Four-Gauge-Particle Scattering Amplitudes and Polyakov String Path Integral in the proper-time gauge

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We evaluate four-gauge-particle scattering amplitudes using the Polyakov string path integral in the proper-time gauge, where the string path integral can be cast into the Feynman-Schwinger proper-time representation. We compare the resultant scattering amplitudes, which include $\alpha'$-corrections, with the conventional ones that may be obtained by substituting local vertex operators for the external string states. In the zero-slope limit, both amplitudes are reduced to the four-gauge-particle scattering amplitude of non-Abelian Yang-Mills gauge theory. However, in the high energy region where the string corrections become relevant, the scattering amplitude in the proper-time gauge differs from the conventional one: The Polyakov string path integral in the proper-time gauge, equivalent to the deformed cubic string field theory, systematically provides the alpha prime corrections. In addition, we find that the scattering amplitude in the proper-time gauge contains tachyon poles in a manner consistent with three-particle-scattering amplitudes. The scattering amplitudes evaluated using the Polyakov string path integral in the proper-time gauge may be more suitable than conventional ones for exploring string corrections to the quantum field theories and high energy behaviors of open string.

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I. INTRODUCTION

Scattering amplitude, which has yielded many important new discoveries, has long been one of the most ubiquitous tools in both experimental and theoretical physics. In string theory, theoretical studies of string scattering amplitudes are also expected to lead to new findings in high energy physics, where string theory is considered the most promising candidate for a unified framework of the fundamental forces, including gravity. In some scenarios [1, 2] involving embedding the standard model in the framework of string theory, the string scale may be as low as the weak scale. These theoretical proposals allow for new possibilities that we may directly study string physics at high energy colliders. Therefore, it becomes an important and urgent task to accurately calculate the multi-particle scattering amplitudes in string theory.

Conventionally, the string scattering amplitudes are calculated by substituting the local vertex operators for external string states. This procedure is based on the one-to-one correspondence between string states and local operators [3]. This method of calculation for string scattering amplitude has proven successful for producing the four-tachyon scattering amplitudes known as the Veneziano amplitude [4] for open string and the Virasoro-Shapiro amplitude [5, 6] for closed string. However, it is not clear whether we can apply this vertex operator technique to more general cases, such as evaluating high energy scattering amplitudes or alpha-prime corrections. In fact, the validity of this procedure has never been examined thoroughly.

An alternative method to evaluate the string scattering amplitudes, which does not make use of vertex operators, is the Polyakov string path integral [7]. By evaluating the Polyakov string path integral defined on the string worldsheet with appropriately chosen boundary conditions for the external string states, we may obtain the string scattering amplitudes. Although this technique is more complicated than the vertex operator technique, it offers a number of advantages: The external string states do not need to be on-shell and the momenta of external strings may not be restricted to the low energy region. If we choose the proper-time gauge [5] in fixing the reparametrization invariance on the string world-sheet covariantly, we can cast string scattering amplitudes into those of second quantized theory in a fashion similar to the Feynman-Schwinger representation of quantum field theory. The string field theory [9, 10] defined by the Polyakov string path integral in the proper-time gauge has been shown to be equivalent to the deformed cubic string field theory [11, 12].

As for the four-tachyon-scattering amplitudes, both methods yield the same result: The well-known Veneziano amplitude and the Virasoro-Shapiro amplitude. However, a recent work [12] has pointed out that the two methods

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may produce different results if we apply them to more general external string states. A simple extension of the Veneziano amplitude is the scattering amplitude of three tachyons and one arbitrary string state, which has been studied extensively in Refs. [15–17, 31, 35] to explore symmetric properties of the string scattering amplitudes in the high energy limit [18–22]. When calculating the string scattering amplitude defined by the Polyakov string path integral, we must map the string world-sheet onto upper half complex plane by the Schwarz-Christoffel transformation. On the other hand, if we adopt the conventional vertex operator technique, the string scattering amplitude is readily defined on upper-half complex plane. It follows then that the two methods may yield the same result only when the scattering amplitude is invariant under the conformal transformation generated by the Schwarz-Christoffel mapping.

In this present work, we shall study the four-gauge-particle scattering amplitude in bosonic open string theory explicitly evaluating the Polyakov string path integral in the proper-time gauge and compare the resultant scattering amplitude with the conventional expression obtained by the vertex operator technique. We will examine their high energy behaviors and the difference of the singularity structures.

\[ \text{FIG. 1: Four-open string scattering amplitudes. Two external strings are replaced by local vertex operators.} \]

**II. OPEN STRING FIELDS ON MULTIPLE SPACE-FILLING BRANES**

The multiple string scattering amplitude may be written in terms of the Polyakov string path integral defined on the corresponding string world-sheet [23]

\[ A_{[N]} = \int D[X] D[h] \exp \left( iS + i \int_{\partial M} \sum_{r=1}^{N} P^{(r)} \cdot X^{(r)} d\sigma \right), \]  
\[ S = -\frac{1}{4\pi} \int_{M} d\tau d\sigma \sqrt{-h} h^{IJ} \frac{\partial X^I}{\partial \sigma^I} \frac{\partial X^J}{\partial \sigma^J} \eta_{IJ}, \quad I, J = 0, \ldots, d - 1 \]  

where \( \sigma^1 = \tau, \sigma^2 = \sigma \) and \( d = 26 \) for open bosonic string and \( d = 10 \) for open super-string. On a space-filling brane, the string coordinates \( X^I \), satisfying the Neumann boundary condition

\[ \frac{\partial X^I}{\partial \sigma} \bigg|_{\sigma = 0, \pi} = 0, \quad \text{for } I = 0, 1, \ldots, d - 1, \]  

may be expanded in terms of normal modes as

\[ X^I(\sigma) = x^\mu + 2 \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} x_n^\mu \cos(n\sigma), \quad I = 0, 1, \ldots, d - 1. \]  

If we choose the proper-time gauge where the proper-time on the string world-sheet is defined properly [8],

\[ \partial_\tau N_{10} = 0, \quad N_{1n} = 0, \quad N_{2n} = 0, \quad n \neq 0, \]
we may recast the string scattering amplitudes $A_{[N]}$ into the Feynman-Schwinger proper-time representaion, which may help obtain a covariant second quantized string theory. Here $N_{\alpha n}$, $\alpha = 1, 2$ are normal modes of the lapse and shift functions $N_{\alpha} = \sum_n N_{\alpha n} e^{i n \sigma}$ of the two-dimensional metric on the world-sheet

$$\sqrt{-h} h^{\alpha \beta} = \frac{1}{N_1} \left( \frac{-1}{N_2} \left( \frac{N_3}{(N_1)^2 - (N_2)^2} \right) \right).$$

(5)

