Geometrical phase effects in biaxial nanomagnetic particles

Sahng-Kyoon Yoo

Department of Physics, Chongju University, Chongju, 360-764, Korea

Soo-Young Lee

Department of Physics, Kyungnam University, Masan, 631-701, Korea

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The oscillation of tunnel splitting are obtained by geometrical analysis of the topological Wess-Zumino phase on the basis of tunneling paths in biaxial nanomagnetic particles with magnetic field along the hard anisotropy axis. This theory not only just yields the previous quantum interference results for the ground state tunneling, but also gives the excited level splittings, all of which agree well with the numerical diagonalization. Furthermore, the parity effect in the asymmetric double well system which was recently discovered in experiment, can be derived by similar arguments, and is also certified by using the complex periodic orbit theory. The possibility of improving the discrepancy with the experiment in the periods of oscillations is discussed.

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I. INTRODUCTION

For last decade, many physicists have explored the nanoscale magnetic particles to study the extrapolation of quantum mechanics to macroscopic realm. The magnetizations of these particles show the macroscopic quantum tunneling (MQT) from a metastable state to a stable one, or macroscopic quantum coherence (MQC) between two degenerate ground states separated by potential energy barrier [1–3]. In the latter case, the geometrical phase, known as the Wess-Zumino phase [4], plays a crucial role in tunnel splitting. In particular, the biaxial nanomagnetic particles present some exotic phenomena related to this topological phase. In the absence of external magnetic field, the ground state tunneling vanishes only for half-integer spins, which is called as spin parity effect and related to well-known Kramer degeneracy [5]. When the magnetic field is applied along the hard anisotropy axis, however, the tunnel splitting oscillates with magnetic field regardless of spin values [6]. These spin parity effect and oscillations of tunnel splittings are interpreted as the interferences of geometrical phases between two opposite tunneling paths in the instantonic approach [5–9].

Most theoretical treatments so far about the oscillations of level splittings have been restricted to the instanton method for the mapped effective one-dimensional potential by performing the cos $\theta$ integration in spin coherent state representation [7,8] or by direct particle mapping method [9]. In these approaches, although the ground state tunneling can be obtained well, it is not possible to treat the tunneling between excited energy levels exactly. In fact, the result of numerical diagonalization for excited states is that the number of oscillations decreases one by one as the quantum number increases. In addition, due to the mapping onto one-dimension, the dominant tunneling path between two wells in $(\theta, \phi)$ space is not clear. This makes it difficult to perform the direct geometrical analysis of Wess-Zumino phase. Recently, Garg [10] derived the splitting oscillations in high energy levels using the discrete WKB approximation. Another semiclassical approach is the complex periodic orbit theory [11,12], the extension of the trace formula in periodic orbit theory [13–15] to tunneling systems. We obtained successfully the energy levels for not only the ground state but also all the higher excited states by applying this method to spin system for the first time, and showed the complete agreement with numerical diagonalization [16]. Since this theory also uses the effective mapping potential obtained by cos $\theta$-integration of original Hamiltonian, the dominant tunneling path in phase space remains to be unspecified yet. The level splitting oscillations were confirmed by experiment very recently in the Fe$_8$ molecular clusters with spin $S = 10$ [17]. They showed that, although the oscillations occurs as predicted by theory, in quantitative aspect the oscillation periods is about 1.6 times larger than previous theoretical results, and this discrepancy can be resolved introducing the fourth-order terms in spin Hamiltonian.

In this paper, we find the dominant tunneling paths in phase space, and analyze geometrically the Wess-Zumino phase without any mapping in order to obtain the ground and excited level splitting oscillations with magnetic field along the hard axis. Furthermore, the experimentally discovered parity effect [17] which is very similar to spin parity effect in the asymmetric system is derived theoretically using this geometrical analysis and complex periodic orbit theory. We also discuss the possibility of improving the discrepancy with experiments by adding the fourth-order terms in spin Hamiltonian. In section
II, after finding the dominant tunneling path, we obtain the splitting oscillations for the tunneling between both the ground and the excited states in symmetric system by investigating the relation between the geometrical phase and the tunnel path in phase space without mapping onto a particle. In Section III, the parity effect occurring in the asymmetric system is derived using the argument of Section II, and furthermore is also certified from the complex periodic orbit theory. We discuss the possibility of improving the quantitative discrepancy with experiment and give conclusions in Section IV.

