ON CONVOLUTION PROPERTY OF HY TRANSFORM AND ITS APPLICATIONS

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Abstract. In this paper, we have given a new application of HY transform. The convolution property for HY transform is obtained. We used this new result to solve integral equations and fractional integral equation. Few examples have been presented to illustrate the efficiency of the property.

Keywords: integral transform; convolution.

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1. INTRODUCTION

Integral transform methods are convenient mathematical methods for solving advance problems of engineering and sciences which are mathematically expressed in terms of differential equations, partial differential equations, integro differential equations, fractional differential equations, etc. During last two decades, many integral transforms are introduced such as Shehu [1], Sumudu [2], Elzaki [3], Natural [4], Aboodh [5], Pourreza [6], Mohand [7] and Sawi [8]. In 2019, Ahmadi [9] defined a new integral transform which is called HY transform. However, some properties of this integral transform are not given such as convolution property. Then, the aim of this paper is to prove convolution property of HY transform which is an important property used to solve integral equations. The basic definition of HY transform is given in

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Section 2. The convolution property is discussed in Section 3, several test examples to show the effectiveness of the proposed property are given in Section 4, and finally the conclusion is summarized in Section 5.

2. MATHEMATICAL PRELIMINARIES

In this section, we present some basic idea about $HY$ transform [8].

**Definition 2.1.** The $HY$ transform of the function $f(t)$ of exponential order is defined over the set of functions

$$A = \{ \exists M > 0, |f(t)| < Me^{\alpha t}, t \in [0, \infty) \}$$

by the following integral

$$HY[f(t)] = F(v) = v \int_0^\infty e^{-vt} f(t) dt.$$  

where $HY[f(t)]$ is called the $HY$ transform of time function. Variables $v$ is the $HY$ transform variable. It converges if the limit of the integral exists, and diverges if not. The $HY^{-1}$ will be the inverse of the $HY$ transform.

The following useful formulas follow directly from equation (1):

(i) $HY[1] = \frac{1}{v}$.

(ii) $HY[t^n] = \frac{n!}{v^{2n+1}}, n = 1, 2, 3,\ldots$

(iii) $HY[t^p] = \frac{\Gamma(p+1)}{v^{2p+1}}, p > -1$.

(iv) $HY[f(t) + g(t)] = HY[f(t)] + HY[g(t)]$. (Linearity property).

**Theorem 2.2.** Let $HY[f(t)] = F(v)$. Then

$$HY[f^{(n)}(t)] = v^{2n}F(v) - \sum_{k=0}^{n-1} v^{2(n-k)-1} f^{(k)}(0), \ n \geq 1.$$  

**Definition 2.3.** The function $f_1 \ast f_2 = \int_\mathbb{R} f_1(t - \tau)f_2(\tau)d\tau$ is called the convolution of both functions $f_1$ and $f_2$ defined on $\mathbb{R}$. 
3. **Convolution Property for HY Transform**

**Theorem 3.1.** Let $HY[f(t)] = F(v)$ and $HY[g(t)] = G(v)$. Then HY transform of $(f * g)(t)$ is

\[ HY[(f * g)(t)] = \frac{1}{v} F(v) G(v). \]

**Proof.** The convolution of two function $f(t)$ and $g(t)$ is

\[ (f * g)(t) = \int_0^t f(t - \tau) g(\tau) d\tau. \]

Using HY transform of equation (1), we get

\[ HY[(f * g)(t)] = HY\left[ \int_0^t f(\tau) g(t - \tau) d\tau \right], \]

\[ = v \int_0^\infty e^{-v^2 t} \left( \int_0^t f(\tau) g(t - \tau) d\tau \right) dt, \]

\[ = v \int_0^\infty \int_0^t e^{-v^2 t} f(\tau) g(t - \tau) d\tau dt, \]

\[ = v \int_0^\infty \int_0^\tau e^{-v^2 \tau} f(\tau) g(t - \tau) d\tau dt. \]

Now setting $b = t - \tau$, we have

\[ HY[(f * g)(t)] = v \int_0^\infty f(\tau) \int_0^\infty e^{-v^2 (b + \tau)} g(b) db d\tau, \]

\[ = v \int_0^\infty e^{-v^2 \tau} f(\tau) d\tau \int_0^\infty e^{-v^2 b} g(b) db, \]

\[ = F(v) \int_0^\infty e^{-v^2 b} g(b) db. \]

(4)

Multiplying both sides of equation (4) by $v$, we obtain

\[ vHY[(f * g)(t)] = F(v) G(v). \]

Thus

\[ HY\{(f * g)(t)\} = \frac{1}{v} F(v) G(v). \]

This proves the theorem of convolution. \[\square\]

**Corollary 3.2.** Let $HY[f(t)] = F(v)$ and $HY[g(t)] = G(v)$. Then HY transform of $(f * g)'(t)$ is

\[ HY[(f * g)'(t)] = vF(v) G(v). \]

(5)
**Proof.** From equation (2), we have

\[ HY[f'(t)] = v^2 HY[f(t)] - vf(0). \]

That is,

\[ HY[(f \ast g)'(t)] = v^2 HY[(f \ast g)(t)] - v(f \ast g)(0). \]

By Theorem 3.1, we obtain

\[ HY[(f \ast g)'(t)] = v^2 \left( \frac{1}{v} F(v) G(v) \right). \]

Therefore

\[ HY[(f \ast g)'(t)] = vF(v)G(v). \]

\[ \square \]

**Theorem 3.3.** Let \( HY[f(t)] = F(v) \) and \( HY[g(t)] = G(v) \). Then \( HY \) transform of \( (f \ast g)^{(n)}(t) \) is

