The Big Bang is a Coordinate Singularity for \( k = -1 \)
Inflationary FLRW Spacetimes

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Abstract

We show that the big bang is just a coordinate singularity for \( k = -1 \) inflationary FLRW spacetimes. That is, it can be removed by introducing a set of coordinates in which the big bang appears as a past Cauchy horizon where the metric is no longer degenerate. In fact this past Cauchy horizon is just the future lightcone at the origin of a spacetime conformal to Minkowski space. For these \( k = -1 \) inflationary FLRW spacetimes, we show that the cosmological constant appears as an initial condition, and the Lorentz group acts by isometries.

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1 Introduction

The big bang, \( \tau = 0 \), in FLRW spacetimes is widely believed to be a genuine singularity. That is, there should exist some infinite energy density or infinite curvature quantity at the big bang. Indeed the scalar curvature is

\[
R(\tau) = 6 \left[ \frac{a''(\tau)}{a(\tau)} + \left( \frac{a'(\tau)}{a(\tau)} \right)^2 + \frac{k}{a(\tau)^2} \right].
\] (1.1)

Therefore if one assumes the universe is in a radiation dominated era, \( a(\tau) \sim \sqrt{\tau} \), all the way down to the big bang (i.e. no inflation), then the scalar curvature diverges as \( \tau \to 0 \). Moreover, assuming the strong energy condition, the Hawking-Penrose singularity theorems [3, 5] show that a singularity is generically unavoidable.

However inflationary eras, \( a''(\tau) > 0 \), imply that the strong energy condition must be violated and so the Hawking-Penrose singularity theorems don’t apply. Therefore a natural question to ask is: is the big bang still singular? In terms of the energy density \( \rho(\tau) \) and pressure function \( p(\tau) \), the scalar curvature is

\[
R(\tau) = 8\pi \rho(\tau) - 24\pi p(\tau).
\] (1.2)

Since the pressure in inflationary eras is negative, we see that the scalar curvature diverges provided the energy density diverges. The divergence of the energy density seems like a physically reasonable assumption, but it is known to not always hold. For example, de Sitter space is partially covered by the so-called ‘open slicing’ coordinates, and in these coordinates is seen to be a \( k = -1 \) FLRW spacetime with scale factor \( a(\tau) = \sinh(\tau) \), i.e. it’s always inflating. In this case we have \( \rho = -p = \Lambda/8\pi \) is constant. Here \( \Lambda = 3 \) is the cosmological constant. In section 3.2 we show that for \( k = -1 \) inflationary FLRW spacetimes, we have \( \lim_{\tau \to 0} \rho(\tau) = (3/8\pi)a''(0) \) provided \( a'(0) = 0 \). Since \( a''(0) = 1 \) for \( a(\tau) = \sinh(\tau) \), we see de Sitter space is a special case of this limit.

Therefore, assuming inflationary theory, there is no reason to believe the big bang is singular except as a coordinate singularity. Indeed in this paper we show that the big bang for \( k = -1 \) inflationary spacetimes (which are defined precisely in section 2.3) is just a coordinate singularity. The situation is analogous to how the \( r = 2m \) event horizon in Schwarzschild is just a coordinate singularity. We also show some physically interesting cosmological properties of these spacetimes which we list below.

**Cosmological properties of** \( k = -1 \) **inflationary FLRW spacetimes**

- They offer a new geometrical viewpoint on how they solve the horizon problem.
- The comoving observers all emanate from a single point \( O \) in the extension.
- The cosmological constant \( \Lambda \) appears as an initial condition.
- An era of slow-roll inflation follows if the initial condition of the potential \( V(\phi) \) is determined by \( \Lambda \).
- The Lorentz group acts on these spacetimes by isometries.

\(^1\)See [2] for a cosmological singularity theorem without the strong energy condition.
2 The Coordinate Singularity

The $k = -1$ FLRW metric is

$$g = -dt^2 + a^2(\tau)\left[d\tau^2 + \sinh^2(R)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

(2.3)

We will show that the big bang, $\tau = 0$, is a coordinate singularity for scale factors which obey an inflationary condition. But first we will demonstrate how the big bang is a coordinate singularity in two familiar examples: the Milne universe and de Sitter space.

