STABILITY CONDITIONS ON MORPHISMS IN A CATEGORY

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Abstract. Let $\mathcal{C}$ be the homotopy category of a stable infinity category $\mathcal{C}$. Then the homotopy category $h\mathcal{C}$ is also triangulated. Hence the space $\text{Stab}(h\mathcal{C})$ of stability conditions on $h\mathcal{C}$ is well-defined though the non-emptiness is not obvious. Our basic motivation is a comparison of the homotopy type of $\text{Stab}(h\mathcal{C})$ and that of $\text{Stab}(h\mathcal{C}^\Delta_1)$. Under the motivation we show that functors $d_0$ and $d_1$ induce continuous maps from $\text{Stab}(h\mathcal{C})$ to $\text{Stab}(h\mathcal{C}^\Delta_1)$ contravariantly where $d_0$ (resp. $d_1$) takes a morphism to the target (resp. source) of the morphism. As a consequence, if $\text{Stab}(h\mathcal{C})$ is nonempty then so is $\text{Stab}(h\mathcal{C}^\Delta_1)$.

1. Introduction

Let $\mathcal{D}$ be a triangulated category. The space $\text{Stab}\mathcal{D}$ of locally finite stability conditions on $\mathcal{D}$ was introduced by Bridgeland [6]. One of remarkable features of $\text{Stab}\mathcal{D}$ is that each connected component of $\text{Stab}\mathcal{D}$ is a complex manifold unless $\text{Stab}\mathcal{D}$ is empty. In general the non-emptiness of $\text{Stab}\mathcal{D}$ is not obvious and the connectedness is an open problem. For instance, if $\mathcal{D}$ is the bounded derived category $\mathcal{D}^b(\text{coh} \mathcal{A}_1^1)$ of coherent sheaves on the affine line over a field $k$, $\text{Stab}\mathcal{D}^b(\text{coh} \mathcal{A}_1^1)$ is empty by Proposition 3.5.

If a scheme is projective, there are many non-empty examples. Let $\mathcal{D}^b(\text{coh} M)$ be the bounded derived category of coherent sheaves on a smooth projective variety $M$. If $\dim M = 1$, the space $\text{Stab}\mathcal{D}^b(\text{coh} M)$ is not empty and is connected by [6], [19], and [22]. If $\dim M = 2$, $\text{Stab}\mathcal{D}^b(\text{coh} M)$ is non-empty by [1] and the connectedness is open. If $\dim M = 3$, Bayer-Macri-Toda [3] shows that a generalized Bogomolov-Gieseker type inequality implies the non-emptiness of $\text{Stab}\mathcal{D}^b(\text{coh} M)$.

It is very difficult to describe $\text{Stab}\mathcal{D}$ globally. Concerned with the description the following working hypothesis states that $\text{Stab}\mathcal{D}$ should be globally simple in a homotopical view:

**Hypothesis:** The space $\text{Stab}\mathcal{D}$ of stability conditions on $\mathcal{D}$ is contractible unless $\text{Stab}\mathcal{D}$ is empty.

If $\mathcal{D} = \mathcal{D}^b(\text{coh} M)$ with $\dim M = 1$, then the hypothesis holds by [6], [19] and [22]. Moreover there are no counter-example to the best of my knowledge.

Specializing $\mathcal{D}$, we can specify the origin of the hypothesis. Let $X$ be the minimal resolution of a Kleinian singularity and $Z$ the schematic fiber of the singularity. Suppose that $\mathcal{D}$ is the category $\mathcal{D}_Z^b(\text{coh} X)$ spanned by bounded complexes in $\mathcal{D}^b(\text{coh} X)$ supported in $Z$. The space...
$\text{Stab} \mathcal{D}^b_2(\text{coh} X)$ is conjecturally$^2$ the universal covering space over a certain configuration space (see also [7] and [8]). It seems natural to expect that $\text{Stab} \mathcal{D}$ is contractible in general since the configuration space should be an Eilenberg MacLane space by the $K(\pi, 1)$ conjecture deriving from Brieskron [9] and Arnol’d (see also [23]).

Stimulated by the hypothesis, it would be interesting to study the homotopy type of $\text{Stab} \mathcal{D}$. Our basic motivation is a comparison of homotopy types of the spaces of stability conditions on $\mathcal{D}$ and on the category $\text{Mor}(\mathcal{D})$ of morphisms in $\mathcal{D}$. Unfortunately the category $\text{Mor}(\mathcal{D})$ is not triangulated in general, so we start with a stable infinity category (see also §2.1) which is a candidate of “enhanced” triangulated categories.

Let $\mathcal{C}$ be a stable infinity category. Then the infinity category $\mathcal{C}^{\Delta^1}$ of morphisms in $\mathcal{C}$ is also stable. Since the homotopy category of a stable infinity category is triangulated, we could introduce a triangulated structure on $h(\mathcal{C}^{\Delta^1})$. Thus the space of stability conditions on $h(\mathcal{C}^{\Delta^1})$ is well-defined though the non-emptiness of $\text{Stab} h(\mathcal{C}^{\Delta^1})$ is not obvious. The following is a natural question motivated by the hypothesis:

**Problem 1.1.** Is $\text{Stab} h(\mathcal{C}^{\Delta^1})$ homotopy equivalent to $\text{Stab} h(\mathcal{C})$?

If the answer of Problem 1.1 is negative we might find an interesting counter-example of the hypothesis. We note that the answer to the easiest case of the problem is affirmative. Precisely if $\mathcal{C}$ is the infinity category $\mathcal{D}^b_{\text{coh}}(\text{Spec} k)$ of $\text{Spec} k$ (see Definition 2.7) then $\text{Stab} h(\mathcal{C}^{\Delta^1})$ is homotopy equivalent to $\text{Stab} h(\mathcal{C})$ by Proposition 3.7.

It is difficult to generalize the argument of the easiest case since the answer comes from calculating $\text{Stab} h(\mathcal{C})$ and $\text{Stab} h(\mathcal{C}^{\Delta^1})$ independently. Our aim is a construction of maps between $\text{Stab} h(\mathcal{C})$ and $\text{Stab} h(\mathcal{C}^{\Delta^1})$ to study Problem 1.1 in more general cases.

Before main theorem, let us recall that there exist functors $d_0$ and $d_1$: $\mathcal{C}^{\Delta^1} \longrightarrow \mathcal{C}$ which take a morphism $f$ in $\mathcal{C}$ to the target and to the source of $f$ respectively. Though an exact functor between triangulated categories does not induce a map between the spaces of stability conditions in general, we show that both functors $d_0$ and $d_1$ respectively induce continuous maps $d_0^*$ and $d_1^*$ contravariantly. The following theorem states some basic properties of these continuous maps.

**Theorem 1.2.** Let $\mathcal{C}$ be a stable infinity category. Assume that the rank of the Grothendieck group $K_0(\text{h}(\mathcal{C}))$ is finite.

1. If there exists a reasonable stability condition on $\text{h}(\mathcal{C})$ then there exists a reasonable stability condition on $\text{h}(\mathcal{C}^{\Delta^1})$. Moreover both functors $d_0$ and $d_1$ induce continuous and injective maps $d_0^*$ and $d_1^*$ from the space $\text{Stab}^f \text{h}(\mathcal{C})$ of reasonable stability conditions on $\text{h}(\mathcal{C})$ to that of $\text{h}(\mathcal{C}^{\Delta^1})$:

   $d_0^*, d_1^* : \text{Stab}^f \text{h}(\mathcal{C}) \longrightarrow \text{Stab}^f \text{h}(\mathcal{C}^{\Delta^1})$.

2. The images $\text{Im} d_0^*$ and $\text{Im} d_1^*$ do not intersect each other.

3. Both images $\text{Im} d_0^*$ and $\text{Im} d_1^*$ are closed in $\text{Stab}^f \text{h}(\mathcal{C}^{\Delta^1})$.

4. A stability condition $\sigma$ is full if and only if $d_0^* \sigma$ (or $d_1^* \sigma$) is full.

Collins-Polishchuk [11] constructed a “glued” stability condition from a semiorthogonal decomposition of a triangulated category. Since the category $\text{h}(\mathcal{C}^{\Delta^1})$ of morphisms has semiorthogonal decompositions with $\text{h}(\mathcal{C})$ and $\text{h}(\mathcal{C}^{\Delta^1})$ (the details are in §2.3), the gluing construction is effective in Theorem 1.2. Reasonable stability conditions (see Definition 4.1) are necessary for the continuity of $d_0$ and $d_1$. In addition full stability conditions are most basic stability conditions (see also Section 3.1), and we do not know whether there exists a non full stability condition does exists or not. Since a full stability condition is reasonable, reasonable stability conditions are sufficiently “reasonable”.

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$^2$If the singularity is type $A_n$ then $\text{Stab} \mathcal{D}^b_2(\text{coh} X)$ is actually the universal cover by [15].
Assertions (1) and (2) are consequences of the gluing construction. We use an “inducing construction” developed in [20] to prove the third assertion (3) in Theorem 1.2. Roughly speaking the inducing construction asserts that a faithful exact functor $F: \mathcal{D} \to \mathcal{D}'$ between triangulated categories induces a continuous map from a closed subset of $\text{Stab} \mathcal{D}'$ to $\text{Stab} \mathcal{D}$. The fourth assertion (4) follows from a necessary and sufficient condition for semistable objects in $h(\mathcal{C}^{\Delta^1})$ with respect to $d^0_\sigma$ and $d^1_\sigma$.

The following problem is derived from the fourth assertion in Theorem 1.2:

**Problem 1.3.** Is the image $\text{Im} d^0_\sigma$ path connected to $\text{Im} d^1_\sigma$?

Unfortunately it is quite difficult to prove the connectedness of the space of stability conditions and there are no counter-example such that the subspace of full stability conditions is connected. Thus it is natural to expect that both images $\text{Im} d^0_\sigma$ and $\text{Im} d^1_\sigma$ are path connected and the following gives an evidence.

**Theorem 1.4.** If $\mathcal{C}$ is the infinity category $\mathcal{D}^b_{\text{coh}}(\mathbb{P}^1)$ of projective line $\mathbb{P}^1$ over a field $k$, then the image $\text{Im} d^0_\sigma$ is path connected to $\text{Im} d^1_\sigma$.

In the proof of Theorem 1.4 above we use an algebraic stability condition which does not exist in $\text{Stab} h(\mathcal{D}^b_{\text{coh}}(C))$ when the genus of the curve $C$ is grater than 0. It is natural to study Theorem 1.4 for positive genus cases. The case of $g(C) = 1$ are discussed in [21] independently and their result gives an affirmative answer to Problem 1.3.

We note that the same argument in Theorem 1.4 is effective for the full component $\text{Stab} h(\mathcal{D}^b_{\text{coh}}(\mathbb{P}^2))$ of the space of stability conditions on the projective plane $\mathbb{P}^2$. Thus Theorem 1.4 gives an evidence of Problem 1.3.

The organization of this article is the following. Section 2 is basically a summary on the infinity category of morphisms. We also describe the Serre functor on $h(\mathcal{C}^{\Delta^1})$ under some assumptions on $\mathcal{C}$. The description gives an answer to [21, Conjecture 3.17] (see also Remark 6.3). In Section 3 we observe Problem 1.1 in the easiest case. In the observation, we show that the space $\text{Stab} \mathcal{D}^b(\text{mod } k)$ for arbitrary field $k$ is isomorphic to $\mathcal{C}$ (Corollary 3.8) where $\text{mod } k$ is the category of finite dimensional $k$-spaces. One of key ingredients is the indecomposability of $\sigma$-stable objects. As an application, we show that the bounded derived category of finite generated modules on a principle ideal domain has no stability condition (Proposition 3.5).

The first and second assertion in Theorem 1.2 is proven in Propositions 4.5 and 4.7. The third assertion of Theorem 1.2 is Theorem 4.15. The fourth assertion of Theorem 1.2 is Theorem 4.11. The section 5 is devoted to prove Theorem 1.4 which is just Theorem 5.7.

In the last section, we discuss another construction of $h(\mathcal{C}^{\Delta^1})$ when $\mathcal{C}$ is the infinity category $\mathcal{D}^b_{\text{coh}}(X)$ for a Noetherian scheme $X$. Since the homotopy category $h(\mathcal{D}^b_{\text{coh}}(X))$ is equivalent to the bounded derived category $\mathcal{D}^b(\text{coh } X)$, there is a natural bounded $t$-structure on $h(\mathcal{D}^b_{\text{coh}}(X))$ whose heart is the abelian category $\mathcal{B} = \text{coh}(X)$ of coherent sheaves on $X$. Then the category $\text{Mor}(\mathcal{B})$ of morphisms in $\mathcal{B}$ is also an abelian category. Thus one could define the bounded derived category $\mathcal{D}^b(\text{Mor}(\mathcal{B}))$ via localization of quasi-isomorphisms. Corollary 6.2 claims that two categories $h(\mathcal{D}^b_{\text{coh}}(X)^{\Delta^1})$ and $\mathcal{D}^b(\text{Mor}(\mathcal{B}))$ are equivalent. As a consequence, $h(\mathcal{D}^b_{\text{coh}}(\text{Spec } k)^{\Delta^1})$ is equivalent to the bounded derived category of $A_2$ -quiver (see also Remark 6.3).

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Let \( \text{Mor}(\mathcal{D}) \) be the category of morphisms in a category \( \mathcal{D} \). Unfortunately \( \text{Mor}(\mathcal{D}) \) is not a triangulated category in general when \( \mathcal{D} \) is a triangulated category. To solve the problem, we start with a stable infinity category.

### 2. A Triangulated Category of Morphisms in \( \mathcal{C} \)

#### 2.1. Stable infinity categories. Let \([n] = \{0 < 1 < \cdots < n\} \) be the finite ordinal \((n \geq 0)\) and let \( \Delta \) be the category of finite ordinals. The \( i \)-th face map is denote by \( d^i: [n] \to [n+1] \) \((0 \leq i \leq n+1)\) and the \( j \)-th degeneracy map is denoted by \( s^j: [n+1] \to [n] \) \((0 \leq j \leq n)\). The standard \( n \)-simplex is denoted by \( \Delta^n \). Given a simplicial set \( K \), the set of \( n \)-simplices is denoted by \( K_n \). We denote by \( d_i: K_{n+1} \to K_n \) (resp. \( s_j: K_n \to K_{n+1} \)) the map induced from \( d^i \) (resp. \( s^j \)).

Let \( \mathcal{C} \) be an infinity category in the sense of [17, Definition 1.1.2.4]. Namely, an infinity category is a simplicial set \( \mathcal{C} \) satisfying the following lifting property:

- For any inclusion \( i: \Lambda^n_k \to \Delta^n \) of the \( k \)-th horn where \( 0 < k < n \) and for any morphism \( p: \Lambda^n_k \to \mathcal{C} \), there exists a morphism \( \bar{p}: \Delta^n \to \mathcal{C} \) such that \( \bar{p} \circ i = p \).

A \( 0 \)-simplex of the simplicial set \( \mathcal{C} \) is called an object of the infinity category \( \mathcal{C} \) and a \( 1 \)-simplex of \( \mathcal{C} \) is called a morphism in \( \mathcal{C} \). The homotopy category \( h(\mathcal{C}) \) of \( \mathcal{C} \) is a usual category whose object is the same as that of \( \mathcal{C} \) and whose morphism is an equivalence class \([f]\) of a morphism \( f \) in \( \mathcal{C} \) (the details are in [17, §1.2.3]). One of descriptions of the equivalence relation is the following:

**Proposition-Definition 2.1 ([17, §1.2.3]).** Let \( \mathcal{C} \) be an infinity category.

1. Two morphisms \( f \) and \( g \) from \( x \) to \( y \) in an infinity category \( \mathcal{C} \) is homotopic if there exists a 2-simplex \( h: \Delta^2 \to \mathcal{C} \) satisfying \( d_0 h = \text{id}_y, d_1 h = g \) and \( d_2 h = f \). Then the relation of homotopy is an equivalence relation.

2. The homotopy category \( h(\mathcal{C}) \) has the same objects of \( \mathcal{C} \). A morphism in \( h(\mathcal{C}) \) is given by the equivalence class of 1-simplices with respect to the relation defined by (1). A functor \( F: \mathcal{C} \to \mathcal{D} \) between infinity categories is nothing but a morphism as simplicial sets. The functor \( F \) naturally induces the functor \( h(\mathcal{C}) \to h(\mathcal{D}) \) between homotopy categories. Though the induced functor should be written as \( h(F) \), we write simply \( F \) by abusing notation.

