Throughout this paper, \( S \) will be a ring (not necessarily commutative) with an identity element \( 1_s \neq 0_s \). We shall use \( R \) to denote a second ring, and \( \phi: S \to R \) will be a fixed ring homomorphism for which \( \phi 1_s = 1_R \).

1. Introduction. In (7), Higman generalized the Casimir operator of classical theory and used his generalization to characterize relatively projective and injective modules. As a special case, he obtained a theorem which contains results of Eckmann (3) and of Higman himself (5), and which also includes Gaschütz’s generalization (4) of Maschke’s theorem. (For a discussion of some of the developments of Maschke’s idea of averaging over a finite group, we refer the reader to (2, Chapter IX).) In the present paper, we define the Casimir operator of a family of \( S \)-homomorphisms of one \( R \)-module into another, and we again use this operator to characterize relatively projective and injective modules. In § 4, we give some special cases, the first of which covers the result of Higman (7) referred to above.

In § 5, we extend (7, Theorem 6) for a special class of pairs \( R, S \). Our result contains a theorem of Popescu (9, Proposition 1.3) which in turn generalizes a result of Cartan and Eilenberg (1, Chapter IV, Proposition 2.3) on the ring of dual numbers.

2. Relatively projective and injective modules. An abelian group \( M \) which is both a left and a right \( S \)-module and for which

\[(su)s' = s(us'), \quad s, s' \in S, \quad u \in M,
\]

will be referred to as an \( S \)-bimodule.

A left \( R \)-module \( M \) may be treated as a left \( S \)-module by putting

\[su = (\phi s)u, \quad s \in S, \quad u \in M,
\]

and similarly for right modules. In particular, \( R \) itself may be regarded as a left or right \( S \)-module.

When \( M \) is a left \( S \)-module and \( X \) is an \( S \)-bimodule, the tensor product \( X \otimes_S M \) may be considered as a left \( S \)-module by taking

\[s(x \otimes u) = sx \otimes u, \quad s \in S, \quad x \in X, \quad u \in M.
\]
Furthermore, the abelian group $\text{Hom}_S(X, M)$ of $S$-homomorphisms of $X$ into $M$ may be regarded as a left $S$-module by putting

$$(sf)x = f(xs), \quad s \in S, f \in \text{Hom}_S(X, M), \quad x \in X.$$ 

Suppose now that $M$ is a left $S$-module; it follows from what we said above that $R \otimes_S M$ may be considered as a left $S$-module. In fact, it may be regarded as a left $R$-module by taking, in addition,

$$r_1(r_2 \otimes u) = r_1 r_2 \otimes u, \quad r_1, r_2 \in R, \quad u \in M.$$ 

When $M$ is a left $R$-module, the mapping

$$t: R \otimes_S M \rightarrow M$$

given by the relation

$$t(r \otimes u) = ru$$

is easily checked to be an $R$-homomorphism. If $\kappa: M \rightarrow R \otimes_S M$ is the $S$-homomorphism under which $u \rightarrow 1_R \otimes u$, then the composition

$$\kappa t: M \rightarrow R \otimes_S M \rightarrow M$$

is the identity mapping, which proves that $\ker t$ is an $S$-direct summand of $R \otimes_S M$.

**Definition.** The left $R$-module $M$ will be said to be $\phi$-projective if $\ker t$ is an $R$-direct summand of $R \otimes_S M$. Clearly, if $M$ is $R$-projective, then it is $\phi$-projective. Our first result forms part of (1, Chapter II, Proposition 6.3).

(1) **Theorem.** For any left $S$-module $M$, $R \otimes_S M$ is $\phi$-projective.

If $M$ is a left $S$-module, then the left $S$-module $\text{Hom}_S(R, M)$ may be regarded as a left $R$-module by setting

$$(r_1f)r_2 = f(r_2r_1), \quad r_1, r_2 \in R, \quad f \in \text{Hom}_S(R, M).$$

When $M$ is a left $R$-module, the mapping $t': M \rightarrow \text{Hom}_S(R, M)$ for which $(t'u)r = ru$ is an $R$-homomorphism, and if $\kappa': \text{Hom}_S(R, M) \rightarrow M$ is the $S$-homomorphism under which $f \rightarrow f1_R$, then the composition

$$t' \text{Hom}_S(R, M) \rightarrow M$$

is the identity mapping, which proves that $\text{Im} t'$ is an $S$-direct summand of $\text{Hom}_S(R, M)$.

