UPPER BOUNDS FOR THE MOMENTS OF $\zeta'(\rho)$

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Abstract. Assuming the Riemann Hypothesis, we obtain an upper bound for the $2k$th moment of the derivative of the Riemann zeta-function averaged over the non-trivial zeros of $\zeta(s)$ for every positive integer $k$. Our bounds are nearly as sharp as the conjectured asymptotic formulae for these moments.

1. Introduction & statement of the main results

Let $\zeta(s)$ denote the Riemann zeta-function. This article is concerned with estimating discrete moments of the form

$$ J_k(T) = \frac{1}{N(T)} \sum_{0<\gamma\leq T} |\zeta'(\rho)|^{2k} $$

where $k \in \mathbb{N}$ and the sum runs over the non-trivial (complex) zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. As usual, the function

$$ N(T) = \sum_{0<\gamma\leq T} 1 = \frac{T}{2\pi} \log \frac{2\pi}{2\pi} - \frac{T}{2\pi} + O(\log T) $$

denotes the number of zeros of $\zeta(s)$ up to a height $T$ counted with multiplicity.

It is an open problem to determine the behavior of $J_k(T)$ as $k$ varies. Independently, Gonek [7] and Hejhal [10] have conjectured that

$$ J_k(T) \asymp (\log T)^{k(k+2)} $$

for fixed $k \in \mathbb{R}$ as $T \to \infty$. Though widely believed for positive values of $k$, there is evidence to suggest that this conjecture is false for $k \leq -3/2$.

Until recently, estimates in agreement with (3) were only known in a few cases. Assuming the Riemann Hypothesis (which asserts that $\beta = \frac{1}{2}$ for each non-trivial zero of $\zeta(s)$), Gonek [5] has shown that $J_1(T) \sim \frac{1}{12}(\log T)^3$ and Ng [17] has proved that

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Confirming a conjecture of Conrey and Snaith (section 7.1 of [1]), the author [15] has calculated the lower-order terms in the asymptotic expression for $J_1(T)$. Under the additional assumption that the zeros of $\zeta(s)$ are simple, Gonek [7] has shown that $J_{-1}(T) \gg (\log T)^{-1}$ and conjectured [9] that $J_{-1}(T) \sim \frac{6}{\pi^2}(\log T)^{-1}$.

In addition, there are a few related unconditional results where the sum in (1) is restricted to the simple zeros of $\zeta(s)$ with $\beta = \frac{1}{2}$. See, for instance, [3, 4, 14, 21].

By using a random matrix model to study the behavior of the Riemann zeta-function and its derivative on the critical line, Hughes, Keating, and O’Connell [12] have refined Gonek’s and Hejhal’s conjecture in (3). In particular, they conjectured a precise constant $D_k$ such that $J_k(T) \sim D_k(\log T)^k(k+2)$ as $T \to \infty$ for fixed $k \in \mathbb{C}$ with $\Re k > -3/2$. Their conjecture is consistent with the results mentioned above.

Very little is known about the moments $J_k(T)$ when $k > 2$. However, assuming the Riemann Hypothesis, one may deduce from well-known results of Littlewood (Theorems 14.14 A-B of Titchmarsh [23]) that for $\sigma \geq 1/2$ and $t \geq 10$, the estimate

$$\zeta'(\sigma+it) \ll \exp\left(\frac{C \log t}{\log \log t}\right)$$

holds for some constant $C > 0$. It immediately follows that

$$J_k(T) \ll \exp\left(\frac{2kC \log T}{\log \log T}\right)$$

for any $k \geq 0$. The goal of this paper is to improve this estimate by obtaining a conditional upper bound for $J_k(T)$ (when $k \in \mathbb{N}$) very near the conjectured order of magnitude. In particular, we prove the following result.

**Theorem 1.1.** Assume the Riemann Hypothesis. Let $k \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. Then for sufficiently large $T$ we have

$$\frac{1}{N(T)} \sum_{0 < \gamma \leq T} \left| \zeta'(\rho) \right|^{2k} \ll (\log T)^{(k+2)+\varepsilon},$$

where the implied constant depends on $k$ and $\varepsilon$.

Under the assumption of the Riemann Hypothesis, N. Ng and the author [16] have shown that $J_k(T) \gg (\log T)^{(k+2)}$ for each fixed $k \in \mathbb{N}$. Combining this result with Theorem 1.1 lends strong support for the conjecture of Gonek and Hejhal concerning the behavior of $J_k(T)$ in the case when $k$ is a positive integer.
Our proof of Theorem 1.1 is based upon a recent method of Soundararajan [22] that provides upper bounds for the frequency of large values of $|\zeta(\frac{1}{2}+it)|$. His method relies on obtaining an inequality for $\log|\zeta(\frac{1}{2}+it)|$ involving a “short” Dirichlet polynomial which is a smoothed approximation to the Dirichlet series for $\log \zeta(s)$. Using mean-value estimates for high powers of this Dirichlet polynomial, he deduces upper bounds for the measure of the set \{t $\in [0, T] : \log|\zeta(\frac{1}{2}+it)| \ge V$\} and from this is able to conclude that, for arbitrary positive values of $k$ and $\varepsilon$,

\begin{equation}
\frac{1}{T} \int_{0}^{T} |\zeta(\frac{1}{2}+it)|^{2k} \ll_{k, \varepsilon} (\log T)^{k^2+\varepsilon}
\end{equation}

Soundararajan’s techniques build upon the work of Selberg [18, 19, 20] who studied the distribution of values of $\log \zeta(\frac{1}{2}+it)$ in the complex plane.

Since $\log \zeta'(s)$ does not have a Dirichlet series representation, it is not clear that $\log|\zeta'(\frac{1}{2}+it)|$ can be approximated by a Dirichlet polynomial. For this reason, we do not study the distribution of the values of $\zeta'(\rho)$ directly, but instead examine the frequency of large values of $|\zeta(\rho+\alpha)|$, where $\alpha \in \mathbb{C}$ is a small shift away from a zero $\rho$ of $\zeta(s)$. This requires deriving an inequality for $\log|\zeta(\sigma+it)|$ involving a short Dirichlet polynomial that holds uniformly for values of $\sigma$ in a small interval to the right of, and including, $\sigma = \frac{1}{2}$. Using a result of Gonek (Lemma 4.1 below), we estimate high powers of this Dirichlet polynomial averaged over the zeros of the zeta-function and are able to derive upper bounds for the frequency of large values of $|\zeta(\rho+\alpha)|$. Using this information we prove the following theorem.

**Theorem 1.2.** Assume the Riemann Hypothesis. Let $\alpha \in \mathbb{C}$ with $|\alpha| \le 1$ and $|\Re \alpha - \frac{1}{2}| \le (\log T)^{-1}$. Let $k \in \mathbb{R}$ with $k > 0$ and let $\varepsilon > 0$ be arbitrary. Then for sufficiently large $T$ the inequality

$$\frac{1}{N(T)} \sum_{0 < \gamma \le T} |\zeta(\rho+\alpha)|^{2k} \ll_{k, \varepsilon} (\log T)^{k^2+\varepsilon}$$

holds uniformly in $\alpha$.

Comparing the result of Theorem 1.2 with the estimate in (4), we see that our theorem provides essentially the same upper bound (up to the implied constant) for $\frac{1}{T} \int_{0}^{T} |\zeta(\frac{1}{2}+it)|^{2k} \ll_{k, \varepsilon} (\log T)^{k^2+\varepsilon}$.

