1. Introduction

The impact interaction between solid bodies typically occurs over a short period of time and is accompanied by large dynamic loads, which could result in the possible destruction of structures' elements. Thus, it is only natural that the simplest theories for calculating the canonical bodies for strength upon impact are highlighted in the resistance of materials [1, 2]. They consider an impact to be instantaneous and, rather than the magnitude of force, apply its momentum. Actually, they consider not the process of a mechanical impact, but its consequences, that is a post-impact motion.
More complex variations in the theories of a mechanical impact are outlined in many publications, among which we emphasize [3–6].

Paper [3] sets out the synthesis of wave and quasi-static theories, as well as certain experimental methods for studying impact processes. Monograph [4] described the impact theory that was developed by the author of that work. It also contains a critical review of known experimental results. Study [5] is a literary review of achievements in the impact theory in the past century. Monographic publication [6] addressed studying the performance of different structures upon impact and compared the theory with experiment.

The above publications consider the development of the impact process over time, similar to [7], take into consideration local deformations, using a solution to the contact problem from the theory of elasticity. Thus, studying the impact processes has for a long time attracted attention of scientists and is a relevant task in mechanics.

2. Literature review and problem statement

The theory, initiated in [7], plays a key role in the research into impact processes of solid bodies. It was modernized in works [8, 9], where it was also suggested that the bodies subjected to an impact are, in a contact zone, restricted by surfaces of the second order, but they additionally considered oscillatory processes.

In addition to the above publications, a review of publications on a mechanical impact was performed in [5]. There was a marked tendency to an increase in the flow of publications that address an analysis of the dynamics of layered composite structures during impact.

Paper [10] describes analysis of performance, upon impact, of layered plates and cylindrical shells using the method of finite elements. The positive point is the presence of a large number of numerical results and geometric illustrations. The disadvantage of the work is the lack of simple calculation formulae, convenient for engineering calculations, because the research was conducted by purely numerical methods.

Study [11] considered an impact at a little velocity along a plane with the preliminary load by pressure. Structures of this type are common in engineering and are widely used. In the work, a body that hits was an elastic ball; no cases were considered for bodies that hit with a more complex shape because it complicates the theory.

Paper [12] examined the performance of layered plates at low-velocity impact using the method of finite elements. However, the work lacks analytical solutions. The obtained results are not universal, because they relate to specific numerical parameters for plates. For the case of change in the parameters, it is necessary to carry out new calculations, which is a disadvantage of numerical methods.

The influence of compressing forces on the performance of composite panels under impact was studied in [13]. It was established that the presence of such forces increases dynamic deflections. However, the problem was solved in a linear statement that rules out a possibility to model a change in the form of equilibrium upon impact.

Study [14] examined the dynamics of a composite plate with a hole upon impact. The authors calculated dynamic concentration of stresses in the zone of a hole. The analysis was carried out by numerical methods, but the study also lacks analytical solutions, which would make it possible to analyze the influence of different factors (size and shape of the hole, the properties of a material, etc.) on the magnitude of stress concentrations.

The above papers [10–14] considered a mechanical impact of bodies restricted by smooth surfaces, which have, at a contact point, a limited curvature; typically, it was a ball. However, in practice, there are other shapes of elastic bodies exposed to an impact. Only paper [4] considered the case of a tighter contact between bodies at dynamic compression, constrained by boundary surfaces that have an order higher than two. However, remain unexplored are the cases of an impact between bodies, whose boundary surfaces’ order is less than two (ogive, cone, and others). These bodies have a particular point at the contact surface. Therefore, existing theories for the case of such bodies are not applicable and require new mathematical models; construction of new theories related to the impact between bodies with a particular point at their surfaces is a promising task.

3. The aim and objectives of the study

The aim of this study is to derive and verify formulae to calculate the variability in the parameters for an impact compression of bodies over time, in the presence of a particular point at the surface of one of the bodies.

To accomplish the aim, the following tasks have been set:
– to derive a formula to calculate a coefficient in the equation of impact;
– to build a solution to the equation of impact at compression of bodies;
– to perform calculations and run a comparative analysis of numerical results.

