Stock markets and quantum dynamics: a second quantized description

F. Bagarello

Dipartimento di Metodi e Modelli Matematici, Facoltà di Ingegneria,
Università di Palermo, I - 90128 Palermo, Italy
E-mail: bagarell@unipa.it
home page: www.unipa.it/~bagarell

Abstract

In this paper we continue our descriptions of stock markets in terms of some non abelian operators which are used to describe the portfolio of the various traders and other observable quantities. After a first prototype model with only two traders, we discuss a more realistic model of market with an arbitrary number of traders. For both models we find approximated solutions for the time evolution of the portfolio of each trader. In particular, for the more realistic model, we use the stochastic limit approach and a fixed point like approximation.
I  Introduction

In a recent paper, [1], we have discussed how a quantum mechanical framework can be used in the analysis of stock markets. The conservation of the total number of shares and of the total amount of cash in any closed marked, i.e. in a market which does not interact with the environment, as well as the discrete nature of the number of the shares and of the monetary unit, suggested the use of some typical tools of QM∞, i.e. of quantum mechanics for systems with infinite degrees of freedom, in this different context. In particular, we have shown that a second quantized vision of the stock market produces in a natural way a set of differential equations describing the time evolution of the portfolio of each trader of the market. These results are on the same line as those given in [2] and [3], as well as in [4] and references therein. We should also mention that the use of tools coming from physics for economical problems, or more generally for dealing with complex systems, is a well established procedure, for which we refer to [5].

The paper is organized as follows:

- in the next section we briefly review the results in [1].
- In Section III, contrarily with what has been made in [1], where the price of the share \( P(t) \) has no real dynamics since it is replaced by its mean value, we introduce a prototype model with only two traders were we show how to keep into account the time evolution of the price \( P(t) \). We prove that many integrals of motion exist. The equations of motion are solved using a perturbative expansion, well known in QM∞.
- In Section IV we consider another model, which generalize the previous one in the sense that it consists of \( N \) traders with arbitrary \( N \geq 2 \), and for which we consider two different approximations: the stochastic limit, which is useful to analyze the equilibrium of the model, and what we call a fixed point-like (FPL) approximation, which we use to deduce the approximated time evolution of the portfolio of any given trader of the stock market associated to the model.
- Section V contains our conclusions and plans for the future, while we discuss in the first Appendix some delicate mathematical points and in Appendix B a general introduction to the stochastic limit.
II The genesis of the model

In this section we review some results and ideas first introduced in [1] which have produced an interesting toy model of a stock market based on the following assumptions:

1. Our market consists of \( L \) traders exchanging a single kind of share;

2. the total number of shares, \( N \), is fixed in time;

3. a trader can only interact with a single other trader: i.e. the traders feel only a *two-body interaction*;

4. the traders can only buy or sell one share in any single transaction;

5. the price of the share changes with discrete steps, multiples of a given monetary unit;

6. when the tendency of the market to sell a share, i.e. the *market supply*, increases then the price of the share decreases;

7. for our convenience the supply is expressed in terms of natural numbers;

8. to simplify the notation, we take the monetary unit equal to 1.

We refer to [1] for the analysis of these conditions, which however, in our opinion, look quite natural and self-explanatory. The *formal* hamiltonian of the model is the following operator:

\[
\tilde{H} = H_0 + \tilde{H}_I, \quad \text{where} \quad
\begin{align*}
H_0 &= \sum_{l=1}^{L} \alpha_l a_l^\dagger a_l + \sum_{l=1}^{L} \beta_l c_l^\dagger c_l + o^\dagger o + p^\dagger p, \\
\tilde{H}_I &= \sum_{i,j=1}^{L} p_{ij} \left( a_i^\dagger a_j (c_i^\dagger c_j)^\hat{P} + a_i^\dagger a_j^\dagger (c_j^\dagger c_i)^\hat{P} \right) + (o^\dagger p + p^\dagger o),
\end{align*}
\]  

where \( \hat{P} = p^\dagger p \) and the following commutation rules are assumed:

\[
[a_i, a_n^\dagger] = [c_i, c_n^\dagger] = \delta_{ln}\mathbb{I}, \quad [p, p^\dagger] = [o, o^\dagger] = \mathbb{I},
\]  

\[ (2.1) \]
while all the other commutators are zero. We further assume that \( p_{ij} = 0 \). Here the operators \( a_i^\dagger, p^\dagger, c_i^\dagger, o^\dagger \) are respectively the number, the price, the cash and the supply operators. \([1]\). The states of the market are

\[
\omega_{\{n\};\{k\};\Omega;M}(\cdot) = \langle \varphi_{\{n\};\{k\};\Omega;M}, \cdot \varphi_{\{n\};\{k\};\Omega;M} > ,
\]

where \( \{n\} = n_1, n_2, \ldots, n_L, \{k\} = k_1, k_2, \ldots, k_L \) and

\[
\varphi_{\{n\};\{k\};\Omega;M} \triangleq \frac{(a_1^\dagger)^{n_1} \cdots (a_L^\dagger)^{n_L} (c_1^\dagger)^{k_1} \cdots (c_L^\dagger)^{k_L} (o^\dagger)^{\Omega} (p^\dagger)^{M} }{\sqrt{n_1! \cdots n_L! k_1! \cdots k_L! \Omega! M!}} \varphi_0.
\]

Here \( \varphi_0 \) is the vacuum of the model: \( a_j\varphi_0 = c_j\varphi_0 = p\varphi_0 = o\varphi_0 = 0 \), for \( j = 1, 2, \ldots, L \). Again we refer to \([1]\) or to any quantum mechanical textbook, see \([6]\) for instance, for further details on second quantization.

The interpretation of the hamiltonian is a key point in our approach and has been discussed in details in \([1]\): just as an example, the presence of the term \( o^\dagger p \) in \( \hat{H}_I \) implies that when the supply increases then the price must decrease. Moreover, because of \( a_i^\dagger a_j (c_i c_j)^{\hat{P}} \), trader \( \tau_i \) increases of one unit the number of shares in his portfolio but, at the same time, his cash decreases because of \( c_i^{\hat{P}} \), that is it must decrease of as many units of cash as the price operator \( \hat{P} \) demands. Clearly, trader \( \tau_j \) behaves in the opposite way: he loses one share because of \( a_j \) but his cash increases because of \( (c_j^{\dagger})^{\hat{P}} \).

However we have discussed in \([1]\) that the hamiltonian in (2.1) suffers of a technical problem: since \( c_j \) and \( c_j^{\dagger} \) are not self-adjoint operators, it was not obvious how to define the operators \( c_j^{\hat{P}} \) and \( (c_j^{\dagger})^{\hat{P}} \), and for this reason we have replaced \( \hat{H} \) with an effective hamiltonian, \( H \), defined as

\[
\begin{align*}
H &= H_0 + H_I, \text{ where} \\
H_0 &= \sum_{l=1}^L \alpha_l a_l^\dagger a_l + \sum_{l=1}^L \beta_l c_l^\dagger c_l + o^\dagger o + p^\dagger p \\
H_I &= \sum_{i,j=1}^L p_{ij} \left( a_i^\dagger a_j (c_i c_j)^{M} + a_i^\dagger a_j (c_j c_i)^{M} \right) + (o^\dagger p + p^\dagger o),
\end{align*}
\]

where \( M = \omega_{\{n\};\{k\};\Omega;M}(\hat{P}) \). In this way, however, we are essentially freezing the price of our action, removing one of the (essential) degrees of freedom out from our market. This strong limitation will be removed in the next two sections of this paper and, in our opinion, this is really a major improvement.
Three integrals of motion for our model trivially exist:

\[
\hat{N} = \sum_{i=1}^{L} a_i^\dagger a_i, \quad \hat{K} = \sum_{i=1}^{L} c_i^\dagger c_i \quad \text{and} \quad \hat{\Gamma} = o^\dagger o + p^\dagger p.
\]  

This can be easily checked since the canonical commutation relations in (2.2) imply that 

\[
[H, \hat{N}] = [H, \hat{\Gamma}] = [H, \hat{K}] = 0.
\]

The fact that \( \hat{N} \) is conserved clearly means that no new share is introduced in the market. Of course, also the total amount of money must be a constant of motion since the cash is assumed to be used only to buy shares. Since also \( \hat{\Gamma} \) commutes with \( H \), moreover, if the mean value of \( o^\dagger o \) increases with time then necessarily the mean value of the price operator \( \hat{P} = p^\dagger p \) must decrease and vice-versa. This is exactly the mechanism assumed in point 6. at the beginning of this section. Moreover, also the following operators commute with \( H \) and, as a consequence, are constant in time:

\[
\hat{Q}_j = a_j^\dagger a_j + \frac{1}{M} c_j^\dagger c_j,
\]  

for \( j = 1, 2, \ldots, L \).

The hamiltonian (2.5) contains a contribution, \( h_{po} = o^\dagger o + p^\dagger p + (o^\dagger p + p^\dagger o) \), which is decoupled from the other terms. For this reason it is easy to deduce the time dependence of both the price and the supply operators, as well as of their mean values. We get, [1],

\[
\begin{align*}
P_r(t) &= \frac{1}{2}\{P_r + O + (P_r - O) \cos(2t)\} \\
o(t) &= \frac{1}{2}\{P_r + O - (P_r - O) \cos(2t)\},
\end{align*}
\]

where we have called \( O(t) = \omega_{\{n\};\{k\};O;M}(\hat{o}(t)\hat{o}(t)) \) and \( P_r(t) = \omega_{\{n\};\{k\};O;M}(\hat{o}(t)\hat{o}(t)) \). Recall that \( P_r = P_r(0) = M \). Equations (2.8) show that, if \( O = P_r \) then \( O(t) = P_r(t) = O \) for all \( t \) while, if \( O \simeq P_r \) then \( O(t) \) and \( P_r(t) \) are almost constant. In the following we will replace \( P_r(t) \) with an integer value, the value \( M \) which appears in the hamiltonian (2.5), which is therefore fixed after the solution (2.8) is found. This value is obtained by taking a suitable mean of \( P_r(t) \) or working in one of the following assumptions: (i) \( O = P_r \); or (ii) \( O \simeq P_r \) or yet (iii) \( |O + P_r| \gg |P_r - O| \). In these last two situations we may replace \( P_r(t) \), with a temporal mean, \( < P_r(t) > \), since there is not much difference between these two quantities. However, in this way we are essentially removing the dynamics of the
price from the model: no price variation occurs within this model after the replacement $P_r(t) \to M$! How already anticipated, this restriction will be removed in the next section.

Our main result in [1] was to deduct the time evolution of the portfolio operator, which we have defined as

$$\hat{\Pi}_j(t) = \gamma \hat{n}_j(t) + \hat{k}_j(t). \quad (2.9)$$

Here we have introduced the value of the share $\gamma$ as decided by the market, which does not necessarily coincides with the amount of money which is payed to buy the share. As it is clear, $\hat{\Pi}_j(t)$ is the sum of the complete value of the shares, plus the cash. The fact that for each $j$ the operator $Q_j$ is an integral of motion allows us to rewrite the operator $\hat{\Pi}_j(t)$ only in terms of $\hat{n}_j(t)$ and of the initial conditions. We find:

$$\hat{\Pi}_j(t) = \hat{\Pi}_j(0) + (\gamma - M)(\hat{n}_j(t) - \hat{n}_j(0)), \quad (2.10)$$

In order to get the time behavior of the portfolio, therefore, it is enough to obtain $\hat{n}_j(t)$. We refer to [1] for a simple perturbative expansion of $\Pi_j(t)$ for $L = 2$. Here we prefer to show the other results, also contained in [1], concerning the semiclassical thermodynamical limit of the model, i.e. a suitable limit for $L \to \infty$.

