A Tight Bound on the Projective Dimension of Four Quadrics

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K a field
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$S = K[X_1, X_2, \ldots, X_n]$
Notation

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\[ S = K[X_1, X_2, \ldots, X_n] \]
\[ S = \bigoplus_{i=0}^{\infty} S_i \text{ is graded} \]
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$S = \bigoplus_{i=0}^{\infty} S_i$ is graded
$S(-d)_i = S_{i-d} = \text{rank one free module with generator in degree } d$
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$I = (f_1, \ldots, f_N) \subset S$ a homogeneous ideal
(i.e. each $f_j$ is in some $S_i$)
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$\text{pd}(S/I) = \text{length of the minimal graded free resolution of } S/I$
Notation

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$$0 \leftarrow S/I \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_p \leftarrow 0,$$

where $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$.
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\[ \text{pd}(S/I) = \max\{i \mid \beta_{i,j} \neq 0\} = p \]

\[ \text{reg}(S/I) = \max\{j - i \mid \beta_{i,j} \neq 0\} \]
An Example: $I = (x^2, y^2, z^2, ax + by + cz)$

Betti Table of $S/I$:

|       | 0  | 1  | 2  | 3  | 4  | 5  | 6  |
|-------|----|----|----|----|----|----|----|
| 0:    | 1  | -  | -  | -  | -  | -  | -  |
| 1:    | -  | 4  | -  | -  | -  | -  | -  |
| 2:    | -  | -  | 6  | -  | -  | -  | -  |
| 3:    | -  | -  | 4  | 17 | 15 | 6  | 1  |
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$pd(S/I) = 6$
An Example: \( I = (x^2, y^2, z^2, ax + by + cz) \)

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\( \text{pd}(S/I) = 6 \)

\( \text{reg}(S/I) = 3 \)
Stillman’s Question

Question (Stillman (2003))

Is there a bound (independent of \( n \)) on \( \text{pd}(S/I) \) depending only on the degrees of the generators of \( I \)?
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Is there a bound (independent of $n$) on $\text{pd}(S/I)$ depending only on the degrees of the generators of $I$?

Still open in full generality.
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Motivation:

- Equivalent to bounding \( \text{reg}(S/I) \) in terms of \( d_1, \ldots, d_N \) (Caviglia)
Stillman’s Question

Known affirmative answers:
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- $\text{pd}(S/(3 \text{ quadrics })) \leq 4$ (Eisenbud-Huneke)
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- \( pd(S/(3 \text{ cubics})) \leq 36 \) (Engheta)
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Known affirmative answers:

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- \( \text{pd}(S/(N \text{ quadrics })) \leq O(2N^{2N}) \) (Ananyan-Hochster)
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- \( \text{pd}(S/(N \text{ quadrics of height 2})) \leq 2N - 2 \) (HMMS)
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- $\text{pd}(S/(4 \text{ quadrics })) \leq 8$ (HMMS)
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- $\text{pd}(S/(3 \text{ cubics})) \leq 8(5)$ (MM)
Engheta’s Method

If $\text{ht}(I) = 1$ or $4$, then $\text{pd}(S/I) \leq 4$. 
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If $\text{ht}(I) = 3$, reduce the problem to bounding $\text{pd}(S/J)$ for $J$ unmixed over $K = \overline{K}$:
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Let $I = (q_1, q_2, q_3, q_4)$. 
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May assume \( q_1, q_2, q_3 \) form a regular sequence.
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Consider the short exact sequence:

$$0 \rightarrow \frac{S}{(q_1, q_2, q_3) : (q_4)} \xrightarrow{q_4} \frac{S}{(q_1, q_2, q_3)} \rightarrow \frac{S}{I} \rightarrow 0$$
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$$

$$
\text{pd}(S/I) \leq \max\{3, \text{pd}(S/((q_1, q_2, q_3) : q_4)) + 1\}.
$$
Engheta’s Method

- If $J$ is unmixed and $ht(J) = 1$, then $J = (f)$ and $pd(S/J) = 1$.

