A Robust Three-Level Time Split High-Order Leapfrog/Crank-Nicolson Scheme For Two-Dimensional Sobolev and Regularized Long Wave Equations Arising In Fluid Mechanics

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Abstract. This paper develops a robust three-level time split high-order Leapfrog/Crank-Nicolson technique for solving the two-dimensional unsteady sobolev and regularized long wave equations arising in fluid mechanics. A deep analysis of the stability and error estimates of the proposed approach is considered using the $L^{\infty}(0, T; H^2)$-norm. Under a suitable time step requirement, the theoretical studies indicate that the constructed numerical scheme is strongly stable (in the sense of $L^{\infty}(0, T; H^2)$-norm), temporal second-order accurate and convergence of order $O(h^3)$ in space, where $h$ denotes the grid step. This result suggests that the proposed algorithm is less time consuming, fast and more efficient than a broad range of numerical methods widely discussed in the literature for the considered problem. Numerical experiments confirm the theory and demonstrate the efficiency and utility of the three-level time split high-order formulation.

Keywords: Sobolev and regularized long wave equations, Leapfrog scheme, Crank-Nicolson method, three-level time-split high-order Leapfrog/Crank-Nicolson approach, stability analysis, error estimates.

AMS Subject Classification (MSC). 65M12, 65M06.

1 Introduction

The exploration of analytical solutions of the time-dependent partial differential equations (PDEs) plays a vital role in describing the behavior of different physical and biological phenomena arising in the areas of mathematical biology, fluid dynamics, engineering, chemical theory, bio-modeling and fluid mechanics. A large class of unsteady systems of PDEs have been used to model various problems in chemistry, physics, biology and engineering such as: chemical kinematics, fluid mechanics, electricity, nonstationary process in semiconductors in the presence of sources, propagation of wave and shallow water waves, optical fibers, flow of heat, plasma physics, immunology, quantum mechanics, sobolev and regularized long wave models and biology \cite{17, 13, 32, 33, 5, 31, 11, 15, 8, 16} and references therein. The evolutionary two-dimensional sobolev and regularized long wave problems usually arise in the flow of fluids to explaining the motion of wave in media. This model is associated with the Rossy and drift waves in rotating fluids and plasmas, respectively, and it describes a broad range of applications in different branches in engineering and science \cite{31}. Developing both exact and efficient numerical solutions for different types of sobolev and regularized long wave equations is an attractive area of research in applied science.

In this paper, we consider the two-dimensional sobolev and regularized long wave equations defined in \cite{31} as

\[ u_t - \alpha \Delta u_t - \gamma \Delta u + (\beta, \beta) \cdot \nabla u = f(x, y, t, u, u_x, u_y), \quad \text{on} \quad \Omega \times (0, T], \tag{1} \]
with initial condition
\[ u(x, y, 0) = u_0(x, y), \quad \text{on} \quad \Omega \times \partial \Omega, \tag{2} \]
and boundary condition
\[ u(x, y, t) = g(x, y, t) \quad \text{on} \quad \partial \Omega \times [0, T], \tag{3} \]
where \( f(x, y, t, u, u_x, u_y) = f_1(x, y, t, u, u_x) + f_2(x, y, t, u, u_y), \) and \( f_m, \) for \( m = 1, 2, \) are nonlinear functions. For the sake of stability analysis and error estimates, we assume that the functions \( f_m \) are locally Lipschitz with respect to the unknown \( u. \) \( \Delta \) and \( \nabla \) designate the Laplacian and gradient operators, respectively. \( u_z \) represents \( \frac{\partial u}{\partial z}, \) for \( z = x, y, t. \) \( \alpha, \beta \) and \( \gamma \) are nonnegative constants less than one, with \( \alpha \neq 0, u_0 \) and \( g \) denote the initial and boundary conditions, respectively. Equation (1) is a third-order mixed derivative in both time and space and it is referred to the Sobolev equations arising in flow of liquid through the theory of heat conduction, fissured rocks and non-study flow \([4, 34, 1]\). When \( \alpha = 0, \) equation (1) becomes a nonlinear convection-diffusion-reaction model which has been widely studied in the literature \([27, 26, 30, 25, 23, 19, 28, 22]\).

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For \( \alpha \neq 0, \) a large set of numerical methods have been developed in an approximate solution of the initial-boundary value problem \([1]-[3]\), such as: Galerkin finite element methods, split least-square plan, Runge Kutta method, conservative scheme, a computational approach, Lumped Galerkin procedure, etc.... For more details, we refer the readers to \([10, 11, 9, 12, 3, 2, 6, 7, 8]\) and references therein. For these methods, either the stability analysis or the error estimates has not been considered. In this work, we develop a three-level time split high-order Leapfrog/Crank-Nicolson approach for solving the partial differential equation (1) subjects to initial-boundary conditions (2)-(3). Under an appropriate time step limitation, the proposed formulation is strongly stable (in the sense of \( L^\infty(0, T; H^2) \)-norm), temporal second-order accurate and convergence in space with order \( \mathcal{O}(h^2) \) in the \( L^\infty(0, T; H^2) \)-norm, where \( h \) denotes the space step. This result suggests that the developed approach is faster and more efficient than a broad range of numerical schemes \([31, 11, 11, 10, 9, 6, 2, 8]\) widely studied in the literature for the considered problem \([1]-[3]\). Furthermore, the new three-level time split technique should be considered as a strong numerical technique for integrating a general system of nonlinear PDEs.

The highlights of the paper is the following items:

i) development of the three-level time split high-order Leapfrog/Crank-Nicolson scheme for solving the initial-boundary value problem \([1]-[3]\),

ii) stability analysis and error estimates of the proposed numerical approach,

iii) some numerical examples that confirm the theoretical studies.

The remainder of the paper is organized as follows. In Section 2, we construct the three-level time split high-order Leapfrog/Crank-Nicolson technique for solving the model problem \([1]-[3]\). Section 3 provides a deep analysis of the stability and error estimates of the new algorithm whereas some numerical evidences are considered in Section 4. Section 5 presents the general conclusions together with our future investigations.

2 Development of the three-level time split approach

In this section, we construct a three-level time split high-order Leapfrog/Crank-Nicolson scheme in a numerical solution of the two-dimensional time dependent Sobolev and regularized long wave equation (1) with initial condition (2) and boundary condition (3). The splitting method describes in this work is as follows: separate equation (1) into two distinct equations allows to write

\[ u_t - \alpha u_{txx} - \gamma u_{xx} + \beta u_x = f_1(x, y, t, u, u_x), \tag{4} \]
\[ u_t - \alpha u_{tyy} - \gamma u_{yy} + \beta u_y = f_2(x, y, t, u, u_y). \tag{5} \]

Let \( M \) and \( N \) be positive integers and \( L_l, \) for \( l = 1, 2, 3, 4, \) be four real numbers such that: \( L_1 < L_2 \) and \( L_3 < L_4. \) We set \( \Omega = (L_1, L_2) \times (L_3, L_4), \) be the region of fluid, \( k = \frac{T}{N}, \) \( h_x = \frac{L_2-L_1}{M}, \) and \( h_y = \frac{L_4-L_3}{M}, \) be the time step and mesh steps in the \( x \)-direction and \( y \)-direction, respectively. For the convenience of writing we set \( u_{ij} = u(x_i, y_j, t_n) \) and \( \hat{U}_{ij} = U(x_i, y_j, t_n), \) be the analytical solution and the approximate
one, respectively, at the discrete point \((x_i, y_j, t_n)\), where \(x_i = L_1 + i h_x\), \(y_j = L_3 + j h_y\), and \(t_n = n k\), for \(i, j = 0, 1, 2, \ldots, M\), and \(n = 0, 1, 2, \ldots, N\). In addition, suppose \(\Omega_k = \{t_n, n = 0, 1, 2, \ldots, N\}\), \(\Omega_{h_y} = \{x_i, i = 0, 1, \ldots, M\}\), and \(\partial \Omega_{h_y} = \partial \Omega_{h_y} \cap \Omega\), be regular partitions of domains: \([0, T]\), \(\Omega\), and \(\partial \Omega\), respectively. Thus, the space of mesh functions defined over the domain \(\Omega_{h_y} \times \Omega_k\) is given by \(\mathcal{V}_{h_y \times k}\). Furthermore, the value of the source term \(f_n(x, y, t, u, u_x, u_y, u_{xx}, u_{yy})\), for \(n = 1, 2, \ldots, M\), \(x, y \in (0, L)\), at the grid point \((x_i, y_j, t_n)\) is represented by \(f_n(x, y, t, u, u_x^n, u_y^n, u_{x,x}^n, u_{x,y}^n)\). Lastly, we consider the following linear operators

\[
\begin{align*}
\delta u_{ij}^n &= \frac{u_{ij}^{n+1} - u_{ij}^n}{\delta t}, \\
\delta^2 u_{ij}^n &= \frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{\delta t^2}, \\
\delta u_{i,j}^n &= \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\delta x}, \\
\delta^2 u_{i,j}^n &= \frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\delta x^2}, \\
\delta u_{i,j}^n &= \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\delta y}, \\
\delta^2 u_{i,j}^n &= \frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\delta y^2}.
\end{align*}
\]

Furthermore, we introduce the following discrete norms

\[
\|u^n\|^2 = \sum_{i,j} (u^n_{i,j})^2, \\
\|\delta u^n\|^2 = \sum_{i,j} \left(\delta x u^n_{i,j}\right)^2, \\
\|\delta^2 u^n\|^2 = \sum_{i,j} \left(\delta x^2 u^n_{i,j}\right)^2, \\
\|\delta u^n\|^2 = \sum_{i,j} \left(\delta y u^n_{i,j}\right)^2, \\
\|\delta^2 u^n\|^2 = \sum_{i,j} \left(\delta y^2 u^n_{i,j}\right)^2.
\]

In addition, the Hilbert space \(L^2(\Omega)\) is equipped with the scalar product \((\cdot, \cdot)_2\) defined as

\[
\begin{align*}
(u^n, v^n)_2 &= \int_{\Omega} u^n v^n \, dx \, dy, \\
(\delta x u^n, v^n)_2 &= \int_{\Omega} \delta x u^n v^n \, dx \, dy, \\
(\delta y u^n, v^n)_2 &= \int_{\Omega} \delta y u^n v^n \, dx \, dy, \\
(\delta x^2 u^n, v^n)_2 &= \int_{\Omega} \delta x^2 u^n v^n \, dx \, dy, \\
(\delta y^2 u^n, v^n)_2 &= \int_{\Omega} \delta y^2 u^n v^n \, dx \, dy.
\end{align*}
\]

In a similar manner, one defines the terms \((u^n, \delta u^n)_2\), \((u^n, \delta^2 u^n)_2\) and \((\delta u^n, \delta^2 u^n)_2\). The spaces \(L^2(\Omega), H^2(\Omega)\) and \(L^\infty(0, T; H^2)\) are equipped with the norms: \(\| \cdot \|_2\), \(\| \cdot \|_{H^2}\) and \(\| \|_{H^2, \infty}\), respectively, whereas the Hilbert space \(L^2(\Omega)\) is endowed with the scalar product \((\cdot, \cdot)_2\).

