INVARIANT GIBBS MEASURES AND GLOBAL STRONG SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS IN DIMENSION TWO

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Dedicated to the memory of Professor Jean Bourgain

Abstract. We consider the defocusing nonlinear Schrödinger equation on $\mathbb{T}^2$ with Wick ordered power nonlinearity, and prove almost sure global well-posedness with respect to the associated Gibbs measure. The heart of the matter is the uniqueness of the solution as limit of solutions to canonically truncated systems. The invariance of the Gibbs measure under the global dynamics follows as a consequence.

1. Introduction

In this paper we study the (defocusing) Wick ordered nonlinear Schrödinger equation on the torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$,

\begin{equation}
\begin{cases}
(i\partial_t + \Delta)u = W^{2r+1}(u), \\
u(0) = u_{in},
\end{cases}
\end{equation}

where $r$ is a given positive integer, $W^{2r+1}$ is the Wick ordered power nonlinearity of degree $2r+1$, which will be defined below. We prove that, almost surely with respect to the associated Gibbs measure, the equation (1.1) has a global strong solution, which is the unique limit of solutions to the canonical finite dimensional truncations. Moreover, this solution map keeps the Gibbs measure invariant.

For $r = 1$ (cubic nonlinearity) this was proved by Bourgain \cite{bourgain}; the results for $r \geq 2$ are new. We remark that in \cite{oh-thomann} Oh and Thomann constructed almost sure global weak solutions to (1.1) with respect to the Gibbs measure, such that at any time the law of these random solutions is again given by the Gibbs measure. The main point of the current paper is the almost sure uniqueness of solution with respect to the Gibbs measure.

1.1. Setup and the main theorem. In this section we setup the problem and state our main theorem. For a review of the background and previous works, see Section 1.2.

1.1.1. Wick ordering and Gibbs measure. We will fix a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, and a set of independent complex Gaussian random variables $\{g_k\}_{k \in \mathbb{Z}^2}$ defined on $\Omega$ that are normalized, i.e. $\mathbb{E}g_k = 0$ and $\mathbb{E}|g_k|^2 = 1$, and the law of $g_k$ is rotationally symmetric.

Let $\mathcal{V} = \mathcal{S}'(\mathbb{T}^2)$ be the space of distributions on $\mathbb{T}^2$. We define the $\mathcal{V}$-valued random variable

\begin{equation}
f = f(\omega) : \omega \mapsto \sum_{k \in \mathbb{Z}^2} \frac{g_k(\omega)}{\langle k \rangle} e^{ik \cdot x}, \quad \omega \in \Omega.
\end{equation}

Let $d\rho$ be the Wiener measure on $\mathcal{V}$, defined for Borel sets $E \subset \mathcal{V}$ by

\begin{equation}
\rho(E) = \mathbb{P}(f^{-1}(E)),
\end{equation}

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so $d\rho$ is the law of the random variable $f$. This measure $d\rho$ is a countably additive Gaussian measure supported in $\cap_{\varepsilon>0} H^{-\varepsilon}(\mathbb{T}^2)$, which we henceforth denote by $H^{0-}(\mathbb{T}^2)$ (similarly $H^{s+}(\mathbb{T}^2) = \cap_{\varepsilon>0} H^{s+\varepsilon}(\mathbb{T}^2)$), but not in $L^2(\mathbb{T}^2)$ (see e.g. [7]). Define the spectral truncation $\Pi_N$ by

$$\mathcal{F}_x \Pi_N u(k) = 1_{(k) \leq N} \cdot \mathcal{F}_x u(k),$$

(1.4)

where $\mathcal{F}_x$ is the space Fourier transform, $\langle k \rangle = \sqrt{|k|^2 + 1}$ and $1_P$ denotes the indicator function of a set or property $P$, and define the expectation of truncated mass,

$$\sigma_N := \frac{1}{(2\pi)^2} \mathbb{E} \|\Pi_N f(u)\|_2^2 \sim \sum_{\langle k \rangle \leq N} \frac{1}{\langle k \rangle^2} \sim \log N.$$ 

(1.5)

For each $N$ and each $p \geq 0$, define the Wick ordered powers,

$$W_{2^p}^2(u) = \sum_{j=0}^{p} (-1)^{p-j} \binom{p}{j} \frac{\sigma_{N}^{p-j} p!}{j!} |u|^{2j},$$

$$W_{2^{p+1}}^2(u) = \sum_{j=0}^{p} (-1)^{p-j} \binom{p+1}{p-j} \frac{\sigma_{N}^{p-j} p!}{j!} |u|^{2j},$$

(1.6)

and the canonical finite dimensional truncations for (1.1),

$$\begin{align*}
(i\partial_t + \Delta) u_N &= \Pi_N W_{2^{p+1}}^2(u_N), \\
\varepsilon u_N(0) &= \Pi_N u_0.
\end{align*}$$

(1.7)

The following proposition ensures the convergence of the right hand side of (1.7) as $N \to \infty$, and provides the definition of $W_{2^{p+1}}^2(u)$ in (1.1).

**Proposition 1.1.** Let $n$ be a nonnegative integer. Then almost surely in $u$ with respect to the Wiener measure $d\rho$, the limit

$$\lim_{N \to \infty} W_N^n(\Pi_N u) = \lim_{N \to \infty} \Pi_N W_N^n(\Pi_N u)$$

exists in $H^{0-}(\mathbb{T}^2)$. We will denote this limit by $W^n(u)$.

For each $N$, we also define the truncated potential energy

$$V_N[u] = \frac{1}{r+1} \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} W_{2^{p+2}}^2(\Pi_N u) \, dx.$$ 

(1.8)

By Proposition 1.1, the limit quantity

$$V[u] = \lim_{N \to \infty} V_N[u] = \frac{1}{r+1} \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} W_{2^{p+2}}^2(u) \, dx$$

(1.9)

is defined $d\rho$-almost surely in $u$. One can verify that (1.7) is a finite dimensional Hamiltonian system with Hamiltonian

$$H_N[u] := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} |\nabla u|^2 \, dx + V_N[u].$$

(1.10)

This $H_N[u]$, as well as the mass $M[u] := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} |u|^2 \, dx$, is conserved under the flow (1.7).

**Proposition 1.2.** Define the measure $d\mu$ by

$$d\mu = Z^{-1} e^{-V[u]} \, d\rho, \quad Z = \int_{\mathcal{V}} e^{-V[u]} \, d\rho(u)$$

(1.11)

Then it is mutually absolutely continuous with $d\rho$, and the Radon-Nikodym derivative $Z^{-1} e^{-V[u]}$ belongs to $L^q(d\rho)$ for any $1 \leq q < \infty$. We call this $d\mu$ the Gibbs measure for (1.1).
Propositions 1.1 and 1.2 stem from seminal works of Glimm and Jaffe \[30\], Simon \[65\] and Nelson \[57, 58\] in the context of quantum field theory. See also Da Prato-Tubaro \[22\]. As stated, a proof of these propositions can be found in \[60\].

1.1.2. The main theorem. We can now state our main theorem.

**Theorem 1.3.** There exists a Borel set $\Sigma \subset \mathcal{V}$ with $\mu(\mathcal{V}\setminus\Sigma) = 0$, such that $W^{2r+1}(u) \in H^{0-}(\mathbb{T}^2)$ is well-defined for $u \in \Sigma$. Furthermore:

1. For each $u_{in} \in \Sigma$ and each $t \in \mathbb{R}$, the solution $u_N(t)$ to (1.7) converges to a unique limit
   \[
   \lim_{N \to \infty} u_N(t) = u(t)
   \]
   in $H^{0-}(\mathbb{T}^2)$, and $u(t) \in \Sigma$ for each $t \in \mathbb{R}$. This $u(t)$ solves (1.7) in the distributional sense.

2. The limit $u(t)$ in (1.12) defines, for each $t \in \mathbb{R}$, a map from $\Sigma$ to itself: $u(t) = \Phi_t u_{in}$. These maps then satisfy the usual group properties, and keep the Gibbs measure $d\mu$ invariant, namely
   \[
   \mu(E) = \mu(\Phi_t(E))
   \]
   for any Borel set $E \subset \Sigma$.

**Remark 1.4.** In proving Theorem 1.3, we will replace $H^{0-}(\mathbb{T}^2)$ by $H^{-\epsilon}(\mathbb{T}^2)$ where $0 < \epsilon \ll 1$ and throughout the proof we will fix this $\epsilon$ (one then takes a countable intersection in $\epsilon$).

**Remark 1.5.** (1) Since the Gibbs measure $d\mu$ is mutually absolutely continuous with the Gaussian measure $d\rho$, part (1) of Theorem 1.3 can also be viewed as an almost sure global well-posedness result with random initial data $u_{in} = f(\omega)$ as in (1.2).

(2) The uniqueness in Theorem 1.3 is in the sense of unique limit of the solutions $u_N$ to the canonical truncations (1.7). This is consistent with the way similar results are stated in the stochastic setting, see \[31, 39, 40\]. The truncation frequency $N$ in (1.7) can be any positive number. For simplicity, in the proof below we will assume that $N$ is a power of two. The general case follows from placing $N$ between two adjacent powers of two, say $\frac{N'}{2}$ and $N'$, and analyzing the difference $u_{N'} - u_N$ in the same way as $u_{N'} - u_{\frac{N'}{2}}$.

(3) The advantage of the truncation (1.7) is that it preserves its own (finite dimensional) Gibbs measure, and is thus suitable for global-in-time arguments. In establishing the local theory, it is possible to replace (1.7) with other canonical truncations, say by using a smooth Fourier multiplier instead of $\Pi_N$, or by truncating only the initial data but not the nonlinearity. The proof will essentially be the same, but needs slight adjustments in a few places. We will not pursue these matters here.

**Remark 1.6.** (1) Theorem 1.3 is part of the program of constructing invariant Gibbs measures and their dynamics for the (renormalized) defocusing nonlinear Schrödinger equation (1.1), see Section 1.2.1 below. With Theorem 1.3 completing all case \[3\] with $d = 2$, the remaining cases that are at least expected to be solvable are the invariance of Gibbs measure for $(d, r) = (3, 1)$, and invariance of white noise for $(d, r) = (1, 1)$. We expect both to be strictly harder than Theorem 1.3 as they are critical in the probabilistic scaling (though the $d = 1$ case may be easier due to integrability), see Section 1.3.

\[1\] Brydges and Slade \[14\] proved that the natural construction of an invariant Gibbs measure is not possible for the focusing cubic NLS when $d = 2$, unlike the case $d = 1$ as proved by Lebowitz, Rose and Speer \[50\].

\[2\] Gibbs measures in the cubic case of (1.1) are not available for $d \geq 5$. This is intimately related to the nonexistence of a $\phi^4$ theory when $d \geq 5$ \[21, 29\]. We have learned that Copin and Aizenman recently also ruled out the existence of a $\phi^4$ theory in $d = 4$. 

(2) One may also try to construct invariant Gibbs measures for nonlinear wave equations with power nonlinearity. This is in general much easier than Schrödinger due to the derivative gain in Duhamel’s formula; for example the \( d = 2 \) case was solved earlier by Oh-Thomann [62]. Here the only remaining case is \((d, r) = (3, 1)\), which is subcritical in the probabilistic scaling and likely can be done by applying the methods of this paper.

1.2. A review of previous works. We start by reviewing previous results and methods on PDEs in the probabilistic setting. As the literature is now extensive, we will put emphasis on the works most relevant to the current paper.

1.2.1. Invariant measures. Since the pioneering works of Lebowitz-Rose-Speer [50] and Bourgain [8, 9], there has been numerous results regarding invariant measures for nonlinear dispersive equations. Generally speaking, for any Hamiltonian dispersive equation one may construct the associated Gibbs measure

\[
d\mu \sim e^{-\beta H} \prod_x dx,
\]

where \( \beta > 0 \) and \( H \) is the Hamiltonian. The definition (1.14) is only formal; in some cases it can be justified by using the Gaussian measure as a reference measure and writing \( d\mu \) as a weighted Wiener measure. For example the Hamiltonian for (1.1) is

\[
H = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} (|\nabla u|^2 + \frac{1}{r+1} W^{2r+2}(u)) \, dx
\]

and the Gibbs measure

\[
d\mu \sim \exp \left[ \frac{-1}{r+1} \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} W^{2r+2}(u) \, dx \right] \cdot \exp \left[ \frac{-1}{(2\pi)^2} \int_{\mathbb{T}^2} \nabla u^2 \, dx \right] \prod_{x \in \mathbb{T}^2} dx
\]

can be rigorously defined as a weighted Wiener measure, as in Proposition 1.2. Defining such Gibbs-type measures and studying their properties under various dynamics is a major problem in constructive quantum field theory.

The Gibbs measure \( d\mu \) for a given dispersive equation is formally invariant due to a ‘formal Liouville’s Theorem’ and the conservation of Hamiltonian. It is of great interest to establish this invariance rigorously, as this would be the first step in studying the global dynamics from the statistical ensemble point of view. In [8, 9], Bourgain developed a systematic way of showing the invariance of \( d\mu \) from the invariance of finite dimensional Gibbs measures, provided one has local well-posedness or almost sure local well-posedness with respect to \( d\mu \).

Therefore, justifying the invariance of \( d\mu \) (and other similar formally invariant measures) basically reduces to proving almost sure local well-posedness on the support of \( d\mu \). As this support is very rough in high dimensions (namely \( H^{1-\frac{d}{2r}} \) for (1.1) in dimension \( d \)), most known results are limited to one dimension, or requires strong symmetry. For the Schrödinger equation (1.1) on the torus \( \mathbb{T}^d \), Bourgain [8] solved the case \( d = 1 \) and any \( r \), and [9] extended this to \( d = 2 \) and \( r = 1 \). These are the only results known to date. For wave equations slightly more is known; Oh-Thomann [60] solved the case \( d = 2 \) and any \( r \) (\( d = 1 \) being much easier). In dimension \( d = 3 \) both problems remain open for the cubic equation (see the scaling calculations in Section 1.3).

Apart from the standard Schrödinger and wave models on tori, there are many results, again mostly in one dimension or on manifolds under radial symmetry, where invariance of Gibbs measure (or of associated weighted Wiener measures) is justified for various dispersive models on various

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1 Strictly speaking the measure defined in Proposition 1.2 involves an additional weight which is an exponential of the \( L^2 \) mass. As the mass is also conserved, this does not affect any invariance properties.
background manifolds (see e.g. [71, 59, 72, 70, 53, 55, 67, 23, 24, 26, 64, 69, 68] and references therein). We also mention the recent developments of the compactness methods of Albeverio and Cruzeiro [2] where one explores the tightness of the sequence of finite dimensional measures and apply the theorems of Prokhorov and Skorokhod to obtain existence of weak solutions (see e.g. [15, 54, 60, 73]). These are less related to the current paper and we will not elaborate further.

1.2.2. Probabilistic well-posedness theory. It has long been known that PDEs with randomness generally behave better in terms of local well-posedness (i.e. probabilistic well-posedness goes below the deterministic well-posedness threshold). Progress have been made in two parallel directions: random initial data problems and stochastically forced problems.

The first results along this line are due to the seminal works by Bourgain [9, 11] in the random data setting and later to Da Prato-Debussche [20, 21] in the stochastic setting. The idea in both works is to make a linear-nonlinear decomposition and observe the effect of probabilistic smoothing.

For example, in [9], the equation (1.1) with \( r = 1 \) on \( \mathbb{T}^2 \) is studied with random initial data in \( H^{-\varepsilon} \) for some \( 0 < \varepsilon \ll 1 \), in which (1.1) is ill-posed. However with randomness one may construct solution to (1.1) that has the form \( u = e^{it\Delta}u(0) + v \), where \( u(0) \in H^{-\varepsilon} \) is the random initial data, and \( v \) belongs to some positive Sobolev space in which (1.1) is well-posed. In other words, this solution contains a rough random part \( u_{\text{lin}} := e^{it\Delta}u(0) \) and a smooth remainder \( v \). The point here is that, even though \( u_{\text{lin}} \) is rough, it has the explicit random structure which allows one to control the nonlinear interactions between \( u_{\text{lin}} \) and \( u_{\text{lin}} \), and between \( u_{\text{lin}} \) and \( v \), in a more regular space.

Until recently the method of Bourgain, as well as its higher order variants which include some nonlinear interactions of \( u_{\text{lin}} \) with itself into the rough random part\(^1\), has been the dominant strategy of exploiting randomness in local well-posedness theory for dispersive and wave equations with random data. After Bourgain’s pioneering work, there has been substantial success (for a sample of works, we refer the readers to [8, 9, 16, 19, 23, 12, 56, 74, 13, 47, 6, 27, 46] and references therein). However this method by itself has its limitations and does not lead to optimal results in most cases.

A few years ago, a series of important works emerged, which greatly advanced the study of local well-posedness for stochastically forced PDEs, in fact reaching the optimal exponents in the parabolic case. These include the theory of regularity structures of Hairer [39, 40, 41, 42] and the para-controlled calculus of Gubinelli-Imkeller-Perkowski [31, 32]. A third method based on Wilsonian renormalization group analysis was independently proposed by Kupiainen in [49].

The theory of regularity structures is based on the local-in-space properties of solutions at fine scales (so it is particularly suitable for parabolic equations); it builds a general theory of distributions which includes the profiles coming from the noise, and allows one to perform multiplications and thus analyze the nonlinearity. Since its success with the KPZ equation [39] and the \( \Phi^4_3 \) model [40], this theory has been developed by Hairer and collaborators and is now powerful enough to solve a wide range of problems that are subcritical according to a suitable parabolic scaling. We will not get into the details, but we refer the reader to [28, 42, 43, 44, 45, 18, 52, 51] and references therein for nice expositions of these ideas.

The theory of para-controlled calculus, which is in spirit the point of departure of the present paper, takes a different approach and is based on the following idea. In the approach of Bourgain and of Da Prato-Debussche mentioned above, some nonlinear interactions between \( u_{\text{lin}} \) and \( v \) may not have enough regularity despite \( v \) being more regular than \( u_{\text{lin}} \). A key observation however, is that the only bad terms here are the high-low interactions where the high frequencies come from \( u_{\text{lin}} \) and the low frequencies come from \( v \), and such terms can be para-controlled by the high-frequency inputs (which are nonlinear interactions of \( u_{\text{lin}} \) with itself). Here \( f \) being para-controlled

\(^1\)This usually results in a finite or infinite tree expansion.
by $g$ simply means that $f$ equals the high-low paraproduct of $g$ with some other function $h$, up to a smoother remainder. With such structure one can show that these para-controlled terms have similar randomness structures as the nonlinear interactions of $u_{\text{lin}}$ with itself and can then be handled similarly as in Bourgain’s or Da Prato-Debussche’s approach, leaving an even smoother remainder. An example \cite{31} is the cubic heat equation with additive white noise on $\mathbb{T}^3$, 
$$(\partial_t - \Delta)u = u^3 + \xi,$$
where $\xi$ is the spacetime white noise (the actual equation involves some subtle renormalization which we omit for simplicity). The solution one constructs has the form 
$$u = u_{\text{lin}} + u_{\text{cubic}} + X + Y,$$
where 
$$u_{\text{lin}} = J\xi, \quad u_{\text{cubic}} = J(u_{\text{lin}}^3), \quad X = \sum_N J(P_N(u_{\text{lin}}^2) \cdot P_{\ll N}(u - u_{\text{lin}})),$$
$J$ is the Duhamel operator associated with the heat equation, and $P_N$ etc. are standard Littlewood-Paley projections. The term $X$ para-controlled by the bilinear Gaussian $u_{\text{lin}}^2$ will be constructed in some less regular space, which allows the remainder $Y$ to be constructed in a more regular space. The para-controlled calculus also has a higher order variant, see \cite{3, 5}, which is believed to have comparable power to the theory of regularity structures. We refer the reader to \cite{17, 35, 36, 37, 38, 51, 63, 3, 4, 5} and references therein for a nice exposition of these ideas and some other recent developments on this method.

Finally, we would like to mention two very recent results of Gubinelli-Koch-Oh \cite{33} and Bringmann \cite{13}. In \cite{33} the authors applied a version of para-controlled calculus to the stochastic wave equation setting, and obtained almost sure local well-posedness for a quadratic wave equation with additive white noise on $\mathbb{T}^3$. In \cite{13} the author studied the nonlinear wave equation with quadratic derivative nonlinearity on $\mathbb{R}^3$ and improved the known well-posedness threshold with random initial data, again by analyzing high-low interactions. The interesting observation made in \cite{13} is that, in the para-controlled scheme above (or anything similar), one may in fact reduce matters to the high-frequency and low-frequency parts being independent, which allows one to use more powerful probabilistic tools and obtain better estimates.

1.2.3. Discussion. From the results in Section 1.2.2 one notices that, for Schrödinger equations, the probabilistic improvement (defined as the difference between exponents of the deterministic $H^s$ well-posedness threshold and the obtained probabilistic $H^s$ well-posedness threshold) is much smaller compared to wave and heat equations\footnote{There are a few results \cite{23, 27} which obtain a larger improvement for Schrödinger, but these are all in the non-compact setting where much better estimates (such as local smoothing) are available.}. For example, the probabilistic improvement in \cite{9} for $(1.1)$ with $(d,r) = (2,1)$ is $\varepsilon \ll 1$ and the improvement in \cite{56} with $(d,r) = (3,2)$ is $\frac{1}{12}$, compared to the improvement $\approx 1$ for wave obtained in \cite{60} (and similarly for heat).

There are two reasons for this. First, heat equations are compatible with Hölder spaces $C^s$, which scales much higher than $H^s$, but a function with independent Gaussian Fourier coefficients that belongs to $H^s$ will automatically belong to $C^{s_\varepsilon}$ due to Khintchine’s inequality. This allows one to have a scaling at a higher regularity and hence be in a better situation when studying the heat equations. Such advantage is absent in Schrödinger and wave equations, since $C^s$ spaces are not compatible even with linear evolution, and cannot be used in any well-posedness theory. Second, the Duhamel evolution for heat equation gains two derivatives, and wave equation gains one. This allows for room to apply Sobolev embedding, and also reduces the task of controlling the nonlinearity to the task of making sense of products, which is still hard but at least more
manageable. In comparison, the Schrödinger Duhamel evolution has no smoothing effect, and it can be challenging to close the estimate even when the relevant products are well-defined.

1.3. Probabilistic scaling and a general conjecture. The proof of Theorem 1.3 consists of two parts: (a) proving almost sure local well-posedness for (1.1) on the support of the Gibbs measure, and (b) applying invariance of truncated measures to extend local solutions to global ones. Apart from a few technical subtleties, part (b) is essentially an adaptation of the classical Bourgain’s proof [9] and nothing is fundamentally new, so let us focus on the local theory.

The obvious difficulty here is that the Gibbs measure $d\mu$ is supported in $H^{0-}$, while the (deterministic) scaling threshold, below which (1.1) is ill-posed, is $s_c = 1 - \frac{1}{r} \to 1$ as $r \to \infty$. In the language of Section 1.2.3 one needs to obtain a probabilistic improvement $\approx 1$. Therefore, it is important to understand exactly how randomness allows us to beat scaling. Before describing our method in Section 1.3 below, we will first perform a heuristic analysis which, to the best of our knowledge, has not appeared in the literature before.

1.3.1. The probabilistic scaling. Consider the Wick ordered nonlinear Schrodinger equation

$$(i\partial_t + \Delta)u = W^{2r+1}(u), \quad u(0) = u_{in}$$

(1.15)

on $\mathbb{T}^d$. For simplicity we will replace $W^{2r+1}(u)$ by the pure power $|u|^{2r} u$ below. The scaling critical threshold for (1.15) is

$$s_c = d - \frac{1}{r},$$

(1.16)

and (1.15) is expected to be locally well-posed in $H^s$ only if $s \geq s_c$. This can be demonstrated in different ways, with the one most relevant to us as follows: suppose the initial data $u_{in}$ has Fourier transform $\mathcal{F}_x u_{in}(k)$ supported in $|k| \sim N$ with $|\mathcal{F}_x u_{in}(k)| \sim N^{-\alpha}$ with $\alpha = s + \frac{d}{2}$, then $\|u_{in}\|_{H^s} \sim 1$.

If local well-posedness holds then one should expect that the second iteration (say at time $t = 1$),

$$u^{(2)} := \int_0^1 e^{i(1-t')\Delta}(|u|^{2r} u) e^{it'\Delta} u_{in} dt',$$

satisfies $\|u^{(2)}\|_{H^s} \lesssim 1$. By making Fourier expansions, we essentially get

$$\mathcal{F}_x u^{(2)}(k) \sim \sum_{k_1 + \cdots + k_{2r+1} = k, |k_j| \lesssim N} \frac{1}{|\Sigma|} \prod_{j=1}^{2r+1} \hat{u}_{in}(k_j), \quad \Sigma = |k|^2 - |k_1|^2 + \cdots - |k_{2r+1}|^2,$$

(1.17)

where complex conjugates are omitted. In the worst scenario this gives, up to log factors,

$$|\mathcal{F}_x u^{(2)}(k)| \lesssim N^{-(2r+1)\alpha} \sup_{m \in \mathbb{Z}} \#S_m,$$

where $S_m = \{(k_1, \ldots, k_{2r+1}) : k_1 - \cdots + k_{2r+1} = k, |k_j| \lesssim N, \Sigma = m\}$. By dimension counting one expects $\#S_m \lesssim N^{2rd-2}$, so in order for $\|u^{(2)}\|_{H^s} \lesssim 1$ we need $-(2r + 1)\alpha + 2rd - 2 \leq -\alpha$, or equivalently $s \geq s_c$.

Now what happens if one assumes the Fourier coefficients of initial data $\{\mathcal{F}_x u_{in}(k)\}$ are independent Gaussians of size $N^{-\alpha}$? The sum (1.17) will then be a sum of products of independent Gaussian random variables, which is reminiscent of the classical Central Limit Theorem. Recall that in the latter we have a sum of $M$ independent random objects of unit size, and under certain general conditions, this sum scales only like $\sqrt{M}$ as opposed to $M$ if without randomness. In the same way, we would expect essentially a ‘square root gain’ here, that is,

$$|\mathcal{F}_x u^{(2)}(k)| \lesssim N^{-(2r+1)\alpha} \sup_{m \in \mathbb{Z}} (\#S_m)^{1/2} \lesssim N^{-(2r+1)\alpha + rd - 1},$$

where $\mathcal{F}_x u^{(2)}(k)$ is the Fourier transform of $u^{(2)}(k)$.
so in order for $\|u^{(2)}\|_{H^s} \lesssim 1$ it suffices to have $-(2r + 1)\alpha + rd - 1 \leq -\alpha$, or equivalently

$$s \geq s_p := -\frac{1}{2r}. \quad (1.18)$$

Note that $s_p$ is independent of the dimension and that we always have $s_p \leq s_c$. We will call this $s_p$ the critical threshold for probabilistic scaling.

Of course the above argument is purely heuristic, and in particular ignores the important issue of high-low interactions (which will be a main difficulty in the current paper, see Section 1.4 below). Nevertheless we believe the following conjecture is natural:

**Conjecture 1.7.** Let $r$ and $d$ be positive integers and $s > s_p$. Then (1.15) is almost surely locally well-posed (in the sense similar to Theorem 1.3) with random initial data

$$u_{in} = \sum_{k \in \mathbb{Z}^d} g_k e^{ik \cdot x}, \quad \alpha = s + \frac{d}{2}, \quad (1.19)$$

where $\{g_k\}$ are independent Gaussian random variables with $\mathbb{E} g_k = 0$ and $\mathbb{E} |g_k|^2 = 1$. Note that almost surely the initial data $u_{in}$ belongs to $H^{s - (T_d)}$.

**Remark 1.8.** Since the Gibbs measure for (1.1), defined in (1.11), is supported in $H^{0-}$ and $s_p = -\frac{1}{2r}$, (the local part of) Theorem 1.3 is a special case of Conjecture 1.7. Moreover when $r \to \infty$ we have $s_p \to 0$, so the result of Theorem 1.3 gets to be almost sharp (i.e. almost reaches the conjectured exponent) when $r \to \infty$.

This leads us to believe that Conjecture 1.7 is reasonable, and may not be too far from reach; on the other hand, trying to prove it for fixed values of $r$ and $d$, even in the simplest case $(r,d) = (1,2)$, requires new ideas.

**1.3.2. More general settings.** The heuristic discussions in the previous subsection can be extended to more general situations. These include, but are not limited to, the followings:

1. **Wave equations.** For wave equation (say with a power nonlinearity as in (1.15)) one can apply the same heuristics as in Section 1.3.1, where instead of (1.17) we have essentially

$$|F_x u^{(2)}(k)| \sim \frac{1}{\langle k \rangle} \sum_{k_1, \ldots, k_{2r+1} = k} \frac{1}{\Sigma} \prod_{j=1}^{2r+1} \hat{u}_{in}(k_j), \quad \Sigma = |k| - |k_1| + \cdots - |k_{2r+1}|. \quad (1.20)$$

Assume now $|k| \sim N$, then compared to (1.17) one gains an extra factor $N^{-1}$ due to the antiderivative, while in the in the dimension counting argument one gains one less power of $N$ as $\Sigma$ is now linear instead of quadratic. In the deterministic setting this leads to the same scaling condition as the Schrödinger equation, but in the probabilistic setting this trade-off leads to a *better* bound than in the Schrödinger case as the one-dimension disadvantage gets ‘square-rooted’ by exploiting randomness as explained above. This then gives

$$|F_x u^{(2)}(k)| \lesssim N^{-(2r+1)\alpha - 1} N^{-rd - \frac{d}{4}},$$

which leads to a *lower* probabilistic scaling threshold, namely $s_p^{\text{wave}} = -\frac{3}{4r}$.

However, unlike Schrödinger, there is also a ‘high into low’ interaction, namely $|k| \sim 1$ in (1.20), that needs to be addressed. A similar calculation using randomness and counting bounds yields heuristically that

$$|F_x u^{(2)}(k)| \lesssim N^{-(2r+1)\alpha} N^{-rd - \frac{d}{4}},$$

This is associated with Gaussian random variables, but by the Central Limit Theorem, the scaling should be the same for more general types of random variables.
which leads to the restriction $s \geq s_p' := \frac{d-1}{2(2r+1)}$. Thus it is reasonable to conjecture that the wave equation is almost surely locally well-posed in $H^s \times H^{s-1}$ for

$$s > \max(s^\text{wave}, s_p') = \max\left(\frac{-3}{4r}, -\frac{d+1}{4r+2}\right);$$

in particular when $(r,d) = (1,3)$ the conjectured threshold is $H^{-\frac{3}{4}}$, which is below $H^{-\frac{1}{2}}$ where the Gibbs measure is supported.

(2) Other dispersion relations and/or nonlinearities. For more general dispersion relations, say $\Lambda = \Lambda(k)$, the only thing above that changes is the counting bound for the set

$$S_m = \{(k_1, \ldots, k_{2r+1}) : k_1 - \cdots + k_{2r+1} = k, \Sigma := \Lambda(k) - \Lambda(k_1) + \cdots - \Lambda(k_{2r+1}) \in [m, m+1]\}.$$

In contrast to parabolic equations (see [40]) where the exact form of the elliptic part is irrelevant once the order is fixed, here the properties of $S_m$ depend crucially on the choice of $\Lambda$, and have to be analyzed on a case by case basis. For simple dispersive relations like Schrödinger, wave or gravity water wave (where $\Lambda$ is also possible but has worse behavior), say of form

$$-\Delta u = |u|^{2^*}u + \zeta,$$

which belongs to the same space. By similar arguments, this time also taking into account the time Fourier variable, one can show that this leads to the restriction $r(d-2) < 2$, which coincides with the subcriticality condition introduced in [40] in the case of (1.21).
1.4. The random averaging operator method. We now turn to the proof of Theorem 1.3. To bring about the main ideas we focus our attention here to the question of almost sure local well-posedness of (1.1) in the support of the Gibbs measure \(d\mu\), namely \(H^0(\mathbb{T}^2)\), and assume \(u_{in} = f(\omega)\) where \(f\) is as in (1.2). We will also replace \(W^{|2r+1}(u)\) by the pure power \(|u|^{2r} u\), like in Section 1.3.1.

1.4.1. Main challenges. A naive attempt would be to follow Bourgain’s approach [2] and look for solutions to (1.1) of form \(u(t) = e^{it\Delta}u_{in} + w\) where \(w\) belongs to \(C^0_tH^s_x\), or more precisely \(X^{s, \frac{1}{2}+}\) (see Section 2.3 for relevant definitions) for some positive \(s\); this \(w\) in particular will contain components of form

\[
u^{(2)}(t) = \mathcal{I}(|e^{it\Delta}f(\omega)|^{2r} e^{it\Delta}f(\omega)) = IF(t) := \int_0^t e^{i(t-t')\Delta} F(t') dt'.
\]

(1.22)

However, it is shown in [9] that even when \(r = 1\) (and obviously also for larger \(r\)), the \(u^{(2)}\) defined in (1.22) belongs to \(X^{s, \frac{1}{2}+}\) only for \(s < \frac{1}{2}\). As the space \(X^{\frac{1}{2} - \frac{1}{2}+}\) is still supercritical with respect to deterministic scaling for \(d = 2\) and \(r \geq 2\), there will be no hope in solving (1.1) using the above ansatz. One may perform higher order Picard iterations, but it turns out that regardless of the order, there is always some contribution in the remainder that has regularity \(X^{\frac{1}{2} - \frac{1}{2}+}\).

Here we make the first observation, namely that the poor regularity of \(u^{(2)}\) is only due to high-low interactions. In fact, by carrying out the analysis of Section 1.3.1 one can show that if the two highest frequencies in the input factors of (1.22) are comparable, for example consider

\[
u^{(2)}_{(1)}(t) = \sum_N \mathcal{I}(|e^{it\Delta}P_N f(\omega)|^2 \cdot |e^{it\Delta}P_{\leq N} f(\omega)|^{2r-2} e^{it\Delta}P_{\leq N} f(\omega)),
\]

where \(P_N\) and \(P_{\leq N}\) are standard Littlewood-Paley projections, then \(u^{(2)}_{(1)}\) will belong to \(X^{1 - \frac{1}{2}+}\) which is (deterministically) subcritical for any \(r\). Therefore to solve (1.1) it suffices to control the high-low interactions, say the ones of the form

\[
X = \sum_N \mathcal{I}(e^{it\Delta}P_N f(\omega) \cdot |P_{\leq N} u|^{2r}),
\]

(1.23)

and to show that this term somehow has similar behavior as the linear evolution \(e^{it\Delta}f(\omega)\).

The above idea is inspired by the paracontrolled analysis of Gubinelli-Imkeller-Perkowski [31], see Section 1.2.2, however due to the different nature of Schrödinger equation compared to heat and wave equations, their method does not work here. For example, suppose one defines a term \(X\) by (1.23), which is paracontrolled by \(e^{it\Delta}f(\omega)\), then one would need to have reasonable bounds on the low-frequency component \(|P_{\leq N} u|^{2r}\). In the paracontrolled ansatz, one would like to construct \(X\) in some less regular space (say \(X^{\frac{1}{2} - \frac{1}{2}+}\)) and the remainder \(Y = u - e^{it\Delta}f(\omega) - X\) in a more regular space (say \(X^{1 - \frac{1}{2}+}\)). Then the low frequency component can be expanded as

\[
|P_{\leq N} u|^{2r} = |P_{\leq N} e^{it\Delta}f(\omega)|^2 + P_{\leq N} X + P_{\leq N} Y|^{2r}.
\]

1This may be improved a little by switching to Fourier-Lebesgue spaces, but is still far from deterministic scaling when \(r\) is large. On the other hand the Hölder spaces \(C^s\), as pointed out in Section 1.2.3 are not suitable for Schrödinger equations.

