Filtrations of random processes
in the light of classification theory.
I. A topological zero-one law.

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Abstract

Filtered probability spaces (called “filtrations” for short) are shown
to satisfy such a topological zero-one law: for every property of filtra-
tions, either the property holds for almost all filtrations, or its negation
does. In particular, almost all filtrations are conditionally nonatomic.

An accurate formulation is given in terms of orbit equivalence re-
lations on Polish G-spaces. The set of all isomorphic classes of fil-
trations may be identified with the orbit space \( X/G \) (such a space is
sometimes called a singular space in the modern descriptive set the-
ory, see Kechris \cite{Kechris}, \S2\)) for a special Polish G-space \( X \). A “property
of filtrations” means a G-invariant subset of \( X \) having the Baire prop-
erty. “Almost all filtrations” means a comeager subset of \( X \) (the Baire
category approach). The zero-one law is a kind of ergodicity of \( X \). It
holds for filtrations both in discrete and continuous time.

The interplay between probability theory and descriptive set the-
ory could be interesting for both parties.

1 Preliminaries on the classification theory
(for probabilists)

1a Topological counterparts of some probabilistic ideas

Objects to be classified (say, filtrations) often form an infinite-dimensional
space with a natural topology but without a natural measure. We cannot say
‘for almost all filtrations’ in the measure-theoretic sense. Instead we can say

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‘for comeager many filtrations’ in the sense explained below. Fubini theorem, zero-one law and other nice probabilistic tools have topological counterparts based on the Baire category theorem.

1.1. Definition. (a) A Polish metric space is a complete separable metric space.
   (b) A Polish topological space (or just ‘Polish space’) is a topological space that admits a compatible metric turning it into a Polish metric space.

   (‘Compatible’ means that the topology corresponding to the metric is the same as the given topology.)

1.2. Theorem. (Baire; see [14, 2.5.5] or [10, 8.4 and 8.1(iii)] or [5, 2.5.2]) Let \( U_1, U_2, \ldots \) be dense open sets in a Polish space \( X \). Then their intersection \( U_1 \cap U_2 \cap \ldots \) is dense in \( X \).

1.3. Definition. Let \( X \) be a Polish space and \( A \subset X \) a set.
   (a) \( A \) is comeager, if \( A \supset U_1 \cap U_2 \cap \ldots \) for some dense open sets \( U_1, U_2, \ldots \).
   (b) \( A \) is meager (or ‘of first category’) if \( X \setminus A \) is comeager, that is, if \( A \subset F_1 \cup F_2 \cup \ldots \) for some closed sets \( F_1, F_2, \ldots \) with no interior points.
   (c) ‘For comeager many \( x \in X \)’ (symbolically, \( \forall^* x \in X \)) means: for all \( x \) of a comeager set \( A \subset X \).
   (d) ‘For non-meager many \( x \in X \)’ (symbolically, \( \exists^* x \in X \)) means: for all \( x \) of a non-meager set \( A \subset X \).

   (See [14, Sect. 8.J.]) The Baire category theorem ensures that a comeager set cannot be meager. Thus,

   \[
   \forall x (\ldots) \implies \forall^* x (\ldots) \implies \exists^* x (\ldots) \implies \exists x (\ldots).
   \]

   We know that Lebesgue measurable sets are Borel sets modulo sets of measure zero. Similarly, sets having the Baire property are Borel sets modulo meager sets, which is equivalent to the next definition.

1.4. Definition. Let \( X \) be a Polish space.
   (a) A set \( A \subset X \) has the Baire property if there exists an open set \( U \subset X \) such that \( A \setminus U \) and \( U \setminus A \) are meager.
   (b) A function \( f \) from \( X \) to another Polish space \( Z \) is called Baire measurable if \( f^{-1}(V) \) has the Baire property for every open set \( V \subset Z \).
1.5. Proposition. Sets having the Baire property are a σ-field. It is the σ-field generated by all open (or equivalently, Borel) sets and all meager sets. (See [10, 8.22].)

A surprise: Borel sets are open modulo meager. In contrast, they are $G_\delta$ (or $F_\sigma$, but generally not open) sets modulo sets of measure zero. The boundary of an open set is always meager, but not always of measure zero.

Here is a counterpart of Fubini theorem, and then — of Kolmogorov’s zero-one law.

1.6. Theorem. (Kuratowski-Ulam; see [10, 8.41(iii)] or [14, 3.5.16]) Let $X, Y$ be Polish spaces, and a set $A \subset X \times Y$ have the Baire property. Then

$$ \forall^* x \forall^* y \ (x, y) \in A \iff A \text{ is comeager} \iff \forall^* y \forall^* x \ (x, y) \in A. $$

1.7. Theorem. (A topological zero-one law; see [10, 8.47].) Let $X_1, X_2, \ldots$ be Polish spaces, and a set $A \subset X_1 \times X_2 \times \ldots$ have the Baire property. If $A$ is a tail set then $A$ is either meager or comeager.

1b Constructing Polish spaces

1.8. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $L_2(\Omega, \mathcal{F}, P)$ the Hilbert space of square integrable random variables, $L_0(\Omega, \mathcal{F}, P)$ the linear topological space of all random variables (equipped with the topology of convergence in probability), MALG$(\Omega, \mathcal{F}, P)$ the set (in fact, complete Boolean algebra) of equivalence classes of $\mathcal{F}$-measurable sets, equipped with the metric $\text{dist}(A, B) = P(A \setminus B) + P(B \setminus A)$. Assume that $(\Omega, \mathcal{F}, P)$ is separable in the sense that one of the three spaces ($L_2$, $L_0$, MALG) is separable, then others are also separable. All the three spaces are Polish ($L_2$ and MALG being Polish metric spaces).

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2In other words, σ-algebra.

3Baire property holds for all Borel sets, and moreover, for all analytic sets (Lusin-Sierpinski theorem, see [10, 21.6]). For projective (and even more general) sets, Baire property is proven using additional (to ZFC) axioms (see [10, 26.3, 36.20; 38.15, 38.17(ii); 39.17]), which is considered a fault of ZFC axiomatics (see [1, p. 16]), not of Baire property!

4 Saying “$\forall^* x \forall^* y$” we mean that the comeager set of these $y$ may depend on $x$.

5That is, the relation $(x_1, x_2, \ldots) \in A$ is insensitive to any change of a finite number of $x_k$.

6By a random variable I mean here an equivalence class of measurable functions $\Omega \to \mathbb{R}$.

7Also $L_0$ becomes a Polish metric space, being equipped with one of several well-known equivalent metrics, see [10, 17.45], [3, 9.2.3].
1.9. Let \((\Omega, \mathcal{F}, P)\) be a separable probability space and \(X\) a Polish space, then \(X\)-valued random variables on \((\Omega, \mathcal{F}, P)\) form a Polish space \(L_0(\Omega, \mathcal{F}, P; X)\) (but not a linear space, in general). Equivalent metrics on \(X\) induce equivalent metrics on \(L_0(\Omega, \mathcal{F}, P; X)\), which follows easily from the well-known relation between convergence in probability and convergence almost everywhere; namely, \(f_n \to f\) in probability if and only if every subsequence of \((f_n)\) contains a subsequence converging to \(f\) almost everywhere. See [3, 9.2.1].

1.10. Let \(X\) be a Polish space. Denote by \(\text{Prob}(X)\) the set of all probability distributions on \(X\) (that is, probability measures on the Borel \(\sigma\)-field of \(X\)), equipped with the weak topology, called also the weak\(^*\) topology. It is the unique metrizable topology such that for every \(\mu, \mu_1, \mu_2, \cdots \in \text{Prob}(X)\)

\[\mu_n \to \mu \quad \text{if and only if} \quad \forall \varphi \quad \int \varphi d\mu_n \to \int \varphi d\mu,\]

where \(\varphi\) runs over all bounded continuous functions \(X \to \mathbb{R}\). See also [3, Sect. 9.3], [10, 17.20]. Some countable set of functions \(\varphi\) is enough for generating the same topology, see [10, 17.20(ii)].

The topological space \(\text{Prob}(X)\) is Polish. See [10, 17.23]. Note also the continuous map

\[L_0(\Omega, \mathcal{F}, P; X) \ni f \mapsto P_f \in \text{Prob}(X);\]

the distribution \(P_f\) of \(f\) is defined by \(P_f(A) = P(f^{-1}(A))\). See [3, 9.3.5, 11.3.5].

1.11. Let \(X\) be a Polish space, and \(K\) a metrizable compact topological space. Denote by \(C(K, X)\) the space of all continuous maps \(K \to X\) with the metric \(\text{dist}(f, g) = \sup_{a \in K} \text{dist}(f(a), g(a))\); equivalent metrics on \(X\) induce equivalent metrics on \(C(K, X)\), and \(C(K, X)\) is a Polish space. See [10, 4.19] or [14, 2.4.3].

1.12. Let \(X\) be a Polish space. Denote by \(K(X)\) the topological space of all compact subsets of \(X\), equipped with the Vietoris topology. It is the topology that corresponds to Hausdorff metric

\[\text{dist}(K_1, K_2) = \sup_{x \in X} |\text{dist}(x, K_1) - \text{dist}(x, K_2)|;\]

here \(\text{dist}(x, K) = \inf_{y \in K} \text{dist}(x, y)\). We assume that \(\text{dist}(x, y) \leq 1\) for all \(x, y\) (otherwise one can use \(\min(1, \text{dist}(x, y))\) instead of \(\text{dist}(x, y)\)) and let
dist(x, ∅) = 1. A choice of a (compatible) metric on X influences the Hausdorff metric on K(X) but not the topology on K(X). (See [10, Sect. 4.F], see also [4, p. 1124].)

For every Polish space X, the topological space K(X) is Polish. (See [10, 4.25].)

1.13. Let X be a Polish metric space. Denote by F(X) the topological space of all closed subsets of X, equipped with the Wijsman topology; it is the unique metrizable topology such that for every F,F_1,F_2,... ∈ F(X)

$$F_n \to F \quad \text{if and only if} \quad \forall x \in X \ \text{dist}(x,F_n) \to \text{dist}(x,F)$$

provided that dist(x,y) ≤ 1 for all x,y (otherwise we use min(1,dist(x,y)) instead of dist(x,y)); still, dist(x,∅) = 1. Any sequence (x_n) dense in X gives rise to a compatible metric on F(X), say,

$$\text{dist}(F_1,F_2) = \max_n \frac{1}{n} |\text{dist}(x_n,F_1) - \text{dist}(x_n,F_2)|.$$  

(See [4, pp. 25–26] and [2, Sect. 3].)

Theorem (Beer [2]). For every Polish metric space X, the topological space F(X) is Polish.

1.14. Note that F(X) is partially ordered (by inclusion) and the corresponding set of pairs, \{(F_1,F_2) : F_1 ⊂ F_2\}, is closed in F(X) × F(X). Note also the sandwich argument: if E_n ⊂ F_n ⊂ G_n, E_n → F and G_n → F then F_n → F. It follows immediately from monotonicity of the correspondence between a set F ∈ F(X) and its distance function x ↦ dist(x,F). The same holds for K(X).

1.15. Lemma. Let X be a Polish metric space and f : X × X → X a continuous function. Then the following condition on a closed set F ⊂ X selects a closed subset of F(X):

$$\forall x,y \in F \quad f(x,y) \in F.$$  

Proof. Let F_n satisfy the condition, F_n → F, and x,y ∈ F. We have dist(x,F_n) → 0, dist(y,F_n) → 0. Take x_n,y_n ∈ F_n such that x_n → x, y_n → y. Then F_n ⊃ f(x_n,y_n) → f(x,y), therefore dist(f(x,y),F_n) → 0. So, f(x,y) ∈ F.

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8The topology of X does not determine uniquely the Wijsman topology on F(X); equivalent metrics on X may lead to different topologies on F(X).

9Not complete in general.
The binary operation \((x,y) \mapsto f(x,y)\) may be replaced with a unary operation \(x \mapsto f(x)\), or even a constant (0-ary operation) \(f \in X\). Note that a closed subset \(F\) of a separable Hilbert space \(H\) is a subspace if and only if it is closed under the binary operation \((x,y) \mapsto x + y\), every one of unary operations \(x \mapsto ax\), and contains 0. The conclusion follows.

1.16. Corollary. Let \(H\) be a separable Hilbert space. The set \(L(H)\) of all (closed linear) subspaces of \(H\), equipped with the Wijsman topology, is a Polish space.

1.17. Lemma. The following two conditions are equivalent for all \(L, L_1, L_2, \ldots \in L(H)\):

(a) \(L_n \to L\) in \(L(H)\);

(b) \(P_n x \to Px\) in \(H\) for every \(x \in H\); here \(P_n\) is the orthogonal projection onto \(L_n\), and \(P \rightleftharpoons \) onto \(L\).

Proof. Condition (a) means that \(\|P_n x - x\| \to \|Px - x\|\) for all \(x\).

(b) \(\implies\) (a): \(P_n x \to Px\), therefore \(\|P_n x - x\| \to \|Px - x\|\).

(a) \(\implies\) (b): We have \(\|P_n x - x\| \to \|Px - x\|\) and \(\|P_n Px - Px\| \to 0\).

Note that \(\|y - x\|^2 = \|y - P_n x\|^2 + \|P_n x - x\|^2\) for all \(y \in L_n\); in particular,

\[
\|P_n Px - x\|^2 = \|P_n Px - P_n x\|^2 + \|P_n x - x\|^2.
\]

However, \(P_n Px \to Px\), thus \(\|P_n Px - x\|^2 \to \|Px - x\|^2\) and \(\|P_n Px - P_n x\|^2 =\|Px - P_n x\|^2 + o(1)\). So, \(\|Px - x\|^2 + o(1) = \|Px - P_n x\|^2 + \|Px - x\|^2 + o(1)\), thus \(\|Px - P_n x\|^2 \to 0\).

