Prospects for determination of thermal history after inflation with future gravitational wave detectors

Sachiko Kuroyanagi,1 Kazunori Nakayama,2 and Shun Saito3

1Institute for Cosmic Ray Research, The University of Tokyo, Chiba 277-8582, Japan
2Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan
3Department of Astronomy, University of California at Berkeley, California 94720, USA

(Dated: October 20, 2011)

Abstract

Thermal history of the Universe between inflation and big-bang nucleosynthesis has not yet been revealed observationally. It will be probed by the detection of primordial gravitational waves generated during inflation, which contain information on the reheating temperature as well as the equation of state of the Universe after inflation. Based on Fisher information formalism, we examine how accurately the tensor-to-scalar ratio and reheating temperature after inflation can be simultaneously determined with space-based gravitational wave detectors such as the DECI-hertz Interferometer Gravitational-wave Observatory (DECIGO) and the Big-Bang Observer (BBO). We show that the reheating temperature is best determined if it is around $10^7$ GeV for tensor-to-scalar ratio of around 0.1, and explore the detectable parameter space. We also find that equation of state of the early Universe can be also determined accurately enough to distinguish different equation-of-state parameters if the inflationary gravitational waves are successfully detected. Thus future gravitational wave detectors provide a unique and promising opportunity to reveal the thermal history of the Universe around $10^7$ GeV.

*skuro@icrr.u-tokyo.ac.jp
I. INTRODUCTION

The standard paradigm in cosmology is well established and explains cosmological evolution after the big-bang nucleosynthesis (BBN). An epoch of primordial inflation at the very early Universe now also becomes an essential part of the standard paradigm, and is supported by accurate measurements of the cosmic microwave background (CMB) anisotropies. The nearly scale-invariant power spectrum of the curvature perturbation, as well as its Gaussian nature, strongly indicates the inflation as a source of the density fluctuation of the Universe [1].

However, thermal history *between* inflation and BBN has not yet revealed observationally. There must be an epoch of reheating after inflation, where the inflaton decays and the radiation-dominated universe starts. The process of reheating crucially depends on the properties of the inflaton: its mass, potential and couplings to standard model particles. Before the inflaton decays, inflaton oscillates around its potential minimum. In the simplest class of inflation models, the inflaton oscillation behaves as matter after inflation ends. But the detail of the inflaton oscillation, or the equation of state of the Universe during the inflaton oscillation depends on the nature of the inflaton. Therefore, it is extremely important to probe the thermal history after inflation. It would determine the nature of the inflaton and might be a direct hint for the underlying high-energy theory such as supergravity or string theory.

Then, how can we probe the very early Universe observationally? Perhaps gravitational waves may be a unique signal that directly carries information on the thermal history of these early epochs, since the Universe is transparent to gravitational waves up to the Planck epoch in principle. Fortunately, there is a promising source of the gravitational wave: primordial gravitational waves generated by inflation [2]. Quantum fluctuations during inflation are expanded and frozen outside the Hubble horizon. Tensor fluctuations generated in this way can be observed as gravitational waves. These gravitational waves form a stochastic background having a very wide range of wavelength from the present horizon scale to the terrestrial one. If the inflationary energy scale is sufficiently large, inflationary gravitational waves will be detected through on-going or future CMB B-mode polarization measurements [3, 4] and/or space laser interferometers [2, 6]. Proposed future gravitational wave measurements such as the DECI-hertz Interferometer Gravitational-wave Observatory (DECIGO)
and the Big-Bang Observer (BBO) [9, 10] aim at the detection of the inflationary gravitational wave background around 0.1-1 Hz as well as the determination of the nature of the present accelerated expansion of the Universe with binary sources as standard sirens [10–12].

Interestingly, the spectrum of the inflationary gravitational wave background directly reflects the expansion history of the Universe [13–16]. This is because the gravitational wave amplitude frozen at super-horizon scale begins to damp inversely proportional to the scale factor of the Universe when the corresponding mode enters the horizon. As we have already discussed, the matter-dominated reheating period is followed by the radiation-dominated era. Therefore, the spectrum shows a knee-like feature at the frequency corresponding to the comoving horizon scale at the end of reheating, which is given by

\[ f_R \approx 0.26 \text{ Hz}(T_R/10^7 \text{ GeV}) \]

(see Eq. (13) below) where \( T_R \) is the temperature of the Universe at the end of reheating defined by

\[ T_R = \left( \frac{10}{\pi^2 g_*(T_R)} \right)^{1/4} \sqrt{\frac{\Gamma_\phi}{M_P}}, \]

(1)

with \( \Gamma_\phi \) being the inflaton decay rate and \( M_P \) being the reduced Planck scale. If future gravitational wave detectors see the knee shape at this frequency, it will be a direct measurement of the reheating temperature.

The prospects for direct detection of the inflationary gravitational wave background with the effect of reheating have been investigated in Refs. [17–19]. The signal-to-noise ratio is calculated taking into account the effect that reheating induces a change of the frequency dependence in the spectrum, and the detectability of the gravitational wave background is discussed in the previous works. However, direct detection of the gravitational wave amplitude does not necessarily mean accurate determination of the reheating temperature. In order to estimate the detectability of the reheating signature more realistically, we need to examine if the experiment can distinguish the change of the characteristic frequency dependence induced by reheating. For that purpose, we employ the Fisher matrix formalism, which enables us to evaluate the sensitivity to the shape of the spectrum with properly taking into account parameter degeneracies.

