Generalization of Kato’s decomposition

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Abstract

The Kato’s decomposition \[\text{Theorem 4}\] is generalized to semi-B-Fredholm operators.

1 Introduction and preliminaries

Let \(T \in L(X)\); where \(L(X)\) is the Banach algebra of bounded linear operators acting on an infinite dimensional complex Banach space \(X\). We denote by \(T^*\) the dual of \(T\), by \(\alpha(T)\) the dimension of the kernel \(\mathcal{N}(T)\) and by \(\beta(T)\) the codimension of the range \(\mathcal{R}(T)\). A subspace \(M\) of \(X\) is \(T\)-invariant if \(T(M) \subseteq M\) and in this case \(T_M\) means the restriction of \(T\) on \(M\). We say that \(T\) is completely reduced by a pair \((M, N)\) (\(M, N \in \text{Red}(T)\) for brevity) if \(M\) and \(N\) are closed \(T\)-invariant subspaces of \(X\) and \(X = M \oplus N\); here \(M \oplus N\) means that \(M \cap N = \{0\}\). Let \(n \in \mathbb{N}\), we denote by \(T_{[n]} := T_{\mathcal{R}(T^n)}\) (in particular, \(T_{[0]} = T\)). It is clear that \((\alpha(T_{[n]}))_n\) and \((\beta(T_{[n]}))_n\) are decreasing sequences. We call \(m_T := \inf\{n \in \mathbb{N} : \inf\{\alpha(T_{[n]}), \beta(T_{[n]})\} < \infty\}\) (with inf\(\emptyset = \infty\)) the essential degree of \(T\). Following \[7\] we say that \(T\) has finite essential ascent (resp., descent) if \(a_e(T) := \inf\{n \in \mathbb{N} : \alpha(T_{[n]}) < \infty\} < \infty\) (resp., \(d_e(T) := \inf\{n \in \mathbb{N} : \beta(T_{[n]}) < \infty\} < \infty\)). Note that \(m_T = \inf\{a_e(T), d_e(T)\}\). Moreover, if \(T\) has finite essential ascent and essential descent then \(m_T = a_e(T) = d_e(T)\). Operators with finite essential ascent or descent seem to have been first studied in \[4\], these operators played an important role in many papers, see for example \[1, 3, 5, 7\].

It is easily seen that the definition given in \[1\] of an upper semi-B-Fredholm (resp., lower semi-B-Fredholm) operator \(T\) is equivalent to the following definition given in \[7\]: \(a_e(T) < \infty\) and \(\mathcal{R}(T^{a_e(T)+1})\) is closed (resp., \(d_e(T) < \infty\) and \(\mathcal{R}(T^{d_e(T)})\) is closed). \(T\) is called semi-B-Fredholm (resp., \(B\)-Fredholm) if it is an upper or (resp., and) a lower semi-B-Fredholm. According to Corollary \[23\] below, if \(T\) is a semi-B-Fredholm then \(\mathcal{R}(T^{m_T})\) is closed and \(m_T = \min\{n \in \mathbb{N} : \mathcal{R}(T^n)\) is closed and \(\inf\{\alpha(T_{[n]}), \beta(T_{[n]})\} < \infty\}\}, and in this case the index of \(T\) is defined by \(\text{ind}(T) := \alpha(T_{[m_T]}) - \beta(T_{[m_T]})\). If \(T\) is an upper semi-B-Fredholm (resp., lower semi-B-Fredholm, semi-B-Fredholm, \(B\)-Fredholm) with essential degree \(m_T = 0\), then \(T\) is said to be an upper semi-Fredholm (resp., lower semi-Fredholm, semi-Fredholm, Fredholm) operator.

The degree of stable iteration \(d := \text{dis}(T)\) of \(T\) is defined as \(d = \inf \Delta(T)\); where

\[
\Delta(T) := \{m \in \mathbb{N} : \alpha(T_{[m]} = \alpha(T_{[r]}), \forall r \in \mathbb{N} r \geq m\}.
\]

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We say that $T$ is semi-regular if $\mathcal{R}(T)$ is closed and $d = 0$, and we say that $T$ is quasi-Fredholm if there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and $T^n$ is semi-regular. From [7] Lemma 12], it follows that if $T$ is a quasi-Fredholm operator then $d = \min\{n \in \mathbb{N} : \mathcal{R}(T^n)$ is closed and $T^n$ is semi-regular}. For more details about these definitions, see for examples [7, 10]. Note that every semi-B-Fredholm operator is quasi-Fredholm \[3, \text{Proposition 2.5}\]. We recall \[11, \text{p. 288}\] that if $\mathcal{T}$ composition) \[11, \text{Theorem 4}\] which establishes that if $\mathcal{T}$ is semi-regular then there exists $(M, N) = \min\{n \in \mathbb{N} : \mathcal{N}(T^n) \not\subseteq \mathcal{R}(T^n) \}$ which play a crucial role in several studies of this direction \[11, 14\]. Hereafter we call $v \equiv v(T)$ the degree of the semi-regularity of $T$.

In \[13, 14\], West simplified the proof of a special case of a Kato’s result (named Kato’s decomposition) \[11, \text{Theorem 4}\] which establishes that if $T$ is a semi-Fredholm operator which is not semi-regular then there exists $(M, N) \in \text{Red}(T)$ such that $\dim N < \infty$, $T_M$ is semi-regular (i.e. $\text{dis}(T_M) = m_T = 0$) and $T_N$ is nilpotent of degree $v$. To prove this result \[13, 14\], West used the notion of jumping ”jump(T)” of a semi-Fredholm operator $T$ introduced by himself, and the degree $v$ of the semi-regularity of $T$. In this paper we generalize the concept of jumping to semi-B-Fredholm operators. By using our new concept of jumping, the degree of stable iteration $d$ and the essential degree $m_T$ of an operator $T$, we generalize the Kato’s decomposition to semi-B-Fredholm operators. In other words, we prove in Theorem \[3, 3\] that if $T$ is semi-B-Fredholm then there exists $(M, N) \in \text{Red}(T)$ such that $\dim N < \infty$, $(T_M)_{m_T}$ is semi-regular (i.e. $\text{dis}(T_M) = m_T$) and $T_N$ is nilpotent of degree $d$. Among other things, some additional results for semi-B-Fredholm operators are given, and some known results in Kato’s theory are generalized (Theorem \[2, 10\], Corollary \[2, 10\] and Theorem \[2, 21\]).

2 Additional results for semi-B-Fredholm operators

The following proposition extend \[7, \text{Lemma 11}\].

**Proposition 2.1.** Let $T \in L(X)$ and let $n, m \in \mathbb{N}$. If $\mathcal{R}(T^n)$ is closed and $\mathcal{R}(T^m) + \mathcal{N}(T^n)$ is closed then $\mathcal{R}(T^{m+n})$ is closed.

**Proof.** Let $(x_k)_k \subseteq X$ be a sequence such that $T^{m+n}x_k \rightarrow y \in X$. Since $\mathcal{R}(T^n)$ is closed then there exists $z \in X$ such that $y = T^nz$ and the operator $T^n \in L(X/\mathcal{N}(T^n))$ induced by $T^n$ is bounded below. Hence $d(T^nx_k - z, \mathcal{N}(T^n)) \rightarrow 0$; where $d(a, A)$ is the distance between $a \in X$ and a subspace $A$ of $X$. As by hypothesis $\mathcal{R}(T^m) + \mathcal{N}(T^n)$ is closed then $z \in \mathcal{R}(T^m) + \mathcal{N}(T^n)$ and thus $y \in \mathcal{R}(T^n)$. Consequently $\mathcal{R}(T^{m+n})$ is closed.

