On Capital Allocation under Time and Information Constraints

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Abstract
Attempts to allocate capital to a selection of different investment objects often face the problem that investors’ decisions are made under limited information (no historical return data) and an extremely limited timeframe. Nevertheless, in some cases, rational investors with a certain level of experience are able to ordinally rank investment alternatives through relative assessments of the probability that an investment will be successful. However, to apply traditional portfolio optimization models, analysts must use historical (or simulated/expected) return data as the basis for their calculations. Our paper develops an alternative portfolio optimization framework that is able to handle this kind of information (given by the ordinal ranking of investment alternatives) and to calculate an optimal capital allocation based on a Cobb-Douglas function. Considering risk-neutral investors, we show that the results of this portfolio optimization model usually outperform the output generated by the (intuitive) Equally Weighted Portfolio (EWP) of the different investment alternatives, which is the result of optimization when one is unable to incorporate additional data (the ordinal ranking of the alternatives). In a further extension, we show that our model is also able to address risk-averse investors to capture diversification benefits.

Keywords: Capital Allocation, Cobb-Douglas Utility Function, Decision Theory, Uncertainty, Portfolio Selection Theory

JEL Classification: C44, D24, D81, E22, G11.

1. Introduction
Decision-makers, e.g., financial investors, often face alternatives that do not differ at first glance. This may be due to the availability of too little or too
According to the Laplace criterion, the distribution of future outcomes should be considered the same for all these alternatives. From a Bayesian perspective, this reflects the prior distribution (Bayes, 1763). A risk-neutral decision-maker would choose any of the alternatives. For risk-averse decision-makers, the stochastic dependency between the alternatives would be important. However, decision-makers often have their own information at their disposal, e.g. from experience. If we assume that this information is well founded, it can be interpreted as a signal that updates the a priori probability to the posterior probability. This signal does not lead to a decision under certainty but a new distribution for each alternative. The posterior distribution is then the rational basis for decision-making.

This paper builds on the concept of Bayes’ theorem but models a specific decision situation that has not yet been researched. The key point here is that the decision-maker has decision-relevant information, but this information is available only in a particular, limited way. The decision-maker’s knowledge thus makes it possible to ordinally rank the alternatives with respect to assessments of their relative contribution to overall success.

However, the basic consideration in this context is that decision-makers often have to decide quickly and without statistical reflection. In the context of Bayes’ theorem, this means that they cannot determine the type I or type II error of their information (or the conditional probabilities). Therefore, we ask how rational decisions are made when the experience of decision-makers leads to relevant but incomplete information.

The above-mentioned problem arises in many different financially related and non-financially related use cases. For example, consider a situation in which rational investors have to decide about possible investments within a very short time and with a limited range of information, such as (so-called) elevator pitches, pitches in the context of seed money or other pitch-like situations, e.g., at specialized investment fairs where investors have the opportunity to allocate their money among a number of more- or less-talented entrepreneurs (e.g., gold miners with differently located claims).

Our research topic combines three different strands of the literature: (i) Modern Portfolio Theory (Asset Management), (ii) Decision Theory and (iii) Pro-

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1The use cases of our innovative methodology are not limited to the asset management context. For example, consider a situation in which human resource managers have to allocate money among different (more- or less-talented) people – whether hiring new personnel using their impressions during interviews or allocating bonus payments (incentives) among existing workers. As a consequence, our methodology is applicable to all situations in which allocation decisions have to be made within a limited time and with a limited range of information.
duction Theory. Modern Portfolio Theory is based on the works of Markowitz (1952, 1959), who develops a portfolio optimization framework for given (historical/simulated/expected) return data (Elton et al., 2017). More recent works on portfolio optimization have sought to overcome the main shortcomings of Markowitz’s traditional mean-variance approach. For example, Rockafellar and Uryasev (2000) develop a Mean Conditional Value at Risk (CVaR) optimization framework to overcome the traditional assumption of normally distributed returns in the mean-variance framework. In contrast to existing portfolio optimization models, we assume the presence of time pressure and informational restrictions. In other words, we address a situation that occurs in reality whenever financial investment decisions cannot be based on long data series (historical/expected/simulated return data), and therefore, a subjectivistic “Bayesian” formulation of probabilities becomes important for rational decision-making. Subsequently, rational investors are able to create an ordinal ordering of the different investment alternatives based on their experience. However, existing portfolio optimization models cannot cope with an ordinal ranking\(^2\) (as an alternative kind of information). As a consequence, we develop an innovative optimization model based on a Cobb-Douglas production function, which is also applicable to decision situations with this kind of informational restriction and time pressure. In this sense, the article also offers advice for practice. To more concretely explain the key points of our optimization strategy, we refer to capital allocation between different early-stage companies as an illustrative use case below. To the best of our knowledge, this is the first paper that extends traditional Portfolio Selection Theory (Markowitz, 1952, 1959) to a model that assumes limited information (only an ordinal ranking of the investment alternatives).

