Abstract—In this article we investigate the solvability of infinite-dimensional differential-algebraic equations. Such equations often arise as partial differential-algebraic equations (PDAEs). A decomposition of the state-space that leads to an extension of the Hille-Yosida Theorem on Hilbert spaces for these equations is described. For dissipative partial differential equations the famous Lumer-Phillips generation theorem characterizes solvability and also boundedness of the associated semigroup. An extension of the Lumer-Phillips generation theorem to dissipative differential-algebraic equations is given. The results is illustrated by coupled systems and the Dzektsir equation.

I. INTRODUCTION

We consider infinite-dimensional differential-algebraic equations (DAEs)
\[
\begin{align*}
\frac{d}{dt} [Ex(t)] = Ax(t), & \quad t \geq 0, \quad Ex(0) = z_0.
\end{align*}
\]
(1)

Here \(A\) and \(E\) are linear operators from \(\mathcal{X}\) to \(\mathcal{Z}\) and \(z_0 \in \mathcal{Z}\), where \(\mathcal{X}\) and \(\mathcal{Z}\) are complex Hilbert spaces. The operator \(E\) is bounded from \(\mathcal{X}\), but \(A\) is densely defined and closed on \(\mathcal{X}\). Such equations arise from the coupling of partial differential equations where one sub-system is in equilibrium, as well as in some other situations.

Establishing well-posedness of these equations, particularly when \(E\) is not invertible, is non-trivial. Solvability of infinite-dimensional differential-algebraic equations has been intensively studied, see [Rei08], [Tro20], [TT01], [TT96], [FY04], [Sho10], [Yag91], [FY09]. For example, Trostorff [Tro20] and Reis and Tischendorf [RT05] provide sufficient conditions in terms of Hille-Yosida type resolvent estimates. In [TT96] the splitting \(\mathcal{X} = \ker E \oplus \overline{\text{ran} E}\) and \(\mathcal{Z} = \ker E^* \oplus \overline{\text{ran} E}\), and the restriction of equation (1) to this splitting and the solvability is investigated.

In [SF03] a concept called \((E, r)\)-radiality is introduced that also leads to Hille-Yosida type conditions for generation of a semigroup. Under associated conditions, there exists a splitting of \(\mathcal{X}\) and \(\mathcal{Z}\) into the kernel of \(E\) and a non-orthogonal complement. With respect to this splitting the differential-algebraic equations (1) are written as
\[
\begin{align*}
\frac{d}{dt} [Ex(t)] = Ax(t), & \quad t \geq 0, \quad Ex(0) = z_0.
\end{align*}
\]
(2)

where \(E_1 : \mathcal{X} \rightarrow \mathcal{Z}\) is bounded and invertible, \(A_0 : \mathcal{D}(A) \cap \mathcal{X}^0 \rightarrow \mathcal{Z}^0\) is closed and invertible, \(A_1 : \mathcal{D}(A) \cap \mathcal{X}^1 \rightarrow \mathcal{Z}^1\) is closed, and \(A_1 E_1^{-1}\) generates a \(C_0\)-semigroup in \(\mathcal{Z}^1\). Here \(\mathcal{D}(A)\) denotes the domain of the operator \(A\).

As we work with Hilbert spaces instead on general Banach spaces, we are able to simplify and weaken the required conditions considerably.

The difficulty with this approach, as with the classical Hille-Yosida Theorem, and with other resolvent estimates, is that it can be difficult to confirm that the assumptions are satisfied. The Lumer-Phillips Theorem e.g. [CZ95] is a very useful tool in the standard \(E = I\) situation for establishing that an operator \(A\) generates a \(C_0\)-semigroup. Favini and Yagi [FY99, Page 37] show that if \((\lambda E - A)^{-1} \in \mathcal{L}(\mathcal{Z}, \mathcal{X})\) for some \(\lambda > 0\) and \(\text{Re} (Ax, Ex)_Z \leq 0\) for all \(x \in \mathcal{D}(A)\), then for every \(x_0 \in \mathcal{E}(\mathcal{D}(A))\) the DAE (1) has a unique classical solution; that is \(x : [0, T] \rightarrow \mathcal{D}(A)\) and \(Ex \in C^1([0, T]; \mathcal{Z})\), \(Ax \in C^0([0, T]; \mathcal{Z})\), and DAE (1) is satisfied. They further provide results for parabolic DAEs. However, they do not investigate a splitting of the state space as in (2), nor do they show generation of a \(C_0\)-semigroup. The main result of this paper is a generalization of the Lumer-Phillips Theorem to dissipative infinite-dimensional DAEs.