The description of para-controlled calculus in Section 1.2.2 actually corresponds to \(|P_{\leq N} u|^{2r} - |P_{\leq N} e^{it\Delta}f(\omega)|^{2r}\) instead of \(|P_{\leq N} u|^{2r}\) in (1.23), but as will be clear below this does not matter. The choice of (1.23) is simpler for our purpose.
Here, if all the factors are \( P_{< N} e^{it\Delta} f(\omega) \), then the corresponding product can be bounded using estimates for multilinear Gaussians; however we also have the situation where all the factors are \( P_{\leq N} X \). As \( X \) is supposed to be constructed in the less regular space \( X^{\frac{1}{2} - \frac{1}{r} +} \), the only thing we know about \( X \) is its regularity, and as this regularity is supercritical, this alone will not be able to guarantee any bound for \( |P_{< N} X|^{2r} \) better than (say) \( |P_{< N} X|^{2r} \in X^{-10, \frac{1}{r} +} \) when \( r \) is large, which is clearly not enough.

Another attempt would be to strengthen the definition of paracontrolling by requiring the low-frequency part to have frequency \( \ll N^\alpha \) for some \( \alpha \in (0,1) \), like in [13]. In this case, however, this \( \alpha \) cannot be too small since otherwise the high-high interaction (say)

\[
\sum_{N} I(e^{it\Delta} P_{N} f(\omega)) \cdot |e^{it\Delta} P_{\leq N} f(\omega)|^{2r-2} e^{it\Delta} P_{\leq N} f(\omega)
\]

(1.24)
cannot be placed in a subcritical space. In fact, calculations show that \( u^{(2)}_\alpha \) defined by (1.24) belongs to \( X^{\frac{1}{2} - \frac{1}{r} +} \), so subcriticality implies \( \alpha \geq 1 - \frac{2}{r} \). When \( r \) is large \( \alpha \) has to be close to 1 and the above issue persists.

Note that when \( r = 2 \), it might be possible to carry out the above with a small \( \alpha \) by doing some refined analysis (which is by no means immediate in view of all the difficulties for the Schrödinger equation; see Section 1.2.3). Our approach, which is described below, allows instead for a unified treatment for all values of \( r \) by synthesizing the main underlying ideas and capturing the true randomness structure of the solution.

1.4.2. Random averaging operators. We propose a new method to resolve the issues above which goes beyond the para-controlled calculus [31, 34, 3] and is based on the following two key observations.

The first observation is that, in (1.23), we may replace the low-frequency part \( |P_{< N} u|^{2r} \) by \( |u_{< N}|^{2r} \), where \( u_{< N} \) is the solution to (1.1) with initial data \( P_{< N} f(\omega) \), in the sense that the difference \( |P_{< N} u|^{2r} - |u_{< N}|^{2r} \) contains at least one high-frequency factor and can be regarded as high-high interaction. The point here is that \( |u_{< N}|^{2r} \), by definition, is a measurable function of \( (g_k(\omega))_{|k|< N} \), and is hence independent with the linear factor \( e^{it\Delta} P_{N} f(\omega) \). This independence will allow us to apply all the large deviation estimates, and will play an important role in the proof. Note that the same idea is also used in [13].

The second and more important observation is that, although the low-frequency factor \( |P_{< N} X|^{2r} \) cannot be controlled with \( X \) only assumed to be in \( X^{\frac{1}{2} - \frac{1}{r} +} \), in reality this \( X \) is not an arbitrary function in \( X^{\frac{1}{2} - \frac{1}{r} +} \). This \( X \) has its own structure - namely it is para-controlled in itself. It is vital to be able to exploit this structure in order to complete the proof of Theorem 1.3. The idea of exploiting the randomness structure of low-frequency components was present in Bourgain [10], though in that case one has substantial smoothing and only the simplest structure is needed.

In order to apply these two observations, especially the second one, one possibility is to perform an iteration: in order to control \( |P_{< N} X|^{2r} \) we need to use the para-controlled structure of \( X \), for example (where we identify \( P_{< L} u \) and \( u_{< L} \))

\[
|P_{< N} X| = \sum_{L \leq N} I(e^{it\Delta} P_{L} f(\omega)) \cdot |P_{< L} u|^{2r},
\]

(1.25)

\[1\]We remark that the situation here is different from the high-order paracontrolled calculus described in for example [3]. In the latter one also has para-controlling formulas (with \( \pi_\leq \) being the standard paraproduct) \( X = \pi_\leq (v, Z) \), where \( v \) has its own paracontrolled structure, but this structure is only used in bounding remainder terms involving things like \( \pi_\geq (v, Z) \), instead of the para-controlled term \( X \).
which reduces the task to controlling \(|P_{\leq L} u|^{2r}\) and \(|P_{\leq L} X|^{2r}\). We then continue this process until the frequency reaches 1 and analyze the resulting multilinear expression. Unfortunately the structure of such multilinear expression is so complicated, that it is hard to extract from it anything useful.

In this paper we take a different approach, in the spirit of our previous work \cite{25} in the deterministic setting, namely we will extract all the randomness properties of \(X\), as well as properties of the multilinearity \(| \cdot |^{2r}\) when applied to these random objects, and turn them into two particular norm bounds for the operator

\[ P : y \mapsto \sum_N I(P_N y \cdot |P_{\leq N} X|^{2r}). \]  

The norms we choose are the operator norm \(\| \cdot \|_{\text{OP}}\) and the Hilbert-Schmidt norm \(\| \cdot \|_{\text{HS}}\) with \(P\) viewed as a linear operator from the space \(X^{s, \frac{1}{2}+}\) to itself (this does not depend on \(s\) so we may in fact choose \(s = 0\)); when the input \(y\) in (1.26) is a linear Schrödinger flow, we get an operator from \(H^s\) to \(X^{s, \frac{1}{2}+}\), and will also measure the corresponding operator and Hilbert-Schmidt norms. Suppose the maximum frequency of \(P_{\leq N} X\) in (1.26) is \(L \ll N\), then roughly speaking, these norm bounds will look like

\[ \|P\|_{\text{OP}} \lesssim L^{-\delta_0}, \quad \|P\|_{\text{HS}} \lesssim N^{\frac{1}{2}+\delta_1} L^{-\frac{1}{2}}, \]  

where \(\delta_1 \ll \delta_0 \ll 1\); note that these bounds are obviously false for arbitrary functions \(X \in X^{s, \frac{1}{2}+}\), so they really encode the randomness structure of \(X\). See Proposition 3.2 for details.

This operator \(P\), which will be of central importance in our proof, depends on the quantity \(X\) that has an implicit randomness structure. Moreover, in the Fourier variables, this operator can be viewed as a weighted average on smaller scales \(L \ll N\). We are thus calling it a \textit{random averaging operator}, which explains the name of our method.

1.4.3. The full ansatz. We can now write down the full ansatz of the solution \(u\) to (1.1), namely that

\[ u = e^{it \Delta} f(\omega) + P(e^{it \Delta} f(\omega)) + z, \]  

where \(z \in X^{1-\frac{1}{2}+}\) is a smooth remainder, and the random averaging operator \(P\) is of form

\[ P = \sum_N \sum_{L \ll N} P_{NL}, \]  

where \(P_{NL}\) has the form (1.26) with the maximum frequency of \(P_{\leq N} X\) in (1.26) being \(L\). This \(P_{NL}\) is a Borel function of \((g_k(\omega))_{(k) \leq L}\) - which is independent with \(e^{it \Delta} P_N f(\omega)\) - and satisfies (1.27). See Section 3.1 for the precise formulas.

With the ansatz (1.28), the proof of local well-posedness then goes by inducting on frequencies to show (1.27) and to bound \(z\) in \(X^{1-\frac{1}{4}+}\). In fact, suppose these are true for components of frequency \(\ll N\), then the bounds (1.27) imply that the part of \(P_{\leq N} u\) (or \(u_{\leq N}\)) involving the random averaging operators really behaves like a linear Schrödinger flow, so in \(P_{NL}\), see (1.26), we can effectively assume \(X\) is a linear flow. Hence (1.27) follows from large deviation estimates for multilinear Gaussians and a \(T^*T\) argument like the one of Bourgain \cite{9}, and the estimate for \(w\) follows from standard contraction mapping arguments. See Sections 3 and 5 for details.

In Section 1.5 we collect some of the notations and conventions that will be used in the proof. The rest of the paper is organized as follows. In Section 2 we introduce the gauge transform and reduce to a favorable nonlinearity, and define the norms that will be used in the proof below. In Section 3 we identify the precise structure of the solution according to the ideas of Section 1.4, and reduce local well-posedness to some multilinear estimates, namely Proposition 3.3. Section 4 then sets up the necessary tools (large deviation and counting estimates) needed in the proof of
Proposition 3.3 and Section 5 contains the proof itself. Finally in Section 6 we apply an adapted version of Bourgain’s argument to extend local solutions to global ones and finish the proof of Theorem 1.3.

1.5. Notations and choice of parameters. Throughout the paper, the space and time Fourier transforms will be respectively fixed as

$$(F_x u)(k) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{-ik \cdot x} f(x) \, dx, \quad (\mathcal{F}_t u)(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi t} f(x) \, dt.$$  \hspace{1cm} (1.29)

We will be working in $(k, t)$ or $(k, \xi)$ variables, instead of the $x$ variable; so we will abbreviate $(F_x u)(k)$ simply as $u_k$, and will abuse notations and write $u = u_k(t)$. The symbol $\bar{u}$ will always represent time Fourier transform, so $(\mathcal{F}_{t,x} u)(k, \xi) = \hat{u}_k(\xi)$.

Let the space mean $\mathcal{A}$ be defined by $\mathcal{A} u = (F_x u)(0) = u_0$ (this may depend on time $t$ if $u$ does). Define the twisted spacetime Fourier transform

$$\bar{u}_k(\lambda) = \bar{u}(k, \lambda) = \bar{u}_k(\lambda - |k|^2).$$  \hspace{1cm} (1.30)

We also need to study functions $h_{k,k^*}(t)$ of variables $k, k^* \in \mathbb{Z}^2$ and $t \in \mathbb{R}$, and $h_{k,k'}(t, t')$ of variables $k, k' \in \mathbb{Z}^2$ and $t, t' \in \mathbb{R}$; for these we will define

$$\bar{h}_{k,k^*}(\lambda) = \bar{h}_{k,k^*}(\lambda - |k|^2), \quad \text{and} \quad \bar{h}_{k,k'}(\lambda, \lambda') = 2\pi \bar{h}_{k,k^*}(\lambda - |k|^2, |k'|^2 - \lambda'),$$  \hspace{1cm} (1.31)

where $\lambda$ and $\lambda'$ are Fourier variables corresponding to $t$ and $t'$ respectively.

Recall that $(k) := \sqrt{|k|^2 + 1}$, and $1_P$ is the indicator function. The cardinality of a finite set $E$ will be denoted by $|E|$ or by $\# E$. We will be using smooth cutoff functions $\chi = \chi(z)$ which equal 1 for $|z| \leq 1$ and equal 0 for $|z| \geq 2$. For any Schwartz function $\varphi$ and any $0 < \tau \ll 1$, we will define $\varphi_\tau(t) = \varphi(\tau^{-1} t)$.

For a complex number $z$ define $z^+ = z$ and $z^- = \overline{z}$; we will also use the notation $z^i$ where $i$ will always be ±. In the proof we will encounter tuples $(k_1, \ldots, k_n)$, or maybe $(k^*_1, \ldots, k^*_n)$, with associated signs $\iota_1, \ldots, \iota_n \in \{\pm\}$; they are usually linked by some equation $\iota_1 k_1 + \cdots + \iota_n k_n = d$ or $\iota_1 |k_1|^2 + \cdots + \iota_n |k_n|^2 = \alpha$, where $d$ and $\alpha$ are given, or by some expression $g_{k_1} \cdots g_{k_n}$.

Definition 1.9. In the above context we say $(k_i, k_j)$ is a pairing in $\{k_1, \ldots, k_n\}$, if $k_i = k_j$ and $\iota_i = -\iota_j$. We say a pairing is over-paired if $k_i = k_j = k_\ell$ for some $\ell \notin \{i, j\}$. Pairings and over-pairings in $\{k^*_1, \ldots, k^*_n\}$ are defined similarly.

For example, suppose $k = k_1 - k_2 + k_3 + d$. If $k_1 = k_2$, then $(k_1, k_2)$ is a pairing in $\{k_1, k_2, k_3\}$; if $k = k_1$ then $(k, k_1)$ is a pairing in $\{k, k_1, k_2, k_3\}$. If $k = k_1 = k_2 \neq k_3$, then $(k_1, k_2)$ is over-paired if considered as a pairing in $\{k, k_1, k_2, k_3\}$, but not if considered a pairing in $\{k_1, k_2, k_3\}$.

Recall Remark 1.5 (2) that for the truncation $\Pi_N$ defined in (14), $N$ will be a power of two that is also ≥ 1. The same applies to other capital letters like $M$, $L$, $R$, etc.. Define also $\Pi_N^0 = \text{Id} - \Pi_N$ and $\Delta_N = \Pi_N - \Pi_N^0$, so that

$$(\Delta_N u)_k = 1_{N/2 \leq |k| \leq N} \cdot u_k.$$  \hspace{1cm} (1.32)

Let $\mathcal{V}_N$ and $\mathcal{V}_N^0$ be the ranges of $\Pi_N$ and $\Pi_N^0$. For $N_1, \ldots, N_n$, we will define $\max^{(j)}(N_1, \ldots, N_n)$ to be the $j$-th maximal element among them, and denote it by $N^{(j)}$.

Definition 1.10. For any $N$ as above, we denote by $\mathcal{B}_{\leq N}$ the $\sigma$-algebra generated by the random variables $g_k$ for $|k| \leq N$, and by $\mathcal{B}_{N}^+$ the smallest $\sigma$-algebra containing both $\mathcal{B}_{\leq N}$ and the $\sigma$-algebra generated by the random variables $|g_k|^2$ for $k \in \mathbb{Z}^2$. 

Recall that \( \varepsilon \) is fixed by Remark 1.4. Let \( 1 \gg \delta_0 \gg \delta \) be two fixed small positive constants depending on \( r \) and \( \varepsilon \) (think of \( \delta_0 = \delta^{1/50} \)). Define the parameters

\[
\gamma = \delta^2, \quad \gamma_0 = \delta^3, \quad \kappa = \delta^{-4}, \quad b = \frac{1}{2} + \delta^4, \quad b_1 = b + \delta^4, \quad b_2 = b - \delta^6, \quad a_0 = 2b - 10\delta^6, \quad (1.32)
\]

then we have the following hierarchy:

\[
\varepsilon \gg \delta_0 \gg \gamma \gg \delta_0 \gg \gamma_0 \gg b - \frac{1}{2} = b_1 - b = \kappa^{-1} \gg \delta^6. \quad (1.33)
\]

Denote by \( \theta \) any positive quantity that is small enough depending on \( \delta \) (for example \( \theta \ll \delta^{50} \)). This \( \theta \) may take different values at different places. Let \( C \) be any large absolute constant depending only on \( r \), and \( C_0 \) be any large constant depending on \( \theta \). Unless otherwise stated, The constants in \( \lesssim, \ll \) and \( O(\cdot) \) symbols will depend on \( C_0 \). Finally, if some statement \( S \) about a random variable holds with probability \( \mathbb{P}(S) \geq 1 - C_0 e^{-A\theta} \) for some \( A > 0 \), we will say this \( S \) is \( A \)-certain.

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2. Equations, measures and norms

2.1. Wick ordering and a gauge transform. Consider a general polynomial \( \mathcal{M}_n(u) \) of degree \( n \), defined by

\[
[\mathcal{M}_n(u)]_k = \sum_{i_1 k_1 + \cdots + i_n k_n = k} a_{kk_1 \cdots k_n} u_{k_1}^{i_1} \cdots u_{k_n}^{i_n}, \quad (2.1)
\]

and similarly consider

\[
[\mathcal{H}_n(u)]_{kk'} = \sum_{i_1 k_1 + \cdots + i_n k_n + i' k' = k} a_{kk' k_1 \cdots k_n} u_{k_1}^{i_1} \cdots u_{k_n}^{i_n}, \quad (2.2)
\]

where \( a_{kk_1 \cdots k_n} \) and \( a_{kk' k_1 \cdots k_n} \) are constants. Recall the definition of pairings in Definition 1.9.

Definition 2.1. We say the polynomial in (2.1) is input-simple, if \( a_{kk_1 \cdots k_n} = 0 \) unless each pairing in \( \{k_1, \ldots, k_n\} \) is over-paired. Similarly we say it is simple, if \( a_{kk_1 \cdots k_n} = 0 \) unless each pairing in \( \{k, k_1, \ldots, k_n\} \) is over-paired, and we say the polynomial in (2.2) is simple, if \( a_{kk' k_1 \cdots k_n} = 0 \) unless each pairing in \( \{k, k', k_1, \ldots, k_n\} \) is over-paired. These notions also apply to multilinear forms.

For \( m := A|u|^2 \), define the following polynomials of degree \( n \in \{2p, 2p + 1\} \) (this \( u \) may also be replaced by \( v \)):

\[
|u|^{2p} = \sum_{j=0}^{p} (-1)^{p-j} \binom{p}{j} \frac{m^{p-j}p!}{j!} |u|^{2j}, \quad (2.3)
\]

\[
|u|^{2p+1} = \sum_{j=0}^{p} (-1)^{p-j} \binom{p+1}{p-j} \frac{m^{p-j}p!}{j!} |u|^{2j}u.
\]

We will see in the proof of Proposition 2.2 that each of these is input-simple.

Define a gauge transform \( v_N = \mathcal{G}_N u_N \) associated with (1.7) by

\[
v_N(t) = u_N(t) \cdot \exp \left( (r + 1)i \int_0^t \mathcal{A} W_{N(t)}^{2p}(u_N) \, dt' \right), \quad (2.4)
\]

Then \( u_N \) solves (1.7) if and only if \( v_N \) solves the gauged equation

\[
\left\{ \begin{array}{l}
(id_t + \Delta) v_N = \Pi_N Q_N(v_N), \\
v_N(0) = \Pi_N u_{in},
\end{array} \right. \quad (2.5)
\]
where
\[ Q_N(v) = W^{2r+1}_N(v) - (r+1)A[W^{2r}_N(v)]v. \] (2.6)

Since the gauge transform does not change the \( t = 0 \) data, we will write \( v_{in} = u_{in} \). The inverse of \( G_N \) is given by
\[ u_N(t) = v_N(t) \cdot \exp \left( -(r+1)i \int_0^t A[W^{2r}_N(v_N)] \, dt' \right), \] (2.7)
since by (1.6), if \( v = e^{i\alpha}u \) where \( \alpha \in \mathbb{R} \), then \( W^n(u) = W^n(v) \) for even \( n \) and \( W^n(u) = e^{i\alpha}W^n(v) \) for odd \( n \). Now assume \( v_N \) is a solution to (2.5). Let \( m_N \) be the truncated mass, which is conserved under (2.5),
\[ m_N = A|v_N|^2 = \sum_{(k) \leq N} |(u_{in})_k|^2, \] (2.8)
and let \( m_N^* := m_N - \sigma_N \) where \( \sigma_N \) is as in (1.5). Note that \( m_N \) and \( m_N^* \) are random terms if \( u_{in} = f(\omega) \) as in (1.2). We have the following formula for \( Q_N \):

**Proposition 2.2.** We have
\[ Q_N(v_N) = \sum_{l=0}^r \frac{(r+1)(m_N^*)^{r-l}l!}{r-l} \mathcal{N}_{2l+1}(v_N), \] (2.9)
where
\[ \mathcal{N}_{2l+1}(v) = |v|^{2l}v : -(l+1)(A : |v|^{2l}v). \] (2.10)

Here \( \mathcal{N}_{2l+1} \) is a simple polynomial of degree \( 2l+1 \). By standard procedure, we can define a \( (2l+1) \)-multilinear form, which we still denote by \( \mathcal{N}_{2l+1} \), such that it reduces to \( \mathcal{N}_{2l+1}(v) \) when all inputs equal to \( v \).

**Proof.** First we prove (2.9). By the definition of \( Q_N(v) \), see (2.6), it will suffice to obtain that
\[ W^{2r+1}_N(v_N) = \sum_{l=0}^r \frac{(r+1)(m_N^*)^{r-l}l!}{r-l} |v_N|^{2l}v_N: \] (2.11)
and
\[ (r+1)W^{2r}_N(v_N) = \sum_{l=0}^r \frac{(r+1)(m_N^*)^{r-l}l!}{r-l} (l+1)|v_N|^{2l}: . \] (2.12)

By the definition of \( |v|^{2l}v \), see (2.3), and combinatorial identities, we have
\[ \text{RHS of (2.11)} = \sum_{l=0}^r \frac{(r+1)(m_N^*)^{r-l}l!}{r-l} \sum_{k=0}^l (-1)^{l-k} \binom{l+1}{l-k} \frac{m_N^{l-k}}{k!} |v_N|^{2k}v_N, \] (2.13)
which implies (2.11) due to binomial expansion.

Similarly we can calculate
\[ \text{RHS of (2.12)} = \sum_{l=0}^r \frac{(r+1)(m_N^*)^{r-l}l!}{r-l} \sum_{k=0}^l (-1)^{l-k} \binom{l+1}{l-k} \frac{m_N^{l-k}}{k!} |v_N|^{2k} \] (2.14)
which implies (2.12).
Next we prove that \(|\alpha|^2p\) and \(|\alpha|^2pv\) are input-simple. Working in Fourier space, for any monomial
\[ \mathcal{X} := (v_{k_1})^{a_1} (\overline{v_{k_1}})^{b_1} \cdots (v_{k_n})^{a_n} (\overline{v_{k_n}})^{b_n}, \]
where the \(k_j\)'s are different, \(a_j\) and \(b_j\) are nonnegative integers, we will calculate the coefficient of \(\mathcal{X}\) in the polynomial \(|\alpha|^{2p}\) and \(|\alpha|^2pv\), and will prove that this coefficient is zero provided \(a_1 = b_1 = 1\). Now clearly the coefficient of \(\mathcal{X}\) in \(|\alpha|^{2p}\) and \(|\alpha|^2pv\), denoted by \([\mathcal{X}](|\alpha|^{2p})\) and \([\mathcal{X}](|\alpha|^2pv)\), are
\[ [\mathcal{X}](|\alpha|^{2p}) = \frac{(p!)^2}{a_1! \cdots b_1!}, \quad [\mathcal{X}](|\alpha|^2pv) = \frac{p!(p+1)!}{a_1! b_1! \cdots b_n!}, \]
under the assumptions \(b_1 + \cdots + b_n = p\) and \(a_1 + \cdots + a_n = p\) (or \(a_1 + \cdots + a_n = p + 1\)). Recall that \(m = \mathcal{A}|\alpha|^2\), we can calculate that
\[ [\mathcal{X}](|\alpha|^2p) = \sum_{l=0}^{p} (-1)^{p-l} \binom{p}{l} \frac{p!}{l! (l!)^2} \sum_{c_1 + \cdots + c_n = p-l} \prod_{s=1}^{n} c_1! (a_s - c_s)! (b_s - c_s)! \]
\[ = (-1)^p (p!)^2 \sum_{l=0}^{p} (-1)^{l} \sum_{c_1 + \cdots + c_n = p-l} \prod_{s=1}^{n} c_1! (a_s - c_s)! (b_s - c_s)!. \tag{2.15} \]
Now suppose \(a_1 = b_1 = 1\), then \(c_1 \in \{0, 1\}\); clearly the terms for \(l\) and with \(c_1 = 0\) exactly cancel the terms for \(l + 1\) and with \(c_1 = 1\), so \([\mathcal{X}](|\alpha|^2p) = 0\). Similarly we can prove \([\mathcal{X}](|\alpha|^2pv) = 0\).

Finally, we prove that \(N_{2p+1} = |\alpha|^2pv - (p+1)(\mathcal{A}|\alpha|^2pv)v\) is simple. By definition it suffices to prove that
\[ \mathcal{A}(N_{2p+1}(v\overline{v})) = \mathcal{A}(\overline{v}|\alpha|^2pv) - (p+1)m \mathcal{A}|\alpha|^2pv:\]
is input-simple. We will actually show that this equals \(\mathcal{A}|\alpha|^2p+2\); whence the result will follow. In fact, by (2.3) we have
\[ \mathcal{A}|\alpha|^2p+2 = \sum_{l=0}^{p+1} (-1)^{p-l+1} \binom{p+1}{l} \frac{p+1}{l!} m^{p-l+1} \mathcal{A}|\alpha|^{2l}, \]
\[ \mathcal{A}(\overline{v}|\alpha|^2pv) = \sum_{l=1}^{p} (-1)^{p-l} \binom{p+1}{l} \frac{p}{(l-1)!} m^{p-l+1} \mathcal{A}|\alpha|^{2l}, \]
\[ -(p+1)m \mathcal{A}|\alpha|^2pv = \sum_{l=0}^{p} (-1)^{p-l} (p+1) \binom{p}{l} \frac{p}{l!} m^{p-l+1} \mathcal{A}|\alpha|^{2l}, \]
so the first line equals the sum of the second and third lines by direct calculation. \(\square\)

Remark 2.3. Later on we will consider general multilinear forms \(N_n\) which are simple, and can be written as
\[ [N_n(v^{(1)}, \ldots, v^{(n)})] = \sum_{i_1 k_1 + \cdots + i_n k_n = k} a_{k_1 \cdots k_n} (v_{k_1}^{(1)})^{i_1} \cdots (v_{k_n}^{(n)})^{i_n}. \tag{2.16} \]
We may assume the coefficient \(a_{k_1 \cdots k_n}\) is symmetric in the \(k_j\)'s for which \(i_j = +\), and also symmetric in the \(k_j\)'s for which \(i_j = -\). Moreover, we assume that this coefficient only depends on the set of pairings among \(\{k_1, k_2, \ldots, k_n\}\).

The multilinear form \(N_{2l+1}\) corresponding to (2.10) satisfies the above properties, and we will assume without any loss of generality that \(i_j = +\) (i.e. \(N_{2l+1}\) is linear in \(v^{(j)}\)) for \(j\) odd, and \(i_j = -\) (i.e. \(N_{2l+1}\) is conjugate linear in \(v^{(j)}\)) for \(j\) even.
2.2. Finite and infinite dimensional measures. Here we will summarize some properties of the infinite dimensional and finite dimensional (or truncated) Gaussian and Gibbs measures, that will be used later in the proof.

Recall that \( \mathcal{V}_N \) and \( \mathcal{V}_N^\perp \) are respectively the ranges of the projections \( \Pi_N \) and \( \Pi_N^\perp \). We will identify \( \mathcal{V} \) with \( \mathcal{V}_N \times \mathcal{V}_N^\perp \). Let \( d\rho_N \) and \( d\rho_N^\perp \) be the Gaussian measures defined on \( \mathcal{V}_N \) and \( \mathcal{V}_N^\perp \) respectively, such that \( d\rho = d\rho_N \times d\rho_N^\perp \). Define the measures \( d\mu_N \) on \( \mathcal{V}_N \) and \( d\mu_N \) on \( \mathcal{V} \) by

\[
d\mu^0_N = Z_N^{-1} e^{-\mathcal{V}[u]} d\rho_N, \quad d\mu_N = Z_N^{-1} e^{-\mathcal{V}[u]} d\rho; \quad Z_N = \int_{\mathcal{V}_N} e^{-\mathcal{V}[u]} d\rho_N(u),
\]

then we have that \( d\mu_N = d\mu^0_N \times d\rho_N^\perp \). Recall also the measure \( d\mu \) defined in Proposition 1.2.

These are probability measures.

**Proposition 2.4.** When \( N \to \infty \) we have \( Z_N \to Z \), with \( 0 < Z < \infty \). The sequence \( Z_N^{-1} e^{-\mathcal{V}[u]} \) converges to \( Z^{-1} e^{-\mathcal{V}[u]} \) almost surely, and also in \( L^q(d\rho) \) for any \( 1 \leq q < \infty \). The measure \( d\mu_N \) converges to \( d\mu \) in the sense that the total variation of \( \mu_N - \mu \) converges to 0. Finally, the measure \( d\mu_N^\perp \) is invariant under the flows of (1.7) and (2.7).

**Proof.** The convergence results are proved in [60]. The measure \( d\mu^0_N \) is invariant under (2.5), because the latter is a finite dimensional Hamiltonian system, and

\[
d\mu^0_N(u_N) = \frac{1}{E_N} e^{-H_N[u_N] - M[u_N]} d\mathcal{L}_N(u_N)
\]

is its Gibbs measure (weighted by another conserved quantity), where \( E_N \) is some constant, \( \mathcal{H}_N \) and \( \mathcal{M} \) are as in (1.10), and \( d\mathcal{L}_N \) is the Lebesgue measure on the finite dimensional space \( \mathcal{V}_N \).

To prove that \( d\mu^0_N \) is invariant under (2.5), it suffices to show that it is preserved\(^1\) by the gauge transform \( \mathcal{G}_N \). In fact, by (1.6) and (1.10) we know \( \mathcal{H}_N[u_N] = \mathcal{H}_N[u_N] \) and \( \mathcal{M}[u_N] = \mathcal{M}[u_N] \), so it suffices to prove that \( \mathcal{G}_N \) preserves the Lebesgue measure \( d\mathcal{L}_N \). Working in the coordinates \( (r_k, \theta_k)_{(k) \leq N} \) and \( (r^*_k, \theta^*_k)_{(k) \leq N} \), which are defined by \( (u_N)_k = r_k e^{i\theta_k} \) and \( (v_N)_k = r^*_k e^{i\theta^*_k} \), we can write the measure \( d\mathcal{L}_N \) as

\[
d\mathcal{L}_N = \prod_{(k) \leq N} r_k dr_k d\theta_k.
\]

If \( v_N = \mathcal{G}_N u_N \), then we have \( r^*_k = r_k \) and \( \theta^*_k = \theta_k + F((r_j, \theta_j)_{(j) \leq N}) \), where \( F \) may also depend on \( t \), but does not depend on \( k \). Moreover, by (1.6) and (2.4) we know that \( F \) actually depends only on \( r_j \) and on the differences \( \theta_j - \theta_k \), which are invariant under the mapping \( \theta_k \to \theta^*_k \). It then follows that the transformation \( (r_k, \theta_k) \to (r^*_k, \theta^*_k) \) preserves the measure (2.18), by a simple calculation of its Jacobian. \( \square \)

2.3. Function spaces and linear estimates. From now on we will work with the equation (2.5) with the nonlinearity defined by (2.9) and (2.10), which has the form (2.16). Recall the well-known \( X^{s,b} \) spaces (where \( b \) may be replaced by \( b_1 \) or \( b_2 \))

\[
\|u\|_{X^{s,b}} = \|\langle k \rangle^s \langle \lambda \rangle^b \tilde{a}_k(\lambda)\|_{\ell^2_k L^2_\lambda}.
\]

(2.19)

We will mostly consider \( s = 0 \) and will denote \( X^{0,b} = X^b \). In addition we introduce matrix norms which measure the functions \( h = h_{kk^*}(t) \) and \( b = b_{kk^*}(t, s) \), namely

\[
\|h\|_{Y^b} = \|\langle \lambda \rangle^b \tilde{a}_{kk^*}(\lambda)\|_{\ell^2_k \to \ell^2_k L^2_\lambda}, \quad \|b\|_{Y^{0,b}} = \|\langle \lambda \rangle^{0,b} \tilde{a}_{kk^*}(\lambda,\lambda')\|_{\ell^2_k L^2_N \to \ell^2_k L^2_\lambda},
\]

(2.20)

\[
\|h\|_{Z^b} = \|\langle \lambda \rangle^b \tilde{a}_{kk^*}(\lambda)\|_{\ell^2_k L^2_\lambda}, \quad \|b\|_{Z^{0,b}} = \|\langle \lambda \rangle^{0,b} \tilde{a}_{kk^*}(\lambda,\lambda')\|_{\ell^2_k L^2_N},
\]

(2.21)

---

\(^1\)For fixed time \( t \), we can view \( \mathcal{G}_N \) as a mapping from \( \mathcal{V}_N \) to itself, by requiring \( u_N \) to solve (1.7).
where $\tilde{b} \in \{b, b_1\}$, $\| \cdot \|_{\ell_2^k \to \ell_2^k L^2_\lambda}$ and $\| \cdot \|_{\ell_4^k L^2_\lambda \to \ell_4^k L^2_\lambda}$ represent the operator norms of linear operators with the given kernels, for example

$$||b||_{Y^{b,b}} = \sup \left\{ \left. \left\| \sum_{k'} \int d\mu \cdot (\lambda)^b (\lambda')^{-b} b_{kk'}(\lambda, \lambda') y_{k'}(\lambda) \right\|_{\ell_2^k L^2_\lambda} : \|y_{k'}(\lambda')\|_{\ell_2^k L^2_\lambda} = 1 \right\}. \quad (2.22)$$

By definition one can verify that

$$||b||_{Y^{b,b}} = \sup_{\|y\|_{\chi^b} = 1} \left\| \sum_{k'} \int d\mu \cdot b_{kk'}(t, s) y_{k'}(s) \right\|_{\chi^b}. \quad (2.23)$$

For any of the above spaces, we can localize them in the standard way to a time interval $J$,

$$\|u\|_{Z(J)} = \inf \{\|v\|_Z : v \equiv u \text{ on } J\}. \quad (2.24)$$

We will need the following simple estimates.