It is worth mentioning that a three-level time split Leapfrog/Crank-Nicolson technique splits the combined numerical scheme into a sequence of one-dimensional operators, thus providing a less time stability requirement. Specifically, the splitting should advance the solution in each direction with a maximum allowable time step. This suggests that the proposed scheme has to be more efficient than the time-split MacCormack procedure. For more details about the time-split MacCormack method, the readers can consult the works discussed in [20, 18, 24, 29].

Applying equation (4) at the grid point \((x_i, y_j, t_n)\) to obtain

\[
u^n_{i,j} - \alpha u^n_{x,x,i,j} - \gamma u^n_{x,y,i,j} + \beta u^n_{x,i,j} = f_1(x_i, y_j, t_n, u^n_{i,j}, u^n_{x,i,j}).
\]
Expanding the Taylor series for the function $u$ at the grid point $(x_i, y_j, t_n)$ using forward and backward difference representations to get

$$u^{n+\frac{1}{2}}_{ij} = u^n_{ij} + \frac{k}{2}u^n_{ij,x} + \frac{k^2}{8}u^n_{ij,xx} + O(k^3) \quad \text{and} \quad u^{n-\frac{1}{2}}_{ij} = u^n_{ij} - \frac{k}{2}u^n_{ij,x} + \frac{k^2}{8}u^n_{ij,xx} + O(k^3).$$

Subtracting the second equation from the first one, this gives

$$u^{n+\frac{1}{2}}_{ij} - u^{n-\frac{1}{2}}_{ij} = ku^n_{ij,x} + O(k^3).$$

This equation can be rewritten as

$$u^n_{ij} = \frac{1}{k}(u^{n+\frac{1}{2}}_{ij} - u^{n-\frac{1}{2}}_{ij}) + O(k^2).$$

Plugging this equation and (9), it is easy to see that

$$\frac{1}{k}\left(u^{n+\frac{1}{2}}_{ij} - u^{n-\frac{1}{2}}_{ij}\right) = \frac{\alpha}{k}\left(u^{n+\frac{1}{2}}_{x,ij} - u^{n-\frac{1}{2}}_{x,ij}\right) - \gamma u^n_{x,ij} + \beta u^n_{ij,x} = f_1(x_i, y_j, t_n, u^n_{ij}, u^n_{ij,x}) + O(k^2). \quad (10)$$

In [21], with the use of the Taylor series the author has established that

$$u^n_{x,ij} = \delta^n_{x}u^n_{ij} + O(h^3), \quad u^n_{ij} = \delta^n_{ij} + O(h^4), \quad (11)$$

where $z = x, y,$ and $q$ is any nonnegative rational number. For $z = x$, substituting equation (11) into (10) provides

$$\frac{1}{k}\left(u^{n+\frac{1}{2}}_{ij} - u^{n-\frac{1}{2}}_{ij}\right) = \frac{\alpha}{k}\left(\delta^n_{x}u^n_{ij,x} - \delta^n_{x}u^n_{x,ij}\right) - \gamma \delta^n_{x}u^n_{ij} + \beta \delta^n_{ij}u^n_{ij,x} = f_1(x_i, y_j, t_n, u^n_{ij}, \delta^n_{ij}u^n_{ij,x} + O(h^3)) + O(k^2 + k^{-1}h^4_x),$$

which is equivalent to

$$(\mathcal{I} - \delta^n_{x}u^n_{ij}) u^{n+\frac{1}{2}}_{ij} = (\mathcal{I} - \delta^n_{x}u^n_{ij}) u^{n-\frac{1}{2}}_{ij} + \gamma \delta^n_{x}u^n_{ij} - \beta \delta^n_{ij}u^n_{ij,x} + kf_1(x_i, y_j, t_n, u^n_{ij}, \delta^n_{ij}u^n_{ij,x} + O(h^3)) + O(k^3 + h^4_x)). \quad (12)$$

where $\mathcal{I}$ denotes the identity operator. In addition, the application of the Taylor expansion for the function $f_m$, $(m = 1, 2)$, about the discrete point $(x_i, y_j, t_n, u^n_{ij}, u^n_{ij,x})$ gives

$$f_m(x_i, y_j, t_n, u^n_{ij}, \delta^n_{ij}u^n_{ij,x} + O(h^3)) = f_m(x_i, y_j, t_n, u^n_{ij}, \delta^n_{ij}u^n_{ij,x} + O(h^3)). \quad (13)$$

Combining (13) and (12) yield

$$(\mathcal{I} - \delta^n_{x}u^n_{ij}) U^{n+\frac{1}{2}}_{ij} = (\mathcal{I} - \delta^n_{x}u^n_{ij}) U^{n-\frac{1}{2}}_{ij} + k \left(\gamma \delta^n_{x} - \beta \delta^n_{ij}\right) U^n_{ij} + kf_1(x_i, y_j, t_n, U^n_{ij}, \delta^n_{ij}U^n_{ij}). \quad (15)$$

We recall that the operators $\delta^n_{x}$ and $\delta^n_{ij}$ are defined in relation (6).

Applying equation (5) at the discrete points $(x_i, y_j, t_{n+\frac{1}{2}})$ and $(x_i, y_j, t_{n+1})$, we obtain

$$u^{n+\frac{1}{2}}_{ij} - \alpha u^n_{yij} - \gamma u^n_{yij,x} + \beta u^n_{ij,x} = f_2(x_i, y_j, t_n, u^n_{ij,\frac{1}{2}}), \quad (16)$$

$$u^{n+1}_{ij} - \alpha u^{n+1}_{yij} - \gamma u^{n+1}_{yij,x} + \beta u^{n+1}_{ij,x} = f_2(x_i, y_j, t_{n+1}, u^{n+1}_{ij,\frac{1}{2}}). \quad (17)$$

The application of the Taylor series for the function $u$ at the mesh point $(x_i, y_j, t_{n+\frac{1}{2}})$ with time step $k/2$ using forward and backward difference schemes results in

$$u^{n+1}_{ij} = u^n_{ij} + \frac{k}{2}u^n_{ij,x} + \frac{k^2}{8}u^n_{ij,xx} + O(k^3) \quad \text{and} \quad u^{n+\frac{1}{2}}_{ij} = u^n_{ij} - \frac{k}{2}u^n_{ij,x} + \frac{k^2}{8}u^n_{ij,xx} + O(k^3).$$

Subtracting the second equation from the first one and rearranging terms, this gives

$$u^{n+1}_{ij} - u^{n+\frac{1}{2}}_{ij} = \frac{k}{4}(u^n_{ij} + u^n_{ij,x}) + \frac{k^2}{16}(u^n_{ij,x} - u^n_{ij,xx}) + O(k^3). \quad (18)$$
Utilizing the Mean Value theorem, it holds \(u_{t,ij}^{n+\frac{1}{2}} - u_{t,ij}^{n}\frac{1}{2} = O(k)\). This fact combined with equation (18) provide

\[
u_{t,ij}^{n+\frac{1}{2}} - u_{t,ij}^{n}\frac{1}{2} = \frac{k}{4} \left( u_{t,ij}^{n+1} + u_{t,ij}^{n}\frac{1}{2} \right) + O(k^3).
\]

Solving this equation for \(u_{t,ij}^{n+\frac{1}{2}} + u_{t,ij}^{n}\frac{1}{2}\), we get

\[
u_{t,ij}^{n+\frac{1}{2}} + u_{t,ij}^{n}\frac{1}{2} = \frac{4}{k} \left( u_{t,ij}^{n+1} - u_{t,ij}^{n}\frac{1}{2} \right) + O(k^2).
\]

Plugging equations (16), (17) and (19), simple calculations yield

\[
\frac{4}{k} \left( u_{t,ij}^{n+1} - u_{t,ij}^{n}\frac{1}{2} \right) - \frac{4\alpha}{k} \left( u_{yy,ij}^{n+1} - u_{yy,ij}^{n}\frac{1}{2} \right) - \gamma \left( u_{t,ij}^{n+1} + u_{t,ij}^{n}\frac{1}{2} \right) = f_2(x, y, t_{n+1}, u_{t,ij}^{n+1}, u_{t,ij}^{n}).
\]

Utilizing equations (11) and (13), this becomes

\[
\frac{4}{k} \left( u_{t,ij}^{n+1} - u_{t,ij}^{n}\frac{1}{2} \right) + \frac{4\alpha}{k} \left( u_{yy,ij}^{n+1} - u_{yy,ij}^{n}\frac{1}{2} \right) = f_2(x, y, t_{n+1}, u_{t,ij}^{n+1}, u_{t,ij}^{n}).
\]

This is equivalent to

\[
(\mathcal{I} - \alpha \delta_{2y}^4) u_{t,ij}^{n+1} - \frac{k}{4} (\gamma \delta_{2y}^4 - \beta \delta_{2x}^4) u_{t,ij}^{n+\frac{1}{2}} = (\mathcal{I} - \alpha \delta_{2y}^4) u_{t,ij}^{n}\frac{1}{2} - \frac{k}{4} (\gamma \delta_{2y}^4 - \beta \delta_{2x}^4) u_{t,ij}^{n}\frac{1}{2} + \frac{k}{4} f_2(x, y, t_{n+1}, u_{t,ij}^{n+1}, \delta_y u_{t,ij}^{n+\frac{1}{2}}) + \frac{k}{4} f_2(x, y, t_{n+1}, u_{t,ij}^{n+\frac{1}{2}}, \delta_y u_{t,ij}^{n+\frac{1}{2}}).
\]

