LIPSCHITZ-FREE SPACES OVER MANIFOLDS AND THE METRIC APPROXIMATION PROPERTY

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Abstract. Let $\|\cdot\|$ be a norm on $\mathbb{R}^N$ and let $M$ be a closed $C^1$-submanifold of $\mathbb{R}^N$. Consider the pointed metric space $(M, d)$, where $d$ is the metric given by $d(x, y) = \|x - y\|$, $x, y \in M$. Then the Lipschitz-free space $\mathcal{F}(M)$ has the Metric Approximation Property.

1. Introduction

For a metric space $(M, d)$ and a point $x_0 \in M$ let $\text{Lip}_0(M, x_0)$ be the Banach space of all real-valued Lipschitz functions $f$ on $M$ vanishing at $x_0$, equipped with the norm

$$\|f\| = \sup\left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in M, x \neq y \right\}.$$

For every $x \in M$ let $\delta_x \in \text{Lip}_0(M, x_0)^*$ be the bounded functional defined by $\delta_x(f) = f(x)$ for all $f$. The Lipschitz-free space $\mathcal{F}(M)$ over $M$ is defined to be the closed linear span of $\{\delta_x : x \in M\} \subseteq \text{Lip}_0(M, x_0)^*$. The map $\delta : M \to \mathcal{F}(M), \delta(x) = \delta_x$ is an injective isometry of $M$ into $\mathcal{F}(M)$. The free space $\mathcal{F}(M)$ is an isometric predual to $\text{Lip}_0(M, x_0)$ and it satisfies the fundamental property that any Lipschitz map $F$ from $M$ to another pointed metric space $N$, preserving the base point, uniquely extends, via the map $\delta$, to a bounded linear map between $\mathcal{F}(M)$ and $\mathcal{F}(N)$ whose norm equals the Lipschitz constant of $F$. This can be used as a tool for linearization of Lipschitz maps and therefore transferring nonlinear problems to a linear setting.

The book [20] by Weaver is devoted to Lipschitz spaces and investigates their Banach space, lattice and algebraic structure, among other topics.

One of the directions of research has been approximation properties of Lipschitz-free spaces. In their seminal paper [11], Godefroy and Kalton showed that if there exists a Lipschitz bijection between $X$ and $Y$ with a Lipschitz inverse, and $X$ has the bounded approximation property (BAP), then $Y$ has the BAP. To this end, they prove that $X$ has the $\lambda$-bounded approximation property ($\lambda$-BAP) if and only if $\mathcal{F}(X)$ has the $\lambda$-BAP, and that free spaces over finite-dimensional Banach spaces have the metric approximation property (MAP). Many results have since

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appeared that prove the ($\lambda$-bounded) approximation property of free spaces over various classes of metric spaces. We present a short survey of them.

Regarding compact metric spaces, it is known that if $M$ is a sufficiently ‘thin’ totally disconnected metric space then $\mathcal{F}(M)$ has the MAP. More precisely, if $M$ is a countable metric space such that its closed balls are compact, then $\mathcal{F}(M)$ has the MAP $[3]$. Also, $\mathcal{F}(M)$ has the MAP when $M$ is compact and uniformly disconnected $[20$, Corollary 4.39], meaning that there exists $0 < a \leq 1$ such that for every distinct $p, q \in M$ there exist complementary clopen sets $C, D \subseteq M$ such that $p \in C, q \in D$ and $d(C, D) \geq ad(p, q)$, where $d(C, D) = \inf\{d(x, y) : x \in C, y \in D\}$. For example, the middle-third Cantor set $C$ and the Cantor dust $C^2 \subseteq \mathbb{R}^2$ satisfy the previous hypotheses. On the other hand, there exists a compact convex subset $K$ of a Banach space such that $\mathcal{F}(K)$ fails the approximation property (AP) $[12]$. Moreover, in $[13]$ it is shown that there exist metric spaces homeomorphic to the Cantor set whose free spaces fail the AP. The recent result $[6]$ shows that free spaces over compact groups equipped with a left-invariant metric have the MAP. A characterisation of the BAP on free spaces in terms of uniformly bounded sequences of Lipschitz ‘almost extension’ operators is given in $[9]$.

Apart from the compact case, it is known that if $M$ is a separable ultrametric space then $\mathcal{F}(M)$ has a monotone Schauder basis (and hence the MAP), and is linearly isomorphic to $l^1$ $[3]$. Also, the free space over the Urysohn space has the MAP $[8]$. In $[15]$ it is proved that $\mathcal{F}(M)$ has the AP for a uniformly discrete metric space $M$, although the question whether it has the BAP is still open and of particular interest.

In his survey of Lipschitz free spaces $[10]$, Godefroy asked whether the free space over any closed subset of a finite-dimensional Banach space has the MAP. It is known that $\mathcal{F}(M)$ has the $CN$-BAP for any $M \subseteq \mathbb{R}^N$, where $C$ is a universal constant, and $\mathbb{R}^N$ is equipped with an arbitrary norm $[16]$. This follows from the more general result that free spaces over doubling metric spaces have the BAP $[16$, Corollary 2.2]. In the case of the Euclidean norm on $\mathbb{R}^N$, we have the improvement that $\mathcal{F}(M)$ has the $C\sqrt{N}$-BAP, for an arbitrary subset $M \subseteq (\mathbb{R}^N, \|\cdot\|_2)$ $[16$, Corollary 2.3]. In $[19]$ it is shown that free spaces over compact subsets of $\mathbb{R}^N$ onto which there are certain Lipschitz ‘almost retractions’ defined on slightly bigger subsets (see below for the precise formulation) have the MAP, where again $\mathbb{R}^N$ is equipped with an arbitrary norm. This includes the case of compact convex subsets of $\mathbb{R}^N$, which contrasts starkly with the infinite-dimensional case mentioned above. Also, it is known that $\mathcal{F}(M)$ has the MAP when $M$ is purely 1-unrectifiable, which is equivalent to the condition that $M$ contains no bi-Lipschitz image of a compact subset of $\mathbb{R}^N$ of positive measure. That $\mathcal{F}(M)$ has the MAP in this case follows because it is separable, has the BAP and is a dual space $[11$, Theorem B]. In $[16]$ it is shown that $\mathcal{F}(\mathbb{R}^N, \|\cdot\|_1)$ and $\mathcal{F}(l^1)$ have monotone finite-dimensional Schauder decompositions. In $[14]$ it is also proved that $\mathcal{F}(\mathbb{R}^N)$ and $\mathcal{F}(l^1)$ have Schauder bases, where $\mathbb{R}^N$ is equipped with an arbitrary norm.