1.18. Corollary. Let \((\Omega, \mathcal{F}, P)\) be a separable probability space. The set of all sub-\(\sigma\)-fields of \(\mathcal{F}\), equipped with the Wijsman topology, is a Polish space. (Each \(\sigma\)-field must contain all sets of measure zero, and is treated as a closed subset of \(\text{MALG}(\Omega, \mathcal{F}, P)\). The proof of the latter is left to the reader. (Similar to 1.16, but Boolean operations are used instead of linear operations.)

1c Polish groups, their actions, orbits

1.19. Definition. (a) A Polish group is a topological group whose topological space is Polish.

\[\text{Well, you may reduce the list.}\]

\[\text{However, it may be also treated as a subspace of } L_2(\Omega, \mathcal{F}, P), \text{ see (2.2).}\]
(b) Let \( G \) be a Polish group. A \textit{Polish \( G \)-space} is a Polish topological space \( X \) equipped with a continuous map \((g, x) \mapsto g \cdot x \) from \( G \times X \) to \( X \) such that \( 1 \cdot x = x \) and \((gh) \cdot x = g \cdot (h \cdot x) \) for all \( x \in X \) and \( g, h \in G \).

(c) Let \( G \) be a Polish group. A \textit{Polish metric \( G \)-space} is a Polish metric space \( X \) equipped with a map \((g, x) \mapsto g \cdot x \) satisfying (b) and in addition, \( \text{dist}(g \cdot x, g \cdot y) = \text{dist}(x, y) \) for all \( x, y \in X \), \( g \in G \).

(d) Let \( X \) be a Polish \( G \)-space. Its \textit{orbit equivalence relation} \( E_G \) is defined by

\[ (x, y) \in E_G \quad \text{if and only if} \quad \exists g \in G \quad g \cdot x = y . \]

The \textit{orbit} of \( x \) is its equivalence class \( [x]_G = G \cdot x \).

From now on, the phrase ‘Let \( X \) be a Polish \( G \)-space’ means ‘Let \( G \) be a Polish group and \( X \) a Polish \( G \)-space’. The same for Polish metric \( G \)-spaces.

1.20. Conjecture. (Topological Vaught Conjecture) Let \( X \) be a Polish \( G \)-space. If \( X \) has uncountably many orbits then \( X \) contains an uncountable closed set \( C \) with no equivalent points (that is, if \( y = g \cdot x \) and \( x, y \in C \) then \( x = y \)). (See [4, Sect. 6.2 and 3.3]; [14, Sect. 5.13].)

The subset \( E_G \) of \( X \times X \) need not be a Borel set (see [4, Sect. 3.2]), even though every orbit is a Borel subset of \( X \) (see [4, 2.3.3]). However, \( E_G \) is an analytic subset of \( X \times X \), since it is the projection to \( X \times X \) of the closed set \( \{(g, x, y) : g \cdot x = y\} \subset G \times X \times X \). All analytic sets have the Baire property, and are universally measurable (that is, measurable w.r.t. every Borel measure); see [14, 4.3.1, 4.3.2] or [14, 21.6, 21.10, Sect. 29.B], see also [3, 13.2.6].

1.21. Lemma. Let \( X \) be a Polish \( G \)-space. Then there exists a function \( g_1 : E_G \to G \) Baire measurable, universally measurable, and such that

\[ g_1(x, y) \cdot x = y \quad \text{for all } (x, y) \in E_G . \]

\textbf{Proof.} Follows from Von Neumann's theorem on measurable selection, see [14, 5.5.3], [10, 29.9], see also [3, 13.2.7]. \( \square \)

1.22. Definition. Let \( X \) be a Polish \( G \)-space.

(a) A \textit{\( G \)-invariant set} in \( X \) is \( A \subset X \) such that \( \forall g \in G \forall x \in A \ g \cdot x \in A \).

(b) A \textit{\( G \)-invariant function} on \( X \) is a map \( f : X \to Z \) (\( Z \) being an arbitrary set) such that \( f(g \cdot x) = f(x) \) for all \( g \in G \), \( x \in X \).
(c) A selector for $E_G$ is a $G$-invariant map $f : X \to X$ such that $(x, f(x)) \in E_G$ for all $x \in X$.

(d) A transversal for $E_G$ is a set $T \subset X$ such that for every $x \in X$ the set $[x]_G \cap T$ contains exactly one point.

(e) The orbit equivalence relation $E_G$ is called smooth if there exist a Polish space $Z$ and a ($G$-invariant) Borel function $\theta : X \to Z$ such that for all $x, y \in X$\

$$(x, y) \in E_G \text{ if and only if } \theta(x) = \theta(y).$$

For (c), (d) see [10, 12.15], [14, p. 186]. For (e) see [10, 18.20], [8, 0.1].

1.23. Theorem. Let $X$ be a Polish $G$-space, then the following four conditions are equivalent.

(a) $E_G$ is smooth;
(b) there exists a Borel selector;
(c) there exists a Borel transversal;
(d) there exist $G$-invariant Borel sets $A_1, A_2, \cdots \subset X$ that separate orbits in the sense that for all $x, y \in X$

$$(x, y) \in E_G \text{ if and only if } \forall n \left( x \in A_n \iff y \in A_n \right).$$

See [14, 5.6.1: Burgess theorem] and [11, 18.20(i,iii)]. The transition (c) \implies (b) uses the Lusin separation theorem for analytic sets (see [11, 28.1] or [14, 4.4.1]) in a way similar to (but simpler than) [14, 4.4.5].

If $E_G$ is smooth then $E_G$ evidently is a Borel subset of $X \times X$. The converse is wrong; for a counterexample, consider the natural action of the additive group $G = \mathbb{Q}$ of rational numbers on the space $X = \mathbb{R}$ of real numbers.

A Borel selector $f$ maps bijectively the set of orbits onto the Borel set $T_f = \{ x \in X : f(x) = x \}$. Let $\theta : X \to Z$ be as in [12, 22](e), then the restriction $\theta|_{T_f}$ maps bijectively $T_f$ onto $\theta(X)$, and is a Borel map. It follows that $\theta(X)$ is a Borel set (see [11, 15.2]). Therefore $Z$ and $\theta$ may be chosen so that $\theta(X) = Z$ (see [11, 13.1]; see also [3, 2.27(iii)]).

1.24. Definition. A Polish $G$-space $X$ will be called ergodic if every $G$-invariant set $A \subset X$ having the Baire property is either meager or comeager.

\[12\] Or ‘section’.
\[13\] Of course, $G$-invariance of $f$ means $f(g \cdot x) = f(x)$, not $f(g \cdot x) = g \cdot f(x)$.
\[14\] Or ‘cross-section’.
\[15\] Or ‘tame’.
\[16\] Not a standard terminology.
1.25. Theorem. The following conditions are equivalent for every Polish $G$-space $X$.

(a) $X$ is ergodic.
(b) Every nonempty open $G$-invariant set is dense.
(c) There exists a dense orbit (at least one).
(d) The union of all dense orbits is a dense $G_{\delta}$-set.\footnote{A $G_{\delta}$ set is, by definition, the intersection of a sequence of open sets.}
(e) For every Polish space $Z$, every $G$-invariant Baire measurable function $f : X \to Z$ is constant on a comeager set.
(f) For every Polish space $Z$ and every Baire measurable function $f : X \to Z$, if $f$ is almost invariant in the sense that\footnote{Recall Sect. 1a about “$\forall^*$. Of course, saying “$\forall^* g \forall^* x$” we mean that the comeager set of these $x$ may depend on $g$.}

$$\forall g \in G \ \forall^* x \in X \ f(g \cdot x) = f(x),$$

then $f$ is almost constant in the sense that

$$\exists z \in Z \ \forall^* x \in X \ f(x) = z.$$ 

See also [9, 3.2, 3.4].

Proof. (a) $\implies$ (b): a meager set cannot have interior points; a comeager set must be dense.

(e) $\implies$ (b): the open set must contain the dense orbit.
(d) $\implies$ (c): trivial.
(e) $\implies$ (a): just take $Z$ of only two points.
(f) $\implies$ (e): trivial.
(b) $\implies$ (a): Given a $G$-invariant set $A \subset X$ that has the Baire property, we need a $G$-invariant open set $U \subset X$ such that $U \setminus A$ and $A \setminus U$ are meager. However, the canonical construction of $U$ given by [10, 8.29] is evidently $G$-invariant.

(b) $\implies$ (d): It is easy to see that the union of all dense orbits is the intersection of all nonempty open $G$-invariant sets. Without affecting the intersection we can restrict ourselves to a sequence of such sets (namely, $G$-saturations of sets of a countable base).

(a) $\implies$ (e): For every open $U \subset Z$ the set $f^{-1}(U)$ is either meager or comeager. Consider the union $V$ of all open sets $U \subset Z$ such that $f^{-1}(U)$ is meager. The set $f^{-1}(V)$ is meager (since we may restrict ourselves to a countable base). It remains to prove that $Z \setminus V$ contains only a single point. However, for two disjoint open sets $U_1, U_2$ the sets $f^{-1}(U_1), f^{-1}(U_2)$, being disjoint, cannot be comeager simultaneously.
(b) $\implies$ (f): Similarly to “(b) $\implies$ (a) $\implies$ (e)” we have (b) $\implies$ (a') $\implies$ (f), where (a') says that every almost invariant set $A \subset X$ (that is, such that $A$ and $g \cdot A$ differ by a meager set) having Baire property is either meager or comeager. The construction used when proving “(b) $\implies$ (a)” still works, giving an invariant $U$ for an almost invariant $A$.

1.26. Corollary. Let $X$ be an ergodic Polish $G$-space such that $E_G$ is smooth. Then there exists a comeager orbit. (See also [8, 4.3], [9, 3.3, 3.5].)

1d Constructing Polish groups and Polish $G$-spaces

Recall constructions of 1b: $C(K, X)$, $K(X)$, $F(X)$, $\text{MALG}(\Omega, \mathcal{F}, P)$ and others.

1.27. Let $X$ be a Polish $G$-space. Then $K(X)$ is a Polish $G$-space.

See [3, item (ii) in Sect. 2.4]. Here and henceforth we do not specify the action, having in mind the evident, natural action.

1.28. Let $X$ be a Polish $G$-space, and $K$ a metrizable compact topological space. Then $C(K, X)$ is a Polish $G$-space.

The proof is left to the reader.

Let $X$ be a Polish metric space. Denote by $\text{Iso}(X)$ the group of isometric invertible transformations $\alpha : X \to X$. Equip $\text{Iso}(X)$ with the unique metrizable topology such that for every $\alpha, \alpha_1, \alpha_2, \cdots \in \text{Iso}(X)$

$$\alpha_n \to \alpha \quad \text{if and only if} \quad \forall x \in X \quad \alpha_n(x) \to \alpha(x).$$

Any sequence $(x_n)$ dense in $X$ gives rise to a compatible metric on $\text{Iso}(X)$, say,

$$\text{dist}(\alpha, \beta) = \max_n \frac{1}{n} \text{dist}(\alpha(x_n), \beta(x_n)).$$

1.29. Theorem. Let $X$ be a Polish metric space, then

(a) $\text{Iso}(X)$ is a Polish group,

(b) $X$ is a Polish metric $\text{Iso}(X)$-space.

For Item (a) see [10, item (9) in Sect. 9.B]. Item (b) is left to the reader. (Continuity of $g \cdot x$ may be checked in $g$ and $x$ separately due to [10, 9.14].)

19For some $X$ the group $\text{Iso}(X)$ may be small, say, the unit only. For some other $X$, however, $\text{Iso}(X)$ is quite large.
1.30. Let $X$ be a Polish metric $G$-space. Then $F(X)$ is a Polish (topological) $G$-space.

The proof is left to the reader. (Once again, separate continuity is enough.)

1.31. Lemma. Let $X$ be a Polish metric space and $f : X \times X \to X$ a continuous map. Then the following condition on $\alpha \in \text{Iso}(X)$ selects a closed subset of $\text{Iso}(X)$:

$$\forall x, y \in X \ f(\alpha(x), \alpha(y)) = \alpha(f(x, y)).$$

Proof. Similarly to [1.13], if $\alpha_n \to \alpha$ then $f(\alpha_n(x), \alpha_n(y)) = \lim f(\alpha_n(x), \alpha_n(y)) = \lim \alpha_n(f(x, y)) = \alpha(f(x, y)).$

The same holds for unary operations and constants. Some applications follow.

1.32. Let $H$ be a separable Hilbert space. The group $U(H)$ of all isometric linear invertible operators $H \to H$ is a Polish group; and $H$ is a Polish metric $U(H)$-space. Also the set $L(H)$ of all (closed linear) subspaces of $H$ is a Polish $U(H)$-space.

1.33. Let $(\Omega, \mathcal{F}, P)$ be a separable probability space. The group $\text{Aut}(\Omega, \mathcal{F}, P)$ of all measure preserving automorphisms of $\text{MALG}(\Omega, \mathcal{F}, P)$ is a closed subgroup of $\text{Iso}(\text{MALG}(\Omega, \mathcal{F}, P))$, therefore a Polish group, and $\text{MALG}(\Omega, \mathcal{F}, P)$ is a Polish metric $\text{Aut}(\Omega, \mathcal{F}, P)$-space. Therefore the set of all sub-$\sigma$-fields of $\mathcal{F}$ (recall [1.18]) is a Polish $\text{Aut}(\Omega, \mathcal{F}, P)$-space. (See also [10, 17.46(i)].)

