Data-Driven Abstraction-Based Control Synthesis

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Abstract—This paper studies formal synthesis of controllers for continuous-space systems with unknown dynamics to satisfy requirements expressed as linear temporal logic formulas. Formal abstraction-based synthesis schemes rely on a precise mathematical model of the system to build a finite abstract model, which is then used to design a controller. The abstraction-based schemes are not applicable when the dynamics of the system are unknown. We propose a data-driven approach that computes the growth bound of the system using a finite number of trajectories. The growth bound together with the sampled trajectories are then used to construct the abstraction and synthesise a controller.

Our approach casts the computation of the growth bound as a robust convex optimisation program (RCP). Since the unknown dynamics appear in the optimisation, we formulate a scenario convex program (SCP) corresponding to the RCP using a finite number of sampled trajectories. We establish a sample complexity result that gives a lower bound for the number of sampled trajectories to guarantee the correctness of the growth bound computed from the SCP with a given confidence. We also provide a sample complexity result for the satisfaction of the specification on the system in closed loop with the designed controller for a given confidence. Our results are founded on estimating a bound on the Lipschitz constant of the system and provide guarantees on satisfaction of both finite and infinite-horizon specifications. We show that our data-driven approach can be readily used as a model-free abstraction refinement scheme by modifying the formulation of the growth bound and providing similar sample complexity results. The performance of our approach is shown on three case studies.

I. INTRODUCTION

One of the major objectives in the design of safety-critical systems is to ensure their safe operation while satisfying high-level requirements. Examples of safety-critical systems include power grids, autonomous vehicles, traffic control, and battery-powered medical devices. Automatic design of controllers for such systems that can fulfil the given requirements have received significant attention recently. These systems can be represented as control systems with continuous state spaces. Within these continuous spaces, it is challenging to leverage automated control synthesis methods that provide satisfaction guarantees for high-level specifications, such as those expressed in Linear Temporal Logic [2], [4], [32], [13].

A common approach to tackle the continuous nature of the state space is to use abstraction-based controller design (ABCD) schemes [32], [4], [21], [29]. The first step in the ABCD scheme is to compute a finite abstraction by discretising the state and action spaces. Finite abstractions are connected to the original system via an appropriate behavioural relation such as feedback refinement relations or alternating bisimulation relations [25], [32]. Under such behavioural relations, trajectories of the abstraction are related to the ones of the original system. Therefore, a controller designed for the simpler finite abstract system can be refined to a controller for the original system. The controller designed by the ABCD scheme is described as being formal due to the guarantees on satisfaction of the specification by the original system in closed loop with the designed controller.

ABCD schemes generally rely on a precise mathematical model of the system. This stems from the fact that establishing a behavioural relation between the original system and its finite abstraction uses reachability analysis over the dynamics of the original system that require knowledge of the dynamical equations. Although such equations can in principle be derived for instance by using physics laws, the real-world control systems are a mixture of differential equations, block diagrams, and lookup tables. Therefore, extracting a clean analytical model for systems of practical interest could be infeasible. A promising approach to tackle this issue is to develop data-driven control synthesis schemes with appropriate formal (probabilistic) guarantees.

The main contribution of this paper is to provide a data-driven approach for formal synthesis of controllers to satisfy temporal specifications. We focus on continuous-time nonlinear dynamical systems whose dynamics are unknown but sampled trajectories are available. Our approach constructs a finite abstract model of the system using only a finite number of sampled trajectories and the growth bound of the system. We formulate the computation of the growth bound as a robust convex program (RCP) that has infinite uncountable number of constraints. We then approximate the solution of the RCP with a scenario convex program (SCP) that has a finite number of constraints and can be solved using only a finite set of sampled trajectories. We establish a sample complexity result that gives a lower bound for the required number of trajectories to guarantee the correctness of the growth bound over the whole state space with a given confidence. We also provide a sample complexity result for the satisfaction of the specification on the system in closed loop with the designed controller for a given confidence. Our result requires estimating a bound on the Lipschitz constant of the system with respect to the initial state, that we obtain using extreme value theory. As our last contribution, we show that our approach can be extended to a model-free abstraction refinement scheme by modifying the formulation of the growth bound and providing similar sample complexity results. We demonstrate the performance...
of our approach on three case studies.

The remainder of this paper is organised as follows. After discussing the related work, Section II covers preliminaries on dynamical systems and finite abstractions, and provides the problem statement. In Section III, we present the assumptions and theoretical results needed for computing RCPs and their corresponding SCPs. In Section IV, we present our approach on data-driven computation of the growth bound and the abstraction, and prove our sample complexity result. Estimation of the Lipschitz constant of the system for computing the number of samples is also discussed in this section. Section V discusses the extension of our approach to a data-driven abstraction refinement scheme. Several numerical examples are provided in Section VI that support the theoretical findings of our paper. Finally, Section VII contains concluding remarks and future research directions.

Related Work. There is an extensive body of literature on model-based formal synthesis for both deterministic and probabilistic systems. We refer the reader to the books [2], [32], [4] and seminal papers [13], [1]. Data-driven approaches for analysis, verification, and synthesis of systems have received significant attention recently to improve efficiency and scalability of model-based approaches, and to study problems in which a model of the system is either not available or costly and time-consuming to construct.

Given a prior inaccurate knowledge about the model of the system, a research line is to use data for refining the model and then synthesise a controller. Such approaches assume a class of models and improve the estimation of the uncertainty within the model class. These approaches range from using Gaussian processes [23], [3], differential inclusions [10], rapidly-exploring random graphs [15], piecewise affine models [27], and model-based reinforcement learning algorithms [8]. A data-driven framework is proposed in [12] for performing data-driven verification and synthesis. Inspired by Esfahani et al. [11], where the optimality of the robust approximately feasible solution for the associated chance constrained program is studied in [30] by assuming additional properties of the underlying probability distributions. We will use the results by Esfahani et al. [11], where the optimality of the robust program is connected directly to the scenario program. These results are also used recently in the papers [18], [28] for performing data-driven verification and synthesis. Inspired by the works [37], [36], we will use extreme value theory to estimate the Lipschitz constant needed for the sample complexity results.

In our approach, we formulate the synthesis problem as a robust convex program and approximate it with a scenario program. Such approximations have been studied for the past two decades. Calafiore and Campi [6] provide an approximately feasible solution for the associated chance constrained program by solving a scenario program, and give a sample complexity result. Relaxing the convexity assumption is studied in [30] by assuming additional properties of the underlying probability distributions. We will use the results by Esfahani et al. [11], where the optimality of the robust program is connected directly to the scenario program. These results are also used recently in the papers [18], [28] for performing data-driven verification and synthesis. Inspired by the works [37], [36], we will use extreme value theory to estimate the Lipschitz constant needed for the sample complexity results.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Preliminaries

Notation. We denote the set of natural, real, positive real, and non-negative real numbers by \( \mathbb{N} \), \( \mathbb{R} \), \( \mathbb{R}_{>0} \), and \( \mathbb{R}_{\geq 0} \), respectively. The set of natural numbers including zero is denoted by \( \mathbb{N}_{\geq 0} \). We use superscript \( n > 0 \) with these sets to denote the Cartesian product of \( n \) copies of these sets. The power set of a set \( A \) is denoted by \( 2^A \) and includes all the subsets of \( A \). For any \( x, y \in \mathbb{R}^n \) with
where \( u(t) \) represents the additive disturbance.

We consider the class of input and disturbance signals \( u : \mathbb{R}_{\geq 0} \to U \) and \( w : \mathbb{R}_{> 0} \to W \) to be piecewise constant with respect to a sampling time \( \tau \geq 0 \), i.e., \( u(t) = u(k\tau) \) and \( w(t) = w(k\tau) \) for every \( k\tau \leq t < (k+1)\tau \) and \( k \in \mathbb{N}_{\geq 0} \). Given a sampling time \( \tau > 0 \), a constant input \( u \in U \), and a constant disturbance \( w \in W \), we define the continuous-time trajectory \( \zeta_{x_0,u,w} \) of the system on the time interval \([0,\tau]\) as an absolutely continuous function \( \zeta_{x_0,u,w} : [0,\tau] \to X \) such that \( \zeta_{x_0,u,w}(0) = x_0 \), and \( \zeta_{x_0,u,w} \) satisfies the differential equation \( \dot{\zeta}_{x_0,u,w}(t) = f(\zeta_{x_0,u,w}(t), u) + w \) for almost all \( t \in [0,\tau] \). The solution of (1) from \( x_0 \) for the constant control input \( u \) with \( u(t) \equiv 0 \) for all \( t \geq 0 \) is called the nominal trajectory of the system. For a fixed \( \tau \), we define the operators

\[
\varphi(x,u,w) := \zeta_{x,u,w} \quad \text{and} \quad \Phi(x,u) := \{ \varphi(x,u,w) : w \in W \}
\]

respectively for the trajectory at time \( \tau \) and the set of such trajectories starting from \( x \).

In this paper, we consider control systems \( \Sigma = (X,x_{in},U,W,f) \) whose vector field \( f \) is not known, but we can observe their time-sampled trajectories. A sequence \( x_0, x_1, x_2, \ldots \) is a time-sampled trajectory of \( \Sigma \) if for each \( i \geq 0 \), we have \( x_{i+1} = \Phi(x_i, u_i) \) for some \( u_i \in U \).

