FIBONACCI NUMBERS AND IDENTITIES

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Abstract. By investigating a recurrence relation about functions, we first give alternative proofs for various identities on Fibonacci numbers and Lucas numbers, and then, make certain well known identities visible via certain trivalent graph associated to the recurrence relation.

1. Introduction

A function $x(n)$ defined over $\mathbb{N} \cup \{0\}$ is called an $\mathcal{F}$-function if $x(n)$ satisfies the following recurrence relation.

$$x(n + 3) = 2(x(n + 2) + x(n + 1)) - x(n).$$

One sees easily that the following are $\mathcal{F}$-functions:

$$x(n) = (-1)^n, \; x(n) = F^2_n, \; x(n) = F_{n+r}F_n, \; x(n) = F_{2n},$$

$$x(n) = L^2_n, \; x(n) = L_{n+r}L_n, \; x(n) = L_{2n}, \; x(n) = F_nL_{n+r},$$

where $r \in \mathbb{Z}$ and $F_n, L_n$ are the $n$-th Fibonacci and Lucas numbers respectively. Note that sum and difference of $\mathcal{F}$-functions are $\mathcal{F}$-functions. A search of the literature turns out that a great deal of identities involve $\mathcal{F}$-functions only. For instance, all the terms in the Cassini’s identity $F_n - 1 F_{n+1} - F^2_n = (-1)^n$ are $\mathcal{F}$-functions. For our convenience, we shall call such an identity $\mathcal{F}$-identity. Identities with other recurrence relations are less frequent. The main purpose of this article is to avoid the intricate case-by-case analysis, thereby obtaining a unified proof of the $\mathcal{F}$-identities. Since these identities involve $\mathcal{F}$-functions only, our proof will make usage of (1.1) and (i), (ii) of the following only.

(i) (1.2) and (1.3) of the above are $\mathcal{F}$-functions,

(ii) $F_{n+2} = F_{n+1} + F_n, \; F_{-m} = (-1)^{m+1}F_m, \; L_{n+2} = L_{n+1} + L_n, \; L_{-m} = (-1)^mL_m.$

Note that our proof can be applied easily to all $\mathcal{F}$-identities. Identities involve other recurrence relations such as (iii) and (iv) of the following will be discussed in section 6.

(iii) $A(n + 2) = -A(n + 1) + A(n),$

(iv) $A(n + 3) = -A(n + 2) + A(n + 1) + A(n).$

The rest of the article is organised as follows. In section 2 we give some basic properties about $\mathcal{F}$-function. Section 3 gives alternative proofs of the well known Catalan’s identity and Melham’s identity. Section 4 lists a few more identities (including d’Ocagne’s, Tagiure’s and Gelin-Cesàro identities) that can be proved by applying our technique presented in section 3. They are the $\mathcal{F}$-identities involved functions we listed in (1.2) and (1.3). In other words, they use functions in (1.2) and (1.3) as building blocks (see Lemma 2.1). Since product of $\mathcal{F}$-functions are not necessarily $\mathcal{F}$-functions, our idea cannot be applied to all identities (see Appendix B). Section 5 is devoted to the possible visualisation of identities via the recurrence relation (1.1). After all, there is nothing to prove if one cannot see the identities in the first place. The last section gives a very brief discussion about identities that involve other recurrence relations.
2. Basic Properties about \( F \)-functions

**Lemma 2.1.** Functions defined in (1.2) and (1.3) are \( F \)-functions. Let \( A(n) \) and \( B(n) \) be \( F \)-functions and let \( r_0 \in \mathbb{Z} \) be fixed. Then \( X(n) = A(n + r_0), Y(n) = r_0A(n) \) and \( A(n) \pm B(n) \) are \( F \)-functions.

**Proof.** Let \( x(n) \) be given as in (1.2) or (1.3). To show \( x(n) \) is an \( F \)-functions, it suffices to show that \( x(n + 3) = 2(x(n + 2) + x(n + 1)) - x(n) \), which can be verified easily. \( \Box \)

**Lemma 2.2.** Let \( A(n) \) and \( B(n) \) be \( F \)-functions. Then \( A(n) = B(n) \) if and only if \( A(k) = B(k) \) for \( k = 0, 1 \) and 2.

Note that the fact that the following functions are \( F \)-functions has been treated as identities in the literature ([HB], identities (31)-(34) of [L]).