Definition 2.1. For an FLRW spacetime, we say $\tau = 0$ is a coordinate singularity if there exist coordinates which can extend the FLRW spacetime into a larger spacetime manifold (i.e. a proper isometric embedding) where the metric at $\tau = 0$ is no longer degenerate.

2.1 The Milne Universe

The Milne universe is a $k = -1$ FLRW spacetime with $a(\tau) = \tau$. The metric is

$$g = -d\tau^2 + \tau^2\left[dR^2 + \sinh^2(R)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

(2.4)

We introduce new coordinates $(t, r)$ via

$$t = \tau \cosh(R) \quad \text{and} \quad r = \tau \sinh(R).$$

(2.5)

Then we have $-dt^2 + dr^2 = -d\tau^2 + \tau^2dR^2$, so that the metric in coordinates $(t, r, \theta, \phi)$ is

$$g = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

(2.6)

which is just the usual Minkowski metric. The constant $\tau$ slices are hyperboloids sitting inside the future lightcone of the origin. As $\tau \to 0$, these slices approach the lightcone where the metric is nondegenerate. Therefore $\tau = 0$ is a coordinate singularity.

![Figure 1: The Milne universe sits inside the future lightcone of the origin $O$ in Minkowski space. It’s foliated by constant $\tau$ slices which are hyperboloids.](image)
2.2 De Sitter Space

De Sitter space in open slicing coordinates is a $k = -1$ FLRW spacetime with $a(\tau) = \sinh(\tau)$. The metric is

$$\begin{align*}
g &= -d\tau^2 + \sinh^2(\tau) \left[ dR^2 + \sinh^2(R) (d\theta^2 + \sin^2 \theta \, d\phi^2) \right].
\end{align*}$$

(2.7)

We introduce new coordinates $(t, r)$ via

$$\begin{align*}
t &= b(\tau) \cosh(R) \quad \text{and} \quad r = b(\tau) \sinh(R)
\end{align*}$$

(2.8)

where $b(\tau) = \tanh(\tau/2) = \sinh \tau/(1 + \cosh \tau)$. Then $b'(\tau) = b(\tau)/a(\tau)$, and so we have the following relationship between $(t, r)$ and $(\tau, R)$.

$$\begin{align*}
\left( \frac{a(\tau)}{b(\tau)} \right)^2 (- dt^2 + dr^2) &= -d\tau^2 + a^2(\tau)dR^2.
\end{align*}$$

(2.9)

Thus the metric is conformal to the Minkowski metric

$$\begin{align*}
g &= \left( \frac{a(\tau)}{b(\tau)} \right)^2 \left[ - dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \right].
\end{align*}$$

(2.10)

Using $b(\tau) = \tanh(\tau/2)$ and $b^2(\tau) = t^2 - r^2$, we have $\tau = 2 \tanh^{-1}(\sqrt{t^2 - r^2})$. Therefore $1/b'(\tau) = a(\tau)/b(\tau) = 2/(1 - t^2 + r^2)$. Thus the metric in $(t, r, \theta, \phi)$ coordinates is

$$\begin{align*}
g &= \left( \frac{2}{1 - t^2 + r^2} \right)^2 \left[ - dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \right].
\end{align*}$$

(2.11)

The constant $\tau$ slices are hyperboloids that sit inside a spacetime which is conformal to Minkowski space. As $\tau \to 0$, these slices approach the lightcone where the metric is non-degenerate. Therefore $\tau = 0$ is a coordinate singularity.