**Definition.** The left $R$-module $M$ is said to be $\phi$-injective if $\text{Im} t'$ is an $R$-direct summand of $\text{Hom}_S(R, M)$.

If $M$ is $R$-injective, then it is obviously $\phi$-injective. Dual to (1) we have the following result.
(1') Theorem. For any left $S$-module $M$, $\text{Hom}_S(R, M)$ is $\phi$-injective.

If $M$ is a left $R$-module and there exists an $R$-isomorphism $\text{Hom}_S(R, M) \cong R \otimes_S M$, then $M$ is $\phi$-projective if and only if it is $\phi$-injective.

3. Casimir operators. Throughout this paper, $I$ will denote an indexing set and $\{R_i\}_{i \in I}, \{R'_i\}_{i \in I}$ will be families of $S$-bimodules which are contained in $R$.

Definition. Let $M$ and $H$ be left $S$-modules and $R$-modules, respectively. For each $i \in I$, an $S$-homomorphism $\delta_i: R_i \otimes_S M \rightarrow H$ will be said to be quasi-$R$-linear if

$$\delta_i(rr' \otimes u) = r\delta_i(r' \otimes u)$$

whenever $r \in R, r', rr' \in R_i$, and $u \in M$.

Dually, an $S$-homomorphism $\bar{\epsilon}_i: H \rightarrow \text{Hom}_S(R'_i, M)$ will also be said to be quasi-$R$-linear if

$$(\bar{\epsilon}_i h)r' = (\bar{\epsilon}_i h)(r'r)$$

whenever $r \in R, h \in H, r', r'r' \in R'_i$.

For each $i \in I$, we suppose that to every left $S$-module $M$ there corresponds an $S$-homomorphism

$$\kappa_i: M \rightarrow R_i \otimes_S M$$

which is such that, if $H$ is a left $R$-module and $\delta_i: M \rightarrow H$ is an $S$-homomorphism, then there exists a unique quasi-$R$-linear homomorphism $\delta_i: R_i \otimes_S M \rightarrow H$ for which $\delta_i = \delta_i \kappa_i$, i.e. for which the diagram

\[
\begin{array}{ccc}
R_i \otimes_S M & \xrightarrow{\kappa_i} & M \\
\downarrow{\delta_i} & & \downarrow{\delta_i} \\
M & \xrightarrow{\delta_i} & H
\end{array}
\]

is commutative.

We shall also suppose that, for each $i \in I$, there corresponds to every $M$ an $S$-homomorphism

$$\kappa'_i: \text{Hom}_S(R'_i, M) \rightarrow M$$

which is such that, if $H$ is a left $R$-module and $\epsilon_i: H \rightarrow M$ is an $S$-homomorphism, then there exists a unique quasi-$R$-linear homomorphism $\bar{\epsilon}_i: H \rightarrow \text{Hom}_S(R'_i, M)$ for which $\epsilon_i = \kappa'_i \bar{\epsilon}_i$, i.e. for which the diagram

\[
\begin{array}{ccc}
\text{Hom}_S(R'_i, M) & \xrightarrow{\kappa'_i} & M \\
\downarrow{\bar{\epsilon}_i} & & \downarrow{\epsilon_i} \\
M & \xleftarrow{\epsilon_i} & H
\end{array}
\]

is commutative.
Let $M$ be a fixed left $R$-module, and, for each $i \in I$, let

$$\rho_i: R_i \otimes_S M \to R \otimes_S M, \quad \rho'_i: \text{Hom}_S(R, M) \to \text{Hom}_S(R'_i, M)$$

be the $S$-homomorphisms induced by the inclusion mappings

$$R_i \to R, \quad R'_i \to R,$$

respectively.

We shall suppose that, for each $i \in I$, there exists an $S$-homomorphism

$$\lambda_i: \text{Hom}_S(R'_i, M) \to R_i \otimes_S M.$$

**Definitions.** Let $M$ and $N$ be left $R$-modules and let $\{\alpha_i\}_{i \in I}$ be a family of $S$-homomorphisms of $N$ into $M$. If, for each $v \in N$, $\rho_i \lambda_i \alpha_i \theta = 0$ for almost all $i$, then the $S$-homomorphism

$$\sum_{i \in I} \rho_i \lambda_i \alpha_i: N \to M,$$

is called a *first Casimir operator* of the family $\{\alpha_i\}_{i \in I}$ and is denoted by $c(\alpha_i)$.

