Loop Variables and the Virasoro Group

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Abstract

We derive an expression in closed form for the action of a finite element of the Virasoro Group on generalized vertex operators. This complements earlier results giving an algorithm to compute the action of a finite string of generators of the Virasoro Algebra on generalized vertex operators. The main new idea is to use a first order formalism to represent the infinitesimal group element as a loop variable. To obtain a finite group element it is necessary to thicken the loop to a band of finite thickness. This technique makes the calculation very simple.
1 Introduction

Understanding the Virasoro group is of central importance in string theory since the gauge invariance of the theory is a consequence of the conformal invariance of the two dimensional world sheet field theory and the generators of infinitesimal conformal transformations obey the Virasoro Algebra. A lot of work has been done on the representation theory of the Virasoro Algebra and applications to various branches of physics [1, 2, 3, 4, 5, 6, 7, 8, 9]. For many purposes it is sufficient to consider the algebra rather than the group. In particular in string theory attention has been focussed on infinitesimal gauge transformations and for this the algebra is all one needs to consider. Experience with Yang-Mills theories and gravity teaches us however that it is essential to consider the group in order to appreciate the geometrical basis underlying gauge symmetries. Another motivation for studying the Virasoro group is that the string evolution operator $e^{L_0 \tau}$ is obviously an element of the Virasoro group. It is plausible that a gauge invariant generalization of this would be $e^{\sum_n L_n \tau_n}$ which is a general element of the Virasoro group. The Virasoro group also has been studied from different points of view [10, 11, 12, 13, 14]. Our goal in this paper is to approach these issues using the loop variable formalism. This will also improve our understanding of the formalism which we feel will be useful for developing computational techniques for string theory. We derive an expression for the action of a Virasoro group element on generalized vertex operators, i.e. we write down an expression for $e^{\sum_n L_n V(z)} \vert >$ where $V(z)$ is a loop variable or generalized vertex operator (gvo).

$$e^{i(k_0 X + \int dt k(t) \partial_z X(z+t))} = e^{i \sum_n \frac{k_n \partial^n X(z)}{(n-1)!}} \equiv e^{ik_n Y_n}$$  (1.1)

(A sum over the index $n$ is understood.)

A loop variable contains in it all the vertex operators in the bosonic string theory [16, 17]. The Taylor expansion in $t$ is allowed because there are no singularities in $t$. This will not be the case if we consider correlation functions with other operators located inside the contour. Thus in general we must make a distinction between the LHS of (1.1), which is a loop variable and the RHS, which is a generalized vertex operator. The loop variable is more general than the generalized vertex operator. In [17] we described a simple algorithm for calculating the action of an arbitrary finite sequence of $L_n$'s
on the loop variable. Here instead of an algorithm we have a mathematical formula involving some definite matrices and their products for the action of a finite group element. The formula is as follows: We define

\[ \lambda_{n,m}(\sigma) = \lambda_{m,n}(\sigma) = \lambda_{n+m} : 0 \leq \sigma \leq 1; n, m \in \mathbb{Z} \]  

(1.2)

\[ \tilde{\theta}_{n,m}(\sigma) = \lambda_{m,n}(\sigma) = \lambda_{n+m} : 0 \leq \sigma \leq 1; n, m \in \mathbb{Z} \]  

(1.3)

where \( N \) is an infinite dimensional matrix acting on infinite dimensional vectors of the form \( y_n, n \in \mathbb{Z} \). If we consider the subspace \( \langle y_n \rangle \), \( n \geq 0 \) it has the form \( N = \left( \begin{array}{cc} 0 & 0 \\ n & 0 \end{array} \right) \). When \( n = 0 \), \( N = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \). We make a distinction between \( y_{+0} \) and \( y_{-0} \) and set the former equal to zero i.e. \( y_{+0} = 0 \). Finally, define the (infinite) column vector \( \mathbf{Y} = \left( \begin{array}{c} Y_n \\ Y_{-n} \end{array} \right) \)

where \( Y_{-n} \equiv -ink_n \quad n > 0 \) and \( Y_{+n} \equiv Y_n \equiv \partial^n X/(n-1)! \), \( n > 0 \) and \( Y_{+n} \equiv 0, n = 0 \)

In terms of these variables:

\[ e^{\sum_n \lambda_{n} L_{-n}} : e^{i \sum_{n \geq 0} q_n Y_n} : =: e^{-1/2 \mathbf{Y}^T P^\lambda \mathbf{Y}} e^{i \sum_{n \geq 0} q_n Y_n} : Det[1 - \lambda \tilde{\theta}] \]  

(1.4)

with

\[ P^\lambda = \int d\sigma_1 \left\{ \frac{1}{\delta_{\sigma_1 \sigma_2} - \lambda(\sigma_1)\tilde{\theta}(\sigma_1, \sigma_2)} \right\} \lambda(\sigma_2) \]  

(1.5)

The object in curly brackets is a matrix with continuous indices \( \sigma_1, \sigma_2 \) and multiplies a column vector \( \lambda(\sigma_2) \). Thus \( \sigma_2 \) is summed over. Inside the curly bracket there is no sum on \( \sigma_1 \). \( P^\lambda \) can be represented as an infinite series in \( \lambda \) by expanding the expression in curly brackets.

\[ P^\lambda = \int_0^1 d\sigma_1 1 + \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \lambda(\sigma_1)\tilde{\theta}(\sigma_1, \sigma_2)\lambda(\sigma_2) + \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \int_0^1 d\sigma_3 \lambda(\sigma_1)\tilde{\theta}(\sigma_1, \sigma_2)\lambda(\sigma_2)\tilde{\theta}(\sigma_2, \sigma_3)\lambda(\sigma_3) + \ldots \]

\[ = \lambda + 1/2 \lambda(N + N^T) + 1/3![\lambda N \lambda N \lambda + \lambda N^T \lambda N \lambda + 2\lambda N \lambda N^T \lambda + 2\lambda N \lambda N^T \lambda N \lambda] + \ldots \]  

(1.6)
Thus $e^{1/2Y^T P^\lambda Y}$ can be regarded as a representation of the group element $e^{\lambda_n L_{-n}}$. There is a well defined group composition law $P^\lambda \circ P^\mu = P^{\rho(\lambda,\mu)}$. When $\lambda$ and $\mu$ are infinitesimal this law satisfies the Virasoro Algebra. Equations (1.4,1.5) summarize the main result of this paper. They describe the action of an element of the Virasoro group on an arbitrary vertex operator.

