Excitation of a nonradial mode in a millisecond X-ray pulsar XTE J1751-305

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Abstract

We discuss possible candidates for non-radial modes excited in a mass accreting and rapidly rotating neutron star to explain the coherent frequency identified in the light curves of a millisecond X-ray pulsar XTE J1751-305. The spin frequency of the pulsar is \( \nu_{\text{spin}} \approx 435 \text{Hz} \) and the identified coherent frequency is \( \nu_{\text{osc}} = 0.5727595 \times \nu_{\text{spin}} \). Assuming the frequency identified is that observed in the corotating frame of the neutron star, we examine \( r \)- and \( g \)-modes in the surface fluid layer of accreting matter composed mostly of helium, and inertial modes and \( r \)-modes in the fluid core and toroidal crust modes in the solid crust. We find that the surface \( r \)-modes of \( l' = m = 2 \) excited by \( \epsilon \)-mechanism due to helium burning in the thin shell can give the frequency ratio \( \kappa = \nu_{\text{osc}} / \nu_{\text{spin}} \approx 0.57 \) at \( \nu_{\text{spin}} = 435 \text{Hz} \), where \( m \) is the azimuthal wave number of the modes. As another candidate for the observed ratio \( \kappa \), we suggest a toroidal crustal mode that has penetrating amplitudes in the fluid core and is destabilized by gravitational wave emission.

Since the surface fluid layer is separated from the fluid core by a solid crust, the amplitudes of an \( r \)-mode in the core, which is destabilized by emitting gravitational waves, can be by a large factor different from those in the fluid ocean. We find that the amplification factor defined as \( f_{\text{amp}} = \alpha_{\text{surface}} / \alpha_{\text{core}} \) is as large as \( f_{\text{amp}} \approx 10^2 \) for the \( l' = m = 2 \) \( r \)-mode at \( \nu_{\text{spin}} = 435 \text{Hz} \) for a typical \( M = 1.4M_\odot \) neutron star model, where \( \alpha \)'s are the parameters representing the \( r \)-mode amplitudes, and \( l' \) is the harmonic degree of the mode. Because of this significant amplification of the \( r \)-mode amplitudes in the surface fluid layer, we suggest that, when proper corrections to the \( r \)-mode frequency such as due to the general relativistic effects are taken into consideration, the core \( r \)-mode of \( l' = m = 2 \) can be a candidate for the detected frequency, without leading to serious contradictions to, for example, the spin evolution of the underlying neutron star.

Key words: stars: oscillations – stars: rotation

1 Introduction

A recent report of the detection of a coherent frequency from a mass accreting millisecond X-ray pulsar XTE J1751-305 (Strohmayer & Mahmoodifar 2014) suggests the existence of a nonradial mode excited in the neutron star. The spin frequency of the pulsar is \( \nu_{\text{spin}} \approx 435 \text{Hz} \) and the identified frequency is \( \nu_{\text{osc}} = 0.5727595 \times \nu_{\text{spin}} = 249.332609 \text{Hz} \). If the frequency is really associated with a non-radial mode of a neutron star, we may be able to rule out \( p \)-modes for the frequency, since their oscillation frequencies are higher than kHz in the case of neutron stars and are too high to be consistent with the detected frequency. We may also rule out the \( g \)-modes residing in the core, since they usually have much lower frequencies than the spin frequency of the star because of nearly isentropic structure of the core (e.g., McDermott et al 1988). Therefore, possible candidates remained for the detected frequency will be a \( g \)-mode or a rotational mode propagating in the surface fluid layer, or a rotational mode in the fluid core, or a toroidal crust mode in the solid crust. Note that low frequency \( g \)-modes and crust modes can be strongly modified by the rapid rotation of the star.

Accretion powered millisecond pulsars show small amplitude X-ray oscillations with periods equal to their spin periods, which are assumed to be produced by a hot spot on the surface of the star (e.g., Lamb et al 2009). Numata & Lee (2010) suggested that global oscillations of neutron stars can periodically perturb such a hot spot so that the oscillation mode periods could be observable as X-ray flux oscillations. They also suggested that since the hot spot on
the neutron star surface is corotating with the star, the oscillation frequencies should be equal to those observed in the corotating frame of the star.