Hejhal [10] studied the distribution of $\log |\zeta'(\frac{1}{2}+it)|$ by a method that does not directly involve the use of Dirichlet polynomials.
discrete averages of the Riemann zeta-function near its zeros as can be obtained for continuous moments of $|\zeta(1_2+it)|$ by using the methods in [22]. There has been some previous work on discrete mean-value estimates of the zeta-function that are of a form that is similar to the sum appearing in Theorem 1.2. For instance, see the results of Gonek [5], Fujii [2], and Hughes [11].

We deduce Theorem 1.1 from Theorem 1.2 since, by Cauchy’s integral formula, we can use bounds for $\zeta(s)$ near its zeros to recover bounds on the values for $\zeta'(\rho)$. For a precise statement of this idea, see Lemma 7.1 below. Our proof allows us only to establish Theorem 1.1 when $k$ is a positive integer despite the fact that Theorem 1.2 holds for all $k > 0$.

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2. An inequality for $\log |\zeta(\sigma+it)|$ when $\sigma \geq 1_2$.

Throughout the remainder of this article, we use $s = \sigma + it$ to denote a complex variable and use $p$ to denote a prime number. We let $\lambda_0 = .5671...$ be the unique positive real number satisfying $e^{-\lambda_0} = \lambda_0$. Also, we put $\sigma_\lambda = \sigma_{\lambda,x} = \frac{1}{2} + \frac{\lambda}{\log x}$ and let

$$\log^+ |x| = \begin{cases} 0, & \text{if } |x| < 1, \\ \log |x|, & \text{if } |x| \geq 1. \end{cases}$$

As usual, we denote by $\Lambda(\cdot)$ the arithmetic function defined by $\Lambda(n) = \log p$ when $n = p^k$ and $\Lambda(n) = 0$ when $n \neq p^k$. The main result of this section is the following lemma.

**Lemma 2.1.** Assume the Riemann Hypothesis. Let $\tau = |t| + 3$ and $2 \leq x \leq \tau^2$. Then, for any $\lambda$ with $\lambda_0 \leq \lambda \leq \log x/4$, the estimate

\[
\log^+ |\zeta(\sigma+it)| \leq \sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma_\lambda+it} \log n} \log x/n + \left(1+\lambda\right)\frac{\log \tau}{2} \log x + O(1)
\]

holds uniformly for $1/2 \leq \sigma \leq \sigma_\lambda$.

In [22], Soundararajan proved an inequality similar to Lemma 2.1 for the function $\log |\zeta(1_2+it)|$. In his case, when $\zeta(1_2+it) \neq 0$, an inequality slightly stronger than (5)
holds with the constant $\lambda_0$ replaced by $\delta_0 = .4912...$ where $\delta_0$ is the unique positive real number satisfying $e^{-\delta_0} = \delta_0 + \frac{1}{2}\delta_0^2$. Our proof of the above lemma is a modification of his argument.

**Proof of Lemma 2.1.** We assume that $|\zeta(\sigma + it)| \geq 1$, as otherwise the lemma holds for a trivial reason. In particular, we are assuming that $\zeta(\sigma + it) \neq 0$. Assuming the Riemann Hypothesis, we denote a non-trivial zeros of $\zeta(s)$ as $\rho = \frac{1}{2} + i\gamma$ and define the function

$$F(s) = \Re \sum_{\rho} \frac{1}{s - \rho} = \sum_{\rho} \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + (t-\gamma)^2}.$$ 

Notice that $F(s) \geq 0$ whenever $\sigma \geq \frac{1}{2}$ and $s \neq \rho$. The partial fraction decomposition of $\zeta'(s)/\zeta(s)$ (equation (2.12.7) of Titchmarsh [23]) says that for $s \neq 1$ and $s$ not coinciding with a zero of $\zeta(s)$, we have

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) - \frac{1}{2} \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} s + 1 \right) - \frac{1}{s - 1} + B$$

where the constant $B = \log 2\pi - 1 - 2\gamma_0$; $\gamma_0$ denotes Euler’s constant. Taking the real part of each term in (6), we find that

$$-\Re \frac{\zeta'}{\zeta}(s) = -\Re \frac{1}{2} \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} s + 1 \right) - F(s) + O(1).$$

Stirling’s asymptotic formula for the gamma function implies that

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + O(|s|^{-2})$$

for $\delta > 0$ fixed, $|\arg s| < \pi - \delta$, and $|s| > \delta$ (see Appendix A.7 of Ivić [13]). By combining (7) and (8) with the observation that $F(s) \geq 0$, we find that

$$-\Re \frac{\zeta'}{\zeta}(s) = \frac{1}{2} \log \tau - F(s) + O(1) \leq \frac{1}{2} \log \tau + O(1).$$
uniformly for $\frac{1}{2} \leq \sigma \leq 1$. Consequently, the inequality

$$\log |\zeta(\sigma+it)| - \log |\zeta(\sigma+i\lambda+it)| = \Re \int_{\sigma}^{\sigma+i\lambda} \left[ -\frac{\zeta'}{\zeta}(u+it) \right] du$$

(10)

holds uniformly for $\frac{1}{2} \leq \sigma \leq \sigma_\lambda$.

To complete the proof of the lemma, we require an upper bound for $\log |\zeta(\sigma+i\lambda+it)|$ which, in turn, requires an additional identity for $\zeta'(s)/\zeta(s)$. Specifically, for $s \neq 1$ and $s$ not coinciding with a zero of $\zeta(s)$, we have

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n \leq x} \frac{\Lambda(n)}{n^s} \log(x/n) \log x + \frac{1}{\log x} \left( \frac{\zeta'}{\zeta}(s) \right)' + \frac{1}{\log x} \sum_{\rho} \frac{x^{\rho-s}}{(\rho-s)^2}$$

(11)

$$- \frac{1}{\log x} \frac{x^{1-s}}{(1-s)^2} + \frac{1}{\log x} \sum_{k=1}^{\infty} \frac{x^{-2k-s}}{(2k+s)^2}.$$

This identity is due to Soundararajan (Lemma 1 of [22]). Integrating over $\sigma$ from $\sigma_\lambda$ to $\infty$, we deduce from the above identity that

$$\log |\zeta(\sigma+i\lambda+it)| = \Re \sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma+i\lambda+it} \log n} \log x + \frac{1}{\log x} \Re \frac{\zeta'}{\zeta}(\sigma+i\lambda+it)$$

(12)

$$+ \frac{1}{\log x} \sum_{\rho} \Re \int_{\sigma_\lambda}^{\infty} \frac{x^{\rho-s}}{(\rho-s)^2} d\sigma + O\left( \frac{1}{\log x} \right).$$

We now estimate the second and third terms on the right-hand side of this expression. Arguing as above, using (6) and (8), we find that

$$\Re \sum_{\sigma_\lambda}^{\infty} (\sigma+i\lambda+it) = \frac{1}{2} \log \tau - F(\sigma+i\lambda+it) + O(1).$$

(13)

Also, observing that

$$\sum_{\rho} \left| \int_{\sigma_\lambda}^{\infty} \frac{x^{\rho-s}}{(\rho-s)^2} d\sigma \right| \leq \sum_{\rho} \int_{\sigma_\lambda}^{\infty} \frac{x^{1/2-s}}{|\rho-s|^2} d\sigma$$

(14)

$$= \sum_{\rho} \frac{x^{1/2-\sigma_\lambda}}{|\rho-\sigma_\lambda-it|^2 \log x} = \frac{x^{1/2-\sigma_\lambda} F(\sigma_\lambda+it)}{(\sigma_\lambda+it) \log x},$$
and combining \((13)\) and \((14)\) with \((12)\), we see that
\[
\log |\zeta(\sigma + it)| \leq \Re \sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma+it} \log n} \frac{\log x/n}{\log x} + \frac{1 \log \tau}{2 \log x} + \frac{\log x^{1/2 - \sigma}}{(\sigma - \frac{1}{2}) \log x - 1} \times O\left(\frac{1}{\log x}\right).
\]

If \(\lambda \geq \lambda_0\), then the term on the right-hand side involving \(F(\sigma + it)\) is less than or equal to zero, so omitting it does not change the inequality. Thus,
\[
(15) \quad \log |\zeta(\lambda + it)| \leq \Re \sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma+it} \log n} \frac{\log x/n}{\log x} + \frac{1 \log \tau}{2 \log x} + O\left(\frac{1}{\log x}\right).
\]

Since we have assumed that \(|\zeta(\lambda + it)| \geq 1\), the lemma now follows by combining the inequalities in \((10)\) and \((15)\) and then taking absolute values.

