LOCAL SYMPLECTIC ALGEBRA OF QUASI-HOMOGENEOUS CURVES

WOJCIECH DOMITRZ

ABSTRACT. We study the local symplectic algebra of parameterized curves introduced by V. I. Arnold in [A1]. We use the method of algebraic restrictions to classify symplectic singularities of quasi-homogeneous curves. We prove that the space of algebraic restrictions of closed 2-forms to the germ of a \(K\)-analytic curve is a finite dimensional vector space. We also show that the action of local diffeomorphisms preserving the quasi-homogeneous curve on this vector space is determined by the infinitesimal action of liftable vector fields. We apply these results to obtain the complete symplectic classification of curves with the semigroups \((3, 4, 5), (3, 5, 7), (3, 7, 8)\).

1. INTRODUCTION

We study the problem of classification of parameterized curve-germs in a symplectic space \((K^{2n}, \omega)\) up to the symplectic equivalence (for \(K = \mathbb{R}\) or \(\mathbb{C}\)). The symplectic equivalence is a right-left equivalence (or \(A\)-equivalence) in which the left diffeomorphism-germ is a symplectomorphism of \((K^{2n}, \omega)\) i.e. it preserves the given symplectic form \(\omega\) in \(K^{2n}\).

The problem of \(A\)-classification of singularities of parameterized curves-germs was studied by J. W. Bruce and T. J. Gaffney, C. G. Gibbson and C. A. Hobbs. Bruce and Gaffney ([BG]) classified the \(A\)-simple plane curves and in [GH] the classification of the \(A\)-simple space curves was given. The singularity (an \(A\)-equivalence class) is called simple if it has a neighbourhood intersecting only finite number of singularities. V. I. Arnold ([A2]) classified stably simple singularities of curves. The singularity is stably simple if it is simple and remains simple after embedding into a larger space.

The main tool and the invariant separating the singularities in \(A\)-classification of curves is the semigroup of a curve singularity \(t \mapsto f(t) = (f_1(t), \ldots, f_m(t))\) (see [GH] and [A2]). It is the subsemigroup of the additive semigroup of natural numbers formed by the orders of zero at the origin of all linear combinations of the products of \(f_i(t)\).

In [A1] V. I. Arnold discovered new symplectic invariants of parameterized curves. He proved that the \(A_{2k}\) singularity of a planar curve (the orbit with respect to standard \(A\)-equivalence of parameterized curves) split into exactly \(2k + 1\) symplectic singularities (orbits with respect symplectic equivalence of parameterized curves). Arnold posed a problem of expressing these invariants in terms of the

1991 Mathematics Subject Classification. Primary 53D05. Secondary 14H20, 58K50, 58A10.
Key words and phrases. symplectic manifold, curves, local symplectic algebra, algebraic restrictions, relative Darboux theorem, singularities.

The work of the author was supported by Institute of Mathematics, Polish Academy of Sciences.
local algebra’s interaction with the symplectic structure. He proposed to call this interaction \textit{local symplectic algebra}.

In \cite{IJ1} G. Ishikawa and S. Janeczko classified symplectic singularities of curves in the 2-dimensional symplectic space. All simple curves in this classification are quasi-homogeneous.

Symplectic singularity is \textit{stably simple} if it is simple and remains simple if the ambient symplectic space is symplectically embedded (i.e. as a symplectic submanifold) into a larger symplectic space. In \cite{K} P. A. Kolgushkin classified the stably simple symplectic singularities of curves (in the \(\mathbb{C}\)-analytic category). All stably simple symplectic singularities of curves are quasi-homogeneous too.

In \cite{DJZ2} new symplectic invariants of singular quasi-homogeneous subsets of a symplectic space were explained by the algebraic restrictions of the symplectic form to these subsets.

The algebraic restriction is an equivalence class of the following relation on the space of differential \(k\)-forms:

Differential \(k\)-forms \(\omega_1\) and \(\omega_2\) have the same \textit{algebraic restriction} to a subset \(N\) if \(\omega_1 - \omega_2 = \alpha + d\beta\), where \(\alpha\) is a \(k\)-form vanishing on \(N\) and \(\beta\) is a \((k-1)\)-form vanishing on \(N\).

The algebraic restriction of a \(k\)-form \(\omega_1\) to a subset \(N_1\) and the algebraic restriction of a \(k\)-form \(\omega_2\) to a subset \(N_2\) are \textit{diffeomorphic} if there exists a diffeomorphism \(\Phi\) of \(\mathbb{K}^m\) which maps \(N_1\) to \(N_2\) such that \(\Phi^*\omega_2\) and \(\omega_1\) have the same algebraic restriction to \(N_1\) (for details see section \(\mathbf{3}\)).

The results in \cite{DJZ2} were obtained by the following generalization of Darboux-Givental theorem.

\textbf{Theorem 1 (\cite{DJZ2}).} Quasi-homogeneous subsets of a symplectic manifold \((M, \omega)\) are locally symplectomorphic if and only if algebraic restrictions of the symplectic form \(\omega\) to them are locally diffeomorphic.

This theorem reduces the problem of symplectic classification of quasi-homogeneous subsets to the problem of classification of algebraic restrictions of symplectic forms to these subsets.

In \cite{DJZ2} the method of algebraic restrictions is applied to various classification problems in a symplectic space. In particular the complete symplectic classification of classical A-D-E singularities of planar curves is obtained, which contains Arnold’s symplectic classification of \(A_{2k}\) singularity.

In this paper we return to Arnold’s original problem of local symplectic algebra of a parameterized curve. We show that the method of algebraic restrictions is a very powerful classification tool for quasi-homogeneous parameterized curves. This is due to the several reasons. The most important one is that the space of algebraic restrictions of germs of closed 2-forms to a \(\mathbb{K}\)-analytic parameterized curve is a finite dimensional vector space. This fact follows from the following more general result conjectured in \cite{DJZ2}, which we prove in this paper.

\textbf{Theorem 2.} Let \(C\) be the germ of a \(\mathbb{K}\)-analytic curve. Then the space of algebraic restrictions of germs of closed 2-forms to \(C\) is a finite dimensional vector space.

By a \(\mathbb{K}\)-analytic curve we understand a subset of \(\mathbb{K}^m\) which is locally diffeomorphic to a 1-dimensional (possibly singular) \(\mathbb{K}\)-analytic subvariety of \(\mathbb{K}^m\). Germs of \(\mathbb{K}\)-analytic parameterized curves can be identified with germs of irreducible \(\mathbb{K}\)-analytic curves.
The tangent space to the orbit of an algebraic restriction \( a \) to the germ \( f \) of a parameterized curve is given by the Lie derivative of \( a \) with respect to germs of liftable vector fields over \( f \). We say that the germ \( X \) of a liftable vector field acts **trivially** on the space of algebraic restriction if the Lie derivative of any algebraic restriction with respect \( X \) is zero.

**Theorem 3.** The space of germs of liftable vector fields over the germ of a parameterized quasi-homogeneous curve which act nontrivially on the space of algebraic restrictions of closed 2-forms is a finite dimensional vector space.

Theorem 2 is proved in section 5. In section 6 we prove Theorem 3 using the quasi-homogeneous grading on the space of algebraic restrictions. We show that there exist quasi-homogeneous bases of the space of algebraic restrictions of closed 2-forms and of the space of liftable vector fields which act nontrivially on the space of algebraic restrictions to a quasi-homogeneous parameterized curve. These bases are allowed us to prove Theorems 6,13 that states that the linear action on the space of algebraic restrictions of closed 2-forms to the germ of a quasi-homogeneous parameterized curve by Lie derivatives with respect to liftable vector fields determines the action on this space by local diffeomorphisms preserving this germ of the curve.

Both the space of algebraic restrictions of symplectic forms and this linear action are determined by the semigroup of the curve singularity.

We apply the method of algebraic restrictions and results of Section 6 to obtain the complete symplectic classification of curves with the semigroups \((3, 4, 5), (3, 5, 7)\) and \((3, 7, 8)\) in Sections 7,8 and 9. The classification results are presented in Table 1, Table 5 and Table 9. All normal forms are given in the canonical coordinates \((p_1, q_1, \cdots , p_n, q_n)\) in the symplectic space \((\mathbb{R}^{2n}, \sum_{i=1}^{n} dp_i \wedge dq_i)\). The parameters \(c, c_1, c_2\) are moduli. The different singularity classes are distinguished by discrete symplectic invariants: the symplectic multiplicity \(\mu_{sympl}(f)\), the index of isotropness \(i(f)\) and the Lagrangian tangency order \(Lt(f)\), which are considered in Section 4.

We consider only quasi-homogeneous parameterized curves in this paper. But there are \(\mathcal{A}\)-simple singularities of curves which are not quasi-homogeneous. For example the curve \( f(t) = (t^3, t^7 + t^8) \) is not quasi-homogeneous. Then Theorem 4 cannot be applied for such curves. But there exists a generalization of this theorem to any subsets \( N \) of \( \mathbb{R}^n \) ([DJZ2], section 2.6). In general there is one more invariant for the symplectic classification problem which can be represented as a cohomology class in the second cohomology group of the complex of 2-forms with zero algebraic restrictions to \( N \). This cohomology groups vanish for quasi-homogeneous subsets ([DJZ1]). They are finite dimensional for \(\mathcal{C}\)-analytic varieties with an isolated singularity ([BH]). It implies that they are finite dimensional for non quasi-homogeneous \(\mathcal{C}\)-analytic curves. The space of algebraic restriction of closed 2-forms to a \(\mathbb{K}\)-analytic curve is finite dimensional too by Theorem 2. But the description of the action of diffeomorphisms preserving a non quasi-homogeneous curve on algebraic restrictions is much more complicated.

**Acknowledgements.** The author wishes to express his thanks to M. Zhitomirskii for suggesting the subject and for many helpful conversations and remarks during the writing of this paper. The author thanks Z. Jelonek for very useful remarks on the proof of Theorem 2 and the referee of this paper for many valuable suggestions.
2. QUASI-HOMOGENEITY

In this section we present the basic definitions and properties of quasi-homogeneous germs.

**Definition 2.1.** A curve-germ \( f : (\mathbb{R}, 0) \to (\mathbb{R}^m, 0) \) is quasi-homogeneous if there exist coordinate systems \( t \) on \((\mathbb{R}, 0)\) and \((x_1, \cdots, x_m)\) on \((\mathbb{R}^m, 0)\) and positive integers \((\lambda_1, \cdots, \lambda_m)\) such that

\[
 df \left( \frac{d}{dt} \right) = E \circ f,
\]

where \( E = \sum_{i=1}^{m} \lambda_i x_i \frac{d}{dx_i} \) is the germ of the Euler vector field on \((\mathbb{R}^m, 0)\). The coordinate system \((x_1, \cdots, x_m)\) is called quasi-homogeneous, and numbers \((\lambda_1, \cdots, \lambda_m)\) are called weights.

**Definition 2.2.** Positive integers \(\lambda_1, \cdots, \lambda_m\) are linearly dependent over non-negative integers if there exists \(j\) and non-negative integers \(k_i\) for \(i \neq j\) such that \(\lambda_j = \sum_{i \neq j} k_i \lambda_i\). Otherwise we say that \(\lambda_1, \cdots, \lambda_m\) are linearly independent over non-negative integers.

It is easy to see that quasi-homogeneous curves have the following form in the quasi-homogeneous coordinates.

**Proposition 2.3.** A curve-germ \( f \) is quasi-homogeneous if and only if \( f \) is \(A\)-equivalent to

\[
 t \mapsto (t^{\lambda_1}, \cdots, t^{\lambda_k}, 0, \cdots, 0),
\]

where \(\lambda_1 < \cdots < \lambda_k\) are positive integers linearly independent over non-negative integers.

\(\lambda_1, \cdots, \lambda_k\) generate the semigroup of the curve \( f \), which we denote by \((\lambda_1, \cdots, \lambda_k)\).

The weights \(\lambda_1, \cdots, \lambda_k\) are determined by \( f \), but weights \(\lambda_{k+1}, \cdots, \lambda_m\) can be arbitrary positive integers. Actually in the next sections we study the projection of \( f \) to non zero components: \( \mathbb{R} \ni t \mapsto (t^{\lambda_1}, \cdots, t^{\lambda_k}) \in \mathbb{R}^k \).

