Correlation Bound for a One-Dimensional Continuous Long-Range Ising Model

David Hasler∗ Benjamin Hinrichs† Oliver Siebert‡
Friedrich-Schiller-University Jena
Department of Mathematics
Ernst-Abbe-Platz 2
07743 Jena
Germany

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Abstract

We consider a measure given as the continuum limit of a one-dimensional Ising model with long-range translationally invariant interactions. Mathematically, the measure can be described by a self-interacting Poisson driven jump process. We prove a correlation inequality, estimating the magnetic susceptibility of this model, which holds for small \(L^1\)-norm of the interaction function. The bound on the magnetic susceptibility has applications in quantum field theory and can be used to prove existence of ground states for the spin boson model.

1 Introduction and Result

The spin boson model describes a two-level quantum mechanical system linearly coupled to a quantized bosonic field. If the bosons have relativistic dispersion relation the spin boson model provides a caricature of a confined non-relativistic quantum mechanical system interacting with the quantized electromagnetic field. This is one of the reasons the spin boson model has been intensively investigated and questions about existence of a ground state of the spin boson Hamiltonian are of interest. In a recent paper [HHS21], we showed that the spin boson model has a square integrable ground state in situations where the coupling function can have infrared singularities as long as the Hamiltonian stays bounded from below. The result assumed a resolvent bound. It is well-known that the ground state energy of the spin boson model can be expressed in terms of an expectation of a one-dimensional continuous Ising model with long range couplings [EL74]. In this paper we prove a bound about this continuous Ising model, which can be used to obtain the resolvent bound needed in [HHS21].

The Ising model is a mathematical model of ferromagnetism and has been intensively investigated. The magnetic dipole moments are approximated by the values \(+1, -1\) often referred to as Ising spins. The Ising spins are typically arranged on a lattice. In this paper we consider a

∗david.hasler@uni-jena.de
†benjamin.hinrichs@uni-jena.de
‡Present Affiliation: École polytechnique fédérale de Lausanne, oliver.siebert@epfl.ch
one-dimensional continuous Ising model which is described in terms of a jump process and a long range interaction given by a nonnegative symmetric integrable function. The main result of this paper is a correlation bound, which has the the physical interpretation of a bound on the magnetic susceptibility. Thus, the bound which we prove is of its own physical interest. In fact, to prove the bound we will consider a scaling limit of an Ising model on the one-dimensional lattice, where the nearest neighbor coupling becomes arbitrarily large. Thus to obtain the main result we will prove a correlation bound for the Ising model on $\mathbb{Z}$. That bound is of its own interest and can be viewed as a result between the results of Dyson [Dys69] and of Rogers and Thompson [RT81] for the Ising model on the one-dimensional lattice.

Let us now give the explicit definition. Let $N(t)$ with $t \in \mathbb{R}$ be a two-sided Poisson process with unit intensity and let $B$ be an independent Bernoulli random variable with $\mathbb{P}(B = 1) = \mathbb{P}(B = -1) = \frac{1}{2}$. Then define $X(t)$ to be the jump process

$$X(t) = B(-1)^{N(t)}.$$  

We give an overview of the connection between the jump process and the spin boson model, as motivated in the beginning of this introduction, in Section 2. This allows us to describe the desired second derivative of the ground state energy as an expectation value (cf. (2.9)).

To state the main result of this paper, assume $W : \mathbb{R} \to \mathbb{R}$ is continuous. We then define the canonical partition function

$$Z(W, T) = \mathbb{E} \left[ \exp \left( \int_{-T}^{T} \int_{-T}^{T} W(t-s)X(t)X(s)dtds \right) \right] \text{ for } T > 0. \quad (1.2)$$

Remark 1.1. The integral occurring in (1.2) is a Riemann integral. If the jump process is realized as a random variable on a space $\Omega$, then for almost every $\omega \in \Omega$ the function $t \mapsto X(t)(\omega)$ has only finitely many discontinuities on compact subsets and is hence Riemann integrable.

Our central result is the following.

**Theorem 1.2.** There exist constants $\varepsilon > 0$ and $C > 0$, such that for all continuous and even $W \in L^1(\mathbb{R})$ with $W \geq 0$ and $\|W\|_1 \leq \varepsilon$, we have

$$\limsup_{T \to \infty} \frac{1}{Z(W, T)} \mathbb{E} \left[ \frac{1}{T} \left( \int_{-T}^{T} X(t)dt \right)^2 \exp \left( \int_{-T}^{T} \int_{-T}^{T} X(t)X(s)W(t-s)dtds \right) \right] \leq C.$$

Remark 1.3. This result equivalently holds, if we choose an arbitrary intensity $\lambda$ of the Poisson process $N$ in (1.1). Note that the constant $C$, however, is not independent of $\lambda$. This can be seen by a simple scaling argument.

Remark 1.4. In the special case, where the integrable $W \geq 0$ satisfies the additional condition $W(t) \sim t^{-2}$ as $t \to \infty$, a bound as in Theorem 1.2 follows from [Spo89, Proposition 8.1] for the conditioned process with boundary conditions $X(T) = X(-T)$. The proof given in [Spo89] is based on results from percolation theory [AN86].

Remark 1.5. The bound in Theorem 1.2 is in general not expected to hold for arbitrary large $\varepsilon > 0$, as the following results indicate. The one-dimensional long-range Ising model with spins $\sigma_i = \pm 1$, $i \in \mathbb{Z}$ and interaction energy $\sum_{i,j} J(i-j)\sigma_i\sigma_j$ with $J(n) = n^{-\alpha}$ has a phase transition if $1 < \alpha \leq 2$. In that case the magnetic susceptibility diverges for sufficiently small temperatures. This was shown in [Dys69] for $1 < \alpha < 2$ and in [ACCN88] for $\alpha = 2$. It is reasonable to believe that such a divergence carries over to the continuous model, since the continuous model can be obtained by a scaling limit of the discrete model if an additional nearest neighbor coupling is imposed. For details on the scaling limit, we refer the reader to Section 4 and also [SD85, Spo89].

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The paper is organized as follows. In Section 2, we describe the connection between the jump process $X$ and the spin boson model. This illustrates our motivation to study the problem, but is not relevant for the proof of the result. In Section 3, we prove an upper bound for correlation functions in the one-dimensional Ising model on $\mathbb{Z}$. This is the main technical ingredient of our proof and can also be seen in a long line of such estimates, as described in the beginning of that section. In Section 4, we then prove the jump process is the continuum limit of a specific Ising model. We summarize our proof of Theorem 1.2 in Section 5.

2 Magnetic Susceptibility and the Spin Boson Model

This section is not needed for the proof of Theorem 1.2, but rather puts the result into a broader context. First, we show that the expression in Theorem 1.2 is equal to the magnetic susceptibility. Then, we relate the result to the spin boson model. In particular, we sketch how Theorem 1.2 can be used to show that the ground state energy satisfies a derivative bound, which was used in [HHS21] to prove the existence of a ground state for the spin boson model. This was our main motivation to prove Theorem 1.2.

Adding a constant magnetic field $\mu \in \mathbb{R}$ to the interaction, we obtain the canonical partition function

$$Z_\mu(W, T) = E \left[ \exp \left( \int_{-T}^{T} \int_{-T}^{T} W(t-s)X(t)X(s)dtds + \mu \int_{-T}^{T} X(t)dt \right) \right].$$

The magnetization is then defined as

$$M_\mu(W, T) = 1/T \partial_\mu \ln Z_\mu(W, T)$$

and the magnetic susceptibility is defined as

$$\chi_\mu(W, T) = \partial_\mu M_\mu(W, T).$$

A straightforward calculation shows that the the magnetic susceptibility at zero satisfies

$$\chi_\mu(W, T) |_{\mu=0} = \frac{1}{E(W, T)} E \left[ \frac{1}{T} \left( \int_{-T}^{T} X(t)dt \right)^2 \exp \left( \int_{-T}^{T} \int_{-T}^{T} X(t)X(s)W(t-s)dtds \right) \right],$$

which is the expression estimated in Theorem 1.2.

Now, let us consider the spin boson model with an external magnetic field. We sketch the relation of the second order derivative of the ground state energy with respect to the magnetic field to the expression estimated in Theorem 1.2. In [HHS21], we proved that an upper bound on the magnetic susceptibility in the spin boson model implies existence of ground states, if it is uniform in the photon mass. It is well-known that the ground state energy of the spin boson model can be equivalently described as a jump process, which itself is the continuum limit of the one-dimensional Ising model [EL74]. This duality has been used to study the spin boson model in the past [FN88, SD85, Spo89, Abd11, HHL14]. In this spirit, our result is formulated as a bound on the expectation value of a Poisson-driven jump process.

