CPT Symmetries and the Bäcklund Transformations

Hans J. Wospakrik* and Freddy P. Zen†

Theoretical Physics Laboratory,
Department of Physics, Institute of Technology Bandung,
Jalan Ganesha 10, Bandung 40132, Indonesia

Abstract

We show that the auto-Bäcklund transformations of the sine-Gordon, Korteweg-deVries, nonlinear Schrödinger, and Ernst equations are related to their respective CPT symmetries. This is shown by applying the CPT symmetries of these equations to the Riccati equations of the corresponding pseudopotential functions where the fields are allowed to transform into new solutions while the pseudopotential functions and the Bäcklund parameter are held fixed.

*email: hansjw@fi.itb.ac.id
†Address after August 1, 1999: Optical Sciences Centre, Institute of Advanced Studies, The Australian National University, Canberra, Australia.
I. Introduction

It has been well-known that the usual (auto-) Bäcklund transformation (BT), of an integrable equation increases or decreases an integral number of solitons. Precisely, within the inverse scattering method (ISM), BT changes an integral number of poles and (or) zeros of the scattering data. This suggests a suspicion that the BT may be related to the discrete symmetries of the corresponding integrable equation, i.e., the "CPT" symmetries. Note that, by CPT symmetries, we mean for either complex-conjugation (C), parity (P), time-reversal (T), or their appropriate products.

It is the purpose of this paper to pursue this suspicion through some specific examples. In fact, by using the CPT symmetries of the following four popular integrable equations as case studies: the sine-Gordon (sG), Korteweg-deVries (KdV), nonlinear Schrödinger (NLS), and Ernst equations, we are able to rederive their corresponding BT.

The key equation for the derivation of these BTs, by using this CPT method, is the Riccati equation of the corresponding pseudopotential function. The derivation of the BT from this Riccati equation had also been considered by Konno and Wadati as transformations that leave the Riccati equation invariant. In contrast to Konno-Wadati approach, the present method does not preserve the invariancy of the Riccati equation under the above mentioned CPT transformations. This is due to the "rule" that under these CPT transformations only the fields that are allowed to transform into new solutions while the pseudopotential functions and the Bäcklund parameter are held unchanged. In this case, the method could be considered as related to Chen’s method. However, the relation with the CPT symmetries was not explicitly stressed in Ref. 3.

In the next section, we will apply this method to derive the BT of the sG equation. The BTs of the KdV, NLS, and Ernst equation will be derived in Secs. 3, 4, and 5, respectively.

II. The Sine-Gordon Equation

The sine-Gordon equation is given by

\[ \phi_{uv} = \sin \phi, \]

where \( u = x + t, v = x - t \), and a comma preceding a subscript denotes partial derivative. Henceforth, the \( x \) and \( t \) variables will be referred to the space
and time coordinates, respectively. The Riccati equations for the corresponding pseudopotential function \( \Gamma(u, v) \), as one derives by using the Wahlquist-Estabrook prolongation method, read,

\[
\Gamma_{,u} = \left( \frac{1}{2} \lambda \right) (\Gamma^2 - 1) \sin \phi + \lambda \Gamma \cos \phi, \tag{2.2a}
\]

\[
\Gamma_{,v} = \left( \frac{1}{\lambda} \right) \Gamma + \frac{1}{2} \left( 1 + \Gamma^2 \right) \phi_{,v}, \tag{2.2b}
\]

where \( \lambda \) is the B"acklund parameter.

Following Chen, this set of Riccati equations may be considered as a transformation between \( \phi \) and \( \Gamma \). Thus, if the same \( \Gamma \) and \( \lambda \) are also related to another solution \( \tilde{\phi} \), then Eq. (2.2) provides a relation between two solution \( \phi \) and \( \tilde{\phi} \) of Eq. (2.1), that is a B"acklund transformation.

It is our purpose to find the other set of Ricatti equations for the same \( \Gamma \), with the same \( \lambda \), which express a transformation between this same \((\Gamma, \lambda)\) pair and the other solution \( \tilde{\phi} \) as the new solution that is generated by the discrete ”CPT” symmetries of the sG equation (2.1).

