Laughlin’s wave functions, Coulomb gases and expansions of the discriminant.

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In the context of the fractional quantum Hall effect, we investigate Laughlin’s celebrated ansatz for the ground state wave function at fractional filling of the lowest Landau level. Interpreting its normalization in terms of a one component plasma, we find the effect of an additional quadrupolar field on the free energy, and derive estimates for the thermodynamically equivalent spherical plasma. In a second part, we present various methods for expanding the wave function in terms of Slater determinants, and obtain sum rules for the coefficients. We also address the apparently simpler question of counting the number of such Slater states using the theory of integral polytopes.

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1. Introduction

These notes originate from an attempt to understand the normalization – and other properties – of the many body fermionic wave functions suggested by Laughlin as candidates for the ground states of the fractional quantum Hall effect. Similar quantities appear in the context of the two–dimensional classical one–component plasma (sometimes called jellium) and in statistical problems related to matrices or random polynomials.

Our work is divided into three parts. In the first one, after recalling the simplest cases of quantum wave functions at odd fractional filling of the lowest Landau level proposed by Laughlin, we discuss their expansion in terms of Slater determinants which would allow to obtain- among other things - their normalization (section 2). We then record the standard interpretation of the normalization problem as computing the partition function of a two dimensional - neutralized - one component plasma. In a variant we show that an added quadrupolar compensating field has no effect on the thermodynamic properties (at complete filling) except to modify the shape of the ”quantum liquid drop”. It is possible to substitute to the planar geometry a spherical one with higher symmetry (section 3). This permits one to obtain bounds for the residual free energy - in the Coulomb interpretation- using Hölder’s and Hadamard’s inequalities. These bounds can be conjecturally improved by using the behaviour at zero temperature of some special symmetric configurations for finitely many particles (up to 42). We devote appendix C to an interesting instability of the cuboctahedral configuration.

Part two is centered around one of the most fundamental objects of algebra: the discriminant of a polynomial

\[ D = \prod_{i \neq j}(x_i - x_j), \quad (1.1) \]

where \( i \) runs over \( N \) distinct values (conveniently chosen as the integers \( 0, 1, 2, ..., N - 1 \) ) and \( x_i \) are indeterminates which may be thought of as the roots of the monic polynomial

\[ P(x) = \prod_i (x - x_i) = \sum_{k=0}^{N} (-1)^k \sigma_k x^{N-k}; \quad \sigma_0 \equiv 1. \quad (1.2) \]

The discriminant \( D \) as well as its powers are symmetric polynomial functions of the indeterminates \( x_i \) and as such admit a unique polynomial representation in terms of the
elementary symmetric functions $\sigma_k$, a well known fact since the days of Newton. The subsequent representation for $D$ is unfortunately not very useful for our purposes. On the other hand the ring of symmetric polynomial functions with rational coefficients can be viewed as a graded vector space over $\mathbb{Q}$. Each homogeneous subspace admits a basis in terms of Jacobi–Schur functions $ch_Y(x_i)$, indexed by partitions or Young tableaux. Thus with $s$ a positive integer

$$(-1)^{sN(N-1)/2}D^s(x_i) = \sum_{|Y|=sN(N-1)} g_Y^{(s)} \cdot ch_Y(x_i). \quad (1.3)$$

We were unable to find in the immense literature on symmetric functions and representations of linear groups a discussion of such an expansion. In section 4 we give various expressions for the coefficients $g_Y$ (eqs. (4.16),(4.20),(4.22) and (4.57)) none of them being very effective. We also derive a number of their properties and present some tabulations in appendix D. As already admitted, even for $s = 1$, we could not find a useful expression for the general integral coefficients $g_Y^{(1)}$ simple enough for our purposes. This is left as a challenge for a clever reader. However in section 5 we exhibit a remarkable sum rule

$$\sum_Y |g_Y^{(s)}|^2 = \frac{[(2s+1)N]!}{N!((2s+1)!)^N} \quad (1.4)$$

derived from a formula conjectured by F. Dyson, and proved by Wilson, Gunson, and Good.

The $g_Y$ coefficients also satisfy an overdetermined linear system (section 6) which turns out in practice to be the fastest means of computation for small enough number of particles. Finally we consider $q$-specializations in section 7.

We devote the third part to yet another problem, at first sight elementary, which is to count the number of terms in the expansion (1.3). It turns out that this is not trivial at all. We will elaborate in section 8 and appendix E on our findings in this direction.

As this work was in progress we received an article by G.W. Dunne [1] which partly overlaps with ours.

2. From the Laughlin wave function to the classical Coulomb gas

In the fractional quantum Hall effect one observes plateaux in the Hall conductance indexed by filling fractions of Landau levels, the infinitely degenerate non–relativistic energy levels for a charged particle interacting with a magnetic field (see for instance [2] for a
review). Assuming the particles constrained in a transverse plane, and choosing a suitable
scale for the field, an orthogonal complete set of states in the lowest Landau level is given
by
\[ \psi_l(x) = \frac{1}{\sqrt{\pi}} x^l e^{-\frac{x^2}{\pi}}, \]  
(2.1)
where \( x \) is a complex variable and \( l = 0, 1, \ldots \), can be interpreted as the angular momentum
eigenvalue. The normalization of these wave functions is
\[ \langle \psi_l | \psi_l' \rangle = \int \frac{d^2x}{\pi} e^{-x\bar{x}} x^l x'^l = l! \delta_{ll'} \]  
(2.2)

For a system of \( N \) non–interacting particles with Fermi statistics in the lowest Landau level
we take Slater determinants as an orthogonal basis of wave functions
\[ \psi_{l_0, l_1, \ldots, l_{N-1}}(x_0, \ldots, x_{N-1}) = \frac{1}{\sqrt{\pi N/2}} \prod_{i=0}^{N-1} x_i^{l_i} \]  
(2.3)

We may assume ordered indices \( 0 \leq l_0 < l_1 < \ldots < l_{N-1} \), since the multiparticle wave
function is antisymmetric in the permutations of the \( l \)'s. We have then
\[ \langle \psi_{l_0, \ldots, l_{N-1}} | \psi_{l'_0, \ldots, l'_{N-1}} \rangle = N! \prod_{i=0}^{N-1} l_i! \delta_{l_i, l'_i} \]  
(2.4)

The state with lowest total angular momentum, corresponds to the choice \( l_i = i \), in which
case the Slater determinant reduces up to a factor to the Vandermonde determinant
\[ \Delta(x_0, \ldots, x_{N-1}) = \frac{1}{\pi N/2} \prod_{i=0}^{N-1} x_i^{l_i} \]  
(2.5)

We use the notation \( D(x_0, \ldots, x_{N-1}) \) for the discriminant, equal up to a sign to the square
of the Vandermonde determinant
\[ D(x_0, \ldots, x_{N-1}) = \prod_{i \neq j} (x_i - x_j) = (-1)^{N(N-1)/2} \Delta(x_0, \ldots, x_{N-1})^2, \]  
(2.6)
and identify the normalization of the “densest state” (filling fraction 1) as a partition
function through
\[ Z_N(1) = \int \prod_{i=0}^{N-1} \left( \frac{d^2x_i}{\pi} e^{-x_i \bar{x}_i} \right) \Delta(x) \Delta(x) = \prod_{j=1}^{N} j! \]  
(2.7)
Electrons subject to an intense magnetic field are nevertheless still interacting via Coulomb forces and also with the substrate as well as impurities. To this day it is still a difficult problem to understand the properties of such a system even at very low temperature. Nevertheless the remarkable stability of such quantum fluids exhibited in particular by the existence of the Hall plateaux led Laughlin to assume simple trial wave functions at fractional fillings. In the simplest cases of filling fractions of the form \( \nu = \frac{1}{2s+1}; \ s = 0, 1, \ldots \) (where the odd denominators are required by Fermi statistics), his guess takes the form

\[
\phi_s(x_0, \ldots, x_{N-1}) = \frac{1}{\pi^{N/2}} \Delta(x_0, \ldots, x_{N-1})^{2s+1} e^{-\sum_{i=0}^{N-1} \frac{x_i^2}{2}}, \tag{2.8}
\]

such that as \( x_i \to x_j \) the wave functions have zeroes of order \( 2s + 1 \). It is then natural to ask the following questions:

(a) compute the normalization

\[
Z_N(2s + 1) = \langle \phi_s | \phi_s \rangle = \int \prod_{i=0}^{N-1} \left( \frac{d^2 x_i}{\pi} e^{-x_i^2/\pi} \right) \left( \Delta(x) \Delta(x) \right)^{2s+1}, \tag{2.9}
\]

with \( Z_N(1) = \langle \phi_1 | \phi_1 \rangle \) given above.

(b) Find the expansion of Laughlin’s wave functions \( \phi_s \) on the complete basis of (free particle) Slater determinants

\[
\phi_s(x_0, \ldots, x_{N-1}) = \sum_{0 \leq l_0 < \ldots < l_{N-1} \leq (2s+1)(N-1)} g^{(s)}_{l_0, \ldots, l_{N-1}} \psi_{l_0, \ldots, l_{N-1}}(x_0, \ldots, x_{N-1}), \tag{2.10}
\]

or equivalently

\[
\Delta(x_0, \ldots, x_{N-1})^{2s+1} = \sum_{0 \leq l_0 < \ldots < l_{N-1} \leq (2s+1)(N-1)} g^{(s)}_{l_0, \ldots, l_{N-1}} |x^{l_0}_{l_0} \ldots x^{l_{N-1}}_{l_{N-1}}|, \tag{2.11}
\]

with the short hand notation \( |x^{l_0}_{l_0} \ldots x^{l_{N-1}}_{l_{N-1}}| \) for the determinant obtained by replacing \( x \) successively by \( x_0, x_1, \ldots \) in the first, second ... line.

This in turn amounts to

\[
\Delta(x_0, \ldots, x_{N-1})^{2s} = (-1)^{\frac{sN(N-1)}{2}} D(x_0, \ldots, x_{N-1})^s = \sum_{0 \leq l_0 < \ldots < l_{N-1} \leq (2s+1)(N-1)} g^{(s)}_{l_0, \ldots, l_{N-1}} ch_{l_0, \ldots, l_{N-1}}(x_0, \ldots, x_{N-1}) \tag{2.12}
\]
The notation \( ch \) stands for the polynomial characters of the general linear group \( GL_N \), expressed here for a diagonal matrix \( X = \text{diag}(x_0, \ldots, x_{N-1}) \) (we assume for the time being that none of the differences among \( x \)'s vanishes) as

\[
ch_{l_0, \ldots, l_{N-1}}(x_0, \ldots, x_{N-1}) = \frac{|x^{l_0}x^{l_1}\cdots x^{l_{N-1}}|}{|x^1\cdots x^{N-1}|}.
\] (2.13)

From L’Hospital’s rule this remains meaningful when any sub sets of \( x \)'s coincide. We call them general Schur functions – although they were introduced by Jacobi – and shall express them in terms of the (normalized) traces

\[
\theta_k = \frac{t_k}{k} = \frac{1}{k} \sum_{i=0}^{N-1} x_i^k \quad k > 0.
\] (2.14)

Parenthetically we will also have use for \( t_0 = N \). Defining the sequence

\[
0 \leq f_0 \leq f_1 \leq \cdots \leq f_{N-1} \leq 2s(N-1)
\] (2.15)

through

\[
l_i = f_i + i \quad i = 0, 1, \ldots, N-1,
\] (2.16)

one can record the increasing sequence \( l_i \) in a Young tableau with \( f_0 \) boxes in the \( N \)-th line, \( f_1 \) boxes in the \( N-1 \)-th line, \ldots, \( f_{N-1} \) boxes in the first line. We will sometimes switch from the notation \( \{l_i\} \) to \( Y \equiv \{f_i\} \). We observe that the expression (2.13) extends naturally as an antisymmetric quantity under permutations of \( l_0, \ldots, l_{N-1} \) and correspondingly for \( g^{(s)}_{l_0, \ldots, l_{N-1}} \).

The expansion (2.10) allows one to perform the integral in the partition function

\[
Z_N(2s+1) = N! \sum_{0 \leq l_0 < \cdots < l_{N-1} \leq (2s+1)(N-1)} |g^{(s)}_{l_0, \ldots, l_{N-1}}|^2 \prod_{i=0}^{N-1} l_i!.
\] (2.17)

This formula is useful only insofar we can obtain some mastery of the coefficients \( g^{(s)}_Y \).

The integral in (2.9) continues to make sense for arbitrary (positive) exponent \( p \) instead of the odd integer \( 2s+1 \), in which case

\[
Z_N(p) = \int \prod_{i=0}^{N-1} \left( \frac{d^2 x_i}{\pi} e^{-x_i \bar{x}_i} \right) \left( \Delta(x) \overline{\Delta(x)} \right)^p
\]

\[
= \int \prod_{i=0}^{N-1} \left( \frac{d^2 x_i}{\pi} e^{-\sum_{i=0}^{N-1} x_i \bar{x}_i + 2p \sum_{0 \leq i < j \leq N-1} \ln|x_i - x_j|} \right),
\] (2.18)
This suggests two alternative descriptions of our problem. In the first, consider the set of monic polynomials of degree \( N \) (with complex coefficients) identified with \( (\mathbb{C}^N)^{\text{symm}} \) and define a probability distribution on this space by considering the roots as independent Gaussian variables

\[
d\mu_N(x_0, \ldots, x_{N-1}) = \prod_{i=0}^{N-1} \left( \frac{d^2x_i}{\pi} e^{-x_i \bar{x}_i} \right)
\]  

(2.19)

This is indeed symmetric in any permutation of the \( x \)'s. The first form of (2.18) at least for \( p \) a positive integer involves the computation of the moments of the distribution of the module of the discriminant \(|D|\), thus for any positive \( p \)

\[
Z_N(p) = \int_0^\infty d\sigma_N(u) \ u^p
\]

\[
d\sigma_N(u) = \int d\mu_N(x_0, \ldots, x_{N-1}) \delta(|D(x)| - u).
\]

(2.20)

For instance, for \( p \) integral

\[
Z_2(p) = p! \ 2^p, \quad d\sigma_2(u) = \frac{1}{2} du e^{-u/2}
\]

\[
Z_3(p) = 2^{-p} (3p)! \ \sum_{0 \leq a \leq p} 3^{2a} \left( \frac{p}{2a} \right)^2 \left( \frac{3p}{2} \right)
\]

(2.21)

The second and more useful interpretation of the integral (2.18) is as a canonical partition function of a one component plasma (logarithmic repulsive two body potential), a classical two–dimensional Coulomb fluid, with neutralizing background. In this framework one can avail oneself of a considerable body of knowledge in the thermodynamic limit \( N \to \infty \) [3], and in particular of some evidence for a first order “fluid–solid” transition in the parameter \( \beta = 2p \) of the order of

\[
\beta_{\text{trans.}} = 2p_{\text{trans.}} \approx 142,
\]

(2.22)

according to Choquard and Clérouin [4]. It would be of great interest to be able to understand from first principles the mechanism of this transition to a classical Wigner solid.

Following Jancovici and Alastuey, the plasma interpretation justifies a large \( N \) behavior for \( \ln Z_N(p) \) obtained as follows. Consider \( N \) classical particles of charge \( e \) in a disk of radius \( R \) interacting via a repulsive two–dimensional Coulomb potential

\[
v_{ij}(x_i - x_j) = \frac{e^2}{2\pi} \ln \frac{1}{|x_i - x_j|},
\]

(2.23)
choosing an arbitrary length scale inside the logarithm (which should anyhow disappear from the final result). The contribution of the charges to the total energy is
\[ E_{\text{ch.}} = -\frac{e^2}{4\pi} \ln |\Delta(x_0, ..., x_{N-1})|^2. \] (2.23)

We assume a uniform neutralizing background with density \( \frac{N}{\pi R^2} \). Its self–interaction is
\[ E_{\text{back.}} = \left( \frac{N}{\pi R^2} \right)^2 \frac{e^2}{2} \int_{|x_1|,|x_2|<R} d^2x_1 \, d^2x_2 \, \frac{1}{2\pi} \ln \frac{1}{|x_1 - x_2|}. \] (2.24)

Set
\[ \phi(x) = \frac{1}{2\pi} \int_{|y|<R} d^2y \, \ln \frac{1}{|y - x|} = \frac{R^2 - x^2}{4} + \frac{R^2}{2} \ln \frac{1}{R} \quad |x| \leq R. \] (2.25)

This expression follows from the fact that \(-\Delta \phi(x) = 1\) for \(|x| \leq R\), while for \(|x| = R\), \(\phi\) is equal to the potential created by the total charge \(\pi R^2\) located at the center, i.e. \(\frac{\pi R^2}{2\pi} \ln \frac{1}{R}\).

After a short calculation
\[ E_{\text{back.}} = \frac{N^2 e^2}{4\pi R^2} \left( \frac{1}{8} + \frac{1}{2} \ln \frac{1}{R} \right). \] (2.26)

The last contribution to the energy (ignoring kinetic energy which factorizes from the partition function) is the attractive interaction between charges and background
\[ E_{\text{int.}} = -\frac{e^2 N}{\pi R^2} \sum_i \phi(x_i) = \frac{e^2 N}{4\pi R^2} \sum_i x_i x_i - \frac{e^2 N^2}{\pi} \left( \frac{1}{4} + \frac{1}{2} \ln \frac{1}{R} \right). \] (2.27)

Thus the total energy divided by \(k_B T\) (\(T\) the temperature) is
\[ \frac{E}{k_B T} = (E_{\text{ch.}} + E_{\text{back.}} + E_{\text{int.}})/k_B T \]
\[ = -\frac{e^2}{4\pi k_B T} \left( \frac{N}{R^2} \sum_i x_i x_i - \ln |\Delta(x_0, ..., x_{N-1})|^2 + N^2 \ln R - \frac{3}{4} N^2 \right). \] (2.28)

We set
\[ \frac{e^2}{4\pi k_B T} = \frac{\beta}{2} = p, \quad R^2 = pN \] (2.29)
in such a way that,
\[ \frac{E}{k_B T} = \sum_i x_i x_i - 2p \ln |\Delta(x_0, ..., x_{N-1})| + \frac{p N^2}{2} \ln (pN) - \frac{3p N^2}{4}. \] (2.30)
The excess free energy $F_{\text{exc}}$ of this system, as compared to the perfect gas (i.e. deriving only from the kinetic energy), is given by

$$-\frac{F_{\text{exc}}}{k_B T} = \ln \int_{|x_i| \leq R} \prod_{i=0}^{N-1} \frac{d^2 x_i}{\pi R^2} e^{-\frac{E}{k_B T}}$$

$$= -\frac{pN^2}{2} \ln(pN) + \frac{3}{4} pN^2 - N\ln(pN) + \ln Z_N(p). \quad (2.31)$$

According to thermodynamics, in the limit of large $N$ (i.e. large $R$) we have extended the integrals to the full plane up to errors of order at most $\sqrt{N}$ in $F_{\text{exc}}$. (i.e. arising from boundary terms). Since we expect an extensive excess free energy, we obtain the estimate

$$\ln Z_N(p) = \frac{pN^2}{2} \ln(pN) - \frac{3}{4} pN^2 + N\ln(pN) + O(N). \quad (2.32)$$

It is the term of order $N$, namely $-\frac{F_{\text{exc}}}{k_B T}$, depending on $p$, which is shown to undergo a first order phase transition, namely the limit

$$\lim_{N \to \infty} \frac{1}{N} (\ln Z_N(p) - \frac{pN^2}{2} \ln(pN) + \frac{3}{4} pN^2 - N\ln(pN)) \quad (2.33)$$

should exist and have a discontinuous derivative for a value of $p$ close to 71. In principle eqn. (2.32) answers question (a) up to the unknown term of order $N$. But its status is at best heuristic. It is therefore reassuring that at least for $p = 1$ ($s = 0$) where we have an exact result, we can justify (2.32). Note that in the thermodynamic reasoning the value of $p$ did not play a crucial role. We devote appendix A to the asymptotic evaluation of $\ln Z_N(1)$ using Euler–Mac Laurin’s formula. It reads

$$\ln Z_N(1) = \ln \prod_{j=1}^{N} j!$$

$$= \frac{N^2}{2} \ln N - \frac{3}{4} N^2 + N\ln N + N \left( \frac{\ln 2\pi}{2} - 1 \right) + \frac{5}{12} \ln N$$

$$+ \frac{1 - \gamma + 5\ln 2\pi}{12} + \frac{\zeta'(2)}{2\pi^2} + \frac{1}{12N} - \frac{1}{720N^2} - \frac{1}{360N^3} + O\left( \frac{1}{N^4} \right), \quad (2.34)$$

We add a few remarks.

(i) The thermodynamic ansatz suggests that

$$\ln Z_N(p) - \frac{1}{p} \ln Z_{NP}(1) = O(N), \quad (2.35)$$
while (trivially) Hölder’s inequality yields

\[ p \ln Z_N(1) \leq \ln Z_N(p). \]  

(2.36)

(ii) One can also obtain a crude upper–bound as follows. From the symmetry of the integrand in \( Z_N(p) \) we have

\[ Z_N(p) = N! \int_{|x_0| \geq |x_1| \geq \ldots \geq |x_{N-1}|} \prod_{i=0}^{N-1} \left( \frac{d^2 x_i}{\pi} e^{-x_i x_i} \right) |\Delta(x_0, \ldots, x_{N-1})|^{2p}. \]  

(2.37)

In this domain

\[ |\Delta(x_0, \ldots, x_{N-1})|^{2p} \leq 2^{p N (N-1)} \prod_{i=0}^{N-1} |x_i|^{2p(N-i-1)}. \]

The integral over the sector is smaller than the full integral, hence

\[ Z_N(p) \leq N! 2^{p N (N-1)} \prod_{k=1}^{N-1} (pk)! , \]  

(2.38)

where the factor \( N! 2^{p N (N-1)} \) is probably a gross overestimate. For further discussion of such bounds see the next section.

3. Coulomb Gases.

3.1. Quadrupolar mean field.

In the interpretation of the norm (2.9) as a one component plasma canonical partition function, the gaussian measure in (2.19) is due to the mean potential created by the neutralizing background charge. This potential is not only determined by the locally constant density, but also by the shape of the bounding domain which, in the case studied so far, is a disk and therefore induces a rotational symmetry around the center, the minimum of the harmonic potential. Far from this minimum the global form of the potential has no consequence on the local correlations at a given density, neither on the residual thermodynamics. This suggests to add to the quadratic \( z \bar{z} \) term an harmonic function (equivalently the real part of an entire function) which does not modify the constant charge density. To ensure convergence of the partition function we are limited to the real part of a quadratic
polynomial, hence up to a translation, to a multiple of a quadrupolar mean field \( V = Rez^2 \), created by far away charges.

It is rather remarkable that the plasma model with a quadrupolar component of the mean field can also be completely solved for \( \beta = 2 \) \((s = 0)\). We give here a short description of the method and of the corresponding results. We want to evaluate the partition function depending on the intensity of the quadrupolar field that will be called \( th\mu \) \((\mu \text{ real})\) so that the total potential

\[
V(z, \bar{z}) = z\bar{z} - \frac{1}{2} th\mu(z^2 + \bar{z}^2) \tag{3.1}
\]

confines the charges. To compute

\[
Z_N(1; \mu) = \int \prod_j \frac{d^2z_j}{\pi} e^{-\Sigma z_j\bar{z}_j + \frac{1}{2} th\mu(z_j^2 + \bar{z}_j^2)} |\Delta|^2 \tag{3.2}
\]

we just need to be able to construct the polynomials \( P_n(z) \) \((z \in \mathbb{C})\) with real coefficients verifying the unitary orthogonal relations

\[
\int \int \frac{d^2z}{\pi} w(z, \bar{z}) P_n(z) P_m(\bar{z}) = \delta_{nm}
\]

\[
w(z, \bar{z}) = e^{-z\bar{z} + \frac{1}{2} th\mu(z^2 + \bar{z}^2)}
\]

\[
deg(P_n) = n = (0, 1, 2, ...)
\]

\[
\lim_{\mu \to 0} P_n(z) = (n!)^{-1/2} z^n
\]

We start from the identity

\[
\int \frac{d^2z}{\pi} e^{-Z^\dagger AZ/2} = \frac{2}{\sqrt{(a + d)^2 - 4cc}} \tag{3.4}
\]

where \( Z, Z^\dagger \) are the column and row vectors \((\bar{z}, z)\) and \( A = \begin{pmatrix} a & c \\ \bar{c} & d \end{pmatrix} \) is a hermitian matrix such that \((a + d)^2 \geq 4cc\). Set

\[
z\bar{z} - \frac{1}{2} th\mu(z^2 + \bar{z}^2) = \frac{1}{2} Z^\dagger AZ, \tag{3.5}
\]

where

\[
A = A^\dagger = \begin{pmatrix} 1 & -th\mu \\ -th\mu & 1 \end{pmatrix}. \tag{3.6}
\]
Now introduce the vector \( X = (x) \) and write the following identity

\[
1 = \frac{\int d^2z \exp -\frac{1}{2}(Z - A^{-1}X)^\dagger A(Z - A^{-1}X)}{\int d^2z \exp -\frac{1}{2}Z^\dagger AZ}
\]

\[
= \langle \exp \left( \frac{1}{2}X^\dagger Z + Z^\dagger X \right) - \frac{1}{2}X^\dagger A^{-1}X \rangle >
\]

\[
= \langle \exp(\bar{x} z + x\bar{z} - \frac{1}{1 - (th\mu)^2}(x\bar{x} + \frac{1}{2}th\mu(x^2 + \bar{x}^2))) \rangle >
\]

Rescaling \( x = \sqrt{\frac{1-(th\mu)^2}{th\mu}} u = \sqrt{\frac{2}{sh2\mu}} u \) and moving the term \( \exp(-\frac{x\bar{x}}{1-(th\mu)^2}) \) in (3.7) on the left hand side, we get

\[
\exp \frac{\bar{u} u}{th\mu} = \langle \exp \left[ \bar{u}z \sqrt{\frac{2}{sh2\mu}} - \frac{\bar{u}^2}{2} \right] \exp \left[ u\bar{z} \sqrt{\frac{2}{sh2\mu}} - \frac{u^2}{2} \right] \rangle >
\]

The r.h.s. of this expression involves generating functions for monic Hermite polynomials

\[
\exp(xu - \frac{u^2}{2}) = \sum_{0}^{\infty} H_n(x) \frac{u^n}{n!}.
\]

Expanding both sides we have

\[
\sum_{n=0}^{\infty} \left( \frac{u\bar{u}}{th\mu} \right)^n \frac{1}{n!} = \sum_{k,l=0}^{\infty} \frac{\bar{u}^k u^l}{l!k!} < H_k(\bar{z}) \sqrt{\frac{2}{sh2\mu}} H_l(z) \sqrt{\frac{2}{sh2\mu}} >
\]

and since in the average \( < ... > \) the integral of the denominator is \( ch\mu \), we get

\[
\int \frac{d^2z}{\pi} e^{-\frac{1}{2}zh(\bar{z} + z^2)} H_k(\bar{z}) \sqrt{\frac{2}{sh2\mu}} H_l(z) \sqrt{\frac{2}{sh2\mu}} = ch\mu \frac{l!}{(th\mu)^l} \delta_{k,l}
\]

The polynomials \( P_n \) are therefore given by

\[
P_n(z) = \frac{(th\mu)^n/2 H_n(\sqrt{\bar{z}})}{\sqrt{n!(ch\mu)^{1/2}}} = (ch\mu)^{-(n+1/2)} z^n \sqrt{n!} + ...
\]

The first few \( P \)'s read

\[
P_0(z) = (ch\mu)^{-1/2}
\]

\[
P_1(z) = (ch\mu)^{-3/2} z
\]

\[
P_2(z) = \frac{(ch\mu)^{-5/2}}{\sqrt{2!}} (z^2 - sh\mu ch\mu)
\]

\[
P_3(z) = \frac{(ch\mu)^{-7/2}}{\sqrt{3!}} (z^3 - 3sh\mu ch\mu z)
\]
Substituting in the Vandermonde determinant \( \sqrt{n!}(ch\mu)^{(n+1/2)}P_n(z) \) for \( z^n \) we readily evaluate the partition function as

\[
\frac{Z_N(1; \mu)}{Z_N(1; 0)} = (ch\mu)^{N^2}
\]  

(3.14)

the superimposed field is contributing only to the dominant term of the free energy by \( N^2 \log ch\mu \), this term being non extensive. After subtraction, the residual energy is therefore independent of \( \mu \), that is of the mean field at large distance as was to be expected.

It is also interesting to compute the local density of the Coulomb system of \( N \) charges in that neutralizing background. We have

\[
\rho_N(z, \bar{z}) = w(z, \bar{z}) \sum_{n=0}^{N-1} P_n(z)P_n(\bar{z})
\]

\[
\equiv e^{-z\bar{z} + \frac{1}{2}th\mu(z^2 + \bar{z}^2)} \sum_{n=0}^{N-1} \frac{(th\mu)^n}{ch\mu n!} |H_n(\frac{\sqrt{2z}}{\sqrt{sh2\mu}})|^2
\]

(3.15)

which for \( \mu = 0 \) reduces to

\[
\rho_N(z, \bar{z})|_{\mu=0} = e^{-z\bar{z}} \sum_{n=0}^{N-1} \frac{(z\bar{z})^n}{n!}.
\]  

(3.16)

The asymptotic behaviour in \( N \) for \( |z|^2 \) of order \( N \) can easily be found outside a transition region, for which a finer analysis is required. Since for fixed \( z \)

\[
\lim_{N \to \infty} \rho_N(z, \bar{z}) \equiv 1 \ \forall z,
\]  

(3.17)

there exists a curve \( C_N(\mu) \), being a continuous deformation of the circle \( |z|^2 = N \), such that the asymptotic density \( \rho_N \) is equal to 1 in the interior of the domain bounded by \( C_N \) and zero outside. To evaluate \( C_N \) asymptotically, we write that the variation of density

\[
\rho_{N+1}(z, \bar{z}) - \rho_N(z, \bar{z})
\]

(3.18)

is maximal for \( z \in C_N \).