As we have mentioned, the change of the frequency dependence is attributed to the fact that reheating causes the change of the Hubble expansion rate. Thus, the inflationary gravitational wave background is useful not only determining the reheating temperature but
also constraining the expansion history of the very early Universe. We evaluate the ability of future direct detection experiments to determine the equation of state of the early Universe, keeping in mind that many applications on cosmological models are expected with direct detection of the inflationary gravitational wave background [13–23].

In this paper, we study prospects for determination of the reheating temperature, as well as the equation of state of the early Universe, at future gravitational wave detectors based on the Fisher matrix method. In Sec. II, basic formulations are provided. In Sec. III, we derive the accuracy of the reheating temperature measurement with future gravitational wave detectors. In Sec. IV, we study how accurately the equation of state of the early Universe will be determined in the same setup. The final section is devoted to conclusions.

The fiducial cosmological parameters are taken to be the maximum likelihood values from the combined analysis of the WMAP 7-year, baryon acoustic oscillation, and supernova data with a flat $\Lambda$CDM universe [1]: matter density $\Omega_m h^2 = 0.1344$, amplitude of curvature perturbation $\Delta^2_{R,\text{prim}} = 2.45 \times 10^{-9}$ and the Hubble parameter $h = 0.702$. Throughout the paper we choose the natural units, $c = \hbar = 1$.

II. FISHER MATRIX METHOD

A. Definitions of gravitational wave observables

Before discussing the Fisher matrix approach, let us first summarize the observable quantities of the gravitational wave detectors. Gravitational waves in the expanding Universe are described as tensor perturbations in the Friedmann-Robertson-Walker metric, $ds^2 = a^2(t)[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j]$, with $a(t)$ being the scale factor of the Universe. The conformal time $\tau$ is defined as $d\tau \equiv dt/a(t)$, and the subscript $i$ and $j$ denote spatial indices which run over 1, 2, 3 or $x, y, z$. The tensor perturbation $h_{ij}$ satisfies the transverse-traceless conditions, $\partial^i h_{ij} = h^i = 0$. It is expanded into its Fourier series as

$$h_{ij}(t, \mathbf{x}) = \sum_{\lambda=+,\times} \int \frac{d^3k}{(2\pi)^3/2} \epsilon^\lambda_{ij}(k) h^\lambda_k(t) e^{i\mathbf{k} \cdot \mathbf{x}},$$

where the polarization tensors $\epsilon^\lambda_{ij}$ satisfy symmetric and transverse-traceless conditions and are normalized as $\sum_{i,j} \epsilon^\lambda_{ij} (\epsilon^\lambda_{ij})^* = 2\delta^{\lambda\lambda'}$. Then the observable intensity of a stochastic gravitational wave background is characterized by its density parameter per logarithmic
wavenumber (or frequency) as
\[ \Omega_{\text{GW}} \equiv \frac{1}{\rho_c} \frac{d \rho_{\text{GW}}}{d \ln k} = \frac{1}{12} \left( \frac{k}{aH} \right)^2 \frac{k^3}{\pi^2} \sum_{\lambda} |h^2_{\lambda}|, \]  
(3)
where \( \rho_{\text{GW}} \) denotes the energy density of the gravitational waves, \( \rho_{\text{GW}} = \langle (\partial_\tau h_{ij})^2 + (\vec{\nabla} h_{ij})^2 \rangle / (64\pi G a^2) \), \( \rho_c \equiv 3H^2/8\pi G \) is the critical density of the Universe, and the Hubble parameter is defined as \( H \equiv (da/dt)/a \). One can also express such a statistical quantity in terms the power spectrum,
\[ \Delta^2_h(k) \equiv \frac{d \langle h_{ij} h^{ij} \rangle}{d \ln k} = \frac{k^3}{\pi^2} \sum_{\lambda} |h^2_{\lambda}|. \]  
(4)

B. Fisher matrix for gravitational wave measurement

Here, we briefly review and discuss the Fisher information matrix approach, focusing on the future gravitational wave detectors. The Fisher matrix is a powerful method to theoretically forecast the constraining power on parameters of interest for a given survey, and is commonly used in observational cosmology (see e.g., [24–26]). An essential but unique assumption in the formalism is a Gaussian likelihood, and the Fisher matrix is defined by the second derivative of the likelihood around its maximum (or fiducial parameters) with respect to the parameters of interest. Then the inverse of the Fisher matrix provides a lower bound on the covariance matrix via the Cramer-Rao bound, and is regarded as the best achievable accuracy for the parameters.

The Fisher matrix generally depends on the covariance matrix of signals, i.e., the noise properties of the survey configurations. In the case of the stochastic gravitational wave background, direct detection can be attempted by cross-correlating the output signals between two detectors [27]. For the cross-correlation analysis, the Fisher information matrix is given by [28]
\[ \mathcal{F}_{ij} = \left( \frac{3H_0^2}{10\pi^2} \right)^2 2 T_{\text{obs}} \sum_{(I,J)} \int_{f_{\text{cut}}}^{f_{\text{max}}} df \frac{|\gamma_{IJ}(f)|^2 \partial_{p_i} \Omega_{\text{GW}}(f) \partial_{p_j} \Omega_{\text{GW}}(f)}{f^6 S_I(f) S_J(f)}, \]
with \( f = k/(2\pi) \) and \( H_0 \) being the frequency of the gravitational wave and the present Hubble parameter. We choose a lower cutoff of \( f_{\text{cut}} = 0.1 \)Hz, below which the signal may be contaminated by noise from cosmological white dwarf binaries [29]. As for \( f_{\text{max}} \), we set \( f_{\text{max}} = \infty \) unless otherwise stated, though the high-frequency range is limited by the noise spectrum which we will see in detail later. For a given survey, we need to assign three
observational ingredients; a total observational \( T_{\text{obs}} \), overlap reduction functions \( \gamma_{IJ}(f) \), and noise spectra \( S_I(f) \). These functional forms rely on the type of interferometry that we choose (for a review, see [30]).