Evaluating $A_{[2]}$ defined as the Polyakov path integral over a strip, we can obtain the covariant free string propagator of the open string. The Polyakov string path integral, $A_{[3]}$ in the proper-time gauge has been calculated in Refs. \[9\] \[11\] and the three-string scattering amplitude $I_{[3]}$ has been found to be

$$I_{[3]} = \frac{2g}{3} \int \prod_{r=1}^{3} dp^{(r)} \delta \left( \sum_{r=1}^{3} p^{(r)} \right) \text{tr} \left( \Psi^{(1)}(\bar{\Psi}^{(2)}, \Psi^{(3)}) \exp \{ E_{[3][1,2,3]} \} |0; p\right);$$

(6a)

$$E[1,2,3] = \frac{1}{2} \sum_{n,m=1}^{\infty} \sum_{r,s=1}^{3} \bar{N}_{nm}^{rs} \alpha_{-n}^{(r)} \cdot \alpha_{-m}^{(s)} + \frac{1}{2} \sum_{n=1}^{3} \bar{N}_{n}^{r} \alpha_{-n}^{(r)} \cdot \alpha_{-n}^{(r)} + \frac{1}{2} \sum_{r=1}^{3} \frac{1}{\alpha_r} \left( \frac{(p^{(r)})^2}{2} - 1 \right),$$

(6b)

where $P = p^{(2)} - p^{(1)}$ and $|\Psi^{(r)}\rangle$, $r = 1, 2, 3$ denote the external string states which carry $U(N)$ group indices. For the proper-time gauge,

$$\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = -2, \text{ and } \tau_0 = -2 \ln 2.$$

(7)

Explicit expressions of the Neumann functions for the three-string scattering, $\bar{N}_{nm}^{rs}$ and $\bar{N}_{n}^{r}$ can be found in Ref. \[9\]:

$$\bar{N}_{11}^{11} = \frac{1}{2}, \quad \bar{N}_{11}^{22} = \frac{1}{2}, \quad \bar{N}_{11}^{33} = 2^2,$$

(8a)

$$\bar{N}_{1}^{11} = \bar{N}_{1}^{22} = \frac{1}{2}, \quad \bar{N}_{1}^{33} = 1, \quad \bar{N}_{1}^{31} = \frac{1}{2}, \quad \bar{N}_{1}^{12} = \bar{N}_{1}^{21} = \frac{1}{2},$$

(8b)

$$\bar{N}_{1}^{1} = \bar{N}_{1}^{2} = \frac{1}{4}, \quad \bar{N}_{1}^{3} = -1.$$

(8c)

If we expand the external string states in terms of mass eigen-states,$$

|\Psi^{(r)}\rangle = \phi(p^{(r)})|0\rangle + A_{\mu}(p^{(r)}) a_{\mu}^{(r)}|0\rangle + \cdots,$$

(9)

we may obtain various three-particle interaction terms. Fig. 2 depicts an expansion of three-string scattering into those of various three-particle scatterings.

![Three-string scattering](image)

FIG. 2: Three-open string scattering and three-particle scattering amplitudes.

By choosing $\prod_{r} \phi(p^{(r)})|0; p\rangle$ as the external three-string state, we obtain the three-tachyon interaction from Eq. (6a)

$$I_{\phi\phi\phi} = \frac{2g}{3} \int \prod_{r=1}^{3} dp^{(r)} \delta \left( \sum_{r=1}^{3} p^{(r)} \right) \text{tr} \phi(p^{(1)}) \phi(p^{(2)}) \phi(p^{(3)}).$$

(10)
If we are interested in the three-particle interaction terms between tachyon and gauge particle, we may choose the external three-string external state as follows

$$\langle \Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)} \rangle = \langle 0 \prod_r (\phi(r) + A(r)) \rangle$$  \hspace{1cm} (11)$$

where $\phi(r) = \phi(p^{(r)})$, $A(r) = A_{\mu}(p^{(r)})a^{(r)\mu}_1$, $r = 1, 2, 3$. Expanding the external string state in terms of component fields, from Eq. [6a] we may get three-particle interactions between tachyons and gauge particles

$$I_{\phi} = \frac{2g}{3} \int \prod_{r=1}^3 dp^{(r)} \delta \left( \sum_{r=1}^3 p^{(r)} \right) \text{tr} \left( 0 \sum_{r=1}^3 (\phi(1)\phi(2)A(3) + \phi(1)A(2)\phi(3) + A(1)\phi(2)\phi(3)) \right)$$

$$\exp \left( \tau_0 \sum_{r=1}^3 \frac{1}{2r} \left( \frac{(p^{(r)})^2}{r} - 1 \right) \right) \left( \sum_{r=1}^3 N_{1r}^r a^{(r)\dagger}_1 \cdot P \right) \right) |0\rangle,$$  \hspace{1cm} (12a)$$

$$I_{AA} = \frac{2g}{3} \int \prod_{r=1}^3 dp^{(r)} \delta \left( \sum_{r=1}^3 p^{(r)} \right) \text{tr} \left( 0 \sum_{r=1}^3 A(1)A(2)A(3) e^{\tau_0 \sum_{r=1}^3 \frac{1}{2r} \left( \frac{(p^{(r)})^2}{r} - 1 \right)} \left( \sum_{r=1}^3 N_{1r}^r a^{(r)\dagger}_1 \cdot P \right) \right) |0\rangle.$$  \hspace{1cm} (12b)$$

$$I_{AAA} = \frac{2g}{3} \int \prod_{r=1}^3 dp^{(r)} \delta \left( \sum_{r=1}^3 p^{(r)} \right) \text{tr} \left( 0 \sum_{r=1}^3 (\phi(1)\phi(2)A(3) + 4\phi(1)A(2)\phi(3) + 4A(1)\phi(2)\phi(3)) \right)$$

$$\exp \left( \tau_0 \sum_{r=1}^3 \frac{1}{2r} \left( \frac{(p^{(r)})^2}{r} - 1 \right) \right) \left( \sum_{r=1}^3 N_{1r}^r a^{(r)\dagger}_1 \cdot P \right) \right) |0\rangle.$$  \hspace{1cm} (12c)$$

The second term in $I_{\phi AA}$ and the second term in $I_{AAA}$ correspond to alpha-prime corrections to three-particle interactions. Here, we note that there exists a three-particle interaction of two gauge particles and one tachyon, which may generate a four-gauge particle scattering mediated by a tachyon.