Our model is defined by the same hamiltonian as in (2.5) but with $M = 1$. This is not a major requirement here since it corresponds to a renormalization of the price of the share, which we take equal to 1: if you buy a share, then your liquidity decreases of one unit while it increases, again of one unit, if you sell a share. Needless to say, this is strongly related to the fact that the original time-dependent price operator $\hat{P}(t)$ has been replaced by a certain weak mean value, $M$.

It is clear that all the same integrals of motion as before exist: $\hat{N}, \hat{K}, \hat{\Gamma}, \hat{\Delta} := \gamma - p$ and $Q_j = \hat{n}_j + \hat{k}_j, \ j = 1, 2, \ldots, L$. They all commute with $H$, which we now write as

$$\begin{cases} H = h + h_{po}, \text{ where} \\ h = \sum_{i=1}^{L} \alpha_i \hat{n}_i + \sum_{i=1}^{L} \beta_i \hat{k}_i + \sum_{i,j=1}^{L} p_{ij} \left( a_i^\dagger a_j c_i c_j^\dagger + a_i^\dagger a_j^\dagger c_i c_j \right) \\ h_{po} = o^\dagger o + p^\dagger p + (o^\dagger p + p^\dagger o) \end{cases} \quad (2.11)$$

For $h_{po}$ we can repeat the same argument as above and an explicit solution can be found which is completely independent of $h$. In particular we have $\omega_{\{n\};\{k\};O;M}(\hat{P}) = 1$. For this
reason, from now on, we will identify $H$ only with $h$ in (2.11) and we will work only with this hamiltonian. Let us introduce the operators

$$X_i = a_i c_i^\dagger,$$

(2.12)

$i = 1, 2, \ldots, L$. The hamiltonian $h$ can be rewritten as

$$h = \sum_{i=1}^L \left( \alpha_i \hat{n}_i + \beta_i \hat{k}_i \right) + \sum_{i,j=1}^L p_{ij} \left( X_i \dagger X_j + X_j \dagger X_i \right).$$

(2.13)

The following commutation relations hold:

$$[X_i, X_j^\dagger] = \delta_{ij} (\hat{k}_i - \hat{n}_i), \quad [X_i, \hat{n}_j] = \delta_{ij} X_i \quad [X_i, \hat{k}_j] = -\delta_{ij} X_i,$$

(2.14)

which show how the operators $\{X_i, X_i^\dagger, \hat{n}_i, \hat{k}_i\}, i = 1, 2, \ldots, L$ are closed under commutation relations. This is quite important, since, introducing the operators $X_i^{(L)} = \sum_{i=1}^L p_{ii} X_i, l = 1, 2, \ldots, L$, we get the following system of differential equations, see [1]:

$$\begin{cases}
\dot{X}_l = i(\beta_l - \alpha_l) X_l + 2i X_l^{(L)} (2\hat{n}_l - Q_l) \\
\dot{\hat{n}}_l = 2i \left( X_l X_l^{(L)} \dagger - X_l^{(L)} X_l \dagger \right)
\end{cases}$$

(2.15)

This system, as $l$ takes all the values $1, 2, \ldots, L$, is a closed system of differential equations for which an unique solution surely exists. Indeed, we have found such a solution in [1] by introducing the so-called mean-field approximation which essentially consists in replacing $p_{ij}$ with $\tilde{p}_{L}$, with $\tilde{p} \geq 0$. After this replacement we have that

$$X_i^{(L)} = \sum_{i=1}^L p_{ii} X_i \rightarrow \frac{\tilde{p}}{L} \sum_{i=1}^L X_i,$$

whose limit, for $L$ diverging, only exists in suitable topologies, [7, 8], like, for instance, the strong one restricted to a set of relevant states. Let $\tau$ be such a topology. We define

$$X^\infty = \tau - \lim_{L \rightarrow \infty} \frac{\tilde{p}}{L} \sum_{i=1}^L X_i,$$

(2.16)

where, as it is clear, the dependence on the index $l$ is lost because of the replacement $p_{ii} \rightarrow \frac{\tilde{p}}{L}$. This is a typical behavior of transactionally invariant quantum systems, where
\( p_{l,i} = p_{l,-i} \). The operator \( X^\infty \) commutes with all the elements of \( \mathfrak{A} \), the algebra of the observables of our stock market: \( [X^\infty, A] = 0 \) for all \( A \in \mathfrak{A} \). In this limit system (2.15) above becomes

\[
\begin{align*}
\dot{X}_l &= i(\beta_l - \alpha_l)X_l + 2iX^\infty(2\hat{n}_l - Q_l) \\
\dot{\hat{n}}_l &= 2i \left( X_l X^\infty\dagger - X^\infty X_l\dagger \right)
\end{align*}
\]  

(2.17)

This system has been solved in [1] under the hypothesis that

\[ \beta_l - \alpha_l =: \Phi \neq \nu \]  

(2.18)

for all \( l = 1, 2, \ldots, L \) (but also in other and more general situations). We again refer to [1] for the details. Here we just write the final result, which is

\[
n_l(t) = \frac{1}{\omega^2} \left\{ n_l(\Phi - \nu)^2 - 8|X^\infty_0|^2 (k_l(\cos(\omega t) - 1) - n_l(\cos(\omega t) + 1)) \right\},
\]  

(2.19)

where we have introduced \( \omega = \sqrt{(\Phi - \nu)^2 + 16|X^\infty_0|^2} \). This allows also to find the time evolution for the portfolio, since \( \Pi_l(t) = \Pi_l(0) + (\gamma - 1)(n_l(t) - n_l(0)) \). Again, we refer to [1] for further comments and results. Here we just want to stress that our point of view has really produced, as an output, the time evolution of the portfolio of each trader of the market, which was indeed our original aim.

### III A two trader model

As we have already discussed the model analyzed in the previous section has a very strong limitation: the time evolution of the price of the share, even if formally appears in the hamiltonian of the system, is frozen in order to get a well defined energy operator (i.e. in moving from \( \tilde{H} \) to \( H \)). Therefore, and in particular when we consider the thermodynamical limit of the model, such a dynamical behavior of the price operator completely disappears!

In this section we will cure this anomaly, and for that we will discuss in many details a model based essentially on the same assumptions listed at the begin of Section II but in which the market consists of only two traders, \( \tau_1 \) and \( \tau_2 \). Of course, more than a realistic stock market, this can be seen as a sort of two-components physical system \( (\tau_1 + \tau_2) \).
changing two different kind of particles (the shares and the money) and subjected to an
external control (the price of the share and the supply of the system itself). However,
even in view of the generalization which we will discuss in the next section, we will still
refer to this physical system as a (toy model of a) stock market.

The hamiltonian looks very much as the one in (2.1):

\[
H = H_0 + H_I,
\]

where

\[
H_0 = \sum_{l=1}^{2} \alpha_l a_l^\dagger a_l + \sum_{l=1}^{L} \beta_l c_l^\dagger c_l + o^\dagger o + p^\dagger p
\]

\[
H_I = \left( a_1^\dagger a_2 c_1^\dagger c_2^\dagger p + a_1 a_2^\dagger c_1^\dagger c_2 p \right) + (o^\dagger p + p^\dagger o),
\]

with the standard commutation relations

\[
[o, o^\dagger] = [p, p^\dagger] = 1, \quad [a_i, a_j^\dagger] = [a_i, a_j] = \delta_{i,j},
\]

while all the other commutators are zero.

The states of the system are defined as in (2.3) and (2.4) with

\[
\varphi_{n_1,n_2;k_1,k_2;O,M} := \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}(c_1^\dagger)^{k_1}(c_2^\dagger)^{k_2}(o^\dagger)^{O}(p^\dagger)^{M}}{\sqrt{n_1! n_2! k_1! k_2! O! M!}} \varphi_0.
\]

(3.3)
Here \( n_j, k_j, O \) and \( M \) are non negative integers, \( \varphi_0 \) is the *vacuum* of the model, \( a_j \varphi_0 = c_j \varphi_0 = p \varphi_0 = o \varphi_0 = 0 \), for \( j = 1, 2 \), and \( \mathcal{H} \) is the closure of the linear span of all these vectors. Then we have, for instance, \( a_1 \varphi_{n_1,n_2;k_1,k_2;O;M} = \sqrt{n_1} \varphi_{n_1-1,n_2;k_1,k_2;O;M} \) if \( n_1 > 0 \) and \( a_1 \varphi_{n_1,n_2;k_1,k_2;O;M} = 0 \) if \( n_1 = 0 \), \( a_1^\dagger \varphi_{n_1,n_2;k_1,k_2;O;M} = \sqrt{n_1 + 1} \varphi_{n_1+1,n_2;k_1,k_2;O;M} \).

Analogous expressions for the action of \( a_2 \), \( a_2^\dagger \), \( c_j \), \( c_j^\dagger \), \( o \), \( o^\dagger \), \( p \) and \( p^\dagger \) on \( \varphi_{n_1,n_2;k_1,k_2;O;M} \) can also be recovered, see [6]. Moreover we have, see Appendix A,

\[
c_1^\dagger \varphi_{n_1,n_2;k_1,k_2;O;M} := \begin{cases} 
\varphi_{n_1,n_2;k_1,k_2;O;M}, & \text{if } M = 0, \forall k_1 \geq 0; \\
0, & \text{if } M > k_1, \forall k_1 \geq 0; \\
\sqrt{k_1^{(-M)}} \varphi_{n_1,n_2;k_1-M,k_2;O;M}, & \text{if } k_1 \geq M > 0
\end{cases}
\]  
(3.4)

and

\[
c_1^\dagger \varphi_{n_1,n_2;k_1,k_2;O;M} := \begin{cases} 
\varphi_{n_1,n_2;k_1,k_2;O;M}, & \text{if } M = 0, \forall k_1 \geq 0; \\
\sqrt{k_1^{(+M)}} \varphi_{n_1,n_2;k_1+M,k_2;O;M}, & \text{if } M > 0,
\end{cases}
\]  
(3.5)

where we have defined

\[
\begin{align*}
k_1^{(-M)} &:= k_1(k_1 - 1) \cdots (k_1 - M + 1) \\
k_1^{(+M)} &:= (k_1 + 1)(k_1 + 2) \cdots (k_1 + M)
\end{align*}
\]  
(3.6)

Analogous formulas hold for \( c_2^\dagger \) and \( c_2^\dagger \). These definitions have a clear *economical* interpretation: acting with \( c_1^\dagger \) on \( \varphi_{n_1,n_2;k_1,k_2;O;M} \) returns \( \varphi_{n_1,n_2;k_1,k_2;O;M} \) itself when \( M = 0 \) since, in this case, the action of \( c_1^\dagger \) coincides with that of the identity operator: the price of the share is zero so you don’t need to pay for it and hence your cash does not change! Moreover, if \( M > k_1 \), \( c_1^\dagger \) destroys more *quanta of money* than \( \tau_1 \) really possesses. Therefore, the result of its action on the vector is zero. A similar problem does not occur when we consider the action of \( c_1^\dagger \) on \( \varphi_{n_1,n_2;k_1,k_2;O;M} \), since in this case the cash is created!