One may also consider transformations that send $P$ into equivalent measures (not just into itself). (See [10, 17.46(ii)].)

1.34. Another construction (unrelated to 1.29, 1.31). Let $X$ be a metrizable compact topological space, and $\text{Homeo}(X)$ the group of all homeomorphisms $X \to X$. Being equipped with a natural topology, $\text{Homeo}(X)$ is a Polish group, and $X$ is a Polish $\text{Homeo}(X)$-space, as well as $\text{K}(X)$. (See [10, item (8) in Sect. 9.B] and [3, 4.2].)

---

20 May we say ‘measure preserving maps $\Omega \to \Omega$’? We’ll return to the question in Sect. 2.
2 Preliminaries on filtrations (for specialists in the classification theory)

2a Two equivalent languages

Probability theory speaks two equivalent languages, ‘pointful’ and ‘pointless’. In the ‘pointful’ language, a morphism between two probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ is defined as a measure preserving map $\alpha : \Omega_1 \to \Omega_2$ (or rather, equivalence class of such maps). Evidently, it generates maps

$$L_2(\Omega_2, \mathcal{F}_2, P_2) \to L_2(\Omega_1, \mathcal{F}_1, P_1),$$

$$L_0(\Omega_2, \mathcal{F}_2, P_2) \to L_0(\Omega_1, \mathcal{F}_1, P_1),$$

$$\text{MALG}(\Omega_2, \mathcal{F}_2, P_2) \to \text{MALG}(\Omega_1, \mathcal{F}_1, P_1).$$

These maps are continuous, linear (for $L_2$ and $L_0$), preserve structural operations (‘max’ and ‘min’ for $L_2$ and $L_0$; ‘$\cup$’ and ‘$\cap$’ for MALG), distributions of random variables (for $L_2$ and $L_0$) or probabilities of events (for MALG). Given a map (between $L_2$, or $L_0$, or MALG) having such properties, one can reconstruct the corresponding $\alpha$ (it is unique mod 0 and exists) provided, however, that the probability spaces are good enough. Namely, each probability space is assumed to be a Lebesgue-Rokhlin space, that is, isomorphic (mod 0) to an interval with Lebesgue measure, or a (finite or countable) set of atoms, or a combination of the two.

The ‘pointless’ language defines a morphism as a map between spaces $L_2$ (or $L_0$, or MALG) satisfying the properties mentioned above. Thus, it avoids any restrictions on probability spaces, except for separability. Moreover, one can escape $\Omega$ at all, since MALG (or $L_0$) can be axiomatized. (See [10, 17.F] and [1, Sect. 1]; see also [3,])

Both languages describe the same category. An object is either a Lebesgue-Rokhlin probability space $(\Omega, \mathcal{F}, P)$ or, equivalently, a separable measure algebra MALG (as defined in [10, 17.44]); also a linear lattice $L_2$ or $L_0$ may be used. A morphism is either $\Omega_1 \to \Omega_2$ or $\text{MALG}_2 \to \text{MALG}_1$ (see above). In the ‘pointful’ language, a morphism from the first object to the second is surely $\Omega_1 \to \Omega_2$. In the ‘pointless’ language we have no consensus about the direction; some authors define a morphism from the first object to the second as $\text{MALG}_1 \to \text{MALG}_2$. However, I prefer to define the direction according to $\Omega_1 \to \Omega_2$ in all cases, even if $\Omega_1, \Omega_2$ are implicit. That is, for me (2.1) is a morphism from the first object to the second.

A morphism $\Omega_1 \to \Omega_2$ selects a closed (linear) sublattice in $L_2(\Omega_1)$, or $L_0(\Omega_1)$, or $\text{MALG}(\Omega_1)$, that is, a topologically closed set, closed under linear (for $L_2, L_0$) and structural operations. Every subset closed in that sense
corresponds to some morphism. Such a subset is a sub-σ-field, — just by definition, if we are speaking the ‘pointless’ language. The corresponding notion in the ‘pointful’ language is a sub-σ-field of \( F \) containing all sets of measure 0. Or equivalently, it is a measurable partition (see [12]). Every sub-σ-field gives rise to a quotient space. The latter is again a Lebesgue-Rokhlin space. If \((\Omega, \mathcal{F}, P)\) is an object and \(\mathcal{E} \subset \mathcal{F}\) a sub-σ-field then \((\Omega, \mathcal{E}, P|_{\mathcal{E}})\) is another object. I treat the latter as a quotient of the former, since the latter is naturally isomorphic to the quotient space \(\Omega/\mathcal{E}\) equipped with its natural σ-field and measure; in that sense, \((\Omega, \mathcal{E}, P|_{\mathcal{E}}) = (\Omega, \mathcal{F}, P)/\mathcal{E}\). Note the natural morphism from the object to its quotient-object.\(^{21}\)

From now on, every probability space is (by assumption or construction) a Lebesgue-Rokhlin space.

Given a probability space \((\Omega, \mathcal{F}, P)\), the set \(\text{SσF}(\Omega, \mathcal{F}, P)\) of all sub-σ-fields \(\mathcal{E} \subset \mathcal{F}\) may be treated as a closed subset of \(L(L_2(\Omega, \mathcal{F}, P)) \subset L(L_2(\mathcal{L}(\Omega, \mathcal{F}, P)))\), or \(F(L_0(\Omega, \mathcal{F}, P))\), or \(F(\text{MALG}(\Omega, \mathcal{F}, P))\). Probably it is all the same, but anyway, I prefer the first option,

\[(2.2) \quad \text{SσF}(\Omega, \mathcal{F}, P) \subset L(L_2(\Omega, \mathcal{F}, P)).\]

Thus, \(\text{SσF}(\Omega, \mathcal{F}, P)\) is a Polish space. Every monotone sequence \((\mathcal{F}_n)\) of σ-fields converges to some σ-field \(\mathcal{F}\); if \((\mathcal{F}_n)\) decreases then \(\mathcal{F}\) is the intersection; if \((\mathcal{F}_n)\) increases then \(\mathcal{F}\) is the least σ-field containing the union. The least among all sub-σ-fields is the trivial σ-field \(0_{\text{SσF}}\) containing sets of measure 0 or 1 only; the greatest one is \(1_{\text{SσF}} = \mathcal{F}\).

Let \(\mathcal{E}\) be a sub-σ-field, then \(\mathcal{E}\)-measurable random variables are a subspace \(L_0(\mathcal{E})\) of the space \(L_0 = L_0(\mathcal{F})\). Also, \(L_2(\mathcal{E}) = L_0(\mathcal{E}) \cap L_2\). The orthogonal projection \(L_2 \to L_2(\mathcal{E})\) has a special notation

\[f \mapsto \mathbb{E}(f \mid \mathcal{E}).\]

It has a continuous extension (evidently unique) to an operator \(L_1 \to L_1(\mathcal{E})\), still denoted by \(f \mapsto \mathbb{E}(f \mid \mathcal{E})\). The random variable \(\mathbb{E}(f \mid \mathcal{E})\) is called the conditional expectation of \(f\) w.r.t. \(\mathcal{E}\). Note that

\[(2.3) \quad \mathcal{E}_n \to \mathcal{E} \text{ in } \text{SσF}(\Omega, \mathcal{F}, P) \quad \text{if and only if} \quad \forall f \in L_2(\Omega, \mathcal{F}, P) \quad \mathbb{E}(f \mid \mathcal{E}_n) \to \mathbb{E}(f \mid \mathcal{E}) \text{ in } L_2(\Omega, \mathcal{F}, P)\]

by Lemma 1.17. An equivalent condition:

\[\forall A \in \mathcal{F} \quad \mathbb{P}(A \mid \mathcal{E}_n) \to \mathbb{P}(A \mid \mathcal{E}) \text{ in } L_2(\Omega, \mathcal{F}, P);\]

\(^{21}\)The direction of the morphism shows that the smaller object should not be called a sub-object of the larger object (unless you prefer the other direction of morphisms).
the conditional probability \( \mathbb{P}(A \mid \mathcal{E}) \) is, by definition, \( \mathbb{E}(1_A \mid \mathcal{E}) \), where \( 1_A \) is the indicator, \( 1_A(\omega) = 1 \) for \( \omega \in A \), otherwise 0. See also \( [3, 10.2.9] \).

The conditional distribution \( P_{f|\mathcal{E}} \) is an element of \( L_0(\Omega, \mathcal{F}, P; \text{Prob}(\mathbb{R})) \),\(^{22}\) such that \( P_{f|\mathcal{E}}(A) = \mathbb{P}(A \mid \mathcal{E}) \) for every Borel set \( A \subset \mathbb{R} \), or equivalently, \( \int_{\mathbb{R}} \varphi \, dP_{f|\mathcal{E}}(\cdot) = \mathbb{E}(\varphi \circ f \mid \mathcal{E}) \) for every bounded Borel function \( \varphi : \mathbb{R} \to \mathbb{R} \). Such \( P_{f|\mathcal{E}} \) exists, is unique, and is \( \mathcal{E} \)-measurable.\(^{23}\) The same for \( f \in L_0(\Omega, \mathcal{F}, P; X) \) and \( P_{f|\mathcal{E}} \in L_0(\Omega, \mathcal{F}, P; \text{Prob}(X)) \), where \( X \) is a Polish space.

2b Filtrations

2.4. Definition. (a) A filtration\(^{24}\) (on a probability space \( (\Omega, \mathcal{F}, P) \)) is a family \( (\mathcal{F}_t)_{t \in [0, \infty)} \) of sub-\( \sigma \)-fields \( \mathcal{F}_t \subset \mathcal{F} \) satisfying \( \mathcal{F}_s \subset \mathcal{F}_t \) whenever \( 0 \leq s \leq t < \infty \).

(b) A filtration is called continuous if the function \( t \mapsto \mathcal{F}_t \) is continuous on \( [0, \infty) \). The same for ‘right-continuous’ and ‘left-continuous’.

One-sided limits \( \mathcal{F}_{t-}, \mathcal{F}_{t+} \) arise naturally. Note that \( \mathcal{F}_{0+} \) need not be the trivial \( \sigma \)-field \( 0_{\mathcal{SF}} \), and \( \mathcal{F}_\infty \) need not be the whole \( 1_{\mathcal{SF}} = \mathcal{F} \).

2.5. Definition. (a) A random process (on a probability space \( (\Omega, \mathcal{F}, P) \)) is a family \( (f_t)_{t \in [0, \infty)} \) of random variables \( f_t \in L_0(\Omega, \mathcal{F}, P) \).

(b) The natural filtration \( (\mathcal{F}_t) \) of a random process \( (f_t) \) (or the filtration generated by a random process \( (f_t) \)) is defined as follows: for each \( t \), \( \mathcal{F}_t \) is the sub-\( \sigma \)-field generated by \( (f_s)_{s \in [0,t]} \); that is, the least sub-\( \sigma \)-field such that \( f_s \in L_0(\mathcal{F}_t) \) for all \( s \in [0, t] \).

(c) A random process \( (f_t) \) is a martingale w.r.t. a filtration \( (\mathcal{F}_t) \), if

\[
 f_t \in L_1(\mathcal{F}_t) \quad \text{and} \quad f_s = \mathbb{E}(f_t \mid \mathcal{F}_s)
\]

whenever \( 0 \leq s \leq t < \infty \).

(d) A filtration \( (\mathcal{E}_t) \) is immersed into another filtration \( (\mathcal{F}_t) \), if every martingale w.r.t. \( (\mathcal{E}_t) \) is a martingale w.r.t. \( (\mathcal{F}_t) \).

Condition (d) may be written as \( \mathbb{E}(\mathbb{E}(f \mid \mathcal{E}_t) \mid \mathcal{F}_s) = \mathbb{E}(f \mid \mathcal{E}_s) \) whenever \( 0 \leq s \leq t < \infty \), or in operator form, \( \mathcal{F}_s \mathcal{E}_t = \mathcal{E}_s \) if we identify sub-\( \sigma \)-fields with their operators of conditional expectation. An implication of

\(^{22}\)Recall \( 1.9, 1.10 \).

\(^{23}\)You may derive it from \( [3, 10.2.9] \). See also \( [3, 10.2.2] \).

\(^{24}\)The measure \( P \) is given. Sometimes a filtration is considered rather on a measure type space, that is, only an equivalence class of the measure is given. Maybe I should say ‘filtered probability space’ instead.

\(^{25}\)Some non-equivalent definitions are in use; say, one may stipulate a measurable function \( [0, \infty) \times \Omega \to \mathbb{R} \).
(d): $\mathcal{E}_t \mathcal{F}_s = \mathcal{F}_s \mathcal{E}_t$ (since $\mathcal{E}_s \mathcal{F}_s = (\mathcal{F}_s \mathcal{E}_s)^* = (\mathcal{F}_s \mathcal{E}_t)^* = \mathcal{E}_s = \mathcal{F}_s \mathcal{E}_t$), thus, all the operators belong to a commutative algebra. Another implication: $\mathcal{E}_s = \mathcal{F}_s \cap \mathcal{E}_t$, especially, $\mathcal{E}_t = \mathcal{F}_t \cap \mathcal{E}_\infty$. An immersed filtration $(\mathcal{E}_t)$ is uniquely determined by $\mathcal{E}_\infty$ (and $(\mathcal{F}_t)$).²⁶

An isomorphism between two filtrations is defined in such a way that interrelations between all $\mathcal{F}_t$ are relevant, but interrelations between $\mathcal{F}_t$ and $\mathcal{F}$ are not.

