The Locally Finite Part of the Dual Coalgebra of Quantized Irreducible Flag Manifolds

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Abstract

For quantized irreducible flag manifolds the locally finite part of the dual coalgebra is shown to coincide with a natural quotient coalgebra $U$ of $U_q(g)$. On the way the coradical filtration of $U$ is determined.

A graded version of the duality between $U$ and the quantized coordinate ring is established. This leads to a natural construction of several examples of quantized vector spaces.

As an application covariant first order differential calculi on quantized irreducible flag manifolds are classified.

1 Introduction

Let $g$ denote a complex simple Lie algebra and $G$ the corresponding connected simply connected algebraic group. The $q$-deformed universal enveloping algebra $U_q(g)$ can essentially be recovered from the $q$-deformed coordinate ring $\mathbb{C}_q[G]$ as the dual Hopf algebra $\mathbb{C}_q[G]^\circ$, [Jos95]. Thus $U := U_q(g)$ and $\mathbb{C}_q[G]$ constitute dual realizations of the same mathematical object. It is the main aim of this paper to establish an analogous duality in the case of quantized irreducible flag manifolds.

Let $P \subset G$ denote a parabolic subgroup with Levi factor $L$. The $q$-deformed coordinate ring [DK92], [DS99]

$$B := \mathbb{C}_q[G/L] := \{ b \in \mathbb{C}_q[G] | b(1) b(2)(k) = \varepsilon(k)b \text{ for all } k \in K \}$$

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where $K := U_q(l) \subset U$ and $l$ is the Lie algebra of $L$, is a right $U$-module algebra. There exists a natural pairing

$$\mathcal{B} \times \mathcal{U} \rightarrow \mathbb{C} \tag{1}$$

between the right $U$-module algebra $\mathcal{B}$ and the left $U$-module coalgebra $\mathcal{U} = U/U_K^+$ where $K^+ = \{ k \in K \mid \varepsilon(k) = 0 \}$. The dual coalgebra $\mathcal{B}^\circ$ generated by the matrix coefficients of all finite dimensional representations of $\mathcal{B}$ is a left $U$-module coalgebra. The main result of the present paper (Theorem 6.5) is the following refinement of the duality (1) for quantized irreducible flag manifolds:

$$\mathcal{B} = \{ f \in \mathcal{U}^* \mid \dim(fU) < \infty \},$$

$$\mathcal{U} = F(\mathcal{B}^\circ, K) := \{ f \in \mathcal{B}^\circ \mid \dim(Kf) < \infty \}$$

where $\mathcal{U}^*$ denotes the dual vector space of $\mathcal{U}$. Note that $F(\mathcal{B}^\circ, K)$ can be considered as an analogue of the locally finite part $F(\mathbb{C}_q[G]^\circ, U) = \{ f \in \mathbb{C}_q[G]^\circ \mid \dim(\text{ad}(U)f) < \infty \}$ where $\text{ad}$ denotes the adjoint action. Indeed $(kf)(b) = (k_{(1)}fS(k_{(2)}))(b)$ for all $b \in \mathcal{B}$, $k \in K$ and $f \in \mathbb{C}_q[G]^\circ$. The locally finite part $F(\mathbb{C}_q[G]^\circ, U)$ has in principle been determined in [JL92] (cp. also [Jos95], [HS98]).

It has been explained in [HK03a] that determining the $K$-module $F(\mathcal{B}^\circ, K)$ is the main step to classify covariant first order differential calculi over $\mathcal{B}$ in the sense of Woronowicz [Wor89]. As an application of Theorem 6.5 it is shown that there exist precisely two nonisomorphic irreducible finite dimensional covariant first order differential calculi over $\mathcal{B}$.

To determine $F(\mathcal{B}^\circ, K)$ several auxiliary results are proven which are of interest on their own. First, it is necessary to write $\mathcal{B}$ in terms of generators and relations. It has been shown in [DS99], [Sto02] that $\mathcal{B}$ is generated by certain products of matrix coefficients. Here, a new proof for all generalized flag manifolds is given relying mainly on the fact that this statement is equivalent to its classical analogue.

Moreover, in the case of irreducible generalized flag manifolds the coradical filtration of the connected coalgebra $\mathcal{U}$ is determined. It is proved that the associated graded coalgebra is the graded dual of the graded algebra $\bigoplus_{k=0}^\infty (\mathcal{B}^+)^k/(\mathcal{B}^+)^{k+1}$ where $\mathcal{B}^+ = \{ b \in \mathcal{B} \mid b(1) = 0 \}$. This yields a natural construction of several examples of quantized vector spaces such as quantum $n \times m$-matrices or quantum orthogonal vector spaces [FRT89].

The assertions of this paper concerning the locally finite part $F(\mathcal{B}^\circ, K)$ and differential calculi over $\mathcal{B}$ are generalizations of results obtained in the case of quantum complex Grassmann manifolds in [Kol02].
The ordering of the paper is as follows. Classical generalized flag mani-
Ifolds are recalled in Section 2 from a point of view also suitable in the
quantum case. In particular the coordinate algebra \( \mathbb{C}[G/L] \) is given in terms
of generators and relations.

The third section is devoted to the relevant notions of quantum groups
and quantum flag manifolds. \( R \)-matrices corresponding to certain represen-
tations of \( U \) appear in the relations of \( \mathbb{C}_q[G/L] \). For explicit calculations it is
useful to know how weight spaces are transformed under the action of these
\( R \)-matrices. Finally \( \mathbb{C}_q[G/L] \) is identified with the subalgebra \( A_\lambda \subset \mathbb{C}_q[G] \)
generated by certain products of matrix coefficients.

In Section 4 it is shown that \( F(B^\circ, K) \) is connected, i.e. the coradical of
\( F(B^\circ, K) \) is spanned by the counit \( \varepsilon \in U \). This result is obtained by explicit
calculations making consequent use of the properties of the \( R \)-matrices.

In Section 5 the coradical filtration of \( U \) is calculated in the irreducible
case. To determine \( F(B^\circ, K) \) in Section 6 it is shown that the pairing
\( B/(B^+)^{k+1} \otimes C_k U \to \mathbb{C} \) is nondegenerate where \( C_k U \) denotes the elements of
degree \( k \) with respect to the coradical filtration. Combined with the results
of Section 4 this implies that \( F(B^\circ, K) = U \). The graded duality is estab-
ish ed and the resulting coordinate algebras of quantized vector spaces are
discussed.

In the final Section 7 the notion of covariant first order differential calculus
is recalled and irreducible covariant first order differential calculi over \( B \) are
classified.

Throughout this paper several filtrations are defined in the following way.
Let \( A \) denote an algebra generated by the elements of a set \( Z \) and \( S \) a totally
ordered abelian semigroup. Then any map \( \deg : Z \to S \) defines a filtration
\( F \) of the algebra \( A \) as follows. An element \( a \in A \) belongs to \( F_n, n \in S \),
if and only if it can be written as a polynomial in the elements of \( Z \) such
that every occurring summand \( a_1 \ldots k z_1 \ldots z_k \), \( a_1 \ldots k \in \mathbb{C} \), \( z_i \in Z \), satisfies
\( \sum_{j=1}^k \deg(z_j) \leq n \). Instead of \( a \in F_n \) by slight abuse of notation we will also
write \( \deg(a) = n \).

For any Hopf algebra \( H \) the symbols \( \Delta, \varepsilon, \) and \( \kappa \) will denote the coproduct,
counit, and antipode, respectively. Sweedler notation for coproducts
\( \Delta a = a(1) \otimes a(2) \), \( a \in H \), will be used. If the antipode \( \kappa \) is invertible we will
frequently identify left and right \( H \)-module structures on a vector space \( V \)
by \( \nu h = \kappa^{-1}(h)v, v \in V, h \in H \).

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2 Generalized Flag Manifolds

First, to fix notations some general notions related to Lie algebras are recalled. Let \( \mathfrak{g} \) be a finite dimensional complex simple Lie algebra and \( \mathfrak{h} \subset \mathfrak{g} \) a fixed Cartan subalgebra. Let \( R \subset \mathfrak{h}^* \) denote the root system associated with \( (\mathfrak{g}, \mathfrak{h}) \). Choose an ordered basis \( \pi = \{\alpha_1, \ldots, \alpha_r\} \) of simple roots for \( R \). Let \( R^+ \) and \( R^- \) be the set of positive and negative roots with respect to \( \pi \), respectively. Moreover, let \( \mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \) be the corresponding triangular decomposition. Identify \( \mathfrak{h} \) with its dual via the Killing form. The induced non-degenerate symmetric bilinear form on \( \mathfrak{h}^* \) is denoted by \( (\cdot, \cdot) \). The root lattice \( Q = \mathbb{Z}R \) is contained in the weight lattice \( P = \{\lambda \in \mathfrak{h}^* | (\lambda, \alpha_i)/d_i \in \mathbb{Z} \forall \alpha_i \in \pi\} \) where \( d_i := (\alpha_i, \alpha_i)/2 \). In order to avoid roots of the deformation parameter \( q \) in the following sections we rescale \( (\cdot, \cdot) \) such that \( (\cdot, \cdot) : P \times P \to \mathbb{Z} \).

For \( \mu, \nu \in P \) we will write \( \mu \succ \nu \) if \( \mu - \nu \) is a sum of positive roots and \( \mu \not\succ \nu \) if \( \mu > \nu \) and \( \mu \not\succeq \nu \). As usual we define \( Q^+ := \{\mu \in Q | \mu > 0\} \). The height \( \text{ht} : Q_+ \to \mathbb{N}_0 \) is given by \( \text{ht}(\sum_{i=1}^r n_i \alpha_i) = \sum_{i=1}^r n_i \). Let \( \succeq \) denote the lexicographic ordering on \( \mathbb{Q} \otimes \mathbb{Z} Q \) with respect to the ordered set \( \pi \) of simple roots: \( \mu \succeq 0, \mu \in \mathbb{Q} \otimes \mathbb{Z} Q \), if and only if there exists \( j \in \{1, \ldots, r\} \) such that \( \mu = \sum_{i=j}^r \mu_i \alpha_i, \mu_i \in \mathbb{Q}, \text{ and } \mu_j > 0 \). We write \( \mu \succ 0 \) if \( \mu \not\succeq 0 \) or \( \mu = 0 \). Note in particular that \( \succ \) induces a total ordering on \( P \).

The fundamental weights \( \omega_i \in \mathfrak{h}^*, i = 1, \ldots, r \) are characterized by \( (\omega_i, \alpha_j)/d_j = \delta_{ij} \). Let \( P^+ \) denote the set of dominant weights, i.e. the \( \mathbb{N}_0 \)-span of \( \{\omega_i | i = 1, \ldots, r\} \). Recall that \( (a_{ij}) := (2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)) \) is the Cartan matrix of \( \mathfrak{g} \) with respect to \( \pi \).

For \( \mu \in P^+ \) let \( V(\mu) \) denote the uniquely determined finite dimensional irreducible left \( \mathfrak{g} \)-module with highest weight \( \mu \). More explicitly there exists a nontrivial vector \( v_\mu \in V(\mu) \) satisfying

\[
Ev_\mu = 0, \quad Hv_\mu = \mu(H)v_\mu \quad \text{for all } H \in \mathfrak{h}, E \in \mathfrak{n}_+.
\]  

(2)

For any weight vector \( v \in V(\mu) \) let \( \text{wt}(v) \in P \) denote the weight of \( v \), that is \( Hv = \text{wt}(v)(H)v \) for any \( H \in \mathfrak{h} \). In particular \( \text{wt}(v_1) - \text{wt}(v_2) \in Q \) for all weight vectors \( v_1, v_2 \in V(\mu) \).

Let \( W \) denote the Weyl group of \( \mathfrak{g} \). Recall that for any finite dimensional \( \mathfrak{g} \)-module \( V \) the Weyl group permutes the weight spaces \( V_\mu \) of \( V \) and \( \dim V_{w\mu} = \dim V_\mu \) for all \( w \in W \). If \( w_0 \) is a longest element in \( W \) then \( w_0 \mu \) is the lowest weight of the irreducible representation \( V(\mu) \).

Let \( G \) denote the connected, simply connected complex Lie group with Lie algebra \( \mathfrak{g} \). Recall that \( G \) is complex algebraic and let \( \mathbb{C}[G] \) denote its coordinate ring. The finite dimensional representations of \( \mathfrak{g} \) are in one-to-one
correspondence to the finite dimensional rational $G$-modules. For $v \in V(\mu)$, $f \in V(\mu)^*$ the matrix coefficient $c_{f,v}^{\mu} \in \mathbb{C}[G]$ is defined by

$$c_{f,v}^{\mu}(g) = f(gv).$$

The linear span of matrix coefficients of $V(\mu)^*$

$$C^V(\mu) := \text{Lin}_\mathbb{C}\{c_{f,v}^{\mu} \mid v \in V(\mu), f \in V(\mu)^*\}$$

obtains a $G$-bimodule structure by

$$(hc_{f,v}^{\mu}k)(g) = f(kghv) = c_{f,hv}^{\mu}(g), \quad g, h, k \in G. \quad (4)$$

Here $V(\mu)^*$ is considered as a right $G$-module. In the same way $C^V(\mu)$ can be endowed with a $U(\mathfrak{g})$-bimodule structure. As $G$ is a closed subgroup of $\text{GL}(n)$ for some $n \in \mathbb{N}$ one obtains

$$\mathbb{C}[G] \cong \bigoplus_{\mu \in P^+} C^V(\mu). \quad (5)$$

For any set $S \subset \pi$ of simple roots define $R_S^\pm := \mathbb{Z}S \cap R^\pm$ and $R_S^- := R^\pm \setminus R_S^\pm$. Let $P_S$ and $P_S^{\text{op}}$ denote the corresponding standard parabolic subgroups of $G$ with Lie algebra

$$\mathfrak{p}_S = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+ \cup R_S^-} \mathfrak{g}_\alpha, \quad \mathfrak{p}_S^{\text{op}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^- \cup R_S^+} \mathfrak{g}_\alpha \quad (6)$$

The generalized flag manifold $G/P_S$ is called irreducible if the adjoint representation of $\mathfrak{p}_S$ on $\mathfrak{g}/\mathfrak{p}_S$ is irreducible. Equivalently, $S = \pi \setminus \{\alpha_i\}$ where $\alpha_i$ appears in any positive root with coefficient at most one. For a complete list of all irreducible flag manifolds consult e.g. [BE89, p. 27].

One can associate several algebras of functions to the generalized flag manifold $G/P_S$. We identify the fixed subset $S = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$ with the corresponding subset $\{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}$.

First consider the irreducible representation $V(\lambda)$ of $\mathfrak{g}$ of highest weight $\lambda = \sum_{i \in \pi \setminus S} \omega_i$. Then $G/P_S$ is isomorphic to the $G$-orbit of the highest weight vector $v_{\lambda} \in V(\lambda)$ in projective space $\mathbb{P}(V(\lambda))$. Therefore the homogeneous coordinate ring $S[G/P_S]$ of $G/P_S$ coincides with the subalgebra of $\mathbb{C}[G]$ generated by the matrix coefficients \{ $c_{f,v_{\lambda}}^\lambda \mid f \in V(\lambda)^*$ \}.