In the rest of the section, however, these formulas will be significantly simplified by assuming that, as it is reasonable, during the transactions between \( \tau_1 \) and \( \tau_2 \) the price of the share never reach the zero value and, moreover, that no trader try to buy a share if he has not enough money to pay for it. Therefore we simply rewrite (3.4) and (3.5) as

\[
\begin{align*}
c_1^\dagger \varphi_{n_1,n_2;k_1,k_2;O;M} &= \sqrt{k_1^{(-M)}} \varphi_{n_1,n_2;k_1-M,k_2;O;M} \\
c_1^\dagger \varphi_{n_1,n_2;k_1,k_2;O;M} &= \sqrt{k_1^{(+M)}} \varphi_{n_1,n_2;k_1+M,k_2;O;M}
\end{align*}
\]  
(3.7)
The commutation rules are the standard ones, see (2.2), plus the ones which extend the rules in (A.4):

\[
[\hat{P}, c_j \hat{P}] = [\hat{P}, c_j^\dagger \hat{P}] = 0,
\]

and

\[
[\hat{k}_j, c_j \hat{P}] = -\delta_{j,l} \hat{P} c_j \hat{P}, \quad [\hat{k}_j, c_j^\dagger \hat{P}] = \delta_{j,l} \hat{P} c_j^\dagger \hat{P}
\]

for \( j = 1, 2 \).

Since our market is closed it is not surprising that the total number of shares and the total amount of cash are preserved. This is indeed proved simply computing the commutators of the total number of shares and the total cash operators, \( \hat{N} = \hat{n}_1 + \hat{n}_2 \) and \( \hat{K} = \hat{k}_1 + \hat{k}_2 \), with the hamiltonian \( H \). Indeed one can check that \([H, \hat{K}] = [H, \hat{N}] = 0\). Moreover, we can also check that \( \hat{\Gamma} := \hat{\Omega} + \hat{P} \) commutes with the hamiltonian. This is, as already discussed in the previous section, the mechanism which fixes the price of the share within our simplified market: the more the market supply increases the less is the value of the share, i.e. its price.

As already stressed before, one big difference between the model we are considering here and the one considered in Section II and in [1] is that now the price operator is not replaced by its mean value. This has an important consequence: the operators extending \( Q_j \) in (2.7) for this model, which are proportional to \( \hat{P} a_j^\dagger a_j + c_j^\dagger c_j \), \( j = 1, 2 \), are no longer constants of motion, and they cannot be used to facilitate the computation of the portfolios of \( \tau_1 \) and \( \tau_2 \). Nevertheless we will still be able to deduce, with an easy perturbative approach, the time behavior of both portfolios at least for small values of \( t \).

The first step consists in deducing the time evolution of the price of the share. This computation is completely analogous to that of [1] and will not be repeated here. Again, we can deduce that \( \hat{\Delta} := o - p \) is another constant of motion and we find that, see (2.5),

\[
\begin{align*}
P(t) &= \omega_{n_1,n_2; k_1,k_2; O; M}(p^I(t)p(t)) = \frac{1}{2}\{M + O + (M - O) \cos(2t)\} \\
O(t) &= \omega_{n_1,n_2; k_1,k_2; O; M}(o^I(t)o(t)) = \frac{1}{2}\{M + O - (M - O) \cos(2t)\}
\end{align*}
\]

In [1] the absence of a true dynamics for \( \hat{P} \) suggested to define the portfolio of the \( j \)-th trader by introducing another parameter, \( \gamma \), which was interpreted as the price of the share as decided by the market, which does not necessarily coincides with \( M \). However, there was no direct link between \( \gamma \) and \( M \) in [1], and this is not completely satisfying, of
course! Here we have no need for introducing such an extra parameter since we are now in a position to consider directly \( P(t) \) instead of its mean value \( M \). Therefore we replace formula (2.9) by defining the portfolio of the trader \( \tau_j \) as

\[
\hat{\Pi}_j(t) = \hat{P}(t) \hat{n}_j(t) + \hat{k}_j(t)
\]  

with \( j = 1, 2 \), which is just the sum of the total price of the shares and the cash of \( \tau_j \).

Of course, due to the fact that \( P(t) \) is known, \( \hat{\Pi}_1(t) \) is known when we both know \( \hat{n}_1(t) \) and \( \hat{k}_1(t) \). Moreover, if we now \( \hat{n}_1(t) \) and \( \hat{k}_1(t) \), then we also know \( \hat{n}_2(t) \) and \( \hat{k}_2(t) \) since their sum must be constant, so that we can also find the analytic form of \( \hat{\Pi}_2(t) \). However, this is not the only way to find \( \hat{\Pi}_1(t) \). Another possibility follows from the fact that, as it is easy to check,

\[
\dot{\hat{\Pi}}_1(t) = \dot{\hat{P}}(t) \hat{n}_1(t),
\]  

which shows again, even without any need of using \( Q_j \) as in the previous section, that it is enough to know \( \hat{n}_1(t) \) to find the time evolution of the portfolio of \( \tau_1 \).

However, even for this two-traders model, it is not easy to deduce the exact expression for \( \hat{\Pi}_1(t) \). Nevertheless, a lot of information can be deduced, mainly for short time behavior, using different perturbative strategies. Here we just consider the most direct technique, i.e. the following perturbative expansion

\[
\hat{\Pi}_1(t) = e^{iHt}\hat{\Pi}_1(0)e^{-iHt} = \hat{\Pi}_1(0) + it[H, \hat{\Pi}_1(0)] + \frac{(it)^2}{2!}[H, [H, \hat{\Pi}_1(0)]] + \cdots,
\]  

leaving to the next section a more detailed analysis of other strategies to produce \( \hat{\Pi}_j(t) \).

The computation of the various terms of this expansion, and of their mean values on the state \( \omega_{n_1, n_2; k_1, k_2; O; M}(\cdot) \), is based on the commutation rules we have seen before and produce, up to the second order in \( t \), the following result:

\[
\Pi_1(t) = \omega_{n_1, n_2; k_1, k_2; O; M}(\hat{\Pi}_1(t)) = \Pi_1(0) + t^2 n_1(O - M),
\]  

which shows that, for sufficiently small values of \( t \), the value of \( \Pi_1(t) \) increases with time if \( O > M \), i.e. if at \( t = 0 \) and in our units the supply of the market is larger than the price of the share. It is further possible to check that the next term in the expansion above is proportional to \( t^4 A_{n_1, n_2; k_1, k_2; M} \) where

\[
A_{n_1, n_2; k_1, k_2; M} = n_1 k_1^{(+M)} k_2^{(-M)} - n_2 k_1^{(-M)} k_2^{(+M)} + n_1 n_2 (k_1^{(+M)} k_2^{(-M)} - k_1^{(-M)} k_2^{(+M)}).
\]
We avoid the details of this computation here since they are not very interesting, mainly because this is just a toy model which is more important for its general structure than for a real financial interpretation. Here we just want to stress that the expansion in (3.13) gives, in principle, the expression of $\hat{\Pi}_1(t)$ at any desired approximation.

IV Many traders

In the previous section we have learned how to define the operators $c^\hat{P}$ and its adjoint and we have used this definition in the analysis of a simple hamiltonian which was essentially already introduced in [1]. We devote this section to a more realistic model, where the stock market is made of $N$ different traders with $N$ arbitrarily large.

In our approach we will focus our attention on a single trader, $\tau$, which interact with an ensemble of other traders in a way that extends the interaction introduced in (3.1). In other words we divide the stock market, which as before is defined in terms of the number of a single type of shares, the cash, the price of the shares and the supply of the market, in two main ingredients: we call system, $\mathcal{S}$, all the dynamical quantities which refer to a fixed trader $\tau$: its shares number operators, $a$, $a^\dagger$ and $\hat{n} = a^\dagger a$, the cash operators of $\tau$, $c$, $c^\dagger$ and $\hat{k} = c^\dagger c$ as well as the price operators of the shares, $p$, $p^\dagger$ and $\hat{P} = p^\dagger p$. On the other hand, we associate to the reservoir, $\mathcal{R}$, all the other quantities, that is first of all, the shares number operators, $A_k$, $A_k^\dagger$ and $\hat{N}_k = A_k^\dagger A_k$ and the cash operators, $C_k$, $C_k^\dagger$ and $\hat{K}_k = C_k^\dagger C_k$ of the other traders. Here $k \in \Lambda$ and $\Lambda$ is a subset of $\mathbb{N}$ which labels the traders of the market (other than $\tau$). It is clear that the cardinality of $\Lambda$ is $N - 1$.

Moreover we associate to the reservoir also the supply of the market, which is described by the following operators $o_k$, $o_k^\dagger$ and $\hat{O}_k = o_k^\dagger o_k$, $k \in \Lambda$. The stock market is given by the union of $\mathcal{S}$ and $\mathcal{R}$, and the hamiltonian, which extends the one in (3.1), is assumed here to be

$$H = H_0 + \lambda H_I,$$

where

$$H_0 = \omega_a \hat{n} + \omega_c \hat{k} + \omega_p \hat{P} + \sum_{k \in \Lambda} \left( \Omega_A(k) \hat{N}_k + \Omega_C(k) \hat{K}_k + \Omega_O(k) \hat{O}_k \right)$$

$$H_I = \left( z^\dagger Z(f) + z Z^\dagger(\overline{f}) \right) + (p^\dagger o(g) + p o^\dagger(\overline{g}))$$

(4.1)

Here $\omega_a$, $\omega_c$ and $\omega_p$ are positive real numbers and $\Omega_A(k)$, $\Omega_C(k)$ and $\Omega_O(k)$ are real valued non negative functions, whose interpretation was first discussed in [1]: they describe the
free time evolution of the different operators of the market. We have also introduced the following *smeared fields* of the reservoir:

\[
\begin{align*}
Z(f) &= \sum_{k \in \Lambda} Z_k f(k) = \sum_{k \in \Lambda} A_k C_k^\dagger \hat{P} f(k), \\
Z^\dagger(f) &= \sum_{k \in \Lambda} Z_k^\dagger \overline{f}(k) = \sum_{k \in \Lambda} A_k^\dagger \hat{P} \overline{f}(k) \\
o(g) &= \sum_{k \in \Lambda} o_k g(k) \\
o^\dagger(g) &= \sum_{k \in \Lambda} o_k^\dagger \overline{g}(k),
\end{align*}
\]

(4.2)

as well as the operators \( z = a c^\dagger \hat{P} \), \( Z_k = A_k C_k^\dagger \hat{P} \) and their conjugates, since for instance \( A_k \) and \( C_k \) appear always in this combination both in \( H_I \) and in all the computations we will perform in the following. This is natural because of the *physical* meaning of, e.g., \( z \): the action of \( z \) on a fixed vector number destroys a share in the portfolio of \( \tau \) and, at the same time, creates as many monetary units as \( \hat{P} \) prescribes! Of course, in \( H_I \) such an operator is associated to \( Z^\dagger(f) \) which acts exactly in the opposite way on the traders of the reservoir: one share is created in the cumulative portfolio of \( \mathcal{R} \) while \( \hat{P} \) quanta of money are destroyed, since they are used to pay for the share. The following non trivial commutation rules are assumed:

\[
[c, c^\dagger] = [p, p^\dagger] = [a, a^\dagger] = 1, \quad [o_i, o_j^\dagger] = [A_i, A_j^\dagger] = [C_i, C_j^\dagger] = \delta_{i,j} 1 \quad (4.3)
\]

which implies

\[
[\hat{K}_k, C_q^\dagger \hat{P}] = -\hat{P} C_q^\dagger \delta_{k,q}, \quad [\hat{K}_k, C_q^\dagger \hat{P}] = \hat{P} C_q^\dagger \delta_{k,q} \quad (4.4)
\]

Finally, the functions \( f(k) \) and \( g(k) \) in (4.1) and (4.2) are sufficiently regular to allows for the sums in (4.2) to be well defined, as well as the quantities which will be defined below, see (4.10).