**Example 2.3.** By Lemma 2.1, \( F_{n+3}^2, F_{n+3}F_{n+4}, L_{n+3}^2 \) and \( F_{n+3}L_{n+3} \) are \( F \)-functions.

The following lemma is straightforward and will be used in sections 3 and 6.

**Lemma 2.4.** Let \( A(n) \) and \( B(n) \) be functions defined over \( \mathbb{N} \cup \{0\} \). Suppose that both \( A(n) \) and \( B(n) \) satisfy either

(i) \( x(n + 2) = -x(n + 1) + x(n) \), or

(ii) \( x(n + 3) = -2x(n + 2) + 2x(n + 1) + x(n) \).

Then \( A(n) = B(n) \) if and only if \( A(k) = B(k) \) for \( k = 0, 1, 2 \).

2.1. Discussion. Let \( \{x_n\} \) be a sequence satisfies the recurrence relation \( x_{n+2} = x_{n+1} + x_n \). Then \( A(n) = x_{2n}, B(n) = x_nx_{n+r}, C(n) = x_n^2 \) are \( F \)-functions.

3. An alternative proof for Catalan’s Identities

In [H], Howard studied generalised Fibonacci sequence and proved that Catalan’s identity is equivalent to an identity discovered and proved by Melham ([M]) (see section 3.1). We give in the following our alternative proof which uses Lemmas 2.1 and 2.2 only.

**Lemma 3.1.** \( 4(-1)^{3-r} + F_{r+3}F_{r-3} - F_r^2 = 0 \).

**Proof.** We assume that \( r \geq 0 \). The case \( r \leq 0 \) can be dealt with similarly. Let \( A(r) = 4(-1)^{3-r} + F_{r+3}F_{r-3} - F_r^2 \). By Lemma 2.1, \( A(r) \) is an \( F \)-function. By Lemma 2.2, \( A(r) = 0 \) for all \( r \). This completes the proof of the Lemma. \( \Box \)

**Remark.** Any \( F \)-identity with one variable \( n \) can be proved by applying the proof of Lemma 3.1. For instance, the Cassini’s identity.

**Theorem 3.2 (Catalan’s Identity).** \( F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2 \).

**Proof.** Recall first that \( F_{-m} = (-1)^{m+1}F_m \). As a consequence, we may assume without loss of generality that \( n, r \geq 0 \). As a matter of fact, we may assume that \( n \geq 1 \) as the case \( n = 0 \) is trivial. Let \( A(n) = F_n^2 - F_{n+r}F_{n-r}, B(n) = (-1)^{n-r}F_r^2 \). By Lemma 2.1, both \( A(n) \) and \( B(n) \) are \( F \)-functions (in \( n \)). Note that

\[
A(1) = 1 - F_{1+r}F_{1-r}, \quad A(2) = 1 - F_{2+r}F_{2-r}, \quad A(3) = 4 - F_{3+r}F_{3-r},
\]
and

\[
B(1) = (-1)^{1-r}F_r^2, \quad B(2) = (-1)^{2-r}F_r^2, \quad B(3) = (-1)^{3-r}F_r^2.
\]

One sees easily that if \( r \leq 3 \), then \( A(n) = B(n) \) for \( n = 1, 2, 3 \). By Lemma 2.2, we have \( A(n) = B(n) \). Hence the theorem is proved. We shall therefore assume that \( r \geq 4 \). Recall
that $F_{-m} = (-1)^{m+1}F_m$. This allows us to rewrite $A(3)$ into $A(3) = 4 + (-1)^{3-r}F_{r+3}F_{r-3}$. Hence

$$A(3) - B(3) = 4 + (-1)^{3-r}F_{r+3}F_{r-3} - (-1)^{3-r}F_r^2. \quad (3.3)$$

By Lemma 3.1, $A(3) = B(3)$. One can show similarly that $A(1) = B(1)$ and $A(2) = B(2)$. Applying Lemma 2.2, we have $A(n) = B(n)$ for all $n$. This completes the proof of the theorem. □

