

A $q$-BRAUER ALGEBRA

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Abstract. We define a new $q$-deformation of Brauer’s centralizer algebra which contains Hecke algebras of type $A$ as unital subalgebras. We determine its generic structure as well as the structure of certain semisimple quotients. This is expected to have applications for constructions of subfactors of type $II_1$ factors and for module categories of fusion categories of type $A$ corresponding to certain symmetric spaces.

In his paper [Br], Richard Brauer introduced a series of algebras, specializations of which describe the decomposition of tensor powers of the defining vector representation of an orthogonal or symplectic group. More recently, $q$-deformations of these algebras have been defined in [BW] and [Mu] in connection with knot theory and quantum groups. They found a number of applications, such as in the study of subfactors and tensor categories (see e.g. [W2], [TW], [TuW]).

In this paper we introduce another $q$-deformation of Brauer’s centralizer algebras motivated by the following problem: Let $V$ be the $N$-dimensional representation of $Gl(N)$. Restricting the action of $Gln(N)$ on tensor powers $V^\otimes n$ to $O(n)$ leads to embeddings of the centralizer algebras $CS_n$, where $S_n$ is the symmetric group, into the Brauer algebra $D_n(N)$. Our idea now is very simple: Find a $q$-deformation of $D_n(N)$ which extends the $q$-deformation of $CS_n$, the Hecke algebra $H_n(q)$ of type $A_{n-1}$, subject to certain compatibility conditions with respect to taking tensor products. This can also be stated in the language of module categories (see the beginning of Section 2). We shall see that these conditions completely determine a $q$-deformation of the Brauer algebra $D_n(N)$. This approach also carries over comparatively easily to the setting of fusion tensor categories, i.e. for certain quotients of Hecke algebras at roots of unity. This will be important for one of the main motivations of this work, the constructions of examples of subfactors of $II_1$ von Neumann factors. They were, at least in part, inspired by work in conformal field theory in connection with twisted affine loop groups and boundary conformal field theory (see e.g. [GG] and references therein).

It is well-known that in this context the Hecke algebras correspond to Jimbo-Drinfeld quantum groups $U_q sl_N$ via an extension of Schur duality. So our new algebras should correspond to a $q$-deformation of the subalgebra $Uso_N \subset UsN$. Such algebras were defined as coideal algebras in work by Letzter (see [L1], [L2]), and also in work by Gavrilik and Klimyk and by Noumi (see [GK], [N]). This could give another, potentially more conceptual approach to derive our algebras, at least for the generic case with $q$ not a root of unity. Related work in this direction has already appeared earlier in [Mo], see the remarks at the end of this paper. So our algebras can also be viewed as part of a categorical construction of quantum analogs of
certain symmetric pairs. Our approach also works in the context of fusion categories, which, so far, would not be so clear in the context of coideal algebras.

Here is a brief outline of the contents of this paper. In the first section, we review results about Brauer’s centralizer algebras and Hecke algebras. This will also serve as a model for our approach of defining and proving results about our $q$-deformation of Brauer’s centralizer algebra. In the second section, we motivate our definitions via an approach to find module categories of quantum groups from subalgebras of the classical Lie algebra. We then give the definition of our algebras via generators and relations in the following section. We show that they have bases labeled by the basis graphs of Brauer’s algebras. Moreover, they also have the same decomposition into full matrix rings in the generic case as Brauer’s. In the fourth section, we define a trace functional on our algebras with certain properties. It is an extension of certain important trace functionals defined on Hecke algebras, which are often referred to as Markov traces. We will use our results on these Markov traces in Section 5 to determine for which values of the parameters our algebras will be semisimple. Moreover, we also determine certain semisimple quotients in the non-semisimple case. One can also see at these quotients that the algebras in this paper are different from the $q$-deformations of Brauer algebras in [BW] and [Mu]. We then discuss several applications of our algebras such as the construction of module categories, subfactors and representations of fusion rings.

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1. Brauer and Hecke algebras

1.1. Basic definitions. In this paper Brauer’s centralizer algebra $D_n$ is defined over the ring $\mathbb{Z}[x]$ via a basis given by graphs with $2n$ vertices, arranged at two levels, and $n$ edges, where each vertex belongs to exactly one edge. We will call an edge vertical if its vertices are on different levels, and horizontal if they are on the same level. Concatenation of two basis graphs $a$ and $b$ is given similarly as with braids. One puts $a$ on top of $b$ such that the $n$ lower vertices coincide with the $n$ upper vertices of $b$. One then removes all cycles, i.e. parts of the resulting graph which are not connected to an upper or lower vertex. The element $ab$ is then defined to be this resulting graph without cycles, multiplied by $x$ taken to the power of the number of removed cycles; here $x$ is a variable. To give an example, let $e_{(k)}$ be the element of $D_n$ given by a graph with $2k$ horizontal edges on the left, and the remaining $n - 2k$ edges vertical. E.g. see below the graph for $e_{(2)} \in D_7$:

\[= = = = =\]

Figure 1
Then it is easy to check that \( e_{(k)}e_{(m)} = e_{(m)}e_{(k)} = x^m e_{(k)} \) for any \( m \leq k \); here the horizontal edges of \( e_{(k)} \) should be drawn slightly concave to obtain cycles. In the following, Brauer’s centralizer algebra \( D_n \) is the free \( \mathbb{Z}[x] \)-module spanned by the above mentioned basis graphs. It is clear from the definition that the multiplication of \( D_n \) is well-defined over \( \mathbb{Z}[x] \) and associative. It is also clear that its rank is \( n!! = 1 \cdot 3 \cdot \ldots \cdot (2n - 1) \).

Observe that \( D_n \) contains a subalgebra which is isomorphic to \( \mathbb{Z}[x]S_n \), where \( S_n \) is the symmetric group of all permutations of \( n \) symbols. It is spanned by the basis graphs which only have vertical edges. Then we get a decomposition of \( D_n(x) \) in terms of \( S_n - S_n \) bimodules as

\[
D_n(x) \cong \bigoplus_{k=0}^{[n/2]} \mathbb{Z}[x]S_n e_{(k)} S_n;
\]

informally, \( S_n e_{(k)} S_n \) can be viewed as the set of all graphs with exactly \( 2k \) horizontal edges. Moreover, as the product of two graphs has at least as many horizontal graphs as either of them, it is easy to see that \( I(m) = \bigoplus_{k \geq m} \mathbb{Z}[x]S_n e_{(k)} S_n \) is a two-sided ideal in \( D_n \) for each \( m \) with \( 2m \leq n \).

It is clear from the pictures that multiplication of a graph of \( D_n \) from the left (i.e. from above pictorially) does not change the position of the lower horizontal edges. This defines a decomposition of \( \mathbb{Z}[x]S_n e_{(k)} S_n \) into \( S_n \)-modules. Combinatorially, the position of the lower horizontal edges of a graph in \( S_n e_{(k)} S_n \) is determined as follows: We choose a subset of \( 2k \) elements from \( \{1, n\} \) (only integers) and partition it into \( k \) subsets of \( 2 \) elements each. Let \( P(n, k) \) be the set of all those partitions. Then

\[
\mathbb{Z}[x]S_n e_{(k)} S_n \cong \bigoplus_{j \in P(n,k)} \mathbb{Z}[x]S_n e_{(k)} w_j,
\]

where \( w_j \in S_n \) such that \( e_{(k)} w_j \) is the graph whose lower horizontal edges are given by the partition \( j \in P(n,k) \) and such that no vertical edges intersect. This completely determines \( e_{(k)} w_j \). The permutation \( w_j \) is not uniquely determined. We shall later make the choice of \( w_j \) more precise.

We shall also consider the Brauer algebra \( D_n(N) \), \( N \in \mathbb{Z} \) which is defined over \( \mathbb{Z} \) by the same graphs as before. The only difference is that now the variable \( x \) is replaced by the integer \( N \).

1.2. The module \( V_n^{(k)} \) for Brauer algebras. It is also easy to see that multiplication of a graph \( d \in \mathbb{Z}[x]S_n e_{(k)} w_j \) by an element in \( D_n \) from the left/above leaves the lower horizontal edges unchanged, but may add additional lower horizontal edges. Hence the factor module \( \mathbb{Z}[x]S_n e_{(k)} w_j + I(k+1)/I(k+1) \) is a \( D_n \)-module with a basis given by the basis graphs of \( \mathbb{Z}[x]S_n e_{(k)} w_j \). In particular, we obtain

\[
I(k)/I(k+1) \cong \bigoplus_{j \in P(n,k)} \mathbb{Z}[x](S_n e_{(k)} w_j + I(k+1))/I(k+1)
\]
As multiplication from the right by $w_1$ commutes with the $D_n$-action, it follows that each summand on the right hand side is isomorphic to the module
\[(1.4) \quad V_n^{(k)} = \mathbb{Z}[x]S_ne_{(k)} + I(k+1)/I(k+1). \]

Combinatorially, it is spanned by graphs with exactly $k$ horizontal edges in the lower part, where the $i$-th edge connects the lower vertices $2i-1$ and $2i$. As additional notation, let $s_i = (i, i+1)$ be the transposition of the numbers $i$ and $i+1$, and let $W(B_k)$ be the subgroup of $S_n$ generated by the elements $s_{2i-1}, 1 \leq i \leq k$ and by $s_{2i}s_{2i-1}s_{2i+1}s_{2i} = (i, i+2)(i+1, i+3), 1 \leq i < k$. It is well-known that $W(B_k)$ is isomorphic to the semidirect product of $(\mathbb{Z}_2)^k$ with $S_k$. We have the following simple properties.

**Lemma 1.1.** (a) The $\mathbb{Z}[x]$-rank of $V_n^{(k)}$ is equal to $n!/2^k k!$. Moreover, as an $S_n$-module, $V_n^{(k)} \cong \mathbb{Z}[x]/(S_n/W(B_k))$.

(b) $\mathbb{Z}[x]S_ne_{(k)}S_n$ is isomorphic to $\mathbb{Z}[x]S_ne_{(k)} \otimes \mathbb{Z}[x]S_{2k+1,n}e_{(k)}S_n$ as $\mathbb{Z}[x]S_{n} - \mathbb{Z}[x]S_n$-bimodule.

(c) The commutant of $D_n$ on $V_n^{(k)}$ is given by $\mathbb{Z}[x]S_{2k+1,n}$.

(d) The algebra $D_n$ is faithfully represented on $\bigoplus_{k=0}^{[n/2]} V_n^{(k)}$.

**Proof.** The first statement is straightforward to prove. The second statement follows from the fact that $S_{2k+1,n}$, which leaves the numbers 1 until $2k$ fixed, commutes with $e_{(k)}$, from which one can deduce that $\mathbb{Z}[x]S_ne_{(k)}$ is a free $\mathbb{Z}[x]S_{2k+1,n}$ right module, and that $\mathbb{Z}[x]e_{(k)}S_n$ is a free $\mathbb{Z}[x]S_{2k+1,n}$ left-module. As to the statement (c), it is easy to see that $\mathbb{Z}[x]S_{2k+1,n}$ is contained in the commutant. As $e_{(k)}$ is a cyclic vector for $V_n^{(k)}$, any element $b$ in the commutant of $D_n$ is already completely determined by its action on $e_{(k)}$. It is easy to inspect by multiplying graphs that $e_{(k)}d$ is in $e_{(k)}S_{2k+1,n} + I(k+1)$ for any $d \in S_n e_{(k)}$. Hence it follows
\[ x^k be_{(k)} = b e_{(k)}^2 = e_{(k)}(be_{(k)}) = \pi e_{(k)} \]
for some $\pi \in \mathbb{Z}[x]S_{2k+1,n}$. To prove the last statement, we use the fact that $\mathbb{Q}(x) \otimes \mathbb{Z}[x] D_n$ is semisimple (see e.g. [HW]). Hence its left regular representation is faithful. But by the discussion in this section, see 1.3 and 1.4, $D_n$ has a filtration of $D_n$-modules, each of whose factors is isomorphic to a $V_n^{(k)}$. By semisimplicity, we can replace this by a direct sum of modules each of which is isomorphic to a $V_n^{(k)}$.

1.3. **Decomposition.** In the following we are primarily interested in the $S_n$-action on $V_n^{(k)}$. For simplicity, we do this over the ring $\mathbb{Z}$; the results are exactly the same for the ring $\mathbb{Z}[x]$. We shall need the decomposition of the module $V_n^{(k)}$ as a $\mathbb{Z}S_{3,n}$-module, where $S_{3,n}$ is the group of permutations of letters 3 until $n$. In view of the last lemma, it is clear that we obtain a decomposition in terms of $S_{3,n}$-orbits of $S_n/W(B_k)$, i.e. in terms of cosets $S_{3,n}wW(B_k)$. We shall describe these double cosets in terms of specially chosen elements $w$ whose meaning will become clear later. If $i \leq j$, we shall use the notation $s_{i,j} = s_is_{i+1} \ldots s_j$. Not surprisingly, the size of such double cosets depends on the intersection $w^{-1}\{1, 2\} \cap [2k+1,n]$. We list the decomposition of $V_n^{(k)}$ into $S_{3,n}$-modules in the table below.
1.4. Length function. Similarly as for elements in reflection groups, one can define a length function for basis graphs of the Brauer algebra. Recall that for a permutation \( w \in S_n \), its length \( \ell(w) \) is the minimum number of factors in an expression of \( w \) as a product of simple reflections; interpreting \( w \) as a graph as above, \( \ell(w) \) would be the number of crossings in that graph with the following caveat: The element \( e_{(k)} \) is drawn fixed and must be left unchanged; e.g. the element \( s_1s_2e_{(k)}s_1 \) has length 4, even though the corresponding graph in the Brauer algebra could be drawn without any crossings. The precise definition of the length \( \ell(d) \) of a basis graph \( d \in D_n \) with exactly \( 2k \) horizontal edges is given by

\[
\ell(d) = \min \{ \ell(w_1) + \ell(w_2), \ w_1e_{(k)}w_2 = d, \ w_1, w_2 \in S_n \}.
\]

We will also call graphs of the form \( we_{(k)} \) basis graphs of the module \( V_{n,k}^{(k)} \). For given \( d \), there can be more than one \( w \) with \( we_{(k)} = d \) and \( \ell(w) = \ell(d) \), e.g. \( s_1s_2e_{(k)} = s_3s_2e_{(k)} \) for \( k \geq 2 \).