Recall that there is an isomorphism of right $U(\mathfrak{g})$-module algebras

$$S[G/P_S] \xrightarrow{\cong} \bigoplus_{n=0}^\infty V(n\lambda)^*$$
where the multiplicative structure on the right hand side is given by the Cartan multiplication

\[ V(n_1\lambda)^* \otimes V(n_2\lambda)^* \rightarrow V((n_1 + n_2)\lambda)^*. \]

Moreover by a theorem of Kostant (cf. [FH91]) the algebra \( S[G/P_S] \) is quadratic, i.e.

\[ S[G/P_S] \cong \bigoplus_{n=0}^{\infty} (V(\lambda)^*)^\otimes^n / \mathcal{I}(\lambda), \]

where \( \mathcal{I}(\lambda) \) denotes the ideal in the tensor algebra generated by the subspace

\[ \bigoplus_{\mu \neq 2\lambda} V(\mu)^* \subset (V(\lambda)^*)^\otimes 2. \]

Similarly \( G/P_S^{op} \) is isomorphic to the \( G \)-orbit of the lowest weight vector \( f_{-\lambda} \in V(-w_0\lambda) \cong V(\lambda)^* \) in \( \mathbb{P}(V(\lambda)^*) \). Therefore \( S[G/P_S^{op}] \) is isomorphic to the subalgebra of \( \mathbb{C}[G] \) generated by the matrix coefficients \( \{ c_{v,f_{-\lambda}}^{-w_0\lambda} \mid v \in V(-w_0\lambda)^* \cong V(\lambda) \} \). Again there exists an isomorphism

\[ S[G/P_S^{op}] \cong \bigoplus_{n=0}^{\infty} V(n\lambda) \]

of right \( U(\mathfrak{g}) \)-module algebras and \( S[G/P_S^{op}] \) is quadratic.

Let \( \mathbb{A}_\lambda \subset \mathbb{C}[G] \) denote the subalgebra generated by the elements

\[ \left\{ z_{fv} := \frac{c^\lambda_{f,v\lambda}c^{-w_0\lambda}_{v,f_{-\lambda}}}{f_{-\lambda}(v\lambda)} \right\} | f \in V(\lambda)^*, v \in V(\lambda) \}. \]  

By construction the space of \( n \)-fold products of the generators \( z_{fv} \) of \( \mathbb{A}_\lambda \) is isomorphic to \( V(n\lambda)^* \otimes V(n\lambda) \). Define \( N := \dim V(\lambda) \) and \( I := \{1, \ldots, N\} \). If \( \{v_i \mid i \in I\} \) and \( \{f_i \mid i \in I\} \) are dual bases of \( V(\lambda) \) and \( V(\lambda)^* \), respectively, then

\[ \sum_{i \in I} z_{f_i v_i} = f_{-\lambda}(v\lambda)^{-1} \sum_{i \in I} c^\lambda_{f_i,v\lambda} c^{-w_0\lambda}_{v_i,f_{-\lambda}} = 1. \]

Therefore there is a natural inclusion of the space of \( n \)-fold products of the generators \( z_{fv} \) of \( \mathbb{A}_\lambda \) into the space of \( (n+1) \)-fold products given by multiplication with \( \sum_{i \in I} z_{f_i v_i} \). With respect to these inclusions \( \mathbb{A}_\lambda \) can be written as a direct limit

\[ \mathbb{A}_\lambda \cong \lim_{\rightarrow} V(n\lambda)^* \otimes V(n\lambda). \]
Thus a complete set of defining relations of $A_{\lambda}$ is given by linearity in the first and second index of $z$ and

\[
\sum_i z_{g_i u_i z_{h_i w_i}} = 0 \quad \text{if} \quad \sum_i g_i \otimes h_i \in \bigoplus_{\mu \neq 2\lambda} V(\mu)^*, \\
\sum_i z_{g_i u_i z_{h_i w_i}} = 0 \quad \text{if} \quad \sum_i u_i \otimes w_i \in \bigoplus_{\mu \neq 2\lambda} V(\mu), \\
\sum_{i \in I} z_{f_i v_i} = 1,
\]

where as above $\{v_i \mid i \in I\}$ is a basis of $V(\lambda)$ with dual basis $\{f_i \mid i \in I\}$.

The algebra $A_{\lambda}$ has a straightforward geometric interpretation. Consider

\[I_S = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathcal{Z}_S} \mathfrak{g}_\alpha\]

of $p_S$ and let $L_S = P_S \cap P_{s_{\text{op}}} \subset G$ denote the corresponding subgroup. As $L_S$ is reductive the quotient $G/L_S$ is an affine algebraic variety with coordinate ring [HK62]

\[\mathbb{C}[G/L_S] = \mathbb{C}[G]^{L_S} := \{a \in \mathbb{C}[G] \mid a(g) = a(gl) \forall g \in G, l \in L_S\}.
\]

Note that by construction $A_{\lambda} \subset \mathbb{C}[G/L_S]$. It is our next aim to show that $A_{\lambda} = \mathbb{C}[G/L_S]$.

Being reduced $A_{\lambda}$ is the coordinate ring of an affine algebraic variety. The corresponding algebraic set $Z(A_{\lambda})$ can be identified with

\[
\{z \in \text{End}(V(\lambda)) \mid \text{trace}(z) = 1, \text{rank } z \leq 1, \text{Im } z \in G v_{\lambda} \subset \mathbb{P}(V(\lambda)), \text{Im } z^t \in G f_{-\lambda} \subset \mathbb{P}(V(\lambda)^*)\}
\]

where $z^t$ denotes the transposed map. Indeed, rank $z \leq 1$ is equivalent to (10) for antisymmetric elements $\sum_i g_i \otimes h_i$. Moreover, in view of (7) Equation (10) with $u = v$ also implies that $\text{Im } z \in G v_{\lambda} \subset \mathbb{P}(V(\lambda))$.

One now verifies that the map

\[
Z(A_{\lambda}) \to \{(v, f) \in G v_{\lambda} \times G f_{-\lambda} \subset \mathbb{P}(V(\lambda)) \times \mathbb{P}(V(\lambda)^*) \mid f(v) \neq 0\}
\]

\[z \mapsto (\text{Im } z, \text{Im } z^t)\]

is an isomorphism of quasiprojective varieties with inverse morphism

\[
(v, f) \mapsto z = (z_{ij})_{i,j=1,\ldots,N}, \quad z_{ij} = \frac{f_i(v) v_j(f)}{f(v)}
\]
where \( z \in \text{End}(V(\lambda)) \) is given with respect to the basis chosen above. Using the identification (15) one obtains a morphism
\[
\psi : G/L_S \to Z(A_\lambda), \quad g \mapsto (gv_\lambda, gf^-_\lambda).
\]
Note that by (16) this morphism corresponds to the inclusion \( A_\lambda \hookrightarrow \mathbb{C}[G/L_S] \) of coordinate rings.

**Proposition 2.1.** The map \( \psi : G/L_S \to Z(A_\lambda) \) is an isomorphism of affine algebraic varieties. In particular \( A_\lambda \hookrightarrow \mathbb{C}[G/L_S] \) is an isomorphism.

**Proof.** Let \( W_S \subset W \) denote the Weyl group of the Levi factor \( L_S \) of \( P_S \) and \( W_S := W/W_{S_0} \). Recall the Bruhat decomposition (cf. e.g. [Be89])
\[
G = \bigsqcup_{w \in W^{-w_0S}} B_\text{op}^w B_{-w_0S} \quad (17)
\]
of \( G \) with respect to the parabolic subgroup \( P_{-w_0S}^o \supset B_\text{op} \) of \( G \). To verify surjectivity of \( \psi \) consider any \( (ghv_\lambda, gf^-_\lambda) \in Gv_\lambda \times Gf^-_\lambda, g, h \in G \), such that \( f^-_\lambda( hv_\lambda) \neq 0 \). Then \( hv_\lambda = hw_0v_{w_0\lambda} \) for a lowest weight vector \( v_{w_0\lambda} \in V(\lambda) \) and hence (17) implies that
\[
hw_0 = bw_0p \in B_\text{op}^o w_0 P_{-w_0S}^o.
\]
Therefore \( hw_0 = bw_0v_{w_0\lambda} \in B_\text{op}^o v_\lambda \subset \mathbb{P}(V(\lambda)) \) and hence
\[
(ghv_\lambda, gf^-_\lambda) = (gbv_\lambda, gf^-_\lambda) = (gbv_\lambda, gbff^-_\lambda) \in \mathbb{P}(V(\lambda)) \times \mathbb{P}(V(\lambda)^*)
\]
which implies surjectivity of \( \psi \).

Since \( L_S = P_S \cap P_S^o \) the morphism \( \psi \) is a bijection and by [Hum95, Thm. 4.6] also birational. Now [Hum95, Prop. 4.7] implies that there exists a nonempty open set \( U \subset Z(A_\lambda) \) such that \( \psi \) induces an isomorphism of \( \psi^{-1}(U) \) onto \( U \). As \( \psi \) is compatible with the transitive action of \( G \) on \( G/L_S \) and \( Z(A_\lambda) \) this implies that \( \psi \) is an isomorphism of affine varieties. \( \square \)

### 3 Quantum Groups and Quantum Flag Manifolds

We keep the notations of the previous section. Let \( 0 \neq q \in \mathbb{C} \) be not a root of unity. The \( q \)-deformed universal enveloping algebra \( U = U_q(\mathfrak{g}) \) associated to
\( g \) can be defined to be the complex algebra with generators \( K_i, K_i^{-1}, E_i, F_i, \) \( i = 1, \ldots, r, \) and relations

\[
K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \\
K_i E_j = q^{(\alpha_i, \alpha_j)} E_j K_i, \quad K_i F_j = q^{-(\alpha_i, \alpha_j)} F_j K_i, \\
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},
\]

\[\sum_{k=0}^{1-a_{ij}} (-1)^k \left( \begin{array}{c} 1 - a_{ij} \\ 1 \end{array} \right) \frac{q_i^{1-a_{ij}} - k}{q_i} E_i^{1-a_{ij} - k} E_j E_i^{k} = 0, \quad i \neq j, \tag{18}\]

\[\sum_{k=0}^{1-a_{ij}} (-1)^k \left( \begin{array}{c} 1 - a_{ij} \\ 1 \end{array} \right) \frac{q_i^{1-a_{ij}} - k}{q_i} F_i^{1-a_{ij} - k} F_j F_i^{k} = 0, \quad i \neq j, \]

where \( q_i := q^{d_i} \) and the \( q \)-deformed binomial coefficients are defined by

\[
\binom{n}{k}_q = \frac{[n]_q [n-1]_q \cdots [n-k+1]_q}{[1]_q [2]_q \cdots [k]_q}, \quad \text{where } [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.
\]

The algebra \( U \) obtains a Hopf algebra structure by

\[
\Delta K_i = K_i \otimes K_i, \quad \Delta E_i = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i, \\
\epsilon(K_i) = 1, \quad \epsilon(E_i) = 0, \quad \epsilon(F_i) = 0, \\
\kappa(K_i) = K_i^{-1}, \quad \kappa(E_i) = -E_i K_i^{-1}, \quad \kappa(F_i) = -F_i K_i^{-1}. \tag{19}
\]

Let \( U_q(n_+), U_q(b_+), U_q(n_-), U_q(b_-) \subset U \) denote the subalgebras generated by \( \{ E_i | i = 1, \ldots, r \}, \{ E_i, K_i, K_i^{-1} | i = 1, \ldots, r \}, \{ F_i | i = 1, \ldots, r \}, \) and \( \{ F_i, K_i, K_i^{-1} | i = 1, \ldots, r \}, \) respectively.

For \( \mu \in P^+ \) let \( V(\mu) \) denote the uniquely determined finite dimensional irreducible left \( U \)-module with highest weight \( \mu \). More explicitly there exists a nontrivial highest weight vector \( v_\mu \in V(\mu) \) satisfying

\[E_i v_\mu = 0, \quad K_i v_\mu = q^{(\mu, \alpha_i)} v_\mu \quad \text{for all } i = 1, \ldots, r. \tag{20}\]

A finite dimensional \( U \)-module \( V \) is called of type 1 if \( V \cong \bigoplus \bigoplus \bigoplus V(\mu_i) \) is isomorphic to a direct sum of finitely many \( V(\mu_i), \mu_i \in P^+ \). The category \( \mathcal{C} \) of \( U \)-modules of type 1 is a tensor category. By this we mean that \( \mathcal{C} \) contains the trivial \( U \)-module \( V(0) \) and satisfies

\[X, Y \in \mathcal{C} \Rightarrow X \oplus Y, X \otimes Y, X^* \in \mathcal{C}\]

where \((uf)(x) := f(\kappa(u)x)\) for all \( u \in U, f \in X^*, x \in X\).
Moreover, $\mathcal{C}$ is a braided tensor category \cite{CP94, KS97}. For all $V, W \in \mathcal{C}$ the braiding

$$\hat{R}_{V,W} : V \otimes W \to W \otimes V$$

satisfies

$$\hat{R}_{V,W}(v \otimes w) = q^{(\text{wt}(v), \text{wt}(w))} v \otimes w + \sum_{j=1}^{r} q^{(\text{wt}(v)+\alpha_j, \text{wt}(w)-\alpha_j)} (q^{d_j} - q^{-d_j}) F_j w \otimes E_j v + \sum w_i \otimes v_i$$

where $\text{wt}(v_i) \geq \text{wt}(v)$, $\text{wt}(w) \geq \text{wt}(w_i)$, and $\text{ht}(\text{wt}(v_i) - \text{wt}(v)) \geq 2$, $\text{ht}(\text{wt}(w) - \text{wt}(w_i)) \geq 2$. To simplify notation we will also write $\hat{R}_{\mu,\nu} := \hat{R}_{V(\mu),V(\nu)}$ if $\mu, \nu \in P^+$.