We use notation similar to [HHS21] and refer the reader to that paper for more rigorous definitions. We fix a measurable function $\omega : \mathbb{R}^d \rightarrow [0, \infty)$ with $\omega > 0$ almost everywhere, an element...
\[ v \in L^2(\mathbb{R}^d) \text{ such that } v/\sqrt{\omega} \in L^2(\mathbb{R}^d) \text{ and a coupling constant } \lambda \in \mathbb{R}. \]  

Let \( \mathcal{F} \) be the Fock space over \( L^2(\mathbb{R}^d) \) and denote by \( d\Gamma(\omega) \) and \( a^*(v) \), \( a(v) \) the usual second quantization operator of \( \omega \) and the creation/annihilation operators corresponding to \( v \), respectively. Further, assume \( \sigma_x \) and \( \sigma_z \) are the usual Pauli matrices. Then, we define the spin boson Hamiltonian with an external field of strength \( \mu \in \mathbb{R} \) as the selfadjoint lower-bounded operator acting on \( \mathbb{C}^2 \otimes \mathcal{F} \) as

\[
H(\mu) = (\sigma_z + 1) \otimes 1 + 1 \otimes d\Gamma(\omega) + \sigma_x \otimes (\lambda(a^*(v) + a(v)) + \mu 1).
\] (2.1)

We investigate properties of the ground state energy

\[
E(\mu) = \inf \sigma(H(\mu)). \tag{2.2}
\]

To that end, we use Bloch’s formula. Let \( \Omega_\downarrow = \begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \Omega \), where \( \Omega \) denotes the Fock space vacuum. Then Bloch’s formula states that for

\[
E_T(\mu) = -\frac{1}{T} \ln \left< \Omega_\downarrow, e^{-TH(\mu)}\Omega_\downarrow \right> \tag{2.3}
\]

one has

\[
E(\mu) = \lim_{T \to \infty} E_T(\mu). \tag{2.4}
\]

A rigorous proof of (2.4) can be obtained by a straightforward application of the spectral theorem, provided \( 1_{H(\mu) \leq E(\mu) + \varepsilon} \Omega_\downarrow \neq 0 \) for all \( \varepsilon > 0 \). This last assumption can be shown using that \( e^{-TH(\mu)} \) is positivity improving, which has been shown for example in [HH11] for \( \mu = 0 \) and follows for arbitrary \( \mu \in \mathbb{R} \) by a simple modification. Now the right hand side of (2.3) can be calculated using the Feynman-Kac formula and integrating out the quantum field in the so called Schrödinger representation [Sim79, LHB11]. Such a calculation yields

\[
\left< \Omega_\downarrow, e^{-TH(\mu)}\Omega_\downarrow \right> = Z_\mu(W, T), \tag{2.5}
\]

where

\[
W(t) = \frac{\lambda^2}{8} \int_{\mathbb{R}^d} |v(k)|^2 e^{-|t|\omega(k)} \, dk, \quad t \in \mathbb{R}. \tag{2.6}
\]

We note that (2.5) has been shown in the literature for \( \mu = 0 \) [HHL14] and a similar formula is derived in [Spo89] for KMS states. Note that the function \( W(t) \) defined in (2.6) is symmetric, continuous, and in \( L^1(\mathbb{R}) \). Since we have not found an explicit proof of (2.5) in the literature, we plan to address this in a forthcoming paper.

Now, inserting (2.5) into (2.3) we find

\[
E_T(\mu) = -\frac{1}{T} \ln Z_\mu(W, T). \tag{2.7}
\]

Differentiating this expression twice with respect to \( \mu \) and evaluating it at zero, we find from the calculation in the first part of this section that

\[
E''_T(0) = -X_\mu(W, T)|_{\mu=0}. \tag{2.8}
\]

Now, let us consider the limit \( T \to \infty \). Provided one can show that the limit

\[
E''(0) = \lim_{T \to \infty} E''_T(0), \tag{2.9}
\]

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exists, one obtains
\[ E''(0) = -\lim_{T \to \infty} \mathcal{X}_\mu(W, T)|_{\mu=0}. \] (2.9)
Given (2.9), Theorem 1.2 yields a bound on the second derivative of the ground state energy of the spin boson model with respect to an external magnetic field. This bound is uniform in the $L^1$-norm of $W$. We note that (2.8) can be shown to hold for example if the ground state energy is isolated from the rest of the spectrum. We plan to address this in a forthcoming paper. It is well-known that there exists such a gap if \( \inf_{k \in \mathbb{R}^d} \omega(k) > 0 \) \cite{AH95}.

3 A Correlation Bound for the Ising Model

In this section, we introduce the Ising model and prove an upper bound on correlation functions, which will be stated in Proposition 3.1 below. The novel aspect of this bound is that it can accommodate arbitrarily large nearest neighbor couplings. This result is the main technical ingredient to our proof of Theorem 1.2. The connection between the jump process and the Ising model will be treated in Section 4. Bounds on correlation functions of the Ising model have been studied throughout the literature, cf. \cite{Gri67a, KS68, Gin70, Tho71, RT81} and references therein. They are for example used to prove the existence of the thermodynamic limit and of phase transitions in the Ising model, cf. \cite{Gri67b, GMS67, Rue68, Dys69, KT69, FILS78, AN86}.

Let \( L \in \mathbb{N} \) and \( \Lambda_L = \mathbb{Z} \cap [-L, +L] \). We define the spin configuration space \( S_L = \{-1, 1\}^{\Lambda_L} \). For \( \sigma = (\sigma_i)_{i \in \Lambda_L} \in S_L \) and \( A \subset \Lambda_L \), we write
\[ \sigma_A = \prod_{i \in A} \sigma_i, \] (3.1)
where we use the convention that \( \sigma_\emptyset = 1 \). For \( J : \mathcal{P}(\mathbb{Z}) \to \mathbb{R} \), we define the corresponding Ising energy
\[ E_{J,L}(\sigma) = -\sum_{A \subset \Lambda_L} J(A)\sigma_A \] (3.2)
and the partition function
\[ Z_{J,L} = \sum_{\sigma \in S_L} \exp(-E_{J,L}(\sigma)). \] (3.3)
In contrast to the standard definitions in statistical mechanics, we absorb the thermodynamic parameter \( \beta \) in the interaction function \( J \). The expectation value of a function \( f : S_L \to \mathbb{R} \) is now defined as
\[ \langle f \rangle_j^{(L)} = \frac{1}{Z_{J,L}} \sum_{\sigma \in S_L} f(\sigma) \exp(-E_{J,L}(\sigma)). \] (3.4)
For given \( f : S_L \to \mathbb{R} \) and \( L \geq L \), we denote the function \( f^\ast : S_L \to \mathbb{R} \) with \( f^\ast(\sigma) = f(\sigma|_{\Lambda_L}) \) again by the same symbol \( f \). Then, if the thermodynamic limit \( L \to \infty \) exists, we will drop the superscript \( (L) \) and write
\[ \langle f \rangle_j = \lim_{L \to \infty} \langle f \rangle_j^{(L)}. \] (3.5)
Especially, we note that the existence of the thermodynamic limit of correlation functions \( \langle \sigma_i \sigma_j \rangle_j \) for \( J : \mathcal{P}(\mathbb{Z}) \to [0, \infty) \) is well-known (cf. \cite{Gri67b} or Corollary 3.6).

For a sequence \( w = (w_k)_{k \in \mathbb{N}} \subset \mathbb{R} \), we define the associated pair interaction
\[ J_w : \mathcal{P}(\mathbb{Z}) \to \mathbb{R} \quad \text{with} \quad \begin{cases} \{i, j\} &\mapsto w_{|i-j|} \quad \text{for} \; i, j \in \mathbb{Z}, i \neq j, \\ A &\mapsto 0 \quad \text{for any other} \; A \subset \mathbb{Z}. \end{cases} \] (3.6)

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In this section we prove the following proposition.

**Proposition 3.1.** For every \( \varepsilon \in (0, \frac{1}{10}) \) there exists a \( C_\varepsilon > 0 \), such that for any \( w = (w_k)_{k \in \mathbb{N}} \in \ell^1(\mathbb{N}) \) with \( w \geq 0 \) and

\[
\sum_{l=2}^{\infty} \tanh w_l \leq \varepsilon (1 - \tanh w_1),
\]

we have

\[
\sum_{i \in \mathbb{Z}} \langle \sigma_i, \sigma_j \rangle_{w} \leq \frac{C_\varepsilon}{1 - \tanh w_1} \quad \text{for all } j \in \mathbb{Z}.
\]

**Remark 3.2.** We note that for \( v \in \ell^1(\mathbb{N}) \) the sequence \( w = \beta v \) satisfies the relation \((3.7)\) for sufficiently small \( \beta > 0 \). Hence, our bound describes absence of long range order in the Ising model for any summable pair interaction provided the temperature is large enough.