In this paper, we pursue the possibility that the detected frequency in the pulsar is caused by an unstable non-radial mode of the rapidly rotating neutron star. To obtain the oscillation frequency $\omega$ of pulsationally unstable non-radial modes of neutron stars, we calculate the surface $r$-modes and $g$-modes excited by nuclear helium burning in the surface layer for $|m| = 1$ and 2, and toroidal crust modes in the solid crust and rotational modes such as inertial modes and $r$-modes in the fluid core for $m = 2$, where $m$ is the azimuthal wave number of the modes. Here, $\omega$ denotes the frequency observed in the corotating frame of the star and is given by $\omega = \sigma + m\Omega$, where $\sigma$ is the oscillation frequency in an inertial frame and $\Omega = 2\pi\nu_{\text{spin}}$ is the angular spin frequency of the star. To calculate surface $r$-modes and $g$-modes, we construct mass accreting and nuclear burning thin shells in steady state. On the other hand, we use a neutron star model composed of a surface fluid ocean, a solid crust, and a fluid core, to compute crust modes in the solid crust and rotational modes in the fluid core. Note that the crust mode and the core $r$-mode are expected to be destabilized by emitting gravitational waves. Assuming $\nu_{\text{esc}} = \omega/2\pi$ and $\nu_{\text{spin}} = \Omega/2\pi$, we look for non-radial oscillation modes that are pulsationally unstable and give the ratio $\kappa \equiv \omega/\Omega \simeq 0.57$ at $\nu_{\text{spin}} = 435\text{Hz}$.

2 NUMERICAL RESULTS

2.1 Surface $r$-modes and $g$-modes

The method of calculation used for $r$- and $g$-modes in the surface fluid ocean is the same as that given in Strohmayer & Lee (1996) and Lee (2004). Following Strohmayer & Lee (1996), assuming steady nuclear burning of hydrogen and helium for a given mass accretion rate $\dot{M}$, we compute the surface fluid shell of mass $10^{-13}M_{\odot}$, which may be regarded as the outermost layer of a mass accreting neutron star. We assume that the surface layer is radiative, and that accreting matter is composed mostly of helium with a small fractional mixture of hydrogen. We obtain the temperature distributions in the shell similar to those calculated by Lee (2004). In the region where helium burning takes place, the mean molecular weight rapidly changes with depth, which yields a bump in the Brunt-Väisälä frequency $N$. The existence of a small fraction of hydrogen in accreting matter slightly enhances the bump, particularly for high mass accretion rates. When the mass accretion rate is low, the burning layers of helium and hydrogen are well separated in the shell. The pulsational stability of oscillation modes propagating in the thin surface layer is examined by non-adiabatic oscillation calculation assuming uniform rotation as described by Lee & Saio (1987). For example, the displacement vector $\xi$ in rotating stars is represented by a finite series expansion in terms of spherical harmonic function $Y_l^m(\theta, \phi)$ for a given $m$:

$$\xi = \sum_{j=1}^{j_{\text{max}}} S_{lj}(r) Y_l^m(\theta, \phi) e^{i\omega t},$$

where $l_j = |m| + 2(j - 1)$ and $\xi_j = \xi_j + 1$ for even modes and $l_j = |m| + 2j - 1$ and $\xi_j = \xi_j - 1$ for odd modes, and $j = 1, \ldots, j_{\text{max}}$. For even mode, the function $\kappa_r$, for example, is symmetric about the equator of the star, while it is antisymmetric for odd modes. Note that in this paper no general relativistic effects are considered for the shell and mode computations. For the length of the expansions, we usually take $j_{\text{max}} = 10$.

