Lightning Stars: Anomalous Photoproduction in Neutron Stars via Parametric Resonance Mechanism

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In this work we propose a new mechanism for photoproduction inside a neutron star based on Parametric Resonance phenomenon as firstly applied to Inflationary Cosmology. Our assumptions are based on the pion condensation model by Harrington and Shepard. We show that a huge number of photons are created which, on turns, reheats the matter in the star’s core. Thus, we argue that Parametric Resonance can be effective during a brief period out of a neutron star lifetime leading to an anomalous uprising variation of its brightness departing from the black body radiation at regularly spaced frequencies. In addition, a time periodic signal is obtained in moderate (not exponential) regimes. We argue also that our PR mechanism offers a simple and feasible explanation for some recent observations of giant flares from neutron stars.

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I. INTRODUCTION

It is well known that in Inflationary Cosmology the early Universe turns out be depleted of its matter content at the end of inflation era. So a mechanism is necessary in order to fill up the Universe with baryonic matter (baryogenesis), as well dark matter. In current models, matter must be generated at a very high temperature, after the inflation era, in order to start baryogenesis. A mechanism, firstly proposed by Traschen and Brandenberger, \(^1\) and independently by Dolgov and Kirilova, \(^2\) based on Parametric Resonance (PR) phenomenon \(^3\), succeed to provide such requirements as a possible way to generate matter in post-inflationary Universe with an exponential particle production. Few years later, Kofman et al \(^4\) showed that this mechanism is effective mainly in the so-called preheating phase of the inflation era.

The simplest model in Field Theory consists of two boson fields, namely, a self-interacting source field \(\phi\) and a resonant (or matter) field \(\chi\). From the mathematical point of view, the source field \(\phi\) can be obtained as solution of an ordinary second order nonlinear differential equation. Periodic solutions are founded when the field \(\phi\) is in the oscillatory regime around the minimum of its potential (for example, in \(\lambda \phi^4\) model). In the simplest (spatial homogeneous) case, one looks for periodic solutions which in turns enter in the ordinary linear (time) differential equation of the matter field \(\chi\) as a periodic potential. These linear equations are particular cases of the so called Hill’s Equation \(^4\) \(^5\). The important point for these models is that Hill’s Equation presents, in its parameter space, instability regions, also named resonant bands. In a resonance band the amplitude of a \(\chi\) solution can grow exponentially and, since the particle number is proportional to this amplitude one obtains a huge number of \(\chi\) quanta \(^4\). If \(\chi\) is a boson field, the particle number occupation with a specific momentum is exponentially large. On the other hand, Dolgov and Kirilova \(^2\) showed that the PR mechanism is not very effective (particle production is not exponential) for fermions field due to Pauli blocking and the solutions presents only a polynomial growth.

Few years ago, García-Bellido and Kusenko \(^7\) proposed that the same PR mechanism could be used to explain the huge energy coming from "point-like" sources in the sky, i.e. the so-called cosmic ray bursts. In order to obtain the needed potential, bounded from below, the above authors took the collision of two neutron stars (NS) which, merging in each other produce a very compact object whose proton matter inside turns out to be superconductor and can be modelled by a Ginzburg-Landau potential \(^5\). In addition, they took electromagnetic field as a resonant field interacting with the superconductor matter. So, when the system is in the PR regime a huge burst of energy is transferred from superconductor matter to the electromagnetic field in form of high energy photons and these, in principle, could be detected on the Earth as cosmic ray bursts. As far as we known this was the first application of the PR mechanism in order to explain the bursts coming from compact objects in the sky.

Nevertheless we wonder whether PR can be applied, not to the spectacular collision of two NS, but to a single star itself in a more moderate energy scale, of course. This was motivated, after Walecka’s pioneering work \(^6\) on quantum field theory applied to stellar structure. So, differently from the above cited researchers, we apply the PR mechanism to a single NS, whose part of its matter content in the core has supranuclear densities. We are

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mainly interested in the regime in which matter in the star medium is populated with negative pion described by a field in a condensate phase \[\psi\] which oscillates around the minimum of its potential as required by the PR mechanism. This oscillation is a natural consequence of the energy excess coming from the formation of the pion condensate, that is, the system to turns in a condensate phase must reduces its energy and therefore is plausible to suppose that this excess is then transferred to the condensate field causing its motion. As in García-Belido and Kusenkos’s model, the role of resonant field is played by the electromagnetic field interacting with this oscillatory field.

We shall show below that the proposed model leads to a very efficient, although not thermal, photoproduction. We expect that in a second phase (quite the same as in Inflationary Cosmology) the production will thermalize with the surrounding matter and consequently changing the equation of state (EoS) of matter inside the star. Of course, a careful study of energy transport is needed in order to evaluate the impact of this kind of photoproduction to the star evolution. In this work we just show that Parametric Resonance is a plausible mechanism for the pion-photon model leading to an anomalous and fast energy production in the interior of compact stars. The new, nevertheless important, questions above mentioned deserve, of course, a careful further study.

II. PION CONDENSATION IN NEUTRON STARS

Neutron stars have provided us with information on the properties of high density nuclear matter. Accumulated data have given useful information about reliable EoS and thermal properties of the NS nuclear matter which seem to suggest the appearing of some phase transitions there, e.g, pion condensation. The EoS then can be used to determine NS parameters, most notably the mass-radius relationship and the maximum allowable mass \[M_{\text{max}}\].

An interesting phase transition that can take place inside a NS is the pion condensation. This possible condensation is due, to a large extent, to the high density of the neutron star medium. The possibility of pion condensation in nuclear or neutron matter was initially proposed by Migdal \[11\], Sawyer \[12\] and Scalapino \[13\]. Whether pion condensate exist or not is of great interest since it can significantly enhance the cooling rates of NS \[14, 15\]. A very fast cooling for NS may be explained by a possible pion condensation \[16\]. PR mechanism start to work just after the condensate is formed as we shall show in the following sections.