Using the large \( N \) asymptotic behaviour of the Hermite polynomial \( H_N \) when \( \frac{z\bar{z}}{N} \) is finite

\[
\frac{1}{N}ln \left| \frac{H_N((e^\phi + e^{-\phi})\sqrt{N})}{N!} \right|^2 \simeq Re(2\phi + e^{-2\phi} + 1),
\]  

(3.19)
with the definition of $\phi$

$$
ch\phi = \frac{z}{\sqrt{2N\text{sh}2\mu}}, \quad (Re\phi > 0), \quad (3.20)
$$

the variation of density (3.18) is written, in the "exponential" approximation:

$$
\exp\left[-z\bar{z} + \frac{1}{2}th\mu(z^2 + \bar{z}^2) + N(lnth\mu + 1 + Re(2\phi + e^{-2\phi}))\right] \quad (3.21)
$$

The maximum over $z$ is given by the vanishing of the derivatives with respect to $z, \bar{z}$, i.e.

$$
-z + th\mu z + N\frac{\partial\phi}{\partial z}(1 - e^{-2\phi}) = 0 \quad (3.22)
$$

and its complex conjugate. According to (3.20),

$$
\text{sh}\phi\frac{\partial\phi}{\partial z} = \frac{1}{\sqrt{2N\text{sh}2\mu}}, \quad (3.23)
$$

so that (3.22) amounts to

$$
th\mu = |e^{-2\phi}|. \quad (3.24)
$$

Putting $\phi = \xi + i\eta$, the equation of the curve is

$$
z = \sqrt{N}(e^{\mu} \cos \eta + ie^{-\mu} \sin \eta) \quad (3.25)
$$

which is an ellipse with semi-axis $a = \sqrt{N}e^{\mu}$, $b = \sqrt{N}e^{-\mu}$. The area of this ellipse is $\pi N$ independently of $\mu$ the intensity of the quadrupolar field. The effect of an added quadrupolar field is therefore to deform the Fermi liquid disc into an ellipse with the same area.

### 3.2. $N$ charges on the sphere.

The thermodynamic equivalence between the harmonic plasma and the system of charges on a sphere has been proven for $\beta = 2$ and is presumably valid for all temperatures. The spherical geometry avoids giving a special status to the center of the disk thereby increasing the symmetry and allows a clear definition of the pressure since the potential function does not depend explicitly on the density.

The Coulomb potential created at point $\hat{\mu}(\theta, \phi)$ by a unit charge located at the north pole ($\theta = 0$) of the unit sphere, is defined, up to a constant, by the Green function of the spherical Laplacian. The source term must be supplemented by a neutralizing background which can be taken to be uniformly distributed (to respect the spherical symmetry) and
the total charge of the source must vanish on the sphere, a surface without boundary. The potential satisfies therefore the inhomogeneous equation

\[ \nabla \hat{u} V = -2\pi \delta(\hat{u}) + \frac{1}{2} \]

(3.26)

with

\[ \int \delta(\hat{u}) d\sigma = 1 \]

(3.27)

\[ d\sigma = d(cos\theta) \wedge d\phi, \quad \int d\sigma = 4\pi \]

(3.28)

and the constant \( \frac{1}{2} \) represents the neutralizing density. Putting \( \eta = \cos\theta \), we have

\[ \frac{\partial}{\partial \eta} (1 - \eta^2) \frac{\partial V}{\partial \eta} = \frac{1}{2} - 2\delta(1 - \eta) \]

(3.29)

Up to a constant the solution is,

\[ V = -\frac{1}{2} \ln \frac{1 - \eta}{2} = -\ln \left( \frac{\sin \theta}{2} \right) = \ln \left( \frac{r}{\sqrt{e} \sin \frac{\theta}{2}} \right) \]

(3.30)

where \( r \) is the euclidian distance between the charge and the point on the sphere. We fix an additive constant requiring a zero average value of \( V(\hat{u}) \), giving the potential

\[ V_0 = \ln \left( \frac{2}{r} \right) - \frac{1}{2} = -\ln \left( \sqrt{e} \sin \frac{\theta}{2} \right) \]

(3.31)

which satisfies \( \int V_0 d\sigma = 0 \). The correspondence between the spherical system and the original planar one is obtained by a stereographic projection from the south pole on the tangent plane to the north pole. The complex coordinate in this plane is \( 2z \), with

\[ z = tg\frac{\theta}{2} e^{i\phi} \]

(3.32)

\[ \frac{d\sigma}{4\pi} = \frac{d\bar{z} \wedge dz}{2\pi i} \frac{1}{(1 + |z|^2)^2} = \frac{d^2z}{\pi} \frac{1}{(1 + |z|^2)^2} \]

(3.33)

while the angular distance \( \gamma_{12} \) between two points \( z_1 \) and \( z_2 \) is given by

\[ \sin^2 \frac{\gamma_{12}}{2} = \frac{|z_1 - z_2|^2}{(1 + |z_1|^2)(1 + |z_2|^2)} \]

(3.34)
On a sphere with unit area, the partition function at inverse temperature $\beta = 2p$ and with two-body potential $V(\gamma_{12})$ is defined by the average

$$I_N(p) = \left\langle \prod_{i<j} \sin^{2p} \frac{\gamma_{ij}}{2} \right\rangle$$

$$< A > = \int \prod_{i=1}^{N} \frac{d\sigma_i}{4\pi} A$$

For a sphere of area $S$, the partition function $I_N$ is to be multiplied by $S^N$. We can also write

$$I_N(p) = \int \prod_{j=1}^{N} \frac{d^2 z_j}{\pi} \frac{1}{(1 + z_j \bar{z}_j)^{N-1}p+2} \prod_{i<j} |z_i - z_j|^{2p}. \quad (3.36)$$

With the choice of potential $V_0(\gamma_{12})$ defined in (3.31), the canonical partition function is

$$Q_N(p) = e^{\frac{p}{2} N(N-1)} I_N(p). \quad (3.37)$$

We note the exact results

$$I_N(1) = \frac{1}{(N!)^{N-1}} \left( \prod_{p=0}^{N-1} p! \right)^2 \quad (3.38)$$

and

$$I_3(p) = \frac{(p!)^3(3p + 1)!}{((2p + 1)!)^3} = \left( \frac{\Gamma(p + 1)}{\Gamma(2p + 2)} \right)^3 \Gamma(3p + 2), \quad \forall p, \quad Re p > -\frac{2}{3}. \quad (3.39)$$

The partition functions on the plane $Z_N(p)$ eqn.(2.9), and on the sphere $I_N(p)$ eqn.(3.36), differ only by the weight functions. They become equivalent in the central region $|z| = O(\frac{1}{\sqrt{Np}})$.

In appendix A, we find from (3.37) and (3.38) the asymptotic free energy per unit charge $F_N(1)$ ($\beta = 2$)

$$-2F_N(1) = \frac{\text{ln}Q_N(1)}{N} = \frac{1}{2} \text{ln}2\pi N - \frac{3}{2} + O\left(\frac{\text{ln}N}{N}\right). \quad (3.40)$$

In contrast with $\text{ln}Z_N$ for the neutral plasma, we note the absence of terms in $N^2 \text{ln}N$ and $N^2$ in $\text{ln}Q_N$ due to the choice of potential $V_0$ having a spherical average zero. If we want to compare the two systems, planar and spherical, we have to renormalize the scale of the potential, which is equivalent to introduce an appropriate continuous background following the convention of Caillol and al. [3] (formula 3.4 in their paper).
The residual free energy is now given by the following form
\[-2f(1) = \lim_{N \to \infty} \frac{1}{N} \ln \left( \frac{Q_N(1)}{\sqrt{N!}} \right)\]
= \lim \left( \frac{1}{2} \ln 2\pi N - \frac{3}{2} - \frac{1}{2}(\ln N - 1) \right)
= \left( \frac{1}{2} \log 2\pi - 1 \right) = \approx -0.0810614...
(3.41)
which agrees with the residual free energy of the neutral plasma i.e. the term proportional
to \(N\) in (2.34).

Remark on the canonical pressure
The following section is devoted to obtaining bounds for \(f(p)\) when \(p \geq 1\), to confirm
the dependence in \(N\) observed for \(p = 1\). On the sphere, it is the quantity
\[
\left( e^{\frac{N-1}{2}} \frac{1}{\sqrt{N!}} |\Delta_N|^2 \right)^p, \quad |\Delta_N|^2 = \prod_{i<j} \sin^2 \frac{\gamma_{ij}}{2}
(3.42)
\]
that produces the correct extensive property.

Assuming for the time being that for any \(p\) the same \(\ln N\) term as in formula (3.40),
this gives the free energy per particle for \(N \geq 1\) on a sphere of unit area, \(S = 1\). For \(S\)
arbitrary we must add \(N\ln S\) to \(\ln Q_N\). Although the energy of the system of \(N\) charges
on the sphere is not extensive, a global pressure denoted by \(\tilde{p}\) not to be confused with the
notation \(p\), is defined by \(\beta \tilde{p} = \frac{\partial (\ln Q_N + N\ln S)}{\partial S} = \frac{N}{S} = \text{density} = \rho\). The non-trivial term in
(3.40) is not contributing to the global pressure. On the other hand if we interpret the free
energy \(F_N(p)\) as that of an extensive system of density \(\rho\), eq. (3.40) gives a supplementary
contribution to the pressure
\[
\rho^2 \frac{\partial}{\partial \rho} \left( -\frac{1}{4} \log \rho \right) = -\frac{1}{4} \rho.
(3.43)
\]
The resulting equation of state \(\beta \tilde{p} = \rho(1 - \frac{\beta}{4})\) is that of a neutral plasma, a system that
probably cannot remain homogeneous for \(\beta \geq 4\).
3.3. Hölder’s and Hadamard’s inequalities.

As we noted in (2.36), for the plasma on a plane Hölder’s inequality requires $F_N(\beta)$ to be a non-increasing function of $\beta$, that is the "potential entropy" $S(\beta) = \beta^2 \frac{\partial F}{\partial \beta}$ is negative.

This inequality reads

$$<a^s> \geq (<a>)^s, \, \forall s \geq 1$$  \hspace{1cm} (3.44)

where

$$<a> = \frac{1}{N} \sum_{j=1}^{N} a_j, \, a_j \geq 0$$  \hspace{1cm} (3.45)

is an average value, (possibly a weighted average), of non-negative quantities. It follows that $<a^\beta>^{1/\beta}$ is a non-decreasing function of $\beta$ for $\beta \geq 0$. We simply need to apply these relations to $a = e^{-\beta_0 V}$, $s = \frac{\beta}{\beta_0} \geq 1$

$$Z_N(\beta) = <e^{-\beta V_N} > \Rightarrow F_N(\beta) = -\frac{1}{\beta N} lnZ_N(\beta) \text{ non–increasing}$$  \hspace{1cm} (3.46)

From this we derive the inequalities

$$F_N(\beta) \leq F_N(2), \, \beta \geq 2$$

$$F_N(\beta) \geq F_N(2), \, 0 \leq \beta \leq 2.$$  \hspace{1cm} (3.47)

Similarly on the sphere, if the residual free energy is defined for all $p$ by the limit

$$-2f(p) = \lim_{N \to \infty} \frac{1}{N} \frac{lnQ_N(p)}{(N!)^{p/2}}$$  \hspace{1cm} (3.48)

we obtain

$$f(p) \leq f(1), \, p \geq 1$$

$$f(p) \geq f(1), \, 0 \leq p \leq 1.$$  \hspace{1cm} (3.49)

Now let us look at Hadamard’s inequality to obtain a lower limit to the free energy.

We start from the identity

$$|\Delta^2| = \det_{i,j} \left| \sum_{n=0}^{N-1} z_j^n z_i^n \right| = \frac{\det_{i,j} \left| \sum_{n=0}^{N-1} c_n z_j^n z_i^n \right|}{\prod_{n=0}^{N-1} c_n}$$  \hspace{1cm} (3.50)

where $c_0, c_1, ..., c_{N-1}$ are $N$ positive indeterminates. The required inequality reads

$$|\Delta^2| \leq \frac{a_1 \cdots a_{N-1} a_N}{c_0 c_1 \cdots c_{N-1}},$$  \hspace{1cm} (3.51)
where the $a_i$‘s are the diagonal elements of the positive matrix, the determinant of which appears in the numerator of the r.h.s. of (3.50)

$$a_i = \sum_{n=0}^{N-1} c_n |z_i|^{2n}, \quad i = 1, 2, \ldots, N$$  \hspace{1cm} (3.52)

Define the polynomial of degree $N-1$

$$a(\rho) = \sum_{n=0}^{N-1} c_n \rho^n$$  \hspace{1cm} (3.53)

Substituting in the expression (3.36) for the integral $I_N(p)$ the inequality (3.51) and writing $|z_j|^2 = \rho_j$, we get

$$I_N(p) \leq \inf_{[c]} \left[ \prod_{j=1}^{N} \frac{a(\rho_j)}{(1+\rho_j)^{(N-1)p+1}} \frac{d\rho_j}{\prod_{n=0}^{N-1} c_n} \right]$$  \hspace{1cm} (3.54)

that is

$$I_N(p) \leq \inf_{[c]} \frac{(A_p(c))^N}{(\prod_n c_n)^p}$$  \hspace{1cm} (3.55)

with the convergent integral $A_p(c)$ given by

$$A_p(c) = \int_0^\infty \frac{a^p(\rho)d\rho}{(1+\rho)^{(N-1)p+2}}$$  \hspace{1cm} (3.56)

This integral is a homogeneous polynomial of total degree $p$ in the $N$ parameters $[c]$. Therefore, for every $p \geq 0$, we have

$$\frac{\ln I_N(p)}{N} \leq \inf_{[c]} \left[ \ln A_p(c) - \frac{p}{N} \sum_n \ln c_n \right]$$  \hspace{1cm} (3.57)

The r.h.s. of (3.55) being homogeneous in $[c]$ of degree zero, we have to seek the minimum according to the direction of the vector $c_0, \ldots, c_{N-1}$. In the first ”quadrant” of $R_N$ ($c_n > 0$), this continuous positive function tends to infinity on all coordinates planes. It therefore has an absolute minimum in an interior direction, and at that point it verifies

$$\frac{1}{A} \frac{\partial A}{\partial c_n} - \frac{p}{N} \frac{1}{c_n} = 0 \quad ; \quad n = 0, 1, \ldots, N - 1.$$  \hspace{1cm} (3.58)

That is

$$\int_0^\infty \frac{a^{p-1}(\rho)c_n \rho^p d\rho}{(1+\rho)^{(N-1)p+2}} = \frac{A}{N}, \quad \forall n.$$  \hspace{1cm} (3.59)
From homogeneity, only \( N - 1 \) equations are independent since the sum is given by Euler’s identity. When \( p = 1 \), the solution is obvious since (3.59) gives

\[
c_n \int_0^\infty \frac{\rho^n d\rho}{(1 + \rho)^{N+1}} = \frac{A}{N}
\]

i.e.

\[
c_n = A \binom{N - 1}{n}
\]

where the constant \( A \) is left undetermined. This yields the polynomial

\[
a(\rho) \equiv A(1 + \rho)^{N-1}
\]

and we get the following upper bound

\[
\frac{\ln I_N(1)}{N} \leq \ln A - \frac{1}{N} \sum_n \ln(A \binom{N - 1}{n})
\]

\[
\leq -\frac{1}{N} \sum_{n=0}^{N-1} \ln \binom{N - 1}{n}
\]

or

\[
I_N(1) \leq \left( \frac{\prod_{n=0}^{N-1} n!}{(N - 1)!^N} \right)^2.
\]

Let us denote by \( B_N(p) \) the Hadamard upper bound (3.49) (given explicitly by the r.h.s. of (3.64) for \( p = 1 \)). Comparing with the exact result (3.38)

\[
\frac{B_N(1)}{I_N(1)} = \frac{(N!)^{N-1}}{((N - 1)!)^N} = \frac{N^N}{N!} \approx \frac{e^N}{\sqrt{2\pi N}}
\]

from which we get

\[
\frac{\ln B_N}{N} = \frac{\ln I_N}{N} + 1 + O(\frac{\ln N}{N})
\]

This shows that the upper bound of \(-2F_N(1)\) given by Hadamard’s inequality and the exact value are only differing by 1, meaning from (3.41)

\[
f(1) = -\frac{1}{4} \ln 2\pi + \frac{1}{2} \geq -\frac{1}{4} \ln 2\pi.
\]

Let us turn to the general case. The polynomial \( a(\rho) \) given in (3.62) for \( p = 1 \) is still a solution for arbitrary \( p \) up to a multiplicative constant. More precisely the ansatz

\[
c_n = A^{1/p} \binom{N - 1}{n}
\]
verifies the stationarity equations (3.58). Indeed, after dividing by A,

\[
\left( \frac{N-1}{n} \right) \int_0^\infty \frac{(1+\rho)^{(N-1)(p-1)}\rho^n}{(1+\rho)^{(N-1)p+2}} d\rho = \frac{1}{N} \tag{3.69}
\]

Since the dependence on \( p \) drops out, this is an identity in view of our previous discussion. The only doubt that remains is whether this stationary point is the absolute minimum. This is certainly the case in the neighbourhood of \( p = 1 \) by continuity if we note that the matrix \( H_{n,m} = c_n c_m \frac{\partial}{\partial c_n} \frac{\partial}{\partial c_m} A_p(c) \) is positive definite and that the function \( A_p^N(c) \) is convex in \( R^+_N \) hence admits only one minimum on the plane \( \sum_n ln c_n = 0 \).

The inequality (3.57) can now be written

\[
\frac{ln I_N(p)}{N} \leq ln A - \frac{p}{N} \sum_n ln (A^{1/p} \left( \begin{array}{c} N-1 \\ n \end{array} \right)) \\
\leq - \frac{p}{N} \sum_{n=0}^{N-1} ln \left( \begin{array}{c} N-1 \\ n \end{array} \right) \tag{3.70}
\]

The upper bound for \( \frac{ln I_N(p)}{N_p} \) is according to (3.63) independent of \( p \) and the same property holds for

\[
\frac{ln \left( \frac{Q_N(p)}{(N!)^{p/2}} \right)}{N_p} = \frac{ln I_N(p)}{N_p} + \frac{N}{2} - \frac{1}{2} ln N + ...
\]

Consequently the residual free energy is bounded by

\[ f(1) - \frac{1}{2} \leq f(p) \leq f(1), \quad p \geq 1. \tag{3.72} \]

where \( f(1) = -\frac{1}{4} ln 2\pi + \frac{1}{2} \). Numerically we get

\[-0.4594692... \leq f(p) \leq 0.0405307 \tag{3.73} \]

The lower bound is of no interest for \( 0 < p < 1 \) since \( f(p) > f(1) \) in the high temperature region (i.e. \( p < 1 \)).

3.4. The zero temperature limit for finite \( N \).

Before going to the thermodynamic limit, i.e. keeping \( N \) finite, the free energy \(-\frac{1}{\beta} ln I_N(\beta)\) of the system with \( N \) unit charges on the sphere has as a limiting value, when \( \beta \to \infty \) (or equivalently \( p \to \infty \)), the absolute minimum of the potential function

\[
\mathcal{V}_N = - \sum_{i<j} ln(\sin^2 \frac{\gamma_{ij}}{2}) \tag{3.74}
\]
or
\[ 2\mathcal{V}_N = -\ln|\Delta_N|^2 + (N - 1) \sum_{j=1}^{N} \ln(1 + z_j \bar{z}_j) \]  
(3.75)

With the definition (3.48) of the residual free energy, we have for the finite system
\[ \lim_{p \to \infty} 2f(p) = \frac{1}{N} \inf(2\mathcal{V}_N) - \frac{N - 1}{2} + \frac{1}{2N} \ln N!. \]  
(3.76)

This result remains valid in the thermodynamic limit provided the residual entropy vanishes. Up to an overall rotation a stable equilibrium configuration does exist on the sphere since \( \mathcal{V}_N \) reaches its minimum. The equilibrium configurations are solutions of the \( N \) equations \( \frac{\partial \mathcal{V}}{\partial z_j} = 0 \) (and c.c.)
\[ \sum_{j=1, j \neq i}^{N} \frac{1}{z_i - z_j} - (N - 1) \frac{\bar{z}_i}{1 + z_i \bar{z}_i} = 0 \]  
(3.77)

having in the planar limit the simplified form
\[ \sum_{j=1, j \neq i}^{N} \frac{1}{z_i - z_j} - (N - 1) \bar{z}_i = 0 \]  
(3.78)

From (3.77) we get the sum rule
\[ \sum_{j=1}^{N} \frac{z_j \bar{z}_j}{1 + z_j \bar{z}_j} = \frac{N}{2} \]  
(3.79)

which reads in polar coordinates
\[ \sum_{j} \cos \theta_j = 0 \]  
(3.80)

This result is true irrespective of the location of the pole, therefore we have \( \sum_j \hat{u}_j = 0 \) (\( \hat{u}_j \) being the unit vector with polar angle \( \theta_j \)), which means that the center of gravity of charge is the center of the sphere as one would naturally expect. In the thermodynamic limit the relation (3.79) allows to show that the radius of the confinement disk of the plasma is 1 (with gaussian weight \( e^{-Npz\bar{z}} \)).

The equations (3.77) are formally invariant under rotation \( z_j \to \xi_j = \frac{\alpha z_j + \gamma}{\gamma z_j + \alpha}, \quad \alpha \bar{\alpha} + \gamma \bar{\gamma} = 1 \). It is easy to find solutions for configurations invariant under a finite rotation group. As an example let us consider the case \( N = 12 \), where 2 configurations come to mind:
-12_{(60)}: The 12 charges are at the vertices of an icosahedron. The order of the symmetry group is 60.

-12_{(24)}: The 12 charges are at the vertices of a cuboctahedron (center of the edges of a cube). The order of the symmetry group is 24.

From symmetry we see that the electric field created by the 11 charges on the twelfth located at the pole is zero: \( \sum_j \frac{1}{\xi_j} = 0 \) and the equations (3.77) are all satisfied. Turning to the question of stability one finds that the configuration with highest symmetry group has lowest energy. In appendix B we show that the configuration of the cuboctahedron is unstable. By a similar method we can show that the icosahedron is stable, that is the Hessian is semi-positive. The Hessian cannot be strictly positive since it admits three zero eigenvalues corresponding to the three generators of global rotations. The equation for the vibration modes \( \lambda, (\lambda = \omega^2) \) is written as

\[
\sum_{j \neq i} \frac{\delta_i - \delta_j}{(z_i - z_j)^2} + (N - 1)(\bar{\delta}_i(1 - \lambda) - z_i^2 \delta_i) = 0
\]

(3.81)

for the \( 2N \) eigenvalues \( (\delta_i, \bar{\delta}_i) \). We exhibit in appendix B one ternary unstable mode of the cuboctahedron.

Let us briefly present the method of calculation of the potential function \( V_N \) for some symmetric polyhedral configurations belonging to the octahedral group or the icosahedral group with \( N = 8, 12, 14, 20, 30, 42 \). We will also give a table of residual energies to compare with the previous bounds.

Given formula (3.74) we only need to know distances between the vertices of the polyhedron, only 3 distinct ones in the case of the icosahedron. For higher configurations it is more appropriate to compute systematically the discriminant \( \Delta^2 \) (formula (3.73)) knowing the invariant polynomials (having roots \( z_j \) in the complex plane corresponding to the vertices on the sphere). Let us take for instance the configuration 12_{(24)} of mid-edges of a cube (or the cuboctahedron). Klein [4] gives the following polynomial

\[
f(z) = z^{12} + 1 - 33(z^8 + z^4) \\
\equiv (z^4 + 1)(z^4 - 1)(z^4 - a^{-4})
\]

(3.82)

with

\[
a^2 + a^{-2} = 6, \quad a^{-1} - a = 2
\]
We get \(a^2 = 3 - \sqrt{8}\) and \(a^{-2} = 3 + \sqrt{8}\). We have \(\Delta^2 = \prod_{j=1}^{12} f'(z_j)\). The roots appear in four quartets. We give the value \(f'\) for a typical root, then the product over the quartet

\[
f'(e^{i\pi/4}) = 4e^{-i\pi/4}(1 + a^4)(1 + a^{-4}) = 4e^{-i\pi/4}6^2
\]

\[
\prod_{4\text{terms}} f'(e^{i\pi/4}) = (4 \times 6^2)^4
\]

\[
f'(a) = 4a^3(a^4 + 1)(a^4 - a^{-4}) = 4a^3 \times 6a^2 \times 6 \times -2\sqrt{8}
\]

\[
\prod_{4\text{terms}} f'(a) = (a^5 2^3\sqrt{8} \times 6^2)^4
\]

\[
\prod_{4\text{terms}} f'(a^{-1}) = (a^{-5} 2^3\sqrt{8} \times 6^2)^4
\]

\[
\Delta^2 = (4 \times 6^2 \times 2^6 \times 2^3 \times 6^4)^4 = (2^{17} 3^6)^4
\]

\[
\frac{1}{12} \ln |\Delta|^2 = 6.125058601
\]

also

\[
(N - 1) \sum_j \ln(1 + |z_j|^2) = 11 \times 4(\ln 2 + \ln(4 + \sqrt{8})(4 - \sqrt{8}))
\]

\[
= 11 \times 2^4 \ln 2
\]

\[
\frac{1}{12} (N - 1) \sum_j \ln(1 + |z_j|^2) = 10.16615865...
\]

\[
e^{-2\mathcal{V}_{12}} = \frac{|\Delta|^2}{\prod_j (1 + |z_j|^2)^{11}} = \frac{(2^{17} 3^6)^4}{2^{16} \times 11} = (2^{-9} 3^2)^{12}
\]

\[
\frac{2\mathcal{V}_{12}}{12} = -2\ln 3 + 9\ln 2 = 4.041100047
\]

The same method applies in the following cases:

-12(60): Vertices of an icosahedron

\[
f(z) = z(z^{10} + 11z^5 - 1) \equiv z(z^5 - a^5)(z^5 + a^{-5})
\]

(3.83)

with

\[
a^{-5} - a^5 = 11
\]

\[
a = 2 \cos \frac{2\pi}{5} = 2\xi - 1
\]

\[
a^{-1} = 2 \cos \frac{\pi}{5} = 2\xi
\]

\[
|\Delta|^2 = 5^{25}
\]

\[
2\mathcal{V}_{12} = 55\ln 5 - 25\ln 5 = 30\ln 5.
\]
-20(60): Vertices of the dodecahedron

\[ f(z) = z^{20} + 1 - 228(z^{15} - z^5) + 494z^{10} \equiv (z^5 + a^5)(z^5 - a^{-5})(z^5 + b^5)(z^5 - b^{-5}). \]  (3.84)

with

\[
\begin{align*}
a^{-5} - a^5 &= 114 + 50\sqrt{5} \\
b^{-5} - b^5 &= 114 - 50\sqrt{5}
\end{align*}
\]

\[ 0 < a < b < 1, \quad |\Delta|^2 = 2^{60}3^{10}5^{95} \]

\[ 2\mathcal{V}_{12} = 190ln(a + a^{-1})(b + b^{-1}) - (60ln2 + 10ln3 + 95ln5). \]

-30(60): Mid-edges of the dodecahedron (or the icosahedron).

\[ f(z) = z^{30} + 1 + 522(z^{25} - z^5) - 10005(z^{20} + z^{10}) \equiv (z^{10} + 1)(z^5 - a^5)(z^5 + a^{-5})(z^5 + b^5)(z^5 - b^{-5}) \]  (3.85)

with

\[
\begin{align*}
a^{-5} - a^5 &= 261 + 5^3\sqrt{5} \\
b^{-5} - b^5 &= -261 + 5^3\sqrt{5}
\end{align*}
\]

\[ |\Delta|^2 = 2^{90}3^{60}5^{205} \]

\[ 2\mathcal{V}_{12} = 290ln2(a + a^{-1})(b + b^{-1}) - (90ln2 + 60ln3 + 205ln5) \]

-42(60): Configuration with the 12 vertices and the 30 mid-edges of the icosahedron.

\[ |\Delta|^2 = 2^{90}3^{60}5^{305} \]

For this mixed configuration the discriminant is obtained by multiplying \(|\Delta_{12}|^2 \times |\Delta_{30}|^2\) by the factor \(\prod_{i=1}^{11} f_{30}(z_i)\), the product being on the 11 roots of \(f_{12}\) which gives us the extra factor

\[ (11^2 + 4)^5(261 + 5^3\sqrt{5} - 11)^{10}(-261 + 5^3\sqrt{5} + 11)^{10} = 5^{75} \]

\[ 2\mathcal{V}_{12} = 410ln2(a + a^{-1})(b + b^{-1}) - (100ln5 + 60ln3 + 90ln2) \]

The results for the free energy per particle \(\frac{\mathcal{V}}{N}\) and the residual free energy \(2f(\infty) = \frac{2\mathcal{V}}{N} - \frac{N-1}{2} + \frac{1}{2N}lnN!\) are shown in the following table. We note that the approximate value
$f(\infty) \simeq -0.32$ is in the limits defined in (3.73) namely : $-0.46 < -0.32 < 0.4$. Finally Hölder’s inequality restricts the width of the interval from 0.5 to approximately 0.37.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$N$ & $N^{-1}ln|\Delta|^2$ & $X$ & $F = N^{-1}\mathcal{V}$ & $Y$ & $-f$ \\
\hline
$8_{(24)}$ & 6.1250586 & 10.166158 & 1.128058 & 1.4186061 & 0.290548 \\
$12_{(24)}$ & 3.3529956 & 7.376590 & 2.0117973 & 2.3335996 & 0.3218023 \\
$12_{(60)}$ & 10.273577 & 18.081646 & 3.904034 & 4.2208047 & 0.3167702 \\
$20_{(60)}$ & 15.274491 & 27.880218 & 6.3028631 & 6.627848 & 0.3249848 \\
$30_{(60)}$ & 14.742346 & 33.202164 & 9.2299092 & 9.548976 & 0.3190676 \\
\hline
\end{tabular}
\caption{Residual free energy at zero temperature on the sphere for some symmetrical configurations. (with $X = (1 - N^{-1})\sum ln(1 + |z|^2)$ and $Y = \frac{1}{4}(N - 1 - N^{-1}lnN)$)}
\end{table}

As already mentioned, we show in appendix B that for $N = 12$ the configuration of the cuboctahedron is unstable. Its potential energy is larger by an amount 0.008752... than the one of the stable icosahedron configuration. This energy difference can be interpreted as an ”activation energy” associated to the cuboctahedral configuration as compared to the icosahedron. It is interesting to note that the associated inverse temperature $\beta_c = 2p_c = 1/0.0087... \simeq 114$ is of the order of magnitude of the estimated fusion temperature corresponding to the transition from a quasi-solid phase to a disordered phase.

