Efficient and Parsimonious Agnostic Active Learning

Tzu-Kuo Huang† Alekh Agarwal† Daniel J. Hsu‡
tkhuang@microsoft.com alekha@microsoft.com djhsu@cs.columbia.edu

John Langford† Robert E. Schapire†
jcl@microsoft.com schapire@microsoft.com

Microsoft Research† Department of Computer Science‡
New York, NY Columbia University, New York, NY

Abstract

We develop a new active learning algorithm for the streaming setting satisfying three important properties: 1) It provably works for any classifier representation and classification problem including those with severe noise. 2) It is efficiently implementable with an ERM oracle. 3) It is more aggressive than all previous approaches satisfying 1 and 2. To do this we create an algorithm based on a newly defined optimization problem and analyze it. We also conduct the first experimental analysis of all efficient agnostic active learning algorithms, discovering that this one is typically better across a wide variety of datasets and label complexities.

1 Introduction

How can you best learn a classifier given a label budget?

Active learning approaches are known to yield exponential improvements over supervised learning under strong assumptions [Cohn et al., 1994]. Under much weaker assumptions, streaming-based agnostic active learning [Balcan et al., 2006, Beygelzimer et al., 2009, 2010, Dasgupta et al., 2007, Zhang and Chaudhuri, 2014] is particularly appealing since it is known to work for any classifier representation and any label noise distribution with an i.i.d. data source. Here, a learning algorithm decides for each unlabeled example in sequence whether or not to request a label, never revisiting this decision. Restated then: What is the best possible active learning algorithm which works for any classifier representation, any label noise distribution, and is computationally tractable?

Computational tractability is a critical concern, because most known algorithms for this setting [e.g., Balcan et al., 2006, Koltchinskii, 2010, Zhang and Chaudhuri, 2014] require explicit enumeration of classifiers, implying exponentially-worse computational complexity compared to typical supervised learning algorithms. Active learning algorithms based on empirical risk minimization (ERM) oracles [Beygelzimer et al., 2009, 2010, Hsu, 2010] can overcome this intractability by using passive classification algorithms as the oracle to achieve a computationally acceptable solution.

Achieving generality, robustness, and acceptable computation has a cost. For the above methods [Beygelzimer et al., 2009, 2010, Hsu, 2010], a label is requested on nearly every unlabeled example where two empirically good classifiers disagree. This results in a poor label complexity, well short of information-theoretic limits [Castro and Nowak, 2008] even for general robust solutions [Zhang and Chaudhuri, 2014]. Until now.

In Section 3, we design a new algorithm ACTIVE COVER (AC) for constructing query probability functions that minimize the probability of querying inside the disagreement region—the set of points where good classifiers disagree—and never query otherwise. This requires a new algorithm that maintains a parsimonious cover of the set of empirically good classifiers. The cover is a result of solving an optimization problem (in Section 5) specifying the properties of a desirable query probability function. The cover size provides a practical knob between computation and label complexity, as demonstrated by the complexity analysis we present in Section 5.

In Section 4, we provide our main results which demonstrate that AC effectively maintains a set of good classifiers, achieves good generalization error, and has a label complexity bound tighter than previous approaches. The label complexity bound depends on the disagreement coefficient [Hanneke, 2009], which does not completely capture the advantage of the algorithm. In Appendix 4.2.2, we provide an example of a hard active learning problem where AC is...
substantially superior to previous tractable approaches. Together, these results show that AC is better and sometimes substantially better in theory. The key aspects in the proof of our generalization results are presented in Section 7 with more technical details and label complexity analysis presented in the appendix.

Do agnostic active learning algorithms work in practice? No previous works have addressed this question empirically. Doing so is important because analysis cannot reveal the degree to which existing classification algorithms substantially better in theory. The key aspects in the proof of our generalization results are presented in Section 7, with substantially superior to previous tractable approaches. Together, these results show that AC is better and sometimes better in theory. The key aspects in the proof of our generalization results are presented in Section 7, with more technical details and label complexity analysis presented in the appendix.

2 Preliminaries

Let $\mathcal{H} \subseteq \{\pm 1\}^X$ be a set of binary classifiers, which we assume is finite for simplicity. Let $E_X[\cdot]$ denote expectation with respect to $X \sim P_X$, the marginal of $P$ over $\mathcal{X}$. The error of a classifier $h \in \mathcal{H}$ is $\text{err}(h) := \Pr_{(X,Y) \sim P}(h(X) \neq Y)$, and the error minimizer is denoted by $h^* := \arg \min_{h \in \mathcal{H}} \text{err}(h)$. The (importance weighted) empirical error of $h \in \mathcal{H}$ on a multiset $S$ of importance weighted and labeled examples drawn from $\mathcal{X} \times \{\pm 1\} \times \mathbb{R}_+$ is $\text{err}(h,S) := \sum_{(x,y,w) \in S} w \cdot 1(h(x) \neq y)/|S|$. The disagreement region for a subset of classifiers $A \subseteq \mathcal{H}$ is $\text{DIS}(A) := \{x \in \mathcal{X} \mid \exists h, h' \in A \text{ such that } h(x) \neq h'(x)\}$. The regret of a classifier $h \in \mathcal{H}$ relative to another $h' \in \mathcal{H}$ is $\text{reg}(h,h') := \text{err}(h) - \text{err}(h')$, and the analogous empirical regret on $S$ is $\text{reg}(h,h',S) := \text{err}(h,S) - \text{err}(h',S)$. When the second classifier $h'$ in (empirical) regret is omitted, it is taken to be the (empirical) error minimizer in $\mathcal{H}$.

A streaming-based active learner receives i.i.d. labeled examples $(X_1,Y_1), (X_2,Y_2), \ldots$ from $P$ one at a time; each label $Y_i$ is hidden unless the learner decides on the spot to query it. The goal is to produce a classifier $h \in \mathcal{H}$ with low error $\text{err}(h)$, while querying as few labels as possible.

In the IWAL framework [Beygelzimer et al. 2009], a decision whether or not to query a label is made randomly: the learner picks a probability $p \in [0,1]$, and queries the label with that probability. Whenever $p > 0$, an unbiased error estimate can be produced using inverse probability weighting [Horvitz and Thompson 1952]. Specifically, for any classifier $h$, an unbiased estimator $E$ of $\text{err}(h)$ based on $(X,Y) \sim P$ and $p$ is as follows: if $Y$ is queried, then $E = 1(h(X) \neq Y)/p; else, E = 0$. It is easy to check that $E = \text{err}(h)$. Thus, when the label is queried, we
Algorithm 1 ACTIVE COVER (AC)

**input:** Constants $c_1, c_2, c_3$, confidence $\delta$, error radius $\gamma$, parameters $\alpha, \beta, \xi$ for (OP), epoch schedule $0 = \tau_0 < \tau_1 < \tau_2 < \tau_3 < \ldots < \tau_M$ satisfying $\tau_{m+1} \leq 2\tau_m$ for $m \geq 1$.

**initialize:** epoch $m = 0$, $\tilde{Z}_0 := \emptyset$, $\Delta_0 := c_1\sqrt{\epsilon_1} + c_2\epsilon_1\log 3$, where

$$\epsilon_m := \frac{32(|\mathcal{H}|/\delta) + \log \tau_m}{\tau_m}.$$ 

1: for $i = 4, \ldots, n$, do
2: if $i = \tau_m + 1$ then
3: Set $\tilde{Z}_m = \tilde{Z}_{m-1} \cup S$, and $S = \emptyset$.
4: Let

$$h_{m+1} := \arg\min_{h \in \mathcal{H}} \text{err}(h, \tilde{Z}_m),$$

(1)

$$\Delta_m := c_1\sqrt{\epsilon_m \text{err}(h_{m+1}, \tilde{Z}_m)} + c_2\epsilon_m\log \tau_m,$$

(2)

$$A_{m+1} := \{h | \text{err}(h, \tilde{Z}_m) - \text{err}(h_{m+1}, \tilde{Z}_m) \leq \gamma \Delta_m\}.$$ 

(3)

5: Compute the solution $P_{m+1}(\cdot)$ to the optimization problem [5].
6: $m := m + 1$.
7: end if
8: Receive unlabeled data point $X_i$.
9: if $X_i \in D_m := \text{DIS}(A_m)$, then
10: Draw $Q_i \sim \text{Bernoulli}(P_m(X_i))$.
11: Update the set of examples [4]

$$S := \begin{cases} S \cup \{(X_i, Y_i, 1/P_m(X_i))\}, & Q_i = 1 \\ S \cup \{X_i, 1\}, & \text{otherwise}. \end{cases}$$

12: else
13: $S := S \cup \{(X_i, h_m(X_i), 1)\}$.
14: end if
15: end for
16: $h_{M+1} := \arg\min_{h \in \mathcal{H}} \text{err}(h, \tilde{Z}_M)$.

produce the importance weighted labeled example $(X, Y, 1/p)$ [3]

3 Algorithm

Our new algorithm, shown as Algorithm [1] breaks the example stream into epochs. The algorithm admits any epoch schedule so long as the epoch lengths satisfy $\tau_{m-1} \leq 2\tau_m$. For technical reasons, we always query the first 3 labels to kick-start the algorithm. At the start of epoch $m$, AC computes a query probability function $P_m: X \rightarrow [0, 1]$ which will be used for sampling the data points to query during the epoch. This is done by maintaining a few objects

---

3 If the label is not queried, we produce an ignored example of weight zero; its only purpose is to maintain the correct count of querying opportunities. This ensures that $1/|S|$ is the correct normalization in $\text{err}(h, S)$.

4 See Footnote. Adding an example of importance weight zero simply increments $|S|$ without updating other state of the algorithm, hence the label used does not matter.
of interest during each epoch:

1. In step 1, we compute the best classifier on the sample \( \hat{Z}_m \) that we have collected so far. Note that the sample consists of the queried, true labels on some examples, while predicted labels for the others.

2. A radius \( \Delta_m \) is computed in step 2 based on the desired level of concentration we want the various empirical quantities to satisfy.

3. The set \( A_m \) in step 3 consists of all the hypotheses which are good according to our sample \( \hat{Z}_m \), with the notion of good being measured as empirical regret being at most \( \Delta_m \).

Within the epoch, \( P_m \) determines the probability of querying an example in the disagreement region for this set \( A_m \) of “good” classifiers; examples outside this region are not queried but given labels predicted by \( h_m \). Consequently, the sample is not unbiased unlike some of the predecessors of our work. The various constants in Algorithm 1 must satisfy:

\[
\alpha \geq 1, \quad \eta \geq 864, \quad \xi \leq \frac{1}{8n\epsilon M \log n}, \quad \beta^2 \leq \frac{\eta}{864\gamma n\epsilon M \log n}, \quad \gamma \geq \eta/4, \\
c_1 \geq 2\alpha\sqrt{6}, \quad c_2 \geq \eta c_1^2/4, \quad c_3 \geq 1.
\]

\[\text{(4)}\]

Epoch Schedules: The algorithm as stated takes an arbitrary epoch schedule subject to \( \tau_m < \tau_{m+1} \leq 2\tau_m \). Two natural extremes are unit-length epochs, \( \tau_m = m \), and doubling epochs, \( \tau_{m+1} = 2\tau_m \). The main difference comes in the number of times (OP) is solved, which is a substantial computational consideration. Unless otherwise stated, we assume the doubling epoch schedule so that the query distribution and ERM classifier are recomputed only \( O(\log n) \) times.

Optimization problem (OP) to obtain \( P_m \): AC computes \( P_m \) as the solution to the optimization problem (OP). In essence, the problem encodes the properties of a query probability function that are essential to ensure good generalization, while maintaining a low label complexity. As we will discuss later, some of the previous works can be seen as specific ways of construction feasible solutions to this optimization problem. The objective function of (OP) encourages small query probabilities in order to minimize the label complexity. It might appear odd that we do not use the more obvious choice for objective which would be \( \mathbb{E}_X [ P(X) ] \), however our choice simultaneously encourages low query probabilities and also provides a barrier for the constraint \( P(X) \leq 1 \)--an important algorithmic aspect as we will discuss in Section 6.

The constraints \[\text{(5)}\] in (OP) bound the variance in our importance-weighted regret estimates for every \( h \in \mathcal{H} \). This is key to ensuring good generalization as we will later use Bernstein-style bounds which rely on our random variables having a small variance. Let us examine these constraints in more detail. The LHS of the constraints measures the variance in our empirical regret estimates for \( h \), measured only on the examples in the disagreement region \( D_m \). This is because the importance weights in the form of \( 1/P_m(X) \) are only applied to these examples; outside this region we use the predicted labels with an importance weight of 1. The RHS of the constraint consists of three terms. The first term ensures the feasibility of the problem, as \( P(X) \equiv 1/(2\alpha^2) \) for \( X \in D_m \) will always satisfy the constraints. The second empirical regret term makes the constraints easy to satisfy for bad hypotheses—this is crucial to rule out large label complexities in case there are bad hypotheses that disagree very often with \( h_m \). A benefit of this is easily seen when \( -h_m \in \mathcal{H} \), which might have a terrible regret, but would force a near-constant query probability on the disagreement region if \( \beta = 0 \). Finally, the third term will be on the same order as the second one for hypotheses in \( A_m \), and is only included to capture the allowed level of slack in our constraints which will be exploited for the efficient implementation in Section 5.

Of course, variance alone is not adequate to ensure concentration, and we also require the random variables of interest to be appropriately bounded. This is ensured through the constraints \[\text{(6)}\], which impose a minimum query probability on the disagreement region. Outside the disagreement region, we use the predicted label with an importance weight of 1, so that our estimates will always be bounded (albeit biased) in this region. Note that this optimization
Essentially for all epochs $m$.

Theorem 1. Pick any type error bounds as we see next. Since $X$ is a population counterpart of the quantity $m^*$, we have

$$\Delta_0 := \Delta_0^*$$

and

$$\Delta_m^* := c_1 \sqrt{\epsilon_m \reg_m(h^*)} + c_2 \epsilon_m \log \tau_m \text{ for } m \geq 1.$$  

Essentially $\Delta_m^*$ is a population counterpart of the quantity $\Delta_m$ used in Algorithm 1 and crucially relies on $\reg_m(h^*)$, the true error of $h^*$ restricted to the disagreement region instead of the empirical error of the ERM at epoch $m$. This quantity captures the inherent noisiness of the problem, and modulates the transition between $O(1/\sqrt{n})$ to $O(1/n)$ type error bounds as we see next.

**Theorem 1.** Pick any $0 < \delta < 1/e$ such that $|\mathcal{H}|/\delta > \sqrt{192}$. Then recalling that $h^* = \arg\min_{h \in \mathcal{H}} \err(h)$, we have for all epochs $m = 1, 2, \ldots, M$, with probability at least $1 - \delta$

$$\reg(h, h^*) \leq 16\gamma \Delta_m^* \text{ for all } h \in A_{m+1}, \text{ and}$$

$$\reg(h^*, h_{m+1}, \hat{Z}_m) \leq 2\Delta_m/4.$$  

4 Generalization and Label Complexity

We now present guarantees on the generalization error and label complexity of Algorithm 1 assuming a solver for (OP), which we provide in the next section.

4.1 Generalization guarantees

Our first theorem provides a bound on generalization error. Define

$$\reg_m(h) := \frac{1}{\tau_m} \sum_{j=1}^{m} (\tau_j - \tau_{j-1}) \mathbb{E}_{X,Y \sim D_j}[\mathbb{I}(h(X) \neq Y) \wedge X \in D_j)],$$

$$\Delta_0^* := \Delta_0^*$$

and

$$\Delta_m^* := c_1 \sqrt{\epsilon_m \reg_m(h^*)} + c_2 \epsilon_m \log \tau_m \text{ for } m \geq 1.$$  

Essentially $\Delta_m^*$ is a population counterpart of the quantity $\Delta_m$ used in Algorithm 1 and crucially relies on $\reg_m(h^*)$, the true error of $h^*$ restricted to the disagreement region instead of the empirical error of the ERM at epoch $m$. This quantity captures the inherent noisiness of the problem, and modulates the transition between $O(1/\sqrt{n})$ to $O(1/n)$ type error bounds as we see next.

**Theorem 1.** Pick any $0 < \delta < 1/e$ such that $|\mathcal{H}|/\delta > \sqrt{192}$. Then recalling that $h^* = \arg\min_{h \in \mathcal{H}} \err(h)$, we have for all epochs $m = 1, 2, \ldots, M$, with probability at least $1 - \delta$

$$\reg(h, h^*) \leq 16\gamma \Delta_m^* \text{ for all } h \in A_{m+1}, \text{ and}$$

$$\reg(h^*, h_{m+1}, \hat{Z}_m) \leq 2\Delta_m/4.$$  

5
The theorem is proved in Section 7.2.2 using the overall analysis framework described in Section 7.

Since we use $\gamma \geq \eta/4$, the bound \( \hat{O} \) implies that $h^* \in A_m$ for all epochs $m$. This also maintains that all the predicted labels used by our algorithm are identical to those of $h^*$, since no disagreement amongst classifiers in $A_m$ was observed on those examples. This observation will be critical to our proofs, where we will exploit the fact that using labels predicted by $h^*$ instead of observed labels on certain examples only introduces a bias in favor of $h^*$, thereby ensuring that we never mistakenly drop the optimal classifier from our version space $A_m$.

The bound \( \epsilon \) shows that every hypothesis in $A_{m+1}$ has a small regret to $h^*$. Since the ERM classifier $h_{m+1}$ is always in $A_{m+1}$, this yields our main generalization error bound on the classifier $h_{\tau_{m+1}}$ output by Algorithm 1. Additionally, it also clarifies the definition of the sets $A_m$ as the set of good classifiers: these are classifiers which have small population regret relative to $h^*$ indeed. In the worst case, if $\mathbb{Err}_m(h^*)$ is a constant, then the overall regret bound is $O(1/\sqrt{n})$. The actual rates implied by the theorem, however depend on the properties of the distribution and below we illustrate this with two corollaries. We start with a simple specialization to the realizable setting.

**Corollary 1.** Under the conditions of Theorem 7, suppose further that $\text{err}(h^*) = 0$. Then $\Delta_m = \Delta_m^* = c_2 \tau_m \log \tau_m$ for all hypotheses $h \in A_{m+1}$.

In words, the corollary demonstrates a $\hat{O}(1/n)$ rate after seeing $n$ unlabeled examples in the realizable setting. Of course the use of $\mathbb{Err}_m(h^*)$ in defining $\Delta_m^*$ allows us to retain the fast rates even when $h^*$ makes some errors but they do not fall in the disagreement region of good classifiers. One intuitive condition that controls the errors within the disagreement region is the low-noise condition of Tsybakov [2004], which asserts that there exist constants $\zeta > 0$ and $0 < \omega \leq 1$ such that

$$\Pr(h(X) \neq h^*(X)) \leq \zeta \cdot (\text{err}(h) - \text{err}(h^*))^{\omega}, \quad \forall h \in \mathcal{H} \text{ such that } \text{err}(h) - \text{err}(h^*) \leq \varepsilon_0. \quad (10)$$

Under this assumption, the extreme $\omega = 0$ corresponds to the worst-case setting while $\omega = 1$ corresponds to $h^*$ having a zero error on disagreement set of the classifiers with regret at most $\varepsilon_0$. Under this assumption, we get the following corollary of Theorem 1.

**Corollary 2.** Under conditions of Theorem 7, suppose further that Tsybakov’s low-noise condition (10) is satisfied with some parameters $\zeta, \omega$, and $\varepsilon_0 = 1$. Then after $m$ epochs, we have $\text{reg}(h, h^*) = \hat{O}(\tau_m^{-\frac{1}{\omega}} \log(|\mathcal{H}|/\delta))$.

The proof of this result is deferred to Appendix E. It is worth noting that the rates obtained here are known to be unimprovable for even passive learning under the Tsybakov noise condition of Castro and Nowak [2008]. Consequently, there is no loss of statistical efficiency in using our active learning approach. The result is easily extended for other values of $\varepsilon_0$ by using the worst-case bound until the first epoch $m_0$ when $16 \gamma \Delta_m^* \tau_{m_0}$ drops below $\varepsilon_0$ and then apply our analysis above from $m_0$ onwards. We leave this development to the reader.

### 4.2 Label complexity

Generalization alone does not convey the entire quality of an active learning algorithm, since a trivial algorithm queries always with probability 1, thereby matching the generalization guarantees of passive learning. In this section, we show that our algorithm can achieve the aforementioned generalization guarantees, despite having a small label complexity in favorable situations. We begin with a worst-case result in the agnostic setting, and then describe a specific example which demonstrates some key differences of our approach from its predecessors.

