A property of a partial theta function

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Abstract

The series \( \theta(q, x) := \sum_{j=0}^{\infty} q^{(j+1)/2} x^j \) converges for \( |q| < 1 \) and defines a partial theta function. For any fixed \( q \in (0, 1) \) it has infinitely many negative zeros. It is known that for \( q \) taking one of the spectral values \( \tilde{q}_1, \tilde{q}_2, \ldots \) (where \( 0.3092493386 \ldots = \tilde{q}_1 < \tilde{q}_2 < \cdots < 1 \), \( \lim_{j \to \infty} \tilde{q}_j = 1 \) the function \( \theta(q, .) \) has a double zero which is the rightmost of its real zeros (the rest of them being simple). For \( q \neq \tilde{q}_j \) the partial theta function has no multiple real zeros. We prove that:

1) for \( q \in (\tilde{q}_j, \tilde{q}_{j+1}] \) the function \( \theta \) is a product of a degree \( 2j \) real polynomial without real roots and a function of the Laguerre-Pólya class \( \mathcal{LP} - \mathcal{I} \);

2) for \( q \in \mathbb{C}\setminus\{0\}, |q| < 1, \theta(q, x) = \prod_{i}(1 + x/x_i), \) where \( -x_i \) are the zeros of \( \theta \);

3) for any fixed \( q \in \mathbb{C}\setminus\{0\}, |q| < 1, \) the function \( \theta \) has at most finitely-many multiple zeros;

4) for any \( q \in (-1, 0) \) the function \( \theta \) is a product of a real polynomial without real zeros and a function of the Laguerre-Pólya class \( \mathcal{LP} \).

5) for any fixed \( q \in \mathbb{C}\setminus\{0\}, |q| < 1, \) and for \( k \) sufficiently large, the function \( \theta \) has a zero \( \zeta_k \) close to \(-q^{-k}\). These are all but finitely-many of the zeros of \( \theta \).

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1 Introduction

The series in two variables \( \theta(q, x) := \sum_{j=0}^{\infty} q^{(j+1)/2} x^j \) defines an entire function in \( x \) for each fixed \( q, |q| < 1 \). In the present paper we consider three situations: \( q \in \mathbb{C}, |q| < 1, x \in \mathbb{C} \) or \( q \in [0, 1), x \in \mathbb{R} \) or \( q \in (-1, 0], x \in \mathbb{R} \). We say that the series defines a partial theta function (the Jacobi theta function is defined by the series \( \Theta(q, x) := \sum_{j=-\infty}^{\infty} q^j x^j \) and one has \( \theta(q^2, x/q) = \sum_{j=0}^{\infty} q^{j^2} x^j \), i.e. in the definition of \( \theta \) only partial summation is performed). We regard \( q \) as a parameter and \( x \) as a variable.

There are several domains in which the partial theta function is an object of interest: the theory of (mock) modular forms (\([3]\)), asymptotic analysis (\([2]\)), statistical physics and combinatorics (\([11]\)) and Ramanujan type q-series (\([12]\)). Additional information about \( \theta \) can be found in \([1]\).

A recent study relates the function \( \theta \) to a problem considered by Hardy, Petrovitch and Hutchinson, see \([4], [5], [10], [6], [8] \) and \([7]\). In the article \([6]\) the existence of a constant \( \tilde{q} \in (0, 1) \) is proved such that for \( q \in (0, \tilde{q}) \) the function \( \theta(q, .) \) has only real negative simple zeros while \( \theta(\tilde{q}, .) \) has a double negative zero the rest of the zeros being negative and simple. The more accurate value \( 0.3092493386 \ldots = \tilde{q} \) is given in \([8]\). The function \( \theta \) belongs to the Laguerre-Pólya class \( \mathcal{LP} - \mathcal{I} \) exactly for \( q \in (0, \tilde{q}) \). We remind that a function of this class is either representable in the form
\[
\psi(x) = c x^m e^{\sigma x} \prod_{k=1}^{\omega} (1 + x/x_k),
\]

where \( \omega \in \mathbb{N} \cup \infty, c \in \mathbb{R}, \sigma \geq 0, x_k > 0, m \in \mathbb{N} \cup 0 \) and \( \sum 1/x_k < \infty \) or \( \psi(-x) \) is representable in this form. For each compact set in \( \mathbb{C} \), the restriction to it of each function of this class is a uniform limit of polynomials with all roots real and negative (or real and positive).