4. Expansion in characters

We return to the expansion of Laughlin’s wave functions in terms of Slater determinants. Using the notations of section 2 we would like to get a better understanding of the coefficients $g_Y^{(s)}$. For short, when $s = 1$ or $p = 3$ corresponding to filling fraction $\frac{1}{3}$ we set $g_Y^{(1)} \equiv g_Y$.

Consider the monic polynomial of degree $N$, $P(x)$ with roots $x_i$, $0 \leq i \leq N - 1$,

$$P(x) = \prod_{i=0}^{N-1} (x - x_i) = \sum_{k=0}^{N} (-1)^{k} \sigma_k x^{N-k}, \sigma_0 \equiv 1.$$  \hspace{1cm} (4.1)
It has a double root if and only if its discriminant vanishes or equivalently if it has a common root with its derivative

\[ P'(x) = \sum_{k=0}^{N-1} (-1)^k \sigma_k (N-k)x^{N-k-1}. \]

(4.2)

The discriminant, as a symmetric function of the roots is obtained by eliminating \( x \) between (4.1) and (4.2). A systematic way to perform this elimination is to think of the powers of \( x \) as independent variables and to consider the polynomials,

\[ x^m P(x), \quad 0 \leq m \leq N - 2, \]

\[ x^n P'(x), \quad 0 \leq n \leq N - 1, \]

as \( 2N - 1 \) linear forms in the \( 2N - 1 \) variables \( x^0, x^1, \ldots, x^{2N-2} \). A root common to \( P(x) \) and \( P'(x) \) then requires the vanishing of the corresponding determinant which is therefore proportional to the discriminant, in fact with our conventions equal up to a sign. For instance for the classical cases,

\[ P(x) = x^2 - \sigma_1 x + \sigma_2 \]

\[ D = \begin{vmatrix} 1 & -\sigma_1 & \sigma_2 & -\sigma_3 & 0 \\ 0 & 1 & -\sigma_1 & \sigma_2 & -\sigma_3 \\ 3 & -2\sigma_1 & \sigma_2 & 0 & 0 \\ 0 & 3 & -2\sigma_1 & \sigma_2 & 0 \\ 0 & 0 & 3 & -2\sigma_1 & \sigma_2 \end{vmatrix} = -\sigma_1^2 \sigma_2^2 - 18\sigma_1 \sigma_2 \sigma_3 + 4\sigma_1^3 \sigma_3 + 4\sigma_2^3 + 27\sigma_3^2 = -\Delta^2. \]

(4.4)

and in general,

\[ D = \begin{vmatrix} 1 & -\sigma_1 & \sigma_2 & \cdots & (\Delta)^{N-1} \sigma_{N-1} & (\Delta)^N \sigma_N & 0 & \cdots \\ 0 & 1 & -\sigma_1 & \cdots & (\Delta)^{N-1} \sigma_{N-1} & (\Delta)^N \sigma_N \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ N & -(N-1)\sigma_1 & (N-2)\sigma_2 & \cdots & 0 & \cdots \cdots \\ 0 & N & -(N-1)\sigma_1 & \cdots & (\Delta)^{N-1} \sigma_{N-1} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & N & -(N-1)\sigma_1 & \cdots & (\Delta)^{N-1} \sigma_{N-1} \end{vmatrix}. \]

(4.6)
Alternatively knowing that
\[ P'(x) = \sum_{i=0}^{N-1} (x - x_0) \cdots (x - x_i) \cdots (x - x_{N-1}) \] (4.7)
we have, \( P'(x_i) = \prod_{j:j \neq i} (x_i - x_j) \), and,
\[ D(x_0, \cdots, x_{N-1}) = \prod_{0 \leq i \leq N-1} \prod_{j:j \neq i} (x_i - x_j) = \prod_{i=0}^{N-1} P'(x_i). \] (4.8)

In particular for \( N \)-th roots of unity \( x_i = e^{\frac{i \pi}{N}} \), \( 0 \leq i \leq N - 1 \),
\[ P(x) = x^N - 1 \]
\[ D(x_0, \cdots, x_{N-1}) = N^N \prod_{i=0}^{N-1} (x_i)^{N-1} = (-1)^{N-1} N^N \] (4.9)

One verifies that this does agree up to a sign with the above determinant.

From equation (2.10) we have also
\[ (-1)^{\frac{N(N-1)}{2}} D(x_0, \cdots, x_{N-1}) = \Delta^2(x_0, \cdots, x_{N-1}) \]
\[ = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{N-1} & x_1^{N-1} & \cdots & x_{N-1}^{N-1} \end{vmatrix} \times \begin{vmatrix} 1 & x_0 & \cdots & x_0^{N-1} \\ 1 & x_1 & \cdots & x_1^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N-1} & \cdots & x_{N-1}^{N-1} \end{vmatrix} \] (4.10)

\[ = \begin{vmatrix} t_0 & t_1 & \cdots & t_{N-1} \\ t_1 & t_2 & \cdots & t_N \\ \vdots & \vdots & \ddots & \vdots \\ t_{N-1} & \cdots & \cdots & t_{2N-2} \end{vmatrix} \]

where,
\[ t_k = x_0^k + x_1^k + \cdots + x_{N-1}^k \]
\[ \frac{P'(x)}{P(x)} = \sum_{k=0}^{\infty} \frac{t_k}{x^{k+1}} \] (4.11)

In particular \( t_0 = N \). For a generalization (due to Shiota) of formula (4.10) for all \( s \) see appendix C. In the example of \( N \)-th roots of unity \( t_k = 0 \) except when \( k \) is a multiple of \( N \), in which case it is equal to \( N \), the above reduces to
\[ (-1)^{\frac{N(N-1)}{2}} D = N^N \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & \cdots \end{vmatrix} = N^N (-1)^{\frac{(N-1)(N-2)}{2}} \] (4.12)
in agreement with (4.3).

Let us apply (4.10) to give a concise - but not very effective - formula for the coefficients \( g_Y \) in the expansion over Schur functions. For this purpose we recall Frobenius reciprocity formula expressing a product of \( t \)'s as a combination of Schur functions \( ch_Y \) having coefficients \( \chi_Y \), the characters of the symmetric group of order \(|Y|\) pertaining to the same set of Young tableaux. Recall that \( \chi_Y \) is only a class function on \( S_{|Y|} \). We denote the classes \((1)^{\alpha_1}, (2)^{\alpha_2}, \ldots\), i.e. \( \alpha_1 \) cycles of length 1, \( \alpha_2 \) cycles of length 2, etc... then

\[
t_1^{\alpha_1}t_2^{\alpha_2} \cdots = \sum_{Y: |Y|=\sum k\alpha_k} \chi_Y((1)^{\alpha_1}(2)^{\alpha_2} \cdots) \ ch_Y(x_0, \cdots, x_{N-1}) \tag{4.13}
\]

For each class \((1)^{\alpha_1}(2)^{\alpha_2} \cdots\) we have a number - in fact a relative integer - that we write \( \chi_Y((1)^{\alpha_1}(2)^{\alpha_2} \cdots) \). Define symbols \((1)(2) \cdots\) that can be multiplied and linearly combined (with coefficients in \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) or \( \mathbb{C} \)). Attribute to \( (k) \) the degree \( k \) and consider any linear combination of fixed degree \(|Y|\). Then we can extend the definition of \( \chi_Y \) by linearity

\[
\chi_Y \left( \sum_{\alpha_1+2\alpha_2+\cdots=|Y|} a_{\alpha_1,\alpha_2,\ldots}(1)^{\alpha_1}(2)^{\alpha_2} \cdots \right) = \sum a_{\alpha_1,\alpha_2,\ldots} \chi_Y((1)^{\alpha_1}(2)^{\alpha_2} \cdots) \tag{4.14}
\]

If the coefficients are integers so will be the value of \( \chi_Y \). With this notation and for \( s = 1 \), \( g_Y^{(1)} = g_Y \), we find by combining (4.10) and (4.13),

\[
\Delta^2(x_0, \cdots, x_{N-1}) = \sum_{|Y|=N(N-1)} \chi_Y \begin{vmatrix} N & (1) & (2) & \cdots & (N-1) \\ (1) & (2) & (3) & \cdots & (N) \\ \vdots & \ddots & \ddots & \vdots \\ (N-1) & \cdots & \cdots & (2N-2) \end{vmatrix} ch_Y(x_0, \cdots, x_{N-1}), \tag{4.15}
\]

where the sum on the r.h.s. extends over Young tableaux with \( N(N-1) \) boxes, \textit{at most} \( N \) lines (otherwise \( ch_Y(x_0, \ldots, x_{N-1}) \) vanishes) \textit{and at most} \( 2(N-1) \) columns (since the leading term in \( \Delta^3(x_0, \cdots, x_{N-1}) \) is \( \Delta(x_0^3, \cdots, x_{N-1}^3) \). We call these Young tableaux \textit{weakly admissible} and generalize this notion for arbitrary \( s \geq 1 \) to those tableaux of area \( sN(N-1) \) contained in a rectangle of size \( 2s(N-1) \times N \). The reason for the qualifier \textit{weakly} is that some of the tableaux have necessarily vanishing coefficients as will be shown below. In any case for (weakly) admissible tableaux,

\[
g_Y = \chi_Y \begin{vmatrix} N & (1) & \cdots & (N-1) \\ (1) & (2) & \cdots & (N) \\ \vdots & \ddots & \vdots \\ (N-1) & (N) & \cdots & (2N-2) \end{vmatrix} . \tag{4.16}
\]
Even though this formula solves in principle the problem of finding an explicit expression in the case \( s = 1 \) it is not very manageable. Indeed the r.h.s. is a linear combination of \( N! \) values of characters of \( S_{N(N-1)} \), a formidable task to evaluate. It is however possible to dispense altogether with the characters \( \chi_Y \) of the symmetric group as follows. The characters \( ch_Y \) of the linear group are expressed in terms of the normalized traces \( \theta_k = \frac{t_k}{k} \) (2.14) as

\[
ch_Y(f_0, \ldots, f_N) = \begin{vmatrix}
p_{f_N-1}(\theta_1) & \cdots & * \\
\vdots & \ddots & \vdots \\
* & \cdots & p_{f_0}(\theta_1)
\end{vmatrix}
= \sum_{\sum k \geq 1 \alpha_k = |Y|} \chi_Y((1)^{\alpha_1}(2)^{\alpha_2}) \frac{\theta_1^{\alpha_1} \theta_2^{\alpha_2}}{\alpha_1! \alpha_2!} \cdots
\] (4.17)

Here \( p_n(\theta_1) \) stands for the elementary Schur functions, the characters of the \( n \)-th symmetric power given by

\[
p_n(\theta) = \sum_{\sum k \geq 1 \alpha_k = n} \frac{\theta_1^{\alpha_1} \theta_2^{\alpha_2} \cdots}{\alpha_1! \alpha_2!} \cdots
\] (4.18)

The * in the determinant indicate that moving a unit step to the right (left) the index of \( p \) increases (decreases) by unity, and one agrees that \( p_0 = 1 \) as well as \( p_n = 0 \) for \( n \) negative.

The second part of (4.17) (from which (4.18) follows) is obtained by inverting eq. (4.13) using the orthogonality and reality of the \( \chi_Y \)’s. For short set \( \partial_k \equiv \frac{\partial}{\partial \theta_k} \) in such a way that

\[
\partial_k p_n(\theta) = p_{n-k}(\theta) \quad (4.19)
\]

Using these notations we rewrite (4.16) as

\[
g_Y = \begin{vmatrix}
N & \partial_1 & \cdots & \partial_{N-1} \\
\partial_1 & \partial_2 & \cdots & \partial_N \\
\vdots & \vdots & \ddots & \vdots \\
\partial_{N-1} & \partial_N & \cdots & \partial_{2N-2}
\end{vmatrix} \times \begin{vmatrix}
p_{f_N-1}(\theta) & \cdots & * \\
\vdots & \ddots & \vdots \\
* & \cdots & p_{f_0}(\theta)
\end{vmatrix}
\] (4.20)

This gives a more manageable way of computing these coefficients. Notice that here we treat the \( \theta \)’s as independent variables and that the amount of derivatives is such as the right hand side is a pure number (indeed an integer).

Finally if we observe that

\[
\exp \sum_1^{\infty} z^k \theta_k \cdot p_n(\theta) = \frac{1}{n!} \left( \frac{\partial}{\partial z} \right)^n \exp \sum_1^{\infty} z^k \theta_k
\] (4.21)
we can even dispense with the \( p_n \)'s and reach a simple expression

\[
g_Y = \begin{pmatrix}
\partial^{l_{N-1}}_{N-1} & \partial^{l_{N-1}+2}_{N-1} & \partial^{l_N-1}_{N-1} \\
(l_{N-1}-N+1)! & (l_{N-1}-N+2)! & (l_{N-1}-1)! \\
\partial^{l_{0}}_{0} & \partial^{l_{2}}_{0} & \partial^{l_{0}}_{0} \\
(l_{0}-N+1)! & (l_{0}-N+2)! & (l_{0})! \\
\end{pmatrix}
\begin{pmatrix}
t_0 & t_1 & \cdots & t_{N-1} \\
\vdots & \ddots & \ddots & \vdots \\
t_{N-1} & \cdots & \cdots & t_{2N-2} \\
\end{pmatrix}
\] (4.22)

As before \( t_0 = N \) and on the right hand side the terms \( \frac{1}{k!} \partial^k_x \) where \( k < 0 \) or \( k > 2N - 2 \) are set equal to zero.

From these expressions we derive a number of properties, starting from the most obvious ones which follow immediately from the definition, with \( p = 2s + 1 \)

**Property 0**

(i) All coefficients \( g_Y^{(s)} \) are integers.

(ii) \( g_{l_0,...,l_{N-1}}^{(s)} = g_{p(N-1)-l_{N-1},p(N-1)-l_{N-2},...,p(N-1)-l_0}^{(s)} \)

(iii) \( g_{l_0,...,l_{N-1}=pN}^{(s),N} = g_{l_0,...,l_{N-1}}^{(s),N} \)

\( g_{l_0,l_0+p,...,l_{N-1}+p}^{(s),N} = g_{l_0,...,l_{N-1}}^{(s),N} \) \[1\].

Assume that \( s = 1 \) and suppose that \( Y \) is contained in an \( N \times N \) square (Its area being \( N(N-1) \) this is indeed possible). This requires \( f_{N-1} \leq N \) or \( l_{N-1} \leq 2N - 1 \). If this is the case \( Y \) and its symmetric \( \tilde{Y} \) with rows and columns interchanged, are both weakly admissible. Since \( \chi_Y \) and \( \chi_{\tilde{Y}} \) differ only by the sign of the permutation and since a cycle \( (k) \) has parity \((-1)^k\), we have,

\[
g_{\tilde{Y}} = \chi_{\tilde{Y}} \begin{pmatrix}
N & (1) & -(2) & \cdots & (-1)^{N-2}(N-1) \\
(1) & -2 & (3) & \cdots & (-1)^{N-1}(N) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
(-1)^{N-2}(N-1) & \cdots & \cdots & -(2N-2) \\
\end{pmatrix}
\] (4.23)

If we set with \( A \) and \( B \) integers,

\[
g_Y = AN + B
\] (4.24)

then,

\[
(-1)^Ng_{\tilde{Y}} = -AN + B
\] (4.25)

\[1\] where the extra index, \( N \) or \( N+1 \) refers to the number of variables.
consequently \( g_Y - (-1)^N g_{\bar{Y}} = 2AN \) and,

**Property 1**

For \( Y \) and \( \bar{Y} \) weakly admissible \( g_Y \equiv (-1)^N g_{\bar{Y}} \mod 2N \).

**Example:**

\[
\Delta^2(x_0, x_1) = ch(x_0, x_1) - 3ch(x_0, x_1)
\]

and \( 1 \equiv -3 \mod 4 \). For more convincing cases one may look at the tables in appendix D.

Now we have a slight refinement of Property 0. Suppose that \( p = 2s + 1 \) is an odd prime, then from Fermat’s theorem \( (x_i - x_j)^p \equiv x_i^p - x_j^p \mod p \). Thus,

\[
\Delta^p(x_0, \ldots, x_{N-1}) \equiv \Delta(x_0^p, \ldots, x_{N-1}^p) \mod p
\]

and

**Property 2**

Apart from \( g^{(s)}_{0, p, 2p, \ldots, (N-1)p} = 1 \), all coefficients \( g^{(s)}_{l_0, \ldots, l_{N-1}} \) vanish mod \( p \) if \( p = 2s + 1 \) is prime.

That \( g^{(s)}_{0, p, \ldots, (N-1)p} = 1 \) follows from the definition (2.10) there being a unique term \( x_0^p x_1^p \cdots x_{N-1}^p \) with coefficient 1 in the expansion of \( \left[ \prod_{i>j} (x_i - x_j) \right]^p \). In particular Property 2 applies for \( s = 1 \) where apart from the leading one, all coefficients are multiples of 3 (see appendix D). One may generalize Property 2 when \( p \) is composite using that if \( p = p_1 p_2 \) with \( p_1 \) prime then \( \Delta(x_0, \ldots, x_{N-1})^p \equiv \Delta(x_0^{p_1}, \ldots, x_{N-1}^{p_1})^{p_2} \mod p_1 \).

The expansion (2.10) over Slater determinants ranges from the most extended one with the sequence of \( l \)’s being \( (0, p, 2p, \ldots, (N-1)p) \) to the most compact one leaving a huge hole around the origin (and physically very unlikely) with every level from \( s(N-1) \) to \( (s+1)(N-1) \) occupied i.e. the sequence

\[
l_0 = s(N-1), \quad l_1 = s(N-1) + 1, \ldots, l_{N-1} = (s+1)(N-1)
\]

For general \( s \) let us describe the admissible sequences of \( l \)’s i.e. those for which the corresponding coefficient \( g^{(s)}_{l_0, \ldots, l_{N-1}} \) is not a priori zero. Subtracting successively 0, 1, \ldots, \( N-1 \) we get the admissible sequence of \( f_0 \leq f_1 \leq \cdots \leq f_{N-1} \) of Young tableaux.

**Property 3**

For \( s \geq 1 \) the set of admissible Young tableaux is of the form:

\[
f_k = 2sk + n_{k+1} - n_k, \quad 0 \leq k \leq N-1
\]
or equivalently,
\[ l_k = (2s + 1)k + n_{k+1} - n_k \]
with non negative integers \( n_k \), such that
\[ n_0 = n_N = 0 \text{ and } 0 \leq n_k \leq \frac{1}{2}(n_{k+1} + n_{k-1}) + s. \]
The last inequality follows from the requirement that \( f_k \geq f_{k-1} \). Clearly all admissible \( Y \) are weakly admissible since \( f_0 \leq \cdots \leq f_{N-1} \leq 2s(N-1) - n_{N-1} \) and the last bound is smaller or equal to \( 2s(N-1) \).

To prove this property we consider for odd \( p \)
\[
\Delta(x_0, \cdots, x_{N-1})^p = \prod_{i > j} (x_i - x_j)^p = \prod_{i > j} (1 - \frac{x_j}{x_i})^p \times x_{N-1}^{p(N-1)} x_{N-2}^{p(N-2)} \cdots x_0^p. \tag{4.29}
\]
For \( i > j \) write \( \frac{x_j}{x_i} = \frac{x_{i-1}}{x_i} \times \frac{x_{i-2}}{x_{i-1}} \cdots \frac{x_j}{x_{j+1}} \) and expand the last product in (4.29). One obtains a sum of terms of the form
\[
\equiv x_{N-1}^{p(N-1)-n_{N-1}} x_{N-2}^{p(N-2)+n_{N-1}-n_{N-2}} \cdots x_k^{p(k+n_{k+1}-n_k)} \cdots x_0^{n_1}, \tag{4.30}
\]
with integer coefficients and non-negative powers. Both sides in (4.29) being antisymmetric we will find on the right hand side for each monomial the antisymmetric sum of its permutations. To identify these determinants it is sufficient to retain those monomials for which the exponents \( pk + n_{k+1} - n_k \) form a strictly increasing sequence. This is Property 3. We denote by \( A_N^{(s)} \) the number of admissible tableaux.

For \( 2 \leq N \leq 29 \) we present in the following table the total number \( A_N^{(1)} \equiv A_N \) of admissible tableaux for \( p = 3, s = 1 \).
The last column gives the ratio of successive terms. The above reasoning does not however insure that this is exactly the total number of terms in the expansion of $\Delta^2$ in characters as some coefficients might still vanish. However experience up to $N=5$ (appendix D) seems to indicate that these accidents do not happen. It is curious to notice that the
condition \( n_k \leq \frac{1}{2}(n_{k+1} + n_{k-1}) + s \) is a discrete generalization of subharmonic functions.

The number of admissible tableaux is smaller or equal to the number of weakly admissible ones

\[
\Pi_{N,2s(N-1)}(sN(N-1))
\]

where \( \Pi_{N,M}(f) = \Pi_{M,N}(f) \) stands for the number of partitions of the integer \( f \) in at most \( N \) (non-empty) parts of size at most equal to \( M \). Equivalently this is the number of tableaux in a \( M \times N \) rectangle of area \( f \). It is given in terms of the generating function,

\[
\sum_{N,f \geq 0} \Pi_{N,M}(f) x^N q^f = \frac{1}{(1-x)(1-xq)(1-xq^2) \cdots (1-xq^M)} \quad (4.31)
\]

Recall that in terms of an infinite sequence of variables \( \theta_1, \theta_2, \cdots \) the (elementary) Schur polynomials (in the \( \theta \)'s) are defined through,

\[
\exp \sum_{n=1}^{\infty} x^n \theta_n = \sum_{n=0}^{\infty} x^n p_n(\theta.) \quad (4.32)
\]

Since

\[
\prod_{k=0}^{M} \frac{1}{(1-xq^k)} = \exp \sum_{n=1}^{\infty} x^n \frac{1}{n} \frac{1 - q^n(M+1)}{1 - q^n} \quad (4.33)
\]

we find,

\[
\sum_{f \geq 0} \Pi_{N,M}(f)q^f = p_N(\theta_k = \frac{1}{k} \frac{1 - q^k(M+1)}{1 - q^k}) \quad (4.34)
\]

An elementary fermion-boson equivalence gives an alternative generating function for the restricted number of partitions, namely

\[
\sum_{N,f \geq 0} \Pi_{N,M}(f)y^N q^{f+\frac{N(N-1)}{2}} = (1+y)(1+yq) \cdots (1+yq^{N+M-1}) \quad (4.35)
\]

Hence

\[
\sum \Pi_{N,M}(f)q^{f+\frac{N(N-1)}{2}} = p_N(\theta_k = \frac{(-1)^k}{k} \frac{1 - q^{k(N+M)}}{1 - q^k}) \quad (4.36)
\]

thus

**Property 4**

The number of admissible tableaux is smaller than the number of weakly admissible ones given by \( \Pi_{N,2s(N-1)}(sN(N-1)) \).
Property 3 suggests to write the expansion over characters using the operator

$$\Delta(\tau_i) = \prod_{1 \leq i \leq j \leq N-1} (1 - \tau_i \tau_{i+1} \cdots \tau_j)$$  \hspace{1cm} (4.37)$$

where the “box shifting” operator $\tau_i$ acts on characters as

$$\tau_i \ ch_{l_0, \ldots, l_{N-1}} = ch_{l_0, \ldots, l_{i-1}, l_i-1, \ldots, l_{N-1}}. \hspace{1cm} (4.38)$$

The expansion of $\Delta^{2s}$ over the characters is also obtained by retaining the admissible part of the action of $\Delta^{2s+1}(\tau_i)$ on the leading character $ch_{0,(2s+1),2(2s+1),\ldots,(2s+1)(N-1)}$, namely that of the monomials $\tau_1^{n_1} \cdots \tau_{N-1}^{n_{N-1}}$ which produce a strictly increasing sequence of indices $n_1 < (2s+1) - n_1 + n_2 < \cdots < (2s+1)^{N-2} - n_{N-2} + n_{N-1} < (2s+1)^{N-1} - n_{N-1}$, i.e.

$$\Delta^{2s+1}(\tau_i) \bigg|_{adm} \ ch_{0,(2s+1),\ldots,(2s+1)(N-1)} = \sum_{n_1, \ldots, n_{N-1}} C^{(s)}_{\{n_1, \ldots, n_{N-1}\}} [\tau_1^{n_1} \cdots \tau_{N-1}^{n_{N-1}} \ ch_{0,(2s+1),\ldots,(2s+1)(N-1)}]$$

$$= \sum_Y g_Y^{(s)} \ ch_Y \hspace{1cm} (4.39)$$

where the $n$’s in the summation are as in Property 3, and $C^{(s)}_{\{n\}} = g_Y^{(s)}$ if $Y$ is obtained from the tableau $\{0, (2s+1), \ldots, (2s+1)(N-1)\}$ by action of a monomial $\tau_1^{n_1} \cdots \tau_{N-1}^{n_{N-1}}$.

This provides us with another compact formula for the $g_Y^{(s)}$, by straightforward expansion of $\Delta(\tau_i)^{2s+1}$:

$$g_Y^{(s)} = C^{(s)}_{\{n\}} = \sum_{m_{ij} = m_{ji} \geq 0 \atop 1 \leq i < j \leq N-1} (-1)^{\Sigma_i n_i + \Sigma_{i<j} m_{ij}} \prod_{i<j} (2s+1) \ m_{ij}^{N-1} \ n_i - \sum_{k<i<j} m_{kj} - \sum_{l \neq i} m_{il} \right),$$

$$= \sum_Y g_Y^{(s)} \ ch_Y \hspace{1cm} (4.40)$$

where the sum over $m_{ij} = m_{ji}$ is restricted to the values for which the combinatorial factors make sense ($\binom{2s+1}{n}$, for $0 \leq n \leq 2s+1$) we list here the first two cases

$$N=2:\hspace{1cm} g_Y^{(s)} = C^{(s)}_{n} = (-1)^{n} \binom{2s+1}{n}; \hspace{1cm} 0 \leq n \leq s$$

2 The elementary action of $\tau_i$ on a Young tableau is to move a box from line $N - i$ to line $N - i + 1$. The actions of various $\tau$’s commute with each other.
N=3:

\[ g_Y^{(s)} = C_{n_1, n_2}^{(s)} = (-1)^{n_1 + n_2} \sum_{m \geq 0} (-1)^m \binom{2s + 1}{m} \binom{2s + 1}{n_1 - m} \binom{2s + 1}{n_2 - m} \]  

(4.42)

with respectively \( Y = \{n, 2s + 1 - n\}, 0 \leq n \leq s \) for N=2 and \( Y = \{n_1, 2s + 1 - n_1 + n_2, 4s + 2 - n_2\}, 0 \leq n_1, n_2; 2n_1 - n_2 \leq 2s, 2n_2 - n_1 \leq 2s \). More coefficients are presented in appendix D. The expression (4.40), although increasingly complicated with \( N \), provides us with a factorization property of the \( g_Y^{(s)} \)

**Property 5**

For \( g_Y^{(s)} = C_{\{n_0, \ldots, n_{N-1}\}}^{(s)} \) as above, we have the factorization property:

\[ C_{\{n_0, \ldots, n_{j-1}, 0, n_{j+1}, \ldots, n_{N-1}\}}^{(s)} = C_{\{n_0, \ldots, n_{j-1}\}}^{(s)} \times C_{\{n_{j+1}, \ldots, n_{N-1}\}}^{(s)} \]  

(4.43)

This is a straightforward consequence of eqn (4.40), as the vanishing of \( n_j \) implies that of all \( m_{kl}, k \leq j \leq l \), therefore the sum over \( m_{ij} \geq 0 \) breaks up into a product of two sums over \( m_{kl} \) with respectively \( k < l < j \) and \( j < k < l \).