**Proposition 2.2.** Let $T \in L(X)$. If there exists $n \in \mathbb{N}$ such that $\min\{\alpha(T^n), \beta(T^n)\} < \infty$, then $\mathcal{R}(T^n)$ is closed if and only if there exists an integer $m \geq n + 1$ such that $\mathcal{R}(T^m)$ is closed. Moreover, in this case $T$ is a semi-B-Fredholm.

**Proof.** Suppose that $\mathcal{R}(T^n)$ is closed, as $\min\{\alpha(T^n), \beta(T^n)\} < \infty$ then $T^n$ is semi-Fredholm operator, and so $T^{m-n}$ is semi-Fredholm for every integer $m \geq n$. Thus $\mathcal{R}(T^{m-n}) = \mathcal{R}(T^m)$ is closed in $\mathcal{R}(T^n)$, and then $\mathcal{R}(T^m)$ is closed. Conversely, we have $k_n(T) \leq \min\{\alpha(T^n), \beta(T^n)\} < \infty$; where $k_n(T) := \dim \mathcal{N}(T^n)/\mathcal{N}(T^{n+1})$. As it is already mentioned that $(\alpha(T^n))_k$ and $(\beta(T^n))_k$ are
Proof. Let $k_i(T) < \infty$ for every $i \geq n$. Let $m$ be an integer such that $m \geq n+1$ and $\mathcal{R}(T^m)$ is closed, then [M Lemma 12] implies that $\mathcal{R}(T^n)$ is closed. Thus $T_{[n]}$ is semi-Fredholm. \hfill \Box

For $T \in L(X)$, we denote by $m_T$ the essential degree of $T$ defined by $m_T := \inf \{ n \in \mathbb{N} : \inf \{ \alpha(T_{[n]}), \beta(T_{[n]}) \} < \infty \}$, with $\inf \emptyset = \infty$. Corollary 2.7 below shows also that

$$m_T = \begin{cases} a_e(T) & \text{if } a_e(T) < \infty \\ d_e(T) & \text{if } d_e(T) < \infty \\ \infty & \text{else} \end{cases}$$

where $a_e(T)$ and $d_e(T)$ are, respectively, the essential ascent and the essential descent of $T$.

**Corollary 2.3.** If $T \in L(X)$ is a semi-B-Fredholm operator then $\mathcal{R}(T^{m_T})$ is closed and $m_T = \min \{ n \in \mathbb{N} : \mathcal{R}(T^n) \text{ is closed and } T_{[n]} \text{ is semi-Fredholm} \}$.

**Proof.** Since $T$ is a semi-B-Fredholm then $\min \{ \alpha(T_{[m_T]}), \beta(T_{[m_T]}) \} < \infty$. On the other hand, there exists an integer $n \geq m_T$ such that $\mathcal{R}(T^n)$ is closed. Then by Proposition 2.2, $\mathcal{R}(T^{m_T})$ is closed and $T_{[m_T]}$ is a semi-Fredholm. Thus $m_T = \min \{ n : \mathcal{R}(T^n) \text{ is closed and } T_{[n]} \text{ is semi-Fredholm} \}$. \hfill \Box

According to the properties of the index of semi-B-Fredholm operators [M Proposition 2.1], we immediately obtain the next corollary.

**Corollary 2.4.** Let $T \in L(X)$. If there exist $n, m \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and the operator $T_{[n]}$ is an upper (resp., a lower) semi-Fredholm and $\beta(T_{[m]}) < \infty$ (resp., $\alpha(T_{[m]}) < \infty$), then $\mathcal{R}(T^m)$ is closed, $T_{[m]}$ and $T_{[n]}$ are Fredholm. Consequently, if $T$ is a B-Fredholm operator then $m_T = \min \{ n : \mathcal{R}(T^n) \text{ is closed and } T_{[n]} \text{ is Fredholm} \} = \min \{ n : \max \{ \alpha(T_{[n]}), \beta(T_{[n]}) \} < \infty \} = a_e(T) = d_e(T)$.

**Proposition 2.5.** Let $T \in L(X)$ and let $n$ be a strictly positive integer. Then $T$ is a semi-B-Fredholm operator if and only if $T^n$ is a semi-B-Fredholm operator. If this is the case then $\text{ind}(T^n) = n . \text{ind}(T)$.

**Proof.** From [M Proposition 4.2], it follows that $T$ is semi-B-Fredholm if and only if $T^n$ is semi-B-Fredholm. Suppose that $T$ is semi-B-Fredholm then by [M Proposition 2.1] $\mathcal{R}(T^{nm})$ is closed, $T_{[nm]}$ is semi-Fredholm and $\text{ind}(T) = \text{ind}(T_{[m]}) = \text{ind}(T_{[nm]})$: where $m = m_T$. As $\mathcal{R}((T^n)^m) = \mathcal{R}(T^{nm})$ is closed and $(T^n)_{[m]} = (T_{[nm]})^n$ is semi-Fredholm, then $\text{ind}(T^n) = \text{ind}((T^n)_{[m]}) = \text{ind}((T_{[nm]})^n) = n . \text{ind}(T_{[nm]}) = n . \text{ind}(T)$. \hfill \Box

**Proposition 2.6.** Let $T \in L(X)$ and let $n \in \mathbb{N}$.

(i) If $\alpha(T_{[n]}) < \infty$ and $\beta(T_{[n+1]}) < \infty$ then $\beta(T_{[n]}) < \infty$.

(ii) If $\alpha(T_{[n+1]}) < \infty$ and $\beta(T_{[n]}) < \infty$ then $\alpha(T_{[n]}) < \infty$.

**Proof.** Since $\alpha(T_{[n]}) = \alpha(T_{[n+1]}) + k_n(T) \leq \alpha(T_{[n+1]}) + \beta(T_{[n]})$ and $\beta(T_{[n]}) = \beta(T_{[n+1]}) + k_n(T) \leq \beta(T_{[n+1]}) + \alpha(T_{[n]})$, then the proof is complete. Remark that for $n = 0$, the assertion (ii) is also an immediate consequence of Corollary 2.3. \hfill \Box
As an immediate consequence of Proposition 2.6, we find [9] Lemma 11, p.204].

**Corollary 2.7.** If \( T \in L(X) \) and \( \max\{d_e(T), d_\alpha(T)\} < \infty \) then \( m_T = d_e(T) = d_\alpha(T) \).

We denote by \( F(X) \) (resp., \( SF_+(X), SF_-(X), SBF_+(X), SBF_-(X) \)) the class of Fredholm (resp., upper semi-Fredholm, lower semi-Fredholm, upper semi-B-Fredholm, lower semi-B-Fredholm) operators.

**Proposition 2.8.** Let \( T \in L(X) \). The following are equivalent.

(i) \( T \) is a Fredholm operator;

(ii) \( \alpha(T) < \infty \) and \( \beta(T_{[n]}) < \infty \) for some integer \( n \);

(iii) \( \beta(T) < \infty \) and \( \alpha(T_{[n]}) < \infty \) for some integer \( n \).

Consequently, \( F(X) = SF_+(X) \cap SBF_-(X) = SF_-(X) \cap SBF_+(X) \).