To write the $q$-analogues of $\mathbb{C}[G/L_S]$ in terms of generators and relations similar to \cite{10-12} it will be helpful to introduce additional notations for certain special cases of $\hat{R}$. Recall that $\lambda = \sum_{i \in S} \omega_i$, $N = \text{dim} V(\lambda)$, and $I = \{1, \ldots, N\}$. Choose a basis $\{v_i \mid i \in I\}$ of weight vectors of $V(\lambda)$ and let $\{f_i \mid i \in I\}$ be the corresponding dual basis. Define matrices $\hat{R}$, $\hat{R}^+$, $\hat{R}^-$, and $\hat{R}^-$ by

$$\hat{R}_{\lambda,\lambda}(v_i \otimes v_j) =: \sum_{k,l \in I} \hat{R}_{ijkl}^{kl} v_k \otimes v_l, \quad \hat{R}_{-w_0\lambda,-w_0\lambda}(f_i \otimes f_j) =: \sum_{k,l \in I} \hat{R}_{ijkl}^{kl} f_k \otimes f_l,$$

$$\hat{R}_{-w_0\lambda,\lambda}(f_i \otimes v_j) =: \sum_{k,l \in I} \hat{R}_{ijkl}^{-kl} v_k \otimes f_l, \quad \hat{R}_{\lambda,-w_0\lambda}(v_i \otimes f_j) =: \sum_{k,l \in I} \hat{R}_{ijkl}^{-kl} f_k \otimes v_l.$$

Alternatively

$$(f_i \otimes f_j) \circ \hat{R}_{\lambda,\lambda} = \sum_{k,l \in I} \hat{R}_{ijkl}^{ij} f_k \otimes f_l, \quad (v_i \otimes v_j) \circ \hat{R}_{-w_0\lambda,-w_0\lambda} = \sum_{k,l \in I} \hat{R}_{ijkl}^{ij} v_k \otimes v_l,$$

$$(f_i \otimes v_j) \circ \hat{R}_{-w_0\lambda,\lambda} = \sum_{k,l \in I} \hat{R}_{ijkl}^{-ij} v_k \otimes f_l, \quad (v_i \otimes f_j) \circ \hat{R}_{\lambda,-w_0\lambda} = \sum_{k,l \in I} \hat{R}_{ijkl}^{-ij} f_k \otimes v_l,$$

where the elements of $V(\lambda)$ are considered as functionals on $V(\lambda)^*$. Let $\hat{R}^+$, $\hat{R}^-$, $\hat{R}^+$, and $\hat{R}^-$ denote the inverse of the matrix $\hat{R}$, $\hat{R}^+$, $\hat{R}$, and $\hat{R}^-$, respectively.

By \cite{22} the matrix $\hat{R}$ has the property that $\hat{R}_{kl}^{ij} \neq 0$ implies that $i = l$, $j = k$ or both $\text{wt}(v_j) \geq \text{wt}(v_k)$ and $\text{wt}(v_i) \geq \text{wt}(v_i)$. Therefore we associate to $\hat{R}$ the symbol $\prec$ which denotes the positions of the larger weights. Similar properties are fulfilled for the other types of $R$-matrices. For example, the
relation $\hat{R}^{-ij}_{kl} \neq 0$ implies that $i = l, j = k$ or both $\text{wt}(v_k) \geq \text{wt}(v_j)$ and $\text{wt}(v_i) \geq \text{wt}(v_l)$. We collect these properties in the following table.

| $\hat{R}$ | $\hat{R}^-$ | $\check{R}$ | $\check{R}^-$ | $\hat{R}^-$ | $\check{R}$ | $\check{R}^-$ |
|-----------|------------|-------------|-------------|-----------|------------|-------------|
| $<\Delta>$| $>\Delta$  | $\land$     | $>\Delta$   | $<\Delta$ | $\land$    | $\lor$      |

(23)

For $j \in \{1, 2, \ldots, r\}$ let $U_j$ denote the Hopf subalgebra of $U$ generated by $E_j, F_j, K_l, K_l^{-1}, l = 1, 2, \ldots, r$. Moreover, let $V(\lambda) = \bigoplus_m V(\lambda)_m$ denote the decomposition of $V(\lambda)$ into irreducible $U_j$-modules. The following Lemma will be used only in Step 5 of the proof of Proposition 4.4.

**Lemma 3.1.** Let $\{v_i \mid i \in I\}$ be a weight basis of $V(\lambda)$ respecting the decomposition $\bigoplus_m V(\lambda)_m$ and let $\{f_i \mid i \in I\}$ denote the dual basis. Assume that there exist $i, k$ such that $F_j v_i = 0$ and $E_j v_i = v_k$. Set $\mu = \text{wt}(v_i)$. Then the following relations hold:

(i) $F_j v_k = (q^{-(\mu, \alpha_j)} - q^{(\mu, \alpha_j)})/(q^{d_j} - q^{-d_j}) v_i, \quad E_j f_k = -q^{-(\mu, \alpha_i)} f_i, \quad K_j f_i = q^{-(\mu, \alpha_j)} f_i, \quad E_j f_i = 0, \quad F_j f_i = (q^{2(\mu, \alpha_j)} - 1)/(q^{d_j} - q^{-d_j}) f_k.$

(ii) $\hat{R}^{-kk}_{ii} = q^{-(\mu+\alpha_j, \mu+\alpha_j)} (q^{2(\mu, \alpha_j)} - 1), \quad R^{kk}_{ii} = -\delta_{ia} q^{(\mu, \mu)} (q^{2(\mu, \alpha_j)} - 1).$

(iii) $\hat{R}^{kk}_{ai} = -\delta_{ai} q^{(\mu, \mu)} (q^{2(\mu, \alpha_j)} - 1), \quad R^{-kk}_{ai} = \delta_{ia} q^{-(\mu, \mu)} (1 - q^{-2(\mu, \alpha_j)}).$

**Proof.** (i) follows from $x f_m(v_n) = f_m(\kappa(x)v_n)$ for all $x \in U$ and the fact that the basis $\{v_i \mid i \in I\}$ respects the decomposition $V(\lambda) = \bigoplus_m V(\lambda)_m$ into irreducible $U_j$-modules.

(ii) follows from (i) and Equation (22) for $\hat{R}^-$ and $\check{R}$.

(iii) One has $0 = \delta_{ka} \delta_{ki} = \sum_{l, m} \hat{R}^{-kk}_{lm} \hat{R}^{-kk}_{ai}$. Since $v_k = E_j v_i$ it follows from table (23) that only summands with $(l, m) = (i, i)$ or $(l, m) = (k, k)$ can be nonzero. Thus $\hat{R}^{kk}_{ai} = -q^{(\mu+\alpha_j, \mu+\alpha_j)} (q^{2(\mu, \alpha_j)} - 1) \delta_{ai}$.

The $q$-deformed coordinate ring $\mathbb{C}_q[G]$ is defined to be the subspace of the linear dual $U^*$ spanned by the matrix coefficients of the finite dimensional irreducible representations $V(\mu), \mu \in P^+$. For $v \in V(\mu), f \in V(\mu)^*$ the matrix coefficient $c^\mu_{f,v} \in U^*$ is defined by

$$c^\mu_{f,v}(X) = f(Xv).$$

The linear span of the matrix coefficients of $V(\mu)$

$$C^{V(\mu)} := \text{Lin}_\mathbb{C}\{c^\mu_{f,v} \mid v \in V(\mu), f \in V(\mu)^*\}$$

(24)
obtains a $U$-bimodule structure by

$$
(Yc^\mu_{f,v}Z)(X) = f(ZXYv) = c^\mu_{fZ,Yv}(X).
$$

Here $V(\mu)^*$ is considered as a right $U$-module. Note that by construction

$$
\mathbb{C}_q[G] \cong \bigoplus_{\mu \in P^+} C^{V(\mu)}
$$

is a Hopf algebra and the pairing

$$
\mathbb{C}_q[G] \otimes U \rightarrow \mathbb{C}
$$

is nondegenerate.

The algebras $S[G/P_S], A_\lambda$ and $\mathbb{C}[G/L_S]$ have natural analogues in the $q$-deformed setting. One defines $S_q[G/P_S] \subset \mathbb{C}_q[G]$ as the subalgebra generated by the matrix coefficients \(\{c^\mu_{f,v} | f \in V(\lambda)^*\}\) \cite{CP94, LR92, TT91, Soi92}. Again $S_q[G/P_S] \cong \bigoplus_{n=0}^\infty V(n\lambda)^*$ endowed with the Cartan multiplication and

$$
S_q[G/P_S] \cong \bigoplus_{n=0}^\infty (V(\lambda)^*)^n / \mathcal{I}(\lambda),
$$

where the $U$-module $\mathcal{I}(\lambda)$ is defined as in Section 2 \cite{TT91, Bra94}. The matrix coefficients \(\{c^\mu_{f,v} | i \in I\}\) satisfy the relations

$$
c^\lambda_{f_i,v} c^\lambda_{f_j,v} = q^{-(\lambda,\lambda)} \sum_{k,l \in I} \hat{R}_{k,l}^{ij} c^\lambda_{f_k,v} c^\lambda_{f_l,v}
$$

As the eigenvalues of $\hat{R}_{\lambda,\lambda}$ are different from $q^{(\lambda,\lambda)}$ on all subspaces $V(\mu) \subset V(\lambda) \otimes V(\lambda), \mu \neq 2\lambda$, the relations \((28)\) form a complete set of defining relations of $S_q[G/P_S]$.

Let $A_{\lambda}^q \subset \mathbb{C}_q[G]$ denote the subalgebra generated by the elements $z_{f,v}$ as in \((3)\). To shorten notation define $z_{ij} = z_{f_i,v_j}$ with $v_j$ and $f_i$ as above. It follows from \((28)\) and

$$
c^{-v_0}_i f_j c^\lambda_{f_j,v} = q^{(\lambda,\lambda)} \sum_{k,l \in I} (\hat{R}^-)_{k,l}^{ij} c^\lambda_{f_k,v} c^{-v_0}_l f_j
$$

that the following relations hold in $A_{\lambda}^q$:

$$
\sum_{m,n,p,t \in I} \hat{R}_{nm}^{ij} \hat{R}_{pt}^{mk} z_{np} z_{tl} = q^{(\lambda,\lambda)} \sum_{p,t \in I} \hat{R}^-_{pt} z_{ip} z_{tl},
$$

$$
\sum_{m,n,p,t \in I} \hat{R}_{nt}^{kl} \hat{R}_{np}^{im} z_{pt} z_{tl} = q^{(\lambda,\lambda)} \sum_{p,t \in I} \hat{R}^-_{pt} z_{ip} z_{tl},
$$

$$
q^{(\lambda,\lambda)} \sum_{i,j \in I} C_{ij} z_{ij} = 1, \quad \text{where } C_{kl} := \sum_{i \in I} (\hat{R}^-)^{ii}_{kl}.
$$

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Instead of (30) and (31) one can also write
\[
\sum_{m,n,p,t \in I} \hat{R}^{-ij}_{nm} \hat{R}^{mk}_{pt} z_{np} z_{tl} = q^{-(\lambda,\lambda)} \sum_{p,t \in I} \hat{R}^{jk}_{pt} z_{ip} z_{tl},
\]
(33)
\[
\sum_{m,n,p,t \in \hat{I}} \hat{R}^{-kl}_{mt} \hat{R}^{lm}_{np} z_{in} z_{pt} = q^{-(\lambda,\lambda)} \sum_{p,t \in I} \hat{R}^{jk}_{pt} z_{ip} z_{tl},
\]
(34)
As the right sides of (33), (34) coincide one obtains
\[
\sum_{m,n,p,t \in \hat{I}} \hat{R}^{-ij}_{nm} \hat{R}^{mk}_{pt} z_{np} z_{tl} = \sum_{m,n,p,t \in \hat{I}} \hat{R}^{-kl}_{mt} \hat{R}^{lm}_{np} z_{in} z_{pt}.
\]
(35)
As in the classical case
\[
A^q_\lambda \cong \lim_{\longrightarrow} V(n\lambda)^* \otimes V(n\lambda).
\]
(36)
Indeed, relation (29) allows one to write any \(n\)-fold product of the generators \(z_{ij}\) as a linear combination of \(e^{n\lambda}_{F(\nu)} \otimes c^{-n\lambda}_{G(\nu)}\), with \(F \in V(n\lambda)^*\) and \(G \in V(n\lambda)\). Therefore the arguments preceding (29) can be repeated to verify (36).

As \(S_q[G/P_S]\) and \(S_q[G/P^{op}_S]\) are quadratic algebras and (36) holds, Equations (30) – (32) form a complete set of defining relations for \(A^q_\lambda\).

The \(q\)-deformed analogue of \(C_q[G/L_S]\) is defined by
\[
C_q[G/L_S] = \{ a \in C_q[G] \mid a_{(1)} a_{(2)}(k) = \varepsilon(k)a \quad \forall k \in K \},
\]
(37)
where \(K := U_q(l_S)\) is the Hopf subalgebra of \(U\) generated by the elements \(\{K_i, K_i^{-1}, E_j, F_j \mid i = 1, \ldots, r, j \in S\}\). By construction \(C_q[G/L_S]\) is a left \(C_q[G]\)-comodule algebra containing \(A^q_\lambda\). The following proposition was proved for special cases by M. S. Dijkhuizen and J. V. Stokman [DS99] and in full generality by J. V. Stokman [Sto02] using \(C^*\)-algebra techniques. Here we give an alternative proof based on the corresponding classical result.

**Proposition 3.2.** \(A^q_\lambda \cong C_q[G/L_S]\) as left \(C_q[G]\)-comodule algebras.

**Proof.** Recall that the spaces \(C_q[G]\) and \(C[G]\) are isomorphic cosemisimple coalgebras with finite dimensional isotypical components (cf. (5) and (26)). The decompositions of \(C_q[G/L_S]\) and of \(C[G/L_S]\) into irreducible \(C[G]\)-subcomodules coincide. This follows from the fact, that the dimensions of the weight spaces of irreducible representations \(V(\mu)\) of \(U\) are the same as in the classical case. On the other hand by (30) and (36) the \(C[G]\)-comodules \(A_\lambda\) and \(A^q_\lambda\) are isomorphic. Now the assertion follows from Proposition 2.1 and from \(A^q_\lambda \subset C_q[G/L_S]\).

\[\square\]
For $q \in \mathbb{R}$ it is well known that $\mathbb{C}_q[G/L_S]$ can be endowed with a $*$-structure induced by the compact real form of $U$. For general $q \in \mathbb{C}$ there remains a $\mathbb{C}$-linear algebra antiautomorphism, which will prove useful in later arguments.

Consider the $\mathbb{C}$-linear algebra antiautomorphism coalgebra homomorphism $\varphi_U : U \to U$ given by

$$K_i \mapsto K_i, \quad E_i \mapsto K_i F_i, \quad F_i \mapsto E_i K_i^{-1}, \quad i = 1, \ldots, r.$$ 

The composition

$$\varphi := \kappa \circ \varphi_U^* : \mathbb{C}_q[G] \to \mathbb{C}_q[G], \quad a \mapsto (u \mapsto a(\varphi_U(\kappa(u)))) \forall u \in U$$

is a $\mathbb{C}$-linear algebra antiautomorphism coalgebra homomorphism satisfying $\varphi^2 = \text{Id}$. As $K$ is a Hopf subalgebra of $U$ satisfying $\varphi_U(K) \subset K$ one obtains $\varphi(B) \subset B$. For any multi-index $J = (j_1, \ldots, j_n)$ of length $|J| := n$ of nonnegative integers $j_i \leq r$ define $E^J := E_{j_1} \cdots E_{j_n}$ and $\text{wt}(J) := \sum_{i=1}^n j_i$, similarly define $K^J$ and $F^J$.