Using algebra we obtain

$$I_{\phi A} = \frac{2g}{3} \int \prod_{r=1}^3 dp^{(r)} \delta \left( \sum_{r=1}^3 p^{(r)} \right) \text{tr} \left( 0 \frac{1}{2} (\phi(1)\phi(2)A(3) + 4\phi(1)A(2)\phi(3) + 4A(1)\phi(2)\phi(3)) \right)$$

$$\exp \left( \tau_0 \sum_{r=1}^3 \frac{1}{2r} \left( \frac{(p^{(r)})^2}{r} - 1 \right) \right) \left( \sum_{r=1}^3 N_{1r}^r a^{(r)\dagger}_1 \cdot P \right) \right) |0\rangle$$

$$= g \int \prod_{r=1}^3 dp^{(r)} \delta \left( \sum_{r=1}^3 p^{(r)} \right) \text{tr} \left\{ A(1) \cdot (p^{(2)} - p^{(3)})\phi(2)\phi(3) \right\}$$

$$= g \int \prod_{r=1}^3 dp^{(r)} \delta \left( \sum_{r=1}^3 p^{(r)} \right) \text{tr} P_{\mu}^{(1)}(\phi(p(1)) A^{\mu}(p^{(2)}), \phi(p^{(3)})) \right\}. \hspace{1cm} (13)$$

In configuration space, it contributes to the action as

$$I_{\phi A} = -gi \int d^d x \text{tr} \partial_{\mu} \phi [A^{\mu}, \phi]. \hspace{1cm} (14)$$
Similarly, we have

\[ I_{\phi AA} = \frac{2g}{3} \int \prod_{r=1}^{3} dp^{(r)} \delta \left( \sum_{r=1}^{3} p^{(r)} \right) \text{tr} \langle 0 | \left( 2\phi(1)A(2)A(3) + 2A(1)\phi(2)A(3) + 2A(1)A(2)\phi(3) \right) \right] \]

\[ \left\{ \frac{1}{2} \sum_{r,s=1}^{3} \vec{N}_{rs} a_{1}^{(r)t} \cdot a_{1}^{(s)t} + \frac{1}{24} \left( \sum_{r=1}^{3} \vec{N}_{r} a_{1}^{(r)t} \cdot P \right)^{2} \right\} |0) \]

\[ = \frac{2g}{3} \int \prod_{r=1}^{3} dp^{(r)} \delta \left( \sum_{r=1}^{3} p^{(r)} \right) \text{tr} \left\{ \frac{3}{2} \phi(1)A(2) \cdot A(3) - \frac{3}{2} \phi(1)A(2) \cdot p^{(3)}A(3) \cdot p^{(2)} \right\} \]

\[ = g \int d^{d}x \text{tr} \left( \phi A_{\mu} A^{\mu} + \phi \partial_{\mu} A^{\nu} \partial_{\nu} A^{\mu} \right). \] (15)

The three-gauge particle interaction may be written as

\[ I_{AAA} = I_{AAA}^{(0)} + I_{AAA}^{(1)}, \] (16a)

\[ I_{AAA}^{(0)} = \frac{2g}{3} \int \prod_{r=1}^{3} dp^{(r)} \delta \left( \sum_{r=1}^{3} p^{(r)} \right) \text{tr} \langle 0 | A(1)A(2)A(3) \right] \]

\[ \left\{ \frac{1}{2} \sum_{r,s=1}^{3} \vec{N}_{rs} a_{1}^{(r)t} \cdot a_{1}^{(s)t} \left( \sum_{r=1}^{3} \vec{N}_{r} a_{1}^{(r)t} \cdot P \right) \right\} |0), \] \] (16b)

\[ I_{AAA}^{(1)} = \frac{2g}{3} \int \prod_{r=1}^{3} dp^{(r)} \delta \left( \sum_{r=1}^{3} p^{(r)} \right) \text{tr} \langle 0 | A(1)A(2)A(3) \right] \]

\[ \left\{ \frac{1}{3} \left( \sum_{r=1}^{3} \vec{N}_{r} a_{1}^{(r)t} \cdot P \right)^{3} \right\} |0). \] \] \] (16c)

Using the Neumann functions for three-string scattering, we get

\[ I_{AAA}^{(0)} = g \int \prod_{i=1}^{3} dp^{(i)} \delta \left( \sum_{i=1}^{3} p^{(i)} \right) p_{\mu}^{(i)} \text{tr} \left( A^{\nu}(p_{1}) [A_{\nu}(p_{2}), A_{\mu}(p_{3})] \right) \] \] (17a)

\[ I_{AAA}^{(1)} = \frac{g}{3} \int \prod_{i=1}^{3} dp^{(i)} \delta \left( \sum_{i=1}^{3} p^{(i)} \right) \text{tr} \left\{ \left( A^{(1)} \cdot p^{(2)} \right) \left( A^{(2)} \cdot p^{(3)} \right) \left( A^{(3)} \cdot p^{(1)} \right) \right\} \]

\[ - \left( A^{(1)} \cdot p^{(3)} \right) \left( A^{(2)} \cdot p^{(1)} \right) \left( A^{(3)} \cdot p^{(2)} \right) \} \] \] (17b)

In configuration space they may be represented by three-gauge-field interaction terms of the \( U(N) \) non-Abelian group Yang-Mills gauge theory:

\[ I_{AAA}^{(0)} = g \int \text{tr} \left( \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \right) [A^{\mu}, A^{\nu}], \] \] (18a)

\[ I_{AAA}^{(1)} = \frac{g}{3} i \int \text{tr} \left( \partial_{\mu} A^{\nu} - \partial^{\nu} A_{\mu} \right) \left( \partial_{\nu} A^{\lambda} - \partial^{\lambda} A_{\nu} \right) \left( \partial_{\lambda} A^{\mu} - \partial^{\mu} A_{\lambda} \right). \] \] (18b)

It is worth mentioning that \( I_{AAA}^{(1)} \) is completely consistent with the \( \alpha' \)-correction to the three-gauge field interaction term, which has been obtained by previous approaches \[24][27].

## III. FOUR-GAUGE-PARTICLE SCATTERING AMPLITUDE ON SPACE-FILLING BRANES

The four-gauge-particle scattering amplitude has been discussed previously in Refs. \[9][10] in the framework of string field theory in the proper-time gauge. However, in the previous works, we only studied the four-gauge-particle...
scattering amplitude in the zero-slope limit. Here, we shall evaluate the amplitude without taking the limit so that the resultant amplitude is valid for the full range of the energy scale. By mapping the string world-sheet for the four-string scattering onto the upper half complex plane by the Schwarz-Christoffel function, we may find that the four-string scattering amplitude on multiple space-filling branes may be written at tree level \cite{9} as

$$I_{[4]} = 2g^2 \int \left[ \prod_{r=1}^{4} \frac{dZ_r}{dV_{abc}} \prod_{r<s} |Z_r - Z_s|^{p_r - p_s} \exp \left[ -\sum_{r=1}^{4} \tilde{N}^{[4]rr}_{00} \right] \right] \text{tr} \langle \Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)}, \Psi^{(4)} | \exp \left[ E_{[4]} \right] | 0 \rangle, \quad (19a)$$

$$E_{[4]} = \frac{1}{2} \sum_{r,s=1}^{4} \sum_{m,n>0} N^{[4]rs}_{mn} \alpha^{(r)}_{mn} \alpha^{(s)}_{nu} \eta^{\mu\nu} + \sum_{r,s=1}^{4} \sum_{n>0} N^{[4]rs}_{0n} \alpha^{(r)}_{n0} \eta^{\mu\nu} \quad (19b)$$

where $Z_r, r = 1, 2, 3, 4$ denote the Koba-Nelson variables, corresponding to the locations of four external strings on upper half complex plane,

$$Z_1 = 0, \quad Z_2 = x, \quad Z_3 = 1, \quad Z_4 = \infty. \quad (20)$$

In the proper-time gauge, we choose $\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = -1, \quad \alpha_4 = -1.$