Condensed phase of negative pion \((\pi^-)\) can take place if we neglect the strong pion correlations with ordinary matter in modifying the pion self energy. It is possible to show (see ref. \[17\]) that the favorable condition to this assumption is satisfied for a neutron on the top of the Fermi sea to turn a proton and \(\pi^-\) when

\[\mu_n - \mu_p = \mu_e > m_\pi,\]

where \(m_\pi \approx 140\) MeV is the \(\pi^-\) rest mass. When this occurs, we say that the negative pion macroscopically occupy a single mode in its ground state, that is, they are in a condensate state.

The \(\pi^0\) condensation can take place also, since its effective mass in the medium is zero \[17\]. Thus, the decay reaction occurs

\[n \rightarrow n + \pi^0.\]

Possible condensation of positive pion \((\pi^+)\) will not taken into account here since Sawyer and Yao \[18\] had showed that in NS medium the number of \(\pi^+\) particles is much smaller than the \(\pi^-\) particles. Therefore, even when \(\pi^+\) condensation occurs it could be neglected when compared with the \(\pi^-\) condensation. Neutral pion does not interact with the electromagnetic field of the NS and the \(\pi^+\) condensate will does not affect the electromagnetic field in significant way, considering the above argument. So, for the photoproduction, in our case, we can take into account only the interaction between the negative pion condensate and the electromagnetic field of the NS. Henceforth, we will indistinctly use the wording pion condensate as \(\pi^-\) condensate.

III. GENERAL FORMALISM IN THE SIGMA MODEL

In order to get a \(\pi^-\) condensate we will use the Lagrangian formalism for the linear \(\sigma\) model. Although the linear \(\sigma\) model is a toy model it is enough, for our purposes, to do a qualitative study of the PR phenomenon.

We start by considering the Lagrangian density for the electromagnetic field \(A^\mu\), the isoscalar \(\sigma\), the isotriplet \(\pi\), and the nucleon isodoublet \(\Psi\) \[14, 20\]:

\[\mathcal{L} = \mathcal{L}_A + \mathcal{L}_\sigma + \mathcal{L}_{\pi-\pi^+} + \mathcal{L}_{\pi^0} + \mathcal{L}_\Psi - \mathcal{L}_{\text{int}},\]

where

\[\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e^2 (A^\mu)^2,\]

is the electromagnetic field contribution,

\[\mathcal{L}_\sigma = \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} m_\sigma^2 \sigma^2 - \frac{\lambda^2}{4} \sigma^4 + c_1 \sigma,\]

is the \(\sigma\) contribution. Note that \(c_1\sigma\) term is responsible for the spontaneous symmetry breaking in this model.
\[ L_{\pi^-\pi^+} = \partial_\mu \pi^- \partial^\mu \pi^+ + \frac{1}{2} m_0^2 (2\pi^-\pi^+) - \frac{\lambda_2}{4} (2\pi^-\pi^+)^2, \]
is the charged pion contribution,

\[ L_{\pi^\alpha} = \frac{1}{2} (\partial_\mu \pi^\alpha)^2 + \frac{1}{2} m_0^2 \pi^\alpha^2 - \frac{\lambda_2}{4} \pi^\alpha^2, \]

represents the neutral pion contribution,

\[ L_\Psi = \bar{\Psi} [i\gamma_\mu \partial^\mu] \Psi, \]
denote the nucleon contribution and finally, the interaction Lagrangian density reads

\[ L_{\text{int}} = -ie(\pi^+ \partial_\mu \pi^- - \pi^- \partial_\mu \pi^+) A^\mu - \bar{\Psi} [e\gamma_\mu A^\mu \frac{1}{2} (1 + \tau_3) - g(\sigma + i\vec{\tau} \cdot \vec{\pi} \gamma_5)] \Psi + \frac{\lambda_2}{4} [\sigma^2 \pi^\alpha^2 + \sigma^2 (2\pi^-\pi^+) + \pi^\alpha^2 (2\pi^-\pi^+)] \]

where \( \tau_j = 1, 2, 3 \) are the Pauli matrices and

\[ \gamma_5 = - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

The nucleon isodoublet \( \Psi \) contains four-component proton and neutron spinor fields and can be written as

\[ \Psi = \begin{pmatrix} \Psi_p \\ \Psi_n \end{pmatrix}, \]

and \( \pi \) denotes the pion field operator that annihilates a \( \pi^- \) (or creates a \( \pi^+ \)), which we can write as

\[ \pi^\pm = \frac{\pi_1 \pm i\pi_2}{\sqrt{2}}, \]

where \( \pi_1 \) and \( \pi_2 \) are real components of the pion field and \( \vec{\pi} = (\pi^0, \pi^+, \pi^-) \).

The constants in the Lagrangian density can be related (in tree approximation) to the physical quantities \( f_\pi \approx 93 \) MeV, \( m_\pi \approx 140 \) MeV \cite{21, 22},

\[ c_1 = f_\pi m_\pi^2, \]
\[ 2\lambda_2 f_\pi^2 = m_\sigma^2 - m_\pi^2, \]
\[ 2m_0^2 = m_\sigma^2 - 3m_\pi^2. \]

Note that if we add a symmetry breaking term in the \( \pi \) Lagrangian density, instead in \( \sigma \) Lagrangian, then different constants must be defined \cite{21}. Finally \( e = \frac{\alpha}{\sqrt{2}} \) is the electromagnetic coupling constant and \( g = f_\pi m, \) where \( m \) is the nucleon mass (typically \( m \approx 940 \) MeV). Usually the mass of \( \sigma \) meson is taken as bigger than 1 GeV \cite{23, 24}. Without loss of generality, we take \( m_\sigma = 1020 \) MeV, and then we get \( m_0 \approx 700 \) MeV, \( \lambda \approx 7.7 \) and \( g \approx 87420 \) MeV. \textit{A priori} we do not have information about the \( \sigma \) meson and in the NS medium we have a lot of candidates to be the “\( \sigma \) meson” \cite{10}. Since this is a qualitative study we choose this value for \( m_\sigma \) due the mass of \( \phi \) meson, whose mass is around 1019 MeV.