Motivated by Godefroy’s question, in this paper we extend the class of subsets of $\mathbb{R}^N$ whose free spaces have the MAP by proving that the free space over a closed
C$^1$-submanifold of $\mathbb{R}^N$ has the MAP, with respect to any norm. More precisely, we prove the following result.

**Theorem 1.1.** Let $\|\cdot\|$ be a norm on $\mathbb{R}^N$ and $M$ be a closed $C^1$-submanifold of $\mathbb{R}^N$ with an arbitrary distinguished point $x_0$. Equip $M$ with the metric $d(x, y) = \|x - y\|$, $x, y \in M$. Then $F(M)$ has the MAP.

We repeat the main result of [19] as it is necessary to show that it cannot be applied to all $C^1$-submanifolds (even compact ones).

**Theorem 1.2 ([19 Theorem 1.1]).** Let $N \geq 1$ and consider $\mathbb{R}^N$ equipped with some norm $\|\cdot\|$. Let a compact set $\hat{M} \subseteq \mathbb{R}^N$ have the property that given $\xi > 0$, there exists a set $M \subseteq \mathbb{R}^N$ and a Lipschitz map $\Psi : \hat{M} \to M$, such that $M \subseteq \text{int}(\hat{M})$, $\text{Lip}(\Psi) \leq 1 + \xi$ and $\|x - \Psi(x)\| \leq \xi$ for all $x \in \hat{M}$. Then the Lipschitz-free space $\mathcal{F}(M)$ has the MAP.

The rest of the paper is organised as follows. In Section 2 we present a number of ancillary results that support the proof of Theorem 1.1 and in Section 3 we make some remarks on open problems.

We conclude the introduction by showing that there exists a $C^\infty$-norm on $\mathbb{R}^3$ whose unit sphere (which is a compact $C^\infty$-submanifold of $\mathbb{R}^3$) does not satisfy the assumptions of Theorem 1.2. More precisely, we will show that there exist two norms $\|\cdot\|, \|\cdot\|$ in $\mathbb{R}^3$, where $\|\cdot\|$ is $C^\infty$, so that for the subset $M = S_1$ of $(\mathbb{R}^3, \|\cdot\|)$, the following condition does not hold:

\[ (*) \text{ for every } \xi > 0 \text{ there exists } M \subset \mathbb{R}^3, M \subseteq \text{int}(M) \text{ and a Lipschitz map } \Psi : \hat{M} \to M \text{ such that Lip}(\Psi) \leq 1 + \xi \text{ and } \|\Psi(x) - x\| \leq \xi \text{ for all } x \in \hat{M}. \]

It is known that if a Banach space $X$ is of dimension at least 3 then $X$ is isometrically isomorphic to a Hilbert space if and only if every subspace of $X$ is 1-complemented [18, 9.3]. We can therefore fix a norm $\|\cdot\|$ on $\mathbb{R}^3$ such that the subspace $\mathbb{R}^2 \times \{0\}$ is not 1-complemented. To define $\|\cdot\|$, let $\phi : [0, \infty) \to [0, \infty)$ be a convex $C^\infty$-function such that $\phi(1) = 1$ and $\phi(t) = 0$ for $t \in [0, \frac{1}{2}]$. Define the convex even $C^\infty$-function $\Phi : \mathbb{R}^3 \to [0, \infty)$ by

\[ \Phi(x) = \sum_{i=1}^{3} \phi(\frac{1}{2}|x_i|), \]

the convex set $B = \{ x \in \mathbb{R}^N : \Phi(x) \leq 1 \}$ and finally let $\|\cdot\|$ be the Minkowski functional of $B$. An application of the Implicit Function Theorem demonstrates that $\|\cdot\|$ is a $C^\infty$-norm; for more details see e.g. [5] Section V.1].