An infinity category \( \mathcal{C} \) is said to be stable if \( \mathcal{C} \) has a zero object, admits finite limits and colimits, and pushout squares coincide with pullback squares. If \( \mathcal{C} \) is stable then \( h(\mathcal{C}) \) is triangulated by [18, Theorem 1.1.2.14]. A zero object in a stable infinity category is denoted by \( 0 \). The mapping cone of a morphism \( f: x \to y \) in the triangulated category \( h(\mathcal{C}) \) is given by the pushout\(^3\) \( \emptyset \sqcup_{x,y} \) of \( f: x \to y \) in the infinity category \( \mathcal{C} \).

Before examples of stable infinity categories, we fix the following notation for differential graded categories (shortly dg categories):

**Notation 2.2.** Let \( \mathcal{D} \) be a dg category over a commutative ring.

- The cochain complex of morphisms in \( \mathcal{D} \) is denoted by \( \text{Map}_\mathcal{D}(x,y) \) for \( x \) and \( y \in \mathcal{D} \).
- The degree \( p \)-th part of \( \text{Map}_\mathcal{D}(x,y) \) is denoted by \( \text{Map}^p_\mathcal{D}(x,y) \).
- The differential of \( \text{Map}^p_\mathcal{D}(x,y) \) is denoted by \( \delta^p: \text{Map}^p_\mathcal{D}(x,y) \to \text{Map}^{p+1}_\mathcal{D}(x,y) \) or shortly \( \delta^p \).
- The kernel of \( \delta^p \) is denoted by \( \text{Map}^p_\mathcal{D}(0,x,y) \) or \( \text{Map}^p_\mathcal{D}(x,0,y) \).
- If \( \mathcal{A} \) is an additive category, the dg-category of cochain complexes in \( \mathcal{A} \) is denoted by \( \text{Ch}(\mathcal{A}) \).
- The category of cochain complexes in the additive category \( \mathcal{A} \) is denoted by \( \text{Kom}(\mathcal{A}) \).

Note that the objects in \( \text{Ch}(\mathcal{A}) \) and \( \text{Kom}(\mathcal{A}) \) are the same, and we have \( \text{Hom}_{\text{Kom}(\mathcal{A})}(x,y) = Z^0_{\text{Ch}(\mathcal{A})}(x,y) \).

\(^3\)The push out is calculated in the infinity category \( \mathcal{C} \), not in the triangulated category \( h(\mathcal{C}) \).
One of good examples of stable infinity categories is a “derived infinity category” $\mathcal{D}(\mathcal{A})$ of a Grothendieck abelian category $\mathcal{A}$. The following is necessary for a relation with $\mathcal{D}(\mathcal{A})$ and the dg category $\text{Ch}(\mathcal{A})$.

**Proposition-Definition 2.3** ([18, §1.3.1, §1.3.2]). Let $\mathcal{D}$ be a dg category over a commutative ring. Define a simplicial set $N_{dg}(\mathcal{D})$ whose $n$-simplex $\sigma \in N_{dg}(\mathcal{D})_n$ is the set of ordered pairs $((x_i)_{i=0}^n, \{f_I\})$ where

- Each $x_i$ is an object in $\mathcal{D}$.
- For any subset $I = \{i_1 < i_2 < \cdots < i_k\}$ of the ordinal $[n]$ where $m \geq 0$, $f_I$ is an element in $\text{Map}_{\mathcal{D}}(x_{i_1}, x_{i_k})$ satisfying

$$\delta f_I = \begin{cases} \sum_{1 \leq j \leq m} (-1)^j \left( f_I - \sum_{i \leq j} f_{\{i, j, \ldots, j\}} \circ f_{\{i, \ldots, i\}} \right) & (m > 0) \\ 0 & (m = 0). \end{cases}$$

For a nondecreasing map $\alpha : [m] \to [n]$, the induced map $\alpha^* : N_{dg}(\mathcal{D})_n \to N_{dg}(\mathcal{D})_m$ is given by $((x_i)_{0 \leq i \leq n}, \{f_I\}) \mapsto ((x_{\alpha(j)})_{0 \leq j \leq m}, \{f_J\})$ where

$$g_J = \begin{cases} f_{\alpha(j)} & \text{if } J = \{j, j', \ldots, j\} \text{ with } \alpha(j) = \alpha(j') = i \\ \text{id}_{x_j} & \text{if } J = \{j, j'\} \text{ with } \alpha(j) = \alpha(j') = i \\ 0 & \text{otherwise}. \end{cases}$$

Then the simplicial set $N_{dg}(\mathcal{D})$ is an infinity category. In particular, the simplicial set $N_{dg}(\mathcal{D})$ is referred as the differential graded nerve of $\mathcal{D}$.

**Remark 2.4.** Keep the notation as in Proposition-Definition 2.3. From the definition, the set $N_{dg}(\mathcal{D})_1$ of 1-simplices of $N_{dg}(\mathcal{D})$ is

$$\{f \in Z^0_{\mathcal{D}}(x, y) \mid x, y \in \mathcal{D}\}.$$ 

Similarly the set $N_{dg}(\mathcal{D})_2$ is

$$\{(f_0, f_1, f_2, f) \in Z^0_{\mathcal{D}}(y, z) \times Z^0_{\mathcal{D}}(x, y) \times Z^0_{\mathcal{D}}(x, z) \times \text{Map}_{\mathcal{D}}(x, z) \mid \delta(h) = f_0 f_2 - f_1 \text{ and } x, y, z \in \mathcal{D}\}.$$ 

Thus a 2-simplex of $\mathcal{C} = N_{dg}(\mathcal{D})$ could be regarded as a 4-tuple of morphisms in $\mathcal{D}$.

**Proposition-Definition 2.5** ([18, Proposition 1.3.5.3]). Suppose $\mathcal{A}$ is a Grothendieck abelian category. $\text{Kom}(\mathcal{A})$ has a left proper combinatorial model structure described as follows:

- (w) A map $\{f^p : E^p \to F^p\}_{p \in \mathbb{Z}}$ in $\text{Kom}(\mathcal{A})$ is a weak equivalence if $f$ is a quasi-isomorphism.
- (c) A map $\{f^p : E^p \to F^p\}_{p \in \mathbb{Z}}$ is a cofibration if $f^p : E^p \to F^p$ is a monomorphism for each degree $p \in \mathbb{Z}$.
- (f) A map $\{f^p : E^p \to F^p\}_{p \in \mathbb{Z}}$ is a fibration if it has the right lifting property with respect to every map which is a cofibration and a weak equivalence.

We refer to the model structure as the injective model structure on $\text{Kom}(\mathcal{A})$.

**Remark 2.6.** Any object in $\text{Kom}(\mathcal{A})$ is cofibrant with respect to the injective model structure.

**Definition 2.7.** Let $\mathcal{A}$ be a Grothendieck abelian category.

- (1) Let $\text{Ch}(\mathcal{A})^0$ be the full sub-dg-category of $\text{Ch}(\mathcal{A})$ consisting of fibrant-cofibrant objects in $\text{Kom}(\mathcal{A})$ with respect to the injective model structure. Define the infinity category $\mathcal{D}(\mathcal{A})$ by the infinity category $N_{dg}(\text{Ch}(\mathcal{A})^0)$.
- (2) If $\mathcal{A}$ is the category $\text{Qcoh}(X)$ of quasi-coherent sheaves on a Noetherian scheme $X$, we denote $\mathcal{D}(\text{Qcoh}(X))$ by $\mathcal{D}(X)$.
- (3) Keep the notation as in (2). Define $\mathcal{D}^\text{b}_{\text{coh}}(X)$ by the full sub-infinity-category of $\mathcal{D}(X)$ consisting of bounded complexes with coherent cohomologies.
Proposition 2.8 ([18, Proposition 1.3.5.9]). Let \( A \) be a Grothendieck abelian category. Then the infinity category \( D(A) \) is stable.

Remark 2.9. The homotopy category \( h(D(A)) \) is isomorphic to the homotopy category \( Ho(Kom(A)) \) with respect to the injective model structure on \( Kom(A) \). Hence the homotopy category \( h(D(X)) \) is equivalent to the unbounded derived category \( D(Qcoh(X)) \) of \( Qcoh(X) \).

Note that \( D_{coh}^b(X) \) is also stable by [18, Lemma 1.1.3.3] and \( h(D_{coh}^b(X)) \) is equivalent to the bounded derived category \( D^b(Qcoh(X)) \) of coherent sheaves on a Noetherian scheme \( X \).

Remark 2.10. Basically we use the script font \( \mathcal{C} \) (ex. \( \mathcal{C} \)) for infinity categories. Capital letters with the bold font \( \mathbf{D} \) (ex. \( D \)) are used for usual categories.

Through this article we use homotopical notation for mapping cones in the triangulated category \( h(\mathcal{C}) \). Namely the mapping cone of a morphism \( f: x \to y \) in \( h(\mathcal{C}) \) is denoted by \( fib \ f \). We also denote \( cof \ f[-1] \) by \( \text{fib} f \). Thus we obtain a distinguished triangle in \( h(\mathcal{C}) \) as follows:

\[
\begin{array}{c}
\text{fib} f \\
\downarrow \\
\downarrow f \\
y \\
\downarrow \text{cof} f \\
\end{array}
\]

We sometimes omit \( \text{cof} f \) (or \( \text{fib} f \)) in the above diagram when the triangle is clear from context.

2.2. The infinity category of morphisms in \( \mathcal{C} \). Let \( \mathcal{C}^{\Delta^1} = \text{Fun}(\Delta^1, \mathcal{C}) \) be the simplicial set of morphism from the standard simplicial set \( \Delta^1 \) to \( \mathcal{C} \). Then \( \mathcal{C}^{\Delta^1} \) is also an infinity category by [17, Proposition 1.2.7.3]. Moreover if \( \mathcal{C} \) is stable then so is \( \mathcal{C}^{\Delta^1} \) by [18, §1.1.1]. Since the objects in \( \mathcal{C}^{\Delta^1} \) are the morphisms in \( \mathcal{C} \), we refer to \( \mathcal{C}^{\Delta^1} \) as the infinity category of morphisms in \( \mathcal{C} \).

Since \( \mathcal{C} \) is isomorphic to \( \text{Fun}(\Delta^0, \mathcal{C}) \), face maps and degeneracy maps contravariantly induce functors between infinity categories respectively: \( \mathcal{C}^{\Delta^1} \xrightarrow{d_i} \mathcal{C} \) and \( \mathcal{C} \xleftarrow{s} \mathcal{C}^{\Delta^1} \). Here \( d_i \) is the induced functor from \( d^i \) \((i = 0, 1)\) and \( s \) is the functor induced by \( s^0 \) by abusing notation.

If a morphism \( f \) in \( \mathcal{C} \) satisfies \( d_0 f = y \) and \( d_1 f = x \), then we write \( f \) as \([f: x \to y]\) to emphasize the source and the target of the morphism. Since there exists a unit transformation \( \text{id}_{\mathcal{C}^{\Delta^1}} \to s \circ d_0 \), \( d_0 \) is left adjoint to \( s \) by [17, Proposition 5.2.2.8]. Similarly \( d_1 \) is left adjoint to \( s \). Thus we obtain the following diagram:

\[
\begin{array}{c}
\mathcal{C} \\
\downarrow \\
\downarrow d_0 \xleftarrow{s} d_1 \\
\end{array}
\]

Let \( \text{Mor}(h(\mathcal{C})) \) be the category of morphisms in \( h(\mathcal{C}) \). Then the objects of \( \text{Mor}(h(\mathcal{C})) \) are the same as that of \( h(\mathcal{C}^{\Delta^1}) \) but the morphisms are different. To explain the difference, take \( f \) and \( g \) in \( \mathcal{C}^{\Delta^1} \).

A morphism \( \tau: f \to g \) in \( \text{Mor}(h(\mathcal{C})) \) is a pair \(([\tau_1], [\tau_0])\) of morphisms \( \tau_i: d_i f \to d_i g \) in \( h(\mathcal{C}) \) \((i \in \{0, 1\})\) satisfying the equality \([g] \circ [\tau_1] = [\tau_0] \circ [f] \) in \( \text{Hom}_{h(\mathcal{C})}(d_1 f, d_0 g) \). On the other hand a morphism \( \varphi: f \to g \) in \( \mathcal{C}^{\Delta^1} \) (not in the homotopy category) is a pair \((h_1, h_0)\) of 2-simplices \( \Delta^2 \to \mathcal{C} \) such that

- \( d_0 h_1 = g, d_2 h_0 = f, \) and
- \( d_1 h_1 = d_1 h_0 (= \psi) \).

\[
\begin{array}{c}
d_1 f \\
\downarrow \\
d_2 h_0 = f \\
\end{array}
\]

\[
\begin{array}{c}
d_1 h_1 \\
\downarrow \\
d_0 f \\
\end{array}
\]

\[
\begin{array}{c}
d_1 g \\
\downarrow \\
d_0 h_0 = g \\
\end{array}
\]

\[
\begin{array}{c}
d_1 h_0 \\
\downarrow \psi \\
h_1 \\
\downarrow h_0 \\
g = d_0 h_1 \\
\end{array}
\]
Hence a morphisms in the homotopy category is obtained by the the equivalence class $[(h_1, h_0)]$ of the relation in Proposition-Definition 2.1.

The following lemma is concerned with the difference between $h(C^3)$ and $\text{Mor}(h(C))$:

**Lemma 2.11.** There exists a natural functor $F: h(C^3) \to \text{Mor}(h(C))$ where $F$ is identity on objects and forgets higher morphisms in $C$. More precisely, for a morphism $[\varphi] = [(h_1, h_0)]$ from $f$ to $g$ in $h(C^3)$, $F([\varphi])$ is given by the pair $([d_2h_1], [d_0h_0])$ of morphisms in $h(C)$. Moreover the functor $F$ is full.

**Proof.** The same argument in the proof of [17, Proposition 1.2.3.7] implies that $F$ is well-defined. Let $([\tau_1], [\tau_0])$ be a morphism from $f$ to $g$ in $\text{Mor}(h(C))$. Choose 1-simplices $\psi_1$ and $\psi_0$ such that $[\psi_1] = [g] \circ [\tau_1]$ and $[\psi_0] = [\tau_0] \circ [f]$. Note that there exist 2-simplices $h_0: \Delta^2 \to C$ satisfying $d_0h_0 = \tau_0, d_1h_0 = \psi_0$, and $d_2h_0 = f$ and $\bar{h}_1: \Delta^2 \to C$ satisfying $d_0\bar{h}_1 = g, d_1\bar{h}_1 = \psi_1$ and $d_2\bar{h}_1 = \tau_1$. By the equation $[g] \circ [\tau_1] = [\tau_0] \circ [f]$, we have a 2-simplex $\rho: \Delta^2 \to C$ such that $d_0\rho = \text{id}_{d_0g}, d_1\rho = \psi_0$ and $d_2\rho = \psi_1$.

Consider the diagram:

$$
\Lambda^3 \xrightarrow{(s_1g,opo,\bar{h}_1)} C
$$

Since $C$ is the infinity category there exists a 3-simplex $\sigma: \Delta^3 \to C$ as indicated in the above diagram. Put $h_1 = d_2\sigma$, then we obtain the desired diagram:

$$
\begin{array}{c}
d_1f \xrightarrow{\tau_1} d_1g \\
\downarrow f \quad \downarrow g \\
d_0f \xrightarrow{\tau_0} d_0g
\end{array}
$$

Hence $F$ is full. $\square$

**Remark 2.12.**

1. A morphism in $h(C^3)$ should be depicted as the diagram $(2.2)$, but we frequently omit $\psi$, $h_1$ and $h_0$.

2. Let $f \to g \to h$ be a distinguished triangle in $h(C^3)$. We obtain the following diagram of distinguished triangles in $h(C)$:

$$
\begin{array}{c}
d_1f \quad d_1g \quad d_1h \\
\downarrow f \quad \downarrow g \quad \downarrow h \\
d_0f \quad d_0g \quad d_0h \\
\downarrow \text{cof } f \quad \downarrow \text{cof } g \quad \downarrow \text{cof } h.
\end{array}
$$

In particular the third row is also distinguished triangle in $h(C)$.

2.3. **Semiorthogonal decompositions of $h(C^3)$**. The aim of this section is to show that $h(C^3)$ has two semiorthogonal decompositions coming from two adjoint pairs $d_0 \dashv s$ and $s \dashv d_1$. Before the proof, let us recall a semiorthogonal decomposition.