**Proposition 2.5.** The norms $||h||_{Y^b}$ and $||b||_{Y^{b,b}}$ do not increase, when $\tilde{h}$ or $\tilde{b}$ is multiplied by a function of $(k, \lambda)$, or a function of $k^*$ (or $(k', \lambda')$ for $\tilde{h}$), which is at most 1 in $l^\infty L^\infty$ or $l^\infty$ norms. Next, if $H$ is defined by

$$\tilde{H}_{kk^*}(\lambda) = \sum_{k'} d\lambda' \cdot \tilde{b}_{kk'}(\lambda, \lambda') \tilde{h}_{kk^*}(\lambda'), \quad (2.25)$$

where $\tilde{b}_{kk'}(\lambda, \lambda')$ is supported in $|k - k'| \lesssim L$, then for any $\alpha > 0$ we have

$$\left\| \left( 1 + \frac{|k - k^*|}{L} \right)^\alpha H \right\|_{Z^b} \lesssim ||b||_{Y^{b,b}} \cdot \left\| \left( 1 + \frac{|k' - k^*|}{L} \right)^\alpha b \right\|_{Z^b}. \quad (2.26)$$

**Proof.** The first statement follows directly from definition (2.22). Now let us prove (2.26). We may fix $k^*$ and by translation invariance, we may assume $k^* = 0$. Relabeling $\tilde{H}_{k0}(\lambda) =: \tilde{H}_k(\lambda)$ and $\tilde{h}_{k0}(\lambda) =: \tilde{h}_k(\lambda)$ we may decompose

$$\tilde{H}_k(\lambda) = \sum_{M \geq L} (\tilde{H}^M)_k(\lambda); \quad (\tilde{H}^M)_k(\lambda) = \begin{cases} 1_{|k| \sim M} \tilde{H}_k(\lambda), & M > L, \\ 1_{|k| \lesssim L} \tilde{H}_k(\lambda), & M = L, \end{cases}$$

and similarly for $\tilde{h}$, so that we have

$$\left\| (\lambda)^b \left( 1 + \frac{|k|}{L} \right)^\alpha \tilde{H}_k(\lambda) \right\|^2_{\ell_2^k L^2_\lambda} \sim \sum_{M \geq L} L^{-2\alpha} M^{2\alpha} \| (\lambda)^b (\tilde{H}^M)_k(\lambda) \|^2_{\ell_2^k L^2_\lambda} \quad (2.27)$$

and similarly for $h$. Since $\tilde{b}$ is supported in $|k - k'| \lesssim L$, we have

$$\|(\tilde{H}^M)_k(\lambda)\| \lesssim \sum_{M' \sim M} \left\| \sum_{k'} d\lambda' \cdot \tilde{b}_{kk'}(\lambda, \lambda') (\tilde{H}^{M'})_{k'}(\lambda') \right\|_{\ell_2^k L^2_\lambda},$$

therefore

$$\left\| (\lambda)^b (\tilde{H}^M)_k(\lambda) \right\|^2_{\ell_2^k L^2_\lambda} \lesssim \left\| (\lambda)^b (\lambda')^{-b} \tilde{b}_{kk'}(\lambda, \lambda') \right\|^2_{\ell_2^k L^2_\lambda} \sum_{M' \sim M} \left\| (\lambda')^b (\tilde{H}^{M'})_{k'}(\lambda') \right\|^2_{\ell_2^k L^2_\lambda},$$

which, combined with (2.27), implies (2.26). \qed

Let $\chi$ be a smooth cutoff as in Section 1.3 and define the time truncated Duhamel operator

$$\mathcal{I}F(t) = \chi(t) \int_0^t e^{i(t-t')\Delta} \chi(t') F(t') \, dt'. \quad (2.28)$$
Lemma 2.6. We have $2\mathcal{F}(t) = \mathcal{F}(t) - \chi(t)e^{it\Delta}\mathcal{F}(0)$, where $\mathcal{F}$ is defined by
\[
\mathcal{F}(t) = \chi(t) \left( \int_{-\infty}^{t} - \int_{t}^{\infty} \right) e^{i(t-t')\Delta} \chi(t') F(t') \, dt'.
\]
Moreover we have the formula
\[
\mathcal{F}(k, \lambda) = \int_{\mathbb{R}} \mathcal{F}(\lambda, \mu) \, d\mu, \quad |\partial_{\lambda, \mu}^{\alpha} \mathcal{F}(\lambda, \mu)| \lesssim_{\alpha, \lambda} \frac{1}{|\lambda - \mu|^{\alpha}} \frac{1}{|\mu|^{\lambda}}.
\]

For the proof of Lemma 2.6 see the calculations in [25, Lemma 3.1].

Proposition 2.7. Let $\varphi$ be any Schwartz function, recall that $\varphi_{\tau}(t) = \varphi(\tau^{-1}t)$ for any $0 < \tau \ll 1$. Then for any $u = u_k(t)$ and $h = h_{kk'}(t, t')$ we have
\[
\|\varphi_{\tau} \cdot u\|_{L^2} \lesssim \tau^{b_1-b} \|u\|_{L^2}, \quad \|\varphi_{\tau}(t) \cdot h\|_{L^2} \lesssim \tau^{b_1-b} \|h\|_{L^2},
\]
provided that $u_k(0) = h_{kk'}(0, t') = 0$.

Proof. Using definition of $Z_{\tau}^{b_1,b}$ norms and fixing the $(k', \lambda')$ variables, we can reduce the second inequality in (2.31) to the first, and by fixing $k$ and conjugating by the linear Schrödinger flow, we can reduce the first to
\[
\|\langle \xi \rangle^{b_1} (\mathcal{P}_{\tau}^{*} \vec{v})(\xi)\|_{L^2} \lesssim \tau^{b_1-b} \|\langle \eta \rangle^{b_1} \vec{v}(\eta)\|_{L^2}
\]
for $\nu$ satisfying $\nu(0) = 0$. Let $\vec{v} = g_1 + g_2$ where
\[
g_1(\xi) = 1_{|\xi| \geq \tau^{-1}}(\xi) \vec{v}(\xi), \quad g_2(\sigma) = 1_{|\xi| < \tau^{-1}}(\xi) \vec{v}(\xi).
\]
We will prove that
\[
\|\langle \xi \rangle^{b_1} (\mathcal{P}_{\tau}^{*} g_1)(\xi)\|_{L^2} \lesssim \tau^{b_1-b} \|\langle \eta \rangle^{b_1} \vec{v}(\eta)\|_{L^2},
\]
\[
\|\langle \xi \rangle^{b_1} (\mathcal{P}_{\tau}^{*} g_2)(\xi)\|_{L^2} \lesssim \tau^{b_1-b} \|\langle \eta \rangle^{b_1} \vec{v}(\eta)\|_{L^2}.
\]
To prove (2.32), we can reduce it to the $L^2 \to L^2$ bound for the operator
\[
g(\eta) \mapsto \int_{\mathbb{R}} R(\xi, \eta) g(\eta) \, d\eta, \quad R(\xi, \eta) = 1_{|\eta| \geq \tau^{-1}} \cdot \tau \mathcal{P}(\xi - \eta) \frac{\langle \xi \rangle^{b_1}}{\langle \eta \rangle^{b_1}}.
\]
Since
\[
1_{|\eta| \geq \tau^{-1}} \frac{\langle \xi \rangle^{b_1}}{\langle \eta \rangle^{b_1}} \lesssim \tau^{b_1-b} \frac{\langle T \xi \rangle^{b_1}}{\langle T \eta \rangle^{b_1}} \lesssim \tau^{b_1-b} \langle T(\xi - \eta) \rangle^{b_1},
\]
it follows from Schur’s estimate that this $L^2 \to L^2$ bound is at most
\[
\tau^{b_1-b} \|\tau \mathcal{P}(\xi) \langle T \xi \rangle^{b_1} \|_{L^2} \lesssim \tau^{b_1-b},
\]
which proves (2.32).

To prove (2.33), note that by $\varphi(0) = 0$ we have $\int_{\mathbb{R}} \vec{v}(\eta) \, d\eta = 0$, so
\[
\|\mathcal{P}_{\tau}^{*} g_2(\xi)\| = \left| - \tau \mathcal{P}(\tau \xi) \int_{|\eta| \geq \tau^{-1}} \vec{v}(\eta) \, d\eta - \int_{|\eta| < \tau^{-1}} \tau \vec{v}(\eta) \left[ \mathcal{P}(\tau \xi) - \mathcal{P}(\tau \xi - \eta) \right] \, d\eta \right|
\]
\[
\lesssim \tau^{-1} \int_{\mathbb{R}} \min(1, |\tau \eta|) |\vec{v}(\eta)| \, d\eta,
\]
and by Hölder we have
\[
\int_{\mathbb{R}} \min(1, |\tau \eta|) |\vec{v}(\eta)| \, d\eta \lesssim \|\langle \eta \rangle^{b_1} \vec{v}(\eta)\|_{L^2} \cdot \|\min(1, |\tau \eta|) \langle \eta \rangle^{-b_1} \|_{L^2} \lesssim \tau^{b_1-\frac{1}{2}} \|\langle \eta \rangle^{b_1} \vec{v}(\eta)\|_{L^2}.
\]
Using also the elementary bound
\[
\|\tau^{-1} \langle \xi \rangle^{b_1} \|_{L^2} \lesssim \tau^{rac{1}{2}-b},
\]
we deduce \( 2.33 \) and hence \( 2.31 \).

3. Structure of solution: random averaging operators

3.1. The decomposition. We now fix a short time \( 0 < \tau \ll 1 \), and establish the local theory for \( 2.25 \), with initial data distributed according to the Gaussian measure \( d\rho_N \), on \( J := [-\tau, \tau] \). By definition, this is equivalent to considering \( 2.23 \) with random initial data \( u_{in} = v_{in} = f(\omega) \), which we will assume from now on, until the end of Section 5. All functions that appear in the proof will be random (i.e. depends on \( \omega \)) we will assume from now on, until the end of Section 5. All functions that appear in the proof will be random (i.e. depends on \( \omega \)), whether or not we explicitly write \( \omega \) in their expressions.

Recall that the truncated mass \( m_N \) defined in \( 2.8 \) and the corresponding \( m_N^* \) are random variables given by

\[
m_N = \sum_{\langle k \rangle \leq N} \frac{|g_k|^2}{\langle k \rangle^2}, \quad m_N^* = \sum_{\langle k \rangle \leq N} \frac{|g_k|^2 - 1}{\langle k \rangle^2}. \tag{3.1}
\]

Note that they are Borel functions of \( g \).\( \| \). In particular, by removing a set of measure \( \leq C_0 e^{-r_0^{-\theta}} \) (which will be done before proving any estimates) we may assume the following bounds, which are used below without any further mentioning:

\[
|g_k| \lesssim \tau^{-\theta} \langle k \rangle^\theta, \quad |m_N^*| \lesssim \tau^{-\theta}, \quad |\nu_N| \lesssim \tau^{-\theta} N^{-1+\theta}. \tag{3.3}
\]

Our goal here is to obtain a quantitative estimate for the difference \( y_N := v_N - \frac{v_N}{2} \). By \( 2.25 \), this \( y_N \) satisfies the equation

\[
(i\partial_t + \Delta)y_N = \Pi_N \mathcal{Q}_N(y_N + \frac{v_N}{2}) - \Pi_N \mathcal{Q}_N(\frac{v_N}{2}), \quad y_N(0) = \Delta_N f(\omega). \tag{3.4}
\]

By \( 2.28 \) we can rewrite the above equation as

\[
\begin{aligned}
(i\partial_t + \Delta)y_N &= \sum_{l=0}^{r} c_{rl} (m_N^*)^{r-l} \left\{ \Pi_N \left[ \mathcal{N}_{2l+1}(y_N + \frac{v_N}{2}) - \mathcal{N}_{2l+1}(\frac{v_N}{2}) \right] - \Delta_N \mathcal{N}_{2l+1}(\frac{v_N}{2}) \right\} \\
&\quad + \sum_{j=0}^{r} c_{jl} \left[ (m_N^*)^{r-j} - (m_N^*)^{r-l} \right] \cdot \Pi_N \mathcal{N}_{2l+1}(\frac{v_N}{2}),
\end{aligned} \quad \tag{3.5}
\]

where \( c_{rl} \) are constants that will not be important in the proof.

Let the set

\[
\mathcal{K} := \{(N, L) \in (2\mathbb{Z})^2 : 2^{-1} \leq L < N^{1-\delta}\}. \tag{3.6}
\]

For each \( (N, L) \in \mathcal{K} \), we define the function \( \psi_{N, L} \) as the solution to the (linear) equation

\[
\begin{aligned}
(i\partial_t + \Delta)\psi_{N, L} &= \sum_{l=0}^{r} (l + 1) c_{rl} (m_N^*)^{r-l} \Pi_N \mathcal{N}_{2l+1}(\psi_{N, L}, v_L, \cdots, v_L), \\
\psi_{N, L}(0) &= \Delta_N f(\omega). \tag{3.7}
\end{aligned}
\]

It is important to place \( \psi_{N, L} \) in the first position of \( \mathcal{N}_{2l+1} \) in \( 3.7 \), see Remark 3.5. By linearity we have,

\[
(\psi_{N, L})_k = \sum_{k'} H_{kk'}^{N, L} g_{k'}(\omega) \frac{|g_{k'}(\omega)|^2}{\langle k' \rangle^2}. \tag{3.8}
\]
where for $N < \langle k^* \rangle \leq N$ and $\langle k \rangle \leq N$, $H_{kk^*}^{N,L} = \varphi_k$ is the $k$-th mode of the solution $\varphi$ to the equation

$$
(i\partial_t + \Delta)\varphi = \sum_{l=0}^{r}(l+1)c_r(m^*_N)^{r-l}\Pi_N\mathcal{N}_{2l+1}(\varphi, v_L, \cdots, v_L),
$$

$$
\varphi(0) = e^{ik^*x},
$$
and for other $(k, k^*) \in \mathbb{Z}^2$ define $H_{kk^*}^{N,L} = 0$. By definition these $H_{kk^*}^{N,L}$, as well as the $h_{kk^*}^{N,L}$ defined below, are $\mathcal{B}_{\leq N}$ measurable and $\mathcal{B}_{\leq L}^2$ measurable in the sense of Definition 1.10.

For any $N$, let $L_0$ be the largest $L$ satisfying $(N, L) \in \mathcal{K}$. We further define

$$
\zeta_{N,L} := \psi_{N,L} - \psi_{N,L}, \quad h^{N,L} := H^{N,L} - H^{N,\frac{L}{2}}, \quad z_N := y_N - \psi_{N,L_0}.
$$

Note that $\psi_{N,L} = e^{it\Delta}(\Delta_N f(\omega))$, and that $H_{kk^*}^{N} = e^{-i|k|^2t}1_{k=k^*}$ restricted to the frequency band $N < \langle k \rangle \leq N$. Moreover $z_N$ is $\mathcal{B}_{\leq N}$ measurable, $z_N(0) = 0$ and satisfies the equation

$$
(i\partial_t + \Delta)z_N = \sum_{l=0}^{r} c_r(m^*_N)^{r-l}\Pi_N\left[2l+1(z_N + \psi_{N,L_0} + v_N) - \mathcal{N}_{2l+1}(v_N) + \Delta_N\mathcal{N}_{2l+1}(v_N)\right] - \sum_{l=0}^{r} c_r(m^*_N)^{r-l}\Pi_N\left[(l+1)\mathcal{N}_{2l+1}(\psi_{N,L_0}, v_L, \cdots, v_L)\right] + \sum_{l=0}^{r} c_r[(m^*_N + \nu_N)^{r-l} - (m^*_N)^{r-l}]\Pi_N\mathcal{N}_{2l+1}(v_N).
$$

Remark 3.1. With the above construction, if we let $v = \lim_{N \to \infty} v_N$ be the gauged version of the solution $u$ to (1.1), then we have

$$
v = e^{it\Delta}f(\omega) + \sum_{(N,L) \in \mathcal{K}} \zeta_{N,L} + z, \quad \text{where} \quad z = \sum_N z_N.
$$

This is the ansatz (1.28) in Section 1.4.3 where $\zeta_{N,L}$ can be viewed as a random averaging operator $\mathcal{P}_{NL}$, whose kernel is essentially given by $h_{NL}$, applied to the Gaussian free field $e^{it\Delta}f(\omega)$. There are however two differences: (1) our $\mathcal{P}_{NL}$ is not exactly the one in (1.26), but an infinite iteration of the latter, because (1.26) has no smoothing effect; and (2) our $\mathcal{P}_{NL}$ is not exactly a Borel function of $(g_k)_{k \leq L}$ as it also depends on $m^*_N$, but as it turns out this does not affect any probabilistic estimates, see Lemma 4.1.

3.2. The a priori bounds. We now state the local well-posedness result for (2.3). Its proof will occupy the rest of this section and Sections 4 and 5.

Proposition 3.2. Recall the relevant constants defined in (1.32), and that $\tau \ll 1$, $J = [-\tau, \tau]$. Then, $\tau^{-1}$-certainly, i.e. with probability $\geq 1 - C_6e^{-\tau^{-\theta}}$, the following estimates hold for all $(N, L) \in \mathcal{K}$:

$$
\|h_{NL}\|_{Y^\theta(J)} \leq L^{-\delta_0}, \quad \|h_{NL}\|_{Z^\theta(J)} \leq N^{\frac{1}{2} + \delta + \delta_0}L^{-\frac{1}{2}}, \quad \|z_N\|_{X^\theta(J)} \leq N^{-1+\gamma}.
$$

3.2.1. The extensions. In proving Proposition 3.2 we will restrict $z_N$ and $h_{NL}$ to $J$ and construct extensions of these restrictions that are defined for all time. This has to be done carefully so as to maintain the correct independence properties. We define these extensions inductively, as follows.

First, let $z_{1\downarrow}(t) := z_1(t)\chi(t)$ and $\psi_{N,L}^\dagger(t) := \chi(t)e^{it\Delta}(\Delta_N f(\omega))$, and define $H_{NL}^{N,2\dagger}$ accordingly.

Suppose $M \geq 1$ is a dyadic number and we have defined $z_{N}^\dagger$ for all $N \leq M$ and $h_{NL}^{N,L^\dagger}$ for all
We will prove Proposition 3.2 by induction in $M$. Recall the relevant constants defined in (1.32), and that Proposition 3.3 estimates. More precisely, we will prove the following result, which contains Proposition 3.2:

$$
\varphi^\dagger(t) = \chi(t)e^{it\Delta_x}(e^{ik^\ast \cdot x}) - i\chi_r(t) \sum_{l=0}^r (l+1) c_r (m_N^*)^{r-l} \cdot \mathcal{I} \Pi_{N}N_{2l+1}(\varphi^\dagger, v^\dagger_{M_1}, \cdots, v^\dagger_{M_r})
$$

(3.14)

provided this solution exists and is unique; otherwise simply define $\varphi^\dagger = 0$. Next, for $(N, L) \in \mathcal{K}$ and $L = M$, we can define $H^{N, M^\dagger}$ such that for $\langle k \rangle \leq N$, $H^{N, M^\dagger}_{kk^*} = \varphi^\dagger_k$ is the $k$-th mode of the solution $\varphi^\dagger$ to the equation

$$
\varphi^\dagger(t) = \chi(t)e^{it\Delta_x}(e^{ik^\ast \cdot x}) - i\chi_r(t) \sum_{l=0}^r (l+1) c_r (m_N^*)^{r-l} \cdot \mathcal{I} \Pi_{N}N_{2l+1}(\varphi^\dagger, v^\dagger_{M_1}, \cdots, v^\dagger_{M_r})
$$

(3.14)

Finally we will define $z_{2M}^\dagger$. As $z_{2M, L_0}^\dagger$ is already defined, where $L_0 \leq M$ is the largest $L$ such that $(2M, L) \in \mathcal{K}$, we can define $z_{2M}^\dagger$ to be the unique fixed point of the mapping

$$
z \mapsto -i\chi_r(t) \sum_{l=0}^r c_r (m_{2M}^*)^{r-l} \cdot \mathcal{I} \Pi_{2M} \left\{ N_{2l+1} (z + \psi_{2M, L_0}^\dagger + v^\dagger_{M_1}) - N_{2l+1}(v^\dagger_{M_1}) \right\}
$$

$$
+ i\chi_r(t) \sum_{l=0}^r (l+1) c_r (m_N^*)^{r-l} \cdot \mathcal{I} \Pi_{2M}N_{2l+1}(\psi^\dagger_{2M, L_0}, v^\dagger_{M_1}, \cdots, v^\dagger_{M_r})
$$

$$
- i\chi_r(t) \sum_{l=0}^r c_r (m_N^* + \nu_{2M})^{r-l} \cdot \mathcal{I} \Pi_{M}N_{2l+1}(v^\dagger_{M_1})
$$

(3.15)

on the set $Z = \{ z : \| z \|_{\mathcal{K}^\dagger} \leq (2M)^{-1+\gamma} \}$, provided that this mapping is a contraction mapping from $Z$ to itself. If it is not a contraction mapping, then simply define $z_{2M}^\dagger = z_{2M} \cdot \chi_r(t)$. This completes the inductive construction. One may then easily verify that:

- The $z_N^\dagger$ and $h^{N, L, \dagger}$ we constructed are indeed extensions of $z_N$ and $h^{N, L}$;
- The $z_N^\dagger$ is supported in $\langle k \rangle \leq N$, and $h^{N, L, \dagger}_{kk^*}$ is supported in $\langle k \rangle \leq N$ and $N^2 < \langle k^* \rangle \leq N$;
- The random variable $h^{N, L, \dagger}$ is $\mathcal{B}_{\leq L}^+$ measurable, and $z_N^\dagger$ and $h^{N, L, \dagger}$ are $\mathcal{B}_{\leq N}$ measurable;
- All the above are smooth and compactly supported in time $t \in [-2, 2]$.

We will prove Proposition 3.2 by induction in $M$, but in the process we will need some auxiliary estimates. More precisely, we will prove the following result, which contains Proposition 3.2.

**Proposition 3.3.** Recall the relevant constants defined in (1.32), and that $\tau \ll 1$. Consider the following statement which we call $\text{Loc}(M)$ for $M \geq 1$: for any $(N, L) \in \mathcal{K}$ with $L < M$, we have

$$
\|h^{N, L, \dagger}\|_{Y^b} \leq L^{-\delta_0};
$$

(3.16)

$$
\|h^{N, L, \dagger}\|_{Z^b} \leq N^{\delta + \eta_0} L^{-\frac{\delta}{2}};
$$

(3.17)

$$
\left\| \left( 1 + \frac{|k - k^*|}{L} \right)^{k} h^{N, L, \dagger}_{kk^*} \right\|_{Z^b} \leq N.
$$

(3.18)
Define the operators (where \(0 \leq l \leq r\))

\[
P^+(w) := \chi_{\tau}(t) \cdot \Pi_N [N_{2l+1}(w, v_L^1, \cdots, v_L^1) - N_{2l+1}(w, v_L^1, \cdots, v_L^1)],
\]

\[
P^-(w) := \chi_{\tau}(t) \cdot \Pi_N [N_{2l+1}(v_L^1, w, v_L^1, \cdots, v_L^1) - N_{2l+1}(v_L^1, w, v_L^1, \cdots, v_L^1)],
\]

(3.19) (3.20)

then for any \((N, L) \in \mathcal{K}\) as defined in (3.6) with \(L < M\) we have

\[
\|P^\pm\|_{X^b \to X^b} \leq \tau^\theta L^{-\theta_0}. \tag{3.21}
\]

Let the kernel of \(P^+\) be \(h_{kk'}^{N,L}(t, t')\), then for any \((N, L) \in \mathcal{K}\) with \(L < M\) we have

\[
\|1_{|k|,|k'| \geq N} \cdot h_{kk'}^{N,L}(t, t')\|_{Z^b,b} \leq \tau^\theta N^{\frac{1}{2} + \gamma_0 - \delta L^{-\frac{1}{2}}}. \tag{3.22}
\]

Finally for any \(N \leq M\) we have

\[
\|z^\dagger_N\|_{X^b} \leq N^{-1+\gamma}. \tag{3.23}
\]

Now suppose the statement \(\text{Loc}(M)\) is true for \(\omega \in \Xi\), where \(\Xi\) is a set, then the statement \(\text{Loc}(2M)\) is true for \(\omega \in \Xi'\) where \(\Xi'\) is another set such that \(\mathbb{P}(\Xi \setminus \Xi') \leq C e^{-\theta_1 M\theta}\). In particular, apart from a set of \(\omega\) with probability \(\leq C e^{-\tau\gamma}\), the statement \(\text{Loc}(M)\) is true for all \(M\).

3.3. The proof of Proposition 3.3: reduction to multilinear estimates. The heart of the proof of Proposition 3.3 is a collection of (probabilistic) multilinear estimates for \(N_{2l+1}\). We will state them in Proposition 3.4 below and show that they imply Proposition 3.3. We leave the proof of Proposition 3.3 to Section 5.

Proposition 3.4. Recall the relevant constants defined in (1.32), and that \(\tau \ll 1\). Let the multilinear form \(N_n\) be as in (2.16), where \(1 \leq n \leq 2r + 1\). We will also consider \(N_{n+1}\), in which we assume \(\tau_1 = +\). For each \(1 \leq j \leq n\), the input function \(v^{(j)}\) satisfies one of the followings:

(i) **Type (G),** where we define \(L_j = 1\), and

\[
\langle \tilde{v}^{(j)} \rangle_{k_j}(\lambda_j) = 1_{N_j/2 < \langle k_j \rangle \leq N_j} \frac{g_{k_j}(\omega)}{\langle k_j \rangle} \tilde{\chi}(\lambda_j). \tag{3.24}
\]

(ii) **Type (C),** where

\[
\langle \tilde{v}^{(j)} \rangle_{k_j}(\lambda_j) = \sum_{N_j/2 < \langle k_j \rangle \leq N_j} h_{k_j k_j'}^{(j)}(\lambda_j, \omega) \frac{g_{k_j'}(\omega)}{\langle k_j' \rangle}, \tag{3.25}
\]

with \(h_{k_j k_j'}^{(j)}(\lambda_j, \omega)\) supported in the set \(\{\langle k_j \rangle \leq N_j, N_j^{1/2} < \langle k_j' \rangle \leq N_j\}\), \(\mathcal{B} \leq N_j\) measurable and \(B_{L_j}^+\) measurable for some \(L_j \leq N_j^{-1/\delta}\), and satisfying the bounds

\[
\|\langle \lambda_j \rangle^{b_h^{(j)}_{k_j k_j'}(\lambda_j)}\|_{L_{2}^{k_j} L_{2}^{k_j'}} \lesssim L_{2}^{-\theta_0}, \quad \|\langle \lambda_j \rangle^{b_h^{(j)}_{k_j k_j'}(\lambda_j)}\|_{L_{2}^{k_j} L_{2}^{k_j'}} \lesssim N_j^{\frac{1}{2} + \gamma} L_{j}^{-\frac{1}{2}}, \tag{3.26}
\]

\[
\|\langle \lambda_j \rangle^{b_h^{(j)}_{k_j k_j'}(\lambda_j)}\|_{L_{2}^{k_j} L_{2}^{k_j'}} \lesssim \langle k_j \rangle \langle k_j' \rangle L_j \lesssim N_j. \tag{3.27}
\]

(iii) **Type (D),** where \(\langle \tilde{v}^{(j)} \rangle_{k_j}(\lambda_j)\) is supported in \(\{\langle k_j \rangle \lesssim N_j\}\), and satisfies

1In fact we will prove stronger bounds where the low frequency inputs in (3.19) are replaced by \(v_{L_1}^1 \cdots v_{L_{2r}}^1\), with \(\max(L_j) = L\), and similarly for (3.20). But for simplicity we will just write (3.19) and (3.20).
In each case, we will assume that derivatives in $\lambda_j$ of these functions satisfy the same bounds. This can always be guaranteed, since in practice everything will be compactly supported in time.

Assume for $n_1 \leq n$ that $v^{(j)}$ are of type (D) for $n_1 + 1 \leq j \leq n$, and of type (G) or (C) for $1 \leq j \leq n_1$. Let $G$ and $C$ be the sets of $j$ such that $v^{(j)}$ are of type (G) and (C) respectively, similarly denote by $D := \{n_1 + 1, \ldots, n\}$. Let $N^{(j)} = \max^{(j)}(N_1, \ldots, N_n)$ as before, and let $1 \leq a \leq n$ be such that $N^{(1)} \sim N_a$. Given $N_a \geq 1$, the followings hold $\tau^{-1}N_a$-certainly. We emphasize that the exceptional set of $\omega$ removed does not depend on the choice of the functions $v^{(j)}(j \geq n_1 + 1)$.

1. If $a \geq n_1 + 1$ (say $a = n$) and $N_a \geq N^{(2)}$, then we have (recall $b_0 = b + \delta^4$)

$$
\|I_{N_a}(v^{(1)}, \ldots, v^{(n)})\|_{X^{b_1}} \lesssim \tau^{-\theta}(N_a)^{\delta^{-1}}(N^{(1)})^{-1+\gamma(N^{(2)})^{-\delta_0}}.
$$

(3.28)

Here the exceptional set does not depend on $N^{(1)}$.

2. If $a \leq n_1$ and $N_a \geq N^{(1)}$, then we have

$$
\|I_{N_a}(v^{(1)}, \ldots, v^{(n)})\|_{X^{b_1}} \lesssim \tau^{-\theta}(N_a)^{\delta^{-1}}(N^{(1)}N^{(2)})^{-\frac{1}{2}(1-\gamma_0)}.
$$

(3.29)

If moreover $I_a = -$, then we have the stronger bound

$$
\|I_{N_a}(v^{(1)}, \ldots, v^{(n)})\|_{X^{b_1}} \lesssim \tau^{-\theta}(N_a)^{\delta^{-1}}(N^{(1)}N^{(2)})^{-\frac{1}{2}(1-\gamma_0)}.
$$

(3.30)

If moreover $I_a = +$ and $N^{(2)} \lesssim (N^{(1)})^{1-\delta}$, then we have stronger bound for the projected term

$$
\|\Pi_{N^{(1)}}I_{N_a}(v^{(1)}, \ldots, v^{(n)})\|_{X^{b_1}} \lesssim \tau^{-\theta}(N_a)^{\delta^{-1}}(N^{(1)})^{1-\frac{1}{2}\delta}.
$$

(3.31)

3. Now consider the operator

$$
\mathcal{Q}^+(w) := \Pi_{N_a}I_{N^{(1)}}(\Pi_{N_a}w, v^{(1)}, \ldots, v^{(n)}),
$$

(3.32)

and let its kernel be $\mathcal{h}_{kk'}(t, t')$. If $N^{(1)} \lesssim N_0^{1-\delta}$ and $N_a \gtrsim N_0$, then we have

$$
\|1_{|k|, |k'| \geq \frac{N_0}{4}} \mathcal{h}_{kk'}(t, t')\|_{Z^{b_1}} \lesssim \tau^{-\theta}(N_a)^{\delta^{-1}}N_0^{-\frac{1}{2}}(N^{(1)})^{-\frac{1}{2}+\gamma_0}.
$$

(3.33)

Remark 3.5. The improvement (3.31) is due to the exact projection $\Pi_{N^{(1)}}$. In fact this implies that in the expression (3.16) there exists some $1 \leq a \leq n$ and some $\Gamma$, namely $\Gamma = (N^{(1)})^2 - 1$, such that

$$
|k|^2 \geq \Gamma \geq |k_a|^2 \quad \text{or} \quad |k|^2 \leq \Gamma \leq |k_a|^2, \quad \text{and} \quad N^{(1)} \sim N_a.
$$

(3.34)

We call (3.34) the $\Gamma$-condition. If we put some other projections in $N_a$ that also guarantee (3.34), for example $\Pi_M\mathcal{N}_a(\ldots, \Pi_Mv^{(a)}, \ldots)$ where $N^{(1)} \sim N_a$, then the same improvement (3.31) will remain true.

In the proof below we will see that the $\Gamma$ condition provides the needed improvements in the case $N^{(1)} \sim N_a$ and $I_a = +$. This is the reason why we place $\psi_{N,L}$ in the first position of $N_{2L+1}$ in (3.7). On the other hand, the term where $\psi_{N,L}$ is placed in the second position can be handled using the improvement (3.30).

Proof of Proposition 3.3 assuming Proposition 3.4. To prove the statement $\text{Loc}(2M)$ we start with (3.21) and (3.22), and may assume $L = M$. The proof for $\mathcal{P}^-$ in (3.21) will be similar, so let us consider $\mathcal{P}^+$. Since $v_M^\dagger = \sum_{L \leq M} y_L^\dagger$, by definition we can write $\mathcal{P}^+$ as a superposition of forms

$$
w \mapsto \chi_\tau(t) \cdot \Pi N\mathcal{N}_{2L+1}(w, y_{N_2}^\dagger, \ldots, y_{N_{2L+1}}^\dagger),
$$

where $\max(N_2, \ldots, N_{2L+1}) = M$. As we have the decomposition

$$
y_L^\dagger = \chi(t)e^{it\Delta}(|\Delta_L f(\omega)| + \sum_{(L,R) \in K} \zeta_{L,R}^\dagger + z_L^\dagger)
$$

(3.35)
by Loc(M) we know that each $y_{N_j}^\dagger$ can be decomposed into terms of type (G), type (C) (corresponding to some $L_j \lesssim N_j^{1-\delta}$), and type (D). The bound (3.21) is then a consequence of (2.31) and (3.28), after removing a set of $\omega$ with measure $\leq C_0 e^{-c_1(M)\theta}$, that is independent of $N$. Note that by (2.23), the $L = M$ case of (3.21) is equivalent to

$$\|h_{N,M}^\theta\|_{Y^b,b} \leq \tau^\theta M^{-\delta_0^b}.$$  (3.36)

Similarly (3.22) follows from (2.31) and (3.33), because we have

$$\tau^{b_1-b} M^{-\frac{1}{2}+\gamma_0} N^\frac{1}{2} M^{-\frac{1}{2}+\gamma_0} \ll \tau^\theta N^{-\frac{1}{2}+\gamma_0-2\delta^3} M^{-\frac{1}{2}}$$

using the fact that $M \lesssim N^{-\delta}$. The set of $\omega$ removed here will depend on $N$, but it will have measure $\leq C_0 e^{-c_1(M)\theta}$, so summing in $N \geq M$ we still get a set of measure $\leq C_0 e^{-c_1(M)\theta}$.

Next we prove (3.16)–(3.18), again assuming $\Delta = 0$. By (3.21) and Loc(M) we already know that the right hand side of (3.14) gives a contraction mapping in $X_b$, so (3.14) does have a unique solution. Subtracting the equations (3.14) with $M = N, M, k', k,$ we get

$$h_{k,k'}^N M^\dagger(t) = -i \sum_{l=0}^r (l+1) c_l (m^s_N)^{t-l} \left\{ \sum_{L \leq M} \sum_{k'} \int dt' \cdot h_{k,k'}^N M^\dagger(t') + \sum_{L \leq M} \sum_{k'} \int dt' \cdot h_{k,k'}^N M^\dagger(t') \right\},$$

where $h_{k,k'}^N M^\dagger(t,t')$ is the kernel corresponding to $P^+$ in the $(k,k',t,t')$ variables. Recall that we are already in a set where (3.3) is true, which allows us to control $m_N^s$. Now by the definition of $Y^b$ and $Y^\dagger$, the statement Loc(M) and (3.36), we conclude that

$$\|h_{N,M}^\theta\|_{Y^b} \lesssim \sum_{L \leq M} \|h_{N,L}^\theta\|_{Y^b} \cdot \|h_{N,M}^\theta\|_{Y^b} + \|h_{N,M}^\theta\|_{Y^b} \left( \sum_{L \leq M} \|h_{N,L}^\theta\|_{Y^b} + 1 \right)$$

$$\lesssim \|h_{N,M}^\theta\|_{Y^b} \cdot \sum_{L \leq M} \tau^\theta L^{-\frac{1}{2}} + \sum_{L \leq M} \tau^\theta M^{-\frac{1}{2}} L^{-\delta_0} \lesssim \tau^\theta \|h_{N,M}^\theta\|_{Y^b} + \tau^\theta M^{-\delta_0^b},$$

which implies (3.16) as desired. In the same way we can prove (3.18) by using (2.28), noting that $\tilde{h}_{N,L}^\theta$ is supported in $|k-k'| \leq L$. As for (3.17), recall that for

$$\tilde{H}_{k,k'}(\lambda) = \sum_{k'} \int d\lambda' \cdot \tilde{h}_{k,k'}(\lambda,\lambda') h_{k'}(\lambda'),$$

we have, by definition of the relevant norms, that

$$\|H\|_{L_{k,k'},L_{\lambda,\lambda}^\infty} \leq \min(\|h\|_{L_{k,k'}^\infty \cdot L_{\lambda,\lambda}^\infty}, \|h\|_{L_{k,k'}^\infty \cdot L_{\lambda,\lambda}^\infty}, \|h\|_{L_{k,k'}^\infty \cdot L_{\lambda,\lambda}^\infty}, \|h\|_{L_{k,k'}^\infty \cdot L_{\lambda,\lambda}^\infty}).$$

Now in (3.37) we may assume $|k-k'| \leq 2^{-10} N$ and $|k'-k^*| \leq 2^{-10} N$ (otherwise the bound follows trivially from (3.18)), so in particular $|k|, |k'| \geq \frac{N}{2}$ as $|k^*| \geq \frac{N}{2}$. Using the statement Loc(M) and (3.22) we get

$$\|h_{N,M}^\theta\|_{Z^b} \lesssim \sum_{L \leq M} \|h_{N,L}^\theta\|_{Y^b} \cdot \|h_{N,M}^\theta\|_{Z^b} + \|1_{|k|,|k'| \geq \frac{N}{2}} \cdot h_{k,k'}^N M^\dagger(t,t')\|_{Z^b} \left( \sum_{L \leq M} \|h_{N,L}^\theta\|_{Y^b} + 1 \right)$$

$$\lesssim \|h_{N,M}^\theta\|_{Z^b} \cdot \sum_{L \leq M} \tau^\theta L^{-\frac{1}{2}} + \sum_{L \leq M} \tau^\theta N^{-\frac{1}{2}+\gamma_0-2\delta^3} M^{-\frac{1}{2}} L^{-\delta_0}$$

$$\lesssim \tau^\theta \|h_{N,M}^\theta\|_{Y^b} + \tau^\theta N^{-\frac{1}{2}+\gamma_0-2\delta^3} M^{-\frac{1}{2}},$$

and thereby conclude (3.18) as desired.
which proves (3.17).