**Remark:**– Of course, since \( \tau \) can be chosen arbitrarily, the asymmetry of the model is just apparent. In fact, changing \( \tau \), we will be able, in principle, to find the time evolution of the portfolio of each trader of the stock market.

The interpretation suggested above concerning \( z \) and \( Z(f) \) are also based on the following results: let

\[
\hat{N} := \hat{n} + \sum_{k \in \Lambda} \hat{N}_k, \quad \hat{K} := \hat{k} + \sum_{k \in \Lambda} \hat{K}_k, \quad \hat{\Gamma} := \hat{P} + \sum_{k \in \Lambda} \hat{O}_k \quad (4.5)
\]
Of course $\hat{N}$ is associated to the total number of shares in our closed market and, therefore, is called the *total number operator*. $\hat{K}$ is the total amount of money present in the market and is called the *total cash operator*. $\hat{\Gamma}$ has not a direct interpretation so far, since is just the sum of the price and the *total supply* operators, $\hat{O} = \sum_{k \in \Lambda} \hat{O}_k$. It may be worth recalling that the supply operators are only related to the reservoir $\mathcal{R}$, because of our initial choice. This is the reason why there is no contribution to the operator $\hat{O}$ coming from $\mathcal{S}$.

**Proposition 1** The operators $\hat{N}$, $\hat{K}$ and $\hat{\Gamma}$ are constants of motion.

The proof of this proposition is a simple exercise based on the commutation rules above. Indeed, it is not hard to check that $H$ commutes with $\hat{N}$, $\hat{K}$ and with $\hat{\Gamma}$. This proves that our main motivation for introducing the hamiltonian in (4.1) is correct: with this choice we are constructing a closed market in which the total amount of money and the total number of shares are preserved and in which, if the total supply increases, then the price of the share must decrease in order for $\hat{\Gamma}$ to stay constant. Of course, it would be interesting to relate the changes of $\hat{O}$ to other (maybe external) conditions, but this will problem will be considered elsewhere: here we just consider the simplified point of view for which $\hat{O}$ may change in time, but we don’t analyze the reason why this happens.

The next step of our analysis should be to recover the equations of motion for the portfolio of the trader $\tau$, defined in analogy with (3.11) as

$$\hat{\Pi}(t) = \hat{P}(t) \hat{n}(t) + \hat{k}(t).$$

(4.6)

It is not surprising that this cannot be done exactly so that some perturbative technique is needed. We will consider in the following sub-sections two orthogonal approaches, orthogonal in the sense that they give different information under different conditions which, together, help in a better understanding of the model. In particular we will first consider the so-called *stochastic limit* of the system: this approximation will produce the explicit form of the generator of the semigroup arising from the hamiltonian (4.1), and this will give some interesting condition for the stationarity of the model, i.e. for $\hat{\Pi}(t)$ to be constant in time. We will see that this is possible under certain conditions on the parameters defining the model. The second approach will make use of a sort of FPL
approximation which will produce a system of differential equation for the mean value of $\hat{\Pi}(t)$ whose solution can be explicitly found.

IV.1 the stochastic limit of the model

The stochastic limit of a quantum system is a perturbative strategy widely discussed in [9] and which proved to be quite useful in the analysis of several quantum mechanical systems, see [10] for a recent review of some applications of this procedure to many-body systems.

Here we adopt this procedure pragmatically, i.e. without discussing any detail, while, to keep the paper self-contained, we postpone to Appendix B the list of some basic facts of this approach.

The first step consists in obtaining the free time evolution of the interaction hamiltonian which we still call, with a small abuse of language, $H_I(t)$. Due to the commutation rules (4.3) and (4.4) we find that

$$H_I(t) := e^{i\hat{H}_0 t} H_I e^{-i\hat{H} t} = z^\dagger Z(f e^{it\varepsilon_Z}) + z Z^\dagger(\bar{f} e^{-it\varepsilon_Z}) + p^\dagger o(g e^{it\varepsilon_0}) + p o^\dagger(\bar{g} e^{-it\varepsilon_0}), \quad (4.7)$$

where we have defined

$$\varepsilon_Z(k) := \hat{P}(\Omega_C(k) - \omega_c) - (\Omega_A(k) - \omega_a), \quad \varepsilon_0(k) := \omega_p - \Omega_O(k) \quad (4.8)$$

and, for instance, $Z(f e^{it\varepsilon_Z}) = \sum_{k \in \Lambda} f(k) e^{it\varepsilon_Z(k)} Z_k$.

The next step consists in computing first $\omega(H_I(t)H_I(t))$, then

$$I_\lambda(t) = \left(-\frac{i}{\lambda}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \omega(H_I \left(\frac{t_1}{\lambda^2}\right) H_I \left(\frac{t_2}{\lambda^2}\right)),$$

and finally the limit of $I_\lambda(t)$ for $\lambda \to 0$. Here $\omega$ is a state of the market, which we take as a product state $\omega = \omega_{\text{sys}} \otimes \omega_{\text{res}}$ with $\omega_{\text{sys}}$ a gaussian state, that is it satisfies $\omega_{\text{sys}}(a^2) = \omega_{\text{sys}}(c^2) = \omega_{\text{sys}}(p^2) = 0$ and $\omega_{\text{sys}}(a a) = \omega_{\text{sys}}(c c) = \omega_{\text{sys}}(a^\dagger a^\dagger) = \omega_{\text{sys}}(p p) = \omega_{\text{sys}}(p^\dagger p^\dagger) = 0$. Here $a^\sharp$ can be $a$ or $a^\dagger$ and the same notation is adopted for $c^\sharp$ and $p^\sharp$. These conditions are obviously satisfied if $\omega_{\text{sys}}$ is a vector state analogous to that in (2.3). We don’t give here the details of the computation, which is rather straightforward, but only the final result which is obtained under the assumptions that the two functions
\( \varepsilon_Z(k) := \omega(\hat{\varepsilon}_Z(k)) \) and \( \varepsilon_O(k) \) are not identically zero. Moreover, it is convenient to assume that
\[
\varepsilon_Y(k) = \varepsilon_Y(q) \iff k = q,
\]
where \( Y = Z, O \). Then, if we define the following complex constants
\[
\begin{align*}
\Gamma^{(a)}_Z &= \sum_{k \in \Lambda} |f(k)|^2 \omega_{\text{res}}(Z_k \hat{Z}_k) \int_{-\infty}^{0} d\tau e^{-i\varepsilon_Z(k)\tau} \\
\Gamma^{(b)}_Z &= \sum_{k \in \Lambda} |f(k)|^2 \omega_{\text{res}}(Z_k^\dagger Z_k) \int_{-\infty}^{0} d\tau e^{i\varepsilon_Z(k)\tau} \\
\Gamma^{(a)}_O &= \sum_{k \in \Lambda} |g(k)|^2 \omega_{\text{res}}(o_k o_k^\dagger) \int_{0}^0 d\tau e^{-i\varepsilon_O(k)\tau} \\
\Gamma^{(b)}_O &= \sum_{k \in \Lambda} |g(k)|^2 \omega_{\text{res}}(o_k^\dagger o_k) \int_{0}^0 d\tau e^{i\varepsilon_O(k)\tau}
\end{align*}
\]
which surely exist if \( f(k) \) and \( g(k) \) are regular enough, we get
\[
I(t) = -t \left\{ \omega_{\text{sys}}(z^\dagger z) \Gamma^{(a)}_Z + \omega_{\text{sys}}(z z^\dagger) \Gamma^{(b)}_Z + \omega_{\text{sys}}(p^\dagger p) \Gamma^{(a)}_O + \omega_{\text{sys}}(p p^\dagger) \Gamma^{(b)}_O \right\}
\]
Next we need to find the expression of a self-adjoint, time dependent operator \( H^{(ts)}(t) \),
the so-called *stochastic limit* hamiltonian, which reproduces this result in a sense that we
will specify in a moment.

Let us take
\[
H^{(ts)}(t) = z^\dagger \left( Z^{(a)}(t) + Z^{(b)}(t)^\dagger \right) + z \left( Z^{(a)}(t)^\dagger + Z^{(b)}(t) \right) + \\
+ p^\dagger \left( o^{(a)}(t) + o^{(b)}(t)^\dagger \right) + p \left( o^{(a)}(t)^\dagger + o^{(b)}(t) \right)
\]
where the new operators introduced here are assumed to satisfy the following commutation rules:
\[
\left[ Z^{(a)}(t), Z^{(a)}(t)^\dagger (t') \right] = \Gamma^{(a)}_Z \delta_+(t - t'), \quad \left[ Z^{(b)}(t), Z^{(b)}(t)^\dagger (t') \right] = \Gamma^{(b)}_Z \delta_+(t - t'),
\]
and
\[
\left[ o^{(a)}(t), o^{(a)}(t)^\dagger (t') \right] = \Gamma^{(a)}_O \delta_+(t - t'), \quad \left[ o^{(b)}(t), o^{(b)}(t)^\dagger (t') \right] = \Gamma^{(b)}_O \delta_+(t - t'),
\]
if \( t \geq t' \). The time ordering is crucial here and \( \delta_+ \) is essentially the Dirac delta functions
but for a normalization which arises because of the time ordering we consider here, \[\Box\].
The only property of \( \delta_+ \) which we will need is the following:
\[
\int_{0}^{t} \delta_+(t - \tau) h(\tau) d\tau = h(t).
\]
Now, let $\Psi_0$ be the vacuum of the operators $Z^{(a)}(t)$, $Z^{(b)}(t)$, $o^{(a)}(t)$ and $o^{(b)}(t)$. This means that $Z^{(a)}(t)\Psi_0 = Z^{(b)}(t)\Psi_0 = o^{(a)}(t)\Psi_0 = o^{(b)}(t)\Psi_0 = 0$ for all $t \geq 0$. Then, if we consider $\Omega(\cdot) = \omega_{sys}(\cdot) \otimes <\Psi_0, \cdot \Psi_0>$ and we compute

$$J(t) = (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \Omega \left( H^{(ls)}(t_1) H^{(ls)}(t_2) \right),$$

we conclude that $J(t) = I(t)$. This means that, at a first order, $H^{(ls)}(t)$ allows us to get the same wave operator $U_t$ which describes the time evolution of the systems. We use $H^{(ls)}(t)$ to construct the wave operator as $U_t = 1 - i \int_0^t H^{(ls)}(t') U_t'$, and then to deduce the following commutation rules:

$$[Z^{(a)}(t), U_t] = -i \Gamma^{(a)}_Z z U_t, \quad [Z^{(b)}(t), U_t] = -i \Gamma^{(b)}_Z z^\dagger U_t \quad (4.14)$$

and

$$[o^{(a)}(t), U_t] = -i \Gamma^{(a)}_O p U_t, \quad [o^{(b)}(t), U_t] = -i \Gamma^{(b)}_O p^\dagger U_t \quad (4.15)$$

by making use of the time consecutive principle. [9]