An entire function belongs to the Laguerre-Pólya class \( \mathcal{LP} \) if it is of the form

\[
\psi(x) = c x^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\omega} (1 + x/x_k) e^{-x/x_k},
\]

where \( \omega \in \mathbb{N} \cup \infty, c, \beta \in \mathbb{R}, \alpha \geq 0, m \in \mathbb{N} \cup 0 \) and \( \sum x_k^{-2} < \infty \). For each compact set in \( \mathbb{C} \), the restriction to it of each function of this class is a uniform limit of polynomials with all roots real.

The spectrum of \( \theta \) is the set \( \Gamma \) of values of \( q \) for which \( \theta(q,.) \) has a multiple real zero. (This terminology has been proposed by B.Z. Shapiro, see [8].) It is shown in [7] that \( \Gamma \) consists of finitely-many of the zeros of \( \theta \).

For \( \tilde{q} \in \mathbb{R} \) the function \( \theta(\tilde{q},.) \) has exactly one multiple real zero which is negative, of multiplicity 2 and is the rightmost of its real zeros.

In the present paper we prove the following theorem:

**Theorem 1.** (1) For any fixed \( q \in \mathbb{C} \setminus 0, |q| < 1 \), and for \( k \) sufficiently large, the function \( \theta(q,.) \) has a zero \( \zeta_k \) close to \(-q^{-k}\) (in the sense that \( |\zeta_k + q^{-k}| \to 0 \) as \( k \to \infty \)). These are all but finitely-many of the zeros of \( \theta \).

(2) For any \( q \in \mathbb{C} \setminus 0, |q| < 1 \), one has \( \theta(q,x) = \prod_k(1 + x/x_k) \), where \( -x_k \) are the zeros of \( \theta \) counted with multiplicity.

(3) For \( q \in (\tilde{q}_j,\tilde{q}_{j+1}] \) the function \( \theta(q,.) \) is a product of a degree \( 2j \) real polynomial without real roots and a function of the Laguerre-Pólya class \( \mathcal{LP} \) - \( \mathbb{I} \). Their respective forms are \( \prod_{k=1}^{2j}(1 + x/\eta_k) \) and \( \prod_k(1 + x/\xi_k) \), where \( -\eta_k \) and \( -\xi_k \) are the complex and the real zeros of \( \theta \) counted with multiplicity.

(4) For any fixed \( q \in \mathbb{C} \setminus 0, |q| < 1 \), the function \( \theta(q,.) \) has at most finitely-many multiple zeros.

(5) For any \( q \in (-1,0) \) the function \( \theta(q,.) \) is a product of the form \( R(q,.)\Lambda(q,.) \), where \( R = \prod_{k=1}^{2j}(1 + x/\tilde{\eta}_k) \) is a real polynomial with constant term 1 and without real zeros and \( \Lambda = \prod_k(1 + x/\xi_k), \xi_k \in \mathbb{R}^* \), is a function of the Laguerre-Pólya class \( \mathcal{LP} \). One has \( \xi_k \xi_{k+1} < 0 \).

The sequence \( \{ |\xi_k| \} \) is monotone increasing for \( k \) large enough.

**Remarks 2.** (1) In the present paper differentiation w.r.t. \( q \) is not used, therefore for a function \( f(q,x) \) we write \( f'(q,x) \) instead of \( (\partial f/\partial x)(q,x) \).

(2) To prove part (1) of the theorem we use properties of the (true) Jacobi theta function \( \Theta(q,x) := \sum_{j=-\infty}^{\infty} q^j x^j \). It is known that \( \Theta \) has only simple zeros (see [13]). Hence this is also the case of the function \( \Theta^*(q,x) = \Theta(\sqrt{q}, \sqrt{x}) = \sum_{j=-\infty}^{\infty} q^{(j+1)/2} x^j \). The zeros of \( \Theta^* \) are all the numbers \( \mu_k := -q^{-k}, k \in \mathbb{Z} \) (this follows from the formula for the zeros of \( \Theta \), see [13]). One has

\[
\Theta^*(q,x) = qx \Theta^*(q,qx).
\]

The function \( \Theta \) satisfies the following identity (known as the Jacobi triple product, see [13]):
\[ \Theta(q, x^2) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + x^2q^{2m-1})(1 + x^{-2}q^{2m-1}) \]

This implies the equality

\[ \Theta^*(q, x^2) = \prod_{m=1}^{\infty} (1 - q^m)(1 + x^2q^m)(1 + x^{-2}q^m) \quad (4) \]

(3) We write \( x \in \Omega_k(\delta) \) instead of \( |x - \mu_k| \leq \delta, \delta > 0 \).