II. RADIALITY AND SEMIGROUP GENERATION

Let \(\mathcal{X}, \mathcal{Z}\) be complex Hilbert spaces, \(E \in \mathcal{L}(\mathcal{X}, \mathcal{Z})\), \(A : \mathcal{D}(A) \subset \mathcal{X} \rightarrow \mathcal{Z}\) densely defined and closed, and \(z_0 \in \mathcal{Z}\). By
\[
\varrho(E, A) := \{s \in \mathbb{C} \mid (sE - A)^{-1} \in \mathcal{L}(\mathcal{Z}, \mathcal{X})\}
\]
we denote the resolvent set of the operator pencil \((E, A)\). For \(s \in \varrho(E, A)\), we define the right- and left-E resolvents of \(A\) (with respect to \(E\)) by
\[
R^E(s, A) = (sE - A)^{-1}E, \quad L^E(s, A) = E(sE - A)^{-1},
\]
In particular, if \(0 \in \varrho(E, A)\),
\[
R^E(0, A) = -A^{-1}E, \quad L^E(0, A) = -EA^{-1}.
\]
Definition 2.1: The operator $A$ is E-radial if

1. $s \in \rho(E, A)$ for all real $s > 0$,
2. there exists $K > 0$ such that for all $n \in \mathbb{N}$ and for all real $s > 0$
   \[
   \| (R_E(s, A))^n \|_{L(X,X)} \leq \frac{K}{s^n}, \quad (3)
   \]
   \[
   \| (L_E(s, A))^n \|_{L(Z,Z)} \leq \frac{K}{s^n}, \quad (4)
   \]

In the case $E = I$ statements (3) and (4) are equivalent; and each statement implies generation of a $C_0$-semigroup by the Hille-Yosida theorem. In [SF03] a more general concept $(E, p)$ radiality is considered. In that framework, an E-radial operator is $(E, 0)$-radial. Here, only $p = 0$ is considered and also the spaces are assumed to be Hilbert spaces.

Definition 2.2: The operator $A$ is weakly E-radial if $s \in \rho(E, A)$ for all $s > 0$, and (3)-(4) holds with $n = 1$.

Clearly any E-radial operator is weakly E-radial. The converse holds if $K \leq 1$.

Definition 2.3: The operator $A$ is strongly E-radial if it is E-radial and there is a linear and dense subspace $\tilde{Z}$ of $Z$ such that

\[
\| R_E(s, A)(\lambda E - A)^{-1}Ax \|_{X} \leq \frac{\text{const}(x)}{\lambda s}, \quad x \in D(A),
\]

\[
\| A(\lambda E - A)^{-1}L_E(s, A)z \|_{Z} \leq \frac{\text{const}(z)}{\lambda s}, \quad z \in \tilde{Z}.
\]

Define for some $\alpha \in \rho(E, A)$,

$X^0 = \ker R_E(\alpha, A) = \ker E$,

$Z^0 = \ker L_E(\alpha, A) = \{ Ax \mid x \in D(A) \cap \ker E \}$,

$X^1 = \overline{\text{ran} R_E(\alpha, A)}$,

$Z^1 = \overline{\text{ran} L_E(\alpha, A)}$.

These spaces are independent of the choice of $\alpha$ ([SF03, Lem. 2.1.2, pg. 18]) and also if $A$ is weakly E-radial, then

\[
\lim_{s \to \infty} sR_E(s, A)x = x, \quad \text{for all } x \in X^1,
\]

\[
\lim_{s \to \infty} sL_E(s, A)z = z, \quad \text{for all } z \in Z^1,
\]

see [SF03, Lem. 2.2.6]. If $A$ is weakly E-radial, then since the Hilbert spaces $X$ and $Z$ are reflexive, [SF03, Theorem 2.5.1] implies

$X = X^0 \oplus X^1$ and $Z = Z^0 \oplus Z^1$.

Thus if $A$ is weakly E-radial,

1. $P : X \to X$ defined by
   \[
P x := \lim_{s \to \infty} sR_E(s, A)x
   \]
   is a projection onto $X^1$ with $\ker P = X^0$ and $\text{ran} P = X^1$,

2. $Q : X \to X$ defined by
   \[
   Q z := \lim_{s \to \infty} sL_E(s, A)z
   \]
   is a projection onto $Z^1$ with $\ker Q = Z^0$ and $\text{ran} Q = Z^1$.

In general, both $P$ and $Q$ are non-orthogonal projections.