In the case of the Time-Delay Interferometry (TDI) that is expected to be adopted in BBO, the subscript \( I \) or \( J \) denotes the TDI channel output index \( (I, J = A, E, T) \). We compute the overlap reduction function \( \gamma_{IJ}(f) \) for the TDI data combinations with the method in Ref. [31]. The noise transfer functions for the TDI variables are assumed as [32]

\[
S_A(f) = S_E(f) = 8 \sin^2\left(\frac{\hat{f}}{2}\right)\left[2(2 + \cos \hat{f})S_{\text{shot}} + 2(3 + 2 \cos \hat{f} + \cos(2 \hat{f}))S_{\text{accel}}\right], 
\]

(5)

\[
S_T(f) = 2\left[1 + 2 \cos \hat{f}\right]^2\left[S_{\text{shot}} + 4 \sin^2(\hat{f}/2)S_{\text{accel}}\right], 
\]

(6)

where \( S_{\text{shot}} \) and \( S_{\text{accel}} \) are the photon shot-noise and the proof-mass acceleration noise for the laser interferometers. The noise spectrum is designed so as to be rescaled with respect to the pivot frequency \( \hat{f} = 2\pi L f \) with \( L \) being the arm length. Unlike BBO, DECIGO would install a Fabry-Perot type interferometer, and hence its noise functions can be different from those of BBO discussed above. However, both DECIGO and BBO are basically designed to aim at the detection of the inflationary gravitational waves with similar frequency ranges around \( 0.1-1 \)Hz. In addition, the purpose of this paper is to quantify the potential of future gravitational wave experiments for determination of model parameters. Based on these facts, we simply assume the TDI-type noise functions, Eqs. (5) and (6), throughout the paper. Even with such a simple assumption, we do not believe that detailed configurations can significantly affect our conclusion.

In table [3] we summarize the survey parameters adopted in the Fisher matrix calculation. We consider two types of detectors. The first one is corresponding to the proposed DECIGO or BBO, quoted by ”BBO/FP-DECIGO”, whose parameter values are taken from Ref. [33]. We also investigate the case of an ideal experiment whose sensitivity is limited only by quantum noises. We quote it as ”Ultimate-DECIGO” and its parameters are taken from Ref. [34].

In summary, the Fisher matrix can be calculated with Eq. (5), once the detector parameters are assigned and theoretical predictions for \( \Omega_{\text{GW}} \) is provided. Then the marginalized \( 1\sigma \) error is easily computed with the inverse of the Fisher matrix,

\[
\sigma(p_i) = \sqrt{(F^{-1})_{ii}}. 
\]

(7)
TABLE I: detector parameters

| Detectors            | $S_{\text{shot}}$ [(L/km)$^{-2}$Hz$^{-1}$] | $S_{\text{accel}}$ [(2$\pi f$/Hz)$^{-1}$] | $L$ [km] |
|----------------------|------------------------------------------|------------------------------------------|----------|
| BBO/FP-DECIGO        | $2 \times 10^{-40}$                     | $9 \times 10^{-40}$                     | $5 \times 10^4$ |
| Ultimate DECIGO      | $9 \times 10^{-44}$                     | $9 \times 10^{-44}$                     | $5 \times 10^4$ |

The Fisher matrix is a product of the signal-to-noise-ratios, $\Omega_{\text{GW}}/S$, and the derivative, $\partial_p \ln \Omega_{\text{GW}}$, and hence depends on the parameter response, namely the parameter degeneracy as well as the signal detectability. This makes a difference from the previous work [18].

III. DETERMINATION OF THE REHEATING TEMPERATURE

As discussed in [17–19], direct detection of the inflationary gravitational wave background has a potential to constrain or even to determine the reheating temperature via the characteristic frequency dependence. Inflationary gravitational wave spectrum has a frequency dependence of $f^{-2}$ for modes which enter the horizon during a matter-dominated universe, and $f^0$ for modes which enter the horizon during a radiation-dominated universe. If the Universe behaves like a matter-dominated universe during reheating, the transition from reheating to the radiation domination is seen as a change of the frequency dependence of the spectrum. If this signature exists in the frequency band of direct detection sensitivity, 0.1 – 1Hz, we may be able to determine the reheating temperature by measuring the knee-like feature where the frequency dependence changes from $f^{-2}$ to $f^0$. In this section, we apply the Fisher matrix analysis to investigate to what extent the future gravitational wave experiments can determine the reheating temperature.

A. The gravitational wave spectrum in the presence of reheating

The spectrum of the inflationary gravitational wave background is often expressed in terms of the initial tensor power spectrum $\Delta_{h,\text{prim}}^2(k)$ and the transfer function $T_h(k)$,

$$
\Omega_{\text{GW}} = \frac{1}{12} \left( \frac{k}{aH} \right)^2 \Delta_{h,\text{prim}}^2(k) T_h^2(k).
$$

In a single field slow-roll inflation, the tensor-to-scalar ratio $r \equiv \Delta_{h,\text{prim}}^2(k_0)/\Delta_{R,\text{prim}}^2(k_0)$ can be related to the tilt of the tensor mode spectrum $n_T \equiv d \ln \Delta_{h,\text{prim}}^2(k_0)/d \ln k$ as $r = -8n_T$. 

