Unitary Representations and BRST Structure of the Quantum Anti–de Sitter Group at Roots of Unity

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It is shown that for suitable roots of unity, there exist finite–dimensional unitary representations of $U_q(so(2,3))$ corresponding to all classical one–particle representations with (half)integer spin, with the correct low–energy limit. In the massless case for spin $\geq 1$, a subspace of “pure gauges” appears which must be factored out, as classically. Unitary many–particle representations are defined, with the same low–energy states as classically. Furthermore, a remarkable element of the center of $U_q(so(2,3))$ is identified which plays the role of the BRST operator, for any spin. The corresponding ghosts are an intrinsic part of indecomposable representations.

1 One–Particle Representations of $U_q(so(2,3))$

The quantum Anti–de Sitter (AdS) group $\mathcal{U} := U_q(so(2,3))$ is defined as a real form of the Drinfeld–Jimbo quantized universal enveloping algebra $U_q(so(5, C))$, which is the Hopf algebra

$$ [H_i, H_j] = 0, \quad [H_i, X_j^\pm] = \pm A_{ij} X_j^\pm, \quad [X_i^+, X_j^-] = \delta_{ij} [H_i]_{q_i} $$

for $i = 1, 2$, plus the quantum Serre relations, where $A_{ij}$ is the Cartan matrix of $B_2$, $q_i = q^{\alpha_i - \alpha_1}$, and $[n]_{q_i} = \frac{q^n - q^{-n}}{q - q^{-1}}$. Coproduct, antipode and counit are defined as usual. Root vectors corresponding to $\alpha_3 = \alpha_1 + \alpha_2, \alpha_4 = \alpha_1 + 2\alpha_2$ can be defined as well. The Drinfeld–Casimir $v = (SR_2)R_1 q^{-2\rho}$ is constructed using the universal $R \in \mathcal{U} \otimes \mathcal{U}$, where $\rho = \frac{1}{2}H_1 + H_2$.

Since we are interested in the root of unity case, the reality structure of $\mathcal{U}$ for $|q| = 1$ is chosen as $H_i = H_i, \quad X_1^+ = X_1^-, \quad X_2^+ = X_2^-$, which is compatible with the coproduct if it also flips the factors in a tensor product. Then $E = H_1 + \frac{1}{2}H_2$ is the energy, and $J_z = \frac{1}{2}H_2$ is a component of angular momentum.

Denote the irreducible representations (irreps) with lowest weight $\mu = E_0 \alpha_3 - s \alpha_2 \equiv (E_0, s)$ as $V(\mu)$. Elementary particles on AdS space are unitary $V(\mu)$ with $E_0 > 0$. The most important ones are the $V(\mu)$ with integral $\mu$ and $E_0 \geq s + 1 \geq 0$.

In the classical case, they are of course infinite–dimensional. In the massless case $E_0 = s + 1$, the ”naive” lowest weight representations are not com-
pletely reducible, they develop an invariant subspace of "pure gauge" states with lowest weight \((E_0 + 1, s - 1)\). To obtain unitarizable irreps (photon, graviton,...), one has to factor them out.

In the quantum case for \(|q| = 1\), the corresponding representations can only be unitarizable if \(q\) is a root of unity. Thus from now on \(q = e^{2\pi i/m}\).

Then the \(V(\mu)\) can be obtained from compact representations by a "shift": Let \(V(\lambda)\) be a highest weight irrep with compact highest weight \(\lambda\), i.e. \(\lambda\) is integral with \(0 \leq (\lambda, \alpha_i) < m\) for \(i = 1,\ldots,4\). Then \(\omega := V(\lambda_0)\) with \(\lambda_0 = \frac{2\pi i}{m}\alpha_3\) is one-dimensional, and \(V(\mu) := V(\lambda) \otimes \omega\) is a positive energy irrep with lowest weight \(\mu = -\lambda + \lambda_0\).

We call \(V(\mu)\) physical if it is unitarizable w.r.t. \(U_q(\mathfrak{so}(2,3))\). If \(n = 1\), \(V(\mu)\) is called \(D_i\) for \(\mu = (1,1/2)\), and \(\text{Rac}\) for \(\mu = (1/2,0)\). Now one can show

**Theorem 1.1** All \(V(\mu)\) where \(\mu = -\lambda + \frac{2\pi i}{m}\alpha_3\) is compact are physical, in particular the massless irreps, as well as the singletons \(D_i\) and \(\text{Rac}\). For \(E \leq \frac{2\pi i}{m}\), they are obtained from a (lowest weight) Verma module by factoring out the submodule with lowest weight \((E_0, -(s+1))\) only, except for the massless case, where an additional lowest weight state with weight \((E_0 + 1, s - 1)\) appears, and for the \(D_i\) resp. \(\text{Rac}\), where an additional lowest weight state with weight \((E_0 + 1, s)\) resp. \((E_0 + 2, s)\) appears. This is the same as classically.

For the singletons, this was already shown in \(5\). All these representations are now finite–dimensional (we essentially work in the unrestricted specialization).

### 2 Tensor Product and Many–Particle Representations

In general, the full tensor product of 2 physical irreps is not unitarizable, and indecomposable at roots of unity. The idea is to keep the appropriate physical lowest weight quotient modules only. Consider two physical irreps \(V(\mu)\) and \(V(\mu')\). For a basis \(\{u_{\lambda'}\}\) of the physical lowest weight states in \(V(\mu) \otimes V(\mu')\), let \(Q_{\mu,\mu'}\) be the quotient of \(\sum \mathcal{U} \cdot u_{\lambda'}\) after factoring out all submodules of the \(\mathcal{U} \cdot u_{\lambda'}\). Then \(Q_{\mu,\mu'} = \bigoplus V(\lambda'')\) where \(V(\lambda'')\) are physical lowest weight irreps.

Therefore we can define

\[
V(\mu) \otimes V(\mu') := \bigoplus_{\lambda''} V(\lambda'').
\]

If \(\frac{2\pi i}{m}\) is not integer, then the physical states have non–integral weights, and \(V(\mu) \otimes V(\mu')\) is zero. Now we have

**Theorem 2.1** If all weights \(\mu, \mu', \ldots\) involved are integral, then \(\otimes\) is associative, and \(V(\mu) \otimes V(\mu')\) is unitarizable w.r.t. \(U_q(\mathfrak{so}(2,3))\).

In particular for \(q = e^{2\pi i/m}\) with \(m\) even, none of the low–energy states have been discarded, and our definition is physically sensible.
3 BRST structure

To describe spin 1 particles, consider $V(\lambda) \otimes V_5$, where $V_5$ is the 5 dimensional representation of $U$ ("basic one–forms"), for $q$ as above. In the massive case, this is completely reducible as classically, see figure 1a). In the massless case however, the would–be irrep with lowest weight $\lambda - \alpha_2$ has an invariant subspace as in Theorem 1.1, which in fact combines with $V(\lambda + \alpha_3)$ into one reducible, but indecomposable representation, with structure as in figure 1 b).

The main observation now is that this "pure gauge" subspace is precisely the image of a "BRST" operator

$$Q := (v^{2m} - v^{-2m}),$$

acting on $V(\lambda) \otimes V_5$. $Q$ vanishes on any irrep, and $Q^2 = 0$ on $V(\lambda) \otimes V_5$. Thus the physical Hilbert space can be defined as $\text{Ker}(Q)/\text{Im}(Q)$. The same $Q$ works similarly for massless particles with higher spin.

References

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