Define restrictions of $E$ and $A$ as follows:

$E_0 := E|_{X^0}$, \quad $A_0 := A|_{D(A_0)}$, \quad $D(A_0) = X^0 \cap D(A)$,

$E_1 := E|_{X^1}$, \quad $A_1 := A|_{D(A_1)}$, \quad $D(A_1) = X^1 \cap D(A)$.

In [SF03, Lem. 2.2.1, pg. 20] it is shown that $E_0 \in L(X^0, Z^0)$ and $A_0 : D(A_0) \to Z^0$. Further, if $A$ is weakly E-radial, then $A_0$ is boundedly invertible; that is,

$A_0^{-1} \in L(Z^0, X^0)$,

see [SF03, Lem. 2.2.4, pg. 22] and also

$A_0^{-1} E_0 = 0$, \quad $E_0 A_0^{-1} = 0$

on $X^0$ and $Z^0$, respectively, by [SF03, Lem. 2.2.5, pg. 22].

The following proposition has been proved in [SF03, Cor. 2.5.1, pg 38] if $A$ is strongly radial. We are able to weaken this assumption because we deal with Hilbert spaces.

Proposition 2.4: If $A$ is weakly E-radial, then

1) for all $x \in D(A)$, $P x \in D(A)$ and $AP x = QAx$,

2) for all $x \in X$, $EP x = QEx$.

Proof: Recall that the operator $P$ is defined by

$P x := \lim_{s \to \infty} sR_E(s, A)x$.

For any $x \in D(A) \subset X$, by [SF03, Equation (2.1.8), pg. 17]

$AR_E(s, A)x = L_E(s, A)Ax$.

Let $x \in D(A)$. Since $R_E(s, A)x \in D(A)$, and $A$ is closed, $P x \in D(A)$. Thus

$AP x = A\left(\lim_{s \to \infty} sR_E(s, A)x\right)
= \lim_{s \to \infty} AsR_E(s, A)x
= \lim_{s \to \infty} sL_E(s, A)Ax
= QAx$.

This proves Part 1). Part 2) follows easily using the fact that $E \in L(X, Z)$. For any $x \in X$,

$EP x = E \lim_{s \to \infty} sR_E(s, A)x
= \lim_{s \to \infty} sER_E(s, A)x
= \lim_{s \to \infty} sL_E(s, A)Ex
= QEx$.

This concludes the proof. \[\square\]

Thus, if $A$ is weakly E-radial, then the operators $A_0$, $A_1$, $E_0$ and $E_1$ are invariant with respect to the projected spaces. More precisely, if $A$ is weakly E-radial, by [SF03, Lem. 2.2.1, pg. 20 and Cor. 2.5.2, pg. 39]

1. $E_0 \in L(X^0, Z^0)$,

$E_1 \in L(X^1, Z^1)$,

$A_0 : D(A_0) \subset X^0 \to Z^0$ is densely defined, closed, and boundedly invertible,

$A_1 : D(A_1) \subset X^1 \to Z^1$ is densely defined and closed.

The following proposition was proven in [SF03, Cor. 2.5.3, pg. 40] with an assumption that $A$ is strongly radial.
Proposition 2.5: If $A$ is weakly $E$-radial and $\text{ran } E$ is closed in $Z$, then $E_1 \in \mathcal{L}(\mathcal{X}^1, Z^1)$ is boundedly invertible.

Proof: The fact that $E_1 \in \mathcal{L}(\mathcal{X}^1, Z^1)$ follows from [SF03, Cor. 2.5.2, pg. 39]. Since $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$, and $\mathcal{X}^0 = \ker E$, it follows that $E_1$ is injective. Thus it remains to show that $E_1$ is surjective. By the definition of $Z_1$,

$$Z^1 = \overline{E(D(A))} \subset \text{ran } E = \text{ran } E_1.$$ 

Since $\text{ran } E_1 \subset Z^1, E_1$ is surjective. Thus $E_1$ has an inverse defined on all of $Z^1$ and so by the Closed Graph Theorem this inverse is bounded. □