Harrington and Shepard \cite{11, 20}, using (1), showed that the pion condensate and the strong electromagnetic field of a NS may co-exist in a particular region (shell) inside of the compact star, as well nucleons. They showed also that the pion condensate can assume a superconducting behavior, even in the presence of an intense electromagnetic field. According, we assume in this work that the pion condensate and the electromagnetic field may co-exist inside of the NS, as showed by the authors of \cite{12, 20}. In terms of Harrington and Shepard’s work the pion condensation, without a superconducting phase transition, occurs when the neutron star magnetic field \( H \) is weaker than some critical low value, \( H_{c1} \). This sets the basis for our physical model.

Now, Parametric Resonance phenomenon belongs to a class of ordinary linear second order differential equation called Hill’s Equation \cite{6}. This class possess two distinct important cases (but they are not the only ones, of course): the Mathieu’s and Lamé’s ones. In the context of particle production by the PR mechanism, Mathieu’s Equation was firstly used by Traschen and Brandenberger \cite{1} and Dolgov and Kirillova \cite{2} and Lamé’s Equation by Kofman et al \cite{3}. Following the above authors we find the equation of motion (EoM) for the pion condensate and its periodic solutions which, in turn, enter as periodic potentials in the EoM for the electromagnetic field. These are Hill’s Equations whose the most useful ones are the Mathieu and Lamé types as mentioned above.

The specific EoM for \( A^\mu \) depends only on the form of periodic solution for the pion condensate EoM. If its periodic solution is a simple trigonometric function then
the $A^\mu$’s EoM is classified as Mathieu’s type; if its solution is an elliptic function, the EoM is classified as Lamé’s type. For both types, solutions for $A^\mu$’s EoM are guaranteed by a Floquet’s theorem [6] and the behavior (growth) of a particular solution is determined, basically, by its Floquet exponent. If the Floquet exponent is a real number, the solutions are unstable, i.e., they present growth (or decay) exponentially. In the case of imaginary number, the solutions are stable (bounded).

So, in order to find the EoM for the electromagnetic field we must find firstly the EoM to the $\pi^-$ condensate, applying the Euler-Lagrange equations in (1). We get,

\[
\square \pi^- + \left[ -m_0^2 + \frac{\lambda^2}{2} \sigma^2 + \frac{\lambda^2}{2} \pi \sigma^2 \right] \pi^- + \lambda^2 \pi^+(\pi^-)^2 + ig\sqrt{2}\Psi|\pi^-\gamma^5\pi^+|^2 = 0,
\]  

(3)

where $\tau_- = (\tau_1 - i\tau_2)/\sqrt{2}$ and the back reaction of $A^\mu$ field over $\pi^-$ condensate was neglected. As mentioned above, our purpose is to study the effects of $\pi^-$ condensate over the $A^\mu$ field and then investigate the PR phenomenon inside of an isolated NS. Back reaction of $A^\mu$ field can affect the evolution of $\pi^-$ condensate, but we expect, in a first order approximation, that this contribution may be neglected as in the preheating phase of the Inflationary Cosmology [1, 4].

The $\pi^0$ field does not interact with the $A^\mu$ field, so we can take it in its ground state

$$\langle \pi^0 \rangle_{\text{vac}} = 0$$

and this implies a non condensation phase to $\pi^0$ meson [17]. The $\pi^-$ condensate is not affected by this assumption since conditions on both condensations are independent, as seen above. Now taking the approximation $\langle \pi^0 \rangle^2 \approx \langle \pi^0 \rangle^2$ we can neglect the $\pi^0$ contribution in equation (3).

Symmetry breaking term presents in $\mathcal{L}_\sigma$ (equation (11)) leads to a $\sigma$ condensation prevent us to consider $\langle \sigma \rangle_{\text{vac}} = 0$ [21, 22]. In order to estimate the $\sigma$ contribution we take $\langle \sigma \rangle = b\langle \sigma \rangle_{\text{vac}}$, where $b$ is a positive real number. Following [18, 21, 22] we may write

$$\langle \sigma \rangle_{\text{vac}} \approx f_\pi \cos \theta$$

where $\theta \neq 0$ is the chiral symmetry angle that gives information about the $\sigma$ and $\pi$ condensation. So,

$$\langle \sigma \rangle = b\langle \sigma \rangle_{\text{vac}} \approx af_\pi,$$

where $a = b \cos \theta > 0$ is a real parameter.

Then, equation (3) reads, after some algebra with the nucleon components,

\[
\square \pi^- + \left[ \frac{\lambda^2 f_\pi^2 a^2}{2} - m_0^2 \right] \pi^- + \lambda^2 \pi^+(\pi^-)^2 - ig\sqrt{2}|\Psi_n|^2 = 0.
\]  

(4)

If we take values previously obtained for $m_0, \lambda$ and $f_\pi$, we obtain two distinct cases for equation (4):

\[
a^2 < \frac{2m_0^2}{\lambda^2 f_\pi^2} \approx 1.92, \\
a^2 > \frac{2m_0^2}{\lambda^2 f_\pi^2} \approx 1.92.
\]

For the sake of simplicity we can take into account these two inequalities defining

$$\frac{\lambda^2 f_\pi^2 a^2}{2} - m_0^2 = \pm 2M^2,$$

where we have case $-2M^2$ corresponds to $a^2 < 1.92$ and case $+2M^2$ to $a^2 > 1.92$. So we can write equation (4) as

\[
\square \pi^- \pm 2M^2 \pi^- + \lambda^2 \pi^+(\pi^-)^2 - ig\sqrt{2}|\Psi_n|^2 = 0.
\]  

(5)

Also, as a first approximation, we can consider here a region inside the NS in which $\pi^-$ condensate has a homogeneous and isotropic distribution. So, we can neglect spatial contributions and then equation (5) may be rewritten as

\[
\frac{d^2}{dt^2} \pi^- \pm 2M^2 \pi^- + \lambda^2 \pi^+(\pi^-)^2 - ig\sqrt{2}|\Psi_n|^2 = 0.
\]  

(6)
We will analyze the above equation (with ± signs) in two cases: Homogeneous ($|\Psi_n|^2 = 0$) and Inhomogeneous ($|\Psi_n|^2 \neq 0$) equation. We will see that for both signs we get periodic solutions which lead to PR for the electromagnetic field.

