LIOUVILLE TYPE THEOREM FOR FRACTIONAL LAPLACIAN SYSTEM

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Abstract. In this paper, using the method of moving planes combined with integral inequality to handle the fractional Laplacian system, we prove Liouville type theorems of nonnegative solution for the nonlinear system.

1. Introduction. This paper is devoted to study the Liouville type theorem of solutions for the nonlinear fractional Laplacian system

\begin{align}
(-\Delta)^s u &= f(u,v), \quad \text{in } \mathbb{R}^n, \\
(-\Delta)^s v &= g(u,v), \quad \text{in } \mathbb{R}^n,
\end{align}

where $0 < s < 1$, $n > 2s$, $f, g$ are continuous functions. Here $(-\Delta)^s$ is the fractional Laplacian operator defined via its multiplier $|\xi|^{2s}$ in Fourier space, that is

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi),$$

where $\mathcal{F}$ is the Fourier transform. Another equivalence of fractional Laplacian operator given in the form of difference quotients,

$$(-\Delta)^s u = C_{n,s}PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy,$$

where $C_{n,s}$ is a positive constant, $PV$ stands for the Cauchy principle value. The above two definitions are equivalent for $u$ belongs to Schwartz space, can refer to [10, 17].

The fractional Laplacian is a nonlocal pseudo-differential operator. The nonlocality makes it difficult to investigate. To circumvent this difficulty, Caffarelli and Silvestre [3] introduced the extension method which can reduce the nonlocal problem to $(-\Delta)^s$ to a local one in higher dimensions. For a function $u : \mathbb{R}^n \to \mathbb{R}$, consider the extension $U : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ that satisfies

$$\begin{cases}
\text{div}(y^{1-2s}\nabla U(x,y)) = 0, & (x,y) \in \mathbb{R}^n \times [0, \infty), \\
U(x,0) = u(x), & x \in \mathbb{R}^n,
\end{cases}$$

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then
\[ (-\Delta)^s u = -C_{n,s} \lim_{y \to 0^+} y^{1-2s} \frac{\partial U}{\partial y}. \]

This method has been applied successfully to study equations involving the fractional Laplacian, and then a series of fruitful results appeared, see [1, 9] and the references therein. Wan and Xiang [26] considered the Neumann problem
\[
\begin{cases}
\text{div}(y^a \nabla u(x,y)) = 0, & x \in \mathbb{R}^n, y > 0, \\
\lim_{y \to 0^+} y^a u(x,y) = -f(u(x,0)), & x \in \mathbb{R}^n,
\end{cases}
\]
where \( a \in (-1, 1), \ u \geq 1, \ \nabla = (\partial x_1, \ldots, \partial x_n, \partial y), \ f \) is a nonnegative function. They got a Liouville type theorem for nonnegative solutions of the nonlinear Neumann problem. In fact, in [26] they also pointed out that this Neumann problem is closely related to the fractional Laplacian equation
\[ (-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^n, \]
where \( s = \frac{1-n}{2} \).

The Liouville type theorems for the scalar equation
\[ -\Delta u = f(u) \quad \text{in } \mathbb{R}^n \]
have been investigated by a large number of authors and there are many famous results. Caffarelli, Gidas and Spruck [2] proved the asymptotic symmetry and local behavior of the semilinear elliptic equations with critical Sobelov growth. Gidas and Spruk [13] showed that there is no positive classical solution for \( f(u) = u^p \) (where \( 0 < p < \frac{n+2}{n-2} \)). And then Chen and Li also considered this problem in [4], they give a simple proof based on the Kelvin transform and got the same result as [13].

In the case \( a = 0 \), the above Neumann problem in [26] is reduced to
\[
\begin{cases}
-\Delta u(x,y) = 0, & (x, y) \in \mathbb{R}^{n+1}, \\
\frac{\partial u}{\partial \nu} = f(u), & x \in \mathbb{R}^n,
\end{cases}
\]
where \( \Delta \) is the Laplacian operator in \( \mathbb{R}^{n+1} \) and \( \nu \) is the unit outward normal. Hu established nonexistence of positive solutions for \( f(u) = u^p \) (where \( 1 \leq p < \frac{n}{n-1} \)) in [16]. Ou [20] extended the result of [16] to the range \( -\infty \leq p < \frac{n}{n-1} \) by using the method of moving planes. Recently, [26] studied the general case that \( a \neq 0 \).

For the Laplacian system
\[
\begin{cases}
-\Delta u = v^p, & \text{in } \mathbb{R}^n, \\
-\Delta v = u^q, & \text{in } \mathbb{R}^n,
\end{cases}
\]
there are also some similar conclusions. De Figueiredo and Felmer in [11] conjectured that the hyperbola
\[ \frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{n}, p > 0, q > 0, \]
is the dividing curve between existence and nonexistence for the above system. This conjecture was supported by the results that there are radial solutions, see Serrin and Zou [22, 23]. In [11], the authors proved the above system has no positive solutions which provided that
\[ 0 < p, q \leq \frac{n+2}{n-2}, (p,q) \neq \left( \frac{n+2}{n-2}, \frac{n+2}{n-2} \right). \]
Guo and Liu [15] established Liouville type results for positive solutions of the elliptic system

\[
\begin{cases}
-\Delta u = f(u, v), & \text{in } \mathbb{R}^n, \\
-\Delta v = g(u, v), & \text{in } \mathbb{R}^n,
\end{cases}
\]

where \( n \geq 3 \). One key tool to prove of the result is the method of moving planes combined with integral inequalities.

For the equations which contain the fractional Laplacian operator, Chen, Li and Li [6] given a direct method of moving planes to study the properties of the solutions for the system (1.1). Firstly we consider the corresponding extension problems. Here based on the extension result, we give a new proof of the properties of the solutions for system (1.1). Firstly we consider the corresponding extension system

\[
\begin{cases}
div(y^{1-2s}\nabla U(x, y)) = 0, & \text{in } \mathbb{H}, \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial U(x, y)}{\partial y} = -f(U(x, 0), V(x, 0)), & \text{on } \partial \mathbb{H} \setminus \{0\}, \\
div(y^{1-2s}\nabla V(x, y)) = 0, & \text{in } \mathbb{H}, \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial V(x, y)}{\partial y} = -g(U(x, 0), V(x, 0)), & \text{on } \partial \mathbb{H} \setminus \{0\},
\end{cases}
\]

(1.2)

where \( \mathbb{H} = \{(x, y) : x \in \mathbb{R}^n, y > 0\} \), and denote

\[
\frac{\partial U}{\partial \nu^s} = \lim_{y \to 0^+} y^{1-2s} \frac{\partial U(x, y)}{\partial y}.
\]

And by the results of (1.2), we obtain the properties of the solutions to system (1.1).