**Definition 2.4.** The germ of a function, a differential \(k\)-form, or a vector field \(\alpha\) on \((\mathbb{R}^m, 0)\) is quasi-homogeneous in a coordinate system \((x_1, \cdots, x_m)\) on \((\mathbb{R}^m, 0)\) with positive weights \((\lambda_1, \cdots, \lambda_m)\) if \(\mathcal{L}_E \alpha = \delta \alpha\), where \( E = \sum_{i=1}^{m} \lambda_i x_i \frac{d}{dx_i} \) is the germ of the Euler vector field on \((\mathbb{R}^m, 0)\) and \(\delta\) is a real number called the quasi-degree.

It is easy to show that \(\alpha\) is quasi-homogeneous in a coordinate system \((x_1, \cdots, x_m)\) with weights \((\lambda_1, \cdots, \lambda_m)\) if and only if \( F^* \alpha = t^\delta \alpha\), where \( F(x_1, \cdots, x_m) = (t^{\lambda_1} x_1, \cdots, t^{\lambda_m} x_m) \). Then germs of quasi-homogeneous functions of quasi-degree \(\delta\) are germs of weighted homogeneous polynomials of degree \(\delta\). The coefficient \(f_{i_1, \cdots, i_k}\) of the quasi-homogeneous differential \(k\)-form \( \sum f_{i_1, \cdots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \) of quasi-degree \(\delta\) is a weighted homogeneous polynomial of degree \(\delta - \sum_{j=1}^{k} \lambda_{i_j}\). The coefficient \(f_i\) of the quasi-homogeneous vector field \( \sum_{i=1}^{m} f_{i} \frac{d}{dx_i} \) of quasi-degree \(\delta\) is a weighted homogeneous polynomial of degree \(\delta + \lambda_i\).

**Proposition 2.5.** If \( X \) is the germ of a quasi-homogeneous vector field of quasi-degree \(i\) and \( \omega \) is the germ of a quasi-homogeneous differential form of quasi-degree \(j\) then \( \mathcal{L}_X \omega \) is the germ of a quasi-homogeneous differential form of quasi-degree \(i + j\).
Proof. Since $L_{E}X = [E, X] = iX$ and $L_{E}\omega = j\omega$, we have

\[ L_{E}(L_{X}\omega) = L_{X}(L_{E}\omega) + L_{[E, X]}\omega = L_{X}(j\omega) + L_{iX}\omega = jL_{X}\omega + iL_{X}\omega = (i + j)L_{X}\omega. \]

It implies that $L_{X}a$ is quasi-homogeneous of quasi-degree $i + j$. \qed

3. The method of algebraic restrictions

In this section we present basic facts on the method of algebraic restrictions. The proofs of all results of this section can be found in [DJZ2].

Given the germ of a smooth manifold $(M, p)$ denote by $\Lambda^{p}(M)$ the space of all germs at $p$ of differential $k$-forms on $M$. Given a curve-germ $f : (\mathbb{R}, 0) \to (M, p)$ introduce the following subspaces of $\Lambda^{p}(M)$:

\[ \Lambda_{imf}^{p}(M) = \{ \omega \in \Lambda^{p}(M) : \omega_{|f(t)} = 0 \text{ for any } t \in \mathbb{R} \}; \]
\[ \mathcal{A}_{0}^{p}(Imf, M) = \{ \alpha + d\beta : \alpha \in \Lambda_{imf}^{p}(M), \beta \in \Lambda_{imf}^{p-1}(M) \}. \]

The relation $\omega_{|f(t)} = 0$ means that the $p$-form $\omega$ annihilates any $p$-tuple of vectors in $T_{f(t)}M$, i.e. all coefficients of $\omega$ in some (and then any) local coordinate system vanish at the point $f(t)$.

Definition 3.1. The algebraic restriction of $\omega$ to a curve-germ $f : \mathbb{R} \to M$ is the equivalence class of $\omega$ in $\Lambda^{p}(M)$, where the equivalence is as follows: $\omega$ is equivalent to $\tilde{\omega}$ if $\omega - \tilde{\omega} \in \mathcal{A}_{0}^{p}(Imf, M)$.

Notation. The algebraic restriction of the germ of a form $\omega$ on $(M, p)$ to a curve-germ $f$ will be denoted by $[\omega]_{f}$. Writing $[\omega]_{f} = 0$ (or saying that $\omega$ has zero algebraic restriction to $f$) we mean that $[\omega]_{f} = [0]_{f}$, i.e. $\omega \in \mathcal{A}_{0}^{p}(Imf, M)$.

Remark 3.2. If $g = f \circ \phi$ for a local diffeomorphism $\phi$ of $\mathbb{R}$ then the algebraic restrictions $[\omega]_{f}$ and $[\omega]_{g}$ can be identified, because $Imf = Img$.

Let $(M, p)$ and $(\tilde{M}, \tilde{p})$ be germs of smooth equal-dimensional manifolds. Let $f : (\mathbb{R}, 0) \to (M, p)$ be a curve-germ in $(M, p)$. Let $\tilde{f} : (\mathbb{R}, 0) \to (\tilde{M}, \tilde{p})$ be a curve-germ in $(\tilde{M}, \tilde{p})$. Let $\omega$ be the germ of a $k$-form on $(M, p)$ and $\tilde{\omega}$ be the germ of a $k$-form on $(\tilde{M}, \tilde{p})$.

Definition 3.3. Algebraic restrictions $[\omega]_{f}$ and $[\tilde{\omega}]_{\tilde{f}}$ are called diffeomorphic if there exists the germ of a diffeomorphism $\Phi : (\tilde{M}, \tilde{p}) \to (M, p)$ and the germ of a diffeomorphism $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\Phi \circ \tilde{f} \circ \phi = f$ and $\Phi^{*}([\omega]_{f}) := [\Phi^{*}\omega]_{\Phi^{-1}\circ f} = [\tilde{\omega}]_{\tilde{f}}$.

Remark 3.4. The above definition does not depend on the choice of $\omega$ and $\tilde{\omega}$ since a local diffeomorphism maps forms with zero algebraic restriction to $f$ to forms with zero algebraic restrictions to $\tilde{f}$. If $(M, p) = (\tilde{M}, \tilde{p})$ and $f = \tilde{f}$ then the definition of diffeomorphic algebraic restrictions reduces to the following one: two algebraic restrictions $[\omega]_{f}$ and $[\tilde{\omega}]_{\tilde{f}}$ are diffeomorphic if there exist germs of diffeomorphisms $\Phi$ of $(M, p)$ and $\phi$ of $(\mathbb{R}, 0)$ such that $\Phi \circ f \circ \phi = f$ and $[\Phi^{*}\omega]_{f} = [\tilde{\omega}]_{\tilde{f}}$.

The method of algebraic restrictions applied to singular quasi-homogeneous curves is based on the following theorem.
Theorem 3.5 (Theorem A in [DJZ2]). Let \( f : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^{2n}, 0) \) be the germ of a quasi-homogeneous curve. If \( \omega_0, \omega_1 \) are germs of symplectic forms on \((\mathbb{R}^{2n}, 0)\) with the same algebraic restriction to \( f \) then there exists the germ of a diffeomorphism \( \Phi : (\mathbb{R}^{2n}, 0) \rightarrow (\mathbb{R}^{2n}, 0) \) such that \( \Phi \circ f = f \) and \( \Phi^* \omega_1 = \omega_0 \).

Two germs of quasi-homogeneous curves \( f, g \) of a fixed symplectic space \((\mathbb{R}^{2n}, \omega)\) are symplectically equivalent if and only if the algebraic restrictions of the symplectic form \( \omega \) to \( f \) and \( g \) are diffeomorphic.

Theorem 3.5 reduces the problem of symplectic classification of singular quasi-homogeneous curves to the problem of diffeomorphic classification of algebraic restrictions of symplectic forms to a singular quasi-homogeneous curve.

In the next section we prove that the set of algebraic restrictions of 2-forms to a singular quasi-homogeneous curve is a finite dimensional vector space. We now recall basic properties of algebraic restrictions which are useful for a description of this subset ([DJZ2]).

Let \( f \) be a quasi-homogeneous curve on \((\mathbb{R}^{2n}, 0)\).

First we can reduce the dimension of the manifold we consider due to the following propositions.

Proposition 3.6. Let \((M, 0)\) be the germ of a smooth submanifold of \((\mathbb{R}^{m}, 0)\) containing \(Imf\). Let \( \omega_1, \omega_2 \) be germs of \( k \)-forms on \((\mathbb{R}^{m}, 0)\). Then \([\omega_1]_f = [\omega_2]_f \) if and only if \([\omega_1|_{TM}]_f = [\omega_2|_{TM}]_f \).

Proposition 3.7. Let \( f_1, f_2 \) be curve-germs in \((\mathbb{R}^{m}, 0)\) whose images are contained in germs of equal-dimensional smooth submanifolds \((M_1, 0), (M_2, 0)\) respectively. Let \( \omega_1, \omega_2 \) be germs of \( k \)-forms on \((\mathbb{R}^{m}, 0)\). The algebraic restrictions \([\omega_1]_{f_1}\) and \([\omega_2]_{f_2}\) are diffeomorphic if and only if the algebraic restrictions \([\omega_1|_{TM_1}]_{f_1}\) and \([\omega_2|_{TM_2}]_{f_2}\) are diffeomorphic.

To calculate the space of algebraic restrictions of 2-forms we will use the following obvious properties.

Proposition 3.8. If \( \omega \in A_k^b(Imf, \mathbb{R}^{2n}) \) then \( d\omega \in A_{k+1}^b(Imf, \mathbb{R}^{2n}) \) and \( \omega \wedge \alpha \in A_0^{k+p}(Imf, \mathbb{R}^{2n}) \) for any \( p \)-form \( \alpha \) on \( \mathbb{R}^{2n} \).

The next step of our calculation is the description of the subspace of algebraic restriction of closed 2-forms. The following proposition is very useful for this step.

Proposition 3.9. Let \( a_1, \ldots, a_k \) be a basis of the space of algebraic restrictions of 2-forms to \( f \) satisfying the following conditions

1. \( da_1 = \cdots = da_j = 0 \),
2. the algebraic restrictions \( da_{j+1}, \ldots, da_k \) are linearly independent.

Then \( a_1, \ldots, a_j \) is a basis of the space of algebraic restriction of closed 2-forms to \( f \).

Then we need to determine which algebraic restrictions of closed 2-forms are realizable by symplectic forms. This is possible due to the following fact.

Proposition 3.10. Let \( r \) be the minimal dimension of germs of smooth submanifolds of \((\mathbb{R}^{2n}, 0)\) containing \( Imf \). Let \((S, 0)\) be one of such germs of \( r \)-dimensional smooth submanifolds. Let \( \theta \) be the germ of a closed 2-form on \((\mathbb{R}^{2n}, 0)\). There exists the germ of a symplectic form \( \omega \) on \((\mathbb{R}^{2n}, 0)\) such that \([\theta]_f = [\omega]_f \) if and only if \( \text{rank} \theta|_{T_0S} \geq 2r - 2n \).
4. Discrete symplectic invariants.

Some new discrete symplectic invariants can be effectively calculated using algebraic restrictions. The first one is a symplectic multiplicity \( \mu_{\text{sympl}}(f) \) introduced in [IJ1] as a symplectic defect of a curve \( f \).

**Definition 4.1.** The *symplectic multiplicity* \( \mu_{\text{sympl}}(f) \) of a curve \( f \) is the codimension of a symplectic orbit of \( f \) in an \( A \)-orbit of \( f \).

The second one is the index of isotropness [DJZ2].

**Definition 4.2.** The *index of isotropness* \( \iota(f) \) of \( f \) is the maximal order of vanishing of the 2-forms \( \omega|_{TM} \) over all smooth submanifolds \( M \) containing \( \text{Im} f \).

They can be described in terms of algebraic restrictions [DJZ2].

**Proposition 4.3.** The symplectic multiplicity of a quasi-homogeneous curve \( f \) in a symplectic space is equal to the codimension of the orbit of the algebraic restriction \( [\omega]_f \) with respect to the group of local diffeomorphisms preserving \( f \) in the space of algebraic restrictions of closed 2-forms to \( f \).

**Proposition 4.4.** The index of isotropness of a quasi-homogeneous curve \( f \) in a symplectic space \((\mathbb{R}^{2n}, \omega)\) is equal to the maximal order of vanishing of closed 2-forms representing the algebraic restriction \( [\omega]_f \).

The above invariants are defined for the image of \( f \). They have the natural generalization to any subset of the symplectic space [DJZ2].