**Proof.** The implications “(i) \( \Rightarrow \) (ii) and “(i) \( \Rightarrow \) (iii)” are obvious.

(ii) \( \Rightarrow \) (i) We denote by \( p = d_e(T) \) and let us prove that \( p = 0 \). Suppose to the contrary, that’s \( p \geq 1 \). We take \( S = T_{[p-1]} \), then \( S_{[1]} = T_{[p]} \) and so \( \beta(S_{[1]}) < \infty \). On the other hand, since \( \alpha(T) < \infty \) then \( \alpha(S) < \infty \). It follows from the assertion (i) of Proposition 2.6 that \( \beta(S) < \infty \) and this is a contradiction.

(iii) \( \Rightarrow \) (i) It is an immediate consequence of Corollary 2.3 \( \square \)

Let \( n, m \in \mathbb{N} \). Note that the space \( X \times \mathcal{N}(T^n) \) equipped with the norm defined by \( \| (x, y) \| = \|x\| + \|y\| \) for every \( (x, y) \in X \times \mathcal{N}(T^n) \), is a Banach space. The map \( S_{n,m} : X \times \mathcal{N}(T^n) \to X \) defined by \( S_{n,m}(x, y) = T^m x + y \) is a bounded linear operator.

**Lemma 2.9.** For every \( T \in L(X) \), the following hold.

(i) If there exists \( n \in \mathbb{N} \) such that \( \beta(T_{[n]}) < \infty \) then \( \mathcal{R}(T^m) + \mathcal{N}(T^n) \) is closed, for all \( m \in \mathbb{N} \) and \( k \in \mathbb{N} \) such that \( k \geq n \).

(ii) \( \forall n \in \mathbb{N} \) and \( \forall m \in \mathbb{N}^* \), we have \( T \) is a Fredholm if and only if \( S_{n,m} \) is a Fredholm.

**Proof.** (i) Let \( n \in \mathbb{N} \) such that \( \beta(T_{[n]}) < \infty \) and \( m \in \mathbb{N} \). The case of \( m = 0 \) is trivial. Suppose that \( m \neq 0 \), we have \( \mathcal{R}(S_{n,m}) = \mathcal{R}(T^m) + \mathcal{N}(T^n) \). From [6, Lemma 3.2] we have \( \beta(S_{n,m}) = \sum_{i=n}^{n+m-1} \beta(T_{[i]}) \). As \( \beta(T_{[n]}) < \infty \) and \( \beta(T_{[k]}) \) is a decreasing sequence then \( \beta(S_{n,m}) \leq m \beta(T_{[n]}) < \infty \), thus \( \mathcal{R}(T^m) + \mathcal{N}(T^n) \) is closed. Let \( k \in \mathbb{N} \) such that \( k \geq n \) then \( \beta(T_{[k]}) < \infty \) and this is a contradiction.

(ii) Let \( n \in \mathbb{N} \) and \( m \in \mathbb{N}^* \). It is easily seen that \( \mathcal{N}(S_{n,m}) = \mathcal{N}(T^{m+n}) \). We conclude from Proposition 2.8 that

\[
\alpha(T) < \infty \quad \text{and} \quad \beta(T) < \infty \iff \alpha(T^{m+n}) < \infty \quad \text{and} \quad \beta(T_{[n]}) < \infty \quad \iff \alpha(S_{n,m}) < \infty \quad \text{and} \quad \beta(S_{n,m}) < \infty.
\]

Hence \( T \) is a Fredholm if and only if \( S_{n,m} \) is a Fredholm. \( \square \)
The next theorem shows that the condition “$R(T^n)$ is closed” cited in Corollary 2.4 can be omitted.

**Theorem 2.10.** Let $T \in L(X)$. If there exist $n, m \in \mathbb{N}$ such that $\alpha(T_{[n]}) < \infty$ and $\beta(T_{[m]}) < \infty$, then $R(T^{mT})$ is closed and $T_{[mT]}$ is a $B$-Fredholm operator.

**Proof.** Let $n, m \in \mathbb{N}$ such that $\alpha(T_{[n]}) < \infty$ and $\beta(T_{[m]}) < \infty$. From Corollary 2.7 we have $\max\{\alpha(T_{[mT]}), \beta(T_{[mT]})\} < \infty$. The case of $m_T = 0$ is clear. Suppose that $m_T \geq 1$, from Lemma 3.1, Lemma 3.2 we have $\frac{N(T^{m_T+k})}{N(T^{m_T})} \cong N(T^k) \cap R(T^{m_T})$ and $\frac{R(T^{m_T})}{R(T^{m_T+k})} \cong \frac{X}{R(T^k) + N(T^{m_T})}$, $\forall k \in \mathbb{N}$.

Hence $\dim(N(T^{m_T}) \cap R(T^{m_T})) = \sum_{k=m_T}^{2m_T-1} \alpha(T_{[k]})$ and $\text{codim}(R(T^{m_T}) + N(T^{m_T})) = \sum_{k=m_T}^{2m_T-1} \beta(T_{[k]})$.

Since $(\alpha(T_{[n]}))_n$ and $(\beta(T_{[n]}))_n$ are decreasing sequences, then $\dim(N(T^{m_T}) \cap R(T^{m_T})) < \infty$ and $\text{codim}(R(T^{m_T}) + N(T^{m_T})) < \infty$. So $N(T^{m_T}) \cap R(T^{m_T})$ is closed and by Lemma 2.9, $R(T^{m_T}) + N(T^{m_T})$ is closed. Since $R(T^{m_T})$ is a paracomplete space, then it follows from Neubauer Lemma 10 Proposition 2.1.1] that $R(T^{m_T})$ is closed. Consequently, $T$ is a $B$-Fredholm operator.

From the proof of Theorem 2.10 it follows that $T \in L(X)$ is $B$-Fredholm if and only if $\dim(N(T^n) \cap R(T^n)) < \infty$ and $\text{codim}(N(T^n) + R(T^n)) < \infty$ for some integer $n \in \mathbb{N}$.

**Definition 2.11.** We say that $T \in L(X)$ is quasi upper semi-$B$-Fredholm (resp., quasi lower semi-$B$-Fredholm) operator if there exists $(M, N) \in \text{Red}(T)$ such that $T_M$ is upper semi-Fredholm (resp., lower semi-Fredholm) and $T_N$ is nilpotent. If $T$ is quasi upper semi-$B$-Fredholm or (resp., and) quasi lower semi-$B$-Fredholm then $T$ is called quasi semi-$B$-Fredholm (resp., quasi $B$-Fredholm).

Every nilpotent operator and every semi-Fredholm operator are quasi semi-$B$-Fredholm. Moreover, every quasi semi-$B$-Fredholm operator is pseudo semi-$B$-Fredholm. For more details about pseudo semi-$B$-Fredholm operators, we refer the reader to [12].

**Proposition 2.12.** If $T \in L(X)$ is quasi upper semi-$B$-Fredholm (resp., quasi lower semi-$B$-Fredholm) operator then $T$ is an upper semi-$B$-Fredholm (resp., a lower semi-$B$-Fredholm).