This paper is organized as follows: In section II we motivate the result in an intuitive way. In section III we give a more complete proof. In section IV we discuss the group composition property. This can be used to construct an alternate proof of the result. A sample calculation is also given. In section V we give some concluding remarks.
2 Loop Variable Representation of Group Element

Consider the integral

$$\int \mathcal{D}k(t)e^{i\int k(t)\partial_z X(z+t)dt} \Phi[k(t)]$$ \hspace{1cm} (2.1)

with $k(t) = k_0 + \frac{k_1}{t} + \frac{k_2}{t^2} + \ldots$ which is a loop variable representation of (D-dimensional) fields of the bosonic string as considered in \[16\]. (Later we will generalize to include positive powers of $t$ also). If we choose

$$\Phi[k(t)] = e^{-1/2 \int dt k^2(t)\lambda^{-1}(t)}$$ \hspace{1cm} (2.2)

and perform the functional integral over $k(t)$ we get, up to an overall normalization factor:

$$e^{-1/2 \int dt \partial X(z+t)\partial X(z+t)\lambda(t) [\text{Det}\lambda]^{D/2}}$$ \hspace{1cm} (2.3)

The exponent is nothing but the energy momentum tensor multiplied by a gauge parameter. If we choose

$$\lambda(t) = \lambda_0 t + \lambda_1 + \lambda_2/t + \ldots + \lambda_{-1}t^2 + \lambda_{-2}t^3 + \ldots + \lambda_n t^{-n+1} + \ldots$$ \hspace{1cm} (2.4)

then (2.3) becomes $e^{\sum_n \lambda_n L_n}$ which is an element of the Virasoro group. The $\lambda_n$ in (2.4) is the same as in (1.2). The above discussion has to be modified to include the effects of normal ordering when we make $X$ a two dimensional quantum field. Furthermore $k(t)$ will also include positive powers of $t$ when we consider operator products of the loop variable (2.1) with other vertex operators at the point $z$. Nevertheless the basic idea illustrated above will be used. Namely we will use the first order representation (2.1) instead of (2.3).

In order to calculate the action of a group element on a vertex operator we first perform the functional integral over $X$ to get the terms in the operator product expansion of (2.1) with the vertex operator and then do the $k(t)$ integration.

Now we have to incorporate quantum mechanics since otherwise the $L_n$'s in (2.3) will commute. That means we have to include normal ordering effects. We can try the following: Write

$$e^{i \int k(t)\partial_z X(z+t)dt} = e^{i \int k(t)\partial_z X(z+t)dt} \cdot e^{1/2 \int dt k(t)<\partial X(z+t)\frac{\partial^n X(z+t)}{(n-1)!}\partial_n X(z+t)>k_n}$$ \hspace{1cm} (2.5)
In order to avoid the ambiguity at \( t = t' \) we will take one loop to have a slightly larger radius as shown in Fig.1. On the inner loop \( X(z + t) \) can be expanded in a Taylor series in \( t \) (i.e. in positive powers of \( t \)) since there is no singularity as the radius of the loop shrinks to zero. We have used

\[ < X(z)X(w) > = \ln(z - w) \]

to get for the second factor

\[ e^{\sum_{m>0}^{+\infty} \theta_{m}} \]

where

\[ k(t) = \sum_{m=-\infty}^{+\infty} k_{m} t^{m} \]

Thus (2.5) becomes

\[ :e^{i \int_{c} k(t) \partial_{x} X(z+t) dt} : e^{\sum_{-\infty}^{+\infty} \frac{1}{2} \theta_{m}} \]

where

\[ \theta_{m} = m \delta_{m,-m}, m \neq 0 \]

and (2.1) becomes

\[ \int \mathcal{D}k(t) : e^{i \int_{c} k(t) \partial_{x} X(z+t) dt} : e^{\lambda_{m}^{-1}} \theta_{m} \theta_{m} \]

where \( \lambda_{m}^{-1} \) is the inverse of \( \lambda_{m} \), the matrix defined in (1.2). It satisfies

\[ \lambda_{m}^{-1} = (\lambda_{n}^{-1})_{n+m} \]

\[ \lambda_{n}^{-1}(t) = (\lambda_{n}^{-1})_{n} t^{-n+1} \]

In fact it will turn out that (2.9) is not the correct matrix to be used nor is (2.10) the correct final answer. But it is quite close. Let us try to use (2.10) in calculating the action of a Virasoro group element on a vertex operator. In the process we will see both the facility of this approach as well as the ordering subtleties that necessitate a further refinement of the steps leading from (2.5) to (2.10). We want to calculate the operator product of (2.10) with a generalized vertex operator

\[ :e^{i \sum_{n \geq 0} \theta_{n} Y_{n}} : \]

where

\[ Y_{n}(z) = \frac{\partial^{n} X (z)}{(n-1)!}; Y_{0} = X(z) \]

Thus we have to evaluate the operator product (OP):

\[ \int \mathcal{D}k(t) : e^{i \int_{c} k(t) \partial_{x} X(z+t) dt} : e^{\lambda_{m}^{-1}} \theta_{m} \theta_{m} \]
Using (we assume that the vertex operator is located at $z$)

$$< \partial_z X(z + t) \partial^n X(z) > = \frac{n!}{t^{n+1}}$$  \hspace{1cm} (2.13)

we get for (2.12):

$$\int Dk(t) : e^{i \int k(t) \partial_z X(z(t)) dt} e^{i \sum_{n \geq 0} q_n Y_n} : e^{i \sum_{n \geq 0} k_{-n} q_n n + k_0 q_0} e^{1/2 \sum_{n,m \geq 0} k_n [\theta_{nm} - \lambda_{nm}^{-1}]} k_m$$  \hspace{1cm} (2.14)

