Algebraic surfaces determine analyticity of functions

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Abstract. Let $f : X \to \mathbb{R}$ be a function defined on a nonsingular real algebraic set $X$ of dimension at least 3. We prove that $f$ is an analytic (resp. a Nash) function whenever the restriction $f|_S$ is an analytic (resp. a Nash) function for every nonsingular algebraic surface $S \subset X$ whose each connected component is homeomorphic to the unit 2-sphere. Furthermore, the surfaces $S$ can be replaced by compact nonsingular algebraic curves in $X$, provided that $\dim X \geq 2$ and $f$ is of class $C^\infty$.

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By [3, Theorem 1], a real-valued function on a real analytic manifold of dimension at least 3 is analytic whenever all its restrictions to analytic submanifolds homeomorphic to the unit 2-sphere are analytic. In the present note, we prove a variant of this result for functions defined on nonsingular real algebraic sets. Henceforth, we abbreviate real analytic to analytic. Besides analytic functions, we also consider Nash functions. We refer to [1] for the general theory of the latter class of functions.

Unless explicitly stated otherwise, all subsets of $\mathbb{R}^n$ are endowed with the Euclidean topology, induced by the standard norm.

Recall that $M \subset \mathbb{R}^m$ is a Nash manifold if it is a semialgebraic subset and an analytic submanifold (in particular, $M$ is a closed subset of some open subset of $\mathbb{R}^m$). A function $f : M \to \mathbb{R}$ is called a Nash function if it is analytic and its graph is semialgebraic. Equivalently, $f$ is a Nash function if and only if it is analytic and there is a relation

$$\sum_{i=0}^{k} \varphi_i(x)f(x)^{k-i} = 0 \text{ for all } x \in M,$$
where \( k \geq 1 \) and the \( \varphi_i : \mathbb{R}^m \to \mathbb{R} \) are polynomial functions, with \( \varphi_0 \) not identically 0 on any connected component of \( M \). For this reason, Nash functions are called \textit{algebraic functions} in the older literature, see for example [5].

Let \( X \subset \mathbb{R}^m \) be an irreducible nonsingular algebraic set of dimension \( n \geq 0 \). We define \( d(X) \) to be the supremum of the number of points in the intersection \( X \cap L \), where \( L \) runs through the family of all affine \((m-n)\)-planes in \( \mathbb{R}^m \) that are transverse to \( X \). Clearly, \( d(X) \) is a positive integer, see [11, Theorem 11.5.3]. Given an integer \( k \) with \( 1 \leq k \leq n-1 \), we denote by \( \mathcal{F}_k(X) \) the collection of all nonsingular algebraic subsets \( Z \subset X \) having at most \( d(X) \) connected components, each of which is homeomorphic to the unit \( k \)-sphere \( S^k \). In what follows, we only make use of \( \mathcal{F}_k(X) \) with \( k = 1 \) and \( k = 2 \).

**Theorem 1.** Let \( f : X \to \mathbb{R} \) be a function defined on an irreducible nonsingular algebraic set \( X \subset \mathbb{R}^m \) of dimension \( n \geq 3 \). Assume that the restriction \( f|_S \) is an analytic (resp. a Nash) function for every algebraic surface \( S \in \mathcal{F}_2(X) \). Then \( f \) is an analytic (resp. a Nash) function.

The proof of Theorem 1 requires some preparation. Along the way, we establish results which are of independent interest. We emphasize that the function \( f \) in Theorem 1 is not assumed to be continuous.

Let \( \mathbb{B}^n \) be the open unit ball in \( \mathbb{R}^n \). For any integer \( k \) with \( 1 \leq k \leq n-1 \), we denote by \( \mathcal{E}_k(\mathbb{B}^n) \) the collection of all Euclidean \( k \)-spheres in \( \mathbb{B}^n \) passing through the origin, that is, all algebraic sets \( \Sigma^k \subset \mathbb{B}^n \) of the form

\[
\Sigma^k = \{ x \in \mathbb{B}^n : \|x - c\| = \|c\| \} \cap V,
\]

where \( c \in \mathbb{R}^n, 0 < \|c\| < \frac{1}{2} \), and \( V \subset \mathbb{R}^n \) is a vector subspace of dimension \( k+1 \). In our results only \( \mathcal{E}_k(\mathbb{B}^n) \) with \( k = 1 \) and \( k = 2 \) are relevant.