In our paper, we show that an investor can improve his benefits by considering the entire set of available information (ordinal ranking of the alternatives). Considering risk-neutral investors, we show that the results of this portfolio optimization model usually outperform the output generated by the (intuitive) Equally Weighted Portfolio (EWP) of the different investment alternatives, which would be the strategy of an investor who is unable or unwilling to use all the relevant information. In conclusion, we provide instructions on how to behave in situations with low information quality that contrast with the traditional portfolio optimization framework developed by Markowitz. To best address the work of Markowitz, we present a further extension by modifying the assumptions on the investor’s the risk attitude. We show that our model is also able to address risk-averse investors, and hence we are able to capture diversification effects.

\(^2\)Ordinal rankings are traditionally used in the context of utility rankings in household theory in microeconomics (e.g., Hicks and Allen, 1934). However, in portfolio theory, there are no existing portfolio optimization models that use ordinally ranked variables as input factors.
The remainder of this paper is structured as follows: Sec. 2 contains an introduction to the above-mentioned use case and develops a suitable optimization model for this capital allocation problem. Next, we focus on the illustrative solution of the special case of allocating seed money between two (or more) early-stage companies for risk-neutral investors who maximize their (monetary) output. Moreover, in Sec. 3, the investors are assumed to be risk-averse to capture diversification benefits in the optimization model. The last section finally discusses the results and summarizes the key points.

2. Portfolio Selection for Risk-Neutral Investors

2.1. Methodological Framework

Consider a use case in which a rational investor has the opportunity to invest his capital $C$ in $n$ different early-stage companies that are operated by different (more- or less-talented) entrepreneurs. These are smaller individual companies for which the respective entrepreneur needs seed money from an investor. Suitably scaled, the companies $i = 1, \ldots, n$ are assumed to have the same maximum absolute output. All companies are then considered to be equivalent in the sense that each basically offers the opportunity to achieve a (monetary) output $x_i$, between 0% and 100%. Without additional information, it is assumed that this (monetary) output follows a standard uniform distribution: $x_i \sim U(0, 1)$. In addition, there is hardly any other valid information available to the investor. However, a rational investor can sort the companies based on the information available. Based on his or her experience, the investor decides that the output of company $i$ must be greater than or equal to that of company $j$ (by assumption). Hence, $0 \leq \ldots \leq x_j \leq x_i \leq \ldots \leq 1$. In our simplified model, this ordinal ranking is based on the investor’s evaluation of the entrepreneur’s ability $Z$, which is an unobservable stochastic variable, leading to the (monetary) output $x$. This gives rise to the question of how the investor should distribute his or her capital $C = \sum_{i=1}^{n} c_i$ among the considered companies to maximize his or her expected total (monetary) output. Thus, concretely, what cash amount $c_i$ should be invested in company $i$?

To answer this question, we need to develop a methodological framework that is able to handle all the relevant information from the above-mentioned use case. As a first consideration, we assume that the investor’s benefit from a company $i$ depends only on the combination of two factors: 1. his or her financial commitment to the company and 2. the output of the company. In addition, we assume that the resulting amount of (monetary) benefit depends on the absolute amounts of both input factors. According to our example, it seems intuitively obvious that the benefit the investor can derive from the potential of a high-ability entrepreneur increases in his or her initial seed investment. In other words, a highly talented entrepreneur who can use a relatively large amount of seed money will generate a higher output than an identical entrepreneur with a
hypothetically lower capital provision. Moreover, investors do not benefit from an entrepreneur’s high ability level if they do not invest any capital in the company in question \((c_i = 0)\). With respect to this relationship, the (monetary) benefit as output in toto would be a function of the multiplication of the input factors. Furthermore, the contribution of each input factor to the total output is determined by an individual partial elasticity.

To model such an interdependency, the literature employs the Cobb-Douglas functional form. Charles W. Cobb and Paul H. Douglas formulated a production theory in the early 20th century by combining the input factors of labor and capital to explain overall economic output (Cobb and Douglas, 1928). Their approach can be appropriately translated to the context considered in this paper. In the application below, the (monetary) output \(Y_i\) of company \(i\) is defined by the following function:

\[
Y_i = ax_i^{\nu}c_i^{1-\nu},
\]

where \(a > 0\) is a scale parameter with an appropriate unit of measurement to express the output value in a desired unit. We denote by \(\nu\) the partial elasticity of factor \(x_i\), which determines the intensity of the impact of factor \(x_i\) on output \(Y_i\). The value \(\nu\) is constant and influenced by the available technology. The model in Eq. (1) arises under the assumption, that the sum of the elasticities of the factors \(x_i\) and \(c_i\) equals 1.

We extend the original model to harmonize the Cobb-Douglas function with our application, which is characterized by a stochastic input factor representing entrepreneurial ability. Hence, the input factor \(x\) is determined by an (unknown) underlying stochastic factor \(Z\), which is lower than or equal to some quantity \(z\). Note that the random variable \(Z\) – and with it the internal company processes – is not under the investor’s control, who instead chooses only an amount of seed money to finance the entrepreneur. With respect to the (unknown) distribution function \(F_Z(z)\) and density function \(f(z)\), the input factor \(x\) is defined as

\[
x = P(Z \leq z) = F_Z(z).
\]

Assume that the success (here, output) of a venture is normally influenced by many different (observable and unobservable) factors. In addition to the size of the capital stock, there are also additional factors, such as the quality of human resources, the vulnerability to financial, operational, logistical or environmental risks, and efficiency issues. In this paper, we assume that the stochastic variable \(Z\) pools all of these success factors to simplify the comparison of different companies.