Singularities in $t$ arise only when (2.14) is inserted in correlators with other vertex operators at $z$. As long as we refrain from doing that one can Taylor expand $\partial X(z + t)$ in (positive) powers of $t$ to get

$$\int [dk_n] : e^{i \sum_{n \geq 0} k_n Y_n} e^{i \sum_{n \geq 0} q_n Y_n} : e^{i \sum_{n \geq 0} k_{-n} q_n n + k_0 q_0} e^{1/2 \sum_{n,m \geq 0} k_n [\theta_{nm} - \lambda_{nm}^{-1}]} k_m$$  \hspace{1cm} (2.15)

which becomes (on doing the $k$ integration)

$$: e^{1/2 \sum_{n,m \geq 0} Y_n (\theta - \lambda^{-1})^{-1}_{n,m} Y_m - 1/2 \sum_{n,m \geq 0} n q_n (\theta - \lambda^{-1})^{-1}_{n,m} m q_m - i \sum_{n \geq 0, m \geq 0} n q_n (\theta - \lambda^{-1})^{-1}_{n,m} Y_m} : \det (1 - \lambda \theta)^{D/2}$$  \hspace{1cm} (2.16)

In (2.16) to save space we have assumed implicitly that $n q_n$ is to be replaced by $q_0$ when $n = 0$. If one defines the column vector

$$\mathbf{Y} \equiv \begin{pmatrix} Y_n \\ -i n q_n \end{pmatrix} \equiv \begin{pmatrix} Y_n \\ Y_{-n} \end{pmatrix}$$  \hspace{1cm} (2.17)

with the understanding that $n = 0$ is counted as a negative index (i.e. there is no $Y_{+0} = X$ in (2.17)), then (2.15) can be written compactly as

$$: e^{1/2 \mathbf{Y}^T (\theta - \lambda^{-1})^{-1} \mathbf{Y} + \sum_{n \geq 0} q_n Y_n} : \det (1 - \lambda \theta)^{-D/2}$$  \hspace{1cm} (2.18)

We have dropped an overall factor of $\det \lambda^{D/2}$. Using $(\theta - \lambda^{-1})^{-1} = - \left( \frac{1}{(1 - \lambda \theta)} \right) \lambda \equiv -P^\Lambda$ we finally get:

$$e^{\lambda_{n} L_{-n}} : e^{i \sum_{n \geq 0} q_n Y_n} : = e^{-1/2 \mathbf{Y}^T P^\Lambda \mathbf{Y}} e^{i \sum_{n \geq 0} q_n Y_n} : \det (1 - \lambda \theta)^{-D/2}$$  \hspace{1cm} (2.19)

This is a simple looking result. Furthermore it required very little effort using the loop variable formalism. Although as pointed out earlier it is not quite
correct, the correct result is very similar to this and is also easy to derive and we will do this in the next section. (We need a more precise treatment of the normal ordering.)

Let us first see why the present result cannot be correct. This becomes evident when one checks the group composition property by calculating $e^{\mu_n L_{-n}} e^{\lambda_n L_{-n}} e^{i \sum_{n \geq 0} q_n Y_n}$. This requires us to calculate the action of $e^{\sum_n \mu_n L_{-n}}$ on (2.19). This is not very difficult if one realizes that written in the form (2.15), (2.19) is just another generalized vertex operator $e^{i (k_n + q_n) Y_n}$, multiplied by some $Y$-independent factors. So we first calculate the action of $e^{\sum_n \mu_n L_{-n}}$ on $e^{i (k_n + q_n) Y_n}$, and then perform the $k$ integration. And of course for $e^{\sum_n \mu_n L_{-n}}$ we can again use a first order representation:

$$e^{\sum_n \mu_n L_{-n}} = \int \mathcal{D} p(t) : e^{i \int p(t) \frac{\partial}{\partial t} X(z + t) dt} : e^{1/2p_n} \mu_{nm} \mu_{pm}$$

(2.20)

Acting on (2.15) gives

$$\int \mathcal{D} p(t) : e^{i \int p(t) \frac{\partial}{\partial t} X(z + t) dt} : e^{i \sum_{n > 0} (k_n + q_n) Y_n + q_0 Y_0} : e^{k_{-n} \mu_{qn + k_0 q_0}}$$

$$= \int \mathcal{D} p(t) : e^{i \int p(t) \frac{\partial}{\partial t} X(z + t) dt} : e^{i \sum_{n > 0} (k_n + q_n) Y_n + q_0 Y_0} : e^{k_{-n} \mu_{qn + k_0 q_0}}$$

$$e^{+1/2k_n} \mu_{nm} \mu_{pm + 1/2p_n}$$

(2.21)

As before $n q_n$ becomes $q_0$ for $n = 0$. We calculate the OPE using (2.13) again to get

$$\int \mathcal{D} p(t) : e^{i \int p(t) \frac{\partial}{\partial t} X(z + t) dt} : e^{i \sum_{n > 0} (k_n + q_n) Y_n + q_0 Y_0} : e^{k_{-n} \mu_{qn + k_0 q_0}}$$

$$= e^{+1/2k_n} \mu_{nm} \mu_{pm + 1/2p_n}$$

(2.22)

We can Taylor expand $\partial X(z + t)$ as before to get (reinserting $\int [dk_n]$):

$$\int [dk_n] \int d[p_n] : e^{i \sum_{n > 0} (p_n + k_n + q_n) Y_n + i q_0 Y_0} : e^{k_{-n} \mu_{qn + k_0 q_0}}$$

$$= e^{+1/2k_n} \mu_{nm} \mu_{pm + 1/2p_n}$$

(2.23)

This can be written compactly if we define the column vectors

$$\mathcal{P} = \begin{pmatrix} p_{+n} \\ p_{-n} \\ k_{+n} \\ k_{-n} \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} Y_{+n} \\ Y_{-n} \\ Y_{+n} \\ Y_{-n} \end{pmatrix}, \quad \begin{pmatrix} Y_{+n} \\ Y_{-n} \end{pmatrix} = \begin{pmatrix} \mathbf{y} \\ \mathbf{y} \end{pmatrix}$$

(2.24)
and the matrix
\[ N = \begin{pmatrix} N_{n+m, n+m} & N_{n, n-m} \\ N_{n, n+m} & N_{n, n-m} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ n \delta_{n,m} & 0 \end{pmatrix} \] (2.25)

(2.23) becomes
\[ \int [dP] e^{i \frac{1}{2} P^T \left( \begin{array}{cc} (\theta - \mu^{-1}) & N \\ NT & (\theta - \lambda^{-1}) \end{array} \right) P} e^{i \sum_{n \geq 0} q_n Y_n} \] (2.26)