Finite-state Abstraction of Control Systems. Let \( \Sigma = (X,x_{in},U,W,f) \) be a control system with a sampling time \( \tau > 0 \). We consider abstract models constructed by using uniformly sized rectangular partitioning of \( X \) and \( U \). We select representative points from these partition sets to obtain \( \hat{X} \) and \( \hat{U} \). We assume that the radius of these partition sets are provided as vectors \( \eta_x, \eta_u \in \mathbb{R}^n_{\geq 0} \) and \( \eta_t \in \mathbb{R}_{\geq 0} \), respectively. Parameters \( \eta_x, \eta_u, \eta_t \) are inputs to the abstraction procedure. A finite-state abstraction of \( \Sigma \) is characterised by the tuple \( \hat{\Sigma} = (\hat{X}, \hat{U}, \hat{f}) \), where \( \hat{X} \) is the set of representative points from a finite partition of \( X \), \( \hat{U} \) is the set of representative points from a finite partition of \( U \), and \( \hat{f} : \hat{X} \times \hat{U} \to \mathbb{R}^n \) is a set-valued map. For any \( x \in \hat{X} \) and \( u \in \hat{U} \), \( x' \in f(\hat{x}, \hat{u}) \) if there is a pair of states \( x \in \Omega_{\eta_x}(\hat{x}) \) and \( x' \in \Omega_{\eta_t}(\hat{u}) \) such that \( x' = \Phi(x,u) \). Note that, the larger \( \eta_x, \eta_t \) is (where comparison is made dimension-wise), the smaller is the cardinality of \( \hat{X} \) resulting in a coarser abstraction. On the other hand, the smaller \( \eta_x \) is, the more precise the abstraction \( \hat{\Sigma} \) will be, increasing the chance of a successful controller synthesis (see, e.g., [32] for more details on this construction).

Feedback Controller. A feedback controller for \( \hat{\Sigma} \) is a function \( \hat{C} : \hat{X} \to \hat{U} \). We denote by \( \hat{C} \parallel \hat{\Sigma} \) the feedback composition of \( \hat{\Sigma} \) and \( \hat{C} \). The set of trajectories of the closed-loop system \( \hat{C} \parallel \hat{\Sigma} \) consists of all finite trajectories \( \bar{x}_0, \bar{x}_1, \ldots \) such that for all \( i \in \mathbb{N}_{\geq 0} \), we have \( \bar{x}_{i+1} = f(\bar{x}_i, \hat{C}(\bar{x}_i)) \).

We can relate a finite abstraction \( \hat{\Sigma} \) to \( \Sigma \) for control synthesis purposes. Simulation relations or feedback refinement relations [32], [25] established between \( \Sigma \) and \( \hat{\Sigma} \) enable us to refine a controller \( \hat{C} \) designed for \( \hat{\Sigma} \) to a controller \( C \) for \( \Sigma \). In its general form, such a refined controller \( C \) maps the current states \( x \in \Omega_{\eta_x}(\hat{x}) \) into an input \( u = \hat{C}(\bar{x}) \). The purpose of designing \( C \) is that the closed-loop system \( C \parallel \Sigma \) satisfies the given objective. Our synthesis objective is expressed as Linear Temporal Logic (LTL) specifications. We refer to [2] and references therein for detailed syntax and semantics of LTL. For the details of the controller synthesis and tool implementation using abstract models we refer to [25] and [26], respectively.

B. Problem Statement

We study abstraction-based control design (ABCD) for systems with unknown dynamics using available data from the system such that a given specification is satisfied with high confidence on the closed-loop system.

Assumption 1. The vector field \( f \) of the control system \( \Sigma = (X,x_{in},U,W,f) \) in unknown, but sampled trajectories of the system can be obtained in the form of \( S_N := \{ (x_k, u_k, x_{k+1}) | x_k \in \Phi(x_k, u_k), k = 1,2,\ldots,N \} \).

Problem 1 (Data-driven ABCD). Inputs: Control system \( \Sigma = (X,x_{in},U,W,f) \) with unknown vector field \( f \), specification \( \Psi \), sampled trajectories \( S_N \), and confidence parameter \( \beta \in (0,1) \).

Outputs: Abstract model \( \hat{\Sigma}, \) abstract controller \( \hat{C} \), and refined controller \( C \) for \( \Sigma \), such that \( C \parallel \Sigma \) satisfies \( \Psi \) with confidence \( (1 - \beta) \).

The first step of the ABCD is to compute a finite abstraction \( \hat{\Sigma} \) for \( \Sigma \). Once such an abstraction is computed, synthesis of the controller \( \hat{C} \) and refining it to \( C \) follow the model-based ABCD scheme. Therefore, the main challenge is to provide a data-driven computation of the abstraction \( \hat{\Sigma} \) that is a true overapproximation of \( \Sigma \) with confidence \( (1 - \beta) \).

x = (x_1, \ldots, x_n) and y = (y_1, \ldots, y_n), and a relational symbol > ∈ {≤, <, =, >, ≥}, we write x > y if x_i > y_i for every i ∈ {1, 2, ..., n}. A matrix M ∈ ℝ^{n×n} is said to be non-negative if all of its entries are non-negative. We use the operators |·| and ∥·∥ to denote the element-wise absolute value and the infinity norm, respectively. We use the notation Ω_ε(c) := \{ x ∈ ℝ^n | ∥x - c∥ ≤ ε \} to denote the ball with respect to infinity norm centred at c ∈ ℝ^n with radius ε ∈ ℝ_{>0}. We consider a probability space \((Ω,F_Ω,P_Ω)\), where Ω is the sample space, \(F_Ω\) is a sigma-algebra on Ω comprising its subsets as events, and \(P_Ω\) is a probability measure that assigns probabilities to events.
Problem 2 (Data-driven Abstraction). Inputs: Control system \( \Sigma = (X, x_m, U, W, f) \) with unknown vector field \( f \), sampled trajectories \( S_N \), discretisation parameters \( \eta_x \) and \( \eta_u \), and confidence parameter \( \beta \in (0, 1) \). Outputs: Finite model \( \hat{\Sigma} \) that is an abstraction of \( \Sigma \) with confidence \((1 - \beta)\).

In this paper, we tackle Problem 2 by showing how to construct \( \Sigma \) from sampled trajectories \( S_N \), and provide a lower bound on the data size \( N \) in order to ensure correctness of the abstraction with confidence \((1 - \beta)\). The required theoretical tools are presented in the next section.

III. ROBUST CONVEX PROGRAMS

In this section, we describe robust convex programs (RCPs) and data-driven approximation of their solution. In Sections IV and V, we show how such an approximation can be used for solving the data-driven abstraction in Problem 2.

Let \( T \subset \mathbb{R}^q \) be a compact convex set for some \( q \in \mathbb{N} \) and \( c \in \mathbb{R}^q \) be a constant vector. Let \( (D, \theta, P) \) be the probability space of the uncertainty and \( g: T \times D \rightarrow \mathbb{R} \) be a measurable function, which is convex in the first argument for each \( d \in D \), and bounded in the second argument for each \( \theta \in T \). The robust convex program (RCP) is defined as

\[
\text{RCP: } \begin{cases} 
\min \ c^T \theta \\
\text{s.t. } \theta \in T \text{ and } g(\theta, d) \leq 0 \quad \forall d \in D.
\end{cases}
\]  

(2)

Computationally tractable approximations of the optimal solution of the RCP (2) can be obtained using scenario convex programs (SCPs) that only require gathering finitely many samples from the uncertainty space [24]. Let \( (d_i)_{i=1}^N \), be \( N \) independent and identically distributed (i.i.d.) samples drawn according to the probability measure \( P \). The SCP corresponding to the RCP (2) strengthened with \( \gamma \geq 0 \) is defined as

\[
\text{SCP}_\gamma: \begin{cases} 
\min \ c^T \theta \\
\text{s.t. } \theta \in T, \text{ and } g(\theta, d_i) + \gamma \leq 0 \quad \forall i \in \{1, 2, \ldots, N\},
\end{cases}
\]  

(3)

We denote the optimal solution of RCP (2) as \( \theta_{\text{RCP}} \), and the optimal solution of SCP \( \gamma \) as \( \theta_{\text{SCP}}^\gamma \). Note that \( \theta_{\text{SCP}}^\gamma \) is a single deterministic quantity but \( \theta_{\text{SCP}}^0 \) is a random quantity that depends on the i.i.d. samples \( (d_i)_{i=1}^N \) drawn according to \( P \). The RCP (2) is a challenging optimisation problem since the cardinality of \( D \) is infinite and the optimisation has infinite number of constraints. In contrast, the SCP (3) is a convex optimisation with finite number of constraints for which efficient optimisation techniques are available [5]. The following theorem provides a sample complexity result for connecting the optimal solution of the SCP \( \gamma \) to that of the RCP.

