(V_t) > \mu(V_{t-1}/V_t)$ implies $W_t \subseteq V_t$.

If $W_t = V_t$ then

$$\mu_{\min}(V_t) = \mu(W_t/W_{t+1}) > \mu(V_{t-1}/W_t) = \mu_{\min}(V_{t-1})$$

and the HN filtration for $V_t$ is $\cdots \subset W_{t+1} \subset W_t = V_t$.

If $W_t \subseteq V_t$ then the exact sequence

$$0 \longrightarrow V_t/W_t \longrightarrow V_{t-1}/W_t \longrightarrow V_{t-1}/V_t \longrightarrow 0$$

implies $\mu(V_t/W_t) = \mu(V_{t-1}/W_t)$ and $V_t/W_t$ is semistable. Hence the HN filtration for $V_t$ is $\cdots \subset W_{t+1} \subset W_t \subset V_t$. \hfill $\square$

**Remark 3.7.** If $V_t$ is the $\mu$-reduction bundle of $V_0$ such that $t \geq 1$ then, by Lemma 3.6 (2), we have $\mu_{\min}(M_{t-1}) \leq \mu_{\min}(V_t) < \mu_{\min}(M_t)$.
3.2. The strong \( \mu \)-reduction bundle. Let \( X \) be a nonsingular curve over an algebraically closed field \( k \) of char \( p > 0 \). Let

\[
0 \rightarrow V_0 \rightarrow M_0 \rightarrow \mathcal{L} \rightarrow 0
\]

be the sequence (3.1). Since this is an exact sequence of locally free sheaves, for any \( s > 0 \) if \( F^s : X \rightarrow X \) is the \( s \)-th-iterated Frobenius map then the induced map (here \( q = p^s \))

\[
(3.2) \quad 0 \rightarrow F^{s*}V_0 \rightarrow F^{s*}M_0 = \bigoplus_{i=1}^{s} \mathcal{O}_X(q - qd_i) \rightarrow F^{s*}\mathcal{L} = \mathcal{O}_X(q) \rightarrow 0
\]
is exact.

**Remark 3.8.** The filtration

\[
0 \subset F^{s*}M_{l_1-1} \subset \cdots \subset F^{s*}M_0 = F^{s*}M
\]
is the HN filtration of \( F^{s*}M_0 \). Moreover, \( X \) being nonsingular implies that the map \( F^s \) is flat and therefore \( F^{s*}M_i \cap F^{s*}V_0 = F^{s*}V_i \). In particular the induced filtration on \( F^{s*}V_0 \) is

\[
F^{s*}V_{l_1-1} \subset \cdots \subset F^{s*}V_1 \subset F^{s*}V_0.
\]

**Definition 3.9.** The bundle \( V_{t_0} \) is the strong \( \mu \)-reduction bundle of \( V_0 \) if the bundle \( F^{m_1}V_{t_0} \) is the \( \mu \)-reduction bundle of \( F^{m_1}V_0 \), where \( m_1 \geq 0 \) is an integer such that the HN filtration of \( F^{m_1}V_0 \) is the strong HN filtration (this exists by [L]). By Lemma 3.5, the strong \( \mu \)-reduction bundle \( V_{t_0} \) does exist. and \( t_0 < l_1 \) is the integer such that \( a_{min}(V_{t_0}) < a_{min}(M_0) \) and \( a_{min}(V_i) = a_{min}(M_i) \), for every \( 0 \leq i < t_0 \).

**Remark 3.10.** All the succeeding results of this section hold true (with exactly the same proofs) for any short exact sequence of locally free sheaves of \( \mathcal{O}_X \)-modules

\[
0 \rightarrow V_0 \rightarrow M_0 \rightarrow \mathcal{L} \rightarrow 0,
\]

where \( \mathcal{L} \) is a line bundle, satisfying the following properties (P1) and (P2),

(P1) The induced map \( M_{l_1-1} \rightarrow \mathcal{L} \) is nonzero and \( \mu_{max}(M_0) < \mu(\mathcal{L}) \), where \( M_{l_1-1} \) is the first nonzero bundle occurring in the HN filtration of \( M_0 \).

(P2) If char \( k = p > 0 \) then the HN filtration of \( M_0 \) is the strong HN filtration.

The following lemma implies that the strong \( \mu \)-reduction bundle always contains the \( \mu \)-reduction bundle.

**Lemma 3.11.** For \( s \geq 1 \), if \( F^{s*}V_{t_1} \) is the \( \mu \)-reduction bundle of \( F^{s*}V_0 \) and \( V_t \) is the \( \mu \)-reduction bundle of \( V_0 \) then \( t_1 \leq t \).

In particular if \( V_{t_0} \) is the strong \( \mu \)-reduction bundle of \( V_0 \) then \( t_0 \leq t \).

**Proof.** We know \( t_1 < l_1 \). By Remark 3.8, \( F^{s*}M_i/F^{s*}M_{i+1} \simeq F^{s*}(M_i/M_{i+1}) \) and \( F^{s*}M_i \cap F^{s*}V_0 = F^{s*}V_i \). By definition, \( \mu_{min}(V_t) < \mu_{min}(M_t) \) therefore (see Remark 3.3)

\[
\mu_{min}(F^{s*}V_t) \leq p^s \mu_{min}(V_t) < p^s \mu_{min}(M_t) = \mu_{min}(F^{s*}M_t),
\]

which implies \( t_1 \leq t \).