![Open slicing of de Sitter space](image)

Figure 2: The open slicing coordinates of de Sitter space sits inside the future lightcone of the origin $O$ in a spacetime conformal to Minkowski space.
2.3 Inflationary Spacetimes

Now we wish to perform the same extension but with a scale factor \( a(\tau) \) that can model the dynamics of our universe. That is, we wish to show \( \tau = 0 \) is a coordinate singularity for suitably chosen scale factors \( a(\tau) \) which

- begin inflationary \( a(\tau) \sim \sinh(\tau) \)
- then transitions to a radiation dominated era \( a(\tau) \sim \sqrt{\tau} \)
- then transitions to a matter dominated era \( a(\tau) \sim \tau^{2/3} \)
- and ends in a dark energy dominated era \( a(\tau) \sim e^{\Lambda \tau} \)

If we assume for small \( \tau \), the scale factor satisfies \( a(\tau) \sim \tau \), then, by curve fitting, we can use \( a(\tau) \) to represent each of the above eras, thus modeling the dynamics of our universe. To make this precise, we assume for small \( \tau \), the scale factor satisfies \( a(\tau) = \tau + o(\tau^{1+\varepsilon}) \) for some \( \varepsilon > 0 \) (i.e. \( [a(\tau) - \tau] / \tau^{1+\varepsilon} \to 0 \) as \( \tau \to 0 \)). In particular any Taylor expansion \( a(\tau) = \sum_{n=1}^{\infty} c_n \tau^n \) (with \( c_1 = 1 \)) will satisfy this condition.

**Definition 2.2.** By an inflationary FLRW spacetime, we mean one where the scale factor for small \( \tau \) satisfies \( a(\tau) = \tau + o(\tau^{1+\varepsilon}) \) for some \( \varepsilon > 0 \).

The main motivation for this definition comes in the next section where we show that these FLRW spacetimes solve the horizon problem. The next theorem improves and refines Theorem 3.4 in [1].

**Theorem 2.3.** The big bang is a coordinate singularity in \( k = -1 \) inflationary spacetimes.

**Proof.** The metric is

\[
g = -d\tau^2 + a^2(\tau) [dR^2 + \sinh^2(R)(d\theta^2 + \sin^2 \theta d\phi^2)].
\] (2.12)

Fix any \( \tau_0 > 0 \). The specific choice does not matter; any \( \tau_0 \) will do. Define new coordinates \((t, r, \theta, \phi)\) by

\[
t = b(\tau) \cosh(R) \quad \text{and} \quad r = b(\tau) \sinh(R)
\] (2.13)

where, for \( \tau > 0 \),

\[
b(\tau) = \exp \left( \int_{\tau_0}^{\tau} \frac{1}{a(s)} ds \right).
\] (2.14)

Note that \( b(\tau) \) is an increasing function and hence it’s invertible, and so \( \tau \) as a function of \( t \) and \( r \) is given by

\[
\tau = b^{-1}(\sqrt{t^2 - r^2}).
\] (2.15)
With respect to these coordinates, the metric takes the form
\[ g = \Omega^2(\tau(t,r)) \left[ -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \] (2.16)

where
\[ \Omega(\tau) = \frac{1}{b'(\tau)} = \frac{a(\tau)}{b(\tau)}. \] (2.17)

Just like with the open slicing coordinates of de Sitter space, we see that these inflationary \( k = -1 \) FLRW spacetimes are sitting inside a spacetime conformal to Minkowski space. Now we prove that \( \tau = 0 \) is a coordinate singularity. For this it suffices to show \( \Omega(0) := \lim_{\tau \to 0} \Omega(\tau) \) exists and is a finite positive number. Indeed this will imply the Lorentzian metric given by equation (2.16) extends continuously through \( \tau = 0 \) which corresponds to the lightcone \( t = r \).

To show \( 0 < \Omega(0) < \infty \), put \( b'(0) = \lim_{\tau \to 0} b'(\tau) = \lim_{\tau \to 0} b(\tau)/a(\tau) \). By our definition of an inflationary spacetime, there is an \( \alpha > 0 \) such that \( a(\tau) = \tau + o(\tau^{1+\alpha}) \). Therefore \( \lim_{\tau \to 0} f(\tau)/\tau^{1+\alpha} = 0 \) where \( f(\tau) \) is given by \( a(\tau) = \tau + f(\tau) \). Therefore for any \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( 0 < \tau < \delta \), we have \( f(\tau) < \varepsilon \tau^{1+\alpha} \). Hence \( \tau - \varepsilon \tau^{1+\alpha} < \tau + f(\tau) < \tau + \varepsilon \tau^{1+\alpha} \). Thus \( b(\tau)/a(\tau) \) is squeezed between
\[ \frac{1}{a(\tau)} \exp \left( - \int_\tau^{\tau_0} \frac{1}{(\tau - \varepsilon \tau^{1+\alpha})} \, ds \right) < \frac{b(\tau)}{a(\tau)} < \frac{1}{a(\tau)} \exp \left( - \int_\tau^{\tau_0} \frac{1}{(\tau + \varepsilon \tau^{1+\alpha})} \, ds \right) \] (2.18)