$(S \xleftarrow{f} S$ is a free resolution.)
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• If $J$ is unmixed and $\text{ht}(J) = 1$, then $J = (f)$ and $\text{pd}(S/J) = 1$. ($S \xleftarrow{f} S$ is a free resolution.)
• If $J$ is unmixed and has $e(S/J) = 1$, then $J = (x_1, \ldots, x_h)$ and $\text{pd}(S/J) = h$. 
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• If $J$ is unmixed and $e(S/J) = ht(J) = 2$, then....
Proposition (Engheta)

If $J$ is unmixed and $K = \bar{K}$, $\text{ht}(J) = e(S/J) = 2$, then $\text{pd}(S/J) \leq 3$ and $J$ is one of the following:
Engheta’s method

**Proposition (Engheta)**

*If* $J$ *is unmixed and* $K = \overline{K}$, $\text{ht}(J) = e(S/J) = 2$, *then* $\text{pd}(S/J) \leq 3$ *and* $J$ *is one of the following:*

- *(prime)* $J = (x, q)$
Engheta’s method

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If $J$ is unmixed and $K = \overline{K}$, $\text{ht}(J) = e(S/J) = 2$, then $\text{pd}(S/J) \leq 3$ and $J$ is one of the following:

- (prime) $J = (x, q)$
- (intersection of linear primes)
  - $(w, x) \cap (y, z) = (wy, wz, xy, xz)$
  - $(x, y) \cap (x, z) = (x, yz)$
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- (primary to $(x, y)$)
  - $J = (x, y^2)$
  - $J = (x, y)^2 + (ax + by)$ where $x, y, a, b$ form a regular sequence
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Similar statement for height 3 multiplicity 2?
Primary Ideals with Large Projective Dimension

Theorem (Huneke-Mantero-M-Seceleanu)

Let $K = \overline{K}$. For any integers $h, e \geq 2$ with $(h, e) \neq (2, 2)$ and for any integer $p$, there exists a primary ideal $I = I$ with

- $ht(I) = h$
- $e(S/I) = e$
- $\sqrt{I} = (x_1, x_2, \ldots, x_h)$
- $pd(S/I) \geq p$. 
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- $pd(S/I) \geq p$.

No finite classification of primary ideals, even for height 3 multiplicity 2.
Classification of \((x, y, z)\)-primary ideals

However, we do get a finite list if we bound the generating degree!
Classification of \((x, y, z)\)-primary ideals

However, we do get a finite list if we bound the generating degree!

**Proposition (HMMS)**

\[
\text{Let } J \text{ be } (x, y, z)\text{-primary with } e(S/J) = 2. \text{ Then one of the following holds:}
\]
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Let \(J\) be \((x, y, z)\)-primary with \(e(S/J) = 2\). Then one of the following holds:

- \(J = (x, y, z^2)\)
- \(J = (x, y^2, yz, z^2, ay + bz)\), where \(ht(x, y, z, a, b) = 5\)
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- \(J = (x, y, z)^2 + (ax + by + cz, dx + ey + fz), \text{ where } \text{ht}(x, y, z, l_2) \geq 5\)
- All quadrics in \(J\) can be written in terms of at most 6 variables.
Classification of \((x, y, z)\)-primary ideals

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Let \(J\) be \((x, y, z)\)-primary with \(e(S/J) = 3\). Then one of the following holds:
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*All quadrics in \(J\) can be written in terms of at most 6 variables.*

*All quadrics in \(J\) generate an ideal of height at most 2.*
Classification of \((x, y, z)\)-primary ideals

**Proposition (HMMS)**

Let \(J\) be \((x, y, z)\)-primary with \(e(S/J) = 4\). Then one of the following holds:
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Classification of \((x, y, z)\)-primary ideals

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- All quadrics in \(J\) can be written in terms of at most 6 variables.
- All quadrics in \(J\) generate an ideal of height at most 2.
- \(J\) contains a linear form.
Classification of \((x, y, z)\)-primary ideals

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- All quadrics in \(J\) can be written in terms of at most 6 variables.
- All quadrics in \(J\) generate an ideal of height at most 2.
- \(J\) contains a linear form.
- \(\text{pd}(S/J) \leq 5\).
Classification of \((x, y, q)\)-primary ideals

