A mass transport model with a simple non-factorized steady-state distribution

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Abstract. We study a mass transport model on a ring with a sublattice-parallel update, where a continuous mass is randomly redistributed along distinct links of the lattice. The redistribution process on a given link depends on the masses on both sites, in contrast to the zero range process and its continuous mass generalizations. We show that the steady-state distribution takes a simple non-factorized form that can be written as a sum of two inhomogeneous product measures. A factorized measure is recovered for symmetric mass redistribution, corresponding to an equilibrium process. A non-equilibrium free energy can be explicitly defined by the partition function. For a certain class of transition rates, a condensation transition is obtained, with a critical density which depends on the driving force. We also evaluate different characterizations of the ‘distance’ to equilibrium, either dynamic or static: the mass flux, the entropy production rate, the Gibbs free-energy difference between the equilibrium and non-equilibrium stationary states, and the derivative of the non-equilibrium free energy with respect to the applied driving force. The connection between these different non-equilibrium parameters is discussed.

Keywords: exact results, zero-range processes, stochastic particle dynamics, stationary states
1. Introduction

One of the goals of non-equilibrium statistical physics is to be able to describe the statistical properties of systems driven in a non-equilibrium steady state by an external non-conservative force. As no general statistical formalism is available to deal with driven systems, exactly solvable models have played an important role in the development of this field. One paradigmatic exactly solvable model is the asymmetric simple

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exclusion process (ASEP) [1], either with periodic [2] or open boundary conditions [3–6]. Generalizations with several types of particles have also been proposed, with periodic [7–9] or open geometries [10–13]. The ABC model [14], which includes three types of particles, also falls into this class. The solution of the ASEP model requires, in most cases, the use of matrix product states [15], often with infinite size matrices, making its analysis relatively involved. Such matrix product state solutions are required even with a periodic geometry, when the model includes several types of particles [7–9]—except if some restrictive conditions are imposed [16].

Simpler models, like the zero-range process (ZRP) [1, 17, 18] and related mass transport models [19–22], have also been considered, often in relation to condensation transitions [17, 23–25]. Multispecies generalizations of these models have also been proposed [23, 26, 27]. When the transition rates satisfy certain conditions [19–21, 25], these models have the advantage that their steady-state distribution factorizes, making their analytical study much easier. However, in a closed geometry, the drawback is that the distribution does not depend on the driving force, and that it thus remains identical to the equilibrium distribution obtained for unbiased dynamics. Note that the same property also holds for the (single-species) ASEP on a ring [2].

In this paper, we propose a class of mass transport models for which the steady-state distribution takes a simple form (a sum of two inhomogeneous product measures) and explicitly depends on the local driving force. The present model is inspired by the equilibrium model considered in [28], though it differs from the latter in several respects, notably in the presence of a driving force and of a synchronous dynamics. The simple form of the steady-state probability distribution makes calculations easy, as illustrated below on several examples including the evaluation of the non-equilibrium free energy. For a certain class of transition rates, our model exhibits a condensation transition similar to the one appearing in mass transport models with factorized steady states [25], but with a condensation threshold depending on the driving force. In addition, the dependence of the steady-state probability distribution on the forcing allows us to compare the dynamical characterizations of the ‘degree of non-equilibrium’ (mass flux and entropy production rate) with static characterizations, like the difference between the non-equilibrium and the corresponding equilibrium distribution for the Gibbs free energy functional (or the Kullback–Leibler divergence [29]). We also evaluate the non-equilibrium order parameter introduced by Sasa and Tasaki [30], defined as a derivative of the non-equilibrium free energy with respect to the driving force, and discuss the relationship between these different measures of the ‘distance’ to equilibrium.

2. Definition of the model

We consider a one-dimensional lattice with \( N \) sites, labelled by \( i = 1, \ldots, N \), with periodic boundary conditions \( (i \pm N \equiv i) \); \( N \) is assumed to be even, namely \( N = 2N' \) with the \( N' \) integer. On each site \( i \), one defines a real positive mass \( m_i \). The model is endowed
with a sublattice parallel-update dynamics\(^1\). The dynamics proceeds at each discrete time step \( t = 0, 1, 2, \ldots \) by parallel redistributions of the mass between neighboring sites \( i \) and \( i + 1 \) on one of the two partitions \( \mathcal{P}_1 = \{(2k, 2k + 1)\} \) and \( \mathcal{P}_2 = \{(2k + 1, 2k + 2)\} \), randomly chosen with equal probability. Once a partition \( \mathcal{P}_j \) has been selected, all links belonging to the partition \( \mathcal{P}_j \) are simultaneously updated. To update a link \((i, i + 1)\), a new value \( m'_i \) of the mass on site \( i \) is randomly drawn from the distribution

\[
K(m'_i|S_i) = \frac{v(m'_i)w(S_i - m'_i)}{v \ast w(S_i)}, \quad S_i \equiv m_i + m_{i+1}
\]

where \( v(m) \) and \( w(m) \) are arbitrary positive functions, and \( v \ast w(S) \) is the convolution product of \( v \) and \( w \),

\[
v \ast w(S) = \int_0^S dm \, v(m)w(S - m),
\]

which has to be non-zero for any \( S > 0 \) for the model to be well defined. From mass conservation, the mass on site \( i + 1 \) is, after redistribution, \( m'_{i+1} = S_i - m'_i \).

3. Master equation and steady-state solution

3.1. Discrete time master equation

To describe the statistical evolution of the system under the above dynamics, we write down the corresponding master equation. The configuration of the system is given by the ordered list \( \mathbf{m} = (m_1, \ldots, m_N) \) of all the masses in the system. The probability density \( P(\mathbf{m}, t) \) evolves according to the discrete time master equation

\[
P(\mathbf{m}', t + 1) = \int d\mathbf{m} \, T(\mathbf{m}'|\mathbf{m}) \, P(\mathbf{m}, t)
\]

with \( d\mathbf{m} = \prod_{i=1}^N dm_i \), and where \( T(\mathbf{m}'|\mathbf{m}) \) is the probability (density) to jump from configuration \( \mathbf{m} \) to configuration \( \mathbf{m}' \) in a single time step. This transition probability is normalized according to

\[
\int d\mathbf{m}' T(\mathbf{m}'|\mathbf{m}) = 1.
\]

For the present mass transport model, the transition probability is given by

\[
T(\mathbf{m}'|\mathbf{m}) = \frac{1}{2} T_1(\mathbf{m}'|\mathbf{m}) + \frac{1}{2} T_2(\mathbf{m}'|\mathbf{m})
\]

where

\[
T_1(\mathbf{m}'|\mathbf{m}) = \prod_{k=1}^{N'} K(m'_{2k}|S_{2k}) \delta(S'_{2k} - S_{2k}),
\]

\[\text{Note that although the asynchronous, continuous time dynamics is most often used in this context, synchronous dynamics has also been used in the ASEP [31–33] and related particle models [27], as well as in mass transport models [19, 26].}

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\[ T_{2}(m' | m) = \prod_{k=1}^{N'} K(m'_{2k+1} | S_{2k+1}) \delta(S'_{2k+1} - S_{2k+1}), \]  

(7)

with the shorthand notations \( S_i \equiv m_i + m_{i+1} \) and \( S'_i \equiv m'_i + m'_{i+1} \).

