Further Inference on Categorical Data - A
Bayesian Approach

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Abstract
Three different inferential problems related to a two dimensional categorical data from a Bayesian perspective have been discussed in this article. Conjugate prior distribution with symmetric and asymmetric hyper parameters are considered. Newly conceived asymmetric prior is based on perceived preferences of categories. An extension of test of independence by introducing a notion of measuring association between the parameters has been shown using correlation matrix. Probabilities of different parametric combinations have been estimated from the posterior distribution using closed form integration, Monte-Carlo integration and MCMC methods to draw further inference from categorical data. Bayesian computation is done using R programming language and illustrated with appropriate data sets. Study has highlighted the application of Bayesian inference exploiting the distributional form of underlying parameters.

Keywords: Bayesian Inference, Categorical Data, Preference Prior, Association of Parameters, Posterior Probabilities

1 Introduction
In statistical inference, we mostly deal with the Methods of Estimation and Testing of Hypothesis. But many modern day problems we face today, demand more inference from the data beyond mere
point and interval estimates. Analyzing categorical data is very common in statistical practice. And doing a Bayesian analysis gives some extra flexibility to explore the data. Many statisticians, Agresti and Hitchcock, 2005, Good et al., 1966, Good, 1967, Lindley, 1964, Altham, 1969, Congdon, 2005, etc. over the years have shown an alternative way of doing categorical data analysis from a Bayesian perspective. The most interesting part is that, the unknown parameter is considered to be a random variable with a probability distribution. This is called prior distribution. We must have a state of knowledge before constructing a prior distribution. The prior can be convenient, informative, non informative or sometimes improper. But selecting a suitable prior for a given problem is a statistician's responsibility. Then we have the likelihood of the data. And using Bayes formula we obtain a posterior distribution of the unknown parameter. In simple words, we have some prior information (state of knowledge) about the unknown parameters, then we update that information using the data in hand and draw some inference about the unknown parameters. With Bayesian analysis, we can directly answer questions regarding the population parameters and that is why it is a convenient way to deal with unknown parameters. The procedure is pretty much straight-forward for a categorical data, but beside estimating the parameters and testing independence between the factors, we can address more inferential problems taking advantage of the Bayesian assumption.

Now let us consider a data set on voters’ preference discussed in Gelman et al., 2013. In late October 1988, a survey was conducted by CBS News of 1447 adults in the United States to find out their preferences in the upcoming Presidential election. Out of 1447 persons, \( x_1 = 727 \) supported George Bush, \( x_2 = 583 \) supported Michael Dukakis, and \( x_3 = 137 \) supported other candidates or expressed no opinion. They have estimated the probability of the difference of the population proportion of Bush and Dukakis supporters \( (\theta_1 - \theta_2) \) by MCMC methods. But, in this situation other relevant questions may arise. So, the exercise can be extended to answer more probabilistic questions regarding the parameters. Suppose, the persons (or a fraction of them) who expressed no opinion, decide to vote for Dukakis, then it changes the whole political equation. So, we may want to know what is the probability that \( (\theta_2 + \theta_3) \) is greater than \( \theta_1 \) or \( (\theta_2 + \frac{\theta_3}{n}) \) is greater than \( \theta_1 \). It can also happen that some people (probably, Dukakis supporters) are also interested to know that,
given the condition Bush got more than 50% support, what is the probability that Dukakis has at least 40% support.

If we imagine the same situation in Indian political system and there are three parties viz. Party A, Party B and Party C, and $P(\theta_1 - \theta_2) = 0.999$ (almost certain), then also, it is not certain that Party A can form the Government. Because after election Party B and Party C can form alliance and if they have more support combined than Party A, they can form the Government. So, now it is of our interest to know that, given Party A has at least 40% support, what is the probability that party B has at least 30% support and party C has at least 20% support. Also, we would like to know, given Party A has at least 40% support what is the probability that Party B and Party C has at least 50% support combined. In other words, we would like to know $P(\theta_2 \geq 0.3, \theta_3 \geq 0.2|\theta_1 \geq 0.4), P(\theta_2 + \theta_3 \geq 0.5|\theta_1 \geq 0.4) \text{ and } P(\theta_2 + \theta_3 \geq \theta_1)$ . We may also want to know $P(0.2 \leq \theta_2 \leq 0.4, 0.1 \leq \theta_3 \leq 0.3)$.