**Remark.** Any $F$-identity with two variables $m$ and $n$ can be proved by applying the proof of Theorem 3.2.

3.1. **Melham’s Identity.** In [M], Melham proved among some very general results the identity $F_{n+r+1}^2 + F_{n-r}^2 = F_{2r+1}F_{2n+1}$. We shall give our alternative proof as follows. Denoted by $A(n)$ and $B(n)$ the left and right hand side of the identity. By Lemma 2.1, $A(n)$ and $B(n)$ are $F$-functions (in $n$). The cases $n = 0, 1$ and 2 of the above identity are given by

$$F_{r+1}^2 + F_r^2 = F_{2r+1}F_1, \quad F_{r+2}^2 + F_{r+1}^2 = F_{2r+1}F_3, \quad F_{r+3}^2 + F_{2r-r}^2 = F_{2r+1}F_5. \quad (3.4)$$

By Lemma 2.1, the functions in (3.4) are $F$-functions (in $r$) and that the identities can be verified by applying Lemma 2.2. Consequently, we have $A(0) = B(0), A(1) = B(1)$ and $A(2) = B(2)$. By Lemma 2.2, we have $A(n) = B(n)$ for all $n$.

3.2. **Discussion.** Our method can be generalised to functions such as $x(n) = F_n^3, y(n) = F_{3n}$ which satisfy the recurrence relation (see Appendix B)

$$x(n+4) = 3x(n+3) + 6x(n+2) - 3x(n+1) - x(n). \quad (3.5)$$

4. **More Identities**

4.1. **d’Ocagne’s Identity.** The proof we presented in section 3 can be applied to all $F$-identities. A search of the literature turns out that there are many such identities. However, as the identities may be described in different manner, it is important to get equivalent forms of the identities. Take d’Ocagne’s identity for instance. It is, in many occasion, given as follows (see [W]).

$$F_mF_{n+1} - F_nF_{m+1} = (-1)^nF_{m-n}. \quad (4.1)$$

The very first look of the left and right hand side does not reveal the fact that they are $F$-function. However, one has the following. Let $r = m - n$. Then (4.1) can be rewritten as

$$F_{n+1}F_{n+r} - F_nF_{n+r+1} = (-1)^nF_r, \quad (4.2)$$

where both the left and right hand side in (4.2) are $F$-functions in terms of $n$. One may now apply the proof of Theorem 3.2 to give a proof of (4.2). As the proof is identically the same, we will not include it here.
4.2. Some more identities. A search of the literature turns up that there are many identities can be verified by Lemmas 2.1 and 2.2 (for instance, out of the 44 identities given by Long [L], 35 of them involve \(F\)-functions). We shall list a few which we pick mainly from [W] ((c1)-(c8), (c11), (c12), (d1)-(d6)).

\[
\begin{align*}
(\text{c1}) & \quad F_{n+a}F_{n+b} - F_nF_{n+a+b} = (-1)^n F_nF_b : \quad F_{2n} = F_{n+1}^2 - F_{n-1}^2 \quad (d1) \\
(\text{c2}) & \quad F_{n+1}^2 = 4F_nF_{n-1} + F_{n-2}^2 : \quad F_{2n+1} = F_{n+1}^2 + F_n^2 \quad (d2) \\
(\text{c3}) & \quad L_n^2 - 5F_n^2 = 4(-1)^n : \quad F_{n+2}F_{n-1} = F_{n+1}^2 - F_n^2 \quad (d3) \\
(\text{c4}) & \quad F_{n-1}F_{n+1} - F_n^2 = (-1)^{n-1} : \quad F_n^2 - F_{n-2}F_{n+2} = (-1)^n \quad (d4) \\
(\text{c5}) & \quad F_nF_n = \frac{1}{5}(L_{m+n} - (-1)^n L_{m-n}) : \quad \sum_{n=1}^n F_n^2 = F_nF_{n+1} \quad (d5) \\
(\text{c6}) & \quad F_n^2 = \frac{1}{5}(L_{2n} - 2(-1)^n) : \quad F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1} \quad (d6) \\
(\text{c7}) & \quad F_{n+m} = F_{n-1}F_m + F_nF_{m+1} : \quad F_nF_{n+3} = F_{n+1}F_{n+2} + (-1)^{n-1} \quad (d7) \\
(\text{c8}) & \quad F_{m+n} = \frac{1}{2}(F_mF_n + L_mF_n) : \quad F_n^2 - F_{n-1}^2 = F_nF_{n+1} - (-1)^{n-1} \quad (d8) \\
(\text{c9}) & \quad F_{n+k+1}^2 + F_{n-k+1}^2 = F_{2k+1}F_{2n+1} : \quad F_{2n+1}^2 + (-1)^n = F_{n-1}F_{n+1} + F_{n+1}^2 \quad (d9) \\
(\text{c10}) & \quad L_{n+k+1}^2 + L_{n-k+1}^2 = 5L_{2k+1}L_{2n+1} : \quad L_{2n+1}^2 - F_{n+1}^2 = (-1)^{n-1} \quad (d10) \\
(\text{c11}) & \quad F_n^2 + (-1)^{n+r-1}F_r^2 = F_{n-r}F_{n+r} : \quad L_{n-1}^2 - F_{n-4}F_n - F_{n+1}^2 = F_{n-2}^2 \quad (d11) \\
(\text{c12}) & \quad F_n^4 - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1 : \quad F_{2n+1}^2 = F_{n+3}F_n - F_{n+1}F_n - 1 \quad (d12)
\end{align*}
\]

