From Dark Energy to Dark Matter via Non-Minimal Coupling

Andrzej Borowiec�,*

�Institute of Theoretical Physics, University of Wroclaw
Pl. Maksy Borna 9, 50-204 Wroclaw, Poland.

and

Joint Institute for Nuclear Research, Dubna,
Moscow region 141980, Russia

Toy cosmological models based on non-minimal coupling between gravity and scalar dilaton-like field are presented in the framework of Palatini formalism. They have the following property: preceding to a given cosmological epoch is a dark energy epoch with an accelerated expansion. The next (future) epoch becomes dominated by some kind of dark matter.

A. Preliminaries and notation

Modification of Einstein’s General Relativity becomes viable candidate to address accelerated expansion, dark matter and dark energy problems in modern cosmology (see e.g. [1, 2] and references therein). This includes modified theories with non-trivial gravitational coupling [3–7]. Particularly, viable non-minimal models unifying early-time inflation with late-time acceleration have been discussed in [5].

Main object of our considerations in this note is cosmological applications of some non-minimally coupled scalar–tensor Lagrangians of the type

\[ L = \sqrt{g} \left( f(R) + F(R) L_d \right) + L_{\text{mat}} \]  

(1)
treated within Palatini approach as in [4]. Hereafter we set \( L_d = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \) the Lagrangian for a scalar (massless) dilaton-like field \( \phi \) and \( L_{\text{mat}} \) represents any matter Lagrangian. Because of Palatini formalism \( R \) is a scalar \( R = R(g, \Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma) \) composed of the metric \( g \) and the Ricci tensor \( R_{\mu\nu}(\Gamma) \) of the symmetric (≡ torsionless) connection \( \Gamma \) (for more details concerning Palatini formalism see e.g. [8–10]). Therefore \( (g, \Gamma) \) are dynamical variables. Particularly, the metric \( g \) will be used for raising and lowering indices.

*Electronic address: borow@ift.uni.wroc.pl, borovec@theor.jinr.ru
We began with recalling some general formulae already developed in [4]: both \( f(R) \) and \( F(R) \) are assumed to be analytical functions of \( R \). Dynamics of the system (1) is controlled by the so-called master equation

\[
2 f(R) - f'(R) R + \tau = (F'(R) R - F(R)) L_d \tag{2}
\]

where prime denotes derivative with respect to \( R \). We set \( T^{\text{mat}}_{\mu\nu} = g^{\mu\nu} T^{\text{mat}}_{\mu\nu} \) and \( T^d_{\mu\nu} = g^{\mu\nu} T^d_{\mu\nu} = L_d \) for traces of the stress-energy tensors: matter \( T^{\text{mat}}_{\mu\nu} = \frac{\delta L_{\text{mat}}}{\delta g_{\mu\nu}} \) and dilaton \( T^d_{\mu\nu} = -\frac{1}{2} \partial_{\mu} \phi \partial_{\nu} \phi \).

Equations of motion for gravitational fields \((\Gamma, g)\) can be recast [4] into the form of generalized Einstein equations

\[
R_{\mu\nu}(h) \equiv R_{\mu\nu}(bg) = g_{\mu\alpha} P^\alpha_{\nu} \tag{3}
\]

(see also [9, 10]), where \( R_{\mu\nu}(\Gamma) \) is now the Ricci tensor of the new conformally related metric \( h = bg \). The conformal factor \( b \) is specified below and a \((1, 1)\) tensor \( P^\mu_{\nu} \) is defined by:

\[
P^\mu_{\nu} = c \frac{g^\mu_{\nu}}{b} - \frac{F(R)}{b} T^d_{\mu\nu} + \frac{1}{b} T^{\text{mat}}_{\mu\nu}
\]

Here one respectively has:

\[
\begin{cases}
c = \frac{1}{2} (f(R) + F(R) L_d) = (L - L_{\text{mat}}) / 2 \sqrt{g} \\
b = f'(R) + F'(R) L_d
\end{cases} \tag{4}
\]

Field equations for the scalar (dilaton-like) field \( \phi \) is

\[
\partial_\nu (\sqrt{g} F(R) g^{\mu\nu} \partial_\mu \phi) = 0 \tag{5}
\]

which reproduces the same field equations as treated in [4].

