We discuss some applications of higher symmetry groups to condensed matter systems. We give special attention to the groups $SO(n)$ ($n = 4, 5, 6, 8$) in the two-dimensional Hubbard model and its generalizations, which model the high $T_c$ cuprate superconductors.

This talk is intended to be a brief introduction to the interesting roles played by higher symmetries in the theory of the cuprate high $T_c$ superconductors and other quasi-two-dimensional condensed matter systems. Since this is mainly a particle physics audience, I will give a quick introduction to the systems we are dealing with. A broad modern survey of superconductivity is given by Tinkham\(^1\), for example.

### 1 High $T_c$ superconductors

The typical cuprate superconductor has $CuO_2$ layers sandwiched between various layers, which serve as charge reservoirs to provide carriers (holes, in fact) to the conducting $CuO_2$ planes, as shown in fig.\(\text{fig.1}\). A typical charge reservoir is $La_{2-x}Sr_xO_2$, in which replacing $La^{3+}$ by $Sr^{2+}$ leads to a hole density $x$ in the conducting layers. High $T_c$ is typically obtained for doping fractions $x \sim 0.15 - 0.2$. Since superconductivity is mainly in the $CuO_2$ planes, it is useful to treat the superconducting system as two-dimensional, with effects due to coupling between the layers ignored at first.

### 2 Two-dimensional lattice

A realistic model Hamiltonian for the $CuO_2$ planes involves electrons (or holes, as is the case in practice) hopping between $p$-shell orbitals in $O$ and $d$-shell orbitals in $Cu$ together with interaction terms which model the Coulomb repulsion between two electrons on the same atom, or on adjacent $Cu$ and $O$ atoms. Detailed models are reviewed by Kresin, Morawitz and Wolf\(^2\). However, it has been suggested\(^3\) that much of the essential physics of the $CuO_2$ planes is captured by a simpler model in which holes only hop between $Cu$ sites; a more detailed exposition of this view is given in the recent work of Anderson\(^4\).
Figure 1: Schematic views of the cuprate high $T_c$ superconductors. The left figure (a) shows the conducting $CuO_2$ layers (white) between the cross-hatched layers which serve as charge reservoirs, as explained in the text. The right figure (b) shows a typical $CuO_2$ layer, with $Cu$ atoms (dark circles) at the vertices of a square lattice and $O$ atoms (open circles) at the midpoints of the lines joining adjacent $Cu$ atoms.

Thus we consider first the hopping Hamiltonian

$$H_0 = -t \sum_{\langle xy \rangle,\alpha} a_{x\alpha}^\dagger a_{y\alpha} - t' \sum_{\langle \langle xy \rangle \rangle,\alpha} a_{x\alpha}^\dagger a_{y\alpha}$$  \hspace{1cm} (1)$$

Here $\langle \rangle$ ($\langle \langle \rangle \rangle$) denotes sum over (next) nearest neighbor pairs, and $\alpha$ is a spin index. The spectrum of $H_0$ consists of a single energy band, with energies

$$E_k = -2t(\cos k_x + \cos k_y) - 4t' \cos k_x \cos k_y$$  \hspace{1cm} (2)$$
as $k$ runs over the first Brillouin zone, which is depicted in fig. 2(a). At the points indicated by large circles in the figure, we have $\nabla_k E = 0$, and hence a singularity in the density of states, the Van Hove singularity (VHS), which is logarithmic in two dimensions. One of us (RSM) has written an extensive review describing the importance of the VHS in high $T_c$ physics.

Another feature important in understanding this lattice is the existence of a nesting vector $Q$ such that $\exp(iQ \cdot x) = \pm 1$ for every lattice site $x$. This vector connects the Van Hove singularities, and it also joins points near the Fermi surface for electron (or hole) densities near one per site. This creates the possibility of various instabilities associated with wave vector $Q$, including antiferromagnetism, $d$-wave superconductivity, and others. Here the vector $Q = (\pm \pi, \pm \pi)$ when length is expressed in units of the lattice spacing.