7
From this relation, the initial power spectrum can be written as
\[ \Delta^2_{h,\text{prim}}(k) \simeq r \Delta^2_{R,\text{prim}}(k_0) \exp \left[ -\frac{r}{8} \ln \frac{k}{k_0} + \cdots \right], \] (9)
where the pivot scale is taken as \( k_0 = 0.002 \text{ Mpc}^{-1} \). \(^1\) The effects of the cosmological evolution after inflation are all included in the transfer function, which is given as \([18, 21]\)
\[ T_h^2(k) = \Omega_m^2 \left( \frac{g_*(T_{\text{in}})}{g_{s0}} \right) \left( \frac{g_{ss}(T_{\text{in}})}{g_{ss}(T_{\text{in}})} \right)^{4/3} \left( \frac{3 j_1(k\tau_0)}{k\tau_0} \right)^2 T^2_1(x_{eq}) T^2_2(x_R). \] (10)

The subscript “0” denotes the present time and “in” denotes the time when the mode \( k \) crosses the horizon. The effective number of degrees of freedom at the end of reheating is taken to be the sum of the standard model particles, \( g_*(T_R) = g_{ss}(T_R) = 106.75 \). The values at present are \( g_{s0} = 3.36 \) and \( g_{ss0} = 3.90 \). Here, \( \tau_0 \) is the present conformal time calculated assuming the Universe is matter dominated: \( \tau_0 = 2H_0^{-1} \). The effect of the cosmological constant is accounted for by the factor of \( \Omega_m = 1 - \Omega_\Lambda \).

In the limit of \( k\tau_0 \ll 1 \), the spherical Bessel function \( j_1(x) = (\sin x - x \cos x)/x^2 \) is replaced as \( j_1(k\tau_0) \rightarrow \cos(k\tau_0)/(k\tau_0) = 1/\sqrt{2k\tau_0} \). The first transfer function \( T_1(x_{eq}) \) describes the change of the frequency dependence of the spectrum which arises from the change of the expansion rate of the Universe at the matter-radiation equality \( t = t_{eq} \) \([5]\),
\[ T^2_1(x_{eq}) = (1 + 1.57x_{eq} + 3.42x_{eq}^2), \] (11)
where \( x_{eq} = k/k_{eq} \) and \( k_{eq} \equiv \tau_{-1}^{-1} = 7.1 \times 10^{-2} \Omega_m h^2 \text{ Mpc}^{-1} \). The second transfer function \( T_2(x_R) \) corresponds to the change of the expansion rate at the end of reheating \( t = t_R \) \([18]\),
\[ T^2_2(x_R) = (1 - 0.32x_R + 0.99x^2_R)^{-1}, \] (12)
where \( x_R = k/k_R \) and \( k_R \simeq 1.7 \times 10^{14} \text{ Mpc}^{-1} (g_{ss}(T_R)/106.75)^{1/6} (T_R/10^7 \text{ GeV}) \). This can be rewritten in terms of frequency as
\[ f_R = \frac{k_R}{2\pi} \simeq 0.26 \text{ Hz} \left( \frac{g_{ss}(T_R)}{106.75} \right)^{1/6} \left( \frac{T_R}{10^7 \text{ GeV}} \right), \] (13)
\(^1\) The contribution from the higher order of \( \ln(k/k_0) \) is sometimes non-negligible, depending on the inflation model, and may cause a wrong estimate of the amplitude of the spectrum \([35]\). It can affect determination of the value of \( r \), which is an important parameter to determine the gravitational wave amplitude, but is not crucial for determination of the reheating temperature, because the reheating temperature is determined basically by the characteristic frequency dependence of the spectrum.
which is the frequency where the change of the frequency dependence due to reheating arises. We show the spectra for different values of the reheating temperature in Fig. 1. As clearly seen from the figure, the knee shape around \( f_R \) can be observed with BBO/FP-DECIGO if the observational time is sufficiently long. Since there are no observable to probe reheating so far, the future gravitational wave experiments may provide us an unique opportunity to reveal the reheating of the Universe.

**B. Result**

Based on the theoretical prediction presented above, let us estimate the detectability of the reheating temperature using the Fisher matrix, which is calculated by substituting Eqs. (8)-(12) into Eq. (5). We take \( r \) and \( T_R \) as free parameters, which correspond to the amplitude of the spectrum and the frequency of the reheating signature.

In Fig. 2 we present an example of the expected future constraints, in which the fiducial
parameters are chosen as \( r = 0.1 \) and \( T_R = 10^7 \text{GeV} \). Each ellipse represents the 2\( \sigma \) error contours expected from 1, 3 and 10 years of observation with BBO/DECIGO. The error ellipse shrinks more for longer observations due to the fact that the signal-to-noise ratio scales as \( \sqrt{T_{\text{obs}}} \). As naturally expected, there is a degeneracy between \( T_R \) and \( r \). This is simply because, as long as the frequency dependence of the spectrum is measured with a good accuracy we cannot distinguish the spectrum with larger \( T_R \) from that with smaller amplitude, i.e., with smaller \( r \).

An interesting and nontrivial question is what frequency range actually carries information on the reheating temperature. In other words, how wide of a band width is necessary to detect the knee shape with a good accuracy. We study this issue in Fig. 3 in which errors in \( T_R \) are plotted as a function of the upper frequency limit in the calculation of the Fisher matrix (\( f_{\text{max}} \) in Eq. (5)). Apparently frequencies above \( f \sim 0.3 \text{Hz} \) do not contribute to detection of the reheating temperature. This is because both the suppression of the signal amplitude due to reheating and the increase of the noise spectrum intensity prevents us from reaching the spectrum information. Thus, a moderate kink around \( f_R \) is enough to reveal the reheating signature.