\[\text{\textsuperscript{4}}\text{In this paper, we generally use the (monetary) output generated by a Cobb-Douglas function. In this context, by connecting Portfolio Theory and Production Theory, we can also interpret the results of the Cobb-Douglas function as returns. To demonstrate such an interpretation, we use the logarithmic transformation of the Cobb-Douglas function: }\ln(y_i) = \ln(a) + \nu \ln(x_i) + (1 - \nu) \ln(c_i). \text{ Consequently, a } 1\% \text{ change in the invested capital } (c_i) \text{ leads to a } (1 - \nu)\% \text{ change in the (monetary) output } (y_i). \text{ This consideration is identical to the classical definition of a return.}
\]

\[\text{\textsuperscript{5}}\text{Assume that the success (here, output) of a venture is normally influenced by many different (observable and unobservable) factors. In addition to the size of the capital stock, there are also additional factors, such as the quality of human resources, the vulnerability to financial, operational, logistical or environmental risks, and efficiency issues. In this paper, we assume that the stochastic variable } Z \text{ pools all of these success factors to simplify the comparison of different companies.}\]
Our simple model in Eq. (1) can then obviously describe situations in which a hidden, random input factor $Z$ is given and only its (unknown) distribution function $F_Z(z)$ is present in the sense that $x$ is a result of stochastics. Recall that no time-series data are available for the hidden variables to fit a suitable estimate of the distribution. This should apply in practice to the vast majority of cases – particularly investing in startup projects, where the key figures from the balance sheet required for the investment decision are often given only sparsely.

Now, let us consider not just one business but a number $i = 1, \ldots, n$ of companies for which the extended model in Eq. (1) holds. Then, let $Z_1, Z_2, \ldots, Z_n$ be a sample of random variables with a distribution function $F_Z(z)$ assumed to be independent and identically distributed. While all $Z_i$ are random, uncontrollable quantities of the enterprise itself, the $c_i$ are deterministic, predeterminable quantities. The input factor $c_i$ can describe the amount of capital that an investor invests in company $i$. In this framework, we use scaled capital provisions to consider a budget constraint: $1 = \sum_{i=1}^{n} c_i$. Because the sum of the individual capital provisions of an investor is scaled to 1, we can interpret them as portfolio weights (traditional notation $\omega_i$) in the asset management context and are allowed to set $\omega_i = c_i$, which leads to the re-formulated budget constraint $1 = \sum_{i=1}^{n} \omega_i$.

We distinguish two cases for a rational investor: one, the investments are identical, and neither is preferred (Case 1); two, we consider the situation in which the investor can sort the investment objects with respect to the factor $x$ (Case 2).

Case 1: If the available information about the investment objects is very scarce, then in many cases, there is little option but to regard the objects as equivalent. Thus, a realization $z_1, z_2, \ldots, z_n$ of the random variables cannot be sorted by size. The transformation $x_i = F_Z(z_i)$ produces a sample $x$ that will be uniformly distributed between 0 and 1: $x \sim U(0, 1)$. For $x_1, x_2, \ldots, x_n$, also applies that a sorting is not possible. To describe the individual (monetary) output $Y_i$ of enterprise $i$, we can re-write Eq. (1) while replacing $c_i$ with $\omega_i$, which leads to

$$Y_i = ax_i^{\nu} \omega_i^{1-\nu}. \quad (3)$$

Thus, the total (monetary) output of the investment can be defined as follows:

$$Y = a \sum_{i=1}^{n} x_i^{\nu} \omega_i^{1-\nu}. \quad (4)$$

Note that because $x_i$ is a random number, the individual (monetary) output $Y_i$ for a company $i$ and the total (monetary) output $Y$ are also random numbers.
We assume that the investor cannot influence the size \( x \). The only input factor that the investor can influence is the portfolio weight \( \omega_i \) of a single investment object \( i \). The question is how he or she should choose the respective portfolio weights to allocate his capital among the \( i = 1, \ldots, n \) possible investment objects such that he or she maximizes his expected total (monetary) output.

To optimize the portfolio weights \( \omega_i \) of the individual investments to maximize the resulting (monetary) output, for technical reasons, we need to transform this total (monetary) output into utility units. Therefore, we define

\[
U_i = Y_i = ax^\nu_i \omega_i^{1-\nu}.
\]  
(5)

Hence,

\[
U = \sum_{i=1}^{n} U_i = \sum_{i=1}^{n} ax^\nu_i \omega_i^{1-\nu} = a \sum_{i=1}^{n} x^\nu_i \omega_i^{1-\nu}.
\]  
(6)

This means that one unit of (monetary) output equals one unit of utility and that the variance of the (monetary) output does not influence risk-neutral investors' utility. Intuitively, such an investor has a higher utility if the (monetary) output (resulting from his investments) is higher.