Our main results are the following

**Theorem 1.1.** For \( 0 < s < 1 \), let \( (u, v) \in H^s_{\text{loc}}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \times H^s_{\text{loc}}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \) be a nonnegative solution of problem

\[
\begin{cases}
(-\Delta)^s u = f(v), & \text{in } \mathbb{R}^n, \\
(-\Delta)^s v = g(u), & \text{in } \mathbb{R}^n.
\end{cases}
\]

(1.3)

Suppose that \( f, g : [0, +\infty) \to \mathbb{R}^+ \) are continuous functions satisfying:

(i) \( f(t), g(t) \) are nondecreasing in \( (0, +\infty) \),

(ii) \( h(t) = \frac{f(t)}{t}, k(t) = \frac{g(t)}{t} \) are nonincreasing in \( (0, +\infty) \).

Then either \( (u, v) = (c_1, c_2) \) for some constants \( c_1, c_2 \) and \( f(c_2) = g(c_1) = 0 \) or there exist positive constants \( A, B \) such that \( h(t) = A, k(t) = B \) and \( (u, v) \) is radially symmetric about some point.

**Theorem 1.2.** For \( 0 < s < 1 \), let \( (u, v) \in H^s_{\text{loc}}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \times H^s_{\text{loc}}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \) be a positive solution of problem (1.1). Suppose that \( f, g : [0, +\infty) \times [0, +\infty) \to \mathbb{R}^+ \) are continuous functions satisfying:

(i) \( f(t_1, t_2) \) is nondecreasing in \( t_2 \) and \( f \geq 0 \) for \( t_1, t_2 > 0 \),
Sobolev spaces, define $W.\,$ Palatucci and Valdinoci in [10] showed the relations among some of the fractional is the so called Gagliardo smeinorm of $u$ where the term $-\triangle \Delta$ claimed that the operator \((\frac{1}{t_1^2} + \frac{1}{t_2^2})\) is nonincreasing in \((t_1, t_2)\).

(iii) $g(t_1,t_2)$ is monotonically decreasing in $t_1$ and $g \geq 0$ for $t_1, t_2 > 0$

(iv) there exist $p_2 > 0$, $q_2 \geq 0$, $p_2 + q_2 = \frac{n + 2s}{n - 2s}$, such that $\frac{g(t_1,t_2)}{t_1^{p_2} t_2^{q_2}}$ is nonincreasing in \((t_1, t_2)\).

Then either \((u,v) = (c_1, c_2)\) for some constants $c_1, c_2$ and $f(c_1, c_2) = g(c_1, c_2) = 0$ or there exist positive constants $\tilde{A}, \tilde{B}$ such that $f(t_1, t_2) = \tilde{A} t_1^{p_1} t_2^{q_1}$, $g(t_1, t_2) = \tilde{B} t_1^{p_2} t_2^{q_2}$ and \((u,v)\) is radially symmetric about some point.

In this paper, we use the integral inequality, which was first introduced by Terracini in [24, 25], and then was widely used in [15, 26, 30] etc.. The main idea of this paper comes from those above works. Quaas and Xia [21] based on a monotonicity argument for suitable transformed functions and the method of moving planes in an infinity half cylinder rely to some maximum principles which obtained by some barrier functions and a coupling argument using fractional Sobolev trace inequality, given the Liouville theorem for the fractional Lane-Emden system. This method gives us some guidance to work on the extension problem. Be different from Zhang and Yu [31] which a direct method of moving planes apply to the fractional Laplacian, here we borrow the integral inequality which can instead of the usual maximum principle in the process of moving planes. Our results contain partially of the conclusion in Li and Li [19], and extend the theorems in Guo and Liu [15] from elliptic system to fractional Laplacian system.

For more articles concerning the method of moving planes for elliptic and fractional elliptic equations and fractional theory, please see [5, 7, 13, 14] and the references therein. There are also some fractional and its extension results on the Heisenberg group, see [27, 28] etc..

The paper is organized as follows. Section 2 collects some well known results, and the proof of a integral inequalities. Section 3 prove the Theorem 1.1. In Section 4, we consider the extension system (1.1), a Liouville type result (Theorem 1.2) is obtained under stronger assumptions on $f, g$.

2. Preliminaries. In [6], Chen, Li and Li gave the $L_{2s}$ space that is

$$L_{2s}(\mathbb{R}^n) = \left\{ u : \mathbb{R}^n \to \mathbb{R} : \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}}\,dx < \infty \right\},$$

and claimed that the operator $(-\Delta)^s$ is well defined if $u \in L_{2s} \cap C^{1,1}_{loc}$. Di Nezza, Palatucci and Valdinoci in [10] showed the relations among some of the fractional Sobolev spaces, define $W^{s,p}$ as follows

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{n+s}} \in L^p(\Omega \times \Omega) \right\},$$

it is a Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} := \left( \int_{\Omega} |u(x)|dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\,dxdy \right)^\frac{1}{p},$$

where the term

$$[u]_{W^{s,p}(\Omega)} := \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\,dxdy \right)^\frac{1}{p}$$

is the so called Gagliardo smeinorm of $u.$
If $p = 2$, the fractional Sobolev space $W^{s,p}$ is Hilbert space, and denoted by $H^s$. This space also can be defined via the Fourier transform. Precisely,

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^{2s})|\mathcal{F}u(\xi)|^2 d\xi < \infty \right\},$$

where $\mathcal{F}u$ is the Fourier transform of $u$.

The authors in [10] proved that

$$|u|^2_{H^s(\mathbb{R}^n)} = 2C(n,s)^{-1} \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi = 2C(n,s)^{-1} \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^n)}.$$

In this paper, we don’t care the constant $C(n,s)$, so we use the notation in [26], that $\dot{H}^s(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ under the quadratic form

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi = \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^n)}^2.$$

In fact, the above two cases of the definition of fractional space are equivalent. At the same time, the extension principle of Caffarelli and Silvestre [3] is work out in $\dot{H}^s(\mathbb{R}^n)$. As [26] mentioned that they consider the variational problem

$$S = \inf \left\{ \int_{\mathbb{H}} y^n |\nabla \phi(x,y)|^2 dx dy : \phi \in C_0^\infty(\mathbb{H}), \int_{\partial\mathbb{H}} |\phi(x,0)|^{2(n-2s/a)} dx = 1 \right\}. \quad (2.1)$$

The constant $S$ in problem (2.1) is well defined, due to the trace inequality

$$\int_{\partial\mathbb{H}} |\phi(x,0)|^{2(n-2s/a)} dx \leq C_{n,a} \int_{\mathbb{H}} y^n |\nabla \phi(x,y)|^2 dx dy, \quad \forall \phi \in C_0^\infty(\mathbb{H}), \quad (2.2)$$

where $C_{n,a}$ is a positive constant depending only on $n$ and $a$. In fact, this result comes from Frank et al. [12], they introduced the sharp trace inequality

$$\|(-\Delta)^{s/2} Tu\|_2 \leq C_{n,a} \int_{\mathbb{H}} y^n |\nabla u(x,y)|^2 dx dy, \quad (2.3)$$

where $T$ is a trace operator, such that $Tu(x) = u(x,0)$. There are also some related results of the trace inequality can refer to [1].