Evaluating the integrals we find
\[ \frac{1}{\tau_0} \left( \frac{\tau}{a(\tau)} \right) \left( \frac{\tau - \varepsilon \tau^{\alpha}}{\tau + \varepsilon \tau^{\alpha}} \right)^{-1/\alpha} < \frac{b(\tau)}{a(\tau)} < \frac{1}{\tau_0} \left( \frac{\tau}{a(\tau)} \right) \left( \frac{\tau + \varepsilon \tau^{\alpha}}{\tau + \varepsilon \tau^{\alpha}} \right)^{-1/\alpha} \] (2.19)

Since this holds for all \( 0 < \tau < \delta \), we have \( \Omega(0) = 1/b'(0) = \tau_0 \).

\[ \Box \]

Figure 3: A \( k = -1 \) inflationary FLRW spacetime sitting inside conformal Minkowski space. The slices of constant \( \tau \) are hyperboloids which foliate the future lightcone of the origin \( \mathcal{O} \).

Remark. The \( k = -1 \) FLRW spacetimes inherit a past Cauchy horizon given by the future lightcone at \( \mathcal{O} \) in the larger spacetime conformal to Minkowski space. Therefore what lies
beneath this lightcone is only speculation. However, just like with Schwarzschild, when $\Omega$ is analytic, then one can consider the maximal analytic extension. For $a(\tau) = \tau$ the maximal analytic extension is Minkowski space. For $a(\tau) = \sinh(\tau)$, the maximal analytic extension is de Sitter space.

3 Cosmological Properties

3.1 The solution to the horizon problem

Our definition for an inflationary FLRW spacetime was one whose scale factor satisfies $a(\tau) = \tau + o(\tau^{1+\varepsilon})$ for some $\varepsilon > 0$. Our motivation is that these spacetimes solve the horizon problem, and this is true for $k = +1, 0, \text{or} -1$. However, what’s unique about the $k = -1$ case is that it extends into a larger spacetime because the big bang is just a coordinate singularity. This offers a new picture of how the $k = -1$ inflationary spacetimes solve the horizon problem as we discuss below.

We briefly recall the horizon problem in cosmology. It is the main motivating reason for inflationary theory \cite{6}. The problem comes from the uniform temperature of the CMB radiation. From any direction in the sky, we observe that the CMB temperature is 2.7 K. The uniformity of this temperature is puzzling: if we assume the universe exists in a radiation dominated era all the way down to the big bang (i.e. no inflation), then the points $p$ and $q$ on the surface of last scattering don’t have intersecting past lightcones. So how can the CMB temperature be so uniform if $p$ and $q$ were never in causal contact in the past?

By using conformal time $\tilde{\tau}$ given by $d\tilde{\tau} = d\tau/a(\tau)$, it is an elementary exercise to show that there is no horizon problem provided the particle horizon at the moment of last scattering is infinite:

$$\int_0^{\tau_{\text{decoupling}}} \frac{1}{a(\tau)} d\tau = \infty.$$ 

This condition widens the past lightcones of $p$ and $q$ so that they intersect before $\tau = 0$. 

