Abstract

In this work, we propose a local Fourier analysis for multigrid methods with coarsening by a factor of three for the staggered finite-difference method applied to the Stokes equations. In [21], local Fourier analysis has been applied to a mass-based Braess-Sarazin relaxation, a mass-based $\sigma$-Uzawa relaxation, and a mass-based distributive relaxation, with standard coarsening on staggered grids for the Stokes equations. Here, we consider multigrid methods with coarsening by three for these relaxation schemes. We derive theoretically optimal smoothing factors for this coarsening strategy. The optimal smoothing factors of coarsening by three are nearly equal to those obtained from standard coarsening. Thus, coarsening by three is superior computationally. Moreover, coarsening by three generates a nested hierarchy of grids, which simplifies and unifies the construction of grid-transfer operators.

Keywords.
Multigrid, Stokes equations, local Fourier analysis, staggered finite-difference method, three-coarsening

1 Introduction

We are interested in multigrid methods [30] for the numerical solution of the Stokes equations. In the literature, different types of discretizations have been applied to the Stokes equations, for example, finite element methods [15, 11], finite difference methods [20, 9], and finite volume methods [32, 12]. As is well-known, the marker and cell method (MAC), a finite difference discretization, is one of the most effective numerical schemes for solving the Stokes equations [26, 22] and in this paper we focus on this method. It was introduced in [23, 29] for solving viscous, incompressible and transient fluid-flow problems, and has been extended to other problems. Many studies of multigrid methods focus on standard coarsening for the MAC scheme for the Stokes equations. The unknowns (velocity and pressure) are located at different locations: the velocity components are placed at the cell faces and the pressure is placed in the center of the cells, and, as a result, for standard coarsening, the locations of unknowns for coarse-grid problems are not subsets of these on the fine grid.
The choice of multigrid components, such as relaxation schemes and grid-transfer operators, plays an important role in designing fast algorithms. Block-structured relaxation schemes are often used for the Stokes equations, such as Braess-Sarazin relaxation [2, 21], Uzawa relaxation [24, 18, 14, 13], and distributive relaxation [1, 26, 28, 8, 25]. In our recent work [21], we presented three block-structured relaxation schemes for solving the Stokes equations discretized by MAC method: a mass-based Braess-Sarazin relaxation (Q-BSR), a mass-based \( \sigma \)-Uzawa relaxation (Q-\( \sigma \)-Uzawa), and a mass-based distributive relaxation (Q-DR), where we used a mass matrix \( Q \) derived from bilinear finite elements in two dimensions to approximate the inverse of a scalar Laplacian discretized by five-point finite difference method. Local Fourier analysis (LFA) was used to study smoothing with standard coarsening. We obtained an optimal smoothing factor of \( \frac{1}{3} \) for the mass-based distributive and Braess-Sarazin relaxation schemes, and \( \sqrt{\frac{17}{3}} \) for the mass-based \( \sigma \)-Uzawa relaxation. These relaxation schemes show high efficiency.

However, for standard coarsening in MAC scheme, different types of grid-transfer operators are needed for velocity components and pressure [25]. In contrast, coarsening by three generates a nested hierarchy of grids, which simplifies the definition of grid-transfer operators. A coarsening-by-three strategy has the potential advantage of coarsening more quickly and reducing the number of levels. Multigrid methods for the generalized Stokes equations with coarsening by three with distributive Gauss-Seidel smoothing is presented in [6]. For multigrid methods with coarsening by a factor of three applied to other problems, see [10, 17, 31]. Unfortunately, there are few studies on coarsening by three for the Stokes equations. Thus, we study \( 3h \)-coarsening for the MAC scheme.

Motivated by the advantages of multigrid with coarsening by three and the high efficiency of mass-based relaxation schemes, we wish to explore what the optimal smoothing factors are for these relaxation schemes. Choosing appropriate algorithmic parameters is challenging. Thus, we apply LFA to help us identify proper parameters and quantitatively predict multigrid convergence speed.

The main contribution of this work is the presentation of a theoretical analysis of optimal smoothing factors of three mass-based block-structured multigrid relaxation schemes for staggered grids using a three-coarsening strategy for the Stokes equations. We derive an optimal smoothing factor and show that it is \( \frac{17}{47} \approx 0.362 \) for Q-BSR and Q-DR, and an optimal smoothing factor of \( \sqrt{\frac{17}{47}} \approx 0.601 \) for Q-\( \sigma \)-Uzawa relaxation. Thus, Q-BSR outperforms other two relaxation schemes. Note that the optimal smoothing factors for coarsening by three are very close to those \( \frac{1}{3} \) and \( \sqrt{\frac{17}{47}} \approx 0.577 \) for standard coarsening. It means that the computation work of coarsening is competitive with standard coarsening.

To avoid solving Schur complement system exactly in Q-BSR, we propose an inexact version of Q-BSR, called Q-IBSR, where one sweep of weighted-Jacobi iteration is applied to the Schur complement system. Numerically, we find that Q-IBSR achieves the same convergence factor as that of exact version, that is, \( \frac{17}{47} \). Moreover, we study the influence of different types of grid-transfer operators on the actual multigrid convergence. We test the Stokes equations with Dirichlet boundary conditions, and numerical results show that Q-IBSR is not too sensitive to the boundary conditions, but we see degradation on convergence for Q-DR Q-\( \sigma \)-Uzawa relaxation.

The remainder of the paper is organized as follows. In Section 2, we review staggered finite-difference discretization for the Stokes equations. In Section 3, we derive optimal smoothing factors of LFA for three mass-based block-structured multigrid relaxation schemes proposed in [21] with coarsening by three. In Section 4, we study different grid-transfer operators by LFA and present some numerical results to validate our theoretical results. Some conclusions are drawn in Section 5.
2 Discretization

Consider the following Stokes equations in two dimensions

$$\begin{align*}
-\Delta u + \nabla p &= f, \\
\nabla \cdot u &= 0,
\end{align*}$$

where $u = \begin{pmatrix} u \\ v \end{pmatrix}$ is the velocity vector, and $p$ is the scalar pressure of a viscous fluid.

For discretization, we consider uniform meshes with $h_{x_1} = h_{x_2} = h$ and apply the MAC scheme \[27\] to equations (1) and (2). The discrete unknowns $u, v, p$ are defined at different positions on the grid: the velocity components $u, v$ and pressure $p$ are defined at the middle points of vertical edges ($\square$), at the middle points of horizontal edges ($\circ$), and in the center of each cell ($\star$), respectively, shown in Figure 1.

The MAC scheme of the Stokes equations is represented by the stencils \[27\]

$$\mathcal{L}_h = \begin{pmatrix}
-\Delta_h & 0 & (\tilde{e}_{x_1})_{h/2} \\
0 & -\Delta_h & (\tilde{e}_{x_2})_{h/2} \\
-(\tilde{e}_{x_1})_{h/2} & -(\tilde{e}_{x_2})_{h/2} & 0
\end{pmatrix},$$

where

$$-\Delta_h = \frac{1}{h^2} \begin{pmatrix}
-1 & -1 & 1 \\
-4 & 4 & -4 \\
-1 & -1 & 1
\end{pmatrix}, \quad (\tilde{e}_{x_1})_{h/2} = \frac{1}{h} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad (\tilde{e}_{x_2})_{h/2} = \frac{1}{h} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$
Discretizations (1) and (2) lead to the following saddle-point system

\[
\mathcal{L}_h x = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u_h \\ p_h \end{pmatrix} = \begin{pmatrix} f_h \\ 0 \end{pmatrix} = b_h,
\]

where \( A \) corresponds to the discretized vector Laplacian, \( B \) stands for the negative of the discrete divergence operator, \( B^T \) is the discrete gradient, and \( u_h = \begin{pmatrix} u_h \\ v_h \end{pmatrix} \).

Here, we are interested in multigrid methods for solving linear system (4). In multigrid, there are two important processes: smoothing and coarse-grid correction. For a given approximation \( x_k \) and a smoother \( \mathcal{M}_h \), an approximation of \( \mathcal{L}_h \), the smoothing process or relaxation scheme is

\[
x_{k+1} = x_k + \omega \mathcal{M}_h^{-1} (b_h - \mathcal{L}_h x_k),
\]

where \( \omega \) is a damping parameter to be determined. Then, the error-propagation operator for relaxation scheme (5) is

\[
S_h = I - \omega \mathcal{M}_h^{-1} \mathcal{L}_h.
\]

Here, we consider mass-based distributive weighted-Jacobi relaxation, Braess-Sarazin relaxation, and Uzawa-type relaxation proposed in [21] and employ LFA to investigate these mass-based relaxation schemes for coarsening by three in next section.