Because of the previously mentioned substitution, we now maximize the investors' total utility, which leads to the following optimization problem:

\[
\max_{\{\omega_1, \omega_2, \ldots, \omega_n\}} \mathbb{E}[U] \quad \text{subjected to} \quad 0 = 1 - \sum_{i=1}^{n} \omega_i.
\]  
(7)

Thus, we must maximize the Lagrange function

\[
\mathcal{L} = \mathbb{E}[U] + \lambda \left( 1 - \sum_{i=1}^{n} \omega_i \right)
\]

\[
= \sum_{i=1}^{n} \mathbb{E}[U_i] + \lambda \left( 1 - \sum_{i=1}^{n} \omega_i \right)
\]

\[
= a \sum_{i=1}^{n} \mathbb{E}[x^\nu_i] \omega_i^{1-\nu} + \lambda \left( 1 - \sum_{i=1}^{n} \omega_i \right)
\]  
(8)

with respect to \( \omega_i \), where \( \lambda \) is the so-called Lagrange multiplier.

The expected value \( \mathbb{E}[x^\nu_i] \) is determined with respect to the uniform distribution and leads to \( \mathbb{E}[x^\nu_i] = \frac{1}{\nu+1} \). Thus, the expected total utility is simply:

\[
\mathbb{E}[U] = \frac{a}{1+\nu} \sum_{i=1}^{n} \omega_i^{1-\nu}.
\]  
(9)
Finally, the Lagrange function in Eq. (8) becomes:

$$L = \frac{a}{1+\nu} \sum_{i=1}^{n} \omega_i^{1-\nu} + \lambda \left( 1 - \sum_{i=1}^{n} \omega_i \right).$$  \hspace{1cm} (10)$$

As usual, the optimization problem is determined by the partial derivatives of $L$ and solving the system of equations. Note that the second derivative of $L$ with respect to $\omega_i$ is negative and leads to a negative definite Hessian matrix if $\omega_i > 0$ and $\nu > 0$. Then, the optimal solution

$$\omega_i^* = \frac{1}{n}$$  \hspace{1cm} (11)

for $i = 1, \ldots, n$ describes the maximum of $E[U]$ in Eq. (9) depending only on the number of companies $n$. This solution is equal to the EWP.

Substituting Eq. (11) in Eq. (9) provides the maximum total utility for Case 1:

$$B_1 = \max_{\{\omega_1, \omega_2, \ldots, \omega_n\}} E[U] = \frac{a}{1+\nu} n^\nu.$$  \hspace{1cm} (12)$$

A special, degenerate case arises when $\nu = 0$ is used. In the present model, this is not an ecologically sensible choice. This eliminates the first input factor in Eq. (9). By doing so, Eq. (4) describes perfect substitutes, meaning that the investor can invest all his capital in any single object to achieve the maximum expected total utility: $\max E[U] = a$. In other words, if the individual absolute output $x_i$ no longer influences investors’ utility, the individual weightings of the investment alternatives also become irrelevant for the investor because from his perspective, all alternatives are equal to one another in their contribution to utility.

Case 2:

We now consider the case in which a rational investor can sort the investment objects based on the little information available. He or she still cannot influence the (hidden) random factors $Z$. Based on his assessment, however, he or she can derive a pairwise forecast of which investment has a higher factor $z$. This leads to an ordinal ranking of the eligible investment objects. Thus, a realization $z_1, z_2, \ldots, z_n$ of the random variables can be sorted by size, and we can conduct further analysis within the framework of the so-called order statistics [Kendall and Stuart, 1976]:

$$z^{(1)} \leq z^{(2)} \leq \ldots \leq z^{(n)}.$$  \hspace{1cm} (13)$$

As in Case 1, the transformation $x_i = F_Z(z_i)$ produces a sample $x$ that will be uniformly distributed between 0 and 1: $x \sim \mathcal{U}(0,1)$. Samples $x$ and $z$ are of the
same order because \( x = F_Z(z) \) is a non-decreasing function of \( z \). If we consider the ordered sample Eq. (13), the transformation leads to

\[
0 \leq x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)} \leq 1. \tag{14}
\]

To describe the investors’ (monetary) output \( Y_i \) of the investment in enterprise \( i \), we can re-write Eq. (1), which leads to

\[
Y_i = ax_i^\nu \omega_i^{1-\nu}. \tag{15}
\]

Thus, the total (monetary) output of all investments is again a random variable and given by \( Y = \sum_{i=1}^{n} Y_i \). Note that the \( Y_i \) also depends on \( \omega_i \) and therefore generally has no sorting.

A similar question as in Case 1 arises here: how should the input factors \( \omega_i \) be chosen such that the expected total utility \( \mathbb{E}[U] \) of the investor is maximized (von Neumann and Morgenstern (1944)). As a starting point, we have only sparse information: a limited budget, uniformly distributed \( x \) and ordinal ranking.