Tracking the error term \(O(k^3 + h_y^2 + h_x^2)\) into equation (20) and replacing the analytical solution \(u\) with the approximate one \(U\) to get the second-step of the developed technique, that is,

\[
(\mathcal{I} - \alpha \delta_{2y}^4) U_{t,ij}^{n+1} - \frac{k}{4} (\gamma \delta_{2y}^4 - \beta \delta_{2x}^4) U_{t,ij}^{n+\frac{1}{2}} = (\mathcal{I} - \alpha \delta_{2y}^4) U_{t,ij}^{n}\frac{1}{2} - \frac{k}{4} (\gamma \delta_{2y}^4 - \beta \delta_{2x}^4) U_{t,ij}^{n}\frac{1}{2} + \frac{k}{4} f_2(x, y, t_{n+1}, U_{t,ij}^{n+1}, \delta_y U_{t,ij}^{n+\frac{1}{2}}) + \frac{k}{4} f_2(x, y, t_{n+1}, U_{t,ij}^{n+\frac{1}{2}}, \delta_y U_{t,ij}^{n+\frac{1}{2}}).
\]

where the operators \(\delta_{2y}^4\) and \(\delta_{2x}^4\) are defined in relation (5).

Now, for \(n = 0\), expanding the Taylor series and using the Mean Value theorem, it is not hard to show that

\[
\left( u_{t,ij}^{0} + u_{t,ij}^{0}\frac{1}{2} \right) = \left( u_{t,ij}^{0} - u_{t,ij}^{0}\frac{1}{2} \right) + O(k^2).
\]

Substituting this equation into (9) and utilizing equations (11) and (13) result in

\[
\frac{1}{4} \left( \frac{1}{4} - \alpha \frac{1}{4} \right) u_{ij}^{0} - \gamma \frac{k}{4} \left( \frac{1}{4} + \frac{0}{4} \right) u_{ij}^{0} + \frac{\beta k}{4} \left( \frac{1}{4} + \frac{0}{4} \right) = \frac{k}{4} f_1(x, y, t, U_{ij}^{0}, \delta_x U_{ij}^{0}) + \frac{k}{4} f_1(x, y, t, U_{ij}^{0}, \delta_x U_{ij}^{0}) + O(k^3 + h_y^2 + kh_x^2).
\]

Truncating the error term \(O(k^3 + h_y^2 + kh_x^2)\), replacing the exact solution \(u\) with the numerical one \(U\) and rearranging terms, this gives,

\[
(\mathcal{I} - \alpha \delta_{2x}^4) U_{ij}^{0} - \frac{k}{4} (\gamma \delta_{2y}^4 - \beta \delta_{2x}^4) U_{ij}^{0} = (\mathcal{I} - \alpha \delta_{2x}^4) U_{ij}^{0} - \frac{k}{4} (\gamma \delta_{2y}^4 - \beta \delta_{2x}^4) U_{ij}^{0} + \frac{k}{4} f_1(x, y, t, U_{ij}^{0}, \delta_x U_{ij}^{0}),
\]

A combination of equations (23), (15) and (21) provides a new three-level time split high-order Leapfrog/Crank-Nicolson technique for solving the two-dimensional sobolev and regularized long wave equation (11) with initial-boundary conditions (2-8). That is, for \(i, j = 2, 3, ..., M - 2\),

\[
(\mathcal{I} - \alpha \delta_{2x}^4) U_{ij}^{0} - \frac{k}{4} (\gamma \delta_{2y}^4 - \beta \delta_{2x}^4) U_{ij}^{0} = (\mathcal{I} - \alpha \delta_{2x}^4) U_{ij}^{0} - \frac{k}{4} (\gamma \delta_{2y}^4 - \beta \delta_{2x}^4) U_{ij}^{0} + \frac{k}{4} f_1(x, y, t, U_{ij}^{0}, \delta_x U_{ij}^{0}),
\]

\[
\frac{1}{4} \left( \frac{1}{4} - \alpha \frac{1}{4} \right) u_{ij}^{0} - \gamma \frac{k}{4} \left( \frac{1}{4} + \frac{0}{4} \right) u_{ij}^{0} + \frac{\beta k}{4} \left( \frac{1}{4} + \frac{0}{4} \right) = \frac{k}{4} f_1(x, y, t, U_{ij}^{0}, \delta_x U_{ij}^{0}) + \frac{k}{4} f_1(x, y, t, U_{ij}^{0}, \delta_x U_{ij}^{0}) + O(k^3 + h_y^2 + kh_x^2).
\]
3 Stability analysis and error estimates of the proposed three-level approach

This section deals with the analysis of stability and error estimates of the constructed technique (24–28) for solving the initial-boundary value problem (11–13), under the time step restriction

\[
k = \min \left\{ h_x^{1/3}, h_y^{4/3} \right\}.
\]

(29)

For the sake of stability analysis and error estimates, we assume that the mesh spaces \(h_x\) and \(h_y\) are equal \((h := h_x = h_y)\) and the analytical solution "\(u\)" satisfies the following regularity condition: \(u \in L^\infty(0, T; H^2(\Omega))\). That is, there is a positive constant \(C_0\) independent of the time step \(k\) and the space step \(h\), so that

\[
\|u\|_{H^2, \infty} \leq \widehat{C}_0.
\]

(30)

where \(\| \cdot \|_{H^2, \infty}\) is the discrete norm defined in relation (7).

**Theorem 3.1. (Stability and error estimates).** Consider \(U\) to be the approximate solution obtained from the new algorithm (24–28) and let \(u\) be the analytical solution of the two-dimensional evolutional sobolev and regularized long wave equation (11) with initial and boundary conditions (12) and (13), respectively. Under the time step restriction (29), the computed solution \(U\) satisfies

\[
\|U\|_{H^2, \infty} \leq \widehat{C}_0 + \widehat{C}(k^2 + h^{8/3} + h^4).
\]

(31)

Setting: \(e^n = u^n - U^n\), \(e = u^n - U^n\) and \(e^{n+1} = u^{n+1} - U^{n+1}\), be the errors at the time levels \(n + \frac{1}{2}\) and \(n + 1\), respectively, it holds

\[
\|e\|_{H^2, \infty} \leq \widehat{C}(k^2 + h^{8/3}).
\]

(32)

where \(\widehat{C}_0 > 0\) is the constant given in estimate (30), \(\widehat{C}\) and \(\widehat{C}\) are two positive constant that do not dependent on the grid size \(h\) and time step \(k\).

The following Lemmas 3.1.3.2 plays an important role in the proof of the main result of this paper (namely, Theorem 3.1).

**Lemma 3.1.** Suppose \(w, v \in C_D^{0,0}\), be two functions defined over the domain \(D = \Omega \times [0, T]\) and satisfying:

\(w(x, y, t) = v(x, y, t) = w(x, y, t) = v(x, y, t) = 0\), for \(t \in \{0, 1, M - 1, M\}\), \(0 \leq i, j \leq M\), and any \(t \in [0, T]\). Let \(w(x, y, t) = w_{ij}(t)\) and \(v(x, y, t) = v_{ij}(t)\), then it holds

\[
\left(\delta_{iz} w(t), v(t)\right)_2 \leq C_3 \|\delta_z w(t)\|_2 \|\delta_z v(t)\|_2,
\]

(33)

\[
\left(\delta_{mz} w(t), v(t)\right)_2 \leq \frac{1}{2} \left[ \|\delta_z w(t)\|_2^2 + \|\delta_z v(t)\|_2^2 + \frac{h^2}{12} \left( \|\delta_z^2 w(t)\|_2^2 + \|\delta_z^2 v(t)\|_2^2 \right) \right],
\]

(34)

\[
\left(\delta_{iz}^2 w(t), v(t)\right)_2 = -\left(\delta_{iz}^2 v(t), w(t)\right)_2,
\]

(35)