**Remark 3.12.** Though \( V_t \) may not occur in the HN filtration of \( V_0 \), the number \( \mu_{min}(V_t) \) is equal to one of the HN slopes of \( V_0 \), by Lemma 3.5. Similarly, if \( V_{t_0} \) is the strong \( \mu \)-reduction bundle of \( V_0 \) then the number \( a_{min}(V_{t_0}) \) is equal to one of the strong HN slopes of \( V_0 \).
Lemma 3.13. Let $X$ be a nonsingular projective curve over a field of char $p > 0$ with an ample line bundle $O_X(1)$ of degree $d$. Let (where $d_i \geq 1$)

$$0 \to V_0 \to M_0 = \oplus_i O_X(1 - d_i) \to O_X(1) \to 0,$$

be a short exact sequence of locally free sheaves of $O_X$-modules. If $V_{t_0}$ is the strong $\mu$-reduction bundle of $V_0$ then

1. $f_{V_0, O_X(1)} - f_{M_0, O_X(1)} = f_{V_0, O_X(1)} - f_{M_0, O_X(1)}$, and
2. $\max \{ x \mid f_{V_0, O_X(1)}(x) - f_{M_0, O_X(1)}(x) \neq 0 \} = 1 - \frac{a_{\min}(V_{t_0})}{d}.$

Proof. If $t_0 = 0$ then the assertion (2) follows from (2.2) and the assertion (1). Hence we can assume $t_0 \geq 1$. Let $n_1 > 0$ such that the HN filtration of $F^{n_1}V_0$ is the strong HN filtration. If $V_0$ is the strong $\mu$-reduction bundle of $V_0$ then, by definition, $F^{n_1}V_{t_0}$ is the $\mu$-reduction bundle of $F^{n_1}V_0$. Hence

1. $\mu_{\min}(F^{n_1}V_{t_0}) < \mu_{\min}(F^{n_1}M_{t_0})$ and, by Lemma 3.6,
2. the HN filtration of $F^{n_1}V_0$ is

$$0 \subset \cdots \subset \tilde{W}_{l+1} \subset \tilde{W}_l \subset F^{n_1}V_{t_0-1} \subset F^{n_1}V_{t_0-2} \subset \cdots \subset F^{n_1}V_0$$

and

3. (a) either the HN filtration of $F^{n_1}V_{t_0}$ is $\cdots \subset \tilde{W}_{l+1} \subset \tilde{W}_l = F^{n_1}V_{t_0}$
4. (b) or the HN filtration of $F^{n_1}V_{t_0}$ is $\cdots \subset \tilde{W}_{l+1} \subset \tilde{W}_l \subset F^{n_1}V_{t_0}.$

Moreover, in both the cases,

$$\frac{F^{n_1}V_0}{F^{n_1}V_1} \simeq \frac{F^{n_1}V_0}{F^{n_1}M_0}, \cdots, \frac{F^{n_1}V_{t_0-1}}{F^{n_1}V_{t_0}} \simeq \frac{F^{n_1}V_{t_0-1}}{F^{n_1}M_{t_0}}$$

and $\mu(F^{n_1}V_{t_0-1}/F^{n_1}V_{t_0}) = \mu(F^{n_1}V_{t_0-1}/\tilde{W}_l)$.

It is easy to check that the HN filtration of $F^{n_1}V_{t_0}$ is the strong HN filtration.

Moreover, if $\{\{a_1, q_1, \ldots, a_{k+1}, q_1\}, \{r_1, \ldots, r_{k+1}\}\}$ is the strong HN data of $F^{n_1}V_{t_0}$ then $\{\{a_1, \ldots, a_{k+1}\}, \{r_1, \ldots, r_{k+1}\}\}$ is the strong HN data of $V_{t_0}$. Let the HN data (which is same as the strong HN data) for $M_{t_0}$ be $\{\{b_1, \ldots, b_{l-1}, t_0\}, \{s_1, \ldots, s_{l-1}\}\}.$

Let

$$A_n(m) = h^1(X, F^{n_1+n_1}V_0(m)) - h^1(X, F^{n_1+n_1}M_0(m)),
B_n(m) = h^1(X, F^{n_1+n_1}V_{t_0}(m)) - h^1(X, F^{n_1+n_1}M_{t_0}(m)).$$

Claim.

1. For $q = p^n$ and there is a constant $C$ such that $|C| \leq (\text{rank } M_0)d(d-3)$ and
   $$A_n(m) = B_n(m) + C, \quad \text{for } m \in \left[0, \frac{(d-3)q_1}{q_1} - \frac{ak+1}{d}\right],$$
   $$A_n(m) = B_n(m) = 0, \quad \text{for } m \in \left[\frac{(d-3)q_1}{q_1} - \frac{ak+1}{d}, \infty\right].$$
2. $A_n(m) = B_n(m) = h^1(X, F^{n_1+n_1}V_{t_0}(m)) = -r_{k+1}[a_{k+1}q_1 + md + d(d-3)],$
   $$\text{for } m \in \left(\frac{(d-3)q_1}{q_1} - \frac{\min\{a_k, b_{l-1}, t_0\}}{d}, \frac{a_{k+1}}{d}\right).$$

Proof of the claim: We prove the claim when $\tilde{W}_l \subset F^{n_1}V_{t_0}$. The case $\tilde{W}_l = F^{n_1}V_{t_0}$ can be argued similarly. Since

$$a_{k+1}q_1 = \mu(F^{n_1}V_{t_0}/\tilde{W}_l) = \mu(F^{n_1}V_{t_0-1}/\tilde{W}_l) = \mu(F^{n_1}(M_{t_0-1}/M_{t_0}))$$
the strong HN data of $V_0$ is \[ \{ \{a_1, \ldots, a_{k+1}, a_{k+2}, \ldots, a_{k+l_0} \}, \{r_1, \ldots, r_k, r_{k+1}, \ldots, r_{k+l_0} \} \} \]. Hence the strong HN data of $M_0$ is given by
\[ \{ \{b_1, \ldots, b_{l_0}, a_{k+1}, a_{k+2}, \ldots, a_{k+l_0} \}, \{s_1, \ldots, s_{l_1-t_0}, r_{k+1}-r_{k+1}, r_{k+2} \ldots, r_{k+l_0} \} \} \]
as
\[ s_{l_1-t_0} = \text{rank}(M_{l_1}-1/M_{l_0}) = \text{rank}(V_{l_0-1}/V_{l_0}) = \frac{r_{k+1}-r_{k+1}}. \]
Now the claim follows from the formula given in Remark 2.3 (1).

Therefore we have
\[ \lim_{q \to \infty} \frac{1}{q(q)} A_n([xq]) = \lim_{q \to \infty} \frac{1}{q(q)} B_n([xq]). \]
This proves assertion (1) of the lemma. The part (1) of the claim also implies that
\[ f_{V_0, \mathcal{O}_X(1)}(x) - f_{M_0, \mathcal{O}_X(1)}(x) = 0, \quad \text{for} \quad x \in \left[ 1 - \frac{a_{k+1}}{d}, \infty \right). \]
Note that $a_{k+1} < a_{k}$ and $a_{k+1} = a_{\text{min}}(V_{l_0}) < b_{l_0-t_0} = a_{\text{min}}(M_{l_0})$.
Hence if $x \in (1 - \min\{a_{k}/d, a_{\text{min}}(M_{l_0})/d\}, 1 - a_{\text{min}}(V_{l_0}/d)$ then
\[ f_{V_0, \mathcal{O}_X(1)}(x) - f_{M_0, \mathcal{O}_X(1)}(x) = -r_{k+1} \lfloor a_{k+1} + d(x - 1) \rfloor > 0. \]
This proves the second assertion and hence the lemma.