For a contradiction, assume that $M = S_{\|\cdot\|}$ possesses the property labelled $\ast$ above. It is easy to check that $\Phi(x) = 1$ whenever $|x_1|, |x_2| \leq \frac{1}{2}$ and $x_3 = 2$, and thus all such points $x$ belong to $M$. Define $C = [0, 1]^2 \times \{0\}$ and $D = [-\frac{1}{2}, \frac{3}{2}] \times \{0\}$. By translating $M$ we can see that for all sufficiently small $\xi > 0$ there exists an open set $U \subseteq \mathbb{R}^3$ that includes $C$ and a map $\Psi : U \to D$ such that $\text{Lip}(\Psi) \leq 1 + \xi$ and $\|\Psi(x) - x\| \leq \xi$ for every $x \in U$. By taking a convolution of $\Psi$ and a $C^\infty$-mollifier
supported on a small enough neighbourhood of 0, we can assume that \( \Psi \) is \( C^\infty \) and still satisfies the previous two inequalities. Choose some sufficiently small \( \xi > 0 \) and let \( U \) and \( \Psi \) be defined as previously. The total derivative \( D\Psi(x, y, z) \) of \( \Psi \) at \((x, y, z)\) maps \((\mathbb{R}^3, \|\cdot\|)\) linearly into \( \mathbb{R}^2 \times \{0\} \) and satisfies \( \|D\Psi(x, y, z)\|_\text{op} \leq 1 + \xi \). Define the linear operator \( T : \mathbb{R}^3 \to \mathbb{R}^2 \times \{0\} \) by
\[
T(a, b, c) = \int_C D\Psi(x, y, 0)(a, b, c) \, dx dy.
\]
We estimate
\[
\|T(1, 0, 0) - (1, 0, 0)\|
= \left\| \int_0^1 \int_0^1 D\Psi(x, y, 0)(1, 0, 0) \, dx dy - (1, 0, 0) \right\|
\leq \int_0^1 \left[ \left\| \Psi(x, y, 0) - \Psi(0, y, 0) \right\| \, dy \right] + \int_0^1 \left[ \left\| \Psi(0, y, 0) - (0, y, 0) \right\| \, dy \right] \leq 2\xi.
\]
Similarly, we can show that \( \|T(0, 1, 0) - (0, 1, 0)\| \leq 2\xi \). Now for all sufficiently large \( n \in \mathbb{N} \), we choose \( \Psi_n \) corresponding to \( \xi_n = \frac{1}{n} \) and let \( T_n \) be defined with respect to \( \Psi_n \) as above. Since \( \|T_n\|_\text{op} \leq 1 + \frac{1}{n} \), \( (T_n) \) has a subsequence converging to some linear map \( T \). Obviously \( \|T\|_\text{op} \leq 1 \) and the range of \( T \) is \( \mathbb{R}^2 \times \{0\} \). Also, by the above computations, the restriction of \( T \) to \( \mathbb{R}^2 \times \{0\} \) is the identity. Therefore \( T \) is a norm-one projection onto \( \mathbb{R}^2 \times \{0\} \), a contradiction.

2. Ancillary results

Let \( \mathbb{R}^N \) be equipped with the Euclidean norm \( \|\cdot\|_2 \) and let \( \|\cdot\| \) be some other norm. Fix \( K \geq 1 \) such that
\[
K^{-1} \|\cdot\|_2 \leq \|\cdot\| \leq K \|\cdot\|_2.
\]
Denote the closed Euclidean ball of radius \( R > 0 \) centered at \( y \in \mathbb{R}^N \) by \( B_R(y) \) and write \( B_R = B_R(0) \). For a subset \( U \subseteq \mathbb{R}^N \), write \( \overline{U} \) for the closure of \( U \). For a Lipschitz function \( f : U \to \mathbb{R} \), where \( U \subseteq \mathbb{R}^N \) is some set, write \( \text{Lip}_f(f) \) and \( \text{Lip}_{\|\cdot\|_2}(f) \) for its Lipschitz constants measured with respect to \( \|\cdot\| \) and \( \|\cdot\|_2 \), respectively. By \( \text{Lip}(f) \) we will mean \( \text{Lip}_f(f) \). Define also \( \|f\|_\infty = \sup \{ |f(x)| : x \in U \} \). Given a differentiable map \( \phi : U \to \mathbb{R}^N' \), where \( U \subseteq \mathbb{R}^N \) is open, we denote by \( D\phi(x) \) the total derivative of \( \phi \) at \( x \), which is a linear operator from \( \mathbb{R}^N \) to \( \mathbb{R}^N' \). The following two definitions are the same as in [17, Definition 2].
Lemma 2.3. By compactness of the orthogonal projection onto $T$, if $x, y \in M$, then there exists an $\epsilon > 0$ such that $\|y - x - P_x(y - x)\|_2 \leq \epsilon \|y - x\|_2$ whenever $x, y \in M \cap B_R$ satisfy $\|x - y\|_2 \leq \delta$.

Proof. By compactness of $M \cap B_R$, it suffices to prove the lemma locally, that is, that for every $q \in M$ there exists a neighbourhood $U$ of $q$ in $M$ such that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$
\|y - x - P_x(y - x)\|_2 \leq \epsilon \|y - x\|_2
$$

for $x, y \in U, \|x - y\|_2 \leq \delta$.

Let $q \in M$ and let $\phi$ be a $C^1$-submanifold chart around $q$, i.e. $\phi: V \to U$ is a $C^1$ diffeomorphism between the open sets $V, U \subseteq \mathbb{R}^N$ such that $0 \in V, q \in U, \phi(0) = q$, $\phi(\mathbb{R}^d \cap V) = M \cap U$, and $\mathbb{R}^d = \mathbb{R}^d \times \{0\} \subseteq \mathbb{R}^N$. It is not hard to see that the $D\phi(v)$ maps $\mathbb{R}^d$ to $T_{\phi(v)}$ for every $v \in V$.

Let $\epsilon > 0$. Using the fact that $\phi$ is locally bi-Lipschitz and $D\phi$ is continuous, we shrink $V$ and $U$, if necessary, so that $V$ is convex and there exists $a \geq 1$ such that

$$
\frac{1}{a} \|v - w\|_2 \leq \|\phi(v) - \phi(w)\|_2 \leq a \|v - w\|_2 \quad \text{and} \quad \|D\phi(v) - D\phi(w)\|_2 \leq \frac{\epsilon}{a}
$$

for all $v, w \in V$.