This framework allows us to decompose the system using the non-orthogonal projections $P$ and $Q$ if $A$ is weakly $E$-radial and $\text{ran } E$ is closed. Defining

$$\tilde{P} = \left[ \begin{bmatrix} I & -P \end{bmatrix} \right] \in \mathcal{L}(\mathcal{X}, \mathcal{X}^0 \times \mathcal{X}^1),$$ 

$$\tilde{Q} = \left[ \begin{bmatrix} I & -Q \end{bmatrix} \right] \in \mathcal{L}(Z, Z^0 \times Z^1),$$

we obtain

$$\tilde{P}^{-1} = \left[ \begin{bmatrix} I & I \end{bmatrix} \right] \in \mathcal{L}(\mathcal{X}^0 \times \mathcal{X}^1, \mathcal{X}),$$ 

$$\tilde{Q}^{-1} = \left[ \begin{bmatrix} I & I \end{bmatrix} \right] \in \mathcal{L}(Z^0 \times Z^1, Z),$$

where $I$ above indicates the natural injection on the various spaces; the different spaces are not explicitly indicated. Let $z \in Z$ and $\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} := \tilde{P}z$. Then we obtain the chain of equivalences

$$E \frac{d}{dt} z = Az \iff \tilde{Q}EP^{-1} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \tilde{Q}AP^{-1} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \iff \begin{bmatrix} E_0 & 0 \\ 0 & E_1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \iff \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A_1 E_1^{-1} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}.$$ 

Our main result in this section is as follows.

Theorem 2.6: If $A - \alpha E$ is $E$-radial and $\text{ran } E$ is closed, then the operator $E_1^{-1} A_1$ with domain $D(A) \cap \mathcal{X}^1$ generates a $C_0$-semigroup $(S(t))_{t \geq 0}$ on $\mathcal{X}^1$ with bound $K e^{\alpha t}$.

Proof: First define $A = (A - \alpha E)$. By our assumption the operator $E_1^{-1} A_1$ with domain $D(A) \cap \mathcal{X}^1$ is well-defined, closed and densely defined. The definition of $E$-radiality further implies $(0, \infty) \in g(E_1^{-1} A_1)$ and there exists $K > 0$ such that for all $n \in \mathbb{N}$ and for all real $s > 0$

$$\|(s I - E_1^{-1} A_1)^n\|_{\mathcal{L}(\mathcal{X}^1, \mathcal{X}^1)} \leq \frac{K}{s^n}.$$ 

Thus from the Hille-Yosida Theorem, $E_1^{-1} A_1$ generates a bounded $C_0$-semigroup on $Z^1$ with bound $K$.

Now note $A_1 = (A - \alpha E) = A_1 - \alpha E_1$. The statement of the theorem now follows. □

Alternatively, the same projections can be applied to

$$\frac{d}{dt} Ez = Az.$$ 

This yields

$$\frac{d}{dt} PEP^{-1} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \tilde{Q}AP^{-1} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \iff \begin{bmatrix} E_0 & 0 \\ 0 & E_1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \iff \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A_1 E_1^{-1} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix},$$ 

where $\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} A_0^{-1} x_0 \\ E_1^{-1} x_1 \end{bmatrix}.$

Corollary 2.7: If $A - \alpha I$ is $E$-radial and $\text{ran } E$ is closed, then the operator $A_1 E_1^{-1}$ with domain $E_1(D(A) \cap \mathcal{X}^1)$ generates a $C_0$-semigroup on $Z_1$ with bound $Ke^{\alpha t}$.

Thus well-posedness of the abstract Cauchy problem $\frac{d}{dt} x = Ax$ reduces to well-posedness of $\frac{d}{dt} \tilde{x}_1 = E_1^{-1} A_1 \tilde{x}_1$. Further, well-posedness of $\frac{d}{dt} \tilde{x}_1 = A_1 E_1^{-1} \tilde{x}_1$.

III. DISSIPATIVE PENCILS

The fact whether an operator $A$ is $E$-radial is in general not easy to verify. Therefore, in this section we aim to generalize the Lumer-Phillips theorem to dissipative differential-algebraic equations. We start with the following definition.

As in the previous section $\mathcal{X}$ and $Z$ are complex Hilbert spaces, $E \in \mathcal{L}(\mathcal{X}, Z)$, and $A : D(A) \subset \mathcal{X} \rightarrow Z$ is densely defined and closed.

Definition 3.1: The operator pencil $(E, A)$ is called dissipative, if

$$\|(\lambda E - A)x\|_Z \geq \lambda \|Ex\|_Z, \quad \lambda > 0, x \in D(A).$$

In the following three propositions we define dissipative operator pencils and prove some implications.

Proposition 3.2: The following statements are equivalent:
1) The operator pencil $(E, A)$ is dissipative.
2) $\text{Re} \langle Ax, Ex \rangle_Z \leq 0, x \in D(A)$.

Proof: For $\lambda > 0$ and $x \in D(A)$ equation (5) is equivalent to

$$\frac{1}{2\lambda} \|Ax\|^2 \geq \text{Re} \langle Ax, Ex \rangle.$$ 

This implies the statement of the proposition. □

Remark 3.3: If $\frac{d}{dt} Ex(t) = Ax(t), t \geq 0$, has a classical solution and satisfies $\frac{d}{dt} \|Ex(t)\|^2 \leq 0$ for every classical solution $x$, then the operator pencil is dissipative.