Next we list the comparison principle in [26].

**Lemma 2.1.** Let $\Omega \subset \mathbb{H}$ be an open set with a part of flat boundary $\Gamma \subset \partial \mathbb{H}$. Let $u \geq 0, u \not\equiv 0$, be classical solution to equation

$$\begin{cases}
\text{div}(y^n \nabla u(x,y)) = 0, & \text{in } \Omega, \\
\lim_{y \to 0^+} y^n u_y(x,y) \leq 0, & x \in \Gamma,
\end{cases}
$$

Then

$$u > 0 \quad \text{on } \Omega \cup \Gamma.$$

Indeed, this conclusion is also right for the system (1.2) on the $\Omega$. And it is also a strong maximum principle for the system (1.2) on the $\mathbb{H}$.

Let $(U, V)$ be nonnegative in $\mathbb{H}$, we introduce their Kelvin transform centered at the origin as

$$w(X) = \frac{1}{|X|^{n-2s}} U(\frac{X}{|X|^2}), \quad z(X) = \frac{1}{|X|^{n-2s}} V(\frac{X}{|X|^2}), \quad X = (x,y) \in \mathbb{H} \setminus \{0\}. \quad (2.4)$$

Obviously, $w, z$ are continuous and nonnegative in $\mathbb{H} \setminus \{0\}$. And a direct computation yields:
Lemma 2.2. Let \((U, V) \in W^{1,2}_{\text{loc}}(\mathbb{H}) \cap C(\mathbb{H}) \times W^{1,2}_{\text{loc}}(\mathbb{H}) \cap C(\mathbb{H})\) be a nonnegative (weak) solution of system:

\[
\begin{align*}
\text{div}(y^{1-2s}\nabla U(x, y)) &= 0, & \text{in } \mathbb{H}, \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial U(x, y)}{\partial y} &= -f(V(x, 0)), & \text{on } \partial \mathbb{H} \setminus \{0\}, \\
\text{div}(y^{1-2s}\nabla V(x, y)) &= 0, & \text{in } \mathbb{H}, \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial V(x, y)}{\partial y} &= -g(U(x, 0)), & \text{on } \partial \mathbb{H} \setminus \{0\}.
\end{align*}
\]

Then \((w, z)\) satisfies (weakly) the following system:

\[
\begin{align*}
\text{div}(y^{1-2s}\nabla w(x, y)) &= 0, & \text{in } \mathbb{H}, \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial w(x, y)}{\partial y} &= -\frac{1}{|x|^{n-2s}} f(|x|^{n-2s} z(x)), & \text{on } \partial \mathbb{H} \setminus \{0\}, \\
\text{div}(y^{1-2s}\nabla z(x, y)) &= 0, & \text{in } \mathbb{H}, \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial z(x, y)}{\partial y} &= -\frac{1}{|x|^{n-2s}} g(|x|^{n-2s} w(x)), & \text{on } \partial \mathbb{H} \setminus \{0\}.
\end{align*}
\]

(2.5)

Moreover \((w, z)\) has decay at infinity as

\[
\lim_{|X| \to \infty} |X|^{n-2s} w(X) = U(0), \quad \lim_{|X| \to \infty} |X|^{n-2s} z(X) = V(0),
\]

and hence \(w, z \in L^{\frac{2n}{n-2s}}(\Sigma_\lambda)\) for any \(n + 1 < q \leq \infty\).

Let \(\lambda \in \mathbb{R}\) and \(X = (x_1, x_2, \ldots, x_n, y) \in \mathbb{H}\). We denote

\[
T_\lambda = \{X \in \mathbb{H} : x_1 = \lambda\}, \quad \Sigma_\lambda = \{X \in \mathbb{H} : x_1 > \lambda\},
\]

\(p_\lambda = (2\lambda, 0, \cdots, 0, 0) \in \partial \mathbb{H}\), \(X_\lambda = (2\lambda - x_1, x_2, \cdots, x_n, y)\).

We define the reflected functions by

\(w_\lambda(X) = w(X_\lambda), \quad z_\lambda(X) = z(X_\lambda)\).

Set

\(W_\lambda(X) = w(X) - w(X_\lambda), \quad Z_\lambda(X) = z(X) - z(X_\lambda)\).

In order to use the method of moving planes, we give the key integral inequalities.

Lemma 2.3. For any fixed \(\lambda > 0\), \(w, z \in L^{\frac{2n}{n-2s}}(\Sigma_\lambda)\), such that

\[
\int_{\Sigma_\lambda} y^{1-2s} |\nabla W_\lambda^+|^2 dX \leq C_\lambda \left( \int_{\partial \mathbb{H} \cap \partial A_\lambda^+} z^{\frac{2n}{n-2s}}(x) dx \right)^\frac{4s}{n} \int_{\Sigma_\lambda} y^{1-2s} |\nabla Z_\lambda^+|^2 dX, \tag{2.7}
\]

\[
\int_{\Sigma_\lambda} y^{1-2s} |\nabla Z_\lambda^+|^2 dX \leq C_\lambda \left( \int_{\partial \mathbb{H} \cap \partial A_\lambda^+} w^{\frac{2n}{n-2s}}(x) dx \right)^\frac{4s}{n} \int_{\Sigma_\lambda} y^{1-2s} |\nabla W_\lambda^+|^2 dX, \tag{2.8}
\]

where \(A_\lambda^+ = \{X \in \Sigma_\lambda : Z_\lambda(X) \geq 0\}, \quad A_\lambda^- = \{X \in \Sigma_\lambda : W_\lambda(X) \geq 0\}, \quad W_\lambda^+ = \max\{W_\lambda, 0\}, \quad Z_\lambda^+ = \max\{Z_\lambda, 0\}, \quad C_\lambda > 0\) is a constant which is bounded when \(\lambda\) is away from zero.

Proof. We only prove (2.7), the proof of (2.8) is similar. For any fixed \(\lambda > 0\), thus \(w\) and \(W_\lambda^+ \leq w \in L^{\frac{2n}{n-2s}}(\Sigma_\lambda)\).