Again, let $\{\beta_i\}_{i \in I}$ be a family of $S$-homomorphisms of $M$ into $N$; if, for each $u \in M$, $\beta_i \lambda_i \rho_i \theta' u = 0$ for nearly all $i$, then the $S$-homomorphism

$$\sum_{i \in I} \beta_i \lambda_i \rho_i \theta': M \to N$$

is called a *second Casimir operator* of $\{\beta_i\}_{i \in I}$ and is denoted by $c'(\beta_i)$. (The terminology is that used in (8, § 8); for a justification of the use of “Casimir operator”, see the Remark following (4) in § 4.)

**Note.** The sets $\{\alpha_i\}_{i \in I}$ and $\{\beta_i\}_{i \in I}$ possess first and second Casimir operators, respectively, whenever the indexing set $I$ is finite.

(2) **Theorem.** Suppose that, as an $S$-bimodule, $R = \sum_{i \in I} R_i$ (direct sum) and let $M$ be a left $R$-module. If

(a) $M$ possesses a family $\{\alpha_i\}_{i \in I}$ of $S$-endomorphisms such that

$$\sum_{i \in I} \rho_i \lambda_i \alpha_i: M \to R \otimes_S M$$

is an $R$-homomorphism and $c(\alpha_i) = \text{id}_M$, the identity mapping of $M$,

then

(b) $M$ is $\phi$-projective.

For each $i \in I$, let $\sigma_i$ be the $S$-homomorphism $R \otimes_S M \to R_i \otimes_S M$ induced by the projection mapping $R \to R_i$. If each $\lambda_i$ is an $S$-isomorphism and each $\lambda_i^{-1} \sigma_i$ is quasi-$R$-linear, then (a) and (b) are equivalent.

**Proof.** (a) implies (b) at once. Suppose then that each $\lambda_i$ is an $S$-isomorphism, that each $\lambda_i^{-1} \sigma_i$ is quasi-$R$-linear, and that $M$ is $\phi$-projective. There exists an $R$-homomorphism $g: M \to R \otimes_S M$ such that $tg = \text{id}_M$. Let $\alpha_i = \kappa_i \lambda_i^{-1} \sigma_i g$; since $\lambda_i^{-1} \sigma_i$ is quasi-$R$-linear, then so is $\lambda_i^{-1} \sigma_i g$, and it follows that $\lambda_i = \lambda_i^{-1} \sigma_i g$. 


Hence
\[ \sum_{i \in I} \rho_i \lambda_i \alpha_i = \sum_{i \in I} \rho_i \sigma_i g = g, \]
since \( \sum_{i \in I} \rho_i \sigma_i = \text{id}_{R \otimes_S M} \). Also,
\[ c[\alpha_i] = \sum_{i \in I} t \rho_i \lambda_i \alpha_i = \sum_{i \in I} t \rho_i \sigma_i g = tg = \text{id}_M. \]

Dual to (2), we have the following result.

(2') Theorem. Let the indexing set \( I \) be finite. Suppose also that, as an \( S \)-bimodule, \( R = \sum_{i \in I} R'_i \) (direct sum), and that \( M \) is a left \( R \)-module. If
\( (a') \) \( M \) possesses a family \( \{ \beta_i \}_{i \in I} \) of \( S \)-endomorphisms such that
\[ \sum_{i \in I} \beta_i \lambda_i \rho'_i : \text{Hom}_S(R, M) \to M \]
is an \( R \)-homomorphism and \( c'[\beta_i] = \text{id}_M \), then
\( (b') \) \( M \) is \( \phi \)-injective.

For each \( i \in I \), let \( \sigma'_i : \text{Hom}_S(R'_i, M) \to \text{Hom}_S(R, M) \) be the \( S \)-homomorphism induced by the projection \( R \to R'_i \). If each \( \lambda_i \) is an \( S \)-isomorphism and each \( \sigma'_i \lambda_i^{-1} \) is quasi-\( R \)-linear, then \( (a') \) and \( (b') \) are equivalent.