2 Proof of Theorem [1]

Represent the function \( \Theta^* \) in a neighbourhood of its zero \( \mu_k = -q^{-k} \) in the form

\[ \Theta^*(q, x) = K(x+q^{-k})\tau(q, x) \]

where \( \tau = 1 + d_1(x+q^{-k}) + d_2(x+q^{-k})^2 + \cdots \), \( K \in \mathbb{C}\setminus 0 \), \( d_j \in \mathbb{C} \).

Noticing that \( x = x + q^{-k-s} - q^{-k-s} = -q^{-k-s}(1 - q^{k+s}(x + q^{-k-s})) \) and using \( s \) times equation (3) one finds the following representation of \( \Theta^* \) in a neighbourhood of \( \mu_{k+s} \):

\[ \Theta^*(q, x) = q^{s(s+1)/2}x^sKq^s(x + q^{-k-s})\tau(q, q^sx) \]

\[ \tau(q, q^sx) = 1 + d_1q^s(x + q^{-k-s}) + d_2q^{2s}(x + q^{-k-s})^2 + \cdots . \]

Introduce the new variable \( X := x + q^{-k-s} \). If one assumes that for all \( s \in \mathbb{N} \cup 0 \) the variable \( X \) takes values in one and the same disk \( |X| \leq \delta \), then as \( s \to \infty \) the functions \( (1-q^{k+s}X)^s \) and \( \tau(q, q^sx) \) tend uniformly to 1. Thus for \( |X| \leq \delta \) (i.e. for \( x \in \Omega_{k+s}(\delta) \)) one has

\[ \Theta^* = K(-1)^s q^{s(-2k-s+3)/2}(1 + o(1))X \]

For any \( B > 0 \) there exists \( k_0 \in \mathbb{N} \cup 0 \) such that for \( k \geq k_0 \) one has \( |q^{s(-2k-s+3)/2}| \geq B \) for any \( s \in \mathbb{N} \). Hence there exists \( \kappa > 0 \) such that for \( |\eta| \leq \kappa \), for \( k \geq k_0 \) and for any \( s \in \mathbb{N} \) the equation \( \Theta^* = \eta \) has a unique solution \( X = X(\eta) \) with \( |X| \leq \delta \).

Set \( \theta(q, x) = \Theta^*(q, x) + \Xi(q, x) \), where \( \Xi(q, x) = -\sum_{j=-\infty}^{-1} q^{j(j+1)/2}x^j \). For fixed \( q \), \( |q| < 1 \), the series of \( \Xi \) and \( \Xi' \) both converge for \( |x| > 1 \), and for any \( \varepsilon > 0 \) there exists \( A \geq 1 \) such that if \( |x| \geq A \), then \( |\Xi| \leq \varepsilon \) and \( |\Xi'| \leq \varepsilon \). (Indeed, both series as series of \( 1/x \) are without constant term and the moduli of all coefficients of the series of \( \Xi \) are less than 1.)

For \( k \in \mathbb{N} \) sufficiently large the equation

\[ \theta(q, x) = 0 \] i.e. \( \Theta^*(q, x) = -\Xi(q, x) \)

has a unique solution \( x = x(q) \in \Omega_k(\delta) \). Indeed, for such \( k \) and for \( x \in \Omega_k(\delta) \) the equation \( \Theta^*(q, x) = \eta \) has a solution \( x(q, \eta) = \mu_k + O(\eta) \) which is holomorphic in \( \eta \) for \( |\eta| \) sufficiently small. Substituting \( -\Xi(q, x) \) for \( \eta \) one obtains an equation of the form \( x = \mu_k + \Delta(q, x) \), where \( \max_{x \in \Omega_k(\delta)} |\Delta(q, x)| \) and \( \max_{x \in \Omega_k(\delta)} |\Delta'(q, x)| \) can be made arbitrarily small by choosing \( A \) and \( k_0 \) sufficiently large. Hence this equation has a unique solution in \( \Omega_k(\delta) \).
To complete the proof of part (1) we show that for $k_0$ large enough there remain only finitely-many zeros of $\theta$ outside the set $\Sigma := \bigcup_{k=k_0}^\infty \Omega_k(\delta)$. We use equation (4). (From now till the end of the proof of part (1) we consider $\theta(q, x^2)$ instead of $\theta(q, x)$, similarly for $\Theta$, $\Theta^*$ and $\Xi$ which is not restrictive.) Suppose that $|x|^2 \geq |q|^{-6}$ (this is not a restriction since the function $\theta$ is holomorphic and hence has finitely-many zeros in any compact set). Then for the products $\Pi_1 := \prod_{m=1}^{\infty} (1 - q^m)$ and $\Pi_2 := \prod_{m=1}^{\infty} (1 + x^{-2}q^{m-1})$ one has

