REMARKS ON MUKAI THREEFOLDS ADMITTING $\mathbb{C}^*$ ACTION

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Abstract. We investigate geometric invariants of the one parameter family of Mukai threefolds that admit $\mathbb{C}^*$ action. In particular we find the invariant divisors in the anticanonical system, and thus establish a bound on the log canonical thresholds. Furthermore we find an explicit description of such threefolds in terms of the quartic associated to the variety-of-sum-of-powers construction. This yields that any such threefold admits an additional symmetry which anticommutes with the $\mathbb{C}^*$ action, a fact that was previously observed near the Mukai-Umemura threefold in [RST]. As a consequence the Kähler-Einstein manifolds in the class form an open subset in the standard topology.

Introduction

The family $V_{12}$ of compact complex threefolds with genus 12 found by Iskovskih [Is1, Is2], (widely known as Mukai threefolds) is the source of many interesting examples in complex geometry. In fact this class of threefolds was missed by Fano in his classification list (see [Is1, Is2]). We refer to [M] for their basic properties.

The interest of complex differential geometry towards Mukai threefolds stems from the fact that a subclass of these appears in the famous example of Tian ([Ti2]) of a compact Fano manifold with no nonzero holomorphic vector fields yet with no Kähler-Einstein metric. In fact before Tian’s paper a folklore conjecture expected holomorphic vector fields to be the only obstruction of existence of such metrics. This example partially motivated a suitable notion of stability ($K$-stability defined by Tian in [Ti2]) which turned out to be the right algebro-geometric condition equivalent to the existence of Kähler Einstein metrics. We refer to [Ti2, Ti3] for the history of the problem, and to [CDS1, CDS2, CDS3, Ti4] for its final solution. In a sense the family of Mukai threefolds provides an excellent test polygon for such investigations.

Below we list what is known on the Kähler-Einstein problem with regard to the family $V_{12}$ so far. First of all of crucial importance is the (identity component of the) automorphism group of such threefolds. This was an object of intensive research (see [P] and references therein). It turns out that a generic Mukai threefold has a discrete group of automorphisms (thus the identity component is just a singleton). The remaining cases were classified by Prokhorov ([P]). If $G$ is the identity component in $\text{Aut}(X)$ then all the possibilities are as follows:

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Theorem 0.1. The identity component of the automorphism group $G$ of $V_{12}$ is a singleton except in the following cases:

1) $X = V_{MU}$ is the Mukai-Umemura manifold, then $G = SL(2)$;
2) $X$ is a member of the family $V^a$, then $G = \mathbb{C}^*$;
3) $X = V^m$ then $G = \mathbb{C}^+$.

The existence of Kähler-Einstein metrics was investigated in detail near the Mukai-Umemura manifold (see [Ti2, D1, D2, RST]). In particular the Mukai-Umemura threefold itself admits such a metric ([D2]) whereas some small deformations of it do not ([Ti2]).

On the other hand it is also interesting to consider the problem globally i.e. in the whole family rather than near a single member. It should be emphasized that the deformation theory near the Mukai-Umemura example is considerably different than around a generic element of the family. One of the reasons is exactly because this example is the unique one having a maximal symmetry group. Below we briefly recall what is known in the global setting.

The case of discrete group of automorphisms is reasonably understood. Indeed, coupling the results of Tian ([Ti2]) and Donaldson ([D1]) with the theorem of Odaka [O] (see also the recent paper [D3]) we get the following observation:

Observation 0.2. The subset of $M$ parametrizing Mukai threefolds with discrete group of automorphisms that admit Kähler-Einstein metrics is Zariski open and nonempty. However, there are special Mukai threefolds (for example class 4 and 5 in Donaldson classification, see [D1]) with discrete automorphism group that do not admit Kähler-Einstein metrics.

In this paper we focus on the remaining cases i.e. when the automorphism group is infinite.

First, as $\mathbb{C}^+$ is not a reductive group the manifold $V^m$ does not admit a Kähler-Einstein metric by the classical Matsushima theorem [Ma].

Secondly, we already mentioned above that the Mukai-Umemura manifold admits a Kähler-Einstein metric.

Thus the only remaining case is the family of manifolds $V^a$ with the identity component of the group of automorphisms isomorphic to $\mathbb{C}^*$. Small deformations of the Mukai-Umemura example living in this class were investigated by Donaldson [D1] and Rollin-Simanca-Tipler [RST] (they correspond to the class 3 in the list in [D1]), and it turns out that some deformations of the Mukai-Umemura example living in this class do possess a Kähler-Einstein metric. A natural question, raised by Donaldson in [D1], is to classify the Kähler-Einstein examples in the family.

Our goal in the present note is to study the geometry of the generic element in the family $V^a$. We shall present three classical constructions of this manifold: as variety of sums of powers (see [M1]), by birational transformations of the Fano threefold $V_5$ of degree 5 (this was the original construction of Prokhorov), finally as a subset of the the Grassmannian $G(4, 7)$. We describe the relations between these constructions. In particular in Section 6 a Macaulay 2 program is presented. This program describes the quartic being the Hilbert scheme of lines on a Fano manifold in $V_{12}$ constructed as a subset.
of $G(4,7)$. The Hilbert scheme of lines is the covariant quartic of another quartic that gives rise through the variety of sums of powers construction the Fano threefold in $V_{12}$ back.

First of all we discuss the ”standard” approach to the problem of existence of Kähler-Einstein metrics by analyzing the log-canonical thresholds (equivalently: alpha invariants) and exploiting the symmetries such manifolds have. Our result, as expected, shows that the natural symmetries are not enough to solve affirmatively the Kähler-Einstein problem.

**Theorem 0.3.** For any element of the family $V^a$ one has

$$\text{lct}(V^a, \mathbb{C}^*) \leq \frac{1}{2}.$$  

More importantly we were able to construct all the $\mathbb{C}^*$-invariant divisors in the linear system $| - K_V|$ for any $V \in V^a$.

The construction by variety of sum of powers, due to Mukai ([M]) was used to characterize the Mukai-Umemura threefold. It turns our that the associated plane quartic in this case is just a double conic. Our next result is the following characterization of Mukai threefolds admitting $\mathbb{C}^*$ action, which can be seen as a generalization of Mukai theorem:

**Proposition 0.4.** Let $V$ be a member of the family $V^a$ of Mukai threefolds admitting (nontrivial) $\mathbb{C}^*$ action. Then the associated plane quartic in the variety of sum of powers construction is formed by two tangent conics. In the case of Mukai-Umemura threefold these conics coincide (i.e. we get a double conic), while for $V \in V^a$ the conics are tangent to each other at two points.

We are also able to compute the quartic associated to the manifold $V^m$:

**Proposition 0.5.** The plane quartic associated to $V^m$ in the VSP construction is the sum of two conics tangent to each other at one point with equation

$$P_4(a, b, c) = (c^2 - ab)^2 + a^4.$$  

Our next observation is that any member of $V^a$ has an additional holomorphic involution $\iota$ which does not commute with the $\mathbb{C}^*$ action. This was observed earlier in [RST] for small deformations of the Mukai-Umemura threefold.

Finally we consider the group $H = < \mathbb{C}^*, \iota > = \mathbb{Z}_2 \rtimes \mathbb{C}^*$, and its compact subgroup $W$ generated by the circle action and $\iota$. Note that by Bando-Mabuchi theorem [BM] if a Kähler-Einstein metric exists then one can find a $W$-invariant one. Then an argument that we learned from [PSSW] yields the following:

**Theorem 0.6.** The set of elements in $V^a$ admitting $W$-invariant Kähler-Einstein metrics is open in the Euclidean topology.

**Remark 0.7.** In view of the recent paper of Donaldson [D3] it seems reasonable that the set above is actually Zariski open.