4. Examples.

Example 1. We suppose that the indexing set \( I \) consists of a single element, and we take \( R_i = R_i' = R \). If \( M \) is a left \( S \)-module, \( H \) is a left \( R \)-module and \( \delta : M \to H \), \( \epsilon : H \to M \) are \( S \)-homomorphisms, then there exist unique \( R \)-homomorphisms \( \delta : R \otimes_S M \to H \), \( \epsilon : H \to \text{Hom}_S(R, M) \) such that \( \delta = \delta \kappa \), \( \epsilon = \kappa' \epsilon \), namely the mappings under which \( r \otimes u \to r(\delta u) \) and \( h \to f \), where \( fr = \epsilon(\delta h) \).
We shall assume that, when \( M \) is a left \( R \)-module, there exists an \( R \)-homomorphism
\[ \lambda : \text{Hom}_S(R, M) \to R \otimes_S M. \]
From (2) and (2') we have the following results.

(3) Corollary. Let \( M \) be a left \( R \)-module. If
\( (a) \) \( M \) possesses an \( S \)-endomorphism \( \alpha \) such that \( c[\alpha] = \text{id}_M \),
then
\( (b) \) \( M \) is \( \phi \)-projective.
If \( \lambda \) is an \( R \)-isomorphism, then \( (a) \) and \( (b) \) are equivalent.

(3') Corollary. Let \( M \) be a left \( R \)-module. If
\( (a') \) \( M \) possesses an \( S \)-endomorphism \( \beta \) such that \( c'[\beta] = \text{id}_M \),
then
\( (b') \) \( M \) is \( \phi \)-injective.
If \( \lambda \) is an \( R \)-isomorphism, then \( (a') \) and \( (b') \) are equivalent.

Note. When \( \lambda \) is an \( R \)-isomorphism, it follows from the remark at the end of § 2 that the conditions \( (a) \), \( (b) \), \( (a') \), \( (b') \) are equivalent.
The results (3) and (3') above were proved by Higman (7, Theorem 5) for a situation similar to the present one. As an application, he considered the situation in which \( S \) is a subring of \( R \) and \( R \) possesses a right \( S \)-basis \( \{r_1, \ldots, r_n\} \) and a set \( \{r'_1, \ldots, r'_n\} \) of elements such that

\[(i) \quad rr_j = \sum_{k=1}^n r_k s_{jk} \quad (r \in R, s_{jk} \in S) \text{ implies that } r'_j r = \sum_{k=1}^n s_{jk} r'_k.\]

In this case, for any left \( R \)-module \( M \), the mapping

\[\lambda: \text{Hom}_S(R, M) \to R \otimes_S M,\]

under which

\[f \to \sum_{j=1}^n r_j \otimes f r'_j,\]

is an \( R \)-homomorphism. If \( N \) is a second \( R \)-module and \( \alpha: N \to M \) is an \( S \)-homomorphism, then it is easily checked that

\[c(\alpha) = c'(\alpha) = \sum_{j=1}^n r_j \alpha r'_j.\]

Furthermore, when \( \{r'_1, \ldots, r'_n\} \) is a left \( S \)-basis of \( R \), \( \lambda \) is an \( R \)-isomorphism.

The following result is then an immediate consequence of (3) and (3').

(4) Corollary. Suppose that \( S \) is a subring of \( R \) and let \( \phi: S \to R \) be the inclusion mapping. Let \( \{r_1, \ldots, r_n\} \) be a right \( S \)-basis of \( R \) and let \( \{r'_1, \ldots, r'_n\} \) be a set of elements of \( R \) which satisfy (i). Suppose also that \( M \) is a left \( R \)-module. The condition (a) \( M \) possesses an \( S \)-endomorphism \( \alpha \) such that

\[\sum_{j=1}^n r_j \alpha r'_j = \text{id}_M\]

implies (3)(b) and (3')(b'). If \( \{r'_1, \ldots, r'_n\} \) is a left \( S \)-basis of \( R \), then each of these conditions is equivalent to (a).

Remark. Let \( R \) be a separable algebra over a field \( S \), and suppose that \( \{r_1, \ldots, r_n\} \) is a basis of \( R \) and that \( \{r'_1, \ldots, r'_n\} \) is a dual basis of \( R \) with respect to some discriminant matrix. If \( \alpha \) is a linear transformation of a representation module for \( R \) over \( S \), then \( c(\alpha) \) is the Casimir operator of classical theory; see (6).