## Local Fourier analysis

LFA or local mode analysis [27] was first introduced by Brandt [3] to study smoothing of multigrid methods for boundary value problems. It has since been extended to many other problems. LFA is used to quantitatively analyze and predict the convergence speed of multigrid methods. There are two important factors in LFA: smoothing factor and two-grid convergence factor. In many cases, the smoothing factor of LFA, assuming an ideal coarse grid operator that annihilates the low frequency error components and leaves the high frequency components unchanged, gives a sharp prediction of actual multigrid convergence. The two-grid convergence factor of LFA estimates the effect of the real coarse grid operator and relaxation scheme, which sometimes poses challenges to derive analytically optimal parameters in multigrid. Thus, we focus on smoothing analysis.

We employ LFA to study the smoothing property of three mass-based block-structured multigrid relaxation schemes proposed in [21] for staggered discretizations with coarsening by three. High and low frequencies for coarsening by three are defined as

\[
\theta \in T^L = \left[ -\frac{\pi}{3}, \frac{\pi}{3} \right], \quad \theta \in T^H = \left[ -\frac{\pi}{2}, \frac{3\pi}{2} \right] \setminus \left[ -\frac{\pi}{3}, \frac{\pi}{3} \right].
\]

In the literature, there are some studies on LFA for coarsening by three. For example, an LFA [17] was proposed to design efficient geometric multigrid methods on hierarchical triangular grids using a three-coarsening strategy for a three-color and block-line type smoothers for the discrete Laplace operator. In [5], LFA was used to determine automatically the optimal values for the parameters involved in defining the polynomial smoothers for multigrid methods with aggressive coarsening to study Poisson equation.

Let us introduce the definition of symbol for a discrete operator [27, 22, 17] in LFA.
Definition 3.1. Let \( L_h = [s_\kappa]_h \) be a scalar stencil operator acting on grid \( G_h \) as
\[
L_h w_h(x) = \sum_{\kappa \in V} s_\kappa w_h(x + \kappa h),
\]
where \( s_\kappa \in \mathbb{R} \) is constant, \( w_h(x) \in l^2(G_h) \), and \( V \) is a finite index set. Then, the symbol of \( L_h \) is defined as:
\[
\tilde{L}_h(\theta) = \sum_{\kappa \in V} s_\kappa e^{i \theta \kappa}, \quad i^2 = -1. \tag{7}
\]

Definition 3.2. The LFA smoothing factor for the error-propagation operator \( S_h \) is defined as
\[
\mu_{\text{loc}}(S_h(p)) = \max_{\theta \in T_h} \{ \rho(\tilde{S}_h(\theta, p)) \}, \quad \tag{8}
\]
where \( p \) is algorithmic parameters, such as a damping parameter or some parameters involved in smoother \( M_h \), and \( \rho(\tilde{S}_h(\theta, p)) \) denotes the spectral radius of symbol \( \tilde{S}_h(\theta, p) \).

For the MAC scheme considered here, \( \tilde{S}_h \) is a 3 \times 3 matrix, due to the block structure of \( L_h \), see (3). We often need to minimize \( \mu_{\text{loc}}(S_h(p)) \) over algorithmic parameters \( p \) to obtain fast convergence speed. We define the LFA optimal smoothing factor as follows.

Definition 3.3. The LFA optimal smoothing factor for the error-propagation operator \( S_h \) is defined as
\[
\mu_{\text{opt}} = \min_{p \in \mathbb{R}^+} \mu_{\text{loc}}(S_h(p)). \tag{9}
\]

Our main goal of this work is to solve (9) analytically to identify optimal parameters to obtain optimal smoothing factors for three mass-based block-structured relaxation schemes introduced in the following.

In a two-grid method, the two-grid error operator can be expressed as
\[
E_h(\nu_1, \nu_2) = S_h^{\nu_2}(I - P_h(L_h^*)^{-1}R_h L_h)S_h^{\nu_1},
\]
where integers \( \nu_1 \) and \( \nu_2 \) are the number of pre- and postsmoothing steps, respectively. \( L_h^* \) is the coarse-grid operator, and \( P_h \) and \( R_h \) are interpolation and restriction operators, respectively. The choices of \( L_h^* \), \( R_h \), and \( P_h \) are very important for designing a good coarse-grid correction. As to the discrete operators on the coarser grids in the hierarchy, direct discretization of the continuous operators is used.

For the interpolation and restriction operators, we will consider the following choices, (10) and (11), and apply LFA to study the corresponding two-grid convergence factors.

\[
P_{h,25} = \frac{1}{9} \begin{bmatrix}
1 & 2 & 3 & 2 & 1 \\
2 & 4 & 6 & 4 & 2 \\
3 & 6 & 9 & 6 & 3 \\
2 & 4 & 6 & 4 & 2 \\
1 & 2 & 3 & 2 & 1
\end{bmatrix}, \tag{10}
\]
and the restriction is taken to be
\[
R_{h,1} = [1], \quad R_{h,9} = \frac{1}{9} \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}, \quad R_{h,9b} = \frac{1}{16} \begin{bmatrix}
1 & 2 & 1 \\
2 & 4 & 2 \\
1 & 2 & 1
\end{bmatrix}. \tag{11}
\]
Remark 3.1. Some adaption is needed to compute the symbols of \( R_h \) and \( P_h \) compared with these defined on collected grids (all discrete unknowns are defined at the same location). [16, Section 3.4] gives a general formula to compute symbols of grid-transfer operators defined on general grids, which can be used here.

Definition 3.4. Let \( \widehat{E}_h \) be the two-grid symbol of \( E_h \). Then, two-grid LFA convergence factor of \( E_h \) is defined as

\[
\rho_h(\nu_1, \nu_2) = \max_{(\theta, p) \in T_L} \rho(\widehat{E}_h(\theta, p)),
\]

where \( \rho(\widehat{E}_h(\theta, p)) \) denotes the spectral radius of matrix \( \widehat{E}_h(\theta, p) \).

In the two-grid method, coarse and fine grid operators are involved. The dimension of the space of 3\( h \)-harmonics for \( \theta \) is \( 3 \times 3 = 9 \), see [17]. The discrete Stokes equations is a \( 3 \times 3 \) block system.

Finally, \( \widehat{E}_h(\theta, p) \) is a \( 27 \times 27 \) matrix. Note that for fixed \( \nu_1 \) and \( \nu_2 \), \( \rho_h \) in (12) is a function of \( p \). In general, it is hard to theoretically minimize \( \rho_h \) over \( p \). Thus, our focus will be on solving (9).

Then, we use the corresponding optimal parameters \( p \) to compute the two-grid LFA convergence factor \( \rho_h \), and compare with actual multigrid performance. For simplicity, throughout the rest of this paper, we drop the subscript \( h \), except when necessary for clarity.

### 3.1 Mass matrix approximation to scalar Laplacian

In order to design good smoothers for (4), the approximation to the discrete Laplacian \( A \) plays an important role. Our recent work [19] shows that mass matrix obtained from bilinear elements in two dimensions is a good smoother for solving discrete Laplacian with standard coarsening. Here, we analyze “mass approximation” in multigrid with coarsening by three. The mass stencil for bilinear discretization in 2D is given by

\[
Q = \frac{h^2}{36} \begin{bmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{bmatrix}.
\]

Using (7), the symbol of \( Q \) is

\[
\tilde{Q}(\theta_1, \theta_2) = \frac{h^2}{9} (4 + 2 \cos \theta_1 + 2 \cos \theta_2 + \cos \theta_1 \cos \theta_2).
\]

Recall that the standard five-point scheme for \( -\Delta_h = A_s \) in (3), whose symbol is

\[
\tilde{A}_s = \frac{4 - 2 \cos \theta_1 - 2 \cos \theta_2}{h^2}.
\]

From (14) and (15), we have

\[
\tilde{Q} \tilde{A}_s = \frac{2}{9} (4 + 2 \cos \theta_1 + 2 \cos \theta_2 + \cos \theta_1 \cos \theta_2)(2 - \cos \theta_1 - \cos \theta_2).
\]

For mass-based relaxation applied to the scalar Laplacian with coarsening by three, we now present the optimal smoothing factor.