In our note we heavily rely on Macaulay 2 computations. The relevant scripts are incorporated into the note.
When finishing our note we learned about Rollin-Simanca-Tipler paper [RST]. There, among other things, the case of small deformations near the Mukai-Umemura example was considered. In particular we would like to point out that the openness of Kähler-Einstein manifolds in the family $V^a$ follows from their arguments if one knows the existence of the symmetry $\iota$ for all elements of $V^a$. On the other hand our argument seems to be a bit simpler albeit much less general (of course both approaches rely on the implicit function theorem). More importantly it seems to emphasize the group action viewpoint so we decided to include it.

1. Notation and basic definitions

Denote as usual by $\mathbb{C}^+$ the group of complex numbers with addition, and by $\mathbb{C}^*$ of non-zero complex numbers with multiplication.

The Grassmannian of $k$-planes in $\mathbb{C}^n$ will be denoted by $G(k, n)$.

Recall that a compact complex manifold $X$ is Fano if its first Chern class $c_1(X)$ is positive or, equivalently, its anti-canonical line bundle $O_X(-K_X)$ is ample.

Given any Fano threefold the integer $g = 1/2(-K_X)^3 + 1$ is called the genus of $X$. If the Picard group of $X$ is generated by $O_X(-K_X)$ we call such a Fano 3-fold prime. Prime Fano threefolds were classified by Iskovskih [Is1, Is2] with respect to their genus. In particular we have either $g \leq 10$ or $g = 12$.

Given an effective $\mathbb{Q}$-divisor $D$ on a compact complex manifold $X$ the multiplier ideal sheaf $J(X, D)$ is defined as follows: if $\mu : X' \to X$ is a log resolution then

$$J(X, D) := \mu_* O_{X'}(K_{X'/X} - \lfloor \mu^* D \rfloor),$$

where $K_{X'/X}$ denotes the relative canonical bundle, whereas $\lfloor A \rfloor$ stands for the round down of the $\mathbb{Q}$-divisor $A$, i.e., $\lfloor A \rfloor = \sum_i [a_i] A_i$ if $A = \sum_i a_i A_i$.

An alternative analytic way to define this sheaf (see [De]) is as follows: take a local trivializing section $\sigma$ of the line bundle $O(-D)$ and equip it with a (singular) metric $h$ such that in any trivialization $||\zeta||_h^2 = ||\zeta(z)||^2 |\sigma(z)|^2$.

Then the stalk of $J$ at $z$ is defined by

$$J(X, D)_z = \{ f \in O_{X,z} | \exists \text{ neighborhood } U \text{ of } z : \int_U |f|^2 e^{-2\log|\sigma|} < \infty \}.$$

The log canonical threshold associated with the pair $(X, D)$ is defined by

$$lct(X, D) := \sup \{ \lambda \in \mathbb{Q} | \text{ the pair } (X, \lambda D) \text{ is log canonical} \},$$

where log canonicity is defined by the condition that $J(X, (1-\varepsilon)\lambda D) = O_X$ for all $0 < \varepsilon < 1$. Alternatively, the pair $(X, D)$ is log canonical if there is a log resolution $\mu : X' \to X$, such that for a normal crossing divisor $D' = \sum_i E_i + \mu_*^{-1} D$ ($E_i$ are the exceptional divisors of the resolution) we have

$$K_{X'} + D' = \mu^*(K_X + D) + \sum a_i E_i,$$

with all of the coefficients $a_i$ satisfying $a_i \geq -1$. 
Given a Fano manifold $X$ the associated global log canonical threshold can be computed simply by taking the infimum over the all log canonical thresholds associated to the divisors that are $\mathbb{Q}$-numerically equivalent to $-K_X$; that is

$$lct(X) := \inf \{lct(X, \frac{1}{n}D) \mid D \in \llbracket -nK_X \rrbracket, \ n \in \mathbb{N} \}.$$ 

Analogously one can define a global log canonical threshold $lct(X, L)$ on any polarized manifold $(X, L)$.

Finally these notions have their $G$-invariant counterparts if $G$ is any subgroup of $\text{Aut}(X)$:

the global $G$ invariant log canonical threshold is defined by

$$lct(X, G) = \sup \{\lambda \in \mathbb{Q} \mid \text{the pair } (X, \frac{\lambda}{n}D) \text{ is log canonical for any } D \text{ in any } G \text{ invariant linear subsystem } D \subset | -nK_X |, \ n \in \mathbb{N} \}.$$ 

It should be noted that the stalks of the sheaf $\mathcal{J}(X, D)$ crucially depend on the singularity of the divisor $D$ at a given point. Roughly speaking the log canonical threshold for a single divisor measures “how singular” the divisor can be at a point. The computation of such thresholds is in general quite subtle but there are special cases where fairly general formulas are available. In particular below we recall the following inequality due to Kollár ([K], Propositions 8.13 and 8.14) which is particularly useful when one deals with weighted homogenous divisors:

**Theorem 1.1.** Let $f$ be a holomorphic function near $0 \in \mathbb{C}^n$ and let $D = \{f = 0\}$. Assign integral weights $w(x_i)$ to the variables and let $w(f)$ be the lowest weight of all monomials occurring in the Taylor expansion of $f$. Then

$$lct(D) \leq \sum_i \frac{w(x_i)}{w(f)}.$$

We refer to [La] and [K] for more background and to the recent article [ChS] for many explicit computations of log canonical thresholds.

The log canonical threshold can also be computed analytically (see the appendix to [ChS] by J. P. Demailly) and it equals the so-called $\alpha$ invariant defined by Tian ([Ti1]).

The fundamental fact, proven by Tian ([Ti1]), is that if the alpha invariant is large enough then the manifold admits a Kähler-Einstein metric.

**Theorem 1.2.** If for a compact group $G$ the log canonical threshold (equivalently the alpha invariant) satisfies

$$lct(X, G) > \frac{n}{n+1}, \ n = \dim_{\mathbb{C}} X,$$

then $X$ admits a $G$-invariant Kähler-Einstein metric.

### 2. Prime Fano threefolds $V$ of genus 12.

In this section we recall several equivalent constructions of Mukai threefolds:
2.1. Constructions of $V_{12}$ I: Variety of sum of powers. Let us start by recalling an algebraic fact proved by Kleppe [Kl], see also [Pa]:

Any homogenous polynomial of three complex variables of degree 4 can be written as at most 7 fourth powers of linear forms. Moreover, there is only one (up to a linear change of variables) polynomial $x_0x_1^3 + x_1^2x_2^2$ that cannot be written as six fourth powers of linear forms.

This Waring type decomposition will be crucial in the construction of the manifolds in $V_{12}$ that we present below.

Let $F_1 = 0 = C \subset \mathbb{P}^2$ be a quartic curve such that the quartic defining $C$ cannot be written as the sum of five (or less) fourth powers of linear forms or a sum of six fourth powers of linear forms defining a quadrangle with its diagonals. Then we construct the associated manifold $VSP(C, 6)$ by

$$VSP(C, 6) := \text{cl}\{([l_1], \ldots, [l_6]) \in \mathbb{P}^{2*} | F_1 \leq l_1^4, \ldots, l_6^4 \}/S_6,$$

where $\text{cl}$ denotes the closure in the Hilbert scheme of six points in the dual projective plane $\mathbb{P}^{2*}$ and the whole construction is made invariant under the permutation group $S_6$.

Mukai observed [M] that each Fano threefold $V_{12}$ is isomorphic to $VSP(C, 6)$ for some quartic as above, moreover such $C$ is unique up to isomorphism. In particular the Mukai-Umemura threefold corresponds to $C$ being a double conic (see [M]).