Theorem 1 ([24]). Assume that the mapping \( d \mapsto g(\theta, d) \) in (2) is Lipschitz continuous uniformly in \( \theta \in T \) with Lipschitz constant \( L_d \) and let \( h: [0, 1] \rightarrow \mathbb{R}_{\geq 0} \) be a strictly increasing function such that

\[
\mathbb{P}(\Omega_\varepsilon(d)) \geq h(\varepsilon),
\]  

(4)

for every \( d \in D \) and \( \varepsilon \in [0, 1] \). Let \( \theta_{\text{RCP}}^\gamma \) be the optimal solution of the RCP (2) and \( \theta_{\text{SCP}}^\gamma \) the optimal solution of SCP \( \gamma \) with

\[
\gamma = L_d h^{-1}(\varepsilon)
\]  

(5)

computed by taking \( N \) i.i.d. samples \( (d_i)_{i=1}^N \) from \( P \). Then \( \theta_{\text{SCP}}^\gamma \) is a feasible solution for the RCP with confidence \((1 - \beta)\) if the number of samples \( N \geq N(\varepsilon, \beta, \gamma) \), where

\[
N(\varepsilon, \beta, \gamma) := \min \left\{ N \in \mathbb{N} \mid \frac{q-1}{N} \sum_{i=0}^{N-1} \left( \frac{N}{i} \right) e^\gamma (1 - e)^{N-i} \leq \beta \right\},
\]  

(6)

with \( q \) being the dimension of the decision vector \( \theta \in T \).

IV. DATA-DRIVEN ABSTRACTION

In this section, we first discuss the steps required for model-based abstraction of control systems. We then show how this can be formulated as an RCP and present its associated SCP. Finally, we use the connection between the RCPs and SCPs in Theorem 1 to provide a lower bound for number of required samples to certify a desired confidence. The simplifying assumption used in this section is that samples from the nominal trajectories of the system \( \Sigma \) are available in the form of \( \{(x_k, u_k, \bar{x}_k) \mid x_k = \varphi(x_k, u_k, 0), k = 1, 2, \ldots, N\} \). We discuss in the next section how this assumption can be relaxed by modifying the inequality of the growth bound.

A. Growth bound for reachable sets

Consider a control system \( \Sigma = (X, x_m, U, W, f) \) with the disturbance set \( W = [-\bar{w}, \bar{w}] \) for some vector \( \bar{w} \in \mathbb{R}_+^n \). Let \( \eta_x \) and \( \eta_u \) be discretisation parameters for the state and input spaces \( X \) and \( U \), respectively. The first step of ABCD is to compute a finite abstraction \( \hat{\Sigma} = (\hat{X}, \hat{U}, \hat{f}) \) using overapproximations of the reachable sets for every pair of abstract state and input. The reachable set for every pair \( (\hat{x}, \hat{u}) \) in \( \hat{X} \times \hat{U} \) is defined as

\[
\text{Reach}(\hat{x}, \hat{u}) := \{ x' \in \Phi(\hat{x}, \hat{u}) \mid x \in \Omega_{\eta_x}(\hat{x}) \}.
\]

The set \( \text{Reach}(\hat{x}, \hat{u}) \) is usually overapproximated using the growth bound of the system dynamics [25].

Definition IV.1. The growth bound of a control system \( \Sigma \) with abstract state and input spaces \( \hat{X}, \hat{U} \) is a function \( \kappa: \mathbb{R}_{\geq 0}^2 \times \hat{X} \times \hat{U} \rightarrow \mathbb{R}_{\geq 0} \) that satisfies

\[
| \varphi(\hat{x}, \hat{u}, w) - \varphi(\hat{x}, \hat{u}, 0) | \leq \kappa(|x - \hat{x}|, \hat{x}, \hat{u})
\]  

\[
\forall \hat{x} \in \hat{X}, \forall u \in \hat{U}, \forall x \in \Omega_{\eta_x}(\hat{x}), \forall w \in W.
\]  

(7)

Note that \( \varphi(\hat{x}, \hat{u}, 0) \) is the nominal (disturbance-free) trajectory of the system. Using this definition, for every abstract state-input pair \( (\hat{x}, \hat{u}) \in \hat{X} \times \hat{U} \), the reachable set \( \text{Reach}(\hat{x}, \hat{u}) \) is overapproximated with a ball centered at \( \varphi(\hat{x}, \hat{u}, 0) \) with radius \( \lambda(\hat{x}, \hat{u}) := \kappa(\eta_x, \hat{x}, \hat{u}) \).

When the system dynamics are known, it is shown in [25] that the growth bound can be computed as

\[
\kappa(r, \hat{x}, \hat{u}) = e^{L_d^\tau} r + \int_0^\tau e^{L_d s} w ds,
\]  

(8)
for all $r \in \mathbb{R}^n_{\geq 0}$, $\hat{x} \in \hat{X}$, and $\hat{u} \in \hat{U}$, where $L: \hat{U} \to \mathbb{R}^{n \times n}$ is a matrix such that the entries of $L(\hat{u})$ satisfy the following inequality for all $x \in \hat{X}$:

$$L_{i,j}(\hat{u}) \geq \begin{cases} D_j f_i(x, \hat{u}) & i = j \\ D_j f_i(x, \hat{u}) & i \neq j, \end{cases}$$

(9)

for all $i, j \in \{1, 2, \ldots, n\}$, where $f_i(x, u)$ is the $i$th element of the vector field $f(x, u)$ and $D_j f_i$ is its partial derivative with respect to the $j$th element of $x$.

B. SCP for the computation of growth bound

When the model of the system is unknown, the matrix $L(\hat{u})$ defined using (9) is not computable, thus the growth bound in (8) is not available. To tackle this bottleneck, we use the parameterisation

$$\kappa(\theta)(r, \hat{x}, \hat{u}) := \theta_1(\hat{x}, \hat{u}) r + \theta_2(\hat{x}, \hat{u}), \forall r \in \mathbb{R}^n_{\geq 0}, \hat{x} \in \hat{X}, \hat{u} \in \hat{U},$$

(10)

where $\theta_1 \in \mathbb{R}^{n \times n}$ and $\theta_2 \in \mathbb{R}^n$. We denote by $\theta \in \mathbb{R}^{n^2 + n}$ the concatenation of columns of $\theta_1$ and $\theta_2$.

Remark 1. The parameterised growth bound in (10) is linear with respect to $r$ similar to (8), but is more general and less conservative by allowing $\theta_1, \theta_2$ to depend on $\hat{x}$ (i.e., they are defined locally for each abstract state).

Theorem 2. The inequality (7) with the parameterised growth bound (10) can be written as the robust convex program

$$\begin{align*}
\min_{\theta} & \quad c^T \theta \\
\text{s.t.} & \quad 0 \leq \theta \leq \bar{\theta}, \quad \forall x \in \Omega_{\eta_i}(\hat{x}), \forall w \in W, \\
& \quad |\varphi(x, \hat{u}, w) - \varphi(\hat{x}, \hat{u}, 0)| - \kappa(\theta)(|x - \hat{x}|, \hat{x}, \hat{u}) \leq 0,
\end{align*}$$

(11)

where $c = [1, 1, \ldots, 1] \in \mathbb{R}^{n^2 + n}$ and $\bar{\theta}$ is a sufficiently large positive vector.

Proof. We first show that the optimisation (11) is in fact a robust convex programme. Let $D = \Omega_{\eta_i}(\hat{x}) \times W$ be the uncertainty space and

$$g(\theta, x, w) := |\varphi(x, \hat{u}, w) - \varphi(\hat{x}, \hat{u}, 0)| - \kappa(\theta)(|x - \hat{x}|, \hat{x}, \hat{u})$$

for all $x \in \Omega_{\eta_i}(\hat{x})$ and $w \in W$ and fixed $(\hat{x}, \hat{u}) \in \hat{X} \times \hat{U}$. We need to show that $g$ is convex in $\theta$ for each $(x, w) \in D$ and bounded in $(x, w)$ for every $\theta \in [0, \bar{\theta}]$. The convexity holds due to the parameterisation of $\kappa(\theta)$ in (10) being linear with respect to the optimisation variables in $\theta$. The boundedness holds due to the set $D$ being compact and trajectories of the system being continuous.

We note that any feasible solution for the optimisation (11) gives a function $\kappa$ that satisfies the inequality (7) for $\Sigma$. Such a system will also have a growth bound of the form (8) that is a feasible solution for (11). To see this, we show that $\theta_1 = e^{L(\hat{u})}$ and $\theta_2 = \int_{0}^{\bar{\theta}} e^{L(\hat{u})} \bar{w} ds$ are always non-negative. By definition, all the entries of $L(\hat{u})$ are non-negative except the diagonal entries. We decompose this matrix as $L(\hat{u}) = Q + D$, where $D$ is a diagonal matrix with all diagonal entries equal to the constant $\max_j \sum_i |L_{i,j}(\hat{u})|$ and $Q = L(\hat{u}) - D$ is a sub-stochastic matrix as (i) its non-diagonal entries are non-negative ($Q_{i,j} = L_{i,j}(\hat{u}) \geq 0$ for $i \neq j$), (ii) its diagonal entries are non-positive ($Q_{i,i} \leq 0$), and finally (iii) all of its row sums are non-positive. Note that $D$ is a multiple of identity matrix and therefore, $DQ = QD$ and $e^{(Q + D)t} = e^{Qt} e^{Dt}$. Further, we define the matrix

$$\tilde{Q} = \begin{bmatrix} Q : & -Q1 & \vdots \\
\vdots & \ddots & \vdots \\
0^T & \vdots & 1 \end{bmatrix},$$

where 0 and 1 represent $n$-dimensional vectors with all entries equal to zero and one, respectively. Note that $\tilde{Q}$ is a stochastic matrix since $Q_{i,i} = -\sum_{j \neq i} Q_{i,j}$ for every $1 \leq i \leq n + 1$ and $Q_{i,j} \geq 0$ for $i \neq j$. Therefore, matrix $\tilde{Q}$ correspond to the transition probability matrix of a continuous-time Markov chain with state space $\{1, 2, \ldots, n + 1\}$ (see, e.g., [2] for more details). Therefore, the entry $(i, j)$ of $e^{\tilde{Q}t}$ is the probability that the Markov chain reaches the $i$th state from the $j$th state at time $\tau$, which is a non-negative quantity.