We are now ready to get the expression of the generator. Let $X$ be a generic observable of the system, that is, in our present context, some dynamical variable related to the trader $\tau$. Let $1_\tau$ be the identity operator of the reservoir. Then the time evolution of $X \otimes 1_\tau$ in the interaction picture is given by $j_t(X \otimes 1_\tau) = U_t^\dagger (X \otimes 1_\tau) U_t$, so that

$$\partial_t j_t(X \otimes 1_\tau) = i U_t^\dagger [H^{(ls)}(t), X \otimes 1_\tau] U_t$$

Using now the commutators in (4.14) and (4.15), and recalling that $\Psi_0$ is annihilated by all the new reservoir operators, we find that

$$\Omega(\partial_t j_t(X \otimes 1_\tau)) = \Omega(U_t^\dagger \{ \Gamma^{(a)}_Z [z^\dagger, X] z - \Gamma^{(a)}_Z z^\dagger [z, X] + \Gamma^{(b)}_Z [z, X] z^\dagger - \Gamma^{(b)}_Z z [z^\dagger, X] + \Gamma^{(a)}_O [p^\dagger, X] p - \Gamma^{(a)}_O p^\dagger [p, X] + \Gamma^{(b)}_O [p, X] p^\dagger - \Gamma^{(b)}_O p [p^\dagger, X] \}) U_t$$

which, together with the equality $\Omega(\partial_t j_t(X \otimes 1_\tau)) = \Omega(j_t(L(X \otimes 1_\tau)))$, gives us the following expression of the generator:

$$L(X \otimes 1_\tau) = \Gamma^{(a)}_Z [z^\dagger, X] z - \Gamma^{(a)}_Z z^\dagger [z, X] + \Gamma^{(b)}_Z [z, X] z^\dagger - \Gamma^{(b)}_Z z [z^\dagger, X] + \ldots$$

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\[ + \Gamma^{(a)}_O [p^\dagger, X] p - \Gamma^{(a)}_O p^\dagger [p, X] + \Gamma^{(b)}_O [p, X] p^\dagger - \Gamma^{(b)}_O p [p^\dagger, X] \]  

(4.16)

Therefore we find, after few computations,

\[ L(\hat{n} \otimes \mathbb{I}_r) = 2 \Re \{ \Gamma^{(b)}_Z \} z z^\dagger - 2 \Re \{ \Gamma^{(a)}_Z \} z^\dagger z, \]  

(4.17)

and

\[ L(\hat{k} \otimes \mathbb{I}_r) = -2 \hat{P} \Re \{ \Gamma^{(b)}_Z \} z z^\dagger + 2 \hat{P} \Re \{ \Gamma^{(a)}_Z \} z^\dagger z, \]  

(4.18)

which in particular shows that \( L(\hat{k} \otimes \mathbb{I}_r) + \hat{P} L(\hat{n} \otimes \mathbb{I}_r) = 0 \). Finally we find, using these results and recalling that \( \hat{\Pi}(t) = \hat{P}(t) \hat{n}(t) + \hat{k}(t) \),

\[ L(\hat{\Pi} \otimes \mathbb{I}_r) = 2 (\Re \{ \Gamma^{(b)}_O \} - \Re \{ \Gamma^{(a)}_O \}) \hat{P} \hat{n} + 2 \Re \{ \Gamma^{(b)}_O \} \hat{n}. \]  

(4.19)

The first remark is that, in the stochastic limit, even if the time dependence of \( \hat{n} \) and \( \hat{k} \) depend on \( \Gamma^{(a)}_Z \) and \( \Gamma^{(b)}_Z \), the time evolution of \( \hat{\Pi} \) in a first approximation do not! In fact, formula (4.19) shows that it only depends on \( \Gamma^{(a)}_O \) and \( \Gamma^{(b)}_O \). However, since the time evolution of \( \hat{n} \) depends on \( \Gamma^{(a,b)}_Z \) because of (4.17) and (4.18), this dependence will necessarily play a role also in \( \hat{\Pi}(t) \).

The above equations show that, even after the stochastic limit has been taken, it is quite difficult to produce a closed set of differential equations. On the contrary it is quite easy to deduce conditions for the stationarity of the market. This is exactly what we will discuss next.

We begin noticing that, for instance, we have

\[ 2 \Re \{ \Gamma^{(a)}_O \} = \sum_{k \in \Lambda} |g(k)|^2 \omega_{\text{res}}(o_k o_k^\dagger) \int_{\mathbb{R}} d\tau e^{-i\tau \varepsilon_O(k)} = 2\pi \sum_{k \in \Lambda} |g(k)|^2 \omega_{\text{res}}(o_k o_k^\dagger) \delta(\varepsilon_O(k)) \]  

(4.20)

and analogously we find that \( \Re \{ \Gamma^{(b)}_O \} = \pi \sum_{k \in \Lambda} |f(k)|^2 \omega_{\text{res}}(Z_k Z_k^\dagger) \delta(\varepsilon_Z(k)) \) and \( \Re \{ \Gamma^{(a)}_Z \} = \pi \sum_{k \in \Lambda} |f(k)|^2 \omega_{\text{res}}(Z_k^\dagger Z_k) \delta(\varepsilon_Z(k)) \).

Therefore, since \([o_k, o_k^\dagger] = \mathbb{I}\) and, as a consequence, \( \omega_{\text{res}}(o_k o_k^\dagger) - \omega_{\text{res}}(o_k^\dagger o_k) = \omega_{\text{res}}(\mathbb{I}) = 1 \), we find that \( \Re \{ \Gamma^{(b)}_O \} - \Re \{ \Gamma^{(a)}_O \} = -\pi \sum_{k \in \Lambda} |g(k)|^2 \delta(\varepsilon_O(k)) \). The conclusion now follows from (4.19): the portfolio of \( \tau \) is stationary (in our approximation) when the function \( \varepsilon_O(k) \) has no zero for \( k \in \Lambda \). Indeed, if this is the case, we deduce that \( L(\hat{\Pi} \otimes \mathbb{I}_r) = 0 \). Since \( \varepsilon_O(k) = \omega_p - \Omega_O(k) \) this means that if the free dynamics of the price and the supply
are based on substantially different quantities then the portfolio of \( \tau \) keeps its original value, even if the operators \( \hat{n}(t) \) and \( \hat{k}(t) \) may separately change with time. This is an interesting result since it can be summarized just stating that, within the approximation we are considering here, the fact that \( \hat{\Pi}(t) \) depends or not on time is only related to a given equilibrium, if any, between the free price hamiltonian, \( \omega_p p^\dagger p \), and the free supply hamiltonian, \( \sum_{k \in \Lambda} \Omega_O(k) c_k^\dagger c_k \): again, the interplay between these two ingredients of the model play an interesting role!

A similar analysis can be carried out also to get conditions for the equilibrium of \( \hat{n}(t) \) and \( \hat{k}(t) \). Because of (4.17) and (4.18), and because of the known time evolution of \( \hat{P}(t) \), \( \dot{\hat{n}}(t) \) is constant if and only if \( \dot{\hat{k}}(t) \) is constant, and for this to be true the function \( \varepsilon_Z(k) \) must be different from zero for each \( k \in \Lambda \). On the other hand, if at least one zero of \( \varepsilon_Z(k) \) exists in \( \Lambda \), then a non-stationary condition for \( \dot{\hat{n}}(t) \) and \( \dot{\hat{k}}(t) \) is possible.

IV.2 a different approximation

The approach we have discussed so far produced some interesting information about the stationarity of the portfolio of \( \tau \) but no concrete insight about its time evolution. In other words, if we try to deduce the time behavior of \( \hat{\Pi}(t) \) we get no significant simplification if we adopt the form of the generator in (4.16) or if we look directly to the Heisemberg expression for \( \hat{\Pi}(t) \), \( \hat{\Pi}(t) = e^{iHt} \hat{\Pi}(0) e^{-iHt} \). However, also this last attempt does not produce directly a closed system of differential equations: some different approximation must be assumed. This different approximation will be discussed in this subsection.

We first remind that given a generic operator \( X \) its time evolution, in the Heisemberg representation, is (formally) given by \( X(t) = e^{iHt} X e^{-iHt} \), and it satisfies the following Heisemberg equation of motion: \( \dot{X}(t) = i e^{iHt}[H,X] e^{-iHt} = i[H,X(t)] \). In the attempt of deducing the analytic expression for \( \hat{\Pi}(t) \), the following differential equations can be
where we have defined $Z(f,t) := e^{iHt}Z(f)e^{-iHt}$, $Z((\hat{P}(t)\Omega_C - \Omega_A)f,t) = \sum_{k\in\Lambda}(\hat{P}(t)\Omega_C(k) - \Omega_A(k))f(k)Z_k(t)$, $o(g,t) = e^{iHt}o(g)e^{-iHt}$, and so on.