Our proposition classifies the family $V^a$ and the manifold $V^m$ in this moduli:

**Proposition 2.1.** The manifolds from the family $V^a$ correspond to $VSP(\Gamma_1, 6)$, with the curve $\Gamma_1$ being a pair of conics tangent at two points. The manifold $V^m$ corresponds to a pair of conics tangent at one point.

The proof is given in Section 3.

2.2. Constructions of $V_{12}$ II: Birational morphisms to $V_5$. Let $V_5$ be a smooth Fano threefold with $Pic(V_5) = Z[H]$, such that $H^3 = 5$. Then $K_{V_5} = -2H$. Explicitly $V_5$ can be realized exploiting the embedding given by $H$ and it is defined as a generic codimension 3 linear section of the Grassmanian $G(2, 5) \subset \mathbb{P}^{35}$.

In such a way we can see that $V_5$ is rigid and can be described as follows: consider the natural action of $SL_2(\mathbb{C})$ on $\mathbb{P}(\mathbb{C}_6[x, y])$. Then $V_5$ can be seen as the closure of the orbit $U$ of $xy(x^4 - y^4)$ by this action. One can check that the manifold admits a stratification

$$V_5 = U \cup R \cup C,$$

where $R$ is the orbit of $x^5y$ and $C$ is the orbit of $x^6$.

Denote by $(x_0, x_1, x_2, x_3, x_4, x_5, x_6)$ the coordinates corresponding to $(x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6)$ in $\mathbb{P}(\mathbb{C}_6[x, y])$. Consider the divisors $F_1$ and $F_2$ defined by $x_0 = 0$ and $x_6 = 0$ respectively.

The orbits of the action of $\mathbb{C}^*$ on $V_5$ have four fixed points corresponding to $x^6, x^5y, y^5x$ and $y^6$. Let us describe the other orbits:

- the closure of the generic orbit of $\mathbb{C}^*$ is a rational curve of degree 6 such that the closure of such a curve contains the points $x^6$ and $y^6$.

These orbits cover the set $V_5 - (F_1 \cup F_2)$. 

The line passing through $x^5y$ and $y^6x$ is the closure of an orbit that we will denote by $f$.

The closure of the remaining orbits are rational normal curves of degree 5, such that $F_2 \setminus F_1$ (resp. $F_1 \setminus F_2$) is covered by orbits whose closure contains the points $x^6$ and $y^5x$ (resp. $y^6$ and $x^5y$). Moreover, $F_1 \cap F_2$ is the sum of $f$ and a rational normal curve of degree 5 that we denote by $t$.

Recall now the Prokhorov construction of threefolds from the family $V^a[7]$. Let us choose $r \subset F_1 \subset V_5$ a rational normal curve of degree 5 different from $t$ being the closure of an orbit of the $\mathbb{C}^*$ action.

Next we define the following diagram of birational maps: first $\varphi: W \to V_5$ is the blow-up of $V_5$ at the curve $r$, $Y = \varphi^{-1}(Y)$. Let $\Sigma \subset W$ be the strict transform of $t$ and $\rho: W \to W'$ be a flop of $\Sigma$. Finally let $\pi: W' \to V$ be a contraction of an extremal ray. Then $V$ is a smooth prime Fano threefold of genus 12 with identity component of the automorphism group isomorphic to $\mathbb{C}^*$. Furthermore the second Betti number $b_2$ equals 1. The morphism $\pi$ contracts the divisor $F' \subset W'$, being the strict transform of $F_1$, to a line $Z \subset V^a$ with normal bundle $\mathcal{O}_Z(1) \oplus \mathcal{O}_Z(-2)$. Moreover, $F'$ is isomorphic to the Hirzebruch surface $F^3$ with negative section $\Sigma'$. The map $\rho^{-1}: W' \to W$ is the flop of $\Sigma'$. There is a one parameter space of degree 5 orbits as above. The important observation stemming from this construction is the following corollary:

**Corollary 2.2.** All the threefolds arising from this construction, form a one parameter family of smooth Fano threefolds with automorphism group containing $\mathbb{C}^*$.

These threefolds are parameterized by the choice of the degree 5 orbit contained in $F_1$. It is interesting to study the “boundary” of this family. In particular it is natural to ask what kind of singularities can occur there.

### 2.3. Constructions of $V_{12}$ III: Zero sections of a homogenous bundle on $\mathcal{G}(4,7)$

Below a third construction is presented (compare [Ti3, D1]): Let $T$ denote the universal quotient vector bundle on the Grassmanian $\mathcal{G}(4,7)$. Let $H$ be the (very ample) line bundle $detT$. Consider a section $s$ of the bundle $\Lambda^2T \oplus \Lambda^3T \oplus \Lambda^2T$. It can be computed that $c_1(3\Lambda^2E) = 6H$. If the subvariety $X_s$ cut out by $s$ is smooth then by adjunction formula

$$K_{X_s} = K_{\mathcal{G}(4,7)} + c_1(3\Lambda^2E)|_{X_s} = -7H + c_1(E)|_{X_s} = -H,$$

thus $X_s$ is a Fano variety.

Alternatively such a section can be constructed as follows: take a 3-dimensional plane $P \subset \Lambda^2\mathbb{C}^7$, then $X$ is defined by

$$X_P = \{ M \in \mathcal{G}(4,7) | P|_M = 0 \}.$$

It is known (see [Ti2]) that the space of holomorphic vector fields of $X_P$ can be identified with the set of matrices $A$ in $sl(7, \mathbb{C})$ whose induced action on $\Lambda^2\mathbb{C}^7$ preserves $P$. Also the automorphisms of $X_P$ correspond to elements in $SL(7, \mathbb{C})$ which preserve $P$. Then the orbit of $SL(7, \mathbb{C})P$ will

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1Recall that the group $sl(7, \mathbb{C})$ acts on $\Lambda^2\mathbb{C}^7$ by $g(v_1 \wedge v_2) = g(v_1) \wedge v_2 + v_1 \wedge g(v_2)$.

2The action is given by $g(v_1 \wedge v_2) = g(v_1) \wedge g(v_2)$. 

be generically of dimension $\text{dim}(\mathfrak{sl}(7, \mathbb{C}))$, except in the special cases when the stabilizer is positive dimensional.

Below we shall describe the equations of elements of $V^a$.

Let $W_7$ denote a vector space of dimension 7 and let $L \subset \wedge^2 W_7$ be a generic subspace of dimension 3. Then we obtain a Fano threefold of genus 12 as:

$$V_{12}^L = \{ \alpha \in G(4, 7) \subset \mathbb{P}(\bigwedge^4 W_7) : \alpha \wedge \omega = 0 \ \forall \omega \in L \} = G(4, 7) \cap P_L.$$  

Hence the equations defining the $\mathbb{P}^13$ section of $G(4, 7)$ are given by elements of the form $\omega_1 \wedge \in \bigwedge^3 W_7 = (\bigwedge^4 W_7)^*$, where $(\omega_1, \omega_2, \omega_3)$ is a basis of $L$.

Any such basis vector can be written as $\omega = \sum_{i<j} a_{ij} e_i \wedge e_j$, with $e_i$, $i = -3, -2, \cdots, 3$ denoting the standard basis of 1 forms dual to the canonical basis of $W_7$ and $a_{ij}$ are constants. Thus a generic form has 21 components. The following technical lemma states that if $V_{12}^L$ is smooth then none of the basis vectors can have less than three nonzero components:

**Lemma 2.3.**  
1. The singular locus of a submanifold $F_\omega \subset G(4, 7)$ given by the zero section of the bundle $\bigwedge^2 T$ corresponding to a form $\omega = e_i \wedge e_j + \epsilon_k \wedge e_l$ for some $i, j, k, l$ is of dimension $\geq 6$.
2. The locus of a submanifold $F_\omega \subset G(4, 7)$ given by the zero section of the bundle $\bigwedge^2 T$ corresponding to a form $\omega = e_i \wedge e_j$ is of dimension $\geq 10$.
3. If $L = \text{span}\{\omega_1, \omega_2, \omega_3\}$ produces a smooth Fano manifold, then none of the basis vectors $\omega_i$ can be of the form above.

