Horndeski gravity without screening in binary pulsars

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ABSTRACT

We test the subclasses of Horndeski gravity without Vainshtein mechanism in the strong field regime of binary pulsars. We find the rate of energy losses via the gravitational radiation predicted by such theories and compare our results with observational data from quasi-circular binaries PSR J1738+0333, PSR J0737-3039, PSR J1012+5307. In addition, we consider few specific cases: the hybrid metric-Palatini f(R)-gravity and massive Brans-Dicke theory.

Key words: gravitation – pulsars: general – gravitational waves – methods: analytical

1 INTRODUCTION

The General Relativity (GR) is the universally recognized theory of gravity. It successfully describes a wide range of scales and gravitational regimes (weak field limit in Solar System and strong field regime of binary black holes). Together with Standard model, they represent two pillars of modern physics.

Unfortunately, some phenomena cannot be explained completely in the frameworks of these two approaches. The accelerated expansion of our Universe has been found from the Supernovae Type Ia (SN Ia) observations (Riess et al. 1999, 2004; Perlmutter et al. 1999; Spergel et al. 2007). So an extra component called “Dark Energy” (DE) has been introduced by Turner (1999), but the nature of this phenomenon is not fully understood. The other problem is dark matter (Oort 1932; Zwicky 1933). It is the invisible matter, which fills up galaxies and manifests itself only in the gravitational interaction. Also, this phenomenon can be described (apart from “new physics”) by changing the gravitational theory at galaxy scales (Capozziello et al. 2013; Borka Jovanovic et al. 2016; Katsuragawa & Matsuzaki 2017; Shi, Li & Han 2017). Furthermore, there is no any complete self-consistent quantum theory of gravity. All these facts lead to an increasing number of modified gravitational theories. One of the most widespread approaches to create the modified gravity is to extend GR with higher order curvature corrections and additional degrees of freedom (Alexeyev & Pomazanov 1997; Alexeyev & Rannu 2012). But the simplest way to modify GR remains adding of a scalar field.

The Horndeski gravity is the most general scalar-tensor theory providing the second-order field equations which evades Ostrogradski instabilities (Horndeski 1974). It represents a covariant generalization of Galileon gravity. Horndeski gravity suggests solutions for some GR’s problems. For example, the scalar field can play the role of DE and explain the accelerating expansion of the Universe (De Felice & Tsujikawa 2012). Therefore during last few years in connection with all these circumstances, the Horndeski gravity attracts a large number of researchers. This theory has recently been studied extensively in the context of cosmology (Germani & Martin-Moruno 2017; Kennedy, Lombriser & Taylor 2017; Nunes, Martn-Moruno & Lobo 2017) and physics of black holes (Treykakova 2017; Treykakova & Lateh 2018). Taking into account the generality and importance of Horndeski model, it is natural to ask how this theory pass different experimental gravitational tests and impose restrictions on its parameters. The Horndeski gravity has already been tested in many experiments (cluster lensing (Narikawa et al. 2013), the cosmic microwave background (CMB) data (Salvatelli, Piazza & Marinoni 2016; Renk, Zumalacarregui & Montanari 2016) and so on). Special attention should be paid to the recent works of Ezquiaga & Zumalacarregui (2017) and Baker et al. (2017) related to the verification of the Horndeski theory using LIGO data for event GW170817 (Abbott et al. 2018a).
2017) and the concomitant gamma-ray burst GRB 170817A (Abbott et al. 2017). In these papers authors investigate the speed of gravitational waves in various theories and show that data of the binary neutron star merger GW170817 (Abbott et al. 2017) and the concomitant gamma-ray burst GRB 170817A (Abbott et al. 2017) allow to restrict the parameters of the Horndeski gravity.

The most general form of Horndeski gravity predicts the existence of a fifth force which is strongly constrained by precision tests at Solar System scales. If a theory involves a scalar field for description of DE, it should contain a mechanism for suppressing of the scalar interaction with visible matter on small scales, that it relates only to cosmological scales. The Vainshtein mechanism represents such a possibility (Vainshtein 1972). Originally it was used in application to massive gravity (Vainshtein 1972). Now this mechanism is actively applied to the Horndeski models due to the presence of non-linear derivative interactions (Kimura, Kobayashi & Yamamoto 2012; Koyama, Niz & Tasinato 2013). The Vainshtein mechanism claims that it is not possible to ignore the effect of non-linearity within the so-called Vainshtein radius $r_V$ from the center of the matter source. Beyond $r_V$, the linearization can be applied. In this paper we consider the subclass of Horndeski theories which do not imply Vainshtein mechanism in the strong field regime of binary pulsars. Such subclass of Horndeski gravity reduces to the standard massive scalar-tensor theories. A similar task was investigated in the work of Hou & Gong (2018) where authors restrict their consideration with massless case.

Investigations of other types of screening mechanisms, such as the chameleon (Khoury & Weltman 2004a,b), the symmetron (Hinterbichler & Khoury 2010; Hinterbichler et al. 2011) and the dilaton (Damour & Polyakov 1994a,b), in the context of scalar-tensor theories are widespread. The question about screening effects manifestations in binary pulsars data was considered earlier by Brax, Davis & Sarkstein (2014); Zhang, Liu & Zhao (2017). In this paper we focus only on scalar-tensor models that do not imply any types of screening mechanisms.

The discovery of the first binary pulsar system PSR B1913 + 16 by Hulse & Taylor (1975) has opened a new testing ground for GR and its extensions. It is important to emphasize that in binary pulsars one deals with a gravitational field, which is stronger than in the Solar System. Moreover, due to the high stability of the pulse arrival it is possible to extract the dynamics of the orbital motion with such an accuracy at which the effects of gravitational waves emission could appear. The observable orbital decay of binary pulsars became the first experimental proof of the gravitational radiation existence. Now observational data of the orbital period change have a high accuracy. All these facts make the binary pulsars an indispensable laboratory for studying the behaviour of gravitational radiation in different models of gravity (Pshirkov, Tuntsov & Postnov 2008). In addition, pulsars allow to understand other physical processes better (Ivanov, Pshirkov & Rubtsov 2016).

The scalar dipole radiation dominates in the expressions for the orbital decay of the binary pulsars predicted by scalar-tensor theories (Eardley 1975; Will & Zaglauer 1989; Alsing et al. 2012; Zhang, Liu & Zhao 2017). This dipole contribution to the gravitational radiation is produced due to violations of the gravitational weak equivalence principle (GWEP) (Di Casola, Liberati & Sonego 2015). This effect becomes more pronounced in mixed binaries (binary systems whose members have different gravitational binding energy). The fact is that the dipole radiation is produced when the system’s centre of mass is offset with respect to the centre of inertia. So mixed binaries and eccentric ones seems to be the best target to constrain scalar-tensor theories (Alsing et al. 2012). In this work, we test subclass of Horndeski gravity (without Vainshtein mechanism as a first step) in mixed binary systems and impose restrictions on the parameters of this model.

The structure of the paper is the following. In section 2 we discuss the action of the Horndeski gravity and reduce it to the standard massive scalar-tensor action. The section 3 contains the field equations in the weak-field limit. In section 4, we solve the post-Newtonian equations for the tensor and scalar fields. Further, in section 5 we find the motion equations of binary systems. After that, in section 6 we obtain the stress-energy pseudotensor using the Noether current method, calculate the rate of the energy loss due to the tensor and scalar gravitational radiations and derive their contributions to the orbital period change. In section 7, we derive the constraints on the parameters of the standard massive scalar-tensor theories by the current observations; also in this section we consider two specific models: the hybrid metric-Palatini (f(R)-gravity and massive Brans-Dicke theory, and impose restrictions on these models. We conclude in section 8 with a summary and discussion.

Throughout this paper the Greek indices $(\mu, \nu, ...)$ run over 0, 1, 2 and the signature is $(-, +, +, +)$. All calculations are performed in the CGS system.

## 2 MASSIVE SCALAR-TENSOR GRAVITY

### 2.1 Action

We start our consideration from the action of the Horndeski theory, which is presented by Kobayashi, Yamaguchi & Yokoyama (2011),

$$S = \frac{c^4}{16\pi} \sum_{i=2}^5 \int d^4x \sqrt{-g} L_i + \int d^4x \sqrt{-g} \mathcal{L}_m + \mathcal{L}_{m}^{(j)},$$

where $c$ is the speed of light, $g$ is the determinant of the metric, and $L_m$ is the Lagrangian density for the matter fields $\Psi_m$ labeled by $j$. $L_i$ are the gravitational Lagrangian densities:

\begin{align*}
L_2 &= G_2(\phi, X), \\
L_3 &= -G_3(\phi, X) \Box \phi, \\
L_4 &= G_4(\phi, X) R + G_4 X \Box \phi, \\
L_5 &= G_5(\phi, X) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{G_5 X}{6} (\Box \phi)^3 + 2(\nabla^\mu \nabla^\nu \phi)^3 - 3(\nabla^\mu \nabla^\nu \phi)^2 \Box \phi.
\end{align*}

where $G_{\mu\nu}$ is the Einstein tensor, $R$ is Ricci scalar, $\phi$ is the scalar field, $X = -1/2 \nabla^\mu \phi \nabla^\nu \phi$. $\nabla^\mu$ is the covariant derivative, $\Box \phi = g^{\mu\nu} \nabla^\mu \nabla^\nu \phi, G_i(\phi, X)$ are arbitrary functions of the scalar field $\phi$ and its kinetic term $X, G_i = \frac{dG_i}{dX}$.
of the specific type of arbitrary functions determines the particular gravitational theory.

In this work we consider the matter Lagrangian density which depends on the gravitational fields according to

$$L_m = L_m(A^2(\phi)g_{\mu\nu}, \Psi_{(i)}^l)$$

(3)

where $A(\phi)$ is an arbitrary function of $\phi$. Using the conformal transformation $g_{\mu\nu} \rightarrow A^2(\phi)g_{\mu\nu}$, we move from the Einstein frame to the Jordan one, where the matter fields do not couple directly to the scalar field but the indirect coupling occurs via the metric (Fujii & Maeda 2003; Esposito-Farèse 2014; Clifton et al. 2012).

In our work we investigate only subclass of Horndeski gravity without screening mechanisms. The general action reduces to the considering model with the following set of the gravitational Lagrangian densities $L_i$:

$$L_2 = G_2(\phi, X), \quad L_3 = -G_3(\phi)\Box\phi, \quad L_4 = G_4(\phi)R, \quad L_5 = 0,$$

(4)

and the function $G_2(\phi, X)$ includes only zero and linear contributions of $X$.

Note that according to the GW170817 and GRB 170817A data $G_{X\phi} = 0$ and $G_5 = \text{constant}$ (Esquiaga & Zumalacarregui 2017; Baker et al. 2017) and the discussed subclass of Horndeski gravity is consistent with these constraints.