for \(m = 1, 2\), and \(z = x, y\). \(C_3\) is a positive constant independent of the mesh size \(h\) and the time step \(k\) and where we set \(\delta_{1z}^2 = \delta_z^2\).
Proof. It follows from relation (36) that
\[ \delta^2_{ix} w_{ij}(t) = \frac{1}{12h^2} \left[ -w_{i-1,j}(t) + 16w_{i,j}(t) + 16w_{i+1,j}(t) - w_{i+2,j}(t) \right] = \]
\[ \frac{1}{12h} \left[ -(\delta_x w_{i-1,j}(t) - \delta_x w_{i-\frac{3}{2},j}(t)) + 14(\delta_x w_{i+\frac{1}{2},j}(t) - \delta_x w_{i+\frac{3}{2},j}(t)) - (\delta_x w_{i+\frac{5}{2},j}(t) - \delta_x w_{i+2,j}(t)) \right]. \]
(36)
Utilizing (36) and the definition of scalar product defined in (38), we obtain
\[ (\delta^2_{ix} w(t), v(t)) = h^2 \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta^2_{ix} w_{ij}(t)) v_{ij}(t) = \frac{h}{12} \left\{ -M-2 \sum_{i=2}^{M-2} \delta_x w_{i-\frac{1}{2},j}(t) - \delta_x w_{i+\frac{1}{2},j}(t) \right\} v_{ij}(t) + \]
\[ 14h \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i+\frac{1}{2},j}(t)) v_{ij}(t) = -14h^2 \sum_{i=1}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i+\frac{1}{2},j}(t)) (\delta_x v_{i+\frac{1}{2},j}(t)). \]
(37)
Utilizing the assumption: \( w_{ij}(t) = v_{ij}(t) = w_{ij}(t) = v_{ij}(t) = 0 \), for any \( i = 0, 1, M - 1, M; j = 0, 1, 2, \ldots, M \), and applying the summation by parts, straightforward computations give
\[ -h \sum_{i=1}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i-\frac{1}{2},j}(t)) v_{ij}(t) = h^2 \sum_{i=1}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i+\frac{1}{2},j}(t)) v_{ij}(t), \]
(38)
\[ 14h \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i+\frac{1}{2},j}(t)) v_{ij}(t) = -14h^2 \sum_{i=1}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i+\frac{1}{2},j}(t)) (\delta_x v_{i+\frac{1}{2},j}(t)). \]
(39)
Substituting equations (35) - (40) into (37) to get
\[ (\delta^2_{ix} w(t), v(t)) = h^2 \left\{ \sum_{i=1}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i-\frac{1}{2},j}(t)) (\delta_x v_{i+\frac{1}{2},j}(t)) + \sum_{i=1}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i+\frac{1}{2},j}(t)) (\delta_x v_{i+\frac{1}{2},j}(t)) \right\} \]
(41)
Using the discrete norm given by (7) together with the Cauchy-Schwarz inequality provide
\[ h^2 \sum_{i=1}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i-\frac{1}{2},j}(t)) (\delta_x v_{i+\frac{1}{2},j}(t)) \leq \left[ \sum_{i=1}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i-\frac{1}{2},j}(t))^2 \right] \left[ \sum_{i=1}^{M-2} \sum_{j=2}^{M-2} (\delta_x v_{i+\frac{1}{2},j}(t))^2 \right] \leq ||\delta_x w(t)||_2 ||\delta_x v(t)||_2. \]
(42)
Analogously
\[ -14h^2 \sum_{i=1}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i+\frac{1}{2},j}(t)) (\delta_x v_{i+\frac{1}{2},j}(t)) \leq 14||\delta_x w(t)||_2 ||\delta_x v(t)||_2, \]
(43)
\[ h^2 \sum_{i=1}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i+\frac{1}{2},j}(t)) (\delta_x v_{i+\frac{1}{2},j}(t)) \leq ||\delta_x w(t)||_2 ||\delta_x v(t)||_2. \]
(44)
Plugging estimates (42)-(43) and (11), this results in
\[ (\delta^2_{ix} w(t), v(t)) \leq \frac{4}{3} ||\delta_x w(t)||_2 ||\delta_x v(t)||_2. \]
(45)
This proves estimate (33), for \( z = x \) and \( m = 2 \). Furthermore, utilizing equation (11), it is easy to observe that
\[ (\delta^2_{ix} w(t), v(t)) = -h^2 \sum_{i=1}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i+\frac{1}{2},j}(t)) (\delta_x v_{i+\frac{1}{2},j}(t)) + \frac{h^2}{12} \sum_{i=1}^{M-2} \sum_{j=2}^{M-2} \left\{ (\delta_x w_{i-\frac{1}{2},j}(t)) - 2(\delta_x w_{i+\frac{1}{2},j}(t)) + (\delta_x w_{i+\frac{3}{2},j}(t)) \right\} (\delta_x v_{i+\frac{1}{2},j}(t)). \]
(46)
But \( \delta_x w_{i+\frac{1}{2},j}(t) = 2(\delta_x w_{i+\frac{1}{2},j}(t)) + \delta_x w_{i+\frac{3}{2},j}(t) = h(\delta^2_{ix} w_{i+1,j}(t) - \delta^2_{ix} w_{ij}(t)) \). This fact, together with (40) and the scalar product given in (38) yield
\[ (\delta^2_{ix} w(t), v(t)) = - (\delta_x w(t), \delta_x v(t)) + \frac{h^2}{12} \sum_{i=1}^{M-2} \sum_{j=2}^{M-2} (\delta^2_{ix} w_{i+1,j}(t) - \delta^2_{ix} w_{ij}(t)) (\delta_x v_{i+\frac{1}{2},j}(t)). \]

Applying the summation by part and utilizing the scalar product \((\cdot,\cdot)\) along with the assumption \(w_{ij}(t) = w_{il}(t) = 0 = v_{ij}(t) = v_{il}(t)\), for \(l \in \{0, 1, M - 1, M\}\), \(0 \leq i, j \leq M\), and for every \(t \in [0, T]\), to get

\[
(\delta^4_x w(t), v(t))_2 = - (\delta_x w(t), \delta_x v(t))_2 - \frac{h^2}{12} (\delta^2_x w(t), \delta^2_x v(t))_2.
\]  

(47)

The application of Cauchy-Schwarz inequality together with estimate \(2ab \leq a^2 + b^2\), for every real numbers \(a\) and \(b\), gives estimate \((48)\) for \(z = x\).

In a similar way, one easily establishes that

\[
(\delta^4_x w(t), v(t))_2 \leq \frac{4}{3} \|\delta_y w(t)\|_2 \|\delta_y v(t)\|_2,
\]  

(48)

and

\[
(\delta^4_x w(t), v(t))_2 \leq \frac{1}{2} \left[ \|\delta_y w(t)\|^2_2 + \|\delta_y v(t)\|^2_2 + \frac{h^2}{12} \left( \|\delta^2_y w(t)\|^2_2 + \|\delta^2_y v(t)\|^2_2 \right) \right].
\]  

(49)

Utilizing the definition of the linear operator \(\delta^4_x\) defined in relation \((40)\), it is not hard to see that

\[
\delta^4_x w_{ij}(t) = \frac{1}{12h^3} [-w_{i-2,j}(t) + 8w_{i-1,j}(t) - 8w_{i+1,j}(t) + w_{i+2,j}(t)] = \frac{1}{12} \left[ \delta_x w_{i-\frac{1}{2},j}(t) - 7(\delta_x w_{i+\frac{1}{2},j}(t) + \delta_x w_{i-\frac{3}{2},j}(t)) + \delta_x w_{i+\frac{3}{2},j}(t) \right].
\]

So

\[
(\delta^4_x w(t), v(t))_2 = \frac{h^2}{12} \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} \left[ \delta_x w_{i-\frac{1}{2},j}(t) - 7(\delta_x w_{i+\frac{1}{2},j}(t) + \delta_x w_{i-\frac{3}{2},j}(t)) + \delta_x w_{i+\frac{3}{2},j}(t) \right] v_{ij}(t).
\]  

(50)

Applying the Cauchy-Schwarz and Poincaré-Friedrich inequalities and using the the hypothesis: \(w_{ij}(t) = v_{ij}(t) = w_{il}(t) = v_{il}(t) = 0\), for any \(l = 0, 1, M - 1, M\); \(i, j = 0, 1, 2, ..., M\), it is not difficult to observe that

\[
\frac{h^2}{12} M^{-2M-2} \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} \left( \delta_x w_{i-\frac{1}{2},j}(t) v_{ij}(t) \right) \leq \frac{h^2}{12} \left( \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i-\frac{1}{2},j}(t))^2 \right)^{\frac{1}{2}} \left( \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (v_{ij}(t))^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{C_p}} \|\delta_x w(t)\|_2 \|\delta_y v(t)\|_2.
\]  

(51)

Analogously,

\[
\frac{h^2}{12} M^{-2M-2} \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i-\frac{1}{2},j}(t)) v_{ij}(t) \leq \frac{1}{\sqrt{C_p}} \|\delta_x w(t)\|_2 \|\delta_y v(t)\|_2 \leq \frac{1}{\sqrt{C_p}} \|\delta_x w(t)\|_2 \|\delta_x v(t)\|_2,
\]  

(52)

\[
\frac{h^2}{12} M^{-2M-2} \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i+\frac{1}{2},j}(t)) v_{ij}(t) \leq \frac{1}{\sqrt{C_p}} \|\delta_x w(t)\|_2 \|\delta_x v(t)\|_2 \leq \frac{1}{\sqrt{C_p}} \|\delta_x w(t)\|_2 \|\delta_x v(t)\|_2,
\]  

(53)

\[
\frac{h^2}{12} M^{-2M-2} \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i+\frac{3}{2},j}(t)) v_{ij}(t) \leq \frac{1}{\sqrt{C_p}} \|\delta_x w(t)\|_2 \|\delta_x v(t)\|_2 \leq \frac{1}{\sqrt{C_p}} \|\delta_x w(t)\|_2 \|\delta_x v(t)\|_2,
\]  

(54)

where \(C_p > 0\), is a constant independent of the mesh grid \(h\) and time step \(k\). In the remainder of this paper, we denote by \(C_p\) be the Poincaré-Friedrich constant. Substituting estimates \((51)\)-(54) into equation \((50)\) provides

\[
(\delta^4_x w(t), v(t))_2 \leq \frac{4}{3} \|\delta_x w(t)\|_2 \|\delta_y v(t)\|_2.
\]

Similarly, it is not hard to show that

\[
(\delta^4_x w(t), v(t))_2 \leq \frac{4}{3} \|\delta_y w(t)\|_2 \|\delta_y v(t)\|_2.
\]

Furthermore, the application of the summation by parts along with the hypothesis: \(w_{ij}(t) = v_{ij}(t) = w_{il}(t) = v_{il}(t) = 0\), for \(l = 0, 1, M - 1, M; i, j = 0, 1, 2, ..., M\), and any \(t \in [0, T]\), yield

\[
\sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i-\frac{1}{2},j}) v_{ij} = - \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i+\frac{1}{2},j}) v_{ij} - \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta_x w_{i+\frac{3}{2},j}) v_{ij} = - \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta_x v_{i-\frac{1}{2},j}) v_{ij}.
\]  

(55)
\[ \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta w_{i,j-\frac{1}{2}}(t)) w_{ij}(t) = \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} w_{ij}(t) \delta x_{i-j,\frac{1}{2}}(t), \quad \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta y_{i,j-\frac{1}{2}}(t)) w_{ij}(t) = \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} w_{ij}(t) \delta y_{i-j,\frac{1}{2}}(t). \] 

\[ \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta w_{i,j-\frac{1}{2}}(t)) w_{ij}(t) = \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} w_{ij}(t) \delta y_{i-j,\frac{1}{2}}(t), \quad \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} (\delta y_{i,j-\frac{1}{2}}(t)) w_{ij}(t) = \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} w_{ij}(t) \delta y_{i-j,\frac{1}{2}}(t). \] 

A combination of equations (56) and (58) gives

\[ (\delta^2 w(t), v(t))_2 = \frac{h^2}{12} \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} \left[ -\delta_x v_{i-j,\frac{1}{2}}(t) + 7(\delta_x v_{i-j,\frac{1}{2}}(t) + \delta_x v_{i-j,\frac{1}{2}}(t)) - \delta_x v_{i-j,\frac{1}{2}}(t) \right] w_{ij}(t) = - (\delta^2 v(t), w(t))_2. \]

In a similar manner, utilizing the linear operator \( \delta^y \) defined in (6) and equations (57)–(58), one shows that

\[ (\delta^y w(t), v(t))_2 = \frac{h^2}{12} \sum_{i=2}^{M-2} \sum_{j=2}^{M-2} \left[ -\delta_y v_{i,j-\frac{1}{2}}(t) + 7(\delta_y v_{i,j-\frac{1}{2}}(t) + \delta_y v_{i,j-\frac{1}{2}}(t)) - \delta_y v_{i,j-\frac{1}{2}}(t) \right] w_{ij}(t) = - (\delta^y v(t), w(t))_2. \]

This ends the proof of Lemma 3.1.