**Proof.** 1. Let us denote by $A_\omega := \{ [\alpha] \in \mathbb{P}(\bigwedge^4 W_7) | \alpha \wedge \omega = 0 \}$. Note that $F_\omega := A_\omega \cap G(4, W_7)$. It is not hard to check (one can also use Macaulay 2) that $\text{dim} F_\omega = 9$, which is the expected dimension for the zero locus of a rank 3 vector bundle. Consider also the Schubert cycle $B_\omega := \{ [U] \in G(3, W_7) | \text{dim}(U \cap < e_i, e_j, e_k, e_l>) \geq 3 \}$. Clearly $B_\omega \subset F_\omega$ and $B_\omega$ is of dimension 6. We claim that $B_\omega \subset \text{Sing}F_\omega$. Indeed, let $\alpha \in B_\omega$ and let $T_{[\alpha]}(G(3, W_7))$ be the projective tangent space to $G(3, W_7)$ at $\alpha$. Then the projective tangent space to $T_{[\alpha]}(F_\omega) = T_{[\alpha]}(G(3, W_7)) \cap A_\omega$ equals

$$\langle [\beta \in G(3, W_7)] | [\beta] \wedge \omega \wedge \alpha = 0 \text{ and } \text{dim} [\beta] \wedge [\alpha] \geq 3 \rangle.$$ 

To prove that $F_\omega$ is singular in $[\alpha]$, it is enough to prove that $\text{dim}(T_{[\alpha]}(G(3, W_7)) \cap A_\omega) \geq 10$. For this, let us consider $A_{\omega, v} = \{ [\alpha] \in \mathbb{P}(\bigwedge^4 W_7) | \alpha \wedge \omega \wedge v = 0 \}$, for each $v \in W_7$. We observe that $T_{[\alpha]}(G(3, W_7)) \subset A_{\omega, v}$ for each $v \in < e_i, e_j, e_k, e_l> + < \alpha >$. Since

$$A_\omega = \bigcap_{v \in W_7} A_{\omega, v}$$
it follows that $T_{[a]}(G(3,W_7)) \cap A_\omega$ has codimension at most 2 in $T_{[a]}(G(3,W_7))$. Hence $\dim(T_{[a]}(G(3,W_7)) \cap A_\omega) \geq 10$ and $F_\omega$ is singular in $[a]$ proving the claim.

2. Take without loss of generality the form $e_{-3} \wedge e_{-2}$. Then clearly the 4-plane $< x_{-3}, x_{-2}, x_{-1}, x_0 >$ belongs to $F_\omega$, and if we look for 4-planes in $F_\omega$ being graphs of linear maps from $< x_{-3}, x_{-2}, x_{-1}, x_0 >$ to $< x_1, x_2, x_3 >$ represented by a matrix

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix}$$

it is easy to see that the necessary and sufficient condition is that the first two columns have to be proportional. This yields a 10-dimensional family within $F_\omega$.

3. It is easy to see that $F_{\omega_1} \cap F_{\omega_2} \cap F_{\omega_3}$ is a smooth Fano threefold only if these $F_{\omega_j}$ intersect transversally. In the case when one of the forms is of the type $e_i \wedge e_j$ the dimension of the intersection is higher than three. In the case when one of the forms is of the type $e_i \wedge e_j + e_k \wedge e_l$ then dimension count tells us that the dimension of the singular locus of the Fano 3-fold is at least zero i.e. it should be a finite (nonzero) set of points. Thus in this case the constructed space must be singular.

On the other hand the weights on the standard $\mathbb{C}^*$ action $t : e_j \to t^j e_j$ induce weights $t^{j+k}$ on $e_j \wedge e_k$ and thus forms on which $t$ acts with weight 5, 4, $-4$ and $-5$ are simply multiples of monomial forms, whereas forms on which the action has weights 3, 2, 2 or $-3$ are spanned by sums of two monomial terms. In turn forms of weight $-1, 0$ or 1 are spanned by three monomial terms. Thus the only possibility to get a smooth $\mathbb{C}^*$-invariant Fano threefold is that the weights are $(-1,0,1)$ and in each case the corresponding basis vector form has to be of full rank i.e. no coefficient can be equal to zero.

Next, following [D1], we specify a $\mathbb{C}^*$-invariant 3 plane $L$ by making use of a $\mathbb{C}^*$ action on $W_7$. By Lemma 2.3 in order to obtain a smooth Fano 3-fold, we have to choose $L$ to be spanned by three 2-forms $(\omega_1, \omega_0, \omega_1)$ on which $\mathbb{C}^*$ acts by weights $(-1,0,1)$. As explained in [D1], such a triple of 2 forms can be written as

$$\omega_1 = u_{01} e_0 \wedge e_1 + u_{-12} e_{-1} \wedge e_2 + u_{-23} e_{-2} \wedge e_3$$

$$\omega_0 = v_{-11} e_{-1} \wedge e_1 + v_{-22} e_{-2} \wedge e_2 + v_{-33} e_{-3} \wedge e_3$$

$$\omega_{-1} = w_{-10} e_{-1} \wedge e_0 + w_{-21} e_{-2} \wedge e_1 + w_{-32} e_{-3} \wedge e_2,$$

for some constants $u_{ij}, v_{ij}, w_{ij} \in \mathbb{C} \setminus \{0\}$. Of course the constants are not uniquely defined, as rescaling the coordinates $x_i$, as well as the whole 2-forms yield the same invariant space $L$. Exploiting the rescalings of the
coordinates \( x_i \) by factors \( \lambda_i \) given by

\[
\lambda_3 = \sqrt{\frac{u_{01}}{w_{-10} w_{-32}^2}} \sqrt{\frac{v_{-22} u_{-12}}{v_{-11} w_{-21}}}
\]

\[
\lambda_2 = \sqrt{\frac{u_{01}}{w_{-10}}} \sqrt{\frac{v_{-11} u_{-12}}{v_{-22} w_{-21}^2}}
\]

\[
\lambda_1 = \frac{v_{-22}}{w_{-10} u_{-11} w_{-21} u_{-12}}
\]

\[
\lambda_0 = \frac{1}{u_{01} w_{-10}} \sqrt{\frac{v_{-22}}{v_{-11} w_{-21} u_{-12}}}
\]

\[
\lambda_1 = \sqrt{\frac{w_{-10}}{u_{01}} \sqrt{\frac{v_{-22}}{v_{-11} w_{-21} u_{-12}}}}
\]

\[
\lambda_2 = \sqrt{\frac{w_{-10}}{u_{01}} \sqrt{\frac{v_{-11} w_{-21}}{v_{-22} u_{-12}^2}}}
\]

\[
\lambda_3 = \sqrt{\frac{w_{-10}}{u_{01} w_{-23}^2} \sqrt{\frac{v_{-22} w_{-21}^3}{v_{-11} u_{-12}}}}
\]

followed by rescaling of \( \omega_0 \) by

\[
\sqrt{\frac{u_{-12} w_{-21}}{v_{-11} v_{-22}}}
\]

we end up with the following basis of \( L \):

\[
\omega_1^L = e_0 \land e_1 + e_{-1} \land e_2 + e_{-2} \land e_3
\]

\[
\omega_0^L = e_{-1} \land e_1 + e_{-2} \land e_2 + \tau e_{-3} \land e_3
\]

\[
\omega_{-1}^L = e_{-1} \land e_0 + e_{-2} \land e_1 + e_{-3} \land e_2,
\]

where \( \tau = \frac{u_{-12} v_{-33} w_{-21}}{u_{-23} v_{-11} w_{-22}} \) (a quantity equivalent to the modulus defined by Donaldson in [D1]). Exactly like in [D1] this \( \tau \) parametrizes the family of 3 planes \( L_\tau \). In particular we obtain the Mukai-Umemura case for \( \tau = 1 \).