2.6. Definition. Let $(\mathcal{F}_t)$ be a filtration on a probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t')$ a filtration on another probability space $(\Omega', \mathcal{F}', P')$.

(a) An isomorphism between the two filtrations is an invertible morphism between quotient spaces $(\Omega, \mathcal{F}, P) / \mathcal{F}_\infty$ and $(\Omega', \mathcal{F}', P') / \mathcal{F}_\infty'$ such that the image of $\mathcal{F}_t$ is $\mathcal{F}_t'$ for every $t$.

(b) A morphism from $(\mathcal{F}_t)$ to $(\mathcal{F}_t')$ is an isomorphism between $(\mathcal{F}_t')$ and a filtration immersed into $(\mathcal{F}_t)$.

See also [3, Sect. 1,2], [15, Sect. 1].

2c Few examples of random processes and filtrations

2.7. Sub-$\sigma$-fields $\mathcal{E}_1, \mathcal{E}_2, \ldots$ are called independent, if $P(A_1 \cap A_2 \cap \ldots) = P(A_1)P(A_2)\ldots$ for all $A_1 \in \mathcal{E}_1, A_2 \in \mathcal{E}_2, \ldots$ Random variables $f_1, f_2, \ldots$ are called independent, if they generate independent sub-$\sigma$-fields.

2c1 Poisson process

Take a sequence of independent random variables $\xi_1, \xi_2, \ldots$, each distributed exponentially, namely, for every $k$

$$P(\xi_k \leq c) = 1 - e^{-c} \quad \text{for all } c \in [0, \infty).$$

Define a random process $(f_t)$ by

$$f_t(\omega) = \max\{k : \xi_1(\omega) + \cdots + \xi_k(\omega) \leq t\},$$

then increments $f_{t_n} - f_{t_{n-1}}, \ldots, f_{t_1} - f_{t_0}$ are independent whenever $0 \leq t_0 \leq \cdots \leq t_n < \infty$, and have Poisson distributions:

$$\mathbb{P}(f_t - f_s = k) = \frac{1}{k!} (t-s)^k e^{-(t-s)} \quad \text{for } k = 0, 1, 2, \ldots \text{ and } s \leq t.$$ ²⁷

²⁶ Another equivalent form of (d): $\mathcal{E}_\infty$ and $\mathcal{F}_t$ are conditionally independent, given $\mathcal{E}_t$.

²⁷ No need to specify the probability space and the choice of measurable functions $\xi_k$ on it, since the filtration $(\mathcal{F}_t)$ constructed below is determined uniquely up to isomorphism.
Such \((f_t)\) is called Poisson process. (See also [3, Problem 12.1.12].) Its natural filtration \((\mathcal{F}_t)\) is continuous. Every \(\mathcal{F}_t\) has a single atom (of probability \(e^{-t}\)) and a nonatomic part. Here are two examples of martingales w.r.t. \((\mathcal{F}_t)\):

\[
M_t = f_t - t; \quad N_t = M_t^2 - t.
\]

Another process \((g_t)\) defined by

\[
g_t = \text{entier}\left(\frac{1}{2}f_t\right) = \max\{k : \xi_1 + \cdots + \xi_{2k} \leq t\}
\]

has its natural filtration immersed into \((\mathcal{F}_t)\). On the other hand, the ‘slowed down’ filtration \((\mathcal{E}_t)\) defined by

\[
\mathcal{E}_t = \mathcal{F}_{t/2} \quad \text{for} \quad t \in [0, \infty)
\]

is not immersed into \((\mathcal{F}_t)\), even though \(\mathcal{E}_t \subset \mathcal{F}_t\) for all \(t\). In fact, the three filtrations are pairwise non-isomorphic.

### 2c2 Brownian motion

Take a sequence of independent random variables \(\xi_1, \xi_2, \ldots\), each distributed normally, namely, for every \(k\)

\[
P(\xi_k \leq c) = (2\pi)^{-1/2} \int_{-\infty}^{c} e^{-u^2/2} du \quad \text{for all} \quad c \in \mathbb{R}.
\]

The (closed linear) subspace \(G \subset L_2(\Omega, \mathcal{F}, P)\) spanned by \((\xi_k)\) consists of normally distributed random variables, and independence is equivalent to orthogonality within \(G\) (see [3, 9.5.14]). Take any linear isometric operator

\[
U : L_2(0, \infty) \to G
\]

and define a random process \((B_t)\) by

\[
B_t = U(1_{(0,t)}) \quad \text{for} \quad t \in [0, \infty);
\]

here \(1_{(0,t)} \in L_2(0, \infty)\) is the indicator of the interval \((0, t)\). Increments \(X_{t_n} - X_{t_{n-1}}, \ldots, X_{t_1} - X_{t_0}\) are independent whenever \(0 \leq t_0 \leq \cdots \leq t_n < \infty\), and have normal distributions:

\[
\mathbb{P}\left( B_t - B_s \leq \sqrt{t-s} \cdot c \right) = (2\pi)^{-1/2} \int_{-\infty}^{c} e^{-u^2/2} du \quad \text{for all} \quad c \in \mathbb{R}, s \leq t.
\]

Such \((B_t)\) is called Brownian motion. (See also [3, 12.1.5].) Its natural filtration \((\mathcal{F}_t)\) is continuous. Every \(\mathcal{F}_t\) is non-atomic (except for \(\mathcal{F}_0\)). Here are two examples of martingales w.r.t. \((\mathcal{F}_t)\):

\[
M_t = B_t; \quad N_t = M_t^2 - t.
\]

The ‘slowed down’ filtration \((\mathcal{F}_{t/2})_{t \in [0, \infty)}\) is isomorphic to \((\mathcal{F}_t)_{t \in [0, \infty)}\), but not immersed.
3 Isomorphism of probability spaces as an orbit equivalence relation

3a Rokhlin’s theory revisited

Rokhlin’s theory [12] gives us, first, a classification of probability spaces (up to isomorphisms mod 0), and second, a classification of sub-$\sigma$-fields, or rather, pairs $((\Omega, \mathcal{F}, P), \mathcal{E})$, where $(\Omega, \mathcal{F}, P)$ is a probability space and $\mathcal{E} \subset \mathcal{F}$ a sub-$\sigma$-field. In other words, the second is a classification of morphisms $\alpha : (\Omega_1, \mathcal{F}_1, P_1) \to (\Omega_2, \mathcal{F}_2, P_2)$, or rather, triples $((\Omega_1, \mathcal{F}_1, P_1), (\Omega_2, \mathcal{F}_2, P_2), \alpha)$. These results are formulated below (without proofs). Orbit equivalence relations do not appear in Sect. 3a. A part of Rokhlin’s theory (roughly, the classification of probability spaces, but not morphisms) is presented also in [7], [13].

Let us choose once and for all our ‘favorites’: an uncountable standard Borel space $(\Omega_{\text{fav}}, \mathcal{F}_{\text{fav}})$, say, $(0, 1)$ with the Borel $\sigma$-field; and a nonatomic probability measure $P_{\text{fav}}$ on $(\Omega_{\text{fav}}, \mathcal{F}_{\text{fav}})$, say, the Lebesgue measure on $(0, 1)$; and a sequence of different points $\omega_1, \omega_2, \ldots \in \Omega_0$, say, $\omega_k = 2^{-k}$. We introduce the set $\mathcal{M}_{\text{fav}}$, consisting of all probability measures $\mu$ on $(\Omega_{\text{fav}}, \mathcal{F}_{\text{fav}})$ of the form

$$\mu(A) = m_0 P_{\text{fav}}(A) + \sum_{k=1}^{\infty} m_k 1_A(\omega_k),$$

where $m_k \in [0, 1]$, $m_0 + m_1 + m_2 + \cdots = 1$ and in addition, $m_1 \geq m_2 \geq \ldots$. In other words, $\mu$ is a convex combination of atoms at $\omega_k$ (with decreasing weights $m_k$), and the nonatomic measure $P_{\text{fav}}$.

3.1. Theorem. Every probability space is isomorphic (mod 0) to $(\Omega_{\text{fav}}, \mu)$ for one and only one measure $\mu \in \mathcal{M}_{\text{fav}}$.

See [12, Sect. 2.4]. Of course, the $\sigma$-field of $(\Omega_{\text{fav}}, \mu)$ is $\mathcal{F}_{\text{fav}}$ completed by adding $\mu$-negligible sets. We’ll usually suppress $\sigma$-fields in the notation like $(\Omega, P)$ when it is just the natural $\sigma$-field on $\Omega$, completed by $P$-negligible sets.

3.2. Note. All nonatomic probability spaces are isomorphic to $(\Omega_{\text{fav}}, P_{\text{fav}})$.

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28 Do not forget our convention proclaimed in Sect. 2a: every probability space is (by assumption or construction) a Lebesgue-Rokhlin space.

29 Containing all $P$-negligible sets (recall Sect. 2a).

30 That is, measure preserving maps (recall Sect. 2a).
Probability spaces are thus classified. In order to classify morphisms (in other words, sub-σ-fields, or measurable partitions) we introduce another set $\mathcal{M}_2$, consisting of all probability measures $\mu$ on $(\Omega^{\text{fav}}, \mathcal{F}^{\text{fav}}) \times (\Omega^{\text{fav}}, \mathcal{F}^{\text{fav}})$ of the form

$$\mu(A \times B) = \int_A \mu_a(B) \, d\mu_1(a),$$

where $\mu_1 \in \mathcal{M}^{\text{fav}}$, and $\mu_a \in \mathcal{M}^{\text{fav}}$ for $\mu_1$-almost all $a$, and the map $a \mapsto \mu_a$ is $\mu_1$-measurable. (In other words: both the marginal distribution $\mu_1$ and conditional distributions $\mu_a$ must belong to the class $\mathcal{M}^{\text{fav}}$.)

Especially, if $\mu_a$ is nonatomic for all $a$, then $\mu_a = P^{\text{fav}}$ and so, $\mu = \mu_1 \otimes P^{\text{fav}}$.

Every measure $\mu \in \mathcal{M}_2$ determines a morphism

$$((\Omega^{\text{fav}} \times \Omega^{\text{fav}}, \mu)) \xrightarrow{\text{pr}} (\Omega^{\text{fav}}, \mu_1),$$

just the projection, $(a, b) \mapsto a$.

3.3. **Theorem.** For every morphism $(\Omega_1, P_1) \xrightarrow{\alpha} (\Omega_2, P_2)$ of probability spaces there exist $\mu \in \mathcal{M}_2$ and isomorphisms $\beta, \gamma$ such that the diagram

$$
\begin{array}{ccc}
(\Omega_1, P_1) & \xleftarrow{\beta} & (\Omega^{\text{fav}} \times \Omega^{\text{fav}}, \mu) \\
\alpha \downarrow & & \downarrow \text{pr} \\
(\Omega_2, P_2) & \xrightarrow{\gamma} & (\Omega^{\text{fav}}, \mu_1)
\end{array}
$$

is commutative.

See [12, Sect. 4.1]. Unlike [11], $\mu$ is (in general) not uniquely determined by $\alpha$. If $\mu_1$ is purely atomic and all atoms have different probabilities, then $\mu$ is unique. However, if there are atoms of equal probability, say, $\mu_1(\{\omega_1\}) = \mu_1(\{\omega_2\})$ (that is, $m_1 = m_2$ for $\mu_1$), then conditional measures $\mu_a$ for $a = \omega_1$ and $a = \omega_2$ may be interchanged (correcting isomorphisms $\beta, \gamma$ accordingly). Though, uniqueness may persist, if these two conditional measures are equal. Similarly, if $\mu_1$ has a nonatomic part, corresponding conditional measures $\mu_a$ can be interchanged (but may happen to be equal). This is why $\mu \in \mathcal{M}_2$ cannot be used as an invariant of $\alpha$.

Especially, the case of $\mu = \mu_1 \otimes P^{\text{fav}}$ is the so-called *conditionally nonatomic* case. Here, of course, $\mu$ is uniquely determined by $\alpha$.

We’ll return to morphisms after classifying measurable functions.

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31I mean, by the triple $((\Omega_1, \mathcal{F}_1, P_1), (\Omega_2, \mathcal{F}_2, P_2), \alpha)$. 

18
A measurable function $f : \Omega \to \mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, P)$ determines a morphism $(\Omega, P) \to (\mathbb{R}, P_f)$ where $P_f$ is the distribution of $f$ (that is, $P_f(A) = P(f^{-1}(A))$). However, an isomorphism between two such functions, $f_1 : (\Omega_1, P_1) \to \mathbb{R}$ and $f_2 : (\Omega_2, P_2) \to \mathbb{R}$, is defined by a commutative diagram

$$
\begin{array}{ccc}
(\Omega_1, P_1) & \xrightarrow{\alpha} & (\Omega_2, P_2) \\
\downarrow f_1 & & \downarrow f_2 \\
(\mathbb{R}, P_f) & & (\mathbb{R}, P_f)
\end{array}
$$

Only the domain is transformed; the range ($\mathbb{R}$) is not. The equality $P_{f_1} = P_{f_2}(= P_f)$ is necessary but not sufficient for existence of an isomorphism. Conditional distributions are indexed by values $x$ of $f$ and cannot be interchanged.