4. The maximum support $\alpha(R, I)$ and the $F$-threshold $c^f(m)$

Throughout this section fix the following

**Notations 4.1.** Let $(R, I)$ be a standard graded pair, where $R$ is a two dimensional domain over an algebraically closed field $k$. Let $d = e_0(R, m)$ be the multiplicity of $R$ with respect to $m$. In the rest of this section we fix a set of homogeneous generators $h_1, \ldots, h_\mu$ of degrees $d_1, \ldots, d_\mu$, respectively, of $I$. Let $S$ be the integral closure of $R$ in its quotient field. Then $X = \text{Proj} S$ is a nonsingular curve with the ample line bundle $\mathcal{O}_X(1)$ of degree $d$ and the short exact sequence
\[ 0 \longrightarrow V_0 \longrightarrow M_0 = \bigoplus_{i=1}^{\mu} \mathcal{O}_X(1 - d_i) \longrightarrow \mathcal{O}_X(1) \longrightarrow 0, \]
where the map $\mathcal{O}_X(1 - d_i) \longrightarrow \mathcal{O}_X(1)$ is the multiplication map given by the element $h_i$.

Let the HN filtration of $M$ be
\[ 0 = M_{l_1} \subset M_{l_1-1} \subset \cdots M_1 \subset M_0 = M, \quad \text{and let} \quad V_t = V \cap M_t. \]

By Proposition 4.1 the bundle $V_0$ has the $\mu$-reduction bundle $V_t$ for some $t < l_1$ and the sequence of canonical maps
\[ 0 \longrightarrow V_t \longrightarrow M_t \longrightarrow \mathcal{O}_X(1) \longrightarrow 0. \]
is a short exact sequence of sheaves of $\mathcal{O}_X$-modules

In case $\text{char} k = p > 0$, the bundle $V_0$ has the strong $\mu$-reduction bundle $V_{l_0}$, for some $t_0 \leq t$ with the short exact sequence of $\mathcal{O}_X$-sheaves
\[ 0 \longrightarrow V_{l_0} \longrightarrow M_{l_0} \longrightarrow \mathcal{O}_X(1) \longrightarrow 0. \]

**Remark 4.2.** Note that for a given choice of generators of $I$, the sequence \[ 4.1 \] and hence the bundles $V_t$ and $V_{l_0}$ are unique, but need not be unique for the pair $(R, I)$. 
4.1. The maximum support \( \alpha(R, I) \) of the HK density function \( f_{R, I} \).

**Theorem 4.3.** Following the Notations \[4.1\], if \((R, I)\) is a standard graded pair over a perfect field of characteristic \( p > 0 \) and \( V_{t_0} \) is a strong \( \mu \)-reduction bundle for \( V_0 \) then

1. \[ f_{R,I}(x) = f_{V_{t_0},O_X(1)}(x) - f_{M_{t_0},O_X(1)}(x), \text{ for } x \geq 1. \]

2. Moreover

\[
\alpha(R, I) := \text{Sup} \{ x \mid f_{R,I}(x) > 0 \} = 1 - a_{\min}(V_{t_0})/d.
\]

**Proof.** Both the assertions follow from Lemma \[3.13\] and \[2.4\]. \( \square \)

**Remark 4.4.** In the following two cases the bundle \( V_0 \) itself is the strong \( \mu \)-reduction bundle of \( V_0 \).

1. If \( I \) has a set of generators of the same degrees. Then \( \mu_{\min}(V_0) < \mu_{\min}(M_0) \) and therefore \( a_{\min}(V_0) < a_{\min}(M_0) \).

2. Suppose \( h_1, \ldots, h_\mu \) is a set of minimal homogeneous generators of \( I \). By Theorem \[4.3\], if \( V_{t_0} \neq V_0 \) is the strong \( \mu \)-reduction bundle then there is a graded ideal \( J \subset I \) such that \( I^* = J^* \), where \( J \) is generated by a proper subset of the set \( \{h_1, \ldots, h_\mu\} \). Therefore if \( I \) itself is the minimal graded tight closure reduction for \( I \), i.e.,

\[
\{I\} = \min\{J \subseteq I \mid J \text{ graded, } J^* = I^*\}
\]

then by choosing a minimal generating set \( \{h_1, \ldots, h_\mu\} \) in the short exact sequence \[4.1\], we can ensure that \( V_0 \) itself is a strong \( \mu \)-reduction bundle. In particular, if \( R \) is a \( F \)-regular ring then \( V_0 = V_{t_0} \).

In the following example we show that \( V_0 \) is not always a strong \( \mu \)-reduction bundle of itself, which is equivalent to showing \( a_{\min}(V_0) = a_{\min}(M_0) \). Moreover, in the example, the functions \( f_{M_0,O_X(1)} \) and \( f_{V_0,O_X(1)} \) are the same functions in the neighbourhood of their maximum common support. In particular \( \alpha(R, I) < 1 - a_{\min}(V_0)/d \).

**Example.** Let \( R = \mathbb{k}[x, y, z]/(x^d + y^d + z^d) \) and \( I = (x^2, y^2, z^2) \). Then, by Lemma 3.2 of [S], \( I \) is in the tight closure of \( (x^2, y^2) \). Hence \( \alpha(R, I) = \alpha(R, (x^2, y^2)) = 4 \), where the last equality follows by Theorem 4.10 of [TrW].

Now, for the pair \((R, I)\), the sequence \[4.1\] is given by

\[
0 \rightarrow V_0 \rightarrow M_0 = O_X(-1) \oplus O_X(-1) \oplus O_X(-4) \rightarrow O_X(1) \rightarrow 0
\]

and the strong HN data of \( M_0 \) is \((-d, -4d), \{2, 1\}\) and \( \mu(V) = -7d \).