where \(A = (L_n^2 - F_{n-3}F_{n+1}) + F_{2n-2}\) and \(L_n\) is the \(n\)-th Lucas number. (d10) and (d11) are less standard. We decide to include them in the table as they are visible via certain trivalent graph (see section 5).

Proof. We note first that functions in (c12) are not \(F\)-functions but the identity can be proved by applying (c4) and (d4). In (c5), (c7) and (c8), one needs to do a rewriting to see that the functions are \(F\)-functions (let \(m + n = r\)). One may now apply Lemmas 2.1 and 2.2 and our proof presented in Theorem 3.2 to verify these identities. \(\Box\)

Remark. As identities may be described differently, the technique of rewriting identities into equivalent forms is crucial (see (4.1) and (4.2)). (c9) and (c10) were first found and proved by Melhem ([M1]). (c12) is the Gelin-Cesàro identity.

4.3. Discussion. Note that in our proof, we do not use any existing identities such as Binet’s formula or any identities listed in [W] except (1.1), (1.2) and (1.3) of this article, which is what we promise in our introduction. Note also that one has to apply Lemma 2.2 three times to prove identity (c1), known as the Tagiuri’s identity.

5. How Far can (1.1) go?

We have demonstrated that the recurrence relation (1.1) can be used to verified various identities. In this section, we will present a trivalent graph (see the graph given in Appendix A) which is closely related to (1.1) that enables us to visualise identities in the following.
Let \( e_3, e_2, e_1 \) be vectors placed in the following trivalent graph and let \( e_4 \) be the vector given by \( e_4 = 2(e_3 + e_2) - e_1 \). Such a vector \( e_4 \) is said to be \( F \)-generated by \( e_3, e_2, \) and \( e_1 \) (in this order). For our convenience, we use the following notation for the vector \( e_4 \):

\[
  e_4 = \langle e_3, e_2 : e_1 \rangle = 2(e_3 + e_2) - e_1.
\]

Note that \( \langle e_3, e_2 : e_1 \rangle \neq \langle e_2, e_1 : e_3 \rangle \neq \langle e_1, e_3 : e_2 \rangle \). Note also that (5.1) can be viewed as a generalisation (in the form of vectors) of the recurrence relation (1.1). We may construct an infinite sequence of vectors given as follows.

\[
  e_1, e_2, e_3, e_4 = 2(e_3 + e_2) - e_1, \ldots, e_{n+1} = 2(e_n + e_{n-1}) - e_{n-1}, \ldots
\]

Denoted by \( F(e_1, e_2, e_3) \) the above sequence. In the case \( \{e_1, e_2, e_3\} \) is the canonical basis of \( \mathbb{R}^3 \), the first nine vectors are given as follows. \( e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1), e_4 = (-1, 2, 2), e_5 = (-2, 3, 6), \ldots \). Note that \( e_2, e_4, e_6, \ldots, e_{2n} \) take the top half of the graph and \( e_1, e_3, e_5, \ldots, e_{2n+1} \) take the bottom half of the graph.

\[
\begin{array}{c|c|c|c|c|c|c|c}
\hline
 & e_2 & & e_3 & & e_4 & & \\
\hline
 & \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} & & \begin{bmatrix} -6 \\ 10 \\ 15 \end{bmatrix} & & \begin{bmatrix} -40 \\ 65 \\ 104 \end{bmatrix} & & \\
\hline
 & -2 & & 3 & & 6 & & \\
\hline
 & -15 & & 24 & & 40 & & \\
\hline
 & -104 & & 168 & & 273 & & \\
\hline
\end{array}
\]