To pin down a specific choice, it will be convenient to use the notation \( s_{i,j} = s_is_{i+1}...s_j \) for \( i < j \). It is well-known that the elements \( w \) of \( S_n \) can be written uniquely in the form \( w = t_{n-1}t_{n-2}...t_1 \), where \( t_j = 1 \) or \( t_j = s_{i,j} \) with \( 1 \leq i,j \leq i \leq i < n \). This can be easily seen as follows: For given \( w \in S_n \), there exists a unique \( t_{n-1} \) such that \( t_{n-1}(n) = w(n) \).

Hence \( w' = t_{n-1}^{-1}W(n) = n \) and we can view \( w' \) as an element of \( S_{n-1} \). The general claim now follows by induction on \( n \). We will apply a similar strategy for defining basis elements for \( V_{n,k}^{(k)} \). Using the notation for the \( t_j \)’s, we now define for \( k \leq n/2 \) the set

\[
B_{n,k} = \{ (t_{n-1}t_{n-2}...t_k) | \ell = t_2...t_{2k-2} \}
\]

Observe that \( B_{n,k} \) has \( n!/2^k k! \) elements.

**Lemma 1.2.**  
(a) The module \( V_{n,k}^{(k)} \) has a basis \( \{ wv_1 = v_{we_{(k)}} \}, \ w \in B_{n,k} \) with \( \ell(we_{(k)}) = \ell(w) \). Here \( \ell(w) \) is the number of factors for \( w \) in Def. 1.5, and \( v_1 = e_{(k)} + I(k+1) \in V_{n,k}^{(k)} \).

(b) We have \( |\ell(s_d) - \ell(d)| \leq 1 \) for any basis graph for \( V_{n,k}^{(k)} \). Equality of lengths holds only if \( s_d = d \).
(c) Let $S_3^{(i)}$ be the subgroup of $S_n$ generated by $s_i$ and $s_{i+1}$. Then each $S_3^{(i)}$-orbit $O$ of basis graphs in $V_n^{(k)}$ has the order structure of $S_3^{(i)}/W$, where $W$ is either the trivial subgroup or the subgroup generated by $s_i$ or by $s_{i+1}$.

Proof. Let $d = w_{e(k)}$ be a graph in $S_n e(k)$. Using exactly the same arguments as given before Def 1.5, we determine $t_{n-1}, \ldots, t_{2k}$ such that $d' = (t_{n-1}t_{n-2} \ldots t_{2k})^{-1}d$ is a graph in $S_{2k} e(k)$, i.e. $d'$ can be viewed as a graph in $D_{2k}$ with only horizontal edges to which we add $n - 2k$ strictly vertical edges to the right. Let $i_{k-2}$ be the label of the upper vertex of $d'$ which is connected with the upper $2k$-th vertex and set $t_{2k-2} = s_{i_{k-2}k-2}$. Then the upper $2k$-th and $(2k - 1)$-st vertices of $d'' = t_{2k-2}^{-1}d'$ are connected by a horizontal edge. Proceeding in this way, we eventually transform $d$ into $e(k)$. Hence every graph in $S_n e(k)$ can be written as $w e(k)$, with $w \in B_{n,k}$.

To show that the $w$ constructed in the last paragraph has minimal length, let $v \in S_n$ be such that $ve(k) = d$. Let $1 \leq r \neq s \leq k$. Then it is easy to see, e.g. by drawing pictures, that we have at least zero, one or two intersections between edges emanating from $2r - 1, 2r, 2s - 1, 2s$ for $1 \leq r < s \leq k$ if $[v(2r - 1), v(2r)] \cap [v(2s - 1), v(2s)]$ is empty, is a proper subinterval of both intervals, or is equal to one of the two intervals respectively. Let us call this minimum number $c(r, s)$. Moreover, we get an additional crossing for each inversion, i.e. for each pair $1 \leq a < b \leq n$ with $b > 2k$ for which $v(a) > v(b)$. It is not hard to check that the number of inversions (with $b > 2k$) is independent of the choice of $v$. Hence

$$\ell(d) \geq \sum_{1 \leq a < b, b > 2k} \text{inv}(a, b) + \sum_{1 \leq r < s \leq k} c(r, s),$$

where $\text{inv}(a, b)$ is equal to 0 if $v(a) < v(b)$ and equal to 1 if $v(a) > v(b)$. It remains to check that the right hand side is equal to $\ell(t_{n-1}t_{n-2} \ldots t_{2k}) + \ell(t_{2k-2}t_{2k-4} \ldots t_2) = \ell(w)$ for $w$ as constructed in the previous paragraph. This is easy. Hence we have equality in Eq. 1.6.

Part (b) can now be checked in a fairly straightforward way using the explicit formula for the length. Also part (c) is either known from the symmetric group case, or it can be checked in a straightforward way. E.g. if the numbers $i, i + 1$ and $i + 2$ label vertices belonging to three different horizontal edges of $d$, say $(i, j_1), (i + 1, j_2)$ and $(i + 2, j_3)$, the action of $S_3(i)$ results in permuting the second coordinates, and it is easy to see that the lowest element is given if $j_1 < j_2 < j_3$. In this case, it can be explicitly checked, for instance via pictures, that the map $w \mapsto w(i, j_1)(i + 1, j_2)(i + 2, j_3)$ is order-preserving. The other cases are similar and easier.

1.5. Braids and Hecke algebras. Recall that Artin’s braid group $AB_n$ is defined via generators $\sigma_i, 1 \leq i \leq n-1$ and relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$. It will also be convenient to introduce the notation $\sigma_{k,l}^+ = \sigma_k \sigma_{k+1} \ldots \sigma_l$ if $k < l$ and $\sigma_{k,l}^- = \sigma_k \sigma_{k-1} \ldots \sigma_{l-1} \sigma_l$ if $k > l$. Similarly, the expressions $\sigma_{k,l}^-$ are defined as above, with $\sigma_i$ replaced by $\sigma_i^{-1}$ for $k \leq i \leq l$. Similarly, one defines elements $g_{k,l}^+$ and $g_{k,l}^-$ in terms of the generators $g_i$ of the Hecke algebra (see below). We have the following simple lemma, which is easy to prove.
Lemma 1.3. (a) The map $\Phi : \sigma_i \mapsto \sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}^{-1}\sigma_{2i}^{-1}$ defines a homomorphism of the braid group $AB_n$ into $AB_{2n}$.

(b) $\sigma_j^\pm \sigma_k^\pm = \sigma_k^\pm \sigma_j^\pm$ if $k < l$ and $k < j \leq l$.

(c) $\sigma_j^\pm \sigma_k^\pm = \sigma_k^\pm \sigma_j^\pm$ if $l < k$ and $l \leq j < k$.

The Hecke algebra $H_n$ of type $A_{n-1}$ is the $\mathbb{Z}[q,q^{-1}]$-algebra defined by generators $g_i$, $1 \leq i < n$ and relations $g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}$ and $g_ig_j = g_jg_i$ for $|i-j| > 1$. It has a basis $(g_w)_{w \in S_n}$ such that

\begin{equation}
(1.7) \quad g_i g_w = \begin{cases} 
q v_d & \text{if } \ell(s_iw) > \ell(w), \\
(q-1)g_w + qg_{s_iw} & \text{if } \ell(s_iw) < \ell(w).
\end{cases}
\end{equation}

It will be convenient to define the module $V_n^{(k)}$ as a $\mathbb{Z}[q,q^{-1}]$-module with a basis $(v_d, d = we(k), w \in B_{n,k})$. We will subsequently define actions of the Hecke algebra and of a $q$-deformation of the Brauer algebra on this module which will specialize to the known actions if we restrict to the classical Brauer algebra. So no confusion should arise from this slight abuse of notation. We now define an action of the generators $g_i$ of $H_n$ on $V_n^{(k)}$ as follows:

\begin{equation}
(1.8) \quad g_i v_d = \begin{cases} 
q v_d & \text{if } s_i d = d, \\
v_{s_i d} & \text{if } \ell(s_i d) > \ell(d), \\
(q-1)v_d + qv_{s_i d} & \text{if } \ell(s_i d) < \ell(d).
\end{cases}
\end{equation}

Proposition 1.4. The action defined in 1.8 makes the $\mathbb{Z}[q,q^{-1}]$-module $V_n^{(k)}$ into an $H_n$-module.

Proof. This could be checked by identifying $V_n^{(k)}$ with a quotient of $H_n$, see Lemma 1.5. Here we check the relations directly as follows: For given $g_i$ and $g_{i+1}$, this only needs to be done on the subspaces spanned by the $S_3(i)$-orbits of the basis graphs. These are either 6 or 3-dimensional. As the definition of the action only depends on the order structure of the basis elements, it follows from Lemma 1.2 that the actions on these subspaces coincides with the left regular representation of $H_3(i)$ in the 6-dimensional case, and with a representation on a coset space in the 3-dimensional case. It is not hard to check that in the latter case we obtain the same matrices as the ones for $g_1$ and $g_2$ in Section 3.2. The relation $g_ig_j = g_jg_i$ can be checked in a similar way and is easier.

Let $1 \leq i \leq j$ and let $n, m \geq j$. We will later need the following relations, which can be proved by straightforward calculations, similar to the ones in Lemma 1.3.

\begin{equation}
(1.9) \quad g_{i,n}^{-1}g_{j,m}^{-1} = \begin{cases} 
g_{j+1,m+1,1}g_{i,n}^{-1} & \text{if } m < n, \\
g_{i,n}^{-1}g_{3,n}^{-1}g_{2,n-1}^{-1} & \text{if } m \geq n.
\end{cases}
\end{equation}

\begin{equation}
(1.10) \quad g_{i,n}^{1}g_{j,m}^{1} = \begin{cases} 
g_{j+1,m+1,1}g_{i,n}^{1} & \text{if } m < n, \\
(q-1)g_{j+1,m}^{1}g_{i,m}^{1} + qg_{i,m}^{1}g_{i,m-1}^{1} & \text{if } m \geq n.
\end{cases}
\end{equation}
Moreover, the same relations hold if we simultaneously replace all + signs with − signs and vice versa, in each of the formulas above.

1.6. **Other versions.** Obviously, we also obtain other $S_n$-modules in the Brauer algebra after conjugating $e(k)$ by a permutation. These modules can be generalized to Hecke algebra modules as before. However, as already remarked at the beginning of Section 1.4, we may get different length functions for the resulting graphs. We deal here with the special case where $e(k)$ is replaced by the same graph except that the two leftmost horizontal edges are replaced by two vertical edges to keep notation simpler. We denote this element by $e(2,k)$. Similarly, we can also define the module $V_n^{(2,k)}$ both for the Brauer algebra, and for the Hecke algebra; we denote the vector corresponding to the element $e(2,k)$ by $v_1^{(2,k)}$. The length function for basis elements of the module $V_n^{(2,k)}$ is defined as before for $V_n^{(k)}$, except that $e(k)$ is replaced by $e(2,k)$.

We shall need the following technical lemma:

**Lemma 1.5.** Let $L = L_{(n,k)}$ be the left ideal in $H_n$ generated by $g_{2i−1} − q$, $1 ≤ i ≤ k$ and by $g_{2i+1}g_{2i} − g_{2i−1}g_{2i}$, $1 ≤ i ≤ k − 1$. Then $H_n/L$ is a free $\mathbb{Z}[q,q^{-1}]$-module of rank $n!/2^k k!$

*Proof.* For $k = 0$, the module $V_n^{(0)}$ is just the left regular representation of $H_n$, and there is nothing to show. If $k > 0$, it follows from the definitions that $L$ is contained in the annihilator of the vector $v_1$. Hence $H_n/L$ has at least dimension $n!/2^k k!$. So it suffices to show that $H_n$ is equal to the span of $B_{n,k}$ and $L$. We shall show this by induction on $n$ and $k$. Let us first show that it suffices to prove this for $n = 2k$. Indeed, in this case the claim for $n > k$ follows by induction on $n$ by observing that

\[(1.11)\]

\[H_{n+1} = \bigoplus_{a=1}^{n+1} g_{a,n}H_n = \text{span} \bigcup_{a=1}^{n+1} g_{a,n}B_{n,k} \cup g_{a,n}L_n^{(r,k)},\]

where we set $g_{n+1,n} = 1$. The claim now follows from the fact that $B_{n+1,k} = \bigcup_{a=1}^{n+1} g_{a,n}B_{n,k}^a$, see Def. 1.5.

It remains to show the claim for $n = 2k$ and $r = 0$, which we again do by induction on $k$, with $k = 1$ being trivially true. By the above, the claim also holds for $n = 2k + 1$, with $B_{2k+1}^{(k)} = \bigcup g_{a,2k}B_{2k}^{(k)}$. Let $b = g_{i_{2k},2k}b' \in B_{2k+1}^{(k)}$. If $g_{i_{2k},2k} = 1$ then we have

\[(1.12)\]

\[g_{i_{2k+1,2k+1}}b − qg_{i_{2k+1,2k}}b \in L\]

while if $g_{i_{2k},2k} \neq 1$, we have

\[(1.13)\]

\[g_{i_{2k+1,2k+1}}b − g_{i_{2k+1,2k}}g_{i_{2k−1,2k}}b'g_{2k−1,2k} \in L\]

It follows that the elements in 1.12 and 1.13 together with the ones in $L_{2k+1}^{(2k)}$ and the ones in $B_{2k+1}^{(k)} = B_{2k+2}^{(k+1)}$ span $H_n$, as required.
Corollary 1.6. Let \( L_{n,k}^{(r)} \) be the ideal generated by the elements \( g_{r+2i-1} - q, 1 \leq r \leq k \) and by \( g_{r+2i} - g_{r+2i+1}, 1 \leq r \leq k-1 \). Then again \( H_n/L_{n,k}^{(r)} \) is a free \( \mathbb{Z}[q, q^{-1}] \)-module of rank \( n!/2^k k! \).