It is clear that the system (4.21) is not closed, since for instance the differential equation for $\hat{P}(t)$ involves $p(t)$, $o(g,t)$ and their adjoint. This is not a major problem since, as in Sections II and III and in [1], it is quite easy to deduce the time evolution of the price operator $\hat{P}$ with no approximation at all. This is because $p(t)$ (and $o(g,t)$) can be found explicitly. Even if these operators can be found under more general conditions, we will now restrict the model requiring that the coefficients in $H$ satisfy some extra requirement, which are only useful to simplify the computations. For instance, we will assume that $\Omega_O(k)$ is constant in $k$, $\Omega_O = \Omega_O(k)$ for all $k \in \Lambda$, and that $\omega_p = \sum_{k\in\Lambda}|g(k)|^2 = \lambda = \Omega_O$. Then we get $p(t) = \frac{1}{2}(p(e^{-2i\lambda t} + o(g)(e^{-2i\lambda t} - 1))$ and $\hat{P}(t) = p^\dagger(t)p(t)$. Since $\hat{P}(t)$ depends only on the operators $p$ and $o$, and not on $a, c$, and so on, and since we are interested to the mean value of the operators in (4.21) in a vector state $\omega$ generalizing (2.3), we replace this system with its semiclassical approximation

$$
\begin{align*}
\frac{dp(t)}{dt} &= i\lambda ( -z^\dagger(t) Z(f,t) + z(t) Z^\dagger(f,t) ) , \\
\frac{dP_c(t)}{dt} &= i\lambda P_c(t) ( z^\dagger(t) Z(f,t) - z(t) Z^\dagger(f,t) ) , \\
\frac{dz(t)}{dt} &= i ( P_c(t)\omega_c - \omega_a ) z(t) + i\lambda z^\dagger(t), \\
\frac{dZ(f,t)}{dt} &= i Z ((P_c(t)\Omega_C - \Omega_A)f,t) + i\lambda z(t) [Z^\dagger(f,t),Z(f,t)] ,
\end{align*}
$$

where

$$
P_c(t) = \omega(\hat{P}(t)) = \frac{1}{2} [ (M + O) + (M - O) \cos(2\lambda t) ]
$$

where we have defined $Z(f,t) := e^{iHt}Z(f)e^{-iHt}$, $Z((\hat{P}(t)\Omega_C - \Omega_A)f,t) = \sum_{k\in\Lambda}(\hat{P}(t)\Omega_C(k) - \Omega_A(k))f(k)Z_k(t)$, $o(g,t) = e^{iHt}o(g)e^{-iHt}$, and so on.

It is clear that the system (4.21) is not closed, since for instance the differential equation for $\hat{P}(t)$ involves $p(t)$, $o(g,t)$ and their adjoint. This is not a major problem since, as in Sections II and III and in [1], it is quite easy to deduce the time evolution of the price operator $\hat{P}$ with no approximation at all. This is because $p(t)$ (and $o(g,t)$) can be found explicitly. Even if these operators can be found under more general conditions, we will now restrict the model requiring that the coefficients in $H$ satisfy some extra requirement, which are only useful to simplify the computations. For instance, we will assume that $\Omega_O(k)$ is constant in $k$, $\Omega_O = \Omega_O(k)$ for all $k \in \Lambda$, and that $\omega_p = \sum_{k\in\Lambda}|g(k)|^2 = \lambda = \Omega_O$. Then we get $p(t) = \frac{1}{2}(p(e^{-2i\lambda t} + o(g)(e^{-2i\lambda t} - 1))$ and $\hat{P}(t) = p^\dagger(t)p(t)$. Since $\hat{P}(t)$ depends only on the operators $p$ and $o$, and not on $a, c$, and so on, and since we are interested to the mean value of the operators in (4.21) in a vector state $\omega$ generalizing (2.3), we replace this system with its semiclassical approximation

$$
\begin{align*}
\frac{dp(t)}{dt} &= i\lambda ( -z^\dagger(t) Z(f,t) + z(t) Z^\dagger(f,t) ) , \\
\frac{dP_c(t)}{dt} &= i\lambda P_c(t) ( z^\dagger(t) Z(f,t) - z(t) Z^\dagger(f,t) ) , \\
\frac{dz(t)}{dt} &= i ( P_c(t)\omega_c - \omega_a ) z(t) + i\lambda z^\dagger(t), \\
\frac{dZ(f,t)}{dt} &= i Z ((P_c(t)\Omega_C - \Omega_A)f,t) + i\lambda z(t) [Z^\dagger(f,t),Z(f,t)] ,
\end{align*}
$$

where

$$
P_c(t) = \omega(\hat{P}(t)) = \frac{1}{2} [ (M + O) + (M - O) \cos(2\lambda t) ]
$$

We refer to [1] for a more complete discussion of the two-fold role of the state $\omega$. Here we just want to remark that a given vector state allows us to pass from the quantum dynamics
of the model to its \textit{classical} counterpart, since we use $\omega$ to replace the time dependent operators with their mean values, which are functions of time. At the same time, moreover, a vector state is used to fix the initial conditions of the differential equations, that is the initial number of shares, the initial cash and so on.

In order to simplify further the analysis of this system, it is also convenient to assume that both $\Omega_C(k)$ and $\Omega_A(k)$ are constant for $k \in \Lambda$. Indeed, under this assumption, the last two equation in (4.22) forms by themselves a closed system of differential equations in the non abelian variables $z(t)$ and $Z(f, t)$:

$$
\begin{align*}
\frac{dz(t)}{dt} &= i \left( P_c(t) \omega_c - \omega_a \right) z(t) + i \lambda Z(f, t) [z(t), z(t)], \\
\frac{dZ(f, t)}{dt} &= i \left( P_c(t) \Omega_C - \Omega_A \right) Z(f, t) + i \lambda z(t) [Z(f, t), Z(f, t)].
\end{align*}
$$

(4.24)

Getting the exact solution of the system (4.22), with (4.24) as the two last equations, is an hard job. However, this is a good starting point for finding an approximated solution of the dynamical problem. Indeed, a natural approach consists in taking the first non trivial contribution of the system, as usually done in perturbation theory. This means that, in system (4.24), the contributions containing the commutators must be neglected since they are proportional to $\lambda$ while $i \left( P_c(t) \omega_c - \omega_a \right) z(t)$ and $i \left( P_c(t) \Omega_C - \Omega_A \right) Z(f, t)$ which, on the other way do not depend on $\lambda$, give a relevant contribution. On the other way, in order not to trivialize the system, we have to keep the first two equations in (4.22) as they are: if we simply put $\lambda = 0$ here, in fact, we would trivialize the time evolution of both $\hat{n}(t)$ and $\hat{k}(t)$. With this choice we get

$$
\begin{align*}
\frac{d\hat{n}(t)}{dt} &= i \lambda \left( -z(t)^\dagger Z(f, t) + z(t) Z(f, t)^\dagger \right), \\
\frac{d\hat{k}(t)}{dt} &= i \lambda P_c(t) \left( z(t)^\dagger Z(f, t) - z(t) Z(f, t)^\dagger \right), \\
\frac{dz(t)}{dt} &= i \left( P_c(t) \omega_c - \omega_a \right) z(t), \\
\frac{dZ(f, t)}{dt} &= i \left( P_c(t) \Omega_C - \Omega_A \right) Z(f, t).
\end{align*}
$$

(4.25)

However, we will now show that this approximation is too rude, meaning with this that, even if the operators $\hat{n}(t)$ and $\hat{k}(t)$ have a non trivial dynamics, at the classical level we deduce that both $n(t) = \omega(\hat{n}(t))$ and $k(t) = \omega(\hat{k}(t))$ are constant in time, so that the time behavior of the portfolio $\Pi(t) = P_c(t) n(t) + k(t) = P_c(t) n + k$ is uniquely given by $P_c(t)$. We first observe that $z(t)$ and $Z(f, t)$ in (4.25) are

$$
\begin{align*}
z(t) &= z e^{i\chi(t)}, \\
Z(f, t) &= Z(f) e^{i\chi(t)}.
\end{align*}
$$

(4.26)
where

\[ \chi(t) = \alpha t + \beta \sin(2\lambda t), \quad \tilde{\chi}(t) = \tilde{\alpha} t + \tilde{\beta} \sin(2\lambda t) \]

(4.27)

with

\[
\begin{align*}
\alpha &= \frac{1}{2}((M + O)\omega_c - 2\omega_a), & \beta &= \frac{\delta}{4\lambda}(M - O) \\
\tilde{\alpha} &= \frac{1}{2}((M + O)\Omega_C - 2\Omega_A), & \tilde{\beta} &= \frac{\delta C}{4\lambda}(M - O)
\end{align*}
\]

(4.28)

Our claim is now an immediate consequence of (4.26) above. Indeed, from the first equation in (4.25), taking its mean value on the number vector state \( \omega \) we find

\[ \hat{n}(t) = \frac{d}{dt} \omega(\hat{n}(t)) = \omega \left( \frac{d}{dt} \hat{n}(t) \right) = i\lambda \left\{ -\omega \left( z^\dagger(t)Z(f, t) \right) + \omega \left( z(t)Z^\dagger(f, t) \right) \right\} = 0 \]

since, for instance, \( \omega \left( z^\dagger(t)Z(f, t) \right) = e^{-i(\chi(t) - \tilde{\chi}(t))} \omega(z^\dagger Z(f)) = 0 \). Analogously we find that \( \hat{k}(t) = \frac{d}{dt} \omega(\hat{k}(t)) = 0 \). Therefore we have \( n(t) = n \) and \( k(t) = k \), as claimed above.

**Remark:** in a certain sense this result relates the two approximations considered so far. Indeed, replacing (4.22) with (4.25) we obtain a stationary behavior for \( n(t) \) and \( k(t) \). An analogous behavior was deduced, in the previous subsection, if \( \varepsilon_Z(k) \) has no zero. However, these two different approximations cannot be directly compared. The reason is the following: in the stochastic limit approach we need to require that \( \varepsilon_Z(k) \) and \( \varepsilon_O(k) \) are not identically zero. This is crucial to ensure the existence of \( \lim_{\lambda \to 0} I_\lambda(t) \).

In the present approximation we are requiring that both \( \Omega_C(k) \) and \( \Omega_A(k) \) are constant in \( k \) so that, see (4.8), we would get \( \varepsilon_Z(k) = P_c(t)(\Omega_C - \omega_c) - (\Omega_A - \omega_a) \), which may have some zero in \( k \) only if it is identically zero in \( k \). In other words, in the conditions in which we are working here the stochastic limit approach does not work. Viceversa, if we are in the assumptions of the previous subsection, then system (4.22) cannot be easily solved! Hence the two approximations cover different situations.

A better approximation can be constructed. Again the starting point is the system (4.24), for which we now construct iteratively a solution, stopping at the first relevant order. In other words, we take \( z_0(t) \) and \( Z_0(f, t) \) as in (4.26), \( z_0(t) = z e^{i\chi(t)} \) and \( Z_0(f, t) = Z(f) e^{i\tilde{\chi}(t)} \), and then we look for the next approximation by considering the following system:

\[
\begin{align*}
\frac{dz_1(t)}{dt} &= i(P_c(t)\omega_c - \omega_a) z_0(t) + i\lambda Z_0(f, t) [z_0^\dagger(t), z_0(t)], \\
\frac{dZ_1(f, t)}{dt} &= i(P_c(t)\Omega_C - \Omega_A) Z_0(f) + i\lambda z_0(t) [Z_0^\dagger(f, t), Z_0(f, t)].
\end{align*}
\]
which can be still written as

\[
\begin{align*}
\frac{dz(t)}{dt} &= \left( P_c(t) \omega_c - \omega_a \right) z(t) + i \lambda Z_0(f, t) \left[ z(t), z_0 \right], \\
\frac{dZ_1(f, t)}{dt} &= i \left( P_c(t) \Omega_C - \Omega_A \right) Z_0(f) + i \lambda z_0(t) \left[ Z_0(\bar{f}), Z_0(f) \right].
\end{align*}
\tag{4.29}
\]

These equations can be solved and the solution can be written as

\[
\begin{align*}
z_1(t) &= z \eta_1(t) + Z(f) [z^\dagger, z] \eta_2(t), \\
Z_1(f, t) &= Z(f) \tilde{\eta}_1(t) + z [Z(\bar{f})^\dagger, Z(f)] \tilde{\eta}_2(t),
\end{align*}
\tag{4.30}
\]

where we have introduced the following functions

\[
\begin{align*}
\eta_1(t) &= 1 + i \int_0^t (P_c(t') \omega_c - \omega_a) e^{ix(t')} dt', \\
\eta_2(t) &= i \lambda \int_0^t e^{ix(t')} dt', \\
\tilde{\eta}_1(t) &= 1 + i \int_0^t (P_c(t') \Omega_C - \Omega_A) e^{ix(t')} dt', \\
\tilde{\eta}_2(t) &= i \lambda \int_0^t e^{ix(t')} dt'.
\end{align*}
\tag{4.31}
\]

It is not a big surprise that this approximated solution does not share with \(z(t)\) and \(Z(f, t)\) all their properties. In particular, while for instance \([z(t), Z(f, t)] = 0\) for all \(t\), \(z_1(t)\) and \(Z_1(f, t)\) do not commute. For this reason we consider the equations for \(\hat{n}(t)\) and \(\hat{k}(t)\) as in \((4.25)\) as far as possible, replacing \(z(t)\) and \(Z(f, t)\) with \(z_1(t)\) and \(Z_1(f, t)\) only at the last step.