For applications of (4) to algebras, separable algebras, and groups, the reader is referred to (7, Part III).

In § 5 we extend (4) for a special class of pairs \( R, S \).

Example 2. Let \( J \) be an indexing set which is partitioned into a family \( \{J_i\}_{i \in I} \) of finite subsets. Suppose also that \( \{r_j\}_{j \in J} \) is a right \( S \)-basis of \( R \) and that \( \{r'_j\}_{j \in J} \) is a family of elements of \( R \), the members of which are not necessarily distinct, such that

\[r_j(\phi s) = (\phi s)r_j, \quad r'_j(\phi s) = (\phi s)r'_j, \quad j \in J, s \in S.\]
For each $i \in I$, let $R_i$ be the right $S$-submodule of $R$ generated by the set $\{r_j\}_{j \in J_i}$; we note that $R_i$ is an $S$-bimodule and that $R = \sum_{i \in I} R_i$ (direct sum). Also, for each $i$, let $R'_i$ be an $S$-bimodule which is contained in $R$ and which contains the set $\{r'_j\}_{j \in J_i}$. Finally, we assume that, for each left $S$-module $M$ and each $i \in I$, there exists an $S$-homomorphism

$$\kappa'_i: \text{Hom}_S(R'_i, M) \to M$$

with the properties specified in § 3. For each $i \in I$, we define an $S$-homomorphism

$$\lambda_i: \text{Hom}_S(R'_i, M) \to R_i \otimes_S M$$

by

$$\lambda_i f = \sum_{j \in J_i} r_j \otimes f r'_j.$$

(5) Lemma. Let $M$ be a left $R$-module and let $\{\alpha_i\}_{i \in I}$ be a family of $S$-endomorphisms of $M$ such that

(ii) for each $u \in M$, $\alpha_i u = 0$ for almost all $i \in I$.

A necessary and sufficient condition for $\sum_{i \in I} \rho_i \lambda_i \alpha_i: M \to R \otimes_S M$ to be an $R$-homomorphism is the following:

if $r \in R$ and if, for all $j \in J_i$, $rr_j = \sum_{k \in J} r_k s_{jk}$, where $s_{jk} \in S$, then, for $k \in J_i$,

(iii) $\sum_{i \in I} (\alpha_i u) \left( \sum_{j \in J_i} s_{jk} r'_j \right) = (\alpha_i r u) r'_k, \quad u \in M$.

Proof. Suppose that $r \in R$, that $rr_j = \sum_{k \in J} r_k s_{jk}$ for all $j \in J$, and that (iii) holds; then

$$r \left( \sum_{i \in I} \rho_i \lambda_i \alpha_i \right) u = \sum_{i \in I} r \rho_i \lambda_i \alpha_i u = \sum_{i \in I} r \sum_{j \in J_i} r_j (\alpha_i u) r'_j$$

$$= \sum_{i \in I} \sum_{j \in J_i} \left( \sum_{k \in J} r_k s_{jk} \right) (\alpha_i u) r'_j = \sum_{i \in I} \sum_{j \in J_i} \left( \sum_{k \in J} r_k \otimes (\alpha_i u) (s_{jk} r'_j) \right)$$

$$= \sum_{i \in I} \sum_{j \in J_i} r_k \otimes (\alpha_i u) (s_{jk} r'_j) = \sum_{i \in I} \sum_{j \in J_i} (\alpha_i u) \left( \sum_{j \in J_i} s_{jk} r'_j \right)$$

$$= \sum_{i \in I} \sum_{k \in J_i} \rho_i \lambda_i \alpha_i u = \left( \sum_{i \in I} \rho_i \lambda_i \alpha_i \right) (ru),$$

and thus $\sum_{i \in I} \rho_i \lambda_i \alpha_i$ is an $R$-homomorphism.

Since $\{r_j\}_{j \in J}$ is a right $S$-basis of $R$, each element of $R \otimes_S M$ can be expressed uniquely in the form $\sum_{j \in J} r_j \otimes v_j$, where the $v_j$ belong to $M$. That (iii) is a necessary condition for $\sum_{i \in I} \rho_i \lambda_i \alpha_i$ to be an $R$-homomorphism can be seen from the first part of this proof.