It should be emphasized however, that this parametrization is non-effective i.e. there might be different \( \tau \)'s corresponding to isomorphic manifolds. In particular Tian’s choice of coordinates for the Mukai-Umemura example [T1] yields \( \tau = 5 \).

Our aim is now to describe the equations defining \( V_{L_\tau}^{12} \subset \mathbb{P}(\Lambda^4 W_7) \).

Starting from a one parameter family \( L_\tau \) as above we describe the linear space \( L_{12} \subset \Lambda^4 W_7 \) describing the minimal generators of the Fano threefold \( V_{L_\tau}^{12} = G(4, 7) \cap L_{12} \subset \mathbb{P}(\Lambda^4 W_7) \). This is done by exploiting the following program that runs in Macaulay 2. In particular mingens \( L_{12} \) above finds the linear equations describing \( L_{12} \subset \Lambda^4 W_7 \) and mingens \( V_{12} \) finds
the equations defining $V^{L^*}_{12} \subset \mathbb{P}(\Lambda^4 W_7)$ and *mingens* $V_{12}^{inPL12}$ find the equations of $V^{L^*}_{12} \subset \mathbb{P}(L12)$. Note that this generalizes the results obtained in [F] concerning the equations of the Mukai-Umemura threefold. In the following we put $\tau = -d$

S=\text{frac}\ (\text{QQ}[d])
R=S[ x_0 \ldots x_6, \text{SkewCommutative=\text{true}}]
\begin{align*}
u &= x_0*x_5+x_1*x_4+x_2*x_3 \\
w &= -d*x_0*x_6+x_1*x_5+x_2*x_4 \\
v &= x_1*x_6+x_2*x_5+x_3*x_4 \\
MING3 &= \text{mingens} \ (\text{ideal}(\text{vars}(R)))^3 \\
MING4 &= \text{rsort mingens} \ (\text{ideal}(\text{vars}(R)))^4 \\
\text{NORM34} &= \text{substitute} \ (\text{transpose MING4})*(MING3), \ \{x_0=1, x_1=1, x_2=1, x_3=1, x_4=1, x_5=1, x_6=1\} \\
MU &= \text{matrix} \ \{\text{apply}(7, i\rightarrow(\text{coefficients}(u*x_i, \text{Monomials=\text{MING3}}))_1)\} \\
MV &= \text{matrix} \ \{\text{apply}(7, i\rightarrow(\text{coefficients}(v*x_i, \text{Monomials=\text{MING3}}))_1)\} \\
MW &= \text{matrix} \ \{\text{apply}(7, i\rightarrow(\text{coefficients}(w*x_i, \text{Monomials=\text{MING3}}))_1)\} \\
RG &= S[p0123, p0124, p0134, p0234, p1234, p0125, p0135, p0235, p1235, p0145, p0245, p1245, p0345, p1345, p2345, p0126, p0136, p0236, p1236, p0146, p0246, p1246, p0346, p1346, p2346, p0156, p0256, p1256, p0356, p1356, p2356, p0456, p1456, p2456, p3456]; \\
KOND &= (\text{map}(RG,R)) \ (\text{NORM34})*(\text{MU})*\text{MV}) \\
GG &= \text{Grassmannian}(3,6, RG); \\
L12 &= \text{ideal (vars}(RG)*KOND); \\
V12 &= GG+L12; \\
P13 &= \text{substitute} \ (V22, \{p2356=\rightarrow p1456, p2346=\rightarrow p0456, p1346=\rightarrow d*p2345, p1246=\rightarrow d*p1345, p0156=\rightarrow p1236, p0236=\rightarrow d*p1235, p0146=\rightarrow d*p1235, p0346=\rightarrow d*p1345, p0345=\rightarrow d*p1236, p0245=\rightarrow d*p1235, p0136=\rightarrow d*p1234, p0235=\rightarrow d*p1234, p0234=\rightarrow p0126, p0135=\rightarrow p0126, p0134=\rightarrow p0125, p1356=\rightarrow p0456, p0356=\rightarrow d*p2345, p0135=\rightarrow p0126, p0145=\rightarrow (d+1)*p1234, p1256=\rightarrow (d+1)*p2345, p0246=\rightarrow d*p1245–p1236\}
T=S[p0123, p0124, p0125, p0126, p1456, p0456, p1345, p1236, p1235, p1234, p2345, p1245, p2456, p3456] \\
V12inPL12 &= \text{map}(TT, RG)) \ P13; \\
\text{degree} \ V12inPL12 \\
\text{dim} \ V12inPL12
3. $\text{VSP}(C,6)$

Recall that in the previous section a construction of $V_{12}$ using an appropriate plane quartic was explained. Thus the “moduli” space of $V_{12}$ is birational to the moduli space of plane quartics. We will be interested in the one parameter family parametrized by two tangent conics i.e.

$$\Gamma_t = \{[a : b : c] \in \mathbb{P}^2 \mid (a^2 + bc)(ta^2 + bc) = 0\}.$$ 

Below we prove that $\Gamma_t$ parametrize the threefolds $V^a$.

Let $C$ be the Hilbert scheme of lines on $V_{12}$. Suppose that there is a quartic curve $\Gamma$ such that $C$ is the covariant quartic of $\Gamma$. Then (see [M1, §5]) we have $V_{12} = \text{VSP}(\Gamma, 6)$. We shall need the following lemma:

**Lemma 3.1.** The covariant quartic of a reducible quartic being the sum of two tangent conics is also a quartic being the sum two tangent conics. The number of tangencies is preserved.

**Proof.** This follows from a straightforward computation using [DK, §8]. □

**Proposition 3.2.** All Fano threefolds $V^a_t$ can be constructed as $\text{VSP}(\Gamma_t, 6)$, where $\Gamma_t$ are two tangent conics.

**Proof.** First observe that the quartics $\Gamma_t$ admit $\mathbb{C}^*$ as a subgroup of the automorphism group. Indeed

$$(7) \quad \mathbb{C}^* \ni \lambda \mapsto \theta_\lambda : (a, b, c) \rightarrow (a, \lambda b, \lambda^{-1} c)$$

fixes $\Gamma_t$.

The second step is to show that the above automorphisms of $\Gamma_t$ induce an one parameter family of automorphisms of $\text{VSP}(\Gamma_t, 6)$. Indeed, if $\theta$ is such an automorphism and

$$(a^2 + bc)(ta^2 + bc) = l_1^4 + \cdots + l_6^4,$$ 

then $(a^2 + bc)(ta^2 + bc) = l_1^4(\theta) + \cdots + l_6^4(\theta)$, where $l_j(\theta)$ denotes the action of $\theta$ on the linear form $l_j$. We claim that the induced automorphism is not trivial for $\theta \neq 1$. Indeed, suppose that it is trivial for a generic $\lambda$ (by Theorem 0.1 it suffices to check for generic $\lambda$). Then $l_1(\theta_\lambda) = e$, for some $1 \leq i \leq 5$ and some $e \in \mathbb{C}$. It follows that if $l_1 = xa + yb + zc$ then two of the numbers $x, y, z$ are zero (the same holds for $l_2, \ldots, l_5$). However, we see that $(a^2 + bc)(ta^2 + bc)$ is not a linear combination of $a^4, b^4, c^4$ the claim follows.