In order to evaluate the four-gauge particle scattering amplitude, we choose the external string states as

$$\langle \Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)}, \Psi^{(4)} | = \langle 0 | \prod_{r=1}^{4} A_{\mu} (p^{(r)}) a^{(r)}_{1\mu} \eta^{\mu\nu} \rangle. \quad (21)$$

Expanding the four-string vertex operator $E_{[4]}$ in terms of oscillator operators, we may find that the four-gauge particle scattering amplitude is given by

$$I_{AAAA} = 2g^2 \int \left[ \prod_{r=1}^{4} \frac{dZ_r}{dV_{abc}} \prod_{r<s} |Z_r - Z_s|^{p_r - p_s} \exp \left[ -\sum_{r=1}^{4} \tilde{N}^{[4]rr}_{00} \right] \text{tr} \langle 0 | A(1)A(2)A(3)A(4) \right]$$

$$\left\{ \frac{1}{2!} \left( \frac{1}{2} \sum_{r,s=1}^{4} \tilde{N}^{[4]rs}_{11} a^{(r)\dagger}_{1} \cdot a^{(s)\dagger}_{1} \right)^2 + \left( \frac{1}{2} \sum_{r,s=1}^{4} \tilde{N}^{[4]rs}_{11} a^{(r)\dagger}_{1} \cdot a^{(s)\dagger}_{1} \right) \right\} \frac{1}{2!} \left( \frac{1}{2} \sum_{r,s=1}^{4} \tilde{N}^{[4]rs}_{10} a^{(r)\dagger}_{1} \cdot p^{(s)} \right)^2 \right\} | 0 \rangle. \quad (22)$$

If we explicitly reintroduce $\alpha'$, the momenta $p^{(r)}$ may be replaced by $\sqrt{\alpha'} p^{(r)}$, and this expansion can also be understood as a series expansion of the four-gauge particle scattering amplitude in powers of $\alpha'$. The third term in $I_{AAAA}$ yields the alpha prime correction, which would have been missed if we had employed the vertex operator technique. It may be convenient to separately calculate each of the three terms in $I_{AAAA}$.

The first term in $I_{AAAA}$ may be defined by

$$I_{AAAA}^{(0)} = 2g^2 \int \left[ \prod_{r=1}^{4} \frac{dZ_r}{dV_{abc}} \prod_{r<s} |Z_r - Z_s|^{p_r - p_s} \exp \left[ -\sum_{r=1}^{4} \tilde{N}^{[4]rr}_{00} \right] \text{tr} \langle 0 | A(1)A(2)A(3)A(4) \right]$$

$$\frac{1}{2!} \left( \frac{1}{2} \sum_{r,s=1}^{4} \tilde{N}^{[4]rs}_{11} a^{(r)\dagger}_{1} \cdot a^{(s)\dagger}_{1} \right)^2 \right\} | 0 \rangle. \quad (23)$$

Using the Neumann functions given in the Appendix, we may evaluate $I_{AAAA}^{(0)}$ as follows:

$$I_{AAAA}^{(0)} = \text{tr} \left\{ A(1)\mu_1 A(2)\mu_2 A(3)\mu_3 A(4)\mu_4 \right\} \frac{\Gamma \left( -\frac{1}{2} \right) \Gamma \left( -\frac{1}{2} \right) \Gamma \left( \frac{3}{2} \right)}{\Gamma \left( \frac{3}{2} + 1 \right)}$$

$$\left\{ \frac{tu}{4} s/2 + 1 \eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_4} + \frac{st}{4} u/2 + 1 \eta^{\mu_1 \mu_3} \eta^{\mu_2 \mu_4} + \frac{us}{4} \frac{1}{t/2 + 1} \eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_3} \right\} \quad (24)$$
where $A(p^{(r)})$, $r = 1, 2, 3, 4$ are abbreviated as $A^a(r)T^a$, $r = 1, 2, 3, 4$. It can be compared with the corresponding sub-amplitude \cite{28} obtained by using the vertex operator technique

$$I_{AAAA}^{\text{vertex}(0)} = \text{tr} \left\{ A(1)_{\mu_1} A(2)_{\mu_2} A(3)_{\mu_3} A(4)_{\mu_4} \right\} \frac{\Gamma(-\frac{t}{2}) \Gamma(-\frac{s}{2})}{\Gamma(\frac{u}{2} + 1)} \left\{ \frac{tu}{4} \eta^{\mu_1 \mu_2 \mu_3 \mu_4} + \frac{st}{4} \eta^{\mu_1 \mu_3 \mu_2 \mu_4} + \frac{us}{4} \eta^{\mu_1 \mu_4 \mu_2 \mu_3} \right\} . \tag{25}$$

In the zero slope limit, both scattering amplitudes reduce to the corresponding sub-amplitude of non-Abelian gauge theory

$$I_{AAAA}^{\text{YM}(0)} = \text{tr} \left\{ A(1)_{\mu_1} A(2)_{\mu_2} A(3)_{\mu_3} A(4)_{\mu_4} \right\} \frac{4}{st} \left\{ \frac{tu}{4} \eta^{\mu_1 \mu_2 \mu_3 \mu_4} + \frac{st}{4} \eta^{\mu_1 \mu_3 \mu_2 \mu_4} + \frac{us}{4} \eta^{\mu_1 \mu_4 \mu_2 \mu_3} \right\} . \tag{26}$$

However, we also notice the difference between two amplitudes: $I_{AAAA}^{\text{vertex}(0)}$ does not contain the tachyon poles whereas $I_{AAAA}^{(0)}$ has the tachyon poles in all three channels. As we have observed in the last section, the three-open-string vertex gives rise to various three-particle interaction terms, including a coupling between two-gauge field and one tachyon when external string states are expanded in terms of mass eigenstates. Thus, the four-string scattering amplitude which is generated by the three-string vertex should contain the tachyon poles. Fig. 4 depicts the difference between the sub-amplitudes $I_{AAAA}^{(0)}$ in $s$-channel at a fixed angle. Thanks to the tachyon pole, the sub-amplitude of four-gauge particle scattering amplitude in $s$-channel changes significantly.

