ON THE J-FLOW IN SASAKIAN MANIFOLDS

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Abstract. We study the space of Sasaki metrics on a compact manifold $M$ by introducing an odd-dimensional analogue of the $J$-flow. That leads to the notion of critical metric in the Sasakian context. In analogy to the Kähler case, on a polarised Sasakian manifold there exists at most one normalised critical metric. The flow is a tool for testing the existence of such a metric. We show that some results proved by Chen in [7] can be generalised to the Sasakian case. In particular, the Sasaki $J$-flow is a gradient flow which has always a long-time solution minimising the distance on the space of Sasaki potentials of a polarized Sasakian manifold. The flow minimises an energy functional whose definition depends on the choice of a background transverse Kähler form $\chi$. When $\chi$ has nonnegative transverse holomorphic bisectional curvature, the flow converges to a critical Sasakian structure.

1. Introduction

Sasakian manifolds are the odd-dimensional counterpart of Kähler manifolds and are defined as odd-dimensional Riemannian manifolds $(M, g)$ whose Riemannian cone $(M \times \mathbb{R}^+, t^2 g + dt^2)$ admits a Kähler structure. These manifolds are important for both geometric and physical reasons. In geometry they can be used to produce new examples of complete Kähler manifolds, manifolds with special holonomy and Einstein metrics. Moreover, Sasakian manifolds play a role in the study of orbifolds since many Kähler orbifolds can be desingularised by using Sasakian spaces. In theoretical physics these manifolds play a central role in the AdS/CFT correspondence (see e.g. [12, 16, 17, 30, 31, 32, 33]). We refer to [3, 39] for general theory and recent advanced in the study of these manifolds.

Given a Sasakian manifold, the choice of a Kähler structure on the Riemannian cone determines a unitary Killing vector field $\xi$ of the metric $g$ and an endomorphism $\Phi$ of the tangent bundle to $M$ such that

$$\Phi^2 = -\text{Id} + \eta \otimes \xi, \quad g(\Phi \cdot, \Phi \cdot) = g(\cdot, \cdot) - \eta \otimes \eta, \quad g = \frac{1}{2} d\eta \circ (\text{Id} \otimes \Phi) + \eta \otimes \eta,$$

$\eta$ being the 1-form dual to $\xi$ via $g$. It turns out that $\eta$ is a contact form and that $\Phi$ induces a CR-structure $(D, J)$ on $M$. Moreover, $\Phi(X) = D_X \xi$ for every vector field $X$ on $M$, where $D$ is the Levi-Civita connection of $g$. The quadruple $(\xi, \Phi, \eta, g)$ is usually called a Sasakian structure and the pair $(\xi, J)$ can be seen as a polarization of $M$.

The research of this paper is mainly motivated by [4, 21, 22, 23] where it is approached the study of Riemannian and symplectic aspects of the space of Sasakian potentials $\mathcal{H}$ on a polarised Sasakian manifold. Our approach consists in using an analogue of the $J$-flow in the context of Sasakian Geometry obtaining some results similar to the ones proved in the Kähler case by Chen in [7]. The $J$-flow is a gradient geometric flow of Kähler structures introduced and firstly studied by Donaldson in [13] from the point of view of moment maps and by Chen in [7] in relation to the Mabuchi energy. It is defined as the gradient flow of a functional $J_\chi$ defined on the space of normalised Kähler potentials whose definition depends on a fixed background Kähler structure $\chi$. Chen proved in [7] that the flow has always a unique long time solution which, in the special case when $\chi$ has nonnegative biholomorphic curvature, converges to a critical Kähler metric. Further results about the flow are obtained in [25, 38, 42, 43].
As far as we know, the interest for geometric flows in foliated manifolds comes from [27] where it is introduced a foliated version of the Ricci flow. Subsequently, Smoczyk, Wang and Zhang proved in [30] that the transverse Ricci flow preserves the Sasakian condition and study its long time behavior generalising the work of Cao in [25] to the Sasakian case. Some deep geometric and analytic aspects of the Sasakian Ricci flow were further investigated in [8, 9, 10, 11].

In analogy to the Kähler case, the Sasaki J-flow introduced in this paper (see Section 4 for the precise definition) is the gradient flow of a functional $J_\chi: \mathcal{H} \to \mathbb{R}$ whose definition depends on the choice of a transverse Kähler structure $\chi$. Sasakian metrics arising from critical points of the restriction of $J_\chi$ to the space of normalized Sasakian potentials $\mathcal{H}_0$, are natural candidates to be canonical Sasakian metrics.

The main result of the paper is the following

**Theorem 1.1.** Let $(M, \xi, \Phi, \eta, g)$ be a $(2n+1)$-dimensional Sasakian manifold and let $\chi$ be a transverse Kähler form on $M$. Then the functional $J_\chi: \mathcal{H}_0 \to \mathbb{R}$ has at most one critical point and the Sasaki J-flow has a long-time solution $f$ for every initial datum $f_0$. Furthermore, the length of any smooth curve in $\mathcal{H}_0$ and the distance between any two points decrease under the flow and when the transverse holomorphic bisectional curvature of $\chi$ is nonnegative, $f$ converges to a critical point of $J_\chi$ in $\mathcal{H}_0$.

The last sentence in the statement of Theorem 1.1 implies that if the transverse Kähler structure $\chi$ has nonnegative transverse holomorphic bisectional curvature, then $J_\chi$ has a critical point in $\mathcal{H}_0$. We remark that Sasakian manifolds having nonnegative transverse holomorphic bisectional curvature are classified in [24], but in the definition of the Sasaki J-flow, $\chi$ is just a transverse Kähler structure not necessarily induced by a Sasaki metric.

From the local point of view, a solution to the Sasaki J-flow can be seen as a collection of solutions to the Kähler J-flow on open sets in $\mathbb{C}^n$. This fact allows us to use all the local estimates about the Kähler J-flow provided in [27]. What is necessary modifying from the Kähler case is the proof of the existence of a short-time solution to the flow (since the flow is parabolic only along transverse directions) and the global estimates. The short-time existence is obtained in Section 4 by using a trick introduced in [30], while the global estimates are obtained by using a transverse version of the maximal principle for transversally elliptic operators (see section 5).

2. Preliminaries

In this section we recall some basic facts about Sasakian Geometry declaring the notation which will be adopted in the rest of the paper.

Let $(M, \xi, \Phi, \eta, g)$ be a $(2n+1)$-dimensional Sasaki manifold. Then the Reeb vector field $\xi$ specifies a Riemannian foliation on $M$, which is usually denoted by $\mathcal{F}_\xi$, and the vector bundle to $M$ splits in $TM = D \oplus L_\xi$, where $L_\xi$ is the line bundle generated by $\xi$ and $D$ has as fiber over a point $x$ the vector space $\ker \eta_x$. The metric $g$ splits accordingly in $g = g^T + \eta^2_x$, where the degenerate tensor $g^T$ is called the transverse metric of the Sasaki structure. In the following we denote by $\nabla^T$ the transverse Levi-Civita connection defined on the bundle $D$ in terms of the Levi-Civita connection $\nabla$ of $g$ as

$$
\nabla^T_X Y = \begin{cases} 
\nabla_X Y & \text{if } X \in \Gamma(D) \\
[\xi, Y]^D & \text{if } X = \xi,
\end{cases}
$$

(1)

where the superscript $D$ denotes the orthogonal projection onto $D$. This connection induces the transverse curvature

$$
R^T(X, Y)Z = \nabla^T_X \nabla^T_Y Z - \nabla^T_Y \nabla^T_X Z - \nabla^T_{[X,Y]^D} Z,
$$