The results of non-adiabatic calculation of $r$-modes propagating in the surface thin fluid layer are given in Figures 1 and 2 for $l = |m| = 1$ and 2, respectively, where the ratio $\kappa = \omega/\Omega$ and the mode growth timescale $\tau \equiv 1/\text{Im}(\omega)$ of the $r$-mode are plotted as functions of $\nu_{\text{spin}} = \Omega/2\pi$ for the mass accretion rates $\dot{M}/\dot{M}_{\text{Edd}} = 0.7$ (black curves) and 0.1 (red curves) for a neutron star model of the mass $M = 1.4M_{\odot}$ and radius $R = 1.179 \times 10^6\text{cm}$, where $\dot{M}_{\text{Edd}} \equiv 4\pi c R \kappa_e = 1.88 \times 10^{18}(1 + X)^{-1}(R/10^6)\text{g s}^{-1}$ is the Eddington mass accretion rate with $\kappa_e$ being the electron scattering opacity. Here, the notation $\xi$ indicates that the $r$-mode has one radial node of the eigenfunction. Note that only $r$-mode of $l = m$ are found pulsationally unstable and $r$-modes that have radial nodes of the eigenfunctions more than one are all stable. As discussed by Lee (2004), the frequency $\omega$ of the $r$-mode propagating in the surface thin shell becomes insensitive to $\Omega$ for rapid rotation rates and is approximately given by the formula:

$$\omega \simeq \frac{mN_0(D/R)}{(2j + 1)\sqrt{\lambda}},$$

where $j$ is an integer associated with the mode, $\lambda$ is the separation constant used to separate the horizontal motions from the vertical motions in the shell, $N_0$ is the representative value of the Brunt-Väisälä frequency in the shell, and $D$ is the depth of the fluid ocean (Lee 2004, see also Pedlosky 1987). We may take $N_0$ as the value at the helium burning layer where the mode excitation takes place. Because of this insensitiveness of $\omega$ to $\Omega$ the ratio $\kappa$ decreases as $\nu_{\text{spin}} = \Omega/2\pi$ increases. The existence of a small amount of hydrogen enhances the value of $N_0$ at the helium burning layer, particularly for high mass accretion rates $\dot{M}$, and this enhancement leads to an increase in the ratio $\kappa$ at a given value of $\nu_{\text{spin}}$. The destabilization of the $r$-modes takes place because of the strong temperature dependence of helium burning in the thin shell. As indicated by the right panels of Figures 1 and 2, the existence of a small amount of hydrogen tends to weaken the destabilization, that is, the mode growth timescale becomes longer as the hydrogen content is increased. Because the temperature in the helium burning region in the shell becomes higher for higher mass accretion rate $\dot{M}$, the growth timescale $\tau$ becomes shorter as $\dot{M}$ increases.

As shown by Figures 1 and 2, if we assume high mass accretion rates $\dot{M} \sim \dot{M}_{\text{Edd}}$ and a small mixture of hydrogen $X$ in the accreting matter, for low $m$ values we can find pulsationally unstable $r$-modes of $l = m$ whose oscillation
frequency in the corotating frame is consistent with the ratio \( \kappa \simeq 0.57 \) at \( \nu_{\text{spin}} \simeq 435 \text{Hz} \).

We have carried out similar calculations for the \( r \)-modes of \( l' = m + 1 = 2 \) propagating in helium burning shells, and we have found that the inertial frame frequency \( \sigma = \omega - m\Omega \) of the \( r_1 \)-mode, which is driven by helium burning, can give the ratio \( \tilde{\kappa} \equiv \sigma/\Omega \) consistent with the observed value.

For \( g \)-modes in the surface helium burning shells, only retrograde \( g_1 \)-modes of \( l = m = 1 \) are found to be pulsationally unstable to give the ratio \( \kappa \) consistent with the observed value.

2.2 Crust modes and Core \( r \)-modes

It is now well known that the \( r \)-mode of \( l' = |m| = 2 \) is most strongly destabilized by gravitational wave radiation (e.g., Andersson 1998; Friedman & Morsink 1998; Lindblom et al. 1998). For the \( l' = |m| \) \( r \)-mode of entirely fluid stars, the ratio \( \kappa = \omega/\Omega \), which tends to \( \kappa \to 2m/l'(l' + 1) = 2/3 \) for \( m = 2 \) as \( \Omega \to 0 \), only weakly depends on \( \Omega \) and on the stratification of the fluid (e.g., Yoshida & Lee 2000). This fact may suggest that the \( r \)-mode of \( l' = |m| = 2 \) is unlikely to be responsible for the observed ratio \( \kappa \simeq 0.57 \). But, Andersson et al. (2014) recently suggested that a general relativistic effect can reduce the ratio \( \kappa \) such that the value.
of $\kappa$ for the $l' = m = 2$ r-mode becomes consistent with the observed value, depending on the mass $M$, the radius $R$, and the equation of state of the star (e.g., Lockitch et al 2003; see also Yoshida & Lee 2002). However, they also argued that the amplitudes of the r-mode suggested by the detection of the coherent frequency is too large to be consistent with the spin evolution of the star.