It is easy to find that the mean values of the first two equations in \((4.25)\) can be written as

\[
\begin{align*}
\hat{n}(t) &= \frac{dn(t)}{dt} = -2 \lambda \Im \left\{ \omega (z(t) \bar{Z}(\bar{f}, t)) \right\}, \\
\hat{k}(t) &= \frac{dk(t)}{dt} = 2 \lambda P_c(t) \Im \left\{ \omega (z(t) \bar{Z}(\bar{f}, t)) \right\},
\end{align*}
\tag{4.32}
\]

which in particular implies a well known identity: \(P_c(t) \hat{n}(t) + \hat{k}(t) = 0\) for all \(t\), which in turns implies that \(\hat{\Pi}(t) = \hat{P}_c(t) n(t)\). It should be remarked that, because of this relation, since \(M = O\) implies \(P_c(t) = P_c(0) = M\), then when \(M = O\) the dynamics of the portfolio of \(\tau\) is trivial, \(\Pi(t) = \Pi(0)\), even if both \(n(t)\) and \(k(t)\) may change in time.

It is now at this stage that we insert \(z_1(t)\) and \(Z_1(f, t)\) in the differential equations. If \(\omega\) is the usual number state, and if we call for simplicity

\[
\begin{align*}
\omega(1) &= \omega (z \bar{Z}(\bar{f}), Z(f)), \\
\omega(2) &= \omega (Z(f) \bar{Z}(\bar{f}) [z^\dagger, z]), \\
r(t) &= \omega(1) \eta_1(t) \tilde{\eta}_2(t) + \omega(2) \eta_2(t) \tilde{\eta}_1(t)
\end{align*}
\tag{4.33}
\]

then we get

\[
\begin{align*}
n(t) &= n - 2 \lambda \Im \left\{ \int_0^t r(t') dt' \right\}, \\
k(t) &= k + 2 \lambda \Im \left\{ \int_0^t P_c(t') r(t') dt' \right\}.
\end{align*}
\tag{4.34}
\]
The time dependence of the portfolio can now be written as
\[ \Pi(t) = \Pi(0) + \delta \Pi(t), \]  
(4.35)
with
\[ \delta \Pi(t) = n(O - M) \sin^2(\lambda t) + \]
\[ + \left( -2\lambda \Im \left\{ \int_0^t r(t') dt' \right\} \left( M + (O - M) \sin^2(\lambda t) \right) + 2\lambda \Im \left\{ \int_0^t P_c(t') \, r(t') \, dt' \right\} \right), \]  
(4.36)
which gives the variation of the portfolio of \( \tau \) in time. We observe that, as it is expected, \( \delta \Pi(t) = 0 \) if \( \lambda = 0 \).

In the last part of this subsection we look for particular solutions of this system under special conditions. A more detailed analysis of these results will be discussed in another paper, which is now in preparation and where a more general settings will be considered.

A first remark concerning (4.36) is the following: if \( O > M \), it is more likely for \( \tau \) to have a positive \( \delta \Pi(t) \) if the number of the shares \( n \) in his portfolio at time \( t = 0 \) is large: if at \( t = 0 \) the supply of the market is larger than the price of the share then for a trader with many shares it is easier to become even richer! If, on the contrary, \( O < M \), having a large number of shares does not automatically produce an increment of the portfolio.

Coefficients \( \omega(1) \) and \( \omega(2) \) can be found explicitly and depend on the initial conditions of the market. If, for simplicity’s sake, we consider \( \Lambda = \{ k_o \} \), that is if the reservoir consists of just another trader interacting with \( \tau \), then we get
\[ \omega(1) = |f(k_o)|^2 (1 + n) k_o \{-(M)\} \left( n' k_{o}^{+M} \right) - (1 + n') k_{o}^{-(M)} \right) \]
and
\[ \omega(2) = |f(k_o)|^2 (1 + n') k_{o}^{-(M)} \left( n k^{+M} \right) - (1 + n) k^{-(M)} \right). \]
It is clear that these coefficients coincide if \( k = k_o \) and \( n = n' \).

Let us first fix \( M = 1, O = 2, \lambda = 1, \omega_a = \omega_c = 1, \Omega_A = \Omega_C = 2 \). Then the plots of \( \delta \Pi(t) \) below, in which \( n \) is fixed to be 10, are related to the following different values of \( \omega(1) \) and \( \omega(2) \): \( (\omega(1), \omega(2)) = (1, 1), (1, 10), (10, 1) \).
Figure 1: $\delta \Pi(t)$ for $n = 10$ and $(\omega(1), \omega(2)) = (1, 1)$ (left), $(\omega(1), \omega(2)) = (1, 10)$ (middle), $(\omega(1), \omega(2)) = (10, 1)$ (right)

The plots do not change much if we fix $n = 5$ and, surprisingly enough, also the ranges of variations of $\delta \Pi(t)$ essentially coincide with those above: $n$ seems to play no crucial role here! In Figure 2 we plot $\delta \Pi(t)$ in the same conditions as before, but for $n = 5$.

Figure 2: $\delta \Pi(t)$ for $n = 5$ and $(\omega(1), \omega(2)) = (1, 1)$ (left), $(\omega(1), \omega(2)) = (1, 10)$ (middle), $(\omega(1), \omega(2)) = (10, 1)$ (right)

From both these figures we see that, for trader $\tau$, the most convenient situation is $(\omega(1), \omega(2)) = (1, 10)$: in this case there is only a small range of time in which $\delta \Pi(t)$ is negative. For all other times $\delta \Pi(t)$ is positive and $\Pi(t)$ increases its original value. The situation is a bit less favorable for other choices of $(\omega(1), \omega(2))$. This is not surprising since $\omega(1)$ and $\omega(2)$ are related to the initial values of the stock market we are considering and, how it is well known, different initial conditions may correspond to quite different dynamical behaviors!
Now we change the relation between $M$ and $O$. Therefore we fix $M = 2$, $O = 1$, $\lambda = 1$, $\omega_a = \omega_c = 1$, $\Omega_A = \Omega_C = 2$. Again the plots of $\delta \Pi(t)$ below are related to the following values of: $(\omega(1), \omega(2)) = (1, 1), (1, 10), (10, 1)$, and we fix $n = 10$.

![Plots of $\delta \Pi(t)$ for different values of $(\omega(1), \omega(2))$.](image)

Figure 3: $\delta \Pi(t)$ for $n = 10$ $(\omega(1), \omega(2)) = (1, 1)$ (left), $(\omega(1), \omega(2)) = (1, 10)$ (middle), $(\omega(1), \omega(2)) = (10, 1)$ (right)

We see that these plots look very much as those in Figure 1 reflexed with respect to the horizontal axis. This means that, for $n = 10$, the main contribution in (4.36) is the term $n(O - M) \sin^2(\lambda t)$. Of course this is even more evident if $n$ is larger than 10, while for small values of $n$ the role of the other contributions in (4.36) is in general more relevant.

We have already stressed that, if $M = 0$, then $\delta \Pi(t) = 0$ for all $t \geq 0$. Therefore we don’t plot $\delta \Pi(t)$ in this condition. Instead of this, we finish considering what happens if we change the values of $\omega_a$ and $\omega_c$ with $\Omega_A$ and $\Omega_C$. For that we fix, as in the first case, $M = 1$, $O = 2$, $\lambda = 1$, while we take $\omega_a = \omega_c = 2$ and $\Omega_A = \Omega_C = 1$. The related plots are

![Plots of $\delta \Pi(t)$ for different values of $\omega_a$ and $\omega_c$.](image)

This result is particularly interesting since it shows that, if $\omega(1) = \omega(2) = 1$ and also for $n$ small enough, trader $\tau$ can only improve the value of his portfolio, no matter the
Figure 4: $\delta \Pi(t)$ for $n = 10$ $(\omega(1), \omega(2)) = (1, 1)$ (left), $(\omega(1), \omega(2)) = (1, 10)$ (middle), $(\omega(1), \omega(2)) = (10, 1)$ (right)

value of $t$. Remember now that $\omega(1) = \omega(2)$ is true if the initial conditions for the trader $\tau$ and for the trader of the reservoir, $\sigma$, coincide. Therefore, this suggests that the relation between the parameters $\omega_a, \omega_c$ and $\Omega_A, \Omega_C$ is crucial to determine $\Pi(t)$ and in particular that if we take $\Omega_A = \Omega_C > \omega_a = \omega_c$, $\tau$ is, in a certain sense, in a better condition with respect to trader $\sigma$. It is therefore natural to associate these parameters, for instance, to a sort of information reaching the traders, in analogy with the interpretation discussed in [1]. We avoid details here since a deeper analysis is needed for a better understanding of the role of all the parameters in $H$. In a forthcoming paper we will focus our interest exactly on this point: we will discuss the solution of system (4.25) under several different conditions, and we will also consider different expressions of $P_c(t)$, arising from some different economically reasonable hamiltonian or by experimental data.

V Conclusions and outcome

In this paper we have carried on the analysis of a stock market in terms of Heisemberg dynamics which we began in [1]. In particular, we have generalized the model introduced in [1] by introducing a real dynamical behavior for the price of the shares. This is, in our opinion, a big achievement with respect to our previous results.

Section III is just a pedagogical two-traders model which is useful to fix some general ideas and giving some definitions. The same model is further generalized in Section IV where a non trivial market has been introduced. We have considered two different approximations of this model. The first approximation, the so-called stochastic limit approach, is useful to get conditions for the staticity of the portfolio of a given trader. The second approximation, more useful for the analysis of the general time evolution of the portfolio, produces many results and is quite interesting in view of future applications.

In particular, in a close future we plan to add more kind of shares within the model, and to use system (4.22) with different functions $P_c(t)$, deduced from other hamiltonian models or by experimental data. A more long-distance program also includes, for a market
with more shares, an analysis of the role of the Heisenberg dynamics in the analysis of stock markets. This will be undertaken clearly via a comparison between our results and the experimental data.