For \(\varepsilon > 0\) small, choose a cut-off function \(\eta_\varepsilon \in C_0^\infty(\mathbb{H})\) such that \(0 \leq \eta_\varepsilon \leq 1\), \(\eta_\varepsilon(X) = 1\) for \(2\varepsilon \leq |X - p_\lambda| \leq \varepsilon^{-1}\); \(\eta_\varepsilon = 0\) for \(|X - p_\lambda| \leq \varepsilon\) or \(|X - p_\lambda| \geq 2\varepsilon^{-1}\),
and $|\nabla \eta| \leq C\varepsilon^{-1}$ for $\varepsilon \leq |X-p_{\lambda}| \leq 2\varepsilon$, $|\nabla \eta| \leq C\varepsilon$ for $\varepsilon^{-1} \leq |X-p_{\lambda}| \leq 2\varepsilon^{-1}$, here $C>0$ is independent of $\varepsilon$. Since $h(t) = \frac{t}{t^{\frac{n-2s}{2}}}$, then for $X \in \Sigma_{\lambda}$, we rewrite (2.5) as

$$
\begin{align*}
\begin{cases}
\text{div}(y^{1-2s}\nabla w(x,y)) = 0, & \text{in } \mathbb{H}, \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial w(x,y)}{\partial y} = -h(|x|^{n-2s}z(x))z_{\lambda}^{\frac{n+2s}{n-2s}}(x), & \text{on } \partial \mathbb{H} \setminus \{0\},
\end{cases}
\end{align*}
$$

(2.9)

and

$$
\begin{align*}
\begin{cases}
\text{div}(y^{1-2s}\nabla w_{\lambda}(x,y)) = 0, & \text{in } \mathbb{H}, \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial w_{\lambda}(x,y)}{\partial y} = -h(|x_{\lambda}|^{n-2s}z_{\lambda}(x))z_{\lambda}^{\frac{n+2s}{n-2s}}(x), & \text{on } \partial \mathbb{H} \setminus \{p_{\lambda}\}.
\end{cases}
\end{align*}
$$

(2.10)

Multiply both sides of the above equations by $\phi_{\varepsilon} = W_{\lambda}^{+}\eta_{\varepsilon}$, we deduce that

$$
\begin{align*}
\int_{\Sigma_{\lambda} \cap \{2\varepsilon \leq |X-p_{\lambda}| \leq \varepsilon^{-1}\}} y^{1-2s} |\nabla W_{\lambda}^+|^2 dX \\
\leq \int_{\Sigma_{\lambda}} y^{1-2s} |\nabla (W_{\lambda}^+ \eta_{\varepsilon})|^2 dX \\
= \int_{\Sigma_{\lambda}} y^{1-2s} \nabla W_{\lambda}^+ \cdot \nabla \phi_{\varepsilon} dX + \int_{\Sigma_{\lambda}} y^{1-2s} |W_{\lambda}^+|^2 |\nabla \eta_{\varepsilon}|^2 dX = I + I_{\varepsilon}.
\end{align*}
$$

(2.11)

We estimate $I_{\varepsilon}$ at first. Write $R_{r} = \{X \in \mathbb{H} : r \leq |X-p_{\lambda}| \leq 2r\}$ for $r > 0$, then

$$
\begin{align*}
I_{\varepsilon} \leq C\varepsilon^{-2} \int_{\Sigma_{\lambda} \cap R_{r}} y^{1-2s} |W_{\lambda}^+|^2 dX + C\varepsilon^{2} \int_{\Sigma_{\lambda} \cap R_{r}^{-1}} y^{1-2s} |W_{\lambda}^+|^2 dX \\
\leq C\varepsilon^{-2} \int_{\Sigma_{\lambda} \cap R_{r}} y^{1-2s}((w-w_{\lambda})^+)^2 dX + C\varepsilon^{2} \int_{\Sigma_{\lambda} \cap R_{r}^{-1}} y^{1-2s}((w-w_{\lambda})^+)^2 dX \\
\leq C\varepsilon^{-2} \int_{R_{r}} y^{1-2s} w^2 dX + C\varepsilon^{2} \int_{R_{r}^{-1}} y^{1-2s} w^2 dX,
\end{align*}
$$

where $C > 0$ independent of $\varepsilon$. For $\varepsilon > 0$ sufficiently small, we derive from (2.6) that

$$
\varepsilon^{-2} \int_{R_{r}} y^{1-2s} w^2 dX \leq C_{\lambda}\varepsilon^{-2} \int_{\{X \in \mathbb{H} : |X-p_{\lambda}| \leq 2\varepsilon\}} y^{1-2s} dX = O(\varepsilon^{n-2s})
$$

and

$$
\varepsilon^{2} \int_{R_{r}^{-1}} y^{1-2s} w^2 dX \leq C_{\lambda}\varepsilon^{2} \int_{\{X \in \mathbb{H} : \varepsilon^{-1} \leq |X-p_{\lambda}| \leq 2\varepsilon^{-1}\}} y^{1-2s} |X|^{2(2s-n)} dX \\
\leq C_{\lambda}\varepsilon^{2+2(n-2s)} \int_{\{X \in \mathbb{H} : |X-p_{\lambda}| \leq 2\varepsilon^{-1}\}} y^{1-2s} dX = O(\varepsilon^{n-2s}),
$$

for some constants $C_{\lambda} > 0$. Hence, as $\varepsilon \to 0$, we have

$$
I_{\varepsilon} = O(\varepsilon^{n-2s}) \to 0.
$$

Now we give the estimate of $I$. Since $|x| > |x_{\lambda}|$, $h$ is nonincreasing, if $z(x) \geq z(x_{\lambda}) \geq 0$, then $-h(|x_{\lambda}|^{n-2s}z_{\lambda}(x)) \geq -h(|x|^{n-2s}z(x))$. By (2.9) and (2.10), we have

$$
I = \int_{\Sigma_{\lambda}} y^{1-2s} \nabla W_{\lambda}^+ \cdot \nabla \phi_{\varepsilon} dX
$$

$$
= \int_{\partial(\Sigma_{\lambda} \cap \text{supp} \eta_{\varepsilon})} y^{1-2s} \phi_{\varepsilon} \nabla W_{\lambda}^+ \cdot \nu \, dx
$$
Lemma 3.1. There exist $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, $W_\lambda(X) \leq 0$ and $Z_\lambda(X) \leq 0$ for any $X \in \Sigma_\lambda$.

Proof. If $\lambda > 0$ large enough, for $w, z \in L^{\frac{2n}{n-2\lambda}}(\Sigma_\lambda)$, we have

$$C_{\lambda}\left(\int_{\partial H \cap \partial A^{1}_{\lambda}} z^\frac{2n}{n-2\lambda} dx\right)^\frac{4n}{n} < 1,$$

for all $\lambda \geq \lambda_0$, and

$$C_{\lambda}\left(\int_{\partial H \cap \partial A^{2}_{\lambda}} w^\frac{2n}{n-2\lambda} dx\right)^\frac{4n}{n} < 1,$$

for all $\lambda \geq \lambda_0$.

By Lemma 2.3, we deduce that

$$\int_{\Sigma_\lambda} y^{1-2s}|\nabla W_\lambda^+|^2 dx = 0 \quad \text{and} \quad \int_{\Sigma_\lambda} y^{1-2s}|\nabla Z_\lambda^+|^2 dx = 0$$

for all $\lambda \geq \lambda_0$. Thus for $\lambda > 0$ large enough we obtain that for any $X \in \Sigma_\lambda$

$$W_\lambda(X) \leq 0 \quad \text{and} \quad Z_\lambda(X) \leq 0.$$

Step 2. Step 1 provides a starting point, now we can move the planes. Define

$$\Lambda = \inf\{\lambda > 0| W_\mu(X) \leq 0, Z_\mu(X) \leq 0, \forall X \in \Sigma_\mu, \mu > \lambda\}.$$

Lemma 3.2. If $\Lambda > 0$ then $W_\Lambda(X) \equiv 0$ and $Z_\Lambda(X) \equiv 0$ for any $X \in \Sigma_\Lambda$. 