Finally we prove (3.23) with $L = 2M$, by showing that the mapping defined in (3.15) is indeed a contraction mapping from the given set $\mathcal{Z} = \{ z : \| z \|_{X^b} \leq (2M)^{-1+\gamma} \}$ to itself. Actually we will only prove that this mapping sends $\mathcal{Z}$ to $\mathcal{Z}$, as the difference estimate is done in the same way.

We will separate the right hand side of (3.15) into six groups, each of which has the form

$$\chi_r(t) \cdot (m^{s}_{2M})^{r-t} T\Pi_{2M} N_{2l+1}(v^{(1)}, \ldots, v^{(2l+1)}),$$

where

(a) At least two of the $v^{(j)}$ are equal to $z + \psi^{\dagger}_{2M,L_0}$, and others are either $z + \psi^{\dagger}_{2M,L_0}$ or $v^{\dagger}_{M}$;

(b) We have $v^{(2)} = \psi^{\dagger}_{2M,L_0}$, and all others equal $v^{\dagger}_{M}$;

(c) One of $v^{(1)}$ or $v^{(2)}$ equals $z$, and all others equal $v^{\dagger}_{M}$;

(d) We have $v^{(1)} = \psi^{\dagger}_{2M,L_0}$, another $v^{(j)}$ equals $v^{\dagger}_{M} - v^{\dagger}_{L_0}$, and all others equal either $v^{\dagger}_{M}$ or $v^{\dagger}_{L_0}$;

(e) The factor $(m^{s}_{2M})^{r-t}$ is replaced by $(m^{s}_{2M} + v_{2M})^{r-t} - (m^{s}_{2M})^{r-t}$ and all $v^{(j)}$ equal $v^{\dagger}_{M}$;

(f) Same as (a), but with $\Delta_{2M}$ instead of $\Pi_{2M}$, and all $v^{(j)}$ equal $v^{\dagger}_{M}$.

By (2.31), it will suffice to prove that each of these terms in (a) through (f), but without the $\chi_r(t)$ factor, is bounded in $X^{b_1}$ by $\tau^{-\theta}(2M)^{-1+\gamma}$. Let one such term be denoted by $\mathcal{M}$, and notice that we can decompose

$$v^{\dagger}_{M} = \sum_{L \leq M} y^{\dagger}_{L}, \quad v^{\dagger}_{L_0} = \sum_{L \leq L_0} y^{\dagger}_{L}, \quad v_{M} - v_{L_0} = \sum_{L_0 < L \leq M} y^{\dagger}_{L},$$

$$\psi^{\dagger}_{2M,L_0} = \chi(t)e^{it\Delta}(\Delta_{2M}f(\omega)) + \sum_{L \leq L_0} \zeta_{2M,L}.$$ 

Moreover by what we have proved so far, we know that $y^{\dagger}_{L}$ for $L \leq M$ can be decomposed into terms of types (G), (C) and (D), and that $\chi(t)e^{it\Delta}(\Delta_{2M}f(\omega))$ is of type (G), $\zeta_{2M,L}$ is of type (C), and $z$ is of type (D). By such decomposition we can reduce $\mathcal{M}$ to the terms studied in Proposition 3.4, with various choices of $N_j$ and $L_j$. We now proceed case by case.

Case (a): Here we have at least two inputs $v^{(j)}$ with $N_j = 2M$, so by either (3.28) or (3.29) we can bound

$$||\mathcal{M}||_{X^{b_1}} \leq \tau^{-\theta}(2M)^{-1+\gamma} e^{C_6}$$

by removing a set of measure $\leq C_6 e^{-(\tau - 1)M^\theta}$, which suffices.

Case (b): Here we have $v^{(2)} = N^{(1)} = 2M$, while $v^{(2)} =$. By (3.29) we have the same bound as above.

Case (c): This term, with the $\chi_r(t)$ factor, can be written as

$$\sum_{L \leq M} P^{\pm}_{L}(\omega),$$

where $P^{\pm}_{L}$ are defined as in (3.19) and (3.20) (with subscript $L$ to indicate $L$ dependence). If $L \leq L_0$ then by (3.21) we can bound

$$||\chi_r(t) \cdot \mathcal{M}||_{X^{b}} \leq (2M)^{-1+\gamma} \sum_{L \leq L_0} \tau^\theta L^{-\delta_0} \leq \tau^\theta (2M)^{-1+\gamma},$$

which suffices. Note that here no further set of $\omega$ needs to be removed; if $L > L_0$ then this term can be bounded in the same way as in case (d) below, by removing a set of measure $\leq C_6 e^{-(\tau - 1)M^\theta}$.

Case (d): Here we have, due to the factor $v^{\dagger}_{M} - v^{\dagger}_{L_0}$, that $N^{(1)} = 2M$ and $N^{(2)} \geq M^{1-\delta}$; so by either (3.28) or (3.29) we can bound

$$||\mathcal{M}||_{X^{b_1}} \leq \tau^{-\theta} M^{-1+\frac{\delta}{2}} + C_7.$$
by removing a set of measure \( \leq C_\theta e^{-\gamma M} \), which suffices.

Case (e): The bound for this term follows from the bound for \( \nu_{2M} \) and the trivial bound (say (3.28) or (3.29)) for the \( \mathcal{N}_{2l+1} \) term.

Case (f): We may assume \( N(1) = N_a \sim M \). If either \( v^{(a)} \) is of type (D) or \( N(2) \geq M^{1-\delta} \) or \( \nu_a = - \), we can reduce to one of the previous cases (namely (c) or (d) or (b)) and close as before; if \( v^{(a)} \) is of type (G) or (C), \( \nu_a = + \) and \( N(2) \ll M^{1-\delta} \), then the \( \Delta_{2M} \) projection allows us to apply the improvement (3.31), which leads to

\[
\|M\|_{\mathcal{N}_{2l}} \leq \tau^{-\gamma} M^{-1} + C_\gamma e^{-\gamma M}
\]

by removing a set of measure \( \leq C_\theta e^{-\gamma M} \), which suffices. This completes the proof.

\[\square\]

4. LARGE DEVIATION AND COUNTING ESTIMATES

Proposition 3.4 will be proved in Section 5. In this section we make some preparations for the proof, namely we introduce two large deviation estimates and some counting estimates for integer lattice points.

4.1. Large deviation estimates. We first prove the following large deviation estimate for multi-linear Gaussians, which as far as we know is new.

**Lemma 4.1.** Let \( E \subset \mathbb{Z}^2 \) be a finite subset, and let \( B \) be the \( \sigma \)-algebra generated by \( \{g_k : k \in E\} \). Let \( C \) be a \( \sigma \)-algebra independent with \( B \), and let \( C^+ \) be the smallest \( \sigma \)-algebra containing both \( C \) and the \( \sigma \)-algebra generated by \( \{|g_k|^2 : k \in E\} \). Consider the expression

\[
F(\omega) = \sum_{(k_1, \ldots, k_n) \in E^n} a_{k_1 \ldots k_n}(\omega) \prod_{j=1}^n g_{k_j}(\omega)^{\eta_j},
\]

where \( n \leq 2r + 1 \), \( \eta_j \in \{\pm\} \) and the coefficients \( a_{k_1 \ldots k_n}(\omega) \) are \( C^+ \) measurable. Let \( A \geq \#E \), then \( A \)-certainly we have

\[
|F(\omega)| \leq A^\gamma M(\omega)^{1/2},
\]

where

\[
M(\omega) = \sum_{(X,Y)} \sum_{(k_1, \ldots, k_p) : m \notin X \cup Y} \left( \sum_{\text{pairing} (k_{s_1}, k_{s_2}) : 1 \leq s \leq p} |a_{k_1 \ldots k_n}(\omega)| \right)^2.
\]

In the summation (4.3) we require that all \( k_j \in E \), and that \( X := \{i_1, \ldots, i_p\} \) and \( Y := \{j_1, \ldots, j_p\} \) are two disjoint subsets of \( \{1, 2, \ldots, n\} \). Recall also the definition of pairing in Definition 1.7.

**Proof.** Write in polar coordinates \( g_k(\omega) = \rho_k(\omega) \eta_k(\omega) \) where \( \rho_k = |g_k| \) and \( \eta_k = \rho_k^{-1} g_k \), then all the \( \rho_k \) and \( \eta_k \) are independent, and each \( \eta_k \) is uniformly distributed on the unit circle of \( \mathbb{C} \). We may write

\[
F(\omega) = \sum_{(k_1, \ldots, k_n) \in E^n} b_{k_1 \ldots k_n}(\omega) \prod_{j=1}^n \eta_{k_j}(\omega)^{\eta_j}, \quad b_{k_1 \ldots k_n}(\omega) := a_{k_1 \ldots k_n}(\omega) \prod_{j=1}^n \rho_{k_j}(\omega).
\]

Since \( a_{k_1 \ldots k_n}(\omega) \) are \( C^+ \) measurable, we know that the collection \( \{b_{k_1 \ldots k_n}\} \) is independent with the collection \( \{\eta_k : k \in E\} \). The goal is to prove that

\[
\mathbb{P}(|F(\omega)| \geq BM_1(\omega)^{1/2}) \leq Ce^{-B^{1/n}},
\]

where \( C \) is an absolute constant, and \( M_1(\omega) \) is the same as \( M(\omega) \) but with the coefficients \( a \) replaced by the coefficients \( b \). In fact, as \( A \geq \#E \) we have \( A \)-certainly that \( |b_{k_1 \ldots k_n}(\omega)| \leq A^\gamma |a_{k_1 \ldots k_n}(\omega)| \), so (4.5) implies the desired bound.

We now prove (4.5). By independence, we may condition on the \( \sigma \)-algebra generated by \( \{b_{k_1 \ldots k_n}\} \) and prove (4.5) for the conditional probability, then take another expectation; therefore we may
assume that $b_{k_1...k_n}$ are constants (so $M_1(\omega) = M_1$ is a constant). Now let $\{h_k : k \in E\}$ be another set of i.i.d. normalized complex Gaussian random variables and define

$$G = \sum_{(k_1, \ldots, k_n) \in E^n} |b_{k_1...k_n}| \prod_{j=1}^n h_{k_j}^{e_j}, \quad (4.6)$$

we want to compare $F$ and $G$ and show $\mathbb{E}|F|^{2d} \leq \mathbb{E}|G|^{2d}$ for any positive integer $d$. In fact,

$$\mathbb{E}(|F|^{2d}) = \sum_{(k_1, e_1; k_2, e_2; \ldots; k_n, e_n) \in E^n} \prod_{i=1}^d b_{k_i}^{e_i} \prod_{i=1}^d \mathbb{E} \left( \prod_{j=1}^n \eta_{k_j}^{e_j} \bar{\eta}_{k_j}^{\bar{e}_j} \right), \quad (4.7)$$

$$\mathbb{E}(|G|^{2d}) = \sum_{(k_1, e_1; k_2, e_2; \ldots; k_n, e_n) \in E^n} \prod_{i=1}^d |b_{k_i}| \prod_{i=1}^d \mathbb{E} \left( \prod_{j=1}^n h_{k_j}^{e_j} \bar{h}_{k_j}^{\bar{e}_j} \right). \quad (4.8)$$

The point is that we always have

$$\left| \mathbb{E} \left( \prod_{i=1}^d \prod_{j=1}^n \eta_{k_j}^{e_j} \bar{\eta}_{k_j}^{\bar{e}_j} \right) \right| \leq \text{Re} \mathbb{E} \left( \prod_{i=1}^d \prod_{j=1}^n h_{k_j}^{e_j} \bar{h}_{k_j}^{\bar{e}_j} \right).$$

In fact, by collecting all different factors we can write the expectations as

$$\mathbb{E} \left( \prod_{\alpha} (\eta_{k(\alpha)})^{x_{\alpha}} (\bar{\eta}_{k(\alpha)})^{y_{\alpha}} \right) \quad \text{and} \quad \mathbb{E} \left( \prod_{\alpha} (h_{k(\alpha)})^{x_{\alpha}} (\bar{h}_{k(\alpha)})^{y_{\alpha}} \right),$$

where the $k^{(\alpha)}$ are pairwise distinct. If $x_{\alpha} \neq y_{\alpha}$ for some $\alpha$, both expectations will be 0; if $x_{\alpha} = y_{\alpha}$ for each $\alpha$, then the first expectation will be 1 and the second expectation will be $\prod_{\alpha} x_{\alpha}! \geq 1$.

Now, since $G$ is an exact multilinear Gaussian expression, by the standard hypercontractivity estimate, see [60], we have

$$\mathbb{E}|F|^{2d} \leq \mathbb{E}|G|^{2d} \leq (2d - 1)^{nd} (\mathbb{E}|G|^2)^d,$$

so for any $D > 0$ by using Chebyshev’s inequality and optimizing in $d$ we have

$$\mathbb{P}(|F(\omega)| \geq D) \leq \min_d \left\{ (2d - 1)^{nd} \left( \frac{\mathbb{E}|G|^2}{D^2} \right)^d \right\} \leq C \exp \left\{ -\frac{1}{2e} \left( \frac{\mathbb{E}|G|^2}{D^2} \right)^{\frac{1}{n}} \right\}$$

with some constant $C$ depending only on $n$. It then suffices to prove $\mathbb{E}|G|^2 \lesssim M_1$ with constants depending only on $n$.

By dividing the sum (4.6) into finitely many terms and rearranging the subscripts, we may assume

$$k_1 = \cdots = k_{j_1}, \quad k_{j_1+1} = \cdots = k_{j_2}, \ldots, k_{j_{r-1}+1} = \cdots = k_{j_r}, \quad 1 \leq j_1 < \cdots < j_r = n,$$

and the $k_{j_s}$ are different for $1 \leq s \leq r$. Such a monomial that appears in (4.6) has the form

$$\prod_{s=1}^r h_{k_{j_s}}^{\beta_s} (\overline{h_{k_{j_s}}})^{\gamma_s}, \quad \beta_s + \gamma_s = j_s - j_{s-1} (j_0 = 0),$$

where the factors for different $s$ are independent. We may also assume $\beta_s = \gamma_s$ for $1 \leq s \leq q$ and $\beta_s \neq \gamma_s$ for $q + 1 \leq s \leq r$, and that $\nu_j$ has the same sign as $(-1)^j$ for $1 \leq j \leq j_q$. Then we can further rewrite this monomial as a linear combination of

$$\prod_{s=1}^r \beta_s! \prod_{s=p+1}^q (|h_{k_{j_s}}|^{2\beta_s} - \beta_s!) \prod_{s=q+1}^r h_{k_{j_s}}^{\beta_s} (\overline{h_{k_{j_s}}})^{\gamma_s}$$
for $1 \leq p \leq q$. Therefore, $G$ is a finite linear combination of expressions of the form

$$\sum_{k_{j_1}, \ldots, k_{j_r}} |b_{k_{j_1}, \ldots, k_{j_r}}| \prod_{s=1}^{p} \beta_{s} \prod_{s=p+1}^{q} (|h_{k_{j_s}}|^{2\beta_{s}} - \gamma_{s}) \prod_{s=q+1}^{r} h_{k_{j_s}}^{\beta_{s}} (\mathcal{f}_{k_{j_s}}) \gamma_{s}.$$ 

Due to independence and the fact that $\mathbb{E}(|h|^{2\beta} - \beta!) = \mathbb{E}(h^{\beta} (\mathcal{h})^{\gamma}) = 0$ for a normalized Gaussian $h$ and $\beta \neq \gamma$, we conclude that

$$\mathbb{E}|G|^2 \lesssim \sum_{k_{j_p+1}, \ldots, k_{j_r}} \left( \sum_{k_{j_1}, \ldots, k_{j_p}} |b_{k_{j_1}, \ldots, k_{j_r}}| \right)^2,$$

which is bounded by $M$ choosing $X = \{1,3,\ldots,j_p - 1\}$ and $Y = \{2,4,\ldots,j_p\}$, since by our assumptions $(k_{2i-1}, k_{2i})$ is a pairing for $2i \leq j_p$. This completes the proof. \qed

For the purpose of Section 5 we will also need the following lemma, which is a more general large deviation-type estimate restricted to the no-pairing case:

**Lemma 4.2.** Let $\delta$ be as in Section 7.5, $n \leq 2r + 1$ and consider the following expression

$$M(\omega) = \sum_{(k_1, \ldots, k_n)} \sum_{(k_1^*, \ldots, k_n^*)} \int d\lambda_1 \cdots d\lambda_n \cdot a_{k_1, \ldots, k_n} (\lambda_1, \ldots, \lambda_n) \prod_{j=1}^{n} g_{k_j}^{(j)} (\omega)^{\gamma_j} h_{k_j}^{(j)} (\lambda_j, \omega)^{\pm}, \quad (4.9)$$

where $a_{k_1, \ldots, k_n} (\lambda_1, \ldots, \lambda_n)$ is a given function of $(k_1, \ldots, k_n, \lambda_1, \ldots, \lambda_n)$. Moreover in the summation we assume that there are no pairings among $(k_1^*, \ldots, k_n^*)$, that $|k_j| \leq N_j$, $\frac{N_j}{2} < |k_j^*| \leq N_j$, and that $h_{k_j}^{(j)} (\lambda_j, \omega)$, as a random variable, is $\mathcal{B}^{\mp}_{\leq N_{1,\delta}}$ measurable. Let $N_s \geq \max(n_1, \ldots, n_n)$, then $N_s$-certainly (the exceptional set removed will depend on the coefficients $a$), we have

$$|M(\omega)| \lesssim (N_s)^{\varphi} \prod_{j=1}^{n} \|h_{k_j}^{(j)} (\lambda_j, \omega)\|_{L_{\lambda_j}^{2}} \cdot \|a_{k_1, \ldots, k_n} (\lambda_1, \ldots, \lambda_n)\|_{\mathcal{L}}, \quad (4.10)$$

where $\mathcal{L}$ is an auxiliary norm defined by

$$\|a_{k_1, \ldots, k_n} (\lambda_1, \ldots, \lambda_n)\|_{\mathcal{L}}^2 := \sum_{k_1, \ldots, k_n} \int d\lambda_1 \cdots d\lambda_n \cdot (\max_{1 \leq j \leq n} \langle \lambda_j \rangle)^{\delta_0} (|a|^2 + |\partial \lambda a|^2). \quad (4.11)$$

**Proof.** Consider the big box $\{ |\lambda_j| \leq (N_s)^{-\delta - 1}\}$ and divide it into small boxes of size $(N_s)^{-\delta - 1}$. By exploiting the weight $\max_{1 \leq j \leq n} (\lambda_j)^{\delta_0}$ in (4.11) and using Poincaré’s inequality, we can find a function $b$ which is supported in the big box and is constant on each small box, such that

$$\sup_{k_j, k_j^*} \|a - b\|^{2}_{L_{\lambda_1, \ldots, \lambda_n}^{2}} \lesssim (N_s)^{-\delta - 1} \|a\|_{\mathcal{L}}. \quad (4.12)$$

Exploiting this $(N_s)^{-\delta - 1}$ gain, summing over $|k_j|, |k_j^*| \lesssim N_s$ and using the simple bound

$$\sup_{k_j, k_j^*} \left| \int d\lambda_1 \cdots d\lambda_n \cdot a_{k_1, \ldots, k_n} (\lambda_1, \ldots, \lambda_n) \prod_{j=1}^{n} h_{k_j}^{(j)} (\lambda_j)^{\pm} \right| \lesssim \sup_{k_j, k_j^*} \|a\|_{L_{\lambda_1, \ldots, \lambda_n}^{2}} \prod_{j=1}^{n} \sup_{k_j, k_j^*} \|h_{k_j}^{(j)} (\lambda_j)\|_{L_{\lambda_j}^{2}}$$

suffices to bound the contribution with $a$ replaced by $a - b$; thus we may now replace $a$ by $b$ (or equivalently, assume $a$ is supported in the big box and is constant on each small box) and will prove (4.10) $N_s$-certainly, with the $\mathcal{L}$ norm replaced by the $L^2$ norm, by induction.
By symmetry we may assume \( N_1 \geq \cdots \geq N_n \). Choose the smallest \( q \) such that \( N_q > 2^{10N_{q+1}} \), then \( N_1 \sim N_q \) with constant depending only on \( n \). Unless \( N_1 \leq C \), in which case (4.10) is trivial, we can conclude that

\[
h_{k_j}^j(\lambda_j, \omega)\pm, \quad 1 \leq j \leq n, \text{ are } B^+_{\leq N_1^{1-\delta}} \text{ measurable}; \quad N_1^{1-\delta} \leq 2^{-10N_q},
\]

\[
g_{k_j}^j(\omega)\pm, \quad q+1 \leq j \leq n, \text{ are } B_{\leq N_{q+1}} \text{ measurable}; \quad N_{q+1} \leq 2^{-10N_q}.
\]

Note that in this case, there is no pairing among \( \{k_1^*, \ldots, k_q^*\} \) if and only if there is no pairing among \( \{k_1^*, \ldots, k_q^*\} \) and no pairing among \( \{k_{q+1}^*, \ldots, k_n^*\} \). We can then write \( M(\omega) \) as

\[
M(\omega) = \sum_{k_1^*, \ldots, k_q^*} b_{k_1^*, \ldots, k_q^*}(\omega) \cdot \prod_{j=1}^q g_{k_j}^j(\omega)^{t_j}, \tag{4.13}
\]

where

\[
b_{k_1^*, \ldots, k_q^*}(\omega) = \sum_{k_1, \ldots, k_q} \int d\lambda_1 \cdots d\lambda_q \prod_{j=1}^q h_{k_j, k_j}^j(\lambda_j, \omega) \pm
\]

\[
\times \sum_{k_{q+1}, \ldots, k_n} \sum_{k_{q+1}^*, \ldots, k_n^*} \int d\lambda_{q+1} \cdots d\lambda_n \cdot a_{k_1 \cdots k_n}(\lambda_1, \ldots, \lambda_n) \prod_{j=q+1}^n g_{k_j}^j(\omega)^{t_j} h_{k_j, k_j}^j(\lambda_j, \omega)\pm. \tag{4.14}
\]

are \( B^+_{\leq 2^{-10N_q}} \) measurable. We then apply Lemma 4.11 and conclude that, after removing a set of \( \omega \) with probability \( \leq Cqe^{-N_*^\theta} \), we have

\[
|M(\omega)|^2 \lesssim (N_*)^\theta \sum_{k_1^*, \ldots, k_q^*} |b_{k_1^*, \ldots, k_q^*}(\omega)|^2. \tag{4.15}
\]

Now by (4.11) we have

\[
\sum_{k_1^*, \ldots, k_q^*} |b_{k_1^*, \ldots, k_q^*}(\omega)|^2 \lesssim \prod_{j=1}^q \|h_{k_j, k_j}^j(\lambda_j, \omega)\|^2_{\ell_2^2} \sum_{k_1, \ldots, k_q} \int d\lambda_1 \cdots d\lambda_q \times \sum_{k_{q+1}, \ldots, k_n} \sum_{k_{q+1}^*, \ldots, k_n^*} \int d\lambda_{q+1} \cdots d\lambda_n \cdot a_{k_1 \cdots k_n}(\lambda_1, \ldots, \lambda_n) \prod_{j=q+1}^n g_{k_j}^j(\omega)^{t_j} h_{k_j, k_j}^j(\lambda_j, \omega)\pm^2; \tag{4.16}
\]

by induction hypothesis, we get that

\[
\left| \sum_{k_{q+1}, \ldots, k_n} \sum_{k_{q+1}^*, \ldots, k_n^*} \int d\lambda_{q+1} \cdots d\lambda_n a_{k_1 \cdots k_n}(\lambda_1, \ldots, \lambda_n) \prod_{j=q+1}^n g_{k_j}^j(\omega)^{t_j} h_{k_j, k_j}^j(\lambda_j, \omega)\pm \right|^2 \lesssim (N_*)^\theta \prod_{j=q+1}^n \|h_{k_j, k_j}^j(\lambda_j, \omega)\|^2_{\ell_2^2} \sum_{k_{q+1}, \ldots, k_n} \int d\lambda_{q+1} \cdots d\lambda_n \cdot |a_{k_1 \cdots k_n}(\lambda_1, \ldots, \lambda_n)|^2, \tag{4.17}
\]

up to a set of \( \omega \) with probability \( \leq Cqe^{-N_*^\theta} \), for any fixed \((k_j, \lambda_j)\) for \( 1 \leq j \leq q \). By our assumption on the coefficients \( a \), the function

\[
(k_{q+1}, \ldots, k_n, \lambda_{q+1}, \ldots, \lambda_n) \mapsto a_{k_1 \cdots k_n}(\lambda_1, \ldots, \lambda_n)
\]

which depends on the parameter \((k_j, \lambda_j)\) for \( 1 \leq j \leq q \), has only \((N_*)^{C^7}\) different possibilities, so by removing a set of \( \omega \) with probability \( \leq Cqe^{-N_*^\theta} \), we may assume (4.17) holds for all \((k_j, \lambda_j)\),
1 \leq j \leq q. Thus we can sum (4.17) over $k_j$ and integrate over $\lambda_j$, and combine with (4.15) and (4.16) to get that

$$|M(\omega)|^2 \lesssim (N_*)^\theta \prod_{j=1}^n \|h_{k_j}^{(j)}(\lambda_j, \omega)\|_{L^2_{k_j}}^2 \sum_{k_1, \ldots, k_n} \int d\lambda_1 \cdots d\lambda_n \cdot |a_{k_1\ldots k_n}(\lambda_1, \ldots, \lambda_n)|^2. \quad (4.18)$$

This completes the proof. □

4.2. Counting estimates for lattice points. We start with a simple lemma, and then state the main integer lattice point counting bounds that will be used in the proof below.

**Lemma 4.3.**

(1) Let $R = \mathbb{Z}$ or $\mathbb{Z}[i]$. Then, given $0 \neq m \in R$, and $a_0, b_0 \in \mathbb{C}$, the number of choices for $(a, b) \in R^2$ that satisfy

$$m = ab, \ |a - a_0| \leq M, \ |b - b_0| \leq N$$

is $O(M^6N^6)$ with constant depending only on $\theta > 0$.

(2) Given dyadic numbers $N_1 \gtrsim N_2 \gtrsim N_3$, consider the set

$$S = \{ (x, y, z) \in (\mathbb{Z}^2)^3 : \iota_1 x + \iota_2 y + \iota_3 z = d, \ \iota_1 |x|^2 + \iota_2 |y|^2 + \iota_3 |z|^2 = \alpha, \ |x - a| \lesssim N_1, \ |y - b| \lesssim N_2, \ |z - c| \lesssim N_3 \}. \quad (4.20)$$

Assume also there is no pairing in $S$. Then, uniformly in $(a, b, c, d) \in (\mathbb{Z}^2)^4$ and $\alpha \in \mathbb{Z}$, we have $\#S \lesssim N_1^{1+\theta}N_3^\theta$. Moreover, if $\iota_1 = \iota_2$, then we have the stronger bound $\#S \lesssim N_2^\theta N_3^\theta$.

**Proof.** (1) This strengthened divisor estimate is essentially proved in [25, Lemma 3.4]. We know that $R$ has unique factorization and satisfies the standard divisor estimate, namely the number of divisors of $0 \neq m \in R$ is $O(|m|^\theta)$. Now suppose $\max(|a_0|, M) \geq \max(|b_0|, N)$, then $|m| \lesssim \max(|a_0|, M)^2$. We may assume $M_1 \sim |a_0| \gg M^4$, and hence $|m| \lesssim M_1^2$.

We then claim that the number of divisors of $a$ of $m$ that satisfies $|a - a_0| \leq M$ is at most two. In fact, suppose $a, b, c$ are different divisors of $m$ that belong to the ball $|x - a_0| \leq M$, then by unique factorization we have $\text{lcm}(a, b, c)|m$, hence

$$abc \quad \frac{\gcd(a, b) \gcd(b, c) \gcd(c, a)}{\gcd(a, b) \gcd(b, c) \gcd(c, a)}$$
divides $m$. As $|a| \sim M_1$ etc., and $|\gcd(a, b)| \leq |a - b| \lesssim M$ etc., we conclude that

$$M_1^2 \gtrsim |m| \gtrsim \frac{abc}{\gcd(a, b) \gcd(b, c) \gcd(c, a)} \gtrsim M_1^2 M^{-3},$$

contradicting the assumption $M_1 \gg M^4$.

(2) Let $x = (x_1, x_2)$, etc. If $\iota_1 = \iota_2$, then with fixed $z$ (which has $O(N_3^2)$ choices), $x + y$ will be constant. Let $x = y = w$, then

$$(w_1 + iw_2)(w_1 - iw_2) = |w|^2 = 2(|x|^2 + |y|^2) - |x + y|^2$$
is constant. As $w$ belongs to a ball of radius $O(N_2)$ in $\mathbb{R}^2$, by (1) we know that the number of choices for $w$ is $O(N_3^2)$, hence $\#S = O(N_2^2N_3^2)$. Below we will assume that $\iota_1 = +$ and $\iota_2 = -$.

(a) Suppose $\iota_3 = +$, then we have that

$$(d_1 - z_1)(z_1 - y_1) + (d_2 - z_2)(z_2 - y_2) = (d - z) \cdot (z - y) = \frac{|d|^2 - \alpha}{2}$$
is constant. If $(d_1 - z_1)(z_1 - y_1) \neq 0$ (or similarly if $(d_2 - z_2)(z_2 - y_2) \neq 0$), then with fixed $(y_2, z_2)$ (which has $O(N_2N_3)$ choices), $(d_1 - z_1)(z_1 - y_1)$ will be constant. As $d_1 - z_1$ belongs to an interval of size $O(N_3)$ in $\mathbb{R}$, and $z_1 - y_1$ belongs to an interval of size $O(N_2)$ in $\mathbb{R}$, by (1) we know that the number of choices for $(y_1, z_1)$ is $O(N_2^2)$, so $\#S \lesssim N_2^\theta N_2^2 N_3$. 


If $(d_1 - z_1)(z_1 - y_1) = 0$ and $(d_2 - z_2)(z_2 - y_2) = 0$, as there is no pairing, we may assume that $d_1 = z_1$ and $z_2 = y_2$ (or $z_1 = y_1$ and $d_2 = z_2$, which is treated similarly), so $z_1 = d_1$ and $x_2 = d_2$ are fixed, $z_2$ has $O(N_3)$ choices and $x_1 = y_1$ has $O(N_2)$ choices, which implies \( \#S \lesssim N_2N_3 \).

(b) Suppose $\nu_3 = -$, then similarly we have that

\[
(d_1 + z_1)(d_1 + y_1) + (d_2 + z_2)(d_2 + y_2) = (d + z) \cdot (d + y) = \frac{|d|^2 + \alpha}{2}
\]

is constant. If $(d_1 + z_1)(d_1 + y_1) \neq 0$ (or similarly if $(d_2 + z_2)(d_2 + y_2) \neq 0$), then with fixed $(y_2, z_2)$ (which has $O(N_2N_3)$ choices), $(d_1 + z_1)(d_1 + y_1)$ will be constant. As $d_1 + z_1$ belongs to an interval of size $O(N_3)$ in $\mathbb{R}$, and $d_1 + y_1$ belongs to an interval of size $O(N_2)$ in $\mathbb{R}$, by (1) we know that the number of choices for $(y_1, z_1)$ is $O(N_2^\theta)$, so \( \#S \lesssim N_2^\theta N_3 \).

If $(d_1 + z_1)(d_1 + y_1) = 0$ and $(d_2 + z_2)(d_2 + y_2) = 0$, as there is no pairing, we may assume that $d_1 + z_1 = 0$ and $d_2 + y_2 = 0$ (or $d_2 + z_2$ and $d_1 + y_1 = 0$, which is treated similarly), so $z_1 = -d_1$ and $y_2 = -d_2$ are fixed, $z_2$ has $O(N_3)$ choices and $y_1$ has $O(N_2)$ choices, which implies \( \#S \lesssim N_2N_3 \).

\[\square\]

**Proposition 4.4.** Recall the relevant constants defined in (4.32). The following bounds are uniform in all parameters. Given $d, d', k^0 \in \mathbb{Z}^2$ and $k_j^0 \in \mathbb{Z}^2(1 \leq j \leq n)$, $\alpha, \Gamma \in \mathbb{R}$, $\nu, \iota_j \in \{\pm\}(1 \leq j \leq n)$, and $2p \leq n$, as well as $M$, $N_j(0 \leq j \leq n)$ and $R_i(1 \leq i \leq p)$, such that for $1 \leq i \leq p$ we have $N_{2i-1} \sim N_{2i}$, \( \nu_{2i-1} = -\nu_{2i} \) and \( R_i \lesssim N_{n-1}^{1/4} \). Let \( N^{(j)} = \max(N_1, \cdots, N_n), N_{PR} = \max(N_1, \cdots, N_{2p}) \) and let \( N_\ast \gtrsim \max(N_0, N^{(1)}) \). Also fix a subset $A$ of $\{1, \cdots, n\}$ that contains \( \{1, \cdots, 2p\} \), and recall the definition of the $\Gamma$-condition (3.35). Consider the sets

\[
S_1 = \left\{ (k, k_1, \cdots, k_n) \in (\mathbb{Z}^2)^{n+1} : \sum_{j=1}^{n} \nu_j k_j = k + d, \quad \sum_{j=1}^{n} \nu_j |k_j|^2 = |k|^2 + \alpha, \quad |k_j - k_j^0| \lesssim N_j (1 \leq j \leq n), \quad |k_{2i-1} - k_{2i}| \lesssim R_i(N_\ast)^{C_{\kappa}-1} (1 \leq i \leq p) \right\},
\]

(4.21)

\[
S_2 = \left\{ (k, k', k_1, \cdots, k_n) \in (\mathbb{Z}^2)^{n+2} : t k' + \sum_{j=1}^{n} \nu_j k_j = k + d, \quad t|k'|^2 + \sum_{j=1}^{n} \nu_j |k_j|^2 = |k|^2 + \alpha, \quad |k|, |k'| \lesssim N_0, \quad |k_j - k_j^0| \lesssim N_j (1 \leq j \leq n), \quad |k_{2i-1} - k_{2i}| \lesssim R_i(N_\ast)^{C_{\kappa}-1} (1 \leq i \leq p) \right\},
\]

(4.22)

\[
S_3 = \left\{ (k, k_1, \cdots, k_n) \in (\mathbb{Z}^2)^{n+1} : \sum_{j=1}^{n} \nu_j k_j = k + d, \quad |k|^2 - \sum_{j=1}^{n} \nu_j |k_j|^2 - \alpha \lesssim M, \quad |k| \lesssim N_0,
\]

(4.24)

Assume that there is no pairing among the variables $k$, $k'$ and $k_j$ in the sets above. Let $S_j^+ = S_j \cap S^+$. Then, for $S_1$ we have

\[
(\#S_1) \prod_{i=1}^{p} \frac{N_{2i-1}^{1+2\gamma_0}}{R_i} \lesssim (N_{PR})^{2\gamma_0}(N_\ast)^{C_{\kappa}-1}(N^{(1)} N^{(2)})^{-1} \prod_{j=1}^{n} N_j^{2}.
\]

(4.25)
If $N^{(1)} \sim N_a$ and $\iota_a = -$, then we have

\[(\#S_1) \cdot \prod_{i=1}^{p} \frac{N_{2i-1}^{1+2\gamma_0}}{R_i} \lesssim (N_{PR})^{2\gamma_0} (N_a)^{C\kappa-1} (N^{(1)})^{-2} \prod_{j=1}^{n} N_j^2. \tag{4.26}\]

For $S_2$ and $S_3$ we have

\[(\#S_2) \cdot \prod_{i=1}^{p} \frac{N_{2i-1}^{1+2\gamma_0}}{R_i} \lesssim (N_{PR})^{2\gamma_0} (N_a)^{C\kappa-1} N_0(N^{(1)})^{-1} \prod_{j=1}^{n} N_j^2, \tag{4.27}\]

\[(\#S_3) \cdot \prod_{i=1}^{p} \frac{N_{2i-1}^{1+2\gamma_0}}{R_i} \lesssim (N_{PR})^{2\gamma_0} (N_a)^{C\kappa-1} M \max((N(2)^2, |\alpha|)) (N^{(1)})^{-2} \prod_{j=1}^{n} N_j^2. \tag{4.28}\]

Finally, suppose we replace any of these $S_j$’s by the set $S_j^+$. Then $(4.25)$–$(4.27)$ hold with the right hand side multiplied by an extra factor

\[\left[ \min \left( (N^{(2)}), \max_{j \in A, j \geq 2p+1} N_j \right) \right]^{-1}. \tag{4.29}\]

If $N^{(1)} \sim N_a$ and $a \in A$, then the stronger bound $(4.26)$ holds for $S_j^+$ with right hand side multiplied by an extra factor $(4.29)$, regardless of whether $\iota_a = -$ or not. If $N^{(1)} \sim N_a$ and $2p+1 \leq a \in A$, then $(4.27)$ holds for $S_j^+$ with right hand side multiplied by an extra factor $(N^{(1)})^{-1}$. As for $S_j^+$, either it satisfies $(4.28)$ with the same extra factor $(4.29)$, or it satisfies

\[(\#S_j^+) \cdot \prod_{i=1}^{p} \frac{N_{2i-1}^{1+2\gamma_0}}{R_i} \lesssim (N_a)^{C\kappa-1} M(N_{PR})^{2\gamma_0} \min \left( \frac{\max((N^{(2)}), |\alpha|)}{(N(1)N^{(2)})} \right) \prod_{j=1}^{n} N_j^2. \tag{4.30}\]

Proof. Let $a, b, c$ be such that $N^{(1)} \sim N_a$, $N^{(2)} \sim N_b$, and $\max(\{N_j : 2p+1 \leq j \in A\}) \sim N_c$. In the proof below any factor that is $\lesssim (N_a)^{C\kappa-1}$ will be negligible, so we will pretend they are 1. For simplicity, let us first also ignore all $N_a^{2\gamma_0}$ factors; at the end of the proof we will explain how to put them back.