The presence of a solid crust in a neutron star, however, makes the modal properties of the star quite complicated. Because of a solid crust, for example, we have toroidal shear waves propagating in the solid crust, which affect the r-modes and inertial modes in the fluid core. In fact, using neutron star models with a solid crust, Yoshida & Lee (2001) calculated toroidal crust modes and rotational modes (r- and inertial-modes) and showed that mode crossings (avoided crossings) between the crust modes and the r- and inertial modes in the core are quite common. In addition to mode crossing, if there is a surface fluid ocean on the solid crust, it is important to note that there exist r-modes propagating in the fluid ocean besides the r-modes in the fluid core (Lee & Strohmayer 1996; Yoshida & Lee 2001) and that the amplitudes of a core r-mode penetrate through the solid crust and are amplified in the surface ocean since the r-modes in the ocean and in the core have similar frequencies and the mass density in the ocean is much smaller than in the core. This means that even if the amplitudes of the r-mode is large at the surface so that a surface hot spot is perturbed with appreciable amplitudes, the r-mode amplitudes in the fluid core can be much smaller than those inferred from the detection of a coherent frequency in the X-ray light curves. This reduction of the r-mode amplitude in the core will render the difficulty in the r-mode interpretation for the identified frequency less serious when the spin evolution of the star is almost exclusively determined by the core r-mode.

The mode crossings mentioned above may be easily understood (Yoshida & Lee 2001). In the corotating frame of the star, the oscillation frequency of a toroidal crust mode may be given by (e.g., Strohmayer 1991)

$$\omega_{\text{crust}}(\Omega) \approx \omega_{\text{crust}}(0) + \frac{m\Omega}{l'(l' + 1)}$$

where $\omega_{\text{crust}}(0)$ is the oscillation frequency of the crust mode at $\Omega = 0$, while the oscillation frequency of an r-mode may be given by

$$\omega_r(\Omega) \approx \frac{2m\Omega}{l'(l' + 1)}$$

For $l' \lesssim 10$, $\omega_{\text{crust}}(0)$ of the fundamental toroidal crustal mode at $\Omega = 0$ is less than the critical frequency $\Omega_{\text{crit}} \equiv \sqrt{GM/R^3}$ and it increases with increasing $l'$, while the frequencies $\omega_{\text{crust}}(0)$ of overtone modes of the toroidal crustal mode are rather insensitive to the values of $l'$ (e.g., Lee 2008). For given $m$ and $l'$, since the frequency ratio $\omega_r(\Omega)/\omega_{\text{crust}}(0)$ can be smaller than $\omega_r/\Omega$ for large $\Omega$, the frequencies of the two modes cross with each other at

$$\Omega_{\text{cross}} \approx \frac{l'(l' + 1)}{m} \omega_{\text{crust}}(0).$$

Because of the mode crossing, which usually results in an avoided crossing, the eigenfrequencies of the r-mode and the toroidal mode as well as their eigenfrequencies are significantly modified. If the $l' = |m| = 2$ r-mode remains unstable because of gravitational wave emission even around the crossing point, we expect that the crustal modes coupled with the r-mode are also destabilized as a result of the coupling. This suggests a possibility for the existence of an unstable crustal mode with the frequency ratio $\kappa \approx 0.57$ at the spin rate $\Omega_{\text{spin}} = 435$Hz, although the mode crossing between the r-mode and the fundamental crustal mode for $l' = m = 2$ may take place at rather slow rotation rates (Yoshida & Lee 2001).