Choose $x, y \in M \cap U$ and let $v = \phi^{-1}(x)$ and $w = \phi^{-1}(y)$. Define $f: [0, 1] \to M$ by $f(t) = \phi(v + t(w - v))$. Then

$$
y - x = f(1) - f(0) = \int_0^1 f'(t) \, dt = \int_0^1 D\phi(v + t(w - v))(w - v) \, dt,
$$
meaning
\[ \|y - x - D\phi(v)(w - v)\|_2 \leq \int_0^1 \|(D\phi(v + t(w - v)) - D\phi(v))(w - v)\|_2 \, dt \]
\[ \leq \frac{C}{a} \|w - v\|_2 \leq \epsilon \|y - x\|_2. \]
Since \( D\phi(v)(w - v) \in T_{\phi(v)} = T_x \) and \( P_x(y - x) \) is the closest point to \( y - x \) in \( T_x \), we have
\[ \|y - x - P_x(y - x)\|_2 \leq \|y - x - D\phi(v)(w - v)\|_2 \leq \epsilon \|y - x\|_2. \]
\[ \square \]

According to \([21] \text{ Lemma 23}\), there exists an open neighbourhood \( E(M) \) of \( M \) and a \( C^1 \) map \( \psi : E(M) \to M \) such that \( \psi(x) = x \) for \( x \in M \) (in \([21] \text{ Lemma 23}\) \( E(M) \) is called \( R(M) \) and \( \psi \) is called \( H \)). Given \( \epsilon > 0 \), let \( M^\epsilon = \{ y \in \mathbb{R}^N : d(y, M) < \epsilon \} \), where \( d(y, M) = \inf_{x \in M} \|y - x\|_2 \). We will fix the map \( \psi \) for the rest of the paper. Next is a straightforward but important observation about \( \psi \) for use later on.

**Lemma 2.4.** For \( x \in M \) we have \( (D\psi(x) - I)|_{T_x} = 0 \).

**Proof.** Let \( x \in M \) and \( v \in T_x \). We need to show that \( D\psi(x)(v) = v \). Let \( \phi : V \to U \) be a \( C^1 \)-submanifold chart around \( x \), where \( V, U \subseteq \mathbb{R}^N \), \( 0 \in V, x \in U \), \( \phi(0) = x \). As \( D\phi(0) \) maps \( \mathbb{R}^d \) onto \( T_x \), there exists \( u \in \mathbb{R}^d \) such that \( D\phi(0)(u) = v \). Since \( (\psi \circ \phi)|_{\mathbb{R}^d} = \phi|_{\mathbb{R}^d} \), we indeed have
\[ D\psi(x)(v) = D(\psi \circ \phi)(0)(u) = D((\psi \circ \phi)|_{\mathbb{R}^d})(0)(u) = D(\phi|_{\mathbb{R}^d})(0)(0) = v. \] \[ \square \]

The following lemma will be of key importance when we mollify Lipschitz functionals defined on bounded subsets of \( M \).

**Lemma 2.5.** For every \( R, \epsilon > 0 \) there exists \( \delta > 0 \) such that
\[ \|\psi(x + z) - \psi(y + z) - (x - y)\|_2 \leq \epsilon \|x - y\|_2 \]
for all \( x, y \in M \cap B_R \) and \( z \in \mathbb{R}^N \) such that \( \|x - y\|_2, \|z\|_2 \leq \delta \).

**Proof.** Let \( R, \epsilon > 0 \). By uniform continuity of \( D\psi \) on compact sets, we choose \( \delta_1 > 0 \) such that \( \overline{M}^{2\delta_1} \cap B_{R + 2\delta_1} \subseteq E(M) \) and
\[ \|D\psi(q) - D\psi(u)\|_2 \leq \frac{\epsilon}{2} \] whenever \( q, u \in \overline{M}^{2\delta_1} \cap B_R, \|q - u\|_2 \leq 2\delta_1 \). \[ (2.1) \]
Set
\[ B = \max\{\|D\psi(q) - I\|_2 : q \in \overline{M}^{2\delta_1} \cap B_{R + 2\delta_1}\}. \]
We apply Lemma 2.3 to \( R \) and \( \frac{\epsilon}{2B} \) to find a corresponding \( \delta_2 > 0 \). Set \( \delta = \min(\delta_1, \delta_2) \) and let \( x, y \in M \cap B_R, \|x - y\|_2, \|z\|_2 \leq \delta \).

For \( t \in [0, 1] \) we have \( \|x + z + t(y - x) - x\|_2 \leq 2\delta_1 \) and so \( x + z + t(y - x) \in \overline{M}^{2\delta_1} \cap B_{R + 2\delta_1} \). Therefore we can define \( f(t) = \psi(x + z + t(y - x)) - (x + t(y - x)) \), \( t \in [0, 1] \). We have
\[ \|\psi(x + z) - \psi(y + z) - (x - y)\|_2 = \|f(1) - f(0)\|_2 = \left\| \int_0^1 f'(t) \, dt \right\|_2 \]
\[
\leq \int_0^1 \|f'(t)\|_2 \, dt.
\]

Furthermore,
\[
\|f'(t)\|_2 \\
= \|D\psi(x + z + t(y - x))(y - x) - (y - x)\|_2 \\
\leq \|(D\psi(x + z + t(y - x)) - D\psi(x))(y - x)\|_2 + \|(D\psi(x) - I)(y - x)\|_2 \\
\leq \frac{\epsilon}{2} \|y - x\|_2 + \|(D\psi(x) - I)(y - x - P_x(y - x))\|_2 \\
\leq \frac{\epsilon}{2} \|y - x\|_2 + B \frac{\epsilon}{2B} \|y - x\|_2 = \epsilon \|y - x\|_2.
\]

Proof. We next define a certain convolution of a given Lipschitz function \(f\) on \(M\) with a standard mollifier. First, define \(\nu: \mathbb{R}^N \to [0, +\infty)\) by
\[
\nu(x) = \begin{cases} 
A \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } \|x\|_2 < 1, \\
0 & \text{if } \|x\|_2 \geq 1,
\end{cases}
\]
where \(A > 0\) is chosen so that \(\int_{\mathbb{R}^N} \nu(x) \, dx = 1\). Next, for each \(s > 0\) we put
\[
\nu_s(x) = \frac{1}{s^N} \nu\left(\frac{x}{s}\right).
\]
The function \(\nu_s\) is \(C^\infty\) and satisfies \(\int_{\mathbb{R}^N} \nu_s(x) \, dx = 1\) and \(\text{supp}(\nu_s) \subseteq B_s(x)\).