2.2 Matter action

Different modified gravitational models predict violations of equivalence principles which are basic for GR (Di Casola, Liberati & Sonego 2015). In scalar-tensor theories, the inertial mass and internal structure of a self-gravitating body depend on the local value of the scalar field. As a result, the laws of a self-gravitating body’s motion depend on its internal structure and the GWEVP violates (Di Casola, Liberati & Sonego 2015). Eardley (1975) first considered the interaction of two point-like masses in the scalar-tensor theory and showed that in this case the influence of the scalar field on the internal structure of the body can be expressed through the assumption that the mass of the body is an arbitrary function of the scalar field. So the matter action for a system of point-like masses can be written as

$$S_m = -c^2 \sum_a m_a(\phi) d\tau_a,$$

(5)

where $m_a(\phi)$ are inertial masses of particles labeled by $a$ and $\tau_a$ is the proper time of the particle $a$ measured along its world-line $\xi^a$. From (5) it is clear that the mass $m_a(\phi)$ is position-dependent (because $\phi$ depends upon position) and hence the GWEVP is violated. The stress-energy tensor of such action (5) and its trace take the forms

$$T^{\mu\nu} = \frac{c^4}{\sqrt{-g}} \sum_a m_a(\phi) \frac{u^\mu u^\nu}{u^0} \delta^3(r - r_a(t)),$$

$$T = g_{\mu\nu} T^{\mu\nu} = -\frac{c^3}{\sqrt{-g}} \sum_a m_a(\phi) \frac{u^\mu}{u^0} \delta^3(r - r_a(t)),$$

(6)

where $u^\mu = dx^\mu / d\tau_a$ is four-velocity of the $a$-th particle, $d\tau = \sqrt{-ds^2/c}$, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ is an interval, $u^\mu u^\nu = -c^2$, and $\delta^3(r - r_a(t))$ is the three-dimensional Dirac delta function.

3 FIELD EQUATIONS IN THE WEAK-FIELD LIMIT

The purpose of our work is to study the gravitational radiation of binary pulsars. A pulsar is a strongly magnetized neutron star. The surface gravitational potential of such object $\Phi_S = (G_NM_N)/c^2R_N = 0.2$ (where $G_N$ is Newtonian gravitational constant, $M_N$ is the mass of the neutron star, $R_N$ is its radius). Therefore, the gravitational field around neutron star is very strong. However we are intend to calculate the energy flux carried away by gravitational radiation from binary pulsars at large distances from the source (e.g. at the position of the detector). At such large distances, the value of the surface potential of the source does not have a significant influence on the metric $g_{\mu\nu}$ and the scalar field $\phi$. Therefore, we can consider the perturbed field equations in a Minkowskian background (Will 1981, 2014; Poisson & Will 2014). Hence one expands the scalar and tensor fields in the limit of small velocities ($v/c \ll 1$):

$$\phi = \phi_0 + \phi, \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

(7)

where $\eta_{\mu\nu}$ is the Minkowski background, $h_{\mu\nu}$ and $\phi$ are the small perturbations of tensor and scalar fields of order $O(v^2/c^2)$, respectively, $\phi_0$ is the asymptotic constant value of the scalar field far away from the source system (determined by the cosmological background solution). Note that, in this paper we do not consider the effect of the cosmological evolution of the scalar field. The interesting aspects of time-dependent scalar field background were investigated earlier by Babichev & Esposito-Farèse (2013); Brax, Davis & Sakstein (2014); Sakstein (2014); Galiaudon & Kopeikin (2016); Arnoux de Pirey Saint Alby & Yunes (2017). Also here we take a quasi-Minkowskian coordinate system. Such approximation of weak field limit and small velocities ($v/c \ll 1$) is the post-Newtonian (PN) expansion (Alsing et al. 2012).

Taking into account expressions (7) and the fact that we consider case of Horndeski gravity without screening, the arbitrary functions $G_i(\phi, X)$ can be expanded in Taylor’s series around the scalar asymptotic value:

$$G(\phi, X) = \sum_{m,n=0} G_{(m,n)} \phi^m X^n, \quad n \leq 1,$$

$$G_{(m,n)} = \left. \frac{1}{m!n!} \frac{\partial^{m+n} G(\phi, X)}{\partial \phi^m \partial X^n} \right|_{\phi = \phi_0, X = 0}.$$

(8)

Here $G_{(m,n)}$ are constants.

In matter action (5) the inertial mass also is an arbitrary function of the scalar field, which can be expanded in Taylor’s series around $\phi_0$ too:

$$m_a(\phi) = m_a(\phi_0) \left[ 1 + s_a^\phi \frac{\phi - \phi_0}{\phi_0} - \frac{1}{2} \left( \frac{\phi}{\phi_0} \right)^2 (s_a^2 - s_a^2 - s_a^2) + O(\phi^3) \right].$$

(9)

Further we denote $m_a(\phi_0)$ as $m_a$ which is the inertial mass at the scalar asymptotic value. The quantities $s_a$ and $s_a^\phi$ are the “first and second sensitivities”. These parameters firstly
where the terms $G_{a(0)}$ and $G_{a(1)}$ are responsible for effects of DE. Ashtekar, Bonga & Kesavan (2016) show that such effects on gravitational waves from isolated systems are insignificant, so we can neglect these terms.

### 4 POST-NEWTONIAN SOLUTIONS

Before investigating the model in the far zone and studying gravitational radiation at the point of the detector, we must solve the field equations in the near zone, where the gravitational radiation is generated.

The field equations (11) and (12) within the post-Newtonian (PN) approximation in the 1st PN order $O(v/c)^4$ take the forms (Hohmann 2015):

$$\begin{align*}
\Box \varphi &= \varphi - G_{a(0)}(\Box - \partial_\mu \partial_\mu \varphi) = 16\pi \frac{\partial T}{c^4} \varphi, \\
\Box \theta &= h_\mu \mu h^\mu + \frac{16\pi}{c^2} \Box \varphi, \\
\Box \varphi - m_\varphi^2 \varphi &= 16\pi c^2 \Box \varphi S, \\
\theta &= -h - \frac{G_{a(0)}}{G_{a(0)}} \varphi.
\end{align*}$$

(15)

The choice of the transverse gauge $\partial_\mu \theta^{\mu \nu} = 0$ reduces the field equations as follows:

$$\begin{align*}
\Box \theta_{\mu \nu} &= -\frac{16\pi}{c^4} G_{a(0)} T_{\mu \nu}, \\
\Box \varphi - m_\varphi^2 \varphi &= \frac{16\pi}{c^2} \Box \varphi S, \\
\varphi &= -\frac{G_{a(1)}}{2G_{a(0)}} \left( \frac{2G_{a(1)}}{2G_{a(1)} - 2G_{a(1)} - 3G_{a(1)}} \right). \\
S &= T - \frac{2G_{a(0)}}{G_{a(1)}} \partial T \frac{\partial T}{\partial \varphi}.
\end{align*}$$

(19)

The equation (17) is an analogue of inhomogeneous Klein-Gordon one, where the parameter $m_\varphi$ is the inverse Compton wavelength of the scalar field. In this paper we work in the CGS system, thus here and further the scalar field mass $m_\varphi$ has the dimension of inverse length [cm$^{-1}$].

The next step is obtaining the leading order of the static solution for the scalar field. Using (17), (13) and (20) it is possible to derive the following expression (Hohmann 2015):

$$\varphi = \frac{4c_\varphi^4}{G_{a(0)}} \sum m_a \left( \frac{G_{a(1)}}{G_{a(0)}} \right) e^{-m_\varphi r},$$

(21)

where $r_a = |r - a(t)|$. 

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Thus, the leading order of metric perturbation is defined as (Hohmann 2015):

\[ h_{00} = \frac{2}{c^2 G(0,0)} \sum_a m_a r_a + \frac{4c \phi}{c^2 G(0,0)} \sum_a m_a e^{-m_a r_a} \times \left( 1 - \frac{2 \delta G(0,0)}{\phi_0 G(1,0)} \right) + O \left( \frac{v}{c} \right)^4, \]

\[ h_{ij} = \delta_{ij} \frac{2}{c^2 G(0,0)} \sum_a m_a r_a + \frac{4c \phi}{c^2 G(0,0)} \sum_a m_a e^{-m_a r_a} \times \left( 1 - \frac{2 \delta G(0,0)}{\phi_0 G(1,0)} \right) + O \left( \frac{v}{c} \right)^4, \]

\[ h = \frac{4}{c^2 G(0,0)} \sum_a m_a r_a + 16c \phi G(1,0) \sum_a m_a e^{-m_a r_a} \times \left( 1 - \frac{2 \delta G(0,0)}{\phi_0 G(1,0)} \right) + O \left( \frac{v}{c} \right)^4, \]

with \( h_{00} = O(v/c)^3 \). Here \( \delta_{ij} \) is the Kronecker delta.

5 EIH EQUATIONS OF MOTION

The gravitational weak equivalence principle states (Di Casola, Liberati & Sonego 2015):

Test particles behave, in a gravitational field and in vacuum, independently of their properties.

The GWEP is one of the principles, which works in GR but can be violated in alternative theories of gravity (Di Casola, Liberati & Sonego 2015). In particular, in the scalar-tensor models the dependence of the inertial mass upon the sensitivity leads to the violation of GWEP. Sensitivity shows the changing of a compact object’s mass as it moves relatively to the additional field. Therefore, different bodies react not in the same manner to the motion relative to the ambient field. Thus, they move along different trajectories. Due to the violation of GWEP the conservative orbital dynamics of compact systems modifies. To find the explicit form of the sensitivity influence on the equations of motion, we use the method suggested by Einstein, Infeld & Hoffmann (1938).

The equations of motion for the mass \( m_a \) can be obtained from the matter Lagrangian:

\[ L_{EIH} = - \frac{2}{c^2} \sum_a m_a \left( \frac{dr_a}{dt} \right)^2 G(0,0), \]

\[ = - \sum_a m_a \left( \frac{dr_a}{dt} \right)^2 \left( -g_{00} - 2g_{00} \frac{v^2}{c^2} - g_{ij} \frac{v_i v_j}{c^2} \right) \]

\[ = - \sum_a m_a \sum_{b \neq a} \left[ 1 - \frac{v^2}{c^2} \right] \delta_{ab} G(0,0) r_{ab} + \frac{2c \phi G(1,0) m_b}{c^2 G(0,0)} \left( 1 - 2 \delta_{ab} G(0,0) \frac{G(1,0)}{\phi_0} \right) e^{-m_a r_{ab}} \]

\[ - \frac{4c \phi G(1,0) \delta_{ab}}{c^2 G(0,0)} \left( 1 - 2 \delta_{ab} G(0,0) \frac{G(1,0)}{\phi_0} \right) e^{-m_a r_{ab}} + O \left( \frac{v}{c} \right)^4, \]

where \( r_{ab} = |r_a(t) - r_b(t)| \). From the equation of motion (24) we can identify the effective gravitational "constant":

\[ G_{ab} = \left[ \frac{1}{G(0,0)} + 2c \phi G(1,0) e^{-m_a r_{ab}} \right] \left( 1 - 2 \delta_{ab} G(0,0) \frac{G(1,0)}{\phi_0} \right) \]

\[ - \frac{4c \phi G(1,0) \delta_{ab}}{c^2 G(0,0)} \left( 1 - 2 \delta_{ab} G(0,0) \frac{G(1,0)}{\phi_0} \right) e^{-m_a r_{ab}}. \]

This result is symmetric under interchange of all particle pairs (Hou & Gong 2018).