We would also like to comment that, as already briefly discussed in [11], the same general strategy seems of some utilities in contexts which are apparently very far from stock markets and particle physics. Indeed, the mechanism analyzed here is natural whenever we are interested in describing exchanges between different active components of our (physical, biological, economical,...) system. Indeed, this is exactly the original remark which produced second quantization in elementary particle physics, [6]. Just as a different example, we may also use the Hamiltonian in [11], or some modification of this, for a predator-prey system. In this case, \( \hat{n} \) represents the number of predator operator while \( \hat{k} \) is the number of prey operator. The mechanism in \( H_I \) implies that when the number of predator increases of one unit the number of preys decreases of \( \langle \hat{P} \rangle \) units, and \( \hat{P} \) can now be interpreted as a sort of ability of the predator to catch its victims. We refer to [11], and to a paper in preparation [12], for more applications to sociological contexts.

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**Appendix A: the definition of \( c^{\hat{P}} \)**

In this Appendix we will discuss in detail how to define the operators \( c^{\hat{P}} \) and \( c^{\dagger \hat{P}} \), and some useful formulas related to them.

To make the situation simpler, we just neglect here the role of the other operators appearing in Section III, i.e. the number of share and the supply operators, since they
play no role in the definition of, say, $c \hat{P}$. We will just consider two sets of bosonic operators $c$ and $p$, with $[c, c^\dagger] = [p, p^\dagger] = 1$, and the common vacuum vector $\varphi_0$: $c \varphi_0 = p \varphi_0 = 0$. In a standard fashion we call

$$\varphi_{k,m} = \frac{1}{\sqrt{k! m!}} (c^\dagger)^k (p^\dagger)^m \varphi_0,$$  \hspace{1cm} (A.1)

where $k, m \geq 0$. It is well known, \[6\], that $\varphi_{k,m}$ is an eigenstate of $\hat{k} = c^\dagger c$ and $\hat{P} = p^\dagger p$: $\hat{k} \varphi_{k,m} = k \varphi_{k,m}$ and $\hat{P} \varphi_{k,m} = m \varphi_{k,m}$. Since we have, if $k$ is large enough,

$$c \varphi_{k,m} = \sqrt{k} \varphi_{k-1,m}, \quad c^2 \varphi_{k,m} = \sqrt{k(k-1)} \varphi_{k-2,m},$$

and so on, it is natural to define

$$c \hat{P} \varphi_{k,m} := \begin{cases} 
\varphi_{k,m}, & \text{if } m = 0, \forall k \geq 0; \\
0, & \text{if } m > k, \forall k \geq 0; \\
\sqrt{k(k-1) \cdots (k-m+1)} \varphi_{k-m,m}, & \text{if } k \geq m > 0
\end{cases} \hspace{1cm} (A.2)$$

Analogously, since $c^\dagger \varphi_{k,m} = \sqrt{k+1} \varphi_{k+1,m}$, $(c^\dagger)^2 \varphi_{k,m} = \sqrt{(k+1)(k+2)} \varphi_{k+2,m}$, and so on, for all $k$ and $m \geq 0$, we put

$$c^\dagger \hat{P} \varphi_{k,m} := \begin{cases} 
\varphi_{k,m}, & \text{if } m = 0, \forall k \geq 0; \\
\sqrt{(k+1)(k+2) \cdots (k+m)} \varphi_{k+m,m}, & \text{if } m > 0,
\end{cases} \hspace{1cm} (A.3)$$

**Remark:** We could use a different name for the operators $c \hat{P}$ and $c^\dagger \hat{P}$. For instance we could call $\hat{Y}$ and $\hat{W}$ the operators defined as in (A.2) and (A.3): however we have decided to keep this notation to stress the role of both $c$ and $\hat{P}$ in the definition of these ladder operators.

These definitions, other than natural, have two nice consequences: (i) they really define the operators $c \hat{P}$ and $c^\dagger \hat{P}$ since they are now defined on the vectors of an orthonormal basis in the Hilbert-Fock space $\mathcal{H}$ of the system, which is the closure of the linear span of the set $\{\varphi_{k,m}, k, m \geq 0\}$. In this way we by-pass the problems raised in \[1\], and we can avoid replacing the hamiltonian (2.1) with the approximated hamiltonian (2.5); (ii) we get an extra bonus which suggests that (A.2) and (A.3) are good definitions: indeed we find that

$$(c^\dagger)^\dagger \hat{P} = (c \hat{P})^\dagger,$$

and we omit the proof of this claim here.

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More relevant for us is to deduce some commutation rules which involve the operators \((c^\dagger)^{\hat{P}}\), where \(c^\dagger\) can be \(c\) or \(c^\dagger\). We claim that

\[
\begin{cases}
[\hat{P}, c^{\hat{P}}] = [\hat{P}, c^{\dagger}\hat{P}] = 0, \\
[k, c^{\hat{P}}] = [c^{\dagger}c, c^{\hat{P}}] = -\hat{P}c^{\hat{P}} = -c^{\hat{P}}\hat{P}, \\
[k, c^{\dagger}\hat{P}] = \hat{P}c^{\dagger}\hat{P} = c^{\dagger}\hat{P}\hat{P}
\end{cases}
\]  

(A.4)

Again, we omit the proof of these rules here since they can be easily deduced applying both sides of each line above to a vector \(\varphi_{k,m}\) of our orthonormal basis. We simply remark that, for instance, \([\hat{k}, c^{\hat{P}}] = -\hat{P}c^{\hat{P}}\) is an extended version of \([\hat{k}, c^{l}] = -l c^{l}\), while \([\hat{k}, c^{\dagger}\hat{P}] = \hat{P}c^{\dagger}\hat{P}\) extends \([\hat{k}, c^{\dagger l}] = l c^{\dagger l}\).

**Appendix B: Few results on the stochastic limit**

In this Appendix we will briefly summarize some of the basic facts and properties concerning the SLA which are used in Section IV. We refer to [9] and references therein for more details.

Given an open system \(S + R\) we write its hamiltonian \(H\) as the sum of two contributions, the free part \(H_0\) and the interaction \(\lambda H_I\). Here \(\lambda\) is a coupling constant, \(H_0\) contains the free evolution of both the system \(S\) and the reservoir \(R\), while \(H_I\) contains the interaction between \(S\) and \(R\). Working in the interaction picture, we define \(H_I(t) = e^{iH_0 t}H_I e^{-iH_0 t}\) and the so called wave operator \(U_\lambda(t)\) which is the solution of the following differential equation

\[
\partial_t U_\lambda(t) = -i\lambda H_I(t)U_\lambda(t),
\]

with the initial condition \(U_\lambda(0) = \mathbb{1}\). Using the van-Hove rescaling \(t \rightarrow \frac{t}{\lambda^2}\), see [9] for instance, we can rewrite the same equation in a form which is more convenient for our perturbative approach, that is

\[
\partial_t U_\lambda\left(\frac{t}{\lambda^2}\right) = -\frac{i}{\lambda} H_I\left(\frac{t}{\lambda^2}\right) U_\lambda\left(\frac{t}{\lambda^2}\right),
\]

(B.2)
with the same initial condition as before. Its integral counterpart is

\[ U_\lambda \left( \frac{t}{\lambda^2} \right) = \mathbb{1} - \frac{i}{\lambda} \int_0^t H_I \left( \frac{t'}{\lambda^2} \right) U_\lambda \left( \frac{t'}{\lambda^2} \right) dt', \tag{B.3} \]

which is the starting point for a perturbative expansion, which works in the following way.

We will limit ourself here to consider the zero temperature situation. Then let \( \varphi_0 \) be the ground vector of the reservoir and \( \xi \) a generic vector of the system. Now we put \( \varphi_0(\xi) = \varphi_0 \otimes \xi \). We want to compute the limit, for \( \lambda \) going to 0, of the first non trivial order of the mean value of the perturbative expansion of \( U_\lambda(t/\lambda^2) \) above in \( \varphi_0(\xi) \), that is the limit of

\[ I_\lambda(t) = \left( -\frac{i}{\lambda} \right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \langle H_I \left( \frac{t_1}{\lambda^2} \right) H_I \left( \frac{t_2}{\lambda^2} \right) \rangle \varphi_0(\xi), \tag{B.4} \]

for \( \lambda \rightarrow 0 \). Under some regularity conditions on the functions which are used to smear out the (typically) bosonic fields of the reservoir, this limit is shown to exist for many relevant physical models, see [9] and [10]. We define \( I(t) = \lim_{\lambda \rightarrow 0} I_\lambda(t) \). In the same sense of the convergence of the (rescaled) wave operator \( U_\lambda(t/\lambda^2) \) (the convergence in the sense of correlators), it is possible to check that also the (rescaled) reservoir operators converge and define new operators which do not satisfy canonical commutation relations but a modified version of these, [10]. Moreover, these limiting operators depend explicitly on time and they live in a Hilbert space which is different from the original one. In particular, they annihilate a vacuum vector, \( \eta_0 \), which is no longer the original one, \( \varphi_0 \).

It is not difficult to deduce the form of a time dependent self-adjoint operator \( H_I^{(sl)}(t) \), which depends on the system operators and on the limiting operators of the reservoir, such that the first non trivial order of the mean value of the expansion of \( U_t = \mathbb{1} - i \int_0^t H_I^{(sl)}(t') U_t dt' \) on the state \( \eta_0(\xi) = \eta_0 \otimes \xi \) coincides with \( I(t) \). The operator \( U_t \) defined by this integral equation is called again the wave operator.

The form of the generator follows now from an operation of normal ordering. More in details, we start defining the flux of an observable \( \tilde{X} = X \otimes \mathbb{1}_r \), where \( \mathbb{1}_r \) is the identity of the reservoir and \( X \) is an observable of the system, as \( j_t(\tilde{X}) = U_t^\dagger \tilde{X} U_t \). Then, using the equation of motion for \( U_t \) and \( U_t^\dagger \), we find that \( \partial_t j_t(\tilde{X}) = iU_t^\dagger [H_I^{(sl)}(t), \tilde{X}] U_t \). In order to compute the mean value of this equation on the state \( \eta_0(\xi) \), so to get rid of the reservoir operators, it is convenient to compute first the commutation relations between \( U_t \) and the
limiting operators of the reservoir. At this stage the so called *time consecutive principle* is used in a very heavy way to simplify the computation. This principle, which has been checked for many classes of physical models, [9], states that, if $\beta(t)$ is any of these limiting operators of the reservoir, then

$$[\beta(t), U_{t'}] = 0, \text{ for all } t > t'. \quad (B.5)$$

Using this principle and recalling that $\eta_0$ is annihilated by the limiting annihilation operators of the reservoir, it is now a simple exercise to compute $\langle \partial_t j_t(\tilde{X}) \rangle_{\eta_0(\xi)}$ and, by means of the equation $\langle \partial_t j_t(\tilde{X}) \rangle_{\eta_0(\xi)} = \langle j_t(L(\tilde{X})) \rangle_{\eta_0(\xi)}$, to identify the form of the generator of the physical system, which allows us to obtain equations of motion in general much easier than the original ones, since the reservoir *disappear*.

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