4. Materials and methods to study an elastic impact of bodies with a particular point at the contact surface

When carrying out a mathematical modeling of the process of the dynamic compression of solid bodies, we shall use assumptions from work [7] where it was believed that the entire kinetic energy of relative motion is converted into potential energy of the elastic deformations of bodies in the zone of their interaction while disregarding other forms of energy, specifically the energy of elastic waves, thermal energy, etc. This imposes certain limitations on the velocity of an impact, which, in accordance with [4], should not exceed 4 m/s. Therefore, we build a mathematical model of a purely elastic impact, employing the theory of nonlinear differential equations and special functions.

Determining coefficients in the equation of impact. When deriving the basic equation of impact, we shall, in addition to basic provisions for theory [7], use a known solution to the axisymmetric contact problem from the elasticity theory [15]. According to this solution, the convergence between the centers of masses of elastic bodies, one of which is a half-space, and the second is restricted by the surface of rotation \( z = A r^{3/2} \), \((A > 0)\) (Fig. 1), is described by expression [15]:

\[
x = \frac{3}{2} f A a^{3/2},
\]
in which:

\[ a = \left[ \frac{P(Q_1 + Q_2)}{3AJ_f} \right]^{3/5}; \quad Q_1 = \frac{1-\mu_1^2}{E_1}; \quad Q_2 = \frac{1-\mu_2^2}{E_2}. \]  \tag{2}

\[ J_1 = \frac{1}{\sqrt{1-\xi^2}} \int_0^\infty \frac{E_1 d\xi}{\sqrt{1-\xi^2}}; \quad J_2 = \frac{1}{\sqrt{1-\xi^2}} \int_0^\infty \frac{E_2 d\xi}{\sqrt{1-\xi^2}}. \]

\( E_1, \mu_1, E_2, \mu_2 \) are, accordingly, the modulus of elasticity and the coefficient of transverse deformation of bodies' materials; \( P \) is the force of bodies' compression; \( a \) is the radius of the contact area.

Pressure distribution at it is governed by law [15]:

\[ p(r) = 1.25 \frac{P}{\pi a^2} \int \frac{r}{a} d\xi, \]  \tag{3}

where

\[ f \left( \frac{r}{a} \right) = \frac{r}{a} \int_0^1 \frac{d\xi}{\sqrt{1-\xi^2}}. \]  \tag{4}

Integrals \( J_1 \) and \( J_2 \) are represented through gamma-function \( \Gamma(z) \), tabulated in [16, 17], because, according to reference [18]:

\[ J_1 = \int_0^1 \sin^2 \phi d\phi = 2^{-1/2} \left[ \Gamma\left( \frac{3}{4} \right) \right]^2 / \Gamma\left( \frac{3}{2} \right), \]  
\[ J_2 = \int_0^1 \sin^2 \phi d\phi = 2^{3/2} \left[ \Gamma\left( \frac{7}{4} \right) \right]^2 / \Gamma\left( \frac{7}{2} \right). \]

These quadratures were obtained through a transition in (2) to the new variable of integration \( \xi = \sin \phi \). If one takes into consideration the tabular value [16]: \( \Gamma(7/4) = 0.9190625 \), then

\[ \Gamma(3/4) = \frac{4}{3} \Gamma(7/4) = \frac{4}{3} \times 0.9190625 = 1.2254167; \]  
\[ \Gamma(3/2) = \frac{1}{2} \sqrt{\pi}; \quad \Gamma(7/2) = \frac{15}{8} \sqrt{\pi}. \]

Thus, approximately, \( J_1 \approx 1.198140; J_2 \approx 0.718884. \)

Values for \( J_1 \) and \( J_2 \), but with a less accuracy, were also calculated in [15].

Expression (4) at \( r \to 0 \) has the uncertainty of type \( [0, \infty[ \). To reveal it, compute the integral in (4). Through a transition to the new variable: \( \xi = \eta^2 \), we obtain:

\[ f \left( \frac{r}{a} \right) = 2^{3/2} \frac{1}{\sqrt{\pi}} \int_0^1 \frac{d\eta}{\sqrt{1-\eta^2}}. \]

This integral is expressed through incomplete elliptic integrals \( F(\delta, \xi) \), \( E(\delta, \xi) \) of first and second kind, respectively, because in [18]:

\[ \int_0^r \frac{d\eta}{\sqrt{1-\eta^2}} = \frac{1}{2} \left[ F(\delta, \xi) - 2E(\delta, \xi) \right] + \frac{1}{\delta} \sqrt{1-u^4}, \]

and \( \delta = \arccos u \), \( \xi = \frac{1}{\sqrt{2}} \).