To describe the interactions of the electrons, we can start with the Hubbard model interaction

$$V_{Hubbard} = U \sum_x n_{x\uparrow} n_{x\downarrow}$$  \hspace{1cm} (3)$$

which represents the Coulomb interaction between two electrons on the same site. Schulz has considered additional interaction terms.
3 Pairing algebras

Consider fermion creation and annihilation operators $a_i^\dagger, a_i$ ($i = 1, \ldots, N$). The $N^2$ number conserving operators $a_i^\dagger a_j$ generate an $SU(N) \oplus U(1)$ Lie algebra, and when we combine them with the $N(N-1)$ pairing operators

$$\Pi_{jk} \equiv a_j a_j^\dagger \quad \text{and} \quad \Pi_{jk}^\dagger \equiv a_j^\dagger a_j^\dagger$$

we obtain a \textit{pairing algebra} which generates a pairing group $SO(2N)$. Some elements of this algebra may be conserved; other elements $\mathcal{O}$ may satisfy a commutation relation of the form

$$[H, \mathcal{O}] = \lambda \mathcal{O}$$

with the Hamiltonian $H$. These elements define a \textit{spectrum generating algebra}, since for an operator $\mathcal{O}$ which satisfies (3),

$$H \psi = E \psi \Rightarrow H(\mathcal{O} \psi) = (E + \lambda)(\mathcal{O} \psi)$$

Thus the operator $\mathcal{O}$ transforms one energy eigenstate into another with eigenvalue different by $\lambda$. The first systematic study of these algebras in condensed matter physics appears to have been done by Birman and Solomon[3].

---

Figure 2: First Brillouin zone for the square lattice. (a) The dashed lines show the Fermi surface for the simplest nearest neighbor hopping Hamiltonian (1) with $t' = 0$ and one electron per site (half filling of the band). The large circles at $k = (0, \pm \pi), (\pm \pi, 0)$ give the location of the Van Hove singularities in the density of states (see text). (b) Division into two subzones $\mathcal{X}$ and $\mathcal{Y}$: $\mathcal{X}$ contains the unshaded region together with the line $k_x + k_y = 0$ and $\mathcal{Y}$ contains the shaded region and the dark line $k_x = k_y$. 
4 A simplified SO(8) algebra

A simplified SO(8) algebra can be constructed if we let $a^\dagger_\alpha, a_\beta$ be creation and annihilation operators for a fermion with momentum near $(\pm \pi, 0)$ ($\alpha, \beta$ are now spin indices), and $b^\dagger_\alpha, b_\beta$ the corresponding operators for momentum near $(0, \pm \pi)$. This choice is motivated by the existence of the Van Hove singularities in the density of states, which lead us to suspect that electrons (or holes) with momenta near these singularities will be most important (since there are more of them). As explained in the previous section, we obtain from these operators an $SU(4) \oplus U(1)$ algebra of number conserving operators and an SO(8) pairing algebra, some elements of which were used by Schulz to analyze pairing instabilities in generalized Hubbard models.

The operators

$$\vec{S} = \frac{1}{2} \left( a^\dagger_\alpha \vec{\sigma}_{\alpha\beta} a_\beta + b^\dagger_\alpha \vec{\sigma}_{\alpha\beta} b_\beta \right) , \quad \vec{A} = \frac{1}{2} \left( a^\dagger_\alpha \vec{\sigma}_{\alpha\beta} a_\beta - b^\dagger_\alpha \vec{\sigma}_{\alpha\beta} b_\beta \right)$$

and

$$Q = \frac{1}{2} (a^\dagger_\alpha a_\alpha + b^\dagger_\alpha b_\alpha - 1)$$

generate an SO(4) $\oplus$ U(1) algebra which is a symmetry of the nearest neighbor hopping Hamiltonian \(\overline{H}\) (with \(t' = 0\)). The remaining number conserving operators

$$\tau = \frac{1}{2} (a^\dagger_\alpha a_\alpha - b^\dagger_\alpha b_\alpha)$$

$$\mathcal{O}_{CDW} = \frac{1}{2} (a^\dagger_\alpha b_\alpha + b^\dagger_\alpha a_\alpha) , \quad \mathcal{O}_{JC} = \frac{1}{2t} (a^\dagger_\alpha b_\alpha - b^\dagger_\alpha a_\alpha)$$

$$\mathcal{O}_{SDW} = \frac{1}{2} (a^\dagger_\alpha \vec{\sigma}_{\alpha\beta} b_\beta + b^\dagger_\alpha \vec{\sigma}_{\alpha\beta} a_\beta) , \quad \mathcal{O}_{JS} = \frac{1}{2t} (a^\dagger_\alpha \vec{\sigma}_{\alpha\beta} b_\beta - b^\dagger_\alpha \vec{\sigma}_{\alpha\beta} a_\beta)$$

generate instabilities (charge density wave, spin density wave, etc.).