3. Proof of Theorem 1.1. Step 1. We begin moving planes from the infinity.

Step 1. We begin moving planes from the infinity.
Proof. By continuity, we have \( W_\Lambda(X) \leq 0 \) and \( Z_\Lambda(X) \leq 0 \) for any \( X \in \Sigma_\Lambda \).

Suppose on the contrary that \( W_\Lambda(X) \neq 0 \) in \( \Sigma_\Lambda \), then for \((x,0) \in \partial \Omega \cap \partial \Sigma_\Lambda\), we have that
\[
\frac{h(|x|^{n-2s}z(x))z^{\frac{n+s}{n-2s}}(x)}{|x|^{n+2s}} \leq \frac{f(|x|^{n-2s}z_\Lambda(x))}{|x|^{n+2s}}.
\]
Applying the Lemma 2.1 to this implies that
\[
\frac{1}{|x|^{n+2s}}w_\Lambda(x) \leq h(|x|^{n-2s}z_\Lambda(x))z^{\frac{n+s}{n-2s}}_\Lambda(x).
\]

Applying the Lemma 2.1 to \( W_\Lambda(X) \), we get that \( W_\Lambda(X) \leq 0 \), by the strong maximum principle, \( W_\Lambda(X) < 0 \) in \( \Sigma_\Lambda \). The strict inequality shows that the characteristic function \( \lambda_\partial A^{\lambda}_\Sigma \to 0 \) a.e. in \( \mathbb{R}^n \) as \( \lambda \to \Lambda \). The dominated convergence theorem shows
\[
\lim_{\lambda \to \Lambda} C_\Lambda(\int_{\partial \Omega \cap \partial A^{\lambda}_\Sigma} w^{\frac{2n}{n-2s}}dx)^{\frac{n}{2n}} = 0,
\]
and hence for \( \lambda \in (\Lambda - \delta, \Lambda) \)
\[
C_\Lambda(\int_{\partial \Omega \cap \partial A^{\lambda}_\Sigma} z^{\frac{2n}{n-2s}}dx)^{\frac{n}{2n}} C_\Lambda(\int_{\partial \Omega \cap \partial A^{\lambda}_\Sigma} w^{\frac{2n}{n-2s}}dx)^{\frac{n}{2n}} < 1,
\]
where \( \delta \) is a sufficiently small positive constant. Recalling the previous argument, this implies that \( W_\Lambda(X) \leq 0 \) and \( Z_\Lambda(X) \leq 0 \) for any \( X \in \Sigma_\Lambda \), which against the definition of \( \Lambda \).

If \( \Lambda = 0 \), for any \((x_1, x_2, \cdots, x_n, y) \in \Sigma_0\), we get
\[
w(x_1, x_2, \cdots, x_n, y) \leq w(-x_1, x_2, \cdots, x_n, y)
\]
and
\[
z(x_1, x_2, \cdots, x_n, y) \leq z(-x_1, x_2, \cdots, x_n, y).
\]
We also can move the planes from the left to right, and obtain that
\[
w(x_1, x_2, \cdots, x_n, y) \geq w(-x_1, x_2, \cdots, x_n, y)
\]
and
\[
z(x_1, x_2, \cdots, x_n, y) \geq z(-x_1, x_2, \cdots, x_n, y).
\]
Hence, we have
\[
w(x_1, x_2, \cdots, x_n, y) = w(-x_1, x_2, \cdots, x_n, y)
\]
and
\[
z(x_1, x_2, \cdots, x_n, y) = z(-x_1, x_2, \cdots, x_n, y).
\]
Therefore, we claim that
\[
w(x, y) = w(|x|, y) \quad \text{and} \quad z(x, y) = z(|x|, y).
\]

Since we can choose any point as the center of Kelvin transform, then \( w, z \) must be independent of \( x \). That is, \( U, V \) are only dependent of \( y \). When go back to the \( u, v \), we have that \((u, v) = (c_1, c_2)\) for some constants \( c_1, c_2 \) with \( f(c_2) = g(c_1) = 0 \).

If \( \Lambda > 0 \), then we have \( w = w_\Lambda \) and \( z = z_\Lambda \). This implies that \( w, z \) are regular at the origin, and hence \( U, V \) are regular at infinity. Since \( w = w_\Lambda \) and \( z = z_\Lambda \), from (2.9) and (2.10) we have that
\[
h(|x|^{n-2s}z(x)) = h(|x_\Lambda|^{n-2s}z_\Lambda(x)).
\]
Noting that for any \( x \), \( |x| > |x_\Lambda| \), and \( h \) is nonincreasing, it follows that \( h \) must be a constant, that is exist positive constant \( A \) such that \( h(t) = A \). For the same
reason, we know there exist positive constant $B$ such that $g(t) = b$. In this case, we use the above Lemmas only show that $w, z$ are symmetric about some point for $x$, so does to $U, V$. But this is enough for our result, because we only need to get the information about $x$ of the solutions $u$ and $v$. Therefore, we prove that $(u, v)$ is radially symmetric about some point.

4. Proof of Theorem 1.2. We use the same notation as in Section 2. Let $(U, V) \in W^{1,2}_0(\mathbb{H}) \times W^{1,2}_0(\mathbb{H}) \cap C(\mathbb{H})$ be a positive (weak) solution of system (1.2), then $(w, z)$ satisfies (weakly) the following system:

\[
\begin{align*}
\text{div}(y^{1-2s}\nabla w(x, y)) &= 0, & \text{in } \mathbb{H}, \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial w(x, y)}{\partial y} &= -\frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z(x)), & \text{on } \partial \mathbb{H} \setminus \{0\}, \\
\text{div}(y^{1-2s}\nabla z(x, y)) &= 0, & \text{in } \mathbb{H}, \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial z(x, y)}{\partial y} &= -\frac{1}{|x|^{n+2s}} g(|x|^{n-2s} w(x), |x|^{n-2s} z(x)), & \text{on } \partial \mathbb{H} \setminus \{0\},
\end{align*}
\]

(4.1)

and then

\[
\begin{align*}
\text{div}(y^{1-2s}\nabla \lambda_{W}(x, y)) &= 0, & \text{in } \mathbb{H}, \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial \lambda_{W}(x, y)}{\partial y} &= -\frac{1}{|x|^{n+2s}} f(|x|^{n-2s} \lambda_{W}(x), |x|^{n-2s} \lambda_{Z}(x)), & \text{on } \partial \mathbb{H} \setminus \{p_{\lambda}\}, \\
\text{div}(y^{1-2s}\nabla \lambda_{Z}(x, y)) &= 0, & \text{in } \mathbb{H}, \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial \lambda_{Z}(x, y)}{\partial y} &= -\frac{1}{|x|^{n+2s}} g(|x|^{n-2s} \lambda_{W}(x), |x|^{n-2s} \lambda_{Z}(x)), & \text{on } \partial \mathbb{H} \setminus \{p_{\lambda}\}.
\end{align*}
\]

