Microcausality of Dirac field on noncommutative spacetime

Zheng Ze Ma

Department of Physics, Southeast University, Nanjing, 210096, P. R. China

Abstract

We study the microcausality of free Dirac field on noncommutative spacetime. We calculate the vacuum and non-vacuum state expectation values for the Moyal commutator \( [\psi_\alpha(x) \star \psi_\beta(x'), \psi_\sigma(x') \star \psi_\tau(x')] \star \) of Dirac field on noncommutative spacetime. We find that they do not vanish for some cases of the indexes for an arbitrary spacelike interval, no matter whether \( \theta^{0i} = 0 \) or \( \theta^{0i} \neq 0 \). However for the physical observable quantities of Dirac field such as the Lorentz scalar \( \psi(x) \star \psi(x) \star \) and the current \( j^\mu(x) = \psi(x) \gamma^\mu \star \psi(x) \star \) etc., we find that they still satisfy the microcausality. Therefore microcausality is satisfied for Dirac field on noncommutative spacetime.

PACS numbers: 11.10.Nx, 03.70.+k

1 Introduction

In recent years, noncommutative field theories (NCFTs) have caused a lot of researches [1-5]. NCFTs have many different properties from that of the commutative spacetime field theories, such as the locality and unitarity. The spacetime noncommutativity is described by the commutation relations of the spacetime coordinates

\[ [x^\mu, x^\nu] = i\theta^{\mu\nu}, \]

where \( \theta^{\mu\nu} \) is a constant real antisymmetric matrix with the dimension of square of length. For NCFTs, their Lagrangians can be obtained through replacing the ordinary products between field functions by the Moyal star-products. The Moyal star-product of two functions is given by

\[ f(x) \star g(x) = e^{i\theta^{\mu\nu} \partial_\mu x \partial_\nu x} f(x + \alpha) g(x + \beta) |_{\alpha = \beta = 0} \]

*Electronic address: z.z.ma@seu.edu.cn
\[ f(x)g(x) + \sum_{n=1}^{\infty} \left( \frac{i}{2} \right)^n \frac{1}{n!} \theta^{\mu_1 \nu_1} \cdots \theta^{\mu_n \nu_n} \partial_{\mu_1} \cdots \partial_{\mu_n} f(x) \partial_{\nu_1} \cdots \partial_{\nu_n} g(x) . \]

(1.2)

In noncommutative spacetime, there may exist the interactions and waves with the propagation speed faster than the speed of light up to infinite [6-8]. This is because of the UV/IR mixing [9] and nonlinear interactions in NCFTs. We hope to study for the free fields on noncommutative spacetime whether there exist the information and interaction with the transmit speed faster than the speed of light. This can be acquainted from the microcausality of quantum fields on noncommutative spacetime. If the microcausality of free quantum fields on noncommutative spacetime is violated, then for the free fields on noncommutative spacetime, there exist the information and interaction with the transmit speed faster than the speed of light.

According to superstring theories, people usually consider that the noncommutative parameters \( \theta^{\mu \nu} \) are invariant constants. This will make the Lorentz invariance be violated for noncommutative field theories generally, except that a subgroup \( SO(1,1) \times SO(2) \) of the usual Lorentz group can be maintained for certain special forms of the parameters \( \theta^{\mu \nu} \) [10]. In Ref. [10], the authors constructed the \( SO(1,1) \times SO(2) \) invariant spectral measure for the Fourier expansions of quantum fields on noncommutative spacetime. This makes the microcausality of quantum fields on noncommutative spacetime be formulated with respect to the \( SO(1,1) \) light wedge. Because inside the light wedge, there are areas outside the light cone. Thus microcausality of quantum fields on noncommutative spacetime will be broken in the area of the light wedge outside the light cone. This results the existence of infinite propagation speed of free fields inside the light wedge. In Ref. [11], the authors pointed out that even the \( SO(1,1) \) microcausality may be violated to consider the renormalization of the propagators. In Ref. [12], the authors generalized the axiom of Ref. [10] for the microcausality of quantum fields on noncommutative spacetime in accordance with the \( SO(1,1) \times SO(2) \) invariance.

It is necessary to point out that the infinite propagation speed of free fields on noncommutative spacetime of Ref. [10] is a necessary result of the breakdown of the Lorentz invariance, i.e., the breakdown of the usual Lorentz invariance of the spectral measure for the Fourier expansions of quantum fields. However there are doubts that whether the spectral measure for free fields on noncommutative spacetime is really in the form of \( SO(1,1) \times SO(2) \) invariance as that constructed in Ref. [10]. Certes if the noncommutative parameters \( \theta^{\mu \nu} \) are invariant constants, then Lorentz invariance of NCFTs will be destroyed [13,14]. However, there are possibilities that Lorentz invariance is maintained for NCFTs if we suppose that \( \theta^{\mu \nu} \) carries tensor indexes. In Ref. [15], the authors suppose that \( \theta^{\mu \nu} \) is a tensor operator and have constructed the Lorentz invariant NCFTs. In Refs. [16,17], the authors demonstrated the Lorentz invariance and unitarity for NCFTs to take \( \theta^{\mu \nu} \) to be a tensor operator as that proposed in Ref. [15]. In Ref. [18], the authors demonstrated that Lorentz invariance can be maintained for the classical field equations of NCFTs if one take \( \theta^{\mu \nu} \) to be a \( c \)-number tensor while not as an operator. In fact we can also demonstrate that Lorentz invariance is maintained for the \( S \)-matrixes of NCFTs if we take \( \theta^{\mu \nu} \) to be a \( c \)-number tensor. On the other hand, the breakdown of Lorentz invariance has not been discovered yet in the experiments.
up to now [19]. It seems that Lorentz invariance is a more fundamental principle of physics, although it may not be so for a large scale structure of the universe, it should be satisfied in the local area of the universe. From the superstring theories, to take $\theta^{\mu\nu}$ to be a c-number tensor means that the background NS-NS $B$-field changes as a second-order antisymmetric tensor when the reference system changes. On the other hand, it is necessary to point out that the nonlocality of NCFTs can be exist in accordance with the Lorentz invariance. The nonlocality of NCFTs does not mean that Lorentz invariance may be necessarily broken for NCFTs.

Therefore we do not suppose that the spectral measure for the Fourier expansions of free fields on noncommutative spacetime is in the form of $SO(1,1) \times SO(2)$ invariance as that of Ref. [10]. As discovered in Ref. [20], in fact the light wedge can be resulted from the nonlocal interactions of NCFTs. For example for a six-dimensional theory, the usual Lorentz invariance can be reduced to $SO(3,1) \times SO(2)$ from the nonlocal interactions of NCFTs. Similarly, the light wedge of $SO(1,1) \times SO(2)$ can be resulted from the nonlocal interactions of a four-dimensional NCFT. Therefore we would rather think that the spectral measure of the $SO(1,1) \times SO(2)$ invariance for the Fourier expansions of free fields on noncommutative spacetime is an effective result that resulted from the nonlocal interactions of NCFTs.

As pointed above, we hope to study whether there exist the information and interaction with the transmit speed faster than the speed of light for free fields on noncommutative spacetime through the microcausality property of quantum fields on noncommutative spacetime. We expand the free fields according to their usual form as that in the ordinary commutative spacetime. Another reason for us to take the Fourier expansion of quantum fields on noncommutative spacetime in their usual form is that in most occasions for the perturbative calculations of NCFTs in the literature, the propagators of quantum fields on noncommutative spacetime are obtained based on the usual Lorentz invariant spectral measures for the expansion of quantum fields. For the microcausality of the free scalar field on noncommutative spacetime, some results are obtained in Refs. [21-23]. In Ref. [22], Greenberg obtained that microcausality is violated for scalar field on noncommutative spacetime generally even if $\theta^{0i} = 0$. However we have pointed out in Ref. [23] that there are some problems for the arguments of Ref. [22] for the result of the microcausality violation. In Ref. [23] we obtained that microcausality is violated for the quadratic operators of scalar field on noncommutative spacetime only when $\theta^{0i} \neq 0$. In this paper, we will study the microcausality of free Dirac field on noncommutative spacetime.

The spacetime noncommutative relations (1.1) is defined for coordinates of the same spacetime point. Because we will calculate the commutators of two operators on two different spacetime points in the following, we need to generalize the noncommutative relations (1.1) to two different spacetime points. Therefore we suppose

$$[x^\mu_1, x^\nu_2] = i\theta^{\mu\nu}. \quad (1.3)$$

This makes us possible define the Moyal star-product of two functions on two different spacetime points as [5]

$$f(x_1) \star g(x_2) = e^{i\frac{\theta^{\mu\nu}}{2} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x'^\nu}} f(x_1 + \alpha)g(x_2 + \beta)|_{\alpha = \beta = 0}$$
\begin{align*}
\mathcal{A}(x_1) \mathcal{B}(x_2) & = f(x_1)g(x_2) + \sum_{n=1}^{\infty} \left( \frac{i}{2} \right)^n \frac{1}{n!} \theta_{\mu_1 \nu_1} \cdots \theta_{\mu_n \nu_n} \partial_{\mu_1} \cdots \partial_{\nu_n} f(x_1) \partial_{\nu_1} \cdots \partial_{\nu_n} g(x_2).
\end{align*}

(1.4)

A demonstration for the self-consistency of the commutative relations (1.3) with (1.1) is given in the Appendix.

The content of this paper is organized as follows. In Sec. II, we analyze the criterion of microcausality violation for quantum fields on noncommutative spacetime. In Sec. III, we calculate the vacuum state expectation value for the Moyal commutator \( [\psi(\mathbf{x}), \psi(\mathbf{x}') \star \psi(\mathbf{x}')] \cdot \) of free Dirac field on noncommutative spacetime. We find that they do not vanish in some cases for an arbitrary spacelike interval, no matter whether \( \theta_{0i} = 0 \) or \( \theta_{0i} \neq 0 \). However for the physical observable quantities of Dirac field such as the Lorentz scalar \( \psi(\mathbf{x}) \star \psi(\mathbf{x}) \) and the current \( J^\mu(\mathbf{x}) = \psi(\mathbf{x}) \gamma^\mu \psi(\mathbf{x}) \), we find that they still satisfy the microcausality. In Sec. IV, we generalize the result of Sec. III to the non-vacuum state expectation values and obtain that microcausality is satisfied for free Dirac field on noncommutative spacetime generally. Sec. V is the conclusion. In the Appendix, we give a demonstration for the self-consistency of the commutative relations (1.3) with (1.1).

