QUANTITATIVE STATISTICAL PROPERTIES OF TWO-DIMENSIONAL PARTIALLY HYPERBOLIC SYSTEMS

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Abstract. We study a class of two dimensional partially hyperbolic systems, not necessarily skew products, trying to establish the germ of a general theory. To illustrate the scope of the theory, we apply our results to the case of fast-slow partially hyperbolic systems.

1. Introduction

One of the main challenges of the field of Dynamical Systems is to understand the ergodic properties of partially hyperbolic systems. Substantial progresses have been made in the study of ergodicity starting with [31, 45, 51] till establishing very general results, e.g. [12], in the case of volume preserving diffeomorphisms. Nevertheless, if the invariant measure is not a priori known, then establishing the existence of SRB measures is a serious challenge by itself, see [11, 1, 48] for some important partial results. Moreover, it is well known, at least since the work of Krylov [40], that for many applications ergodicity does not suffice and mixing (usually in the form of effective quantitative estimates on the decay of correlations) is of paramount importance. Some results on correlation decay exist in the case of mostly expanding central direction [2], and mostly contracting central direction [22, 15]. Such results, albeit important, are often not easy to apply since it is very difficult to estimate the central Lyapunov exponent.

For a central direction with zero Lyapunov exponents (or close to zero) there exist quantitative results on exponential decay of correlations only for group extensions of Anosov maps and Anosov flows [23, 16, 21, 41, 49], but none of them apply to an open class (with the notable exception of [14, 50]; also some form of rapid mixing is known to be typical for large classes of flows [28, 43]). Hence, the problem of effectively studying the quantitative mixing properties of partially hyperbolic systems is wide open.

Recently, motivated by deep physical reasons [24, 8, 42], the second author has proposed the study of a simple class of partially hyperbolic systems with the goal of developing a
theory applicable to a large class of fast-slow systems. Some encouraging results have been obtained [17, 18, 20]. However, the amount of work needed to prove the above partial results has proven rather daunting and to extend such an approach to more realistic systems seems extremely challenging. To attain substantial progresses it seems necessary to introduce new ideas supplementing the approaches developed so far.

In the last years, starting with [10, 30, 6], an extremely powerful method to investigate the statistical properties of hyperbolic systems has been developed: the functional approach. It consists in the study in the spectral properties of the transfer operator on appropriate Banach spaces. Although the basic idea can be traced back, at least, to Von Neumann ergodic theorem, the new ingredient consists in the understanding that non standard functional spaces must be used and in the insight of how to embed the key geometrical properties of the system in the topology of the Banach space. See [4] for a recent review of this approach.

This point of view has produced many important results, e.g. see [41, 38, 39, 29, 27, 25, 7] just to cite a few. It is then natural to investigate if the functional approach can be extended to partially hyperbolic systems. Some result that hint at this possibility already exist (e.g. [3, 26]), however, a general approach is totally missing. Nonetheless, the idea that some quantitative form of accessibility should play a fundamental role has slowly emerged, e.g. see [46, 44, 13].

In this paper we attempt to further the latter point of view combining ideas from [3] and [30]. We find checkable conditions that imply the existence of finitely many physical measures for a large class of two dimensional endomorphisms, see Theorem 2.7; we also show that such conditions are fulfilled for an open set of physically relevant systems, see Theorem 2.10. Moreover, for such systems, we are able to obtain some quantitative information on the regularity of the eigenvectors of the transfer operator (Theorem 2.11), which hopefully should allow further progress. In addition, we show how the results obtained here can be combined with averaging results, e.g. [20], to provide a very detailed description of the physical measures, see Theorem 2.13. We believe that this approach can be further refined and extended to produce results in a much more general class of systems.

The attempt to obtain precise quantitative information is responsible for much of the length of the paper, as it entails a strenuous effort to keep track of many constants. Indeed, it is customary to think that the constants appearing in Lasota-Yorke type inequalities are largely irrelevant. This is certainly not the case in the context discussed in section 8, as the possibility to consider the class of maps discussed there as a perturbation of a limiting case depends crucially on the size of such constants. It was then essential to try to push the estimates to their extreme in order to find out if perturbative ideas could be applied. It turns out that our estimates are not sharp enough to do so. However, we have identified precisely the obstructions to this approach, hence clarifying the focus of future research.

The plan of the paper is as follows: in the next section we describe the systems we consider and we state our results. In section 3 we introduce the necessary notation and prove several facts needed to define the Banach spaces we are interested in. In section 4 we prove a first Lasota-Yorke inequality. Unfortunately, the spaces considered in this section do not embed compactly in each other and hence one cannot deduce the quasi-compactness of the operator from such inequalities. Sections 5 and 6 are the core of the paper where some inequalities relating the previous norms to the Sobolev norms $H^s$ are
obtained. In section 7 we collect the work done to prove our main Theorem 2.7. In section 8 we show that fast-slow systems satisfy the hypotheses of Theorem 2.7, hence our results apply. Also, we take advantage of the peculiarities of the fast-slow systems to prove some sharper results on the spectral projections.

**Remark 1.1.** In order to make the reading more fluid, we will use the notation \( f \lesssim g \) to mean that there exists a constant \( C_f > 0 \), depending only on the norm of the derivatives of the map \( F \), such that \( f \leq C_f g \). The values the constants \( C_f \) can change from one occurrence to the next. Moreover, in the following we will use \( C_{a,b,\ldots} \) to designate constants that depend on the quantities \( a, b, \ldots \).

Finally, to simplify notations, we use \( \{a,b,\ldots\}^+ \) to designate the maximum between the quantities \( a, b, \ldots \).

Note that \( \chi_c, \chi_u \), which determined the size of the central and unstable cone, respectively, are not uniquely determined by the map, hence we must keep track of how the constants depend on \( \chi_c^{-1}, \chi_u^{-1} \) and we cannot hide such a dependency inside a constant \( C_f \). In the next sections it will be apparent that it may be convenient to choose \( \chi_u \) as small as possible while it is convenient to choose \( \chi_c \) as large as possible.

**2. The systems and the results**

In this section we introduce the class of systems we are interested in, the main assumptions and some definitions necessary to present the results. In this work \( T^2 \) and \( T \) represent the quotients \( \mathbb{R}^2/\mathbb{Z}^2 \) and \( \mathbb{R}/\mathbb{Z} \) respectively. For a local diffeomorphism \( F : T^2 \to T^2 \) we define the functions \( m_F, m^*_F : T^2 \times \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}_+ \) as

\[
m_F(z,v) = \frac{\|D_z F v\|}{\|v\|}; \quad m^*_F(z,v) = \frac{\|(D_z F)^{-1} v\|}{\|v\|}.
\]

### 2.1. Partially hyperbolic systems

Let \( r \geq 2 \) and \( F : T^2 \to T^2 \) be a surjective \( C^r \) local diffeomorphism. We call \( F \) a partially hyperbolic system \(^2\) if there exist a continuous splitting, not necessarily invariant, of the tangent bundle into subspaces \( T_{T^2} = E_c \oplus E_u \), \( \sigma > 1 \) and \( c > 0 \) such that for each \( n \in \mathbb{N} \)

\[
\|DF^n_{|E_c}\| > c\sigma^n \\
\|DF^n_{|E_u}\| < c^{-n}\|DF^n_{|E_u}\|.
\]

Notice that for non-invertible map the unstable direction is not necessarily unique, nor invariant. It is then more convenient to work with cones instead than distributions. Indeed, it is well known (see e.g.\([35]\)) that the above conditions are equivalent to the existence of smooth invariant transversal cone fields \( C_u(z), C_c(z) \), which satisfy conditions equivalent to (2.1). To simplify the following arguments we will restrict ourselves to maps without critical points. We can thus assume, without further loss of generality.

\((H_0)\) for all \( p \in T^2 \) we have \( \det(D_p F) > 0 \).

In addition, to simplify notations, we make the assumption that the cone fields can be chosen constant since this hypothesis applies to all the examples we have in mind. Hence

\(^1\)By \( \| \cdot \| \) we mean the Riemannian metric in \( T^2 \) induced by the Euclidean norm in \( \mathbb{R}^2 \).

\(^2\)In the present case the term partially expanding would be more appropriate, as there is only an expanding direction which is dominant.
we assume:

(H1) There exists $\chi_u, \chi_c \in (0, 1)$ and $0 < \mu_- < 1 < \mu_+ < \lambda_- \leq \lambda_+$ such that, setting

\[ C_u := \{ (\xi, \eta) \in T^2 : |\eta| \leq \chi_u |\xi| \} , \]

\[ C_c := \{ (\xi, \eta) \in T^2 : |\xi| \leq \chi_c |\eta| \} , \]

defining

\[ \lambda_n^-(z) := \inf_{v \in \mathbb{R}^2 \setminus C_c} m_{F^n}(z, v) \quad \lambda_n^+(z) := \sup_{v \in \mathbb{R}^2 \setminus C_c} m_{F^n}(z, v) , \]

\[ \mu_n^-(z) := \inf_{v \in C_c \setminus \{0\}} m_{F^n}(F^n(z), v) \quad \mu_n^+(z) := \sup_{v \in C_c \setminus \{0\}} m_{F^n}(F^n(z), v) , \]

and letting $\lambda_n^- = \inf_z \lambda_n^-(z)$ and $\lambda_n^+ = \sup_z \lambda_n^+(z)$ we have the following:

There exists $C_* \geq 1$ such that, for all $z \in T^2$ and $n \in \mathbb{N},$

\[ D_z F C_u \subset C_u \quad D_z F^{-1} C_c \subset C_c , \]

\[ C_*^{-1} \mu_n^-(z) \leq \mu_n^+(z) \leq C_* \mu_n^+ ; \quad C_*^{-1} \lambda_n^- \leq \lambda_n^+ \leq C_* \lambda_n^+ , \]

(2.6) $0 < \mu_- < 1 < \mu_+ < \lambda_- \leq \lambda_+$.

From now on we set $\mu := \{ \mu_-, \mu_+^\ast \}^+ > 1$. Note that the above conditions imply, in particular, $\det(DF) \neq 0$.

(H2) Let $\Upsilon$ be the family of closed curve $\gamma \in C^\ast (\mathbb{T}, \mathbb{T}^2)$ such that 4

\begin{enumerate}
  \item $\gamma' \neq 0$, \label{c0}
  \item $\gamma$ has homotopy class $(0, 1)$, \label{c1}
  \item $\gamma'(t) \in C_c$, for each $t \in \mathbb{T}$, \label{c2}
\end{enumerate}

then $F^{-1}(\gamma)$ is the disjoint union of closed curves and $\Upsilon \subset F^{-1}(\Upsilon)$.

(H3) Let

\[ \zeta_r := \frac{1}{3} [ (r + 1)! (6r - 1) + 1] . \]

Then we say that $F$ satisfies the pinching condition if

\[ \mu \zeta_r < \lambda_- . \]

A partially hyperbolic system satisfying (2.8) will be called strongly dominated.

\textbf{Remark 2.1.} Note that, since $F$ is a local diffeomorphism, then it can be lifted to a diffeomorphism $\overline{F}$ of $\mathbb{R}^2$ with the projection $\pi$ map being mod 1, so that $\pi(0, 0) = 0$. Then we can define $G(x, \theta) = \overline{F}(x, \theta) - (0, \theta)$ and write $F \circ \pi(x, \theta) = \pi(G(x, \theta) + (0, \theta))$.

Thus in the following, with a slight abuse of notation, we will often confuse the map with his covering and write

\[ F(x, \theta) = (f(x, \theta), \theta + \omega(x, \theta)) , \]

\[ A \Subset B \text{ means } \overline{A} \subset \text{int}(B) \cup \{0\} . \]

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In addition, note that if the map satisfies condition (H2) then for each \( x \in \mathbb{R}^2 \) the curve \( \gamma_x(t) = (x, t) \), \( t \in \mathbb{T} \) has a preimage \( \nu \in \mathbb{Y} \) homotop to the curve \( \bar{\gamma}_p(t) = p + (0, t) \), \( p \in \nu \), \( F(p) = (x, 0) \). This implies that \( F(\bar{\gamma}_p(t)) \) is a curve homotop to \( \gamma_x \). Thus for each \( (x, \theta) \in \mathbb{R}^2 \) the lift has the property \( F(x, \theta + 1) = F(x, \theta) + (0, 1) \), which implies that \( \omega \) lifts to a periodic function in the second variable.

In the following we will need some quantitative information on the Lipschitz constant of the graphs associated to “unstable manifolds.” To simplify matters, we prove the needed results in Lemma D.1. We require then that our maps satisfy the hypotheses of such a Lemma. However, be aware that such hypotheses are not optimal and the following condition is used only in Lemma D.1, hence it becomes superfluous if in a given system one can prove Lemma D.1 independently.

(H4) With the notation (2.9) we require, for each \( p \in \mathbb{T}^2 \),
\[
\partial_f f(p) > \{2(1 + \|\partial_x \omega\|_{\infty}), |\partial_\theta f(p)|\}^+.
\]

**Definition 2.2.** We call a map \( F \) a strongly dominated vertical partially hyperbolic system (SVPH for simplicity) if it satisfies assumptions (H0),.., (H4).

**Remark 2.3.** Note that if \( F \) satisfies (H1) and (H2), then so does \( F^n, n \in \mathbb{N} \). Thus one can consider \( F^n \), instead of \( F \), to check (H3) and (H4), which makes such conditions rather weak.

From now on we will write a SVPH in the form (2.9) when convenient.

2.2. **Transversality of unstable cones.** In [46] Tsujii introduces the following notion of transversality.

**Definition 2.4.** Given \( n \in \mathbb{N} \), \( y \in \mathbb{T}^2 \) and \( z_1, z_2 \in F^{-n}(y) \), we say that \( z_1 \) is transversal to \( z_2 \) (at time \( n \)) if \( Dz_1 F^n C_{\nu} \cap Dz_2 F^n C_{\mu} = \{0\} \), and we write \( z_1 \cap z_2 \).

For each \( y \in \mathbb{T}^2 \) and \( z_1 \in F^{-n}(y) \), we define
\[
\mathcal{N}_F(n, y, z_1) := \sum_{\substack{z_2 \not\in z_1 \cap \nu \cap F^{-1}(y) \\text{ \( z_2 \in F^{-n}(y) \)}}} |\det Dz_2 F^n|^{-1}
\]
and set \( \mathcal{N}_F(n) = \sup_{y \in \mathbb{T}^2} \sup_{z_1 \in F^{-n}(y)} \mathcal{N}_F(n, y, z_1) \).

**Remark 2.5.** Note that if all the preimages are non-transversal, then the sum in (2.10) corresponds to the classical transfer operator applied to one \( (\mathcal{L}_F 1) \).

In essence, \( \mathcal{L}_F 1 - \mathcal{N}_F(n) \) provides a quantitative version of the notion of accessibility in our systems.

As \( \mathcal{N}_F \) is difficult to estimate we also introduce a related quantity, inspired by [46]. Given \( y \in \mathbb{T}^2 \) and a line \( L \) in \( \mathbb{R}^2 \) passing through the origin, define
\[
\tilde{\mathcal{N}}_F(n, y, L) := \sum_{\substack{z \in F^{-n}(y) \\text{ \( Df(z)C_{\nu} \cap L \)}}} |\det DF^n(z)|^{-1}.
\]
As before we set \( \tilde{\mathcal{N}}_F(n) = \sup_{y \in \mathbb{T}^2} \sup_L \tilde{\mathcal{N}}_F(n, y, L) \). Section 5.2 provides the properties of \( \tilde{\mathcal{N}}_F \) and Lemma 5.5 explains the relation between \( \mathcal{N}_F \) and \( \tilde{\mathcal{N}}_F \).
2.3. Result for SVPH. A physical measure is an $F$-invariant probability measure $\nu$ such that the set

$$B(\nu) := \{ p \in \mathbb{T}^2 : \frac{1}{n} \sum_{k=0}^{n-1} \delta_{F^k(p)} \to \nu \text{ weakly as } n \to \infty \}$$

has positive Lebesgue measure. One way to obtain information on the physical measures of the system is to study the spectral properties of the Transfer operator.

**Definition 2.6.** Given a map $F : \mathbb{T}^2 \to \mathbb{T}^2$, we define $\mathcal{L}_F : L^1(\mathbb{T}^2) \to L^1(\mathbb{T}^2)$, the transfer operator associated to $F$, as

$$(2.12) \quad \mathcal{L}_F u(z) = \sum_{y \in F^{-1}(z)} \frac{u(y)}{|\det(D_yF)|}. $$

Iterating (2.12) yields

$$(2.13) \quad \mathcal{L}_F^n u(z) = \sum_{y \in F^{-n}(z)} \frac{u(y)}{|\det(D_yF^n)|}, \quad n \in \mathbb{N}. $$

It is a well known fact that $\| \mathcal{L}_F u \|_{L^1} \le \| u \|_{L^1}$. For each integer $1 \le s \le r - 1$ we define

$$\alpha = \frac{\log(\lambda_-\mu^{-1})}{\log(\lambda_+\mu)}$$

$$\alpha_s := 2(2 + s - \alpha) ; \quad \beta_s := 2(s + 2) ; \quad \zeta_s := \frac{1}{3} + (s + 2)! \left(2s + \frac{5}{6}\right).$$

We are now ready to state the main result for SVPH, which proof is given in Section 7.

**Theorem 2.7.** Let $F \in C^r(\mathbb{T}^2, \mathbb{T}^2)$ be SVPH, and let $\alpha, \alpha_s, \beta_s, \zeta_s$, as in (2.14). We assume that there exist $n_0 \in \mathbb{N}$ and $\nu_s < 1$ such that, for some $1 \le s \le r - 1$,

$$(2.15) \quad \left\{ \mu^s \lambda_-^s, \sqrt[n]{N_F([\alpha n_0]) \mu^{\alpha_s n_0 + \beta_s m_s}} \right\}^+ < \nu_s < 1,$$

where $m_s$ is defined in (6.30). Then there exists Banach spaces $B_{s,s}, C^{r-1}(\mathbb{T}^2) \subset B_{s,s} \subset H^r(\mathbb{T}^2)$ such that $\mathcal{L}_F(B_{s,s}) \subset B_{s,s}$. The restriction of $\mathcal{L}_F$ to $B_{s,s}$ is a bounded quasiconpact operator, with spectral radius one and essential spectral radius smaller than $\nu_s$.

In particular, Theorem 2.7 implies that the map has finitely many physical measures and that if it is topologically mixing, then it mixes exponentially fast for all Hölder observables. Note that the condition involves only a finite power of the map and it is, at least in principle, checkable for a given map. Of course checking it may be quite laborious and may entail some computer assisted strategy. It is then interesting to consider less general models in which the previous condition can be explicitly verified.

2.4. A general class of models. It is natural to ask when a map of the form (2.9) satisfies (H0),..., (H4). Here we provide checkable conditions implying (H0),... (H4).

**Lemma 2.8.** Let $\lambda := \inf_{\mathbb{T}^2} \partial_x f, \Lambda := \sup_{\mathbb{T}^2} \partial_x f$ and suppose that:

1. $\partial_x f(p) > \left\{ 2(1 + \| \partial_x \omega \|_\infty), |\partial f(p)| \right\}^+ \quad \forall p \in \mathbb{T}^2$,
2. $\| \partial_x \omega \|_\infty + \| \partial_\theta \omega \|_\infty < 1$,
3. $\| \partial_\theta \omega \|_\infty < \frac{1 + \| \partial_x \omega \|_\infty}{\lambda - 1}$. 


unstable slope field

(2.17) \( D_p F(1, u) = (\partial_x f + u \partial_b f)(1, \Xi(u, p)), \) \( \Xi(u, p) = \frac{\partial_x \omega(p) + u \partial_b \omega(p) + u}{\partial_x f(p) + u \partial_b f(p)}. \)

Notice that

\[
\frac{d}{du} \Xi(p, c) = \frac{\partial_x f + (\partial_b \omega \partial_x f - \partial_b f \partial_x \omega)}{(\partial_x f + u \partial_b f)^2} = \frac{\det DF(x, \theta)}{(\partial_x f + u \partial_b f)^2} > 0,
\]

since \( \det DF > 0 \) by (1). Hence, checking the invariance of \( C_u \) under \( DF \) is equivalent to showing that, for each \( p \in T^2 \), \( |\Xi(p, \pm \chi_u)| \leq \chi_u \). That is

\[
|\partial_b f| \chi_u^2 - (\lambda - |\partial_b \omega| - 1) \chi_u + |\partial_x \omega| \leq 0.
\]

Setting \( \phi = \lambda - |\partial_b \omega| - 1 \), inequality (2.19) has positive solutions since \( \phi > 0 \) by (4), which also implies

\[
\phi^2 - 4|\partial_b f| |\partial_x \omega| \geq (|\partial_b f| - |\partial_x \omega|)^2 > 0.
\]

Setting \( \Phi_\pm = \phi \pm \sqrt{\phi^2 - 4|\partial_b f||\partial_x \omega|} \), we can choose

\[
\chi_u \in \left( \frac{\Phi_+}{2|\partial_b f|}, 1 \right).
\]

Note that the interval it is not empty due to (4).

On the other hand, if \( (c, 1) \in C_c \) we consider the center slope field

\[
\Xi^- (c, p) = \frac{(1 + \partial_b \omega(p)) c - \partial_b f(p)}{\partial_x f(p) - \partial_b f(p)c},
\]

and by an analogous computation we obtain \( |\Xi^- (c, \pm \chi_c)| \leq \chi_c \) if

\[
\chi_c \in \left( \frac{\Phi_-}{2|\partial_x \omega|}, 1 \right).
\]

Again, the interval it is not empty due to (4), we have thus proved (2.4).

Next, by the invariance of the cones we can define real quantities \( \lambda_n, \mu_n, u_n \) and \( c_n \) such
that, for each \( p \in \mathbb{T}^2 \),\(^5\)
\[
D_p F^n(1, 0) = \lambda_n(p) \{(1, u_n(p)) \} \quad D_p F^n(c_n(p), 1) = \mu_n(p)(0, 1),
\]
with \( \|u_n\|_\infty \leq \chi_u \), \( \|c_n\|_\infty \leq \chi_c \). Moreover, by definition
\[
D_p F(c_n(p), 1) = \frac{\mu_n(p)}{\mu_{n-1}(F(p))} (c_{n-1}(F(p)), 1),
\]
from which it follows, by (2.9),
\[
\mu_n(p) = \mu_{n-1}(F(p)) (1 + \partial_0 \omega(p) + c_n(p) \partial_x \omega(p)).
\]
Since \( \|c_n\|_\infty \leq \chi_c \), setting \( b := \|\partial_0 \omega\|_\infty + \chi_c \|\partial_x \omega\|_\infty \), we have
(2.23)
\[
(1 - b)^n \leq \mu_n(p) \leq (1 + b)^n.
\]
Note in particular that, by (2.23), we can make the choice (2.16) which immediately implies (H3) by (5). Similarly,
\[
\lambda_n(p) = \lambda_{n-1}(F(p)) (\partial_x f(p) + \partial_0 f(p) u_n(p))
\]
\[
= \prod_{k=0}^{n-1} \partial_x f(F^k p) \left( \partial_x f(F^k p) + \frac{\partial_0 f(F^k)}{\partial_x f(F^k p)} u_{n-k}(F^k p) \right),
\]
which, setting \( a := \chi_u \|\partial_x f\|_\infty \), implies
(2.24)
\[
(1 - a)^n \prod_{k=0}^{n-1} \partial_x f(F^k(p)) \leq \lambda_n(p) \leq (1 + a)^n \prod_{k=0}^{n-1} \partial_x f(F^k(p)).
\]
By (2.23) and (2.24) we have, for each \( n \in \mathbb{N} \) and \( p \in \mathbb{T}^2 \),
(2.25)
\[
\frac{\|D_p F^n(c_n, 1)\|}{\|D_p F^n(1, 0)\|} = \frac{|\mu_n(p)|}{|\lambda_n(p)|} \leq \frac{(1 + b)^n}{(1 - a)^n \lambda_n}.
\]
To conclude, we need to check that \( \frac{(1+b)}{(1-a)\lambda} < 1 \), form which we deduce (H1). This is implied by
\[
1 + \|\partial_0 \omega\|_\infty + \|\partial_x \omega\|_\infty + \|\partial_0 f\|_\infty < \lambda
\]
which correspond to equation (4).
It remains to prove (H2). Since \( \lambda > 2 \), \( F \) has rank at least two at each point, hence it is a covering map and each point has the same number of preimages, says \( d \). Let then \( \gamma : [0, 1] \to \mathbb{T}^2 \) be a smooth closed curve \( \gamma(t) = (c(t), t) \) such that \( \gamma ' \in C_c \) with homotopy class \((0, 1)\). If \( p = (x, \theta) \in \gamma(t) \) then \( F^{-1}(p) = \{q_1, \ldots, q_d\} \). Note that, by the implicit function theorem, locally \( F^{-1}\gamma \) is a curve, also, due to the above discussion, it belongs to the central cone. If we call \( \eta \) the local curve in \( F^{-1}\gamma \) such that \( \eta(0) = q_i \) we can prolong it uniquely to a curve \( \nu : [0, 1] \to \mathbb{T}^2 \). We will prove that \( \nu(1) = q_i = \nu(0) \). In turn this implies that \( F^{-1}\gamma \) is the union of \( d \) closed curves \( \nu_1, \ldots, \nu_d \) with \( \nu'_j \in C_c \), each one with homotopy class \((0, 1)\), by the lifting property of covering maps (see [32, Proposition 1.30]).
We argue by contradiction: assume that \( \nu(1) = q_j \neq q_i \). Let \( q_k = (x_k, \theta_k), k \in \{1, \ldots d\} \), then
\[
\theta_i + \omega(x_i, \theta_i) = \theta_j + \omega(x_j, \theta_j)
\]
\(^5\) Note that the definition of \( \lambda_n \) differs from the one of \( \lambda_n^3 \) in (2.5), since we are considering iteration of vectors inside the unstable cone. Nevertheless, they are related since there exists an integer \( m \) such that \( F^m(\mathbb{R}^2 \setminus C_c) \subseteq C_u \).
implies

\[ (2.26) \quad |\theta_i - \theta_j| \leq \frac{\|\partial_x \omega\|_\infty}{1 - \|\partial_\theta \omega\|_\infty} |x_i - x_j|. \]

Hence the segment joining \( q_i \) and \( q_j \) belong to the unstable cone if

\[ (2.27) \quad \chi_u \geq \frac{\|\partial_x \omega\|_\infty}{1 - \|\partial_\theta \omega\|_\infty} \]

which is possible since \( 2 \) implies that this condition is compatible with \((2.20)\). It follows that the image of the segment \( \ell = \{tq_i + (1 - t)q_j\} \) is an unstable curve and hence it cannot join \( p \) to itself without wrapping around the torus. In particular, if \( q_i \neq q_j \), then the horizontal length of \( F(\ell) \) must be larger than one. Then, setting \( \delta = |x_i - x_j| \),

\[ (2.28) \quad 1 \leq \int_0^1 \left| (\epsilon_1, D_{\epsilon(t)} F'\epsilon(t)) \right| \leq \|\partial_x f\|_\infty \left( 1 + \chi_u \frac{\|\partial_\theta f\|_\infty}{\|\partial_x f\|_\infty} \right) |x_i - x_j| \leq (1 + a) \Lambda \delta. \]

To conclude we must show that \( \nu \) cannot move horizontally by \( \delta \) whereby obtaining the wanted contradiction. Let \( \nu(t) = (\alpha(t), \beta(t)) \), then

\[ \left( \begin{array}{c} \epsilon'(t) \\ 1 \end{array} \right) = \gamma'(t) = D F \alpha' = \left( \begin{array}{c} \alpha' \partial_x f + \beta' \partial_\theta f \\ \alpha' \partial_x \omega + (1 + \partial_\theta \omega) \beta' \end{array} \right). \]

Since we know that \( |\epsilon'| \leq \chi_c \) and \( |\alpha'| \leq \chi_c |\beta'| \) we have

\[ |\beta'| \leq (1 - \chi_c \|\partial_x \omega\|_\infty - \|\partial_\theta \omega\|_\infty)^{-1} \]

\[ |\alpha'| \leq \frac{\chi_c}{\lambda} + \frac{\|\partial_\theta f\|_\infty}{\lambda(1 - \chi_c \|\partial_x \omega\|_\infty - \|\partial_\theta \omega\|_\infty)}. \]

I follows that it must be

\[ \frac{1}{(1 + a) \Lambda} \leq \delta \leq \int_0^1 |\alpha'(t)| dt \leq \frac{\chi_c}{\lambda} + \frac{\|\partial_\theta f\|_\infty}{\lambda(1 - \chi_c \|\partial_x \omega\|_\infty - \|\partial_\theta \omega\|_\infty)}. \]

We thus have a contradiction if we can choose \( \chi_c \) such that

\[ \left( 1 + \frac{\|\partial_\theta f\|_\infty}{\lambda} \right) \Lambda \left[ \frac{\chi_c}{\lambda} + \frac{\|\partial_\theta f\|_\infty}{\lambda(1 - \|\partial_x \omega\|_\infty - \|\partial_\theta \omega\|_\infty)} \right] < 1 \]

which, by \((2.22)\), is possible only if

\[ \frac{\Phi_-}{2\|\partial_x \omega\|_\infty} < \left( 1 + \frac{\|\partial_\theta f\|_\infty}{\lambda} \right)^{-1} \frac{\lambda}{\lambda} - \frac{\|\partial_\theta f\|_\infty}{\lambda(1 - \|\partial_x \omega\|_\infty - \|\partial_\theta \omega\|_\infty)} =: A. \]

Note that if \( A \geq 1 \), then the inequality is trivially satisfied. We must consider then only the case \( A < 1 \). A direct computation shows that the above inequality is implied by

\[ (2.29) \quad \|\partial_\theta f\|_\infty < A [\phi - A \|\partial_x \omega\|_\infty] = A [\lambda - \|\partial_\theta \omega\|_\infty - 1 - A \|\partial_x \omega\|_\infty] \]

Let us set for simplicity \( \omega := \|\partial_\theta \omega\|_\infty + \|\partial_x \omega\|_\infty \). Since \( A < 1 \) the above equation is in turn implied by the following inequality

\[ (2.30) \quad \|\partial_\theta f\|_\infty < \left( 1 + \frac{\|\partial_\theta f\|_\infty}{\lambda} \right)^{-1} \frac{\lambda}{\lambda} - \frac{\|\partial_\theta f\|_\infty}{\lambda(1 - \omega)} (\lambda - (1 + \omega)). \]

By elementary algebra \((2.30)\) is equivalent to

\[ (2.31) \quad \|\partial_\theta f\|_\infty (\|\partial_\theta f\|_\infty + 1) < \frac{\lambda^2}{\lambda} \left( 1 - \frac{1}{\lambda + \omega} \right). \]
Since \( \lambda > 2 \), (2.31) is implied by \( \| \partial_\theta f \|_\infty (\| \partial_x f \|_\infty + 1) < \frac{1}{2} \lambda^2 \Lambda^{-1} \), which is true if \( \| \partial_x f \|_\infty < \frac{1}{2} \left(-1 + \sqrt{1 + 2 \lambda^2 \Lambda^{-1}}\right) \). Hence the conclusion by condition \((5)\). \( \square \)

We have thus explicit conditions that imply \((H0), (H4)\). It remains to investigate how to check condition (2.15), which is, by far, the hardest to verify. One can directly investigate (2.15) in any concrete example (possibly via a computer assisted strategy), however to verify it for an explicit open set of maps we further restrict the class of systems under consideration. Note however that the endomorphisms we are going to consider still include a large class of physically relevant systems.

2.5. Fast slow systems. We consider a class of systems given by the following model introduced in [29] (and inspired by the more physically relevant model introduced in [24]). Let \( F_0(x, \theta) = (f(x, \theta), \theta) \) be \( C^r(T^2, T^2) \), for \( r \geq 2 \), such that \( \inf_{(x, \theta) \in T^2} \partial_x f(x, \theta) \geq \lambda > 2 \).

For any \( \omega \in C^r(\mathbb{R}^2, \mathbb{R}) \), periodic of period one, and \( \varepsilon > 0 \), we define
\[
F_\varepsilon(x, \theta) = (f(x, \theta), \theta + \varepsilon \omega(x, \theta)).
\]
Before stating our result we need the following definition.

**Definition 2.9.** The function \( \omega \in C^0(T^2, \mathbb{R}) \) is called \( x \)-constant with respect to \( F_0 \) if there exist \( \theta \in T \), \( \Phi_\theta \in C^0(T, \mathbb{R}) \) and a constant \( c \in \mathbb{R} \) such that, for each \( x \in T \),
\[
\omega(x, \theta) = \Phi_\theta(f(x, \theta)) - \Phi_\theta(x) + c.
\]

Note that it is fairly easy to check that a function is not \( x \)-constant by looking at the periodic orbits. Hence, the condition that \( \omega \) is not \( x \)-constant is considerably easier to check than (2.15). The following theorem is proven in section 8.

**Theorem 2.10.** Under condition \((5)\) of Lemma 2.8, there exists \( \varepsilon_* \) such that the map \( F_\varepsilon \) is SVPH for any \( \varepsilon < \varepsilon_* \). In addition, if \( \omega \) is not \( x \)-constant, then the transfer operator \( \mathcal{L}_{F_\varepsilon} \) is quasi compact on the spaces \( B_{\nu_*} \) with spectral radius one and essential spectral radius \( \nu_* < 1 \), uniformly in \( \varepsilon \).

The above Theorem is much stronger than the results in [48] (where only the existence of the physical measure is discussed and the results hold only generically) or [11, 1] (where the existence of SRB measures is obtained under an additional condition on the contraction or the expansion in the center foliation, even though for more general systems). However, the papers [18, 19] show that, using the standard pair technology and investigating limit theorems, it is possible to obtain considerably more detailed information on the system. Unfortunately, on the one hand the arguments in [18] are rather involved and, on the other hand, the conclusions pertaining the physical measure in [17] hold only for mostly contracting systems (contrary to the present ones). It is then very important to investigate if the present strategy can provide further information.

First of all we have an explicit bound on the regularity of the eigenfunctions. The reader can find the proof of the following theorem at the end of section 8.4.

**Theorem 2.11.** If \( \omega \) is not \( x \)-constant, then there exist \( \varepsilon_* > 0 \) such that, for each \( \varepsilon > 0 \) small enough, and \( r \in (0, 1) \), if \( \nu \in \sigma_{B_0}(\mathcal{L}_{F_\varepsilon}) \cap \{ z \in \mathbb{C} : 1 - r c_* \ln \varepsilon^{-1} \leq |z| \} \), and \( u \) is an eigenvector with eigenvalue \( \nu \) with \( \| u \|_{B_0} = 1 \),\(^6\) then for all \( \alpha > \frac{11}{2} \),
\[
\| u \|_{H^1} \leq C_{\alpha} \varepsilon^{-(1+r)\alpha}.
\]

\(^6\)See Section 4 for the definition of the space \( B_0 \).
Remark 2.12. It is not clear if the above Theorem is sharp. Certainly some form of blow-up is inevitable. For example: let $f_\theta(\cdot) = f(x, \theta)$ and call $h_\theta(\cdot, \theta)$ the unique invariant probability density of $f_\theta$. Let $\bar{\omega}(\theta) = \int_T \omega(x, \theta) h_\theta(x, \theta) dx$. If $\bar{\omega}$ has non degenerate zeroes $\{\theta_i\}_{i=1}^N$ such that $\bar{\omega}'(\theta_i) < 0$, then [20] (see also Theorem 2.13 below) implies that there must exist an eigenfunction $u$ essentially concentrated in the $\sqrt{\varepsilon}$ neighborhood of each $\theta_i$. This implies that $\|u\|_{H^l} \geq C_1 \varepsilon^{-\frac{1}{4}}$. However, there is a large gap between such a lower bound and the upper bound provided by Theorem 2.11. In particular, much more information on the spectrum could be obtained if one could establish an upper bound of the type $\varepsilon^{-\beta}$ with $0 < \beta < 1$.

By the above results it follows that $\mathcal{L}_{F_{\varepsilon}} = \Pi + Q$ where $\Pi Q = Q \Pi = 0$, $\Pi$ is a finite rank operator with spectrum either zero or of modulus one and $Q$ has spectral radius strictly smaller than one. We define the finite rank operator $\hat{P}$

$$
\int_{\mathbb{T}^d} \varphi \hat{P} h = \sum_j \int_{\mathbb{T}} \varphi(x, \theta_j) h_\theta(x, \theta_j) \int_{U_i \times \mathbb{T}} h
$$

where $U_i$ is the basin of attraction of the stable equilibrium points $\{\theta_j\}$ of the averaged dynamics

$$
\dot{\bar{\theta}} = \bar{\omega}(\bar{\theta}) \quad \bar{\theta}(0, \theta) = \theta.
$$

(2.33)

In section 8.5 we prove the following.

Theorem 2.13. The eigenvectors associated to the eigenvalue one are the physical measures of $F_{\varepsilon}$. The operator $\Pi$ is a projection. The spectrum on the unit circle form a finite group. Finally, in the setting of Remark 2.12,

$$
\|\Pi - \hat{P}\|_{g_{\varepsilon, \gamma} \to (C^1)} \leq C_1 \varepsilon^{\frac{1}{4}}
$$

$$
\|Q_{\varepsilon, \gamma}^n \ln^{-1} \|_{g_{\varepsilon, \gamma} \to (C^1)} \leq C_1 \varepsilon^{\frac{1}{2}}.
$$

Remark 2.14. Theorem 2.13 suggests the conjecture that the rank of $\Pi$ is $N$ and the only eigenvalue on the unit circle is one. We believe this to be the case but our estimates, in particular Theorem 2.11, are not strong enough to prove it.

Remark 2.15. Theorem 2.13 may seem weaker than the results in [20]. However, it should be remarked that a) the results of [20] are conditional to the existence of the physical measure which has been previously proven only for the generic case [46] (and hence may not apply to the present concrete situation) or in the case in which the central Lyapunov exponent is negative, see [17]. On the contrary here the existence of the physical measures is ensured by Theorems 2.10, 2.13. b) the results in [20] use the full force of [18], while here we invoke [18] only for the few pages pertaining averaging. This leaves open the very exciting possibility to obtain the results in [18] using a simplified argument which relies on some improved version of the present results.

3. Preliminary estimates

In this Section we start discussing vertical partially hyperbolic systems. We provide several basic definitions and we prove many estimates that will be extensively used in the following.
3.1. \( C^r \)-norm. Since we will need to work with high order derivatives, it is convenient to choose a norm \( \| \cdot \|_{C^r} \) equivalent to the standard one, which ensures our spaces to be Banach Algebras. We thus define the weighted norm in \( C^r(\mathbb{T}^2, \mathcal{M}(m, n)) \), where \( \mathcal{M}(m, n) \) are the \( m \times n \) matrices,\(^7\)

\[
\| \varphi \|_{C^0} = \sup_{x \in \mathbb{T}^2} \sup_{i \in \{1, \ldots, n\}} \sum_{j=1}^m |\varphi_{i,j}(x)|
\]

(3.1)

\[
\| \varphi \|_{C^r} = \sum_{k=0}^p 2^{\rho - k} \sup_{|\alpha| = k} \| \partial^\alpha \varphi \|_{C^0}
\]

(3.2)

where, for a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_k) \) with \( \alpha_k \in \{1, 2\} \), and we will use the notation \( |\alpha| = k \) and \( \partial^\alpha = \partial_{\alpha_1} \cdots \partial_{\alpha_k} \).\(^8\) The above definition implies

\[
\| \varphi \|_{C^{r+1}} = 2^{\rho+1} \| \varphi \|_{C^0} + \sup_i \| \partial_x \varphi \|_{C^0}.
\]

We will often need to compute the \( C^0 \) norm of \( \varphi \) along a curve \( \nu \in C^r(\mathbb{T}, \mathbb{T}^2) \). In this case we use the notation \( \| \varphi \|_{C^r_\nu} := \| \varphi \circ \nu \|_{C^r} \).

The following Lemma is proven in Appendix A. Note that the estimate in the Lemma are not sharp, however they try to optimize the balance between simplicity and usefulness.\(^9\)

**Lemma 3.1.** For every \( \rho, n, m, s \in \mathbb{N}_0 \), \( \psi \in C^\rho(\mathbb{T}^2, \mathcal{M}(n, m)) \) and \( \varphi \in C^\rho(\mathbb{T}^2, \mathcal{M}(m, s)) \) we have

\[
\| \varphi \circ \psi \|_{C^\rho} \leq \| \varphi \|_{C^\rho_\nu} \| \psi \|_{C^\rho_\nu}.
\]

Moreover, there exists \( C^*_\rho > 0 \) such that, if \( \varphi \in C^\rho(\mathbb{T}^2, \mathcal{M}(n, m)) \) and \( \psi \in C^\rho(\mathbb{T}^2, \mathbb{T}^2) \),

\[
\| \varphi \circ \psi \|_{C^\rho} \leq C^*_\rho \sum_{s=0}^\rho \| \varphi \|_{C^\rho} \sum_{k \in \mathcal{K}_{\rho, s}} \prod_{i \in \mathbb{N}} \| D \psi \|_{C^{|l_i| - 1}}
\]

(3.3)

where \( \mathcal{K}_{\rho, s} = \{ k \in \mathbb{N}_0^d : \sum_{i=1}^d k_l \leq s, \sum_{l=1}^d l k_l \leq \rho \} \).

Using the above Lemma it follows that there exists a constant \( \Lambda > 1 \) such that

\[
\| DF^\rho \|_{C^r} + \| (DF^\rho)^{-1} \|_{C^r} \leq \Lambda^s, \quad \forall n \in \mathbb{N}.
\]

(3.4)

3.2. Admissible curves. In this section we introduce the notion of admissible curve in order to define important auxiliary spaces and norms in the next section. We start by fixing some notations and defining exactly what we mean by inverse branch.

**Lemma 3.2.** Let \( \gamma \) be a differentiable close curve in the homotopy class \((0, 1)\) such that \( \gamma' \notin \mathcal{C}_u \) and \( F^{-1}\gamma = \bigcup_{k=1}^d \nu_k \), where the \( \nu_k \) are disjoint closed curves in the homotopy class \((0, 1)\). Then, there exist open sets \( \Omega_\gamma, \Omega_{\nu_k} \), with \( \Omega_\gamma = \mathbb{T}^2 \), and diffeomorphisms (the inverse branches) \( \eta_{\nu_k} : \Omega_\gamma \to \Omega_{\nu_k} \) satisfying,

- \( F \circ \eta_{\nu_k} = Id|_{\Omega_{\nu_k}} \)
- \( \Omega_{\nu_k} \cap \Omega_{\nu_j} = \emptyset \)
- \( \bigcup_{\nu_k \in F^{-1}\gamma} \overline{\Omega_{\nu_k}} = \mathbb{T}^2 \)

---

\(^7\)According with the previous notations we set \( x_1 = x \) and \( x_2 = \theta \).

\(^8\)Notice that this is at odd with the usual multi-index definition in PDE, however we prefer it for homogeneity with the case, treated later, of non-commutative vector fields.

\(^9\) See [5, 34] for precise, but much more cumbersome, formulae.
Remark 3.3. Note that if $\gamma \in \Upsilon$, then the hypotheses of the Lemma are satisfied thanks to hypothesis (H2).

Proof of Lemma 3.2. The circle $q = \{(a, 0)\}_{a \in \Upsilon}$ intersects each $\nu_k$ in only one point $p_k = \nu_k \cap q$. Indeed, by the backward invariance of the complement of $C_u$, $\nu_k$ is locally monotone so it can meet twice $q$ only if it wraps around the torus more than once, which cannot happen since $\nu_k$ belongs to the homotopy class $(0, 1)$. We can then label the $\nu_k$ so that the map $k \to p_k$ is orientation preserving (mod $d$), let us call it positively oriented.\footnote{This definition is ambiguous if $d = 2$, but in such a case the ambiguity is irrelevant.}

Also, calling $\tilde{\gamma}$ the curve obtained by translating $\gamma$ by $\frac{1}{d}$ in the horizontal direction, we consider $A := F^{-1}(\tilde{\gamma}) \cap q$. Since $F$ is a local diffeomorphism, if $\tilde{p} \in A$, in a neighborhood of $\tilde{p}$ the set $F^{-1}(\tilde{\gamma})$ consists of a curve with derivative outside $C_u$, hence transversal to $q$. Accordingly $A$ is a finite collection of points. Suppose that $\tilde{p} \in A$ is between $p_k$ and $p_{k+1}$, then $\tilde{T}^2 \backslash \nu_k$ is a cylinder and $\nu_k \cap \tilde{T}^2$ separates the cylinder in two disjoint regions (by Jordan curve theorem), thus $\tilde{p}_k$ belongs to a cylinder defined by the curves $\nu_k, \nu_{k+1}$. We can then follow the curve in $F^{-1}(\tilde{\gamma})$ starting from $\tilde{p}_k$, such curve cannot exit the cylinder (since $\gamma$ and $\tilde{\gamma}$ are disjoint). If it intersects again $q$ at a point $p'$ then the image, under $F$, of the segment of $q$ between $\tilde{p}_k$ and $p'$ is an unstable curve that starts and ends at $\tilde{\gamma}$, hence it must cross $\gamma$, contrary to the hypothesis. It follows that $p' = \tilde{p}_k$, that is $F^{-1}(\tilde{\gamma}) = \bigcup_{k=1}^d \nu_k$, where the $\nu_k$ are disjoint closed curves, of homotopy type $(0, 1)$, and $\tilde{p}_k = \nu_k \cap q$. As before, we can label the curves so that the $\tilde{p}_k$ are positively oriented and $\tilde{p}_k, k, k'p$, where the indexes are mod $d$. Next, for $i \in \{1, \cdots, d\}$ and $q \in \nu_i$, we define the horizontal segment $\{\xi(t)\}_{t \in (-\delta_-(q), \delta_+(q))}$ where $\xi(t) = q + e_1t$, $\xi(\delta_+(q)) \in \nu_i$ and $\xi(-\delta_-(q)) \in \nu_{i-1}$. We then define the regions

$$\Omega_\nu = \bigcup_{q \in \nu_i} \xi_q.$$  

Clearly, $\Omega_{\nu_i} \cap \Omega_{\nu_j} = \emptyset$ if $i \neq j$, and $\bigcup \Omega_{\nu_i} = \tilde{T}^2$. Note that $F : \Omega_{\nu_i} \cup \tilde{\nu}_{i-1} \to \tilde{T}^2$ is a bijection, although the inverse is not continuous. However, if we restrict the map to the set $\Omega_{\nu_i}$ then it is a diffeomorphism between $\Omega_{\nu_i}$ and $\Omega_\gamma = \tilde{T}^2 \backslash \{\tilde{\gamma}\}$. Thus it is well defined the diffeomorphism $h_{\nu_i} : \Omega_\gamma \to \Omega_{\nu_i}$ such that $F \circ h_{\nu_i} = Id|_{\Omega_\gamma}$. $\square$

We call $h_\gamma$ the inverse branch of $F$ associated to $\nu$ and simply $h$ when the curve $\nu$ is clear from the context. We denote by $\mathcal{H}$ the set of inverse branches of $F$. Likewise, for each $n \in \mathbb{N}$ we denote with $\mathcal{H}_n$ the set of inverse branches of $F^n$. As usual, we wish to identify the elements of $\mathcal{H}_n$ as compositions of elements of $\mathcal{H}$. Unfortunately, Lemma 3.2 tells us that each $h \in \mathcal{H}$ is defined on a domain obtained by removing a curve in $T$ from $\tilde{T}^2$. Therefore the composition of two inverse branches in $\mathcal{H}$ may not be well defined. We can however consider the following sets: denoting as $D_h$ and $R_h$ the domain and the range of $h$ respectively. For a curve $\gamma \in \Upsilon$ and $n \in \mathbb{N}$ we define

$$\mathcal{H}_{\gamma, n} := \{h \in \mathcal{H}^n : D_h = \tilde{T}^2 \backslash \{\gamma\}\},$$

$$\mathcal{H}_{n, \gamma} := \{h_n = (h_1^*, \cdots, h_n^*) \in \mathcal{H}^n : D_{h_j^*} \subset R_{h_{j-1}^*}, j \in \{2, \cdots, n\}, D_{h_1^*} \cap \{\gamma\} \neq \emptyset\}.$$

In $\mathcal{H}_{n, \gamma}$ there exists the obvious equivalence relation $h_n \sim \tilde{h}^*_n$ if $h_1^* \circ \cdots \circ h_n^* = \tilde{h}_1^* \circ \cdots \circ \tilde{h}_n^*$ and the quotient of $\mathcal{H}_{n, \gamma}$ is naturally isomorphic to $\mathcal{H}_{\gamma, n}$. In the following we will use the
two notations interchangeably. Finally, we define
\[ \mathcal{H}^\infty = \left\{ \mathfrak{h} = (\mathfrak{h}_1, \ldots) \in \mathcal{H}^1 : D_{\mathfrak{h}_{j+1}} \subset \mathbb{R}_{\mathfrak{h}_j}, j \in \mathbb{N} ; D_{\mathfrak{h}_1} \cap \{ \gamma \} \neq \emptyset \right\}. \]
For \( \mathfrak{h} \in \mathcal{H}^\infty \), the symbol \( \mathfrak{h}_n \) will denote the restriction of \( \mathfrak{h} \) to \( \mathcal{H}^n \) and we will say that \( \mathfrak{h} \sim \mathfrak{h}' \) iff their restrictions are equivalent for each \( n \in \mathbb{N} \).