(2)

and the transverse Ricci curvature $\text{Ric}^T$ obtained as the trace of the map $X \mapsto R^T(X, \cdot)$ on $D$ with respect to $g^T$. We further recall that a real $p$-form $\alpha$ on $(M, \xi, \Phi, \eta, g)$ is called basic if

$$
\iota_\xi \alpha = 0, \quad \iota_\xi d \alpha = 0,
$$

where $\iota_\xi$ denotes the contraction along $\xi$. The set of basic $p$-forms is usually denoted by $\Omega^p_B(M)$ and $\Omega^0_B(M) = C^0_{T\pi}(M)$. Since the exterior differential operator takes basic forms into basic forms, its restriction $d_B$ to $\Omega_B(M) = \oplus \Omega^p_B(M)$ defines a cohomological complex. Moreover, $\Phi$ induces a transverse
complex structure $J$ on $(M, \xi)$ and a splitting of the space of complex basic forms in forms of type $(p, q)$ in the usual way. Furthermore, the complex extension of $d_B$ to $\Omega_B(M, C)$ splits as $d_B = \partial_B + \bar{\partial}_B$ and $\bar{\partial}_B = 0$ (see e.g. [2] for details). A basic $(1, 1)$-form $\chi$ on $(M, \xi, \Phi, \eta, g)$ is said to be positive if
\begin{equation}
\chi(Z, \bar{Z}) > 0, \tag{3}
\end{equation}
for every non-zero section $Z$ of $\Gamma(D^{1,0})$. If further $\chi$ is closed, we refer to $\chi$ as to a transverse Kähler form. Note that condition (3) depends only on the transverse complex structure $J$ and on $\xi$, since $\chi$ is basic. Every such a $\chi$ induces the global metric
\[ g_\chi(\cdot, \cdot) = \chi(\cdot, \Phi \cdot) + \eta(\cdot) \eta(\cdot), \]
on $M$. The metric $g_\chi$ induces a transverse Levi-Civita connection $\nabla^x$ and a transverse curvature $R^x$ as in [1] and [2] (here it is important that $\chi$ is basic in order to define $\nabla^x$).

### 2.1. Adapted coordinates

Let $(M, \xi, \Phi, \eta, g)$ be a Sasakian manifold. We can always find local coordinates $\{z^1, \ldots, z^n, z\}$ taking values in $\mathbb{C}^n \times \mathbb{R}$ such that
\begin{equation}
\xi = \partial_z, \quad \Phi(dz^i) = i \, dz^i, \quad \Phi(d\bar{z}^i) = -i \, d\bar{z}^i. \tag{4}
\end{equation}
A function $h$ is basic if and only if does not depend on the variable $z$ and we usually denote by $h_{1t}, \ldots, h_{nt}$ the space derivatives of $h$ along $\partial_{z^1}, \ldots, \partial_{z^n}$, $\partial_{\bar{z}^1}, \ldots, \partial_{\bar{z}^n}$. We denote by $A_{1t}, \ldots, A_{nt}$ (without "_t") the components of the basic tensor $A$. Furthermore, when a function $f$ depends also on a time variable $t$, we use notation $\dot{f}$ to denote its time derivative. In the case when $f$ depends on two time variables $(t, s)$, we write $\partial_t f$ and $\partial_s f$, to distinguish the two derivatives.

For instance, the metric $g$ and the transverse symplectic form $d\eta$ locally write as
\[ g = g_{ij} \, dz^i d\bar{z}^j + \eta^2, \quad d\eta = 2i \, g_{ij} \, dz^i \wedge d\bar{z}^j, \]
where the $g_{ij}$ are all basic functions. In particular the transverse metric $g_T$ writes as $g_T = g_{ij} \, dz^i d\bar{z}^j$ and a Sasakian structure can be regarded as a collection of Kähler structures each one defined on an open set of $\mathbb{C}^n$. Observe that conditions (4) depend only $(\xi, J)$ and therefore they hold for every Sasakian structure compatible with $(\xi, J)$. This fact is crucial in the proof of Theorem [1].

In this paper we make sometimes use of special foliated coordinates with respect to a transverse Kähler form $\chi$. Indeed, once a transverse Kähler form $\chi$ on the Sasakian manifold $(M, \xi, \Phi, \eta, g)$ is fixed, we can always find foliated coordinates $\{z^1, \ldots, z^n, z\}$ around any fixed point $x$ such that if $\chi = \chi_{ij} \, dz^i \wedge d\bar{z}^j$, then
\[ \chi_{ij} = \delta_{ij}, \quad \partial_z \chi_{ij} = 0, \quad at \ x. \]
Moreover, we can further require that the transverse metric $g_T$ takes a diagonal expression at $x$.

### 2.2. The space of the Sasakian potentials and the definition of $J$-flow

Following [4, 21, 22, 23], given a Sasakian manifold $(M, \xi, \Phi, \eta, g)$, we consider
\[ \mathcal{H} = \{ h \in C^0_\Phi(M, \mathbb{R}) : \ h = \eta + d^c h \text{ is a contact form} \}, \]
where $d^c h$ is the 1-form on $M$ defined by $(d^c h)(X) = -\frac{1}{2} d h(\Phi(X))$. Every $h \in \mathcal{H}$ induces the Sasakian structure $(\xi, \Phi_h, \eta_h, g_h)$ where
\[ \Phi_h = \Phi - (\xi \otimes (\eta_h - \eta)) \circ \Phi, \quad g_h = \frac{1}{2} d \eta_h \circ (\text{Id} \otimes \Phi_h) + \eta_h \otimes \eta_h. \]
Notice that
\[ \eta_h \wedge (d \eta_h)^n = \eta \wedge (d \eta_h)^n. \]
All the Sasakian structures induced by the functions in $\mathcal{H}$ have the same Reeb vector field and the same transverse complex structure. It is rather natural to restrict our attention to the space of $\mathcal{H}_0$ of normalized Sasakian potentials. $\mathcal{H}_0$ is defined as the zero set of the functional $I : \mathcal{H} \to \mathbb{R}$ defined trough its first variation by
\[ \frac{\partial}{\partial t} I(f) = \frac{1}{2^{n+1}} \int_M \dot{f} \, \eta \wedge d \eta^n, \quad I(0) = 0, \]
where \( f \) is a smooth curve in \( \mathcal{H} \) (see \cite{21} formula (14)] for an explicit formulation of \( I \). The pair \((\xi,J)\) can be seen as a polarisation of the Sasakian manifold (see \cite{4}). Notice that \( \mathcal{H} \) is open in \( C^\infty_B(M,\mathbb{R}) \) and has the natural Riemannian metric

\[
(\varphi,\psi)_h := \frac{1}{2^n n!} \int_M \varphi \psi \eta \wedge (d\eta)_h^n.
\]

The covariant derivative of (5) along a smooth curve \( f = f(t) \) in \( C^\infty_B(M,\mathbb{R}) \) takes the following expression

\[
D_t \psi = \psi - \frac{1}{4} \langle dB \psi, dB \dot{f} \rangle_f,
\]

where \( \psi \) is an arbitrary smooth curve in \( C^\infty_B(M,\mathbb{R}) \) and \( \langle \cdot, \cdot \rangle_f \) is the pointwise scalar product induced by \( g_f \) on basic forms (see \cite{21} \cite{24}). Note that \( D_t \) can be alternatively written as

\[
D_t \psi = \dot{\psi} - \frac{1}{2} \text{Re}\langle \partial_B \psi, \partial_B \dot{f} \rangle_f
\]

which has the following local expression

\[
D_t \psi = \dot{\psi} - \frac{1}{4} g_f^{jk} (\psi, j \dot{f}_j + \dot{\psi} j \dot{f}_k).
\]

Moreover, a curve \( f = f(t) \) in \( \mathcal{H} \) is a geodesic if and only if it solves

\[
\ddot{f} - \frac{1}{4} |dB \dot{f}|_f^2 = 0.
\]

Furthermore, W. He proved in \cite{23} that \( \mathcal{H} \) is an infinite dimensional symmetric space whose curvature can be written as

\[
R_h(\psi_1,\psi_2)\psi_3 = -\frac{1}{16} \{\{\psi_1,\psi_2\}_f,\psi_3\}_h,
\]

where \( \{\cdot,\cdot\}_h \) is the Poisson bracket on \( C^\infty_B(M,\mathbb{R}) \) induced by the contact form \( \eta_h \).