A. Homogeneous Case ($|\Psi_n|^2 = 0$)

Neglecting nucleon contributions we can write (6) as

$$\frac{d^2}{dt^2} \Pi - 2 M^2 \Pi + \lambda^2 \Pi (\Pi)^2 = 0.$$  \hfill (7)

This ordinary nonlinear second order differential equation is not in any way a simple one. It can be solved in terms of Weierstrass’s Elliptic Functions which are periodic. We can use relation (2) and write (7) as a real ordinary nonlinear second order differential coupled system:

\[
\begin{align*}
\frac{d^2}{dt^2} \Pi_1 & = 2 M^2 \Pi_1 + \frac{\lambda^2}{2} \Pi_1^3 + \frac{\lambda^2}{2} \Pi_2^2 \Pi_1 = 0 \\
\frac{d^2}{dt^2} \Pi_2 & = 2 M^2 \Pi_2 + \frac{\lambda^2}{2} \Pi_2^3 + \frac{\lambda^2}{2} \Pi_1^2 \Pi_2 = 0,
\end{align*}
\hfill (8)
\]

where the same sign must be taken simultaneously in both equations for the factor $2 M^2$.

It is easy to see that system described by (8) is symmetric under $\Pi_1 \leftrightarrow \Pi_2$ exchange. This fact suggests similarities between amplitudes, wavelength and frequencies of $\Pi^-$ and $\Pi^+$ solutions. In the next subsection we will make a numerical analysis of this system.

1. Numerical Solution

Figure 1 shows the temporal evolution of the system (8) for the case $+2 M^2$ with $M = 140 \ MeV$ and $M = 420 \ MeV$. These values were obtained when $a^2 = 1.78$ and $a^2 = 0.54$, respectively. Since $a^2 < 1.92$, these values can be completely arbitrary (recall, $a > 0$ always). We took some values for $M$ as one pion mass and three pion mass, approximately, just for numerical analysis. A priori, any real positive value for $M$ may be used.

Figure 2 shows the temporal evolution of the system (8) for the case $-2 M^2$ with $M = 140 \ MeV$ and $M = 420 \ MeV$. These values were obtained when $a^2 = 2.1$ and $a^2 = 3.3$, respectively.

In both Figures 1 and 2, the auxiliary $\Pi_1$ and $\Pi_2$ have a periodic behavior. Therefore, as a direct consequence of this fact, the pion condensate will have a periodic behavior by (2). But it is still necessary to know what kind of regularity we have: a trigonometric one or an elliptic one.

Figure 3 shows the phase space $(\dot{\Pi}_1, \Pi_1)$ for the case $-2 M^2$. Figure 4 shows the phase space for $(\dot{\Pi}_1, \Pi_1)$ for the case $+2 M^2$. Both pictures confirm the periodic behavior of solutions for $\Pi_1$ and $\Pi_2$ and suggest us that periodic solutions shown in Figure 1 and 2 have an elliptic function behavior, that is, they could be described by Jacobi’s Elliptic Functions. Therefore, as we will see, Hill’s Equation that comes out from (1) for the $\mathcal{A}_n$’s EoM is, in fact, a Lamé’s Equation whose solutions are of elliptic type.

It must be observed that (8) is an ordinary second order differential nonlinear coupled system and this may imply a strong dependence on a given set of initial conditions. Alongside this paper we adopt, without loss of generality, $\Pi_1(0) = 0$ and $\Pi_1(0) = 1$ (the same values for $\Pi_2$, of course).

It is possible to obtain, numerically, an estimative of wavelength ($\lambda$) and frequency ($\nu$) for the cases $-2 M^2$ and $+2 M^2$, if we consider roughly $\nu \simeq \lambda^{-1}$. The Table I show some values of $\lambda$ and $\nu$.

Particle production by the PR mechanism depends strongly on the amplitude and frequency oscillation of the source field, that is, on the behavior of the $\Pi^-$ field. Determination of these parameters provide a roughly quantitative estimate of the particles number produced in the NS medium.

2. Exact Solutions

The symmetry of system (8) and its numerical analysis suggest us the ansatz

$$\Pi_2 = \Pi_1,$$

and the system is now reduced to a single ordinary nonlinear differential equation written as

$$\frac{d^2}{dt^2} \Pi_1 + 2 M^2 \Pi_1 + \lambda^2 \Pi_1^3 = 0.$$  \hfill (9)

The above equation is the so-called Duffing’s Equation and can be solved in terms of Jacobi’s Elliptic Functions. This fact is in complete agreement with the above numerical analysis. The Duffing’s theory allow us to understand the behavior of exact solutions as follow below.

Equation (9) can be transformed in a following system:

\[
\begin{align*}
\frac{d\Pi_1}{dt} &= \eta \\
\frac{d\eta}{dt} &= -\lambda^2 \Pi_1^3 + 2 M^2 \Pi_1,
\end{align*}
\]
where \( \eta = \eta(t) \) is a real function. The field direction of this system in the phase plane \((\eta, \pi_1)\) can be written as

\[
\frac{d\eta}{dt} \frac{dt}{d\pi_1} = \frac{d\eta}{d\pi_1} = -\frac{\lambda^2 \pi_1^3 \pm 2M^2\pi_1}{\eta}.
\]

Direct integration of this equation results

\[
\eta^2 + \frac{\lambda^2}{2}\pi_1^4 \pm 2M^2\pi_1^2 = 2c,
\]

where \( c \) is an integration constant with dimensions of MeV\(^4\). Figure 1 show a pictorial representation of \((\eta, \pi_1)\) phase space. In Duffing’s theory context, this \( c \) parameter can be interpreted as an energy level to a given closed trajectory. For the case \( \eta^2 + \frac{\lambda^2}{2}\pi_1^4 \pm 2M^2\pi_1^2 = 2c \), we get the so-called hard spring case. The solutions are closed trajectories centered at the origin of the coordinate system, as showed in Figure 1a. In this case, only positive values for \( c \) are permitted. For \( \eta^2 + \frac{\lambda^2}{2}\pi_1^4 - 2M^2\pi_1^2 = 2c \), we have the soft spring case and, like the previous one, the closed trajectories, which exist only near the center of the potential, represents yet periodic solutions. For these situations, each value of \( c \) represents an energy level for a given trajectory, as showed in Figure 1b. In this case, far from the center of the potential, the solutions are non-periodic, for an arbitrary but fixed value of \( c \) [26].