There is one more discrete symplectic invariant introduced in [A1] which is defined specifically for a parameterized curve. This is the maximal tangency order of a curve \( f \) to a smooth Lagrangian submanifold. If \( H_1 = \ldots = H_n = 0 \) define a smooth submanifold \( L \) in the symplectic space then the tangency order of a curve \( f : \mathbb{R} \rightarrow M \) to \( L \) is the minimum of the orders of vanishing at 0 of functions \( H_1 \circ f, \ldots, H_n \circ f \). We denote the tangency order of \( f \) to \( L \) by \( t(f, L) \).

**Definition 4.5.** The *Lagrangian tangency order* \( Lt(f) \) of a curve \( f \) is the maximum of \( t(f, L) \) over all smooth Lagrangian submanifolds \( L \) of the symplectic space.

For a quasi-homogeneous curve \( f \) with the semigroup \((\lambda_1, \ldots, \lambda_k)\) the Lagrangian tangency order is greater than \( \lambda_1 \).

\( Lt(f) \) is related to the index of isotropness. If the index of isotropness of \( \omega \) to \( f \) is 0 then there does not exist a closed 2-form vanishing at 0 representing algebraic restriction of \( \omega \). Then it is easy to see that the order of tangency of \( f \) to \( L \) is not greater than \( \lambda_k \).

The Lagrangian tangency order of a quasi-homogeneous curve in a symplectic space can also be expressed in terms of algebraic restrictions.

The order of vanishing of the germ of a 1-form \( \alpha \) on a curve-germ \( f \) at 0 is the minimum of the orders of vanishing of functions \( \alpha(X) \circ f \) at 0 over all germs of smooth vector fields \( X \). If \( \alpha = \sum_{i=1}^{m} g_i dx_i \) in local coordinates \((x_1, \ldots, x_m)\) then the order of vanishing of \( \alpha \) on \( f \) is the minimum of the orders of vanishing of functions \( g_i \circ f \) for \( i = 1, \ldots, m \).
Proposition 4.6. Let \( f \) be the germ of a quasi-homogeneous curve such that the algebraic restriction of a symplectic form to it can be represented by a closed 2-form vanishing at 0. Then the Lagrangian tangency order of the germ of a quasi-homogeneous curve \( f \) is the maximum of the order of vanishing on \( f \) over all 1-forms \( \alpha \) such that \( [\omega]_f = [d\alpha]_f \).

Proof. Let \( L \) be the germ of a smooth Lagrangian submanifold in a standard symplectic space \( (\mathbb{R}^{2n}, \omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i) \). Then there exist disjoint subsets \( J, K \subset \{1, \cdots, n\} \), \( J \cup K = \{1, \cdots, n\} \) and a smooth function \( S(p_J, q_K) \) such that

\[
L = \{ q_j = -\frac{\partial S}{\partial p_j}(p_J, q_K), p_k = \frac{\partial S}{\partial q_k}(p_J, q_K), j \in J, k \in K \}. 
\]

It is obvious that the order of tangency of \( f \) to \( L \) is equal to the order of vanishing of the following 1-form:

\[
\alpha = \sum_{k \in K} p_k dq_k - \sum_{j \in J} q_j dp_j - dS(p_J, q_K) \quad \text{and} \quad d\alpha = \omega_0.
\]

If closed 2-forms have the same algebraic restrictions to \( f \) then their difference can be written as a differential of a 1-form vanishing on \( f \) by relative Poincare lemma for quasi-homogeneous varieties \cite{DJZ1}. That implies that the maximum of orders of vanishing of 1-forms \( \alpha \) on \( f \) depends only on the algebraic restriction of \( \omega = d\alpha \). Let \( f(t) = (t^{\lambda_1}, \cdots, t^{\lambda_k}, 0, \cdots, 0) \). We may assume that \( [\omega]_f \) may be identified with \([d\alpha]_f\), where \( \alpha \) is a 1-form on \( \{ x_{k+1} = \cdots = x_{2n} = 0 \} \) and \( d\alpha|_0 = 0 \).

In local coordinates \( \alpha = \sum_{i=1}^{k} g_i dx_i \) where \( g_i \) are smooth function-germs. Let \( \sigma \) be the following germ of a symplectic form

\[
\sigma = d\alpha + \sum_{i=1}^{k} dx_i \wedge dx_{k+i} + \sum_{i=1}^{n-k} dx_{2k+i} \wedge dx_{n+k+i}.
\]

Let \( L \) be the following germ of a smooth Lagrangian submanifold (with respect to \( \sigma \))

\[
\{ x_{k+i} = g_i, i = 1, \cdots, k, x_{2k+j} = 0, j = 1, \cdots, n-k \}.
\]

The tangency order of \( f \) to \( L \) is the same as the order of vanishing of \( \alpha \) on \( f \).

It is obvious that the pullback of \( \sigma \) to \( \{ x_{k+1} = \cdots = x_{2n} = 0 \} \) is \( d\alpha \). Then by Darboux-Givental theorem \((\text{AG})\) there exists a local diffeomorphism which is the identity on \( \{ x_{k+1} = \cdots = x_{2n} = 0 \} \) and maps \( \sigma \) to \( \omega \). \( L \) is mapped to a smooth Lagrangian submanifold (with respect to the symplectic form \( \omega \)) with the same tangency order to \( f \).

\[\blacksquare\]

5. The proof of Theorem 2

In this section we prove Theorem 2. The proof is based on the following lemmas.

Lemma 5.1. Let \( N \) be the germ of a subset of \( \mathbb{K}^m \) at 0. Let \( (x_1, \cdots, x_m) \) be a local coordinate system on \( \mathbb{K}^m \).

The space of algebraic restrictions of 2-forms to \( N \) is finite dimensional if and only if there exists a non-negative integer \( L \) such that \( x_i^L dx_j \wedge dx_k \) has zero algebraic restriction to \( N \) for any \( i, j, k = 1, \cdots, m \).

Proof of Lemma 5.1. To prove the ”only if” part notice that there exists an non-negative integer \( K \) such that algebraic restrictions

\[
[x_i dx_j \wedge dx_k]_N, \ [x_i^2 dx_j \wedge dx_k]_N, \ [x_i^3 dx_j \wedge dx_k]_N, \cdots, \ [x_i^K dx_j \wedge dx_k]_N
\]
are linearly dependent, since the space of algebraic restrictions of 2-forms to $N$ is finite dimensional. Therefore there exist a non-negative integer $M$ and $c_1, \ldots, c_s \in \mathbb{K}$ such that $[x_i^M(1 + \sum_{j=1}^s c_j x_j^j) dx_j \wedge dx_k] = 0$. It implies that $[x_i^M dx_j \wedge dx_k] = 0$.

Now it is easy to see that $\lambda$ is the maximum of $M$ for all choices of $i,j,k$.

To prove the "if" part first notice that any germ of a 2-form can be written in the local coordinates as $\sum_{1 \leq j,k \leq m} F_{j,k}(x) dx_j \wedge dx_k$, where $F_{j,k}(x)$ are function-germs on $\mathbb{K}^m$. Using Taylor expansions of $F_{j,k}(x)$ with the reminder of degree greater than $mL$ we obtain the result, since $x_1^i \cdots x_m^i dx_j \wedge dx_k$ has zero algebraic restriction to $N$ for $i_1 + \cdots + i_m \geq mL$.

\[ \square \]

**Lemma 5.2.** Let $f : (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ be the germ of a $\mathbb{K}$-analytic parameterized curve in $\mathbb{K}^2$. Let $(y, z)$ be a local coordinate system on $\mathbb{K}^2$, such that the line $\{ y = 0 \}$ does not contain $f(\mathbb{K})$. Then there exists a $\mathbb{K}$-analytic function-germ $H$ vanishing on $f$ of the following form $H(y, z) = z^p - G(y, z)y^l$, where $G$ is a $\mathbb{K}$-analytic function-germ on $\mathbb{K}^2$, and $p, l$ are positive integers.

**Proof of Lemma 5.2.** We use the method of a construction of $H$ described in [W] (the proof of Lemma 2.3.1 on page 28). $f$ is $\mathbb{K}$-analytic then there exists a coordinate system $t$ on $\mathbb{K}$ such that $f(t) = (t^m, \sum_{i=1}^{\infty} a_i t^i)$. We write it in the following way $y = t^m$, $z = \sum_{i=k}^{\infty} a_i t^i$. Any non-negative integer $i$ can be written in the following way $i = qm + r$, where $r, q$ are integers such that $0 \leq r \leq m - 1$ and $q \geq 0$. Thus $z = \sum_{r=0}^{m-1} t^a \phi_r(y)$, where $\phi_r(y) = \sum_{r=0}^{\infty} a_{qm+r}y^q$ is $\mathbb{K}$-analytic for $r = 0, 1, \cdots, m - 1$. Then regard the equations

\[
(5.1) \quad t^a z = \sum_{r=0}^{m-a-1} t^{a+r} \phi_r(y) + \sum_{r=m-a}^{m-1} t^{a+r-m} \phi_r(y) \quad \text{for} \quad a = 0, 1, \cdots, m - 1.
\]

as a system of linear equations for the unknowns $t^r$ $r = 0, \cdots, m - 1$ with coefficients in $\mathbb{K}\{y, z\}$. The determinant $D(x, y)$ of this system has the following form

\[
\det \begin{bmatrix}
    z - \phi_0(y) & -\phi_1(y) & -\phi_2(y) & \cdots & -\phi_{m-1}(y) \\
    -z\phi_m-1(y) & z - \phi_0(y) & -\phi_1(y) & \cdots & -\phi_{m-2}(y) \\
    -z\phi_{m-2}(y) & -z\phi_{m-1}(y) & z - \phi_0(y) & \cdots & -\phi_{m-3}(y) \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    -z\phi_1(y) & -z\phi_2(y) & -z\phi_3(y) & \cdots & z - \phi_0(y)
\end{bmatrix} = z^m + \psi_1(y)z^{m-1} + \cdots + \psi_{m-1}(y)z + \psi_m(y),
\]

where $\psi_1, \ldots, \psi_m$ are $\mathbb{K}$-analytic function-germs. Since the values $t^r$ for $r = 0, \cdots, m - 1$ provide non-zero solutions of $(5.1)$, the determinant $D(y, z)$ vanishes on the image of the curve $f$. Since $f(0) = 0$ we have that $\psi_m(0) = 0$.

Thus we can decompose $D(y, z)$ in the following form

\[
D(y, z) = z^m + \psi_1(0)z^{m-1} + \cdots + \psi_{m-1}(0)z + y^l F(y, z) = z^p h(z) + y^l F(y, z),
\]

where $h$ is a polynomial of degree $m - k$ that does not vanish at $0$, $p, l$ are positive integers and $F$ is a $\mathbb{K}$-analytic function-germ. Now we take $H(y, z) = \frac{D(y, z)}{mz}$. \[ \square \]

**Lemma 5.3.** Let $C$ be the germ of a $\mathbb{K}$-analytic curve on $\mathbb{K}^2$ at $0$. Let $(y, z)$ be a local coordinate system on $\mathbb{K}^2$, such that the line $\{ y = 0 \}$ does not contain any branch of $C$. Then there exists a $\mathbb{K}$-analytic function-germ $H$ vanishing on $f$ of the
following form \( H(y, z) = z^p - G(y, z)y^l \), where \( G \) is a \( \mathbb{K} \)-analytic function-germ on \( \mathbb{K}^2 \) and \( p, l \) are positive integers.

**Proof.** We decompose \( C \) into branches \( C_1, \cdots, C_s \). Then we apply Lemma 5.2 to each branch \( C_i \). We obtain a \( \mathbb{K} \)-analytic function-germ vanishing on \( C_i \) of the form \( H_i(y, z) = z^p_i - G_i(y, z)y^{l_i} \), where \( p_i, l_i \) are positive integers and \( G_i \) is a \( \mathbb{K} \)-analytic function-germ for \( i = 1, \cdots, s \). Now we may take \( H = H_1 \cdots H_s \), which vanishes on \( C \) and has the desired form. \( \square \)

**Lemma 5.4.** Let \( N \) be the germ of a subset of \( \mathbb{K}^2 \) at \( 0 \). Let \( H \) be a \( \mathbb{K} \)-analytic function-germ on \( \mathbb{K}^2 \) vanishing on \( N \).

If \( H \) has a regular point at 0 or an isolated critical point at 0 then the space of algebraic restrictions of 2-forms to \( N \) is finite dimensional.