### 3.2. Steady-state distribution

In the following, we show that the distribution

\[ P(m) = \frac{1}{Z_N(M)} \left( \prod_{k=1}^{N'} v(m_{2k})w(m_{2k+1}) + \prod_{k=1}^{N'} w(m_{2k})v(m_{2k+1}) \right) \delta \left( \sum_{i=1}^{N} m_i - M \right) \]  

(8)

is a stationary solution of the master equation, equation (3). In equation (8), \( M \) is the (constant) total mass, and \( Z_N(M) \) is a normalization factor. In some cases, it may be convenient to write \( P(m) \) in the form

\[ P_j(m) = \frac{2}{Z_N(M)} Q_j(m) \delta \left( \sum_{i=1}^{N} m_i - M \right), \]  

(9)

having defined

\[ Q_1(m) = \prod_{k=1}^{N'} v(m_{2k})w(m_{2k+1}), \quad Q_2(m) = \prod_{k=1}^{N'} w(m_{2k})v(m_{2k+1}). \]  

(10)

The partition function \( Z_N(M) \) then reads

\[ Z_N(M) = \int \text{d}m \left[ Q_1(m) + Q_2(m) \right] \delta \left( \sum_{i=1}^{N} m_i - M \right). \]  

(11)

Using equation (8), taking into account the fact that the dynamics conserves the total mass, the master equation (3) reads

\[ Q_1(m') + Q_2(m') \]  

\[ = \frac{1}{2} \int \text{d}m \left[ T_1(m'|m) + T_2(m'|m) \right] \left[ Q_1(m) + Q_2(m) \right] \]  

(12)

where, to lighten the notation, the Dirac delta function accounting for the total mass conservation is understood.

Expanding the r.h.s. of equation (12) into four terms, we evaluate these terms separately, obtaining for \( j, k \in \{1, 2\} \) (see appendix A)

\[ \int \text{d}m T_k(m'|m) Q_j(m') = Q_k(m). \]  

(13)

The sum of the four contributions appearing on the r.h.s. of equation (12) is thus equal to \( Q_1(m') + Q_2(m') \), so that equation (12) is satisfied. Hence the distribution \( P(m) \) given in equation (8) is the stationary solution of the model.
3.3. Physical interpretation of the dynamics

Without loss of generality, one can rewrite the functions \( v(m) \) and \( w(m) \) as
\[
v(m) = e^{-\beta \varepsilon(m) - \beta h(m)}, \quad w(m) = e^{-\beta \varepsilon(m) + \beta h(m)}
\]
where we have defined
\[
e^{-\beta \varepsilon(m)} = \sqrt{v(m)w(m)}, \quad e^{-\beta h(m)} = \sqrt{\frac{v(m)}{w(m)}}.
\]

The parameter \( \beta > 0 \), to be thought of as an inverse temperature, is arbitrary here, and has only been introduced to facilitate comparison with the equilibrium. A symmetric redistribution process, obtained for \( v(m) = w(m) \), corresponds to \( h(m) = 0 \), and the stationary distribution equation (8) boils down to an equilibrium distribution,
\[
P(m) = \frac{2}{Z_N(M)} e^{-\beta \varepsilon(m)} e^{-\beta \sum_{i=1}^{\mathcal{N}} h(m_i)} \delta \left( \sum_{i=1}^{\mathcal{N}} m_i - M \right).
\]
The function \( \varepsilon(m) \) thus appears as effective local energy associated with a local density \( m \). The function \( h(m) \) describes the asymmetry of the dynamics. In the linear case \( h(m) = h_0 m \), keeping in mind the local detailed balance, the term \( 2h_0 (m_i - m'_i) \) that enters the ratio \( K(m_i'|S_i)/K(m_i|S'_i) \) (with \( S_i = S'_i \)) can be interpreted as the work done by a driving force \( f = 2h_0 \) associated with a displaced mass \( m_i - m'_i \) on a unit distance (one lattice spacing). This case is thus physically meaningful, and we will focus on it when dealing with specific examples (keeping \( f \) rather than \( h_0 \) as the driving parameter).

When \( h(m) \neq 0 \), the non-equilibrium steady-state distribution \( P(m) \) given in equation (12) can be rewritten as
\[
P(m) = \frac{2}{Z_N(M)} e^{-\beta E(m)} \cosh[\beta H(m)] \delta \left( \sum_{i=1}^{\mathcal{N}} m_i - M \right)
\]
where one has introduced the global observables
\[
E(m) = \sum_{i=1}^{\mathcal{N}} \varepsilon(m_i), \quad H(m) = \sum_{i=1}^{\mathcal{N}} (-1)^{i} h(m_i).
\]

The presence of the hyperbolic cosine in equation (17) yields long-range correlations as can be seen explicitly by a calculation of the two-point spatial correlation function \( G_j = \langle m_i m_{i+j} \rangle - \rho^2 \rho = \langle m_i \rangle \) for a particular choice of \( \varepsilon(m) \) and \( h(m) \) (appendix B). In more intuitive terms, these correlations are generated by the synchronous dynamics over two different partitions of the lattice.

Note that more details on the evaluation of the correlation function and on the expression of the pair and single mass distributions can be found in appendix B.

In the following, the arbitrary inverse temperature scale \( \beta \) is set to unity, unless stated otherwise.
4. Partition function: the non-equilibrium free energy and condensation transition

4.1. Expression of the partition function in the thermodynamic limit

It is natural to define the non-equilibrium (intensive) free energy $\phi(\rho)$ at an average density $\rho$ from the partition function given in equation (11) as

$$\phi(\rho) = -\lim_{N \to \infty} \frac{1}{N} \ln Z(N \rho), \quad (19)$$

if this limit exists.

To evaluate $\phi(\rho)$, we apply the classical saddle-node approximation. Plugging the Laplace representation of the delta function into equation (11),

$$\delta(s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\zeta e^{s\zeta} \quad (20)$$

with $a$ as an arbitrary real number, we end up with

$$Z_N(N \rho) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\zeta e^{N(\lambda(\zeta) - \rho \zeta)} \quad (21)$$

where we have introduced the functions $^2$

$$\lambda(\zeta) = \frac{1}{2} \ln[\hat{v}(\zeta)\hat{w}(\zeta)] \quad (22)$$

and

$$\hat{v}(\zeta) = \int_0^\infty dm e^{\zeta m} v(m), \quad \hat{w}(\zeta) = \int_0^\infty dm e^{\zeta m} w(m). \quad (23)$$

Note that the real part of $\zeta$ (equal to $a$) is chosen small enough for the integrals to converge. Note that a possible restriction on $v$ and $w$ may appear at this stage: $\hat{v}$ and $\hat{w}$ should exist for the free energy to be properly defined.