Thus, pressure distribution is described by expression:

\[ p(r) = 2.5 \frac{P}{\pi a^2} \left[ \frac{r}{2a} \left\{ \frac{1}{\sqrt{a^2 - r^2}} \left[ F\left( \arcsin \left( \frac{r}{\sqrt{a^2 - r^2}} \right) \right) \right] - \frac{1}{\sqrt{a^2 - r^2}} + \frac{1}{\delta} \sqrt{1-u^4} \right\} - 2E\left( \arcsin \left( \frac{r}{\sqrt{a^2 - r^2}} \right) \right) \right] \]  \tag{5}

Since

\[ F\left( 0; \frac{1}{\sqrt{2}} \right) = E\left( 0; \frac{1}{\sqrt{2}} \right) = 0, \]

then at the edge of the contact area, at \( r = a \), pressure \( p(a) = 0 \). At the center of the area, the pressure is maximum. There,

\[ p(0) = 2.5 \frac{P}{\pi a^2} = 2.5 \cdot p_a. \]

Therefore, maximum pressure is 2.5 times greater than the average over the zone. Graph of pressure distribution along the radius of the zone is shown in Fig. 2.

In addition to [15], other authors later built the generalized solutions to the static contact problem [19].

The following formula to calculate the strength of an impact follows from (1), (2):

\[ P = \beta \cdot a^{5/3}. \]

Here
If the mass of a body that hits is equal to \( M \), then the impact process, according to [7], will be described by a differential equation:

\[
M \ddot{x} = -P = -\beta x^{5/3},
\]

where a dot above \( x \) indicates a time-dependent derivative \( t \).

For the further integration of equation (6), we shall use it in the form:

\[
\frac{d}{dx} \left( \frac{d \ddot{x}}{dx} \right) = -\frac{\beta}{M} x^{5/3}.
\]

Construction of the solution to the equation of impact at compression of bodies. It is matched by time \( t \in (0; t_c) \). Differential equation (7) is to be solved under initial conditions:

\[
x(0) = 0, \quad \dot{x}(0) = \upsilon_0,
\]

we denote via symbol \( \upsilon_0 \) the initial velocity of collision between bodies.

Upon integrating (7), taking into consideration (8), we obtain:

\[
\dot{x} = \sqrt{\upsilon_0^2 - \frac{3 \beta}{4 M} x^{5/3}}; \quad \int_0^t \frac{d y}{\sqrt{\upsilon_0^2 - \frac{3 \beta}{4 M} y^{5/3}}} = t.
\]

At the time of maximum compression, radicand \( x = x_c \) in (9) is equal to zero. Therefore, the maximum compression is:

\[
x_c = \left( \frac{4 M \upsilon_0^2}{3 \beta} \right)^{2/3}.
\]

Through a transition to the new variable in the integration \( \xi = \frac{y}{x_c} \), the integral in (9) is reduced to the following:

\[
\int_0^{x_c^{5/3}} \frac{d \xi}{\sqrt{1 - \xi^{5/3}}} = \frac{\upsilon_0 I}{x_c}
\]

where the upper bound is the Ateb-sine [20–23]. Thus,

\[
x(t) = x_c \cdot \text{Sa} \left( \frac{5}{3} \frac{4 \upsilon_0 t}{3 x_c} \right),
\]

A time-dependent change in the force of an impact is governed by law:

\[
P(t) = \beta x_c^{5/3} \left[ \text{Sa} \left( \frac{5}{3} \frac{4 \upsilon_0 t}{3 x_c} \right) \right]^{5/3}.
\]

The radius of the contact area, as well as pressure at its center, depend on the values for Ateb-sine as well, because, according to (2) and (5):

\[
a(t) = \left( \frac{\beta (Q + Q_c)}{3 A f} \right) x_c^{2/3} \left[ \text{Sa} \left( \frac{5}{3} \frac{4 \upsilon_0 t}{3 x_c} \right) \right]^{2/3}.
\]

Maximum values are acquired by the characteristics of an impact interaction at \( t = t_c \) or when:

\[
\text{Sa} \left( \frac{5}{3} \frac{4 \upsilon_0 t_c}{3 x_c} \right) = 1.
\]