Let $D$ be a triangulated category. A pair $(D_1, D_2)$ of full triangulated subcategories $D_1$ and $D_2$ of $D$ is said to be a **semiorthogonal decomposition** of $D$ if the pair satisfies

1. $\hom_D(x_2, x_1) = 0$ for any $x_i \in D_i$ ($i = 1, 2$), and
Thus \( \text{Hom}_{\mathcal{D}}(x_1, x_2) = 0 \) holds, the semiorthogonal decomposition is said to be orthogonal.

**Remark 2.13.** If \( \mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle \) is a semiorthogonal decomposition of \( \mathcal{D} \), then \( \mathcal{D}_1 \) (resp. \( \mathcal{D}_2 \)) is left admissible (resp. right admissible) by [4, Lemma 3.1]. Namely the inclusion functor \( \iota_1: \mathcal{D}_1 \to \mathcal{D} \) (resp. \( \iota_2: \mathcal{D}_2 \to \mathcal{D} \)) has the left adjoint functor \( \tau_1: \mathcal{D} \to \mathcal{D}_1 \) (resp. the right adjoint functor \( \tau_2: \mathcal{D} \to \mathcal{D}_2 \)). Thus we have adjoint pairs of functors \( \iota_2 \dashv \tau_2 \) and \( \iota_1 \dashv \tau_1 \).

**Lemma 2.14.** Let \( \mathcal{C} \) be a stable infinity category. Set full subcategories of \( \mathcal{C}^{\Delta^1} \) by

\[
\mathcal{C}_{/0} = \left\{ [x \to 0] \in \mathcal{C}^{\Delta^1} \mid x \in \mathcal{C} \right\}, \quad \mathcal{C}_{0/} = \left\{ [0 \to y] \in \mathcal{C}^{\Delta^1} \mid y \in \mathcal{C} \right\}
\]

and

\[
\mathcal{C}_s = \left\{ [z \xrightarrow{id} z] \in \mathcal{C}^{\Delta^1} \mid z \in \mathcal{C} \right\}.
\]

Furthermore put \( \mathcal{C}_{/0} := \text{h}(\mathcal{C}_{/0}), \mathcal{C}_{0/} := \text{h}(\mathcal{C}_{0/}) \) and \( \mathcal{C}_s := \text{h}(\mathcal{C}_s) \) Then the pairs \( (\mathcal{D}_{/0}^L, \mathcal{D}_{/0}^R) := (\mathcal{C}_{/0}, \mathcal{C}_{0/}) \) and \( (\mathcal{D}_s^L, \mathcal{D}_s^R) := (\mathcal{C}_s, \mathcal{C}_s) \) are semiorthogonal decompositions of \( \text{h}(\mathcal{C}^{\Delta^1}) \) respectively.

**Proof.** Note that an adjoint pair of functors between infinity categories induces the adjoint pair between corresponding homotopy categories by [17, Proposition 5.2.12].

We first prove \( \text{h}(\mathcal{C}^{\Delta^1}) = \langle \mathcal{C}_s, \mathcal{C}_{/0} \rangle \). Put \( [b: y \to 0] \in \mathcal{C}_{/0} \) and \( [id_z: z \to z] \in \mathcal{C}_s \). Then we see

\[
\text{Hom}_{\text{h}(\mathcal{C}^{\Delta^1})}(b, id_z) = \text{Hom}_{\text{h}(\mathcal{C}^{\Delta^1})}(b, s(z)) \cong \text{Hom}_{\text{h}(\mathcal{C})}(d_0(b), z) = \text{Hom}_{\text{h}(\mathcal{C})}(0, z) = 0.
\]

Thus \( \text{Hom}_{\text{h}(\mathcal{C}^{\Delta^1})}(b, id_z) = 0 \) for any \( b \in \mathcal{C}_{/0} \) and any \( id_z \in \mathcal{C}_s \).

Take an object \( [f: x \to y] \in \text{h}(\mathcal{C}^{\Delta^1}) \) arbitrary. Then the adjunction \( d_0 \dashv s \) implies the canonical morphism \( \tau: f \to s \circ d_0(f) = \text{id}_y \) in \( \text{h}(\mathcal{C}^{\Delta^1}) \), and we obtain the following distinguished triangle in \( \text{h}(\mathcal{C}^{\Delta^1}) \):

\[
\begin{align*}
\text{fib} f & \quad x \quad f \quad y \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
0 \quad y \quad \text{id} \quad y
\end{align*}
\]

Hence we have \( \text{h}(\mathcal{C}^{\Delta^1}) = \langle \mathcal{C}_s, \mathcal{C}_{/0} \rangle \).

Let \( \mathcal{D} \) be the opposite category \( \mathcal{C}^{\text{op}} \) of \( \mathcal{C} \). Then \( \mathcal{D} \) is also stable (see also [18, \$1.1.1]) and any colimit in \( \mathcal{D} \) is equivalent to the limit in \( \mathcal{C} \). Since \( \mathcal{D}^{\Delta^1} = \text{Fun}(\Delta^1, \mathcal{D}) \) is equivalent to the opposite category of \( \mathcal{C}^{\Delta^1} \) by the canonical equivalence \( (\Delta^1)^{\text{op}} \cong \Delta^1 \), the functor \( d_0: \mathcal{D}^{\Delta^1} \to \mathcal{D} \) is the opposite functor of \( d_1: \mathcal{C}^{\Delta^1} \to \mathcal{C} \). Hence the argument above for the adjoint pair \( d_0 \dashv s \) also implies \( \text{h}(\mathcal{C}^{\Delta^1}) = \langle \mathcal{C}_{/0}, \mathcal{C}_s \rangle \).

**Remark 2.15.** Note that the functor \( s: \mathcal{C} \to \mathcal{C}_s \) gives an equivalence. Since the natural projection \( d_1: \mathcal{C}_{/0} \to \mathcal{C} \) is a trivial Kan fibration by [17, \$1.2.12], \( d_1 \) has a section denoted by \( j_1: \mathcal{C} \to \mathcal{C}_{/0} \). Similarly a section of \( d_0: \mathcal{C}_{0/} \to \mathcal{C} \) is denoted by \( j_0: \mathcal{C} \to \mathcal{C}_{0/} \). Taking homotopy categories, the functor \( j_1: \text{h}(\mathcal{C}) \to \mathcal{C}_{/0} \) (resp. \( j_0: \text{h}(\mathcal{C}) \to \mathcal{C}_{0/} \)) gives the inverse functor of the restriction of \( d_1: \mathcal{C}_{/0} \to \text{h}(\mathcal{C}) \) (resp. \( d_0: \mathcal{C}_{0/} \to \text{h}(\mathcal{C}) \)). Throughout this article we always identify
C_0, C_{0j} and C_s with h(\mathcal{C}) via j_1, j_2 and s respectively. In particular we have the following explicit descriptions of j_1 and j_2:

\begin{equation}
\begin{aligned}
j_1 &: h(\mathcal{C}) \xrightarrow{\sim} C_0; j_1(x) = [x \to 0], \\
j_2 &: h(\mathcal{C}) \xrightarrow{\sim} C_{0j}; j_2(x) = [0 \to x].
\end{aligned}
\end{equation}

2.4. Serre functors on \(h(\mathcal{C}^\Delta_1)\). We observe the Serre functor on \(h(\mathcal{C}^\Delta_1)\) under certain assumption on \(\mathcal{C}\). As a consequence if \(\mathcal{C}\) is the infinity category \(D_{coh}^b(X)\) of a smooth projective variety \(X\), we have an explicit description of the Serre functor on \(h(\mathcal{C}^\Delta_1)\). Before the argument, recall that \(k\)-linear category \(D\) is said to be finite if any vector space \(\text{Hom}_D(x, y)\) is finite dimensional for all \(x, y \in D\).

**Proposition 2.16.** Let \(D\) be a dg category over a field \(k\).

1. Then the homotopy category \(h(N_{dg}(D)^{\Delta_1})\) is a \(k\)-linear category.

2. If \(N_{dg}(D)\) is stable and \(h(N_{dg}(D))\) is finite, then \(h(N_{dg}(D)^{\Delta_1})\) is also finite.

**Proof.** Put \(\mathcal{C} = N_{dg}(D)\) and let \(f \) and \(g\) be in \(\mathcal{C}^{\Delta_1}\). Then \(f\) is in \(Z^0_D(d_1f, d_0f)\) and so does \(g\) by Remark 2.4. Note that an \(n\)-simplex of \(\mathcal{C}^{\Delta_1}\) is a morphism \(\Delta^n \times \Delta^1 \to \mathcal{C}\) as simplicial sets. According to Proposition-Definition 2.3, a morphism from \(f\) to \(g\) in \(\mathcal{C}^{\Delta_1}\) is a 5-tuple \((\tau_1, \tau_0, \psi, h_1, h_0)\) where

\begin{itemize}
  \item \(\tau_1 \in Z^0_D(d_1f, d_0g), \psi \in Z^0_D(d_1f, d_0g),\)
  \item \(h_1 \in \text{Map}_D^{-1}(d_1f, d_0g)\) with \(\delta(h_1) = g\tau_1 - \psi,\) and
  \item \(h_0 \in \text{Map}_D^{-1}(d_1f, d_0g)\) with \(\delta(h_0) = \tau_0f - \psi.\)
\end{itemize}

Let \(\text{Hom}(f, g)\) be the set of 5-tuples \((\tau_1, \tau_0, \psi, h_1, h_0)\) satisfying the above relation. Then \(\text{Hom}(f, g)\) is naturally a \(k\)-linear space via termwise addition and \(k\)-action.

By Proposition-Definition 2.1, we see that two 5-tuples \((\tau_1, \tau_0, \psi, h_1, h_0)\) and \((\tilde{\tau}_1, \tilde{\tau}_0, \tilde{\psi}, \tilde{h}_1, \tilde{h}_0)\) are equivalent if there exists a 7-tuple \(h_{12} \in \text{Map}_D^{-1}(d_1f, d_1g)\), \(\{h_{12}, h_{121}, h_{122}\} \subset \text{Map}_D^{-1}(d_1f, d_0g),\)

\begin{itemize}
  \item \(\delta(h_{12}) = \tau_1 - \tilde{\tau}_1,\)
  \item \(\delta(h_{121}) = \tau_0 - \tilde{\tau}_0,\)
  \item \(\delta(h_{122}) = g\tau_1 - \tilde{\psi},\)
  \item \(\delta(h_{121}) = \psi - \tilde{\psi},\)
  \item \(\delta(h_{122}) = -h_{12} + gh_{121} + \tilde{h}_1,\)
  \item \(\delta(h_{121}) = -h_{12} + h_1 + h_{122},\) and
  \item \(\delta(h_{12}) = \tilde{h}_0 + h_0 + h_{122} - h_{121}f.\)
\end{itemize}

Since the set of collections \((\tau_1, \tau_0, \psi, h_1, h_0)\) which are equivalent to zero of \(\text{Hom}(f, g)\) is a \(k\)-linear subspace of \(\text{Hom}(f, g)\), \(h(\mathcal{C}^{\Delta_1})\) is a \(k\)-linear category.

To prove the second assertion, consider the semiorthogonal decomposition of \(h(\mathcal{C}^{\Delta_1}) = \langle D_0^L, D_0^R \rangle\) in Lemma 2.14. Then we have the distinguished triangle

\[ j_1 \circ \text{fib}(f) \longrightarrow f \longrightarrow s \circ d_0(f) \longrightarrow j_1 \circ \text{fib}(f)[1] \]

for any \(f \in \mathcal{C}^{\Delta_1}\), and the exact sequence of \(k\)-vector spaces

\begin{equation}
\begin{aligned}
\text{Hom}_{h(\mathcal{C}^{\Delta_1})}(s \circ d_0f, g) &\longrightarrow \text{Hom}_{h(\mathcal{C}^{\Delta_1})}(f, g) \longrightarrow \text{Hom}_{h(\mathcal{C}^{\Delta_1})}(j_1 \circ \text{fib}(f), g)
\end{aligned}
\end{equation}

for any \(g \in \mathcal{C}^{\Delta_1}\). The right term in (2.5) is isomorphic to \(\text{Hom}_{h(\mathcal{C})}(\text{fib} f, \text{fib} g)\). Moreover the left term in (2.5) is isomorphic to \(\text{Hom}_{h(\mathcal{C})}(d_0f, d_0g)\) since the functor \(d_1\) is the right adjoint of \(s\). Then the assumption implies that the category \(h(N_{dg}(D)^{\Delta_1})\) is finite. \(\square\)
Proposition 2.17. Let $\mathcal{D}$ be a $k$-linear differential graded category and put $\mathcal{C} = N_{dg}(\mathcal{D})$. Suppose that $\mathcal{C}$ is stable and $h(\mathcal{C})$ is finite. If $h(\mathcal{C})$ has a Serre functor $S_\mathcal{C}$ then $h(\mathcal{C}^{\Delta^1})$ has also a Serre functor $S_{\mathcal{C}^{\Delta^1}}$ given by

\begin{equation}
S_{\mathcal{C}^{\Delta^1}}(f) = [S_\mathcal{C}(u): S_\mathcal{C}(d_0 f) \to S_\mathcal{C}(\text{cof } f)],
\end{equation}

where $u$ is the universal morphism $y \to \text{cof } f$ in $h(\mathcal{C})$.

Proof. According to Bondal-Kapranov [5, Proposition 3.8, Theorem 2.10], let $\mathcal{B}$ be the essential image $\mathcal{C}_s$ of $s: h(\mathcal{C}) \to h(\mathcal{C}^{\Delta^1})$. Since the right (resp. left) adjoint of $s$ is $d_1$ (resp. $d_0$), $\mathcal{B}$ is right and left admissible in $h(\mathcal{C}^{\Delta^1})$. Moreover the right adjoint of the inclusion $j_s: h(\mathcal{C}) \to \mathcal{B}^\perp \to h(\mathcal{C}^{\Delta^1})$ is given by $d_0: h(\mathcal{C}^{\Delta^1}) \to h(\mathcal{C})$. Thus $\mathcal{B}^\perp$ is also right and left admissible. Since $\mathcal{B}$ and $\mathcal{B}^\perp$ are both equivalent to $h(\mathcal{C})$, the triangulated category $h(\mathcal{C}^{\Delta^1})$ has the Serre functor by [5, Proposition 3.8].

It is enough to show that the functor $h(-) = \text{Hom}_{h(\mathcal{C}^{\Delta^1})}(F, -)^\vee: h(\mathcal{C}^{\Delta^1}) \to \text{Vect}_k$ is represented by $S_\mathcal{C}(u)$ in (2.6). Since $\mathcal{B} \cong h(\mathcal{C})$ has the Serre functor, the contravariant functor $\text{Hom}_{h(\mathcal{C})}(x, -)^\vee: h(\mathcal{C}) \to \text{Vect}_k$ is represented by $S_\mathcal{C}(x)$ where $\text{Vect}_k$ is the category of $k$-vector spaces.

Take the semiorthogonal decomposition $h(\mathcal{C}^{\Delta^1}) = \langle \mathcal{B}^\perp, \mathcal{B} \rangle = \langle \mathcal{C}_s, \mathcal{C}/0 \rangle$. Then we obtain the following distinguished triangle in $h(\mathcal{C}^{\Delta^1})$:

\[
\begin{array}{ccc}
\text{fib } f & \rightarrow & x \\
\downarrow & & \downarrow \\
0 & \rightarrow & y
\end{array}
\]

Then the restriction $h|_\mathcal{B}$ is represented by the object $E = [\text{id}: S_\mathcal{C}(y) \to S_\mathcal{C}(y)]$ in $h(\mathcal{C}^{\Delta^1})$.

Take the semiorthogonal decomposition $\langle \mathcal{B}^\perp, \mathcal{B} \rangle = \langle \mathcal{C}_{0/}, \mathcal{C}_s \rangle$. Then we obtain the following:

\[
\begin{array}{ccc}
x & \rightarrow & 0 \\
\downarrow & & \downarrow \\
x & \rightarrow & \text{cof } f
\end{array}
\]

The same argument above implies that the restriction $h|_{\mathcal{B}^\perp}$ of $h$ to $\mathcal{B}^\perp$ is represented by the object $F = j_s(S_\mathcal{C}(\text{cof } f)) = [0 \to S_\mathcal{C}(\text{cof } f)]$ in $h(\mathcal{C}^{\Delta^1})$.

The adjunction $d_0 \dashv s$ implies that the restriction functor of the representable functor by the morphism $E$ to $\mathcal{B}^\perp$ is represented by $F' = [0 \to S_\mathcal{C}(y)]$. Namely we have

\begin{equation}
\text{Hom}_{h(\mathcal{C}^{\Delta^1})}(-, E)|_{\mathcal{B}^\perp} \cong \text{Hom}_{\mathcal{B}^\perp}(-, F').
\end{equation}

Then evaluation of (2.7) at $F'$ implies the canonical morphism $\gamma: F' \to E$ which satisfies $d_0(\gamma) = \text{id}$.