Let us derive for \( p = 3 \) yet an alternative expression for the coefficients \( g_Y \) starting from equation (4.8) which can be rewritten

\[ (-1)^{N(N-1)/2} \Delta^3(x_0, \ldots, x_{N-1}) = |P'(x)x^0, P'(x)x^1, \ldots, P'(x)x^{N-1}| \]  

(4.44)

Recall that this is understood as a determinant with \( x_0 \) substituted in the first line, \( x_1 \) in the second, \ldots. Introduce another set of variables \( \mu_0, \ldots, \mu_{N-1} \). One readily sees that

\[ (-1)^{N(N-1)/2} \Delta^3(x_0, \ldots, x_{N-1}) = \frac{\partial^N}{\partial \mu_0 \partial \mu_1 \cdots \partial \mu_{N-1}} |P(\mu)\mu^0, \ldots, P(\mu)\mu^{N-1}| \bigg|_{\mu_k = x_k} \]  

(4.45)

since \( x_0, \ldots, x_{N-1} \) are the roots of \( P \). Set for simplicity \( \phi_m \equiv (-1)^{N-m}\sigma_{N-m}, (\phi_N = \sigma_0 \equiv 1) \) in such a way that

\[ P(x) = \sum_{0 \leq k \leq N} (-1)^k \sigma_k x^{N-k} \equiv \sum_{0 \leq m \leq N} \phi_m x^m \]  

(4.46)
\[-1 \frac{N(N-1)}{2} \Delta^3(x_0, \ldots, x_{N-1})
\]
\[= \frac{\partial^n}{\partial \mu_0 \cdots \partial \mu_{N-1}} \sum_{m_0, \ldots, m_{N-1}} \phi_{m_0} \phi_{m_1-1} \cdots \phi_{m_{N-1}-N+1} \left| \mu^{m_0} \cdots \mu^{m_{N-1}} \right|_{\mu_k = x_k}
\]
\[= \frac{\partial^n}{\partial \mu_0 \cdots \partial \mu_{N-1}} \sum_{m_0 < \ldots < m_{N-1}} \begin{vmatrix} \phi_{m_0} & \phi_{m_0-1} & \cdots & \phi_{m_0-N+1} \\ \phi_{m_1} & \phi_{m_1-1} & \cdots & \phi_{m_1-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{m_{N-1}} & \phi_{m_{N-1}-1} & \cdots & \phi_{m_{N-1}-N+1} \end{vmatrix} \times \left| \mu^{m_0} \cdots \mu^{m_{N-1}} \right|_{\mu_k = x_k}
\]
\[= \sum_{m_0 < \ldots < m_{N-1}} m_0 \ldots m_{N-1} \begin{vmatrix} \phi_{m_0} & \phi_{m_0-1} & \cdots & \phi_{m_0-N+1} \\ \phi_{m_1} & \phi_{m_1-1} & \cdots & \phi_{m_1-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{m_{N-1}} & \phi_{m_{N-1}-1} & \cdots & \phi_{m_{N-1}-N+1} \end{vmatrix} \times |x^{m_0-1} \cdots x^{m_{N-1}-1}|
\]
(4.47)

We now change \(m_k \rightarrow m_k + 1\), substitute \(\phi_m \rightarrow (-1)^{N-m} \sigma_{N-m}\), and collect the signs, with the result that

\[\Delta^3(x_0, \ldots, x_{N-1}) = \sum_{m_0 < \ldots < m_{N-1}} (-1)^{\sum m_k} (m_0 + 1) \cdots (m_{N-1} + 1) \times
\]
\[\begin{vmatrix} \sigma_{N-1-m_0} & \sigma_{N-m_0} & \sigma_{2(N-1)-m_0} \\ \vdots & \ddots & \vdots \\ \sigma_{N-1-m_{N-1}} & \cdots & \sigma_{2(N-1)-m_{N-1}} \end{vmatrix} \times |x^{m_0} \cdots x^{m_{N-1}}|
\]
(4.48)

Set

\[m_k = f_k + k ; \; 0 \leq k \leq N - 1,
\]
\[\sum m_k = \sum f_k + \frac{N(N-1)}{2},
\]
\[\prod_k (m_k + 1) = \prod_k (f_k + k + 1).
\]
(4.49)

Define the Young tableaux

\[Y(f) = \{0 \leq f_0 \leq f_1 \leq \cdots \leq f_N \leq N - 1\}
\]
(4.50)
and $Y(f'')$ given by the sequence $0 \leq f_0'' \leq \cdots \leq f_{N-1}'' \leq N - 1$ with

\[
f''_{N-1} = N - 1 - m_0 = N - 1 - f_0
\]
\[
\vdots \quad \vdots
\]
\[
f''_{N-k} = N - 1 + k - m_k = N - 1 - f_k
\]
\[
\vdots \quad \vdots
\]
\[
f_0'' = 2(N - 1) - m_{N-1} = N - 1 - f_{N-1}.
\]

Then

\[
\frac{|x^{m_0} \cdots x^{m_{N-1}}|}{\Delta(x_0, \cdots, x_{N-1})} = \text{ch}_Y(f)(x_0, \cdots, x_{N-1})
\]

while according to a standard property of characters [5],

\[
\sigma_{N-1 - m_0} \cdots \sigma_{2(N-1) - m_{N-1}} \\
\vdots \quad \vdots \\
\sigma_{N-1 - m_{N-1}} \cdots \sigma_{2(N-1) - m_{N-1}}
\equiv
\begin{vmatrix}
\sigma_{f''_{N-1}} & \sigma_{f''_{N-1}+1} & \cdots & \sigma_{f''_{N-1}+N-1} \\
\sigma_{f''_{N-2}} & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots \\
\sigma_{f''_{0}-(N-1)} & \cdots & \cdots & \sigma_{f''_0}
\end{vmatrix}
= \text{ch}_{Y_{(f'')}}(x_0, \cdots, x_{N-1})
\]

where $Y \rightarrow \widetilde{Y}$ corresponds to the transposition (i.e. the interchange of lines and columns) of Young tableaux (a notation already used) in which elementary Schur polynomials ($p_k \equiv$ trace of $k$-symmetric power) get interchanged with elementary symmetric functions ($\sigma_k \equiv$ trace of $k$-antisymmetric power). This is recorded in the following picture where $Y(f)$ and $Y^c(f') \equiv \widetilde{Y}(f'')$ are two complementary parts of an $N \times (N - 1)$ rectangle.

**Fig. 1:** Notations used for Young tableaux

This gives the final result

\[
\Delta^2(x_0, \cdots, x_{N-1}) = \sum_{Y \in N \times (N-1)} (-1)^{\frac{N(N-1)}{2} + \sum f_k} \\
\times (f_0 + 1) \cdots (f_{N-1} + N) \text{ch}_Y(x_0, \cdots, x_{N-1}) \text{ch}_{Y^c}(x_0, \cdots, x_{N-1})
\]

(4.54)
where the short hand notation $Y \in N \times (N - 1)$ indicates that the tableau is inscribed in the rectangle $N \times (N - 1)$ and if $Y$ is given by the sequence of $0 \leq f_1 \leq \cdots \leq f_{N-1} \leq N-1$, $Y^c$ corresponds to the complementary sequence $f'_0 = 0 \leq f'_1 \leq \cdots \leq f'_{N-1} \leq N$.

As a (trivial) example using the notation $<l_0, l_1, \ldots, l_{N-1}>$ for $ch_{f_0=0, \ldots, f_{N-1}=l_{N-1}-N+1}$ we have,

$$\Delta^2(x_0, x_1) = -2ch\Box + 3ch\Box ch\Box - 6ch\Box$$

$$= -2ch\Box + 3(ch\Box + ch\Box) - 6ch\Box$$

$$= ch\Box - 3ch\Box = <0, 3> - 3 <1, 2>$$

and the slightly less obvious one,

$$\Delta^2(x_0, x_1, x_2) = -6ch\Box + 8ch\Box ch\Box - 10ch\Box ch\Box$$

$$-12ch_{0} ch_{1} + 15ch_{2} ch_{2} - 20ch_{0} ch_{2} + 24ch_{2} ch_{2}$$

$$-30ch_{0} ch_{1} + 40ch_{2} ch_{1} - 60ch_{1} ch_{1}$$

$$= -6ch_1 + 8(ch_1 + ch_1 + ch_1) - 10(ch_1 + ch_1 + ch_1)$$

$$-12(ch_0 + ch_0 + ch_0) + 15(ch_2 + ch_2 + ch_2 + 2ch_2 + ch_2)$$

$$-20(ch_0 + ch_0) + 24ch_2 ch_2 - 30(ch_1 + ch_1) + 40(ch_1 + ch_1)$$

$$-60ch_0$$

$$=<0, 3, 6> - 3 <1, 2, 6> - 3 <0, 4, 5> + 6 <1, 3, 5> - 15 <2, 3, 4>$$

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Since for given \( N \) the Clebsch Gordan rule yields,

\[
ch_{Y_1}ch_{Y_2} = \sum_{Y_3} N_{Y_1,Y_2}^Y ch_{Y_3} \tag{4.55}
\]

with \( N_{Y_1,Y_2}^Y \) non negative multiplicities, and \(|Y_3| = |Y_1| + |Y_2|\), we rewrite equation (4.54) as

\[
\Delta^2(x_0, \cdots, x_{N-1}) = \sum_{Y_1 \in N \times (N-1)} \sum_{Y_2 \in N \times (N-1)} N_{Y_1,Y_2}^Y \left( -1 \right)^{\frac{N(N-1)}{2} + \sum_{0}^{N-1} f_k(Y_1)(f_0(Y_1) + 1) \cdots (f_{N-1}(Y_1) + N) \right) \times ch_{Y}(x_0, \cdots, x_{N-1}) \tag{4.56}
\]

Equivalently,

\[
g_Y = \sum_{Y_1 \in N \times (N-1)} N_{Y_1,Y_2}^Y \left( -1 \right)^{\frac{N(N-1)}{2} + \sum_{0}^{N-1} f_k(Y_1)(f_0(Y_1) + 1) \cdots (f_{N-1}(Y_1) + N) \right) \tag{4.57}
\]

The computation of \( g_Y \) is here reduced to the application of the Hall-Richardson rule to yield the Clebsch Gordan series over the \( \mathcal{N} = \sum_f \Pi_{N,N-1}(f) \) products of characters pertaining respectively to a Young tableau inscribed in an \( N \times (N - 1) \) rectangle and its complementary.

5. Sum rules

A famous identity due to Dixon (1891) states that \( \Box \)

\[
\sum_{0 \leq s \leq 2p} (-1)^s p! \binom{2p}{s}^3 = \frac{(3p)!}{(p!)^3} \tag{5.1}
\]

A considerably generalized form first conjectured by Dyson was proved by Gunson and Wilson independently. Let as before,

\[
P(x) = \prod_{0 \leq i \leq N-1} (x - x_i) \tag{5.2}
\]

Set

\[
P(x_0 \cdots x_{N-1}; a_0 \cdots a_{N-1}) = \prod_{0 \leq i \leq N-1} \left( \frac{P'(x_i)}{x_i^{N-1}} \right)^{a_i} = \prod_{0 \leq i \neq j \leq N-1} (1 - \frac{x_j}{x_i})^{a_i} \tag{5.3}
\]
Consider the constant term in $P$ understood as a Laurent series in the variables $x_i$

$$G_N(a_0, \ldots, a_{N-1}) = \oint \prod_{0 \leq i \leq N-1} \frac{d\theta_i}{2\pi} P(e^{i\theta_0}, \ldots, e^{i\theta_{N-1}}; a_0, \ldots, a_{N-1})$$  \hspace{1cm} (5.4)$$

then

**Theorem** (Dyson, Gunson, Wilson [7]):

$$G_N(a_0, \ldots, a_{N-1}) = (\sum a_i)! \prod a_i!$$  \hspace{1cm} (5.5)$$

Dixon's identity is nothing else than the particular case where $N = 3$ and $a_0 = a_1 = a_2 = p$. The remarkable proof due to Good [8] goes as follows. Remarki ng from Cauchy's formula that

$$\sum_i x_i^{N-1} P'(x_i) = \oint \frac{dz}{2\pi i} z^{N-1} P(z) = 1$$  \hspace{1cm} (5.6)$$

we multiply $P(x_0, \ldots x_{N-1}; a_0, \ldots a_{N-1})$ by 1 written as above to get for the constant term,

$$G_N(a_0, \ldots, a_{N-1}) = \sum_{0 \leq i \leq N-1} G_N(a_0, \ldots, a_i - 1, \ldots, a_{N-1})$$  \hspace{1cm} (5.7)$$

Since when a particular $a_i$ vanishes the variable $x_i$ only appears with non negative power in $P(x_0, \ldots x_{N-1}; a_0, \ldots, a_i = 0, \ldots a_{N-1})$ we can let $x_i = 0$ in the computation of the constant term, i.e.

$$G_N(a_0, \ldots, a_i = 0, \ldots, a_{N-1}) = G_{N-1}(a_0, \ldots, \hat{a}_i, \ldots, a_{N-1})$$  \hspace{1cm} (5.8)$$

Furthermore

$$G_N(0, \ldots, 0) = 1$$  \hspace{1cm} (5.9)$$

Finally $G_N$ is obviously symmetric in the $a_i$'s. This property together with (5.7) (5.8) and (5.9) uniquely determines $G_N$ recursively and it is clear that the r.h.s. in (5.3) satisfies the same conditions. This suffices to prove the required equality.

Let us apply Dyson’s theorem in the following form. The integral

$$\oint \frac{d\theta_0}{2\pi} \ldots \frac{d\theta_{N-1}}{2\pi} |\Delta(e^{i\theta_0}, \ldots, e^{i\theta_{N-1}})|^{2p}$$  \hspace{1cm} (5.10)$$

is equal for $p = 2s + 1$ odd to ,

$$N! \sum_{0 \leq l_0 < \ldots < l_{N-1} \leq p(N-1)} |g^{(s)}_{l_0, \ldots, l_{N-1}}|^2$$  \hspace{1cm} (5.11)$$

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according to its expansion (2.11). On the other hand it is also equal to
\[
\int 2\pi \cdots \int 2\pi \frac{\Delta(e^{i\theta_0} \cdots e^{i\theta_{N-1}})^{2p}}{\Delta(e^{i\theta_0} \cdots e^{i\theta_{N-1}})N(1)p} (-1)^{N(1)p} e^{i(\theta_0 + \cdots + \theta_{N-1})} \prod_{0 \leq k \leq N-1} \left[ \frac{P'(e^{i\theta_k})}{e^{i(N-1)\theta_k}} \right]^p
\]
\[
= \frac{(Np)!}{(p!)^N}
\]
where the last equality is obtained from (5.5) by setting all \(a_i = p\). Therefore for \(p\) odd we conclude that we have the sum rule,

**Property 6**
\[
\sum_{0 \leq l_0, \ldots, l_{N-1} \leq p(N-1)} |g_{l_0, \ldots, l_{N-1}}^{(s)}|^2 = \frac{(Np)!}{N!(p!)^N}
\]
where both sides are integers (i.e. \(N!\) divides \(\frac{(Np)!}{(p!)^N}\)). For example when \(s=1\) we have
\[
\sum_{0 \leq l_0, \ldots, l_{N-1} \leq 3(N-1)} g_{l_0, \ldots, l_{N-1}}^2 = \frac{(3N)!}{N!3!N}
\]
In a similar manner we can compute the coefficient \(g_{s(N-1), (s+1)(N-1)}^{(s)}\) of the most compact term noting that the corresponding character is \(\sigma_{s(N-1)}^{(s)} = (\prod_{0 \leq k \leq N-1} x_{k}^{N-1})^s\), hence
\[
N! g_{s(N-1), (s+1)(N-1)}^{(s)} = \int 2\pi \cdots \int 2\pi \frac{\Delta(e^{i\theta_0} \cdots e^{i\theta_{N-1}})^{2s+1}\Delta(e^{-i\theta_0} \cdots e^{-i\theta_{N-1}})}{\left(\prod_{0 \leq k \leq N-1} e^{i\theta_k(N-1)}\right)^s} \prod_{k=0}^{N-1} \left[ \frac{P'(e^{i\theta_k})}{e^{i(N-1)\theta_k}} \right]^{s+1}
\]
This gives
\[
g_{s(N-1), (s+1)(N-1)}^{(s)} = (-1)\frac{N(N-1)}{2} \frac{(s+1)!}{N!(s+1)!N!}
\]
In particular when \(s = 1\),
\[
g_{N-1, \ldots, 2N-2} = (-1)\frac{N(N-1)}{2} (2N-1)!!
\]
From equation (2.11) and (5.13) it follows that,
\[
\frac{(Np)!}{(p!)^N} \inf_{Y_{adm}} l_0! \cdots l_{N-1}! \leq Z_N(p) \leq \frac{(Np)!}{(p!)^N} \sup_{Y_{adm}} l_0! \cdots l_{N-1}!
\]
The sup and inf are taken over admissible tableaux. We suspect that these correspond respectively to the most extended and most compact terms, in other words that

\[
\frac{(Np)!}{(p!)^N} \prod_{k=s(N-1)}^{(s+1)(N-1)} k! \leq Z_N(p) \leq \frac{\prod_{k=1}^{N}(pk)!}{p!^N}
\]  

(5.19)

Referring to equation (A.18) and our expectation (2.32) we see that these bounds are unfortunately quite miserable. We add the following remark. Consider for simplicity in \(Z_N(3)\) the ratio \(u_N\) of the contribution of the most extended to the most compact term,

\[
u_N = \frac{\prod_{k=0}^{N}(3k)!}{\prod_{k=N-1}^{2(N-1)} k!((2N - 1)!!)^2}
\]

(5.20)

Then,

\[
\frac{u_{N+1}}{u_N} = 2\frac{N!(3N)!}{(2N+1)!^2} \sim_{N \to \infty} \frac{\sqrt{3}}{(2N+1)^2} \left(\frac{27}{16}\right)^N
\]

(5.21)

Therefore for \(N \to \infty\), \(u_N\) goes to infinity which suggests that in the large \(N\) limit, \(Z_N(3)\) is dominated by the contributions from extended states in agreement with "physical intuition".

6. Linear system for the coefficients

We now come to linear equations satisfied by the expansion coefficients \(g\), where for simplicity we will set \(s = 1\) although all that will be said admits a generalization to arbitrary \(s\). We start with the following obvious observation. Let \(Q(x_0, \cdots, x_{N-1})\) be an arbitrary symmetric polynomial in \(x_0, \cdots, x_{N-1}\), homogeneous in these variables of degree \(N(N-1)\) and vanishing when \(x_0 = x_1\). Then \(Q(x_0, \cdots, x_{N-1})\) is proportional to \(\Delta(x_0, \cdots, x_{N-1})^2\). Indeed it admits the factor \(x_0 - x_1\), hence from symmetry is divisible by \(\Delta\). But \(\frac{Q}{\Delta}\) is antisymmetric, thus is divisible by \(\Delta\). From the comparison of degrees, \(\frac{Q}{\Delta^2}\) is a constant. Similarly an antisymmetric polynomial in \(x_0, \cdots, x_{N-1}\), of degree \(\frac{3}{2}N(N-1)\) vanishes obviously at \(x_0 = x_1\). If its derivatives also vanish at \(x_0 = x_1\) then it is proportional to \(\Delta(x_0, \cdots, x_{N-1})^3\). Let us apply this to the expansion (2.10) - (2.11). By requiring that the sum

\[
\sum_{0 \leq l_0 < \cdots < l_{N-1} \leq 3(N-1) \atop \sum l_i = 3N(N-1)/2} \ g_{l_0,\cdots,l_{N-1}} |x^{l_0} \cdots x^{l_{N-1}}|
\]

(6.1)
obviously antisymmetric in $x_0, \ldots, x_{N-1}$ has vanishing derivative at $x_0 = x_1$, we will obtain a quantity proportional to $\Delta^3$. The condition reads (by multiplying it by $x_0$ assumed different from zero),

\[
\sum_{0 \leq l_0 < \cdots < l_{N-1} \leq 3(N-1) \atop \sum l_i = 3N(N-1)/2} g_{l_0, \ldots, l_{N-1}} \begin{vmatrix}
  l_0 x_0^{l_0} & l_1 x_0^{l_1} & \cdots & l_{N-1} x_0^{l_{N-1}} \\
  x_0^{l_0} & x_0^{l_1} & \cdots & x_0^{l_{N-1}} \\
  x_2^{l_0} & x_2^{l_1} & \cdots & x_2^{l_{N-1}} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{N-1}^{l_0} & x_{N-1}^{l_1} & \cdots & x_{N-1}^{l_{N-1}}
\end{vmatrix} = 0 \quad (6.2)
\]

Let us expand this in powers of $x_0$ with coefficients which are determinants in $x_2, \ldots, x_{N-1}$. We set equal to zero the coefficient of

\[
x_0^{l_0+l_1} x_2^{l_2} \cdots x_2^{l_{N-1}} \cdot (x_2, \ldots, x_{N-1})
\]

extending $g_{l_0, \ldots, l_{N-1}}$ as an antisymmetric tensor in its indices to get

**Property 7**

*Given non negative integers $0 \leq l_0 < \cdots < l_{N-3} \leq 3(N-1)$ with sum equal to $\frac{3N(N-1)}{2} - L$, $L > 0$, the coefficients $g$. extended as antisymmetric tensors are uniquely defined by the conditions*

\[
\sum_{l \geq l' \geq 0 \atop l+l' = L} (6.3d) g_{l_0, \ldots, l_{N-3}, l', l} = 0
\]

\[
(6.3b) \quad g_{0,3,\ldots,3(N-1)} = 1.
\]

**Examples:**

$N = 2$
\[
\sum_{l > l' \geq 0 \atop l + l' = 3} (l - l') g_{l,l'} = 3g_{0,3} + g_{1,2} \equiv 0
\]  
(6.4)

i.e. \(g_{0,3} = 1\) and \(g_{1,2} = -3\).

\[N = 3\]

\begin{align*}
(a) \quad & 3g_{0,3,6} + g_{0,4,5} = 0 \\
(b) \quad & 4g_{1,2,6} + 2g_{1,3,5} = 0 \\
(c) \quad & 5g_{2,1,6} + g_{2,3,4} \\
& \equiv -5g_{1,2,6} + g_{2,3,4} = 0 \\
(d) \quad & 6g_{3,0,6} + 4g_{3,1,5} + 2g_{3,2,4} \\
& \equiv -6g_{0,3,6} - 4g_{1,3,5} - 2g_{2,3,4} = 0 \\
(e) \quad & 5g_{4,0,5} + g_{4,2,3} \\
& \equiv -5g_{0,4,5} + g_{2,3,4} = 0 \\
(f) \quad & 4g_{5,0,4} + 2g_{5,1,3} \\
& \equiv 4g_{0,4,5} + 2g_{1,3,5} = 0 \\
(g) \quad & 3g_{6,0,3} + g_{6,1,2} \\
& \equiv 3g_{0,3,6} + g_{1,2,6} = 0
\end{align*}

Conditions (a) and (g) yield from the normalization

\[g_{0,3,6} = 1, \quad g_{0,4,5} = g_{1,2,6} = -3\]

Conditions (b) and (c) give the remaining coefficients

\[g_{1,3,5} = 6, \quad g_{2,3,4} = -15\]

One checks that (d), (e) and (f) are identically fulfilled. This example illustrates the fact that we have an over-determined set of equations. To make the algorithm really useful would require to extract a subset of equations of co-rank equal to 1. This means that if \(A_N \equiv A_N^{(1)}\) is the number of admissible Young tableaux to find a subset of \(A_N\) equations of rank \(A_N - 1\). Should this be possible we could solve for the \(g\)'s in terms of minors of the corresponding matrix up to normalization.
A weak consequence follows from the fact that $\Delta(x_0, \ldots, x_{N-1})^2$ vanishes when all $x_i = 1$ hence, since $ch_Y(1, \ldots, 1) \equiv \dim Y$.

$$\sum_{Y \text{ adm}} g_Y \dim Y = \sum_{0 \leq l_0 < \cdots < l_{N-1} \leq 3(N-1)} g_{l_0, l_1, \ldots, l_{N-1}} \prod_{i > j} \left( \frac{l_i - l_j}{i - j} \right) = 0 \quad (6.6)$$

where $\dim Y$ is the dimension of the corresponding representation of the linear group. When $N = 2$ this coincides with our previous condition, while for $N = 3$ one finds

$$27 g + 10 g + 10 g + 8 g + g \equiv 27 - 10 \times 3 - 10 \times 3 + 8 \times 6 - 15 = 0 \quad (6.7)$$

This may be interpreted as saying that $\Delta(x_0, \ldots, x_{N-1})^2$ is the difference of two characters of $GL_N$ operating in two vector spaces of equal dimension

$$\sum_{Y \text{ adm}, g_Y > 0} g_Y \dim Y$$

7. Specializations

In this section we study specializations of the discriminant leading to sum rules on the coefficients $g_Y^{(s)}$. For simplicity we stick again to the case $s=1$.

As we saw when $x_k = e^{\frac{2i\pi k}{N}}$,

$$\Delta^2(1, e^{\frac{2i\pi}{N}}, \ldots, e^{\frac{2i\pi(N-1)}{N}}) = (-1)^{(N-1)(N-2)} N^N \quad (7.1)$$

For any integer $l$ let $< l >$ denote its representative mod $N$ in the range $0, 1, \ldots, N-1$. If the sequence $0 \leq l_0 < \cdots < l_{N-1} \leq 3(N-1)$ is such that in the sequence $< l_0 >, \ldots, < l_{N-1} >$ two numbers coincide, the corresponding character $ch_{l_0, \ldots, l_{N-1}}(1, e^{\frac{2i\pi}{N}}, \cdots)$ vanishes. If all $< l_0 >, \ldots, < l_{N-1} >$ are distinct let $\epsilon(l_0, \ldots, l_{N-1})$ denote the sign of the permutation

$$\left( \begin{array}{c} < l_0 > \\ 0 \\ \vdots \\ < l_{N-1} > \end{array} \right) \quad \frac{N}{N-1} \quad (7.2)$$

in which case we have

$$ch_{l_0, \ldots, l_{N-1}}(1, e^{\frac{2i\pi}{N}}, \ldots, e^{\frac{2i\pi(N-1)}{N}}) = \epsilon(l_0, \ldots, l_{N-1}). \quad (7.3)$$

Thus,

$$N^N (-1)^{(N-1)(N-2)} = \sum_{\{< l_0 > \cdots < l_{N-1} >\}} g_{l_0, \ldots, l_{N-1}} \epsilon(l_0, \ldots, l_{N-1}) \quad (7.4)$$
The reader can check this sum rule using the listings in appendix D.

More generally we can specialize the \( x_i \)’s to be the successive powers of a single variable \( q \), the so called principal specialization,

\[
x_i = q^i \quad 0 \leq i \leq N - 1
\]

Thus

\[
\Delta(1, q, \ldots, q^{N - 1}) = (-1)^{\frac{N(N - 1)}{2}} \prod_{0 \leq i < j \leq N - 1} (q^i - q^j)
\]

\[
= (-1)^{\frac{N(N - 1)}{2}} q^{\sum_{n=0}^{N-1} n(N-1-n)} \times [1]_q[2]_q \cdots [N-1]_q
\]

where the \( q \)-factorial is defined as

\[
[n]_q = (1 - q)(1 - q^2) \cdots (1 - q^n); \quad \lim_{q \to 1} \frac{[n]_q}{(1 - q)^n} = n!
\]

and

\[
\sum_{n=0}^{N-1} n(N - 1 - n) = \frac{N(N - 1)(N - 2)}{6}
\]

Note that the symbol \([\infty]_q\) makes sense for \(|q| < 1\) being equal to \(q^{-1/24} \eta(q)\) where \(\eta\) is Dedekind’s function, and that Jacobi’s identity reads

\[
\prod_{n>0} (1 - q^n)^3 = \sum_{j>0} (-1)^j (2j + 1) q^{\frac{j(j+1)}{2}}
\]

Under the same specialization,

\[
|x|_0 \cdots x|_{N-1} = \det_{0 \leq a, b \leq N - 1} q^{a b}
\]

\[
= \Delta(q^0, \ldots, q^{N-1}) = (-1)^{\frac{N(N - 1)}{2}} \prod_{0 \leq i < j \leq N - 1} (q^i - q^j)
\]

\[
= (-1)^{\frac{N(N - 1)}{2}} q^{\sum_{k=0}^{N-1} l_k(N-1-k)} \prod_{0 \leq i < j \leq N - 1} (1 - q^{l_j-l_i})
\]

It follows from (7.6) and (7.10) that

\[
([1]_q[2]_q \cdots [N-1]_q)^{2s+1} = \sum_{0 \leq l_0 < \cdots < l_{N-1} \leq (2s+1)(N-1), \Sigma l_i = (2s+1)\frac{N(N - 1)}{2}} g_{l_0}^{(s)} q^{\sum_{k=0}^{N-1} l_k(N-1-k)} \prod_{0 \leq i < j \leq N - 1} (1 - q^{l_j-l_i})
\]

\(47\)
Set

$$\mu_k = l_{N-1-k} - (2s+1)(N-1-k) \quad 0 \leq k \leq N-1$$  \hspace{1cm} (7.12)

which represents, counted from the ”top” the departures of the $l$’s from their values in the most extended case. The exponent in the explicit $q$-factor on the r.h.s. of (7.10) reads

$$E(l_0, \cdots, l_{N-1}) = \sum_{k=1}^{N-1} k \mu_k = \sum_{k=1}^{N-1} n_k \geq 0$$  \hspace{1cm} (7.13)

in terms of the notation introduced in section 4. Its extreme values are,

$$E(0, 2s+1, 2s+1, \cdots, (N-1)(2s+1)) = 0$$

$$E(s(N-1), \cdots, (s+1)(N-1)) = \frac{s(N-1)N(N+1)}{6}$$  \hspace{1cm} (7.14)

Since the power of $(1 - q)$ occurring on the l.h.s. is higher than the one in any term on the r.h.s. we recover (7.9) and its generalization to all $s$ by letting $q \rightarrow 1$ on both sides of (7.11). Of course one can equate further terms to zero as $q \rightarrow 1$ (in fact as many as $sN(N-1)$) to require a similar leading behavior on both sides, but this would be less effective than the equations discussed in section 6.

Finally we can rewrite (7.11) as

$$([1]_q \cdots [N-1]_q)^{2s} = \sum_{Y \text{ admiss}} g_{Y}^{(s)} q^{\sum_{k=1}^{N-1} n_k(Y)} \prod_{b \in Y} \left[ \frac{1 - q^{c(b)}}{1 - q^{h(b)}} \right]$$  \hspace{1cm} (7.15)

where following Macdonald \cite{3} $b$ runs over the boxes of a tableau, $h(b)$ is the hook length at $b$, i.e. one plus the number of boxes at the right or below $b$, and $c(b)$ is its ”label” or ”content”, i.e. the sum $i + j$ where $i$ is the line index starting with zero from the $N$-th bottom line and $j$ its column index starting at 1 at the left as shown in the illustration for $N = 4$ where the shaded box has $h(b) = 7$ and $c(b) = 6$. Equivalently the label of the left uppermost box is $N$ and labels increase (decrease) by a unit when moving one step to the right (down).