A separation of variables followed by an integration take equation (10), for both cases (+2\( M^2 \) and −2\( M^2 \)), into the form:

\[
\int_0^z \frac{dz'}{\sqrt{-z'^4 \pm z'^2 + \frac{p}{2}}} = \sqrt{2}Mt,
\]

where, for the sake of simplicity, we took \( t_0 = 0 \), and defined \( z = \frac{\lambda}{2M}\pi_1 \) and \( \tilde{p} = \frac{\lambda^2 c}{4M^4} \).

The above integral depends strongly on the values of \( p \). So we split it in several cases as showed below.
FIG. 3: Phase space for the case $-2M^2$: (a) $M = 140$ MeV and (b) $M = 420$ MeV. In both cases, $\lambda = 7.7$. The shape of these curve suggests periodic, bounded solutions as elliptic functions, as can be seen in [20].

FIG. 4: Phase space for the case $+2M^2$: (a) $M = 140$ MeV and (b) $M = 420$ MeV. In both cases, $\lambda = 7.7$. Also the elliptic behavior is suggested by these graphs.

**Exact Solutions: Case $-2M^2$**

For $p < -1/2$: In this case we do not have real solutions for the integral presents in (11).

For $p > 0$: It is possible to write the roots of $-z'^4 - z'^2 + \frac{p}{2} = 0$ as

$$r^2 = -\frac{1}{2} + \frac{1}{2}\sqrt{1+2p}, \quad s^2 = +\frac{1}{2} + \frac{1}{2}\sqrt{1+2p},$$

and using the table of integrals in [27] we write the integral in (11) as

$$\int_0^z \frac{dz'}{\sqrt{(z'^2 + r^2)(s^2 - z'^2)}} = \frac{1}{\sqrt{r^2 + s^2}}K[\gamma, q^2], \quad (12)$$

where $K[\gamma, q^2]$ is the elliptic integral of first kind given by

$$\int_0^{\gamma'} \frac{d\gamma}{\sqrt{(1-q^2)(1-\sin^2\gamma)}}$$

and

$$\gamma = \arcsin\left(\frac{z}{s}\frac{\sqrt{r^2 + s^2}}{\sqrt{r^2 + z'^2}}\right), \quad q^2 = \frac{s^2}{r^2 + s^2},$$

and $q^2$ is the elliptic function modulus ($0 \leq q \leq 1$).

From (11) and (12) and after some algebra one obtains

$$\frac{\lambda}{2M} \pi_1 = \frac{\sqrt{2psn}}{\sqrt{1+2p(2+sn^2)-sn^2}}, \quad (13)$$

where, for short,

$$sn = sn \left[4(1+2p)\frac{1}{2}Mt, \frac{1}{2}\frac{1+\sqrt{1+2p}}{2\sqrt{1+2p}}\right],$$

is the Jacobi’s elliptic sine that possess a double period. The behavior of this solution can be fixed by an appropriate choice of the parameters $M$ and $p$.

For $-1/2 \leq p \leq 0$: Again, using [27] we obtain two distinct solutions which depends on the roots

$$r^2 = +\frac{1}{2} + \frac{1}{2}\sqrt{1+2p}, \quad s^2 = +\frac{1}{2} - \frac{1}{2}\sqrt{1+2p},$$

FIG. 5: Pictorial representation of (a) hard spring case and (b) soft spring case. Note that in the soft spring case we have two instability points \( \pi_1 = \pm \sqrt{2M} \).

Using [27] we can write

\[
\int_0^z \frac{dz'}{\sqrt{-z'^4 + z'^2 + \frac{p^2}{2}}} = \frac{1}{r} K[\zeta, q^2],
\]

where now

\[
\zeta = \arcsin \left( \frac{r}{z} \sqrt{\frac{r^2 - s^2}{r^2 - s^2}} \right), \quad q^2 = \frac{r^2 - s^2}{r^2}.
\]

Again, from equations (14) and (11) we get

\[
\frac{\lambda}{2M} \pi_1 = \frac{\sqrt{-2p}}{1 + \sqrt{1 + 2p(1 - 2sn^2)}},
\]

and we obtain the same solution expressed by equation (13). Solutions obtained until now does not possess singularities and have a time periodic evolution. Jacobian sine behaves, roughly speaking, as a trigonometric sine.

In our numerical analysis we noted that for an appropriate fixed \( M \) (140 MeV or 420 MeV), frequencies oscillation versus \( p \) of equations in (13), (15), (17) presents a logarithm form, that is, for some fixed value of \( p \), the frequency have a slow, but growth behavior. The PR phenomenon here depends on frequency oscillation of \( \pi^- \) field and it indicates that, from a determined value of \( p \), the particle creation rate itself have a slow, but continuous growth.

B. Inhomogeneous Case (|\( \Psi_n \)|^2 ≠ 0)

The |\( \Psi_n \)|^2 field will be treated here, for the sake of simplicity, as an isotropic, homogeneous medium where interactions between the pion condensate and the \( A^\mu \) field can take place. In other words, |\( \Psi_n \)|^2 = \( \Psi_n^\dagger \Psi_n \) possess a constant value in the NS medium.