Under the PT transformation:

\[
u \rightarrow v' = -v, \tag{2.3b}\]

the sG equation (2.1) is left invariant if the field \( \phi \) transforms as an eigenstate with eigenvalues \( \pm 1 \), i.e.,

\[
\phi(u, v) \rightarrow \phi'(u', v') = \pm \phi(u, v). \tag{2.4}
\]

Let us assume that there exist a nontrivial transformed field \( \tilde{\phi} (u, v) \neq \phi(-u, -v) \), such that,

\[
\phi(u, v) \rightarrow \phi'(u', v') = \pm \tilde{\phi}(u, v), \tag{2.5}
\]

is another solution of the sG equation (2.1). In this case, the sG equation (2.1) will also be considered by us, as invariant under the PT transformations (2.3) and (2.5). Our claim is that the relation between \( \phi(u, v) \) and \( \tilde{\phi}(u, v) \) is given by the corresponding BT, as we will show by using the following method.

Since the new solution \( \tilde{\phi}(u, v) \) satisfies the sG equation (2.1), i.e.,

\[
\tilde{\phi}_{,uv} = \sin \tilde{\phi}, \tag{2.6}
\]
so within the Wahlquist-Estabrook prolongation method\(^4\), we may choose to hold the pseudopotential function \(\Gamma\) and the Bäcklund parameter \(\lambda\) unchanged under the PT transformations (2.3) and (2.5), i.e.,

\[
\lambda \rightarrow \lambda' = \lambda, \quad \Gamma(u, v) \rightarrow \Gamma'(u', v') = \Gamma(u, v).
\] (2.7)

Equations (2.5) and (2.7) are essential to derive the corresponding Riccati equations that connect the same \((\Gamma, \lambda)\) pair of Eq. (2.2) with the new solution \(\tilde{\phi}\).

However, it must be emphasized that since the CPT transformations are discrete, i.e., they yield identity transformation when operated twice, so the condition (2.7) is only valid for two solutions, \(\phi\) and \(\tilde{\phi}\), which are related by one particular pair of \((\lambda, \Gamma)\).

Thus, under the PT transformations (2.3), (2.5), and (2.7), the Riccati equations (2.3a, b) become

\[
- \Gamma_u = \left(\frac{1}{2}\lambda\right)(\Gamma^2 - 1) \sin \tilde{\phi} + \lambda \Gamma \cos \tilde{\phi}, \quad (2.8a)
\]

\[
- \Gamma_v = \left(\frac{1}{\lambda}\right)\Gamma - \frac{1}{2}(1 + \Gamma^2)\phi_v. \quad (2.8b)
\]

These Riccati equations could also be derived, using the Wahlquist-Estabrook prolongation method, directly from Eq. (2.6), in terms of the primed coordinates \((u', v')\) and the field \(\phi'(u', v')\), with one condition that the PT transformation (2.3) and the eigenvalue equation (2.5) should only be applied to the resulting Riccati equations.

Add Eq. (2.2a) to (2.8a) to eliminate \(\Gamma_u\), gives

\[
\tan \frac{1}{2}(\tilde{\phi} + \phi) = 2\Gamma/(1 - \Gamma^2). \quad (2.9)
\]

Solve for \(\Gamma\), gives the following well-known relation between \(\Gamma\), \(\phi\) and \(\tilde{\phi}\),

\[
\Gamma = \tan \frac{1}{4}(\tilde{\phi} + \phi), \quad (2.10a)
\]

or

\[
\Gamma = -\cot \frac{1}{4}(\tilde{\phi} + \phi). \quad (2.10b)
\]

The Riccati equations (2.2) together with Eq. (2.10) constitute the BT of the sG equation (2.1). To reduce this BT into the usual form, first of all we take the derivatives of Eq. (2.10a), with respect to \(u\), and \(v\), respectively.
Use the Riccati equations (2.2) again to eliminate $\Gamma_u$ and $\Gamma_v$ from the resulting equation, then after performing simple trigonometric manipulations, we obtain the following familiar looking BT of the sG equation:

\[
\tilde{\phi}_u = -\phi_u + (2\lambda) \sin \frac{1}{2}(\tilde{\phi} - \phi), \quad (2.11a)
\]

\[
\tilde{\phi}_v = \phi_v + \left(\frac{2}{\lambda}\right) \sin \frac{1}{2}(\tilde{\phi} + \phi). \quad (2.11b)
\]

The relation (2.10b) gives a similar transformation which differs only in sign of the Bäcklund parameter $\lambda$.