Maxima are represented by concise formulae:

\[
\max x(t) = x_c = \left( \frac{4 M \upsilon_0^2}{3 \beta} \right)^{3/8}; \quad \max P(t) = P_c = \beta x_c^{5/3};
\]

\[
\max a(t) = a_c = 0.67650 \left( \frac{x_c}{A} \right)^{2/3}.
\]

The root of equation (12) is related to integral:

\[
\frac{\upsilon_0 I}{x_c} = I = \frac{1}{\sqrt{8}} \int_0^{\frac{3}{8} \Gamma(3/8)} \Gamma(7/8).
\]

Given \( \Gamma(3/8) = 2.370437, \Gamma(7/8) = 1.089653 \), we find:

\[
I = 1.445927.
\]

The result is the obtained formula for the calculation of compression process duration:

\[
t_c = 1.445927 \frac{x_c}{\upsilon_0}.
\]

To simplify the numerical realization of solution derived, we give a table of Ateb-sine whose values, with an acceptable accuracy, can be found through a linear interpolation of data from Table 1.

| \( 10\eta \) | \( 10\text{Sa} \left( \frac{5}{3} \frac{4 \upsilon_0 t}{3 x_c} \right) \) | \( 10\eta \) | \( 10\text{Sa} \left( \frac{5}{3} \frac{4 \upsilon_0 t}{3 x_c} \right) \) |
|---|---|---|---|
| 0.0 | 0.00 (0.00) | 7.5 | 7.04 (7.02) |
| 0.5 | 0.50 (0.50) | 8.0 | 7.42 (7.42) |
| 1.0 | 1.00 (1.00) | 8.5 | 7.78 (7.78) |
| 1.5 | 1.50 (1.50) | 9.0 | 8.12 (8.12) |
| 2.0 | 2.00 (2.00) | 9.5 | 8.42 (8.43) |
| 2.5 | 2.49 (2.50) | 10.0 | 8.72 (8.72) |
| 3.0 | 2.98 (3.00) | 10.5 | 8.98 (8.98) |
| 3.5 | 3.47 (3.49) | 11.0 | 9.22 (9.22) |
| 4.0 | 3.95 (3.97) | 11.5 | 9.43 (9.43) |
| 4.5 | 4.43 (4.44) | 12.0 | 9.60 (9.60) |
| 5.0 | 4.89 (4.89) | 12.5 | 9.75 (9.75) |
| 5.5 | 5.35 (5.34) | 13.0 | 9.86 (9.86) |
| 6.0 | 5.79 (5.78) | 13.5 | 9.94 (9.94) |
| 6.5 | 6.22 (6.20) | 14.0 | 9.99 (9.99) |
| 7.0 | 6.64 (6.62) | 14.0 | 10.00 (10.00) |
Here in parentheses we give approximate values, which are derived via approximation:

$$S_a\left(\frac{5}{3}, \frac{4}{3}, \eta\right) = \frac{\eta}{1-1.2\sin^2\left(\sqrt{5}/3\right)(I-\eta)} \left(0.1996 + 1.0277(\eta-0.2) - 0.2064(\eta-0.2)^2\right)$$

at $0 \leq \eta \leq 0.2$,

$$t_p = t_f + \frac{x}{v_o} - 2t_c = 2.891854 \frac{x}{v_o}$$

at $0.2 < \eta < 0.8$,

$$\eta \leq \eta \leq I.$$ at $0.8 \leq \eta \leq I$.

In order to simplify calculation of the decompression process, we give the values for Ateb-cosine in Table 2.