[Note that since \( Y_{+0} \equiv 0 \), we can also assume \( p_{+0} \equiv k_{+0} \equiv 0 \) and \( p_{-0} \equiv p_0; k_{-0} \equiv k_0 \) Let us define
\[ K = \left( \begin{array}{cc} (\theta - \mu^{-1}) & 0 \\ 0 & (\theta - \lambda^{-1}) \end{array} \right) \text{ and } M = \left( \begin{array}{cc} 0 & N \\ NT & 0 \end{array} \right) \] (2.27)

Then
\[ K^{-1} = \left( \begin{array}{cc} (\theta - \mu^{-1})^{-1} & 0 \\ 0 & (\theta - \lambda^{-1})^{-1} \end{array} \right) = \left( \begin{array}{cc} -P^\mu & 0 \\ 0 & -P^\lambda \end{array} \right) \equiv -P \] (2.28)

Performing the Gaussian integrals we get
\[ : e^{-1/2 Y^T (K + M)^{-1} Y} e^{i \sum_{n \geq 0} q_n Y_n} : \text{Det}[K + M]^{-D/2} \] (2.29)

Using
\[ (K + M)^{-1} = \left( \frac{1}{1 - PM} \right) P = P + PMP + PMPMP + \cdots \] (2.30)

this becomes
\[ : e^{-1/2 Y^T (P^\mu + P^\lambda + P^\mu N P^\lambda + P^\lambda N^T P^\mu + P^\mu N P^\lambda N^T P^\mu + P^\lambda N^T P^\mu + P^\lambda \cdots) Y} \]
\[ e^{i \sum_{n \geq 0} q_n Y_n} : \text{Det}[K + M]^{-D/2} \] (2.31)

If we set \( \mu = \lambda \) in (2.31) we should just end up with \( P^{2\lambda} \). But one can check that \( P^\lambda \) as defined in (2.18) and (2.19) does not satisfy this. This shows that this result is not quite correct. It turns out with some modification of \( \theta \) the final result (see (1.5)) has the same general form and is not much harder to derive. Except for this modification of the form of \( \theta \) (and therefore of \( P^\lambda \))
the calculation proceeds \textit{exactly as above}. The argument that leads to the correct expression for $\theta$ can be motivated by looking at (2.31). Note that $\theta$ in (2.18) is $1/2(N + N^T)$. From (2.31) we see that $N$ and $N^T$ always occur either as $P^\mu N P^\lambda$ or as $P^\lambda N^T P^\mu$ i.e. they keep track of the order of $P^\mu$ vis a vis $P^\lambda$. We can generalize this rule as follows. We imagine rewriting $e^{\sum_n \lambda_n (L - n)}$ as $e^{\lambda_1 L} e^{\lambda_2 L} \cdots e^{\lambda_N L}$ for some large number $N$ (not to be confused with the matrix $N$). In the end we set $\lambda_1 = \lambda_2 = \cdots = \lambda_N$ We would expect based on the above that in a particular sequence of $\lambda N \lambda N \lambda N^T \cdots \lambda$ whether we should have an $N$ or $N^T$ is dependent on which factor of $e^{\lambda L}$ the particular $\lambda$ (adjacent to that $N$ or $N^T$) came from. Thus the rule is to have $\lambda_i N \lambda_j$ if $i < j$ and $\lambda_i N^T \lambda_j$ if $i > j$. Thus we have a counting problem: The coefficient of a term with a particular sequence of $N$'s and $N^T$'s is equal to the number of ways we can arrange the $\lambda$'s (in accordance with the above rule) in order to get this particular sequence of $N$'s and $N^T$'s. Remember that in the end the $\lambda$'s are all set equal to each other. Implementing this rule modifies $P^\lambda$ defined in (2.19) to the one defined in (1.5) with a more elaborate $\theta$ matrix. We elaborate on this in the next section and prove that (1.5) is the right result.
3 Proof

We first outline the steps involved in the proof and then fill in the details:

**Step 1:** To linear order in $\lambda$ the expression (1.4) and (1.5) for $\sum_n \lambda_n L_n$ is correct.

**Step 2:** To bilinear order in $\lambda \mu$, expression (2.31) for $\sum_n \lambda_n L_n \mu_n L_n$ is correct and the Virasoro Algebra is satisfied.

**Step 3:** To trilinear order in $\lambda, \mu, \rho$ the expression for $\sum_n \lambda_n L_n \sum_n \mu_n L_n \sum_n \rho_n L_n$ is correct and the Virasoro Algebra is satisfied.

**Step 4:** To an $p$th order term of $\lambda^i \lambda^j$, by the expression (1.4) and (1.5) for $\sum_n \lambda_n L_n$, $\sum_n \mu_n L_n$ and the Virasoro Algebra.

**Step 5:** If we set $\lambda^1_n = \lambda^2_n = \ldots = \lambda^p_n$ then step 4 gives us the $p$th order term of $\sum_n \lambda_n L_n e^{i \sum_n q_n Y_n}$ upt a factor of $1/p!$, i.e. it gives $(\lambda_n L_n)^p e^{i \sum_n q_n Y_n}$.

**Step 6:** Using steps 4 and 5 we get a prescription for writing down the $p$th order term of $\sum_n \lambda_n L_n e^{i \sum_n q_n Y_n}$. Take all possible orderings $\lambda^1 \ldots \lambda^p$
placing $N$’s and $N^T$’s as described in step 4. When we set $\lambda^1 = \lambda^2 = \ldots = \lambda^p$ all terms with a given sequence of $N$’s and $N^T$’s become identical and add together. Thus the number of ways we can order the $\lambda^i$’s to give that particular sequence (of $N$’s and $N^T$’s), divided by $p!$, is the coefficient of that particular $p$th order term in $e^{\sum_n \lambda_n L_n - n e^{i \sum_n q_n Y_n}}$.