### III. The Korteweg-de Vries Equation

The second equation we shall be considering in this section is the KdV equation:

\[
\phi_t + \phi_{xxx} + 12\phi\phi_x = 0. \quad (3.1)
\]

The Riccati equations for the corresponding pseudopotential function $\Gamma(x, t)$ are

\[
\Gamma_x = -(2\phi + \Gamma^2 - \lambda), \quad (3.2a)
\]

\[
\Gamma_t = 4 \left[ (\phi + \lambda)(2\phi + \Gamma^2 - \lambda) + \frac{1}{2}\phi_{xx} - \phi_x \Gamma \right], \quad (3.2b)
\]

where $\lambda$ is the Bäcklund parameter.

The KdV equation (3.1) remains invariant under the PT transformation:

\[
x \to x' = -x, \quad (3.3a)
\]

\[
t \to t' = -t, \quad (3.3b)
\]

\[
\phi \to \phi' (x', t') = \tilde{\phi} (x, t), \quad (3.3c)
\]

where $\tilde{\phi}(x, t)$ is a new solution of the KdV equation (3.1). Holding the pseudopotential function $\Gamma(x, t)$ and the Bäcklund parameter $\lambda$ unchanged under the PT transformation (3.3), for the same reason as in Sec. 2, the Riccati equations (3.2a, b) transform to

\[
-\Gamma_x = - \left( 2\tilde{\phi} + \Gamma^2 - \lambda \right), \quad (3.4a)
\]

\[
-\Gamma_t = 4 \left[ (\tilde{\phi} + \lambda)(2\tilde{\phi} + \Gamma^2 - \lambda) + \frac{1}{2}\tilde{\phi}_{xx} + \tilde{\phi}_x \Gamma \right]. \quad (3.4b)
\]
Add Eq. (3.2) to (3.4) to eliminate $\Gamma_{,x}$ gives the well known relation:}

$$\tilde{\phi} = -\phi - \Gamma^2 + \lambda. \quad (3.5)$$

Subtraction, on the other hand, gives

$$\Gamma_{,x} = \left( \tilde{\phi} - \phi \right), \quad (3.6)$$

which is exactly Eq. (3.1-19) of Ref. 6, up to some constants.

Equations (3.2) and (3.5) constitute the BT of the KdV equation (3.1). In fact, the usual form of this BT is obtained by eliminating the pseudopotential function $\Gamma(x, t)$ from Eqs. (3.2) and (3.5) then expressing the final result in terms of the potential $w(x, t)$ through the relation $\phi = -w_{,x}$.

**IV. The Non-linear Schrödinger Equation**

In this section, we will be considering the BT of the NLS equation,

$$i\Psi_{,t} = -\Psi_{,xx} + \frac{1}{2}\varepsilon\Psi^*\Psi^2; \quad (4.1a)$$

where $\Psi$ is a complex field, and $^*$ is complex conjugation. Beside the equation (4.1a) we also have the complex conjugate equation:

$$-i\Psi^*_{,t} = -\Psi^*_{,xx} + \frac{1}{2}\varepsilon\Psi\Psi^2. \quad (4.1b)$$

The NLS equations (4.1a, b) remain invariant under the PT transformation,

$$x \rightarrow x' = -x, \quad (4.2a)$$

$$t \rightarrow t' = -t, \quad (4.2b)$$

if the complex field $\Psi$ and its complex conjugate $\Psi^*$ transform according to the following rule:

$$\Psi(x, t) \rightarrow \Psi'(x', t') = \tilde{\Psi}^*(x, t), \quad (4.3a)$$

$$\Psi^*(x, t) \rightarrow \Psi^{**}(x', t') = \tilde{\Psi}(x, t), \quad (4.3b)$$

where $\tilde{\Psi}^*(x, t)$ and $\tilde{\Psi}(x, t)$ are new solutions of Eqs. (4.1a) and (4.1b), respectively. The transformation (4.3) is the anticipated $C$ transformation. Thus, the NLS equation (4.1) remains invariant under the CPT transformations (4.2) and (4.3).
The Riccati equations for the corresponding complex pseudopotential function $\Gamma(x, t)$ of Eq. (4.1), as derived by Estabrook and Wahlquist, are

\[ 2 \Gamma, x = -\Gamma^2 \Psi + \varepsilon \Psi^* - 2\Lambda \Gamma \]  
\[ 2 \Gamma, t = -i\lambda \left( \Gamma^2 \Psi - \varepsilon \Psi^* - 2\Lambda \Gamma \right) + i \left( -\Psi, x \Gamma^2 - \varepsilon \Psi, x + \varepsilon \Psi^* \Psi \Gamma \right) \]  
and for the complex conjugate pseudopotential function $\Gamma^*(x, t)$:

\[ 2 \Gamma^*, x = -\Gamma^*^2 \Psi^* + \varepsilon \Psi - 2\Lambda^* \Gamma^* \]  
\[ 2 \Gamma^*, t = -i\lambda^* \left( \Gamma^*^2 \Psi^* - \varepsilon \Psi - 2\Lambda^* \Gamma^* \right) - i \left( -\Psi^*, x \Gamma^*^2 - \varepsilon \Psi^*, x + \varepsilon \Psi^* \Psi \Gamma^* \right) \]

where $\lambda$ is the (complex) Bäcklund parameter. Comparing with Eqs. (9a) and (9b) for NLS in Ref. 2, one finds that, up to the overall factor 2, their pseudopotential function $\Gamma$ corresponds to our complex conjugate pseudopotential function $\Gamma^*$ in the coordinates $(-x, -t)$, and that their Bäcklund parameter is real. These differences are not essential since both lead to the same BT of NLS, and we will use this freedom to fix our convention for the transformation rules of a complex pseudopotential function $\Gamma$.

We observe that, since the new field $\Psi'(x', t') = \tilde{\Psi}^*(x, t)$ satisfies the NLS equation (4.1b), so within the Wahlquist-Estabrook prolongation method\(^4\), we choose to interchange the pseudopotentials $\Gamma$ and $\Gamma^*$ according to the rule:

\[ \Gamma(x, t) \rightarrow \Gamma'(x', t') = \Gamma^*(x, t), \]  
\[ \Gamma^*(x, t) \rightarrow \Gamma''(x', t') = \Gamma(x, t), \]

as was used in Ref. 2.

In order to derive the usual relation between $\Gamma$ and $\Psi$ we fix our convention by the following rule: “If the complex field $\Psi$ and its conjugate $\Psi^*$ satisfy different equations, then under the CPT transformation, the pseudopotential functions $\Gamma$ and $\Gamma^*$ are chosen to obey the transformation rule (4.5), otherwise, they are kept unchanged”.

Since $\Psi$ and $\Psi^*$ satisfy different equations, i.e. (4.1a) and (4.1b) respectively, so according to the above rule, we choose to transform $\Gamma$ and $\Gamma^*$ according to the rule (4.5).

Thus, under the CPT transformations (4.2), (4.3), and (4.5), the Riccati equations (4.4a, b) transform to

\[ -2 \Gamma^*, x = -\Gamma^*^2 \tilde{\Psi}^* + \varepsilon \tilde{\Psi} - 2\Lambda \Gamma^* \]  
\[ -2 \Gamma^*, t = -i\lambda \left( \Gamma^*^2 \tilde{\Psi}^* - \varepsilon \tilde{\Psi} - 2\Lambda^* \Gamma^* \right) + i \left( \tilde{\Psi}^*, x \Gamma^*^2 + \varepsilon \tilde{\Psi}^* \Gamma^* \right), \]
and the Riccati equations (4.4c, d) to

\[-2 \Gamma_x = -\Gamma^2 \Psi + \varepsilon \Psi^* - 2\lambda^* \Gamma, \tag{4.6c}\]

\[-2 \Gamma_t = i\lambda \left( \Gamma^2 \Psi - \varepsilon \Psi^* - 2\lambda^* \Gamma \right) - i \left( \Psi_{,x} \Gamma^2 + \varepsilon \Psi_{,x} + \varepsilon \Psi \Psi^* \Gamma \right). \tag{4.6d}\]

Note that the Bäcklund parameter \( \lambda \) is held unchanged under these CPT transformations.

To derive the BT of the NLS equation, we first eliminate \( \Gamma_x \) and \( \Gamma^*_{,x} \) from Eqs. (4.4a), (4.4c), (4.6a), and (4.6c). The final result is the following well-known relation\(^2,4\):

\[\Psi = -\Psi - 2 (\lambda + \lambda^*) \Gamma^*/(\Gamma^* \Gamma - \varepsilon)\] \(\tag{4.7}\)

Eqs. (4.4) and (4.7) constitute the BT of the NLS equation. The usual form of this BT can be derived by solving Eq. (4.7) for \( \Gamma \) then substituting the result into the Riccati equation (4.4) to eliminate \( \Gamma_x \) and \( \Gamma_t \). (The minus sign in front of the field \( \Psi \) in the right hand side of Eq. (4.7) is irrelevant since \((-\Psi)\) is also a solution of Eq. (4.1)).