(4.2)

Lemma 4.1. For any fixed $\lambda > 0$, $w, z \in L^{\frac{2n}{n-2s}}(\Sigma_{\lambda})$, such that

\[
\int_{\Sigma_{\lambda}} y^{1-2s} |\nabla W_{\lambda}^+|^2 dX \leq C_{\lambda} \left( \int_{\partial \mathbb{H} \cap \partial A_{\lambda}^+} \frac{1}{x^{2n}} d\sigma \right)^{\frac{2}{n}} \int_{\Sigma_{\lambda}} y^{1-2s} |\nabla W_{\lambda}^+|^2 dX 
\]

(4.3)

\[
+ C_{\lambda} \left( \int_{\partial \mathbb{H} \cap \partial A_{\lambda}^+ \cap \partial A_{\lambda}^+} \frac{1}{x^{2n}} d\sigma \right)^{\frac{2}{n}} \left( \int_{\Sigma_{\lambda}} y^{1-2s} |\nabla W_{\lambda}^+|^2 dX \right)^{\frac{1}{2}} \left( \int_{\Sigma_{\lambda}} y^{1-2s} |\nabla Z_{\lambda}^+|^2 dX \right)^{\frac{1}{2}},
\]

\[
\int_{\Sigma_{\lambda}} y^{1-2s} |\nabla Z_{\lambda}^+|^2 dX \leq C_{\lambda} \left( \int_{\partial \mathbb{H} \cap \partial A_{\lambda}^+} \frac{1}{x^{2n}} d\sigma \right)^{\frac{2}{n}} \int_{\Sigma_{\lambda}} y^{1-2s} |\nabla Z_{\lambda}^+|^2 dX
\]

(4.4)

\[
+ C_{\lambda} \left( \int_{\partial \mathbb{H} \cap \partial A_{\lambda}^+ \cap \partial A_{\lambda}^+} \frac{1}{x^{2n}} d\sigma \right)^{\frac{2}{n}} \left( \int_{\Sigma_{\lambda}} y^{1-2s} |\nabla Z_{\lambda}^+|^2 dX \right)^{\frac{1}{2}} \left( \int_{\Sigma_{\lambda}} y^{1-2s} |\nabla W_{\lambda}^+|^2 dX \right)^{\frac{1}{2}},
\]

where $A_{\lambda}^+ = \{ X \in \Sigma_{\lambda} : Z_{\lambda}(X) \geq 0 \}$, $A_{\lambda}^- = \{ X \in \Sigma_{\lambda} : W_{\lambda}(X) \geq 0 \}$, $W_{\lambda}^+ = \max\{W_{\lambda}, 0\}$, $Z_{\lambda}^+ = \max\{Z_{\lambda}, 0\}$, $C_{\lambda} > 0$ is a constant which is bounded when $\lambda$ is away from zero.

Proof. We just prove (4.3), the proof of (4.4) is similar. For any fixed $\lambda > 0$, thus $w$ and $W_{\lambda}^+ \leq w \in L^{\frac{2n}{n-2s}}(\Sigma_{\lambda})$. For $\varepsilon > 0$ small, choose a cut-off function $\eta_{\varepsilon} \in C_0^\infty(\mathbb{H})$ such that $0 \leq \eta_{\varepsilon} \leq 1$, $\eta_{\varepsilon}(X) = 1$ for $2\varepsilon \leq |X - p_{\lambda}| \leq \varepsilon^{-1}$; $\eta_{\varepsilon} = 0$ for $|X - p_{\lambda}| \leq \varepsilon$ or $|X - p_{\lambda}| \geq 2\varepsilon^{-1}$, and $|\nabla \eta_{\varepsilon}| \leq C \varepsilon^{-1}$ for $\varepsilon \leq |X - p_{\lambda}| \leq 2\varepsilon$, $|\nabla \eta_{\varepsilon}| \leq C \varepsilon$ for $\varepsilon^{-1} \leq |X - p_{\lambda}| \leq 2\varepsilon^{-1}$, here $C > 0$ independent of $\varepsilon$. We will test the equations in
(4.1) and (4.2) with the function \( \phi = W^+_{\lambda} \eta^2 \). Hence one can assume \( w \geq w_\lambda \). So that \( |x|^{n+2s} w \geq |x\lambda|^{n+2s} w_\lambda \) for any \( \lambda > 0 \).

If \( z \leq z_\lambda \), by assumptions in Theorem 1.2, we know that

\[
\begin{align*}
  f(|x\lambda|^{n-2s} w_\lambda(x), |x\lambda|^{n-2s} z_\lambda(x)) \\
  \geq f(|x\lambda|^{n-2s} w_\lambda(x), |x\lambda|^{n-2s} z(x) \frac{w_\lambda(x)}{w(x)}) \\
  \geq f(|x|^{n-2s} w(x), |x|^{n-2s} z(x)) \left| \frac{x\lambda|^{n+2s} w_\lambda(x)}{|x|^{n+2s} z(x)} \right| \frac{w_\lambda(x)}{w(x)} p_1, \\
  \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z(x)) (1 - \left| \frac{w_\lambda(x)}{w(x)} \right| p_1) \\
  \leq \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z(x)) (1 - \left| \frac{w_\lambda(x)}{w(x)} \right| \frac{n+2s}{n-2s}) \\
  \leq \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z(x)) \frac{n+2s}{n-2s} (1 - \frac{w_\lambda(x)}{w(x)}) \\
  = \frac{n+2s}{n-2s} \frac{f(|x|^{n-2s} w(x), |x|^{n-2s} z(x))}{|x|^{n-2s} w(x)} \left( \frac{w_\lambda(x)}{w(x)} \right) \frac{1}{|x|^{n-2s} z(x)} (w(x) - w_\lambda(x)) \leq \frac{C_\lambda}{|x|^{n-2s} z(x)} (w(x) - w_\lambda(x)),
\end{align*}
\]

for some constant \( C_\lambda \).