(6)

\[ HY[(f \ast g)^{(n)}(t)] = v^{2n-1} F(v) G(v). \]

**Proof.** Assuming that equation (6) is true for \( n = k \). From equation (6) and by mathematical induction, we have that

(7)

\[
HY[((f \ast g)^{(k)}(t))'] = HY \left[ \frac{d}{dt} (f \ast g)^{(k)}(t) \right],
\]

\[ = v^2 HY[(f \ast g)^{(k)}(t)] - (f \ast g)^{(k)}(0), \]

\[ = v^{2k-1+2} F(v) G(v), \]

\[ = v^{2k+1} F(v) G(v). \]

Therefore

\[ HY[(f \ast g)^{(n)}(t)] = v^{2n-1} F(v) G(v). \]

\[ \square \]
4. EXAMPLES

Example 4.1 Consider the following Volterra integral equation of first kind

\begin{equation}
    f(x) = \int_{0}^{x} h(x-t)g(t)dt.
\end{equation}

Taking \( HY \) transform on both sides of equation (8), we have

\begin{align*}
    HY[f(x)] &= HY \left[ \int_{0}^{x} h(x-t)g(t)dt \right], \\
    F(v) &= HY[h(x) \ast g(x)].
\end{align*}

By Theorem 3.1, we obtain

\begin{align*}
    F(v) &= \frac{1}{v} H(v) G(v), \\
    G(v) &= v \cdot \frac{F(v)}{H(v)}.
\end{align*}

Thus,

\begin{equation}
    g(x) = HY^{-1} \left[ v \cdot \frac{F(v)}{H(v)} \right].
\end{equation}

Example 4.2 Consider the following Volterra integral equation of second kind

\begin{equation}
    g(x) = f(x) + \int_{0}^{x} h(x-t)g(t)dt.
\end{equation}

Taking \( HY \) transform on both sides of equation (10), we have

\begin{align*}
    HY[g(x)] &= HY[f(x)] + HY \left[ \int_{0}^{x} h(x-t)g(t)dt \right], \\
    G(v) &= F(v) + HY[h(x) \ast g(x)].
\end{align*}

By Theorem 3.1, we obtain

\begin{align*}
    G(v) &= F(v) + \frac{1}{v} H(v) G(v), \\
    G(v) &= \frac{vF(v)}{v-H(v)}.
\end{align*}

Then,

\begin{equation}
    g(x) = \left. \frac{vF(v)}{v-H(v)} \right|_{x}.
\end{equation}
Example 4.3 Consider the following Volterra integral equation

(12) \[ \int_0^x e^{2(x-t)} f(t) dt = x. \]

Taking \( HY \) transform both sides of (12) and by Theorem 3.1, we obtain

\[ HY \left[ \int_0^x e^{2(x-t)} f(t) dt \right] = HY[x], \]

\[ \frac{1}{v} \left( \frac{v}{v^2 + 2} \right) F(v) = \frac{1}{v^3}, \]

\[ F(v) = \frac{v^2 + 2}{v^3}. \]

Then,

(13) \[ f(x) = 1 + 2x. \]

Example 4.4 Consider the following Volterra integral equation

(14) \[ g(x) = x + \int_0^x g(t) \sin(x-t) dt. \]

Taking \( HY \) transform on both sides of equation (14) and by Theorem 3.1, we obtain

\[ G(v) = HY[x] + HY[g(x) \ast \sin(x)], \]

\[ = \frac{1}{v^3} + \frac{1}{v} \left( \frac{v}{v^4 + 1} \right) G(v), \]

\[ = \frac{v^4 + 1}{v^3}. \]

Then,

(15) \[ g(x) = x + \frac{1}{6}x^3. \]

Example 4.5 Consider the following integro-differential equation

(16) \[ f'(x) = 1 - \int_0^x f(t) dt, \quad f(0) = 0. \]
Taking \( HY \) transform on both sides of equation (16) and by Theorem 3.1, we obtain

\[
HY[f'(x)] = HY[1] - HY[1 * f(x)],
\]

\[
v^2 F(v) - v f(0) = \frac{1}{v} - \frac{1}{v} \left( \frac{1}{v} F(v) \right),
\]

\[
v^2 F(v) = \frac{1}{v} - \frac{1}{v^2} F(v),
\]

\[
F(v) = \frac{v}{v^4 + 1}.
\]

Then,

\[
f(x) = \sin x.
\]

**Example 4.6** Consider the following fractional integral equation

\begin{equation}
y(t) = g(t) + I^\alpha y(t), \quad \alpha \in \mathbb{R}^+,
\end{equation}

where \( I^\alpha \) is the well known Riemann–Liouville fractional integral operator. It is defined by

\[
I^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} y(\tau) d\tau.
\]

By substituting \( \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} y(\tau) d\tau \) instead of \( I^\alpha \) in equation (18) and applying the convolution Theorem 3.1, we have

\[
Y(v) = G(v) + \frac{1}{\Gamma(\alpha)} HY[t^{\alpha-1}] HY[y(t)],
\]

\[
= G(v) + \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{v^{2\alpha}} Y(v),
\]

\[
Y(v) = \frac{v^{2\alpha}}{v^{2\alpha} - 1} G(v).
\]

Then,

\begin{equation}
y(t) = HY^{-1} \left[ \frac{v^{2\alpha}}{v^{2\alpha} - 1} G(v) \right]
\end{equation}

where \( Y(v) \) is \( HY[y(t)] \).
5. Conclusion

In this paper, convolution property of _HY_ transform of is obtained. We have successfully applied _HY_ transform for the solution of integral equations and fractional integral equations. For further study, _HY_ transform can be applied for solving other singular integral equations and their systems.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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