II. LEVEL SPLITTING OSCILLATIONS IN THE SYMMETRIC SYSTEM

In this section, we calculate the both ground and excited level splitting oscillations by analyzing the topological Wess-Zumino phase geometrically in phase space. The tunneling rate in spin system is

$$\Gamma = \int D[\cos \theta] D[\phi] \exp(-S_E)$$

(2.1)

in the path integral formalism, where $D$ means the integration over all paths and $S_E$ is the Euclidean action which is given as

$$S_E = i S_{WZ} + \int d\tau H(\theta, \phi).$$

(2.2)

Here, $\tau$ is the imaginary time, $H(\theta, \phi)$ is the spin Hamiltonian in the $(\theta, \phi)$ representation, and

$$S_{WZ} \equiv S \int d\tau \phi \left[ 1 - \cos \theta(\tau) \right]$$

$$= \int_0^\pi d\phi \left[ S - p(\phi(\tau))] \right]$$

(2.3)

is the Wess-Zumino action with total spin number $S$. Here $p \equiv S \cos \theta$ and corresponds to the conjugate momentum to coordinate $\phi$, since $\phi$ and $p$ satisfy the Hamilton’s equations of motion in the spin system. When we consider the tunneling from $\phi = 0$ to $\pi$, Eq. (2.3) means the area surrounded by $p = S$, $\phi = 0$, $\pi$ lines and the tunneling path $p(\phi)$ in $(\phi, p)$ phase space. [See the Fig. 1 (a).] Since the energy splitting due to tunneling, $\Delta E$, is determined by Wess-Zumino action (the area mentioned above), i.e.,

$$\Delta E \sim \cos(S_{WZ}),$$

(2.4)

whenever this area becomes $(n + 1/2)\pi$ with integer $n$ the tunnel splitting vanishes. In order to perform the geometrical evaluation of Wess-Zumino phase without mapping, it is essential to find the dominant tunneling path $p(\phi)$ in the phase space. It will be shown below that in the case of $k$th state, the Wess-Zumino phase continuously decreases from $(S - k)\pi$ to zero as the external field $h$ increases, so that the moment of area being the half-integral multiple of $\pi$, at which the tunnel splitting quenches, occurs $(S - k)$ times.

Let us consider the Hamiltonian describing the biaxial spin system which is given by

$$H = -DS^2_z + E(S^2_z - S^2_y) + g \mu_B H \cdot S,$$

(2.5)

where $D$ and $E$ are longitudinal and transverse anisotropy constants, respectively, $g$ is the gyromagnetic ratio and $\mu_B$ is the Bohr magneton. This Hamiltonian represents that the system has an easy axis in the $x$-direction, and hard axis in the $z$-direction. When the magnetic field $H$ is applied along the $z$-direction, the system is symmetric and the reduced Hamiltonian is given in the spin coherent state representation by

$$H' = [2(1 - \lambda) + (2\lambda - 1)\cos^2 \phi]p^2 - 2Shp$$

$$- (1 - \lambda)S^2 - (2\lambda - 1)S^2 \cos^2 \phi,$$

(2.6)

where $H' \equiv H/(D + E)$, $\lambda = D/(D + E)$ and $h \equiv H/H_e = g \mu_B H/2(D + E)S$ with coercive field $H_e$. In $(p, \phi)$ phase space, $H'$ has two degenerate minima when $p = Sh$ and $\phi = 0, \pi$. At $\phi = \frac{\pi}{2}$, however, $p = Sh/h_1$ with $h_1 \equiv 2(1 - \lambda)$ gives the saddle point for $0 < h < h_1$, local minimum for $h_1 < h < h_2$ with $h_2 \equiv \sqrt{2(1 - \lambda)}$ and global minimum for $h_2 < h < 1$, respectively, if $p$ can be also defined beyond the range $-S \leq p \leq S$. These pictures of various external fields are shown in Fig. 1 as the energy contour plots.

In order to get an information about the dominant tunneling path, we revisit, for a moment, the mapping onto a one-dimensional particle problem by integration over $\cos \theta$ in Eq. (2.1). The resulting Euclidean action is given by

$$S_E(\phi) = iA(\phi) + S\sqrt{2\lambda - 1} \int d\tau \left[ \frac{1}{2} M(\phi) \dot{\phi}^2 + V(\phi) \right],$$

(2.7)

where the dot means the Euclidean time derivative and the imaginary part of Euclidean action which is responsible for the phase effect is

$$A(\phi) = S \int d\phi \left[ 1 - \frac{h}{1 - (2\lambda - 1)\sin^2 \phi} \right],$$