**Proof.** The space of algebraic restrictions of 2-forms to \( \{ H = 0 \} \) is isomorphic to \( \mathcal{C}_2 \big/ [H, \mathbb{N}H] \), where \( \mathcal{C}_2 \) is the space of \( \mathbb{K} \)-analytic function-germs on \( \mathbb{K}^2 \). Thus its dimension is finite and equal to the Tjurina number of \( \{ H = 0 \} \). \( N \) is a subset of \( \{ H = 0 \} \). Hence the dimension of the space of algebraic restriction of 2-forms to \( N \) is smaller than the Tjurina number of \( \{ H = 0 \} \), and consequently it is finite. \( \square \)

**Proof of Theorem 2.** Let \( C \) be the germ of a \( \mathbb{K} \)-analytic curve in \( \mathbb{K}^m \) at \( 0 \). In fact we prove that the vector space of algebraic restrictions of all 2-forms to \( C \) is finite dimensional. It is obvious that the set of algebraic restrictions of closed 2-forms is a vector subspace of the vector space of algebraic restrictions of all 2-forms.

Let \( (x_1, \cdots, x_m) \) be a coordinate system on \( \mathbb{K}^m \) and let

\[ \pi_{j,k} : \mathbb{K}^m \ni (x_1, \cdots, x_m) \mapsto (x_j, x_k) \in \mathbb{K}^2 \]

be the standard projection. We choose a coordinate system in such way that for any \( j \neq k \) \( \pi_{j,k}(C) \) is the germ of a \( \mathbb{K} \)-analytic curve on \( \mathbb{K}^2 \) at \( 0 \) such that the lines \( \{ x_j = 0 \} \) and \( \{ x_k = 0 \} \) do not contain any branch of \( \pi_{j,k}(C) \).

Then the space of algebraic restrictions of 2-forms to \( \pi_{j,k}(C) \) is finite dimensional by Lemma 5.4 since \( \pi_{j,k}(C) \) may have a non-singular point at 0 or an isolated singular point at 0. By Lemma 5.1 there exists a positive integer \( K \) such that \( x^K_j dx_j \land dx_k \) has zero algebraic restriction to \( \pi_{j,k}(C) \) and consequently it has zero algebraic restriction to \( C \).

By Lemma 5.3 there exist positive integers \( p, l \) and a \( \mathbb{K} \)-analytic function-germ \( G \) on \( \mathbb{K}^2 \) such that the function-germ \( H(x_j, x_k) = x^K_j + G(x_j, x_k)x^l_j \) vanishes on \( \pi_{j,i}(C) \) and consequently it vanishes on \( C \). It implies that

\[ x^K_j dx_j \land dx_k = (-G(x_j, x_k))^K x^K_j dx_j \land dx_k \]

has zero algebraic restriction to \( C \) too.

Hence by Lemma 5.1 we obtain that the space of algebraic restrictions of 2-forms to \( C \) is finite dimensional. \( \square \)

6. **Quasi-Homogeneous Algebraic Restrictions**

In this section we prove that the action by diffeomorphisms preserving the curve is totally determined by infinitesimal action by liftable vector fields and the space of such vector fields which act nontrivially on algebraic restrictions is a finite dimensional vector space spanned by quasi-homogeneous liftable vector fields of bounded quasi-degrees.
The proof of Theorem 2 is very easy in the case of quasi-homogeneous parameterized curves. Let \( f \) be the germ of a quasi-homogeneous curve. Then \( f \) is an \( \mathcal{A} \)-equivalent to \( f(t) = (t^{\lambda_1}, \ldots, t^{\lambda_k}, 0, \ldots, 0) \). By Proposition 3.6 we consider forms in \( x_1, \ldots, x_k \) coordinates only. We may also assume that the greatest common divisor \( g(\lambda_1, \ldots, \lambda_k) \) is 1. If it is not 1 we introduce weights \( \lambda_i/gcd(\lambda_1, \ldots, \lambda_k) \) for \( x_i, i = 1, \ldots, k \). The proof of Theorem 2 in this special case is based on the following easy observation.

**Lemma 6.1.** The function-germ \( h(x) = x_i^{\lambda_j} - x_j^{\lambda_i} \) vanishes on \( f \).

The above lemma implies the following facts.

**Lemma 6.2.** The 2-form \( x_i^{\lambda_j} - 1 dx_i \wedge dx_j \) has zero algebraic restriction to \( f \)

*Proof of Lemma 6.2.* By Lemma 6.1 \( dh \) has zero algebraic restriction to \( f \). It implies that \( \frac{1}{\lambda_j} dh \wedge dx_i = x_i^{\lambda_j} dx_i \wedge dx_j \) has zero algebraic restriction to \( f \).

**Lemma 6.3.** If the monomials \( s(x) = \prod_{i=1}^{k} x_i^{s_i} \) and \( p(x) = \prod_{i=1}^{k} x_i^{p_i} \) have the same quasi-degree then the forms \( s(x) dx_i \wedge dx_j \) and \( p(x) dx_i \wedge dx_j \) have the same algebraic restrictions to \( f \).

*Proof of Lemma 6.3.* The function-germ \( s(x) - p(x) \) vanishes on \( f \).

The above lemmas imply that we can choose the quasi-homogeneous bases of the space of algebraic restrictions of 2-forms to \( f \). Thus as a corollary of Theorem 2 and the above lemmas we obtain the following theorem.

**Theorem 6.4.** The space of algebraic restrictions of closed 2-forms to the germ of a quasi-homogeneous curve \( f \) is a finite dimensional vector space spanned by algebraic restrictions of quasi-homogeneous closed 2-forms of bounded quasi-degrees.

We will use quasi-homogeneous grading on the space of algebraic restrictions. Therefore we define quasi-homogeneous algebraic restrictions.

Let \( f \) be the germ of a quasi-homogeneous curve on \((\mathbb{R}^m, 0)\). Let \( \omega \) be the germ of a \( k \)-form on \((\mathbb{R}^m, 0)\). By \( \omega^{(r)} \) we denote a quasi-homogeneous part of quasi-degree \( r \) in the Taylor series of \( \omega \). It is easy to see that if \( h \) is a function-germ on \((\mathbb{R}^m, 0)\) and \( h \circ f = 0 \) then \( h^{(r)} \circ f = 0 \) for any \( r \). This simple observation implies the following proposition.

**Proposition 6.5.** If \( [\omega]_f = 0 \) then \( [\omega^{(r)}]_f = 0 \) for any \( r \).

Proposition 6.5 allows to define quasi-homogeneous algebraic restriction.

**Definition 6.6.** Let \( a = [\omega]_f \) be an algebraic restriction to \( f \). The algebraic restriction \( a^{(r)} = [\omega^{(r)}]_f \) is called the quasi-homogeneous part of quasi-degree \( r \) of an algebraic restriction \( a \). \( a \) is quasi-homogeneous of quasi-degree \( r \) if \( a = a^{(r)} \).

We consider the action on the space of algebraic restrictions of closed 2-forms by the group of diffeomorphism-germs which preserve the curve \( f \) to obtain a complete symplectic classification of curves (Theorem 3.3). The tangent space at the identity to this group is given by the space of vector fields liftable over \( f \).
Definition 6.7 ([Za, BPW]). The germ $X$ of a vector field on $(\mathbb{R}^m, 0)$ is called liftable over $f$ if there exists a function germ $g$ on $(\mathbb{R}, 0)$ such that

$$g \left( \frac{df}{dt} \right) = X \circ f.$$ 

The tangent space to the orbit of an algebraic restriction $\alpha$ is given by $L_X \alpha$ for all vector field $X$ liftable over $f$. The Lie derivative of an algebraic restriction with respect to a liftable vector field is well defined due to the following proposition.

Proposition 6.8. Let $X$ be the germ of a vector field on $(\mathbb{R}^m, 0)$ liftable over $f$ and $\omega$ be the germ of a $k$-form on $(\mathbb{R}^m, 0)$. If $[\omega]_f = 0$ then $[L_X \omega]_f = 0$.

Proof. This is a consequence of the Cartan formula and the following fact: $dh(X) \circ f = 0$ for any function-germ $h$ on $(\mathbb{R}^m, 0)$ vanishing on $f$. To prove the above fact let us notice that $dh(X) \circ f = (dh \circ f)(X \circ f) = (dh \circ f)(df/dt)$ for any function-germ $f$.

By the Cartan formula we also obtain the following proposition.

Proposition 6.9. If $X$ is a vector field vanishing on the image of $f$ then $L_X \alpha = 0$ for any algebraic restriction $\alpha$ to $f$.

If $f$ is quasi-homogeneous then the Euler vector field $E$ is liftable over $f$. The following proposition describes its infinitesimal action on quasi-homogeneous algebraic restrictions.

Proposition 6.10. If an algebraic restriction $\alpha$ to $f$ is quasi-homogeneous of quasi-degree $\delta$ then $L_E \alpha = \delta \alpha$.

Let $X$ be a smooth vector field. By $X^{(r)}$ we denote the quasi-homogeneous part of quasi-degree $r$ in the Taylor series of $X$. We have the following result.

Proposition 6.11. If $X$ is liftable over $f$ then $X^{(r)}$ is liftable over $f$.

Proof. We assume that $f(t) = (t^{\lambda_1}, \ldots, t^{\lambda_k}, 0, \ldots, 0)$. Then $X \circ f = g(t) df/dt$ for some function-germ $g$ on $\mathbb{R}$. It implies that

$$X^{(r)} \circ f = \frac{1}{(r+1)!} \frac{d^{r+1} g}{dt^{r+1}} (0) \frac{df}{dt}.$$ 

Let $K(f)$ be the minimal natural number such that all quasi-homogeneous algebraic restrictions to $f$ of closed 2-forms of quasi-degree greater than $K(f)$ vanish. By Theorem 6.12 $K(f)$ is finite.

Theorem 6.12. Let $f(t) = (t^{\lambda_1}, \ldots, t^{\lambda_k}, 0, \ldots, 0)$. Let $X_s$ be the germ of a vector field such that $X_s \circ f = t^{s+1} df/dt$. Then the tangent space to the orbit of the quasi-homogeneous algebraic restriction $\alpha_r$ of quasi-degree $r$ is spanned by $L_X \alpha_r$ for $s$ that are $\mathbb{Z}_{\geq 0}$-linear combinations of $\lambda_1, \ldots, \lambda_k$ and smaller than $K(f) - r$.

Proof. Let $Y$ be a quasi-homogeneous vector field liftable over $f$. Then $Y \circ f = c t^{s+1} df/dt$ where $s$ is the quasi-degree of $Y$ and $c \in \mathbb{R}$. By Proposition 6.9 we obtain that $L_Y \alpha_r = c L_X \alpha_r$, since $(Y - c X_s) \circ f = 0$. If $Z$ is a liftable vector field we can decompose it to $\sum_{s=0}^{K(f) - r - 1} Z^{(s)} + V$, where $V$ is a liftable vector field such that $V^{(s)}(0) = 0$ for $s < K(f) - r$. Then $L_Z \alpha_r = \sum_{s=0}^{K(f) - r - 1} c_s L_X \alpha_r + L_V \alpha_r$, where $c_s \in \mathbb{R}$ for $s = 0, \ldots, K(f) - r - 1$. Proposition 2.9 implies that $(L_V \alpha_r)^{(s)}(0) = 0$ for
Let \( s < K(f) \). By Taylor expansion we can decompose \( L_Va_r \) to an algebraic restrictions of \( f \) to \( f \) for \( j = 1, \cdots, p \) and \( c_j \neq 0 \).

**Theorem 6.12.** \( \sum_{i=1}^{m} f_i b_i \), where \( f_i \) are function-germs and \( b_i \) are quasi-homogeneous algebraic restrictions of quasi-degree greater than \( K(f) - 1 \). Thus \( L_Va_r = 0 \).