In the following we will often suppress the subscripts \( \gamma, \nu \) if it does not create confusion.

3.2.1. Some further notation. For technical reason it is convenient to work with cones which are slightly smaller than \( C_u \) and \( C_c \). Take \( \epsilon > 0 \) arbitrarily small but fixed\(^{12}\) and, setting \( \epsilon^* = 1 - \epsilon \), let us consider the cone
\[ C_{\epsilon,u} = \{(x, y) \in \mathbb{R}^2 : |y| \leq \chi_u \epsilon^* |x|\}, \]
which is strictly contained in \( C_u \). Moreover the difference between the angle of \( C_u \) and the angle of \( C_{u,\epsilon} \) is smaller than \( \epsilon \). In the same way it is defined \( C_{\epsilon,c} \). For each \( p \in \mathbb{T}^2 \) let \( \mathcal{H}^n := \{ \mathfrak{h} \in \mathcal{H}^n : p \in D_{\mathfrak{h}} \} \). By the expansion of the unstable cone under backward dynamics and the backward invariance of the central cone we can define \( m_{\chi_u}(p, \mathfrak{h}) \) : \( \mathbb{T}^2 \times \mathcal{H}^\infty \rightarrow \mathbb{N} \) and \( m_{\chi_u} \in \mathbb{N} \) as
\[ m_{\chi_u}(p, \mathfrak{h}) = \min\{ n \in \mathbb{N} : D_{\mathfrak{h}} \mathfrak{h}_n(\mathbb{R}^2 \setminus C_u) \subset C_{\epsilon,c} \} \]
\[ m_{\chi_u}(p) = \sup_{\mathfrak{h} \in \mathcal{H}^\infty} m_{\chi_u}(p, \mathfrak{h}) \]
\[ m_{\chi_u} = \sup_{p \in \mathbb{T}^2} \sup_{\mathfrak{h} \in \mathcal{H}^\infty} m_{\chi_u}(p, \mathfrak{h}). \]
To guarantee that the above quantities are finite, we choose \( \epsilon \) such that \( C_{\epsilon,c} \supset D_p \mathfrak{h} C_c \), where \( \mathfrak{h} \circ F(p) = p \). Note that the latter condition is possible because of \((2.4)\), the continuity of \( D_p \mathfrak{h} C_c \) and the compactness of \( \mathbb{T}^2 \).

By a direct computation (see Sub-Lemma 3.12 for the details) equation \((3.8)\) implies
\[ \lambda_{m_{\chi_u}}(p, \mathfrak{h})(p)^{-1} \mu^{m_{\chi_u}} < \epsilon^* \chi_u \epsilon^*, \quad \forall p \in \mathbb{T}^2, \mathfrak{h} \in \mathcal{H}^\infty, \]
\[ m_{\chi_u} < \tilde{c}_2 \log \chi_u^{-1}, \]
for some fixed constant \( \tilde{c}_2 > 0 \). Next, consider a vector \( v = (1, u_0) \in C_u \), so that \( |u_0| \in [-\chi_u, \chi_u] \). By forward invariance of the unstable cone, there exist continuous functions \( \Upsilon_n, \Xi_n : \mathbb{N} \times \mathbb{T}^2 \times [-\chi_u, \chi_u] \rightarrow \mathbb{R} \) such that
\[ D_p F^n v = \Upsilon_n(p, u_0)(1, \Xi_n(p, u_0)), \]
where \( \|\Xi_n\|_{\infty} \leq \chi_u \). We are interested in the evolution of the slope field \( \Xi_n \). For this purpose it is convenient to introduce the dynamics \( \Phi(p, u_0) = (F(p), \Xi(p, u_0)) \), for \( p \in \mathbb{T}^2 \), \( u_0 \in [-\chi_u, \chi_u] \) and where we use the notation \( \Xi = \Xi_1 \). The map \( \Phi \) will describe how the slopes of the cones change while iterating \( F \). Note that
\[ \Phi^n(p, u_0) = (F^n(p), \Xi_n(p, u_0)). \]
Finally, for \( n \in \mathbb{N} \) and \( \mathfrak{h} \in \mathcal{H}^\infty \), let us define the function
\[ u_{\mathfrak{h},n}(p, u_0) = \pi_2 \circ \Phi^n(\mathfrak{h}_n(p), u_0) : \mathbb{T}^2 \times [-\chi_u, \chi_u] \rightarrow [-\chi_u, \chi_u], \]
\(^{11}\) As it is not obvious how to make sense of infinite compositions, we define the equivalence relation indirectly.
\(^{12}\) During the following sections \( \epsilon \) will have to satisfies different conditions. However, it is important to note that, once the conditions are satisfied, the value of \( \epsilon \) is fixed once and for all.
where $\pi_2$ is the projection on the second coordinate. By Lemma D.1, applied with $u = u^c = u_0$, $\varepsilon_0 = 1$ and $A = \frac{1}{2}$, there exists $C_2$ such that
\[
(3.13) \quad |u_{h,n}(p_1,u_0) - u_{h,n}(p_2,u_0)| \leq L_*(n)||p_1 - p_2||, \quad \forall h \in \mathcal{H}^\infty, n \in \mathbb{N}, p_1, p_2 \in \mathbb{T}^2, \]
where $L_*(n)$ is the Lipschitz constant.

### 3.2.2. Admissible central and unstable curves.
In the following $\pi_k : \mathbb{T}^2 \rightarrow \mathbb{T}$ will denote the projection on the $k$th component, for $k = 1, 2$. Also, for $\varphi \in C^r(\mathbb{T}, \mathbb{C})$ we use the notation $\langle \varphi \rangle^{(j)}(t) = \frac{d^j}{dt^j} \varphi(t)$ and $\varphi'$ in the case $j = 1$.

**Definition 3.4.** Let $c$ be a positive constant, then $\Gamma_j(c)$ is the set of the $C^r$ closed curves $\gamma : \mathbb{T} \rightarrow \mathbb{T}^2$ which are parametrized by vertical length, i.e. $\gamma(t) = (\gamma_1(t), t)$, satisfy conditions $c1)$ and $c2)$ of assumption (H2), and:

- $c3)$ for every $2 \leq \ell \leq j$: $|(\pi_1 \circ \gamma(t))(t)| \leq c^{(\ell - 1)!}$.

Given $c > 0$ and $j \leq r$ we will call $\gamma \in \Gamma_j(c)$ a $(j, c)$-admissible central curve (or simply admissible curve if the context is clear). We will choose $c$ in Corollary (3.10).

Similarly, a curve $\eta \in C^r(I, \mathbb{T}^2)$ of length $\delta$ defined on a compact interval $I = [0, \delta]$ of $\mathbb{T}$ is called an admissible unstable curve if $\eta'(t) \in C_u$, it is parametrized by horizontal length and its $j$-derivative is bounded by $c^{(j-1)!}$.

The basic objects used in the paper are integrals along admissible (or pre-admissible) curves. To estimate precisely such objects are necessary several technical estimates that are developed in the next subsections.

### 3.3. Preliminary estimates on derivatives.
We start with the following simple, but very helpful, propositions.

**Proposition 3.5.** There exists a constant $C_* > 0$ such that, for every $z \in \mathbb{T}^2$, any $n \in \mathbb{N}$, any vectors $v^u \in C_u$ and $v^c \in C_c$ such that $(a, b) := D_z F^n v^c \notin C_u$, we have:
\[
C_*^{-1} \frac{||D_z F^n v^u||}{||v^u||} \leq |\det D_z F^n| \leq C_* \frac{||D_z F^n v^u||}{||v^c||}.
\]

**Proof.** Recall that for a matrix $D \in GL(2, \mathbb{R})$ and vectors $v_1, v_2 \in \mathbb{R}^2$ linearly independent
\[
(3.14) \quad |\det D| = \frac{|Dv_1 \wedge Dv_2|}{|v_1 \wedge v_2|} = \frac{|Dv_1||Dv_2|}{|v_1|} \sin(\angle(Dv_1, Dv_2)) \frac{\sin(\angle(v_1, v_2))}{|v_2|}.
\]

Let $\theta = \angle(D^F n v^u, D^F n v^c)$, $\theta_1 = \angle(D^F n v^u, e_1)$, $\theta_2 = \angle(D^F n v^c, e_1)$ and $\theta_u = \arctan \chi_u$. Since $D_z F^n v^u \in DFC_u$ we have $|\theta_1| < c \theta_u$, for some fixed $c \in (0, 1)$. On the other hand, by hypothesis, $|\theta_2| \geq \theta_u$. Thus
\[
\begin{align*}
\frac{\theta}{|\theta|} &= \frac{\theta_2 - \theta_1}{|\theta_2|} \leq \frac{|\theta_2| + |\theta_1|}{|\theta_2|} \leq 1 + c \\
\frac{\theta}{|\theta|} &\geq \frac{|\theta_2| - |\theta_1|}{|\theta_2|} \geq 1 - c.
\end{align*}
\]

The Lemma follows since $|DF^n v^c| \sin \theta_2 = b$. \hfill $\square$

We introduce the following quantities for each $n \in \mathbb{N}$:
\[
(3.15) \quad C_{\mu,n} := C_2 \frac{1 - \mu^{-n}}{\mu - 1} \leq C_2 \min\{n, (\mu - 1)^{-1}\}; \quad C_{\mu,0} = 0,
\]
\[
(3.16) \quad \varsigma_{n,m}(p) = \{1, C_{\mu,n} + (\chi_u + |||\diamondsuit|||_c^2)\{C_{\mu,n}, \lambda^+(p)\}^+\}; \quad \varsigma_{n,n} := \varsigma_n.
\]
Remark 3.6. Note that we can always estimate $C_{\mu,n}$ with $(\mu - 1)^{-1}$, which is independent on $n$, and we will do it in the general case (SVPH) if we need estimates uniform in $n$. However, such a bound will deteriorate when $\mu$ approaches one, a case we want to investigate explicitly in Section 8, and for which (3.15) is more convenient.

Next, we provide sharper estimates of various quantities relevant in the next sections.

**Proposition 3.7.** For any $m \leq n \in \mathbb{N}$ and $e > 0$, $p \in \mathbb{T}^2$ and $\nu \in \Gamma_2(e)$, such that $DF^{n-m}v \in C_c$, we have:

\[
\lambda^+(p) \lesssim \lambda^-(p)
\]

(3.17)

\[
\|DF^n\|_{C^0_v} \leq C_2 \lambda^+
\]

In addition,\(^\dagger\)

\[
\begin{align*}
\|DF^n\|_{C^1_v} & \leq C_4 \lambda^+ + \lambda^- \mu^{n-m} \\
\|DF^n\|_{C^2_v} & \leq C_4 \lambda^+ + \zeta \|DF^n\|_{C^0_v} \\
\|\frac{d}{dt}(DF^n)\|_{C^0_v} & \leq C_2 \mu^{2n-m} \nu(t) \\
\|\frac{d^2}{dt^2}(DF^n)\|_{C^0_v} & \leq C_2 \mu^{2n-m} \nu(t) + C_2 \zeta \nu(t) \lambda^+(\nu(t)) \mu^{n-m} + \zeta.
\end{align*}
\]

(3.18)

**Proof.** Let $v^c \in T_{F^n(p)}\mathbb{T}^2$ with $v^c \in C_c$ unitary, and $w_u \in C_u$. Define

\[
\tilde{w}_u = \frac{DF^n F^n w_u}{\|DF^n F^n w_u\|} \in C_u.
\]

For each $v \in T_{F^n(p)}\mathbb{T}^2$ we can write $v = \alpha v^c + \beta \tilde{w}_u$, then

\[
\|DF^n v\| \leq |\alpha|\|DF^n v^c\| + |\beta|\|DF^n \tilde{w}_u\|
\]

By (2.3) and (2.5) we have the following

\[
\begin{align*}
(1) \quad & \|DF^n v^c\| \leq C \lambda_-^n, \\
(2) \quad & \|DF^n \tilde{w}_u\| \leq C \mu^n.
\end{align*}
\]

Hence,

\[
\|DF^n v\| \leq C \mu^n |\alpha| + C \lambda_-^n |\beta|.
\]

A direct computation shows

\[
\{|\alpha|, |\beta|\} \leq \frac{1 + |\langle v^c, \tilde{w}_u \rangle|}{1 - |\langle v^c, \tilde{w}_u \rangle|^2} \|v\| \leq \frac{1 + \cos \vartheta}{1 - (\cos \vartheta)^2} \|v\|
\]

where

\[
\cos \vartheta := \cos \left[ \inf_{v \in C_c, w \in C_c} \{ \angle (v,w) \} \right] \leq \frac{1}{\sqrt{1 + \chi^2}} < 1.
\]

From the above first statement of the Lemma, limited to $\rho = 0$, follows. The strategy for proving the second of (3.17) is similar. We take $u_1, w_2 \notin C_c$ unitary and $\nu = (0,1) \in C_c$, and we set $\tilde{v}_c = \frac{(DF^n v^c) - \nu}{\|DF^n v^c - \nu\|} \in C_c$. Notice that $\|DF^n \tilde{v}_c\| \leq C \mu^n$. Let

\(^\dagger\)Recall Section 3.1 for the definition of $\|\cdot\|_{C^0_v}$.\)
w_2 = \alpha w_1 + \beta \tilde{v}^c. By (2.4) it follows that there exists a minimal angle between w_1 \not\in C_c and \tilde{v}^c \in (DF)^{-1} C_c, thus |\alpha| + |\beta| \leq C for some constant C > 0. Hence,

$$\|D_p F^n w_1 - D_p F^n w_2\| \leq |1 - \alpha| \|D_p F^n w_1\| + C_\mu \mu^n \leq (1 + C_\mu) \|D_p F^n w_1\| + C_\mu \mu^n.$$  

Since \|D_p F^n w_1\| \geq C\lambda_n^-(p), it follows that

$$1 - \frac{\|D_p F^n w_2\|}{\|D_p F^n w_1\|} \leq \frac{\|D_p F^n w_1\| - \|D_p F^n w_2\|}{\|D_p F^n w_1\|} \leq (1 + C_\mu) + C_\mu \frac{\mu^n}{\lambda_n}(p).$$

Equation (3.17) follows by the arbitrariness of w_1, w_2 and since \mu < \lambda_. To conclude we must compute the derivatives of DF^n, (DF^n)^{-1}. By (2.3), we have

$$(3.19) \quad \|D_x F^k\| \leq C_\lambda \lambda_\mu^+(x).$$

Moreover, for each n, k \in \mathbb{N}, we have

$$\frac{d}{dt} D_{\nu(t)} F^n = \sum_{s=1}^{n-1} \sum_{k=0}^{n-1} D_{F^{n-1} F^{n-k}} \partial_{x_s} (DF^{n-1}) \partial_{w_s} (DF^{n-k}) D_{\nu(t)} F^k (D_{\nu(t)} F^k \nu_s')$$

$$(3.20) \quad \frac{d}{dt} (D_{\nu(t)} F^n)^{-1} = \sum_{s=1}^{n-1} (D_{\nu(t)} F^k)^{-1} [\partial_{x_s} (DF)^{-1} (DF^{n-k-1})^{-1}] \circ F^k (\nu(t))$$

$$\cdot (D_{\nu(t)} F^k \nu_s').$$

The above, also differentiating once more, implies that

$$\|\frac{d}{dt} (D_{\nu(t)} F^n)\| \leq C_\lambda \lambda_\mu^+ \lambda_\mu^{n-m}$$

$$(3.21) \quad \|\frac{d^2}{dt^2} (D_{\nu(t)} F^n)\| = \|\sum_{\ell,s} (\partial_{x_\ell} \partial_{w_s} D_x F^n) \nu_s' \nu_s' + \sum_s \partial_{x_s} D_x F^n \nu_s''\|$$

$$\leq C_\lambda \lambda_\mu^+ (\mu^{n-m} \lambda_\mu^+)^2 + C_\lambda (\lambda_\mu^+)^2 \omega.$$  

To estimate the second of (3.20), note that for each p \in \mathbb{T}^2, there exists \xi \in \mathcal{C}^{r-2}(\mathbb{T}^2, \mathbb{R}^2), \|\xi\|_{\mathcal{C}^{r-2}} \leq C_{\xi}, such that, for all w \in \mathbb{R}^2, and |\alpha| \leq r - 2,

$$(3.22) \quad \|\partial_\alpha (DF)^{-1} w - e_1 \langle \partial_\xi, w \rangle\| \leq C_\lambda \|w\| \|\omega\|_{\mathcal{C}^{r+2}}.$$  

Thus, setting \eta_k(p) = D_p F^k e_1 \|D_p F^k e_1\|^{-1}, we have \|\eta_k - e_1\| \leq C_\mu \mu^k and, for all w \in \mathbb{R}^2,

$$\|(D_x F^k)^{-1} \partial_{x_s} (DF)^{-1} w\| \leq \|(D_x F^k)^{-1} \eta_k(x) \partial_{x_s} (DF)^{-1} w\|$$

$$\quad + \|(D_x F^k)^{-1} \partial_{x_s} (DF)^{-1} w - (D_x F^k)^{-1} \eta_k(x) \partial_{x_s} (DF)^{-1} w\|$$

$$\leq C_\lambda \|w\| \|\omega\|_{\mathcal{C}^2} + C_\mu \mu^k \|\omega\|_{\mathcal{C}^2}.$$
For simplicity we set $C_F := \chi_u + \|\omega\|_{C^r}$. Hence, using the above and (3.17),
\[
\left\| \frac{d}{dt}((D_\nu F^n)^{-1}) \right\| \leq C_2 \sum_{k=0}^{n-m-1} \mu^{n-k}\{ (\lambda_k^+ \circ \nu)^{-1} + C_F \mu^k \}
+ C_2 \sum_{k=n-m}^{n-1} \mu^{n-k}\{ (\lambda_k^+ \circ \nu)^{-1} + C_F \mu^k \} \mu^{n-m} \lambda_{m-n+k}^+ \circ \nu
\leq \mu^{2n-m} [C_{\mu,m} + C_F \{C_{\mu,n-m}, \lambda_m^+ \circ \nu\}^+] .
\]
Therefore
\[
(3.23) \quad \left\| \frac{d}{dt}((D_\nu(t) F^n)^{-1}) \right\| \leq C_2 \mu^{2n-m} \varsigma_{n,m} \circ \nu(t),
\]
which yields the statement for the first derivative. Next, differentiating once more the second of (3.20),
\[
\frac{d^2}{dt^2}(D_\nu F^n)^{-1} = \sum_{s=1}^{2n-1} \sum_{k=0}^{n-1} \left\{ \frac{d}{dt}(D_\nu(t) F^k)^{-1} \right\} \left[ \partial_{x_s} (DF)^{-1} (D_{F(\nu)} F^{n-k-1})^{-1} \right] \circ F^k(\nu)
\cdot (D_\nu F^k \nu')(\nu) + \sum_{s=1}^{2n-1} \sum_{k=0}^{n-1} \left\{ (D_\nu F^k)^{-1} \right\} \left( \partial_{x_t} (DF)^{-1} (D_{F(\nu)} F^{n-k-1})^{-1} \right) \circ F^k(\nu)
\cdot (D_\nu F^k \nu')(\nu) + \sum_{s=1}^{2n-1} \sum_{k=0}^{n-1} \left\{ (D_\nu F^k)^{-1} \right\} \left( \partial_{x_s} (DF)^{-1} (D_{F(\nu)} F^{n-k-1})^{-1} \right) \circ F^k(\nu)
\cdot (D_\nu F^k \nu')(\nu) .
\]
We estimate the three sums above separately. By (3.22) and (3.23), the first one is bounded by
\[
C_3 \sum_{k=0}^{n-m-1} \mu^{2k} \varsigma_{k,0} \circ \nu \mu^{n-k-1} \mu^k + C_3 \sum_{k=n-m}^{n-1} \mu^{n-k-1} \mu^k \varsigma_{k,k} \circ \nu \mu^{n-k-1} \mu^{n-m} \lambda_{m-n+k}^+ \circ \nu
\leq C_3 \mu^{2n} C_{\mu,n-m} \varsigma_{n-m,0} \circ \nu + \mu^{2n-m} \varsigma_{n,n} \lambda_m^+ \circ \nu \leq \mu^{2n} \varsigma_{n,n} \circ \nu \varsigma_m^+ \circ \nu .
\]
The second one is equal to
\[
\sum_{k=0}^{n-1} (D_\nu F^k)^{-1} \left\{ \partial_{x_t} (DF)^{-1} \right\} \circ F^k(\nu) \cdot (D_\nu F^k \nu')(\nu)s + (D_\nu F^k)^{-1} \left\{ \partial_{x_s} (DF)^{-1} \right\} \circ F^k(\nu) \cdot (D_\nu F^k \nu')(\nu)s,
\]
so we can use again (3.22) to get the bound
\[
C_4 \sum_{k=0}^{n-m-1} [(\lambda_k^+ \circ \nu(t))^{-1} + C_F \mu^k] \mu^{n-k} \{ 1 + [C_{\mu,n-k} + C_F \lambda_{n-k}^+ \circ \nu] \}
\cdot \mu^{2\min\{n-k,n-m\}} (\lambda_{[0,n-k+n]}^+ \circ \nu) \leq C_4 \mu^{n} \varsigma_{n,n} \circ \nu \lambda_m^+ \circ \nu .
\]
For the last term we use the estimates above and, recalling (3.21), we obtain the bound
\[
\{C_{\mu,n} + C_F \lambda_m^+(\nu(t))\} (\lambda_m^+ \mu^{n-m} + \varepsilon) \leq C_5 \varsigma_{n,n} \circ \nu(t) (\lambda_m^+(\nu(t)) \mu^{n-m} + \varepsilon) .
\]
Collecting the above estimates, the last of the (3.18) readily follows. \(\square\)
3.4. Iteration of curves. We first check how the above curves behave under iteration. The following is a more quantitative version of [48, Lemma 3.2] adapted to our case.

**Lemma 3.8.** Let $F$ be SVPH. Then there exist $\tilde{n} \in \mathbb{N}$, $c > 2$, $C_\alpha > 1$ and $\eta < 1$ such that, for each $c_\alpha > c/2$, $\gamma \in \Gamma_r(c_\alpha)$, $\ell \leq r$, and $n \geq \tilde{n}$, setting $\nu_0 \in F^{-n}\gamma$, there exist diffeomorphisms $h_{n,\nu} = h_n \in C^r(\mathbb{T})$ such that:

(a) The curve $\hat{\nu}_n = \nu_n \circ h_n$ is in $\Gamma_\ell(n^\circ c_\alpha + c/2)$ and

$$
||h_n||_{C^\ell} \leq \begin{cases} 
C_\alpha \mu_n & \text{if } \ell = 1 \\
C_\alpha^2 c_\alpha C_{\mu,n} ||h_n||^{2n} & \text{if } \ell = 2 \\
(C_\alpha^2 c_\alpha)\ell \mu_{n} ||h_n||^{\ell n} & \text{if } \ell > 2, 
\end{cases}
$$

where $a_\ell = (\ell - 1)! \sum_{k=0}^{\ell-1} \frac{1}{k!}$, and $C_{\mu,n}$ as in (3.15).

**Proof.** Fix $\gamma \in \Gamma_\ell(c_\alpha)$ and $n \in \mathbb{N}$. Let $\nu_n$ be a pre-image of $\gamma$ under $F^n$ and consider $\mathfrak{h} \in \mathfrak{H}_\infty$ such that $\nu_n = \mathfrak{h}_n \circ \gamma$. Let $h_n : \mathbb{T} \to \mathbb{T}$ be the diffeomorphism such that $\hat{\nu}_n = \nu_n \circ h_n$ is parametrized by vertical length. We then want to check properties (c1), ..., (c3) for $\nu_n$. The first two follow immediately by assumption (H2), thus we only have to check property (c3). By definition we have

$$
F^n \hat{\nu}_n = \gamma \circ h_n.
$$

Differentiating equation (3.25) twice we obtain

$$
(\partial_s D_{\nu_n} F^n) \hat{\nu}_n' + D_{\nu_n} F^n \hat{\nu}_n'' = \gamma'' \circ h_n (h_n')^2 + \gamma' \circ h_n h''_n.
$$

Similarly, if we differentiate equation (3.25) $j$-th times,

$$
R_j (F^n, \hat{\nu}_n) + D_{\nu_n} F^n \hat{\nu}_n^{(j)} = \gamma^{(j)} \circ h_n (h_n')^j + Q_j (h_n, \gamma) + \gamma' \circ h_n \cdot h''_n^{(j)},
$$

where $R_j$ is the sum of monomials, with coefficients depending only of $(\partial^s F^n) \circ \hat{\nu}_n$ with $|\alpha| \leq j$, in the variables $\hat{\nu}_n^{(s)}$, $s \in \{0, \ldots, j - 1\}$, where if $k_s$ is the degree of $\hat{\nu}_n^{(s)}$ we have $\sum_{s=0}^{j-1} s k_s = j$. Likewise the $Q_j$ are the sum of monomials that are linear in $\gamma^{(\sigma)}$, $\sigma \in \{2, \ldots, j - 1\}$, and of degree $p_s$ in $h_n^{(s)}$, $s \in \{1, \ldots, j - \sigma + 1\}$, such that $\sum_{s=1}^{j-\sigma+1} s p_s = j$.  

In order to obtain an estimate for $||\hat{\nu}_n^{(j)}||$ it is convenient to introduce the vectors $\eta_{n,j} = D_{\nu_n} F^n \hat{\nu}_n^{(j)}$. We then define the unitary vectors $\eta_{n,j}^\perp, \hat{n}_{n,j}$ such that $\langle \eta_{n,j}^\perp, \eta_{n,j} \rangle = 0$ and $\hat{n}_{n,j} = \frac{\eta_{n,j}}{||\eta_{n,j}||}$. Multiplying equation (3.27) by $\eta_{n,j}^\perp$ and $\eta_{n,j} \eta_{n,j}^\perp$ respectively, we obtain the system of equations

$$
\begin{align*}
\langle \eta_{n,j}^\perp, R_j (F^n, \hat{\nu}_n) \rangle &= \langle \eta_{n,j}^\perp, \gamma^{(j)} \circ h_n (h_n')^j + Q_j (h_n, \gamma) + \gamma' \circ h_n \cdot h''_n \rangle \\
\langle \hat{n}_{n,j}, R_j (F^n, \hat{\nu}_n) \rangle + ||\eta_{n,j}|| &= \langle \hat{n}_{n,j}, \gamma^{(j)} \circ h_n (h_n')^j + Q_j (h_n, \gamma) + \gamma' \circ h_n \cdot h''_n \rangle.
\end{align*}
$$

14 The reader can check this by induction (equation (3.26) gives the case $j = 2$). E.g., if a term $Q$ in $R_j$ has the form $P = \prod_{i=0}^{j-1} \alpha_i (\hat{\nu}_n^{(i)})$ where $\alpha_i(x)$ is homogeneous of degree $k_s$, in $x$, then $\partial_i Q$ will be a sum of terms of the same type with homogeneity degrees $k'_s$. Let us compute such homogeneity degrees: if the derivative does not hit a $\hat{\nu}_n^{(i)}$, $s > 0$, then, by the chain rule, we will get a monomial with $k'_s = k_s + 1$ while all the other homogeneity degree are unchanged: $k'_s = k_s$ for $s > 0$. Hence, $\sum_{s=0}^{j} k'_s = j + 1$. If the derivative hits one $\hat{\nu}_n^{(i)}$, then it produces a monomial with $k'_s = k_s$ for $s \notin \{i, i + 1\}$ while $k'_s = k_s - 1$ and $k'_{i+1} = k_{i+1} + 1$. Then $\sum_{s=0}^{j} k'_s = j - ik_i - (i + 1)k_{i+1} + (i(i + 1) + (i + 1)(k_{i+1} + 1) = j + 1$. 

Notice that, since $\hat{v}_n^{(j)}$, $j > 1$, is a horizontal vector, by the invariance of the unstable cone $\eta_n, j \in C_n$. Moreover $\gamma' \in C_{c}$ by assumption and $\|\eta_n^{j}\| = 1$, thus there exists $\vartheta \in (0, 1)$ such that

$$\|\eta_{n,j}^{\dagger} \gamma' \circ h_n\| \geq \vartheta \|\gamma' \circ h_n\| \geq \vartheta. \tag{3.29}$$

Using (3.29) and setting $R_{j,n} := \|R_j(F^n, \hat{\nu}_{n})\| + \|Q_j(h_n, \gamma)\|$, equation (3.28) yields

$$|h_n^{(j)}| \leq \frac{|h_n^{(j)}| \gamma(\gamma') \circ h_n + R_{j,n}}{\vartheta \|\gamma' \circ h_n\|}, \tag{3.30}$$

$$\|\eta_{n,j}\| \leq \|\gamma(\gamma') \circ h_n\| |h_n^{(j)}| + \|\gamma' \circ h_n\| |h_n^{(j)}| + R_{j,n}. \tag{3.31}$$

By equation (3.25) it follows that

$$\|\hat{\nu}_n\| = |h_n^{(j)}|(D_{\hat{\nu}_n} F^n)^{-1} \gamma' \circ h_n|| \leq \sqrt{\frac{\mu_{-n}}{\sqrt{1 + \chi_{c}^2}}} C_{*} \mu_{n} =: \hat{C}_{*} \mu_{n}. \tag{3.32}$$

Using this in (3.30) and observing that $\|\eta_{n,j}\| = \|D_{\hat{\nu}_n} F^n \hat{n}_{n}^{(j)}\| \geq \lambda_{n}^{-1} \|\hat{n}_{n}^{(j)}\|$, we obtain

$$\|\hat{n}_{n}^{(j)}\| \leq \|\gamma(\gamma') \circ h_n\| |(\lambda_{n}^{-1})(\hat{C}_{*} \mu_{n})^{j} A + R_{j,n}^{*}|| \leq 1, \tag{3.33}$$

$$\eta := (3^{(1 + \vartheta})(\hat{C}_{*} \mu_{n})^{j} \lambda_{n}^{-1})^{\frac{1}{2n}}. \tag{3.34}$$

Therefore we have

$$3^{j}(1 + \vartheta)(\hat{C}_{*} \mu_{n})^{j} \lambda_{n}^{-1} \leq \eta^{2n} \leq 1. \tag{3.35}$$

We are ready to conclude. For $j = 1$ the Lemma is trivial since $\|\hat{\nu}_n\| \leq \sqrt{1 + \chi_{c}^2}$ and $h_n^{(j)}$ can be bounded by (3.32), provided $C_{*} \geq \hat{C}_{*}$.

Equation (3.25) implies that $\|R_2(F^{2n}, \hat{\nu}_{2n})\| \leq C_{2}$ and $Q_2 = 0$, thus $R_{2,2n} \leq C_{2}$. Then the first of (3.30), remembering (3.26), and (3.31) together with equation (3.33) imply

$$h_n^{(j)} \leq C_{2} \hat{C}_{2} \mu_{n}^{2} \quad \forall n \leq 2 \bar{n}. \tag{3.36}$$

$$\|\hat{\nu}_n^{(j)}\| \leq A(\lambda_{n})^{-1} \left\{ \|\gamma(\gamma') \circ h_n\| (\hat{C}_{*} \mu_{n})^{2} + C_{2} \right\}. \tag{3.37}$$

Next, we proceed by induction on $j < \ell$ to prove that for each $\bar{n} \leq n \leq 2 \bar{n}$

$$\|h_n^{(j)}\| \leq C_{2} c_{*}^{(j-1)!} \mu_{n} \tag{3.38}$$

$$\|\hat{\nu}_n^{(j)}\| \leq \eta^{n} c_{*} + c/2^{(j-1)!}. \tag{3.39}$$

By (3.36) we have the case $j = 2$, let us assume it for all $s \leq j > 2$. Recalling the structure of $R_{j}, Q_{j}$, see after (3.27), and setting $c_{n} := \eta^{n} c_{*} + 2(1 - \eta^{n}) c_{*} \leq c_{*}$ we have

$$R_{j+1,n} \leq C_{2} \left\{ \sum_{k} \frac{\sum_{s=1}^{j}(s-1)!k_{s}}{c_{n}} + C_{2}^{j+1} \sum_{\sigma=2}^{j} \sum_{p} \left( \begin{array}{c} \sigma - 1 \sigma - 1 \end{array} \right) \mu_{n} \sum_{k=0}^{j+2-s} \mu_{k} \right\}. \tag{3.40}$$

\(^{15}\text{Note in particular that both } \bar{n} \text{ and } \eta \text{ depend only on the bound of the derivative of } F.\)
Note that \( \sum_{s=0}^{j} (s-1)!k_s \leq (j-2)! \sum_{s=1}^{j} s k_s = (j-2)!(j+1) \). If \( \sigma = j \), then
\[
(\sigma - 1)! + \sum_{s=1}^{j+1-\sigma} p_s s! = (j-1)! + j.
\]

On the other hand if \( \sigma < j \), then we have
\[
(\sigma - 1)! + \sum_{s=1}^{j+1-\sigma} p_s s! \leq (j-2)! + (j-\sigma)! j \leq (j-2)!(j+1).
\]

Accordingly, since the sums in \( k \) and \( p \) have at most \( j \) terms, setting \( \tau_j = \{(j-1)! + j, (j-2)! (j+1)\} \),
\[
R_{j+1,n} \leq C \left\{ j^{(j-2)!(j+1)} + j^{j+1} C^j_{(j+1)} \mu^n (j-1)! (j+1) \right\}
\]
(3.38)
\[
R_{j+1,n} \leq 3^{-j} \eta^{2\eta^{\alpha}} (\bar{C} \mu^n)^{-j-1} R_{j+1,n}.
\]

Let us show the first of (3.37). Substituting the above in the first of (3.30) and using (3.32) we have
\[
\|h_n^{(j+1)}\| \leq \frac{(\bar{C} \mu^n)^{j+1}}{\sigma} c_3 + C j^{j+1} \left\{ \left( \frac{1}{2} + C j^{j+1} C^j_{(j+1)} \right) \mu^n (j+1) \right\}.
\]

We can finally choose \( C_\alpha = \{2 C_3, 1\} \) and write
\[
\|h_n^{(j+1)}\| \leq C_\alpha c_\alpha \left\{ \frac{1}{2} + C_\alpha j^{j+1} C^j_{(j+1)} \right\} \mu^n (j+1).
\]

Note that for \( j = 3 \) we have \( \tau_3 = 5 \), which yields the wanted estimate if \( \varepsilon \geq C_3 2^{5} C_3^2 \). If \( j > 3 \), then \( \tau_j = (j-2)! (j+1) \) and the first of (3.37) follows. Next, we substitute (3.38) in (3.33) and, using (3.35), write
\[
\|\hat{\rho}_n^{(j+1)}\| \leq 3^{-j} \eta^{n^{2j}} \left\{ c_3 + C_\alpha j^{j+1} C_3^j_{(j+1)} (\bar{C} \mu^n)^{j+1} \right\}
\leq \left\{ \eta^{2j} / c_\alpha + \frac{C_\alpha}{c_\alpha} + \frac{C_3}{c_3} \right\}^{j+1} \mu^n (j+1).
\]

Observing that \( C_3^{2/3} \leq (\eta^\alpha c_\alpha + 2 \varepsilon)^{2/3} \leq \eta^{3\alpha/2} c_\alpha^2 + (2 \varepsilon)^{2/3} \) we have, for each \( j > 2 \),
\[
\|\hat{\rho}_n^{(j+1)}\| \leq \left\{ \eta^\alpha \left[ \frac{1}{3} + C_\alpha \eta^{3\alpha/2} c_\alpha^{-1} + c_\alpha \eta^{3\alpha/2} \mu^{3\alpha/2} C_3^{2/3} + C_\alpha \sqrt{\varepsilon} \right] \right\}^{j+1}
\]

Hence the second of (3.37) will follows if the term in the round brackets is smaller than \( c_\alpha + \varepsilon / 2 \). This is the case, provided we have chosen \( \tilde{n} \) large enough and
\[
\varepsilon \geq C_\varepsilon \{1, C_\alpha^{3/2} \mu^{3\alpha}, \mu^\alpha \}.
\]

In particular \( \hat{n} \in \Gamma_\ell (\varepsilon c_\alpha) \) for each \( \ell \leq r \) and \( \tilde{n} \leq n \leq 2 \tilde{n} \). Next, let \( c_{\alpha+1} = c_{\tilde{n}} \leq c_\alpha \), we have for each \( k \in \mathbb{N} \)
\[
\frac{\varepsilon}{2} \leq c_{\alpha,k} = \eta^\beta c_{\alpha,k-1} + \frac{\varepsilon}{2} \leq \eta^\beta c_\alpha + \frac{\varepsilon}{2(1 - \eta^\beta)}.
\]

\[\]
It follows that $\hat{\nu}_{k\bar{n}} \in \Gamma(\ell (c_*, k))$ where, for all $m \in \{\bar{n}, \ldots, 2\bar{n}\}$,

$$\hat{\nu}_{k\bar{n}+m} = h_{k\bar{n}+m-1}^* \circ \cdots \circ h_{k\bar{n}+1}^* \circ \hat{\nu}_{k\bar{n}} \circ h_{m,k+1}^*.$$ 

$h_{n,1}^* = h_n$, and

$$\|h_{m,(k+1)}^*\|_{C^j} \leq 2C_5c_*^{(j-1)!}\mu^{jm}.$$ 

Hence, applying iteratively the above argument to $\hat{\nu}_n$ for $k\bar{n} \leq n \leq (k+1)\bar{n}$, we obtain the second of (3.37) for each $n \geq \bar{n}$. It remains to prove the estimate for $h_n$, $n \geq \bar{n}$. We write $n = m + k\bar{n}$, $m \in \{\bar{n}, \ldots, 2\bar{n}\}$ and

$$h_n = h_{m,k+1}^* \circ h_{n,k}^* \circ \cdots \circ h_{n,1}^* = h_{m,k+1}^* \circ h_{k\bar{n}}.$$ 

Note that (3.32) yields $\|h_n\|_{C^j} \leq C_5 \mu^{n}$, provided we choose $C_5 \geq 3C_*$. It is then natural to start by investigating the second derivative. In fact, it turns out to be more convenient to study the following ratio

$$\frac{h''_n}{h'_n} = \log((h_{m,k+1}^* \circ h_{k\bar{n}}))' + \frac{h''_{k\bar{n}}}{h'_{k\bar{n}}} = Q_1 + Q_2.$$ 

Since (3.17) and (3.31) imply $|h_{n,i}^*| \geq c_0 \mu^{-n}$ for each $i$, for some constant $c_0$, formula (3.3) and (3.40) yield $\|\log(h_{m,k}^* \circ f)^{\epsilon} \leq C_5^{f^{(f-1)!}} \mu^{(f+1)m}$, provided $C_5$ has been chosen large enough. It then follows immediately that $\|Q_1\|_{C^0} \leq C_5C_5^2c_*\mu^{2m}\mu^{k\bar{n}} \leq C_5C_5^2c_*\mu^n$. To estimate $\|Q_2\|_{C^0}$ we write

$$\frac{h''_{k\bar{n}}}{h'_{k\bar{n}}} = \left(\prod_{i=1}^{k} h_{n,i}^* \circ h_{i\bar{n}}\right)' = \left(\log \prod_{i=1}^{k} h_{n,i}^* \circ h_{i\bar{n}}\right)' = \sum_{i=1}^{k} (\log h_{n,i}^* \circ h_{i\bar{n}})'.$$ 

Using formulae (3.32), (3.36) and (3.42) we have, since $\bar{n} \leq m$,

$$\|Q_1\|_{C^0} \leq C_5^2 \sum_{i=1}^{k} \|\log h_{n,i}^* \circ h_{i\bar{n}}\|_{C^1} \leq C_5^2 \sum_{i=1}^{k} \|\log h_{n,i}^*\|_{C^j} \|h_{i\bar{n}}\|_{C^0} \leq C_5^2 \sum_{i=1}^{k} \|\log h_{n,i}^*\|_{C^j} \|h_{i\bar{n}}\|_{C^0}$$

$$\leq C_5^2C_5^2c_*\mu^{2n} \sum_{i=1}^{k} \mu^{-n} \leq C_5^2C_5^2c_*\mu^{2n} \left(1 - \mu^{-k\bar{n}}\right) \mu^{-k\bar{n}}$$

$$\leq \mu^{2n}C_5^2c_*\mu^{k}\mu^{k\bar{n}} \leq C_5^2c_*\mu^{k}.$$ 

Hence, using the above and (3.32), it follows by (3.43)

$$\|h_n\|_{C^2} \leq C_5^2 \left(\log(h_n')\right)_{C^0} \|h'_n\|_{C^0} \leq C_5^2 \mu^n \left(\log(h_n')\right)_{C^0} \|h'_n\|_{C^0} \leq C_5^2c_*\mu^{n}.$$ 

This proves the second of (3.24). Next we prove the general case by induction on $j \leq \ell$. Assume it true for all $i \leq j$. Using again (3.3), by the inductive assumption we have

$$\|Q_1\|_{C^{j-1}} = \|\log((h_{m,k+1}^* \circ h_{k\bar{n}}))_{C^j} \leq C_5^{j+1}C_5^{m(j+1)!}c_*^{m(j+1)!}\mu^{k\bar{n}}.$$ 

17Recall the definition of $h_{n}^*$ in (3.6).
On the other hand, by formulae (3.43), (3.3) and the inductive assumption
\[ \|Q_2\|_{C_{j-1}} \leq C_2 \sum_{i=1}^{k} \| \log h_{n,i}^{\ast} \|_{C_{j}} \sum_{q=0}^{j-1} \| h_{nn} \|_{C_{j}}^{q} \]
(3.47)
\[ \leq C_2 C_b^{j+1+2(j+1)!} C^{(j+1)!n}_{\mu,j} \sum_{i=1}^{k} \sum_{q=0}^{j-1} (C^{\mu_{j,i}}_{\mu,n} h_{n}^{j+1})^{q} \]

To estimate the last sum, notice that by definition
\[ i \leq C_{\mu,n} \leq C_{\mu,n}, \quad \forall i \leq k, \]
\[ C_{\mu,n} \leq C_{\mu,n}, \quad \forall a > 1, \]

Hence,
\[ \sum_{i=1}^{k} \sum_{q=0}^{j-1} (C^{\mu_{j,i}}_{\mu,n} h_{n}^{j+1})^{q} \leq C_{\mu,n}^{j+1+2(j+1)!} C^{(j+1)!n}_{\mu,n} C_{\mu,n}^{j+1} h_{n}^{j+1} \]

Using this in (3.47) we obtain
\[ \|Q_2\|_{C_{j-1}} \leq C_2 C_b^{j+1+2(j+1)!} C^{(j+1)!n}_{\mu,n} C_{\mu,n}^{j+1} h_{n}^{j+1} \]

Therefore, by the inductive assumption, equations (3.46), (3.48) and (3.42), and provided we choose \( C_b \) large enough, we finally have\(^1\)
\[ \|h_n\|_{C_{j+1}} \leq C_2 \|h_n^{\ast}\|_{C_{j-1}} \leq C_2 \|h_n^{\ast}\|_{C_{j-1}} \|h_n\|_{C_j} \]
\[ \leq C_2 2^{j+2)!} (j+1)! C^{(j+1)!n}_{\mu,n} h_{n}^{j+1} \]

In Section 8 we will need much sharper estimates (but limited to the first derivatives) than the ones provided by Lemma 3.8; we prove them next.