Proof. Conjugating the ideal \( L_{n,k} \) by the element \( g_{1,2k} g_{2,2k+1} \ldots g_{r,2k+r-1} \) gives us the ideal \( L_{n,k}^{(r)} \).

1.7. \( H_{3,n} \)-modules. We can now use these results to define certain \( H_{3,n} \)-module morphisms in \( V_n^{(k)} \) which will be needed later. First of all, we replace the elements \( w \) in the table of Section 1.3 by elements \( g_w \) by replacing \( s_{2,j_2} \) by \( g_{2,j_2}^+ \) and replacing \( s_{1,j_1} \) by \( g_{1,j_1}^- \). Then we can show the following:

Lemma 1.7. Let \( w \) be an element as in the table of Section 1.3, and let \( g_w \) be as just defined. Then we get a decomposition \( V_n^{(k)} \cong \bigoplus H_{3,n} g_w v_1^{(k)} \) as \( H_{3,n} \)-modules analogous to the one in Section 1.3. In particular, we have the following well-defined \( H_{3,n} \) homomorphisms:

(a) \( h g_{1,j_1} v_1^{(k)} \mapsto h g_{3,j_1} v_1^{(k)}, h \in H_{3,n}, j_1 > 2k, \)
(b) \( h g_{2,j_2}^+ v_1^{(k)} \mapsto h g_{3,j_2}^+ v_1^{(k)}, h \in H_{3,n}, j_2 > 2k, \)
(c) \( h g_{2,j_2}^- v_1^{(k)} \mapsto h v_1^{(k)}, h \in H_{3,n}, \)
(d) \( h g_{2,j_2}^+ v_1^{(k)} \mapsto h g_{2,j_2}^- v_1^{(k)} \) and \( h g_{2,j_2}^+ v_1^{(k)} \mapsto h g_{2,j_2}^- v_1^{(k)}, h \in H_{3,n}. \)

Proof. The only nontrivial part in the proof is to show that the maps are well-defined. Observe that in case (c) the annihilator of \( g_2 v_1^{(k)} \) in \( H_{3,n} \) contains the elements \( g_{2i-1} - q, 3 \leq i \leq k \) and \( g_{2i+1} - g_{2i}, 3 \leq i < k \). By Lemma 1.5, the quotient of \( H_{3,n} \) with the left ideal \( L \) generated by these elements has rank \( (n-2)!/2^{k-2} (k-2)! \), which coincides with the rank of the module \( H_{3,n} g_2 v_1^{(k)} \), see the table in Section 1.3. Hence the annihilator coincides with \( L \), which is obviously contained in the annihilator of \( v_1^{(k)} \). It follows that the homomorphism is well-defined. One similarly determines annihilator ideals in the other cases, using Lemma 1.5, Corollary 1.6 and the table in Section 1.3. The claim follows as before.

2. Deformation of module tensor categories

2.1. Motivation and deformation conditions. This and the subsequent subsection only serve to motivate the following definitions. They are less self-contained and less rigorous than the other parts of this paper, which can be read independently of this section. For background for categorical notions see e.g. the book [Ks] and references therein, and the paper [Os].

It is well-known that for groups \( H \subset G \), we can make the representations of \( H \) into a module category of \( \text{Rep}(G) \). The right module action is defined for \( V \) an \( H \)-module, \( W \) a \( G \)-module by \( V \otimes W = V \otimes \text{Res}(W) \), where \( \text{Res}(W) \) is \( W \) viewed as an \( H \)-module. In particular, we obtain embeddings

\[
\text{End}_H(V) \otimes \text{End}_G(W) \to \text{End}_H(V \otimes \text{Res}(W)).
\]
The idea for the construction of the new $q$-Brauer algebra can now be stated very easily, which we will do on the level of Lie algebras. Let $\mathfrak{h} \subset \mathfrak{g}$ be semisimple Lie algebras. There exist canonical $q$-deformations of their universal enveloping algebras due to Drinfeld and Jimbo. It is known that these deformations usually are not compatible with the inclusion $\mathfrak{h} \subset \mathfrak{g}$. Hence we weaken the problem and ask for a compatible deformation of $\text{Rep}(\mathfrak{h})$ as a module category over $\text{Rep}(\mathfrak{g})$. More precisely, we require the following conditions:

(A) Same restriction rules: If $\mathcal{C}$ is the (finite-dimensional) representation category of a Drinfeld-Jimbo quantum group corresponding to $\mathfrak{g}$, we would like to find a module category $\mathcal{D}$ with the same Grothendieck semigroup as $\text{Rep}(\mathfrak{h})$ and with a right tensor module action as in 2.1 which should be compatible with the identifications of Grothendieck semigroups.

(B) Compatible traces In addition $\mathcal{C}$ is a spherical category, i.e. it has canonical duality morphisms which lead to canonical traces for $\text{End}(X)$, for any object $X$ in $\mathcal{C}$ (see e.g. the chapter on duality in [Ks]). We also require that these extend in a compatible way to our module category. This condition is equivalent to a fundamental notion in the study of subfactors known as the commuting square condition. We will state it in this context as follows:

In a spherical category, there exists for every object $Z$ in $\mathcal{C}$ a canonical trace $Tr_Z$ on $\text{End}_\mathcal{C}(Z)$; we will denote by $tr_Z$ the multiple of $Tr_Z$ such that $tr_Z(1) = 1$. We now require extensions of $Tr_Z$ to $\text{End}_\mathcal{D}(Z)$ such that the following holds:

\[(2.2)\quad E(a) \in \text{End}_\mathcal{C}(X) \text{ for any } a \in \text{End}_\mathcal{C}(X \otimes Y), \ X, Y \in \text{Ob}(\mathcal{C});\]

here $E$ is the orthogonal projection onto the subalgebra $\text{End}_\mathcal{D}(X) \cong \text{End}_\mathcal{D}(X) \otimes 1 \subset \text{End}_\mathcal{D}(X \otimes Y)$ with respect to the bilinear form $(b, c) = tr(bc)$; for more details see Section 5.

2.2. Some relations. We give some examples how Cond. 2.2 forces relations for a deformation of Brauer’s centralizer algebra, if we take for $\mathfrak{g} = \mathfrak{sl}_N$ and for the subalgebra $\mathfrak{h} = \mathfrak{so}_N$, with $N$ odd to avoid needless complications. We denote by $V$ the object corresponding to the vector representation of $\mathfrak{sl}_N$ resp. of $\mathfrak{so}_N$ both in $\mathcal{C}$ and in the module category $\mathcal{D}$. It is well-known that $\text{End}_\mathcal{C}(V^{\otimes n})$ is generated by a representation of the Hecke algebra $H_n$. We shall denote the images of the generators again just by $g_i$. The canonical traces mentioned before are known under the name Markov traces; see Section 4 for details. In this context, Cond. 2.2 translates for $X = V^{\otimes n}$ and $Y = V$ to the condition

\[(2.3)\quad tr(bg_n) = tr(b)tr(g_n), \quad b \in \text{End}_\mathcal{D}(V^{\otimes n}) \otimes 1.\]

Let $\bar{e}$ denote the projection in $\text{End}_\mathcal{D}(X^{\otimes 2})$ onto the object in $X^{\otimes 2}$ corresponding to the trivial representation of $\mathfrak{so}_N$, which is a subrepresentation of the symmetrization of the vector representation. One deduces from this that $\bar{e}g_1 = gg_1$, as the eigenprojection of $g_1$ with eigenvalue $q$ projects onto the object corresponding to the symmetrization of the vector representation.

We shall also denote the embedding $\bar{e} \otimes 1$ of $\bar{e}$ into $\text{End}_\mathcal{D}(X^{\otimes 3})$ just by $\bar{e}$. Then $\bar{e}$ also projects onto a simple object in $X^{\otimes 3}$, and hence $\bar{e}g_2\bar{e} = \alpha \bar{e}$ for a scalar $\alpha$. To calculate this scalar, we use the requirements concerning the conditional expectation: By definition, $E(\bar{e}g_2)$ is the unique element in $\text{End}_\mathcal{D}(X^{\otimes 2})$ such that \[tr_{X^{\otimes 3}}(\alpha \bar{e}g_2) = tr_{X^{\otimes 2}}(aE(\bar{e}g_2))\] for all $a \in \text{End}_\mathcal{D}(X^{\otimes 2})$.
Remark 2.1. It is possible to derive relation (E3) in Section 3.1 from condition 2.2 and relations (H), (E1) and (E2). More precisely, these conditions and relations essentially determine the matrices of $g_3$ in all irreducible representations of $Br_4$ with respect to the path basis, see e.g. [W1] (for the Hecke algebra part) and [RW]. From this one can check that relation (E3) has to be satisfied as well. The proof is not very instructive, so we do not give the details here.

3. q-BRAUER ALGEBRAS

3.1. Definitions. Fix $N \in \mathbb{Z}\backslash\{0\}$ and let $[N] = (1 - q^N)/(1 - q) \in \mathbb{Z}[q, q^{-1}]$. The q-Brauer algebra $Br_n(N)$ is defined over the ring $\mathbb{Z}[q, q^{-1}]$ via generators $g_1, g_2, \ldots, g_{n-1}$ and $e$ and relations

(H) The elements $g_1, g_2, \ldots, g_{n-1}$ satisfy the relations of the Hecke algebra $H_n$.
(E1) $e^2 = [N]e$,
(E2) $eg_i = g_i e$ for $i > 2$, $eg_1 = qe$, $eg_2 = q^N e$ and $eg_2^{-1} e = q^{-1} e$.
(E3) $g_2 g_3 g_4^{-1} g_2^{-1} e_{(2)} = e_{(2)} g_2 g_3 g_4^{-1} g_2^{-1}$, where $e_{(2)} = e(g_2 g_3 g_4^{-1} g_2^{-1}) e$.

We shall need a second version of the q-Brauer algebra, denoted by $Br_n(r, q)$ or just $Br_n$ by carrying the information of the parameter $N$ in the variable $r = q^N$. More precisely, the algebra $Br_n(r, q)$ is defined over the ring $\mathcal{R} = \mathbb{Z}[q^\pm 1, r^\pm 1, (r - 1)/(q - 1)]$ via the same generators as before, with relations (H) and (E3) unchanged, and with

(E1)$'$ $e^2 = \frac{r-1}{q-1} e$,

(E2)$'$ $eg_i = g_i e$ for $i > 2$, $eg_1 = qe$, $eg_2 = re$ and $eg_2^{-1} e = q^{-1} e$.

Remark 3.1. 1. It should be clear that we get back the algebra $Br_n(N)$ from $Br_n(r, q)$ by setting $r = q^N$. In particular, we can use this to also define $Br_n(0)$ as one of those specializations, where the direct definition would cause some (presumably minor) technical difficulties (see e.g. Lemma 3.3,(g)); the author would like to thank Dung Tien Nguyen for pointing this out to him. It is also easy to see that we get the Brauer algebra $D_n(N)$ for $r = q^N$ in the limit $q \to 1$. In this case $g_i$ becomes the simple reflection $s_i$ and the element $e$ can be identified with the graph $e_{(1)}$. In general, we prefer the algebra $Br_n(N)$ as its defining ring is more natural, and it is closer to the intended applications. However, as the algebra $Br_n(r, q)$ is generically semisimple, it is sometimes more convenient to work with. In many cases, the proofs are the same for both versions and we will only give them for one version, sometimes without explicitly mentioning the other version.
2. It is easy to see that the assignment \( g_i \mapsto g_i^T = g_i \) and \( e \mapsto e^T = e \) defines a linear anti-
automorphism \( a \mapsto a^T \) of \( Br_n(r, q) \). Similarly, the map \( g_i \mapsto g_i^* = g_i^{-1} \) and \( e \mapsto e^* = q^{1-N} e \)
defines an anti-linear antiautomorphism with respect to the involution of the ring \( \mathcal{R} \) defined
by \( \bar{q} = q^{-1} \) and \( \bar{r} = r^{-1} \).

3. We shall later show that the subalgebra of \( Br_n(N) \) resp. \( Br_n(r, q) \) generated by the
generators \( g_1, g_2, \ldots, g_{n-1} \) is indeed isomorphic to \( H_n \). If the reader feels uncomfortable with
this, he should use different notation for the generators of the Hecke algebras.

4. It may be instructive to some readers to visualize the relations via graphical calculus for
ribbon tensor categories (see e.g. [Ks], [Tu]), with \( e \) given by the composition \( \cup \circ \cap \), and \( g_i \) given
by a standard braid generator \( \sigma_i \). While this may give a somewhat better intuitive feel about
the relations, it does not provide a topological interpretation for our algebra. E.g. in this
usual tangle interpretation, \( e(2) \) would describe the same topological object as \( g_2g_3g_1g_2e(2) \),
while it can be checked that these are different elements in \( Br_4 \). It would be interesting if one
could find a topological interpretation of our algebra.

3.2. Low-dimensional examples. One checks directly for \( n = 2 \) that \( Br_n(N) \) is spanned by
the elements \( 1, g_1 \) and \( e \). If \( n = 3 \) one also easily shows that \( Br_n(N) \) is spanned by the basis
elements \( g_w \) of \( H_3 \) and the elements \( h_1e h_2 \), where \( h_1 \in \{1, g_1, g_2\} \) and \( h_2 \in \{1, g_2, g_2g_1\} \).
Hence its rank is at most 15. On the other hand, consider the assignments

\[
\begin{align*}
g_1 & \mapsto \begin{pmatrix} q & 0 & 0 \\ 0 & 0 & q \\ 0 & 1 & q-1 \end{pmatrix}, \\
g_2 & \mapsto \begin{pmatrix} 0 & q & 0 \\ 1 & q-1 & 0 \\ 0 & 0 & q \end{pmatrix} \quad \text{and} \quad e_1 & \mapsto \begin{pmatrix} \frac{r^2 - 1}{q - 1} & r & rq \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

It is easy to check that these matrices define a representation of \( Br_3(r, q) \) whose image is a free
\( \mathcal{R} \)-module of rank 9. By calculating the determinant of the matrix formed from the nonzero
rows of the matrices representing the elements \( e, e_2 \) and \( e_2g_2g_1 \), one can also determine for
which algebraic relations for \( r \) and \( q \) this representation is not semisimple. We have the
following Lemma:

Lemma 3.2. (a) The algebra \( Br_3(r, q) \) is a free \( \mathcal{R} \)-module of rank 15.