**Fig. 2**: Hook length and label of a box

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The expressions occurring on the right hand side of equation (7.15) generalize the $q$-binomial coefficients. For a tableau with a unique column $Y_k$ of height $k \leq N$ we have,

$$\prod_{b \in Y_k} \left( \frac{1 - q^{c(b)}}{1 - q^{h(b)}} \right) = \frac{(1 - q^N)(1 - q^{N-1}) \cdots (1 - q^{N-k+1})}{(1 - q)(1 - q^2) \cdots (1 - q^k)} = \frac{[N]_q}{[k]_q[N-k]_q}$$

while for a single row of length $k$, call it $\bar{Y}_k$,

$$\prod_{b \in \bar{Y}_k} \left( \frac{1 - q^{c(b)}}{1 - q^{h(b)}} \right) = \frac{(1 - q^N)(1 - q^{N+1}) \cdots (1 - q^{N+k-1})}{(1 - q)(1 - q^2) \cdots (1 - q^k)} = \frac{[N + k - 1]_q}{[k]_q[N - 1]_q}$$

and as $q \to 1$, they reduce to

$$\prod_{b \in Y} \frac{c(b)}{h(b)} = \text{dim} Y(GL_N)$$

which gives one of the fastest means to compute these dimensions. Even though equations (7.11) or (7.13) depend on a single variable $q$ it is still a tremendous task to extract from these equalities the coefficients $g$ for instance by expanding at small $q$.

8. The number of admissible tableaux

The number of admissible tableaux $A_N^{(s)}$ is according to property 3 of section 4. the number of $N + 1$-uples of non-negative integers $(n_0 = 0, n_1 \geq 0, \ldots, n_{N-1} \geq 0, n_N = 0)$ such that

$$2s - 2n_k + n_{k-1} + n_{k+1} \geq 0 \quad k = 1, 2, \ldots, N - 1. \tag{8.1}$$

Fig. 3: The polytope $\Pi_4^{(1)}$ and its 16 integer points. The black dots belong to vertices or edges, the half-filled to faces, and the empty one is strictly inside the polytope.

This counts the number of points with integral coordinates in the closed polytope $\pi_N^{(s)}$ defined by the equations (8.1). For illustration, we represent the 3-dimensional polytope $\pi_4^{(1)}$ and its 16 points with integer coordinates on Fig.3. We first concentrate on the number $A_N \equiv A_N^{(1)}$, for $s = 1$.

Let $f_N(x_0, \ldots, x_N)$ be the Laurent series

$$f_N(x_0, \ldots, x_N) = \frac{x_1^2 x_2^2 \cdots x_{N-1}^2}{\prod_{j=1}^{N-1} (1 - \frac{x_{j-1} x_{j+1}}{x_j})},$$

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and expand its denominators as infinite power series to find

\[ f_N = \sum_{n_1, \ldots, n_N \geq 0} x_0^{n_0} x_N^{n_N-1} \prod_{j=1}^{N-1} \left( \frac{x_j x_{j+1}}{x_j^2} \right)^{n_j} \]

\[ = \sum_{n_1, \ldots, n_{N-1} \geq 0} x_0^{n_1} x_N^{n_N-1} \prod_{k=1}^{N-1} x_k^{2-2n_k+n_{k-1}+n_{k+1}}, \]

therefore the desired number \( A_N \) is equal to the number of monomials appearing in \([f_N]_+\), the polynomial part of \( f_N \).

**Examples**

1) \( N=2 \):

We have

\[ f_2(x_0, x_1, x_2) = \frac{x_1^2}{(1 - \frac{x_0 x_2}{x_1^2})} \]

and

\[ [f_2(x_0, x_1, x_2)]_+ = x_1^2 + x_0 x_2, \]

thus \( A_2 = 2 \).

2) \( N=3 \):

We have

\[ f_3(x_0, x_1, x_2, x_3) = \frac{x_1^2 x_2^2}{(1 - \frac{x_0 x_2}{x_1^2})(1 - \frac{x_1 x_3}{x_2^2})} \]

and

\[ [f_3(x_0, x_1, x_2, x_3)]_+ = x_1^3 x_2^2 + x_0 x_2^3 + x_3 x_1^3 + x_0 x_1 x_2 x_3 + x_0^2 x_3^2, \]

which implies \( A_3 = 5 \).

3) \( N=4 \):
\[ [f_4+] = x_1^2 x_2^2 x_3 + x_0 x_2^2 x_3^2 + x_0 x_1 x_2 x_3^2 + x_1 x_2 x_3^4 + x_0 x_1 x_2 x_3^3 + x_0 x_2^4 \]
\[ + x_0 x_1 x_2 x_3 x_4 + x_0 x_1 x_2 x_3 x_4 + x_0 x_1 x_2 x_3 x_4 + x_0 x_1 x_2 x_3 x_4 \]
\[ + x_1^2 x_3^2 + x_0 x_1 x_2 x_4^2 + x_0 x_2 x_3 x_4 + x_0 x_1 x_2 x_4^2 + x_0 x_1 x_2 x_4^2 \]

hence \( A_4 = 16 \).

We now derive an integral formula for \( A_N \). Let \( f \) be a Laurent series of the form
\[ f = \sum_{k=-\infty}^{m} a_k x^k, \]
and \( f_+ = \sum_{k=0}^{m} a_k x^k \). By Cauchy’s theorem, we can write
\[ f_+(1) = \oint_{|x|>1} \frac{dx}{2i\pi} f(x) = \sum_{k=0}^{m} a_k. \]

This gives an integral formula for \( A_2 \), by replacing \( f \) by \( f_+ \), and taking \( x_0 = x_2 = 1 \). More generally, we have a similar formula for
\[ A_N = [f_N]+(1, 1, \ldots, 1), \]
which reads
\[ A_N = \oint_{|x|=\rho^{(N-i)}} \frac{dx_1}{2i\pi} \cdots \frac{dx_{N-1}}{2i\pi} \frac{f_N(1, x_1, \ldots, x_{N-1}, 1)}{\prod_{j=1}^{N-1} (x_j - 1)}, \]
for any \( \rho > 1 \). The choice of contours ensures that
\[ \left| \frac{x_{k-1}x_{k+1}}{x_k^2} \right| < 1 \quad \text{and} \quad |x_k| > 1 \quad k = 1, \ldots, N-1. \]

The number of states \( A_N^{(s)} \) corresponding to the expansion of \( \Delta^{2s} \) is obtained by using as generating function
\[ f_N^{(s)}(x_0, x_1, \ldots, x_N) = \frac{x_1^{2s} x_2^{2s} \cdots x_N^{2s}}{\prod_{k=1}^{N-1} (1 - \frac{x_{k-1}x_{k+1}}{x_k^2})}, \]
and
\[ A_N^{(s)} = [f_N^{(s)}]+(1, 1, \ldots, 1) \]
\[ = \oint_{|x|=\rho^{(N-i)/2s}} \frac{dx_1}{2i\pi} \cdots \frac{dx_{N-1}}{2i\pi} \frac{f_N^{(s)}(1, x_1, \ldots, x_{N-1}, 1)}{\prod_{j=1}^{N-1} (x_j - 1)}, \]
for arbitrary $\rho > 1$. The computation of $A_N^{(s)}$ is consequently reduced to a straightforward but tedious extraction of residues. The result for $A_N^{(s)}$ is a polynomial of degree $(N - 1)$ in $s$ with rational coefficients. This is a general property as discussed below.

We list below the first few polynomials $A_N^{(s)}$.

\begin{align*}
A_1^{(s)} &= 1 \\
A_2^{(s)} &= s + 1 \\
A_3^{(s)} &= 2s^2 + 2s + 1 \\
A_4^{(s)} &= \frac{16}{3} s^3 + \frac{13}{2} s^2 + \frac{19}{6} s + 1 \\
A_5^{(s)} &= \frac{50}{3} s^4 + 24s^3 + \frac{40}{3} s^2 + 4s + 1 \\
A_6^{(s)} &= \frac{288}{5} s^5 + 96s^4 + \frac{385}{6} s^3 + 23s^2 + \frac{157}{30} s + 1 \\
A_7^{(s)} &= \frac{9604}{45} s^6 + \frac{6076}{15} s^5 + \frac{2858}{9} s^4 + 134s^3 + \frac{1531}{45} s^2 + \frac{267}{45} s + 1. \\
A_8^{(s)} &= \frac{262144}{315} s^7 + \frac{26624}{15} s^6 + \frac{71992}{45} s^5 + \frac{6371}{8} s^4 + \frac{43657}{120} s^3 + \frac{5783}{140} s^2 + \frac{971}{140} s + 1. \\
A_9^{(s)} &= \frac{118098}{35} s^8 + \frac{279936}{45} s^7 + \frac{367144}{45} s^6 + 4696s^5 + \frac{75724}{45} s^4 + \frac{5911}{15} s^3 \\
&\quad + \frac{19927}{315} s^2 + \frac{163}{21} s + 1 \\
A_{10}^{(s)} &= \frac{8000000}{567} s^9 + \frac{2320000}{63} s^8 + \frac{39666608}{945} s^7 + \frac{1236328}{45} s^6 + \frac{12340889}{1080} s^5 \\
&\quad + \frac{228025}{72} s^4 + \frac{13641133}{22680} s^3 + \frac{204457}{2520} s^2 + \frac{10957}{1260} s + 1. \\
A_{11}^{(s)} &= \frac{857435524}{14175} s^{10} + \frac{162983612}{945} s^9 + \frac{205427098}{945} s^8 + \frac{50279276}{315} s^7 \\
&\quad + \frac{51136132}{675} s^6 + \frac{365944}{15} s^5 + \frac{61904509}{11340} s^4 + \frac{1628849}{1890} s^3 \\
&\quad + \frac{622849}{6300} s^2 + \frac{323}{35} s + 1.
\end{align*}

(8.2)

These quantities take a neat form when expressed in terms of the polynomials

\[\sigma_k(s) = \frac{s(s-1)(s-2)...(s-k+1)}{k!}.\]
instead of powers $s^k$. We have

\[
\begin{align*}
A_1^{(s)} & = 1 \\
A_2^{(s)} & = \sigma_1 + 1 \\
A_3^{(s)} & = 4\sigma_2 + 4\sigma_1 + 1 \\
A_4^{(s)} & = 32\sigma_3 + 45\sigma_2 + 15\sigma_1 + 1 \\
A_5^{(s)} & = 400\sigma_4 + 744\sigma_3 + 404\sigma_2 + 58\sigma_1 + 1 \\
A_6^{(s)} & = 6912\sigma_5 + 16128\sigma_4 + 12481\sigma_3 + 3503\sigma_2 + 246\sigma_1 + 1 \\
A_7^{(s)} & = 153664\sigma_6 + 432768\sigma_5 + 437776\sigma_4 + 188244\sigma_3 + 30702\sigma_2 + 1110\sigma_1 + 1 \\
A_8^{(s)} & = 4194304\sigma_7 + 13860864\sigma_6 + 17367872\sigma_5 + 10162473\sigma_4 + 2731521\sigma_3 \\
& \quad + 275599\sigma_2 + 5301\sigma_1 + 1 \\
A_9^{(s)} & = 136048896\sigma_8 + 516481920\sigma_7 + 773038912\sigma_6 + 574771360\sigma_5 + 218829232\sigma_4 \\
& \quad + 39174798\sigma_3 + 2537596\sigma_2 + 26375\sigma_1 + 1 \\
A_{10}^{(s)} & = 5120000000\sigma_9 + 21964800000\sigma_8 + 38261688576\sigma_7 + 34587246976\sigma_6 \\
& \quad + 17164897361\sigma_5 + 4532740981\sigma_4 + 561999521\sigma_3 + 23932596\sigma_2 + 135669\sigma_1 + 1 \\
A_{11}^{(s)} & = 219503494144\sigma_{10} + 1050351430656\sigma_9 + 2088303502080\sigma_8 + 2225685070528\sigma_7 \\
& \quad + 1366844046336\sigma_6 + 482748121856\sigma_5 + 92013182474\sigma_4 \\
& \quad + 8109500738\sigma_3 + 230617119\sigma_2 + 716541\sigma_1 + 1.
\end{align*}
\]

(8.3)

These triangular expressions for $A_N^{(s)}$ in terms of the $\sigma_k(s)$ are easily inverted as follows. If

\[
A_N^{(s)} = \sum_{k=0}^{N-1} a_{N,k} \sigma_k(s),
\]

then one has

\[
a_{N,k} = \sum_{s=0}^{k} A_N^{(s)} \binom{k}{s} (-1)^{k-s}
\]

due to the orthogonality relation:

\[
\sum_{k=m}^{s} \binom{s}{k} \binom{k}{m} (-1)^{k-m} = \delta_{m,s}
\]

This shows why the coefficients in (8.3) are integers. We have no clue as to why they turn out to be positive, implying inequalities among $A_N^{(s)}$ for fixed $N$. 

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The problem of counting the number of integral points in an integral polytope (i.e. a polytope with all its vertices at integer points) is a classical (and difficult) one. We base the following on the work of Ehrhart [9], who, starting from an integral $d$–dimensional polytope $\Pi$ with vertices in $\mathbb{Z}^d$ considered the number of lattice (integral) points in the dilated polytope $s\Pi$, $s$ integer,

$$E_\Pi(s) = \text{card}(\mathbb{Z}^d \cap s\Pi).$$

He showed that this number is a polynomial of degree $d$ in $s$, hence deserves to be called the Ehrhart polynomial of $\Pi$. Our number $A_N^{(s)}$ of admissible tableaux is precisely the Ehrhart polynomial of the $(N-1)$–dimensional polytope $\Pi_N^{(1)}$, defined by the equations (8.1), with $n_i \in \mathbb{R}_+$. In the following, we derive a simple expression for the Ehrhart polynomial of a specific class of polytopes, which we conjecture is valid for $\Pi_N^{(1)}$. We will make use of the Ehrhart reciprocity theorem, which reads as follows. Let $\overline{E}_\Pi(s)$ denote the number of lattice points which lie strictly inside the dilated polytope $s\Pi$, i.e. the number $E_\Pi(s)$ minus the number of lattice points on the boundary $\partial(s\Pi)$. Then one has the reciprocity relation

**Theorem (Ehrhart):**

$$E_\Pi(s) = (-1)^d E_\Pi(-s). \quad (8.4)$$

Given an integral polytope $\Pi$ in $d$ dimensions, we wish to perform a decomposition into “elementary cells”, namely non–flat polyhedral integral simplices with $d+1$ vertices, which do not contain any other lattice point (the intersection of an elementary cell with $\mathbb{Z}^d$ is reduced to its $d+1$ vertices). We have the following

**Proposition 1.** Such a decomposition exists but is in general not unique.

To prove the existence, we rely on a classical theorem which guarantees the existence of a decomposition in any integral polytope. Let us consider a simplex in this decomposition. If it is not an elementary cell, it contains at least one lattice point $P$. If we draw all the edges linking this point $P$ to the $(N+1)$ vertices of the simplex, this defines a decomposition of the simplex into smaller simplices.

**Fig. 4:** The three possible new decompositions of a tetrahedron, according to the position of the point $P$ on an edge (2 tetrahedra), on a face (3 tetrahedra) or strictly inside (4 tetrahedra).
Retaining only the non-flat ones, we end up with $(d + 1)$ simplices if the point $P$ is strictly inside the initial simplex, $d$ simplices if it lies strictly inside a $(d - 1)$ dimensional face,..., 2 simplices if it lies strictly inside a 1 dimensional edge of the initial simplex (we illustrate the case $d = 3$ on Fig.4). We can iterate this procedure with the new decomposition, until the decomposition is into elementary cells.

For any such elementary cell decomposition of $\Pi$, the total numbers of vertices (0–faces), edges (1–faces),..., $k$–faces,..., cells ($d$–faces) are not in general invariant. However, the situation gets much better in the following particular case

**Proposition 2.** If the polytope $\Pi$ is such that one of its elementary cell decompositions is only made of “basic cells”, i.e. elementary cells with volume equal to the minimal value $\frac{1}{d!}$, then the numbers of $j$–faces of the decomposition, which we denote by $F_0$, $F_1$,..., $F_d$, are arithmetical invariants of the polytope $\Pi$. Such a polytope will be called basic in the following.

To prove the above, we will derive a formula for the Ehrhart polynomial of $\Pi$, involving only the $F_j$, $j = 0, ..., d$, which will consequently appear as invariants of $\Pi$. The issue of whether an elementary cell is basic or not becomes relevant in 3 or more dimensions. For $d = 2$, an elementary cell is a triangle with vertices in $\mathbb{Z}^2$, which contains no other lattice point. Any two of its edges define two basis vectors of $\mathbb{Z}^2$, giving rise to a parallelogram of area 1, therefore the triangle, half of the parallelogram, has area $\frac{1}{2}$. So in 2 dimensions, all elementary cells are basic. This is no longer the case in 3 dimensions. Take for instance the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,p,q)$, $p$ and $q$ co-prime, it defines an elementary cell, with volume $\frac{q}{6}$, in general not basic. Nevertheless one checks that the closed tetrahedron intersects the lattice $\mathbb{Z}^3$ only at its vertices! However, in the particular case of $\Pi_4^{(1)}$ (of Fig.3), it is possible to show that it decomposes into 32 elementary cells (tetrahedra) which turn out to be all basic (volume $\frac{1}{6}$). We are thus led to the conjecture that the polytopes $\Pi_N^{(1)}$ are basic, for all $N$.

Before treating the general case of a basic polytope, let us first concentrate on the simplest example of the basic simplex $\Sigma_d$, with vertices at $(n_1, ..., n_d)$, where either all $n_i = 0$, or all $n_i = 0$ except one, which is 1. The simplex itself is its own basic decomposition, and we compute very easily the numbers $F_j^{(d)}$ of $j$–faces

$$F_j^{(d)} = \binom{d + 1}{j + 1} \quad j = 0, 1, ..., d.$$
On the other hand, the computation of the Ehrhart polynomial $E_{\Sigma_d}(s) = \text{card}(s\Sigma_d \cap \mathbb{Z}^d)$ is very simple, we find

$$E_{\Sigma_d}(s) = \sum_{k=0}^{s} \sum_{\substack{n_1, n_2, \ldots, n_d \geq 0 \atop n_1 + n_2 + \ldots + n_d = k}} 1.$$ 

The number of partitions of the integer $k$ into $d$ arbitrary integers is just $\binom{d+k-1}{k}$, and

$$E_{\Sigma_d}(s) = \sum_{k=0}^{s} \binom{d+k-1}{k} = \frac{(s+1)(s+2)(s+3)\ldots(s+d)}{d!},$$

which can be recast into

$$E_{\Sigma_d}(s) = \sum_{j=0}^{d} F_j^{(d)} \binom{s-1}{j}.$$ 

We are now ready to generalize this to any basic polytope $\Pi$. The result reads

**Proposition 3.** Let $\Pi$ be a basic $d$–dimensional integral polytope, and $F_j, j = 0, 1, \ldots, d$ denote the numbers of $j$–faces in a given basic decomposition. Then the Ehrhart polynomial of $\Pi$ reads

$$E_{\Pi}(s) = \sum_{j=0}^{d} F_j^{(d)} \binom{s-1}{j}.$$ 

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**Proposition 3.** Let $\Pi$ be a basic $d$–dimensional integral polytope, and $F_j, j = 0, 1, \ldots, d$ denote the numbers of $j$–faces in a given basic decomposition. Then the Ehrhart polynomial of $\Pi$ reads

$$E_{\Pi}(s) = \sum_{j=0}^{d} F_j^{(d)} \binom{s-1}{j}.$$ 

(8.5)

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**Proposition 3.** Let $\Pi$ be a basic $d$–dimensional integral polytope, and $F_j, j = 0, 1, \ldots, d$ denote the numbers of $j$–faces in a given basic decomposition. Then the Ehrhart polynomial of $\Pi$ reads

$$E_{\Pi}(s) = \sum_{j=0}^{d} F_j^{(d)} \binom{s-1}{j}.$$ 

(8.6)

Proposition 2 above appears as a simple corollary: due to the invariant definition of $E_{\Pi}(s)$, the numbers $F_j$ are also invariant, i.e. do not depend on the particular basic decomposition performed. Note also that the intuitive fact that $E_{\Pi}(s = 0) = 1$ for any polytope $\Pi$, because the zero–dilated polytope is reduced to a point, translates into the non–trivial Euler’s relation linking the numbers $F_j$ to the zero genus of the $d$–dimensional decomposed space

$$\sum_{j=0}^{d} (-1)^j F_j = E_{\Pi}(0) = 1.$$
Let us proceed to the proof of proposition 3. Equation (8.3) was just derived in the case of the particular basic simplex $\Sigma_d$, and extends trivially to any elementary basic cell. For a given polytope $\Pi$, we first perform an elementary cell decomposition as in sect.1. We want to count the number $E_{\Pi}(s)$ of lattice points in the dilated polytope $s\Pi$. This number is equal to a sum of $(d + 1)$ terms.

(0) the number of dilated vertices, $F_0$.

(1) the number of dilated (1 dimensional) edges of the initial elementary basic cell decomposition, $F_1$, multiplied by the number of lattice points which lie strictly inside each dilated edge. This last number is $(s - 1)$, the same for all the dilated edges.

\vdots

(j) the number of dilated $j$–dimensional faces of the initial elementary basic cell decomposition, $F_j$, multiplied by the number of lattice points which lie strictly inside each dilated $j$–face. By Ehrhart reciprocity (8.4), the number of lattice points inside any dilated $j$–face is

$$(-1)^j E_{\Sigma_j}(-s) = (-1)^j \frac{(-s + 1)(-s + 2)\ldots(-s + j)}{j!} = \frac{(s - 1)(s - 2)\ldots(s - j)}{j!} = \binom{s - 1}{j}.$$

\vdots

(d) the number of dilated $d$–dimensional basic cells of the elementary basic decomposition, $F_d$, multiplied by the number of lattice points strictly inside each cell, namely $\frac{(s - 1)(s - 2)\ldots(s - N)}{N!}$.

Summing all these contributions, we get the desired result (8.6)

$$E_{\Pi}(s) = \sum_{j=0}^{d} \binom{s - 1}{j} F_j.$$

When applied to our case of (conjecturally) basic polytopes $\Pi_N^{(1)}$, the above formula (8.6) suggests to rewrite the Ehrhart polynomials $A_N^{(s)}$ in the basis $\nu_k(s) = \frac{(s - 1)(s - 2)\ldots(s - k)}{k!} = \binom{s - 1}{k}$, $\nu_0(s) = 1$ instead of $\sigma_k(s)$ as in (8.3). We get again positive integers, which reinforce our hope that our conjecture might be true. We obtain the following table
A_1^{(s)} = 1
A_2^{(s)} = \nu_1 + 2
A_3^{(s)} = 4\nu_2 + 8\nu_1 + 5
A_4^{(s)} = 32\nu_3 + 77\nu_2 + 60\nu_1 + 16
A_5^{(s)} = 400\nu_4 + 1144\nu_3 + 1148\nu_2 + 462\nu_1 + 59
A_6^{(s)} = 6912\nu_5 + 23040\nu_4 + 28609\nu_3 + 15984\nu_2 + 3749\nu_1 + 247
A_7^{(s)} = 153664\nu_6 + 586432\nu_5 + 870544\nu_4 + 626020\nu_3 + 218946\nu_2 + 31812\nu_1 + 1111
A_8^{(s)} = 4194304\nu_7 + 18055168\nu_6 + 31228736\nu_5 + 27530345\nu_4 + 12893994\nu_3
+ 3007120\nu_2 + 280900\nu_1 + 5302
A_9^{(s)} = 136048896\nu_8 + 652530816\nu_7 + 1289520832\nu_6 + 1347810272\nu_5 + 793600592\nu_4
+ 258004030\nu_3 + 41712394\nu_2 + 2563971\nu_1 + 6376
A_{10}^{(s)} = 5120000000\nu_9 + 27084800000\nu_8 + 60226488576\nu_7 + 72848935552\nu_6
+ 5175214337\nu_5 + 21697638342\nu_4 + 5094740502\nu_3 + 585932117\nu_2 + 24068265\nu_1 + 135670
A_{11}^{(s)} = 21950349144\nu_{10} + 12698544924800\nu_9 + 3138654932736\nu_8 + 4313988572608\nu_7
+ 3592529116864\nu_6 + 1849592168192\nu_5 + 574761304330\nu_4
+ 100122683212\nu_3 + 8340117857\nu_2 + 231333660\nu_1 + 716542.

(8.7)

For \( N = 3 \), we read that the 2–dimensional polytope \( \Pi_3^{(1)} \) has a basic decomposition into 4 basic triangles, with 8 edges and 5 vertices. For \( N = 4 \), we have 32 basic tetrahedra, with 77 triangular faces, 60 edges, and 16 vertices, etc...

In particular, if our conjecture is true, the volume of the polytope \( \Pi_N^{(1)} \) should read

\[ V_N = \frac{F_{N-1}}{(N-1)!} \]

where \( F_{N-1} \) is the number of basic cells of any basic decomposition of \( \Pi_N^{(1)} \). In the remainder of this section, we will derive explicitly the first few leading coefficients of \( A_N^{(s)} \) when \( s \) is large. The leading coefficient is intuitively the volume \( V_N = \frac{2^{N-1}N^{N-3}}{(N-1)!} \) (see appendix E for the calculation) of the polytope \( \Pi_N^{(1)} \). A more surprising feature is that more can be said on the subleading coefficients.

We apply a particular case of the combinatorial Riemann–Roch theorem of Kantor and Khovanski [10]. The latter relates the number of points with integer coordinates in
a given integral polytope to the volume of the polytope deformed by slightly shifting its faces, parallelly to themselves. To simplify notations it proves more convenient to work in \( N \) dimensions, i.e. to consider the polytope \( \Pi_{N+1}^{(1)} \) and its Ehrhart polynomial \( A^{(s)}_{N+1} \).

Let \( V_{N+1}^{(s)}(h_1, \ldots, h_N; \epsilon_1, \ldots, \epsilon_N) \) denote the volume of the deformed polytope defined by the inequalities

\[-2\epsilon_k \leq 2n_k \leq n_{k-1} + n_{k+1} + 2s + h_k \quad k = 1, \ldots, N,\]

with as usual \( n_0 = n_{N+1} = 0 \), and \( \mathcal{T}(x) \) the generating function of Bernoulli numbers

\[\mathcal{T}(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \frac{1}{30240}x^6 + \ldots\]

Consider the polynomial \( B_N(s) \) of degree \( N \) in \( s \)

\[B_N(s) = \left[ \prod_{k=1}^{N} \mathcal{T}(\partial_{h_k})\mathcal{T}(\partial_{\epsilon_k}) \right] V_{N+1}^{(s)}(h_i; \epsilon_i) \bigg|_{h_i=\epsilon_i=0} = \sum_{k=0}^{N} b_k^{(N)} s^k, \quad (8.8)\]

On the other hand, the desired number of points, \( A^{(s)}_{N+1} \), is also a polynomial in \( s \)

\[A^{(s)}_{N+1} = \sum_{k=0}^{N} a_k^{(N)} s^k, \quad (8.9)\]

The theorem relates the coefficients of both polynomials, we have

\[a_N^{(N)} = b_N^{(N)} = V_{N+1}\]
\[a_{N-1}^{(N)} = b_{N-1}^{(N)} = \mu_{N-1}\]
\[a_{N-2}^{(N)} = b_{N-2}^{(N)} + \sum_{F_{N-2}} \mu_{N-2}(F_{N-2})\tau_k(\hat{F}_2), \quad (8.10)\]

where \( V_{N+1} \) is the volume of the polytope \( \Pi_{N+1}^{(1)} \), the last sum extends over the \((N-2)\)–dimensional faces \( F_{N-2} \) of the polytope \( \Pi_{N+1}^{(1)} \), \( \mu_{N-1} \) is the so–called \((N-1)\)–dimensional relative measure of \( \Pi_{N+1}^{(1)} \) (the sum of the \((N-1)\)–dimensional measures of the \((N-1)\)–dimensional faces of \( \Pi_{N+1}^{(1)} \), such that the unit cell of the restriction of \( \mathbb{Z}^N \) to the given face be taken as unity), \( \mu_{N-2}(F_{N-2}) \) the relative \((N-2)\)–dimensional measure of the face \( F_{N-2} \) defined analogously, and \( \tau_2 \) a certain invariant of the two–dimensional polar cone

---

\footnote{This definition is only valid for \( N > 1 \). When \( N = 1 \), one has to take the deformation \(-\epsilon_1 \leq n_1 \leq s + h_1\), in which \( h_1 \) is multiplied by 2 in the definition.}
\( \hat{F}_2 \) to \( F_{N-2} \) (the cone generated by the normals to the two \( (N - 1) \)-dimensional faces of which \( F_{N-2} \) is the intersection).

The crucial hypothesis which enables to apply the above mentioned theorem is that the polytopes we are considering have so-called 2–primitive fans. The fan of a polytope is the set of cones generated by subsets of the (integer valued) normal vectors to its \( (N - 1) \)-dimensional faces: it is a kind of dual envelope of the polytope. The \( k \)–primitivity means that for all \( m \)–dimensional cones of the fan, \( m < k \), the generators of the cone, i.e. a set of \( m \) (integer valued) normals to the \( (N - 1) \)-dimensional faces of the polytope, can be completed into a \( \mathbb{Z} \)–basis of the lattice \( \mathbb{Z}^N \). In our particular case, the normals to the \( 2N \) \( (N - 1) \)-dimensional faces read

\[
(0, ..., 0, -1, 0, ..., 0) - 1 \text{ in } i^{th} \text{ position } i = 1, .., N
\]
\[
(0, ..., -1, 2, -1, 0, ..., 0) \quad \text{2 in } i^{th} \text{ position } i = 1, .., N
\]

It is clear that for instance the two normals

\[(2, -1, 0, ..., 0) \quad \text{and } (0, -1, 0, ..., 0)\]

cannot be completed into a \( \mathbb{Z} \)–basis of \( \mathbb{Z}^N \). Otherwise one could find \( (N - 2) \) vectors of \( \mathbb{Z}^N \) \( v_1, v_2, ..., v_{N-2} \), with integer valued coordinates \( v_i^{(j)} \in \mathbb{Z} \), such that the determinant (volume of the unit cell associated to the basis)

\[
\begin{vmatrix}
2 & 0 & v_1^{(1)} & \cdots & v_{N-2}^{(1)} \\
-1 & -1 & v_1^{(2)} & \cdots & v_{N-2}^{(2)} \\
0 & 0 & v_1^{(3)} & \cdots & v_{N-2}^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & v_1^{(N)} & \cdots & v_{N-2}^{(N)}
\end{vmatrix}
\]

would be equal to one. But after subtracting the second column from the first one, we can factor out 2, hence the determinant cannot be equal to 1. Therefore our polytopes are 2–primitive, as any single normal can be completed into a \( \mathbb{Z} \)–basis of \( \mathbb{Z}^N \).