Note that $\varphi_U \circ \kappa$ is an algebra homomorphism such that

$$\varphi_U \circ \kappa(E^J) = (-1)^{|J|} F^J, \quad \varphi_U \circ \kappa(F^J) = (-1)^{|J|} E^J, \quad \varphi_U \circ \kappa(K^J) = (K^J)^{-1}.$$ 

Then for any irreducible finite dimensional representation $V(\mu)$ with highest weight vector $v_\mu$ and any $f \in V(\mu)^*$ one obtains

$$\varphi(\delta_{f,\nu}^\mu)(E^J K^J F^{J'}) = \delta_{J',0} (-1)^{|J|} q^{-(\text{wt}(J'),\mu)} f(F^J v_\mu).$$

On the other hand for the dual representation $V(\mu)^* \cong V(-w_0 \mu)$ with lowest weight vector $f_{-\mu}$ and any $v \in V(\mu)$

$$\delta_{v,f_{-\mu}}(E^J K^J F^{J'}) = \delta_{J',0} q^{-(\text{wt}(J'),\mu)} v(F^J f_{-\mu}).$$

Recall that for $\mu = \sum_{i=1}^r n_i \omega_i$ the left ideal in $U_q(n_-)$ which annihilates $v_\mu$ is generated by $\{F^{n_i+1} | i = 1, \ldots, r\}$ and the left ideal in $U_q(n_+)$ which annihilates $f_{-\mu}$ is generated by $\{E^{n_i+1} | i = 1, \ldots, r\}$, \cite{Jos95} 4.3.6. Therefore there is a well defined isomorphism of vector spaces $\tilde{\varphi} : V(\mu)^* \to V(\mu)$ such that

$$(E^J f_{-\mu})(\tilde{\varphi}(f)) = (-1)^{|J|} f(F^J v_\mu)$$

for $f \in V(\mu)^*$ and all multi-indices $J$. Note that $\tilde{\varphi}$ maps weight spaces to their duals and by definition

$$\varphi(\delta_{f,\nu}^\mu) = \delta_{\tilde{\varphi}(f),f_{-\mu}}^\mu, \quad \varphi(\delta_{v,f_{-\mu}}^\mu) = \delta_{\tilde{\varphi}(v),v_\mu}^\mu.$$ 

For later reference we collect the above considerations in the following lemma.
Lemma 3.3. For $0 \neq q \in \mathbb{C}$ not a root of unity there exist a $\mathbb{C}$-linear map $\varphi : \mathbb{C}_q[G/L_S] \to \mathbb{C}_q[G/L_S]$ and a vector space isomorphism $\tilde{\varphi} : V(\lambda)^* \to V(\lambda)$ with the following properties:

1. The map $\varphi$ is an algebra antiautomorphism and a coalgebra automorphism.

2. The map $\tilde{\varphi}$ maps weight spaces to their duals.

3. The relation $\varphi(z_{fv}) = z_{\varphi^{-1}(v)\varphi(f)}$ holds for the generators $z_{fv}$ of $\mathbb{C}_q[G/L_S]$.

4 Finite dimensional irreducible graded representations of $\mathbb{C}_q[G/L_S]$

To shorten notation we will write $B := \mathbb{C}_q[G/L_S]$ from now on. The right action of $U$ endows $B$ with a natural $Q$-grading given by

$$\deg z_{ij} = \text{wt}(f_i) + \text{wt}(v_j) = \text{wt}(v_j) - \text{wt}(v_i) \in Q.$$ 

Let $B_+, B_0$, and $B_-$ denote the unital subalgebras of $B$ generated by the sets $\{z_{ij}\}$ with $\deg z_{ij} \geq 0$, $\deg z_{ij} = 0$ and $0 \geq \deg z_{ij}$, respectively. We will also write $B_+^0 := \{b \in B_+ \mid \varepsilon(b) = 0\}$.

Lemma 4.1. $B = B_- B_0 B_+$.

Proof. It suffices to find a filtration $F$ on $B$ and $\lambda_{ijkl} \in \mathbb{C} \setminus \{0\}$ such that $z_{ij} z_{kl} = \lambda_{ijkl} z_{kl} z_{ij}$ holds in the associated graded algebra.

Recall the total ordering $\succ$ on $P$ defined at the beginning of Section 2. Consider the totally ordered abelian semigroup

$$\mathcal{N} = \{(k, \mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_k) \mid k \in \mathbb{N}_0, \mu_i, \nu_i \in P, \mu_j \succ \mu_i \text{ or } (\mu_j = \mu_i, \nu_j \succ \nu_i) \forall i < j\}$$

with the lexicographic ordering with respect to the ordering $\succ$ of $P$. The sum of two elements of $\mathcal{N}$ is defined by

$$(k, \mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_k) + (l, \mu'_1, \ldots, \mu'_k, \nu'_1, \ldots, \nu'_k) = (k + l, \mu''_1, \ldots, \mu''_{k+l}, \nu''_1, \ldots, \nu''_{k+l})$$

where $(\mu''_i, \nu''_i), i = 1, \ldots, k + l$, are the elements of

$$\{(\mu_j, \nu_j), (\mu'_p, \nu'_p) \mid j = 1, \ldots, k \text{ and } p = 1, \ldots, l\}$$
in lexicographically increasing order.

There exists an $\mathcal{N}$-filtration $\mathcal{F}$ on $\mathcal{B}$ defined by (recall the remark at the end of the introduction)

$$\text{deg}_{\mathcal{F}}(z_{ij}) = (1, \text{wt}(v_i), \text{wt}(v_j)).$$

By (35) one has

$$z_{ij}z_{kl} = \sum_{a, b, c, d, m, n, p, t \in I} \hat{R}_{ab}^{jk} \hat{R}_{mc}^{ia} \hat{R}_{dt}^{bl} \hat{R}_{np}^{cd} z_{mn}z_{pt}.$$  

Suppose that $\text{wt}(v_i) \nleq \text{wt}(v_k)$ or $\text{wt}(v_i) = \text{wt}(v_k)$, $\text{wt}(v_j) \nleq \text{wt}(v_l)$. Since on the right hand side $\text{wt}(v_k) \triangleright \text{wt}(v_a) \triangleright \text{wt}(v_m)$ holds in each nonzero summand by (23), one obtains $\text{deg}_{\mathcal{F}}(z_{mn}z_{pt}) < \text{deg}_{\mathcal{F}}(z_{kl}z_{ij})$ whenever $m \neq a$ or $a \neq k$. Moreover if $m = a$ then $i = c$. Thus again by (23) one has $\text{wt}(v_i) = \text{wt}(v_c) \triangleright \text{wt}(v_p)$ and therefore $\text{deg}_{\mathcal{F}}(z_{mn}z_{pt}) < \text{deg}_{\mathcal{F}}(z_{kl}z_{ij})$ for $c \neq p$. Finally, if $c = p$ then $n = d$ and therefore $\text{wt}(v_i) \triangleright \text{wt}(v_d) = \text{wt}(v_n)$. Thus $\text{deg}_{\mathcal{F}}(z_{mn}z_{pt}) < \text{deg}_{\mathcal{F}}(z_{kl}z_{ij})$ in all cases different from $i = c$, $j = b = t$, $k = a = m$, $l = d = n$. This yields

$$z_{ij}z_{kl} = q^{(\text{wt}(v_i) - \text{wt}(v_j), \text{wt}(v_k) + \text{wt}(v_l))} z_{mn}z_{pt}$$

up to terms of lower degree with respect to $\mathcal{F}$.  

Let $\mathcal{B}_\mu$, $\mu \in Q$, denote the subspace of $\mathcal{B}$ consisting of elements of $Q$-degree $\mu$. In the following all irreducible finite dimensional graded representations $V$ of $\mathcal{B}$ will be determined. Here, a graded representation is a graded vector space $V = \bigoplus_{\mu \in Q} V_\mu$ with a $\mathcal{B}$ action such that $\mathcal{B}_\mu V_\nu \subset V_{\mu+\nu}$ for all $\mu, \nu \in Q$. A graded representation of $\mathcal{B}$ will be called an irreducible graded representation if it does not possess any nontrivial invariant graded subrepresentation.

**Lemma 4.2.** Let $V$ be a finite dimensional irreducible graded representation of $\mathcal{B}$ and let $\lambda_0$ denote the highest weight of $V$ with respect to $\nleq$. Then $\dim V_{\lambda_0} = 1$.

**Proof.** Let $V$ be a finite dimensional irreducible graded representation with highest weight $\lambda_0 \in Q$. Then $\lambda_0 + \text{deg } z_{ij} \nleq \lambda_0$ for $\text{deg } z_{ij} \nleq 0$ and hence $\mathcal{B}_+ V_{\lambda_0} = 0$. By Lemma 4.1 and the irreducibility of $V$, $V_{\lambda_0}$ is an irreducible representation of $\mathcal{B}_0$. Given $v \in V_{\lambda_0}$, $v \neq 0$, act with both sides of Equation (35) with $\text{wt}(v_k) = \text{wt}(v_j)$, $\text{wt}(v_i) = \text{wt}(v_i)$ on $v$. Using $\text{wt}(v_k) \triangleright \text{wt}(v_m) \triangleright$
Since the generators of $B$ on the right hand side, and Lemma 4.3.

Let presentation of $B$

Proposition 4.4.

Any finite dimensional irreducible graded representation $V_\lambda$ exists $\mu \in B$

Proof. Let $\mu \in W\lambda$ if and only if $(\mu, \mu) = (\lambda, \lambda)$.

(ii) If $\mu \in W\lambda$ then $(\mu, \mu) = (\mu, \nu)$ if and only if $\mu = \nu$.

(iii) For any $\mu \in W\lambda$, $\mu \not= \lambda$, there exists $i \in \{1, \ldots, r\}$ with $(\mu, \alpha_i) < 0$. In this case $\mu - \alpha_i$ is not a weight of $V(\lambda)$.

Proof. The first two statements can be found in [Kac90, 11.4]

(iii) If $(\mu, \alpha_i) \geq 0$ for all $i$ then $\mu$ is dominant. As $\lambda$ is the only dominant weight in $W\lambda$ this proves the first statement. If $\mu = w\lambda$, $w \in W$, and $(\lambda, w^{-1}\alpha_i) = (\mu, \alpha_i) < 0$ then $w^{-1}\alpha_i \in R^-$. Hence $w^{-1}(\mu - \alpha_i) = \lambda - w^{-1}\alpha_i$ is not a weight of $V(\lambda)$ which proves the second statement.

Let $V^0 = \mathbb{C}v$ denote the trivial representation, i.e. $bv = \varepsilon(b)v$ for all $b \in B$.

Proposition 4.4. Any finite dimensional irreducible graded representation of $B$ is isomorphic to $V^0$.

Proof. Let $V$ be such a representation and let $v \in V$ be a highest weight vector. Lemma 4.2 implies that for any $i, j \in I$ satisfying $\text{wt}(v_i) = \text{wt}(v_j)$ there exists $\mu_{ij} \in \mathbb{C}$ such that $z_{ij}v = \mu_{ij}v$. Moreover by [22] there exist $i_0, j_0 \in I$, $\text{wt}(v_{i_0}) = \text{wt}(v_{j_0})$, such that $\mu_{i_0j_0} \neq 0$. The proof is now performed in several steps. The first term on the right hand side of (22) and the properties of the $R$-matrices collected in (23) are frequently used.

Step 1. If $\text{wt}(v_i) \neq \text{wt}(v_{i_0})$ or $\text{wt}(v_{i_0}) \nless \text{wt}(v_k)$ then $z_{kl}v = 0$.

The action of (33) and (34) on $v$ yields

$$z_{kl}v = 0 \text{ for } \text{wt}(v_{i_0}) \nless \text{wt}(v_k) \text{ or } \text{wt}(v_i) \gtrsim \text{wt}(v_{i_0}). \quad (39)$$
It remains to prove that \( z_{il}v = 0 \) whenever \( \text{wt}(v_{i_0}) \not\leq \text{wt}(v_l) \) and \( \text{wt}(v_i) \gg \text{wt}(v_{i_0}) \). We proceed by induction over \( \text{wt}(v_l) \) with respect to \( \gg \). Suppose that \( i \in I \) such that \( \text{wt}(v_i) \gg \text{wt}(v_{i_0}) \) and \( z_{al}v = 0 \) for all \( a \) with \( \text{wt}(v_i) \gg \text{wt}(v_a) \). By \( (32) \) this is fulfilled if \( \text{wt}(v_i) = \text{wt}(v_{i_0}) \). For \( i, l \) as above and any \( j, k \) such that \( \text{wt}(v_k) \ll \text{wt}(v_j) \), the action of \( (35) \) on \( v \) yields \( q^{(\text{wt}(v_i),\text{wt}(v_k) - \text{wt}(v_j))} z_{jk} z_{il}v = 0 \) by induction hypothesis and \( B^+z_{il}v = 0 \). Thus \( z_{il}v \) is a weight vector of \( V \) with \( B^+z_{il}v = 0 \). Since \( 0 \leq \deg \text{deg} \), by Lemma \( 4.3(i) \) the representation \( Bz_{il}v = B_0z_{il}v \) is a graded subrepresentation of \( V \) which does not contain \( v \). Since \( V \) is irreducible, \( z_{il}v = 0 \).

Step 2. \( \text{wt}(v_{i_0}) \in W\lambda \). In particular \( i_0 = j_0 \).

Insert \( i = j = i_0, k = l = j_0 \) in \( (33) \). By Step 1 one gets

\[
q^{(\text{wt}(v_{i_0}),\text{wt}(v_{i_0}) - \text{wt}(v_{i_0}))} z_{i_0j_0}^2 v = q^{-(\lambda,\lambda) + (\text{wt}(v_{i_0}),\text{wt}(v_{i_0}))} z_{i_0j_0}^2 v.
\]

Since \( z_{i_0j_0}v = \mu_{i_0j_0}v \) and \( \mu_{i_0j_0} \neq 0 \), one has \( (\lambda,\lambda) = (\text{wt}(v_{i_0}),\text{wt}(v_{i_0})) \). Then Lemma \( 4.3(i) \) applies.

Step 3. \( z_{kl}V = 0 \) for any \( k, l \) with \( \text{wt}(v_{i_0}) \ll \text{wt}(v_k) \) or \( \text{wt}(v_{i_0}) \ll \text{wt}(v_l) \).

To shorten notation we will write \( v_i := \text{wt}(v_i) \) in this step. Equation \( (35) \) yields

\[
q^{(\text{wt}(v_i),\text{wt}(v_k) - \text{wt}(v_j))} z_{jk} z_{il} + \sum_{n, p, t \in I} \alpha_{npt} z_{np} z_{it} =
\]

\[
= q^{(\text{wt}(v_i),\text{wt}(v_j) - \text{wt}(v_k))} z_{il} z_{jk} + \sum_{n, p, t \in I} \beta_{npt} z_{in} z_{pt}
\]

for all \( i, j, k, l \) and some complex numbers \( \alpha_{npt}, \beta_{npt} \) (depending on \( i, j, k, l \)). By induction over \( \text{wt}(v_l) \) with respect to \( \gg \) this implies that

\[
z_{il} z_{jk} = q^{(\text{wt}(v_i) + \text{wt}(v_l),\text{wt}(v_k) - \text{wt}(v_j))} z_{jk} z_{il} + \sum_{m, n, p, t \in I} \alpha_{mnpt} z_{mn} z_{pt}
\]  

(40)

where \( \alpha_{mnpt} \in \mathbb{C} \) depend on \( i, j, k, l \). Since \( z_{il}v = 0 \) for \( \text{wt}(v_{i_0}) \ll \text{wt}(v_i) \) or \( \text{wt}(v_{i_0}) \ll \text{wt}(v_l) \) Equation \( (40) \) immediately gives \( z_{il}V = z_{il}Bv = 0 \) in this case.