The second term in $I_{AAAA}$ which is of order $\alpha'$ is written as

$$I_{AAAA}^{(1)} = 2g^2 \int \frac{dZ_r}{dV_{abc}} \prod_{r<s} |Z_r - Z_s|^{p_r \cdot p_s} \exp \left[ -\sum_{r=1}^{4} N^{[4]}_{00} \right] \text{tr} \langle 0| A(1) A(2) A(3) A(4) |0 \rangle . \tag{27}$$

Using the Neumann functions in the proper-time gauge given in the Appendix, we find that $I_{AAAA}^{(1)}$ may be expressed
as an integral over the real Koba-Nelson variable $x$,

\[
\mathcal{I}_{AAA}^{(1)} = \text{tr}\left\{ A_{\mu_1}(1)A_{\mu_2}(2)A_{\mu_3}(3)A_{\mu_4}(4) \right\} \int_0^1 dx \, x^{-\frac{5}{2}} (1-x)^{-\frac{5}{2}}
\]

\[
\begin{align*}
&\left\{ -\eta^{\mu_1\mu_2} \frac{1}{4} \frac{1}{(1-x)x^2} \left( xp^{(1)\mu_3} + p^{(4)\mu_3} \right) \left( xp^{(2)\mu_4} + p^{(3)\mu_4} \right) \\
&+ \eta^{\mu_1\mu_3} \frac{1}{4} \frac{1}{(1-x)x} \left( p^{(1)\mu_2} + xp^{(4)\mu_2} \right) \left( xp^{(2)\mu_4} + p^{(3)\mu_4} \right) \\
&-\eta^{\mu_1\mu_4} \frac{1}{4} \frac{1}{(1-x)x^2} \left( (1-x)p^{(3)\mu_1} + p^{(4)\mu_1} \right) \left( (1-x)p^{(1)\mu_3} + p^{(2)\mu_3} \right) \\
&-\eta^{\mu_2\mu_3} \frac{1}{4} \frac{1}{(1-x)x^2} \left( (1-x)p^{(3)\mu_1} + p^{(4)\mu_1} \right) \left( p^{(1)\mu_3} + (1-x)p^{(2)\mu_3} \right) \\
&+ \eta^{\mu_2\mu_4} \frac{1}{4} \frac{1}{(1-x)x} \left( xp^{(3)\mu_1} + p^{(2)\mu_1} \right) \left( p^{(1)\mu_3} + p^{(4)\mu_3} \right) \\
&-\eta^{\mu_3\mu_4} \frac{1}{4} \frac{1}{(1-x)x^2} \left( xp^{(3)\mu_1} + p^{(2)\mu_1} \right) \left( p^{(1)\mu_2} + xp^{(4)\mu_2} \right) \right\}. 
\end{align*}
\]
Integrating out the Koba-Nielsen variable $x$ leads us to

\[
I_{AAAA}^{(1)} = \text{tr}\left\{ A_{\mu_1} (1) A_{\mu_2} (2) A_{\mu_3} (3) A_{\mu_4} (4) \right\} \frac{\Gamma \left( -\frac{\xi}{2} \right) \Gamma \left( -\frac{s}{2} \right)}{\Gamma \left( \frac{n}{2} + 1 \right) \Gamma \left( \frac{n}{2} + 1 \right)}
\]

\[
\left\{ -\frac{1}{4} \eta^{\mu_1 \mu_2} \left( -\frac{s}{2} p^{(1) \mu_2} p^{(2) \mu_4} + \frac{u}{2} p^{(1) \mu_2} p^{(3) \mu_4} + \frac{u}{2} p^{(4) \mu_2} p^{(2) \mu_4} - \frac{u (u - 2)}{4} \frac{1}{2} + 1 p^{(4) \mu_2} p^{(3) \mu_4} \right) \\
+ \frac{1}{4} \eta^{\mu_1 \mu_3} \left( -\frac{s}{2} p^{(1) \mu_2} p^{(2) \mu_4} + \frac{u}{2} p^{(1) \mu_2} p^{(3) \mu_4} + \frac{s (s - 2)}{4} \frac{1}{2} + 1 p^{(4) \mu_2} p^{(2) \mu_4} - \frac{s}{2} p^{(4) \mu_2} p^{(3) \mu_4} \right) \\
- \frac{1}{4} \eta^{\mu_1 \mu_4} \left( -\frac{t}{2} p^{(4) \mu_2} p^{(2) \mu_3} + \frac{u}{2} p^{(4) \mu_2} p^{(2) \mu_3} + \frac{u}{2} p^{(3) \mu_2} p^{(1) \mu_4} - \frac{u (u - 2)}{4} \frac{1}{2} + 1 p^{(4) \mu_2} p^{(3) \mu_4} \right) \\
- \frac{1}{4} \eta^{\mu_2 \mu_3} \left( -\frac{t}{2} p^{(4) \mu_2} p^{(2) \mu_3} + \frac{u}{2} p^{(4) \mu_2} p^{(2) \mu_3} + \frac{u}{2} p^{(3) \mu_2} p^{(1) \mu_4} - \frac{u (u - 2)}{4} \frac{1}{2} + 1 p^{(4) \mu_2} p^{(3) \mu_4} \right) \\
+ \frac{1}{4} \eta^{\mu_2 \mu_4} \left( -\frac{u (u - 2)}{4} \frac{1}{2} + 1 p^{(3) \mu_2} p^{(1) \mu_4} - \frac{s}{2} p^{(3) \mu_2} p^{(4) \mu_4} - \frac{u (u - 2)}{4} \frac{1}{2} + 1 p^{(3) \mu_2} p^{(1) \mu_4} + \frac{u}{2} p^{(2) \mu_1} p^{(4) \mu_3} \right) \\
- \frac{1}{4} \eta^{\mu_3 \mu_4} \left( -\frac{u (u - 2)}{4} \frac{1}{2} + 1 p^{(3) \mu_2} p^{(4) \mu_3} - \frac{s}{2} p^{(3) \mu_2} p^{(4) \mu_4} - \frac{u (u - 2)}{4} \frac{1}{2} + 1 p^{(3) \mu_2} p^{(4) \mu_3} \right) \right\}. \tag{29}
\]

Again, we find that the scattering amplitude $I_{AAAA}^{(1)}$ contains the tachyon poles in all three channels. It may be interesting to compare this sub-amplitude with the corresponding one, which can be obtained using the conventional vertex operator technique. If we apply the vertex operator technique to evaluate the corresponding sub-amplitude, we obtain \[28\]

\[
I_{AAAA}^{\text{vertex}(1)} = \text{tr}\left\{ A_{\mu_1} (1) A_{\mu_2} (2) A_{\mu_3} (3) A_{\mu_4} (4) \right\} \frac{\Gamma \left( -\frac{\xi}{2} \right) \Gamma \left( -\frac{s}{2} \right)}{\Gamma \left( \frac{n}{2} + 1 \right) \Gamma \left( \frac{n}{2} + 1 \right)}
\]

\[
\left\{ \eta^{\mu_1 \mu_2} \left( t p^{(1) \mu_2} p^{(2) \mu_4} + u p^{(2) \mu_2} p^{(1) \mu_4} \right) + \frac{\eta^{\mu_1 \mu_3}}{2} \left( t p^{(1) \mu_2} p^{(3) \mu_4} + s p^{(3) \mu_2} p^{(1) \mu_4} \right) \\
+ \frac{\eta^{\mu_1 \mu_4}}{2} \left( u p^{(4) \mu_2} p^{(2) \mu_3} + s p^{(4) \mu_2} p^{(1) \mu_3} \right) + \frac{\eta^{\mu_2 \mu_3}}{2} \left( u p^{(4) \mu_2} p^{(3) \mu_4} + s p^{(3) \mu_2} p^{(4) \mu_4} \right) \\
+ \frac{\eta^{\mu_2 \mu_4}}{2} \left( u p^{(4) \mu_2} p^{(3) \mu_4} + s p^{(3) \mu_2} p^{(4) \mu_4} \right) + \frac{\eta^{\mu_3 \mu_4}}{2} \left( t p^{(3) \mu_1} p^{(4) \mu_2} + u p^{(4) \mu_1} p^{(3) \mu_2} \right) \right\}. \tag{30}
\]