**Lemma 3.9.** In the hypotheses of Lemma 3.8, there exist \( C_3, C_4, \bar{c}_3, \bar{c}_3, c_3, c_3 > 0 \) such that, for all \( n \in \{ \bar{n}, \ldots, \bar{c}_2 \ln \chi_n^{-1} \} \), setting \( a_{n *} = (\bar{c}_3)^{n_*}, c_{n *} = (\bar{c}_1)^{n_*} \) and\(^2\)
\[ b_{n_*} = (C_{4n_*} C_{3n_*})^{n_*} \]
\[ s_{n_*} = \{ C_{2n_*} C_{3n_*} C_{4n_*} C_{5n_*} C_{6n_*} \}^{n_*} \]
we have, for all \( n \geq \bar{n}, \)
\[ \|Q_n^{\ast}(t)\| \leq c_2 c_n^{a_{n_*}} h_{n_*}^{j+1} \lambda_n^{b_{n_*}} (\gamma \circ h_n(t))^{-1} + C_{\mu,n} \mu^{3n_*} \]
\[ \|Q_n^{\ast}(t)\| \leq c_3 h_{n_*}^{c_{n_*}} \lambda_n^{b_{n_*}} (\gamma \circ h_n(t))^{-1} + c_3 h_{n_*}^{c_{n_*}} \mu^{3n_*} \lambda_n^{b_{n_*}} (\gamma \circ h_n(t))^{-1} + c_3 + s_{n_*} \]

---

\(^1\) Here we are using the following elementary facts:
- \((j+1)! + (j+1)! + 2(j+1)! + 2(j+1)! \leq 2(j+2)!
- \( j(j+1)! + (j+1)! \leq (j+2)!
- \( a_j(j-1) + a_j + 1 = a_{j+1} \)

\(^2\) Recall (3.10) for the definition of \( c_{2} \) and (3.16) for the definition of \( c_{n} \).
**Proof.** To prove the first of (3.50) it is convenient to go back to equation (3.26) and, recalling (3.20), for each \( v \in \mathbb{R}^2 \), \( \|v\| = 1 \), we have

\[
\left| \langle v, \hat{\nu}_n'' \rangle - \langle v, \hat{\nu}_n' \rangle \frac{h''_n}{h'_n} \right| \leq \| \langle v, (D\hat{\nu}_n F^n)^{-1} \gamma'' \circ h_n(h'_n)^2 \rangle \|
\]

(3.51)  

\[
+ \sum_{k=0}^{n-1} \sum_{i=1}^{2} \left| \langle v, (D\hat{\nu}_n F^{k+1})^{-1} \left[ \partial_{x_i} D F^k (\hat{\nu}_n) F\right] D\hat{\nu}_n F^k \hat{\nu}_n' \rangle \right| \| (D\hat{\nu}_n F^k) \hat{\nu}_n' \| \| F \| c_2 \| (D\hat{\nu}_n F^k) \hat{\nu}_n' \|^2.
\]

Note that, recalling (3.10), for each \( n \leq n_\ast \leq \tilde{c}_2 \log \chi_n^{-1} \) we have \( (D\hat{\nu}_n t F^n)^{-1} e_1 \notin C_c \). Consequently

(3.52)  

\[ \left| \langle v, (D\hat{\nu}_n F^n)^{-1} \gamma'' \circ h_n \rangle \right| \leq (\lambda_n^- (\hat{\nu}_n(t)))^{-1} \| \gamma'' \circ h_n(t) \|, \quad \forall n \leq n_\ast. \]

Next, if \( v \) is perpendicular to \( \hat{\nu}_n' \), then it must be \( |v_2| \leq \chi_c |v_1| \), hence

(3.53)  

\[ \left| \langle v, \hat{\nu}_n'' \rangle \right| = |v_1| \| \hat{\nu}_n'' \| \geq (1 + \chi_c^2)^{-\frac{1}{2}} \| \hat{\nu}_n'' \|. \]

On the other hand, if \( v \) is perpendicular to \( \hat{\nu}_n'' \), then \( v = e_2 \) and \( \| \langle v, \hat{\nu}_n' \rangle \| = 1 \). Accordingly, recalling Proposition 3.7 and equations (2.5), (3.32) we have for \( n \leq n_\ast \)

(3.54)  

\[
\| \hat{\nu}_n''(t) \| \leq (1 + \chi_c^2)^{-\frac{1}{2}} (\lambda_n^- (\hat{\nu}_n(t)))^{-1} C_5^2 \mu_2^n \| \gamma'' \circ h_n(t) \| + \sum_{k=0}^{n-1} C_3^* \mu_3^k C_2^*.
\]

Setting \( c_{n*} = \left[ (1 + \chi_c^2)^{-\frac{1}{2}} C_5^2 \right]^{-\frac{1}{2}} \) we obtain

(3.55)  

\[
\| \hat{\nu}_n''(t) \| \leq c_{n*} \mu_2^n (\lambda_n^- (\hat{\nu}_n(t)))^{-1} \| \gamma'' \circ h_n(t) \| + \sum_{k=0}^{n-1} C_3^* \mu_3^k C_2^*.
\]

We can now proceed by induction since, setting \( h_{l,m}^* = h_{l,m} \circ h_{l,m}^{-1} \), if \( n = l n_\ast + m, \) \( m \leq n_\ast \), then

(3.56)  

\[
\| \hat{\nu}_n''(t) \| \leq c_{n*} \mu_2^m (\lambda_m^- (\hat{\nu}_m(t)))^{-1} \| \hat{\nu}_m'' \circ h_{l,m}^{-1}(t) \| + \sum_{k=0}^{n_\ast-1} C_3^* \mu_3^k C_2^*
\]

\[
\leq c_{n*} \mu_2^m (\lambda_m^- (\hat{\nu}_m(t)))^{-1} \| \hat{\nu}_m'' \circ h_{l,m}(t) \| \cdots \| \lambda_2^- (\gamma \circ h_n(t)) \|^{-1} C_*
\]

\[
+ \sum_{s=1}^{l} c_{n*} \mu_2^s \lambda^{-s} \sum_{k=0}^{n_\ast-1} C_3^* \mu_3^k C_2^*
\]

\[
\leq c_{n*} \mu_2^m c_{n*} \lambda^m (\lambda_n^+ (\gamma \circ h_n(t)))^{-1} C_* + C_{\mu,n*} \mu_3 n_\ast
\]

where \( c_{n*} \) is the constant implicit in (3.17). It remains to bound the third derivative of \( \hat{\nu}_n \). The strategy is basically the same. Recalling that \( \hat{\nu}_n' = (D\hat{\nu}_n F^n)^{-1} \gamma' \circ h_n h'_n, \) we
differentiate this expression twice and multiply by a unitary vector \(v\) orthogonal to \(\hat{v}'\):

\[
\langle \hat{v}''', v \rangle = \left(\left[(D_{\hat{v}'''} F^n)^{-1}\right]'\right)''' \circ h_n h_n' + 2\left[(D_{\hat{v}'''} F^n)^{-1}\right]'\right)'' \circ h_n (h_n')^2 + \gamma' \circ h_n h_n'' \\
(3.57)
\]

We will estimate the norms of the terms in the first line of the above equation one at a time, for each \(n \leq n_*\). First, using (3.18) with \(m = 0\) and \(\epsilon = \|\hat{v}''\|\) (where the latter is estimated using (3.54)), and (3.24) we have, for some \(A_1 > 0\)

\[
\|\left[(D_{\hat{v}'''} F^n)^{-1}\right]'\right)''' \circ h_n h_n'\| \leq \mu^2 \nu_n + A_1 C_5 \nu_n c_n^3 \mu^3 (\lambda_n, (\hat{v}_n(t)))^{-1} \|\gamma'' \circ h_n(t)\| \\
+ C_4 C_5 \nu_n c_n^3 \mu^3 n^3.
\]

Next, notice that \((D_{\hat{v}'''} F^n)^{-1}\gamma'' \notin C_{\epsilon n}\), hence by the second of (3.20) and subsequent, there is \(A_2 > 0\) such that

\[
(3.58) \quad \|([D_{\hat{v}'''} F^n]^{-1})'\gamma'' \circ h_n(h_n')^2\| \leq A_2 C_5^2 \mu^3 (\lambda_n, (\hat{v}_n(t)))^{-1} \nu_n \|\gamma'' \circ h_n(t)\|
\]

It is convenient to write the third term as

\[
[D_{\hat{v}'''} F^n]^{-1}' \gamma' \circ h_n h_n' = \frac{h_n''}{h_n} (D_{\hat{v}'''} F^n)^{-1} \gamma' \circ h_n h_n' \\
= \frac{h_n''}{h_n} \left(\hat{v}_n'' - (D_{\hat{v}'''} F^n)\gamma' \circ h_n (h_n')^2 - \hat{v}_n' h_n''\right).
\]

The last term vanishes when we multiplied by \(v\); hence, by (3.52) and (3.54), we have\footnote{Recall also the lower bound for \(|h_n''|\) in (3.32).}

\[
\left|\langle (D_{\hat{v}'''} F^n)\gamma' \circ h_n h_n', v \rangle\right| \leq \frac{h_n''}{h_n} \left\{\|\hat{v}_n''\| + \|D_{\hat{v}'''} F^n\|^{-1} \|\gamma'' \circ h_n (h_n')^2\|\right\} \\
\leq c_n^3 C_5^2 \mu^4 (\lambda_n, (\hat{v}_n(t)))^{-2} \|\gamma'' \circ h_n(t)\| + c_n^3 \nu_n \mu^5 (\lambda_n, (\hat{v}_n(t)))^{-1} \|\gamma'' \circ h_n(t)\| + C_5^2 \mu^6 n.
\]

For the two terms in the second line of (3.57), when the matrix hits \(\gamma''\) or \(\gamma'''\), we can use (3.52) for \(n \leq n_*\) and (3.24) with \(\|\gamma' \circ h_n(t)\|\) instead of \(c_*\). Collecting all the above estimates in (3.57) we finally have, recalling also (3.53),

\[
(1 + \chi^2)^{\frac{1}{2}} \hat{v}_n''' \leq C_5 \mu^2 \nu_n + A_1 C_5 \nu_n c_n^3 \mu^3 (\lambda_n, (\hat{v}_n(t)))^{-1} \|\gamma'' \circ h_n(t)\| \\
+ C_5 \nu_n \mu^3 C_\mu \mu^3 n^3 \\
+ A_2 C_5^2 \mu^3 \nu_n (\lambda_n, (\hat{v}_n(t)))^{-1} \|\gamma'' \circ h_n(t)\| \\
+ c_n^3 C_5^2 \mu^4 (\lambda_n, (\hat{v}_n(t)))^{-2} \|\gamma'' \circ h_n(t)\| + c_n^3 \nu_n \mu^5 (\lambda_n, (\hat{v}_n(t)))^{-1} \|\gamma'' \circ h_n(t)\| + C_5^2 \mu^6 n \\
+ C_5^3 \mu^3 (\lambda_n, (\hat{v}_n(t)))^{-1} \|\gamma'' \circ h_n(t)\| \\
+ (\lambda_n, (\hat{v}_n(t)))^{-1} C_5^2 \mu^3 n - \gamma' \circ h_n(t))\|
\]

Hence, setting \(\tilde{a}_n = [(1 + \chi^2)^{1/2} C_5^3]^{1/n}, \tilde{b}_n = [(A_1, A_2)^{1/2} + (1 + \chi^2)^{1/2} C_5 \nu_n]^{1/n}, c_n\), and recalling the second of (3.49) we

\[
\hat{v}_n''' \leq \tilde{a}_n^3 \mu^3 (\lambda_n, (\hat{v}_n(t)))^{-1} \|\gamma'' \circ h_n(t)\| + \tilde{c}_n^3 \mu^4 (\lambda_n, (\hat{v}_n(t)))^{-2} \|\gamma'' \circ h_n(t)\| \\
+ \tilde{b}_n^3 (\lambda_n, (\hat{v}_n(t)))^{-1} \|\gamma'' \circ h_n(t)\| + \lambda_n (\hat{v}_n(t))^{-1} c_n^3 \mu^3 C_\mu \mu^3 n - \gamma' \circ h_n(t) + s_n.
\]
We can now iterate as in (3.56), using the latter to estimate the terms involving $\gamma''$ and $\gamma$ and, proceeding by induction, we obtain\footnote{Here we use again that $\mu^t \lambda^{-1} < 1.$}
\[
\|\nu''\| \leq a_n^t \mu^{3n} (\lambda_\nu (\gamma \circ h_n))^{-1} c_n^t + b_n^t \mu^{3n} (\lambda_\nu (\gamma \circ h_n))^{-1} c_n + s_n,
\]
from which the third of (3.59) follows setting $C_4 = \{A_1, A_2\}^+, \bar{c}_1 = \bar{c}_0 (1 + \lambda_\nu^2)^2 C_2^3, \bar{c}_3 = \bar{c}_0 (1 + \lambda_\nu^2)^2 C_3^3$, and the Lemma is proved. \hfill $\Box$

Lemma 3.8 and 3.9 imply immediately the following important result.

**Corollary 3.10.** If $n > \bar{n}$ and $\eta^n \leq \frac{1}{2}$, then $F^{-n} \Gamma_{\ell}(\bar{s}) \subset \Gamma_{\ell}(\bar{s})$. In addition, if $\ell \in \{2, 3\}$, then we can choose $\bar{s} = s_{\bar{n}} = \{\mu^{2n}, c_0^{2n}, C_0, C_{\mu, n}, \mu^{4n}, C_2^0, \mu^{6n}\}^+.$

From now until the end of this section we fix $\bar{n}$ as in Lemma 3.8.

The above results tell us that the space of admissible central curves is stable under backward iteration of the map. Arguing as above, but forward in time, we can prove that the space of admissible unstable curves is stable under the iteration of $F^n$, for $n$ greater than $\bar{n}$. In particular, if $\eta : I \to T^2$ is an admissible unstable curve, and $\eta_n$ is the image of $\eta$ under $F^n$, then there exists a diffeomorphism $p_{n, t} = p_n$ such that
\[
\pi \circ \gamma(t) = \frac{D_{\eta_n} F^n \cdot \eta(t)}{\|\eta(t)\|},
\]
and $\eta_n \circ p_n = F^n \circ \eta$ is an admissible unstable curve. Moreover, as $F$ acts as an expanding map along those curves, we have the following standard distortion estimate for each $n \geq 1$:
\[
\frac{p_n'(t)}{p_n'(s)} \leq 1, \quad \forall t, s \in I.
\]

In the following we will need to control the evolution also of curves not in the center cone. To this end it is convenient to introduce a further quantity. Given a smooth curve $\gamma$ such that $\pi_1 \circ \gamma(t) \neq 0$ for each $t \in T$, let
\[
\vartheta_\gamma(t) = \left\{ \frac{\pi_2 \circ \gamma(t)}{\pi_1 \circ \gamma(t)}, \chi_u \right\}^+ \quad \vartheta_\gamma = \inf_t \{\vartheta_\gamma(t)\}.
\]

**Lemma 3.11.** Let $F$ be a SVPH. For all $\gamma \in C^r \subset C^r$ closed curve homotop to $0, 1$ with $\|\gamma''\| = 1$ and $\|\gamma''(t)\| \leq \Delta_\gamma(t),$\footnote{We will apply this Lemma with $\Delta_\gamma(t)$ given by (E.1).} for all $j \in \{1, \ldots, r\}$ and $t \in T$, if $\bar{b} \in \bar{H}^\infty$, $m \geq \bar{m}$ and $m \geq \bar{m}_n > 0$ are the smallest integers such that, for all $t \in T$, \[
D_{\gamma(t)} h_{m} \gamma(t) \notin C_u \quad \text{and} \quad D_{\gamma(t)} h_{m} \gamma(t) \in \text{Int}(C_c)
\]
then
\[a)\] Let $\eta < 1,$ given inLemma 3.11,\footnote{See (3.34) for a precise definition of $\eta.$} and $\Lambda$ as in (3.4); setting $\bar{m} = \sigma m$, where
\[
\sigma = \left\lceil \frac{\ln \Lambda^{-1}}{\ln \eta} \right\rceil,
\]
for each curve $\nu_F \in F^{-\bar{m}} \gamma$, there exists a diffeomorphism $h_F$ such that $h_F := \nu_F \circ h_F \in \Gamma_\nu(\bar{s})$ and $\gamma - \text{norm of } h_F$ satisfies (3.24) with $c_* = \chi_u^{-1} \|\Delta_\gamma\|_{\infty} (\mu^t \Lambda)^m$.

In the case $j \in \{1, 2\}$ we have the following sharper version:
b) For each \( p \in \gamma \) and \( n_* \in \{ \bar{n}, \cdots, \bar{c}_2 \log \chi_u^{-1} \} \), let \( \mathbf{m}(p, b) \equiv \mathbf{m} \) be the minimum integer such that

\[
\eta_{n_*}(\mathbf{m}, m; t)M_{n_0}(m, t) \leq C_{n_*}\mu^{3n_*},
\]

\[
\tilde{\eta}_{n_*}(\mathbf{m}, m; t)\overline{M}_{n_0}(m, t) = s_{n_*},
\]

where

\[
\eta_{n_*}(\mathbf{m}, m; t) := c_0\{b_{n_*}, c_{n_*}\}^\perp \mu^3\nu_{n_0}C_{\nu_{n_0}}(\mathbf{m} + h_{\mathbf{m}}(t))^{-1},
\]

\[
\tilde{\eta}_{n_*}(\mathbf{m}, m; t) := c_{0\nu_{n_*}}\mu^3\nu_{n_0}C_{\nu_{n_0}}(\mathbf{m} + h_{\mathbf{m}}(t))^{-1},
\]

\[
M_{n_0}(m; t) := \{ \Lambda^{2n_0}\mu^m\Delta_{\gamma}(t), (1 + \mu^{2m}\varphi_{\nu_{n_0}}(t)\|C_2\Gamma\nu_{n_0}(\mathbf{m} + h_{\mathbf{m}}(t))) \}
\]

\[
\overline{M}_{n_0}(m, t) := \{ \mu^m\Lambda^{3n_0}\Delta_{\gamma}^2(t), M_{n_0}(m, t)\varphi^{-1}_{\nu_{n_0}}, \varphi^{-2}_{\nu_{n_0}}, \nu_{n_0}(\mathbf{m} + h_{\mathbf{m}}(t)) \}.
\]

and \( a_{n_*}, b_{n_*}, c_{n_*}, s_{n_*} \) are defined in Lemma 3.9. Then \( \nu_{\mathbf{m}} \in \Gamma_3(\mathbf{z}) \) and

\[
C_4\nu_{\mathbf{m}}^{-1}\varphi_{\nu_{n_0}}(t)^{-1}\mu^m \leq |h_{\mathbf{m}}(t)| \leq C_4\nu_{\mathbf{m}}^{-1}\varphi_{\nu_{n_0}}(t)^{-1}\mu^m
\]

\[
|h_{\mathbf{m}}^\prime(t)| \leq C_4\nu_{\mathbf{m}}^{-1}\varphi_{\nu_{n_0}}(t)^{-1}\mu^m
\]

Proof. We set \( \Delta_{\gamma} := \|\Delta_{\gamma}\| \). Let us start proving item a) first. Let \( \mathbf{h} \in \mathcal{H}_Z \) such that \( \nu_{\mathbf{m}} = \mathbf{h}_\gamma \). Recalling (3.4), we can apply (3.3) and we have for each \( j \leq r \)

\[
\|\nu_{\mathbf{m}}\|_{C^{j+1}} = \|h_{\mathbf{m}} \circ \gamma\|_{C^{j+1}} \leq C_4(\Delta_{\gamma}\mu^m)^j.
\]

We set \( \phi(t) := (\pi_2 \circ \nu_{\mathbf{m}})(t) \). By (3.62) there exists \( c_{u, \gamma} \geq \chi_u \mu^{-m} \) such that we have \( |\phi'| > c_{u, \gamma} \) for \( u, \gamma > 0 \), so it is well defined the diffeomorphism \( \mathbf{h}_{\mathbf{m}}(t) = \phi^{-1}(t) \), so that \( \nu_{\mathbf{m}} = \nu_{\mathbf{m}} \circ \mathbf{h}_{\mathbf{m}} \) is parametrized by vertical length. We want to estimate the higher order derivatives of \( \mathbf{h}_{\mathbf{m}} \) using a formula for inverse functions given in [37]. For the reader convenience we write it down here for our case:

\[
h_{m}^{(j+1)}(t) = \frac{d^{j+1}\phi^{-1}(t)}{d\phi^{-1}^2} = \sum_{k=0}^{j} \frac{\phi'(t)^{-j-k-1}}{k!} \sum_{b_1+\cdots+b_k=j+k \atop b_i \geq 2} B_{j,k,\{b_i\}}^k \prod_{l=1}^{k} \phi(b_l)(t),
\]

where \( B_{j,k,\{b_i\}}^k = \frac{(j+k)!}{k!b_1!\cdots b_k!} \). It follows by (3.67) and (3.68) that for each \( t \)

\[
|h_{\mathbf{m}}^{(j+1)}(t)| \leq C_4(\Delta_{\gamma}^{2} \mu^m)^j.
\]

By (3.67), (3.69) and formula (3.3) for the composition,

\[
\|\nu_{\mathbf{m}}\|_{C^{j+1}} = \|h_{\mathbf{m}} \circ \gamma\|_{C^{j+1}} \leq C_4 \sum_{k=0}^{j+1} \|\nu_{\mathbf{m}}\|_{C^k} \sum_{k \in \mathcal{K}_{s, \gamma}, t \in \mathbb{N}} \|h_{\mathbf{m}}\|_{C^t}^{-1}
\]

\[
\leq C_4(\Delta_{\gamma}\mu^{2m} \mu^m)^j \leq (\Delta_{\gamma}\mu^{m})^{(j+1)!},
\]

where \( \Delta_{\gamma} = \{ \Delta_{\gamma}^2, \gamma \} \). Hence, setting \( \mathbf{c}_{s}(m) = \Delta_{\gamma}\Delta_{\gamma} \mu^m \) we have that \( \nu_{\mathbf{m}} \in \Gamma_j(\mathbf{c}_{s}(m)) \). Since \( \mathbf{m} > \mathbf{n} \) we can apply Lemma 3.8 and we have that the curve \( \nu_{\mathbf{m}} = \nu_{\mathbf{m}} \circ \mathbf{h}_{\mathbf{m}} \) belongs to \( \Gamma_j(\mathbf{c}_{s}(m) + \frac{\mathbf{n}}{2}) \). The statement follows choosing \( \mathbf{m} = \mathbf{m} \) with \( s \geq 2 \Delta_{\gamma}\Delta_{\gamma} \). Let us prove item b). Let \( \nu_{\mathbf{n}} = \mathbf{h}_{n} \circ \gamma \) for each \( n \in \mathbb{N} \). Then, \( C_2 \nu_{\mathbf{n}}(t) = \nu_{\mathbf{n}}(h_{\mathbf{n}}(t)) \geq |\pi_2 \circ \nu_{\mathbf{n}}(t)| \geq \varphi_\gamma(t)|\pi_1 \circ \nu_{\mathbf{n}}(t)| > 0 \), and we can reparametrize \( \nu_{\mathbf{n}}, n \geq \mathbf{n}_0 \), by vertical length \( \nu_{\mathbf{n}}(t) = \nu_{\mathbf{n}}(h_{\mathbf{n}}(t)) \). Note that \( |\nu_{\mathbf{n}}(t)| \leq C_2 \nu_\gamma(t)^{-1} \).
If \( n_0 = 0 \), then \( C_2 \gamma \vartheta_{\rho_0}(t)^{-1} = C_2 \gamma \circ h_0(t)^{-1} \leq |h_0'(t)| \leq C_2 \gamma \vartheta_{\rho_0}(t)^{-1} \) and (3.51) yields
\[
|h''_{\rho_0}(t)| \leq \left[ \frac{\| \gamma'' \circ h_{\rho_0}(t) \| |h'_{\rho_0}(t)|^2}{\langle c_1, \nu_{\rho_0}(t) \rangle} \right] \leq C_2 \Delta \gamma \circ h_{\rho_0}(t) \partial_{\nu_{\rho_0}}(t)^{-2}
\]
\[
\|h''_{\rho_0}(t)\| \leq C_2 \Delta \gamma \circ h_{\rho_0}(t) \partial_{\nu_{\rho_0}}(t)^{-1}.
\]
If \( n_0 > 0 \), then \( C_2 A^{-n_0} \partial_{\nu_{\rho_0}}(t)^{-1} \leq |h'_{\rho_0}(t)| \leq C_2 A^{n_0} \partial_{\nu_{\rho_0}}(t)^{-1} \), \( \|h''_{\rho_0}(t)\| \leq C_2 A^{2n_0} \| \gamma''(t)\| \) and
\[
|h''_{\rho_0}(t)| \leq C_2 A^{3n_0} \Delta \gamma \circ h_{\rho_0}(t) \partial_{\nu_{\rho_0}}(t)^{-2}
\]
(3.71)
\[
\|\hat{\nu}''_{\rho_0}(t)\| \leq C_2 A^{2n_0} \Delta \gamma \circ h_{\rho_0}(t) \partial_{\nu_{\rho_0}}(t)^{-1}
\]
\[
\|\hat{\nu}'''_{\rho_0}(t)\| \leq C_2 A^{3n_0} \Delta^2 \gamma \circ h_{\rho_0}(t) \partial_{\nu_{\rho_0}}(t)^{-2}.
\]
Remark that
\[
\| (D_{\hat{\nu}_m}(F^k) \hat{\nu}'_m(t) \| \leq \sqrt{1 + \chi_2^2 C_2 \lambda_k^+(\hat{\nu}_m(t))},
\]
and, setting \( F^{m-n_0} \hat{\nu}_m = \hat{\nu}_m \circ \hat{h}_{m-n_0} \), we have
\[
|\hat{h}'_{m-n_0}(t)| = |\langle e_2, D_{\hat{\nu}_m}(F^{m-n_0} \hat{\nu}'_m(t)) \rangle| \leq C_2 \lambda_k^+ \lambda_{m-n_0} \partial_{\nu_{\rho_0}}(\hat{h}_{m-n_0}(t))
\]
\[
|\hat{h}'_{m-n_0}(t)| \geq C_2 \lambda_k^- \lambda_{m-n_0} \partial_{\nu_{\rho_0}}(\hat{h}_{m-n_0}(t)).
\]
Next, we want to use equation (3.51), with \( \gamma \) replaced by \( \hat{\nu}_m \). Note that there exists \( \xi_t \in C^{r-2} \), \( \| \xi_t \|_{C^{r+2}} \leq C_2 \), such that, for all \( w \in \mathbb{R}^2 \),
\[
\| \partial_{\nu} (D_{\hat{\nu}_m} F^k \hat{\nu}'_m(t) \| \leq C_2 \| w \|_{C^2}.
\]
In addition, it must be \( (D_{\hat{\nu}_m} F^k)^{-1} e_1 \not\in C_c \), for all \( k < m - n_0 \), otherwise, by the monotonicity of the dynamics in the tangent bundle, it would be that \( \hat{\nu}_{m-1} \in C_c \) contrary to the hypothesis. Accordingly, recalling (2.3), (3.18), (3.17) and setting \( m_0 = m - n_0 \),
\[
\| (D_{\hat{\nu}_m} F^{k+1})^{-1} \left[ \partial_{\nu} (D_{F^k} \hat{\nu}_m) F \right] D_{\hat{\nu}_m} F^k \hat{\nu}'_m(t) \| \leq \frac{\lambda_k^+(\hat{\nu}_m(t))}{\lambda_{k+1}^+(\hat{\nu}_m(t))} + C_2 \mu^{k+1} \| \omega \|_{C^2} \lambda_k^+(\hat{\nu}_m(t))
\]
\[
\leq C_2 \left( 1 + \mu^{k+1} \lambda_k^+(\hat{\nu}_m(t)) \| \omega \|_{C^2} \right).
\]
Arguing as in the proof of Lemma 3.8, just after (3.51), the above and (3.71) yields,
\[
\| \hat{\nu}''_m(t) \| \leq C_2 \lambda_{m_0}^+(\hat{\nu}_m(t)) \partial_{\nu_{\rho_0}}(\hat{h}_{m_0}(t)) \lambda_{m_0} \Delta \gamma \circ h_{m_0}(t)
\]
\[
+ \sum_{k=0}^{m_0-1} C_2 \left\{ 1 + \mu^k \lambda_k^+(\hat{\nu}_m(t)) \| \omega \|_{C^2} \right\} \lambda_k^+(\hat{\nu}_m(t))
\]
\[
|\hat{h}''_{m_0}(t)| \leq C_2(\lambda_{m_0}^+(\hat{\nu}_m(t)))^2 \partial_{\nu_{\rho_0}}(\hat{h}_{m_0}(t)) \lambda_{m_0}^2 \Delta \gamma \circ h_{m_0}(t)
\]
\[
+ \sum_{k=0}^{m_0-1} C_2 \left\{ 1 + \mu^k \lambda_k^+(\hat{\nu}_m(t)) \| \omega \|_{C^2} \right\} \lambda_k^2(\hat{\nu}_m(t))^2 \partial_{\nu_{\rho_0}}(\hat{h}_{m_0}(t)).
\]
To continue we need the following

**Sublemma 3.12.** If \( m_0 \) is the smallest integer for which \( \hat{\nu}'_{m_0}(t) \not\in C_c \) for each \( t \), then
\[
\chi_u \lambda_{m_0}^+(\hat{\nu}_m(t)) \leq C_2 \chi_c^{-1} \mu^{m_0}, \quad \forall t \in \mathbb{T}^2.
\]
\[^{24}\text{Note that (3.51) holds also if } \gamma \text{ is not parametrized vertically.}\]
Proof. If we define $w$, $||w|| = 1$, such that $DF^m_0w = ||DF^m_0w||_e$, then $\hat{\nu}_m = \alpha e_1 + \beta w$, with $c_2 \leq |\alpha|, |\beta| \leq C_4$. Then, since $D\nu_m F^m_0e_1 \in C_u$, $w \in C_c$ and using (2.3) again,

$$
C_2\lambda_m^m \circ \hat{\nu}_m \geq |\langle e_1, D\nu_m F^m_0\hat{\nu}_m' \rangle| \geq C_2\lambda_m^m \circ \hat{\nu}_m \geq 1, \quad |\langle e_2, D\nu_m F^m_0\hat{\nu}_m' \rangle| \leq C_2(\mu_m^m + \lambda_m^m \circ \hat{\nu}_m \chi_u).
$$

Next, let $v \in \mathbb{R}^2$, $||v|| = 1$ such that $DF^m_0v = ||DF^m_0v||(1, \chi_u)$. Note it must be $v \notin C_c$, otherwise we would have $\hat{\nu}_m \in C_c$, contrary to the hypothesis. We can then write again $v = \alpha e_1 + \beta v$. Note that $w \in (DF)^{-1}C_c$, moreover the uniform cone contraction implies that there exists $\theta_* \in (0, 1)$ such that, for all $p \in \mathbb{T}^2$, $D\nu F C_u \subset \{(x, y) \in \mathbb{R}^2 : |y| \leq \theta_* \chi_u|x|\}$ and $(DF)^{-1}C_c c \subset \{(x, y) \in \mathbb{R}^2 : |x| \leq \theta_* \chi_c|y|\}$. It follows $|w_1| \leq \theta_* \chi_c|w_2|$ while $|v_1| \geq \chi_c|v_2|$, thus $v_2 = bw_2$ and

$$
|a| \geq \chi_c|v_2| - |bw_1| \geq \chi_c(1 - \theta_\ast)|b||w_2| \geq \chi_c(1 - \theta_\ast)(1 + \chi_2^2\theta_*^2)^{-\frac{1}{2}}|b|
$$

which implies $|b| |a| \leq C_2 \chi_c^{-1}$. Finally, by equations (2.3) and (3.17), we can write

$$
\chi_u = \left|\frac{\langle e_2, D\nu F^m_0v \rangle}{\langle e_1, D\nu F^m_0v \rangle}\right| \leq \frac{|b|\mu_m^m + |a|}{|a|} \frac{|\langle e_2, D\nu F^m_0e_1 \rangle|}{|\langle e_1, D\nu F^m_0e_1 \rangle|} \leq C_2 \frac{|b|}{|a|} \mu_m^m (\lambda_m^m \circ \hat{\nu}_m)^{-1} + \theta_* \chi_u,
$$

that is (3.75). \qed

By the above Sub-Lemma it follows that

$$
\chi_c \geq \theta_{\hat{\nu}_m}(t) \geq \mu_m^{-m} \lambda_m^m \circ \hat{\nu}_m \theta_{\hat{\nu}_m}(t).
$$

Thus

$$
\|\hat{\nu}_m''(t)\| \leq C_2 \lambda_m^m \mu_m^m \delta_\gamma \circ h_m(t) + C_2 \left\{1 + \mu_m^m \delta_\gamma^{-1} \|\omega\|_c^2\right\} \lambda_m^m (\hat{\nu}_m(t)),
$$

$$
\|\hat{h}_m''(t)\| \leq C_2 \lambda_m^m \mu_m^m \delta_\gamma \circ h_m(t) + C_2 \left\{1 + \mu_m^m \delta_\gamma^{-1} \|\omega\|_c^2\right\} \lambda_m^m (\hat{\nu}_m(t)) \mu_m^m
$$

To estimate $\hat{\nu}_m''$, we use (3.57) where $\hat{\nu}_m, \gamma, h_m$ are replaced by $\hat{\nu}_n, \gamma_n, h_m$. In this case the curve $\hat{\nu}_n \notin C_c$, and so is $\hat{h}_n(t)$ for each $k < m_0, k \in \mathbb{N}$. Therefore, using Proposition 3.7, we have the following estimates

$$
\|\left[(D\nu_m F^m_0)^{-1}\right]'\hat{\nu}_m''(t)\| \leq \mu_m^{-m} (\hat{\nu}_m(t) + \chi_m \hat{\nu}_m''(t)) \|\hat{\nu}_m''\| \|\hat{h}_m''\|,
$$

$$
\|\left[(D\nu_m F^m_0)^{-1}\right]'\hat{\nu}_m''(t)\| \leq \lambda_m^m (\hat{\nu}_m(t))^{-1} \mu_m^{-m} \chi_m \|\hat{h}_m''\|.
$$

Additionally, again by Proposition 3.7,

$$
\|\left[(D\nu_m F^m_0)^{-1}\right]'\hat{\nu}_m''(t)\| \leq \mu_m^{-m} \chi_m \|\hat{\nu}_m''\| \|\hat{h}_m''\|,
$$

$$
\|\left[(D\nu_m F^m_0)^{-1}\right]'\hat{\nu}_m''(t)\| \leq \mu_m^{-m} \chi_m \|\hat{\nu}_m''\| \|\hat{h}_m''\|,
$$

Using the above estimates in (3.57) and recalling (3.72), (3.78), and (3.71) we conclude

$$
\|\hat{\nu}_m''\| \leq C_4 M_m (m, t) \delta_{\hat{\nu}_m}^{-1} \left[\mu_m^m \chi_m + \mu_m^m \lambda_m^m \Delta_\gamma \circ \hat{h}_m(t)\right]
$$

$$
+ C_5 \mu_m^{-2} \mu_m^m \lambda_m^m \Delta_\gamma \circ \hat{h}_m(t) + C_4 \mu_m^m \chi_m \lambda_m^m (\hat{\nu}_m(t)) \delta_{\hat{\nu}_m}^{-1} \leq A_0 \overline{M}_m (m, t),
$$

where $A_0 \overline{M}_m (m, t)$ is the right hand side of (3.71).
for some $A_0 > 0$. Next we set $\bar{m} \equiv \bar{m}(\bar{h}, p)$ and $F^{\bar{m}_n}\hat{\nu}_m = \hat{\nu}_m \circ \bar{h}_{\bar{m}_n-m}$. First,
\begin{align}
|\hat{h}'_{\bar{m}_{n-m}}(t)| &= |(e_2, D_{\bar{m}_{n-m}}F^{\bar{m}_n}\hat{\nu}_m(t))| \leq C_2\chi_c^{-1}\mu^{-m} \\
|\hat{h}_{\bar{m}_{n-m}}(t)| &\geq C_1\chi_c^{-1}\mu^{-m}.
\end{align}
\hspace{1cm} (3.79)

We can now apply Lemma 3.8, in particular (3.50), to $\hat{\nu}_m$ and $h_{\bar{m}_n-m}$ with $\gamma$ replaced by $\hat{\nu}_m$, and $c_*$ and $c_*^2$ replaced by $M_{n_0}(m, t)$ and $\bar{M}_{n_0}(m, t)$ respectively, defined in (3.65).

We thus obtain
\begin{align}
\|\hat{\nu}_m''\| &\leq c_4c_n\mu^{-m}\lambda_n^+(\hat{\nu}_m \circ \hat{h}_{\bar{m}_{n-m}})^{-1}M_{n_0}(m, \cdot) + C_{\mu_n, \mu}^3n_* \\
\|\hat{\nu}_m''\| &\leq c_4c_n\mu^{-m}\lambda_n^+(\hat{\nu}_m \circ \hat{h}_{\bar{m}_{n-m}})^{-1}M_{n_0}(m, \cdot) + b_n^{-1}\mu^{-m}\lambda_n^+(\gamma \circ h_{n_*})^{-1}M_{n_0}(m, \cdot) + s_{n_*} \\
|\hat{h}_{\bar{m}_{n-m}}| &\leq C_3M_{n_0}(m, \cdot)\mu^{-2\mu\bar{m}M_{n_0}(m)}.
\end{align}
\hspace{1cm} (3.80)

We are ready to conclude. Recalling Corollary 3.10, the first two of the above equations plus condition (3.64) give $\bar{h}_{m} \in \Gamma_{3}(\varepsilon)$. Next we set $m_1 = \bar{m} - m$. If $F^{\bar{m}_n}\gamma = \gamma \circ h_{\bar{m}_n}$, by definition we have
\begin{align}
h_{\bar{m}_n} = h_{n_0} \circ \hat{h}_{m_0} \circ \hat{h}_{m_1}.
\end{align}
\hspace{1cm} (3.81)

Hence, differentiating (3.81) and recalling (3.72), (3.79) and $C_4\mu^{-m_0}\theta_{\bar{v}_{n_0}}(t)^{-1}|h_{n_0}'(t)| \leq C_4\mu^{-m_0}\theta_{\bar{v}_{n_0}}(t)^{-1}$, we have the first of (3.66). Taking two derivatives of (3.81) and using the second lines of (3.71), (3.78) and the third of (3.80), we have
\begin{align}
|h_{\bar{m}_n}''| &\leq |h_{n_0}'' \circ \hat{h}_{m_0} \circ \hat{h}_{m_1} \circ \hat{h}_{m_0} \circ \hat{h}_{m_1}| \\
&+ |\hat{h}_{n_0}' \circ \hat{h}_{m_0} \circ \hat{h}_{m_1} \circ \hat{h}_{m_0} \circ \hat{h}_{m_1} \circ \hat{h}_{m_0} | \\
&\leq C_3 \left( \partial^{-1}_{\bar{v}_{n_0}}\mu^{-1} M_{n_0}(m) + \partial^{-1}_{\bar{v}_{n_0}}C_{\mu, \bar{m}M_{n_0}(m)} \right),
\end{align}
form which the second of (3.66) follows and the Lemma is proven.

\hspace{1cm} \Box

Remark 3.13. From now on we will use $\Gamma$ to denote $\Gamma_{r}(\varepsilon)$ where $\varepsilon$ is defined in Lemma 3.8 and has thus the invariance property stated in Corollary 3.10.

3.5. Distortion. We conclude this section with some technical distortion results needed in the following.

Lemma 3.14. For all $n \in \mathbb{N}$, $c \geq 1$, $\nu \in F^{-n}(\Gamma(c))$ and $x, y \in \nu$, we have
\begin{align}
e^{-\mu^n c_{\mu-n}\|x - y\|} &\leq \frac{\lambda_n^+(x)}{\lambda_n^+(y)} \leq e^{\mu^n c_{\mu-n}\|x - y\|}.
\end{align}
\hspace{1cm} (3.82)

Proof. We prove it by induction. To start with, let $x = \nu(t_1), y = \nu(t_2)$ such that $\|x - y\| \leq \tau_n$ for some $\tau_n$ to be chosen shortly. For $n = 1$ we have, for all unit vector $v \notin C_{\varepsilon}$,
\begin{align}
\frac{\|D_x F v\|}{\|D_y F v\|} &\leq e\left[1 + \left\|\frac{D_x F v - D_y F v}{\|D_x F v\|}\right\|\right] \leq e\left\|\frac{D_x F v - D_y F v}{\|D_x F v\|}\right\|.
\end{align}
\hspace{1cm} (3.83)

\begin{align}
|D_x F v - D_y F v| &\leq \int_{t_1}^{t_2} \|\frac{d}{ds} D_{\nu(s)} F v\| ds \leq C_3|t_2 - t_1| \leq C_4\|x - y\|,
\end{align}

\hspace{1cm} \Box

\footnote{Here we drop the dependence on $t$ to ease notations.}
Since the induction hypothesis imply

\[ \nu \in F^{-n}(\Gamma(\varepsilon) \cap \Gamma(x)), \quad \| D_{\nu} F^x v - D_{\nu} F^{x+k} v \| \leq C_\nu \| x - y \|. \]

Also remark that (3.17) and the induction hypothesis imply

\[ \lambda^{+}_{n-k}(F^k y) \lambda^{+}_{n}(x) \leq e^{\mu k} C_{\mu,n} \| x - y \| \lambda^{+}_{n-k}(F^k y) \lambda^{+}_{n}(y) \leq 2 \lambda^{+}_{n}(y), \]

provided we have chosen \( \tau_n \) small enough. Accordingly, since \( \| D_{\nu} F^x v \| \geq \lambda^{+}_{n}(y) \),

\[ \| D_{\nu} F^x v \| \leq \frac{\| D_{\nu} F^x v - D_{\nu} F^{x+i} v \|}{\| D_{\nu} F^{x+i} v \|} \leq C_{\nu} \sum_{i=0}^{n} \| x - y \|. \]

We can now choose \( v \) such that \( \| D_{\nu} F^x v \| = \lambda^{+}_{n}(x) \) so

\[ \frac{\lambda^{+}_{n}(x)}{\lambda^{+}_{n}(y)} \leq \frac{\| D_{\nu} F^x v \|}{\| D_{\nu} F^x v \|} \leq C_{\nu} \sum_{i=0}^{n} \| x - y \|, \]

which proves the upper bound, for points close enough. Next, for all \( x, y \in \nu \) we can consider close intermediate points \( \{ x_i \}_{i=0}^{l} \), \( x_0 = x, x_l = y \), to which the above applies, hence

\[ \frac{\lambda^{+}_{n}(x)}{\lambda^{+}_{n}(y)} \leq \prod_{i=0}^{l-1} \frac{\| D_{\nu} F^x v \|}{\| D_{\nu} F^{x+i} v \|} \leq C_{\nu} \sum_{i=0}^{l-1} \| x - y \|. \]

Taking the limit for \( l \to \infty \) we have the distance, along the curve, between \( x \) and \( y \) which is bounded by \( C_{\nu} \| x - y \| \). This proves the upper bound. The lower bound is proven similarly.

Next, we prove two more distortion Lemmata, inspired by Lemma 6.2 in [30]. Even though the basic idea of the proof is the same, the presence of the central direction creates some difficulties.

**Lemma 3.15.** For each \( \gamma \in \Gamma(x) \), \( n > \tilde{n} \) and \( 0 \leq \rho \leq r - 1 \), we have

\[ \sum_{\nu \in F^{-n} \gamma} \left\| \frac{h'_\nu}{\det D_{\nu} F^n} \right\|_{C^\rho(T)} \leq C_{\nu} \epsilon \alpha_{\mu,n} H^{b_{\rho,n}} \]

(3.84)

\[ \sum_{\nu \in F^{-n} \gamma} \left\| \frac{1}{\det D_{\nu} F^n} \right\|_{C^\rho(T)} \leq C_{\nu} \epsilon \alpha_{\mu,n} H^{(b_{\rho,n}+1)} \]

where\(^{26}\) \( \alpha_{\rho} = a_{\rho} \rho(\rho + 1)/2 + 1 \) and \( b_{\rho} = \rho! \rho(\rho + 1)/2 + 1 \).

