CHARACTERIZATIONS OF REGULAR LOCAL RINGS IN POSITIVE CHARACTERISTICS

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ABSTRACT. In this note, we provide several characterizations of regular local rings in positive characteristics, in terms of the Hilbert-Kunz multiplicity and its higher Tor counterparts $t_i = \lim_{n \to \infty} \ell(\mathcal{Tor}_i(R/I, f^n R))/p^{nd}$. We also apply the characterizations to improve a recent result by Bridgeland and Iyengar in the characteristic $p$ case. Our proof avoids using the existence of big Cohen-Macaulay modules, which is the major tool in the proof of Bridgeland and Iyengar.

1. INTRODUCTION

Let $(R, \mathfrak{m}, k)$ be a $d$-dimensional local ring of characteristic $p > 0$. The Frobenius endomorphism $f_R : R \to R$ is defined by $f_R(r) = r^p$ for $r \in R$. Each iteration $f^n_R$ defines a new $R$-module structure on $R$, denoted $f^n_R$, for which $a \cdot b = a^{p^n} b$. For any $R$-module $M$, $F^n_M(M)$ stands for $M \otimes_R f^n R$, the $R$-module structure of which is given by the base change along the Frobenius endomorphism. When $M$ is a cyclic module $R/I$, it is easy to show that $F^n_R(R/I) \cong R/I[p^n]$, where $I[p^n]$ denotes the ideal generated by the $p^n$-th power of the generators of $I$.

In what follows, $\ell(-)$ denotes the length function.

For any $\mathfrak{m}$-primary ideal $I$, the Hilbert-Kunz multiplicity of $R$ with respect to $I$ was first introduced by Monsky in [Mo]:

$$e_{HK}(I, R) = \lim_{n \to \infty} \ell(F^n(R/I))/p^{nd}.$$  

The Hilbert-Kunz multiplicity of $R$ is $e_{HK}(R) = e_{HK}(\mathfrak{m}, R)$. We also frequently write $e_{HK}(I) = e_{HK}(I, R)$. It has been shown by many authors that the Hilbert-Kunz multiplicity encodes subtle information about the singularity of $R$. One such example is the following characterization of the regularity due to Watanabe and Yoshida; see [W-Y], [H-Y].

**Theorem 1.1.** If $R$ is unmixed, then it is regular if and only if $e_{HK}(R) = 1$.

It is natural to ask whether the higher Tor counterparts of the Hilbert-Kunz multiplicity, which are defined below, can encode similar information on the singularity of the ring.

**Definition.** Let $R$ be a $d$-dimensional local ring of characteristic $p > 0$. Let $I$ be any $\mathfrak{m}$-primary ideal. Define

$$t_i(I, R) = \lim_{n \to \infty} \ell(\mathcal{Tor}_i(R/I, f^n R))/p^{nd}.$$
Seibert has shown that such limits always exist [Se]. In the sequel, we also write $t_i(R) = t_i(m, R)$. The main result of this note is the following theorem in Section 2:

**Main Theorem** (Theorem 2.2). Let $(R, m, k)$ be a $d$-dimensional local ring of characteristic $p > 0$. Then the following are equivalent:

(i) $R$ is regular,
(ii) $t_1(R) = 0$,
(iii) $t_2(R) = 0$,
(iv) $e_{HK}(R) - 1 = t_1(R)$.

The proof of this theorem is inspired by the work of Huneke and Yao [H-Y] 2.1 and Blickle and Enescu [B-E].

In Section 3, we apply our main theorem to slightly generalize the positive characteristic case of a recent result by Bridgeland and Iyengar [B-I] 1.1. The result of Bridgeland and Iyengar states that if $R$ contains a field and if $k$ is a direct summand of $H^0(C_\bullet)$, where $C_\bullet$ is a perfect complex of length exactly $d$ such that $\ell(H_i(C_\bullet)) < \infty$ for all positive $i$, then $R$ must be regular. Using Theorem 2.2, we are able to show that in the positive characteristic case, not only $k$, but also the first syzygy of $k$ cannot be a direct summand of $H^0(C_\bullet)$ for such $C_\bullet$ unless $R$ is regular.

Our proof of this result is quite different from that of Bridgeland and Iyengar. In particular, we avoid using the existence of big Cohen-Macaulay modules.

## 2. The Main Result

The following fact plays a crucial role in this paper. It is contained in [D] 1.1. We provide a sketch of the proof here for the completeness of this paper.

**Lemma 2.1.** Let $(R, m, k)$ be a $d$-dimensional local ring of characteristic $p > 0$. Let $I = (x)$ be an ideal generated by a system of parameters $x = x_1, \ldots, x_d$. Then $t_1(I, R) = 0$.

**Proof.** There is a surjection

$$H_1(x^{[p^n]}; R) \twoheadrightarrow \text{Tor}_1(R/I, f^\infty R)$$

and it is well known (see [R] 7.3.5 or [D] 1.1) that

$$\lim_{n \to \infty} \ell(H_i(x^{[p^n]}; R)) / p^{nd} = 0, \text{ for } i > 0.$$

The following is the main theorem of this paper.