**Lemma 3.2.** Suppose \( w \in C_D^{0,0} \), be a function defined on \( D = \Omega \times [0,T] \), that satisfies \( w_{ij}(t) = w_{ij}(t) = 0 \), for every \( l = 0, 1, M - 1, M; i, j = 0, 1, ..., M \) and \( t \in [0,T] \). Letting \( w(x, y, t) = w_{ij}(t) \), the following estimates are satisfied:

\[ (\delta_x^2 w(t), w(t))_2 = 0, \]

\[ (-\delta_x^2 w(t), w(t))_2 = \| \delta_x w(t) \|_2^2 + \frac{h^2}{12} \| \delta_x^2 w(t) \|_2^2, \]

for every \( t \in [0,T] \), where \( z = x, y \).

**Proof.** In equation (55) of Lemma 3.1 replacing \( v(t) \) with \( w(t) \) to get

\[ (\delta_x^2 w(t), w(t))_2 = - (\delta^2_x w(t), w(t))_2, \]

which is equivalent to

\[ 2 (\delta^2_x w(t), w(t))_2 = 0. \]

Thus, \( (\delta_x^2 w(t), w(t))_2 = 0 \). This proves equation (60). Now, replacing \( v(t) \) with \( w(t) \) in equation (57), this provides

\[ (\delta_x^2 w(t), w(t))_2 = - (\delta_x w(t), \delta_x w(t))_2 - \frac{h^2}{12} (\delta_x^2 w(t), \delta_x^2 w(t))_2. \]

This is equivalent to

\[ (-\delta_x^2 w(t), w(t))_2 = \| \delta_x w(t) \|_2^2 + \frac{h^2}{12} \| \delta_x^2 w(t) \|_2^2. \]

Similarly, it is not difficult to show that

\[ (-\delta_y^2 w(t), w(t))_2 = \| \delta_y w(t) \|_2^2 + \frac{h^2}{12} \| \delta_y^2 w(t) \|_2^2. \]

We recall that the operators: \( \delta_x, \delta_y, \delta_x^2 \) and \( \delta_y^2 \), for \( z \in \{x, y\} \), are defined in relation (6). This completes the proof of Lemma 3.2.

**Lemma 3.3.** Suppose \( \{d_n\}_n \) and \( \{q_n\}_n \) be sequences of nonnegative real numbers. Let \( c_0 \geq 0 \), be a constant and \( l_0 \) be a nonnegative rational number which is an integer multiple of \( 2^{-1} \), that is, \( l_0 = 2^{-1}k \), where \( k \) is a nonnegative integer. For \( \sigma \in \left(\frac{1}{2}, 1\right) \), if

\[ q_{n+\sigma} \leq c_0 + \sum_{i=l_0}^{n+\sigma} d_i q_i, \]

then

\[ q_{n+\sigma} \leq c_0 \exp \left( \sum_{i=l_0}^{n+\sigma} d_i \right), \]

where the summation index "\( l \)" varies in the range: \( l = l_0, l_0 + \frac{1}{2}, l_0 + 1, ..., n + \sigma - \frac{1}{2}, n + \sigma \).
Proof. The proof of this Lemma is similar to the well known proof of the Gronwall Lemma discussed in the literature by replacing the integer summation index with the ones having a step size $2^{-1}$.

With Lemmas 5.2, 5.3 we are ready to prove the main result of this paper (namely Theorem 5.4).

Proof. (of Theorem 5.4)

Subtracting \( \frac{1}{4} \) and \( \frac{1}{2} \) from (14) and (21), respectively, and utilizing the fact that $h_x = h_y = h$, simple calculations result in

\[
(I - \delta_{2x}^4) e_i^{n+\frac{1}{2}} = (I - \delta_{2x}^4) e_i^{n-\frac{1}{2}} + k \left( \sum_{j} f_i(x, y, t, u^n_{ij}, \delta_{ij}^{n+1} u^n_{ij}) - f_i(x, y, t, u^n_{ij}, \delta_{ij}^{n} u^n_{ij}) \right) + O(k^3 + h^4 + kh^4).
\]

Let $e_i^{n+\frac{1}{2}} = u_i^{n+\frac{1}{2}} - U_i^{n+\frac{1}{2}}$, be the temporal error obtained at the time level $n + \frac{1}{2}$ and let \( e_i^{n+1} = u_i^{n+1} - U_i^{n+1} \), be the error provided by the proposed formulation (22)-(28) at the discrete time $t_{n+1}$. This fact combined with equations \( (\ref{eq:5.5}) \) and \( (\ref{eq:5.6}) \), after rearranging terms give

\[
(I - \delta_{2x}^4) e_i^{n+\frac{1}{2}} = (I - \delta_{2x}^4) e_i^{n-\frac{1}{2}} + k \left( f_i(x, y, t, u^n_{ij}, \delta_{ij}^{n} u^n_{ij}) - f_i(x, y, t, u^n_{ij}, \delta_{ij}^{n-\frac{1}{2}} u^n_{ij}) \right) + O(k^3 + h^4 + kh^4).
\]

Multiplying both sides of equations \( (\ref{eq:5.5}) \) and \( (\ref{eq:5.6}) \) by $h^2 e_i^{n+\frac{1}{2}}$ and $h^2 e_i^{n+1}$, respectively, summing up from $i, j = 2, 3, \ldots, M - 2$, and utilizing the definition of the scalar product and the $L^2$ norm defined in relations \( (\ref{eq:5.8}) \) and \( (\ref{eq:5.9}) \), respectively, we obtain

\[
\left( e_i^{n+\frac{1}{2}} - e_i^{n-\frac{1}{2}} \right)_{2} + \alpha \left( -\delta_{2x}^4 e_i^{n+\frac{1}{2}} - e_i^{n+\frac{1}{2}} \right)_{2} = -\alpha \left( \delta_{2x}^4 e_i^{n-\frac{1}{2}} - e_i^{n-\frac{1}{2}} \right)_{2} + k \left( \sum_{j} f_i(x, y, t, u^n_{ij}, \delta_{ij}^{n} u^n_{ij}) - f_i(x, y, t, u^n_{ij}, \delta_{ij}^{n-\frac{1}{2}} u^n_{ij}) \right)_{2} + O(k^3 + h^4 + kh^4).
\]

(67)

\[
\left( e_i^{n+1} - e_i^{n+\frac{1}{2}} \right)_{2} + \alpha \left( -\delta_{2x}^4 e_i^{n+\frac{1}{2}} - e_i^{n+\frac{1}{2}} \right)_{2} = -\alpha \left( \delta_{2x}^4 e_i^{n+\frac{1}{2}} - e_i^{n+\frac{1}{2}} \right)_{2} + k \left( \sum_{j} f_i(x, y, t, u^n_{ij}, \delta_{ij}^{n+1} u^n_{ij}) - f_i(x, y, t, u^n_{ij}, \delta_{ij}^{n+\frac{1}{2}} u^n_{ij}) \right)_{2} + O(k^3 + h^4 + kh^4).
\]

(68)

where $\alpha = \frac{k}{4} \left( \gamma - \delta_{2x}^4 e_i^{n+\frac{1}{2}} \right)_{2} + \beta \left( \delta_{2x}^4 e_i^{n-\frac{1}{2}} \right)_{2} + O(k^3 + h^4 + kh^4)$.

Plugging estimates \( (\ref{eq:5.10}) \) - \( (\ref{eq:5.11}) \) together with estimates \( (\ref{eq:5.5}) \) and \( (\ref{eq:5.6}) \) of Lemma 5.5, equations \( (\ref{eq:5.5}) \) - \( (\ref{eq:5.6}) \) of Lemma 5.2 and multiplying the resulting relations by 2, this provides

\[
2 \left( e_i^{n+\frac{1}{2}} - e_i^{n-\frac{1}{2}} \right)_{2} + 2 \alpha \left( \delta_{2x} e^{n+\frac{1}{2}} \right)_{2} + \frac{\alpha h^2}{6} \left( \delta_{2x} e^{n+\frac{1}{2}} \right)_{2} \leq \alpha \left( \delta_{2x} e^{n-\frac{1}{2}} \right)_{2} + \| \delta_{2x} e^{n+\frac{1}{2}} \|_{2} + h^2 \left( \| \delta_{2x} e^{n-\frac{1}{2}} \|_{2} + \right.
\]

\[
\left. \frac{h^2}{6} \left( \| \delta_{2x} e^{n-\frac{1}{2}} \|_{2} \right) \right).
\]

for $z \in \{ x, y \}$, $l = 1, 2$, and $q \in \{ n + \frac{1}{2}, n + 1 \}$.
\[
\begin{align*}
\| & \delta_x e^{n+\frac{1}{2}} \|_2^2 + 2C_3k(\beta + \gamma)\| \delta_x e^n \|_2 \| e^{n+\frac{1}{2}} \|_2 + 2C_1Lk h^2 \sum_{i=2}^{M-2M-2} \sum_{j=2}^{M-2M-2} |e^{i}_{ij}||e^{i+\frac{1}{2}}_{ij}| + \left( O(k^3 + h^4 + kh^4), e^{n+\frac{1}{2}} \right)_2, \\
2 \left( e^{n+1} - e^{n+\frac{1}{2}}, e^{n+\frac{1}{2}} \right)_2 + 2\alpha \left( \| \delta_y e^{n+\frac{1}{2}} \|_2 + \frac{h^2}{12} \| \delta_x e^{n+\frac{1}{2}} \|_2 \right) + \gamma \left( \| \delta_y e^{n+\frac{1}{2}} \|_2 + \frac{h^2}{6} \| \delta_x e^{n+\frac{1}{2}} \|_2 \right) \leq \\
\alpha \left( \| \delta_x e^{n+\frac{1}{2}} \|_2^2 + \| \delta_y e^{n+\frac{1}{2}} \|_2^2 + \frac{h^2}{12} \left( \| \delta_x e^{n+\frac{1}{2}} \|_2^2 + \| \delta_y e^{n+\frac{1}{2}} \|_2^2 \right) \right) + C_2((\beta + \gamma)k) \frac{1}{2} \| \delta_x e^{n+\frac{1}{2}} \|_2^2.
\end{align*}
\]