**Proof.** For every \( \nu \in F^{-n} \gamma \) define

\[ \Psi_{\nu}(t) = \frac{h'_\nu(t)}{\det D_{\nu}(t) F^n}, \]

\(^{26}\)Recall the definition of \( a_{\rho} \) in Lemma 3.8
and recall that in dimension one holds \( \|\Psi_{t,\nu_n}\|_{C^0} \leq \|\Psi_{t,\nu_n}\|_{L^1} + \|\Psi'_{t,\nu_n}\|_{L^1} \). We then first look for a bound of the \( W^{1,1}(\mathbb{T}) \)-norm of \( \Psi_{t,\nu_n} \). Since \( \nu_1 = (1, 0) \in C_u, D_{\nu_n} F^n \nu'_n \notin C_u \) and recalling that \( F^n \nu_n = \gamma \circ h_n \), we have
\[
  h'_n D_{\nu_n} F^n e_1 \wedge \gamma' \circ h_n = D_{\nu_n} F^n e_1 \wedge D_{\nu_n} F^n \nu'_n = \det(D_{\nu_n} F^n)e_1 \wedge \nu'_n.
\]
Thus we have the equation
\[
  \frac{h'_n(t)}{\det D_{\nu_n(t)} F^n} = \frac{e_1 \wedge \nu'_n(t)}{D_{\nu_n(t)} F^n e_1 \wedge \gamma' \circ h_n(t)}.
\]
Arguing as in Proposition 3.5 and since \( \|\gamma'\| \geq 1 \) we have, recalling definition (3.61),
\[
  |D_{\nu_n} F^n e_1 \wedge \gamma' \circ h_n| \geq C_2 \|h\|_n \|D_{\nu_n} F^n e_1\|.
\]
Therefore, since \( \|\nu'_n\| \leq 1 + \chi^2 \), we have
\[
  \sum_{\nu_n \in P^{n-\gamma}} \|\Psi_{\nu_n}\|_{L^1} \lesssim \sum_{\nu_n \in P^{n-\gamma}} \frac{1}{\|\theta_{\gamma} \circ h_n\|D_{\nu_n} F^n \cdot e_1\|_{L^1}}.
\]
Recall that, by Lemma 3.2, for each \( \nu_n \) we have an inverse branch \( h_{\nu_n}: \Omega_{\gamma} \rightarrow \Omega_{\nu_n} \) such that \( F^n \circ h_{\nu_n} = \text{Id}_{\Omega_{\nu_n}} \). More precisely, the domain \( \Omega_{\nu_n} = \bigcup_{t \in \mathbb{T}} \xi_{t,\nu_n} \), where \( \xi_{t,\nu_n}(s) = \tilde{\nu}_n(t) + s\tilde{c}_1 \) are horizontal segments defined on an interval \( I_t \) of length \( \delta_{\nu_n(t)} \) whose images are unstable curves \( \xi^u_{t,\gamma} \) with length \( \|\xi^u_{t,\gamma}\| = \delta^u_{t,\gamma} \geq 1 \). Let \( p_{n,\xi_{t,\nu_n}} \) be the diffeomorphism associated to \( \xi_{t,\nu_n} \), see formula (3.59). By equation (3.60), \( p'_{n,\xi_{t,\nu_n}}(s) \lesssim p'_{n,\xi_{t,\nu_n}}(0) = \|D_{\nu_n(t)} F^n e_1\| \). It follows
\[
  1 \lesssim \delta^u_{t,\gamma} = \int_{I_t} \left\| \frac{d}{ds} F^n (\xi_{t,\tilde{\nu}_n}(s)) \right\| ds \leq C_4 \delta_{\nu_n(t)} p'_{n,\xi_{t,\nu_n}}(0),
\]
from which
\[
  \|D_{\nu_n(t)} F^n e_1\| \lesssim \frac{1}{\delta_{\nu_n(t)}}.
\]
Since by Lemma 3.2 the \( \Omega_{\nu_n} \) are all disjoint and the \( \nu_n \) are parametrized vertically, by (3.88) we have\(^{27}\)
\[
  \sum_{\nu_n \in P^{n-\gamma}} \|D_{\nu_n} F^n e_1\|_{L^1} \lesssim \sum_{\nu_n \in P^{n-\gamma}} \int_{I_t} \delta_{\nu_n(t)} \lesssim \sum_{\nu_n \in P^{n-\gamma}} m(\Omega_{\nu_n}) \lesssim m(\mathbb{T}^2) \lesssim 1.
\]
Using this in (3.87) yields
\[
  \sum_{\nu_n \in P^{n-\gamma}} \|\Psi_{\nu_n}\|_{L^1} \leq C_2 \|\Psi'_{\nu_n}\|_{L^1} \leq C_4,
\]
since \( |\pi_1 \circ \gamma'(t)|^{-1} \geq \chi^{-1}_c > 1 > \chi^{-1}_u \) implies \( \theta^{-1}_{\gamma'} \leq 1 \). To bound the \( L^1 \) norm of the derivative we can notice that:
\[
  \|\Psi'_{\nu_n}\|_{L^1} \leq \left\| \Psi'_{\nu_n} \right\|_{C^0} \|\Psi_{\nu_n}\|_{L^1}.
\]
To continue it is useful to see \( \tilde{\nu}_n = \nu_n \circ h_n \) as the time evolution of curves parametrized by vertical length. For each \( 0 \leq i \leq n \), let \( \nu_{n-i} = F^i \nu_n \) and \( h_i \) the diffeomorphism such that \( \tilde{\nu}_i = \nu_i \circ h_i \) is parametrized by vertical length. Define the diffeomorphisms \( h^*_i \) by
\[
  \tilde{\nu}_i = F \circ \tilde{\nu}_{i+1} \circ (h^*_i)^{-1},
\]
\(^{27}\) Here \( m(A) \) is the Lebesgue measure of a set \( A \).
Thus, setting $\psi$ where
\[ \nu(t) = \frac{\frac{d}{dt} h(t)}{\det D(\nu(t))} = \frac{\prod_{i=1}^{n} (h_{i}^*)' \circ h_{i+1}^* \circ \cdots \circ h_{n}^*)}{\prod_{i=1}^{n} (\det D(\nu) \circ h_{i+1}^* \circ \cdots \circ h_{n})} (t) \]
where $\psi(t) = (h_{i}^*)'(t) \cdot (\det D(\nu(t))^{-1}$. Hence,
\[ \left| \frac{\psi'}{\psi} \right| \leq \sum_{i=1}^{n} \left| \left( (\psi_i^* \circ h_{i+1}^* \circ \cdots \circ h_{n}^*) (h_{i+1}^* \circ \cdots \circ h_{n})' \right) \right|. \]
By (3.91), since $\nu \in \Gamma(\ell)$, it follows by (3.3) that $\| \psi \|_{C^\ell} \leq C_{2}\ell^{(\ell-1)!}$ for each $\ell \leq \rho$. Thus, setting $b_{\ell} \equiv \ell!$ and $h_{i,n} = h_{i+1}^* \circ \cdots \circ h_{n}^*$, by (3.3) and (3.24) we have
\[ \sum_{i=0}^{n} \| \psi_i \|_{L^1} \leq C_{2}C_{\mu,n}^{\mu} \sum_{\nu \in F^{n}} \| \psi \|_{L^1} \leq C_{2}C_{\mu,n}^{\mu} n! , \]
In particular the above estimates in the case $\ell = 1$ and (3.90) gives
\[ \sum_{\nu \in F^{n}} \| \psi \|_{L^1} \leq C_{2}C_{\mu,n}^{\mu} \sum_{\nu \in F^{n}} \| \psi \|_{L^1} \leq C_{2}C_{\mu,n}^{\mu} n! , \]
which gives the result for $\rho = 0$. Once we have the bound of the $C^0$-norm, we can obtain the general case $\rho \in [1, r - 1]$ as follows:
\[ \sum_{\nu \in F^{n}} \| \psi \|_{C^\rho} \leq \sum_{\nu \in F^{n}} \| \psi \|_{C^{\rho-1}} \leq \sum_{\nu \in F^{n}} \left| \frac{\psi'}{\psi} \right|_{C^{\rho-1}} \leq \sum_{\nu \in F^{n}} \| \psi \|_{C^{\rho-1}} \leq C_{2}C_{\mu,n}^{\mu} n! \]
The procedure to prove the second of (3.84) is analogous, with the difference that, by (3.85) and (3.32), the estimate for $\rho = 0$ gives another $C_{\mu} n^0$, while the computation for $\rho \geq 1$ is exactly the same, but using $\psi_i = (\det D(\nu(t))^{-1}$. □

The next result is a refinement of the the previous Lemma in the more general case in which the curve $\gamma$ is simply not contained in $C_{\mu}$. To state the result it is convenient to
define the following quantities

$$J_{\gamma,n} = \int_{\mathbb{R}} \frac{1}{\|1 - \omega\|_\infty \partial_{\bar{v}_{n_0}}(s)^{-1}, \chi_u \partial_{\bar{v}_{n_0}}(s)^{-1}} ds,$$

$$L_{\gamma,n,m} = \left[ \gamma + \chi_u \partial_{\bar{v}_{n_0}}(s)^{-1} \right]$$

(3.93)

**Lemma 3.16.** In the same hypothesis of Lemma 3.11 with \( n_0 = 0 \), we have

$$\sum_{m \in F - \gamma} \left\| \frac{\partial J_{\gamma,m}}{\partial \bar{m}_{\gamma}} \right\|_{C^0(\mathbb{T})} \leq C_T \left( \chi + \partial_{\gamma,n,m} \partial_{\gamma}^{-1} \right) \mu \bar{c} J_{\gamma,m}$$

(3.94)

$$\sum_{m \in F - \gamma} \left\| \frac{\partial J_{\gamma,m}}{\partial \bar{m}_{\gamma}} \right\|_{C^1(\mathbb{T})} \leq \left( C_{\mu, \bar{m}_{\gamma}} \right) \left( \chi + \partial_{\gamma,n,m} \partial_{\gamma}^{-1} \right)^2 \mu \bar{c} J_{\gamma,m}$$

$$\sum_{m \in F - \gamma} \left\| \frac{\partial J_{\gamma,m}}{\partial \bar{m}_{\gamma}} \right\|_{C^2(\mathbb{T})} \leq O_x M_0(m) \left\{ \partial_{\gamma}^2, M_0(m), (\lambda_m^+)^2 \right\}$$

$$\sum_{m \in F - \gamma} \left\| \frac{\partial J_{\gamma,m}}{\partial \bar{m}_{\gamma}} \right\|_{C^\rho(\mathbb{T})} \leq C^\rho \bar{c} J_{\gamma,m}, \quad \rho > 2,$$

where, recalling (3.65), \( M_0(m) = \| M_0(m, \cdot) \|_\infty \), and

$$O_x = O_x(\bar{m}, m) = C^4_{\mu, \bar{m}_{\gamma}} \left( \chi + \partial_{\gamma,n,m} \partial_{\gamma}^{-1} \right) \bar{c} J_{\gamma,n}$$

(3.95)

Proof. We use the same notations of the proof of Lemma 3.15. In the case \( \rho > 2 \) we content ourselves with a rough estimate, so we can proceed exactly as in the proof of the above Lemma and, using (3.67) and (3.69), the estimate is immediate. In the other cases we need to be more careful in the estimation of (3.87). Setting \( J_k^*(x) = \det D_x F^k \), we write, recalling (3.81) and \( m_1 = \bar{m} - m \),

$$\sum_{m \in F - \gamma} \left\| \frac{\partial J_{\gamma,m}}{\partial \bar{m}_{\gamma}} \right\|_{C^\rho(\mathbb{T})} \leq \sum_{m_1 \in F - \gamma} \sum_{m_2 \in F - m_1} \left\| \frac{\partial J_{\gamma,m_1} \circ \hat{h}_{m_2}}{\partial \bar{m}_{\gamma}} \right\|_{C^\rho(\mathbb{T})}$$

(3.96)

First we are going to estimate the last sum, for \( \rho = 2 \). By the results of Lemma 3.11, \( \nu_m \) is an admissible central curve and, by equation (3.78), \( \| \nu_m(t) \| \leq M_0(t, m) \). Therefore we can apply Lemma 3.15 with \( c^{a_2} \) replaced by \( M_0(t, m) \) and we have

$$\sum_{m_1 \in F - m_1} \left\| \Psi \hat{\nu}_{m_1} \right\|_{C^2} \leq \| M_0(\cdot, m) \|_\infty C^{a_2}_{\mu, m_1} \mu \hat{b}_{m_1}$$

(3.97)

Next, arguing as in (3.85) we have

$$\frac{\hat{h}_{m} \circ \hat{h}_{m_1}}{J_{m_1}(\nu_m \circ h_{m_1})} = \frac{e_1 \wedge \hat{\nu}_{m_1}^{(t)}}{D_{\bar{m}_{\gamma}} (t) E_1 \wedge \gamma \circ h_{m_1}} = \psi_{\bar{m}_{\gamma}}$$

(3.98)
By (3.88) we have
\[
\sum_{\nu_m \in F^{-m} \gamma} \|\tilde{\Psi}_{\nu_m}\|_{L^1} \leq C_{\delta} \sum_{\nu_m \in F^{-m} \gamma} \int_{T_1} \frac{\delta_{\nu_m}(s)}{\theta_{\gamma \circ h_{\nu_m}(s)}} |\hat{h}_{\nu_m}'(s)| ds \\
\leq C_{\delta} m \int_{T_1} \frac{\delta_{\nu_m}(s)}{\theta_{\gamma \circ h_{\nu_m}(s)}} ds.
\]
Since \(|\pi_2(F^m(x, \theta)) - \theta| \leq m\|\omega\|_{\infty} \) it follows that, given \(\hat{\nu}_{*,m} \in F^{-m} \gamma\), for each \(\hat{\nu}_m \in F^{-m} \gamma\)
\[
|\pi_2(F^m(\hat{\nu}_{*,m}(t))) - \pi_2(F^m(\hat{\nu}_m(t)))| \leq m\|\omega\|_{\infty},
\]
accordingly, since \(\gamma' \notin C_u\), we have, calling \(h_{\hat{\nu}_m}\) the reparametrization associated to \(\hat{\nu}_m\),
\[
\theta_{\gamma \circ h_{\hat{\nu}_m}}(t) \geq \{\theta_{\gamma \circ h_{\hat{\nu}_{*,m}}}(t) - m\|\omega\|_{\infty}, \chi_u\}_+.
\]
Hence,
\[
\sum_{\nu_m \in F^{-m} \gamma} \|\tilde{\Psi}_{\nu_m}\|_{L^1} \leq C_{\delta} \int_{T_1} \frac{\sum_{\nu_m \in F^{-m} \gamma} \delta_{\nu_m}(t)}{\{\theta_{\gamma \circ h_{\hat{\nu}_{*,m}}}(t) - m\|\omega\|_{\infty}, \chi_u\}_+} dt \\
\leq C_{\delta} \int_{T_1} \frac{1}{\{\theta_{\gamma \circ h_{\hat{\nu}_{*,m}}}(s)\}_+} \{\theta_{\gamma}(s) - m\|\omega\|_{\infty}, \chi_u\}_+ dt.
\]
Recalling (3.66) we obtain (3.99)
\[
\sum_{\nu_m \in F^{-m} \gamma} \|\tilde{\Psi}_{\nu_m}\|_{L^1} \leq \mu m C_{\delta} \int_{T_1} \frac{1}{1 - n\|\omega\|_{\infty}, \chi_u\theta_{\gamma}(s)\}_+} dt = C_{\delta} \mu m \|\gamma\|_{\gamma,m}.
\]
Next, we want to compute, using (3.98),
\[
\frac{\hat{\Psi}_{\nu_m}'}{\hat{\Psi}_{\nu_m}} = e_1 \wedge \hat{\nu}_m' - \frac{\partial_1 \left(D_{\nu_m}F^m e_1\right) \wedge \gamma' \circ \hat{h}_m \cdot \hat{\nu}_m'}{D_{\nu_m}F^m e_1 \wedge \gamma' \circ \hat{h}_m} \\
(3.100) \quad + \frac{\left(D_{\nu_m}F^m\right)^{-1} \partial_1 \left(D_{\nu_m}F^m\right) e_1 \wedge \hat{\nu}_m' + e_1 \wedge \left(D_{\nu_m}F^m\right)^{-1} \gamma'' \circ \hat{h}_m \cdot \left(h_m'\right)^2}{e_1 \wedge \hat{\nu}_m'}
\]
where we have used equation (3.85). Next, note that \(\gamma''(s) = \alpha e_1\) with \(|\alpha| \leq \Delta\), and \(e_1 = a\eta + bc\) with \(|b| \leq \chi_u\) and \((DF^m)^{-1} \eta \wedge e_1 = 0\). Using (3.20), arguing as in (3.74), we have
\[
\| \left(D_{\nu_m}F^m\right)^{-1} \partial_1 \left(D_{\nu_m}F^m\right) e_1 \| \leq \sum_{k=0}^{m-1} \| (D_{\nu_m}F^{k+1})^{-1} [\partial_{x_k} D_{\nu_m}(\hat{h}_m') F] D_{\nu_m}F^k e_1 \| \\
\cdot \left| D_{\nu_m}F^k \hat{\nu}_m' \right| \leq \sum_{k=0}^{m-1} C_{\delta} \left( 1 + \mu^k (\|\omega\|_{\infty}, \chi_u) \lambda_k^+ \right) \|D_{\nu_m}F^k \hat{\nu}_m'\| \\
\leq C_{\delta} m \| \hat{h}_m' \|.
\]
Thus, using (3.66),
\[
\left| \frac{\hat{\Psi}_{\nu_m}'}{\hat{\Psi}_{\nu_m}} \right| \leq C_{\delta} \left( c + m \| \hat{h}_m' \| + \chi_u \mu m \Delta \gamma |\hat{h}_m'|^2 \right) \leq C_{\delta} \left( c + \left| \hat{\nu}_m' + \chi_u \mu m \Delta \gamma \eta' \right| \gamma' \right).
\]
The first of (3.94) follows by (3.90) and (3.99). While,

\[ \| \tilde{\Psi}'_m \|_{C^0} \leq \left\| \frac{\tilde{\Psi}'_m}{\tilde{\Psi}_m} \right\|_{C^0} \| \tilde{\Psi}_m \|_{C^0}. \]

leads immediately to the second of (3.94). To conclude the lemma we must compute \( \tilde{\Psi}''_m \), which can be obtained by (3.100):

\[ \begin{align*}
\frac{\tilde{\Psi}''_m}{\tilde{\Psi}_m} &= e_1 \land \tilde{\nu}''_m - \frac{(e_1 \land \tilde{\nu}''_m)^2}{(e_1 \land \tilde{\nu}''_m)^2} \left[ e_1 \land \tilde{\nu}''_m - \frac{\tilde{\Psi}'_m}{\tilde{\Psi}_m} \right] e_1 \land \tilde{\nu}''_m - 2e_1 \land (D_{\nu_m \nu_m} (D_{\nu_m \nu_m})^{-1} \frac{\partial}{\partial z_x} (D_{\nu_m \nu_m} F^k_\nu) (D_{\nu_m \nu_m} F^k_\nu)_{t,s}) \\
&= e_1 \land \tilde{\nu}''_m - \frac{(e_1 \land \tilde{\nu}''_m)^2}{(e_1 \land \tilde{\nu}''_m)^2} \left[ e_1 \land \tilde{\nu}''_m - \frac{\tilde{\Psi}'_m}{\tilde{\Psi}_m} \right] e_1 \land \tilde{\nu}''_m - 2e_1 \land (D_{\nu_m \nu_m} (D_{\nu_m \nu_m})^{-1} \frac{\partial}{\partial z_x} (D_{\nu_m \nu_m} F^k_\nu) (D_{\nu_m \nu_m} F^k_\nu)_{t,s}) \\
&\leq \sum_{s=1}^{2} \sum_{k=0}^{n-1} (D_{\nu_m \nu_m} F^k_\nu)^{(s)} (D_{\nu_m \nu_m} F^k_\nu)_{t,s}. 
\end{align*} \]

We estimate the lines of (3.102) one at a time. The first line is bounded by

\[ \begin{align*}
&\sum_{s=1}^{2} \sum_{k=0}^{n-1} (D_{\nu_m \nu_m} F^k_\nu)^{(s)} (D_{\nu_m \nu_m} F^k_\nu)_{t,s}. 
\end{align*} \]

To estimate the second line we first note that

\[ (D_{\nu_m \nu_m} F^k_\nu)^{(s)} (D_{\nu_m \nu_m} F^k_\nu)_{t,s} \leq \sum_{s=1}^{2} \sum_{k=0}^{n-1} (D_{\nu_m \nu_m} F^k_\nu)^{(s)} (D_{\nu_m \nu_m} F^k_\nu)_{t,s}. \]

We can thus use the fourth (3.18) and (3.74) to bound the second line of (3.102) with

\[ \sum_{s=1}^{2} \sum_{k=0}^{n-1} (D_{\nu_m \nu_m} F^k_\nu)^{(s)} (D_{\nu_m \nu_m} F^k_\nu)_{t,s}. \]

To estimate the third line we use the second line of (3.20), arguing as above, and (3.66)

\[ \begin{align*}
&\sum_{s=1}^{2} \sum_{k=0}^{n-1} (D_{\nu_m \nu_m} F^k_\nu)^{(s)} (D_{\nu_m \nu_m} F^k_\nu)_{t,s}. 
\end{align*} \]

Finally, again by (3.66), the last line is estimated by

\[ \begin{align*}
&\sum_{s=1}^{2} \sum_{k=0}^{n-1} (D_{\nu_m \nu_m} F^k_\nu)^{(s)} (D_{\nu_m \nu_m} F^k_\nu)_{t,s}. 
\end{align*} \]

Collecting the above estimates we obtain

\[ \begin{align*}
&\sum_{t,s=1}^{2} \sum_{k=0}^{n-1} (D_{\nu_m \nu_m} F^k_\nu)^{(s)} (D_{\nu_m \nu_m} F^k_\nu)_{t,s}. 
\end{align*} \]

We finally have, setting \( M_{0}(m) := ||M_{0}(m, \cdot)||_{\infty}, \)

\[ \begin{align*}
&| | \tilde{\Psi}'_m | |_{C^2} \leq C_{4} \left| | \tilde{\Psi}'_m | |_{C^0} \sum_{t,s=1}^{2} \sum_{k=0}^{n-1} (D_{\nu_m \nu_m} F^k_\nu)^{(s)} (D_{\nu_m \nu_m} F^k_\nu)_{t,s}. \end{align*} \]
By the above equation and (3.97) we then obtain the statement. □

Corollary 3.17. For each \( n \in \mathbb{N} \)

\[
\| L^1 \|_{L^\infty(\mathbb{T}^2)} \leq C_{\mu,n} \mu^{2n}.
\]

Proof. For any \( x \in \mathbb{T}^2 \) we want to estimate the quantity

\[
L^1(x) = \sum_{y \in F^{-n}x} \frac{1}{|\det D_y F^n|}.
\]

Recall the notation in Section 3.2 and take \( y \in \gamma \), where \( \gamma \in \Gamma \) is an admissible central curve. Then, for every \( x \in F^{-n} \gamma \), there exist \( t \in \mathbb{T} \) and \( \nu \in F^{-n} \gamma \) such that \( x = \nu(h_n(t)) = \nu(t) \). Hence

\[
\sup_{y \in \gamma} \sum_{x \in F^{-n}(y)} \left| \frac{1}{\det D_x F^n} \right| \leq \sum_{\nu \in F^{-n} \gamma} \left| \frac{h'_{\nu,n}}{\det D_\nu F^n} \right| \| (h'_\nu)^{-1} \|_{C^0}.
\]

By equations (3.32) and (2.5) we know that \( \| (h'_\nu)^{-1} \|_{C^0} \leq C_{\nu} \mu^{n} \), for every \( \nu \) and \( n \). Moreover, Lemma 3.15 gives the bound

\[
\sum_{\nu \in F^{-n} \gamma} \left| \frac{h'_{\nu}}{\det D_\nu F^n} \right| \|_{C^0} \leq C_{\mu,n} \mu^{2n}.
\]

\[ \square \]

Remark 3.18. With some extra work the estimate (3.107) can be made sharper, however the above bound is good enough for our current purposes. We will need an improvement, provided in Lemma 8.5, in Section 8.

4. A first Lasota-Yorke inequality

We define a class of geometric norms inspired by [30] and [3]. Given \( u \in C^r(\mathbb{T}^2, \mathbb{R}) \) and an integer \( \rho < r \), we denote by \( B_{\rho} \) the completion of \( C^r(\mathbb{T}^2, \mathbb{R}) \) with respect to the norm:

\[
\| u \|_\rho := \max_{|\alpha| \leq \rho} \sup_{\gamma \in \Gamma} \sup_{\phi \in C^{\alpha}(\mathbb{T})} \| \phi(t) (\partial^\alpha u)(\gamma(t)) \| dt.
\]

This defines a decreasing sequence of Banach spaces continuously embedded in \( L^1 \), namely

\[
\| u \|_{L^1} \leq C \| u \|_{\rho_1} \leq C \| u \|_{\rho_2}, \quad \text{for every} \quad 0 \leq \rho_1 \leq \rho_2 \leq r - 1.
\]

To see this we observe that, since \( \sigma_x(t) = (x, t) \in \Gamma \),

\[
\| u \|_{L^1} = \sup_{\| \phi \|_{C^0(\mathbb{T})} \leq 1} \int_\mathbb{T} dx \int_\mathbb{T} dy \phi(x,y)u(x,y) \leq \int_\mathbb{T} dx \sup_{\| \phi \|_{C^0(\mathbb{T})} \leq 1} \int_\mathbb{T} dy \phi(x,y)u(x,y)
\]

\[
\leq \int_\mathbb{T} dx \sup_{\| \phi \|_{C^0(\mathbb{T})} \leq 1} \int_\mathbb{T} dt \phi(t)u(\sigma_x(t)) \leq \int_\mathbb{T} dx \| u \|_0 = \| u \|_0.
\]

The above proves the first inequality of (4.2), the others being trivial. We start with a Lasota-Yorke type inequality between the spaces \( B_{\rho} \) and \( B_{\rho-1} \).
Theorem 4.1. Let $F \in C^r(T^2, T^2)$ be a SVPH. Let $\mathcal{L} := \mathcal{L}_F$ be the transfer operator defined in (2.12), and $\bar{n}$ be the integer given in Lemma 3.8. For each $\rho \in [1, r - 1]$ and $n > \bar{n}$, there exists $C_{n, \rho}$ such that

\begin{align}
\|\mathcal{L}^n u\|_0 & \leq C_{\mu, n, \mu} \|u\|_0 \\
\|\mathcal{L}^n u\|_\rho & \leq \frac{C_{\mu, n, \mu} b_{n, \rho}}{\lambda_{\rho}^n} \|u\|_{\rho} + C_{n, \rho} \|u\|_{\rho - 1}
\end{align}

where $\bar{a}_{\rho} = 1 + a_{\rho}(\rho^2 + \rho(\rho + 1)/2 + 1)$ and $\bar{b}_{\rho} = 1 + \rho(2\rho^2 + \rho/2 + 1)$.

We postpone the proof of Theorem 4.1 to section 4.2. First we need to develop several results on the commutators between differential operators and transfer operators which will be needed throughout the paper.

4.1. Differential Operators. For $s, \rho \in \mathbb{N}$ we denote by $P_{s, \rho}$ a differential operator of order at most $\rho$ defined as a finite linear combination of compositions of at most $\rho$ vector fields, and we write

\begin{equation}
P_{s, \rho} = \sum_{j=0}^{s} \sum_{a \in A \subset \mathbb{N}} v_{j, a_{1}} \cdots v_{j, a_{j}} u,
\end{equation}

where $A$ is a finite set and for every $i \leq j$, $v_{j, a_{i}}$ are vector fields in $C^{p+j-s}$, with the convention that $v_{j, a_{1}} \cdots v_{j, a_{j}} u = u$ if $j = 0$. We denote by $\Psi^{s, \rho}$ the set of differential operators $P_{s, \rho}$. Finally, for a function $u \in C^r(T^2, \mathbb{R})$ and a smooth vector field $v$, we denote $\partial_v u(x) = \langle \nabla_x u, v(x) \rangle$.

Proposition 4.2. Given smooth vector fields $v_1, \ldots, v_s \in C^p$, we have

\[\partial_{v_1} \cdots \partial_{v_s} \mathcal{L}^n = \mathcal{L}^n \partial_{F^{n}v_1} \cdots \partial_{F^{n}v_s} + \mathcal{L}^n P_{s-1, \rho},\]

where $F^s v(x) := (D_x F)^{-1} v(F(x))$ is the pullback of $v$ by the map $F$ and $P_{s-1, \rho} \in \Psi^{s-1, \rho}$ whose coefficients may depend on $n$.

Proof. Let us start with $s = 1$. Let $v_1 \in C^p(T^2, T^2)$ and define

\begin{equation}
J_n(p) = (\det D_p F^n)^{-1}; \quad \phi_n(p) = \log |\det D_p F^n|.
\end{equation}

For each $\mathfrak{h} \in H^n$ we have

\[\langle \nabla [J_n \circ \mathfrak{h} \cdot u \circ \mathfrak{h} \cdot v_1] , v_1 \rangle = \langle (J_n \circ \mathfrak{h}(D \mathfrak{h}))^n \nabla u \circ \mathfrak{h} \circ v_1 \rangle - \langle (D \mathfrak{h})^n \nabla(\det D F^n) \circ \mathfrak{h} J_n^2 \circ \mathfrak{h} u \circ \mathfrak{h} \circ v_1 \rangle \]

\[= J_n \circ \mathfrak{h}(\langle (D \mathfrak{h})^n \nabla u \circ \mathfrak{h} \circ v_1 \rangle - J_n \circ \mathfrak{h}(\langle (D \mathfrak{h})^n \nabla \phi_n \circ \mathfrak{h} u \circ \mathfrak{h} \circ v_1 \rangle).
\]

Then, since $DF^n \circ \mathfrak{h} D \mathfrak{h} = \text{Id}_{R_{\mathfrak{h}}}$, for each $\mathfrak{h} \in H^n$ and $x \in D_0$

\begin{equation}
\langle \nabla [J_n \circ \mathfrak{h} \cdot u \circ \mathfrak{h} \cdot v_1] (x) , v_1 (x) \rangle = J_n \circ \mathfrak{h}(x) [\partial_{F^{n}v_1} u - \partial_{F^{n}v_1} \phi_n u] \circ \mathfrak{h}(x).
\end{equation}

Observing that

\[\mathcal{L}^n u = \sum_{\mathfrak{h} \in H^n} u \circ \mathfrak{h} J_n \circ \mathfrak{h} 1_R_{\mathfrak{h}} \circ \mathfrak{h},\]

it follows

\begin{equation}
\langle \nabla_x \mathcal{L}^n u , v_1 (x) \rangle = \mathcal{L}^n \left( \partial_{F^{n}v_1} u \right)(x) - \mathcal{L}^n (\partial_{F^{n}v_1} \phi_n u)(x),
\end{equation}

\[\text{Recall that } D_0, R_{\mathfrak{h}} \text{ indicate respectively the domain and the range of } \mathfrak{h}.\]
which prove the result since the multiplication operator $P_{0,\rho} := -\partial_{F_{n}^{*}v_{1}}\phi_{n} \in \Psi^{0,\rho}$. Next, we argue by induction on $s$:

$$
\partial_{v_{s+1}} \cdots \partial_{v_{1}} \mathcal{L}^{n} u = \partial_{v_{s+1}} \left[ \mathcal{L}^{n} \partial_{F_{n}^{*}v_{s}} \cdots \partial_{F_{n}^{*}v_{1}} u + \mathcal{L}^{n} P_{s-1,\rho} u \right]
$$

(4.10)

$$
= \mathcal{L}^{n} \partial_{F_{n}^{*}v_{s+1}} \cdots \partial_{F_{n}^{*}v_{1}} u + \mathcal{L}^{n} \left( \partial_{F_{n}^{*}v_{s+1}} \phi_{n} \cdot \partial_{F_{n}^{*}v_{s}} \cdots \partial_{F_{n}^{*}v_{1}} u \right) + \mathcal{L}^{n} \partial_{F_{n}^{*}v_{s+1}} P_{s-1,\rho} u + \mathcal{L}^{n} \left( \partial_{F_{n}^{*}v_{s+1}} \phi_{n} \cdot P_{s-1,\rho} u \right),
$$

which yields the Lemma with

$$
P_{s,\rho} = \partial_{F_{n}^{*}v_{s+1}} P_{s-1,\rho} + \partial_{F_{n}^{*}v_{s+1}} \phi_{n} \cdot \left[ \partial_{F_{n}^{*}v_{s}} \cdots \partial_{F_{n}^{*}v_{1}} + P_{s-1,\rho} \right] + \partial_{F_{n}^{*}v_{s+1}} P_{s-1,\rho}.
$$

(4.11)

In the case $v_{j} \in \{e_{1}, e_{2}\}$ for each $j$, we have the following Corollary as an immediate iterative application of formulae (4.7) and (4.9).

**Corollary 4.3.** For each $t \geq 1$, $n \in \mathbb{N}$ $\alpha = (\alpha_{1}, \ldots, \alpha_{t}) \in \{1, 2\}^{t}$ and $\hbar \in \mathcal{H}^{n},$

$$
\partial^{\alpha}[J_{n} \circ \hbar \circ u \circ \hbar] = J_{n} \circ \hbar \cdot \left[ P_{n,t}^{\alpha} \right] \circ \hbar,
$$

(4.12)

in particular

$$
\partial^{\alpha} \mathcal{L}^{n} u = \mathcal{L}^{n} P_{n,t}^{\alpha} u,
$$

(4.13)

the operators $P_{n,t}^{\alpha}$ being defined by the following relations, for each $u \in C^{t},$

$$
\begin{cases}
P_{n,t}^{\alpha} u = u, \\
P_{n,t}^{\alpha} u = A_{n,t}^{\alpha} u - A_{n,t}^{\alpha} \phi_{n} \cdot u, \\
P_{n,t}^{\alpha} u = A_{n,t}^{\alpha} u - \sum_{k=1}^{t} A_{n,k}^{\alpha} (A_{n,k}^{\alpha} \phi_{n}) \cdot P_{n,k-1}^{\alpha} u & \text{for } t \geq 2,
\end{cases}
$$

(4.14)

where $A_{n,t}^{\alpha} = \partial_{F_{n}^{*}e_{\alpha}}, A_{n,k}^{\alpha} := A_{n}^{\alpha_{k}} \cdots A_{n}^{\alpha_{1}}, A_{n,t+1}^{\alpha} = \text{Id}$ and $\phi_{n}$ is defined in (4.6).

**Proposition 4.4.** For each $n \in \mathbb{N}$ let $P_{n,t}^{\alpha} \in \Psi^{1,t}$ given by (4.14). For any $1 \leq t < r$, $\psi \in C^{t}(\mathbb{T}^{2}, \mathbb{C})$ with $\text{supp} \psi \subset U = \hat{U} \subset \mathbb{T}^{2}, \nu \in \Gamma(\varphi)$ such that $DF_{n-m} \psi \in C_{c, \varphi} \in C^{t}(\mathbb{T}^{2}, \mathbb{C})$ with $\|\varphi\|_{C^{t}} \leq 1$, multi-index $\alpha$, $|\alpha| = t$ and $u \in C^{t}(\mathbb{T}^{2})$ we have

$$
\int_{\mathbb{T}} \varphi(\tau) P_{n,t}^{\alpha} (\psi u)(\nu(\tau)) d\tau \leq \tilde{C}(t, n, m) \|\psi\|_{C^{t}(U)} \|u\|_{t},
$$

(4.15)

where

$$
\tilde{C}(t, n, m) = \begin{cases}
C_{t} \mu^{2n} \sup_{\varphi} \varphi \{ \zeta_{\varphi}^{2} \circ \nu(t) \lambda_{n}^{+} \circ \nu(t) + \mu^{n} \zeta_{\varphi} \circ \nu(t) \zeta \}
& \text{if } t = 2,
C_{t} \Lambda^{n}_{\zeta} & \text{if } t > 2.
\end{cases}
$$

(4.16)

Proof. For simplicity we set $\partial_{k} = \partial_{x_{k}}$ for $k \in \{1, 2\}$. First of all notice that, if we set $d_{k,i} = \langle (DF_{n})^{-1} e_{k}, e_{i} \rangle$, then $A_{n}^{\alpha} = \sum_{i=1}^{2} d_{\alpha,i} \partial_{x_{i}}$. Furthermore, by formula (3.4), $\|d_{k,i}\|_{C^{t}} \leq \|(DF_{n})^{-1}\|_{C^{t}} \leq A_{n}^{\alpha}$, for each $2 \leq t \leq r$. We are going to prove (4.15) by induction on $t$. For $t = 0$ it is obvious, let us assume it for any $k \leq t - 1$. By (4.14) the integral in (4.15) splits into

$$
\int_{\mathbb{T}} \varphi(\tau) P_{n,t}^{\alpha} (\psi u)(\nu(\tau)) d\tau
$$

(4.17)

$$
= \int_{\mathbb{T}} \varphi \left[ A_{n}^{\alpha} \cdots A_{n}^{\alpha} (\psi u) \right] \circ \nu - \int_{\mathbb{T}} \varphi \sum_{k=1}^{t} A_{n,k+1}^{\alpha} \left( (A_{n,k}^{\alpha} \phi_{n}) \cdot P_{n,k-1}^{\alpha} (\psi u) \right) \circ \nu.
$$

29Unless differently specified, in the following all the integrals are on $T$. 
The first integral is equal to

\[ \sum_{i_1, \ldots, i_t \in \{1, \ldots, j \}} \sum_{j_e \in \{1, \ldots, j \}} \int \varphi \cdot \left( \prod_{j \in J} \partial_j u \right) \left( \prod_{j \in J} \partial_j \psi \right) \left( \prod_{j \in J} \partial_j d_{\alpha_1, i_1} \right) \cdots \left( \prod_{j \in J} \partial_j d_{\alpha_t, i_t} \right), \tag{4.18} \]

where the second sum is made over all the partitions \( J, J_0, J_1, \ldots, J_t \) of \( \{1, \ldots, t\} \) such that \( J_j \subset \{j + 1, \ldots, t\}, j \geq 1 \).\(^{30}\) Note that

\[ \left\| \prod_{j=1}^t \Pi_{j \in J} \partial_j d_{\alpha_j, i_j} \right\|_{C^{t+1}_j} \leq \Lambda^{n \sum_{j=1}^t \sharp J_j} \text{ and } \left\| \Pi_{j \in J_0} \partial_j \psi \right\|_{C^{t+1}_{J_0}} \leq \| \psi \|_{C^t}. \]

Consequently, from (4.18) and the definition (4.1), we have

\[ \left| \int \varphi(\tau) A^\alpha_n \psi u(\nu(\tau)) d\tau \right| \leq C_2 \Lambda^{c_n} \| \psi \|_{C^t} \| u \|_t. \tag{4.19} \]

To bound the second integral in (4.17) we first note that

\[ A_n^\alpha \phi_n(x) = \sum_{j=0}^{n-1} \langle (D_x F^j)^* \nabla \phi_1 \circ F^j(x), (D_x F^n)^{-1} e_{\alpha_k} \rangle \]

\[ = \sum_{j=0}^{n-1} \langle \nabla \phi_1, (DF^{n-j})^{-1} e_{\alpha_k} \rangle \circ F^j(x), \tag{4.20} \]

thus (3.3) implies

\[ \| A_n^\alpha \phi_n \|_{C^t} \leq C_2 \sum_{j=0}^{n-1} \| (DF^{n-j})^{-1} \|_{C^t} A^{n+1} \leq C_2 \sum_{j=0}^{n-1} \Lambda^{c_t (n-j+l)} \leq C_2 \Lambda^{c_n}. \tag{4.21} \]

We can then use (4.19) to estimate

\[ \left| \int \varphi A_{n, k+1} \left( (A_{n, k}^\alpha \phi_n) \cdot P_n^{\alpha, k-1} u \right) \right| \leq C_2 \Lambda^{c_n} \| A_n^\alpha \phi_n \|_{C^{t-k-1}} \| P_n^{\alpha, k-1} u \|_{t-k-1} \]

\[ \leq C_2 \Lambda^{c_n} \| P_n^{\alpha, k-1} u \|_{t-k-1}. \tag{4.22} \]

To bound the last term we take \( \phi \in C^{l-k-1}, \| \phi \|_{C^{l-k-1}} = 1, \gamma \in \Gamma, \) and we consider

\[ \int \phi(\tau) \partial^{t-k-1} \left[ P_n^{\alpha, k-1} (u) \right] \circ \gamma. \]

We can then split the integral as in (4.17), although this time \( \alpha = (\alpha_1, \cdots, \alpha_k, \cdots, \alpha_{k-1}) \). For the first term we take \( t-k-1 \) derivatives in (4.18) and, arguing as we did to prove (4.19), we have

\[ \left| \int \varphi(\tau) \partial^{t-k-1} A_{n, 1}^\alpha (u) (\gamma(\tau)) d\tau \right| \leq C_2 \Lambda^{c_n} \| \psi \|_{C^t} \| u \|_t. \]

The second term is estimated in the same way, using the inductive assumption. The first statement of the Lemma then follows using this in (4.22).

\(^{30}\) We use the conventions \( \prod_{j \in B} \partial_j A = A \) and \( \sharp B \) denote the cardinality of the set \( B \).
In the special case $t = 2$, for $\alpha = (\alpha_1, \alpha_2)$,\(^{31}\)

\[ P_{n,2}^2(\psi u) = A_{n,1}^\alpha(\psi u) - A_{n,2}^\alpha(\phi_n)A_{n,1}^\alpha(\psi u) - A_{n,1}^\alpha(\phi_n)\psi u - A_{n,1}^\alpha(\phi_n)A_{n,2}^\alpha(\psi u) \]
\[ - A_{n,1}^\alpha(\phi_n)A_{n,2}^\alpha(\phi_n)u \]
\[ = \{ A_{n,1}^\alpha(\psi) - A_{n,2}^\alpha(\phi_n)A_{n,1}^\alpha(\psi) - (A_{n,1}^\alpha(\phi_n)A_{n,2}^\alpha(\phi_n) + A_{n,1}^\alpha(\phi_n))\} \psi u \]
\[ - \{ A_{n,2}^\alpha(\psi + \psi A_{n,2}^\alpha(\phi_n))A_{n,1}^\alpha(\psi) u - A_{n,1}^\alpha(\psi + \psi A_{n,2}^\alpha(\phi_n))A_{n,2}^\alpha(\psi) u + \psi A_{n,1}^\alpha(\psi) u \} \]
\[ := \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4. \]

We then want to integrate the above terms along the curve $\nu$ against a test function $\varphi \in C^2$. Recalling that the coefficients of the differential operators $A_n^\alpha$ have $C^r$ norm bounded by $\|(DF^n)^{-1}\|_{C^r}$, we thus have

\[
\int \varphi \Phi_1 \circ \nu \leq C_2 \max_{i,j} \{|\langle \nu, P_{n,1}^\alpha \psi \rangle |_{C^0}, |\langle \nu, A_{n,1}^\alpha \phi_n \rangle |_{C^0}, (1 + |\langle \nu, A_{n,1}^\alpha \phi_n \rangle |_{C^0})^2 |\langle \nu, \psi A_{n,1}^\alpha A_{n,2}^\alpha \varphi \rangle |_{C^0}, |\langle \nu, A_{n,1}^\alpha \phi_n A_{n,2}^\alpha \psi \rangle |_{C^0} \} u \|_1. \]

The bounds for $\Phi_2$ and $\Phi_3$ are similar:

\[
\int \varphi \Phi_2 \circ \nu \leq C_2 \|\langle (DF^n)^{-1} \|_{C^r} \max_i \{|\langle \nu, P_{n,1}^\alpha \psi \rangle |_{C^0}, |\langle \nu, A_{n,1}^\alpha \phi_n \rangle |_{C^0} |\psi \rangle |_{C^1} \} u \|_1. \]

Next, for any two vector $v, w \in \mathbb{R}^2$, $i, j \in \{1, 2\}$ and $x = (x_1, x_2) \in \mathbb{T}^2$,\(^{32}\)

\[
\partial_{F^v} \partial_{F^w} u \cdot \partial_{F^v} \partial_{F^w} \psi \cdot \partial_{F^v} \partial_{F^w} \nu = \sum_{j,k} \partial_{x_j} u \cdot [(DF)^{-1} v]_j \cdot [(DF)^{-1} w]_k \cdot [(DF)^{-1} v]_j. \]

Recalling the properties of the $\| \cdot \|_\rho$ norm and (3.18) we have

\[
\int \varphi \Phi_4 \circ \nu \leq C_4 \{ \mu^{2n} \| \psi \|_{C^2}, \mu^n \| (DF^n)^{-1} \|_{C^2} \| \psi \|_{C^1}, \| (DF^n)^{-1} \|_{C^2} \| \psi \|_{C^1} \} u \|_2. \]

It follows by the property of the $C^r$ norm and (4.2) that

\[
\int \varphi P_{n,2}^2(\psi u) \circ \nu \leq C_2 \{ \max_{i \in \{1,2\}} \| A_{n,1}^\alpha \phi_n \|_{C^0} \| \psi \|_{C^0}, \| A_{n,1}^\alpha \phi_n \|_{C^0} \| \psi \|_{C^1}, \mu^n \| (DF^n)^{-1} \|_{C^2} \| \psi \|_{C^1}, \| (DF^n)^{-1} \|_{C^2} \| \psi \|_{C^1} \} u \|_2. \]

We have thus proved that

\[
\bar{C}(2, n) = C_4 \{ \max_{i \in \{1,2\}} \| A_{n,1}^\alpha \phi_n \|_{C^0} \| \psi \|_{C^0}, \| (DF^n)^{-1} \|_{C^2} \| \psi \|_{C^1}, \mu^n \| (DF^n)^{-1} \|_{C^2} \| \psi \|_{C^1} \}^+. \]

To conclude we give a bound of the above quantity. By Proposition 3.7 it is enough to find estimates for $\| A_{n,1}^\alpha \phi_n \|_{C^0}$ and $A_{n,1}^\alpha \phi_n \|_{C^1} \| (DF^n)^{-1} \|_{C^2}$. First we can use formulae (4.20) and (3.18),

\[
(4.23) \quad |\partial_{F^x} \psi \cdot \phi_n(x)| \leq C_4 \sum_{j=0}^{n-1} \mu^{n-j} \leq C_{\mu,n} \mu^n. \]

\(^{31}\) We use the following notation: $\Phi_1$ equals the third line from the bottom, the other $\Phi_i$ are, ordered, the terms in the second line from the bottom.

\(^{32}\)Here we denote $[(DF)^{-1} w]_k := [(DF)^{-1} w, e_k]$. 
In particular $\|A^{n,1}_n \phi_n\|_{C^2} \leq C_{\mu,n} \mu^n$. Next we take another derivative of (4.23) in the direction of $F^{n,1} e_q$ and, setting $g_{t,n,j}(x) = (\nabla \phi_1, (DF^{n,j})^{-1} e_q)(x)$, we have

$$\partial_{F^{n,1} e_q} (\partial_{F^{n,1} e_q} \phi_n(x)) = \sum_{j=0}^{n-1} \langle \nabla (g_{t,n,j} \circ F^j(x)), (D_x F^n)^{-1} e_q \rangle$$

(4.24)

$$= \sum_{j=0}^{n-1} \langle (D_x F^j)^* \nabla g_{t,n,j} \circ F^j(x), (D_x F^n)^{-1} e_q \rangle$$

$$= \sum_{j=0}^{n-1} \langle \nabla g_{t,n,j} \circ F^j(x), (D_x F^{n,j})^{-1} e_q \rangle.$$ 

By a direct computation we see that, recalling (3.22),

$$\|\nabla g_{t,n,j}\| \leq C_1 \max_i \{\|\partial_{x_i} (DF^{n,j})^{-1}\|\} \leq C_2 s_{n,j}(x) \mu^{n-j}.$$ 

We use this in (4.24) obtaining

$$\|\partial_{F^{n,1} e_q} (\partial_{F^{n,1} e_q} \phi_n(x))\| \leq C_2 \mu^{2s_n}(x).$$ 

Hence, $\|A^{n,1}_n \phi_n\|_{C^2} \leq C_2 \mu^{2s_n}$. Finally, we compute

$$\left| \frac{d}{dt} (A^{n,1}_n \phi_n \circ \nu) \right| \leq \sum_{j=0}^{n-1} \left| \langle (D_x F^j)^* D(\nabla \phi_1) \circ F^j \nu, (D_x F^n)^{-1} e_{\alpha_1} \rangle \right|$$

$$+ \left| \langle \nabla \phi_1 \circ F^j(\nu), [(DF^n)^{-1}] e_{\alpha_1} \rangle \right|$$

$$\leq C_{\mu,n} \mu^n + C_2 s_{n,m} \circ \nu,$$

so that, using (3.18) and the definition of $s_{n,m}$ in (3.15), we obtain

$$\|A^{n,1}_n \phi_n\|_{C^2} \cdot \|(DF^n)^{-1}\|_{C^2} \leq C_2 \mu^{2n-m} s_{n,m}^2.$$ 

The Lemma follows collecting all the above estimates and recalling again (3.18) for the estimate of $\mu^n \|(DF^n)^{-1}\|_{C^2}$.  

4.2. Proof of Theorem 4.1.

Proof. Given Lemma 3.15 the proof of Theorem 4.1 is almost exactly the same as in [30], hence we provide the full proof for the case $1 < \rho \leq r - 1$.

Let us prove (4.3) first, since it is an immediate consequence of Lemma 3.15 and Definition 4.1 in the case $\rho = 0$. Indeed, by changing the variables and recalling the notation of Section 3.2 and Lemma 3.15, we have

$$\int_T \phi(t) L^n u(\gamma(t)) dt = \sum_{\nu \in F^{-n,\gamma}} \int_T |\det D_{\nu(t)} F^n|^{-1} \cdot (u \circ \nu)(t) \cdot \phi(t) dt$$

$$= \sum_{\nu \in F^{-n,\gamma}} \int_T |\det D_{\nu} F^n|^{-1} \cdot (u \circ \nu)(t) \cdot (\phi \circ h_n)(t) h_n'(t) dt$$

$$\leq \sum_{\nu \in F^{-n,\gamma}} \|h_n' \| \|\det D_{\nu} F^n|^{-1}\|_{C^0} \cdot \|u\| \leq C_{\mu,n} \mu^n \|u\|_0.$$
Let us now proceed with the case $\rho = 1$, from which we deduce the general case by similar computations. We must bound the quantity

$$\int_T \phi(t)(\partial_t \mathcal{L}^n u)(\gamma(t))dt = \int_T \phi(t)\langle \nabla (\mathcal{L}^n u)(\gamma(t)), v \rangle dt,$$

where now $\phi \in C^1(T)$ with norm one and $v$ is a unitary $C^r$ vector field. From Proposition 4.2 the above quantity is equal to the sum over $\nu \in F^{-n}\gamma$ of

$$(4.25) \quad \int |\det D_{\nu} F^n| \phi \cdot \partial_{F \gamma^* \nu} u(\nu) + \int \mathcal{L}^n (P_0 u) \phi,$$

where $P_0$ is an operator of multiplication by a $C^0$ function. By Proposition 4.4 applied with $\psi = 1$, plus the result for $\rho = 0$, the last term is then bounded by $C_n \| u \|_1$. In order to bound the first term of $(4.25)$ we need an analogous of Lemma 6.5 in [30]. The idea is to decompose the vector field $v$ into a vector tangent to the central curve $\gamma$ and a vector field approximately in the unstable direction so that the first one can be integrated by parts, while for the other we can exploit the expansion. The proof of the following Lemma follows that of the aforementioned paper, since the key point is the splitting of the tangent space in two directions, one of which is expanding. Once more, however, the presence of the central direction creates difficulties. For completeness we give the proof adapted to our case in Appendix B.