3. A variation of Lemma 2.1

In this section, we prove a version of Lemma 2.1 in which the sum on the right-hand side of the inequality is restricted just to the primes. A sketch of the proof of the lemma appearing below has been given previously by Soundararajan (see [22], Lemma 2). Our proof is different and the details are provided for completeness.

Lemma 3.1. Assume the Riemann Hypothesis. Put \(\tau = |t| + e^{30}\). Then, for \(\sigma \geq \frac{1}{2}\) and \(2 \leq x \leq \tau^2\), we have
\[
\left| \sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma+it} \log n} \frac{\log x/n}{\log x} - \sum_{p \leq x} \frac{1}{p^{\sigma+it} \log x} \right| = O\left(\log \log \log \tau\right).
\]

As a consequence, for any \(\lambda\) with \(\lambda_0 \leq \lambda \leq \frac{\log x}{4}\), the estimate
\[
\log^+ |\zeta(\lambda + it)| \leq \left| \sum_{p \leq x} \frac{1}{p^{\sigma+it} \log x} \right| + \frac{(1+\lambda) \log \tau}{2 \log x} + O\left(\log \log \log \tau\right)
\]
holds uniformly for \(\frac{1}{2} \leq \sigma \leq \sigma_\lambda\) and \(2 \leq x \leq \tau^2\).
Proof. First we observe that, for $\sigma \geq \frac{1}{2}$,

$$
\sum_{n \leq x} \frac{\Lambda(n)}{n^s \log n} \frac{\log x/n}{\log x} - \sum_{p \leq x} \frac{1}{p^s} \frac{\log x/p}{\log x} = \frac{1}{2} \sum_{p \leq \sqrt{x}} \frac{1}{p^{2s}} \frac{\log \sqrt{x}/n}{\log \sqrt{x}} \cdot 1 + O(1).
$$

Thus, if we let $w = u + iv$ and $\nu = |v| + e^{30}$, the lemma will follow if we can show

$$
\sum_{n \leq z} \frac{\Lambda(n)}{n^w \log n} \frac{\log z/n}{\log z} = O(\log \log \log \nu)
$$

uniformly for $u \geq 1$ and $2 \leq z \leq \nu$. In what follows, we can assume that $z \geq (\log \nu)^2$ as otherwise

$$
\sum_{n \leq z} \frac{\Lambda(n)}{n^w \log n} \frac{\log z/n}{\log z} \ll \sum_{p < \log^2 \nu} \frac{1}{p} \ll \log \log \log \nu.
$$

Let $c = \max(2, 1 + u)$. Then, by expressing $\zeta'(s+w)$ as a Dirichlet series and interchanging the order of summation and integration (which is justified by absolute convergence), it follows that

$$
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[ - \frac{\zeta'}{\zeta}(s+w) \right] z^s \frac{ds}{s^2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[ \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{s+w}} \right] z^s \frac{ds}{s^2}
$$

$$
= \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^w} \int_{c-i\infty}^{c+i\infty} \left( \frac{z}{n} \right)^s \frac{ds}{s^2}
$$

$$
= \sum_{n \leq z} \frac{\Lambda(n)}{n^w} \log(z/n).
$$

Here we have made use of the standard identity

$$
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^s \frac{ds}{s^2} = \begin{cases} \log x, & \text{if } x \geq 1, \\ 0, & \text{if } 0 \leq x < 1, \end{cases}
$$
which is valid for \( c > 0 \). By moving the line of integration in the integral left to \( \Re s = \sigma = \frac{3}{4} - u \), we find by the calculus of residues that

\[
\sum_{n \leq z} \frac{\Lambda(n)}{n^w} \log(z/n) = -(\log z) \frac{\zeta'}{\zeta}(w) - \left( \frac{\zeta'}{\zeta}(w) \right)' + \frac{z^{1-w}}{(w-1)^2} + \frac{1}{2\pi i} \int_{\frac{3}{4} - u - i\infty}^{\frac{3}{4} - u + i\infty} \left[ - \frac{\zeta'}{\zeta}(s+w) \right] z^s \frac{ds}{s^2}.
\]

(17)

That there are no residues obtained from poles of the integrand at the non-trivial zeros of \( \zeta(s) \) follows from the Riemann Hypothesis. To estimate the integral on the right-hand side of the above expression, we use Theorem 14.5 of Titchmarsh [23], namely, that if the Riemann Hypothesis is true, then

\[
\left| \frac{\zeta'}{\zeta}(\sigma + it) \right| \ll (\log \tau)^{2-2\sigma} - \frac{1}{\log z} \left( \frac{\zeta'}{\zeta}(w) \right)' + O \left( \frac{z^{1-u}}{\log z} \sqrt{\log \nu} \right).
\]

(18)

uniformly for \( \frac{5}{8} \leq \sigma \leq \frac{7}{8} \), say. Using (18), it immediately follows that

\[
\int_{\frac{3}{4} - u - i\infty}^{\frac{3}{4} - u + i\infty} \left[ - \frac{\zeta'}{\zeta}(s+w) \right] z^s \frac{ds}{s^2} \ll z^{3/4 - u} \sqrt{\log \nu}.
\]

Inserting this estimate into equation (17) and dividing by \( \log z \), it follows that

\[
\sum_{n \leq z} \frac{\Lambda(n)}{n^w} \log(z/n) \frac{\log(z/n)}{\log z} = -\frac{\zeta'}{\zeta}(w) - \frac{1}{\log z} \left( \frac{\zeta'}{\zeta}(w) \right)' + \frac{z^{1-w}}{(w-1)^2} \log z + O \left( \frac{z^{3/4 - u}}{\log z} \sqrt{\log \nu} \right).
\]

(19)

Integrating the expression in (19) from \( \infty \) to \( u \) (along the line \( \sigma + i\nu \), \( u \leq \sigma < \infty \)), we find that

\[
\sum_{n \leq z} \frac{\Lambda(n)}{n^w} \log(z/n) \frac{\log z}{\log n} \log n = \log \zeta(w) + \frac{1}{\log z} \zeta'(w) + O \left( \frac{z^{1-u}}{\nu^2 (\log z)^2} + \frac{z^{3/4 - u}}{\nu^2 (\log z)^2} \sqrt{\log \nu} \right).
\]
Assuming the Riemann Hypothesis, we can estimate the terms on the right-hand side of the above expression by invoking the bounds

\[ |\log \zeta(\sigma + it)| \ll \log \log \tau \quad \text{and} \quad \left| \frac{\zeta'}{\zeta}(\sigma + it) \right| \ll \log \log \tau \]

which hold uniformly for \( \sigma \geq 1 \) and \( |t| \geq 1 \). (For a discussion of such estimates see Heath-Brown’s notes following Chapter 14 in Titchmarsh [23].) Using the estimates in (20) and recalling that we are assuming that \( u \geq 1 \) and \( z \geq (\log \nu)^2 \), we find that

\[
\sum_{n \leq z} \Lambda(n) \frac{\log(z/n)}{\log z} \ll \log \log \nu + \frac{\log \log \nu}{\log z} + \frac{z^{1-u}}{\nu^2(\log z)^2} + z^{-1/4} \sqrt{\log \nu} \frac{\log \nu}{(\log z)^2} \\
\ll \log \log \nu.
\]

This establishes (16) and, thus, the lemma.

\[ \square \]

4. A sum over the zeros of \( \zeta(s) \)

In this section we prove an estimate for the mean-square of a Dirichlet polynomial averaged over the zeros of \( \zeta(s) \). Our estimate follows from the Landau-Gonek explicit formula.