The first estimate comes from the Cauchy-Schwarz inequality while the second and third ones follow from the Poincaré-Friedrichs inequality and estimate \(2ab \leq a^2 + b^2\), respectively.

\[
\left( O(k^3 + h^4 + kh^4), e^{n+\frac{1}{2}} \right)_2 \leq 2C_4(k^3 + h^4 + kh^4)\| e^{n+\frac{1}{2}} \|_2 \leq 2C_4\sqrt{C_p}(k^3 + h^4 + kh^4)\| \delta_x e^{n+\frac{1}{2}} \|_2 \leq \\
\sqrt{C_p}k\| \delta_x e^{n+\frac{1}{2}} \|_2^2 + 12C_2^2k(k^2 + h^4 + h^4)^2,
\]

where \(C_4\) is a positive constant independent of \(h\) and \(k\).

\[
\begin{align*}
2 \left( e^{n+1} - e^{n+\frac{1}{2}}, e^{n+\frac{1}{2}} \right)_2 & = \| e^{n+1} \|_2^2 + \| e^{n+\frac{1}{2}} \|_2^2 + \| e^{n+\frac{1}{2}} \|_2^2, \\
\frac{C_2(\beta + \gamma)k}{2} \| \delta_y e^{n+\frac{1}{2}} \|_2 \| \delta_x e^{n+\frac{1}{2}} \|_2 & \leq \frac{C_2(\beta + \gamma)k}{4} \left( \| \delta_y e^{n+\frac{1}{2}} \|_2^2 + \| \delta_x e^{n+\frac{1}{2}} \|_2^2 \right), \\
\frac{C_2(\delta + \gamma)k}{2} \sum_{i=2}^{M-2M-2} \sum_{j=2}^{M-2M-2} \left( c^{i+1}_{ij} \right)^2 + \left| e^{i+\frac{1}{2}}_{ij} \right| \left| e^{i+\frac{1}{2}}_{ij} \right| & \leq \frac{C_2(\kappa + \gamma)k}{2} \left( \| \delta_y e^{n+\frac{1}{2}} \|_2^2 + \| \delta_x e^{n+\frac{1}{2}} \|_2^2 \right) \leq \frac{C_2(\kappa + \gamma)k}{2} \left( \| \delta_y e^{n+\frac{1}{2}} \|_2^2 + \| \delta_x e^{n+\frac{1}{2}} \|_2^2 \right), \\
\| \delta_x e^{n+\frac{1}{2}} \|_2 \| \delta_x e^{n+\frac{1}{2}} \|_2 & \leq \frac{C_2(\kappa + \gamma)k}{2} \left( \| \delta_y e^{n+\frac{1}{2}} \|_2^2 + \| \delta_x e^{n+\frac{1}{2}} \|_2^2 \right), \\
\left( O(k^3 + h^4 + kh^4), e^{n+\frac{1}{2}} \right)_2 & \leq C_4(k^3 + h^4 + kh^4)\| e^{n+\frac{1}{2}} \|_2 \leq C_4\sqrt{C_p}(k^3 + h^4 + kh^4)\| \delta_y e^{n+\frac{1}{2}} \|_2 \leq \\
k\| \delta_y e^{n+\frac{1}{2}} \|_2^2 + \frac{C_2^2C_p^2k}{4}(k^2 + k^{-1}h^4 + h^4)^2,
\end{align*}
\]

A combination of (70) and (72)-(75) results in

\[
\| e^{n+\frac{1}{2}} \|_2^2 + \alpha \left( \| \delta_x e^{n+\frac{1}{2}} \|_2^2 + \frac{h^2}{12} \| \delta_x e^{n+\frac{1}{2}} \|_2^2 + \alpha \left( \| \delta_x e^{n-\frac{1}{2}} \|_2^2 + \frac{h^2}{12} \| \delta_x e^{n-\frac{1}{2}} \|_2^2 \right) + \\
24k\| \delta_x e^n \|_2^2 \right) + \frac{1}{12} \left( C_2^2(\beta + \gamma) + C_2L^2C_p^2 \right) k\| \delta_x e^{n+\frac{1}{2}} \|_2^2 + 12C_2^2k(k^2 + k^{-1}h^4 + h^4)^2.
\]

Setting

\[
E_{(x)} = \| e^n \|_2^2 + \alpha \left( \| \delta_x e^n \|_2^2 + \frac{h^2}{12} \| \delta_x e^n \|_2^2 \right),
\]

where \(z = x, y\). Since \(0 < \alpha \leq 1\), for \(z = x\), a combination of (81) and (80) implies

\[
E^{n+\frac{1}{2}}_x + \| e^{n+\frac{1}{2}} - e^n \|_2^2 \leq E^{n+\frac{1}{2}}_x + C_\alpha \left( E^{n+\frac{1}{2}}_x + E^{n+\frac{1}{2}}_x \right) + 12C_2^2k(k^2 + k^{-1}h^4 + h^4)^2,
\]

(82)
where $C_\alpha = \alpha^{-1} \max \{ 24, \frac{1}{15} |C_2^2(\beta + \gamma) + C_1^2 C_0^2 \}$. But estimate (82) holds for all $1 \leq p \leq n$. Summing up this estimate from $p = 1, 2, \ldots, n$, provides

$$E_{(x)}^{n+\frac{1}{2}} \leq \frac{n}{p=1} \left\| e_{\frac{p}{2}} - e_{\frac{p-1}{2}} \right\|^2 \leq E_{(x)}^{\frac{1}{2}} + C_n k \sum_{p=1}^{n} \left( E_{(x)}^{\frac{p}{2}} + E_{(x)}^{\frac{p+1}{2}} \right) + 12 C_n^2 k n (k^2 + k^{-1} h^4 + h^4)^2.$$ 

Since $kn \leq kN = T$, introducing a summation index $l$ which varies in the range: $l = 1, \frac{\pi}{2}, 2, \ldots, n, n + \frac{\pi}{2}$, it is not difficult to observe that this estimate implies

$$E_{(x)}^{n+\frac{1}{2}} \leq E_{(x)}^{\frac{1}{2}} + C_n k \sum_{l=1}^{n+\frac{1}{2}} E_{(x)}^{l} + 12 C_n^2 T (k^2 + k^{-1} h^4 + h^4)^2.$$ 

Utilizing estimate (62) given in Lemma 3.3, this becomes

$$E_{(x)}^{n+\frac{1}{2}} \leq \frac{1}{\pi} \left( E_{(x)}^{\frac{1}{2}} + 12 C_n^2 T (k^2 + k^{-1} h^4 + h^4)^2 \right) \exp \left( \sum_{l=1}^{n+\frac{1}{2}} C_n k \right) \leq \left( E_{(x)}^{\frac{1}{2}} + 12 C_n^2 T (k^2 + k^{-1} h^4 + h^4)^2 \right) \exp (2nC_n k).$$

Because $n = 1, 2, \ldots, N - 1$, and $k = \frac{T}{N}$, so $2nk \leq 2Nk = 2T$. Thus

$$E_{(x)}^{n+\frac{1}{2}} \leq e^{2C_n T} \left( E_{(x)}^{\frac{1}{2}} + 12 C_n^2 T (k^2 + k^{-1} h^4 + h^4)^2 \right).$$

Now, subtracting equation (24) from (22) and using the initial condition (28), this results in

$$(T - \delta_{ij}) e_{ij}^{\frac{1}{2}} - k (\gamma \delta_{ij} e_{ij}^{\frac{1}{2}} - e_{ij}^{\frac{1}{2}}) e_{ij}^{\frac{1}{2}} = k \left( f_1(x_i, y_j, t_{\frac{1}{2}}, U_{i,j}^{\frac{1}{2}}, \delta_{ij} U_{i,j}^{\frac{1}{2}}) - f_1(x_i, y_j, t_{\frac{1}{2}}, U_{i,j}^{\frac{1}{2}}, \delta_{ij} U_{i,j}^{\frac{1}{2}}) \right) + O(k^3 + h^4 + kh^4).$$

Multiplying side by side this equation by $2h_e e_{ij}^{\frac{1}{2}}$, summing up from $i, j = 2, 3, \ldots, M - 2$, utilizing Lemma 5.2 together with estimate (69) and rearranging terms to obtain

$$\| e_{\frac{1}{2}} \|^2 + \alpha \left( \| \delta_x e_{\frac{1}{2}} \|^2 + \frac{h^2}{12} \| \delta_x^2 e_{\frac{1}{2}} \|^2 \right) + \frac{1}{4} \left( \| \delta_y e_{\frac{1}{2}} \|^2 + \frac{h^2}{12} \| \delta_y^2 e_{\frac{1}{2}} \|^2 \right) \leq \frac{C_1 T}{4} \| e_{\frac{1}{2}} \|^2 + C_2 k (k^2 + k^{-1} h^4 + h^4) \| e_{\frac{1}{2}} \|_2,$$

where $C_5 = 0$ is a constant that does not depend on the time step $k$ and mesh size $h$. But it is easy to see that

$$C_5 k (k^2 + k^{-1} h^4 + h^4) \| e_{\frac{1}{2}} \|_2 \leq \frac{k}{4} \| e_{\frac{1}{2}} \|^2 + C_5^2 k (k^2 + k^{-1} h^4 + h^4)^2.$$ 