**Theorem 3.1.** Consider the relaxation error operator \( S_s = I - \omega QA_s \). For \( \theta \in T^3 \), we have

\[
(\tilde{Q} \tilde{A}_s)_{\min} = \frac{5}{6}, \quad (\tilde{Q} \tilde{A}_s)_{\max} = \frac{16}{9}.
\]
Moreover, the smoothing factor for $S_s$ with coarsening by three is

$$
\mu_{\text{opt}}(S_s) = \min_{\omega} \max_{\theta \in T^3} \{1 - \omega \tilde{Q} \tilde{A}_s\} = \frac{17}{47} \approx 0.362,
$$

provided that $\omega = \frac{36}{47}$.

Before giving the proof, we comment that the optimal smoothing for $S_s = I - \omega Q A_s$ with standard coarsening is $\frac{1}{3}$, as shown in [19]. From Theorem 3.1, we see that the optimal smoothing factor of $\frac{17}{47} \approx 0.362$ for coarsening by three is very close to $\frac{1}{3} = 0.333$.

**Proof.** Let $x = \cos \theta_1$, $y = \cos \theta_2$. Then, formula (16) can be rewritten as $\hat{Q} \hat{A}_s = \frac{2}{3} g(x, y)$, where

$$
g(x, y) = (2 - x - y)(4 + 2x + 2y + xy). \quad (18)
$$

For $\theta \in T^3$, see (6), it is easy to show that $(x, y) \in [-1, 1] \times [-1, \frac{1}{2}] \cup [-1, \frac{1}{2}] \times [\frac{1}{2}, 1] =: D$. To find minimum and maximum of $\hat{Q} \hat{A}_s$ with $\theta \in T^3$, we start by computing the partial derivatives of $g(x, y)$:

$$
g'_x = -2(2 + y)(2x + y),
g'_y = -(2 + x)(2y + x).
$$

Let $g'_x = g'_y = 0$ with $\theta \in T^3$. We have $x = y = 0$, that is, $(\cos \theta_1, \cos \theta_2) = (0, 0)$, and $g(0, 0) = 8$ which might be an extreme value.

Next, we only to find the extreme values of $g(x, y)$ at the boundary of $D$. However, due to the symmetry of (18), we only need to consider the following three boundaries:

$$
\partial D_1 = \{-1\} \times [-1, 1], \partial D_2 = \{1\} \times [-1, \frac{1}{2}], \partial D_3 = \left[\frac{1}{2}, 1\right] \times \{1\}.
$$

1. For $(x, y) \in \partial D_1$,

   $$
g(x, y) = g(-1, y) = (3 - y)(2 + y).
$$

   For $y \in [-1, 1]$, we have

   $$
g(-1, y)_{\text{max}} = g(-1, 1/2) = \frac{25}{4},
   g(-1, y)_{\text{min}} = g(-1, -1) = 4.
$$

2. For $(x, y) \in \partial D_2$,

   $$
g(x, y) = g(1, y) = 3(1 - y)(2 + y).
$$

   For $y \in [-1, 1/2]$, we have

   $$
g(1, y)_{\text{max}} = g(1, -1/2) = \frac{27}{4},
   g(1, y)_{\text{min}} = g(1, 1/2) = \frac{15}{4}.
$$
3. For \( (x, y) \in \partial D_3 \),
\[
g(x, y) = g(x, 1/2) = \frac{5}{2}(3/2 - x)(2 + x).
\]

For \( x \in [1/2, 1] \), we have
\[
\begin{align*}
g(x, 1/2)_{\text{max}} &= g(1/2, 1/2) = \frac{25}{4}, \\
g(x, 1/2)_{\text{min}} &= g(1, 1/2) = \frac{15}{4}.
\end{align*}
\]

Based on the above discussions, when \( \theta \in T^H \), \( g(x, y)_{\text{max}} = g(0, 0) = 8 \) and \( g(x, y)_{\text{min}} = g(1, 1/2) = \frac{15}{4} \).

It follows that when \( \theta \in T^H \),
\[
(\tilde{Q}\tilde{A})_{\text{max}} = \frac{2}{9} \times 8 = \frac{16}{9}, \quad (\tilde{Q}\tilde{A})_{\text{min}} = \frac{2}{9} \times \frac{15}{4} = \frac{5}{6}.
\]

Thus,
\[
\max_{\theta \in T^H} |1 - \omega \tilde{Q}\tilde{A}| = \max \left\{ \left| 1 - \frac{16}{9} \omega \right|, \left| 1 - \frac{5}{6} \omega \right| \right\}.
\]

To minimize \( \max_{\theta \in T^H} |1 - \omega \tilde{Q}\tilde{A}| \) over \( \omega \), it requires that
\[
\left| 1 - \frac{16}{9} \omega \right| = \left| 1 - \frac{5}{6} \omega \right|,
\]
which gives \( \omega = \frac{2}{16/9 + 5/6} = \frac{36}{47} \). Furthermore,
\[
\mu_{\text{opt}} = 1 - \frac{5}{6} \times \frac{36}{47} = \frac{17}{47},
\]
completing the proof.

Based on mass matrix, \( Q \), we proposed in [21] three mass-based block-structured multigrid relaxation schemes with standard coarsening for the Stokes equations. Here, we consider coarsening by three with the same mass-based block-structured relaxation schemes for the Stokes equations. We are interested in deriving optimal smoothing factors for these relaxation schemes with coarsening by three. Since the Laplacian appears in Stokes equations, Theorem 3.1 is very useful to carry out our analysis of mass-based block-structured multigrid relaxation schemes for the Stokes equations in the next subsections.

For simplicity, let \( m_s = (\tilde{Q}/h^2)^{-1} \) and \( m(\theta) = \sin^2(\theta_1/2) + \sin^2(\theta_2/2) \). Then, from (16) and (17), we have
\[
\tilde{Q}\tilde{A}_s = \frac{4m}{m_s} = m_r \in \left[ \frac{5}{6}, \frac{16}{9} \right] \text{ for } \theta \in T^H.
\] (19)

It can be shown that the symbol of \( \mathcal{L} \) defined in (3) is
\[
\tilde{\mathcal{L}}(\theta_1, \theta_2) = \frac{1}{h^2} \begin{pmatrix}
4m & 0 & i2h \sin \frac{\theta_1}{2} \\
0 & 4m & i2h \sin \frac{\theta_2}{2} \\
-i2h \sin \frac{\theta_1}{2} & -i2h \sin \frac{\theta_2}{2} & 0
\end{pmatrix}.
\]
3.2 Mass-based distributive relaxation

We review distributive relaxation following [4, 26]. To relax $\mathcal{L}\mathbf{x} = b$, we consider a transformed system $\mathcal{LP}\hat{\mathbf{x}} = \mathbf{b}$, where $\mathcal{P}\hat{\mathbf{x}} = \mathbf{x}$. Then, we can solve the new system with coefficient matrix $\mathcal{LP}$ efficiently. Here, $\mathcal{P}$ is given by

$$
\mathcal{P} = \begin{pmatrix}
I_h & 0 & (\partial_{x_1})_{h/2} \\
0 & I_h & (\partial_{x_2})_{h/2} \\
0 & 0 & \Delta h
\end{pmatrix}.
$$

The discrete matrix form of $\mathcal{P}$ is

$$
\mathcal{P} = \begin{pmatrix}
I & B^T \\
0 & -A_p
\end{pmatrix},
$$

where $-A_p$ is the standard five-point stencil of the Laplacian operator discretized at cell centers. It follows that

$$
\mathcal{K} = \mathcal{LP} = \begin{pmatrix}
-\Delta h & 0 & 0 \\
0 & -\Delta h & 0 \\
-(\partial_{x_1})_{h/2} & -(\partial_{x_2})_{h/2} & -\Delta h
\end{pmatrix}.
$$

(20)

We can apply block relaxation to the transformed system $\mathcal{K}\hat{\mathbf{x}} = \mathbf{b}$, for example, distributive weighted-Jacobi relaxation [22] and distributive Gauss-Seidel relaxation [4, 26]. The former is simple, but has a convergence factor of 0.6 [22]. The latter is not suitable for parallel computation although it has a two-grid convergence factor of 0.4 shown in [22]. Here, we consider mass-based distributive relaxation (refer to $Q$-DR) proposed in [21], where $\mathcal{K}$ is approximated by $\mathcal{M}_D$ given by

$$
\mathcal{M}_D = \begin{pmatrix}
\alpha_D C & 0 \\
B & \alpha_D E
\end{pmatrix}
$$

(21)

with

$$
C^{-1} = \begin{pmatrix}
Q & 0 \\
0 & Q
\end{pmatrix}
$$

(22)

and $E = Q_p^{-1}$, where $Q_p$ is the discrete mass matrix for the pressure unknowns.