**Lemma 4.1.** Let \( x, T > 1 \) and let \( \rho = \beta + i\gamma \) denote a non-trivial zero of \( \zeta(s) \). Then

\[
\sum_{0 < \gamma \leq T} x^\rho = -\frac{T}{2\pi} \Lambda(x) + O \left( x \log(2xT) \log \log(3x) \right) \\
+ O \left( \log x \min \left(T, \frac{x}{\langle x \rangle} \right) \right) + O \left( \log(2T) \min \left(T, \frac{1}{\log x} \right) \right),
\]

where \( \langle x \rangle \) denotes the distance from \( x \) to the nearest prime power other than \( x \) itself, \( \Lambda(x) = \log p \) if \( x \) is a positive integral power of a prime \( p \), and \( \Lambda(x) = 0 \) otherwise.

**Proof.** This is due to Gonek [6, 8].

**Lemma 4.2.** Assume the Riemann Hypothesis and let \( \rho = \frac{1}{2} + i\gamma \) denote a non-trivial zero of \( \zeta(s) \). For any sequence of complex numbers \( \mathcal{A} = \{a_n\}_{n=1}^{\infty} \) define, for \( \xi \geq 1 \),

\[
m_\xi = m_\xi(\mathcal{A}) = \max_{1 \leq n \leq \xi} (1, |a_n|).
\]
Then for \( 3 \leq \xi \leq T (\log T)^{-1} \) and any complex number \( \alpha \) with \( \Re \alpha \geq 0 \) we have

\[
\sum_{0 < \gamma \leq T} \left| \sum_{n \leq \xi} \frac{a_n}{n^{\rho + \alpha}} \right|^2 \ll m_\xi T \log T \sum_{n \leq \xi} \frac{|a_n|}{n},
\]

where the implied constant is absolute (and independent of \( \alpha \)).

**Proof.** Assuming the Riemann Hypothesis, we note that \( 1 - \rho = \bar{\rho} \) for any non-trivial zero \( \rho = \frac{1}{2} + i\gamma \) of \( \zeta(s) \). This implies that

\[
\left| \sum_{n \leq \xi} \frac{a_n}{n^{\rho + \alpha}} \right|^2 = \sum_{m \leq \xi} \sum_{n \leq \xi} \frac{a_m}{m^{\rho + \alpha}} \frac{a_n}{n^{1 - \rho + \alpha}},
\]

and, moreover, that

\[
\sum_{0 < \gamma \leq T} \left| \sum_{n \leq \xi} \frac{a_n}{n^{\rho + \alpha}} \right|^2 = N(T) \sum_{n \leq \xi} \frac{|a_n|^2}{n^{1 + 2\Re \alpha}} + 2 \Re \sum_{m \leq \xi} \frac{a_m}{m^\alpha} \sum_{m < n \leq \xi} \frac{a_n}{n^{1 + \alpha}} \sum_{0 < \gamma \leq T} \left( \frac{n}{m} \right)^\rho,
\]

where \( N(T) \sim \frac{T}{2\pi} \log T \) denotes the number of zeros \( \rho \) with \( 0 < \gamma \leq T \). Since \( \Re \alpha \geq 0 \), it follows that

\[
N(T) \sum_{n \leq \xi} \frac{|a_n|^2}{n^{1 + 2\Re \alpha}} \ll T \log T \sum_{n \leq \xi} \frac{|a_n|^2}{n} \ll m_\xi T \log T \sum_{n \leq \xi} \frac{|a_n|}{n}.
\]

Appealing to Lemma 4.1, we find that

\[
\sum_{m \leq \xi} \frac{a_m}{m^\alpha} \sum_{n < m} \frac{a_n}{n^{1 + \alpha}} \sum_{0 < \gamma \leq T} \left( \frac{n}{m} \right)^\rho = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4,
\]

where

\[
\Sigma_1 = \frac{T}{2\pi} \sum_{m \leq \xi} \frac{a_m}{m^{\alpha}} \sum_{m < n \leq \xi} \frac{a_n}{n^{1 + \alpha}} \Lambda \left( \frac{n}{m} \right),
\]

\[
\Sigma_2 = O \left( \log T \log \log T \sum_{m \leq \xi} \frac{|a_m|}{m^{1 + \Re \alpha}} \sum_{m < n \leq \xi} \frac{|a_n|}{n^{\Re \alpha}} \right),
\]

\[
\Sigma_3 = O \left( \sum_{m \leq \xi} \frac{|a_m|}{m^{1 + \Re \alpha}} \sum_{m < n \leq \xi} \frac{|a_n| \log \frac{m}{n}}{n^{\Re \alpha}} \right),
\]

and

\[
\Sigma_4 \ll \frac{1}{\xi} \sum_{m \leq \xi} \frac{|a_m|}{m^{1 + \Re \alpha}} \sum_{m < n \leq \xi} \frac{|a_n|}{n^{\Re \alpha}}.
\]
and
\[ \Sigma_4 = O \left( \log T \sum_{m \leq \xi} \frac{|a_m|}{m^{\Re \alpha}} \sum_{n \leq \xi} \frac{|a_n|}{n^{1+\Re \alpha} \log \frac{n}{m}} \right). \]

We estimate \( \Sigma_1 \) first. Making the substitution \( n = mk \), we re-write our expression for \( \Sigma_1 \) as
\[-\frac{T}{2\pi} \sum_{m \leq \xi} \frac{a_m}{m^\alpha} \sum_{k \leq \frac{\xi}{m}} \frac{a_{mk} \cdot \Lambda(k)}{(mk)^{1+\alpha}} = -\frac{T}{2\pi} \sum_{m \leq \xi} \frac{a_m}{m^{1+2\Re \alpha}} \sum_{k \leq \frac{\xi}{m}} \frac{a_{mk} \cdot \Lambda(k)}{k^{1+\alpha}}.\]

Again using the assumption that \( \Re \alpha \geq 0 \), we find that
\[ \Sigma_1 \ll m_\xi T \sum_{n \leq \xi} \frac{|a_n|}{n} \sum_{m \leq \xi} \frac{\Lambda(m)}{m} \ll m_\xi T \log T \sum_{n \leq \xi} \frac{|a_n|}{n}. \]
Here we have made use of the standard estimate \( \sum_{m \leq \xi} \frac{\Lambda(m)}{m} \ll \log \xi \). We can replace \( \Re \alpha \) by 0 in each of the sums \( \Sigma_i \) (for \( i = 2, 3, \) or 4), as doing so will only make the corresponding estimates larger. Thus, using the assumption that \( 3 \leq \xi \leq T/\log T \), it follows that
\[ \Sigma_2 \ll m_\xi \log T \log \log T \sum_{n \leq \xi} \frac{|a_n|}{n} \sum_{m \leq \xi} 1 \ll m_\xi T \log T \sum_{n \leq \xi} \frac{|a_n|}{n}. \]