The situation leads to the same optimization problem as in Eq. (7) – with the difference that in the Lagrange function of Eq. (8), another expectation value must be considered. However, the transformation of (monetary) output units into utility units \( (U = Y) \) is used analogously; see Eq. (6). The expected value \( \mathbb{E}[x_{(i)}^\nu] \) is now determined with respect to the order statistics. Since the distribution of the \( i \)th-order statistic \( x_{(i)} \) in a random sample of size \( n \) from the uniform distribution over the interval \([0, 1]\) is a beta distribution of the first kind (Kendall and Stuart, 1976)

\[
\rho(x; i, n) = \frac{1}{B(i, n - i + 1)} x^{i-1}(1-x)^{n-i} \tag{16}
\]

The expectation value can easily be calculated as follows:

\[
\mathbb{E}[x_{(i)}^\nu] = \int_0^1 x^\nu \rho(x; i, n) \, dx
= \int_0^1 x^{i+\nu-1}(1-x)^{n-i} \, dx
= \frac{B(i + \nu, n - i + 1)}{B(i, n - i + 1)}, \tag{17}
\]

with \( B(\alpha, \beta) \) being the beta function, also called the Euler integral of the first kind (Abramowitz and Stegun (2014)). With the discrete probability distribution

\[
p_i(\nu) = \frac{1 + \nu}{n} \frac{B(i + \nu, n - i + 1)}{B(i, n - i + 1)} \tag{18}
\]
for $i = 1, \ldots, n$ with integer $n > 0$ and real $\nu > -1$ (Hoffmann and Börner 2018, Corollary 4) the expectation value can be re-written:

$$E[x_{(i)}^\nu] = \frac{n}{1 + \nu} \cdot p_i(\nu).$$ (19)

Therefore, the expectation of the total utility is simply:

$$E[U] = a \frac{n}{1 + \nu} \sum_{i=1}^{n} p_i(\nu) \omega_i^{1-\nu}.$$ (20)

The sum thus corresponds to the expected value of the transformed input factors $\omega_i$. Hence, the Lagrange function for Case 2 becomes:

$$L = a \frac{n}{1 + \nu} \sum_{i=1}^{n} p_i(\nu) \omega_i^{1-\nu} + \lambda \left( 1 - \sum_{i=1}^{n} \omega_i \right).$$ (21)

For Eq. (10), the optimization problem is determined by the partial derivatives of $L$ and solving the system of equations. Note, as in Case 1, the second derivative from $L$ with respect to $\omega_i$ is negative and leads to a negative definite Hessian matrix if $\omega_i > 0$ and $\nu > 0$. Then, the optimal solutions

$$\omega_i^* = \frac{p_i(\nu)^{\frac{1}{\nu}}}{\sum_{j=1}^{n} p_j(\nu)^{\frac{1}{\nu}}}$$ (22)

for $i = 1, \ldots, n$ describe a maximum of $E[U]$ in Eq. (20) depending only on the elasticity $\nu$ and the number of companies $n$.

Substituting Eq. (22) into Eq. (20) provides the maximum total utility for Case 2:

$$B_2 = \max_{\{\omega_1, \omega_2, \ldots, \omega_n\}} E[U]$$

$$= a \frac{n}{1 + \nu} \left( \sum_{i=1}^{n} p_i(\nu)^{\frac{1}{\nu}} \right)^{\nu}$$

$$= a \frac{n}{1 + \nu} \|p(\nu)\|^{\frac{1}{\nu}}.$$ (23)

where $\|p(\nu)\|^{\frac{1}{\nu}}$ denotes fixed $\nu \in [0, 1]$ the $\frac{1}{\nu}$-norm of the probability vector $p(\nu) = (p_1(\nu), \ldots, p_n(\nu))$.

Comparing Case 1 and Case 2, the following theorem can be proven.

**Theorem 2.1.**

$$B_2 \geq B_1$$ (24)

for $n > 1$ and $\nu \in [0, 1] \subset \mathbb{R}$. Equality applies if $\nu = 0, 1$. 

Proof. With Eq. (12) and (23) Eq. (24) becomes:

\[ n^{1-\nu} \| p(\nu) \|_b \geq 1. \]  

(25)

Eq. (25) follows directly from the Hölder inequality (Abramowitz and Stegun, 2014):

\[ \sum_{k=1}^{n} |a_k b_k| \leq \left( \sum_{k=1}^{n} |a_k|^u \right)^{\frac{1}{u}} \left( \sum_{k=1}^{n} |b_k|^v \right)^{\frac{1}{v}} \]  

(26)

when \( a_k = p_k(\nu), \quad b_k = \frac{1}{n}, \quad u = \frac{1}{\nu}, \quad \text{and} \quad v = \frac{1}{1-\nu}. \) The properties of \( p(\nu) \) show that equality holds when \( \nu = 0, 1. \)

\[ \square \]

Theorem 2.1 is consistent with the expectation that the investors’ total utility (and therefore also the total (monetary) output of their investments) should increase if more information can be used in the investment decision.

2.2. Applications

2.2.1. Special Case – Two Investments

As an example, consider a special formulation of Case 2 in which seed money has to be distributed between two investment objects available for selection. Hence, we \( n = 2 \) is the number of different investment objects, \( \omega_1 + \omega_2 = 1 \) is the budget constraint and the only further information assumed is \( 0 \leq x^{(1)} \leq x^{(2)} \leq 1 \) for the first input factor. With \( x^{(i)} \) being the order statistic of a uniformly distributed random variable. Thus, with Eq. (18):

\[ p_1(\nu) = \frac{1 + \nu}{2} \frac{B(1 + \nu, 2)}{B(1, 2)} = \frac{1}{2 + \nu} \]

\[ p_2(\nu) = \frac{1 + \nu}{2} \frac{B(2 + \nu, 2)}{B(2, 1)} = \frac{1 + \nu}{2 + \nu} \]  

(27)