The above proves rigorously the intuitively obvious fact that the volume \( V_{N+1} \) of the polytope \( \Pi_{N+1}^{(1)} \) appears as leading \( s^N \) term in \( A_{N+1}^{(s)} \), and gives us compact formulae for the next two subleading coefficients.

We summarize the conjectures in the following proposition, and leave the detailed arguments to appendix E.
Proposition 4. For large $s$ the polynomial $A_{N+1}^{(s)}$ behaves like

$$A_{N+1}(s) = a_N^{(N)} s^N + a_{N-1}^{(N)} s^{N-1} + a_{N-2}^{(N)} s^{N-2} + O(s^{N-3}), \quad (8.11)$$

where

$$a_N^{(N)} = V_{N+1} = \frac{2^N(N+1)^{N-2}}{N!}$$

$$a_{N-1}^{(N)} = \frac{N}{4} V_{N+1} + \frac{1}{2} \sum_{k=0}^{N} V_k V_{N-k} = \frac{2^{N-1}(N+1)^{N-4}}{(N-1)!2!}(N^2 + 3N + 8)$$

$$a_{N-2}^{(N)} = b_{N-2}^{(N)} + \frac{1}{4} V_{N-1} = \frac{2^{N-2}}{(N-2)!4!} [(N+1)^{N-6}(3N^4 + 17N^3 + 72N^2 + 144N + 266) + 6(N-1)^{N-4}]. \quad (8.12)$$

Our conjecture that $\Pi_{N+1}^{(1)}$ is basic would imply that the various numbers $F_N$, $F_{N-1}$ and $F_{N-2}$ of respectively basic cells, $(N-1)$–faces and $(N-2)$–faces in the basic cell decomposition of $\Pi_{N+1}^{(1)}$, are given by

$$F_N = 2^N(N+1)^{N-2}$$

$$F_{N-1} = 2^{N-2}(N+1)^{N-4}(2N^3 + 7N^2 + 9N + 10)$$

$$F_{N-2} = \frac{2^{N-4}}{6} [(N+1)^{N-6}(12N^6 + 64N^5 + 143N^4 + 237N^3 + 264N^2 + 212N + 258) + 6(N-1)^{N-4}]. \quad (8.13)$$

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Appendix A. Asymptotic evaluation of $lnZ_N(1)$ using Euler-Mac Laurin’s formula

As indicated in eqn.(2.7) we have

$$Z_N(1) = \prod_{j=1}^{N} j! \quad (A.1)$$
This section is devoted to a proof that asymptotically as $N \to \infty$

$$lnZ_N(1) = \frac{N^2}{2}lnN - \frac{3}{4}N^2 + NlnN + N\left(\frac{1}{2}ln2\pi - 1\right) + \frac{5}{12}lnN$$
$$+ \frac{1 - \gamma + 5ln2\pi}{12} + \frac{\zeta'(2)}{2\pi^2} + \frac{1}{12N} - \frac{1}{720N^2} - \frac{1}{360N^3} + O\left(\frac{1}{N^4}\right).$$

(A.2)

Here $\gamma$ is Euler’s constant, $\zeta(t) = \sum_{n\geq1} n^{-t}$ is Riemann’s zeta–function, and the numerical constant is

$$\frac{1 - \gamma + 5ln2\pi}{12} + \frac{\zeta'(2)}{2\pi^2} = .75351738...$$

(A.3)

Such an expression vindicates the estimate (2.32) with

$$-2f(1) = \frac{1}{2}ln2\pi - 1.$$ 

(A.4)

Note that beyond the term of order $N$ we find corrections of order $lnN$ (and not $\sqrt{N}$ as crudely expected from the thermodynamic reasoning). We guess that this is not peculiar to the case $p = 1$.

We rederive Euler–Mac Laurin’s formula for our purpose [11]. Let $f(z)$ be an analytic function for $Re(z) > 0$ such that in this half plane $|f(x + iy)| = o(e^{2\pi y})$. We have in mind a function such as $f(z) = z^\alpha$, $\alpha$ real positive, and we take the principal determination.

**Fig. 5:** The contour of integration for Euler–Mac Laurin’s formula.

Remarking that in the vicinity of any integer $k$, cotan $\pi z$ behaves as

$$1/\pi (z - k)$$

we have for a sum $I_N$,

$$I_N = f(1) + f(2) + \ldots + f(N) = \frac{1}{2i} \oint_C dz f(z) \cotan \pi z,$$

(A.5)

where the contour $C$ is indicated on figure 5, $\theta$ being an arbitrary real parameter between 0 and 1 and $\Gamma$ a half circle of arbitrary small radius $\rho$. Call $C_+$ and $C_-$ the upper and lower parts of the contour $C$, excluding the arc $\Gamma$ (see fig.5 ). Noting that for any fixed $x$

$$\lim_{|y| \to \infty} \cotan \pi (x + iy) = \frac{1}{i} sgn(y),$$
where $\text{sgn}(y)$ is the sign of $y$, we add and subtract the integrals $\frac{1}{2} \int_{c_{\pm}} f(z) dz$. Thus

\[
I_N = \frac{1}{2i} \int_{c_{+}} dz f(z) \cotan \pi z + i + \frac{1}{2i} \int_{c_{-}} dz f(z) (\cotan \pi z - i) + \frac{1}{2i} \int_{\Gamma} dz f(z) \cotan \pi z - \frac{1}{2} \int_{c_{+}} dz f(z) + \frac{1}{2} \int_{c_{-}} dz f(z).
\]

(A.6)

As $\rho$ shrinks to zero, the third integral over $\Gamma$ tends to $\frac{1}{2} f(N)$, while the fourth and fifth combine to yield the integral of $f$ along the real axis from $\theta$ to $N$. Letting $M \to \infty$, the two horizontal branches in the first and second integral vanish and we are left with

\[
I_N = \int_{\theta}^{N} dx f(x) + \frac{1}{2} f(N) + \int_{\theta}^{\theta+\infty} dz \frac{f(z)}{e^{-2i\pi z} - 1} + \int_{\theta}^{\theta-i\infty} dz \frac{f(z)}{e^{2i\pi z} - 1} + \int_{\theta}^{\theta+i\infty} dz \frac{f(z)}{e^{-2i\pi z} - 1} + \int_{\theta}^{\theta-i\infty} dz \frac{f(z)}{e^{2i\pi z} - 1}.
\]

(A.7)

In the sense of asymptotic series we replace $[f(N + iy) - f(N - iy)]/i$ in the last integral by its Taylor series at $y = 0$ to the effect that

\[
I_N \simeq \int_{\theta}^{N} dx f(x) dx + C(\theta) + \frac{1}{2} f(N) + \sum_{k \geq 0} \frac{2(-1)^k}{(2k+1)!} f^{(2k+1)}(N) \int_{0}^{\infty} dy \frac{y^{2k+1}}{e^{2\pi y} - 1},
\]

(A.8)

This is indeed an asymptotic expansion provided

\[
\lim_{N \to \infty} \frac{f^{(k+1)}(N)}{f^{(k)}(N)} = 0
\]

\[
|f^{(k)}(N + iy)| \leq A e^{(2\pi - \epsilon)|y|},
\]

which will be satisfied in our applications. By using a specific function $f(z) = e^{tz}$ and comparing $I_{N+1} - I_N = f(N)$ to its asymptotic expansion, we find for the coefficient of $f^{(2k+1)}(N)$

\[
\frac{2(-1)^k}{(2k+1)!} \int_{0}^{\infty} dy \frac{y^{2k+1}}{e^{2\pi y} - 1} = \frac{B_{2k+2}}{(2k+2)!},
\]

(A.9)

where the Bernoulli numbers $B_k$ are defined through

\[
\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{t^k B_k}{k!}
\]

(A.10)

and vanish for $k$ odd larger than 1. If $f(0)$ exists, one can let $\theta \to 0$ in which case

\[
C(0) = -\frac{1}{2} f(0) + \int_{0}^{\infty} dy \frac{f(iy) - f(-iy)}{i e^{2\pi y} - 1}.
\]

(A.11)
On applying eqn. (A.8) to \( f(z) = z^\alpha, \alpha > 0, \) one obtains

\[
\sum_{j=1}^{N} j^\alpha \simeq \frac{N^{\alpha+1}}{\alpha + 1} + \frac{N^\alpha}{2} + C_\alpha(\theta) \frac{\theta^\alpha}{\alpha + 1} + \sum_{k \geq 0} \frac{B_{2k+2}}{(2k+2)!} \alpha^{\alpha-1} \ldots (\alpha - 2k) N^{\alpha-2k-1},
\]

(A.12)

where

\[
C_\alpha(\theta) - \frac{\theta^\alpha}{\alpha + 1} = C_\alpha(0) = -2 \sin \frac{\pi \alpha}{2} \int_0^\infty dy \frac{y^\alpha}{e^{2\pi y} - 1}
\]

\[
= -2 \sin \frac{\pi \alpha}{2} \sum_{m=1}^{\infty} \int_0^\infty dy y^\alpha e^{-2m\pi y}
\]

\[
= -2 \sin \frac{\pi \alpha}{2} \sum_{m=1}^{\infty} \frac{\Gamma(\alpha + 1)}{(2\pi m)^{\alpha+1}}
\]

\[
= -2 \sin \frac{\pi \alpha}{2} \frac{\Gamma(\alpha + 1)}{(2\pi)^{\alpha+1}} \zeta(\alpha + 1)
\]

\[
= \zeta(-\alpha),
\]

since the zeta function satisfies the reciprocity relation

\[
\zeta(s) = 2\Gamma(1-s)(2\pi)^{s-1} \sin \frac{\pi s}{2} \zeta(1-s).
\]

(A.13)

The case \( f(z) = \ln z \) obtained by taking a derivative at \( \alpha = 0 \) of the previous estimate yields Stirling’s formula as well as \( \partial_\alpha C_\alpha(0)|_{\alpha = 0} = -\zeta'(0) = \frac{1}{2} \ln 2\pi. \) Thus

\[
\ln N! \simeq (N + \frac{1}{2}) \ln N - N + \frac{1}{2} \ln 2\pi + \sum_{k \geq 0} \frac{B_{2k+2}N^{-2k-1}}{(2k+1)(2k+2)},
\]

(A.15)

where of course the asymptotic series diverges! Similarly when \( f(z) = z\ln z \) we take a derivative at \( \alpha = 1, \) use \( \Gamma(2) = 1, \Gamma'(2) = 1 - \gamma, \zeta(2) = \frac{\pi^2}{6} \) and find

\[
\sum_{j=1}^{N} j \ln j \simeq \frac{N^2}{2} \ln N - \frac{N^2}{4} + \frac{N}{2} \ln N + \frac{\gamma + \ln 2\pi}{12} - \frac{\zeta'(2)}{2\pi^2}
\]

\[
- \sum_{k \geq 1} \frac{B_{2k+2}N^{-2k}}{2k(2k+1)(2k+2)}.
\]

(A.16)
Finally from equations (A.15) and (A.16)

\[ \ln Z_N(1) = \ln \prod_{j=1}^{N} j! \]

\[ = \sum_{j=1}^{N} (N + 1 - j) \ln j \]

\[ = (N + 1) \ln N! - \sum_{j=1}^{N} j \ln j \] \hspace{1cm} (A.17)

\[ = \frac{N^2}{2} \ln N - \frac{3}{4} N^2 + N \ln N + N \left( \frac{\ln 2\pi}{2} - 1 \right) + \frac{5}{12} \ln N \]

\[ + \frac{1 - \gamma + 5 \ln 2\pi}{12} + \frac{\zeta'(2)}{2\pi^2} + \frac{1}{12N} - \frac{1}{720N^2} - \frac{1}{360N^3} + O\left( \frac{1}{N^4} \right), \]

as stated in equation (A.2).

In view of the above remarks, we add here the following estimate

\[ \ln \prod_{j=1}^{N} (jp)! = \frac{1}{p} \ln \prod_{j=1}^{Np} j! + \frac{p+1}{2} \ln (p^N N!) - \frac{1}{p} \ln ((Np)!) + O(\ln N). \] \hspace{1cm} (A.18)

The first term on the r.h.s. is \( \frac{1}{p} \ln Z_{Np}(1) \). Indeed

\[ \ln \prod_{j=1}^{N} (jp)! = \sum_{r=0}^{N-1} \sum_{k=1}^{p} (N - r) \ln (k + rp) \]

\[ = \frac{1}{p} \left[ Np \sum_{q=1}^{Np} \ln q - \sum_{r=0}^{N-1} \sum_{k=1}^{p} (rp + k - k) \ln (k + rp) \right] \]

\[ = \frac{1}{p} \left[ \sum_{q=1}^{Np} (Np - q) \ln q + \sum_{k=1}^{N-1} \sum_{r=0}^{p} k \ln (k + rp) \right] \]

\[ = \frac{1}{p} \ln \prod_{j=1}^{Np} j! + R_N(p), \]

where

\[ R_N(p) = \frac{1}{p} \sum_{k=1}^{p} \sum_{r=0}^{N-1} \ln (k + rp). \] \hspace{1cm} (A.19)

With \( 1 \leq k \leq p \) we have

\[ \sum_{r=1}^{N-1} \ln (rp) \leq \sum_{r=0}^{N-1} \ln (k + rp) \leq \sum_{r=1}^{N} \ln (rp) \] \hspace{1cm} (A.20)
thus
\[ \frac{p+1}{2} \ln(p^{N-1}(N-1)!) \leq R_N(p) \leq \frac{p+1}{2} \ln(p^N N!) \] (A.22)
which yields the required result \( (A.18) \).

**Appendix B. Instability of the cuboctahedron configuration**

In the stereographic representation of the sphere on the plane \( \mathcal{C} \), the equilibrium configurations, stable or unstable, \( \{z_j\} \) are solutions of the equations \( \frac{\partial V}{\partial z_i} = \frac{\partial V}{\partial \bar{z}_i} = 0 \), \( \forall i \), that is
\[ \sum_{j=1 \atop j \neq i}^{N} \frac{1}{z_i - z_j} = (N-1) \frac{\bar{z}_i}{1 + z_i \bar{z}_i}, \quad i = 1, 2, \ldots, N \] (B.1)
expressing that the potential
\[ V = -\sum_{i<j} \ln z_{ij} \bar{z}_{ij} + (N-1) \sum_j \ln(1 + \bar{z}_j z_j) \] (B.2)
is stationary. Since \( V \) is invariant under an overall rotation of the configuration, that is by a unitary homographic substitution
\[ z_j \to \xi_j = \frac{\alpha z_j + \gamma}{-\bar{\gamma} z_j + \bar{\alpha}} \]
\[ \alpha \bar{\alpha} + \gamma \bar{\gamma} = 1 \] (B.3)
the system (B.1) is formally invariant in this substitution. We can easily then show the existence of equilibrium configurations on the sphere defined not only by the regular polyhedron but by their composites in diverse polyhedra, so that we have by symmetry
\[ \sum_{j=1}^{N-1} \frac{1}{\xi_j} = 0 \] (B.4)
by putting the vertex \( \xi_N \) at the pole (\( \xi_N = 0 \)).

Around a solution \( \{z_j\} \) of (B.1) the potential is approximated, up to third order in the variations \( \{\delta z_j\} \) by the real quadratic form \( Q(\delta z, \delta \bar{z}) \)
\[ 2Q \equiv \sum_{i,j} A_{ij} \delta z_i \delta z_j + \bar{A}_{ij} \delta \bar{z}_i \delta \bar{z}_j + 2D_i \delta z_i \delta \bar{z}_i \] (B.5)

66
with
\[ A_{ij} = \frac{\partial^2 V}{\partial z_i \partial z_j} = -\frac{1}{(z_i - z_j)^2} + \delta_{ij} \left( \sum_{l \neq i} \frac{1}{(z_l - z_j)^2} - (N - 1) \frac{z_i^2}{(1 + z_i \bar{z}_i)^2} \right) \]  
(B.6)

\[ D_i = \frac{\partial^2 V}{\partial z_i \partial \bar{z}_i} = \frac{N - 1}{(1 + z_i \bar{z}_i)^2} \]  
(B.7)

This is a real form in the 2N variables \( \delta x_j, \delta y_j \) (\( \delta z = \delta x + i \delta y \)) with rank not greater than 2N - 3 since the variations \( \{ \delta z_i \} \) associated with an overall rotation of the configuration \( \{ z_i \} \) annihilate \( Q \) and depends on 3 real parameters as shown by (B.3). The configuration is stable if \( Q \) is semi-positive.

To prove that the configuration 12(24) of the cuboctahedron is unstable we must show the existence of a negative eigenvalue of the real 2N \( \times \) 2N symmetric matrix
\[ \begin{bmatrix} D + ReA & -ImA \\ -ImA & D - ReA \end{bmatrix} \]  
(B.8)

but the spectrum of this matrix is not invariant under the rotations. However the signature is invariant and this is all we need. To obtain the intrinsic modes of vibration we consider the form \( Q(\delta u, \delta \bar{u}) \)
\[ 2Q \equiv \sum B_{ij} \delta u_i \delta u_j + c.c. + 2\delta u_i \delta \bar{u}_i \]  
(B.9)

with
\[ B_{ij} = \frac{1}{\sqrt{D_i D_j}} A_{ij} = B_{ji} \]  
(B.10)

and
\[ \delta u_i = \sqrt{D_i} \delta z_i = \sqrt{N - 1} \frac{\delta z_i}{1 + z_i \bar{z}_i} \]  
(B.11)

where
\[ \frac{4}{N - 1} \delta u \delta \bar{u} = \frac{1}{4} \frac{\delta z \delta \bar{z}}{(1 + z \bar{z})^2} = \sin^2 \theta d\phi^2 + d\theta^2 \]  
(B.12)

is the invariant metric on the sphere. The secular equation associated with (B.10) reads
\[ B_{ij} \delta u_j + (1 - \lambda) \delta \bar{u}_i = 0 \]  
(B.13)

and its conjugate. Denoting by \( \rho^2 \) the spectrum of the non-negative hermitian matrix \( BB \) of order \( N \), we have the 2N eigenvalues
\[ \lambda_n = 1 - \rho_n \text{ and } 2 - \lambda_n = 1 + \rho_n \]  
(B.14)
and the diagonal form of $Q$

$$Q \simeq \sum_{n=1}^{N} (1 + \rho_n)\delta \xi_n^2 + (1 - \rho_n)\delta \eta_n^2$$ (B.15)

With the exception of the three zero modes ($\rho_{1,2,3} = 1$), the configuration is stable if $\rho_n^2 < 1, \forall n > 3$ and it is unstable if $\max_n \rho_n^2 > 1$.

The spectrum of $B\bar{B}$ is invariant under (B.3) since the hessian, at the stationary point, is transforming as

$$\frac{\partial^2 V}{\partial z_i \partial z_j} = \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} \frac{d \xi_i}{d z_i} \frac{d \xi_j}{d z_j}$$ (B.16)

with

$$\left| \frac{d \xi}{d z} \right| = \frac{1}{|\alpha - \gamma z|^2} = \frac{1 + \xi \bar{\xi}}{1 + z \bar{z}}$$ (B.17)

Consequently if we call $\chi_j$ the argument of $\frac{d \xi_i}{d z_j}$, the similarity property

$$B\bar{B}(x) = e^{ix} B\bar{B}(z)e^{-ix}$$ (B.18)

implies the invariance of the spectrum. The eigenvalue equation can also be written

$$A\delta z + (1 - \lambda)D\delta \bar{z} = 0, \quad \text{and \ c.c.}$$ (B.19)

and differs from the naive form related to (B.8) which was written $A\delta z + (D - \lambda)\delta \bar{z} = 0$.

Explicitly, equation (B.19) reads

$$\sum_{j=1}^{N} \frac{\delta z_i - \delta z_j}{z_{ij}^2} + \frac{N - 1}{(1 + z_i \bar{z}_i)^2} ((1 - \lambda)\delta \bar{z}_i - z_i^2 \delta z_i) = 0$$ (B.20)

With the help of (B.1), it can be reexpressed using $\delta z_j = z_j \xi_j$ as

$$\sum_{j \neq i} \frac{z_i z_j}{z_{ij}^2} (\xi_i - \xi_j) + \frac{N - 1 z_i \bar{z}_i}{(1 + z_i \bar{z}_i)^2} (\xi_i - \rho \bar{\xi}_i) = 0$$ (B.21)

The 3 zero modes ($\rho = 1$) are associated with the 3 eigenvectors

$$\delta z_j = i z_j \delta (\text{Arg} \alpha) \quad \text{or} \quad \xi_j \propto i$$

$$\delta z_j = (z_j^2 + 1) \delta (\text{Re} \gamma) \quad \text{or} \quad \xi_j \propto (z_j + z_j^{-1})$$ (B.22)

$$\delta z_j = i(z_j^2 - 1) \delta (\text{Im} \gamma) \quad \text{or} \quad \xi_j \propto i(z_j - z_j^{-1})$$
Let us exhibit one negative eigenvalue in the case of the cuboctahedron. The invariance of the system \([\text{B.21]}\) by the octahedral group reduces the degree of the equations to 3 or 4. In the cyclic representation of order \(m = 4\), that is taking as a polar axis \(m\) the axis having a quaternary symmetry, the \(z_j\)'s separate in 4 cosets: \(\sqrt{i}^{\nu}, ai^{\nu}, a^{-1}i^{\nu}\) with \(\nu = 1, 2, 3, 4(\text{mod} \ 4)\) belonging to 3 modules

\[
1, a, a^{-1}, \quad a = \sqrt{2} - 1 = tg\frac{\pi}{8}
\]

so that

\[
f(z) = \prod_{1}^{12}(z - z_j) \equiv (z^4 + 1)(z^4 - a^4)(z^4 - a^{-4})
\]

\[
= z^{12} + 1 - 33(z^8 + z^4)
\] \(\text{(B.23)}\)

We can also choose the ternary representation in 4 cosets

\[
ij^{\nu}, -ij^{\nu}, -b j^{\nu}, \ b^{-1} j^{\nu}, \ \nu = 0, 1, 2 \text{ mod } 3
\]

\[
b = \sqrt{3} - \sqrt{2} = tg\frac{\chi}{2}
\] \(\text{(B.24)}\)

with \(\chi\) the angle between ternary and binary axis. This yields the polynomial

\[
g(\xi) = (\xi^6 + 1)(\xi^3 + b^3)(\xi^3 - b^{-3})
\]

\[
\equiv \xi^{12} - 1 - 22\sqrt{3}(\xi^9 + \xi^3)
\] \(\text{(B.25)}\)

We switch from \(z\) to \(\xi\) by the rotation

\[
\xi = \frac{z\sqrt{i} + \rho}{1 - \rho z\sqrt{i}}, \quad \rho = tg\frac{\psi}{2} = \frac{\sqrt{3} - 1}{\sqrt{2}} = 2\sin\frac{\pi}{12}
\] \(\text{(B.26)}\)

where \(\psi\) is the angle between ternary and quaternary axis. Although we did not want to do the complete reduction according to the octahedral group, let us note that this is the group of permutations of the 4 diagonals of the cube named 1, 2, 3, 4. The elements of the proper group fall in various classes according to the table

| classes | (1) | (13) | (134) | (13)(24) | (1234) |
|---------|-----|------|------|---------|-------|
| order   | 1   | 2    | 3    | 2       | 4     |
| number of elements | 1  | 6    | 8    | 3       | 6     |
| name    | \(E\) | \(J^{-1}\) | \(I^2\) | \(I\)   |
| realization | \(z \to \frac{i}{z}\) | \(z \to \frac{z-i}{z+i}\) | \(z \to -z\) | \(z \to iz\) |
And the reflection with respect to the center is \( z \rightarrow -\frac{1}{z} \).

We are now going to seek the proper modes (B.21) of the system corresponding to invariant vectors by \( I(J) \). The form (B.21) is adapted to Fourier transformation. We limit ourselves to the zero axial moment \( \xi_j \equiv \xi_c \) where \( c \) is the index of the coset of \( j \)

\[
c = \sqrt{i}, \ a, \ a^{-1} \text{ for } I m = 4 \\
c = i, \ i^{-1}, \ -b, \ b^{-1} \text{ for } J m = 3
\]

We only need to compute

\[
\sum_{j \in b} \frac{z_i z_j}{(z_i - z_j)^2} = m^2 \frac{a^m b^m}{(a^m - b^m)^2}; \ i \in a
\]  \( \text{(B.27)} \)

where \( a, b \) distinguishes here 2 indices of arbitrary cosets. The system (B.21) is then reduced to

\[
\sum_b (\xi_a - \xi_b) \frac{m^2 a^m b^m}{(a^m - b^m)^2} + \frac{N - 1}{(|a| + |a^{-1}|)^2}(\xi_a - \rho \xi_a) = 0
\]

\( \text{(B.28)} \)

In the case \( m = 4 \) the coefficients are all real because of (B.23)-(B.25) and the system decouples in \( Re\xi_a = 0 \) or \( Im\xi_a = 0 \) according to the sign of \( \rho \). We are left with one equation of second degree in \( \lambda \), after elimination of the root \( \lambda = 0 \) in the space of cosets

\[
A = 16 \begin{vmatrix} 
\frac{11}{36} & \frac{-1}{36} & \frac{-1}{36} \\
\frac{1}{36} & \frac{36 \times 32}{2 \times 24^2} & \frac{36 \times 32}{2 \times 24^2} \\
\frac{1}{36} & \frac{1}{36} & \frac{36 \times 32}{2 \times 24^2} \\
\end{vmatrix} \\
D = \frac{11}{8} \begin{vmatrix} 
2 & \cdot & \cdot \\
\cdot & 1 & \cdot \\
\cdot & \cdot & 1 \\
\end{vmatrix}
\]

\( \text{(B.29)} \)

If we write \( \lambda' = \frac{9 \times 11}{2^7} \lambda \), the secular equation

\[
\begin{vmatrix} 
2 - 2\lambda' & -1 & -1 \\
-1 & 1 - \lambda' - 2^{-5} & 2^{-5} \\
-1 & 2^{-5} & 1 - \lambda' - 2^{-5} \\
\end{vmatrix} = 0
\]

\( \text{(B.30)} \)

has the three roots \( \lambda' = 0, 2, 1 - 2^{-4} \) which give the five stable modes

| \( \lambda \) | \( 2 - \lambda \) | \( \rho \) |
|---|---|---|
| * | 2 | 1 |
| 10 | 56 | 33 | 33 |
| 33 | 134 | 35 | 99 |
| 64 | 99 | 99 | 99 |
We would obtain the other modes relative to the moments $|k| = 2, 1$ with the ansatz
\[ \xi_j = \xi_c i^{k \nu}, \quad j \equiv (c, \nu), \quad \nu = 0, 1, 2, 3 \text{(mod 4)} \]  
(B.31)
but we will now turn to the ternary representation $m = 3$ limiting ourselves again to the total zero moment (states invariant under $J$). Taking into account (B.28), naming $\xi_{1,2,3,4}$ the components of the vector $\xi$ relative to the 4 cosets (axial triangles) $i, -i, -b, b^{-1}$, the equations (B.28) reads
\[ \begin{align*}
-\frac{1}{4} (\xi_1 - \xi_2) - \frac{1}{2} i 3\overline{5} (1 + 11i\sqrt{2}) (2\xi_1 - \xi_3 - \xi_4) + \frac{11}{36} (\xi_1 - \rho \xi) &= 0 \\
-\frac{1}{4} (\xi_1 - \xi_2) - \frac{1}{2} i 3\overline{5} (1 - 11i\sqrt{2}) (2\xi_2 - \xi_3 - \xi_4) + \frac{11}{36} (\xi_2 - \rho \bar{\xi}) &= 0 \\
-\frac{1}{4} i 3\overline{5} (\xi_3 - \xi_4) - \frac{1}{2} i 2\overline{3} ((2\xi_3 - \xi_1 - \xi_2) - 11i\sqrt{2}(\xi_1 - \xi_2)) + \frac{11}{36} (\xi_3 - \rho \bar{\xi}) &= 0 \\
-\frac{1}{4} i 3\overline{5} (\xi_3 - \xi_4) - \frac{1}{2} i 2\overline{3} ((2\xi_4 - \xi_1 - \xi_2) - 11i\sqrt{2}(\xi_1 - \xi_2)) + \frac{11}{36} (\xi_4 - \rho \bar{\xi}) &= 0
\end{align*} \]  
(B.32)
Setting
\[ \xi_1 + \xi_2 = \xi, \quad \xi_3 + \xi_4 = \eta \]  
(B.33)
The ternary system (B.32) decouples, as predicted, according to whether the variables (B.33) are all real or all imaginary corresponding to 2 values of opposite sign of $\rho$ and can be written
\[ \begin{align*}
-\frac{1}{4} i 3\overline{5} (\eta - \xi - 11i\sqrt{2} \xi) &= \frac{11}{4 \times 3\overline{2}} (\xi - \rho \bar{\xi}) \\
-\frac{1}{4} i 3\overline{5} (\eta - \xi + 11i\sqrt{2} \bar{\xi}) &= \frac{11}{4 \times 3\overline{2}} (\bar{\eta} - \rho \bar{\xi}) \\
\left(\frac{1}{2} + \frac{1}{3\overline{5}}\right) \xi' - \frac{11i\sqrt{2}}{3\overline{5}} (\xi - \eta) &= \frac{11}{4 \times 3\overline{2}} (\xi' + \rho \bar{\xi}') \\
\frac{1}{2} 3\overline{4} \eta' &= \frac{11}{4 \times 3\overline{2}} (\eta' - \rho \bar{\eta}')
\end{align*} \]  
(B.34)
which gives assuming that $Re(\xi, ..., \eta') = 0$, the zero root $\lambda = 0$ ($\rho = -1$), the obvious root $\lambda = \frac{11}{3 \times 11}$ and two other solutions of an equation of second degree in $\mu = \frac{11 \times 3^2}{4} \lambda$
\[ (\mu - 4)(\mu - 26) - 8 \times 11^2 = 0 \]  
(B.35)
So $\mu = 15 \pm 33$. We therefore obtain 7 eigenvalues of which one is negative

| \[ \lambda \] | \[ 2 - \lambda \] |
|-----------------|----------------|
| \[ \frac{2}{33} \] | 0.0606... | 1.9393... |
| \[ \frac{2^6}{11 \times 3^2} \] | 0.6464... | 1.3535... |
| \[ -\frac{2^3}{11 \times 3} \] | -0.2424... | 2.2424... |
We note that the 15 eigenvalues computed (from the 21 non zero ones) are rational numbers. Would this be a general property of hessian matrices of equilibrium configuration on the sphere? We should first check this on the icosahedron.