In this article we have mainly focused to answer these kinds of questions theoretically and also using simulation techniques. And again, it can also be of our interest to know the association between the population parameters, where chi-square test of independence can only give an idea of overall independence between the categorical variables. In that case, we have used correlation matrix of the posterior distribution to measure the association in parameter level. We have also tried to give a notion of a prior based on perceived preference to stop the predominant use of symmetric priors where possible. Bayesian Computation has been done with closed form integration, Monte-Carlo integration and MCMC methods using R programming language.

2 Inference For Categorical Data

A detail study of categorical data is done by [Agresti, 2003]. We have considered general two-dimensional categorical data, consisting of two categorical variables for our study. A categorical
variable has a measurement scale consisting of a set of categories. A general structure of categorical
data is given below.

| Table 1: A Two Dimensional $I \times J$ Categorical Data |
|-----------------------------------------------------------|
| X \ Y | $B_1$ | $B_2$ | $B_3$ | ... | $B_J$ | Total |
|-------|-------|-------|-------|-----|-------|-------|
| $A_1$ | $x_{11}$ | $x_{12}$ | $x_{13}$ | ... | $x_{1J}$ | $r_1$ |
| $A_2$ | $x_{21}$ | $x_{22}$ | $x_{23}$ | ... | $x_{2J}$ | $r_2$ |
| $A_3$ | $x_{31}$ | $x_{32}$ | $x_{33}$ | ... | $x_{3J}$ | $r_3$ |
| ...   | ...   | ...   | ...   | ... | ...   | ...  |
| $A_I$ | $x_{I1}$ | $x_{I2}$ | $x_{I3}$ | ... | $x_{IJ}$ | $r_I$ |
| Total | $c_1$ | $c_2$ | $c_3$ | ... | $c_J$ | $n$ |

A general $I \times J$ categorical data can be considered as a multinomial model, where cell counts
$x_{ij}, (i = 1, 2, 3, ..., I; j = 1, 2, 3, ..., J)$ follows a multinomial distribution with unknown population
proportions $\theta_{ij}$.

Let us define, row total, $r_i = \sum_j x_{ij}$, column total, $c_j = \sum_i x_{ij}$ and total observation, $n = \sum_i \sum_j x_{ij}$
Then the multinomial likelihood is given by,

$$f(x|\theta) = \frac{n!}{\prod_i \prod_j x_{ij}} \theta_{ij}^{x_{ij}}, \text{ where } \sum_i \sum_j \theta_{ij} = 1$$

For Bayesian analysis, the common choice of a conjugate prior distribution is Dirichlet Distribution,
as suggested by, [Gelman et al., 2013], where $\theta_{ij}$ follows Dirichlet($\alpha_{ij}$). And posterior distribution
of $\theta_{ij}$ follows Dirichlet($x_{ij} + \alpha_{ij}$); $\alpha_{ij} > 0$.

Now obtaining point and interval estimates or conducting tests of hypothesis is a common
practice in Bayesian analysis, but here in our study we have focused on three different problems.
• Choices of hyper parameters of Dirichlet Prior to make use of prior knowledge.

• Association in parameter level, going beyond test of overall independence of categorical variables.

• Answering different types of complex probabilistic questions directly related to the parameters to draw more inference from the data.

Let us discuss on these three topics one by one.

2.1 A Prior Based on Preference

Predominant use of symmetric prior in every problem sometimes limits the scope of Bayesian Inference and sometimes, it is quite misleading. For example, in a Dirichlet prior distribution, if we use Jeffreys prior \( \alpha_{ij} = 0.5 \forall i, j \), the marginal beta distributions become positively skewed and it gives high probabilities for the population proportion to be near zero. Where, in reality we may not want to make such assumptions or we may have different prior knowledge about the population proportions. So, it becomes necessary to incorporate relevant information in the prior construction. As shown by [Ng et al., 2011], for a Dirichlet distribution the density function and the marginals are as follows.