Step 4. If \( \text{wt}(v_{i_0}) \gg \text{wt}(v_{i_0}) \) fails for some \( k_0 \) then \( z_{k_0i_0}V = 0 \).

Suppose that \( \text{wt}(v_{i_0}) \gg \text{wt}(v_{i_0}) \) does not hold. Then the same is true for any \( k_1 \) with \( \text{wt}(v_{k_1}) \ll \text{wt}(v_{k_1}) \). Thus one can perform the proof by induction on \( \text{wt}(v_{i_0}) \) with respect to \( \gg \). First let both sides of \( (33) \) with \( (i, j, k, l) = (k_0, i_0, i_0, i_0) \) act on \( v \). By \( \text{wt}(v_i) \gg \text{wt}(v_l) \) and the induction hypothesis on the left hand side and by \( \text{wt}(v_j) \gg \text{wt}(v_i) \) and Steps 1 and 2 on the
right hand side one gets $z_{i\alpha_0} z_{k\alpha_0} v = z_{k\alpha_0} z_{i\alpha_0} v$. Next let both sides of (33) with $(i, j, k, l) = (i_0, k_0, i_0, i_0)$ act on $v$. Now $\text{wt}(v_i) \succ \text{wt}(v_l)$ and Step 1 on the left and $\text{wt}(v_k) \succ \text{wt}(v_p)$ and Step 3 on the right side imply that 

$$q^\text{(wt}(v_i_0),\text{wt}(v_{i_0})−\text{wt}(v_{k_0})) z_{k\alpha_0} z_{i\alpha_0} v = q^{−(\lambda, \lambda)+\text{wt}(v_{k_0})} z_{i\alpha_0} z_{k\alpha_0} v.$$ 

These two equations give

$$0 = (1 - q^{2\text{wt}(v_{i_0}),\text{wt}(v_{i_0})−\text{wt}(v_{k_0})) z_{k\alpha_0} z_{i\alpha_0} v.$$ 

Since $\mu_{i\alpha_0} \neq 0$, Lemma 4.3(ii) yields $z_{k\alpha_0} v = 0$. Finally, set $i = k_0$ and $l = i_0$ in (40). Then from Step 3 together with the induction hypothesis one obtains $z_{k\alpha_0} V = 0$.

**Step 5.** One has $\text{wt}(v_{i_0}) = \lambda$. By Step 1 and (33) this implies that $V = \mathbb{C} v = V^0$.

Suppose that $\text{wt}(v_{i_0}) \neq \lambda$. By Lemma 4.3(iii) there exists $j \in \{1, 2, \ldots, r\}$ such that $(\text{wt}(v_{i_0}), \alpha_j) < 0$ and therefore $E_j v_{i_0} \neq 0$. In particular the second statement of Lemma 4.3(iii) implies that $F_j v_{i_0} = 0$. Since by Step 2 the weight space of $V(\lambda)$ of weight $\text{wt}(v_{i_0})$ is one-dimensional there exists a weight basis of $V(\lambda)$ respecting the decomposition $V(\lambda) = \bigoplus_m V(\lambda)_m$ into irreducible $U_j$-modules. As the previous steps hold for an arbitrary weight basis, we can assume that the conditions of Lemma 3.1 are satisfied and that there exists $i_1 \in I$ such that $v_{i_1} = E_j v_{i_0}$.

For $m \geq 0$ set $v_{(m)} := z_{i_1}^{m} v$. Then by (40) with $i = l = k = i_0$, $j = i_1$ and Step 3 one obtains for all $m \geq 0$ the equation

$$z_{i_0 i_0} v_{(m)} = q^{-2m(\text{wt}(v_{i_0}), \alpha_j)} \mu_{i_0 i_0} v_{(m)}.$$ (41)

Now let both sides of (34) with $(i, j, k, l) = (i_1, i_1, i_0, i_0)$ act on $v_{(m)}$. By Steps 3 and 4 and Lemma 3.1(iii) one gets

$$z_{i_1 i_0} z_{i_1 i_0} v_{(m)} = q^{\text{wt}(v_{i_1}),\text{wt}(v_{i_0})−(\lambda, \lambda)} z_{i_1 i_0} z_{i_1 i_0} v_{(m)} − (q^{2\text{wt}(v_{i_0}),\alpha_j} - 1) z_{i_1 i_0} z_{i_0 i_0} v_{(m)}.$$ (42)

Since $z_{i_1 i_1} v = 0$ by Step 1, one obtains from (41) by induction over $m$ the formula

$$z_{i_1 i_0} v_{(m)} = q^{-2m(\text{wt}(v_{i_0}), \alpha_j)} (q^{2\text{wt}(v_{i_0}),\alpha_j} - 1)(1 - q^{-2md_j})/(q^{2d_j} - 1) \mu_{i_0 i_0} v_{(m)}.$$ 

Consider (35) with $i = k = i_1$, $j = l = i_0$. Then in view of the Steps 3 and 4 and Lemma 3.1(iii) one obtains

$$q^{\text{wt}(v_{i_1}),\alpha_j} z_{i_0 i_1} z_{i_0 i_0} v_{(m)} + (q^{\text{wt}(v_{i_0}),\alpha_j} - q^{-\text{wt}(v_{i_0}),\alpha_j})(z_{i_1 i_0} z_{i_0 i_0} v_{(m)} - z_{i_0 i_0}^{2} v_{(m)}) = q^{\text{wt}(v_{i_0}),\alpha_j} z_{i_1 i_0} z_{i_0 i_1} v_{(m)}.$$ 

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Since \( z_{i_0 i_1} v = 0 \) by Step 1, one obtains by induction for all \( m \geq 1 \) the formula
\[
z_{i_0 i_1} v(m) = q^{-4(m-1)(\mathrm{wt}(v_{i_0}),\alpha_j)}(1 - q^{-2(\mathrm{wt}(v_{i_0}),\alpha_j)}) \frac{1 - q^{-2md_j}}{q^{2d_j} - 1} \mu_{i_0 i_0}^2 v(m-1). \tag{43}
\]
Since \( V \) is finite dimensional and \( 0 \geq \text{deg} z_{i_1 i_0} \) there exists \( m > 0 \) such that \( v(m) = 0 \) but \( v(m-1) \neq 0 \). Then \( z_{i_0 i_1} v(m) = 0 \) which is a contradiction to (43).

5 The coalgebra \( \overline{U} = U/UK^+ \)

From now on for the remaining sections of this paper we restrict to the case of irreducible flag manifolds. Let \( s \in \{1, \ldots, r\} \) denote the missing index, i.e. \( S \cup \{\alpha_s\} = \pi \) and therefore \( \lambda = \omega_s \). Fix a reduced decomposition of the longest element \( w_0 \in W \) of the Weyl group. Let \( E_\beta, F_\beta, \beta \in R^+ \), denote the corresponding root vectors in \( U \) [CP94, 8.1], [KS97, 6.2].

Lemma 5.1. Let \( g_i, i = 1, 2, \ldots, \#R + r \), denote the generators \( E_\beta, F_\beta, K_j \) of \( U \) with respect to \( w_0 \) in an arbitrary order. Then the elements
\[
\prod_{i=1}^{\#R + r} g_i^{n_i},
\]
\( n_i \in \mathbb{N}_0 \) if \( g_i = E_\beta, F_\beta \) and \( n_i \in \mathbb{Z} \) if \( g_i = K_j \) form a vector space basis of \( U \).

Proof. This follows from the \( q \)-commutativity of the generators in the graded algebra associated to a certain filtration of \( U \) [CK90, Prop. 1.7 d]. \( \square \)

Recall the decomposition \( R^+ = R^+_S \cup \overline{R}^+_S \) of the positive roots of \( g \). In the following we will use the abbreviation \( M := \dim g/p_S = \#\overline{R}^+_S \).

Proposition 5.2. Let \( \beta_1, \beta_2, \ldots, \beta_M \) and \( \beta'_1, \beta'_2, \ldots, \beta'_M \) denote the elements of \( \overline{R}^+_S \) in arbitrary fixed orders. Then the elements
\[
\prod_{i=1}^M (E_{\beta_i})^{m_i} \prod_{j=1}^M (F_{\beta'_j})^{n_j},
\]
\( m_i, n_j \in \mathbb{N}_0 \), form a vector space basis of \( U = U/UK^+ \).
Proof. Let $\gamma_1, \gamma_2, \ldots, \gamma_{#R_S^+}$ and $\gamma'_1, \gamma'_2, \ldots, \gamma'_{#R_S^+}$ denote the elements of $R_S^+$ in arbitrary fixed orders. By Lemma 5.1 the elements

$$
\prod_{k=1}^M (E_{\beta_k})^{m_k} \prod_{l=1}^M (F_{\gamma_l})^{n_l} \prod_{i=1}^\#R_S^+ (F_{\gamma_i})^{s_i} K_1^{i_1} \ldots K_r^{i_r},
$$

(45)

$i_k \in \mathbb{Z}, r_i, s_j, n_k, m_l \in \mathbb{N}_0$, form a vector space basis of $U$. Thus it suffices to show that the elements

$$
\prod_{k=1}^M (E_{\beta_k})^{m_k} \prod_{l=1}^M (F_{\gamma_l})^{n_l} \prod_{i=1}^\#R_S^+ (E_{\gamma_i})^{r_i} \prod_{j=1}^\#R_S^+ (F_{\gamma'_j})^{s'_j} K_1^{i_1} \ldots K_r^{i_r} - 1,
$$

(46)

$$
\prod_{k=1}^M (E_{\beta_k})^{m_k} \prod_{l=1}^M (F_{\gamma_l})^{n_l} \prod_{i=1}^\#R_S^+ (E_{\gamma_i})^{r_i} \prod_{j=1}^\#R_S^+ (F_{\gamma'_j})^{s'_j} K_1^{i_1} \ldots K_r^{i_r},
$$

(47)

$i_k \in \mathbb{Z}, r_i, s_j, n_k, m_l \in \mathbb{N}_0, \sum_{j=1}^\#R_S^+ (r_i + s_i) \geq 1$, form a vector space basis of $UK^+$. The expressions (46) and (47) form a set of linearly independent elements of $UK^+$. Any element of $UK^+$ can be written as a sum of expressions of the form

$$
\prod_{k=1}^M (E_{\beta_k})^{m_k} \prod_{l=1}^M (F_{\gamma_l})^{n_l} \prod_{i=1}^\#R_S^+ (E_{\gamma_i})^{r_i} \prod_{j=1}^\#R_S^+ (F_{\gamma'_j})^{s'_j} K_1^{i_1} \ldots K_r^{i_r} G
$$

(48)

where $G \in \{ K_j - 1, F_i, E_i | i \neq s \}$ and $i_k \in \mathbb{Z}, r_i, s_j, n_k, m_l \in \mathbb{N}_0$. For $G = E_i$, $i \neq s$, we show that (48) is a linear combination of elements of the form (46) and (47). The cases $G = F_i$, $i \neq s$, and $G = K_i - 1$ are dealt with in a similar way.

First observe that $G = E_i q$-commutes with $K_1^{i_1} \ldots K_r^{i_r}$. If $\prod_{j=1}^\#R_S^+ (F_{\gamma'_j})^{s'_j} \neq F_i$ then by reordering $\prod_{j=1}^\#R_S^+ (F_{\gamma'_j})^{s'_j} E_i$ according to the above basis (45) one obtains a linear combination of monomials of the form

$$
E_i^\delta \prod_{j=1}^{\#R_S^+} (F_{\gamma'_j})^{s'_j} K_1^{i_1} \ldots K_r^{i_r}
$$

where $\delta = 1$ or ($\delta = 0$ and $\sum_{j=1}^{\#R_S^+} s'_j > 0$). As the elements $E_{\gamma_i}$ generate a sub-algebra of $U$ with basis $\prod_{i=1}^\#R_S^+ (E_{\gamma_i})^{r_i}$ and $E_i$ is an element of this subalgebra the expression (48) for $G = E_i$ can indeed be written as a linear combination

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of elements of the form (47). If on the other hand \( \prod_{j=1}^{#R^+} (F_{\gamma_j})^{s_j} = F_i \) then the relation

\[
F_i E_i = E_i F_i - \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}
\]

implies the claim. \( \square \)

The canonical projection \( \Phi : U \rightarrow \overline{U} \) is a surjective coalgebra map. Recall that the coradical \( U_0 \) of \( U \) is the subalgebra generated by the elements \( K_i, K_i^{-1}, i = 1, \ldots, r \) \cite[Lem. 5.5.5]{Mon93}. For any surjective coalgebra map \( f : C \rightarrow D \) the coradical of \( D \) is contained in the image of the coradical of \( C \) \cite[Cor. 5.3.5]{Mon93}. Therefore \( \overline{U} \) is connected, i.e. the coradical of \( \overline{U} \) is one-dimensional.

Set \( \overline{U}_+ = \text{Lin}_C \{ \prod_{i=1}^{M} E_{\beta_i}^{m_i}, F_{\beta_i'}^{m_i} \} \subset \overline{U} \), \( \overline{U}_- = \text{Lin}_C \{ \prod_{i=1}^{M} F_{\beta_i}^{n_i}, E_{\beta_i}'^{n_i} \} \subset \overline{U} \), where the products are taken over all \( i \) such that \( \beta_i, \beta_i' \in R_S^+ \). As \( \overline{U}_+ \) and \( \overline{U}_- \) are the images of the Hopf subalgebras \( U_q(b_+) \) and \( U_q(b_-) \), respectively, under the canonical projection \( \Phi \), the subspaces \( \overline{U}_+ \subset \overline{U} \) and \( \overline{U}_- \subset \overline{U} \) are subcoalgebras. Note that \( \overline{U}_+ \) and \( U/U(K^+ + CE_s) \) are isomorphic coalgebras and hence \( \overline{U}_+ \) admits a left \( U \)-module structure. Similarly, the coalgebras \( \overline{U}_- \) and \( U/U(K^+ + CE_s) \) are isomorphic.

Let \( F^s \) denote the filtration on \( U \) defined by

\[
\text{deg}^s(E_i) = \delta_{i,s} = \text{deg}^s(F_i), \quad \text{deg}^s(K_i) = 0.
\]

Then \( F^s \) induces a filtration on the \( U \)-module \( \overline{U} \) by

\[
\text{deg}^s \left( \prod_{i=1}^{M} (E_{\beta_i})^{m_i} \prod_{j=1}^{M} (F_{\beta_j})^{n_j} \right) = \sum_{i=1}^{M} m_i + n_i
\]

and on \( \overline{U}_+ \) and \( \overline{U}_- \) all of which will also be denoted by \( F^s \). Further let \( C \) denote the respective coradical filtrations. Note that \( F^s \overline{U} \subset \overline{C} \overline{U} \) and \( F^s \overline{U}_\pm \subset \overline{C} \overline{U}_\pm \). On \( \overline{U}_\pm \) the filtration \( F^s \) is induced by a grading. The homogeneous components of degree \( k \) of this grading will be denoted by \( \overline{U}_{\pm,k} \), i.e. \( F^s_k(\overline{U}_{\pm}) = \bigoplus_{i=0}^{k} \overline{U}_{\pm,i} \).