**Theorem 2.** Let \( f : \mathbb{B}^n \to \mathbb{R} \) be a function defined on the open unit ball \( \mathbb{B}^n \subset \mathbb{R}^n \). Assume that \( n \geq 3 \) and the restriction \( f|_{\Sigma^2} \) is an analytic (resp. a Nash) function for every Euclidean 2-sphere \( \Sigma^2 \in \mathcal{E}_2(\mathbb{B}^n) \). Then \( f \) is an analytic (resp. a Nash) function.

The analytic case in Theorem 2 is already settled in [3, Theorem 2]. It plays the key role in the proof of the Nash case.

The inversion

\[
\mu : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}, \quad \mu(x) = \frac{x}{\|x\|^2},
\]

is a biregular isomorphism. It maps \( \mathbb{B}^n \setminus \{0\} \) onto the complement \( \mathbb{R}^n \setminus \mathbb{B}^n \) of the closed unit ball \( \mathbb{B}^n \) and gives a one-to-one correspondence between the Euclidean \( k \)-spheres in \( \mathcal{E}_k(\mathbb{B}^n) \) and the affine \( k \)-planes contained in \( \mathbb{R}^n \setminus \mathbb{B}^n \).

**Proof of Theorem 2.** As indicated above, we may assume that \( f \) is an analytic function and the restriction \( f|_{\Sigma^2} \) is a Nash function for every Euclidean 2-sphere \( \Sigma^2 \in \mathcal{E}_2(\mathbb{B}^n) \). Hence the function

\[
g := f \circ (\mu|_{\mathbb{R}^n \setminus \mathbb{B}^n}) : \mathbb{R}^n \setminus \mathbb{B}^n \to \mathbb{R}
\]

is analytic and its restriction to any affine 2-plane contained in \( \mathbb{R}^n \setminus \mathbb{B}^n \) is a Nash function. Evidently, the restriction of \( g \) to any affine line contained in
\( \mathbb{R}^n \setminus \overline{B^n} \) is a Nash function. It follows from [2, Theorem 2.4] that \( g \) is a Nash function on \( \mathbb{R}^n \setminus \overline{B^n} \), and therefore so is \( f \) on \( \mathbb{B}^n \). \( \square \)

**Proposition 3.** Let \( X \subset \mathbb{R}^m \) be an irreducible nonsingular real algebraic set of dimension \( n \geq 1 \), and let \( p \) be a point in \( X \). Then there exists a linear map \( \lambda : \mathbb{R}^m \to \mathbb{R}^n \) for which the following hold:

(i) The restriction \( \lambda|_X : X \to \mathbb{R}^n \) is a proper map with finite fibers (some fibers may be empty).

(ii) The map \( \lambda|_X \) is transverse to \( \lambda(p) \).

**Proof.** Let \( Y := X - p \) be the translate of \( X \). For each linear map \( \alpha : \mathbb{R}^m \to \mathbb{R}^n \), the restriction \( \alpha|_Y : Y \to \mathbb{R}^n \) induces a homomorphism of the coordinate rings (=rings of polynomial functions)

\[
(\alpha|_Y)^* : A(\mathbb{R}^n) \to A(Y).
\]

Let \( L(m, n) \) be the space of all linear maps from \( \mathbb{R}^m \) to \( \mathbb{R}^n \), which we identify with the space \( M(n, m) \) of all \( n \)-by-\( m \) matrices with real entries. By a suitable version of the Noether normalization theorem, see [6, Theorem 13.3] and its proof, there is a nonempty Zariski open subset \( \Omega \subset L(m, n) \) such that for each \( \beta \in \Omega \), the homomorphism \( (\beta|_Y)^* \) is injective and the ring \( A(Y) \) is integral over \( A(\mathbb{R}^n) \cong \text{Im}(\beta|_Y)^* \) (equivalently, \( A(Y) \) is a finitely generated \( A(\mathbb{R}^n) \)-module). It follows that \( \beta \) is surjective and the restriction \( \beta|_Y \) is a proper map with finite fibers. Now we choose a linear map \( \gamma \in \Omega \) such that the derivative of \( \varphi := \gamma|_Y : Y \to \mathbb{R}^n \) at the origin \( 0 \in Y \) is an isomorphism. After a coordinate change, we may assume that \( \gamma \) is the canonical projection \( \mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n \).