Proposition 3.4: If the operator pencil $(E, A)$ is dissipative, then the following are equivalent
1) $\ker A \cap \ker E = \{0\}$,
2) $\ker (\lambda E - A) = \{0\}$ for some $\lambda > 0$,
3) $\ker (\lambda E - A) = \{0\}$ for every $\lambda > 0$.

Proof: Clearly statement (3) implies (2) and (2) implies (1). Thus it remains to show that (1) implies (3). Assume that (3) does not hold, that is, there exists $\lambda > 0$ and $x \in \ker (\lambda E - A), x \neq 0$. This implies

$$\lambda^2 \|Ex\|^2 - 2\lambda \text{Re} \langle Ex, Ax \rangle + \|Ax\|^2 = 0.$$
Since \((E, A)\) is dissipative, all terms of the right hand side are non-negative. Thus \(Ex = 0\) and \(Ax = 0\), which implies that (1) does not hold. Hence statement (1) implies (3). □

**Proposition 3.5:** If the operator pencil \((E, A)\) is dissipative and \(\varrho(E, A) \cap (0, \infty) \neq \emptyset\), then \((0, \infty) \subset \varrho(E, A).\)

**Proof:** Let \(\lambda_0 \in \varrho(E, A) \cap (0, \infty).\) Proposition 3.2 implies \(\ker(\lambda - A) = \{0\}\) for every \(\lambda > 0.\)

Next we show that \(\ran(\lambda E - A)\) is dense in \(Z\) for every \(\lambda > 0.\) Let \(\lambda > 0\) be arbitrary and \(z \in Z\) be orthogonal to \(\ran(\lambda E - A).\) Since \(\ran(\lambda_0 E - A) = Z\) there exists \(x \in D(A)\) such that \(z = (\lambda_0 E - A)x.\) This implies

\[
0 = \Re \langle z, (\lambda E - A)x \rangle = \Re \langle (\lambda_0 E - A)x, (\lambda E - A)x \rangle = \lambda_0 \lambda \|Ex\|^2 - (\lambda + \lambda_0) \Re \langle Ex, Ax \rangle + \|Ax\|^2.
\]

Because \((E, A)\) is dissipative, all terms of the right hand side are non-negative. Thus \(Ex = 0\) and \(Ax = 0\), which implies \(\lambda_0 E x - Ax = 0.\) Since \(\ker(\lambda_0 E - A) = \{0\}\), it follows that \(x = 0\) and therefore \(z = 0.\) Thus \(\ran(\lambda E - A)\) is dense in \(Z\) for every \(\lambda > 0.\)

It remains to show that \(\ran(\lambda E - A)\) is closed in \(Z\) for every \(\lambda > 0.\) We note that for an injective, closed and densely defined operator \(T: D(T) \subset X \to Z\) the following statements are equivalent

- \(\ran(T)\) is closed in \(Z.\)
- There exists \(c > 0\) such that \(\|Tx\| \geq c\|x\|\) for every \(x \in D(T).\)

Thus there exists \(c_{\lambda_0} > 0\) such that \(\|(\lambda_0 E - A)x\| \geq c_{\lambda_0} \|x\|\) for every \(x \in D(A).\) This implies that for any \(x \in D(A)\) with \(\|x\| = 1:\)

\[
\lambda_0^2 \|Ex\|^2 - 2\lambda_0 \Re \langle Ex, Ax \rangle + \|Ax\|^2 \geq c_{\lambda_0}. \tag{6}
\]

It remains to show that for every \(\lambda > 0\) there exists \(c_{\lambda} > 0\) such that

\[
\lambda^2 \|Ex\|^2 - 2\lambda \Re \langle Ex, Ax \rangle + \|Ax\|^2 \geq c_{\lambda} \tag{7}
\]

for all \(x \in D(A)\) with \(\|x\| = 1.\) Assume that this is not true: that is, there exists \(\lambda > 0\) and a sequence \((x_n) \subset D(A)\) with \(\|x_n\| = 1\) such that

\[
\lambda^2 \|Ex_n\|^2 - 2\lambda \Re \langle Ex_n, Ax_n \rangle + \|Ax_n\|^2 \to 0
\]

as \(n \to \infty.\) As \((E, A)\) is dissipative, all three terms are non-negative and thus \(\|Ex_n\|^2 \to 0,\) \(\|Ax_n\|^2 \to 0\) and \(\langle Ex_n, Ax_n \rangle \to 0\) as \(n \to \infty,\) which implies that (6) does not hold. Thus, statement (7) holds.