For $m = 2$, we calculate r-modes, inertial modes and toroidal crust modes for a 1.4$M_\odot$ neutron star model. The neutron star model, which is composed of a surface fluid ocean, a solid crust, and a fluid core, is computed by using a cooling evolution code of neutron stars, where the equation of state (EOS) for the core is that by Douchin & Haensel (2001), EOS for the crust by Nelg & Vautherin (1973) and Baym, Pethick & Sutherland (1971), and the surface ocean is assumed to be made of iron. We have picked up a neutron star model having the central temperature $T_c \approx 2 \times 10^8$K for mode computation. For the solid crust, we assume the average shear modulus $\mu_{\text{crust}} = \mu_0$, where $\mu_0 = 0.1194(Ze)^2n/a$ and $n$ is the number density of the nuclei and $a$ is the separation between the nuclei defined by $4\pi a^3/n = 1$ (Strohmayer et al 1991). The method of calculation for oscillation modes of the tree component neutron star model is the same as that in Lee & Strohmayer (1996), who apply Newtonian dynamics in the Cowling approximation, employ the finite series expansions similar to equations (1)~(3), and assume adiabatic oscillations. Using the eigenfunctions of the modes obtained numerically, we compute the mode growth timescale $\tau$ defined by $\tau^{-1} = (2E)^{-1}dE/dt$, where $E$ is the oscillation energy defined by

$$E = \frac{1}{2} \int \left( \rho \delta v^r \delta v^r + \frac{\delta p}{\rho} \right) d^3 x,$$

where $\delta v^r$, $\delta p$, and $\delta \rho$ represent the Eulerian perturbations of the fluid velocity, the pressure, and the mass density, respectively, and the asterisk (*) implies complex conjugation. The energy gain rate $dE/dt$ is determined by the sum of various excitation and dissipation mechanisms (Yoshida & Lee 2000; see also Ipser & Lindblom 1991; Lindblom et al 1998), that is,

$$\frac{dE}{dt} = (\frac{dE}{dt})_S + (\frac{dE}{dt})_B + (\frac{dE}{dt})_{GD} + (\frac{dE}{dt})_{GJ},$$

where

$$\frac{dE}{dt} = -2 \int \eta \delta \sigma^{ij} \delta \sigma^{ij} d^3 x \quad (10)$$

is the dissipation rate due to the shear viscosity with $\eta$ being the shear viscosity coefficient, and

$$\frac{dE}{dt} = - \int \zeta \delta \theta \delta \theta^* d^3 x \quad (11)$$

is the dissipation rate due to the bulk viscosity with $\zeta$ being the bulk viscosity coefficient, and

$$\frac{dE}{dt} = - \sigma \sum_{l=2}^{\infty} N_l \sigma^2 |\delta D_{lm}|^2 \quad (12)$$

$$\frac{dE}{dt} = - \sigma \sum_{l=2}^{\infty} N_l \sigma^2 |\delta J_{lm}|^2 \quad (13)$$

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are the dissipation or excitation rates associated with gravitational wave radiation, and the definitions of the various quantities such as $\delta \sigma$, $\delta \theta$, $\delta D_m$, $\delta J_m$, $N_l$, $\eta$, and $\zeta$ are given in Yoshida & Lee (2000). For the details, see also, e.g., Ipser & Lindblom (1991), Cutler & Lindblom (1987), and Sawyer (1989). Since we are interested in the $r$-modes which are destabilized by emitting gravitational waves, we expect that $dE/dt$ and $\tau$ are positive for the modes interested. Note that we ignore the effects of rotational deformation on the oscillation calculations, and that no effects of superfluidity in the core on the modal properties and on the viscosity coefficients are included.

As indicated by equations (12) and (13), gravitational wave emission yields destabilizing contributions to the oscillation modes with $\omega > 0$, which can be rewritten as

$$0 < \kappa = \omega / \Omega < m, \quad (14)$$

and we have $0 < \kappa < 2$ for $m = 2$. For a mode to be globally unstable due to the gravitational wave emission, however, the destabilizing contributions need to dominate the sum of all dissipative contributions.