The optimal allocation of capital is then described by the following proportions (cf. Eq. (22)):

\[ \omega_1^* = \frac{1}{1 + (1 + \nu)^+} \]

\[ \omega_2^* = \frac{1}{1 + (1 + \nu)^-} \]  

(28)

Depending on the elasticity \( \nu \in [0, 1] \), three special cases can be considered:

a) The seed money \( \omega \) dominates the utility: \( \nu \to 0. \)

b) There is indifference between the input factors related to utility: \( \nu = 0.5. \)

c) The random number \( x \) dominates the utility: \( \nu \to 1. \)
Tab. 1
Allocation of capital between \( n = 2 \) investments for selected elasticities \( \nu \)

| Elasticity | Asset 1 | Asset 2 |
|------------|---------|---------|
| \( \nu \) | \( \omega_1 \) | \( \omega_2 \) |
| \( \nu \to 0 \) | \( \frac{1}{1+e} \) | \( \frac{e}{1+e} \) |
| \( \nu = 0.5 \) | \( \frac{12}{39} \) | \( \frac{27}{39} \) |
| \( \nu \to 1 \) | \( \frac{13}{39} \) | \( \frac{26}{39} \) |

Fig. 1 The dependence of the capital allocation on the elasticity \( \nu \) considering two assets
Tab. 1 reports the optimal capital allocation for these special cases, and Fig. 1 depicts the allocation of capital between the two assets depending on the elasticity $\nu$.

Solving the allocation problem with unknown elasticity $\nu$: as a rule of thumb, the capital $C = 1$ can be distributed as seed money in a proportion $\frac{1}{3} : \frac{2}{3}$ between the two assets (usually, (minority) stakes in the two early-stage companies), if no further information is known except the ordinal ranking of the first input factor (production rate $x(1) \leq x(2)$) and its assumed uniform distribution between 0% and 100%.

### 2.2.2. Applications to More than Two Investments

In the case of $n > 2$ investments, the (risk-neutral) investors face a more complex decision situation that is not covered by the rule of thumb developed above. Nevertheless, using the formula in Eq. (22), investors can easily calculate the optimal portfolio weights for arbitrary $n$.

Considering the ordinal ranking $0 \leq x(1) \leq x(2) \leq \ldots \leq x(n) \leq 1$, then Tab. 2 provides the optimal weights for $n = 3, 4, 5$ investments as an example. Since the weights $\omega_i$ of the individual investments for fixed $n$ do not change substantially as the elasticity $\nu$ changes (see Tab. 2), we can focus on the limiting case $\nu \to 1$ in Eq. (22) and derive a rule of thumb for an unknown $\nu$ in practice:

$$\omega_i = \frac{2i}{n(n + 1)}.$$  \hspace{1cm} (29)

As a rough approximation, the capital allocation $\omega_i$ depends only on the position $i$ of the investment in the sorted alternatives and the total number $n$ of investment alternatives.

### 3. Extension: Portfolio Selection for Risk-Averse Investors

Modern Portfolio Theory describes the situation in which an investor considers the return and risk of each investment when making his capital allocation. In addition to the return and the risk, the dependency structure of the investment opportunities in the optimization is used to construct an efficient portfolio (Markowitz 1952). Usually, these three characteristics are estimated from historical data (Elton et al. 2017).

When allocating seed money or venture capital, we assumed in Sec. 2.1 that there is no market data available to estimate the return and risk from historical data. In the following, we show how, nevertheless, the basic idea of modern portfolio theory can be transferred to our described use case – the optimal allocation of seed money. To do so, we consider the total utility as defined in Eq. (6).

In practice, an investor will not only consider the expected value $E[U]$ of the total utility but also the variance $\text{Var}[U]$ of the total utility (as a measure
Tab. 2
Sorted Weighted Portfolios considering $n = \{3, 4, 5\}$ investments and selected elasticities $\nu$

| Investments | Elasticity | Weights |  |   |   |   |
|-------------|------------|---------|---|---|---|---|
| $n$         | $\nu$     | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ |
| 3           | 0          | 0.1223   | 0.3315   | 0.5462     |   |   |
|             | $\frac{1}{4}$ | 0.1360   | 0.3321   | 0.5319     |   |   |
|             | $\frac{2}{4}$ | 0.1478   | 0.3326   | 0.5196     |   |   |
|             | $\frac{3}{4}$ | 0.1579   | 0.3330   | 0.5091     |   |   |
|             | 1          | 0.1667   | 0.3333   | 0.5000     |   |   |
| 4           | 0          | 0.0694   | 0.1881   | 0.3100     | 0.4325 |   |
|             | $\frac{1}{4}$ | 0.0785   | 0.1917   | 0.3070     | 0.4228 |   |
|             | $\frac{2}{4}$ | 0.0866   | 0.1948   | 0.3044     | 0.4143 |   |
|             | $\frac{3}{4}$ | 0.0937   | 0.1976   | 0.3021     | 0.4067 |   |
|             | 1          | 0.1000   | 0.2000   | 0.3000     | 0.4000 |   |
| 5           | 0          | 0.0446   | 0.1210   | 0.1993     | 0.2781 | 0.3570 |
|             | $\frac{1}{4}$ | 0.0510   | 0.1245   | 0.1995     | 0.2748 | 0.3502 |
|             | $\frac{2}{4}$ | 0.0568   | 0.1278   | 0.1997     | 0.2718 | 0.3440 |
|             | $\frac{3}{4}$ | 0.0620   | 0.1307   | 0.1998     | 0.2691 | 0.3384 |
|             | 1          | 0.0667   | 0.1333   | 0.2000     | 0.2667 | 0.3333 |
of risk) in the capital allocation. As we have seen, the uncertainty in the total utility results only from the factor \( x_i \) (random variable) of the individual utility function Eq. (3). We consider a constant elasticity \( \nu \) for the input factor \( x_i \). Thus, the random variable becomes \( x_i^\nu \).