$$|\Pi_1| \geq \prod_{m=1}^{\infty} (1 - |q|^m) =: r > 0 , \quad |x^{-2}q^{m-1}| < |q|^m \quad \text{and} \quad |\Pi_2| \geq \prod_{m=1}^{\infty} (1 - |q|^m) = r .$$

To estimate the product $\prod_{m=1}^{\infty} |1 + x^2q^m|$ we define $l \in \mathbb{N} \cup 0$ by the condition $|q|^{-l} \leq |x|^2 < |q|^{-l-1}$, $l \geq 6$. Thus for $m \geq l + 2$ one has

$$|x^2q^m| < |q^{m-l-1}| \quad \text{and} \quad \prod_{m=1}^{l+2} |1 + x^2q^m| \geq \prod_{m=1}^{\infty} (1 - |q|^m) = r , \quad \text{hence}$$

$$\prod_{m=1}^{l-1} |1 + x^2q^m| \geq \prod_{m=1}^{l-1} (|q^{m-l}| - 1) = |q|^{-l(l-1)/2} \prod_{m=1}^{l-1} (1 - |q|^{l-m}) \geq |q|^{-l(l-1)/2r} .$$

There remains to consider the product

$$|1 + x^{-2}q^l| |1 + x^{-2}q^{l+1}| = (1/|x|^4) |x^2 + q^l| |x^2 + q^{l+1}| .$$

The second and the third factor are not less than $\delta$ for $x^2 \in (\mathbb{C}\setminus 0) \setminus \Sigma$. Hence the whole product is not less than $|q|^{2l+2\delta^2}$. Thus

$$|\Theta(q, x^2)| \geq r^3 |q|^{-l(l-1)/2+2l+2\delta^2} .$$

The exponent of $|q|$ is negative for $l \geq 6$. For $h > 0$ set $B_h := \{ x \in \mathbb{C} \mid |x|^2 \geq h \}$. Hence $|\Theta(q, x^2)| \geq r^3 \delta^2$ for $x \in B_{|q|^{-6}} \setminus \Sigma$. On the other hand $\max_{B_h} |\Xi(q, x^2)|$ tends to $0$ as $h \to \infty$. Hence for $h$ sufficiently large the equation $\theta(q, x^2) = 0$, i.e. $\Theta^*(q, x^2) = -\Xi(q, x^2)$, has no solution in $B_{|q|^{-6}} \setminus \Sigma$. Part (1) is proved.

To prove part (2) consider the function $h(q, x) := \prod (1 + x/x_k)$, where $-x_k$ are the zeros of $\theta$ counted with multiplicity their moduli forming a non-decreasing sequence. (The convergence of the infinite product follows from part (1) – for large values of $k$ the numbers $x_k$ are approximated by a geometric progression with ratio $1/q$.) Hence one has $\theta(q, x) = h(q, x) \Phi(q, x)$, where for each fixed $q$ the function $\Phi$ is an entire function without zeros. By Theorem 3 of Chapter I, Section 3 of [9] one has $\Phi = e^\varphi$, where for each fixed $q$, $\varphi$ is an entire function.

The order of the product of two entire functions of different (of equal) orders is the greater of the two orders (is their common order, see Theorem 12 of Chapter I, Section 9 of [9]). The order of $\theta$ is defined by the formula (see Theorem 2 of Chapter I, Section 2 of [9])

$$\lim_{k \to \infty} (k \ln k / \ln(1/|q|^{k(k+1)/2})) = 0 .$$

Hence $\Phi$ and $h$ must be (for each fixed $q$) both of order $0$. This implies that $\varphi$ must be a constant. As $\theta(q, 0) = 1 = h(q, 0)$, one must have $\Phi = 1$ (i.e. $\varphi = 0$). This proves part (2) of the theorem.