Next, turning to \( \Sigma_3 \), we find that
\[ \Sigma_3 \ll m_\xi \sum_{m \leq \xi} \frac{|a_m|}{m} \sum_{m \leq \xi} \log \frac{m}{\langle m \rangle}. \]
Writing \( n = qm + \ell \) with \( -\frac{m}{2} < \ell \leq \frac{m}{2} \), we have
\[ \Sigma_3 \ll m_\xi \sum_{m \leq \xi} \frac{|a_m|}{m} \sum_{q \leq \frac{\xi}{m} + 1} \sum_{\frac{\xi}{m} < \ell \leq \frac{m}{2}} \frac{\log (q + \frac{\ell}{m})}{\langle q + \frac{\ell}{m} \rangle}, \]
where, as usual, \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \). Now \( \langle q + \frac{\ell}{m} \rangle = \lfloor \frac{q}{m} \rfloor \) if \( q \) is a prime power and \( \ell \neq 0 \), otherwise \( \langle q + \frac{\ell}{m} \rangle \) is \( \geq \frac{1}{2} \). Using the
estimate \( \sum_{n \leq \xi} \Lambda(n) \ll \xi \), we now find that

\[
\Sigma_3 \ll m_\xi \sum_{m \leq \xi} \frac{|a_m|}{m} \sum_{q \leq \left\lfloor \frac{\xi}{m} \right\rfloor + 1} \Lambda(q) \sum_{1 \leq \ell \leq m} \frac{m}{\ell} + m_\xi \sum_{m \leq \xi} \frac{|a_m|}{m} \sum_{q \leq \left\lfloor \frac{\xi}{m} \right\rfloor + 1} \log(q+1) \sum_{1 \leq \ell \leq m} \frac{1}{\ell} \\
\ll m_\xi \sum_{m \leq \xi} |a_m| \log m \sum_{q \leq \left\lfloor \frac{\xi}{m} \right\rfloor + 1} \Lambda(q) + m_\xi \sum_{m \leq \xi} |a_m| \sum_{q \leq \left\lfloor \frac{\xi}{m} \right\rfloor + 1} \log(q+1) \\
\ll m_\xi (\xi \log \xi) \sum_{m \leq \xi} \frac{|a_m|}{m} \\
\ll m_\xi T \log T \sum_{m \leq \xi} \frac{|a_m|}{m}.
\]

It remains to consider the contribution from \( \Sigma_4 \) which is

\[
\ll m_\xi \log T \sum_{m \leq \xi} |a_m| \sum_{m < n \leq \xi} \frac{1}{m \log \frac{n}{m}} \ll m_\xi \log T \sum_{m \leq \xi} \frac{|a_m|}{m} \sum_{m < n \leq \xi} \frac{1}{\log \frac{n}{m}},
\]

since \( \frac{1}{m} > \frac{1}{n} \) if \( n > m \). Writing \( n = m + \ell \), we see that

\[
\sum_{m < n \leq \xi} \frac{1}{m \log \frac{n}{m}} = \sum_{1 \leq \ell \leq \xi - m} \frac{1}{\ell \log \left( \frac{1 + \ell}{m} \right)} \ll \sum_{1 \leq \ell \leq \xi - m} \frac{m}{\ell} \ll m \log \xi \ll \xi \log \xi.
\]

Consequently,

\[
\Sigma_4 \ll m_\xi T \log T \sum_{m \leq \xi} \frac{|a_m|}{m}.
\]

Now, by combining estimates, we obtain the lemma. \( \square \)

5. The Frequency of Large Values of \( |\zeta(\rho + \alpha)| \)

Our proof of Theorem 1.2 requires the following lemma concerning the distribution of values of \( |\zeta(\rho + \alpha)| \) where \( \rho \) is a zero of \( \zeta(s) \) and \( \alpha \in \mathbb{C} \) is a small shift. In what follows, \( \log_3 (\cdot) \) stands for \( \log \log \log (\cdot) \).
Lemma 5.1. Assume the Riemann Hypothesis. Let $T$ be large, $V \geq 3$ a real number, and $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $0 \leq \Re \alpha - \frac{1}{2} \leq (\log T)^{-1}$. Consider the set 

$$S_{\alpha}(T; V) = \{ \gamma \in (0, T] : \log |\zeta(\rho + \alpha)| \geq V \}$$

where $\rho = \frac{1}{2} + i \gamma$ denotes a non-trivial zero of $\zeta(s)$. Then, the following inequalities for $\#S_{\alpha}(T; V)$, the cardinality of $S_{\alpha}(T; V)$, hold.

(i) When $\sqrt{\log \log T} \leq V \leq \log \log T$, we have 

$$\#S_{\alpha}(T; V) \ll N(T) \frac{V}{\sqrt{\log \log T}} \exp \left( -\frac{V^2}{\log \log T} \left( 1 - \frac{4V}{(\log \log T) \log_3 T} \right) \right).$$

(ii) When $\log \log T \leq V \leq \frac{1}{2}(\log \log T) \log_3 T$, we have 

$$\#S_{\alpha}(T; V) \ll N(T) \frac{V}{\sqrt{\log \log T}} \exp \left( -\frac{V^2}{\log \log T} \left( 1 - \frac{4V}{(\log \log T) \log_3 T} \right) \right).$$

(iii) Finally, when $V > \frac{1}{2}(\log \log T) \log_3 T$, we have 

$$\#S_{\alpha}(T; V) \ll N(T) \exp \left( -\frac{V}{201} \log V \right).$$

Here, as usual, the function $N(T) \sim \frac{T^{1/2}}{2\pi} \log T$ denotes the number of zeros $\rho$ of $\zeta(s)$ with $0 < \gamma \leq T$.

Proof. Since $\lambda_0 < \frac{3}{5}$, by taking $x = (\log \tau)2^{-\varepsilon}$ in Lemma 3.1 (where $\varepsilon > 0$ arbitrary) and estimating the sum over primes trivially, we find that 

$$\log^+ |\zeta(\sigma + i\tau)| \leq \left( 1 + \frac{\lambda_0}{4} + o(1) \right) \frac{\log \tau}{\log \log \tau} \leq \frac{2}{5} \frac{\log \tau}{\log \log \tau}$$

for $|\tau|$ sufficiently large. Therefore, we may suppose that $V \leq \frac{2}{5} \frac{\log T}{\log \log \tau}$, for otherwise the set $S_{\alpha}(T; V)$ is empty.