**Proof.** We use the Moser homotopy method. Let \( a_t = \sum_{j=s}^{p} c_j a_j + \sum_{j=k+1}^{p} b_j(t)a_j \) where \( b_j(t) \) are smooth functions \( b_j : [0; 1] \to \mathbb{R} \) such that \( b_j(1) = c_j \) for \( j = k + 1, \cdots, p \). Let \( \Phi_t, t \in [0; 1] \) be a flow of the vector field \( \frac{d}{dt} a_t \) over \( \mathbb{R} \). We show that there exist such functions \( b_j \) that

\[
\Phi_t^* a_t = a
\]

for \( t \in [0; 1] \). Differentiating (6.1) we obtain

\[
(6.2) \quad \mathcal{L}_{\frac{d}{dt} a_t} a_t = c_k a_k - \sum_{j=k+1}^{p} \frac{db_j}{dt} a_j.
\]

Since \( \mathcal{L}_{\frac{d}{dt} a_t} a_t = ra_k \), the quasi-degree of \( X \) is \( \delta_k - \delta_s \). Hence the quasi-degree of \( \mathcal{L}_{\frac{d}{dt} a_t} a_t \) is greater than \( \delta_k \) for \( j > s \). Then \( b_j \) are solutions of the system of \( p - k \) first order linear ODEs defined by (6.2) with the initial data \( b_j(0) = c_j \) for \( j = k + 1, \cdots, p \). It implies that \( a_0 = a \) and \( a_1 = \sum_{j=s}^{k-1} c_j a_j + \sum_{j=k+1}^{s} b_j(1)a_j \) are diffeomorphic. \( \square \)

### 6.1. Remarks on the algorithm for a quasi-homogeneous parameterized curve with an arbitrary semigroup

The results of section 6.1 allow us to formulate an algorithm for the classification of the symplectic singularities of an arbitrary quasi-homogeneous parameterized curve-germ \( f \). It is obvious that this algorithm depends only on the semigroup of the curve singularity.

Let us assume that the semigroup have the following form:

\[
(\lambda_1, \cdots, \lambda_k),
\]

where \( \lambda_1 < \cdots < \lambda_k \) are positive integers linearly independent over non-negative integers. We use the quasi-homogeneous grading on the space of algebraic restrictions of 2-forms with weights \( (\lambda_1, \cdots, \lambda_k) \) for coordinates \( (x_1, \cdots, x_k) \). We may also assume that \( \lambda_1, \cdots, \lambda_k \) are relatively prime. If they are not we introduce weights \( \lambda_i / \gcd(\lambda_1, \cdots, \lambda_k) \) for \( x_i, i = 1, \cdots, k \), where \( \gcd(\lambda_1, \cdots, \lambda_k) \) is the greatest common divisor of \( \lambda_1, \cdots, \lambda_k \).

We fixed the quasi-degree \( \delta \) starting with \( \lambda_1 + \lambda_2 \) since there are no quasi-homogeneous 2-forms with a smaller quasi-degree.

2-forms of the quasi-degree \( \delta \) (together with the zero 2-form) form a finite dimensional vector subspace of the space of differential 2-forms.
By Lemma 6.3 algebraic restrictions of all forms of the quasi-degree $\delta$ of the form

\[
(6.3) \quad s(x)\,dx_i \wedge dx_j
\]

for fixed $i \neq j$ are linearly dependent. Hence for all $i < j$ we need to check if the equation

\[
(6.4) \quad a_1\lambda_1 + \cdots + a_k\lambda_k = d - \lambda_i - \lambda_j
\]

has a solution $a_1, \cdots, a_k$ in non-negative integers.

For fixed $i < j$ we take one of the solutions of equation (6.4) (if it exists). All other algebraic restrictions of the 2-forms of the form (6.3) are linear combinations of $[x_1^{a_1} \cdots x_k^{a_k} \,dx_i \wedge dx_j]_f$.

To find a basis of algebraic restrictions of quasi-homogeneous 2-forms with the quasi-degree $\delta$ we are looking for quasi-homogeneous functions vanishing on $f$ of a quasi-degree $\delta$.

To find them we need to find all solutions of the equation

\[
(6.5) \quad a_1\lambda_1 + \cdots + a_k\lambda_k = d - \lambda_i
\]

If $(a_1, \cdots, a_k)$ and $(b_1, \cdots, b_k)$ are distinct solutions of (6.5) then a function-germ

\[
(6.6) \quad H(x_1, \cdots, x_k) = x_1^{a_1} \cdots x_k^{a_k} - x_1^{b_1} \cdots x_k^{b_k}
\]

vanishes on $f$ and the form $dH \wedge dx_i$ has zero algebraic restriction to $f$. In this way we obtain all relations between algebraic restrictions of quasi-homogeneous forms of quasi-degree $\delta$ and consequently we find a basis of this vector space.

Then we proceed to algebraic restrictions with quasi-degree $\delta + 1$.

For some quasi-degrees we obtain that all quasi-homogeneous 2-forms have zero algebraic restriction to $f$. Then using the fact that quasi-homogeneous forms of a sufficiently high quasi-degree can be obtained by multiplication by functions of quasi-homogeneous forms of lower degrees we get that all 2-forms of a sufficiently high quasi-degree have zero algebraic restriction. In this way we construct the quasi-homogeneous basis of the space of algebraic restriction of all 2-forms.

Then by Proposition 3.9 we get the quasi-homogeneous basis of the space of algebraic restriction of closed 2-forms from the quasi-homogeneous basis of the space of algebraic restriction of all 2-forms.

Then we calculate the number $K(f)$ and we find germs of vector field such that $X_s \circ f = t^{s+1} \, df/dt$ for $s$ that are representable as a non-negative integers combinations of $\lambda_1, \cdots, \lambda_k$ and smaller than $K(f) - \lambda_1 - \lambda_2$. By Theorem 6.12 the tangent space to the orbit of the quasi-homogeneous algebraic restriction $a_r$ of quasi-degree $r$ is spanned by $L_{X_s} a_r$.

Finally we apply Theorem 6.13 to get the classification of algebraic restrictions. From this classification we easily obtain the symplectic singularities normal forms.

In the next sections we apply the above algorithm for curves with semigroups $(3, 4, 5)$, $(3, 5, 7)$ and $(3, 7, 8)$.

Although the algorithm works very well for concrete examples, the problem of calculations of the dimension of the space of algebraic restrictions of closed 2-forms to a quasi-homogeneous parameterized curve in terms of the semigroup of this curve is complicated. It is obvious that it is related to the classical Frobenius coin problem (the diophantine Frobenius problem) $\mathbb{R}$.
**Frobenius Coin Problem.** Given $k$ relatively prime positive integers $\lambda_1, \cdots, \lambda_k$, find the largest natural number (denoted by $g(\lambda_1, \cdots, \lambda_k)$ and called **Frobenius number**) that is not representable as a non-negative integer combination of $\lambda_1, \cdots, \lambda_k$.

By Schur’s theorem Frobenius number is finite ([R]). The formula for Frobenius number for $k = 2$ was found by J. J. Sylvester: $g(\lambda_1, \lambda_2) = \lambda_1\lambda_2 - \lambda_1 - \lambda_2$ ([S]). Sylvester also demonstrated that there are $(\lambda_1-1)(\lambda_2-1)/2$ non-representable natural numbers. More complicated formulas and fast algorithms to calculate Frobenius number for $k = 3$ are known, but the general problem for arbitrary $k$ is known to be NP-hard ([R]).

### 7. Symplectic singularities of curves with the semigroup $(3, 4, 5)$

In this section we apply the results of the previous section to prove the following theorem.

**Theorem 7.1.** Let $(\mathbb{R}^{2n}, \omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i)$ be the symplectic space with the canonical coordinates $(p_1, q_1, \cdots, p_n, q_n)$.

Then the germ of a curve $f : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^{2n}, 0)$ with the semigroup $(3, 4, 5)$ is symplectically equivalent to one and only one of the curves presented in the second column of the Table 1 (on page 12) for $n > 2$ and $f$ is symplectically equivalent to one and only one of the curves presented in the second column and rows 1-3 for $n = 2$.

The symplectic multiplicity, the index of isotropness and the Lagrangian tangency order are presented in the third, fourth and fifth columns of Table 1.

| normal form of $f$ | $\mu_{sympl}(f)$ | $\iota(f)$ | $Lt(f)$ |
|---------------------|------------------|-----------|---------|
| 1 $t \mapsto (t^3, t^4, t^5, 0, \cdots, 0)$ | 0 | 0 | 4 |
| 2 $t \mapsto (t^3, \pm t^2, t^4, 0, \cdots, 0)$ | 1 | 0 | 5 |
| 3 $t \mapsto (t^3, 0, t^4, t^5, 0, \cdots, 0)$ | 2 | 0 | 5 |
| 4 $t \mapsto (t^3, \pm t^2, t^4, 0, t^6, 0, \cdots, 0)$ | 3 | 1 | 7 |
| 5 $t \mapsto (t^3, t^4, 0, t^4, 0, \cdots, 0)$ | 4 | 1 | 8 |
| 6 $t \mapsto (t^3, 0, t^4, 0, t^6, 0, \cdots, 0)$ | 5 | $\infty$ | $\infty$ |

Table 1. Symplectic classification of curves with the semigroup $(3, 4, 5)$.

We use the method of algebraic restrictions. The germ of a curve $f : \mathbb{R} \ni t \mapsto f(t) \in \mathbb{R}^{2n}$ with the semigroup $(3, 4, 5)$ is $\mathcal{A}$-equivalent to $t \mapsto (t^3, t^4, t^5, 0, \cdots, 0)$. First we calculate the space of algebraic restrictions of 2-forms to the image of $f$ in $\mathbb{R}^{2n}$.

**Proposition 7.2.** The space of algebraic restrictions of differential 2-forms to $f$ is the 6-dimensional vector space spanned by the following algebraic restrictions:

\[ a_7 = [dx_1 \wedge dx_2]_g, \quad a_8 = [dx_3 \wedge dx_1]_g, \quad a_9 = [dx_2 \wedge dx_3]_g, \]

\[ a_{10} = [x_1dx_1 \wedge dx_2]_g, \quad a_{11} = [x_2dx_1 \wedge dx_2]_g, \quad a_{12} = [x_1dx_2 \wedge dx_3]_g, \]

where $\delta$ is quasi-degree of $a_8$. 


Proof. The image of $f$ is contained in the following 3-dimensional smooth submanifold $\{x_4 = 0\}$. By Proposition 8.6 we can restrict our consideration to $\mathbb{R}^3$ and the curve $g : \mathbb{R} \ni t \mapsto (t^3, t^4, t^5) \in \mathbb{R}^3$. $g$ is quasi-homogeneous with weights $3, 4, 5$ for variables $x_1, x_2, x_3$. We use the quasi-homogeneous grading on the space of algebraic restrictions of differential 2-forms to $g(t) = (t^3, t^4, t^5)$ with these weights. It is easy to see that the quasi-homogeneous functions or 2-forms of a fixed quasi-degree form a finite dimensional vector space. The same is true for quasi-homogeneous algebraic restrictions of 2-forms of a fixed quasi-degree.

There are no quasi-homogeneous function-germs on $\mathbb{R}^3$ vanishing on $g$ of quasi-degree less than 8. The vector space of quasi-homogeneous function-germs of degree $i = 8, 9, 10$ vanishing on $g$ is spanned by $f_i$ presented in Table 2 together with their differentials. We do not need to consider quasi-homogeneous function-germs of higher quasi-degree, since using $f_8, f_9$ and $f_{10}$ we show that algebraic restrictions of quasi-homogeneous 2-forms of quasi-degree greater than 12 are zero (see Table 3) and all possible relations of algebraic restrictions of quasi-homogeneous 2-forms of quasi-degree less than 13 are generated by quasi-homogeneous functions vanishing on $g$ of quasi-degree less than $13 - 3 = 10$. Now we can calculate the space of algebraic restrictions of 2-forms. The scheme of the proof is presented in Table 3.

| quasi-degree $\delta$ | $f_\delta$ | differential $df_\delta$ |
|-----------------------|------------|--------------------------|
| 8                     | $x_1 x_3 - x_2^2$ | $x_1 dx_3 + x_3 dx_1 - 2x_2 dx_2$ |
| 9                     | $x_2 x_3 - x_1^2$ | $x_2 dx_3 + x_3 dx_2 - 3x_1^2 dx_1$ |
| 10                    | $x_1 x_2 - x_3^2$ | $x_1^2 dx_2 + 2x_1 x_2 dx_1 - 2x_3 dx_3$ |

Table 2. Quasi-homogeneous function-germs of quasi-degree 8, 9, 10 vanishing on the curve $t \mapsto (t^3, t^4, t^5)$.
restrictions of quasi-degree 13 are zero. The same arguments we use for quasi-degree 14 and 15.