#### 4.2.1 Disagreement-based label complexity bounds

In order to quantify the extent of gains over passive learning, we measure the hardness of our problem using the disagreement coefficient [Hanneke, 2014], which is defined as

---

\[ \omega \] in our statement of the low-noise condition (10) corresponds to $1/\kappa$ in the results of Castro and Nowak [2008].
\[ \theta = \theta(h^*) := \sup_{r > 0} \frac{\mathbb{P}_X \{ x \mid \exists h \in \mathcal{H} \text{ s.t. } h^*(x) \neq h(x), \mathbb{P}_X \{ x' \mid h(x') \neq h^*(x') \} \leq r \} \] \]

Intuitively, given a set of classifiers \( \mathcal{H} \) and a data distribution \( \mathbb{P} \), an active learning problem is easy if good classifiers disagree on only a small fraction of the examples, so that the active learning algorithm can increasingly restrict attention only to this set. With this definition, we have the following result for the label complexity of Algorithm 1.

**Theorem 2.** Under conditions of Theorem [7] with probability at least \( 1 - \delta \), the expected number of label queries made by Algorithm 1 after \( n \) examples over \( M \) epochs is at most

\[ 2\theta \mathbb{E}_M(h^*) n + \theta \cdot \tilde{O}(\sqrt{\mathbb{E}_M(h^*) \log(|\mathcal{H}|/\delta)} + \log(|\mathcal{H}|/\delta)). \]

The proof is in Appendix [D]. The dominant first term of the label complexity bound is linear in the number of unlabeled examples, but can be quite small if \( \theta \) is small, or if \( \mathbb{E}_M(h^*) \approx 0 \)—it is indeed 0 in the realizable setting.

We illustrate this aspect of the theorem with a corollary for the realizable setting.

**Corollary 3.** Under the conditions of Theorem [2] suppose further that \( \mathbb{E}(h^*) = 0 \). Then the expected number of label queries made by Algorithm 1 is at most \( \theta \tilde{O}(\log(|\mathcal{H}|/\delta)). \)

In words, we attain a logarithmic label complexity in the realizable setting. We contrast this with the label complexity of IWAL [Beygelzimer et al., 2010], which grows as \( \theta \sqrt{n} \) independent of \( \mathbb{E}(h^*) \). This leads to an exponential difference in the label complexities of the two methods in low-noise problems. A much closer comparison is with respect to the Oracular CAL algorithm [Hsu, 2010], which does have a dependence on \( \sqrt{\mathbb{E}(h^*)} \) in the second term, but has a worse dependence on \( \theta \).

Just like Corollary [2], we can also obtain improved bounds on label complexity under the Tsybakov noise condition.

**Corollary 4.** Under conditions of Theorem [2] suppose further that Tsybakov’s low-noise condition (10) is satisfied with some parameters \( \zeta, \omega, \text{ and } \varepsilon_0 = 1 \). Then after \( m \) epochs, the expected number of label queries made by Algorithm 1 is at most \( \tilde{O} \left( \frac{2(1-\omega)}{\tau_m} \log(|\mathcal{H}|/\delta) \right) \).

The proof of this result is deferred to Appendix [E]. The label complexity obtained above is indeed optimal in terms of the dependence on \( n \), the number of unlabeled examples, matching known information-theoretic rates of Castro and Nowak [2008]. This can be seen since the regret from Corollary [2] falls as a function of the number of queries at a rate of \( \tilde{O}(q_m \frac{2(1-\omega)}{\tau_m} \log(|\mathcal{H}|/\delta)) \) after \( m \) epochs, where \( q_m \) is the number of label queries. This is indeed optimal according to the lower bounds of Castro and Nowak [2008], after recalling that \( \omega = 1/\kappa \) in their results. Once again, the corollary highlights our improvements on top of IWAL, which does not attain this optimal label complexity.

These results, while strong, still do not completely capture the performance of our method. Indeed the proofs of these results are entirely based on the fact that we do not query outside the disagreement region, a property shared by the previous Oracular CAL algorithm [Hsu, 2010]. Indeed we only improve upon that result as we use more refined error bounds to define the disagreement region. However, such analysis completely ignores the fact that we construct a rather non-trivial query probability function on the disagreement region, as opposed to using any constant probability of querying over this entire region. This gives our algorithm the ability to query much more rarely even over the disagreement region, if the queries do not provide much information regarding the optimal hypothesis \( h^* \). The next section illustrates an example where this gain can be quantified.

### 4.2.2 Improved label complexity for a hard problem instance

We now present an example where the label complexity of Algorithm [1] is significantly smaller than both IWAL and Oracular CAL by virtue of rarely querying in the disagreement region. The example considers a distribution and a
classifier space with the following structure: (i) for most examples a single good classifier predicts differently from the remaining classifiers (ii) on a few examples half the classifiers predict one way and half the other. In the first case, little advantage is gained from a label because it provides evidence against only a single classifier. ACTIVE COVER queries over the disagreement region with a probability close to $P_{\text{min}}$ in case (i) and probability 1 in case (ii), while others query with probability $\Omega(1)$ everywhere implying $O(\sqrt{n})$ times more queries.

Concretely, we consider the following binary classification problem. Let $H$ denote the finite classifier space (defined later), and distinguish some $h^* \in H$. Let $U\{-1, 1\}$ denote the uniform distribution on $\{-1, 1\}$. The data distribution $D(\mathcal{X}, \mathcal{Y})$ and the classifiers are defined jointly:

- With probability $\epsilon$,
  $$y = h^*(x), \quad h(x) \sim U\{-1, 1\}, \forall h \neq h^*.$$

- With probability $1 - \epsilon$,
  $$y \sim U\{-1, 1\}, \quad h^*(x) \sim U\{-1, 1\},$$
  $$h_r(x) = -h^*(x) \text{ for some } h_r \text{ drawn uniformly at random from } H \setminus h^*,$$
  $$h(x) = h^*(x) \forall h \neq h^* \land h \neq h_r.$$

Indeed, $h^*$ is the best classifier because $\text{err}(h^*) = \epsilon \cdot 0 + (1 - \epsilon)(1/2) = (1 - \epsilon)/2$, while $\text{err}(h) = 1/2 \forall h \neq h^*$. This problem is hard because only a small fraction of examples contain information about $h^*$. Ideally we want to focus label queries on those informative examples while skipping the uninformative ones. However, algorithms like IWAL, or more generally, active learning algorithms that determine label query probabilities based on error differences between a pair of classifiers, query frequently on the uninformative examples. Let $u(h, h') := \mathbb{I}(h(x) \neq y) - \mathbb{I}(h'(x) \neq y)$ denote the error difference between two different classifiers $h$ and $h'$. Let $C$ be a random variable such that $C = 1$ for the $\epsilon$ case and $C = 0$ for the $1 - \epsilon$ case. Then it is easy to see that

$$E[u(h, h') \mid C = 1] = \begin{cases} 0, & h \neq h^*, h' \neq h^*, \\ -1/2, & h = h^*, h' \neq h^*, \\ 1/2, & h \neq h^*, h' = h^*. \end{cases}$$

$$E[u(h, h') \mid C = 0] = 0, \forall h \neq h'.$$

Therefore, IWAL queries all the time on uninformative examples ($C = 0$).

Now let us consider the label complexity of Algorithm 1 on this problem. Let us focus on the query probability inside the $1 - \epsilon$ region, and fix it to some constant $p$. Let us also allow a query probability of 1 on the $\epsilon$ region. Then the left hand side in the constraint (5) for any classifier $h$ is at most $\epsilon + P(h(X) \neq h_m(X))/p \leq \epsilon + 2/(p(|H| - 1))$, since $h$ and $h_m$ disagree only on those points in the $1 - \epsilon$ region where one of them is picked as the disagreeing classifier $h_r$ in the random draw. On the other hand, the RHS of the constraints is at least $\xi \rho_{\tau_m - 1} \Delta_{m-1}^2 \geq \xi \text{err}(h_m, \tilde{Z}_{m-1})$, which is at least $\xi/4$ as long as $\epsilon$ is small enough and $\tau_m$ is large enough for empirical error to be close to true error. Consequently, assuming that $\epsilon \leq \xi/8$, we find that any $p \geq 16/(\xi(|H| - 1))$ satisfies the constraints. Of course we also have that $p \geq P_{\text{min},m}$, which is $O(1/\sqrt{m})$ in this case since $\overline{\text{err}}(h^*)$ is a constant. Consequently, for $|H|$ large enough $p = P_{\text{min},m}$ is feasible and hence optimal for the population (op). Since we find an approximately optimal solution based on Theorem 4, the label complexity at epoch $m$ is $O(1/\sqrt{m})$. Summing things up, it can then be checked easily that we make $O(\sqrt{n})$ queries over $n$ examples, a factor of $\sqrt{n}$ smaller than baselines such as IWAL and Oracular CAL on this example.

5 Efficient implementation

In Algorithm 1 the computation of $h_m$ is an ERM operation, which can be performed efficiently whenever an efficient passive learner is available. However, several other hurdles remain. Testing for $x \in D_m$ in the algorithm, as well
as finding a solution to \((\text{OP})\) are considerably more challenging. The epoch schedule helps, but \((\text{OP})\) is still solved \(O(\log n)\) times, necessitating an extremely efficient solver.

Starting with the first issue, we follow Dasgupta et al. \[2007\] who cleverly observed that \(x \in D_m\) can be efficiently determined using a single call to an ERM oracle. Specifically, to apply their method, we use the oracle to find \(h' = \arg \min \{\text{err}(h, Z_{m-1}) \mid h \in \mathcal{H}, h(x) \neq h_m(x)\}\). It can then be argued that \(x \in D_m = \text{DIS}(A_m)\) if and only if the easily-measured regret of \(h'\) (that is, \(\text{reg}(h', h_m, Z_{m-1})\)) is at most \(\gamma \Delta_{m-1}\).

Solving \((\text{OP})\) efficiently is a much bigger challenge because, as an optimization problem, it is enormous: There is one variable \(P(x)\) for every point \(x \in \mathcal{X}\), one constraint \((5)\) for each classifier \(h\) and bound constraints \((6)\) on \(P(x)\) for every \(x\). This leads to infinitely many variables and constraints, with an ERM oracle being the only computational primitive available. Another difficulty is that \((\text{OP})\) is defined in terms of the true expectation with respect to the example distribution \(P_X\), which is unavailable.

In the following we first demonstrate how to efficiently solve \((\text{OP})\) assuming access to the true expectation \(E_X[\cdot]\), and then discuss a relaxation that uses expectation over samples.

### 5.1 Solving \((\text{OP})\) with the true expectation

The main challenge here is that the optimization variable \(P(x)\) is of infinite dimension. We deal with this difficulty using Lagrange duality, which leads to a dual representation of \(P(x)\) in terms of a set of classifiers found through successive calls to an ERM oracle. As will become clear shortly, each of these classifiers corresponds to the most violated variance constraint \((5)\) under some intermediate query probability function. Thus at a high level, our strategy is to expand the set of classifiers for representing \(P(x)\) until the amount of constraint violation gets reduced to an acceptable level.

We start by eliminating the bound constraints using barrier functions. Notice that the objective \(E_X[1/(1 - P(x))]\) is already a barrier at \(P(x) = 1\). To enforce the lower bound \((6)\), we modify the objective to

\[
E_X \left[ \frac{1}{1 - P(X)} \right] + \mu^2 E_X \left[ \frac{1(X \in D_m)}{P(X)} \right],
\]

where \(\mu\) is a parameter chosen momentarily to ensure \(P(x) \geq P_{\text{min},m}\) for all \(x \in D_m\). Thus, the modified goal is to minimize \((12)\) over non-negative \(P\) subject only to \((5)\).

We solve the problem in the dual where we have a large but finite number of optimization variables, and efficiently maximize the dual using coordinate ascent with access to an ERM oracle over \(\mathcal{H}\). Let \(\lambda_h \geq 0\) denote the Lagrange
multiplier for the constraint \((\ref{eq:cone})\) for classifier \(h\). Then for any \(\lambda\), we can minimize the Lagrangian
\[
\mathcal{L}(\lambda) := \mathbb{E}_X \left[ \frac{1}{1 - P(X)} \right] + \mu^2 \mathbb{E}_X \left[ \frac{\mathbb{1}(X \in D_m)}{P(X)} \right] - \sum_{h \in \mathcal{H}} \lambda_h \left( b_m(h) - \mathbb{E}_X \left[ \frac{\mathbb{1}(h(X) \neq h_m(X) \land X \in D_m)}{P(X)} \right] \right)
\]
over each primal variable \(P(X)\) yielding the solution
\[
P_\lambda(x) = \frac{\mathbb{1}(x \in D_m) q_\lambda(x)}{1 + q_\lambda(x)}, \quad \text{where } q_\lambda(x) = \sqrt{\mu^2 + \sum_{h \in \mathcal{H}} \lambda_h T_h^m(x)}
\]
and \(T_h^m(x) = \mathbb{1}(h(x) \neq h_m(x) \land x \in D_m)\). Clearly, \(\mu/(1 + \mu) \leq P_\lambda(x) \leq 1\) for all \(x \in D_m\), so all the bound constraints \((\ref{eq:cone})\) in (OP) are satisfied if we choose \(\mu = 2P_{\min,m}\). Plugging the solution \(P_\lambda\) into the Lagrangian, we obtain the dual problem of maximizing the dual objective
\[
\mathcal{D}(\lambda) = \mathbb{E}_X \left[ \mathbb{1}(X \in D_m) (1 + q_\lambda(X))^2 \right] - \sum_{h \in \mathcal{H}} \lambda_h b_m(h) + C_0
\]
over \(\lambda \geq 0\). The constant \(C_0\) is equal to \(1 - \text{Pr}(D_m)\) where \(\text{Pr}(D_m) = \text{Pr}(X \in D_m)\). An algorithm to approximately solve this problem is presented in Algorithm \((\ref{alg:approx})\). The algorithm takes a parameter \(\varepsilon > 0\) specifying the degree to which all of the constraints \((\ref{eq:cone})\) are to be approximated. Since \(\mathcal{D}\) is concave, the rescaling step can be solved using a straightforward numerical line search. The main implementation challenge is in finding the most violated constraint \((\ref{eq:cone})\). Fortunately, this step can be reduced to a single call to an ERM oracle. To see this, note that the constraint violation on classifier \(h\) can be written as
\[
\mathbb{E}_X \left[ \frac{T_h^m(X)}{P(X)} \right] - b_m(h) = \mathbb{E}_X \left[ \mathbb{1}(X \in D_m) \left( \frac{1}{P(X)} - 2\alpha^2 \right) \mathbb{1}(h(X) \neq h_m(X)) \right]
\]
\[
- 2\beta^2 \gamma \Delta_{m-1}(\text{err}(h, Z_{m-1}) - \text{err}(h_m, Z_{m-1})) - \xi \Delta_{m-1} \Delta_{m-1}^2.
\]
The first term of the right-hand expression is the risk (classification error) of \(h\) in predicting samples labeled according to \(h_m\) with importance weights of \(1/P(x) - 2\alpha^2\) if \(x \in D_m\) and 0 otherwise; note that these weights may be positive or negative. The second term is simply the scaled risk of \(h\) with respect to the actual labels. The last two terms do not depend on \(h\). Thus, given access to \(P_X\) (or samples approximating it, discussed shortly), the most violated constraint can be found by solving an ERM problem defined on the labeled samples in \(Z_{m-1}\) and samples drawn from \(P_X\) labeled by \(h_m\), with appropriate importance weights detailed in Appendix \((\ref{app:ERM})\).

When all primal constraints are approximately satisfied, the algorithm stops. Consequently, we can execute each step of Algorithm \((\ref{alg:approx})\) with one call to an appropriately defined ERM oracle, and approximate primal feasibility is guaranteed when the algorithm stops. More specifically, we can prove the following guarantee on the convergence of the algorithm.

**Theorem 3.** When run on the \(m\)-th epoch, Algorithm \((\ref{alg:approx})\) has the following guarantees.

1. It halts in at most \(\frac{\text{Pr}(D_m)}{\mathbb{P}_{\text{min,m}}^2} \varepsilon^2\) iterations.

2. The solution \(\hat{\lambda} \geq 0\) it outputs has bounded \(\ell_1\) norm: \(\|\hat{\lambda}\|_1 \leq \text{Pr}(D_m)/\varepsilon\).

3. The query probability function \(P_\lambda\) satisfies:
   - The variance constraints \((\ref{eq:cone})\) up to an additive factor of \(\varepsilon\), i.e.,
     \[
     \forall h \in \mathcal{H} \quad \mathbb{E}_X \left[ \frac{\mathbb{1}(h(x) \neq h_m(x) \land x \in D_m)}{P_\lambda(X)} \right] \leq b_m(h) + \varepsilon,
     \]
   - The simple bound constraints \((\ref{eq:cone})\) exactly.
Approximate primal optimality:
\[
\mathbb{E}_X \left[ \frac{1}{1 - P_\lambda(X)} \right] \leq \mathbb{E}_X \left[ \frac{1}{1 - P^*(X)} \right] + 4P_{\min,m}\Pr(D_m),
\]
where \( P^* \) is the solution to \((\text{OP})\).

That is, we find a solution with small constraint violation to ensure generalization, and a small objective value to be label efficient. If \( \varepsilon \) is set to \( \xi \tau_{m-1} \Delta_{m-1}^2 \), an amount of constraint violation tolerable in our analysis, the number of iterations in Theorem 3 varies between \( O(\tau_{m-1}^{3/2}) \) and \( O(\tau_{m-1}^2) \) as the \( \text{err}(h_m, \tilde{Z}_{m-1}) \) varies between a constant and \( O(1/\tau_{m-1}) \). The theorem is proved in Appendix F.3.

### 5.2 Solving \((\text{OP})\) with expectation over samples

So far we considered solving \((\text{OP})\) defined on the unlabeled data distribution \( P_X \), which is not available in practice. A simple and natural substitute for \( P_X \) is an i.i.d. sample drawn from it. Here we show that solving a properly-defined sample variant of \((\text{OP})\) leads to a solution to the original \((\text{OP})\) with similar guarantees as in Theorem 3.

More specifically, we define the following sample variant of \((\text{OP})\): Let \( S \) be a large sample drawn i.i.d. from \( P_X \), and \((\text{OP}_S)\) be the same as \((\text{OP})\) except with all population expectations replaced by empirical expectations with respect to \( S \). Now for any \( \varepsilon \geq 0 \), define \((\text{OP}_{S,\varepsilon})\) to be the same as \((\text{OP}_S)\) except that the variance constraints \( (5) \) are relaxed by an additive slack of \( \varepsilon \).

Every time \textsc{Active Cover} needs to solve \((\text{OP})\) (Step 5 of Algorithm 1), it draws a fresh unlabeled i.i.d. sample \( S \) of size \( u \) from \( P_X \), which can be done easily in a streaming setting by collecting the next \( u \) examples. It then applies Algorithm 2 to solve \((\text{OP}_{S,\varepsilon})\) with accuracy parameter \( \varepsilon \). Note that this is different from solving \((\text{OP}_S)\) with accuracy parameter \( \varepsilon \). We establish the following convergence guarantees.

**Theorem 4.** Let \( S \) be an i.i.d. sample of size \( u \) from \( P_X \). When run on the \( m \)-th epoch for solving \((\text{OP}_{S,\varepsilon})\) with accuracy parameter \( \varepsilon \), Algorithm 2 satisfies the following.

1. It halts in at most \( \hat{\Pr}(D_m) \) iterations, where \( \hat{\Pr}(D_m) := \sum_{X \in S} 1(X \in D_m)/u. \)
2. The solution \( \hat{\lambda} \geq 0 \) has bounded \( \ell_1 \) norm: \( \|\hat{\lambda}\|_1 \leq \hat{\Pr}(D_m)/\varepsilon. \)
3. If \( u \geq O(\langle 1/(P_{\min,m}\varepsilon)^4 + \alpha^4/\varepsilon^2 \rangle \log(|\mathcal{H}|/\delta)), \) then with probability \( \geq 1 - \delta \), the query probability function \( P_{\hat{\lambda}} \) satisfies:
   - All constraints of \((\text{OP})\) except with an additive slack of \( 2.5\varepsilon \) in the variance constraints \( (5) \).
   - Approximate primal optimality:
     \[
     \mathbb{E}_X \left[ \frac{1}{1 - P_{\hat{\lambda}}(X)} \right] \leq \mathbb{E}_X \left[ \frac{1}{1 - P^*(X)} \right] + 8P_{\min,m}\Pr(D_m) + (2 + 4P_{\min,m})\varepsilon,
     \]
where \( P^* \) is the solution to \((\text{OP})\).