Figure 4: The horizon problem. Without inflation the past lightcones of $p$ and $q$ never intersect. But then why does the Earth measure the same 2.7 K temperature from every direction?
\[ \tau = 0 \]

Proposition 3.1. *The particle horizon in an inflationary spacetime is infinite.*

*Proof.* From the definition of an inflationary spacetime, we have

\[ \lim_{\tau \to 0} \frac{a(\tau)}{\tau} = 1. \quad (3.21) \]

Therefore for any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( |a(\tau)/\tau - 1| < \varepsilon \) for all \( 0 < \tau < \delta \). Hence \( 1/a(\tau) > 1/(1 + \varepsilon)\tau \) for all \( 0 < \tau < \delta \). Then the particle horizon at the moment of last scattering is

\[ \int_{0}^{\tau_{\text{decoupling}}} \frac{1}{a(\tau)} d\tau \geq \int_{0}^{\delta} \frac{1}{a(\tau)} d\tau \geq \int_{0}^{\delta} \frac{1}{(1 + \varepsilon)\tau} d\tau = \infty \quad (3.22) \]

Thus the particle horizon is infinite. \( \square \)

In the \( k = -1 \) case, the origin \( \mathcal{O} \) plays a unique role. The lightcones of any two points must intersect above \( \mathcal{O} \). This follows from the metric being conformal to Minkowski space, \( g_{\mu\nu} = \Omega^2(\tau)\eta_{\mu\nu} \). As such the lightcones are given by 45 degree angles; see Figure 6, which, in a certain way, clarifies in the \( k = -1 \) case the situation depicted in Figure 5.

![Figure 5: Inflation solves the horizon problem by widening the past lightcones.](image)

![Figure 6: A \( k = -1 \) inflationary FLRW spacetime modeling our universe. The points \( p \) and \( q \) have past lightcones which intersect at some point above \( \mathcal{O} \).](image)
Also we observe that the comoving observers all emanate from the origin \( O \). Indeed a comoving observer \( \gamma(\tau) \) is specified by a point \( (R_0, \theta_0, \phi_0) \) on the hyperboloid.

\[
\gamma(\tau) = (\tau, R_0, \theta_0, \phi_0)
\]  

(3.23)

In the \((t, r, \theta, \phi)\) coordinates introduced in equation (2.13), the comoving observer is given by

\[
\gamma(\tau) = (t(\tau), r(\tau), \theta_0, \phi_0)
\]

(3.24)

where

\[
t(\tau) = b(\tau) \cosh(R_0) \quad \text{and} \quad r(\tau) = b(\tau) \sinh(R_0).
\]

(3.25)

Thus the relationship between \( t \) and \( r \) for \( \gamma \) is \( t = \coth(R_0)r \). Therefore for any comoving observer, we have \( t = Cr \) for some \( C > 1 \). Thus the comoving observers emanate from the origin.

Figure 7: The comoving observers in a \( k = -1 \) inflationary spacetime. They all emanate from the origin \( O \).

3.2 The cosmological constant appears as an initial condition

In this section we show how the cosmological constant \( \Lambda \) appears as an initial condition for \( k = -1 \) inflationary FLRW spacetimes. Moreover, an era of slow-roll inflation follows if the initial condition for the potential is determined by the cosmological constant.

Consider the Einstein equation with a cosmological constant

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}.
\]

(3.26)

Let \( u^\mu \) denote the four-velocity of the comoving observers and \( e^\mu \) be any unit spacelike orthogonal vector (its choice does not matter by isotropy). We define the energy density \( \rho \) and pressure function \( p \) in terms of the Einstein tensor

\[
\rho(\tau) = \frac{1}{8\pi} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) u^\mu u^\nu = T_{\mu\nu} u^\mu u^\nu + \frac{\Lambda}{8\pi}
\]

(3.27)

\[
p(\tau) = \frac{1}{8\pi} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) e^\mu e^\nu = T_{\mu\nu} e^\mu e^\nu - \frac{\Lambda}{8\pi}
\]

(3.28)
If $T_{\mu\nu} = 0$ (e.g. de Sitter), then the equation of state for the cosmological constant is fixed for all $\tau$.

$$\rho = -p = \frac{\Lambda}{8\pi}. \tag{3.29}$$

We show that this equation of state appears as an initial condition for $k = -1$ inflationary FLRW spacetimes.

**Theorem 3.2.** Consider a $k = -1$ inflationary FLRW spacetime. If $a''(0) = 0$, then

$$\rho(0) = -p(0) = \frac{3}{8\pi} a'''(0).$$

Before proving Theorem 3.2 we first understand its implications. If the cosmological constant $\Lambda$ is the dominant energy source during the Planck era, then we have the following connection between the scale factor and $\Lambda$.