B. Cosmology from the generalized Einstein equations

We assume the physical metric \( g \) to be a standard Friedmann-Robertson-Walker (FRW) metric \( g \):

\[
g = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right) \tag{6}
\]

where \( a(t) \) is a scale factor. We also suppose the Cosmological Principle to hold. The matter content \( T^{\text{mat}}_{\mu\nu} \) of the universe is described by a non-interacting mixture of perfect fluids. We denote by \( w_i \) the barotropic coefficients. Each species is represented by the stress-energy tensor
\( T_{\mu\nu}^{(i)} = (\rho_i + p_i) u_\mu u_\nu + p_ig_{\mu\nu} \) satisfying a metric (with the Christoffel connection of \( g \)) conservation equation \( \nabla(\rho)_{\mu} T^{(i)}_{\mu\nu} = 0 \) (see [11]). This gives rise to the standard relations between pressure and density (equation of state) \( p_i = w_i \rho_i \) and \( \rho_i = \eta_i a^{-3(1+w_i)} \).

It follows thence that the generalized Einstein equations lead to the generalized Friedmann equation under the form:

\[
\left( \frac{\dot{a}}{a} + \frac{b}{2b} \right)^2 = \frac{F(R) L_d}{6b} + c \frac{b}{3b} + \sum_i \frac{(1 + 3w_i) \eta_i}{6b} a^{-3(1+w_i)}
\] (7)

where \((1 + 3w_i) \eta_i a^{-3(1+w_i)}\) represents a perfect fluid component with an equation of state (EoS) parameter \( w_i \).

Let us observe that for standard cosmological model based on the standard Einstein-Hilbert variational principle

\[
L_{EH} = \sqrt{g}R + L_{mat}
\] (8)

(considered both in purely metric as well as in Palatini formalism) the corresponding Friedmann equation takes the form

\[
H^2 \equiv \frac{\dot{a}}{a} = \frac{1}{3} \sum \eta_i a^{-3(1+w_i)}
\] (9)

when coupled to (non-interacting) multi-component perfect fluid. This is due to the fact that geometry contributes to the r.h.s. of the Friedmann equation through

\[
R = -T_{\mu\nu}^{\text{mat}} = \sum_i \frac{(1 - 3w_i) \eta_i}{6b} a^{-3(1+w_i)}
\]

For example, the preferred ΛCDM model requires three fluid components: cosmological constant \( w_\Lambda = -1 \), dust \( w_{\text{dust}} = 0 \) and radiation \( w_{\text{rad}} = \frac{1}{3} \) and can be obtained from (7) provided \( \alpha = \beta = \gamma = 0 \).

On the other hand we have that the field equation for the scalar field \( \phi \equiv \phi(t) \) is \( \frac{d}{dt}(\sqrt{g}F(R)\dot{\phi}) = 0 \), so that \( \sqrt{g}F(R)\dot{\phi} = \text{const} \) and consequently \( gF(R)^2 L_d = A^2 = \text{const} \). This simply implies that:

\[
F(R)^2 L_d = A^2 a^{-6}
\] (10)

with an arbitrary positive integration constant \( A^2 \) (see (5)).
C. Toy cosmological models

Our objective here is to investigate a possible cosmological applications of the following subclass of gravitational Lagrangians (1)

\[ L = \sqrt{g} \left( R + \alpha R^2 + \beta R^{1+\delta} + \gamma R^{1+2\delta} L_d \right) + L_{mat} \]  

(11)

where \( \alpha, \beta, \gamma, \delta, \) are free parameters of the theory. It should be observed that the gravitational part \( f(R) \) contains Starobinsky term [12] with some \( R^{1+\delta} \) contribution. In the limit \( \alpha, \beta, \gamma \to 0 \) our Lagrangians reproduce General Relativity. The numerical value for the constant \( \gamma \) (when \( \neq 0 \)) is unessential since it can be always incorporated (by re-scaling) into the field \( \phi \). As matter contribution we assume two non-interacting most natural components: pressureless dust \( (w_{dust} = 0) \) and radiation \( w_{rad} = \frac{1}{3} \).