The proof is in Appendix F.3. Intuitively, the optimal solution \( P^* \) to \((\text{OP})\) is also feasible in \((\text{OP}_{S,\varepsilon})\) since satisfying the population constraints leads to approximate satisfaction of sample constraints. Since our solution \( P_{\hat{\lambda}} \) is approximately optimal for \((\text{OP}_{S,\varepsilon})\) (this is essentially due to Theorem 3), this means that the sample objective at \( P_{\hat{\lambda}} \) is not much larger than \( P^* \). We now use a concentration argument to show that this guarantee holds also for the population objective with slightly worse constants. The approximate constraint satisfaction in \((\text{OP})\) follows by a similar concentration argument. Our proofs use standard concentration inequalities along with Rademacher complexity to provide uniform guarantees for all vectors \( \lambda \) with bounded \( \ell_1 \) norm.

The first two statements, finite convergence and boundedness of \( \|\hat{\lambda}\|_1 \), are identical to Theorem 3 except \( \Pr(D_m) \) is replaced by \( \hat{\Pr}(D_m) \). When \( \varepsilon \) is set properly, i.e., to be \( \xi^2 \tau_{m-1} \Delta_{m-1}^2 \), the number of unlabeled examples \( u \) in the third statement varies between \( O(\tau_{m-1}^2) \) and \( O(\tau_{m-1}^4) \) as the \( \text{err}(h_m, \tilde{Z}_{m-1}) \) varies between a constant and \( O(1/\tau_{m-1}) \). The third statement shows that with enough unlabeled examples, we can get a query probability function almost as good as the solution to the population problem \((\text{OP})\).
6 Experiments with Agnostic Active Learning

While AC is efficient in the number of ERM oracle calls, it needs to store all past examples, resulting in large space complexity. As Theorem 3 suggests, the query probability function (13) may need as many as $O(\tau_i^2)$ classifiers, further increasing storage demand. In Section 6.1 we discuss a scalable online approximation to ACTIVE COVER, ONLINE ACTIVE COVER (OAC), which we implemented and tested empirically with the setup in Section 6.2. Experimental results and discussions are in Section 6.3.

6.1 Online Active Cover (OAC)

Algorithm 3 gives the online approximation that we implemented, which uses an epoch schedule of $\tau_i = i$, assigning every new example to a new epoch. It involves some new notations which we explain below. To make explicit the dependence of a query probability function on both a weight vector $\lambda$ over classifiers and the current epoch, we write them as subscripts:

$$q_{\lambda,i}(x) := \sqrt{(2P_{\min,i})^2 + \sum_h \lambda_h \mathbb{1}(h(x) \neq h_i(x))},$$

(20)

$$P_{\lambda,i}(x) := \mathbb{1}(x \in D_i) \frac{q_{\lambda,i}(x)}{1 + q_{\lambda,i}(x)}.$$  

(21)

We use $\mathbb{1}_h$ for a $|H|$-dimensional binary vector with 1 in the entry corresponding to the classifier $h$ and 0 elsewhere.

To explain the connections between Algorithms 1 (AC) and 3 (OAC), we start with the update of the ERM classifier and thresholds, corresponding to Step 1 of AC and Step 6 of OAC. Instead of batch ERM oracles, OAC invokes online importance weighted ERM oracles that are stateful and process examples in a streaming fashion without the need to store them. The specific importance weighted oracle we use is a reduction to online importance-weighted logistic regression [Karampatziakis and Langford, 2011] implemented in Vowpal Wabbit (VW).

Instead of computing the query probability function by solving a batch optimization problem as in Step 5 of AC, OAC maintains a fixed number $l$ of classifiers that are intended to be a cover of the set of good classifiers. On every new example, this cover undergoes a sequence of online, importance weighted updates (Steps 7 to 13 of OAC), which are meant to approximate the coordinate ascent steps in Algorithm 2. The importance structure (16) is derived from (57), accounting for the fact that the algorithm simply uses the incoming stream of examples to estimate $\mathbb{E}_{X}[\cdot]$ rather than a separate unlabeled sample. The same approximation is also present in the updates (17) and (18), which are online estimates of the numerator and the denominator of the additive coordinate update in Step 7 of Algorithm 2. Because (17) is an online estimate, we need to explicitly enforce non-negativity.

Finally, Steps 9 to 14 of AC and Steps 14 to 26 of OAC perform the querying of labels. As pointed out in Section 5, the test in Step 16 of OAC is done via an online technique detailed in Appendix F of Karampatziakis and Langford [2011].

6.2 Setup

We conduct an empirical comparison of OAC with the following active learning algorithms.

- IWAL: A slight modification of Algorithm 1 of Beygelzimer et al. [2010], which performs importance-weighted sampling of labels and maintains an unbiased estimate of classification error. In computing the query probability, rather than using a conservative, problem-independent threshold as in Beygelzimer et al. [2010], we use the following error-dependent quantity:

$$\sqrt{C_0 \log k \frac{e_{k-1}}{k-1} + C_0 \log \frac{k}{k-1}},$$

(22)

where $e_{k-1}$ is the importance-weighted error estimate after the algorithm processes $k-1$ examples. $C_0$ is the only active learning hyper-parameter. The query probability is set to 1 if the error difference $G_k$, defined in Step 2 of Algorithm 1 of Beygelzimer et al. [2010], is smaller than the threshold (22), and otherwise a decreasing function of $G_k$.
Algorithm 3 Online Active Cover

input: cover size \( l \), parameters \( c_0 \) and \( \alpha \).

1. Initialize online importance weighted minimization oracles \( \{O_i\}_{i=0}^l \), each controlling a classifier and some associated weights \( \{(h(t), \lambda(t), \nu(t), \omega(t))\}_{t=1}^l \) with all weights initialized to 0.
2. Query the first three \( \{(X_1, Y_1)\}_{i=1}^3 \) and stream \( \{(X_i, Y_i, 1)\}_{i=1}^2 \) through \( O_0 \).
3. Get classifier \( h_3 \) and error estimate \( c_2 \) from \( O_0 \), and compute \( P_{\min,3} \).
4. Let \( Y_3^* := Y_3, \tilde{Y}_3 := h_3(X_3) \) and \( W_3 := 1 \). Set \( \beta := (\sqrt{\alpha/c_0})/10 \).
5. for \( i = 4, \ldots, n \) do
   6. Update the ERM, the error estimate and the threshold
      \[
      h_i := O_0((X_{i-1}, Y_{i-1}^*, W_{i-1})), \\
      e_{i-1} := \frac{(i - 2)e_{i-2} + 1(Y_{i-1} \neq Y_{i-1}^*)W_{i-1}}{i - 1}, \\
      \Delta_{i-1} := \sqrt{c_0 e_{i-1}/(i - 1)} + \max(2\alpha/4, c_0 \log(i - 1)/(i - 1)).
      \]
   7. for \( t = 1, \ldots, l \) do
      8. \( \lambda_t := \sum_{t' < t} \lambda(t') 1_{h(t')} \).
      9. \( S_t := 2\alpha^2 - 1/P_{\lambda_t,i-1}(X_{i-1}) \).
      10. Set up a vector \( c \)
          \[
          c_y := \mathbb{1}(X_{i-1} \in D_{i-1})|S_t| \mathbb{1}(y \neq \text{sign}(S_t)\tilde{Y}_{i-1}) + \\
               2\beta^2(i - 2)\Delta_{i-2} \left( \mathbb{1}(X_{i-1} \notin D_{i-1}) \mathbb{1}(y \neq \tilde{Y}_{i-1}) + \mathbb{1}(X_{i-1} \in D_{i-1}) \frac{Q_{i-1} \mathbb{1}(y \neq Y_{i-1})}{P_{i-1}} \right). \]  
      11. Set \( Y_{i(t)} := \arg \min_y c_y \) and \( W_{i(t)} := |c_1 - c_{-1}| \).
      12. Update \( h(t), \lambda(t), \nu(t), \omega(t) \) as follows:
          \[
          h(t) := O_t((X_{i-1}, Y_{i(t)}, W_{i(t)})), \\
          \nu(t) := \max \left( \nu(t) - 2(c_{h(t)}(X_{i-1}) + \min(S_t, 0) \mathbb{1}(X_{i-1} \in D_{i-1})), 0 \right), \\
          \omega(t) := \omega(t) + \mathbb{1}(h(t)(X_{i-1}) \neq Y_{i-1} \wedge X_{i-1} \in D_{i-1}) \frac{Q_{\lambda_t,i}(X_{i-1})^3}{P_{\lambda_t,i}(X_{i-1})^3}, \\
          \lambda(t) := \lambda(t) + \frac{\nu(t)}{\omega(t)} \mathbb{1}(\nu(t), \omega(t) \neq (0, 0)).
          \]
      13. end for
      14. Receive data point \( X_i \) and predict \( \tilde{Y}_i := h_3(X_i) \).
      15. Compute \( P_{\min,i} := \min \left( (\sqrt{(i - 1)e_{i-1} + \log(i - 1)})^{-1}, 1/2 \right) \).
      16. if \( X_i \in D_i := \text{DIS}(A_i) \), then
          17. Compute \( P_i := P_{\lambda_t,i}(X_i) \), where \( \lambda := \sum_{t=1}^l \lambda(t) 1_{h(t)} \).
          18. Draw \( Q_i \sim \text{Bernoulli}(P_i) \).
          19. if \( Q_i = 1 \) then
              20. Query \( Y_i \) and set \( Y_i^* := Y_i, W_i := 1/P_i \).
          21. else
              22. Set \( Y_i^* := 1, W_i := 0 \).
          23. end if
      24. else
          25. Set \( Y_i^* := \tilde{Y}_i, W_i := 1 \).
      26. end if
      27. end for
• ORA-I: An Oracular-CAL [Hsu, 2010] style variant of Algorithm 3. If the test in Step 16 of Algorithm 3 is true, meaning the new example $X_i$ is in the current disagreement region, the query probability $P_i$ is set to 1. This algorithm does not need to maintain a cover, but still uses two tuning parameters $c_0$ and $\alpha$ to compute the threshold (19). Note that both the generalization and label complexity guarantees (Theorems 1 and 2) apply to this variant if a batch ERM oracle is used.

• ORA-II: An Oracular-CAL [Hsu, 2010] style variant of IWAL, where the query probability is set to 1 if the error difference $G_k$, defined in Step 2 of Algorithm 1 of Beygelzimer et al. [2010], is smaller than the threshold
\[
\sqrt{\frac{C_0 \log k}{k-1}} e_{k-1} + \frac{C_0 \log k}{k-1}.
\]
Otherwise, the algorithm uses predicted labels by the current ERM hypothesis. Note that the error estimate $e_{k-1}$ now uses both the queried labels and predicted labels, and is no longer unbiased. Like IWAL, it only has one hyper-parameter $C_0$.

• RANDOM: Passive learning on a labeled sub-sample drawn uniformly at random.

We implemented these algorithms\(^8\) in Vowpal Wabbit (VW) and performed experiments on 23 binary classification datasets with varying sizes ($10^3$ to $10^6$) and diverse feature characteristics. Details about the datasets are in Appendix G.1. For each dataset we performed a random 80/20 training/testing split, ran the five algorithms under various hyper-parameter settings each for one pass over the training data, and evaluated the learned classifiers on testing data. Our goals are:

1. Understanding the trade-offs between test error and query rate achieved by different algorithms;
2. Comparing different algorithms when each uses the best fixed hyper-parameter setting.

With regard to the second goal, note that it is in general very difficult to select active learning hyper-parameters on a per-task basis because labeled validation data are not available. However, with a variety of classification datasets, it might still be reasonable to look for the single hyper-parameter setting that performs the best on average across datasets, thereby reducing over-fitting to any individual dataset, and compare different algorithms under such fixed parameter settings. More details about hyper-parameters are in Appendix G.2.

6.3 Results and Discussions

Figure 2 gives a summary of the performances of different algorithms. At a fixed query rate, defined as the fraction of label queries at the end of one pass over the training data, we consider the minimum test error achievable with at most that query rate. Taking this for each algorithm on each dataset, we compute the fraction of datasets on which an algorithm wins against RANDOM. Figure 2(a), which is the same as Figure 1, plots the win fractions\(^9\) at different query rates. OAC dominates all other agnostic active learning algorithms, and outperforms RANDOM except at low query rates where it is on parity. This result also shows that RANDOM is a strong baseline at low query rates.

Figure 2(b) reveals the magnitude of the performance differences. Here we only show results for OAC and IWAL because ORA-I and ORA-II are significantly worse. To account for the varying hardness of different datasets, we scale the test error on a per-dataset basis. Formally, let $\text{error}(a, p, d)$ denote the test error achieved by algorithm $a$ with hyper-parameter setting $p$ on dataset $d$. Let $\text{error}_{\text{max}}(d)$ and $\text{error}_{\text{min}}(d)$ denote the maximum and minimum test errors on dataset $d$ any algorithm can achieve with any hyper-parameter setting. Then we define the relative test error as
\[
\text{rel}_\text{err}(a, p, d) = \frac{\text{error}(a, p, d) - \text{error}_{\text{min}}(d)}{\text{error}_{\text{max}}(d) - \text{error}_{\text{min}}(d)}.
\]
As in Figure 2(a), at a fixed query rate we consider the minimum relative test error achievable with at most that query rate. Taking this for each algorithm on each dataset, we compute the improvement in relative test error with respect to

\(^8\)For IWAL we use the existing implementation by the authors in VW.

\(^9\)A tie counts as 0.5.
Figure 2: Summary of results on 23 datasets. In the right figure, the medians over datasets are connected across query rates, while the bars extend from the 25-th to the 75-th quantile. OAC outperforms RANDOM on most datasets at most query rates (see text for more details).

Figure 3: Minimum test error vs. query rate with different noise regimes. Full results on remaining datasets are in Appendix G.3.

\[ \text{OAC is on par with RANDOM across the datasets at query rates lower than } 10^{-2}, \text{ and outperforms RANDOM at higher query rates. The magnitude of improvement is not very large, but it should be noted that on many of these datasets a very large improvement is not necessarily possible. In contrast, IWAL has higher relative test errors at query rates lower than } 10^{-2}, \text{ and only becomes competitive with RANDOM at higher query rates.} \]

\[ \text{Figure 3 shows minimum test errors at different query rates for three specific datasets. They are relatively large in size (more than } 10^5 \text{ examples) and possess different levels of difficulties. The advantage of OAC over other algorithms is clear especially at low query rates. Results for the remaining 20 datasets are in Appendix G.3.} \]

\[ \text{Figure 4 gives results obtained by different algorithms under fixed hyper-parameter settings, which are selected to optimize trade-offs between test errors and query rates across all datasets. Let } \text{query}(a, p, d) \text{ be the query rate of algorithm } a \text{ with hyperparameters } p \text{ on dataset } d. \text{ Given a weight parameter } w \in [0, 1], \text{ we define for each algorithm } a \text{ the best hyper-parameter setting as} \]

\[ p_w^*(a) := \arg \min_w \text{median}_{d} \text{perf}(w, a) := \{ w \cdot \text{query}(a, p, d) + (1 - w) \cdot \text{rel.err}(a, p, d) \}, \quad (24) \]

15
Figure 4: Fraction of datasets for which an algorithm reaches different levels of \( \text{perf}(w, a) \) with parameters \( p^*_w(a) \) at \( w = 0.1 \) (left) and \( w = 0.9 \) (right). Intercepts with a horizontal line reveals the \( \text{perf} \) advantage of one algorithm over another, typically in the range of 2 to 10 between best and worst. Intercepts with a vertical line reveals the reliability of the algorithm with a typical 20% gap observed between best and worst.

Figure 5: Relative test errors vs. query rates under fixed hyper-parameter settings where \( \text{rel_err} \) is as defined in Equation \( 23 \).

Figures 4(a) and 4(b) demonstrate how well each algorithm optimizes the combined metric \( 24 \) by plotting the curve of cumulative fraction of datasets on which an algorithm, with parameters fixed at \( p^*_w(a) \) across datasets, achieves no more than a certain value of the weighted sum \( \text{perf}(w, a) \) in \( 24 \). A higher curve thus means a better performance. When the weight on query rate is 0.1, \( \text{perf}(w, a) \) tends to care mostly about test error, and the unbiased nature of IWAL possibly helps it outperform other algorithms at high query rates, while OAC is competitive at low query rates. At the other extreme when the weight is 0.9, query rate matters much more and OAC is superior to others.

Figure 5 plots relative test errors and query rates across all datasets achieved by each algorithm using its best single parameter \( p^*_w(a) \) in \( 24 \), with \( w \) varying from 0.1 to 0.9. The markers are plotted at
\[
(\text{median}_d \{ \text{query}(a, p^*_w(a), d) \}, \text{median}_d \{ \text{rel_err}(a, p^*_w(a), d) \})
\]
for each algorithm \( a \), and the vertical bars extend from the 25th to the 75th quantile of each algorithm \( a \)’s relative test errors achieved with \( p^*_w(a) \) across datasets. OAC in general achieves test errors comparable to the other algorithms, but at lower query rates.

How much headroom for improvement is there by a better automatic tuning of hyperparameters? In addition to
Figure 6: Fraction of datasets for which an algorithm reaches different levels of \( \text{perf}(w, a) \) with parameters \( p^*_w(a, d) \) optimized on a per-dataset basis at \( w = 0.1 \) (left) and \( w = 0.9 \) (right). Note that when \( w = 0.9 \), because hyper-parameters are chosen for each dataset independently and \( \text{rel}_\text{err} \leq 1 \), an algorithm can easily achieve \( \text{perf} \leq 0.1 \) on any dataset by using the hyper-parameter that results in a nearly zero query rate on that dataset.

results obtained with a fixed hyper-parameter setting, we examine performances of different algorithms when each uses the best hyper-parameter on a per-dataset basis. Figure 6 is the counterpart of Figure 4 in this setting, showing the cumulative fractions of datasets for which each algorithm \( a \) achieves a certain value of the performance metric \( \text{perf} \) in (24) when using dataset-dependent best hyper-parameters:

\[
p^*_w(a, d) := \arg \min_p \min_d \text{perf}(w, a) := \left\{ w \cdot \text{query}(a, p, d) + (1 - w) \cdot \frac{\text{error}(a, p, d) - \text{error}_{\min}(d)}{\text{error}_{\max}(d) - \text{error}_{\min}(d)} \right\}.
\]

Figure 6 suggests the possibility that with the right hyper-parameter settings, OAC may dominate all other algorithms at both extremes of the query-rate vs. test-error trade-off. Overall, we find that OAC achieves reasonable performance gains in terms of good generalization with a few labeled examples, compared with a number of baselines on a diverse collection of datasets.

7 Analysis of generalization ability

In this section we present the main framework and analysis for the results on the generalization properties of the ACTIVE COVER algorithm. Our analysis is broken up into several steps. We start by setting up some additional notation for the proofs. Our analysis relies on two deviation bounds for the empirical regret and the empirical error of the ERM classifier. These are obtained by appropriately applying Freedman-style concentration bounds for martingales. Both these bounds depend on the variance and range of our error and regret estimates for all classifiers \( h \in \mathcal{H} \), and these quantities are controlled using the constraints (5) and (6) in the definition of the optimization problem (OP). Since our data consists of examples from different epochs, which use different query probabilities \( P_m \), the above steps with appropriate manipulations yield bounds for the epoch \( m \), in terms of various quantities involving the previous epochs. Theorem 1 and its corollaries are then obtained by setting up appropriate inductive claims. We make this intuition precise in the following sections.

7.1 Framework for generalization analysis

Before we can prove our main results, we recall some notations and introduce a few additional ones. We also prove some technical lemmas in this section which are used to prove our main results.