In Case 2 of Sec. 2.1, a rational investor was able to sort the investment objects based on an assessment of the factors \( x_i \). This Case 2 is further examined below.

First, we calculate the variance of the total utility with respect to the order statistic \( x^{(i)} \), the capital allocation \( \omega_i \) for \( i = 1, \ldots, n \) and \( n \) investment objects:

\[
\text{Var}[U] = E[U^2] - E[U]^2 = a^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \left( E[x_i^\nu x_j^\nu] - E[x_i^\nu]E[x_j^\nu] \right) \omega_i^{1-\nu} \omega_j^{1-\nu} - a^n \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij} \omega_i^{1-\nu} \omega_j^{1-\nu}.
\]  

(30)

The calculation and representation of the covariance matrix \( V_{ij} = V_{ij}(\nu, n) \) for \( i, j = 1, \ldots, n \) is shown in Appendix A Eq. (A.6). Note that the order statistics are not necessarily stochastically independent; thus, we also have entries in the minor diagonals of the covariance matrix (in contrast to Case 1).

With the variance, the objective function to be maximized can now be specified for Case 2, and together with the budget function, the Lagrange function is simply:

\[
L = E[U] - \frac{1}{2} b \text{Var}[U] + \lambda \left( 1 - \sum_{i=1}^{n} \omega_i \right) = a \frac{n}{1+\nu} \sum_{i=1}^{n} p_i(\nu) \omega_i^{1-\nu} - \frac{1}{2} a^2 b \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij} \omega_i^{1-\nu} \omega_j^{1-\nu} + \lambda \left( 1 - \sum_{i=1}^{n} \omega_i \right).
\]  

(31)

As in Sec. 2.1, \( a > 0 \) is the efficiency parameter, and \( b > 0 \) denotes an individual risk-aversion parameter.

The optimization problem is to find the maximum of the Lagrange function.
dependent on the capital allocation $\omega_i$ for $i = 1, \ldots, n$:

$$\max_{\{\omega_1, \omega_2, \ldots, \omega_n\}} \mathcal{L}. \quad (32)$$

The optimal solution $\omega^*_i$ depends on the number of investment objects $n$, the parameters $a$ and $b$ and the elasticity $\nu$. In a numerical determination of the optimal solution, the $\omega$-space should be limited to the positive orthant reflecting the long-only case, where no short positions are possible. The maximization of the Lagrange function Eq. (31) thus takes place with respect to the inequality constraint $\omega_i > 0$ for all $i$.

If the expected total utility $E[U]$ is ignored in the Lagrange function Eq. (31) and only the variance $\text{Var}[U]$ of the total utility is considered in the objective function, then the optimal solution of the optimization problem describes the capital allocation in a minimum variance portfolio.

### 4. Discussion and Conclusions

Allocating capital to a selection of different investment objects often posses the problem that investors’ decisions are made under limited information (no historical return data) and within an extremely limited timeframe. Nevertheless, in some cases, rational investors with a certain level of experience are able to ordinally rank these investment alternatives.

In this paper, we developed an innovative portfolio optimization framework that uses such ordinal rankings as the foundation for determining the optimal portfolio weights for a certain investment alternative. Under the assumption of a risk-neutral investor, we provided a closed-form solution for the optimal weight vector – which depends on the partial elasticity $\nu$. For an unknown $\nu$, we developed a rule of thumb that capital $C = 1$ should be distributed across the alternatives in a certain proportion, depending only on $n$. For $n = 2$ investment opportunities and the assessment $x_{(1)} < x_{(2)}$, we found the approximate distribution $\frac{1}{3} : \frac{2}{3}$ for capital.

We showed that, in general, the Sorted Weighted Portfolio (SWP) outperforms the intuitive solution of an Equally Weighted Portfolio (EWP), which is traditionally the benchmark for portfolio optimization strategies in the literature and in this special case also the result of an optimization when it is not possible to account for the additional information (ordinal ranking of the investment alternatives).

The extension of the model to risk-averse investors revealed that we were able to formulate $E[U]$ and $\text{Var}[U]$ as the classical input factors of Lagrangian optimization and to address diversification effects. However, in this case, there is no algebraic closed-form solution available for the optimization model. Consequently, our model has important implications for practice and is also a useful
starting point for the development of further extensions and practical applications in research.