If \( a_{\min}(V_0) \neq a_{\min}(M_0) \) then \( V_0 \) is the strong \( \mu \)-reduction of \( V_0 \). Hence \( a_{\min}(V_0) = -3d \) which would imply the strong HN filtration of \( V_0 \) is \( 0 \subset L_1 \subset F_s^*V_0 \), for some \( s \geq 0 \), where \( L_1 \) is a line bundle. But then

\[
a_{\max}(V_0) = \deg(L_1/p^s) = \deg V_0 - a_{\min}(V_0) = -4d < a_{\min}(V_0).
\]

Hence \( a_{\min}(V_0) = a_{\min}(M_0) = -4d \). In particular the strong HN data of \( V_0 \) is \((-3d, -4d), \{1, 1\}\). Now the HK density functions \( f_{M_0,O_X(1)} \) and \( f_{V_0,O_X(1)} \) can be written as follows:

\[
f_{M_0,O_X(1)}(x) = \begin{cases} 3d(3-x) & \text{if } x < 2 \\ d(5-x) & \text{if } 2 \leq x < 5 \\ 0 & \text{if } 5 \leq x, \end{cases}
\]

\[
f_{V_0,O_X(1)}(x) = \begin{cases} d(9-2x) & \text{if } x < 4 \\ d(5-x) & \text{if } 4 \leq x < 5 \\ 0 & \text{if } 5 \leq x. \end{cases}
\]
Remark 4.5. We can give a bound on the strong HN slope \( a_{\min}(V_0) \) in terms of the degrees of the generators \( h_1, \ldots, h_\mu \) of \( I \) as follows.

Let \( d_1 < d_2 < \ldots < d_t \) be the degrees of these generators (see Notations [3.1]). If 
\( V \) is the strong \( \mu \)-reduction bundle then \( a_{\min}(V)/d < a_{\min}(M)/d = 1 - d_i \).
Moreover, if \( V \) is not a strong \( \mu \)-reduction of itself, i.e., if \( t_0 \geq 1 \) then, by Remark 3.7,
\[
a_{\min}(M_{t_0-1})/d = 1 - d_i - t_0 + 1 \leq a_{\min}(V)/d < a_{\min}(M)/d = 1 - d_i - t_0.
\]

4.2. The F-threshold \( c^f(m) \) and \( \alpha(R, I) \) in char \( p > 0 \). Here we prove \( c^f(m) = \alpha(R, I) \).
This equality is known to hold when \( R \) itself is a normal domain. Though we know \( \alpha(R, I) = \alpha(S, IS) \), we can not deduce the equality by considering the normalization of \( R \) as we do not know if \( c^f(m) = c^f(S(mS)) \).

Let \( Y = \text{Proj} R \) and let \( \pi : X \to Y \) be the canonical map then, by construction, the sequence \((4.1)\) descends to the canonical sequence
\[
0 \to W_0 \to N_0 \to \mathcal{O}_Y(1-d_i) \to \mathcal{O}_Y(1) \to 0
\]
of \( \mathcal{O}_Y \)-modules. In fact the following lemma implies that the exact sequences \((4.2)\) and \((4.3)\) also descend to similar exact sequences of sheaves of \( \mathcal{O}_Y \)-modules.

Lemma 4.6. If for any \( i \leq l_1 \) the sequence
\[
(4.4) \quad 0 \to V_i \to M_i \to \mathcal{O}_X(1) \to 0
\]
is exact then it descends to a short exact sequence
\[
0 \to W_i \to N_i \to \mathcal{O}_Y(1) \to 0
\]
of \( \mathcal{O}_Y \)-modules. In particular \( V_i = \pi^*(W_i) \), where \( W_i \) is a vector bundle on \( Y \).

Proof. By definition \( M_i = \sum_j \mathcal{O}_X(-n_j) \), where \( n_j \) are nonnegative integers. Let \( N_i = \sum_j \mathcal{O}_Y(-n_j) \). Then the map \( g_i : M_i \to \mathcal{O}_X(1) \) descends to the canonical map \( g_i : N_i \to \mathcal{O}_Y(1) \).

We claim that the map \( g_i \) is surjective: Otherwise there is a closed point \( y \in Y \) such that the map \( g_i : (N_i)_y \to (\mathcal{O}_Y(1))_y \) factors through the map \( m_{Y,y} \to (\mathcal{O}_Y(1))_y \). But then for any \( x \in \pi^{-1}(y) \), the map \( g_i : (M_i)_x \to (\mathcal{O}_X(1))_x = \mathcal{O}_{X,x} \)
factors through \( m_{X,x} \to \mathcal{O}_{X,x} \), which contradicts the surjectivity of \( M_i \to \mathcal{O}_X(1) \).

Now we have a short exact sequence
\[
0 \to W_i \to N_i \to \mathcal{O}_Y(1) \to 0
\]
of \( \mathcal{O}_Y \)-modules, which is locally split exact. Hence
\[
0 \to \pi^*W_i \to \pi^*N_i = M_i \to \pi^*\mathcal{O}_Y(1) = \mathcal{O}_X(1) \to 0
\]
is an exact sequence of \( \mathcal{O}_X \)-modules and therefore is the same as the sequence \((4.4)\). \( \Box \)

In the following lemma \( F^n_X : X \to X \) denotes the \( n \)-th iterated Frobenius map on \( X \) (ditto for \( Y \)). The sheaf \( K \) is a 0-dimensional coherent sheaf of \( \mathcal{O}_Y \)-modules given by the canonical exact sequence
\[
(4.5) \quad 0 \to \mathcal{O}_Y \to \pi_*\mathcal{O}_X \to K \to 0.
\]

Lemma 4.7. Let \( W \) be a vector bundle on \( Y \) and \( V = \pi^*W \) then
\[
h^1(X, (F^n_XV)(m)) \leq h^1(Y, (F^n_YW)(m)) \leq h^1(X, (F^n_XV)(m)) + s \cdot h^0(Y, K),
\]
for all \( m, n \geq 0 \), where \( s = \text{rank } W \).
In particular, for every \( q \) and therefore for \( \alpha \), the elements \( h \) and \( V \) sequence (4.3) descends to the short exact sequence that

\[ H^0(Y, K_{\oplus s}) \rightarrow H^1(Y, (F^n_{Y}W)(m) \otimes \pi_*O_X) \rightarrow 0. \]

By the projection formula

\[ (F^n_{Y}W)(m) \otimes \pi_*O_X = \pi_*((F^n_{Y}W)(m)) = \pi_*((F^n_XV)(m)) \]

which implies

\[ h^1(Y, (F^n_{Y}W)(m) \otimes \pi_*O_X) = h^1(Y, \pi_*((F^n_XV)(m))) = h^1(X, (F^n_XV)(m)). \]

\[ \square \]

**Theorem 4.8.** If \((R, I)\) is a standard graded pair, where \( R \) is a 2-dimensional domain then

\[ \alpha(R, I) = c^I(m). \]

In particular \( c^I(m) = c^S(mS) \), where \( R \rightarrow S \) is a finite graded degree 0 morphism of rings.