2 The criterion of microcausality violation

In this section, we first analyze the measurement of quantum fields on noncommutative spacetime and the criterion of microcausality violation. We need to point out that in the left hand sides of Eqs. (1.2) and (1.4), the coordinates \( x^\mu \) are regarded as noncommutative operators that satisfying the commutation relations (1.1) and (1.3). In the right hand sides of Eqs. (1.2) and (1.4), after the expansion of Moyal star-products, the coordinates \( x^\mu \) are treated as the ordinary commutative \( c \)-numbers, i.e., the experientially observable spacetime coordinates. To be clearer, we may write the coordinates in the left hand sides of Eqs. (1.2) and (1.4) as \( \hat{x}^\mu \), in order to indicate that they are operators. However we do not use the sign \( \hat{x}^\mu \) for the left hand sides of Eqs. (1.2) and (1.4), as well as in Eqs. (1.1) and (1.3). We do not discriminate them in signs and throughout this paper. We consider that their prescribed meanings in different places are clear.

For quantum field theories, as well as quantum mechanics, what the observer measures are certain expectation values. We suppose that there are two observers A and B situated at spacetime points \( x \) and \( y \), they proceed a measurement separately on the state vector \( |\Psi\rangle \) for the locally observable quantity \( \mathcal{O}(x) \) in the same occasion. However the time \( x_0 \) may not equal to the time \( y_0 \) generally. For the observer A, the state vector \( |\Psi\rangle \) has been affected by the measurement of the observer B at the spacetime point \( y \). Or we can say the observer B’s observation instrument has taken an action on the state vector \( |\Psi\rangle \). The state vector has become \( \mathcal{O}(y)|\Psi\rangle \). When the observer A takes his or her measurement on the state vector, his or her observation instrument will act on the state vector \( \mathcal{O}(y)|\Psi\rangle \) again. These two sequent actions should be represented by the product operation of the operators. However because now the spacetime is noncommutative, the product operation should be the Moyal star-product, while not the ordinary product. Or we regard that in noncommutative
spacetime, the basic product operation is the Moyal star-product. Thus what the measuring result the observer A obtained from his or her instrument is $\langle \Psi | \mathcal{O}(y) \star \mathcal{O}(y) | \Psi \rangle$. Similarly for the observer B, the state vector $|\Psi\rangle$ has been affected by the action of the observer A’s instrument at the spacetime point $x$. The state vector becomes $\mathcal{O}(x)|\Psi\rangle$. What the measuring result the observer B obtained from his or her instrument is $\langle \Psi | \mathcal{O}(y) \star \mathcal{O}(x) | \Psi \rangle$.

Supposing that microcausality is satisfied for NCFTs, this means that there do not exist the physical information and interaction with the transmit speed faster than the speed of light. Thus when the spacetime interval between $x$ and $y$ is spacelike, the affection of the observer B’s measurement or the action of observer B’s instrument at spacetime point $y$ on the state vector $|\Psi\rangle$ has not propagated to the spacetime point $x$ when the observer A takes his or her measurement on the state vector $|\Psi\rangle$ at the spacetime point $x$. These two physical measurements do not interfere with each other. For the observer A, the state vector is still $|\Psi\rangle$. The same reason as the observer A, the measuring result what the observer B obtained at spacetime point $x$ is $\langle \Psi | \mathcal{O}(x) | \Psi \rangle$.

We use $P^\mu$ to represent the eigenvalues of the energy-momentum of the state vector $|\Psi\rangle$. Therefore we obtain

$$
\langle \Psi | \mathcal{O}(y) | \Psi \rangle = \langle \Psi | \mathcal{O}(y) \star \mathcal{O}(x) | \Psi \rangle = \langle \Psi | \exp(i a_\mu P^\mu) \star \mathcal{O}(x) \star \exp(-i a_\mu P^\mu) | \Psi \rangle.
$$

Therefore Eq. (2.3) is satisfied.

\[\text{We note } y - x = a. \text{ From the Heisenberg relations and the translation transformation, we have}
\]

$$
\mathcal{O}(y) = \exp(i a_\mu P^\mu) \mathcal{O}(x) \exp(-i a_\mu P^\mu).
$$

Because $a_\mu$ now is a constant four-vector, from Eq. (1.2) we can also write the above expression as

$$
\mathcal{O}(y) = \exp(i a_\mu P^\mu) \star \mathcal{O}(x) \star \exp(-i a_\mu P^\mu).
$$

\[\text{We use } P^\mu \text{ to represent the eigenvalues of the energy-momentum of the state vector } |\Psi\rangle. \text{ Therefore we obtain}
\]

$$
\langle \Psi | \mathcal{O}(y) | \Psi \rangle = \langle \Psi | \exp(i a_\mu P^\mu) \star \mathcal{O}(x) \star \exp(-i a_\mu P^\mu) | \Psi \rangle = \langle \Psi | \exp(i a_\mu P^\mu) \star \mathcal{O}(x) \star \exp(-i a_\mu P^\mu) | \Psi \rangle
\]

$$
= \exp(i a_\mu P^\mu) \star \langle \Psi | \mathcal{O}(x) | \Psi \rangle \star \exp(-i a_\mu P^\mu) = \exp(i a_\mu P^\mu) \langle \Psi | \mathcal{O}(x) | \Psi \rangle \exp(-i a_\mu P^\mu) = \langle \Psi | \mathcal{O}(x) | \Psi \rangle.
$$

Therefore Eq. (2.3) is satisfied.

5
If microcausality is violated for a NCFT, then there may exist the physical information and interaction with the transmit speed faster than the speed of light. For the two measurements of the observer A and observer B located at \( x \) and \( y \) separated by a spacelike interval, the affection of the observer B’s measurement at spacetime point \( y \) on the state vector \( |\Psi\rangle \) will propagate to the spacetime point \( x \) when the observer A takes his or her measurement on the state vector \( |\Psi\rangle \) at the spacetime point \( x \), and the affection of the observer A’s measurement at spacetime point \( x \) on the state vector \( |\Psi\rangle \) will propagate to the spacetime point \( y \) when the observer B takes his or her measurement on the state vector \( |\Psi\rangle \) at the spacetime point \( y \). These two physical measurements will interfere with each other. For such a case, Eqs. (2.1) and (2.2) cannot be satisfied, while we still have \( \langle \Psi|O(x)|\Psi\rangle = \langle \Psi|O(y)|\Psi\rangle \) as that of Eq. (2.3). Therefore generally we have

\[
\langle \Psi|[O(x), O(y)]_\star|\Psi\rangle \neq 0 \quad \text{for} \quad (x - y)^2 < 0
\]

for a NCFT to violate the microcausality. Therefore we can judge whether the microcausality is maintained or violated for a NCFT according to Eq. (2.5).

Now we suppose that \( O_1(x) \) and \( O_2(y) \) are two different observable field operators, \( x \) and \( y \) are separated by a spacelike interval, two observers A and B situate at \( x \) and \( y \), and microcausality is satisfied for the field theory on noncommutative spacetime. Supposing that the observers A and B proceed a measurement separately on the state vector \( |\Psi\rangle \) for the locally observable quantities \( O_1 \) and \( O_2 \) at \( x \) and \( y \) respectively, then from the above analysis, we have for the observer A

\[
\langle \Psi|O_1(x) \star O_2(y)|\Psi\rangle = \langle \Psi|O_1(x)|\Psi\rangle \quad \text{for} \quad (x - y)^2 < 0 .
\]

And we have for the observer B

\[
\langle \Psi|O_2(y) \star O_1(x)|\Psi\rangle = \langle \Psi|O_2(y)|\Psi\rangle \quad \text{for} \quad (x - y)^2 < 0 .
\]

Because now \( O_1(x) \) and \( O_2(y) \) are two different operators, we obtain generally

\[
\langle \Psi|O_1(x)|\Psi\rangle \neq \langle \Psi|O_2(y)|\Psi\rangle .
\]

Therefore from Eqs. (2.6)-(2.8) we have

\[
\langle \Psi|[O_1(x), O_2(y)]_\star|\Psi\rangle \neq 0 \quad \text{for} \quad (x - y)^2 < 0
\]

generally, even if \( x \) and \( y \) are separated by a spacelike interval, and the field theory satisfies the microcausality. Therefore we cannot deduce that a NCFT violates microcausality from Eq. (2.9) from the expectation values of the Moyal commutator of two different operators. In order to judge whether a NCFT violates microcausality, we must analyze the expectation values of the Moyal commutator of the same operator as that of Eq. (2.5).

### 3 Vacuum state expectation values

For the free Dirac field on noncommutative spacetime, its Lagrangian is given by

\[
\mathcal{L} = \overline{\psi} \star i\gamma^\mu \partial_\mu \psi - m \overline{\psi} \star \psi .
\]

(3.1)
We suppose that in noncommutative spacetime, quantum fields can still be expanded in the usual Lorentz invariant form. Therefore the Fourier expansions for the free Dirac field and its conjugate field are given by

\[
\psi(x, t) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}} \sum_{s=1,2} [b(p, s)u(p, s)e^{-ipx} + d^\dagger(p, s)v(p, s)e^{ipx}],
\]

\[
\bar{\psi}(x, t) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}} \sum_{s=1,2} [b^\dagger(p, s)\bar{u}(p, s)e^{ipx} + d(p, s)\bar{v}(p, s)e^{-ipx}],
\]

where \( E_p = p_0 = +\sqrt{|p|^2 + m^2} \) and \( px = p_\mu x^\mu \). In Eq. (3.2), the spacetime coordinates are treated as noncommutative operators. They satisfy the commutation relations (1.1) and (1.3). The anticommutation relations for the creation and annihilation operators are still the same as that in the commutative spacetime. They are

\[
\{b(p, s), b^\dagger(p', s')\} = \delta_{ss'}\delta^3(p - p'),
\]

\[
\{d(p, s), d^\dagger(p', s')\} = \delta_{ss'}\delta^3(p - p'),
\]

\[
\{b(p, s), b(p', s')\} = \{d(p, s), d(p', s')\} = 0,
\]

\[
\{b^\dagger(p, s), b^\dagger(p', s')\} = \{d^\dagger(p, s), d^\dagger(p', s')\} = 0,
\]

\[
\{b(p, s), d(p', s')\} = \{d(p, s), b(p', s')\} = 0.
\]

The spinors \( u(p, s) \) and \( v(p, s) \) satisfy the completeness relations

\[
\sum_{s=1,2} u_\alpha(p, s)\bar{u}_\beta(p, s) = \left( \frac{\not{p} + m}{2m} \right)^{\alpha\beta},
\]

\[
\sum_{s=1,2} v_\alpha(p, s)\bar{v}_\beta(p, s) = \left( \frac{\not{p} - m}{2m} \right)^{\alpha\beta}.
\]

They are the same as that of the commutative spacetime case.