Let $\varphi: F' \to F$ be the morphism defined by $h(\gamma)(\text{id}_E) \in h(F') \cong \text{Hom}_{h(\mathcal{C}^{\Delta^1})}(F', F)$. Since $h(E)$ (resp. $h(F')$) is canonically isomorphic to $\text{Hom}_{h(\mathcal{C})}(y, y)$ (resp. $\text{Hom}_{h(\mathcal{C})}(y, \text{cof } f)$), we see $d_0\varphi = S_\mathcal{C}(u)$ where $u$ is a universal morphism $u: y \to \text{cof } f$ in $\mathcal{C}$. Then we obtain the following distinguished triangle in $h(\mathcal{C}^{\Delta^1})$:

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
S_\mathcal{C}(y) & \rightarrow & S_\mathcal{C}(\text{cof } f)
\end{array}
\]

\[
\begin{array}{ccc}
& & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & S_\mathcal{C}(x)[1]
\end{array}
\]

\[
\begin{array}{ccc}
& & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & S_\mathcal{C}(y)[1]
\end{array}
\]
Put \( F'' = [0 \to S_C(x)[1]] \). Let \( \delta : F'' \to E[1] \) be the composite \( F'' \to F'[1] \to E[1] \) and \( X \) the mapping cone of \( \delta \). Then \( X \) is isomorphic to \( S_C(u) \) in (2.6). Hence the functor \( h(-) = \text{Hom}_{\mathcal{C}_\Delta^1}(f, -)^\vee \) is represented by \( S_C(u) \) via the proof of [5, Theorem 2.10]. □

**Remark 2.18.** If the Serre functor \( S_\mathcal{D} \) on a \( k \)-linear triangulated category \( \mathcal{D} \) is isomorphic to the \( d \)-times shifts \([d]\), then \( \mathcal{D} \) is said to be \( d \)-dimensional Calabi-Yau category. Moreover if \( k \)-times composite \( S^k_\mathcal{D} \) is isomorphic to the shifts \([d]\), then \( \mathcal{D} \) is said to be \( d/k \)-dimensional Calabi-Yau or a fractional Calabi-Yau. One of the simplest example of fractional Calabi-Yau categories is the bounded derived category of the finite dimensional representations of \( A_2 \)-quiver. Such a category is obtained by \( h(\mathcal{D}^b_{\text{coh}}(\text{Spec } k)) \) of a field \( k \) (See also Remark 6.3).

From Proposition 2.17, we see

\[
S^3_{\mathcal{C}_\Delta^1}([f : x \to y]) = [S^3_C(f[1]) : S^3_C(x[1]) \to S^3_C(y[1])].
\]

Thus \( h(\mathcal{C}_\Delta^1) \) is \((3d + 1)/3\)-dimensional Calabi-Yau category if \( h(\mathcal{C}) \) is \( d \)-dimensional Calabi-Yau.

### 3. An Observation on Problem 1.1

#### 3.1. Notation for stability conditions

Let \( \mathcal{D} \) be a triangulated category such that the rank of Grothendieck group \( K_0(\mathcal{D}) \) of \( \mathcal{D} \) is finite. We follow notation and a basic definition of stability conditions on a triangulated category \( \mathcal{D} \) from the original article [6] due to Bridgeland. For instance \( \text{Stab}_\mathcal{D} \) is the set of locally finite stability conditions on \( \mathcal{D} \). The central charge of \( \sigma \in \text{Stab}_\mathcal{D} \) is denoted by \( Z \), \( A \), and \( \mathcal{P} \) respectively the heart and the slicing of \( \sigma \). The set \( \text{Stab}_\mathcal{D} \) has a topology and each non-empty connected component is a complex manifold whose dimension is smaller than or equal to the rank of \( K_0(\mathcal{D}) \). We also recall that \( \text{Stab}_\mathcal{D} \) has a left action of \( \text{Aut}_\mathcal{D} \) and a right action of the universal cover \( \widetilde{\text{GL}}^+_2(\mathbb{R}) \) of \( \text{GL}^+_2(\mathbb{R}) \). The right action of \( \widetilde{\text{GL}}^+_2(\mathbb{R}) \) is continuous.

A stability condition \( \sigma \) on \( \mathcal{D} \) is said to be full if the tangent space of \( \text{Stab}_\mathcal{D} \) at \( \sigma \) has the maximal dimension rank \( K_0(\mathcal{D}) \). A connected component of full stability conditions on \( \mathcal{D} \) is said to be a full component.

One of basic properties of stability conditions is the support property:

**Definition 3.1** (Support property). Let \( \mathcal{D} \) be a triangulated category with rank \( K_0(\mathcal{D}) < \infty \), and let \( \| - \| : K_0(\mathcal{D}) \otimes \mathbb{R} \to \mathbb{R} \) be a norm. Then a stability condition \( \sigma \) on \( \mathcal{D} \) satisfies the support property if the following holds:

\[
\exists C > 0 \text{ such that } \sup \left\{ \left| \frac{\|E\|}{\|Z(E)\|} \right| : \text{E is } \sigma \text{-semistable} \right\} \leq C.
\]

Since any norm on \( K_0(\mathcal{D}) \otimes \mathbb{R} \) is equivalent, the support property is independent of the choice of the norm. The following lemma gives us a transparent understanding of fullness and support property.

**Lemma 3.2** ([2, Appendix B.4]). Let \( \mathcal{D} \) be a triangulated category. Assume rank \( K_0(\mathcal{D}) \) is finite. Then a locally finite stability condition \( \sigma \) is full if and only if \( \sigma \) has the support property.

#### 3.2. A fundamental observation

We give an observation of Problem 1.1 in Proposition 3.7. The following is a key ingredient of the observation.

**Lemma 3.3.** Let \( \mathcal{D} \) be an additive category. If any non-zero endomorphism of an object \( E \in \mathcal{D} \) is invertible, then \( E \) is indecomposable.

**Remark 3.4.** In the earlier version of Lemma 3.3, the category \( \mathcal{D} \) was assumed to be a bounded derived category. The generalization to additive categories was given by the referee.
Proposition 3.7. Let \( R \) be a field (cf. Definition 2.7). Then both \( (3.1) \) of shifts of torsion \( R \)-modules implies that the module \( \sigma \) itself has a non-invertible endomorphism as \( R \)-modules by the assumption. Thus \( R \) never be \( \sigma \)-stable since any non-zero endomorphism of a \( \sigma \)-stable object is invertible.

Then any \( \sigma \)-stable object is isomorphic to a torsion \( R \)-module up to shifts. The existence of the Harder-Narasimhan filtrations implies that the module \( R \) itself is given by a successive extension of shifts of torsion \( R \)-modules. This clearly gives a contradiction since the rank of \( R \) is positive but the rank of torsion modules is zero.

Lemma 3.3 supplies an example of a derived category which has no stability condition.

Proposition 3.5. Suppose that a principle integral domain \( R \) is not a field. Let \( D^b(\text{mod } R) \) be the bounded derived category of the abelian category \( \text{mod } R \) of finitely generated \( R \)-modules. Then \( \text{Stab } D^b(\text{mod } R) \) is empty.

Proof. Suppose to the contrary that there exists a locally finite stability condition \( \sigma \) on \( D^b(\text{mod } R) \). We can take a \( \sigma \)-stable object \( E \in D^b(\text{mod } R) \) since \( \sigma \) is locally finite. Then any non-zero endomorphism of \( E \) is invertible and Lemma 3.3 implies \( E \) is indecomposable. Since the global dimension of \( R \) is 1, any indecomposable object in \( D^b(\text{mod } R) \) is a shifts of an indecomposable object in \( \text{mod } R \). Moreover any indecomposable object in \( \text{mod } R \) is \( R \) or \( R/(r) \) for some non zero element \( r \in R \) by the assumption for \( R \).

Without loss of generality, we assume \( E = R \) or \( E = R/(r) \) where \( r \in R \setminus \{0\} \). Recall that \( R \) itself has a non-invertible endomorphism as \( \text{mod } R \)-modules by the assumption. Thus \( R \) never be \( \sigma \)-stable since any non-zero endomorphism of a \( \sigma \)-stable object is invertible.

The category \( D^b(\text{mod } R) \) has no locally finite stability condition.

Corollary 3.6. Let \( \{R_i\}_{i=1}^n \) be the finite set of principle ideal domains which are not fields and put \( R = \prod_{i=1}^n R_i \). Then \( \text{Stab } D^b(\text{mod } R) \) is empty.

Proof. From the assumption, the derived category \( D^b(\text{mod } R) \) has the orthogonal decomposition: \( D^b(\text{mod } R) = \bigoplus_{i=1}^n D^b(\text{mod } R_i) \). By [12, Proposition 5.2], we see \( \text{Stab } D^b(\text{mod } R) \) is isomorphic to the product \( \prod_{i=1}^n \text{Stab } D^b(\text{mod } R_i) \). Thus the category \( D^b(\text{mod } R) \) has no locally finite stability condition.

Now we go back to an observation for Problem 1.1.

Proposition 3.7. Let \( \mathcal{C} \) be the infinity category \( D^b_{\text{coh}}(\text{Spec } k) \) of the scheme \( \text{Spec } k \) where \( k \) is a field (cf. Definition 2.7). Then both \( \text{Stab } h(\mathcal{C}) \) and \( \text{Stab } h(\mathcal{C}^{\Delta^1}) \) are contractible. In particular the answer for Problem 1.1 is affirmative.

Proof. Since any nonzero complex number gives an orientation preserving \( \mathbb{R} \)-liner isomorphism on \( \mathbb{R}^2 \), the multiplicative group \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) is a subgroup of \( GL_2^+ \). The universal cover \( \mathbb{C} \) of \( \mathbb{C}^* \) is a subgroup of \( GL_2^+(\mathbb{R}) \). Thus \( \mathbb{C} \) acts on \( \text{Stab } D \) for any triangulated category \( D \). The right action of \( \mathbb{C} \) on \( \text{Stab } D \) is explicitly given by

\[
\sigma \cdot z = (W, Q), W(E) = \exp(-z)Z(E) \quad \text{and} \quad Q(\phi) = P(\phi + y/\pi),
\]

where \( \sigma = (Z, P) \) and \( z = x + \sqrt{-1}y \). In particular the \( \mathbb{C} \) action on \( \text{Stab } D \) is holomorphic.

We show that \( \text{Stab } h(\mathcal{C}) \) is isomorphic to \( \mathbb{C} \) as complex manifolds. It is enough to show that the \( \mathbb{C} \) action on \( \text{Stab } h(\mathcal{C}) \) is free and transitive. Let \( E \) be a stable object for a stability condition \( \sigma \in \text{Stab } h(\mathcal{C}) \). Since \( E \) is indecomposable by Lemma 3.3, \( E \) should be isomorphic to \( k \) up to shifts. Hence \( k \) is stable for all \( \sigma \in \text{Stab } h(\mathcal{C}) \) and the \( \mathbb{C} \) action on \( \text{Stab } h(\mathcal{C}) \) is free by the \( \mathbb{C} \) action (3.1).

\footnote{Thought the article [12] has been written with assuming a triangulated category is linear over algebraically closed field, one can prove [12, Proposition 5.2] without the assumption. Hence we can apply the proposition.
Take stability conditions $\sigma_1$ and $\sigma_2$. Let $\phi_i$ be the phase of $k$ in $\sigma_i$ ($i \in \{1, 2\}$). Put $z = \log m_1 - \log m_2 + \sqrt{-1}\pi(\phi_1 - \phi_2)$. Then we see $\sigma_1 \cdot z = \sigma_2$, since $k$ is stable for any $\sigma \in \text{Stab}(C)$. Thus the $C$-action is transitive.

By Corollary 6.2, $h(C^{\Delta^1})$ is the bounded derived category $D^b(\text{mod} (\bullet \to \bullet))$ of the finite dimensional representations of the $A_2$ quiver $\bullet \to \bullet$. The argument due to Macrì [19] essentially implies that $\text{Stab}(h(C^{\Delta^1}))$ is contractible as follows.

According to [19], $\text{Stab}(h(C^{\Delta^1}))$ has an open covering $U = \{\Theta_i\}_{i=0}^2$ such that each $\Theta_i$ is contractible and any two intersections are the same $GL_2^+(\mathbb{R})$-orbit $O_{-1}$ of a stability condition $\Theta_i \cap \Theta_j = O_{-1}$.

Then any finite intersection is contractible since $O_{-1}$ is homeomorphic to $GL_2^+(\mathbb{R})$. The nerve theorem [24] (or [14, Corollary 4.3]) implies that $\text{Stab}(C^{\Delta^1})$ is homotopy equivalent to $N(U)$ where $N(U)$ is the nerve of the covering $U$. Clearly $N(U)$ is the standard simplicial complex $\Delta^2$ (here $\Delta^2$ is not a simplicial set but a simplicial complex). Hence $\text{Stab}(h(C^{\Delta^1}))$ is contractible.

Through the proof above, we have proved the following:

**Corollary 3.8.** Let $D^b(\text{mod} k)$ be the bounded derived category of finite dimensional $k$-vector spaces where $k$ is a field. Then $\text{Stab} D^b(\text{mod} k)$ is isomorphic to $C$.

**Remark 3.9.** It is difficult to generalize Proposition 3.7 to arbitrary case since we only calculate the homotopy types of $\text{Stab}(h(C))$ and $\text{Stab}(h(C^{\Delta^1}))$ independently. Thus our basic motivation of this note is a construction of continuous maps between $\text{Stab}(h(C))$ and $\text{Stab}(h(C^{\Delta^1}))$ for (arbitrary) stable infinity categories.

### 4. Stability conditions on morphisms

The aim of this section is the construction of continuous maps from $\text{Stab}(h(C))$ to $\text{Stab}(h(C^{\Delta^1}))$ and we wish to study some properties of the morphisms. A key ingredient of the construction is the “gluing construction” developed by [11].

**4.1. Gluing construction of stability conditions.** As observed in §2.3, $h(C^{\Delta^1})$ has semiorthogonal decompositions. In particular each component is equivalent to $h(C)$ itself. Collins–Polishchuck [11] proposed a construction of stability conditions on a triangulated category $D$ coming from a semiorthogonal decomposition $D = \langle D_1, D_2 \rangle$. A key ingredient of the construction is a reasonable stability condition on $h(C)$:

**Definition 4.1 ([11, pp. 568]).** A stability condition $\sigma = (A, Z)$ on a triangulated category $D$ is **reasonable** if $\sigma$ satisfies

$$0 < \inf\{|Z(E)| \in \mathbb{R} | \text{E is semistable in } \sigma\}.$$

We also denote by $\text{Stab}' D$ the set of reasonable stability conditions on $D$.

**Remark 4.2.** Since a reasonable stability condition is locally finite by [11, Lemma 1.1], $\text{Stab}' D$ is a subset of $\text{Stab} D$. Unfortunately we do not know whether $\text{Stab}' D = \text{Stab} D$ or not.

If a stability condition $\sigma = (Z, P)$ satisfies the support property then $|Z(E)|$ has lower bound since a norm on $K_0(D)$ is discrete. Hence $\text{Stab}' D$ contains any full components of $\text{Stab} D$ by Lemma 3.2. Thus a reasonable stability condition is sufficiently “reasonable”.

**Proposition 4.3 ([11]).** Suppose that the triangulated category $D$ has a semiorthogonal decomposition $\langle D_1, D_2 \rangle$. The left adjoint of the inclusion $D_1 \to D$ is denoted by $\tau_1$ and the right adjoint of the inclusion $D_2 \to D$ is denoted by $\tau_2$. Let $\sigma_i = (Z_i, P_i)$ be a reasonable stability condition on $D_i$ for $i \in \{1, 2\}$. Suppose that $\sigma_1$ and $\sigma_2$ satisfy the following conditions

1. $\text{Hom}^0_D (P_i(0, 1), P_j(0, 1)) = 0$ and
Proposition 4.7. The construction of

\[ \text{Proof.} \]

Let \( x, y \) be non-zero stable objects in \( \mathcal{A} \). Then there exists a unique reasonable stability condition \( \mathfrak{gl}(\sigma_1, \sigma_2) \) on \( \mathcal{D} \) glued from \( \sigma_1 \) and \( \sigma_2 \) whose heart \( \mathcal{A} \) of the t-structure of \( \mathfrak{gl}(\sigma_1, \sigma_2) \) is given by

\[ \mathcal{A} = \{ E \in \mathcal{D} \mid \tau_i(E) \in \mathcal{P}_i((0,1]) \ (i = 1,2) \} \]

and whose central charge \( Z \) is given by \( Z(E) = Z_1(\tau_1(E)) + Z_2(\tau_2(E)) \). Moreover the gluing construction is continuous on pairs \((\sigma_1, \sigma_2)\) satisfying the above conditions (1) and (2).