**Step 7:** The expression (1.5) implements this rule to any order in $\lambda$.

Let us fill in the details now:

**Step 1:**

\[
: 1/2 \partial_z X(z + t) \partial_z X(z + t) :: e^{i \sum_n q_n Y_n} :=
\]

\[
: 1/2 \partial_z X(z + t) \partial_z X(z + t) e^{i \sum_n q_n Y_n} : + : \left[ \sum_{n>0} (q_n \partial_z X(z + t) \frac{n}{n!}) \right] + q_0 \partial_z X(z + t) \frac{1}{2} e^{i \sum_n q_n Y_n} : + \sum_{n,m} \frac{nq_nmq_m}{t^{n+m+2}}. \tag{3.4}
\]

gives the O.P.E. with the energy momentum tensor. Using

\[
\partial_z X(z + t) = \partial_z X(t^2 \partial^2 X + t^2 \partial^3 X/2!) + \ldots + t^{n-1} \partial^n X/(n-1)! + \ldots = \sum_n t^{n-1} Y_n \tag{3.5}
\]

we get

\[
\partial_z X(z + t) \partial_z X(z + t) = \sum_{n,m} t^{n+m-2} Y_n Y_m \tag{3.6}
\]

\[
q_n \partial_z X(z + t) nt^{-n-1} = \sum_m nq_n Y_m t^{m-n-2}. \tag{3.7}
\]

Multiplying (3.4) by $\int_{c}^{d} d\lambda(t) = \int_{c}^{d} dt \lambda P t^{-p+1}$

we get for the RHS:

\[
\sum_{n=1}^{p} \lambda_p Y_{n}.Y_{p-n} + \sum_{n=0}^{p} \lambda_p nq_n.Y_{p+n} + \sum_{n=1}^{p} \lambda_{-p} nq_n.(p-n)q_{p-n} + \sum_{n=p}^{p} \lambda_{-p} nq_n.Y_{n-p} \tag{3.8}
\]

(As always $nq_n$ is replaced by $q_0$ when $n = 0$). Since $\lambda_{m,n} = \lambda_{m+n}$ this term is exactly the exponent of (1.4) with $P\lambda = \lambda$. Thus the term $e^{1/2 Y^T P \lambda Y}$ is correct to linear order in $\lambda$. This concludes step 1. At this juncture we should point out one fact. At higher orders in $\lambda$ there are two kinds of terms. One of them is of the form $Y^T \lambda N \lambda \ldots \lambda Y$. These are the ”non-trivial” terms.
we are concerned about. Then there are terms of the form \((YY^T)Y^m/m!
\)
that come from expanding the exponential in (1.4). This term comes from
the repeated (m-fold) action of \(L_{-n}\) on \(e^{\sum \lambda_n L_{-n}}\) directly (i.e. not from
commutation of the \(L_n\)'s amongst themselves). In the language of the
operator product expansion these are terms that do not involve contraction of
\(X\)'s from different \(T_{zz}\)'s with each other. This is a "trivial" higher order \(L\)
-dependence (i.e. trivial in that it does not require any effort to prove that
these terms are present in the right way in (1.4)).

**Step 2:** This step is obvious since the action of \(e^{\sum \lambda_n L_{-n}}\) results in a
generalized vertex operator just like the one we started with, as can be seen
from (2.15). If this is correct to \(O(\lambda)\), and the subsequent action of \(e^{\sum \mu_n L_{-n}}\)
is correct to \(O(\mu)\) then the composition \(e^{\sum \lambda_n L_{-n}}e^{\sum \mu_n L_{-n}}\) must
be correct to \(O(\mu\lambda)\). For completeness we present an explicit verification of
the Virasoro algebra with central charge, in the Appendix. This concludes
step 2.

**Step 3:** The end point of step 2 is (2.31). (2.31) is equivalent to (2.23)
which is a generalized vertex operator. One can act on this operator by
\(e^{\sum \rho_n L_{-n}}\) in exactly the same way as in in the preceding two steps. If we
repeat (with three parameters) the steps leading from (2.23) to (2.31) we
end up with (3.1). The algebra is extremely straightforward so we will not
repeat it here. By the same logic that was used in step 2, this procedure is
guaranteed to give the right answer to (non-trivial) order \(\rho\mu\lambda\). This concludes
step 3.

**Step 4:** This is a \(p\)-parameter generalization of step 3 and there is nothing
new here. This concludes step 4.

Step 5 and Step 6 are obvious.

**Step 7:** Consider a typical term in (1.5):

\[
\int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \cdots \int_0^1 d\sigma_m \lambda(\sigma_1)\hat{\theta}(\sigma_1, \sigma_2)\lambda(\sigma_2)\hat{\theta}(\sigma_2, \sigma_3)\lambda(\sigma_3) \cdots \hat{\theta}(\sigma_{p-1}, \sigma_p)\lambda(\sigma_p).
\]

(3.9)

Pick a particular sequence of \(N\) and \(N^T\). The \(\theta\) functions enforce that
\(\sigma_2 > \sigma_1, \sigma_2 > \sigma_3\) etc.

\[
\int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \cdots \int_0^1 d\sigma_m \lambda(\sigma_1)N\theta(\sigma_1-\sigma_2)\lambda(\sigma_2)N^T\theta(\sigma_3-\sigma_2)\lambda(\sigma_3) \cdots \theta(\sigma_{p-1}-\sigma_p)\lambda(\sigma_p).
\]

(3.10)
The region $0 \leq \sigma_1, \sigma_2, \cdots, \sigma_p \leq 1$ can be decomposed into $p!$ regions each with a particular ordering of the $\sigma$’s:

$$0 \leq \sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_p} \leq 1.$$  \hspace{1cm} (3.11)

Thus (3.10) can be decomposed into $p!$ integrals. Each of the $p!$ terms can only give one of two possible values. The $\theta$-functions either vanish in the entire range of one of these integrals or equal one in the entire range, for if you have $\theta(\sigma_i - \sigma_j)$ the integration region either has $\sigma_i > \sigma_j$ over the entire range, in which case $\theta(\sigma_i - \sigma_j) = 1$ or it has $\sigma_i < \sigma_j$ over the entire range in which case $\theta(\sigma_i - \sigma_j) = 0$. If any of the $\theta$-functions are zero the integral vanishes. If all the $\theta$-functions are equal to one we get (Note that $\lambda$ does not depend on $\sigma$)

$$\int_0^1 d\sigma_{i_1} \int_0^{\sigma_{i_1}} d\sigma_{i_2} \cdots \int_0^{\sigma_{i_p}} d\sigma_{i_p} (\lambda N \lambda N^T \lambda \cdots) = \frac{1}{p!} (\lambda N \lambda N^T \lambda \cdots)$$  \hspace{1cm} (3.12)