**Proof.** Let $(M, N) \in \text{Red}(T)$ such that $T_M$ is semi-Fredholm and $T_N$ is nilpotent. From Corollary 3.7 below there exists $(A, B) \in \text{Red}(T_M)$ such that $T_A$ is semi-Fredholm and semi-regular and $T_B$ is nilpotent. So $(A, B \oplus N) \in \text{Red}(T)$ and $T_{B \oplus N}$ is nilpotent of degree $d$. Hence $R(T^d) = R(T'_{A})$ and $N(T_A) = N(T_{[d]})$ and $T(A \oplus (B \oplus N) = N(T^d) + R(T)$. Therefore $\alpha(T_A) = \alpha(T_{[d]})$ and $\beta(T_A) = \beta(T_{[d]})$. So $R(T^d)$ is closed and $T_{[d]}$ is semi-Fredholm. Thus $T$ is semi-$B$-Fredholm.

The converse of Proposition 2.12 is true if $X$ is a Hilbert space, see [3] Theorem 2.6].

**Open question:** Does there exist a semi-$B$-Fredholm operator $T$ acting on a Banach space $X$ which is not quasi semi-$B$-Fredholm?

We recall that $T \in L(X)$ is Drazin invertible if $T$ is a direct sum of an invertible operator and a nilpotent operator, and $T$ is meromorphic if $T - \lambda I$ is Drazin invertible for every $\lambda \in \mathbb{C} \setminus \{0\}$. Note that every nilpotent operator is meromorphic.
Definition 2.13. \[ T \in L(X) \] is generalized Drazin-meromorphic semi-Fredholm operator if \( T = T_1 \oplus T_2 \); where \( T_1 \) is semi-Fredholm and \( T_2 \) is meromorphic.

Hereafter, we denote by \( \sigma_{se}(T) \), \( \sigma_{usf}(T) \), \( \sigma_{lsf}(T) \), \( \sigma_{d}(T) \), \( \sigma_{sf}(T) \), \( \sigma_{sbf}(T) \) and \( \sigma_{d}(T) \) respectively, the semi-regular spectrum, the upper semi-Fredholm spectrum, the lower semi-Fredholm spectrum, the semi-Fredholm spectrum, the semi-regular spectrum, the upper semi-Fredholm spectrum, and the Drazin invertible spectrum of \( T \). We also denote by \( A^C \) the complementary of a given complex subset \( A \).

Lemma 2.14. Let \( T \in L(X) \) such that there exist \( (M, N), (M', N') \in \text{Red}(T) \) with \( T_M \) and \( T_{M'} \) are semi-Fredholm, \( T_N \) and \( T_{N'} \) are meromorphic. Then \( \text{ind}(T_M) = \text{ind}(T_{M'}) \).

Proof. Since \( T_M \) and \( T_{M'} \) are semi-Fredholm operators then from punctured neighborhood theorem for semi-Fredholm operators, there exists \( \epsilon > 0 \) such that \( B(0, \epsilon) \subset \sigma_{sf}(T_M)^C \cap \sigma_{sf}(T_{M'})^C \), \( \text{ind}(T_M - \lambda I) = \text{ind}(T_{M'}) \) for every \( \lambda \in B(0, \epsilon) \). As \( T_N \) and \( T_{N'} \) are meromorphic then \( B_0 := B(0, \epsilon) \setminus \{0\} \subset \sigma_{sf}(T_M)^C \cap \sigma_{sf}(T_{M'})^C \cap \sigma_d(T_N)^C \cap \sigma_d(T_{N'})^C \subset \sigma_{sbf}(T)^C \).

Let \( \lambda \in B_0 \), then \( T - \lambda I \) is semi-B-Fredholm and \( \text{ind}(T - \lambda I) = \text{ind}(T_{M}) + \text{ind}(T_{N'} - \lambda I) = \text{ind}(T_{M'}) + \text{ind}(T_{N'} - \lambda I) \). Thus \( \text{ind}(T_M) = \text{ind}(T_{M'}) \), since the index of a Drazin invertible operator is equal to zero.

The previous lemma gives meaning to the following definition.

Definition 2.15. Let \( T = T_1 \oplus T_2 \) be a generalized Drazin-meromorphic semi-Fredholm operator; where \( T_1 \) is semi-Fredholm and \( T_2 \) is meromorphic. Then we define the index of \( T \) as the usual index of the semi-Fredholm operator \( T_1 \). In particular, if \( T \) is semi-Fredholm then we find its usual index.

In the following corollary, we extend a particular case of [2] Theorem 2.4] to the general case of Banach spaces.

Corollary 2.16. \( T \in L(X) \) is quasi-B-Fredholm if and only if \( T \) is B-Fredholm. Moreover, if this is the case then the index of \( T \) as a quasi B-Fredholm coincides with its usual index as a B-Fredholm.

Proof. From [8] Theorem 7], we have \( T \) is B-Fredholm if and only if \( T = T_1 \oplus T_2 \); where \( T_1 \) is Fredholm and \( T_2 \) is nilpotent. So if \( T \) is B-Fredholm then \( T \) is quasi-B-Fredholm. Suppose that \( T \) is quasi-B-Fredholm, then there exist \( (M, N), (M', N') \in \text{Red}(T) \) such that \( T_M \) is an upper semi-Fredholm, \( T_{M'} \) is a lower semi-Fredholm, \( T_N \) and \( T_{N'} \) are nilpotent. From Lemma 2.14 we have \( \text{ind}(T_M) = \text{ind}(T_{M'}) \), and so \( (\alpha(T_M) + \beta(T_{M'})) - \alpha(T_{M'}) = \beta(T_M) \geq 0 \). Thus \( T_M \) and \( T_{M'} \) are Fredholm operators. Again by [8] Theorem 7] we deduce that \( T \) is B-Fredholm. Suppose that \( T = T_1 \oplus T_2 \) is a B-Fredholm operator. Since \( T_1 \) is Fredholm, then [5] Corollary 4.7] implies that there exists \( \epsilon > 0 \) such that \( B_0 := B(0, \epsilon) \setminus \{0\} \subset (\alpha(T))^C \), \( \text{ind}(T_{M}) = \text{ind}(T_{N'} + j) \) for all \( \lambda \in B_0 \). Let \( \lambda \in B_0 \), as \( T_2 \) is nilpotent then \( T - \lambda I = (T_1 - \lambda I) \oplus (T_2 - \lambda I) \) is a Fredholm operator and \( \text{ind}(T - \lambda I) = \text{ind}(T_1 - \lambda I) \). Consequently,
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\[ \text{ind}(T) = \text{ind}(T_1) = \text{ind}(T_{1^{\text{red}}}) \]. So the index of \( T \) as a quasi B-Fredholm coincides with its usual index as a B-Fredholm.

**Definition 2.17.** \([10][11]\) \( T \in L(X) \) is said to be of Kato-type of degree \( d \) if there exists \((M,N) \in \text{Red}(T)\) such that \( T_m \) is semi-regular and \( T_N \) is nilpotent of degree \( d \).

**Remark 2.18.** The degree \( d \) of a Kato-type operator \( T \in L(X) \) is unique (see [10] Remark p.205).

It is easy to see that if \( T \in L(X) \) is of Kato-type of degree \( d \) then \( T \) is quasi-Fredholm of degree \( \text{dis}(T) \), \( \alpha(T_M) = \alpha(T_{[n]}) \) and \( \beta(T_M) = \beta(T_{[n]}) \), for every \( n \geq d \) and \((M,N) \in \text{Red}(T)\) such that \( T_M \) is semi-regular and \( T_N \) is nilpotent. Recall that the reduced minimal modulus \( \gamma(T) \) of an operator \( T \) is defined by \( \gamma(T) := \inf_{x \in N(T)} \frac{\|Tx\|}{d(x,N(T))} \). The following proposition gives some characterizations of quasi semi-B-Fredholm operators.