Let \(R \geq 1\) be fixed. Choose \(\delta_0 > 0\) such that \(\overline{M^{2\delta_0}} \cap B_{R+2\delta_0} \subseteq \mathcal{E}(M)\). Fix \(s \in (0, \delta_0]\) and let \(f \in \text{Lip}_0(M)\). For \(x \in \overline{M^{2\delta_0}} \cap B_{R+\delta_0}\) we define
\[
f_s(x) = \int_{B_s} \nu_s(z) f(\psi(x + z)) \, dz = \int_{x + B_s} \nu_s(z - x) f(\psi(z)) \, dz.
\]

In the next two results, we fix \(f \in \text{B}_{\text{Lip}_0(M)}\) and estimate the Lipschitz constant of \(f_s\) by first considering points that are close together and then far from each other. Since \(\psi\) is \(C^1\) on \(\mathcal{E}(M)\), it is Lipschitz on the compact set \(\overline{M^{2\delta_0}} \cap B_{R+2\delta_0}\). Observe that the next result holds even if, when restricted to this set, the Lipschitz constant of \(\psi\) (relative to \(\|\cdot\|\)) is large.

**Proposition 2.6.** Let \(\epsilon > 0\). There exists \(\delta \in (0, \delta_0]\) such that
\[
|f_s(x) - f_s(y)| \leq (1 + \epsilon) \|x - y\|
\]
whenever \(x, y \in M \cap B_{R+\delta_0}, \|x - y\| \leq \delta\) and \(s \in (0, \delta_0]\).

Proof. Let \(\epsilon > 0\). According to Lemma 2.5 we can choose \(\delta \in (0, \delta_0]\) such that
\[
\|\psi(x + z) - \psi(y + z) - (x - y)\|_2 \leq \frac{\epsilon}{K^2} \|x - y\|_2
\]
for all \(x, y \in M \cap B_{R+\delta_0}, \|x - y\|_2 \leq K\delta\) and \(z \in \mathbb{R}^N, \|z\|_2 \leq \delta\). Let \(s \in (0, \delta]\) and pick \(x, y \in M \cap B_{R+\delta_0}, \|x - y\| \leq \delta\). We have \(\|x - y\|_2 \leq K\delta\) and if \(\|z\|_2 \leq s\) then
\[
\|\psi(x + z) - \psi(y + z) - (x - y)\|_2 \leq K \|\psi(x + z) - \psi(y + z) - (x - y)\|_2
\]
\[ \leq \frac{\epsilon}{K} \|x - y\| \leq \epsilon \|x - y\|. \]

Hence \( \|\psi(x + z) - \psi(y + z)\| \leq (1 + \epsilon) \|x - y\|. \)

We conclude that
\[ |\hat{f}_s(x) - \hat{f}_s(y)| \leq \int_{B_s} \nu_s(z) |f(\psi(x + z)) - f(\psi(y + z))| \, dz \]
\[ \leq \int_{B_s} \nu_s(z) \|\psi(x + z) - \psi(y + z)\| \, dz \leq (1 + \epsilon) \|x - y\|. \]

Next, we make an estimate in the case where points are far apart. Let \( L \geq 1 \) be the Lipschitz constant of the restriction of \( \psi \) to \( \overline{M^{2\delta>} \cap B_{R+2\delta}} \), with respect to \( \|\cdot\|_2 \).

**Proposition 2.7.** Let \( \epsilon, \delta > 0 \). Then
\[ |\hat{f}_s(x) - \hat{f}_s(y)| \leq (1 + \epsilon) \|x - y\| \]
whenever \( x, y \in M \cap B_{R+\delta}, \|x - y\| \geq \delta \) and \( s \in (0, \min \left( \frac{\epsilon \delta}{LK}, \delta_0 \right)) \).

**Proof.** Let \( x, y \in M \cap B_{R+\delta} \) be such that \( \|x - y\| \geq \delta \), and let \( s \in (0, \min \left( \frac{\epsilon \delta}{LK}, \delta_0 \right)) \).

We have
\[ |\hat{f}_s(x) - f(x)| \leq \int_{B_s} \nu_s(z) |f(\psi(x + z)) - f(x)| \, dz \]
\[ \leq \int_{B_s} \nu_s(z) \|\psi(x + z) - x\| \, dz \leq LKs, \]
and similarly for \( y \). Therefore
\[ |\hat{f}_s(x) - \hat{f}_s(y)| \leq |\hat{f}_s(x) - f(x)| + |f(x) - f(y)| + |f(y) - \hat{f}_s(y)| \]
\[ \leq 2LKs + \|x - y\| \leq \epsilon \delta + \|x - y\| \leq (1 + \epsilon) \|x - y\|. \]

We require some estimates concerning \( D\hat{f}_s \). In the next two lemmas, \( \|D\nu_s(z)\|_2 \) and \( \|D\hat{f}_s(x)\|_2 \) denote the norms of the operators \( D\nu_s(z) \) and \( D\hat{f}_s(x) \) with respect to \( \|\cdot\|_2 \), respectively.

**Lemma 2.8.** There exists \( G > 0 \) (depending only on \( N \)) such that
\[ \int_{B_s} \|D\nu_s(z)\|_2 \, dz \leq \frac{G}{s} \]
for all \( s > 0 \).