**Remark 3.3.** We note that correlations estimates have been shown already a long time ago in [Dys69, RT81]. We generalize the result of [Dys69], in the sense that we can accommodate arbitrary large nearest neighbor couplings and obtain an analogous correlation bound. On the other hand the assumptions in [RT81] or weaker but their assertion is weaker as well. Explicitly, Rogers and Thompson prove the estimate \( \lim_{N \to \infty} \frac{1}{N^2} \sum_{i,j=1}^{N} \langle \sigma_i, \sigma_j \rangle_{w} = 0 \) under the assumption \( \sum_{k=1}^{N} kw_k = o((\ln N)^{1/2}) \), which shows the absence of long-range order. Note that under the stronger assumption \((3.7)\), Proposition 3.1 implies the correlation estimate

\[
\limsup_{N \to \infty} \frac{1}{N} \sum_{i,j=1}^{N} \langle \sigma_i, \sigma_j \rangle_{w} < \infty,
\]

which is stronger.

Let us begin with recalling some well-known inequalities on correlation functions in the Ising model, which go back to Griffiths [Gri67a, Gri67c] (later generalized in [KS68, Gin70] and referred to as the GKS (Griffiths-Kelly-Sherman) inequalities) and Thompson [Tho71]. To that end, we write the symmetric set difference as \( AB = A \cup B \setminus (A \cap B) \) for \( A, B \subset \mathbb{Z} \). Further, if \( \mathcal{A} \subset \mathcal{P}(\mathbb{Z}) \), we define

\[
\langle \cdot \rangle_{\mathcal{A}}^{(L)} := \langle \cdot \rangle_{I_{\mathcal{A}}}^{(L)} \quad \text{and} \quad \langle \cdot \rangle_{\mathcal{A}} := \langle \cdot \rangle_{I_{\mathcal{A}}}, \quad \text{where} \quad I_{\mathcal{A}}(A) = \begin{cases} J(A) & \text{for } A \notin \mathcal{A}, \\ 0 & \text{for } A \in \mathcal{A}. \end{cases}
\]

**Lemma 3.4.** Let \( J : \mathcal{P}(\mathbb{Z}) \to [0, \infty) \) and assume \( A, B \subset \mathbb{Z}, L \in \mathbb{N} \). Then the following holds.

(i) \( \langle \sigma_A \rangle_{\mathcal{A}}^{(L)} \geq 0 \) (Griffiths’ first inequality)

(ii) \( \langle \sigma_{AB} \rangle_{\mathcal{A}}^{(L)} \geq \langle \sigma_A \rangle_{\mathcal{A}}^{(L)} \langle \sigma_B \rangle_{\mathcal{A}}^{(L)} \) (Griffiths’ second inequality)

(iii) \( \langle \sigma_A \rangle_{\mathcal{A}}^{(L)} \leq \langle \sigma_A \rangle_{\mathcal{A} \setminus \{B\}}^{(L)} + \tanh(J(B)) \langle \sigma_{AB} \rangle_{\mathcal{A} \setminus \{B\}}^{(L)} \) (Griffiths’ third inequality)

(iv) \( \langle \sigma_A \rangle_{\mathcal{A} \setminus \{B\}}^{(L)} \leq \langle \sigma_A \rangle_{\mathcal{A}}^{(L)} \)

(v) \( \langle \sigma_A \rangle_{\mathcal{A}}^{(L)} \leq \tanh(J(B)) \langle \sigma_{AB} \rangle_{\mathcal{A}}^{(L)} + (1 - \tanh^2(J(B))) \langle \sigma_A \rangle_{\mathcal{A} \setminus \{B\}}^{(L)} \)

**Proof.** Parts (i) and (ii) follow from the main theorem in [KS68]. Parts (iii) and (v) are shown in [Tho71] in (3.1), (1.6), and (2.5), respectively. \( \square \)

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We will also utilize the well-established simple fact that expectations involving uncoupled Ising spins always vanish. This is the content of the next lemma.

**Lemma 3.5.** Let $L \in \mathbb{N}$, $i \in \Lambda_L$ and assume $J : \mathcal{P}(\mathbb{Z}) \to \mathbb{R}$ satisfies $J(A) = 0$ for all $A \subset \Lambda_L$ with $i \in A$. Then $\langle \sigma_B^{(L)} \rangle_j = 0$ for any $B \subset \Lambda_L$ with $i \in B$.

**Proof.** We define $\phi_i : S_L \to S_L$ as $(\phi_i(\sigma))_k = -\sigma_k$, if $k = i$, and $(\phi_i(\sigma))_k = \sigma_k$, if $k \neq i$. By the assumptions, it follows that $E_{i,L}(\phi_i(\sigma)) = E_{i,L}(\sigma)$ for all $\sigma \in S_L$. Further, if $i \in B$, we have $\sigma_B \circ \phi_i = -\sigma_B$. Together, we obtain

$$\langle \sigma_B^{(L)} \rangle_j = \langle \sigma_B \circ \phi_i^{(L)} \rangle_j = \langle -\sigma_B^{(L)} \rangle_j = -\langle \sigma_B^{(L)} \rangle_j.$$ 

This implies the claim. \hfill $\square$

The existence of the thermodynamic limit immediately follows from Lemma 3.4 and is well-known since [Gri67b].

**Corollary 3.6.** Let $J : \mathcal{P}(\mathbb{Z}) \to [0, \infty)$ and assume $A \subset \mathbb{Z}$. Then the thermodynamic limit $\langle \sigma_A \rangle_j$ exists.

**Proof.** By Lemma 3.4 the expectation $\langle \sigma_A \rangle_j^{(L)}$ is nonnegative (Part (i)), increasing in $L$ (Part (iv)), and bounded above by 1. Thus the statement follows by monotone convergence. \hfill $\square$

The major ingredient of the proof of Proposition 3.1 is the following correlation bound for finite Ising spin chains.

**Lemma 3.7.** Let $L \in \mathbb{N}$ and $w = (w_k)_{k \in \mathbb{N}} \subset [0, \infty)$. We set $\tau_k = \tanh(w_k)$.

If $i, j \in \Lambda_L$ with $i \leq j$, we have

$$\langle \sigma_i \sigma_j \rangle_{j,w}^{(L)} \leq \tau_1 \langle \sigma_i \sigma_{j+1} \rangle_{j,w}^{(L)} + \sum_{l=2}^{\infty} \tau_l \langle \sigma_i \sigma_{j+l} \rangle_{j,w}^{(L)} + (1 - \tau_1^2) \sum_{l=2}^{\infty} \tau_1 \sum_{s=\pm 1} \tau_s \langle \sigma_i \sigma_{j+b+l} \rangle_{j,w}^{(L)},$$

where we use the convention that $\langle \sigma_i \sigma_k \rangle_{j,w}^{(L)} = 0$ if $l$ or $k$ is not an element of $\Lambda_L$.

**Proof.** The philosophy of our proof is sketched in Fig. 1. We use the estimates in Lemma 3.4 to reduce the number of interaction edges, in which $j$ contributes. To that end, for $j \in \Lambda_L$, we define the sets

$$S_j^\pm = \{j, j \pm 1\} \quad \text{and} \quad O_j = \{\{j, k\} : k \in \mathbb{Z} \setminus \{j, j - 1, j + 1\}\}.$$ 

Note that $S_j^\pm$ contain the nearest neighbors of $j$, while $O_j$ are all long-range pairs involving $j$. Throughout this proof, we drop the superscript $(L)$ and the subscript $J_w$ of expectation values. Moreover we assume $i < j$. The statement in the case $i > j$ can be treated completely analogous.

By twice applying Lemma 3.4(iii) we obtain

$$\langle \sigma_i \sigma_j \rangle \leq \langle \sigma_i \sigma_j \rangle_{\{j, j-2\}} + \tau_2 \langle \sigma_i \sigma_{j-2} \rangle_{\{j, j-2\}}$$

$$\leq \langle \sigma_i \sigma_j \rangle_{\{j, j-2\}, \{j, j+2\}} + \tau_2 \langle \sigma_i \sigma_{j+2} \rangle_{\{j, j-2\}, \{j, j+2\}} + \tau_2 \langle \sigma_i \sigma_{j-2} \rangle_{\{j, j-2\}}.$$ 

Combined with Lemma 3.4(iv) this implies

$$\langle \sigma_i \sigma_j \rangle \leq \tau_2 \left( \langle \sigma_i \sigma_{j-2} \rangle + \langle \sigma_i \sigma_{j+2} \rangle \right) + \langle \sigma_i \sigma_j \rangle_{\{j, j-2\}, \{j, j+2\}}.$$ 

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Figure 1: Illustration of the set $\mathcal{E}_L$, consisting of all edges with vertices in $\Lambda_L$, without the edges of the indicated sets.