Recall the notation \( \text{reg}(h, h') := \text{err}(h) - \text{err}(h'), h^* \in \arg \min_{h \in \mathcal{H}} \text{err}(h), \text{reg}(h) := \text{reg}(h, h^*) \). Let \( Z_m \) denote the set of importance-weighted examples in \( \tilde{Z}_m \), and the corresponding empirical error is denoted as:
\[
err(h, Z_m) := \frac{1}{\tau_m} \sum_{j=1}^{m} \sum_{i=\tau_{j-1}+1}^{\tau_j} \left( \frac{Q_i \mathbb{I}(h(X_i) \neq Y_i \land X_i \in D_j)}{P_j(X_i)} \right).
\]  

(25)

Taking expectations, we define the following quantities with respect to the sequence of regions \(\{D_m\}\):

\[
\begin{align*}
err_m(h) & := \mathbb{E}_{X,Y}[\mathbb{I}(h(X) \neq Y \land X \in D_m)], \\
err_m(h^*) & := \mathbb{E}_{X,Y}[\mathbb{I}(h(X) \neq Y \land X \in D_m)].
\end{align*}
\]

(26)

Intuitively, \(err_m\) captures the population error of \(h\), restricted to only the examples in the disagreement region. This is also the expectation of the sample error restricted to the importance-weighted examples in epoch \(m\). Averaging these quantities, we obtain \(\overline{err}_m\) which is the expectation of the sample error over \(Z_m\). Centering around the corresponding errors of \(h^*\), we obtain the following regret terms:

\[
\begin{align*}
\overline{reg}_m(h) & := err_m(h) - err_m(h^*), \\
\overline{reg}_m(h) & := \frac{1}{\tau_m} \sum_{j=1}^{m} (\tau_j - \tau_{j-1}) \overline{reg}_j(h).
\end{align*}
\]

While the above quantities only concern the importance-weighted examples, it is also useful to measure error and regret terms over the entire biased sample. We define the empirical error and regret on \(Z_m\) as follows:

\[
\begin{align*}
err(h, \tilde{Z}_m) & := \frac{1}{\tau_m} \sum_{j=1}^{m} \sum_{i=\tau_{j-1}+1}^{\tau_j} \left( \mathbb{I}(h(X_i) \neq h_j(X_i) \land X_i \notin D_j) + \frac{Q_i \mathbb{I}(h(X_i) \neq Y_i \land X_i \in D_j)}{P_j(X_i)} \right), \\
\overline{reg}(h, h', \tilde{Z}_m) & := err(h, \tilde{Z}_m) - err(h', \tilde{Z}_m),
\end{align*}
\]

and the associated expected regret:

\[
\begin{align*}
\overline{\overline{reg}}_m(h, h') & := \mathbb{E}_X[(\mathbb{I}(h(X) \neq h_m(X)) - \mathbb{I}(h'(X) \neq h_m(X)))\mathbb{I}(X \notin D_m)] + \\
& \quad \mathbb{E}_{X,Y}[(\mathbb{I}(h(X) \neq Y) - \mathbb{I}(h'(X) \neq Y))\mathbb{I}(X \in D_m)], \\
\overline{\overline{reg}}_m(h, h') & := \frac{1}{\tau_m} \sum_{j=1}^{m} (\tau_j - \tau_{j-1}) \overline{\overline{reg}}_j(h, h').
\end{align*}
\]

(27)

(28)

To simplify notation, we sometimes use the shorthand \(\overline{\overline{\overline{reg}}}_m(h, \tilde{Z}_m) := \overline{\overline{\overline{reg}}}_m(h, h_{m+1}, \tilde{Z}_m)\). The quantity \(\overline{\overline{\overline{reg}}}_m(h, h')\) will play quite a central role in our analysis as it is the expectation of the empirical regret of \(h\) relative to \(h'\) on our biased sample \(\tilde{Z}_m\). We also recall the earlier notations

\[
\begin{align*}
\Delta_m & := c_1 \sqrt{\epsilon_m \overline{\overline{\overline{reg}}}(h_{m+1}, \tilde{Z}_m)} + c_2 \epsilon_m \log \tau_m, \\
A_{m+1} & := \{h \mid err(h, \tilde{Z}_m) - err(h_{m+1}, \tilde{Z}_m) \leq \gamma \Delta_m\}, \quad \text{and} \\
\Delta_m^* & := \begin{cases} 
\left( c_1 \sqrt{\epsilon_m \overline{\overline{\overline{reg}}}(h^*)} + c_2 \epsilon_m \log \tau_m \right), & m \geq 1, \\
\Delta_0, & \quad m = 0.
\end{cases}
\end{align*}
\]
Throughout the paper, we adopt the convention that the quantities \[26\] to \[3\] take the value of zero when \(m = 0\). We use the shorthand \(m(i)\) to denote the epoch containing example \(i\).

With the notations in place, we start with an extremely important lemma, which shows that the biased sample \(\tilde{Z}\) which we create introduces a bias in the favor of good hypotheses, overly penalizing the bad hypotheses while favorably evaluating the optimal \(h^*\).

**Lemma 1** (Favorable Bias). \(\forall m \geq 1, \forall \tilde{h} \in A_m, \forall h \in \mathcal{H}\), the following holds:

\[
\text{reg}^m_{\cdot h}(\tilde{h}, \tilde{h}) \geq \text{reg}(h, \tilde{h}).
\]

The next key ingredient for our proofs is a deviation bound, which will be appropriately used to control the deviation of the empirical regret and error terms.

**Lemma 2** (Deviation Bounds). Pick \(0 < \delta < 1/e\) such that \(|\mathcal{H}|/\delta > \sqrt{192}\). With probability at least \(1 - \delta\) the following holds. For all \((h, h') \in \mathcal{H}^2\) and \(\forall m \geq 1\),

\[
|\text{reg}_{\cdot m}^m(h, h') - \text{reg}(h, h', \tilde{Z}_m)| \leq \sqrt{\frac{\epsilon_m}{T_m} \sum_{i=1}^m (\tau_i - \tau_{i-1}) \mathbb{E}_X \left[ \left( \mathbb{I}(X \notin D_i) + \frac{\mathbb{I}(X \in D_i)}{P_i(X)} \right) \mathbb{I}(h(X) \neq h'(X)) \right] + \frac{\epsilon_m}{P_{\text{min}, m}}, \tag{29}
\]

\[
|\text{err}(h, Z_m) - \text{err}_{\cdot m}(h)| \leq \sqrt{\frac{\epsilon_m}{T_m} \sum_{i=1}^m (\tau_i - \tau_{i-1}) \mathbb{E}_{X,Y} \left[ \frac{\mathbb{I}(X \in D_i \land h(X) \neq Y)}{P'_i(X)} \right] + \frac{\epsilon_m}{P_{\text{min}, m}}, \tag{30}
\]

where \(\epsilon_m := 32 \left( \frac{\log(|\mathcal{H}|/\delta) + \log \tau_m}{\tau_m} \right)\).

The lemma is obtained by applying a form of Freedman’s inequality presented in Appendix \[\text{A}\]. Intuitively, the deviations are small so long as the average importance weights over the disagreement region and the minimum query probability over the disagreement region are well-behaved. This lemma also highlights why \(\tilde{Z}\) concentrates around it.

To keep the handling of probabilities simple, we assume for the bulk of this section that the conclusions of Lemma \[\text{2}\] hold deterministically. The failure probability is handled once at the end to establish our main results. Let \(\mathcal{E}\) denote the event that the assertions of Lemma \[\text{2}\] hold deterministically, and we know that \(\Pr(\mathcal{E}^C) \leq \delta\). Based on the above lemma, we obtain the following propositions for the concentration of empirical regret and error terms.

**Proposition 1** (Regret concentration). Fix an epoch \(m \geq 1\). Suppose the event \(\mathcal{E}\) holds and assume that \(h^* \in A_j\) for all epochs \(j \leq m\).

\[
|\text{reg}(h, h^*, \tilde{Z}_m) - \tilde{\text{reg}}_{\cdot m}^m(h, h^*)| \leq \frac{1}{4} \tilde{\text{reg}}_{\cdot m}^m(h) + 2\alpha \sqrt{\frac{\epsilon_m}{T_m} \sum_{i=1}^m (\tau_i - \tau_{i-1}) \text{reg}_i(h_i) + 2\alpha \sqrt{3\text{err}_{\cdot m}(h^*)\epsilon_m}} + \sqrt{2\gamma \epsilon_m \Delta_m \sum_{i=1}^m (\tau_i - \tau_{i-1}) \left( \text{reg}(h, \tilde{Z}_{i-1}) + \text{reg}(h^*, \tilde{Z}_{i-1}) \right) + 4\Delta_m}
\]

\[19\]
We need an analogous result for the empirical error of the ERM at each epoch.

**Proposition 2 (Error concentration).** Fix an epoch \( m \geq 1 \). Suppose the event \( \mathcal{E} \) holds and assume that \( h^* \in A_j \) for all epochs \( j \leq m \).

\[
\left| \text{err}_m(h^*) - \text{err}(h_{m+1}, \tilde{Z}_m) \right| \leq \frac{\text{err}_m(h^*)}{2} + \frac{3\Delta_m}{2} + \text{reg}(h^*, h_{m+1}, \tilde{Z}_m).
\]

We now present the proofs of our main results based on these propositions.

### 7.2 Proofs of main results

We prove a more general version of the theorem. Theorem 1 and its corollaries follow as consequences of this more general result.

**Theorem 5.** For all epochs \( m = 1, 2, \ldots, M \) and all \( h \in \mathcal{H} \), we have with probability at least \( 1 - \delta \)

\[
|\text{reg}(h, h^*, \tilde{Z}_m) - \text{reg}_m(h, h^*)| \leq \frac{1}{2} \text{reg}_m(h, h^*) + \frac{\eta\Delta_m}{4},
\]

\[
\text{reg}(h^*, h_{m+1}, \tilde{Z}_m) \leq \frac{\eta\Delta_m}{4},
\]

\[
|\text{err}_m(h^*) - \text{err}(h_{m+1}, \tilde{Z}_m)| \leq \frac{\text{err}_m(h^*)}{2} + \frac{\eta\Delta_m}{2}.
\]

The theorem is proved inductively. We first give the proof outline for this theorem, and then show how Theorem 1 and its corollaries follow.

#### 7.2.1 Proof of Theorem 5

The theorem is proved via induction. Let us start with the base case for \( m = 1 \). Clearly,

\[
|\text{reg}(h, h^*, \tilde{Z}_1) - \text{reg}_1(h, h^*)| \leq 1 \leq \eta\Delta_1/4,
\]

since \( \eta_{\text{min},1} = 1 \). The conclusions for the second and third statements follow similarly. This establishes the base case.

Let us now assume that the hypothesis holds for \( i = 1, 2, \ldots, m - 1 \) and we establish it for the epoch \( i = m \). We start from the conclusion of Proposition 1 which yields

\[
|\text{reg}(h, h^*, \tilde{Z}_m) - \text{reg}_m(h, h^*)| \leq \frac{1}{2} \text{reg}_m(h, h^*) + 2\alpha \sqrt{\frac{\epsilon_m}{\tau_m}} \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \text{reg}(h_i) + 2\alpha \sqrt{3\text{err}_m(h^*)\epsilon_m / \tau_2}
\]

\[
+ \beta \sqrt{2\gamma\epsilon_m \Delta_m \sum_{i=1}^{m} (\tau_i - \tau_{i-1})(\text{reg}(h, \tilde{Z}_{i-1}) + \text{reg}(h^*, \tilde{Z}_{i-1})) + 4\Delta_m / \tau_3}
\]

We now control \( \tau_1, \tau_2 \) and \( \tau_3 \) in the sum using our inductive hypothesis and the propositions in a series of lemmas. To state the lemmas cleanly, let \( \mathcal{E}_m \) refer to the event where the bounds (31), (33) hold at epoch \( m \). Then we have the following lemmas. The first lemma gives a bound on \( \tau_1 \).
Lemma 3. Suppose that the event $\mathcal{E}$ holds and that the events $\mathcal{E}_i$ hold for all epochs $i = 1, 2, \ldots, m - 1$. Then we have

$$2\alpha \sqrt{2^{m} \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \text{reg}_m(h_i)} \leq \frac{\eta \Delta_m}{12} + 24\alpha^2 \epsilon_m \log \tau_m.$$ 

Intuitively, the lemma holds since Lemma 1 allows us to bound $\text{reg}_m(h_i)$ with $\text{reg}_m(h_i)$. The latter is then controlled using the event $\mathcal{E}_i$. Some algebraic manipulations then yield the lemma, with a detailed proofs in Appendix C.

Lemma 4. Suppose that the event $\mathcal{E}$ holds and that the events $\mathcal{E}_i$ hold for all epochs $i = 1, 2, \ldots, m - 1$. Then we have

$$2\alpha \sqrt{\frac{3\epsilon_m \epsilon_m}{\tau_m} \epsilon_m} \leq 2\alpha \sqrt{6\epsilon_m \epsilon_m \epsilon_m \epsilon_m \epsilon_m \epsilon_m \epsilon_m} + \Delta_m + \frac{1}{4} \text{reg}(h^*, h_m, Z_m) + 33\alpha^2 \epsilon_m.$$ 

The lemma follows more or less directly from Proposition 2 combined with some algebra. Finally, we present a lemma to bound $T_2$.

Lemma 5. Suppose that the event $\mathcal{E}$ holds and that the events $\mathcal{E}_i$ hold for all epochs $i = 1, 2, \ldots, m - 1$. Then we have

$$\beta \sqrt{2\gamma \epsilon_m \Delta_m \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \text{reg}(h, Z_i - 1) + \text{reg}(h^*, \tilde{Z}_i - 1)} \leq \frac{1}{4} \text{reg}_m(h, h^*) + \frac{7\eta \Delta_m}{72}.$$ 

The $\text{reg}(h^*, h_i, \tilde{Z}_i - 1)$ terms in the lemma are bounded directly due to the event $\mathcal{E}_i$. For the second term, we observe that the empirical regret of $h$ relative to $h^*$ is not too different from the empirical regret to $h^*$ (since $h^*$ has a small empirical regret by $\mathcal{E}_i$). Furthermore, the empirical regret to $h^*$ is close to $\text{reg}_i(h, h^*)$ by the event $\mathcal{E}_i$. These observations, along with some technical manipulations, yield the lemma.

Given these lemmas, we can now prove the theorem in a relatively straightforward manner. Given our inductive hypothesis, the events $\mathcal{E}_i$ indeed hold for all epochs $i = 1, 2, \ldots, m - 1$ which allows us to invoke the lemmas. Substituting the above bounds on $T_1$ from Lemma 5, $T_2$ from Lemma 4, and $T_3$ from 5 into Proposition 1 yields

$$|\text{reg}(h, h^*, \tilde{Z}_m) - \text{reg}_m(h, h^*)|$$

$$\leq \frac{1}{4} \text{reg}_m(h) + \frac{\eta \Delta_m}{12} + 24\alpha^2 \epsilon_m \log \tau_m + 2\alpha \sqrt{6\epsilon_m \epsilon_m \epsilon_m \epsilon_m \epsilon_m \epsilon_m} + \Delta_m + \frac{1}{4} \text{reg}(h^*, h_m, \tilde{Z}_m) + 33\alpha^2 \epsilon_m + \frac{1}{4} \text{reg}_m(h, h^*) + \frac{7\eta \Delta_m}{72} + 4\Delta_m$$

$$\leq \frac{1}{2} \text{reg}_m(h, h^*) + 57\alpha^2 \epsilon_m \log \tau_m + \frac{13\eta \Delta_m}{72} + 2\alpha \sqrt{6\epsilon_m \epsilon_m \epsilon_m \epsilon_m \epsilon_m \epsilon_m} + 5\Delta_m$$

Further recalling that $c_1 \geq 2\alpha \sqrt{6}$ and $c_2 \geq 57\alpha^2$ by our assumptions on constants, we obtain
\[ |\text{reg}(h, h^*, \tilde{Z}_m) - \tilde{\text{reg}}_m(h, h^*)| \leq \frac{1}{2}\tilde{\text{reg}}_m(h, h^*) + \frac{13\eta}{72}\Delta_m + 6\Delta_m + \frac{1}{4}\text{reg}(h^*, h_{m+1}, \tilde{Z}_m). \]  

(34)

To complete the proof of the bound (31), we now substitute \( h = h_{m+1} \) in the above bound, which yields

\[ \frac{1}{2}\tilde{\text{reg}}_m(h_{m+1}, h^*) - \frac{5}{4}\text{reg}(h, h^*, \tilde{Z}_m) \leq \frac{13\eta}{72}\Delta_m + 6\Delta_m. \]

Since \( h^* \in A_i \) for all epochs \( i \leq m \), we have \( \tilde{\text{reg}}_m(h, h^*) \geq \text{reg}(h, h^*) \geq 0 \) for all classifiers \( h \in \mathcal{H} \). Consequently, we see that

\[ \text{reg}(h^*, h_{m+1}, \tilde{Z}_m) = -\text{reg}(h_{m+1}, h^*, \tilde{Z}_m) \leq \frac{52\eta}{360}\Delta_m + \frac{24}{5}\Delta_m \leq \frac{\eta}{4}\Delta_m, \]

(35)

where the last inequality uses the condition \( 38\eta \geq 1728 \). We can now substitute this back into our earlier bound (34) and obtain

\[ |\text{reg}(h, h^*, \tilde{Z}_m) - \tilde{\text{reg}}_m(h, h^*)| \]

\[ \leq \frac{1}{2}\tilde{\text{reg}}_m(h, h^*) + \frac{13\eta}{72}\Delta_m + 6\Delta_m + \frac{\eta}{16}\Delta_m \leq \frac{1}{2}\tilde{\text{reg}}_m(h, h^*) + \frac{\eta}{4}\Delta_m, \]

where we use the condition \( \eta/144 \geq 6 \). This completes the proof of the first part of our inductive claim.

For the second part, this is almost a by product of the first part through Equation (35). Recalling that \( \gamma \geq \eta/4 \) by assumption, this ensures that \( h^* \in A_{m+1} \).

We next establish the third part of the claim. This is obtained by combining our bound (35) with Proposition 2. We have

\[ |\text{err}_m(h^*) - \text{err}(h_{m+1}, \tilde{Z}_m)| \leq \frac{\text{err}_m(h^*)}{2} + \frac{3\Delta_m}{2} + \text{reg}(h^*, h_{m+1}, \tilde{Z}_m) \]

\[ \leq \frac{\text{err}_m(h^*)}{2} + \frac{3\Delta_m}{2} + \frac{\eta\Delta_m}{4} \]

\[ \leq \frac{\text{err}_m(h^*)}{2} + \frac{\eta\Delta_m}{2}, \]

since \( \eta \geq 6 \). This completes the third part.

Finally, note that our analysis has been conditioned on the event \( \mathcal{E} \) so far. By Lemma 2, \( \Pr(\mathcal{E}^C) \leq \delta \), which completes the proof of the theorem.

We now provide a proof for Theorem 1.

7.2.2 Proof of Theorem 1

We only prove the first part of the theorem. The second part is simply a restatement of the inequality (32) in Theorem 5. The first part is essentially a restatement of (31) in Theorem 5 except the bound uses \( \Delta_m \) instead of \( \Delta_m \). In order to prove the theorem, pick any epoch \( m \leq M \) and \( h \in A_{m+1} \). Because \( h^* \in A_j \), \( 1 \leq j \leq m + 1 \), we have by Lemma 1 that

\[ \text{reg}(h) \leq \tilde{\text{reg}}_m(h, h^*). \]

It then suffices to bound \( \tilde{\text{reg}}_m(h, h^*) \). By the deviation bound (31), we have

\[ \tilde{\text{reg}}_m(h, h^*) \leq \text{reg}(h, h^*, \tilde{Z}_m) + \frac{1}{2}\tilde{\text{reg}}_m(h, h^*) + \frac{\eta}{4}\Delta_m \]

\[ \leq \text{reg}(h, h_{m+1}, \tilde{Z}_m) + \frac{1}{2}\tilde{\text{reg}}_m(h, h^*) + \frac{\eta}{4}\Delta_m \]

\[ \leq \frac{1}{2}\tilde{\text{reg}}_m(h, h^*) + \left( \gamma + \frac{\eta}{4} \right) \Delta_m. \]
Rearranging terms leads to
\[ \widetilde{\text{reg}}_m(h, h^*) \leq 4\gamma \Delta_m \]
because \( \gamma \geq \eta/4 \). Now we show that \( \Delta_m \leq 4\Delta^*_m \), which leads to the desired result. It is trivially true for \( m = 1 \) because \( \Delta^*_1 = \Delta_1 \). For \( m \geq 2 \), by the deviation bound on the empirical error (33) we have
\[
\Delta_m \leq c_1 \sqrt{\epsilon_m \left( \frac{3}{2} \text{err}_m(h^*) + \frac{\eta}{2} \Delta_m \right)} + c_2 \epsilon_m \log \tau_m
\]
\[
\leq 2c_1 \sqrt{\epsilon_m \text{err}_m(h^*)} + \sqrt{\frac{c_1^2 \epsilon_m \eta}{2}} \Delta_m + c_2 \epsilon_m \log \tau_m
\]
\[
\leq 2c_1 \sqrt{\epsilon_m \text{err}_m(h^*)} + \frac{c_1^2 \epsilon_m \eta}{4} + \frac{\Delta_m}{2} + c_2 \epsilon_m \log \tau_m
\]
\[
\leq 2\Delta^*_m + \frac{\Delta_m}{2},
\]
where the last inequality uses our choice of constants \( c_1^2 \eta/4 \leq c_2 \). Rearranging terms completes the proof.