The main difference between two sub-amplitudes $I_{AAAA}^{(1)}$ and $I_{AAAA}^{\text{vertex}(1)}$ is the presence of tachyon poles. In the zero-slope limit, both scattering amplitudes reduce to the corresponding one of Yang-Mills gauge theory as expected

\[
I_{AAAA}^{\text{YM}(1)} = \text{tr}\left\{ A_{\mu_1} (1) A_{\mu_2} (2) A_{\mu_3} (3) A_{\mu_4} (4) \right\} \frac{4}{s^2}
\]

\[
\left\{ \eta^{\mu_1 \mu_2} \left( t p^{(1) \mu_2} p^{(2) \mu_4} + u p^{(2) \mu_3} p^{(1) \mu_4} \right) + \frac{\eta^{\mu_1 \mu_3}}{2} \left( t p^{(1) \mu_2} p^{(3) \mu_4} + s p^{(3) \mu_2} p^{(1) \mu_4} \right) \\
+ \frac{\eta^{\mu_1 \mu_4}}{2} \left( u p^{(4) \mu_2} p^{(2) \mu_3} + s p^{(4) \mu_2} p^{(1) \mu_3} \right) + \frac{\eta^{\mu_2 \mu_3}}{2} \left( u p^{(4) \mu_2} p^{(3) \mu_4} + s p^{(3) \mu_2} p^{(4) \mu_4} \right) \\
+ \frac{\eta^{\mu_2 \mu_4}}{2} \left( u p^{(4) \mu_2} p^{(3) \mu_4} + s p^{(3) \mu_2} p^{(4) \mu_4} \right) + \frac{\eta^{\mu_3 \mu_4}}{2} \left( t p^{(3) \mu_1} p^{(4) \mu_2} + u p^{(4) \mu_1} p^{(3) \mu_2} \right) \right\}. \tag{31}
\]

IV. THE ALPHA PRIME CORRECTIONS

One of outstanding advantages of the string field theory in the proper-time gauge is that we may be able to systematically calculate the $\alpha'$-corrections. An expansion of $N$-string vertex operator $\exp[E_{\mathrm{I}}]$ in terms of oscillator operators naturally yields a series expansion of $\alpha'$ with a unique ordering of non-Abelian operators. When we expand the vertex operator $\exp[E_{\mathrm{I}}]$ Eq. \[19a\] in terms of oscillator operators, we find that the third term is proportional
to $\alpha'^2$. This term may be considered as $\alpha'$-corrections to the four-gauge-particle scattering amplitude, which do not have counterparts in the conventional calculation using vertex operators:

$$I_{AAAA}^{(2)} = 2g^2 \left[ \prod_{r=1}^{N} dZ_r \right] \prod_{r<s} |Z_r - Z_s|^{p_r - p_s} \exp \left[ -4 \sum_{r=1}^{N} \tilde{N}_{00}^{(r)} \right] \text{tr} \langle 0| A(1)A(2)A(3)A(4) \rangle \left( \frac{1}{4!} \left( \sum_{r,s=1}^{N} \tilde{N}_{10}^{(r)rs} A(r) p_s \right) \right)^4 \langle 0 \rangle$$

$$= 2g^2 \int_0^1 dx x^{-\frac{3}{4} + 2(1 - x)^{-\frac{2}{4} - 2}} e^{2(\tau_2 - \tau_1)} \prod_{r=1}^{N} \left( \sum_{s=0}^{N} \tilde{N}_{10}^{(r)rs} A(r) p_s \right).$$

(32)

Using the Neumann functions $\tilde{N}_{10}^{(r)rs}$ evaluated in the Appendix and integrating out the real Koba-Nielsen variable $x$, we find that

$$I_{AAAA}^{(2)} = (2g^2) \text{tr} \left\{ A(1)^{\mu_1}A(2)^{\mu_2}A(3)^{\mu_3}A(4)^{\mu_4} \right\} \frac{\Gamma \left( -\frac{3}{4} \right) \Gamma \left( -\frac{1}{4} \right)}{\Gamma \left( \frac{1}{4} + 1 \right)} \left( \frac{\frac{t}{2} - 1}{\frac{t}{2} + 1} \right) \left( \frac{\frac{s}{2} - 1}{\frac{s}{2} + 1} \right)$$

$$+ \frac{1}{4} \left( \frac{\frac{s}{2} - 1}{\frac{s}{2} + 1} \right) \left( \frac{\frac{t}{2} - 1}{\frac{t}{2} + 1} \right) \left( \frac{\frac{t}{2} + 1}{\frac{s}{2} + 1} \right) \left( \frac{\frac{s}{2} + 1}{\frac{t}{2} + 1} \right)$$

$$+ \frac{1}{4} \left( \frac{\frac{t}{2} - 1}{\frac{t}{2} + 1} \right) \left( \frac{\frac{s}{2} - 1}{\frac{s}{2} + 1} \right) \left( \frac{\frac{t}{2} + 1}{\frac{s}{2} + 1} \right) \left( \frac{\frac{s}{2} + 1}{\frac{t}{2} + 1} \right) \left( \frac{\frac{t}{2} - 1}{\frac{t}{2} + 1} \right) \left( \frac{\frac{s}{2} - 1}{\frac{s}{2} + 1} \right) \left( \frac{\frac{t}{2} + 1}{\frac{s}{2} + 1} \right) \left( \frac{\frac{s}{2} + 1}{\frac{t}{2} + 1} \right).$$

(33)

