Lennard-Jones quark matter and massive quark stars

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1 INTRODUCTION

To understand the nature of pulsars we need to know the state of cold quark matter, in which the dominant degree of freedom is quarks, and their Fermi energy is much larger than their thermal energy. However, this is a difficult task because of (i) the non-perturbative effect of the strong interaction between quarks at low energy scale and (ii) the many-body problem of vast assemblies of interacting particles.

On one hand, some efforts have been made for understanding the behavior of quantum chromo-dynamics (QCD) at high density, among which a color super-conductivity (CSC) state is currently focused on in perturbative QCD as well as in QCD-based effective models (e.g., Alford et al. 2003). On the other hand, it is phenomenologically conjectured that astrophysical cold quark matter could be in a solid state (Xu 2003), since the strong interaction may render quarks grouped in clusters and the ground state of realistic quark matter might not be that of Fermi gas (see a recent discussion given by Xu 2009). If the residua interaction between quark clusters is stronger than their kinetic energy, each quark cluster could be trapped in the potential well and cold quark matter will be in a solid state. Solid quark stars still cannot be ruled out in both astrophysics and particle physics (Horvath 2005; Owen 2005). Additionally, there is evidence that the interaction between quarks is very strong in hot quark-gluon plasma (i.e., the strongly coupled quark-gluon plasma, Shuryak 2004), according to the recent achievements of relativistic heavy ion collision experiments. When the temperature goes down, it is reasonable to conjecture that the interaction between quarks should be stronger than that in the hot quark-gluon plasma.

Because of the difficulty to obtain a realistic state equation of cold quark matter at a few nuclear densities, we try to apply some phenomenological models, which would have some implications about the properties of QCD at low energy scale if the astronomical observations can provide us with some limitations on such models. In our previous paper (Lai & Xu 2009), a polytropic quark star model has been suggested in order to establish a general framework in which theoretical quark star models could be tested by observations. This model can help us to understand the observed masses of pulsars and the energy released during some extreme bursts; however, this is a phenomenological model and does not include the form of interaction between quarks. To calculate the interaction between quarks and predict the state of matter for quark stars by QCD calculations is a difficult task; however, it is still meaningful for us to consider some models to explore the properties of quarks at the low energy scale.

We can compare the interaction between quark clusters with the interaction between inert molecules. A single quark cluster inside a quark star is assumed to be colorless, just like each molecule in a bulk of inert gas is electric neutral. The interaction potential between two inert gas molecules can be well described by the Lennard-Jones potential (Lennard-Jones 1924)

\[ u(r) = 4U_0 \left( \frac{r_0}{r} \right)^6 - \left( \frac{r_0}{r} \right)^{12}, \]  

where \( U_0 \) is the depth of the potential and \( r_0 \) can be considered as the range of interaction. This form of potential has the property of short-distance repulsion and long-distance attraction. We assume that the interaction between the quark clusters in quark stars can also be described by...
the form of Lennard-Jones potential. If the inter-cluster potential is deep enough to trap the clusters in the potential wells, the quark matter would crystallize and form solid quark stars. Under such potential, we can get the equation of state for solid quark stars, where the pressure comes from both the inter-cluster potential and lattice vibrations. Because the chromo-interaction is stronger than the electromagnetic interaction that is responsible for the intermolecular forces, the values of parameters, $U_0$ and $r_0$ used in the cold quark matter should be different from that in the inert gas, and should be determined in the context of quark stars.

The model of quark stars composed of Lennard-Jones matter is much different from the conventional models (e.g., MIT bag model) in which the ground state is of Fermi gas. In the former case the quark-clusters are non-relativistic particles, whereas in the latter case quarks are relativistic particles. Consequently, the equations of state in this two kinds of models are different, and we found that the Lennard-Jones model has some more stiffer equations of state, which lead to higher maximum masses for quark stars. Certainly, a quark star can be very low massive ($<10^{-3} M_{\odot}$) due to self color interaction. On the other hand, we find that under some reasonable values of parameters, a quark star could also be very massive ($>2 M_{\odot}$).

This paper is arranged as follows. The details of lattice thermodynamics are listed in Section 2. The forms of equations of state are given in Section 3, including the comparison with the MIT bag model, and we show the corresponding mass-radius curves in Section 4. We make conclusions and discussions in Section 5.