**Proof.** Set \( G = (e \int_0^1 \exp \left( \frac{1}{\rho^2 - 1} \right) r^{N-1} dr)^{-1} \) and let \( \Gamma \) denote the area of the Euclidean unit sphere \( S^{N-1} \subseteq \mathbb{R}^N \). Given \( z \in \mathbb{R}^N, \|z\|_2 < 1 \), we have
\[ D\nu(z) = -A \exp \left( \frac{1}{\|z\|_2^2 - 1} \right) \frac{2 \langle z, \cdot \rangle}{\|z\|_2^2 - 1^2}, \]
where \(\langle \cdot | \cdot \rangle\) denotes the inner product in \(\mathbb{R}^N\). Let \(s > 0\). As \(\nu_s(z) = \frac{1}{s^N} \nu\left(\frac{z}{s}\right)\), we obtain \(D\nu_s(z) = \frac{1}{s^{N+1}} D\nu\left(\frac{z}{s}\right)\) and thus

\[
\|D\nu_s(z)\|_2 = \frac{A}{s^{N+1}} \exp\left(\frac{1}{\|z\|_2^2 - 1}\right) \frac{2 \|z\|_2}{\left(\|z\|_2^2 - 1\right)^2}.
\]

This function of \(z\) is radially symmetric, hence

\[
\int_{B_s} \|D\nu_s(z)\|_2 \, dz = \frac{\Gamma}{s^{N+1}} \int_0^s \exp\left(\frac{1}{(\frac{r}{s})^2 - 1}\right) \frac{2 \left(\frac{r}{s}\right)}{\left((\frac{r}{s})^2 - 1\right)^2} r^{N-1} \, dr
\]

\[
\leq \frac{\Gamma}{s} \int_0^1 \exp\left(\frac{1}{u^2 - 1}\right) \frac{2u}{(u^2 - 1)^2} \, du = \frac{\Gamma}{es}.
\]

The result now follows from the fact that

\[
\frac{1}{A} = \int_{B_1} \exp\left(\frac{1}{\|z\|_2^2 - 1}\right) \, dz = \frac{\Gamma}{e^G}.
\]

**Lemma 2.9.** For \(f \in \text{Lip}_0(M)\), the function \(\hat{f}_s\) is \(C^\infty\) on \(M^{\delta_0} \cap \text{int}(B_{R+\delta_0})\) and satisfies

\[
\|D\hat{f}_s(x)\|_2 \leq KL \text{Lip}(f),
\]

and

\[
\|D\hat{f}_s(x) - D\hat{f}_s(y)\|_2 \leq \frac{GKL}{s} \text{Lip}(f) \|x - y\|_2,
\]

for every \(x, y \in M^{\delta_0} \cap \text{int}(B_{R+\delta_0})\).

**Proof.** By [7] Appendix C.4, Theorem 6(i) we have that \(\hat{f}_s\) is \(C^\infty\) on \(M^{\delta_0} \cap \text{int}(B_{R+\delta_0})\) and

\[
D\hat{f}_s(x) = \int_{B_s} D\nu_s(z) f(\psi(x + z)) \, dz.
\]

If \(x, y \in M^{\delta_0} \cap \text{int}(B_{R+\delta_0})\) then

\[
|\hat{f}_s(x) - \hat{f}_s(y)| \leq \int_{B_s} \nu_s(z) |f(\psi(x + z)) - f(\psi(y + z))| \, dz \leq KL \text{Lip}(f) \|x - y\|_2.
\]

Therefore

\[
\|D\hat{f}_s(x)\|_2 \leq \text{Lip}_{\|\cdot\|_2}(\hat{f}_s) \leq KL \text{Lip}(f).
\]

Also,

\[
\|D\hat{f}_s(x) - D\hat{f}_s(y)\|_2 \leq \int_{B_s} \|D\nu_s(z)\|_2 \|f(\psi(x + z)) - f(\psi(y + z))\| \, dz
\]

\[
\leq KL \text{Lip}(f) \int_{B_s} \|D\nu_s(z)\|_2 \, dz \|x - y\|_2 \leq \frac{GKL}{s} \text{Lip}(f) \|x - y\|_2.
\]
In order to obtain suitable finite-rank operators that witness the MAP, we make use of an interpolation process first used in [16] and again in [19].

Fix $w \in \mathbb{R}^d$. We define a closed hypercube $C \subseteq \mathbb{R}^d$ having edge length $\delta > 0$ and vertices $v_\gamma \in \mathbb{R}^d, \gamma \in \{0,1\}^d$, given by $v_\gamma = w + \delta \gamma$. Let $f$ be a real-valued function whose domain of definition includes the set of vertices of $C$. We define the ‘interpolation function’ $\Lambda(f,C)$ on $\mathbb{R}^d$ by

$$\Lambda(f,C)(x) = \sum_{\gamma \in \{0,1\}^d} \left( \prod_{i=1}^d \left( 1 - \gamma_i + (-1)^{\gamma_i+1} \frac{x_i - u_i}{\delta} \right) \right) f(v_\gamma).$$

Note that $\Lambda(f,C)$ is the unique function that agrees with $f$ on the vertices of $C$ and is coordinatewise affine, i.e. $t \mapsto \Lambda(f,C)(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_d)$ is affine whenever $1 \leq i \leq d$.

The following lemma is very similar to [19, Lemma 3.3], with the important difference that the estimate

$$\text{Lip}_{1,2}((\Lambda(f,C) - f)|_C) \leq (1 + \sqrt{d})\epsilon.$$ 

is in place of [19, Lemma 3.3 (14)]. Despite the similarity between the two results, we provide a proof for completeness and ask the reader to excuse any redundancy.

**Lemma 2.10.** Let $f$ be a $C^1$ function defined on a convex neighbourhood $V$ of a given hypercube $C \subseteq (\mathbb{R}^d, \|\cdot\|_2)$ as above. Let $\epsilon > 0$ and suppose that there is $\delta > 0$ such that $\|Df(u) - Df(u')\|_2 \leq \epsilon$ if $\|u - u'\|_2 \leq \sqrt{d}\delta$. If $C$ has sidelength $l \leq \delta$ then

$$\text{Lip}_{1,2}((\Lambda(f,C) - f)|_C) \leq (1 + \sqrt{d})\epsilon,$$

and

$$\|\Lambda(f,C) - f\|_\infty \leq \sqrt{d}\delta \text{Lip}_{1,2}(f).$$

**Proof.** First we show that for $u \in V, h \in \mathbb{R}^d$ such that $u + h \in V$ and $\|h\|_2 \leq \sqrt{d}\delta$ we have