The corresponding n-body equations of motion up to Newtonian order are defined as follows

\[ a_a = - \sum \frac{\mathcal{G}_{ab} m_b}{r_{ab}^2} \hat{r}_{ab}, \]

with

\[ \mathcal{G}_{ab} = \frac{1}{G(0,0)} \left[ 1 + (1 + m_a r_{ab}) e^{-m_a r_{ab}} \right] \frac{2c \phi G(1,0)}{G(0,0)} \left( 1 - 2 \delta_{ab} G(0,0) \frac{G(1,0)}{\phi_0} \right) \]

\[ \times \left( 1 - \frac{2 \delta G(0,0)}{\phi_0 G(1,0)} \right) - \frac{4c \phi G(1,0)}{c^2 G(0,0)} \left( 1 - 2 \delta_{ab} G(0,0) \frac{G(1,0)}{\phi_0} \right) \right], \]

where \( a_a \) is the acceleration of the \( a \)-th object, \( \hat{r}_{ab} \) is the unit direction vector from the \( b \)-th object to the \( a \)-th one.

The scalar field mass is responsible for DE effect. Therefore, the manifestations of influence of this effect start from the distances much larger than the distance between components in binary pulsars. Thus we use the approximation \( m_a r_{ab} \ll 1 \) and \( e^{-m_a r_{ab}} \rightarrow 1 \). In this case, the effective gravitational constant between components in binary pulsars takes the following form:

\[ \mathcal{G}_{ab} = \frac{1}{G(0,0)} \left[ 1 + 2c \phi G(1,0) \left( 1 - 2 \delta_{ab} G(0,0) \frac{G(1,0)}{\phi_0} \right) \right. \]

\[ - \frac{4c \phi G(1,0)}{c^2 G(0,0)} \left( 1 - 2 \delta_{ab} G(0,0) \frac{G(1,0)}{\phi_0} \right) \right). \]

Now let us consider an orbital dynamics of a binary system with compact objects. A motion in binary system obeys the Kepler’s third law:

\[ a^3 (2\pi/P_b)^2 = \mathcal{G}_{12} m \]

and the orbital binding energy of such system is

\[ E = -\frac{\mathcal{G}_{12} m \mu}{2a}. \]
m = m_1 + m_2, and μ = m_1m_2/m, \( P_\text{b} \) is the orbital period of a binary system.

The most significant dissipative effect is the orbital period decay due to the emission of gravitational radiation. The energy loss can be expressed via the first derivative of the orbital period using equations (29) and (30):

\[
\dot{E} \over E = - \frac{2}{3} \frac{P_\text{b}}{P_\text{b}}
\]

Thus, we find that the orbital decay of the binary pulsars is directly determined by the energy loss of the system.

6 GRAVITATIONAL RADIATION FROM BINARY PULSARS

In this section, we focus on the dissipative effects, calculate the rate of the energy loss due to the emission of gravitational radiations (including monopole, dipole, quadrupole, and dipole-octupole radiations), and derive their contributions to the change of the orbital period. Nevertheless, before we move to direct computations, we need to obtain the stress-energy pseudotensor.

6.1 Effective stress-energy pseudotensor

Far away from the local system the stress-energy tensor of the source \( T_{\mu\nu} \) vanishes. However, the influence of the local system on the flat space-time remains in the form of gravitational radiation. This radiation has energy and momentum and is described by effective stress-energy pseudotensor.

There are many different methods to define the energy-momentum pseudotensor (Petrov 2008; Saffer, Yunes & Yagi 2018). Authors of the last work investigate four ones: the second variation of the action under short-wavelength averaging, the second perturbation of the field equations in the short-wavelength approximation, the construction of an energy complex leading to a Landau-Lifshitz tensor, and the using of Noether’s theorem in field theories about a flat background. All these ways lead to different results but yield the same rate of energy loss. We apply the Noether current method. It is suitable for our purposes and the fact that the method yields a not symmetric stress-energy pseudotensor is not significant. However, the discussed approach has the serious problems with definition of angular momentum of an isolated gravitating system. Fortunately, in our work, the angular momentum is not required and chosen method provides the correct result.

First, we consider a general action:

\[
S = \int d^4x \sqrt{-g} \left[ L_\text{g}(g_{\mu\nu}, \partial_\mu g_{\nu\alpha}, \partial_\mu \partial_\nu g_{\alpha\beta}, \partial_\mu \phi, \partial_\mu \partial_\nu \phi) + L_\text{m}(q, \partial_\mu q, \partial_\mu \partial_\nu q, \partial_\mu \phi, \partial_\mu \partial_\nu \phi) \right]
\]

where \( L_\text{g} \) is a gravitational Lagrangian density, \( L_\text{m} \) is a matter Lagrangian density, \( q \) is the matter fields.

Far away from the local system the conservation laws take the following form:

\[
\partial_\mu (\sqrt{-g}(T^\mu_\nu + \rho \gamma)) = 0.
\]

The part with \( L_\text{m} \) in (32) gives the canonical matter stress-energy tensor which is equivalent to Hilbert stress-energy tensor \( T_{\mu\nu} \). The remaining quantity \( \rho \frac{\partial}{\partial \phi} \) is our sought-for pseudotensor. The Noether current method defines a pseudotensor as:

\[
T_\gamma^\mu = - \partial_\nu \left( \frac{\partial L_\text{g}}{\partial (\partial_\mu \partial_\nu g_{\alpha\beta})} \right) \partial_\alpha \partial_\beta g_{\mu\nu} + \partial_\mu \left( \frac{\partial L_\text{g}}{\partial (\partial_\mu \partial_\nu g_{\alpha\beta})} \right) \partial_\alpha \partial_\beta g_{\nu\mu}
\]

\[
- \partial_\nu \left( \frac{\partial L_\text{g}}{\partial (\partial_\mu \partial_\nu g_{\alpha\beta})} \right) \partial_\alpha \partial_\beta g_{\mu\nu} - \partial_\mu \left( \frac{\partial L_\text{g}}{\partial (\partial_\mu \partial_\nu g_{\alpha\beta})} \right) \partial_\alpha \partial_\beta g_{\nu\mu}
\]

\[
+ \partial_\nu \left( \partial_\mu \partial_\nu \phi \right) \partial_\alpha \partial_\beta \phi + R_{\mu\nu} \left( \frac{\partial L_\text{g}}{\partial (\partial_\mu \partial_\nu \phi)} \right) \partial_\alpha \partial_\beta \phi + \delta_\gamma^\alpha L_\text{m}.
\]

Now we return to the considering subclass of Horndeski theory. The gravitational part of Lagrangian density is given by the expressions (4). Using the expansions (7) the Lagrangian densities are reduced to

\[
L_2 = \frac{c^4}{16\pi} \left( G_{2(2.0)} \phi^2 - \frac{1}{2} G_{2(0.1)} \partial_\mu \phi \partial^\mu \phi \right),
\]

\[
L_3 = \frac{c^4}{16\pi} G_{3(1.0)} \partial_\mu \phi \partial^\mu \phi,
\]

\[
L_4 = \frac{c^4}{16\pi} \left[ G_{4(0.0)} \left( 4 \partial_\mu \partial_\nu h_{\mu\nu} - 4 \Box h - 8 h^{\mu\nu} \partial_\mu \partial_\nu h^\alpha_{\alpha} 
\right.
\]

\[
+ 4 h^{\mu\nu} \Box h_{\mu\nu} - 4 \partial_\mu h^{\alpha\mu} \partial_\nu h^\alpha_{\mu} + 4 \partial_\mu h_{\mu\nu} \partial_\nu h + 4 h^{\mu\nu} \partial_\mu \partial_\nu h + 3 \partial_\mu h^{\alpha\mu} \partial_\nu h_{\mu\nu} - \partial_\mu h \partial^\mu h - 2 \partial_\mu h^{\alpha\mu} \partial_\nu h^\alpha_{\nu} 
\]

\[
+ G_{4(1.0)} \phi \left( \partial_\mu h^{\mu\nu} \partial_\nu h - \Box h \right) \right],
\]

\[
L_5 = 0.
\]

According to the four-dimensional analogue of the Ostrogradskii-Gauss theorem we can throw out the total derivatives from the action (1). The remaining part is

\[
\sqrt{-g} L_2 = \frac{c^4}{16\pi} \left( G_{2(2.0)} \phi^2 - \frac{1}{2} G_{2(0.1)} \partial_\mu \phi \partial^\mu \phi \right),
\]

\[
\sqrt{-g} L_3 = \frac{c^4}{16\pi} G_{3(1.0)} \partial_\mu \phi \partial^\mu \phi,
\]

\[
\sqrt{-g} L_4 = \frac{c^4}{16\pi} \left[ G_{4(0.0)} \left( 2 \partial_\mu h^{\alpha\mu} \partial_\nu h^\alpha_{\mu} - 2 \partial_\mu h^{\mu\nu} \partial_\nu h - \partial_\mu h^{\alpha\mu} \partial_\nu h_{\mu\nu} + \partial_\mu h \partial^\mu h 
\right.
\]

\[
- G_{4(1.0)} \phi \left( \partial_\mu h^{\mu\nu} \partial_\nu h - \partial_\mu h \right) \right],
\]

\[
\sqrt{-g} L_5 = 0.
\]

Now we express equation (34) in respect the equa-
In the last step the expression for matter is absent and scalar and tensor modes. In the wave zone (far zone) the tensor theories, gravitational radiation comes from both scalar and tensor modes. The angular brackets represent a time average over the interval of stress-energy pseudotensor. The component of stress-energy pseudotensor is expressed in terms of tensor and fluxes) of the gravitational waves are expressed in terms of tensor and scalar energy fluxes. Gravitational waves carry energy and bend the space-time.

6.2 Tensor and scalar energy fluxes

Gravitational waves carry energy and bend the space-time. Different momentum and energy characteristics (densities and fluxes) of the gravitational waves are expressed in terms of stress-energy pseudotensor. The component of stress-energy pseudotensor is expressed in terms of tensor and scalar energy fluxes. Gravitational waves carry energy and bend the space-time.

6.2.1 Tensor energy flux

According to (39) the average energy flux radiated in gravitational waves due to tensor part is

\[
\langle E_\gamma \rangle = -c^2 \int d\Omega \langle \theta_{\mu \nu}^{TT} \rangle,
\]

where the angular brackets represent a time average over a period of the system's motion, \( \Omega \) is the solid angle, TT means the transverse-traceless gauge.

In GR, the energy flux appears only due to the propagation of tensor mode, but in the standard massive scalar-tensor theories, gravitational radiation comes from both scalar and tensor modes. In the wave zone (far zone) the matter is absent and \( T^{\mu \nu}_{TT} = 0 \), so the conservation law is

\[
\partial_\mu T^{\mu \nu}_{TT} = 0.
\]

Since there are no mixed components inside the pseudotensor \( T^{\mu \nu}_{TT} \) (\( \theta_{\mu \nu} \) and \( \phi \) are decoupled). The energy-momentum pseudotensors (i.e., Noether currents) of the tensor

\[
\left. \left( \theta^T_{\mu \nu} \right) \right|_{TT} = c^4 \frac{G_4(0,0)}{16\pi} \frac{\partial_\mu \theta^T_{\nu \rho}}{\partial_\nu \theta^T_{\rho \sigma}} \frac{\partial_\sigma \theta^T_{\chi \chi}}{\partial_\chi \theta^T_{\chi \chi}}
\]

and scalar gravitational waves

\[
\left. \left( \varphi \right) \right|_{TT} = c^4 \frac{G_4(0,0)}{2G_0(0,0)\epsilon_\varphi} \frac{\partial_\mu \theta_{\nu \rho}}{\partial_\nu \theta_{\rho \sigma}} \frac{\partial_\sigma \theta_{\chi \chi}}{\partial_\chi \theta_{\chi \chi}}
\]

are respectively conserved and we can investigate them separately.