Lemma 4.4. Notation is the same as in Proposition 4.3. Suppose that \( \mathfrak{gl}(\sigma_1, \sigma_2) = (Z, \mathcal{P}) \) is the glueing stability condition. Then we have

\[ \mathcal{P}(\phi) \cap \mathbf{D}_i = \mathcal{P}_i(\phi) \]

Proof. According to [11, Proposition 2.2], we have \( \mathcal{P}_i(\phi) \subset \mathcal{P}(\phi) \). Thus we see \( \mathcal{P}_i(\phi) \subset \mathcal{P}(\phi) \cap \mathbf{D}_i \).

Conversely let \( E \in \mathcal{P}(\phi) \cap \mathbf{D}_i \). Suppose to the contrary that \( E \) is not \( \sigma_i \)-semistable. Take the first term \( A \) of the Harder-Narasimhan filtration of \( E \) with respect to \( \sigma_i \). Then \( A \) is \( \sigma_i \)-semistable with \( \arg Z_i(A) > \phi \). Since \( A \) is also \( \mathfrak{gl}(\sigma_1, \sigma_2) \)-semistable in phase \( \arg Z_i(A) \) by [11, Proposition 2.2], there is no morphism from \( A \) to \( E \). Hence \( E \) should be \( \sigma_i \)-semistable and this give the proof of the opposite inclusion \( \mathcal{P}_i(\phi) \supset \mathcal{P}(\phi) \cap \mathbf{D}_i \).

Proposition 4.5. Let \( \mathcal{E} \) be a stable infinity category with rank \( K_0(h(\mathcal{E})) < \infty \). Then there exists continuous maps as follows:

\[ d_0^*, d_1^* : \text{Stab}^{\text{f}} h(\mathcal{E}) \longrightarrow \text{Stab}^{\text{f}} h(\mathcal{E}^{\Delta^1}) \]

(1) Let \( h(\mathcal{E}^{\Delta^1}) = (\mathbf{D}_0^*, \mathbf{D}_0^R) \) be the semiorthogonal decomposition in Lemma 2.14. We define the pull back of \( \sigma \) along the functor \( d_0 \) by

\[ d_0^* \sigma := \mathfrak{gl}(\sigma_L, \sigma_R) = \mathfrak{gl}(\sigma, [1] \sigma) \]

(2) Let \( h(\mathcal{E}^{\Delta^1}) = (\mathbf{D}_1^*, \mathbf{D}_1^R) \) be the semiorthogonal decomposition in Lemma 2.14. We define the pull back of \( \sigma \) along the functor \( d_1 \) by

\[ d_1^* \sigma := \mathfrak{gl}(\sigma_L, \sigma_R) = \mathfrak{gl}(\sigma, [1] \sigma, \sigma) \]

Proof. Let \( \mathcal{P} \) be the slicing of \( \sigma \in \text{Stab}^{\text{f}} h(\mathcal{E}) \). Then the heart of \( \sigma \) and \([1] \sigma \) are respectively \( \mathcal{P}((0,1]) \) and \( \mathcal{P}((-1,0]) \). Thus the conditions (1) and (2) in Proposition 4.3 holds since

\[ \text{Hom}_{h(\mathcal{E}^{\Delta^1})}(s(x), j_i(y)) \cong \text{Hom}_{h(\mathcal{E})}(x, d_i \circ j_i(y)) = \text{Hom}_{h(\mathcal{E})}(x, y) \]

for \( x, y \in h(\mathcal{E}) \). Hence \( d_0^* \sigma = \mathfrak{gl}(\sigma, [1] \sigma) \) does exists. Continuity of the gluing construction follows from [11, Theorem 4.3]. The second assertion is similar.

Lemma 4.6. Let \( \sigma = (\mathcal{A}, Z) \) be in \( \text{Stab}^{\text{f}} h(\mathcal{E}) \) and let \( d_i^* \sigma = (d_i^* \mathcal{A}, d_i^* Z) \) be the pull back of \( \sigma \) by \( d_i \) for \( i \in \{1,2\} \).

(1) An object \( f \in h(\mathcal{E}^{\Delta^1}) \) is in \( d_0^* \mathcal{A} \) if and only if both \( d_0 f \) and \( \text{cof} f \) are in \( \mathcal{A} \).

(2) If an object \( x \) is in \( \mathcal{A} \), then \( \arg d_0^* Z(s(x)) = \arg Z(x) \) and \( \arg d_0^* Z(j(x[-1])) = \arg Z(x) \).

(3) An object \( f \in h(\mathcal{E}^{\Delta^1}) \) is in \( d_1^* \mathcal{A} \) if and only if both \( d_1 f \) and \( \text{fib} f \) are in \( \mathcal{A} \).

(4) If an object \( x \) is in \( \mathcal{A} \), then \( \arg d_1^* Z(s(x)) = \arg Z(x) \) and \( \arg d_1^* Z(j(x[x[-1]])) = \arg Z(x) \).

Proof. The construction of \( d_0^* \sigma \) and \( d_1^* \sigma \) directly implies the assertions.

Proposition 4.7. Let \( \mathcal{E} \) be a stable infinity category. Then the induced maps

\[ \text{Stab}^{\text{f}} h(\mathcal{E}) \longrightarrow \text{Stab}^{\text{f}} h(\mathcal{E}^{\Delta^1}) \]

satisfy the following:
(1) $d_0^*$ and $d_1^*$ are both injective.

(2) The image $d_0^*$ does not intersect the image of $d_1^*$.

**Proof.** Since the proof is similar, we only prove the assertion (1) for $d_0^*$.

Let $\sigma = (Z, \mathcal{P})$ and $\tau = (W, \mathcal{Q})$ be in $\text{Stab}^b h(\mathcal{C})$, and suppose $d_0^* \sigma = d_0^* \tau$. Then they have the same central charge and this implies $Z = W$. It is enough to show $\mathcal{P} = \mathcal{Q}$ for the conclusion.

Recall the semiorthogonal decomposition $h(\mathcal{C}^{\Delta^1}) = (D_0^L, D_0^R)$ in Lemma 2.14. Since $D_0^L = C_s$, Proposition 4.3 implies

$$\mathcal{P}(\phi) = d_0^* \mathcal{P}(\phi) \cap C_s = d_0^* \mathcal{Q}(\phi) \cap C_s = \mathcal{Q}(\phi).$$

This gives the proof of the first assertion.

Let $h(\mathcal{C}^{\Delta^1}) = (D_0^L, D_0^R)$ (resp. $(D_1^L, D_1^R)$) be the semiorthogonal decomposition in Lemma 2.14. Recall $D_0^L = D_1^R = C_s$. Take $d_0^* \sigma \in \text{Im} d_0^*$ and $d_1^* \tau \in \text{Im} d_1^*$. Suppose that $d_0^* \sigma = d_1^* \tau$ for some $\sigma = (Z, \mathcal{P})$ and $\tau = (W, \mathcal{Q}) \in \text{Stab}^b h(\mathcal{C})$. Lemma 4.4 implies

$$\mathcal{P}(\phi) = d_0^* \mathcal{P} \cap D_0^L = d_1^* \mathcal{Q} \cap D_0^L = d_1^* \mathcal{Q} \cap D_1^R = \mathcal{Q}(\phi).$$

Furthermore we see

$$Z = d_0^* Z|_{D_0^L} = d_1^* W|_{D_0^L} = d_1^* W|_{D_1^R} = W.$$

Hence $Z$ is the same as $W$. For $[f: x \to y] \in h(\mathcal{C}^{\Delta^1})$, we have

$$d_0^* Z(f) = Z(y) - Z(\text{fib} f) \quad \text{and} \quad d_1^* W(f) = W(x) - W(\text{cof} f).$$

Since $\text{fib} f[1] = \text{cof} f$, we see $d_1^* W(f) = W(x) + W(\text{fib} f) = W(y) + 2W(\text{fib} f)$. This gives a contradiction. \(\square\)

**4.2. Fullness of $d_0^* \sigma$ and $d_1^* \sigma$.** The aim is to prove the fullness of $d_0^* \sigma$ and of $d_1^* \sigma$ are equivalent to that of $\sigma$. The following is a key ingredient in the proof.

**Proposition 4.8.** Let $\mathcal{C}$ be a stable infinity category and let $\sigma = (A, Z)$ be a reasonable stability condition on $h(\mathcal{C})$.

(1) Suppose that $f \in \mathcal{C}^{\Delta^1}$ is semistable in $d_0^* \sigma$ with the phase $\phi$. Then we have

1a) $d_0 f = y$ is $\sigma$-semistable,

1b) $\text{cof} f = \text{fib} f[1]$ is $\sigma$-semistable, and

1c) Moreover $\arg Z(y) = \arg Z(\text{cof} f) = \phi$.

(2) Suppose that $f \in \mathcal{C}^{\Delta^1}$ is semistable in $d_1^* \sigma$ with the phase $\phi$. Then we have

2a) $d_1 f = x$ is $\sigma$-semistable,

2b) $\text{fib} f = \text{cof} f[-1]$ is $\sigma$-semistable, and

2c) Moreover $\arg Z(x) = \arg Z(\text{fib} f) = \phi$.

**Proof.** Put $d_0 f = y, d_1 f = x$. The central charge of $d_0^* \sigma$ is denoted by $d_0^* Z$ and the heart of $d_0^* \sigma$ is denoted by $d_0^* A$.

We first show the assertion (1a). We may assume that the phase $\phi$ is in $(0, 1] \subset \mathbb{R}$ without loss of generality. Then the object $f$ is in the heart $d_0^* A$. In particular both $y$ and $\text{cof} f$ belong to the heart $A$ by Lemma 4.6.

Suppose to the contrary that $y$ is not $\sigma$-semistable. Then there is a subobject $a$ of $y$ such that $a$ is $\sigma$-semistable with

$$\arg Z(a) > \arg Z(y).$$

From the definition of $d_0^* A$, we have the following short exact sequence in $d_0^* A$:

$$0 \longrightarrow j_!(\text{fib } f) \longrightarrow f \longrightarrow \text{id}_y \longrightarrow 0.$$
(4.2) \[ \arg Z(\text{cof}) = \arg d_0^* Z(j_i(\text{fib} f)) \leq \arg d_0^* Z(f) \leq \arg d_0^* Z(\text{id}_y) = \arg Z(y). \]

Let \( g \) be the composition of morphisms \( g: a \to \text{cof} f \). Then we have a commutative diagram in \( h(\mathcal{C}) \):

\[
\begin{array}{ccc}
\text{ker } g & \longrightarrow & x \\
\downarrow h & & \downarrow f \\
\bullet & \longrightarrow & y.
\end{array}
\]

Then we have a morphism \( \tau: h \to f \) in \( h(\mathcal{C}^{\Delta^1}) \) by Lemma 2.11. Since \( \text{cof} h = \text{im } g \) is a colimit in the infinity category \( \mathcal{C} \), a morphism \( \text{cof} \tau: \text{cof } h \to \text{cof } f \) in \( \mathcal{C} \) induces the monomorphism \( \text{im } g \longrightarrow \text{cof } f \) in \( h(\mathcal{C}) \). Since \( d_0 \tau: d_0 h \to d_0 f \) is also monomorphism in \( \mathcal{A} \), the mapping cone \( \text{cof}(h \to f) \) in \( h(\mathcal{C}^{\Delta^1}) \) is in \( d_0^* \mathcal{A} \) by Lemma 4.6. Thus \( [h: \text{ker } g \to a] \) is a subobject of \( f \). Since \( f \) is \( d_0^* \sigma \)-semi-stable we see

\[ \arg d_0^* Z(h) \leq \arg d_0^* Z(f). \]

Since the morphism \( a \to \text{im } g \) is an epimorphism in \( \mathcal{A} \) from the semi-stable object \( a \) we see \( \arg Z(a) \leq \arg Z(\text{im } g) \). Moreover the definition of \( d_0^* Z \) implies

\[ \arg Z(a) \leq \arg d_0^* Z(h) \leq \arg Z(\text{im } g). \]

Hence the inequalities (4.1), (4.4) and (4.2) imply the following inequality

\[ \arg d_0^* Z(f) \leq \arg Z(y) < \arg Z(a) \leq \arg d_0^* Z(h) \]

which contradicts (4.3). Hence \( y \) is \( \sigma \)-semi-stable.

For the proof of (1b), suppose to the contrary that \( \text{cof} f \) is not \( \sigma \)-semi-stable. Then there exists a quotient \( a' \) of \( \text{cof } f \) in \( \mathcal{A} \) such that \( a' \) is \( \sigma \)-semi-stable with

\[ \arg Z(a') < \arg Z(\text{cof } f). \]

The inequality (4.2) implies

\[ \arg Z(a') < \arg Z(y). \]

Since \( y \) is semi-stable, we have \( \text{Hom}_{h(\mathcal{C})}(y, a') = 0 \). Thus we obtain the following diagram of distinguished triangles in \( h(\mathcal{C}) \) by \( 3 \times 3 \) Lemma:

\[
\begin{array}{ccc}
z & \longrightarrow & x \\
\downarrow f & & \downarrow h' \\
y & \longrightarrow & 0 \\
\downarrow d & \quad & \downarrow \text{cof } f \\
\bullet & \longrightarrow & a'
\end{array}
\]

In the diagram above, \( d \) is the kernel of the morphism \( \text{cof } f \to a' \) in \( \mathcal{A} \). By Lemma 2.11, the diagram above gives a morphism \( f \to h' \) in \( h(\mathcal{C}^{\Delta^1}) \). Since \( \text{fib}(f \to h') \) is isomorphic to \([z \to y]\), the object \( \text{fib}(f \to h') \) is in \( d_0^* \mathcal{A} \) by Lemma 4.6 and we see that \( h' \) is a quotient of \( f \) in \( d_0^* \mathcal{A} \). The following inequality holds since \( f \) is \( d_0^* \sigma \)-semi-stable:

\[ \arg d_0^* Z(f) \leq \arg d_0^* Z(h') = \arg Z(a'). \]

On the other hand the above inequality contradicts (4.2) and (4.5). Hence \( \text{cof } f \) is semi-stable.
Finally we prove (1c). If \( \arg Z(\cof f) \neq \arg Z(y) \) then we have \( \Hom_{h(\mathcal{C})}(y, \cof f) = 0 \) by the inequality (4.2). Hence we have \( \Hom_{h(\mathcal{C}^{\Delta^1})}(\id_y, [\cof f \rightarrow 0]) = 0 \) and this implies
\[
f \cong \id_y \oplus [(\cof f)[-1] \rightarrow 0].
\]
Since \( f \) is semistable \( (\cof f)[-1] = \fib f = 0 \) or \( y = 0 \).

The same argument for the opposite infinity category \( \mathcal{C}^{\text{op}} \) of \( \mathcal{C} \) implies the opposite second part (2). □

**Proposition 4.9.** Let \( \sigma = (A, Z) \) be a reasonable stability condition on \( h(\mathcal{C}) \).

1. For an object \( f \in \mathcal{C}^{\Delta^1} \), \( f \) is \( d_0^*\sigma \)-semistable in phase \( \phi \) if and only if \( d_0 f \) and \( \cof f \) are \( \sigma \)-semistable in phase \( \phi \).

2. For an object \( f \in \mathcal{C}^{\Delta^1} \), \( f \) is \( d_1^*\sigma \)-semistable in phase \( \phi \) if and only if \( d_1 f \) and \( \fib f \) are \( \sigma \)-semistable in phase \( \phi \).

**Proof.** “Only if” part follows from Proposition 4.8. “If” part is a direct consequence of Lemma 4.4 as follows. Lemma 4.4 implies both \( s(y) \) and \( j!(\fib f) \) are \( d_0^*\sigma \)-semistable. Since \( f \) is given by the extension of \( s(y) \) and \( j!(\fib f) \), \( f \) is \( d_0^*\sigma \)-semistable. This give the proof of the first assertion and the proof of the second assertion is similar. □

**Proposition 4.10.** Let \( \sigma = (A, Z) \) be a reasonable stability condition on \( h(\mathcal{C}) \). The stability condition \( \sigma \) satisfies the support property, if and only if \( d_0^*\sigma \) (resp. \( d_1^*\sigma \)) satisfies the support property.