**Lemma 4.5.** Let $\bar{n}$ be the integer provided by Lemma 3.8. For every $n > \bar{n}$, $\gamma \in \Gamma(\varepsilon)$, $\nu \in F^{-n}\gamma$, and any vector field $v \in C^r$, with $\| v \|_{C^r} \leq 1$, defined in some neighborhood $M(\gamma)$ of $\gamma$, there exist a neighborhood $M'(\gamma)$ of $\gamma$ and a decomposition

$$(4.26) \quad v = \hat{v}^c + \hat{v}^u,$$

where $\hat{v}^c$ and $\hat{v}^u$ are $C^r(M'(\gamma))$ vector fields such that, setting $F^n (N(\nu)) = M'(\gamma)$,

- $\gamma'(t) = \hat{v}^c(\gamma(t))$,
- $\| (F^n)^* \hat{v}^c \|_{C^r(N(\nu))} \leq \lambda^{-n} C_{\mu,n} \mu^{\rho n} n!
- \| (F^n)^* \hat{v}^u \|_{C^r(N(\nu))} \leq C_{\mu,n} (\nu + 1)(2\rho + 1)n!
- \| \hat{v}^u \|_{C^r(M(\gamma))} + \| \hat{v}^c \|_{C^r(M'(\gamma))} \leq C_n.$

By the above decomposition, the first term in $(4.25)$ becomes

$$(4.27) \quad \int |\det D_{\nu} F^n| \phi \cdot \partial_{F \gamma^* \nu} u(\nu) + \int |\det D_{\nu} F^n| \phi \cdot \partial_{F \gamma^* \nu} u(\nu) dt = \int |\det D_{\nu} F^n| \phi \cdot \partial_{F \gamma^* \nu} u(\nu) dt.$$

Since $\gamma(t) = F^n \nu(t)$ we have $D_{\nu(t)} F^n \cdot \nu'(t) = \hat{v}^c(F^n \nu(t))$, hence:

$$\nu'(t) = (D_{\nu(t)} F^n)^{-1} \cdot \hat{v}^c(F^n \nu(t)) = F^n \hat{v}^c(\nu(t)).$$

Accordingly,

$$\int |\det D_{\nu(t)} F^n| \phi \cdot \partial_{F \gamma^* \nu} u(\nu(t)) dt = \int |\det D_{\nu(t)} F^n| \frac{d}{dt} (u(\nu(t))) dt$$

$$= \int |\det D_{h n}\nu(t)} F^n| \frac{d}{dt} (u \circ \hat{v}^-)(h n^{-1}(t)) dt$$

$$= \int |\det D_{\nu(t)} F^n| (u \circ \hat{v}^-)(t) dt = \int |\det D_{\nu(t)} F^n| \frac{d}{dt} (\phi \circ h_n(t)) u(\hat{v}^-)(t) dt$$

$$\leq \| \frac{\phi \circ h_n}{\det D_{F^n}} \|_{C^1} \| u \|_{C^1}.$$
Summing over \( \nu \in F^{-n,\gamma} \) and using Lemma 3.15 we obtain

\[
(4.28) \quad \sum_{\nu \in F^{-n,\gamma}} \int \phi(t) \partial_{F^n,\nu} u(\nu(t)) dt \lesssim \varphi C_{\mu,n} \mu^{2n} \|u\|_0.
\]

The second term of (4.27) is

\[
\int \frac{\phi}{| \det D\nu F^n |} \partial_{F^n,\nu} u(\nu) = \int \frac{\phi}{| \det D\nu F^n |} (\nabla u, F^n, \nu) \circ \nu \leq C_{\nu} \left\| \phi \circ h_{\nu} h'_{\nu} \right\| _{C^1} \left\| F^n, \nu \right\| _{C^1} \left\| u \right\|_1
\]

\[
\leq C_{\nu} \left\| h_{\nu} \right\| _{C^1} \left\| \frac{h'_{\nu}}{\det D\nu F^n} \right\| _{C^1} \left\| \lambda_{-n} C_{\mu,n} \mu^n \|u\|_1, \right.
\]

where we made the usual change of variables \( t = h_{\nu}(s) \) and used Lemma 4.5. Finally, using (4.28) and (4.29) in (4.27), and recalling (3.24), we have by Lemma 3.15, with \( \rho = 1 \),

\[
(4.30) \quad \|C^n u\|_1 \leq \lambda_{-n} C_{\mu,n} \mu^{2n} \|u\|_1 + C_{\mu} \|u\|_0.
\]

For the general case \( 1 \leq \rho \leq r-1 \) one has to control the term \( \int \phi(t) \partial_{\nu_j} \cdots \partial_{\nu_s} C^n u(\nu(t)) dt \), for vector fields \( \nu_j \in C^\rho, j = 1, \ldots, s \) and \( s \leq \rho \). Using again Propositions 4.2 and 4.4, the latter is bounded by

\[
(4.31) \quad \sum_{\nu \in F^{-n,\gamma}} \int \frac{1}{| \det D\nu F^n |} \phi \cdot \partial_{F^n,\nu_s \cdots F^n,\nu_1} u(\nu) + C_{\mu,\rho} \|u\|_{\rho-1}.
\]

Now the strategy is exactly the same as before. We use Lemma 4.5 to decompose each \( \nu_j = \tilde{\nu}_j^{\mu} + \tilde{\nu}_j^\rho \). We take \( \sigma \in \{u, e\}^s \), \( k = \# \{ i | \sigma_i = e \} \) and let \( \pi \) be a permutation of \( \{1, \ldots, s\} \) such that \( \pi \{1, \ldots, k\} = \{i | \sigma_i = e\} \). Using integration by parts, we can write the integral in (4.31) as

\[
\int \frac{\phi}{\det D\nu F^n} \partial_{F^n,\nu_s \cdots F^n,\nu_1} u(\nu) = \sum_{\sigma \in \{u, e\}^s} \int \frac{\phi}{\det D\nu F^n} \left( \prod_{s=1}^{s} \partial_{F^n,\nu_1} \right) u(\nu)
\]

\[
= \sum_{\sigma \in \{u, e\}^s} \int \frac{\phi}{\det D\nu F^n} \left( \prod_{s=1}^{s} \partial_{F^n,\nu_1} \right) u(\nu) + C_{\mu,\rho} \|u\|_{\rho-1}
\]

By Lemma 4.5, \( \|F^n,\nu \|_{C^\rho(\nu)} \leq C_{\mu,n}^{\rho+1} \mu^{(\rho+1)(2\rho+1)n} \) while \( \| \prod_{s=1}^{s} \partial_{F^n,\nu_1} \|_{C^\rho(\nu)} \leq C \lambda_{-n}^{(s-k)n} (C_{\mu,n}^{\rho+1} \mu^{\rho+1})^{s-k} \). It follows by Lemma 3.15, equation (3.24) and the fact that
$\|\phi\|_{C^r} \leq 1$, that\footnote{Notice that the coefficient in front of the strong norm is obtained in the case $s = \rho$ and $k = 0$, while all the other terms are bounded again by $C_{n, \rho} \|u\|_{\rho - 1}$.} \[ \sum_{\nu \in F^{-n, \gamma}} \int \frac{\phi}{\det D F_{\nu}} \partial F_{\nu} \cdots \partial F_{\nu, n} u \circ \nu \]
\[ \leq \lambda_{-\rho n} C_{\mu, n}^2 \mu^2 \rho n \|h_n\|_{C^0} \sum_{\nu \in F^{-n, \gamma}} \left\| \frac{h_{\nu}'}{\det D F_{\nu}} \right\|_{C^0} \|u\|_\rho + C_{n, \rho} \|u\|_{\rho - 1} \]
\[ \leq \lambda_{-\rho n} C_{\mu, n}^2 \mu^2 \rho n \|h_n\|_{C^0} \sum_{\nu \in F^{-n, \gamma}} \left\| \frac{h_{\nu}'}{\det D F_{\nu}} \right\|_{C^0} \|u\|_\rho + C_{n, \rho} \|u\|_{\rho - 1}, \]

hence (4.4) with $\bar{a}_\rho = 1 + a_\rho (\rho^2 + \rho (\rho + 1)/2 + 1)$ and $\bar{b}_\rho = 1 + \rho (2\rho^2 + \rho/2 + 1)$. \qed

The last result of this section is a Corollary of Theorem 4.1 which provide the inequality we are truly interested in.

**Corollary 4.6.** Let us assume that, for every integer $1 \leq \rho \leq r - 1,$
\[ \mu^{\bar{b}_\rho} \lambda_{-\frac{r}{\rho}} < 1, \]
where $\bar{b}_\rho$ given in Theorem 4.1. Let $\delta_* \in (\lambda_{-\frac{1}{\rho}}, 1)$. Then, for each $n \in \mathbb{N}$,
\[ \|L^n u\|_\rho \leq C_{\delta_*}^{\delta_{\rho n}} \|u\|_\rho + C_{\mu, n} \|u\|_0. \]

**Proof.** Let us set $\delta := \lambda_{-\frac{1}{\rho}}$ and take $\bar{n} \in \mathbb{N}$ large enough such that $C_{\mu, n} \delta_{\rho n} \lambda_{-\rho \bar{n}} < \delta_{\rho \bar{n}}$ for every $\rho \in [1, r - 1]$. Notice that this is possible by the definition of $C_{\mu, n}$ and (4.32).

Let us proceed by induction on $\rho$. For $\rho = 1$ the statement is simply (4.3). Let us assume it true for each integer smaller then or equal to $\rho - 1$. By Theorem 4.1 and (4.32), we have
\[ \|L^n u\|_\rho \leq C_{\delta_*}^{\delta_{\rho \bar{n}}} \|u\|_\rho + C_{\bar{n}} \|u\|_{\rho - 1}. \]

For every $m \in \mathbb{N}$ we write $m = \bar{n} q + r$, $0 \leq r < \bar{n}$, and iterate (4.34) to have
\[ \|L^m u\|_\rho = \|L^{\bar{n}}(L^{m - \bar{n}} u)\|_\rho \leq C_{\delta_*}^{\delta_{\rho \bar{n}}} \|L^{m - \bar{n}} u\|_\rho + C_{\bar{n}} \|L^{m - \bar{n}} u\|_{\rho - 1} \leq \cdots \]
\[ \cdots \leq C_{\delta_*}^{\delta_{\rho \bar{n}}} \|L^r u\|_\rho + C_{\mu, n} \sum_{k=0}^{q-1} \delta_{\rho \bar{n}}^k \|L^{m - (k+1) \bar{n}} u\|_{\rho - 1} \leq C_{\delta_*}^{\delta_{\rho \bar{n}}} \|u\|_\rho + C_{\mu, m} \|u\|_{\rho - 1}, \]

where we used $\|L^{m - (k+1) \bar{n}} u\|_{\rho - 1} \leq C_{\mu, m} \|L^{m - (k+1) \bar{n}} u\|_{\rho - 1}$ by the inductive assumption. We iterate the last inequality $\rho$ times and obtain
\[ \|L^m u\|_\rho \leq C_{\mu, m}^{(\mu^{\bar{b}_\rho} \lambda_{-\frac{r}{\rho}})^m} \|u\|_\rho + C_{\mu, m} \|u\|_0 \]
\[ \leq C_{\mu, m}^{(\mu^{\bar{b}_\rho} \lambda_{-\frac{r}{\rho}})^m} \|u\|_\rho + C_{\mu, m} \|u\|_0. \]

We then consider the above inequality for $m$ such that $\rho m = \bar{n}$, so that $C_{\mu, m}^{(\mu^{\bar{b}_\rho} \lambda_{-\frac{r}{\rho}})^m} < \delta_{\rho \bar{n}}$, as $\mu^{\bar{b}_\rho} \leq \mu^{\bar{b}_\rho} \lambda_{-\frac{r}{\rho}} < 1$ by assumption. Hence,
\[ \|L^n u\|_\rho \leq \delta_{\rho \bar{n}} \|u\|_\rho + C_{\mu, \bar{n}} \|u\|_0. \]

Finally, we iterate once again (4.35) and we obtain the result for some $\delta_* \in (\delta, 1) = (\lambda_{-\frac{r}{\rho}}, 1)$. \qed

**Remark 4.7.** Although Corollary 4.6 provides a Lasota-Yorke inequality, a fundamental ingredient is missing. Indeed the embedding of $\mathcal{B}_\rho$ in $\mathcal{B}_0$ is not compact.
5. A second Lasota-Yorke inequality: preliminaries

The main result of the following two sections is the second step towards the proof of Theorem 2.7, namely a Lasota-Yorke type inequality between the Hilbert space $\mathcal{H}^s$ and $B_{\rho}$. We will see in Corollary 3.17 that this solves the compactness problem mentioned in Remark 4.7. First we state some result on the $\mathcal{H}^s$-norm of the transfer operator.

5.1. $\mathcal{H}^s$-norm of $L$.

**Lemma 5.1.** Let $F \in C^r(\mathbb{T}^2, \mathbb{T}^2)$ satisfying (H1). For each $n \in \mathbb{N}$ and $1 \leq s \leq r$, there exist $A_s, Q(n, s) > 0$ such that, for every $u \in \mathcal{H}^s(\mathbb{T}^2, \mathbb{R})$,

\[
\begin{align*}
(5.1) \quad & \|L^nu\|_{L^2} \leq \|L^1 u\|_{L^2}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{1}{2}} \\
(5.2) \quad & \|L^nu\|_{\mathcal{H}^s}^2 \leq A_s \mu^{2sn} \|L^1 u\|_{\mathcal{H}^s}^2 + Q(n, s)\|u\|_{\mathcal{H}^{s-1}}^2,
\end{align*}
\]

where $Q(n, 1) \leq C_{\mu, n}\mu^{2n}$.

**Proof.** First of all notice that

\[
\|L^nu\|_{L^2}^2 \leq \|u\|_{L^2} \left( \int (L^n u \circ F^n)^2 \right)^{\frac{1}{2}} \leq \|u\|_{L^2} \left( \int (L^n u)^2 L^1 u \right)^{\frac{1}{2}},
\]

hence (5.1). Next, by (4.13) and (4.14) we have, for each $v_i \in \{e_1, e_2\}$,

\[
\|\partial_{v_1} \cdots \partial_{v_1} L^nu\|_{L^2}^2 \leq \|L^n (\partial_{F^n v_1} \cdots \partial_{F^n v_1}) u\|_{L^2}^2 + \sum_{k=1}^s \|L^n (A_{n,k}((A_n^{\alpha k} \phi_n) \cdot P_{n,k-1}^- u))\|_{L^2}^2.
\]

Let us analyse the first term above when $s = 2$. Notice that

\[
\partial_{F^n v_2} (\partial_{F^n v_1} u) = \langle \nabla \left( (\nabla u, (DF^n)^{-1} v_1) \right), (DF^n)^{-1} v_2 \rangle = \langle (DF^n)^{-1} v_1 \nabla u, (DF^n)^{-1} v_2 \rangle + \langle D((DF^n)^{-1} v_1) \nabla u, (DF^n)^{-1} v_2 \rangle.
\]

where $D^2 f$ indicates the Hessian of a function $f$ and $D(V)$ is the Jacobian of the vector field $V$. The term with higher derivatives of $u$ has coefficients bounded by $\|(DF^n)^{-1}\|^2$, while the other term is a differential operator of order one applied to $u$. In the general case we can find some $P_{s-1, 0}$ such that

\[
|L^n (\partial_{F^n v_1} \cdots \partial_{F^n v_1}) u| \leq \|(DF^n)^{-1}\|^s L^n (|\partial_{v_1} \cdots \partial_{v_1} u|) + L^n u P_{s-1, 0} u|.
\]

Hence, by (5.1), (C.4) and (3.17), there exists a constant $C_1(n, s)$ such that

\[
\|\partial_{v_1} \cdots \partial_{v_1} L^nu\|_{L^2}^2 \leq C_2 \|L^1 u\|_{\mathcal{H}^s} \|u\|_{\mathcal{H}^{s-1}}^2 + C_1(n, s)\|u\|_{\mathcal{H}^{s-1}}^2.
\]

Similarly there exists $C_2(s, n)$ such that

\[
\sum_{k=1}^s \|L^n (A_{n,k}((A_n^{\alpha k} \phi_n) \cdot P_{n,k-1}^- u))\|_{L^2}^2 \leq C_2(n, s)\|u\|_{\mathcal{H}^{s-1}}^2.
\]

By (5.4), (5.6) and (5.7) we obtain

\[
\|L^n (\partial_{F^n v_1} \cdots \partial_{F^n v_1}) u\|_{L^2}^2 \leq C_2 \|L^1 u\|_{\mathcal{H}^s} \|u\|_{\mathcal{H}^{s-1}}^2 + Q(n, s)\|u\|_{\mathcal{H}^{s-1}}^2.
\]

---

34 See Appendix C for definitions and properties of $\mathcal{H}^s(\mathbb{T}^2)$. 


It remains to prove that in the case $s = 1$ we have an explicit bound on $Q(n, 1)$. Recall that by (4.9) and (5.1) we have, for any $v \in \{e_1, e_2\}$,
\begin{equation}
\|\langle \nabla L^n, v \rangle \|_{L^2} \leq \| \mathcal{L}^n (\nabla u, (DF^n)^{-1} v) \|_{L^2} + \| \mathcal{L}^n (\nabla \phi_n, (DF^n)^{-1} v) u \|_{L^2},
\end{equation}
(5.8)
\begin{equation}
\leq \| \mathcal{L}^n_e \|_{L^2} \left( \| \langle \nabla u, (DF^n)^{-1} v \rangle \|_{L^2} + \| \langle \nabla \phi_n, (DF^n)^{-1} v \rangle u \|_{L^2} \right).
\end{equation}
A bound for the first term is straightforward, since by (3.17)
\begin{equation}
\| \langle \nabla u, (DF^n)^{-1} v \rangle \|_{L^2} \leq C \| u \|_{L^2}.
\end{equation}
(5.9)
For the second term we use formula (4.20) and we have
\begin{equation}
\| \langle \nabla \phi_n, (DF^n)^{-1} v \rangle u \|_{L^2} \leq \sum_{j=0}^{n-1} \| \langle \nabla \phi_j \circ F^j (x), (DF^j)^{-1} v \rangle u \|_{L^2}
\end{equation}
(5.10)
\begin{equation}
\leq C \mu^n \| u \|_{L^2} \leq C \mu^n \| u \|_{L^2}.
\end{equation}
By (5.8), (5.9), (5.10) and (3.107) we obtain (5.2) for $s = 1$.

5.2. Transversality. In this Section we give some useful definitions and results related to the quantities $\mathcal{N}_F, \tilde{\mathcal{N}}_F$ defined in section 2.2. Recall that
\begin{equation}
\mathcal{N}_F(n) = \sup_{y \in \mathbb{T}^2} \sup_{z_1 \in F^{-n}(y)} \mathcal{N}(n, y, z_1)
\end{equation}
(5.11)
\begin{equation}
\tilde{\mathcal{N}}_F(n) = \sup_{y \in \mathbb{T}^2} \sup_{L} \tilde{\mathcal{N}}(n, y, L).
\end{equation}
Assume that there exists an integer $n_0 \in \mathbb{N}$ such that
\begin{equation}
n_0 := \min \{ n \in \mathbb{N} : \forall p \in \mathbb{T}^2 \exists z_1, z_2 \in F^{-n} p : z_1 \cap z_2 \} < \infty.
\end{equation}
(5.12)

Remark 5.2. In [48] it is proven that assumption (5.12) is generic. In addition, in Section 8 we will introduce a large set of systems for which (5.12) is satisfied.

Both $\mathcal{N}_F$ and $\tilde{\mathcal{N}}_F$ depend on the map $F$, however in the following we will drop the $F$ dependence to ease notation. An important advantage of $\tilde{\mathcal{N}}$ over $\mathcal{N}$ is the following

Proposition 5.3. $\tilde{\mathcal{N}}(n)$ is sub-multiplicative, i.e $\tilde{\mathcal{N}}(n + m) \leq \tilde{\mathcal{N}}(n) \tilde{\mathcal{N}}(m)$, for every $n, m \in \mathbb{N}$.

Proof. For any $z \in \mathbb{T}^2$, let us call $L'$ the line obtained applying $(DF^n(z))^{-1}$ to $L$. Then
\begin{equation}
\tilde{\mathcal{N}}(y, L, n + m) = \sum_{z \in F^{-n-m}(y)} | \det DF^{n+m}(z) |^{-1}
\end{equation}
\begin{equation}
= \sum_{\tilde{z} \in F^{-n}(y)} \sum_{z \in F^{-m}(\tilde{z})} \frac{1}{| \det DF^m(z) | \det DF^n(z) | \det DF^m(\tilde{z}) |} \sum_{z \in F^{-m}(\tilde{z})} \frac{1}{| \det DF^m(z) | | \det DF^m(\tilde{z}) |}
\end{equation}
taking the sup over $y \in \mathbb{T}^2$ and $L$ we get the claim. \qed
Remark 5.4. The above Proposition, in spite of its simplicity, turns out to be pivotal. The sub-multiplicativity of the sequence $\tilde{N}(n)$ implies the existence of $\lim_{n\to\infty} \tilde{N}(n)^\frac{3}{2}$. Also, an estimate of $\tilde{N}(n_0)$ for some $n_0 \in \mathbb{N}$ yields an estimate for all $n \in \mathbb{N}$.

The result below, inspired by [13], provides the relation between $N$ and $\tilde{N}$.

Lemma 5.5. Let $\alpha = \frac{\log(\lambda - \mu - 1)}{\log(\lambda + \mu)} \in (0, 1)$ and $m_0 = m_0(n) = \lceil \alpha n \rceil$ we have, for all $n \in \mathbb{N}$

$$N(n)^\frac{3}{2} \leq \|L^{n-m_0}\|\tilde{N}(m_0)^\frac{3}{2}.$$

Proof. Given $y \in \mathbb{T}^2$, we consider $z_1, z_2 \in F^{-n}(y)$ such that $DF^n(z_1)C_u \cap DF^n(z_2)C_u \neq \{0\}$ and the line $L := L(z_1) := DF^n(z_1) \cap (\mathbb{R} \times \{0\})$. Note that in the projective space $\mathbb{R}P^2$ the cones are canonically identified with two intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$, while the line is a point that we also denote by $L$. From the assumption on the cones, and the definition of $L(z_1)$ we have that the distance between $L$ and each one of the extremal points of $I_2$ is bounded by

$$(5.13) \quad \{\text{dist}(L, a_2), \text{dist}(L, b_2)\}^+ \leq C_1 \lambda^{-\mu} m^n.$$

Let us now take $m < n$ to be chosen later and, for $z = F^{-m}(z_2)$, consider the cone $DF^m(z)C_u$ corresponding to the interval $I_3$ in the projective space. By the forward invariance of the unstable cone it is clear that $DF^m(z)C_u \cap DF^n(z_2)C_u$, meaning that $I_3 \supseteq I_2$. Now, again from (2.5) the length of $I_3$ is bounded from below by $C_2 \lambda^{-\mu} m^{-m}$. Then if we choose $m$ such that

$$(5.14) \quad C_2 \lambda^{-\mu} m^{-m} \geq C_1 \lambda^{-\mu} m^n,$$

we obtain from (5.13) that $L \in I_3$. Let us define $\alpha := \frac{\log(\lambda - \mu - 1)}{\log(\lambda + \mu)}$, $C = \frac{C_2}{C_1}$, and $\beta := \frac{\log C}{\log(\lambda + \mu)}$, then inequality (5.14) is satisfied choosing $m = \lceil \alpha n + \beta \rceil$. The above computation shows that, given $z_1 \in F^{-n}(y)$, for every $z_2 \in F^{-n}(y)$ which is non-transversal to $z_1$, the line $L$ is contained in the cone $DF^m(z)C_u$, for $z = F^{-m}(z_2)$. In particular, for every $y \in \mathbb{T}^2$, one has

$$\sup_{z_1 \in F^{-n}(y)} \sum_{z_2 \in F^{-n}(y), \ z_2 \neq z_1} |\det DF^n(z_2)|^{-1} \leq \sup_{L \in \mathbb{R}P^2} \sum_{z_2 \in F^{-n}(y), \ DF^m(z)C_u \supseteq L} |\det DF^n(z_2)|^{-1} \leq \sup_{L \in \mathbb{R}P^2} \sum_{z \in F^{-m}(y), \ DF^m(z)C_u \supseteq L} |\det DF^m(z)|^{-1} \leq \mathcal{L}^{n-m}1(z) \sup_{L \in \mathbb{R}P^2} \sum_{z \in F^{-m}(y), \ DF^m(z)C_u \supseteq L} |\det DF^m(z)|^{-1},$$

where we have used (2.13). The above inequality then implies

$$N(n) \leq \|\mathcal{L}^{n-m}\|\infty \tilde{N}(m) \leq \|\mathcal{L}^{n(1-\alpha)}\|\infty \tilde{N}(\lceil \alpha n \rceil). \quad \square$$
6. A second Lasota-Yorke inequality: Results

To state the main result we need a few definitions. From Appendix C we recall that, for positive integers \( N \in \mathbb{N} \) and \( s \geq 1 \), and for \( u \in C^r(\mathbb{T}^2) \),
\[
\| \mathcal{L}^N u \|_{H^s} = \sum_{\xi \in \mathbb{Z}^2} |\langle \xi \rangle^s \mathcal{F} \mathcal{L}^N u(\xi)|^2,
\]
where \( \langle \xi \rangle = \sqrt{1 + |\xi|^2} \). Since we will work in Fourier space, it is convenient to introduce the notion of the dual of a cone in \( \mathbb{R}^2 \) by:
\[
\mathcal{C}^\perp = \{ v \in \mathbb{R}^2 \text{ s.t. } \exists u \in \mathcal{C} : <v, u> = 0 \},
\]
and if \( \xi \in \mathbb{Z}^2 \) we define \( \xi^* := (\xi^*_1, \xi^*_2) \) to be the unit vector normal to \( \xi \) with the usual orientation. In addition, we define \( \rho(\xi^*) = |\xi^*_1|/|\xi^*_1| \), for \( \xi^*_1 \neq 0 \), and \( \rho(\pm e_2) = \infty \), and
\[
\vartheta(\xi^*) := \{ \rho(\xi^*), \chi_u \}^\perp.
\]
Finally, we define the sequence
\[
\Pi_n := \| \mathcal{L}^n 1 \|_{\infty}.
\]
The scope of this Section is to prove the following Theorem.

**Theorem 6.1.** Let \( m_{\chi_u} \) and \( n_0 \) be the integers given in (3.8) and (5.12) respectively. There exist: \( C_1, C_{q_0}, c_t > 0 \), \( \Lambda > 2 \) and \( \sigma > 1 \) such that, for any \( 1 \leq s < r \) and \( q_0 > n_0 \), if \( M = \sigma m_{\chi_u} \) and \( N = M + q_0 \),
\[
\| \mathcal{L}^N u \|_{H^s} \leq C_1 \left( \sqrt{[\Pi_M N(q_0)]^{\frac{1}{4} \mu^{2s}}} \right)^N \| u \|_{H^s} + \Theta_{\chi_u}(M, s) \| u \|_{s+2}.
\]
where \( \Theta_{\chi_u}(N, s) \lesssim C_{q_0} (M, s) C_{\mu, M}^{q_0 M} \) and \( Q(M, s) \) is the constant given in Lemma 5.1. In addition, if the map \( F \) satisfies the following condition
\[
\chi_u^{-1} \| \omega \|_{C^r} = C_4,
\]
then there exist real numbers \( \beta_3, \beta_4 > 0 \) such that
\[
\Theta_{\chi_u}(M, 1) \leq C_4 C_{q_0} \chi_u^{-\frac{1}{4} \mu^{2s}} C_{\mu, M}^{\beta_3} M^\frac{1}{2}.
\]
We will prove Theorem 6.1 in Section 6.4, after several steps.

6.1. Partitions of unity. We will use notations and definitions given in Section 3.2.1. First of all we want to decompose the transfer operator using suitable partitions of unity. For each point \( z \in \mathbb{T}^2 \), and \( q_0 \geq n_0 \), let us set \( \delta_{q_0}(z) := \mu_{q_0}(z) \lambda_{q_0}^{-1} \), \(^{35}\) and define
\[
\mathcal{U}_{z, q_0} = \{ y \in \mathbb{T}^2 : \| y - z \| \leq d \delta_{q_0}(z) \},
\]
where\(^{36}\)
\[
d = d(\chi_u) = L_{\ast}(q_0, \chi_u)^{-1} \lambda_{0} \chi_u.
\]
By Besicovitch covering theorem there exists a finite subset \( \mathcal{A} \) and points \( \{ z_0 \}_{\alpha \in \mathcal{A}} \) such that \( \mathbb{T}^2 \subset \bigcup_{\alpha} \mathcal{U}_\alpha \) where \( \mathcal{U}_\alpha = 5 \mathcal{U}_{z_\alpha, q_0} \), and such that the number of intersections is bounded by some fix constant \( C_4 \). We then define a family of smooth function \( \{ \psi_\alpha \}_{\alpha} \) supported on \( \mathcal{U}_\alpha \) such that \( \sum_\alpha \psi_\alpha = 1 \). Next we construct a refinement of the above partition using the inverse branches introduced in Section 3.2. For \( \alpha \in \mathcal{A} \) we pick two

\(^{35}\)The functions \( \mu_{q_0} \) and \( \lambda_{q_0} \) are defined in (2.3).

\(^{36}\)Recall that \( L_{\ast} \) is the Lipschitz constant of the unstable cone field given in (E.2).
curves $\gamma_\alpha, \tilde{\gamma}_\alpha \in Y$ such that $U_\alpha \cap \gamma_\alpha = \{0\}$ and, recalling $F_{\gamma_\alpha,1} = \{ h \in F : D_h = T^2 \setminus \gamma_\alpha \}$, for each $h \in F_{\gamma_\alpha,1} \cup F_{\tilde{\gamma}_\alpha,1}$ either $h(T^2) \cap \gamma_\alpha = \emptyset$ or $h(T^2) \cap \tilde{\gamma}_\alpha = \emptyset$. Note that the cardinality of $F_{\gamma_\alpha,0} := F_{\gamma_\alpha,1}$ and $F_{\tilde{\gamma}_\alpha,1}$ is exactly $d$.

We can then consider the set $F^n_{\alpha} = \{ (h_1, \ldots, h_n) \in F^n : h_j \in F_{\gamma_j-1, \alpha}, j \in \{1, \ldots, n\} \}$ where $\gamma_j = \gamma_\alpha$ if $h_j(T^2) \cap \gamma_\alpha = \emptyset$ and $\gamma_j = \tilde{\gamma}_\alpha$ if $h_j(T^2) \cap \tilde{\gamma}_\alpha \neq \emptyset$. Note that the $F^n_{\alpha}$ has an element for each equivalence class of $F^n_{\alpha, \gamma_\alpha}$, defined in equation (3.6), hence it is isomorphic to $F^n_{\alpha, \gamma_\alpha}$ and has exactly $d^n$ elements.

Next, let

$$\psi_{\alpha,h}(z) = \psi_\alpha \circ F^{y_0}(z) I_{h,\alpha}(z), \quad \forall h \in F^{y_0}, z \in T^2,$$

where $I_{h,\alpha} := I_{U_{h,\alpha}}$, and $U_{h,\alpha} := h(U_{\alpha})$. Notice that (6.10) defines again a $C^\infty$ partition of unity, supported on $\{U_{h,\alpha}\}_{h \in F^{y_0}}$, which have intersection multiplicity bounded by $C_2$.

We have the following result from [3, Lemma 9], whose proof is adapted to our diagram.

**Lemma 6.2.** For each $u \in C^r(T^2)$

$$\sum_{\alpha \in A} \sum_{h \in F^{y_0}} \left\| u \psi_{\alpha,h} \right\|_{H^r}^2 \leq C_2 \sum_{\alpha \in A} \left\| u \psi_\alpha \right\|_{H^r}^2,$$

where $C_\psi(s)$ depends on $\psi_\alpha$, while $C > 0$ does not. However, $C_\psi(1) \lesssim C_{y_0}(\delta e)^{-2}$.

**Proof.** For the first inequality note that

$$\left\| u \right\|_{H^r}^2 = \left\| \sum_{\alpha \in A} u \psi_{\alpha} \right\|_{H^r}^2 = \sum_{(\alpha, \alpha') \in A \times A} \left\langle \psi_{\alpha} u, \psi_{\alpha'} u \right\rangle.$$  

By the definition of the $\langle \cdot, \cdot \rangle_s$ the above sum is zero if the supports of $\psi_\alpha$ and $\psi_{\alpha'}$ do not intersect. For the other terms, denoting with $A^*$ the set of elements in $A \times A$ for which the above supports intersect, we have:

$$\sum_{A^*} \left\langle \psi_{\alpha} u, \psi_{\alpha'} u \right\rangle_s \leq \sum_{A^*} \frac{\left\| \psi_{\alpha} u \right\|_{H^r}^2 + \left\| \psi_{\alpha'} u \right\|_{H^r}^2}{2} \leq C_2 \sum_{\alpha \in A} \left\| \psi_{\alpha} u \right\|_{H^r}^2.$$  

We now prove (6.12). By formula (C.4) we have

$$\sum_{\alpha, h} \left\| u \psi_{\alpha,h} \right\|_{H^r}^2 \lesssim \sum_{\alpha, h, |\beta| \leq s} \left\| \partial^\beta (u \psi_{\alpha,h}) \right\|_{L^2}^2 \lesssim \sum_{\alpha, h, |\beta| \leq s} \int_{T^2} \left| \partial^\beta u \right|^2 |\psi_{\alpha,h}|^2 + \sum_{\alpha, h, |\beta| \leq s} \sum_{|\gamma| < |\beta|} C_{\beta, \gamma} \int_{T^2} \left| \partial^\beta u \right|^2 |\partial^\gamma - \gamma| \psi_{\alpha,h}|^2 \leq C_2 \left\| u \right\|_{H^r}^2 + C_\psi(s) \| u \|_{L^2}^2 \lesssim C_2 \| u \|_{H^r}^2 + C_\psi(s) \| u \|_{L^2}^2,$$

where in the last line we used the fact that the $\psi_{\alpha,h}$ are partitions of unity and Lemma C.1. This proves (6.12) in the general case $s \geq 1$. Next we compute explicitly the second summation in the second line above for $s = 1$, which is bounded by:

$$\sum_{\alpha, h} \int_{T^2} |u|^2 |\nabla (\psi_{\alpha,h})|^2 \leq \sum_{\alpha, h} \int_{T^2} |u|^2 |(DF^{y_0})^t \nabla \psi_\alpha \circ F^{y_0} I_{h,\alpha}|^2 \leq \| (DF^{y_0}) \|_{C^\infty} \sum_{\alpha} \int_{h \in F^{y_0}} \left| \nabla \psi_\alpha \right|^2 \circ (DF^{y_0}) I_{h,\alpha} |u|^2 \leq C_{y_0} \sup_{\alpha} \left\| \psi_\alpha \right\|_{C^1}^2 \| u \|_{L^2}^2.$$.  


Finally, since $\psi_0$ is supported on $U_0$ which has diameter bounded by $d_0 \mu^{-\theta_0} \lambda_+^{-\theta_0}$, it is easy to see that there exists $C_{q_0}$ such that

$$\|\psi_0\|_{C^1} \lesssim C_{q_0} \left(4e\right)^{-1},$$

hence $C_{\psi}(1) \lesssim C_{q_0} \left(4e\right)^{-1}$, from which we conclude. □

**Remark 6.3.** Note that, under condition (6.6), recalling (E.2), we have

$$L_\epsilon(\chi_u, q_0) = C_\epsilon C_{q_0} \lambda_1^{\frac{1-\epsilon}{\epsilon}},$$

which implies that, by (6.9), $C_{\psi}(1) \leq \epsilon^{-2} C_{\psi} \lambda_1^{\frac{1-\epsilon}{\epsilon}}$.

The next Proposition is the main ingredient for the proof of Theorem 6.1.

**Proposition 6.4.** For each $\xi \in \mathbb{Z}^2$, $m \in \mathbb{N}$, $q_0 \geq n_0$, $\tilde{h} \in \tilde{h}^0$, $\alpha \in A$ and $h \in H^0$ such that $\tilde{h} = h \circ h$ is well-defined, $D_p \tilde{h}_m \xi^* \in C_{\epsilon, \kappa}$ for each $p \in \text{supp} \psi_{\alpha, h}$ and $D_p \tilde{h}_m \xi^* \notin C_u$, there exists $M_\ell \geq \sigma m$, with $\sigma$ as in (3.63), such that, for each $t \geq 2$,

$$\left(\psi_{\alpha, h}(\xi)\right)^{\frac{1}{\epsilon}} \mathcal{L}^{\psi_{\alpha, h}}(\xi) \lesssim K_1(t, M_\ell, m) \|u\|_t,$$

where $K_1(t, M_\ell, m) \leq C_t C_{\epsilon, \kappa} C_{\alpha, \kappa} M_\ell$, with $C_{\psi, q_0}$ a constant which depends on $\psi_{\alpha, h}$. In addition, if the map satisfies condition (6.6), then there exist $C_{c_{q_0}}, \beta_1, \beta_2 > 0$ such that

$$K_1(t, 2, M_\ell, m) \leq C_t C_{\epsilon, \kappa} C_{\alpha, \kappa} \lambda_1^{\frac{1-\epsilon}{\epsilon}}.$$  

**Proof.** Let $\xi = (\xi_1, \xi_2)$, let $j \in \{1, 2\}$ such that $\|\xi\| \leq 2|\xi_j|$, and $M_\ell > 0$ be chosen later. Since $\xi_j \mathcal{F} = -i \mathcal{F} \partial_{\xi_j} u$, $\mathcal{F} u = \|u\|_\infty \lesssim \|u\|_L^\infty$ and using (4.1) we have, for each $t \geq 1$ and setting $u_{\alpha, h}^M = \psi_{\alpha, h} \mathcal{L}^{M_\ell} u$,

$$\left(\psi_{\alpha, h}(\xi)\right)^{\frac{1}{\epsilon}} \mathcal{L}^{\psi_{\alpha, h}}(\xi) \lesssim \mathcal{L}^{\psi_{\alpha, h}}(u_{\alpha, h}^M) \|u\|_t + \|\xi_j\| \mathcal{F}^{\psi_{\alpha, h}}(u_{\alpha, h}^M) \lesssim \|u\|_t + \|\mathcal{F} \partial_{\xi_j} \mathcal{L}^{\psi_{\alpha, h}}(u_{\alpha, h}^M)\|_t.$$  

Let us estimate the last term. Letting $J_0(p) = (\det D_p F^p)^{-1}$ we have

$$\left[\mathcal{F} \partial_{\xi_j} \mathcal{L}^{\psi_{\alpha, h}}(u_{\alpha, h}^M)\right](\xi) = \int_{U_0} dz e^{-2\pi i (z, \xi)} \partial_{\xi_j} [J_{q_0} \psi_{\alpha, h} \mathcal{L}^{M_\ell} u \circ h(z)]$$

$$+ \sum_{|\beta_1| + |\beta_2| = t} C_{\beta_1, \beta_2} \int_{U_0} dz e^{-2\pi i (z, \xi)} \partial_{\beta_1} [\psi_{\alpha, h}] \circ h(z) \cdot \partial_{\beta_2} [J_{q_0} \mathcal{L}^{M_\ell} u \circ h(z)].$$

Operating the change of variables $\gamma(\tau) = z + \ell \xi + \tau \xi^\perp$, where $\xi^\perp$ is the unit vector perpendicular to $\xi$ and $\ell, \tau \in I_{q_0} = [-d_0 \delta_{\theta_0} (z_0), d_0 \delta_{\theta_0} (z_0)],$ we have

$$\left[\mathcal{F} \partial_{\xi_j} \mathcal{L}^{\psi_{\alpha, h}^M}(u_{\alpha, h}^M)\right] \lesssim C_{\beta_1, \beta_2} \int_{I_{q_0}} d\ell \int_{I_{q_0}} d\tau \left\{ \partial_{\beta_1} \psi_{\alpha, h} \cdot \partial_{\beta_2} [J_{q_0} \mathcal{L}^{M_\ell} u \circ h(\gamma(\tau)) \circ h(z)].$$

Let $\overline{m}(z, \tilde{h} \circ h)$ satisfy (3.64) and set $\overline{m}(\tilde{h} \circ h) = \sup_{\gamma \in \gamma_\ell} \sup_{z \in \gamma_\ell} \overline{m}(\tilde{h} \circ h, z).$ We then define

$$M_\ell = \sup_{\tilde{h} \in \tilde{h}^0} \overline{m}(\tilde{h}).$$

---

37 Recall the definition of $\vartheta(\xi^*)$ in (6.3).

38 This is because $\gamma_\ell$ is supported in some $\mathcal{U}\mathcal{M}_{\alpha, q_0}$ given in (6.8), i.e. the integrand is supported on an interval depending on $q_0$.

39 Notice that $\overline{m}$ depends on $\xi$ through $\gamma_\ell$. Also, it would be more precise to call it $\overline{m}(\tilde{h} \circ h)$, but we keep the notation as simple as possible.
and we observe that, by Lemma (3.11), \( M_\xi \geq \sigma m \), where \( \sigma \) is defined in (3.63). Next, we define \( \tilde{\mathcal{F}}_\alpha = \{ \tilde{h} : \tilde{h} \circ \tilde{h} \in \tilde{\mathcal{F}}_\alpha \} \), \( \nu_{\alpha,\tilde{h}} = \mathcal{L}^{M_\xi - \overline{m}(\tilde{h}) + \psi u} \) and write
\[
\mathcal{J}^{M_\xi} = \bigcup_{\tilde{h} \in \tilde{\mathcal{F}}_\alpha} \{ \tilde{h}' \circ \tilde{h} : \tilde{h}' \in \mathcal{J}^{M_\xi - \overline{m}(\tilde{h}) + \psi_0 u} \}
\]
which allows to define the decomposition
\[
\mathcal{L}^{M_\xi} u = \sum_{\tilde{h} \in \tilde{\mathcal{F}}_\alpha} J^{M_\xi - \overline{m}(\tilde{h}) - \psi_0 u} \circ \tilde{h} \cdot \mathcal{L}^{M_\xi - \overline{m}(\tilde{h}) + \psi_0 u} \circ \tilde{h} = \sum_{\tilde{h} \in \tilde{\mathcal{F}}_\alpha} \mathcal{J}^{M_\xi - \overline{m}(\tilde{h}) - \psi_0 u} \circ \tilde{h} \cdot \nu_{\alpha,\tilde{h}} \circ \tilde{h}.
\]
Thus, recalling (4.12),
\[
\left| \mathcal{F} \partial_x \mathcal{L}^{M_\xi}(u_{M_\xi}) \right| 
\leq C_T \sup_{|\beta_1| + |\beta_2| = t} \sum_{\tilde{h} \in \tilde{\mathcal{F}}_\alpha} \int_{t_0}^t \int_{t_0}^t d\tau \left\| \partial^{\beta_1} \psi_{\alpha} \cdot J^{M_\xi - \overline{m}(\tilde{h})} \circ \tilde{h} \circ \tilde{h} \left[ P^\beta_{\overline{m}(\tilde{h})} \nu_{\alpha,\tilde{h}} \right] \circ \tilde{h} \circ \tilde{h} \right\| (\gamma_\ell(\tau))
\]
Next, we apply Lemma E.1 to \( \gamma_\ell \) with \( \delta = d\epsilon \delta_{q_0}(z_\alpha) \), note that the hypotheses of the Lemma are satisfied thanks to the assumptions on \( \xi \). We thus obtain closed curves \( \tilde{\gamma}_\ell \) with \( j + 1 \) derivative bounded by \( C_{q_0, \Delta_\gamma} \). It follows
\[
\left| \mathcal{F} \partial_x \mathcal{L}^{q_0}(u_{M_\xi}) \right| 
\leq C_T \sup_{|\beta_1| + |\beta_2| = t} \sum_{\tilde{h} \in \tilde{\mathcal{F}}_\alpha} \int_{t_0}^t \int_{t_0}^t d\tau \left\| \partial^{\beta_1} \psi_{\alpha} \cdot J^{M_\xi - \overline{m}(\tilde{h})} \circ \tilde{h} \circ \tilde{h} \left[ P^\beta_{\overline{m}(\tilde{h})} \nu_{\alpha,\tilde{h}} \right] \circ \tilde{h} \circ \tilde{h} \right\| (\gamma_\ell(\tau))
\]
Next, we apply, for each inverse branch \( \tilde{h} \circ \tilde{h} \), Lemma 3.11 to the curves \( \tilde{\gamma}_\ell \) and obtain admissible central curves \( \tilde{\nu}_\ell = \nu_\ell \circ \tilde{h}_\ell \circ \overline{m} \).\(^{40}\) Thus, we can rewrite the integrals in the right hand side of the above equation as follows
\[
\int_{T} d\tau \left\{ \partial^{\beta_1} \psi_{\alpha} \cdot J^{M_\xi - \overline{m}(\tilde{h})} \circ \tilde{h} \circ \tilde{h} \left[ P^\beta_{\overline{m}(\tilde{h})} \nu_{\alpha,\tilde{h}} \right] \circ \tilde{h} \circ \tilde{h} \right\} (\tilde{\gamma}_\ell(\tau))
\]
where \( \Psi_{\nu_\ell}(\tau) = h'_\ell \overline{m} \left[ \det D_{\nu_\ell}(\tau) \right]^{-1} \). By Proposition 4.4 applied with \( n = \overline{m}(\tilde{h}) \), \( \phi = \Psi_{\nu_\ell}(\partial^{\beta_1} \psi_{\alpha}) \circ \overline{m}(\tilde{h}) \circ \tilde{h}_\ell \| \Psi_{\nu_\ell}(\partial^{\beta_1} \psi_{\alpha}) \circ \overline{m}(\tilde{h}) \|^{-1} \), \( \psi = 1 \), \( u = \nu_{\alpha,\tilde{h}} \) and \( U = T^2 \), the above integral is bounded by
\[
(6.19) \quad \tilde{C}(t, \overline{m}(\tilde{h}), m) \| \Psi_{\nu_\ell}(\partial^{\beta_1} \psi_{\alpha}) \circ \overline{m}(\tilde{h}) \| c_{\overline{m}} m_{\nu_\ell} \| I_{q_0} \|
\]
where \( |I_{q_0}| \leq 2d\epsilon \delta_{q_0}(z_\alpha) \leq 2d\epsilon \lambda_{q_0} \mu^{q_0} \). Accordingly,
\[
(6.20) \quad \left| \mathcal{F} \partial_x \mathcal{L}^{q_0}(u_{M_\xi}) \right| 
\leq C_T \sup_{|\beta_1| + |\beta_2| = t} \sum_{\tilde{h} \in \tilde{\mathcal{F}}_\alpha} \tilde{C}(t, \overline{m}(\tilde{h}), m) \| \Psi_{\nu_\ell}(\partial^{\beta_1} \psi_{\alpha}) \circ \overline{m}(\tilde{h}) \| c_{\overline{m}} m_{\nu_\ell} \| I_{q_0} \|
\]
\(^{40}\)Notice that \( \nu_\ell \) depends on \( \tilde{h} \), but we drop this dependence for simplicity.
By (4.16), $\tilde{C}(t, \overline{m}(h), m) \leq C_2 \Lambda e^{-M \xi}$ and

$$\tilde{C}(2, \overline{m}(h), m) \leq C_2 \mu^{2M \xi} \sup_{s \in \text{supp} \phi} [\lambda^+_m \circ \dot{v}_t(s) \overline{m}(h) \circ \dot{v}_t(s) + s \overline{m}(h) \circ \dot{v}_t C_{q_0} \Delta \gamma].$$

(6.21)

Note that, by Corollary 4.6

$$\|v_{\alpha, h}\|_t = \|L^{M \xi - \overline{m}(h), + q_0 u}\|_t \leq C_\mu M \xi \mu^{M \xi} \|u\|_t$$

and, by Lemma 3.16, for each $\alpha \in A$

$$\sum_{h \in \overline{h}} \|\Psi_{\alpha}(h)\|_{c_t} \leq \sum_{h \in \overline{h}} \|\Psi_{\alpha}(h)\|_{c_t} \leq A_u(t, M_\xi, m),$$

(6.23)

where

$$A_u(\tau, M_\xi, m) := \left\{ \begin{array}{ll}
C_2 \left( \Delta \gamma + I_{\gamma, \overline{m}(h), m} \right) \mu^{M \gamma, m} & \tau = 0 \\
(C_\mu \overline{m}(h)) \mu^{M \gamma, m} \left( \Delta \gamma + I_{\gamma, \overline{m}(h), m} \right)^2 \mu^{M \gamma, m} & \tau = 1 \\
O_4(\lambda^+_m) \cdot \{\varphi^2, [\overline{M}_0(m, \cdot)]_{\infty}, (\lambda^+_m)^2\}^+ & \tau = 2 \\
C_2 \mu^{2M \xi} & \tau > 2.
\end{array} \right.$$

(6.24)

Since $\|\partial^{\beta_1} \psi_{\alpha} \circ F^{\overline{m}(h)}\|_{c_t} \leq C_2 C_{\mu, \psi_{\alpha}} A_{c^1 \mu^m}$, this concludes the case $t > 2$.