Now we return to the equation (42). The formal solution is

\[
\theta_{\mu \nu} = \frac{4}{c^4 G_4(0,0) \epsilon_\varphi} \int \frac{d^3 r}{\rho} T^{\mu \nu}(t - |r - r'|/c, |r - r'|) \times
\]

where, source point \( r' \) belongs to the near zone \( N \), whereas the field point \( r \) is located in the far zone (wave zone), such that \(|r'| < |r|\). Taking into account this condition, we can expand the integrand in powers of \((n \times r')\) in the slow-motion approximation

\[
\theta_{\mu \nu} = \frac{4}{c^4 G_4(0,0) \epsilon_\varphi} \sum_{l=0}^{\infty} \int d^3 r \frac{1}{r} T^{\mu \nu}(t - |r - r'|/c, |r - r'|) \times
\]
tensor gravitational radiation because the tensor graviton is a massless spin-2 particle.

The quantity $T_{00}$ is the energy density. The leading PN order contribution from $T_{00}$ is

$$T_{00} = \sum n_c c^2 \delta^3(r - r_c(t)).$$  

(47)

Substituting this expression into equation (46) we obtain

$$\theta_{ij} = \frac{2}{rc^5 G_{4(0,0)}} \frac{d^2}{dr^2} M_{ij} \bigg|_{r(t)} ,$$  

(48)

where

$$M_{ij} = \sum \sigma^2(\theta) r_0^2(t) r_0^2(t)$$  

(49)

is the mass quadrupole moment. The subscript "ret" means that the quantity $M_{ij}$ is evaluated at the retarded time $t - r/c$.

Now it is possible to find the average energy flux of tensor sector in terms of the mass quadrupole moments:

$$\langle E_\theta \rangle = -\frac{1}{5c^3 G_{4(0,0)}} \left\langle \frac{\partial}{\partial t} M_{ij} - \frac{1}{3} \left( M_{ij} \right)^2 \right\rangle .$$  

(50)

For the integration over the solid angle we use the fact that $\theta_{ij}^{\theta \theta} = \Lambda_{ij,k} \theta^k$. Here the projector $\Lambda_{ij,k}$ is the Lambda tensor (see Alting et al. (2012)). The overdots represent derivatives with respect to coordinate time.

In our work, we consider only binary systems with quasi-circular orbits which can be parameterized by

$$x_1(t) = -R_1 \cos(\omega t), \quad y_1(t) = -R_1 \sin(\omega t), \quad z_1 = 0;$$

$$x_2(t) = R_2 \cos(\omega t), \quad y_2(t) = R_2 \sin(\omega t), \quad z_2 = 0,$$  

(51)

here by $R_k$ we denote the orbital radii of the binary system components, and $\omega$ is the orbital frequency. Using the Kepler’s third law (29) we find the final form of the average energy flux radiated in gravitational waves due to tensor sector:

$$\langle E_\theta \rangle = -\frac{32 \mu^2 (\Delta z)^3}{5c^3 G_{4(0,0)} R^3 \theta} - \frac{32 \mu^2 \mathbf{m}^3}{5c^3 G_{4(0,0)} R^3 \theta} \times \left\{ 1 + \frac{2c G_{4(0,0)}}{G_{(4,0)}} \left( 1 - \frac{G_{4(0,0)}}{G_{4(0,0)}} \right) \right\} ,$$  

(52)

where $R = R_1 + R_2$ and in the case of quasi-circular orbit in equation (29) $R = a$. The quantity $\theta_{ij}$ is the effective gravitational constant between components of binary system (28).

6.2.2 Scalar energy flux

Now we obtain the change of the binding energy due to the scalar radiation. Our consideration starts from a formal solution of equation (17). The obtaining of such type solution by using Green’s function method is described in detail in papers of Morse & Feshbach (1953); Alting et al. (2012); Zhang, Liu & Zhao (2017). We fully follow them and present the final expression for the formal solution

$$\varphi = -\frac{4 \pi^2}{rc^5} \int_0^\infty dJ_1(z) \sum_{l=1}^\infty \frac{1}{c^2 l^2} \frac{d^l}{dN} \int d^3 r' \left( n \times r' \right)^l \times \left[ S(t - r/c, r') - S(t - ru(r,z)/c, r') \right] .$$  

(53)

where $J_1$ is the Bessel function of the first kind, $S(t, r)$ is the source function from (20), $z = m_{\phi} \sqrt{c^2 (t - r/c)^2 - |r - r'|^2}$, and $u(r, z) = \sqrt{1 + (z/m_{\phi})^2}$. Here the integration region $N$ is taken over the near zone, and $|r'| \ll |r|$. Substituting the source term $S(t, r)$ in the explicit form (20) to (53) and performing the integration over $r'$ we find $\varphi$ in terms of the scalar multipole moments $\mathcal{M}_l^k$:

$$\varphi = \frac{4 \pi^2}{rc^5} \int_0^\infty dJ_1(z) \sum_{l=1}^\infty \frac{1}{c^2 l^2} \frac{d^l}{dN} \mathcal{M}_l^k .$$  

(54)

where

$$\mathcal{M}_l^k = \mathcal{M}_l^{k+1}(t, r, z) = \sum_{a} \left( M_a(t - r/c) \mathcal{L}_l^a(t - r/c) - u^{-l+1}(r,z) M_a(t - ru(r,z)/c) \mathcal{L}_l^a(t - ru(r,z)/c) \right) .$$  

(55)

and

$$M_a(t) = m_a \left[ 1 - \frac{2G_{4(0,0)}}{G_{4(1,0)}} \frac{s_0}{\phi_0} \frac{r_0^2}{c^2} \left( 1 - \frac{2G_{4(0,0)}}{G_{4(1,0)}} \frac{s_0}{\phi_0} \right) \right] - 3 \sum_{b \neq a} \frac{m_b}{c^2 G_{4(0,0)}} \left[ 1 - \frac{2G_{4(0,0)}}{G_{4(1,0)}} \frac{s_0}{\phi_0} \right] \left( \sum_{b \neq a} \frac{mb}{c^2 G_{4(0,0)}} \sum_{b \neq a} \frac{mb}{c^2 G_{4(0,0)}} \right) \left( 1 - \frac{2G_{4(0,0)}}{G_{4(1,0)}} \frac{s_0}{\phi_0} \right) \times \left( 8 \pi \phi_{d} \frac{s_0}{\phi_0} \frac{G_{4(1,0)}}{G_{4(0,0)}} \left( s_0^2 - s_0 + \phi_{d} \right) \right).$$  

(56)

Here $n_a = n_1, n_1, ..., n_{l+1} = r_0^l(t)/r_0^l(t)...r_0^l(t)$.

The average rate of the binding energy change due to the scalar radiation is given by

$$\langle E_\varphi \rangle = -\frac{c^2 r}{rc^5} \int d\Omega \left( \langle \partial_\varphi \rangle \right) = \frac{c^2 r}{16 \pi} \int d\Omega \left( \frac{G_{4(1,0)}}{G_{4(0,0)} c^2} \partial_\varphi \partial_s \varphi \right) .$$  

(57)

In the case of the scalar energy flux, we cannot change derivatives from spatial to temporal as in the tensor case because $\varphi$ is not a function of the argument $(t - r/c)$. Dependence upon $r$ is more complicated. This occurs due to the presence of the scalar field mass in equation (17). So in the explicit form the temporal and spatial derivatives of $\varphi$ are

$$\partial_s \varphi = -\frac{4 \pi^2}{rc^5} \int_0^\infty dJ_1(z) \sum_{l=1}^\infty \frac{1}{c^2 l^2} \frac{d^l}{dN} \mathcal{M}_l^k \left( t - r/c \right),$$  

(58)

$$\partial_s \varphi = -\frac{4 \pi^2}{rc^5} \int_0^\infty dJ_1(z) \sum_{l=1}^\infty \frac{1}{c^2 l^2} \frac{d^l}{dN} \mathcal{M}_l^k \left( t - r/c \right) .$$  

(59)

where a scalar multipole moments $\mathcal{M}_l^k$ are defined as

$$\mathcal{M}_l^k = \mathcal{M}_l^{k+1}(t, r, z) = \sum_{a} \left( M_a(t - r/c) \mathcal{L}_l^a(t - r/c) - u^{-l+1}(r,z) M_a(t - ru(r,z)/c) \mathcal{L}_l^a(t - ru(r,z)/c) \right) .$$  

(60)

There are monopole, dipole, and dipole-octupole radiations in the scalar sector, besides quadrupole one, and the
average rate of the binding energy change due to scalar radiation in terms of scalar multipole moments takes the form

\[
\langle \dot{E}_\Phi \rangle = - \frac{2c^2 G_{4(1.0)} \epsilon_\Phi}{G_{4(1.0)}} \int dz_1 dz_2 J_1(z_1) J_2(z_2) \left( \frac{1}{2} \hat{\mathcal{M}}_0 \hat{\mathcal{M}}_1 \right) \\
+ \frac{1}{6k^4} \left( \hat{\mathcal{M}}_1 \hat{\mathcal{M}}_2 + \hat{\mathcal{M}}_0 \hat{\mathcal{M}}_3 + \hat{\mathcal{M}}_1 \hat{\mathcal{M}}_3 \right) \\
+ \frac{1}{60k^6} \left( \hat{\mathcal{M}}_2 \hat{\mathcal{M}}_3 + \hat{\mathcal{M}}_2 \hat{\mathcal{M}}_3 \right) \\
+ \frac{1}{30k^8} \left( \hat{\mathcal{M}}_3 \hat{\mathcal{M}}_3 + \hat{\mathcal{M}}_3 \hat{\mathcal{M}}_3 \right),
\]

(61)

where we have used the identity

\[
\int \frac{d^3q}{(2\pi)^3} \delta_k \delta_{k_1} \delta_{k_2} \ldots \delta_{k_n} = \begin{cases} 0, & \text{for } k = \text{odd}, \\
\delta_{k_1} \delta_{k_2} \ldots \delta_{k_n} \frac{1}{(k+1)!}, & \text{for } k = \text{even},
\end{cases}
\]

the final dots denote all possible pairing of indices.