**Remark 3.3.** A more tedious analysis of the various cases easily shows that the action is nontrivial unless $\lambda^2 = 1, \lambda^3 = 1$ or $\lambda^6 = 1$. Each of these cases can be ruled out by hand exploiting careful computations.

We know from [P1] that the manifolds $V^a_t$ form a one parameter family, like the family of two tangent conics. In order to conclude we should show that the threefolds $\text{VSP}(\Gamma_t, 6)$ are smooth. It follows from [D1, §5.3] that the one parameter family $V^a_t$ is the same as the one parameter family $V^a_\tau$ parameterized by $\tau$ in [D1]. Thus the Hilbert scheme of lines on $V^a_\tau$ are two tangent conics. We conclude with Lemma 3.1 since each pair of conics can be obtained as a covariant quartic of a pair of conics. □
The two components of the quartic above parameterize two families of lines of \(V_t^a\).

**Corollary 3.4.** There are two divisors spanned by lines on a generic Fano threefold \(V_t^a\).

Note that from \([M]\) the intersection of the conics parameterizing the Hilbert scheme of lines corresponds to lines with special normal bundle \((1,-2)\). From now on we will identify the families \(V^a, V_t^a, V_t^a\) and \(VSP(Γ_t,6)\).

We finish this section by analyzing the manifold \(V^m\). In order to find a \(\mathbb{C}^+\)-invariant 3-plane in \(\Lambda \mathbb{C}^7\), we consider the standard \(\mathbb{C}^+\) action on one forms given by the matrix

\[
\begin{bmatrix}
1 & 6t & 15t^2 & 20t^3 & 15t^4 & 6t^5 & t^6 \\
0 & 1 & 5t & 10t^2 & 10t^3 & 5t^4 & t^5 \\
0 & 0 & 1 & 4t & 6t^2 & 4t^3 & t^4 \\
0 & 0 & 0 & 1 & 3t & 3t^2 & t^3 \\
0 & 0 & 0 & 0 & 1 & 2t & t^2 \\
0 & 0 & 0 & 0 & 0 & 1 & t \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

By direct computation the following 3-forms yield a \(\mathbb{C}^+\)-invariant 3-plane

\[
\begin{align*}
ω_1 &= e_1 ∧ e_6 - 5e_2 ∧ e_5 + 10e_3 ∧ e_4 + e_5 ∧ e_6 \\
ω_2 &= e_0 ∧ e_6 - 4e_1 ∧ e_5 + 5e_2 ∧ e_4 + (2e_3 ∧ e_6 - 6e_4 ∧ e_5) \\
&+ e_4 ∧ e_6 + 2e_5 ∧ e_6 \\
ω_3 &= e_0 ∧ e_5 - 5e_1 ∧ e_4 + 10e_2 ∧ e_3 + (e_2 ∧ e_6 - 2e_3 ∧ e_5) \\
&+ (e_3 ∧ e_6 - 2e_4 ∧ e_5) + e_4 ∧ e_6 + e_5 ∧ e_6.
\end{align*}
\]

The corresponding variety turns out to be a smooth Fano threefold. This example is an element of Tian's family of deformations (see \([T]\)). The corresponding plane quartic in the VSP construction can be computed in Macaulay 2: it reads

\[
P_4(a, b, c) = 32a^4 + 16a^2b^2 - 8abc^2 + c^4 = (c^2 - 4ab)^2 + 32a^4.
\]

It is obvious that this \(P_4\) is a deformation of the double conic and it is a sum of two conics intersecting at one point.

4. **Special divisors**

Similarly as in \([F]\) our goal is to describe the singularities of invariant divisors on \(V_{12}^{L_4}\). Let us first describe divisors in the system \(|−K_{V_{12}^{L_4}}|\) which are invariant under the \(\mathbb{C}^*\) action. Recall that \(V_{12}^{L_4}\) is obtained as the intersection \(\mathcal{G}(4,7) \cap \mathbb{P}(L)\), where \(\mathbb{P}(L) \simeq \mathbb{P}^{13}\). Observe now that each invariant divisor in \(|−K_{V_{12}^{L_4}}|\) can be obtained as the intersection in \(\mathbb{P}(\Lambda^4 W_7)\) of \(\mathcal{G}(4,7) \cap \mathbb{P}(L)\) with an invariant hyperplane in \(\mathbb{P}(\Lambda^4 W_7)\). The latter hyperplanes correspond to fixed points of the action of \(\mathbb{C}^*\) on \(\mathbb{P}(\Lambda^4 W_7)\). The set of fixed points of the latter action is a sum of 13 linear spaces \(P_k\),
Remark 4.2. The set of divisors fixed by the $\mathbb{C}^*$ action consists of 12 points and a line.

Proposition 4.1. On the other hand we have the following invariant divisors on $V^a$: First recall from \[F1\] that on a generic $V^a$ we have two special lines $l_1, l_2$ having the normal bundle $(-1, 2)$.

- the divisors spread by lines on $V^a$: we have two components each belonging to $|−K_{V_{12}}^*$|. There are two families of lines;
- the two divisors spread by conics on $V^a$ cutting the line $l_i$ for $i = 1, 2$. We denote these divisors by $C^1, C^2$.

We denote by $p_{ijkl}$ for $0 \leq i, j, k, l \leq 6$ the natural coordinates of $\wedge^4 W_7$. Our aim is to understand the singularities of the invariant divisors. To this end we compute some affine parts (containing the most singular points on these divisors) of the twelve invariant divisors and the one parameter family on $V_{12}^{L_{12}}$. These affine parts can be described by an elimination of variables as hypersurfaces with the following equations:

1. the divisor $p_{0123} = 0$
   with equation $0 = p_{1456}^5P_{2456}^2 + 2p_{1356}^3P_{1456}^2P_{2456}^2 + p_{1356}^2P_{1456}P_{2456}^2 - 1/(d + 1)p_{1456}^6 + (2d^2 + 2d^2 + d - 2)/(d + 1)p_{1356}^2P_{1456}^2P_{2456}^2 + (2d^2 - 1)/(d + 1)p_{1356}^2P_{1456}^2P_{2456}^2 + (-2d^2 - d)/(d + 1)p_{1356}P_{2456} + (2d^2 + d + 2d)p_{1356}^2P_{1456}^2 + (2d^2 + 2d)p_{1356}P_{1456}P_{2456}^2 + (-d^2 - d - d^2)p_{1356}^2$

2. the divisor $p_{0124} = 0$
   with equation $0 = p_{1456}^2P_{2456}^2 + 2p_{1356}^2P_{1456}P_{2456}^2 + p_{1356}^2P_{1456}^2 + (d - 1)p_{1456}^4P_{2456}^2 + (2d^2 + 4d - 2)p_{1356}P_{1456}P_{2456}^2 + (2d^2 + 3d - 1)p_{1356}P_{1456}^2 + (d^1 - d^2 - d)p_{1356}P_{1456} + (d^4 + 4d^3 + d^2 + d^4)p_{1356}P_{1456}^2 + (2d^2 + 2d + d)p_{1356}P_{1456}^2 + (d^1 + d^2 + d^1 + 2d^3 + d^2 + d^2)p_{1356}P_{1456}^2$

3. the divisor $p_{0125} = 0$
   with equation $0 = p_{1456}^2P_{2456}^2 + 2p_{1356}^2P_{1456}P_{2456}^2 + p_{1356}^2P_{1456}^2 + (d^2 + 2d - 2)p_{1356}P_{1456}P_{2456} + (2d^2 + 2d - 1)p_{1356}P_{1456}^2P_{2456} + (d^1 + 2d^3 + 2d^2 + d)p_{1356}P_{1456}^2 + (d^1 + 2d^3 + 2d^2 + d)p_{1356}P_{1456}^2$