Further, we have

$$e^{\tilde{Q}t} = \begin{bmatrix} e^{Q_1} & 1 - e^{Q_1} & \vdots \\
\vdots & \ddots & \vdots \\
0^T & \vdots & 1 \end{bmatrix},$$

Therefore, $e^{\tilde{Q}t}$ is non-negative, which gives $e^{L(\hat{u})t} = e^{\tilde{Q}t} e^{Dt}$ since $e^{Dt} \geq 0$. This naturally results in $\theta_1$ and $\theta_2$ being non-negative as the integral of non-negative functions. 

To construct the SCP, associated with the RCP (11), we fix $\hat{x} \in \hat{X}$ and $\hat{u} \in \hat{U}$, consider a uniform distribution on the space $D = \Omega_{\eta_i}(\hat{x}) \times W$ and obtain $N$ i.i.d. sample trajectories $\Sigma_N = \{(x_i, \hat{u}, x'_i) : x'_i \in \Phi(x_i, \hat{u}), i = 1, 2, \ldots, N\}$. Note that every $x'_i$ corresponds to a random disturbance $w_i \in W$. The SCP, is

$$\begin{align*}
\min_{\theta} & \quad c^T \theta \\
\text{s.t.} & \quad 0 \leq \theta \leq \bar{\theta} \quad \text{and} \quad \forall i \in \{1, \ldots, N\}, \\
& \quad |x'_i - x'_{i,\text{nom}}| - \theta_1(\hat{x}, \hat{u}) |x_i - \hat{x}| + \theta_2(\hat{x}, \hat{u}) + \gamma \leq 0,
\end{align*}$$

(12)

where $x'_{i,\text{nom}} := \varphi(\hat{x}, \hat{u}, 0)$ and $\gamma \in \mathbb{R}_{\geq 0}$.

Theorem 3. For any $\hat{x} \in \hat{X}$ constructed with discretisation size $\eta_i$, any $\hat{u} \in \hat{U}$, and the disturbance set $W = [-\bar{\omega}, \bar{\omega}]$, the optimal solution of (12) gives a growth bound for the system $\Sigma$ corresponding to $(\hat{x}, \hat{u})$ with confidence $1 - \beta$, when the number of samples $N \geq N(\varepsilon, \beta)$ and

$$\gamma = 4 L^\varepsilon(\hat{u}) 2^n \prod_{i=1}^{n} \eta_i(\varepsilon) \prod_{i=1}^{n} \bar{\omega}(i),$$

(13)

where $\varepsilon \in [0, 1]$, $n$ is the dimension of the state space and $L^\varepsilon(\hat{u})$ is the Lipschitz constant of the system trajectories $\varphi(x, \hat{u}, w)$ with respect to $(x, w)$. 
Proof. We apply Theorem 2 to the RCP (11) for fixed \( \hat{x} \in \mathring{X} \) and \( \hat{u} \in \mathring{U} \). Define

\[
g(\theta, x, w) := \max\{|\varphi(x, \hat{u}, w) - \varphi(x, \hat{u}, 0)| - \theta_1(\hat{x}, \hat{u})|x - \hat{x}| - \theta_2(\hat{x}, \hat{u}), \}
\]

where the \( \max\{\cdot\} \) is applied to the elements of its argument that belong to \( \mathbb{R}^n \). Since the distribution on \( D = \Omega_{\eta_x}(\hat{x}) \times W \) is uniform, we choose

\[
h(\varepsilon) = P(\Omega(\varepsilon)) = \frac{(\varepsilon/2)^{2n}}{\prod_{i=1}^{n} \eta_x(i) \prod_{i=1}^{n} w(i)}
\]
to satisfy the inequality (4). Note that \( h(\varepsilon) \) gives the probability of choosing a point within the \( 2n \)-ball \( \Omega(\varepsilon) \) uniformly at random. We use Equation (5) as \( h(\varepsilon) \) to get the value of \( \gamma \) in (13). It only remains to show that \( g(\theta, x, w) \) is Lipschitz continuous with constant \( \Lambda = 2L \varphi(\hat{u}) \). Note that \( L \varphi(\hat{u}) \) is the Lipschitz constant of \( \varphi(x, \hat{u}, w) \) with respect to \( (x, w) \), and satisfies

\[
\|\varphi(x, \hat{u}, w) - \varphi(x', \hat{u}, w')\| \leq L \varphi(\hat{u}) \|(x, w) - (x', w')\|
\]

for all \( x, x' \in \Omega_{\eta_u}(\hat{x}) \) and \( w, w' \in W \). Since \( \|\theta_1(\hat{x}, \hat{u})\| \) can be bounded by \( L \varphi(\hat{u}) \), we get that

\[
\|g(\theta, x, w) - g(\theta, x', w')\| \
\leq \|\varphi(\theta, x, w) - \varphi(x', \hat{u}, w')\| + \|\theta_1(\hat{x}, \hat{u})\||x - x'| \
\leq L \varphi(\hat{u}) \|(x, w) - (x', w')\| + L \varphi(\hat{u}) \|x - x'| \
\leq 2L \varphi(\hat{u}) \|(x, w) - (x', w')\|.
\]

Therefore, \( g(\theta, x, w) \) is Lipschitz continuous with constant \( 2L \varphi(\hat{u}) \). This completes the proof.

Remark 2. The value of \( \gamma \) in (13) depends on the Lipschitz constant \( L \varphi(\hat{u}) \). We provide an algorithm in the next subsection for estimating this constant using sampled trajectories of the system. Note that as the above proof shows, the estimated quantity \( \hat{\theta}_1 = L \varphi(\hat{u}) \) is non-zero, and we can use the result of Corollary 1 to provide an algorithmic solution for Problem 2. This algorithm uses a confidence parameter \( \theta^* \), divides it by the cardinality of \( X \times U \), and constructs the abstraction using these growth bounds.

Algorithm 1: Data-Driven Abstraction

Data: \((X, U, W)\) of a control system \( \Sigma \), confidence \( \beta \), discretisation parameters \( \eta_x, \eta_u \)

1. Compute the finite state and input sets \( \mathring{X} \) and \( \mathring{U} \) using \( \eta_x, \eta_u \);
2. Define \( n_x \) and \( n_u \) as cardinalities of \( \mathring{X} \) and \( \mathring{U} \);
3. Choose \( \varepsilon \in [0, 1] \);
4. Set \( N = \frac{\eta_x^{\alpha} \eta_u^{\beta}}{\varepsilon} \) using Eq. (6);
5. Compute \( \gamma \) using Eq. (13);
6. for \( \hat{x} \in \mathring{X} \) do
   7. for \( \hat{u} \in \mathring{U} \) do
      8. \( f(\hat{x}, \hat{u}) = \emptyset \);
      9. Consider the uncertainty space \( D = \Omega_{\eta_x}(\hat{x}) \times W \);
      10. Select \( N \) i.i.d sample trajectories using uniform distribution over \( D \);
      11. Simulate the nominal trajectory \((\hat{x}, \hat{u}, x'_\text{nom}) \);
      12. Solve the SCP, (12) to get the optimiser \( \theta^* (\hat{x}, \hat{u}) \);
      13. \( z \leftarrow x'_\text{nom} \);
      14. \( L \leftarrow \frac{\kappa(\theta^*)}{\eta_x} \) and add them to \( f(\hat{x}, \hat{u}) \);
      15. Find all states \( \hat{x}' \) for which \( \Omega_{\eta_x}(\hat{x}') \cap \Omega_{\eta_x}(z) \neq \emptyset \) and add them to \( f(\hat{x}, \hat{u}) \);
   16. end
   17. end

Result: \( \mathring{\Sigma} = (\mathring{X}, \mathring{U}, \mathring{f}) \) as a finite abstraction of \( \Sigma \) with confidence \((1 - \beta), \theta^*(\hat{x}, \hat{u}) \) as a growth bound for \( \hat{x} \in \mathring{X}, \hat{u} \in \mathring{U} \).
The finite abstraction \( \hat{\Sigma} \) constructed by Algorithm 1 is a valid abstraction for \( \Sigma \) with confidence \((1 - \beta)\). This means any controller \( \hat{C} \) synthesised on \( \hat{\Sigma} \) and refined to a controller \( C \) for \( \Sigma \) will satisfy the desired specification with confidence \((1 - \beta)\) on the closed loop system \( \Sigma \parallel C \). In the next section, we extend our approach to make it suitable for abstraction refinement in case there is no controller \( \hat{C} \) satisfying the specification due to the conservatism of the approach.