Now we obtain the time derivatives of monopole, dipole, quadrupole, and octupole scalar moments for the quasicircular orbit (51):

1. Monopole.
\[
\mathcal{M}_0 = \mathcal{M}_1 = 0.
\]

(62)

2. Dipole.
\[
\hat{\mathcal{M}}_1^k = \mu R \omega^3 \left( A_d + \tilde{A}_d \frac{\mu}{c^2 R} \right) \left[ \cos(\omega(t-r/c)) - u^{-2} \cos(\omega(t-r_u/c)), \right. \\
\sin(\omega(t-r/c)) - u^{-2} \sin(\omega(t-r_u/c)), \left. 0 \right],
\]

\[
\hat{\mathcal{M}}_2^k = \mu R \omega^3 \left( A_d + \tilde{A}_d \frac{\mu}{c^2 R} \right) \left[ \cos(\omega(t-r/c)) - u^{-3} \cos(\omega(t-r_u/c)), \right. \\
\sin(\omega(t-r/c)) - u^{-3} \sin(\omega(t-r_u/c)), \left. 0 \right],
\]

(63)

3. Quadrupole.
\[
\hat{\mathcal{M}}_3^k = \begin{pmatrix} \hat{\mathcal{M}}_3^{11} & \hat{\mathcal{M}}_3^{12} \\\n\hat{\mathcal{M}}_3^{12} & \hat{\mathcal{M}}_3^{11} \end{pmatrix} - \hat{\mathcal{M}}_3^{00}, \\
\hat{\mathcal{M}}_3^{11} = 4A_d \mu^4 R^3 \sin(2\omega(t-r/c)) - u^{-3} \sin(2\omega(t-r_u/c)), \\
\hat{\mathcal{M}}_3^{12} = -4A_d \mu^4 R^3 \cos(2\omega(t-r/c)) - u^{-3} \cos(2\omega(t-r_u/c)), \\
\hat{\mathcal{M}}_3^{00} = 4A_d \mu^4 R^3 \sin(2\omega(t-r/c)) - u^{-4} \sin(2\omega(t-r_u/c)), \\
\hat{\mathcal{M}}_3^{00} = -4A_d \mu^4 R^3 \cos(2\omega(t-r/c)) - u^{-4} \cos(2\omega(t-r_u/c)), \\
\]

(64)

where

\[
\hat{\mathcal{M}}_3^{11} = 4A_d \mu^4 R^3 \sin(2\omega(t-r/c)) - u^{-3} \sin(2\omega(t-r_u/c)), \\
\hat{\mathcal{M}}_3^{12} = -4A_d \mu^4 R^3 \cos(2\omega(t-r/c)) - u^{-3} \cos(2\omega(t-r_u/c)), \\
\hat{\mathcal{M}}_3^{00} = 4A_d \mu^4 R^3 \sin(2\omega(t-r/c)) - u^{-4} \sin(2\omega(t-r_u/c)), \\
\hat{\mathcal{M}}_3^{00} = -4A_d \mu^4 R^3 \cos(2\omega(t-r/c)) - u^{-4} \cos(2\omega(t-r_u/c)).
\]

3. Octupole.
\[
\hat{\mathcal{M}}_4^{kk} = A_0 \mu^4 R^4 \sin(\omega(t-r/c)) - u^{-4} \sin(\omega(t-r_u/c)), \\
\hat{\mathcal{M}}_4^{kk} = A_0 \mu^4 R^4 \cos(\omega(t-r/c)) - u^{-4} \cos(\omega(t-r_u/c)), \\
\hat{\mathcal{M}}_4^{kk} = A_0 \mu^4 R^4 \sin(\omega(t-r/c)) - u^{-5} \sin(\omega(t-r_u/c)), \\
\hat{\mathcal{M}}_4^{kk} = A_0 \mu^4 R^4 \cos(\omega(t-r/c)) - u^{-5} \cos(\omega(t-r_u/c)).
\]

(65)

We use the following definitions:

\[
A_d = \frac{2G_{4(1.0)} (s_2 - s_1)}{G_{4(1.0)} \Phi_0}, \\
A_q = 1 - \frac{2G_{4(1.0)} (s_2 m_1 + s_1 m_2)}{G_{4(1.0)} \Phi_0 m}, \\
A_o = \frac{m_1 - m_2}{m} - \frac{2G_{4(1.0)} (s_2^2 m_1^2 - s_1 m_2^2)}{G_{4(1.0)} \Phi_0 m^2}
\]

and

\[
\hat{\lambda}_d = - \frac{7}{2G_{4(0)} \Phi_0} \left( m_2^2 - m_2 m_1 - m_2 m_1 \right) + \frac{7}{G_{4(1.0)} \Phi_0} \left( m_2 m_1 - m_2 m_1 \right) \\
+ \frac{6}{G_{4(1.0)} \Phi_0} (s_1 - s_2) + \frac{23}{4} \epsilon_\Phi G_{4(1.0)} \left( m_2^2 - m_1 m_1 \right) \\
+ 15 \epsilon_\Phi \left( m_2 m_1 + m_2 m_1 \right) + 12 \epsilon_\Phi \left( m_1 s_1 - m_2 s_2 \right) \\
+ 14 G_{4(1.0)} \Phi_0 \left( s_1^2 m_2 - s_2^2 m_1 \right) \left( m_2 - m_1 \right) \\
+ 4 \epsilon_\Phi \left( s_1 - s_2 \right) + 18 G_{4(1.0)} \Phi_0 \epsilon_\Phi \left( s_2^2 m_2 + s_2^2 m_1 \right) \\
+ 16 G_{4(1.0)} \Phi_0 \epsilon_\Phi \left( s_1^2 m_2 + s_2^2 \right) \\
+ 8 G_{4(1.0)} \Phi_0 \epsilon_\Phi \left( m_2^2 m_1 + m_1^2 m_1 \right) \\
+ 8 G_{4(1.0)} \Phi_0 \epsilon_\Phi \left( s_1^2 m_2 + s_2^2 m_1 \right) \\
- 16 G_{4(1.0)} \Phi_0 \epsilon_\Phi \left( s_1^2 m_2 + s_2^2 m_1 \right) \\
- 16 G_{4(1.0)} \Phi_0 \epsilon_\Phi \left( s_1^2 m_2 + s_2^2 m_1 \right).
\]

(66)

We can divide the scalar sector on the dipole, quadrupole and dipole-octupole components:

\[
\langle \dot{E}_\Phi \rangle = \left( \dot{E}_D \right) + \left( \dot{E}_Q \right) + \left( \dot{E}_D \dot{E}_O \right),
\]

(67)

where the scalar dipole part is

\[
\dot{E}_D = - \frac{2G_{4(1.0)} \epsilon_\Phi}{3c^2 G_{4(0)}^2} \int dz_1 dz_2 J_1(z_1) J_2(z_2) \left( A_d^2 + A_d \tilde{A}_d \frac{\mu}{c^2 R} \right) \left[ \cos(\omega r/c) \cos(\omega r_u/c) \right. \\
\left. - \sin(\omega r/c) \sin(\omega r_u/c) \right].
\]

(68)
the scalar quadrupole part is

\[
\langle E_\varphi^{(Q)} \rangle = - \frac{G(4,0)c_\varphi}{15c^2G(4,0)} \int dz_1 dz_2 J_1(z_1) J_2(z_2) \left\langle \frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial t} \right\rangle \\
= - \frac{32G(4,0)c_\varphi^2 \mu^3 m^3}{15c^6G(4,0)} R^3 \\
x \left[ 1 - \cos(2\omega r/c) / \cos(2\omega r/c) \right] \\
- \sin(2\omega r/c) / \sin(2\omega r/c) - \cos(2\omega r/c) \\
- \left( \cos(2\omega r/c) \right)^3 / \sin(2\omega r/c) - \sin(2\omega r/c) \\
- \left( \sin(2\omega r/c) \right)^3 / \sin(2\omega r/c) \right], \tag{69}
\]

and the scalar dipole-octupole part is

\[
\langle E_\varphi^{(DO)} \rangle = - \frac{G(4,0)c_\varphi}{15c^2G(4,0)} \int dz_1 dz_2 J_1(z_1) J_2(z_2) \\
x \left( \frac{A^0_1, A^0_2 + A^0_4, A^0_5} \right) \\
= \frac{G(4,0)c_\varphi^2 \mu^3 m^3}{15c^6G(4,0)} R^3 A_1 A_0 \\
x \left[ 2 - \cos(\omega r/c) / \left( \cos(\omega r/c) \right)^2 + \cos(\omega r/c) \right] \\
+ \cos(\omega r/c)^4 / \sin(\omega r/c) \\
+ \left( \sin(\omega r/c)^2 / \sin(\omega r/c)^2 \right)^3 + \sin(\omega r/c)^4 \\
+ \left( \cos(\omega r/c)^3 / \cos(\omega r/c)^2 \right)^3 + \cos(\omega r/c)^5 \\
+ \left( \sin(\omega r/c)^2 / \sin(\omega r/c)^2 \right)^3 + \sin(\omega r/c)^4 \\
+ \left( \sin(\omega r/c)^2 / \sin(\omega r/c)^2 \right)^3 + \sin(\omega r/c)^4 \right]. \tag{70}
\]

Here we have used the Kepler’s third law (29) and introduced the following notation for integrals:

\[
\int_0^{\infty} \sin \left( \frac{\omega r}{c} \right) \sqrt{1 + \left( \frac{z}{m_\phi R} \right)^2} - \frac{J_1(z) dz}{1 + \left( \frac{z}{m_\phi R} \right)^2} \tag{71}
\]

\[
\int_0^{\infty} \cos \left( \frac{\omega r}{c} \right) \sqrt{1 + \left( \frac{z}{m_\phi R} \right)^2} - \frac{J_1(z) dz}{1 + \left( \frac{z}{m_\phi R} \right)^2} \tag{72}
\]

Let us obtain the total power of scalar radiation and perform these integrals in the limit \( r \to \infty \). The detailed calculations can be found in the papers of Alsing et al. (2012); Zhang, Liu & Zhao (2017). We present only the final result as follows:

\[
\lim_{r \to \infty} \sin \left( \frac{\omega r}{c} \right) \right] = \begin{cases} 
\sin \left( \frac{\omega r}{c} \right) - \left( \frac{\nu(\omega)}{c} \right)^{-1} e^{-i \omega r c}, & \text{for } \omega < c m_\phi, \\
\sin \left( \frac{\omega r}{c} \right) - \left( \frac{\nu(\omega)}{c} \right)^{-1} e^{-i \omega r c}, & \text{for } \omega > c m_\phi, 
\end{cases} \tag{73}
\]

and

\[
\lim_{r \to \infty} \cos \left( \frac{\omega r}{c} \right) \right] = \begin{cases} 
\cos \left( \frac{\omega r}{c} \right) - \left( \frac{\nu(\omega)}{c} \right)^{-1} \cos(\omega r c), & \text{for } \omega > c m_\phi, \\
\cos \left( \frac{\omega r}{c} \right) - \left( \frac{\nu(\omega)}{c} \right)^{-1} \cos(\omega r c), & \text{for } \omega < c m_\phi, 
\end{cases} \tag{74}
\]

where \( \nu(\omega) = c \sqrt{1 - \frac{m_\phi^2 c^2}{\omega^2}} \) is the propagation speed of the scalar gravitational radiation.

Finally, we obtain the expression for the dipole

\[
\langle E_\varphi^{(D)} \rangle = - \frac{2G(4,0)c_\varphi^2 \mu^3 m^3}{15c^6G(4,0)} \left( \frac{\nu(\omega)}{c} \right)^5 \Theta(\omega - c m_\phi), 
\tag{75}
\]

the quadrupole

\[
\langle E_\varphi^{(Q)} \rangle = - \frac{32G(4,0)c_\varphi^2 \mu^3 m^3}{15c^6G(4,0)} \left( \frac{\nu(\omega)}{c} \right)^5 \Theta(\omega - c m_\phi), 
\tag{76}
\]

and the dipole-octupole

\[
\langle E_\varphi^{(DO)} \rangle = \frac{G(4,0)c_\varphi^2 \mu^3 m^3}{15c^6G(4,0)} \left( \frac{\nu(\omega)}{c} \right)^5 \Theta(\omega - c m_\phi). 
\tag{77}
\]

parts of the scalar radiation power. Here \( \Theta(\omega - c m_\phi) \) is the Heaviside function.