**Theorem 2.2.** Let $(R, m, k)$ be a $d$-dimensional local ring of characteristic $p > 0$. Then

(a) $e_{HK}(R) - 1 \leq t_1(R)$;
(b) $t_1(R) - e_{HK}(R) + 1 \leq t_2(R)$.

Moreover, the following are equivalent:

(i) $R$ is regular,
(ii) $t_1(R) = 0$,
(iii) $t_2(R) = 0$,
(iv) $e_{HK}(R) - 1 = t_1(R)$.
Proof. We first prove (a) and (b). Let $q$ be a power of $p$. Let $I$ be an $m$-primary ideal generated by a system of parameters of $R$. Consider the following filtration:

$$0 \to Q_1 \to R/I[q] \to k \to 0,$$

$$0 \to Q_2 \to Q_1 \to k \to 0,$$

$$\vdots$$

$$0 \to k \to Q_t \to k \to 0.$$

Applying $- \otimes f^pR$ to the above short exact sequences, we obtain the long exact sequences

$$\cdots \to \text{Tor}_2(k, f^pR) \to \text{Tor}_1(Q_1, f^pR) \to \text{Tor}_1(R/I[q], f^pR) \to \text{Tor}_1(k, f^pR) \to F^n(Q_1) \to F^n(R/I[q]) \to F^n(k) \to 0,$$

$$\cdots \to \text{Tor}_2(k, f^pR) \to \text{Tor}_1(Q_2, f^pR) \to \text{Tor}_1(Q_1, f^pR) \to \text{Tor}_1(k, f^pR) \to F^n(Q_2) \to F^n(Q_1) \to F^n(k) \to 0,$$

$$\vdots$$

$$\cdots \to \text{Tor}_2(k, f^pR) \to \text{Tor}_1(k, f^pR) \to \text{Tor}_1(Q_t, f^pR) \to \text{Tor}_1(k, f^pR) \to F^n(k) \to F^n(Q_t) \to F^n(k) \to 0.$$

It follows that

1. $(\ell(R/I[q]) - 1) \cdot \ell(\text{Tor}_1(k, f^pR)) + \ell(F^n(R/I[q])) \geq \ell(R/I[q]) \cdot \ell(F^n(k))$

and

2. $(\ell(R/I[q]) - 1) \cdot \ell(\text{Tor}_2(k, f^pR)) + \ell(\text{Tor}_1(R/I[q], f^pR)) + \ell(R/I[q]) \cdot \ell(F^n(k))$

$\geq \ell(R/I[q]) \cdot \ell(\text{Tor}_1(k, f^pR)) + \ell(F^n(R/I[q])).$

Divide both sides of (1) by $p^{nd}$ and let $n \to \infty$ to obtain

3. $(\ell(R/I[q]) - 1) \cdot t_1(R) + q^d \cdot e_{HK}(I) \geq \ell(R/I[q])e_{HK}(R).$

Dividing (3) by $q^d$ and letting $q \to \infty$ then yields

$$e_{HK}(I) \cdot t_1(R) + e_{HK}(I) \geq e_{HK}(I) \cdot e_{HK}(R).$$

Hence

$$e_{HK}(R) \leq 1 + t_1(R).$$

Similarly, from inequality (2) we can get (we need to apply Lemma 2.1 here)

4. $(\ell(R/I[q]) - 1) \cdot t_2(R) + \ell(R/I[q])e_{HK}(R) \geq \ell(R/I[q]) \cdot t_1(R) + q^d \cdot e_{HK}(I).$

Therefore

$$t_1(R) - e_{HK}(R) + 1 \leq t_2(R).$$
For the second part of the theorem, it is clear that (i) ⇒ (ii), (iii) and (iv) due to the exactness of Frobenius. We now prove (ii) ⇒ (i) and (iii) ⇒ (iv) ⇒ (ii).

(ii)⇒(i). Note that inequality (3) is valid for any \( m \)-primary ideal \( I \) (not just for ideals generated by a system of parameters). Since \( t_1(R) = 0 \), inequality (3) becomes the equality

\[
q^d \cdot e_{HK}(I) = \ell(R/I^{[q]})e_{HK}(R).
\]

Taking \( I = m \) immediately gives \( q^d = \ell(R/m^{[q]}) \), which forces \( R \) to be regular by Kunz’s Theorem [K].

(iii)⇒(iv). Since \( t_2(R) = 0 \), inequality (4) becomes the equality

\[
\ell(R/I^{[q]})e_{HK}(R) = \ell(R/I^{[q]}) \cdot t_1(R) + q^d \cdot e_{HK}(I).
\]

We therefore obtain (iv) by dividing both sides by \( q^d \) and taking the limits.