Following common strategy particularly applicable within Palatini formalism (see [8–10]) one firstly finds out an exact solution of the master equation (2). In the case under consideration it can be chosen as

\[ R = \left[ \frac{\eta}{(1-\delta)\beta} \right]^\frac{1}{1+\delta} a^{-\frac{3}{1+\delta}} \equiv \xi a^{-\frac{3}{1+\delta}} \]  

(12)

where \( \xi \equiv \left[ \frac{\eta}{(1-\delta)\beta} \right]^\frac{1}{1+\delta} \) provided the integration constant \( A \) (see 10) takes the value

\[ A^2 = \frac{\gamma}{2\delta} \left[ \frac{\eta}{(1-\delta)\beta} \right]^2 \]  

(13)

which can vanish only in the case \( \gamma = 0 \) (no dilaton) and/or \( \eta = 0 \) (no matter).

Then the conformal factor \( b \) reads:

\[ b = \frac{1 + 4\delta}{2\delta} + 2\alpha \xi a^{-\frac{3}{1+\delta}} + \beta(1+\delta)\xi \delta a^{-\frac{3\delta}{1+\delta}} \]  

(14)

As a consequence, we have obtained the generalized Friedmann equations under the form:

\[ \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{2b} \right)^2 = (6b)^{-1} \left[ \frac{1+\delta}{\delta} \xi a^{-\frac{3}{1+\delta}} + \alpha \xi^2 a^{-\frac{6}{1+\delta}} + \right. \]

\[ \left. + \frac{\delta}{1-\delta} \eta a^{-3} + 2\eta_{rad} a^{-4} \right] \]  

(15)

We are now in position to calculate the generalized Hubble factor as

\[ \frac{\dot{a}}{a} + \frac{\dot{b}}{2b} \equiv \frac{\dot{a}}{a} B \equiv H B \]  

(16)
where $H$ denotes ordinary Hubble "constant" and

$$B = \frac{1+4\delta}{\delta} - \frac{2+2\delta}{1+\delta} \alpha \xi a^{-\frac{3}{1+\delta}} + (2 - \delta) \beta \xi^4 a^{-\frac{3\delta}{1+\delta}} + \frac{\delta}{1 - \delta} \eta a^{-3} + 2 \eta_{rad} a^{-4} \tag{17}$$

Before proceeding further let us observe that scaling properties of (7) are analogical to that in standard cosmology (9). More exactly, choosing some reference epoch $a_e \equiv a(t_e)$, e.g. the current cosmological epoch, one can rewrite the generalized Friedmann equation (18) as

$$H^2 B^2 = (6b)^{-1} \left[ \frac{1+\delta}{\delta} \xi e \left[ \frac{a}{a_e} \right]^{-\frac{3}{1+\delta}} + \alpha \xi a^{-\frac{6}{1+\delta}} + \frac{\delta}{1 - \delta} \eta a^{-3} + \frac{\delta}{1 + 4\delta} \eta_{rad} a^{-4} \right] \tag{19}$$

where one has $\eta_e = \eta a_e^{-\frac{3}{1+\delta}}$, $\eta_{rad,e} = \eta_{rad} a_e^{\frac{4}{1+\delta}}$, $\xi_e = \left[ \frac{\eta_e}{(1-\delta)^{1/3}} \right]^{\frac{1}{1+\delta}}$,

$$b = \frac{1+4\delta}{2\delta} + 2\alpha \xi e \left[ \frac{a}{a_e} \right]^{-\frac{3}{1+\delta}} + \beta (1+\delta) \xi^4 e \left[ \frac{a}{a_e} \right]^{-\frac{3\delta}{1+\delta}} + 2 \eta_{rad} a^{-4} \tag{20}$$

eetc..

Assume now that $0 < \delta < 1$.

Thus for $\frac{a}{a_e} \gg 1$, one can approximate (19) by the following Hubble law

$$H^2 \approx \frac{1}{3} \left[ \frac{1+\delta}{1+4\delta} \xi e \left[ \frac{a}{a_e} \right]^{-\frac{3}{1+\delta}} + \frac{\delta}{1 + 4\delta} \xi e \left[ \frac{a}{a_e} \right]^{-\frac{3\delta}{1+\delta}} + \frac{\delta^2}{(1+4\delta)(1-\delta)} \eta e \left[ \frac{a}{a_e} \right]^{-3} + \frac{2\delta}{1 + 4\delta} \eta_{rad,e} \left[ \frac{a}{a_e} \right]^{-4} \right] \tag{20}$$

Due to the factor $\delta$ the "true matter" and radiation decay as $\delta \to 0$. From the other hand the first term on the r.h.s of (20) plays a role of matter: it can be considered as "dark matter" which amount is controlled by the factor $\xi_e$. In the regime $\delta \to 1$ the first term gives a bit of acceleration expansion while the second mimics matter (dark matter).