C. Lipschitz Constant Estimation

For estimating the Lipschitz constant \( L_\varphi \) in (15), we estimate an upper bound for the fraction

\[
\Delta(\hat{u}) := \frac{\| \varphi(x, \hat{u}, w) - \varphi(x', \hat{u}, w') \|}{\| (x, w) - (x', w') \|}
\]

that holds for all \( x, x' \in X \) and \( w, w' \in W \). We follow the line of reasoning in [37], [36] and use the extreme value theory for the estimation.

Let us fix a \( \delta > 0 \) and assign uniform distribution to the pairs \((x, w)\) and \((x', w')\) over the domain

\[
\{x, x' \in X, w, w' \in W \mid ||(x, w) - (x', w')|| \leq \delta \}.
\]

Then \( \Delta(\hat{u}) \) is a random variable with an unknown cumulative distribution function (CDF). Based on the assumption of Lipschitz continuity of the system, the support of the distribution of \( \Delta(\hat{u}) \) is bounded from above, and we want to estimate an upper bound for its support. We take \( n \) sample pairs \((x, w)\) and \((x', w')\), and compute \( n \) samples \( \Delta_1, \Delta_2, \ldots, \Delta_n \) for \( \Delta(\hat{u}) \). The CDF of \( \max \{\Delta_1, \Delta_2, \ldots, \Delta_n\} \) is called the limit distribution of \( \Delta(\hat{u}) \). Fisher-Tippett-Gnedenko theorem says that if the limit distribution exists, it can only be one of the three family of extreme value distributions – the Gumbel class, the Fréchet class, and the reverse Weibull class.

These CDF’s have the following forms:

**Gumbel class:** \( G(s) = \exp \left[ -\exp \left( \frac{s - a}{b} \right) \right], \ s \in \mathbb{R} \)

**Fréchet class:** \( G(s) = \begin{cases} 0 & \text{if } s < a \\ \exp \left[ -\left( \frac{s-a}{c} \right)^{-1} \right] & \text{if } s \leq a \end{cases} \)

**Reverse Weibull class:** \( G(s) = \begin{cases} \exp \left[ -\left( \frac{s-a}{c} \right)^{1} \right] & \text{if } s < a \\ 1 & \text{if } s \leq a \end{cases} \)

where \( a \in \mathbb{R}, b > 0, c > 0 \) are respectively the location, scale and shape parameters of the distributions.

Among the above three distributions, only the reverse Weibull class has a support bounded from above. Therefore, the limit distribution of \( \Delta(\hat{u}) \) will be from this class and the location parameter \( a \) is such an upper bound. As a result, we can estimate the location parameter of the limit distribution of \( \Delta(\hat{u}) \) to get an estimation of the Lipschitz constant.

The approach is summarised in Algorithm 2. The most inner loop computes samples of \( \Delta(\hat{u}) \). The middle loop computes samples of \( \max \{\Delta_1, \ldots, \Delta_n\} \). The outer loop estimates the Lipschitz constant for each \( \hat{u} \) by fitting a reverse Weibull distribution.

---

**Algorithm 2: Lipschitz Constant Estimation**

**Data:** \((X, U, W)\) of a control system \( \Sigma \), abstract input space \( \hat{U} \)

1. Select number of samples \( n \) and \( m \) for the estimation
2. Select \( \delta > 0 \)
3. for \( \hat{u} \in \hat{U} \) do
   4. for \( j = 1 : m \) do
      5. for \( i = 1 : n \) do
         6. Sample pairs \((x, w), (x', w')\) uniformly from the domain in (16)
         7. Run \( \Sigma \) to get trajectories \( \varphi(x, \hat{u}, w) \) and \( \varphi(x', \hat{u}, w') \)
         8. Compute \( \Delta_i := \frac{||\varphi(x, \hat{u}, w) - \varphi(x', \hat{u}, w')||}{||(x, w) - (x', w')||} \)
      end
   end
   9. \( \Gamma_j := \max \{\Delta_1, \ldots, \Delta_n\} \)
end
11. Fit a reverse Weibull distribution to the sample set \( \{\Gamma_1, \Gamma_2, \ldots, \Gamma_m\} \)
12. \( L_\varphi(\hat{u}) \) is the location parameter of the fitted distribution
13. **Result:** Estimated value of \( L_\varphi(\hat{u}) \) for all \( \hat{u} \in \hat{U} \)

---

V. SYNTHESIS VIA ABSTRACTION REFINEMENT

The data-driven synthesis discussed in Section IV inherits the soundness property from the ABCD approach: they both work with overapproximations of the dynamics and may not return a controller despite one may exists. Therefore, there is a need for refining the abstraction in order to check for controllers using less conservative abstractions. While the method of Section IV is good for a given fixed discretisation parameter \( \eta_x \), it is not suitable for reducing \( \eta_x \), which requires re-computing all local parameters of the growth bounds \( \theta_1(\hat{\chi}, \hat{u}), \theta_2(\hat{\chi}, \hat{u}) \). A another shortcoming of the method is related to the data collection: the nominal trajectories of the system should be available and are used in the constraints of the SCP. In this section, we discuss an extension of the approach of Section IV, in order to

- enable removing \( \eta_x \) without the need for re-computing the growth bound, and
- relax the assumption of having access to the nominal trajectories of the system.

Let us define a modified growth bound as a function \( \kappa_e : \mathbb{R}^+_0 \times X \times U \rightarrow \mathbb{R}^+_0 \) that is strictly increasing in its first argument and satisfies

\[
|\varphi(x_1, \hat{u}, w_1) - \varphi(x_2, \hat{u}, w_2)| \leq \kappa_e(|x_1 - x_2|, \hat{x}, \hat{u})
\]

\( \forall \hat{x} \in \hat{X}, \forall \hat{u} \in \hat{U}, \forall x_1, x_2 \in \Omega_{\eta_x}(\hat{x}), \forall w_1, w_2 \in W \). (17)

This definition is more conservative than (7) in comparing trajectories under two arbitrary disturbances, and we always have that \( \kappa_e \) satisfies (7). Using this new definition, for every pair of abstract state and input \( (\hat{x}, \hat{u}) \), the corresponding overapproximation of the reach set can be computed as a
For any \( \mathbf{z}(\hat{x}, \hat{u}) \in \Phi(\hat{x}, \hat{u}) \) with radius \( \lambda(\hat{x}, \hat{u}) = \kappa_c(\hat{\eta}_2, \hat{x}, \hat{u}) \).

we choose a parametrisation for \( \kappa_c \) similar to (10), i.e.,

\[
\kappa(c)(\hat{\theta}), \hat{x}, \hat{u}) = \theta_1(\hat{x}, \hat{u})r + \theta_2(\hat{x}, \hat{u}),
\]

where \( r \in \mathbb{R}_{\geq 0}, \theta_1 \in \mathbb{R}^{n \times n}, \theta_2 \in \mathbb{R}^n, \) and \( \theta \in \mathbb{R}^{n^2+n} \) is constructed by concatenating columns of \( \theta_1 \) and \( \theta_2 \). The SCP associated with this growth bound is constructed by considering a uniform distribution over \( \Omega_{\eta_2}(\hat{x}) \times W \) and obtain \( 2N \) i.i.d. sample trajectories \( \Sigma_{2N} = \{(x_i, \hat{u}_i, x'_i) \mid x'_i \in \Phi(x_i, \hat{u}_i), i = 1, 2, \ldots, 2N \} \) so that every \( x'_i \) corresponds to a random disturbance \( w_i \in W \). The modified SCP, \( \gamma \) is defined as

\[
\gamma = 8L_{\psi}^4 \epsilon^2 \left( \prod_{i=1}^n \eta_c(i) \prod_{i=1}^n \hat{w}(i) \right)^2,
\]

where \( \epsilon \in [0, 1], n \) is the dimension of the state space, and \( L_{\psi}(\hat{u}) \) is the Lipschitz constant of the system trajectories \( \varphi(x, \hat{u}, w) \) with respect to \( (x, w) \).

Proof. The proof of this theorem is similar to that of Theorem 4. Define

\[
g(\theta, x_1, w_1, x_2, w_2) := \max \{||\varphi(x_1, \hat{u}, w_1) - \varphi(x_2, \hat{u}, w_2)|| - \theta_1(\hat{x}, \hat{u})||x_1 - x_2|| - \theta_2(\hat{x}, \hat{u})\}.
\]

To satisfy the inequality (4), we can choose

\[
h(\epsilon) = P(\Omega_c(d)) = \frac{1}{(\epsilon/2)^{2n}} \prod_{i=1}^n \eta_c(i) \prod_{i=1}^n \hat{w}(i)^2,
\]

since the distribution on \( (\Omega_{\eta_2}(\hat{x}) \times W)^2 \) is uniform. Using Equation (5), we have \( \gamma = L_{\psi} h^{-1}(\epsilon) \).