It can be shown that the error propagation operator for the distributive relaxation scheme is given by $\mathcal{S}_D = I - \omega_D \mathcal{P} \mathcal{M}_D^{-1} \mathcal{L}$. Next, we examine the smoothing factor for $\mathcal{S}_D$.

The symbol of operator $\mathcal{K} = \mathcal{LP}$, see (20), is given by

$$
\tilde{\mathcal{K}}(\theta_1,\theta_2) = \frac{1}{h^2} \begin{pmatrix}
4m(\theta) & 0 & 0 \\
0 & 4m(\theta) & 0 \\
-i2h \sin \frac{\theta_1}{2} & -i2h \sin \frac{\theta_2}{2} & 4m(\theta)
\end{pmatrix}.
$$

The symbol of the block relaxation operator (21) with $C^{-1}$ given by the mass approximation (22) is

$$
\tilde{\mathcal{M}}_D(\theta_1,\theta_2) = \frac{1}{h^2} \begin{pmatrix}
\alpha_D m_s(\theta) & 0 & 0 \\
0 & \alpha_D m_s(\theta) & 0 \\
-i2h \sin \frac{\theta_1}{2} & -i2h \sin \frac{\theta_2}{2} & \alpha_D m_s(\theta)
\end{pmatrix}.
$$

Then, the eigenvalues of $\tilde{\mathcal{S}}_D(\alpha_D,\omega_D,\theta) = I - \omega_D \tilde{\mathcal{P}} \tilde{\mathcal{M}}_D^{-1} \tilde{\mathcal{L}}$ are $1 - \omega_D \frac{4m(\theta)}{\alpha_D m_s(\theta)}$. Now, we are able to give the optimal smoothing factor for $Q$-DR.
Theorem 3.2. The optimal smoothing factor for $Q$-DR with coarsening by three is

$$
\mu_{\text{opt},D} = \min_{\alpha_D, \omega_D} \max_{\theta \in T^h} \{ \rho(\tilde{S}_D(\alpha_D, \omega_D, \theta)) \} = \frac{17}{47} \approx 0.362,
$$

where the minimum is uniquely achieved at $\frac{\omega_D}{\alpha_D} = \frac{36}{47}$.

Proof. Since all eigenvalues of $\tilde{S}_D$ are $1 - \omega_D \frac{4m}{\alpha_D m_s} = 1 - \frac{\omega_D}{\alpha_D} \tilde{Q} \tilde{A}_s$, using Theorem 3.1, we know that

$$
\mu_{\text{opt}}(I - \frac{\omega_D}{\alpha_D} QA) = \frac{17}{47} \text{ with } \frac{\omega_D}{\alpha_D} = \frac{36}{47}.
$$

Thus, $\min_{\alpha_D, \omega_D} \max_{\theta \in T^h} \{ \rho(\tilde{S}_D(\alpha_D, \omega_D, \theta)) \} = \frac{17}{47}$. \qed

In [21], for the standard coarsening, we have shown that the optimal smoothing factor for $Q$-DR for MAC discretization of the Stokes equations is $\frac{1}{3}$. From Theorem 3.2, we see that the optimal smoothing factor for coarsening by three is competitive with that for standard coarsening.

Assume that the cost of one cycle standard multigrid (using one pre-smoothing and no post-smoothing step) is $W$ and the cost of coarsening by three is roughly $W/3$. To achieve tolerance $\epsilon$, the total cost of standard coarsening and coarsening by three are

$$
T_1 = W \log_{1/3} \epsilon, \quad T_2 = \frac{W}{3} \log_{17/47} \epsilon,
$$

respectively. Then,

$$
\frac{T_1}{T_2} \approx 2.78.
$$

(23)

So coarsening by three is computationally beneficial.

### 3.3 Mass-based Braess-Sarazin relaxation

We consider mass-based Braess-Sarazin relaxation scheme proposed in [21], named $Q$-BSR, where $M_h$ in (5) is given by

$$
M_B = \begin{pmatrix}
\alpha_B C & B^T \\
B & 0
\end{pmatrix},
$$

(24)

in which $C^{-1}$ is defined in (22), and $\alpha_B > 0$ is to be determined. In (5), let $\delta x = M_B^{-1}(b_h - L_h x_k)$. Then, the update $\delta x = (\delta_u, \delta_p)$ is given by

$$
(BC^{-1}B^T)\delta_p = BC^{-1}r_u - \alpha_B r_p, \\
\delta_u = \alpha_B^{-1}C^{-1}(r_u - B^T \delta_p),
$$

(25)

where $(r_u, r_p) = b_h - L_h x_k$.

For the Schur complement system (25), we will consider exact solve and inexact solve discussed.

#### 3.3.1 Exact Braess-Sarazin relaxation

We first consider solving (25) exactly, and derive optimal smoothing factor for the corresponding exact $Q$-BSR. Using $m_s = (\bar{Q}/h^2)^{-1}$, the symbol of $M_B$ is

$$
\tilde{M}_B(\theta_1, \theta_2) = \frac{1}{h^2} \begin{pmatrix}
\alpha_B m_s & 0 & i2h \sin \frac{\theta_1}{2} \\
0 & \alpha_B m_s & i2h \sin \frac{\theta_2}{2} \\
-i2h \sin \frac{\theta_1}{2} & -i2h \sin \frac{\theta_2}{2} & 0
\end{pmatrix}.
$$
It can be easily shown that the determinant of \( \tilde{\mathcal{L}} - \lambda \tilde{\mathcal{M}}_B \) is
\[
4\alpha_B m_m (\lambda - 1)^2 \left( \lambda - \frac{4m}{\alpha_B m_m} \right).
\]
It follows that the eigenvalues of \( \tilde{\mathcal{M}}_B^{-1} \tilde{\mathcal{L}} \) are 1, 1, and \( \frac{4m}{\alpha_B m_m} \). Note that \( \frac{4m}{\alpha_B m_m} \) is the eigenvalue of \( \tilde{\mathcal{Q}}_A \), and \( \frac{2}{9} < 1 < \frac{16}{7} \). From Theorem 3.1, we directly have the following result.

**Theorem 3.3.** The optimal smoothing factor for exact Q-BSR with coarsening by three is
\[
\mu_{\text{opt},B} = \min_{\alpha_B, \omega_B} \max_{\theta \in T} \rho(\tilde{S}_B(\alpha_B, \omega_B, \theta)) = \frac{17}{47},
\]
where the minimum is uniquely achieved at \( \frac{\omega_B}{\alpha_B} = \frac{36}{47} \) with \( \omega_B \in [30/47, 64/47] \).

We point out that for standard coarsening, the optimal smoothing factor for exact Q-BSR for MAC discretization of the Stokes equations is \( \frac{1}{3} \) [21]. Thus, using (23) for exact Q-BSR, coarsening by three is better than standard coarsening.

### 3.3.2 Inexact Braess-Sarazin relaxation

Solving the Schur complement system (25) with coefficient matrix \( BC^{-1}B^T \) directly is expensive. Many studies have shown that a good approximation for (25) is sufficient [33], for example, using a simple sweep of a Gauss-Seidel [25] or weighted Jacobi iteration [22, 21]. Moreover, our previous work [21] exhibits that for standard coarsening inexact Q-BSR can achieve the same convergence factor as the exact Q-BSR for MAC discretization of the Stokes equations. Thus, we also consider mass-based inexact Braess-Sarazin relaxation (refer to Q-IBSR), where we apply a single sweep of weighted (\( \omega_J \)) Jacobi iteration to approximate the solution of (25). Theoretical analysis of the optimal smoothing factor is challenging due to complex eigenvalues of \( \tilde{S} \), the inexact version, so we numerically study the performance of inexact BSR under the condition that \( \frac{\omega_B}{\alpha_B} = \frac{36}{47} \) because \( 1 - \frac{\omega_B}{\alpha_B} m_m \) is an eigenvalue for both exact and inexact version. Our findings in Section 4 show that inexact BSR of a two-grid method can obtain the same convergence factor, \( \frac{17}{47} \), as the exact version with \( \omega_B = 1, \alpha_B = \frac{47}{36} \omega_B \) and \( \omega_J = 0.9 \).