We define the Moyal anticommutators of the Dirac field to be

\[
\{\psi_\alpha(x), \bar{\psi}_\beta(x')\}_\ast = \psi_\alpha(x) \ast \bar{\psi}_\beta(x') + \bar{\psi}_\beta(x') \ast \psi_\alpha(x),
\]

\[
\{\psi_\alpha(x), \psi_\beta(x')\}_\ast = \psi_\alpha(x) \ast \psi_\beta(x') + \psi_\beta(x') \ast \psi_\alpha(x),
\]

\[
\{\bar{\psi}_\alpha(x), \bar{\psi}_\beta(x')\}_\ast = \bar{\psi}_\alpha(x) \ast \bar{\psi}_\beta(x') + \bar{\psi}_\beta(x') \ast \bar{\psi}_\alpha(x).
\]

We have shown in Ref. [24] that the Moyal anticommutators of the Dirac field are not c-number functions. In order to obtain the c-number results for these Moyal anticommutators, we need to evaluate their vacuum and non-vacuum state expectation values. We have obtained in Ref. [24] that

\[
\langle 0 | \{\psi_\alpha(x), \bar{\psi}_\beta(x')\}_\ast | 0 \rangle = \langle \Psi | \{\psi_\alpha(x), \bar{\psi}_\beta(x')\}_\ast | \Psi \rangle = -iS_{\alpha\beta}(x - x'),
\]

\[
\langle 0 | \{\psi_\alpha(x), \psi_\beta(x')\}_\ast | 0 \rangle = \langle \Psi | \{\psi_\alpha(x), \psi_\beta(x')\}_\ast | \Psi \rangle = 0,
\]

\[
\langle 0 | \{\bar{\psi}_\alpha(x), \bar{\psi}_\beta(x')\}_\ast | 0 \rangle = \langle \Psi | \{\bar{\psi}_\alpha(x), \bar{\psi}_\beta(x')\}_\ast | \Psi \rangle = 0,
\]

\[
\langle 0 | \{\bar{\psi}_\alpha(x), \psi_\beta(x')\}_\ast | 0 \rangle = \langle \Psi | \{\bar{\psi}_\alpha(x), \psi_\beta(x')\}_\ast | \Psi \rangle = 0.
\]
where $|\Psi\rangle$ is a state vector of the Dirac field quantum system, and the singular function $S(x - x')$ is given by [25]

$$S_{\alpha\beta}(x - x') = -(i \not\partial_x + m)_{\alpha\beta} \Delta(x - x') .$$

(3.11)

$S(x - x')$ is zero for a spacelike interval.

For the free Dirac field on ordinary commutative spacetime, its observable quantities are constructed from the fundamental bilinear forms $\bar{\psi}_\alpha(x) \psi_\beta(x)$ and the $\gamma$-matrixes. For the fundamental bilinear forms $\bar{\psi}_\alpha(x) \psi_\beta(x)$, we have [25]

$$\begin{align*}
[\bar{\psi}_\alpha(x) \psi_\beta(x), \bar{\psi}_\sigma(x') \psi_\tau(x')] &= \bar{\psi}_\alpha(x) \{\psi_\beta(x), \bar{\psi}_\sigma(x')\} \psi_\tau(x') - \{\bar{\psi}_\alpha(x), \bar{\psi}_\sigma(x')\} \psi_\beta(x) \psi_\tau(x') \\
&+ \bar{\psi}_\sigma(x') \bar{\psi}_\alpha(x) \{\psi_\beta(x), \psi_\tau(x')\} - \bar{\psi}_\sigma(x') \{\bar{\psi}_\alpha(x), \psi_\tau(x')\} \psi_\beta(x) .
\end{align*}$$

(3.12)

From the properties of the fundamental anticommutators of Dirac field, we have [25]

$$[\bar{\psi}_\alpha(x) \psi_\beta(x), \bar{\psi}_\sigma(x') \psi_\tau(x')] = 0 \quad \text{for} \quad (x - x')^2 < 0 .$$

(3.13)

Therefore microcausality is satisfied for Dirac field on ordinary commutative spacetime.

For the free Dirac field on noncommutative spacetime, its observable quantities such as the current $j^\mu(x) =: \bar{\psi}(x) \gamma^\mu \psi(x)$ are constructed from $\bar{\psi}_\alpha(x) \psi_\beta(x)$ and $\gamma$-matrixes. Therefore in order to investigate its microcausality property, we need to analyze the Moyal anticommutator $[\bar{\psi}_\alpha(x) \psi_\beta(x), \bar{\psi}_\sigma(x') \psi_\tau(x')]$. We have

$$\begin{align*}
[\bar{\psi}_\alpha(x) \psi_\beta(x), \bar{\psi}_\sigma(x') \psi_\tau(x')] &= \bar{\psi}_\alpha(x) \{\psi_\beta(x), \bar{\psi}_\sigma(x')\} \psi_\tau(x') - \{\bar{\psi}_\alpha(x), \bar{\psi}_\sigma(x')\} \psi_\beta(x) \psi_\tau(x') \\
&+ \bar{\psi}_\sigma(x') \bar{\psi}_\alpha(x) \{\psi_\beta(x), \psi_\tau(x')\} - \bar{\psi}_\sigma(x') \{\bar{\psi}_\alpha(x), \psi_\tau(x')\} \psi_\beta(x) \\
&\quad - \bar{\psi}_\sigma(x') \bar{\psi}_\alpha(x) \{\psi_\tau(x'), \psi_\beta(x)\} - \{\bar{\psi}_\alpha(x), \bar{\psi}_\sigma(x')\} \psi_\beta(x) \psi_\tau(x') \\
&= \bar{\psi}_\alpha(x) \psi_\beta(x) \bar{\psi}_\sigma(x') \psi_\tau(x') - \bar{\psi}_\sigma(x') \psi_\beta(x) \bar{\psi}_\alpha(x) \psi_\tau(x') .
\end{align*}$$

(3.14)

As obtained in Ref. [24], the fundamental Moyal anticommutators of the Dirac field are not $c$-number functions, in order to examine the microcausality property for the operator $\bar{\psi}_\alpha(x) \psi_\beta(x)$, we need to calculate its expectation values, to see whether they are vanished or not for a spacelike interval. As demonstrated in Sec. II, this is also the demand of physical measurements. In order to simplify the calculation, we can adopt the normal orderings for the Moyal star-product. Therefore we need to calculate the function

$$B_{\alpha\beta\sigma\tau}(x, y) = \langle \Psi | : \bar{\psi}_\alpha(x) \psi_\beta(x) : : \bar{\psi}_\sigma(y) \psi_\tau(y) : | \Psi \rangle ,$$

(3.15)

where $|\Psi\rangle$ is a state vector of free Dirac field quantum system. To adopt the normal orderings for the field operators $\bar{\psi}_\alpha(x) \psi_\beta(x)$ and $\bar{\psi}_\sigma(y) \psi_\tau(y)$ means that an infinite charge of the vacuum with all of the negative energy states occupied has been eliminated in the corresponding commutative spacetime field theory. To be the limit of physical measurements,
we take the state vector $|\Psi\rangle$ in Eq. (3.15) to be the vacuum state $|0\rangle$. Therefore in this section, we first study the vacuum expectation value

$$B_{0,\alpha\beta\sigma\tau}(x, y) = \langle 0 | [ : \bar{\psi}_\alpha(x) \star \psi_\beta(x) : , : \bar{\psi}_\sigma(y) \star \psi_\tau(y) : ] | 0 \rangle .$$  (3.16)

For the non-vacuum state expectation value of Eq. (3.15), we will analyze it in Sec. IV.