To prove that all algebraic restrictions of quasi-degree 16 are 0 we notice that they can have the following forms of quasi-degree 16: \( x_1\beta_{13} \) or \( x_2\beta_{12} \) or \( x_3\beta_{11} \). In the first case the algebraic restriction \( b_{13} = [\beta_{13}]_g \) has quasi-degree 13, so it is 0. In the second case the quasi-degree of \( b_{12} = [\beta_{12}]_g \) is 12. So the algebraic restriction \( b_{12} \) can be presented in the form \( cx_2a_9 \), where \( c \in \mathbb{R} \). But then \( x_2b_{12} = x_1(cx_2a_9) \). The quasi-degree of \( cx_2a_9 \) is 13 and it implies that \( cx_2a_9 \) is 0. We use a similar argument to prove that \( x_3b_{11} \) is 0. Using the same arguments and induction by the quasi-degree we show that all algebraic restrictions of higher quasi-degree are 0.

Any smooth 2-form \( \omega \) can be decomposed to \( \omega = \sum_{i=7}^{12} \omega_i + \sum_{j=1}^{k} f_j \sigma_j \), where \( k \) is a positive integer, \( \omega_i \) is a quasi-homogeneous 2-form of quasi-degree \( i \) for \( i = 7, \cdots, 12 \) and \( f_j \) are smooth function-germs and \( \sigma_j \) are quasi-homogeneous 2-forms of quasi-degree greater than 12 for \( j = 1, \cdots, k \). Thus the space of algebraic restrictions of 2-forms is spanned by \( a_7, \cdots, a_{12} \).

\[
\begin{array}{|c|c|c|c|}
\hline
\delta & \text{basis} & \text{forms} & \text{relations} & \text{proof} \\
\hline
7 & a_7 & \alpha_7 = dx_1 \land dx_2 & \alpha_7 := [\alpha_7]_g & \text{proof} \\
8 & a_8 & \alpha_8 = dx_3 \land dx_1 & \alpha_8 := [\alpha_8]_g & \text{proof} \\
9 & a_9 & \alpha_9 = dx_2 \land dx_3 & \alpha_9 := [\alpha_9]_g & \text{proof} \\
10 & a_{10} & x_1\alpha_7 & a_{10} := x_1\alpha_7 & \text{proof} \\
11 & a_{11} & x_2\alpha_7, x_1a_8 & a_{11} = -2x_1a_8 & [df_8 \land dx_1]_g = 0 \\
12 & a_{12} & x_3\alpha_7, x_2a_8, x_1a_9 & a_{12} = x_3\alpha_7, a_{12} = x_2a_8, a_{12} = x_1a_9 & [df_9 \land dx_1]_g = 0, [df_8 \land dx_2]_g = 0 \\
13 & 0 & x_1^2\alpha_7, x_3a_8, x_2a_9 & x_1^2\alpha_7 = 0, x_3a_8 = 0, x_2a_9 = 0 & [df_{10} \land dx_1]_g = 0, [df_9 \land dx_2]_g = 0, [df_8 \land dx_3]_g = 0 \\
14 & 0 & x_1x_2\alpha_7, x_2^2a_8, x_2a_{10} & x_1x_2\alpha_7 = 0, x_2^2a_8 = 0, x_2a_{10} = 0 & [df_{10} \land dx_2]_g = 0, [df_9 \land dx_3]_g = 0, x_1[df_8 \land dx_1]_g = 0 \\
15 & 0 & x_1x_3\alpha_7, x_1x_2a_8, x_2^2a_9 & x_1x_3\alpha_7 = 0, x_1x_2a_8 = 0, x_2^2a_9 = 0 & [df_{10} \land dx_3]_g = 0, x_1[df_9 \land dx_1]_g = 0, x_1[df_8 \land dx_2]_g = 0 \\
\geq 16 & 0 & x_1\beta_{\geq 13}, x_2\beta_{\geq 12}, x_3\beta_{\geq 11} & b_{\geq 13} := [\beta_{\geq 13}]_g, b_{\geq 12} := [\beta_{\geq 12}]_g, b_{\geq 11} := [\beta_{\geq 11}]_g & x_2b_{\geq 12} = x_1b_{\geq 13}, x_3b_{\geq 11} = x_1b_{\geq 13}, \delta(b_{\geq 13}) \geq 13, b_{\geq 13} = 0 \\
\hline
\end{array}
\]

Table 3. The quasi-homogeneous basis of algebraic restrictions of 2-forms to the curve \( t \mapsto (t^3, t^4, t^5) \).
Table 4. Infinitesimal actions on algebraic restrictions of closed 2-forms to the curve $t \mapsto (t^3, t^4, t^5)$.

| $\mathcal{L}_{X,a_j}$ | $a_7$ | $a_8$ | $a_9$ | $a_{10}$ | $a_{11}$ |
|-----------------------|-------|-------|-------|----------|----------|
| $X_0 = E$             | $7a_7$| $8a_8$| $9a_9$| $10a_{10}$| $11a_{11}$|
| $X_1$                 | $-4a_8$| $-3a_9$| $-10a_{10}$| $11a_{11}$| $0$       |
| $X_2$                 | $-3a_9$| $-5a_{10}$| $11a_{11}$| $0$       | $0$       |
| $X_3 = x_1 E$         | $10a_{10}$| $-22a_{11}$| $0$       | $0$       | $0$       |
| $X_4 = x_2 E$         | $11a_{11}$| $0$       | $0$       | $0$       | $0$       |

Proposition 7.3. The space of algebraic restrictions of closed differential 2-forms to the image of $f$ is the 5-dimensional vector space spanned by the following algebraic restrictions:

$$a_7, a_8, a_9, a_{10}, a_{11}.$$  

Proof. It is easy to see that $da_i = 0$ for $i < 12$ and $da_{12} \neq 0$. Then we apply Theorem 5.49. □

Proposition 7.4. Any algebraic restriction of a symplectic form to $f$ is diffeomorphic to one and only one of the following $a_7, a_8, -a_8, a_9, -a_9, a_{10}, -a_{10}, a_{11}, 0$.

Proof. By Theorem 6.12 we consider vector fields $X_s$ such that $X_s \circ f = t^{s+1} df/dt$ for $s = 0, \ldots, 5$. They have the following form

$$X_0 = E = 3x_1 \frac{\partial}{\partial x_1} + 4x_2 \frac{\partial}{\partial x_2} + 5x_3 \frac{\partial}{\partial x_3}, \quad X_1 = 3x_2 \frac{\partial}{\partial x_1} + 4x_3 \frac{\partial}{\partial x_2} + 5x_4 \frac{\partial}{\partial x_3},$$

$$X_2 = 3x_3 \frac{\partial}{\partial x_1} + 4x_4 \frac{\partial}{\partial x_2} + 5x_5 \frac{\partial}{\partial x_3}, \quad X_3 = x_1 E, \quad X_4 = x_2 E.$$  

The infinitesimal action of these germs of quasi-homogeneous liftable vector fields on the basis of the vector space of algebraic restrictions of closed 2-forms to $f$ is presented in Table 4. Using the data of Table 4 we obtain by Theorem 6.13 that an algebraic restriction of the form $\sum_{i \geq 1} c_i a_i$ for $c_s \neq 0$ is diffeomorphic to $c_s a_s$. Finally we reduce $c_s a_s$ to $a_s$ if the quasi-degree $s$ is odd or to $\text{sgn}(c_s) a_s$ if $s$ is even by a diffeomorphism $\Phi_t(x_1, x_2, x_3) = (t^3 x_1, t^4 x_2, t^5 x_3)$ for $t = e_s^+$ or for $t = |c_s|^+$ respectively.

The algebraic restrictions $a_8, -a_8$ are not diffeomorphic. Any diffeomorphism $\Phi = (\Phi_1, \cdots, \Phi_{2n})$ of $(\mathbb{R}^{2n}, 0)$ preserving $f(t) = (t^3, t^4, t^5, 0, \ldots, 0)$ has the following linear part

$$A^3 x_1 + A_{12} x_2 + A_{13} x_3 + A_{14} x_4 + \cdots + A_{1,2n} x_{2n}$$

$$A^4 x_2 + A_{23} x_3 + A_{24} x_4 + \cdots + A_{2,2n} x_{2n}$$

$$A^5 x_3 + A_{34} x_4 + \cdots + A_{3,2n} x_{2n}$$

$$A_{44} x_4 + \cdots + A_{4,2n} x_{2n}$$

$$\vdots$$

$$A_{2n,4} x_4 + \cdots + A_{2n,2n} x_{2n}$$

where $A, A_{i,j} \in \mathbb{R}$.

Assume that $\Phi^*(a_8) = -a_8$. It implies that $A^8 dx_3 \wedge dx_1 |_0 = -dx_3 \wedge dx_1 |_0$, which is a contradiction.

One can similarly prove that $a_{10}, -a_{10}$ are not diffeomorphic. □
Proof of Theorem 7.1. Let $\theta_i$ be a 2-form on $\mathbb{R}^3$ such that $a_i = [\theta_i]_{\mathbb{R}}$. Then $\text{rank}(\theta_i|_0) \geq 2$ if $n = 2$ and $\text{rank}(\theta_i|_0) \geq 0$ if $n > 2$ by Proposition 3.10. It is easy to see that $a_7, \pm a_8, a_9$ are realizable by the following symplectic forms
\[ dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots, \pm dx_3 \wedge dx_1 + dx_2 \wedge dx_4 + \cdots, \]
respectively. The algebraic restrictions $\pm a_{10}, a_{11}$ and $a_{\infty} = 0$ are realizable by the following forms
\[ \pm x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 + dx_3 \wedge dx_6 + \cdots, \]
\[ x_2 dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + \cdots, \]
\[ dx_1 \wedge dx_3 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + \cdots \]
respectively. By a simple coordinate change we map the above forms to the Darboux normal form and we obtain the normal forms of the curve.

By Propositions 4.3, 7.3, 7.4 and using the data in Table 5, we obtain the symplectic multiplicities of curves in Table 4. The indexes of isotropness for these curves are calculated by Propositions 4.4 and 7.4. The Lagrangian tangency orders for the curves in rows $1-3$ are obtained using the fact that any Lagrangian submanifold can be represented in the form $(4, 1)$. By Propositions 4.6 and 7.4 we obtain this invariant for other curves in Table 4.

\[ \square \]

8. Symplectic Singularities of Curves with the Semigroup $(3, 5, 7)$

In this section we present the symplectic classification of curves with the semigroup $(3, 5, 7)$.

Theorem 8.1. Let $(\mathbb{R}^{2n}, \omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i)$ be the symplectic space with the canonical coordinates $(p_1, q_1, \ldots, p_n, q_n)$.

Then the germ of a curve $f : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^{2n}, 0)$ with the semigroup $(3, 5, 7)$ is symplectically equivalent to one and only one of the curves presented in the second column of the Table 2 (on page 16) for $n > 2$ and $f$ is symplectically equivalent to one and only one of the curves presented in the second column and rows $1-3$ and $5$ for $n = 2$. The parameter $c$ is a modulus.

The symplectic multiplicity, the index of isotropness and the Lagrangian tangency order are presented in the third, fourth and fifth columns of Table 5.

| normal form of $f$ | $\mu_{\text{sympl}}(f)$ | $\iota(f)$ | $Lt(f)$ |
|---------------------|---------------------|----------|---------|
| $1 \ t \mapsto (t^3, \pm t^5, t^7, 0, \ldots, 0)$ | 0 | 0 | 5 |
| $2 \ t \mapsto (t^3, \pm t^5, t^7, ct^6, \ldots, 0)$ | 2 | 0 | 7 |
| $3 \ t \mapsto (t^3, t^5, t^7, ct', \ldots, 0), c \neq 0$ | 3 | 0 | 7 |
| $4 \ t \mapsto (t^3, t^5, t^7, 0, t', 0, \ldots, 0)$ | 3 | 1 | 8 |
| $5 \ t \mapsto (t^3, ct^10, \pm t^5, t', 0, \ldots, 0)$ | 4 | 0 | 7 |
| $6 \ t \mapsto (t^3, ct^{11}, t^5, t^7, t', 0, \ldots, 0)$ | 5 | 1 | 10 |
| $7 \ t \mapsto (t^3, \pm t^{11}, t^5, 0, t', 0, \ldots, 0)$ | 5 | 2 | 11 |
| $8 \ t \mapsto (t^3, \pm t^{13}/2, t^5, 0, t', 0, \ldots, 0)$ | 6 | 2 | 13 |
| $9 \ t \mapsto (t^3, 0, t^5, 0, t', 0, \ldots, 0)$ | 7 | $\infty$ | $\infty$ |

Table 5. Symplectic classification of curves with the semigroup $(3, 5, 7)$. 
The germ of a curve \( f : \mathbb{R} \ni t \mapsto f(t) \in \mathbb{R}^{2n} \) with the semigroup \((3, 5, 7)\) is \(A\)-equivalent to \(t \mapsto (t^3, t^5, t^7, 0, \ldots, 0)\). We use the same method as in the previous section to obtain symplectic classification of these curves. We only present the main steps with all calculation results in tables.