One sees immediately the following:

(a) The entries of the vectors \( e_n = (a, b, c) \) are product of two Fibonacci numbers. To be more accurate, for the first nine terms, the vectors take the form

\[
  (-F_n F_{n+1}, F_n F_{n+2}, F_{n+1} F_{n+2}).
\]

(b) The norm of \( e_n = (a, b, c) \) is just the sum \( a + b + c \), where the norm \( N((x_1, x_2, x_3)) \) is defined to be \((x_1^2 + x_2^2 + x_3^2)^{1/2}\) (see Appendix A).

(c) The absolute value of the entries of \( e_{2n} - e_{2n-2} \) (the top half of the trivalent graph) and \( e_{2n+1} - e_{2n-1} \) (the bottom half of the trivalent graph) are Fibonacci numbers. For instance, \((-40, 65, 104) - (-6, 10, 15) = (-F_9, F_10, F_{11})\). This allows one to write each entry of the vectors as a sum of Fibonacci numbers.

(a) and (b) of the above implies that for \( n \leq 6 \)

\[
  (F_n F_{n+1})^2 + (F_n F_{n+2})^2 + (F_{n+1} F_{n+2})^2 = (-F_n F_{n+1} + F_n F_{n+2} + F_{n+1} F_{n+2})^2,
\]

which leads us to (i) of the following lemma. Note that \(-F_n F_{n+1} + F_n F_{n+2} + F_{n+1} F_{n+2} = F_n^2 + F_{n+1} F_{n+2}\). A careful study of (a) and (c) implies that each entry of the nine vectors can be written as sum as well as product of Fibonacci numbers which leads us to (ii)-(v) of the following lemma.

**Lemma 5.1.** Let \( F_n \) be the \( n \)-th Fibonacci number. Then the following hold.

(i) \((F_n F_{n+1})^2 + (F_n F_{n+2})^2 + (F_{n+1} F_{n+2})^2 = (F_n^2 + F_{n+1} F_{n+2})^2\).

(ii) \( F_{2n-3} F_{2n-2} = F_1 + F_5 + \cdots + F_{4n-7} \).
(iii) \(F_{2n-3}F_{2n-1} = 1 + F_2 + F_6 + \cdots + F_{4n-6}\).
(iv) \(F_{2n-2}F_{2n-1} = F_3 + F_7 + \cdots + F_{4n-5}\).
(v) \(F_{2n-2}F_{2n} = F_4 + F_8 + \cdots + F_{4n-4}\).

5.1. Discussion. (i) of the above lemma can be viewed as generalisation of Raine's results on Pythagorean's triple. (i)-(v) must be well known. As they are not included in [L] or [W], we have them here for the reader's reference. Proofs of (i)-(v) are not included here as they can be proved easily. As the identities in Lemma 5.1 are from our observation of the trivalent graph. We consider that the recurrence relation (1.1), (5.1) and the trivalent graph \(F(e_1,e_2,e_3)\) make those identities visible.

To one's surprise, the trivalent graph actually tells us more.

(i) The sum of the first entries (staring from \(e_4\)) of the first \(2k-1\) consecutive vectors is the negative of a perfect square of a Fibonacci numbers.
(ii) The sum of the second entry (starting from \(e_4\)) of the first \(k\) vectors is a product of two Fibonacci numbers.
(iii) The entries of every vector is a product of two Fibonacci numbers. Let \((-a,b,c)\) be such a vector. Then \(c-b-a=\pm 1\).
(iv) Take any two consecutive vectors of the top half of the trivalent graph (such as \((e_2,e_4)\), \((e_4,e_6),\cdots\)). Label them as \((-a,b,c)\) and \((-A,B,C)\). Then \(C-c=(B-b)+(A-a)\).
(v) Take two consecutive vectors of the top half (likewise the bottom half) and label them as \((-a,b,c)\) and \((-C,B,A)\). One sees that all the entries are product of two Fibonacci numbers and the product of \(a\) and \(A\) is one less than a fourth power of a Fibonacci number! \((1 \cdot 15 = 2^4 - 1, 6 \cdot 104 = 5^4 - 1, 2 \cdot 40 = 3^4 - 1, 15 \cdot 273 = 12^4 - 1)\).