**Proof.** We only prove the assertion for \( d_0^* \) since the proof is similar.

Fix a norm \( \| - \| \) on \( K_0(h(\mathcal{C})) \otimes \mathbb{R} \). The Grothendieck group \( K_0(h(\mathcal{C}^{\Delta^1})) \) is isomorphic to \( K_0(h(\mathcal{C}))^{\otimes 2} \) since \( h(\mathcal{C}^{\Delta^1}) \) has a semiorthogonal decomposition by \( h(\mathcal{C}) \). Consider the semiorthogonal decomposition \( h(\mathcal{C}^{\Delta^1}) = (D_0^L, D_0^R) \). Let \( \| - \|_{d_0} \) be a norm induced from the norm \( \| - \| \) on \( K_0(h(\mathcal{C})) \otimes \mathbb{R} \), that is, \( \| f \|_{d_0} := \sqrt{\|d_0 f\|^2 + \|\cof f\|^2} \).

By Proposition 4.9, \( f \in \mathcal{C}^{\Delta^1} \) is \( d_0^*\sigma \)-semistable if and only if \( d_0 f \) and \( \cof f \) are \( \sigma \)-semistable in the same phase. Hence a \( d_0^*\sigma \)-semistable object \( f \) satisfies
\[
|d_0^*Z(f)| = |Z(d_0 f) + Z(\cof f)|
= |Z(d_0 f)| + |Z(\cof f)|
\geq \frac{1}{C}(\|d_0 f\| + \|\cof f\|)
\geq \frac{1}{C}\| f \|_{d_0}.
\]

In the first inequality above we use the support property for \( \sigma \). Hence \( d_0^*\sigma \) satisfies the support property.

Conversely suppose that \( d_0^*\sigma \) satisfies the support property. Then there is a constant \( C' \) such that \( C'\|Z(f)\| \geq \| f \|_{d_0} \) for any \( d_0^*\sigma \)-semistable object in \( h(\mathcal{C}^{\Delta^1}) \). Take \( f \) as the image \( s(x) \) of \( s: \mathcal{C} \to \mathcal{C}^{\Delta^1} \) where \( x \in \mathcal{C} \) is \( \sigma \)-semistable. Since \( s(x) \) is also \( d_0^*\sigma \)-semistable and the fact \( Z(x) = d_0^*Z(s(x)) \), we have \( C'|d_0^*Z(s(x))| \geq \| s(x) \|_{d_0} = \| x \| \). Hence \( \sigma \) satisfies the support property. □

**Theorem 4.11.** Let \( \sigma \) be a reasonable stability condition on \( h(\mathcal{C}) \). Then the following are equivalent:

1. \( \sigma \) is full, (2) \( d_0^*\sigma \) is full, and (3) \( d_1^*\sigma \) is full.

**Proof.** By Proposition 4.10 both \( d_1^*\sigma \) and \( d_0^*\sigma \) satisfy the support property if so does \( \sigma \). By Lemma 3.2, the support property on a stability condition is equivalent to the fullness of it. Thus we complete the proof. □
4.3. **On the images** $\text{Im} d_0^* \text{ and } \text{Im} d_1^*$. Now we prove that both images $\text{Im} d_0^*$ and $\text{Im} d_1^*$ are closed in $\text{Stab}^* h(\mathcal{E}^A)$ by using the “inducing” construction due to [20]. Let us recall the construction.

Let $F: D \to D'$ be an exact functor between triangulated categories. Assume that $F$ satisfies the following additional condition

\[(\text{Ind}) \quad \text{Hom}_D(F(a), F(b)) = 0 \text{ implies } \text{Hom}_D(a, b) = 0 \text{ for any } a, b \in D.\]

**Remark 4.12.** Recall functors $j_!$ and $j_*: h(\mathcal{E}) \to h(\mathcal{E}^A)$ in (2.4). Then three functors $s, j_!$ and $j_*$ from $h(\mathcal{E})$ to $h(\mathcal{E}^A)$ satisfy the condition (Ind) since they are faithful.

Let $\sigma' = (Z', \mathcal{P}') \in \text{Stab} D'$. Define $F^{-1}\sigma'$ by the pair $(Z, \mathcal{P})$ where

\[Z = Z' \circ F, \quad \mathcal{P}(\phi) = \{ x \in D \mid F(x) \in \mathcal{P}'(\phi) \}.\]

Then one can easily see that the pair $F^{-1}\sigma'$ is a stability condition on $D$ if and only if $F^{-1}\sigma'$ has the Harder-Narasimhan property.

**Lemma 4.13** ([20, Lemmas 2.8 and 2.9]). *Notation is the same as above.*

1. The set

\[\text{Dom}(F^{-1}) = \{ \sigma' \in \text{Stab} D' \mid F^{-1}\sigma' \in \text{Stab} D \}\]

is closed in $\text{Stab} D'$.

2. The map $F^{-1}: \text{Dom}(F^{-1}) \to \text{Stab} D$ is continuous.

**Remark 4.14.** We note that $F^{-1}\sigma'$ is reasonable if $\sigma \in \text{Stab} D'$ is reasonable. Hence the restriction of $F^{-1}$ to reasonable stability conditions is well-defined.

By using Lemma 4.13 we show that both images $\text{Im} d_0^*$ and $\text{Im} d_1^*$ are closed.

**Theorem 4.15.** Let $\mathcal{E}$ be a stable infinity category. Then

1. Images $\text{Im} d_0^*$ and $\text{Im} d_1^*$ are given by

\[(4.6) \quad \text{Im} d_0^* = \{ \sigma \in \text{Stab}^* h(\mathcal{E}^A) \mid \sigma \in \text{Dom}(s^{-1}) \cap \text{Dom}(j_!^{-1}), \quad s^{-1}\sigma = [1] \cdot j_!^{-1}\sigma \} \text{ and} \]

\[\text{Im} d_1^* = \{ \sigma \in \text{Stab}^* h(\mathcal{E}^A) \mid \sigma \in \text{Dom}(s^{-1}) \cap \text{Dom}(j_*^{-1}), \quad s^{-1}\sigma = [-1] \cdot j_*^{-1}\sigma \}.\]

2. Both $\text{Im} d_0^*$ and $\text{Im} d_1^*$ are closed.

**Proof.** Since the proof is similar we only prove the assertion for $d_0^*$.

Let $\tau$ be in $\text{Stab}^* h(\mathcal{E})$ and put $\sigma = d_0^* \tau$. Take the Harder-Narasimhan filtration of an object $x \in h(\mathcal{E})$ with respect to $\tau$:

\[
\begin{array}{cccccc}
0 & \rightarrow & x_1 & \rightarrow & x_2 & \cdots & \rightarrow & x_{n-1} & \rightarrow & x_n = x \\
& \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow & \\
& a_1 & a_2 & & a_{n-1} & & & a_n. & \\
\end{array}
\]

Then $s(a_i)$ is $d_0^* \tau$-semistable by Lemma 4.4. Since the Harder-Narasimhan filtration is unique, the value of the filtration by the functor $s: h(\mathcal{E}) \to h(\mathcal{E}^A)$ gives the Harder-Narasimhan filtration of $s(x)$ with respect to $d_0^* \tau$. Thus $s^{-1}\sigma$ has the Harder-Narasimhan property and $\sigma \in \text{Dom}(s^{-1})$.

The same argument for $j_!$ implies that $\text{Im} d_0^*$ is a subset of $\text{Dom}(j_!^{-1})$. The condition $s^{-1}\sigma = [1] \cdot j_!^{-1}\sigma$ is obvious. Thus $\text{Im} d_0^*$ is a subset of the right hand side in (4.6).

Conversely take $\sigma = (Z, \mathcal{P})$ from the right hand side in (4.6). Put $\tau = (W, Q) = s^{-1}\sigma = [1]j_!^{-1}\sigma$. We wish to prove $d_0^* \tau = \sigma$.

Any object $f \in h(\mathcal{E}^A)$ has the following decomposition:

\[(4.7) \quad j_! \circ \text{fib}(f) \rightarrow f \rightarrow s \circ d_0(f) \rightarrow j_! \circ \text{fib}(f)[1].\]
Hence we have $Z(f) = Z(j_1 \circ \text{fib}(f)) + Z(s \circ d_0(f))$. Moreover we see $Z(s \circ d_0(f)) = W(d_0f)$ since $W$ is the central charge of $s^{-1} \sigma$. Similarly we have $Z(j_1 \circ \text{fib}(f)) = W(\text{fib} f[1])$. Thus we see

$$Z(f) = Z(j_1 \circ \text{fib}(f)) + Z(s \circ d_0(f)) = W(\text{fib} f[1]) + W(s \circ d_0(f)) = d_0^* W(f).$$

Hence $d_0^* \tau$ and $\sigma$ have the same central charge.

Thus it is enough to show that $d_0^* \tau$ and $\sigma$ have the same heart. Take an object $f \in d_0^* Q(\phi)$. Then Proposition 4.8 implies $\text{cof} f$ and $d_0f$ are both $\tau$-semistable in phase $\phi$. Since $\tau$ is $s^{-1} \sigma$, the object $s \circ d_0(f)$ is in $P(\phi)$. Similarly the equality $\tau = [1]j^{-1}_1 \sigma$ implies that $j_1(\text{cof} f)$ is in $P(\phi + 1)$. In particular we see $j_1(\text{fib} f) \in P(\phi)$.

Thus $s \circ d_0(f)$ and $j_1(\text{fib} f)$ are both $\tau$-semistable in the phase $\phi$ if $f \in d_0^* Q(\phi)$. Hence $f$ is $\tau$-semistable in phase $\phi$ by (4.7) and we see $d_0^* Q(\phi) \subset P(\phi)$. Then the heart of the $t$-structure of $d_0^* \tau$ is contained in the heart of $\sigma$. Then both hearts should be the same since they are hearts of bounded $t$-structures on $h(\mathbb{C}^{\Delta^1})$. Thus we prove the assertion (4.6).

Define $\pi : \text{Stab}' h(\mathbb{C}^{\Delta^1}) \to \text{Stab}' h(\mathbb{C}) \times \text{Stab}' h(\mathbb{C})$ by $\pi(\sigma) = (s^{-1} \sigma, [1] \cdot j^{-1}_1 \sigma)$. Then $\pi$ is continuous by Lemma 4.13. The right hand side in (4.6) is the inverse image of the diagonal $\Delta_{\text{Stab}}$ by the map $\pi$. Since the diagonal is closed, $\pi^{-1} \Delta_{\text{Stab}} = \text{Im} d_0^*$ is closed. \qed

5. An example of $\mathbb{C}^{\Delta^1}$

The aim of this section is the proof of Theorem 1.4.

**Definition 5.1** (Construction of path). Put $g_\theta \in \tilde{\text{GL}}_2(\mathbb{R})$ by

$$(5.1) \quad g_\theta = (\exp(\sqrt{-1} \pi \theta), f(t) = t + \theta) : \theta \in [0, 1).$$

Suppose that a reasonable stability condition $\sigma = (\mathcal{A}, Z)$ on $h(\mathbb{C})$ satisfies the following condition:

**Deg** The image of $Z : K_0(h(\mathbb{C})) \to \mathbb{C}$ is contained in the subset $\mathbb{R}$ of real numbers in $\mathbb{C}$.

(1) Let $h(\mathbb{C}^{\Delta^1}) = (D_0^L, D_0^R)$ be the semiorthogonal decomposition in Lemma 2.14. We define $d_0^*(\theta)\sigma$ by $\text{gl}(\sigma, [-1] \sigma g_\theta) = \text{gl}(\sigma, \sigma g_{-1+\theta})$. The central charge and the heart of $d_0^*(\theta)\sigma$ are respectively denoted by $Z_0^\theta$ and $\mathcal{A}_0^\theta$.

(2) Let $h(\mathbb{C}^{\Delta^1}) = (D_1^L, D_1^R)$ be the semiorthogonal decomposition in Lemma 2.14. We define $d_1^*(\theta)\sigma$ by $\text{gl}([1] \sigma, \sigma g_\theta)$. The central charge and the heart of $d_1^*(\theta)\sigma$ are respectively denoted by $Z_1^\theta$ and $\mathcal{A}_1^\theta$.

**Lemma 5.2.** Let $\sigma$ be a stability condition on $h(\mathbb{C})$. If $\sigma$ satisfies the condition (Deg) then any object in the heart of $\sigma$ is $\sigma$-semistable.

**Proof.** The condition (Deg) implies that any object in $\mathcal{A}$ has phase 1. Then any object in $\mathcal{A}$ is $\sigma$-semistable by the definition of stability conditions. \qed

**Remark 5.3.** If $\theta = 0$, then $d_0^*(0)\sigma$ and $d_1^*(0)\sigma$ are the same as respectively $d_0^*\sigma$ and $d_1^*\sigma$.

The condition (Deg) is necessary since it guarantees the conditions (1) and (2) in Proposition 4.3. We note that these conditions fail if $\theta = 1$.

**Lemma 5.4.** Suppose that a reasonable stability condition $\sigma = (\mathcal{A}, Z) \in \text{Stab}(h(\mathbb{C}))$ satisfies the condition (Deg).

(1) Then any object $j_*(x) = [0 \to x]$ for $x \in \mathcal{A}$ is $d_0^*(\theta)\sigma$-semistable for any $\theta \in [0, 1)$.

(2) Then any object $j_!(x) = [x \to 0]$ for $x \in \mathcal{A}$ is $d_1^*(\theta)\sigma$-semistable for any $\theta \in [0, 1)$.

**Proof.** Since the proof is similar, we only prove the first assertion.

We first note that the heart $\mathcal{A}_0^\theta$ is constant for any $\theta \in [0, 1)$, since the stability condition $\sigma = (\mathcal{A}, Z)$ on $h(\mathbb{C})$ satisfies the condition (Deg). If $x \in \mathcal{A}$, Lemma 4.6 implies that the object $j_*(x) \in \mathbb{C}^{\Delta^1}$ is in $\mathcal{A}_0^\theta$ since $\mathcal{A}_0^\theta$ is constant.
We can see that an object \([g: y \to z] \in A^0 \) is a subobject of \(j_*(x)\) if and only if \(z \subset \operatorname{cof} g \subset x \in A\) by \(3 \times 3\) lemma and Lemma 2.11. Moreover \(y\) is given by \((\operatorname{cof} g/z)[-1]\) by if \(g \subset j_*(x)\). The semiorthogonal decomposition \(h(C^\Delta) = (D^0_\alpha, D^1_\beta)\) implies the following exact sequence:

\[
\begin{array}{ccc}
\operatorname{cof} g[-1] & \longrightarrow & (\operatorname{cof} g/z)[-1] \\
\downarrow h & & \downarrow g \\
0 & \longrightarrow & z \\
\end{array}
\]

Put \(Z(z) = -\beta, Z(x) = -\alpha, Z(\operatorname{cof} g) = -\gamma\) where \(\alpha, \beta\) and \(\gamma\) are positive real numbers. Then we have

\[
Z^0_\alpha(g) = Z^0_\alpha(h) + Z^0_\alpha(\operatorname{id}_x) = -\beta - \gamma \cdot \exp(-\pi \sqrt{-1} \theta)
\]

\[
Z^0_\beta(j_*(x)) = Z^0_\beta(\operatorname{id}_x) + Z^0_\beta(x[-1] \to 0) = -\alpha - \alpha \exp(-\pi \sqrt{-1} \theta).
\]

Since \(z \subset \operatorname{cof} g \subset x\), we have \(\beta \leq \gamma \leq \alpha\) which simply implies the inequality \(\arg Z^0_\beta(j_*(x)) \leq \arg Z^0_\beta(g)\) as follows. In fact it is enough to show that the imaginary part of \(Z^0_\beta(j_*(x))Z^0_\alpha(g)\) is non-negative where \(Z^0_\beta(g)\) is a complex conjugate of \(Z^0_\alpha(g)\). Since the imaginary part is given by \(\alpha(\gamma - \beta) \sin(\pi \theta)\), we finish the proof.

**Lemma 5.5.** Let \(d^*_0(\theta)\sigma\) and \(d^*_1(\theta)\sigma\) be the stability conditions on \(h(C^\Delta)\) defined in Definition 5.1. Assume \(\theta \neq 0\).

(1) Any object \([f: x \to y] \in h(C^\Delta)\) in the heart \(A^0\) has the Harder-Narasimhan filtration as follows:

\[
\begin{array}{ccc}
\ker \delta & \longrightarrow & \ker \delta \\
\downarrow \operatorname{id} & & \downarrow \ker \delta \to y \\
[0 \to \operatorname{im} \delta] & & [x \to y] \\
\end{array}
\]

where \(\operatorname{im} \delta, \ker \delta, \operatorname{cok} \delta\) are respectively the image, the kernel and the cokernel of the morphism \([\delta: y \to \operatorname{cof} f]\).