Assuming that these restrictions are satisfied and that $\lambda(\zeta) - \rho \zeta$ has a unique real saddle-point for a fixed $\rho = M/N$, $\zeta^*(\rho)$, given by

$$\frac{d\lambda}{d\zeta}(\zeta^*) = \rho, \quad (24)$$

one gets

$$Z_N(N \rho) \sim e^{-N[\rho \zeta^*(\rho) - \lambda(\zeta^*(\rho))]} \quad (25)$$

The intensive free energy $\phi(\rho)$ introduced in equation (19) is then given by

$$\phi(\rho) = \rho \zeta^*(\rho) - \lambda(\zeta^*(\rho)). \quad (26)$$

As in equilibrium, one can define a non-equilibrium chemical potential $\mu$ and a non-equilibrium pressure $p$, derived as usual as derivatives of the extensive free energy $F(M, N) = N\phi(M/N) \quad [30, 34]$. Moving to the intensive free energy, one gets

2 We take here a determination of the logarithm in the complex plane such that the integration path does not cross the branch cut.
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\[
\mu(\rho) = \varphi'(\rho) = \zeta^*(\rho), \quad p(\rho) = -\varphi(\rho) + \rho\varphi'(\rho).
\]

(27)

Assuming that the mass \( M \) and the volume \( N \) are the only extensive parameters, one shows, using the Euler relation for a homogeneous function \([30]\), that

\[
\varphi(\rho) = -p(\rho) + \rho\mu(\rho).
\]

(28)

As an example, we evaluate explicitly the free energy in the specific case of linear functions

\[
\varepsilon(m) = \varepsilon_0 m \quad \text{and} \quad h(m) = \frac{1}{2} f m,
\]

with the parameterization equation \((14)\) of the functions \( v(m) \) and \( w(m) \). For \(-\infty < \zeta < \varepsilon_0 - f/2\), one obtains

\[
\hat{v}(\zeta) = \frac{1}{\varepsilon_0 - \zeta + \frac{f}{2}}, \quad \hat{w}(\zeta) = \frac{1}{\varepsilon_0 - \zeta - \frac{f}{2}}
\]

and

\[
\lambda(\zeta) = -\frac{1}{2} \ln \left( (\varepsilon_0 - \zeta)^2 - \frac{f^2}{4} \right).
\]

(30)

The saddle-point \( \zeta^*(\rho) \), as defined in equation \((24)\), is given by

\[
\zeta^*(\rho) = \varepsilon_0 - \frac{1 + \sqrt{1 + \rho^2 f^2}}{2\rho},
\]

(31)

which is always strictly lower than \( \varepsilon_0 - f/2 \) for any positive \( \rho \). Then, the free energy reads, from equation \((26)\),

\[
\varphi(\rho, f) = \varepsilon_0 \rho - \frac{1 + \sqrt{1 + \rho^2 f^2}}{2} + \frac{1}{2} \ln \left( \frac{1 + \sqrt{1 + \rho^2 f^2}}{2\rho^2} \right).
\]

(32)

According to equation \((28)\), one reads from the last equation

\[
\mu(\rho) = \varepsilon_0 - \frac{1 + \sqrt{1 + \rho^2 f^2}}{2\rho}
\]

(33)

\[
p(\rho) = -\frac{1}{2} \ln \left( \frac{1 + \sqrt{1 + \rho^2 f^2}}{2\rho^2} \right).
\]

(34)

The chemical potential \( \mu(\rho) \) and the pressure \( p(\rho) \) are plotted in figure 1 for the sake of illustration.

Note that here and in what follows, we emphasize the \( f \)-dependence of the free energy density by denoting it as \( \varphi(\rho, f) \) when considering the specific case \( h(m) = \frac{1}{2} f m \). At equilibrium, for \( f = 0 \), one recovers the equilibrium free energy \( \varphi(\rho, 0) = \varepsilon_0 \rho - 1 - \ln \rho \) (we recall that the temperature is set to unity).

4.2. Condensation transition

Another choice of \( v(m) \) and \( w(m) \) can lead to the phenomenon of condensation, which has been well studied for different versions of the zero range process \([17]\). Indeed, it may happen that the saddle-point equation \((24)\) has no real solution \( \zeta^* \) for a density \( \rho \) that is greater than a critical value \( \rho_c \). The ensemble equivalence (between 'grand...
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canonical' and 'canonical' ensembles) is therefore broken and the condensation of a macroscopic fraction of the total mass then occurs on a randomly selected site [35]. For our model, the exact single-site probability distribution is given by (see appendix B)

\[
p(m) = v(m) \int_{0}^{+\infty} dm' \frac{Z_{N-2}(M - m - m')}{Z_N(M)} w(m') + w(m) \int_{0}^{+\infty} dm' \frac{Z_{N-2}(M - m - m')}{Z_N(M)} v(m').
\]  

(35)

As explained in [35], the phenomenon of condensation is closely related to the existence of a saddle-point for the evaluation of \(Z_N(M)\), leading to different distributions of \(p(m)\).

As a specific example, we consider \(\epsilon(m) = \epsilon_{0}m + \ln(1 + m^{\gamma})\) and \(h(m) = \frac{1}{2}fm\). One obtains,

\[
\hat{v}(\zeta) = \int_{0}^{\infty} \frac{dm}{1 + m^{\gamma}} e^{-(\epsilon_{0} + \frac{f}{2} - \zeta)m}, \quad \hat{w}(\zeta) = \int_{0}^{\infty} \frac{dm}{1 + m^{\gamma}} e^{-(\epsilon_{0} - \frac{f}{2} - \zeta)m},
\]

(36)

and the saddle-point equation (24) reads

\[
\rho = \frac{1}{2} \left( \frac{\hat{v}'(\zeta^{\ast})}{\hat{v}(\zeta^{\ast})} + \frac{\hat{w}'(\zeta^{\ast})}{\hat{w}(\zeta^{\ast})} \right).
\]

(37)

Clearly, as in the previous example, \(\zeta\) has to be lower than or equal to \(\epsilon_{0} - f/2\) for \(\hat{w}\) to exist\(^3\). Now, the existence of a critical value \(\rho_{c} < +\infty\) in equation (37) depends on \(\gamma\). Indeed, for \(\zeta\) tending to its upper bound \(\epsilon_{0} - f/2\), the associated \(\rho\) in equation (37) can be infinite or finite, depending on the actual value of \(\gamma\).

If \(\gamma \leq 2\), one has

\[
\lim_{\zeta \to \epsilon_{0} - f/2} \frac{\hat{w}'(\zeta)}{\hat{w}(\zeta)} = +\infty,
\]

(38)

\(^3\) Note that \(\hat{v}\) automatically exists if this restriction is satisfied.