We define a parameter 

$$A = A(T, V) = \begin{cases} \frac{1}{2} \log_3(T), & \text{if } V \leq \log \log T, \\ \frac{\log T}{2V} \log_3(T), & \text{if } \log \log T < V \leq \frac{1}{2}(\log \log T) \log_3 T, \\ 1, & \text{if } V > \frac{1}{2}(\log \log T) \log_3 T, \end{cases}$$

set $x = \min(T^{1/2}, T^{A/V})$, and put $z = x^{1/\log \log T}$. Further, we let 

$$S_1(s) = \sum_{p \leq x} \frac{1}{p^{s+\frac{\lambda_0}{\log x}}} \frac{\log(x/p)}{\log x} \quad \text{and} \quad S_2(s) = \sum_{z < p \leq x} \frac{1}{p^{s+\frac{\lambda_0}{\log x}}} \frac{\log(x/p)}{\log x}.$$
Then Lemma 3.1 implies that

\begin{equation}
\log^+ |\zeta(\rho + \alpha)| \leq |S_1(\rho)| + |S_2(\rho)| + \frac{(1+\lambda_0)}{2A} V + O(\log_3 T)
\end{equation}

for any non-trivial zero \( \rho = \frac{1}{2} + i\gamma \) of \( \zeta(s) \) with \( 0 < \gamma \leq T \). Here we have used that \( \lambda_0 \geq \frac{1}{2}, x \leq T^{1/2}, \) and \( 0 \leq \Re \alpha - \frac{1}{2} \leq (\log T)^{-1} \) which together imply that

\[
\frac{1}{2} \leq \Re(\rho + \alpha) \leq \frac{1}{2} + \frac{1}{\log T} \leq \frac{1}{2} + \frac{\lambda_0}{\log x}.
\]

Since \( \lambda_0 < 3/5 \), it follows from the inequality in (22) that

\[
\log^+ |\zeta(\rho + \alpha)| \leq |S_1(\rho)| + |S_2(\rho)| + \frac{4V}{5A} + O(\log_3 T).
\]

Therefore, if \( \rho \in S_\alpha(T; V) \), then either

\[
|S_1(\rho)| \geq V \left( 1 - \frac{9}{10A} \right) \quad \text{or} \quad |S_2(\rho)| \geq \frac{V}{10A}.
\]

For simplicity, we put \( V_1 = V \left( 1 - \frac{9}{10A} \right) \) and \( V_2 = \frac{V}{10A} \).

Let \( N_1(T; V) \) be the number of \( \rho \) with \( 0 < \gamma \leq T \) such that \( |S_1(\rho)| \geq V_1 \) and let \( N_2(T; V) \) be the number of \( \rho \) with \( 0 < \gamma \leq T \) such that \( |S_2(\rho)| \geq V_2 \). We prove the lemma by obtaining upper bounds for the size of the sets \( N_i(T; V) \) for \( i = 1 \) and \( 2 \) using the inequality

\begin{equation}
N_i(T; V) \cdot V_i^{2k} \leq \sum_{0 < \gamma \leq T} |S_1(\rho)|^{2k},
\end{equation}

which holds for any positive integer \( k \). With some restrictions on the size of \( k \), we can use Lemma 4.2 to estimate the sums appearing on the right-hand side of this inequality.

We first turn our attention to estimating \( N_1(T; V) \). If we define the sequence \( \alpha_k(n) = \alpha_k(n, x, z) \) by

\[
\sum_{n \leq x^k} \frac{\alpha_k(n)}{n^s} = \left( \sum_{p \leq z} \frac{1}{p^s} \log \frac{x}{p} \right)^k,
\]

then...
then it is easily seen that $|\alpha_k(n)| \leq k!$. Thus, Lemma 4.2 implies that the estimate

$$
\sum_{0 < \gamma \leq T} |S_1(\rho)|^{2k} \ll N(T) \frac{k!}{\log x} \left( \sum_{p \leq z} \frac{1}{p} \log \frac{x}{p} \right)^k
$$

holds for any positive integer $k$ with $z^k \leq T(\log T)^{-1}$ and $T$ sufficiently large. Using (23), we deduce from this estimate that

(24) \[ N_1(T; V) \ll N(T) \sqrt{k} \left( \frac{k \log \log T}{e} \right)^k \]

It is now convenient to consider separately the case when $V \leq (\log \log T)^2$ and the case $V > (\log \log T)^2$. When $V \leq (\log \log T)^2$ we choose $k = \lfloor V_1^2/\log \log T \rfloor$ where, as before, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$. To see that this choice of $k$ satisfies $z^k \leq T(\log T)^{-1}$, we notice from the definition of $A$ that

$$
V A \leq \max \left( V, \frac{1}{2} (\log \log T) \log T \right).
$$

Therefore, we find that

$$
z^k \leq z^{V_1^2/\log \log T} = \exp \left( \frac{V A \log \log T}{(\log \log T)^2} \left( 1 - \frac{9}{10A} \right)^2 \right)
$$

$$
\leq \exp \left( \log T \left( 1 - \frac{9}{10A} \right)^2 \right)
$$

$$
\leq T/\log T.
$$

Thus, by (24), we see that for $V \leq (\log \log T)^2$ and $T$ large we have

(25) \[ N_1(T; V) \ll N(T) \frac{V}{\sqrt{\log \log T}} \exp \left( - \frac{V_1^2}{\log \log T} \right). \]

When $V > (\log \log T)^2$ we choose $k = \lfloor 10V \rfloor$. This choice of $k$ satisfies $z^k \leq T(\log T)^{-1}$ since $z^{10V} = T^{10/\log \log T} \leq T(\log T)^{-1}$ for large $T$. With this choice of $k$, we conclude
from (24) that
\[ N_1(T; V) \ll N(T) \exp \left( \frac{1}{2} \log V - 10V \log \left( \frac{eV}{1000 \log \log T} \right) \right) \]
\[ \ll N(T) \exp \left( -10V \log V + 11V \log_3(T) \right) \]
for \( T \) sufficiently large. Since \( V > (\log \log T)^2 \), we have that \( \log V \geq 2 \log_3(T) \) and thus it follows from (26) that
\[ N_1(T; V) \ll N(T) \exp \left( -4V \log V \right). \]

By combining (25) and (27), we have shown that, for any choice of \( V \),
\[ N_1(T; V) \ll N(T) \frac{V}{\sqrt{\log \log T}} \exp \left( -\frac{V^2}{\log \log T} \right) + N(T) \exp \left( -4V \log V \right). \]

We now turn our attention to estimating \( N_2(T; V) \). If we define the sequence \( \beta_k(n) = \beta_k(n, x, z) \) by
\[ \sum_{n \leq x^k} \frac{\beta_k(n)}{n^s} = \left( \sum_{z < p \leq x} \frac{1}{p^k \log x} \right)^k, \]
then it can be seen that \( |\beta_k(n)| \leq k! \). Thus, Lemma 4.2 implies that
\[ \sum_{0 < \gamma \leq T} |S_2(\rho)|^{2k} \ll N(T) k! \left( \sum_{z < p \leq x} \frac{1}{p^k \log x} \right)^k \]
\[ \ll N(T) k! \left( \sum_{z < p \leq x} \frac{1}{p} \right)^k \]
\[ \ll N(T) \left( \log_3(T) + O(1) \right)^k \]
\[ \ll N(T) \left( 2 \log_3(T) \right)^k \]
\[ \ll N(T) \left( 2k \log_3(T) \right)^k \]
for any natural number \( k \) with \( x^k \leq T/\log T \) and \( T \) sufficiently large. The choice of \( k = \lfloor \frac{V}{A} - 1 \rfloor \) satisfies \( x^k \leq T/\log T \) when \( T \) is large. To see why, recall that \( A \geq 1, x = T^{A/V} \), and \( V \leq 2^\frac{\log T}{\log \log T} \). Therefore,
\[ x^k \leq x^{(V/A - 1) \leq T^{1-4/V} \leq T^{1-4/V} = T(\log T)^{-5/2} \leq T(\log T)^{-1}. \]
Also, observing that $A \leq \frac{1}{2} \log_3(T)$ and recalling that $V \geq \sqrt{\log \log T}$, with this choice of $k$ and $T$ large, it follows from (23) that

$$N_2(T; V) \ll N(T) \left( \frac{10A}{V} \right)^{2k} (2k \log_3(T))^k$$

$$\ll N(T) \exp \left( -2k \log \left( \frac{V}{10A} \right) + k \log (2k \log_3(T)) \right)$$

$$\ll N(T) \exp \left( -2 \frac{V}{A} \log \left( \frac{V}{10A} \right) + 2 \log \frac{V}{10A} + \frac{V}{A} \log \left( \frac{2V}{A} \log_3(T) \right) \right)$$

$$\ll N(T) \exp \left( -\frac{V}{2A} \log V \right).$$

(30)