**Proposition 8.2.** The space of algebraic restrictions of differential 2-forms to \(g\) is the 8-dimensional vector space spanned by the following algebraic restrictions:

\[
\begin{align*}
a_8 &= [dx_1 \wedge dx_2]_g, a_{10} = [dx_3 \wedge dx_1]_g, a_{11} = [x_1 dx_1 \wedge dx_2]_g, a_{12} = [dx_2 \wedge dx_3]_g, \\
a_{13} &= [x_2 dx_1 \wedge dx_2]_g, a_{14} = [x_2^2 dx_1 \wedge dx_2]_g, a_{15} &= [x_3 dx_1 \wedge dx_2]_g, a_{16} &= [x_1 x_2 dx_1 \wedge dx_2]_g,
\end{align*}
\]

where \(\delta\) is the quasi-degree of \(a_8\).

The sketch of the proof. We use the same method as in the previous section. The sketch of the proof is presented in Tables 6 and 7. \(\square\)

**Proposition 8.3.** The space of algebraic restrictions of closed differential 2-forms to the image of \(f\) is the 7-dimensional vector space spanned by the following algebraic restrictions:

\[
a_8, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{16}.
\]

**Proof.** By Proposition 8.2 it is easy to see that \(da_i = 0\) for \(i \neq 15\) and \(da_{15} \neq 0\). By Theorem 3.9 we get the result. \(\square\)

**Proposition 8.4.** Any algebraic restriction of a symplectic form \(\omega\) to \(f\) is diffeomorphic to one of the following \(\pm a_8, \pm a_{10} + ca_{11}, a_{11} + ca_{12}, a_{11} + ca_{12} + ca_{13}, a_{13} + ca_{14}, a_{14}, a_{16}, a_{16}, 0\), where the parameter \(c \in \mathbb{R}\) is a modulus.

**Sketch of the proof.** The vector fields \(X_s\) (see Theorem 6.12) have the following form:

\[
\begin{align*}
X_0 &= E = 3x_1 \frac{\partial}{\partial x_1} + 5x_2 \frac{\partial}{\partial x_2} + 7x_3 \frac{\partial}{\partial x_3}, \\
X_2 &= 3x_2 \frac{\partial}{\partial x_1} + 5x_3 \frac{\partial}{\partial x_2} + 7x_3^2 \frac{\partial}{\partial x_3}, \\
X_3 &= x_1 E, \\
X_4 &= 3x_3 \frac{\partial}{\partial x_1} + 5x_1^2 \frac{\partial}{\partial x_2} + 7x_1^2 x_2 \frac{\partial}{\partial x_3}, \\
X_5 &= x_2 E, \\
X_6 &= x_1^2 E, \\
X_7 &= x_3 E, \\
X_8 &= x_1 x_2 E.
\end{align*}
\]

Their actions on the space of algebraic restrictions of closed 2-forms are presented in Table 8. From these data we obtain the classification of algebraic restrictions as in the previous section.

From Table 8 and Theorem 6.12 we also see that the tangent space to the orbit of \(\pm a_{10} + ca_{11}\) at \(\pm a_{10} + ca_{11}\) is spanned by \(\pm 10a_{10} + 11ca_{11}, a_{12}, a_{13}, a_{14}, a_{16}, a_{11}\) does not belong to it. Therefore parameter \(c\) is the modulus in the normal form \(\pm a_{10} + ca_{11}\).

In the same way we prove that \(c\) is the modulus in the other normal forms. \(\square\)
| $\delta$ | basis $a_\delta$ | forms $\alpha_\delta$ | relations $\alpha_\delta = [\alpha_\delta]_g$ | proof $[d\alpha_\delta \wedge dx_1]_g = 0$ |
|-------|-----------------|-----------------|-----------------|-----------------|
| 8     | $a_8$           | $\alpha_8 = dx_1 \wedge dx_2$ | $a_8 := [\alpha_8]_g$ |                  |
| 10    | $a_{10}$        | $\alpha_{10} = dx_3 \wedge dx_1$ | $a_{10} := [\alpha_{10}]_g$ |                  |
| 11    | $a_{11}$        | $x_{10}a_8$     | $a_{11} := x_1a_8$ |                  |
| 12    | $a_{12}$        | $\alpha_{12} = dx_2 \wedge dx_3$ | $a_{12} := [\alpha_{12}]_g$ |                  |
| 13    | $a_{13}$        | $x_{2\alpha_8}$, $x_{1\alpha_{10}}$ | $a_{13} := x_2a_8$, $x_1a_{10} = -2a_{13}$ | $[d\alpha_{10} \wedge dx_1]_g = 0$ |
| 14    | $a_{14}$        | $x_7^2a_8$      | $a_{14} := x_7^2a_8$ |                  |
| 15    | $a_{15}$        | $x_{3\alpha_8}$, $x_{2\alpha_{10}}$, $x_{1\alpha_{12}}$ | $a_{15} := x_3a_8$, $x_2a_{10} = a_{15}$, $x_1a_{12} = a_{15}$ | $[d\alpha_{12} \wedge dx_1]_g = 0$, $[d\alpha_{10} \wedge dx_2]_g = 0$ |
| 16    | $a_{16}$        | $x_{1\alpha_{10}}a_8$, $x_{3\alpha_{10}}$, $x_{2\alpha_{12}}$, $x_{1\alpha_{12}}$ | $a_{16} := x_1x_2a_8$, $x_3a_{10} = 0$, $x_2a_{12} = 0$, $x_1a_{12} = 0$ | $x_1[d\alpha_{14} \wedge dx_1]_g = 0$ |
| 17    | 0               | $x_{3\alpha_8}$, $x_{3\alpha_{10}}$, $x_{3\alpha_{12}}$ | $x_3^2a_8 = 0$, $x_3a_{10} = 0$, $x_3a_{12} = 0$ | $[d\alpha_{12} \wedge dx_1]_g = 0$, $[d\alpha_{10} \wedge dx_3]_g = 0$ |
| 18    | 0               | $x_{1\alpha_{10}}a_8$, $x_{1\alpha_{12}}$, $x_{3\alpha_{12}}$, $x_{3\alpha_{10}}$, $x_{3\alpha_{12}}$, $x_{2\alpha_{12}}$, $x_{1\alpha_{12}}$ | $x_1x_3a_8 = 0$, $x_1x_2a_{10} = 0$, $x_1x_3a_{12} = 0$, $x_1x_3a_{10} = 0$, $x_1x_3a_{12} = 0$, $x_1x_3a_{12} = 0$ | $x_1[d\alpha_{14} \wedge dx_1]_g = 0$, $x_1[d\alpha_{12} \wedge dx_2]_g = 0$, $x_2[d\alpha_{10} \wedge dx_1]_g = 0$ |
| 19    | 0               | $x_{7}^2x_2a_8$, $x_{7}^2a_{10}$, $x_{3}a_{12}$, $x_{3}a_{12}$, $x_{3}a_{10}$, $x_{3}a_{12}$ | $x_7^2x_2a_8 = 0$, $x_7^2a_{10} = 0$, $x_3a_{12} = 0$, $x_3a_{12} = 0$, $x_3a_{10} = 0$, $x_3a_{12} = 0$ | $x_1[d\alpha_{14} \wedge dx_2]_g = 0$, $[d\alpha_{12} \wedge dx_3]_g = 0$ |
| $\geq 20$ | 0               | $x_{1}\beta_{\geq 17}, x_{2}\beta_{\geq 15}, x_{3}\beta_{\geq 13}$, $b_{\geq 17} := [\beta_{\geq 17}]_g$, $b_{\geq 15} := [\beta_{\geq 15}]_g$, $b_{\geq 13} := [\beta_{\geq 13}]_g$, $b_{\geq 17} = 0$, $b_{\geq 15} = 0$, $b_{\geq 13} = 0$ | $x_1b_{17} = 0$, $x_2b_{15} = x_1b_{17}^2$, $x_3b_{13} = x_1b_{17}^2$, $\delta(b_{\geq 17}) \geq 17$, $b_{\geq 17} = 0$ | $x_2b_{15} = x_1b_{17}^2$, $x_3b_{13} = x_1b_{17}^2$, $\delta(b_{\geq 17}) \geq 17$, $b_{\geq 17} = 0$ |

Table 7. The quasi-homogeneous basis of algebraic restrictions of 2-forms to the curve $t \mapsto (t^3, t^5, t^7)$.

| $L_{\mathcal{X}}a_j$ | $a_8$ | $a_{10}$ | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | $a_{16}$ |
|---------------------|-------|---------|---------|---------|---------|---------|---------|
| $X_0 = E$           | $8a_8$ | $10a_{10}$ | $11a_{11}$ | $12a_{12}$ | $13a_{13}$ | $14a_{14}$ | $16a_{16}$ |
| $X_2 = -5a_{10}$    | $-3a_{12}$ | $13a_{13}$ | $-21a_{14}$ | 0 | $16a_{16}$ | 0 | 0 |
| $X_3 = x_1E$        | $11a_{11}$ | $-26a_{13}$ | $14a_{14}$ | 0 | $16a_{16}$ | 0 | 0 |
| $X_4 = -3a_{12}$    | $-7a_{14}$ | 0 | $6a_{16}$ | 0 | 0 | 0 | 0 |
| $X_5 = x_2E$        | $13a_{13}$ | 0 | $16a_{16}$ | 0 | 0 | 0 | 0 |
| $X_6 = x_1^2E$      | $11a_{14}$ | $-32a_{16}$ | 0 | 0 | 0 | 0 | 0 |
| $X_7 = x_3E$        | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X_8 = x_1x_2E$     | $4a_{16}$ | 0 | 0 | 0 | 0 | 0 | 0 |

Table 8. Infinitesimal actions on algebraic restrictions of closed 2-forms to the curve $t \mapsto (t^3, t^5, t^7)$.
9. Symplectic singularities of curves with the semigroup \((3, 7, 8)\)

In this section we present the symplectic classification of curves with the semigroup \((3, 7, 8)\).

**Theorem 9.1.** Let \((\mathbb{R}^{2n}, \omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i)\) be the symplectic space with the canonical coordinates \((p_1, q_1, \ldots, p_n, q_n)\).

Then the germ of a curve \(f : (\mathbb{R}, 0) \to (\mathbb{R}^{2n}, 0)\) with the semigroup \((3, 7, 8)\) is symplectically equivalent to one and only one of the curves presented in the second column of the Table 9 (on page 19) for \(n > 2\) and \(f\) is symplectically equivalent to one and only one of the curves presented in the second column and rows 1-3, 5 and 7 for \(n = 2\). The parameters \(c, c_1, c_2\) are moduli.

The symplectic multiplicity, the index of isotropness and the Lagrangian tangency order are presented in the third, fourth and fifth columns of Table 9.

| \(\text{normal form of } f\) | \(\mu_{\text{symp}}(f)\) | \(\iota(f)\) | \(\text{Lt}(f)\) |
|-----------------------------|---------------|---------------|---------------|
| 1 \(t \mapsto (t^3, t^2, t, t^0, 0, \ldots, 0)\) | 1 | 0 | 7 |
| 2 \(t \mapsto (t^3, t^2, t, c^0, 0, \ldots, 0)\) | 2 | 0 | 8 |
| 3 \(t \mapsto (t^3, t^0 + c_1 t^1, t^0, 0, \ldots, 0), c_2 \neq 0\) | 4 | 0 | 8 |
| 4 \(t \mapsto (t^3, t^0, t^0, t^0, 0, \ldots, 0)\) | 4 | 1 | 10 |
| 5 \(t \mapsto (t^3, \pm t^1, t^0, c_1 t^0, 0, \ldots, 0), c_1 \neq 0\) | 5 | 0 | 8 |
| 6 \(t \mapsto (t^3, \pm t^1, t^0, c^0, t^0, 0, \ldots, 0)\) | 5 | 1 | 11 |
| 7 \(t \mapsto (t^3, c_1 t^1 + c_2 t^0, t^0, 0, \ldots, 0)\) | 6 | 0 | 8 |
| 8 \(t \mapsto (t^3, \pm t^1, t^0, t^0, c_1 t^0, 0, \ldots, 0), c_2 \neq 0\) | 7 | 1 | 11 |
| 9 \(t \mapsto (t^3, \pm t^1, t^0, c_1 t^0, 0, \ldots, 0)\) | 7 | 2 | 13 |
| 10 \(t \mapsto (t^3, t^0, t^0, t^0, c_1 t^0, 0, \ldots, 0), c_1 \neq 0\) | 8 | 1 | 11 |
| 11 \(t \mapsto (t^3, t^0, 0, t^0, c_1 t^0, 0, \ldots, 0)\) | 8 | 2 | 14 |
| 12 \(t \mapsto (t^3, c_1 t^0, t^0, \pm t^1, t^0, c_1 t^0, 0, \ldots, 0)\) | 9 | 1 | 11 |
| 13 \(t \mapsto (t^3, t^0, t^0, c_1 t^0, 0, \ldots, 0)\) | 9 | 3 | 16 |
| 14 \(t \mapsto (t^3, \pm t^1, t^0, 0, t^0, 0, \ldots, 0)\) | 9 | 3 | 17 |
| 15 \(t \mapsto (t^3, 0, t^0, t^0, 0, \ldots, 0)\) | 10 | \(\infty\) | \(\infty\) |

Table 9. Symplectic classification of curves with the semigroup \((3, 7, 8)\).