**Proposition**

Let \(q = (x, y, q)\) be a height 3 multiplicity 2 prime and let \(J\) be a \(q\)-primary ideal with \(e(S/J) = 4\). Then \(J\) has one of the following forms:
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- \( J = (x^2, xy, y^2, q) + (\geq \deg 3) \),
- \( J = (x^2, xy, y^2, ax + by, q) + (\geq \deg 3) \), where \( \text{ht}(x, y, a, b, q) = 5 \),
- \( J = (x^2, xy, y^2, ax + by, cx + dy, ad - bc + ex + fy = q) \), where \( \text{ht}(x, y, a, b) = \text{ht}(x, y, c, d) = 4 \) and \( \text{ht}(x, y, ad - bc) = 3 \).
Classification of \((x, y, q)\)-primary ideals

**Proposition**

Let \(q = (x, y, q)\) be a height 3 multiplicity 2 prime and let \(J\) be a \(q\)-primary ideal with \(e(S/J) = 4\). Then \(J\) has one of the following forms:

- \(J = (x, y^2, q)\),
- \(J = (x^2, xy, y^2, q) + (\geq \text{deg 3})\),
- \(J = (x^2, xy, y^2, ax + by, q) + (\geq \text{deg 3})\), where \(\text{ht}(x, y, a, b, q) = 5\),
- \(J = (x^2, xy, y^2, ax + by, cx + dy, ad - bc + ex + fy = q)\), where \(\text{ht}(x, y, a, b) = \text{ht}(x, y, c, d) = 4\) and \(\text{ht}(x, y, ad - bc) = 3\).

*All quadrics in \(J\) generate an ideal of height at most 2.*
Problem case:
\[ I = (ax + y^2, bx + yz, cx + z^2, dx) \text{ in } S = k[a, b, c, d, x, y, z] \]
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Still need to prove \( \text{pd}(S/I) \leq 6 \) if \( (q_1, q_2, q_3) : I \) extended from 6 variable polynomial ring.
Lemma

Suppose $q_1, q_2, q_3 \in K[z_1, \ldots, z_6]$. If $I = (q_1, q_2, q_3, xy)$, then $\text{pd}(S/I) \leq 6$. 
Lemma

Suppose $q_1, q_2, q_3 \in K[z_1, \ldots, z_6]$. If $I = (q_1, q_2, q_3, xy)$, then $\text{pd}(S/I) \leq 6$.

Proof: May assume $I = (xy, q_1, q_2, q_3)$ with $\text{ht}(q_1, q_2, q_3) = 3$. 
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If $x$ regular on $S/(q_1, q_2, q_3)$, then $I + (x)$ and $I : x$ have form $(q_1, q_2, q_3, \ell)$.
Still not done

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$\text{pd}(S/(3\text{ quadrics} + 1\text{ linear form})) \leq 5 \implies \text{pd}(S/I) \leq 5$
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Lemma

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$\implies x, y \in K[z_1, \ldots, z_6]$ Done
Main Result

4 more Lemmas gives...
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**Theorem (HMMS)**

Let $S$ be a polynomial ring over a field $K$, and let $I = (q_1, q_2, q_3, q_4)$ be an ideal of $S$ minimally generated by 4 homogeneous polynomials of degree 2. Then $\text{pd}(S/I) \leq 6$. Moreover, this bound is tight.
Additivity Formula

**Proposition**

*If $J$ is an ideal of $S$, then*

$$e(S/J) = \sum_{\text{primes } p \supseteq J} e(S/p) \lambda(S_p/J_p).$$

*ht$(p) = \text{ht}(J)$
Additivity Formula

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if $J$ is an unmixed ideal, we say it is of type

$$\langle e_1, \ldots, e_m; \lambda_1, \ldots, \lambda_m \rangle$$

if $J$ has $m$ associated prime ideals $p_1, \ldots, p_m$ with $e_i = e(S/p_i)$ and $\lambda_i = \lambda(S_{p_i}/J_{p_i})$ for all $i$. 
Additivity Formula

**Proposition**

*If $J$ is an ideal of $S$, then*

$$e(S/J) = \sum_{\text{primes } p \supseteq J \atop \text{ht}(p) = \text{ht}(J)} e(S/p)\lambda(S_p/J_p).$$