As in the Kähler case, it is still an open problem to establish when two points in \( \mathcal{H} \) can be connected by a geodesic path. Fortunately, Guan and Zhang proved in \cite{22} that this can be always done in a weak sense. More precisely, the role of \( \mathcal{H} \) is replaced with its completion \( \bar{\mathcal{H}} \) with respect to the \( C^1_\infty \)-norm (see \cite{22} for details) and the geodesic equation (6) with

\[
\left( \ddot{f} - \frac{1}{4} |dB \dot{f}|_f^2 \right) \eta \wedge d\eta_f^n = \epsilon \eta \wedge d\eta_f^n.
\]

Then, by definition a \( C^{1,1} \)-geodesic is a curve in \( \bar{\mathcal{H}} \) obtained as weak limit of solutions to (7), and from \cite{22} it follows that for every two points in \( \mathcal{H} \) there exists a \( C^{1,1} \)-geodesic connecting them.

Now we can introduce the Sasakian version of the J-flow. The definition depends on the choice of a transverse Kähler form \( \chi \). Note that

\[
\eta_h \wedge \chi^n = \eta \wedge \chi^n \neq 0,
\]

for every \( h \in \mathcal{H} \), since \( \chi \) and \( dB \eta_h \) are both basic forms.

**Proposition 2.1.** Let \( f_0,f_1 \in \mathcal{H} \) and \( f: [0,1] \to \mathcal{H} \) be a smooth path satisfying \( f(0) = f_0, f(1) = f_1 \). Then

\[
A_\chi(f) := \int_0^1 \int_M \dot{f} \chi \wedge \eta \wedge (d\eta_f)^{n-1} dt,
\]

depends only on \( f_0 \) and \( f_1 \).

**Proof.** Following the approach of Mabuchi in \cite{25}, let \( \psi(s,t) := sf(t) \) and let \( \Psi \) be the 2-form on the square \( Q = [0,1] \times [0,1] \) defined as

\[
\Psi(s,t) := \left( \int_M \partial_s \psi \eta \wedge \chi \wedge (d\eta_\psi)^{n-1} \right) dt + \left( \int_M \partial_t \psi \eta \wedge \chi \wedge (d\eta_\psi)^{n-1} \right) ds.
\]
We show that $\Psi$ is closed as 2-form on $Q$:

$$
\begin{align*}
\int_{\partial Q} \Psi &= 0,
\end{align*}
$$

and the claim follows.

In view of the last proposition, we can write $A_\chi(f_0, f_1)$ instead of $A_\chi(f)$.

**Definition 2.2.** The Sasaki $J$-functional is the map $J_\chi : H \to \mathbb{R}$ defined as

$$
J_\chi(h) = \frac{1}{2^{n-1}(n-1)!} A_\chi(0, h).
$$

Alternatively we can define $J_\chi$ through its first variation by

$$
\partial_t J_\chi(f) = \int_M \frac{1}{2^{n-1}(n-1)!} \dot{f} \chi \wedge \eta \wedge (d\eta_f)^{n-1}, \quad J_\chi(0) = 0,
$$

and then apply Proposition 2.1 to show that the definition is well-posed. Note that

$$
\partial_t J_\chi(f) = \frac{1}{2^n n!} \int_M \dot{f} \sigma_f \eta \wedge d\eta_f^n,
$$

where for $h \in H$

$$
\sigma_h = g^{ba}_h \chi_{ab},
$$

the components and the derivatives are computed with respect to transverse holomorphic coordinates and with the upper indices in $g_h$ we denote the components of the inverse matrix.

If we restrict $J_\chi$ to $H_0$, then $h \in H_0$ is a critical point of $J_\chi : H_0 \to \mathbb{R}$ if and only if

$$
\int_M k \eta \wedge \chi \wedge d\eta_h^{n-1} = 0,
$$

for every $k$ in the tangent space to $H_0$ at $h$, i.e. if and only if $2n \eta \wedge \chi \wedge d\eta_h^{n-1} = c \eta \wedge d\eta_h^n$, where

$$
c = \frac{2n \int_M \chi \wedge \eta \wedge d\eta_h^{n-1}}{\int_M \eta \wedge d\eta_h^n}.
$$

Given $h \in H_0$, we can rewrite the condition of being a critical point of $J_\chi$ as

$$
\sigma_h = c.
$$

Therefore, if $f_0 \in H_0$ is fixed, the evolution equation

$$
\dot{f} = c - \sigma_f, \quad f(0) = f_0,
$$

is transcendental.
can be seen as the gradient flow of $J_\chi: \mathcal{H}_0 \to \mathbb{R}$.

**Definition 2.3.** A Sasakian structure $(\xi, \Phi, \eta, g)$ is called critical if $h$ satisfies \([11]\). We will refer to \([11]\) as to the Sasaki $J$-flow.

### 3. Technical Results and Critical Sasaki Metrics

Let $(M, \xi, \Phi, \eta, g)$ be a $(2n+1)$-dimensional compact Sasakian manifold and let $f = f(t)$ be a smooth curve in the space of normalized Sasakian potentials $\mathcal{H}_0$. Then

$$\frac{\partial}{\partial t} \eta \wedge (d\eta)^n = \Delta_f \eta \wedge (d\eta)^n,$$

where for $h \in \mathcal{H}_0$, $\Delta_h$ denotes the basic Laplacian

$$\Delta_h \psi = -\partial^*_B \partial_B \psi = g^r_h \psi_{rj}, \quad \text{for } \psi \in C^\infty_B(M, \mathbb{R}).$$

A direct computation yields

$$\dot{\sigma}_f = -g^m_f \int_m g^q_f \chi q = -\langle i \partial_B \partial_B \dot{f}, h \rangle,$$

where, given $\alpha$ and $\beta$ in $\Omega^{(p,q)}_B(M, \mathbb{C})$, we set

$$(\alpha, \beta)_h = \alpha_1 \ldots \alpha_{d \eta}, \beta_1 \ldots \beta_{d \eta}, \ldots \alpha_{d \eta}, \beta_1 \ldots \beta_{d \eta},$$

and

$$(\alpha, \beta)_h = \int_M (\alpha, \beta)_h \eta \wedge d\eta^n.$$

In particular, if $\alpha = \alpha_i dz^i$ and $\beta = \beta_j dz^j$ are transverse forms of type $(1,0)$, by writing $\chi = i \chi_{a\bar{b}} dz^a \wedge z^b$, we have

$$\langle \chi, \alpha \wedge \beta \rangle_h = i \chi_{a\bar{b}} \bar{\alpha}_j \beta_j g^a_h g^b_h.$$

The following technical lemma will be useful in the sequel.