It is apparent that the sub-amplitude $I_{AAAA}^{(2)}$ also contains the tachyon poles. It would have been difficult to obtain these $\alpha'$-corrections to the four-gauge particle scattering amplitude if we employ the vertex operator technique. In the zero slope limit, $I_{AAAA}^{(2)}$ reduces to

$$I_{AAAA}^{(2)} \rightarrow (2g^2)^2 \text{tr} \left\{ A(1)^{\mu_1}A(2)^{\mu_2}A(3)^{\mu_3}A(4)^{\mu_4} \right\} \left( \frac{1}{8} \left( \frac{2p_1^{(2)}p_2^{(4)}p_3^{(4)}p_4^{(4)}}{p_1^{(2)}p_2^{(4)}p_3^{(4)}p_4^{(4)}} + p_1^{(2)}p_2^{(4)}p_3^{(4)}p_4^{(4)} + p_1^{(2)}p_2^{(4)}p_3^{(4)}p_4^{(4)} + p_1^{(2)}p_2^{(4)}p_3^{(4)}p_4^{(4)} \right) \right\}$$

$$+ \frac{1}{2} \left( \frac{2p_1^{(2)}p_2^{(2)}p_3^{(2)}p_4^{(2)}}{p_1^{(2)}p_2^{(2)}p_3^{(2)}p_4^{(2)}} + p_1^{(2)}p_2^{(2)}p_3^{(2)}p_4^{(2)} + p_1^{(2)}p_2^{(2)}p_3^{(2)}p_4^{(2)} \right)$$

$$+ \frac{1}{2} \left( \frac{2p_1^{(2)}p_2^{(2)}p_3^{(2)}p_4^{(2)}}{p_1^{(2)}p_2^{(2)}p_3^{(2)}p_4^{(2)}} + p_1^{(2)}p_2^{(2)}p_3^{(2)}p_4^{(2)} + p_1^{(2)}p_2^{(2)}p_3^{(2)}p_4^{(2)} \right)$$

$$+ 2p_1^{(2)}p_2^{(2)}p_3^{(2)}p_4^{(2)} \right\}.$$

(34)

From this expression of $I_{AAAA}^{(2)}$, we may infer that the corresponding $\alpha'$-corrections to the non-Abelian Yang-Mills action are of order $F^4$. 


V. DISCUSSIONS AND CONCLUSIONS

We conclude this work with a few remarks on possible extensions. We calculated the four-gauge-particle scattering amplitudes on multiple space-filling D-branes using the Polyakov string path integral in the proper-time gauge. Although the resultant scattering amplitudes are completely consistent with the conventional ones obtained by substituting local vertex operators for external strings in the low energy region, they significantly differ from the conventional ones in the high energy region where string corrections become relevant, as they contain tachyon poles in all three scattering channels in a manner consistent with three-particle interactions and $\alpha'$-corrections, which are of order $F^4$.

The string scale may be as low as the weak scale $[1,2]$ in some phenomenological models based on string theory, so it is important to accurately evaluate particle scattering amplitudes, which are valid for the full range of the energy scale. The Polyakov string path integral in the proper-time gauge, which is equivalent to the deformed cubic string field theory $[11]$ may be the right theoretical tool to handle this request. The momenta of external strings are not restricted to the low energy region and the multi-particle scattering amplitudes evaluated in the proper-time gauge yield series expansions of $\alpha'$ with an unambiguously defined ordering of non-Abelian field operators.

Particle scattering amplitudes satisfy various relationships between themselves such as the Kleiss-Kuijf (KK) relation $[29]$ and the Bern-Carrasco-Johansson (BCJ) relation $[30]$, and there has been much effort toward understanding the origins of these relations in string theory $[31,32]$. Because the multi-particle scattering amplitudes obtained by evaluating Polyakov string path integral in the proper-time gauge can be interpreted as the Feynman-Schwinger proper-time representation of open string field theory, we may be able to study relations between scattering amplitudes of massive higher spin particles in string theory by extending this work. It may also be interesting to explore the scattering amplitudes of massless scalars and non-Abelian gauge fields by defining the scattering amplitudes of open strings on $Dp$-branes $[10]$. It may also be worth noting that the scattering amplitudes in the proper-time gauge are valid for the full range of energy scales and expanded in a power series of $\alpha'$. These properties make them useful tools for probing high energy limits of string theory $[18,22]$.

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[1] J. D. Lykken, Phys. Rev. D 54, 3693 (1996).
[2] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B 436, 257 (1998).
[3] J. Polchinski, String Theory (Cambridge Univ. Press, Cambridge 1998) Vols. 1, 2.
[4] G. Veneziano, Nuovo Cimento A 57, 190 (1968).
[5] M. Virasoro, Phys. Rev. 177, 2309 (1969).
[6] J. A. Shapiro, Phys. Rev. 179, 1345 (1969).
[7] A. M. Polyakov, Phys. Lett. B 103, 207 (1981).
[8] T. Lee, Ann. Phys. 183, 191 (1988).
[9] T. Lee, Jour. Kor. Phys. Soc. 71, 886 (2017).
[10] T. Lee, Chinese Phys. C 42, 113105 (2018).
[11] T. Lee, Phys. Lett. B 768, 248 (2017).
[12] S. H. Lai, J. C. Lee, Y. Yang, and T. Lee, Phys. Lett. B 776, 150 (2018).
[13] C. T. Chan, J. C. Lee, and Y. Yang, Nucl. Phys. B 749, 280 (2006).
[14] S. H. Lai, J. C. Lee, and Y. Yang, JHEP 05, 186 (2016).
[15] C. T. Chan, P. M. Ho, J. C. Lee, S. Teraguchi and Y. Yang, Nucl. Phys. B 725, 352 (2005).
[16] S. H. Lai, J. C. Lee, and Y. Yang, JHEP 11, 062 (2016).
[17] S. H. Lai, J. C. Lee, T. Lee, and Y. Yang, JHEP 09, 130 (2017).
[18] D. J. Gross and P. F. Mende, Phys. Lett. B 197, 129 (1987).
[19] D. J. Gross and P. F. Mende, Nucl. Phys. B 303, 407 (1988).
[20] D. J. Gross, Phys. Rev. Lett. 60, 1229 (1988).
[21] D. J. Gross, Philos. Trans. R. Soc. London, Ser. A 329, 401 (1989).
[22] D. J. Gross and J. L. Manes, Nucl. Phys. B 326, 73 (1989).
[23] M. B. Green, J. H. Schwarz and E. Witten, Superstring Theory Volume 1 and 2, (Cambridge University Press 1987).
[24] A. Neveu and J. Scherk, Nucl. Phys. B 36, 155 (1972).
[25] J. Scherk and J. H. Schwarz, Nucl. Phys. B 81, 118 (1974).
[26] A. A. Tseytlin, Nucl. Phys. B 276, 391 (1986).
Appendix A: Neumann Functions for Four-String Scattering Amplitudes

In order to evaluate the Polakov string path integral, we map the world sheet of four-string scattering onto upper half complex plane: Four external strings are located on the boundary of upper half complex plane

\[ Z_1 = 0, \quad Z_2 = x, \quad Z_3 = 1, \quad Z_4 = \infty, \quad 0 \leq x \leq 1. \]  \hfill (A1)

Accordingly, the Schwarz-Christoffel transformation which maps the four-string-scattering world sheet onto upper half complex plane is constructed as

\[ \rho = \sum_{r=1}^{4} \alpha_r \ln(z - Z_r) = \ln z + \ln(z - x) - \ln(1 - z) = \ln \frac{z(z - x)}{(1 - z)} \]  \hfill (A2)

where \( \alpha_1 = \alpha_2 = 1, \alpha_3 = \alpha_4 = -1 \). It follows from this mapping that relations between the global coordinate \( \rho \) and the local coordinates \( \zeta_i, i = 1, 2, 3, 4 \) on individual string patches as given by

\[ e^{-\zeta_1} = e^{\tau_1} \frac{(1 - z)}{z(z - x)}, \quad e^{-\zeta_2} = -e^{\tau_1} \frac{(1 - z)}{z(z - x)}, \]
\[ e^{-\zeta_3} = e^{-\tau_2} \frac{z(z - x)}{(1 - z)}, \quad e^{-\zeta_4} = -e^{-\tau_2} \frac{z(z - x)}{(1 - z)}, \]  \hfill (A3)

where \( \tau_1 \) and \( \tau_2 \) are two interaction times on the world sheet.