**Proposition 3.3.** If $T_{\mu\nu} \to 0$ as $\tau \to 0$ and $a''(0) = 0$, then

$$\Lambda = 3a'''(0).$$

*Proof.* This follows from Theorem 3.2 and equation (3.27). \qed

**Remark.** In $(3+1)$-dimensional de Sitter space we have $T_{\mu\nu} = 0$ and $\Lambda = 3$. In the open slicing coordinates of de Sitter, we have $a(\tau) = \sinh(\tau)$. Hence $a'''(0) = 1$. Therefore de Sitter space is a special example of Proposition 3.3.

Now we examine how an inflaton scalar field behaves in the limit $\tau \to 0$. We will demonstrate that slow-roll inflation follows if the initial condition for the potential is given by the cosmological constant: $V|_{\tau=0} = \Lambda/8\pi$. Recall the energy-momentum tensor for a scalar field $\phi$ is

$$T^\phi_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} (\nabla_\sigma \phi \nabla_\sigma \phi + V(\phi)) g_{\mu\nu}. \tag{3.30}$$

And its energy density is

$$\rho_\phi(\tau) = \frac{1}{2} \dot{\phi}'(\tau)^2 + V(\phi(\tau)). \tag{3.31}$$

**Proposition 3.4.** If $T_{\mu\nu} \to T^\phi_{\mu\nu} \to 0$ as $\tau \to 0$ and $a''(0) = 0$, then the initial condition $V(\phi(0)) = \Lambda/8\pi$ implies $\phi'(0) = 0$. Hence it yields an era of slow-roll inflation.

*Proof.* Since $T_{\mu\nu} \to T^\phi_{\mu\nu}$ as $\tau \to 0$, Theorem 3.2 implies $\rho_\phi(0) = (3/8\pi)a'''(0)$. Since $T^\phi_{\mu\nu} \to 0$ as $\tau \to 0$, Proposition 3.3 implies $\rho_\phi(0) = \Lambda/8\pi$. Thus the initial condition $V(\phi(0)) = \Lambda/8\pi$ implies $\phi'(0) = 0$. \qed
Put \( \rho(0) = \lim_{\tau \to 0} \rho(\tau) \) and likewise for \( p(0) \). Friedmann’s equations are \( (8\pi/3)\rho = H^2 + k/a^2 \) and \( 8\pi p = -2a''/a + (8\pi/3)\rho \) where \( H = a'/a \) is the Hubble parameter. Using \( k = -1 \) and \( a(\tau) = \tau + f(\tau) \), the Friedmann equations become

\[
\frac{8\pi}{3} \rho(\tau) = \left( \frac{a'(\tau)}{a(\tau)} \right)^2 - \frac{1}{a(\tau)^2} \frac{2f'(\tau) + f'(\tau)^2}{[\tau + f(\tau)]} \left( \frac{f'(\tau)/\tau}{1 + f(\tau)/\tau} \right)^2. \tag{3.32}
\]

and

\[
-8\pi p(\tau) = 2a''(\tau)/a + \frac{8\pi}{3} \rho(\tau) = \frac{2f''(\tau)/\tau}{1 + f(\tau)/\tau} + \frac{8\pi}{3} \rho(\tau). \tag{3.33}
\]

By definition of an inflationary spacetime, we have \( f'(0) := \lim_{\tau \to 0} f(\tau)/\tau = 0 \). Since \( 0 = a''(0) = f''(0) = \lim_{\tau \to 0} f'(\tau)/\tau \) and \( \alpha := a''(0) = \lim_{\tau \to 0} f''(\tau)/\tau \), for all \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( |f''(\tau)/\tau - \alpha| < \varepsilon \) for all \( 0 < \tau < \delta \). Integrating this expression gives \( (\alpha - \varepsilon)\tau/2 < f'(\tau)/\tau < (\alpha + \varepsilon)\tau/2 \). Plugging this into the first Friedmann equation yields \( 8\pi\rho(0)/3 = \alpha \). Using this for the second Friedmann equation yields \(-8\pi\rho(0) = 3\alpha\). \( \square \)

### 3.3 Elements in the Lorentz group act by isometries

Let \( \eta_{\mu\nu} \) be the Minkowski metric. The Lorentz group is

\[
L = O(1, 3) = \{ \Lambda \mid \eta_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu \eta_{\alpha\beta} \}. \tag{3.34}
\]

A Lorentz transformation \( \Lambda \) shifts elements in Minkowski space, but it leaves the hyperboloids fixed.