**Proof.** By Proposition 4.4 of [TrW], we have \( \alpha(R, I) \leq c^I(m) \). We only need to prove that \( c^I(m) \leq \alpha(R, I) \). Let \( x_0 = \alpha(R, I) = 1 - a_{\min}(V_{t_0})/d \). By Lemma 4.6 the sequence (4.3) descends to the short exact sequence

\[ 0 \rightarrow W_{t_0} \rightarrow N_{t_0} \rightarrow O_Y(1) \rightarrow 0 \]

and \( V_{t_0} = \pi^*W_{t_0} \). If \( N_{t_0} = \oplus_j O_Y(1 - d_1j) \rightarrow O_Y(1) \) is the multiplication map given by the elements \( h_{11}, \ldots, h_{1a} \in I \) of degrees \( d_{11}, \ldots, d_{1a} \), respectively, then for \( q = p^n \gg 0 \) and \( m \in \mathbb{N} \) we have

\[ 0 \rightarrow (F^{n*}W_{t_0})(m - q) \rightarrow \oplus_j O_Y(m - qd_{1j}) \rightarrow O_Y(m) \rightarrow 0 \]

and therefore for \( J = (h_{11}, \ldots, h_{1a}) \),

\[ \ell(R/J^{[q]}m) \leq \ell(R/J^{[q]}m) \leq h^1(Y, F^{n*}W_{t_0}(m - q)). \]

Let \( q_1 = p^{n_1} \) be such that the HN filtration of \( F^{n_1*}V_{t_0} \) is the strong HN filtration. Then, by Remark 2.3 (1),

\[ h^1(X, F^{(n+n_1)*}V_{t_0}(m - qq_1)) = 0 \quad \text{for} \quad m \geq (d - 3) + x_0qq_1. \]

By Lemma 4.7 there is a constant \( C_0 \) such that \( h^1(Y, F^{n+n_1*}W_{t_0}(m - qq_1)) \leq C_0 \), for every \( m \geq (d - 3) + x_0qq_1 \). This implies (see Proposition 4.6 of [TrW])

\[ h^1(Y, F^{n+n_1*}W_{t_0}(m - qq_1)) = 0 \quad \text{for} \quad m \geq C_0 + (d - 3) + x_0qq_1. \]

In particular \( \ell(R/J^{[qq_1]}m) = 0 \), in other words \( m \in I^{[qq_1]} \). Now

\[ c^J(m) \leq \lim_{q \rightarrow \infty} \frac{1}{qq_1} C_0 + (d - 3) + x_0qq_1 = x_0. \]

The second assertion follows as we have \((S, I)\) is considered as an \( R\)-module here

\[ \alpha(S, I) \leq c^S(mS) \leq c^J(m) = \alpha(R, I) = \alpha(S, I), \]

where the first inequality and the last equality follow from Proposition 4.4 of [TrW] and Proposition 2.14 of [T2], respectively. \[ \square \]
4.3. The $F$-threshold $c^i(m)$ and $\alpha(R,I)$ in characteristic 0.

**Notations 4.9.** In this section we consider the sequence (4.1), where char $k = 0$. The bundle $V_t$ denotes the $\mu$-reduction bundle of $V_0$ and the filtration

\[(4.6) \quad \cdots \subset W_{i+1} \subset W_i \subset V_t \subset \cdots \subset V_0\]

denotes the HN filtration of $V_0$.

Now, by Lemma 3.6

1. $W_i = V_t$, if $\mu_{\text{min}}(V_t) > \mu_{\text{min}}(V_{i-1})$ and
2. $W_i \subset V_t$, if $\mu_{\text{min}}(V_t) = \mu_{\text{min}}(V_{i-1})$.

Moreover

1. $V_0/V_1 \simeq M_0/M_1, \ldots, V_{i-1}/V_i \simeq M_{i-1}/M_i$.

For the notion of spread the reader can refer to subsection 6.3 of [TrW] (or [EGA] [4] for details). We choose a finitely generated $k$-algebra $A \subset k$ such that $(A, I_A, A)$, $(A, S_A, IS_A)$, $(A, X_A, O_{X_A}(1))$ and $(A, X_A, O_{X_A}(1))$ are spreads for $(R, I)$, $(S, IS)$, $(X, O_X(1))$ and $(X, V_0)$, respectively.

Restricting to the fiber $X_s$, where $s \in \text{Spec } A$ is a closed point, we have the following exact sequence of locally free sheaves of $O_{X_s}$-modules (where $X_s = X_A \otimes_A k(s)$ and $V_0^s = V_{0A} \otimes_A k(s)$). Let $p_s = \text{char } k(s)$

\[(4.7) \quad 0 \to V_0^s \to \oplus_{i=1}^n O_{X_s}(1-d_i) \to O_{X_s}(1) \to 0.\]

Since $V_i = \ker(V_0 \to M_0/M_i)$, the sheaf $V_i$ is the kernel of the map $V_0 \to M_{0A}/M_{iA}$ and hence $V_i^s := V_i \otimes_A k(s) = (V_0^s)_{s} = V_0^s \cap (M_i)_{s}$, that is
denotes the reduction mod $p_s$ of $V_i = (\text{the reduction mod } p_s \text{ of } V_0) \cap M_i$.

As a consequence of the openness of the semistability property of sheaves ([Ma]), we can further choose $A$ such that the spread of the HN filtration of $V_0$ can be defined similarly. In particular, there are spreads $(A, W_i)$ of $V_i$ such that for every $s \in \text{Spec } A$, the HN filtration of $V^s$ is

\[\cdots \subset W_{i+1}^s \subset W_i^s \subset V_{i}^s \subset \cdots \subset V_0^s = V^s\]

and therefore the bundle $V_0^s$ has the $\mu$-reduction bundle $V_t^s$, where $t$ is independent of the point $s$ and where the underlying sequence is

\[0 \to V_0^s = V^s \to M_0^s = M^s \to O_{X_s}(1) \to 0.\]

We recall the following result (Lemma 1.8 and Lemma 1.16 from [T1]).