**Proposition 2.19.** Let \( T \in L(X) \). The following statements are equivalents.

(i) \( T \) is quasi semi-B-Fredholm [resp., quasi upper semi-B-Fredholm, quasi lower semi-B-Fredholm, quasi B-Fredholm];

(ii) \( T \) is of Kato-type of degree \( d \) and \( \min \{\alpha(T_{[d]}), \beta(T_{[d]})\} < \infty \) [resp., \( T \) is of Kato-type of degree \( d \) and \( \alpha(T_{[d]}) < \infty \), \( T \) is of Kato-type of degree \( d \) and \( \beta(T_{[d]}) < \infty \), \( \max \{\alpha(T_{[n]}), \beta(T_{[n]}))\} < \infty \) for an integer \( n \)];

(iii) \( T \) is of Kato-type and \( 0 \notin \text{acc} \sigma_{\text{usf}}(T) \) [resp., \( T \) is of Kato-type and \( 0 \notin \text{acc} \sigma_{\text{usf}}(T) \), \( T \) is of Kato-type and \( 0 \notin \text{acc} \sigma_{\text{lsf}}(T) \), \( T \) is of Kato-type and \( 0 \notin \text{acc} \sigma_{\text{lsf}}(T) \)].

**Proof.** Remark firstly from Theorem 2.10 and Corollary 2.16 that if \( \max \{\alpha(T_{[n]}), \beta(T_{[n]})\} < \infty \) for an integer \( n \), then \( T \) is quasi B-Fredholm.

(i) \( \iff \) (ii) Suppose that \( T \) is quasi semi-B-Fredholm, then there exists \((A,B) \in \text{Red}(T)\) such that \( T_A \) is semi-Fredholm and \( T_B \) is nilpotent. According to Corollary 3.7, there exists a pair \((M,N) \in \text{Red}(T)\) such that \( T_M \) is semi-Fredholm which is semi-regular and \( T_N \) is nilpotent of degree \( d \). Thus \( T \) is of Kato-type of degree \( d \) and \( \min \{\alpha(T_{[d]}), \beta(T_{[d]})\} = \min \{\alpha(T_M), \beta(T_M)\} < \infty \). The converse is obvious. The other equivalence cases go similarly.

(iii) \( \iff \) (i) Assume that \( T \) is of Kato-type and \( 0 \notin \text{acc} \sigma_{\text{usf}}(T) \); where \( \text{acc}A \) is the accumulation points of a given complex subset \( A \). Then there exists \((M,N) \in \text{Red}(T)\) such that \( T_M \) is semi-regular and \( T_N \) is nilpotent of degree \( d \). From the punctured neighborhood theorem for semi-regular operators, \( B_0 := B(0, \epsilon) \setminus \{0\} \subset (\sigma_{\text{se}}(T) \cup \sigma_{\text{usf}}(T))^C \); for some \( \epsilon \leq \gamma(T_M) \). On the other hand, it is easily seen that \( \alpha(T_M) = \alpha(T_{[d]}) = \alpha(T - \lambda I) \) and \( \beta(T_M) = \beta(T_{[d]}) = \beta(T - \lambda I) \), thus \( \min \{\alpha(T_M), \beta(T_M)\} = \min \{\alpha(T - \lambda I), \beta(T - \lambda I)\} < \infty \) for every \( \lambda \in B_0 \). Thus \( T_M \) is semi-Fredholm and so \( T \) is quasi semi-B-Fredholm. The converse is an immediate consequence of the punctured neighborhood theorem for semi-Fredholm operators. The other equivalence cases go similarly.

In the sequel we denote by \( \sigma_{\text{usf}}(T), \sigma_{\text{quusf}}(T), \sigma_{\text{qlusf}}(T) \) and \( \sigma_{\text{usf}}(T) \) respectively, the quasi semi-B-Fredholm spectrum, the quasi upper semi-B-Fredholm spectrum, the quasi lower semi-B-
Fredholm spectrum and the quasi B-Fredholm spectrum of $T \in L(X)$. The second point of the next corollary is a consequence of the classical Heine-Borel theorem.

**Corollary 2.20.** For every $T \in L(X)$ we have

(i) $\sigma_{qbf}(T)$, $\sigma_{qubf}(T)$, $\sigma_{qbf}(T)$ and $\sigma_{qbf}(T)$ are compact.

(ii) If $\Omega$ is a component of $(\sigma_{qubf}(T))^C$ or $(\sigma_{qbf}(T))^C$, then the index $\text{ind}(T - \lambda I)$ is constant as $\lambda$ ranges over $\Omega$.

In [10] Theorem 3.2.2, Labrousse showed that the degree of a Kato-type operator $T$ acting on a Hilbert space is exactly its degree of stable index $\text{dis}(T)$. In the next theorem we extend this result to the general case of Banach space Kato-type operators.

**Theorem 2.21.** The degree of a Kato-type operator $T \in L(X)$ is equal to $\text{dis}(T)$.

**Proof.** Let $(M, N) \in \text{Red}(T)$ such that $T_M$ is semi-regular and $T_N$ is nilpotent of degree $d$. Let $m \geq 1$, since $T_M$ is semi-regular then $R(T) + N(T^m) = R(T_M) + N(T^m) + R(T_N) + N(T^N) = R(T_M) \oplus N = R(T_M) + N(T^M) + R(T_N) + N(T_N^T) = R(T) + N(T^d)$. Thus $d \geq \text{dis}(T)$. On the other hand, $T_N$ is a quasi B-Fredholm operator then there exists according to [8] Theorem 7 a pair $(A, B) \in \text{Red}(T_N)$ such that $T_A$ is semi-regular and $T_B^{\text{dis}(T_N)} = 0$. The same reasoning as above shows that $T_B$ is nilpotent of degree $\text{dis}(T_N)$. It is easily seen that $(M \oplus A, B) \in \text{Red}(T)$ and $T_M \oplus A$ is semi-regular. Since the degree of a Kato-type operator is unique then $d = \text{dis}(T_N) \leq \text{dis}(T) = \max \{\text{dis}(T_M), \text{dis}(T_N)\}$. Hence $d = \text{dis}(T)$. \qed

**Corollary 2.22.** The degree of a nilpotent operator $T \in L(X)$ is exactly $\text{dis}(T)$.

Let $M$ be a subset of $X$ and $N$ a subset of $X^*$. The annihilator of $M$ and the pre-annihilator of $N$ are the closed subspaces defined respectively, by

$$M^\perp := \{f \in X^* : f(x) = 0 \text{ for every } x \in M\},$$

and

$$N^\perp := \{x \in X : f(x) = 0 \text{ for every } f \in N\}.$$
From the proof of Proposition 2.23 it follows that if $T \in L(X)$ is quasi-Fredholm then $T^*$ is also quasi-Fredholm and $\text{dis}(T) = \text{dis}(T^*)$.