(b) We obtain a representation of \( Br_4(r, q) \) from the representation in (a) by assigning to \( g_3 \)
the matrix of \( g_1 \). It is equivalent to the representation of \( Br_4(r, q) \) on \( Br_4(r, q)e(g_2g_3g_1^{-1}g_2^{-1})e \).
In particular, the ideal generated by \( e(2) \) has rank 9.

(c) We also have \( e(2) = e(g_2g_3g_1^{-1}g_2^{-1})e = e(g_2^{-1}g_3g_1^{-1}g_2)e = e(g_2^{-1}g_3^{-1}g_1g_2)e \).

Proof. We have already shown part (a). The fact that we also obtain a representation of
\( Br_4(r, q) \) as described in (b) is almost immediate. It only remains to show that \( Br_4(r, q)e(2) \)
is spanned by \( e(2), g_2e(2) \) and \( g_1g_2e(2) \), which follows from the \( Br_3(r, q) \) case and relation
\((E3)\). Part (c) can be shown by a direct calculation using \((E2)\), \( g_i^{-1} = q^{-1} g_i + (g^{-1} - 1) \) and
\( g_i = q g_i^{-1} + (q - 1) \) as well as the identity \( q^{-1}(q-1)e g_2 e + q(q^{-1} - 1) e g_2^{-1} e + (r-1)(q^{-1} - 1) e = 0 \).
3.3. Elements $e(k)$. In the following, we define elements $e(k)$ in $Br_n(N)$ inductively by $e(1) = e$ and by

$$e(k+1) = e g^+_{2,2k+1} g^-_{1,2k} e(k) = e \Phi(g^+_{1,k}) e(k) = e(k) \Phi(g^+_{k,1}) e$$

where $\Phi$ is defined as in Lemma 1.3 with $\sigma$'s replaced by $g_i$'s. The equivalence of these and additional expressions for $e(k)$ will be proved in the following lemma. For $q = 1$, it is not hard to show that both definitions produce the same graph in the usual Brauer algebra. The following lemmas will indicate how the Brauer relations will extend to these new algebras.

Lemma 3.3. (a) The elements $e(k)$ are well-defined.

(b) $g_{2j-1} g_{2j} e(k) = g_{2j+1} g_{2j} e(k)$ and $g_{2j-1}^{-1} g_{2j} e(k) = g_{2j+1}^{-1} g_{2j} e(k)$ for $1 \leq j < k$.

(c) $g^+_{1,2l} e(k) = g^+_{2l+1,2} e(k)$ and $g^-_{1,2l} e(k) = g^-_{2l+1,2} e(k)$ for $l < k$.

(d) For any $j \leq k$ we have $e(j) e(k) = e(k) e(j) = [N]^{j} e(k)$.

(e) $[N]^{j-1} e_{(k+1)} = e(j) g^+_{2,2j+1} g^-_{1,2j} g^-_{2j-1,2} e(k)$ for $1 \leq j < k$.

(f) $e(j) g_{2j} e(k) = q^{N} [N]^{j-1} e(k)$ for $1 \leq j \leq k$.

(g) $(N^g) e_{(k)} = e_{(k)}$ for $N \neq 0$ and $k \geq 1$.

Proof. Part (a) is shown by induction on $k$, using the fact that $\Phi(g_i)$ commutes with $e$ for $i > 1$. For part (b), the claim follows for $j = 1$ from the definitions. If $j > 1$, we use $g_{2j+1,2j} g^-_{1,2k} g_{2j-1,2} e(k)$ for $l = 2j - 1, 2j$, by Lemma 1.3, and induction assumption to show the claim. Part (c) follows easily from (b) by induction on $l$. For part (d), we use induction on $j$ and part (c) as follows:

$$e_{(j+1)} e(k) = [N]^{j-1} e_{(2,j+1)} g^-_{1,2j} e(k)$$

Part (e) is shown by induction on $j$ with $j = 1$ being the first definition of $e_{(k)}$. Moreover, we have

$$e_{(j+1)} g^+_{2,2j+1} g^-_{1,2j} g^-_{2j-1,2} e(k) = [N]^{j-1} e_{(2,j+1)} g^-_{1,2j} g^-_{2j-1,2} e(k)$$

which proves (e) using the induction assumption and part (d). For part (f), observe that the left hand side of the equation is equal to

$$e_{(2,j+1)}$g^-_{1,2j} g^-_{2j-1,2} e(k) = [N]^{j-1} e_{(2,j+1)} g^-_{1,2j} g^-_{2j-1,2} e(k) = [N]^{j-1} e_{(2,j+1)} g^-_{1,2j} g^-_{2j-1,2} e(k) = q^{N} [N]^{j-1} e(k)$$

where we used (c), Lemma 1.3 (b) and (c), and relations (E2). Part (g) follows from the definitions and Lemma 3.2 (c) for $k = 1, 2$, and by induction and part (e) (with $j = k$) for $k > 2$. The difficulty for $N = 0$ and a complete proof in the other cases was pointed out to the author by D. T. Nguyen in [Ng].

Lemma 3.4. We have $e(j) H_n e(k) \subset H_{2j+1,n} e(k) + \sum_{m \geq k+1} H_n e(m) H_n$, where $H_{r,s}$ is generated by $g_r, g_{r+1}, \ldots, g_{s-1}$ and $j \leq k$. Moreover, if $j_1 \geq 2k$ and $j_2 \geq 2k+1$, we also have:

(a) $g^+_{1,j_1} g^-_{1,j_1} e(k) = e_{(k+1)} g^+_{2k+2,j_2} g^-_{2k+1,j_1}$, if $j_1 \geq 2k$ and $j_2 \geq 2k+1$, ...
(b) \(eg_{2,j}^-g_{1,j_1}^+e(k)\) is equal to

\[
e^{(k+1)}g_{2k+2,j}^+g_{2k+1,j_2}^+ + q^{N+1}(q-1)\sum_{l=1}^{k} q^{2l-2}(g_{2l+1}^1 + 1)g_{2l+2,j}^+g_{2l+1,j_1}^+e(k).
\]

Proof. We will use the analogous decomposition of \(H_ne(k)\) into \(H_{3,n}^\ast\)-modules as in Section 1.3, with the adjustments for the Hecke algebra case as explained at the beginning of the next subsection. Let us first prove the claims for \(j = 1\). Claim (a) follows almost immediately from Lemma 3.3, (e). This proves the first statement of the Lemma for elements in the \(H_{3,n}^\ast\) submodules in the first case of the table in Section 1.3. For submodules in the second case, the claim follows from relations (E2), and the remaining cases are easy.

To prove part (a) for \(j > 1\) observe that the left hand side of the statement is equal to

\[
e_{g_{2,j}^-g_{1,j_2}^+}^1e(k) = [N]^{j-1}eg_{2,j}^+g_{2,j-1,2}^-e(k) = [N]^{j-1}eg_{2,j}^+g_{2,j}^+e(k) = q^N[N]^{j-1}e(k),
\]

where we used Lemma 3.3(c), Lemma 1.3 (b) and (c), and relations (E2). The first statement of the lemma for \(j > 1\) can now be done by a fairly straightforward induction on \(j\), using (a), Lemma 3.3(e),(f) and the inductive definition of \(e(j)\).

To prove part (b) we first observe that

\[
e_{g_{2,j}g_{2,j+1}g_{2,j-1}g_{2,j}^-}e(j) = [N]^{j-2}q^2e(j+1) + q^{N+1}(q-1)(g_{2,j+1}^1 + 1)e(j),
\]

which follows from \(g_i = qg_i^+ + (q-1)\) and the relations proved so far. We deduce from this

\[
e_{g_{2,j}g_{2,j}^+g_{2,j-1,j_1}^+}e(k) = \frac{1}{[N]}q^2g_{2j+2,j_2}^+g_{2j+1,j_1}^+e(k) + [N]^{j-1}q^N(q-1)(g_{2,j+1}^1 + 1)g_{2j+2,j_2}^+g_{2j+1,j_1}^+e(k),
\]

where we use Lemma 3.3(e). This shows, among other things that the first term on the right hand side is an element in \(Br_n(N)\). We can now show by downwards induction on \(j\), starting with \(j = k\), that

\[
\frac{1}{[N]^{j-1}}e_{g_{2,j}^1,j_1}^+e(k) = \frac{1}{[N]^{j-1}}e_{g_{2,j}^1,j_1}^+(k+1)g_{2k+2,j_2}^+g_{2k+1,j_2}^+e(k) + q^{N+1}(q-1)\sum_{l=j}^{k} q^{2l-2}(g_{2l+1}^1 + 1)g_{2l+2,j_2}^+g_{2l+1,j_1}^+e(k)
\]

This follows for \(j = k\) almost immediately from Eq 3.4, and for \(j < k\) again from Eq 3.4 and induction assumption. The desired identity now follows for \(j = 1\). We note again that even though some of the expressions do not look like it, all the elements involved are indeed in \(Br_n(N)\).
In analogy to the Brauer case, we can now define

\( I(j) = \sum_{k=j}^{n/2} H_n e_{(k)} H_n. \)

It follows from the Lemma that \( I(j) \) forms a two-sided ideal in \( Br_n(N) \) for \( j = 1, 2, \ldots \) and we have the inclusions of two-sided ideals \( Br_n(N) \supset I(1) \supset I(2) \supset \ldots \).

**Proposition 3.5.** The algebra \( Br_n(r, q) \) is spanned by \( \sum_{k=0}^{n/2} H_n e_{(k)} H_n. \) In particular, its dimension is at most the one of the Brauer algebra \( D_n. \)

**Proof.** To prove the first statement, it suffices to show that the right hand side is invariant under multiplication by the generators of \( Br_n(r, q) \). This is obvious for the Hecke algebra generators \( g_i \). It follows for left multiplication by \( e \) from Lemma 3.4 for \( j = 1 \). The same proof works for right multiplication, using the involution \( T \), see Remark 2 after the definitions.

To prove the estimate for the dimension, observe that the annihilator of \( e_{(k)} \) in \( H_n \), acting via left multiplication, contains the left ideal \( L_n k \) (see Lemma 1.5). Hence the dimension of \( H_n e_{(k)} \) is at most equal to the dimension of \( V_n^{(k)} \), which is equal to the number of graphs \( S_n e_{(k)} \) in the Brauer algebra. One similarly shows that the dimension of \( e_{(k)} H_n \) is \( \leq \) the number of graphs in \( e_{(k)} S_n \). Finally, it follows as in Lemma 1.1 that \( H_n e_{(k)} H_n \) is a quotient of \( H_n e_{(k)} \otimes H_{2k+1, n} e_{(k)} H_n \), where the latter has dimension \( \leq \dim \mathbb{Z}[x] S_n e_{(k)} S_n \). Hence the dimension of \( Br_n(r, q) \equiv \bigoplus H_n e_{(k)} H_n \) is at most the one of the Brauer algebra. This proves the other inequality.

3.4. The \( Br_n(N) \)-module \( V_n^{(k)} \). The results in the last section show that \( H_n e_{(k)} \) is a \( Br_n(N) \)-module modulo \( I(k+1) \). We will show that it is isomorphic to the Hecke algebra module \( V_n^{(k)} \) after making it into a \( Br_n(N) \)-module by defining an action of \( e \) on it. We will again use the decomposition of \( V_n^{(k)} \) into a direct sum of \( H_{3,n} \)-modules as in Lemma 1.7 using the table in Section 1.3. As before, we will replace the elements \( s_{2,j_2} \) and \( s_{1,j_1} \) in the first column of the table in that section by \( g_{2,j_2}^+ \) and \( g_{1,j_1}^- \) respectively to obtain elements \( g_{w} \) as before Lemma 1.7, and we write \( V_n^{(k)} = \bigoplus_{w} H_{3,n} g_{w} v_1 \) as a direct sum of \( H_{3,n} \)-modules. We now define the action of \( e \) on \( V_n^{(k)} \) by

\[
(3.7) \quad e h g_{2,j_2}^+ g_{1,j_1}^- v_1 = \begin{cases} q^N h g_{3,j_2}^+ v_1 & \text{if } g_{1,j_1}^- = 1, \\ q^{-1} h g_{2,j_2+1,j_1}^- v_1 & \text{if } g_{2,j_2}^- = 1, \\ 0 & \text{if } j_1 \geq 2k \text{ and } j_2 \geq 2k + 1; \end{cases}
\]

moreover, we define \( e h v_1 = [N] h v_1 \) for \( h \in H_{3,n} \) and \( e h g_{2} v_1 = q^N h v_1 \) for the remaining two cases. It follows from Lemma 1.7 that the action of \( e \) commutes with the action of \( H_{3,n} \); this implies that it is well-defined. Moreover, observe that the image of \( e \) on \( V_n^{(k)} \) is equal to \( H_{3,n} v_1^{(k)} \). From this it follows easily that \( e g_{2} e \) and \( q^N e \) act via the same map on \( V_n^{(k)} \); the
same goes for \( e g_2^{-1} e \) and \( q^{-1} e \). We have proved the following proposition, except for part of relation \((E1)\) and \((E3)\), which will be proved in the following subsections.

**Proposition 3.6.** The action of the elements \( g_i, 1 \leq i < n \) and \( e \) on \( V_n^{(k)} \), as given in Eq 3.7 defines a representation of \( Br_n(N) \).