The first main result of this section is summarized in the following theorem.

**Theorem 3.6:** If \(\lambda \in \varrho(E, A)\) for some \(\lambda > 0\) and

\[
\Re \langle Ax, Ex \rangle \leq 0, \quad x \in D(A),
\]

\[
\Re \langle A^* x, E^* x \rangle \leq 0, \quad x \in D(A^*),
\]

then

1) \((0, \infty) \subset \varrho(E, A)\) and \((0, \infty) \subset \varrho(E^*, A^*).\)
2) \(\|E(\lambda E - A)^{-1}\| \leq \frac{1}{\lambda}\) for \(\lambda > 0,\)
3) \(\|\lambda E - A\|^{-1} \leq \frac{1}{\lambda}\) for \(\lambda > 0,\)
4) \(A\) is \(E\)-radial.

**Proof:** The first statement follows from Proposition 3.5 and the second from Proposition 3.2. Moreover, Proposition 3.2 implies

\[
\|E^*(\lambda E^* - A^*)^{-1}\| \leq \frac{1}{\lambda}
\]

for \(\lambda > 0,\) which implies the third statement. The last statement now follows directly from the definition of \(E\)-radiality.

**Theorem 3.6** together with Theorem 2.6 now implies the second result of this section.

**Theorem 3.7:** If the operator \(E\) has closed range, \(\lambda \in \varrho(E, A)\) for some \(\lambda > 0\) and

\[
\Re \langle Ax, Ex \rangle \leq 0, \quad x \in D(A),
\]

\[
\Re \langle A^* x, E^* x \rangle \leq 0, \quad x \in D(A^*),
\]

Then the Hilbert spaces \(X, Z\) can be split as \(X = X^0 \oplus X^1\) and \(Z = Z^0 \oplus Z^1,\) where

\[
X^0 = \ker E, \quad Z^0 = \{Ax \mid x \in D(A) \cap \ker E\},
\]

\[
X^1 = \overline{\ran R^E(\alpha, A)}, \quad Z^1 = \ran E
\]

for some (and hence every) \(\alpha \in \varrho(E, A).\) Further, \(P : X \to X\) defined by \(Px := \lim_{n \to \infty} sR^E(x, A)x\) is a projection onto \(X^1\) with \(\ker P = X^0,\) and \(Q : X \to X\) defined by \(Qz := \lim_{n \to \infty} sL^E(s, A)z\) is a projection onto \(Z^1\) with \(\ker Q = Z^0.\) For

\[
E_1 := E|_{X^1}, \quad A_1 := A|_{D(A_1)} : D(A_1) \subset X^1 \to Z^1,
\]

\[
D(A_1) = X^1 \cap D(A),
\]

we have \(E_1\) is boundedly invertible, and \(A_1\) is closed and densely defined.

The equation \(\frac{d}{dt}(Ex) = Ax\) is equivalent to

\[
\frac{d}{dt} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & E_1^{-1}A_1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix},
\]

where \(x = \begin{bmatrix} x^0 \\ x^1 \end{bmatrix},\) and \(E_1^{-1}A_1\) generates a contraction semigroup on \(X^1.\)

**IV. Coupled systems**

In this section we study an application of Theorem 2.6 to a class of differential-algebraic systems defined on a Hilbert space \(Z,\)

\[
\frac{d}{dt} x(t) = A_1 x(t) + A_2 y(t)
\]

\[
0 = A_3 x(t) + A_4 y(t),
\]

where for \(i = 1 \ldots 4, A_i : D(A_i) \subset Z \to Z\) are closed and densely defined. This class of systems is of the form (1):

\[
\frac{d}{dt} \begin{bmatrix} I \\ E \end{bmatrix} x(t) = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} x(t), \quad t > 0,
\]

with \(X = Z = Z \times Z\) and

\[
D(A) = (D(A_1) \cap D(A_3)) \times (D(A_2) \cap D(A_4)).
\]
Although this is a very particular class of systems, a number of applications fit this class. For examples see [MO14] for piezoelectric beams with quasi-static magnetic effects, [CKC+10] for lithium-ion cell models and [HHOS07] for a model with convection-diffusion dynamics.

In order for \( \varrho(E, A) \) to be non-empty, it is necessary that \( A_4 \) have a bounded inverse. This assumption will be made throughout this section, as well as several other assumptions that will guarantee well-posedness.