### 3.4 Mass-based \( \sigma \)-Uzawa relaxation

Uzawa-type relaxation is a popular family of algorithms for solving saddle-point systems [14, 24]. We consider mass-based \( \sigma \)-Uzawa, called Q-\( \sigma \)-Uzawa, proposed in [21], that is, \( \mathcal{M}_h \) in (5) is taken to be
\[
\mathcal{M}_U = \begin{pmatrix}
\alpha_U C & 0 \\
B & -\sigma^{-1}I
\end{pmatrix},
\]
where \( C^{-1} \) is defined in (22).

To identify the optimal smoothing factor for \( S_U = I - \omega_U \mathcal{M}^{-1}_U \mathcal{L} \), we first compute the eigenvalues of \( \mathcal{M}_U^{-1} \mathcal{L} \). The symbol of \( \mathcal{M}_U \) is
\[
\mathcal{M}_U(\theta_1, \theta_2) = \frac{1}{h^2} \begin{pmatrix}
\alpha_U m_m & 0 & 0 \\
0 & \alpha_U m_m & 0 \\
-i2h \sin \frac{\theta_1}{2} & -i2h \sin \frac{\theta_2}{2} & -\sigma^{-1}h^2
\end{pmatrix}.
\]
It can be shown that the determinant of $\tilde{L} - \lambda M_U$ is
\[
\frac{(\alpha_U m_s)^2}{\sigma} \left( \lambda - \frac{4m}{m_s \alpha_U} \right) \left( \lambda^2 - \frac{1 + \sigma}{\alpha_U m_s} 4m\lambda + \frac{4m\sigma}{\alpha_U m_s} \right),
\] (26)

Recall that $m_r = \frac{4m}{m_s}$, see (19). We rewrite (26) as
\[
\frac{(\alpha_U m_s)^2}{\sigma} \left( \lambda - \frac{m_r}{\alpha_U} \right) \left( \lambda^2 - \frac{(1 + \sigma)m_r}{\alpha_U} \lambda + \frac{m_r \sigma}{\alpha_U} \right).
\] (27)

From (27), we know that the eigenvalues of $\tilde{M}_U^{-1} \tilde{C}$ are $\lambda_{1,2}$, the two roots of
\[
T(\lambda) = \lambda^2 - \frac{(1 + \sigma)m_r}{\alpha_U} \lambda + \frac{m_r \sigma}{\alpha_U},
\] (28)

and $\lambda_3 = \frac{m_r}{\alpha_U}$.

Our goal is to theoretically solve
\[
\min_{(\alpha_U, \sigma, \omega_U)} \max_{\theta \in T^n} \{|1 - \omega_U \lambda_{1,2}|, |1 - \omega_U \lambda_3|\}.
\] (29)

For $\lambda_3$, from Theorem 3.2 we have known that the corresponding optimal smoothing factor is $\frac{17}{47}$. Thus, the optimal result of (29) is not less than $\frac{17}{47}$. The theoretical result is given below. We note that it is less effective than the other two discussed previously.

**Theorem 3.4.** The optimal smoothing factor for $Q-\sigma$-Uzawa relaxation with coarsening by three is
\[
\mu_{opt,U} = \min_{(\alpha_U, \omega_U, \sigma)} \max_{\theta \in T^n} \{|1 - \omega_U \lambda_{1,2}|, |1 - \omega_U \lambda_3|\} = \sqrt{\frac{17}{47}} \approx 0.601,
\]

with
\[
\frac{225}{47(16\mu_{opt,U} - 1)} \leq \omega_U \leq \frac{30}{47(1 - \mu_{opt,U})},
\]
\[
\alpha_U = \frac{376\omega_U^2}{9(47\omega_U - 15)},
\]
\[
\sigma = \frac{15}{47\omega_U - 15}.
\]

We comment that parameters $\omega_U = 1, \alpha_U = \frac{47}{36}, \sigma = \frac{15}{32}$ are in the domain of the above optimal parameters. Our previous work [21] proved that for standard coarsening, the optimal smoothing factor for $Q-\sigma$-Uzawa relaxation for MAC discretization of the Stokes equations is $\sqrt{\frac{3}{2}}$. Note that $\sqrt{\frac{17}{47}} \approx 0.601$ is very close to $\sqrt{\frac{3}{2}} \approx 0.577$. Now, we have optimal smoothing factors for the mass-based Braess-Sarazin, $\sigma$-Uzawa, and distributive relaxation schemes. It is clear that mass-based Braess-Sarazin and distributive relaxation schemes outperform $\sigma$-Uzawa relaxation in terms of smoothing factor. Moreover, in Section 4, we see degradation in convergence of actual $Q-\sigma$-Uzawa multigrid method, and the proof of Theorem 3.4 is tedious. Thus, we present it in the Appendix.
4 Numerical experiments

In this section, we first use optimal parameters obtained from our smoothing analysis to compute two-grid LFA convergence factors \( \rho(\nu_1, \nu_2) = \rho(\nu) \) with \( \nu = \nu_1 + \nu_2 \) defined in (12). We also study the influence of different grid-transfer operators defined in (10) and (11) on two-grid methods by LFA. Finally, multigrid methods (two-grid and V-cycles) performances are presented to validate our LFA predictions.

4.1 LFA results

We use optimal parameters obtained from Section 3 that minimize the smoothing factor and \( h = \frac{1}{3^4} \) to conduct the test. We study the influence of different grid-transfer operators defined in (10) and (11) on the performance of two-grid methods. Specifically, we consider four pairs of grid-transfer operators:

- \((\mathcal{P}_h, \mathcal{R}_h) = (\mathcal{P}_h, 25), (\mathcal{R}_h, 1)\),
- \((\mathcal{P}_h, 25), (\mathcal{R}_h, 9)\),
- \((\mathcal{P}_h, 25), (\mathcal{R}_h, 9b)\),
- \((\mathcal{P}_h, 25), (\mathcal{P}_T h, 25)/9\).

For all two-grid LFA prediction tests, we use \( \alpha_D = 1, \omega_D = \frac{36}{47} \) for \( Q\)-DR, \( \alpha_B = 1, \omega_B = \frac{36}{47} \) for \( Q\)-BSR, and \( \omega_U = 1, \alpha_U = \frac{15}{32}, \sigma = \frac{15}{32} \) for \( Q\)-\(\sigma\)-Uzawa.

In Table 1, we report two-grid LFA convergence factors using grid-transfer operators \((\mathcal{P}_h, \mathcal{R}_h) = (\mathcal{P}_h, 25), (\mathcal{R}_h, 1)\) for three relaxation schemes. We see that there is a degradation of \( \rho_h(\nu) \) compared with \( \mu_{opt}^\nu \), especially for \( \nu = 1 \). This degradation can be mitigated by using more points for the restriction operators, which we observe in Tables 2, 3 and 4, where we use \((\mathcal{P}_h, \mathcal{R}_h) = (\mathcal{P}_h, 25), (\mathcal{R}_h, 9)\), \((\mathcal{P}_h, 25), (\mathcal{R}_h, 9b)\), \((\mathcal{P}_h, 25), (\mathcal{P}_T h, 25)/9\), respectively. Overall, using \((\mathcal{P}_h, \mathcal{R}_h) = (\mathcal{P}_h, 25), (\mathcal{P}_T h, 25)/9\) gives better results compared with other choices. From LFA predictions, we see that \( Q\)-BSR outperforms the other two relaxation schemes when \( \nu \) increases. The results also suggest that it is important to select proper grid-transfer operators to design fast multigrid methods, and LFA is helpful to identify good grid-transfer operators before we do actual numerical tests.