Note that our analysis is performed imposing the consistency relation, \( r = -8n_T \). Since the tilt of the spectrum is defined at the CMB scale, it can largely affect the amplitude at the direct detection scale as seen in Eq. (2). Therefore, an additional degeneracy between \( r \) and \( n_T \) arises and causes larger uncertainties in parameters if we take \( n_T \) as a free parameter. For instance, in the case of \( r = 0.1 \) and \( T_R = 10^7 \text{GeV} \) with BBO/FP-DECIGO for 3-year observation, the uncertainty on the reheating temperature is degraded to \( \sigma_{T_R} = 2.9 \times 10^7 \text{GeV} \) by a factor of 8 compared to \( \sigma_{T_R} = 3.7 \times 10^6 \text{GeV} \) with fixed \( n_T \). However, combining with CMB B-mode polarization constraints would help to relax the degradation.

So far, we have fixed fiducial values of \( r \) and \( T_R \). In the following, we discuss the fiducial-value dependence and predict the parameter space where the signature of reheating can be successfully detected. Fig. 4 shows dependence on fiducial value of \( r \). The marginalized error in \( T_R \) (\( \sigma_{T_R} \)) is calculated by changing \( r \) with the fixed value of \( T_R = 10^7 \text{GeV} \). The error becomes smaller as the gravitational wave background is detected with larger signal-to-noise ratio. According to Eq. (9), the amplitude of the spectrum at the frequency band of the experiment \( f \sim 0.1 \text{Hz} \) takes the maximum value when \( r \simeq 0.2 \) in the balance between the factors \( r \) (which increases the amplitude) and \( \exp[-r \ln(k/k_0)/8] \) (which decreases the
FIG. 2: The 2σ confidence level contours in the $T_R - r$ plane for 1-year (dotted), 3-year (dashed) and 10-year (solid) observation by BBO/FP-DECIGO. The fiducial parameters are set as $r = 0.1$ and $T_R = 10^7$ GeV, which is shown by a cross mark.

This results in the smallest error on $T_R$ around $r \simeq 0.2$ in Fig. 4. Notice that the spikes originate from the fact that the Fisher matrix $\mathcal{F}_{rr} \propto (\partial \Omega_{GW}/\partial r)^2 \propto \{1 - r \ln(k/k_0)/8\}$ goes to zero around $r \simeq 0.23$. This is an artificial effect due to our choice of parametrization.

Similarly in Fig. 5 we show dependence on the fiducial value of $T_R$ with the fixed value of $r = 0.1$. The error becomes smaller when the signature of reheating comes into the range of the sensitivity, which corresponds to the reheating temperature of about $10^6$ GeV to $10^8$ GeV. We should comment on our results at high $T_R$ where the resultant $\sigma_{T_R}$ is underestimated. This is because the term of $-0.32x_R$ in our approximated transfer function, Eq. (12), causes a small but an artificial “bump” in the spectrum, which does not arise in the full numerical calculation. For this reason, we highlight such questionable results at $T_R > 10^8$ GeV as
FIG. 3: The 1σ marginalized errors are shown as a function of $f_{\text{max}}$, calculated assuming $r = 0.1$ and $T_R = 10^7\text{GeV}$ (corresponding to $f_R = 0.26\text{Hz}$) with 3-year observations. For clarity, the best-sensitivity frequency is plotted as a vertical dotted line.

gray-shaded regions. The right panel of Fig. 5 shows a very promising fact in the case of Ultimate-DECIGO that the reheating temperature could be determined with 1% accuracy if $1.2 \times 10^6\text{GeV} < T_R < 3.3 \times 10^8\text{GeV}$ \(^2\) for $r = 0.1$, and if $2.1 \times 10^6\text{GeV} < T_R < 7.0 \times 10^7\text{GeV}$ for $r = 0.01$.

Finally in Fig. 6 we present the parameter space of $r$ and $T_R$ where the reheating signature is detected at greater than 2σ level (i.e., $T_R/\sigma_{T_R} > 2$) with 3-year observation. Similarly to Fig. 5 the region affected by our incorrect transfer function are indicated by the gray shade, and may shrink in a more realistic case. We also show the parameter region for the detection with a signal-to-noise ratio higher than 5 ($S/N > 5$) that is the criterion adopted in the previous work [18]. We cannot instantly compare two areas in the sense that the areas largely depend on variant criteria. However, the shape of parameter regions are at least different, originating from the fact parameter degeneracies are neglected in the previous work. We conclude that the parameter space presented here are more realistic and worth

\(^2\) The upper value may be smaller by a several factor since the value is evaluated in the shaded region of the figure.
FIG. 4: The marginalized 1σ uncertainty in $T_R$ as a function of $r$ for BBO/FP-DECIGO (left panel) and Ultimate-DECIGO (right panel). The fiducial value of the reheating temperature is fixed to be $T_R = 10^7$ GeV. The upper horizontal axis represents the values of $\Omega_{GW}$ corresponded to $r$ by Eq. (8).

while being pursued for detection of the reheating temperature with future gravitational wave detectors.

IV. PROBING THE EQUATION OF STATE OF THE EARLY UNIVERSE

In the previous section, we have focused on the determination of the reheating temperature by observing the change of the frequency dependence of the spectrum. On another front, direct detection of the inflationary gravitational wave background can be used to determine the equation of state of the very early Universe, at the cosmic temperature of around $10^7$ GeV. So we could ask how sensitive the detectors are to different equation-of-state parameters. Here, we calculate the Fisher matrix assuming that the frequency dependence of the spectrum is uniform over the range of sensitivity, and investigate how accurately direct detection can determine the equation of state of the early Universe.
FIG. 5: The marginalized 1σ uncertainty in $T_R$ as a function of $T_R$ for BBO/FP-DECIGO (left panel) and Ultimate-DECIGO (right panel). The fiducial value of the tensor-to-scalar ratio is fixed to be $r = 0.1$.