**Proposition 5.3.** The coradical filtrations on \( \overline{U}_+ \) and \( \overline{U}_- \) coincide with \( F^s \).

**Proof.** We will prove the proposition for \( \overline{U}_+ \). Define a grading \( G \) on \( U_q(b_+) \) by

\[
\text{deg}_G(E_i) = 1, \quad \text{deg}_G(K_i) = \text{deg}_G(K_i^{-1}) = 0, \quad i = 1, \ldots, r.
\]
The induced grading on $U_+$ will also be denoted by $\mathcal{G}$. Define linear functionals $e_{\beta_i} \in (\overline{U}_+)^*$ by

$$e_{\beta_i}(E_{\beta_j}) = \delta_{ij}, \quad e_{\beta_i}(X) = 0$$

for all $X \in U_{+,k}$ with $k \neq 1$. Assume that the elements of $\overline{R}_S^+$ are ordered by increasing height, i.e.

$$\text{ht}(\beta_i) > \text{ht}(\beta_j) \Rightarrow i > j.$$ 

It suffices to verify the following statement:

(*) The set of functionals

$$\left\{ \prod_{i=1}^{M}(e_{\beta_i})^{n_i} \mid (n_1, \ldots, n_M) \in \mathbb{N}_0, \sum_{i=1}^{M} n_i = k \right\}$$

is a basis of the homogeneous component $(\overline{U}_{+,k})^*$ of the graded dual algebra $\overline{U}_+^{\text{Gr}} = \bigoplus_{k=0}^{\infty} (\overline{U}_{+,k})^*$. Indeed, $\mathcal{F}_k^{*}\overline{U}_+ \subset C_k\overline{U}_+$. On the other hand if $x \in \mathcal{F}_k^{*}\overline{U}_+ \setminus \mathcal{F}_{k-1}^{*}\overline{U}_+$ then by (*) there exists $f = \prod_{i=1}^{M}(e_{\beta_i})^{n_i}, \sum_{i=1}^{M} n_i = k$, such that $f(x) \neq 0$. As $f|_{C_{k-1}\overline{U}_+} = 0$ this implies that $x \notin C_{k-1}\overline{U}_+$.

To verify (*) note first that as the elements of $\overline{R}_S^+$ are ordered by increasing height one has $e_{\beta_i}(X E_{\beta_t}) = 0$ for all $X \in U_q(b_+), i < t$, and $e_{\beta_t}(X E_{\beta_t}) = 0$ for all $X \in \sum_{j=1}^{r} E_j U_q(b_+)$. This implies that for $1 \leq t \leq M, m_t \geq n_t$ the relation

$$\left( \prod_{i=1}^{t}(e_{\beta_i})^{n_i} \right) \left( \prod_{i=1}^{t}(E_{\beta_i})^{m_i} \right) = q^{n_t \beta_t, \sum_{i=1}^{t} m_i \beta_i - n_t \beta_t} \times$$

$$\times \left( \prod_{i=1}^{t-1}(e_{\beta_i})^{n_i} \right) \left( \prod_{i=1}^{t-1}(E_{\beta_i})^{m_i} \right) \cdot e_{\beta_t}^{n_t} (E_{\beta_t}^{n_t})$$

holds. For the same reason

$$\left( \prod_{i=1}^{t}(e_{\beta_i})^{n_i} \right) \left( \prod_{i=1}^{t+1}(E_{\beta_i})^{m_i} \right) = 0$$

if $m_{t+1} \neq 0$. Therefore

$$\left( \prod_{i=1}^{M}(e_{\beta_i})^{n_i} \right) \left( \prod_{i=1}^{M}(E_{\beta_i})^{m_i} \right) = 0$$

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whenever \((n_M, \ldots, n_2, n_1) < (m_M, \ldots, m_2, m_1)\) with respect to the lexicographic ordering. To prove (*) it now suffices to show that for all \(\beta \in \widetilde{R}_S^+\) and \(n \in \mathbb{N}\)

\[
e^n_\beta (E^n_\beta) = e^n_\beta(\Delta^{n-1}(E^n_\beta)) \neq 0.
\]

For \(\beta = \sum_{i=1}^r n_i \alpha_i \in Q\) define \(K^n_\beta = \prod_{i=1}^r K^n_{i_i}\). To evaluate the above expression one verifies by induction over \(n\) that

\[
\Delta^{n-1}(E^n_\beta) = \left[\sum_{i=1}^n 1 \otimes \cdots \otimes 1 \otimes E^n_\beta \otimes K_\beta \otimes \cdots \otimes K_\beta\right]^n + \sum_i X_{ij} \otimes \cdots \otimes X_{in}
\]

where \(X_{ij} \in \mathcal{U}_q(b_+)^e\) and for all \(i\) there exists \(j \in \{1, \ldots, n\}\) such that \(X_{ij} \in \sum_{k=1}^r U_q(b_+)E_kE^n_\beta + \sum_{k \neq s} U_q(b_+)E_kE^n_\beta\) holds. Since \(K_\beta E^n_\beta = q^{(\beta, \beta)}[E^n_\beta K_\beta\] and \(e^n_\beta\) vanishes on \(\sum_{k=1}^r U_q(b_+)E_kE^n_\beta\) one obtains

\[
e^n_\beta (E^n_\beta) = e^n_\beta \left(\Phi^n \left(\left[\sum_{i=1}^n 1 \otimes \cdots \otimes 1 \otimes E^n_\beta \otimes K_\beta \otimes \cdots \otimes K_\beta\right]^n\right)\right)
\]

\[
= \prod_{k=1}^n q^{(k, \beta)} - 1
\]

where \(\beta \in \widetilde{R}_S^+\) and \(\Phi : \mathcal{U}_q(b_+) \rightarrow \mathcal{U}_+\) denotes the canonical projection.

\[\square\]

**Lemma 5.4.** \(KC_lU \subset C_l\mathcal{U}\) for all \(l \geq 0\).

*Proof.* Since \(K\) is a coalgebra, for \(x \in \mathcal{U}\), \(k \in K\) we get

\[(\Delta k)(x \otimes 1 + 1 \otimes x) = kx \otimes 1 + 1 \otimes kx \in \mathcal{U} \otimes \mathcal{U}.
\]

Thus if \(x \in C_l\mathcal{U}\) and \(k \in K\) then by [Mon93, Lem. 5.3.2(2)]

\[
\Delta(kx) - kx \otimes 1 - 1 \otimes kx \in (\Delta k) \sum_{i=1}^{l-1} C_i\mathcal{U} \otimes C_{l-i}\mathcal{U}
\]

which proves the statement by induction over \(l\).

\[\square\]

**Proposition 5.5.** The coradical filtration of \(\mathcal{U}\) coincides with \(\mathcal{F}^s\).

*Proof.* Recall that \(\mathcal{F}_k^s\mathcal{U} \subset C_k\mathcal{U}\) and \(\Delta x - 1 \otimes x - x \otimes 1 \in \sum_{i=1}^{k-1} \mathcal{F}_i\mathcal{U} \otimes \mathcal{F}_{k-i}\mathcal{U}\) for all \(x \in \mathcal{F}_k^s\mathcal{U}\). Contrary to the assertion of the proposition assume that \(\mathcal{F}_k^s\mathcal{U} \neq C_k\mathcal{U}\) for some \(k \in \mathbb{N}_0\) and that \(k\) is minimal with this property. Choose
\[ u = \sum \lambda_{m_1...m_M} \prod_{i=1}^{M} (E_{\beta_i})^{m_i} (F_{\beta_i'})^{n_i} \in C_k \mathcal{U} \setminus F_k \mathcal{U}. \]

Note that \( \mathcal{U} \) possesses a \( \mathbb{Z}^r \)-grading induced by the standard \( \mathbb{Z}^r \)-grading of \( U \). By Lemma 5.4 one may assume that \( u \) is homogeneous with respect to the \( \mathbb{Z}^r \)-grading. Moreover, without loss of generality we can assume that \( \lambda_{m_1...m_M} = 0 \) whenever \( \sum_{i=1}^{M} m_i + n_i \leq k \). By Proposition 5.3 one has \( u \notin U_+ \oplus U_- \).

Let \( S_u \subset \mathbb{N}_0^M \) denote the subset defined by

\[ S_u := \{(m_1, ..., m_M) | \exists (n_1, ..., n_M) \text{ such that } \lambda_{m_1...m_M} \neq 0 \}. \]

Choose a multi-index \( (k_1, ..., k_M) \in S_u \) such that \( \prod_{i=1}^{M} (E_{\beta_i})^{k_i} \) is maximal among the \( \prod_{i=1}^{M} (E_{\beta_i})^{m_i}, (m_1, ..., m_M) \in S_u, \) with respect to the grading \( \mathcal{G} \) defined in the proof of Proposition 5.3. The assumption \( u \notin U_- \) implies that \( \prod_{i=1}^{M} (E_{\beta_i})^{k_i} \neq 1 \). Pick \( (k'_1, ..., k'_M) \) such that \( \lambda_{k_1...k_M}k'_1...k'_M \neq 0 \). Observe that then \( \prod_{i=1}^{M} (F_{\beta_i'})^{k'_i} \neq 1 \). Indeed, otherwise \( u \in U_+ \), since \( u \) is homogeneous and \( \prod_{i=1}^{M} (E_{\beta_i})^{k_i} \) is maximal with respect to the grading \( \mathcal{G} \). Write \( \Delta u \in U \otimes \mathcal{U} \) with respect to the basis given in Proposition 5.2 in the first tensor factor. The second tensor factor corresponding to \( \prod_{i=1}^{M} (F_{\beta_i'})^{k'_i} \) is given by

\[ \sum_{(m_1, ..., m_M) \in S_u} \lambda_{m_1...m_M}k'_1...k'_M \prod_{i=1}^{M} (E_{\beta_i})^{m_i} \neq 0 \]

as \( u \) is homogeneous with respect to the \( \mathbb{Z}^r \)-grading. But this means that

\[ \Delta(u) - 1 \otimes u - u \otimes 1 \notin U_s \otimes U \otimes U = \sum_{i=1}^{k-1} C_i \mathcal{U} \otimes C_{k-i} \mathcal{U}. \]

This contradicts \( u \in C_k \mathcal{U} \). \( \square \)

### 6 Duality

By definition of \( \mathcal{B} = \mathbb{C}_q[G/L_{S}] \) the nondegenerate pairing (27) induces a well defined pairing

\[ \langle \cdot, \cdot \rangle : \mathcal{B} \otimes \mathcal{U} \rightarrow \mathbb{C} \quad (49) \]

of the algebra \( \mathcal{B} \) and the coalgebra \( \mathcal{U} \).

**Proposition 6.1.** The pairing (49) is nondegenerate.
Proof. Since (27) is nondegenerate  \( \overline{U} \) separates \( \mathcal{B} \) in (49). To show that \( \mathcal{B} \) separates \( \overline{U} \) take any \( x \in \overline{U} \setminus \{0\} \). By Proposition 5.2 the element \( x \) can be represented by linear combination of elements of \( U \) of the form (44). Choose a basis vector

\[
\prod_{i=1}^{M} (E_{\beta_i})^{m_i} \prod_{j=1}^{M} (F_{\beta_j})^{n_j} \in U
\]

occurring with non-vanishing coefficient \( a_0 \) in such a linear combination such that \( (\sum_{j=1}^{M} n_j, \sum_{i=1}^{M} m_i) \) is maximal with respect to the lexicographic order. For given \( n \in \mathbb{N} \) the elements \( \prod_{j=1}^{M} (F_{\beta_j})^{n_j} v_{\omega_s}, \sum_{j=1}^{M} l_j \leq n, \) where \( f_{-\omega_s} \) denotes a lowest weight vector of \( V(n\omega_s)^* \), are linearly independent in \( V(n\omega_s)^* \). Indeed, by [Jos95, 4.3.6] one has

\[
V(n\omega_s) \cong U_q(n_+) \bigg/ \left( \sum_{i \neq s} U_q(n_+) E_i + U_q(n_+) E_1^{n+1} \right) \cong \overline{U}_+/U_q(n_+) E_1^{n+1}
\]

as \( U_q(n_+) \)-modules, where \( U_q(n_+) E_1^{n+1} \subset \overline{U}_+ \). By similar reasoning the elements \( \prod_{j=1}^{M} (F_{\beta_j})^{n_j} v_{\omega_s}, \sum_{j=1}^{M} l_j \leq n, \) where \( v_{n\omega_s} \) denotes a highest weight vector of \( V(n\omega_s) \), are linearly independent in \( V(n\omega_s) \). Thus for any \( n > \max\{\sum_{j=1}^{M} n_j, \sum_{i=1}^{M} m_i\} \) one obtains

\[
x(v_{n\omega_s} \otimes f_{-n\omega_s}) = a_0 \left( \prod_{j=1}^{M} (F_{\beta_j})^{n_j} v_{\omega_s} \right) \otimes \left( \prod_{i=1}^{M} (E_{\beta_i})^{m_i} f_{-\omega_s} \right) + \ldots
\]

where \( \ldots \) denote terms which are linearly independent of the first expression. As \( c^e_{n\omega_s}, c_{v_{n\omega_s}} \) \( \in \mathcal{B} \) for all \( f \in V(n\omega_s)^* \) and \( v \in V(n\omega_s) \) one obtains \( \langle \mathcal{B}, x \rangle \neq 0 \).

To shorten notation we introduce index sets

\[
I_{(k)} := \{ i \in I \mid (\omega_s - \text{wt}(v_i), \omega_s) = kd_s \}.
\]

Moreover, assume that the basis vector \( v_N \) is a highest weight vector of \( V(\omega_s) \). Recall that \( M = \dim \mathfrak{g}/\mathfrak{p}_s = \# R^+_s \).

Lemma 6.2. For all \( i, j \) such that \( (i, j) \notin (I_{(1)} \times \{N\}) \cup (\{N\} \times I_{(1)}) \) one has

\[
z_{ij} - \delta_{iN} \delta_{jN} \in (\mathcal{B}^+)^2.
\]

In particular \( \dim \mathcal{B}^+/(\mathcal{B}^+)^2 \leq 2M \).
Proof. By definition the generators $z_{ij}$ of $\mathcal{B}$ satisfy the relation $\sum_{k \in I} z_{ik} z_{kj} = z_{ij}$. For $i \neq N \neq j$ or $i = N = j$ using $\varepsilon(z_{ij}) = \delta_i \delta_{jN}$ one obtains

$$z_{ij} - \delta_i \delta_{jN} \in (\mathcal{B}^+)^2.$$ 

Since $V(\omega_s) = \text{Lin}_\mathbb{C}\{\prod_{i=1}^M (F_{\beta_i})^{n_i} v_N | \beta_i \in \overline{R_\mathcal{S}^+}, n_i \in \mathbb{N}_0\}$ it suffices to show that $z_{Nj}, z_{jN} \in (\mathcal{B}^+)^2$ if $\text{wt}(v_j) = \omega_s - 2\alpha_s - \beta$ for some $\beta \in Q, \beta > 0$.