8 Conclusion

In this paper, we proposed a new algorithm for agnostic active learning in a streaming setting. The algorithm has strong theoretical guarantees, maintaining good generalization properties while attaining a low label complexity in favorable settings. Specifically, we show that the algorithm has an optimal performance in a disagreement-based analysis of label complexity, as well in special cases such as realizable problems and under Tsybakov’s low-noise condition. Additionally, we present an interesting example that highlights the structural difference between our algorithm and some predecessors in terms of label complexities. Indeed a key improvement of our algorithm is that we do not always need to query over the entire disagreement region—a limitation of most computationally efficient predecessors. This is achieved through a careful construction of an optimization problem defining good query probability functions, which relies on using refined data-dependent error estimates.

The strong theoretical properties of our algorithm are also mirrored in the extensive empirical evaluation of an online variant, which performs well against a number of strong baselines across a suite of 23 datasets. Indeed this comprehensive empirical evaluation on a range of diverse datasets has not been previously done for agnostic active learning algorithms before to our knowledge, and is a key contribution of this work.

We believe that our work naturally leads to several interesting directions for future research. As the example in Section 4.2.2 reveals, the worst-case label complexity analysis in Theorem 2 is rather pessimistic. It would be interesting to obtain sharper characterization of the label complexity, by exploiting the structure of the query probability function over the disagreement region. This would likely involve understanding more fine-grained properties that make a problem easy or hard for active learning beyond the disagreement coefficient, and such a development might also lead to better algorithms. A limitation of the current theory is the somewhat poor dependence in Theorem 4 on the number of unlabeled examples needed to solve the optimization problem. Ideally, we would like to be able to use \( O(\tau_m) \) unlabeled examples to solve (OP) at epoch \( m \), and improving this dependence is perhaps the most important direction for future work. Finally, while AC is extremely attractive from a theoretical standpoint, a direct implementation still seems somewhat impractical. Obtaining theory for an algorithm even closer to the practical variant OAC would be an important step in bringing the theory and implementation closer.

Acknowledgements

The authors would like to thank Kamalika Chaudhuri for helpful initial discussions.
References

Maria-Florina Balcan and Phil Long. Active and passive learning of linear separators under log-concave distributions. In Conference on Learning Theory, pages 288–316, 2013.

Maria-Florina Balcan, Alina Beygelzimer, and John Langford. Agnostic active learning. In Proceedings of the 23rd international conference on Machine learning, pages 65–72. ACM, 2006.

Maria-Florina Balcan, Andrei Broder, and Tong Zhang. Margin based active learning. In Proceedings of the 20th annual conference on Learning theory, pages 35–50. Springer-Verlag, 2007.

P. Bartlett and S. Mendelson. Gaussian and Rademacher complexities: Risk bounds and structural results. Journal of Machine Learning Research, 3:463–482, 2002.

A. Beygelzimer, S. Dasgupta, and J. Langford. Importance weighted active learning. In ICML, 2009.

A. Beygelzimer, D. Hsu, J. Langford, and T. Zhang. Agnostic active learning without constraints. In NIPS, 2010.

R.M. Castro and R.D. Nowak. Minimax bounds for active learning. Information Theory, IEEE Transactions on, 54 (5):2339 –2353, 2008.

D. Cohn, L. Atlas, and R. Ladner. Improving generalization with active learning. Machine Learning, 15:201–221, 1994.

S. Dasgupta. Coarse sample complexity bounds for active learning. In Advances in Neural Information Processing Systems 18, 2005.

S. Dasgupta, D. Hsu, and C. Monteleoni. A general agnostic active learning algorithm. In NIPS, 2007.

D. A. Freedman. On tail probabilities for martingales. The Annals of Probability, 3(1):100–118, February 1975.

S. Hanneke. Theoretical Foundations of Active Learning. PhD thesis, Carnegie Mellon University, 2009.

Steve Hanneke. Theory of disagreement-based active learning. Foundations and Trends in Machine Learning, 7(2-3):131–309, 2014.

D. G. Horvitz and D. J. Thompson. A generalization of sampling without replacement from a finite universe. J. Amer. Statist. Assoc., 47:663–685, 1952. ISSN 0162-1459.

Daniel J. Hsu. Algorithms for Active Learning. PhD thesis, University of California at San Diego, 2010.

S. M. Kakade and A. Tewari. On the generalization ability of online strongly convex programming algorithms. In Advances in Neural Information Processing Systems 21, 2009.

Sham M Kakade, Karthik Sridharan, and Ambuj Tewari. On the complexity of linear prediction: Risk bounds, margin bounds, and regularization. In Advances in neural information processing systems, pages 793–800, 2009.

Nikos Karampatziakis and John Langford. Online importance weight aware updates. In UAI 2011, Proceedings of the Twenty-Seventh Conference on Uncertainty in Artificial Intelligence, Barcelona, Spain, July 14-17, 2011, pages 392–399, 2011.

Vladimir Kolchinskii. Rademacher complexities and bounding the excess risk in active learning. J. Mach. Learn. Res., 11:2457–2485, December 2010.

A. B. Tsybakov. Optimal aggregation of classifiers in statistical learning. Ann. Statist., 32:135–166, 2004.

Chicheng Zhang and Kamalika Chaudhuri. Beyond disagreement-based agnostic active learning. In Advances in Neural Information Processing Systems, pages 442–450, 2014.
A Deviation bound

We use an adaptation of Freedman’s inequality [Freedman, 1975] as the main concentration tool.

**Lemma 6.** Let $X_1, X_2, \ldots, X_n$ be a martingale difference sequence adapted to the filtration $F_i$. Suppose there exists a function $b_n$ of $X_1, \ldots, X_n$ that satisfies

\[
\forall 1 \leq i \leq n, \quad |X_i| \leq b_n, \\
1 \leq b_n \leq b_{\max},
\]

where $b_{\max}$ is a non-random quantity that may depend on $n$. Define

\[
S_n := \sum_{i=1}^{n} X_i, \\
V_n := \sum_{i=1}^{n} \mathbb{E}[X_i^2 \mid F_{i-1}].
\]

Pick any $0 < \delta < 1/e^2$ and $n \geq 3$. We have

\[
\Pr \left( S_n \geq 2 \sqrt{V_n \log(1/\delta)} + 3b_n \log(1/\delta) \right) \leq 4\sqrt{\delta}(2 + \log_2 b_{\max}) \log n.
\]

**Proof.** Define $r_j := 2^j$ for $-1 \leq j \leq m := \lceil \log_2 b_{\max} \rceil$. Then we have

\[
\Pr \left( S_n \geq 2 \sqrt{V_n \log(1/\delta)} + 3b_n \log(1/\delta) \right) \\
= \sum_{j=0}^{m} \Pr \left( S_n \geq 2 \sqrt{V_n \log(1/\delta)} + 3b_n \log(1/\delta) \land r_{j-1} < b_n \leq r_j \right) \\
\leq \sum_{j=0}^{m} \Pr \left( S_n \geq 2 \sqrt{V_n \log(1/\delta)} + 3r_{j-1} \log(1/\delta) \land b_n \leq r_j \right) \\
\leq \sum_{j=0}^{m} \Pr \left( S_n \geq 2 \sqrt{V_n \log(1/\delta)} + 3r_j \log(1/\delta) / 2 \land b_n \leq r_j \right) \\
\leq \sum_{j=0}^{m} 4(\log n) \sqrt{\delta} \\
\leq 4\sqrt{\delta}(2 + \log_2 b_{\max}) \log n,
\]

where (36) is a direct consequence of Lemma 3 of Kakade and Tewari [2009] and the others result from simple algebra. \(\square\)

B Auxiliary results for Theorem 1

Before presenting our regret analysis, we first establish several useful results.

**Lemma 7.** The threshold defined in (2) and the minimum probability $P_{\min, m}$ defined in (7) satisfy the following for all $m \geq 1$,

\[
\tau_{m-1} \Delta_{m-1} \leq \tau_m \Delta_m, \\
P_{\min, m} \geq P_{\min, m+1}, \\
\frac{\epsilon_m}{P_{\min, m}} \leq \Delta_m.
\]

25
Proof. Notice that
\[ \tau_{m-1} \epsilon_{m-1} = 32(\log(|\mathcal{H}|/\delta) + \log \tau_{m-1}) \]
\[ \leq 32(\log(|\mathcal{H}|/\delta) + \log \tau_{m}) \]
\[ = \tau_{m} \epsilon_{m}. \] (40)

We first prove (37). It holds trivially for \( m = 1 \). For \( m \geq 2 \) we have
\[ \tau_{m-1} \Delta_{m-1} = c_1 \sqrt{\tau_{m-1} \epsilon_{m-1} \text{err}(h_m, \tilde{Z}_{m-1})} + c_2 \tau_{m-1} \epsilon_{m-1} \log \tau_{m-1} \]
\[ \leq c_1 \sqrt{(\tau_{m-1} \epsilon_{m-1}) \tau_{m-1} \text{err}(h_{m+1}, \tilde{Z}_{m-1})} + c_2 \tau_{m-1} \epsilon_{m-1} \log \tau_{m-1} \]
\[ \leq c_1 \sqrt{(\tau_{m} \epsilon_{m}) \tau_{m} \text{err}(h_{m+1}, \tilde{Z}_{m})} + c_2 \tau_{m} \epsilon_{m} \log \tau_{m} \]
\[ = \tau_{m} \Delta_{m}, \]
where the first inequality is by the fact that \( h_m \) minimizes the empirical error on \( \tilde{Z}_{m-1} \) and the second inequality is by \( \tau_{m-1} \epsilon_{m-1} \leq \tau_{m} \epsilon_{m} \). Then for (38), it is easy to see
\[ \sqrt{\tau_{m-1} \epsilon_{m-1} \text{err}(h_m, \tilde{Z}_{m-1})} \leq c_1 \sqrt{(\tau_{m} \epsilon_{m}) \tau_{m} \text{err}(h_{m+1}, \tilde{Z}_{m})} + c_2 \tau_{m} \epsilon_{m} \log \tau_{m} \]
\[ \leq \Delta_{m}, \]
for \( m \geq 1 \), implying \( P_{\min, m} \geq P_{\min, m+1} \). Finally to prove (39), we have that
\[ \frac{\epsilon_{m}}{P_{\min, m}} \leq \frac{\epsilon_{m}}{P_{\min, m+1}} \]
\[ = \max \left( \frac{\sqrt{\epsilon_{m}^2 \text{err}(h_{m+1}, \tilde{Z}_{m})/(ne_M) + \epsilon_{m} \log \tau_{m}}}{c_3}, 2\epsilon_{m} \right) \]
\[ \leq \max \left( \frac{\epsilon_{m} \text{err}(h_{m+1}, \tilde{Z}_{m}) + \epsilon_{m} \log \tau_{m}}{c_3}, 2\epsilon_{m} \right) \]
\[ \leq \Delta_{m}, \]
where the second inequality is by \( \tau_{m} \epsilon_{m} \leq n \epsilon_M \), and the third inequality is by our choices of \( c_1, c_2 \) and \( c_3 \). \( \square \)

We also need a lemma regarding the epoch schedule.

Lemma 8. Let \( \tau_{m-1} < \tau_{m} \leq 2\tau_{m-1} \) for all \( m > 1 \). Then we have for all \( m \geq 1 \),
\[ \sum_{i=1}^{m} \frac{\tau_{i+1} - \tau_{i}}{\tau_{i}} \leq 4 \log \tau_{m+1}, \]
\[ \sum_{i=1}^{m} (\tau_{i} - \tau_{i-1}) \Delta_{i-1} \leq 4 \tau_{m} \Delta_{m} \log \tau_{m}. \]
Proof. Note that we can rewrite the summation in question as

\[
\sum_{i=1}^{m} \frac{\tau_{i+1} - \tau_i}{\tau_i} = \sum_{i=1}^{m} \sum_{j=\tau_i+1}^{\tau_{i+1}} \frac{1}{\tau_i} \\
\leq \sum_{i=1}^{m} \sum_{j=\tau_i+1}^{\tau_{i+1}} \frac{2}{\tau_{i+1}},
\]

where the second inequality uses our assumption on epoch lengths. The summation can then be further bounded as

\[
\sum_{i=1}^{m} \frac{\tau_{i+1} - \tau_i}{\tau_i} \leq \sum_{i=1}^{m} \sum_{j=\tau_i+1}^{\tau_{i+1}} \frac{2}{j} \leq \sum_{i=1}^{m+1} \frac{2}{i} \\
\leq 2(1 + \log \tau_{m+1}) \leq 4 \log \tau_{m+1},
\]

where the third inequality is by the bound \(\sum_{i=1}^{n} 1/i \leq 1 + \log n\), and the final inequality is by \(1 \leq \log \tau_m, m \geq 1\).

To prove the second bound in the lemma, we write

\[
\sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \Delta_{i-1} = \tau_1 \Delta_0 + \sum_{i=1}^{m-1} (\tau_{i+1} - \tau_i) \Delta_i \\
= \tau_1 \Delta_0 + \sum_{i=1}^{m-1} \frac{\tau_{i+1} - \tau_i}{\tau_i} \tau_i \Delta_i \\
\leq \tau_1 \Delta_0 + (2 + 2 \log \tau_m) \tau_m \Delta_m \\
\leq (2 \log \tau_1 - 2) \tau_1 \Delta_1 + (2 + 2 \log \tau_m) \tau_m \Delta_m \\
\leq (2 \log \tau_m - 2) \tau_m \Delta_m + (2 + 2 \log \tau_m) \tau_m \Delta_m \\
= 4 \tau_m \Delta_m \log \tau_m,
\]

where the first inequality is by (41), and \(\tau_i \Delta_i \leq \tau_m \Delta_m\) (Lemma 7), the second inequality is by our choice of \(\Delta_0\) and the fact that \(\tau_1 \Delta_1 \leq 1\), and the third inequality again uses \(\tau_i \Delta_i \leq \tau_m \Delta_m\). \(\square\)

C Proofs omitted from Section 7.2

We now provide the proofs of the lemmas and propositions from Section 7.2 that were used in proving Theorem 1. We start with proofs of Lemmas 1 and 2.

Proof of Lemma 1

Pick any \(m \geq 1, h \in \mathcal{H}\) and \(\tilde{h} \in A_m\). Note that the definitions of \(\text{reg}^+_{m}(h, \tilde{h})\) and \(\text{reg}(h, \tilde{h})\) only differ on \(X \notin D_m := \text{DIS}(A_m)\), and \(\forall X \notin D_m, \tilde{h}(X) = h_m(X)\). We thus have

\[
\text{reg}^+_{m}(h, \tilde{h}) - \text{reg}(h, \tilde{h}) \\
= \mathbb{E}_{X,Y} \left[ I(X \notin D_m) \left( (1(h(X) \neq h_m(X)) - (1(\tilde{h}(X) \neq h_m(X))) \\
- (1(h(X) \neq Y) - (1(\tilde{h}(X) \neq Y)) \right) \right] \\
= \mathbb{E}_{X,Y} [I(X \notin D_m)(1(h(X) \neq h_m(X)) - (1(h(X) \neq Y) - (1(h_m(X) \neq Y))].
\]

The desired result then follows from the inequality that

\[
1(h(X) \neq Y) - 1(h_m(X) \neq Y) \leq 1(h(X) \neq h_m(X)).
\]
Proof of Lemma 2

Our proof strategy is to apply Lemma 6 to establish concentration of properly defined martingale difference sequences for fixed classifiers \( h, h' \) and some epoch \( m \), and then use a union bound to get the desired statement. First we look at the concentration of the empirical regret on \( Z_m \). To avoid clutter, we overload our notation so that \( D_i = D_{m(i)} \), \( h_i = h_{m(i)} \) and \( P_i = P_{m(i)} \) when \( i \) is the index of an example rather than a round.

For any pair of classifiers \( h \) and \( h' \), we define the random variables for the instantaneous regrets:

\[
\tilde{R}_i := \mathbb{1}(X_i \notin D_i)(\mathbb{1}(h(X_i) \neq h_i(X_i)) - \mathbb{1}(h'(X_i) \neq h_i(X_i))) + \\
\mathbb{1}(X_i \in D_i)(\mathbb{1}(h(X_i) \neq Y_i) - \mathbb{1}(h'(X_i) \neq Y_i))Q_i/P_i(X_i)
\]

and the associated \( \sigma \)-fields \( \mathcal{F}_i := \sigma(\{X_j, Y_j, Q_j\}_{j=1}^i) \). We have that \( \tilde{R}_i \) is measurable with respect to \( \mathcal{F}_i \). Therefore \( \tilde{R}_i - \mathbb{E}[\tilde{R}_i | \mathcal{F}_{i-1}] \) forms a martingale difference sequence adapted to the filtrations \( \mathcal{F}_i, i \geq 1 \), and

\[
\mathbb{E}[\tilde{R}_i | \mathcal{F}_{i-1}] = \text{reg}_{m(i)}^\dagger(h, h')
\]

according to (27) and the fact that \( X_i, Y_i, Q_i \) are independent from the past. To use Lemma 6 we first identify an upper bound on elements in the sequence:

\[
|\tilde{R}_i - \mathbb{E}[\tilde{R}_i | \mathcal{F}_{i-1}]| = |\tilde{R}_i - \text{reg}_{m(i)}^\dagger(h, h')| \leq \max(\tilde{R}_i, \text{reg}_{m(i)}^\dagger(h, h')) \\
\leq \frac{1}{P_{\text{min},m(i)}} \leq \frac{1}{P_{\text{min},m}}
\]

(42)

for all \( i \) such that \( m(i) \leq m \), where the last inequality is by Lemma 7. The definition of \( P_{\text{min},m} \) implies that

\[
\frac{1}{P_{\text{min},m}} \leq \max(\sqrt{\tau_{m-1}/(n\epsilon_M)} + \log \tau_{m-1}, 2) \leq 2\sqrt{\tau_{m-1} + 1}
\]

(43)

because \( n\epsilon_M \geq 1 \). Then we consider the conditional second moment. Using the fact that

\[
(\mathbb{1}(h(X_i) \neq Y_i) - \mathbb{1}(h'(X_i) \neq Y_i))^2 \leq \mathbb{1}(h(X_i) \neq h'(X_i))
\]

we get

\[
\mathbb{E}[(\tilde{R}_i - \mathbb{E}[\tilde{R}_i | \mathcal{F}_{i-1}])^2 | \mathcal{F}_{i-1}] = \mathbb{E}[\tilde{R}_i^2 | \mathcal{F}_{i-1}] \\
\leq \mathbb{E}\left[ \left( \mathbb{1}(X_i \notin D_i) + \frac{\mathbb{1}(X_i \in D_i)Q_i}{P_i(X_i)} \right)^2 \mathbb{1}(h(X_i) \neq h'(X_i)) | \mathcal{F}_{i-1} \right] \\
= \mathbb{E}\left[ \left( \mathbb{1}(X_i \notin D_i) + \frac{\mathbb{1}(X_i \in D_i)Q_i}{P_i(X_i)^2} \right) \mathbb{1}(h(X_i) \neq h'(X_i)) | \mathcal{F}_{i-1} \right] \\
= \mathbb{E}\left[ \left( \mathbb{1}(X_i \notin D_i) + \frac{\mathbb{1}(X_i \in D_i)}{P_i(X_i)} \right) \mathbb{1}(h(X_i) \neq h'(X_i)) | \mathcal{F}_{i-1} \right] \\
= \mathbb{E}_X \left[ \left( \mathbb{1}(X \notin D_{m(i)}) + \frac{\mathbb{1}(X \in D_{m(i)})}{P_{m(i)}(X)} \right) \mathbb{1}(h(X) \neq h'(X)) \right]
\]

(45)
where the last two equalities are from the fact that $X_i$ is independent from the past and replacing our overloaded notation respectively. Lemma 6 with (42), (43), and (45) then implies for any $0 < \delta_m < 1/e^2$ and $m \geq 1$, the following holds with probability at most $8\sqrt{\delta_m}(2 + \log_2(2\sqrt{\tau_{m-1}} + 1)) \log \tau_m$:

$$|\text{reg}(h, h', Z_m) - \bar{\text{reg}}_m(h, h')| \geq \sqrt{\frac{4\log(1/\delta_m)}{\tau_m^2} \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \mathbb{E}_{X, Y} \left[ \left( \mathbb{I}(X \in D_i) + \mathbb{I}(X \notin D_i) \right) \mathbb{I}(h(X) \neq h'(X)) \right]}$$

$$+ \frac{4\log(1/\delta_m)}{n\bar{P}_{\min, m}}.$$

(46)

Then we consider the concentration of the empirical error on the importance-weighted examples. Define the random examples for the empirical errors:

$$E_i := \frac{Q_i \mathbb{I}(X_i \notin D_i, X_i \in D_i)}{P_i(X_i)}$$

and the associated $\sigma$-fields $F_i := \sigma(\{X_j, Y_j, Q_j\}_{j=1}^i)$. By the same analysis of the sequence of instantaneous regrets, we have $E_i - \mathbb{E}[E_i \mid F_{i-1}]$ is a martingale difference sequence adapted to the filtrations $F_i, i \geq 1$, with the following properties:

$$\mathbb{E}[E_i \mid F_{i-1}] = \mathbb{E}[\mathbb{I}(X_i \in D_i \land h(X_i) \neq Y_i) \mid F_{i-1}] = \bar{\text{err}}_{m(i)}(h),$$

$$|E_i - \mathbb{E}[E_i \mid F_{i-1}]| \leq \frac{1}{P_{\min, m(i)}} \leq \frac{1}{P_{\min, m}} \leq 2\sqrt{\tau_{m-1}} + 1,$$

for all $i$ such that $m(i) \leq m$. Furthermore,

$$\mathbb{E}[(E_i - \mathbb{E}[E_i \mid F_{i-1}])^2 \mid F_{i-1}] \leq \mathbb{E} \left[ \frac{\mathbb{I}(X_i \in D_i \land h(X_i) \neq Y_i)}{P_i(X_i)} \mid F_{i-1} \right]$$

$$= \mathbb{E}_{X, Y} \left[ \frac{\mathbb{I}(X \in D_i \land h(X) \neq Y)}{P_i(X)} \right].$$

With these properties, Lemma 6 then implies for any $0 < \delta_m < 1/e^2$ and $m \geq 1$, the following holds with probability at most $8\sqrt{\delta_m}(2 + \log_2(2\sqrt{\tau_{m-1}} + 1)) \log \tau_m$:

$$|\text{err}(h, Z_m) - \bar{\text{err}}_m(h)| \geq \sqrt{\frac{4\log(1/\delta_m)}{\tau_m^2} \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \mathbb{E}_{X, Y} \left[ \frac{\mathbb{I}(X \in D_i \land h(X) \neq Y)}{P_i(X)} \right]}$$

$$+ \frac{4\log(1/\delta_m)}{n\bar{P}_{\min, m}}.$$

(47)

Setting

$$\delta_m = \left( \frac{192|H|^2\tau_m^2(\log \tau_m)^2}{\delta} \right)^2$$

ensures that the probability of the union of the bad events (46), and (47) over all pairs of classifiers $h, h'$ and $m \geq 1$ is bounded by $\delta > 0$. Choosing $\delta \leq |H|/\sqrt{192}$, we have

$$\log(1/\delta_m) = 2\log \left( \frac{192|H|^2\tau_m^2(\log \tau_m)^2}{\delta} \right)$$

$$\leq 2(2\log(|H|/\delta) + 4\log \tau_m + \log 192)$$

$$\leq 8(\log(|H|/\delta) + \log \tau_m),$$

29
leading to the desired statement.