(1) We start with $(4.25)$. If $a, b \geq 2p+1$, we may fix all $k_j, j \not\in \{a, b\}$, and then apply Lemma 4.3 (2) to count the triple $(k, k_a, k_b)$. This gives

\[(\#S_1) \prod_{i=1}^{p} \frac{N_{2i-1}}{R_i} \lesssim \left( \prod_{i=1}^{3} N_{2i-1}^3 R_i \prod_{2p+1 \leq j_\neq \{a,b\}} N_j^2 \right) N_a N_b, \tag{4.31}\]

which proves $(4.25)$ as $R_j \lesssim N_a^{-\delta}$. If $a \geq 2p+1$ and $b \leq 2p$ (say $b = 1$), we may fix all $k_j, j \not\in \{1, a\}$, and then apply Lemma 4.3 (2) to count the triple $(k, k_1, k_a)$, noticing that $k_1$ belongs to a disc of radius $O(R_1)$ once $k_2$ is fixed. This gives

\[(\#S_1) \prod_{i=1}^{p} \frac{N_{2i-1}}{R_i} \lesssim \left( \prod_{i=2}^{2p} N_{2i-1}^3 R_i \prod_{2p+1 \leq j \neq \{a, b\}} N_j^2 \right) N_a R_1 \frac{N_1}{R_1}, \tag{4.32}\]

which proves $(4.25)$. Finally, if $a \leq 2p$ (say $a = 1$) then we may assume $b = 2$. We may fix all $k_j, j \geq 3$, and then apply Lemma 4.3 (2) to count the triple $(k, k_1, k_2)$, noticing that $k$ belongs to a disc of radius $O(R_1(N_a)^{C\kappa-1})$ once all $k_j, j \geq 3$ are fixed. This gives

\[(\#S_1) \prod_{i=1}^{p} \frac{N_{2i-1}}{R_i} \lesssim \left( \prod_{i=2}^{p} N_{2i-1}^3 R_i \prod_{j \geq 2p+1} N_j^2 \right) N_1 R_1 \frac{N_1}{R_1}, \tag{4.33}\]
which proves (4.25).

As for (4.26) and (4.27) we only need to consider $a$. If $a \geq 2p + 1$ we may fix all $k_j (j \neq a)$, and then apply Lemma 4.3 (2) to count the pair $(k, k_a)$ for (4.26) (using the fact $i_a = -$) and the triple $(k, k', k_a)$ for (4.27), and get

\[
(#S_1) \prod_{i=1}^{p} \frac{N_{2i-1} - R_i}{R_i} \lesssim \left( \prod_{i=1}^{2p} N_{2i-1} R_i \prod_{2p+1 \leq j \neq a} N_j^2 \right),
\]

(4.34)

\[
(#S_2) \prod_{i=1}^{p} \frac{N_{2i-1} - R_i}{R_i} \lesssim \left( \prod_{i=1}^{2p} N_{2i-1} R_i \prod_{2p+1 \leq j \neq a} N_j^2 \right) N_0 N_a,
\]

(4.35)

which proves (4.26) and (4.27). If $a \leq 2p$ (say $a = 1$) we may assume $b = 2$, in particular $N^{(1)} \sim N^{(2)}$ and (4.26) follows from (4.25); for (4.27) we may fix all $k_j (j \geq 2)$ and then apply Lemma 4.3 (2) to count the triple $(k, k', k_1)$, noticing that $k_1$ belongs to a disc of radius $O(R_1(N)_{C_{k-1}})$ once $k_2$ is fixed, and get

\[
(#S_2) \prod_{i=1}^{p} \frac{N_{2i-1} - R_i}{R_i} \lesssim \left( \prod_{i=2}^{p} N_{2i-1} R_i \prod_{i \geq 2p+1} N_i^2 \right) N_0 R_1 \frac{N_a}{R_1},
\]

(4.36)

which proves (4.27).

(2) Next we prove the improvements to (4.25)–(4.27) for $S_j^+$. We start with (4.25). If $a, b \not\in A$, we may fix all $k_j (c \neq j \not\in A)$ and apply (4.25) to the rest variables and get

\[
(#S_1^+) \prod_{i=1}^{p} \frac{N_{2i-1} - R_i}{R_i} \lesssim (N_a N_b)^{-1} \prod_{j \not\in A} N_{j}^2 \prod_{j \in A} N_{2j-1} R_i \prod_{j \in A, c \neq j \geq 2p+1} N_j^2,
\]

(4.37)

which gains a factor $N_{c}^{-2}$ upon (4.25). If $a \not\in A$ and $2p + 1 \leq b \in A$, we may fix all $k_j (b \neq j \in A)$ and apply (4.25) to the rest variables and get

\[
(#S_1^+) \prod_{i=1}^{p} \frac{N_{2i-1} - R_i}{R_i} \lesssim N_a^{-1} \prod_{j \not\in A} N_{j}^2 \prod_{j \in A} N_{2j-1} R_i \prod_{b \not\in A, j \geq 2p+1} N_j^2,
\]

(4.38)

which gains a factor $N_{b}^{-1}$ upon (4.25). If $a \not\in A$ and $b \leq 2p$ (say $b = 1$), we may fix all $k_j (2 \leq j \neq a)$, noticing that $k_c$ belongs to a ball of radius $\min(N_c, R_1(N)_{C_{-1}})$ once all $k_j (3 \leq j \not\in \{a, c\})$ are fixed, and then apply Lemma 4.3 (2) to count the pair $(k, k_a)$ and get

\[
(#S_1^+) \prod_{i=1}^{p} \frac{N_{2i-1} - R_i}{R_i} \lesssim \left( N_1^2 \min(N_{c}^2, R_{1}^2) \prod_{i=2}^{p} N_{2i-1} R_i \prod_{2p+1 \leq j \not\in \{a, c\}} N_j^2 \right) N_0 \frac{N_a}{R_1},
\]

(4.39)

which gains a factor $N_{c}^{-1}$ upon (4.25).

Now if $2p + 1 \leq a \in A$ and either $b \not\in A$ or $2p + 1 \leq b \in A$, we may fix all $k_j (j \not\in \{a, b\})$ and apply Lemma 4.3 (2) to count the pair $(k, k_b)$ (if $b \not\in A$) or $(k_a, k_b)$ (if $2p + 1 \leq b \in A$), and get

\[
(#S_1^+) \prod_{i=1}^{p} \frac{N_{2i-1} - R_i}{R_i} \lesssim \left( \prod_{i=1}^{p} N_{2i-1} R_i \prod_{2p+1 \leq j \not\in \{a, b\}} N_j^2 \right) N_b,
\]

(4.40)

which gains a factor $N_{b}^{-1}$ upon the stronger bound (4.26). If $2p + 1 \leq a \in A$ and $b \leq 2p$ (say $b = 1$), we may fix all $k_j (j \not\in \{a, 1, 2\})$ and apply Lemma 4.3 (2) to count the triple $(k_a, k_1, k_2)$, noticing that $k_a$ belongs to a disc of radius $O(R_1(N)_{C_{-1}})$ once all $k_j (j \not\in \{a, 1, 2\})$ are fixed, and get

\[
(#S_1^+) \prod_{i=1}^{p} \frac{N_{2i-1} - R_i}{R_i} \lesssim \left( \prod_{i=2}^{p} N_{2i-1} R_i \prod_{2p+1 \leq j \neq a} N_j^2 \right) N_1 R_1 \frac{N_a}{R_1},
\]

(4.41)
which gains a factor $N_p^{-2}$ upon the stronger bound \((4.26)\). Finally, if $a \leq 2p$ (say $a = 1$) then we may assume $b = 2$. We may fix all $k_j(j \geq 3)$, noticing that $k_c$ belongs to a disc of radius $\min(N_c, R_1(N_c)^{C\kappa^{-1}})$ once all $k_j(3 \leq j \neq c)$ are fixed, and then apply Lemma \ref{lem:bdy} (2) to count the pair $(k_1, k_2)$. This gives
\begin{equation}
(#S_1^+) \prod_{i=1}^{p} \frac{N_{2i-1}}{R_i} \lesssim \left( \min(N_c, R_1)^2 \prod_{i=2}^{p} N_{2i-1}^3 R_i \prod_{2p+1 \leq j \neq c} N_j^2 \right) N_1 N_1 R_1, \tag{4.42}
\end{equation}
which gains a factor $N_c^{-1}$ upon the stronger bound \((4.26)\).

As for \((4.26)\) and \((4.27)\) we only need to consider $a$. If $a \notin A$ we may fix all $k_j(c \neq j \in A)$ and apply \((4.26)\) or \((4.27)\) to the rest variables and get
\begin{equation}
(#S_1^+) \prod_{i=1}^{p} \frac{N_{2i-1}}{R_i} \lesssim N_0 N_a^{-1} \prod_{j \notin A} N_j^2 \prod_{j \in A, c \neq j \geq 2p+1} N_j^2, \tag{4.43}
\end{equation}
\begin{equation}
(#S_2^+) \prod_{i=1}^{p} \frac{N_{2i-1}}{R_i} \lesssim N_0 N_a^{-1} \prod_{j \notin A} N_j^2 \prod_{j \in A, c \neq j \geq 2p+1} N_j^2, \tag{4.44}
\end{equation}
which gains a factor $N_c^{-2}$ upon \((4.26)\) or \((4.27)\). If $a \in A$ then \((4.26)\) follows from the above proof for \((4.25)\); for \((4.27)\), if $2p+1 \leq a \in A$ we may fix all $k_j(a \neq j \in A)$ and apply \((4.27)\) to the rest variables and get
\begin{equation}
(#S_2^+) \prod_{i=1}^{p} \frac{N_{2i-1}}{R_i} \lesssim N_0 \prod_{j \notin A} N_j^2 \prod_{j \in A, a \neq j \geq 2p+1} N_j^2, \tag{4.45}
\end{equation}
which gains a factor $N_a^{-1}$ upon \((4.27)\); if $a \leq 2p$ (say $a = 1$) we may fix all $k_j(2 \leq j \in A)$, noticing that $k_c$ belongs to a ball of radius $\min(N_c, R_1(N_c)^{C\kappa^{-1}})$ once all $k_j(j \in A\{1, 2, c\})$ are fixed, and apply \((4.27)\) to the rest variables and get
\begin{equation}
(#S_2^+) \prod_{i=1}^{p} \frac{N_{2i-1}}{R_i} \lesssim \min(N_c, R_1)^2 N_1 \prod_{j \notin A} N_j^2 \prod_{j \in A, c \neq j \geq 2p+1} N_j^2, \tag{4.46}
\end{equation}
which gains a factor $N_c^{-1}$ upon \((4.27)\).

(3) Now we consider \((4.28)\) and its improvement. We may assume $\epsilon_a = +$ and $N^{(1)} \gg N^{(2)}$ (so $a \geq 2p+1$), since otherwise \((4.28)\) follows from \((4.26)\) or \((4.25)\) and similarly for the improvement. Now let $M_0 = \max(\epsilon_a, (N^{(2)})^2)$, if $M \gg M_0$ then we have
\[ ||k|^2 - |k_a|^2| \leq |\epsilon_a| + \sum_{j \neq a} |k_j|^2 + M \lesssim M; \]
combining with \((3.34)\) and Lemma \ref{lem:bdy} (1) we conclude that the number of choices for $|k_a|^2$, and thus $k_a$, is $O(M)$. We may fix $k_a$ and then count $k_j(j \neq a)$ to get
\begin{equation}
(#S_3) \prod_{i=1}^{p} \frac{N_{2i-1}}{R_i} \lesssim M \prod_{i=1}^{p} N_{2i-1}^3 R_i \prod_{2p+1 \leq j \neq a} N_j^2, \tag{4.47}
\end{equation}
which proves \((4.28)\). As for $S_3^+$, if $a \in A$ then the improvement of \((4.28)\) follows from the improvement of \((4.26)\); if $a \notin A$ we may fix $k_a$ and count $k_j(j \notin \{a, c\})$ to get
\begin{equation}
(#S_3^+) \prod_{i=1}^{p} \frac{N_{2i-1}}{R_i} \lesssim M \prod_{i=1}^{p} N_{2i-1}^3 R_i \prod_{2p+1 \leq j \notin \{a, c\}} N_j^2, \tag{4.48}
\end{equation}
which gains a factor $N_c^{-2}$ upon \((4.28)\).
Assume now $M \lesssim M_0$, then just like above we have $|\{k\}^2 - |k_a|^2| \lesssim M_0$, so $k$ has at most $O(M_0)$ choices, similarly $k_a$ has at most $O(M_0)$ choices. If $b \geq 2p + 1$, we may assume $\nu = +$ (otherwise switch the roles of $k$ and $k_a$), then fix $k$ and $k_j(j \notin \{a, b\})$ and apply Lemma 4.3 (2) to count the pair $(k_a, k_b)$ to get

$$\left(\#S_3\right) \prod_{i=1}^{p} \frac{N_{2i+1}}{R_i} \lesssim M_0 \left( \prod_{i=1}^{p} N_{2i-1}^3 R_i \prod_{2p+1 \leq j \notin \{a,b\}} N_j^2 \right) M,$$

which proves (4.28); if $b \leq 2p$ (say $b = 1$), we may fix $k$ and $k_j(j \notin \{1, 2, a\})$ and apply Lemma 4.3 (2) to count the triple $(k_1, k_2, k_a)$, noticing that $k_a$ belongs to a disc of radius $O(R_1(N_a)^{\nu - 1})$ once $k$ and $k_j(j \notin \{1, 2, a\})$ are fixed, and get

$$\left(\#S_3\right) \prod_{i=1}^{p} \frac{N_{2i+1}}{R_i} \lesssim \left( M_0 \prod_{i=2}^{p} N_{2i-1}^3 R_i \prod_{2p+1 \leq j \neq a} N_j^2 \right) N_1 R_1 M \frac{N_1}{R_1},$$

which proves (4.28).

It remains to prove the improvement of (4.28) for $S_3^+$. We may assume $a \notin A$, since otherwise it follows from the improvement of (4.28). Now if $b \notin A$ we may fix all $k_j(c \neq j \in A)$ and apply (4.28) to the rest variables and get

$$\left(\#S_3^+\right) \prod_{i=1}^{p} \frac{N_{2i+1}}{R_i} \lesssim M M_0 (N_a N_b)^{-2} \prod_{j \notin A} N_j^2 \prod_{i=1}^{p} N_{2i-1}^3 R_i \prod_{2p+1 \leq j \neq c} N_j^2,$$

which gains a factor $N_2^{-2}$ upon (4.28). If $b \leq 2p$, say $b = 1$, we may fix $k$ and $k_j(3 \leq j \neq a)$, noticing that $k_c$ belongs to a disc of radius $\min(N_c, R_1(N_a)^{\nu - 1})$ once all $k_j(3 \leq j \notin \{a, c\})$ are fixed, and then apply Lemma 4.3 (2) to count the pair $(k_1, k_2)$ and get

$$\left(\#S_3^+\right) \prod_{i=1}^{p} \frac{N_{2i+1}}{R_i} \lesssim M M_0 \left( \min(N_{2i}, R_1) \prod_{i=2}^{p} N_{2i-1}^3 R_i \prod_{2p+1 \leq j \notin \{a, c\}} N_j^2 \right) N_1 \frac{N_1}{R_1},$$

which gains a factor $N_2^{-1}$ upon (4.28). Finally, assume $2p + 1 \leq b \in A$, then we will prove (4.30). Let $\max(2) \{N_j : 2p + 1 \leq j \in A\} \sim N_d$, we may fix $k$ and $k_j(j \notin \{a, b, d\})$, then apply Lemma 4.3 (2) to count the pair $(k_b, k_d)$ and get

$$\left(\#S_3^+\right) \prod_{i=1}^{p} \frac{N_{2i+1}}{R_i} \lesssim M M_0 \left( \prod_{i=1}^{p} N_{2i-1}^3 R_i \prod_{2p+1 \leq j \notin \{a, b, d\}} N_j^2 \right) N_d;$$

alternatively we may choose to fix $k_j(j \notin \{a, b\})$ then apply Lemma 4.3 (2) to count the pair $(k, k_a)$ and get

$$\left(\#S_3^+\right) \prod_{i=1}^{p} \frac{N_{2i+1}}{R_i} \lesssim M \left( \prod_{i=1}^{p} N_{2i-1}^3 R_i \prod_{2p+1 \leq j \notin \{a, b\}} N_j^2 \right) N_a,$$

and combining (4.53) and (4.54) yields (4.30).

In the last part we will explain how to put back the powers $N_{2i-1}^{2\gamma}$. In fact, in each estimate above we have the product $\prod_{i \geq 2} N_{2i-1}^3$. As $R_i \lesssim N_{2i-1}^{1-\delta}$ and $N_{2i-1} \sim N_{2i}$ we have

$$N_{2i-1}^3 R_i \lesssim N_{2i-1}^2 N_{2i} \cdot N_{2i-1}^{2\gamma},$$

which allows us to incorporate the extra factor $N_{2i-1}^{2\gamma}$ for $i \geq 2$. Thus we lose at most a factor $N_{2i}^{2\gamma}$, which is acceptable as $N_1 \lesssim N_{PR}$. \hfill $\square$
Corollary 4.5. Recall $a_0 > 1$ defined in (1.32), and let all the parameters $(d, N_j, \iota_j \text{ etc.})$ be as in Proposition 4.4. From the sets $S_j(1 \leq j \leq 3)$ in Proposition 4.4 we may construct the quantities $\mathcal{E}_j$ as follows: each $\mathcal{E}_j$ is a sum over a set $S^\text{lin}_j$. This $S^\text{lin}_j$ is formed from $S^+_j$ by removing from its defining properties the one that involves the quadratic algebraic sum $\Sigma$ (this $\Sigma$ is $\iota_1 |k_1|^2 + \cdots + \iota_n |k_n|^2 - |k|^2$ for $S_1$ and $S_3$, and $\iota_1 |k_1|^2 + \cdots + \iota_n |k_n|^2 + \iota_k' |2 - |k|^2$ for $S_2$), and the summand is simply $\langle \Sigma - \alpha \rangle - a_0$. Similarly define $\mathcal{E}_j^+$ by replacing $S_j$ with $S_j^+$. Therefore, the inequalities (4.25)–(4.30), as well as their improvements, hold with $\#S_j$ replaced by $\mathcal{E}_j$ ($\#S^+_j$ replaced by $\mathcal{E}_j^+$), and with the factor $M$ on the right hand sides of (4.28) and (4.30) removed.

Proof. This is straightforward, by applying Proposition 4.4 for each value of $\Sigma$ and summing up using $a_0 > 1$ for (4.25)–(4.27), and by dyadically decomposing $\langle \Sigma - \alpha \rangle$ and applying Proposition 4.4 for each dyadic piece for (4.28)–(4.30). □

5. PROOF OF THE MULTILINEAR ESTIMATES

In this section we will prove Proposition 3.4 thus completing the local theory. We will start with an estimate for general multilinear forms without pairing. Given $d \in \mathbb{Z}^2$ and $\alpha \in \mathbb{R}$, consider the following expressions:

$$\mathcal{X} = \sum_{(k,k_1,\ldots,k_n)} \int d\lambda d\lambda_1 \cdots d\lambda_n \cdot \eta \left( \lambda, \lambda - |k|^2 - \sum_{j=1}^n \iota_j (\lambda_j - |k_j|^2) - \alpha \right) v_k(\lambda) \prod_{j=1}^n [v_{k_j}^{(j)}(\lambda_j)]^{\iota_j}, \quad (5.1)$$

$$\mathcal{Y} = \sum_{(k,k',k_1,\ldots,k_n)} \int d\lambda d\lambda' d\lambda_1 \cdots d\lambda_n$$

$$\times \eta \left( \lambda, \lambda - |k|^2 - \iota (\lambda' - |k'|^2) - \sum_{j=1}^n \iota_j (\lambda_j - |k_j|^2) - \alpha \right) y_{kk'}(\lambda, \lambda') \prod_{j=1}^n [v_{k_j}^{(j)}(\lambda_j)]^{\iota_j}, \quad (5.2)$$

where $d \in \mathbb{Z}^2$ and $\alpha \in \mathbb{R}$ are fixed, $\eta$ is a function that satisfies

$$|\eta(\lambda, \mu)| + |\partial_{\lambda,\mu} \eta(\lambda, \mu)| \lesssim (\mu)^{-10}. \quad (5.3)$$

In the summation we always assume that there is no pairing among the variables $k, k'$ and $j$.

We assume that the input functions $v^{(j)}$ are as in Proposition 3.4 where $v^{(j)}$ are of type (G) or (C) for $1 \leq j \leq n_1$, and of type (D) for $n_1 + 1 \leq j \leq n$. Since we are working exclusively in the $\lambda_j$ spaces, we will abuse notations here and write $(v^{(j)}_{k_j})(\lambda_j)$ instead of $(\tilde{v}^{(j)}_{k_j})(\lambda_j)$. Let the parameters $N_j, L_j, N^{(j)} \text{ etc.}$, and the sets $\mathcal{G}$ and $\mathcal{C}$ be as in that proposition. We further assume that the functions $v_k(\lambda)$ and $y_{kk'}(\lambda, \lambda')$ satisfy

$$\|\langle \lambda \rangle v_k(\lambda)\|_{L^2}^2 \lesssim 1, \quad \|\langle \lambda \rangle \langle \lambda' \rangle y_{kk'}(\lambda, \lambda')\|_{L^2}^2 \lesssim 1, \quad (5.4)$$

and that $v_k(\lambda)$ is supported in $\{|k| \lesssim N_0\}$ and $y_{kk'}(\lambda, \lambda')$ is supported in $\{|k|, |k'| \lesssim N_0\}$.

Proposition 5.1. Recall the relevant constants defined in (1.32), and that $\tau \ll 1$. Under all the above assumptions, there exist $p$ and $q$, and $N_{2p+1} \gtrsim R_{2p+1} \gtrsim L_{2p+1}(1 \leq l \leq q)$ such that $2p + q \leq n_1$, that for $1 \leq i \leq p$ we must have $N_{2i-1} \sim N_{2i}$, and $\iota_{2i-1} = -\iota_{2i}$, and that $2i - 1$ and $2i$ do not both

\footnote{This requirement appears in the form of coefficients which are indicator functions of sets of form $\{k_j \neq k_i\}$. Such coefficients may lead to slightly different multilinear Gaussian expressions in the estimates below, but there will be at most $(N_j)^C$ possibilities where $N_j$ is a parameter to be defined below, and will not affect any estimates since our exceptional sets will always have measure at most $C_y e^{-(N_j)^y}$.}
belong to $\mathcal{G}$. Define $R_i = \max(L_{2i-1}, L_{2i})$ and let $N_*$ be fixed, then the following estimates hold \( \tau^{-1} N_* \)-certainly. Here, as in Proposition 3.4, the exceptional set of $\omega$ removed does not depend on the choice of the functions $v_j (j \geq n_1 + 1)$ or $v$ or $w$.

1. Assume $N_* \gtrsim \max(N_0, N^{(1)})$. Then we have
\[
|X|^2 \lesssim \tau^{-\theta} (N_*)^{C\kappa^{-1}} \mathcal{E}_1 \left( \prod_{i=1}^{p} \frac{N_i^{1+2\gamma}}{R_i} \right) \prod_{j=1}^{n} N_j^{-2} \prod_{j \geq n_1 + 1} N_j^{2\gamma} \prod_{2p+1 \leq j \leq n_1} L_j^{-2\theta_0}, \tag{5.5}
\]
and similarly
\[
|Y|^2 \lesssim \tau^{-\theta} (N_*)^{C\kappa^{-1}} \mathcal{E}_2 \left( \prod_{i=1}^{p} \frac{N_i^{1+2\gamma}}{R_i} \right) \prod_{j=1}^{n} N_j^{-2} \prod_{j \geq n_1 + 1} N_j^{2\gamma} \prod_{2p+1 \leq j \leq n_1} L_j^{-2\theta_0}, \tag{5.6}
\]
where $\mathcal{E}_1$ and $\mathcal{E}_2$ are the quantities defined in Corollary 4.5 with $a_0 = 2b - 10\delta^6$, and for some choice of the parameters in that corollary that do not appear in the assumptions of the current proposition. Moreover, if in the sum defining $X$ we also assume the $\Gamma$-condition (3.34), then (5.5) holds with $\mathcal{E}_1$ replaced by $\mathcal{E}_3$ (see Corollary 4.9 for the relevant definitions).

2. Assume $N_* \gtrsim \max(N_0, N^{(1)})$. Then we have
\[
|X|^4 \lesssim \tau^{-\theta} (N_*)^{C\kappa^{-1}} \mathcal{E}_1^{\pm} \left( \prod_{i=1}^{p} \frac{N_i^{1+2\gamma}}{R_i} \right) \prod_{j=1}^{n} N_j^{-4} \prod_{j \geq n_1 + 1} N_j^{4\gamma} \prod_{2p+1 \leq j \leq n_1} L_j^{4\theta_0^2}, \tag{5.7}
\]
and similarly
\[
|Y|^4 \lesssim \tau^{-\theta} (N_*)^{C\kappa^{-1}} \mathcal{E}_2^{\pm} \left( \prod_{i=1}^{p} \frac{N_i^{1+2\gamma}}{R_i} \right) \prod_{j=1}^{n} N_j^{-4} \prod_{j \geq n_1 + 1} N_j^{4\gamma} \prod_{2p+1 \leq j \leq n_1} L_j^{4\theta_0^2}, \tag{5.8}
\]
where $\mathcal{E}_j$ and $\mathcal{E}_j^{\pm}$ are the quantities defined in Corollary 4.5, again for some choice of the parameters in that corollary that do not appear in the assumptions of the current proposition. In the set $S^+$ in (4.22) the set $A$ will contain $\{1, 2, \ldots, 2p\} \cup \{n_1 + 1, \ldots, n\}$. Moreover, if in the sum defining $X$ we also assume the $\Gamma$-condition (3.34), then (5.7) holds with $\mathcal{E}_1^{\pm}$ replaced by $\mathcal{E}_3^{\pm}$.

3. Assume in addition that $N^{(1)} \sim N_n$ and $N_* \gtrsim N^{(2)}$. Then (5.7) is true, with $N_n$ replaced by $N^{(2)}$ in both quantities $\mathcal{E}_1$ and $\mathcal{E}_1^{\pm}$. Moreover we have
\[
|X|^4 \lesssim \tau^{-\theta} (N_*)^{C\kappa^{-1}} (N^{(1)})^{-4(1-\gamma)} (N^{(2)}) C\kappa^{-1} \mathcal{E}_1^{\pm} \left( \prod_{i=1}^{p} \frac{N_i^{1+2\gamma}}{R_i} \right) \prod_{j=1}^{n-1} N_j^{-4} \prod_{j=2p+1}^{n} N_j^{2\gamma} R_j^{-2}, \tag{5.9}
\]
where $\mathcal{E}_1$ is the quantity defined in Corollary 4.5 for some choice of the parameters in that corollary that do not appear in the assumptions of the current proposition, but with $N_n$ replaced by $N^{(2)}$. Similarly $\mathcal{E}_1^{\pm}$ is the quantity $\mathcal{E}_1^{\pm}$ defined in Corollary 4.5 with $A = \{1, \ldots, n\}$, but with $N_n$ replaced by $N^{(2)}$ and $N_{2p+1}$ replaced by $R_{2p+1}(N_*)^{C\kappa^{-1}}$ for $1 \leq l \leq q$. Moreover, the exceptional set of $\omega$ removed is independent of $N^{(1)}$.

5.1. Proof of Proposition 5.1 We will prove Proposition 5.1 in the following three subsections. We will only prove the bounds for $X$ without $\Gamma$-condition; with obvious modifications the proof also works for $Y$ and for the version with $\Gamma$-condition. For simplicity we will omit the $\omega$ dependence, and may ignore any factors that are $\lesssim \tau^{-\theta} (N_*)^{C\kappa^{-1}}$.

Our proof will roughly follow an algorithm, indicated by the following steps. (1) Distinguish between the inputs $j \in D$, where $v(j)$ are bounded in $\ell^2 L^2$, with $j \in \mathcal{G} \cup \mathcal{C}$. (2) Identify the pairings among $k_j (j \in \mathcal{G})$ and $k_j^\ell (j \in \mathcal{C})$ and reduce the sum of products of the $h^{(j)}$ functions over the paired variables to some functions $P^{(i)}$, see (5.11), that are also bounded in $\ell^2 L^2$. (3) Estimate the sum in
unpaired variables using Lemma 4.2 (in Section 5.1.2 we will skip step (2) and estimate the whole sum including paired and unpaired variables using Lemma 4.1). (4) Apply Cauchy-Schwartz to handle all the factors in $L^2L^2$, and colorblue then reduce to the $E_j$ type quantities in Corollary 4.3.

(5) When necessary, apply a $T^*T$ argument and repeat the previous steps for the resulting kernel.

As the proof will be notation heavy, the reader may do a first reading making the following simplifications without missing the core parts of the proof: (1) omit integration in any $\lambda_j$ and pretend $\lambda_j = 0$ (so $v^{(j)}$ is a function of $k_j$ only and $h^{(j)}$ is a function of $k_j$ and $k_j^*$ only); (2) when identifying pairings, restrict to only simple pairings where (say) $k_j^* = k_j^*$ and does not equal any other $k_j^*$. These will make formulas like (5.12) simpler and the proofs more transparent.

Throughout the proof we will fix the sets $U = \{1, 2, \cdots, n\}$ and $V = \{1, 2, \cdots, n-1\}$. We will (in this section only) introduce a shorthand notation for vectors: for a finite set $X$, define $k_{[X]}$ to be the vector $(k_j : j \in X)$; similarly define $\lambda_{[X]}$, $k_{[X]}^*$, etc., and define $d\lambda_{[X]} = \prod_{j \in X} d\lambda_j$.

5.1.1. A simple bound. We first prove (5.5). By definition we expand

$$
X = \sum_{\ell_1 k_1 + \cdots + \ell_n k_n = k+d} \int d\lambda d\lambda_{[U]} \times \eta\left(\lambda, \lambda - |k|^2 - \sum_{j=1}^n \ell_j (\lambda_j - |k_j|^2) - \alpha\right) v_k(\lambda) \prod_{j=1}^{n_1} g_{k_j}^{(j)} \frac{h^{(j)}_{k_j k_j^*}(\lambda_j)}{k_j^*} \prod_{j=n_1+1}^n [v_{k_j}^{(j)}(\lambda_j)]^{t_j},
$$

(5.10)

recall that $k_{[U]}$ means $(k_1, \cdots, k_n)$, etc. The sum in $k_{[U]}^*$ is restricted to $\frac{N_j}{2} < \langle k_j^* \rangle \leq N_j$, and $h^{(j)}_{k_j k_j^*}(\lambda_j, \omega)$ is defined as in (3.25) for $j \in C$, and is defined to be $1_{k_j = k_j^*} \tilde{\chi}(\lambda_j)$ for $j \in G$.

Consider now the sum in $k_{[U]}^*$. By identifying all pairings among them (recall the definition of pairings in Definition 3.19), we may assume there are $p$ sets $Y_i (1 \leq i \leq p, 2p \leq n_1)$ and a set $Z$ that partitions $\{1, \cdots, n_1\}$, such that: (i) each $Y_i$ contains a pairing, (ii) the $k_j^*$ takes a single value for $j$ in each $Y_i$, (iii) this value is different for different $Y_i$ and is different from $k_j^*$ for $j \in Z$, and (iv) there is no pairing in $\{k_j^* : j \in Z\}$. Then we manipulate this sum and rewrite it as a combination of two types of sums, namely (1) where we only require\(^1\) that $k_j^*$ takes a single value for $j$ in each $Y_i$ and that there is no pairing in $\{k_j^* : j \in Z\}$, and (2) where there are more pairings in addition to case (1), namely when the value for $Y_i$ equals the value for some other $Y_i'$ or some $k_j^*$ for $j \in Z$. Since there are strictly more pairings in case (2) than in the sum we started with, we may repeat this process and eventually reduce to sums of type (1) only. The purpose of this manipulation is to ensure that the sum in $k_j^*$ $(j \in Y_i)$ gives exactly

$$
P^{(i)}_{k_{[Y_i]}^*}(\lambda_{[Y_i]}) = \sum_{k^*} \prod_{j \in Y_i} h^{(j)}_{k_j k_j^*}(\lambda_j)^{\pm} \langle k_j^* \rangle^{-q_3} (g_{k_j^*})^{q_1} (\overline{g_{k_j^*}})^{q_2},
$$

(5.11)

where $q_1 + q_2 = q_3 = |Y_i|$.