\[
p(\theta) = \frac{\Gamma(\alpha)}{\prod_i \Gamma(\alpha_{ij})} \prod_i \prod_j \theta_{ij}^{\alpha_{ij}-1}, \alpha = \sum_i \sum_j \alpha_{ij}, \sum_i \sum_j \theta_{ij} = 1, \alpha_{ij} > 0 \forall i, j
\]

And \( \theta_{ij} \sim Beta(\alpha_{ij}, \sum_{k=1}^{I} \sum_{l=1}^{J} \alpha_{kl}), k \neq i, l \neq j \)

So, when we use any symmetric prior, all the marginal distributions become positively skewed as 
\[
\sum_{k=1}^{I} \sum_{l=1}^{J} \alpha_{kl} \gg \alpha_{ij}, k \neq i, l \neq j
\]

Alternatively, we can order the parameters of the prior distribution based on preference or choice. If there are \( m \) parameters in the prior distribution, we can form \( k \) groups \((k \leq m)\) of size \( m_1, m_2, \ldots, m_k \), where, \( m_1 + m_2 + \ldots + m_k = m \) and order them from least preferred to highly preferred. So, higher preferred groups will be assigned more weights than less preferred groups. But, this is highly context based and will vary for different categorical variables. The preference
or choice can be formed based on the type of the problem, relevant assumptions or some prior knowledge.

For example, if we have data on income groups (levels: low, medium, high) and expenditure (levels: low, medium, high), then the Dirichlet prior will have 9 parameters. From the problem, it can be easily perceived that diagonal cells are expected to have higher weights than the others. This assumption can be incorporated in the prior construction. We can form three groups viz. higher, moderate and low.

Hyper-parameters corresponding to the diagonal cells, $\alpha_1, \alpha_5, \alpha_9$ can be assigned higher values, hyper-parameters corresponding to low income with high expenditure and high income with low expenditure $\alpha_3, \alpha_7$ can be assigned less values and the rest of the hyper-parameters can be assigned moderate values. In that way, the prior construction becomes more meaningful and the analysis can yield relevant results.

### 2.2 Association Between Population Proportions

A most common interest of doing a two dimensional categorical data analysis is to know, if the two variables involved are related to each other. This is a fundamental question, when the data is about a bio-medical or social problem. There are many ways to deal with this question. We get some ideas by looking at the values of Difference of Proportion, Relative Risk, Odds Ratio, or we can conduct a test of independence.

Chi-Square test is most famous in these kinds of situations, although it has some limitations. Sangeetha et al., 2014, Nandram and Choi, 2007, Gunel and Dickey, 1974 have provided an alternative way to test independence between two categorical variables in a I$\times$J table, using Bayes Factor. But a test of independence gives the idea of overall independence between the variables. Even if the null hypothesis is rejected and it implies that the variables are not independent, it doesn’t give any idea about the extent of association or the association between the population proportions. Also for very small or very large data the test can not be applied or reliable. Many statisticians (eg. Berkson, 1938, Cochran, 1954 etc.) has shown concerns regarding chi-square test
and felt that it can be dangerous to solely depend on the test result.

To resolve these issues many improvements have been proposed over the years. Pearson and Standardized Residuals, Partitioning Chi-Squared, Linear Trend (Mantel, 1963), Monotone Trend etc., are duly notable. We would like to introduce another way of understanding the association in parameter level from a Bayesian perspective, exploiting the posterior distribution.

The Posterior Distribution of the population proportions follow a Dirichlet Distribution with parameters \((x_{11} + \alpha_{11}, x_{12} + \alpha_{12}, ..., x_{IJ} + \alpha_{IJ})\) [Kruschke, 2014]

\[
\text{Var}(\theta_{ij}) = \frac{(x_{ij} + \alpha_{ij})(x + \alpha - (x_{ij} + \alpha_{ij}))}{(x + \alpha)(x + \alpha + 1)} \quad \text{Cov}(\theta_{ij}, \theta_{kl}) = \frac{-(x_{ij} + \alpha_{ij})(x_{kl} + \alpha_{kl})}{(x + \alpha)(x + \alpha + 1)}
\]

Using these results, subsequently the correlation matrix of the posterior distribution can be formed.