The next result follows from (2) and (5).
6. **Theorem.** Let $M$ be a left $R$-module. If
   (a) $M$ possesses a family $\{\alpha_i\}_{i \in I}$ of $S$-endomorphisms which satisfy conditions (ii) and (iii) and such that $c[\alpha_i] = \text{id}_M$,
then
   (b) $M$ is $\phi$-projective.

   For each $i \in I$ let $\sigma_i: R \otimes_S M \to R_i \otimes_S M$ be the mapping induced by the projection $R \to R_i$. If each $\lambda_i$ is an isomorphism and each $\lambda_i^{-1}\sigma_i$ is quasi-$R$-linear, then (a) and (b) are equivalent.

5. Throughout this section, $S$ will be a subring of $R$ and $\phi: S \to R$ will be the inclusion mapping. We shall suppose that the elements $r_1, \ldots, r_n, r_1', \ldots, r_n'$ of $R$ commute with every member of $S$, and that $\{r_1, \ldots, r_n\}, \{r_1', \ldots, r_n'\}$ are $S$-bases of $R$ which satisfy condition (i). We assume also that
   \[ r_1'r_1 = r_2'r_2 = \ldots = r_n'r_n = a, \] say,
and that
   (iv) $r_j'r_k = 0$ when $j < k$.

7. **Theorem.** For any left $R$-module $M$, the following conditions are equivalent:
   (a) $M$ is $\phi$-projective;
   (a') $M$ is $\phi$-injective;
   (b) $M$ possesses an $S$-endomorphism $\alpha$ such that

   (v) $\sum_{j=1}^{n} r_j \alpha r_j' = \text{id}_M$;
   (c) $M \cong^h R \otimes_S aM$;
   (c') $M \cong^b \text{Hom}_S(R, aM)$.

**Proof.** The equivalence of (a), (a'), and (b) follows from (4).

(b) $\implies$ (c). Multiplying both sides of (v) on the left by $r_k'$ and using (iv), we see that
   (vi) $\sum_{j=1}^{k} r_j' r_j \alpha r_j' = r_k'$.

The relation
   \[ \psi u = \sum_{j=1}^{n} r_j \otimes a\alpha r_j' u \]
defines a mapping, namely
   \[ \psi: M \to R \otimes_S aM. \]

If $r \in R$ and $rr_j = \sum_{k=1}^{n} r_k s_{jk}$ ($j = 1, \ldots, n$), then
   \[ r(\psi u) = \sum_{j=1}^{n} rr_j \otimes a\alpha r_j' u = \sum_{j=1}^{n} \left( \sum_{k=1}^{n} r_k s_{jk} \right) \otimes a\alpha r_j' u \]
   \[ = \sum_{k=1}^{n} r_k \otimes a\alpha \left( \sum_{j=1}^{n} s_{jk} r_j' \right) u = \sum_{k=1}^{n} r_k \otimes a\alpha (r_k r) u, \]

since condition (i) is satisfied, and hence
\[ r(\psi u) = \sum_{j=1}^{n} r_{j} \otimes a_{\alpha r_{j}}'(ru) = \psi(ru), \]
thus proving that \( \psi \) is an \( R \)-homomorphism. If \( \psi u = 0 \), then, since \( \{r_1, \ldots, r_n\} \) is an \( S \)-base for \( R \), we can infer that \( a_{\alpha r_{j}}'(u) = 0 \) for each \( j \). Replacing \( k \) in (vi) by \( 1, \ldots, n \) in succession, we see that \( r_{j}'u = 0 \) for each \( j \). It follows from (v) that \( u = 0 \). Thus \( \psi \) is injective.

We next show that \( \psi \) is surjective. Multiplying both sides of (vi) on the right by \( r_{k} \) and using (iv), we have
\[ (vii) \quad a_{\alpha a}a = a. \]
Suppose now that \( v \in aM \). We can put \( v = au \), where \( u \in M \), and then
\[ a_{\alpha r_{k}}'(r_{k}a_{\alpha}) = a_{\alpha a}a_{\alpha}u = au, \quad \text{by (vii)}, \]
and hence
\[ a_{\alpha r_{k}}'(r_{k}a_{\alpha}) = v. \]
In addition, when \( j < k \), \( a_{\alpha r_{j}}'(r_{j}a_{\alpha}) = 0 \). Thus, \( \psi \) is surjective, and hence is an \( R \)-isomorphism.