Iterating this argument, we arrive at

$$\langle \sigma_i \sigma_j \rangle \leq \sum_{l=2}^{\infty} \tau_l \sum_{s=\pm} \langle \sigma_i \sigma_{j+sl} \rangle + \langle \sigma_i \sigma_j \rangle_{O_J}.$$  \hfill (3.11)

Then, Lemma 3.4(v) yields

$$\langle \sigma_i \sigma_j \rangle_{O_J} \leq \tau_1 \langle \sigma_i \sigma_{j-1} \rangle_{O_J} + (1 - \tau_1^2) \langle \sigma_i \sigma_j \rangle_{O_J \cup \{S_j^-\}}.$$  \hfill (3.12)

The second term on the right hand side can be estimated by Lemma 3.4(iii) and Lemma 3.5

$$\langle \sigma_i \sigma_j \rangle_{O_J \cup \{S_j^-\}} \leq \langle \sigma_i \sigma_{j+1} \rangle_{O_J \cup \{S_j^-, S_j^+\}} + \tau_1 \langle \sigma_i \sigma_{j+1} \rangle_{O_J \cup \{S_j^-, S_j^+, S_{j+1}^+\}}.$$  \hfill (3.13)

Now applying (3.11) with $j$ replaced by $j+1$ and using Lemma 3.4(iv), we obtain

$$\langle \sigma_i \sigma_{j+1} \rangle_{O_J \cup \{S_j^-, S_j^+\}} \leq \sum_{l=0}^{\infty} \sum_{s=\pm} \tau_l \langle \sigma_i \sigma_{j+1+sl} \rangle + \langle \sigma_i \sigma_{j+1} \rangle_{O_J \cup \{S_j^-, S_j^+, S_{j+1}^+\}}.$$  \hfill (3.14)

As in (3.13), we use Lemma 3.4(iii) and Lemma 3.5 which yield

$$\langle \sigma_i \sigma_{j+1} \rangle_{O_J \cup \{S_j^-, S_j^+\}} \leq \langle \sigma_i \sigma_{j+1} \rangle_{O_J \cup \{S_j^-, S_j^+, S_{j+1}^+\}} + \tau_1 \langle \sigma_i \sigma_{j+1} \rangle_{O_J \cup \{S_j^-, S_j^+, S_{j+1}^+, S_{j+1}^+\}}.$$  \hfill (3.15)

Note that we hereby used $S_j^+ = S_{j+1}^-$. We now insert (3.15) into (3.14) and iterate the same arguments. As a result

$$\langle \sigma_i \sigma_{j+1} \rangle_{O_J \cup \{S_j^-, S_j^+\}} \leq \sum_{l=0}^{\infty} \sum_{s=\pm} \tau_l \langle \sigma_i \sigma_{j+b+sl} \rangle.$$  \hfill (3.16)

The statement now follows by combining (3.11), (3.12), (3.13) and (3.16). □

We use the previous lemma to prove the central result of this section.

**Correlation Bound for a One-Dimensional Continuous Long-Range Ising Model**
Proof of Proposition 3.1. For the proof of the statement, we will use the estimate from Lemma 3.7. We need to take the limit $L \to \infty$ and sum over all $i \in \mathbb{Z}$. To show finiteness we will make use of translation invariance of the model. Let us first assume that $w \in \ell^1(\mathbb{N})$ with $w \geq 0$ has compact support and let $K > 0$ be such that

$$w_k = 0, \quad k \geq K. \tag{3.17}$$

As in Lemma 3.7 we shall use the notation $\tau_k = \tanh(w_k)$. We introduce a regularization parameter $\eta > 0$ and define

$$\tau_k,\eta = e^{\eta k} \tau_k \quad \text{and} \quad \langle \sigma_i \sigma_j \rangle^{(\text{L},\eta)}_w = e^{-\eta|i-j|} \langle \sigma_i \sigma_j \rangle^{(\text{L})}_w. \tag{3.18}$$

Further, we define

$$M^+_{j,L}(\eta) = \sum_{i=-L}^{j-1} \langle \sigma_i \sigma_j \rangle^{(\text{L},\eta)}_w, \quad M^-_{j,L}(\eta) = \sum_{i=j+1}^{L} \langle \sigma_i \sigma_j \rangle^{(\text{L},\eta)}_w, \quad \text{and} \quad M_{j,L}(\eta) = \sum_{i=-L}^{j} \langle \sigma_i \sigma_j \rangle^{(\text{L},\eta)}_w = 1 + M^+_{j,L}(\eta) + M^-_{j,L}(\eta).$$

By the regularization (3.18) and Corollary 3.6 the limits

$$M^+_{j}(\eta) = \lim_{L \to \infty} M^+_{j,L}(\eta) = \sum_{i \geq j} \langle \sigma_i \sigma_j \rangle^{(\text{L},\eta)}_w \quad \text{and} \quad M^{-}_{j}(\eta) = \lim_{L \to \infty} M^{-}_{j,L}(\eta) = \sum_{i \in \mathbb{Z}} \langle \sigma_i \sigma_j \rangle^{(\text{L},\eta)}_w$$

exist for any $\eta > 0$. By translation invariance of $J_w$, i.e., $\langle \sigma_i \sigma_j \rangle_{J_{w}} = \{\sigma_{i+k} \sigma_{j+k}\}_{J_{w}}$ for any $k \in \mathbb{Z}$, it follows that $M_{j}(\eta) \quad \text{and} \quad M^+_{j}(\eta)$ are independent of $j$ and we shall write $M(\eta)$ for $M_{j}(\eta)$.

For $L \in \mathbb{N}$, we now multiply the inequalities in Lemma 3.7 with $e^{-\eta|i-j|}$ and use the triangle inequality, to obtain for $i \leq j$

$$\langle \sigma_i \sigma_j \rangle^{(\text{L},\eta)}_w \leq \tau_1 \langle \sigma_i \sigma_{j+1} \rangle^{(\text{L},\eta)}_w + \sum_{l=2}^{\infty} \sum_{s=\pm 1} \tau_{l,\eta} \langle \sigma_i \sigma_{j+s} \rangle^{(\text{L},\eta)}_w$$

$$+ (1 - \tau_1^2) \sum_{b=1}^{\infty} \tau_{1,\eta}^b \sum_{l=2}^{\infty} \sum_{s=\pm 1} \tau_{l,\eta} \langle \sigma_i \sigma_{j+b+s} \rangle^{(\text{L},\eta)}_w.$$

Adding the above expression for the cases $i > j$ and $j > i$, summing over all $i \in \Lambda_L$, using $\sigma_r^2 = 1$ for any $r \in \mathbb{Z}$ as well as Lemma 3.4(i) we find

$$M_{j,L}(\eta) \leq 1 + \tau_1 \left( M^-_{j-1,L}(\eta) + 2 + M^+_{j+1,L}(\eta) \right) + \sum_{l=2}^{K} \tau_{l,\eta} \sum_{s=\pm 1} M_{j+s,L}(\eta)$$

$$+ \sum_{b=1}^{\infty} \tau_{1,\eta}^b (1 - \tau_1^2) \sum_{l=2}^{K} \tau_{l,\eta} \sum_{s=\pm 1} (M_{j+b+s,L}(\eta) + M_{j-b+s,L}(\eta)). \tag{3.19}$$

Now we can take the limit $L \to \infty$. Since $\tau$ has compact support and $\eta > 0$, expressions on the right hand side stay finite. Then, using the translation invariance of $J_w$ we can drop the index $j$, and summing the geometric series $\sum_{b \in \mathbb{N}} \tau_{1,\eta}^b$, we obtain

$$M(\eta) \leq 1 + \tau_1 + M(\eta) \left( \tau_1 + 2 \sum_{l=2}^{K} \tau_{l,\eta} \left( 1 + \frac{1 - \tau_1^2}{1 - \tau_1 \eta} \right) \right). \tag{3.20}$$
Fix $D > 1$, such that $\varepsilon \in (0, (10D)^{-1})$. Since

$$1 < \frac{1 - \tau_1^2}{1 - \tau_1} = 1 + \tau_1 < 2,$$  \hspace{1cm} (3.21)

we can choose $\eta_0 > 0$, such that $\frac{1 - \tau_1^2}{1 - \tau_1} < 2$ and $e^{K\eta_0} < D$. Then, for any $\eta \in (0, \eta_0)$ we obtain

$$\sum_{l=2}^{K} \tau_{l,\eta} \leq D \sum_{l=2}^{K} \tau_l \leq \frac{1 + \tau_1}{1 - \tau_1 - 2 \sum_{l=2}^{K} \tau_{l,\eta} \left(1 + 2 \frac{1 - \tau_1^2}{1 - \tau_1,\eta}\right)} \leq \frac{2}{1 - \tau_1 - 10D \sum_{l=2}^{\infty} \tau_l}. \hspace{1cm} (3.22)$$

and

$$\tau_1 + 2 \sum_{l=2}^{K} \tau_{l,\eta} \left(1 + 2 \frac{1 - \tau_1^2}{1 - \tau_1,\eta}\right) < \tau_1 + 10D \sum_{l=2}^{\infty} \tau_l. \hspace{1cm} (3.23)$$