In order to prove that \( \gamma \) takes the value in (19), we must show that \( g \) is Lipschitz continuous with constant \( L_4 = 4L_{\psi}(\hat{u}) \). Bounding \( \|\theta_1(\hat{x}, \hat{u})\| \) by \( L_{\psi} \), for all \( (x_1, w_1, x_2, w_2) \) and \( (x'_1, w'_1, x'_2, w'_2) \) we have

\[
g(\theta, x_1, w_1, x_2, w_2) - g(\theta, x'_1, w'_1, x'_2, w'_2) \\
\leq ||\varphi(x_1, \hat{u}, w_1) - \varphi(x'_1, \hat{u}, w'_1)|| \\
+ ||\varphi(x_2, \hat{u}, w_2) - \varphi(x'_2, \hat{u}, w'_2)|| \\
+ ||\theta_1(\hat{x}, \hat{u})\||||x_1 - x'_1|| + ||x_2 - x'_2|| \\
\leq 4L_{\psi}(\hat{u})||x_1, w_1, x_2, w_2) - (x'_1, w'_1, x'_2, w'_2)||.
\]

TABLE I: Results for the DC-DC boost converter.

| Case-study | Dimension X-U | Disturbance W | Fixed Discretisation time (min) | \( |V| \) |
|------------|---------------|---------------|-------------------------------|------|
| DC-DC boost converter | 2 | 1 | \([-0.01, 0.01]\) | 1,807 | 22.2 | 37,783 |
| | | | | 2.285 | 30.6 | 37,414 |

Therefore, \( g \) is Lipschitz continuous with constant \( 4L_{\psi}(\hat{u}) \). This completes the proof.

A statement similar to Corollary 1 holds for the growth bound computed using (19).

VI. EXPERIMENTAL EVALUATION

To demonstrate our approach, we apply it to a DC-DC boost converter and a path planning problem. These case studies are taken from [26], [14] and will be used as black-box models to generate sample trajectories. We also introduce a case study from power systems based on [20], that is implemented in the Power System Toolbox (PST) [7]. We will use both trajectories from the black-box reduced model of the 30 state power system model. We apply our approach to construct finite abstractions of these systems and employ SCOTS [26] to design controllers. Our algorithms are implemented in C++ on a 64-bit Linux cluster machine with two Intel Xeon E5 v2 CPUs, 1866 MHz, and 50GB RAM.

A. DC-DC boost converter

The objective in the DC-DC boost converter problem is to design a controller to enforce a reach and stay specification. The DC-DC boost converter can be modelled as a two dimensional linear switching system with two functional modes. The state vector of the system at time \( t \in \mathbb{R}_{\geq 0} \) is \( x(t) = (i(t), v_c(t)) \), where \( i(t) \) is the inductor current and \( v_c \) is the capacitor voltage. The system’s evolution can be controlled by selecting the appropriate mode \( u(t) \) at every time \( t \in \mathbb{R}_{\geq 0} \). The system’s dynamics under the two modes can be represented as \( \dot{x} = A_{u_1}(x(t)) + b + c w(t), u \in \{1, 2\} \), with matrices \( A_{1}, A_{2}, b, c \) as reported in [14]. The state and input spaces are \( X = [0.65, 1.65] \times [4.95, 5.95] \) and \( U = \{1, 2\} \). The initial state is \( (i_{01}(t), v_{c0}(t)) = (0.7, 5.4) \) and the target set is \([1.1, 1.6] \times [5.4, 5.9]\). The target set is shown in red colour in Figure 1.

Our implementation results are reported in Table I for the system without disturbance \( \omega = (0, 0) \) and with disturbance bound \( \omega = (0.01, 0) \). These results are obtained with discretisation parameters \( \eta_x = (0.005, 0.005) \) and \( \eta_u = 1, \) confidence parameter \( \beta = 0.01, \) \( \epsilon = 0.01 \) and estimation for \( L_{\psi} = 0.9935 \). The resulted finite abstraction has cardinality \( n_x = 40,000 \) and \( n_u = 2 \). The required number of sample trajectories, \( N \), for each \( (\hat{x}, \hat{u}) \in \hat{X} \times \hat{U} \) is computed using equation (6). Runtimes and the resulting winning region sizes, \( |V| \), for the DC-DC boost converter are given in Table I.

We have used Algorithm 1 to compute the finite-state abstraction by collecting sample trajectories of the system. Subsequently, SCOTS is used for designing the controller. The performance of the controller is shown in Figures 1.
We consider a path planning problem for a vehicle that is modelled as
\[
\dot{x} = v \cos(\alpha + \theta) / \cos(\alpha) + w \\
\dot{y} = v \sin(\alpha + \theta) / \cos(\alpha) \\
\dot{\theta} = v \tan(\omega),
\]  
where the state variables \(x, y, \theta\) represent the position of the vehicle in the 2-dimensional space and the orientation of the vehicle, respectively. Inputs are \((v, \omega)\), the disturbance is \(w\), and \(\alpha := \arctan(\tan(\omega)/2)\). The state and input spaces are \(X = [0, 10] \times [0, 10] \times [-\pi - 0.4, \pi + 0.4]\) and \(U = [-1, 1]^2\), respectively. The goal is to find a controller to steer the vehicle from the initial state \((x_0, y_0, \theta_0) = (0, 1, 2, 0)\) to the target set \((x, y) \in [9, 9.51] \times [0, 0.51]\) while avoiding the obstacles. These obstacles are shown in blue colour in Figures 3 and 4.

We computed the growth bounds with a coarse discretisation \(\eta_x = (1.6, 1.6, 1.6)\) and reduced it iteratively with the factor of two. The algorithm successfully finds a controller for the system after five iterations. The implementation results are reported in Table II. These results are obtained with \(\eta_\psi = (0.3, 0.3)\), the confidence parameter \(\beta = 0.01\), \(\varepsilon = 0.01\) and estimated constant \(L_\varphi = 1.46\). The resulted abstraction has cardinalities \(n_x = 88, 500\) and \(n_u = 24\). For the case of disturbance-free model we set \(\bar{w} = (0, 0)\), and for the case of dynamics with disturbance, we set \(\bar{w} = (0.01, 0)\). The required number of sample trajectories for each \((\bar{\varphi}, \bar{u})\) is computed using Equation (6) and marked with \(N\) in the table. Finally, runtimes and size of the winning regions \(|V|\) are reported.

We have used the synthesis method based on abstraction refinement presented in Section V, to construct the finite-state abstraction by collecting sample trajectories of the system. We used SCOTS to design the controller fulfilling the given specification. The performance of the controller is shown in Figures 3 and 4 for the system without and with the disturbance, respectively. These figures compare the closed-loop trajectories of the system under the controllers designed by our data-driven abstraction approach (black) and by the model-based approach of SCOTS (red). Our data-driven approach successfully finds a controller for the system that satisfies the specification without the need for knowing the dynamics of the system.

### C. Three Area Three Machine Power System

We consider a three area three machine (3A3M) power system adapted from [20] and is shown in Figure 5. The system consists of three buses, which are each connected to a power source (generator) and a load. At bus 1 we consider a load which is bidirectional, meaning it can both draw power and inject power into the system. The loads at buses 2 and 3 can only draw power from the system; when these loads increase, more power will be drawn from the system, causing an imbalance between generation and consumption which may result in reduction of the network frequency. The nominal frequency of the network is set to 60 Hz.

![Fig. 1: The closed-loop trajectory of the DC-DC boost converter with \(\bar{w} = (0, 0)\) under the controller designed by our data-driven abstraction approach.](image1)

![Fig. 2: The closed-loop trajectory of the DC-DC boost converter with \(\bar{w} = (0.01, 0)\) under the controller designed by our data-driven abstraction approach.](image2)

### TABLE II: Results for the path planning case study.

| Case-study | Dimension | Disturbance | Abstraction Refinement |
|------------|-----------|-------------|------------------------|
| Path planning | 3, 2 | \((0, 0, 0)\) | 3.127 225 405, 493 |
| | | \((0.01, 0)\) | 4.277 513 447, 212 |

![Figures 3 and 4.](image3)
Fig. 3: Comparison between the closed-loop trajectories of the system (20) without disturbance under the controllers designed by our data-driven abstraction refinement approach (black) and by the model-based approach of SCOTS (red). Blue blocks represent the obstacles, the green dot represents the initial state, and the orange rectangle shows the target region.

Fig. 4: Comparison between the closed-loop trajectories of the system (20) with disturbance bound \( \bar{w} = (0.01, 0, 0) \) under the controllers designed by our data-driven abstraction refinement approach (black) and by the model-based approach of SCOTS (red).