Here, |\( \Psi_n \)|^2 describe a neutron matter in its condensate phase. The condensation of neutrons can take place in the usual way, the neutrons comprise Cooper pairs which obey the Bose-Einstein statistics. In the condensed phase its ground state assume a non-zero value, that is

\[
\langle \Psi_n \rangle_{\text{vac}} \neq 0.
\]
In this phase it is well-known that neutrons can be described by a standing wave function. We take the approximation \( \Psi_n = \langle \Psi_n \rangle \) and write
\[
\langle \Psi_n \rangle = \langle \Psi_n \rangle_{\text{vac}} e^{i \vec{k} \cdot \vec{r}},
\]
where \( \vec{k} \) is the momentum of neutron condensate. If we assume isotropy of the condensate neutron we can write it as
\[
\langle \Psi_n \rangle = \langle \Psi_n \rangle_{\text{vac}} \exp[i \vec{k} \cdot \vec{r}].
\]

As we will see in a moment below, it is responsible for the PR phenomenon. We need to make some approximations in order to get an EoM which can be handled more easily. We start by neglecting nucleon contributions, i.e., we firstly study the Homogeneous Case \( |\Psi_n|^2 = 0 \). This can be made to first order approach \[13, 20\]. In the next subsection we study this case.

A. Photoproduction in the Homogeneous Case
\( |\Psi_n|^2 = 0 \)

For \( |\Psi_n|^2 = 0 \) the equation \[19\] reads
\[
\Box A^\mu - \partial^\mu \partial_\nu A^\nu + ie(\pi^+ \partial^\mu \pi^- - \pi^- \partial^\mu \pi^+) + 2e^2(\pi^- \pi^+) A^\mu - e^2 \frac{1}{2} \bar{\Psi}(1 + \tau_3) \Psi = 0. \tag{19}
\]

Note that the interaction term \( 2e^2(\pi^- \pi^+) A^\mu \) is the most interesting in the above equation to our purposes. As we will see in a moment below, it is responsible for the PR phenomenon. We need to make some approximations in order to get an EoM which can be handled more easily. We start by neglecting nucleon contributions, i.e., we firstly study the Homogeneous Case \( |\Psi_n|^2 = 0 \). This can be made to first order approach \[13, 20\]. In the next subsection we study this case.

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\Box A^\mu - \partial^\mu \partial_\nu A^\nu + ie(\pi^+ \partial^\mu \pi^- - \pi^- \partial^\mu \pi^+) + 2e^2(\pi^- \pi^+) A^\mu - e^2 \frac{1}{2} \bar{\Psi}(1 + \tau_3) \Psi = 0. \tag{19}
\]

Now, from the symmetry of pion system in \( \pi \) we can assume, for the sake of simplicity, the ansatz \( \pi_2 = \pi_1 \) the term \( ie(\pi^+ \partial^\mu \pi^- - \pi^- \partial^\mu \pi^+) \) vanishes. This ansatz was suggested by the analysis done above of the solutions for the pion condensate solutions obtained from the homogeneous system for the case \( |\Psi_n|^2 = 0 \). Thus, the above equation reduces to
\[
\Box A^\mu - \partial^\mu \partial_\nu A^\nu + 2e^2(\pi^- \pi^+) A^\mu = 0. \tag{20}
\]

Introducing the gauge field transformation
\[
A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu \zeta(\vec{r}, t), \tag{21}
\]

where \( g \) is the coupling constant presents in the interaction Lagrangian. In order to get a qualitative study we take \( h = g |\Psi_n|^2 \) as a free parameter and analyze the behavior of the pion condensate for some values of \( h \).

Virtually, the above system does not have an exact solution. So, we can make only a numerical study of its temporal evolution. Figure \[18\] shows the temporal behavior of \( \pi_1(a) \) and \( \pi_2(b) \) for \( (-2M^2) \). Note that regularity persists even for a wide range of \( h \); Figure \[19\] shows the time behavior of \( \pi_1(a) \) and \( \pi_2(b) \) for \( (+2M^2) \). Again, the regularity persists. Some irregularities in the figures are due to numerical approximation.

In this paper we will only analyze below the PR phenomenon through exact solutions of the pion condensate EoM. The fact that inhomogeneous system \[18\] possess periodic solutions suggests that the PR phenomenon can persists even in this situation.

IV. PHOTOPRODUCTION BY THE A\( ^\mu \) FIELD OSCILLATIONS

In this section we study solutions obtained for the time dependent electromagnetic field coupled to the pion condensate field. From the Lagrangian density \[11\] one obtains the EoM of the electromagnetic field which reads:

\[
\Box A^\mu - \partial^\mu \partial_\nu A^\nu + ie(\pi^+ \partial^\mu \pi^- - \pi^- \partial^\mu \pi^+) + 2e^2(\pi^- \pi^+) A^\mu - e^2 \frac{1}{2} \bar{\Psi}(1 + \tau_3) \Psi = 0. \tag{19}
\]
FIG. 6: Figures above show the behavior of $\pi_1$ and $\pi_2$ for the case $-2M^2$. For (a) and (b) $M = 140$ MeV: continuous-line ($h = 1000$) and dashed-line ($h = 0.001$). For (c) and (d) $M = 420$ MeV: continuous-line ($h = 1000$) and dashed-line ($h = 0.001$). Note that even under extreme variation of $h$ the regularity still persists ($h = g\langle \Psi_{n\text{vac}} \rangle^2$).

FIG. 7: Figures above show the behavior of $\pi_1$ and $\pi_2$ for the case $+2M^2$. For (a) and (b) $M = 140$ MeV: continuous-line ($h = 1000$) and dashed-line ($h = 0.001$). For (c) and (d) $M = 420$ MeV: continuous-line ($h = 1000$) and dashed-line ($h = 0.001$). Note that even under extreme variation of $h$ the regularity still persists ($h = g\langle \Psi_{n\text{vac}} \rangle^2$).

where $\zeta(\vec{r}, t)$ is an arbitrary function of $\vec{r}$ and $t$ and using the Lorentz gauge condition

$$\partial_\mu A^\mu = \partial_\mu A^\mu + \square \zeta(\vec{r}, t) = 0,$$

since $-\partial^\mu \partial_\nu A^\nu$ vanishes. An isotropic gauge field $A^\mu$ can be written as

$$A^\mu(x) = \chi(x)e^\mu,$$

where $\chi(x)$ can be expanded in Fourier modes and $e^\mu$ is a polarization vector.