In view of (23) relation (30) for $i, j < N, k = l = N$ implies that

$$\sum_{n \in I} \hat{R}^{ij}_{nN} z_{nN} \in (\mathcal{B}^+)^2.$$ 

More precisely,

$$\sum_{n \in I} \hat{R}^{ij}_{nN} z_{nN} + \sum_{n, m \in I \setminus \{N\}} q^{(\omega_s, \text{wt}(v_m))} \hat{R}^{ij}_{nm} z_{nN} z_{mN} - q^{(\omega_s, \text{wt}(v_j))} z_{iN} z_{jN} \in (\mathcal{B}^+)^3.$$ 

(50)

Consider now $v_n \in V(\omega_s)$ with $\text{wt}(v_n) = \omega_s - 2\alpha_s - \beta$ for some $\beta \in Q, \beta > 0$. We will show that there exist $i, j < N$ such that $\hat{R}^{ij}_{nN} \neq 0$. This implies that

$$(P_\prec \otimes P_\prec) \circ \hat{R}_{\omega_s, \omega_s} : V(\omega_s - 2\alpha_s - \beta) \otimes \mathbb{C}v_N \to P_\prec V(\omega_s) \otimes P_\prec V(\omega_s)$$ 

(52)

is injective. Here $P_\prec : V(\omega_s) \to V(\omega_s)$ denotes the projection given by

$$P_\prec | v_\mu := (1 - \delta_{\mu, \omega_s}) \text{Id} | v_\mu$$

for any weight space $V_\mu \subset V(\omega_s)$. Therefore $z_{nN} \in (\mathcal{B}^+)^2$.

If $\hat{R}^{ij}_{nN} = 0$ for all $i, j < N$ then

$$\hat{R}_{\omega_s, \omega_s}(v_n \otimes v_N) - q^{(\text{wt}(v_n), \omega_s)} v_N \otimes v_n \in V(\omega_s) \otimes v_N.$$ 

(53)

As $\hat{R}_{\omega_s, \omega_s}$ is a $U$-module homomorphism $E_k v_n$ also satisfies (53) for all $k \neq s$. Therefore we can assume that $E_k v_n = 0$ for all $k \neq s$. But in this case $E_s v_n \neq 0$ and with $F_s v_N \neq 0$ one obtains from (22)

$$\hat{R}^{ij}_{nN} = q^{(\omega_s - \alpha_s, \text{wt}(v_n) + \alpha_s)} (q^{d_s} - q^{-d_s}) \neq 0$$

for $v_i = F_s v_N$ and $v_j = E_s v_n$.

To verify $z_{Nn} \in (\mathcal{B}^+)^2$ for all $n$ with $\text{wt}(v_n) = \omega_s - 2\alpha_s - \beta$ and $\beta \in Q, \beta > 0$, apply the algebra antiautomorphism $\varphi$ from Lemma 3.3 to $z_{nN}$. □
The relations $V(\omega_s) = U_q(n_-)v_N$ and $V(\omega_s)^* = U_q(n_+)f_N$ induce direct sum decompositions

\[
V(\omega_s) = \bigoplus_k V(\omega_s)(k), \quad V(\omega_s)^* = \bigoplus_k V(\omega_s)^*(k) \tag{54}
\]

where

\[
V(\omega_s)(k) := \text{Lin}_C \left\{ \prod_{j=1}^{M} (F_{\beta_j})^{n_j} v_N \left| \sum_{j=1}^{M} n_j = k \right. \right\} = \text{Lin}_C \{ v_i \mid i \in I(k) \}
\]

\[
V(\omega_s)^*(k) := \text{Lin}_C \left\{ \prod_{i=1}^{M} (E_{\beta_i})^{m_i} f_N \left| \sum_{i=1}^{M} m_i = k \right. \right\} = \text{Lin}_C \{ f_i \mid i \in I(k) \}
\]

and $\beta_i, \beta'_j$ denote the elements of $R^+_S$ in the order fixed in the beginning of Section 5.

Note that (30) for $i = l = N$ and $j, k < N$ yields

\[
z_{jk} = q^{(\omega_s, \omega_s) - \text{wt}(v_j) - \text{wt}(v_k)} \sum_{p,t \in I \setminus \{N\}} \hat{R}_{pt}^{jk} z_{Np} z_{tN} \mod (B^+)^3.
\]

Using $\sum_{m \in I} z_{jm} z_{mk} = z_{jk}$ one obtains

\[
z_{jN} z_{Nk} = q^{(\omega_s, \omega_s) - \text{wt}(v_j) - \text{wt}(v_k)} \sum_{p,t \in I \setminus \{N\}} \hat{R}_{pt}^{jk} z_{Np} z_{tN} \mod (B^+)^3. \tag{55}
\]

Therefore Lemma 6.2 leads to an isomorphism of $\mathbb{Z}^r$-graded vector spaces

\[
(B^+)^2 / (B^+)^3 = D^{2,0} \oplus D^{1,1} \oplus D^{0,2}
\]

where

\[
D^{2,0} = \text{Lin}_C \{ z_{iN} z_{jN} \in (B^+)^2 / (B^+)^3 \mid i, j \in I(1) \}
\]

\[
D^{1,1} = \text{Lin}_C \{ z_{Ni} z_{jN} \in (B^+)^2 / (B^+)^3 \mid i, j \in I(1) \}
\]

\[
D^{0,2} = \text{Lin}_C \{ z_{Ni} z_{Nj} \in (B^+)^2 / (B^+)^3 \mid i, j \in I(1) \}.
\]

Recall from Section 5 that $U_{+,i}$ denotes the homogeneous component of degree $i$ of the graded coalgebra $U_+$. 

**Lemma 6.3.** $\dim D^{2,0} \leq \dim U_{+,2}$. 

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Proof. Consider the vector space
\[ \tilde{V} := \text{Lin}_\mathbb{C} \{ f_i \otimes f_j \in V(\omega_s)^* \otimes V(\omega_s)^* | (2\omega_s - \text{wt}(v_i) - \text{wt}(v_j), \omega_s) = 2d_s \} \]
- \[V(\omega_s)_{(1)}^* \otimes V(\omega_s)_{(1)}^* \oplus V(\omega_s)_{(0)}^* \otimes V(\omega_s)_{(2)}^* \oplus V(\omega_s)_{(2)}^* \oplus V(\omega_s)_{(0)}^* \]
and the linear maps
\[ \Psi_1 : \tilde{V} \rightarrow \tilde{V}, \quad \Psi_1(f) := f \circ \tilde{R}_{\omega_s,\omega_s} - q^{(\omega_s,\omega_s)} f, \]
\[ \Psi_2 : \tilde{V} \rightarrow D^{2,0}, \quad \Psi_2(f_i \otimes f_j) := \begin{cases} q^{(\omega_s,\text{wt}(v_j))} z_{jN} & \text{if } i = N, j \in I_{(2)}, \\ q^{(\omega_s,\omega_s)} z_{iN} & \text{if } i \in I_{(2)}, j = N, \\ q^{(\omega_s,\text{wt}(v_j))} z_{iN} z_{jN} & \text{if } i, j \in I_{(1)}. \end{cases} \]
Recall from (51) and from the injectivity of (52) that \( z_{jN} \in D^{2,0} \) for all \( j \in I_{(2)} \) and therefore \( \Psi_2 \) is well defined. Moreover, \( \Psi_2 \) is surjective. Now we claim that
\[ \Psi_2 \circ \Psi_1 = 0. \] (56)
Indeed, if \( i, j \in I_{(1)} \) then in \((B^+)^2/(B^+)^3\) one calculates using (50) and Lemma (52)
\[ \Psi_2 \circ \Psi_1(f_i \otimes f_j) = \Psi_2 \left( \sum_{n,m \in I} \hat{R}_{nm}^{ij} f_n \otimes f_m - q^{(\omega_s,\omega_s)} f_i \otimes f_j \right) \]
\[ = \sum_{n \in I_{(2)}} \hat{R}_{nm}^{ij} (\omega_s,\omega_s) z_{nN} + \sum_{n,m \in I_{(1)}} q^{(\omega_s,\text{wt}(v_m))} \hat{R}_{nm}^{ij} z_{mN} z_{nN} \]
\[ - q^{(\omega_s,\omega_s + \text{wt}(v_j))} z_{iN} z_{jN} \]
\[ = \sum_{n,m,p,t \in I} \hat{R}_{nm}^{ij} \hat{R}_{pt}^{mN} z_{nN} z_{pN} z_{tN} - q^{(\omega_s,\omega_s)} \sum_{p,t \in I} \hat{R}_{pt}^{mN} z_{ip} z_{tN} = 0. \]
The cases \( i = N, j \in I_{(2)} \) and \( i \in I_{(2)}, j = N \) are dealt with in a similar manner.

Note that for \( V(\mu)^* \subset V(\omega_s)^* \otimes V(\omega_s)^* \) the restriction \( \Psi_1 \vert_{\tilde{V} \cap V(\mu)^*} \) is multiplication by a nonzero scalar if \( \mu \neq 2\omega_s \) and 0 if \( \mu = 2\omega_s \). As \( f_N \otimes f_N \) is the lowest weight vector of the submodule \( V(2\omega_s)^* \subset V(\omega_s)^* \otimes V(\omega_s)^* \) one obtains \( \tilde{V} \cap V(2\omega_s)^* = \overline{U}_{+,2}(f_N \otimes f_N) \), where \( \overline{U}_{+} \) is interpreted as \( U_q(\mathfrak{n}_+)/\sum_{i \neq s} U_q(\mathfrak{n}_+)E_i \). Therefore \( \dim \ker \Psi_1 \leq \dim \overline{U}_{+,2} \). Combining this estimate with (56) one gets
\[ \dim D^{2,0} = \dim \text{Im} \Psi_2 = \dim \tilde{V} - \dim \ker \Psi_2 \]
\[ \leq \dim \tilde{V} - \dim \text{Im} \Psi_1 = \dim \ker \Psi_1 \leq \dim \overline{U}_{+,2}. \]
Proposition 6.4. The pairing $\langle \cdot, \cdot \rangle : \mathcal{B}/(\mathcal{B}^+)^{k+1} \otimes C_k \mathcal{U} \to \mathbb{C}$ is nondegenerate.

Proof. By Proposition 6.1 $\mathcal{B}$ separates $C_k \mathcal{U}$. Therefore it suffices to verify that $\dim \mathcal{B}/(\mathcal{B}^+)^{k+1} \leq \dim C_k \mathcal{U}$. By (55) one obtains a decomposition

$$(\mathcal{B}^+)^k/(\mathcal{B}^+)^{k+1} = \sum_{i=0}^{k} D^{i,k-i}$$

where $D^{i,k-i} = \text{Lin}_C \{ \prod_{m=1}^{k-i} z_{N j_m} \prod_{m=1}^{i} z_{l_{m},N} \mid j_m, l_m \in I_{(1)} \}$. Thus it suffices to show that

$$\dim D^{0,i} \leq \dim \mathcal{U}_{+,i}. \quad (57)$$

The estimate $\dim D^{i,0} \leq \dim \mathcal{U}_{-,i}$ then follows by application of the antiautomorphism $\varphi$ from Lemma 3.3.

For $i = 1$ equality holds in (57) as both $\text{Lin}_C \{ z_i | i \in I_{(1)} \}$ and $\mathcal{U}_{-,1}$ are irreducible $K$-modules with highest weight $-\alpha_s$. To verify (57) for general $i$ consider an ordering $\sqsubseteq$ of the set $I_{(1)}$ such that

$$\text{ht}(\omega_s - \text{wt}(v_i)) > \text{ht}(\omega_s - \text{wt}(v_j)) \implies i \sqsubseteq j.$$ 

Note then that the elements

$$\{ z_{N i} z_{N j} \in (\mathcal{B}^+)^2/(\mathcal{B}^+)^3 \mid i, j \in I_{(1)}, j \sqsubseteq i \} \quad (58)$$

form a basis of $D^{0,2}$. Indeed, recall the functionals $e_\beta \in (\mathcal{U}_+)^*$ defined in the proof of Proposition 5.3 and note that $z_{N i} v_\beta$ is a nonzero multiple of $e_\beta$ if $E_\beta f_N = f_i, i \in I_{(1)}$. Hence by $(\ast)$ from the proof of Proposition 5.3 the set (58) consists of $\dim \mathcal{U}_{+,2}$ linear independent elements which by Lemma 6.3 span $D^{0,2}$. Thus the elements

$$\left\{ \prod_{m=1}^{i} z_{N j_m} \mid j_m \in I_{(1)}, j_{m+1} \sqsubseteq j_m \right\}$$

span $D^{0,i}$ and (57) follows from Proposition 5.2. \qed

In view of the results of the previous sections the proposition above allows us to refine the duality between $\mathcal{B}$ and $\mathcal{U}$ of Proposition 6.1. Let $\mathcal{B}^\circ$ denote the dual coalgebra of $\mathcal{B}$, i.e. the coalgebra generated by the matrix coefficients of all finite dimensional representations of $\mathcal{B}$ (cf. e.g. [Mon93, Sect. 1.2]). As $\mathcal{B}$ is a right $\mathcal{U}$-module algebra the dual coalgebra $\mathcal{B}^\circ$ is a left $\mathcal{U}$-module.
coalgebra. In the simplest case of quantized $\mathbb{C}P^1$, the so called “Podleś’ quantum sphere”, the dual coalgebra has been determined in [HK03b]. The left $U$-action on $B^\circ$ permits to ask for elements of $B^\circ$ with an additional finiteness property, which we will call the locally finite part

$$F(B^\circ, K) := \{ f \in B^\circ \mid \dim(Kf) < \infty \}.$$  

Note that $\overline{U} \subset F(B^\circ, K)$ via the nondegenerate pairing (49), cf. Lemma 5.4. The first statement of the following theorem is the main result of this paper. Together with the second statement which is merely a reformulation of the definition of $B$ it furnishes the duality between $B$ and $\overline{U}$. Recall that $\overline{U}$ is a left $U$-module coalgebra and therefore the dual algebra $\overline{U}^*$ of linear functionals on $\overline{U}$ is a right $U$-module algebra.

**Theorem 6.5.** 1) $F(B^\circ, K) = \overline{U}$  2) $B = \{ b \in \overline{U} \mid \dim(bU) < \infty \}$.

**Proof.** 1) Let $K' \subset K$ denote the subalgebra generated by the elements $K_i, K_i^{-1}, i = 1, \ldots, r$. As $F(B^\circ, K) \subset F(B^\circ, K')$ it suffices to show that $F(B^\circ, K') \subset \overline{U}$. Note that the coradical filtration of $F(B^\circ, K')$ is invariant under the left $K'$-action, i.e.

$$K' C_n F(B^\circ, K') \subset C_n F(B^\circ, K'),$$

as the generators $K_i, K_i^{-1}$ of $K'$ act on $F(B^\circ, K')$ by coalgebra automorphisms. Thus by Proposition 6.4 the coalgebra $F(B^\circ, K')$ is connected, i.e. $C_0 F(B^\circ, K') = \mathbb{C}\epsilon$, and therefore any $f \in C_n F(B^\circ, K'), n \in \mathbb{N}_0$, vanishes on $(B^\circ)^{n+1}$. By Proposition 6.4 this implies that $f \in C_n \overline{U}$.