We consider the class $\mathcal{M}_2(P_f)$ of all measures $\mu$ on $\mathbb{R} \times \Omega^{\text{fav}}$ of the form

$$
\mu(A \times B) = \int_A \mu_x(B) dP_f(x)
$$

where $\mu_x \in \mathcal{M}^{\text{fav}}$ for $P_f$-almost all $x \in \mathbb{R}$, and the map $x \mapsto \mu_x$ is $P_f$-measurable.\footnote{Thus, $\mathcal{M}_2$ introduced before is the union of sets $\mathcal{M}_2(\mu_1)$ over all $\mu_1 \in \mathcal{M}^{\text{fav}}$.}

3.4. Theorem. For every probability space $(\Omega, \mathcal{F}, P)$ and every measurable function $f : \Omega \to \mathbb{R}$ there exists one and only one measure $\mu \in \mathcal{M}_2(P_f)$ such that there exists an isomorphism $\alpha$ such that the diagram

$$
\begin{array}{ccc}
(\Omega, P) & \xrightarrow{\alpha} & (\mathbb{R} \times \Omega^{\text{fav}}, \mu) \\
\downarrow f & & \downarrow \text{pr} \\
(\mathbb{R}, P_f) & & (\mathbb{R}, P_f)
\end{array}
$$

is commutative.

The proof is left to the reader (hint: (re)read \cite[Sect. 4]{12}). Note that $\mu$ is uniquely determined by $f$, but $\alpha$ is not. Say, if some $\mu_x$ has two atoms of equal probability, their interchange influences $\alpha$ (but not $\mu$). In some sense, $\mu_x$ describes the multiplicity of the value $x$ of $f$.

Note also the conditionally nonatomic case: $\mu = P_f \otimes P^{\text{fav}}$; here, $\alpha$ is highly non-unique.
There is nothing special in $\mathbb{R}$ as the range of $f$; any other Polish (or standard Borel) space may be used instead. Especially, $\mathcal{M}^{\text{fav}}$-valued measurable functions will be used in Theorem 3.3.

Now we return to morphisms. Recall that the measure $\mu \in \mathcal{M}_2$ is not uniquely determined by a morphism $\alpha : (\Omega_1, P_1) \to (\Omega_2, P_2)$. Note however that the projection $\mu_1$ of $\mu$ is uniquely determined by the morphism, since $(\Omega_2, P_2)$ is isomorphic to $(\Omega^{\text{fav}}, \mu_1)$.

3.5. Theorem. Let two measures $\mu', \mu''$ on $\Omega^{\text{fav}} \times \Omega^{\text{fav}}$ belong to $\mathcal{M}_2$, both having the same projection $\mu_1 \in \mathcal{M}^{\text{fav}}$. Then existence of isomorphisms $\alpha, \beta$ making commutative the diagram

\[
\begin{array}{ccc}
(\Omega^{\text{fav}} \times \Omega^{\text{fav}}, \mu') & \xrightarrow{\alpha} & (\Omega^{\text{fav}} \times \Omega^{\text{fav}}, \mu'') \\
\downarrow \text{pr} & & \downarrow \text{pr} \\
(\Omega^{\text{fav}}, \mu_1) & \xrightarrow{\beta} & (\Omega^{\text{fav}}, \mu_1)
\end{array}
\]

is equivalent to existence of an isomorphism $\gamma$ making commutative the diagram

\[
\begin{array}{ccc}
(\Omega^{\text{fav}}, \mu_1) & \xrightarrow{\gamma} & (\Omega^{\text{fav}}, \mu_1) \\
\downarrow \text{f} & & \downarrow \text{f'} & \downarrow \text{f''} \\
\mathcal{M}^{\text{fav}} & \xrightarrow{f'} & \mathcal{M}^{\text{fav}} & \xrightarrow{f''} & \mathcal{M}^{\text{fav}}
\end{array}
\]

where a measurable function $f' : \Omega^{\text{fav}} \to \mathcal{M}^{\text{fav}}$ is defined by $f'(a) = \mu'_a$, $\mu'_a$ being the conditional measure of $\mu'$,\footnote{That is, $\mu'(A \times B) = \int_A \mu'_a(B) \, d\mu_1(a)$.} the same for $f''$ (and $\mu''$, $\mu''$).

See [12, Sect. 4.1]. Existence of $\gamma$ may be checked via the invariant given in Theorem 3.4. Combining 3.3 and 3.4 we see that a complete invariant of a morphism $\alpha : (\Omega_1, P_1) \to (\Omega_2, P_2)$ of probability spaces consists of

- a measure $\mu_1 \in \mathcal{M}^{\text{fav}}$ on $\Omega^{\text{fav}}$, describing the type of the probability space $(\Omega_2, P_2)$;
- a measure $\mu_2$ on the space $\mathcal{M}^{\text{fav}}$ (sic! not a measure belonging to $\mathcal{M}^{\text{fav}}$) describing the distribution of the type of the conditional measure;
- a measure $\mu_3 \in \mathcal{M}_2(\mu_2)$ describing the type of the conditional measure on $\Omega_1$, treated as a measurable function on $\Omega_2$.\footnote{Thus, $(\mu_3)_x$ describes the multiplicity of the type $x \in \mathcal{M}^{\text{fav}}$ of the conditional measure.}

Note the conditionally nonatomic case; here, $\mu_2$ is concentrated at the point $P^{\text{fav}}$ of the space $\mathcal{M}^{\text{fav}}$, and $\mu_3 = \mu_2 \otimes \mu_1$.\footnote{That is, $\mu'(A \times B) = \int_A \mu'_a(B) \, d\mu_1(a)$.}


3b The construction

Theorem 3.1 shows that probability spaces can be classified by points of \( \mathcal{M}^{\text{fav}} \), the latter being a Polish space. It should mean smoothness, as defined by \( \text{[1.22(e)]} \). To this end we want to embed Rokhlin’s theory into the framework of orbit equivalence relations on Polish G-spaces. We need to construct a Polish space \( X \) whose points represent probability spaces, such that all isomorphic classes are available in \( X \), and all their isomorphisms are available in a single Polish group \( G \) acting on \( X \).

Recall our ‘favorite’ probability space \((\Omega^{\text{fav}}, P^{\text{fav}})\).\(^{35}\) We consider the Polish space

\[ S\sigma F^{\text{fav}} = S\sigma F(\Omega^{\text{fav}}, P^{\text{fav}}) \]

of all sub-\( \sigma \)-fields (see \( \text{[2.2]} \)), and the Polish group

\[ G^{\text{fav}} = \text{Aut}(\Omega^{\text{fav}}, P^{\text{fav}}) \]

of all measure preserving automorphisms (see \( \text{[1.33]} \)). Note that

\[ S\sigma F^{\text{fav}} \text{ is a Polish } G^{\text{fav}} \text{-space,} \]

since \( G^{\text{fav}} \) is a closed subgroup of \( \text{U}(L_2(\Omega^{\text{fav}}, P^{\text{fav}})) \), and \( S\sigma F^{\text{fav}} \subset L(L_2(\Omega^{\text{fav}}, P^{\text{fav}})) \); recall \( \text{[1.32]} \).

If \( \mathcal{E}_1, \mathcal{E}_2 \in S\sigma F^{\text{fav}} \) belong to the same orbit then the corresponding quotient spaces are isomorphic; however, the converse is wrong. Say, it may happen that \( \mathcal{E}_1, \mathcal{E}_2 \in S\sigma F^{\text{fav}} \) are both nonatomic, but \( \mathcal{E}_1 = \mathcal{F}^{\text{fav}}, \mathcal{E}_2 \neq \mathcal{F}^{\text{fav}} \). Then the two quotient spaces are isomorphic, however, \( \mathcal{E}_1, \mathcal{E}_2 \) belong to different orbits. For that reason we introduce the set

\[ X_1 \subset S\sigma F^{\text{fav}} \]

consisting of all conditionally nonatomic sub-\( \sigma \)-fields.\(^{36}\) Note that for all \( \mathcal{E}_1, \mathcal{E}_2 \in S\sigma F^{\text{fav}} \)

\[ \text{if } \mathcal{E}_1 \subset \mathcal{E}_2 \text{ and } \mathcal{E}_2 \in X_1 \text{ then } \mathcal{E}_1 \in X_1 . \]
3.8. Lemma. (a) Every probability space is isomorphic to the quotient space \((\Omega^{\text{fav}}, P^{\text{fav}})/\mathcal{E}\) for some \(\mathcal{E} \in X_1\).

(b) For every \(\mathcal{E}_1, \mathcal{E}_2 \in X_1\) the following two conditions are equivalent:
- quotient spaces \((\Omega^{\text{fav}}, P^{\text{fav}})/\mathcal{E}_1\) and \((\Omega^{\text{fav}}, P^{\text{fav}})/\mathcal{E}_2\) are isomorphic;
- \(\mathcal{E}_2 = g \cdot \mathcal{E}_1\) for some \(g \in G^{\text{fav}}\).

Proof. (a) By Theorem 3.1 we may restrict ourselves to probability spaces \((\Omega^{\text{fav}}, \mu)\), where \(\mu \in M^{\text{fav}}\) is parametrized by \(m_0, m_1, m_2, \ldots\). We choose disjoint sets \(A_k \subset \Omega^{\text{fav}}\) such that \(P^{\text{fav}}(A_k) = m_k\) and consider the sub-\(\sigma\)-field \(\mathcal{E}\) generated by all \(A_k\) and all measurable subsets of \(A_0\). Clearly, \((\Omega^{\text{fav}}, P^{\text{fav}})/\mathcal{E}\) is isomorphic to \((\Omega^{\text{fav}}, \mu)\). Though, \(\mathcal{E} \notin X_1\) (unless \(m_0 = 0\)). However, we may use another probability space \((\Omega, P^{\text{fav}}) = (\Omega^{\text{fav}}, P^{\text{fav}}) \times (\Omega^{\text{fav}}, P^{\text{fav}})\) and another sub-\(\sigma\)-field \(\mathcal{E}_2 = \mathcal{E} \otimes \mathcal{0}_{\text{SorF}}\); here \(\mathcal{0}_{\text{SorF}}\) is the trivial \(\sigma\)-field (consisting of sets of probability 0 or 1 only). Indeed, \(\mathcal{E}_2\) is conditionally nonatomic, and \((\Omega, P^{\text{fav}})/\mathcal{E}_2\) is isomorphic to \((\Omega^{\text{fav}}, P^{\text{fav}})/\mathcal{E}\), therefore, to \((\Omega^{\text{fav}}, \mu)\).

(b) If \(\mathcal{E}_2 = g \cdot \mathcal{E}_1\) then \(g\) evidently gives us an isomorphism between the quotient spaces. On the other hand, let the quotient spaces be isomorphic, then Theorem 3.3 (specialized for the conditionally nonatomic case) gives us a commutative diagram

\[
\begin{array}{c}
\Omega^{\text{fav}}, P^{\text{fav}} \overset{\alpha_1}{\longrightarrow} \Omega^{\text{fav}} \times \Omega^{\text{fav}}, \mu_1 \otimes P^{\text{fav}} \overset{\alpha_2}{\longrightarrow} \Omega^{\text{fav}}, P^{\text{fav}} \\
\downarrow \quad \downarrow \quad \downarrow \\
\Omega^{\text{fav}}, P^{\text{fav}}/\mathcal{E}_1 \quad \Omega^{\text{fav}}, \mu_1 \quad \Omega^{\text{fav}}, P^{\text{fav}}/\mathcal{E}_2
\end{array}
\]

Combining \(\alpha_1\) and \(\alpha_2\) we get \(g \in G^{\text{fav}}\) such that \(\mathcal{E}_2 = g \cdot \mathcal{E}_1\).

3.9. Note. In addition to Item (b) of Lemma 3.8:
Every isomorphism between \((\Omega^{\text{fav}}, P^{\text{fav}})/\mathcal{E}_1\) and \((\Omega^{\text{fav}}, P^{\text{fav}})/\mathcal{E}_2\) is induced by some (at least one) \(g \in G^{\text{fav}}\).

(A proof is implicitly contained in the proof of Lemma 3.8)

3.10. Lemma. \(X_1\) is a \(G_\delta\)-set in \(S_{\text{SorF}}^{\text{fav}}\).

Proof. We start with rather general claims; they hold for any probability space \((\Omega, \mathcal{F}, P)\) and any Polish space \(X\).

A. Claim. Let \(F \subset X\) be a closed set. Consider such a function \(\text{Prob}(X) \to \mathbb{R}\):

\[
\text{Prob}(X) \ni \mu \mapsto \mu(F) \in \mathbb{R}.
\]
The function is upper semicontinuous.\(^{37}\)
(See [10], 17.20(iii) or [3], 11.1.1(c); see also [10], 17.29.)

B. CLAIM. For any \(f \in L_0(\Omega, \mathcal{F}, P; X)\), the map (recall the end of 2a)
\[
S_{\sigma F}(\Omega, \mathcal{F}, P) \ni E \mapsto P_{f|E} \in L_0(\Omega, \mathcal{F}, P; \text{Prob}(X))
\]
is continuous.