FIG. 6: 2σ detection region of $T_R$ is shown as a blue shaded region for 3-year observations by BBO/FP-DECIGO (left panel) and Ultimate-DECIGO (right panel). The gray area represents the region where $\sigma_{T_R}$ may be underestimated. In the light blue shaded region, the inflationary gravitational wave background would be detected with a signal to noise ratio higher than 5.
A. The spectrum with a generic equation of state

Let us first relate the equation of state of the Universe $w = p/\rho$ to the tilt of the gravitational wave background spectrum. If the initial power spectrum is assumed to have no tilt, $\Delta^2_{h,\text{prim}} \propto k^0$, the frequency dependence of the spectrum is determined only by the transfer function. Hence, Eq. (8) implies $\Omega_{GW} \propto k^2 T^2(k)$. Since a gravitational wave, which has an initial amplitude of $h_{k,\text{prim}}$, maintains constant amplitude outside the horizon and it starts to decrease inversely proportional to the scale factor when the mode enters the horizon, the transfer function can be written as $T(k) = |h_{k,0}|/|h_{k,\text{prim}}| = (a_0/a_{in})^{-1}$, which means $\Omega_{GW} \propto k^2 a_{in}^2$. Combining the facts that the mode enter the horizon when $k = aH$ and the Hubble expansion rate is given in terms of the equation of state as $H^2 \propto a^{-3(1+w)}$, we obtain $a_{in} \propto k^{-2/(1+3w)}$. Thus, for modes which enter the horizon when the Universe has the equation of state $w$, the spectrum has the frequency dependence of

$$\Omega_{GW} \propto k^{2(3w-1)/(3w+1)}. \quad (14)$$

We parametrize the amplitude of the gravitational wave background spectrum, normalizing at $f = F$, as

$$\Omega_{GW}(f) = \Omega_{GW,F}(f/F)^{2(3w-1)/(3w+1)}. \quad (15)$$

B. Result

The fisher matrix is calculated by substituting Eq. (15) into Eq. (5), taking $\Omega_{GW,F}$ and $w$ as parameters. We investigate three fiducial models of the very early Universe; matter-dominated ($w = 0$), radiation-dominated ($w = 1/3$) and kination-dominated ($w = 1$), which correspond to the frequency dependence of $f^{-2}$, $f^0$ and $f^1$, respectively. Here, the normalization is taken at $F = 0.203$ Hz for BBO/FP-DECIGO and $F = 0.158$ Hz for Ultimate-DECIGO, which is chosen as the covariance matrix to be diagonalized for the flat spectrum [28] and is almost in the middle of the sensitivity range. Fig. 7 is one example of the error contours with the amplitude chosen to be $\Omega_{GW,F} = 1.84 \times 10^{-16}$ corresponding to $r = 0.1$. From the figure, we see the constraint on $w$ is significantly better in the matter-dominated case. This is because the measurable quantity in the experiment is not $w$ but the tilt of the spectrum, $n_T = 2(3w - 1)/(3w + 1)$. In this parametrization, the value of the tilt is more sensitive to the change of $w$ when the fiducial model is $w \sim 0$ than when $w$ has
FIG. 7: The $2\sigma$ confidence level contours in the $w-\Omega_{GW,F}$ plane for 1-year (dotted), 3-year (dashed) and 10-year (solid) observation by BBO/FP-DECIGO. The fiducial value of $\Omega_{GW,F}$ is taken to be $1.84 \times 10^{-16}$ which corresponds to $r = 0.1$. The three panels represent a different model for the equation of state; $w = 0$ (matter dominant, left panel), $w = 1/3$ (radiation dominant, middle panel) and $w = 1$ (kination dominant, right panel). These three fiducial points are shown as cross marks in each panel.

A larger value. This makes it easier for direct detection experiments to distinguish the model with smaller value of $w$.

Fig. 8 shows dependence on the fiducial value of $\Omega_{GW,F}$. The marginalized error $\sigma_w$ is calculated by changing $\Omega_{GW,F}$ assuming the matter-dominated, radiation-dominated and kination-dominated universe, respectively. Note that the vertical axis does not represent $\sigma_w$, but represents $w$. The blue shaded regions represent the $1\sigma$ error ranges for the determination of $w = 0, 1/3$ and 1, calculated assuming 1, 3 and 10-year observation with BBO/DECIGO and Ultimate-DECIGO. Obviously, the errors become smaller as the amplitude of the gravitational wave background increases, which enables us to distinguish the value of $w$ from that of the other models.