We decompose $\psi(x)$ into the annihilation (positive frequency) and creation (negative frequency) part

$$\psi(x) = \psi^+(x) + \psi^-(x) ,$$  (3.17)

where

$$\psi^+(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{m} \sum_{s=1,2} b(p, s) u(p, s) e^{-ipx} ,$$

$$\psi^-(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{m} \sum_{s=1,2} d^\dagger(p, s) v(p, s) e^{ipx} .$$  (3.18)

For the conjugate field we have

$$\bar{\psi}(x) = \bar{\psi}^+(x) + \bar{\psi}^-(x) ,$$  (3.19)

$$\bar{\psi}^+(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{m} \sum_{s=1,2} d(p, s) \bar{u}(p, s) e^{-ipx} ,$$

$$\bar{\psi}^-(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{m} \sum_{s=1,2} b(p, s) \bar{v}(p, s) e^{ipx} .$$  (3.20)

From Eqs. (3.17) and (3.19) we have

$$\bar{\psi}_\alpha(x) \star \psi_\beta(x) = \bar{\psi}^\dagger_\alpha(x) \star \psi^+_\beta(x) + \bar{\psi}^+_\alpha(x) \star \psi^-_\beta(x) + \bar{\psi}^-_\alpha(x) \star \psi^+_\beta(x) + \bar{\psi}^+_\alpha(x) \star \psi^-_\beta(x) .$$  (3.21)

The normal ordering of the operator $\bar{\psi}_\alpha(x) \star \psi_\beta(x)$ is given by

$$\bar{\psi}_\alpha(x) \star \psi_\beta(x) := \bar{\psi}^\dagger_\alpha(x) \star \psi^+_\beta(x) - \psi^-_\beta(x) \star \bar{\psi}^\dagger_\alpha(x) + \bar{\psi}^-_\alpha(x) \star \psi^+_\beta(x) + \psi^+_\beta(x) \star \bar{\psi}^-_\alpha(x) .$$  (3.22)

Here we have made a simplified manipulation for the normal ordering of the Moyal star-product operator $\bar{\psi}^\dagger_\alpha(x) \star \psi^-_\beta(x)$. This is because the result of the Moyal star-product of two functions is related with the order of two functions. In the Fourier integral representation, we can see that $\psi^-_\beta(x) \star \bar{\psi}^\dagger_\alpha(x)$ will have an additional phase factor $e^{ipx'}$ relative to $\bar{\psi}^\dagger_\alpha(x) \star \psi^-_\beta(x)$. However in Eq. (3.22) we have ignored such a difference for the normal ordering of $\psi^-_\beta(x) \star \bar{\psi}^\dagger_\alpha(x)$. The reason is that the terms that contain $\psi^-_\beta(x) \star \bar{\psi}^\dagger_\alpha(x)$ in the expansion of Eq. (3.16) will contribute zero when we evaluate their vacuum expectation values, as it can be seen in the following. Thus we can ignore such a difference equivalently for convenience.

To expand $\bar{\psi}_\alpha(x) \star \psi_\beta(x) : \star : \bar{\psi}_\sigma(y) \star \psi_\tau(y) :$, we obtain

$$\bar{\psi}_\alpha(x) \star \psi_\beta(x) : \star : \bar{\psi}_\sigma(y) \star \psi_\tau(y) : = \bar{\psi}^\dagger_\alpha(x) \star \psi^+_\beta(x) \star \psi^+_\sigma(y) \star \psi^-_\tau(y) - \psi^-_\beta(x) \star \bar{\psi}^\dagger_\alpha(x) \star \psi^+_\sigma(y) \star \psi^-_\tau(y) \star \bar{\psi}^\dagger_\sigma(y) .$$
we need to consider the additional four terms which will contribute non-zero results as that of Eq. (3.14), then in the calculation of the vacuum expectation value for Eq. (3.14),

\[ \langle 0 | : \bar{\psi}_\alpha(x) \times \bar{\psi}_\beta(x) : * : \bar{\psi}_\sigma(y) \times \psi_\tau(y) : 0 \rangle = \langle 0 | \bar{\psi}_\sigma^+ (x) \times \bar{\psi}_\sigma^+(y) \times \bar{\psi}_\sigma^+(y) \times \psi_\tau(y) | 0 \rangle. \]  

(3.24)

Similarly, the non-zero contribution to the vacuum expectation value of \( \bar{\psi}_\sigma(y) \times \psi_\tau(y) : * : \bar{\psi}_\sigma(x) \times \psi_\beta(x) : \) only comes from the part \( \bar{\psi}_\sigma(y) \times \psi_\tau(y) \times \bar{\psi}_\sigma(x) \times \psi_\beta(x) \). We have

\[ \langle 0 | : \bar{\psi}_\sigma(x) \times \psi_\tau(y) : * : \bar{\psi}_\sigma(x) \times \psi_\beta(x) : 0 \rangle = \langle 0 | \bar{\psi}_\sigma^+(y) \times \psi_\tau(y) \times \bar{\psi}_\sigma^+(x) \times \psi_\beta(x) | 0 \rangle. \]  

(3.25)

If we do not use the normal orderings for the operators : \( \bar{\psi}_\alpha(x) \times \psi_\beta(x) : \) and : \( \bar{\psi}_\sigma(y) \times \psi_\tau(y) : \) as that of Eq. (3.14), then in the calculation of the vacuum expectation value for Eq. (3.14), we need to consider the additional four terms which will contribute non-zero results

\[ \bar{\psi}_\alpha^+(x) \times \bar{\psi}_\beta^+(x) \times \bar{\psi}_\sigma^+(y) \times \psi_\tau(y) - \bar{\psi}_\sigma(y) \times \bar{\psi}_\sigma(y) \times \bar{\psi}_\sigma(y) \times \psi_\tau(y), \]

\[ \bar{\psi}_\alpha^+(x) \times \bar{\psi}_\beta^+(x) \times \bar{\psi}_\sigma^+(y) \times \psi_\tau(y) - \bar{\psi}_\sigma(y) \times \bar{\psi}_\sigma(y) \times \bar{\psi}_\sigma(y) \times \psi_\beta(x). \]

However in fact we can obtain that the total vacuum expectation value of these terms cancel at last. Thus to take the normal orderings for the operators : \( \bar{\psi}_\alpha(x) \times \psi_\beta(x) : \) and : \( \bar{\psi}_\sigma(y) \times \psi_\tau(y) : \) has simplified the calculation.

Through calculation we obtain

\[ \langle 0 | : \bar{\psi}_\alpha(x) \times \psi_\beta(x) : * : \bar{\psi}_\sigma(y) \times \psi_\tau(y) : 0 \rangle = \langle 0 | \bar{\psi}_\sigma^+(x) \times \bar{\psi}_\sigma^+(y) \times \bar{\psi}_\sigma^+(y) \times \psi_\tau(y) | 0 \rangle - \langle 0 | \bar{\psi}_\sigma^+(y) \times \bar{\psi}_\sigma^+(y) \times \bar{\psi}_\sigma^+(x) \times \psi_\beta(x) | 0 \rangle \]

\[ = \int \frac{d^3 p_1}{(2\pi)^3} \int \frac{d^3 p_2}{(2\pi)^3} \sum_{s_1,s_2} \left( \frac{m}{E_{p_1}} \right)^{\frac{1}{2}} \bar{u}_\alpha(p_1,s_1)e^{-ip_1 x} \times \left( \frac{m}{E_{p_2}} \right)^{\frac{1}{2}} u_\beta(p_2,s_2)e^{-ip_2 x}. \]
In Eq. (3.26), there are two terms in the second equality. The first term means that two Dirac field quanta $|p_1, s_1\rangle$ and $|p_2, s_2\rangle$ are generated at the spacetime point $y$, and annihilated at the spacetime point $x$. The second term means that two Dirac field quanta $|p_1, s_1\rangle$ and $|p_2, s_2\rangle$ are generated at the spacetime point $x$, and annihilated at the spacetime point $y$.

In Eq. (3.26), \( \frac{d^3p_1}{(2\pi)^32E_{p_1}} \int \frac{d^3p_2}{(2\pi)^32E_{p_2}} \) is the Lorentz invariant volume element, \( p_1 = p_1\mu \gamma^\mu \), \( p_2 = p_2\mu \gamma^\mu \), and \( (p_1+p_2)(x-y) = (p_1+p_2)_\mu (x-y)^\mu \). Thus the total expression is Lorentz invariant. In the above calculation, we have used Eq. (1.4) for the Moyal star-product of two functions defined on two different spacetime points. The result does not rely on the parameters $\theta^{\mu\nu}$. However if $\theta^{\mu\nu} = 0$, we can deduce from Eq. (3.13) directly that the free Dirac field satisfies the microcausality on ordinary commutative spacetime. For such a case, we need not to evaluate the expectation value of Eq. (3.26).

We need to analyze whether the expression of Eq. (3.26) disappears or not for a spacelike interval. This can be seen from the vacuum expectation value of the equal-time commutator. Thus to take $x_0 = y_0$ in Eq. (3.26), we have

\[
\langle 0| : \bar{\psi}_\alpha(x, t) \star \psi_\beta(x, t) : : \bar{\psi}_\alpha(y, t) \star \psi_\tau(y, t) : : |0 \rangle = \int \frac{d^3p_1}{(2\pi)^32E_{p_1}} \int \frac{d^3p_2}{(2\pi)^32E_{p_2}} \left( (\not{\! p}_1 - m)_{\tau\alpha}(\not{\! p}_2 + m)_{\beta\sigma}e^{-i(p_1+p_2)(x-y)} - (\not{\! p}_1 - m)_{\beta\sigma}(\not{\! p}_2 + m)_{\tau\alpha}e^{-i(p_1+p_2)(x-y)} \right).
\]

(3.27)

We can see that in Eq. (3.27), the integral measure does not change when the arguments $(p_1, p_2)$ change to $(-p_1, -p_2)$. The integral space is symmetrical to the integral arguments $(p_1, p_2)$ and $(-p_1, -p_2)$. Therefore the odd function part of the integrand contributes zero to the whole integral. While the even function part will contribute nonzero to the whole integral. Thus to omit the odd function part in the integrand we obtain

\[
\langle 0| : \bar{\psi}_\alpha(x, t) \star \psi_\beta(x, t) : : \bar{\psi}_\alpha(y, t) \star \psi_\tau(y, t) : : |0 \rangle = \int \frac{d^3p_1}{(2\pi)^32E_{p_1}} \int \frac{d^3p_2}{(2\pi)^32E_{p_2}} \left( m_{10} \gamma^\alpha_{\tau\alpha} \delta_{\beta\sigma} - m_{10} \gamma^\alpha_{\beta\sigma} \delta_{\tau\alpha} + m_{20} \gamma^\alpha_{\tau\alpha} \delta_{\beta\sigma} - m_{20} \gamma^\alpha_{\beta\sigma} \delta_{\tau\alpha} \right).
\]

\[11\]


\[ +p_{1i}p_{2j}\gamma^{i}_{\alpha\sigma}\gamma^{j}_{\beta\sigma} - p_{1i}\slashed{p}_{2j}\gamma^{i}_{\alpha\tau}\gamma^{j}_{\beta\tau} \cos(p_{1} + p_{2}) \cdot (x - y) \]

\[-(p_{10}p_{2j}\gamma^{0}_{\alpha\sigma} + p_{1i}\slashed{p}_{20}\gamma^{i}_{\alpha\sigma}) + mp_{1i}\gamma^{i}_{\alpha\tau}\delta_{\beta\sigma} - mp_{2j}\delta_{\beta\sigma}\gamma^{j}_{\alpha\tau} \]