Thus whenever the ordering of the $\sigma$’s is such that it gives the particular sequence of $N$’s and $N^T$’s we get a factor of $1/p!$. The full integral (3.10) thus tries all possible $(p!)$ orderings of $\sigma_i$’s and it therefore counts the number of ways it can be done. This concludes Step 7 and thus the proof of (1.4).
4 Group Composition

We briefly discuss the group composition property and give an example of 
an explicit calculation for concreteness. Equation (2.31) contains the basic 
composition rule: Write

\[ e^{\sum_n \mu_n L_{-n}} e^{\sum_n \lambda_n L_{-n}} = e^{\sum_n \rho_n L_{-n}} e^{\rho_c (\mu, \lambda)} \]  

(4.1)

\( \rho_c \) refers to an overall normalization that depends on \( \mu, \lambda \). Comparing (1.4), 
(2.31) and (4.1) we see that

\[ P^{\rho(\mu, \lambda)} = P^\mu + P^\lambda + P^\mu N P^\lambda + P^\lambda N^T P^\mu + \cdots; \rho_c = D / 2 T r l n [K + M] \]  

(4.2)

This defines \( \rho \) implicitly since (1.5) gives

\[ P^\rho = \int d\sigma_1 \left( \frac{1}{1 - \rho \bar{\theta}} \right)_{\sigma_1 \sigma_2} \rho \sigma_2 \]  

(4.3)

Note that if we set \( \mu = \lambda \) we get

\[ P^{2\lambda} = 2P^\lambda + P^\lambda (N + N^T) P^\lambda + \cdots \]  

(4.4)

(4.4) is a non linear equation for \( P^\lambda \). It can be used to determine \( P^\lambda \) recursively. 
If we set

\[ P^\lambda = \lambda + a\lambda N \lambda + b\lambda N^T \lambda + c\lambda N \lambda N \lambda + \cdots \]  

(4.5)

one can solve recursively for the coefficients \( a, b, c, \ldots \) by substituting (4.5) 
into (4.4). One can check that the result reproduces (1.5). In fact one 
can prove that (1.5) satisfies this equation. We will not do this here but 
the outline of the proof is as follows: Define \( \lambda(\sigma) : 0 < \sigma < 2 \) so that

\[ \lambda(\sigma) = \lambda \quad 0 < \sigma < 1 \]

\[ \lambda(\sigma) = \mu \quad 1 < \sigma < 2 \]

One then shows that (4.2) can be written in the 
form (4.3) except that the range of \( \sigma \) is from 0 to 2. Doubling the range 
of \( \sigma \) is equivalent to scaling \( \lambda \rightarrow 2\lambda \). This proves that (1.5) satisfies (4.4). 
This is an alternate proof of the correctness of (1.5). Finally one can also 
show that setting \( \mu = -\lambda \) gives \( P^\rho = 0 \). To lowest order this is obvious 
since \( P^\mu = \mu \), however it is not at all obvious a priori that this is satisfied 
by the full expression (4.2). We have checked this but will not reproduce the 
argument here.

We conclude this section with an example.
4.1 Example

\[ e^{(\lambda_2 L_{-2} + \lambda_{-2} L_2)} e^{ik_0 Y} |0> \] (4.6)

The exact answer of course is (1.4). We can check this order by order. Some of the non-zero elements of the matrix \( \lambda_{n,m} \) are:

\[
\begin{align*}
\lambda_{1,1} &= \lambda_{0,2} = \lambda_{2,0} = \lambda_{-1,3} = \lambda_{3,-1} = \lambda_{4,-2} = \lambda_{-2,4} = \lambda_2 \\
\lambda_{-1,-1} &= \lambda_{0,-2} = \lambda_{-2,0} = \lambda_{1,-3} = \lambda_{-3,1} \lambda_{-4,2} = \lambda_{2,-4} = \lambda_2 
\end{align*}
\] (4.7)

\[
P^\lambda = \lambda + 1/2 \lambda(N + N^T) \lambda + O(\lambda^3) \] (4.8)

\[
Y^T \lambda Y = (Y_n, -imk_n) \left( \begin{array}{cc} \lambda_{n,m} & \lambda_{+n,-m} \\ \lambda_{-n,m} & \lambda_{n,-m} \end{array} \right) \left( \begin{array}{c} Y_m \\ -imk_m \end{array} \right) \] (4.9)

In this example only \( k_0 \) is non zero. This simplifies (4.9) considerably: All the \( -m \) indices in the matrix can be set to zero. We remind the reader of our convention that '0' belongs to the negative index set and '+m' is assumed to be a positive integer. This fact, along with (4.7), quickly reduces (4.9) to the equation:

\[-1/2 Y^T \lambda Y = -\lambda_2(Y_1^2 + 2Y_2(-ik_0)) \] (4.10)

This is just \( L_{-2} e^{ik_0 X} |0> \). At the next order in \( \lambda \) we have to calculate

\[
Y^T (1/2 \lambda(N + N^T) \lambda) Y = \] (4.11)

\[
(Y_n - ik_0) \left( \begin{array}{cc} \lambda_{n,m} & \lambda_{+n,-m} \\ \lambda_{-n,m} & \lambda_{n,-m} \end{array} \right) \left( \begin{array}{cc} 0 & m \\ m & 0 \end{array} \right) \left( \begin{array}{cc} \lambda_{m,+p} & \lambda_{m,-p} \\ \lambda_{-m,+p} & \lambda_{-m,-p} \end{array} \right) \left( \begin{array}{c} Y_p \\ -ik_0 \end{array} \right) \]

As before the \( -p \) index is forced to be '0', as is the \( -n \) index. Once again using (4.7) we get finally

\[-1/4 Y^T \lambda(N + N^T) \lambda Y + 1/8 (Y^T \lambda Y)^2 = \]

\[-1/4 \lambda_2^2 (2Y_1^2 - 4ik_0 Y_3) + 1/4 \lambda_2 \lambda_2 4k_0^2 + \lambda_2^2/2(Y_1^2 + 2Y_2(-ik_0))^2 \] (4.12)

where we have added the 'trivial' second order contribution to (4.10). One also has to add a contribution from the determinant at this order.