Fig. 5: 3A3M power system with generators (G) and loads (L). L1 represents a bidirectional load such as Electric Vehicles or Energy Storage Systems.

where

\[
A = \begin{bmatrix}
0.00027563 & 0 & 0 \\
0 & -0.3951 & 0.687 \\
0 & -0.6869 & -0.016
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0.00031166 \\
0.1359 \\
0.0230
\end{bmatrix}
\]

\[
E = \begin{bmatrix}
0.00033103 & 0.00031244 \\
0.1309 & 0.1308 \\
0.0250 & 0.0233
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
-0.0115 & -0.2296 & 0.0412
\end{bmatrix}.
\] (22)

The state and input spaces are \( X = [-0.02, 0.02] \times [-0.05, 0.05] \times [-0.12, 0.12] \) and \( U = [0, 0.5] \). Further, we set \( W = [-0.2, 0.2] \times [-0.3, 0.3] \), \( \eta_u = 0.025 \), \( \tau = 0.4 \), \( \eta_e = (0.0015, 0.0015, 0.0015) \), \( \beta = 0.01 \) and \( \varepsilon = 0.01 \). The resulted abstraction has \( n_x = 228 \) and \( n_u = 20 \). The estimated Lipschitz constant is \( L_{\varphi} = 1.5715 \). The target set is given by \( -0.008 < y < 0.008 \) and the avoid set is given by \( y < -0.015 \). Multiplying by the nominal frequency to get the specification in Hertz, the target region is \([59.52, 60.48] \) and the avoid region is \((-\infty, 59.1) \). Figure 6 shows that the specification is violated when no control is applied.

We apply the data-driven approaches of Section IV (fixed discretisation) and Section V (abstraction refinement). Both controllers are synthesised with disturbance \( W = [-0.2, 0.2] \times [-0.3, 0.3] \). A comparison of the two control approaches is shown in Table III. The required number of sample trajectories for each \( (\bar{x}, \bar{u}) \) is computed using equation (6) and marked with \( N \) in the table. The abstraction refinement starts with \( \eta_e = 0.012 \) and refines the discretisation iteratively with a factor of two. The algorithm successfully finds a controller after five iterations. The runtimes and the resulting winning region sizes \( |V| \) are also given in Table III. The abstraction refinement synthesises the controller a factor...
of 100 times faster than the fixed discretisation by iteratively decreasing the value of $\eta_x$.

**TABLE III: Results for the 3A3M power system.**

| Control Approach          | Dimension | Disturbance | $N$ time (min) | $|V|$ |
|---------------------------|-----------|-------------|----------------|------|
| Fixed Discretisation      | 3 1       | (0.2, 0.3)  | 3,290 5.253    | 230,760 |
| Adaptive Refinement       |           | (0.2, 0.3)  | 4,460 50.25    | 314,802 |

The data-driven control approach with fixed discretisation is simulated in PST and is reported in Figures 7 and 8. The controlled system successfully keeps the frequencies of the three areas outside of the avoid set (i.e., always above 59.1 Hz) and bring them back to the target set (i.e., above 59.5 Hz). Figure 8 shows the load changes in the system. Load at bus 1 is able to maintain the frequencies of the three areas above the avoid region and facilitate the system returning to the target set for the maximum disturbances applied at buses 2, 3. Figures 9 and 10 show the results of simulating the system in PST with the control obtained from the abstraction refinement approach. The controlled system has the same performance in satisfying the specification.

**D. Comparison with PAC Learning**

In this subsection, we compare our approach with the results provided by Xue et al. [38] that is based on probably approximately correct (PAC) learning on the 3A3M power system case study. The abstraction approach of [38] has no bias term $\gamma$, but uses confidence parameter $\beta \in (0, 1)$, error level $\epsilon \in (0, 1)$, and cardinality of the parameter vector $\theta$ denoted by $q \in \mathbb{N}$. The required number of samples is

$$N \geq \frac{2}{\epsilon} \left( \ln \frac{1}{\beta} + q \right),$$  \hspace{1cm} (23)

which allows the constructed abstraction to hold for the entire state space except a subset measured by parameter $\epsilon$.

We implement our data-driven robust scenario approach (RSA), the PAC approach in [38] with parameters $\beta = 0.01$ and $\epsilon = 0.01$.

**TABLE IV: Comparing the winning domain of controllers obtained from our RSA method, PAC method of [38], and the model-based approach of [25].** The pairwise comparison is made by computing the intersections (\(\cap\)) and set differences (row \(\setminus\) column). The results are reported both in cardinalities and percentages.

| Winning Domain | RSA \(\cap\) | PAC \(\cap\) | Model-based \(\cap\) |
|----------------|--------------|--------------|---------------------|
| RSA            | 230,760      | 0            | 230,760             | 0                        |
| %              | 100.00%      | 0.00%        | 100.00%             | 0.00%                    |
| PAC            | 230,760 15,664 | 246,424      | 245,345 1,079       |
| %              | 93.64% 6.36% | 100.00%      | 99.56% 0.44%        |
| Model-based    | 230,760 22,216 | 245,345 7.631 | 252,976 0          |
| %              | 91.22% 8.78% | 96.98% 3.02% | 100.00% 0.00%      |
It includes states not identified winning by the model-based approach. In contrast, the PAC approach gives a winning domain that is likely to provide an overly conservative controller. On this case study, note that our sampling approach uses the Lipschitz constant estimated using sample trajectories. This Lipschitz constant can in turn be used to construct the abstraction. The direct use of the estimated Lipschitz constant does not provide a formal guarantee as it is an estimated value that converges to the true value only in the limit (i.e., the number of samples goes to infinity), and is likely to provide an overly conservative controller. On this particular case study, the direct use of the Lipschitz constant gives a controller that covers only 78.8% of the winning domain of the model-based approach.

As a final point on this case study, note that our sampling approach uses the Lipschitz constant estimated using sample trajectories. This Lipschitz constant can in turn be used to construct the abstraction. The direct use of the estimated Lipschitz constant does not provide a formal guarantee as it is an estimated value that converges to the true value only in the limit (i.e., the number of samples goes to infinity), and is likely to provide an overly conservative controller. On this particular case study, the direct use of the Lipschitz constant gives a controller that covers only 78.8% of the winning domain of the model-based approach.

E. Parameter Optimisation

In this subsection, we discuss how selection of different parameters can affect the sample complexity and conservativeness of our method. We fix the path planning case study with the estimated Lipschitz constant 1.46. Figures 11 and 12 illustrate the effect of changing parameters $\epsilon, \beta$ on the number of samples $N$ required for each pair $(\hat{x}, \hat{u})$ in order to compute the growth bound with confidence $(1 - \gamma)$. Figure 11 illustrates the effect of increasing the confidence parameter $\beta$ on reducing the sample complexity, for a fixed $\epsilon = 0.01$. Figure 12 shows that for a fixed $\beta = 0.01$, increasing $\epsilon$ leads to a rapid drop in $N$. In both Figures 11 and 12, the sample complexity increases in the presence of disturbance as the dimension of the sample space becomes larger.

Figure 13 demonstrates the effect of changing $\epsilon$ on the value of the bias term $\gamma$ that makes the inequalities of the SCP more conservative. The bias term $\gamma$ increases for larger values of $\epsilon$. Therefore, increasing $\epsilon$ can decrease the sample complexity while increasing $\gamma$. Finally, it can be observed that the value of $\gamma$ is larger in the presence of disturbance.

VII. DISCUSSION AND FUTURE WORK

We proposed a data-driven method for computing finite abstractions of continuous systems with unknown dynamics. Our approach casts the computation of an overapproximation of reachable sets as a robust convex program (RCP). A feasible solution for the RCP is then obtained with a
given confidence by solving a corresponding scenario convex program (SCP). The SCP does not need the dynamics of the system and requires only a finite set of sample trajectories. We provided a sample complexity result that gives a lower-bound on the number of trajectories to achieve a certain confidence. Our sample complexity results requires knowing a bound on the Lipschitz constant of the system, that we estimated using extreme value theory.

We guaranteed that with high confidence, the computed abstraction is a valid abstraction of the system that over-approximates its behaviours on its entire state space. We showed that our data-driven approach can be embedded into abstraction refinement schemes for designing a controller and enlarging the winning region of the controller with respect to satisfaction of temporal properties. Finally, we evaluated our approach on three case studies.

In the future, we plan to extend our approach by enlarging the class of disturbances beyond piece-wise constant ones (i.e., tackling the issue of infinite dimensional sampling spaces), improve scalability of the approach by providing more efficient parallel implementation of the approach, and apply it to large case studies that are combinations of differential equations, block diagrams, and lookup tables.

REFERENCES

[1] Alessandro Abate, Maria Prandini, John Lygeros, and Shankar Satsry. Probabilistic reachability and safety for controlled discrete time stochastic hybrid systems. *Automatica*, 44(11):2724–2734, 2008.

[2] Christel Baier and Joost-Pieter Katoen. *Principles of model checking*. MIT Press, 2008.

[3] Andrea Bajcsy, Somil Bansal, Eli Bronstein, Varun Tolani, and Claire J Tomlin. An efficient reachability-based framework for provably safe autonomous navigation in unknown environments. In 2019 IEEE 58th Conference on Decision and Control (CDC), pages 1758–1765. IEEE, 2019.

[4] Calin Belta, Boyan Yordanov, and Ebru Aydin Gol. *Formal methods for discrete-time dynamical systems*, volume 15. Springer, 2017.

[5] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[6] Giuseppe Carlo Calafiore and Marco C Campi. The scenario approach to robust control design. *IEEE Transactions on automatic control*, 51(5):742–753, 2006.

[7] J.H. Chow and K.W. Cheung. A toolbox for power system dynamics and control engineering education and research. *IEEE Transactions on Power Systems*, 7(4):1559–1564, 1992.