| $10 \xi$ | $C_a \left(\frac{5}{3}, 1, \frac{4}{3}, \xi\right)$ | $10 \xi$ | $C_a \left(\frac{5}{3}, 1, \frac{4}{3}, \xi\right)$ |
|---------|---------------------------------|---------|---------------------------------|
| 0       | 0.00 (0.00)                      | 7.5     | 6.61 (6.62)                      |
| 0.5     | 0.98 (9.89)                      | 8.00    | 6.19 (6.17)                      |
| 1.0     | 0.93 (9.93)                      | 8.50    | 5.76 (5.74)                      |
| 1.5     | 0.85 (9.85)                      | 9.00    | 5.31 (5.30)                      |
| 2.0     | 0.74 (9.74)                      | 9.50    | 4.86 (4.86)                      |
| 2.5     | 0.59 (9.59)                      | 10.00   | 4.39 (4.40)                      |
| 3.0     | 0.41 (9.41)                      | 10.50   | 3.91 (3.93)                      |
| 3.5     | 0.20 (9.20)                      | 11.00   | 3.43 (3.45)                      |
| 4.0     | 0.89 (8.96)                      | 11.50   | 2.94 (2.96)                      |
| 4.5     | 0.70 (8.70)                      | 12.00   | 2.45 (2.46)                      |
| 5.0     | 0.41 (8.41)                      | 12.50   | 1.96 (1.96)                      |
| 5.5     | 0.29 (8.09)                      | 13.00   | 1.46 (1.46)                      |
| 6.0     | 0.75 (7.76)                      | 13.50   | 0.96 (0.96)                      |
| 6.5     | 0.39 (7.40)                      | 14.00   | 0.46 (0.46)                      |
| 7.0     | 0.70 (7.02)                      | 14.50   | 0.00 (0.00)                      |
| 7.5     | 0.17 (6.76)                      | 15.00   | 0.00 (0.00)                      |
| 8.0     | 0.00 (6.50)                      | 15.50   | 0.00 (0.00)                      |

In brackets, Table 2 gives the approximated values for Ateb-cosine, obtained through approximation:

$$1 - 1.2\sin^2\left(\sqrt{5}/3 - \xi\right)$$

at $0 \leq \xi \leq 0.75$,

$$0.1996 + 1.0277(1.2459 - \xi) - 0.2064(1.2459 - \xi)^2$$

at $0.75 \leq \xi < 1.25$,

$$1.2459$$

at $1.25 \leq \xi \leq I$.

Table 2 shows that an error of the recorded analytical approximation is less than one percent. This indicates the adequacy of the represented formulae.

5. Results of calculations of the impact parameters, their comparative analysis

By using the derived formulae, we shall compute the characteristics of an impact at: $M=0.7$ kg; $v_0=3$ m/s; $A=5$ m$^2$; $E_r=2\times10^{14}$ Pa; $\mu=0.25$. Material of the half-space is rubber for which [25]: $E_r=7.5-10^5$ Pa; $\mu=0.5$. For these initial data $E_r=2.77632-10^6$ N/m$^2$. From formulae (13) and (14), we obtain: $x_c=0.008517$ m; $P_c=986.2176$ N; $v_o=3.919649$ m; $p_c=8429081.797$ Pa. Duration of the compression and impact are, respectively: $t_c=0.004105$ s; $t_p=0.009649$ s. Values for $x(t)$, calculated based on formulae (10), (15), at different moments are given in Table 3.

Values for Ateb functions were computed by the method of linear interpolation of data from Tables 1, 2. For comparison, Table 3 also gives values for $x(t)$, obtained by a computer-based integration of equation (6) in the programming environment “Maple” [26]. There is a good consistency of
results from calculations by two procedures, confirming the probability of analytical solutions to the nonlinear problem.

### Table 3

| \( t/t_0 \) | \( x(t)/x_0 \) | \( x(t)/x_0, \) formulae (10, 15) | \( x(t)/x_0, \) numerical integration |
|------------|----------------|---------------------------------|---------------------------------|
| 0.25       | 0.3615         | 0.3580                          | 0.3582                          |
| 0.50       | 0.7230         | 0.6824                          | 0.6826                          |
| 0.75       | 1.0845         | 0.9146                          | 0.9149                          |
| 1.00       | 1.4459         | 1.0000                          | 1.0000                          |
| 1.25       | 1.8074         | 0.9145                          | 0.9149                          |
| 1.50       | 2.1689         | 0.6826                          | 0.6826                          |
| 1.75       | 2.5303         | 0.3580                          | 0.3582                          |

The diagrams of change in \( x(t) \) and \( P(t) \) over time, obtained from calculations, are shown in the dimensionless coordinates in Fig. 3. Over a compression interval, the diagram of \( x(t) \) is convex, and the diagram for \( P(t) \) changes concavity to convexity.