![Figure 8: A Lorentz transformation \( \Lambda \) based at \( \Theta \) can only shift points \( p \) to other points \( q \) on the same \( \tau = \text{constant slice} \). For \( k = -1 \) inflationary spacetimes, \( \Omega \) is a function of \( \tau \), so \( \Omega(\tau) = \Omega(\Lambda\tau) \), e.g. in this figure we would have \( \Omega(p) = \Omega(q) \).

For a \( k = -1 \) inflationary spacetime, we have \( g_{\mu\nu} = \Omega^2(\tau)\eta_{\mu\nu} \) where \( \eta_{\mu\nu} \) is the usual Minkowski metric. Since a Lorentz transformation leaves hyperboloids invariant, we have

\[
\Omega(\tau) = \Omega(\Lambda\tau). \tag{3.35}
\]

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Recall the Lorentz group $L = O(1, 3)$ has four connected components $L^+_+, L^+_-, L^-_+, L^-_.$ The $\pm$ corresponds to $\det \Lambda = \pm 1$, the $\uparrow$ corresponds to $\Lambda^0_0 \geq 1$, and the $\downarrow$ corresponds to $\Lambda^0_0 \leq -1$. Since we are only interested in transformations which preserves $t > 0$, we only consider the subgroup $L^\uparrow = L^+_+ \cup L^+_-$.

**Theorem 3.5.** The group $L^\uparrow$ acts by isometries in $k = -1$ inflationary spacetimes.

In the case $\Omega$ is $C^2$ we can actually say more. Recall the causal future $J^+(\emptyset)$ is the union of the $k = -1$ inflationary spacetime with the future lightcone at $\emptyset$.

**Theorem 3.6.** If $\Omega$ is $C^2$ in $J^+(\emptyset)$, then $L^\uparrow$ is isomorphic to the group of isometries in $k = -1$ inflationary spacetimes which fix the origin $\emptyset$.

**Proofs of Theorems 3.5 and 3.6**

Let $\Lambda_{\nu}^\mu$ be an element of $L^\uparrow$. It produces a unique map, $x \mapsto \Lambda x$ via $x^\mu \mapsto \Lambda_{\nu}^\mu x^\nu$ where $x^\mu = (t, x, y, z)$ are the conformal Minkowski coordinates introduced in the proof of Theorem 2.3. Since our $k = -1$ inflationary spacetime is only defined for $t > 0$, we must restrict to Lorentz transformations $\Lambda \in L^\uparrow$. Consider a point $p$ in the spacetime and a tangent vector $X = X^\mu \partial_\mu$ at $p$. Then $\Lambda$ acts on $X$ by sending it to $d\Lambda(X) = \Lambda_{\nu}^\mu X^\nu \partial_\mu$ and $d\Lambda(X)$ at the point $\Lambda p$. Since our metric is $g_{\mu\nu} = \Omega^2(\tau) \eta_{\mu\nu}$ and $\Omega(\Lambda p) = \Omega(p)$, we have

$$
g_{\mu\nu}(d\Lambda X)^\mu (d\Lambda Y)^\nu = \Omega^2(p) \eta_{\mu\nu}(\Lambda_{\rho}^\mu \Lambda_{\sigma}^\nu \eta_{\rho\sigma}) = \Omega^2(p) \eta_{\alpha\beta} X^\alpha Y^\beta = g_{\alpha\beta} X^\alpha Y^\beta.
$$

Thus $\Lambda$ is an isometry. This proves Theorem 3.5.