(2) Any object \([f: x \to y] \in h(C^\Delta)\) in the heart \(A^0\) has the Harder-Narasimhan filtration as follows:

\[
\begin{array}{ccc}
[0 \to \ker \epsilon[1]] & \longrightarrow & [\operatorname{im} \epsilon \to \ker \epsilon[1]] \\
\downarrow \operatorname{im} \epsilon & & \downarrow [x \to y] \\
[\operatorname{im} \epsilon \to 0] & & [\operatorname{cok} \epsilon \longrightarrow \operatorname{cok} \epsilon] \\
\end{array}
\]

where \(\operatorname{im} \epsilon, \ker \epsilon, \operatorname{cok} \epsilon\) are respectively the image, the kernel and the cokernel of the morphism \([\epsilon: \operatorname{fib} f \to x]\).

**Proof.** We only prove the first assertion by the same reason in Lemma 5.4.

There is a distinguished triangle in \(h(C)\):

\[
\ker \delta \longrightarrow x \longrightarrow \operatorname{cok} \delta[-1]
\]

The construction of \(d^*_0(\theta)\sigma\) implies that \(\operatorname{id}_{\ker \delta}\) is semistable in \(d^*_0(\theta)\sigma\) with phase 1 and that \([\operatorname{cok} \delta[-1] \to 0]\) is \(d^*_0(\theta)\sigma\)-semistable in phase \(1 - \theta\) by Lemma 5.2.
The following diagram and Lemma 2.11
\[
\begin{array}{ccc}
\ker \delta & \xrightarrow{f_1} & x \\
\downarrow y & f & \downarrow \alpha_2 \\
y & \xrightarrow{\mu} & 0
\end{array}
\]
implies that \(\alpha_2\) is a quotient of \(f\) by Lemma 4.6, since \(\mathcal{A}_1^0\) is constant for \(\theta\).

Similarly, by the following diagram,
\[
\begin{array}{ccc}
\ker \delta & \xrightarrow{id} & \ker \delta \\
\downarrow \mu & f_1 & \downarrow \alpha_1 \\
\ker \delta & \xrightarrow{\mu} & y \\
\downarrow \alpha_2 & \xrightarrow{\delta} & \im \delta
\end{array}
\]
we see that \(\id_{\ker}\) is a subobject of \(f_1\) by Lemma 4.6. Thus we obtain a filtration denoted in (5.2).

By Lemma 5.4, the morphism \([0 \to \im \delta]\) is \(d_0^*(\theta)\sigma\)-semistable. If \(\theta \neq 0\) then the phase of \([0 \to \im \delta]\) is smaller than 1 and is bigger than \(1 - \theta\). Thus the filtration in (5.2) gives an HN filtration of \(f\).

**Corollary 5.6.** Let \(d_0^*(\theta)\sigma\) be the stability condition constructed in Definition 5.1. Suppose that \(\theta \neq 0\).

1. Any semistable object in \(d_0^*(\theta)\sigma\) is one of the following:
\[
\left\{ \begin{array}{l}
[x \xrightarrow{id} x], [y[-1] \to 0], [0 \to z] \\
x, y, z \in \mathcal{A}
\end{array} \right\}
\]
2. The pair \((\mathcal{T}, \mathcal{F})\) gives a torsion pair on the heart \(\mathcal{A}_1^0\) where \(\mathcal{T}\) and \(\mathcal{F}\) are respectively
\[
\mathcal{T} = \left\{ \begin{array}{l}
[x \xrightarrow{id} x] \in \mathcal{A}_1^0 \\
x \in \mathcal{A}
\end{array} \right\}
\quad \text{and} \quad
\mathcal{F} = \left\{ [y \to z] \in \mathcal{A}_1^0 \mid y \in \mathcal{A}[-1], z \in \mathcal{A} \right\}.
\]

**Proof.** The first assertion is obvious from Lemma 5.5. For the second assertion, take any morphism \([f : x \to y] \in \mathcal{A}_1^0\) and let \(\delta\) be the morphism \(y \to \cof f\). Since \(\id_{\ker} \delta\) is a subobject of \(f\), we denote by \(e\) the quotient \(f / \id_{\ker} \delta\). Then \(d_0 e\) is isomorphic to \(\im \delta\) and \(d_1 e\) is isomorphic to \(\cok \delta[-1]\). Thus \(e\) is given by \([e : \cok \delta[-1] \to \im \delta]\). Then the object \([e : \cok \delta[-1] \to \im \delta]\) clearly is in \(\mathcal{F}\). The adjunction \(s \dashv d_1\) implies \(\Hom_{h(\mathcal{C})}((\mathcal{T}, \mathcal{F})) = 0\). Thus we conclude the proof.

**Theorem 5.7.** Suppose that an infinity category \(\mathcal{C}\) is the infinity category \(\mathbb{D}^b_{\text{coh}}(\mathbb{P}^1)\) of \(\mathbb{P}^1\) (cf. Definition 2.7). Then two distinct images \(d_1^*, d_0^* : \text{Stab}(h(\mathcal{C})) \to \text{Stab}(h(\mathcal{C}^1))\) are disjoint but path connected to each other.

**Proof.** Let \(Q\) be a quiver denoted by \(V_1 \xrightarrow{v_1} V_2\) and let \(R\) be the path algebra of \(Q\) over \(k\). The algebra \(R\) is described by exceptional collections \(\mathcal{O}, \mathcal{O}(1)\) in \(h(\mathcal{C})\); \(R \cong \text{End}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1))\). The derived category \(h(\mathcal{C})\) is equivalent to the bounded derived category \(\mathbb{D}^b(\text{mod} R)\) of \(R\) with the functor
\[
\mathbb{R} \Hom_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1), -) : h(\mathcal{C}) \to \mathbb{D}^b(\text{mod} R)
\]
Take a stability condition \(\sigma = (\mathcal{A}, \mathcal{Z})\) on \(h(\mathcal{C})\) such that any simple object in the abelian category \(\text{mod} R\) is stable in phase 1 \(^5\). Then \(\sigma\) satisfies the degenerate condition (Deg) in Definition 5.1. Moreover \(\text{Stab}(h(\mathcal{C}))\) is connected by [19] or [22]. Thus it is enough to show that \(d_0^* \sigma\) and \(d_1^* \sigma\) are path connected.

\(^5\)Equivalently take a stability condition \(\sigma = (\mathcal{A}, \mathcal{Z})\) on \(\mathbb{D}^b(\text{coh} \mathbb{P}^1)\) such that \(\mathcal{O}\) and \(\mathcal{O}(-1)[1]\) are both \(\sigma\)-stable with \(Z(\mathcal{O}) = Z(\mathcal{O}(-1)[1]) = -1\).
Step 1. By Definition 5.1, there is a collection \( \{d^\sigma_0(\theta)\sigma\}_{0<\theta<2/3} \) of stability conditions on \( h(\mathbb{C}^1) \). Since \( \sigma \) is reasonable, the collection \( \{d^\sigma_0(\theta)\sigma\}_{0<\theta<2/3} \) is a continuous family in \( \text{Stab} h(\mathbb{C}^1) \). Thus the collection is a path in \( \text{Stab} h(\mathbb{C}^1) \).

Step 2. Let \( x, y, \) and \( z \) are in the heart \( \mathcal{A} \) and set \( \theta = 2/3 \). By the construction of \( d^\sigma_0(\theta)\sigma \), objects \([id: \ x \to x]\) and \([y[-1] \to 0]\) are \( d^\sigma_0(2/3)\sigma \)-semistable in phase respectively 1 and 1/3. By Lemma 5.4, the object \([0 \to z] \in \mathcal{A}_{2/3}^0 \) is semistable in phase 2/3. Moreover the torsion pair \( (\mathcal{P}^{2/3}_0 ((2/3,1)), \mathcal{P}^{2/3}_0 ((0,2/3))) \) is just \( (\mathcal{P}^{2/3}_0 (1), \mathcal{P}^{2/3}_0 ([1/3,2/3])) \). Recall \( g_{2/3} \in \text{GL}_2(\mathbb{R}) \) defined in (5.1). We denote \( (d^\sigma_0(2/3)\sigma) \cdot g_{2/3} \) by \( \tau \). Then the heart \( B \) of \( \tau \) is \( \mathcal{P}^{2/3}_0 (2/3,5/3) \) and any \( \tau \)-semistable objects in \( B \) is one of the following:

\[
[\gamma: 0 \to z[1]], [\beta: y \to 0] \text{ and } [id_x: x \to x].
\]

We note that the phases of \( \gamma, \beta \) and \( id_x \) in \( \tau \) are respectively 1, 2/3 and 1/3.

Step 3. Take the semiorthogonal decomposition \( h(\mathbb{C}^1) = \langle D^I_1, D^R_1 \rangle \). Similarly to the case of \( d^\sigma_0(\theta)\sigma \), the collection \( \{d^\sigma_1(\theta)\sigma\}_{0<\theta<2/3} \) determines a path in \( \text{Stab} h(\mathbb{C}^1) \). The second part of Lemma 5.5 implies that any semistable object in \( d^\sigma_1(\sigma) \) is one of the following:

\[
\left\{ [0 \to x[1]], [y \to 0], [z \xrightarrow{id} z] \right\} x, y, z \in \mathcal{A} \text{ where } \sigma = (A, Z).
\]

Furthermore if \( \theta = 2/3 \) then the phases of \([0 \to x[1]], [y \to 0] \) and \([z \xrightarrow{id} z] \) are respectively 1, 2/3 and 1/3. Hence \( d^\sigma_1(2/3)\sigma \) is the same as \( \tau \) defined in Step 2. Thus we obtain a path connecting \( d^\sigma_1 \sigma \) to \( d^\sigma_0 \sigma \).

Remark 5.8. The stability condition \( \sigma \) taken in the proof above is not geometric. Namely skyscraper sheaves \( \mathcal{O}_x \) is not \( \sigma \)-stable but \( \sigma \)-semistable. If the genus of a smooth projective curve is positive, such a non-geometric stability condition does not exist.

The same argument in Theorem 1.4 is effective for the distinguished full component \( \text{Stab}^1 \mathbb{D}^b(\text{coh} \mathbb{P}^2) \) of the space of stability conditions on the bounded derived category \( \mathbb{D}^b(\text{coh} \mathbb{P}^2) \simeq h(\mathbb{D}^b_{\text{coh}}(\mathbb{P}^2)) \) of the projective plane \( \mathbb{P}^2 \). Namely we have

Corollary 5.9. Suppose an infinity category \( \mathcal{C} \) is the infinity category \( \mathbb{D}_{\text{coh}}^b(\mathbb{P}^2) \) (cf. Definition 2.7). Let \( \text{Stab}^1 h(\mathcal{C}) \) be the distinguished full component of \( \text{Stab} h(\mathcal{C}) \) and let \( d^\sigma_{0|1} \) (resp. \( d^\sigma_{1|1} \)) be the restriction of \( d^\sigma_0 \) (resp. \( d^\sigma_1 \)) to \( \text{Stab}^1 h(\mathcal{C}) \). Then \( \text{Im} d^\sigma_{0|1} \) is path connected to \( \text{Im} d^\sigma_{1|1} \).

Proof. Due to Li [16], \( \text{Stab}^1 h(\mathcal{C}) \) is the union of algebraic stability conditions and geometric stability conditions. Since \( h(\mathcal{C}) \) is equivalent to the bounded derived category of representations of the quiver \( Q = \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \). Thus there exists \( \sigma \in \text{Stab}^1 h(\mathcal{C}) \) such that any simple module in the abelian category \( \text{mod}(kQ) \) of the path algebra \( kQ \) is \( \sigma \)-stable in phase 1. Since the stability condition \( \sigma \) satisfies the condition (Deg), the same argument in Theorem 5.7 works.

6. Another Construction of \( h(\mathbb{C}^1) \)

Let \( \text{Mor}(\mathcal{B}) \) be the category of morphisms in an abelian category \( \mathcal{B} \). The category \( \text{Mor}(\mathcal{B}) \) is tautologically the same as the category of functors from the ordinal \([1] = \{0 < 1\} \) to \( \mathcal{B} \). Since \( \text{Mor}(\mathcal{B}) \) is also an abelian category, one could define the bounded derived category \( \mathbb{D}^b(\text{Mor}(\mathcal{B})) \) by the localization of quasi-isomorphisms. When \( \mathcal{B} \) is the abelian category \( \text{coh}(X) \) of coherent sheaves on \( X \), we show that the derived category \( \mathbb{D}^b(\text{Mor}(\mathcal{B})) \) is equivalent to the homotopy category \( h(\mathbb{D}^b_{\text{coh}}(X)) \) in Corollary 6.2 below.
Proposition 6.1. Suppose that $A$ is a Grothendieck abelian category. Then the unbounded derived category $D(Mor(A))$ is equivalent to the homotopy category $h(D(A)^{\Delta 1})$ of the infinity category $D(A)^{\Delta 1}$.

Proof. Throughout the proof we identify $Kom(Mor(A))$ with $Mor(Kom(A))$. Since $A$ is the Grothendieck abelian category, so is $Mor(A)$ by [13]. Thus the derived category $D(Mor(A))$ is obtained by the homotopy category $Ho(Kom(Mor(A)))$ with respect to the injective model structure on $Kom(Mor(A))$. Moreover $Kom(A)$ is also the injective model category by Proposition 2.5.

We wish to define a functor $\Phi: Ho(Kom(Mor(A))) \to h(D(A)^{\Delta 1})$ which gives an equivalence. The proof consists of 3 steps. In Step 1, we define a functor $\Phi$ and show that $\Phi$ is essentially surjective. We next show that $\Phi$ is full in Step 2 and show that $\Phi$ is faithful in Step 3.

Step 1. Let $f$ be in $Kom(Mor(A)) = Mor(Kom(A))$. Then $f$ is given by the form $[f: x \to y]$ where $x$ and $y$ are in $Kom(A)$. There exist functors between $Kom(Mor(A))$ and $Kom(A)$:

- $j_*: Kom(A) \to Kom(Mor(A))$; $j_*(z) = [0 \to z]$,
- $d_0: Kom(Mor(A)) \to Kom(A)$; $d_0([f: x \to y]) = y$,
- $s: Kom(A) \to Kom(Mor(A))$; $s(z) = [id: z \to z]$, and
- $d_1: Kom(Mor(A)) \to Kom(A)$; $d_1([f: x \to y]) = x$.

Moreover $j_*$ is the left adjoint of $d_0$ and $s$ is also the left adjoint of $d_1$. Since $j_*$ and $s$ preserve trivial cofibrations, both $d_0$ and $d_1$ preserve fibrations. Hence, if $f$ is fibrant in $Kom(Mor(A))$, then both $d_1f = x$ and $d_0f = y$ are fibrant in $Kom(A)$. Hence $f$ gives an object in $h(D(A)^{\Delta 1})$ and we define $\Phi(f)$ by $f$ itself.

We next define $\Phi([\rho])$ for a morphism $[\rho]: f \to g$ in the category $Ho(Kom(Mor(A)))$. Put $d_1g = z$ and $d_0g = w$. Recall that the morphism $[\rho]$ is a chain homotopy class of a morphism $\rho: f \to g$ in the category $Kom(Mor(A))$ and that $\rho$ is given by a pair $(\rho_1, \rho_0)$ of morphisms $\rho_1: x \to y$ and $\rho_0: y \to w$ in $Kom(A)$ satisfying $\rho_0f = g\rho_1$. On the other hand, a morphism $\Phi(f) \to \Phi(g)$ in $h(D(A)^{\Delta 1})$ is the equivalence class of a 5-tuple $\varphi = (\tau_1, \tau_0, \psi, h_1, h_0)$ satisfying the following relations (see also the proof of Proposition 2.16):

- $\tau_1 \in Z^0_{Ch(A)}(x, z)$, $\tau_0 \in Z^0_{Ch(A)}(y, w)$, $\psi \in Z^0_{Ch(A)}(x, w)$,
- $h_1 \in \text{Map}^{-1}_{Ch(A)}(x, w)$ with $\delta(h_1) = g\tau_1 - \psi$, and
- $h_0 \in \text{Map}^{-1}_{Ch(A)}(x, w)$ with $\delta(h_0) = \tau_0f - \psi$.