We then provide the proofs of Propositions 1 and 2.

**Proof of Proposition 1.** By the inequality (29) of Lemma 2, we have

\[
|\text{reg}(h, h^*, \tilde{Z}_m) - \text{reg}_m(h, h^*)| 
\leq \frac{\epsilon_m}{\epsilon_m} \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \mathbb{E}_X \left[ \left( \mathbb{1}(X \in D_i) + \mathbb{1}(X \notin D_i) \right) \mathbb{1}(h(X) \neq h^*(X)) \right] + \frac{\epsilon_m}{P_{\min,m}} (48)
\]

We now control the term \( \text{dev}_m(h) \) in order to establish the proposition. We have

\[
\frac{\tau_m}{\epsilon_m} \text{dev}_m(h) = \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \mathbb{E}_X \left[ \left( \mathbb{1}(X \in D_i) + \mathbb{1}(X \notin D_i) \right) \mathbb{1}(h(X) \neq h^*(X)) \right]
\]

\[
\leq \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \mathbb{E}_X \left[ 2\alpha^2 \mathbb{1}(X \in D_i) \left( \mathbb{1}(h(X) \neq h_i(X)) + \mathbb{1}(h^*(X) \neq h_i(X)) \right) + 2\beta^2 \gamma_{i-1} \Delta_{i-1} \right]
\]

where the second inequality uses our variance constraints in defining the distribution \( P_i \) for classifiers \( h \) and \( h^* \). Note that

\[
\mathbb{1}(h(X) \neq h^*(X)) \leq \mathbb{1}(h(X) \neq Y) + \mathbb{1}(h^*(X) \neq Y)
\]

\[
= (\mathbb{1}(h(X) \neq Y) - \mathbb{1}(h^*(X) \neq Y)) + 2\mathbb{1}(h^*(X) \neq Y),
\]

so that the final inequality can be rewritten as

\[
\frac{\tau_m}{\epsilon_m} \text{dev}_m(h) \leq \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \left[ 2\alpha^2 \text{reg}_i(h) + 2\text{reg}_i(h_i) + 12\alpha^2 \text{err}_i(h^*) + 2\beta^2 \gamma_{i-1} \Delta_{i-1} \right]
\]

\[
+ \text{reg}(h^*, \tilde{Z}_{i-1}) + 2\xi \tau_{i-1} \Delta_{i-1}^2 + \mathbb{E}_X [\mathbb{1}(h(X) \neq h^*(X) \land X \notin D_i)]]
\]

With the assumptions \( \alpha \geq 1 \) and \( h^* \in A_i \) for all epochs \( i \leq m \), the first term \( \text{reg}_i(h) \) can be combined with the last disagreement term and bounded by \( 2\alpha^2 \text{reg}_i(h) \). Further noting that \( \tau_{i-1} \Delta_{i-1} \leq \tau_m \Delta_m \) by Lemma 7 we can further
simplify the inequality to

\[
\frac{\tau_m}{\epsilon_m} \text{dev}_m(h) \leq 2\alpha^2 \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \text{reg}_i^2(h) + 4\alpha^2 \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \text{reg}_i(h_i) + 12\tau_m \alpha^2 \epsilon_m \epsilon_m(h^*) + 2\beta \tau_m \Delta_m \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) (\text{reg}(h, \tilde{Z}_{i-1}) + \text{reg}(h^*, \tilde{Z}_{i-1})) + 8\xi \tau_m \Delta_m^2 \log \tau_m.
\]

The first summand is simply \(2\alpha^2 \tau_m \epsilon_m \epsilon_m(h)\) by definition. The final summand above can be bounded using Lemmas\(^7\) and\(^8\) since

\[
\sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \tau_{i-1} \Delta_i^2 = \sum_{i=1}^{m-1} (\tau_{i+1} - \tau_i) \tau_i \Delta_i^2 \leq \tau_m \Delta_m \sum_{i=1}^{m-1} (\tau_{i+1} - \tau_i) \Delta_i \\
\leq 4\tau_m \Delta_m^2 \log \tau_m.
\]

Substituting the above inequalities back, we obtain

\[
\frac{\tau_m}{\epsilon_m} \text{dev}_m(h) \leq 2\alpha^2 \tau_m \epsilon_m \epsilon_m(h) + 4\alpha^2 \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \text{reg}_i(h_i) + 12\tau_m \alpha^2 \epsilon_m \epsilon_m(h^*) + 2\beta \tau_m \Delta_m \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) (\text{reg}(h, \tilde{Z}_{i-1}) + \text{reg}(h^*, \tilde{Z}_{i-1})) + 8\xi \tau_m \Delta_m^2 \log \tau_m.
\]

Since \(\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}\), we can further bound

\[
\text{dev}_m(h) \leq \sqrt{2\alpha^2 \tau_m \epsilon_m \epsilon_m(h) + 2\alpha \sqrt{\frac{\epsilon_m}{\tau_m} \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \text{reg}_i(h_i) + 2\alpha \sqrt{3\epsilon_m \epsilon_m(h^*) \epsilon_m}}
\]

\[
+ \beta \sqrt{2\gamma \epsilon_m \Delta_m \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) (\text{reg}(h, \tilde{Z}_{i-1}) + \text{reg}(h^*, \tilde{Z}_{i-1}))} + 2\Delta_m \sqrt{2\xi \tau_m \epsilon_m \log \tau_m}.
\]

Substituting this inequality back into our deviation bound\(^{48}\), we obtain

\[
|\text{reg}(h, h^*, \tilde{Z}_m) - \epsilon_m \epsilon_m(h, h^*)| \\
\leq \frac{\epsilon_m}{\epsilon_m \epsilon_m} + \sqrt{2\alpha^2 \epsilon_m \epsilon_m(h) + 2\alpha \sqrt{\frac{\epsilon_m}{\tau_m} \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \text{reg}_i(h_i) + 2\alpha \sqrt{3\epsilon_m \epsilon_m(h^*) \epsilon_m}}
\]

\[
+ \beta \sqrt{2\gamma \epsilon_m \Delta_m \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) (\text{reg}(h, \tilde{Z}_{i-1}) + \text{reg}(h^*, \tilde{Z}_{i-1}))} + 2\Delta_m \sqrt{2\xi \tau_m \epsilon_m \log \tau_m}.
\]
We can further use Cauchy-Schwarz inequality to obtain the bound

\[
\left| \text{reg}(h, h^*, \hat{Z}_m) - \tilde{\text{reg}}(h, h^*) \right| \\
\leq \frac{1}{4} \tilde{\text{reg}}(h) + 2\alpha^2 \epsilon_m + 2\alpha \left( \frac{\epsilon_m}{\tau_m} \sum_{i=1}^m (\tau_i - \tau_{i-1}) \text{reg}(h_i) + 2\alpha \sqrt{3\text{err}_m(h^*)\epsilon_m} \right) \\
+ \beta \sqrt{2\gamma \epsilon_m \Delta_m \sum_{i=1}^m (\tau_i - \tau_{i-1})(\text{reg}(h, \hat{Z}_{i-1}) + \text{reg}(h^*, \hat{Z}_{i-1})) + 2\Delta_m \sqrt{2\xi \tau_m \epsilon_m \log \tau_m}} \\
+ \frac{\epsilon_m}{P_{\min,m}}
\]

\[
\leq \frac{1}{4} \tilde{\text{reg}}(h) + 2\alpha \left( \frac{\epsilon_m}{\tau_m} \sum_{i=1}^m (\tau_i - \tau_{i-1}) \text{reg}(h_i) + 2\alpha \sqrt{3\text{err}_m(h^*)\epsilon_m} \right) \\
+ \beta \sqrt{2\gamma \epsilon_m \Delta_m \sum_{i=1}^m (\tau_i - \tau_{i-1})(\text{reg}(h, \hat{Z}_{i-1}) + \text{reg}(h^*, \hat{Z}_{i-1})) + \Delta_m + \frac{\epsilon_m}{P_{\min,m}}}
\]

\[
\leq \frac{1}{4} \tilde{\text{reg}}(h) + 2\alpha \left( \frac{\epsilon_m}{\tau_m} \sum_{i=1}^m (\tau_i - \tau_{i-1}) \text{reg}(h_i) + 2\alpha \sqrt{3\text{err}_m(h^*)\epsilon_m} \right) \\
+ \beta \sqrt{2\gamma \epsilon_m \Delta_m \sum_{i=1}^m (\tau_i - \tau_{i-1})(\text{reg}(h, \hat{Z}_{i-1}) + \text{reg}(h^*, \hat{Z}_{i-1})) + 4\Delta_m}
\]

where the last two inequalities use our assumptions on $\xi$ and $\alpha$ respectively.

**Proof of Proposition 2** We start by observing that

\[
|\text{err}_m(h^*) - \text{err}(h_{m+1}, \hat{Z}_m)| \leq |\text{err}_m(h^*) - \text{err}(h^*, \hat{Z}_m)| + \text{reg}(h^*, h_{m+1}, \hat{Z}_m).
\]

Since $h^* \in A_i$ for all epochs $i \leq m$, we know that $h^*$ agrees with all the predicted labels. Consequently, $\text{err}(h^*, \hat{Z}_m) = \text{err}(h^*, Z_m)$, where we recall that $Z_m$ is the set of all examples where we queried labels up to epoch $m$. This allows us to rewrite

\[
|\text{err}_m(h^*) - \text{err}(h^*, \hat{Z}_m)| = |\text{err}_m(h^*) - \text{err}(h^*, Z_m)|.
\]

Under the event $\mathcal{E}$, the above deviation is bounded, according to Lemma[2] by

\[
\sqrt{\frac{\epsilon_m}{\tau_m} \sum_{i=1}^m (\tau_i - \tau_{i-1}) \mathbb{E}_{X,Y} \frac{1(h^*(X) \neq Y, X \in D_i)}{P_i(X)}} + \frac{\epsilon_m}{P_{\min,m}} \leq \sqrt{\epsilon_m \text{err}_m(h^*) \frac{1}{P_{\min,m}}} + \frac{\epsilon_m}{P_{\min,m}},
\]

where the inequality uses the bound $P_i(X) \geq P_{\min,i}$ for all $X \in D_i$ and $P_{\min,i} \geq P_{\min,m}$ for all epochs $i \leq m$ by Lemma[7]. A further application of Cauchy-Schwarz inequality yields the bound

\[
|\text{err}_m(h^*) - \text{err}(h^*, \hat{Z}_m)| \leq \frac{\text{err}_m(h^*)}{2} + \frac{3\epsilon_m}{2P_{\min,m}}
\]

\[
\leq \frac{3\Delta_m}{2}.
\]
Combining the bounds yields

\[ |\text{err}_m(h^*) - \text{err}(h_{m+1}, \tilde{Z}_m)| \leq \frac{\text{err}_m(h^*)}{2} + \frac{3\Delta_m}{2} + \text{reg}(h^*, h_{m+1}, \tilde{Z}_m), \]

which completes the proof of the proposition.

Finally, we prove Lemmas 3 to 5 used in the proof of Theorem 1.

**Proof of Lemma 3** We first bound the \( \text{reg}_i(h_i) \) terms. For \( i = 1 \), we have

\[ \text{reg}_1(h_1) = \text{reg}(h_1) \leq 1 \leq \frac{\eta \Delta_0}{2} \]

by \( P_{\min,1} = 1 \) and our choices of \( \eta \) and \( \Delta_0 \). For \( 2 \leq i < m \), we have

\[ \text{reg}_i(h_i) = \mathbb{E}_{X,Y} [\mathbb{I}(h_i(X) \neq Y, X \in D_i) - \mathbb{I}(h^*(X) \neq Y, X \in D_i)] = \text{reg}(h_i) \leq \text{reg}_{i-1}(h_i, h^*), \]

where the second equality uses the fact that \( h^* \in A_i \) for all \( i \leq m \) by inductive hypothesis \( \text{L}9 \) and the inequality uses Lemma 1. Consequently, we can bound \( \text{reg}_{i-1}(h_i) \) using the event \( \mathcal{E}_i \), since \( \text{reg}(h_i, h^*, Z_{i-1}) = 0 \). The event \( \mathcal{E}_i \) now further implies that

\[ \text{reg}_i(h_i) \leq \text{reg}_{i-1}(h_i, h^*) \leq \text{reg}(h_i, h^*, Z_{i-1}) + \frac{\eta \Delta_{i-1}}{2} \leq \frac{\eta \Delta_{i-1}}{2}. \]

Using this, we can simplify \( T_1 \) as

\[
T_1 = 2\alpha \sqrt{\frac{\epsilon_m}{2\tau_m} \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \text{reg}_i(h_i)} \leq 2\alpha \sqrt{\frac{\epsilon_m}{2\tau_m} \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \frac{\eta \Delta_{i-1}}{2}} \]  
\[
\leq 2\alpha \sqrt{2\eta \epsilon_m \Delta_m \log \tau_m} \leq \frac{\eta \Delta_m}{12} + 24\alpha^2 \epsilon_m \log \tau_m. \quad (50)
\]

Here the second inequality is by Lemma 8 and the third inequality is by Cauchy-Schwarz.

**Proof of Lemma 4** We first invoke Proposition 2 whose assumptions now hold due to the claim \( h^* \in A_i \) in \( \mathcal{E}_i \) for all \( i \leq m \), and obtain

\[ \text{err}_m(h^*) = 2\text{err}(h_{m+1}, \tilde{Z}_m) + 3\Delta_m + 2\text{reg}(h^*, h_{m+1}, \tilde{Z}_m). \]

The above inequality allows us to simplify \( T_2 \) as

\[
T_2 = 2\alpha \sqrt{3\epsilon_m \text{err}_m(h^*)} \leq 2\alpha \sqrt{3\epsilon_m (2\text{err}(h_{m+1}, \tilde{Z}_m) + 3\Delta_m + 2\text{reg}(h^*, h_{m+1}, \tilde{Z}_m))} \]
\[
\leq 2\alpha \sqrt{6\epsilon_m \text{err}(h_{m+1}, \tilde{Z}_m) + 2\alpha \sqrt{9\epsilon_m \Delta_m} + 2\alpha \sqrt{6\epsilon_m \text{reg}(h^*, h_{m+1}, \tilde{Z}_m)} \]
\[
\leq 2\alpha \sqrt{6\epsilon_m \text{err}(h_{m+1}, \tilde{Z}_m) + \Delta_m + \frac{1}{4} \text{reg}(h^*, h_{m+1}, \tilde{Z}_m) + 33\alpha^2 \epsilon_m}, \quad (51)
\]

where the last inequality uses the Cauchy-Schwarz inequality.

**Proof of Lemma 5**

Observe that the event \( \mathcal{E}_i \) gives a direct bound of \( \eta \Delta_{i-1}/4 \) on the \( \text{reg}(h^*, h_i, Z_{i-1}) \) terms. For the other term, recall by the same event that for all \( h \in H \) and for all \( i = 1, 2 \ldots, m-1 \),
\[ \text{reg}(h, h^*, Z_i) \leq \frac{3}{2} \hat{\text{reg}}_m(h, h^*) + \frac{\eta}{4} \Delta_i. \]

Combining with the empirical regret bound for \( h^* \), this implies that

\[ \text{reg}(h, Z_i) \leq \frac{3}{2} \hat{\text{reg}}_m(h, h^*) + \frac{\eta}{2} \Delta_i. \]

Consequently we have the bound

\[ T^2 \leq \beta^2 \gamma \Delta_m \epsilon_m \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \left( 3 \hat{\text{reg}}_{i-1}(h, h^*) + \frac{3\eta}{2} \Delta_{i-1} \right) \]

To simplify further, note that by the definition of \( \hat{\text{reg}}_m(h, h^*) \) and our earlier definition of \( \hat{\text{reg}}_m(h, h^*) \), we have

\[ \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \hat{\text{reg}}_{i-1}(h, h^*) = \sum_{i=1}^{m-1} \frac{\tau_{i+1} - \tau_i}{\tau_i} \sum_{j=1}^{i} (\tau_j - \tau_{j-1}) \hat{\text{reg}}_j(h, h^*) \]

\[ = \sum_{j=1}^{m-1} (\tau_j - \tau_{j-1}) \hat{\text{reg}}_j(h, h^*) \sum_{i=j}^{m-1} \frac{\tau_{i+1} - \tau_i}{\tau_i} \]

\[ \leq 4 \log \tau_m \sum_{j=1}^{m-1} (\tau_j - \tau_{j-1}) \hat{\text{reg}}_j(h, h^*) \]

\[ \leq 4 \tau_m \log \tau_m \hat{\text{reg}}_m(h, h^*), \]

where the first equality uses our convention \( \hat{\text{reg}}_0(h, h^*) = 0 \) and proper index shifting, and the first inequality uses Lemma 8. We also have

\[ \sum_{i=1}^{m} (\tau_i - \tau_{i-1}) \Delta_{i-1} \leq 4 \tau_m \Delta_m \log \tau_m. \]

by Lemma 8. Consequently, we can rewrite

\[ T^2 \leq \beta^2 \gamma \Delta_m \epsilon_m \left( 12 \tau_m \log \tau_m \hat{\text{reg}}_m(h, h^*) + 6 \tau_m \eta \log \tau_m \Delta_m \right) \]

\[ = \beta^2 \gamma \tau_m \epsilon_m \log \tau_m \Delta_m \left( 12 \hat{\text{reg}}_m(h, h^*) + 6 \eta \Delta_m \right) \]

\[ \leq \eta \Delta_m \hat{\text{reg}}_m(h, h^*) + \frac{\eta^2 \Delta_m^2}{72}, \]

where the last inequality is by our choice of \( \beta \) such that \( \beta^2 \gamma n \epsilon_n \log n \leq \eta/864 \). Taking square roots, we obtain

\[ T \leq \sqrt{\eta \Delta_m \hat{\text{reg}}_m(h, h^*) + \frac{\eta^2 \Delta_m^2}{144}} \]

\[ \leq \frac{1}{4} \hat{\text{reg}}_m(h, h^*) + \frac{7 \eta \Delta_m}{72} \]  