The pairing operators

$$\eta \equiv \bar{a} \alpha b_\alpha , \quad \overline{\eta} \equiv \bar{a} \alpha \vec{\sigma}_{\alpha\beta} b_\beta$$

[\(\bar{a}_\alpha \equiv a_\beta (i\sigma^y)_{\beta\alpha}\)] belong to the SO(6) spectrum generating algebra of the nearest neighbor Hamiltonian, and the additional pairing operators

$$\Delta_\alpha \equiv \bar{a} \alpha a_\alpha + \bar{b}_\alpha b_\alpha , \quad \Delta_d \equiv \bar{a} \alpha a_\alpha - \bar{b}_\alpha b_\alpha$$

in the SO(8) pairing algebra generate instabilities corresponding to s-wave and d-wave superconductivity, respectively.

4
5 The full SO(8) algebra

We now describe the full SO(8) algebra constructed using the nesting vector $Q$ introduced above. We start with the usual charge and spin operators

$$Q \equiv \sum_{\mathbf{x},\alpha} \left( a_{\mathbf{x}\alpha}^\dagger a_{\mathbf{x}\alpha} - \frac{1}{2} \right) = \sum_{\mathbf{k},\alpha} \left( \hat{c}_{\mathbf{k}\alpha}^\dagger \hat{c}_{\mathbf{k}\alpha} - \frac{1}{2} \right)$$

(14)

$$\vec{S} \equiv \frac{1}{2} \sum_{\mathbf{x},\alpha\beta} \hat{a}_{\mathbf{x}\alpha}^\dagger \vec{\sigma}_{\alpha\beta} \hat{a}_{\mathbf{x}\beta} = \frac{1}{2} \sum_{\mathbf{k},\alpha\beta} \hat{c}_{\mathbf{k}\alpha}^\dagger \vec{\sigma}_{\alpha\beta} \hat{c}_{\mathbf{k}\beta}$$

(15)

(we use $a^\dagger$, $a$ to denote coordinate space operators, $c^\dagger$, $c$ to denote momentum space operators). Now divide the Brillouin zone for the square lattice into two subzones $\mathcal{X}$ and $\mathcal{Y}$, as shown in fig. 2(b), and introduce the characteristic function (also used by Henley)

$$\tau_k \equiv \begin{cases} +1, & k \in \mathcal{X} \\ -1, & k \in \mathcal{Y} \end{cases}$$

(16)

The operators $T^\pm$, $T^3$ defined by

$$T^+ = \sum_{\mathbf{k} \in \mathcal{X}} \sum_{\alpha} \hat{c}_{\mathbf{k}\alpha}^\dagger \hat{c}_{\mathbf{k}+\mathbf{Q},\alpha} , \quad T^- = \sum_{\mathbf{k} \in \mathcal{Y}} \sum_{\alpha} \hat{c}_{\mathbf{k}\alpha}^\dagger \hat{c}_{\mathbf{k}+\mathbf{Q},\alpha}$$

(17)

$$T^3 = \frac{1}{2} \sum_{\mathbf{k},\alpha} \tau_k \hat{c}_{\mathbf{k}\alpha}^\dagger \hat{c}_{\mathbf{k}\alpha}$$

(18)

generate another $SU(2) \sim SO(3)$ algebra, which we call *isospin*; the 'flavor' here is simply the direction of the largest momentum component. Also define

$$T^1 \equiv \frac{1}{2} \left( T^+ + T^- \right) = \frac{1}{2} \sum_{\mathbf{k},\alpha} \hat{c}_{\mathbf{k}\alpha}^\dagger \hat{c}_{\mathbf{k}+\mathbf{Q},\alpha}$$

(19)

$$T^2 \equiv \frac{1}{2i} \left( T^+ - T^- \right) = \frac{1}{2i} \sum_{\mathbf{k},\alpha} \tau_k \hat{c}_{\mathbf{k}\alpha}^\dagger \hat{c}_{\mathbf{k}+\mathbf{Q},\alpha}$$

(20)

The spin $SU(2)$ and 'isospin' $SU(2)$ can be enlarged to $SU(4)$ in the manner of the supermultiplet models in nuclear theory by including the operators

$$\vec{T}^+ = \sum_{\mathbf{k} \in \mathcal{X}} \sum_{\alpha\beta} \hat{c}_{\mathbf{k}\alpha}^\dagger \vec{\sigma}_{\alpha\beta} \hat{c}_{\mathbf{k}+\mathbf{Q},\beta} , \quad \vec{T}^- = \sum_{\mathbf{k} \in \mathcal{Y}} \sum_{\alpha\beta} \hat{c}_{\mathbf{k}\alpha}^\dagger \vec{\sigma}_{\alpha\beta} \hat{c}_{\mathbf{k}+\mathbf{Q},\beta}$$