Substituting (22) in (20) and taking into account the Lorentz gauge condition one obtains
\[ e^\mu [\Delta \chi(x) + 2\epsilon^2(\pi^-\pi^+)\chi(x)] = 0. \]  

(23)

Now, in a crude, but nevertheless unrealistic approximation, we can take also the electromagnetic field as homogenous inside the NS. So (23) reduces to an ordinary differential equation in the time variable. Taking the Fourier transform to \( k \) space momentum we finally get

\[ \frac{d^2}{dt^2} \chi_k(t) + [k^2 + 2\epsilon^2(\pi^-\pi^+)]\chi_k(t) = 0. \]  

(24)

This is a homogeneous differential equation for an oscillator with frequency (or mass) varying in time. We define the frequency oscillation of the \( \chi_k(t) \) field as

\[ \omega_k = k^2 + 2\epsilon^2(\pi^-\pi^+). \]

For the Homogeneous Case (\(|\Psi_n|^2 = 0\)) and solutions of \( \pi^- \) and \( \pi^+ \) fields, previously obtained, we may write (24) as a Lamé’s type equation.

A general Lamé’s Equation have the form

\[ \frac{d^2}{dt^2} \psi(t) + [A + BQ(t)]\psi(t) = 0, \]

where \( A \) and \( B \) are real numbers and \( Q(t) \) is an elliptic function. Phase space of the parameters \( A \) and \( B \) is not easily defined, and this is exactly the case of equation (24). Nevertheless it still possess an important property, namely, the existence of instability (exponential growth)

\[ \chi_k(t) \sim \exp[\mu_k t], \]  

(25)

where \( \mu_k \in (-\infty, \infty) \) is the so-called Floquet exponent (or characteristic exponent) labelled by the \( k \) momentum. To find exact solutions for the Lamé’s Equation is a very difficult task. For example, if the coefficient of \((\pi^-\pi^+)\) term in (24) may be written as \( n(n+1)q^2 \), where \( n \) is an integer and \( 0 \leq q \leq 1 \) is the elliptic function modulus, then an exact solution can be found. On the other hand for a qualitative study we can use \( \mu_k \) taken from the formal solution of Hill’s Equation [2], which can be written as

\[ \mu_k = \frac{1}{\pi} \cosh^{-1}[1 - \Delta(0) \sin^2(k\pi/2)], \]  

(26)

where \( \Delta(0) \) is an infinite determinant. Figure 8 shows \( \mu_k > 0 \) for arbitrary values of \( k > 0 \) momentum with several fixed \( \Delta(0) \) values.

The Floquet’s theorem [6] gives

\[ \chi_k(t) = \exp[\mu_k t] p(t), \]

where \( p(t) = p(t + T) \) is a real limited periodic function with period \( T \). When \( \mu_k > 0 \), instability corresponds to an exponential growth of occupation number

\[ n_k(t) \sim \exp[2\mu_k t], \]  

(27)

that may be interpreted as number density of particles produced by PR mechanism [2, 4].

In the following subsections will analyze the equation (24) by means solutions (13), (15) and (17) for the \( \pi^- \) condensate.

1. Homogeneous (\(|\Psi_n|^2 = 0\)): Case \(-2M^2\)

In order to obtain the Lamé’s Equations, the solutions obtained for the pion condensate EoM were substituted in equation (24). All solutions that we found for the non-linear field (source) are periodic and substituting them in (24) we got a Hill’s Equation. For the sake of simplicity, below we show some of them.

Using equation (16) and after some simplifications we write (24) as

\[ \frac{d^2}{dt^2} \chi_k(t) + \left[ k^2 + \frac{16M^2e^2p}{\lambda^2[\sqrt{1 + 2p(2 + sn^2) - sn^2}]sn^2} \right] \chi_k(t) = 0, \]  

(28)

where

\[ \omega_k = k^2 + \frac{16M^2e^2psn^2}{\lambda^2[\sqrt{1 + 2p(2 + sn^2) - sn^2}].} \]

The above equation may be classified as a Lamé’s type having a Jacobian periodicity, and, \( a \ priori \), may be write in a more handle way, i.e., as just one Jacobian sine or a sum of them. For our purposes, it is sufficient in the Hill’s theory that the potential (13) has a periodic behavior.
FIG. 8: Floquet exponent $\mu_k$ as a function of $k$ and fixed values of the determinant $\Delta(0)$: (a) $\Delta(0) = -1000$, (b) $\Delta(0) = -100$, (c) $\Delta(0) = -1$ and (d) $\Delta(0) = -0.001$. If $\Delta(0) < 0$, $\mu_k$ is real; if $\Delta(0) > 0$, $\mu_k$ is imaginary.

\[
\frac{d^2}{dt^2} \chi_k(t) + \left[ k^2 + \frac{16M^2e^2p}{\lambda^2[1 + \sqrt{1 + 2p(1 - 2sn^2)}]} \right] \chi_k(t) = 0, \tag{29}
\]

where clearly

\[
\omega_k = k^2 + \frac{16M^2e^2p}{\lambda^2[1 + \sqrt{1 + 2p(1 - 2sn^2)}]},
\]

We got again an equation whose potential is written in terms of elliptic functions and it is of Lamé's type.

Now, using (17), equation (24) can be written as

\[
\frac{d^2}{dt^2} \chi_k(t) + \left[ k^2 + \frac{16M^2e^2p}{\lambda^2[1 + \sqrt{1 + 2p(1 - 2sn^2)}]^2} \right] \chi_k(t) = 0, \tag{30}
\]

with

\[
\omega_k = k^2 + \frac{8M^2e^2(1 + \sqrt{1 + 2p})}{\lambda^2} - \frac{16M^2e^2\sqrt{1 + 2p}}{\lambda^2} sn^2.
\]

This is, clearly, a Lamé's Equation which is easier to handle than the another ones above.