Now we prove Theorem 3.6. By Theorem 3.5 we have $L^\uparrow$ is a subgroup, so it suffices to show it’s the whole group. Suppose $f$ is an isometry which fixes $\emptyset$. The differential map $df_{\emptyset}$ is a linear isometry on the tangent space at $\emptyset$. Therefore $df_{\emptyset}$ corresponds to an element of the Lorentz group, say $\Lambda_{\nu}^\mu$. It operates on vectors $X$ at $\emptyset$ via $df(X) = \Lambda_{\nu}^\mu X^\nu \partial_\mu$. Now we define the isometry $\bar{f}$ by $\bar{f}(x) = \Lambda_{\nu}^\mu x^\nu$. Consider the set

$$
A = \{ p \in J^+(\emptyset) \mid df_p = d\bar{f}_p \}.
$$

Note that if $df_p = d\bar{f}_p$, then $f(p) = \bar{f}(p)$. Hence it suffices to show $A = J^+(\emptyset)$. $A$ is nonempty since $\emptyset \in A$, and $A$ is closed because $df - d\bar{f}$ is continuous. So since $J^+(\emptyset)$ is connected, it suffices to show $A$ is open in the subspace topology. Let $p \in A$. Since $\Omega$ is $C^2$, there is a normal neighborhood $U$ about $p$. If $q \in U$, there is a vector $X$ at $p$ such that $\exp_p(X) = q$. Since isometries map geodesics to geodesics, they satisfy the property $f \circ \exp_p = \exp_{f(p)} \circ df_p$ for all points in $U$ (see page 91 of [B]). Therefore

$$
f(q) = f(\exp_p(X)) = \exp_{f(p)}(df_p X) = \exp_{f(p)}(d\bar{f}_p X) = \bar{f}(\exp_p(X)) = \bar{f}(q).
$$

Thus $f(q) = \bar{f}(q)$ for all $q \in U$; hence $df_q = d\bar{f}_q$ for all $q \in U$. Therefore $A$ is open. \qed
Remark. It would be interesting to understand the implications of the full Lorentz group acting by isometries.

3.4 Open problems

(1) Is $\tau = 0$ a coordinate singularity for $k = 1$ and $k = 0$ inflationary FLRW spacetimes? From [1] it is known that no extension can exist with spherical symmetry.

(2) Is $\tau = 0$ a coordinate singularity for $k = -1$ inflationary FLRW spacetimes with compact $\tau$-slices? The null expansion $\theta$ of the future lightcone in Minkowski space diverges as one approaches $\mathcal{O}$ along the cone. This suggests that in the compact case, the past boundary $\partial^- M$ (as defined in [1]) cannot be compact.

(3) To understand what can lie beyond $\tau = 0$, it is desired to understand the maximal analytic extension whenever $\Omega$ is analytic on $J^+(\mathcal{O})$. Minkowski space is the maximal analytic extension of the Milne universe. De Sitter space is the maximal analytic extension of the $k = -1$ FLRW spacetime with scale factor $a(\tau) = \sinh(\tau)$. Therefore we suggest

Conjecture: Let $(M, g)$ be a $k = -1$ inflationary FLRW spacetime with an analytic $\Omega$ on $J^+(\mathcal{O})$. If $(M, g)$ is asymptotically flat (i.e. admits a smooth null scri structure), then the maximal analytic extension contains a noncompact Cauchy surface. If $(M, g)$ is asymptotically de Sitter (i.e. admits a smooth spacelike scri structure), then the maximal analytic extension contains a compact Cauchy surface.

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References

[1] G. J. Galloway and E. Ling, Some remarks on the $C^0$-inextendibility of spacetimes, Annales Henri Poincaré 18 (2017), no. 10, 3427–3447.

[2] G. J. Galloway and E. Ling, Topology and singularities in cosmological spacetimes obeying the null energy condition, Communications in Mathematical Physics, (Online first, November, 2017).

[3] S. W. Hawking and G. F. R. Ellis, The large scale structure of space-time, Cambridge University Press, London-New York, 1973, Cambridge Monographs on Mathematical Physics, No. 1.

[4] B. O’Neill, Semi-Riemannian geometry, Pure and Applied Mathematics, vol. 103, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983.

[5] R. M. Wald, General relativity, University of Chicago Press, Chicago, IL, 1984.

[6] S. Weinberg, Cosmology, Oxford University Press, Oxford, 2008.