In Figure 3, we plot the ratio $\kappa$ and the growth time $\tau$ in second for oscillation modes of the $M = 1.4 M_\odot$ neutron star model as functions of $\nu_{\text{spin}}$, where the black and red dots represent stable and unstable modes, respectively, and $\tau$ is plotted only for unstable modes. Note that the shear modulus in the crust is set equal to $\mu_{\text{crust}} = \mu_0$. Note also that we have not tried to obtain a complete mode distribution by calculating every detail of mode crossings in the $\nu_{\text{spin}}$-$\kappa$ plane. Almost horizontally running black curves, which experience mode crossings with toroidal crust modes, indicate inertial modes in the core. Figure 3 shows that mode crossings are quite common between the toroidal crust modes and inertial modes or $r$-modes, and that the effects of mode coupling between the modes belonging to the same $l$ are significant so that the crossing appreciably modifies the frequencies and eigenfunctions, which is particularly true between the toroidal crustal modes and $r$-mode of $l' = m = 2$. In other words, the crossings between modes of different $l$'s are not necessarily strong enough to significantly modify the mode properties near the crossing point. The mode along the red curve, running almost parallel to the line of $\kappa \approx 0.65$ after the crossing at $\nu_{\text{spin}} \sim 130$Hz, has the shortest growth timescale $\tau$ at every $\nu_{\text{spin}}$ and can be regarded as the $r$-mode of $l' = m = 2$. The eigenfunctions of the mode with $\kappa \approx 0.6567$ at $\nu_{\text{spin}} = 435$Hz are shown in Figure 4, where the expansion coefficients $xT_{11}$, $xH_{11}$, and $xS_{11}$ are plotted versus $x \equiv r/R$ and, in the inset, versus $\log(1 - x)$, and the amplitude normalization is given by $i T_{00} = 1$ at the surface $x = 1$. The toroidal component is totally dominating over the other components both in the fluid regions and in the solid crust, indicating the mode is an $r$-mode. At the bottom of the solid crust, the amplitude of the toroidal component is only 1% of the amplitude at the surface, indicating that the amplification of the mode amplitude occurs between the fluid core and the surface ocean.

As shown by Figure 3, we find no unstable modes with the ratio $\kappa \approx 0.57$ at $\nu_{\text{spin}} = 435$Hz, but we find another unstable mode with $\kappa \approx 0.4999$, the eigenfunctions of which are plotted in Figure 5. The figure shows that the toroidal component is dominating both in the crust and in the fluid core although the spheroidal components of the displacement vector also have appreciable amplitudes in both of the regions. Note also that the amplitude of the eigenfunctions in the surface ocean is not amplified by a large factor compared with the amplitude in the crust and fluid core, indicating that no strong amplification of the amplitudes takes place between the fluid core and ocean. The properties of the eigenfunctions suggest that the mode is regarded as a toroidal crust mode and that the amplitudes of the mode penetrate into the fluid core as a result of the effects of rapid rotation. This penetration of the amplitudes into the fluid core makes the mode unstable by emitting gravitational waves.

Figure 3 shows that for the case of $\mu_{\text{crust}} = \mu_0$, there exist neither unstable crust modes nor unstable $r$-modes which give the ratio $\kappa \approx 0.57$ at $\nu_{\text{spin}} = 435$Hz. As suggested by

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and the ratio $\kappa \approx 0.57$ at $\nu_{\text{spin}} = 435\text{Hz}$ for a certain reasonable combination of physical quantities such as the mass $M$, the radius $R$, the shear modulus $\mu_{\text{crust}}$, and the equation of state used to construct a neutron star model.

3 CONCLUSION

We have discussed candidates of non-radial modes for the detected frequency $\nu_{\text{osc}} = 5727 \times \nu_{\text{spin}}$ at $\nu_{\text{spin}} = 435\text{Hz}$ found for the millisecond X-ray pulsar XTE J1751-305. We have shown that the $r_1$-modes and $q_1$-modes propagating in the surface fluid layer of accreting matter composed mostly of helium with a small mixture of hydrogen are pulsationally unstable and can be responsible for the frequency detected. We have found that toroidal crustal modes of $l' = m = 2$, which have appreciable amplitudes in the fluid core because of the effects of rapid rotation, are destabilized by emitting gravitational waves, although the strength of the destabilization is weaker than that for the $r$-mode of $l' = m = 2$. We have also suggested a possibility that an unstable toroidal crust mode of a neutron star model can be responsible for the observed periodicity in the pulsar.