Let \(f : \mathbb{R} \ni t \mapsto f(t) \in \mathbb{R}^{2n}\) be the germ of a smooth or \(\mathbb{R}\)-analytic curve \(\mathcal{A}\)-equivalent to \(t \mapsto (t^3, t^7, t^8, 0, \ldots, 0)\). First we calculate the space of algebraic restrictions of 2-forms to the image of \(f\) in \(\mathbb{R}^{2n}\).

**Proposition 9.2.** The space of algebraic restrictions of differential 2-forms to \(g\) is the 12-dimensional vector space spanned by the following algebraic restrictions:

\[
\begin{align*}
a_{10} &= [dx_1 \wedge dx_2]_g, \\
a_{11} &= [dx_3 \wedge dx_1]_g, \\
a_{13} &= [x_1 dx_3 \wedge dx_1]_g, \\
a_{14} &= [x_1 dx_3 \wedge dx_1]_g, \\
a_{15} &= [dx_2 \wedge dx_1]_g, \\
a_{16} &= [x_1 dx_2 \wedge dx_2]_g, \\
a_{17} &= [x_2 dx_1 \wedge dx_2]_g, \\
a_{18} &= [x_1 dx_3 \wedge dx_1]_g, \\
a_{19} &= [x_3 dx_3 \wedge dx_1]_g, \\
a_{20} &= [x_1 x_2 dx_1 \wedge dx_2]_g, \\
a_{21} &= [x_1 x_3 dx_1 \wedge dx_2]_g,
\end{align*}
\]

where \(\delta\) is quasi-degree of \(a_\delta\).

The sketch of the proof. We use the same method as in the previous sections. The sketch of the proof is presented in Tables 10 and 11. \(\square\)
| quasi-degree $\delta$ | $h_8$ | differential $dh_8$ |
|---------------------|--------|-----------------|
| 14                  | $x_2^3 x_3 - x_2^4$ | $2 x_1 x_3 dx_1 + x_2^2 dx_3 - 2 x_2 dx_3$ |
| 15                  | $x_2 x_3 - x_2^2$ | $x_2 dx_3 + x_3 dx_2 - 5 x_2^2 dx_3$ |
| 16                  | $x_2^4 x_3 - x_2^5$ | $3 x_1^2 x_2 dx_1 + x_1^3 dx_2 - 2 x_3 dx_3$ |

Table 10. Quasi-homogeneous function-germs of quasi-degree 14, 15, 16 vanishing on the curve $t \mapsto (t^3, t^7, t^8)$.

| $\delta$ | basis | forms | relations | proof |
|----------|-------|-------|-----------|-------|
| 10       | $a_{10}$ | $\alpha_{10} = dx_1 \wedge dx_2$ | $a_{10} := [\alpha_{10}]_g$ | |
| 11       | $a_{11}$ | $\alpha_{11} = dx_3 \wedge dx_4$ | $a_{11} := [\alpha_{11}]_g$ | |
| 13       | $a_{13}$ | $x_1 a_{10}$ | $a_{13} := x_1 a_{10}$ | |
| 14       | $a_{14}$ | $\alpha_{14} = x_1 a_{11}$ | $a_{14} := x_1 a_{11}$ | |
| 15       | $a_{15}$ | $\alpha_{15} = dx_2 \wedge dx_3$ | $a_{15} := [\alpha_{15}]_g$ | |
| 16       | $a_{16}$ | $x_2^4 a_{10}$ | $a_{16} := x_2^4 a_{10}$ | |
| 17       | $a_{17}$ | $x_2 a_{10}$, $x_2^2 a_{11}$ | $a_{17} := x_2 a_{10}$ | $[dh_{14} \wedge dx_1]_g = 0$ |
| 18       | $a_{18}$ | $x_3 a_{11}$ | $x_2^2 a_{11} = -2 a_{17}$ | |
| 19       | $a_{19}$ | $x_3 a_{11}$, $x_2^3 a_{10}$ | $a_{19} := x_3 a_{11}$ | $[dh_{16} \wedge dx_1]_g = 0$ |
| 20       | $a_{20}$ | $x_1 x_2 a_{10}$, $x_2^3 a_{11}$ | $a_{20} := x_1 x_2 a_{10}$ | $x_1 [dh_{14} \wedge dx_1]_g = 0$ |
| 21       | $a_{21}$ | $x_1 x_3 a_{10}$, $x_1 x_2 a_{11}$, $x_2^3 a_{15}$ | $a_{21} := x_1 x_3 a_{10}$ | $[dh_{14} \wedge dx_2]_g = 0$ |
| 22       | 0     | $x_1 a_{10}$, $x_1 x_2 a_{11}$, $x_2 a_{15}$ | $x_1 a_{10} = 0$ | $[dh_{16} \wedge dx_1]_g = 0$ |
| 23       | 0     | $x_2^2 x_2 a_{10}$, $x_2^2 a_{11}$, $x_3 a_{15}$ | $x_2^2 a_{10} = 0$ | $[dh_{15} \wedge dx_1]_g = 0$ |
| 24       | 0     | $x_2^2 a_{10}$, $x_2^2 x_2 a_{11}$, $x_3 a_{15}$ | $x_2^2 a_{10} = 0$ | $[dh_{14} \wedge dx_2]_g = 0$ |
| $\geq 25$ | 0     | $x_1 b_{22}$, $x_2 b_{22}$, $x_3 b_{17}$ | $b_{22} := [b_{22}]_g$ | $[dh_{15} \wedge dx_1]_g = 0$ |

Table 11. The quasi-homogeneous basis of algebraic restrictions of 2-forms to the curve $t \mapsto (t^3, t^7, t^8)$. 
The vector fields $X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}$ have the following form:

\[
X_0 = E = 3x_1 \frac{\partial}{\partial x_1} + 7x_2 \frac{\partial}{\partial x_2} + 8x_3 \frac{\partial}{\partial x_3}, \quad X_3 = x_1 E, \\
X_4 = 3x_2 \frac{\partial}{\partial x_1} + 7x_1 x_3 \frac{\partial}{\partial x_2} + 8x_1 \frac{\partial}{\partial x_3}, \quad X_5 = 3x_3 \frac{\partial}{\partial x_1} + 7x_1 \frac{\partial}{\partial x_2} + 8x_1^2 x_2 \frac{\partial}{\partial x_3}, \\
X_7 = x_1^2 E, \quad X_8 = x_2 E, \quad X_9 = x_3 E, \quad X_{10} = x_1 x_2 E.
\]

Their actions on the space of algebraic restrictions of closed 2-forms are presented in Table 12. From these data we obtain the classification of algebraic restrictions as in the previous section.

Now we prove that parameters $c, c_1, c_2$ are moduli in the normal forms. The proofs are very similar in all cases. As an example we consider the normal form $a_{13} + c_1 a_{14} + c_2 a_{15}$ - the first normal form with two parameters. From Table 12 and Theorem 6.12 we see that the tangent space to the orbit of $a_{13} + c_1 a_{14} + c_2 a_{15}$ at $a_{13} + c_1 a_{14} + c_2 a_{15}$ is spanned by linearly independent algebraic restrictions $a_{13} + 14 c_1 a_{14} + 15 c_2 a_{15}, a_{16}, a_{17}, a_{18}, a_{19}, a_{20}$. Hence algebraic restrictions $a_{14}$ and $a_{15}$ do not belong to it. Therefore parameters $c_1$ and $c_2$ are independent moduli in the normal form $a_{13} + c_1 a_{14} + c_2 a_{15}$.
LOCAL SYMPLECTIC ALGEBRA

References

[A1] V. I. Arnold, *First step of local symplectic algebra*, Differential topology, infinite-dimensional Lie algebras, and applications. D. B. Fuchs’ 60th anniversary collection. Providence, RI: American Mathematical Society. Transl., Ser. 2, Am. Math. Soc. 194(44), 1999, 1-8.

[A2] V. I. Arnold, *Simple singularities of curves*, Proc. Steklov Inst. Math. 1999, no. 3 (226), 20-28.

[AG] V. I. Arnold, A. B. Givental *Symplectic geometry*, in Dynamical systems, IV, 1-138, Encyclopedia of Matematical Sciences, vol. 4, Springer, Berlin, 2001.

[AVG] V. I. Arnold, S. M. Gusein-Zade, A. N. Varchenko, *Singularities of Differentiable Maps*, Vol. 1, Birhauser, Boston, 1985.

[BH] T. Bloom and M. Herrera, De Rham cohomology of an analytic space, *Invent. Math.* 7 (1969), 275–296.

[BG] J. W. Bruce, T. J. Gaffney, *Simple singularities of mappings* $(\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$, J. London Math. Soc. (2) 26 (1982), 465-474.

[BPW] J. W. Bruce, A. A. du Plessis, L. C. Wilson, *Discriminants and liftable vector fields*, J. Algebraic Geom. 3 (1994), no. 4, 725-753.

[DJZ1] W. Domitrz, S. Janeczko, M. Zhitomirskii, *Relative Poincare lemma, contractibility, quasi-homogeneity and vector fields tangent to a singular variety*, Ill. J. Math. 48, No.3 (2004), 803-835.

[DJZ2] W. Domitrz, S. Janeczko, M. Zhitomirskii, *Symplectic singularities of varieties: the method of algebraic restrictions*, J. reine und angewandte Math. 618 (2008), 197-235.

[GH] C. G. Gibson and C. A. Hobbs, *Simple singularities of space curves*, Math. Proc. Cambridge Philos. Soc. 113 (1993), 297–310.

[IJ1] G. Ishikawa, S. Janeczko, *Symplectic bifurcations of plane curves and isotropic liftings*, Q. J. Math. 54, No.1 (2003), 73-102.

[IJ2] G. Ishikawa, S. Janeczko, *Symplectic singularities of isotropic mappings*, Geometric singularity theory, Banach Center Publications 65 (2004), 85-106.

[K] P. A. Kolgushkin, *Classification of simple multigerms of curves in a space endowed with a symplectic structure*, St. Petersburg Math. J. 15 (2004), no. 1, 103-126.

[R] J. L. Ramírez Alfonsín, *The Diophantine Frobenius problem*, Oxford Lecture Series in Mathematics and its Applications, 30, Oxford University Press, Oxford, 2005.

[S] J. J. Sylvester, *Question 7382*, Mathematical Questions from the Educational Times 37: 26 (1884).

[W] C. T. C. Wall, *Singular points of plane curves*, London Mathematical Society Student Texts, 63, Cambridge University Press, Cambridge, 2004.

[Za] Zakaljukin, V. M. *Rearrangements of wave fronts that depend on a certain parameter*, (Russian) Funkcional. Anal. i Priloen. 10 (1976), no. 2, 69–70.

[Z] M. Zhitomirskii, *Relative Darboux theorem for singular manifolds and local contact algebra*, Can. J. Math. 57, No.6 (2005), 1314-1340.

Warsaw University of Technology, Faculty of Mathematics and Information Science, Plac Politechniki 1, 00-661 Warsaw, Poland, and Institute of Mathematics, Polish Academy of Sciences, Sniadeckich 8, P.O. Box 137, 00-950 Warsaw, Poland

E-mail address: domitrz@mini.pw.edu.pl