\[ +p_{10}p_{2j}\gamma^{0}_{\beta\sigma} + p_{1i}\slashed{p}_{20}\gamma^{i}_{\beta\sigma} + mp_{1i}\gamma^{i}_{\beta\tau}\delta_{\alpha\sigma} - mp_{2j}\delta_{\alpha\sigma}\gamma^{j}_{\beta\tau} \]

\[ i\sin(p_{1} + p_{2}) \cdot (x - y) \]

\[
(3.28)
\]

where \( p_{1i} = (p_{1x}, p_{1y}, p_{1z}) \), \( p_{2i} = (p_{2x}, p_{2y}, p_{2z}) \). We can see that Eq. (3.28) does not vanish generally for an arbitrary interval of \((x - y)\) for some cases of the indexes \(\alpha, \beta, \sigma, \) and \(\tau\), and the result depend on the choice of the representation of \(\gamma\)-matrixes. Because the total expression of Eq. (3.26) is Lorentz invariant, we have

\[
\langle 0| [\overline{\psi}_{\alpha}(x) \star \psi_{\beta}(x) ; : \overline{\psi}_{\sigma}(y) \star \psi_{\tau}(y) :]_{\tau} |0\rangle \neq 0 \quad \text{for} \quad (x - y)^{2} < 0 \quad (3.29)
\]

for some cases of the indexes \(\alpha, \beta, \sigma, \) and \(\tau\), and the result depend on the choice of the representation of \(\gamma\)-matrixes. The result does not depend on whether \(\theta^{\mu}_{\nu}\) vanishes or not.

However we cannot conclude that microcausality is violated necessarily for free Dirac field on noncommutative spacetime from the above result. There are two sides of the reasons. On the one side as pointed out in Sec. II, for the criterion of the violation of microcausality, we must analyze the commutator of the same operator. Therefore in Eqs. (3.26)-(3.28), we must let \(\sigma = \alpha\) and \(\tau = \beta\). For such a case, we can see that in Eq. (3.28), the coefficient of \(\cos(p_{1} + p_{2}) \cdot (x - y)\) is zero, however the coefficient of \(\sin(p_{1} + p_{2}) \cdot (x - y)\) may not be zero for an arbitrary choice of the representation of \(\gamma\)-matrixes. Therefore we have

\[
B_{0,\alpha\beta\sigma\tau}(x, y) = \langle 0| [\overline{\psi}_{\alpha}(x) \star \psi_{\beta}(x) ; : \overline{\psi}_{\sigma}(y) \star \psi_{\tau}(y) :]_{\tau} |0\rangle \neq 0 \quad \text{for} \quad (x - y)^{2} < 0 \quad (3.30)
\]

for an arbitrary choice of the representation of \(\gamma\)-matrixes.

If for Dirac field, the observable quantities are directly \(\overline{\psi}_{\alpha}(x) \star \psi_{\beta}(x)\), then from Eq. (3.30) we can obtain that microcausality is not satisfied for free Dirac field on noncommutative spacetime. However this conclusion is not reasonable because it relies on the representation of \(\gamma\)-matrixes. The reason is that for Dirac field, its physical observable quantities are not \(\overline{\psi}_{\alpha}(x) \star \psi_{\beta}(x)\) directly. They are \(\overline{\psi}(x) \star \psi(x)\) and \(\overline{\psi}(x)\gamma^{\mu} \star \psi(x)\) etc. that constructed from \(\overline{\psi}_{\alpha}(x) \star \psi_{\beta}(x)\) and the \(\gamma\)-matrixes. Therefore we need to analyze the commutators for these observable quantities actually.

For the Lorentz scalar \(\overline{\psi}(x) \star \psi(x)\) we need to analyze whether

\[
B_{0}(x, y) = \langle 0| [\overline{\psi}(x) \star \psi(x) ; : \overline{\psi}(y) \star \psi(y) :]_{\tau} |0\rangle
\]

\[
= \langle 0| [\overline{\psi}_{\alpha}(x) \star \psi_{\beta}(x) ; : \overline{\psi}_{\sigma}(y) \star \psi_{\tau}(y) :]_{\tau} |0\rangle
\]

\[
(3.31)
\]

vanishes or not for a spacelike interval of \((x - y)\). This can be seen from its equal-time commutator

\[
B_{0}(x, t, y, t) = \langle 0| [\overline{\psi}(x, t) \star \psi(x, t) ; : \overline{\psi}(y, t) \star \psi(y, t) :]_{\tau} |0\rangle
\]

\[
= \langle 0| [\overline{\psi}_{\alpha}(x, t) \star \psi_{\beta}(x, t) ; : \overline{\psi}_{\sigma}(y, t) \star \psi_{\tau}(y, t) :]_{\tau} |0\rangle
\].
From Eq. (3.28) we obtain

\[
\langle 0 | [\overline{\psi}_\alpha(x, t) * \psi_\alpha(x, t) : : \overline{\psi}_\sigma(y, t) * \psi_\sigma(y, t) :]_s | 0 \rangle \\
= \int \frac{d^3 p_1}{(2\pi)^3 2E_{p_1}} \int \frac{d^3 p_2}{(2\pi)^3 2E_{p_2}} \left\{ (mp_{10} \gamma^0_\sigma \delta_\alpha^\sigma - mp_{10} \gamma^0_\sigma \delta_\alpha^\sigma + mp_{20} \gamma^0_\sigma \delta_\alpha^\sigma - mp_{20} \gamma^0_\sigma \delta_\alpha^\sigma \\
+ p_{1i} p_{2j} \gamma^i_\sigma \delta_\alpha^\sigma \gamma^j_\sigma \delta_\alpha^\sigma - p_{1i} p_{2j} \gamma^i_\sigma \delta_\alpha^\sigma \gamma^j_\sigma \delta_\alpha^\sigma \cos(p_1 + p_2) \cdot (x - y) \\
- (p_{10} p_{2j} \gamma^0_\sigma \delta_\alpha^\sigma + p_{1i} p_{20} \gamma^i_\sigma \delta_\alpha^\sigma - mp_{20} \delta_\alpha^\sigma \gamma^0_\sigma \delta_\alpha^\sigma \\
+ p_{10} p_{2j} \gamma^0_\sigma \delta_\alpha^\sigma + p_{1i} p_{20} \gamma^i_\sigma \delta_\alpha^\sigma - mp_{20} \delta_\alpha^\sigma \gamma^0_\sigma \delta_\alpha^\sigma) \cdot i \sin(p_1 + p_2) \cdot (x - y) \right\} .
\]

(3.32)

In Eq. (3.33) for \( \alpha \) and \( \sigma \) we need to sum up them from 1 to 4. From the properties of the traces of \( \gamma \)-matrices [26], we can obtain that the coefficients before \( \cos(p_1 + p_2) \cdot (x - y) \) and \( \sin(p_1 + p_2) \cdot (x - y) \) are all zero in Eq. (3.33), and they do not depend on the representation of the \( \gamma \)-matrices. From the Lorentz invariance of the expression we obtain

\[
B_0(x, y) = \langle 0 | [\overline{\psi}(x) * \psi(x) : : \overline{\psi}(y) * \psi(y) :]_s | 0 \rangle = 0 \quad \text{for} \quad (x - y)^2 < 0 .
\]

(3.34)

Therefore microcausality is satisfied for the Lorentz scalar \( \overline{\psi}(x) * \psi(x) \) of free Dirac field on noncommutative spacetime.

For the current \( \overline{\psi}(x) \gamma^\mu * \psi(x) \) of the Dirac field, we need to analyze whether

\[
B^\mu_0(x, y) = \langle 0 | [\overline{\psi}(x) \gamma^\mu * \psi(x) : : \overline{\psi}(y) \gamma^\mu * \psi(y) :]_s | 0 \rangle \\
= \langle 0 | [\overline{\psi}_\alpha(x) \gamma^\mu_\alpha \delta_\beta^\sigma * \psi_\beta(x) : : \overline{\psi}_\sigma(y) \gamma^\mu_\sigma \delta_\tau^\sigma * \psi_\tau(y) :]_s | 0 \rangle
\]

(3.35)

vanishes or not for a spacelike interval of \((x - y)\). This can be seen from its equal-time commutator

\[
B^\mu_0(x, t, y, t) = \langle 0 | [\overline{\psi}(x, t) \gamma^\mu * \psi(x, t) : : \overline{\psi}(y, t) \gamma^\mu * \psi(y, t) :]_s | 0 \rangle \\
= \langle 0 | [\overline{\psi}_\alpha(x, t) \gamma^\mu_\alpha \delta_\beta^\sigma \gamma^\mu_\sigma \delta_\tau^\sigma \psi_\beta(x, t) : : \overline{\psi}_\sigma(y, t) \gamma^\mu_\sigma \delta_\tau^\sigma \psi_\tau(y, t) :]_s | 0 \rangle .
\]

(3.36)