$$|f(u + h) - f(u) - Df(u)(h)| \leq \epsilon \|h\|_2. \quad (2.2)$$

For such $u$ and $h$, using the Mean Value Theorem we can find a vector $u'$ on the line segment between $u$ and $u + h$ such that $f(u + h) - f(u) = Df(u')(h)$. Then $\|u - u'\|_2 \leq \sqrt{d}\delta$ and

$$|f(u + h) - f(u) - Df(u)(h)| = |Df(u')(h) - Df(u)(h)|$$

$$\leq \|Df(u') - Df(u)\|_2 \|h\|_2 \leq \epsilon \|h\|_2.$$

Now let $z \in \text{int}(C)$. The first equality on [19] p.41 is

$$D\Lambda(f,C)(z) = \sum_{\gamma \in \{0,1\}^d} c_\gamma \left( \frac{f(v_\gamma + (-1)^{\gamma_j} l e_j) - f(v_\gamma)}{(-1)^{\gamma_j} l} \right)_{j=1}^d, \quad (2.3)$$
Lemma 2.11. Let $U$ be a subset of $\mathbb{R}^N$ and $f: U \to \mathbb{R}$ be a Lipschitz function. Suppose that $\epsilon > 0$, $(U_i)_{i=1}^m$ is an open cover of $U$, $(\alpha_i)_{i=1}^m$ a partition of unity subordinate to $(U_i)_{i=1}^m$ consisting of $H$-Lipschitz functions, and $f_i: U_i \to \mathbb{R}, 1 \leq i \leq m$ satisfy $\|f_i - f|_{U_i}\|_\infty, \text{Lip}_{1,2}(f_i - f|_{U_i}) < \epsilon$. If $g = \sum_{i=1}^m \alpha_i f_i$ then $\|g - f\|_\infty \leq \epsilon$ and $\text{Lip}_{1,1}(g - f) \leq (1 + mH)\epsilon$. 

Lemma 2.11. This lemma is again similar to [19, Lemma 2.7].
Proof. For any \( x \in U \) we have
\[
|g(x) - f(x)| = \left| \sum_{i=1}^{m} \alpha_i(x)(f_i(x) - f(x)) \right| \leq \varepsilon \sum_{i=1}^{m} \alpha_i(x) = \varepsilon,
\]
and therefore \( \|g - f\|_\infty \leq \varepsilon \). Now pick different \( x, y \in U \). Let \( A = \{ i : x \in U_i \} \) and \( B = \{ i : y \in U_i \} \). We have
\[
|g(x) - f(x) - (g(y) - f(y))| = \left| \sum_{i \in A} \alpha_i(x)(f_i(x) - f(x)) - \sum_{i \in B} \alpha_i(y)(f_i(y) - f(y)) \right|
\leq \sum_{i \in A \cap B} |\alpha_i(x)(f_i(x) - f(x)) - \alpha_i(y)(f_i(y) - f(y))|
+ \sum_{i \in A \setminus B} |\alpha_i(x)|f_i(x) - f(x)| + \sum_{i \in B \setminus A} |\alpha_i(y)|f_i(y) - f(y)|
\leq \sum_{i \in A \cap B} \alpha_i(x)|f_i(x) - f(x) - (f_i(y) - f(y))|
+ \sum_{i \in A \setminus B} |(\alpha_i(x) - \alpha_i(y))(f_i(y) - f(y))|
+ \varepsilon \left( \sum_{i \in A \setminus B} \alpha_i(x) + \sum_{i \in B \setminus A} \alpha_i(y) \right)
\leq \varepsilon \|x - y\| + \text{card}(A \cap B)\varepsilon H \|x - y\|
+ \varepsilon \left( \sum_{i \in A \setminus B} |\alpha_i(x) - \alpha_i(y)| + \sum_{i \in B \setminus A} |\alpha_i(y) - \alpha_i(x)| \right)
\leq (1 + mH)\varepsilon \|x - y\|,
\]
where \( \text{card}(A \cap B) \) is the cardinality of \( A \cap B \). Therefore \( \text{Lip}_{\varepsilon,1}(g - f) \leq (1 + mH)\varepsilon \).

Finally, we define a ‘flattening’ operator \( \Phi \), following an idea taken from \[1,1\] Proposition 5.1. Let \( R > 0 \) be fixed.
Let \( \mu : [0, \infty) \to [0, 1] \) be defined by
\[
\mu(t) = \begin{cases} 
1, & \text{if } t \leq R \\
(\log R)^{-1}(2 \log R - \log t), & \text{if } R \leq t \leq R^2 \\
0, & \text{otherwise.}
\end{cases}
\tag{2.4}
\]
Denote by \( \text{Lip}_0(M \cap B_{R^2}) \) the space of all Lipschitz functions on \( M \cap B_{R^2} \) vanishing at 0, equipped with the norm \( \text{Lip}(\cdot) \). Given \( f \in \text{Lip}_0(M \cap B_{R^2}) \) and \( x \in M \), we define
\[
\Phi(f)(x) = \begin{cases} 
\mu(\|x\|_2)f(x), & \text{if } x \in B_{R^2} \\
0, & \text{otherwise.}
\end{cases}
\]
It is clear that $\Phi(f)$ agrees with $f$ on $M \cap B_R$ and vanishes outside $B_{R^2}$. In particular, $\Phi(f)(0) = 0$.