2 LATTICE THERMODYNAMICS

For an inter-cluster potential which is deep enough to trap the clusters in the potential wells, the quark matter would crystallize to get lower energy and form solid quark stars. In this section we use the results in classical solid physics to discuss the properties of crystallized cold quark matter.

2.1 The inter-cluster potential

Like the inert gas, the interaction potential $u$ between two quark-clusters as the function of their distance $r$ is described by the Lennard-Jones potential, Eq.(1). Let us consider a system containing $N$ clusters with the volume $V$, then the total interaction potential is

$$U = \frac{N}{2} \sum_{i \neq j} u(r_{ij}),$$

(2)

and if we ignore the surface tension, we get

$$U = \frac{N}{2} \sum_{i \neq j} u(r_{ij}) = \frac{N}{2} \sum_{i \neq j} \{4U_0[\left(\frac{r_0}{r_{ij}}\right)^{12} - \left(\frac{r_0}{r_{ij}}\right)^6]\}. \quad (3)$$

The lattice structure of cold quark matter is unknown, and we adopt the simple-cubic structure. The cold quark matter may have other kinds of structures, but that will not make much differences at least quantitatively. If the nearest distance between two quark-clusters is $R$, then the total interaction potential of $N$ quark-clusters is

$$U(R) = 2NU_0[A_{12}(\frac{r_0}{R})^{12} - A_6(\frac{r_0}{R})^6], \quad (4)$$

where $A_{12} = 6.2$, and $A_6 = 8.4$. In the simple cubic structure, the number density of clusters $n$ is

$$n = \frac{3}{2} R^{-3}, \quad (5)$$

so

$$U(R) = 2NU_0(A_{12}r_0^{12} n^4 - A_6 r_0^6 n^3), \quad (6)$$

and the potential energy density is

$$\epsilon_\rho = 2U_0(A_{12}r_0^{12} n^3 - A_6 r_0^6 n^3), \quad (7)$$

2.2 Lattice vibrations

Consider a system of volume $V$ containing $N$ quark-clusters. Each quark-cluster in the crystal lattice undergoes a three-dimensional vibration about its lattice site. Performing a normal-mode analysis in which the vibrations of the lattice are decomposed into 3N independent normal modes of vibrations, the total lattice vibration is a superposition of these 3N decoupled vibrations.

For cold quark matter, the thermal vibration can be neglect compared to the zero-point energy of phonon, so the average energy of an individual mode of vibration with frequency $\omega_j$ is

$$E_j = \frac{1}{2} \hbar \omega_j. \quad (8)$$

Using Debye approximation, at low temperature, the thermodynamics properties of crystal lattice is mainly determined by the long-wavelength sound waves, and the propagation of the wave can be decomposed into one longitudinal mode with velocity $v_\parallel$ and two transverse modes with velocity $v_\perp$, and the total velocity $v$ has the relation

$$\frac{1}{v^3} = \frac{1}{3} (\frac{1}{v_\parallel^3} + \frac{2}{v_\perp^3}), \quad (9)$$

and the total energy of the 3N vibrations is

$$\bar{E} = \int_0^{\hbar \omega_m} \frac{1}{2} h \omega f(\omega) d\omega, \quad (10)$$

where $f(\omega)d\omega$ is the number of modes in the interval from $\omega$ to $\omega + d\omega$, and $\omega_m$ is the maximum frequency related to the non-continuous structure of the solid, and is determined by

$$\omega_m = v(6\pi^2 n)^{1/3}. \quad (11)$$

Under the condition of zero-temperature, the integration can be done and the total energy of the crystal vibrations is

$$\bar{E} = \frac{9V}{8} (6\pi^2)^{\frac{3}{2}} \hbar v n^{\frac{3}{2}}, \quad (12)$$

so the energy density of the lattice vibration is
3 EQUATION OF STATE FOR QUARK STARS