If

$$\sum_{l=2}^{\infty} \tau_l \leq \varepsilon (1 - \tau_1), \hspace{1cm} (3.24)$$

the right hand side of (3.23) is smaller than 1, and we can bring $M(\eta)$ in (3.20) to the left hand side. Thus using (3.21) and (3.23), we find

$$M(\eta) \leq \frac{1 + \tau_1}{1 - \tau_1 - 2 \sum_{l=2}^{K} \tau_{l,\eta} \left(1 + 2 \frac{1 - \tau_1^2}{1 - \tau_1,\eta}\right)} \leq \frac{2}{1 - \tau_1 - 10D \sum_{l=2}^{\infty} \tau_l} \leq \frac{1}{1 - \tau_1}. \hspace{1cm} (3.25)$$

By monotone convergence, the limit $\eta \downarrow 0$ exists and

$$\sum_{i \in \mathbb{Z}} \langle \sigma_i \sigma_j \rangle_{J_w} = \lim_{\eta \downarrow 0} M(\eta) \leq \frac{2}{1 - 10D \varepsilon} \frac{1}{1 - \tau_1}. \hspace{1cm} (3.25)$$

Thus, we have proven (3.25) for all nonnegative $w \in \ell^1(\mathbb{N})$ satisfying (3.17) and (3.24).

Finally, let us consider general $w \in \ell^1(\mathbb{N})$ with $w \geq 0$ satisfying only (3.24). If $i, j \in \Lambda_L$, then as an immediate consequence of the definition (3.4)

$$\langle \sigma_i \sigma_j \rangle_{J_w}^{(L)} = \langle \sigma_i \sigma_j \rangle_{J_w[0,2L+1]}^{(L)}. \hspace{1cm} (3.25)$$

Since $w_{1[0,2L+1]}$ trivially satisfies (3.24) because $w$ does, we find from (3.25) and monotonicity (Lemma 3.4(iii)) the estimate for all $N \in \mathbb{N}$

$$\sum_{i=-N}^{N} \langle \sigma_i \sigma_j \rangle_{J_w}^{(L)} \leq \frac{2}{1 - \tau_1} \frac{1}{1 - \tau_1}. \hspace{1cm} (3.25)$$

Thus the bound (3.3) of the proposition now follows by taking in the above inequality first the limit $L \to \infty$ and then $N \to \infty$. \hfill \Box
4 THE CONTINUUM LIMIT OF THE ISING MODEL

In this section we prove that the jump process $X$ defined in (1.1) is the continuum limit of a one-dimensional Ising model defined as in the previous section. The approach we use is based on the description in [SD85, Spo89]. To that end, we use a parameter $\delta \in (0, \infty)$ as lattice spacing of the discrete Ising model and define the map

$$i_\delta : \mathbb{R} \rightarrow \mathbb{N}, \quad t \mapsto \left\lfloor \frac{t}{\delta} + \frac{1}{2} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ as usually denotes the integer part. Note, the interval $[-T, T]$ is mapped to the lattice $\Lambda_{L_\delta(T)}$ with $L_\delta(T) = i_\delta(T)$. We set the nearest neighbor interaction on this lattice to be

$$j_\delta = -\frac{1}{2} \ln(\delta).$$

For a function $W : \mathbb{R} \rightarrow \mathbb{R}$, we define the corresponding pair interaction (cf. (3.6)) on the lattice as

$$w^{(\delta)}_k = 2 W(\delta k).$$

We define the expectation values in the Ising model given with these interactions as

$$\langle \cdot \rangle^{(n)}_{\delta, T} := \langle \cdot \rangle^{(L_\delta(T))}_{j_\delta(0), \ldots} \quad \text{and} \quad \langle \cdot \rangle_{\delta, T} := \langle \cdot \rangle^{(L_\delta(T))}_{j_\delta(0), \ldots} + w^{(\delta)}.$$

In this section we prove the following proposition.

**Proposition 4.1.** Assume $W : \mathbb{R} \rightarrow \mathbb{R}$ is even and continuous, $T > 0$ and $-T \leq t_1 \leq \cdots \leq t_N \leq T$. Then

$$\lim_{\delta \downarrow 0} \langle \sigma_{i_\delta(t_1)} \cdots \sigma_{i_\delta(t_N)} \rangle_{\delta, T} = \mathbb{E}[X(t_1) \cdots X(t_N)] \exp \left( \int_{-T}^{T} \int_{-T}^{T} W(t-s)X(t)X(s)dt ds \right).$$

As a first step of our proof, we recall the following lemma.

**Lemma 4.2.** Let $t_1 \leq \cdots \leq t_N$ be an increasing sequence of times. Then, we have

$$\mathbb{E}[X(t_1) \cdots X(t_N)] = e^{-2||t_2-t_1|+\cdots+|t_N-t_{N-1}|)} \quad \text{if } N \text{ is even}$$

and $\mathbb{E}[X(t_1) \cdots X(t_N)] = 0$ if $N$ is odd.

**Proof.** For a simple proof see for example [Abd11, Lemma 1].

It is well-known, that the expectation values of Ising models only with nearest neighbor coupling can be calculated explicitly, see also Appendix A. In the limit $\delta \rightarrow 0$, we use this to obtain the jump process $X$.

**Lemma 4.3.** Let $-T \leq t_1 \leq \cdots \leq t_N \leq T$ be an increasing sequence of times. Then

$$\lim_{\delta \downarrow 0} \langle \sigma_{i_\delta(t_1)} \cdots \sigma_{i_\delta(t_N)} \rangle^{(n)}_{\delta, T} = \mathbb{E}[X(t_1) \cdots X(t_N)].$$
Proof. If \( N \) is odd both sides vanish (Lemmas 4.3 and A.1), so we assume \( N \) is even. Then, the definition (4.1) and Lemma A.1(iii) yield

\[
\{ \sigma_{i_1}(t_1) \cdots \sigma_{i_N}(t_N) \}_{\delta,T} = (\tanh j_\delta)^{\| i_2 - i_1 \|_{\delta,T} + \cdots + \| i_N - i_{N-1} \|_{\delta,T} + \| i_1 - i_N \|_{\delta,T} - |i_1 - i_N|_{\delta,T}}.
\]

Since (4.1) also yields \( |u-v|_\delta - 1 \leq |i_\delta(u) - i_\delta(v)| \leq |u-v|_\delta + 1 \) for all \( u, v \in \mathbb{R} \), we obtain

\[
\left( \tanh j_\delta \right)^{\delta^{-1}} |t_2 - t_1| + \cdots + |t_N - t_{N-1}| \leq \| \sigma_{i_1}(t_1) \cdots \sigma_{i_N}(t_N) \|_{\delta,T} \leq \left( \tanh j_\delta \right)^{\delta^{-1}} |t_2 - t_1| + \cdots + |t_N - t_{N-1}| + \delta N.
\]

Using \( \lim_{\delta \to 0} \tanh j_\delta \delta^{-1} = e^{-2} \), the statement follows by Lemma 4.2.

\( \square \)

Lemma 4.3 shows Proposition 4.1 in the case \( W = 0 \). To show the proposition for nonzero \( W \), we will use the notion of weak convergence of measures, as outlined in [SD85, Spo89]. To this end, we introduce the following definitions and recall elementary properties, which can be found in [Bil99, Chapter 3]. We define \( D_T \) to be the set of all right-continuous functions \( \omega : [-T, T] \to \{ \pm 1 \} \) with finitely many jumps. We equip \( D_T \) with the so-called Skorokhod topology. That is, if \( \Phi_T \) denotes the set of all continuous strictly increasing bijections \( \varphi : [-T, T] \to [-T, T] \), we define the metric

\[
d(\omega, \nu) = \inf_{\varphi \in \Phi_T} (\| \varphi - 1 \|_\infty + \| \omega - \nu \circ \varphi \|_\infty)
\]

for \( \omega, \nu \in D_T \).