It is important to emphasize that in our work we take into account PN corrections to the dipole term and dipole-octupole term which are absent if we neglect two terms of order \( \cal O(1/c^4) \) during considering the expression for \( \Phi \). Thus, our consideration of the scalar radiation power is complete.

The total power of the scalar radiation is

\[
\langle E_\varphi \rangle = \langle E_\varphi^{(Q)} \rangle + \langle E_\varphi^{(D)} \rangle + \langle E_\varphi^{(DO)} \rangle \\
- \frac{32G(4,0)c_\varphi^2 \mu^3 m^3}{15c^6G(4,0)} \left[ 1 + \frac{G(4,0)c_\varphi R}{485760} \left( \frac{\nu(\omega)}{c} \right)^5 \Theta(\omega - c m_\phi) \right. \\
+ \left. \frac{G(4,0)c_\varphi^2 \mu^3 m^3}{96} \left( \frac{\nu(\omega)}{c} \right)^5 \Theta(\omega - c m_\phi) \right]. 
\tag{78}
\]

According to the equation (78) in contrast to all other contributions, the dipole-octupole term has the opposite sign and describes the negative modification of the energy flux at the same PN order as the quadrupole radiation contribution.

Using equations (30) and (31) we find the final form of
the orbital decay rate including the tensor and scalar parts:

\[
\frac{\dot{P}_b^{eh}}{P_b} = -\frac{9\mu(m_1m_2)^2}{5c^3G_*(0,0)R^2} \left[ 1 + \frac{5G(4,1,0)\mu}{48G_*(0,0)R} \right] \times \left( \frac{2\mu}{c^2} + \frac{2\mu A_d A_4}{c^2} \right) \left( \frac{\Theta(\omega - \omega_m)}{c} \right)^3 \Theta(\omega - \omega_m) + \frac{G_*(4,1,0)\mu}{3} A_d \left( \frac{\Theta(2\omega - \omega_m)}{c} \right)^5 \Theta(2\omega - \omega_m) - \frac{G_*(4,1,0)\mu}{96} A_d A_0 \left( \frac{\Theta(\omega - \omega_m)}{c} \right)^5 \Theta(\omega - \omega_m). \tag{79}
\]

where index "eh" denotes expression obtained in the framework of considering subclass of the Horndeski gravity. According to the equation (79) the main contribution to scalar radiation is produced by scalar dipole term. Also from expressions (65) and (66) one can see that the contribution of the scalar dipole radiation depends upon the difference (s1 − s2). Thus, the scalar dipole radiation is the most noticeably in mixed binaries where this difference in sensitivities reaches the maximum values.

Here and further we take into account that \( m_0 < \omega/c \), thus \( \Theta(\omega - \omega_m) = 1 \). Using the Kepler’s third law (29) (in the quasi-circular orbit \( \omega = R \)) and explicit form of the scalar propagation speed \( \nu(\omega) \), equation (79) can be rewritten as

\[
\frac{\dot{P}_b^{eh}}{P_b^{GR}} = \frac{9\mu^2}{G^2 G(4,0,0)} \left\{ 1 + \frac{5G(4,1,0)\mu}{48c^2}\right\} \times \left[ \frac{2\mu}{c^2} + \frac{2\mu A_d A_4}{c^2} \right] \left( \frac{\Theta(\omega - \omega_m)}{c} \right)^3 \Theta(\omega - \omega_m) + \frac{G_*(4,1,0)\mu}{3} A_d \left( \frac{\Theta(2\omega - \omega_m)}{c} \right)^5 \Theta(2\omega - \omega_m) - \frac{G_*(4,1,0)\mu}{96} A_d A_0 \left( \frac{\Theta(\omega - \omega_m)}{c} \right)^5 \Theta(\omega - \omega_m). \tag{80}
\]

where \( P_b^{GR} \) is the value of orbital decay predicted by GR:

\[
\frac{\dot{P}_b^{GR}}{P_b^{GR}} = -\frac{192\pi^2}{5c^3 m} \left( \frac{2\pi Gm}{P_b} \right)^2. \tag{81}
\]

Readers who wish to familiarize with the full expression of the orbital decay within the framework of the standard massive scalar-tensor theory after all substitutions can find it in Appendix A.

7 OBSERVATIONAL CONSTRAINTS

At the moment, the GR perfectly describes all observational data from binary pulsars within observational uncertainties (Taylor & Weissberg 1982; Stairs 2005; Kramer et al. 2006; Bhat, Bailes & Verbiest 2008; Freire, Kramer & Wex 2012a; Freire et al. 2012b; Ransom et al. 2014; Desvignes et al. 2016; Archibald et al. 2018). Therefore, all deviations from GR predicted by modified gravity should be smaller than existing observational uncertainties. This fact allows to obtain very strict constraints on the considering subclass of the Horndeski gravity.

The observational value of the orbital period change \( P_b^{obs} \) consists of various components which have the different nature: intrinsic and kinematic effects (Damour & Taylor 1991; Lazaridis et al. 2009). We are interested in the intrinsic part \( P_b^{intrinsic} \) because the dominant element of this component is the orbital period change due to the emission of gravitational waves. Also, intrinsic part includes effects of the mass loss from the binary and from tidal torques (Damour & Taylor 1991; Lazaridis et al. 2009) but at the current stage we consider only such systems where these effects are negligibly small in relation to the effect of gravitational radiation.

The constraints on the Horndeski gravity can be obtained from the comparing of the predicted quantity \( \dot{P}_b^{eh}/P_b^{GR} \) and the observational quantity \( P_b^{intrinsic}/P_b^{GR} \) at 95% confidence level:

\[
\left| \frac{\dot{P}_b^{eh}}{P_b^{GR}} - \frac{P_b^{intrinsic}}{P_b^{GR}} \right| \leq 2\sigma, \tag{82}
\]

where \( \sigma \) is the observational uncertainty.

7.1 Constraints on the massive scalar-tensor theories

The scalar dipole radiation prevails in the predictions of the standard massive scalar-tensor theories for the orbital period decay. The contribution of the scalar dipole part is the most noticeable in the mixed binary systems (Will 1981, 2014; Zaglauer 1992; Poisson & Will 2014). We test the massive scalar-tensor models in mixed binary system PSR J1738+0333. This system has the most accurate observational data among quasi-circular mixed binaries. The mass of the white dwarf and mass ratio were obtained in theory-independent way (under the assumption that non-perturbative strong-field effects are absent and higher-order contributions in powers of the gravitational binding energies of the bodies can be neglected) (Antoniadis et al. 2012; Freire et al. 2012b). This fact allows to test a theory of gravity using only one PPK parameter \( P_b \). The orbital parameters for this system are listed in Table 1.

| Parameter | Physical meaning | Experimental value |
|-----------|-----------------|--------------------|
| \( P_b \) | orbital period | 0.3547907398724(13) day |
| \( e \) | eccentricity | 0.34(11) \times 10^{-6} |
| \( P_b^{obs} \) | observational secular | -0.170(31) \times 10^{-11} |
| \( P_b^{intrinsic} \) | intrinsic secular | -0.259(32) \times 10^{-11} |
| \( P_b^{intrinsic}/P_b^{GR} \) | relation between \( P_b^{intrinsic} \) and \( P_b^{GR} \) | 0.93(13) |
| \( m_1 \) | mass of the pulsar | 1.46^{+0.06}_{-0.05} M_\odot |
| \( m_2 \) | mass of the white dwarf | 0.181^{+0.07}_{-0.08} M_\odot |
| \( m \) | total system mass | 1.65^{+0.06}_{-0.08} M_\odot |

Table 1. Parameters PSR J1738+0333 (Antoniadis et al. 2012; Freire et al. 2012b)
change (A1) takes the form:

$$\frac{\dot{P}_b}{P'_b} = \frac{G^2 G_4(0,0)}{G^7 G_{4(1,0)}} \left[ 1 + \frac{5 e \phi}{12} \left( \frac{P_b c^3}{2 \pi m c \Omega_{12}} \right)^2 \right]^{\frac{1}{2}}$$

where $\phi$ is sensitivity of neutron star and $\phi_{WD}$ is the sensitivity of white dwarf.

Using (82) and (83) we obtain the following bounds on the considering subclass of the Horndeski theory:

$$\left| \frac{\dot{\phi}_{WD}}{G^2 G_{4(0,0)}} \left[ 1 + \frac{5 e \phi}{12} \left( \frac{P_b c^3}{2 \pi m c \Omega_{12}} \right)^2 \right]^{\frac{1}{2}} \times \left( \frac{G^2 G_{4(0,0)} (\phi_{NS} - \phi_{WD})^2}{G_{4(1,0)} m_b^0} \right) \right| - 0.93 \leq 0.26. (84)$$

From the condition $m_{\phi} < \omega / c$ one obtains the constraints on scalar field mass (see Table 1):

$$m_{\phi} < 7 \times 10^{-15} \text{cm}^{-1}. (85)$$

Generally speaking, the longer the system period, the more accurate constraints on the scalar field mass. However, for systems with large orbital period the quantity $P_b$ has very inaccurately measured value. Therefore, the best laboratories for testing scalar-tensor theories are systems combining both a large orbital period and a well-measured value of the orbital period decay.

In the general case of the scalar-tensor theories, the mixed binary system gives the better constraints than neutron star-neutron star binary due to the value of the difference of sensitivities. However, not all scalar-tensor models include the sensitivities. If there is no concept of sensitivity in the theory, then in the expression for the predicted orbital period change $\dot{P}_b$ only quadrupole terms remain, regardless of the type of the binary pulsar. In this case the best restrictions could be found from the binary pulsar with the most accurate value of the quantity $P_b^{\text{obs}} / P_b^{\text{GR}}$. The double binary pulsar PSR J0737-3039 is such system (Burgay et al. 2003; Kramer et al. 2006).

The system PSR J0737-3039 is the only known double binary pulsar. The extraordinary closeness of the system components, small orbital period and the fact that we see almost edge-on system allow to investigate the manifestation of relativistic effects with the highest available precision. This system consists of two pulsars and provides the most accurate data among all binary pulsars. The observational data for this system is listed in the Table 2 (Burgay et al. 2003; Kramer et al. 2006).

In PSR J0737-3039 the kinematic contribution in the orbital period decay is negligibly small (Burgay et al. 2003; Kramer et al. 2006). Thus, the quantities $P_b^{\text{obs}}$ and $P_b^{\text{GR}}$ almost the same (within observable accuracy). Using method (82) and the observational data from Table 2, it is possible to obtain the following constraints for the scalar-tensor theories without sensitivities:

$$\left| 1.003 - \frac{\phi_{12}^2}{G^7 G_{4(0,0)}} \left[ 1 + \frac{G_{4(1,0)} \phi_{WD}}{3 \left( 1 - 4 \times 10^{26} m_{\phi}^2 / c^7 \right) \phi_{12}^2 \right] \right| \leq 0.028.$$

7.2 Scalar-tensor specific cases: hybrid f(R)-gravity and massive Brans-Dicke theory

The action (1) with the set of the gravitational Lagrangian densities (4) is the generic one for the standard massive scalar-tensor theories with the second order field equations. Taking different sets of $G_i$ one can obtain limiting cases of this theory (Kobayashi, Yamaguchi & Yokoyama 2011). In this work we consider two mathematically very similar but physically very different specific massive scalar-tensor cases: the hybrid metric-Palatini f(R)-gravity (Capozziello et al. 2013, 2015) and massive Brans-Dicke theory (Brans & Dicke 1961; Abing et al. 2012).