(21)

$$\vec{T}^3 = \frac{1}{2} \sum_{\mathbf{k},\alpha\beta} \tau_k \hat{c}_{\mathbf{k}\alpha}^\dagger \vec{\sigma}_{\alpha\beta} \hat{c}_{\mathbf{k}\beta}$$

(22)
and, analogous to \([13]\) and \([20]\),

\[
\begin{align*}
\vec{T}^1 &= \frac{1}{2} \left( \vec{T}^+ + \vec{T}^- \right), \\
\vec{T}^2 &= \frac{1}{2i} \left( \vec{T}^+ - \vec{T}^- \right)
\end{align*}
\]  \tag{23}

The number conserving \(SU(2) \oplus SU(2) \sim SO(4)\) algebra generated by the operators \(\vec{S}\) and \(\vec{T}^3\) commutes with the free particle Hamiltonian \(H_0\) of eq. \([1]\) [the commuting \(SU(2)\) generators are actually \(\vec{S}_\pm = \vec{S} \pm \vec{T}^3\), which are the spin generators restricted to the subzones \(X(+)\) and \(Y(-)\)]. The operators \(Q, T^3\) also commute with \(H_0\).

We can also introduce pairing operators

\[
\Delta = \frac{1}{2} \sum_{k, \alpha} \bar{c}_{k\alpha} \sigma_{k\alpha}, \quad \Delta_\tau = \frac{1}{2} \sum_{k, \alpha} \tau_k \bar{c}_{k\alpha} \sigma_{k\alpha}
\]  \tag{24}

and extended pairing operators

\[
\Pi = \frac{1}{2} \sum_{k, \alpha} \bar{c}_{k+Q, \alpha} c_{k\alpha}, \quad \bar{\Pi} = \frac{1}{2} \sum_{k, \alpha, \beta} \tau_k \bar{c}_{k+Q, \alpha} \sigma_{\alpha\beta} c_{k\beta}
\]  \tag{25}

[here \(\bar{c}_{k\alpha} \equiv (i\sigma^2)_{\alpha\beta} c_{k\beta}\) is a ‘charge conjugate’ operator which has the same transformation properties as the creation operator for an antiparticle of momentum \(k\), spin state \(\alpha\)]. The \(\Delta\) and \(\Pi\) operators, together with their adjoints, augment the \(SU(4) \oplus U(1)\) algebra of number-conserving operators to form an \(SO(8)\) pairing algebra. The operators introduced here are equivalent to those introduced in the simplified version, as shown in table \([1]\).

The largest subalgebra of the \(SO(8)\) which commutes with the simplest hopping Hamiltonian \(H_0\) [eq. \([1]\) with \(t' = 0\)] is the \(SO(6) \oplus SO(2)\) algebra generated by \(Q, \vec{S}, \vec{T}^3, \Pi, \Pi^\dagger, \bar{\Pi}, \bar{\Pi}^\dagger\) and \(T^3\) [\(T^3\) is the \(SO(2)\) generator]. Note that the subalgebra can also be described as \(SU(4) \oplus U(1)\), since \(SU(4) \sim SO(6)\) and \(U(1) \sim SO(2)\) (this equivalence of Lie algebras is sometimes overlooked in the literature). This \(SO(6) \oplus SO(2)\) provides a starting point for analyzing collective instabilities.

Table 1: Translation table for elements of \(SO(8)\) in the simplified form of sec. 4 and in the full form of sec. 5. The operators \(Q, \vec{S}\) and \(\Pi\) are common to both presentations.

| Full \(SO(8)\) | \(T^1\) | \(T^2\) | \(T^3\) | \(\vec{T}^1\) | \(\vec{T}^2\) | \(\vec{T}^3\) | \(\Delta\) | \(\Delta_\tau\) | \(\Pi\) |
|---------------|--------|--------|--------|------------|------------|------------|--------|--------|--------|
| Simple \(SO(8)\) | \(\sigma_{CDW}\) | \(\sigma_{JC}\) | \(\tau\) | \(\sigma_{SDW}\) | \(\sigma_{JS}\) | \(\Lambda\) | \(\Delta_s\) | \(\Delta_d\) | \(\eta\) |
6 Symmetries and instabilities