| Method       | \( \mu_{opt} \) | \( \rho_h(1) \) | \( \rho_h(2) \) | \( \rho_h(3) \) | \( \rho_h(4) \) |
|--------------|-----------------|----------------|----------------|----------------|----------------|
| Q-DR         | 0.362           | 0.546          | 0.222          | 0.116          | 0.087          |
| Q-BSR        | 0.362           | 0.515          | 0.245          | 0.181          | 0.098          |
| Q-\(\sigma\)-Uzawa | 0.601           | 0.642          | 0.377          | 0.226          | 0.165          |

| Method       | \( \mu_{opt} \) | \( \rho_h(1) \) | \( \rho_h(2) \) | \( \rho_h(3) \) | \( \rho_h(4) \) |
|--------------|-----------------|----------------|----------------|----------------|----------------|
| Q-DR         | 0.362           | 0.419          | 0.205          | 0.157          | 0.126          |
| Q-BSR        | 0.362           | 0.361          | 0.166          | 0.097          | 0.073          |
| Q-\(\sigma\)-Uzawa | 0.601           | 0.601          | 0.361          | 0.217          | 0.154          |
Table 3 Two-grid LFA convergence factor, $\rho_h(\nu)$, using $(P_h, R_h) = (P_{h, 25}, R_{h, 9})$.

| Method   | $\mu_{opt}$ | $\rho_h(1)$ | $\rho_h(2)$ | $\rho_h(3)$ | $\rho_h(4)$ |
|----------|-------------|-------------|-------------|-------------|-------------|
| Q-DR     | 0.362       | 0.431       | 0.192       | 0.144       | 0.117       |
| Q-BSR    | 0.362       | 0.361       | 0.149       | 0.091       | 0.052       |
| Q-σ-Uzawa| 0.601       | 0.601       | 0.361       | 0.217       | 0.150       |

Table 4 Two-grid LFA convergence factor, $\rho_h(\nu)$, using $(P_h, R_h) = (P_{h, 25}, P_{h, 25}^T/9)$.

| Method   | $\mu_{opt}$ | $\rho_h(1)$ | $\rho_h(2)$ | $\rho_h(3)$ | $\rho_h(4)$ |
|----------|-------------|-------------|-------------|-------------|-------------|
| Q-DR     | 0.362       | 0.387       | 0.257       | 0.197       | 0.160       |
| Q-BSR    | 0.362       | 0.361       | 0.161       | 0.123       | 0.099       |
| Q-σ-Uzawa| 0.601       | 0.601       | 0.361       | 0.240       | 0.197       |

4.2 Multigrid performance

We consider model problems (1) and (2) on a unit domain with Dirichlet boundary conditions with zero solution, since in [21], numerical results of Q-BSR, Q-σ-Uzawa and Q-DR multigrid methods with standard coarsening for the Stokes problems with periodic boundary conditions agree with LFA predictions. The introduction of implementation of MAC scheme for the Stokes systems can be found in [7]. For grid-transfer operators, we choose the combination $(P_h, R_h) = (P_{h, 25}, P_{h, 25}^T/9)$ defined in (10) and (11). The coarsest grid is $3 \times 3$. Experimentally measured convergence factors are computed as

$$\rho_m^{(k)} = \left( \frac{\|r_k\|}{\|r_0\|} \right)^{1/k},$$

where $r_k = b_h - L_h z_k$ is the residual and $z_k$ is the $k$-th multigrid iteration. In our test, we report $\rho_m^{(k)} =: \rho_m$ with the smallest $k$ such that $\|r_k\| \leq 10^{-12}$. Again, we consider $h = \frac{1}{81}$ for all tests.

For Q-DR, Theorem 3.2 shows that the optimal parameters are $\omega_D = 36/47$. Under this condition, we tested a range of parameter values for the multigrid methods, and found that the choice of $\alpha_D = 0.7$ and $\omega_D = 36/47 \times 0.7$ is typically best. For Q-σ-Uzawa relaxation, we use $\omega_U = 1$, $\alpha_U = 37/36$, and $\sigma = 15/32$. We report two-grid and V-cycle multigrid results of Q-DR and Q-σ-Uzawa relaxation in Tables 5 and 6, respectively. For problems with Dirichlet boundary conditions, we notice that the smoothing of Q-σ-Uzawa relaxation and Q-DR remains unsatisfactory near the boundary, which degrades the convergence factors compared with LFA predictions. The degradation has been observed in other studies, see [22, 25], and further pre-relaxation might be needed near the boundaries. We point out that the influence of boundaries and of boundary conditions is not taken into account for LFA.

Table 5 Measured multigrid convergence factors vs. LFA predictions for Q-DR.

| $\nu$ | 1   | 2   | 3   | 4   |
|-------|-----|-----|-----|-----|
| LFA prediction | 0.387 | 0.257 | 0.197 | 0.160 |
| Two-grid $\rho_m$ | 0.525 | 0.507 | 0.443 | 0.394 |
| V-cycle $\rho_m$ | 0.797 | 0.715 | 0.658 | 0.615 |
Table 6 Measured multigrid convergence factors vs. LFA predictions for $Q$-$\sigma$-Uzawa.

| $\nu$   | 1     | 2     | 3     | 4     |
|---------|-------|-------|-------|-------|
| LFA prediction | 0.601 | 0.361 | 0.240 | 0.197 |
| Two-grid $\rho_m$ | 0.745 | 0.602 | 0.480 | 0.382 |
| V-cycle $\rho_m$  | 0.759 | 0.632 | 0.542 | 0.442 |

For BSR, we consider the inexact version, that is, we apply one sweep of the weighted Jacobi iteration to the Schur complement system, (25). For $Q$-IBSR, there is no degradation on the convergence factor, which is observed for $Q$-DR and $Q$-$\sigma$-Uzawa relaxation. We find that using parameters, $\omega_B = 1, \alpha_B = \frac{47}{56}$ and $\omega_J = 0.9$, gives the same convergence factors as LFA two-grid predictions of exact $Q$-BSR, shown in Table 7. This indicates that $Q$-IBSR is more robust with respect to boundary conditions compared with $Q$-DR and $Q$-$\sigma$-Uzawa relaxation.

Table 7 Measured multigrid convergence factors for $Q$-IBSR vs. LFA predictions for $Q$-BSR.

| $\nu$   | 1     | 2     | 3     | 4     |
|---------|-------|-------|-------|-------|
| LFA prediction | 0.361 | 0.161 | 0.123 | 0.099 |
| Two-grid $\rho_m$ | 0.349 | 0.163 | 0.115 | 0.090 |
| V-cycle $\rho_m$  | 0.350 | 0.183 | 0.129 | 0.097 |

From the results shown in Tables 5, 6, and 7, we conclude that $Q$-IBSR outperforms $Q$-DR and $Q$-$\sigma$-Uzawa relaxation in terms of convergence factor, and $Q$-IBSR is not too sensitive to boundary conditions. Therefore, we recommend $Q$-IBSR for practical use.

5 Conclusion

We have considered a staggered finite difference discretization for the Stokes equations. For this discretization, many of multigrid studies focus on standard coarsening. In contrast, we propose highly efficient multigrid methods with coarsening by three to solve the resulting linear system. This coarsening-by-three strategy leads to coarsening more quickly and reducing the number of levels. The unknowns in different levels are nested, and, thus, the construction of grid-transfer operators is simplified, avoiding different ones needed for two components of velocity and pressure when considering standard coarsening. It is well-known that properly selected algorithmic parameters of multigrid methods are very important to design fast algorithms. An LFA is presented to quantitatively analyze three block-structured mass-based relaxation schemes and grid-transfer operators, and help choose algorithmic parameters. We derive LFA optimal smoothing factors for a mass-based Braess-Sarazin relaxation, a mass-based Uzawa relaxation, and a mass-based distributive relaxation with this 3h-coarsening strategy for the Stokes equations.

Our theoretical results show that the optimal smoothing factors of coarsening by three are nearly equal to those obtained from standard coarsening, but the computational cost per iteration is lower due to the three factor, and therefore coarsening by three is superior computationally. Our results also show that the mass-based Braess-Sarazin and distributive relaxation schemes have same optimal smoothing factor, which is smaller than that of the mass-based Uzawa relaxation. Furthermore, we report two-grid and V-cycle multigrid performance for the Stokes equations with Dirichlet boundary conditions. We find that there is a degradation of actual convergence factors
for mass-based Uzawa and distributive relaxation schemes compared with LFA predictions. This might be due to the boundary conditions. However, the actual multigrid performance of mass-based Braess-Sarazin relaxation matches the LFA two-grid convergence factor. As a result, the mass-based Braess-Sarazin relaxation is preferred.