V. The Ernst Equation

In this final section, we proceed to apply this CPT method to rederive the auto-BT of the celebrated Ernst’s equation of the stationary axysymmetric gravitational field or the static self-dual Euclidean SU(2) gauge field\(^7\), i.e. the Neugebauer I\(_1\) BT.

The Ernst equation is given by

\[(\text{Re} \ \mathcal{E}) (2V \mathcal{E}_{,12} + V_{,2} \mathcal{E}_{,1} + V_1 \mathcal{E}_{,2}) = 2V \mathcal{E}_{,1} \mathcal{E}_{,2}, \tag{5.1}\]

where \( \mathcal{E} \) is the complex Ernst potential, \( \text{Re} \ \mathcal{E} \) the real part of \( \mathcal{E} \). Henceforth, a comma preceding a subindex denotes partial derivative with respect to the corresponding characteristic coordinates:

\[x^1 = \rho + iz, \quad x^2 = \rho - iz, \tag{5.2}\]

with \((\rho, z)\) are the cylindrical coordinates. \( V \) is an arbitrary harmonic function, i.e.,

\[V_{,12} = 0. \tag{5.3}\]

Note that the complex conjugate Ernst potential \( \mathcal{E}^* \) does also satisfy the Ernst equation (5.1).
In terms of the Neugebauer field variables \((M_i, N_i), i = 1, 2, 3\), introduced according to
\[ M_1 = (2 \Re \mathcal{E})^{-1} \mathcal{E}, \quad M_2 = (2 \Re \mathcal{E})^{-1} \mathcal{E}^* \quad M_3 = V^{-1} V, \quad (5.4) \]
\[ N_1 = (2 \Re \mathcal{E})^{-1} \mathcal{E}^*, \quad N_2 = (2 \Re \mathcal{E})^{-1} \quad N_3 = V^{-1} V, \]
the Ernst equation (5.1) together with its complex conjugate equation, and

the harmonic equation (5.3) are equivalent to the first order equations\(^8\):
\[ M_{i,2} = C_{ik}^{kl} M_{k,1}, \quad N_{i,1} = C_{ik}^{kl} N_{k,1}, \quad (5.5a) \]

where the non-vanishing \(C_{ik}^{kl}\) are given by
\[ C_{11}^{11} = C_{22}^{22} = C_{33}^{33} = -C_{1}^{2} = -C_{2}^{1} = -1, \quad (5.5b) \]
\[ C_{32}^{12} = C_{13}^{13} = C_{31}^{31} = -C_{2}^{3} = -\frac{1}{2}. \]

It is obvious that the Ernst equation (5.1), or the equivalent first order equations (5.5) are invariant under the following CP symmetries:

(I) :
\[
\begin{align*}
  x^1 & \rightarrow x^1 = x^2, \quad x^2 \rightarrow x^2 = x^1, \\
  \mathcal{E} & \rightarrow \mathcal{E}' (x^1, x^2) = \tilde{\mathcal{E}}^* (x^1, x^2), \\
  V & \rightarrow V' (x^1, x^2) = \tilde{V} (x^1, x^2),
\end{align*}
\]  

(II) :
\[
\begin{align*}
  x^1 & \rightarrow x^1 = x^2, \quad x^2 \rightarrow x^2 = x^1, \\
  \mathcal{E} & \rightarrow \mathcal{E}' (x^1, x^2) = \tilde{\mathcal{E}} (x^1, x^2), \\
  V & \rightarrow V' (x^1, x^2) = \tilde{V} (x^1, x^2),
\end{align*}
\]  

where \(\tilde{\mathcal{E}}\), and \(\tilde{\mathcal{E}}^*\) are new solutions of Eqs. (5.1) or (5.5), and \(\tilde{V}\) of (5.3).