Note that $N_j$ for $j \in Y_i$ are all comparable. Without loss of generality we may assume $\{2i - 1, 2i\} \subset Y_i$ and $\ell_{2i-1} = -\ell_{2i}$. As $(k_{2i-1}, k_{2i})$ is not a pairing, $2i - 1$ and $2i$ cannot both belong to $G$. Now we may assume $|k_{2i-1} - k_j^*| \lessapprox L_{2i-1} (N_s)^{-C_k^{-1}}$ and similarly for $k_{2i}$, since otherwise we gain a power $(N_s)^{-200n^2}$ due to the last bound in (3.26), which cancels any summation in any $(k_j, k_j^*)$ and the estimate will then follow immediately.

---

\(^1\)That is, we relax the requirement (iii) above, keeping only requirements (ii) and (iv).
Let \( R_i = \max(L_{2i-1}, L_{2i}) \), say \( R_i = L_{2i-1} \), then we have \( |k_{2i-1} - k_{2i}| \lesssim R_i (N_*)^{C_k^{-1}} \) for \( 1 \leq i \leq p \). For (5.11) using the first two bounds in (3.26), we have that
\[
\| \langle \lambda_j \rangle^{\frac{b}{2}} \lambda_j \|_{L^2_{\lambda_j} [Y_i]}^2 \lesssim \prod_{j \in Y_i} N_j^{-2} \| \langle \lambda_j \rangle^{\frac{b}{2}} \lambda_j \|_{L^2_{\lambda_j} [Y_i]}^2 \sum_{k^* \neq j \in Y_i} \| \langle \lambda_j \rangle^{\frac{b}{2}} \lambda_j \|_{L^2_{\lambda_j} [Y_i]}^2 \]
\[
\lesssim \| \langle \lambda_j \rangle^{\frac{b}{2}} \lambda_j \|_{L^2_{\lambda_j} [Y_i]}^2 \| \langle \lambda_j \rangle^{\frac{b}{2}} \lambda_j \|_{L^2_{\lambda_j} [Y_i]}^2 \prod_{j \in Y_i} L_j^{-2\delta_0}.
\]

Now we have reduced the expression for \( X \) to
\[
X = \sum_{(k,k[Z]):} \sum_{k^*[Z]} \int d\lambda_2 \cdot \eta \left( \lambda, \sum_{j=1}^n \lambda_j - |k_j|^2 - \alpha \right)
\times \prod_{i=1}^p \langle \lambda_{[U_i]} \rangle \prod_{j \in Z} \langle \lambda_j \rangle \prod_{j=1}^n \langle \lambda_j \rangle^{t_j}.
\]

Compared to (5.10) it is important that there is no pairing in \( k_{*[Z]} \). For simplicity of notations we will write
\[
X = \sum_{(1)} \int \mathcal{F} \cdot \mathcal{G},
\]
where the symbol \( \sum_{(1)} \int \) represents the sum in \( k \) and \( k_{[U \setminus Z]} \) and integration in \( \lambda \) and \( \lambda_{[U \setminus Z]} \), and the factor \( \mathcal{F} \) is
\[
\mathcal{F} := \prod_{i=1}^p \langle \lambda_{[U_i]} \rangle \prod_{j \in Z} \langle \lambda_j \rangle \prod_{j=1}^n \langle \lambda_j \rangle^{t_j},
\]
and the multilinear Gaussian \( \mathcal{G} \) given by
\[
\mathcal{G} := \sum_{k_{[Z]} \cdot k^*_{[Z]}} \int d\lambda_{[Z]} \cdot \prod_{j \in Z} \langle \lambda_j \rangle^{t_j} \cdot \prod_{j=1}^n \langle \lambda_j \rangle^{t_j} \cdot \mathcal{A},
\]
with coefficient \( \mathcal{A} \) of form
\[
\mathcal{A} := 1_{\{ j \in Z: t_j \neq 0 \}} \cdot \eta \left( \lambda, \sum_{j \in Z} t_j (\lambda_j - |k_j|^2) \right) \prod_{j \in Z} \langle \lambda_j \rangle^{-b},
\]
where
\[
d_0 := k + d - \sum_{j \notin Z} t_j k_j, \quad \alpha_0 := \lambda - |k|^2 - \alpha - \sum_{j \notin Z} \lambda_j - |k_j|^2 \in \mathbb{R}.
\]
\((N_\ast)^C\delta^{-7}\) choices. In the end, after removing a set of probability \(\leq C_\theta e^{-(r-1)N_\ast}\) we can apply Lemma 4.2 for all choices of \((k, \lambda, k_{\mathcal{U}\setminus \mathcal{Z}}, \lambda_{\mathcal{U}\setminus \mathcal{Z}})\), and use the first bound in (3.26) to get that

\[
|\mathcal{E}|^2 \leq \prod_{j \in \mathcal{Z}} N_j^{-2} \prod_{j \in \mathcal{Z}} L_j^{-2b_0} \cdot \|\mathcal{A}\|^2 \leq \prod_{j \in \mathcal{Z}} N_j^{-2} \prod_{j \in \mathcal{Z}} L_j^{-2b_0} \sum_{k_{\mathcal{Z}]}: \sum_{j \in \mathcal{Z}} \tau_j k_j = d_0} \left\langle \alpha_0 + \sum_{j \in \mathcal{Z}} \tau_j |k_j|^2 \right\rangle^{-\alpha_0}. \tag{5.19}
\]

Finally applying Cauchy-Schwarz in the variables \((k, \lambda, k_{\mathcal{U}\setminus \mathcal{Z}}, \lambda_{\mathcal{U}\setminus \mathcal{Z}})\), we deduce that

\[
|\mathcal{X}|^2 \lesssim \left( \sum \int \langle \lambda \rangle^{2b} \prod_{j \in \mathcal{U}\setminus \mathcal{Z}} \langle \lambda_j \rangle^{2b} \cdot |\mathfrak{S}|^2 \right) \left( \sum \int \langle \lambda \rangle^{-2b} \prod_{j \in \mathcal{U}\setminus \mathcal{Z}} \langle \lambda_j \rangle^{-2b} \cdot |\mathcal{E}|^2 \right),
\]

where the first parenthesis (together with some factors from the second parenthesis) gives the product of all factors in (5.5) except \(\mathcal{E}_1\), by using (3.27), (5.4) and (5.12); the second parenthesis, after applying (5.19), integrating in \((\lambda, \lambda_{\mathcal{U}\setminus \mathcal{Z}})\) and plugging in (5.18), reduces to

\[
\sum_{(k, k_{\mathcal{U}}): \iota_1 k_1 + \cdots + \iota_n k_n = k + d} \langle \Sigma - \alpha \rangle^{-\alpha_0} \lesssim \mathcal{E}_1,
\]

where \(\Sigma = \iota_1 |k_1|^2 + \cdots + \iota_n |k_n|^2 - |k|^2\) as in Corollary 4.5. This proves (5.5).

5.1.2. A general \(T^* T\) argument. Now we prove (5.7), starting from (5.13). Note that due to (3.27) and (5.4), the bound for \(\mathcal{X}\) would follow from the \(\ell_2^2 L_{\lambda_{\mathcal{Z}]}^2 \to \ell_2^2 L_{\lambda_{\mathcal{Z}]}^2\) bound of the linear operator \(T\) with kernel

\[
T_{\mathcal{K}_n}(\lambda, \lambda_{\mathcal{Z}}) = \sum_{k_{\mathcal{Z}]}: \sum_{j \in \mathcal{Z}} \iota_j k_j = k + d} \sum_{k_{\mathcal{Z}]}: \sum_{j \in \mathcal{Z}} \iota_j k_j = k + d} \int d\lambda_{\mathcal{V}]} \cdot \eta \left( \lambda, \lambda - |k|^2 - \sum_{j = 1}^{n} \iota_j \langle \lambda_j - |k_j|^2 \rangle - \alpha \right)
\times \left( \prod_{i = 1}^{p} P^{(i)}_{k_{\mathcal{V}}} (\lambda_{\mathcal{Y}_i]) \prod_{j \in \mathcal{Z}} \frac{g^{(i)}_{j}}{[k_j]^{2}} h^{(j)}_{k_j k_j^*} (\lambda_j) \right)^{n-1} \prod_{j = n+1}^{n} \left[ v_{k_j}^{(j)} (\lambda_j) \right]^{\tau_j} \langle \lambda \rangle^{-b} \langle \lambda_{\mathcal{Z}} \rangle^{-b}. \tag{5.20}
\]

We then calculate the kernel of \(\mathcal{O} = T^* T\), which (similar to (5.14)) can be written as

\[
\mathcal{O}_{\mathcal{K}_n, k_0} (\lambda, \lambda_{\mathcal{Z}}) = \langle \lambda \rangle^{-b} \langle \lambda_{\mathcal{Z}} \rangle^{-b} \sum_{(2)} \int \mathfrak{F} \cdot \mathfrak{G}, \tag{5.21}
\]

where the symbol \(\sum_{(2)} \int\) represents the sum in \((k_{\mathcal{U}\setminus \mathcal{Z}}, k_{\mathcal{U}\setminus \mathcal{Z}})\) and integration in \((\lambda_{\mathcal{U}\setminus \mathcal{Z}}, \lambda_{\mathcal{U}\setminus \mathcal{Z}})\), the factor \(\mathfrak{F}\) is independent of \((k_0, k_0^*, \lambda, \lambda_0^*)\), and is now defined as

\[
\mathfrak{F} := \prod_{i = 1}^{p} P^{(i)}_{k_{\mathcal{V}}} (\lambda_{\mathcal{Y}_i]) P^{(i)}_{k_{\mathcal{V}}} (\lambda_{\mathcal{Y}_i]) \prod_{j = n+1}^{n} \left[ v_{k_j}^{(j)} (\lambda_j) \right]^{\tau_j} \left[ v_{k_j^*}^{(j)} (\lambda_j) \right]^{\tau_j}, \tag{5.22}
\]

and the multilinear Gaussian \(\mathfrak{G}\) is now given by

\[
\mathfrak{G} := \sum_{(k, k_{\mathcal{Z}}, k_{\mathcal{Z}}^*)} \prod_{j \in \mathcal{Z}} g^{(-\iota_j)}_{k_j} g^{\iota_j}_{k_j^*} \sum_{(k, k_{\mathcal{Z}}, k_{\mathcal{Z}}^*)} \mathcal{C}, \tag{5.23}
\]
with coefficient $C$ of form

$$
C := \int d\lambda d\lambda[Z] |d\lambda[Z]| \cdot \langle \lambda \rangle^{-2b} \eta \left( \lambda, \lambda - |k|^2 - \sum_{j \in Z} \epsilon_j (\lambda_j - |k_j|^2) - \alpha_0 \right) \times \eta \left( \lambda, \lambda - |k|^2 - \sum_{j \in Z} \epsilon_j (\lambda_j' - |k_j'|^2) - \alpha'_0 \right) \prod_{j \in Z} \frac{1}{\langle k_j \rangle} h^{(j)}_{k_j k_j'} (\lambda_j) \pm \frac{1}{\langle k'_j \rangle} h^{(j)}_{k'_j k'_j} (\lambda'_j) \pm, \tag{5.24}
$$

where we now have

$$
d_0 := d - \sum_{j \in Z} \epsilon_j k_j, \quad d'_0 := d - \sum_{j \in Z} \epsilon_j k'_j; \quad \alpha_0 := \alpha + \sum_{j \in Z} \epsilon_j (\lambda_j - |k_j|^2), \quad \alpha'_0 := \alpha + \sum_{j \in Z} \epsilon_j (\lambda'_j - |k'_j|^2). \tag{5.25}
$$

As in Section 5.1.1 we may assume $|k_{2i-1} - k_{2i}| \lesssim R_i (N_* C^{\kappa - 1})$ for $1 \leq i \leq p$ and similarly for $k'_{2i-1}$ and $k'_{2i}$. The goal now is to estimate $|\mathfrak{G}|$ in (5.23). Let $L_+ = \max \{L_j : j \in Z\}$, in view of the power $(L_+)^{40n^2}$ on the right hand side of (5.7), we may assume $N_j \gg (L_+)^2$ for each $j \in Z$, otherwise we simply sum over $(k_j, k_j^*)$ and $(k'_j, k'_j^*)$ and get rid of these variables. By the same arguments as in Section 5.1.1 we may reduce to $\lesssim (N_*)^{C\delta - 7}$ choices for $(k_j[Z \setminus W], k'_j[Z \setminus W], \lambda_j[Z \setminus W], \lambda'_j[Z \setminus W])$; for each single choice, as $C$ is $B^+_{\leq L_+}$ measurable and there is no pairing in $k_j^*$ or $k'_j$, we may apply Lemma 4.1 and get

$$
|\mathfrak{G}|^2 \lesssim \sum_{(k_j[Z \setminus W], k'_j[Z \setminus W])} \left( \sum_{k_j^* = k'_j^* (1 \leq l \leq s)} \sum_{(k, k_j, k'_j) : \sum_{j \in Z} \epsilon_j k_j = k + d_0} \sum_{\sum_{j \in Z} \epsilon_j k'_j = k + d'_0} |C| \right)^2, \tag{5.26}
$$

where $W = \{a_1, \ldots, a_s\}$ and $W' = \{b_1, \ldots, b_s\}$ are subsets of $Z$, and we have $N_{a_l} \sim N_{b_l}$ and $\epsilon_{a_l} = \epsilon_{b_l}$ for $1 \leq l \leq s$. As before we may assume $|k_j - k_j^*| \lesssim L_+(N_*)^{C\kappa - 1}$ and similarly for $k'_j - k'_j^*$, and due to the $(L_+)^{40n^2}$ factor we may then fix the values of $k_j - k_j^* = \epsilon_j$ and $k'_j - k'_j^* = \epsilon'_j$. Therefore $k'_j - k_j = \epsilon'_j - \epsilon_j := f_i$ is also fixed.

Now the outer sum in (5.26) can be viewed as a sum over $k_j[Z \setminus W]$ and $k'_j[Z \setminus W']$, and the inner sums can be viewed as a sum over $(k, k_j, k'_j : 1 \leq l \leq s)$ that satisfies $k'_j - k_j = f_i$. When all these $k$-variables are fixed, we have

$$
\sup_{k_j, k_j^*} \|\langle \lambda_j \rangle^b h^{(j)}_{k_j k_j^*} (\lambda_j)\|_{L^2_{\lambda_j}} \lesssim 1, \quad \sup_{k'_j, k'_j^*} \|\langle \lambda'_j \rangle^b h^{(j)}_{k'_j k'_j^*} (\lambda'_j)\|_{L^2_{\lambda'_j}} \lesssim 1,
$$

due to the first bound in (5.26). Using the algebra property of the norm $\|\langle \lambda \rangle^b h(\lambda)\|_{L^2}$ under convolution, we have

$$
|C| \lesssim \prod_{j \in Z} N_{j}^{-2} \int d\lambda \cdot \langle \lambda \rangle^{-2b} \left( \lambda - |k|^2 + \sum_{j \in Z} \epsilon_j |k_j|^2 - \alpha_0 \right)^{-b} \left( \lambda - |k|^2 + \sum_{j \in Z} \epsilon_j |k'_j|^2 - \alpha'_0 \right)^{-b} \lesssim \prod_{j \in Z} N_{j}^{-2} \left( |k|^2 - \sum_{j \in Z} \epsilon_j |k_j|^2 + \alpha_0 \right)^{-b} \left( |k|^2 - \sum_{j \in Z} \epsilon_j |k'_j|^2 + \alpha'_0 \right)^{-b}. \tag{5.27}
$$
With (5.27) we can now bound \( \mathcal{G} \) by

\[
|\mathcal{G}|^2 \lesssim (L_+)^{4n^2} \prod_{j \in Z} N_j^{-4} \left( \sum_{(k,l) \in \mathbb{C}_j} \left( \sum_{j \in \mathbb{C}'} \left( \sum_{l=1}^{s} \left( \sum_{l=1}^{s} t_{a_l} |k_{a_l}|^2 + \alpha_0 \right)^b \right)^2 \right) \right),
\]

and multiplying out the square we get

\[
|\mathcal{G}|^2 \lesssim (L_+)^{4n^2} \prod_{j \in Z} N_j^{-4} \left( \sum_{(k,l) \in \mathbb{C}_j} \left( \sum_{j \in \mathbb{C}'} \left( \sum_{l=1}^{s} t_{a_l} |k_{a_l}|^2 + \alpha_0 \right)^b \right)^2 \right),
\]

where

\[
\mathcal{Y} = |k|^2 - \sum_{j \in \mathbb{C}'} t_j |j|^2 - \sum_{l=1}^{s} t_{a_l} |k_{a_l}|^2 + \alpha_0, \quad \hat{\mathcal{Y}} = |\hat{k}|^2 - \sum_{j \in \mathbb{C}'} t_j |j|^2 - \sum_{l=1}^{s} t_{a_l} |k_{a_l}|^2 + \alpha_0,
\]

\[
\mathcal{Y}' = |k'|^2 - \sum_{j \in \mathbb{C}'_l} t_j |j|^2 - \sum_{l=1}^{s} t_{b_l} |k_{b_l}|^2 + \alpha_0, \quad \hat{\mathcal{Y}}' = |\hat{k}'|^2 - \sum_{j \in \mathbb{C}'_l} t_j |j|^2 - \sum_{l=1}^{s} t_{b_l} |k_{b_l}|^2 + \alpha_0,
\]

with \( t_{a_l} = t_{b_l}, \alpha_0 \) and \( \alpha_0' \) as in (5.25), and the variables in the summation verify the following linear equations:

\[
\sum_{j \in \mathbb{C}'} t_j k_j + \sum_{l=1}^{s} t_{a_l} k_{a_l} - k = \sum_{j \in \mathbb{C}'} t_j k_j + \sum_{l=1}^{s} t_{a_l} k_{a_l} - \hat{k} = d_0,
\]

\[
\sum_{j \in \mathbb{C}'_l} t_j k'_j + \sum_{l=1}^{s} t_{b_l} k_{b_l} - k = \sum_{j \in \mathbb{C}'_l} t_j k'_j + \sum_{l=1}^{s} t_{b_l} k_{b_l} - \hat{k} = d'_0,
\]

with \( d_0 \) and \( d'_0 \) as in (5.25), as well as \( k'_b - k_{a_l} = k_{b_l} - k_{a_l} = f_l \).

By Cauchy-Schwartz, we may replace the summand on the right hand side of (5.29) by \( \langle \mathcal{Y} \rangle^{-2b} \cdot \langle \hat{\mathcal{Y}} \rangle^{-2b} \) (or by \( \langle \hat{\mathcal{Y}} \rangle^{-2b} \cdot \langle \mathcal{Y} \rangle^{-2b} \), which is treated similarly by symmetry). Now going back to (5.21) and applying Cauchy-Schwartz in the variables \((k,l) \in \mathbb{C}_j, k'_l \in \mathbb{C}'_j, \lambda_n \in \mathbb{C}_j, \lambda'_n \in \mathbb{C}'_j \), we get

\[
|\mathcal{X}|^4 \lesssim N_n^{-4(1-\gamma)} \left( \sum_{k,l} \int d\lambda_n d\lambda'_n |\mathcal{O}_{k,n} (\lambda_n, \lambda'_n)|^2 \right)^2 \lesssim N_n^{-4(1-\gamma)} \left( \prod_{j \in \mathbb{C}} \langle \lambda_j \rangle^{2b} \langle \lambda'_j \rangle^{-2b} \cdot |\mathcal{G}|^2 \right)^2.
\]

The first parenthesis (together with some factors from the second parenthesis) give the product of all factors in (5.27) except \( \epsilon_1 \epsilon_1^* \), by using (5.27) and (5.12). The second parenthesis, after applying (5.29) with the summand \( \langle \mathcal{Y} \rangle^{-b} \cdot \langle \hat{\mathcal{Y}} \rangle^{-b} \cdot \langle \mathcal{Y} \rangle^{-b} \cdot \langle \hat{\mathcal{Y}} \rangle^{-b} \) replaced by \( \langle \mathcal{Y} \rangle^{-2b} \cdot \langle \hat{\mathcal{Y}} \rangle^{-2b} \), integrating in \( \lambda_{[U \setminus Z]} \) and \( \lambda'_{[U \setminus Z]} \), and plugging in (5.25), reduces to

\[
\sum_{(k,l) \in \mathbb{C}_j, k'_l} \sum_{(k,l) \in \mathbb{C}_j, k'_l} \langle \mathcal{X} \rangle^{-2b} \langle \hat{\mathcal{X}} \rangle^{-2b},
\]

as in (5.25).
where $\Sigma$ and $\tilde{\Sigma}'$ are respectively

$$
\Sigma = \sum_{j \not\in W} t_j |k|^2 + \sum_{l=1}^{s} t_{a_l} |k_{a_l}|^2 - |k|^2, \quad \tilde{\Sigma}' = \sum_{j \not\in W'} t_j |k'_j|^2 + \sum_{l=1}^{s} t_{b_l} |k'_b|^2 - |\tilde{k}'|^2,
$$

and the variables in the summation satisfy

$$
\sum_{j \not\in W} t_j k_j + \sum_{l=1}^{s} t_{a_l} k_{a_l} - k = \sum_{j \not\in W} t_j k_j + \sum_{l=1}^{s} t_{a_l} k_{a_l} - \tilde{k} = d,
$$

$$
\sum_{j \not\in W'} t_j k'_j + \sum_{l=1}^{s} t_{b_l} k'_b - k = \sum_{j \not\in W'} t_j k'_j + \sum_{l=1}^{s} t_{b_l} k'_b - \tilde{k} = d,
$$

as well as $k'_b - k_{a_l} = k'_b - \tilde{k}_{a_l} = f_l$. Now, when $k_{[U\setminus W]}$ and $(k, k_{a_l} : 1 \leq l \leq s)$ are fixed, the sum of $\langle \tilde{\Sigma}' - \alpha \rangle^{-2b}$ over $k_{[U\setminus W]}$ and $(k, k'_b : 1 \leq l \leq s)$ can be bounded by $\mathcal{E}_1^+$ with $A = U\setminus W'$ in (5.23) due to the equations (5.33); on the other hand the sum of $\langle \Sigma - \alpha \rangle^{-2b}$ over $k_{[U\setminus W]}$ and $(k, k_{a_l} : 1 \leq l \leq s)$ can be bounded by $\mathcal{E}_1$. This bounds the sum $\langle \tilde{\Sigma}' \rangle$ by $\mathcal{E}_1\mathcal{E}_1^+$ and proves (5.7).

5.1.3. A special $\mathcal{T}^* \mathcal{T}$ argument. Assume now $N_n = N^{(1)}$ and $N_s \gtrsim N^{(2)}$. Again we only need to study the operator $\mathcal{T}$ given by the kernel (5.20); note that $\mathcal{T}_{k|n}(\lambda, \lambda_n)$ is supported in the set $\{(k, k_n) : |k - i_n k_n + d| \lesssim N^{(2)}\}$, by the standard orthogonality argument it suffices to prove the same operator bound for $\tilde{\mathcal{T}}$ which is $\mathcal{T}$ restricted to the set $\{k : |k - f| \leq N^{(2)}\}$, uniformly in $f \in \mathbb{Z}^2$. Below we will fix an $f$ and denote $\tilde{\mathcal{T}}$ still by $\mathcal{T}$, so that in any summations below we may assume $|k - f| \lesssim N^{(2)}$ and $|k_n - i_n (f + d)| \lesssim N^{(2)}$ (same for $k'_n$). At this point the parameter $N^{(1)}$ or $N_s$ no longer explicitly appears in the estimate, so the set of $\omega$ we remove will be independent of it. Also we can prove (5.7), with $N_n$ replaced by $N^{(2)}$ in both $\mathcal{E}_1$ and $\mathcal{E}_1^\infty$, by essentially repeating the proof in Section 5.1.2 above (and making the bound uniform in $f$ in exactly the same way as below); it remains to prove (5.7).

We start with (5.21) and now look for further pairings in $(k|Z_i, k'|Z_i)$ in the expression $\mathcal{G}$ given by (5.23). By repeating the same reduction step in Section 5.1.1 we can find two partitions $(X_1, \cdots, X_q, W)$ and $(X'_1, \cdots, X'_q, W')$ of the set $Z$, where $2p + q \leq n_1$, such that $N_j$ are all comparable for $j$ in each $X_i \cup X'_i$, and further reduce (5.21) to a sum

$$
\mathcal{O}_{k|n, k'|n}(\lambda_n, \lambda'_n) = \langle \lambda_n \rangle^{-b} \langle \lambda'_n \rangle^{-b} \sum_{(3)} \int \mathfrak{S}^+ \cdot \mathfrak{G}^+,
$$

where the symbol $\sum_{(3)} \int$ represents the sum in $k_{[V \setminus W]}$ and $k'_{[V' \setminus W]}$ and integration in $\lambda_{[V \setminus W]}$ and $\lambda'_{[V' \setminus W]}$, the factor $\mathfrak{S}^+$ is independent of $(k, k'_n, \lambda_n, \lambda'_n)$,

$$
\mathfrak{S}^+ = \prod_{i=1}^{p} P_{k|V_i}(\lambda|V_i) P_{k'|V_i, \lambda}(\lambda|V_i) \prod_{j=n+1}^{n-1} [\nu_{k_j}^{(j)}(\lambda_j)]^{\nu_j} [\nu_{k'_j}^{(j)}(\lambda'_j)]^{\nu_j} \prod_{l=1}^{q} Q_{k|X_l, k'|X'_l}(\lambda|X_l, \lambda'|X'_l),
$$

$$
Q_{k|X_l, k'|X'_l}(\lambda|X_l, \lambda'|X'_l) := \sum_{k \not\in X_i} \sum_{j \in X_i} h_{k_j k^*}^{(j)}(\lambda_j) \sum_{j \in X'_i} h_{k'_j k^*}^{(j)}(\lambda'_j) \langle k^* \rangle = q_3 (g_k k^*) q_4 (g_k k^*)^{-2q},
$$

where $q_1 + q_2 = q_3 = |X_l| + |X'_l|$, and the $P$ factors are defined in (5.11). We may also fix $a_l \in X_l$ and $b_l \in X'_l$ such that $t_{a_l} = t_{b_l}$; without loss of generality assume $b_l = 2p + l$ for $1 \leq l \leq q$. We can

---

1This $W$ and $W'$ are different from the $W$ and $W'$ of Section 5.1.2.
bound (5.36) just like we bound (5.11) in (5.12), except that now it is possible to have $X_t \cup X'_t \subset \mathcal{G}$. Let $R_{2p+l} = \max \{ L_j : j \in X_t \cup X'_t \} \geq L_{2p+l}$, then the same argument as in (5.12) gives

$$
\|Q_{k,|X|'}^{(l)}(\lambda|X|', \lambda'|X|') \prod_{j \in X_t} \langle \lambda_j \rangle^b \prod_{j \in X'_t} \langle \lambda'_j \rangle^b \sum_{k|X|', \lambda|X|'}^2 \| \leq \prod_{j \in X_t} \sum_{j \in X_t} N_{j-2} \prod_{j \in X'_t} N_{j-2} \cdot N_{2p+l} R_{2p+l}^{-2}.
$$

(5.37)

Finally, the multilinear Gaussian $\mathcal{G}^+$ is given by

$$
\mathcal{G}^+ = \sum_{(k_1|W|, k_1^*|W|)} \sum_{(k_1^*|W|', k_1'|W|')} \int d\lambda \cdot \eta \prod_{j \in W} \frac{g_{\lambda}}{k_j^*} (\lambda_j^b h_{k_j^* j} (\lambda_j) \pm \prod_{j \in W'} \frac{g_{\lambda}}{k_j} (\lambda_j^b h_{k_j j} (\lambda_j) \pm \mathfrak{A}^+,
$$

(5.38)

where there is no pairing among $(k_1^*|W|, k_1'|W|')$, and coefficient $\mathfrak{A}^+$ of form

$$
\mathfrak{A}^+ = \sum_{k_1|W|, \ldots, k_n|W| = k + d} \int d\lambda \cdot \eta \left( \lambda, \lambda - |k|^2 - \sum_{j=1}^n \lambda_j \right) \prod_{j \in W} \langle \lambda_j \rangle^b \prod_{j \in W'} \langle \lambda'_j \rangle^b,
$$

(5.39)

where the sum is over a single variable $k$. As before we may also assume $|k_j - k^*| \leq R_{2p+l} (N_s)^{-1}$ for $j \in X_t$ in (5.36) and similarly for $k_j'$ and $j \in X'_t$, so in particular $|k_{a_j} - k'_{2p+l}| \leq R_{2p+l} (N_s)^{-1}$.

The goal now is to estimate $\mathcal{G}^+$. As before, we need to reduce to $(N_s)^{C_8^{-7}}$ choices for $(k_1|W|, \lambda_1|W|)$ and $(k_1'|W|, \lambda_1'|W|)$, and $f$. By the same argument as in Sections 5.1.1 and 5.1.2 we may assume $|\lambda| \lesssim (N_s)^{C_8^{-7}}$ and $|\lambda_j| \lesssim (N_s)^{C_8^{-7}}$ for $j \not\in W$ (similarly for $\lambda'_j$) and get rid of these parameters; in the same way we may also fix $k_1|W'|$ and $k_1'|W'|$, as well as $f + d - \epsilon_n k_n$ and $f + d - \epsilon_n k'_n$. Letting $k = f + g$, we can rewrite

$$
\mathfrak{A}^+ = \sum_{|g| \leq N(2)} \sum_{\sum_{j \in W} \lambda_j = g + d_0} \sum_{\sum_{j \in W'} \lambda'_j = g + d'_0} \int d\lambda \cdot \eta \left( \lambda, \lambda - 2f \cdot g - |g|^2 - \sum_{j \in W} \lambda_j \right) \prod_{j \in W} \langle \lambda_j \rangle^b \prod_{j \in W'} \langle \lambda'_j \rangle^b,
$$

(5.40)

where $\gamma, \gamma' \in [0, 1]$ are fixed, $d_0, d'_0 \in \mathbb{Z}^2$, and $\beta(f), \beta'(f)$ are fixed integer-valued functions of $f$. One may assume $|d_0|, |d'_0| \leq N(2)$ since otherwise $\mathfrak{A}^+ \equiv 0$, then we may fix them and see that $f$ enters the whole expression only through the function $-2f \cdot g + \beta(f)$; moreover we may restrict $g$ to the set where $| -2f \cdot g + \beta(f) | \leq (N_s)^{C_8^-7}$ since otherwise either $\lambda$ or one $\lambda_j$ must be large and we close as before. The reduction to finitely many cases can then be done by invoking the following claim, which will be proved at the end of this section:

**Claim 5.2.** Let the function $F_{f, \beta}(g) = -2f \cdot g + \beta$, with the particular domain $\text{Dom}(F_{f, \beta}) = \{ |g| \leq N(2) : | -2f \cdot g + \beta | \leq (N_s)^{C_8^-7} \}$. Then when $f \in \mathbb{Z}^2$ and $\beta \in \mathbb{Z}$ varies, the function $F_{f, \beta}$ (together with its domain) has finitely many, and in fact $\lesssim (N_s)^{C_8^-7}$ possibilities.
From now on we may fix the value of $f$. By removing a set of probability $\leq C_0 e^{-(\tau-1)N_\ast}$, we can apply Lemma 4.2 and conclude that (recall that $a_0 = 2b - 10\delta_0$)

$$|\Theta^+|^2 \lesssim \prod_{j \in W} N_j^{-2} \prod_{j \in W'} N_j^{-2} \prod_{j \in W} \sum_{(k_{[W]}, k'_{[\mathcal{W}']})} \int d\lambda_{[W]} d\lambda'_{[\mathcal{W}']} \prod_{j \in W} (\lambda_j)^{-a_0} \prod_{j \in W'} (\lambda'_j)^{-a_0} \left[ \sum_{k_1 k_1' + \cdots + k_n k_n' = k + d} \int d\lambda \right] \times \langle \lambda \rangle^{-2b} \eta \left( \lambda, \lambda - |k|^2 - \sum_{j=1}^n t_j (\lambda_j - |k_j|^2) - \alpha \right) \eta \left( \lambda, \lambda - |k|^2 - \sum_{j=1}^n t_j (\lambda'_j - |k'_j|^2) - \alpha \right) \right]^2. \quad (5.41)$$

The integral over $\lambda$ gives a factor

$$\left\langle \Sigma - \sum_{j=1}^n t_j \lambda_j - \alpha \right\rangle^{-b} \left\langle \Sigma' - \sum_{j=1}^n t_j \lambda'_j - \alpha \right\rangle^{-b},$$

where $\Sigma$ and $\Sigma'$ are defined as

$$\Sigma = \sum_{j=1}^n t_j |k_j|^2 - |k|^2, \quad \Sigma' = \sum_{j=1}^n t_j |k'_j|^2 - |k|^2.$$

Since there is only one value of $k$ in the summation, we can reduce

$$|\Theta^+|^2 \lesssim \sum_k \sum_{(k_{[W]}, k'_{[\mathcal{W}']})} \int d\lambda_{[W]} d\lambda'_{[\mathcal{W}']} \prod_{j \in W} (\lambda_j)^{-a_0} \prod_{j \in W'} (\lambda'_j)^{-a_0} \times \prod_{j \in W} N_j^{-2} \prod_{j \in W'} N_j^{-2} \left\langle \Sigma - \sum_{j=1}^n t_j \lambda_j - \alpha \right\rangle^{-2b} \left\langle \Sigma' - \sum_{j=1}^n t_j \lambda'_j - \alpha \right\rangle^{-2b} \lesssim \sum_k \sum_{(k_{[W]}, k'_{[\mathcal{W}']})} \left\langle \Sigma - \sum_{j \notin W} t_j \lambda_j - \alpha \right\rangle^{-a_0} \left\langle \Sigma' - \sum_{j \notin W'} t_j \lambda'_j - \alpha \right\rangle^{-a_0}. \quad (5.42)$$

where in the summation over $k$ and $(k_{[W]}, k'_{[\mathcal{W}']})$ we assume that $\iota_1 k_1 + \cdots + \iota_n k_n = \iota_1 k'_1 + \cdots + \iota_n k'_n = k + d$. Returning to (5.31), by applying Cauchy-Schwartz in the variables $(k_{[\mathcal{W}]}, \lambda_{[\mathcal{W}]})$ and $(k'_{[\mathcal{W}']}, \lambda'_{[\mathcal{W}']})$ we conclude as before that

$$|\lambda|^4 \lesssim (N^{(1)})^{-4(1-\gamma)} \left( \sum_{(3)} \int \prod_{j \in V \setminus W} (\lambda_j)^{2b} \prod_{j \in V' \setminus W'} (\lambda'_j)^{2b} \cdot |\Theta|^2 \right) \times \left( \sum_{k_n k'_n} \int d\lambda_{[W]} d\lambda'_{[\mathcal{W}']} \cdot \langle \lambda_n \rangle^{-2b} \langle \lambda'_n \rangle^{-2b} \sum_{(3)} \int \prod_{j \in V \setminus W} (\lambda_j)^{-2b} \prod_{j \in V' \setminus W'} (\lambda'_j)^{-2b} \cdot |\Theta|^2 \right).$$

The first parenthesis (together with some factors from the second parenthesis) give the product of all factors in (5.3) except $\tilde{E}_1 \tilde{E}_1^+$, by using (3.27), (5.12) and (5.31). The second parenthesis, after applying (5.42) and integrating in $\lambda_{[\mathcal{W}]}$ and $\lambda'_{[\mathcal{W}']}$, reduces to

$$\sum_k \sum_{k_{[W]} k'_n} \sum_{k_1 k'_1 + \cdots + k_n k'_n = k + d} \sum_{k_{[U]}} \langle \Sigma - \alpha \rangle^{-a_0} \langle \Sigma' - \alpha \rangle^{-a_0}. \quad (5.43)$$

Now when $(k, k_{[\mathcal{W}]})$ are fixed, the sum of $\langle \Sigma' - \alpha \rangle^{-a_0}$ over $k'_{[\mathcal{W}]}$ can be bounded by $\tilde{E}_1^+$ with $A = \{1, \cdots, n\}$ in (4.23), due to the linear equation $\iota_1 k'_1 + \cdots + \iota_n k'_n = k + d$, the fact that $k'_{2p+1}$ belongs to a disc of radius $O(R_{2p+1}(N_\ast)^{\kappa^{-1}})$ once $k_{[U]}$ is fixed, and the fact that $|\iota_n k'_n - f - d| \lesssim N^{(2)}$. Moreover
the sum of $(\Sigma - \alpha)^{-a_0}$ over $(k, k|\nu|)$ can be bounded by $\tilde{E}_1$, due to the fact that $\|\nu_nk_n - f - d\| \lesssim N^{(2)}$. This then bounds \eqref{eq:5.43} by $\tilde{E}_1^{-1}$ and proves \eqref{eq:5.9}.