The implication (b) \( \Rightarrow \) (c') follows at once once \( \text{Hom}_S(R, aM) \cong^R R \otimes_S aM \); cf. Example 1.

The implications (c) \( \Rightarrow \) (a), (c') \( \Rightarrow \) (a') were cited in (1) and (1').

(8) **Theorem.** The \( R \)-module \( M \) is projective if and only if there exists a projective \( S \)-module \( N \) such that \( M \cong^R R \otimes_S aM \). Dually, \( M \) is injective if and only if there exists an injective \( S \)-module \( N \) such that \( M \cong^R \text{Hom}_S(R, N) \).

**Proof.** If \( M \) is \( R \)-projective, then it is also \( \phi \)-projective, and hence it follows from (7) that there exists an \( S \)-module \( N \) such that \( M \cong^R R \otimes_S N \). Since \( R \) is \( S \)-free, it follows that \( M \) is \( S \)-projective; and thus \( N \), being \( S \)-isomorphic to a direct summand of \( M \), is \( S \)-projective. The converse follows from (1, Chapter II, Proposition 6.1).

6. **Examples.**

**Example 3.** Let \( R \) be the free left \( S \)-module on the set \( \{1, s, d, \ldots, d^{n-1}\} \). We make \( R \) into a ring by means of the identity
\[
(s_01_s + s_1d + \ldots + s_{n-1}d^{n-1})(s_0'1_s + s_1'd + \ldots + s_{n-1}'d^{n-1}) \\
= s_0s_0'1_s + (s_0s_1' + s_1s_0')d + \ldots + (s_0s_{n-1}' + s_{n-1}s_0')d^{n-1} \\
(s_0, \ldots, s_{n-1}, s_0', \ldots, s_{n-1}') \in S,
\]
so that \( d^n = 0 \). We may regard \( S \) as a subring of \( R \) by identifying \( s \) and \( 1_s \) for every \( s \in S \), in which case \( d \) commutes with every member of \( S \). It is clear that
if $M$ is a left $S$-module having an $S$-endomorphism $d$ for which $d^n = 0$, then $M$ is a left $R$-module. In (7), we can take
\[ r_1 = 1_S, r_2 = d, \ldots, r_n = d^{n-1}, \quad r_1' = d^{n-1}, \quad r_2' = d^{n-2}, \ldots, r_n' = 1_S, \]
the identity in (7) (b) then becomes
\[ 1_S \alpha d^{n-1} + d \alpha d^{n-2} + \ldots + d^{n-2} \alpha d + d^{n-1} \alpha 1_S = \text{id}_M, \]
and we have (9, Proposition 1.3). Taking $n = 2$ yields (1, Chapter IV, Proposition 2.3). We remark that, in the former case, $a = d^{n-1}$.

**Example 4.** Let $R$ be the free left $S$-module on the set $\{1_S, d_1, d_2, d_1d_2\}$. We make $R$ into a ring by means of the identity
\[
(s_01_S + s_1d_1 + s_2d_2 + s_3d_1d_2)(s_0'1_S + s_1'd_1 + s_2'd_2 + s_3'd_1d_2) = s_0s_0'1_S + (s_0s'_1 + s_1s'_2 + s_2s'_3)d_1 + (s_0s'_2 + s_2s'_1)d_2 \\
+ (s_0s'_3 + s_1s'_2 + s_2s'_1 + s_3s'_0)d_1d_2
\]
so that
\[ d_1d_1 = d_2d_2 = 0 \quad \text{and} \quad d_3d_1 = d_1d_2, \]
and, when we identify $s1_S$ and $s$ for each $s \in S$, it follows that
\[ d_1s = sd_1, \quad d_2s = sd_2. \]
In (7) we can put
\[ r_1 = 1_S, r_2 = d_1, r_3 = d_2, r_4 = d_1d_2, \quad r_1' = d_2d_2, r_2' = d_2, r_3' = d_1, r_4' = 1_S. \]
The identity in (7) (b) then becomes
\[ 1_S \alpha d_1d_2 + d_1 \alpha d_2 + d_2 \alpha d_1 + d_1d_2 \alpha 1_S = \text{id}_M, \]
and $a = d_1d_2$.

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