If \( z \geq z_\lambda \), we have

\[
\begin{align*}
  f(|x\lambda|^{n-2s} w_\lambda(x), |x\lambda|^{n-2s} z_\lambda(x)) \\
  \geq f(|x|^{n-2s} w(x), |x|^{n-2s} z(x)) \left| \frac{x\lambda|^{n+2s} w_\lambda(x)}{|x|^{n+2s} z(x)} \right| \frac{w_\lambda(x)}{w(x)} q_1, \\
  \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z(x)) (1 - \left| \frac{w_\lambda(x)}{w(x)} \right| q_1) \\
  \leq \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z(x)) (1 - \left| \frac{w_\lambda(x)}{w(x)} \right| \frac{n+2s}{n-2s}) \\
  \leq \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z(x)) \frac{n+2s}{n-2s} (1 - \frac{w_\lambda(x)}{w(x)}) \\
  = \frac{n+2s}{n-2s} \frac{f(|x|^{n-2s} w(x), |x|^{n-2s} z(x))}{|x|^{n-2s} w(x)} \left( \frac{w_\lambda(x)}{w(x)} \right) \frac{1}{|x|^{n-2s} z(x)} (w(x) - w_\lambda(x)) \\
  + \frac{f(|x|^{n-2s} w(x), |x|^{n-2s} z(x))}{|x|^{n-2s} z(x)} \left( \frac{w_\lambda(x)}{w(x)} \right) \frac{1}{|x|^{n-2s} z(x)} (z(x) - z_\lambda(x)) \\
  \leq \frac{C_\lambda}{|x|^{n-2s} z(x)} ((w(x) - w_\lambda(x)) + (z(x) - z_\lambda(x))).
\end{align*}
\]
Hence, for \( w \geq w_\lambda \)
\[
-\left( \frac{\partial w}{\partial \nu^s} - \frac{\partial w_\lambda}{\partial \nu^s} \right) \leq \frac{C_\lambda}{|x|^{4s}} ((w(x) - w_\lambda(x))^+ + (z(x) - z_\lambda(x))^+).
\]

In order to reach the conclusion, we also have (2.11), and the estimate of \( I_\epsilon \) can refer to Lemma 2.3. Next, we only need to work on the \( I \).

\[
I = \int_\Sigma y^{1-2s} \nabla W_\lambda^+ \cdot \nabla \phi_\epsilon dX = \int_{\partial (\Sigma \cap \text{supp} \eta_\epsilon)} y^{1-2s} \phi_\epsilon \nabla W_\lambda^+ \cdot \nu dx
\]
\[
\leq \int_{\{ x \in \mathbb{R}^n : x_1 > \lambda, x_2 \leq |x - \rho \lambda| \leq 2 \epsilon \}} \frac{C_\lambda}{|x|^{4s}} ((w(x) - w_\lambda(x))^+ + (z(x) - z_\lambda(x))^+) \phi_\epsilon dx
\]
\[
\leq C_\lambda \int_{\partial \Omega \cap \partial A_2^\lambda} \frac{1}{|x|^{2n}} \left( (W_\lambda^+)^2 + W_\lambda^+ Z_\lambda^+ \right) dx
\]
\[
\leq C_\lambda \left( \int_{\partial \Omega \cap \partial A_2^\lambda} \frac{1}{|x|^{2n}} dx \right)^{\frac{2n}{n+2s}} \left( \int_{\partial \Omega \cap \partial A_2^\lambda} (W_\lambda^+ \frac{2n}{n+2s} dx) \frac{n+2s}{n} \right)
\]
\[
+ C_\lambda \left( \int_{\partial \Omega \cap \partial A_2^\lambda} \frac{1}{|x|^{2n}} dx \right)^{\frac{2n}{n+2s}} \left( \int_{\partial \Omega \cap \partial A_2^\lambda} (Z_\lambda^+) \frac{2n}{n+2s} dx \right)^{\frac{n+2s}{n}}.
\]

Therefore, by the trace inequality (2.2), we get the desired result. \( \square \)

**Lemma 4.2.** There exist \( \lambda_0 > 0 \) such that for all \( \lambda \geq \lambda_0 \), \( W_\lambda(X) \leq 0 \) and \( Z_\lambda(X) \leq 0 \) for any \( X \in \Sigma_\lambda \).

**Proof.** Since \( \frac{1}{|x|^{2n}} \in L^1(\Sigma_\lambda) \), as \( \lambda \to +\infty \) then
\[
\int_{\partial \Omega \cap \partial A_2^\lambda \cap \partial A_2^\lambda} \frac{1}{|x|^{2n}} \leq \int_{\Sigma_\lambda} \frac{1}{|x|^{2n}} \to 0.
\]
It follows that there exists \( \lambda_0 > 0 \), for all \( \lambda \geq \lambda_0 \) such that
\[
C_\lambda \left( \int_{\partial \Omega \cap \partial A_2^\lambda} \frac{1}{|x|^{2n}} dx \right)^{\frac{2n}{n+2s}} < 1, \quad \text{and} \quad C_\lambda \left( \int_{\partial \Omega \cap \partial A_2^\lambda} \frac{1}{|x|^{2n}} dx \right)^{\frac{2n}{n+2s}} < 1.
\]
By Lemma 4.2, we deduce that
\[
\int_\Sigma y^{1-2s} |\nabla W_\lambda|^2 dX = 0 \quad \text{and} \quad \int_\Sigma y^{1-2s} |\nabla Z_\lambda|^2 dX = 0
\]
for all \( \lambda \geq \lambda_0 \). Thus for \( \lambda \geq \lambda_0 \) and any \( X \in \Sigma_\lambda \), we have
\[
W_\lambda(X) \leq 0 \quad \text{and} \quad Z_\lambda(X) \leq 0.
\]
\( \square \)

For the definition of \( \Lambda \) in Section 3, we also have the following lemma.

**Lemma 4.3.** If \( \Lambda > 0 \) then \( W_\lambda(X) \equiv 0 \) and \( Z_\lambda(X) \equiv 0 \) for any \( X \in \Sigma_\Lambda \).

**Proof.** By continuity, we prove that if \( w = w_\lambda \) at some point \( X_0 \in \Sigma_\lambda \), then in a neighborhood of \( X_0 \), and hence \( w = w_\lambda \) in \( \Sigma_\Lambda \).
In fact, by continuity, we see that \( W_\lambda(X) \leq 0 \) and \( Z_\lambda(X) \leq 0 \) for \( X \in \Sigma_\Lambda \). Since \( w(X_0) = w_\lambda(X_0) \), we have \( |X|^{n-2s} w(X_0) > |X_\lambda|^{n-2s} w_\lambda(X_0) \) for \( X \in B_r(X_0) \) a
neighborhood of $X_0$. By using the same arguments as we used in the proof of Lemma 4.1 and the fact that if $t_1 > t_1', t_2 > t_2'$, then

$$f(t_1', t_2') \geq f(t_1, t_2) \left(\frac{t_1'}{t_1}\right)^p \left(\frac{t_2'}{t_2}\right)^q,$$

and

$$g(t_1', t_2') \geq g(t_1, t_2) \left(\frac{t_1'}{t_1}\right)^p \left(\frac{t_2'}{t_2}\right)^q.$$