**Proof of Claim 5.2.** Let $D = \text{Dom}(F_{f,\beta})$. If $D$ contains three points $g_1, g_2, g_3$ that are not collinear, then we have $|f \cdot (g_1 - g_2)| \lesssim (N_*)^{\delta/7}$, $|f \cdot (g_1 - g_3)| \lesssim (N_*)^{\delta/7}$, and that $g_1 - g_2$ and $g_1 - g_3$ are linearly independent. This implies that $|f| \lesssim (N_*)^{2\delta/7}$ and hence $|\beta| \lesssim (N_*)^{3\delta/7}$ so the result is trivial. Now let us assume $D$ is contained in a line $\ell$; we may assume $\ell$ contains at least two points in the set $\{g : |g| \leq N^{(2)}\}$, otherwise $D$ is at most a singleton and the result is also trivial. Then the integer points in $\ell$ are linearly independent. This implies that $\sum_{j=1}^n |\lambda_j - |k_j||^2 - \sum_{j=1}^n \mu_j - \alpha) \leq 1$. Moreover when $|D| \geq 2$ we know that $|a| \lesssim (N_*)^{2\delta/7}$ and $|b| \lesssim (N_*)^{3\delta/7}$, so $F_{f,\beta}$ indeed has $\lesssim (N_*)^{3\delta/7}$ possibilities, as claimed.

**Remark 5.3.** For later use we will also consider the following variant of $\mathcal{X}$ (same for $\mathcal{Y}$):

$$
\mathcal{X}^+ = \sum_{(k, k_1, \ldots, k_n) \in I_{k1 + \ldots + kn = k+d}} \int d\lambda_1 d\lambda_2 \cdots d\lambda_n d\mu_1 \cdots d\mu_s \times \eta(\lambda, \lambda - |k| - \sum_{j=1}^n \ell_j (\lambda_j - |k_j|), - \sum_{j=1}^n \mu_j - \alpha) v_k(\lambda) \prod_{j=1}^n [v_{k_j}(\lambda_j)]^{\theta_j} \prod_{j=1}^s w_j(\mu_j), \quad (5.44)
$$

where each $w_j$ satisfies

$$
\|\mu_j^b w_j(\mu_j)\|_{L^2_{\mu_j}} \lesssim 1.
$$

Then $\mathcal{X}^+$ will satisfy exactly the same estimates as $\mathcal{X}$ (same for $\mathcal{Y}$). In fact one can introduce a “virtual” variable $l_j$ which take a single value and view $w_j(\mu_j)$ as a function of $l_j$ and $\mu_j$ which has type $D$, and repeat all the above proof with these new variables.

5.2. **Proof of Proposition 3.4.** With Corollary 4.5 and Proposition 5.1 now we can prove Proposition 3.4. Recall that we will abuse notation and write $(v_{k_j}^{(j)})(\lambda_j)$ instead of $(\tilde{v}^{(j)})(k_j)(\lambda_j)$. We will proceed in three steps; note that as before, in the proof below we will ignore any factor $\lesssim \tau^{-\theta_0(N_*)^{C\kappa^{-1}}}$.

**Step 1: reduction to estimating $\mathcal{X}$ and $\mathcal{Y}$.** First notice that, when the set of pairings among the variables involved in $N_n$ is fixed, the coefficient in $N_n$ will be a constant (see Remark 2.3). By Lemma 2.6 we may replace $I$ by $J$ in all estimates. Now by definition of the relevant norms, the kernel bound (2.30) and duality, we can reduce the desired estimates to the estimates of quantities of form $\mathcal{X}$ (for parts (1) and (2)) and $\mathcal{Y}$ (for part (3)) defined in (5.1) and (5.2), in fact with $d = \alpha = 0$, except that the functions $v$ and $y$ introduced by duality only satisfy weaker bounds

$$
\|\langle \lambda \rangle^{1-b_1} v_k(\lambda)\|_{L^2_{\lambda}} \lesssim 1, \quad \|\langle \lambda \rangle^{1-b_1} (X^\mu)^b y_{kk'}(\lambda, \lambda')\|_{L^2_{\lambda, \lambda}} \lesssim 1, \quad (5.45)
$$

instead of (5.4), and that there may be pairings in $\mathcal{X}$ and $\mathcal{Y}$ (but they will always be over-paired). Now if $|\lambda| \lesssim (N_*)^{C_0}$ where $C_0$ is a large constant depending only on $n$, then since $b_1 - b = 2b - 1 \sim \kappa^{-1} \sim \delta^4$, we can replace the power $\langle \lambda \rangle^{1-b_1}$ by $\langle \lambda \rangle^b$ in (5.45), at a price of losing a factor $(N_* )^{C\kappa^{-1}}$ which is acceptable. Now we will assume $|\lambda| \geq (N_*)^{C_0}$; below we will only consider part (1) of Proposition 3.4 since we have $N_0 \gtrsim N^{(1)}$ in part (2) and $N_0 \gtrsim N^{(2)}$ in part (3), and the proof will be similar and much easier.

Here the point is to use the weight $\langle \lambda \rangle^{1-b_1}$ in (5.45) to gain a power $\geq \delta^{1-b_1}$, after which we still can assume $\|v_k(\lambda)\|_{L^2_{\lambda}} \lesssim 1$. In view of this gain and the assumption $N_0 \gtrsim N^{(2)}$, we may fix the values of $k_j$ and/or $k_j'$ for each $1 \leq j \leq n - 1$. Moreover when $k_j$ and $k_j'$ fixed the
resulting function in $\lambda_j$ (let’s call them $w_j(\lambda_j)$) satisfies $\|\langle \lambda_j \rangle^{b} w_j(\lambda_j)\|_{L^2_{\lambda_j}} \lesssim 1$, which implies the corresponding $L^1_{\lambda_j}$ bound, so we may fix $\lambda_j (1 \leq j \leq n - 1)$ also. Finally as
\[
\int \langle \lambda_j \rangle^{b} ||v_{k_{n}}^{(n)}(\lambda_j)\|_{L^2_{k_{n}}} \ d\lambda_n = \|\langle \lambda_j \rangle^{b} v_{k_{n}}^{(n)}(\lambda_j)\|_{L^2_{k_{n}}} \lesssim (N(1))^{-2(1-\gamma)},
\]
we may also fix the value of $\lambda_n$, and reduce to
\[
\mathcal{X} = \sum_{k} \int v_{k}(\lambda) d\lambda \cdot \eta(\lambda, F(k)) G_{k - d'},
\]
where $\|G\|_{L^2} \sim (N(1))^{-1+\gamma}$ and $F(k)$ is a function of $k$ which, as well as $d'$, depends on the choice of the other fixed variables. By first integrating in $\lambda$ using Cauchy-Schwartz and (5.3), then summing in $k$ using Cauchy-Schwartz again, we deduce that
\[
|\mathcal{X}| \lesssim \|v_{k}(\lambda)\|_{L^2} \cdot \|G\|_{L^2} \lesssim (N(1))^{-1+\gamma},
\]
which suffices in view of the gain $(N_{4})^{-C_{0}/3}$.

**Step 2: the no-pairing case.** We have now reduced Proposition 3.4 to the estimates for the quantities $\mathcal{X}$ and $\mathcal{Y}$. If we assume there is no pairing, then we can apply Proposition 5.1 and then Corollary 4.5. Recall the new parameters such as $p, q$ and $R_{j}$ defined in Proposition 5.1 denote $L_{+} = \max(L_{2p+1}, \cdots, L_{n})$ and $N_{+} = \max(N_{n_{1}+1}, \cdots, N_{n})$. Also when we talk about an estimate in Proposition 4.4 we are actually talking about its counterpart in Corollary 4.3.

In part (1), by combining (5.7), (4.25) and (4.25) with the improvement factor (4.29), with $N_{n}$ replaced by $N_{(2)}$ in both places, we obtain that
\[
|\mathcal{X}| \lesssim (N(1))^{-1+\gamma} (N(2))^{C_{\gamma}(L_{+})^{40n^{3}} (N(2))^{-\frac{1}{4}},
\]
on the other hand by combining (5.9), (4.25) and (4.25) with the improvement factor (4.29), with the changes adapted to $\tilde{E}_{1}$ and $E_{1}^{\infty}$ indicated in Proposition 5.1 we obtain that
\[
|\mathcal{X}| \lesssim (N(1))^{-1+\gamma} (N(2))^{C_{\gamma}(L_{+})^{-\frac{1}{4}},
\]
noticing that $R_{2p+1} \geq L_{2p+1}$ for $1 \leq l \leq q$ and $N_{j} \geq L_{j}$ for $2p + q + 1 \leq j \leq n_{1}$. Interpolating the above two bounds then gives (5.28).

In parts (2) and (3), we have $N_{(1)} \sim N_{a}$ and $a \in G \cup C$, in particular the extra factor (4.29) is bounded by $(N_{+})^{-1}$; note that in case (3) one may have $a \in D$ but in this case the extra factor will be replaced by $(N(1))^{-1}$. By combining (5.5) and (4.25) we obtain (noticing that $N_{PR} \lesssim N_{(2)}$
\[
|\mathcal{X}| \lesssim (N(1))^{N(2)} \cdot \frac{1}{2}(N(2))^{\gamma_{0}(N_{+})^{C_{\gamma}(L_{+})^{-\delta_{0}},
\]
and by combining (5.7) and (4.25) together with the improvement factor (4.29) we obtain that
\[
|\mathcal{X}| \lesssim (N(1))^{N(2)} \cdot \frac{1}{2}(N(2))^{\gamma_{0}(N_{+})^{C_{\gamma}(L_{+})^{40n^{3}} (N_{+})^{-\frac{1}{4}},
\]
and interpolating the above two bounds gives (5.29); in the same way (5.30) follows from (5.5), (5.7), (4.24) and (4.29) with the improvement factor (4.29), and (5.33) follows from (5.6), (5.8), (4.27) and (4.27) with the suitable improvement factor.

Finally consider (5.31); here we will define $N' = \max(2)(N_{n_{1}+1}, \cdots, N_{n})$. Note that $\alpha = 0$, so by combining (5.5) and (4.28) we get
\[
|\mathcal{X}| \lesssim (N(1))^{-1+\gamma_{0}(N_{+})^{\gamma} (N')^{C_{\gamma}(L_{+})^{-\delta_{0}},
\]
on the other hand, by combining (5.7) and either (4.28) with the improvement factor (4.29) or (4.30), we get that either
\[
|\mathcal{X}| \lesssim (N(1))^{-1+\gamma_{0}(N_{+})^{\gamma} (N')^{C_{\gamma}(L_{+})^{40n^{3}} (N_{+})^{-\frac{1}{4}},
\]
or

\[ |\mathcal{X}| \lesssim (N^{(1)})^{-1+\gamma} (N_\pm)^{\gamma}(N')^{C\gamma} (L_\pm)^{40n^3 \min \left( \left( (N')^{-\frac{1}{4}}, (N^{(1)})^{-\frac{1}{4}} (N_\pm)^{-\frac{1}{2}} \right) \right)}. \]

Clearly interpolating (5.46) and (5.47) gives (3.31); suppose instead we have (5.46) and (5.48). Now if \( N_+ \geq (N^{(1)})^{\frac{2}{3}} \) and \( L_+ \geq (N_\pm)^{\frac{1}{40n^3}} \) then (5.46) implies (3.31); if \( N_+ \geq (N^{(1)})^{\frac{5}{8}} \) and \( L_+ \leq (N_\pm)^{\frac{1}{40n^3}} \) then (5.48) implies \( |\mathcal{X}| \lesssim (N^{(1)})^{-1+\gamma} \) which implies (3.31); if \( N_+ \leq (N^{(1)})^{\frac{5}{8}} \) then interpolating (5.46) and (5.48) implies

\[ |\mathcal{X}| \lesssim (N^{(1)})^{-1+\gamma} (N_\pm)^{\gamma} \lesssim (N^{(1)})^{-1+\gamma + \frac{2\gamma}{3}} \]

which implies (3.31). Note also that for general \( \alpha \), due to the factor \( \frac{\max((N^{(2)})^2,|\alpha|)}{(N^{(2)})^2} \) on the right hand side of (4.28), the above argument gives the bound

\[ |\mathcal{X}| \lesssim (N^{(1)})^{-1+\frac{\alpha}{2}} \max \left( 1, \frac{|\alpha|^2}{N^{(2)}} \right). \]

Step 3: the over-pairings. We will only consider \( \mathcal{X}', \mathcal{Y}' \) is similar and easier since there cannot be any pairing between \( \{k, k'\} \) and any \( k_j \) due to \( N^{(1)} \lesssim N_\pm^{1-\delta} \) and the restrictions \( |k|, |k'| \geq \frac{N_\pm}{2} \) in (3.33). Now due to simplicity, any pairing in \( \mathcal{X} \) must be an over-pairing; by collecting all these pairings we can find a partition \((A_1, \ldots, A_p, B)\) of \( \{1, \ldots, n\} \) such that \( |A_i| \geq 3 \) and \( k_j \) takes a single value for \( j \) in each \( A_i \); that this value is different for different \( 1 \leq i \leq p \), and there is no over-pairing among \( \{k_j : j \in B\} \). Then we can check that either there is no pairing among \( \{k, k_j : j \in B\} \), or there is a unique over-pairing \( k = k_{j_1} = k_{j_2} \) with \( j_1, j_2 \in B \) and \( (\tau_{j_1}, \tau_{j_2}) \neq (-, -) \). In the latter case denote \( \{j_1, j_2\} = A_0 \) and replace \( B \) by \( B \setminus A_0 \), so that there is no pairing among \( \{k, k_j : j \in B\} \).

Below we will focus on the first case, and leave to the end the necessary changes caused by \( A_0 \).

Now \( \mathcal{X} \) is reduced to

\[ \mathcal{X} = \sum_{l_1, \ldots, l_p} \int \prod_{i=1}^{p} \prod_{j \in A_i} (v^{(j)}_{l_i}(\lambda_j))^\pm d\lambda_j \cdot \mathcal{X}', \]

where \( l_i \) is the common value of \( k_j \) for \( j \in A_i \) (so that \( |l_i| \lesssim N^{(2)} \)), and \( \mathcal{X}' \) is an expression of the same form as \( \mathcal{X} \), but only involves the variables \( (k, k_j) \) and \( (\lambda, \lambda_j) \) for \( j \in B \), with \( d \) being a fixed linear combination of \( l_i \), and \( \alpha \) being a fixed linear combination of \( |l_i|^2 \). This gives

\[ |\mathcal{X}| \lesssim \sum_{l_1, \ldots, l_p} \prod_{i=1}^{p} M_{l_i}^{(i)} \cdot \sup_{l_1, \ldots, l_p} \left| \int \prod_{i=1}^{p} \frac{1}{M_{l_i}^{(i)}} \prod_{j \in A_i} (v^{(j)}_{l_i}(\lambda_j))^\pm d\lambda_j \cdot \mathcal{X}' \right|, \]

where recall that \( b_2 = b - \delta^2 \). When each \( l_i \) is fixed, by Remark 5.3 the expression

\[ \int \prod_{i=1}^{p} \frac{1}{M_{l_i}^{(i)}} \prod_{j \in A_i} (v^{(j)}_{l_i}(\lambda_j))^\pm d\lambda_j \cdot \mathcal{X}' \]

can be estimated in the same way as \( \mathcal{X}' \) (replacing \( b \) by \( b_2 \) will not change the proof), which is done in Step 2 above. We then only need to bound

\[ \sum_{l_1, \ldots, l_p} \prod_{i=1}^{p} M_{l_i}^{(i)} = \prod_{i=1}^{p} \sum_{l_i} M_{l_i}^{(i)}, \]

which we establish in the following claim.

Claim 5.4. Let \( K_i = \max(N_j : j \in A_i) \) and \( K_i' = \max(N_j : j \in A_i) \), then \( \tau^{-1} N_\pm \)-certainly we have that

\[ \sum_{l_i} M_{l_i}^{(i)} \lesssim \begin{cases} K_i^{-1+\gamma}(K_i')^{-\frac{1}{4}}, & K_i \sim N_j, j \in \mathcal{D}; \\ K_i^{-1+\theta}, & K_i \sim N_j, j \in \mathcal{G} \cup \mathcal{C}. \end{cases} \]
Proof of Claim 5.4. Let \( R_i^{(j)} = \| \langle \lambda_j \rangle^{b_2} v_i^{(j)}(\lambda_j) \|_{L^2_{\lambda_j}} \), then we have \( \| R^{(j)} \|_{\ell^\infty_i} \lesssim \| R^{(j)} \|_{\ell^2_i} \lesssim N_j^{-1+\gamma} \) if \( j \in \mathcal{D} \), \( \| R^{(j)} \|_{\ell^\infty_i} \lesssim N_j^{-1+\theta} \) and \( \| R^{(j)} \|_{\ell^2_i} \lesssim N_j^\theta \) if \( j \in \mathcal{G} \). If \( j \in \mathcal{C} \) we will apply Lemma 4.1 and again reduce to finitely many \( \lambda_j \) by restricting the size of \( \lambda_j \) and dividing into small intervals, and using the differentiability in \( \lambda_j \) of \( h^{(j)}_{k_j,k_j^*}(\lambda_j) \), which is assumed in the statement of Proposition 3.4. In the same way as before, by removing a set of probability \( \leq C_\theta e^{-(\tau-1)N_d} \) and omitting any \( \tau^{-\theta}(N_*)^{C\tau-1} \) factors, we conclude that

\[
\| R^{(j)} \|_{\ell^\infty_i} \lesssim \| R^{(j)} \|_{\ell^2_i} \lesssim N_j^{-1} \| \langle \lambda_j \rangle^{b_2} h^{(j)}_{k_j,k_j^*}(\lambda_j) \|_{\ell^2_{k_j,k_j^*} L^2_{\lambda_j}} \lesssim N_j^{-\frac{1}{2}} L_j^{-\frac{1}{2}},
\]

as well as

\[
\| R^{(j)} \|_{\ell^\infty_i} \lesssim N_j^{-1} L_j^2 \sup_{k_j,k_j^*} \| \langle \lambda_j \rangle^{b_2} h^{(j)}_{k_j,k_j^*}(\lambda_j) \|_{L^2_{\lambda_j}} \lesssim N_j^{-1} L_j^2,
\]

which also implies \( \| R^{(j)} \|_{\ell^\infty_i} \lesssim N_j^{-0.55} \). Now let \( K_i \sim N_j \) and \( K_i' \sim N_\tau \), then if \( j \in \mathcal{D} \), (5.51) follows from applying Hölder and measuring \( R^{(j)} \) and another factor other than \( R^{(j)} \) or \( R^{(s)} \) in \( \ell^2_i \), and all other factors in \( \ell^\infty_i \). If \( j \in \mathcal{G} \cup \mathcal{C} \), then we may assume \( |l_i| \sim K_i \) (otherwise (5.51) follows trivially from the third inequality in (3.8)), so the \( N_j \) for \( j \in A_i \) must all be comparable. We may then assume \( j \in \mathcal{G} \cup \mathcal{C} \) for each \( j \in A_i \), and (5.51) follows from applying Hölder and measuring two factors in \( \ell^2_{\lambda_i} \) and the rest in \( \ell^\infty_i \), such that at least one \( R^{(j)} \) with \( j \in \mathcal{G} \) is measured in \( \ell^\infty_i \) if there is any. This completes the proof.

The general case of Proposition 3.4 then follows from the \( \mathcal{X}' \) estimate, namely the no-pairing case in Step 2, combined with Claim 5.4. More precisely, suppose \( N^{(1)} = N_a \) with \( a \in A_i \) for some \( i \), then if \( a \in \mathcal{D} \) the bound (5.51) gives the power \( (N^{(1)})^{-1+\gamma} \), while the power \( (K_i')^{-\frac{1}{2}} \) in (5.51), as well as the no-pairing case of the bounds (3.23) and (3.29), give the power \( (N^{(2)})^{-1+\frac{1}{2}} \). If \( a \in \mathcal{G} \cup \mathcal{C} \), then the power \( (N^{(1)})^{-1+\theta} \) from (5.51) is already enough. If \( N^{(1)} = N_a \) with \( a \in B \) then we simply apply the no-pairing case and use (5.51) to gain decay in \( N^{(2)} \) when \( N^{(2)} = N_j \) and \( j \not\in B \). The only nontrivial case is (3.31), where there is no need to gain decay in \( N^{(2)} \), but we have an extra factor \( \lesssim \max(1,|\alpha|^{1}) \) from (5.49), where \( \alpha \) is a linear combination of \( |l_i|^2 \). By Claim 5.4 we have

\[
\langle \alpha \rangle_{1} \lesssim \max_{1 \leq i \leq p} \sum_{j \in A_i} M^{(i)}_{l_i}(N_{j})^{\theta}(\alpha)^{-\frac{1}{2}},
\]

which cancels this extra factor and proves (3.31).

Finally we consider the case with \( A_0 \), say \( k = k_{j_1} = k_{j_2} \) and \( N_{j_1} \geq N_{j_2} \). Here we can check that \( \mathcal{X}' \) still has the form (5.50), except that in \( \mathcal{X}' \) the input \( v_k(\lambda) \) is replaced by

\[
\overline{v}_k(\lambda) = \int_{\pm \lambda_0 \pm \lambda_1 \pm \lambda_2} v_k(\lambda_0) \pm \cdot v_k^{(j_1)}(\lambda_1) \pm \cdot v_k^{(j_2)}(\lambda_2) \pm d\lambda_1 d\lambda_2,
\]

which satisfies, due to the same proof as in Claim 5.4

\[
\| \langle \lambda \rangle^{b_2} \overline{v}_k(\lambda) \|_{\ell^2_{\lambda_j}} \lesssim \| \langle \lambda_0 \rangle^{b_2} v_k(\lambda_0) \|_{\ell^2_{\lambda_0}} \| \langle \lambda_1 \rangle^{b_2} v_k^{(j_1)}(\lambda_1) \|_{\ell^\infty_{\lambda_1}} \| \langle \lambda_2 \rangle^{b_2} v_k^{(j_2)}(\lambda_2) \|_{\ell^\infty_{\lambda_2}} \| R^{(j_1)} \|_{\ell^\infty_i} \lesssim N_j^{-1+\gamma} N_{j_2}^{-\frac{1}{2}}, \quad j_1 \in \mathcal{D},
\]

\[
N_j^{-1+\theta}, \quad j_1 \in \mathcal{G} \cup \mathcal{C}.
\]

The rest of proof now goes exactly as above using the additional bound (5.52), which has exactly the same gain as in Claim 5.4 and the set \( A_0 \) is treated together with the other sets \( A_i \). This completes the proof of Proposition 3.4.
5.3. Stability and convergence. Recall that $v_N$ is the solution of (2.5). Proposition 3.2 already implies the convergence of $v_N$ on the short time interval $[-\tau, \tau]$. For the purpose of proving global well-posedness, we need some additional results, namely a stability and a commutator estimate. For the notations involved in the proof, see Sections 3.1 and 3.2.

**Proposition 5.5.** Recall the relevant constants defined in (1.32), and that $\tau \ll 1$, $J = [-\tau, \tau]$. The following two statements hold $\tau^{-1}$-certainly.

1. For any $N \leq N'$, we have
   \[
   \|v_N - \Pi_N v_{N'}\|_{X^b(J)} \leq N^{-1+\gamma};
   \]
   (5.53)

2. Let $w = \Pi_N w$ be a solution to (2.25) on $J$, with data $w(t_0)$ assigned at some $t_0 \in J$ such that $\|w(t_0) - v_N(t_0)\|_{L^2} \leq AN^{-1+\gamma}(\log N)^\alpha$, where $\alpha \geq 0$ is an integer, then we have
   \[
   \|w - v_N\|_{X^b(J)} \leq BN^{-1+\gamma}(\log N)^{\alpha+1},
   \]
   (5.54)

where $B$ depends only on $A$ and $\alpha$.

**Proof.** (1) It suffices to prove $\|\Pi_N v^\uparrow_{N'} - v^\uparrow_N\|_{X^b} \leq N^{-1+\gamma}$. We write
   \[
   \Pi_N v^\uparrow_{N'} - v^\uparrow_N = \sum_{N < M \leq N'} \Pi_M v^\uparrow_M = \sum_{N < M \leq N'} (\Pi_M v^\uparrow_M - \Pi_M z^\uparrow_M),
   \]
   where $L_0(M)$ is the largest $L$ satisfying $(M, L) \in \mathcal{K}$. The bound for $\Pi_M Z^\uparrow_M$ follows from Proposition 3.3 so it suffices to bound $\Pi_M \psi_M L_0$, where $L_0 = L_0(M)$. Let $\psi = \psi^\uparrow M L_0$, then we have
   \[
   \psi(t) = \chi(t) e^{i\Delta (\Delta_M f(\omega))} - i\chi(t) \sum_{l=0}^r (l+1)c_{r-l}(m^*_M)^{r-l} \cdot \Pi_M \mathcal{N}^{2l+1}(\psi, v^\uparrow_{L_0}, \ldots, v^\uparrow_{L_0}).
   \]
   (5.55)

Since $N \leq N^2$, $\Pi_N \psi$ will solve the equation
   \[
   \Pi_N \psi(t) = -i\chi(t) \sum_{l=0}^r (l+1)c_{r-l}(m^*_M)^{r-l} \cdot \Pi\mathcal{N}^{2l+1}(\Pi_N \psi, v^\uparrow_{L_0}, \ldots, v^\uparrow_{L_0})
   \]
   \[\quad - i\chi(t) \sum_{l=0}^r (l+1)c_{r-l}(m^*_M)^{r-l} \cdot \Pi\mathcal{N}^{2l+1}(\Pi_N \psi, v^\uparrow_{L_0}, \ldots, v^\uparrow_{L_0}).
   \]
   (5.56)

Now $\tau^{-1}$-certainly we may assume (3.21) and the variant of (3.31) described in Remark 3.5. Note that (3.21) allows us to control the first line of (5.56); the second line of (5.56) is controlled by using (2.31) and the variant of (3.31). In the end we get that
   \[
   \|\Pi_N \psi\|_{X^b} \lesssim \tau^\theta \|\Pi_N \psi\|_{X^b} + \tau^\theta M^{-1+\frac{3\gamma}{8}},
   \]
   which proves (5.53).

(2) If $N \leq N^1 A_1(1)$ there is nothing to prove, so we may assume $N$ is large depending on $(A, \alpha)$. Let $\sigma = A|w|^2 - A|v_N|^2$ (this is conserved), then we have
   \[
   |\sigma| \lesssim (\|v_N(t_0)\|_{L^2} + \|w(t_0)\|_{L^2})\|w(t_0) - v_N(t_0)\|_{L^2} \lesssim \tau^{-\theta} AN^{-1+\gamma}(\log N)^{\alpha+1}.
   \]
   Note the log loss due to the fact that $\|v_N\|_{L^2} \lesssim \tau^{-\theta} \log N$. Recall that $v_N$ and $w$ satisfy the equations
   \[
   \begin{cases}
   (i\partial_t + \Delta) v_N = \sum_{l=0}^r c_{r-l}(m^*_N)^{r-l} \Pi_N \mathcal{N}^{2l+1}(v_N, \ldots, v_N), \\
   (i\partial_t + \Delta) w = \sum_{l=0}^r c_{r-l}(m^*_N + \sigma)^{r-l} \Pi_N \mathcal{N}^{2l+1}(w, \ldots, w)
   \end{cases}
   \]
   (5.57)
on $J$, so $z = w - v_N$ satisfies the equation

\[(i\partial_t + \Delta)z = \sum_{l=0}^r c_r[(m^*_N + \sigma)^{r-l} - (m^*_N)^{r-l}]\Pi_NN_{2l+1}(v_N, \cdots, v_N) + \sum_{l=0}^r c_r(m^*_N + \sigma)^{r-l}\Pi_N[N_{2l+1}(z + v_N, \cdots, z + v_N) - N_{2l+1}(v_N, \cdots, v_N)] (5.58)\]

on $J$, and $z_0 = z(t_0)$ satisfies $\|z_0\|_{L^2} \leq AN^{-1+\gamma}(\log N)^\alpha$. In order to bound $\|z\|_{X^b(J)}$, it will suffice to prove that given $z_0$ and $\sigma$, the mapping

\[z^1 \mapsto \chi(t - t_0)e^{i(t-t_0)\Delta}z_0 - i\chi_2(t - t_0)\sum_{l=0}^r c_r[(m^*_N + \sigma)^{r-l} - (m^*_N)^{r-l}]I_0\Pi_NN_{2l+1}(v_N^\dagger, \cdots, v_N^\dagger) - N_{2l+1}(v_N^\dagger, \cdots, v_N^\dagger)] (5.59)\]

is a contraction mapping from the set $\{z^1 : \|z^1\|_{X^b} \leq AN^{-1+\gamma}(\log N)^{\alpha+1}\}$ to itself, where

\[I_0F(t) = IF(t) - \chi(t)e^{i(t-t_0)\Delta}IF(t_0); \quad I_0F(t) = \chi(t)\int_{t_0}^t \chi(t')e^{i(t-t')\Delta}F(t') \, dt'.\]

To this end we will decompose $v_N^\dagger = \sum_{N' \leq N} h_{N'}^\dagger$, and $(\tau^{-1} \text{-certainly})$ apply the estimates (3.21) and (3.28), in the same way as in the proof of (3.23). More precisely, we may use (3.21) to control the terms in (5.59) that contain only one factor $z^1$ (where we use the $\tau^0$ gain to ensure smallness), and use (3.28) to control the terms in (5.59) that contain at least two factors $z^1$ (where we use the gain of powers of $N$ to ensure smallness, noticing that $N$ is large enough compared to $B$). Note that in applying these estimates we need to replace $I$ by $I_0$ and $\chi_2(t)$ by $\chi_2(t - t_0)$. This can be done because in Section 3.3 all estimates for $\chi_2(t) \cdot I[\cdots]$ are deduced from (2.31) and the corresponding estimates for $I[\cdots]$; here by definition we have $\|I_0F\|_{X^b} \leq \|IF\|_{X^b}$ for $b \in \{b, b_1\}$ which allows us to replace $I$ by $I_0$, and that $I_0F(t_0) = 0$ so (2.31) is still applicable with $I$ replaced by $I_0$ and $\chi_2(t)$ replaced by $\chi_2(t - t_0)$. The rest of the proof will be the same. \[
\]

**Proposition 5.6.** Recall the relevant constants defined in (1.32), the $\epsilon$ fixed as in Remark 1.4. Then the followings hold $\tau^{-1}$-certainly.

1. For any $N \leq N'$ we have

\[\|v_N - v_{N'}\|_{X^{-\theta, b_2}(J)} \leq \tau^{-\theta}N^{-\frac{\theta}{2}}, \quad (5.60)\]

\[\|(v_N - e^{i\Delta v_N(0)}) - (v_{N'} - e^{i\Delta v_{N'}(0)})\|_{X^{\frac{1}{2} - \frac{\theta}{2}, -\theta, b_2}(J)} \leq N^{-\frac{\theta}{2}}. \quad (5.61)\]

Note that the $X^{s, b}(J)$ bounds also imply the corresponding $C^0_t H_x^s(J)$ bounds.

2. Let $N_n(v)$ be a polynomial, also viewed as a multilinear form $N_n(v(1), \cdots, v(n))$, as in (2.10), but is only assumed to be input-simple (instead of simple). Then for any $N \leq N'$, the distance in $C^0_t H_x^{\gamma}(J)$ of any two of the following expressions

\[N_n(v), \Pi_N N_n(v), \Pi_N N_n(v) : v \in \{v_N, v_{N'}, \Pi_N v_{N'}\} (5.62)\]

is bounded by $\tau^{-\theta}N^{-\gamma}$. The same conclusion holds if $N_n$ is replaced by $W^0_N$ or $W^0_N$, or if $v$ is perturbed by any $w_N$ satisfying $\|w_N\|_{X^b(J)} \leq AN^{-1+\gamma}(\log N)^\alpha$. In the latter case the bound will be $O_{A, \tau, \alpha}(1)N^{-\gamma}$.
Proof. (1) We only need to prove (5.61). By taking a summation we may assume $N' = 2N$, and it suffices to prove that $\|y_{N'} - e^{it}\Delta(\Delta_{N'}f(\omega))\|_{X^{b_2}(J)} \leq (N')^{-\frac{1}{2}+\gamma_0}$. Now an extension of this function is given by

$$y^\uparrow = \sum_L \zeta^\uparrow_{N', L} + z^\uparrow_{N'},$$

see Sections 5.1 and 3.2 where $\zeta^\uparrow_{N', L}$ is defined from $h^{N', L, \uparrow}$ by (3.8) and (3.10), and $h^{N', L, \uparrow}$ and $z^\uparrow_{N'}$ satisfy (3.17) and (3.23). This controls the second term; to bound the first term, we use the $B^+_{\leq L}$ measurability of $h^{N', L, \uparrow}$ and Lemma 4.1 and perform the same reduction step as in the proof of Claim 5.4 using differentiability in $\lambda$ of $h_{k\cdot}(\lambda)$ with $h = h^{N', L, \uparrow}$, to bound $\tau^{-1}N'$-certainly that

$$\sum_L \|\zeta^\uparrow_{N', L}\|_{X^{b_2}} \leq \sum_L (N')^{-1}\|h^{N', L, \uparrow}\|_{Z^b} \lesssim (N')^{-\frac{1}{2}+\gamma_0 + \frac{\theta}{2}}. \quad (5.63)$$

The right hand side of (5.63) may involve a $\tau^{-\theta}$ factor, but this loss can always by recovered by adding a $\chi_\tau$ factor since $y^\uparrow(0) = 0$.

(2) The bounds for $W^a$ follows from the bounds for $N_n$ and the formulas (2.11) and (2.12), noticing that $|v|^{2\nu}v$ and $|v|^{2\nu}e$ are input-simple. As for $N_n$, by decomposing

$$v^\uparrow_{N'} = \sum_{N' \leq N} v^\uparrow_{N'}, \quad y^\uparrow_{N'} = \chi(t)e^{it\Delta(\Delta_{N'}f(\omega))} + \sum_L \zeta^\uparrow_{N', L} + z^\uparrow_{N'},$$

it suffices to $\tau^{-1}N^{(1)}$-certainly bound $N_n(v^{(1)}, \ldots, v^{(n)})$ in $C^0_tL^\infty_x(J)'$ by $\tau^{-\theta}(N^{(1)})^{-\gamma}$ for $v^{(j)}$ as in the assumptions of Proposition 3.4. The proof is a much easier variant of the arguments in Section 5.1.1 so we will only sketch the most important points.