Now we prove part (3). It is shown in [7] that for $q \in (\bar{q}_j, \bar{q}_{j+1}]$ the function $\theta(q, .)$ has $j$ conjugate pairs of roots counted with multiplicity. Hence for these values of $q$ one can set
\[ \theta = P(q, x)\psi(q, x), \] for each fixed \( q \), \( P(q, \cdot) \) is a degree 2\( j \) polynomial in \( x \) without real roots and the zeros of \( \psi(q, \cdot) \) (resp. of \( P(q, \cdot) \)) are the real (resp. the complex) zeros of \( \theta(q, \cdot) \) counted with multiplicity. We assume that \( P(q, 0) = 1 \). Recall that the real zeros of \( \theta \) are all negative.

Consider the infinite product \( g(q, x) := \prod (1 + x/\xi_k) \), where \( -\xi_k \) are the real zeros of \( \theta(q, \cdot) \) given in the decreasing order. Theorem 4 in [7] implies that \( \lim_{k \to \infty} \xi_k q^k = 1 \). Hence there exists \( D > 0 \) such that \( \xi_k \geq Dq^{-k} \). Set \( g := g_0 + g_1 x + g_2 x^2 + \cdots \). One has \( g_k \geq 0 \) and

\[
g_k = \sum_{1 \leq j_1 < j_2 < \cdots < j_k} 1/\xi_{j_1} \xi_{j_2} \cdots \xi_{j_k} \leq D^{-k} S_k,
\]

where \( S_k := \sum_{1 \leq j_1 < j_2 < \cdots < j_k} q^{j_1 + j_2 + \cdots + j_k} \). It is clear that \( S_1 = q/(1 - q) \) and

\[
S_k \leq \sum_{1 \leq j_1 < j_2 < \cdots < j_{k-1}} q^{j_1 + j_2 + \cdots + j_{k-1}} \sum_{j_k = 0}^{\infty} q^{j_k} = S_{k-1} q^k/(1 - q).
\]

Hence by induction on \( k \) one shows that \( S_k \leq q^{k(k+1)/2}/(1 - q)^k \) and \( g_k \leq q^{k(k+1)/2}/(D(1 - q))^k \).

Thus for each fixed \( q \in (0, 1) \) the product \( g \) is an entire function of the Laguerre-Pólya class \( \mathcal{LP} - \mathcal{I} \). Its order equals 0. Indeed, using again Theorem 2 of Chapter I, Section 2 of [9], the order of \( g \) is defined by the formula

\[
\overline{\lim}_{k \to \infty} (k \ln k/\ln(1/|g_k|)) = 0.
\]

For large values of \( k \) one has

\[
1/|g_k| \geq (D(1 - q))^k q^{-k(k+1)/2} \text{ hence } \ln(1/|g_k|) \leq -k(k + 1)(\ln q)/2 + k(\ln D + \ln(1 - q))
\]

and the above limit is 0.

The function \( \psi \) equals \( gg_1 \), where \( g_1 \) is an entire function without zeros. As in the proof of part (1) one shows that \( g_1 = e^{\tilde{g}} \), where for each fixed \( q \), \( \tilde{g} \) is an entire function, and that the order of \( g_1 \) equals 0. Hence \( \tilde{g} \) is a constant and as \( \psi(q, 0) = g(q, 0) = 1 \), one must have \( \tilde{g} \equiv 0 \), i.e. \( g_1 \equiv 1 \).

Part (4) of the theorem results from part (3) – for \( k \) large enough the zero of \( \theta \) which is close to \( \mu_k = -q^{-k} \) is simple and only finitely-many of its zeros are not of this kind.

To prove part (5) notice that for \( q \in (-1, 0) \) the numbers \( \mu_k \) are real and change alternatively sign. As \( \theta(q, x) = \theta(q^\overline{x}, x) \), part of the zeros of \( \theta \) are real and the rest form conjugate pairs. For large \( k \) there is just one zero of \( \theta \) close to \( \mu_k \) hence this zero is real. The condition \( \xi_k \overline{\xi_{k+1}} < 0 \) follows from the alternative changing of sign of \( \mu_k \). As in the proof of part (3) we show that \( \theta = RA_1 A_2 \), where \( R \) is a real polynomial with constant term 1 and without real zeros and \( A_i \) (of the form \( \prod_k (1 + x/\xi_{i,k}) \)) are functions of the Laguerre-Pólya class \( \mathcal{LP} - \mathcal{I} (\xi_{1,k} > 0, \overline{\xi}_{2,k} < 0) \). Hence \( \Lambda := A_1 A_2 \in \mathcal{LP} \). For \( k \) large enough the sequence \( \{ |\xi_k| \} \) is monotone increasing because \( \overline{\xi_k} \) is close to \( \mu_k \). Theorem 5 is proved.

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