Fig. 4 shows the estimated diagrams of change in \( a(t) \) and \( p(t) \) over time. There is a rapid growth in pressure \( p(t) \) in the center of the contact area of bodies.

The course of the impact process is affected by the geometry of the hitting body. This is confirmed by results from the calculation of an impact characteristics, given in Table 4 for different values of \( A. \)

### Table 4

| \( A, \text{m}^{-1/2} \) | \( 10^6 \) \( x_0, \text{m} \) | \( P_0, \text{N} \) | \( 10^3 \) \( a_0, \text{m} \) | \( 10^5 \) \( p_0, \text{Pa} \) | \( 10^3 \) \( t_0, \text{s} \) |
|-----------------|-----------------|----------------|-----------------|----------------|----------------|
| 2               | 67.74           | 1240.24        | 152.6           | 4.23825        | 32.65          |
| 4               | 80.55           | 1042.76        | 107.9           | 7.12741        | 38.82          |
| 6               | 89.15           | 942.36         | 88.1            | 9.66174        | 42.97          |

An increase in \( A \) leads to an increase in \( x_0, \) pressure \( p_0, \) and the impact duration. However, it is accompanied by a decrease in \( P_0 \) and \( a_0. \) Attention should be paid to that at a change in \( A \) the product \( P_0 t_0 \) remains an approximately constant magnitude. To find out why it happens, we shall consider the calculation of impact pulse \( S(P). \) At time interval \( t \in [0, t_0], \) it is represented by integral:

\[
S(P) = \int_0^{t_0} P(t) \, dt
\]

Over the entire impact duration, when \( t = t_0, \) the pulse is equal to:

\[
S(P) = 2P_0 \left[ 5/3 \cdot \left( \frac{4}{3} \cdot x_0 \right)^{5/3} \right]^{1/3}
\]

Compute this integral approximately from the formula of trapezoid [27]:

\[
S(P) = \frac{1}{3} P_0 t_0 \left[ 1 + 4 \left( \frac{5}{3} \cdot \left( \frac{4}{3} \cdot x_0 \right)^{5/3} \right) \right].
\]

Since

\[
\left( \frac{5}{3} \cdot \left( \frac{4}{3} \cdot x_0 \right)^{5/3} \right) = 0.682,
\]

then

\[
\frac{1}{3} \left[ 1 + 4 \left( \frac{5}{3} \cdot \left( \frac{4}{3} \cdot x_0 \right)^{5/3} \right) \right] = 1.03788.
\]

Therefore, strength of pulse \( P(t) \), over an impact duration, is approximately:

\[
S(P) = 1.03788 P_0 \cdot t_0.
\]

By using the results from Table 4 and formula (16), we obtain, at \( A=2, 4, 6 \text{ m}^{-1/2}, \) respectively, \( S(P)=4.2027; 4.2028; 4.2027 \text{ N\cdot m}. \) These numbers are little different from the exact value for pulse \( S(P)\) by \( 2M_0=4.2 \text{ N\cdot m}. \) Thus, the stable product \( P_0 t_0 \) in Table 4 is a consequence of the fact that \( S(P)\) is constant in the considered example.

Next, we determine the influence on the impact parameters exerted by the initial velocity of bodies' collision, which, in accordance with [25], can reach \( 5 \text{ m/s}. \) To conduct calculations, we save the above-specified source data, where \( A=5 \text{ m}^{-1/2}, \beta=2.77632 \times 10^6 \text{ N\cdot m}^{5/3}. \)
The calculations confirm that an increase in \( v_0 \) leads to an increase in \( x_c \), \( P_c \), \( a_0 \), \( p_c \), and a decrease in \( t_c \). At \( v_0 \approx 5 \, \text{m/s} \), force \( P_c \) is close to 1,900 \, \text{N}. Paper [25] noted that force \( P_c \) may exceed 1,960 \, \text{N}.

Information about the effect of a body mass on the impact characteristics is given in Table 6. In the calculations, we used previous numerical data by assigning \( v_0 = 4 \, \text{m/s} \).

Here, an increase in the hitting body mass leads to an increase in both \( x_c \), \( P_c \), \( a_0 \), \( p_c \), and \( t_c \), which was not the case when increasing \( v_0 \).