Figure 1. The chemical potential \(\mu(\rho)\) (left panel) and pressure \(p(\rho)\) (right panel) as a function of the density \(\rho\), for different values of the driving force \(f\) (from top to bottom, \(f = 0, 2, 4, 6\) and 8).
and thus, there exists the solution $\zeta^*$ of equation (37) for each $\rho < +\infty$: a saddle-point always exists.

If $\gamma > 2$, in contrast one has that

$$\lim_{\zeta \to \epsilon_0 - f/2} \frac{\tilde{w}'(\zeta)}{\tilde{w}(\zeta)} < +\infty$$

and thus equation (37) has a solution $\zeta^*$ only for $\rho \leq \rho_c$ where $\rho_c$ is given by

$$\rho_c = \frac{1}{2} \left( \frac{\tilde{v}'(\epsilon_0 - f/2)}{\tilde{v}(\epsilon_0 - f/2)} + \frac{\tilde{w}'(\epsilon_0 - f/2)}{\tilde{w}(\epsilon_0 - f/2)} \right).$$

(40)

This implies that condensation occurs for $\rho > \rho_c$ [35]. The threshold density $\rho_c$ that appears when $\gamma > 2$ can be rewritten more explicitly as

$$\rho_c = \frac{1}{2} \left( \frac{\int_{0}^{+\infty} dm (1 + m \gamma)^{-1} me^{-f m}}{\int_{0}^{+\infty} dm (1 + m \gamma)^{-1} e^{-f m}} + \int_{0}^{+\infty} dm (1 + m \gamma)^{-1} \right).$$

(41)

We emphasize that the dependence on the driving force of the stationary distribution leads to a dependence on the driving force of the threshold (or critical) density $\rho_c$ for condensation to appear. The threshold density $\rho_c$ is plotted as a function of the driving force $f$ in figure 2.

5. Characterization of the ‘distance’ to equilibrium

We have seen in the previous section that the presence of a non-zero driving force has observable consequences, for instance by shifting the threshold density for condensation. This suggests that one should try to further characterize the non-equilibrium character of the dynamics by computing several quantities that can be thought of as different evaluations of a ‘distance’ to equilibrium. The fundamental characterization of a non-equilibrium stationary stochastic process relies on the breaking of detailed balance, which can be macroscopically quantified through observables like the mass flux or the entropy production rate. These two quantities are evaluated in sections 5.1 and 5.2 respectively. We call these observables ‘dynamical’ (although they are time-independent) in the sense that they involve fluxes, either of mass or of probability. In addition, one may compute ‘static’ observables that characterize the distance to equilibrium only based on the stationary measure, by comparing the non-equilibrium stationary measure with that of the equilibrium. Although such static observables do not reveal the intrinsic non-equilibrium character of the system, one expects this out-of-equilibrium dependence of the stationary measure to be rather typical, as suggested, for instance, by the McLennan formula in a close-to-equilibrium regime [36, 37]. It is then
of interest to compare these static observables to the dynamical ones to see whether both are connected. We thus discuss in section 5.3 the Gibbs free energy difference with the equilibrium state, and in section 5.4 the non-equilibrium order parameter is defined as a derivative of the non-equilibrium free energy with respect to the driving force.

5.1. Stationary mass flux

We start by evaluating the stationary mass flux between two sites $i$ and $i+1$ (which, due to mass conservation, is independent of $i$). During a given time step, a mass is transferred between $i$ and $i+1$, only if the link $(i,i+1)$ belongs to the chosen partition ($P_1$ or $P_2$) of the lattice; mass transfer on this link thus occurs with the probability $\frac{1}{2}$. The average flux $\Phi$ then reads

$$\Phi = \frac{1}{2} (\langle m_i \rangle - \langle m'_i \rangle)$$

where $m_i$ is the mass on site $i$ before a redistribution occurs on the link $(i,i+1)$, while $m'_i$ is the mass on site $i$ after the redistribution. The masses $m_i$ and $m_{i+1}$ before redistribution are assumed to follow the steady-state distribution $P(m_i,m_{i+1})$ given in appendix B—see equation (B.3); one thus has $\langle m_i \rangle = \rho$. Note that the time step has been set to unity.

The average mass $\langle m'_i \rangle$ after redistribution can be expressed as

$$\langle m'_i \rangle = \int_0^\infty dm_i \int_0^\infty dm_{i+1} P(m_i,m_{i+1}) \int_0^\infty dm'_i m'_i K(m'_i|m_i + m_{i+1}).$$

After some algebra, one finds

$$\langle m'_i \rangle = 2C_2(\rho) \int_0^\infty S \, e^{-\mu S} \int_0^S dm' \, m' \, v(m') w(S - m').$$

The calculation can be carried out explicitly on the example $\varepsilon(m) = \varepsilon_0 m$ and $h(m) = \frac{1}{2} fm$, yielding

Figure 2. The threshold density $\rho_c$ for condensation, plotted as a function of the driving force $f$, for several values of $\gamma$ (from top to bottom, $\gamma = 2.5, 3, 4$ and 5).
The average mass flux $\Phi$ then reads, using equations (33) and (42),

$$\Phi = \frac{1}{2} \left( \rho - \langle m'_i \rangle \right) = \frac{f}{4(\varepsilon_0 - \mu)^2 - f^2}. \quad (46)$$

Also, using the explicit expression of $\mu(\rho)$ given in equation (33), one finds

$$\Phi = \frac{\rho^2 f}{2 + 2\sqrt{1 + \rho^2 f^2}}. \quad (47)$$

Furthermore, one notices that the flux, which can be interpreted as a response of the system to the driving force $f$ (when $h(m) = \frac{1}{2} fm$), is directly related to the free energy $\varphi$ (19), as explicitly shown in appendix C:

$$\Phi = -\frac{\partial \varphi(\rho, f)}{\partial f}. \quad (48)$$

We will comment on this relation in the next section, in connection to the entropy production rate.