And when we have correlation coefficient of any pair of parameters \((\theta'_{ij}s)\), we get the measure of association between the population proportions. For a Dirichlet posterior distribution we are expected to have negative covariance. But we can measure the extent of association using correlation matrix. Test of independence using Chi-Square test or Bayes Factor can be used to test independence between the categorical variables. If the test rejects the null hypothesis, we can use correlations between the parameters to see what is happening in parameter level. It shows the individual behavior of the parameters with the others. For a categorical data, that is a very essential piece of information as the parameters represent the population proportion of each category. In that situation posterior correlation matrix can be very useful to determine association or dependence between the parameters.

### 2.3 Inference Using Posterior Probabilities

From the posterior distribution as well as joint and marginal distributions of \(\theta'_{ij}s\), now we can estimate posterior probabilities of different parametric combinations like,

- \(P[\theta_{ij} > \theta_{kl}]\)
- \(P[a < \theta_{ij} < b, c < \theta_{kl} < d]\)
\begin{itemize}
\item $P[a \leq \theta_{ij} < b | c \leq \theta_{kl} < d]$
\item $P[a \leq \theta_{ij} < b, c \leq \theta_{kl} < d | e \leq \theta_{mn} < f]$ 
\item $P[\theta_{ij} + \theta_{kl} > a | \theta_{mn} > b]$
\item $P[\theta_{ij} + \theta_{kl} > \theta_{mn}]$
\end{itemize}

and many more.

The above list is not exhaustive. Depending on the problem and need, many parametric combinations can be formed and posterior probability can be obtained subsequently. These probabilities can be estimated using closed form integration. But sometimes, depending on the problem and size of the data, it may not be convenient to carry on closed form integration. We face difficulty to evaluate the integral using R. In that case Monte-Carlo integration or MCMC methods are lot easier to apply. Let us first see how the probabilities can be obtained in the light of closed form integration.

\begin{align*}
P[\theta_{ij} > \theta_{kl}] &= \int_{\theta_{kl}=0}^{\theta_{ij}=0} \int_{\theta_{kl}}^{1} f(\theta_{ij}, \theta_{kl}) \, d\theta_{ij} \, d\theta_{kl} \\
P[a \leq \theta_{ij} < b, c \leq \theta_{kl} < d] &= \int_{\theta_{kl}=c}^{\theta_{ij}=a} \int_{\theta_{kl}}^{b} f(\theta_{ij}, \theta_{kl}) \, d\theta_{ij} \, d\theta_{kl} \\
P[a \leq \theta_{ij} < b | c \leq \theta_{kl} < d] &= \frac{\int_{\theta_{kl}=c}^{\theta_{ij}=a} \int_{\theta_{kl}}^{b} f(\theta_{ij}, \theta_{kl}) \, d\theta_{ij} \, d\theta_{kl}}{\int_{\theta_{kl}=c}^{\theta_{ij}=a} f(\theta_{ij}, \theta_{kl}) \, d\theta_{kl}} \\
P[a \leq \theta_{ij} < b, c \leq \theta_{kl} < d | e \leq \theta_{mn} < f] &= \frac{\int_{\theta_{mn}=e}^{\theta_{ij}} \int_{\theta_{kl}=a}^{\theta_{mn}} \int_{\theta_{kl}=a}^{b} f(\theta_{ij}, \theta_{kl}, \theta_{mn}) \, d\theta_{ij} \, d\theta_{kl} \, d\theta_{mn}}{\int_{\theta_{mn}=e}^{\theta_{ij}} \int_{\theta_{kl}=a}^{\theta_{mn}} f(\theta_{ij}, \theta_{kl}, \theta_{mn}) \, d\theta_{ij} \, d\theta_{kl} \, d\theta_{mn}} \\
P[\theta_{ij} + \theta_{kl} > a | \theta_{mn} > b] &= \frac{\int_{\theta_{mn}=b}^{\theta_{ij}} \int_{\theta_{kl}=a}^{\theta_{mn}} \int_{\theta_{kl}=a}^{\theta_{mn}} f(\theta_{ij}, \theta_{kl}, \theta_{mn}) \, d\theta_{ij} \, d\theta_{kl} \, d\theta_{mn}}{\int_{\theta_{mn}=b}^{\theta_{ij}} \int_{\theta_{kl}=a}^{\theta_{mn}} f(\theta_{ij}, \theta_{kl}, \theta_{mn}) \, d\theta_{ij} \, d\theta_{kl} \, d\theta_{mn}} \\
P[\theta_{ij} + \theta_{kl} > \theta_{mn}] &= \int_{\theta_{mn}=0}^{\theta_{ij}=0} \int_{\theta_{kl}=\theta_{mn}}^{1} \int_{\theta_{kl}=\theta_{mn}}^{1-\theta_{mn}} f(\theta_{ij}, \theta_{kl}, \theta_{mn}) \, d\theta_{ij} \, d\theta_{kl} \, d\theta_{mn} \\
\end{align*}