For any constant \( \varepsilon > 0 \), we set

\[
M_\varepsilon := \{ t = (t_{ij}) \in M(n, m) : |t_{ij}| < \varepsilon \text{ for } 1 \leq i \leq n, 1 \leq j \leq m \}
\]

and consider the map \( \Phi : Y \times M_\varepsilon \to \mathbb{R}^n \) defined by

\[
\Phi(x, t) = (x_1 + \sum_{j=1}^{m} t_{1j}x_j, \ldots, x_n + \sum_{j=1}^{m} t_{nj}x_j)
\]

where \( x = (x_1, \ldots, x_m) \in Y \) and \( t = (t_{ij}) \in M_\varepsilon \). If \( \varepsilon \) is sufficiently small, then \( \Phi \) is a submersion since for each point \( x \neq 0 \), the restriction of \( \Phi \) to \( \{ x \} \times M_\varepsilon \) is a submersion, and \( \varphi \) is a submersion at the origin \( 0 \in Y \). Hence, according to the standard consequence of Sard’s theorem [7, p. 79, Theorem 2.7], the map \( \Phi_t : Y \to \mathbb{R}^n, \Phi_t(x) = \Phi(x, t) \), is transverse to the origin \( 0 \in \mathbb{R}^n \) for some \( t \in M_\varepsilon \). There is a linear map \( \lambda \in L(m, n) \) with \( \lambda|_Y = \Phi_t \). If \( \varepsilon \) is small, then \( \lambda \) belongs to \( \Omega \) and has the required properties. \( \square \)

**Proof of Theorem 1.** Let \( p \) be a point in \( X \) and let \( \lambda : \mathbb{R}^m \to \mathbb{R}^n \) be a linear map as in Proposition 3. We can choose a constant \( r > 0 \) such that

\[
(\lambda|_X)^{-1}(B(\lambda(p), r)) = U_1 \cup \cdots \cup U_l,
\]

where \( B(\lambda(p), r) \subset \mathbb{R}^n \) is the open ball centered at \( \lambda(p) \) with radius \( r \), the \( U_i \) are pairwise disjoint open subsets of \( X \), \( \lambda|_{U_i} : U_i \to B(\lambda(p), r) \) are Nash
isomorphisms, and \( p \in U_1 \). Clearly, \( l \leq d(X) \). Define the map \( \pi : X \to \mathbb{R}^n \) by
\[
\pi(x) = \frac{1}{r}(\lambda(x) - \lambda(p)) \text{ for } x \in X.
\]
Then
\[
\pi^{-1}(\mathbb{B}^n) = U_1 \cup \cdots \cup U_l
\]
and the restriction \( \pi|_{U_i} : U_i \to \mathbb{B}^n \) is a Nash isomorphism for \( i = 1, \ldots, l \).

If \( \Sigma^2 \in \mathcal{E}_2(\mathbb{B}^n) \), then \( S(\Sigma^2) := \pi^{-1}(\Sigma^2) \in \mathcal{F}_2(X) \). Assume that for every \( \Sigma^2 \in \mathcal{E}_2(\mathbb{B}^n) \), the restriction \( f|_{S(\Sigma^2)} \) is an analytic (resp. a Nash) function. Then, by Theorem 2, the composite \( f \circ (\pi|_{U_i})^{-1} : \mathbb{B}^n \to \mathbb{R} \) is an analytic (resp. a Nash) function. It follows that \( f \) is an analytic (resp. a Nash) function, the point \( p \in X \) being arbitrary. \( \square \)

Making use of local coordinate charts and applying Theorem 2, we immediately obtain the following.

**Theorem 4.** Let \( f : M \to \mathbb{R} \) be a function defined on an analytic (resp. a Nash) manifold \( M \) of dimension \( n \geq 3 \). Assume that the restriction \( f|_{N} \) is an analytic (resp. a Nash) function for every analytic (resp. Nash) submanifold \( N \subset M \) homeomorphic to \( \mathbb{S}^2 \). Then \( f \) is an analytic (resp. a Nash) function.

The analytic case in Theorem 4 is contained in [3, Theorem 1].

Replacing in Theorem 1 (resp. Theorem 2) \( \mathcal{F}_2(X) \) by \( \mathcal{F}_1(X) \) (resp. \( \mathcal{E}_2(\mathbb{B}^n) \) by \( \mathcal{E}_1(\mathbb{B}^n) \)), one would get a false statement.

**Counterexample 5.** Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be the function defined by
\[
f(x, y, z) = \begin{cases} 
x^8 + y(x^2 - y^3)^2 + z^4 & \text{for } (x, y, z) \neq (0, 0, 0), \\
x^{10} + (x^2 - y^3)^2 + z^4 & \text{for } (x, y, z) = (0, 0, 0).
\end{cases}
\]
Then the restriction of \( f \) is an analytic (resp. a Nash) function on each nonsingular analytic (resp. Nash) curve in \( \mathbb{R}^3 \), but \( f \) is not even continuous at \( (0, 0, 0) \).