It is interesting to know the direction of the unstable ternary mode \( \lambda = \frac{-8}{33} = -0.2424\ldots \) and also that of the stable mode \( \lambda = \frac{2}{33} = 0.0606\ldots \). With the help of (B.34) we obtain for the unstable ternary mode the proportion

\[
\text{Im}(\xi : \xi' : \eta : \eta') = 1 : -\sqrt{2} : -3 : 0
\]

and for the less stable only \( \text{Im} \eta' \) is non zero. This gives according to (B.33) for the unstable mode

\[
\begin{align*}
\xi_3 &= \xi_4 = -1.5i \\
\xi_1 &= 0.5i - 0.707\ldots \\
\xi_2 &= 0.5i + 0.707\ldots 
\end{align*}
\]

and for the stable mode

\[
\begin{align*}
\xi_1 &= \xi_2 = 0, \\
\xi_3 &= -\xi_4 = i 
\end{align*}
\]

The elementary displacements \( \delta z \) in \( C \) are such that \( \delta z = \xi z \) according to (B.21). To interpret the result (B.35) we see that it is possible, by a global rotation around the ternary axis (\( \delta z_j/z_j = 1.5i \)), to keep fixed the two cosets \(-b\) and \(b^{-1}\) made of the two opposite triangles of the cuboctahedron centered on the ternary axis. Finally this gives us the displacements

\[
\begin{align*}
\delta z_3 = \delta z_4 &= 0 \\
\delta z_1/z_1 &= 2i - \frac{1}{\sqrt{2}} \\
\frac{\delta z_2}{z_2} &= 2i + \frac{1}{\sqrt{2}}
\end{align*}
\]

which describe a deformation of the plane hexagon of the cuboctahedron made of i) a rotation of the hexagon (\( \frac{\delta z}{z} = 2i \)) ii) a deformation of the latter by dedoubling of the plane of the two cosets \( i \) and \(-i\). The value of the angle (Arctg(1/(2\sqrt{2}))) and the direction indicated on the figure suggests that this unstable ternary mode describes well the beginning of the displacement of the cuboctahedron towards the stable icosahedron, the two opposite triangles staying the same to first order. The less stable mode (B.38) is related to the vibration of torsion (opposite rotations) of only these two triangles, the plane hexagon remaining fixed.

**Fig. 6:** Going from the cuboctahedron to the icosahedron
Appendix C. Shiota’s Formula

It is possible to write a formula for $\Delta^{2s}$ in terms of the traces $t_l = \sum_{1 \leq i \leq N} z_i^l$. Noting that

$$\Delta^{2s} = \prod_{1 \leq i < j \leq N} (z_i - z_j)^{2s}$$  \hspace{1cm} (C.1)

is $(-1)^{sN(N-1)/2}$ times the coefficient of $\epsilon^{N(N-1)}$ in

$$\Pi := \prod_{1 \leq i \leq N \atop 1 \leq j \leq N} (1 - \epsilon(z_i - z_j)^s)$$  \hspace{1cm} (C.2)

where $\epsilon$ is a formal scalar parameter close to zero. Let us compute this quantity

$$\Pi = \exp \left( \sum_{1 \leq i \leq N \atop 1 \leq j \leq N} \log(1 - \epsilon(z_i - z_j)^s) \right)$$

$$= \exp \left( -\sum_{n=1}^{\infty} \frac{\epsilon^n}{n} \sum_{1 \leq i \leq N \atop 1 \leq j \leq N} (z_i - z_j)^{ns} \right)$$  \hspace{1cm} (C.3)

$$= \exp \left( -\sum_{n=1}^{\infty} \frac{\epsilon^n}{n} \sum_{l=0}^{ns} \binom{ns}{l} \sum_{1 \leq i \leq N \atop 1 \leq j \leq N} z_i^{n_s-l}(-z_j)^l \right)$$

$$= \exp \left( -\sum_{n=1}^{\infty} \frac{\epsilon^n}{n} \sum_{l=0}^{ns} \binom{ns}{l} (-1)^l t_{ns-l} t_l \right)$$

Taylor expanding the exponential we have

$$\Pi = \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\sum_{n=1}^{\infty} \frac{\epsilon^n}{n} \sum_{l=0}^{ns} \binom{ns}{l} (-1)^l t_{ns-l} t_l \right)^k$$  \hspace{1cm} (C.4)

Taking $(-1)^{sN(N-1)/2}$ times the coefficient of $\epsilon^{N(N-1)}$ we obtain

$$\Delta^{2s} = (-1)^{sN(N-1)/2} \sum_{k \geq 1} \frac{(-1)^k}{k!} \sum_{n_1+\ldots+n_k=N(N-1)} \prod_{i=1}^{k} \frac{1}{2n_i} \sum_{l=0}^{n_i s} \binom{n_i s}{l} (-1)^l t_{ni_s-l} t_l$$  \hspace{1cm} (C.5)

(summing over $k \leq N(N-1)$ suffices since the $n_i \geq 1$ must add up to $N(N-1)$), or

$$\Delta^{2s} = (-1)^{sN(N-1)/2} \times \sum_{k_1+k_2+\ldots \geq 0 \atop k_1+2k_2+3k_3+\ldots = N(N-1)} \prod_{k_1,k_2,\ldots \geq 0} \frac{(-1)^{k_1+k_2+\ldots}}{k_1! k_2! \ldots} \prod_{n \geq 1} \left( \frac{1}{n} \sum_{l=0}^{ns} \binom{ns}{l} (-1)^l t_{ns-l} t_l \right)^{k_n}$$  \hspace{1cm} (C.6)
Appendix D. Decomposition in terms of characters for small $N$

The decomposition of the square of the Vandermonde determinant in terms of characters can be written in the following form,

$$
\Delta^2 = \prod (1 - \tau_i)^3(1 - \tau_i \tau_{i+1})^3 \cdots (1 - \tau_i \cdots \tau_{N-1})^3 \bigg|_{adm} \ ch_{0,3}, \ldots, 3(N-2), 3(N-1) \tag{D.1}
$$

where the operators $\tau_i$ are acting on the labels of the characters at position $i$ shifting $l_{i-1}$ upward by one and $l_i$ downward by one. Doing the decomposition explicitly we find

$N=2$ (2 terms)

$$
\Delta^2 = -3ch_{1,2} + ch_{0,3}. \tag{D.2}
$$

$N=3$ (5 terms)

$$
\Delta^2 = -15ch_{2,3,4} + 6ch_{1,3,5} - 3ch_{0,4,5} - 3ch_{1,2,6} + ch_{0,3,6}. \tag{D.3}
$$

$N=4$ (16 terms)

$$
\Delta^2 = 105ch_{3,4,5,6} - 45ch_{2,4,5,7} - 6ch_{2,3,6,7} + 27ch_{1,4,6,7} \\
- 15ch_{0,5,6,7} + 27ch_{2,3,5,8} - 9ch_{1,4,5,8} - 12ch_{1,3,6,8} \\
+ 6ch_{0,4,6,8} + 9ch_{1,2,7,8} - 3ch_{0,3,7,8} - 15ch_{2,3,4,9} \\
+ 6ch_{1,3,5,9} - 3ch_{0,4,5,9} - 3ch_{1,2,6,9} + ch_{0,3,6,9}. \tag{D.4}
$$
\[ \Delta^2 = 945c h_{4,5,6,7,8} - 420c h_{3,5,6,7,9} - 75c h_{3,4,6,8,9} + 270c h_{2,5,6,8,9} + 45c h_{2,4,7,8,9} - 180c h_{1,5,7,8,9} + 105c h_{0,6,7,8,9} + 270c h_{3,4,6,7,10} - 90c h_{2,5,6,7,10} + 45c h_{3,4,5,8,10} - 144c h_{2,4,6,8,10} + 72c h_{1,5,6,8,10} - 18c h_{2,3,7,8,10} + 81c h_{1,4,7,8,10} - 45c h_{0,5,7,8,10} - 18c h_{2,4,5,9,10} + 111c h_{2,3,6,9,10} - 27c h_{1,4,6,9,10} - 6c h_{0,5,6,9,10} - 54c h_{1,3,7,9,10} + 27c h_{0,4,7,9,10} + 45c h_{1,2,8,9,10} - 15c h_{0,3,8,9,10} - 180c h_{3,4,5,7,11} + 72c h_{2,4,6,7,11} - 36c h_{1,5,6,7,11} + 81c h_{2,4,5,8,11} - 27c h_{2,3,6,8,11} - 36c h_{1,4,6,8,11} + 27c h_{0,5,6,8,11} + 18c h_{1,3,7,8,11} - 9c h_{0,4,7,8,11} - 54c h_{2,3,5,9,11} + 18c h_{1,4,5,9,11} + 24c h_{1,3,6,9,11} - 12c h_{0,4,6,9,11} - 18c h_{1,2,7,9,11} + 6c h_{0,3,7,9,11} + 45c h_{2,3,4,10,11} - 18c h_{1,3,5,10,11} + 9c h_{0,4,5,10,11} + 9c h_{1,2,6,10,11} - 3c h_{0,3,6,10,11} + 105c h_{3,4,5,6,12} - 45c h_{2,4,5,7,12} - 6c h_{2,3,6,7,12} + 27c h_{1,4,6,7,12} - 15c h_{0,5,6,7,12} + 27c h_{2,3,5,8,12} - 9c h_{1,4,5,8,12} - 12c h_{1,3,6,8,12} + 6c h_{0,4,6,8,12} + 9c h_{1,2,7,8,12} - 3c h_{0,3,7,8,12} - 15c h_{2,3,4,9,12} + 6c h_{1,3,5,9,12} - 3c h_{0,4,5,9,12} - 3c h_{1,2,6,9,12} + c h_{0,3,6,9,12}. \]  

(D.5)

If we label the coefficients by \( C(\tau_i \ldots \tau_j) \) the following simple rules can be found directly by looking at (D.1),

\[
C(\tau_i \tau_{i+1} \ldots \tau_{i+n}) = C(\tau_j \tau_{j+1} \ldots \tau_{j+n}) \tag{D.6}
\]

\[
C(\tau_i \ldots \tau_{i+p}) = -2C(\tau_{i+1} \ldots \tau_{i+p}) \tag{D.7}
\]

We also read from these examples the following list of coefficients,

\[
C(\tau_i) = -3 \tag{D.8}
\]

\[
C(\tau_i^2) = 3 \tag{D.9}
\]

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\[ C(\tau_i \cdots \tau_{i+n}) = -3(-2)^n \]  
(D.10)

\[ C(\tau_i^2 \cdots \tau_{i+n}^2) = 3(4^n - 9n) \]  
(D.11)

\[ C(\tau_i \tau_{i+1} \cdots \tau_{n+i-1}) = -3(2^{2n-3} - \frac{27}{2} n + \frac{99}{4} + \frac{1}{4}(-5)^{n-2}) \]  
(D.12)

\[ C(\tau_i^2 \tau_{i+1} \cdots \tau_{n+i-1}^2) = 162 + (24 - \frac{27}{4} n)(-5)^{n-4} + 3 \cdot 2^{2n-4} - \frac{243}{4} n \]  
(D.13)

For higher powers of the Vandermonde determinant we found the following decompositions in terms of characters.

**Fourth power**

N=2 (3 terms)

\[ \Delta^4 = ch_{0,5} - 5ch_{1,4} + 10ch_{2,3} \]  
(D.14)

N=3 (13 terms)

\[ \Delta^4 = ch_{0,5,10} - 5ch_{1,4,10} - 5ch_{0,6,9} + 10ch_{2,3,10} + 10ch_{0,7,8} - 15ch_{2,5,8} + 20ch_{1,5,9} - 25ch_{1,6,8} - 25ch_{2,4,9} + 100ch_{3,4,8} + 100ch_{2,6,7} - 160ch_{3,5,7} + 280ch_{4,5,6} \]  
(D.15)
\[ \Delta^4 = ch_{0,5,10,15} - 5ch_{0,5,11,14} + 10ch_{0,5,12,13} \]
\[- 5ch_{0,6,9,15} + 20ch_{0,6,10,14} - 25ch_{0,6,11,13} + 10ch_{0,7,8,15} \]
\[- 25ch_{0,7,9,14} - 15ch_{0,7,10,13} + 100ch_{0,7,11,12} + 100ch_{0,8,9,13} \]
\[+ 280ch_{0,9,10,11} - 160ch_{0,8,10,12} - 5ch_{1,4,10,15} + 25ch_{1,4,11,14} \]
\[- 50ch_{1,4,12,13} + 20ch_{1,5,9,15} - 80ch_{1,5,10,14} + 100ch_{1,5,11,13} \]
\[- 25ch_{1,6,8,15} + 50ch_{1,6,9,14} + 100ch_{1,6,10,13} - 375ch_{1,6,11,12} \]
\[+ 75ch_{1,7,8,14} - 300ch_{1,7,9,13} + 375ch_{1,7,10,12} + 300ch_{1,8,9,12} \]
\[- 600ch_{1,8,10,11} + 10ch_{2,3,10,15} - 50ch_{2,3,11,14} + 100ch_{2,3,12,13} \]
\[- 25ch_{2,4,9,15} + 100ch_{2,4,10,14} - 125ch_{2,4,11,13} - 15ch_{2,5,8,15} \]
\[+ 100ch_{2,5,9,14} - 290ch_{2,5,10,13} + 475ch_{2,5,11,12} + 100ch_{2,6,7,15} \]
\[- 300ch_{2,6,8,14} + 200ch_{2,6,9,13} + 100ch_{2,6,10,12} + 825ch_{2,7,8,13} \]
\[- 1125ch_{2,7,9,12} + 150ch_{2,7,10,11} + 1200ch_{2,8,9,11} + 100ch_{3,4,8,15} \]
\[- 375ch_{3,4,9,14} + 475ch_{3,4,10,13} - 250ch_{3,4,11,12} - 160ch_{3,5,7,15} \]
\[+ 375ch_{3,5,8,14} + 100ch_{3,5,9,13} - 685ch_{3,5,10,12} + 300ch_{3,6,7,14} \]
\[- 1125ch_{3,6,8,13} + 1425ch_{3,6,9,12} - 750ch_{3,6,10,11} - 200ch_{3,7,8,12} \]
\[+ 800ch_{3,7,9,11} - 4400ch_{3,8,9,10} + 280ch_{4,5,6,15} - 600ch_{4,5,7,14} \]
\[+ 150ch_{4,5,8,13} - 750ch_{4,5,9,12} + 3180ch_{4,5,10,11} + 1200ch_{4,6,7,13} \]
\[+ 800ch_{4,6,8,12} - 3800ch_{4,6,9,11} - 1200ch_{4,7,8,11} + 6600ch_{4,7,9,10} \]
\[- 4400ch_{5,6,7,12} + 6600ch_{5,6,8,11} - 880ch_{5,6,9,10} - 9240ch_{5,7,8,10} \]
\[+ 15400ch_{6,7,8,9} \]

\[ \Delta^6 = ch_{0,7} - 7ch_{1,6} + 21ch_{2,5} - 35ch_{3,4} \]
\[ \Delta^6 = ch_{0,7,14} - 7ch_{0,8,13} + 21ch_{0,9,12} 
- 35ch_{0,10,11} - 7ch_{1,6,14} + 42ch_{1,7,13} - 98ch_{1,8,12} 
+ 98ch_{1,9,11} + 21ch_{2,5,14} - 98ch_{2,6,13} + 119ch_{2,7,12} 
+ 147ch_{2,8,11} - 539ch_{2,9,10} - 35ch_{3,4,14} + 98ch_{3,5,13} \] (D.18)

N=3 (41 terms)
\[ \Delta^8 = ch_{0,9,18} - 9ch_{1,8,17} + 36ch_{2,7} 
- 84ch_{3,6} + 126ch_{4,5} \] (D.19)

Eighth power
N=2 (5 terms)
\[ \Delta^8 = ch_{0,9,18} - 9ch_{1,8,17} + 36ch_{0,11,16} 
- 84ch_{0,12,15} + 126ch_{0,13,14} - 9ch_{1,8,18} + 72ch_{1,9,17} 
- 243ch_{1,10,16} + 432ch_{1,11,15} - 378ch_{1,12,14} + 36ch_{2,7,18} 
- 243ch_{2,8,17} + 603ch_{2,9,16} - 432ch_{2,10,15} - 972ch_{2,11,14} 
+ 2646ch_{2,12,13} - 84ch_{3,6,18} + 432ch_{3,7,17} - 432ch_{3,8,16} 
- 1776ch_{3,9,15} - 5724ch_{3,10,14} - 6048ch_{3,11,13} + 126ch_{4,5,18} \] (D.20)

N=3 (25 terms)
\[ \Delta^8 = ch_{0,7,14} - 7ch_{0,8,13} + 21ch_{0,9,12} 
- 35ch_{0,10,11} - 7ch_{1,6,14} + 42ch_{1,7,13} - 98ch_{1,8,12} 
+ 98ch_{1,9,11} + 21ch_{2,5,14} - 98ch_{2,6,13} + 119ch_{2,7,12} 
+ 147ch_{2,8,11} - 539ch_{2,9,10} - 35ch_{3,4,14} + 98ch_{3,5,13} \] (D.18)

\[ \Delta^8 = ch_{0,9,18} - 9ch_{1,8,17} + 36ch_{2,7} 
- 84ch_{3,6} + 126ch_{4,5} \] (D.19)

N=3 (41 terms)
\[ \Delta^8 = ch_{0,9,18} - 9ch_{1,8,17} + 36ch_{0,11,16} 
- 84ch_{0,12,15} + 126ch_{0,13,14} - 9ch_{1,8,18} + 72ch_{1,9,17} 
- 243ch_{1,10,16} + 432ch_{1,11,15} - 378ch_{1,12,14} + 36ch_{2,7,18} 
- 243ch_{2,8,17} + 603ch_{2,9,16} - 432ch_{2,10,15} - 972ch_{2,11,14} 
+ 2646ch_{2,12,13} - 84ch_{3,6,18} + 432ch_{3,7,17} - 432ch_{3,8,16} 
- 1776ch_{3,9,15} - 5724ch_{3,10,14} - 6048ch_{3,11,13} + 126ch_{4,5,18} \] (D.20)

Coefficients \( C_{\{n\}}^{(s)} \) for the vertices of the admissible polytope.

The admissible monomials in the expansion (D.1) are of the form \( \tau_1^{n_1} \ldots \tau_{N-1}^{n_{N-1}} \), where the \((n_0 = 0, n_1, ..., n_{N-1}, n_N = 0)\) are subject to
\[ 2s + n_{i+1} + n_{i-1} - 2n_i \geq 0; \quad n_i > 0; \quad i = 1, ..., N - 1. \] (D.21)
The equations (D.21) define a polytope or $\mathbb{R}^{N-1}$, which is a certain deformation of an hypercube. It can be shown [12] that its $2^{N-1}$ vertices are in one to one correspondence with strictly increasing sequences of integers:

$$i_0 = 0 < i_1 < i_2 < \ldots < i_k < i_{k+1} = N$$

denoted $[0, i_1, \ldots, i_k, N]$ here. The corresponding points read

$$n_i = s(i - i_k)(i_{k+1} - i), \quad \forall i \in [i_k, i_{k+1}], \quad l = 1, \ldots, k - 1. \quad \text{(D.22)}$$

These points are easily identified as the only ones which vanish at $i = i_1, \ldots, i_k$ (saturate the second inequality of (D.21) at these points), and have $n_{i+1} - n_i - 2n_i + 2s = 0$ (saturate the first inequality of (D.21) ) between those.

The farthest vertex from the origin corresponds to $[0, N]$, and $n_i = si(N - i)$ for $i = 1, \ldots, N - 1$. It yields the most compact character $ch_{s(N-1), s(N-1)+1, \ldots, (s+1)(N-1)}$ with coefficient already given in (5.16)

$$C_{\{si(N-i)\}}^{(s)} = \frac{[(s + 1)N]!}{[(s + 1)!]^N N!} (-1)^s N(N-1)/2 \quad \text{(D.23)}$$

Thanks to the factorization property 5 (eqn (4.43) ) the coefficient pertaining to the vertex $[0, i_1, \ldots, i_k, N]$ reads

$$C_{[0, i_1, \ldots, i_k, N]}^{(s)} = \prod_{l=0}^{k} (-1)^{s(i_{l+1} - i_l)(i_{l+1} - i_{l-1})/2} \times \frac{[(s + 1)(i_{l+1} - i_l)]!}{[(s + 1)!]^{i_{l+1} - i_l}(i_{l+1} - i_l)!} \quad \text{(D.24)}$$

**Appendix E. More on the number of terms in the expansion**

**Computing the volume** $b_N^{(N)} = a_N^{(N)} = V_{N+1}$

If we reorganize the inequalities (8.1) for $s = 1$, we find that

$$\sup(0, 2n_{N-2} - n_{N-3} - 2) \leq n_{N-1} \leq 1 + \frac{n_{N-2}}{2}$$

$$\sup(0, 2n_{N-3} - n_{N-4} - 2) \leq n_{N-2} \leq 2 + \frac{2n_{N-3}}{3}$$

$$\ldots$$

$$\sup(0, 2n_2 - n_1 - 2) \leq n_3 \leq N - 3 + \frac{(N - 3)n_2}{N - 2}$$

$$\sup(0, 2n_1 - 2) \leq n_2 \leq N - 2 + \frac{(N - 2)n_1}{N - 1}$$

$$0 \leq n_1 \leq N - 1$$
Let us define \( \phi_{N-1-k} \) as \( \phi_{N-1} = 1 \), and

\[
\phi_{N-1-k} = \theta(k + 1) \frac{(k+1)n_{N-k-2}}{k+2} - n_{N-1-k} \int_{\sup(0,2n_{N-k}-n_{N-k-2}-2)}^{k+\frac{k_nN_{N-1-k}}{k+1}} dn_{N-k} \phi_{N-k},
\]

where

\[
\theta(x) = 1 \text{ if } x \geq 0 = 0 \text{ otherwise.}
\]

For instance,

\[
\phi_{N-2} = \theta(2 + \frac{2n_{N-3}}{3} - n_{N-2}) \inf f(1 + \frac{n_{N-2}}{2}, \frac{3}{2}(2 + \frac{2n_{N-3}}{3} - n_{N-2})).
\]

The volume of the polytope \( \pi_N^{(1)} \) is

\[ V_N = \phi_0. \]

Let us rewrite

\[
\phi_{N-2} = \frac{3}{2}(2 + \frac{2n_{N-3}}{3} - n_{N-2}) \theta(2 + \frac{2n_{N-3}}{3} - n_{N-2}) - 2(1 + \frac{n_{N-3}}{2} - n_{N-2}) \theta(1 + \frac{n_{N-3}}{2} - n_{N-2}),
\]

and introduce the notation

\[
x^{(j)}_k = j + \frac{jn_{k-1}}{j+1} - n_k
\]

\((x^{(j)}_1 = j - n_1 \text{ and } x^{(j)}_0 = j) \text{ then}

\[
\phi_{N-2} = \frac{3}{2}x^{(2)}_{N-2} \theta(x^{(2)}_{N-2}) - 2x^{(1)}_{N-2} \theta(x^{(1)}_{N-2})
\]

Suppose we know that \( \phi_{N-k} \) has the form

\[
\phi_{N-k} = \sum_{j=1}^{k} p_j^{(k-1)}(x^{(j)}_{N-k}) \theta(x^{(j)}_{N-k}) \tag{E.1}
\]

where the \( p_j^{(k-1)} \)'s are some polynomials of degree \( k - 1 \). Then by definition,

\[
\phi_{N-k-1} = \theta(x^{(k+1)}_{N-k-1}) \int_{\sup(0,2n_{N-k-1}-n_{N-k-2}-2)}^{k+\frac{k_nN_{N-1-k}}{k+1}} dn_{N-k} \phi_{N-k}
\]

\[
= \theta(x^{(k+1)}_{N-k-1}) \sum_{j=1}^{k} \int_{\sup(0,2n_{N-k-1}-n_{N-k-2}-2)}^{k+\frac{k_nN_{N-1-k}}{k+1}} dn_{N-k} p_j^{(k-1)}(x^{(j)}_{N-k}) \theta(x^{(j)}_{N-k})
\]

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which becomes after the changes of variables $n_{N-k} \to x_{N-k}^{(j)}$

$$\phi_{N-k-1} = \theta(x_{N-k-1}^{(k+1)}) \sum_{j=1}^{k} \theta(\inf(j + \frac{jn_{N-k-1} - \frac{j+2}{j+1} x_{N-k-1}^{(j+1)}}{j+1}) \times$$

$$\int_{0}^{\inf(j + \frac{jn_{N-k-1} - \frac{j+2}{j+1} x_{N-k-1}^{(j+1)}}{j+1})} du \ p_{j}^{(k-1)}(u)$$

Note that

$$\frac{j+2}{j+1} x_{N-k-1}^{(j+1)} - (j + \frac{jn_{N-k-1}}{j+1}) = 2x_{N-k-1}^{(1)}$$

so that

$$\phi_{N-k-1} = \sum_{j=1}^{k} [1 - \theta(x_{N-k-1}^{(1)})] \theta(x_{N-k-1}^{(j+1)}) \int_{0}^{\frac{j+2}{j+1} x_{N-k-1}^{(j+1)}} du \ p_{j}^{(k-1)}(u)$$

$$+ \sum_{j=1}^{k} \theta(x_{N-k-1}^{(1)}) \int_{0}^{\frac{j+2}{j+1} x_{N-k-1}^{(j+1)}} du \ p_{j}^{(k-1)}(u)$$

$$= \theta(x_{N-k-1}^{(1)}) \sum_{j=1}^{k} \int_{0}^{\frac{j+2}{j+1} x_{N-k-1}^{(j+1)}} du \ p_{j}^{(k-1)}(u)$$

$$- \sum_{j=1}^{k} \theta(x_{N-k-1}^{(j+1)}) \int_{0}^{\frac{j+2}{j+1} x_{N-k-1}^{(j+1)}} du \ p_{j}^{(k-1)}(u).$$

where we used the inequality $x_{N-k-1}^{(1)} \leq x_{N-k-1}^{(j+1)}$ for $j = 1, \ldots, k$ to rewrite

$$\theta(x_{N-k-1}^{(1)}) \theta(x_{N-k-1}^{(j+1)}) = \theta(x_{N-k-1}^{(1)}).$$

The form of $\phi_{N-k-1}$ will be that of the recursion hypothesis (E.1) iff

$$p_{1}^{(k)}(x) = \sum_{j=1}^{k} \int_{0}^{\frac{j+2}{j+1} x_{N-k-1}^{(j+1)}} du \ p_{j}^{(k-1)}(u)$$

is independent on $y$, and a polynomial of $x$ only. Then we have, for $j = 2, \ldots, k+1$

$$p_{j}^{(k)}(x) = \int_{0}^{\frac{j+2}{j+1} x_{N-k-1}^{(j+1)}} du \ p_{j-1}^{(k-1)}(u).$$

Thanks to these relations, we can write $p_{1}^{(k)}$ as

$$p_{1}^{(k)}(x) = \sum_{j=2}^{k+1} \left[ p_{j}^{(k)} \left( \frac{j(j-1)}{j+1} + \frac{y}{j} \right) - p_{j}^{(k)} \left( \frac{j(j-1)}{j+1} + 1 + \frac{y}{j} \right) + \frac{2x_{N-k-1}^{(1)} y}{j+1} \right].$$
Let us prove by recursion that this expression is independent on $y$. Suppose the property true for $k - 1$, then in the above we can use it to rewrite (E.2)

$$p_1^{(k)}(x) = \sum_{j=1}^{k} \int_{j(1 + \frac{y}{j+1})}^{j(1 + \frac{y}{j+1}) + 2x} du \ p_j^{(k-1)}(u)$$

by replacing

$$p_1^{(k-1)}(u) = \sum_{j=2}^{k} \left[ p_j^{(k-1)}(\frac{j(j-1)}{j+1}(1 + \frac{z}{j})) - p_j^{(k-1)}(\frac{j(j-1)}{j+1}(1 + \frac{z}{j}) + 2uj) \right]$$

valid for any $z$. We get

$$p_1^{(k)}(x) = \sum_{j=2}^{k} \left[ \int_{j(1 + \frac{y}{j+1})}^{j(1 + \frac{y}{j+1}) + 2x} du \ p_j^{(k-1)}(u) + \int_{1 + \frac{y}{j+1} + 2x}^{1 + \frac{y}{j+1}} \left[ p_j^{(k-1)}(\frac{j(j-1)}{j+1}(1 + \frac{z}{j})) - p_j^{(k-1)}(\frac{j(j-1)}{j+1}(1 + \frac{z}{j}) + 2uj) \right] \right]$$

We take the range of the integrations to be $[2x, 0]$ by suitable changes (shifts) of variables, to get

$$p_1^{(k)}(x) = \sum_{j=2}^{k} \left[ \int_{2x}^{0} du \ p_j^{(k-1)}(u + j(1 + \frac{y}{j+1}) + \frac{j(j-1)}{j+1}(1 + \frac{z}{j})) - p_j^{(k-1)}(\frac{j(j-1)}{j+1}(1 + \frac{z}{j}) + 2uj) \right].$$

But this is valid for any $z$, let us take $z = -u$, then the first and last term in the sum cancel out exactly, leaving us with

$$p_1^{(k)}(x) = \sum_{j=2}^{k} \int_{2x}^{0} du \ p_j^{(k-1)}(\frac{j(j-1)}{j+1}(1 + \frac{u}{j})).$$

clearly independent of $y$. This completes the proof of the general formula

$$\phi_{N-k-1} = \sum_{j=1}^{k+1} \theta(x_{N-k-1}^{(j)}p_j^{(k)}(x_{N-k-1}^{(j)}),$$

with the recursions

$$p_1^{(0)}(x) = 1$$

$$p_j^{(k)}(x) = \int_{0}^{\frac{i+1}{j}} du \ p_j^{(k-1)}(u) \quad j = 2, 3, ..., k + 1$$

$$p_1^{(k)}(x) = \sum_{j=2}^{k+1} \left[ p_j^{(k)}(\frac{j(j-1)}{j+1}(1 + \frac{y}{j})) - p_j^{(k)}(\frac{j(j-1)}{j+1}(1 + \frac{y}{j}) + 2xj) \right],$$

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where the last expression is independent of \(y\).