For the morphism $\rho: f \to g$ in $Kom(Mor(A))$, let us denote by $\phi_\rho$ the 5-tuple $(\rho_1, \rho_0, \rho_0f, 0, 0)$ which gives a morphism in $D(A)^{\Delta 1}$. We wish to define $\Phi([\rho])$ by the equivalence class $[\phi_\rho]$. To complete the definition of $\Phi$, let $\hat{\rho} = (\hat{\rho}_1, \hat{\rho}_0)$ be a morphism in $Kom(Mor(A))$ which is chain homotopic to the morphism $\rho$. Then there exists a morphism $\eta \in \text{Map}^{-1}_{Ch(Mor(A))}(f, g)$ such that $\delta(\eta) = \rho - \hat{\rho}$. In particular the morphism $\eta$ is given by a pair $(\eta_1, \eta_0)$ of morphisms $\eta_1 \in \text{Map}^{-1}_{Ch(A)}(x, z)$ and $\eta_0 \in \text{Map}^{-1}_{Ch(A)}(y, w)$ satisfying $\eta_0f = g\eta_1$ and $\delta(\eta_i) = \rho_i - \hat{\rho}_i$ for $i \in \{0, 1\}$. By the relation $\eta_0f = g\eta_1$ and the morphism $\eta$, one can find a 7-tuple which gives an equivalence between $\phi_{\rho}$ and $\phi_{\rho}$. Thus we can define $\Phi([\rho])$ by the equivalence class $[\phi_{\rho}]$ and obtain the functor $\Phi$:

$$\Phi: Ho(Kom(Mor(A))) \to h(D(A)^{\Delta 1}); \Phi(f) = f \text{ and } \Phi(\rho) = [\phi_{\rho}]$$

Take an object $f \in h(D(A)^{\Delta 1})$. Then $f$ can be regarded as an object in $Kom(Mor(A))$. By the fibrant replacement $f^\sharp$ of $f$ in $Kom(Mor(A))$, $\Phi(f^\sharp) = f^\sharp$ is isomorphic to $f$ in $h(D(A)^{\Delta 1})$. Hence the functor $\Phi$ is essentially surjective.

Step 2. We claim that any morphism $[\varphi]: \Phi(f) \to \Phi(g)$ in $h(D(A)^{\Delta 1})$ is given by $\Phi([\rho])$ for some $[\rho]: f \to g \in Ho(Kom(Mor(A)))$. To see this, let $\varphi = (\tau_1, \tau_0, \psi, h_1, h_0)$ be a morphism
\varphi: f \to g \text{ in the infinity category } \mathcal{D}(\mathcal{A})^{\Delta^1}. \text{ Then } \tau_0 f \text{ and } g \tau_1 \text{ are chain homotopic since the relation } \\
\delta(h_0 - h_1) = \tau_0 f - g \tau_1 \text{ holds.}

Take a factorization of the morphism \( g \) in \( \text{Kom}(\mathcal{A}) \) as

\begin{equation}
\begin{array}{ccc}
\tilde{z} & \xrightarrow{\text{tc}} & \tilde{z} \\
\downarrow^g & & \downarrow^g \\
& \to & \to \\
\end{array}
\end{equation}

where \( g_{\text{tc}} \) is a trivial cofibration and \( g_b \) is a fibration in \( \text{Kom}(\mathcal{A}) \). Since \( \tau_0 f \) and \( g \tau_1 \) are chain homotopic, we obtain the following commutative diagram in \( \text{Kom}(\mathcal{A}) \) by using the cylinder object \( C(x) \) of \( x \):

\begin{equation}
\begin{array}{ccc}
x & \xrightarrow{i} & C(x) \\
\downarrow^f & & \downarrow^j \\
y & \xrightarrow{\tau_0} & w \\
\end{array}
\end{equation}

Note that \( j \) is a trivial cofibration in \( \text{Kom}(\mathcal{A}) \). Since \( g_b \) is a fibration, there exists a morphism \( \tilde{H}: C(x) \to \tilde{z} \) as indicated rendering the diagram commutative in \( \text{Kom}(\mathcal{A}) \). Thus there exists a morphism \( h \in \text{Map}_{\text{Ch}(\mathcal{A})}^{-1}(x, \tilde{z}) \) such that \( g_b h = h_0 - h_1 \).

Put \( \tau_1 = g_{\text{tc}} \tau_1 + \delta(h) \) where \( \delta: \text{Map}_{\text{Ch}(\mathcal{A})}^{-1}(x, \tilde{z}) \to \text{Map}_{\text{Ch}(\mathcal{A})}^0(x, \tilde{z}) \). Then we obtain the commutative diagram in \( \text{Kom}(\mathcal{A}) \) which represents morphisms \( f \to g_b \) and \( g_b \leftarrow g \) in \( \text{Mor}(\text{Kom}(\mathcal{A})) = \text{Kom}(\text{Mor}(\mathcal{A})) \):

\begin{equation}
\begin{array}{ccc}
x & \xrightarrow{i} & C(x) \\
\downarrow^f & & \downarrow^j \\
y & \xrightarrow{\tau_0} & w \\
\end{array}
\end{equation}

Let \((g_{\text{tc}}, \text{id}_w): g \to g_b\) be the morphism indicated in the diagram above. Since the object \( g \) is fibrant and the morphism \((g_{\text{tc}}, \text{id}_w)\) is a trivial cofibration, the lifting property in \( \text{Kom}(\text{Mor}(\mathcal{A})) \) implies a morphism

\begin{equation}
(r_1, r_0): g_b \to g
\end{equation}

satisfying \((r_1, r_0) \circ (g_{\text{tc}}, \text{id}_w) = \text{id}_g\). Since \( r_0 = \text{id}_w \), one can check that \( \rho = (r_1 \tau_1, \tau_0) \) satisfies \( \Phi([\rho]) = [\varphi] \).

**Step 3.** Let \( \rho = (\rho_1, \rho_0) \) be a morphism \( \rho: f \to g \) for the objects \([f: x \to y]\) and \([g: z \to w]\) in \( \text{Ho}(\text{Kom}(\text{Mor}(\mathcal{A}))) \). We are going to show that \( \rho \) is null homotopic if \( \Phi([\rho]) \) is zero. The procedure is to find a chain homotopy, that is a pair \((h_1, h_0) \in \text{Map}_{\text{Ch}(\mathcal{A})}^{-1}(x, z) \times \text{Map}_{\text{Ch}(\mathcal{A})}^{-1}(y, w)\) satisfying \( \delta(h_1) = \rho_1, \delta(h_0) = \rho_0, \) and \( gh_1 = h_0 f \).

Unwinding the definitions of \( \phi_{\rho} \) and of \( h(\mathcal{D}(\mathcal{A}))^{\Delta^1} \), there exist morphisms \( h_{012} \in \text{Map}_{\text{Ch}(\mathcal{A})}^{-1}(x, z) \), \( h_{012} \in \text{Map}_{\text{Ch}(\mathcal{A})}^{-1}(y, w) \), and \( h_{012} \in \text{Map}_{\text{Ch}(\mathcal{A})}^{-1}(x, w[1]) \) satisfying \( \delta(h_{012}) = \rho_1, \delta(h_{012}) = \rho_0, \) and \( \delta(h_{012}) = h_{012} f - gh_{012} \). Put \( I = h_{012} f - gh_{012} \). Then the morphism \( I \) is in \( Z_{\text{Ch}(\mathcal{A})}^{-1}(x, w) = Z_{\text{Ch}(\mathcal{A})}^0(x, w[1]) \) and is null homotopic by \( h_{012} \). By the factorization of \( g \) as in (6.1), we have the following commutative diagram in \( \text{Kom}(\mathcal{A}) \):

\begin{equation}
\begin{array}{ccc}
x & \xrightarrow{i} & C(x) \\
\downarrow^l & & \downarrow^j \\
x & \xrightarrow{\text{id}} & x \\
\end{array}
\end{equation}
Similarly as in Step 2, since \( g_b[-1] \) is a fibration, there exists a morphism \( \tilde{h} \in \operatorname{Map}^{-1}_{\operatorname{Ch}(A)}(x, \tilde{z}[-1]) = \operatorname{Map}^2_{\operatorname{Ch}(A)}(x, \tilde{z}) \) such that \( g_b\tilde{h} = h_{012}\).

The pair \((g_c, h_{012} + \delta(h), h_{012})\) satisfies \( g_b\left(g_c h_{012} + \delta(h)\right) = h_{012}f \). Using the morphism \( r_1 \) in (6.2), put \( h_1 = h_{012} + r_1\delta(h) \) and \( h_0 = h_{012} \). Then the pair \((h_1, h_0)\) gives a desired chain homotopy for \( \rho: f \to g \).

**Corollary 6.2.** Suppose \( \mathcal{B} \) is the abelian category \( \operatorname{coh}(X) \) of coherent sheaves on a Noetherian scheme \( X \). The bounded derived category \( D^b(\operatorname{Mor}(\mathcal{B})) \) of \( \operatorname{Mor}(\mathcal{B}) \) is equivalent to the homotopy category \( h(D^b_{\operatorname{coh}}(X)^{\Delta^1}) \).

**Proof.** Let \( A \) be the abelian category \( \operatorname{Qcoh}(X) \) of quasi-coherent sheaves on \( X \) and let

\[
\Phi: \operatorname{Ho}(\operatorname{Kom}(\operatorname{Mor}(A))) \to h(D(A)^{\Delta^1})
\]

be the functor constructed in the proof of Proposition 6.1.

Define \( D^b_A(\operatorname{Mor}(A)) \) to be the full subcategory of \( D(\operatorname{Mor}(A)) = \operatorname{Ho}(\operatorname{Kom}(\operatorname{Mor}(A))) \) consisting of the bounded complexes with coherent cohomologies:

\[
D^b_A(\operatorname{Mor}(A)) = \{ [f]: x \to y \in D(\operatorname{Mor}(A)) \mid x \text{ and } y \in D^b_{\operatorname{coh}}(X) \}.
\]

Then \( D^b_A(\operatorname{Mor}(A)) \) is equivalent to \( h(D^b_{\operatorname{coh}}(A)^{\Delta^1}) \) via the functor \( \Phi \). Thus it is enough to show that \( D^b_A(\operatorname{Mor}(A)) \) is equivalent to \( D^b(\operatorname{Mor}(\mathcal{B})) \).

Let \( g \to f \) be an epi morphism in \( \operatorname{Mor}(A) \) such that \( f \in \operatorname{Mor}(\mathcal{B}) \) and \( g \in \operatorname{Mor}(A) \). Then there exists a subobject \( z \subseteq d_1g \) and \( w \subseteq d_0g \) such that the composites \( z \to d_1g \to d_1f \) and \( w \to d_0g \to d_0f \) are epi and \( z \) and \( w \) are in \( \mathcal{B} \). Let \( \tilde{w} \) be the subobject of \( d_0g \) generated by \( \operatorname{im}(g|_z) \) and \( w \). Then a morphism \( h: z \to \tilde{w} \) is a subobject of \( g \) in \( \operatorname{Mor}(\mathcal{B}) \), and the composite \( h \to g \to f \) is also an epi morphism. Hence we see that the natural functor \( D^b(\operatorname{Mor}(\mathcal{B})) \to D^b_A(\operatorname{Mor}(A)) \) is essentially surjective. Then the functor gives an equivalence by [10, Theorem B].

**Remark 6.3.** Keep the notation as in Corollary 6.2.

1. Suppose that \( X \) is a smooth projective curve \( C \) over \( \mathbb{C} \). Then \( D^b(\operatorname{Mor}(\mathcal{B})) \) is the same as the category \( \mathcal{T}_C \) discussed in [21]. Hence our category \( h(D^b_{\operatorname{coh}}(C)^{\Delta^1}) \) is equivalent to \( \mathcal{T}_C \). Consequently, Proposition 2.17 gives an answer to [21, Conjecture 3.17] related with the Serre functor on \( \mathcal{T}_C \).

2. If \( X \) is the affine scheme \( \operatorname{Spec} \mathbb{k} \) of a field \( \mathbb{k} \), then \( \operatorname{Mor}(\mathcal{B}) \) is nothing but the abelian category \( \operatorname{mod}(\bullet \to \bullet) \) of finite dimensional representations of the \( A_2 \)-quiver \( \bullet \to \bullet \). Hence our category \( h(D^b_{\operatorname{coh}}(\operatorname{Spec} \mathbb{k})^{\Delta^1}) \) is equivalent to the bounded derived category \( D^b(\operatorname{mod}(\bullet \to \bullet)) \) of the abelian category \( \operatorname{mod}(\bullet \to \bullet) \). Thus the homotopy category \( h(D^b_{\operatorname{coh}}(X)^{\Delta^1}) \) is a generalization of the bounded derived category \( D^b(\operatorname{mod}(\bullet \to \bullet)) \) of quiver representations.

3. From a geometrical view, a quiver is nothing but a simplicial set \( K \) whose arbitrary \( n \)-simplices degenerate for \( n > 1 \). Let \( D^b_{\operatorname{coh}}(X)^K = \operatorname{Fun}(K, D^b_{\operatorname{coh}}(X)) \) be the infinity category of maps from any simplicial set \( K \) to the infinity category \( D^b_{\operatorname{coh}}(X) \). Then \( D^b_{\operatorname{coh}}(X)^K \) can be regarded as a further generalization of the bounded derived category of quiver representations. It might be interesting to study \( D^b_{\operatorname{coh}}(X)^K \) from a perspective of the representation theory.

**References**

[1] Daniele Arcara and Aaron Bertram. Bridgeland-stable moduli spaces for \( K \)-trivial surfaces. *J. Eur. Math. Soc. (JEMS)*, 15(1):1–38, 2013. With an appendix by Max Lieblich.

[2] Arend Bayer and Emanuele Macri. The space of stability conditions on the local projective plane. *Duke Math. J.*, 160(2):263–322, 2011.
[3] Arend Bayer, Emanuele Macrì, and Yukinobu Toda. Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities. *J. Algebraic Geom.*, 23(1):117–163, 2014.

[4] A. I. Bondal. Representations of associative algebras and coherent sheaves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(1):25–44, 1989.

[5] Alexey I. Bondal and Mikhail M. Kapranov. Representable functors, Serre functors, and reconstructions. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(6):1183–1205, 1337, 1989.

[6] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. of Math. (2)*, 166(2):317–345, 2007.

[7] Tom Bridgeland. Stability conditions on K3 surfaces. *Duke Math. J.*, 141(2):241–291, 2008.

[8] Tom Bridgeland. Stability conditions and Kleinian singularities. *Int. Math. Res. Not. IMRN*, (21):4142–4157, 2009.

[9] Egbert Brieskorn. Sur les groupes de tresses. pages 21–44. Lecture Notes in Math., Vol. 317, 1973.

[10] Xiao-Wu Chen, Zhe Han, and Yu Zhou. Derived equivalences via HRS-tilting. *Adv. Math.*, 354:106749. 26, 2019.

[11] John Collins and Alexander Polishchuk. Gluing stability conditions. *Adv. Theor. Math. Phys.*, 14(2):563–607, 2010.

[12] George Dimitrov and Ludmil Katzarkov. Some new categorical invariants. *Selecta Math. (N.S.)*, 25(3):Art. 45, 60, 2019.

[13] Alexander Grothendieck. Sur quelques points d’algèbre homologique. *Tohoku Math. J. (2)*, 9:119–221, 1957.

[14] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[15] Akira Ishii, Kazushi Ueda, and Hokuto Uehara. Stability conditions on $A_n$-singularities. *J. Differential Geom.*, 84(1):87–126, 2010.

[16] Chunyi Li. The space of stability conditions on the projective plane. *Selecta Math. (N.S.)*, 23(4):2927–2945, 2017.

[17] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[18] Jacob Lurie. *Higher Algebra*. available on https://www.math.ias.edu/~lurie/, 2017.

[19] Emanuele Macrì. Stability conditions on curves. *Math. Res. Lett.*, 14(4):657–672, 2007.

[20] Emanuele Macrì, Sukhendu Mehrotra, and Paolo Stellari. Inducing stability conditions. *J. Algebraic Geom.*, 18(4):605–649, 2009.

[21] Eva Martínez-Romero, Alejandra Rincón-Hidalgo, and Arne Rüffer. Bridgeland stability conditions on the category of holomorphic triples over curves. arXiv e-prints, page arXiv:1905.04240, May 2019.

[22] So Okada. Stability manifold of $\mathbb{P}^1$. *J. Algebraic Geom.*, 15(3):487–505, 2006.

[23] Luis Paris. $K(\pi,1)$ conjecture for Artin groups. *Ann. Fac. Sci. Toulouse Math. (6)*, 23(2):361–415, 2014.

[24] André Weil. Sur les théorèmes de de Rham. *Comment. Math. Helv.*, 26:119–145, 1952.