It remains to prove (6.14). In this case we assume (6.6) and we estimate the terms in (6.19) for $t = 2$. Arguing as in Remark 6.3 we first have $\|\psi_{\alpha}\|_{c_t} \leq C_4 C_{\mu, \psi_{\alpha}} \overline{\chi} \mu^{c_1 \mu^m}$.

Next, setting temporarily $\overline{m} = \overline{m}(h)$, $g_{\alpha} = \partial^{\beta_1} \psi_{\alpha}$, and $G_{\alpha}(s) = g_{\alpha} \circ \dot{v}_t \circ h_t \overline{m}(s)$, and recalling that $F^{\overline{m}} \dot{v}_t = \gamma_t \circ h_t \overline{m}$,

$$\begin{align*}
C' &= \langle \nabla g_{\alpha} \circ \dot{v}_t \circ h_t \overline{m}, \gamma_t \circ h_t \overline{m} \rangle \\
C'' &= \langle (D \nabla g_{\alpha}) \dot{v}_t \circ h_t \overline{m}, \dot{v}_t \circ h_t \overline{m} \rangle(h_t \overline{m})^2 \\
&+ \langle \nabla g_{\alpha} \circ \dot{v}_t \circ h_t \overline{m}, \dot{v}_t \circ h_t \overline{m} \rangle(h_t \overline{m})^2 + \dot{v}_t \circ h_t \overline{m} h_t \overline{m}^2).
\end{align*}$$

Then, by (3.66) (with $h_{\overline{m}} = h_{\overline{m}, t}$) and since (E.1), (E.2) imply $\Delta \gamma \leq C_{\mu, \psi_{\alpha}} \overline{\chi} \mu^{c_1 \mu^m}$,

$$\|\partial^{\beta_1} \psi_{\alpha} \circ F^{\overline{m}(h)}\|_{c_t} \leq C_{\mu, \psi_{\alpha}} \overline{\chi}^{-1} \mu^{M \xi}$$

(6.25)

Since $(\Psi_{\dot{v}_t} G_{\alpha})' = \Psi_{\dot{v}_t} G_{\alpha} + 2 \Psi_{\dot{v}_t} G_{\alpha} + \Psi_{\dot{v}_t} G_{\alpha}''$, by Lemma 3.16, (6.23) and (6.25)

$$\sum_{h \in \overline{h}} \|\dot{v}_t(\partial^{\beta_1} \psi_{\alpha}) \circ F^{\overline{m}(h)}\|_{c_t} \leq C_{\mu, \psi_{\alpha}} \overline{\chi}^{-1} \mu^{M \xi}$$

(6.26)

To conclude, we need to relate all the quantities to $\partial(\xi^*)$. First we notice that, by (3.61) and Lemma 3.9, $\varphi^2 \leq \overline{\gamma}^{-1} = [(\mu(\xi^*), \overline{\chi})^{-1}]^{-1} = \partial(\xi^*)^{-1}$. Therefore, recalling (3.77), it follows that $\lambda^+_m \circ \dot{v}_t(s) \leq C_2 \partial(\xi^*)^{-1} \mu^m$, for each $s \in \mathbb{T}^2$. Next, choosing $n_\ast = \min\{c_\xi \log \overline{\chi}^{-1}, m\}$ in Lemma 3.9, we can check that

$$a_{\overline{m}} + b_{\overline{m}} \leq C_2; \quad \overline{m}_\ast \leq C_{\mu, n_\ast} = C_{\mu, M_\xi}; \quad \overline{s}_n = C_{\mu, n_\ast} \mu^{6n}.$$
since, by (3.15), \( \zeta_n \leq [C_{\mu, n} + C_{\xi} u \vartheta(\xi)^{-1}] \leq C_{\mu, n} \). Similarly \( \zeta_m \leq C_{\mu, m} \). We can use this to compute, in (3.65),

\[
\| M_{n_0}(m, t) \|_\infty \leq \left\{ C_{q_0, \varepsilon} \chi_u^{-c_5 \ln \mu m^2}, (1 + C_{q_2} \mu^2 m) \vartheta(\xi)^{-1} \mu^m \right\} + \\
\leq \left\{ C_{q_0, \varepsilon} \vartheta(\xi)^{-c_1 \ln \mu m^2}, (1 + C_{q_2} \mu^2 m) \vartheta(\xi)^{-1} \mu^m \right\} + \\
\leq C_{q_0, \varepsilon} \mu^3 \vartheta(\xi)^{-1} \{1, c_5 \ln \mu \}^+.
\]

Consequently, we can also compute

\[
\| M_{n_0}(m, t) \|_\infty \leq \{ \mu^6 C_{q_0, \varepsilon} \chi_u^{-c_5 \ln \mu}, C_{q_0, \varepsilon} \mu^3 \vartheta(\xi)^{-1} \{1, c_5 \ln \mu \}^+, \vartheta(\xi)^{-2}, C_{q_2} \vartheta(\xi)^{-2} \mu^m \} + \\
\leq C_{q_0, \varepsilon} \mu^3 \vartheta(\xi)^{-1} \{1, c_5 \ln \mu \}^+.
\]

Finally, by the above estimates and condition (3.64),

\[
\chi^+_{\mu} \circ \nu_c(s) = \frac{c_q \overline{\mu} M_{n_0}(m, t)}{s_{n_0}} \leq C_{q_0, \varepsilon} \mu^3 \{M_{\xi} + m\} \vartheta(\xi)^{-1} \{1, c_5 \ln \mu \}^+ \\
\leq C_{q_0, \varepsilon} \mu^3 \vartheta(\xi)^{-1} \{1, c_5 \ln \mu \}^+,
\]

so that \( \zeta_{M_{\xi}} \leq C_{q_0, \varepsilon} \mu^3 \vartheta(\xi)^{-1} \{1, c_5 \ln \mu \}^+ \), and we immediately have by (4.16)

\[
(6.27) \quad \hat{C}(2, \overline{\mu}(h), m) \leq C_{q_0, \varepsilon} \mu^3 \vartheta(\xi)^{-3} \{1, c_5 \ln \mu \}.
\]

We can now conclude. Using the above estimates it follows that there are \( a, b > 0 \) such that

\[
J_{\gamma, \mu} \leq C_2, \\
I_{\gamma, \mu, \overline{\mu}, m} \leq C_{q_0, \varepsilon} \mu^6 \chi_u^{-c_5 \ln \mu} + \\
O_4(M_{\xi}, m) \leq C_2 C_{q_0, \varepsilon} C_{\mu, M_{\xi}}^a h M_{\xi},
\]

which imply

\[
(6.28) \quad A_u(0, M_{\xi}, m) \leq C_{q_0, \varepsilon} \mu^{2 M_{\xi}} \chi_u^{-c_5 \ln \mu} \vartheta(\xi)^{-1}, \\
A_u(1, M_{\xi}, m) \leq C_{q_0, \varepsilon} C_{\mu, M_{\xi}} \mu^{7 M_{\xi}} \chi_u^{-c_5 \ln \mu} \vartheta(\xi)^{-2} \\
A_u(2, M_{\xi}, m) \leq C_{q_0, \varepsilon} C_{2} C_{\mu, M_{\xi}}^a h M_{\xi} \vartheta(\xi)^{-1} \{1, c_5 \ln \mu \}^+.
\]

Using this in (6.26) we find \( \beta_1, \beta_2 > 0 \) such that

\[
(6.29) \quad \sum_{b \in \mathcal{B}_n} \| \Psi_{\xi_b}^{(s)} \vartheta(\xi_b) \|_{L^2}^2 \leq C_{q_0, \varepsilon} C_{\mu, M_{\xi}}^{\beta_1} \mu^{2 M_{\xi}} \vartheta(\xi)^{-3} \chi_u^{-c_5 \ln \mu}.
\]

Hence, by (6.20), (6.27), and (6.29), we have

\[
\left| \mathcal{F}_{x_j}^2 \left( C_{\mu, M_{\xi}}^{\beta_1} \mu^{2 M_{\xi}} \vartheta(\xi)^{-6} \chi_u^{-c_5 \ln \mu}, \right. \right| \leq C_{q_0, \varepsilon} C_{\mu, M_{\xi}}^{\beta_1} \mu^{2 M_{\xi}} \vartheta(\xi)^{-6} \chi_u^{-c_5 \ln \mu},
\]

which concludes the proof recalling equation (6.15). \( \square \)

We henceforth consider \( \sigma > 1 \) as in (3.63) and \( m_{x_u} \) as in (3.8), and we define

\[
(6.30) \quad \overline{m}_{x_u} = \sigma m_{x_u}.
\]

\footnote{Recall that \( q_0 \geq n_0 \).}
6.2. Decomposition in Fourier space. Let $\mathcal{Z}_u = \{\xi : \xi^* \in C_u\}$ and $\mathcal{Z}_u^c = \mathbb{Z}^2 \setminus \mathcal{Z}_u$. Recalling that $\rho(\xi^*) = |\xi_1^*||\xi_2^*|^{-1}$, $\rho(e_2) = \infty$,

$$\mathcal{Z}_u = \{\xi : \rho(\xi^*) \leq \chi_u\}; \quad \mathcal{Z}_u^c = \{\xi : \rho(\xi^*) > \chi_u\}.$$  

Next, take $N = q_0 + M$, for some $M \in \mathbb{N}$ to be chosen shortly. For simplicity, it is convenient to introduce the following notation for $A \subset \mathbb{Z}^2$,

$$S_{q_0,M}^\alpha (A, h, b') = \sum_{\xi \in \mathbb{Z}^2} 1_A(\xi)\langle \xi \rangle^{2s}[\mathcal{F}\mathcal{L}^{\zeta_0}(u_{\alpha,b})](\xi)[\hat{\mathcal{F}\mathcal{L}^{\zeta_0}(u_{\alpha,b})}](\xi),$$

where $u_{\alpha,b} = \psi_{\alpha,b}L^M u$. Then, by equation (6.11) we have

$$\|L^N u\|_{H^s}^2 \leq C_2 \sum_{\alpha} \| \sum_{b \in \mathbb{Z}^2} \mathcal{L}^{\zeta_0}(u_{\alpha,b}) \|_{H^s}^2$$

$$= C_2 \sum_{\alpha} \sum_{(h,b') \in \mathbb{Z}^2} \langle \xi \rangle^{2s} \langle \mathcal{F}\mathcal{L}^{\zeta_0}(u_{\alpha,b}) \rangle \langle \mathcal{F}\mathcal{L}^{\zeta_0}(u_{\alpha,b'}) \rangle_s$$

(6.32)

$$= C_2 \sum_{\alpha} \sum_{(h,b') \in \mathbb{Z}^2} \sum_{\xi \in \mathbb{Z}^2} \langle \xi \rangle^{2s} \langle \mathcal{F}\mathcal{L}^{\zeta_0}(u_{\alpha,b}) \rangle \langle \mathcal{F}\mathcal{L}^{\zeta_0}(u_{\alpha,b'}) \rangle \langle \xi \rangle$$

$$= C_2 \sum_{\alpha} \sum_{(h,b') \in \mathbb{Z}^2} S_{q_0,M}^\alpha (\mathcal{Z}_u, h, b') + C_2 \sum_{\alpha} \sum_{(h,b') \in \mathbb{Z}^2} S_{q_0,M}^\alpha (\mathcal{Z}_u^c, h, b').$$

We start estimating the second term in the above equation, in the next section we will treat the term with $\xi \in \mathcal{Z}_u$.

**Lemma 6.5** (Bound on $\mathcal{Z}_u^c$). For each $M \geq \chi_u$, $1 \leq s \leq r-1$, $b' \in \mathcal{N}_0$ and $N = q_0 + M$,

$$\sum_{\xi \in \mathbb{Z}^2} \langle \xi \rangle^{2s} \mathcal{F}\mathcal{L}^{\zeta_0}(u_{\alpha,b}) \langle \xi \rangle \|L^N u\|_{H^s}^2 \lesssim \Theta_s \|u\|_{H^s}^2,$$

(6.33)

where $\Theta_s = (C_{\psi,\zeta_0}C_{\mu,M}M^{\zeta_1+M}c_1)\|u\|_{H^s}^2$ and, under condition (6.6),

$$\Theta_1 := (C_{\psi,\zeta_0}C_{\mu,M}M^{\zeta_1+M}c_1)\|\mathcal{L}^{\zeta_0}(u_{\alpha,b'})\|^2 \lesssim \chi_u^{-11-c_t \log \mu} M.$$

**Proof.** Since $\xi^* \notin C_u$, by the definition of $m_{\chi_u}$ in (3.8), for each $p \in \mathbb{T}^2$, $h \in \mathcal{N}_0$,

$$D_p h_{m_u^x, \alpha} \xi^* \in C_c,$$

$$D_p h_{m_u^z, \alpha} \xi^* \notin C_u,$$

(6.35)

so the hypothesis of Proposition 6.4 are satisfied with $m = m_{\chi_u}$. We will treat the cases $s > 1$ and $s = 1$ separately. In the first case we have, for each $M \geq M_1 \geq \chi_u$,

$$\sum_{\xi \in \mathbb{Z}^2} \langle \xi \rangle^{2s} \mathcal{F}\mathcal{L}^{\zeta_0}(u_{\alpha,b'}) \langle \xi \rangle \|L^N u\|_{H^s}^2 \lesssim \Theta_s \|u\|_{H^s}^2,$$

(6.36)

where we used the fact that $\Lambda > 2$ and the convergence of the series. The statement (6.33) for $s > 1$ then follows since, by Corollary 4.6,

$$\|L^M u\|_{H^s}^2 \leq C_{\mu,M}^2 \|u\|_{H^s}^2, \quad \forall t \geq 1.$$
Let us move on the the $s = 1$ case. For any $R > 0$ let $B_R = \{ \xi \in \mathbb{Z}^2 : ||\xi|| \leq R \}$ and $B_R^* = \mathbb{Z}^2 \setminus B_R$. Then
\[
\sum_{\xi \in \mathbb{Z}^2} |\langle \xi \rangle \mathbb{I}_{\mathbb{Z}_e^*} \mathcal{F} \mathcal{L}^{\varphi_0} (u_{\alpha, b}^M) |^2 \\
= \sum_{\xi \in \mathbb{Z}_e^* \cap B_R} |\langle \xi \rangle|^{-2} |\langle \xi \rangle \mathbb{I}_{\mathbb{Z}_e^*} \mathcal{F} \mathcal{L}^{\varphi_0} (u_{\alpha, b}^M) |^2 + \sum_{\xi \in \mathbb{Z}_e^* \cap B_R^*} |\langle \xi \rangle|^{-3} |\langle \xi \rangle \mathbb{I}_{\mathbb{Z}_e^*} \mathcal{F} \mathcal{L}^{\varphi_0} (u_{\alpha, b}^M) | |\langle \xi \rangle \mathbb{I}_{\mathbb{Z}_e^*} \mathcal{F} \mathcal{L}^{\varphi_0} (u_{\alpha, b}^M) |.
\]
For each $\xi \in \mathbb{Z}_e^*$ we take $M > M_\xi > m_{\chi_u}$ and we apply Proposition 6.4
\[
(6.37) \quad \sum_{\xi \in \mathbb{Z}_e^* \cap B_R} |\langle \xi \rangle | \mathcal{F} \mathcal{L}^{\varphi_0} (u_{\alpha, b}^M) |^2 \leq C_{\mu, \mu}^2 M^2 ||u||_2^2 \sum_{\xi \in \mathbb{Z}_e^* \cap B_R} \langle \xi \rangle^{-2} K_{1} (2, M_\xi, m_{\chi_u})^2
\]
and
\[
(6.38) \quad \sum_{\xi \in \mathbb{Z}_e^* \cap B_R^*} |\langle \xi \rangle | \mathcal{F} \mathcal{L}^{\varphi_0} (u_{\alpha, b}^M) |^2 \leq C_{\mu, \mu}^2 M^2 ||u||_2 ||u||_3 \sum_{\xi \in \mathbb{Z}_e^* \cap B_R^*} \langle \xi \rangle^{-3} K_{1} (2, M_\xi, m_{\chi_u}) K_{1} (3, M_\xi, m_{\chi_u}).
\]
We use the estimate of $K_{1} (2, M_\xi, m_{\chi_u})$ in (6.14) for the sum in (6.37), with $\vartheta (\xi) = \rho (\xi)$, since $\xi \in \mathbb{Z}_e^*$, and we have
\[
(6.39) \quad \sum_{\xi \in \mathbb{Z}_e^* \cap B_R} \langle \xi \rangle^{-2} K_{1} (M_\xi, 2) \lesssim (C_{\epsilon, \eta, \chi_u}^{-c_1 \ln \mu} C_{\mu, \mu}^{\beta_1} \mu^{\beta_2 M_\xi} )^2 \sum_{\xi \in \mathbb{Z}_e^* \cap B_R} \langle \xi \rangle^{-2} \rho (\xi)^{-12} \lesssim (C_{\epsilon, \eta, \chi_u}^{-c_1 \ln \mu} C_{\mu, \mu}^{\beta_1} \mu^{\beta_2 M_\xi}$
\[
\lesssim (C_{\epsilon, \eta, \chi_u}^{-c_1 \ln \mu} C_{\mu, \mu}^{\beta_1} \mu^{\beta_2 M_\xi} )^2 \chi_u^{-11 - c_1 \ln \mu} \log R,
\]
since
\[
(6.40) \quad \sum_{\xi \in \mathbb{Z}_e^* \cap B_R} \langle \xi \rangle^{-2} \rho (\xi)^{-12} \lesssim \int_0^R \int_{\{\tan \theta > \chi_u\}} \frac{1}{1 + \rho^2 (\tan \theta)^{12}} \frac{1}{\rho \rho \theta} \lesssim \chi_u^{-11} \log R.
\]
Similarly, for the sum in (6.38), we have
\[
(6.41) \quad \sum_{\xi \in \mathbb{Z}_e^* \cap B_R^*} \langle \xi \rangle^{-3} K_{1} (2, M_\xi, m_{\chi_u}) K_{1} (3, M_\xi, m_{\chi_u})
\]
\[
\lesssim (C_{\epsilon, \eta, \chi_u}^{-c_1 \ln \mu} C_{\mu, \mu}^{\beta_1} \mu^{\beta_2 M_\xi} ) \sum_{\xi \in \mathbb{Z}_e^* \cap B_R^*} \langle \xi \rangle^{-3} \rho (\xi)^{-6} \chi_u^{c_1 M_\xi}
\]
\[
\lesssim (C_{\epsilon, \eta, \chi_u}^{-c_1 \ln \mu} C_{\mu, \mu}^{\beta_1} \mu^{\beta_2 M_\xi} \chi_u^{-6 - c_1 \ln \mu} R^{-1} \Lambda^{c_2 M_\xi}.
\]
Choosing $R = \Lambda^{c_2 M_\xi}$ by (6.37) and (6.38) we have the following estimate:
\[
(6.42) \quad \sum_{\xi \in \mathbb{Z}^2} |\langle \xi \rangle | \mathbb{I}_{\mathbb{Z}_e^*} \mathcal{F} \mathcal{L}^{\varphi_0} (u_{\alpha, b}^M) |^2 \lesssim (C_{\epsilon, \eta, \chi_u}^{-c_1 \ln \mu} C_{\mu, \mu}^{\beta_1} \mu^{\beta_2 M_\xi} )^2 \chi_u^{-11 - c_2 \ln \mu} M_\xi ||u||_3^2,
\]
from which we conclude the proof of (6.33) also for the case $s = 1$, since $M \geq M_\xi$. □

6.3. The case $\xi^* \in \mathcal{C}_{\alpha, \beta}$. In this case we cannot apply Proposition 6.4 directly as we did in the previous section. The main reason is that there could be “bad” vectors $\xi^*$ which are in an unstable direction, so (6.35) may fail. Here transversality plays a major role.
Lemma 6.6 (Bound on $Z_u$). There exist $C_{q_0}$ such that, for each $M \geq \overline{m}_u$,
\[
\sum_\alpha \sum_{(h, h') \in \mathcal{Y}^{\alpha} \times \mathcal{Y}^{\alpha}, \xi \in \mathbb{Z}^2} \mathbf{1}_{Z_u}'(\xi)(\xi)^{2s} |F\mathcal{L}^{q_0}(u_{\alpha, h})(\xi)| \overline{F\mathcal{L}^{q_0}(u_{\alpha, h'})}(\xi) \leq \mathcal{N}(q_0)\mu^{2s q_0} \sum_{b \in \mathcal{Y}^{\alpha}} \|u_{\alpha, h}\|_{\mathcal{H}^s}^2 + C_{q_0} \sum_{h \in \mathcal{Y}^{\alpha}} \|u_{\alpha, h}\|_{\mathcal{H}^{s+1}}^2 + C_{q_0} Q(M, s) \sqrt{\Theta_s \|u\|_{\mathcal{H}^s}}
\]
where $Q(M, s)$ is given in (5.2) and $\Theta_s$ in Lemma 6.5.

The rest of this Section is devoted to the proof of the above Lemma. We divide the argument in three Steps.

6.3.1. Step I (Local transversality). We need a definition of transversality uniform on the elements the partition of unity (6.10):

Definition 6.7. Given $n \in \mathbb{N}$ and $h, h' \in \mathcal{Y}^n$ we say that $h \cap_\alpha h'$ ($h$ is transversal to $h'$ on $\alpha$ at time $n$) if for every $z \in h(U_\alpha)$ and $w \in h'(U_\alpha)$ such that $F^n(z) = F^n(w) \in U_\alpha$:
\[
(6.43) \quad D_z F^n C_{z, w} \cap D_w F^n C_{z, w} = \{0\}.
\]
Next, we relate the (pointwise) Definition 2.4 to the (local) Definition 6.7.

Lemma 6.8. The constant $C_0$ in (6.9) can be chosen such that: for all $\alpha \in A$, $p \in U_\alpha \subset \mathbb{T}^2$ and $h, h' \in \mathcal{Y}^n$, if $z_1 = h(p)$ and $z_2 = h'(p)$, then $z_1 \cap z_2$ implies $h \cap_\alpha h'$.

Proof. Recall that $z_1 \cap z_2$ means
\[
(6.44) \quad D_{z_1} F^{q_0} C_u \cap D_{z_2} F^{q_0} C_u = \{0\}.
\]
As $C_{u, \epsilon} \subseteq C_u$, clearly $D_{z_1} F^{q_0} C_{u, \epsilon} \subseteq D_{z_1} F^{q_0} C_u$. So the above implies also
\[
D_{z_1} F^{q_0} C_{u, \epsilon} \cap D_{z_2} F^{q_0} C_{u, \epsilon} = \{0\}.
\]
Let $\hat{p} \in U_\alpha$, $\hat{p} \neq p$, and define $\hat{z}_1 = h(\hat{p})$ and $\hat{z}_1 = h'(\hat{p})$. We claim that, for each $v \in C_{u, \epsilon}$, the difference between $D_{z_1} F^{q_0} v$ and $D_{z_2} F^{q_0} v$ is smaller than the opening of $D_{z_1} F^{q_0} C_u$, provided we choose $U_\alpha$ small enough. This suffices to conclude the argument.

We compute a lower bound for the opening of the connected components of $D_{z_1} F^{q_0} C_u \setminus D_{z_1} F^{q_0} C_{u, \epsilon}$. By Proposition 3.5, and by formula (3.14), we deduce that there exists some constant $C_0 > 0$ such that, for each unitary vectors $v \in C_{u, \epsilon}$ and $w \notin C_u \cup C_c$,
\[
\mathcal{L}(D_{z_1} F^{q_0} v, D_{z_1} F^{q_0} w) = \frac{|\det D_{z_1} F^{q_0}| \mathcal{L}(v, w)}{\|D_{z_1} F^{q_0} v\| \|D_{z_1} F^{q_0} w\|} \geq \frac{C_* \chi_u \epsilon}{\mu_{q_0}(z) \Lambda^{q_0}(z)} = C_* \chi_u \epsilon \delta_{q_0}(z_1).
\]
On the other hand let us recall that $u_{b, q_0}(p)$ defined in (3.12) gives the slope of the boundary of the cone $D_{b, q_0}(F^{q_0} C_u$, and it is a Lipschitz function of $p$. In particular Lemma E.1 provides an estimate for the Lipschitz constant $L_*(q_0)$ given in (E.2). Then, by the definition of $U_{c, q_0}$ in (6.8) and (6.9), we have the claim, since
\[
\|D_{z_1} F^{q_0} v - D_{z_1} F^{q_0} w\| \leq L_*(q_0) \|z_1 - \hat{z}_1\| \leq L_*(q_0) L_*(\chi_u, q_0)^{-1} C_0 \chi_u \epsilon \delta_{q_0}(z_1)
\]
\[
\leq C_2 C_0 \chi_u \epsilon \delta_{q_0}(z_1).
\]
Clearly the same is true replacing $z_1, \hat{z}_1, b$ with $z_2, \hat{z}_2, b'$, and the result follows. □
Lemma 6.9. Let \( m_{\chi_u} \) given in (3.8). For every \( p \in \mathcal{U}_u, M > m_{\chi_u} \) and \( \tilde{z}, \tilde{w} \in F^{-q_0}(p) \) such that \( \tilde{z} \cap \tilde{w} \) we have

\[
\mathbb{R}^2 = ((D_z F^M + q_0)^*)^{-1} C_{e,c}^\perp \cup ((D_w F^M + q_0)^*)^{-1} C_{e,c}^\perp,
\]

for every \( z \in \mathfrak{h}(\tilde{z}) \) and \( w \in \mathfrak{h}'(\tilde{w}), \mathfrak{h}, \mathfrak{h}' \in S^M \).

**Proof.** By assumption \( D_z F^{q_0} C_u \cap D_z F^{q_0} C_u = \{0\} \) which, together with condition (3.8) implies that for every \( z_1 \in F^{-M}(\tilde{z}) \) and \( z_2 \in F^{-M}(\tilde{w}) \)

\[
D_z F^{q_0} (D_z F^M(\mathbb{R}^2 \setminus C_c)) \cap D_w F^{q_0} (D_z F^M(\mathbb{R}^2 \setminus C_c)) = \{0\}.
\]

Therefore, setting \( N = q_0 + M \), there are \( z, w \in F^{-N}(p) \) such that

\[
D_z F^N(\mathbb{R}^2 \setminus C_c) \cap D_w F^N(\mathbb{R}^2 \setminus C_c) = \{0\}.
\]

Now we can conclude the argument showing that the above implies the statement. Indeed, equation (6.47) obviously implies \( (D_z F^N(\mathbb{R}^2 \setminus C_c))^\perp \cap (D_w F^N(\mathbb{R}^2 \setminus C_c))^\perp = \{0\} \). For any cone \( \mathcal{K} \subset \mathbb{R}^2 \) and any \( z \in T^2 \), one has \( (D_z F^N \mathcal{K})^* = ((D_z F^N)^*)^{-1} \mathcal{K}^\perp \) and \( (\mathbb{R}^2 \setminus \mathcal{K})^\perp = \mathbb{R}^2 \setminus \mathcal{K}^\perp \). The conclusion then follows using Lemma 6.8 and obtaining the statement for the smaller cones \( C_{e,c}. \)

Using Definition 6.7, and recalling notation (6.31), we have the following decomposition into transversal and non transversal terms:

\[
\sum_{(b, b') \in \mathcal{S}^{q_0} \times \mathcal{S}^{q_0}} S_{q_0, M}^{\alpha}(Z_u, b, b') = \sum_{b \in \mathcal{S}^{q_0}} S_{q_0, M}^{\alpha}(Z_u, b, b) + \sum_{b \in \mathcal{S}^{q_0} \setminus \mathcal{S}^{q_0}} S_{q_0, M}^{\alpha}(Z_u, b, b').
\]

**Step II (Estimate of transversal terms).** In this step we will prove that

\[
\sum_{b \in \mathcal{S}^{q_0} \setminus \mathcal{S}^{q_0}} S_{q_0, M}^{\alpha}(Z_u, b, b') \leq C_d C_{q_0} Q(M, s) \sqrt{\Theta_s} \|u\|_{s+2} \|u\|_{H^s},
\]

where \( \Theta_s \) is given in Lemma 6.5.

If \( M > m_{\chi_u}, \) for \( \mathfrak{h} \cap \mathfrak{h}' \neq \emptyset, \) Lemma 6.8 and Lemma 6.9 imply that, for any \( \xi \in \mathbb{Z}^2, \) either \( (D_z F^N)^* \xi \in (C_{e,c})^\perp \) for every \( z \in \text{supp}(\psi_{\alpha,b}), \) or \( (D_z F^N)^* \xi \in (C_{e,c})^\perp \) for every \( z \in \text{supp}(\psi_{\alpha,b'}). \) We then decompose \( Z_u = Z_1 \cup Z_2, \) where

\[
Z_1 = \{ \xi \in Z_u : (D_z F^N)^* \xi \in C_{e,c}^\perp \land z \in \text{supp}(\psi_{\alpha,b}) \}, \quad Z_2 = Z_u \setminus Z_1,
\]

and we write

\[
S_{q_0, M}^{\alpha}(Z_u, b, b') = S_{q_0, M}^{\alpha}(Z_u \cap Z_1, b, b') + S_{q_0, M}^{\alpha}(Z_u \cap Z_2, b, b').
\]

It is enough to estimate the first addend, the second being analogous. Notice that for each \( \xi \in Z_i, i \in \{1, 2\} \) we can apply Proposition (6.4) with \( m = m_{\chi_u}. \) By the Cauchy-Schwartz
inequality we have

\[(6.51) \quad |S_{q_0, M}^\alpha(Z_u \cap Z_1, \mathfrak{h}, \mathfrak{h}')| \lesssim \left( \sum_{\xi \in \mathbb{Z}^2} |\langle \xi \rangle \mathbf{1}_{Z_u \cap Z_1} F L^{q_0}(u_{\alpha, \beta}^M)^2 \rangle \right)^\frac{1}{2} \|L^{q_0}(u_{\alpha, \beta}^M)\|_{\mathcal{H}^s}.
\]

Moreover, by (5.2), \(\|L^{q_0}(u_{\alpha, \beta}^M)\|_{\mathcal{H}^s} \leq C Q(M, s)\|u\|_{\mathcal{H}^s} \). On the other hand, we can bound the sum inside the square root using the same argument of the proof of Lemma 6.5, since the key condition (6.35) is now replaced by \(\xi \in Z_1\), with the difference that this time \(\phi(\xi) = \chi_u\), so we use the estimate

\[\sum_{\xi \in Z_u \cap Z_1 \cap B_R} \langle \xi \rangle^{-2} \leq C_\chi \chi_u \log R\]

instead of (6.40). We thus have

\[|S_{q_0, M}^\alpha(Z_u \cap Z_1, \mathfrak{h}, \mathfrak{h}')| \lesssim Q(M, s)\sqrt{\tilde{\Theta}} \|u\|_{s+2} \|u\|_{\mathcal{H}^s}.
\]

Of course the same computation is valid for the second term of (6.50) from which, summing over \(h \mathfrak{h}_{q_0}^{M} \mathfrak{h}'\), we conclude the proof of (6.49).

**Step III (Estimate of non-transversal terms).** We now want to estimate the sum in (6.48) for \(h \mathfrak{h}_{q_0}^{M} \mathfrak{h}'\). We are going to prove that, for \(N = q_0 + M\),

\[(6.52) \quad \sum_{h \mathfrak{h}_{q_0}^{M} \mathfrak{h}'} \langle L^{q_0}(u_{\alpha, \beta}^M), L^{q_0}(u_{\alpha, \beta}^M) \rangle_s \lesssim N(q_0)\mu_{2^{q_0}} \sum_{h \in \mathcal{F}_q} \|u_{\alpha, \beta}^M\|_{\mathcal{H}^s}^2 + C_{q_0} \sum_{h \in \mathcal{F}_q} \|u_{\alpha, \beta}^M\|_{\mathcal{H}^{s-1}}^2.
\]

Keeping the same notation used previously, we write

\[(6.53) \quad \sum_{h \mathfrak{h}_{q_0}^{M} \mathfrak{h}'} \langle L^{q_0}(u_{\alpha, \beta}^M), L^{q_0}(u_{\alpha, \beta}^M) \rangle_s = \sum_{h \in \mathcal{F}_q} \sum_{h \mathfrak{h}_{q_0}^{M} \mathfrak{h}} \langle L^{q_0}(u_{\alpha, \beta}^M), L^{q_0}(u_{\alpha, \beta}^M) \rangle_s.
\]

By equation (C.4) and the definition of the inner product (C.3), there are \(C_{\gamma, \beta}\) such that

\[(6.54) \quad \langle L^{q_0}(u_{\alpha, \beta}^M), L^{q_0}(u_{\alpha, \beta}^M) \rangle_s = \sum_{\gamma + \beta = s} C_{\gamma, \beta} \langle \partial_{\gamma x_1} \partial_{x_2}^\beta (L^{q_0}(u_{\alpha, \beta}^M)), \partial_{\gamma x_1} \partial_{x_2}^\beta (L^{q_0}(u_{\alpha, \beta}^M)) \rangle_{L^2}.
\]

We then use equation (5.5) and we have, for every \(\gamma, \beta\) such that \(\gamma + \beta = s\)

\[|\partial_{\gamma x_1} \partial_{x_2}^\beta (L^{q_0}(u_{\alpha, \beta}^M))| \leq \|(DF^{q_0})^{-1}\|_{s}^{s} \mathcal{L}^{q_0}(\langle \partial_{\gamma x_1} \partial_{x_2}^\beta u_{\alpha, \beta}^M \rangle) + \mathcal{L}^{q_0}(P_{-1}^{q_0} u_{\alpha, \beta}^M)
\]

where \(P_{-1}^{q_0}\) is a differential operator of order \(s - 1\). By (5.17) \(\|(DF^{q_0})^{-1}\|_{s}^{s} \leq C \mu_{q_0}\). Clearly the same inequality holds for \(\mathfrak{h}'\) and we use this in (6.54) to obtain

\[(6.55) \quad \sum_{\gamma + \beta = s} C_{\gamma, \beta} \langle \partial_{\gamma x_1} \partial_{x_2}^\beta (L^{q_0}(u_{\alpha, \beta}^M)), \partial_{\gamma x_1} \partial_{x_2}^\beta (L^{q_0}(u_{\alpha, \beta}^M)) \rangle_{L^2}
\]

\[\lesssim \mu_{2^{q_0}} \sum_{\gamma + \beta = s} C_{\gamma, \beta} \langle \mathcal{L}^{q_0}(\langle \partial_{\gamma x_1} \partial_{x_2}^\beta u_{\alpha, \beta}^M \rangle), \mathcal{L}^{q_0}(\langle \partial_{\gamma x_1} \partial_{x_2}^\beta u_{\alpha, \beta}^M \rangle) \rangle_{L^2}
\]

\[+ C_{q_0} \|u_{\alpha, \beta}^M\|_{\mathcal{H}^{s-1}} \|u_{\alpha, \beta}^M\|_{\mathcal{H}^{s-1}}.
\]

Since \(u_{\alpha, \beta}^M\) and \(u_{\alpha, \beta}^M\) are supported on invertibility domains of \(F^{q_0}\),

\[(6.56) \quad \mathcal{L}^{q_0}(\langle \partial_{\gamma x_1} \partial_{x_2}^\beta u_{\alpha, \tau}^M \rangle) = \frac{|\partial_{\gamma x_1} \partial_{x_2}^\beta u_{\alpha, \tau}^M| \circ \tau}{|\det DF^{q_0}| \circ \tau}, \quad \tau \in \{\mathfrak{h}, \mathfrak{h}'\}.
\]
We define \(\chi_\tau := |\partial_x^\gamma \partial_y^\beta u_{\alpha,\tau}| \circ \tau\) and \(g_\tau := |\det DF^N| \circ \tau\) and we have

\[
\langle \mathcal{L}^0(\partial_x^\gamma \partial_y^\beta u_{\alpha,\tau}), \mathcal{L}^0((\partial_x^\gamma \partial_y^\beta u_{\alpha,\tau})) \rangle_{L^2} = \int_{\mathbb{T}^2} \frac{\chi_b \chi_{b'}}{\sqrt{g_{bb'} \sqrt{g_{bb'}}}} \leq \frac{1}{2} \int_{\mathbb{T}^2} \frac{\chi_b^2}{g_{bb'}} + \frac{1}{2} \int_{\mathbb{T}^2} \frac{\chi_{b'}^2}{g_{bb'}}
\]

(6.57)

where we used the elementary inequality \(ab \leq \frac{1}{2}(a^2 + b^2)\) with \(a = \frac{\chi_b}{\sqrt{g_{bb'}}}, b = \frac{\chi_{b'}}{\sqrt{g_{bb'}}}\). In order to obtain (6.52), we need to sum equation (6.55) over \(h \in \mathcal{H}^0\) and \(h' \in \mathcal{H}^0\). Let us begin with the first term. Consider one of the integrals in (6.57), for example the first one. By Definition 5.11 of \(\mathcal{N}(q_0)\) and Lemma 6.8 it follows that

\[
\sum_{h} \sum_{h': h' \in \mathcal{H}^0} \int_{\mathbb{T}^2} \frac{\chi_b^2}{g_{hh'}} \leq \frac{\sqrt{\alpha}}{\sqrt{g_{bb'}}} \leq \mathcal{N}(q_0) \sum_{h} \int_{\mathbb{T}^2} \frac{\chi_b^2}{g_{hh'}} \leq \mathcal{N}(q_0) \sum_{h} \|\partial_x^\gamma \partial_y^\beta u_{\alpha,\tau}\|_{L^2}^2 \leq \mathcal{N}(q_0) \sum_{h} \|\partial_x^\gamma \partial_y^\beta u_{\alpha,\tau}\|_{L^2}^2
\]

(6.58)

By symmetry we have

\[
\mu^{2q_0} \sum_{h \in \mathcal{H}^0} \sum_{\gamma + \beta = s} \sum_{h': h' \in \mathcal{H}^0} C_{\gamma,\beta} \langle \mathcal{L}^0(\partial_x^\gamma \partial_y^\beta u_{\alpha,\tau}), \mathcal{L}^0((\partial_x^\gamma \partial_y^\beta u_{\alpha,\tau})) \rangle_{L^2} \leq \mu^{2q_0} \mathcal{N}(q_0) \sum_{h \in \mathcal{H}^0} \|\partial_x^\gamma \partial_y^\beta u_{\alpha,\tau}\|_{L^2}^2 \leq C_2 \mu^{2q_0} \mathcal{N}(q_0) \sum_{h \in \mathcal{H}^0} \|u_{\alpha,\tau}\|_{L^2}^2
\]

(6.59)

which corresponds to the first addend of the r.h.s. of (6.52).

Finally we sum the second term of (6.55) over \(h \in \mathcal{H}^0\), and we write

\[
C_{q_0} \sum_{h \in \mathcal{H}^0} \|u_{\alpha,\tau}\|_{L^2}^2 \leq C_{q_0} \sum_{h \in \mathcal{H}^0} \|u_{\alpha,\tau}\|_{L^2}^2 \leq C_{q_0} \sum_{h \in \mathcal{H}^0} \|u_{\alpha,\tau}\|_{L^2}^2
\]

(6.60)

which yields the second addend of (6.52) and, together with (6.49), conclude the proof of Lemma 6.6.

We are finally ready to prove Theorem 6.1.

6.4. Proof of Theorem 6.1. By (6.32) and Lemmata 6.5 and 6.6,\(^{42}\) we have

\[
\|\mathcal{L}^N u\|_{L^2}^2 \leq \Theta_s \|u\|_{L^2}^2 + C_{q_0} Q(M, s) \sqrt{\Theta_s} \|u\|_{L^2}^2 + \mathcal{N}(q_0) \mu^{2q_0} \sum_{h \in \mathcal{H}^0} \sum_{\alpha} \|u_{\alpha,\tau}\|_{L^2}^2 + C_{q_0} \sum_{h \in \mathcal{H}^0} \sum_{\alpha} \|u_{\alpha,\tau}\|_{L^2}^2
\]

(6.61)

\(^{42}\)Note that, to use (6.33) in (6.32), we just use an inequality analogous to (6.60).
Recalling that \( u^M_{\alpha,b} = \psi_{\alpha,b} \mathcal{L}^M u \), we can use equations (6.12) and (5.2) to write,

\[
\sum_{\alpha} \sum_{b \in \mathcal{B}_0} \| u^M_{\alpha,b} \|_{h^s}^2 = \sum_{\alpha} \sum_{b \in \mathcal{B}_0} \| \psi_{\alpha,b} \mathcal{L}^M u \|_{h^s}^2 \leq C \| \mathcal{L}^M u \|_{h^s}^2 + C_\psi \| \mathcal{L}^M u \|_{L^1}^2
\]

(6.62)

\[
\leq C A_s \| \mathcal{L}^M 1 \|_\infty \mu^{2s} \| u \|_{h^s}^2 + Q(M,s) \| u \|_{h^{s-1}}^2 + C_\psi \| u \|_{L^1}^2
\]

and

\[
\sum_{\alpha} \sum_{b \in \mathcal{B}_0} \| u^M_{\alpha,b} \|_{h^{s-1}}^2 \leq C \| \mathcal{L}^M u \|_{h^{s-1}}^2 + C_\psi \| \mathcal{L}^M u \|_{L^1}^2
\]

(6.63)

Next, by Lemma C.1

\[
\| u \|_{h^{s-1}}^2 \leq \zeta \| u \|_{h^s}^2 + \zeta^{-1} C \| u \|_{L^1}^2, \quad \forall \zeta > 0.
\]

If we chose \( \zeta = \mathcal{N}(q_0) \mu^{2\rho_0} Q(M,s)^{-1} C_{q_0}^{-1} \), using (6.62) and (6.63) in (6.61), setting \( \overline{Q}(M,s) = \{ Q(M,s), C_\psi \}^+ \), and recalling (6.4) for the definition of \( \mathcal{L}_M \), we obtain

\[
\| \mathcal{L}^N u \|_{h^s}^2 \leq C \| \mathcal{L}_M \mathcal{N}(q_0) \mu^{2sN} \| u \|_{h^s}^2
\]

\[
+ \Theta_s \| u \|_{s+2}^2 + C_{q_0} Q(M,s) \sqrt{\Theta_s} \| u \|_{s+2} \| u \|_{h^s} + C_{q_0} \overline{Q}(M,s) \| u \|_{L^1}^2.
\]

Finally we note that \( \| u \|_{L^1}^2 \lesssim \| u \|_{h^s} \| u \|_{s+2} \) and, as \( \sqrt{a b} \leq \frac{a}{2} + \frac{b}{2} \) for each \( a, b, \varepsilon > 0 \), we have \( \sqrt{\| u \|_{h^s} \| u \|_{s+2}} \leq \sqrt{\frac{\varepsilon}{2}} \| u \|_{h^s} + \sqrt{\frac{1}{2}} \| u \|_{s+2} \). We apply this with \( \varepsilon := \Theta_s^{\frac{1}{4}} Q(M,s)^{-1} C_{q_0} \), for \( \varepsilon \) arbitrarily small so that, taking the square root of (6.65), there exist \( C_1 > 0 \) and \( C_{q_0} > 0 \) such that

\[
\| \mathcal{L}^N u \|_{h^s} \leq C_1 \left( \sqrt{\| \mathcal{L}_M \mathcal{N}(q_0) \mu^{2s} \| \right) \| u \|_{h^s} + C_{q_0} \overline{Q}(M,s) \sqrt{\Theta_s} \| u \|_{s+2},
\]

from which we obtain (6.5) in the case \( s > 1 \).

6.4.1. **The case \( s = 1 \).** It remains to prove (6.7) for \( s = 1 \). First, by Lemma 5.1, \( Q(M,1) \lesssim C_{\beta_0} \mu^{2M} \). Recalling Remark 6.3, \( C_{q_0}(1) \lesssim C_q C_{q_0} \chi_n^{-2\varepsilon_0 \ln \mu} \). Finally, using also (6.34), we can find \( \beta_3, \beta_4 > 0 \) such that

\[
\overline{Q}(M,1) \sqrt{\Theta_1} \lesssim C_{\beta_3} \mu^{\beta_4 M} \chi_n^{\frac{1}{2} - 2\varepsilon_0 \ln \mu M},
\]

which concludes the proof of Theorem 6.1. \( \square \)

7. **The final Lasota–Yorke Inequality**

In this section we state and prove our main technical Theorem which implies the Theorems stated in section 2. For each integer \( 1 \leq s \leq r-1 \) we define the following norm

\[
\| \cdot \|_{s,s} := \| \cdot \|_{h^s} + \| \cdot \|_{s+2}.
\]

---

43We also use repeatedly \( \| \mathcal{L}^n u \|_{L^1} \leq \| u \|_{L^1} \).

44Here we use \( \| u \|_{L^1} \lesssim \| u \|_{s+2} \).
**Theorem 7.1.** Let $F \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{T}^2)$ be SVPH and $\alpha = \frac{\log(\lambda_+^{n^{-1}})}{\log(\lambda_+^{n^{-1}})}$. Let $\mathbf{m}_{n,u}$ as in (3.63), and $C_1 > 0$ provided in Theorem (6.1). We assume that there exist $\tau_0 \geq 1, c \geq 0, K > 0$, where $K$ may depend on $\mu$, and $\kappa, \kappa_0 \in \mathbb{N}$ such that, for some $1 \leq s \leq r - 1$,

\begin{equation}
\sup_{m \leq n} \left\| \mathcal{L}^m u \right\|_{\infty} \leq K \mu^{\kappa\tau_0}, \quad \forall n < \kappa n_0 + \mathbf{m}_{n,u},
\end{equation}

\begin{equation}
\left\{ \mu^{\zeta} \lambda_-^{\frac{1}{2}}, \sqrt{\mathcal{N}_F(\lfloor \alpha n_0 \rfloor)} \mu^\alpha n_0^{\tau_0 + \beta} \mathbf{m}_{n,u} \right\}^+ \leq \nu_0 < 1,
\end{equation}

\begin{equation}
(KC_1)^{\frac{1}{\kappa_0}} \frac{\mu^{n_0\tau_0}}{\mathbf{m}_{n,u}^{n_0\tau_0}} \nu_0^{-\frac{\kappa_0}{n_0\tau_0}} < 1,
\end{equation}

where $n_0$ is defined in (5.12), $\mathcal{N}_F$ is given in (2.11), $\alpha_s = c[(1 - \alpha)^{\tau_0} + 1] + 2s, \beta_s = 2(s + c)$ and $\zeta_s$ given in (2.14). Moreover, for $\kappa > \kappa_0$, choose

\begin{equation}
\sigma_\kappa \in \left\{ \left( \lambda_-^{\frac{1}{2}}, (C_1 K)^{\frac{1}{\kappa_0\tau_0}} \mathbf{m}_{n,u}^{n_0\tau_0} \mu_0^{n_0\tau_0} \right), 1 \right\}.
\end{equation}

Then, for each $n \in \mathbb{N}$ and $\sigma_\kappa \in (\sigma_\kappa, 1)$ we have

\begin{equation}
\left\| \mathcal{L}^n u \right\|_{s,s} \leq C_2 A(\kappa, n_0, \mathbf{m}_{n,u}, s) \sigma_\kappa^n \left\| u \right\|_{s,s} + C_4 A(\kappa, n_0, \mathbf{m}_{n,u}, s) \mu^n \left\| u \right\|_{0,0}
\end{equation}

\begin{equation}
\left\| \mathcal{L}^n u \right\|_{s,s} \leq C_2 A(\kappa, n_0, \mathbf{m}_{n,u}, s) \sigma_\kappa^n \left\| u \right\|_{s,s} + C_4 A(\kappa, n_0, \mathbf{m}_{n,u}, s) \mu^n \left\| u \right\|_{L^1,}
\end{equation}

where $A(\kappa, n_0, \mathbf{m}_{n,u}, s) = C_{q_0} \Theta_{\mathbf{m}_{n,u}}(\kappa n_0 + \mathbf{m}_{n,u}, s), \Theta_{\mathbf{m}_{n,u}}$ is given in (6.7).