**Theorem 4.10.** If $W$ is a vector bundle on a nonsingular projective curve $X$ over a field of char $0$. Then there is a spread $(A, X_A, W_A)$ of $(X, W)$ such that if $s$ is a closed point in Spec $A$ and $p_s > 4(\text{genus } X)(\text{rank } W)^3$ then

1. for every $m \geq 1$, the HN filtration of $F^m(W^s)$ is a refinement of the $m$th Frobenius pull back of the HN filtration of $W^s$. This means, if the HN filtration of $W^s$ is $0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset W^s$ then the HN filtration of $F^m(W^s)$ is of the form

\[0 \subset E_{01} \subset \cdots \subset E_{0t_0} \subset F^m E_1 \subset \cdots \subset F^m E_i \subset E_{i_1} \subset \cdots \subset E_{it_i} \subset F^m E_{i+1} \subset \cdots \subset F^m W^s. \]

In particular, for each $i$, the HN filtration of $F^m(E_{i+1}/E_i)$ is

\[0 \subset E_{i_1}/F^m E_i \subset \cdots \subset E_{it_i}/F^m E_i \subset F^m(E_{i+1}/E_i).\]
On the other hand

\[ \lim_{p_s \to \infty} a_{min}(W^s) = \mu_{min}(W). \]

Now we proceed to give a well defined notion of \( \alpha(R, I) \) in characteristic 0.

**Lemma 4.11.** We have a spread \( A \) such that

1. \( \mu_{min}(V_{i-1}) < \mu_{min}(V_i) \) \( \implies \) \( \alpha(R_s, I_s) = 1 - a_{min}(V^s_i)/d, \forall s \in \text{maxSpec}(A), \)

2. \( \mu_{min}(V_{i-1}) = \mu_{min}(V_i) \) \( \implies \) \( \alpha(R_s, I_s) = 1 - a_{min}(V^s_{i-1})/\mu_{max}(V_{i-1}), \forall s \in \text{maxSpec}(A), \)

where \( V^s_i \) is a reduction mod \( p_s \) of \( V_i \).

**Proof.** We choose a spread \( A \) as in Notations 4.9 such that \( p_s > 4(\text{genus } X)(\text{rank } V_0)^3 \), for every \( s \in \text{maxSpec } A \).

Recall that \( W_t, V_t \subseteq V_{t-1} \).

We fix a closed point \( s \in \text{Spec } A \) and let \( F : X_s \to X_s \) denote the Frobenius map. Let \( m_1 (m_1 \text{ may depend on } s) \) be an integer such that both \( F^{m_1}(V^s_t) \) and \( F^{m_1}(V^s_{t-1}) \) have strong HN filtration. Let \( V^s_t \) be the strong \( \mu \)-reduction bundle of \( V^s_0 \). This means \( F^{m_1}(V^s_0) \) is the \( \mu \)-reduction bundle of \( F^{m_1}(V^s_0) \) and \( t_0 \leq t \).

Case (1) Let \( \mu_{min}(V_{i-1}) < \mu_{min}(V_i) \).

Then \( V_t = W_t \) and the HN filtration of \( V_0 \) is \( \cdots \subset W_{i+1} \subset V_t \subset V_{i-1} \subset \cdots V_0 \).

Now by Theorem 4.10 (1), the HN filtration for \( F^{m_1}(V^s_0) \) is

\[ \cdots \subset F^{m_1}(W^s_{i+1}) \subset \cdots \subset F^{m_1}(V^s_t) \subset F^{m_1}(V^s_{t-1}) \subset \cdots \subset F^{m_1}(V^s_0), \]

as \( V^s_t/V^s_{i-1} \cong M^s_i/M^s_{i-1} \) is strongly semistable on \( X_s \), for \( i < t \).

Hence

\[ \mu_{min}(F^{m_1}(V^s_t)) = \mu_{min}(F^{m_1}(M^s_t)), \text{ for all } i < t. \]

In particular \( t_0 \geq t \) and therefore \( a_{min}(V^s_{t_0}) = a_{min}(V^s_t) \) which implies \( \alpha(R, I) = 1 - a_{min}(V^s_t)/t. \)

Case (2) Let \( \mu_{min}(V_{i-1}) = \mu_{min}(V_i) \).

Then the HN filtration for \( F^{m_1}(V^s_0) \) is

\[ F^{m_1}(V^s_t) \subset \cdots \subset F^{m_1}(V^s_{t-1}) \subset F^{m_1}(V^s_{t-2}) \subset \cdots \subset F^{m_1}(V^s_0). \]

Therefore, we have

\[ \mu_{min}(F^{m_1}(V^s_t)) = \mu_{min}(F^{m_1}(M^s_t)), \text{ for all } i < t - 1. \]

Hence, \( t_0 = t - 1 \) or \( t_0 = t \).

If \( t_0 = t - 1 \) then \( \alpha(R_s, I_s) = 1 - a_{min}(V^s_{t-1})/d. \)

If \( t_0 = t \) then \( \mu_{min}(F^{m_1}(V^s_t)) \geq \mu_{min}(F^{m_1}(V^s_{t-1})) \) and

\[ \mu_{min}(F^{m_1}(V^s_{t-1})) = \mu_{min}(F^{m_1}(M^s_{t-1})) = p^m \mu_{min}(M_{t-1}) = p^m \mu_{min}(V^s_{t-1}). \]

On the other hand \( p^m \mu_{min}(V^s_{t-1}) = p^m \mu_{min}(V^s_t) \geq \mu_{min}(F^{m_1}(V^s_t)). \) Hence

\[ a_{min}(V^s_t) = a_{min}(V^s_{t-1}) \implies \alpha(R_s, I_s) = 1 - a_{min}(V^s_t)/d = 1 - a_{min}(V^s_{t-1})/d. \]

\[ \square \]

**Theorem 4.12.** Let \( (R, I) \) be a standard graded pair defined over a field of characteristic 0 with a spread \( (A, R_A, I_A) \) as in Notations 4.9. Let \( V_t \) be the \( \mu \)-reduction bundle of \( V_0 \). Let \( s \in \text{Spec}(A) \) denote a closed point and \( p_s = \text{char } R_s \). Then

1. for every \( x \geq 0 \), \( f_{R, I}^\infty(x) := \lim_{p_s \to \infty} f_{R_s, I_s}(x) \) exists and the function \( f_{R, I}^\infty : [0, \infty) \to [0, \infty) \) is a continuous compactly supported function such that

\[ \alpha^\infty(R, I) := \text{Sup } \{ x \mid f_{R, I}^\infty(x) \neq 0 \} = 1 - \frac{\mu_{min}(V^s_t)}{d}. \]
(2) \( \lim_{p_s \to \infty} \alpha(R_s, I_s) = \alpha^\infty(R, I) \).