Table 5

| \( v_0 \), m/s | \( x_c \), \( 10^4 \), m | \( P_c \), N | \( 10^6 a_0 \), m | \( 10^5 p_c \), Pa | \( 10^4 t_c \), s |
|-------------|----------------|----------|----------------|----------------|----------------|
| 3           | 85.17          | 986.22   | 96.49          | 8.429          | 41.05          |
| 4           | 105.68         | 1413.03  | 111.42         | 9.058          | 38.20          |
| 5           | 124.94         | 1867.63  | 124.57         | 9.578          | 36.13          |

Table 6

| \( M \), kg | \( x_c \), \( 10^4 \), m | \( P_c \), N | \( 10^6 a_0 \), m | \( 10^5 p_c \), Pa | \( 10^4 t_c \), s |
|------------|----------------|----------|----------------|----------------|----------------|
| 0.7        | 105.68         | 1413.03  | 111.42         | 9.058          | 38.20          |
| 0.8        | 111.11         | 1536.05  | 115.20         | 9.211          | 40.16          |
| 0.9        | 116.13         | 1653.39  | 118.65         | 9.346          | 41.98          |
| 1.0        | 120.81         | 1765.93  | 121.81         | 9.471          | 43.67          |

Here, an increase in the hitting body mass leads to an increase in both \( x_c \), \( P_c \), \( a_0 \), \( p_c \), and \( t_c \), which was not the case when increasing \( v_0 \).

The calculations show that a velocity change in \( v_0 \) has a considerably larger effect on the impact characteristics than a change in \( M \).

6. Discussion of results of studying the derived analytic solutions to an elastic impact of the body with a special point

The built analytical solutions to the impact are quite universal. They do not have analogs in the scientific literature because they relate to the development of the theory of a quasi-static impact of bodies that have a particular point at the contact surface.

The obtained solutions are not linked to specific values for the physical and geometrical parameters of bodies exposed to impact. They make it possible to calculate a change in the impact process parameters over time, as well as extreme values: the strength of an impact, the magnitude of bodies' compression, the radius of the contact area, and the maximum pressure at the center of this plane. An apparatus of the periodic Ateb functions has proven to be an effective means to represent the analytical solutions, owing to these special functions, tabulated in this work. Comparison of numerical results obtained by different methods confirms the reliability of the derived analytical solutions. The results of calculations correspond to the physical essence of the impact process (small duration in time, a large maximum of the impact force, etc.).

The results obtained could be used in hybrid theories where they synthesize the quasi-static and wave theories. They could be applied when calculating a response to the impact from both homogeneous and composite beams, plates, and shells.

The outlined theory applies to low impact velocities, not exceeding 5 \, \text{m/s}, so that the deformations of bodies are within the limits of the elasticity theory.

The considered theory applies to only one case of a particular point at the boundary surface of a body. However, there are cases with points with higher attributes (a vertex of the cone, etc.). Development of the impact theory for such bodies could be a direction for further research.

7. Conclusions

1. We have derived a formula for the calculation of a coefficient in the impact equation, which depends on the geometrical characteristics and materials of bodies exposed to impact. It is easy to calculate when using the formula derived in this work. Calculations are given in the results computation.

2. We have built a solution to the equation of impact at compression of bodies, which is expressed by Ateb-sine, and makes it possible to calculate time-dependent changes in the impact strength, in the convergence of the centers of mass of bodies, the radius of the contact area and pressure at its center, by using the compiled table of this function. Ateb-sine has not been used in the theory of impact before.

3. We have built a solution to the equation of impact at decompression of bodies, which is expressed through Ateb-cosine. We have derived a formula for the calculation of an impact duration. The table of this function makes it easy to use the solution in calculation.

4. The calculations have been performed, which confirmed the adequacy of the built analytical solutions; the effect of various factors on the main impact characteristics has been investigated. It was established:
   - increasing the geometrical parameter leads to an increase in the maximum of compression, impact strength, pressure and duration of the impact, but it is accompanied by a decrease in the maximum of impact strength and the radius of the contact area;
   - a growth of the initial impact velocity is accompanied by the growth of all impact parameters, except for the impact duration, which decreases;
   - a growth in the hitting body mass is accompanied by an increase in all impact parameters due to an increase in the kinetic energy of the hitting body;
   - a change in the initial velocity exerts a more essential, compared to a change in mass, influence on the impact characteristics, which has been represented in tables.

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