To establish the first part of the assertion, it suffices to prove that for any nonsingular analytic curve \( C \subset \mathbb{R}^3 \) passing through \( (0, 0, 0) \), the restriction \( f|_{C} \) is analytic at \( (0, 0, 0) \). Since \( C \) is nonsingular, it has near \( (0, 0, 0) \) a local analytic parametrization
\[
x = x(t), \ y = y(t), \ z = z(t) \text{ for } t \text{ near } 0 \in \mathbb{R},
\]
where \( x(0) = y(0) = z(0) = 0 \), and at least one of the analytic functions \( x(t), y(t), z(t) \) has a zero of order 1 at \( t = 0 \). It is not hard to check that the function \( f(x(t), y(t), z(t)) \) is analytic for \( t \) near \( 0 \in \mathbb{R} \). Thus \( f|_{C} \) is analytic at \( (0, 0, 0) \), as required.

Clearly, the function \( f \) is not continuous at \( (0, 0, 0) \) since on the curve \( x^2 - y^3 = 0, z = 0 \), it is equal to \( \frac{1}{x^2} \) away from \( (0, 0, 0) \).

However, for \( \mathcal{C}^\infty \) functions, we have the following version of Theorem 1.

**Theorem 6.** Let \( f : X \to \mathbb{R} \) be a \( \mathcal{C}^\infty \) function defined on an irreducible nonsingular algebraic set \( X \subset \mathbb{R}^m \) of dimension \( n \geq 2 \). Assume that the restriction \( f|_{C} \) is an analytic (resp. a Nash) function for every algebraic curve \( C \in \mathcal{F}_1(X) \). Then \( f \) is an analytic (resp. a Nash) function.
Proof. We argue as in the proof of Theorem 1, substituting Theorem 7 below for Theorem 2.

Theorem 7. Let \( f: \mathbb{B}^n \to \mathbb{R} \) be a \( C^\infty \) function defined on the unit open ball \( \mathbb{B}^n \subset \mathbb{R}^n \). Assume that \( n \geq 2 \) and the restriction \( f|_{\Sigma^1} \) is an analytic (resp. a Nash) function for every Euclidean 1-sphere \( \Sigma^1 \in \mathcal{E}_1(\mathbb{B}^n) \). Then \( f \) is an analytic (resp. a Nash) function.

Proof. To begin with, we consider the analytic case, assuming that the restriction \( f|_{\Sigma^1} \) is an analytic function for every Euclidean 1-sphere \( \Sigma^1 \in \mathcal{E}_1(\mathbb{B}^n) \).

First we prove analyticity of \( f \) on the punctured unit ball \( \mathbb{B}^n \setminus \{0\} \). This is equivalent to proving analyticity of the function

\[
g := f \circ \mu|_{\mathbb{R}^n \setminus \mathbb{B}^n} : \mathbb{R}^n \setminus \mathbb{B}^n \to \mathbb{R}.
\]

The problem is local, so fix a point \( b \in \mathbb{R}^n \setminus \mathbb{B}^n \). Our goal is to show that \( g \) is analytic at \( b \). Evidently, \( g \) is of class \( C^\infty \) and its restriction to any affine line contained in \( \mathbb{R}^n \setminus \mathbb{B}^n \) is analytic. Let

\[
\mathcal{L} := \text{the set of all affine lines contained in } \mathbb{R}^n \setminus \mathbb{B}^n \text{ and passing through } b.
\]

The union \( C \) of all lines in \( \mathcal{L} \) is a cone in \( \mathbb{R}^n \) and the set \( U := C \setminus \{b\} \) is open in \( \mathbb{R}^n \). Consider the series \( \Sigma_k P_k \) of homogeneous polynomials in \( n \) variables \( x = (x_1, \ldots, x_n) \), where

\[
P_k(x) = \frac{1}{k!} \sum_{|\alpha| = k} \frac{\partial^{|\alpha|} g}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(b)x_1^{\alpha_1} \cdots x_n^{\alpha_n},
\]

\( \alpha = (\alpha_1, \ldots, \alpha_n), |\alpha| = \alpha_1 + \cdots + \alpha_n. \)