**Lemma 3.1.** Let $u \in C^\infty_B(M, \mathbb{R})$ and $f$ be a smooth path in $C^\infty_B(M, \mathbb{R})$. Then

1. $(\Delta_f \dot{f}, u \sigma)_f = -\langle \partial_B \dot{f}, \sigma \partial_B u \rangle_f - \langle u \partial_B \dot{f}, \partial_B \sigma \rangle_f$;
2. $(\partial_B \partial_B \dot{f}, u \chi)_f = -i \langle u \partial_B \dot{f}, \partial_B \sigma \rangle_f - \langle \chi, \partial_B u \wedge \partial_B \dot{f} \rangle_f$;
3. $(\dot{f}, \dot{\sigma})_f = \frac{1}{2} (\partial_B (\dot{f})^2, \partial_B \sigma)_f - i \langle \chi, \partial_B \dot{f} \wedge \partial_B \dot{f} \rangle_f$.

where $\sigma = g^r_f \chi_{r\bar{k}}$.

**Proof.**

1. $(\Delta_f \dot{f}, u \sigma)_f = -\langle \partial_B \dot{f}, \sigma \partial_B u \rangle_f - \langle u \partial_B \dot{f}, \partial_B \sigma \rangle_f$.
2. Since the Laplacian is self-adjoint we have:

$$2^n M (\partial_B \partial_B \dot{f}, u \chi)_f = - \int_M u g^j_f g^{a} \dot{f}^j \chi_{a\bar{c}} \eta \wedge (d\eta)^n = \int_M u g^j_f g^{a} \chi_{a\bar{c}} \dot{f}^j \eta \wedge (d\eta)^n = \int_M u g^j_f g^{a} \chi_{a\bar{c}} \dot{f}^j \eta \wedge (d\eta)^n = \frac{1}{2} \langle \partial_B (\dot{f})^2, \partial_B \sigma \rangle_f - i \langle \chi, \partial_B \dot{f} \wedge \partial_B \dot{f} \rangle_f.$$

(iii) By using \([13]\) and (ii), we have

$$(\dot{f}, \dot{\sigma})_f = -i \langle \partial_B \partial_B \dot{f}, \dot{f} \chi \rangle_f = \langle \dot{f} \partial_B \dot{f}, \partial_B \sigma \rangle_f - i \langle \chi, \partial_B \dot{f} \wedge \partial_B \dot{f} \rangle_f$$

$$= \frac{1}{2} (\partial_B (\dot{f})^2, \partial_B \sigma)_f - i \langle \chi, \partial_B \dot{f} \wedge \partial_B \dot{f} \rangle_f.$$
as required.

The following proposition is about the uniqueness of critical Sasaki metrics in $\mathcal{H}_0$ and it is analogue to the Kähler case.

**Proposition 3.2.** $J_\chi : \mathcal{H}_0 \to \mathbb{R}$ has at most one critical point.

**Proof.** Let $f$ be a curve in the space $\tilde{\mathcal{H}}$ obtained as completion of $\mathcal{H}$ with respect to the $C^2_\omega$-norm. Then taking into account the definition of $J_\chi$, Lemma 3.1 and equations (12), (13), we have

$$\partial^2_t J_\chi (f) = (\ddot{f}, \sigma_f) + \frac{1}{2} (\Delta_f \ddot{f}, \dot{f} \sigma_f) + i (f \bar{\partial_B} \dot{f}, \chi) f$$

$$= \frac{1}{2^{n+1}} \int_M \left( \ddot{f} - \frac{1}{2} \bar{\partial_B} \dot{f}^2 \right) \sigma_f \eta \wedge (d\eta)^n - i (\chi, \partial_B \dot{f} \wedge \bar{\partial_B} \dot{f}) f.$$  

Therefore if $f$ solves the modified geodesic equation (7), then

$$\partial^2_t J_\chi (f) \geq -i (\chi, \partial_B \dot{f} \wedge \bar{\partial_B} \dot{f}) f \geq 0.$$  

Let us assume now to have two critical points $f_0$ and $f_1$ of $J_\chi$ in $\mathcal{H}_0$ and denote by $\mathcal{H}_0$ the competition of $\mathcal{H}_0$ with respect to the $C^2_\omega$-norm. Then, in view of [22], there exists a $C^1$-geodesic $\dot{f}$ in $\mathcal{H}_0$ such that $f(0) = f_0$ and $f(1) = f_1$. Let $h(t) = J_\chi (f(t))$. Then since $f_0$ and $f_1$ are critical points of $J_\chi$, we have $\dot{h}(0) = \dot{h}(1) = 0$. Since $\dot{h} \geq 0$, it as to be $\dot{h} \equiv 0$ which implies $\partial_B \dot{f} = 0$ and $\dot{f}(t)$ is constant for every $t \in [0, 1]$. Finally, since $f$ is a curve in $\mathcal{H}_0$, then $H(f) = 0$ and therefore $\dot{f} = 0$, which implies $f_0 = f_1$, as required.

On a compact 3-dimensional Sasaki manifold, the existence of a critical metric is always guaranteed. Indeed, if $(M, \xi, \Phi, \eta, g)$ is a compact 3-dimensional Sasaki manifold with a fixed background transverse Kähler form $\chi$, then we can write:

$$\chi = \frac{1}{4} \langle \chi, d\eta \rangle \, d\eta, \quad d\eta_h = \left( 1 - \frac{1}{2} \Delta_B h \right) \, d\eta,$$

where the scalar product and the basic Laplacian are computed with respect to the metric induced by $\eta$. Hence, $\eta_h = \eta + d^* h$ induces a critical metric if and only if $h$ solves:

$$\Delta_B h = 2 - \frac{1}{e} \langle \chi, d\eta \rangle, \quad e = \frac{\int_M \eta \wedge \chi}{\int_M \eta \wedge d\eta},$$

which has always a solution since:

$$\int_M \left( 2 - \frac{1}{e} \langle \chi, d\eta \rangle \right) \eta \wedge d\eta = 0.$$  

In higher dimensions there is a cohomological obstruction to the existence of a critical metric similar to the one in the Kähler case.

Recall that if $(M, \omega)$ is a Kähler $2n$-dimensional manifold (with $n > 1$) with a fixed background Kähler metric $\chi$, then the existence of a $J_\chi$-critical normalised Kähler potential on $(M, \omega)$ implies that $|\omega - \chi|$ is a Kähler class in $H^2(M, \mathbb{R})$ (see [14]). In [14], Donaldson conjectured that the condition $|\omega - \chi| > 0$ is also sufficient for the existence of a critical metric and Chen proved in [6] that the conjecture is true on complex surfaces. Some partial results in higher dimensions about this problem have been obtained in [38,12,43]. The same conjecture can be stated also for the Sasakian case. Indeed, given a Sasakian manifold $(M, \xi, \Phi, \eta, g)$ with a fixed background transverse Kähler form $\chi$, then if $h \in \mathcal{H}_0$ is a critical normalised Sasakian potential, then $\frac{e}{2} d\eta_h - \chi$ is a transverse Kähler form. Hence it is rather natural to conjecture that the existence of a Sasakian potential $h$ satisfying $\frac{e}{2} d\eta_h - \chi > 0$, implies the existence of a critical Sasaki metric and we expect that the results in [38,12,43] could be generalised to the Sasakian case.