3.1 Quark stars composed of Lennard-Jones matter

The total energy density for cold quark matter is

\[ \epsilon_q = \epsilon_p + \epsilon_L + n m_c c^2 \]

\[ = 2 U_0 (\rho q A_{12} r_0^{-12} n^5 - A_6 r_0^{-6} n^3) + \frac{9}{8} (6 \pi^2)^{1/3} \hbar c \alpha \]

\[ + \frac{9}{8} (6 \pi^2)^{1/3} \hbar c + n m_c c^2, \tag{14} \]

here \( m_c \) is the mass of each quark-cluster. The pressure can be derived as

\[ P_q = n^2 \frac{d(\epsilon_q/n)}{dn} \]

\[ = 4 U_0 (2 A_{12} r_0^{12} n^5 - A_6 r_0^{-6} n^3) + \frac{3}{8} (6 \pi^2)^{1/3} \hbar c \alpha. \tag{15} \]

Apart from quarks, there are electrons in quark matter. In MIT bag model, the number of electrons per baryon \( N_e/A \) is found for different strange quarks mass \( m_s \) and coupling constant \( \alpha_s \) [Farhi & Jaffe 1984]. In their results, when \( \alpha_s = 0.3 \), \( N_e/A \) is less than \( 10^{-5} \); a larger \( \alpha_s \) means a smaller \( N_e/A \) at fixed \( m_s \), because the interaction between quarks will lead to more strange quarks and consequently less electrons. In our model, we also consider the strong interaction between quarks as well as between quark-clusters, and consequently the required number of electrons per baryon to guarantee the neutrality should be also be very small. Although at the present stage we have not got the exact value for the number density of electrons, we assume that \( N_e/A \) is less than \( 10^{-4} \). We find that even at this value, the pressure of the degenerate electrons is negligible comparing to the pressure of quarks. Therefore, we neglect the contribution of electrons to the equation of state. Then the equation of state for quark stars is

\[ P = P_q, \tag{16} \]

\[ \rho = \epsilon_q/c^2. \tag{17} \]

3.2 Parameters

Up to now, there are a couple of parameters in the equation of state, and in this section we will show how to determine them.

1. The long-wavelength sound speed \( v \) for lattice vibration. For the extremely relativistic systems, the sound velocity is \( 1/\sqrt{3} \), and in general it will be less than \( 1/\sqrt{3} \). We find that the equation of state does not change much when \( v \) goes from the velocity of light \( c \) to \( 10^{-5} c \), so in our calculations we set \( v = c/3 \).

2. The mass of quarks. Quark stars are composed entirely of deconfined light quarks (up, down and strange quarks), the so-called strange stars. Because the deconfined phase transition and the chiral-restoration phase transition might not occur simultaneously in the QCD phase-diagram, we give each quark a constituent mass and assume it is one-third of nuclear mass.

3. The number of quarks in one cluster \( N_q \). Quarks are fermions and they have three flavor (up, down and strange) degrees of freedom and three color degrees of freedom. Pauli’s exclusion principle tells us that if in the inner space quarks are exchange-asymmetric, they are exchange-symmetric in position space and they have a tendency to condensate in position space. We therefore conjecture the existence of quark-clusters in quark matter and leave the number of quarks in one cluster, \( N_q \), as a free parameter. A 18-quark cluster, called quark-alpha [Michel 1988], could be completely asymmetric in spin, flavor and color space, so in our calculation we set \( N_q = 18 \), and we also set \( N_q = 3 \). On the other hand, it has been conjectured that strongly interacting matter at high densities and low temperatures might be in a “quarkyonic” state, which also contains three quarks in one cluster, and is characterized by chiral symmetry and confinement [McLerran & Pisarski 2007, Blaschke et al. 2008]. However, in our model, the state is characterized by chiral symmetry breaking and deconfinement.

4. The depth of the potential \( U_0 \). Given the density of quark matter \( \rho \) and the mass of each individual quark, from Heisenberg’s uncertainty relation we can approximate the kinetic energy of one cluster as

\[ E_k \sim 1 \text{ MeV} (\frac{\rho}{\rho_0})^{1/2} (\frac{N_q}{18})^{-1/2}, \tag{18} \]

where \( \rho_0 \) is the nuclear matter density. To get the quarks trapped in the potential wells to form lattice structure, \( U_0 \) should be larger than the kinetic energy of quarks. Because of the strong interaction between quarks, we adopt \( U_0 = 50 \) MeV and 100 MeV to do the calculations.