Apply the Wahlquist-Estabrook prolongation method to Eq. (5.5), one derives the following Riccati equations\(^5,8\) for the corresponding pseudopotential functions \(\alpha\) and \(\gamma\),
\[ \alpha_{,1} = \alpha (\alpha - 1) M_1 + (\alpha - \gamma) M_2 + \frac{1}{2} \alpha (\gamma - 1) M_3, \quad (5.8a) \]
\[ \alpha_{,2} = (\alpha - 1) N_1 + \left( \frac{\alpha}{\gamma} \right) (\alpha - \gamma) N_2 + \left( \frac{\alpha}{2 \gamma} \right) (\gamma - 1) N_3, \quad (5.8b) \]
\[ \gamma_{,1} = \gamma (\gamma - 1) M_3, \quad (5.8c) \]
\[ \gamma_{,2} = (\gamma - 1) N_3, \quad (5.8d) \]

For real solutions, \( \alpha, \gamma, \) and their complex conjugates satisfy the relation,
\[ \alpha^* = \alpha^{-1}, \quad \gamma^* = \gamma^{-1}. \quad (5.9) \]

Let us consider explicitly the CP transformation (5.6). Under these discrete transformations, the Neugebauer field variables \( M_i, N_i \) transform accordingly as follows:
\[ M_1 \to \tilde{N}_1, \quad M_2 \to \tilde{N}_2, \quad M_3 \to \tilde{N}_3, \quad (5.10) \]
\[ N_1 \to \tilde{M}_1, \quad N_2 \to \tilde{M}_2, \quad N_3 \to \tilde{M}_3. \]

Since the Ernst fields \( E \) and its conjugate \( E^* \) satisfy the same field equation (5.1), so following our rule in Sec. 4, we choose to hold \( \alpha \) and \( \gamma \) unchanged under this transformation. Then, the Riccati equations (5.8) transform to
\[ \alpha_{,2} = \alpha (\alpha - 1) \tilde{N}_1 + (\alpha - \gamma) \tilde{N}_2 + \frac{1}{2} \alpha (\gamma - 1) \tilde{N}_3, \quad (5.11a) \]
\[ \alpha_{,1} = (\alpha - 1) \tilde{M}_1 + \left( \frac{\alpha}{\gamma} \right) (\alpha - \gamma) \tilde{M}_2 + \left( \frac{\alpha}{2\gamma} \right) (\gamma - 1) \tilde{M}_3, \quad (5.11b) \]
\[ \gamma_{,2} = \gamma (\gamma - 1) \tilde{N}_3, \quad (5.11c) \]
\[ \gamma_{,1} = (\gamma - 1) \tilde{M}_3. \quad (5.11d) \]

Eliminating \( \alpha_{,1}, \alpha_{,2}, \gamma_{,1}, \) and \( \gamma_{,2} \) from Eqs. (5.10) and (5.11), we derive:
\[ (\alpha - 1) \left( \tilde{M}_1 - \alpha M_1 \right) + (\alpha - \gamma) \left( \frac{\alpha}{\gamma} \tilde{M}_2 - M_2 \right) = 0, \quad (5.12a) \]
\[ (\alpha - 1) \left( \alpha \tilde{N}_1 - N_1 \right) + (\alpha - \gamma) \left( \tilde{N}_2 - \frac{\alpha}{\gamma} N_2 \right) = 0, \quad (5.12b) \]
\[ \tilde{M}_3 = \gamma M_3, \quad \tilde{N}_3 = \left( \frac{1}{\gamma} \right) N_3. \quad (5.13a) \]

If the solutions are chosen to include the special cases \( \alpha = 1, \) and \( \alpha = \gamma, \) then the (simplest) required solutions are:
\[ \tilde{M}_1 = \alpha M_1, \quad \tilde{M}_2 = \left( \frac{\gamma}{\alpha} \right) M_2, \quad (5.13b) \]
\[ \tilde{N}_1 = \left( \frac{1}{\alpha} \right) N_1, \quad \tilde{N}_2 = \left( \frac{\alpha}{\gamma} \right) N_2. \]

The transformation (5.13) is nothing but the Neugebauer $I_1$ BT\(^8\).

The CP symmetry (5.7) does not lead to an essentially new BT since it is nothing but (5.6) followed by complex conjugation on the new Ernst potentials $\tilde{E}$ and $\tilde{E}^*$ or vice versa.

**Concluding Remarks**

In conclusion, we have shown that the auto-Bäcklund transformations of the sG, KdV, NLS, and Ernst equations are related to their respective CPT symmetries. In fact, the above analysis shows that the explicit field equation is prerequisite for the application of this CPT method for deriving the auto-BT of a two dimensional integrable equation.

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