The topology induced on \( D_T \) by \( d \) is the Skorokhod topology. We equip \( D_T \) with the Borel \( \sigma \)-algebra. There exists a probability measure \( P_X \) on \( D_T \), such that for \( \omega \in D_T \) the jump process is given by \( X(t)(\omega) = \omega(t) \) for \( t \in [-T, T] \) and for any measurable function \( f : D_T \to \mathbb{R} \)

\[
E[f(X_{[-T,T]})] = \int f(\omega) dP_X(\omega).
\]

In the following two lemmas, we will assume this realization of the jump process \( X \). We define \( s_\delta : S_{L_\delta(T)} \to D_T \) by \( s_\delta(\sigma) = [t \mapsto \sigma_{i_\delta(t)}] \).

**Lemma 4.4.** Let \( f : D_T \to \mathbb{R} \) be bounded and continuous, \( N \in \mathbb{N}_0 \) and \( -T \leq t_1 \leq \cdots \leq t_N \leq T \). Then

\[
\lim_{\delta \downarrow 0} \{ \sigma_{i_1}(t_1) \cdots \sigma_{i_N}(t_N) f(s_\delta(\sigma)) \}_{\delta,T}^{(n)} = \mathbb{E}[X(t_1) \cdots X(t_N) f(X)].
\]

**Remark 4.5.** In fact, we prove the stronger statement \( \{ f(s_\delta(\sigma)) \}_{\delta,T}^{(n)} \xrightarrow{\delta \downarrow 0} \mathbb{E}[f(X)] \) for any bounded measurable function \( f : D_T \to \mathbb{R} \) for which the set of discontinuities \( U_f \) satisfies \( P_X(U_f) = 0 \).

**Remark 4.6.** The proof of Lemma 4.4 is based on weak convergence of measures. To obtain weak convergence, we will show tightness of the associated probability measures by a combinatorial estimate. We note that tightness can in fact been shown by reflection positivity [Spo89].

**Proof.** We prove that the measures on \( D_T \) associated to the nearest neighbor Ising model weakly converge to the measure given by the jump process \( X \). Then, the statement follows by the Portmanteau theorem [Kle20, Theorem 3.16]. To prove weak convergence, we need to combine the convergence of moments from Lemma 4.3 and the tightness of the Ising measures, cf. [Bil99, Theorem 13.1].

For \( \delta > 0 \), let \( P_\delta \) be the pushforward measure on \( D_T \) obtained from the Ising probability measure on \( S_{L_\delta(T)} \) through the (obviously measurable) map \( s_\delta \), i.e.,

\[
P_\delta(A) = \sum_{\sigma \in s_\delta^{-1}(A)} \frac{e^{-E_{J_{s_\delta L_\delta(T)}}(\sigma)}}{Z_{J_{s_\delta L_\delta(T)}}} \quad \text{for all measurable sets } A \subset D_T.
\]
Hence,
\[ \langle \sigma_{i(t_1)} \cdots \sigma_{i(t_N)} f(\sigma) \rangle^{(n)}_{S,T} = \int \omega(t_1) \cdots \omega(t_N) f(\omega) dP_\delta(\omega). \]

Now, for any \( k \in \mathbb{N} \) and \( t = (t_1, \ldots, t_k) \in [-T, T]^k \), we define the projections \( \pi_t : D_T \to \{ \pm 1 \}^k \), \( \omega \mapsto (\omega(t_1), \ldots, \omega(t_k)) \). Observe that the expectation values in Lemma 4.3 uniquely determine the probability measures \( P_\delta \circ \pi_t^{-1} \) and \( P_X \circ \pi_t^{-1} \), respectively, since every function on the set \( \{ -1, 1 \} \) is given as a linear combination of the constant function one and the identity function. Hence, Lemma 4.3 implies the weak convergence of \( P_\delta \circ \pi_t^{-1} \) to \( P_X \circ \pi_t^{-1} \). To deduce weak convergence of \( P_\delta \) to \( P_X \) as \( \delta \downarrow 0 \), we need to prove that the family \( (P_\delta) \) is tight (cf. [Bil99, Theorem 13.1]). Let us reformulate this statement similar to [Bil99, Theorem 13.2]. For \( \varepsilon > 0 \), we denote by \( \Omega_\varepsilon \) the set of all \( \omega \in D_T \), having two discontinuities with a distance less than \( \varepsilon \), i.e.,
\[ \Omega_\varepsilon = \left\{ \omega \in D_T : \exists t_1, t_2 \in (-T, T) : |t_2 - t_1| < \varepsilon, \lim_{t \uparrow t_1} \omega(t) \neq \omega(t_1), \lim_{t \uparrow t_2} \omega(t) \neq \omega(t_2) \right\}. \]

The family \( (P_\delta) \) is tight if and only if
\[ \lim_{\varepsilon \downarrow 0} \limsup_{\delta \downarrow 0} P_\delta(\Omega_\varepsilon) = 0. \]

For now fix \( \varepsilon > 0 \) and \( \delta \in (0, \varepsilon) \). For \( \sigma \in S_{L_\delta(T)} \), we denote by \( n_\sigma \) the number of sign changes (cf. Lemma A.1[(i)]). We observe that \( \sigma_\delta(\sigma) \in \Omega_\varepsilon \) if \( n_\sigma > 2T/\varepsilon \). Otherwise, \( \sigma_\delta(\sigma) \notin \Omega_\varepsilon \) if and only if all sign changes have a distance of at least \( \varepsilon/\delta \). If \( n_\sigma = k \) for some fixed \( k \in \mathbb{N} \), then simple combinatorics yield that there are \( \binom{2L_\delta(T) - (k - 1)[\varepsilon/\delta]}{k} \) possibilities to position the sign changes, such that all distances are larger than \( \varepsilon/\delta \). Taking into account that an element \( \sigma \in S_{L_\delta(T)} \) is uniquely determined by the choice of the value \( \sigma_{L_\delta(T)} \in \{ \pm 1 \} \) and the position of its sign changes, we obtain
\[ \# \{ \sigma \in \sigma_\delta^{-1}(\Omega_\varepsilon) : n_\sigma = k \} = \begin{cases} 2 \frac{(2L_\delta(T))}{k} & \text{for } k > \frac{2T}{\varepsilon}, \\ 2 \frac{(2L_\delta(T))}{k} - 2 \frac{(2L_\delta(T) - (k - 1)[\varepsilon/\delta])}{k} & \text{for } 2 \leq k \leq \frac{2T}{\varepsilon}. \end{cases} \]

From the definition of nearest neighbor coupling, it easily follows that (cf. Lemma A.1[(i)])
\[ E_{J_{\delta},L_\delta(T)}(\sigma) = 2J_{\delta}(n_\sigma - L_\delta(T)) \quad \text{for all } \sigma \in S_{L_\delta(T)}. \]

Hence, combining (1.6) and (4.8) and summing over all possible numbers of spin changes, we obtain
\[ P_\delta(\Omega_\varepsilon) = \sum_{k=2}^{\lceil \frac{2T}{\varepsilon} \rceil} \left( \frac{(2L_\delta(T))}{k} - \frac{(2L_\delta(T) - (k - 1)[\varepsilon/\delta])}{k} \right) \frac{2e^{2j_\delta(L_\delta(T) - k)}}{Z_{J_{\delta},L_\delta(T)}} + \sum_{k=\lceil \frac{2T}{\varepsilon} \rceil + 1}^{2L_\delta(T)} \frac{(2L_\delta(T))}{k} \frac{2e^{2j_\delta(L_\delta(T) - k)}}{Z_{J_{\delta},L_\delta(T)}}. \]

Explicitly, the combinatorial argument is as follows: In a chain of \( N + 1 \) Ising spins, there are \( \binom{N}{k} \) possibilities to position \( k \) sign changes. This is equal to the number of possibilities to choose \( k + 1 \) positive integers \( x_1, \ldots, x_{k+1} \), such that \( x_1 + \cdots + x_{k+1} = N + 1 \). Now, if the distance between any two sign changes shall be larger than \( m \), this is equivalent to requiring \( x_2, \ldots, x_k > m \). By the change of variables \( y_1 = x_1, y_i = x_i - m \) for \( i = 2, \ldots, k \), \( y_{k+1} = x_{k+1} \), we find the number of possibilities to be equal to the number of possibilities to choose \( x_1, \ldots, x_{k+1} \), such that \( y_1 + \cdots + y_{k+1} = N + 1 - (k - 1)m \). Recalling the initial argument, this is \( \binom{N-(k-1)m}{k} \). In our case we have \( N = 2L_\delta(T) \) and \( m = \lceil \varepsilon/\delta \rceil \).
Since it is possible to explicitly calculate the partition function for nearest neighbor coupling (cf. Lemma A.1(ii)), we have

\[
2e^{2j_0L_\delta(T)} = \left( \frac{e^{j_0}}{e^{j_0} + e^{-j_0}} \right)^{2L_\delta(T)} < 1. \quad (4.10)
\]