The generators of $SO(8)$ not in the $SO(6) \oplus SO(2)$ subalgebra can be grouped into two 6-dimensional superspins:

$$\Sigma^\pm \equiv \left( T^\pm, T^\mp, \Delta \mp \Delta, \Delta^\dagger \mp \Delta^\dagger \right)$$

which are eigenstates of the $SO(2)$ generator $T^3$ with eigenvalues $\pm 1$. Generalized pairing operators of the form

$$\Delta[\xi] \equiv \frac{1}{2} \sum_{k,\alpha} \xi(k)\bar{c}_k\alpha c_k$$

which generate BCS-like states can be introduced. These can also be grouped into multiplets which transform like vectors under the $SO(6)$ group, and define instabilities which will compete when interactions which break the $SO(6)$ symmetry are included.

Zhang\(^{11}\) has considered the $SO(5)$ algebra obtained from $SO(6)$ by deleting the generators $T^3$, $\Pi$, $\Pi^\dagger$. The $SO(3)$ ‘isospin’ algebra generated by $T^\pm$, $T^3$ commutes with Zhang’s $SO(5)$, and we are left with a triplet of 5-dimensional superspins:

$$Z^\pm \equiv \left( T^\pm, \Delta \mp \Delta, \Delta^\dagger \mp \Delta^\dagger \right), \quad Z^0 \equiv \left( T^3, \Pi, \Pi^\dagger \right)$$

These algebras are not exact symmetries of any ‘natural’ two-dimensional model (though an ad hoc model has been constructed\(^{12}\)). The only known fully two dimensional model with a higher symmetry is the Hubbard model, with nearest neighbor hopping and interaction given by eq. (3). This model has an $SO(4) \sim SU(2) \oplus SU(2)$ spectrum generating algebra\(^{13}\) consisting of the usual spin $SU(2)$ and a second $SU(2)$ generated by $\Pi$, $\Pi^\dagger$ and $Q$. There are other symmetries for two-site models of the type discussed in sec. 4, which will be discussed elsewhere.

7 Conclusions

We have described some higher symmetries and approximate symmetries which may be useful in classifying the phase structure of quasi-two-dimensional systems such as the cuprate high $T_c$ superconductors. Dynamical calculations are required for detailed description – a first step in this direction has been taken\(^{14}\), using methods developed originally by Balseiro and Falicov\(^{15}\).
Acknowledgments

MTV’s research is supported by the US Department of Energy under Grant #DE-FG02-85ER40233. Publication 749 of the Barnett Institute.

References

1. M Tinkham, Introduction to Superconductivity, 2nd ed. (McGraw-Hill, New York, 1996)
2. V Z Kresin, H Morawitz and S A Wolf, Mechanisms of Conventional and High $T_c$ Superconductivity (Oxford University Press, New York, 1993)
3. F C Zhang and T M Rice, Phys. Rev. B 37, 3759 (1988)
4. P W Anderson The Theory of Superconductivity in the High $T_c$ Cuprates (Princeton University Press, Princeton, 1997)
5. R S Markiewicz, J. Phys. Chem. Solids 58, 1179 (1997)
6. A Montorsi (ed.), The Hubbard Model: A Reprint Volume (World Scientific, Singapore, 1992)
7. H J Schulz, Phys. Rev. B 39, 2940 (1989)
8. C Henley, Phys. Rev. Lett. 80, 3590 (1998)
9. A I Solomon and J L Birman, J. Math. Phys. 28, 1526 (1987) and references therein.
10. R S Markiewicz and M T Vaughn, J. Phys. Chem. Solids 59, to be published; Phys. Rev. B 57, 14052 (1998) and unpublished.
11. S C Zhang, Science 275, 1089 (1997)
12. S Rabello, H Kohno, E Demler and S C Zhang, Phys. Rev. Lett. 80, 3586 (1998)
13. C N Yang and S C Zhang, Mod. Phys. Lett. B 4, 759 (1990); C N Yang, Phys. Rev. Lett. 63, 2144 (1989)
14. R S Markiewicz, C Kusko and M T Vaughn, cond-mat/9807067; R S Markiewicz, C Kusko and V Kidambi, cond-mat/9807068
15. C A Balseiro and L M Falicov, Phys. Rev. B 20, 4457 (1979)