4. the divisor $p_{0135} = 0$
   with equation $0 = p_{1456}^2P_{2456}^2 + 2p_{1356}^2P_{1456}P_{2456}^2 + p_{1356}^2P_{1456}^2 + (d^2 - 1)p_{1356}P_{1456}^2 + (d^1 + 2d^3 + 2d^2 + d)p_{1356}P_{1456}^2 + (d^1 + 2d^3 + 2d^2 + d)p_{1356}P_{1456}^2$

5. the divisor $p_{0136} = 0$
   with equation $0 = p_{0125}^2 + p_{0124}P_{0135}$

6. the divisor $p_{0146} = 0$
   with equation $0 = p_{1456}^2P_{2456} + p_{1356}P_{1456}P_{2456}^2 + d^2p_{1356}P_{1456}$

7. the divisor $p_{0256} = 0$
   with equation $0 = p_{1456}^2P_{2456} + p_{1356}P_{2456}^2 + (d^2 + d)p_{1356}P_{1456}$

8. the divisor $p_{0352} = 0$
   with equation $p_{1456}^2 + p_{1356}P_{2456}$
(9) the divisor $p_{1356} = 0$
  smooth in the affine part $p_{3456} = 1$ for $d \neq -1$
(10) the divisor $p_{1456} = 0$
  smooth in the affine part $p_{3456} = 1$ for $d \neq -1$
(11) the divisor $p_{2456} = 0$
  smooth in the affine part $p_{3456} = 1$ for $d \neq -1$
(12) the divisor $p_{3456} = 0$
  with equation $0 = p_{0124}^2 p_{0125}^2 + 2 p_{0124}^3 p_{0125} p_{0135} + p_{0124} p_{0125} p_{0135}^2$
  $+ 1/(d + 1) p_{0125}^6 + (2 d^2 + 2 d^2 + d - 2)/(d + 1) p_{0124} p_{0125} p_{0135}$
  $+(2 d^2 - 1)/(d + 1) p_{0124}^2 p_{0125}^2 p_{0135}^2 + (-2 d^2 - d)/(d + 1) p_{0124} p_{0135}^3$
  $+(d^4 + 2 d^2 + 2 d) p_{0125} p_{0135}^2 + (-2 d^2 + 2 d) p_{0124} p_{0125} p_{0135}^3$
  $+ (-d^2 - 2 d^2 - d^2) p_{0135}^4$
(13) one parameter family of invariant divisors parameterized by
  $b$: $p_{0156} = b * p_{0246}$ with equations $0 = p_{0124}^2 p_{0125}^2 + p_{0123}^3 p_{0135} +$
  $1/b p_{0125}^3 + (d^2 b + d b + 1)/b * p_{0124} p_{0125} p_{0135} + (d^3 b + d^2 b + d^2 - d)/b * p_{0135}^2$
It is interesting that all these divisors are weighted homogeneous. The next proposition summarizes what Kollár's inequality \( \| \) yields for each of them:

**Proposition 4.3.**  
(1) The first divisor is weighted homogenous with weights on $p_{1456}, p_{2456}, p_{3456}$ given by (3, 2, 1), respectively. Then the polynomial becomes homogeneous of degree 12. In particular $\text{lct}(D_1) \leq \frac{2 + 3 + 1}{12} = \frac{1}{2}$;
(2) The second divisor is weighted homogeneous with weights on $p_{1456}, p_{2456}, p_{3456}$ given by (3, 2, 1), respectively. Then the polynomial becomes homogeneous of degree 11. In particular $\text{lct}(D_2) \leq \frac{2 + 3 + 1}{11} = \frac{4}{11}$;
(3) The third divisor is weighted homogeneous with weights on $p_{1456}, p_{2456}, p_{3456}$ given by (3, 2, 1), respectively. Then the polynomial becomes homogeneous of degree 10. In particular $\text{lct}(D_3) \leq \frac{2 + 3 + 1}{10} = \frac{3}{5}$;
(4) The fourth divisor is weighted homogeneous with weights on $p_{1456}, p_{2456}, p_{3456}$ given by (3, 2, 1), respectively. Then the polynomial becomes homogeneous of degree 9. In particular $\text{lct}(D_4) \leq \frac{2 + 3 + 1}{9} = \frac{5}{9}$;
(5) The fifth divisor is homogeneous. The polynomial is of degree 2. In particular $\text{lct}(D_5) \leq \frac{1 + 1 + 1}{2} = \frac{3}{2}$. Of course in this case we get a nodal singularity and much better estimate can be given by different means;
(6) The sixth divisor is weighted homogeneous with weights on $p_{1456}, p_{2456}, p_{3456}$ given by (3, 2, 1), respectively. Then the polynomial becomes homogeneous of degree 7. In particular $\text{lct}(D_6) \leq \frac{2 + 3 + 1}{7} = \frac{6}{7}$;
(7) The seventh divisor is weighted homogeneous with weights on $p_{1356}, p_{1456}, p_{2456}$ given by (3, 2, 1), respectively. Then the polynomial becomes homogeneous of degree 5. In particular $\text{lct}(D_7) \leq \frac{2 + 3 + 1}{5} = \frac{8}{5}$;
(8) The eighth divisor is homogeneous. The polynomial is of degree 2. In particular $\text{lct}(D_8) \leq \frac{1 + 1 + 1}{2} = \frac{3}{2}$. As in the fifth case we get a nodal singularity;
(9) The twelfth divisor is weighted homogeneous with weights on $p_{0124}, p_{0125}, p_{0135}$ given by (1, 2, 3), respectively. Then the
polynomial becomes homogenous of degree 12. In particular 
\[ \text{lct}(D_{12}) \leq \frac{2+3+1}{12} = \frac{1}{2}; \]
(10)
The thirteenth family of divisors are weighted homogenous with 
weights on \( p_{0124}, p_{0125}, p_{0135} \) given by \( (1, 2, 3) \), respectively. Then the 
polynomial becomes homogenous of degree 6. In particular 
\[ \text{lct}(D_{12}) \leq \frac{2+3+1}{6} = 1; \]

**Remark 4.4.** In \[F\] part of this computation was performed for the Mukai-Umemura threefold. In particular the first and the twelfth divisors were computed explicitly.

An immediate consequence is the following upper bound for the log canonical threshold:

**Proposition 4.5.** For any \( V \in V^a \) the log canonical threshold satisfies 
\[ \text{lct}(V, \mathbb{C}^*) \leq \frac{1}{2}. \]

5. **The group of automorphisms of elements of \( V^a \) and the openness of Kähler-Einstein examples**

The aim of this section is to study more precisely the automorphism group of elements \( V \in V^a \) and to use the symmetries in the Kähler-Einstein deformation problem.

**Proposition 5.1.** Every Fano threefold \( V_{12}^a \) admits an additional automorphism being an involution \( \iota \). Furthermore if \( \theta_\lambda, \lambda \in \mathbb{C}^* \) denotes an element of the \( \mathbb{C}^* \) action then we have the identity \( \iota \circ \theta_\lambda = \theta_{\lambda^{-1}} \circ \iota \) i.e. the involution anti-commutes with the action.

**Proof.** It is enough to observe that two mutually tangent conics form a symmetric quartic. Recall that modulo linear change of coordinates such two tangent conics are described by 
\[ \Gamma_\iota = \{ [a : b : c] \in \mathbb{P}^2 | (ta^2 + bc)(a^2 + bc) = 0 \}. \]
Observe that there is a natural symmetry of \( \mathbb{P}^2 \) exchanging the two tangent points given by 
\[ \iota : \mathbb{P}^2 \ni [a : b : c] \mapsto [a : c : b] \in \mathbb{P}^2 \]
which preserves \( \Gamma_\iota \) and hence induces an automorphism on \( V \text{SP}(\Gamma_\iota, 6) \). The identity then follows trivially. \( \square \)

Note that an automorphism of \( V^a \) induces an automorphism on the Hilbert scheme of lines.