These integrations can be easily done using R programming languages. The programming codes are attached in the Appendix section.
In Monte-Carlo integration, we simulate a large number of random samples from the given function and see how many of them fall under the required region. So, that ratio multiplied by the total area gives the value of the integration. For our problem, we are going to simulate random samples from the posterior distribution and see how many of them fall under the given region. For a probability distribution, the total area is always 1. So, the ratio will give the required probability. This procedure can be done with some simple programming. The code is attached in the Appendix section.

To apply MCMC method we do not need supply the posterior distribution. Markov chain Monte Carlo draws those samples by running a cleverly constructed Markov chain over a long period. [Robert and Casella, 2013] has discussed the simulation process in detail.

There are two basic methods of MCMC:

Gibbs sampler is a technique for generating random variables from a marginal distribution indirectly, without having to calculate the density, but it sequentially samples from the collection of full conditional distributions. In addition to an impact and theory, these calculation methodologies focus on the statistical aspects of a problem, thereby freeing statisticians from dealing with complicated calculations [Gelfand and Smith, 1990].

The second method is applicable when simulation from the full conditionals becomes difficult. The Metropolis-Hastings algorithm [Tierney, 1994] simulates from a different Markov chain, having some other stationary distributions, but then modifies it in such a way that a new Markov chain is constructed having the posterior as its stationary distribution.

In our problem, the simulations have been done using R programming language. And like before the probabilities have been calculated with the ratio. The code is attached in the Appendix section. Now, let us consider a real-life problem and see how these techniques can be useful.
3 Data Analysis

A survey was conducted by GSS to know the political affiliation of the people of united states in the year 2016. The data is given below.

| Table 2: Political Affiliation of People of USA, 2016 |
|-------------------------------------------------------|
|                                                     |
| Male        | 5203 | 3966 | 5610 |
| Female      | 3862 | 1685 | 3288 |

The data model is Multinomial, and the conjugate prior is Dirichlet. According to Jeffreys non-informative prior $\alpha_{ij} = 0.5 \forall i = 1, 2; j = 1, 2, 3$. The posterior distribution is also Dirichlet.

$\therefore (\theta|\alpha) \sim Dirichlet(5203.5, 3966.5, 5610.5, 3862.5, 1685.5, 3288.5)$

The correlation matrix of the posterior distribution is given by,

$$
\rho = \begin{pmatrix}
1 & -0.238834 & -0.296733 & -0.235061 & -0.14737 \\
-0.238834 & 1 & -0.250786 & -0.198664 & -0.124551 \\
-0.296733 & -0.250786 & 1 & -0.246824 & -0.154745 \\
-0.235061 & -0.198664 & -0.246824 & 1 & -0.122583 \\
-0.14737 & -0.124551 & -0.154745 & -0.122583 & 1
\end{pmatrix}
$$