By construction, for each vector line \( \Lambda \subset \mathbb{R}^n \) with \( b + \Lambda \in \mathcal{L} \), the series \( \Sigma_k P_k(x) \) converges to \( g(b + x) \) for all \( x \in \Lambda \) in a neighborhood of \( 0 \in \Lambda \). Hence, in view of Lemma 8 below, the series \( \Sigma_k P_k(x) \) converges to an analytic function \( \gamma: B \to \mathbb{R} \) defined on an open ball \( B \subset \mathbb{R}^n \) centered at the origin. Clearly, the function

\[
\gamma_b: B_b \to \mathbb{R}, \gamma_b(b + x) := \gamma(x) \text{ for } x \in B,
\]

is analytic on the ball \( B_b := b + B \) centered at \( b \). Since \( g = \gamma_b \) on \( B_b \cap L \) for all \( L \in \mathcal{L} \), it follows that

\[
g = \gamma_b \text{ on } B_b \cap C.
\]

We claim that \( g = \gamma_b \) in a neighborhood of \( b \) in \( \mathbb{R}^n \). Indeed, fix an affine line \( L_0 \in \mathcal{L} \) and let \( K \) be any affine line contained in \( \mathbb{R}^n \setminus \mathbb{B}^n \) that is parallel to \( L_0 \in \mathcal{L} \) and satisfies \( K \cap B_b \cap U \neq \emptyset \). The functions \( g \) and \( \gamma_b \) coincide on \( K \cap B_b \cap U \), hence the analytic functions \( g|_{K \cap B_b} \) and \( \gamma_b|_{K \cap B_b} \) are equal. The union of the sets \( K \cap B_b \), for all \( K \) as above, is a neighborhood of \( b \) in \( \mathbb{R}^n \). It follows that \( g \) and \( \gamma_b \) are equal in a neighborhood of \( b \) in \( \mathbb{R}^n \), as claimed. So \( f \) is an analytic function on \( \mathbb{B}^n \), except possibly at the origin.

To prove that \( f \) is analytic at the origin, we use the inversion

\[
\mu_a: \mathbb{R}^n \setminus \{a\} \to \mathbb{R}^n \setminus \{a\}, \mu_a(x) = \mu(x - a) + a,
\]
centered at a point \( a \in \mathbb{B}^n \setminus \{0\} \). Let \( U_a := \mu_a(\mathbb{B}^n \setminus \{a\}) \). The function 
\[
h := f \circ (\mu_a|_{U_a})^{-1} : U_a \to \mathbb{R}
\]
is of class \( C^\infty \), analytic except possibly at \( \mu_a(0) \), and its restriction is analytic on every affine line passing through \( \mu_a(0) \) and contained in \( U_a \). Arguing as above, we show that the Taylor series of \( h \) at \( \mu_a(0) \) converges to \( h \) in a neighborhood of \( \mu_a(0) \) in \( \mathbb{R}^n \). Thus \( f \) is analytic at the origin, and therefore everywhere on \( \mathbb{B}^n \).

In the Nash case, we proceed as in the proof of Theorem 2. \( \square \)

We have used the following result, see [4, Lemma 3] for the proof.

Lemma 8. Let \( \sum_k P_k \) be a series of real homogenous polynomials in \( n \) variables, \( \deg P_k = k \). Assume that there exists a nonempty open subset \( \Omega \subset S^{n-1} \) such that for every point \( a \in \Omega \), one can find a constant \( \rho_a > 0 \) such that the series \( \sum_k P_k(x) \) converges at \( x = \rho_a a \). Then there exist constants \( c > 0, r > 0 \) such that
\[
|P_k(z)| \leq \frac{c}{2^k} \text{ for } z \in \mathbb{C}^n, \|z\| \leq r, \ k \geq 0.
\]
In particular, the function \( z \mapsto \sum_k P_k(z) \) is holomorphic in the ball \( \|z\| < r, z \in \mathbb{C}^n \).

Working on local coordinate charts, we derive from Theorem 7 the following.

Theorem 9. Let \( f : M \to \mathbb{R} \) be a \( C^\infty \) function defined on an analytic (resp. a Nash) manifold \( M \) of dimension \( n \geq 2 \). Assume that the restriction \( f|_C \) is an analytic (resp. a Nash) function for every analytic (resp. Nash) submanifold \( C \subset M \) homeomorphic to \( S^1 \). Then \( f \) is an analytic (resp. a Nash) function.

One can compare Theorems 6 and 9 with the following example.

Example 10. The function \( f : \mathbb{R}^2 \to \mathbb{R} \) defined by
\[
f(x, y) = \begin{cases} \xy \exp \left(-\frac{1}{x^2+y^2}\right) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}
\]
is of class \( C^\infty \) and analytic with respect to each variable separately. However, \( f \) is not analytic as a function of two variables.
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