Proof. (1) By [2.4], we have \( f_{R_s, I_s}(x) = f_{V^*, \mathcal{O}_X(1)}(x) - f_{M^*, \mathcal{O}_X(1)}(x) \). On the other hand, for a vector bundle \( E \) on \( X \) there is a spread \((A, E_A)\) such that
\[
f_{E, \mathcal{O}_X(1)}(x) := \lim_{p_s \to \infty} f_{E^*, \mathcal{O}_X(1)}(x)
\]
events, where the function \( f_{E, \mathcal{O}_X(1)} \) can be written in terms of HN data of \( E \) (see Remark 6.6 of [TrW]). In particular we have a well defined function
\[
f^\infty(R, I)(x) := \lim_{p_s \to \infty} f_{R_s, I_s}(x) = f^\infty_{V^*, \mathcal{O}_X(1)}(x) - f^\infty_{M^*, \mathcal{O}_X(1)}(x),
\]
where the functions \( f^\infty_{V^*, \mathcal{O}_X(1)} \) and \( f^\infty_{M^*, \mathcal{O}_X(1)} \) can be written in terms of their respective HN data. Moreover, if \( \{(r_1, \ldots, r_{k+1})\} \) is the HN data for \( V_t \) then
\[
f^\infty(R, I)(x) = -r_{k+1}(\mu_{k+1} + xd) \quad \text{for} \quad x \in (-\min(\mu_k, \mu_{\text{min}}(M_t))/d, -\mu_{\text{min}}(V_t)/d)
\]
and \( f^\infty(R, I)(x) = 0 \), for \( x \in [-\mu_{\text{min}}(V_t)/d, \infty) \).

Hence \( f^\infty_{R_t, I_t} : [0, \infty) \to [0, \infty) \) is a compactly supported continuous function and \( \alpha^\infty(R, I) = 1 - \mu_{\text{min}}(V_t)/d \).

(2) By Theorem 4.10 (2) \( \lim_{p_s \to \infty} a_{\text{min}}(V^*_t) = \mu_{\text{min}}(V_t) \). If \( \mu_{\text{min}}(V_{t-1}) < \mu_{\text{min}}(V_t) \) then, by Lemma 4.11
\[
\lim_{p_s \to \infty} \alpha(R_s, I_s) = \lim_{p_s \to \infty} 1 - a_{\text{min}}(V^*_t)/d = 1 - \mu_{\text{min}}(V_t)/d.
\]
If \( \mu_{\text{min}}(V_{t-1}) = \mu_{\text{min}}(V_t) \) then
\[
\lim_{p_s \to \infty} \alpha(R_s, I_s) = 1 - \mu_{\text{min}}(V_{t-1})/d = 1 - \mu_{\text{min}}(V_{t-1})/d = 1 - \mu_{\text{min}}(V_t)/d.
\]

\( \square \)

Proof of Theorem D : By Theorem 4.8 for \( p_s > 0 \) we have \( \alpha(R_s, I_s) = c^s(\mathfrak{m}_s) \), hence assertion (1) follows from Theorem 4.8.

(2) If \( V_t \) is the \( \mu \)-reduction bundle of \( V_0 \) then \( \mu_{\text{min}}(V_t) \geq \mu_{\text{min}}(V_{t-1}) \).

If \( \mu_{\text{min}}(V_t) > \mu_{\text{min}}(V_{t-1}) \) then
\[
c^I(\mathfrak{m}) = 1 - \mu_{\text{min}}(V_t)/d \leq 1 - a_{\text{min}}(V^*_t)/d = c^s(\mathfrak{m}_s).
\]

If \( \mu_{\text{min}}(V_t) = \mu_{\text{min}}(V_{t-1}) \) then
\[
c^I(\mathfrak{m}) = 1 - \mu_{\text{min}}(V_{t-1})/d = 1 - \mu_{\text{min}}(V^*_t)/d \leq 1 - a_{\text{min}}(V^*_t)/d = c^s(\mathfrak{m}_s).
\]

(3) Suppose \( V_0 \) is semistable. Let \( V_t \) be the \( \mu \)-reduction bundle of \( V_0 \).

Case (1) If \( t = 0 \). Then \( t_0 = 0 \). Hence \( c^I(\mathfrak{m}) = 1 - \mu(V_0)/d \) and \( c^s(\mathfrak{m}_s) = 1 - a_{\text{min}}(V^*_0)/d \).

Case (2) If \( t \geq 1 \). Then \( 0 \neq V_t \subseteq V_{t-1} \) and, by Lemma 3.6, the HN filtration of \( V_0 \) is \( 0 \subseteq W_t \subseteq V_{t-1} \subseteq V_0 \). Hence \( V_{t-1} = V_0 \) and \( W_t = 0 \). So \( V_t \) is the \( \mu \)-reduction bundle of \( V_0 \) such that \( \mu_{\text{min}}(V_t) = \mu_{\text{min}}(V_0) = \mu(V_0) \). Hence again \( c^I(\mathfrak{m}) = 1 - \mu(V_0)/d \) and \( c^s(\mathfrak{m}_s) = 1 - a_{\text{min}}(V^*_0)/d \). Therefore \( c^I(\mathfrak{m}) = c^s(\mathfrak{m}_s) \iff \mu(V_0) = a_{\text{min}}(V^*_0) \iff V^*_0 \) is strongly semistable. \( \square \)
5. F-THRESHOLDS AND REDUCTION MOD p

Lemma 5.1. Let $V$ be a vector bundle of rank $r$ on a nonsingular projective curve $X$ of genus $g$ over a field of char $p > 0$. If $p > \max\{4(g-1)r^3, r!\}$ then

\[ a_{\min}(V) < \mu_{\min}(V) \implies \mu_{\min}(V) = a_{\min}(V) + a/pb, \]

where $a, b$ are positive integers such that $g.c.d.(a, p) = 1$ and $0 < a/b \leq 4(g-1)(r-1)$.