### 3.5. Checking the relations \( eg_1 = qe = eg_1 \). As \( e V_n^{(k)} = H_{3,n} v_1 \), we see easily that the relation \( g_1 e = qe \) is preserved. To check the relation \( eg_1 = qe \), we express the action of \( e \) with respect to the original basis of the Hecke algebra module \( V_n^{(k)} \), which is now easier to do. Indeed, as we have already established that \( e \) commutes with \( H_{3,n} \), it suffices to calculate the action of \( e \) on vectors of the form \( g_{2,j_2}^+ g_{1,j_1}^+ v_1^{(k)} \). It follows that

\[
(3.8) \quad eg_{2,j_2}^+ g_{1,j_1}^+ v_1^{(k)} = q^{N+1}(q - 1) \sum_{l=1}^{k} q^{2l-2}(g_{2l+1} + 1)g_{2+2,j_2}^+ g_{2l+1,j_1}^+ v_1^{(k)}.
\]

This result holds for all \( j_1 \geq 2k \) and \( j_2 > 2k \). Moreover, observe that

\[
(3.9) \quad s_{2l+1}(s_{2l+2,j_2}s_{2l+1,j_1}) > s_{2l+2,j_2}s_{2l+1,j_1} \Leftrightarrow j_2 \geq j_1,
\]

which holds for any \( l \geq 0 \) for which \( 2l + 1 \leq \min(j_1,j_2) \). We leave it to the reader to check, both for \( j_2 \geq j_1 \) and for \( j_2 < j_1 \), using Eq 3.8 and 3.9 that

\[
eg_1(g_{2,j_2}^+ g_{1,j_1}^+ v_1^{(k)}) = q^{N+1}(q - 1) \sum_{l=1}^{k} q^{2l-2}(g_{2l+1} + 1)g_{2l+1,j_1}^+ g_{2l+2,j_2}^+ g_{2l+1,j_1}^+ v_1^{(k)}.
\]

The desired equality now follows from \((g_{2l+1} + 1)g_{2l+1} = q(g_{2l+1} + 1)\).

### 3.6. Checking the relation \((E3)\). Observe that \( e_{(2)} V_n^{(k)} = H_{5,n} v_1^{(k)} \) by Lemma 3.4, from which one easily deduces the first equation of relation \((E3)\). The second equality is more difficult to check. We will first verify it for \( Br_4(r,q) \). We then show that an arbitrary \( V_n^{(k)} \) can be written as a direct sum of \( Br_4(r,q) \)-modules for each of which relation \((E3)\) holds. This is done in several steps:

**Step 1:** We show that relation \((E3)\) is preserved for \( n = 4 \). This is easy, as \( e_{(2)} \) acts as 0 on \( V_4^{(0)} \) and \( V_4^{(1)} \); moreover, on \( V_4^{(2)} \), \( g_1 \) and \( g_3 \) act via the same linear map, which also trivially implies that relation \((E3)\) is preserved. It follows that \( Br_4(r,q) \) has the same dimension as the Brauer algebra \( D_4 \).

**Step 2:** Let \( Br_4(r,q) \) be the algebra defined as \( Br_4(r,q) \), except for the relation \( e_{(2)} = e_{(2)} g_2 g_3 g_1^{-1} g_2^{-1} \). Observe that we also have \( e_{(2)} g_1 = qe_{(2)} = e_{(2)} g_3 \) in \( Br_4(r,q) \). As the subgroup generated by \( s_1 \) and \( s_3 \) has index 6 in \( S_4 \), one deduces that \( e_{(2)} H_4 \) is spanned by the elements \( e_{(2)} g_2, e_{(2)} g_2 g_1^{-1}, e_{(2)} g_2 g_3, e_{(2)} g_2 g_1^{-1} g_3, e_{(2)} g_2 g_1^{-1} g_3 g_2^{-1} \) in \( Br_4(r,q) \). As \( H_4 e_{(2)} \) is three-dimensional also in \( Br_4(r,q) \), it follows that \( H_4 e_{(2)} H_4 \) has at most dimension 18 in \( Br_4(r,q) \). Now one checks directly for the six spanning elements of \( e_{(2)} H_4 \) that also in \( Br_4(r,q) \) we have \( e_{(2)} H_4 e \) is spanned by \( e_{(2)} \); indeed, e.g. we have \( e_{(2)} g_2 g_1^{-1} g_3 g_2^{-1} e = g_2 g_1^{-1} g_3 g_2^{-1} e_{(2)} \).
(by definition of $e(2)$), which is equal to $e(2)$ also in $\tilde{Br}_4(r, q)$. It follows from this that also in $\tilde{Br}_4(r, q)$ the ideal generated by $e(2)$ is equal to $H_4e(2)H_4$.

**Step 3** : Let $v \in V_n^{(k)}$ and let $W = \tilde{Br}_4(r, q)v$ be the $\tilde{Br}_4(r, q)$-submodule generated by $v$. Then $W$ is also a $Br_4(r, q)$-module if $e(2)g_2g_1^{-1}g_3g_2^{-1}v = e(2)v$, $e(2)g_2g_1^{-1}g_3v = e(2)g_2v$ and $e(2)g_2g_3v = e(2)g_2g_1v$. Indeed, if $I$ is the two-sided ideal of $Br_4(r, q)$ such that $\tilde{Br}_4(r, q)/I = Br_4(r, q)$, it is easy to check that $I$ is generated by $e(2)(g_2g_1^{-1}g_3g_2^{-1} - 1)$, $e(2)g_2(g_1^{-1}g_3 - 1)$ and by $e(2)g_2(g_3 - g_1)$ as a $\tilde{Br}_4(r, q)$-left ideal. The claim follows from this and our assumptions.

**Lemma 3.7.** The action of the generators of $Br_n$ on $V_n^{(k)}$ also preserve relation $(E3)$.

**Proof.** We decompose $V_n^{(k)}$ into a sum of cyclic $\tilde{Br}_4(r, q)$-modules of the form $W = \tilde{Br}_4(r, q)v$, for suitable vectors $v$. It then suffices to check that $W$ is also a $Br_4(r, q)$-module by the criterion of Step 3. Indeed, in this case $W \cong Br_4(r, q)/\text{Ann}(v)$ is a $Br_4(r, q)$-module on which obviously also relation $(E3)$ holds.

The explicit checking of the criterion in Step 3 is somewhat tedious as there are several different cases. It is easier to study the combinatorics in the context of the original Brauer algebra. Obviously, there is only something to prove if $e(2)H_4v \neq 0$. This implies that $k \geq 2$, and that among the first four upper vertices at least two belong to distinct horizontal edges. It remains to consider the cases that 0, 1 or 2 of the first four upper vertices belong to vertical edges.

Let us first consider the case with the first four upper vertices belonging to four distinct horizontal edges. Multiplying such a graph by a suitable permutation in $S_{5,n}$, if necessary, we can assume that each of these four edges connect a vertex $v \in \{1, 2, 3, 4\}$ with a vertex $v \in \{5, 6, 7, 8\}$ (observe that $S_{5,n}$ commutes with $D_4$, hence this multiplication induces an isomorphism of $D_4$-modules). Among such graphs, $s_4s_5s_6s_3s_4s_2e(4)$ has the fewest crossings.

It will be convenient to pick the element $v_0 = g_4g_5g_6g_3^{-1}g_4^{-1}g_2^{-1}e(4)$ in $Br_4(r, q)$. We now leave it to the reader to check, using Lemma 3.3, (c) that $e(2)v_0 = q^{N-1}v_0 = e(2)g_2g_3g_1^{-1}g_2^{-1}v_0$, $e(2)g_2v_0 = q^{N-1}g_0v_0 = e(2)g_2g_3g_1^{-1}v_0$ and $e(2)g_2g_3v_0 = e(2)g_2g_3v_0 = e(2)g_2g_1v_0$.

The case with three of the first four vertices connected to three different horizontal edges, and the remaining one connected to one of the lower row is done similarly. Here we can take $v_0 = g_4g_5g_6g_3^{-1}g_4^{-1}g_2e(2)$, with essentially the same calculations as before.

Next let us consider the case where two of the first four vertices belong to horizontal edges which connect them with vertices larger than 4, and that the other two vertices are connected to vertices in the lower row. Again, it suffices to consider the cyclic module generated by the element $v_0 = g_4g_5g_6^{-1}g_4^{-1}g_2e(2)$. Using the relations, one checks that $e(2)v_0 = 0 = e(2)g_2^{-1}g_1^{-1}g_3g_2v_0$, $e(2)g_2^{-1}v_0 = q^{N-1}e(2) = e(2)g_2^{-1}g_1^{-1}g_3v_0$ and $e(2)g_2^{-1}g_1^{-1}v_0 = q^{N-1}g_0^{-1}v_0$.

In the remaining cases, we have at least two of the first four vertices connected by a horizontal edge. We leave it to the reader to check that these cases can be reduced to submodules generated by $ev_0$, with $v_0$ as in one of the previous cases. This finishes the proof of the lemma.
3.7. Dimension. We can now prove the main theorem of this section. We define for each basis graph \(d\) of the Brauer algebra \(D_n\), an element \(g_d \in Br_n(r,q)\) as follows: If \(d\) has \(2k\) horizontal edges, fix a reduced expression \(d = w_1e(k)w_2\) (see Section 1.4) with \(w_1, w_2 \in S_n\). Then we define \(g_d = gw_1e(k)gw_2\); as usual, we abuse notation by denoting by \(e(k)\) both a certain graph, and an element in \(Br_n(r,q)\).

**Theorem 3.8.** (a) The algebra \(Br_n(r,q)\) is a free \(\mathbb{Z}[q^{\pm 1}, r^{\pm 1}, (r - 1)/(q - 1)]\)-module of rank \(n!! = 1 \cdot 3 \cdot \ldots (2n - 1)\) with basis \((g_d)\) labeled by the basis graphs of the Brauer algebra.

(b) The algebra \(Br_n(N)\) is a free \(\mathbb{Z}[q, q^{-1}]\)-module of rank \(n!! = 1 \cdot 3 \cdot \ldots (2n - 1)\) with spanning set \((g_d)\) labeled by the basis graphs of the Brauer algebra.

(c) The algebra \(Br_n(r,q)\) has the same decomposition into a direct sum of simple matrix rings as a \(\mathbb{Q}(r,q)\) algebra as the generic Brauer algebra \(D_n\); this also includes the restriction rules from, say, \(Br_n(r, q)\) to \(Br_{n-1}(r,q)\), see Remark 3.9.

**Proof.** We have seen that there is a faithful representation of the Brauer algebra \(D_n\) on \(\bigoplus_{0 \leq k \leq n/2} V_n^{(k)}\) in Lemma 1.1. As this is a specialization of the representation of \(Br_n(r,q)\) on the same direct sum of modules \(V_n^{(k)}\), the dimension of \(Br_n(r,q)\) must be at least the one of \(D_n\).

To prove the other inequality, observe that the annihilator of \(e(k)\) in \(H_n\), acting via left multiplication, contains the left ideal \(L_{n,k}\) (see Lemma 1.5). Hence the dimension of \(H_n\) is at most equal to the dimension of \(V_n^{(k)}\), which is equal to the number of graphs \(S_n\) in the Brauer algebra. One similarly shows that the dimension of \(e(k)H_n\) is \(\leq\) the number of graphs in \(e(k)S_n\). Finally, it follows as in Lemma 1.1 that \(H_n\) is a quotient of \(H_n \otimes H_{2k+1,n} e(k)H_n\), where the latter has dimension \(\leq \dim \mathbb{Z}[x]S_n\). Hence the dimension of \(qB_n(r,q) \cong \bigoplus H_n e(k)H_n\) is at most the one of the Brauer algebra. This proves the other inequality.

To prove part (b), observe that we obtain a representation of \(Br_n(r,q)\) with respect to the basis \((g_d)\) with coefficients in \(\mathbb{Z}[q^{\pm 1}, r^{\pm 1}, (r - 1)/(q - 1)]\). Specializing \(r = q^N\), these coefficients become elements of \(\mathbb{Z}[q, q^{-1}]\) and we obtain a representation \(\pi\) of \(Br_n(N)\). As \(\pi(g_d)1 = g_d\), it follows that the image has dimension at least \(n!!\). The other inequality follows as before from the fact that \((g_d)\) is a spanning set for \(Br_n(N)\).

The proof of statement (c) follows from standard arguments. Fix a basis \((g_d)\) and consider the left regular representation \(\pi_l\) with respect to this basis. Then the discriminant \(\det(T_r(\pi_l(b q b q)))\) is a polynomial in \(r\) and \(q\). It specializes for \(r = q^N\) and \(q \to 1\) to the discriminant of \(D_n(N)\), which is known to be nonzero for \(N > n\). This shows semisimplicity. Similarly, the decomposition of a \(Br_n(r,q)\)-module into simple ones is already determined by the decomposition of any specialization for \(r\) and \(q\), provided this specialized algebra has the same decomposition into simple matrix algebras.

**Remark 3.9.** If \(V_{n,\nu}\) is a simple \(Br_{k,\nu}\)-module, we have the decomposition

\[
V_{n,\lambda} \cong \bigoplus_{\mu} V_{n-1,\mu},
\]
where $\mu$ runs through diagrams obtained by removing or also, if $|\lambda| < n$, by adding a box to $\lambda$. This follows from the restriction rule for the classical Brauer algebra, essentially going back to Brauer (see also e.g. [W3]). If $|\lambda| = n$, this becomes the restriction rule of modules of $S_n$ and $H_n$.

4. Markov trace

4.1. Definitions. It will be convenient to slightly extend the ground rings. So throughout this section we will consider the algebra $Br_n(N)$ defined over the ring $\mathbb{Z}[q, q^{-1}, [N]^{-1}]$, and the algebra $Br_n(r, q)$ defined over $\mathbb{Z}[q^{\pm 1}, r^{\pm 1}, ((r - 1)/(q - 1))^{\pm 1}]$. For simplicity, we will only formulate the results for $Br_n(N)$; all the proofs will go through as well for $Br_n(r, q)$. We can now define the elements $\bar{e} = \frac{1}{[N]} e$ and $\bar{e}(k) = \frac{1}{[N]} e(k)$; for $Br_n(r, q)$, we replace $[N]^{-1}$ by $(1 - q)/(1 - r)$. Observe that $\bar{e}$ and $\bar{e}(k)$ are idempotents with $\bar{e}(m) \bar{e}(k) = \bar{e}(k)$ for $m \leq k$. Recall that a functional $\phi$ on an algebra $A$ has the trace property if $\phi(ab) = \phi(ba)$ for all $a, b \in A$. It is well-known that one can inductively define a trace functional $tr$ on $H_n$ by $tr(1) = 1$, and $tr(g_{n-1}h) = \frac{q^n}{[N]} tr(h)$ for any $h \in H_{n-1}$. Such a functional on the Hecke algebras $H_n$ is called a Markov trace. It is compatible with the obvious standard inclusion $H_{n-1} \subset H_n$.