(52)

\[ \square \]

### D Label Complexity

Here we prove Theorem 8. Fix any epoch \( m \) and index \( i \leq \tau_m \). Consider \( X_i \in D_m \) and define
The expected number of label queries made by our algorithm after seeing \( n \) examples is upper-bounded w.p. 1 – \( \delta \) by

\[
3 + \sum_{i=4}^{n} \mathbb{E}[\mathbb{I}(X_i \in D_{m(i)})] \leq 3 + \sum_{j=2}^{M} (\tau_j - \tau_{j-1}) \theta (16\gamma \Delta^*_j - 1 + 2\text{err}_m(h^*))
\]

\[
\leq 3 + 2n\theta \text{err}_M(h^*) + 16\gamma \theta \sum_{j=2}^{M} (\tau_j - \tau_{j-1}) \Delta^*_j - 1
\]

\[
= 3 + 2n\theta \text{err}_M(h^*) + 16\gamma \theta \sum_{j=2}^{M} \frac{\tau_j - \tau_{j-1}}{\tau_{j-1}} \tau_{j-1} \Delta^*_j - 1.
\]

A similar argument as Lemma 7 shows that \( \tau_j \Delta^*_j \) is increasing in \( j \), so we have by a further invocation of Lemma 8

\[
3 + \sum_{i=4}^{n} \mathbb{E}[\mathbb{I}(X_i \in D_{m(i)})] \leq 3 + 2n\theta \text{err}_M(h^*) + 128\gamma \theta (n-1) \Delta^*_M - 1 \log(n - 1)
\]

\[
= 3 + 2n\theta \text{err}_M(h^*) + \theta O \left( \sqrt{n\text{err}_M(h^*)} \left( \log \left( \frac{\left| \mathcal{H} \right|}{\delta} \right) \log^2 n + \log^3 n \right) + \log \left( \frac{\left| \mathcal{H} \right|}{\delta} \right) \log^2 n + \log^3 n \right).
\]

**E  Proofs for Tsybakov’s low-noise condition**

We begin with a lemma that captures the behavior of the \( \Delta^*_m \) terms, \( err_m(h^*) \) and the probability of disagreement region under the Tsybakov noise condition (10). The proofs of Corollaries 2 and 4 are immediate given the lemma.
Lemma 9. Under the conditions of Theorem [I] suppose further that the low-noise condition (10) holds. Then we have for all epochs $m = 1, 2, \ldots, M$

$$\text{err}_m(h^*) \leq c\epsilon_m \log \tau_m \frac{2(1-\omega)}{\tau_m^{-\frac{1-\omega}{2}}} \quad \text{and} \quad \text{err}_m(h^*) \leq 5c\epsilon_m \log^2 \frac{\tau_m^{-\frac{2(1-\omega)}{2}}}{\tau_m}.$$ (53)

Proof. We will establish the lemma inductively. We make the following inductive hypothesis. There exists a constant $c > 0$ (dependent on the distributional parameters) such that for all epochs $j \geq 1$, the bounds (53) in the statement of the Lemma hold. The base case for $j = 1$ trivially follows since $\text{err}_1(h^*) = \text{err}_1(h^*) = \text{err}(h^*) \leq 1 \leq c\epsilon_1 \log \frac{\tau_1^{-\frac{2(1-\omega)}{2}}}{\tau_1}$, which is clearly true for an appropriately large value of $c$. Suppose now that the claim is true for epochs $j = 1, 2, \ldots, m - 1$. We will establish the claim at epoch $m$. To see this, first note that we have

$$\text{err}_m(h^*) = \Pr(I(h^*(X) \neq Y, X \in D_m)) \leq \Pr(X \in D_m).$$

Under the noise condition, we can further upper bound the probability of the disagreement region, since by Theorem [I] we obtain

$$\Pr(X \in D_m) = \Pr(X \in \text{DIS}(A_m)) \leq \Pr(X \in \text{DIS}(\{h \in \mathcal{H} : \text{reg}(h) \leq 16\gamma \Delta_{m-1}^*\}))$$

$$\leq \Pr(X \in \text{DIS}(h \in \mathcal{H} : \Pr(h(X) \neq h^*(X)) \leq \zeta (16\gamma \Delta_{m-1}^*)^\omega),$$

where the first inequality follows from Theorem [I] and the second one is a consequence of Tsybakov’s noise condition (10). Recalling the definition of disagreement coefficient (11), this can be further upper bounded by

$$\Pr(X \in D_m) \leq \zeta (16\gamma \Delta_{m-1}^*)^\omega.$$ (54)

Hence, we have obtained the bound

$$\text{err}_m(h^*) \leq \zeta (16\gamma \Delta_{m-1}^*)^\omega.$$ (55)

Note that $\Delta_{m-1}^* = c_1 \sqrt{\epsilon_{m-1} \text{err}_{m-1}(h^*)} + c_2 \epsilon_{m-1} \log \tau_{m-1}$. Our inductive hypothesis (53) allows us to upper bound the $\text{err}_{m-1}$ in this expression for $\Delta_{m-1}^*$ and hence we obtain

$$\Delta_{m-1}^* \leq c_1 \sqrt{\epsilon_{m-1} 5c\epsilon_{m-1} \log^2 \tau_{m-1}^{-\frac{2(1-\omega)}{2}}} + c_2 \epsilon_{m-1} \log \tau_{m-1}$$

$$\leq c_1 \epsilon_{m-1} \log \tau_{m-1} \frac{1-\omega}{5c} + c_2 \epsilon_{m-1} \log \tau_{m-1}$$

$$\leq \frac{\epsilon_{m} \tau_{m}}{\tau_{m-1}} \log \tau_{m} \left(c_1 \sqrt{5c\tau_{m}^{-\frac{1-\omega}{2}}} + c_2\right)$$

$$\leq 2\epsilon_{m} \log \tau_{m} \left(c_1 \sqrt{5c\tau_{m}^{-\frac{1-\omega}{2}}} + c_2\right).$$

Since $\tau_m \geq 3$ and $0 < \omega \leq 1$, we can further write

$$\Delta_{m-1}^* \leq 2\epsilon_{m} \log \tau_{m} \frac{1-\omega}{5c} \left(c_1 \sqrt{5c} + c_2\right).$$ (55)

Substituting this inequality in our earlier bound on $\text{err}_m(h^*)$ yields

$$\text{err}_m(h^*) \leq \zeta \left(32\gamma \epsilon_{m} \log \tau_{m} \frac{1-\omega}{\tau_{m}} \left(c_1 \sqrt{5c} + c_2\right)\right)^\omega.$$
Since \( \epsilon_m \tau_m \log \tau_m \geq 1 \) and \( 0 < \omega \leq 1 \), we can further bound
\[
\text{err}_m(h^*) \leq \theta \zeta \epsilon_m \tau_m \log \tau_m \left( 32 \gamma \frac{ \tau_{m+1} }{ \tau_m } \left( c_1 \sqrt{5c} + c_2 \right) \right)^\omega
\]
\[
= \theta \zeta \epsilon_m \tau_m \log \tau_m \left( 32 \gamma \left( c_1 \sqrt{5c} + c_2 \right) \right)^\omega \frac{ \tau_m^{1-\omega} }{ \tau_m^{2-\omega} }
\]
\[
= \theta \zeta \epsilon_m \tau_m^{2(1-\omega)} \log \tau_m \left( 32 \gamma \left( c_1 \sqrt{5c} + c_2 \right) \right)^\omega \frac{ \tau_m^{2(1-\omega)} }{ \tau_m^{2-\omega} }
\]
\[
\leq c \epsilon_m \log \tau_m \tau_m^{2(1-\omega)}
\]

Here the last bound follows for any choice of \( c \) such that
\[
c \geq \theta \zeta \left( 32 \gamma \left( c_1 \sqrt{5c} + c_2 \right) \right)^\omega.
\]

The above inequality has a solution since the LHS is smaller than the RHS at \( c = 0 \), while for \( c \) large enough, the LHS grows linearly in \( c \), while the RHS grows as \( c^{\omega/2} \), and hence is asymptotically smaller than the LHS.

We now verify the second part of our induction hypothesis for epoch \( m \). Note that we have
\[
\text{err}_m(h^*) = \frac{1}{\tau_m} \sum_{j=1}^{m} (\tau_j - \tau_{j-1}) \text{err}_j(h^*)
\]
\[
\leq \frac{1}{\tau_m} \sum_{j=1}^{m} (\tau_j - \tau_{j-1}) \epsilon_j \log \tau_j \tau_j^{2(1-\omega)}
\]
\[
= \frac{1}{\tau_m} \sum_{j=1}^{m} \left( \frac{\tau_j - \tau_{j-1}}{\tau_j} \right) \epsilon_j \tau_j \log \tau_j \tau_j^{2(1-\omega)}
\]

We now observe that \( \tau_j \) is clearly increasing in \( j \), and so is \( \tau_j \epsilon_j \) by definition. Consequently, we can further upper bound this inequality by
\[
\text{err}_m(h^*) \leq \frac{1}{\tau_m} \epsilon_m \tau_m \log \tau_m \sum_{j=1}^{m} \left( \frac{\tau_j - \tau_{j-1}}{\tau_j} \right) \epsilon_j \tau_j^{2(1-\omega)}
\]
\[
\leq c \epsilon_m \log \tau_m \tau_m^{2(1-\omega)} \left( 1 + \frac{m}{j=2} \left( \frac{\tau_j - \tau_{j-1}}{\tau_j} \right) \right)
\]
\[
= c \epsilon_m \log \tau_m \tau_m^{2(1-\omega)} \left( 1 + \frac{m}{j=1} \left( \frac{\tau_{j+1} - \tau_j}{\tau_{j+1}} \right) \right)
\]
\[
\leq c \epsilon_m \log \tau_m \tau_m^{2(1-\omega)} \left( 1 + \frac{m}{j=1} \left( \frac{\tau_{j+1} - \tau_j}{\tau_j} \right) \right)
\]

where the inequality \( (a) \) holds since \( \tau_j \) is increasing in \( j \) and \( \omega \in (0, 1] \) so that the exponent on \( \tau_j \) is non-negative, and the final inequality follows since \( \tau_j \leq \tau_{j+1} \). Invoking Lemma 8, we obtain

37
where we used the fact that $1 \leq \log \tau_m$. Therefore, we have established the second part of the inductive claim, finishing the proof of the lemma.

Using the lemma, we now prove the corollaries.

**Proof of Corollary 2** Based on the proof of Lemma 9, we see that $\Delta_m^*$ satisfies the bound (55). Plugging this into the statement of Theorem 1 immediately yields the lemma.

**Proof of Corollary 4** Based on the proof of Lemma 9, we see that the probability of the disagreement region follows the bound (54). Substituting the bound (55) yields the stated result.

### F Analysis of the Optimization Algorithm

We begin by showing how to find the most violated constraint (Step 3) by calling an importance-weighted ERM oracle. Then we prove Theorem 3, followed by the framework and proof for Theorem 4.

#### F.1 Finding the Most Violated Constraint

Recall our earlier notation $T^m_m(x) = \mathbb{1}(h(x) \neq h_m(x) \land x \in D_m)$. Consider solving (op) using an unlabeled sample $S$ of size $u$. Note that Step 3 is equivalent to

\[
\arg \min_{h \in \mathcal{H}} \ b_m(h) - \mathbb{E}_X \left[ \frac{T^m_m(X)}{P_X(X)} \right] \quad (56)
\]

\[
= \arg \min_{h \in \mathcal{H}} \ 2\gamma \beta^2 (\tau_m - 1) \Delta_{m-1} \text{err}(h, \hat{Z}_{m-1}) + E_X \left[ \left( 2\alpha^2 - \frac{1}{P_X(X)} \right) T^m_m(X) \right]
\]

\[
= \arg \min_{h \in \mathcal{H}} \ 2\gamma \beta^2 (\tau_m - 1) \Delta_{m-1} \text{err}(h, \hat{Z}_{m-1}) + \mathbb{E}_X \left[ \left( 2\alpha^2 - \frac{1}{P_X(X)} \right) \max \left\{ 0, 1(X \in D_m) \mathbb{1}(h(X) \neq h_m(X)) \right\} \right]
\]

\[
= \arg \min_{h \in \mathcal{H}} \ 2\gamma \beta^2 (\tau_m - 1) \Delta_{m-1} \text{err}(h, \hat{Z}_{m-1}) + \mathbb{E}_X \left[ \left( 2\alpha^2 - \frac{1}{P_X(X)} \right) \max \left\{ 0, 1(X \in D_m) \mathbb{1}(h(X) \neq -h_m(X)) \right\} \right],
\]

where $s_\lambda(X) := 2\alpha^2 - 1/P_X(X)$. In the above derivation, the second equality is by the fact that the extra term added to the objective is independent of $h$ and hence does not change the minimizer. The third equality uses a case analysis on the sign of $s_\lambda(X)$ and the identity $1 - \mathbb{1}(h(X) \neq h_m(X)) = \mathbb{1}(h(X) \neq -h_m(X))$. The last expression suggests that an importance-weighted error minimization oracle can find the desired classifier on examples $\{(X, Y^*, W)\}$ with labels and importance weights defined as:

\[
Y^* := \arg \min_Y c(X, Y),
\]

\[
W := |c(X, 1) - c(X, -1)|,
\]

38
where
\[
c(X, Y) := \left\{ 2\gamma^2 \Delta_{m-1} \left( \frac{1}{n} \sum_{i=1}^{n} h_m(X_i) + \frac{1}{2} s^\lambda(X) \right) \right\},
\]
\[X = X_i \in \tilde{Z}_{m-1},
\]
\[X : S.
\]

**F.2 Proof of Theorem 3**

Where clear from context, we drop the subscript \( m \).

We first show that each coordinate ascent step causes sufficient increase in the dual objective. Pick any \( h \) and \( \lambda \). Let \( \lambda' = \lambda_h + \delta \) for some \( \delta > 0 \). Then the increase in the dual objective \( D \) can be computed directly:

\[
D(\lambda') - D(\lambda) = \delta \mathbb{E}_X [T_1^m(X)] + 2 \mathbb{E}_X [1 \cdot (X \in D_m)] (\sqrt{\mathbb{E}_X [T_1^m(X)]^2 + \delta T_1^m(X) - \mathbb{E}_X [T_1^m(X)]}) - \delta b(h)
\]

\[
\geq \delta \mathbb{E}_X [T_1^m(X)] + 2 \mathbb{E}_X \left[ \mathbb{E}_X \left( \frac{T_1^m(X)}{P(X)} - b(h) \right) \right] - \delta^2 \mathbb{E}_X \left[ \frac{T_1^m(X)^2}{4 \mathbb{E}_X [T_1^m(X)]^3} \right]
\]

\[
\geq \delta \left( \mathbb{E}_X \left[ \frac{T_1^m(X)}{P(X)} - b(h) \right] \right) - \frac{\delta^2}{4} \mathbb{E}_X \left[ \frac{T_1^m(X)^2}{\mathbb{E}_X [T_1^m(X)]^3} \right].
\]

The inequality \((58)\) uses the fact that \( \sqrt{1 + z} \geq 1 + z/2 - z^2/8 \) for all \( z \geq 0 \) (provable, for instance, using Taylor’s theorem). The lower bound \((59)\) on the increase in the objective value is maximized exactly at

\[
\delta = 2 \frac{\mathbb{E}_X [T_1^m(X) / P(X)] - b(h)]}{\mathbb{E}_X [T_1^m(X)]^2 / \mathbb{E}_X [P(X)]^3},
\]

as in Step 7. Plugging into \((59)\), it follows that if \( h \) is chosen on some iteration of Algorithm 2 prior to halting then the dual objective \( D \) increases by at least

\[
\frac{\mathbb{E}_X [T_1^m(X) / P(X)] - b(h)]^2}{\mathbb{E}_X [T_1^m(X)]^2 / \mathbb{E}_X [P(X)]^3} \geq \varepsilon^2 \mu^3
\]

since \( \mathbb{E}_X [P(X)] \geq \mu \), and since \( \mathbb{E}_X [T_1^m(X) / P(X)] - b(h)] \geq \varepsilon \).

The initial dual objective is \( D(\lambda) = (1 + \mu)^2 \mathbb{E}_X [P(X)] \). Further, by duality and the fact that \( P(X) = 1/2 \) is a feasible solution to the primal problem, we have \( D(\lambda) \leq 2(1 + \mu^2) \mathbb{E}_X [P(X)] \). And of course, rescaling can never cause the dual objective to decrease. Combining, it follows that the coordinate ascent algorithm halts in at most \( \mathbb{E}_X [P(X)]/(2(1 + \mu^2) - (1 + \mu^2)/\varepsilon^2 \mu^3) \leq \mathbb{E}_X [P(X)]/(\varepsilon^2 \mu^3) \) rounds proving the bound given in the theorem.

By this same reasoning, the left hand side of \((61)\) is equal to \( \delta \cdot \mathbb{E}_X [T_1^m(X) / P(X)] - b(h)] \), which is at least \( \delta \varepsilon \).

That is, the change on each round in the dual objective \( D \) is at least \( \varepsilon \) times the change in one of the coordinates \( \lambda_h \). Furthermore, the rescaling step can never cause the weights \( \lambda_h \) to increase. Therefore, \( \varepsilon \|\lambda\|_1 \) is upper bounded by the total change in the dual objective, which we bounded above. This proves the bound on \( \|\lambda\|_1 \) given in the theorem.

To see \((15)\), consider first the function \( g(s) = D(s \cdot \lambda) \) for \( \lambda \) as in the algorithm after the rescaling step has been executed. At this point, it is necessarily the case that \( s = 1 \) maximizes \( g \) over \( s \in [0,1] \) (since \( \lambda \) has already been rescaled). This implies that \( g'(1) \geq 0 \) where \( g' \) is the derivative of \( g \); that is,

\[
0 \leq g'(1) = \mathbb{E} \left[ \sum_h \lambda_h T_h^m(X) / P(X) - \sum_h \lambda_h b(h). \right]
\]
Now let $F(P)$ denote the modified primal objective function in (12), and let $\hat{P}$ denote the minimum of this objective over all feasible solutions. Then

$$F(P_\lambda) \leq F(P_\lambda) + \sum_h \lambda_h \left( \mathbb{E}_X \left[ \frac{T_h^m(X)}{P_\lambda(X)} \right] - b(h) \right) \leq \min_P \mathcal{L}(P, \lambda) \leq \max_{\lambda} \min_P \mathcal{L}(P, \lambda) = F(\hat{P}).$$

(63)

Here, (63) follows from (62); (64) by the definition of $P_\lambda(X)$ as the minimizer of the Lagrangian; and (65) is by strong duality. Then we have

$$\mathbb{E} \left[ \frac{1}{1-P_\lambda(X)} \right] \leq F(P_\lambda) \leq F(\hat{P}) \leq F(P^*) \leq \mathbb{E} \left[ \frac{1}{1-P^*(X)} \right] + \mu \Pr(D_m).$$

F.3 Proof of Theorem 4

For $\varepsilon > 0$, define $\Lambda_\varepsilon := \{ \lambda \in \mathbb{R}^H : \lambda \geq 0, \|\lambda\|_1 \leq 1/\varepsilon \}$. We begin with a simple lemma.