The volume reads

\[
V_N = \phi_0 = \sum_{j=1}^{N} p_j^{(N-1)}(j).
\]

The expression for \(p_1^{(k)}(x)\) gives for \(y = 0\)

\[
p_1^{(k)}(x) = \sum_{j=2}^{k+1} \left[ p_j^{(k)} \left( \frac{j(j-1)}{j+1} \right) - p_j^{(k)} \left( \frac{j(j-1)}{j+1} + \frac{2x}{j+1} \right) \right],
\]

hence for \(x = 1\),

\[
\sum_{j=1}^{k+1} p_j^{(k)}(j) = \sum_{j=2}^{k+1} p_j^{(k)} \left( \frac{j(j-1)}{j+1} \right).
\]

The proof of independence of \(y\) gives us another expression for \(p_1^{(k)}\)

\[
p_1^{(k)}(x) = \sum_{j=3}^{k+1} \frac{j}{j-2} \left[ p_j^{(k)} \left( \frac{(j-1)(j-2)}{j+1} \left(1 - \frac{2x}{j-1}\right) \right) - p_j^{(k)} \left( \frac{(j-1)(j-2)}{j+1} \right) \right]
\]

The first few \(p\)'s are listed below.
\[ k = 0 \quad p_1^{(0)} = 1 \]
\[ k = 1 \quad p_2^{(1)} = \frac{3}{2}x \]
\[ p_1^{(1)} = -2x \]
\[ k = 2 \quad p_3^{(2)} = \frac{4}{3}x^2 \]
\[ p_2^{(2)} = -\frac{9}{4}x^2 \]
\[ p_1^{(2)} = x^2 - 2x \]
\[ k = 3 \quad p_4^{(3)} = \frac{125}{144}x^3 \]
\[ p_3^{(3)} = -\frac{16}{9}x^3 \]
\[ p_2^{(3)} = \frac{9}{8}x^3 - \frac{9}{4}x^2 \]
\[ p_1^{(3)} = -\frac{2}{9}x^3 + 2x^2 - 4x \]
\[ k = 4 \quad p_5^{(4)} = \frac{9}{20}x^4 \]
\[ p_4^{(4)} = \frac{625}{576}x^4 \]
\[ p_3^{(4)} = \frac{8}{9}x^4 - \frac{16}{9}x^3 \]
\[ p_2^{(4)} = -\frac{9}{32}x^4 + \frac{9}{4}x^3 - \frac{9}{2}x^2 \]
\[ p_1^{(4)} = \frac{1}{36}x^4 - \frac{2}{3}x^3 + 5x^2 - \frac{32}{3}x \]
\[ k = 5 \quad p_6^{(5)} = \frac{16807}{86400}x^5 \]
\[ p_5^{(5)} = -\frac{27}{50}x^5 \]
\[ p_4^{(5)} = \frac{625}{1152}x^5 - \frac{625}{576}x^4 \]
\[ p_3^{(5)} = \frac{-32}{135}x^5 + \frac{16}{9}x^4 - \frac{32}{9}x^3 \]
\[ p_2^{(5)} = \frac{27}{320}x^5 - \frac{27}{32}x^4 + \frac{45}{8}x^3 - 12x^2 \]
\[ p_1^{(5)} = -\frac{1}{450}x^5 + \frac{1}{9}x^4 - 2x^3 + \frac{44}{3}x^2 - \frac{100}{3}x \]
which lead to the volumes

| $N$ | Volume $V_N$ |
|-----|-------------|
| 2   | 1           |
| 3   | 2           |
| 4   | 16/3        |
| 5   | 50/3        |
| 6   | 288/5       |
| 7   | 9604/45     |
| 8   | 262144/315  |

It is easy to read the actual exact result

$$a_N^{(N)} = b_N^{(N)} = V_{N+1} = 2^N \frac{(N+1)^{N-2}}{N!}$$

(E.3)

hence the dilated polytope $\Pi_{N+1}^{(s)}$ has volume

$$V_{N+1}^{(s)} = a_N^{(N)} s^N = (2s)^N \frac{(N+1)^{N-2}}{N!}.$$  

(E.4)

We guess formula (E.3) on the basis of the previous calculations.

Computing $b_{N-1}^{(N)} = a_{N-1}^{(N)}$

Let us now concentrate on the subleading coefficient $a_{N-1}^{(N)} = b_{N-1}^{(N)}$. By definition, we have

$$b_{N-1}^{(N)} s^{N-1} = \sum_{k=1}^{N} \frac{1}{2} \left( \frac{\partial}{\partial h_k} + \frac{\partial}{\partial \epsilon_k} \right) V_{N+1}^{(s)} \left| \begin{array}{c} h_i = \epsilon_i = 0 \\ h_i = \epsilon_i = 0 \\ \end{array} \right. .$$

(E.5)

Let us write the deformed volume

$$V_{N+1}^{(s)}(h_i; \epsilon_i) = \int_{u \in \mathbb{R}^N} d^N u \prod_{k=1}^{N} \theta(u_k + \epsilon_k) \theta(-2u_k + u_{k+1} + u_{k-1} + 2s + h_k),$$

(E.6)

$\frac{d}{dx} \theta(x) = \delta(x)$. Therefore we have two contributions to (E.5), the one coming from $\epsilon$ derivatives

$$\partial_{\epsilon_k} V_{N+1}^{(s)}(h_i; \epsilon_i) \bigg|_{h_i = \epsilon_i = 0} = \int_{u \in \mathbb{R}^N} d^N u \delta(u_k) \prod_{j \neq k} \theta(u_j) \prod_{l=1}^{N} \theta(-2u_l + u_{l+1} + u_{l-1} + 2s)$$

$$= V_k^{(s)} V_{N-k}^{(s)}$$

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and the one coming from \( h \) derivatives

\[
\sum_{k=1}^{N} \frac{\partial h_k V^{(s)}_{N+1}(h_i; \epsilon_i)}{h_i = \epsilon_i = 0} = \frac{d}{d(2s)} V^{(s)}_{N+1} = \frac{N}{2s} V^{(s)}_{N+1}.
\]

Putting all contributions together, we find

\[
b_{N-1} s^{N-1} = a_{N-1} s^{N-1} = \frac{N}{4s} V^{(s)}_{N+1} + \frac{1}{2} \sum_{k=0}^{N} V^{(s)}_k V^{(s)}_{N-k}.
\]

The latter sum turns out to have a simple form

\[
\sum_{k=0}^{N} V^{(s)}_k V^{(s)}_{N-k} = (2s)^{N-1} (N+1)^{N-4} (N + 7),
\]

which leads to

\[
a^{(N)}_{N-2} = b^{(N)}_{N-2} = \frac{2^{N-1}}{(N - 1)!2!} (N + 1)^{N-4} (6 + (N + 1)(N + 2)).
\]

Applying this to \( N = 2, 3, 4, 5, 6, 7, 8, 9 \) yields respectively \( 2, \frac{13}{2}, 24, 96, \frac{6076}{15}, \frac{26624}{15}, \frac{27936}{35}, \frac{2320000}{63} \), so we get all the subleading coefficients in the list (8.2). Note that the general expression fails to reproduce the corresponding \( N = 1 \) term, due to the different definition of the deformation for \( N = 1 \). We also get the subleading coefficients in the expression of the \( A \)'s in terms of the \( \sigma \)'s (8.3), namely the coefficient of \( \sigma_{N-1} \) in \( A^{(s)}_{N+1} \) reads

\[
2^{N-2}(N + 1)^{N-4}[6 + N(N + 1)(2N + 1)],
\]

which yields for \( N = 2, 3, 4, 5, 6, 7, 8, 9, 10 \) respectively 4, 45, 744, 16128, 432768, 13860864, 516481920, 21964800000, 1050351430656.

**Computing \( b^{(N)}_{N-2} \) and \( a^{(N)}_{N-2} \)**

The next to leading coefficient \( a^{(N)}_{N-2} \) is trickier to obtain, because it differs from \( b_{N-2} \) by a highly non-trivial quantity. The latter involves the two-dimensional cone invariant \( \tau_2 \) computed in [10] and function of the Dedekind sum of the two integers defining the (rational) cone. However in our very particular case, we are able to guess a simple answer.
The first task is to compute $b_{N-2}$ then to proceed and compare its value to that of $a_{N-2}$ which we read from (8.2). Thanks to the definition (8.8), we find a closed formula for $b_{N-2}$

$$b_{N-2}s^{N-2} = \left[ \frac{1}{12} \sum_i (\partial^2_{\varepsilon_i} + \partial^2_{\varepsilon_i}) + \frac{1}{4} \sum_{i<j} (\partial_{h_i} \partial_{h_j} + \partial_{\varepsilon_i} \partial_{\varepsilon_j}) 
+ \frac{1}{4} \sum_{i,j} \partial_{h_i} \partial_{\varepsilon_j} \right] V^{(s)}_{N+1}(h_i, \varepsilon_i) \bigg|_{h_i=\varepsilon_i=0}.$$ 

We use the fact that

$$\sum_i \partial_{h_i} V = \frac{d}{d(2s)} V,$$

$$\sum_{i<j} \partial_{h_i} \partial_{h_j} V = \frac{1}{2} \left( \frac{d}{d(2s)} \right)^2 - \sum_i \partial^2_{h_i} V,$$

to rewrite

$$b_{N-2}s^{N-2} = \left[ \frac{1}{12} \sum_i (\partial^2_{\varepsilon_i} - \frac{1}{2} \partial^2_{h_i}) + \frac{1}{4} \sum_{i<j} \partial_{\varepsilon_i} \partial_{\varepsilon_j} + \frac{1}{32} \frac{d^2}{ds^2} \right] V^{(s)}_{N+1}(h_i, \varepsilon_i) \bigg|_{h_i=\varepsilon_i=0}.$$ 

Each of the above terms is easily derived by using the integral expression for the deformed volume (E.6). We find respectively

$$\sum_{i<j} \partial_{\varepsilon_i} \partial_{\varepsilon_j} V_{N+1} = \sum_{p,q \geq 1} V_p V_q V_{N-p-q},$$

$$\frac{d^2}{ds^2} V_{N+1} = N(N-1)V_{N+1}/s^2,$$

$$\frac{d}{ds} \sum_j \partial_{\varepsilon_j} V_{N+1} = \frac{N-1}{s} \sum_{p=1}^{N-1} V_p V_{N-p}. $$

The only difficult part involves the double derivatives. For $\varepsilon$ derivatives, we have

$$\partial^2_{\varepsilon_i} V_{N+1} = \int_{u \in \mathbb{R}^N} d^N u \delta'(u_i) \prod_{j \neq i} \theta(u_j) \prod_j \theta(-2u_j + u_{j+1} + u_{j-1} + 2s)$$

$$= - \int_{R^N} d\varphi(u_i) \prod_{j \neq i} \theta(u_j) \varphi(u_j),$$
where
\[
\varphi_i(u_j) = \delta(u_i + u_{i+2} + 2s - 2u_{i+1}) \prod_{j \neq i+1} \theta(-2u_j + u_{j+1} + u_{j-1} + 2s) \\
+ \delta(u_i + u_{i-2} + 2s - 2u_{i-1}) \prod_{j \neq i-1} \theta(-2u_j + u_{j+1} + u_{j-1} + 2s) \\
- 2\delta(u_{i+1} + u_{i-1} + 2s - 2u_i) \prod_{j \neq i} \theta(-2u_j + u_{j+1} + u_{j-1} + 2s).
\]

The last term does not contribute because it multiplies \(\delta(u_i)\theta(u_{i-1})\theta(u_{i+1})\) and for \(s > 0\) the various constraints are incompatible. So we are finally left with
\[
\partial^2 \epsilon_i V_{N+1} = - \sum_{p=1}^{N-1} (V_{N-i} \partial_{h_{i-1}} V_i + V_i \partial_{h_{i+1}} V_{N-i}).
\]

Using again the integral definition (E.6), we find
\[
\partial_{h_{i-1}} V_i = \frac{1}{2} \sum_{p=1}^{i-1} V_p V_{i-p},
\]
so that
\[
\sum_i \partial^2 \epsilon_i V_{N+1} = - \sum_{p, q \geq 1, p + q \leq N-1} V_p V_q V_{N-p-q}.
\]

For \(h\) derivatives, we find\(^4\)
\[
\sum_i \partial^2 h_i V_{N+1} = \frac{(2s)^{N-2}(N+1)^{N-3}}{(N-2)!}.
\]

Finally, we use
\[
\sum_{p, q \geq 1, p + q \leq N-1} V_p^{(s)} V_q^{(s)} V_{N-p-q}^{(s)} = \frac{3}{4} \frac{(2s)^{N-2}(N+1)^{N-6}}{(N-2)!}[N^2 + 15N + 74],
\]
so that \(b_{N-2}\) reads
\[
b_{N-2} = \frac{2^{N-2}(N+1)^{N-6}}{4!(N-2)!}[3N^4 + 17N^3 + 72N^2 + 144N + 266]. \tag{E.9}
\]

Now comparing this to the values of \(a_{N-2}\) that we read off from (8.2), we conjecture the following relation
\[
a_{N-2} s^{N-2} = b_{N-2} s^{N-2} + \frac{1}{4} V_{N-1}^{(s)}, \tag{E.10}
\]

\(^4\) See next section for a detailed proof
leading to
\[ a_{N-2} = \frac{2^{N-2}}{4!(N-2)!} [(N+1)^{N-6}(3N^4+17N^3+72N^2+144N+266)+6(N-1)^{N-4}] \]. (E.11)

We check that for \( N = 4, 5, 6, 7, 8, 9, 10 \), we get \( a_{N-2} = \frac{40}{9}, \frac{385}{6}, \frac{2858}{45}, \frac{71992}{45}, \frac{367144}{45}, \frac{3966608}{945}, \frac{205427098}{945} \). Note that the above formula fails to reproduce the \( N = 2, 3 \) cases. This, we suspect, is related to the difference in the definition of the deformation for the boundary case \( N = 1 \).

This leads to the coefficient of \( \sigma_{N-2} \) in \( A_{N+1}^{(s)} \) of (8.3)
\[ \frac{2^{N-2}}{4!} [(N+1)^{N-6}[12N^6+16N^5-25N^4+21N^3+16N^2-100N+114]+6(N-1)^{N-4}] \]. (E.12)

For \( N = 4, 5, 6, 7, 8, 9, 10 \), we get respectively 404, 12481, 437776, 17367872, 773038912,38261688576, 2088303502080.

**Computing the polynomials** \( B_N(s) \)

The computation of the polynomials \( B_{N-1}(s) \) resembles very much that of the volume \( V_N(s) \) of previous appendix. The only difference is that the defining relations for the polytope are decorated with the parameters \( h_i \) and \( \epsilon_i \), \( i = 1, 2, ..., N - 1 \). For simplicity, let us first compute the deformed volume \( W_N(h_i, \epsilon_i) \) defined by the relations
\[-2\epsilon_k \leq 2n_k \leq n_{k-1} + n_{k+1} + h_k.\]

The desired volume is
\[ V_N^{(s)}(h_i, \epsilon_i) = W_N(h_i + 2s, \epsilon_i). \]

Let us again introduce \( \phi_{N-1}(h_i, \epsilon_i) = 1 \) and \( \phi_{N-k-1}(h_i, \epsilon_i) \) by the recursion
\[ \phi_{N-1-k}(h_i, \epsilon_i; x = nN_{N-1-k}) = \theta\left(\sum_{j=1}^{k+1} jh_{N-j} + (k+1)n_{N-k} - (k+2)x\right) \times \]
\[ \int_{\sup(-\epsilon_{N-k}, 2n_{N-1-k} - n_{N-k} - n_{N-2-k} - h_{N-1-k})}^{x + \frac{1}{k+1}} \sum_{j=1}^{k} jh_{N-j} \]
\[ \sup(-\epsilon_{N-k}, 2n_{N-1-k} - n_{N-k} - n_{N-2-k} - h_{N-1-k}) \]
\[ du \ \phi_{N-k}(h_i, \epsilon_i; u). \]

With these definitions, the volume \( W \) reads
\[ W_N(h_i, \epsilon_i) = \phi_0(h_i, \epsilon_i; 0). \]
Just as in the non–deformed case, the \( \phi \)'s turn out to have a simple polynomial form

\[
\phi_{N-k}(h_i, \epsilon_i; x) = \sum_{j=1}^{k} p_j^{(k-1)}(x_{N-k}^{(j)}(h_i, \epsilon_i; x)) \theta(x_{N-k}^{(j)}(h_i, \epsilon_i; x)),
\]

(E.13)

where the \( p_j^{(l)}(u), j = 1, 2, \ldots, l + 1 \) are some polynomials of \( u \), with coefficients themselves polynomial in \( h_i \) and \( \epsilon_i \), and we used some reduced variables

\[
x_{N-k}^{(j)}(h_i, \epsilon_i; x) \equiv \frac{1}{j+1} \sum_{r=1}^{j} rh_{N-k+j-r} - \epsilon_{N-k+j} + \frac{j}{j+1} n_{N-k-1} - x,
\]

with the convention that \( \epsilon_N = 0 \). The proof of (E.13) is identical to that of the undeformed case and leads to the recursion relations between the \( p \)'s

\[
p_j^{(k)}(x) = \int_{0}^{j+1} du \ p_{j-1}^{(k-1)}(u) \quad j = 2, 3, \ldots, k + 1,
\]

(E.14)

and

\[
p_1^{(k)}(u) = \sum_{j=2}^{k+1} p_j^{(k)} \left( \frac{\sum_{r=1}^{j-1} rh_{N-r} + j\epsilon_{N-k} - \epsilon_{N-k+j-1} + (j-1)y}{j+1} \right)
\]

\[
- p_j^{(k)} \left( \frac{\sum_{r=1}^{j-1} rh_{N-r} + j\epsilon_{N-k} - \epsilon_{N-k+j-1} + (j-1)y + 2ju}{j+1} \right),
\]

(E.15)

independent on \( y \).

We list the first few deformed \( p \)'s below
\[ k = 0 \quad p_1^{(0)}(u) = 1 \]
\[ k = 1 \quad p_2^{(1)}(u) = \frac{3}{2}u \]
\[ p_1^{(1)}(u) = -2u \]
\[ k = 2 \quad p_3^{(2)}(u) = \frac{4}{3}u^2 \]
\[ p_2^{(2)}(u) = -\frac{9}{4}u^2 \]
\[ p_1^{(2)}(u) = u^2 - [h_{N-1} + 2\epsilon_{N-1} - \epsilon_{N-2}]u \]
\[ k = 3 \quad p_4^{(3)}(u) = \frac{125}{144}u^3 \]
\[ p_3^{(3)}(u) = -\frac{16}{9}u^3 \]
\[ p_2^{(3)}(u) = \frac{9}{8}u^3 - \frac{9}{8}[h_{N-1} + 2\epsilon_{N-1} - \epsilon_{N-2}]u^2 \]
\[ p_1^{(3)}(u) = -\frac{2}{9}u^3 + \frac{1}{3}[h_{N-2} + 2h_{N-1} + 3\epsilon_{N-1} - \epsilon_{N-3}]u^2 \]
\[ -\frac{1}{6}h_{N-1}^2 + h_{N-2}^2 + 4h_{N-1}h_{N-2} + h_{N-2}(6\epsilon_{N-1} - 2\epsilon_{N-3}) \]
\[ + h_{N-1}(6\epsilon_{N-2} - 4\epsilon_{N-3}) + \epsilon_{N-3}^2 - 3(\epsilon_{N-1}^2 + \epsilon_{N-2}^2) + 12\epsilon_{N-1}\epsilon_{N-2} - 6\epsilon_{N-1}\epsilon_{N-3}u \]

We recover the undeformed \( p \)'s (at \( s = 1 \)) by taking \( h_i = 2, \forall i \). A few remarks are in order in view of this list.

(i) The coefficients of the polynomials \( p_j^{(k)} \) have a simple structure for \( j > 1 \), due to the recursive definition (E.14). Let us denote by \( \alpha_m^{(1,k)} \) the coefficients of \( p_1^{(k)} \)

\[ p_1^{(k)}(x) = \sum_{m=1}^{k} \alpha_m^{(1,k)} x^m, \]

where we used the definition (E.13), to include the fact that \( p_1^{(k)}(0) = 0 \). Then the coefficients \( \alpha_m^{(j,k)} \) of \( p_j^{(k)} \) satisfy the recursion

\[ \alpha_m^{(j,k)} = \frac{1}{m+1} \left( \frac{j+1}{j} \right)^{m+1} \alpha_m^{(j-1,k-1)} \quad m = 1, 2, \ldots, k-1, \]

which implies

\[ \alpha_m^{(k+1,k)} = \delta_{m,k} \left( \frac{k+2}{(k+1)!} \right)^k \alpha_0^{(1,0)} \]
\[ \alpha_m^{(j,k)} = \theta(m-j)\theta(k-m) \left( \frac{j+1}{j!} \right)^{m-j+1} \left( \frac{m-j+1}{j!} \right)^{m-j+1} \alpha_m^{(j-k+1)} \quad j = 2, 3, \ldots, k. \]
So we have only to compute the coefficients $\alpha_m^{(1,k)}$ of $p_1^{(k)}$. For instance, due to $\alpha_0^{(1,0)} = 1$, $\alpha_1^{(1,1)} = -2$, $\alpha_2^{(1,2)} = 1$, $\alpha_1^{(1,2)} = -(h_{N-1} + 2\epsilon_{N-1} - \epsilon_{N-2})$, we have

\[
\begin{align*}
  j &= k + 1 : \quad \alpha_k^{(k+1,k)} = \frac{(k + 2)^k}{(k + 1)!} \\
  j &= k : \quad \alpha_k^{(k,k)} = -\frac{(k + 1)^k}{(k!)^2} \\
  j &= k - 1 : \quad \alpha_k^{(k-1,k)} = \frac{(k)^k}{k!(k - 1)!} \alpha_{k-1}^{(k-1,k)} = -\frac{(k)^{k-1}}{2!((k - 1)!)^2}(h_{N-1} + 2\epsilon_{N-1} - \epsilon_{N-2}).
\end{align*}
\]

The coefficients of $p_1^{(k)}$ satisfy the following recursion relation inherited from the definition (E.15)

\[
\alpha_m^{(1,k)} = -\frac{2^m(k + 1)^m}{m!(k - m)!(k + 1)!} \left[ \sum_{r=1}^{k} rh_{N-r} + (k + 1)\epsilon_{N-k} \right]^{k-m} \]

\[
- \sum_{p=m}^{k} \sum_{j=2}^{p} \frac{j^m(p - j + 1)!2^{m-p+j-1}}{j!(p - m)!m!} \alpha_{p-j+1}^{(1,k-j+1)} \left[ \sum_{r=1}^{j-1} rh_{N-r} + j\epsilon_{N-j+1} - \epsilon_{N-k+j-1} \right]^{p-m},
\]

(E.16)

for $m = 1, 2, ..., k$. As a consequence, we derive the leading coefficient $\alpha_k^{(1,k)}$ of $p_1^{(k)}$

\[
\alpha_k^{(1,k)} = -\frac{2^k(k + 1)^k}{k!(k + 1)!} + \sum_{j=2}^{k} \frac{j^k(k - j + 1)!2^{j-1}}{k!j!} \alpha_{k-j+1}^{(1,k-j+1)}
\]

(E.17)

thanks to the identity

\[
\sum_{j=0}^{k+1} (x-j)^k(-1)^j \binom{k+1}{j} = 0,
\]

valid for any $x$, taken at $x = 0$

\[
\sum_{j=1}^{k} (-1)^j j^k \binom{k+1}{j} = (-1)^k(k + 1)^k.
\]

(ii) The dependence on $h_i$ and $\epsilon_i$ of the coefficients $\alpha_m^{(1,k)}$ is very peculiar too. Actually, the coefficients $\alpha_m^{(1,k)}$ are functions of $h_{N-1}$, ..., $h_{N-k+1}$ and $\epsilon_{N-1}$, ..., $\epsilon_{N-k}$ only, as a consequence of (E.10).
Finally the volume reads

\[ W_N(h_i, \epsilon_i) = \sum_{j=1}^{N} p_j^{(N-1)}(x_0^{(j)}(h_i, \epsilon_i; 0)) \]

\[ = \sum_{j=1}^{N} p_j^{(N-1)} \left( \sum_{r=1}^{j-1} r h_{j-r} - \epsilon_j \right) / j + 1. \]

This provides us with a powerful recursive scheme for computing the deformed volumes.

As an example let us derive exactly the coefficient of \( h_i^{N-1}, i = 1, 2, ..., N - 1 \), in \( W_N \).

The above remark (i) enables to write the leading coefficient of the \( p \)'s as

\[ \alpha_k^{(j,k)} = \frac{(j+1)^k(k-j+1)!}{j!2^{k-j+1}} \alpha_{k-j+1}^{(1,k-j+1)} \]

\[ = (-1)^{k-j+1} \frac{(j+1)^k}{j!k!(k-j+1)!}. \]

The contribution to \( W_N \) of any subleading piece, say of degree \( m < k \), of \( p_j^{(k)} \) involves, thanks to remark (ii) a coefficient which is only a function of \( h_{N-1}, h_{N-k+j-1} \), and which multiplies a power

\[ \left[ \sum_{r=1}^{j-1} r h_{j-r} - \epsilon_j \right]^m, \]

but \( N - k + j - 1 > j - 1 \), therefore such a term does not contribute to any \( h_i^{N-1} \) term of \( W_N \). The only contributions come from the leading pieces of the \( p \)'s

\[ \sum_{j=1}^{N} \frac{\alpha_{N-1}^{(j,N-1)}}{(j+1)^{N-1}} \left( \sum_{r=1}^{j-1} r h_{j-r} - \epsilon_j \right)^{N-1}, \]

and we get the coefficient of \( h_i^{N-1} \) in \( W_N \)

\[ \sum_{j=i+1}^{N} (j-i)^{N-1} \alpha_{N-1}^{(j,N-1)} = \sum_{j=i+1}^{N} \frac{(j-i)^{N-1}(-1)^{N-j}}{j!(N-1)!(N-j)!} \]

\[ = \frac{1}{N!(N-1)!} \sum_{j=0}^{N-i-1} (-1)^j (N - i - j)^{N-1} \binom{N}{j} \]

\[ = \frac{1}{N!(N-1)!} \sum_{j=0}^{i-1} (-1)^j (i - j)^{N-1} \binom{N}{j}. \]
The last identity expresses the symmetry of $W_N$ under the interchange $h_i \leftrightarrow h_{N-i}$ and is a consequence of the identity (E.18), taken at $x = i$ and $k = N - 1$

$$\sum_{j=0}^{N} (i-j)^{N-1}(-1)^j \binom{N}{j} = 0.$$  

Finally we have at leading order in the $h_i$'s

$$W_N \simeq \sum_{i=1}^{N-1} \frac{h_i^{N-1}}{N!(N-1)!} \sum_{j=0}^{i-1} (-1)^j (i-j)^{N-1} \binom{N}{j}.$$  

This leads to the following contribution of the $(N-1)$th derivatives of $W_N$ w.r.t. the $h$’s

$$\sum_{i=1}^{N-1} \partial h_i^{N-1} W_N = \frac{1}{N!} \sum_{i=1}^{N-1} \sum_{j=0}^{i-1} (-1)^j (i-j)^{N-1} \binom{N}{j} \binom{N}{j} = \frac{1}{N}.$$  

Using this recursive method, we computed the first few $B_N$’s

$B_0(s) = 1$

$B_1(s) = s + 1$

$B_2(s) = 2s^2 + 2s + \frac{19}{36}$

$B_3(s) = \frac{16}{3} s^3 + \frac{13}{2} s^2 + \frac{8}{3} s + \frac{107}{288}$

$B_4(s) = \frac{50}{3} s^4 + 24s^3 + \frac{77}{6} s^2 + \frac{109}{36} s + \frac{641}{2400}$

$B_5(s) = \frac{288}{5} s^5 + 96s^4 + \frac{377}{6} s^3 + \frac{1453}{72} s^2 + \frac{509}{160} s + \frac{51103}{259200}$

$B_6(s) = \frac{9604}{45} s^6 + 6076 s^5 + \frac{5641}{18} s^4 + \frac{6821}{54} s^3 + \frac{20111}{720} s^2 + \frac{8663}{2700} s + \frac{189789}{12700800}$.
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