The following proposition is about the existence of a critical Sasaki metric in dimension 5:
Proposition 3.3. Let \((M, \xi, \Phi, \eta, g)\) be a compact 5-dimensional Sasaki manifold. Assume that there exists a map \(h \in \mathcal{H}_0\) such that \(\frac{1}{2}(d\eta + dd^c h) - \chi\) is a transverse Kähler form. Then, there exists a critical Sasaki metric on \(M\).

Proof. Up to rescaling \(\eta\), we may assume \(c = 1\). A function \(h \in \mathcal{H}_0\) is critical if and only if

\[
2\eta \wedge \chi \wedge \left(\frac{1}{2} d\eta + dd^c h\right) = \eta \wedge \left(\frac{1}{2} d\eta + dd^c h\right)^2.
\]

Let \(\Omega = \frac{1}{2}d\eta - \chi\). Then our hypothesis implies that \(\Omega\) is a transverse Kähler form and moreover by substituting we get

\[
(\Omega + dd^c h)^2 = \chi^2.
\]

Finally, the Calabi-Yau theorem in Kähler foliations [15] implies the statement. \(\square\)

4. Well-posedness of the Sasaki J-flow

Theorem 4.1. The Sasaki J-flow is well-posed, i.e., for every initial datum \(f_0\), system (11) has a unique maximal solution \(f\) defined in \([0, \epsilon_{\max})\), for some positive \(\epsilon_{\max}\).

Proof. Since \(\mathcal{H}\) is not open in \(C^\infty(M, \mathbb{R})\), to apply the standard parabolic theory we have to use a trick adopted by Smoczyk, Wang and Zhang for showing the short-time existence of the Sasaki-Ricci flow in [36]. Since the functional \(F: \mathcal{H} \to \mathbb{R}\) defined as

\[
F(f) = \xi^2(f) + \sigma_f,
\]

is elliptic, the standard parabolic theory implies that the geometric flow

\[
\dot{f} = c - \xi^2(f) - \sigma_f, \quad f(0) = f_0,
\]

has a unique maximal solution \(f \in C^\infty(M \times [0, \epsilon_{\max}), \mathbb{R})\), for some \(\epsilon_{\max} > 0\). Of course if \(f(\cdot, t)\) is a solution to (14) which is basic for every \(t\) and \(I(f) = 0\), then \(f\) solves (11). We first show that when \(f_0\) is basic, then the solution \(f\) to (11) holds basic for every \(t \in [0, \epsilon_{\max})\). We have

\[
\partial_t \xi(f) = \xi(\dot{f}) = \xi(\chi) = g^{kr} \chi_{rk},
\]

Moreover, since the components of \(\chi\) are basic, we have

\[
\xi(g^{kr} \chi_{rk}) = -g^{kl} \chi_{lm} \xi(\chi),
\]

i.e.

(15) \[
\partial_t \xi(f) = -\xi^2(f) + \langle dd^c \xi(f), \chi \rangle_f.
\]

Equation (15) is parabolic in \(\xi(f)\) and then, since the solution to a parabolic problem is unique, if \(\xi(f_0) = 0\), \(\xi(f(t)) = 0\) for every \(t \in [0, \epsilon_{\max})\), as required. Finally we show that if \(f_0\) is normalised, then \(I(f) = 0\) for every \(t \in [0, \epsilon_{\max})\). We have

\[
\partial_t I(f) = \frac{1}{2 n!} \int_M \dot{f} \eta \wedge d\eta^n = \frac{1}{2 n!} \int_M (c - \sigma_f) \eta \wedge d\eta^n,
\]

and since \(c \int_M \eta \wedge d\eta^n = \int_M \sigma_f \eta \wedge d\eta^n\), we have \(\partial_t I(f) = 0\). Therefore, since \(I(f_0) = 0\), \(I(f) = 0\) for every \(t \in [0, \epsilon_{\max})\) and the claim follows. \(\square\)

Remark 4.2. Alternatively, the short-time existence of the Sasaki J-flow can be obtained by invoking the short-time existence of any second order transversally parabolic equation on compact manifolds foliated by Riemannian foliations. A proof of the latter result can be founded in [11].

In analogy to the Kähler case, let \(\text{En}: \mathcal{H}_0 \to \mathbb{R}\) be the energy functional

\[
\text{En}(h) = \frac{1}{2 n!} \int_M \sigma_h^2 \eta \wedge (d\eta_h)^n = \langle \sigma_h, \sigma_h \rangle_h^2.
\]

Proposition 4.3. The following items hold:

1. \(\text{En}\) has the same critical points of \(J_\chi\) and it is strictly decreasing along the Sasaki J-flow;
2. any critical point of $E_n$ is a local minimizer;
3. the length of any curve in $H_0$ and the distance of any two points in $H_0$ decrease under the $J$-flow.

Proof. Let $f: [0,1] \to H_0$ be a smooth curve. Then, by using (13) and Lemma 3.1 the first variation of $E$ reads:

$$\partial_t E_n(f) = \frac{1}{2n} \partial_t \int_M \sigma_f^2 \eta \wedge (d\eta_f)^n = 2(\sigma_f, \dot{\sigma}_f)_f + (\sigma_f^2, \Delta_f \dot{\sigma}_f)_f$$

$$= 2(\sigma_f \partial_B \dot{f}, \partial_B \sigma_f) - 2i(\chi, \partial_B \dot{f} \wedge \partial_B \sigma_f)_f - 2(\partial_B \sigma_f, \sigma_f \partial_B \dot{f})_f$$

$$= -2i(\chi, \partial_B \dot{f} \wedge \partial_B \sigma_f)_f.$$

Along the Sasaki $J$-flow one has $\dot{f} = c - \sigma_f$, thus:

$$\partial_t E_n(f) = -2i(\chi, \partial_B \sigma_f \wedge \partial_B \sigma_f)_f \leq 0,$$

and $E_n$ is strictly decreasing along the $J$-flow. Moreover, if $h \in H_0$ is a critical point of $E_n$, then $\partial_B \sigma_h = 0$ which implies that $h$ is critical if and only if $\sigma_h = c$.

2. Now we compute the second variation of $E_n$. Let $f: (-\delta, \delta) \times (-\delta, \delta) \to H_0$ be a smooth map in the variables $(t,s)$. Assume that $f(0,0) = h$ is a critical point of $E_n$ and let $u = \partial_t f_{(0,0)}$, $v = \partial_s f_{(0,0)}$. Then we have

$$\partial_s \partial_t E_n(\alpha) = \frac{1}{2n-1} \int_M \langle \chi, \partial_B \partial_u \wedge \partial_B \sigma_{\alpha(0,0)} \partial_B \sigma_h \eta \wedge (d\eta_h)^n \rangle,$$

and

$$\partial_s \partial_t E_n(\alpha) = \frac{1}{2n-1} \int_M \langle \chi, \partial_B u \wedge \partial_B \sigma_{\alpha(0,0)} \eta \wedge (d\eta_h)^n \rangle = 2(\chi, \partial_B u \wedge \partial_B \sigma_{\alpha(0,0)} h),$$

since $\sigma_h$ is constant. Now

$$2(\chi, \partial_B u \wedge \partial_B \sigma_{\alpha(0,0)} h) = 2(\chi, \partial_s \sigma_{\alpha(0,0)} \partial_B \partial_B u)_h,$$

and formula (13) implies

$$\partial_s \partial_t E_n(f)_{(0,0)} = \frac{1}{2n-1} \int_M \langle i \partial_B \partial_B u, \chi \rangle_h (i \partial_B \partial_B v, \chi)_h \eta \wedge (d\eta_h)^n,$$

which implies $\partial_s \partial_t E_n(f)_{(0,0)}$ is positive definite as symmetric form.