**Lemma 4.1.** (a) There exists an isomorphism $\Psi_k$ between $\bar{e}(k) Br_n(N) \bar{e}(k)$ and $Br_{n-2k}(N)$ such that $\Psi_k(\bar{e}(k) g_i) = g_{i-2k}$ for $i > 2k$ and $\Psi_k(\bar{e}(k+1)) = \bar{e}$.

(b) There exists a functional $\Phi_k : Br_n(N) \rightarrow Br_{n-2k}(N)$ uniquely defined by $\Phi_k(h) = \Psi_k(\bar{e}(k) h \bar{e}(k))$.

This lemma can be fairly easily checked using Lemma 3.4 and the explicit basis for $Br_n$ in Theorem 3.8.

**Lemma 4.2.** There exists a unique extension, also denoted by $tr$ of the Markov trace on $H_n$ to $Br_n(N)$ which is defined via induction on $n$ by $tr(a \bar{e}(k) b) = tr(\bar{e}(k) ba \bar{e}(k)) = \frac{1}{[N]^{2k}} tr(\Phi_k(ba))$. This extension also has the trace property $tr(cd) = tr(dc)$ for all $c, d \in Br_n(N)$.

**Proof.** We will prove well-definedness and the trace property of the functional $tr$ by induction on $n$. This is easy to check for $n = 1, 2$, as the algebras $Br_1(N)$ and $Br_2(N)$ are abelian. As to well-definedness in general, we have to show that $tr(a \bar{e}(k) b) = tr(a \bar{e}(k) cb)$ for all $a, b \in H_n$ and $c \in H_{2k+1,n}$. This is equivalent to showing $tr(\bar{e}(k) ba \bar{e}(k) c) = tr(\bar{e}(k) ba \bar{e}(k))$ by definition of $tr$. But this follows from the trace property of $tr$ for $Br_{n-2k}(N)$, using the homomorphism $\Psi_k$.

Let us now prove the trace property for elements $(a_1 \bar{e}(k_1) b_1)$ and $(a_2 \bar{e}(k_2) b_2)$, with $a_1, a_2, b_1, b_2 \in H_n$. Recall that we already know that $tr(ab) = tr(ba)$ if $a, b \in H_n$. Assuming $k_1 \leq k_2$, we can write

$$\bar{e}(k_1) b_1 a_2 \bar{e}(k_2) = \sum_{j \geq k_2} a^{(j)} \bar{e}(j) b^{(j)}$$
for suitable \(a^{(j)}, b^{(j)} \in H_{2k+1,n}\). So we have
\[
\text{tr}((a_1 \bar{e}(k_1)b_1)(a_2 \bar{e}(k_2)b_2)) = \sum_{j \geq k_2} \text{tr}(a_1 a^{(j)} \bar{e}(j)b^{(j)}b_2) = \sum_{j \geq k_2} \text{tr}(\bar{e}(j)b^{(j)}b_2a_1 a^{(j)} \bar{e}(j)) =
\]
using \(\bar{e}(k_2)\bar{e}(j) = \bar{e}(j)\) for \(j \geq k_2\) and \(\bar{e}(k_2)b^{(j)} = b^{(j)}\bar{e}(k_2)\)
\[
= \sum_{j \geq k_2} \text{tr}(\bar{e}(j)b^{(j)}e_{(k_2)}b_2a_1 e_{(k_2)}a^{(j)}\bar{e}(j)) = \sum_{j \geq k_2} \text{tr}(e_{(k_2)}b_2a_1 e_{(k_2)}a^{(j)}\bar{e}(j)b^{(j)}) =
\]
\[
= \text{tr}((\bar{e}(k_2)b_2a_1 \bar{e}(k_2))(\bar{e}(k_2)b_2a_2 \bar{e}(k_2))) = \text{tr}((\bar{e}(k_2)b_2a_2 \bar{e}(k_2))(\bar{e}(k_2)b_2a_1 \bar{e}(k_2))),
\]
where we used the induction assumption for elements in \(\bar{e}(k_2)Br_{n}(N)\bar{e}(k_2) \cong Br_{n-2k}(N)\). Equality with \(\text{tr}((a_2 \bar{e}(k_2)b_2)(a_1 \bar{e}(k_1)b_1))\) is now shown by the same calculations as above. Checking the trace property for elements \(a \in H_n\) and \(a_2 \bar{e}(k_2)b_2\) goes similarly and is easier. The lemma is proved.

4.2. Markov Property: Preparations. The goal is to prove an analog of the Markov property for the extension of \(tr\) to \(Br_{n}(N)\). We will need the following technical lemmas:

**Lemma 4.3.** (a) If \(j_1 < i_1 < n - 1\), \(eg_{2,n-1}g_{1,j_1}g_{n}^{-1}g_{i_1,1}g_{n-1,2}e = g_{1+1,3}g_{n,4}e(2)g_{4,n}g_{3,j_1+2}\).

(b) If \(i_1 < j_1 < n - 1\), then \(eg_{2,n-1}g_{1,j_1}g_{n}^{-1}g_{i_1,1}g_{n-1,2}e = g_{1+1,3}g_{n,4}e(2)g_{4,n}g_{3,j_1+1}\).

(c) If \(a, b \in H_n\), then \(\text{tr}(ag_{n,2} \bar{e}g_{n,2}b) = \text{tr}(ab)\text{tr}(\bar{e})\).

**Proof.** Using Lemma 1.3 and Eq 1.9, we see that the left hand side of statement (a) is equal to
\[
\text{tr}(eh_{1}g_{2}h_{2} e) = \text{tr}(h_{1} \bar{e}g_{2}h_{2}) = \text{tr}(g_{2})\text{tr}(h_{1} \bar{e}h_{2}) = \text{tr}(\bar{e})\text{tr}(h_{1}g_{2}h_{2}),
\]
using relation (E2) and the Markov property of \(tr\) for Hecke algebras. It follows that \(\text{tr}(\bar{e}h\bar{e}) = \text{tr}(h)\text{tr}(\bar{e})\) for any \(h \in H_{2,n}\). By Lemma 1.3 the map \(h \in H_{n} \mapsto g_{n,1}h g_{n,1}^{-1} \in H_{2,n+1}\) defines a trace-preserving homomorphism from \(H_n\) onto \(H_{2,n+1}\). Claim (c) follows from this and the trace property.

**Lemma 4.4.** Let \(a, b \in H_n\). Then \(\text{tr}(abe \bar{g}_{n}^{-1}) = \text{tr}(g_{n}^{-1})\text{tr}(abe)\).

**Proof.** We are going to prove the theorem by induction on \(n\), with \(n = 1\) and \(n = 2\) easy to check. We will also need the fact that \(eH_{n}e \subset eH_{3,n} + H_{3,n}e(2)H_{3,n}\). Indeed this can be checked easily using the fact that \(H_n\) is the span of elements of the form \(g_{j_1,1}g_{j_2,2}h\) with \(h \in H_{3,n}\). Hence if the claim holds for \(n = 2\), then we also have \(\text{tr}(g_{n}^{-1}ehe) = \text{tr}(g_{n}^{-1})\text{tr}(ehe)\) by using the definition of \(tr\) and induction assumption.
To prove the claim, let us write \(a = g_{1,1}g_{2,2}a''\) and \(b = b''g_{2,3j}g_{1,j_1}\), where \(a'', b'' \in H_{3,n}\). We first observe that the claim follows if both \(i_1, i_2 < n - 1\). Indeed, we have
\[
tr(abg_n^{-1}) = \frac{tr(g_{1,1}g_{2,2}g_n^{-1}ea''b)}{tr(g_n^{-1}tr(aeb))}
\]
where we used the argument of the first paragraph for the beginning of the second line. Similarly, one shows the claim if both \(j_1, j_2 < n - 1\). Hence we can assume that at least one of \(i_1\) or \(i_2\) is equal to \(n - 1\). But as \(g_{n-1}g_{2,2}e = g_{2-1,1}g_{n-1,1}e = qg_{2-1,1}g_{n-1,2}e\), we can assume that \(i_2 = n - 1\) and \(i_1 < n - 1\). One similarly shows that we can assume \(j_2 = n - 1\) and \(j_1 < n - 1\). Using Lemma 4.3 and the isomorphism \(\bar{e}Br_n\bar{e} \cong Br_{n-2}\), we can calculate for the case \(j_1 < i_1\) that
\[
tr(abg_n^{-1}) = tr\left(b''g_{2,n-1}g_{1,j_1}g_{1,1}g_{n-1,2}e''a''\right) = tr\left(b''g_{1,1}g_{2,2}g_{n-1,1}g_{n-1,2}e''a''\right) = \frac{tr(e_{(2)}g_{4,n}(g_{3,j_1+2}a''b''g_{1,1+3}g_{n-1,1}e)(g_{3,j_1+2}a'')g_{3,j_1+2}a'')}{tr(e_{(2)}g_{4,n}(g_{3,j_1+2}a''b''g_{1,1+3}g_{n-1,1}e)(g_{3,j_1+2}a'')g_{3,j_1+2}a'')} = \frac{tr(g_{2})}{} \frac{tr(g_{2})}{tr(g_n^{-1})tr(e_{(2)})}. \]
It remains to calculate \(tr(ab)\). We get
\[
tr(ab) = tr\left(b''g_{2,n-1}g_{1,j_1}g_{1,1}g_{n-1,2}e''a''\right) = tr\left(b''g_{1,1}g_{2,2}g_{n-1,1}g_{n-1,2}e''a''\right) = \frac{tr(g_{2})}{} \frac{tr(g_{2})}{tr(g_n^{-1})tr(e_{(2)})}. \]
The claim now follows from this and the fact that \(tr(e_{(2)}) = tr(g_{2})(tr(g_n^{-1})tr(e_{(2)})).\) The case \(i_1 > j_1\) goes similarly, and \(i_1 = j_1\) is easy.

4.3. Proof of Markov property.

**Theorem 4.5.** The functional \(tr\) satisfies \(tr(cg_n) = tr(c)tr(g_n)\) for all \(c \in Br_n(N)\).

**Proof.** Observe that that the claim follows for \(c \in H_n\) by definition of \(tr\), and for \(c \in H_n e H_n\) by Lemma 4.4. We will prove the general claim by induction on \(n\). It is trivially true for \(n = 1\). If \(n = 2\), we have \(tr(g_1g_2) = tr(g_1)tr(g_2)\) by definition of \(tr\), and \(tr(e_2e) = tr(e_2e) = \frac{g_n}{[N]}tr(e) = tr(g_2)tr(e)\) by relation \((E2)\).

Assuming that the claim holds for \(n - 1\) and \(n - 2\), we also have \(tr(\bar{e}eg_n) = tr(\bar{e}e)tr(g_n)\) for any \(c \in H_n\), using the isomorphism between \(\bar{e}Br_{n+1}\bar{e}\) and \(Br_{n-1}\), see Lemma 4.1. The induction step in our proof will depend on this observation.

Recall that any \(b \in H_n\) can be written as \(b = g_{n,n-1,n-1}b'\) with \(b' \in H_{n-1}\) and \(1 \leq i_n \leq n\); here \(g_{n,n-1}\) stands for 1, i.e. \(b = b' \in H_{n-1}\). But then we have
\[
tr(ae(k)b\mu_n) = tr(ae(k)g_{n,n}b') = tr(b'ae(k)g_{n-1,n}). \]
One deduces that it suffices to show that $tr(ae(k)g_{i_{n-1},n}) = tr(g_n)tr(ae(k)g_{i_{n-1},n-1})$. Now if $i_{n-1} > 2$, $g_{i_{n-1},n-1}$ commutes with $\bar{e}$ and we have

$$tr(ae(k)g_{i_{n-1},n}) = tr(ae(k)g_{i_{n-1},n-1}\bar{e}g_n) = tr(\bar{e}ae(k)g_{i_{n-1},n-1}\bar{e})tr(g_n).$$

The claim now follows after verifying that the first factor in the last expression is indeed equal to $tr(ae(k)g_{i_{n-1},n-1})$. As $e(k)g_1 = qe(k)$, it only remains to consider the case $i_{n-1} = 2$. But then we have for $k \geq 2$, using Lemma 3.3(b) that

$$tr(ae(k)g_{2,n}) = tr((ae(k)g_2q_1)g_{4,n}) = tr((\bar{e}g_1ae(k)g_{4,n-1}\bar{e})g_n).$$

The claim now follows again by the argument mentioned at the beginning of this proof.

4.4. Weights. It is well-known that any trace functional on a full $m \times m$ matrix algebra is equal to the usual trace, i.e. the sum of the diagonal elements, up to a scalar multiple. Hence any trace functional on a direct sum of full matrix algebras is completely determined as soon as one knows this multiple for each summand; these multiples are called the weights of the trace. The weights for the Markov trace on the Hecke algebra $H_n$ for $tr(g_i) = r(q - 1)/(r - 1)$ and $\lambda$ a Young diagrams with $n$ boxes are given by (see [W1])

$$\tilde{\omega}_\lambda = q^{c_1(\lambda)} \left( \frac{q - 1}{r - 1} \right)^n \prod_{(i,j) \in \lambda} \frac{rq^{i-j} - 1}{q^{h(i,j)} - 1} = \frac{q^{c_2(\lambda)}}{[N]^n} \prod_{1 \leq i < j \leq N} \frac{[\lambda_i - \lambda_j + j - i]}{[i - j]}.$$

Here $c_1(\lambda)$ and $c_2(\lambda)$ are determined such that the formulas remain invariant under the simultaneous substitutions $r \mapsto r^{-1}$ and $q \mapsto q^{-1}$, and equality with the second expression holds for $r = q^N$, for Young diagrams with at most $N$ rows. Moreover, $(i,j)$ denotes row and column of a box in the Young diagram $\lambda$, $h(i,j)$ is the length of the hook in $\lambda$ with corner at $(i,j)$ given by

$$h(i,j) = \lambda_i - i + \lambda'_j - j + 1,$$

where $\lambda_i$ and $\lambda'_j$ denote the number of boxes in the $i$-th row and $j$-th column of $\lambda$. For more details, see e.g. [Mac]. Moreover, if $r = q^N$, we also have

$$[N]^n \tilde{\omega}_\lambda = \chi^{GL(N)}(q, q^2, ..., q^{N-1}),$$

where the right hand side is the character of an element of $GL(N)$ with the indicated eigenvalues in the simple representation labeled by $\lambda$. We shall now similarly appeal to the character formulas of orthogonal groups to calculate the weights of $tr$ for the algebras $Br_n(N)$. We will need the following quantities for a given Young diagram $\lambda$

$$d(i,j) = \begin{cases} 
\lambda_i + \lambda_j - i - j & \text{if } i \leq j, \\
-\lambda'_i - \lambda'_j + i + j - 2 & \text{if } i > j.
\end{cases}$$
Theorem 4.6. The weights of the Markov trace $tr$ for $Br_n(r,q)$ are given by

$$
\omega_{\lambda,n} = q^{c_3(\lambda)} \frac{(q-1)}{(r-1)}^n \prod_{(i,j) \in \lambda} \frac{rq^d(i,j) - 1}{q^h(i,j) - 1},
$$

where $\lambda$ runs through all the Young diagrams with $n, n-2, n-4, \ldots$ boxes, and $c_3(\lambda)$ is determined such that the formula is invariant under the substitution $q \mapsto q^{-1}$.