\[
Det[\delta_{\sigma_1 \sigma_2} - \lambda_{\sigma_1} \tilde{\theta}_{\sigma_1} \sigma_2] = e^{-D/2(Tr\ln[1-\lambda \tilde{\theta}])} = e^{D/2(Tr\lambda \bar{\theta} + 1/2 Tr(\lambda \tilde{\theta} \lambda \bar{\theta} ...)} \] (4.13)
The trace includes one over the σ-index as well as the n-index.

\[ T r \lambda \tilde{\theta} = T r \int d\sigma_1 \int d\sigma_2 [\lambda N \theta(\sigma_1 - \sigma_2) + \lambda N^T \theta(\sigma_2 - \sigma_1)] \delta(\sigma_1 - \sigma_2) = 0 \] (4.14)

since \( T r \lambda N = 0 \). At the next order we have

\[
T r \int d\sigma_1 \int d\sigma_2 \lambda [N \theta(\sigma_1 - \sigma_2) + N^T \theta(\sigma_2 - \sigma_1)] [N \theta(\sigma_2 - \sigma_1) + N^T \theta(\sigma_1 - \sigma_2)]
\]

\[ = 2 \text{tr} \int d\sigma_1 \int d\sigma_2 \lambda N \lambda N^T \theta(\sigma_1 - \sigma_2)
\]

\[ = 1/22 T r \lambda N \lambda N^T = \lambda_2 \lambda_{-2}
\]

Thus the contribution of the determinant is

\[ D/4 \lambda_2 \lambda_{-2} \] (4.15)

Since

\[
L_2 L_{-2} e^{ik_0 x} |0> = [L_2, L_{-2}] e^{ik_0 x} |0>
\]

\[
(4L_0 + D/2) e^{ik_0 x} |0> = 2k_0^2 + D/2) e^{ik_0 x} |0>
\]

we see that (4.12) and (4.15) do indeed give the right answer.
5 Conclusions

Let us summarize the results. We have given an expression in closed form for the action of an element of the Virasoro group on generalized vertex operators $e^{i \sum_{n > 0} k_n Y_n}$ in the form $e^{\sum_n \lambda_n L_n - n e^{i \sum_{n > 0} k_n Y_n}} |\Omega\rangle$. The result is given in equation (1.4). The main idea was to use a first order formalism for the energy momentum tensor which recasts the group element in the form of a loop variable $e^{i \int c(t) \partial_z X(z+t) dt}$. This makes the calculation very easy.

The crucial issue that has to be addressed is that of regularizing the loop variable or equivalently taking into account contractions of the $X$ field inside the loop variable. We sidestepped this question by building up a finite group element as a product of infinitesimal group elements, each represented as a loop variable for which we do not have self contractions (since we only keep the linear piece of each infinitesimal element) We were able to derive the expression (1.4) in this manner.

The naive approach described in section II does not give the (right) answer (1.4) although it gives something very similar looking. It would give some insight into the formalism of loop variables if we could identify a procedure that modifies the naive approach to take into account the self interactions of the $X$ fields in a loop variable in a self consistent way and leads to (1.4) directly. It turns out that this is not very difficult. The basic idea is to thicken the loop into a band or an annulus as shown in fig 2. Then we repeat the method of section 2 but now applied to a superposition of all the loops making a band : i.e. to variables of the form $\int d\sigma \int dt \partial_t X(z+t)$. $\sigma$ is assumed to vary from 0 to 1 and parametrizes the different loops in the annulus. Thus we let $\sigma = 0$ correspond to the inner boundary of the annulus and $\sigma = 1$ correspond to the outer boundary. Then (2.5) is modified to

$$e^{i \int c(t,\sigma) \partial_z X(z+t,\sigma) dt} = e^{i \int \int c(t,\sigma) \partial_z X(z+t,\sigma) dt} : e^{i \int dt \int dt' \int d\sigma' k(t,\sigma) k(t',\sigma') < \partial X(z+t) \partial X(Z+t')} (5.1)$$

For $\sigma > \sigma'$ we take $t$ to be the outer loop and $t'$ to be the inner loop and vice versa when $\sigma < \sigma'$. The inner loop can be shrunk to a point since there are no singularities and we can expand $\partial X(z+t)$ in a Taylor series with only positive powers of $t$. The result is

$$e^{i \int \int c(t,\sigma) \partial_z X(z+t,\sigma) dt} = e^{i \int \int c(t,\sigma) \partial_z X(z+t,\sigma) dt} : e^{ \int \int c(t,\sigma) \partial_z X(z+t,\sigma) dt} e^{ \int \int \int c(t,\sigma) c(t',\sigma') < \partial X(z+t) \partial X(Z+t')} (5.2)$$
This can be written as
\[ e \int d\sigma \int d\sigma' k_n(\sigma) k_m(\sigma') [N\theta(\sigma-\sigma') + NT\theta(\sigma'-\sigma')]_{nm} \tag{5.3} \]

The expression in square brackets is \( \tilde{\theta} \) of eq.(1.3). If we include the wave function as before, the complete first order form for the group element is
\[ \int Dk(t, \sigma) : e^{i \int c_k(t, \sigma) \partial \mathcal{X}(z + t, \sigma) dt} : e^{\int d\sigma \int d\sigma' k_n(\sigma) k_m(\sigma') [\tilde{\theta}_1 \sigma_2 - \delta_1 \sigma_2 \lambda(\sigma_1)]_{nm}} \tag{5.4} \]

Taking the OP of this loop (or 'band') variable with a generalized vertex operator \( e^{i \sum_{n \geq 0} q_n Y_n} \) and doing the \( k \)-integration gives the result (1.4). Intuitively the group element \( e^{\sum_n \lambda_n L_{-n}} \) is being written as a product of infinitesimal elements:
\[ \left( e^{\lambda_{nL_{-n}}} \right)^{N\text{-factors}}, N \to \infty \tag{5.5} \]

Each loop in the annulus represents one of these factors. The contractions between the \( X \)-fields of different loops represent the nontrivial commutation between the \( L_n \)'s coming from different factors.