5.2. Entropy production rate

An alternative dynamical measure of the degree of irreversibility is given by the entropy production rate. For a discrete time Markov process, the (time-dependent) entropy production rate (i.e. the entropy production per time step) is defined as [38] \footnote{Note that the notations in [38] do not follow the same convention, as $P(\omega|\omega')$ denotes there the probability of a transition from a configuration $\omega$ to a configuration $\omega'$, while we use here a (somehow more standard) conditional probability notation, where $T(m|m')$ is the transition probability from $m'$ to $m$.}

$$\Delta_{\text{int}} S_t = \frac{1}{2} \int \text{d}m \text{d}m' \left[ T(m'|m)P_t(m) - T(m|m')P_t(m') \right] \ln \frac{T(m'|m)P_t(m)}{T(m|m')P_t(m')} \quad (49)$$

The advantage of this form is that the positivity of $\Delta_{\text{int}} S$ is visible, as it involves the products of factors of equal sign. In a steady state, the entropy production rate simplifies to [38]

$$\Delta_{\text{int}} S = \int \text{d}m \text{d}m' T(m'|m)P(m) \ln \frac{T(m'|m)}{T(m|m')} \quad (50)$$

The entropy production rate $\Delta_{\text{int}} S$ can be evaluated in the present model, yielding (the technical details are reported in appendix D):

$$\Delta_{\text{int}} S = \frac{1}{2} \int \text{d}m \left[ P_1(m) - P_2(m) \right] H(m), \quad (51)$$

where $P_j(m)$ is defined in equation (9). One thus recovers, as expected, $\Delta_{\text{int}} S = 0$ at equilibrium, when $P_1(m) = P_2(m)$. Equation (51) can be rewritten in terms of the observables $E$ and $H$ defined in equation (18), as
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\[ \Delta_{\text{int}} S = \frac{1}{Z_N(M)} \int \text{d}m \, H(m) \, e^{-E(m)} \sinh \left( H(m) \right) \delta \left( \sum_{i=1}^{N} m_i - M \right). \]  (52)

Since the entropy production rate is extensive with the system size, it is convenient to define the density of the entropy production rate \( \sigma = \lim_{N \to \infty} \Delta_{\text{int}} S/N \), when this limit exists. A way to evaluate \( \sigma \) in practice is to introduce the generalized partition function \( Z_N(M, \theta) \), obtained by replacing \( h(m) \) by \( \theta h(m) \) where \( \theta \) is a real parameter, yielding

\[ Z_N(M, \theta) = \int \text{d}m \, e^{-E(m)} \cosh \left( \theta H(m) \right) \delta \left( \sum_{i=1}^{N} m_i - M \right). \]  (53)

Assuming a thermodynamic form \( Z_N(N \rho, \theta) \approx e^{-N \tilde{\varphi}(\rho, \theta)} \), one can then write

\[ \sigma = -f \frac{\partial \tilde{\varphi}}{\partial \theta} (\rho, \theta = 1). \]  (54)

The free energy \( \tilde{\varphi}(\rho, \theta) \) can be evaluated in the same way as \( \varphi(\rho) \), simply replacing \( h(m) \) with \( \theta h(m) \) in the calculation of \( \lambda(\zeta) \) see equation (22).

In the specific case \( h(m) = \frac{1}{2} fm \), one can also write the entropy production rate in terms of the non-equilibrium free energy \( \varphi(\rho, f) \) as

\[ \sigma = -f \frac{\partial \varphi}{\partial f}. \]  (55)

Given that the flux \( \Phi \) is equal to \(-\partial\varphi/\partial f\), the entropy production \( \sigma \) reads

\[ \sigma = f \Phi \]  (56)

using equation (48). One then recovers the usual expression of the local entropy production, interpreted as the average local work injected into the system (times the inverse temperature that is equal to 1 here). Note that if the inverse temperature \( \beta \neq 1 \), one finds \( \sigma = \beta f \Phi \). This result is consistent with the local detailed balance interpretation of the dynamics briefly discussed in section 3.3.

Having discussed the dynamical characterizations of the distance to equilibrium, we now turn to static characterizations of this distance, namely, measures of the ‘degree of non-equilibrium’ that are based only on the steady-state probability distribution \( P(m) \), without any explicit reference to the dynamics.

5.3. Gibbs free energy difference

One possible such measure is the difference of the Gibbs free energy functional between the non-equilibrium and equilibrium distributions, for the same temperature of thermal bath. Note that for the sake of clarity, in this section we explicitly take into account the temperature \( T = \beta^{-1} \) (previously set to \( T = 1 \)). For an arbitrary probability distribution \( P(m) \) over the configuration space of the model, the Gibbs free energy functional \( \mathcal{F}[P] \) is defined as

\[ \mathcal{F}[P] = \int \text{d}m \, P(m) E(m) - T \int \text{d}m \, P(m) \ln P(m). \]  (57)

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Given that the equilibrium distribution \( P_{\text{eq}}(m) \) at a temperature of \( T \) minimizes the functional \( F[P] \), the quantity

\[
\Delta F = \frac{1}{N} \left( F[P] - F[P_{\text{eq}}] \right)
\]

satisfies \( \Delta F \geq 0 \) for any distribution \( P \) (note that we have introduced the factor \( 1/N \) to make \( \Delta F \) an intensive quantity). It is thus natural to interpret \( \Delta F \) as a measure of the distance to equilibrium. Note that \( \beta N \Delta F \) identifies with the Kullback–Leibler divergence

\[
\mathcal{D}[P||P_{\text{eq}}] = \int dm P(m) \ln \frac{P(m)}{P_{\text{eq}}(m)}.
\]

In the present model, a straightforward calculation yields

\[
\Delta F = \frac{1}{N} \ln Z^\alpha_N(M) - \frac{1}{N} \ln Z_N(M) + \frac{1}{N} \int dm P(m) \ln \cosh[\beta H(m)].
\]

The last integral can be evaluated explicitly in the case \( h(m) = \frac{1}{2} f m \), where one has

\[
\int dm P(m) \ln \cosh[\beta H(m)]
\]

\[
= \int dM' \int dm P(m) \delta \left( \sum_{k=1}^{N'} m_{2k} - M' \right) \ln \cosh[\beta f(M - 2M')]
\]

\[
= \int dM' \Psi(M'|M) \ln \cosh[\beta f(M - 2M')]
\]

where \( \Psi(M'|M) \) is the distribution of the total mass over the even sites \( M' = \sum_{k=1}^{N'} m_{2k} \), given the total mass \( M \) in the system. By symmetry, the most probable value of \( M' \) is \( M/2 \), so that by a saddle-point argument, the last integral in equation (61) is equal to zero at order \( N \), with only possible subextensive corrections. One thus finds from equations (60) and (19), for \( N \to \infty \),

\[
\Delta F = \varphi(\rho, f) - \varphi(\rho, 0)
\]

so that \( \Delta F \) also identifies in this case with the difference of the free energy as defined by the non-equilibrium free energy \( \varphi(\rho, f) \) of the partition function \( Z_N(M) \), which is a quantity that is a priori distinct from the Gibbs free energy functional, as seen from equation (60).

5.4. Non-equilibrium order parameter

A non-equilibrium order parameter \( \Psi \) has been introduced by Sasa and Tasaki [30] as (the opposite of) the derivative of the non-equilibrium free energy with respect to the driving force. In the present model with \( h(m) = \frac{1}{2} f m \), this definition leads to

\[
\Psi = -\frac{\partial \varphi}{\partial f}(\rho, f).
\]
Several remarks are in order here. First, this definition is similar to the relation linking, at equilibrium, an order parameter like the magnetization to its conjugate field—hence the name ‘non-equilibrium order parameter’. Second, an alternative definition, involving the derivation with respect to the (mass or particle) flux, has also been proposed in [30]. Third, we use here an intensive order parameter instead of the extensive order parameter originally introduced in [30].