**Proof.** We will use Theorem 6.1 with $N = q_0 + \mathbf{m}_{n,u}$, where $q_0 = \kappa n_0$ and $\kappa > \kappa_0$.

First, by conditions (7.1) and (7.2) and Lemma 5.5, we observe that have

\[ \left\| \mathcal{L}^n u \right\|_{s,s} \leq [K \mu^{\sigma_\kappa \kappa_0} \mathbf{m}_{n,u} (q_0)]^{\frac{1}{\mu^{\alpha_0}} \mu^{2s}} \]

\begin{equation}
\leq \left( K \mu^{\sigma_\kappa \kappa_0} \mathbf{m}_{n,u} (q_0) \mathcal{N}(\lfloor \alpha q_0 \rfloor) \right)^{\frac{1}{\mu^{\alpha_0}}} \mu^{2s}
\end{equation}

\[ \leq \left( K^2 \mu^{\sigma_\kappa \kappa_0} (1 - \alpha)^{\tau_0} + \epsilon \mathbf{m}_{n,u} (q_0) \mathcal{N}(\lfloor \alpha q_0 \rfloor) \right)^{\frac{1}{\mu^{\alpha_0}}} \mu^{2s}
\]

\[ \leq \left( K^2 \mathcal{N}(\lfloor \alpha q_0 \rfloor) \mu^{\sigma_\kappa \kappa_0} (1 - \alpha)^{\tau_0} + \epsilon \mathbf{m}_{n,u} (q_0) \right)^{\frac{1}{\mu^{\alpha_0}}} \mu^{2s}.
\]

Therefore, by equation (6.5),

\begin{equation}
\left\| \mathcal{L}^n u \right\|_{s,s} \leq C_1 K \left( \mathcal{N}(\lfloor \alpha q_0 \rfloor) \mu^{\sigma_\kappa \kappa_0} (1 - \alpha)^{\tau_0} + \epsilon \mathbf{m}_{n,u} (q_0) \right)^{\frac{1}{\mu^{\alpha_0}}} \left\| u \right\|_{s,s} + C_{q_0} \Theta_{\mathbf{m}_{n,u}}(N, s) \left\| u \right\|_{s+2}.
\end{equation}

Moreover by the sub-multiplicativity of $\mathcal{N}$

\[ \mathcal{N}(\lfloor \alpha q_0 \rfloor) = \mathcal{N}(\lfloor \alpha q_0 \rfloor) \leq \mathcal{N}(\lfloor \alpha q_0 \rfloor)^{\kappa}.
\]

It follows by the definition of $\nu_0$ that

\[ \sqrt{\mathcal{N}(\lfloor \alpha q_0 \rfloor) \mu^{\sigma_\kappa \kappa_0} (1 - \alpha)^{\tau_0} + \epsilon \mathbf{m}_{n,u} (q_0)} \leq \mathcal{N}(\lfloor \alpha q_0 \rfloor) \mu^{\sigma_\kappa \kappa_0} (1 - \alpha)^{\tau_0} + \epsilon \mathbf{m}_{n,u} (q_0)^{\kappa} \leq \nu_0^{-\frac{\kappa_0 \kappa_0}{n_0\tau_0}} \mathbf{m}_{n,u}.
\]

Accordingly

\begin{equation}
\left\| \mathcal{L}^n u \right\|_{s,s} \leq C_{q_0} \Theta_{\mathbf{m}_{n,u}}(N, s) \left\| u \right\|_{s+2}.
\end{equation}

On the other hand, the assumption $\mu^{\alpha_0} \lambda_-^{\frac{1}{\mu}} \leq \nu_0$ implies (4.32), so that we can choose $\delta_s$ in (4.33) such that, for all $n \in \mathbb{N}$,

\begin{equation}
\left\| \mathcal{L}^n u \right\|_{s+2} \leq C \sigma_\kappa^{2n} \left\| u \right\|_{s+2} + CC_{\mu,n} \mu^n \left\| u \right\|_{0,0},
\end{equation}
where $C_{\mu,n}$ is defined in \eqref{eq:3.15}. Iterating \eqref{eq:7.9} by multiple of $N$ and using \eqref{eq:7.10} yields
\begin{equation}
\|L^n u\|_{s,s} \leq C_2 \sigma^n \|u\|_{H^s} + A(\kappa, n_0, \overline{m}_{\chi^s}, s) \|u\|_{s+2} + C_2 A(\kappa, n_0, \overline{m}_{\chi^s}, s) \mu^2 \|u\|_0,
\end{equation}
from which we deduce \eqref{eq:7.5}.

Next, we want to compare the norm $\| \cdot \|_0$ with the $L^1$-norm. Let us fix $\ell > 0$. Take an admissible central curve $\gamma$ and notice that, for any $\phi \in C^0(\mathbb{T})$ with $\|\phi\|_{\infty} = 1$, we have
\begin{equation}
\left| \int_T \phi(t)(u)(\gamma(t) + \ell \varepsilon_1)dt - \int_T \phi(t)(u)(\gamma(t))dt \right| = \int_0^{\ell} ds \int_T \phi(t) \partial_{x^1} u(\gamma(t) + se_1)dt.
\end{equation}
Writing $\gamma(t) = (\sigma(t), t)$ we can make the change of variables $\psi(s, t) = (\gamma(t) + se_1) = (\sigma(t) + s, t)$. Since $\det(D\psi) = -1$ and setting $D_{\ell} = \{\psi(s, t) : t \in \mathbb{T}, s \in [0, \ell]\}$, we have
\begin{equation}
\left| \int_T \phi(t)(u)(\gamma(t) + \ell \varepsilon_1)dt - \int_T \phi(t)(u)(\gamma(t))dt \right| = \int_{D_{\ell}} \phi(z) \partial_{x^1} u(x, z)dx dz
\end{equation}
\begin{equation}
\leq \|\phi\|_{L^\infty} \sqrt{\ell} \|u\|_{H^s}.
\end{equation}
Hence
\begin{equation}
\int_T \phi(t)(u)(\gamma(t) + se_1)dt \geq \int_T \phi(t)(u)(\gamma(t))dt - \sqrt{\ell} \|u\|_{H^s}.
\end{equation}
Integrating in $s \in [0, \ell]$ and taking the sup on $\gamma$ and $\phi$ yields
\begin{equation}
\|u\|_0 \leq \ell^{-1} \|u\|_{L^1} + \frac{2\ell^2}{3} \|u\|_{H^s}.
\end{equation}
Applying the above formula to \eqref{eq:7.5} with $\ell = C_2 \sigma^2 \mu^{-2n}$ yields
\begin{equation}
\|L^n u\|_{s,s} \leq C_2 A(\kappa, n_0, \overline{m}_{\chi^s}, s) \sigma^n \|u\|_{s,s} + C_2 A(\kappa, n_0, \overline{m}_{\chi^s}, s) \sigma^2 \mu^2 \|u\|_{L^1}.
\end{equation}

Next, for each $\bar{\sigma}_\kappa \in (\sigma, 1)$, let $n_\kappa$ be the smallest integer such that $C_2 A(\kappa, n_0, \overline{m}_{\chi^s}, s) \sigma^{n_\kappa} \leq \bar{\sigma}_\kappa$. For each $n \in \mathbb{N}$, write $n = kn_\kappa + m$ with $m < \kappa$, then iterating the above equation yields
\begin{equation}
\|L^n u\|_{s,s} \leq \sigma^{kn_\kappa} \|L^m u\|_{s,s} + C_2 A(\kappa, n_0, \overline{m}_{\chi^s}, s) \mu^{-2n_\kappa} \sum_{j=0}^{k-1} \sigma^{kn_\kappa} \|u\|_{L^1} \leq C_2 A(\kappa, n_0, \overline{m}_{\chi^s}, s) \sigma^{n_\kappa} \|u\|_{s,s} + C_2 A(\kappa, n_0, \overline{m}_{\chi^s}, s) \mu^{-2n_\kappa} \sigma^{-2n_\kappa} \|u\|_{L^1},
\end{equation}
which implies \eqref{eq:7.6}. $\square$

**Corollary 7.2.** Under the assumptions of Theorem 7.1 there exists a Banach space $B_{s,s}$ such that $C^{r-1}(\mathbb{T}^2) \subset B_{s,s} \subset H^s(\mathbb{T}^2)$ on which the operator $L : B_{s,s} \to B_{s,s}$ has spectral radius one and is quasi compact with essential spectral radius bounded by $\sigma_\kappa$.

**Proof.** We call $B_{s,s}$ the completion of $C^{r-1}(\mathbb{T}^2)$ with respect to the norm $\| \cdot \|_{s,s}$, then $C^{r-1}(\mathbb{T}^2) \subset B_{s,s} \subset H^s(\mathbb{T}^2)$. Iterating \eqref{eq:7.6}, and since $L$ is a $L^1$ contraction, implies that the spectral radius is bounded by one, but since the adjoint of $L$ has eigenvalue one, so does $L$, hence the spectral radius is one.

To bound the essential spectral radius note that the immersion $B_{s,s} \hookrightarrow H^s$ is continuous by definition of the norm. Moreover the immersion $H^s \hookrightarrow L^1$ is compact for every $s$ by Sobolev embeddings theorems, hence $B_{s,s} \hookrightarrow L^1$ is compact. Hence by \eqref{eq:7.6} and Hennion theorem \cite{35} follows that the essential spectral radius is bounded by $\bar{\sigma}_\kappa$ and hence the claim by the arbitrariness of $\bar{\sigma}_\kappa$. $\square$
Proof of Theorem 2.7. It is enough to check the conditions of Theorem 7.1. The existence of $\kappa_0$ is guaranteed by (5.12). Since $\mu > 1$, Corollary 3.17 implies $\sup_{k \leq \eta} \|L^k(1)\|_\infty \leq K\mu^\kappa n$ for each $n \in \mathbb{N}$, with $K = (\mu - 1)^{-1}$.\(^{45}\) $c = 2$ and $\tau_0 = 1$; in particular we can take any $\kappa \in \mathbb{N}$. Next, $\mu^\kappa n < 1$ is implied by hypothesis (H3). Therefore, condition (2.15) coincide with (7.3) with $\alpha_s, \beta_s, \zeta_s$ given in (2.14). Finally, choosing any $\kappa_0$ such that

\[
\kappa_0 > \frac{\ln(C_1K^{\frac{1}{c}})}{\ln \nu_0},
\]

we have also (7.3), whereby we conclude. □

8. The map $F_\varepsilon$

In this section we check that we can apply Theorem 7.1 to the family of maps $F_\varepsilon$ and we prove Theorems 2.10 and 2.11.

8.1. The $F_\varepsilon$ are SVPH. Let $(\varepsilon u) \in \mathcal{C}_\varepsilon^n$, for $p = (x, \theta) \in T^2$. In this case equation (2.17) yields

\[
D_p\varepsilon F_\varepsilon(1, \varepsilon u) = (\partial_x f + \varepsilon u \partial_\theta f)(1, \varepsilon \Xi_\varepsilon(u, p)),
\]

where

\[
\Xi_\varepsilon(u, p) = \frac{\partial_x \omega + \varepsilon u \partial_\theta \omega + u}{\partial_x f + \varepsilon u \partial_\theta f}.
\]

We have also a more explicit formula for iteration of the map $\Xi_\varepsilon$. For any $k \geq 0$ and $p \in T^2$, let us denote $p_k = F_\varepsilon^k(p)$. Then the recursive formula,

\[
\Xi_\varepsilon^{(n)}(p, u) = \Xi_\varepsilon^{(n-1)}(p, \Xi_\varepsilon(p_{n-1}, u)),
\]

yields

\[
\Xi_\varepsilon^{(n)}(p, u) = \frac{\sum_{k=0}^{n-1} \partial_x \omega(p_k) + u}{\prod_{j=0}^{n-1} \partial_x f(p_j)} + O(\varepsilon),
\]

for every $\varepsilon > 0$. On the other hand, recalling (2.18):

\[
\partial_u \Xi_\varepsilon(p, u) = \frac{\partial_x f + \varepsilon (\partial_\theta \omega \partial_x f - \partial_\theta f \partial_x \omega)}{(\partial_x f + \varepsilon \partial_\theta f)^2}.
\]

Now we use Lemma 2.8, applied with $\omega$ replaced by $\omega_\varepsilon$, to check that the maps given in (2.32) are SVPHS for $\varepsilon$ small enough. Conditions (2) and (3) are immediate. In particular, $\mathcal{C}_\varepsilon^\varepsilon = \{(\xi, \eta) \in \mathbb{R}^2 : |\eta| \leq \varepsilon u, |\xi|\}$ and $\mathcal{C}_\varepsilon^\varepsilon = \{(\xi, \eta) \in \mathbb{R}^2 : |\xi| \leq \chi_{\varepsilon}|\eta|\}$ satisfy $D_p F_\varepsilon(\mathcal{C}_\varepsilon^\varepsilon) \subseteq \mathcal{C}_\varepsilon^\varepsilon$ and $D_p F_\varepsilon^{-1}(\mathcal{C}_\varepsilon^\varepsilon) \subseteq \mathcal{C}_\varepsilon^\varepsilon$ if

\[
u_0 = 2\|\partial_x \omega\|_\infty := \varepsilon^{-1}\chi_{\varepsilon} \quad \text{and} \quad \chi_\varepsilon := \frac{1}{2}.
\]

Next, we note that condition (5) of Lemma 2.8 implies conditions (1) and (4) for each $\varepsilon$ small enough. Moreover, in (2.23) it is shown that for some $\bar{c} > 0$, $\mu_\pm = e^{\pm \bar{c} \varepsilon}$, which implies (6) for sufficiently small $\varepsilon$.

\(^{45}\) Recall that $C_{\mu, n} \leq (\mu - 1)^{-1}$ (see also Remark 3.6).

\(^{46}\) Observe that in this special case $\chi_{\varepsilon}(\varepsilon) = \varepsilon u_\varepsilon$, thus we have an unstable cone of size $\varepsilon$.  
Finally, it is useful to note that, if we set \( \psi(p) = \langle \nabla \omega, (-\frac{\partial f}{\partial y}, 1) \rangle(p) \), for every \( p \in \mathbb{T}^2 \) and \( n \in \mathbb{N} \) we have
\[
\det D_p F^n_\varepsilon = \prod_{k=0}^{n-1} \det D_{F^k_p} F_\varepsilon = \prod_{k=0}^{n-1} \left[ \partial_x f(F^k_\varepsilon p)(1 + \varepsilon \psi(F^k_\varepsilon p)) \right],
\]
hence
\[
e^{-\varepsilon n} \lambda^n \leq \det D_p F^n_\varepsilon \leq e^{\varepsilon n} \Lambda^n, \quad \forall p \in \mathbb{T}^2, \forall n \in \mathbb{N}.
\]

8.2. \textbf{A non-transversality argument.} The aim is to prove the following theorem which guarantees that, after some time \( \varepsilon \) which does not depend on \( \varepsilon \), for each point we have at least one couple of pre-images with transversal unstable cones, provided \( \omega \) satisfies some checkable conditions. We will see that this corresponds to proving the existence of the integer \( n_0 \) defined in (5.12).

In the following we denote as \( \mathcal{S}_\varepsilon \) the set of the inverse branches of \( F_\varepsilon \).\(^{47}\) Moreover, \( \mathcal{S}_\varepsilon^n \) will be the set of elements of the form \( h_1 \circ \cdots \circ h_n \), for \( h_j \in \mathcal{S}_\varepsilon \) and \( \mathcal{S}_\varepsilon^\infty := \mathcal{S}_\varepsilon^\infty \), in particular, for \( h \in \mathcal{S}_\varepsilon^\infty \) the symbol \( h_n \) will denote the restriction of \( h \) on \( \mathcal{S}_\varepsilon^n \).

\textbf{Remark 8.1.} Since \( F_0 \) and \( F_\varepsilon \) are homotopic coverings they are isomorphic, that is there exist \( I_\varepsilon : \mathbb{T}^2 \to \mathbb{T}^2 \) such that \( F_\varepsilon = F_0 \circ I_\varepsilon \). This induces an isomorphism \( \mathcal{I}_\varepsilon : \mathcal{S}_0 \to \mathcal{S}_\varepsilon \) defined by \( \mathcal{I}_\varepsilon h = I_\varepsilon^{-1} \circ h \). Hence the same is true for the sets \( \mathcal{S}_\varepsilon^n = \mathcal{S}_0^n \) and \( \mathcal{S}_\varepsilon^\infty \). In the following we will then identify inverse branches of \( F_\varepsilon^n \) and \( F_0^n \) by these isomorphisms, and drop the script \( \varepsilon \) from the notation when it is not necessary.

The main result of this section is the following.

\textbf{Proposition 8.2.} If \( \omega \) is not \( \varepsilon \)-constant with respect to \( F_0 \) (see Definition 2.9), then there exist \( \varepsilon_0 > 0 \) and \( n_0 \in \mathbb{N} \) such that, for every \( \varepsilon \leq \varepsilon_0 \), \( p \in \mathbb{T}^2 \) and vector \( v \in \mathbb{R}^2 \), there exists \( q \in F_\varepsilon^{-n_0}(p) \) such that \( v \not\in D_q F_\varepsilon^{-n_0} C_\varepsilon^n \).

\textbf{Proof.} We argue by contradiction and suppose that for every \( \varepsilon_0 > 0 \) and \( \ell \in \mathbb{N} \) there exist \( \varepsilon_\ell \in [0, \varepsilon_0] \), \( n_\ell \in \mathbb{N} \), \( p_\ell \in \mathbb{T}^2 \) and \( v_\ell = (1, \varepsilon_\ell u_\ell) \) with \( |u_\ell| \leq u_\ast \) such that\(^{48}\)
\[
D_q F_\varepsilon^{-n_\ell} C_\varepsilon^n \ni v_\ell, \quad \forall q \in F_\varepsilon^{-n_\ell}(p_\ell),
\]
namely, all the above cones have a common direction. Since the sequence \( \{p_\ell, u_\ell\} \subset \mathbb{T}^2 \times [-u_\ast, u_\ast] \), it has an accumulation point \( (p_\ast, u_\ast) \). In analogy with (3.11), for \( p \in \mathbb{T}^2 \) and \( u \in [-u_\ast, u_\ast] \) we define
\[
\Phi_\varepsilon(p, u) = \left( F_\varepsilon^n(p), \Xi_\varepsilon^{(n)}(p, u) \right),
\]
where \( \Xi_\varepsilon^{(n)} \) is given by formula (8.3). Condition (8.7) in terms of this dynamics says that the slope \( u_\ell \) is contained in the interval \( \Xi_\varepsilon^{(n_\ell)}(q, [-u_\ast, u_\ast]) \) for every \( \ell \in \mathbb{N} \) and \( q \in F_\varepsilon(p_\ell) \). Hence, it can be written as:
\[
\forall \ell \in \mathbb{N}, \quad \exists (p_\ell, u_\ell) : \quad \pi_2 \circ \Phi_\varepsilon^{n_\ell}(q, [-u_\ast, u_\ast]) \supset \{u_\ell\}, \quad \forall q \in F_\varepsilon^{-n_\ell}(p_\ell),
\]
\(^{47}\) Accordingly \( \mathcal{S}_0 \) is the set of inverse branches of \( F_0 \).
\(^{48}\) We use the notation with subscript \( \ell \) for a generic object that depends on \( \ell \) through \( n_\ell \) and \( \varepsilon_\ell \), but we keep the notation as simple as possible when there is no need to specify.
where \( \pi_2 : \mathbb{T}^2 \times [-u_*, u_*) \to [-u_*, u_*) \) is the projection on the second coordinate. Now, for \( m \in \mathbb{N}, \varepsilon \in [0, \varepsilon_0], \mu_0 \in [-u_*, u_*) \) and \( h) \in \mathcal{H}^\infty \), let us define
\[
(8.10) \quad u^\varepsilon_{h,m}(p) = \pi_2 \circ \Phi^m_{\varepsilon}(h_m(p), u_0) : \mathbb{T}^2 \to [-u_*, u_*)\)

Next, we prove the following result, which will allow us to conclude the proof.

**Sublemma 8.3.** The sequence of functions defined in (8.10) satisfies:

(i) For every \( \varepsilon \in [0, \varepsilon_0] \) and \( h) \in \mathcal{H}^\infty \), there exists \( u^\varepsilon_{h,\infty}(q) := \lim_{m \to \infty} u^\varepsilon_{h,m}(q) \), and the limit is uniform in \( q \in \mathbb{T}^2 \).

(ii) For every \( h) \in \mathcal{H}^\infty \), the sequence \( \{u^\varepsilon_{h,\infty}\}_\varepsilon \) converges to \( u_{h,\infty} \) uniformly.

(iii) The functions \( u_{h,\infty} \) are independent of \( h \), we call them \( \bar{u} \). In addition, \( \bar{u} \) satisfies
\[
(8.11) \quad \bar{u}(F_0(q)) = \Xi_0(q, \bar{u}(q)), \quad \forall q \in \mathbb{T} \times \{\theta\}.
\]

**Proof.** Applying Lemma D.1 with \( u = u' \equiv u_0 \in [-u_*, u_*) \), \( \varepsilon_0 = 1, A = 2\chi_u \), and \( B = 0 \) we have that there exists \( \nu \in (0, 1) \) such that, for each \( h) \in \mathcal{H}^\infty \), \( q \in \mathbb{T}^2 \), \( \varepsilon, \varepsilon' \in [0, 1) \), \( m \in \mathbb{N} \) and \( n > m, \)
\[
|u^\varepsilon_{h,m}(q) - u^\varepsilon_{h,m}^n(q)| \leq C_2 \mu^{3m} \varepsilon - \varepsilon'
\]
\[
|u^\varepsilon_{h,n}(q) - u^\varepsilon_{h,m}(q)| \leq C_2 \mu^n.
\]

It follows that there exists \( u^\varepsilon_{h,\infty}(q) := \lim_{m \to \infty} u^\varepsilon_{h,m}(q) \), and the limit is uniform in \( q \).

Next, for each \( \delta > 0 \), we choose \( \varepsilon_* \) and \( m \) such that \( C_2 \mu^{3m} \varepsilon_* \leq \frac{\delta}{4} \) and \( \mu^n \leq \frac{\delta}{4} \), then, for each \( \varepsilon, \varepsilon' \leq \varepsilon_* \) and \( q \in \mathbb{T}^2 \)
\[
|u^\varepsilon_{h,\infty}(q) - u^\varepsilon_{h,\infty}^n(q)| \leq |u^\varepsilon_{h,\infty}(q) - u^\varepsilon_{h,m}(q)| + |u^\varepsilon_{h,m}(q) - u^\varepsilon_{h,m}^n(q)| + |u^\varepsilon_{h,m}^n(q) - u^\varepsilon_{h,\infty}^n(q)|
\]
\[
\leq 2\varepsilon + \mu^n \leq 2\varepsilon + C_2 \mu^{3n} \varepsilon - \varepsilon' \leq \delta.
\]

The above proves the first two items. Let us proceed with the third one.

First we claim that, for \( q \in \mathbb{T}^2 \), if \( h_q \) is such that \( q = h_q(F_\varepsilon(q)) \), then
\[
(8.13) \quad u^\varepsilon_{h_{\infty}}(F_\varepsilon(q)) = \Xi_\varepsilon(q, u^\varepsilon_{h,\infty}(q)), \quad \forall q \in \mathbb{T}^2.
\]

Indeed, since \( u^\varepsilon_{h,\infty} \) belongs to the unstable cone, by (8.10), for every \( h) \in \mathcal{H}^\infty \) and \( q \in \mathbb{T}^2 \),
\[
(F_\varepsilon(q), \Xi_\varepsilon(q, u^\varepsilon_{h,\infty}(q))) = \Phi_\varepsilon(q, u^\varepsilon_{h,\infty}(q)) = (F_\varepsilon(q), u^\varepsilon_{h,\infty}(F_\varepsilon(q))),
\]
which implies the claim taking the projection on the second coordinate.

For every \( \ell \in \mathbb{N} \), let us consider \( \varepsilon_\ell, n_\ell, \mu_\ell \) and \( u_\ell \) as given in (8.9) and let \( \ell_j \) so that \( (p_{\ell_j}, u_{\ell_j}) \) is a convergent sequence. Equation (8.7) implies
\[
(8.14) \quad |u_{\ell_j} - u^\varepsilon_{h_{n_{\ell_j}}}(p_{\ell_j})| \leq C_2 \mu^\varepsilon_{\ell_j}.
\]

Taking the limit for \( j \to \infty \) in the above inequality yields
\[
(8.15) \quad u_\ast = \lim_{j \to \infty} u_{\ell_j} = \lim_{j \to \infty} u^\varepsilon_{h_{n_{\ell_j}}}(p_{\ell_j}) = \Xi_{\varepsilon_\ell}(p_\ast),
\]
regardless of the choice of the inverse branch \( h) \in \mathcal{H}^\infty \). Let \( h_q \) be the inverse branch such that \( q = h_q(F_\varepsilon(q)) \), and set \( q_\ell = h_q(p_{\ell}) \) in equation (8.13) to obtain:
\[
(8.16) \quad u^\varepsilon_{h_{\infty}}(p_{\ell}) = \Xi_{\varepsilon_\ell}(q_\ell, u^\varepsilon_{h_{\infty}}(q_\ell)).
\]

\(^{49}\) The second equation of (8.12) is a direct consequence of (D.4) which implies that \( \Xi_{\varepsilon}(p_\ast, \cdot) \) is a contraction.

\(^{50}\) Recall that \( (p_\ast, u_\ast) \) is an accumulation point of the sequence \( (p_{\ell}, u_{\ell}) \) given in (8.9).
By item (ii) above, and by the continuity of the map $F_\epsilon$, we can take the limit as $\ell_j \to \infty$ in the last equation and obtain

$$\varpi_{\epsilon \to \infty}(p_\ast) = \Xi_0(q_\ast, \varpi_{\epsilon, \infty}(q_\ast)),$$

where $q_\ast$ is such that $F_0(q_\ast) = p_\ast$. By (8.15), the above equation becomes $u_\ast = \Xi_0(q_\ast, \varpi_{\epsilon, \infty}(q_\ast))$, and, since $\Xi_0(q_\ast, \cdot)$ is invertible, this implies that there exists $u_\ast(q_\ast)$ independent of $\epsilon \in H^\infty$ such that

$$u_\ast(q_\ast) = \tilde{u}_\ast(q_\ast) = \lim_{j \to \infty} u^{\epsilon}_{j, \ast}(q_\ast).$$

Hence, by induction, $\varpi_{\epsilon, \infty}(q)$ is independent on $\epsilon$ for each $q \in \bigcup_{k \in \mathbb{N}} F_0^{-k}(p_\ast) =: \Lambda_{q_\ast}$, let us call it $u_\ast(q)$. Taking the limit in equation (8.13) we have, for each $q \in \Lambda_{q_\ast}$,

$$u_\ast(F_0(q)) = \Xi_0(q, u_\ast(q)).$$

(8.17)

Note that the $\tilde{u}_{\epsilon, \infty}$ are uniform limits of continuous functions and hence are continuous functions such that $\varpi_{\epsilon, \infty}|_{\Lambda_{q_\ast}} = u_\ast$. Since $\Lambda_{q_\ast}$ is dense in $T \times \{q_\ast\}$. It follows that the $\varpi_{\epsilon, \infty}$ equal some continuous function $\tilde{u}$ defined on $T \times \{q_\ast\}$ and independently of $\epsilon$. In addition, $\tilde{u}$ satisfies (8.11).\]

We can now conclude the proof of Proposition 8.2. By Sub-Lemma 8.3 we can find a function $\tilde{u} : T^2 \to \mathbb{R}$ and $\theta_\ast \in T^1$ such that (8.11) holds, namely:

$$\tilde{u}(F_0(q)) = \frac{\partial_x \omega(q) + \tilde{u}(q)}{\partial_x f(q)}, \quad q \in T^1 \times \{\theta_\ast\}$$

(8.18)

Let us use the notation $g_\theta(x)$ for a function $g(x, \theta)$ and observe that, integrating (8.18) and recalling that $\omega$ is periodic by hypothesis, we have

$$\int_0^1 \tilde{u}_\ast(x)dx = \int_0^1 f'_{\theta_\ast}(x)\tilde{u}_\ast(f_{\theta_\ast}(x))dx - \int_0^1 \partial_x \omega(x, \theta_\ast)dx = \sum_{i=0}^{d-1} \int_{U_i} f'_{\theta_\ast}(x)\tilde{u}_\ast(f_{\theta_\ast}(x))dx$$

$$= d \int_0^1 \tilde{u}_\ast(t)dt,$$

where $U_i$ are the invertibility domains of $f_{\theta_\ast}$, and $d > 1$ its topological degree. Hence $\int_0^1 \tilde{u}_\ast(x)dx = 0$. So there is a potential given by $\Psi_{\theta_\ast}(x) = \int_0^x \tilde{u}_\ast(z)dz$. Finally, integrating equation (8.18) from 0 to $x$, there exists $c > 0$ such that

$$\omega_{\theta_\ast}(x) = \Psi_{\theta_\ast}(f_{\theta_\ast}(x)) - \Psi_{\theta_\ast}(x) + c,$$

which contradicts the assumption on $\omega$ whereby proving the Proposition.\]

For reasons which will be clear in a moment, we introduce a further quantity related to $N_{\mathcal{F}_\epsilon}$ and $\mathcal{N}_{\mathcal{F}_\epsilon}$ which can be interpreted as a kind of normalization of the latter one. The following definition is inspired by [13].

**Definition 8.4.** For each $p = (x, \theta) \in T^2$, $v \in \mathbb{R}^2$, $n \in \mathbb{N}$ and $\epsilon > 0$ we define

(8.19) $\mathcal{R}(x, \theta, v, n) := \frac{1}{\rho(x, \theta)} \sum_{(y, \eta) \in \mathcal{F}_{\epsilon}^{-n}(x, \theta)} \frac{\rho(y, \theta)}{|\det DF_\epsilon^n(y, \eta)|}$.

---

51 It follows from the expansivity of $f(\cdot, \theta_\ast)$ that the preimages of any point form a dense set.

52 Just approximate any point with a sequence $\{q_i\} \subset \Lambda_{q_\ast}$ and take the limit in (8.17).
where, for every \( \theta \in T \), \( \rho(\cdot, \theta) =: \rho_0(\cdot) \) is the density of the unique invariant measure of \( f(\cdot, \theta) \). As before we will denote \( \tilde{N}(n) := \sup_p \sup_x \mathfrak{R}(p, v, n) \).

The motivation to introduce this quantity is twofold. One reason lies in Lemma 8.5 below in which, using a shadowing argument similar to \[18\, ,\, \text{Appendix B}\], we exploit the following fact: for each \( \theta \in T \), setting \( f_\theta(\cdot) = f(\cdot, \theta) \), we have

\[
\frac{1}{\rho_0(x)} \sum_{y \in f_\theta(x)} \frac{\rho_0(y)}{(f_\theta)^\prime(y)} = 1, \quad \forall x \in T.
\]

On the other hand it is easy to see that \( \tilde{N} \) has the same properties of \( \tilde{N}_{F,\epsilon} \). In particular, arguing exactly in the same way as in Proposition 5.3 and Lemma 5.5, one can show that

\[
\tilde{N}(n) \text{ is submultiplicative}, \tag{8.21}
\]

\[
N_{F,\epsilon}(n) \leq C_\epsilon \|L_{F,\epsilon}^{n+1}\|_\infty \left( \tilde{N}([\alpha n]) \right)^{\frac{1}{1-\alpha}} \tag{8.22}
\]

\[
\tilde{N}(n) \leq \sup_{(x,\theta) \in T^2} \frac{1}{\rho(x,\theta)} (L_{F,\epsilon}^n,\rho)(x,\theta). \tag{8.23}
\]

This implies that we can check condition (7.2) of Theorem (7.1) with \( \tilde{N} \) replaced by \( \tilde{N} \).

To ease notation in the following we set \( L_{F,\epsilon} := L_\epsilon \).

**Lemma 8.5.** There are constants \( C, c_* > 0 \) such that, for each \( n < C\epsilon^{-\frac{1}{2}} \),

\[
\sup_{(x,\theta) \in T^2} \frac{1}{\rho(x,\theta)} (L_{F,\epsilon}^n,\rho)(x,\theta) \leq e^{c_*n^2\epsilon}. \tag{8.24}
\]

**Proof.** Let \( F_\epsilon^n(q) = (x,\theta) \) and define \( q_k = (x_k, \theta_k) = F_\epsilon^k(q), \) for every \( 0 \leq k \leq n \). Then,

\[
|\theta - \theta_k| \leq \sum_{j=k}^{n-1} \epsilon||\omega||_\infty \leq C_\epsilon(n-k)\epsilon. \tag{8.25}
\]

Let us set \( f_\theta(y) = f(y,\theta) \). Since \( f_\theta \) is homotopic to \( f_{\theta_k} \), for each \( k \), there is a correspondence between inverse branches, hence there exists \( x_* \) such that \( |f_{\theta_k}^k(x_*) - x_k| = \lambda^{-1} \). Moreover, let \( \xi_k = f_{\theta_k}^k(x_*) - x_k \). Since \( f \) is expanding, by the mean value theorem and (8.25), there is \( (\bar{x},\bar{\theta}) \) such that

\[
|\xi_{k+1}| = |(\nabla f(\bar{x},\bar{\theta}), (\xi_k, \theta_k - \theta))| \geq \lambda|\xi_k| - C_\epsilon n\epsilon.
\]

Since \( \xi_n = 0 \), we find by induction \( |\xi_k| \leq \sum_{j=k}^{n-1} \lambda^{-j+k}C_\epsilon \epsilon n \leq C_\epsilon \epsilon n \). Moreover, since \( \rho \) is differentiable\(^{53}\) we also have

\[
|\rho(x_k, \theta_k) - \rho(f_{\theta_k}^k(x_*))| \leq C_\epsilon \epsilon n.
\]

Next, since \( |\det D f_\theta F_\epsilon - \partial_x f(q)| \leq C_\epsilon \epsilon, \)

\[
\frac{(f_{\theta_k}^k)'(x_*)}{\det D f_\theta F_\epsilon(x_0, \theta_0)} = \prod_{k=0}^{n-1} \frac{f_{\theta_k}^k(x_*)}{\det D f_\theta F_\epsilon(x_k, \theta_k)} \leq \prod_{k=0}^{n-1} \frac{f_{\theta_k}^k(x_*)}{\det D f_\theta F_\epsilon(x_k, \theta)} \left( 1 + C_\epsilon \epsilon n \right) \leq e^{c_*n^2\epsilon}.
\]

It follows that,

\[
\frac{1}{\rho(x,\theta)} \sum_{y \in f_\theta^{-1}(x,\theta)} \left( \frac{\rho(y,\theta)}{\det D f_\theta F_\epsilon(y, \theta)} \right) \leq e^{c_*n^2\epsilon} \sum_{x \in f_\theta^{-1}(x,\theta)} \frac{\rho(y)}{(f_\theta)^\prime(x)} = e^{c_*n^2\epsilon},
\]

\(^{53}\)See \[17\, ,\, \text{for the details.}\)
where we have used (8.20).

8.3. Proof of Theorem 2.10. By the results of section 8.1 \( F_\varepsilon \) is SVPH for \( \varepsilon \) small enough. We now prove conditions of Theorem 7.1 for \( F_\varepsilon \), under the assumption that \( \omega \) is not \( x \)-constant. In this case the existence of \( n_0 \) independent of \( \varepsilon \) is guaranteed by Proposition 8.2. Next, notice that \( \chi_u = u \varepsilon \), i.e. the unstable cone \( C^u_\varepsilon \) is of order \( \varepsilon \) while the center cone \( C^c_\varepsilon \) is of order one. Hence, by (3.9), there exist \( c_0 > 0 \) such that \( m_{\varepsilon} = [c_0 \log \varepsilon^{-1}] \).\(^{54}\) We then take any \( \kappa \leq c_1 \log \varepsilon^{-1} \), for some \( c_1 > 0 \) and, by Lemma 8.5, we have

\[
\sup_{m \leq n} \| L^m_\varepsilon \|_\infty \leq \frac{1}{\inf \rho} \sup_{m \leq n} \| L^m_\varepsilon \rho \|_\infty \leq \frac{1}{\inf \rho} e^{c_\varepsilon n^2}, \quad \forall n \leq \{c_0, c_1 \}^+ \log \varepsilon^{-1},
\]

hence condition (7.1) with \( K = \frac{1}{\inf \rho} \), \( \tau_0 = 2 \) and \( c = c_\varepsilon / \varepsilon \). Next we prove

\[
\left\{ e^{\varepsilon \varepsilon \lambda^\frac{1}{2}}, \tilde{\mathcal{N}}(\{an_0\}) e^{\varepsilon (\alpha, n_0^2 + \beta, m_{\varepsilon}^2)} \right\}^+ \leq \nu_0 < 1,
\]

i.e condition (7.1) with \( \tilde{\mathcal{N}}_F \) replaced by \( \tilde{\mathcal{N}} \) which, as we already observed, implies (7.2) for \( F_\varepsilon \). Obviously there exists \( \varepsilon_1 > 0 \) such that, for each \( 1 \leq s \leq r - 1 \)

\[
\mu^s \lambda^\frac{1}{2} = e^{\varepsilon \varepsilon \lambda^\frac{1}{2}} < 1, \quad \forall \varepsilon \in (0, \varepsilon_1).
\]

Let \( n_0 \) and \( \varepsilon_0 \) be as in Proposition 8.2. Accordingly, for every \( p = (x, \theta) \in \mathbb{T}^2 \) and \( v \in \mathbb{R}^2 \), there exists \( q_* \in F^{-n_0}(p) \) such that

\[
\frac{1}{\rho(x, \theta)} \sum_{(y, \theta) \in F^{-n_0}(p)} \frac{\rho(y, \theta)}{|\det D_q F_\varepsilon^{n_0}|} \leq \frac{1}{\rho(x, \theta)} (L^m_\varepsilon \rho)(x, \theta) - \frac{k}{|\det D_q F_\varepsilon^{n_0}|},
\]

where \( k = \inf \rho / \sup \rho \). By Lemma 8.5 and equation (8.6), the last expression is bounded by

\[
e^{cn_0^2 \varepsilon^2} - \frac{C}{A n_0}.
\]

Choosing \( \varepsilon_2 < \min \left( \varepsilon_0, \sqrt{\frac{1}{cn_0^2}} \log(1 + CA^{-n_0}) \right) \), we have that \( \tilde{\mathcal{N}}(n_0) \leq \tilde{\sigma} < 1 \) for every \( \varepsilon \in [0, \varepsilon_2] \). Consequently there exists \( \varepsilon_3 \) such that

\[
\tilde{\mathcal{N}}(\{an_0\}) e^{\varepsilon \varepsilon \lambda^\frac{1}{2}} \leq \tilde{\sigma} e^{\varepsilon (\alpha, n_0^2 + \beta, m_{\varepsilon}^2)} < 1, \quad \forall \varepsilon \in (0, \varepsilon_3).
\]

By (8.27) and (8.28) we deduce (8.26) taking \( \varepsilon_* = \min\{\varepsilon_1, \varepsilon_3\} \). Finally, condition (7.3) is satisfied choosing \( \kappa_0 \) as in (7.13). Thus Theorem 7.1 applies and Theorem 2.10 follows by Corollary 7.2.

8.4. Eigenfunctions regularity (quantitative). As we have already seen in 7.2, the main consequence of Theorem 6.1 is that there exists a Banach space \( B_{\varepsilon_*} \subset H^s \) on which the transfer operator \( L_\varepsilon \) is quasi compact for each \( \varepsilon < \varepsilon_* \). In addition, using inequality (7.5), we can say much more about the constants, paying the price of having a bigger essential spectral radius. Indeed for each \( n, \kappa \in \mathbb{N} \)

\[
\| L^m_\varepsilon u \|_{s, \kappa} \leq C_4 A(\kappa, n_0, m_{\varepsilon}, s) \sigma^\kappa \| u \|_{s, \kappa} + B_4 A(\kappa, n_0, m_{\varepsilon}, s) \mu^{\kappa} \| u \|_s,
\]

where \( m_{\varepsilon} = c_0 \log \varepsilon^{-1} \) and \( \sigma_\kappa \) given in (7.4). The choice \( \kappa = C_4 \log \varepsilon^{-1} \) yields a spectral radius uniform in \( \varepsilon \), but we have no control on the constant \( A(\kappa, n_0, m_{\varepsilon}, s) \). On the contrary, the choice \( \kappa = 2\kappa_0 \in \mathbb{N} \) (independent of \( \varepsilon \)) implies, for some \( c_* > 0 \),

\[
\sigma_{n_0} \in (1 - (c_* \log \varepsilon^{-1})^{-1}, 1),
\]

\(^{54}\)For simplicity in the following we drop the \( |\cdot| \) notation.
hence a lesser information on the size of the essential spectrum but allows a control of the constants, especially in the case $s = 1$. Indeed, observing that by (3.15)
\[ C_{t, n_0 + m_x} \leq C_1 \min \{ \log \varepsilon^{-1}, \varepsilon^{-1} \} = C_2 \log \varepsilon^{-1}, \]
it follows, by (6.7),\(^{55}\) that we can find $\beta_3, \beta_3, C_4 > 0$ and $\varepsilon > 0$ such that
\[ (8.29) \quad \Theta_{\varepsilon}(n_0 + u\varepsilon, 1) \leq C_2 e^{-\frac{\beta_3}{\varepsilon^2}} \log \varepsilon^{-1} \beta_4 e^{\beta_3 \varepsilon \log \varepsilon^{-1}}. \]
Thus, for $s = 1$ and for each $\alpha > \frac{11}{2}$ and provided $\varepsilon$ is chosen small enough, we have, for all $n \in \mathbb{N}$,
\[ \| L_{\varepsilon} u \|_0 \leq C e^{C \varepsilon} \| u \|_0 \]
\[ \| L_{\varepsilon}^{n} u \|_{1,s} \leq C_2 \varepsilon^{-\alpha} e^{-\frac{\beta_3}{\varepsilon^2}} \| u \|_{1,s} + B_2 \varepsilon^{-\alpha} \| u \|_0. \]

**Proof of Theorem 2.11.** Let $c_s = c_0$ and $L_{\varepsilon} u = \nu u$ with $\nu^n > e^{-\frac{\beta_3}{\varepsilon^2}}$, $\alpha < 1$, then
\[ \| u \|_{1,s} = \nu^{-n} \| L_{\varepsilon}^{n} u \|_{1,s} \leq C_2 \varepsilon^{-\alpha} \| u \|_{1,s} + B_2 \nu^{-n} \varepsilon^{-\alpha} \| u \|_0. \]
We choose $n$ to be the smallest integer such that $C_2 \varepsilon^{-\alpha} e^{-\frac{\beta_3}{\varepsilon^2}} \| u \|_0 \leq \frac{1}{2}$, which yields
\[ \| u \|_{H^1} \leq \| u \|_{1,s} \leq C_2 \varepsilon^{-(1+\alpha)} \| u \|_0 \]
which concludes the proof. \(\square\)

8.5. **Proof of Theorem 2.13.** Let $\sigma_{ph}(L_{F_s}) = \{ z \in \mathbb{C} : |z| = 1 \}$ be the peripheral spectrum. If $e^{i\theta} \in \sigma_{ph}(L_{F_s})$, then by Theorem 2.10 it is point spectrum of finite multiplicity. In addition, since the operator is power bounded, there cannot exists Jordan block, thus the algebraic and geometric multiplicity are equal.

In fact, see [9, Section 5] for a proof which applies verbatim to the present context, the eigenvectors associated to the eigenvalue one are the physical measure.

Hence there is $N \in \mathbb{N}$ and $\{ \phi_j, h_j, \ell_j \}_{j=1}^N$ such that $\phi_0 = 1$, $\ell_0(\phi) = \int_{T^2} \phi$, $\phi_j \in [0, 2\pi)$, $h_j \in B_{s,s}$, $\ell_j \in B_{s,s}$ and $L_{F_s} h_j = e^{i\theta} h_j$, $\ell_j \circ L_{F_s} \phi = e^{i\theta} \ell_j(\phi)$ for all $\phi \in B_{s,s}$. On the other hand, for each $j$ let $\varphi_j \in C^\infty$ be such that $\int_{T^2} h_k \varphi_j = \delta_{k,j}$, then
\[ |\ell_j(h)| = \left| \lim_{n \to \infty} \sum_{k=0}^{n-1} e^{-i\theta j k} \int_{T^2} \varphi_j L_{F_s}^k h \right| \leq \left| \lim_{n \to \infty} \sum_{k=0}^{n-1} \int_{T^2} |\varphi_j \circ F^k_\varepsilon| |h| \right| \leq \| \varphi_j \|_\infty \| h \|_{L^1}. \]
Which implies that there exists $\tilde{\ell}_j \in L^\infty$ such that
\[ \ell_j(h) = \int_{T^2} \tilde{\ell}_j(h). \]
Note that the above also implies $\tilde{\ell}_j \circ F_\varepsilon = e^{i\theta j} \tilde{\ell}_j$. The above means that, for all $l \in \mathbb{N}$,
\[ \int_{T^2} \tilde{\ell}_j L_{F_s}^l h = \int_{T^2} \tilde{\ell}_j \circ F_\varepsilon^l h = e^{i\theta_j \varepsilon} \int_{T^2} \tilde{\ell}_j h \]
This implies that $e^{i\theta_j \varepsilon l}$ belongs to the spectrum of $(L_{F_s})'$, hence of $L_{F_s}$. Since there can be only finitely many elements of $\sigma_{ph}(L_{F_s})$, it must be $\theta_j = \frac{2\pi n}{q}$ for some $p, q \in \mathbb{N}$, that is the $\{ \theta_j \}$ form a finite group.

\(^{55}\)Note that in this case we have $C_\theta = C_{\kappa \alpha \theta} = C_{2\kappa \alpha \theta} \leq C_\theta$. 
It follows that we have the following spectral decomposition

$$\mathcal{L}_{F_n} = \sum_j e^{i\theta_j} \Pi_j + Q$$

where $\Pi_j h = h_t \ell_j(h)$, $\Pi_j \Pi_k = \delta_{jk} \Pi_j$ and $Q$ has spectral radius strictly smaller than one.