**Assumption 4.1:**

(a) Let \( 0 \in \varrho(A_4) \), \( D(A_4) \subset D(A_2) \) and \( D(A_4^\ast) \subset D(A_2^\ast) \). By [Tre08, Rem. 2.2.3.15], this implies that the operator \( A_2 A_4^{-1} A_3 : D(A_3) \rightarrow Z \) is well defined. Assume also \( A_2 A_4^{-1} A_3 \in \mathcal{L}(Z) \).

(b) Let there exist \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that for every \( s > \omega \), \( s \in \varrho(A_1) \) and

\[
\| (s - A_1)^{-n} \| \leq \frac{M}{(s - \omega)^n}, \quad s > \omega, n \in \mathbb{N}.
\]

**Remark 4.2:** If \( A_2, A_3 \in \mathcal{L}(Z) \), then Assumption 4.1 reduces to \( 0 \in \varrho(A_4) \).

**Remark 4.3:** By the Hille-Yosida Theorem, Assumption 4.1b is equivalent to assuming that \( A_1 \) is closed and generates a \( C_0 \)-semigroup on \( Z \).

For \( \mu > \omega \) define the Schur complement

\[
S_1(\mu) : D(A_1) \subset Z \rightarrow Z
\]

by

\[
S_1(\mu) := \mu - A_1 + A_2 A_4^{-1} A_3.
\]

Because \( A_1 \) is closed and densely defined, the Schur complement \( S_1(\mu) \) is closed and densely defined. Moreover, we can factor \( S_1(\mu) \) as

\[
S_1(\mu) := (\mu - A_1)(I + (\mu - A_1)^{-1} A_2 A_4^{-1} A_3).
\]

A Neumann series yields that the operator \( S_1(\mu) \) is invertible for \( \mu > \omega_0 := \omega + M \| A_2 A_4^{-1} A_3 \| \) and also

\[
\| S_1(\mu)^{-n} \| \leq \| (\mu - A_1)^{-n} \| \| I + (\mu - A_1)^{-1} A_2 A_4^{-1} A_3 \|^{-n} \leq \frac{M}{(\mu - \omega)^n} \frac{1}{(1 - M \| A_2 A_4^{-1} A_3 \| / \mu - \omega)^n}.
\]

Using Proposition 4.4,

\[
((\mu E - A)^{-1} E)^n = 
\begin{pmatrix}
S_1(\mu)^{-n} & 0 \\
-A_4^{-1} A_3 S_1(\mu)^{-n} & 0
\end{pmatrix},
\]

\[
= \begin{pmatrix}
I & 0 \\
-A_4^{-1} A_3 & 0
\end{pmatrix}
\begin{pmatrix}
S_1(\mu)^{-n} & 0 \\
0 & 0
\end{pmatrix}
\]

\[
(E(\mu E - A)^{-1})^n = 
\begin{pmatrix}
S_1(\mu)^{-n} & -S_1(\mu)^{-n} A_2 A_4^{-1} \\
0 & 0
\end{pmatrix},
\]

\[
= \begin{pmatrix}
S_1(\mu)^{-n} & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
I & -A_2 A_4^{-1} \\
0 & 0
\end{pmatrix}.
\]

These calculations together with (10) show that \( A - \omega_0 E \) is E-radial. Further, \( \text{ran} \ E \) is closed. Thus Theorem 2.6 is applicable.

The projections \( P \) and \( Q \) will be explicitly calculated for this class of systems. Note that because of Assumption 4.1 \( A_1 \) generates a \( C_0 \)-semigroup, and thus

\[
\lim_{s \to \infty} s (S_1(s))^{-1} z = z.
\]

This implies

\[
P \left( \begin{array}{c} x \\ y \end{array} \right) = \lim_{s \to \infty} s (s E - A)^{-1} \left( \begin{array}{c} x \\ 0 \end{array} \right)
\]

\[
= \lim_{s \to \infty} \left( \begin{array}{c}
- \frac{s S_1(\mu)^{-1} x}{A_4^{-1} A_3}
\end{array} \right)
\]

\[
= \left( \begin{array}{c}
I \\ 0
\end{array} \right) \left( \begin{array}{c}
0 \\ 0
\end{array} \right)
\]

and similarly

\[
Q \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} I \\ 0 \end{array} \right) \left( \begin{array}{c} -A_2 A_4^{-1} \\ 0 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right).
\]