Appendix

To prove Theorem 3.4, we first focus on minimizing $\max_{\theta \in T^H}\{1 - \omega_U \lambda_{1,2}\}$ over $\omega_U$, which gives a lower bound for (29). We follow [22, 21] to accomplish our analysis for $Q - \sigma$-Uzawa in the following. To analyze $\lambda_{1,2}$, we compute the discriminant of $T(\lambda)$, which is

$$\Delta(m_r) = \frac{m_r(1 + \sigma)^2}{\alpha_U^2} \left( m_r - \frac{4\alpha_U \sigma}{(1 + \sigma)^2} \right).$$

Two roots of $\Delta(m_r) = 0$ with respect to $m_r$ are $m_1 = 0$ and $m_2 = \frac{4\alpha_U \sigma}{(1 + \sigma)^2}$. From (28), we have

$$\lambda_{1,2} = \frac{(1 + \sigma)m_r}{2\alpha_U} \left( 1 \pm \sqrt{1 - \frac{m_2}{m_r}} \right).$$

(30)

We see that when $m_r \geq m_2$, $\lambda_1$ and $\lambda_2$ are real, and when $m_r < m_2$, $\lambda_1$ and $\lambda_2$ are complex. Using (28), we have

$$\lambda_1 + \lambda_2 = \frac{m_r(1 + \sigma)}{\alpha_U}, \quad \lambda_1 \lambda_2 = \frac{m_r \sigma}{\alpha_U}.$$ (31)

For complex eigenvalues $\lambda_{1,2}$, we define $\Psi = |1 - \omega_U \lambda_1|$. Then, $\Psi^2 = (1 - \omega_U \lambda_1)(1 - \omega_U \lambda_2)$. Using (31), we can simplify $\Psi^2$ as

$$\Psi^2(m_r) = 1 - (\lambda_1 + \lambda_2)\omega_U + \lambda_1 \lambda_2 \omega_U^2 = 1 + \frac{\omega_U}{\alpha_U}(\omega_U \sigma - \sigma - 1)m_r.$$ Recall that $m_r(\theta) \in [5/6, 16/9]$ for $\theta \in T^H$, see (19). We first derive a general result for $m_2 \geq m_r$.

**Theorem 5.1.** Let $\gamma = \min\{m_2, 16/9\}$, where $m_2 \geq \frac{5}{6}$. Then, the smoothing factor for $\lambda_{1,2}$ with $m_r \in [5/6, \gamma]$ is

$$\mu^C = \max_{\gamma \leq \frac{5}{6}, \gamma} \Psi(m_r) = \sqrt{1 + \frac{5\omega_U (\omega_U \sigma - \sigma - 1)}{6\alpha_U}} \geq \sqrt{1 - \frac{5}{6\gamma}},$$

where the equality is achieved if and only if $\frac{\omega_U}{\alpha_U}(\omega_U \sigma - \sigma - 1) = -\frac{1}{\gamma}$.

**Proof.** When $m_r \in [5/6, \gamma]$, $\Delta(\alpha_U, \sigma) \leq 0$ and $|1 - \omega_U \lambda_1| = |1 - \omega_U \lambda_2| = \Psi(m_r)$. Setting $\Psi^2(m_r) < 1$ gives $\frac{\omega_U}{\alpha_U}(\omega_U \sigma - \sigma - 1) < 0$. Using $\gamma = \min\{m_2, 16/9\}$ gives

$$\Psi^2(\gamma) = 1 + \frac{\omega_U}{\alpha_U}(\omega_U \sigma - \sigma - 1) \geq 1 + \frac{\omega_U}{\alpha_U}(\omega_U \sigma - \sigma - 1)m_2 = \left(1 - \frac{2\omega_U \sigma}{1 + \sigma} \right)^2 \geq 0.$$ This means that $1 + \frac{\omega_U}{\alpha_U}(\omega_U \sigma - \sigma - 1) \geq 0$, that is, $\frac{\omega_U}{\alpha_U}(\omega_U \sigma - \sigma - 1) \geq -\frac{1}{\gamma}$. Furthermore,

$$\max_{\gamma \leq \frac{5}{6}, \gamma} \Psi(m_r) = \Psi(5/6) = \sqrt{1 + \frac{5\omega_U}{6\alpha_U}(\omega_U \sigma - \sigma - 1)} \geq \sqrt{1 - \frac{5}{6\gamma}},$$

where the equality is achieved if and only if $\frac{\omega_U}{\alpha_U}(\omega_U \sigma - \sigma - 1) = -\frac{1}{\gamma}$.

\[\blacksquare\]
Note that \( m_r(\theta) \in [5/6, 16/9] \). We consider the special situation that \( m_2 > \frac{16}{9} \) in Theorem 5.1.

**Corollary 5.1.** For \( m_2 = \frac{4\alpha U \sigma}{(1+\sigma)^2} > \frac{16}{9} \), the optimal smoothing factor for \( Q\sigma\)-Uzawa relaxation is not less than \( \sqrt{\frac{34}{8}} \).

**Proof.** Since \( m_2 > \frac{16}{9} \), from Theorem 5.1 with \( \gamma = \frac{16}{9} \), we know the smoothing factor for the complex modes \( \lambda_{1,2} \) is

\[
\mu_c^C = \Psi(5/6) \geq \sqrt{1 - \frac{5}{6\gamma}} = \sqrt{1 - \frac{5}{6 \cdot \frac{16}{9}}} = \frac{\sqrt{34}}{8}.
\]

It follows that when \( m_2 > \frac{16}{9} \), the optimal smoothing for \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) is not less than \( \frac{\sqrt{34}}{8} \). \( \square \)

Next, we give a general result for \( m_r \geq m_2 \) for the real case. We consider \( m_r \in [\gamma, 16/9] \) and \( m_2 \leq \gamma \). From (30), we have

\[ |1 - \omega_U \lambda_{1,2}| = \left| 1 - \frac{(1 + \sigma) \omega_U}{2\alpha U} m_r \left( 1 \pm \sqrt{1 - \frac{m_2}{m_r}} \right) \right|. \]

For simplicity, let \( \chi_s(m_r) = \frac{m_r}{2} \left( 1 \pm \sqrt{1 - \frac{m_2}{m_r}} \right) \). It is easy to see that \( \chi_s(m_r) \) is an increasing function over \( m_r \in [\gamma, 16/9] \). As to \( \chi_s(m_r) \), [22] has shown that it is a decreasing function. Thus,

\[
\chi_+(m_r)_{\max} = \chi_s(16/9) = \frac{8}{9} \left( 1 + \sqrt{1 - \frac{9m_2}{16}} \right) =: \chi_1, \quad (32)
\]
\[
\chi_-(m_r)_{\min} = \chi_s(16/9) = \frac{8}{9} \left( 1 - \sqrt{1 - \frac{9m_2}{16}} \right) =: \chi_2. \quad (33)
\]

Define

\[
\mu^R = \max_{m_r \in [\gamma, 16/9]} \{ |1 - \omega_U \lambda_1|, |1 - \omega_U \lambda_2| \}
\]

\[
= \max \left\{ \left| 1 - \frac{(1 + \sigma) \omega_U}{\alpha U} \chi_1 \right|, \left| 1 - \frac{(1 + \sigma) \omega_U}{\alpha U} \chi_2 \right| \right\}
\]

\[
= \left\{ \begin{array}{ll}
\frac{(1 + \sigma) \omega_U}{\alpha U} \chi_1 - 1, & \text{if} \quad \frac{(1 + \sigma) \omega_U}{\alpha U} \geq \frac{9}{8}; \\
1 - \frac{(1 + \sigma) \omega_U}{\alpha U} \chi_2, & \text{if} \quad \frac{(1 + \sigma) \omega_U}{\alpha U} \leq \frac{9}{8}.
\end{array} \right. \quad (34)
\]

Note that \( \mu^R \) is a function of \( m_2 \). Now, we can give a lower bound on the optimal smoothing factor for \( Q\sigma\)-Uzawa relaxation for \( m_2 \leq \frac{5}{6} \).