*if $J$ is an unmixed ideal, we say it is of type*

$$\langle e_1, \ldots, e_m; \lambda_1, \ldots, \lambda_m \rangle$$

*if $J$ has $m$ associated prime ideals $p_1, \ldots, p_m$ with $e_i = e(S/p_i)$ and $\lambda_i = \lambda(S_{p_i}/J_{p_i})$ for all $i$. Note: $e(S/J) = \sum_{i=1}^m e_i\lambda_i$.***
## Bounds on $\text{pd}(S/I)$ when $\text{ht}(I) = 2$

| $e(S/I)$ | $\langle e; \lambda \rangle$ for $I^{un}$ | Bound for $\text{pd}(S/I)$ |
|----------|--------------------------------|--------------------------|
| 1        | $\langle 1; 1 \rangle$ | 6                        |
| 2        | $\langle 2; 1 \rangle$ | 4                        |
|          | $\langle 1; 2 \rangle$ | 6                        |
|          | $\langle 1, 1; 1, 1 \rangle$ | 5                        |
| 3        | $\langle 3; 1 \rangle$ | 2                        |
|          | $\langle 1; 3 \rangle$ | 6                        |
|          | $\langle 1, 2; 1, 1 \rangle$ | 4                        |
|          | $\langle 1, 1; 1, 2 \rangle$ | 5                        |
|          | $\langle 1, 1, 1; 1, 1, 1 \rangle$ | 5                        |
Bounds on $\text{pd}(S/I)$ when $\text{ht}(I) = 3$

### Table 1: Bounds on $\text{pd}(S/I)$ for $I$ unbound

| $\text{e}(S/I)$ | $\langle \text{e}; \lambda \rangle$ for $I \text{ unmixed}$ | Bound for $\text{pd}(S/I)$ |
|------------------|-----------------------------------------------------------|-----------------------------|
| 1                | $\langle 1; 1 \rangle$                                     | 4                           |
| 2                | $\langle 2; 1 \rangle$                                     | 4                           |
|                  | $\langle 1; 2 \rangle$                                     | 5                           |
|                  | $\langle 1, 1; 1, 1 \rangle$                               | 5                           |
| 3                | $\langle 3; 1 \rangle$                                     | 5                           |
|                  | $\langle 1, 3 \rangle$                                     | 6                           |
|                  | $\langle 1, 1; 1, 2 \rangle$                               | 6                           |
|                  | $\langle 1, 1, 1; 1, 1 \rangle$                            | 6                           |
| 6                | any                                                         | 3                           |

### Table 2: Bounds on $\text{pd}(S/L)$ for $L = (q_1, q_2, q_3) : I$

| $\text{e}(S/L)$ | $\langle \text{e}; \lambda \rangle$ for $L = (q_1, q_2, q_3) : I$ | Bound for $\text{pd}(S/I)$ |
|------------------|---------------------------------------------------------------|-----------------------------|
| 4                | $\langle 4; 1 \rangle$                                     | 4                           |
|                  | $\langle 2; 2 \rangle$                                     | 5                           |
|                  | $\langle 1; 4 \rangle$                                     | 6                           |
|                  | $\langle 1, 3; 1, 1 \rangle$                               | 5                           |
|                  | $\langle 1, 1; 1, 3 \rangle$                               | 6                           |
|                  | $\langle 2, 2; 1, 1 \rangle$                               | 6                           |
|                  | $\langle 1, 2; 2, 1 \rangle$                               | 6                           |
|                  | $\langle 1, 1, 2, 2 \rangle$                               | 6                           |
|                  | $\langle 1, 1, 2; 1, 1, 1 \rangle$                         | 6                           |
|                  | $\langle 1, 1, 1; 1, 1, 1 \rangle$                         | 6                           |
| 3                | $\langle 3; 1 \rangle$                                     | 5                           |
|                  | $\langle 1; 3 \rangle$                                     | 6                           |
|                  | $\langle 1, 2; 1, 1 \rangle$                               | 5                           |
|                  | $\langle 1, 1; 1, 2 \rangle$                               | 6                           |
|                  | $\langle 1, 1, 1; 1, 1, 1 \rangle$                         | 6                           |
Thank you!

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