We can also compare the result obtained here with that in [17]. The result obtained there was that the action of a sequence of \( L_n \)'s on a vertex operator \( e^{i \sum_{n>0} k_n Y_n} \) can be obtained by starting with
\[ e^{-1/2q_1 q_2 Y + i k_1 k_2 Y - 1/2k_1 k_2 \theta_k \theta_k} Det(1 - q)^{-D/2} \tag{5.6} \]

where \( k_n, q_{nm} \) and \( \theta_{nm} \) were variables (defined in [17]) with simple transformation properties under the Virasoro algebra. Comparing (5.6) with (1.4) and (2.15) shows that they are very similar looking. It must be true then that \( q_{nm}, \theta_{nm} \) are variables that parametrize the group so that they should correspond more or less to \( \lambda_{+n,+m}, \lambda_{-n,-m}, \lambda_{+n,-m} \). While we know the transformation properties of \( q, \theta \) we do not know how they parametrize the group. On the other hand in the case of the \( \lambda \) variables of this paper, we know how they parametrize the group but we have not worked out the transformation properties. Thus the results obtained here complement those of [17]. To make a precise comparison one would have to work out the transformations of the \( \lambda \)'s and compare with those of the \( q, \theta \) variables. This is in progress.
Finally, on a more speculative level we might hope that some of the
techniques used here might be useful for string interactions. The action of
a group element on a vertex operator is related to the kinetic term in string
field theory. Thus if we can write the kinetic term as an OP of two loop
variables as done here we might hope for a more unified treatment of the
kinetic and interaction terms in string field theory.

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cussion.
We show that to linear order in $\lambda$ and $\mu$ (2.31) is consistent with the Virasoro Algebra. We reproduce (2.31) for convenience

$$e^{-1/2Y^TP\rho Y}e^{i\sum_{n \geq 0} q_n Y_n} Det[1 - \rho \tilde{\theta}]^{-D/2} =$$

$$e^{\sum_{n} \mu_n L_{-n}} e^{\sum_{n} \lambda_n L_{-n}} e^{i\sum_{n \geq 0} q_n Y_n} =$$

$$e^{-1/2Y^T(P^\mu + P^\lambda + P^\mu N P^\lambda + P^\lambda N^T P^\mu + P^\mu N P^\lambda N^T P^\mu + P^\lambda N^T P^\mu N P^\lambda + \ldots) Y}$$

$$e^{i\sum_{n \geq 0} q_n Y_n} : Det[K + M]^{-D/2} \quad (A.1)$$

We need to determine $\rho(\mu, \lambda)$ from $P^\rho$ as defined above and then calculate the commutator $\rho(\mu, \lambda) - \rho(\lambda, \mu)$. To second order in $\rho$

$$P^\rho = \rho + 1/2 \rho(N + N^T)\rho + \cdots \quad (A.2)$$

In solving for $\rho$ to $O(\mu \lambda)$ we need to keep terms up to second order in $P^\rho$, since $P^\rho = \mu + \lambda + \ldots$. The answer is

$$\rho = P^\rho - 1/2 P^\rho(N + N^T)P^\rho + O(P^\rho^3) + \cdots \quad (A.3)$$

Substituting for $P^\rho$ from (2.31) we get

$$\rho(\mu, \lambda) = P^\mu + P^\lambda + 1/2(P^\mu N P^\lambda + P^\lambda N^T P^\mu - P^\mu N^T P^\lambda - P^\lambda N P^\mu - P^\mu(N + N^T)P^\mu - P^\lambda(N + N^T)P^\lambda) \quad (A.4)$$

which gives

$$\rho(\mu, \lambda) - \rho(\lambda, \mu) = P^\mu N P^\lambda + P^\lambda N^T P^\mu - P^\mu N^T P^\lambda - P^\lambda N P^\mu$$

$$= \mu N \lambda + \lambda N T \mu - \mu N T \lambda - \lambda N \mu \quad (A.5)$$

to $O(\mu \lambda)$.

Let us choose $\lambda_a$ and $\mu_b$ as the two non-vanishing parameters. This will give us $[L_a, L_b]$. From the definition of $\lambda_{n,m}$ we have:

$$\lambda_{a-n,n} = \lambda_{n,a-n} = \lambda_a; \mu_{b-n,n} = \mu_{n,b-n} = \mu_b; \forall n. \quad (A.6)$$

We then get:

$$(\lambda N \mu)_{nj} = \sum_{m \leq 0} \lambda_{nm} (-m) \delta_{-m,p} \mu_{p,q} =$$
\[
\sum_{n \geq a} \lambda_{n,a-n}(a-n)\mu_{n-a,b-n+a} = -\sum_{n \geq a} \lambda_a \mu_b (a-n) \delta_{q,b+a-n} \quad (A.7)
\]

\[
(\lambda N^T \mu)_{nq} = \sum_{n \leq a} \lambda_a \mu_b (a-n) \delta_{q,b+a-n} \quad (A.8)
\]

\[
(\lambda N^T \mu - \lambda N \mu)_{nq} = \sum_{n} \lambda_a \mu_b (a-n) \delta_{q,b+a-n} \quad (A.9)
\]

\[
(\mu N^T \lambda - \mu N \lambda)_{nq} = \sum_{n} \mu_b \lambda_a (a-b) \delta_{q,b+a-n} \quad (A.10)
\]

which gives

\[
(\rho)_{nq} = \sum_{n} \lambda_a \mu_b (a-b) \delta_{q,b+a-n} \quad (A.11)
\]

Using the definition \(\rho_{n+m} = \rho_{n,m}\) we get \(\rho_{a+b} = \lambda_a \mu_b (a-b)\). This confirms the Virasoro Algebra except for the central charge, to which we now turn.

The central charge is \(D/2 T r ln (K + M) - (\mu \leftrightarrow \lambda)\) which, to this order is :

\[
D/2 T r (P^\mu NP^\lambda N^T) - (\mu \leftrightarrow \lambda) \quad (A.12)
\]

with

\[
T r (P^\mu NP^\lambda N^T) = \sum_{m,p>0} mp \mu_{-p-m} \lambda_{+m+p}
\]

which gives for (A.12)

\[
\sum_{m,p>0} (\mu_{-p-m} \lambda_{+m+p} - \lambda_{-m-p} \mu_{+m+p}) mp \quad (A.13)
\]

If we let \(\lambda_a; a > 1\) and \(\mu_{-a}\) be the two non vanishing parameters, this becomes:

\[
D/2 \sum_{m=1}^{a-1} m(a-m) \mu_{-a} \lambda_a = \frac{(a^3 - a)D}{12} \mu_{-a} \lambda_a \quad (A.14)
\]

which is the central charge.

Thus eq.(2.31) is consistent with the Virasoro Algebra with central extension.
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