Moreover, inserting the definition (4.12), we have \(e^{-2j_0k} = \delta^k\) and hence

\[
\left( \frac{2L_\delta(T)}{k} \right) e^{-2j_0k} \leq \frac{(2L_\delta(T))^k}{k!} \delta^k \leq \frac{(2T + \delta)^k}{k!} \delta^k \quad \text{for all } k \leq 2L_\delta(T), \quad (4.11)
\]

where we used \(L_\delta(T) = \lfloor \frac{T}{\delta} \rfloor + \frac{1}{2} \leq \frac{T}{\delta} + \frac{1}{2}\) in the last step. Along the same lines, we use Bernoulli’s inequality to obtain for \(k \leq T/\varepsilon\)

\[
\left( \frac{2L_\delta(T)}{k} - \left(2L_\delta(T) - (k - 1)[\varepsilon/\delta] \right) \right) e^{-2j_0k} \leq \frac{(2L_\delta(T))^k - (2L_\delta(T) - k(\varepsilon/\delta + 1))^k}{k!} \delta^k 
\leq \frac{(2L_\delta(T))^k k^2(\varepsilon/\delta + 1)}{2L_\delta(T)} \delta^k 
\leq \frac{(2T + \delta)^{k-1}}{(k - 1)!} k(\varepsilon + \delta). \quad (4.12)
\]

We can now insert (4.10), (4.11) and (4.12) into (4.9). Hence, for any \(s_\varepsilon \in [0, \frac{1}{2}]\), we have

\[
P_\delta(\Omega_\varepsilon) \leq \sum_{k=2}^{\lfloor T_{s_\varepsilon} \rfloor} \frac{(2T + \delta)^{k-1}}{(k - 1)!} k(\varepsilon + \delta) + \sum_{k=\lfloor T_{s_\varepsilon} \rfloor + 1}^{2L_\delta(T)} \frac{(2T + \delta)^k}{k!} \leq T s_\varepsilon(\varepsilon + \delta) e^{2T + \delta} + \sum_{k=\lfloor T_{s_\varepsilon} \rfloor + 1}^{\infty} \frac{(2T + \delta)^k}{k!},
\]

where we estimated the first half of the first sum in (4.9) by (4.12) and the second half using (4.11). Taking the limit \(\delta \downarrow 0\), we observe

\[
\limsup_{\delta \downarrow 0} P_\delta(\Omega_\varepsilon) \leq T s_\varepsilon \varepsilon e^{2T} + \sum_{k=\lfloor T_{s_\varepsilon} \rfloor + 1}^{\infty} \frac{(2T)^k}{k!}.
\]

We choose \(s_\varepsilon\) such that both \(\lim_{\varepsilon \downarrow 0} s_\varepsilon = \infty\) and \(\lim_{\varepsilon \downarrow 0} s_\varepsilon \varepsilon = 0\) hold, e.g., \(s_\varepsilon = \varepsilon^{-1/2}\). Then, the summability of the second term proves (4.7) and hence \(P_\delta\) weakly converges to \(P_X\).

Since \(f\) is bounded and continuous, the statement for \(N = 0\) directly follows from the definition of weak convergence. Further, observe that for any fixed \(N \in \mathbb{N}\) and \(t \in \mathbb{R}^N\) the function \(\omega \rightarrow \pi_t(\omega) f(\omega)\) is only discontinuous at those \(\omega\) having jumps exactly at the points given by the \(N\)-tuple \(t\). Hence, the set of discontinuities has \(P_X\)-measure zero and the statement follows from the Portmanteau theorem [Kle20, Theorem 3.16].

We apply above lemma to prove the expectation value in Proposition 4.1 is a limit of expectation values in the nearest neighbor Ising model.

**Lemma 4.7.** Assume \(W : [-T, T] \rightarrow \mathbb{R}\) is even and continuous and \(w^{(\delta)}\) is as defined in (4.3).

For \(N \in \mathbb{N}_0\), let \(-T \leq t_1 \leq \cdots \leq t_N \leq T\). Then

\[
\lim_{\delta \downarrow 0} \left\langle \sigma_{i_1(t_1)} \cdots \sigma_{i_N(t_N)} \exp \left( \sum_{i,j \in \lambda L_\delta(T)} w^{(\delta)}_{i-j} \sigma_i \sigma_j \right) \right\rangle^{(n)}_{\delta, T} = E \left[ X(t_1) \cdots X(t_N) \exp \left( \int_{-T}^T \int_{-T}^T W(t-s)X(s)X(t)dsdt \right) \right].
\]

**Correlation Bound for a One-Dimensional Continuous Long-Range Ising Model**
Proof. We define $f_0 : \mathcal{D}_T \rightarrow \mathbb{R}$ by

$$f_0(\omega) = \omega(t_1) \cdots \omega(t_N)e^{\mathcal{I}_0(\omega)},$$

where $\mathcal{I}_0(\omega) = \int_{-T}^T \int_{-T}^T W(t - s)\omega(t)\omega(s)dsdt$.

It is straightforward to verify that $\mathcal{I}_0 : \mathcal{D}_T \rightarrow \mathbb{R}$ is bounded and continuous. Hence, we can apply Lemma 4.14 and obtain

$$\lim_{\delta \downarrow 0} \langle f_0(\mathcal{g}_\delta(\sigma)) \rangle_{\delta,T}^{(n)} = \mathbb{E} \left[ f_0(X) \right]. \quad (4.13)$$

It remains to consider the left hand side and to analyze $f_0(\mathcal{g}_\delta(\sigma))$. Further, for $\sigma \in \mathcal{S}_{L\delta(T)}$, we define

$$g_\delta(\sigma) = \sigma_{i_1(t_1)} \cdots \sigma_{i_N(t_N)}e^{\mathcal{J}_\delta(\sigma)}, \quad \text{where} \quad \mathcal{J}_\delta(\sigma) = \sum_{i,j \in \Lambda_{L\delta(T)}} w^{(\delta)}_{|i-j|}\sigma_i\sigma_j.$$

Since continuous functions on compact intervals are uniformly continuous, for any $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$, such that

$$|W(t) - W(s)| < \varepsilon \quad \text{for any} \quad t, s \in [-T, T] \text{ with } |t - s| < \delta_\varepsilon.$$

Then, for any $\delta \in (0, \delta_\varepsilon)$ and $\sigma \in \mathcal{S}_{L\delta(T)}$, we use (4.1) and (4.3) to obtain

$$|\mathcal{J}_\delta(\sigma) - \mathcal{I}_0(\mathcal{g}_\delta(\sigma))| = \left| \sum_{i,j \in \Lambda_{L\delta(T)}} \left[ w^{(\delta)}_{|i-j|}\sigma_i\sigma_j - \int_{(i - \frac{1}{2})\delta}^{(i + \frac{1}{2})\delta} - \int_{(j - \frac{1}{2})\delta}^{(j + \frac{1}{2})\delta} W(t - s)\sigma_i\sigma_jdsdt \right] \right| \leq \sum_{i,j \in \Lambda_{L\delta(T)}} \delta^2 \sup \{ |W(\delta t) - W(\delta |i - j|)| : t \in [|i - j| - 1, |i - j| + 1] \} \leq (2L\delta(T) + 1)^2\delta^2\varepsilon \leq 4(T + \delta)^2\varepsilon. \quad (4.14)$$

Now, for all $\sigma \in \mathcal{S}_{L\delta(T)}$ we have the algebraic identity

$$g_\delta(\sigma) - f_0(\mathcal{g}_\delta(\sigma)) = f_0(\mathcal{g}_\delta(\sigma)) \left( e^{\mathcal{J}_\delta(\sigma) - \mathcal{I}_0(\mathcal{g}_\delta(\sigma))} - 1 \right).$$

Using this identity and (4.14) it follows that there exist constants $C_1$ and $C_2$, such that for $\varepsilon > 0$ sufficiently small, $\delta \in (0, \delta_\varepsilon)$ and all $\sigma \in \mathcal{S}_{L\delta(T)}$

$$|g_\delta(\sigma) - f_0(\mathcal{g}_\delta(\sigma))| \leq |f_0(\mathcal{g}_\delta(\sigma))|C_1|\mathcal{J}_\delta(\sigma) - \mathcal{I}_0(\mathcal{g}_\delta(\sigma))| \leq C_2e^{4T^2\|W\|_\infty(T + \delta)^2\varepsilon}.$$

Since $\sigma \in \mathcal{S}_{L\delta(T)}$ was arbitrary, this estimate also holds for the expectation value, i.e.,

$$\left| \langle g_\delta(\sigma) \rangle_{\delta,T}^{(n)} - \langle f_0(\mathcal{g}_\delta(\sigma)) \rangle_{\delta,T}^{(n)} \right| \leq C_2e^{4T^2\|W\|_\infty(T + \delta)^2\varepsilon}. \quad (4.15)$$

Combining (4.13) and (4.15), the statement follows.