(2.8)

the $\phi$-dependent effective mass is

$$M(\phi) = \frac{1}{1 - (2\lambda - 1)\sin^2 \phi},$$

(2.9)

and the effective potential is

$$V(\phi) = \frac{1}{2} \sin^2 \phi \left[ 1 - \frac{h^2}{1 - (2\lambda - 1)\sin^2 \phi} \right].$$

(2.10)
For the ground state tunneling, if we compare Eq. (2.3) and Eq. (2.8), the tunneling path $p(\phi)$ is found to be the second

It is emphasized that the tunneling path $p(\phi)$ in the phase space $(\phi, p)$ is just the path satisfying $\partial H/\partial p = 0$. This means that the tunneling path follows the minimum energy positions at all $\phi$ values. These tunneling paths for various fields are shown in Fig. 1 as thick solid lines. This simple condition for dominant tunneling path in the phase space $(\phi, p)$ works well only when the Gaussian integration is possible as in the present case.

Now, we calculate the Wess-Zumino action (the area) mentioned above as $h$ increases from 0 to 1. When $h = 0$, the phase is just $S\pi$ (whole area of phase space considered) because the dominant tunneling path is $p(\phi) = 0$. It is important to note that, although the quantum mechanical ground state energy is not the well minimum, the path starts and ends at well minima, because the Euclidean action with the energy of the minimum gives the information about the ground state splitting in semiclassical theory. When $0 < h < h_1$ [Fig. 1(a)], the path is confined within a range $0 < p < S$. Therefore, we can easily calculate the area enclosed by the path and $p = S$ line. Until $h$ becomes $h_1$, where the north-pole becomes the saddle point of energy, the area continuously decreases from $S\pi$. For $h_1 < h < h_2$ [Fig. 1(b)], however, the local minimum exists outside the $p = S$ line. In this case the tunneling path runs beyond the $p = S$ line and thus, the Wess-Zumino action becomes the area of (region B + region C - region A). As $h$ increases the action continues to decrease and becomes zero at $h = h_2$. As a result, within a range $0 < h < h_2$, the quenching take places $S$ times for the ground state by Eq. (2.4). The range $h_2 < h < 1$ is more or less subtle [Fig. 1(c)]. The local minimum turns into the global one, i.e., the point $(Sh_1/\pi, \pi/2)$ becomes lower than the well minimum. In this case, the path giving the Wess-Zumino action should be determined by somewhat different way, because there exists the contour of $E_0$ (well minimum energy). If the path obtained by the previous method encounters the contour of $E_0$, it should follow the contour in a real time due to energy conservation. Therefore, the action is equivalent to the area of [region B + region C - region A] of Fig. 2(c) which always vanishes in the range of $h_2 < h < 1$. This vanishing of the action is closely related to the cancellation between resonant tunneling and quenching phenomena [13] and also between the Euclidean action of real time motion and Wess-Zumino action [3].

The extention to the excited states is straightforward. The $k^{th}$ energy level of each well can be calculated from the well-known EBK (Einstein-Brillouin-Keller) quantization rule [18],

$$\int p(\phi)d\phi = 2\pi \left( k + \frac{1}{2} \right), \quad k = 0, 1, 2, \cdots \tag{2.11}$$

In this quantization, whenever the action for closed orbit in one well differs by $2\pi$, the energy level is given. It will be shown that, by this fact, the number of splitting oscillations decreases one by one with increasing quantum number. When $h = 0$, there exist the contours of $E_k$ centered at $p = 0$ and $\phi = 0$, $\pi$ whose areas are $2k\pi$ for $k^{th}$ excited state (see also Fig. 1). If we use the analogy for the case $h_2 < h < 1$ of the ground state tunneling, the area of $k\pi$ is excluded, due to the energy conservation, for the $k^{th}$ excited state (see also Fig. 1(b)) in the phase space considered, i.e., $(S-k)\pi$. As $h$ increases the continuous reduction to zero is same as the ground state case. This leads to the $(S-k)$ oscillations for $k^{th}$ energy level. The calculated periods of oscillations for both ground and excited state levels completely agree with the numerical diagonalizations. Comparing with the previous semiclassical theories [6–9], in this theory the dominant tunneling path in $(\phi, p)$ phase space can be found to make possible to perform the geometrical analysis of the effect of topological phase, and in addition to the ground state case the structures of splitting oscillations for the excited states can be obtained easily on the basis of EBK quantization scheme.