In addition, $|h_j| \leq \mathcal{L}_{F_n}|h_j|$. Since $h_j \in \mathcal{H}^3$ it follows that $h_j \in C^1$, so $|h - J|$ is Lipschitz, hence $h^*_j = |h_j| \in \mathcal{H}^1 \cap C^0$. Hence,

$$0 = \int_{T^2} \mathcal{L}_{F_n} h^*_j - h^*_j$$

which implies $h^*_j = \mathcal{L}_{F_n} h^*_j$. It follows that $h^*_j$ is an eigenvector of $\mathcal{L}_{F_n}$ associated to the eigenvalue one. Next we would like to better understand the structure of the peripheral spectrum.

Let $(x_k, \theta_k) = F^k(x, \theta)$ and $f_\theta(x') = f(x', \theta)$. By [18, Lemma 4.2] there exists $Y_n$ such that $\pi_2(F^n(x, \theta)) = f^n_Y(Y_n(x))$ and, for all $k \leq n$,

$$|x_k - f^n_\theta(Y_n(x))| \leq C_\varepsilon^k$$

$$|\theta_k - \theta| \leq C_\varepsilon^k$$

$$|1 - \theta_k Y_n| \leq C_\varepsilon^k.$$
Since, for each $\varphi \in C^1$, a direct computation shows $\|L^k h\|_{H^{1}} \leq C_2 \varepsilon^{-\alpha} \|h\|_{H^{1}}$, by equation (8.32) we have

$$
\int_{T^2} \varphi \Pi_j h = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{T^2} \varphi e^{-i\theta_j k} L^k_{F^i} h \\
= \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-\varepsilon \ln \varepsilon^{-1}} \int_{T^2} \varphi e^{-i\theta_j k} L^k_{F^i} h \\
(8.33) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-\varepsilon \ln \varepsilon^{-1}} \int_{T^2} \varphi e^{-i\theta_j k} P L^k_{F^i} h + O(\varepsilon [\ln \varepsilon^{-1}]^2 \|\varphi\|_{C^1} \|h\|_{L^1} + \varepsilon^{50} \|h\|_{H^1}) \\
= \int_{T^2} \varphi P \Pi_j h + O(\varepsilon [\ln \varepsilon^{-1}]^2 \|\varphi\|_{C^1} \|h\|_{L^1} + \varepsilon^{50} \|h\|_{H^1}).
$$

Hence,

(8.34) $\|\Pi_j - P \Pi_j\|_{B_{\varepsilon \to \varepsilon^{-1}}(C^1)} \leq C_2 \varepsilon [\ln \varepsilon^{-1}]^2$.

To have more informations we need to know more about $\omega$. A full discussion would need to follow the line of argument developed in [20], but using also the present results. As an example let us discuss an explicit case.

Let $\bar{\omega}(\theta) = \int_{T} \omega(x, \theta) dx$ and $\bar{\theta}(t, \theta)$ be the solution of equation (2.33).

Suppose that $\bar{\omega}$ has $N$ non degenerate zeroes $\{\theta_i\}$ such that $\bar{\omega}'(\theta_i) < 0$. Then any $\theta_i$ has a basin of attraction for the dynamics (2.33) consisting in an interval $U_i$. It follow that in a time $c_\varepsilon \ln \varepsilon^{-1}$ all the points will be in an $\varepsilon$ neighborhood of some $\theta_i$ a part for intervals $\varepsilon^\gamma$ around the boundaries of the basins. Then by equation (8.33) and [18, Theorem 2.1], more precisely using [18, equation (4.5)], we have, for $n = c_\varepsilon \varepsilon^{-\frac{4}{\gamma}}$,

$$
\int_{T^2} \varphi \Pi_0 h = \int_{T^2} \varphi \Pi_0 L^0_{F^i} h = \int_{T^2} \varphi L^0_{F^i} h + O(\varepsilon [\ln \varepsilon^{-1}]^2 \|\varphi\|_{C^1} \|h\|_{H^1}) \\
= \sum_i \varphi(\theta_i) \int_{U_i \times T} h + O(\varepsilon^\frac{50}{\gamma} \|\varphi\|_{C^1} \|h\|_{H^1}).
$$

Accordingly, setting $\int_{T^2} \varphi \hat{P} h = \sum_j \int_{T^2} \varphi(x, \theta_j) h(x, \theta_j) \int_{U_j \times T} h$, we have

(8.35) $\|\Pi_0 - \hat{P}\|_{B_{\varepsilon \to \varepsilon^{-1}}(C^1)} \leq C_2 \varepsilon^\frac{1}{\gamma}$.

The last equation of the Theorem 2.13 follows from (8.32) and (8.35). \[ \square \]

**Appendix A. Proof of Lemma 3.1**

We start considering $\varphi, \psi \in C^\rho(T^2, \mathbb{R})$. First we prove, by induction on $\rho$, that

(A.1) $\sup_{|\alpha| = \rho} \|\partial^\alpha (\varphi \psi)\|_{C^0} \leq \sum_{k=0}^{\rho} \binom{\rho}{k} 2^{\rho-k} \sup_{|\beta| = \rho-k} \|\partial^\beta \varphi\|_{C^0} \sup_{|\gamma| = k} \|\partial^\gamma \psi\|_{C^0}$.
Indeed, it is trivial for $\rho = 0$ and
\[
\|\partial_x, \partial^\alpha (\varphi \psi)\|_{C^0} = \|\partial^\alpha (\varphi \partial_x, \varphi + \varphi \partial_x, \psi)\|
\]
\[
\leq \sum_{k=0}^\rho \binom{\rho}{k} \sup_{|\beta| = \rho - k} \|\partial^\beta \partial_x, \varphi\|_{C^0} \sup_{|\gamma| = k} \|\partial^\gamma \psi\|_{C^0} + \sum_{k=0}^\rho \binom{\rho}{k} \sup_{|\beta| = \rho - k} \|\partial^\beta \partial_x, \psi\|_{C^0} \sup_{|\gamma| = k} \|\partial^\gamma \varphi\|_{C^0}
\]
\[
\leq \sum_{k=0}^\rho \binom{\rho}{k} \sup_{|\beta| = \rho - k} \|\partial^\beta \varphi\|_{C^0} \sup_{|\gamma| = k} \|\partial^\gamma \psi\|_{C^0} + \sum_{k=0}^\rho \binom{\rho}{k} \sup_{|\beta| = \rho - k} \|\partial^\beta \varphi\|_{C^0} \sup_{|\gamma| = k} \|\partial^\gamma \psi\|_{C^0},
\]
from which (A.1) follows taking the sup on $\alpha, i$ and since $\binom{\rho}{k} = \binom{\rho+1}{k-1}$. We then have the first statement of the Lemma, indeed
\[
\|\varphi \psi\|_{C^\rho} = \sum_{k=0}^\rho 2^{\rho-k} \sum_{j=0}^{k} \binom{k}{j} \sup_{|\beta| = k-j} \|\partial^\beta \varphi\|_{C^0} \sup_{|\gamma| = j} \|\partial^\gamma \psi\|_{C^0}
\]
\[
\leq \sum_{j=0}^{\rho-1} \sum_{k=0}^{\rho-j} \binom{\rho}{k} 2^{\rho-j} \sup_{|\beta| = j} \|\partial^\beta \varphi\|_{C^0} \sup_{|\gamma| = j} \|\partial^\gamma \psi\|_{C^0} \leq \|\varphi\|_{C^\rho} \|\psi\|_{C^\rho}
\]
since $\binom{\rho}{k} \leq 2^\rho$. The extension to function with values in the matrices follows trivially since we have chosen a norm in which the matrices form a norm algebra.

To prove the second statement we proceed again by induction on $\rho$. The case $\rho = 0$ is immediate since $K_{0,0}$ contains only the zero string. Let us assume that the statement is true for every $k \leq \rho$ and prove it for $\rho+1$. By equation (A.1) and the inductive hypothesis (3.3), we have, for each $|\alpha| = \rho+1$,
\[
|\partial^\alpha (\varphi \circ \psi)\| \leq C_\rho \sup_{|\beta| = \rho} \sup_{|\tau_1, |\tau_2| = 1} \left|\partial^\beta \left[ (\partial^\tau_1 \varphi) \circ \psi \cdot \partial^\tau_2 \psi \right] \right|
\]
\[
\leq C_\rho \sup_{|\tau_1, |\tau_2| = 1} \sup_{|\alpha_0| = |\alpha_1| = \rho} \|\partial^\alpha_0 \left[ (\partial^\tau_1 \varphi) \circ \psi \right] \|_{C^0} \|\partial^\alpha_1 \partial^\tau_2 \psi\|_{C^0}
\]
\[
\leq C_\rho \sup_{|\tau_1| = 1} \|\partial^\tau_1 \varphi\|_{C^\rho} \circ \psi\|_{C^\rho} \|D\psi\|_{C^{\rho-\rho}}
\]
\[
\leq C_\rho C^*_\rho \sup_{|\alpha_0| \leq \rho} \sum_{s=0}^{\alpha_0} \|\varphi\|_{C^{\rho+s+1}} \sum_{k\in K_{0,0}, \tau \in \mathcal{N}} \|D\psi\|_{C^{\tau-1}} \|D\psi\|_{C^{\rho-s}}
\]
\[
\leq C_\rho C^*_\rho \sup_{|\alpha_0| \leq \rho} \sum_{s=0}^{\rho+1} \|\varphi\|_{C^{\rho+s+1}} \sum_{k\in K_{\rho+1,\tau+1}, \tau \in \mathcal{N}} \|D\psi\|_{C^{\tau-1}}
\]

The result follows by choosing $C^*_\rho$ large enough. \(\square\)

**Appendix B. Proof of Lemma 4.5**

This appendix is devoted to the proof of Lemma 4.5.

As usual we use the notation $F^k \tilde{b}_k = \gamma \circ h_k$, $F^k \nu_k = \gamma$ . As the computation is local it suffices to consider $p_n \in \tilde{b}_n$ and $p_0 \in \gamma$ such that $F^n(p_n) = p_0$. Let $p_k = F^{n-k}p_n$. To ease notation we use a translation to reparametrize the curves so that $\nu_k(0) = \tilde{b}_k(0) = p_k$, note
that \( h_k(0) = 0 \). Before discussing the splitting of the vector field we need some notations and few estimates.

It is convenient to perform the changes of variables \( \phi_k^{-1}(x, y) = (x, 0) + \hat{v}_k(y) \) and set

\[
\tilde{F}^k = \phi_0 \circ F^k \circ \phi_k^{-1}; \quad \tilde{F}_k = \phi_{k-1} \circ F \circ \phi_k^{-1}
\]

Note that \( \tilde{F}_k = \tilde{F}_k \circ \cdots \circ \tilde{F}_1 \) and \( \tilde{F}_n(0, y) = \phi_0 \circ F^n(\hat{v}_n(y)) = \phi_0(\gamma \circ h_n(y)) = (0, h_n(y)) \), this implies that

\[
D(0, y) \tilde{F}^n = \begin{pmatrix} a^n(y) & 0 \\ c^n(y) & d^n(y) \end{pmatrix}; \quad D(0, y) \tilde{F}_k = \begin{pmatrix} a_k(y) & 0 \\ c_k(y) & d_k(y) \end{pmatrix}; \quad D\phi_k^{-1} = \begin{pmatrix} 1 & (\nabla h_k) \end{pmatrix},
\]

with \( d^n(y) = h'_n(y) \) and \( d_k(y) = h'_k(y) \). Thus, we have the estimates on the \( C^\rho \) norms of \( \tilde{F}^k \) by Lemma 3.8, also the changes of coordinates \( \phi_k \) have uniformly bounded \( C^\rho \) norms.

From the above we easily get the formulae:

\[
\begin{align*}
(B.1) \quad a^{k+1}(y) &= a^k(y) a_{k+1}(h_k(y)) \\
(B.2) \quad d^{k+1}(y) &= d_{k+1}(h_k(y)) d^k(y) \\
(B.3) \quad c^k(y) &= \sum_{j=1}^{k} d_k(h_{k-1}(y)) \cdots d_{j+1}(h_j(y)) c_j(h_{j-1}(y)) a_{j-1}(h_{j-2}(y)) \cdots a_1(y).
\end{align*}
\]

Moreover,

\[
DF^k = \begin{pmatrix} a^k + (\nabla h'_k)_1 c^k & (\nabla h'_k)_1 d^k - (\nabla h'_k)_1 c^k \\ d^k - (\nabla h'_k)_1 c^k & 0 \end{pmatrix}
\]

which, setting \( y_k = h_k(y) \), yields the alternative representations and estimates

\[
\begin{align*}
&c_k(y_{k-1}) = (e_2, D(0, y_{k-1}) F e_1) \\
&a_k(y_{k-1}) = (e_1, D(0, y_{k-1}) F e_1) - \nabla h'_{k-1}(y_{k-1}) (e_2, D(0, y_{k-1}) F e_1) \\
&|c^k(y)| = |(e_2, D(0, y) F^k e_1)| \leq \frac{\lambda^+_k}{\sqrt{1 + \chi u}} - \chi c \chi u \lambda^+_k \leq |a^k(y)| \leq \lambda^+_k + \chi c \chi u \lambda^+_k
\end{align*}
\]

Also, for further use,

\[
(B.5) \quad (DF^k)^{-1} = \begin{pmatrix} a^k(y)^{-1} & 0 \\ -d^k(y)^{-1} a^k(y)^{-1} c^k(y) & d^k(y)^{-1} \end{pmatrix}
\]

We are now ready to describe the splitting of the vector field. We do it in the new coordinates. Consider the subspace \( E_n(y) = \{(\nu, u^n(y) \eta)\}_{\eta \in \mathbb{R}} \), where \( u_n(y) = a^n(y)^{-1} c^n(y) \), which is a \( C^\sigma \) approximation of the unstable direction. Given a vector \( v \in \mathbb{R}^2 \) let us call \( \tilde{v} = D\phi_0 v \) the vector in the new coordinates. Next, we decompose a vector \( \tilde{v} \) as

\[
\tilde{v} = (1, u_n \circ \hat{h}_n) \tilde{v}_1 + (\tilde{v}_2 - \tilde{v}_1 u_n \circ \hat{h}_n) e_2
\]

where \( h_n \circ \tilde{F}^n(0, y) = (0, y) \). Thus, setting \( V(t) = v_1(\gamma(t)) - \gamma'(t) \gamma(v_2(\gamma(t)), we have the decomposition (4.26) with

\[
\begin{align*}
v^v(\gamma(t)) &= V(t)(1 + \gamma'(t) \gamma_n(0, t) \gamma_n(0, t)) \\
v^v(\gamma(t)) &= (\gamma'(t)) [v_2(\gamma(t)) - u_n \circ \hat{h}_n(0, t) V(t)] - u_n \circ \hat{h}_n(0, t) V(t).
\end{align*}
\]
To extend the above decomposition in a neighborhood of $\gamma$ we will proceed as in [30, Lemma 6.5]. First, we compute the derivatives along the curve, to this end note that in the new coordinates $t = y_n$. Differentiating (B.1) we have

$$(B.7) \quad \partial_{y}a^{k}(y)^{-1} = [\partial_{y}a^{k-1}(y)^{-1}] a_{k}(y_{k-1})^{-1} + a^{k-1}(y)^{-1}\partial_{y_{k-1}}a_{k}(y_{k-1})^{-1}\tilde{a}^{k-1},$$

and, by (B.4) and and Lemma 3.8,

$$|\partial_{y_{k-1}}a_{k}(y_{k-1})| \leq C_{2}(1 + \|\nu_{k-1}''\|) \leq C_{2}(1 + \epsilon)$$

$$\|\partial_{y}a_{k}\|_{C^{r}} \leq C_{2}\|\nu_{k-1}\|_{C^{r+1}} \leq C_{2}\epsilon\delta.$$ 

Next, using (B.7), we can prove by induction that $\|a^{n}\|^{-1}_{C^{r}} \leq C_{2}\lambda_{n}^{-n}\epsilon^{\rho(n)}C_{\mu,n}^{\rho(n)}\mu^{\rho(n)}$. \(57\)

$$\|a^{n}\|^{-1}_{C^{r}} \leq C_{2}\lambda_{n}^{-n} + \lambda^{-1}\|a^{n-1}\|^{-1}_{C^{r}} + C_{2}\|a^{n-1}\|^{-1}_{C^{r-1}}\epsilon^{\rho(n)}C_{\mu,n-1}^{\rho(n-1)}$$

(B.8)

$$\leq C_{2}\lambda_{n}^{-n} + \epsilon^{\rho}C_{2}\sum_{j=0}^{n-1}\lambda_{j}^{-n}\|a^{j}\|^{-1}_{C^{r-1}}C_{\mu,j}^{\rho}$$

$$\leq C_{2}\lambda_{n}^{-n}\epsilon^{\rho}C_{\mu,n}^{\rho(n)}.$$

To compute $\|d^{n}\|^{-1}_{C^{r}}$ we can use formula (3.3) and recall (3.24) and (3.32):

(B.9) \quad $\|d^{n}\|^{-1}_{C^{r}} = \|\hat{f}_{n}\|^{-1}_{C^{r}} \leq C_{2}\mu_{n}^{(\rho+1)n}C_{\mu,n}^{(\rho+1)n} = C_{2}\mu_{n}^{(\rho+1)n}n.$

Next, by (B.1), (B.2) and (B.3) we have

$$[a^{n}(y)]^{-1}e^{n}(y) = \sum_{j=1}^{n}d_{n}(h_{n-1}(y))\cdots d_{j+1}(h_{j}(y))c_{j}(h_{j-1}(y))[a_{n}(h_{n-1}(y))\cdots a_{j}(h_{j-1}(y))]^{-1},$$

$$[d^{n}(y)a^{n}(y)]^{-1}e^{n}(y) = \sum_{j=1}^{n}[d_{j-1}(h_{j-2}(y))\cdots d_{1}(y)][a_{n}(h_{n-1}(y))\cdots a_{j}(h_{j-1}(y))]^{-1}c_{j}(h_{j-1}(y))[a_{n}(h_{n-1}(y))\cdots a_{j}(h_{j-1}(y))]^{-1}.$$ 

Hence, by (B.8), (B.9) and the first of (B.4), we obtain, using (A.1),

$$\|d^{n}a^{n}\|^{-1}e^{n}, \|a^{n}\|^{-1}e^{n} \leq C_{2}\epsilon^{\rho(n)}C_{\mu,n}^{(\rho+1)n}n.$$ 

(B.10)

We are ready to conclude. Since

$$(D\hat{\phi}(y)F^{n})^{-1} = D_{(0,y)}\hat{\phi}^{-1}(D_{(0,y)}F^{n})^{-1}D_{y=\hat{\phi}(y)}\phi_{0},$$

by (B.6) and (B.5) it follows

$$\left(\begin{array}{c}
(D\hat{\phi}(y)F^{n})^{-1}v^{u}(\gamma \circ h_{n}(y)) = V(h_{n}(y)) (a^{n}(y)^{-1},0), \\
(D\hat{\phi}(y)F^{n})^{-1}v^{c}(\gamma \circ h_{n}(y)) = d^{n}(y)^{-1} \cdot \left[\nu_{2} - v_{u}v_{1}\right] \circ \gamma(h_{n}(y)) \left((\nu^{n}_{u})_{1}(y),1\right).
\end{array}\right.$$ 

Recalling that $u_{n}(y) = a^{n}(y)^{-1}e^{n}(y)$, by (B.8), (B.9), (B.10), and since $\gamma \in \Gamma(x)$ and $\|v\|_{C^{r}} \leq 1$, we have the result for the vector field along the curve. Finally, we extend $v^{u}$ to a neighborhood of $\gamma$. It turns out the be more convenient to define first the extension

$$w(x,y) = F^{n}u^{n}(\hat{\nu}_{n}(y))$$

\(56\)In the mentioned paper the authors need more regularity for the extended vector field. Here it is enough to obtain a vector field which is $C^{r}$.

\(57\)Here $a_{\rho}$ is the one given by Lemma 3.8.
then $\hat{v} = h^*_n w$ and $F^n \hat{v} = w$. By these definitions it follows
\[
\|F^n \hat{v}\|_{L^2(\mathbb{R}^n)} = \|F^n v^n\|_{L^2} \leq \lambda^{-n} C_{\mu,n} \mu^{n+2n}.
\]
The definition of $\hat{v}$ and relative estimates are analogous. □

**APPENDIX C. THE SPACE $\mathcal{H}^s$**

Let $u \in C^\infty(\mathbb{T}^2)$. The *Fourier Transform* of $u$ and its inverse are
\[
(C.1) \quad \mathcal{F} u(\xi) = \int_{\mathbb{T}^2} e^{-i2\pi x \xi} u(x) dx, \quad \xi \in \mathbb{Z}^2,
\]
\[
(C.2) \quad u(x) = \sum_{\xi \in \mathbb{Z}^2} \mathcal{F} u(\xi) e^{i2\pi x \xi}, \quad x \in \mathbb{T}^2.
\]
Then $\mathcal{H}^s$ is the completion of $C^\infty(\mathbb{T}^2)$ with respect to the inner product
\[
(C.3) \quad \langle u, v \rangle = \sum_{\xi \in \mathbb{Z}^2} (\xi)^{2s} \mathcal{F} u(\xi) \overline{\mathcal{F} v(\xi)}, \quad (\xi) := \sqrt{1 + \|\xi\|^2}.
\]
Notice that, by formula (7.9.2) of [36], there is $C > 0$ such that
\[
(C.4) \quad C^{-1} \sum_{\gamma + \beta = s} C_{\gamma,\beta} \|\partial^{\gamma}_{x_1} \partial^{\beta}_{x_2} u\|_{L^2}^2 \leq \|u\|^2_{\mathcal{H}^s} \leq C \sum_{\gamma + \beta = s} C_{\gamma,\beta} \|\partial^{\gamma}_{x_1} \partial^{\beta}_{x_2} u\|_{L^2}^2.
\]

**Lemma C.1.** For every $\zeta \in (0, 1)$ and $1 \leq s < r$ there exists constants $C_s$ such that
\[
\|u\|^2_{\mathcal{H}^s} \leq \zeta \|u\|^2_{\mathcal{H}^r} + \frac{C_s}{\zeta} \|u\|^2_{L^1}, \quad u \in C^r(\mathbb{T}^2).
\]

**Proof.** By definition of the norm we have, for all $\tau \in (1, 2),$
\[
(C.5) \quad \|u\|^2_{\mathcal{H}^{s-1}} = \sum_{\xi \in \mathbb{Z}^2} |\mathcal{F} u(\xi)|^2 \langle \xi \rangle^{2(s-1)} = \sum_{\xi \in \mathbb{Z}^2} |\mathcal{F} u(\xi)|^2 \langle \xi \rangle^{2s-2+\tau} \langle \xi \rangle^{-\tau}
\]
By Young inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, for every $\zeta > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. We apply this with $a = \langle \xi \rangle^{2s-2+\tau}, b = \langle \xi \rangle^{-\tau}$ and $p = \frac{2s}{(2s-2+\tau)}, q = \frac{2s}{2-\tau}$ to obtain:
\[
\langle \xi \rangle^{2s-2+\tau} \langle \xi \rangle^{-\tau} \leq \left( 1 - \frac{2 - \tau}{2s} \right) \zeta \langle \xi \rangle^{2s} + \zeta^{-1} \frac{(2 - \tau) \langle \xi \rangle^{-\frac{2s}{2-\tau}}}{2s}.
\]
Using this fact in (C.5) and recalling that $\|\mathcal{F} u\|_{\infty} \leq C\|u\|_{L^1}$, we get
\[
\|u\|^2_{\mathcal{H}^{s-1}} \leq \zeta \sum_{\xi \in \mathbb{Z}^2} |\mathcal{F} u(\xi)|^2 \langle \xi \rangle^{2s} + \frac{C_s}{\zeta} \|\mathcal{F} u\|^2_{\infty} \leq \zeta \|u\|^2_{\mathcal{H}^r} + \frac{C_s}{\zeta} \|u\|^2_{L^1}.
\] □
Appendix D. Vector Field Regularity

This appendix is devoted to proving the following regularity results on the iteration of a vector field.

**Lemma D.1.** Let \( \varepsilon_0 \in (0, 1] \), \( A \in [0, 1/2] \), \( B > 0 \) and \( u, u' \in C^1(\mathbb{T}^2, \mathbb{R}) \) such that \( \|u\|_\infty, \|u'\|_\infty \leq A\varepsilon_0^{-1} \) and \( \|\nabla u\|_\infty, \|\nabla u'\|_\infty \leq B\varepsilon_0^{-1} \). Consider a family of vertically partially hyperbolic maps \( F_\varepsilon \), \( \varepsilon \leq \varepsilon_0 \) such that

\[
\left\| \frac{\partial f(p)}{\partial x_f(p)} \right\|_\infty \leq 1
\]

(D.1)

\[
\partial_x f(p) \left[ 1 - A \frac{\left\| \frac{\partial f(p)}{\partial x_f(p)} \right\|_\infty}{1 - \varepsilon} \right] \geq 2(1 + \varepsilon_0 \|\partial_x \omega\|_\infty).
\]

For each \( h \in H^\infty \) and \( k \leq n \in \mathbb{N} \), we define the sequence of functions \( u_n(p, \varepsilon) = u(h_n(p)) \)

\[
\tilde{u}_0(p, \varepsilon) = u(h_0(p))
\]

and similarly for \( \tilde{u}_k \). Then, for each \( p, p' \in \mathbb{T}^2 \) and \( \varepsilon, \varepsilon' < \varepsilon_0 \),

\[
|\tilde{u}_n(p, \varepsilon) - \tilde{u}_n(p', \varepsilon')| \leq C_2 \varepsilon^{4A} \mu^{2n} \left( \lambda_n^+(h_n(p))^{-1} \right) \|u - u'\|_\infty
\]

\[
+ (\|\omega\|_{L^2} + \mu \lambda_n^+(h_n(p))^{-1} C_{2\varepsilon} |u'|)\|p - p'\| + [1 + \lambda_n^+(h_n(p))^{-1} |u'|^2] |\varepsilon - \varepsilon'|
\]

Proof. Let \( p_k(p, \varepsilon) = h_k(p) \), for \( h \in H^\infty \), \( p \in \mathbb{T}^2 \). By (3.18) (or see [17] for details) we have

(D.2)

\[
\|\partial_f p_k\| \leq \|(D_{b_k}(p) F_{\varepsilon})^{-1}\| \leq C_2 \varepsilon \mu^k \leq C_2 \varepsilon e^{c_1\varepsilon k}.
\]

For each \( u > 0 \) and for \( k \leq n \) let

\[
\lambda(p, \varepsilon) = \frac{|\partial_x f(p)|}{1 + \varepsilon (\|\partial_\theta \omega\|_\infty + \|\partial_x \omega\|_\infty)} \geq |\partial_x f(p)| \mu^{-1}
\]

\[
u_k(p, \varepsilon, u) = \bar{\Xi}(p_{n-k+1}(p, \varepsilon), u_{k-1}(p, \varepsilon, u)),
\]

where in the first line we have used Remark 2.16. Note that \( \tilde{u}_n(p, \varepsilon) = u_n(p, \varepsilon, u_0(p, \varepsilon)) \).

Using (8.2) and (8.4) a direct computation yields, for \( |u| \leq A\varepsilon_0^{-1} \),

\[
\left| \bar{\Xi}_t(p, u) \right| \leq \frac{|u| (1 + \varepsilon \|\partial_\theta \omega\|_\infty)}{|\partial_x f(p)|} \left[ 1 - \varepsilon |u| \left| \frac{\partial f(p)}{\partial x_f(p)} \right| \|\partial_x \omega\|_\infty \right] + \frac{\|\partial_\theta \omega\|_\infty}{|\partial_x f(p)|} \left[ 1 - A \left| \frac{\partial f(p)}{\partial x_f(p)} \right| \|\partial_\theta \omega\|_\infty \right]
\]

(D.4)

\[
\left| \partial_{\theta_t} \bar{\Xi}_t(p, u) \right| \leq \frac{1}{\lambda(p, \varepsilon)} \left[ 1 - \varepsilon \left| \frac{\partial f(p)}{\partial x_f(p)} \right| \right]^{-2}
\]

\[
\|\partial_x \bar{\Xi}_t(p, u)\| \leq C_2 (|\omega|_{L^2} + |u|)
\]

The first line of the (D.4) and the second of (D.1) imply

\[
|u_k(p, \varepsilon, u)| \leq 2^{-k} |u| + \|\partial_x \omega\|_\infty.
\]

\footnote{See (8.8) for the definition of \( \Phi^\varepsilon \).}
We can get a sharper bound defining
\[ \Lambda_{k,j}(p) := \prod_{i=k+1}^{j} \lambda(p_i, \varepsilon); \quad \Xi_{k,j}(p) := \prod_{i=k+1}^{j} |\partial_x f(p_i)| \]
\[ \Delta := \|\partial_x \omega\|_{\infty} \frac{\|\partial_y f(p)\|}{\|\partial_x f(p)\|_{\infty}}, \]
then
\[ (D.5) \quad |u_k(p, \varepsilon, u)| \leq \Lambda_{n-k,n}(p)^{-1}|u| + \|\partial_x \omega\|_{\infty}. \]
Moreover, setting \( u_j = u_j(p, u, \varepsilon), \) \( u'_j = u_j(p', u', \varepsilon'), \) with \(|u|, |u'| \leq \frac{\Delta}{\varepsilon_0}, \) and recalling \((D.2), (D.3), (D.5),\) we have
\[ |u_{k+1}(p, \varepsilon, u) - u_{k}(p', \varepsilon', u')| = |\Xi_{k}(p_{n-k}, u_k) - \Xi_{k}(p'_{n-k}, u'_k)| \]
\[ \leq C_{\varepsilon}(2\|\varepsilon\|^2 + |u_k|)\|p_{n-k} - p'_{n-k}\| + C_{\varepsilon}(1 + |u'_k|)|u_k|\|\varepsilon - \varepsilon'\| \]
\[ + \Lambda_{n-k-1,n-k}(p)^{-1}\varepsilon e^{2^{k+1}A+2\varepsilon\Delta}|u_k - u'_k| \]
\[ (D.6) \quad \leq C_{\varepsilon}(2\|\varepsilon\|^2 + \Xi_{n-k,n}(p')^{-1}\mu^k|u'|)\mu^{n-k}\|p - p'\| \]
\[ + (1 + \Xi_{n-k,n}(p')^{-1}\mu^k|u'|)|\varepsilon - \varepsilon'| + \Lambda_{n-k-1,n-k}(p)e^{2^{k+1}A+2\varepsilon\Delta}|u_k - u'_k|. \]
We can then iterate the above equation and obtain
\[ |u_n(p, \varepsilon, u) - u_n(p', \varepsilon', u')| = \Xi_{0}(p_{n-k}, u_k) \]
\[ + C_{\varepsilon} \sum_{k=0}^{n-1} \Xi_{0,k,n}(p)^{-1}\mu^{n-k}\varepsilon e^{2^{k+1}A+2\varepsilon\Delta}\||\varepsilon|\|c^2 + \Xi_{n-k,n}(p')^{-1}\mu^k|u'|\mu^{n-k}\|p - p'\| \]
\[ + C_{\varepsilon} \sum_{k=0}^{n-1} \Xi_{0,k,n}(p)^{-1}\mu^{n-k}\varepsilon e^{2^{k+1}A+2\varepsilon\Delta}(1 + \Xi_{n-k,n}(p')^{-2}\mu^k|u'|^2)|\varepsilon - \varepsilon'|. \]
In addition equations \((8.1)\) and \((3.17)\) imply
\[ \Xi_{j,n}(p) \geq C_{\varepsilon}\lambda_{n-j}(p_n) \]
\[ |\partial_p \Xi_{j,n}(p)| \leq \sum_{j=1}^{n} \Xi_{j,n}(p)|\partial_p^2 f(p_n)|\Xi_{j,t-1}(p)|\partial_p p_l| \leq C_{\varepsilon}C_{p,n}\mu^n\Xi_{j,n}(p). \]
Thus,
\[ |u_n(p, \varepsilon, u) - u_n(p', \varepsilon', u')| \leq C_{\varepsilon}e^{2^{n+2A\Delta}}\mu^n \left\{ \lambda_{n}(p_n)^{-1}|u - u'| \right. \]
\[ + \left( |\varepsilon|\|c^2 + \mu^{2n}\lambda_{n}(p_n)^{-1}C_{p,n}|u'|\|p - p'\| + (1 + \lambda_{n}(p_n)^{-1}|u'|^2)|\varepsilon - \varepsilon'| \right\}. \]
The Lemma follows recalling that Remark 2.16 and our hypotheses imply \( e^{\varepsilon_0\Delta} \leq \mu. \]

**Appendix E. Extension of curves**

In this section we explain how to extend a segment to a close curve of homotopy class \((0, 1)\) with precise dynamical properties and explicit bounds on the derivatives.
Lemma E.1. There exist constants $\delta_0$, $C_{n_0,j} > 0$ and $L_* \geq 1$ such that for each line segment $\gamma(t) = \gamma(0) + (1, v)t$ of length $\delta \leq \delta_0$ and $n_0 \in \mathbb{N}$ such that $\gamma'(t) \notin \cup_{z \in F^{n_0}(\gamma(t))} \dot{D}_z F^{n_0} C_u$ we can extend $\gamma$ to a closed curve $\bar{\gamma}$, parametrized by arclength, of homotopy class $(0, 1)$ with the following properties:

- let $\gamma_-(t) = \gamma(0) + \frac{1}{2}e_1 + e_2 t$, then for each $h \in \mathcal{F}_\infty \gamma_-$ and $k \in \mathbb{N}$ we have $\bar{\gamma} \in \text{Dom}(h_k)$ and $h_k \circ \bar{\gamma}$ is a closed curve in the homotopy class $(0, 1)$.
- $\vartheta_j \leq \vartheta_j$.
- For all $p \in T^2$ and $m \geq n_0 \in \mathbb{N} \cup \{0\}$, if $D_p h_{n_0} \gamma' \notin C_u$ and $D_p h_{m} \gamma' \in C_c$, then $D_p h_{m} \gamma' \notin C_{C_c, u}$ and $D_p h_{m} \gamma' \in C_c$.
- For each $j \in \{1, \ldots, r - 1\}$ and $t \in \mathbb{R}$,
  \[
  \|\bar{\gamma}(t)\| \leq C_{n_0,j} \left(\frac{L_* \{L_* \{1 + e^{-1} \mu^m\}^j}{\chi_u + |\pi_2(\gamma'(t)))|}\right) := C_{n_0,j} \Delta_j^j. \tag{E.1}
  \]

Moreover, if the conditions of Lemma D.1 are satisfied, then (E.1) holds true with

\[
L_* (n) = \sup_{|v| \leq 1} L_* (n, v), \tag{E.2}
\]

\[
L_* (n, v) = C_n \tilde{K}^{-e_1 \ln \mu} C_{\mu, n_0} (\|\omega\| c^n + \tilde{K}^{1 - e_1 \ln \mu}) ; \quad \tilde{K} = |v| + \chi_u.
\]

Proof. By an isometric change of variables we can assume, without loss of generality, that $\gamma(0) = 0$. Hence $\gamma(t) = (1, v)t$ for $t \in [-\delta, \delta]$ and $\gamma'(t) = (1, v) =: \bar{v}$. Note that we can assume $|v| \leq 1$ since otherwise the Lemma is trivial.

Before getting to the extension per se, we need some results on the dynamics of the central cone as $(1, \zeta)$, so $\zeta$ can be interpreted as a projective coordinate. Then, in analogy with (2.17), we have, for each $p \in T^2$ and $\zeta \in \mathbb{R}$,

\[
D_p F(1, \zeta) = (\partial_\zeta F_1 + \partial_\theta F_1 \zeta)(1, \Xi(p, \zeta))
\]

\[
\Xi(p, \zeta) = \frac{\partial_\zeta F_2 + \partial_\theta F_2 \zeta}{\partial_\zeta F_1 + \partial_\theta F_1 \zeta}
\]

Also, computing as in (2.18),

\[
\partial_\zeta \Xi(p, \zeta) = \frac{\partial_\zeta (D_p F)}{(\partial_\zeta F_1 + \partial_\theta F_1 \zeta)^2}.
\]

Next, for each $q \in T^2$, let $q_n = F^{n_0}(q)$, $z_0(\zeta) = \zeta$, $z_1(q, \zeta) = \Xi(q, z_0(\zeta))$ and, for $j \geq 1$, $z_{j+1}(q_j, \zeta) = \Xi(q_j, z_j(q_{j-1}, \zeta))$. In particular, if $h \in \mathcal{F}_\infty$, $p \in T^2$ and $\Gamma_j(p) = D_{h_j(p)} F^j C_c$, then $\Gamma_j(p) = \{(1, \bar{z}_j(p, \zeta)) : |\zeta| \leq \chi_c\}$ where $\bar{z}_j(p, \zeta) := z_j(h_j(p), \zeta)$. Note that for all $j$ such that $\bar{z}_j \notin C_u$ we have

\[
|\bar{z}_j(p, \chi_c)| \leq C_n \lambda_j^{-1} (h_j(p))^{-1} \chi_c.
\]

In the following we need an estimate of $|\bar{z}_j(p, \pm \chi_c) - \bar{z}_j(p, \pm \chi_c(1 - \epsilon))|$. Since

\[
\partial_\zeta \bar{z}_j(p, \zeta) = \partial_\zeta \Xi(h_j(p), \bar{z}_{j-1}(p, \zeta)) \partial_\zeta \bar{z}_{j-1}(p, \zeta)
\]

iterating the above identities and recalling Propositions 3.5, 3.7 we have

\[
C_n \frac{\mu^{-j}}{\lambda_j^e(h_j(p))} \leq |\partial_\zeta \bar{z}_j(p, \zeta)| \leq C_n \lambda_j^e(h_j(p)) \mu^j \leq C_n \mu^m \chi^{-n_0}.
\]

It follows that $|\bar{z}_j(p, \pm \chi_c) - \bar{z}_j(p, \pm \chi_c(1 - \epsilon))| \geq C_n \epsilon \mu^{-j} |\bar{z}_j(p, \pm \chi_c)|$. 


If, for some \( v_0 > 0, v_0 \geq \varepsilon \), \( \varepsilon \geq \varepsilon_j(p, \chi) \geq 0 \), then either \( \varepsilon_j(p, \chi) \leq \frac{\varepsilon_0}{v_0} \), then \( |\varepsilon_j(p, \chi) - v_0| \geq \frac{\varepsilon_0}{v_0} \); otherwise
\[
|\varepsilon_j(p, \chi) - v_0| \geq |\varepsilon_j(p, \chi) - \varepsilon_j(p, \chi(1 - \varepsilon))| \geq C_\varepsilon \mu^{-j} |\varepsilon_j(p, \chi)| \geq C_\varepsilon \mu^{-j} v_0.
\]
Accordingly, \( |\varepsilon_j(p, \pm \chi) - v_0| \geq C_\varepsilon \mu^{-j} v_0 \). Let \( L_\ast \) be the maximal Lipschitz constant of the \( \varepsilon_j(p, \pm \chi) \) for \( m - n_0 \leq C_\varepsilon \). If the hypotheses of Lemma D.1 are satisfied, then we can provide and explicit estimate for \( L_\ast \): in a finite number of steps \( n_1 \) (depending only on the derivatives of \( F \)) we can have \( \varepsilon_n \leq 1/2 \), we can thus apply Lemma D.1
\[
\varepsilon_0 = \varepsilon = \varepsilon_1 = 1, A = 1/2, B \leq C_\varepsilon \quad \text{and} \quad u = u' = \varepsilon_n(p), \quad \text{we have}
\]
\[
|\varepsilon_m - \varepsilon_{m-n_0}(p, \pm \chi) - \varepsilon_{m-n_0}(p', \pm \chi)| \leq L_{m-n_0} \|p - p'\|
\]
\[
L_j = C_\mu^3 \mu^{j} \lambda^+(p)^{-1} C_{\mu, j}/2.
\]
Since \( D\mathfrak{h}_n v \not\in \mathcal{C}_a \) we have, for \( n_0 = 0 \),
\[
|\varepsilon_m(p, \pm \chi)| - v_0 \geq C_n \mu^{-n} v_0,
\]
while, for \( n_0 > 0 \), applying the above considerations to \( v_0 = D\mathfrak{h}_n \tilde{v} \) yields
\[
|\varepsilon_m - \varepsilon_{m-n_0}(p', \pm \chi)| \geq \pi_2(D\mathfrak{h}_n \tilde{v}) \geq C_\mu \mu^{-n} v_0 \chi_a.
\]
Hence,
\[
(\varepsilon_1.3) \quad |\varepsilon_m(\gamma(t), \pm \chi)| - v_0 \geq C_n \mu^{-n} \chi_a.
\]
Next, note that, by usual distortion arguments, it must be \( \lambda_{m-n_0}^+ \geq C_\mu \mu^{-n} (\chi \kappa)^{-1} \) and \( m - n \leq C_\mu \ln \kappa^{-1} \), thus
\[
L_{m-n_0} \leq C_\mu^3 \mu^2 \ln \kappa^{-1} (\|\omega\| + C_{\mu, n_0} \mu \kappa^{-1} \kappa^{-1} = L_\ast(v).
\]
We are finally ready to extend our segment. We discuss only the case \( v \geq \varepsilon_m(\gamma(0), \chi) \) and \( t \geq 0 \) since the other cases can be treated similarly.

For \( \varphi \in \mathbb{R} \), let \( w(\varphi) = (\cos \varphi, \sin \varphi) \), \( \theta = \tan v \) and \( a = \tan \varphi \). Then \( \varphi = \omega \theta \). We start by extending the curve to the interval \( [\delta, \delta + A] \), with \( A = \frac{\pi}{2} C_\mu \kappa^{-1} \mu^{-m} \kappa^{-1} \). Next, let \( b \in \mathcal{C}^\infty(\mathbb{R}, [0, 1]) \) be a bump function with \( b(t) = 0 \) for \( t \leq 0 \) and \( b(t) = 1 \) for \( t \geq 1 \). Also, let \( B = \{a L_\ast, 16 \theta, k\} \), for some \( k \geq 1 \) to be chosen later, and where \( \theta := \arctan(2 \chi \kappa)^{-1} \), and define
\[
(\varepsilon_4) \quad \gamma'(t) = \omega(\theta + b(t - \delta) A^{-1})(B(t - \delta)) = : \omega(\hat{\theta}(t)).
\]
Note that, by construction, \( \hat{\theta}(t) \geq \theta \). Moreover, for \( t \in [\delta, \delta + A] \), we have
\[
\|\gamma(t) - \tilde{\gamma}(\delta)\| \leq \int_\delta^{\delta+A} \|\omega(\hat{\theta}(ts))\| ds \leq A a.
\]
Thus, recalling (\varepsilon_3),
\[
\arctan \varepsilon_m(\gamma(t), \chi) \leq \arctan \varepsilon_m(\gamma(\delta), \chi) + L_\ast a A \leq \theta + \frac{\kappa}{2 C_\mu} \mu^{-m} + L_\ast A a < \theta \leq \hat{\theta}(t),
\]
which implies that \( D\tilde{\gamma}(t) \mathfrak{h}_m \gamma'(t) \in \mathcal{C}_a \). In addition, for \( t \geq \delta + A \), we have
\[
\frac{d}{dt} \tan \hat{\theta}(t) \geq B \geq a L_\ast \geq |\frac{d}{dt} \varepsilon_m(\gamma(t))|.
\]
Next, let $T > 0$ be such that $\hat{\theta}(T) = \theta_{c}$ so that $\hat{\gamma}'(T)$ is well inside the central cone. This implies $T \leq \delta + \theta_{c}B^{-1}$ and

$$|\pi_{1}(\hat{\gamma})| \leq C_{g}T \leq C_{g}\delta + B^{-1} \leq C_{g}(\delta_{0} + k^{-1}) < 1/2,$$

provided $\delta_{0}$ and $k^{-1}$ are small enough. It is then a simple exercise to construct an extension $\hat{\gamma} : [0, S] \to T^{2}$ such that $\hat{\gamma}'(t) \in C_{c}$, $\|\hat{\gamma}'\| = a$, for all $t \in [T, S]$ and $\hat{\delta}(S) = (0, 1/2)$, $|\pi_{1}(\hat{\gamma})| \leq C_{g}(\delta_{0} + k^{-1})$, $\hat{\gamma}'(S) = (-\chi_{c}/2, 1)$, $\hat{\gamma}(S) = 0$ for all $j > 1$ and $\sup_{t \in [T, S]}\|\hat{\gamma}(t)\| \leq C_{g}$. By symmetry we have a closed curve $\hat{\gamma}$ of homotopy class $(0, 1)$. It suffices to ask $C_{g}(\delta_{0} + k^{-1}) \leq \frac{1}{4}$, to insure that $\hat{\gamma} \in \text{Dom}(\mathfrak{h}_{k})$ for each $\mathfrak{h} \in \mathfrak{h}_{\infty}$ and $k \in \mathbb{N}$. Then Lemma 3.2 implies that there exists inverse branches $\{\mathfrak{h}_{k,i}\}_{i=1}^{d}$, where $d$ is the degree of $F$, such that $F^{-k}\hat{\gamma} = \bigcup_{i=1}^{d} \mathfrak{h}_{k,i} \circ \hat{\gamma}$. Since $\mathfrak{h}_{k,i}$ is a diffeomorphism, $\mathfrak{h}_{k,i} \circ \hat{\gamma}$ is a closed curve. In addition it must be of homotopy type $(0, 1)$, otherwise it would intersect an horizontal segment in more than one point and the image, under $F^{k}$, of the interval between two intersection points would be an unstable curve going from $\hat{\gamma}$ to itself. Since such a curve would be transversal to $\hat{\gamma}$ by hypothesis, it follows that it would have to wrap around the torus horizontally an hence intersect $\gamma_{-}$ contradicting the fact that it is in the domain of $\mathfrak{h}_{k,i}$.

Recalling (E.4), formula (3.3) gives, for all $j \geq 2$, \footnote{Notice that, as $\|\hat{\theta}\|_{C^{1}} \leq C_{g}A^{-1}B$, recalling the definition of $K_{j,s}$ we have \[ \sum_{k \in K_{j,s}} \prod_{l \in \mathbb{N}} (A^{-1}B)^{k_{l}} \leq \sum_{k \in K_{j,s}} A^{\sum_{l=1}^{\infty} k_{l}} B^{\sum_{l=1}^{\infty} k_{l}} \leq A^{-j}B^{s}. \]}

$$\|\hat{\gamma}\|_{C^{j+1}} \leq C_{g}^{j} \|w \circ \hat{\theta}\|_{C^{j}} \leq C_{g}^{j} \sum_{s=0}^{j} \|w\|_{C^{s}} \prod_{k \in K_{j,s}} \|\hat{\theta}\|_{C^{s}}^{k_{i}} \leq C_{g}^{j} \sum_{s=0}^{j} \sum_{k \in K_{j,s}} \prod_{l \in \mathbb{N}} (A^{-1}B)^{k_{i}} \leq A^{-j} \sum_{s=0}^{j} B^{s}. $$

Thus, since $\|\hat{\gamma}'\| = a$, we can reparametrize the curve by arc-length. Calling $\tilde{\gamma}$ the reparametrized curve we obtain

$$\|\hat{\gamma}'(t)\| \leq \begin{cases} 0 & \text{if } |t| \leq \delta \\ C_{g}A^{-j+1}B^{j-1} & \text{if } \delta \leq |t| \leq \delta + A \\ C_{g}B^{j-1} & \text{if } |t| \geq \delta + A, \end{cases}$$

which yields (E.1) since

$$|\pi_{2}(\hat{\gamma}'(t))| \geq \begin{cases} |v| & \text{if } |t| \leq \delta + A \\ C_{g}(|v| + B(t - \delta)) & \text{if } |t| \geq \delta + A. \end{cases} \quad \square$$

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