Since the non-equilibrium free energy \( \varphi(\rho, f) \) is, from symmetry arguments, an even function of \( f \), \( \Psi(\rho, f) \) is an odd function of \( f \), and thus vanishes for \( f = 0 \), which is consistent with the interpretation of \( \Psi \) as a non-equilibrium order parameter.

Using equation (48), the non-equilibrium order parameter \( \Psi \) simply boils down to the mass flux,

\[
\Psi(\rho, f) = \Phi(\rho, f).
\]  

(64)

Although the non-equilibrium parameter \( \Psi \) turns out to be numerically equal to the mass flux \( \Phi \), the two quantities differ in essence: \( \Psi \) is a static order parameter, while the flux \( \Phi \) is a dynamical quantity. Introducing explicitly a time step \( \Delta t \) in the model (this time step has been set to \( \Delta t = 1 \) up to now), we would have \( \Phi = \Psi / \Delta t \), showing that both quantities have different dimensions. In any case, equation (64) provides an interesting connection between the static and dynamic measures of the ‘distance’ to equilibrium.

6. Discussion and conclusion

In this paper, we have introduced a one-dimensional mass transport model on a ring geometry with sublattice-parallel dynamics for which the steady-state distribution takes a non-factorized form. In contrast to other models with a non-factorized distribution, like the target process [39] (see also [25] for a general discussion of the conditions for factorized steady states), the steady-state distribution can here be determined explicitly—as such constituting the main contribution of this work—and is found to take a simple form as the sum of two inhomogeneous product measures. Knowledge of the steady-state distribution allows for a straightforward evaluation of the local distributions of mass, and, if it exists, of the non-equilibrium free energy. Also, we showed that like for other similar systems, our model may exhibit a condensation transition. As already emphasized, the explicit dependence of the stationary measure on the driving force is the main advantage of this model—at odds with, for instance, the zero range process and related mass transport models [17]—since this behavior is expected to be generic. This has allowed us to exhibit the explicit dependence of different quantities on the driving field, like the threshold density for condensation.

In addition, we have evaluated several quantities, either static or dynamic, that characterize the ‘degree of non-equilibrium’ of the steady state of the system. These include the mass flux \( \Phi \), the entropy production rate per site \( \sigma \), the difference \( \Delta \mathcal{F} \) of the Gibbs free energy functional (per site) between the non-equilibrium and equilibrium states, as well as the non-equilibrium order parameter \( \Psi \) introduced by Sasa and Tasaki [30], as the derivative of the non-equilibrium free energy with respect to the driving
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force. We have found that all these non-equilibrium parameters are closely related to one another, and that (at least in the case of a density-independent driving force \( f \)) the non-equilibrium order parameter \( \Psi \) may be seen as a key parameter from which the others can be evaluated. In particular, we have found that

\[
\Phi(\rho, f) = \Psi(\rho, f), \quad \sigma = f \Psi(\rho, f), \quad \Delta F(\rho, f) = \int_0^f df' \Psi(\rho, f').
\]

(65)

For a non-zero applied force \( f \), all these parameters have a non-zero value. This is to be contrasted, for instance, with the more standard mass transport models \([17, 19]\) (including the ZRP) which, in spite of the presence of a non-zero particle flux, have vanishing values of \( \Psi \) and \( \Delta F \), because their steady-state distribution is independent of the driving.

Future work may consider possible extensions of the model with asynchronous dynamics, where more complicated forms of the steady-state distribution (involving, e.g. matrix-product states) are likely to be needed. Applications of the model to the field of glassy dynamics could also be considered, by including kinetic constraints in the spirit of the model introduced in \([28]\).

Appendix A. Evaluation of the integral terms in the master equation

Calculations of the integrals appearing in the steady-state master equation, as formulated in equation (12), are straightforward. We provide here the explicit calculation for the case \( j = k = 1 \) (see equation (13)), using again the short notation \( S_i = m_i + m_{i+1} \) and \( S_i' = m'_i + m'_{i+1} \):

\[
\int \mathrm{d}m T_1(m'|m) Q_1(m) = \frac{1}{Z} \prod_{k=1}^{N'} \left[ \int_{0}^{\infty} \mathrm{d}m_{2k} \int_{0}^{\infty} \mathrm{d}m_{2k+1} K(m'_{2k}|S_{2k}) v(m_{2k})w(m_{2k+1}) \delta(S'_{2k} - S_{2k}) \right] \]

\[
= \frac{1}{Z} \prod_{k=1}^{N'} \left[ \frac{v(m'_{2k})w(m'_{2k+1})}{v * w(S'_{2k})} \int_{0}^{\infty} \mathrm{d}m_{2k} \int_{0}^{\infty} \mathrm{d}m_{2k+1} v(m_{2k})w(m_{2k+1}) \delta(S'_{2k} - S_{2k}) \right].
\]

(A.1)

Given that

\[
\int_{0}^{\infty} \mathrm{d}m_{2k} \int_{0}^{\infty} \mathrm{d}m_{2k+1} v(m_{2k})w(m_{2k+1}) \delta(S'_{2k} - S_{2k}) = v * w(S'_{2k})
\]

(A.2)

one eventually obtains

\[
\int \mathrm{d}m T_1(m'|m) Q_1(m) = Q_1(m).
\]

(A.3)

Calculations for other values of \( j, k \) follow the same lines. For instance, for \( k = 1 \) and \( j = 2 \), \( v \) and \( w \) are exchanged on the l.h.s. of equation (A.2), but the result is the same since the convolution product is commutative.
Appendix B. One- and two-site mass distributions and two-point correlation function

In this appendix, we derive the two-point correlation function as well as the one- and two-site mass distributions in the thermodynamic limit, provided that the saddle-point approximation discussed in section 4.1 can be performed.