**Proof of Claim.** Let \(\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2, \cdots \in S_{\sigma F}, \mathcal{E}_n \to \mathcal{E}\). It suffices to find a subsequence \(\mathcal{E}_{n_k}\) such that \(P_{f|\mathcal{E}_{n_k}} \to P_{f|\mathcal{E}}\) almost everywhere. For any given \(\varphi\) (bounded, continuous, \(X \to \mathbb{R}\)) we have \(\mathbb{E}\left(\varphi \circ f \big| \mathcal{E}_n\right) \to \mathbb{E}\left(\varphi \circ f \big| \mathcal{E}\right)\) in \(L_2\) by (2.3); convergence almost everywhere holds for some subsequence, however, the subsequence may depend on \(\varphi\). Doing so for a countable set of functions \(\varphi\) that generates the topology of \(\text{Prob}(X)\) (recall 1.10) we get a subsequence such that \(\mathbb{E}\left(\varphi \circ f \big| \mathcal{E}_{n_k}\right) \to \mathbb{E}\left(\varphi \circ f \big| \mathcal{E}\right)\) almost everywhere for all \(\varphi\) simultaneously; thus \(P_{f|\mathcal{E}_{n_k}} \to P_{f|\mathcal{E}}\) almost everywhere. \(\square\)

C. CLAIM. If \(\varphi : X \to [0, 1]\) is upper semicontinuous, then the function\(^{38}\)
\[
L_0(\Omega, \mathcal{F}, P; X) \ni f \mapsto \mathbb{E}\varphi(f) \in \mathbb{R}
\]
is upper semicontinuous.

**Proof of Claim.** Let \(f, f_1, f_2, \cdots \in L_0(\Omega, \mathcal{F}, P; X)\), \(f_n \to f\); we have to prove that \(\mathbb{E}\varphi(f) \geq \lim \sup \mathbb{E}\varphi(f_n)\). We may assume that \(f_n \to f\) almost everywhere (not only in probability), due to the argument of subsequences. Semicontinuity of \(\varphi\) gives \(\varphi(f) \geq \lim \sup \varphi(f_n)\) almost everywhere. Therefore \(\mathbb{E}\varphi(f) \geq \lim \sup \mathbb{E}\varphi(f_n)\) by Fatou’s theorem (applied to \(1 - \varphi(f_n)\)). \(\square\)

D. CLAIM. The function
\[
\text{Prob}(X) \ni \mu \mapsto \sup_{x \in X} \mu(\{x\}) \in [0, 1]
\]
is upper semicontinuous.

**Proof of Claim.** It is easy to check that
\[
\sup_{x \in X} \mu(\{x\}) = \inf_{(U_k)} \max_k \mu(U_k^{cl});
\]
here the infimum is taken over all finite open coverings \((U_k)_{k=1, \ldots, n}\) of \(X\), and \(U_k^{cl}\) stands for the closure of \(U_k\).\(^{39}\) Each \(\mu(U_k^{cl})\) is upper semicontinuous (in \(\mu\)) by Claim A; therefore the infimum is upper semicontinuous. \(\square\)

---

\(^{37}\)That is, \(\{\mu : \mu(F) < x\}\) is an open subset of \(\text{Prob}(X)\) for every \(x \in \mathbb{R}\).

\(^{38}\)Here \(\mathbb{E}\varphi(f)\) means \(\int f \varphi(f) \, dP\).

\(^{39}\)The formula holds also without taking the closure; however, we need closed sets here.
Now we choose some $f \in L_0(\Omega^{\text{fav}}, P^{\text{fav}})$ that generates the whole $\sigma$-field $\mathcal{F}^{\text{fav}}$ a sub-$\sigma$-field $\mathcal{E} \in \mathcal{S}\mathcal{F}^{\text{fav}}$ is conditionally nonatomic if and only if almost all conditional distributions $P_{f|\mathcal{E}}$ are nonatomic, which may be written as $\mathbb{E}\varphi(P_{f|\mathcal{E}}) = 0$, where $\varphi : \text{Prob}(\mathbb{R}) \to [0,1]$ is defined by $\varphi(\mu) = \sup_{r \in \mathbb{R}} \mu(\{r\})$. The map $\mathcal{S}\mathcal{F}^{\text{fav}} \ni \mathcal{E} \mapsto \mathbb{E}\varphi(P_{f|\mathcal{E}})$ is upper semicontinuous, since it is the composition of the map $\mathcal{S}\mathcal{F}^{\text{fav}} \ni \mathcal{E} \mapsto P_{f|\mathcal{E}} \in L_0(\Omega, \mathcal{F}, P; \text{Prob}(\mathbb{R}))$, continuous by Claim B, and the map $L_0(\Omega, \mathcal{F}, P; \text{Prob}(\mathbb{R})) \ni Z \mapsto \mathbb{E}\varphi(Z)$, upper semicontinuous by Claims C and D. Therefore the set $\{\mathcal{E} \in \mathcal{S}\mathcal{F}^{\text{fav}} : \mathbb{E}\varphi(P_{f|\mathcal{E}}) < \varepsilon\}$ is open for any $\varepsilon > 0$, and so, the set $\{\mathcal{E} \in \mathcal{S}\mathcal{F}^{\text{fav}} : \mathbb{E}\varphi(P_{f|\mathcal{E}}) = 0\}$ belongs to the class $G_\delta$.

We may say that comeager many sub-$\sigma$-fields are conditionally nonatomic, in the following sense.

3.11. Corollary. $X_1$ is a dense $G_\delta$, therefore comeager subset of $\mathcal{S}\mathcal{F}^{\text{fav}}$.

Proof. By 3.10, $X_1$ is $G_\delta$; also, $X_1$ is dense in $\mathcal{S}\mathcal{F}^{\text{fav}}$, since $X_1$ contains all finite sub-$\sigma$-fields, these being dense.

3.12. Theorem. (a) $X_1$ is a Polish $G^{\text{fav}}$-space.

(b) The orbit equivalence relation on $X_1$ is smooth.

Proof. (a) A $G_\delta$-subset of a Polish space is a Polish space, see [10, 3.11] or [13, 2.5.4]. We use 3.10 and note that $X_1$ is $G^{\text{fav}}$-invariant.

(b) Similarly to the proof of 3.8(a), for any $\mu \in \mathcal{M}^{\text{fav}}$ parametrized by $m_0, m_1, m_2, \ldots$ we construct $\mathcal{E} \in \mathcal{S}\mathcal{F}^{\text{fav}}$ such that $(\Omega^{\text{fav}}, P^{\text{fav}})/\mathcal{E}$ is isomorphic to $(\Omega^{\text{fav}}, \mu)$, and use an isomorphism between $(\Omega^{\text{fav}}, P^{\text{fav}})$ and its square $(\Omega_2, P_2)$, and the conditionally nonatomic sub-$\sigma$-field $\mathcal{E}_2 = \mathcal{E} \otimes \mathcal{E}$. In contrast to that proof, now we construct $\mathcal{E}$ in a canonical way, using the fact that $\Omega^{\text{fav}} = (0,1)$. Namely, atoms $A_k$ of $\mathcal{E}_k$ are intervals $A_1 = (m_0, m_0 + m_1)$, $A_2 = (m_0 + m_1, m_0 + m_1 + m_2)$, and so on; and $A_0 = (0, m_0)$. Thus, $\mathcal{E}$ depends on $(m_0, m_1, \ldots)$ continuously, and the set of all these sub-$\sigma$-fields $\mathcal{E}$ is closed in $\mathcal{S}\mathcal{F}^{\text{fav}}$ (in fact, it is homeomorphic to the compact space $\mathcal{M}^{\text{fav}}$). The same for the set of corresponding $\mathcal{E}_2$. The latter set, transplanted from $(\Omega_2, P_2)$ to $(\Omega^{\text{fav}}, P^{\text{fav}})$ by an isomorphism, gives us a Borel transversal in $X_1$ due to 3.1; recall 1.22(d) and 1.23(a,c).
So, orbits of $X_1$ are in a natural, Borel measurable, one-one correspondence with points of $\mathcal{M}^{\text{fav}}$. These are isomorphic types of probability spaces. Especially, nonatomic probability spaces are described by a single orbit, consisting of nonatomic $\mathcal{E} \in X_1$.

### 3c Independence and products

The set $\Sigma \mathcal{F}^{\text{fav}}$ of sub-$\sigma$-fields is a lattice; for any $\mathcal{E}, \mathcal{F} \in \Sigma \mathcal{F}^{\text{fav}}$ there exist the least sub-$\sigma$-field $\mathcal{E} \vee \mathcal{F}$ containing $\mathcal{E}$ and $\mathcal{F}$, and the greatest sub-$\sigma$-field $\mathcal{E} \wedge \mathcal{F}$ contained in $\mathcal{E}$ and $\mathcal{F}$. In fact, $\mathcal{E} \wedge \mathcal{F}$ is just $\mathcal{E} \cap \mathcal{F}$; in contrast, $\mathcal{E} \vee \mathcal{F}$ is generated by $\mathcal{E} \cup \mathcal{F}$.

If $\mathcal{E}, \mathcal{F}$ are independent, then $\mathcal{E} \vee \mathcal{F}$ may be called the product of $\mathcal{E}, \mathcal{F}$ and denoted also by $\mathcal{E} \times \mathcal{F}$ (We do not define $\mathcal{E} \times \mathcal{F}$ when $\mathcal{E}, \mathcal{F}$ are dependent.) The same for any finite or countable family of sub-$\sigma$-fields.

Multiplication of sub-$\sigma$-fields is closely related to multiplication of probability spaces, which is well-known (see [1, Prop. 4]).

#### 3.13. Note.

(a) Let $(\Omega_1, P_1), (\Omega_2, P_2), (\Omega_3, P_3)$ be three probability spaces and $(\Omega, P)$ their product. Then $\mathcal{E}_1 \times \mathcal{E}_2 = \mathcal{E}_{12}$, where sub-$\sigma$-fields $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_{12}$ on $(\Omega, P)$ correspond to the first factor, the second factor, and to both factors, respectively.

(b) Let $(\Omega, P)$ be a probability space, $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_{12}$ sub-$\sigma$-fields on $(\Omega, P)$ such that $\mathcal{E}_1 \times \mathcal{E}_2 = \mathcal{E}_{12}$. Then the quotient space $(\Omega, P)/\mathcal{E}_{12}$ is naturally isomorphic to the product of two quotient spaces $((\Omega, P)/\mathcal{E}_1) \times ((\Omega, P)/\mathcal{E}_2)$.

#### 3.14. Lemma.

For every $\mathcal{E} \in \Sigma \mathcal{F}^{\text{fav}}$ the following three conditions are equivalent.

(a) $\mathcal{E} \in X_1$;

(b) there exists a nonatomic $\mathcal{F} \in \Sigma \mathcal{F}^{\text{fav}}$ such that $\mathcal{E} \times \mathcal{F} = 1_{\Sigma \mathcal{F}^{\text{fav}}}$;

(c) the morphism $(\Omega^{\text{fav}}, P^{\text{fav}}) \to (\Omega^{\text{fav}}, P^{\text{fav}})/\mathcal{E}$ is isomorphic to the projection $(\Omega^{\text{fav}} \times \Omega^{\text{fav}}, \mu_1 \otimes P^{\text{fav}}) \to (\Omega^{\text{fav}}, \mu_1)$ for some $\mu_1 \in \mathcal{M}^{\text{fav}}$.

Proof. (a) $\iff$ (c) by Theorem 3.3; and (b) $\iff$ (c) by Note 3.13.

#### 3.15. Lemma.

For every $\mathcal{E}_1, \mathcal{E}_2, \cdots \in X_1$ there exist $\mathcal{E} \in X_1$ and $g_1, g_2, \cdots \in G^{\text{fav}}$ such that

$$\left( g_1 \cdot \mathcal{E}_1 \right) \times \left( g_2 \cdot \mathcal{E}_2 \right) \times \cdots = \mathcal{E}.$$  

---

44 As defined by 2.7.
45 Or maybe $\mathcal{E} \otimes \mathcal{F}$.
46 That is, $\mathcal{E}$ and $\mathcal{F}$ are independent, and $\mathcal{E} \vee \mathcal{F}$ is the whole $1_{\Sigma \mathcal{F}} = \mathcal{F}^{\text{fav}}$. 

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Proof. The countable product of probability spaces

\[(\Omega, P) = (\Omega^{\text{fav}}, P^{\text{fav}}) \times (\Omega^{\text{fav}}, P^{\text{fav}}) \times \ldots\]

is again a probability space isomorphic to \((\Omega^{\text{fav}}, P^{\text{fav}})\). Consider sub-\(\sigma\)-fields \(\mathcal{E}_1 = \mathcal{E}_1 \otimes \mathcal{B}_{SDF} \otimes \mathcal{B}_{SDF} \otimes \ldots, \mathcal{E}_2 = \mathcal{B}_{SDF} \otimes \mathcal{B}_2 \otimes \mathcal{B}_{SDF} \otimes \ldots,\) and so on. They are independent, and \(\tilde{\mathcal{E}} = \mathcal{E}_1 \times \mathcal{E}_2 \times \ldots\) is conditionally nonatomic (since \(\mathcal{E}_1\) is; use 3.14(b)). We transplant \(\mathcal{E}_n\) from \((\Omega, P)\) to \((\Omega^{\text{fav}}, P^{\text{fav}})\) by some isomorphism and apply 3.8(b), getting \(g_n \cdot \mathcal{E}_n, \mathcal{E}\). \(\square\)

3.16. Note. The orbit of \(\mathcal{E}\) is uniquely determined by orbits of \(\mathcal{E}_1, \mathcal{E}_2, \ldots\) Therefore, multiplication of sub-\(\sigma\)-fields induces on the set of orbits, \(X/G^{\text{fav}}\), an associative operation that takes any finite or countable number of operands. The corresponding operation on \(M^{\text{fav}}\) is easy to describe explicitly.