V. CONCLUSIONS AND DISCUSSION

In this paper, we study prospects for direct determination of the reheating temperature after inflation and the equation of state of the early Universe with future gravitational wave
FIG. 8: Dependence of the errors in $w$ on the fiducial value of the gravitational wave amplitude, $\Omega_{GW,F}$. The upper horizontal axis represents the values of $r$ corresponded to $\Omega_{GW,F}$ by Eq. (8). The solid straight lines represent values of $w$ for the matter ($w = 0$), radiation ($w = 1/3$) and kination ($w = 1$)-dominated universe. The blue shaded regions indicate the $1\sigma$ error range on each value of $w$ for 1,3 10-year observations. Each three panels show the case where the fiducial model assumes the matter (left), radiation (middle) and kination (right)-dominated universe. The upper panel is for BBO/FP-DECIGO and the lower panel is for Ultimate-DECIGO.

detectors. The reheating temperature, $T_R$, will be determined accurately for $T_R \sim 10^7$GeV by BBO/FP-DECIGO, if the tensor-to-scalar ratio $r$ is larger than $\sim 0.05$. Therefore, if the CMB B-mode polarization is measured by the Planck satellite or ground-based telescopes at the level of $r \sim 0.1$, we will have a good chance to detect the inflationary gravita-
tional waves at future space laser interferometers and determine/constrain the reheating temperature. Since $T_R$ of $10^{6-9}$GeV is close to the upper bound from the gravitino problem in supergravity \cite{37}, the detection of the gravitational waves will have an impact on the supersymmetric models if the Large-Hadron Collider finds signals of supersymmetry. Determination of the reheating temperature is also closely related to the origin of the ordinary matter in the present Universe, since the amount of baryon asymmetry crucially depends on $T_R$ for many baryogenesis mechanisms. For example, thermal leptogenesis scenario needs $T_R \gtrsim 10^9$GeV \cite{38}, and nonthermal leptogenesis needs $T_R \gtrsim 10^6$GeV \cite{39,40}.

Moreover, future gravitational wave detectors have a potential to determine equation of state of the early Universe to a good accuracy. The matter, radiation and kination dominated universe are clearly distinguished by BBO/FP-DECIGO for $r \sim 0.1$. It will be an important step toward the complete understanding of the whole history of the Universe and the nature of the inflaton itself.

As a final remark, we mention another probe of reheating. Accurate measurements of the CMB anisotropy are also sensitive to the reheating temperature \cite{41,42}. The expansion history of the Universe can be connected to the length of inflation and slightly affects the values of the slow-roll parameters. This is reflected both in the primordial spectra of the scalar and tensor perturbations, and is also applicable to direct detection of the inflationary gravitational wave background. Hence, direct detection helps to constrain the reheating temperature even in the case where the reheating signature is not seen in the frequency coverage of experiments \cite{43}. Also, if there is a process called preheating \cite{44}, gravitational waves generated during preheating can be another probe of reheating \cite{45,46}. These gravitational waves have a peak at the characteristic frequency, corresponding to the comoving Hubble scale at the end of inflation. Thus, they may provide us with complementary information on properties of inflation and reheating.

\textbf{Acknowledgments}

This work is supported in part by Grant-in-Aid for Scientific research from the Ministry of Education, Science, Sports, and Culture (MEXT), Japan, No. 21111006 (K.N.), No. 22244030 (K.N.). S.S. acknowledges the Japan Society for the Promotion of Science (JSPS)
for a support through the postdoctoral fellowship for research abroad, No. 23-529.

[1] E. Komatsu et al. [ WMAP Collaboration ], Astrophys. J. Suppl. 192, 18 (2011).
[arXiv:1001.4538 [astro-ph.CO]].

[2] A. A. Starobinsky, JETP Lett. 30, 682-685 (1979); V. A. Rubakov, M. V. Sazhin, A. V. Veryaskin, Phys. Lett. B115, 189-192 (1982); L. F. Abbott, M. B. Wise, Nucl. Phys. B244, 541-548 (1984).

[3] M. Kamionkowski, A. Kosowsky and A. Stebbins, Phys. Rev. D 55, 7368 (1997)
[arXiv:astro-ph/9611125].

[4] S. Saito, K. Ichiki and A. Taruya, JCAP 0709, 002 (2007) [arXiv:0705.3701 [astro-ph]].

[5] M. S. Turner, M. J. White, J. E. Lidsey, Phys. Rev. D48, 4613-4622 (1993).
[astro-ph/9306029].

[6] T. L. Smith, M. Kamionkowski, A. Cooray, Phys. Rev. D73, 023504 (2006).
[astro-ph/0506422].

[7] N. Seto, S. Kawamura and T. Nakamura, Phys. Rev. Lett. 87, 221103 (2001)
[arXiv:astro-ph/0108011].

[8] S. Kawamura et al., Class. Quant. Grav. 23, S125 (2006).

[9] S. Phinney et al., The big bang observer: direct detection of gravitational waves from the birth of the Universe to the present, NASA Mission Concept Study.

[10] C. Cutler, D. E. Holz, Phys. Rev. D80, 104009 (2009). [arXiv:0906.3752 [astro-ph.CO]].

[11] A. Nishizawa, A. Taruya and S. Saito, Phys. Rev. D 83, 084045 (2011) [arXiv:1011.5000 [astro-ph.CO]].

[12] A. Nishizawa, K. Yagi, A. Taruya and T. Tanaka, arXiv:1110.2865 [astro-ph.CO].

[13] N. Seto, J. Yokoyama, J. Phys. Soc. Jap. 72, 3082-3086 (2003). [gr-qc/0305096].

[14] H. Tashiro, T. Chiba, M. Sasaki, Class. Quant. Grav. 21, 1761-1772 (2004). [gr-qc/0307068].

[15] L. A. Boyle, P. J. Steinhardt, Phys. Rev. D77, 063504 (2008). [astro-ph/0512014].

[16] L. A. Boyle, A. Buonanno, Phys. Rev. D78, 043531 (2008). [arXiv:0708.2279 [astro-ph]].

[17] K. Nakayama, S. Saito, Y. Suwa, J. Yokoyama, Phys. Rev. D77, 124001 (2008).
[arXiv:0802.2452 [hep-ph]].