(iv)⇒(ii). We can make a flat extension of \( R \) to assume that \( k \) is infinite without changing any of the relevant lengths. Let \( I \) be a minimal reduction of \( m \) which is generated by a system of parameters of \( R \). In this case, it is well known (see, for instance, [Ma 14.12]) that the Hilbert-Kunz multiplicity \( e_{HK}(I) \) coincides with the Hilbert-Samuel multiplicity \( e(R) \). Taking \( q = 1 \) in (3), we have

\[
(\ell(R/I) - 1) \cdot t_1(R) + e_{HK}(I) \geq \ell(R/I)e_{HK}(R).
\]

Replacing \( e_{HK}(R) \) by \( 1 + t_1(R) \) in (5), we get

\[
e(R) \geq \ell(R/I) + t_1(R) \geq \ell(R/I).
\]

On the other hand, since \( I \) is a minimal reduction of \( m \), \( e(R) \leq \ell(R/I) \). This forces all the inequalities in (6) to be equalities. Therefore \( t_1(R) = 0 \). \( \Box \)

The inequalities (a) and (b) in Theorem 2.2 are far from being the best possible bounds for the Hilbert-Kunz multiplicity. For example, the following corollary gives better bounds when \( R \) is Cohen-Macaulay.

**Corollary.** Let \((R, m)\) be a Cohen-Macaulay local ring of characteristic \( p > 0 \) and let \( e = e(R) \) be the Hilbert-Samuel multiplicity of \( R \). Then

\[
e_{HK}(R) - 1 \leq \left( \frac{e - 1}{e} \right)t_1(R).
\]

This follows easily from inequality (3) in the proof of Theorem 2.2. One can again assume the residue field of \( R \) is infinite so that there is a minimal reduction \( I \) of \( m \) which is generated by a system of parameters. Since \( R \) is Cohen-Macaulay, we have

\[
e = \ell(R/I) = e_{HK}(I).
\]

Take \( q = 1 \) in (3) and then replace \( \ell(R/I) \) and \( e_{HK}(I) \) in (3) by \( e \). We obtain

\[
(e - 1) \cdot t_1(R) + e \geq e \cdot e_{HK}(R),
\]

which gives the desired inequality.

**Remark.** When \( R \) is Cohen-Macaulay, we can argue exactly the same way as in the proof of Theorem 2.2 to trivially generalize Theorem 2.2 to cases of higher Tor. Namely, we have

\[
t_i(R) - t_{i-1}(R) + \cdots + (-1)^{i-1}t_1(R) + (-1)^it_{HK}(R) + (-1)^{i+1} \geq 0 \text{ for all } i \geq 1
\]
and $R$ being regular can be characterized by either the above inequalities taking “$=$” for some $i \geq 1$, or $t_i(R)$ being zero for some $i \geq 1$. However, the author does not know if this generality can be true without the Cohen-Macaulay assumption. The main obstruction here is, when $R$ is not Cohen-Macaulay, we no longer have the higher Tor (for $i \geq 2$) analog of Lemma 2.1.

3. An improvement of Bridgeland-Iyengar’s Result

Recently, Bridgeland and Iyengar \[B-I, 1.1\] proved the following characterization for regular local rings.

**Theorem 3.1** (Bridgeland-Iyengar). Let $(R, \mathfrak{m}, k)$ be a $d$-dimensional local ring containing a field or of dimension $\leq 3$. Assume $C_\bullet$ is a complex of free $R$-modules with $C_i = 0$ for $i \notin [0, d]$, the $R$-module $H_0(C_\bullet)$ is finitely generated, and $\ell(H_i(C_\bullet)) < \infty$ for $i > 0$. If $k$ is a direct summand of $H_0(C_\bullet)$, then $R$ is regular.

Their proof of Theorem 3.1 uses the existence of balanced big Cohen-Macaulay modules. Here we can apply Theorem 2.2 to give a more direct proof in the positive characteristic case that avoids using the existence of big Cohen-Macaulay modules. Moreover, our proof also yields the same conclusion if (instead of $k$) the first syzygy module of $k$ is a direct summand of $H_0(C_\bullet)$.

**Theorem 3.2.** Let $(R, \mathfrak{m}, k)$ be a $d$-dimensional local ring of characteristic $p > 0$, $C_\bullet$ a complex of free $R$-modules with $C_i = 0$ for $i \notin [0, d]$, the $R$-module $H_0(C_\bullet)$ finitely generated, and $\ell(H_i(C_\bullet)) < \infty$ for $i > 0$. If either $k$ or the first syzygy of $k$ is a direct summand of $H_0(C_\bullet)$, then $R$ is regular.

**Proof.** By the same argument as in \[B-I, Lemma 2.2\], we have a surjection

$$H_1(F^n(C_\bullet)) \twoheadrightarrow \operatorname{Tor}_1(H_0(C_\bullet), f^pR)$$

It is well known that $\lim_{n \to \infty} \ell(H_1(F^n(C_\bullet)))/p^n = 0$; see \[D, 1.7\]. So we are done by Theorem 2.2.

\[\square\]

**Remark.** Theorem 3.2 is still valid for rings containing a field or of dimension $\leq 3$ (the exact same hypothesis on $R$ as in Theorem 3.1) although the proof requires the use of big Cohen-Macaulay modules. To see this, one needs to use a result of Schoutens \[Sc, Proposition 2.5\] to modify the original proof of Bridgeland and Iyengar (their proof of \[B-I, Theorem 2.4\]) slightly. We leave the details here to the readers.

It seems that the mixed characteristic case of the above result remains unknown.

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