It now remains to rewrite the Ising expectation value in above lemma as a correlation function.

**Proof of Proposition 4.1.** By the definition (3.4), we observe

$$\langle \sigma_{i_1(t_1)} \cdots \sigma_{i_N(t_N)} \rangle_{\delta,T} = \left\langle \exp \left( \sum_{i,j \in \Lambda_{L\delta(T)}} w^{(\delta)}_{|i-j|}\sigma_i\sigma_j \right) \right\rangle_{\delta,T}^{(n)}.$$

Hence, the statement follows from Lemma 4.7 and 122.\qed
5 Proof of the Main Result

In this section we combine the central statements from the previous sections to the proof of our main result Theorem 1.2. We use the definitions from the previous section.

**Proof of Theorem 1.2.** Fix $T > 0$. Then using Fubini in the first equality and Proposition 4.1 in the second equality, we find

$$
\frac{1}{Z(W, T)} \mathbb{E} \left[ \frac{1}{T} \left( \int_{-T}^{T} X(t) dt \right)^2 \exp \left( \int_{-T}^{T} \int_{-T}^{T} X(t) X(s) W(t-s) dt ds \right) \right]
$$

$$
= \frac{1}{TZ(W, T)} \int_{-T}^{T} \int_{-T}^{T} \mathbb{E} \left[ X(u) X(v) \exp \left( \int_{-T}^{T} \int_{-T}^{T} X(t) X(s) W(t-s) dt ds \right) \right] dudv
$$

$$
= \frac{1}{T} \lim_{\delta \downarrow 0} \int_{-T}^{T} \int_{-T}^{T} \langle \sigma_{i \delta(u)} \sigma_{i \delta(v)} \rangle_{\delta, T} dudv
$$

$$
= \lim_{\delta \downarrow 0} \frac{1}{T} \sum_{i,j \in \Lambda_{L_{\delta}(T)}} \delta^2 \langle \sigma_i \sigma_j \rangle_{\delta, T}, \tag{5.1}
$$

where in the last step, we calculated the integral using that the integrand is a step function. To estimate (5.1) we want to use Proposition 3.1. First observe that by definition (4.2) we find

$$
\frac{1}{1 - \tanh \frac{y}{\delta}} = \frac{e^{2y \delta} + 1}{2} < \frac{1}{\delta} \quad \text{for any } \delta \in (0, 1). \tag{5.2}
$$

Further, using the definition (4.3), $W \in L^1(\mathbb{R})$ and the continuity of $W$, we have

$$
w^{(\delta)} \in ℓ^1 \quad \text{and} \quad \lim_{\delta \downarrow 0} \frac{1}{\delta} \|w^{(\delta)}\|_1 = \|W\|_1.
$$

Combining the above two relations, we find for any constant $D > 1$

$$
\frac{\|w^{(\delta)}\|_1}{1 - \tanh \frac{y}{\delta}} \leq D\|W\|_1 \quad \text{if } \delta > 0 \text{ is sufficiently small.} \tag{5.3}
$$

Now let $\varepsilon > 0$ and $C_\varepsilon$ be as in Proposition 3.1. If $\|W\|_1 \leq \varepsilon/D$ and $\delta > 0$ is sufficiently small, it follows from (5.3) that the assumption (3.7) of the proposition holds, since $\tanh x \leq x$ for $x \in [0, \infty)$. In that case it hence follows from Proposition 3.1 that

$$
\sum_{i \in \mathbb{Z}} \langle \sigma_i \sigma_j \rangle_{\delta, T} \leq \frac{C}{1 - \tanh \frac{y}{\delta}} \leq \frac{C}{\delta},
$$

where we used (5.2) in the second inequality. The last displayed inequality implies

$$
\sum_{i,j \in \Lambda_{L_{\delta}(T)}} \langle \sigma_i \sigma_j \rangle_{\delta, T} \leq \frac{C}{\delta} L_{\delta}(T).
$$

Inserting this into (5.1) and using $L_{\delta}(T) \leq \frac{T}{\delta} + \frac{1}{2}$ finishes the proof.
A Ising Model with Nearest Neighbor Coupling

In this appendix we consider the Ising model with nearest neighbor coupling and calculate the known partition function and correlation functions. These calculations are well-known and can be found in most textbooks covering the Ising model. Here we use slightly different notation than in the rest of the paper.

We fix the lattice length $L \in \mathbb{N}$, the lattice $\Lambda = \{-L, \ldots, L\} \subset \mathbb{Z}$, the spin configuration space $S = \{\pm 1\}^\Lambda$, and the interaction strength $j \geq 0$. The Ising energy is defined by

$$E(\sigma) = -\sum_{i=-L}^{L-1} j \sigma_i \sigma_{i+1}$$

for $\sigma \in S$.

the partition function by

$$Z = \sum_{\sigma \in S} e^{-E(\sigma)},$$

and the expectation value of $f : S \to \mathbb{R}$ by

$$\langle f \rangle = \frac{1}{Z} \sum_{\sigma \in S} f(\sigma) e^{-E(\sigma)}.$$

We prove the following statements.

Lemma A.1.

(i) For $\sigma \in S$ we write $n_\sigma = \# \{i = -L, \ldots, L - 1 : \sigma_i \sigma_{i+1} = -1\}$. Then $E(\sigma) = 2j(n_\sigma - L)$.

(ii) We have $Z = 2(e^j + e^{-j})^{2L}$.

(iii) For $n \in \mathbb{N}$ and $-L \leq i_1 \leq \cdots \leq i_n \leq L$ we have

$$\langle \sigma_{i_1} \cdots \sigma_{i_n} \rangle = \begin{cases} \tanh(j) \sum_{k=1}^{n} |i_{2k-1} - i_{2k-2}| & \text{if } n = 2N, \\ 0 & \text{else.} \end{cases}$$

Proof. (i) follows directly from the definition. For the proof of (ii) and (iii), we use the change of variables

$$\sigma'_i = \sigma_i \sigma_{i+1} \quad \text{for } i = -L, \ldots, L - 1, \quad \sigma'_L = \sigma_L.$$

Then, we have $E(\sigma) = -j \sum_{i=-L}^{L-1} \sigma'_i$ and hence

$$Z = \sum_{\sigma' \in S} \prod_{i=-L}^{L-1} e^{j\sigma'_i} = 2 \prod_{i=-L}^{L-1} \sum_{\sigma'_i = \pm 1} e^{j\sigma'_i} = 2(e^j + e^{-j})^{2L},$$

so (ii) is proved. Now, if $-L \leq i < j \leq L$, we observe

$$\sigma_i \sigma_j = (\sigma_i \sigma_{i+1}) (\sigma_{i+1} \sigma_{i+2}) \cdots (\sigma_{j-1} \sigma_j) = \sigma'_i \sigma'_{i+1} \cdots \sigma'_{j-1}.$$
Assume $n = 2N$. Then, we have

\[
Z \langle \sigma_{i_1} \cdots \sigma_{i_{2N}} \rangle = \sum_{\sigma \in S} \sigma_{i_1} \cdots \sigma_{i_{2N}} e^{-E(\sigma)} = \sum_{\sigma \in S} e^{-E(\sigma)} \prod_{a=1}^{N} (\sigma_{i_{2a-1}} \sigma_{i_{2a-1}+1}) \cdots (\sigma_{i_{2a-1}} \sigma_{i_{2a}}) = \sum_{\sigma' \in S} \prod_{a=1}^{N} \sigma'_{i_{2a-1}} \cdots \sigma'_{i_{2a-1}+1} \prod_{i=-L}^{L-1} e^{j\sigma'_i}.
\]

Now, for $l \in \Lambda$, we set

\[
s_l = \begin{cases} 
1 & \text{if there is some } a \in \mathbb{N} \text{ with } i_{2a-1} \leq l < i_{2a}, \\
0 & \text{else}.
\end{cases}
\]

and $S = \# \{ l \in \Lambda : s_l = 1 \} = |i_2 - i_1| + \cdots + |i_{2N} - i_{2N-1}|$. Inserting above, we obtain

\[
Z \langle \sigma_{i_1} \cdots \sigma_{i_{2N}} \rangle = 2 \prod_{l=-L}^{L-1} \sum_{\sigma'_l = \pm 1} (\sigma'_l)^{s_l} e^{j\sigma'_l} = 2(e^j - e^{-j})^S (e^j + e^{-j})^{2L-S}.
\]

Combined with [ii], we obtain the identity [iii] for even $n$. It remains to consider the case that $n$ is odd. The statement then, however, follows similar to the proof of Lemma 3.5 by the change of variables $\sigma \mapsto -\sigma$. 

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Correlation Bound for a One-Dimensional Continuous Long-Range Ising Model

D. Hasler, B. Hinrichs, O. Siebert

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