Proof. Recall that the generic structures of $H_n$ and $Br_n(r,q)$ coincide with the ones of the group algebra of the symmetric group and of the Brauer algebra. Moreover, these isomorphisms are compatible with the inclusions. We have faithful representations of $S_n$ and $D_n(N)$ on $V \otimes^\kappa$ if $N = \dim V > n$, where a minimal idempotent of $\mathbb{C}S_n$ projects onto an irreducible representation of $Gl(N)$ in $V \otimes^\kappa$ and a minimal idempotent of $D_n(N)$ projects onto an irreducible representation of $O(N)$. Hence it follows

$$
(4.6) \quad \tilde{\omega}_\lambda = \sum_\mu b^\lambda_\mu \omega_{\mu,n},
$$

where $b^\lambda_\mu$ is the multiplicity of the irreducible $O(N)$-module labeled by $\mu$ in the irreducible $Gl(N)$-module labeled by $\lambda$. Moreover, we have $b^\lambda_\lambda = 1$ and $b^\lambda_\mu \neq 0$ for $\mu \neq \lambda$ only if $\mu$ has fewer boxes than $\lambda$. Hence Eq. 4.6 gives us a triangular system of equations from which we can calculate $\omega_\lambda$ for all $\lambda$s. As

$$
[N]^n \tilde{\omega}_\lambda = \chi^{|Gl(N)|}(1, q, \ldots, q^{N-1}) = q^{n(N-1)/2} \chi^{|Gl(N)|}(q^{(1-N)/2}, q^{(3-N)/2}, \ldots, q^{(N-1)/2})
$$

for $r = q^N$, we obtain the solution

$$
\omega_\lambda = \frac{1}{[N]^n} q^{n(N-1)/2} \chi^{|O(N)|}(q^{(1-N)/2}, q^{(3-N)/2}, \ldots, q^{(N-1)/2}) \quad \text{if } r = q^N.
$$

If $N$ is odd and sufficiently large, the character on the right hand side is what is called the principal character for type $B_{(N-1)/2}$ in [Ko]. It is shown in that paper that

$$
\chi^{|O(N)|}(q^{(1-N)/2}, q^{(3-N)/2}, \ldots, q^{(N-1)/2}) = q^{c_4(\lambda)} \prod_{(i,j) \in \lambda} \frac{q^N + d(i,j) - 1}{q^h(i,j) - 1},
$$

with $c_4(\lambda)$ again chosen such that the formula is invariant under the substitution $q \mapsto q^{-1}$. Substituting $r = q^N$ in the numerators, we obtain the desired expression for the weights. As these equalities hold for $r$ equal to any sufficiently large odd power of $q$, they must hold true in general for rational functions in $q$ and $r$.

Remark 4.7. Contrary to the statement in [Ko], the principal characters for type $B_n$ (and also for other types) do not coincide with the $q$-dimensions of the corresponding quantum group (the computations in the paper are correct, though). The corresponding two-variable polynomials for these $q$-dimensions have been calculated in [W2] as $Q_\lambda(r,q)$ in connection with another $q$-deformation of Brauer’s centralizer algebra and lead to different weights than the $\omega_{\lambda,n}$ in this paper.
4.5. Special values. The formulas for the weights of the Markov trace are valid for the generic case, i.e., when \( r \) and \( q \) are viewed as variables over a ring of rational functions. In this case, our algebras are semisimple. These formulas will also hold if we define the algebras \( Br_n \) over, say, the complex numbers, for any values of \( r \) and \( q \) for which \( Br_n(r,q) \) will have the same decomposition into a direct sum of simple matrix rings as in the generic case. We shall use the weights of the trace to determine these values, and also to determine special semisimple quotients for certain cases when the algebras are not semisimple.

We define special finite sets \( \Lambda(N, \ell) \) of Young diagrams for integers \( N \) and \( \ell \) satisfying \( 1 < |N| < \ell \). These will be related to algebras \( Br_n(r,q) \) where \( r = q^N \) and \( q = \xi \) is a primitive \( \ell \)-th root of unity.

**Definition 4.8.** Fix integers \( N \) and \( \ell \) satisfying \( 1 < |N| < \ell \). The set \( \Lambda(N, \ell) \) consists of all Young diagrams \( \lambda \) with \( \lambda_i \) boxes in the \( i \)-th row and \( \lambda'_j \) boxes in the \( j \)-th column which satisfy

(a) \( \lambda_1 + \lambda'_2 \leq N \) and \( \lambda_1 \leq (\ell - N)/2 \) if \( N > 0 \) and \( \ell - N \) even,
(b) \( \lambda_1 + \lambda'_2 \leq N \) and \( \lambda_1 + \lambda_2 \leq \ell - N \) if \( N > 0 \) and \( \ell - N \) odd,
(c) \( \lambda_1 \leq |N|/2 \) and \( \lambda'_1 + \lambda'_2 \leq \ell - |N| \) if \( N < 0 \) is even,
(d) \( \lambda_1 + \lambda_2 \leq |N| \) and \( \lambda'_1 + \lambda'_2 \leq \ell - |N| \) if \( N < 0 \) is odd.

In each of these cases, we call a Young diagram a boundary diagram of \( \Lambda(N, \ell) \) if it satisfies one inequality of the definition, but misses the other one by 1 (e.g. in case (a) if \( \lambda'_1 + \lambda'_2 \leq N \) and \( \lambda_1 = 1 + (\ell - N)/2 \). We denote by \( \bar{\Lambda}(N, \ell) \) the union of \( \Lambda(N, \ell) \) with its boundary diagrams.

**Proposition 4.9.** (a) The weights \( \omega_{\lambda,n} = \omega_{\lambda,n}(\xi^N, \xi) \) are nonzero and well-defined for any primitive \( \ell \)-th root of unity \( \xi \).

(b) If \( \xi \) is a primitive \( \ell \)-th root of unity, then \( \omega_{\lambda,n}(\xi^N, \xi) \neq 0 \) for \( \lambda \in \Lambda(N, \ell) \), and \( \omega_{\lambda,n}(\xi^N, \xi) = 0 \) for any boundary diagram of \( \Lambda(N, \ell) \).

**Proof.** The statements can be easily checked using the explicit product form of the formulas for \( \omega_{\lambda,n} \).

**Lemma 4.10.** Let \( \xi \) be a primitive \( \ell \)-th root of unity and let \( |N| \geq 2 \).

(a) Every Specht module \( S_\lambda \) of the Hecke algebra \( H_m(\xi) \) labeled by a Young diagram \( \lambda \) in \( \bar{\Lambda}(N, \ell) \) with \( m \) boxes is simple.

(b) If \( V \) is an \( H_n(\xi) \)-module which decomposes as an \( H_{n-1}(\xi) \)-module into a direct sum of simple modules labeled by Young diagrams in \( \Lambda(N, \ell) \) with \( n-1 \) boxes, then \( V \) is also semisimple as an \( H_n(\xi) \)-module, with its simple components labeled by Young diagrams in \( \bar{\Lambda}(N, \ell) \).

**Proof.** It follows from the Nakayama Conjecture for Hecke algebras (a theorem proved in [DJ]) that any Specht module is simple if it is labeled by a Young diagram \( \lambda \) for which \( \lambda_1 + \lambda'_1 < \ell + 1 \). Moreover, the corresponding central idempotent \( z_\lambda \) is well-defined for a primitive \( \ell \)-th root of unity. This can also be easily checked using the explicit representations e.g. in [W1]. Statement (a) can now be fairly easily checked using this criterion.

To prove statement (b), let \( z_{(n-1)} = \sum_\mu z_\mu \), with \( \mu \) in \( \Lambda(N-1, \ell) \), and let \( z_{(n)} = \sum_\lambda z_\lambda \), with \( \lambda \in \bar{\Lambda}(N, \ell) \). It follows from the well-known restriction rule for simple Hecke algebra modules
in the semisimple case that \( z_{(n)}z_{(n-1)} = z_{(n-1)} \). Hence \( V = z_{(n-1)}V = (z_{(n)}z_{(n-1)})V = z_{(n)}V \), also for \( \xi \) a primitive \( \ell \)-th root of unity. This proves part (b).

5. Semisimplicity

We now view our algebras \( Br_n(\rho, \xi) \) defined over a field of characteristic 0. We determine for which values of the parameters \( r = \rho \) and \( q = \xi \) in the chosen field our algebras will be semisimple. This follows the same patterns as in \([W2] \) and \([W3] \), using Jones’ basic construction and our formulas for the weights of the trace from the previous section. The only new complications come from the fact that we will not be able to use the standard embeddings \( Br_n \subset Br_{n+1} \). We will often just write \( Br_n \) instead of \( Br_n(\rho, \xi) \), assuming \( \rho \) and \( \xi \) to be fixed.

5.1. Jones’ construction. Let \( A \subset B \subset C \) be finite dimensional algebras. Moreover, let \( tr \) be a trace functional on \( B \) such that the induced bilinear form \( (b_1, b_2) = tr(b_1b_2) \) is nondegenerate for \( B \), and also for its restriction on \( A \). We can then define a conditional expectation \( E_A : B \rightarrow A \) uniquely determined by

\[
(E_A(b), a) = (b, a) \quad \text{for all } a \in A.
\]

Moreover, we assume that there exists an idempotent \( p \) in \( C \) satisfying the following conditions

(a) \( pa = ap \) for all \( a \in A \), and the map \( a \in A \mapsto ap \) is a monomorphism,

(b) \( pbp = E_A(B)p \) for all \( b \in B \).

Under these conditions we have the following results, going back to Jones’ basic construction (see \([W3] \), Lemma 1.1 or \([W2] \), Theorem 1.1):

**Proposition 5.1.** The ideal \( \langle p \rangle \) in the algebra generated by \( B \) and \( p \) is isomorphic to the commutant \( \text{End}_A(B) \) of \( A \), acting via right multiplication on \( B \). In particular, if \( A \) is semisimple, so is \( \langle p \rangle \). Moreover, the ideal \( \langle p \rangle \) is spanned by elements of the form \( b_1pb_2 \), with \( b_1, b_2 \in B \).

5.2. Embeddings. We define the embeddings \( i_1, i'_1 : Br_{n-1} \rightarrow Br_n \) by \( i_1(b) = g_{1,n-1}^{-1}bg_{1,n-1}^{-1} \) and \( i'_1(b) = g_{1,n-1}^{-1}b(g_{1,n-1}^{-1})^{-1} \) for \( b \in Br_{n-1} \). Moreover, we also define \( i_2, i'_2 : Br_{n-2} \rightarrow Br_n \) by \( i_2(b) = i_1(i'_1(b)) = g_{2,n-1}^{-1}i_1(b)(g_{2,n-1}^{-1})^{-1} \), for \( b \in Br_{n-2} \), and \( i'_2(b) = i'_1(i_1(b)) \). Observe that we have \( i_1(g_j) = g_{j+1} = i'_1(g_j) \) for \( 1 \leq j < n - 1 \). Then we have the following easy lemma:

**Lemma 5.2.** (a) With notations above we have that \( i_2(Br_{n-2}) \) commutes with \( e \) and the map \( b \in i_2(Br_{n-2}) \mapsto b \) defines an injective homomorphism. The statement also holds for \( i'_2(Br_{n-2}) \) instead of \( i_2(Br_{n-2}) \).

(b) Assume that \( Br_{n-1} \) is spanned by elements of the form \( b_1 \chi b_2 \) and that \( tr(b_1 \chi b_2) = tr(\chi)tr(b_1b_2) \), where \( \chi \in \{1, g_1, e\} \), and \( b_1, b_2 \in i'_1(Br_{n-2}) \). Then we have \( e(i_1(b_1 \chi b_2))e = tr(\chi)i_1(b_1b_2)e \) and also \( tr(c_1 \chi c_2) = tr(\chi)tr(c_1c_2) \) for \( c_1, c_2 \in i_1(Br_{n-1}) \).

(c) Under the assumptions and notations of (b), we have \( E_{i'_1(Br_{n-2})}(b_1 \chi b_2) = tr(\chi)b_1b_2 \), assuming that \( tr \) induces nondegenerate bilinear forms on \( Br_{n-1} \) and \( Br_{n-2} \).
Proof. It follows from Lemma 1.3 that \( i_1(g_j) = i'(g_j) = g_{j+1} \) for \( j < n - 1 \) and \( i_2(g_j) = i'_2(g_j) = g_{j+2} \) for \( j < n - 2 \). If we define \( e_2 = i_1(e) \), and \( e_3 = i_2(e) \), then it follows from our relations that

\[
e e_3 = e g_2^{-1} g_1 g_2 e g_2^{-1} g_1^{-1} g_3 g_2 = e_2 g_2^{-1} g_1^{-1} g_3 g_2 = e_2 (2).
\]

One similarly checks that \( e_3 e = e_2 (2) \). This, together with the relation \( e g_j = g_j e \) for \( j > 2 \) shows that \( e \) commutes with \( i_2(Br_{n-2}) = A \). Hence the map \( b \in Br_{n-2} \mapsto \bar{e} i_2(b) \) is an algebra homomorphism. One checks easily at the generators that it is the inverse of the isomorphism \( \Psi_1 \), as defined in Lemma 4.1. The same proof goes through if we replace \( i_2 \) by \( i'_2 \). This proves part (a).