5. The range of the action \( \alpha_s \). Since a quark star could be bound not only by gravity but also by strong interaction due to the confinement between quarks, the number density of quarks on a quark star surface \( \rho_0 \) is non-zero. For a given \( \rho_0 \), we can get \( r_0 \) at the surface where the pressure is zero. We choose \( \rho_0 \) as 2 times of nuclear matter densities, and get the value of \( r_0 \) accordingly, which is found in the range from about 1 to 3 fm.

When \( U_0 \) and \( r_0 \) are given, the inter-cluster potential Eq.(1) is fixed. One should note that it describes the interaction between only two clusters; if we consider other clusters’ influences, a cluster will always be in the minimal potential state. When the cluster deviates from the equilibrium position, it will be pulled back due to the stronger repulsion from one side, just as the case of a chain of springs.

3.3 Comparison with the MIT bag model

In the MIT bag model, quark matter is composed of massless up and down quarks, massive strange quarks, and few electrons. Quarks are combined together by an extra potential, denoted by the bag constant \( B \). For the comparison, we apply the formulae given by Alcock [Alcock et al. 1986] to calculate the equation of state, with strange quark mass \( m_s = 100 \text{ MeV} \), the strong coupling constant \( \alpha_s = 0.3 \), and the bag constant \( B = 60 \text{ MeV fm}^{-3} \) (e.g., Zdunik 2000). The comparison of equation of state in our model and in the MIT bag model is shown in Fig.1.

In our model, quarks are grouped in clusters and these clusters are non-relativistic particles. If the inter-cluster potential can be described as the Lennard-Jones form, the equation of state can be very stiff, because at a small inter-
The comparison of the equations of state, in the case \( N_q = 3 \), including \( U_0 = 50 \) MeV (blue solid lines) and \( U_0 = 100 \) MeV (blue dashed lines), and the corresponding case \( N_q = 18 \) with \( U_0 = 50 \) MeV (red dash-dotted lines) and \( U_0 = 100 \) MeV (red dotted lines), and that derived in the MIT bag model with the mass of strange quark \( m_s = 100 \) MeV and the strong coupling constant \( \alpha_s = 0.3 \) and bag constant \( B = 60 \) MeV fm\(^{-3} \) (thin lines), for a given surface density \( \rho_s = 2 \rho_0 \). Here and in the following figures, \( \rho_0 \) is the nuclear saturation density.

Cluster distance (i.e., the number density is large enough), there is a very strong repulsion. Whereas in MIT bag model quarks are relativistic particles (at least for up and down quarks). For a relativistic system, the pressure is proportional to the energy density, so it cannot have stiff equation of state.

### 3.4 The speed of sound

The adiabatic sound speed is defined as

\[
c_s = \sqrt{\frac{dP}{d\rho}}. \tag{19}\]

If we use the equation of state in our model, the speed of sound will exceed the speed of light not far away from the surface of a quark star. It seems to contradict to the relativity that signals cannot propagate faster than light.

The possibility of speed of sound exceeding the speed of light in ultradense matter have been discussed previously (Bludman & Ruderman 1968) because of using classical potential (i.e., a kind of action at a distance). The physical reasons of apparent superluminal have also been analyzed (Caporaso & Brecher 1979). The authors argued that the adiabatic sound speed can exceed the speed of light, yet signals propagate at speed less than \( c \).

One reason is that the \( P(\rho) \) relation arises from a static calculation, ignoring the dynamics of the medium. The notion that \( c_s \) is a signal propagation speed is a carry-over from Newtonian hydrodynamic, in which one assumes infinite interaction speed but finite temperature, so the static and dynamic calculations give the same result. On the other hand, if one assumes finite interaction speed and zero temperature, the adiabatic sound speed is not a dynamically meaningful speed, but only a measure of the local stiffness.

Another reason is that a lattice does not have an infinite range of allowed frequencies of vibration, but a signal should contain components at all frequencies. Therefore, the adiabatic sound speed is not capable of giving the velocity of propagation of disturbs.

In our model, although we have not make it explicit that how the particles interact with each other, we may assume that the interaction is mediated by some particles with non-zero masses, and the interaction does not propagate instantaneously. We have also use the low frequency approximation to calculate the lattice energy. Therefore, we could conclude that in our model the signal can not propagate faster than light.

Whether the equation of state of cold quark matter can be so stiff that the adiabatic speed of sound is larger than \( c \) could still be an open question. However, in our present paper we do not put the limitation on the adiabatic sound speed, and only treat it as a measurement of the stiffness of the equation of state.