And then

$$f(|x|^{n-2s} w_{\Lambda}(x), |x|^{n-2s} z(x))$$

$$\geq f(|x|^{n-2s} w(x), |x|^{n-2s} z(x))(\frac{|x|^{n-2s} w_{\Lambda}(x)}{|x|^{n-2s} w(x)})^p_1 (\frac{|x|^{n-2s} z(x)}{|x|^{n-2s} z(x)})^{q_1}$$

$$= f(|x|^{n-2s} w(x), |x|^{n-2s} z(x)) (\frac{|x|^{n+2s} w_{\Lambda}(x)}{|x|^{n+2s} w(x)})^p_1,$$

therefore,

$$- (\frac{\partial w}{\partial \nu^s} - \frac{\partial w_{\Lambda}}{\partial \nu^s})$$

$$= \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z(x)) - \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w_{\Lambda}(x), |x|^{n-2s} z_{\Lambda}(x))$$

$$\leq \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z(x)) - \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w_{\Lambda}(x), |x|^{n-2s} z_{\Lambda}(x))$$

$$\leq \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z(x))(1 - (\frac{w_{\Lambda}(x)}{w(x)})^{p_1})$$

$$\leq - C(w(x) - w_{\Lambda}(x)), \text{ in } B_r(X_0),$$

for some constant $C > 0$ which depending on $X_0, r$. Hence,

$$\begin{cases}
\text{div}(y^{1-2s}\nabla W_{\Lambda}(x, y)) = 0 \\
\lim_{y \to 0^+} y^{1-2s} \frac{\partial W_{\Lambda}(x, y)}{\partial y} + CW_{\Lambda} \leq 0 \\
W_{\Lambda} \leq 0, W_{\Lambda}(X_0) = 0 \text{ in } B_r(X_0).
\end{cases}$$

By the maximum principle, $W_{\Lambda}(X) \equiv 0$ in $B_r(X_0)$.

Next, we claim that $W_{\Lambda}(X) \equiv 0$ implies $Z_{\Lambda}(X) \equiv 0$. In fact, by the equations (4.1) and (4.2),

$$\frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z(x))$$

$$= \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z_{\Lambda}(x))$$

$$\geq \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z_{\Lambda}(x))$$

$$\geq \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z_{\Lambda}(x)).$$

Since $f(t_1, t_2)$ is nondecreasing in $t_2$, by the above inequality, we deduce that

$$|x|^{n-2s} z(x) > |x|^{n-2s} z_{\Lambda}(x). \quad (4.5)$$

On the other hand,

$$\frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z(x))$$
As a consequence of (4.6) and (4.8), we have
\[ z_{\lambda} \geq \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z_{\lambda}(x)) \]
\[ \geq \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z(x)) \]
i.e.,
\[ \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z(x)) \]
\[ = \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z_{\lambda}(x)) \]
\[ = \frac{1}{|x|^{n+2s}} f(|x|^{n-2s} w(x), |x|^{n-2s} z_{\lambda}(x)), \] (4.6)
hence
\[ \frac{f(|x|^{n-2s} w(x), |x|^{n-2s} z(x))}{(|x|^{n-2s} w(x))^{\nu_1} (|x|^{n-2s} z(x))^{\nu_1}} = \frac{f(|x|^{n-2s} w(x), |x|^{n-2s} z_{\lambda}(x))}{(|x|^{n-2s} w(x))^{\nu_1} (|x|^{n-2s} z_{\lambda}(x))^{\nu_1}}. \] (4.7)
It follows from (4.5), that
\[ |x|^{n-2s} w(x) \geq |x|^{n-2s} w_{\lambda}(x), |x|^{n-2s} z(x) \geq |x|^{n-2s} z_{\lambda}(x) \geq |x|^{n-2s} z(x). \]

By (4.7) and assumption (ii) of Theorem 1.2,
\[ \frac{f(|x|^{n-2s} w(x), |x|^{n-2s} z(x))}{(|x|^{n-2s} w(x))^{\nu_1} (|x|^{n-2s} z(x))^{\nu_1}} = \frac{f(|x|^{n-2s} w(x), |x|^{n-2s} z_{\lambda}(x))}{(|x|^{n-2s} w(x))^{\nu_1} (|x|^{n-2s} z_{\lambda}(x))^{\nu_1}}. \] (4.8)
As a consequence of (4.6) and (4.8), we have \( z_{\lambda} \geq z_{\lambda}^{q_1} \), and hence \( z = z_{\lambda} \) since \( q_1 > 0 \).

Suppose that \( W_{\lambda}(X) \neq 0 \) and \( Z_{\lambda}(X) \neq 0 \) in \( \Sigma_{\lambda} \), then \( w(X) < w_{\lambda}(X) \) and \( z(X) < z_{\lambda}(X) \) in \( \Sigma_{\lambda} \). Now let \( \chi_{S} \) be the characteristic function of set \( S \). Then \( \frac{1}{|x|^{2s}} \chi_{\partial A_{\lambda}^2} \) converges pointwise to zero as \( \lambda \to \Lambda \) in \( \mathbb{R} \setminus \{p_{A}\} \). Hence if \( 0 < \lambda - \delta < \Lambda \) (here \( \delta \) sufficiently small), then \( \frac{1}{|x|^{2s}} \chi_{\partial A_{\lambda}^2} \leq \frac{1}{|x|^{2s}} \chi_{\Sigma_{\lambda}-\delta} \in L^1(\Sigma_{\lambda}). \)

By the dominate convergence, as \( \lambda \to \Lambda \), we have
\[ \int_{\partial \Omega \cap \partial A_{\lambda}^2} \frac{1}{|x|^{2n}} \to 0, \]
hence
\[ C_{\lambda}(\int_{\partial \Omega \cap \partial A_{\lambda}^2} \frac{1}{|x|^{2n}} dx)^{\frac{2n}{p_1}} < 1 \]
and
\[ C_{\lambda}(\int_{\partial \Omega \cap \partial A_{\lambda}^2} \frac{1}{|x|^{2n}} dx)^{\frac{2n}{p_1}} < 1 \]
for all \( \lambda \in (\Lambda - \delta, \Lambda) \). Also we get
\[ C_{\lambda}(\int_{\partial \Omega \cap \partial A_{\lambda}^2} \frac{1}{|x|^{2n}} dx)^{\frac{2n}{p_1}} < 1 \]
for all \( \lambda \in (\Lambda - \delta, \Lambda) \).

By the Lemma 4.1, we deduce that
\[ \int_{\Sigma_{\lambda}} y^{1-2s} |\nabla W_{\lambda}^z|^2 dX = 0 \]
and
\[ \int_{\Sigma_{\lambda}} y^{1-2s} |\nabla Z_{\lambda}^z|^2 dX = 0 \]
for all \( \lambda \geq \Lambda - \delta \), this implies that \( W_{\lambda} \leq 0 \) and \( Z_{\lambda} \leq 0 \) in \( \Sigma_{\lambda} \) for \( \lambda \geq \Lambda - \delta \), which contradicts with the definition of \( \Lambda \).
5. **Proof of Theorem 1.2.** Suppose that \((u, v)\) is a positive solution of \((1.1)\). Make the Kelvin transform around point \(p \in \mathbb{H}\) and define \(\Lambda\) as Section 3. If \(\Lambda = 0\) for all \(p\), then \((w, z)\) is a radially symmetric with respect to all \(p\), then must be constant, so does to \((u, v)\). If \(\Lambda > 0\), then we get that \((w, z)\) is radially symmetric about some point, we also have that \((u, v)\) is radially symmetric about some point.

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