3 A relation between $m_T$, $v(T)$ and $\text{dis}(T)$

Let $T \in L(X)$. As mentioned in the introduction, the degree of stable iteration $d := \text{dis}(T)$ of $T$ is defined as $d = \inf \Delta(T)$; where

$$\Delta(T) := \{m \in \mathbb{N} : \mathcal{N}(T[m]) = \mathcal{N}(T[r]), \forall r \in \mathbb{N} r \geq m\},$$

with $\inf \emptyset = \infty$. Let $r \in \mathbb{N}$, then $r \geq d$ if and only if $\mathcal{R}(T) + \mathcal{N}(T^m) = \mathcal{R}(T) + \mathcal{N}(T^r), \forall m \in \mathbb{N}$ such that $m \geq r$, see [3, Theorem 3.1].

If $T$ is an upper semi-B-Fredholm operator then from [3, Proposition 2.5], $T$ is quasi-Fredholm of degree $d \geq m_T$. From the punctured neighborhood theorem for semi-B-Fredholm operators [5, Theorem 4.7], there exists $\epsilon := \gamma(T_{\mathcal{R}(T^*)}) > 0$ such that $\lambda \rightarrow \alpha(T - \lambda I)$ is constant on $B(0, \epsilon) \setminus \{0\}$, $\alpha(T - \lambda I) \leq \alpha(T_{m_T})$ and $\text{ind}(T_{m_T}) = \text{ind}(T - \lambda I)$ for every $\lambda \in B(0, \epsilon)$. Analogously, if $T$ is a lower semi-B-Fredholm operator then $\lambda \rightarrow \beta(T - \lambda I)$ is constant on $B(0, \epsilon) \setminus \{0\}$, $\beta(T - \lambda I) \leq \beta(T_{m_T})$ and $\text{ind}(T_{m_T}) = \text{ind}(T - \lambda I)$ for every $\lambda \in B(0, \epsilon)$.

**Definition 3.1.** Let $T \in L(X)$ be a semi-B-Fredholm operator. The jump of $T$ is defined by

$$\text{jump}(T) = \left\{ \begin{array}{ll} \alpha(T_{m_T}) - \alpha(T - \lambda I) & \text{if } T \text{ is an upper semi-B-Fredholm operator} \\ \beta(T_{m_T}) - \beta(T - \lambda I) & \text{if } T \text{ is a lower semi-B-Fredholm operator} \end{array} \right.$$ 

where $\lambda \in B(0, \epsilon) \setminus \{0\}$ be arbitrary and $\epsilon = \gamma(T_{\mathcal{R}(T^*)}) > 0$.

The definition of the jump($T$) of a semi-B-Fredholm operator is well defined since $\text{ind}(T_{m_T}) = \text{ind}(T - \lambda I)$. Furthermore, if $T$ is semi-Fredholm we find its jump given by T. T. West in [14].

The next proposition extends [14, Proposition 3], which establishes that if $T \in L(X)$ is semi-Fredholm then $T$ is semi-regular if and only if $\text{jump}(T) = 0$. Note that here $m_T = 0$.

**Proposition 3.2.** Let $T \in L(X)$ be a semi-B-Fredholm operator. Then $T_{m_T}$ is semi-regular if and only if $\text{jump}(T) = 0$.

**Proof.** Let $\lambda \in B(0, \epsilon) \setminus \{0\}$. From [5, Theorem 4.7], $\alpha(T - \lambda I) = \dim \frac{\mathcal{N}(T[d])}{\mathcal{N}(T)}$. Moreover, by [6, Lemma 3.1] we have $\mathcal{N}(T[d]) = \mathcal{N}(T_{m_T})$ and so $\alpha(T - \lambda I) = \alpha(T[d])$. Therefore $\text{jump}(T) = 0$ if and only if $\mathcal{N}(T[d]) = \mathcal{N}(T_{m_T})$ if and only if $d = m_T$. Since $T[d]$ is semi-regular then we get the desired result. 

**Theorem 3.3.** If $T \in L(X)$ is a semi-B-Fredholm operator then there exists $(M, N) \in \text{Red}(T)$ such that $\dim N < \infty$, $\text{jump}(T_M) = 0$ (i.e. $\text{dis}(T_M) = m_T$) and $T_N$ is nilpotent of degree $d$. Moreover $\text{ind}(T) = \text{ind}(T_M)$.

Before giving the proof of this theorem we need the following lemmas.
Lemma 3.4. Let $T \in L(X)$ be an operator with the degree of stable iteration $0 < d < \infty$. Then for every $y \in \mathcal{N}(T^d) \setminus (\mathcal{N}(T^{d-1}) + \mathcal{R}(T))$ we have $T_i y \in \mathcal{N}(T^{d-i}) \setminus (\mathcal{N}(T^{d-i-1}) + \mathcal{R}(T^{i+1}))$, $i = 0, \ldots, d-1$, and $\{T^i y\}_{i=1}^{d-1}$ are linearly independent modulo $\mathcal{R}(T^d)$.

Proof. Since $0 < d < \infty$ then $\mathcal{N}(T^d) \not\subseteq \mathcal{N}(T^{d-1}) + \mathcal{R}(T)$. Let $y \in \mathcal{N}(T^d) \setminus (\mathcal{N}(T^{d-1}) + \mathcal{R}(T))$ and $i \in \{0, \ldots, d-1\}$, then $T_i y \in \mathcal{N}(T^{d-i})$. Suppose that $T_i y \in \mathcal{N}(T^{d-i-1}) + \mathcal{R}(T^{i+1})$, then $y \in T^{-i}(\mathcal{N}(T^{d-i-1}) + \mathcal{R}(T^{i+1})) = \mathcal{N}(T^{d-i}) + \mathcal{R}(T)$ and this is a contradiction. Thus $T_i y \in \mathcal{N}(T^{d-i}) \setminus (\mathcal{N}(T^{d-i-1}) + \mathcal{R}(T^{i+1})).$

Let $(\alpha_i)_{i=0}^{d-1} \subseteq C$ such that $\sum_{i=0}^{d-1} \alpha_i T^i y \in \mathcal{R}(T^d)$. Then $\alpha_0 T^{d-1} y = T^{d-1} \sum_{i=0}^{d-1} \alpha_i T^i y \in \mathcal{R}(T^{2d-1})$. Thus $\alpha_0 y \in T^{-(d-1)}(\mathcal{R}(T^{2d-1})) = \mathcal{N}(T^{d-1}) + \mathcal{R}(T^{d}) \subseteq \mathcal{N}(T^{d-1}) + \mathcal{R}(T)$, and this implies that $\alpha_0 = 0$. If $d \geq 2$, then by similar arguments we get $\alpha_j = 0$, for $j \in \{0, \ldots, i\}$; where $i = 0, \ldots, d-2$. Therefore $\alpha_{i+1} y \in T^{-(d-1)}(\mathcal{R}(T^{2d-1})) = \mathcal{N}(T^{d-1}) + \mathcal{R}(T^{d-i-1}) \subseteq \mathcal{N}(T^{d-1}) + \mathcal{R}(T)$, since $d - i \geq 2$. Thus $\alpha_{i+1} = 0$ and so $y, T_1 y, \ldots, T^{d-1} y$ are linearly independent modulo $\mathcal{R}(T^d)$. □

Lemma 3.5. Let $T \in L(X)$ be a quasi-Fredholm operator of degree $d > 0$. Then for every $y \in \mathcal{N}(T^d) \setminus (\mathcal{N}(T^{d-1}) + \mathcal{R}(T))$ there exists $f \in X^*$ satisfies the following
(a) $T^i f(T^{d-j-1} y) = \delta_{ij}$, for every $0 \leq i, j \leq d - 1$.
(b) $T^i f \in \mathcal{N}(T^{(d-i)*}) \setminus (\mathcal{N}(T^{(d-i-1)*}) + \mathcal{R}(T^{(i+1)*}))$, for every $0 \leq i \leq d - 1$.
(c) The cascade $\{T^i f\}_{i=0}^{d-1}$ is linearly independent modulo $\mathcal{R}(T^d)$.