2. Homogeneous $|\Psi_n|^2 = 0$: Case $+2M^2$

In this case we get just one equation. Using (13) equation (24) may be written as

\[
\frac{d^2}{dt^2} \chi_k(t) + \left[ k^2 + \frac{16M^2e^2p}{\lambda^2[\sqrt{1 + 2p(2 + sn^2)}]^{-1} - sn^2} \right] \chi_k(t) = 0, \tag{31}
\]

where

\[
\omega_k = k^2 + \frac{16M^2e^2p}{\lambda^2[\sqrt{1 + 2p(2 + sn^2)}]^{-1} - sn^2},
\]
represents its mass oscillation coefficient.

All Lamé’s Equations obtained above have only two free parameters, $M$ and $p$. These two free parameters control frequency and amplitude of pion oscillations and consequently by equation (27), also the rate of particle production of the photon field.

Energy density for each Lamé’s Equation can be calculated by [7]

$$\rho_{\gamma} = \frac{1}{2\pi^2} \int_0^{k_c} dk \omega_k n_k(t) k^2.$$  \hspace{1cm} (32)

We are not taking into account here the back reaction of the particles produced.

\[A^\nu + ie(\pi^+ \partial^\mu \pi^- - \pi^- \partial^\mu \pi^+) + 2e^2(\pi^- \pi^+)A^\nu - e\gamma^\mu \frac{1}{2}(1 + \tau_3)\Psi = 0. \hspace{1cm} (33)\]

In order to obtain a more realistic physical description is necessary introduce by hand a term, in (1) or, yet in (24), that represents the back reaction of particles produced. Since this work aims just to propose a new process of photoproduction in NS, it is sufficient to prove that the PR phenomenon could exist in the NS medium, as it is shown above.

B. Photoproduction in the Inhomogeneous Case \hspace{1cm} (|Ψ_n|^2\neq 0)

Using the gauge transformation (21) with the Lorentz condition, equation (19) can be written as

V. PARAMETRIC RESONANCE AND NEUTRONS STARS FLARES

We had showed that Parametric Resonance phenomenon, under certain conditions, may occur inside of a NS modifying briefly its thermal evolution. It is worth to mention that our approach can be implemented, not only for the Harrington and Shepard’s model, but in fact for any NS model which presents superconductor phase with periodic motions (EoM) and a suitable coupling to Electromagnetic Field to furnish a Hill’s Equation. PR mechanism can be effective enough to produce a huge number of photons, out of equilibrium with the surrounding matter, which must diffuse out from the NS. In this case, clearly, a subsequent reheating of the NS interior must occur, much the same as in the Inflationary Theory, but in a far smaller energy scale. From the mathematic point of view we have two cases to consider, namely, for real Floquet exponent and when it is an imaginary number. Both cases lead to an enhancing of the NS brightness, but the possible observable effects are expected to be very different as detailed below.

Taking for granted the current theories of NS interiors as nearly correct, in our opinion, this kind of PR phenomenon can, in principle, be detected by observation of anomalous enhancing of the NS brightness. The intensity, as well the lapse of time of this higher brightness depends on the details of diffusion of the produced photons to reach the surface of the star. This is a difficult problem since, by its turn, it depends on the model of the NS matter structure. On the other hand, recent works seem to indicate a nearly black body surface radiation from Neutron Stars whose details also depends on the model of their atmospheres [30, 31, 32, 33]. If this is correct, in the case that PR mechanism is active, it leads to some interesting results on the radiation released by a NS. We consider first the case of real Floquet exponent. On a blackbody spectrum background, PR manifests its presence, by the theory outlined above, in a weak regime, as a perturbation on NS spectrum with superimposed bumps on the equally spaced frequencies due to the periodic pattern of the Floquet exponent as a function of the momentum $k$, as is clearly seen in equation (24) and in Fig. 8. This periodicity implies that high energy photons are also produced and their spectral lines should dominate over the blackbody ones, which lead us
to expect that PR effects could be better measured on the tails of a blackbody spectrum. Furthermore, although in a qualitative ground, in the case of a very strong regime (with a large real Floquet exponent) of photon production, the external star layers can not resist against the enormous pressure due to the reheating of matter in the star core by high energy photons. In this case the star ejects its upper layers with a huge burst of energy releasing. In this way our model suggest a feasible and simple explanation for the recent observations of neutron stars giant flares $^{34, 35, 37}$. Shortly, PR mechanism with a real and large Floquet exponent is better to explain non-periodic, but giant flares from NS. On the other hand, if the Floquet exponent is imaginary, Eq. (27) shows that the number density of produced particles is a periodic function in time and so we can expect a periodic perturbation on the brightness of the NS while the PR regime is active. This is quite interesting since giant periodic flares from repeaters have been observed in the past few years. See, for example, $^{38}$. However, for this case, we can have only an incomplete view since for giant flares the unknown coefficient in Eq.(27) must be very large, since the time factor is a trigonometrical function and so a bounded function. So, a further study is needed to solve this quest.

VI. CONCLUSION

Firstly, it must be stressed that Parametric Resonance in NS can be a rare phenomenon, due to the special conditions of the superconductor matter and its coupling with the electromagnetic field. Nevertheless, in the weak PR regime, as mentioned above, it is worth to notice that, in principle, it is possible to measure an uprising regular (equally spaced frequencies) brightness variation on the nearly blackbody spectra, perhaps better measured on their tails, of those neutron stars close enough to us. In addition, in the strong PR regime, the model offers also a simple explanation for some reported rare giant flares coming from these compact objects. On the other hand the signal can also presents a time periodic pattern. In this case, as in Inflationary Theory, it is not expected an explosive behaviour of the star, but only periodic flares with low intensity.

Finally, more importantly perhaps, we are inclined to think that Parametric Resonance is a phenomenon active not only in Inflationary Cosmology but in several much smaller coupled physical systems with extreme conditions of pressure and temperature in our Universe.

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