The existence of \( \iota \) has the following consequence:

**Theorem 5.2.** The set of all \( V \)'s in the family \( V^a \) which admits Kähler-Einstein metrics is open in the Euclidean topology.

**Proof.** The openness near the Mukai-Umemura threefold follows from \[D1\] or \[RST\]. Consider now a Kähler-Einstein element \( V_{10}^a \) and a smooth family \( V_t \) of deformations within \( V^a \). For each such \( V_t \) we consider the space \( C^\infty_W \) of smooth functions invariant under the group 
\[ W = \mathbb{Z}_2 \rtimes S^1. \]
Note that by Bando-Mabuchi theorem [BM] if a Kähler-Einstein metric exists one can also find a $W$-invariant one.

Choose a smooth family of $W$-invariant forms $\omega_t$ representing the first Chern classes of $V_t$ and let $f_t$ be the corresponding Ricci potential i.e.

$$Ric(\omega_t) - \omega_t = i\partial\bar{\partial} f_t, \quad \int_{V_t} e^{f_t} \omega_t^3 = c_1(V_t)^3.$$

As is well known existence of $W$-invariant Kähler-Einstein metric is equivalent to the existence of an $W$-invariant function $u_t$ solving the problem

$$(10) \begin{cases} (\omega_t + i\partial\bar{\partial} u_t)^3 = e^{h-u} \omega_t^3, \\ \omega_t + i\partial\bar{\partial} u_t > 0, \quad u_t \in C_0^{\infty}(V_t). \end{cases}$$

In order to prove the openness it suffices to apply the implicit function theorem. Indeed by assumption the problem $*_t$ is solvable. The linearized operator at $t_0$ is $\Delta_{t_0} + id$, where $\Delta_{t_0}$ denotes the Laplacian with respect to the $W$-invariant Kähler-Einstein metric $\omega_{t_0} + i\partial\bar{\partial} u_{t_0}$. This operator is not invertible as it has a nontrivial one dimensional kernel (corresponding to the holomorphic vector field induced by the $C^*$ action). The operator however is invertible when restricted to $W$-invariant functions (see [PSSW]). Indeed as in Theorem 2 in [PSSW] the group $W$ is a closed subgroup stabilizing $\omega_{t_0} + i\partial\bar{\partial} u_{t_0}$, whose centralizer in $Stab(\omega_{t_0} + i\partial\bar{\partial} u_{t_0})$ is finite by the anti-commutativity of $\iota$. Note that in [PSSW] it is assumed that $Stab(\omega_{KE})$ is a subgroup of the identity component of the automorphisms (which is not the case in our setting) but the proof applies verbatim without this requirement. Then Theorem 2 in [PSSW] implies that the space of $W$-invariant functions is orthogonal to the kernel of $\Delta_{t_0} + id$ with respect to the $L^2(V_{t_0}, (\omega_{t_0} + i\partial\bar{\partial} u_{t_0})^3)$. Thus $\Delta_{t_0} + id$ is an invertible operator from $C^2_W$ to $C^0_W$ which yields $C^{2,\alpha}$ smooth solution of $*_t$ for $t$ sufficiently close to $t_0$. Further $C^\infty$ regularity of the solutions $u_t$ is standard.

\[ \square \]

6. The Hilbert scheme of lines on $V_{12} \subset \mathbb{P}^{13}$

The aim of this section is to give an algorithm to find the Hilbert scheme of lines on any Fano 3-fold $V_{12}$. We present an implementation of the algorithm in Macaulay 2. We proceed as follows. Each line $l \subset V_{12}$ is a line in the Grassmannian $G(4, W_7)$. Hence, there exist subspaces $K_3 \subset T_5 \subset W_7$ of dimension 3 and 5 respectively, such that $l = F(K_3, 4, T_5) \subset G(4, W_7)$, where $F$ denotes the flag variety. Let now $\gamma \in \bigwedge^3 W_7$ be the simple 3-form associated to $K_3$. Consider the map:

$$\varphi_l : L_\lambda \ni \omega \mapsto \omega \wedge \gamma \in \bigwedge^5 W_7.$$  

We claim that the image $\varphi_l$ is a dimension 1 subspace $\bigwedge^5 W_7$ consisting of simple forms. Indeed by the description of $l$ we infer that $\omega \wedge \gamma$ is annihilated by $T_5$. Then for each non-zero element $\theta \in \varphi_l(L_\lambda)$ by reduction with $\gamma$, we associate an element $\tilde{\theta} \in \bigwedge^2(W_7/K_3)$. The latter satisfies $\tilde{\theta} \wedge \alpha = 0$.

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The first author wishes to thank J. Sturm for pointing this out.
for each $\alpha \in T^1_5/K^1_3$. Since $\dim T^1_5/K^1_3 = 2$ and $\dim W_7/K^1_3 = 4$, the latter implies $\tilde{\theta}$ is a simple form and $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are proportional for any two nonzero $\theta_1, \theta_2 \in \varphi_1(L^\lambda)$. This proves the claim. We now have a map
\[ \Phi : \text{Hilb}_{t+1}(V^L_1) \ni l \mapsto [\ker \varphi_l] \in \mathbb{P}(L^\lambda)^* . \]

To perform the computation we use the following Macaulay 2 script. The program below finds the quartic being the Hilbert scheme of lines on $V^L_1$ where $L$ is generated by $u, v, w$. To make the program work we have to fix $u, v, w$ i.e. specify the coefficient $d$.

S=QQ[d]
R=S[x_0..x_6, SkewCommutative=>true]
u=x_1*x_6+x_2*x_5+x_3*x_4
w=-d*x_0*x_6+x_1*x_5+x_2*x_4
v=x_0*x_5+x_1*x_4+x_2*x_3
IL=(gens (intersect (kernel (-u+v), kernel (v-w),
kernel (u-w)))) \{-21\}
KO=IL*matrix\{\{u,v,w\}\}
MAT=((coefficients (KO_0_0))_1)
|((coefficients (KO_1_0))_1)
|((coefficients (KO_2_0))_1)
MING2=mingens (ideal (vars(R))*ideal (vars(R)))
MING5=(coefficients (KO))_0
NORM=substitute ((transpose MING5)*(MING2),
\{x_0=>1, x_1=>1, x_2=>1, x_3=>1, x_4=>1, x_5=>1, x_6=>1\})
UL=MING2*NORM*MAT
ID2=(map(S,R))((coefficients (UL_0_0)^2))_1+
ideal ((coefficients ((UL_1_0)^2))_1)+
ideal ((coefficients ((UL_2_0)^2))_1));
ID=saturate ID2

7. Concluding remarks

Fano threefolds $V$ in the family $V^a$ are examples of $T$-varieties (see [S]). Kähler-Einstein metrics on symmetric $T$-varieties were studied in [S] (our examples are not symmetric and have complexity two). In this context it is natural to study the rational $\mathbb{C}^*$ quotient $V \to Y$ where $Y$ is a surface and to describe the possible targets $Y$. It would be also very interesting to understand $\mathbb{C}^*$-invariant test configurations for elements of $V^a$. Then the central fiber of such a configuration is a $T$-variety with complexity one, which might be singular.

Given the importance of the additional symmetry $\iota$ the following problem seems natural:

Problem 7.1. For $V \in V^a$ find the log canonical threshold $\text{lct}(V, \mathbb{Z}_2 \ltimes S^1)$.

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