[18] K. Nakayama, S. Saito, Y. Suwa, J. Yokoyama, JCAP 0806, 020 (2008). [arXiv:0804.1827].
[astro-ph]]

[19] S. Kuroyanagi, T. Chiba, N. Sugiyama, Phys. Rev. D83, 043514 (2011). arXiv:1010.5246 [astro-ph.CO]]

[20] S. Kuroyanagi, T. Chiba, N. Sugiyama, Phys. Rev. D79, 103501 (2009). arXiv:0804.3249 [astro-ph]]

[21] K. Nakayama, J. Yokoyama, JCAP 1001, 010 (2010). arXiv:0910.0715 [astro-ph.CO]]

[22] K. Nakayama, F. Takahashi, JCAP 1011, 009 (2010). arXiv:1008.2956 [hep-ph]]

[23] S. Mukohyama, K. Nakayama, F. Takahashi, S. Yokoyama, Phys. Lett. B679, 6-9 (2009). arXiv:0905.0055 [hep-th]]

[24] M. Tegmark, A. Taylor and A. Heavens, Astrophys. J. 480, 22 (1997) arXiv:astro-ph/9603021.

[25] D. J. Eisenstein, W. Hu, M. Tegmark, Astrophys. J. 518, 2-23 (1999). astro-ph/9807130.

[26] S. Saito, M. Takada, A. Taruya, Phys. Rev. D80, 083528 (2009). arXiv:0907.2922 [astro-ph.CO]]

[27] B. Allen and J. D. Romano, Phys. Rev. D 59, 102001 (1999) arXiv:gr-qc/9710117.

[28] N. Seto, Phys. Rev. D 73, 063001 (2006) arXiv:gr-qc/0510067.

[29] A. J. Farmer and E. S. Phinney, Mon. Not. Roy. Astron. Soc. 346, 1197 (2003) arXiv:astro-ph/0304393.

[30] M. Maggiore, Oxford University Press, October 2007. 572p. (ISBN-13: 978-0-19-857074-5).

[31] V. Corbin and N. J. Cornish, Class. Quant. Grav. 23, 2435 (2006) arXiv:gr-qc/0512039.

[32] T. A. Prince, M. Tinto, S. L. Larson and J. W. Armstrong, Phys. Rev. D 66, 122002 (2002) arXiv:gr-qc/0209039.

[33] J. Crowder and N. J. Cornish, Phys. Rev. D 72, 083005 (2005) arXiv:gr-qc/0506015.

[34] H. Kudoh, A. Taruya, T. Hiramatsu and Y. Himemoto, Phys. Rev. D 73, 064006 (2006) arXiv:gr-qc/0511145.

[35] S. Kuroyanagi and T. Takahashi, arXiv:1106.3437 [astro-ph.CO].

[36] B. A. Powell, arXiv:1106.5059 [astro-ph.CO]]

[37] M. Kawasaki, K. Kohri, T. Moroi, Phys. Lett. B625, 7-12 (2005). astro-ph/0402490; Phys. Rev. D71, 083502 (2005). astro-ph/0408426; M. Kawasaki, K. Kohri, T. Moroi, A. Yotsuyanagi, Phys. Rev. D78, 065011 (2008). arXiv:0804.3745 [hep-ph]]

[38] M. Fukugita, T. Yanagida, Phys. Lett. B174, 45 (1986); For a review, see W. Buchmuller,
R. D. Peccei, T. Yanagida, Ann. Rev. Nucl. Part. Sci. 55, 311-355 (2005). [hep-ph/0502169].

[39] T. Asaka, K. Hamaguchi, M. Kawasaki, T. Yanagida, Phys. Lett. B464, 12-18 (1999). [hep-ph/9906366]; Phys. Rev. D61, 083512 (2000). [hep-ph/9907559].

[40] K. Hamaguchi, H. Murayama, T. Yanagida, Phys. Rev. D65, 043512 (2002). [hep-ph/0109030].

[41] J. Martin, C. Ringeval, Phys. Rev. D82, 023511 (2010). [arXiv:1004.5525 [astro-ph.CO]].

[42] J. Mielczarek, Phys. Rev. D 83, 023502 (2011) [arXiv:1009.2359 [astro-ph.CO]].

[43] S. Kuroyanagi, C. Gordon, J. Silk and N. Sugiyama, Phys. Rev. D 81, 083524 (2010) [arXiv:0912.3683 [astro-ph.CO]].

[44] L. Kofman, A. D. Linde, A. A. Starobinsky, Phys. Rev. Lett. 73, 3195-3198 (1994). [hep-th/9405187]; Phys. Rev. D56, 3258-3295 (1997). [hep-ph/9704452].

[45] S. Y. Khlebnikov, I. I. Tkachev, Phys. Rev. D56, 653-660 (1997). [hep-ph/9701423]; J. F. Dufaux, A. Bergman, G. N. Felder, L. Kofman, J. -P. Uzan, Phys. Rev. D76, 123517 (2007). [arXiv:0707.0875 [astro-ph]].

[46] R. Easther, E. A. Lim, JCAP 0604, 010 (2006). [astro-ph/0601617]; R. Easther, J. T. Giblin, Jr., E. A. Lim, Phys. Rev. Lett. 99, 221301 (2007). [astro-ph/0612294].

[47] J. Garcia-Bellido, D. G. Figueroa, Phys. Rev. Lett. 98, 061302 (2007). [astro-ph/0701014]; J. Garcia-Bellido, D. G. Figueroa, A. Sastre, Phys. Rev. D77, 043517 (2008). [arXiv:0707.0839 [hep-ph]].