B.1. Joint mass distribution on a pair of sites

The easiest distribution to compute is the joint distribution of masses \( P(m_i, m_{i+1}) \) on neighboring sites. Integrating equation (8) over the \( N-2 \) remaining variables \( m_j \) \( (j \neq i, i+1) \), one finds
\[
P(m_i, m_{i+1}) = \frac{Z_{N-2}(M - m_i - m_{i+1})}{Z_N(M)} [v(m_i)w(m_{i+1}) + w(m_i)v(m_{i+1})]. \tag{B.1}
\]

Using the ‘thermodynamic limit’ form of \( Z_N \), one finds
\[
\lim_{N \to \infty} \frac{Z_{N-2}(M - m_i - m_{i+1})}{Z_N(M)} = \exp \left[ -2\varphi(\rho) + \mu(\rho)(m_i + m_{i+1} - 2\rho) \right]. \tag{B.2}
\]

Hence the distribution \( P(m_i, m_{i+1}) \) can be written as
\[
P(m_i, m_{i+1}) = C_2(\rho) e^{\mu(\rho)(m_i + m_{i+1})} [v(m_i)w(m_{i+1}) + w(m_i)v(m_{i+1})], \tag{B.3}
\]
where \( C_2(\rho) \) is a normalization constant. It is convenient at this stage to introduce the auxiliary distributions \( p_v(m) \) and \( p_w(m) \) defined as
\[
p_v(m) = c_v(\rho) e^{\mu(\rho)m} v(m), \quad p_w(m) = c_w(\rho) e^{\mu(\rho)m} w(m), \tag{B.4}
\]
where \( c_v \) and \( c_w \) are normalization constants. In this way, the distribution \( P(m_i, m_{i+1}) \) given in equation (B.3) can be reformulated as
\[
P(m_i, m_{i+1}) = \frac{1}{2} [p_v(m_i) p_w(m_{i+1}) + p_w(m_i) p_v(m_{i+1})]. \tag{B.5}
\]

The same calculation holds for the joint distribution \( P_j(m_i, m_{i+j}) \) of the masses \( m_i \) and \( m_{i+j} \) on distant sites \( i \) and \( i+j \), as long as \( j \) is odd. One thus has
\[
P_j(m_i, m_{i+j}) = \frac{1}{2} [p_v(m_i) p_w(m_{i+j}) + p_w(m_i) p_v(m_{i+j})] \quad (j = 2k - 1, k > 0). \tag{B.6}
\]

When \( j \) is even, the calculation is slightly more complicated; one has
\[
P_j(m_i, m_{i+j}) = \frac{Z_{N'-2,N'}(M - m_i - m_{i+j})}{Z_N(M)} v(m_i)v(m_{i+j})
+ \frac{Z_{N',N'-2}(M - m_i - m_{i+j})}{Z_N(M)} w(m_i)w(m_{i+j}), \tag{B.7}
\]
with \( N' = N/2 \) and where the quantity \( Z_{N_1,N_2}(M) \) is defined as
\[
Z_{N_1,N_2}(M) = \int \prod_{i=1}^{N_1+N_2} dm_i \prod_{i=1}^{N_1} v(m_i) \prod_{i=N_1+1}^{N_2} w(m_i) \delta \left( \sum_{i=1}^{N_1+N_2} m_i - M \right). \tag{B.8}
\]
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However, in the thermodynamic limit $N' \to \infty$, the two prefactors $Z_{N'-2,N'}/Z_N$ and $Z_{N',N'-2}/Z_N$ have the same limit, again given by equation (B.2). Hence the distribution reduces in the thermodynamic limit to

$$P_j(m_i, m_{i+j}) = \frac{1}{2} \left[ p_v(m_i) p_v(m_{i+j}) + p_w(m_i) p_w(m_{i+j}) \right] \quad (j = 2k, k > 0). \quad (B.9)$$

Using the more physically meaningful parameterization in terms of the functions $\epsilon(\rho)$ and $h(\rho)$, the distribution $P_j(m_i, m_{i+j})$ can also be written for all $j > 0$ in the form

$$P(m_i, m_{i+1}) = 2 C_2(\rho) e^{-\epsilon(m_i) + \epsilon(m_{i+1}) + \mu(m_i + m_{i+1})} \cosh \left[ h(m_i) + (-1)^j h(m_{i+1}) \right]. \quad (B.10)$$

As an explicit example, $P_j(m_i, m_{i+j})$ reads in the specific case $\epsilon(m) = \epsilon_0 m$ and $h(m) = h_0 m$

$$P(m_i, m_{i+j}) = \frac{\left[ (\epsilon_0 - \mu(\rho))^2 - h_0^2 \right]^2}{(\epsilon_0 - \mu(\rho))^2 + (-1)^j h_0^2} e^{-\epsilon_0(\rho)} e^{\mu(\rho)(m_i + m_{i+j})} \cosh \left( h_0 m_i + (-1)^j h_0 m_{i+j} \right) \quad (B.11)$$

where $\mu(\rho)$ is given by equation (33). We recall here that $(\epsilon_0 - \mu(\rho))^2 - h_0^2$ is always strictly positive, as can be checked from the expression of $\mu(\rho)$ in equation (33).

**B.2. Two-point correlation**

The two-point correlation function $G_j$ between the masses $m_i$ and $m_{i+j}$, defined as

$$G_j = \langle m_i m_{i+j} \rangle - \rho^2 \quad (B.12)$$

takes a simple form. From equations (B.6) and (B.9), one has for $k > 0$

$$G_{2k-1} = \langle m \rangle_v \langle m \rangle_w - \rho^2 \quad (B.13)$$

$$G_{2k} = \frac{1}{2} (\langle m \rangle_v^2 + \langle m \rangle_w^2) - \rho^2 \quad (B.14)$$

where $\langle \ldots \rangle_v$ and $\langle \ldots \rangle_w$ are averages over the distributions $p_v(m)$ and $p_w(m)$ respectively. Obviously, $G_j$ is 2-periodic for $j > 0$. In the example $\epsilon(m) = \epsilon_0 m$ and $h(m) = h_0 m$, $G_j$ is given by

$$G_j = \left( \frac{\rho h_0}{\epsilon_0 - \mu(\rho)} \right)^2 \left( \frac{2 + (-1)^j}{\epsilon_0 - \mu(\rho)} \right) \frac{h_0^2 + (2 + (-1)^j) \left( \epsilon_0 - \mu(\rho) \right)^2}{\epsilon_0 - \mu(\rho) + (-1)^j h_0^2}. \quad (B.15)$$

In the limit where $f = 2h_0$ is small, one can expand $G_j$ to the leading order, yielding

$$G_j \rightarrow \frac{2 + (-1)^j}{\epsilon_0 - \mu(\rho)} \frac{h_0^2}{\epsilon_0^2} f^2 + \mathcal{O} \left( f^4 \right). \quad (B.16)$$

**B.3. Single-site distribution**

The single-site distribution $p(m)$ is obtained by integrating the two-site distribution over one of the masses. Using equation (B.5) for instance, we get

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\[ p(m) = \frac{1}{2} [p_u(m) + p_o(m)] \]  
\[ \text{or equivalently, in terms of } \varepsilon(m) \text{ and } h(m), \]

\[ p(m) = c(\rho) e^{-\varepsilon(m) + \mu(\rho)m} \cosh h(m), \]  
\[ \text{where } c(\rho) \text{ is a normalization constant.} \]

Appendix C. The link between the flux $\phi$ and the non-equilibrium free energy $\varphi$