For $\frac{a}{a_e} \ll 1$ (preceding epoch), Friedmann type approximation reads instead

$$H^2 \approx \frac{1}{3} \left[ \frac{(1+\delta)^3}{\delta(1-2\delta)^2} \alpha \xi e \left[ \frac{a}{a_e} \right]^{-\frac{3}{1+\delta}} + \frac{(1+\delta)^2}{(1-2\delta)^2} \xi e \left[ \frac{a}{a_e} \right]^{-\frac{3\delta}{1+\delta}} + \frac{\delta(1+\delta)^2}{((1-2\delta)^2(1-\delta) \alpha \xi e} \left[ \frac{a}{a_e} \right]^{-\frac{3\delta}{1+\delta}} + \frac{2(1+\delta)^2}{(1-2\delta)^2 \alpha \xi e} \left[ \frac{a}{a_e} \right]^{-\frac{3\delta}{1+\delta}} \right] \tag{21}$$
This epoch is dominated by dark energy in the form of cosmological constant which produces Starobinsky inflation. The universe described by Freedmann equation (21) undergoes two additional phases of accelerated expansion (power–law inflation) followed by the matter dominated era when $\delta \to 0$. Similarly, when $\delta \to 1$. In this scenario the evolution goes from dark energy to dark matter dominated eras. More detailed study of such models will be given elsewhere.

Acknowledgment

This note is dedicated to Sergei Odintsov on the occasion of his birthday. I thank Gianluca Allemandi, Salvatore Capozziello and Mauro Francaviglia for helpful discussions.

[1] E. J. Copeland, M. Sami, S. Tsujikawa, Int. J. Mod. Phys. D15:1753-1936 (2006) [hep-th/0603057]; S. Nojiri, S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. 4:115-146 (2007) [hep-th/0601213]; S. Capozziello, M. Francaviglia, Gen. Rel. Grav. 40:357-420 (2008) [astro-ph/0706.1146];
[2] S. Nojiri and S.D. Odintsov, Phys. Rev. D68: 123512 (2003) [hep-th/0307288]; S. Nojiri and S.D. Odintsov, Gen. Rel. Grav. 36:1765 (2004) [hep-th/0308176]; S. Nojiri and S.D. Odintsov, Mod. Phys. Lett. A19:627 (2004) [hep-th/0310045];
[3] S. Nojiri and S. Odintsov, Phys. Lett. B599:137 (2004) [astro-ph/0403622];
[4] G. Allemandi, A. Borowiec, M. Francaviglia and S.D. Odintsov, Phys.Rev. D72:063505 (2005) [gr-qc/0506123];
[5] S. Nojiri, S.D. Odintsov and P. Tretyakov, Progr. Theor. Phys. Suppl. 172:81 (2008) [hep-th/0710.5232];
[6] O. Bertolami and J. Pramos, On the non-trivial gravitational coupling to matter [gr-qc/0805.1241]; O. Bertolami, T. Harko, F.S.N. Lobo and J. Pramos, Non-minimal curvature-matter couplings in modified gravity [gr-qc/0811.2876];
[7] D. Puetzfeld and Y.N. Obukhov, Motion of test bodies in theories with nonminimal coupling [astro-ph/811.0913];
[8] M.Ferraris, M.Francaviglia and I.Volovich, Nouvo Cim. B108:1313 (1993); M.Ferraris, M.Francaviglia and I.Volovich, Class. Quant.Grav. 11:1505 (1994); A. Borowiec, M.Ferraris, M.Francaviglia and I.Volovich, Class. Quant.Grav. 15:43 (1998) [gr-qc/9611067];
[9] G. Allemandi, A. Borowiec and M. Francaviglia, Phys.Rev. D70:043524 (2004) [hep-th/0403264];
[10] G. Allemandi, A. Borowiec and M. Francaviglia, Phys.Rev. D70:103503 (2004) [hep-th/0407090];
[11] T. Koivisto, Class. Quant. Grav. 23:4289-4296 (2006) [gr-qc/0505128];
[12] A. Starobinsky, Sov. Phys. JETP Lett. 37:66, 37:126 (1983); Phys. Lett. B91:99 (1980);