Substituting this into estimate (84) yields

$$\| e_{\frac{1}{2}} \|^2 + \alpha \left( \| \delta_x e_{\frac{1}{2}} \|^2 + \frac{h^2}{12} \| \delta_x^2 e_{\frac{1}{2}} \|^2 \right) + \frac{1}{4} \left( \| \delta_y e_{\frac{1}{2}} \|^2 + \frac{h^2}{12} \| \delta_y^2 e_{\frac{1}{2}} \|^2 \right) \leq \frac{1 + C_1 T}{4} \| e_{\frac{1}{2}} \|^2 + C_5^2 k (k^2 + k^{-1} h^4 + h^4)^2,$$

which implies

$$E_{(x)}^{\frac{1}{2}} \leq \frac{1 + C_1 T}{4} E_{(x)}^{\frac{1}{2}} + C_5^2 k (k^2 + k^{-1} h^4 + h^4)^2.$$ 

Apply estimate (62) of Lemma 3.3 to get

$$E_{(x)}^{\frac{1}{2}} \leq C_5^2 k (k^2 + k^{-1} h^4 + h^4)^2 \exp \left( \frac{1 + C_1 T}{4} k \right).$$

Since $0 < k \leq 1$, this becomes

$$E_{(x)}^{\frac{1}{2}} \leq C_5^2 \exp \left( \frac{1 + C_1 T}{4} k \right).$$

A combination of relations (85) and (82) gives

$$E_{(x)}^{n+\frac{1}{2}} \leq e^{2C_n T} \left( C_5^2 \exp \left( \frac{1 + C_1 T}{4} k \right) + 12 C_n^2 T (k^2 + k^{-1} h^4 + h^4)^2 \right).$$

Plugging estimates (71) and (78) - (79), straightforward computations provide

$$\| e^{n+\frac{1}{2}} \|^2 + 2 \alpha \left( \| \delta_x e^{n+\frac{1}{2}} \|^2 + \frac{h^2}{12} \| \delta_x^2 e^{n+\frac{1}{2}} \|^2 \right) + \frac{1}{4} \left( \| \delta_y e^{n+\frac{1}{2}} \|^2 + \frac{h^2}{12} \| \delta_y^2 e^{n+\frac{1}{2}} \|^2 \right) \leq \| e^{n+\frac{1}{2}} \|^2 +$$

$$+ \| e^{n-\frac{1}{2}} \|^2 \leq \| e^{n+\frac{1}{2}} \|^2 +$$

$$+ \| e^{n-\frac{1}{2}} \|^2.$$
\[
\alpha \left[ \|\delta_v e^{n+\frac{1}{2}}\|^2 + \|\delta_v e^{n+1}\|^2 + \frac{k^2}{12} \left( \|\delta_v e^{n+\frac{1}{2}}\|^2 + \|\delta_v e^{n+1}\|^2 \right) \right] + 4^{-1} k(C_{2L} C_p + \gamma + \beta) \|\delta_v e^{n+\frac{1}{2}}\|^2 + 4^{-1} k(1 + \beta + \gamma + 3C_{2L} C_p) \|\delta_v e^{n+1}\|^2 + 4^{-1} C_{2L} C_p (k^2 + k^{-1} h^4 + h^4)^2.
\]

Utilizing relation \(81\) with \(z = y\), it is not hard to observe that this estimate implies

\[
E_{n+1}^y \leq E_n^y + 4^{-1} k(1 + \beta + \gamma + 3C_{2L} C_p) \left( \|\delta_v e^{n+\frac{1}{2}}\|^2 + \|\delta_v e^{n+1}\|^2 \right) + 4^{-1} C_{2L} C_p (k^2 + k^{-1} h^4 + h^4)^2.
\]

Replacing the discrete times \(t_n + \frac{1}{2}\) and \(t_n + 1\) in the predictor and corrector phases with \(t_n\) and \(t_{n+\frac{1}{2}}\), respectively, using Lemmas \(3.1\) and \(3.3\) and performing straightforward calculations, this results in

\[
E_{n+1}^y \leq E_n^y + 12C_{2L} C_p (k^2 + k^{-1} h^4 + h^4)^2,
\]

for \(n \geq 1\), and

\[
E_{n+1}^y \leq C_0 \exp \left( \frac{1 + C_{2L}}{4} \right) (k^2 + k^{-1} h^4 + h^4)^2,
\]

where \(C_0\) is a positive constant independent of the space step \(h\) and the time step \(k\). Inequalities \(87\) and \(89\) indicate that

\[
E_{n+1}^y \leq E_n^y + 4^{-1} k(1 + \beta + \gamma + 3C_{2L} C_p) \left( \|\delta_v e^{y+\frac{1}{2}}\|^2 + \|\delta_v e^{y+1}\|^2 \right) + 4^{-1} C_{2L} C_p (k^2 + k^{-1} h^4 + h^4)^2,
\]

for \(q = 0, \frac{1}{2}, 1, ..., n - \frac{1}{2}, n\). Summing up this estimate for \(q = 0, \frac{1}{2}, 1, ..., n\), yields

\[
E_{n+1}^y \leq E_0^y + 4^{-1} k(1 + \beta + \gamma + 3C_{2L} C_p) \sum_{q=0}^{n+\frac{1}{2}} \left( \|\delta_v e^{q+\frac{1}{2}}\|^2 + \|\delta_v e^{q+1}\|^2 \right) + 4^{-1} C_{2L} C_p (2n + 1)(k^2 + k^{-1} h^4 + h^4)^2.
\]

But \((2n + 1)k \leq 2T + 1\). This fact combined with this estimate and \(90\) give

\[
E_{n+1}^y \leq \left( C_0 e^{\frac{1 + C_{2L}}{4}} + C_0 C_p (2T + 1) \right) (k^2 + k^{-1} h^4 + h^4)^2 + \frac{1 + \beta + \gamma + 3C_{2L} C_p}{2} k \sum_{q=0}^{n+\frac{1}{2}} \|\delta_v e^{q+1}\|^2.
\]

This can be rewritten after rearranging terms as

\[
E_{n+1}^y \leq \left( C_0 e^{\frac{1 + C_{2L}}{4}} + C_0 C_p (2T + 1) \right) (k^2 + k^{-1} h^4 + h^4)^2 + \frac{1 + \beta + \gamma + 3C_{2L} C_p}{2} k \sum_{q=0}^{n+\frac{1}{2}} \|\delta_v e^{q+1}\|^2.
\]

Indeed, the summation index varies in the range: \(\frac{1}{2}, 1, ..., n, n + \frac{1}{2}\). Applying inequality \(92\) given in Lemma \(3.3\) to get

\[
E_{n+1}^y \leq \left( C_0 e^{\frac{1 + C_{2L}}{4}} + C_0 C_p (2T + 1) \right) (k^2 + k^{-1} h^4 + h^4)^2 \exp \left( (1 + \beta + \gamma + 3C_{2L} C_p)(n + 1)k \right).
\]

Since \((n + 1)k \leq T + 1\), absorbing all the constants into a positive constant \(\tilde{C}_1\), we obtain

\[
E_{n+1}^y \leq \tilde{C}_1 (k^2 + k^{-1} h^4 + h^4)^2.
\]

In addition, it is easy to see that \(89\) implies

\[
E_{n+1}^y \leq \tilde{C}_2 (k^2 + k^{-1} h^4 + h^4)^2,
\]

where all the constant have been absorbed into a positive constant \(\tilde{C}_2\). Analogously, it is not hard to establish that the term \(E_n^y\) and \(E_{n+1}^y\), associated with the predicted error \(e_n^y\) and the corrected one \(e_{n+\frac{1}{2}}^y\), at the discrete points \((x_i, y_j, t_{n+\frac{1}{2}})\) and \((x_i, y_j, t_{n+\frac{1}{2}})\), respectively, satisfy

\[
E_n^y \leq \tilde{C}_1 (k^2 + k^{-1} h^4 + h^4)^2, \quad \text{for } n \geq 1,
\]

\[
E_{n+1}^y \leq \tilde{C}_2 (k^2 + k^{-1} h^4 + h^4)^2, \quad \text{for } n \geq 0.
\]
Plugging estimates (92), (94) and (81) (respectively, (91), (93) and (81)), utilizing the time step requirement (29), that is \( k = h^{4/3} \), along with the \( L^\infty(0,T;H^2) \)-norm given in (7), provides
\[
\|e^{n+\sigma}\|_{H^2}^2 \leq 2\|e^{n+\sigma}\|_2^2 + \alpha \left[ \|\delta_x e^{n+\sigma}\|_2^2 + \|\delta_y e^{n+\sigma}\|_2^2 + \frac{h^2}{12} \left( \|\delta_x^2 e^{n+\sigma}\|_2^2 + \|\delta_y^2 e^{n+\sigma}\|_2^2 \right) \right] = E^{n+\sigma}_0 + E^{n+\sigma}_{\text{new}} \leq \\
\max\{\hat{C}_1 + \hat{C}_2 + \hat{C}_3\}(k^2 + h^{8/3} + h^4)^2,
\]
for \( \sigma = \frac{1}{3} \), and any \( n = 0, 1, 2, ..., N - 1 \). The maximum over \( n \), for \( 0 \leq n \leq N - 1 \), on the left and right sides of these estimates implies
\[
\|e\|_{H^2,\infty} \leq \hat{C}_3(k^2 + h^{8/3} + h^4)^2,
\]
where \( \hat{C}_3 := \max\{\hat{C}_1 + \hat{C}_2 + \hat{C}_3\} \). Taking the square root on both sides of inequality (95), using relation \( \|u\|_{H^2,\infty} - \|U\|_{H^2,\infty} \leq \|e\|_{H^2,\infty} \), together with the regularity condition (30), to get
\[
\|U\|_{H^2,\infty} \leq C_0 + \hat{C}_3(k^2 + h^{8/3} + h^4),
\]
Finally, since \( h < 1 \), so \( h^4 < h^{8/3} \). Substituting this into (95) and taking the square root of the new estimate gives
\[
\|\sigma\|_{H^2,\infty} \leq \hat{C}_4(k^2 + h^{8/3}),
\]
where \( \hat{C}_4 > 0 \), is a constant that does not depend on the time step \( k \) and the grid size \( h \). Thus, the proof of Theorem 3.1 is completed.