### 4 MASSES AND RADII

From the equations of state, we can get the mass-radius curves and mass-central density curves (the central density only includes the rest mass energy density), as are shown in Fig.2.

Because of stiffer equations of state, which we have discussed in §3, the maximum masses of quark stars in our model could be higher. In Fig.2, we can see that (i) a deeper potential well \( U_0 \) means a higher maximum mass; (ii) if there are more quarks in a quark-cluster, the maximum mass of a quark star will be lower.

A stiffer equation of state leading to a higher maximum mass could have very important astrophysical implications. Although we could still obtain high maximum masses under MIT bag model by choosing suitable parameters (Zdunik et al. 2000), a more realistic equation of state in the density-dependent quark mass model (e.g., Dev et al. 1993) is very difficult to reach a high enough maximum stellar mass, which was considered as possible negative evidence for quark stars (Cottam et al. 2002). Some recent observa-
tions have indicated some massive ($\sim 2M_\odot$) pulsars (e.g., Freire et al. 2008); however, because of the uncertain inclination of the binary systems, we are still not sure about the real mass. Though we have not definitely detected any pulsar whose mass is higher than $2M_\odot$ up to now, the Lennard-Jones quark star model could be supported if massive pulsars ($>2M_\odot$) are discovered in the future. Moreover, a high maximum mass for quark stars might be helpful for us to understand the mass-distribution of stellar-mass black holes (Bailyn et al. 1998), since a compact star with a high mass (e.g., $\sim 5M_\odot$) could still be stable in our model presented.

5 CONCLUSIONS AND DISCUSSIONS

In cold quark matter at realistic baryon densities of compact stars (with an average value of $\sim 2-3\rho_0$), the interaction between quarks is so strong that they would condensate in position space to form quark-clusters. Like the classical solid, if the inter-cluster potential is deep enough to trap the clusters in the potential wells, the quark matter would crystallize and form solid quark stars. This picture of quark stars is different from the one in which quarks form Cooper pairs and quark stars are consequently color super-conductive.

In this paper, we argue that quarks in quark stars are grouped in clusters and the quark-clusters form simple-cubic structure, and apply Lennard-Jones potential to describe the interaction potential between quark-clusters. The parameters such as the depth of potential $U_0$ (50 MeV and 100 MeV) and the range of interaction $r_0$ (about 1 to 3 fm) are given by the physical context of quark stars. Under such equations of state, the masses and radii of quark stars are derived, and we find that the mass of a quark star can be higher than $2M_\odot$.

It is surely interesting to experimentally or observationally distinguish between our solid quark star model and other models for quark stars, e.g., the color superconductivity state. Starquakes could naturally occur in solid quark stars and the observations of pulsar glitches and SGR giant flares could qualitatively be reproduced when the solid matter breaks (Zhou et al. 2004; Xu et al. 2006), moreover the post-glitch recovers in the solid quark star model and the color super-conductivity model would be different. Additionally, because the solid quark star model depends on quark clustering, the interaction behaviors between quarks could be tested in sQGP (strongly coupled quark-gluon plasma, see Shuryak 2009) by the LHC and/or FAIR experiments.

The research of compact stars involves two kinds of challenges: particle physics and many-body physics. Nevertheless, if we know about the properties of compact stars from observations, we can get information of the elementary physics. Take the model we discussed in this paper as an example. If we get the masses and radii of some pulsars from accurate enough observations, we can put limits on the parameters such as potential well depth $U_0$, interaction range $r_0$, and the number of quarks that condensate in position space to form a cluster, which could help us to explore the strong interaction between quarks. Although the state of cold quark matter at a few nuclear densities is still an unsolved problem in the low-energy QCD, it would be hopeful for us to use pulsars as idea laboratories to study the nature of strong interaction.

In general, stars are equilibrium bodies with pressure against gravity. The thermal and radiation pressure dominates in main sequent stars, while degenerate pressure of Fermions, originated from Pauli’s principle, dominates in Fermion stars (e.g., while dwarfs). For solid quark stars in the models presented in this paper, the pressure is related to the increase of both potential and lattice vibration energies as the stellar quark matter contracts. The degenerate pressure might be negligible there.

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