**Theorem 5.2.** For \( m_2 = \frac{4\alpha U \sigma}{(1+\sigma)^2} \leq \frac{5}{6} \), the optimal smoothing factor for \( Q\sigma\)-Uzawa relaxation is not less than \( \frac{\sqrt{34}}{8} \).
Proof. When \( m_2 \leq \frac{5}{6} \), \( \lambda_1, \lambda_2 \) are all real. From (32), (33), and (34), we see that \( \mu^R \) is a decreasing function of \( m_2 \). Thus, for \( m_r \in [5/6, 16/9] \),

\[
\mu^R(m_2) \geq \mu^R(5/6) = \begin{cases} 
\frac{8(1+\sigma)\omega_U}{9\alpha_U} \left( 1 + \frac{\sqrt{34}}{8} \right) - 1, & \text{if } \frac{(1+\sigma)\omega_U}{\alpha_U} \geq \frac{9}{8}, \\
\frac{8(1+\sigma)\omega_U}{9\alpha_U} \left( 1 - \frac{\sqrt{34}}{8} \right), & \text{if } \frac{(1+\sigma)\omega_U}{\alpha_U} \leq \frac{9}{8}.
\end{cases}
\]

To minimize \( \mu^R(m_2) \) with respect to \( \alpha_U, \omega_U, \sigma \), it requires that \( \frac{(1+\sigma)\omega_U}{\alpha_U} = \frac{9}{8} \) and \( m_2 = \frac{4\alpha_U\sigma}{(1+\sigma)^2} = \frac{5}{6} \).

It follows that \( \min_{(\alpha_U, \omega_U, \sigma)} \mu^R(m_2) = \frac{\sqrt{34}}{8} \). Since there is another eigenvalue \( \lambda_3 \), the optimal smoothing factor for \( \lambda_{1,2} \) and \( \lambda_3 \) may be not less than \( \frac{\sqrt{34}}{8} \).

The above discussions indicate that when \( m_2 > \frac{16}{9} \) or \( m_2 \leq \frac{5}{6} \), the optimal smoothing factor is at least \( \frac{\sqrt{34}}{8} \). Next, we will show that the global optimal smoothing factor for all choices of \( m_2 \) achieves when \( \frac{5}{6} \leq m_2 \leq \frac{16}{9} \).

Proof. We first consider \( m_2 \in [5/6, 16/9] \). From previous discussions, we know that for \( m_r \in [5/6, m_2] \), \( \lambda_{1,2} \) are complex, and for \( m_r \in [m_2, 16/9] \), \( \lambda_{1,2} \) are real. We consider \( \frac{(1+\sigma)\omega_U}{\alpha_U} = \frac{9}{8} \) such that the two expressions in (34) are the same. Furthermore, we have \( m_2 = \frac{4\alpha_U\sigma}{(1+\sigma)^2} = \frac{47}{36} \frac{\sqrt{34}}{8} \).

For \( m_r \in [5/6, m_2] \), Theorem 5.1 gives

\[
\mu^C = \sqrt{1 + \frac{5\omega_U(\omega_U - \sigma - 1)}{6\alpha_U}} = \sqrt{1 + \frac{5\omega_U^2\sigma}{6\alpha_U}}.
\]

For \( m_r \in [m_2, 16/9] \), using (34) gives

\[
\mu^R = \frac{(1+\sigma)\omega_U}{\alpha_U} \chi_1 - 1 = \sqrt{1 - \frac{9m_2}{16}} = \sqrt{1 - \frac{16\omega_U^2\sigma}{9\alpha_U}}.
\]

Thus, for \( \frac{(1+\sigma)\omega_U}{\alpha_U} = \frac{9}{8} \),

\[
\max_{m_r \in [5/6, 16/9]} \{1 - \omega_U \lambda_{1,2}\} = \max \left\{ \mu^R, \mu^C \right\}.
\]

(35)

Note that \( \mu^R \) is a decreasing function of \( \frac{\omega_U^2\sigma}{\alpha_U} \) and \( \mu^C \) is an increasing function of \( \frac{\omega_U^2\sigma}{\alpha_U} \). The minimum of (35) is achieved if and only if \( \mu^R = \mu^C \) and is

\[
\min_{(\alpha_U, \omega_U, \sigma), \frac{(1+\sigma)\omega_U}{\alpha_U} = \frac{9}{8}} \max \left\{ \mu^R, \mu^C \right\} = \sqrt{\frac{17}{47}} =: \mu_{\text{opt}, U},
\]

under the condition that

\[
\frac{\omega_U^2\sigma}{\alpha_U} = \frac{135}{376}, \quad \frac{(1+\sigma)\omega_U}{\alpha_U} = \frac{9}{8}.
\]

(36)
Since $\sqrt{\frac{17}{47}} < \frac{\sqrt{37}}{8}$, it means that $m_2 \in \left[\frac{5}{6}, \frac{16}{9}\right]$ gives a smaller smoothing factor for $\lambda_{1,2}$. Next, we prove that the optimal smoothing factor for $\lambda_{1,2}$ over all possible parameters is achieved at $\frac{(1+\sigma)\omega_U}{\alpha_U} = \frac{9}{8}$. Then, we include $\lambda_3$.

Let $a = \frac{(1+\sigma)\omega_U}{\alpha_U}$ and $b = \frac{\omega_U \sigma}{\alpha_U}$. Then, $m_2 = \frac{4\omega_U \sigma}{(1+\sigma)^2} = \frac{4b}{\sigma^2}$. Assume that $\mu^C \leq \sqrt{\frac{17}{47}}$, that is,

$$\sqrt{1 + \frac{5\omega_U (\omega_U \sigma - \sigma - 1)}{6\alpha_U}} = \sqrt{1 - \frac{5a}{6} + \frac{5b}{6}} \leq \sqrt{\frac{17}{47}},$$

which gives $b \leq a - \frac{36}{47}$.

Under condition $b \leq a - \frac{36}{47}$, we consider two situations of $a$:

Case 1: If $a > \frac{9}{8}$, using (34) gives

$$\mu^R = \frac{(1+\sigma)\omega_U}{\alpha_U} \frac{8}{9} \left(1 + \sqrt{1 - \frac{9m_2}{16}}\right) - 1$$

$$= \frac{8}{9} \left(a + \sqrt{a^2 - 9b/4}\right) - 1$$

$$> \frac{8}{9} \sqrt{a^2 - \frac{9}{4}(a - 36/47)}$$

$$= \frac{8}{9} \sqrt{(a - 9/8)^2 + (81 \times 17)/(64 \times 47)}$$

$$> \sqrt{\frac{17}{47}}.$$

Case 2: If $a < \frac{9}{8}$, using (34) gives

$$\mu^R = 1 - \frac{(1+\sigma)\omega_U}{\alpha_U} \frac{8}{9} \left(1 - \sqrt{1 - \frac{9m_2}{16}}\right)$$

$$= 1 - \frac{8}{9} a + \frac{8}{9} \sqrt{a^2 - 9b/4}$$

$$> \frac{8}{9} \sqrt{a^2 - \frac{9}{4}(a - 36/47)}$$

$$> \sqrt{\frac{17}{47}}.$$

It means that $a = \frac{(1+\sigma)\omega_U}{\alpha_U} = \frac{9}{8}$ gives the optimal smoothing factor for $\lambda_{1,2}$, and the corresponding optimal smoothing factor is $\mu_{opt,U} = \sqrt{\frac{17}{47}}$.

Next, we consider $\lambda_3 = \frac{m_r}{\alpha_U}$. From Theorem 3.1, we know that the optimal smoothing factor for $\lambda_3$ is $\sqrt{\frac{17}{47}}$, which is less than $\sqrt{\frac{17}{47}}$. Thus, the optimal smoothing factor for $\lambda_{1,2}$ and $\lambda_3$ is not less than $\sqrt{\frac{17}{47}}$. We will show it is $\sqrt{\frac{17}{47}}$. Recall that $m_r \in \left[\frac{5}{6}, \frac{16}{9}\right]$ for $\theta \in \mathcal{T}^4$. Let

$$\left|1 - \frac{5\omega_U}{6\alpha_U}\right| \leq \mu_{opt,U} \quad \text{and} \quad \left|1 - \frac{16\omega_U}{9\alpha_U}\right| \leq \mu_{opt,U}.$$
The above two inequalities give
\[
\frac{6}{5} (1 - \mu_{\text{opt},U}) \frac{1}{\omega_U} \leq \frac{1}{\alpha_U} \leq \frac{9(1 + \mu_{\text{opt},U})}{16} \frac{1}{\omega_U}.
\] (37)

Note that (36) can be expressed as
\[
\alpha_U = \frac{376\omega_U^2}{9(47\omega_U - 15)}, \quad \sigma = \frac{15}{47\omega_U - 15}.
\] (38)

Using (38), we can rewrite (37) as
\[
\frac{225}{47(16\mu_{\text{opt},U} - 1)} \leq \omega_U \leq \frac{30}{47(1 - \mu_{\text{opt},U})},
\]
which gives the desired result. \qed

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