**V. EXAMPLE: DZEKTSER EQUATION**

We consider the Dzekts er equation

\[
\frac{\partial}{\partial t} \left( 1 + \frac{\partial^2}{\partial \zeta^2} \right) x(\zeta, t) = \left( \frac{\partial^2}{\partial \zeta^2} + 2 \frac{\partial^4}{\partial \zeta^4} \right) x(\zeta, t),
\]

\( t > 0 \) and \( \zeta \in (0, \pi) \), with boundary conditions

\[
x(0, t) = x(\pi, t) = 0, \quad t > 0
\]

\[
\frac{\partial^2 x}{\partial \zeta^2}(0, t) = \frac{\partial^2 x}{\partial \zeta^2}(\pi, t) = 0, \quad t > 0.
\]

Let \( \mathcal{Z} = L^2(0, \pi) \) and \( \mathcal{X} = H^2(0, \pi) \cap H_0^1(0, \pi) \) with \( \| x \|_{\mathcal{X}}^2 = \| x'' \|_{L^2}^2, \ E \in \mathcal{L}(\mathcal{X}, \mathcal{Z}) \) and \( A : D(A) \subset \mathcal{X} \rightarrow \mathcal{Z} \) given by

\[
E x = x + x''
\]

\[
A x = x'' + 2x^{(4)},
\]

\[
D(A) = \{ x \in H^4(0, \pi) \cap H_0^1(0, \pi) \mid \text{ran} x''(0) = \text{ran} x''(\pi) = 0 \}.
\]

This system was shown in [GGZ20] to have a splitting \( 4 \) using the eigenfunction expansion. Here the system will
be shown to be dissipative $E-$radial. For $x \in D(A)$ we calculate

$$
\text{Re} \langle Ax, Ex \rangle_Z = \text{Re} \int_0^\pi (x'' + 2x^{(4)})(\pi + \pi')d\zeta \\
= -\|x''\|^2_{L^2(0,\pi)} + \|x''\|^2_{L^2(0,\pi)} \\
- 2\|x^{(3)}\|^2_{L^2(0,\pi)} - 2 \text{Re} \int_0^\pi x^{(3)}(\pi')d\zeta \\
\leq \|x''\|^2_{L^2(0,\pi)} - \|x^{(3)}\|^2_{L^2(0,\pi)} \\
\leq 0,
$$

by the Poincaré inequality. It is easy to see that $\text{ran} (E - A) = Z$ and $\ker A \cap \ker E = \{0\}$. Next we calculate $A^* : D(A^*) \subset Z \to X$.

Note that $S : \mathcal{X} \to Z$ given by $Sf := f''$ is an isometric isomorphism with

$$
(S^{-1}f)(x) = \int_0^x (x-t)f(t)dt - \frac{x}{\pi} \int_0^\pi (\pi - t)f(t)dt.
$$

Then, for $x \in D(A)$ and $z \in \mathcal{X}$

$$
\langle Ax, z \rangle_Z = \int_0^\pi x''(\pi + \pi')d\zeta \\
= \int_0^\pi x''d\zeta + 2x^{(4)}d\zeta \\
= \int_0^\pi x''(S^{-1}z + 2z)''d\zeta \\
= \langle x, A^*z \rangle_x
$$

with $A^*z = S^{-1}z + 2z$ for $z \in \mathcal{X}$. For $x \in D(A^*) = \mathcal{X}$ and $y = S^{-1}x$ we calculate

$$
\text{Re} \langle A^*x, E^*x \rangle_X = \text{Re} \langle EA^*x, x \rangle_Z \\
= \text{Re} \int_0^\pi (S^{-1}x + x + 2x')\pi' d\zeta \\
= \text{Re} \int_0^\pi (y + y'' + 2y')y''d\zeta \\
= -\|y''\|^2_{Z} + \|y''\|^2_{Z} \\
- 2 \text{Re} \int_0^\pi y'(y^{(3)})d\zeta - 2\|y^{(3)}\|^2_{Z} \\
= \|y''\|^2_{Z} - \|y''\|^2_{Z} \\
\leq 0.
$$

The calculations are simpler than using the eigenfunction expansion. Furthermore, with this approach it can be concluded that not only is the system well-posed, but also that the dynamics are a contraction with respect to the $L^2(0,\pi)$-norm.

**Conclusion**

Conditions for the solvability of partial differential-algebraic equations on Hilbert spaces have been presented and a generalization of the Lumer-Phillips generation theorem to dissipative differential-algebraic equations. The results were illustrated with some applications.

It is assumed that $E$ is a linear bounded operator from $\mathcal{X}$ to $Z$. In order to apply the results to some other applications such as dissipative water waves one has to deal with unbounded operators $E$. This will be the subject of future work.

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