3. Given smooth curve $u: [0,1] \to H_0$ in $H_0$ and $h \in H_0$ we denote by

$$L(h, u) = \frac{1}{2^n} \int_0^1 \int_M \dot{u}^2 \eta \wedge (d\eta_h)^n \wedge ds = (\dot{u}, \dot{u})_u,$$

the square of the length of $u$ respect to the Sasaki metric induced by $h$. Let $f: [0, \epsilon] \times [0,1] \to H_0$ and assume that $t \mapsto f(t,s)$ is a solution to the $J$-flow for every $s \in [0,1]$. Then, by using Lemma 3.1 we have

$$2^n \partial_t L(f, f) = \partial_t \left[ \int_0^1 \int_M (\partial_t f)^2 \eta \wedge (d\eta_f)^n \wedge ds \right]$$

$$= \int_0^1 \int_M 2 \partial_s f \partial_f \partial_t f + (\partial_t f)^2 \Delta_f \partial_t f \eta \wedge (d\eta_f)^n \wedge ds$$

$$= \int_0^1 \int_M -2 \partial_s \sigma_f \partial_t f - (\partial_t f)^2 \Delta_f (\sigma_f) \eta \wedge (d\eta_f)^n \wedge ds$$

$$= \int_0^1 [2(\partial_s \sigma_f, \partial_t f)_f + ((\partial_t f)^2, \Delta_f \sigma_f)_f] ds$$

$$= \int_0^1 [2(\partial_s \sigma_f, \partial_t f)_f + (\partial_B (\partial_t f)^2, \partial_B \sigma_f)_f] ds$$

$$= 2i \int_0^1 (\chi, \partial_B \partial_u \wedge \partial_B \partial_u f) ds \leq 0.$$
and the equality holds if and only if \( \partial_s f(t, s) \) is constant in \( s \).

\[\square\]

5. A Maximum principle for basic maps and tensors

In this section we introduce a basic principle for transversally elliptic operators on Sasakian manifolds. The principle will be applied in the next section to compute the \( C^2 \)-estimate about the solutions to equation (11).

Let \((M, \xi, \Phi, \eta)\) be a Sasakian manifold. By a smooth family of basic linear partial differential operators \( \{E_t\}_{t \in [0, \epsilon)} \) we mean a smooth family of operators \( E(\cdot, t) : C^\infty_B(M, \mathbb{R}) \rightarrow C^\infty_B(M, \mathbb{R}) \) which can be locally written as

\[E(h(y), t) = \sum_{1 \leq |k| \leq m} a_k(y, t) \frac{\partial^{|k|}}{\partial y_{k_1} \ldots \partial y_{k_r}} h(y)\]

for every \( h \in C^\infty_B(M, \mathbb{R}) \), where \( \{y^1, \ldots, y^{2n}, z\} \) are real coordinates on \( M \) such that \( \xi = \partial_z \). The maps \( a_k \) are assumed to be smooth and basic in the space coordinates (see [15] for a detailed description of these operators on compact manifolds foliated by Riemannian foliations). Observe that \( E \) can be regraded as a functional \( E : C^\infty_B(M \times [0, \epsilon), \mathbb{R}) \rightarrow C^\infty_B(M \times [0, \epsilon), \mathbb{R}) \) in a natural way. We further make the strong assumption on \( E \) to satisfy

\[(16) \quad E(h(x, t), t) \leq 0,\]

whenever the complex Hessian \( dd^c h \) of \( h \) is nonpositive at the point \((x, t) \in M \times [0, \epsilon)\).

**Proposition 5.1** (Maximum principle for basic maps). Assume that \( h \in C^\infty(M \times [0, \epsilon), \mathbb{R}) \) satisfies

\[\partial_t h(x, t) - E(h(x, t), t) \leq 0.\]

Then

\[\sup_{(x,t) \in M \times [0, \epsilon)} h(x, t) \leq \sup_{x \in M} h(x, 0).\]

**Proof.** Fix \( \epsilon_0 \in (0, \epsilon) \) and let \( h_\lambda : M \times [0, \epsilon_0) \rightarrow \mathbb{R} \) be the map \( h_\lambda(x, t) = h(x, t) - \lambda t \). Assume that \( h_\lambda \) achieves its global maximum at \((x_0, t_0)\) and assume by contradiction that \( t_0 > 0 \). Then \( \partial_t h_\lambda(x_0, t_0) \geq 0 \) and \( dd^c h_\lambda(x_0, t_0) \) is nonpositive. Therefore condition (16) implies \( E(h_\lambda(x_0, t_0), t_0) \leq 0 \) and consequently

\[\partial_t h_\lambda(x_0, t_0) - E(h_\lambda(x_0, t_0), t_0) \geq 0.\]

Since \( \partial_t h_\lambda = \partial_t h - \lambda \) and \( E(h_\lambda(x, t), t) = E(h(x, t), t) \), we have

\[0 \leq \partial_t h(x_0, t_0) - E(h(x_0, t_0), t_0) - \lambda \leq -\lambda,\]

which is a contradiction. Therefore \( h_\lambda \) achieves its global maximum at a point \((x_0, 0)\) and

\[\sup_{M \times [0, \epsilon_0]} h \leq \sup_{M \times [0, \epsilon_0]} h_\lambda + \lambda \epsilon_0 \leq \sup_{x \in M} h(x, 0) + \lambda \epsilon_0.\]

Since the above inequality holds for every \( \epsilon_0 \in (0, \epsilon) \) and \( \lambda > 0 \), the claim follows. \(\square\)

A similar result can be stated for tensors:

**Proposition 5.2** (Maximum principle for basic tensors). Let \( \kappa \) be a smooth curve of basic \((1, 1)\)-forms on \( M \) for \( t \in [0, \epsilon) \). Assume \( \kappa \) nonpositive and such that

\[\partial_t \kappa_{ij}(x, t) - E(\kappa_{ij}(x, t), t) = N_{ij}(x, t),\]

where \( N \) is a nonpositive basic form and the components are with respect to foliated coordinates. Then \( \kappa \) is nonpositive for every \( t \in [0, \epsilon) \).

**Proof.** The proof is very similar to the case of functions. We show that for every positive \( \lambda \), \( \kappa_\lambda = \kappa - t dt \) is nonpositive. Assume by contradiction that this is not true. Then there exists a \( \lambda \), a first point \((x_0, t_0) \in M \times [0, \epsilon)\) and \( g \)-unitary \((1, 0)\)-vector \( Z \in D_{x_0}^{1,0} \) such that \( \kappa_\lambda(Z, \bar{Z}) = 0 \). We extend \( Z \) to a basic unitary vector field in a small enough neighborhood \( U \) of \( x \) and consider the map \( f_\lambda : U \times [0, t_0] \rightarrow \mathbb{R} \) given by \( f_\lambda = \kappa_\lambda(Z, \bar{Z}) \). Then \( f_\lambda \) has a maximum at \((x_0, t_0)\) and so

\[\partial_t f_\lambda \geq 0, \quad E(f_\lambda(x_0, t_0), t_0) \leq 0,\]
at \((x_0, t_0)\). Now since
\[
E(f_\lambda(x, t), t) = E(f_0(x, t), t),
\]
we have
\[
0 \leq \partial_t(f_\lambda) = E(f_\lambda, \cdot) + N(Z, \bar{Z}) - \frac{\lambda}{2} \leq 0,
\]
at \((x_0, t_0)\), which implies
\[
N(Z, \bar{Z}) \geq \frac{\lambda}{2},
\]
at \((x_0, t_0)\), which is a contradiction. 
\[\square\]

In the following we will apply the two propositions when \(E\) is the operator \(\tilde{\Delta}_f\) depending on a smooth curve \(f\) in \(\mathcal{H}\) defined by:
\[
\tilde{\Delta}_f(h, t) = g_f^{kp}g_f^{\bar{q}j} \chi_{jk,h,ab}.
\]

6. Second order estimates

The following two lemmas provide the a priori estimates we need to prove the main theorem.