2) The dual Hopf algebra $U^\circ$ of $U$ satisfies

$$U^\circ = \{ a \in U^* \mid \dim(aU) < \infty \}$$

and contains $C_q[G]$ as the linear span of the matrix coefficients of the representations $V(\mu), \mu \in P^+$. Recall that $U$ is semisimple and any irreducible representation of $U$ can be obtained by tensoring some $V(\mu)$ with a one dimensional representation $D_\nu, \nu \in \{-1, 1\}^r$, given by $K_i v = \nu_i v$ for $v \in D_\nu$. Therefore

$$B = \{ a \in U^\circ \mid a_{(1)} a_{(2)}(k) = \epsilon(k) a \ \forall k \in K \}. \quad (60)$$

Inserting (59) in (60) leads to the desired expression.

Duality of $B$ and $\overline{U}$ also holds in the graded setting. Note that the algebra $B$ admits a decreasing filtration given by $F_n B = (B^\circ)^n$. Let $\text{Gr} B =$
\[ \bigoplus_{n=0}^{\infty} (B^n)/(B^{n+1}) \] denote the associated graded algebra. Define \((\text{Gr}B)_+\) and \((\text{Gr}B)_-\) to be the subalgebras of \(\text{Gr}B\) generated by \(D_{0,1}\) and \(D_{1,0}\), respectively. Let \((\text{Gr}B)_{\pm,n}\) denote the elements of \((\text{Gr}B)_{\pm}\) of degree \(n\).

On the other hand let \(\text{Gr}U\) denote the graded coalgebra associated to the coradical filtration of \(U\). Let further \((\text{Gr}U)^{Gr^+}\) and \((\text{Gr}U)^{Gr^-}\) denote the subalgebras of \(\text{Gr}U\) generated by \(D_{0,1}\) and \(D_{1,0}\), respectively. Let \((\text{Gr}U)_{\pm,n}\) denote the elements of \((\text{Gr}U)_{\pm}\) of degree \(n\).

The following Corollary is an immediate consequence of Proposition 6.4.

**Corollary 6.6.** The pairing (49) induces isomorphisms

\[ \text{Gr}B \cong (\text{Gr}U)^{Gr^+} \quad (\text{Gr}B)_+ \cong U^{Gr^+} \quad (\text{Gr}B)_- \cong U^{Gr^-} \]

of graded right \(K\)-module algebras.

The \(K\)-module algebras \((\text{Gr}B)_+\) and \((\text{Gr}B)_-\) have been constructed in [SV98] as \(q\)-analogues of the polynomial algebra on the prehomogeneous space \(g_{-1}\). Note that the left \(U\)-module structures on \(U_+\) and \(U_-\) induce right \(U\)-module structures on \((\text{Gr}B)_+\) and \((\text{Gr}B)_-\), respectively.

By statement (\(\ast\)) from the proof of Proposition (5.3) the algebra \((\text{Gr}B)_+\) is generated by the elements \((\text{Gr}B)_{+,1}\) of degree one and quadratic relations. The \(K\)-module \((\text{Gr}B)_{+,1}\) is irreducible and each weight space of \((\text{Gr}B)_{+,1}\) is one-dimensional. Therefore we have a decomposition

\[ (\text{Gr}B)_{+,1} \otimes (\text{Gr}B)_{+,1} = \bigoplus_i V_i \]  

(61)

into irreducible \(K\)-modules \(V_i\) and each of these modules occurs with multiplicity one. The dimension of each weight space of \((\text{Gr}B)_{+,2}\) coincides with the dimension of the corresponding weight space of the symmetric elements of \(V(\omega_1) \otimes V(\omega_1)\). Therefore one obtains a complete set of defining relations for \((\text{Gr}B)_+\) if one sets the ”antisymmetric” components of the tensor product (61) equal to zero. More precisely, let \(V(\omega_1) \otimes V(\omega_1) = S(1) \oplus A(1)\) denote the decomposition into ”symmetric” and ”antisymmetric” \(U\)-submodules. Then by the above arguments the following statement is proved.

**Corollary 6.7.** The algebra \((\text{Gr}B)_+\) is isomorphic to the quotient of the tensor algebra \(\bigoplus_{k=0}^{\infty} V(\omega_1)^{\otimes k}\) by the ideal generated by \(A(1) \subset V(\omega_1)^{\otimes 2}\). Moreover,

\[ \dim(\text{Gr}B)_{+,k} = \binom{M+k-1}{k} \]
An analogous result holds for \((\text{Gr}\mathcal{B})_\cdot\). In particular one obtains the following statements (cp. the list [BE89, p. 27], simple roots are ordered as in [Hum72, p. 58]).

1. If \(g = \mathfrak{sl}_N\) and \(S = \pi \setminus \{\alpha_s\}\) then \((\text{Gr}\mathcal{B})_\cdot\) is isomorphic to the \(U_q(\mathfrak{sl}_s \times \mathfrak{sl}_{N-s})\)-module algebra \(C_q[\text{Mat}^s_{N-s},s]\) of quantized \((N-s,s)\)-matrices. Similarly \((\text{Gr}\mathcal{B})_+ \cong C_q[\text{Mat}^s_{N,s},s]\). The construction of \(C_q[\text{Mat}^s_{N,s},s]\) as the graded dual of \(\mathcal{U}_\cdot\) has already been in detail established in \([SSV99]\).

2. If \(g = \mathfrak{so}_{N+2}\) and \(S = \pi \setminus \{\alpha_1\}\) then both \((\text{Gr}\mathcal{B})_\cdot\) and \((\text{Gr}\mathcal{B})_+\) are isomorphic to the \(U_q(\mathfrak{so}_N)\)-module algebra \(O^N_q(C)\) considered in \([FRT89]\), the so called quantum orthogonal vector space.

3. If \(g = \mathfrak{sp}_{2r}\) and \(S = \pi \setminus \{\alpha_r\}\) then \((\text{Gr}\mathcal{B})_\cdot\) and \((\text{Gr}\mathcal{B})_+\) are quantum coordinate algebras generated by the \(U_q(\mathfrak{sl}_r)\)-module \(V(\omega_2)\) and \(V(2\omega_{r-1})\), respectively.

4. If \(g = \mathfrak{so}_{2r}, r > 3\), and \(S = \pi \setminus \{\alpha_r\}\) or \(S = \pi \setminus \{\alpha_{r-1}\}\) then \((\text{Gr}\mathcal{B})_\cdot\) and \((\text{Gr}\mathcal{B})_+\) are quantum coordinate algebras generated by the \(U_q(\mathfrak{sl}_r)\)-module \(V(\omega_2)\) and \(V(\omega_{r-2})\), respectively.

5. If \(g = \mathfrak{e}_6\), and \(S = \pi \setminus \{\alpha_6\}\) then \((\text{Gr}\mathcal{B})_+\) and \((\text{Gr}\mathcal{B})_\cdot\) are quantum coordinate algebras generated by the spin-representation \(V(\omega_6)\) of \(U_q(\mathfrak{so}_{10})\) and its dual, respectively.

6. If \(g = \mathfrak{e}_7\), and \(S = \pi \setminus \{\alpha_7\}\) then \((\text{Gr}\mathcal{B})_\cdot\) and \((\text{Gr}\mathcal{B})_+\) are quantum coordinate algebras generated by the \(U_q(\mathfrak{e}_6)\)-modules \(V(\omega_1)\) and \(V(\omega_6)\), respectively.

\section{Covariant First Order Differential Calculus}

For the convenience of the reader the notion of differential calculus from \([Wor89]\) is recalled. A first order differential calculus (FODC) over an algebra \(\mathcal{B}\) is a \(\mathcal{B}\)-bimodule \(\Gamma\) together with a \(\mathbb{C}\)-linear map
\[
d : \mathcal{B} \rightarrow \Gamma
\]
such that \(\Gamma = \text{Lin}_\mathbb{C}\{a\,db\,c \mid a, b, c \in \mathcal{B}\}\) and \(d\) satisfies the Leibniz rule
\[
d(ab) = a\,db + da\,b.
\]
Let in addition \(\mathcal{A}\) denote a Hopf algebra and \(\Delta_\mathcal{B} : \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B}\) a left \(\mathcal{A}\)-comodule algebra structure on \(\mathcal{B}\). If \(\Gamma\) possesses the structure of a left \(\mathcal{A}\)-comodule
\[
\Delta_\Gamma : \Gamma \rightarrow \mathcal{A} \otimes \Gamma
\]
such that 
\[ \Delta \Gamma(adbc) = (\Delta \Gamma(a))(\text{Id} \otimes d)(\Delta \Gamma(b))(\Delta \Gamma(c)) \]
then \( \Gamma \) is called left covariant. For further details on first order differential calculi consult [KS97].

Let \( U \) denote a Hopf algebra with bijective antipode and \( K \subset U \) a right coideal subalgebra (For the moment \( U \) and \( K \) are arbitrary, in Theorem 7.2 we will again consider the case \( U = U_q(g) \) and \( K = U_q(I_S) \)). Consider a tensor category \( C \) of finite dimensional left \( U \)-modules. By this we mean as in [MS99] that \( C \) is a class of finite dimensional left \( U \)-modules containing the trivial \( U \)-module via \( \varepsilon \) and satisfying (21). Let \( A := U_{0}^{\circ} \) denote the dual Hopf algebra generated by the matrix coefficients of all \( U \)-modules in \( C \). Assume that \( A \) separates the elements of \( U \) and that the antipode of \( A \) is bijective.

Define a left coideal subalgebra \( B \subset A \) by
\[ B := \{ b \in A | b_{(1)} b_{(2)}(k) = \varepsilon(k) b \text{ for all } k \in K \}. \]
(62)
Assume that \( K \) is \( C \)-semisimple, i.e. the restriction of any \( U \)-module in \( C \) to the subalgebra \( K \subset U \) is isomorphic to the direct sum of irreducible \( K \)-modules. In full analogy to [MS99] Theorem 2.2 (2) this implies that \( A \) is a faithfully flat \( B \)-module.

In this situation left covariant first order differential calculi over \( B \) can be classified via certain right ideals of \( B^{+} \) [Her02]. More explicitly the subspace
\[ \mathcal{R} = \left\{ \sum_{i} \varepsilon(a_i) b_i^+ \left| \sum_{i} a_i d b_i = 0 \right. \right\} = \{ b \in B^{+} | d b \in B^{+} \Gamma \} \]
(63)
of \( B^{+} \), where \( b^+ = b - \varepsilon(b) \) for all \( b \in B \), is a right ideal which determines the differential calculus uniquely. To the FODC \( \Gamma \) corresponding to this right ideal one associates the vector space
\[ T_{\varepsilon}^{\circ} = \{ f \in B^{\circ} | f(x) = 0 \text{ for all } x \in \mathcal{R} \} \]
and the so called quantum tangent space
\[ T_{\Gamma} = (T_{\varepsilon}^\circ)^{+} = \{ f \in T_{\varepsilon}^\circ | f(1) = 0 \}. \]
The dimension of a first order differential calculus is defined by
\[ \dim \Gamma = \dim_{\mathbb{C}} \Gamma / B^{+} \Gamma = \dim_{\mathbb{C}} B^{+} / \mathcal{R}. \]

**Proposition 7.1.** [HK03a Cor. 5] Under the above assumptions there is a canonical one-to-one correspondence between \( n \)-dimensional left covariant FODC over \( B \) and \((n+1)\)-dimensional subspaces \( T_{\varepsilon} \subset B^{\circ} \) such that
\[ \varepsilon \in T_{\varepsilon}, \quad \Delta T_{\varepsilon} \subset T_{\varepsilon} \otimes B^{\circ}, \quad KT_{\varepsilon} \subset T_{\varepsilon}. \]
(64)
A covariant FODC $\Gamma \neq \{0\}$ over $\mathcal{B}$ is called \textit{irreducible} if it does not possess any nontrivial quotient (by a left covariant $\mathcal{B}$-bimodule). Note that this property is equivalent to the property that $T_\Gamma^\epsilon$ does not possess any left $K$-invariant right $\mathcal{B}^\epsilon$-subcomodule $\tilde{T}$ such that $\mathbb{C} \cdot \varepsilon \subsetneq \tilde{T} \subsetneq T_\Gamma^\epsilon$.

We now return to the situation $U = U_q(\mathfrak{g})$ and $K = U_q(\mathfrak{ls})$ as in Section 5. As an application of Theorem 6.5 it is possible to determine all finite dimensional irreducible covariant FODC over $\mathcal{B} = \mathbb{C}_q[\mathcal{G}/\mathcal{L}_S]$ for irreducible flag manifolds $\mathcal{G}/\mathcal{P}_S$.

For any coalgebra $C$ let $P(C) = \{x \in C | \Delta x = 1 \otimes x + x \otimes 1\}$ denote the vector space of primitive elements of $C$. Recall from Lemmas 5.3 and 5.5 that $P(U_+) = \text{Lin}_\mathbb{C}\{E_\beta | \beta \in \overline{R}_S^+\}$, $P(U_-) = \text{Lin}_\mathbb{C}\{F_\beta | \beta \in \overline{R}_S^+\}$, and $P(U) = P(U_+) \oplus P(U_-)$.

**Theorem 7.2.** There exist exactly two nonisomorphic finite dimensional irreducible covariant first order differential calculi $\Gamma_+, \Gamma_-$ over the quantized irreducible flag manifold $\mathcal{B} = \mathbb{C}_q[\mathcal{G}/\mathcal{L}_S]$. The corresponding quantum tangent spaces are $T_+ = P(U_+) \subset \overline{U}$ and $T_- = P(U_-)$.

**Proof.** By Proposition 7.1 one has to determine all finite dimensional left $K$-invariant right $\mathcal{B}^\epsilon$-subcomodules $T \subset F(\mathcal{B}^\epsilon, K)$ containing $\varepsilon$ but no other $K$-invariant $\mathcal{B}^\epsilon$-subcomodule. Theorem 6.5 implies that $T \subset \overline{U}$. Since $\overline{U}$ is connected any $x \in C_n \overline{U}$ satisfies $\Delta x - 1 \otimes x - x \otimes 1 \in \sum_{i=1}^{n-1} C_i \overline{U} \otimes C_{n-i} \overline{U}$ (cf. [Mon93] Lem. 5.3.2(2))). Therefore any nontrivial right $\overline{U}$-subcomodule $T \subset \overline{U}$ contains $\varepsilon$ and a primitive element. Now the claim follows from Proposition 5.5 and the fact that $P(U_+)$ and $P(U_-)$ are irreducible $K$-modules. \qed

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