(i)-(v) of the above actually give us five well known identities. Take (v) for instance, our observation shows that \(a\) fourth power of a Fibonacci number \(-1=\text{product of four Fibonacci numbers}\). To be more accurate, one has the remarkable Gelin-Cesàro identity visible.

\[F_n^4 - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1.\]  

(5.5)

We are currently investigate the trivalent graph \(F(u,v,w)\) for arbitrary triples \((u,v,w)\). It turns out that such study makes a lot of identities visible. For example, every identity appears in the right hand side of our table ((d1)-(d11)) in section 4 can be seen from some trivalent graph \(F(u,v,w)\). See [LL] for more detail.

6. Discussion

We have demonstrated in this article that a simple study of the recurrence relation (1.1) ends up with a unified proof for many known identities in the literature. This suggests that one should probably group the identities together based on the recurrence relations (if it exists) and study them as a whole. Note that a given function may satisfy more than one recurrence relations ((\(-1\))^\(n\)\(F_n\) satisfies (i) of the following and (3.5)). The next recurrence relation in line, we believe, should be

(i) \(x(n+2) = -x(n+1) + x(n)\),
(ii) \(x(n+3) = -2x(n+2) + 2x(n+1) + x(n)\).

Identities (in Fibonacci numbers) with such recurrence relations are rare but of great importance. To see our point, one recall that the right hand side of the very elegant identity of
Melham’s \((F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3) = (-1)^n F_n\) satisfies (i) of the above and that the following attractive identities of Fairgrieve and Gould ([FG]) also satisfy (i) and (ii) of the above.

\[F_{n-2}F_{n+1}^2 - F_n^3 = (-1)^n F_{n-1}, \quad (6.1)\]

\[F_{n-3}F_{n+1}^3 - F_n^4 = (-1)^n(F_{n-1}F_{n+3} + 2F_{n+1}^2). \quad (6.2)\]

To end our discussion, we give the following example which suggests how a new identity can be obtained by the study of recurrence relation (i): Since the right hand side of (6.1) satisfies (i) of the above, \(x(n) = F_{n-2}F_{n+1}^2 - F_n^3\) satisfies the same recurrence relation. Namely, \(x(n + 2) = -x(n + 1) + x(n)\). With the help of the famous identity \(F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3\), one has

\[F_nF_{n+3}^2 + F_{n-1}F_{n+2}^2 - F_{n-2}F_{n+1}^2 = F_{3n+3}, \quad (6.3)\]

7. APPENDIX A

Let \(u = (u_i), v = (v_i), w = (w_i) \in \mathbb{Z}^3\) be vectors. We say \(\{u, v, w\}\) is a \(\mathcal{F}\)-triple if \(N(u) = u_1 + u_2 + u_3, N(v) = v_1 + v_2 + v_3\) and \(N(w) = w_1 + w_2 + w_3\) are squares in \(\mathbb{N}\) and

(i) \(2u \cdot v - v \cdot w - w \cdot u = 2N(u)N(v) - N(v)N(w) - N(w)N(u)\),

(ii) \(2u \cdot w - v \cdot w - v \cdot u = 2N(u)N(w) - N(v)N(w) - N(v)N(u)\),

(iii) \(2v \cdot w - v \cdot u - w \cdot u = 2N(v)N(w) - N(v)N(u) - N(w)N(u)\),

where \(u \cdot v\) is the usual dot product. One sees easily that \(\{e_1, e_2, e_3\}\) is an \(\mathcal{F}\)-triple, where \(e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\). The following lemma shows that if \(\{u, v, w\}\) is an \(\mathcal{F}\)-triple, then any vector \((a, b, c)\) in \(F(u, v, w)\) has the property \(N((a, b, c)) = a + b + c\), which proves (b) of section 5.

**Lemma A.** Let \(u = (u_i), v = (v_i), w = (w_i) \in \mathbb{R}^3\) be a \(\mathcal{F}\)-triple and let \(x = (x_i) = 2(u + v) - w, y = (y_i) = 2(w + v) - u, z = (z_i) = 2(w + u) - v\). Then the following hold.