### III. PARITY EFFECT IN THE ASYMMETRIC SYSTEM

Unlike the symmetric case in Sec. II, in this section we treat the asymmetric system having the longitudinal field as well as the transverse field. In such system, we derive the parity effect which was discovered in recent experiment [17], by using the simple geometrical argument developed in section II, and also lead to the same result in the semiclassical complex periodic orbit theory. This parity effect is very similar to the spin parity effect proposed by Loss et. al. and von Delft and Henley [3]. This is observed when one measures the transition between the ground state in upper well and the $k^{th}$ excited state in lower well, by applying the external field $h'$ along the easy axis (longitudinal field) which makes the system asymmetric in addition to $h$ in the hard direction (transverse field). For the case of the tunneling from the ground state of the upper well to even $k^{th}$ state of lower well, the phase of the oscillation is same as that of the symmetric case, while for the case of tunneling to odd $k^{th}$ state of the lower well the phase is shifted by $\pi/2$.

This interesting observation can naturally arises in our above analysis. As shown in Fig. 2, the energy structure of spin system is asymmetric due to $h'$. At certain values of $h'$ the ground state energy level $E_0$ in upper (left) well exactly coincides with the $k^{th}$ excited level $E_k$ in lower (right) well, so that one can consider the tunnel
splittings. The tunneling path starts from the minimum at upper well, and when it exits the barrier it follows the contour of \( k \)th excited state in the lower well due to energy conservation. Therefore, by EBK quantization relation and analogous arguments with the excited states in Section II, the area \( k \pi/2 \) in the \((φ,p)\) phase space is excluded when \( h = 0 \) case. As the transverse field \( h \) increases, total area surrounded by this path continuously decreases from \((Sπ - k \pi/2)\) to zero in the same manner with the symmetric case except the \( k\pi/2 \) shift. This leads to the parity effect in the asymmetric system.

Next, let us consider this effect in complex periodic orbit theory. In quantum mechanics the energy spectrum of the system can be derived from the singularities of the trace of the energy-dependent Green’s function which is given by

\[
g(E) = \text{Tr} \left[ (E - \hat{H})^{-1} \right] = \sum_{n} \frac{d_n}{E - E_n}, \quad (3.1)
\]

where Tr means the trace and \( d_n \) is the degeneracy factor. The semiclassical approximation of Eq. (3.1) was given by Gutzwiller \[14\] in the form

\[
g_{sc}(E) = \frac{1}{i} \sum_{j} T_j \sum_{r=1}^{\infty} e^{i r (S_j - \mu_j \pi/2)}, \quad (3.2)
\]

where \( T_j, S_j \) and \( \mu_j \) are the period, the action and the Maslov index of the \( j \)th primitive classical periodic orbit, and index \( r \) corresponds to the repetition of primitive periodic orbit. In evaluating this sum, we restrict on the one-dimensional effective particle problem. The system with \( h' = 0 \) which are given as Eq. \(2.7\) - Eq. \(2.10\) is recently treated in detail within the complex periodic orbit theory in Ref. \[14\]. When the longitudinal magnetic field \( h' \) is applied, the effective potential of Eq. \(2.10\) becomes asymmetric as shown in Fig. 3. This figure presents the case when the ground state energy of upper well coincides with the \( k \)th excited level of lower well. The classical orbits in two wells and tunneling orbits within the barrier are represented as solid and dotted lines, respectively. The directions denoted by arrows are given by the rules of Ref. \[14\].

Now we should consider all possible periodic orbits in order to perform the summation in Eq. (3.2). Let us take the classical segments in the respective wells as the half of whole orbit having the actions \( W_1 \equiv S_1/2 \) and \( W_2 \equiv S_2/2 \), and similarly for the tunneling segments within the barrier having an action \( Θ \equiv S_c/2 \). Let us consider the case where the ground state energy in the upper well is equal to the \( k \)th excited state in the lower well. Note, then, that the relation between two actions \( W_1 \) and \( W_2 \) is

\[
W_2 = W_1 + k \pi, \quad k = 0, 1, 2, \cdots, \quad (3.3)
\]
due to EBK quantization rule. The possible periodic orbits can be parametrized as follows: i) the paths that start and end at \( φ_A \) or \( φ_A' \) without complete rotations, ii) the paths that start at \( φ_A(φ_A') \) and end at \( φ_A'(φ_A) \), i.e., having complete rotations (See Fig. 3. \( φ_A \) and \( φ_A' \) are the equivalent position in \( φ \) coordinate). The Wess-Zumino phase contributes only to the paths belonging to category ii) as

\[
α = 2πS \left( 1 - \frac{h}{h_2} \right). \quad (3.4)
\]