From Eq. (3.28) we obtain

\[
\langle 0 | [\overline{\psi}_\alpha(x, t) \gamma^\mu_\alpha \delta_\beta^\sigma \psi_\beta(x, t) : : \overline{\psi}_\sigma(y, t) \gamma^\mu_\sigma \delta_\tau^\sigma \psi_\tau(y, t) :]_s | 0 \rangle \\
= \int \frac{d^3 p_1}{(2\pi)^3 2E_{p_1}} \int \frac{d^3 p_2}{(2\pi)^3 2E_{p_2}} \left\{ (mp_{10} \gamma^0_\alpha \delta_\sigma^\beta \gamma^\mu_\beta \delta_\tau^\sigma - mp_{10} \gamma^0_\alpha \delta_\sigma^\beta \gamma^\mu_\beta \delta_\tau^\sigma + mp_{20} \delta_\sigma^\beta \gamma^0_\beta \gamma^\mu_\beta \delta_\tau^\sigma \\
- mp_{20} \delta_\sigma^\beta \gamma^0_\beta \gamma^\mu_\beta \delta_\tau^\sigma + p_{1i} p_{2j} \gamma^i_\beta \delta_\sigma^\beta \gamma^j_\beta \delta_\tau^\sigma - p_{1i} p_{2j} \gamma^i_\beta \delta_\sigma^\beta \gamma^j_\beta \delta_\tau^\sigma \cos(p_1 + p_2) \cdot (x - y) \\
- (p_{10} p_{2j} \gamma^0_\beta \delta_\sigma^\beta \gamma^\mu_\beta \delta_\tau^\sigma + p_{1i} p_{20} \gamma^i_\beta \delta_\sigma^\beta \gamma^\mu_\beta \delta_\tau^\sigma - mp_{20} \delta_\sigma^\beta \gamma^0_\beta \delta_\tau^\sigma \\
+ p_{10} p_{2j} \gamma^0_\beta \delta_\sigma^\beta \gamma^j_\beta \delta_\tau^\sigma + p_{1i} p_{20} \gamma^i_\beta \delta_\sigma^\beta \gamma^j_\beta \delta_\tau^\sigma - mp_{20} \delta_\sigma^\beta \gamma^0_\beta \delta_\tau^\sigma) \cdot i \sin(p_1 + p_2) \cdot (x - y) \right\} .
\]
In Eq. (3.37), the summations of the indexes are traces of the producted \( \gamma \)-matrixes. From the properties of the traces of \( \gamma \)-matrixes [26], we can obtain that the coefficients before \( \cos(p_1 + p_2) \cdot (x - y) \) and \( \sin(p_1 + p_2) \cdot (x - y) \) are all zero in Eq. (3.37), and they do not depend on the representation of the \( \gamma \)-matrixes. From the Lorentz invariance of the expression we obtain

\[
B^\mu_0(x, y) = \langle 0 | [\overline{\psi}(x) \gamma^\mu \psi(x) \vert y] \vert 0 \rangle = 0 \quad \text{for} \quad (x - y)^2 < 0 .
\](3.38)

Therefore microcausality is satisfied for the current \( \overline{\psi}(x) \gamma^\mu \psi(x) \) of free Dirac field on non-commutative spacetime. For the other bilinear forms of free Dirac field on noncommutative spacetime such as \( \overline{\psi}(x) \gamma^5 \psi(x) \), \( \overline{\psi}(x) \gamma_\mu \psi(x) \), and \( \overline{\psi}(x) \sigma_{\mu\nu} \psi(x) \), we can also obtain that they satisfy the microcausality through explicit calculations, and the conclusion does not depend on the representation of \( \gamma \)-matrixes. However for the details of the calculations for these bilinear forms, we omit to write down them here.

From the above analysis, we can also see that for the microcausality of Dirac field on ordinary commutative spacetime, if we examine it from Eq. (3.12), we can obtain that it satisfies the microcausality directly. However if we calculate the vacuum expectation values for the commutators, we will obtain the result which is equal to Eq. (3.26), i.e.,

\[
\langle 0 | [\overline{\psi}(x) \psi(\beta \sigma)(y) \vert y] \vert 0 \rangle = \int \frac{d^3p_1}{(2\pi)^3 2E_{p_1}} \int \frac{d^3p_2}{(2\pi)^3 2E_{p_2}} \left[ (\not{p}_1 - m)_{\tau\alpha}(\not{p}_2 + m)_{\beta\sigma} e^{i(p_1 + p_2)(x-y)} - (\not{p}_1 - m)_{\beta\sigma}(\not{p}_2 + m)_{\tau\alpha} e^{-i(p_1 + p_2)(x-y)} \right].
\](3.39)

Similarly we will obtain the result as that of Eqs. (3.27)-(3.30). Thus from this approach, we cannot obtain that Dirac field satisfies the microcausality on ordinary commutative spacetime directly. However through the calculation of the expectation values for the commutators of \( \overline{\psi}(x) \psi(x) \) and \( \overline{\psi}(x) \gamma^\mu \psi(x) \) etc., we can still obtain that Dirac field on ordinary commutative spacetime satisfies the microcausality, as that of Eqs. (3.31)-(3.38).

### 4 Non-vacuum state expectation values

For the criterion of microcausality violation given by Eq. (2.5), the state vector \(|\Psi\rangle\) is not a vacuum state generally. Therefore we need to calculate the non-vacuum state expectation values of \( B_{\alpha\beta\sigma\tau}(x, y) \) given by Eq. (3.15). For such a purpose we need first to define the state vector \(|\Psi\rangle\) for a Dirac field quantum system. We can write it as

\[
|\Psi\rangle = |N_{p_1}(s, s') N_{p_2}(s, s') \cdots N_{p_i}(s, s') \cdots, 0 \rangle.
\](4.1)

It is in the occupation eigenstate. In Eq. (4.1) we use \( N_{p_i} \) to represent the occupation number for the momentum \( p_i \), and use \( (s, s') \) to represent four kinds of the spinors \( u(p, s) \).
and $V(p, s)$. $N_{p_i}(s, s')$ can only take the values 0 and 1. We suppose that the occupation numbers are nonzero only on some separate momentums $p_i$. For all the other momentums, the occupation numbers are zero. We use 0 to represent that the occupation numbers are zero on all the other momentums and spins in Eq. (4.1). The state vector $|\Psi\rangle$ has the following properties [27]:

\[
\langle N_{p_1}(s, s')N_{p_2}(s, s') \cdots N_{p_{i-1}}(s, s')0_{p_i}(s, s')N_{p_{i+1}}(s, s') \cdots \rangle = 0 ,
\]

\[
\sum_{N_{p_1}(s, s')N_{p_2}(s, s') \cdots} |N_{p_1}(s, s')N_{p_2}(s, s') \cdots N_{p_{i-1}}(s, s')0_{p_i}(s, s')N_{p_{i+1}}(s, s') \cdots \rangle = 1 ,
\]

\[
(4.2)
\]

\[
\begin{align*}
a_{s, s'}(p_i) & |N_{p_1}(s, s')N_{p_2}(s, s') \cdots N_{p_{i-1}}(s, s')0_{p_i}(s, s')N_{p_{i+1}}(s, s') \cdots \rangle = 0 , \\
a_{s, s'}^\dagger(p_i) & |N_{p_1}(s, s')N_{p_2}(s, s') \cdots N_{p_{i-1}}(s, s')1_{p_i}(s, s')N_{p_{i+1}}(s, s') \cdots \rangle = 0 , \\
a_{s, s'}^\dagger(p_i) & |N_{p_1}(s, s')N_{p_2}(s, s') \cdots N_{p_{i-1}}(s, s')0_{p_i}(s, s')N_{p_{i+1}}(s, s') \cdots \rangle = 0 , \\
a_{s, s'}(p_i) & |N_{p_1}(s, s')N_{p_2}(s, s') \cdots N_{p_{i-1}}(s, s')1_{p_i}(s, s')N_{p_{i+1}}(s, s') \cdots \rangle = 0 .
\end{align*}
\]

\[
(4.3)
\]

In Eq. (4.3), we use $a_{s, s'}$ to represent one kind of the annihilation operators $b(p, s)$ and $d(p, s)$, and use $a_{s, s'}^\dagger$ to represent one kind of the creation operators $b^\dagger(p, s)$ and $d^\dagger(p, s)$. We need to calculate the function

\[
B_{\alpha\beta\sigma\tau}(x, y) = \langle \Psi | [ : \psi_\alpha(x) \psi_\beta(x) \cdots : \psi_\sigma(y) \psi_\tau(y) :]_s | \Psi \rangle .
\]

\[
(4.4)
\]

In the scalar field case [23], we have proved that the non-vacuum state expectation value for the Moyal commutator $[ : \varphi(x) \star \varphi(x) \cdots : \varphi(y) \star \varphi(y) :]_s$ is just equal to the corresponding vacuum expectation value. It is a universal function for an arbitrary state vector of scalar field quantum system. We can also prove that such a property also hold for Dirac field. We first need to recognize that the total energy of an actual field quantum system is always finite. This makes the occupation numbers $N_{p_i}(s, s')$ be nonzero only on a set of finite number separate momentums $p_i$, because $N_{p_i}(s, s')$ take values of the integral numbers 0 and 1. If $N_{p_i}$ take nonzero values on infinite number separate momentums $p_i$ or on a continuous interval of the momentum, the total energy of the field quantum system will be infinite.

Through calculation we can write

\[
\langle \Psi | [ : \overline{\psi}_\alpha(x) \ast \psi_\beta(x) \cdots : \overline{\psi}_\sigma(y) \ast \psi_\tau(y) :]_s | \Psi \rangle = \int \frac{d^3p_1}{(2\pi)^32E_{p_1}} \int \frac{d^3p_2}{(2\pi)^32E_{p_2}} G_{\alpha\beta\sigma\tau} (p_1, p_2, x, y) .
\]