The Neumann functions \( \bar{N}_{rs}^{n0} \) are given by contour integrals as follows

\[ \bar{N}_{rs}^{n0} = \frac{1}{n} \oint_{\gamma} \frac{dz}{2\pi i z - Z_s} e^{-n\zeta_r(z)}. \]  \hfill (A4)

Performing the contour integrals and the binormal series expansions we may explicitly evaluate \( \bar{N}_{n0}^{rs} \)

\[ \bar{N}_{11}^{n0} = \frac{e^{n\tau_1}}{n!} \frac{1}{x^n} \sum_{k=0}^{n} \binom{n}{k} \frac{(n + k - 1)!}{k!} \left( \frac{-1}{x} \right)^k, \]  \hfill (A5)

\[ \bar{N}_{12}^{n0} = \frac{e^{n\tau_1}}{n!} \frac{1}{x^{n+1}} \sum_{k=0}^{n-1} \binom{n - 1}{k} \frac{(n + k)!}{(k + 1)!} \left( \frac{-1}{x} \right)^k, \]  \hfill (A6)

\[ \bar{N}_{13}^{n0} = \frac{e^{n\tau_1}}{n!} \frac{1}{x^n} \sum_{k=0}^{n-1} \binom{n - 1}{k} \frac{(n + k - 1)!}{k!} \left( \frac{-1}{x} \right)^k, \]  \hfill (A7)
\[ N_{a0}^{14} = 0, \]
\[ N_{a0}^{21} = -e^{n\tau_1} \frac{1}{n!} \frac{1}{x^n} \sum_{k=0}^{n-1} \binom{n-1}{k} (n+k)! \left( \frac{1}{x} - 1 \right)^{k+1}, \]
\[ N_{a0}^{22} = e^{n\tau_1} \frac{1}{n!} \frac{1}{x^n} \sum_{k=0}^{n} \binom{n}{k} (n+k-1)! \left( \frac{1}{x} - 1 \right)^k, \]
\[ N_{a0}^{23} = \frac{1}{n!} \frac{1}{x^n} \sum_{k=0}^{n-1} \binom{n-1}{k} \left( n+k-1 \right)! \left( \frac{1}{x} - 1 \right)^k, \]
\[ N_{a0}^{24} = 0, \]
\[ N_{a0}^{31} = \frac{(-1)^n e^{-n\tau_2}}{n} (1-x)^n \sum_{k=0}^{n-1} \binom{n-1}{k} \left( \frac{1}{1-x} \right)^k, \]
\[ N_{a0}^{32} = \frac{(-1)^n e^{-n\tau_2}}{n} (1-x)^n \sum_{k=0}^{n-1} \binom{n-1}{k} \left( \frac{1}{1-x} \right)^{k+1}, \]
\[ N_{a0}^{33} = \frac{(-1)^n e^{-n\tau_2}}{n} (1-x)^n \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{1-x} \right)^k, \]
\[ N_{a0}^{34} = 0, \]
\[ N_{a0}^{41} = -\frac{(-1)^n e^{-n\tau_2}}{n!} x^n \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{k!} \right) \left( \frac{1}{x} \right)^k, \]
\[ N_{a0}^{42} = \frac{(-1)^n e^{-n\tau_2}}{n} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{(k-1)!} \right) \frac{1}{x^k}, \]
\[ N_{a0}^{43} = -\frac{(-1)^n e^{-n\tau_2}}{n} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{k!} \right) \left( \frac{1}{x} \right)^k, \]
\[ N_{a0}^{44} = 0. \]

For \( n = 1 \), we have
\[ N_{10}^{11} = -e^{\tau_1} \frac{(1-x)}{x^2}, \quad N_{10}^{12} = e^{\tau_1} \frac{x}{x^2}, \quad N_{10}^{13} = e^{\tau_1} \frac{x}{x}, \quad N_{10}^{14} = 0, \]
\[ N_{10}^{21} = -e^{\tau_1} \frac{(1-x)}{x^2}, \quad N_{10}^{22} = e^{\tau_1} \frac{x}{x^2}, \quad N_{10}^{23} = e^{\tau_1} \frac{x}{x}, \quad N_{10}^{24} = 0, \]
\[ N_{10}^{31} = -e^{-\tau_2} (1-x), \quad N_{10}^{32} = -e^{-\tau_2}, \quad N_{10}^{33} = -e^{-\tau_1} (2-x), \quad N_{10}^{34} = 0, \]
\[ N_{10}^{41} = -e^{-\tau_2} (1-x), \quad N_{10}^{42} = -e^{-\tau_2}, \quad N_{10}^{43} = -e^{-\tau_2} (x-2), \quad N_{10}^{44} = 0. \]

The Neuman functions \( \bar{N}_{rs}^{nm} \) is defined by
\[ \bar{N}_{rs}^{nm} = \frac{1}{nm} \oint_{Z_s} \oint_{Z_r} \frac{dz}{2\pi i} \frac{dz'}{2\pi i} \frac{1}{(z-z')^2} e^{-n\zeta(z)-m\zeta(z')}, \quad n, m \geq 1 \]

We may explicitly calculate \( \bar{N}_{rs}^{11} \):
\[ \bar{N}_{11}^{11} = \oint_{Z_1} \oint_{Z_1} \frac{dz}{2\pi i} \frac{dz'}{2\pi i} \frac{1}{(z-z')^2} e^{-\zeta(z)-\zeta(z')}, \quad \text{for } r < s. \]

Through algebra, we find that
\[ \bar{N}_{11}^{12} = e^{\tau_1} \frac{(1-x)}{x^4}, \quad \bar{N}_{11}^{13} = e^{\tau_1-\tau_2} \frac{(1-x)}{x}, \quad \bar{N}_{11}^{14} = e^{\tau_1-\tau_2} \frac{1}{x}, \]
\[ \bar{N}_{11}^{21} = e^{\tau_1-\tau_2} \frac{1}{x}, \quad \bar{N}_{11}^{22} = e^{\tau_1-\tau_2} \frac{(1-x)}{x}, \quad \bar{N}_{11}^{24} = e^{-2\tau_2} (1-x). \]