[8] Max H. Cohen and Calin Belta. Model-based reinforcement learning for approximate optimal control with temporal logic specifications. In *HSCC ’21: 24th ACM International Conference on Hybrid Systems: Computation and Control*, pages 12:1–12:11. ACM, 2021.

[9] Alex Devonport, Adnane Saoud, and Murat Arcak. Symbolic abstractions from data: A pac learning approach. *arXiv preprint arXiv:2104.13991*, 2021.

[10] Franck Djeumou, Abraham P Vinod, Eric Goubault, Sylvie Putot, and Ufuk Topcu. On-the-fly control of unknown systems: From side information to performance guarantees through reachability. *arXiv preprint arXiv:2011.05324*, 2020.

[11] Peyman Mohajerin Esfahani, Tobias Sutter, and John Lygeros. Performance bounds for the scenario approach and an extension to a class of non-convex programs. *IEEE Transactions on Automatic Control*, 60(1):46–58, 2014.

[12] Chuchu Fan, Bolun Qi, Sayan Mitra, and Mahesh Viswanathan. Dvr: Data-driven verification and compositional reasoning for automotive systems. In *Computer Aided Verification - 29th International Conference, CAV 2017, Heidelberg, Germany, July 24-28, 2017. Proceedings, Part I*, volume 10426 of *Lecture Notes in Computer Science*, pages 441–461. Springer, 2017.
[13] Antoine Girard and George J Pappas. Approximation metrics for discrete and continuous systems. *IEEE Transactions on Automatic Control*, 52(5):782–798, 2007.

[14] Antoine Girard, Giordano Pola, and Paolo Tabuada. Approximately bisimilar symbolic models for incrementally stable switched systems. *IEEE Transactions on Automatic Control*, 55(1):116–126, 2009.

[15] Kush Grover, Fernando dos Santos Barbosa, Jana Tumova, and Jan Kretinsky. Semantic abstraction-guided motion planning for scit missions in unknown environments. In *Robotics: Science and Systems*, 2021.

[16] Kai-Chieh Hsu, Vicenc Rubies-Royo, Claire J Tomlin, and Jaime F Fisac. Safety and liveness guarantees through reach-avoid reinforcement learning. In *Robotics: Science and Systems*, 2021.

[17] Krishna C Kalagarla, Rahul Jain, and Pierluigi Nuzzo. Model-free reinforcement learning for optimal control of markov decision processes under signal temporal logic specifications. *arXiv preprint arXiv:2109.13377*, 2021.

[18] Abolfazl Lavaei, Ameneh Nejati, Pushpak Jagtap, and Majid Zamani. Formal safety verification of unknown continuous-time systems: A data-driven approach. In *Proceedings of the 24th International Conference on Hybrid Systems: Computation and Control*, HSCC ’21, New York, NY, USA, 2021. Association for Computing Machinery.

[19] Benoit Legat, Raphael M Jungers, and Jean Bouchat. Abstraction-based branch and bound approach to q-learning for hybrid optimal control. In *Learning for Dynamics and Control*, pages 263–274. PMLR, 2020.

[20] Minyue Ma and Lingling Fan. Implementing consensus based distributed control in power system toolbox. In *2016 North American Power Symposium (NAPS)*, pages 1–6, 2016.

[21] Rupak Majumdar, Necmiye Ozay, and Anne-Kathrin Schmuck. Approximation metrics for discrete and continuous systems. *IEEE Transactions on Automatic Control*, 52(5):782–798, 2007.

[22] Antoine Girard, Giordano Pola, and Paolo Tabuada. Approximately bisimilar symbolic models for incrementally stable switched systems. *IEEE Transactions on Automatic Control*, 55(1):116–126, 2009.

[23] Kush Grover, Fernando dos Santos Barbosa, Jana Tumova, and Jan Kretinsky. Semantic abstraction-guided motion planning for scit missions in unknown environments. In *Robotics: Science and Systems*, 2021.

[24] Kai-Chieh Hsu, Vicenc Rubies-Royo, Claire J Tomlin, and Jaime F Fisac. Safety and liveness guarantees through reach-avoid reinforcement learning. In *Robotics: Science and Systems*, 2021.

[25] Krishna C Kalagarla, Rahul Jain, and Pierluigi Nuzzo. Model-free reinforcement learning for optimal control of markov decision processes under signal temporal logic specifications. *arXiv preprint arXiv:2109.13377*, 2021.

[26] Abolfazl Lavaei, Ameneh Nejati, Pushpak Jagtap, and Majid Zamani. Formal safety verification of unknown continuous-time systems: A data-driven approach. In *Proceedings of the 24th International Conference on Hybrid Systems: Computation and Control*, HSCC ’21, New York, NY, USA, 2021. Association for Computing Machinery.

[27] Benoit Legat, Raphael M Jungers, and Jean Bouchat. Abstraction-based branch and bound approach to q-learning for hybrid optimal control. In *Learning for Dynamics and Control*, pages 263–274. PMLR, 2020.

[28] Minyue Ma and Lingling Fan. Implementing consensus based distributed control in power system toolbox. In *2016 North American Power Symposium (NAPS)*, pages 1–6, 2016.

[29] Antelope Girard and Laurent Fribourg. Efficient data-driven abstraction of monotone systems with disturbances. In *7th IFAC Conference on Analysis and Design of Hybrid Systems, ADHS 2021, Brussels, Belgium, July 7–9, 2021*, volume 54 of *IFAC-PapersOnLine*, pages 49–54. Elsevier, 2021.

[30] Ioanna Mitsioni, Pouriya Tajvar, Danica Kragic, Jana Tumova, and Christian Pek. Safe data-driven contact-rich manipulation. In *2020 IEEE-RAS 20th International Conference on Humanoid Robots (Humanoids)*, pages 120–127. IEEE, 2020.

[31] Peyman Mohajerin Esfahani, Tobias Sutter, and John Lygeros. Performance bounds for the scenario approach and an extension to a class of non-convex programs. *IEEE Transactions on Automatic Control*, 60(1):46–58, 2015.

[32] G. Reissig, A. Weber, and M. Rungekk. Feedback refinement relations for the synthesis of symbolic controllers. *IEEE TAC*, 62(4):1781–1796, 2016.

[33] Matthias Rungger and Majid Zamani. Scoto: A tool for the synthesis of symbolic controllers. In *Proceedings of the 9th international conference on hybrid systems: Computation and control*, pages 99–104, 2016.

[34] Sadra Sadradini and Calin Belta. Formal guarantees in data-driven model identification and control synthesis. In *Proceedings of the 21st International Conference on Hybrid Systems: Computation and Control (part of CPS Week), HSCC 2018, Porto, Portugal, April 11-13, 2018*, pages 147–156. ACM, 2018.

[35] Ali Salamati, Abolfazl Lavaei, Sadegh Soudjani, and Majid Zamani. Data-driven verification and synthesis of stochastic systems through barrier certificates. *arXiv preprint arXiv:2111.10130*, 2021.

[36] Stanly Samuel, Kaushik Mallik, Anne-Kathrin Schmuck, and Daniel Neider. Resilient abstraction-based controller design. In *HSCC ’20-23rd ACM International Conference on Hybrid Systems: Computation and Control, Sydney, New South Wales, Australia, April 21-24, 2020*, pages 33:1–33:2. ACM, 2020.

[37] Sadegh Soudjani and Rupak Majumdar. Concentration of measure for chance-constrained optimization. *IFAC-PapersOnLine*, 51(16):277–282, 2018.

[38] Dawei Sun, Susmit Jha, and Chuchu Fan. Learning certified control using contraction metric. *arXiv preprint arXiv:2011.12560*, 2020.

[39] Paulo Tabuada. Verification and Control of Hybrid Systems: A Symbolic Approach. Springer Publishing Company, Incorporated, 1st edition, 2009.

[40] Cees F Verdier, Niklas Kochdumper, Matthias Althoff, and Manuel Mazo Jr. Formal synthesis of closed-form sampled-data controllers for nonlinear continuous-time systems under stl specifications. *arXiv preprint arXiv:2006.04260*, 2020.

[41] Xiao Wang, Saasha Nair, and Matthias Althoff. Falsification-based robust adversarial reinforcement learning. In *2020 19th IEEE International Conference on Machine Learning and Applications (ICMLA)*, pages 205–212. IEEE, 2020.

[42] Kandali Watanabe, Nicholas Renninger, Sriram Sankaranarayanan, and Mortezat Lahijanian. Probabilistic specification learning for planning with safety constraints. In *Intelligent Robots and Systems (IROS)*, page TBA. IEEE, 2021.

[43] Tsu-Wei Weng, Huan Zhang, Pin-Yu Chen, Jinfeng Yi, Dong Su, Yu peng Gao, Cho-Jui Hsieh, and Luca Daniel. Evaluating the robustness of neural networks: An extreme value theory approach. In *International Conference on Learning Representations*, 2018.

[44] GR Wood and BP Zhang. Estimation of the lipschitz constant of a function. *Journal of Global Optimization*, 8(1):91–103, 1996.

[45] Bai Xue, Miaomiao Zhang, Arvind Easwaran, and Qin Li. Pac model checking of black-box continuous-time dynamical systems. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 39(11):3944–3955, 2020.