Our starting point is the hybrid metric-Palatini f(R)-gravity. This theory is geometrical but can be reduced to scalar-tensor model as any other f(R)-gravity (Teyssandier & Tourranc 1983). The hybrid f(R)-theory was created as a mixture of metric and Palatini approaches to eliminate the disadvantages of both of them. This model allows to describe the accelerated expansion of the Universe and the galaxies rotation curves in a purely geometric way without introducing new particles. An additional interesting aspect of hybrid f(R)-gravity is the possibility to generate long-range forces without conflict with local tests of gravity and without invoking any kind of screening mechanism (which would however require that at the present time the cosmological evolution reduces to GR) (Capozziello et al. 2015). This theory has already been well studied in cosmology and in different galaxies (Capozziello et al. 2013, 2015). However, it is necessary to test any theory of gravity in the different field limits. So we test the hybrid f(R)-gravity in a strong field of binary pulsars.

To reduce the general scalar-tensor action (1) to hybrid

| Parameter | Physical meaning | Experimental value |
|-----------|------------------|--------------------|
| $P_b$ | orbital period | 0.10225156248(5) day |
| $e$ | eccentricity | 0.007775759(9) |
| $P_b^{\text{obs}}$ | observational secular change of $P_b$ | $-1.252(17) \times 10^{-12}$ |
| $P_b^{\text{GR}}$ | relation between $P_b^{\text{obs}}$ and $P_b^{\text{GR}}$ | 1.003(14) |
| $m_1$ | mass of the first pulsar | 1.338(7) $M_\odot$ |
| $m_2$ | mass of the second pulsar | 1.2489(7) $M_\odot$ |
| $m$ | total system mass | 2.58708(16) $M_\odot$ |

It is important to emphasize that these bounds do not include the masses of the components, since the absence of sensitivities nullifies all such terms. So our estimation is correct regardless of the method used to obtain the masses of companions in the system.

So the observational data of PSR J0737-3039 impose the following restrictions on scalar field mass:

$m_{\phi} < 5 \times 10^{-14} \text{cm}^{-1}$. (87)

These restrictions are connected with the orbital period of the binary system and do not depend on the specific choice of the scalar-tensor theory.
f(R)-gravity, it is necessary to choose the following set of parameters $G$:  
\[ G_2 = -\frac{3\chi}{G\phi} V(\phi), \ G_3 = 0, \ G_4 = \frac{1 + \phi}{G}, \ G_5 = 0 \]  
(88)

and

\[ G_{4(0,0)} = \frac{1 + \phi_0}{G}, \ G_{4(1,0)} = \frac{1}{G}, \ G_{3(1,0)} = 0, \ G_{2(0,1)} = -\frac{3}{G\phi_0} \]  
(89)

The hybrid metric-Palatini f(R)-gravity is pure geometrical theory and sensitivities $\omega_\phi$ do not appear in this model. Thus, the scalar quadrupole term is leading one in both types of pulsars systems: in mixed binaries and in neutron star-neutron star binaries.

The observational data from PSR J0737-3039 provides the following constraints on the hybrid f(R)-gravity (from (86)):

\[ 0.975 \leq \frac{1}{(1 + \phi_0)^2} \left( 1 - \frac{5\phi_0}{18} \left( 1 - 2 \times 10^{-26} m_\phi^3 \right) \right) \leq 1. \]  
(90)

Fig. 1 reflects the dependence $\phi_0$ from the scalar field mass $m_\phi$ for the system PSR J0737-3039.

Since the theory does not contain sensitivity, the quadrupole terms only contribute to the value of orbital period decay. Thus, the mixed binary PSR J1738+0333 gives the following bounds (from the method (82)) in the case of hybrid f(R)-gravity (see Fig. 2):

\[ 0.67 \leq \frac{1}{(1 + \phi_0)^2} \left( 1 - \frac{5\phi_0}{18} \left( 1 - 3 \times 10^{-27} m_\phi^3 \right) \right) \leq 1. \]  
(91)

It is clear from comparison of constraints (90) and (91) the double binary pulsar gives the better bounds on $\phi_0$ in the hybrid f(R)-gravity. On the other hand, the mixed binary system PSR J1738+0333 provides the best constraints on the scalar field mass. On the Fig. 3 we compare the our limits on hybrid f(R)-gravity from binary pulsars with restrictions from the PPN parameter $\gamma_{PPN} = 1 + (2.1 \times 10^{-5}) (2.3 \times 10^{-7})$ (Bertotti, Iess & Tortora 2003; Leanizbarrutia, Lobo & Saez-Gomez 2017). Thus, the $\gamma_{PPN}$ gives the better bounds on the $\phi_0$ than the system PSR J0737-3039. The combined restrictions can be obtained from $\gamma_{PPN}$ and system PSR J1738+0333:

\[ \phi_0 \leq 0.00004, \ m_\phi \leq 1.4 \times 10^{-14} \text{cm}^{-1}. \]  
(92)

The Brans-Dicke model is one of the first scalar-tensor theories which is widespread (Brans & Dicke 1961). In the framework of this model, many interesting results were obtained including bounces, wormholes, and constraints on cosmological parameters (Alexeyev, Rannu & Gareeva 2011; Tretyakova et al. 2012; Novikov et al. 2014). After the discovery of the Universe accelerated expansion (Riess et al. 1999, 2004; Perlmutter et al. 1999; Spergel et al. 2007) scientists consider its massive version as one of the ways to explain this phenomenon (Boisseau 2011). The massive Brans-Dicke theory was considered by Alsing et al. (2012) in the binary pulsars PSR J1012+5307, PSR J1141-6545, and PSR J0737-3039. Authors have shown that the best restrictions on the Brans-Dicke theory are obtained from the Solar System’s PPN parameter $\gamma_{PPN}$. This model is thoroughly studied in the binary pulsars and the expression for the orbital period change is already obtained. It is interesting to compare the restrictions on the massive Brans-Dicke theory obtained as a particular case of our more general consideration with the results of Alsing et al. (2012). We also add two contributions to the expression of the orbital period decay (PPN corrections to the scalar dipole term and scalar dipole-octupole term) unlike the work of Alsing et al. (2012). The other difference of our work from paper of Alsing et al. (2012) is the accounting of the fact that $\partial_\phi \phi \neq \partial_\phi \phi$ (see equations (58) and (59)). Our approach leads to significant deviations in the final constraints.

The standard massive scalar-tensor gravity can be reduced to the massive Brans-Dicke theory by the following choice of parameters $G_i$ (McManus, Lombriser & Penarrubia 2016):

\[ G_2 = \frac{2\omega_{BD}}{\phi} X + V(\phi), \ G_3 = 0, \ G_4 = \phi, \ G_5 = 0. \]  
(93)

\[ \phi_0 \leq 0.00004, \ m_\phi \leq 1.4 \times 10^{-14} \text{cm}^{-1}. \]  
(92)
and hence

\[ G_{4(0,0)} = \phi_0, \ G_{4(1,0)} = 1, \ G_{3(1,0)} = 0, \ G_{2(0,1)} = \frac{2\omega_{BD}}{\phi_0} \]  

(94)

Here \( \omega_{BD} \) is Brans-Dicke parameter.

The dependence of the quantity \( \phi_0 \) upon \( \omega_{BD} \) for massive and massless Brans-Dicke cases (Alsing et al. 2012) is

\[ \phi_0 = \frac{4 + 2\omega_{BD}}{G(3 + 2\omega_{BD})} \]  

(95)

One of the most important feature of the Brans-Dicke model is the presence of sensitivities in structure of the theory (Eardley 1975). Dependence of sensitivity value on a neutron star mass and equations of state was studied in detail earlier by Will & Zaglauer (1989); Zaglauer (1992). For our calculations we use approximate values \( s_{WD} \sim 10^{-4} \) and \( s_{NS} \sim 0.2 \) (Alsing et al. 2012).

Firstly, we obtain the restrictions on the massive Brans-Dicke theory from the double binary pulsar PSR J0737-3039. In the neutron star-neutron star binary \( s_\alpha - s_\beta \approx 0 \) in massive Brans-Dicke theory (Will 1981, 1984; Alsing et al. 2012; Poisson & Will 2014) and hence scalar dipole, PN corrections to the scalar dipole and scalar dipole-octupole terms vanish. There is only quadrupole terms in expression for restrictions:

\[ 1 \leq \left( \frac{3 + 2\omega_{BD}}{4 + 2\omega_{BD}} \right)^2 + \frac{0.4(3 + 2\omega_{BD})^2}{(2\omega_{BD} + 4)^2} \]  

(96)

Further we constrain the massive Brans-Dicke gravity using the observational data of the mixed binary system PSR J1738+0333:

\[ 1 \leq \left( \frac{3 + 2\omega_{BD}}{4 + 2\omega_{BD}} \right)^2 + \frac{0.4(3 + 2\omega_{BD})^2}{(2\omega_{BD} + 4)^2} \]  

\[ + \frac{6 \times 10^{10}}{2\omega_{BD} + 4} \times \frac{2 \times 10^{32} m_\phi^2}{2\omega_{BD} + 4} \leq 1.19 \]  

(97)

The system was not considered by Alsing et al. (2012) because the accurate data for this binary pulsar (Antoniadis et al. 2012; Freire et al. 2012b) has appeared after the publication of Alsing et al. (2012). This system predicts the greatest deviations between our approach and the one of Alsing et al. (2012). This difference also appears due

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**Figure 3.** Hybrid f(R)-gravity. Dependence of the scalar field mass upon scalar field background value. Figures a) and b) represent allowable regions at different scales. The region with oblique dotted lines corresponds to allowable values in the case of the system PSR J0737-3039, the horizontal solid lines are the critical value of scalar mass \( m_{\phi} = 2\omega/c \), the vertical dotted lines describe allowable values in the case of the system PSR J1738+0333, the horizontal dot-dash lines correspond to allowable values from the \( \gamma_{\text{PN}} \) (Leanizbarrutia, Lobo & Saez-Gomez 2017).

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**Figure 4.** Massive Brans-Dicke. Dependence of the \( \omega_{BD} \) upon the scalar field mass in the case of the system PSR J0737-3039. The vertical dotted lines correspond to the results of Alsing et al. (2012). The bold line and region of horizontal dotted lines describe the results of our work. The vertical solid line is the critical value of scalar mass \( m_{\phi} = 2\omega/c \).
to the account of $\partial_0 \phi \neq \partial_\phi$. The PN corrections to the scalar dipole and scalar dipole-octupole terms introduce insignificant deviations since they are smaller than the dipole term for an order of magnitude. All deviations are reflected on the Fig. 5.

Among mixed binary systems considered by Alsing et al. (2012) the system PSR J1012+5307 is the only one with negligibly small eccentricity (Callanan, Garnavich & Koester 1998; Lazaridis et al. 2009). The observational data for this system are reflected in the Table 3. The mass of white dwarf was obtained from spectroscopic observations and then, using pulsar timing data, mass ratio was found (Callanan, Garnavich & Koester 1998). From our approach in the mixed binary PSR J1012+5307 we obtain the following restrictions:

$$1 \leq \frac{(3 + 2\omega_{BD})^\frac{5}{3}}{(4 + 2\omega_{BD})^\frac{5}{3}} + \frac{0.4(3 + 2\omega_{BD})^\frac{4}{3}}{(4 + 2\omega_{BD})^\frac{4}{3}} + \frac{8 \times 10^3}{2\omega_{BD} + 4} \frac{\bar{\mathcal{M}_{\nu}}}{2\omega_{BD} + 4} \leq 4.14.$$  

(98)

The system PSR J1012+5307 gives the best restrictions for the scalar field mass among considering system:

$$m_\varphi < 4 \times 10^{-15}$$

(99)

However, the increasing of the accuracy in determining the scalar field mass, causes the decreasing of accuracy the orbital period change value (98).