Appendix A. Covariance Matrix

In the following section, we determine the covariance matrix $V_{ij}(\nu, n)$. The covariance matrix is needed in Sec. 3 to calculate the variance of the total utility. Following Eq. (30), we have

$$V_{ij}(\nu, n) = E[x_{(i)}^{\nu}x_{(j)}^{\nu}] - E[x_{(i)}^{\nu}]E[x_{(j)}^{\nu}]$$  \hspace{1cm} (A.1)

The last term of this equation can be calculated using Eq. (19). Therefore, we only need to focus our attention on the first term. The evaluation of $E[x_{(i)}^{\nu}x_{(j)}^{\nu}]$ leads to a matrix $M$ with entries $M_{ij}$ and $i, j = 1, \ldots, n$.

Case $j = i$:

If $j = i$ then $M_{ii} = E[x_{(i)}^{\nu}x_{(i)}^{\nu}] = E[x_{(i)}^{2\nu}] = \frac{n}{1+2\nu} p_i(2\nu)$ (cf. Eq. (19)).

Case $j > i$:

This case describes the situation in which the order statistic $x_{(i)}$ is smaller than the order statistic $x_{(j)}$. We therefore first calculate the upper triangle of the matrix $M$. To determine the expected value $E[x_{(i)}^{\nu}x_{(j)}^{\nu}]$, the joint probability distribution of the order statistics is needed. The formulas become shorter if we write $u = x_{(i)}$ and $v = x_{(j)}$ with $u < v$. Then, the expectation value $E[u^\nu v^\nu]$ must be derived with respect to the joint probability distribution (Kendall and Stuart 1970):

$$\rho(u, v; i, j, n) = C(n, i, j) u^{i-1}(v - u)^{j-i-1}(1 - v)^{n-j}$$ \hspace{1cm} (A.2)

with the constant

$$C(n, i, j) = \frac{\Gamma(n + 1)}{\Gamma(i)\Gamma(j - i)\Gamma(n + 1 - j)}$$ \hspace{1cm} (A.3)

and $\Gamma(\alpha)$ being the gamma function (Abramowitz and Stegun 2014). A double
integration then provides the expectation value:

\[ E[u^\nu v^\nu] \]

\[ = \int_0^1 \int_0^v u^\nu v^\nu \rho(u, v; i, j, n) \, du \, dv \]

\[ = C(n, i, j) \int_0^1 \int_0^v u^\nu v^\nu u^{i-1} (v-u)^{j-i-1} (1-v)^{n-j} \, du \, dv. \]

Substituting \( w = 1 - v \) leads to

\[ E[u^\nu v^\nu] = C(n, i, j) \int_0^1 \int_0^{1-w} (1-w)^\nu u^{i-1} w^{n-j} (1-w-u)^{j-i-1} \, d\nu \, dw. \]

Now, using the series representation

\[ (1-w)^\nu = \sum_{k=0}^{\infty} (-w)^k \frac{\Gamma(\nu+1)}{\Gamma(k+1)\Gamma(\nu+1-k)} \quad \text{for } |w| < 1. \]

Since this is a convergent series with a finite limit, it follows:

\[ = C(n, i, j) \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu+1)}{\Gamma(k+1)\Gamma(\nu+1-k)} \times \]

\[ \int_0^1 \int_0^{1-w} u^{i-1} w^{n-j} (1-w-u)^{j-i-1} \, d\nu \, dw. \quad (A.4) \]

The last double integral is the representation of the two-dimensional beta function [Waldron 2003]. Hence, the upper triangle of the matrix \( M \) is given by

\[ M_{ji}(\nu, n) = E[u^\nu v^\nu] = \frac{\Gamma(n+1)\Gamma(\nu+1)\Gamma(\nu+i)}{\Gamma(i)\Gamma(n+1-j)} \times \]

\[ \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(n+k+1-j)}{\Gamma(k+1)\Gamma(\nu+1-k)\Gamma(\nu+n+k+1)} \quad (A.5) \]

Case \( j < i \):

For reasons of symmetry, \( E[v^\nu u^\nu] = E[u^\nu v^\nu] \). Therefore, the lower triangle of the matrix is \( M_{ij}(\nu, n) = M_{ji}(\nu, n) \).
Finally, the entire covariance matrix is given by:

\[
\begin{pmatrix}
V_{ij}(\nu, n)
\end{pmatrix}_{i,j=1,\ldots,n} =
\begin{pmatrix}
\ddots & M_{ji}(\nu, n) - \left(\frac{n}{1+2\nu}\right)^2 p_j(\nu)p_i(\nu) \\
\left(\frac{n}{1+2\nu}\right) p_i(2\nu) - \left(\frac{n}{1+\nu}\right)^2 p_i(\nu)^2 & M_{ij}(\nu, n) - \left(\frac{n}{1+\nu}\right)^2 p_i(\nu)p_j(\nu) & \ddots \\
M_{ij}(\nu, n) - \left(\frac{n}{1+\nu}\right)^2 p_i(\nu)p_j(\nu) & \ddots & \ddots 
\end{pmatrix}
\]  

(A.6)

Here, \(M_{ji}(\nu, n)\) is defined in Eq. (A.5), and \(p_i(\nu)\) is defined in Eq. (18).

By construction, for \(n > 1\), the covariance matrix is positive semi-definite. For \(\nu \neq 0\), the random variables \(x_i^{(\nu)}\) and \(x_j^{(\nu)}\), with \(i \neq j\) in Eq. (A.1) are linearly independent, and the covariance matrix is positive definite and thus invertible [Kendall and Stuart 1976].

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