For the paths that start at \( φ_A \) and end at \( φ_A' \), i.e., proceed from left to right, we take the Wess-Zumino phase as positive, whereas for the paths from right \((φ_A')\) to left \((φ_A)\) we take as negative. The Maslov index is easily evaluated following the method for double well potential in Ref. \[11\], i.e., each classical segment \( W_1, W_2 \) accompanies the additional phase \(+ π/2\), while the tunneling segment \(- π/2\). The all possible primitive periodic orbits are drawn graphically in Fig. 3, where the solid line corresponds to \( \exp[i(W_1 + π/2)] \), thick solid line to \( \exp[i(W_2 + π/2)] \) and dotted line to \( \exp[-θ - iπ/2] \). Therefore, the semiclassical trace of Green’s function can be expressed as the summation over all repetitions of these orbits, and its poles are found to be when

\[
1 - e^{i(2W_1 + π)} + e^{-2θ} + \frac{e^{-2θ}}{1 - e^{i(2W_2 + π)} + e^{-2θ}} \times \left[ e^{i(W_1 + W_2 + π)} (e^{iα} + e^{-iα}) \right.
\]
\[
\left. e^{i(2W_1 + π)} + e^{i(2W_2 + π)} \right] = 0 \quad (3.5)
\]

If we neglect, for the moment, the tunneling probability \( e^{-2θ} \), then the poles give the EBK energies \( E_n \) for upper well where

\[
S_1(E_n) = 2W_1(E_n) = 2π \left( n + \frac{1}{2} \right), \quad (3.6)
\]

with \( n = 0 \) for the ground state. Therefore, we can expand the actions around the EBK energy like

\[
2W_1(E) = π + T_1(E - E_0) + \cdots, \quad 2W_2(E) = π + 2kπ + T_2(E - E_0) + \cdots \quad (3.7)
\]

near the ground state of upper well, where \( T_1 \) and \( T_2 \) are given by \( dS_1/dE \) and \( dS_2/dE \), respectively. Here, we used the fact that the difference of two actions at EBK energy is \( 2πk \) for the transition between ground state in upper well and \( k \)th excited state in lower well. Expanding Eq. (3.5) up to the first-order in \( e^{-2θ} \),

\[
T_1 T_2(E - E_0)^2 - e^{-2θ} \left[ (-1)^k (e^{iα} + e^{-iα}) + 2 \right] = 0 \quad (3.8)
\]

The final energy splitting is obtained as
\[ E = E_0 \pm \frac{2}{\sqrt{1 + T^2}} e^{-\phi \cos \alpha + \frac{\alpha}{2}} \text{ even } k, \]
\[ E = E_0 \pm \frac{2}{\sqrt{1 + T^2}} e^{-\phi \sin \alpha + \frac{\alpha}{2}} \text{ odd } k, \]

which is just the parity effect in the asymmetric system of Ref. [17].

IV. DISCUSSIONS AND CONCLUSIONS

Until now, we have analyzed the Wess-Zumino phase of biaxial spin system in \((p, \phi)\) phase space, and obtained not only the same results with those in the mapping formalism for ground state, but also the excited energy level splitting oscillations. Furthermore, the parity effect in the asymmetric system shown in recent experiment [17] is derived by both the geometrical analysis and the complex periodic orbit theory. However, the quantitative discrepancy with experiment in the period of oscillation exists, i.e., about 1.6 times larger oscillation period than expected in the theory. In order to resolve these discrepancies the authors of Ref. [17] introduced the fourth-order term \(C(S^4_x + S^4_y)\) in spin Hamiltonian (Eq. (2.4)), where \(C\) is the adjustable parameter which is shown to be \(-2.9 \times 10^{-5}\) K in Kelvin unit through numerical diagonalization, when anisotropy constants are given by \(D = 0.292\) K and \(E = 0.046\) K. Then the fourth-order Hamiltonian is expressed in the \(S_z\)-representation as

\[ H_1 = H' + 2C(S^4_x + S^4_y - 6S^2_zS^2_y) \]

within the semiclassical approximation. In fact, if we consider the classical commutation relation satisfied by angular momenta, the cubic and quadratic terms must be added in this Hamiltonian. But we have ignored them since their contributions are negligible. It is not possible to get an analytical effective potential through integration, since Eq. (4.1) beyond the Gaussian approximation. Therefore, as the field \(h\) increases, the saddle point moves more slowly to the north-pole, compared with the \(C = 0\) case. If we use the same analogy about finding the dominant tunneling path, this means that the slower decrease of the area surrounded by tunneling path, and thus the larger oscillation period. Although it is possible to get a qualitative correction to \(C = 0\) case in the desirable direction, in order to be consistent with experiment, the adjustable parameter \(C\) must be about 3 times greater than the present value.