For our chosen problem, the values of the gamma functions are too large to be evaluated in R and hence the integration also gives some error. But for a smaller data closed form integration seems to work smoothly. Now let us list out the results using Monte-Carlo integration and MCMC methods.
| Probabilities from Posterior Distribution | Monte-Carlo Integration | MCMC Methods |
|------------------------------------------|-------------------------|--------------|
| $P[\theta_1 > \theta_2]$                | 1.0000                  | 1.0000       |
| $P[\theta_4 > \theta_5]$                | 1.0000                  | 1.0000       |
| $P[\theta_1 > \theta_4]$                | 1.0000                  | 1.0000       |
| $P[\theta_2 > \theta_5]$                | 1.0000                  | 1.0000       |
| $P[\theta_3 > \theta_6]$                | 1.0000                  | 1.0000       |
| $P[\theta_2 + \theta_3 > \theta_1]$    | 1.0000                  | 1.0000       |
| $P[\theta_5 + \theta_6 > \theta_4]$    | 1.0000                  | 1.0000       |
| $P[\theta_2 + \theta_3 > \theta_1]$    | 0.0845                  | 0.1175       |
| $P[\theta_5 + \theta_6 > \theta_4]$    | 0.9747                  | 0.9545       |
| $P[\theta_2 + \theta_3 > \theta_1]$    | 0.0040                  | 0.0096       |
| $P[\theta_5 + \theta_6 > \theta_4]$    | 0.9508                  | 0.9295       |
| $P[\theta_2 > \theta_4 | \theta_1 > 0.40]$ | 0.4800                  | 0.5000       |
| $P[\theta_5 + \theta_6 > 0.21 | \theta_1 > 0.16]$ | 0.5780                  | 0.5735       |
| $P[0.16 < \theta_1 < 0.32] \cap [0.16 < \theta_4 < 0.32]$ | 0.9310                  | 0.9263       |
| $P[0.16 < \theta_1 < 0.32] \cap [0.16 < \theta_4 < 0.32]$ | 1.0000                  | 1.0000       |
| $P[\theta_2 > 0.16, \theta_3 > 0.2 | \theta_1 > 0.2]$ | 0.9996                  | 0.9994       |

The correlation matrix shows relevant association between the parameters. The correlation between $(\theta_{11}, \theta_{13}), (\theta_{12}, \theta_{13}), (\theta_{13}, \theta_{21}), (\theta_{11}, \theta_{21})$ are little high. So, change in population proportions of those categories will affect significantly to the corresponding categories and the extent is directly interpreted from the result. One interesting result that came out from the analysis is that, if the proportion of male independent voters increases or decreases then the proportion of female democrat voters decreases or increases significantly. And the reason is left to be found out by the experts of the concerned domain. So, the association between parameters can reveal meaningful results regarding the population. When dimension of the categorical data increases and chi square test of independence shows some limitation, correlation matrix of the posterior distribution can give some
idea about the associations between the parameters.

For the above problem we have estimated several types of probabilities from the posterior distribution. Depending on the problem many more probabilistic questions regarding the parameters can be formed and those can be directly answered by the above methods. Our analysis shows some interesting results regarding the problem. From table 3, it can be seen that, it is almost certain that the proportion of male democrats being more than male republicans or female democrats. And proportion of female democrats being more than female republicans is also almost certain. Again, proportion of male republicans being more than female republicans and proportion of male independent voters being more than female independent voters are also certain. Now, if 20% of male independent voters, vote for republicans, then also probability of losing of democrats are very less. But, if 25% male independent voters, vote for republicans then the probability of losing of democrats become very high. Similarly, if 60% of female independent voters vote for republicans, probability of losing of democrats is very less. But, if 70% female independent voters, vote for republicans then the probability of losing of democrats becomes very high. So, even though the probability of winning of democrats seems almost certain, a little shift of independent voters towards republicans can totally reverse the scenario.

Joint and conditional probabilities obtained from the posterior probability is also showing some meaningful results. From table 3, it can be seen that given the male proportion of democrats is more than 23%, the probability of having proportion of male republican and independent voters more than 40% is not so high. But the good news for the democrats is that the probability of the proportions of male democrats and female democrats both being in the range 16% to 32% is very high. And that is again confirmed by the probability of proportions of male democrats being in the range 16% to 32% given that female proportion of democrats is in the range 16% to 32%: which is almost certain.
4 Conclusion

The study has attempted to address few problems regarding categorical data from a Bayesian perspective. Using these techniques, we can go deeper in a categorical data and reveal results directly related to the population parameters. In Bayesian inference, it is possible to form different types of probabilistic questions regarding the population parameters, and the answers can serve the purpose of comparative analysis. Again, going beyond test of overall independence of categorical variables, correlation matrix of the posterior distribution can give some idea about the association between the parameters.