Such cascade $\{T^i f\}_{i=0}^{d-1}$ called adjoint cascade of $\{T^i y\}_{i=0}^{d-1}$.

Proof. We denote by $Y = \langle y, \ldots, T^{d-1} y \rangle$ the subspace spanned by the cascade $\{T^i y\}_{i=0}^{d-1}$. Then Lemma 3.4 implies that $Y \cap \mathcal{R}(T^d) = \{0\}$. As $\mathcal{R}(T^d)$ is closed, according to Hahn-Banach theorem there exists $f \in \mathcal{R}(T^d)^\perp = \mathcal{N}(T^d)$ such that $f(T^{d-i} y) = \delta_{i1}$, for every $0 \leq i \leq d - 1$. It is not difficult to see that $T^i f(T^{d-j-1} y) = \delta_{ij}$, for every $0 \leq i, j \leq d - 1$. Moreover, $f \in \mathcal{N}(T^{d*}) \setminus (\mathcal{N}(T^{d-1*}) + \mathcal{R}(T^*)$). Indeed, if $f \in \mathcal{N}(T^{d-1*}) + \mathcal{R}(T^*)$ then there exists $(g, h) \in \mathcal{N}(T^{d-1*}) \times X^*$ such that $f = g + T^* h$, thus $f(T^{d-1} y) = g(T^{d-1} y) + T^* h(T^{d-1} y) = 0$, and this is impossible. On the other hand, since $0 < \text{dis}(T^*) = d < \infty$ (see Proposition 2.2.3) we conclude from Lemma 3.4 that the linear form $f$ satisfies the points (b) and (c). □

Let $(g, x) \in X^* \times X$ be non-zero. We denote by $g \otimes x$ the rank-one operator defined by $(g \otimes x) z = g(z) x$ for all $z \in X$. Note that every rank-one operator in $L(X)$ can be written in this form.

Lemma 3.6. Let $T \in L(X)$ be a quasi-Fredholm operator of degree $d > 0$, $y \in \mathcal{N}(T^d) \setminus (\mathcal{N}(T^{d-1}) + \mathcal{R}(T))$ and $\{T^i y\}_{i=0}^{d-1}$ be an adjoint cascade of $\{T^i y\}_{i=0}^{d-1}$. Then
(a) $P := \sum_{i=0}^{d-1} T^i f \otimes T^{d-i-1} y$ is a finite-rank projection onto $Y$ which commutes with $T$; where $Y := \langle y, \ldots, T^{d-1} y \rangle$.
(b) $\mathcal{N}(P) = \{x \in X : \langle x, \ldots, T^{d-1} x \rangle \subset \mathcal{N}(f)\}$ is a closed $T$-invariant subspace of $X$.
(c) $T_Y$ is nilpotent of degree $d$ and $\text{jump}(T_Y) = 1$. 

Proof. (a) Let $P_i = T^{i\ast} f \otimes T^{d-i-1} y$ be the rank-one operator; it is easily seen that $P_i \in L(X)$, $R(P_i) = \langle T^{d-i-1} y \rangle$ and $P_i P_j = \delta_{ij} P_i$, for every $0 \leq i, j \leq d - 1$. Thus $P = \sum_{i=0}^{d-1} P_i \in L(X)$ is a projection and $R(P) = Y$. Moreover, $TP_i = T(T^{i\ast} f \otimes T^{d-i-1} y)$ and $PT = T(i+1)\ast f \otimes T^{d-i-1} y$, $i = 0, \ldots, d - 1$. Since $y \in \mathcal{N}(T^d)$ and $f \in \mathcal{N}(T^{d_*})$ then $TP_0 = P_{d-1} T = 0$ and hence $TP = PT$.

(b) It is easily seen that $\mathcal{N}(P) = \{x \in X : \{x, \ldots, T^{d-1} x\} \subset \mathcal{N}(f)\}$. Moreover, it is clear that $\mathcal{N}(P)$ is closed. Let $x \in \mathcal{N}(P)$, as $f \in \mathcal{N}(T^{d_*})$ then $\{T x, \ldots, T^d x\} \subset \mathcal{N}(f)$ and consequently $T(\mathcal{N}(P)) \subset \mathcal{N}(P)$.

(c) We have $R(T^{d-1}_Y) = \mathcal{N}(T_Y) = \langle T^{d-1} y \rangle$ and $R(T^d_Y) = \{0\}$. So $T_Y$ is nilpotent of degree $d$ with $\text{jump}(T_Y) = 1$. 

Now we give the proof of Theorem 3.3.

Proof of Theorem 3.3. Since $T$ is semi-Fredholm then $T$ is quasi-Fredholm of degree $d$. We distinguish two cases:

Case 1: $d = m_T$. In this case if $d = 0$ then the theorem is true with $M = X$ and $N = \{0\}$. If $d > 0$ then let $f$ be the linear form defined in Lemma 3.6 and let $P$ be the projection defined in Lemma 3.6. Hence the proof is complete by taking $M = \mathcal{N}(P) = \bigcap_{i=0}^{d-1} \mathcal{N}(T^{i\ast} f)$ and $N = R(P)$.

Indeed, $m_T = \max\{m_{TM}, m_{TN}\} = m_{TM}$ and $d = \max\{\text{dis}(T_M), \text{dis}(T_N)\} \geq \text{dis}(T_M) \geq m_T$. Thus $d = \text{dis}(T_M) = m_T$ and $\text{jump}(T_M) = 0$.

Case 2: $d > m_T$. Let $P_i \in L(X)$ be the projection defined in Lemma 3.6. If we take $M_i = \mathcal{N}(P_i)$ and $M_1 = \mathcal{N}(P_1)$ and $M_1 = \mathcal{R}(P_1) = \langle y, \ldots, T^{d-1} y \rangle$; where $y \in \mathcal{N}(T^d) \setminus (\mathcal{N}(T^{d-1}) + \mathcal{R}(T))$, then $(M_i, N_i) \in \text{Red}(T)$, $\dim N_i = d < \infty$ and $T_{N_i}$ is a nilpotent operator of degree $d$. Moreover $R(T_{N_i}) = \langle T_{N_i} \rangle = \langle T^{d-1} y, \ldots, T^{d-i} y \rangle$, $i = 0, \ldots, d - 1$; where $\mathcal{N}(T_{N_i}) = \{0\}$. Thus $\alpha(T_{m_T}) = \alpha((T_{m_T})_{|m_T}) + 1$ and $\beta(T_{m_T}) = \beta((T_{m_T})_{|m_T}) + 1$. So $\text{jump}(T_{m_T}) = \text{jump}(T) = 1$. In the sequel we denote by $k := \text{jump}(T) > 0$. Continuing this process $k$ times, we get from Lemma 3.6 a sequence of projections $(P_i)_{i=1}^k$ with $P_{i+1} \in L(M_i)$, $i = 0, \ldots, k - 1$; where $(M_0, N_0) = (X, \{0\})$ and $(M_i, N_i) = (\mathcal{N}(P_i), \mathcal{R}(P_i))$, $i = 1, \ldots, k$. Then $(M_{i+1}, N_{i+1}) \in \text{Red}(T_M)$, $d_i := \dim N_{i+1} = \text{dis}(T_M)$ and $T_{N_{i+1}}$ is nilpotent of degree $d_i$, $i = 0, \ldots, k - 1$, $\text{jump}(T_{M_i}) = \text{jump}(T) - i$, $i = 0, \ldots, k$. Let $M := M_k$ and $N := N_1 \oplus \ldots \oplus N_k$. Then $(M, N) \in \text{Red}(T)$, $\alpha(T_{|d}) = \alpha((T_M)_{|m_T})$, $\beta(T_{|d}) = \beta((T_M)_{|m_T})$, $\text{jump}(T_M) = 0$, $\text{jump}(T_N) = k$, $d + (k - 1)m_T \leq \dim N = \sum_{i=0}^{k-1} d_i \leq kd$ and $T_N$ is nilpotent of degree $d$. Thus $\text{ind}(T) = \text{ind}(T_M)$. 