3.17. Lemma. Let \(\mathcal{E}_1, \mathcal{E}_2, \ldots \in S\sigma F^{\text{fav}}\) be independent. Consider the countable product

\[\{0, 1\}^\infty = \{0, 1\} \times \{0, 1\} \times \ldots\]

of two-point topological spaces. For every \(i = (i_1, i_2, \ldots) \in \{0, 1\}^\infty\) consider the sub-\(\sigma\)-field

\[\mathcal{E}_i = \mathcal{E}_1^{i_1} \times \mathcal{E}_2^{i_2} \times \ldots,\]

where \(\mathcal{E}_n = \mathcal{E}_n, \mathcal{E}_n^0 = \mathcal{B}_{SDF}\). Then the map

\[\{0, 1\}^\infty \ni i \mapsto \mathcal{E}_i \in S\sigma F^{\text{fav}}\]

is continuous.

Proof. Due to (2.3) we have to prove continuity of the function

\[\{0, 1\}^\infty \ni i \mapsto \mathbb{E} \left( f \middle| \mathcal{E}_i \right) \in L_2(\Omega^{\text{fav}}, P^{\text{fav}})\]

for every \(f \in L_2\). We may assume that \(f\) is measurable w.r.t. \(\mathcal{E}_1 \times \mathcal{E}_2 \times \ldots\). Then \(f\) is a sum of functions of the form \(f_{k_1} \ldots f_{k_n}\) where \(k_1 < \ldots < k_n\), and each \(f_{k_j}\) is measurable w.r.t. \(\mathcal{E}_{k_j}\), and \(\mathbb{E} f_{k_j} = 0\) for all \(j\). However, \(\mathbb{E} \left( f_{k_1} \ldots f_{k_n} \middle| \mathcal{E}_i \right) = (i_{k_1} \ldots i_{k_n}) f_{k_1} \ldots f_{k_n}\), which evidently is continuous in \(i\). \(\square\)

\(^{47}\text{Hint: use 3.8(b).}\)
3.18. Corollary. Let $\mathcal{E}_1, \mathcal{E}_2, \cdots \in S\sigma F^{fav}$ be independent. Then $\mathcal{E}_n \to 0_{S\sigma F}$ and $\mathcal{E}_1 \times \mathcal{E}_n \to \mathcal{E}_1$ for $n \to \infty$.

3.19. Lemma. For every $\mathcal{E} \in X_1$ there exist $g_1, g_2, \cdots \in G^{fav}$ such that
$$g_k \cdot \mathcal{E} \to 0_{S\sigma F} \text{ for } k \to \infty.$$ 

Proof. Lemma 3.15 gives us $g_k$ such that $g_k \cdot \mathcal{E}$ are independent. Corollary 3.18 ensures that $g_k \cdot \mathcal{E} \to 0_{S\sigma F}$.

3.20. Corollary. The closure of every orbit in $X_1$ contains the trivial $\sigma$-field $0_{S\sigma F}$.

3d Ergodicity

3.21. Lemma. $X_1$ is ergodic (as defined by 1.24).

Proof. Due to 1.25 it suffices to prove that $X_1$ contains a dense orbit. We’ll see that the orbit of nonatomic sub-$\sigma$-fields is dense. Let $\mathcal{E}_0 \in X_1$ be nonatomic, and $\mathcal{E} \in X_1$ be arbitrary. Lemma 3.15 gives us $g_1, g_2, \cdots \in G^{fav}$ such that $\mathcal{E}, g_1 \cdot \mathcal{E}_0, g_2 \cdot \mathcal{E}_0, \cdots$ are independent. Corollary 3.18 shows that $\mathcal{E} \times g_k \cdot \mathcal{E}_0 \to \mathcal{E}$ for $k \to \infty$. However, $g_k \cdot \mathcal{E}_0$ is nonatomic, therefore $\mathcal{E} \times g_k \cdot \mathcal{E}_0$ is nonatomic. Being also conditionally nonatomic, it belongs to the orbit of nonatomic sub-$\sigma$-fields. Thus, $\mathcal{E}$ belongs to the closure of that orbit.

Combining 3.21, 3.12(b) and 1.26 we see that $X_1$ contains a comeager orbit. It is easy to guess that it is the orbit of nonatomic sub-$\sigma$-fields. The next result confirms the guess.

3.22. Proposition. The orbit of nonatomic sub-$\sigma$-fields is a dense $G_\delta$, therefore comeager subset of $X_1$.

Proof. The orbit is dense (as was shown in the proof of 3.21); we have to prove that it is $G_\delta$. Similarly to the proof of 3.10 we’ll find an upper semicontinuous function vanishing exactly on that orbit.

For a given $f \in L_2(\Omega^{fav}, P^{fav})$ consider the (unconditional) distribution $P_{\mathcal{E}(f|\mathcal{E})}$ of the conditional expectation $\mathbb{E}(f|\mathcal{E})$. The function
$$S\sigma F^{fav} \ni \mathcal{E} \mapsto P_{\mathcal{E}(f|\mathcal{E})} \in \text{Prob}(\mathbb{R})$$
is continuous (recall (2.3) and 1.10). On the other hand, the function
$$\text{Prob}(\mathbb{R}) \ni \mu \mapsto \varphi(\mu) = \sup_{r \in \mathbb{R}} \mu\{r\} \in [0, 1]$$

is continuous (recall (2.3) and 1.10). On the other hand, the function

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is upper semicontinuous (recall Claim 3.10.D). Hence the function

\[ S\sigma^{F_{\text{fav}}} \ni \mathcal{E} \mapsto \varphi(P_{E(f|\mathcal{F})}) \in [0,1] \]

is upper semicontinuous. Therefore, its infimum over all \( f \in L_2(\Omega^{\text{fav}}, P^{\text{fav}}) \) is upper semicontinuous. It is easy to see that the infimum vanishes if and only if \( \mathcal{E} \) is nonatomic. \( \square \)

4 Isomorphism of filtrations as an orbit equivalence relation

4a The construction

Recall that filtrations are defined by 2.4, and their isomorphisms — by 2.6(a).

Lemma 3.8 has a counterpart for filtrations. Denote by \( X \) the set of all filtrations \((\mathcal{E}_t)\) on \((\Omega^{\text{fav}}, P^{\text{fav}})\) such that \( \mathcal{E}_\infty \in X_1 \).

4.1. Lemma. (a) Every filtration on every probability space is isomorphic to some filtration belonging to \( X \).

(b) Two filtrations belonging to \( X \) are isomorphic if and only if some automorphism \( g \in G^{\text{fav}} \) sends one of them to the other.

Proof. (a) Lemma 3.8(a) gives us \( \mathcal{E} \in X_1 \) and an isomorphism \( \alpha \) between the given probability space and \((\Omega^{\text{fav}}, P^{\text{fav}})/\mathcal{E})\). The \( \alpha \)-image of the given filtration is an isomorphic filtration belonging to \( X \).

(b) If \( g \) sends one filtration to the other then they are evidently isomorphic. On the other hand, let two filtrations \((\mathcal{E}_t^{(1)}), (\mathcal{E}_t^{(2)}) \in X \) be isomorphic. Their isomorphism \( \alpha \) is an isomorphism between quotient spaces \((\Omega^{\text{fav}}, P^{\text{fav}})/\mathcal{E}_\infty^{(1)}) \) and \((\Omega^{\text{fav}}, P^{\text{fav}})/\mathcal{E}_\infty^{(2)})\) sending each \( \mathcal{E}_t^{(1)} \) to \( \mathcal{E}_t^{(2)} \). Note 3.8 gives us \( g \in G^{\text{fav}} \) that does the same. \( \square \)

4.2. Note. In addition to Item (b) of Lemma 4.1:

Every isomorphism between two filtrations belonging to \( X \) is induced by some (at least one) \( g \in G^{\text{fav}} \).

(A proof is implicitly contained in the proof of Lemma 4.1.)

Dealing with continuous-time filtrations it is natural to require continuity (recall 2.4(b)). On the other hand, discrete-time filtrations should not be forgotten. It is convenient to deviate from Definition 2.4 as follows.
4.3. Definition. Let $T$ be a compact subset of $[0, +\infty]$\footnote{Or just a linearly ordered compact topological space, embeddable to $[0, +\infty]$ (or equivalently, to $[0, 1]$).} We define $X_T$ as the set of all families $(\mathcal{F}_t)_{t \in T}$ of sub-$\sigma$-fields $\mathcal{F}_t \in X_1$ such that

- $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s \leq t$;
- the map $T \ni t \mapsto \mathcal{F}_t \in X_1$ is continuous;
- the least sub-$\sigma$-field (corresponding to the least element of $T$) is the trivial $\sigma$-field $\mathbf{0}_{0\sigma\mathcal{F}}$ (consisting of sets of probability 0 or 1 only).

The set $X_T$ is a subset of the Polish $G^{\text{fav}}$-space $C(T, X_1)$ (recall 1.28). It is easy to see that $X_T$ is a closed invariant set. Therefore

$$X_T$$

is a Polish $G^{\text{fav}}$-space.

Lemma 4.1 shows that orbits of $X_T$ are in a natural one-one correspondence with isomorphic types of continuous filtrations on $T$ (starting with the trivial $\sigma$-field).

Immersed filtrations and morphisms of filtrations were defined by 2.5, 2.6 in the framework of 2b, but their counterparts for $X_T$ are evident.

The trivial filtration $\mathbf{0}_{X_T} = (\mathbf{0}_{0\sigma\mathcal{F}})_{t \in T}$ is immersed into every filtration. Its orbit is a single point.

4.4. Lemma. For every filtrations $x, y \in X_T$, the following two conditions are equivalent.

1. There exists a morphism from $x$ to $y$.
2. There exists $g \in G^{\text{fav}}$ such that $g \cdot y$ is immersed into $x$.

Proof. Evidently, (2) implies (1). Assume (1). A morphism from $x$ to $y$ is, by definition, an isomorphism between $y$ and some filtration $z$ immersed into $x$; and $z$ necessarily belongs to $X_T$. Note 4.2 gives us $g \in G^{\text{fav}}$ such that $g \cdot y = z$.

4.5. Lemma. Pairs $(x, y)$ of filtrations $x, y \in X_T$ such that $x$ is immersed into $y$ are a closed subset of $X_T \times X_T$.

Proof. As was said after Definition 2.3, immersion can be expressed in operator form as $\mathcal{F}_s \mathcal{E}_t = \mathcal{E}_s$. The operator $\mathcal{E}_t$ of conditional expectation depends continuously on the corresponding sub-$\sigma$-field (recall (2.3)), therefore, on the filtration $x$. The strong operator topology is meant. The product of two operators is jointly continuous (in these operators), as far as all operators are of norm $\leq 1$.  

\footnote{Or just a linearly ordered compact topological space, embeddable to $[0, +\infty]$ (or equivalently, to $[0, 1]$).}
4.6. Lemma. Pairs \((x, y)\) of filtrations \(x, y \in X_T\) satisfying equivalent conditions of Lemma 4.4 are an analytic subset of \(X_T \times X_T\).

**Proof.** We take the closed (by 4.5) set of pairs \((x, y)\) such that \(y\) is immersed into \(x\), multiply it by \(G^{fav}\) and apply the continuous map \((g, x, y) \mapsto (x, g^{-1} \cdot y)\).

Especially, we may take \(T = [0, +\infty]\) and consider all continuous filtrations immersed into Brownian filtrations.\(^{49}\) The set of such filtrations is analytic, therefore it has the Baire property and is universally measurable. Similarly we may consider filtrations that contain immersed Brownian filtrations.

4b Independence and products

Each filtration \(x \in X_T\) has its maximal sub-\(\sigma\)-field (corresponding to the maximal point of \(T\)). Filtrations are called independent, if their maximal sub-\(\sigma\)-fields are independent.

For two independent filtrations \(x, y \in X_T\), \(x = (\mathcal{E}_t)_{t \in T}\), \(y = (\mathcal{F}_t)_{t \in T}\), we define their product \(x \times y\) as the filtration \((\mathcal{E}_t \times \mathcal{F}_t)_{t \in T}\); it is again a continuous filtration;\(^{50}\) it belongs to \(X_T\) provided that its maximal sub-\(\sigma\)-field belongs to \(X_1\). Both \(x\) and \(y\) are immersed into \(x \times y\). The same for the product of any finite or countable number of filtrations.

4.7. Lemma. For every \(x_1, x_2, \cdots \in X_T\) there exist \(x \in X_T\) and \(g_1, g_2, \cdots \in G^{fav}\) such that

\[(g_1 \cdot x_1) \times (g_2 \cdot x_2) \times \cdots = x.\]

**Proof.** Apply Lemma 3.13 to maximal sub-\(\sigma\)-fields of given filtrations. \(\square\)

4.8. Note. Each \(g_k \cdot x_k\) is immersed into \(x\).

4.9. Note. The orbit of \(x\) is uniquely determined by orbits of \(x_1, x_2, \cdots\)\(^{51}\)

Therefore, multiplication of filtrations induces on the set of orbits, \(X_T/G^{fav}\), an associative operation that takes any finite or countable number of operands.

\(^{49}\)These are filtrations isomorphic to the natural filtration of the Brownian motion, see 2c2.

\(^{50}\)Hint: check continuity in \(t\) of \(E \left( f \mid \mathcal{E}_t \times \mathcal{F}_t \right)\) for the special case when \(f\) is the product of an \(\mathcal{E}_\infty\)-measurable function and an \(\mathcal{F}_\infty\)-measurable function.

\(^{51}\)Hint: use 4.1(b).
4.10. Lemma. For every $x \in X_T$ there exist $g_1, g_2, \cdots \in G^{\text{fav}}$ such that 
$$g_k \cdot x \to 0_{X_T} \quad \text{for } k \to \infty.$$ 

Proof. Apply Lemma 3.19 to maximal $\sigma$-fields of given filtrations. \qed

4.11. Corollary. The closure of every orbit in $X_T$ contains the trivial filtration $0_{X_T}$.

4.12. Lemma. For every $x, y \in X_T$ there exist $x_1, x_2, \cdots \in X_T$ and $g_1, g_2, \cdots \in G^{\text{fav}}$ such that $x_k \to x$, and $g_k \cdot y$ is immersed into $x_k$ for each $k$.