**Lemma 6.1.** Let \(f: M \times [0, \epsilon) \to \mathbb{R}\) be a solution to (11), with \(\epsilon < \infty\). Then
\[
\sigma_f \leq \min_{x \in M} \sigma_f(x, 0)
\]
and there exists a uniform constants \(C\), depending only on \(f_0\), such that
\[
\gamma_f(x, t) \leq \sup_{x \in M} \gamma_f(x, 0) e^{Ce}
\]
where \(\gamma_f = \tilde{\chi}^{jk}(g_f)_{kj}\).

**Proof.** The upper bound of \(\sigma_f\) easily follows from the definition of \(J_\chi\) and Proposition 5.1. Indeed, differentiating (11) in \(t\) we have
\[
\dot{f} = -\partial_t \sigma_f = g_f^{\bar{a}b} g_f^{\bar{j}k} \chi_{bk,\dot{f}ja} = -\tilde{\Delta}_f \sigma_f, \text{ i.e.,}
\]
\[
\partial_t \sigma_f = \tilde{\Delta}_f \sigma_f
\]
and Proposition 5.1 implies the first inequality. About the upper bound of \(\gamma_f\), we have
\[
\partial_t \gamma_f = \tilde{\chi}^{jk} \partial_t [(g_f)_{kj}] = \tilde{\chi}^{jk} \dot{f}_{kj}.
\]
Since \(f\) solves (11), we have
\[
\dot{f}_{ja} = g_f^{kp}(g_f)_{p\bar{q},a} g_f^{\bar{q}j} \chi_{jk} - g_f^{\bar{j}k} \chi_{\bar{J}k,a}
\]
and
\[
\dot{f}_{ab} = -2g_f^{ks}(g_f)_{\bar{s}x,b}g_f^{\bar{p}f}(g_f)_{p\bar{q},a} g_f^{\bar{q}j} \chi_{jk} + \tilde{\Delta}_f [(g_f)_{ab}]
\]
\[
+ g_f^{kp}(g_f)_{p\bar{q},a} g_f^{\bar{q}j} \chi_{jk,\bar{b}} + g_f^{ks}(g_f)_{\bar{s}x,b}g_f^{\bar{p}f}(g_f)_{p\bar{q},a} \chi_{jk,a} - g_f^{\bar{j}k} \chi_{\bar{J}k,a,\bar{b}}.
\]

Let \(R^T = R^T(\chi)\) be the transverse curvature of \(\chi\) and \(\text{Ric}^T(\chi)\) its transverse Ricci tensor (see section 2). The components of \(R^T\) with respect to foliated coordinates read as
\[
R^T_{\bar{k}a\bar{b}} = -\chi_{\bar{k}\bar{b},a} + \chi^q_p \chi_{jp,a} \chi_{qk,\bar{b}}.
\]

Fix a point \((x_0, t_0) \in M \times [0, \epsilon)\) and special foliated coordinates for \(\chi\) around it (see subsection 2.1). We may further assume without loss of generality that \((g_f)_{jk} = \lambda_j \delta_{jk}\) at \((x_0, t_0)\). Then
\[
\dot{f}_{ab} = \sum_{k,r=1}^{n} \frac{-2}{\lambda_k^2 \lambda_r} (g_f)_{\bar{k}k,\bar{b}}(g_f)_{r\bar{k},a} + \tilde{\Delta}_f [(g_f)_{ab}] - \sum_{k=1}^{n} \frac{1}{\lambda_k} \chi_{kk,ab} \text{ at } (x_0, t_0)
\]
and
\[
\partial_t \gamma_f = \sum_{a=1}^{n} \dot{f}_{aa} = \sum_{a=1}^{n} \left[ \sum_{k,r=1}^{n} \frac{-2}{\lambda_k^2 \lambda_r} |(g_f)_{kr,a}|^2 + \tilde{\Delta}_f [(g_f)_{aa}] - \sum_{k=1}^{n} \frac{1}{\lambda_k} \chi_{kk,aa} \right] \text{ at } (x_0, t_0).
\]
\[ \partial_t \gamma_f = \sum_{a=1}^{n} \left( \sum_{k,r=1}^{n} \frac{-2}{\lambda_k^2 \lambda_r} [(g_f)_{k\bar{r},a}]^2 + \tilde{\Delta}_f [(g_f)_{a\bar{a}}] \right) - \sum_{k=1}^{n} \frac{1}{\lambda_k} \text{Ric}^T_{kk} \quad \text{at } (x_0, t_0). \]

Now a direct computation yields
\[ \tilde{\Delta}_f \gamma_f = \sum_{a=1}^{n} \tilde{\Delta}_f [(g_f)_{a\bar{a}}] - \sum_{a,k=1}^{n} \frac{\lambda_a}{\lambda_k} \text{Ric}^T_{a\bar{a}kk} \quad \text{at } (x_0, t_0) \]

and therefore
\[ \partial_t \gamma_f - \tilde{\Delta}_f \gamma_f = \sum_{a,k=1}^{n} \left( \sum_{r=1}^{n} \frac{-2}{\lambda_k^2 \lambda_r} [(g_f)_{k\bar{r},a}]^2 + \frac{\lambda_a}{\lambda_k} \text{Ric}^T_{a\bar{a}kk} \right) - \sum_{k=1}^{n} \frac{1}{\lambda_k} \text{Ric}^T_{kk} \quad \text{at } (x_0, t_0). \]

Observe that
\[ \sum_{k=1}^{n} \frac{1}{\lambda_k} = \sigma_f(x_0, t_0) \leq C_1, \quad \sum_{k=1}^{n} \lambda_k = \gamma_f(x_0, t_0), \]

where \( C_1 = \min_{x \in M} \sigma_f(x, 0) \). Thus for all \( k = 1, \ldots, n \) we have
\[ \frac{1}{\lambda_k} \leq C_1, \quad \lambda_k \leq \gamma_f(x_0, t_0). \]

Since \( M \) is compact, there exists a constant \( C_2 \) such that \( \text{Ric}^T - C_2 \chi \) is nonnegative and therefore at \( (x_0, t_0) \) we have
\[ \left| \frac{1}{\lambda_k} \text{Ric}^T_{kk} \right| \leq nC_1C_2, \quad \left| \sum_{a,k=1}^{n} \frac{\lambda_a}{\lambda_k^2} \text{Ric}^T_{a\bar{a}kk} \right| \leq C_2 \sum_{a,k=1}^{n} \lambda_a \text{Ric}^T_{aa} \leq nC_2^2 C_2 \gamma_f, \]

Thus there exists a constant \( C \) such that
\[ \partial_t \gamma_f - \tilde{\Delta}_f \gamma_f \leq C \gamma_f + C. \]

Let \( F := e^{-Ct} \gamma_f - Ct. \) Then
\[ \partial_t F - \tilde{\Delta}_f F = e^{-Ct} \left( -c \gamma_f + \partial_t \gamma_f - \tilde{\Delta} \gamma_f \right) - C, \]

and by Proposition 5.1 we have
\[ \sup_{(x, t) \in M \times [0, \epsilon]} F \leq \sup_{x \in M} F(x, 0) = \sup_{x \in M} \gamma_f(x, 0), \]

which implies
\[ \sup_{(x, t) \in M \times [0, \epsilon]} \gamma_t = \sup_{x \in M} \gamma_f(x, 0) e^{C \epsilon} \]
as required. \( \square \)

In order to get a uniform lower bound for \( d\eta_f \) we need to add an hypothesis on the bisectional curvature of \( \chi \) (see Theorem 6.2 below). Observe that the existence of a uniform lower bound without further assumption would imply the existence of a critical metric in \( H_0 \) for each choice of \( \eta \) and \( \chi \), in contrast with the necessary condition \( \frac{d\eta_f - \chi}{\sqrt{\text{Vol}(\eta_f)}} > 0 \).