Using our estimates for $N_1(T; V)$ and $N_2(T; V)$ we can now complete the proof of the lemma by checking the various ranges of $V$. By combining (28) and (30), we see that

$$\# S_\alpha(T; V) \ll N(T) \frac{V}{\sqrt{\log \log T}} \exp \left( -\frac{V^2}{\log \log T} \left( 1 - \frac{9}{5 \log_3(T)} \right)^2 \right)$$

$$\ll N(T) \frac{V}{\sqrt{\log \log T}} \exp \left( -\frac{V^2}{\log \log T} \left( 1 - \frac{4}{\log_3(T)} \right) \right).$$

(31)

If $\sqrt{\log \log T} \leq V \leq \log \log T$, then $A = \frac{1}{2} \log_3(T)$ and (31) implies that, for $T$ sufficiently large,

$$\# S_\alpha(T; V) \ll N(T) \frac{V}{\sqrt{\log \log T}} \exp \left( -\frac{V^2}{\log \log T} \left( 1 - \frac{9}{5 \log \log T} \right)^2 \right)$$

$$\ll N(T) \frac{V}{\sqrt{\log \log T}} \exp \left( -\frac{V^2}{\log \log T} \left( 1 - \frac{4}{\log \log T} \right) \right).$$

(32)

If $\log \log T < V \leq \frac{1}{2} (\log \log T) \log_3(T)$, then $A = \frac{\log \log T}{2V} \log_3(T)$ and we deduce from (31) that

$$\# S_\alpha(T; V) \ll N(T) \frac{V}{\sqrt{\log \log T}} \exp \left( -\frac{V^2}{\log \log T} \left( 1 - \frac{9}{5 \log \log T \log_3(T)} \right)^2 \right)$$

$$\ll N(T) \frac{V}{\sqrt{\log \log T}} \exp \left( -\frac{V^2}{(\log \log T) \log_3(T)} \right) + N(T) \exp \left( -4V \log V \right).$$

(33)

For $V$ in this range, $\frac{\log \log T \log_3(T)}{(\log \log T) \log_3(T)} > \frac{1}{\log \log T}$ and $\frac{V}{\log \log T} < \log \log T$, so (33) implies that

$$\# S_\alpha(T; V) \ll N(T) \frac{V}{\sqrt{\log \log T}} \exp \left( -\frac{V^2}{\log \log T} \left( 1 - \frac{36}{5 \log \log T \log_3(T)} \right)^2 \right)$$

$$\ll N(T) \frac{V}{\sqrt{\log \log T}} \exp \left( -\frac{V^2}{\log \log T} \left( 1 - \frac{36}{(\log \log T) \log_3(T)} \right) \right).$$

(34)
Finally, if $V \geq \frac{1}{2}(\log \log T) \log_3 T$, then $A = 1$ and we deduce from (31) that

$$\#S_\alpha(T; V) \ll N(T) \exp \left( \log V - \frac{V^2}{100 \log \log T} \right) + N(T) \exp \left( -\frac{V}{2} \log V \right).$$

Certainly, if $V \geq \frac{1}{2}(\log \log T) \log_3 T$ then we have that $\frac{V^2}{100 \log \log T} - \log V > \frac{1}{201} V \log V$ for $T$ sufficiently large and so it follows from (35) that

$$\#S_\alpha(T; V) \ll N(T) \exp \left( -\frac{V}{201} \log V \right).$$

The lemma now follows from the estimates in (32), (34), and (36). \qed

6. The Proof of Theorem 1.2

Using Lemma 5.1, we first prove Theorem 1.2 in the case where $|\alpha| \leq 1$ and $0 \leq \Re \alpha \leq (\log T)^{-1}$. Then, from this result, the case when $-((\log T)^{-1} \leq \Re \alpha < 0$ can be deduced from the functional equation for $\zeta(s)$ and Stirling’s formula for the gamma function. In what follows, $k \in \mathbb{R}$ is fixed and we let $\varepsilon > 0$ be an arbitrarily small positive constant which may not be the same at each occurrence.

First, we partition the real axis into the intervals $I_1 = (-\infty, 3], I_2 = (3, 4k \log \log T]$, and $I_3 = (4k \log \log T, \infty)$ and set

$$\Sigma_i = \sum_{\nu \in I_i \cap \mathbb{Z}} e^{2k \nu} \cdot \#S_\alpha(T, \nu)$$

for $i = 1, 2, 3$. Then we observe that

$$\sum_{0 < \gamma \leq T} |\zeta(\rho + \alpha)|^{2k} \leq \sum_{\nu \in \mathbb{Z}} e^{2k \nu} \left[ \#S_\alpha(T, \nu) - \#S_\alpha(T, \nu - 1) \right] \leq \Sigma_1 + \Sigma_2 + \Sigma_3.$$

Using the trivial bound $\#S_\alpha(T, \nu) \leq N(T)$, which holds for every $\nu \in \mathbb{Z}$, we find that $\Sigma_1 \leq e^{6k} N(T)$. To estimate $\Sigma_2$, we use the bound

$$\#S_\alpha(T, \nu) \ll N(T)(\log T)^\varepsilon \exp \left( -\frac{\nu^2}{\log \log T} \right).$$
which follows from the first two cases of Lemma 5.1 when \( \nu \in I_2 \cap \mathbb{Z} \). From this, it follows that

\[
\Sigma_2 \ll N(T)(\log T)^\varepsilon \int_3^{4k \log \log T} \exp \left( 2ku - \frac{u^2}{\log \log T} \right) du
\]

\[
\ll N(T)(\log T)^\varepsilon \int_0^{4k} (\log T)^u (2k-u) du
\]

\[
\ll N(T)(\log T)^{k^2 + \varepsilon}
\]

When \( \nu \in I_3 \cap \mathbb{Z} \), the second two cases of Lemma 5.1 imply that

\[
\# S_{\alpha}(T, \nu) \ll N(T)(\log T)^\varepsilon e^{-4k\nu}.
\]

Thus,

\[
\Sigma_3 \ll N(T)(\log T)^\varepsilon \int_{4k \log \log T}^{\infty} e^{-2ku} du \ll N(T)(\log T)^{-8k^2 + \varepsilon}.
\]

In light of (37), by collecting estimates, we see that

\[
(38) \quad \sum_{0 < \gamma \leq T} |\zeta(\rho + \alpha)|^{2k} \ll N(T)(\log T)^{k^2 + \varepsilon}
\]

for every \( k > 0 \) when \( |\alpha| \leq 1 \) and \( 0 \leq \Re \alpha \leq (\log T)^{-1} \).