First, since the $\partial_t$ derivative of all the $v^{(j)}$s are bounded by $(N^{(1)})^C$ (by restricting the size of $\lambda_j$ variables as we did in Section 5.1.1), by dividing $J$ into $(N^{(1)})^{\delta-1}$ intervals we may reduce to $(N^{(1)})^C\delta^{-1}$ exceptional sets and thus fix a time $t \in J$. This gets rid of all the $\lambda_j$ variables (so we are considering $v^{(j)}_{k_j}$ and $h^{(j)}_{k_j\cdot}$), and by a simple $H^\frac{1}{2} \rightarrow C^0$ argument, the estimates (3.26) and (3.27) remain true with the obvious changes. Now by repeating the arguments in Section 5.1.1 (in a simplified situation without $\lambda_j$ integrations) and Section 5.2 (which deals with over-pairings) we get that

$$\|\Delta_{N_0}N_n(v^{(1)}, \ldots, v^{(n)})\|_{L^2} \lesssim \tau^{-\theta}(N^{(1)})^{C\kappa^{-1}}(N^{(1)})^{C\gamma} \cdot (\#S) \prod_{j=1}^n N_j^{-\frac{p}{2}} \prod_{i=1}^p \frac{N_{2i-1}}{R_i},$$

where $N_{2i-1} \sim N_{2i} \gtrsim R_i$ and

$$S = \left\{(k, k_1, \ldots, k_n) \in \mathbb{Z}^{n+1} \cap \sum_{j=1}^n \lambda_j k_j = k, \quad |k| \lesssim N_0, \right\}$$

$$|k_j| \lesssim N_j (1 \leq j \leq n), \quad |k_{2i-1} - k_{2i}| \lesssim R_i(N^{(1)})^{C\kappa^{-1}}(1 \leq i \leq p),$$

and a simple counting estimate yields

$$\|\Delta_{N_0}N_n(v^{(1)}, \ldots, v^{(n)})\|_{L^2} \lesssim \tau^{-\theta}(N^{(1)})^{C\gamma} \min(1, (N^{(1)})^{-1}N_0) \lesssim \tau^{-\theta}N_0^\gamma(N^{(1)})^{-\gamma}$$

as by our choice $\gamma \ll \varepsilon$, which concludes the proof. With the $w_N$ perturbations the proof works the same way, except that the constants may depend also on $A$. \qed

Before ending this section, we would like to shift the point of view from the probability space $(\Omega, \mathcal{B}, \mathbb{P})$ to the spaces $V$ and $V_N$. Given $0 < \tau \ll 1$, all the above proof has allowed us to identify
a Borel set $E_\tau$ of $\mathcal{V}$ with $\rho(E_\tau) \geq 1 - C_\theta e^{-\tau^{-\theta}}$, such that $u_{in} = v_{in} \in E_\tau$, all the results in Sections 3 and 5, including Propositions 3.2, 3.3 and 5.5, are true.

In reality we will be using finite dimensional truncations of $E_\tau$, namely $E_\tau^N = \Pi^N E_\tau$. Clearly when $\Pi^N u_{in} \in E_\tau^N$, all the quantitative estimates proved before will remain true if all the frequencies $N, N', L$, etc., are $\leq N$. We moreover know that $\rho_N(E_\tau^N) \geq \rho(E_\tau) \geq 1 - C_\theta e^{-\tau^{-\theta}}$; since the Radon-Nikodym derivative $\frac{d\rho_N}{d\rho_\tau}$ is uniformly bounded in $L^2(d\rho_\tau)$, we have that

$$\mu_\rho^N(E_\tau^N) \geq 1 - C\sqrt{\rho_N(V_N \setminus E_\tau^N)} \geq 1 - C_\theta e^{-\tau^{-\theta}}. \quad (5.64)$$

Finally, due to the gauge symmetry of (1.7) and (2.5), we may assume that $E_\tau$ (and $E_\tau^N$) is rotation invariant, i.e. $e^{\alpha} E_\tau = E_\tau$ for $\alpha \in \mathbb{R}$.

6. Global well-posedness and measure invariance

In this section we will prove Theorem 1.3. Recall the sets $E_\tau^N$ defined at the end of Section 5.3. Denote the solution flow of (1.7) by $\Phi^N_t$ and the solution flow of (2.5) by $\Psi^N_t$, which are mappings from $V_N$ to itself. Define successively the sets

$$F^N_{T,K} = \bigcap \{ (\Phi^N_{t,K})^{-1} E^N_{T,K} \}. \quad (6.1)$$

$$G^N_{T,K,A,D} = \left\{ v \in V_N : \exists t \in [-D, D] \text{ s.t. } \Psi^N_t v = v' + v'' , \, v' \in F^N_{T,K}, \, \|v''\|_{L^2} \leq AN^{\frac{1}{2} + \gamma} (\log N)^\alpha \right\}, \quad (6.2)$$

$$\Sigma = \bigcup_{D \geq 1} \bigcap_{T \geq 2^{10} D} \bigcap_{K \gg T, A, D} \limsup_{N \to \infty} \Pi^{-1}_N G^N_{T,K,A,D}. \quad (6.3)$$

Here $\Pi^{-1}_N G = G \times V_N^1$ is the cylindrical set. We understand that $T, K, A, D$ all belong to some given countable set (say powers of two), and $\alpha$ is an integer. All these sets are Borel, since in (6.2) one may replace the $\leq$ sign by the $<$ sign, and then restrict to rational $t$ by continuity. We will start by proving global well-posedness and then measure invariance.

Proposition 6.1. The set $\Sigma$ satisfies $\mu(\mathcal{V} \setminus \Sigma) = 0$, and $W^{2r + 1}(u) \in H^{-\varepsilon}$ is well-defined for $u \in \Sigma$. For any $u_{in} \in \Sigma$, the solutions $u_N(t) = \Phi^N_t u_{in}$ to (1.7) converge to some $u(t) = \Phi_t u_{in}$ in $C^0_t H^\varepsilon([-T, T])$ for any $T > 0$. This $u$ is a distributional solution to (1.4), and $u(t) \in \Sigma$ for each $t$. The mappings $\Phi_t : \Sigma \to \Sigma$ satisfy $\Phi_0 = \mathrm{Id}$ and $\Phi_{t + t'} = \Phi_t \Phi_{t'}$.

Proof. We first prove $\mu(\mathcal{V} \setminus \Sigma) = 0$. By definition we have

$$\Sigma \supset \bigcap_{T \geq 2^{10}} \bigcup_{K \gg T} \limsup_{N \to \infty} \Pi^{-1}_N F^N_{T,K}, \quad (6.4)$$

so it suffices to prove for any fixed $T \geq 2^{10}$ that

$$\sup_{K \gg T} \mu \left( \limsup_{N \to \infty} \Pi^{-1}_N F^N_{T,K} \right) = 1.$$

Now by Fatou’s lemma and the fact that the total variation of $\mu - \mu^N_\tau$ converges to 0, we have

$$\mu \left( \limsup_{N \to \infty} \Pi^{-1}_N F^N_{T,K} \right) \geq \limsup_{N \to \infty} \mu_N \left( \Pi^{-1}_N F^N_{T,K} \right) = \limsup_{N \to \infty} \mu_N^N (F^N_{T,K}).$$

By invariance of $d\mu^N_\tau$ under the flow $\Psi^N_t$ (Proposition 5.4) we know that

$$\mu_N^N (F^N_{T,K}) \geq 1 - (2K + 1) \mu_N^N (V_N \setminus E^N_{T,K}) \geq 1 - C_\theta K e^{-(K T^{-1})^\gamma}.$$
uniformly in $\overline{N}$, and the right hand side converges to 1 as $K \to \infty$, so $\mu(V \setminus \Sigma) = 0$.

Now suppose $u_{in} \in \Sigma$. By definition we may choose some $D$, then for any $T \geq 2^{10}D$ we can find $(K, A, \alpha)$ such that $\Pi_{T}^{N}u_{in} \in C^{\alpha}_{T,K,A,D}$ for infinitely many $\overline{N}$. We may fix this $T$ (hence also $(K, A, \alpha)$) and this $\overline{N}$, so that $\|\Psi_{t_{0}}^{N} \Pi_{T}^{N}u_{in} - v'\|_{L^{2}} \leq AN^{-1+\gamma}(\log N)^{\alpha}$ for some $t_{0} \in [-D,D]$ and $v' \in F_{T,K}^{N}$. We proceed in three steps.

**Step 1: analyzing $v'$.** We first prove that, for any $N \leq \overline{N}$ and $|j| \leq K$ there holds that

$$\|\Pi_{N}\Psi_{N}^{j}v' - \Psi_{N}^{j} \Pi_{N}v'\|_{L^{2}} \leq BN^{-1+\gamma}(\log N)^{|j|}$$

(6.5)

for some $B$ depending only on $(T, K)$. This is obviously true for $j = 0$; suppose this is true for $j$, since $\Psi_{N}^{j}v' \in E_{N}$, by Proposition 5.5 (1) we have

$$\|\Pi_{N}\Psi_{N}^{j+1}v' - \Psi_{N}^{j+1} \Pi_{N}v'\|_{L^{2}} \leq B'N^{-1+\gamma}(\log N)^{|j|+1}$$

by Proposition 5.5 (2) and (6.5), where $B'$ depends only on $B$ and $(T, K)$, so (6.5) holds also for $j \pm 1$ which concludes the inductive proof. By the same argument, we can show that (6.5) remains true with $\frac{t}{\overline{N}}$ replaced by any $t \in [-T,T]$ and $|j|$ replaced by $K$.

Similarly, since $\Psi_{N}^{j}v' \in E_{N}$ for each $|j| \leq K$, by combining Propositions 5.5 and 5.6 we conclude that for any $N \leq N' \leq \overline{N}$,

$$\sup_{t \in [-T,T]}\|\Psi_{t}^{N} \Pi_{N}v' - \Psi_{t}^{N} \Pi_{N}v'\|_{H^{-\theta}} \leq O_{T,K}(1)N^{-\frac{\theta}{2}},$$

(6.6)

$$\sup_{t \in [-T,T]}\|W_{N}^{j}(\Psi_{t}^{N} \Pi_{N}v' - \Psi_{t}^{N} \Pi_{N}v')\|_{H^{-\varepsilon}} \leq O_{T,K}(1)N^{-\gamma},$$

(6.7)

and the same is true if $W_{N}^{j}$ and $W_{N}^{j}$ in (6.7) is replaced by $\Pi_{N}W_{N}^{j}$ and $\Pi_{N}W_{N}^{j}$.

**Step 2: linking $u_{in}$ to $v'$.** Recall that $\|\Psi_{t_{0}}^{N} \Pi_{N}u_{in} - v'\|_{L^{2}} \leq AN^{-1+\gamma}(\log N)^{\alpha}$. Since $|t_{0}| \leq D \ll T$, by iterating Proposition 5.5 (2) we deduce that $\|\Pi_{T}^{N}u_{in} - \Psi_{t}^{N} \Pi_{N}v'\|_{L^{2}} \leq A'N^{-1+\gamma}(\log N)^{\alpha+K}$ where $A'$ depends only on $(T, K, A, \alpha)$. Writing $-t_{0} = \frac{t}{\overline{N}} + t'$ with $|j| \leq 2^{-\rho}K$ and $|t'| \leq \frac{T}{\overline{N}}$, we may apply Proposition 5.5 (2) again and combine this with (6.5) and similar estimates to deduce for any $N \leq \overline{N}$ that

$$\sup_{t \in [-\frac{T}{\overline{N}}, \frac{T}{\overline{N}}]}\|\Psi_{t}^{N} \Pi_{N}u_{in} - \Psi_{t-t_{0}}^{N} \Pi_{N}v'\|_{L^{2}} \leq BN^{-1+\gamma}(\log N)^{\alpha+2K}$$

(6.8)

with $B$ depending only on $(T, K, A, \alpha)$. By (6.6), (6.7), (6.8) and Proposition 5.6 we conclude for all $N \leq N' \leq \overline{N}$ that

$$\sup_{t \in [-\frac{T}{\overline{N}}, \frac{T}{\overline{N}}]}\|\Psi_{t}^{N} \Pi_{N}u_{in} - \Psi_{t}^{N} \Pi_{N}v'\|_{H^{-\theta}} \leq O_{T,K,A,\alpha}(1)N^{-\frac{\theta}{2}},$$

(6.9)

$$\sup_{t \in [-\frac{T}{\overline{N}}, \frac{T}{\overline{N}}]}\|W_{N}^{j}(\Psi_{t}^{N} \Pi_{N}u_{in} - \Psi_{t}^{N} \Pi_{N}v')\|_{H^{-\varepsilon}} \leq O_{T,K,A,\alpha}(1)N^{-\gamma},$$

(6.10)

and the same is true for projections of $W_{N}^{j}$.

**Step 3: completing the proof.** Now, for fixed $(D, T, K, A, \alpha)$ we know that there exists infinitely many $\overline{N}$ such that (6.9) and (6.10) are true for all $N \leq N' \leq \overline{N}$, so (6.9) and (6.10) are simply true.
for all $N \leq N'$. This implies the convergence of $\Psi_t^N \Pi_N u_{in}$ in $C_0^\infty H_x^{-\varepsilon}([-\frac{T}{2}, \frac{T}{2}])$, and we will define $\Psi_t = \lim_{N \to \infty} \Psi_t^N \Pi_N$. Since by the definition of gauge transform we have

$$
\Phi_t^N \Pi_N u_{in} = \Psi_t^N \Pi_N u_{in} \cdot e^{-iB_N(t)}, \quad B_N(t) = (r + 1) \int_0^t A[W_N^2(\Psi_t^N \Pi_N u_{in})] \, dt',
$$

(6.11) and (6.10) also implies the convergence of $\Phi_t^N \Pi_N u_{in}$ in $C_0^\infty H_x^{-\varepsilon}([-\frac{T}{2}, \frac{T}{2}])$, as well as the convergence of $\Pi_N W_N^{2r+1}(\Phi_t^N \Pi_N u_{in})$ in the same space. As $u_N = \Phi_t^N \Pi_N u_{in}$ solves the equation (1.7) with right hand side being $\Pi_N W_N^{2r+1}(u_N)$, we know that the limit $u = \lim_{N \to \infty} u_N$ solves (1.1) in the distributional sense. Let $u(t) = \Phi_t u$, the group properties of $\Phi_t$ follow from the group properties of $\Psi_t^N$ and limiting arguments similar to the above.

Finally we prove that $\Phi_{t_1} u_{in} \in \Sigma$ for $u_{in} \in \Sigma$ and any $t_1$. Let $D$ be associated with the assumption $u_{in} \in \Sigma$, and fix $D_1 > D + |t_1|$. For any $T \geq 2^{10}D_1$ there exists $(K, A, \alpha)$ such that $\Pi_N u_{in} \in G_{T,K,A}, D$ for infinitely many $N$. It suffices to show that for such $N$ we must have $\Pi_N \Phi_{t_1} u_{in} \in G_{T,K,B,D_1}$ with $B$ depending only on $(T, K, A, \alpha)$. Since $\Pi_N \Phi_{t_1} u_{in}$ and $\Pi_N \Psi_{t_1} u_{in}$ only differs by a rotation and the sets we constructed are rotation invariant, we only need to prove the same thing for $\Psi_{t_1} u_{in}$.

Now, on the one hand we know for some $|t_0| \leq D$ that $\|\Pi_N u_{in} - \Psi_{t_0} u'\|_{L^2} \leq A' N^{-1+\gamma} (\log N)^{\alpha + K}$ for some $A'$ depending only on $(T, K, A, \alpha)$ (see Step 2) and similarly $\|\Psi_{t_1}^N \Pi_N u_{in} - \Psi_{t_0}^N u'\|_{L^2} \leq A' N^{-1+\gamma} (\log N)^{\alpha + 2K}$. On the other hand by taking limits in (6.5) and (6.8) we also get $\|\Pi_N \Psi_{t_1} u_{in} - \Psi_{t_1}^N \Pi_N u_{in}\|_{L^2} \leq A' N^{-1+\gamma} (\log N)^{\alpha + 2K}$, and hence $\|\Pi_N \Phi_{t_1} u_{in} - \Psi_{t_1}^N u'\|_{L^2} \leq A' N^{-1+\gamma} (\log N)^{\alpha + 2K}$. Applying Proposition 5.5 (2) again we get that $\|\Psi_{t_0}^N \Phi_{t_1} u_{in} - u'\|_{L^2} \leq B' N^{-1+\gamma} (\log N)^{\alpha + 3K}$ with $|t_0 - t_1| \leq D_1$ and $B \leq B(T, K, A, \alpha)$, thus by definition $\Pi_N \Psi_{t_1} u_{in} \in G_{T,K,B,D_1}$. This completes the proof.

**Proposition 6.2.** For any Borel subset $E \subset \Sigma$ and any $t_0 \in \mathbb{R}$, we have $\mu(E) = \mu(\Phi_{t_0} E)$.

**Proof.** The map $\Phi_t$ is a limit of continuous mappings, so it is Borel measurable. By taking limits, we may assume the set $E$ is compact in $H_x^{-\varepsilon}$ topology. We may also assume that $|t_0| \leq 1$, and then for some fixed $(T, K, A, \alpha, D)$ with $K > T \geq 2^{10}D$ we have $E \subset \limsup_{N \to \infty} G_{T,K,A,D}$.

By the proof of Proposition 6.1 we can deduce that for $u \in E$ and $|t| \leq 2$,

$$
\|\Psi_{t_1}^N \Pi_N u - \Pi_N \Psi_t u\|_{L^2} \lesssim N^{-1+\gamma} (\log N)^{\alpha + 3K},
$$

(6.12)

with constants depending on $(T, K, A, \alpha)$ (same below). Moreover, concerning the phase $B_N(t)$ involved in the gauge transform, namely

$$
B_N(t) = (r + 1) \int_0^t A[W_N^2(\Psi_t^N \Pi_N u)] \, dt',
$$

one can show that as $N \to \infty$, $B_N(t)$ converges to its limit $B(t)$ at a rate $\|B_N(t) - B(t)\|_{C_0^\infty([-2,2])} \lesssim N^{-1+\gamma} (\log N)^{\alpha + 4K}$. In fact we may first reduce to short time intervals where local theory is applicable, then notice that

$$
\int_0^t A[W_N^2(\Psi_t^N \Pi_N u)] \, dt' = A[\mathcal{A}(W_N^2(\Psi_t^N \Pi_N u))]
$$

for $|t| \ll 1$, and apply Proposition 3.3 more precisely (3.29), with the observation that the mean $\mathcal{A}$ restricts the two highest input frequencies in any multilinear expression $\mathcal{N}_n$ occurring in $W_N^2$ to be comparable, i.e. $N^{1(1)} \sim N^{2}$. We omit the details.
With the explicit convergence rate of $B_N(t)$, we see that (6.12) holds with $\Psi_t^N$ and $\Psi_t$ replaced by $\Phi_t^N$ and $\Phi_t$, and with $3K$ replaced by $4K$. This gives, for $|t| \leq 1$, that
\[
\Pi_N \Phi_t E \subset \Phi_t^N \Pi_N E + \mathfrak{B}_{L^2}(A_1 N^{-1+\gamma}(\log N)^{\alpha+4K}) \subset \Phi_t^N(\Pi_N E + \mathfrak{B}_{L^2}(A_2 N^{-1+\gamma}(\log N)^{\alpha+5K})),
\]
where $A_{1,2}$ are constants depending only on $(T, K, A, \alpha)$, and $\mathfrak{B}_{L^2}(R)$ is the ball of radius $R$ in $L^2$ centered at the origin; note that the second subset relation follows from long-time stability, which is also a consequence of the proof of Proposition 6.1. By invariance of $d\mu^N$ under $\Phi_t^N$ we have that
\[
\mu_N(\Phi_t E) \leq \mu_N^2(\Pi_N \Phi_t E) \leq \mu_N(\Pi_N E + \mathfrak{B}_{L^2}(A_2 N^{-1+\gamma}(\log N)^{\alpha+5K})))
= \mu_N(\Pi_N E + \mathfrak{B}_{L^2}(A_2 N^{-1+\gamma}(\log N)^{\alpha+5K})). \tag{6.13}
\]
It then suffices to prove that
\[
\limsup_{N \to \infty} \Pi_N^{-1}(\Pi_N E + \mathfrak{B}_{L^2}(A_2 N^{-1+\gamma}(\log N)^{\alpha+5K}))) \subset E, \tag{6.14}
\]
which would imply $\mu(\Phi_t E) \leq \mu(E)$, and conclude the proof by time reversibility. To prove (6.14), suppose $u$ is such that $\|\Pi_N(u - u_N)\|_{L^2} \leq A_2 N^{-1+\gamma}(\log N)^{\alpha+5K}$ with $u_N \in E$ for infinitely many $N$, then by compactness we may assume $u_N \to u \in E$ in $H^{-\varepsilon}$, so $u_N \to u$ coordinate-wise and $u_N \to v$ coordinate-wise, hence $u = v \in E$ and the proof is complete. \qed

References

[1] M. Aizenman. Geometric analysis of $\varphi^4$ fields and Ising models. Part I and II, Comm. Math. Phys. 86, issue 1 (1982), 1–48.
[2] S. Albeverio and A. Cruzeiro. Global flows with invariant (Gibbs) measures for Euler and Navier-Stokes two dimensional fluids, Comm. Math. Phys. 129 (1990), 431–444.
[3] I. Bailleul and F. Bernicot. High order paracontrolled calculus, 2016. [arXiv:1609.06966]
[4] I. Bailleul, I. and F. Bernicot. Heat semigroup and singular PDEs, J. Funct. Anal. 270 no. 9 (2016), 3344–3452.
[5] I. Bailleul, F. Bernicot and D. Frey. Space-time paraproducts for paracontrolled calculus, 3D-PAM and multiplicative Burgers equations, Ann. Sci. Éc. Norm. Supér. (4) 51 no. 6 (2018), 1399–1456.
[6] Á. Bényi, T. Oh and O. Pocovnicu. Higher order expansions for the probabilistic local Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^3$, Trans. Amer. Math. Soc. Ser. B 6 no. 4 (2019), 114–160.
[7] V. I. Bogachev. Gaussian measures, Mathematical Surveys and Monographs vol. 62. American Mathematical Society, Providence, RI, (1998), xii+433.
[8] J. Bourgain. Periodic nonlinear Schrödinger equation and invariant measures, Comm. Math. Phys. 166 (1994), 1–26.
[9] J. Bourgain. Invariant measures for the 2D-defocusing nonlinear Schrödinger equation, Comm. Math. Phys. 176 (1996), 421–445.
[10] J. Bourgain. Invariant measures for the Gross-Piatevskii equation. J. Math. Pures Appl. (9) 76 (1997), no. 8, 649–702.
[11] J. Bourgain. Global solutions of nonlinear Schrödinger equations, Amer. Math. Soc. Colloq. Pub. 46, Amer. Math. Soc. (1999).
[12] J. Bourgain and A. Bulut. Almost sure global well-posedness for the radial nonlinear Schrödinger equation on the unit ball $\mathbb{R}^3$: the 3d case, J. Eur. Math. Soc. 16 no. 6 (2014), 1289–1325.
[13] B. Bringmann. Almost sure local well-posedness for a derivative nonlinear wave equation, 2018, to appear in Inter. Math. Res. Notices (IMRN). [arXiv:1809.00220]
[14] D. Brydges and G. Slade. Statistical Mechanics of the 2-Dimensional Focusing Nonlinear Schrödinger Equation, Commun. Math. Phys. 182 (1996), 485-504.
[15] N. Burq, L. Thomann and N. Tzvetkov. Remarks on the Gibbs measures for nonlinear dispersive equations, Ann. Fac. Sci. Toulouse Math., Ser. 6 Vol. 27 no. 3 (2018), 527–597.
[16] N. Burq and N. Tzvetkov. Random data Cauchy theory for supercritical wave equations I: local theory, Invent. Math. 173 no. 3 (2008), 449–475.

\[1\] By repeating the proof of Proposition 6.1, we can show that $\Phi_t^N \Pi_N u \to \Phi_t u$ in $H^{-\varepsilon}$ uniformly for $u \in E$ and $|t| \leq 2$, so $\Phi_t^N E$ is also compact in $H^{-\varepsilon}$ and satisfies similar properties as $E$. 
[17] R. Catellier and K. Chouk. Paracontrolled distributions and the 3-dimensional stochastic quantization equation, *Ann Prob.* 46 no. 5 (2018), 2621–2679.

[18] A. Chandra and H. Weber. Stochastic PDEs, regularity structures, and interacting particle systems, *Ann. Fac. Sci. Toulouse Math.* (6) 26 no. 4 (2017), 847–909.

[19] J. Colliander and T. Oh. Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $L^2(T)$, *Duke Math. J.* 161 no. 3 (2012), 367–414.

[20] G. Da Prato and A. Debussche. Two-dimensional Navier-Stokes equations driven by a space-time white noise, *J. Funct. Anal.* 196 no. 1 (2002), 180–210.

[21] G. Da Prato and A. Debussche. Strong solutions to the stochastic quantization equations, *Ann. Probab.* 31 no. 4 (2003), 1900–1916.

[22] G. Da Prato and L. Tubaru. Wick powers in stochastic PDEs: an introduction, Technical Report UTM 711, 2006, 39 pp. [http://eprints.biblio.unitn.it/1189/1/UTM711.pdf](http://eprints.biblio.unitn.it/1189/1/UTM711.pdf)

[23] Y. Deng. Two dimensional nonlinear Schrödinger equation with random radial data, *Anal. PDE* 5 no. 5 (2012), 913–960.

[24] Y. Deng. Invariance of the Gibbs measure for the Benjamin-Ono equation. *J. Eur. Math. Soc. (JEMS)* 17 no. 5 (2015), 1107–1198.

[25] Y. Deng, A. Nahmod and H. Yue. Optimal local well-posedness for the periodic derivative nonlinear Schrödinger equation, 2019. arXiv:1905.04352.

[26] Y. Deng, N. Tzvetkov, and N. Visciglia. Invariant Measures and Long Time Behavior for the Benjamin-Ono Equation III. *Comm. Math. Phys.* 347 no. 3 (2015), 815–857.

[27] B. Dodson, J. Lührmann and D. Mendelson, Almost sure local well-posedness and scattering for the 4D cubic nonlinear Schrödinger equation, *Advances in Mathematics* 347 (2019), 619-676.

[28] P. K. Friz and M. Hairer. *A course on rough paths, with an introduction to regularity structures*, Universitext. Springer, Cham, 2014. xiv+251 pp.

[29] J. Fröhlich. On the triviality of $\lambda\phi^4$ theories and the approach to the critical point in $d (\sim) > 4$ dimensions, *Nuclear Physics B* 200, issue 2 (1982), 281–296.

[30] J. Glimm and A. Jaffe. *Quantum physics, A functional integral point of view*, Second edition, Springer-Verlag, New York, 1987. xxii+535 pp.

[31] M. Gubinelli, P. Imkeller and N. Perkowski. Paracontrolled distributions and singular PDEs, *Forum Math Pi* 3, e6, 75 pp, 2015.

[32] M. Gubinelli, P. Imkeller and N. Perkowski. A Fourier analytic approach to pathwise stochastic integration, *Electron. J. Probab.* 21, paper No. 2, (2016) 37 pp.

[33] M. Gubinelli, H. Koch and T. Oh. Renormalization of the two-dimensional stochastic nonlinear wave equations, *Trans. Amer. Math. Soc.* 370 no. 10 (2018), 7335–7359.

[34] M. Gubinelli, H. Koch and T. Oh. Paracontrolled approach to the three-dimensional stochastic nonlinear wave equation with quadratic nonlinearity, 2018. arXiv:1811.07308

[35] M. Gubinelli and N. Perkowski. Lectures on singular stochastic PDEs, *Ensaios Matemáticos, Mathematical Surveys*, 29. Sociedade Brasileira de Matemática, Rio de Janeiro. (2015), 89 pp.

[36] M. Gubinelli and N. Perkowski. KPZ reloaded, *Comm. Math. Phys.* 349 no. 1 (2017), 165–269.

[37] M. Gubinelli and N. Perkowski. Energy solutions of KPZ are unique, *J. Amer. Math. Soc.* 31 no. 2 (2018), 427–471.

[38] M. Gubinelli and N. Perkowski. An introduction to singular SPDEs, in *Stochastic partial differential equations and related fields*, 69–99, *Springer Proc. Math. Stat.*, 229, Springer, Cham, 2018.

[39] M. Hairer. Solving the KPZ equation, *Ann. of Math.* 178 no. 2 (2013), 559–664.

[40] M. Hairer. A theory of regularity structures, *Invent. Math.* 198 no. 2 (2014), 269–504.

[41] M. Hairer. Singular Stochastic PDE, *Proceedings of the ICM-Seoul*, Vol. I, (2014), 685–709.

[42] M. Hairer. Introduction to regularity structures, *Braz. J. Probab. Stat.* 29 no. 2 (2015), 175–210.

[43] M. Hairer. Regularity structures and the dynamical $\Phi^3_2$ model, *Current Developments in Mathematics 2014*, Int. Press, Somerville, MA, (2016), 1–49.

[44] M. Hairer. Renormalisation of parabolic stochastic PDEs, *Jpn. J. Math.* 13 no. 2 (2018), 187–233.

[45] M. Hairer and C. Labbé. The reconstruction theorem in Besov spaces, *J. Funct. Anal.* 273 no. 8 (2017), 2578–2618.

[46] C. Kenig and D. Mendelson, The focusing energy-critical nonlinear wave equation with random initial data, 2019, to appear in *Int. Math. Res. Not.* arXiv:1903.07240

[47] R. Killip, J. Murphy and M. Visan, Almost sure scattering for the energy-critical NLS with radial data below $H^1(\mathbb{R}^4)$, *Comm. Partial Differential Equations* 44 (2019), no. 1, 51–71.
[48] S. Klainerman, I. Rodnianski and J. Szeftel. The bounded $L^2$ curvature conjecture, *Invent. Math.* 202 no. 1 (2015), 91–216.

[49] A. Kupiainen. Renormalization group and stochastic PDEs, *Annales Henri Poincaré* 17 no. 3 (2016), 497–535.

[50] J. Lebowitz, R. Rose and E. Speer. Statistical mechanics of the nonlinear Schrödinger equation, *J. Statist. Phys.* 50 (1988), 657–687.

[51] J.C. Mourrat and H. Weber. The dynamic $\Phi^4_3$ model comes down from infinity, *Comm. Math. Phys.* 356 no. 3 (2017), 673–753.

[52] J.C. Mourrat, H. Weber and W. Xu. Construction of $\Phi^4_3$ diagrams for pedestrians, in From particle systems to partial differential equations, 1–46, *Springer Proc. Math. Stat.*, 209, Springer, Cham, 2017.

[53] A. Nahmod, T. Oh, L. Rey-Bellet and G. Staffilani. Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS, *J. Eur. Math. Soc. (JEMS)* 14 no. 4 (2012), 1275–1330.

[54] A. Nahmod, N. Pavlovic, G. Staffilani and N. Totz. Global Flows with Invariant Measures for the Inviscid Modified SQG Equations, *Stoch. Partial Differ. Equ. Anal. Comput.* 6 no. 2 (2018), 184–210.

[55] N. Nahmod and G. Staffilani. Almost sure well-posedness for the periodic 3D quintic nonlinear Schrödinger equation below the energy space, *J. Eur. Math. Soc. (JEMS)* 17 no. 7 (2015), 1687–1759.

[56] E. Nelson. A quartic interaction in two dimensions, 1966 Mathematical Theory of Elementary Particles, *Proc. Conf.*, Dedham, Mass. 1965 69–73, MIT Press, Cambridge, Mass.

[57] E. Nelson. Construction of quantum fields from Markoff fields, *J. Functional Analysis* 12 (1973), 97–112.

[58] T. Oh. Invariant Gibbs measures and a.s. global well-posedness for coupled KdV systems, *Diff. Int. Eq.* 22 no. 7-8 (2009), 637–668.

[59] T. Oh and L. Thomann. A pedestrian approach to the invariant Gibbs measures for the 2-d defocusing nonlinear Schrödinger equations, *Stoch. Partial Differ. Equ. Anal. Comput.* 6 no. 3 (2018), 397–445.

[60] T. Oh, O. Pocovnicu and N. Tzvetkov, Probabilistic local well-posedness of the cubic nonlinear wave equation in negative Sobolev spaces, 2019. [arXiv:1904.06792](https://arxiv.org/abs/1904.06792)

[61] T. Oh and L. Thomann. Invariant Gibbs measure for the 2-d defocusing nonlinear Schrödinger equations, *Stoch. Partial Differ. Equ. Anal. Comput.* 6 no. 3 (2018), 397–445.

[62] A. Nahmod, L. Rey-Bellet, S. Sheffield and G. Staffilani. Absolute continuity of Brownian bridges under certain gauge transformations, *Math. Res. Lett.* 18 no. 5, (2011), 875–887.

[63] N. Nahmod and G. Staffilani. Almost sure well-posedness for the periodic 3D quintic nonlinear Schrödinger equation below the energy space, *J. Eur. Math. Soc. (JEMS)* 17 no. 7 (2015), 1687–1759.

[64] E. Nelson. Para-controlled distributions and singular SPDEs. [https://personal-homepages.mis.mpg.de/perkow/paracontrolled-leipzig.pdf](https://personal-homepages.mis.mpg.de/perkow/paracontrolled-leipzig.pdf)

[65] B. Simon. *The $P(\phi)_2$ Euclidean (quantum) field theory*, Princeton Series in Physics. Princeton University Press, Princeton, N.J., 1974. xx+392 pp.

[66] H. Smith and D. Tataru. Sharp local well-posedness results for the nonlinear wave equation, *Ann. of Math.* (2) 162 no. 1 (2005), 291–366.

[67] A-S. de Suzzoni. Invariant measures for the cubic wave equation on the unit ball in $\mathbb{R}^3$, *Dyn. Partial Differ. Equ.* 8 (2011), no. 2, 127–148.

[68] M. Sy. Invariant measure and long time behavior of regular solutions of the Benjamin-Ono equation, *Anal. PDE* 11 no. 8 (2018), 1841–1879.

[69] L. Thomann. Invariant Gibbs measures for dispersive PDEs, Lecture Notes from Hamiltonian dynamics, PDEs and waves on the Amalfi coast, (2016) [http://www.iecl.univ-lorraine.fr/~Laurent.Thomann/Gibbs_Thomann_Maiori.pdf](http://www.iecl.univ-lorraine.fr/~Laurent.Thomann/Gibbs_Thomann_Maiori.pdf)

[70] L. Thomann and N. Tzvetkov. Gibbs measure for the periodic derivative nonlinear Schrödinger equation, *Nonlinearity* 23 (2010), 2771–2791.

[71] N. Tzvetkov. Invariant measures for the Nonlinear Schrödinger equation on the disc. *Dyn. Partial Differ. Equ.* 3 (2006), 111–160.

[72] N. Tzvetkov. Construction of a Gibbs measure associated to the periodic Benjamin-Ono equation, *Probab. Theory Related Fields* 146 (2010), 481–514.

[73] W. Wang and H. Yue. Almost sure existence of global weak solutions to the Boussinesq equations. *Preprint*.

[74] H. Yue. Almost sure well-posedness for the cubic nonlinear Schrödinger equation in the super-critical regime on $\mathbb{T}^d$, $d \geq 3$, 2018. [arXiv:1808.00657](https://arxiv.org/abs/1808.00657)