\[
(4.5)
\]
From Eq. (3.23) we can see that the non-zero contributions to the integrand $G_{\alpha\beta,\sigma\tau}(p_1, p_2, x, y)$ not only come from the operators $\overline{\psi}_\alpha(x) \star \psi_\beta(x) \star \overline{\psi}_\sigma(y) \star \psi_\tau(y)$ and $\overline{\psi}_\sigma(y) \star \psi_\tau(y) \star \overline{\psi}_\alpha(x) \star \psi_\beta(x)$ as that of Eq. (3.26), but also come from the operators $\psi_\beta(x) \star \overline{\psi}_\alpha(x) \star \psi_\tau(y) \star \overline{\psi}_\sigma(y)$, $\psi_\tau(y) \star \overline{\psi}_\sigma(y) \star \psi_\beta(x) \star \overline{\psi}_\alpha(x)$, $\overline{\psi}_\sigma(y) \star \psi_\tau(y) \star \psi_\beta(x) \star \overline{\psi}_\alpha(x)$, $\psi_\tau(y) \star \overline{\psi}_\sigma(y) \star \psi_\beta(x) \star \overline{\psi}_\alpha(x)$, $\psi_\beta(x) \star \overline{\psi}_\alpha(x) \star \psi_\tau(y) \star \overline{\psi}_\sigma(y)$, $\overline{\psi}_\sigma(y) \star \psi_\tau(y) \star \psi_\beta(x) \star \overline{\psi}_\alpha(x)$, $\psi_\beta(x) \star \overline{\psi}_\alpha(x) \star \psi_\tau(y) \star \overline{\psi}_\sigma(y)$, $\psi_\tau(y) \star \overline{\psi}_\sigma(y) \star \psi_\beta(x) \star \overline{\psi}_\alpha(x)$, $\psi_\beta(x) \star \overline{\psi}_\alpha(x) \star \psi_\tau(y) \star \overline{\psi}_\sigma(y)$, $\overline{\psi}_\sigma(y) \star \psi_\tau(y) \star \psi_\beta(x) \star \overline{\psi}_\alpha(x)$, $\psi_\beta(x) \star \overline{\psi}_\alpha(x) \star \psi_\tau(y) \star \overline{\psi}_\sigma(y)$, $\psi_\tau(y) \star \overline{\psi}_\sigma(y) \star \psi_\beta(x) \star \overline{\psi}_\alpha(x)$, $\psi_\beta(x) \star \overline{\psi}_\alpha(x) \star \psi_\tau(y) \star \overline{\psi}_\sigma(y)$, $\overline{\psi}_\sigma(y) \star \psi_\tau(y) \star \psi_\beta(x) \star \overline{\psi}_\alpha(x)$, $\psi_\beta(x) \star \overline{\psi}_\alpha(x) \star \psi_\tau(y) \star \overline{\psi}_\sigma(y)$, $\psi_\tau(y) \star \overline{\psi}_\sigma(y) \star \psi_\beta(x) \star \overline{\psi}_\alpha(x)$, $\psi_\beta(x) \star \overline{\psi}_\alpha(x) \star \psi_\tau(y) \star \overline{\psi}_\sigma(y)$, $\psi_\tau(y) \star \overline{\psi}_\sigma(y) \star \psi_\beta(x) \star \overline{\psi}_\alpha(x)$. These operators all have the equal numbers of negative frequency and positive frequency components. This makes the integrand $G_{\alpha\beta,\sigma\tau}(p_1, p_2, x, y)$ of Eq. (4.5) be not equal to the integrand of Eq. (3.26) generally. However because in the state vector $|\Psi\rangle$, the occupation numbers $N_{\alpha\beta}(s, s')$ are nonzero only on a set of finite number separate momentums $p_i$ as pointed out above, the integrand $G_{\alpha\beta,\sigma\tau}(p_1, p_2, x, y)$ of Eq. (4.5) only changes its value relative to the integrand of Eq. (3.26) on a set of finite number separate momentums $p_i$ of $|\Psi\rangle$, just like that of the scalar field case. However we omit to write down the detailed analysis for such a fact here. Thus we need not to obtain the explicit form for the integrand $G_{\alpha\beta,\sigma\tau}(p_1, p_2, x, y)$ of Eq. (4.5) on the set of finite number separate momentums $p_i$ of $|\Psi\rangle$. This is because the integrand $G_{\alpha\beta,\sigma\tau}(p_1, p_2, x, y)$ may be a bounded function. The total integral measure for a set of finite number separate momentums $p_i$ is zero. Therefore according to the theory of integration (for example see Ref. [28]), the total integral of Eq. (4.5) is not changed from that of Eq. (3.26). Thus we obtain

$$
\langle \Psi| : \overline{\psi}_\alpha(x) \star \psi_\beta(x) : : \overline{\psi}_\sigma(y) \star \psi_\tau(y) : |\Psi \rangle = \int \frac{d^3p_1}{(2\pi)^3 2E_{p_1}} \int \frac{d^3p_2}{(2\pi)^3 2E_{p_2}} \left[ (p_1 - m)_{r\alpha} (p_2 + m)_{\beta\sigma} e^{-i(p_1 + p_2)(x - y)} - (p_1 - m)_{\beta\sigma} (p_2 + m)_{r\alpha} e^{i(p_1 + p_2)(x - y)} \right].
$$

(4.6)

Or we can write

$$
B_{\alpha\beta\sigma\tau}(x, y) = B_{0,\alpha\beta\sigma\tau}(x, y),
$$

(4.7)

which is a universal function for an arbitrary state vector of Eq. (4.1). Thus from the result of Sec. IV we can derive

$$
\langle \Psi| : \overline{\psi}(x) \star \psi(x) : : \overline{\psi}(y) \star \psi(y) : |\Psi \rangle = 0 \quad \text{for} \quad (x - y)^2 < 0 ,
$$

(4.8)

and similarly

$$
\langle \Psi| : \overline{\psi}(x) \gamma^\mu \star \psi(x) : : \overline{\psi}(y) \gamma^\mu \star \psi(y) : |\Psi \rangle = 0 \quad \text{for} \quad (x - y)^2 < 0 .
$$

(4.9)

For the other bilinear forms of the Dirac field such as $\overline{\psi}(x) \gamma^5 \star \psi(x)$, $\overline{\psi}(x) \gamma^5 \gamma^\mu \star \psi(x)$, and $\overline{\psi}(x) \sigma^{\mu\nu} \star \psi(x)$ on noncommutative spacetime, we can obtain the same result. This means that microcausality is satisfied for free Dirac field on noncommutative spacetime generally, no matter whether $\theta^{01} = 0$ or $\theta^{01} \neq 0$. 

16
5 Conclusion

In this paper, we studied the microcausality of free Dirac field on noncommutative spacetime. As pointed out in the Introduction, we expand quantum fields on noncommutative spacetime according to their usual Lorentz invariant spectral measures. For the $SO(1, 1) \times SO(2)$ invariant spectral measures as that constructed in Ref. [10], we would rather consider that it is an effective result evoked from the nonlocal interactions of NCFTs as pointed out in Ref. [20]. Therefore we consider that the infinite propagation speed of the waves inside the $SO(1, 1)$ light wedge is in fact an effect of the nonlocal interactions of NCFTs, while not the property of free fields. Similarly, we would rather consider that the twisted Poincaré invariance [13] of NCFTs is also an effective property that generated from the nonlocal interactions of NCFTs. For NCFTs, although there exist the nonlocality, it does not mean that Lorentz invariance is necessarily broken. In fact, the nonlocality of NCFTs can be in self-consistency with the Lorentz invariance. For the noncommutative parameters $\theta^{\mu \nu}$, if we take them to be a second-order antisymmetric tensor, it is possible for us to realize the Lorentz invariance of NCFTs [15-18]. On the other hand we know that the breakdown of Lorentz invariance has not been discovered yet in the experiments [19]. It is reasonable to believe that Lorentz invariance is a more fundamental principle of physics in a local area, although it may not be satisfied for a large scale structure of the universe. Therefore we expand quantum fields on noncommutative spacetime according to their usual Lorentz invariant spectral measures, except that we take the coordinates to be noncommutative operators.

Another reason for us to do so is that in most occasions for the perturbative calculations of NCFTs in the literature, the propagators of quantum fields on noncommutative spacetime are obtained based on the usual Lorentz invariant spectral measures for the expansion of fields. Thus it is necessary for us to study whether microcausality is violated or not for quantum fields on noncommutative spacetime if we expand them according to their usual Lorentz invariant spectral measures, i.e., we hope to study whether the breakdown of microcausality can occur in NCFTs in accordance with the Lorentz invariance. In Ref. [22], Greenberg obtained that microcausality is violated for free scalar field on noncommutative spacetime generally no matter whether $\theta^{0i}$ vanishes or not. However we have pointed out in Ref. [23] that there are some problems for the arguments in Ref. [22] for the result of the microcausality violation. We obtained that microcausality is violated for the quadratic operators of scalar field on noncommutative spacetime only when $\theta^{0i} \neq 0$ [23].

For the free Dirac field on ordinary commutative spacetime, we can obtain that microcausality is satisfied directly from Eqs. (3.12) and (3.13) according to the properties of the fundamental anticommutators of Dirac field. However for the Dirac field on noncommutative spacetime, because its fundamental Moyal anticommutators are not the $c$-number functions [24], we cannot obtain the conclusion that microcausality is satisfied for Dirac field on noncommutative spacetime from Eq. (3.14). We need to calculate the vacuum and non-vacuum state expectation values for the Moyal commutator \( [\bar{\psi}_\alpha(x) \star \psi_\beta(x), \bar{\psi}_\sigma(x') \star \psi_\tau(x')] \). As demonstrated in Sec. II, this is also the demand of the physical measurements. In Sec. IV, we argued that because the total energy of an actual field quantum system is always finite, the non-vacuum state expectation value for the Moyal commutator \( [\bar{\psi}_\alpha(x) \star \psi_\beta(x), \bar{\psi}_\sigma(x') \star \psi_\tau(x')] \).
is just equal to its vacuum expectation value. It is a universal function for an arbitrary state vector of the Dirac field quantum system as that of the scalar field case [23].

In Sec. III, we obtained that microcausality is not satisfied generally for the Moyal commutator $[\bar{\psi}_\alpha(x) \star \psi_\beta(x), \bar{\psi}_\sigma(x') \star \psi_\tau(x')]_*$ for some cases of the indexes $\alpha, \beta, \sigma,$ and $\tau$, and the result depends on the choice of the representation of $\gamma$-matrixes, as that given by Eqs. (3.29) and (3.30). However, for the physical observables of Dirac field on noncommutative spacetime such as the Lorentz scalar $\bar{\psi}(x) \star \psi(x)$ and the current $\bar{\psi}(x) \gamma^\mu \star \psi(x)$, we have obtained that microcausality is satisfied. For some other bilinear forms of the Dirac field such as $\bar{\psi}(x) \gamma^5 \star \psi(x)$, $\bar{\psi}(x) \gamma^5 \gamma^\mu \star \psi(x)$, and $\bar{\psi}(x) \sigma^\mu \nu \star \psi(x)$, we can also obtain that microcausality is satisfied for them on noncommutative spacetime. Therefore microcausality is not violated for the free Dirac on noncommutative spacetime generally. We have not found the violation of microcausality of free Dirac on noncommutative spacetime in accordance with the Lorentz invariance, no matter whether $\theta^0i$ vanishes or not. Therefore there does not exist the information and interaction with the transmit speed faster than the speed of light for the free Dirac on noncommutative spacetime.