The quantitative disagreement implies that the tunneling paths which follow all energy minima are no longer the dominant paths in the quartic case. In order to treat the quartic case exactly, new theoretical approach in order to obtain the dominant tunneling path in phase space is needed.

In conclusion, we derive both the ground and excited tunnel splitting oscillations in the biaxial nanomagnetic particle with the magnetic field along the hard anisotropy axis by finding the dominant tunneling path and geometrically analyzing the topological Wess-Zumino phase in the phase space. All the results of analysis are in agreement with the numerical diagonalization. Furthermore, the interesting parity effect in the asymmetric case is naturally clear in this analysis, and also certified within the complex periodic orbit theory. We also discussed the possibility of improving the discrepancies with experiment in the period, by introducing the quartic terms in spin variables into Hamiltonian. In order to resolve the discrepancy quantitatively, the new approach to find the dominant tunneling path in quartic case is required.

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* Electric mail: phystyr@tiger.seonam.ac.kr.
† Present address: Division of Theoretical Mechanics, School of Mathematical Sciences, University of Nottingham, NG7 2RD, U.K.
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FIG. 1. The contour plots of Hamiltonian $H_0$ and the tunneling paths (thick solid lines) which follow all the minima with $p$ for various external magnetic fields $h$. Here, $\lambda = 0.861$ of Fe is used. In this case, $h_1 = 0.278$ and $h_2 = 0.528$. The dashed lines mean $p = S$ lines. a) $h = 0.2$. The area corresponding to the geometrical phase is the region between the tunneling path and $p = S$ line. b) $h = 0.5$. The phase is evaluated as the area of (region B + region C - region A). c) $h = 0.7$. When the path encounters the contour of $E_0$, it follows the contour in real time (thick solid line with arrow). The phase is also the area of (region B + region C - region A) but yields zero. For the excited states, see the text.

FIG. 2. The typical asymmetric contour plot of the Hamiltonian with both longitudinal ($h'$) and transverse ($h$) magnetic fields. The dominant path starts at the left well minimum $E_{l_0}$, and when it encounters the contour of $k$th excited state in right well $E_{r_k} (= E_{l_0})$, it follows the contour (thick solid line with arrows).

FIG. 3. The asymmetric effective potential when the longitudinal field is applied. The actions are denoted by $S_1$, $S_2$ and $S_c$. Actually, $\phi_A$ and $\phi_A'$ are same positions, but we use the different symbols in order to distinguish the direction.

FIG. 4. The diagram representation of all possible periodic orbits in the asymmetric potential. The segments represent $e^{i(W_1 + \pi/2)}$ (solid line), $e^{i(W_2 + \pi/2)}$ (thick solid line) and $e^{-i\Theta - i\pi/2}$ (dotted line).

FIG. 5. The energy barrier of Hamiltonians $\mathcal{H}_1(p, \phi = \pi/2)$ for $C = 0$ K (solid line), and $C = -2.9 \times 10^{-5}$ K (dashed line) at $h = 0.15$. As $h$ increases, the barrier minimum (saddle point) moves to the right. But the movement of that for $C = 0$ K case is faster than for $C = 2.9 \times 10^{-5}$ K case. If we can think these minima as the passage that is taken by the dominant tunneling path, the area reduction in phase space is smaller for $C = 2.9 \times 10^{-5}$ K case.
Fig. 1(a)
Fig. 1(b)
Fig. 2
Fig. 3
\[(\quad)^2 + (\quad)^2\]

\[+ (\quad)e^{i\alpha/2} + (\quad)e^{-i\alpha/2}\]

\[\times \sum_{m=0}^{\infty} \left[ (\quad)^2 + (\quad)^2 \right]^m\]

\[\times (\quad)e^{i\alpha/2} + (\quad)e^{-i\alpha/2}\]

\[\quad\]: classical segment for \(W_1\)

\[\quad\quad\quad\text{classical segment for } W_2\]

\[\quad\quad\quad\text{tunneling segment for } \Theta\]

**Fig. 4**
Energy barrier at $\phi = \pi/2$:

- $h = 0.15$
- $C = 2.9 \times 10^{-5}$ K
- $C = 0$ K

Fig. 5