When one goes from one configuration $\mathbf{m}$ to another $\mathbf{m}'$, the local instantaneous current $\Delta_{i,i+1}(\mathbf{m}, \mathbf{m}')$, which goes to the right on the link $(i,i+1)$ is

\[ \Delta_{i,i+1}(\mathbf{m}, \mathbf{m}') = -(m_i' - m_i) = m_{i+1}' - m_{i+1}. \]  
\[ \text{(C.1)} \]

Summing over all the links, the total mass transferred during the transition $\mathbf{m} \rightarrow \mathbf{m}'$, $\Delta(\mathbf{m}, \mathbf{m}')$, is given by

\[ \Delta(\mathbf{m}, \mathbf{m}') = \begin{cases} -\sum_{k=1}^{N'} (m_{2k}^2 - m_{2k}) = \frac{1}{2} \sum_{i=1}^{N} (-1)^i (m_i - m_i') & \text{for part } \mathcal{P}_1 \\ -\sum_{k=0}^{N'-1} (m_{2k+1}' - m_{2k+1}) = \frac{1}{2} \sum_{i=1}^{N} (-1)^i (m_i' - m_i) & \text{for part } \mathcal{P}_2 \end{cases} \]  
\[ \text{(C.2)} \]

On average,

\[ \langle \Delta(\mathbf{m}, \mathbf{m}') \rangle = \frac{1}{4} \int \mathbf{dm} \mathbf{dm}' \left( \sum_{i=1}^{N} (-1)^i (m_i - m_i') \right) T_1(\mathbf{m}'|\mathbf{m}) P(\mathbf{m}) \]
\[ + \frac{1}{4} \int \mathbf{dm} \mathbf{dm}' \left( \sum_{i=1}^{N} (-1)^i (m_i' - m_i) \right) T_2(\mathbf{m}'|\mathbf{m}) P(\mathbf{m}). \]  
\[ \text{(C.3)} \]

Since $\int \mathbf{dm}' T_1(\mathbf{m}'|\mathbf{m}) = 1$ ($k = 1, 2$), the terms involving $\sum_{i=1}^{N} m_i$ cancel out. Using equation (10) and (13), one gets

\[ \langle \Delta(\mathbf{m}, \mathbf{m}') \rangle = \frac{1}{2Z_N(M)} \int \mathbf{dm}' \left( \sum_{i=1}^{N} (-1)^i m_i' \right) Q_2(\mathbf{m}') \]
\[ - \frac{1}{2Z_N(M)} \int \mathbf{dm}' \left( \sum_{i=1}^{N} (-1)^i m_i' \right) Q_1(\mathbf{m}'). \]  
\[ \text{(C.4)} \]

To go further, one needs to use the physical interpretation of the dynamics, given in section 3.3. Indeed, using equations (10) and (14), one notices that in the (linear) case where $h(m) = \frac{1}{2} f m$,

\[ \int \mathbf{dm} \left( \sum_{i=1}^{N} (-1)^i m_i \right) Q_k(\mathbf{m}) = 2 \int \mathbf{dm} \frac{H(m)}{f} Q_k(\mathbf{m}) = 2(-1)^k \frac{\partial Q_k}{\partial f}(\mathbf{m}). \]  
\[ \text{(C.5)} \]

Eventually, using equation (11), the total averaged mass transferred is equal to

\[ \langle \Delta(\mathbf{m}, \mathbf{m}') \rangle = \frac{\partial \ln Z_N}{\partial f}. \]  
\[ \text{(C.6)} \]
leading to the final expression of the mass current $\Phi$ (mass transferred per link and per time step)

$$\Phi = \frac{\langle \Delta(m, m') \rangle}{N} = \frac{1}{N} \frac{1}{\partial f} = -\frac{\partial \varphi}{\partial f}(\rho, f),$$

thus proving the relation given in equation (48).

Appendix D. Evaluation of the entropy production rate

In this appendix, we evaluate the entropy production rate in the model defined in section 2. From equation (5), the transition rate $T(m'|m)$ takes the form

$$T(m'|m) = \frac{1}{2} T_1(m'|m) + \frac{1}{2} T_2(m'|m)$$

where $T_1(m'|m)$ and $T_2(m'|m)$ respectively describe the redistributions over the partitions $P_1$ and $P_2$ of the lattice. For a given configuration $m$, we define the sets $D_1(m)$ and $D_2(m)$ as the subsets of the configurations $m'$ accessible from $m$ through redistributions over the partitions $P_1$ and $P_2$. More formally, one has for $j \in \{1, 2\}$,

$$D_j(m) = \{m'|\forall k = 1, \ldots, N', m'_{2k+j-1} + m'_{2k+j} = m_{2k+j-1} + m_{2k+j}\}.$$  \hspace{1cm} (D.2)

Using the subsets $D_1(m)$ and $D_2(m)$, one can express the ratio of reciprocal, nonzero transition probabilities, so that the entropy production reads, in the steady state,

$$\Delta_{int}S = \frac{1}{2} \int dm P(m) \left\{ \int_{D_1(m)} dm' T_1(m'|m) \ln \frac{T_1(m'|m)}{T_1(m|m')} + \int_{D_2(m)} dm' T_2(m'|m) \ln \frac{T_2(m'|m)}{T_2(m|m')} \right\}.$$  \hspace{1cm} (D.3)

The ratios of the transition rates can be expressed as

$$\ln \frac{T_1(m'|m)}{T_1(m|m')} = [E(m) - E(m')] + [H(m) - H(m')]$$  \hspace{1cm} (D.4)

$$\ln \frac{T_2(m'|m)}{T_2(m|m')} = [E(m) - E(m')] - [H(m) - H(m')]$$  \hspace{1cm} (D.5)

The restriction of the integration domains to the subsets $D_1(m)$ and $D_2(m)$ in equation (D.3) was only needed so as to be able to properly define the ratio of the reverse transition probabilities. Once equation (D.3) is rewritten in terms of the observables $E(m)$ and $H(m)$, the integration domains no longer need to be restricted to these subsets, since the transition probabilities $T_1(m'|m)$ and $T_2(m'|m)$ that appear in the integrals vanish by definition outside the subsets $D_1(m)$ and $D_2(m)$. Hence one has

$$\Delta_{int}S = \frac{1}{2} \int dm dm' P(m) \left[ T_1(m'|m) \left( E(m) - E(m') + H(m) - H(m') \right) + T_2(m'|m) \left( E(m) - E(m') - H(m) + H(m') \right) \right].$$  \hspace{1cm} (D.6)
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The part of the integral involving $E$ is easily shown to vanish. Using the form $P(m) = \frac{1}{2}[P_1(m) + P_2(m)]$ of the probability distribution—see equation (9)—thanks to equation (13) one has that $\int d m T_k(m|m)P_2(m) = P_k(m)$. The $H$-dependent part in equation (D.6) can then be simplified, after a straightforward calculation, to

$$\Delta_{\text{int}} S = \frac{1}{2} \int d m \left[ P_1(m) - P_2(m) \right] H(m),$$ (D.7)

which is precisely equation (51).

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