4 Numerical experiments and Convergence rate

This section presents some numerical evidences to verify the stability and convergence order of the developed technique (24)-(28), for solving the two-dimensional unsteady Sobolev and regularized long wave equation (11) subjected to initial-boundary conditions (2)- (3). To demonstrate the efficiency of the new algorithm, we set \( k = h^\frac{4}{3} \), for \( h = 2^{-l} \), \( l = 1, 2, 3, 4, 5 \), and we compute the exact solution: \( u^n \), approximate solution: \( U^n \), and the error: \( e^n = u^n - U^n \), at discrete time \( t_n \), in the \( L^\infty(0,T; L^2(\Omega)) \)-norm, denoted \( \|\cdot\|_{L^\infty,\infty} \) and defined in relation (7). The following formula is considered for the calculation of the error
\[
\|e(h)\|_{L^\infty,\infty} = \max_{0 \leq n \leq N} \|u^n - U^n_h\|_2,
\]
where \( U^n_h \) denotes the numerical solution obtained at time level \( n \) and associated with a mesh size \( h \). Furthermore, the convergence rate \( R(k,h) \) of the proposed approach is estimated using the formula
\[
R(k,h) = \log_2(\|e(2h)\|_{L^\infty,\infty}/\|e(h)\|_{L^\infty,\infty}).
\]
Here we set \( k = h^\frac{4}{3} \). Lastly, the numerical results are carried out using MATLAB R2013b.

- **Example 1.** Let \( \Omega = (0,1)^2 \) be the fluid region and let \( T = 1 \) be the final time. Consider the problem given in (11) by
\[
u_t - (u_{txx} + u_{yy}) - (u_{xx} + u_{yy}) + u = 0, \quad \Omega \times (0,1],
\]
with initial and boundary conditions
\[
u(x,y,0) = \sin(\pi x)\sin(\pi y), \quad \text{for} \quad (x,y) \in \Omega \cup \partial \Omega,
\]
\[
u(x,y,t) = 0, \quad \text{for} \quad (x,y,t) \in \partial \Omega \times [0,1],
\]
The exact solution \( u \) of the initial-boundary value problem (11)-(13) is given by
\[
u(x,y,t) = e^{-t}\sin(\pi x)\sin(\pi y).
\]

**Table 1.** Stability and convergence rate \( R(k,h) \) of the new three-level time split technique with varying mesh grid \( h \) and time step \( k \), where \( k = h^{4/3} \).

| \( h \) | \( \|u\|_{L^\infty,\infty} \) | \( ||U||_{L^\infty,\infty} \) | \( ||e(h)||_{L^\infty,\infty} \) | \( R(k,h) \) |
|---|---|---|---|---|
| 2^{-1} | 5.000 \times 10^{-4} | 4.8602 \times 10^{-4} | 1.0113 \times 10^{-2} | - |
| 2^{-2} | 4.9908 \times 10^{-4} | 4.9863 \times 10^{-4} | 1.7103 \times 10^{-3} | 2.5738 |
| 2^{-3} | 4.9998 \times 10^{-4} | 4.9887 \times 10^{-4} | 2.7873 \times 10^{-4} | 2.6173 |
| 2^{-4} | 4.9998 \times 10^{-4} | 4.9779 \times 10^{-4} | 4.4420 \times 10^{-5} | 2.6496 |
| 2^{-5} | 5.0001 \times 10^{-4} | 5.0002 \times 10^{-4} | 6.8298 \times 10^{-5} | 2.7013 |
Example 2. Let $\Omega = (0, 1) \times (0, 1)$ and $[0, T] = [0, 1]$. Consider equation (1) of the form
\[ u_t - (u_{xx} + u_{yy}) - (u_x + u_y) + f(x, y, t, u) = 0, \quad \text{on} \quad \Omega \times (0, 1), \]
subjects to initial and boundary conditions
\begin{align*}
  u(x, y, 0) &= \sin(\pi x) \sin(\pi y)e^{x+y}, \quad \text{on} \quad \Omega \cup \partial \Omega, \\
  u(x, y, t) &= 0, \quad \text{on} \quad \partial \Omega \times [0, 1],
\end{align*}
where $f(x, y, t, u) = (4\pi^2 - 3)u - 4\pi e^{x+y}[\sin(\pi y) \cos(\pi x) + \cos(\pi x) \sin(\pi y)]$. The analytical solution $u$ is defined as
\[ u(x, y, t) = \sin(\pi x) \sin(\pi y)e^{x+y}. \]

Table 2. Analysis of stability and convergence rate $R(k, h)$ of the developed three-level time split Leapfrog/Crank-Nicolson approach with varying space step $h$ and time step $k$. We set $k = h^{3/4}$.

| $h$  | $\|u\|_{\infty, 2}$ | $\|U\|_{\infty, 2}$ | $\|e(h)\|_{\infty, 2}$ | $R(h, k)$ |
|------|----------------|----------------|----------------|-----------|
| $2^{-4}$ | $3.6945 \times 10^9$ | $3.6645 \times 10^9$ | $7.3932 \times 10^{-2}$ | $2.5849$ |
| $2^{-5}$ | $3.9303 \times 10^9$ | $3.9224 \times 10^9$ | $1.2343 \times 10^{-2}$ | $2.6173$ |
| $2^{-6}$ | $3.9417 \times 10^9$ | $3.9369 \times 10^9$ | $2.0011 \times 10^{-3}$ | $2.6594$ |
| $2^{-7}$ | $3.9423 \times 10^9$ | $3.9415 \times 10^9$ | $3.1674 \times 10^{-4}$ | $2.7211$ |

Example 3. Suppose $\Omega = (0, 1) \times (0, 1)$ and $T = 1$. Consider the 2D sobolev and regularized long wave equation given in [14] by
\[ u_t - (u_{txx} + u_{uyy}) - (u_x + u_y) + uu_x + uu_y = 0, \quad \text{on} \quad \Omega \times (0, 1), \]
with initial condition
\[ u(x, y, 0) = \frac{1}{2} \sec h^2 \left[ \frac{1}{\sqrt{2}} (x + y) \right], \quad \text{on} \quad \Omega \cup \partial \Omega, \]
and boundary condition
\[ u(0, y, t) = \sec h^2(y - t), \quad u(1, y, t) = \sec h^2(y - t + 1), \quad u(x, 0, t) = \sec h^2(-x + t), \quad u(x, 1, t) = \sec h^2(-x + t - 1). \]
The exact solution $u$ is given by
\[ u(x, y, t) = \sec h^2(y - x + t). \]

Table 3. Stability analysis and convergence order $R(k, h)$ of the proposed three-level time split Leapfrog/Crank-Nicolson algorithm with varying mesh size $h$ and time step $k$. We take $k = h^{3/4}$.

| $h$  | $\|u\|_{\infty, 2}$ | $\|U\|_{\infty, 2}$ | $\|e(h)\|_{\infty, 2}$ | $R(h, k)$ |
|------|----------------|----------------|----------------|-----------|
| $2^{-4}$ | $1.2069 \times 10^9$ | $1.1828 \times 10^9$ | $2.4124 \times 10^{-2}$ | $-$ |
| $2^{-5}$ | $1.0475 \times 10^9$ | $1.0433 \times 10^9$ | $4.2324 \times 10^{-3}$ | $2.5739$ |
| $2^{-6}$ | $9.6445 \times 10^8$ | $9.6403 \times 10^8$ | $6.9341 \times 10^{-4}$ | $2.6097$ |
| $2^{-7}$ | $9.2189 \times 10^8$ | $9.2197 \times 10^8$ | $1.1183 \times 10^{-4}$ | $2.6324$ |
| $2^{-8}$ | $9.2188 \times 10^8$ | $9.2188 \times 10^8$ | $1.7575 \times 10^{-5}$ | $2.6697$ |

Figures indicate that the developed two-step explicit/implicit approach is stable whereas Tables 1-3 suggest that the new algorithm is temporal second-order convergent and spatial accurate with order $O(k^{8/3})$. These numerical results confirm the theory (see Section 3 Theorem 3.1).

5 General conclusions and future works

This paper has developed a three-level time split high-order Leapfrog/Crank-Nicolson approach in an approximate solution of the two-dimensional time-dependent sobolev and regularized long wave equations arising in fluid mechanics. The stability and error estimates of the proposed numerical technique have been deeply analyzed in $L^\infty(0, T; H^2)$-norm. The theoretical studies have shown that the developed numerical scheme is stable and accurate with order $O(k^2 + h^{8/3})$, where $k$ and $h$ represent the time step and space step, respectively. These results have been confirmed by three numerical examples. Furthermore, both theoretical and numerical results suggest that the proposed algorithm is computationally less expensive, faster and more efficient than a large class of numerical methods.
widely discussed in the literature for solving the initial-boundary value problem. The use of the $L^\infty(0, T; H^2)$-norm indicates that the constructed scheme can be considered as a strong numerical method for the integration of general systems of (2 + 1)-dimensional nonlinear Sobolev problems. Our future works will develop a three-level time split high-order Leapfrog/Crank-Nicolson formulation for solving the generalized (2 + 1)-dimensional nonlinear evolutionary models. Both stability and convergence rate of the constructed technique also will be analyzed in $L^\infty(0, T; H^2)$-norm.

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Stability and convergence of the constructed three-level time split Leapfrog/Crank-Nicolson with $k = h^{4/3}$.

Figure 1: Exact solution ($u$: in green), numerical solution ($U$: in blue) and error ($E$: in red) for Example 1
Stability analysis and convergence of the developed three-level time split Leapfrog/Crank-Nicolson approach with $k = h^{4/3}$.

Figure 2: analytical solution (u: in green), computed solution (U: in blue) and error (E: in red) for Example 2
Stability and convergence of the proposed three-level time split explicit/implicit approach with $k = h^{4/3}$

Figure 3: exact solution (u: in green), approximate solution (U: in blue) and error (E: in red) for Problem 3