Our objective was to do the whole exercise using closed form integration, Monte-Carlo Integration and MCMC methods. But, for large data, closed form integration becomes troublesome and it is easier to use other two methods. The study has shown the analysis of two-dimensional categorical data, but it can be extended to higher dimensions using same techniques as the underlying data model will still be Multinomial. This approach can also be applied for other data models, especially multivariate normal models, but the exercise has not been shown in this study. We leave that exercise for future study.

[Kruschke, 2010] showed his concern about the need to construct mildly informed or consensually informed prior distributions rather than objective priors. So, it was also of our interest to stop the practice of using symmetric priors for every problem and an idea of using preference-based priors has been illustrated in this article. These techniques can be very useful in modern day problems, especially in the fields of bio-statistics, social sciences and management studies.

References

[Agresti, 2003] Agresti, A. (2003). *Categorical data analysis*, volume 482. John Wiley & Sons.

[Agresti and Hitchcock, 2005] Agresti, A. and Hitchcock, D. B. (2005). Bayesian inference for categorical data analysis. *Statistical Methods and Applications*, 14(3):297–330.
[Altham, 1969] Altham, P. M. (1969). Exact bayesian analysis of a $2 \times 2$ contingency table, and fisher’s” exact” significance test. Journal of the Royal Statistical Society. Series B (Methodological), pages 261–269.

[Berkson, 1938] Berkson, J. (1938). Some difficulties of interpretation encountered in the application of the chi-square test. Journal of the American Statistical Association, 33(203):526–536.

[Cochran, 1954] Cochran, W. G. (1954). Some methods for strengthening the common $\chi^2$ tests. Biometrics, 10(4):417–451.

[Congdon, 2005] Congdon, P. (2005). Bayesian models for categorical data. John Wiley & Sons.

[Gelfand and Smith, 1990] Gelfand, A. E. and Smith, A. F. (1990). Sampling-based approaches to calculating marginal densities. Journal of the American statistical association, 85(410):398–409.

[Gelman et al., 2013] Gelman, A., Stern, H. S., Carlin, J. B., Dunson, D. B., Vehtari, A., and Rubin, D. B. (2013). Bayesian data analysis. Chapman and Hall/CRC.

[Good, 1967] Good, I. J. (1967). A bayesian significance test for multinomial distributions. Journal of the Royal Statistical Society. Series B (Methodological), pages 399–431.

[Good et al., 1966] Good, I. J., Hacking, I., Jeffrey, R., and Törnebohm, H. (1966). The estimation of probabilities: An essay on modern bayesian methods.

[Gunel and Dickey, 1974] Gunel, E. and Dickey, J. (1974). Bayes factors for independence in contingency tables. Biometrika, 61(3):545–557.

[Kruschke, 2014] Kruschke, J. (2014). Doing Bayesian data analysis: A tutorial with R, JAGS, and Stan. Academic Press.

[Kruschke, 2010] Kruschke, J. K. (2010). What to believe: Bayesian methods for data analysis. Trends in cognitive sciences, 14(7):293–300.

[Lindley, 1964] Lindley, D. V. (1964). The bayesian analysis of contingency tables. The Annals of Mathematical Statistics, pages 1622–1643.
[Mantel, 1963] Mantel, N. (1963). Chi-square tests with one degree of freedom; extensions of the mantel-haenszel procedure. *Journal of the American Statistical Association*, 58(303):690–700.

[Nandram and Choi, 2007] Nandram, B. and Choi, J. W. (2007). Alternative tests of independence in two-way categorical tables. *Journal of Data Science*, 5(2):217–237.

[Ng et al., 2011] Ng, K. W., Tian, G.-L., and Tang, M.-L. (2011). *Dirichlet and related distributions: Theory, methods and applications*, volume 888. John Wiley & Sons.

[Robert and Casella, 2013] Robert, C. and Casella, G. (2013). *Monte Carlo statistical methods*. Springer Science & Business Media.

[Sangeetha et al., 2014] Sangeetha, U., Subbiah, M., Srinivasan, M., and Nandram, B. (2014). Sensitivity analysis of bayes factor for categorical data with emphasis on sparse multinomial data. *Journal of Data Science*, 12(2):339–357.

[Tierney, 1994] Tierney, L. (1994). Markov chains for exploring posterior distributions. *the Annals of Statistics*, pages 1701–1728.