Proof. Lemma 4.1 gives us $g_k$ such that $x, g_1 \cdot y, g_2 \cdot y, \cdots$ are independent. Consider $x_k = x \times (g_k \cdot y)$. Corollary 3.18 ensures that $x_k \to x$. \qed

4.13. Corollary. For every $x \in X_T$ the set 
$$\{ y \in X_T : \exists g \in G^{\text{fav}} \left( g \cdot x \text{ is immersed into } y \right) \}$$

is dense in $X_T$.

For example, filtrations containing immersed Brownian filtrations are a dense subset of $X_{[0, +\infty]}$.

4c Ergodicity

Consider first the case $T = \{0, 1, 2\}$, denoting $X_T = X_{[0,1,2]}$ simply by $X_2$. An element of $X_2$ may be thought of as a triple $(\mathcal{E}_0 = 0_{\sigma F}, \mathcal{E}_1, \mathcal{E}_2)$ or just a pair $(\mathcal{E}_1, \mathcal{E}_2)$ of sub-$\sigma$-fields $\mathcal{E}_1, \mathcal{E}_2 \in S\sigma F^{\text{fav}}$ such that $\mathcal{E}_1 \subset \mathcal{E}_2$ and $\mathcal{E}_2 \in X_1$. The latter is equivalent to

$$\mathcal{E}_2 \times \mathcal{E}_{2,\infty} = 1_{S\sigma F} \quad \text{for some nonatomic } \mathcal{E}_{2,\infty} \in S\sigma F^{\text{fav}}$$

(recall 3.14). It may happen (or not) that

$$\mathcal{E}_1 \times \mathcal{E}_{12} = \mathcal{E}_2 \quad \text{for some nonatomic } \mathcal{E}_{12} \in S\sigma F^{\text{fav}},$$

in which case we say that $\mathcal{E}_2|\mathcal{E}_1$ is nonatomic. It may also happen (or not) that $\mathcal{E}_1$ is nonatomic.

4.14. Lemma. The set $\{(\mathcal{E}_1, \mathcal{E}_2) \in X_2 : \mathcal{E}_2|\mathcal{E}_1 \text{ is nonatomic } \}$ is a $G_{\delta}$-subset of $X_2$.

\footnote{Though, $\mathcal{E}_2|\mathcal{E}_1$ itself is undefined.}
Proof. First, a general claim strengthening Claim 3.10.B.

A. CLAIM. The map

\[ L_0(\Omega, \mathcal{F}, P; X) \times \sigma\mathcal{F}(\Omega, \mathcal{F}, P) \ni (f, \mathcal{E}) \mapsto P_{f|\mathcal{E}} \in L_0(\Omega, \mathcal{F}, P; \text{Prob}(X)) \]

is continuous.

Proof of Claim. Let \( f_n \to f \) in \( L_0(\Omega, \mathcal{F}, P; X) \), \( \mathcal{E}_n \to \mathcal{E} \) in \( \sigma\mathcal{F}(\Omega, \mathcal{F}, P) \), and \( \varphi : X \to \mathbb{R} \) a bounded continuous function; it suffices to find a subsequence \( (f_{n_k}, \mathcal{E}_{n_k}) \) such that

\[ \mathbb{E} \left( \varphi \circ f_{n_k} \mid \mathcal{E}_{n_k} \right) \to \mathbb{E} \left( \varphi \circ f \mid \mathcal{E} \right) \]

almost everywhere (after that the proof is finished similarly to the proof of Claim 3.10.B).

We choose \( n_k \) satisfying two conditions:

\[ \mathbb{E} \left( \varphi \circ f \mid \mathcal{E}_{n_k} \right) \to \mathbb{E} \left( \varphi \circ f \mid \mathcal{E} \right) \]

almost everywhere, and

\[ \sum_k \mathbb{E} \left| \varphi \circ f_{n_k} - \varphi \circ f \right| < \infty. \]

Thus

\[ \mathbb{E} \left( \varphi \circ f_{n_k} \mid \mathcal{E}_{n_k} \right) - \mathbb{E} \left( \varphi \circ f \mid \mathcal{E}_n \right) \to 0 \]

almost everywhere. So,

\[ \mathbb{E} \left( \varphi \circ f_{n_k} \mid \mathcal{E}_{n_k} \right) - \mathbb{E} \left( \varphi \circ f \mid \mathcal{E}_n \right) = \mathbb{E} \left( \varphi \circ f_{n_k} \mid \mathcal{E}_{n_k} \right) - \mathbb{E} \left( \varphi \circ f \mid \mathcal{E}_n \right) + \mathbb{E} \left( \varphi \circ f \mid \mathcal{E}_n \right) - \mathbb{E} \left( \varphi \circ f \mid \mathcal{E} \right) \to 0 \]

almost everywhere. \( \square \) 

The needed nonatomicity may be written as \( \psi(\mathcal{E}_2|\mathcal{E}_1) = 0 \), where

\[ \psi(\mathcal{E}_2|\mathcal{E}_1) = \inf_P \mathbb{E} \varphi(P_{E(f|\mathcal{E}_2)|\mathcal{E}_1}) \]

and \( \psi \) is the same as in the proof of 3.10 (and 3.22); it remains to prove that \( \psi \) is an upper semicontinuous function on \( X_2 \).

We know from (2.3) that the map \( \sigma\mathcal{F} \ni \mathcal{E} \mapsto \mathbb{E} \left( f \mid \mathcal{E} \right) \in L_2 \) is continuous for every \( f \in L_2 \). Therefore the map \( X_2 \ni (\mathcal{E}_1, \mathcal{E}_2) \mapsto \mathbb{E} \left( f \mid \mathcal{E}_2 \right) \in L_2 \) is continuous. On the other hand, Claim A shows that the map

\[ L_2 \times X_2 \ni (g, (\mathcal{E}_1, \mathcal{E}_2)) \mapsto P_{g|\mathcal{E}_1} \in L_0(\Omega, \mathcal{F}, P; \text{Prob}(\mathbb{R})) \]

is continuous. Combining the two facts we see that the map

\[ X_2 \ni (\mathcal{E}_1, \mathcal{E}_2) \mapsto P_{E(f|\mathcal{E}_2)|\mathcal{E}_1} \in L_0(\Omega, \mathcal{F}, P; \text{Prob}(\mathbb{R})) \]

is continuous. Using Claims 3.10.C,D we see that the map \( X_2 \ni (\mathcal{E}_1, \mathcal{E}_2) \mapsto \mathbb{E} \varphi(P_{E(f|\mathcal{E}_2)|\mathcal{E}_1}) \) is upper semicontinuous for every \( f \in L_2 \). \( \square \)

4.15. Definition. A filtration \( (\mathcal{F}_t)_{t \in T} \in X_T \) is called conditionally nonatomic, if \( \mathcal{F}_t|\mathcal{F}_s \) is nonatomic whenever \( s < t, s \in T, t \in T \). The set of all conditionally nonatomic filtrations belonging to \( X_T \) is denoted by \( X_T^{cna} \).
The next result shows that comeager many filtrations are conditionally nonatomic. It shows also that $X_{cna}^T$ is a Polish $G^{\text{fav}}$-space.

4.16. Theorem. $X_{cna}^T$ is a dense $G_\delta$-set in $X_T$.

Proof. We choose a countable subset $T_0 \subset T$ such that for every $s, t \in T$ satisfying $s < t$ there exist $s_0, t_0 \in T_0$ satisfying $s \leq s_0 < t_0 \leq t$. Nonatomicity of $\mathcal{F}_{s_0}|\mathcal{F}_{s_0}$ ensures nonatomicity of $\mathcal{F}_{t}|\mathcal{F}_{s}$. Therefore it suffices to prove that nonatomicity of $\mathcal{F}_{t}|\mathcal{F}_{s}$ selects a dense $G_\delta$-set for each pair $s, t$ separately.

For given $s, t$ the map $X_T \ni (\mathcal{F}_r)_{r \in T} \mapsto (\mathcal{F}_s, \mathcal{F}_t) \in X_2$ is continuous. Due to Lemma 4.14, the considered set is a $G_\delta$-set in $X_T$. The set is dense, which follows from 4.13 and existence of a conditionally nonatomic filtration. \hfill $\Box$

For a finite $T$, say $T = \{0, 1, \ldots, n\}$, we denote $X_T = X_{\{0,1,\ldots,n\}}$ simply by $X_n$. The simplest infinite $T$ is an increasing sequence (and its limit), say $T = \{0,1,2,\ldots;+\infty\}$; here we denote $X_T$ by $X_\infty$. The next fact is well-known.

4.17. Proposition. Each space $X_n^{cna}$ is a single orbit. Also $X_\infty^{cna}$ is a single orbit.

Proof. We consider $X_\infty^{cna}$ only. Let $(E_n)_n \in X_\infty^{cna}$, then $E_{n+1} = E_n \times E_{n,n+1}$ for some nonatomic $E_{n,n+1}$. Thus $E_n = E_{0,1} \times E_{1,2} \times \cdots \times E_{n-1,n}$, which describes $(E_n)_n$ uniquely up to isomorphism. \hfill $\Box$

We return to arbitrary $T$.

4.18. Corollary. Let $x, y \in X_T^{cna}$ and $T_0 \subset T$ be a finite subset. Then there exists $g \in G^{\text{fav}}$ such that $(g \cdot x)|_{T_0} = y|_{T_0}$.\hfill $\Box$

4.19. Lemma. Let $x \in X_T$ and $U$ be a neighborhood of $x$. Then there exists a finite subset $T_0 \subset T$ such that

$$\forall y \in X_T \quad (x|_{T_0} = y|_{T_0} \implies y \in U).$$

Proof. Recall the sandwich argument stated in [4.14] for $\text{F}(X)$. The space $L(H)$ introduced by [4.16] inherits from $\text{F}(H)$ its topology and (partial) order. Therefore the sandwich argument holds also for $L(H)$. Further, $\sigma \text{F}(\Omega, \mathcal{F}, P)$.

\hfill $53$Take a dense subset and add endpoints of intervals that constitute the complement of $T$.

\hfill $54$For example, a Brownian filtration restricted to $T$.

\hfill $55$That is, denoting $x = (E_t)_{t \in T}$, $y = (F_t)_{t \in T}$ we have $g \cdot E_t = F_t$ for all $t \in T_0$. 

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introduced by (2.2) inherits its topology and order from \( L(L_2(\Omega, \mathcal{F}, P)) \). Therefore the sandwich argument holds for \( \mathcal{S} \). So, for any \( \mathcal{E}, \mathcal{E}_n, \mathcal{E}_n', \mathcal{E}_n'' \in X_1 \), if \( \mathcal{E}_n' \to \mathcal{E}, \mathcal{E}_n'' \to \mathcal{E} \), and \( \mathcal{E}_n' \subset \mathcal{E}_n \subset \mathcal{E}_n'' \) for all \( n \), then \( \mathcal{E}_n \to \mathcal{E} \).

Recalling that \( \text{dist}(x, y) = \sup_{t \in T} \text{dist}(x(t), y(t)) \) (the choice of a compatible metric on \( X_1 \) does not matter), we choose \( \varepsilon \) such that \( \text{dist}(x, y) \leq \varepsilon \implies y \in U \). Using the sandwich argument and continuity of \( x : T \to X_1 \) we see that for every \( \varepsilon \), there exist \( t', t'' \in T \) such that

\[
[t', t''] \text{ is a neighborhood of } t \text{ in } T; \\
\forall \mathcal{E} \in X_1 \quad (x(t') \subset \mathcal{E} \subset x(t'') \implies \text{dist}(\mathcal{E}, x(t)) \leq \varepsilon/2).
\]

(Of course, the neighborhood \([t', t'']\) need not be open. We have \( t' \leq t \leq t'' \); however, do not think that \( t' < t < t'' \); it may fail, when \( t \) is not an interior point of \( T \subset [0, +\infty] \).) We choose a finite subcovering and construct \( T_0 \) as the set of endpoints \( t', t'' \) of elements of the subcovering.

**4.20. Theorem.** In the space \( X^c_{T^c} \), every orbit is dense.

**Proof.** Let \( x, y \in X^c_{T^c} \) and \( U \subset X^c_{T^c} \) be a neighborhood of \( y \). Lemma 4.19 gives us an appropriate finite \( T_0 \subset T \). Corollary 4.18 gives us \( g \in G_{\text{fav}} \) such that \( (g \cdot x)|_{T_0} = y|_{T_0} \), which implies \( g \cdot x \in U \).

**4.21. Theorem.** The space \( X_T \) is ergodic.

**Proof.** Due to 1.25 it suffices to find a dense orbit. Choose any \( x \in X^c_{T^c} \). By 4.20 its orbit is dense in \( X^c_{T^c} \). However, \( X^c_{T^c} \) is dense in \( X_T \) by 1.16.

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\(^{56}\)A citation, say, \([10, 8.4]\) means Item 8.4 (be it a theorem, definition or whatever) in the book \([10]\). That is unambiguous for \([10, 8.4]\). However, \([8, 2.3]\) could mean either Item 2.3 (in fact, Lemma in Sect. 2.1) of \([8]\), or Section 2.3 of \([8]\). I always mean an item, unless ‘Sect.’ is indicated explicitly.
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