Lemma 10. Suppose $\phi : \mathbb{R} \times H \rightarrow \mathbb{R}$ be $L$-Lipschitz with respect to its first argument, and $\phi(\sum_{h \in \mathcal{H}} \lambda_h T_h^m(x), x) \leq R$ for all $\lambda \in \Lambda_\varepsilon$ and $x \in \mathcal{X}$. Let $\hat{\mathbb{E}}_X[\cdot]$ denote the empirical expectation with respect to an i.i.d. sample from $\mathbb{P}_X$. For any $\delta \in (0, 1)$, with probability at least $1 - \delta$, every $\lambda \in \Lambda_\varepsilon$ satisfies

$$\left| \hat{\mathbb{E}}_X \left[ \phi \left( \sum_{h \in \mathcal{H}} \lambda_h T_h^m(X), X \right) \right] - \mathbb{E}_X \left[ \phi \left( \sum_{h \in \mathcal{H}} \lambda_h T_h^m(X), X \right) \right] \right| \leq \frac{2L}{\varepsilon} \cdot \sqrt{\frac{2 \ln |\mathcal{H}|}{u} + R \cdot \sqrt{\frac{\ln(1/\delta)}{u}}},$$

(64)

Proof. Let $x \in \{0, 1\}^H$ denote the vector with $x_h = 1(h(x) \neq h_m(x))$, and define the linear function class

$$\mathcal{F} := \{ x \mapsto \langle \lambda, x \rangle : \lambda \in \Lambda_\varepsilon \}.$$

By a simple variant of the argument by Bartlett and Mendelson (2002), with probability at least $1 - \delta$,

$$\left| \hat{\mathbb{E}}_X \left[ \phi \left( \sum_{h \in \mathcal{H}} \lambda_h T_h^m(X), X \right) \right] - \mathbb{E}_X \left[ \phi \left( \sum_{h \in \mathcal{H}} \lambda_h T_h^m(X), X \right) \right] \right| \leq 2L \cdot \mathcal{R}_u(\mathcal{F}) + R \cdot \sqrt{\frac{\ln(1/\delta)}{u}},$$

for all $\lambda \in \Lambda_\varepsilon$, where $\mathcal{R}_u(\mathcal{F})$ is the expected Rademacher average for the linear function class $\mathcal{F}$ for an i.i.d. sample of size $n$. By Kakade et al. (2009), this Rademacher complexity satisfies

$$\mathcal{R}_u(\mathcal{F}) \leq \frac{1}{\varepsilon} \sqrt{\frac{2 \ln |\mathcal{H}|}{u}}.$$

This completes the proof. \hfill \square

Lemma 11. Pick any $\delta \in (0, 1)$. Let $\hat{\mathbb{E}}_X[\cdot]$ denote the empirical expectation with respect to an i.i.d. sample from $\mathbb{P}_X$. With probability at least $1 - \delta$, every $\lambda \in \Lambda_\varepsilon$ satisfies

$$\left| \mathbb{E}_X \left[ \frac{1}{1-P_\lambda(X)} \right] - \hat{\mathbb{E}}_X \left[ \frac{1}{1-P_\lambda(X)} \right] \right| \leq \sqrt{\frac{2 \ln |\mathcal{H}|}{\mu^2 \varepsilon^2 u} + \frac{\ln(1/\delta)}{u}},$$

(65)
and for all \( h \in \mathcal{H} \),
\[
\left| \mathbb{E}_X \left[ \frac{T^m_h(X)}{P_h(X)} \right] - \hat{\mathbb{E}}_X \left[ \frac{T^m_h(X)}{P_h(X)} \right] \right| \leq \sqrt{\frac{2 \ln |\mathcal{H}|}{\mu^4 \varepsilon^2 u}} + \sqrt{\frac{\ln (3|\mathcal{H}|/\delta)}{\mu^2 u}} + \sqrt{\frac{\ln (6|\mathcal{H}|/\delta)}{2u}}
\]
and
\[
\left| \mathbb{E}_X [T^m_h(X)] - \hat{\mathbb{E}}_X [T^m_h(X)] \right| \leq \sqrt{\frac{6|\mathcal{H}|/\delta}{2u}}.
\]

**Proof.** Observe that \( 1/(1 - P_h(x)) = 1 + q_h(x) \) for all \( \lambda \in \Lambda_\varepsilon \) and \( x \in \mathcal{X} \). Now we apply Lemma 10 to the function \( \phi_1(z, x) := \sqrt{\mu^2 + z} \), which is \((2\mu)^{-1}\)-Lipschitz with respect to its first argument. Since \( q_h(x) = f_1(\sum_{h \in \mathcal{H}} \lambda_x T^m_h(x), x) \leq \sqrt{\mu^2 + 1/\varepsilon} \) for all \( \lambda \in \Lambda_\varepsilon \) and \( x \in \mathcal{X} \), Lemma 10 implies that, with probability at least \( 1 - \delta/3 \),
\[
\mathbb{E}_X \left[ \frac{1}{1 - P_h(X)} \right] - \hat{\mathbb{E}}_X \left[ \frac{1}{1 - P_h(X)} \right] \leq \frac{1}{\mu \varepsilon} \sqrt{\frac{2 \ln |\mathcal{H}|}{u}} + \sqrt{\frac{(\mu^2 + 1/\varepsilon) \ln (3/\delta)}{u}}, \quad \forall \lambda \in \Lambda_\varepsilon.
\]

Next, observe that for every \( h \in \mathcal{H} \) and \( x \in \mathcal{X} \),
\[
\frac{T^m_h(x)}{P_h(x)} = \frac{I^m_h(x)}{q_h(x)}.
\]

By Hoeffding’s inequality and a union bound, we have with probability at least \( 1 - \delta/3 \),
\[
\left| \mathbb{E}_X [T^m_h(X)] - \hat{\mathbb{E}}_X [T^m_h(X)] \right| \leq \sqrt{\frac{6|\mathcal{H}|/\delta}{2u}}, \quad \forall h \in \mathcal{H}.
\]

Now we apply Lemma 10 to the functions \( \phi_h(z, x) := I^m_h(x)/\sqrt{\mu^2 + z} \) for each \( h \in \mathcal{H} \); each function \( \phi_h \) is \((2\mu^2)^{-1}\)-Lipschitz with respect to its first argument. Furthermore, since \( \phi_h(\sum_{h \in \mathcal{H}} \lambda_x I^m_h(x), x) = I^m_h(x)/q_h(x) \leq 1/\mu \) for all \( \lambda \in \Lambda_\varepsilon \) and \( x \in \mathcal{X} \), Lemma 10 and a union bound over all \( h \in \mathcal{H} \) implies that, with probability at least \( 1 - \delta/3 \),
\[
\left| \mathbb{E}_X \left[ \frac{T^m_h(X)}{q_h(X)} \right] - \hat{\mathbb{E}}_X \left[ \frac{T^m_h(X)}{q_h(X)} \right] \right| \leq \sqrt{\frac{2 \ln |\mathcal{H}|}{\mu^4 \varepsilon^2 u}} + \sqrt{\frac{\ln (3|\mathcal{H}|/\delta)}{\mu^2 u}}, \quad \forall \lambda \in \Lambda_\varepsilon, h \in \mathcal{H}.
\]

Finally, by a union bound, all of (66), (67), and (68) hold simultaneously with probability at least \( 1 - \delta \).

We can now prove Theorem 6. We first state a slightly more explicit version of the theorem, which is then proved.

**Theorem 6.** Let \( S \) be an i.i.d. sample of size \( u \) from the \( P_X \). Suppose Algorithm 2 is run on the \( m \)-th epoch for solving \((\text{OPS}_{S, \varepsilon})\) up to slack \( \varepsilon \) in the variance constraints. Then the following holds:

1. Algorithm 2 halts in at most \( \frac{\mathbb{F}_r(D_m)}{87\varepsilon^2} \) iterations, where \( \mathbb{F}_r(D_m) := \sum_{X \in S} \mathbb{I}(X \in D_m)/u \).

2. The solution \( \hat{\lambda} \geq 0 \) it outputs has bounded \( \ell_1 \) norm:
\[
\|\hat{\lambda}\|_1 \leq \mathbb{F}_r(D_m)/\varepsilon.
\]

3. There exists an absolute constant \( C > 0 \) such that the following holds. If
\[
u \geq C \cdot \left( \frac{1}{P_{\text{min}, m}^4 \varepsilon^2} + \alpha^4 \right) \cdot \frac{\log |\mathcal{H}|}{\varepsilon^2} + \left( \frac{1}{P_{\text{min}, m}^4} + \frac{1}{\varepsilon} + \alpha^4 \right) \cdot \frac{\log (1/\delta)}{\varepsilon^2},
\]
then with probability at least \( 1 - \delta \), the query probability function \( P_X(X) \) satisfies
• All constraints of (OP) except with slack $2.5\varepsilon$ in constraints (6).

• Approximate primal optimality:

\[
\mathbb{E}_X \left[ \frac{1}{1 - P_\lambda(X)} \right] \leq \mathbb{E}_X \left[ \frac{1}{1 - P_\ast(X)} \right] + 8P_{\min, m} \Pr(D_m) + (2 + 4P_{\min, m})\varepsilon,
\]

where $P_\ast$ is the solution to (OP).

Theorem 4 is just a result of some simplifications in the $O(\cdot)$ notation in the above result. We now prove the theorem.

**Proof of Theorem 6** The first two statements, finite convergence and boundedness of the solution’s $\ell_1$ norm, can be proved with the techniques in Appendix F.2 that establish the same for Theorem 3. We thus focus on proving the third statement here.

Let $\widehat{\mathbb{E}}_X[\cdot]$ denote empirical expectation with respect to $S$. Hoeffding’s inequality implies that with probability at least $1 - \delta/2$,

\[
\widehat{\mathbb{E}}_X[\mathbb{I}(X \in D_m)] \leq \mathbb{E}_X[\mathbb{I}(X \in D_m)] + \varepsilon. \quad (69)
\]

Also, Lemma 11 implies that with probability at least $1 - \delta/2$,

\[
\left| \widehat{\mathbb{E}}_X \left[ \frac{1}{1 - P_\lambda(X)} \right] - \mathbb{E}_X \left[ \frac{1}{1 - P_\lambda(X)} \right] \right| \leq \varepsilon, \quad \forall \lambda \in \Lambda_{\varepsilon/2}; \quad (70)
\]

\[
\left| \mathbb{E}_X \left[ T_h^{\alpha}(X) \right] - \widehat{\mathbb{E}}_X \left[ T_h^{\alpha}(X) \right] \right| \leq \varepsilon/(8\alpha^2), \quad \forall h \in \mathcal{H}; \quad (71)
\]

\[
\left| \mathbb{E}_X \left[ \frac{T_h^{m}(X)}{P_\lambda(X)} \right] - \widehat{\mathbb{E}}_X \left[ \frac{T_h^{m}(X)}{P_\lambda(X)} \right] \right| \leq \varepsilon/4, \quad \forall \lambda \in \Lambda_{\varepsilon/2}, h \in \mathcal{H}. \quad (72)
\]

Therefore, by a union bound, there is an event of probability mass at least $1 - \delta$ on which Eqs. (69), (70), (71), (72) hold simultaneously. We henceforth condition on this event.

By Theorem 3, $\lambda$ satisfies $\|\lambda\|_1 \leq 1/\varepsilon$, the bound constraints in (6), as well as

\[
\widehat{\mathbb{E}}_X \left[ T_h^{\alpha}(X) \right] \leq b_m(h) + 2\varepsilon, \quad \forall h \in \mathcal{H}, \quad (73)
\]

and

\[
\widehat{\mathbb{E}}_X \left[ \frac{1}{1 - P_\lambda(X)} \right] \leq \widehat{\mathbb{E}}_X \left[ \frac{1}{1 - P_\ast(X)} \right] + 4P_{\min, m}\widehat{\mathbb{E}}_X[\mathbb{I}(X \in D_m)] \quad (74)
\]

where $P_\ast$ is the optimal solution to (OP $\varepsilon$). We use this to show that $P_\lambda$ is a feasible solution for (OP $2.5\varepsilon$), and compare its objective value to the optimal objective value for (OP).

Applying (71) and (72) to (73) gives

\[
\mathbb{E}_X \left[ \frac{T_h^{m}(X)}{P_\lambda(X)} \right] \leq b_m(h) + 2.5\varepsilon, \quad \forall h \in \mathcal{H}. \quad (75)
\]

Since $P_\lambda$ also satisfies the bound constraints in (6), it follows that $P_\lambda$ is feasible for (OP $2.5\varepsilon$).

Now we turn to the objective value. Applying (69) and (70) to (74) gives

\[
\mathbb{E}_X \left[ \frac{1}{1 - P_\lambda(X)} \right] \leq \widehat{\mathbb{E}}_X \left[ \frac{1}{1 - P_\ast(X)} \right] + 4P_{\min, m}\mathbb{E}_X[\mathbb{I}(X \in D_m)] + (1 + 4P_{\min, m})\varepsilon. \quad (75)
\]

We need to relate the first term on the right-hand side to the optimal objective value for (OP).

Let $\lambda^*$ be the output of running Algorithm 2 for solving (OP) up to slack $\varepsilon/2$. By Theorem 3, $\lambda^*$ satisfies $\|\lambda^*\|_1 \leq 2/\varepsilon$, the bound constraints in (6), as well as

\[
\mathbb{E}_X \left[ \frac{T_h^{m}(X)}{P_\ast(X)} \right] \leq b_m(h) + \varepsilon/2, \quad \forall h \in \mathcal{H}, \quad (76)
\]

We therefore have
\begin{equation}
\mathbb{E}_X \left[ \frac{1}{1 - P_{\lambda}(X)} - \frac{1}{1 - P^*(X)} \right] \leq \mathbb{E}_X \left[ \frac{1}{1 - P^*(X)} \right] + 4P_{\text{min},m}\mathbb{E}_X[1(X \in D_m)].
\tag{76}
\end{equation}

Applying (70) to (76), we have
\begin{equation}
\hat{\mathbb{E}}_X \left[ \frac{1}{1 - P_{\lambda}(X)} \right] \leq \mathbb{E}_X \left[ \frac{1}{1 - P^*(X)} \right] + 4P_{\text{min},m}\mathbb{E}_X[1(X \in D_m)] + \varepsilon.
\tag{77}
\end{equation}

And applying (71) and (72) to (76) gives
\begin{equation}
\hat{\mathbb{E}}_X \left[ \frac{I_m(h)(X)P_{\lambda}(X)}{P_{\lambda}(X)} \right] \leq b_m(h) + \varepsilon, \quad \forall h \in \mathcal{H}.
\tag{78}
\end{equation}

This establishes that \( \lambda^* \) is a feasible solution for \((\text{OP}_{S,\varepsilon})\). In particular,
\begin{equation}
\hat{\mathbb{E}}_X \left[ \frac{1}{1 - P_{\lambda}(X)} \right] \leq \mathbb{E}_X \left[ \frac{1}{1 - P^*(X)} \right]
\leq \mathbb{E}_X \left[ \frac{1}{1 - P^*(X)} \right] + 4P_{\text{min},m}\mathbb{E}_X[1(X \in D_m)] + \varepsilon
\end{equation}

where the second inequality follows from (77). We now combine this with (75) to obtain
\begin{equation}
\mathbb{E}_X \left[ \frac{1}{1 - P_{\lambda}(X)} \right] \leq \mathbb{E}_X \left[ \frac{1}{1 - P^*(X)} \right] + 8P_{\text{min},m}\mathbb{E}_X[1(X \in D_m)] + (2 + 4P_{\text{min},m})\varepsilon.
\end{equation}

\section{Experimental Details}

Here we provide more details about the experiments.

\subsection*{G.1 Datasets}

Table \[1\] gives details about the 23 binary classification datasets used in our experiments, where \( n \) is the number of examples, \( d \) is the number of features, \( s \) is the average number of non-zero features per example, and \( r \) is the proportion of the minority class.

\subsection*{G.2 Hyper-parameter Settings}

We start with the actual hyper-parameters used by OAC. Going back to Algorithm\[1\] we note that the tuning parameters get used in mostly the following three quantities: \( \gamma \Delta_{i-1} \), \( \alpha \) and \( \beta \). We use this fact to reduce the number of input parameters. Let \( c_0 := \gamma^2 c_1 32(\log(|\mathcal{H}|/\delta) + \log(i-1)) \) (treating \( \log(i-1) \) as a constant) and set \( \eta = 864, \gamma = \eta/4 \) and \( c_2 = \eta c_1^2/4 \) according to our theory. Then we have
\begin{align*}
\gamma \Delta_{i-1} &= \sqrt{\gamma^2 c_1 \epsilon_{i-1} \text{err}(h_i, \tilde{Z}_{i-1}) + \gamma c_2 \epsilon_{i-1} \log(i-1)} \\
&= \sqrt{\frac{c_0 \text{err}(h_i, \tilde{Z}_{i-1})}{i-1} + c_0 \frac{c_2}{\gamma c_1} \frac{\log(i-1)}{i-1},}
\end{align*}

where \( \frac{c_2}{\gamma c_1} = c_2 = O(\alpha) \). Based on this, we use
\begin{equation}
\hat{\Delta}_{i-1} := \sqrt{\frac{c_0 \text{err}(h_i, \tilde{Z}_{i-1})}{i-1} + \max(2\alpha, 4) c_0 \frac{\log(i-1)}{i-1}}
\tag{79}
\end{equation}
Table 1: Binary classification datasets used in experiments

| Dataset            | n    | s    | d    | r    |
|--------------------|------|------|------|------|
| titanic            | 2201 | 3    | 8    | 0.323|
| abalone            | 4176 | 8    | 8    | 0.498|
| mushroom           | 8124 | 22   | 117  | 0.482|
| eeg-eye-state      | 14980| 13.9901| 14    | 0.449|
| 20news             | 18845| 93.8854| 101631| 0.479|
| magic04            | 19020| 9.98728| 10    | 0.352|
| letter             | 20000| 15.5807| 16    | 0.233|
| ijcnn1             | 24995| 13   | 22   | 0.099|
| nomao              | 34465| 82.3306| 174   | 0.286|
| shuttle            | 43500| 7.04984| 9     | 0.216|
| bank               | 45210| 13.9519| 44    | 0.117|
| a9a                | 48841| 13.8676| 123   | 0.239|
| adult              | 48842| 11.9967| 105   | 0.239|
| w8a                | 49749| 11.6502| 300   | 0.030|
| bio                | 145750| 73.4184| 74    | 0.009|
| maptaskcoref       | 158546| 40.4558| 5944  | 0.438|
| activity           | 165632| 18.5489| 20    | 0.306|
| skin               | 245057| 2.948 | 3     | 0.208|
| vehv2binary        | 299254| 48.5652| 105   | 0.438|
| census             | 299284| 32.0072| 401   | 0.062|
| covtype            | 581011| 11.8789| 54    | 0.488|
| rcv1               | 781265| 75.7171| 43001 | 0.474|
| kdda               | 8407751| 36.349 | 19306083| 0.147|

in Algorithm 3 in place of $\gamma \Delta_{i-1}$. Next we consider

$$
\beta^2 \leq \frac{1}{216ne_n \log n} \\
\approx \frac{\gamma^2 c_1}{216c_0 \log n} \\
: n \epsilon_n \approx c_0/(\gamma^2 c_1) \text{ by treating } \log n \text{ as a constant} \\
= \mathcal{O}\left(\frac{\alpha}{c_0}\right) \text{ by again treating } \log n \text{ as a constant and } c_1 = \mathcal{O}(\alpha).
$$

Based on the last expression, we set $\beta := \sqrt{\alpha/c_0}$. In sum, the actual input parameters boil down to the cover size $l$, $\alpha \geq 1$ and $c_0$, and we use them to set

$$
\gamma \Delta_{i-1} := \sqrt{c_0 \epsilon(h_i, \tilde{Z}_{i-1}) \over i-1} + \max(2\alpha, 4) c_0 \log(i-1) \over i-1, \quad \beta = \sqrt{\alpha/c_0} \over 10.
$$

Finally, we use the following setting for the minimum query probability:

$$
P_{\text{min}, i} = \min\left(1, \frac{1}{\sqrt{(i-1) \epsilon(h_i, \tilde{Z}_{i-1}) + \log(i-1)}}, \frac{1}{2}\right).
$$

Next we describe hyper-parameter settings for different algorithms. A common hyper-parameter is the learning rate of the underlying online oracle, which is a reduction to importance-weighted logistic regression. For all active learning algorithm, we try the following 11 learning rates: $10^{-1}, \{2^{-2}, 2^{-1}, \ldots, 2^8\}$. Active learning hyper-parameter settings are given in the following table:
Good hyper-parameters of the algorithms usually lie in the interior of these value ranges.

### G.3 More Experimental Results

We provide detailed per-dataset results in Figures 7 and 8, which show minimum test errors over hyper-parameter settings that are achievable at different query rates for small (fewer than $10^5$ examples) and large (more than $10^5$ examples) datasets. On small datasets, OAC is generally competitive with other algorithms. On all (including the three shown in Figure 3) but two large datasets, bio and kdda, OAC outperforms other algorithms at most query rates, with a clear advantage at low query rates. Note that both bio and kdda, as shown in Table 1, are imbalanced. The fraction of the minority class in bio is about 1%, and the minimum test error is about 0.4%, a quite significant difference. IWAL strongly dominates other algorithms on this dataset, which suggests that using predicted labels, as done by the other three agnostic active learning algorithms, may be undesirable for highly imbalanced classification problems. There is less class skewness in kdda, but the minimum test error 12% is only slightly lower than the fraction of the minority class 14.7%. On this hard dataset, ORA-I, i.e., the Oracular CAL variant of OAC, outperforms other algorithms.
Figure 7: Minimum test error vs. query rate for datasets with fewer than $10^5$ examples
Figure 8: Minimum test error vs. query rate for datasets with more than $10^5$ examples