**Theorem 6.2.** Assume that the transverse bisectional curvature of \( \chi \) is nonnegative and let \( f : M \times [0, \epsilon) \to \mathbb{R} \) be a solution to (11). Then there exists constant \( C \) depending only on the initial datum \( f_0 \) such that \( C \chi - d\eta_f \) is a transverse Kähler form for every \( t \in [0, \epsilon) \).
Proof. Let $\kappa = \frac{1}{2} d\eta_f - C\chi$ where $C$ is a constant chosen big enough to have $\kappa$ nonpositive at $t = 0$. Then $\kappa$ is a time-dependent basic $(1,1)$-form which is nonpositive at $t = 0$. We apply Proposition 5.2 to show that $\kappa$ is nonpositive for every $t \in [0, \epsilon)$. Once a system of foliated coordinates $(z^k, z)$ is fixed, we have $\partial_t \kappa_{ab} = \tilde{f}_{,ab}$ and formula (12) implies

$$\partial_t \kappa_{ab} = -2g^{\tilde{k}p}(gf)_{\tilde{r}s,\tilde{b}}g^{\tilde{r}p}(gf)_{\tilde{q},\tilde{a}}g^{\tilde{q}j}x_{\tilde{j}k} + g^{\tilde{k}p}(gf)_{\tilde{r}s,\tilde{b}}g^{\tilde{r}p}x_{\tilde{j}k} - g^{\tilde{k}p}x_{\tilde{j}k,\tilde{a}}$$

$$= -2g^{\tilde{k}p}(gf)_{\tilde{r}s,\tilde{b}}g^{\tilde{r}p}(gf)_{\tilde{q},\tilde{a}}g^{\tilde{q}j}x_{\tilde{j}k} + \tilde{\Delta}[(gf)_{ab}]$$

$$+ g^{\tilde{k}p}(gf)_{\tilde{r}s,\tilde{b}}g^{\tilde{r}p}x_{\tilde{j}k,\tilde{b}} + g^{\tilde{k}p}(gf)_{\tilde{r}s,\tilde{b}}g^{\tilde{r}p}x_{\tilde{j}k,a} - g^{\tilde{k}p}x_{\tilde{j}k,ab},$$

i.e.

$$\partial_t \kappa_{ab} - \tilde{\Delta}[(gf)_{ab}] = -2g^{\tilde{k}p}(gf)_{\tilde{r}s,\tilde{b}}g^{\tilde{r}p}(gf)_{\tilde{q},\tilde{a}}g^{\tilde{q}j}x_{\tilde{j}k}$$

$$+ g^{\tilde{k}p}(gf)_{\tilde{r}s,\tilde{b}}g^{\tilde{r}p}x_{\tilde{j}k,\tilde{b}} + g^{\tilde{k}p}(gf)_{\tilde{r}s,\tilde{b}}g^{\tilde{r}p}x_{\tilde{j}k,a} - g^{\tilde{k}p}x_{\tilde{j}k,ab}.$$ (19)

We apply Proposition 5.2 using as $N$ the basic form defined by the right hand part of formula (19). To this end, we have to show that $N$ is nonpositive. That can be easily done as follows: fix a point $(x,t) \in M \times [0, \epsilon)$ and an arbitrary unitary vector field $Z \in \mathcal{D}_x^{1,0}$. Then we can find foliated coordinates $(z, z^k)$ around $x$ which are special for $\chi$ and such that: $Z = \partial_{z^1}z$ and $gf$ takes a diagonal expression with eigenvalues $\lambda_k$ at $(x, t)$. Then we have

$$N(Z, \bar{Z}) = -2\sum_{k,r,s=1}^{n} \frac{1}{\lambda_k^{2} \lambda_r} |(gf)_{kr,\bar{s}|\bar{r}}|^2 - \sum_{k=1}^{n} \frac{1}{\lambda_k^{2}} R^T(\chi)_{k\bar{k}l\bar{l}}$$

at $(x, t)$ and the claim follows. \qed

7. Proof of the main theorem

The proof of Theorem 1.1 is based on the second order estimates provided in Section 5 and on the following result in Kähler geometry.

**Theorem 7.1.** Let $B$ be an open ball about $0$ in $\mathbb{C}^n$ and let $\omega, \chi$ be two Kähler forms on $B$. Let $f: M \times [0, \epsilon) \to \mathbb{R}$ be solution to the Kähler $J$-flow

$$\dot{f} = c - g^{\tilde{k}r}x_{\tilde{r}k},$$

where $g_f$ is the metric associated to $\omega_f = \omega + df^c. \omega_f$ Assume that $\omega_f$ is uniformly bounded in $B \times [0, \epsilon)$. Then $f$ is $C^\infty$-bounded in a small ball about $0$.

As explained in [7], the theorem can be proved by using the well-known Evans and Krylov’s interior estimate (see [19] for a proof of the estimates in the complex case).

**Proof of Theorem 1.1** The proof of the long time existence consists in showing that every solution $f$ to (11) have a $C^\infty$-bound. Let $f: M \times [0, \epsilon_{max}) \to \mathbb{R}$ be the solution to (11) with initial condition $f_0 \in H_0$ and assume by contradiction $\epsilon_{max} < \infty$. Lemma 6.1 implies that the second derivatives of $f$ are uniformly bounded in $M$. Since $f$ can be regarded as a collection of solutions to the Kähler $J$-flow on small open balls in $\mathbb{C}^n$, Theorem 7.1 implies that $f$ is $C^\infty$-uniformly bounded in $M$. Therefore $f$ converges in $C^\infty$-norm to a smooth function $\tilde{f}$ as $t$ tends to $\epsilon_{max}$. Since $\partial_t \tilde{f}$ is basic for every $t \in [0, \epsilon_{max})$, $\tilde{f}$ is basic and by the well-posedness of the Sasaki $J$-flow, the solution $f$ can be extended after $\epsilon_{max}$ contradicting its maximality.

The proof of the long time existence in the case when $\chi$ has nonnegative transverse holomorphic bisectional curvature, is obtained exactly as in the Kähler case. Let $f: M \times [0, \infty) \to \mathbb{R}$ be a solution to the Sasaki $J$-flow. Since $\chi$ has nonnegative holomorphic bisectional curvature, Theorem 6.2 implies that $f$ has a uniform $C^\infty$-bound and Ascoli-Arzelà implies that given a sequence $t_j \in [0, \infty)$, $t_j \to \infty$, $f_{t_j}$ has
a subsequence converging in $C^\infty$-norm to function $f_\infty$ as $t_j \to \infty$. Therefore, $f$ converges to a critical map $f_\infty \in H_0$.\hfill\Box

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