For the free electromagnetic field on noncommutative spacetime, we can expect that its microcausality property is similar to the free scalar field case, i.e., there may exist the violation of microcausality for the free electromagnetic field on noncommutative spacetime when $\theta^0i \neq 0$. As pointed out in Refs. [29,30], unitarity of the $S$-matrixes may be lost for NCFTs when $\theta^0i \neq 0$. Therefore if we exclude the case of $\theta^0i \neq 0$ for the noncommutative parameters $\theta^\mu \nu$ from the demand of the unitarity of the $S$-matrixes, microcausality is also satisfied for free bose fields on noncommutative spacetime. However one can be conscious of that it is rather difficult for us to consider that the time coordinate is in an unequal position to the space coordinates even if the spacetime coordinates are noncommutative. For the unitarity problem of NCFTs with $\theta^0i \neq 0$, some authors have devoted in searching for the retrieving possibilities [31-33].

Although for the free fields on noncommutative spacetime there may not exist the violation of microcausality generally, and therefore there does not exist the information and interaction with the transmit speed faster than the speed of light for the free fields on noncommutative spacetime, except for the free bose fields for the spacetime noncommutativity with $\theta^0i \neq 0$, there can exist the infinite propagation speed for the signals and interactions on noncommutative spacetime as that indicated in Refs. [6-8], no matter whether $\theta^0i$ vanishes or not. On the other hand, the nonlocal interactions of NCFTs can result the existence of lower dimensional light wedge [20]. This can result the effective spectral measures for the expansions of quantum fields on noncommutative spacetime in the form of the light wedge invariance as that constructed in Ref. [10] and therefore results the infinite propagation speed inside the light wedge.

ACKNOWLEDGEMENTS

Thanks very much for C.S. Chu, D.H.T. Franco, and S. Pasquetti to inform me Refs. [20], [12], and [33].
APPENDIX: SELF-CONSISTENCY OF SPACETIME COMMUTATION RELATIONS (1.3) WITH (1.1)

In this Appendix, we give a proof for the self-consistency of Eq. (1.3) with Eq. (1.1) for the spacetime commutation relations. First we have the commutation relations (1.1) for the coordinates at the same spacetime point

\[ [x^\mu, x^\nu] = i \theta^{\mu\nu} . \]  

(A1)

For a different spacetime point \( y \) we also have

\[ [y^\mu, y^\nu] = i \theta^{\mu\nu} . \]  

(A2)

If we generalize the commutation relations (1.1) to two different spacetime points, then we have

\[ [x^\mu, y^\nu] = i \theta^{\mu\nu} . \]  

(A3)

At the same time equivalently we have

\[ [y^\mu, x^\nu] = i \theta^{\mu\nu} . \]  

(A4)

To introduce the difference \( \Delta x^\mu \) between \( x^\mu \) and \( y^\mu \), we have

\[ x^\mu + \Delta x^\mu = y^\mu . \]  

(A5)

Similarly to introduce the difference \( \Delta x^\nu \) between \( x^\nu \) and \( y^\nu \), we have

\[ x^\nu + \Delta x^\nu = y^\nu . \]  

(A6)

From Eqs. (A3) and (A6) we have

\[ [x^\mu, x^\nu + \Delta x^\nu] = i \theta^{\mu\nu} . \]  

(A7)

To combine Eq. (A1) we obtain from Eq. (A7)

\[ [x^\mu, \Delta x^\nu] = 0 . \]  

(A8)

Similarly, from Eqs. (A4) and (A5) we have

\[ [x^\mu + \Delta x^\mu, x^\nu] = i \theta^{\mu\nu} . \]  

(A9)

To combine Eq. (A1) we obtain from Eq. (A9)

\[ [\Delta x^\mu, x^\nu] = 0 . \]  

(A10)

To use Eqs. (A5) and (A6), we can write Eq. (A2) as

\[ [x^\mu + \Delta x^\mu, x^\nu + \Delta x^\nu] = i \theta^{\mu\nu} . \]  

(A11)
To combine Eqs. (A1), (A8), and (A10), we obtain from Eq. (A.11) that the relation
\[ [\Delta x^\mu, \Delta x^\nu] = 0 \] (A12)
should be satisfied, if the relations (A3) or equivalently (A4) are in consistence with the relations (A1) or equivalently (A2). We can verify this equation.

To use Eqs. (A5) and (A6), we can write
\[ [\Delta x^\mu, \Delta x^\nu] = [y^\mu - x^\mu, y^\nu - x^\nu] . \] (A13)
To expand the right hand side of Eq. (A13), we have
\[ [y^\mu - x^\mu, y^\nu - x^\nu] = [y^\mu, y^\nu] - [y^\mu, x^\nu] - [x^\mu, y^\nu] + [x^\mu, x^\nu] . \] (A14)
To use Eqs. (A1)-(A4), we obtain
\[ [y^\mu - x^\mu, y^\nu - x^\nu] = 0 . \] (A15)
Thus Eq. (A12) is satisfied. This means that the relations (A3) or (A4) are in consistence with the relations (A1) or (A2). Thus the generalization of the spacetime commutative relations (1.1) defined at the same spacetime point to the spacetime commutative relations (1.3) defined on two different spacetime points is reasonable. Or we can write Eqs. (A3) and (A4) in the form of Eq. (1.3)
\[ [x^\mu_1, x^\nu_2] = i\theta^{\mu\nu} . \] (A16)

We need to take notice that the origin of coordinates in noncommutative spacetime is not a \( c \)-number either. It is also an operator. We can write it in the form of boldface as \( 0 \). We have
\[ [x^\mu, 0^\nu] = i\theta^{\mu\nu} , \] (A17)
and similarly
\[ [0^\mu, x^\nu] = i\theta^{\mu\nu} . \] (A18)
Thus we need to notice that \( x^\mu - 0^\mu = \Delta x^\mu \neq x^\mu \).

References

[1] H.S. Snyder, Phys. Rev. 71, 38 (1947).
[2] S. Doplicher, K. Fredenhagen, and J.E. Roberts, Phys. Lett. B 331, 39 (1994); Commun. Math. Phys. 172, 187 (1995), [hep-th/0303037].
[3] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032, [hep-th/9908142].
[4] M.R. Douglas and N.A. Nekrasov, Rev. Mod. Phys. 73, 977 (2001), [hep-th/0106048].
[5] R.J. Szabo, Phys. Rep. 378, 207 (2003), [hep-th/0109162].
[6] M. Van Raamsdonk, J. High Energy Phys. 11 (2001) 006, [hep-th/0110093].
[7] A. Hashimoto and N. Itzhaki, Phys. Rev. D 63, 126004 (2001), [hep-th/0102093].
[8] B. Durhuus and T. Jonsson, J. High Energy Phys. 10 (2004) 050, [hep-th/0408190].
[9] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, J. High Energy Phys. 02 (2000) 020, hep-th/9912072.
[10] L. Alvarez-Gaumé and M.A. Vázquez-Mozo, Nucl. Phys. B668, 293 (2003), hep-th/0305093.
[11] L. Alvarez-Gaumé, J.L.F. Barbón, and R. Zwicky, J. High Energy Phys. 05 (2001) 057, hep-th/0103069.
[12] D.H.T. Franco and C.M.M. Polito, J. Math. Phys. 46, 083503 (2005), hep-th/0403028.
[13] M. Chaichian, P. Prešnajder, and A. Tureanu, Phys. Rev. Lett. 94, 151602 (2005), hep-th/0409096.
[14] M. Chaichian, K. Nishijima, and A. Tureanu, Phys. Lett. B 633, 129 (2006), hep-th/0511094.
[15] C.E. Carlson, C.D. Carone, and N. Zobin, Phys. Rev. D 66, 075001 (2002), hep-th/0206035.
[16] H. Kase, K. Morita, Y. Okumura, and E. Umezawa, Prog. Theor. Phys. 109, 663 (2003), hep-th/0212176.
[17] K. Morita, Y. Okumura, and E. Umezawa, Prog. Theor. Phys. 110, 989 (2003), hep-th/0309155.
[18] R. Banerjee, B. Chakraborty, and K. Kumar, Phys. Rev. D 70, 125004 (2004), hep-th/0408197.
[19] A. Anisimov, T. Banks, M. Dine, and M. Graesser, Phys. Rev. D 65, 085032 (2002), hep-ph/0106356.
[20] C.S. Chu, K. Furuta, and T. Inami, Int. J. Mod. Phys. A 21, 67 (2006), hep-th/0502012.
[21] M. Chaichian, K. Nishijima, and A. Tureanu, Phys. Lett. B 568, 146 (2003), hep-th/0209008.
[22] O.W. Greenberg, Phys. Rev. D 73, 045014 (2006), hep-th/0508057.
[23] Z.Z. Ma, hep-th/0603054.
[24] Z.Z. Ma, hep-th/0601094.
[25] J.D. Bjorken and S.D. Drell, Relativistic quantum fields (McGraw-Hill, 1965).
[26] C. Itzykson and J.-B. Zuber, Quantum field theory (McGraw-Hill Inc., 1980).
[27] L.D. Landau and E.M. Lifshitz, Quantum mechanics (Pergamon Press, 1977).
[28] G. de Barra, Measure theory and integration (Halsled Press, New York, 1981).
[29] J. Gomis and T. Mehen, Nucl. Phys. B591, 265 (2000), hep-th/0005129.
[30] A. Bassetto, F. Vian, L. Griguolo, and G. Nardelli, J. High Energy Phys. 07 (2001) 008, hep-th/0105257.
[31] D. Bahns, S. Doplicher, K. Fredenhagen, and G. Piacitelli, Phys. Lett. B 533, 178 (2002), hep-th/0201222.
[32] C.S. Chu, J. Lukierski, and W.J. Zakrzewski, Nucl. Phys. B632, 219 (2002), hep-th/0201143.
[33] N. Caporaso and S. Pasquetti, hep-th/0511127.