For part (b), observe that \( e_2 = i_1(e) = g_1 g_2 e g_2^{-1} g_1^{-1} \). If \( \Delta_k = g_{1,k-1} g_{1,k-2} \ldots g_1 \), then \( \Delta_n^{-1} i_1(e) \Delta_n = \Delta_n^{-1} e \Delta_n^{-1} \in Br_{n-1} \), and \( \Delta_n^{-1} e \Delta_n = \Delta_n^{-1} \). One deduces from this that \( \Delta_n^{-1} i_1(Br_{n-1}) \Delta_n = Br_{n-1} \). But then, if \( b \in i_1(Br_{n-1}) \), we have

\[
tr(g_1 b) = tr(\Delta_n^{-1} g_1 b \Delta_n) = tr(g_{n-1} \Delta_n^{-1} b \Delta_n) = tr(g_1) tr(b),
\]

using the trace property and Theorem 4.5. Hence we only need to prove the last statement of (b) for \( \chi = e \), or, equivalently, \( \chi = \bar{e} \). By our assumptions, we can write \( c_2 c_1 = i_1(b_1 \psi b_2) \), with \( \psi \in \{1, e, g_1\} \) and \( b_1, b_2 \in i'_1(Br_{n-1}) \). But then

\[
tr(c_1 \bar{e} c_2) = tr(\bar{e} i_1(b_1 \psi b_2) \bar{e}) = tr(\psi) tr(i_1(b_1) \bar{e} i_1(b_2)) = tr(e) tr(b_1 \psi b_2),
\]

using our assumptions and already proven claims. It only remains to prove claim (c), which follows from the definitions and from \( tr(b_1 \chi b_2 c) = tr(\bar{e} tr(\chi) b_1 b_2 c) \) for any \( c \in i'_1(Br_{n-2}) \).

**Theorem 5.3.** The algebra \( Br_n(\rho, \xi) \) is semisimple if \( \rho \neq \xi^k \) for \( |k| \leq n \) and if \( \xi \) is not an \( \ell \)-th root of unity, \( \ell \leq n \). In this case, it has the same decomposition into simple matrix rings as the generic Brauer algebra, and the trace \( tr \) is nondegenerate. In particular, the assumptions in Lemma 5.2 hold for all \( n \).

Proof. We will prove the claim by induction on \( n \) together with the spanning assumption in Lemma 5.2(b), with \( n \) replaced by \( n + 1 \) (i.e. when \( b_1, b_2 \) are in \( i'_1(Br_{n-1}) \)). This, as well as the claim in the statement is easy to check for \( n = 1 \) and \( n = 2 \).

By induction assumption, \( tr \) is nondegenerate on \( Br_{n-1} \) and \( Br_n \). Hence, by Lemma 5.2, all the assumptions for Prop. 5.1 are satisfied for \( A = i_2(Br_{n-1}) \), \( B = i_1(Br_n) \) and \( p = \bar{e} \).

Hence the ideal \( \langle e \rangle \) generated by \( e \) in the algebra generated by \( i_1(Br_n) \) and \( e \) is isomorphic to \( End_{Br_{n-1}} Br_n \). It is known from the generic Brauer algebra that the latter algebra has dimension \( (2n+1)! - (n+1)! \); it is spanned by all graphs which have at least one horizontal edge. Using the basis \( (g_d) \) of Theorem 3.8, we see that this ideal coincides with the ideal \( I_{n+1} \) generated by \( e \) in \( Br_{n+1} \), and that it has zero intersection with \( H_{n+1} \). Now both \( I_{n+1} \) and \( H_{n+1} \) are semisimple algebras with mutually nonisomorphic simple modules (as \( e \) acts nonzero on simple \( I_{n+1} \)-modules and zero on simple \( H_{n+1} \)-modules). It follows that \( Br_{n+1} \cong I_{n+1} \oplus H_{n+1} \) as algebras. Nondegeneracy of a trace on a semisimple algebra now can be checked by just showing that its values on minimal idempotents are nonzero. This follows from Theorem 4.6.
Additionally, it follows from Prop. 5.1 and well-known properties of the Hecke algebra \( H_{n+1} \) that \( Br_{n+1} \) is spanned by elements of the form \( b_1 \chi b_2 \), with \( b_1, b_2 \in B = i_1(\mathbb{B}_n) \) and \( \chi \in \{1, e, g_1\} \). To prove the spanning assumption, we observe that everything in this proof so far would have worked as well for the inclusion \( A' = i'_2(Br_{n-1}) \subset B' = i'_1(Br_n) \subset Br_{n+1} \). Hence \( Br_{n+1} \) is also spanned by elements of the form \( b'_1 \chi b'_2 \), with \( b'_1, b'_2 \in B' = i'_1(Br_n) \) and \( \chi \in \{1, e, g_1\} \). This finishes the proof.

**Corollary 5.4.** Let \( \text{Ann}_n(\rho, \xi) = \{a \in Br_n(\rho, \xi), tr(ab) = 0 \text{ for all } b \in Br_n(\rho, \xi)\} \) and let \( \overline{Br}_n(\rho, \xi) = Br_n(\rho, \xi)/\text{Ann}_n(\rho, \xi) \). Then \( \text{Ann}_n(\rho, \xi) \subset \text{Ann}_{n+1}(\rho, \xi) \) for all \( n \).

**Proof.** Let \( \Delta_{n+1} \) be as defined in the proof of Lemma 5.2. We have seen in the proof of Theorem 5.3 that \( Br_{n+1} \) is spanned by elements of the form \( b_1 \chi b_2 \), with \( b_1, b_2 \in i_1(\mathbb{B}_n) \), and \( \chi \in \{1, e, g_1\} \). Conjugating this by \( \Delta_{n+1} \), we see that \( Br_{n+1}(r, q) \) is also spanned by elements of the form \( c_1 \psi c_2 \), with \( c_1, c_2 \in Br_n \) and \( \psi \in \{1, e_n = \Delta_{n+1} e \Delta_{n+1}^{-1}, g_n\} \). If \( a \in \text{Ann}_n(\rho, \xi) \), then we also have \( tr(ac_1 \chi c_2) = tr(\chi) tr(ac_1 \chi c_2) = 0 \). Hence also \( a \in \text{Ann}_{n+1}(\rho, \xi) \).

**Theorem 5.5.** Let \( \xi \) be a primitive \( \ell \)-th root of unity, and let \( N \) be an integer satisfying \( 1 < |N| < \ell \). Then \( \overline{Br}_n(\xi^N, \xi) \) is semisimple for all \( n \in \mathbb{N} \). Its simple components are labeled by the Young diagrams in \( \Lambda(N, \ell) \) with \( n, n-2, n-4, \ldots \) boxes, and the values of the Markov trace for minimal idempotents in \( \overline{Br}_n(\xi^N, \xi) \) are given by the formulas in Theorem 4.6. The restriction rule from \( \overline{Br}_n(\xi^N, \xi) \) to \( \overline{Br}_{n-1}(\xi^N, \xi) \) is as in Remark 3.9, where now only diagrams in \( \Lambda(N, \ell) \) are allowed.

**Proof.** We will only write \( Br_n \) for \( Br_n(\xi^N, \xi) \) in this proof, which will be done by induction on \( n \) similar to the one of Theorem 5.3. For \( n = 1 \) and \( n = 2 \), the claim is easily checked. To prove the induction step \( n \to n+1 \), we obtain from Corollary 5.4 that also \( \overline{Br}_{n+1} \) is semisimple, with the ideal \( \langle e \rangle \cong \text{End}_{\overline{Br}_{n-1}} Br_n \), and \( \overline{Br}_{n+1} \cong \langle e \rangle \oplus \overline{H}_{n+1} \), where \( \overline{H}_{n+1} \) is a quotient of the Hecke algebra \( H_{n+1} \). Moreover, it is well-known in the setting of Section 5.1 that we get minimal idempotents in \( \text{End}_A B \) in the form \( pf \), where \( f \) is a minimal idempotent in \( A \), acting from left on \( A \). Hence we get minimal idempotents in the ideal \( \langle e \rangle \) of the form \( p_\lambda e \), where \( p_\lambda \) is a minimal idempotent in \( i'_2(\overline{Br}_{n-1, \lambda}) \cong \overline{Br}_{n-1, \lambda} \) with \( \lambda \in \Lambda(N, \ell) \) such that \( n - 1 - |\lambda| \) is nonnegative and even. We have \( tr(p_\lambda e) = tr(e) tr(p_\lambda) \), as claimed.

It remains to determine the remaining simple components of \( \overline{H}_{n+1} \). By induction assumption and the restriction rules, see Remark 3.9 and Lemma 4.10, such a simple module must be isomorphic to a Specht module labeled by a Young diagram \( \lambda \) in \( \Lambda(N+1, \ell) \). So now it suffices to show that the trace of a minimal idempotent in the corresponding simple component is again given by \( \omega_{\lambda, n} \). This follows as soon as we can find an explicit expression for a minimal idempotent in \( Br_{N+1, \lambda} \) in terms of basis elements for which the coefficients are rational functions in \( r \) and \( q \) which are well-defined for our special values. This can be done by using the path idempotent approach, as it was done in [RW], as follows: Let \( \mu \) be a diagram in \( \Lambda(N, \ell) \) obtained by removing a box from \( \lambda \). It follows from the restriction rule that the minimal idempotent \( p_\mu \in Br_{n, \mu} \) can be written as a sum of mutually commuting minimal idempotents \( p_\nu \in Br_{n+1, \nu} \) labeled by diagrams \( \nu \) obtained by adding or subtracting a box to/from \( \lambda \). Now if \( \nu \) has one box less than \( \lambda \), \( Br_{n+1, \nu} \) is in the basic construction part of \( Br_{n+1} \), and hence \( p_\nu \).
can be obtained via formulas in [RW]; see [RW], Theorem 1.4 and our explicit formulas for the weights of the trace, Theorem 4.6. In particular, they are well-defined at our given root of unity $q$. Let $p'_\mu$ be the idempotent obtained after subtracting these idempotents $p_\nu$ from $p_\mu$. We then obtain $p_\lambda$ as an eigenprojection from $p'_\mu g_\nu p'_\mu$ using the formulas in [W1], Cor. 2.3. This finishes the proof.

**Remark 5.6.**
1. Essentially by the same method, semisimple quotients were constructed in [W2] for another $q$-deformation of Brauer’s centralizer algebra. As the weights for the Markov traces for these two generalizations of Brauer’s algebras differ, we also obtain different quotients. However, as in [W2], we will be able to construct new subfactors of the hyperfinite II$_1$ factor from our algebras by exhibiting a C*-structure for certain quotients. This analysis will be similar to the one in [W2], but the subfactors will be substantially different. E.g. it is expected that for $N = 2$ we would get the Goodman-de la Harpe-Jones subfactors labeled by Dynkin graphs $D_{2n}$, see [GHJ]. This will be done in a future paper.

2. The semisimple quotients constructed in this paper are not maximum in general. It is expected that the algebras in this paper are cellular in the sense of [GL]. It would be interesting to determine their decomposition series.

3. It is possible to define a $q$-deformation of $U_{sl_N}$ as a subalgebra of $U_q sl_N$, see [L1],[L2], [IK]. It is not a sub Hopf algebra of $U_q sl_N$ but a coideal algebra. Hence its representations can be made into a module category of $Rep(U_q sl_N)$. Taking the commutant of its action on $V^\otimes n$, where $V$ is the vector representation, we obtain a $q$-deformation of the Brauer algebra. This algebra was already studied in [Mo] (see remarks below) and is closely related to our algebras here. In particular, as these coideal algebras were constructed for a wide class of subalgebras of a semisimple Lie algebra, it might be possible to generalize constructions of this paper in this more general context. This would require more detailed studies of their representation theory in the nonsemiple case.

4. (Module categories) It follows from the description via generators and relations that the map $b \otimes g_i \mapsto bg_{i+m}$ defines embeddings of $BR_m(\xi^N, \xi) \otimes \mathcal{H}_n(\xi) \subset BR_{n+m}(\xi^N, \xi)$, with the algebras as defined in this section. This should lead to the construction of a module category of the fusion tensor category of type $A_{|N|-1}$ of level $\ell - |N|$ (see e.g. [Os]), with the objects being idempotents of the algebras $BR_n(\xi^N, \xi)$. Here the fusion tensor category would be defined via idempotents in the Hecke algebra quotients, see e.g. [Bl]. It appears that for $N = 2$, we would obtain the module tensor categories as in [Os] given by Dynkin graphs $D_n$. At least in this case, this category should also be realized via bimodules of von Neumann factors and subfactors as mentioned in Remark 1. Finally, we also mention that we obtain for each set $\Lambda(N, \ell)$ a representation of the fusion ring of type $A_{|N|-1}$ of level $\ell - |N|$ via matrices with nonnegative integer entries whose rows and columns are labeled by the entries of $\Lambda(N, \ell)$. They describes the tensor product rules of the model action. So our paper gives a rigorous derivation of at least some of the NIMREP representations in e.g. [GG] (see also the references in that paper). This was one of the motivations for this paper.

5. It would be interesting to see whether our algebras have any topological meaning. There exist other algebras, motivated by topological considerations, which contain Hecke algebras
as unital subalgebras, see [Ju], [RH]. It is not clear at this point what the relation is between these and our algebras, if any.

6. While putting on the finishing touches on this paper, the author noticed the work [Mo] by A. Molev. It deals with algebras acting on tensor spaces which also are $q$-deformations of quotients of Brauer’s centralizer algebras. The structure analysis in [Mo] was considerably less detailed than in this paper, though. It was conjectured in an earlier version of this paper that those algebras should be related to the ones in this paper. The author would like to thank A. Molev for informing him that indeed the algebras in this paper do satisfy the relations of the ones in [Mo].

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