We have shown that for the $r$-mode of $l' = m = 2$ there occurs a strong amplification of the amplitudes between the fluid core and the surface fluid ocean, an amplification as large as $f_{\text{amp}} \equiv \alpha_{\text{surface}}/\alpha_{\text{core}} \sim 10^2$, where $\alpha$'s are the parameters representing the $r$-mode amplitudes. As discussed by Strohmayer & Mahmoodifar (2014), the strength of the detected frequency indicates the amplitude of $\alpha_{\text{surface}} \sim 10^{-3}$, for which the $r$-mode amplitudes in the fluid core becomes $\alpha_{\text{core}} \sim 10^{-5}$ for $f_{\text{amp}} \sim 10^2$. This significant reduction in the $r$-mode amplitudes in the core will render much less serious the difficulty met in the $r$-mode interpretation for the detected frequency, since the spin change rate and heating rate of the star due to the $r$-mode excitation become much smaller than those inferred from the detection of the frequency. Note that, as Andersson et al (2014) discussed, if various frequency corrections such as due to the general relativity are taken into account (see e.g. Yoshida & Lee 2002; Lockitch, Friedman & Andersson 2003), it is possible to obtain the ratio $\kappa \approx 0.57$ for the $l' = m = 2$ $r$-mode, for which the ratio tends to $\kappa = 2/3$ in the limit of $\Omega \rightarrow 0$ in the Newtonian gravity. Probably, we need a larger amplification factor $f_{\text{amp}}$ to completely remove the difficulty, since Mahmoodifar & Strohmayer (2013), for example, suggested the $r$-mode amplitudes ranging from $\alpha \sim 10^{-8}$ to $\sim 10^{-6}$. We need more careful discussions and calculations for the determination of the factor $f_{\text{amp}}$ for the $r$-modes of neutron star models with a solid crust. The factor $f_{\text{amp}}$ may depend on the structures of the ocean and the crust. We need to compute the $r$-mode in the general relativistic frame work with proper treatments of the jump conditions at the interfaces between the solid crust and the fluid regions. The existence of a weak magnetic field possibly affects the amplification.

To estimate the effects of the viscous boundary layer on the stability we use an extrapolation formula (Bildsten & Ushomirsky 2000; Andersson et al 2000; Yoshida & Lee 2001) given by

\begin{align*}
\kappa &\approx 0.57 \text{ at } \nu_{\text{spin}} = 435\text{Hz} \\
\alpha_{\text{surface}} &\sim 10^{-3} \\
\alpha_{\text{core}} &\sim 10^{-5} \\
f_{\text{amp}} &\sim 10^2
\end{align*}
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\[
\frac{1}{\tau_{\text{VBL}}} = \frac{1}{\tau_{\text{VBL}}^0} \left( \frac{10^8 \text{ K}}{T_c} \right) \left( \frac{\Omega^2}{\pi G \bar{\rho}} \right)^{1/4},
\]

(15)

where \( \bar{\rho} = M/(4\pi R^3/3) \). If we take the value \( \bar{\tau}_{\text{VBL}} = 3.7 \times 10 \)
(e.g., Yoshida & Lee 2001), we have \( \tau_{\text{VBL}} \sim 10^2 \) for \( T_c \sim 2 \times 10^8 \text{K} \), which is shorter than the growth timescales of the \( l' = m = 2 \) r-mode and toroidal crust mode computed in this paper, suggesting that these modes are damped by the viscous boundary layer effects. If the transition between the solid crust and the fluid core is not sharp enough for a thin viscous boundary layer to form, the effects of viscous dissipations at the boundary will be weak and probably the destabilized r- and crust modes by emitting gravitational waves remain unstable (e.g., Bondarescu & Wasserman 2013).

If the frequency detected in the X-ray pulsar XTE J1751-305 is really produced by a non-radial mode of the underlying neutron star and if it is possible to obtain a correct mode identification for the frequency, we will be able to use the mode to probe the physical properties of the star, such as the mass \( M \), the radius \( R \), the shear modulus \( \mu_{\text{crust}} \), and equation of state. If the frequency is due to a toroidal crustal mode or an r-mode destabilized by emitting gravitational waves, the detection of the oscillation frequency can be regarded as an evidence for the existence of a neutron star radiating gravitational waves with detectable amplitudes, which will be useful for understanding the physics expected in strong gravity environment.

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