The functional equation for the zeta-function states that \( \zeta(s) = \chi(s)\zeta(1-s) \) where \( \chi(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin \left( \frac{\pi s}{2} \right) \). Stirling’s asymptotic formula for the gamma function (see Appendix A.7 of Ivic [13]) can be used to show that

\[
|\chi(\sigma + it)| = \left( \frac{|t|}{2\pi} \right)^{1/2 - \sigma} \left( 1 + O\left( \frac{1}{|t|} \right) \right)
\]

uniformly for \(-1 \leq \sigma \leq 2 \) and \(|t| \geq 1 \). Using the Riemann Hypothesis, we see that

\[
|\zeta(\rho + \alpha)| = |\chi(\rho + \alpha)\zeta(1-\rho - \alpha)|
\]

\[
= |\chi(\rho + \alpha)\zeta(\bar{\rho} - \alpha)|
\]

\[
= |\chi(\rho + \alpha)\zeta(\rho - \bar{\alpha})|
\]

\[
\leq C|\zeta(\rho - \bar{\alpha})|
\]
for some absolute constant $C > 0$ when $|\alpha| \leq 1$, $|R\alpha - \frac{1}{2}| \leq (\log T)^{-1}$, and $0 < \gamma \leq T$. Consequently, for $-(\log T)^{-1} \leq \Re \alpha < 0$,

$$
\sum_{0 < \gamma \leq T} |\zeta(\rho + \alpha)|^{2k} \leq C^{2k} \cdot \sum_{0 < \gamma \leq T} |\zeta(\rho - \bar{\alpha})|^{2k}.
$$

Applying the inequality in (38) to the right-hand side of (39) we see that

$$
\sum_{0 < \gamma \leq T} |\zeta(\rho + \alpha)|^{2k} \ll_k N(T)(\log T)^{k^2 + \varepsilon}
$$

for every $k > 0$ when $|\alpha| \leq 1$ and $-(\log T)^{-1} \leq \Re \alpha < 0$. The theorem now follows from the estimates in (38) and (40).

7. **Theorem 1.2 implies Theorem 1.1**

Theorem 1.1 can now be established as a simple consequence of Theorem 1.2 and the following lemma.

**Lemma 7.1.** Assume the Riemann Hypothesis. Let $k, \ell \in \mathbb{N}$ and let $R > 0$ be arbitrary. Then we have

$$
\sum_{0 < \gamma \leq T} |\zeta(\ell)(\rho)|^{2k} \leq \left(\frac{\ell!}{R^\ell}\right)^{2k} \cdot \left[\max_{|\alpha| \leq R} \sum_{0 < \gamma \leq T} |\zeta(\rho + \alpha)|^{2k}\right].
$$

**Proof.** Since the function $\zeta(\ell)(s)$ is real when $s \in \mathbb{R}$, $\zeta(\ell)(\bar{s}) = \overline{\zeta(\ell)(s)}$. Hence, assuming the Riemann Hypothesis, the identity

$$
|\zeta(\ell)(1 - \rho + \alpha)| = |\zeta(\ell)(\bar{\rho} + \alpha)| = |\zeta(\ell)(\rho + \bar{\alpha})|
$$

holds for any non-trivial zero $\rho$ of $\zeta(s)$ and any $\alpha \in \mathbb{C}$. For each positive integer $k$, let $\vec{\alpha}_k = (\alpha_1, \alpha_2, \ldots, \alpha_{2k})$ and define

$$
\mathcal{Z}(s; \vec{\alpha}_k) = \prod_{i=1}^{k} \zeta(s + \alpha_i)\zeta(1-s + \alpha_{i+k}).
$$

If we suppose that each $|\alpha_i| \leq R$ for $i = 1, \ldots, 2k$ and apply Hölder’s inequality in the form

$$
\left| \sum_{n=1}^{N} \left( \prod_{i=1}^{2k} f_i(s_n) \right) \right| \leq \prod_{i=1}^{2k} \left( \sum_{n=1}^{N} |f_i(s_n)|^{2k} \right)^{\frac{1}{2k}},
$$

then

$$
\sum_{0 < \gamma \leq T} |\zeta(\ell)(\rho)|^{2k} \leq \left(\frac{\ell!}{R^\ell}\right)^{2k} \cdot \left[\max_{|\alpha| \leq R} \sum_{0 < \gamma \leq T} |\zeta(\rho + \alpha)|^{2k}\right].
$$
we see that (42) implies that
\[
\left| \sum_{0<\gamma\leq T} \mathcal{Z}(\rho; \vec{\alpha}_k) \right| \leq \prod_{i=1}^{k} \left( \sum_{0<\gamma\leq T} |\zeta(\rho+\alpha_i)|^{2k} \right)^{\frac{1}{2\pi}} \left( \sum_{0<\gamma\leq T} |\zeta(\rho+\alpha_{k+1})|^{2k} \right)^{\frac{1}{2\pi}}
\]
\[
\leq \max_{|\alpha|\leq R} \sum_{0<\gamma\leq T} |\zeta(\rho+\alpha)|^{2k}
\]
(43)

In order to prove the lemma, we first rewrite the left-hand side of equation (41) using the function \( Z(s; \vec{\alpha}_k) \) and then apply the inequality in (43). By Cauchy’s integral formula and another application of (42), we see that
\[
\sum_{0<\gamma\leq T} |\zeta^{(\ell)}(\rho)|^{2k} = \sum_{0<\gamma\leq T} \left( \prod_{i=1}^{k} \zeta^{(\ell)}(\rho) \zeta^{(\ell)}(1-\rho) \right)
\]
\[
= \frac{(\ell!)^{2k}}{(2\pi i)^{2k}} \int_{\mathcal{C}_1} \cdots \int_{\mathcal{C}_{2k}} \left( \sum_{0<\gamma\leq T} \mathcal{Z}(\rho; \vec{\alpha}_k) \right) \prod_{i=1}^{2k} \frac{d\alpha_i}{\alpha_i^{\ell+1}}
\]
(44)

where, for each \( i = 1, \ldots, 2k \), the contour \( \mathcal{C}_i \) denotes the positively oriented circle in the complex plane centered at 0 with radius \( R \). Now, combining (43) and (44) we find that
\[
\sum_{0<\gamma\leq T} |\zeta^{(\ell)}(\rho)|^{2k} \leq \left( \frac{\ell!}{2\pi} \right)^{2k} \cdot \max_{|\alpha|\leq R} \sum_{0<\gamma\leq T} |\zeta(\rho+\alpha)|^{2k} \cdot \int_{\mathcal{C}_1} \cdots \int_{\mathcal{C}_{2k}} \prod_{i=1}^{2k} \frac{d\alpha_i}{|\alpha_i|^{\ell+1}}
\]
\[
\leq \left( \frac{\ell!}{2\pi} \right)^{2k} \cdot \max_{|\alpha|\leq R} \sum_{0<\gamma\leq T} |\zeta(\rho+\alpha)|^{2k} \cdot \left( \frac{2\pi}{R^\ell} \right)^{2k}
\]
\[
\leq \left( \frac{\ell!}{R^\ell} \right)^{2k} \cdot \max_{|\alpha|\leq R} \sum_{0<\gamma\leq T} |\zeta(\rho+\alpha)|^{2k},
\]
as claimed. \( \square \)

**Proof of Theorem 1.1.** Let \( k \in \mathbb{N} \) and set \( R = (\log T)^{-1} \). Then, it follows from Theorem 1.2 and Lemma 7.1 that
\[
\frac{1}{N(T)} \sum_{0<\gamma\leq T} |\zeta^{(\ell)}(\rho)|^{2k} \ll_{k, \ell, \epsilon} (\log T)^{k(2k+2\ell)+\epsilon}
\]
(45)

for any \( \ell \in \mathbb{N} \) and for \( \epsilon > 0 \) arbitrary. Theorem 1.1 now follows by setting \( \ell = 1 \).
UPPER BOUNDS FOR THE MOMENTS OF $\zeta'(\rho)$

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