From Theorem 3.3, we immediately obtain the following corollary which is the main result of [14] (see also [13], Proposition 2.5). And it is also a special case of Kato’s decomposition theorem [11], Theorem 4.

Corollary 3.7. If $T \in L(X)$ is a semi-Fredholm operator then there exists $(M, N) \in \text{Red}(T)$ such that $\dim N < \infty$, $T_M$ is semi-regular operator and $T_N$ is nilpotent of degree $\text{dis}(T)$. 


The next theorem proves that the degree of stable iteration \( d = \text{dis}(T) \) of a semi-Fredholm operator \( T \in L(X) \) is equal to

\[
d = \begin{cases} 
0 & \text{if } T \text{ is semi-regular} \\
v(T) & \text{else}
\end{cases}
\]

where \( v(T) := \inf\{n \in \mathbb{N} : \mathcal{N}(T) \not\subset \mathcal{R}(T^n)\} \) is the degree of the semi-regularity of \( T \). So if \( d > 0 \) then \( \mathcal{N}(T) \subset \mathcal{R}(T^n) \) and \( \mathcal{N}(T) \not\subset \mathcal{R}(T^d) \) for every \( n < d \). Note that if \( T \) is semi-regular then \( v(T) = \infty \) and \( \text{dis}(T) = 0 \).

**Theorem 3.8.** If \( T \in L(X) \) is semi-Fredholm which is not semi-regular then \( \text{dis}(T) = v(T) \).

**Proof.** It is an immediate consequence of [11, Theorem 4] (see also [14, p.609]) and Theorem 2.21. (We also can use [11, Theorem 4], Corollary 3.7 and Remark 2.18).

**Corollary 3.9.** If \( T \in L(X) \) is semi-Fredholm and \( d = \text{dis}(T) \) then

\[
\alpha(T^{[n]}) = \begin{cases} 
\alpha(T^{[d]}) & \text{if } n \geq d \\
\alpha(T) & \text{else}
\end{cases}
\]

and

\[
\beta(T^{[n]}) = \begin{cases} 
\beta(T^{[d]}) & \text{if } n \geq d \\
\beta(T) & \text{else}
\end{cases}
\]

In particular,

\[
\alpha(T^n) = \begin{cases} 
na(T) & \text{if } n \leq d \\
n\alpha(T) + (n-d)\alpha(T^{[d]}) & \text{else}
\end{cases}
\]

and

\[
\beta(T^n) = \begin{cases} 
\beta(T) & \text{if } n \leq d \\
n\beta(T) + (n-d)\beta(T^{[d]}) & \text{else}
\end{cases}
\]

**Proof.** We will assume that \( T \) is not semi-regular, otherwise the assertion is trivial. From Theorem 3.8 we have \( d = v(T) \). So \( \mathcal{N}(T) \subset \mathcal{R}(T^n) \) and therefore [11, Lemma 511] and [6, Lemma 3.1, Lemma 3.2] imply that \( \frac{\mathcal{N}(T^{n+1})}{\mathcal{N}(T^n)} \cong \mathcal{N}(T^n) \) and \( \frac{\mathcal{R}(T^{n+1})}{\mathcal{R}(T^n)} \cong X/\mathcal{R}(T^n) \) for every integer \( n < d \). On the other hand, by the definition of \( \text{dis}(T) \) we have \( \alpha(T^{[d]}) = \alpha(T^{[m]}) \) and \( \beta(T^{[d]}) = \beta(T^{[m]}) \) for every \( m \geq d \). Moreover, it is easily seen that \( \alpha(T^m) = \sum_{k=0}^{m-1} \alpha(T^{[k]}) \) and \( \beta(T^m) = \sum_{k=0}^{m-1} \beta(T^{[k]}) \) for every \( m \in \mathbb{N}^* \). So the proof is complete.

**Corollary 3.10.** If \( T \in L(X) \) is semi-Fredholm then \( T \) is semi-regular if and only if \( \mathcal{N}(T) \subset \mathcal{R}(T^d) \); where \( d = \text{dis}(T) \).

**References**

[1] M. Berkani, *On a class of quasi-Fredholm operators*, Integr. Equ. and Oper. Theory, **34** (1999), 244–249.

[2] M. Berkani, N. Castro, *Unbounded B-Fredholm operators on Hilbert spaces*, Proc. Edinb. Math. Soc., **51** (2008), 285–296.
[3] M. Berkani, M. Sarih, *On semi B-Fredholm operators*, Glasg. Math. J., **43** (2001), 457–465.

[4] S. Grabiner, *Ranges of products of operators*, Can. J. Math., **6** (1974), 1430–1441.

[5] S. Grabiner, *Uniform ascent and descent of bounded operators*, J. Math. Soc. Japan, **34** (1982), 317–337.

[6] M. A. Kaashoek, *Ascent, descent, nullity and defect, a note on a paper by A. E. Taylor*, Math. Annalen, **172** (1967), 105–115.

[7] M. Mbekhta, V. Müller, *On the axiomatic theory of spectrum II*, Studia Math., **119** (1996), 129–147.

[8] V. Müller, *On the Kato-decomposition of quasi-Fredholm and B-Fredholm operators*, Vienna, Preprint ESI 1013 (2001).

[9] V. Müller, *Spectral theory of linear operators in spectral systems in Banach algebras*, Oper. Theory Adv. Appl. **139**, Birkhäuser Verlag, Basel, Second edition (2003).

[10] J. P. Labrousse, *Les opérateurs quasi-Fredholm: une généralisation des opérateurs semi Fredholm*, Rend. Circ. Math. Palermo, **29** (1980), 161–258.

[11] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Anal. Math. **6** (1958), 261–322.

[12] A. Tajmouati, M. Karmouni, M. Abkari, *Pseudo semi-B-Fredholm and Generalized Drazin invertible operators Through Localized SVEP*, Italian Journal of pure and applied mathematics, **37** (2017), 301–314.

[13] T. T. West, *A Riesz-Schauder theorem for semi-Fredholm operators*, Proc. Royal Irish Academy. **87A** (1987), 137–146.

[14] T. T. West, *Removing the jump-Kato’s decomposition*, Rocky Mountain J. Math. **20** (1990), 603–612.

[15] S. Ć. Živković-Zlatanović, B.P. Duggal, *Generalized Kato-meromorphic decomposition, generalized Drazin-meromorphic invertible operators and single-valued extension property*, Banach J. Math. Anal, **14** (2020), 894–914 .

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