(i) \(\{u, v, x\}, \{u, w, z\}\) and \(\{v, w, y\}\) are \(\mathcal{F}\)-triples,

(ii) \(N(x)^2 = (2N(u) + 2N(v) - N(w))^2, N(y)^2 = (2N(w) + 2N(v) - N(u))^2, N(z)^2 = (2N(w) + 2N(u) - N(v))^2\).

**Proof.** The lemma is straightforward and can be proved by direct calculation. \(\square\)

Note that \(x, y, \text{ and } z\) in the above lemma are defined as in (5.1) and can be described as follows:

```
    z
   /|
  / |\
 w /   \\x
  \\
 y
```

Following our lemma, one may extend the above graph to an infinite trivalent graph that takes the whole \(xy\)-plane such that each triple \(\{r, s, t\}\) associated to a vertex is an \(\mathcal{F}\)-triple. In particular, the entries of every vector of this trivalent graph give solution to \(x^2 + y^2 + z^2 = \)
\[(x + y + z)^2\]. Note that a complete set of integral solutions of the above mentioned equation is given by \(\{(mn, m(m + n), n(m + n)) : n, m \in \mathbb{Z}\}\).

### 8. Appendix B: More Recurrence relations

Let \(x(n)\) be a function defined on \(\mathbb{Z}\). Consider the equation

\[x(n + k) = a_{k-1}x(n + k - 1) + \cdots + a_1x(n + 1) + a_0x(n).\]  \hspace{1cm} (B1)

One sees easily that whether \(x(n)\) satisfies some recurrence relation depends on whether there exists some \(k\) and \(a_i\)'s such that (B1) holds for all \(n\). In the case \(x(n)\) indeed admits some recurrence relation, such relation can be obtained by solving system of linear equations.

#### 8.1. The Recurrence relation

\(x(n + 4) = 3x(n + 3) + 6x(n + 2) - 3x(n + 1) - x(n)\). In [M2], Melham proved that

\[F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3 = (-1)^n F_n.\]  \hspace{1cm} (B2)

We shall give our alternative proof as follows. Let \(A(n) = F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3, B(n) = (-1)^n F_n.\) One sees easily that both \(A(n)\) and \(B(n)\) satisfy the above recurrence relation. Since

(i) \(A(n)\) and \(B(n)\) satisfy the above recurrence relation and \(A(n) = B(n)\) for \(n = 0, 1, 2, 3,\)

(ii) function \(x(n)\) satisfies the above recurrence relation is completely determined by \(x(0), x(1), x(2)\) and \(x(3)\),

we conclude that \(A(n) = B(n)\). This completes the proof of (B2). The identity \(F_{3n} = F_{n+1}^3 + F_{n-1}^3 - F_n^3\) and Fairgrieve and Gould’s identities ((11), (12) of [FG]) can be proved by the same method.

#### 8.2. Recurrence relation for \(F_n^4\)

\(F_n^4\) satisfies the following recurrence relation,

\[x(n + 5) = 5x(n + 4) + 15x(n + 3) - 15x(n + 2) - 5x(n + 1) + x(n).\]  \hspace{1cm} (B3)

One sees easily that both the left and right hand side of (6.2) satisfy (B3). As a consequence, identity (6.2) can be verified by applying our technique given in the above subsection.

#### 8.3. Construction of Identities.

Recurrence relations can be used to construct identities. Take (6.3) for example, one can actually construct (6.3) as follows.

\[
\begin{array}{cccc}
  x(n) & x(0) & x(1) & x(2) & x(3) \\
  F_{3n+3} & 2 & 8 & 34 & 144 \\
  F_n F_{n+3}^2 & 0 & 9 & 25 & 128 \\
  F_{n-1} F_{n+2}^2 & 1 & 0 & 9 & 25 \\
  F_{n-2} F_{n+1}^2 & -1 & 1 & 0 & 9 \\
\end{array}
\]

Since \(F_{3n+3}, F_n F_{n+3}^2, F_{n-1} F_{n+2}^2\) and \(F_{n-2} F_{n+1}^2\) satisfy the recurrence relation \(x(n + 4) = 3x(n + 3) + 6x(n + 2) - 3x(n + 1) - x(n)\), applying (ii) of the above, one sees from the above table that

\[F_{3n+3} = F_n F_{n+3}^2 + F_{n-1} F_{n+2}^2 - F_{n-2} F_{n+1}^2.\]  \hspace{1cm} (B4)
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