We choose the system PSR J1012+5307 following to the work of Alsing et al. (2012). The deviations between our results and results of Alsing et al. (2012) in this system are reflected on the Fig. 6. The difference between two approaches is smaller than in PSR J1738+0333, and restrictions on $\omega_{BD}$ are worse but ones on the scalar field mass are better. For all parameters both of systems give the better constraints than PSR J0737-3039. In the rest, the conclusions being correct for PSR J1738+0333 are also true for PSR J1012+5307.

On the Fig. 7 we present all constraints from binary pulsars with restrictions on the massive Brans-Dicke theory obtained from the PPN parameter $\gamma_{PN}$ (Bertotti, Iess & Tortora 2003). It is clear that $\gamma_{PN}$ gives the better constraints on the $\omega_{BD}$ than all of considering binary pulsars. The mixed binary PSR J1012+5307 provides the best bounds on the scalar field mass. Combining the restrictions of these two systems are following:

$$\omega_{BD} \geq 36000, \quad m_\varphi \leq 4 \times 10^{-15}$$  

(100)

8 CONCLUSIONS

We considered the subclass of Horndeski gravity without Vainshtein mechanism in the strong field regime of binary pulsars. Our purpose was to impose restrictions on this subclass in the strong field limit. That is why we found the expression for the orbital period decay $P'_b$ predicted by standard massive scalar-tensor theories in quasi-circular orbit approximation. The expression was obtained using the post-Newtonian expansion (Eddington 1922) of the tensor and...
scalar fields. The stress-energy pseudotensor was derived using the Noether theorem (Saffer, Yunes & Yagi 2018). We showed that the orbital period change consists of two sectors: tensor one and scalar one.

The tensor sector coincides with the value of the orbital period decay predicted by GR (Will 1981) up to the effective gravitational constant $\gamma_{ab}$ between two bodies $a$ and $b$. There is neither monopole nor dipole radiations.

The scalar sector includes monopole, dipole, quadrupole, and dipole-octupole terms. In addition, we considered the PN corrections to the scalar dipole term. The monopole term vanishes in the quasi-circular approximation but all other terms survive. The dipole-octupole contribution adds the negative modification to the scalar energy flux at the same PN order as the scalar quadrupole radiation.

To impose restrictions on the standard massive scalar-tensor theories in binary pulsars we choose the observational data of mixed binary pulsar PSR J1738-0333 (Antoniadis et al. 2012). This system has the most accurate data among mixed binary pulsars with small eccentricity at the moment. In this system, the scalar dipole term has the leading order in scalar sector. So we focused our attention mostly on this contribution in addition to tensor quadrupole term.

All terms of scalar sector depend on the propagation speed of the massive scalar mode $v_p(\omega, m_\phi)$. We showed that constraints on the scalar field mass $m_\phi$ are completely independent of the choice of a specific scalar-tensor theory with the second-order field equations. The scalar field mass depends only on the orbital period of binary pulsar $P_b$. The greater orbital period causes the better scalar field mass restrictions. However, the greater the orbital period provides the worse accuracy of the orbital period decay. The best constraints on the scalar field mass we obtained from the system PSR J1012+5307 but this system has much worse measurement accuracy of the orbital period change (Lazaridis et al. 2009) than PSR J1738+0333 (Antoniadis et al. 2012). However, the latter loses to the former in the magnitude of the orbital period.

Not all scalar-tensor theories predict existence of sensitivities. Regardless of the type of binary pulsar, only quadrupole contributions (tensor and scalar) remain in such theories. We showed that in gravitational models without sensitivities the double binary pulsar PSR J0737-3039 gives the best constraints on all parameters of the theory except the scalar field mass.

We considered contributions of the PN corrections to the scalar dipole term and the scalar cross dipole-octupole term (see also paper of Zhang, Liu & Zhao (2017)). These two terms have the same PN order as scalar quadrupole one therefore they cannot be neglected a priori. However, we showed directly that in two types of binary pulsars contribution of the scalar dipole term and the scalar cross dipole-octupole term are insignificant.

The important aspect of our work is evidence of injustice of expression $\partial_\gamma \phi = - \partial_0 \phi$ (see equations (58) and (59)). Earlier the proof was given in the paper for the screened modified gravity (Zhang, Liu & Zhao 2017) but we proved injustice this equality using the example of the more general case of scalar-tensor theory.

Also two specific cases of massive scalar-tensor gravity were considered: the hybrid metric-Palatini f(R)-gravity theory and the massive Brans-Dicke theory.

The hybrid f(R)-gravity is pure geometrical (Capozziello et al. 2013) and there are no sensitivities. Among two pulsars systems PSR J0737-3039 and PSR J1738+0333 the former gives the most accurate restrictions for the background value of scalar field $\phi_0$. Nevertheless, the latter provides the best constraints on the scalar field mass (due to value of the orbital period). We also tested the hybrid f(R)-gravity in the Solar System using the observational data for PPN parameter $\gamma_{PPN}$. From our consideration it is clear that restrictions of $\phi_0$ obtained from $\gamma_{PPN}$ are ahead of accuracy of the constraints obtained from PSR J0737-3039.

The another specific theory, which was considered in our work, is the massive Brans-Dicke model. Earlier this theory studied in detail in binary pulsars by Alsing et al. (2012) and we were interested in comparison of their and our results. We tested the massive Brans-Dicke theory in three binary pulsars: PSR J0737-3039, PSR J1738+0333, and PSR J1012+5307. The last system was taken into account for direct comparison with the results of Alsing et al. (2012) and ours. The direct comparison was also carried out for the system PSR J0737-3039. Besides we apply the predication for the orbital period change of Alsing et al. (2012) to the system PSR J1738+0333 and compare with our results. We clearly show that the differences are significant especially in the system PSR J1738+0333. The difference between two approaches is that we took into account additional contributions to the scalar sector (the PN corrections to the scalar dipole and the scalar dipole-octupole terms), as well as the injustice of the expression $\partial_\gamma \phi = - \partial_0 \phi$ (see equations (58) and (59)). The contribution of the PN corrections to the scalar field and the cross term are insignificant, but the equality and inequality of $\partial_\gamma \phi$ and $- \partial_0 \phi$ leads to essentially different results. Thus we showed by the example that it is necessary
take into account the inequality $\partial_\tau \phi \neq \partial_\varphi \phi$. In addition, we consider the massive Brans-Dicke theory in the Solar System. After comparison of all constraints our conclusion is the following: the best constraints on the massive Brans-Dicke theory are obtained from mixed test of $\psi_{PV}$ (restrictions on $\omega_{BD}$) and PSR J1012+5307 (restrictions on $m_0$).

As the next step, we plan to investigate the general case of Horndeski theory. One of the most important directions of the further researches is obtaining the expression for sensitivity and determination of its dependence of star mass in the Horndeski gravity. Also other possible generalization of our work is the test of the Horndeski gravity using all the set of post-Keplerian parameters (Damour & Deruelle 1985).

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APPENDIX A: ORBITAL PERIOD CHANGE IN THE MASSIVE SCALAR-TENSOR THEORIES

After all the substitutions of parameters $\gamma_2$ from equation (28), and $A_4, A_3, A_2$ from expressions (65) and (66) the first derivative of the orbital period in the standard massive scalar-tensor theories takes the form:

$$
\begin{align*}
\frac{\partial \mathcal{P}_h}{\partial \mathcal{A}} &= - \frac{192\pi \mu}{5m} \left( \frac{2\pi m}{c^3 G_4(0,0) \mathcal{P}_h} \right)^{\frac{1}{2}} \left\{ 1 + \left[ 2c \varphi G_4(1,0) \left( 1 - \frac{2\gamma_2 G_4(0,0)}{\phi_0 G_4(1,0)} \right) - \frac{4\pi c \varphi G_4(0,0)}{\phi_0} \left( 1 - \frac{2\gamma_2 G_4(0,0)}{\phi_0 G_4(1,0)} \right) \right] \right\}^{\frac{1}{2}} \\
&\times \left\{ 1 + \frac{5c^2 \varphi G_4(0,0) (s_2 - s_1)}{2\phi_0} \left( \frac{2G_4(0,0) (s_2 - s_1)}{G_4(1,0) \phi_0} \right) \left( \frac{\mathcal{P}_h G_4(0,0)}{2\pi m} \right)^{3/2} \left( 1 + \left[ 2c \varphi G_4(1,0) \left( 1 - \frac{2\gamma_2 G_4(0,0)}{\phi_0 G_4(1,0)} \right) \right] \right) \right\}^{\frac{1}{2}} \\
&- \frac{4\pi c \varphi G_4(0,0)}{\phi_0} \left[ 1 - \frac{2\gamma_2 G_4(0,0) (s_2 - s_1)}{2\phi_0 G_4(0,0)} \left( \frac{m_2}{m_1} - \frac{m_1}{m_2} \right) + \frac{1}{G_4(1,0) \phi_0} \left( \frac{m_2}{m_1} - \frac{m_1}{m_2} \right) - \frac{2G_4(0,0) (s_2 - s_1)}{m_1} \right] \\
&+ 6\gamma_2 - 6\gamma_1 + 23 \left[ \frac{G_4(0,0) \phi_0}{G_4(1,0) \phi_0} \left( \frac{m_2}{m_1} - \frac{m_1}{m_2} \right) + \frac{4c \varphi G_4(0,0) (s_2 - s_1)}{\phi_0} \right] \\
&+ \frac{16s_2 \gamma_4 m_2}{m_1} + 16s_1 \gamma_4 m_2 + 8s_1 - 8s_2 + 8 \gamma_4 m_2 \left( \frac{m_2}{m_1} - \frac{m_1}{m_2} \right) - \frac{8s_2 \gamma_4 m_2}{m_1} \\
&+ \frac{16s_2 \gamma_4 m_2}{m_1} + 16s_1 \gamma_4 m_2 + 8s_1 - 8s_2 + 8 \gamma_4 m_2 \left( \frac{m_2}{m_1} - \frac{m_1}{m_2} \right) - \frac{8s_2 \gamma_4 m_2}{m_1} \\
&\times \left[ 1 - \left( \frac{r_0 c \varphi G_4(1,0)}{2\pi} \right)^2 \right]^{\frac{1}{2}} + \frac{G_4(0,0) \phi_0}{G_4(1,0) \phi_0} \left( \frac{m_2}{m_1} - \frac{m_1}{m_2} \right) \left[ 1 - \left( \frac{r_0 c \varphi G_4(1,0)}{2\pi} \right)^2 \right]^{\frac{1}{2}} \\
&\times \left( \frac{m_1 - m_2}{m} \right) \left( \frac{G_4(0,0) (s_2 m_1 + s_1 m_2)}{G_4(1,0) \phi_0} \right) \left[ 1 - \left( \frac{r_0 c \varphi G_4(1,0)}{2\pi} \right)^2 \right]^{\frac{1}{2}} \right\} \\
\end{align*}
(A1)

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