Proof. Let $m$ be an integer such that $F^{m}(V)$ achieves the strong HN filtration. Since $V$ is not strongly semistable, the integer $m \geq 1$. By definition $a_{\min}(V) = \mu_{\min}(F^{m}V)/p^{m}$. By Lemma 1.14 of [TrW],

\[ \mu_{\min}(F^{m}V)/p^{m} + C/p = \mu_{\min}(V), \]

where $0 < C \leq 4(g-1)(r-1)$.

Note that $\mu_{\min}(F^{m}V)$ and $\mu_{\min}(V) \in \mathbb{Z}[1/r!]$. This implies $Cp^{m-1}(r!) \in \mathbb{N}$ and we can write

\[ \mu_{\min}(V) = a_{\min}(V) + \frac{Cp^{m-1}(r!)}{p^{m}(r!)} = a_{\min}(V) + \frac{a}{pb}, \]

where $a, b$ are positive integers such that $g.c.d.(a, p) = 1$. This proves the lemma. \qed

Proof of Theorem E: By Theorem D, if $c^{p}_{I}(m_{p}) \neq c^{\infty}_{I}(m)$ then $c^{p}_{I}(m_{p}) > c^{\infty}_{I}(m)$.

Let $X = \text{Proj} \ S$ where $R \to S$ is the normalization of $R$. By Lemma 4.11, there is a vector bundle $W$ ($W = V_{l}$ or $V_{l-1}$, where $V_{l}$, $V_{l-1}$ are the bundles given as in Notation [4.9] on $X$ such that, for $p_{s} \gg 0$,

\[ c_{s}(m_{s}) = 1 - \mu_{\min}(W)/d \quad \text{and} \quad c_{s}(m_{s}) = 1 - a_{\min}(W^{s})/d \quad \text{and} \quad \mu_{\min}(W^{s}) = \mu_{\min}(W), \]

where $W_{s}$ denotes the reduction mod $p_{s}$ bundle of $W$ on $X_{s}$ and $d = \deg \ X$.

Therefore, by Lemma 5.1 if $g$ denotes the genus of $X$ and $r + 1$ is the number of minimal generator of $I$, then we can write

\[ c^{s}_{I}(m_{s}) = 1 - \frac{\mu_{\min}(W^{s})}{d} + \frac{a}{p_{s}b}, \]

where $a, b \in \mathbb{Z}_{+}$ and $0 < a/b \leq 4(g-1)(r-1)$.

Since $\mu_{\min}(W^{s}) = d_{1}/r_{1}$, where $d_{1}, r_{1} \in \mathbb{Z}_{+}$ such that $r_{1} \leq r$, the theorem follows for $p_{s} \gg 0$. \qed

Remark 5.2. By the above theorem, if $c^{m}_{p}(m_{p}) \neq c^{\infty}_{m}(m)$ then $p$ divides the denominator of $c^{m}_{\infty}(m_{p})$. However, the following example from [TrW] (Example 6.9) shows that the denominator need not always be a power of $p$. 

Let $R_{p} = k[x, y, z]/(h)$ be the Klein curve of degree $d \geq 17$ over a field of characteristic $p \geq d^{2}$. In other words $h = x^{d-1}y + y^{d-1}z + z^{d-1}x$. If, in addition, $d$ is odd integer and $p \equiv \pm 2 \pmod{(d^{2} - 3d + 3)}$ then we know (loc. cit.)

\[ c^{m}_{p}(m_{p}) = (3pd + d^{2} - 9d + 15)/2pd \quad \text{and} \quad c^{m}_{\infty}(m) = 3/2. \]

References

[BS] Bhargav, B., Singh, A., The F-pure threshold of a Calabi-Yau hypersurface, Math. Ann. (2015) 362, 551-567.

[CHSW] Canton, E., Hernández, D., Schwede, K., Witt, E., On the behavior of singularities at the F-pure threshold, Illinois J. Math. 60 (2016), no. 3-4, 669-685.

[DSnP] Stefani, A., Núñez-Betancourt, L., Pérez, F., On the existence of F-thresholds and related limits, Trans. Amer. Math. Soc. 370 (2018), no. 9, 6629-6650.

[EGA IV] Grothendieck, A., Dieudonné, J.A., Eléments de Géométrie Algébrique IV, Pub. Math. IHÉS.
[HY] Hara, N., Yoshida, K., *A generalization of tight closure and multiplier ideals*, Trans. Am. Math. Soc. 355, 3143-3174 (2003).

[HMTW] Huneke, C., Mustaţă, M., Takagi, S., Watanabe, K.I., *F-thresholds, tight closure, integral closure and multiplicity bounds*, Michigan Math. J. 57, in Special Volume in Honor of Melvin Hochster, Univ. Michigan Press, Ann Arbor, (2008), 463-483.

[L] Langer, A., *Semistable sheaves in positive characteristic*, Ann. Math., 159 (2004).

[Ma] Maruyama, M., *Openness of a family of torsion free sheaves*, J. Math. Kyoto Univ., 16-3 (1976), 627-637.

[M] P. Monsky, *The Hilbert-Kunz function*, Math. Ann. 263 (1983) 43-49.

[MTW] Mustaţă, M., Takagi, S., Watanabe, K.I., *F-thresholds and Bernstein-Sato polynomials*, European congress of mathematics, 341-364, Eur. Math. Soc., Zurich, 2005.

[S] Smith, K., *Tight closure in graded rings*, J. Math. Kyoto Univ. 37 (1997), no. 1, 35-53.

[T1] Trivedi, V., *Hilbert-Kunz multiplicity and reduction mod p*, Nagoya Math. Journal 185 (2007), 123-141.

[T2] Trivedi, V., *Hilbert-Kunz density Function and Hilbert-Kunz multiplicity*, Trans. Amer. Math. Soc. 370 (2018), no. 12, 8403-8428.

[TrW] Trivedi, V., Watanabe, K.I., *Hilbert-Kunz density functions and F-thresholds*, arXiv:1808.04093v2 [math.AC].

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai-400005, India

E-mail address: vija@math.tifr.res.in