**Proposition 2.12.** The map $\Phi: \text{Lip}_0(M \cap B_{R^2}) \to \text{Lip}_0(M)$ is a bounded operator satisfying

$$\|\Phi\| \leq 1 + \frac{K^2}{\log R}.$$ 

**Proof.** Let $f \in B_{\text{Lip}_0(M \cap B_{R^2})}$. Extend $f$ to $\mathbb{R}^n$ while preserving its Lipschitz constant, by McShane’s Extension Theorem [20, Theorem 1.33] and define $g(x) = \mu(\|x\|_2) f(x)$, $x \in \mathbb{R}^n$. It suffices to prove that $\text{Lip}(g) \leq 1 + \frac{K^2}{\log R}$. By Rademacher’s Theorem, it is enough to prove $\|Dg(x)\| \leq 1 + \frac{K^2}{\log R}$ whenever $x$ is a point of differentiability of both $\mu \circ \|\cdot\|_2$ and $f$. Given such $x$, if $\|x\|_2 < R$ or $\|x\|_2 > R^2$ then $Dg(x) = Df(x)$ or $Dg(x) = 0$, respectively. If $R \leq \|x\|_2 \leq R^2$ then

$$\|Dg(x)\| = \left\| \mu'((\|x\|_2)/2) \frac{x}{\|x\|_2} f(x) + \mu((\|x\|_2) Df(x) \right\|$$

$$\leq \left\| \mu'((\|x\|_2)/2) \frac{x}{\|x\|_2} f(x) \right\| + \|\mu((\|x\|_2) Df(x)\|$$

$$\leq 1 + K \left\| \mu((\|x\|_2)/2) \frac{x}{\|x\|_2} f(x) \right\| = 1 + \frac{K \|f(x)\|}{\|x\|_2 \log R} \leq 1 + \frac{K^2}{\log R}. \quad \square$$

3. Proof of Theorem 1.1

We will build a sequence of operators in the proof of Theorem 1.1. Before giving the proof, we present three results that supply the components needed to assemble these operators.

First we construct ‘mollifier’ operators $S_n$ defined on $\text{Lip}_0(M)$. Fix an $n \in \mathbb{N}$ and a $\delta_n > 0$ such that $\mathcal{M}^{2\delta_n} \cap B_{n^2 + 2\delta_n} \subseteq \mathcal{E}(M)$. Let $L_n \geq 1$ be the Lipschitz constant of the restriction of $\psi$ to $\mathcal{M}^{2\delta_n} \cap B_{n^2 + 2\delta_n}$, with respect to $\|\cdot\|_2$. Recall the functions $\hat{f}_s$ defined before Proposition 2.6.

**Theorem 3.1.** There exists $s_n \in (0, \delta_n]$ such that the linear map $S_n: \text{Lip}_0(M) \to \text{Lip}_0(M \cap B_{n^2 + \delta_n})$,

$$S_n(f)(x) = \hat{f}_{s_n}(x) - \hat{f}_{s_n}(0),$$

satisfies $\|S_n\| \leq 1 + \frac{1}{n}$ and $\left\| S_n(f) - f|_{M \cap B_{n^2 + \delta_n}} \right\|_\infty \leq \frac{1}{n}$ for all $f \in B_{\text{Lip}_0(M)}$. Moreover, if $(f^{(k)})_k$ is a bounded sequence of functions in $\text{Lip}_0(M)$ converging pointwise to $f \in \text{Lip}_0(M)$ then $S_n(f_k) \to S_n(f)$ pointwise as $k \to \infty$.

**Proof.** Clearly $S_n(f)(0) = 0$. By Proposition 2.6 there exists $\delta \in (0, \delta_n]$ such that $|\hat{f}_s(x) - \hat{f}_s(y)| \leq (1 + \frac{1}{n}) \|x - y\|$ whenever $f \in B_{\text{Lip}_0(M)}$, $x, y \in M \cap B_{n^2 + \delta_n}$, $\|x - y\| \leq \delta$ and $s \in (0, \delta]$. Set $s_n = \frac{\delta}{2nm\log R} < \delta \leq \delta_n$. By Proposition 2.7 we have $|\hat{f}_{s_n}(x) - \hat{f}_{s_n}(y)| \leq (1 + \frac{1}{n}) \|x - y\|$ whenever $x, y \in M \cap B_{n^2 + \delta_n}$ and $\|x - y\| \geq \delta$. 

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Therefore $|\hat{f}_{s_n}(x) - \hat{f}_{s_n}(y)| \leq (1 + \frac{1}{n}) \|x - y\|$ for all $x, y \in M \cap B_{n^2+\delta_n}$. It follows that $\text{Lip}(S_n(f)) \leq 1 + \frac{1}{n}$ and hence $\|S_n\| \leq 1 + \frac{1}{n}$.

If $x \in M \cap B_{n^2+\delta_n}$ and $\|z\|_2 \leq s_n$ then

$$\|x - \psi(x + z)\| \leq K \|\psi(x) - \psi(x + z)\|_2 \leq KL_n \|z\|_2 \leq KL_n s_n.$$ 

Therefore if $\text{Lip}(f) \leq 1$ then

$$|f(x) - \hat{f}_{s_n}(x)| \leq \int B_{s_n} \nu_{s_n}(z)|f(x) - f(\psi(x + z))| \, dz \leq KL_n s_n \leq \frac{1}{2n}.$$ 

Hence

$$|S_n(f)(x) - f(x)| \leq |\hat{f}_{s_n}(x) - f(x)| + |\hat{f}_{s_n}(x_0) - f(x_0)| \leq \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n},$$

for all $x \in M \cap B_{n^2+\delta_n}$.

To prove the last part of the theorem, let $(f^{(k)})_k$ be a bounded sequence in $\text{Lip}_0(M)$ converging to $f$ pointwise. For a fixed $x \in M \cap B_{n^2+\delta_n}$ we get $\hat{f}_{s_n}(x) \to \hat{f}_{s_n}(x)$ by the Dominated Convergence Theorem. Now it is easily seen that $S_n(f^{(k)}(x)) \to S_n(f)(x)$ pointwise. \hfill \Box

The next two results stipulate how to construct of finite-rank operators that can closely approximate the mollified functions furnished by the $S_n$ in both a Lipschitz and uniform sense.

**Theorem 3.2.** Let $x \in M \cap B_{n^2}$ and let $E_x : \mathbb{R}^d \to T_x$ be a linear $\|\cdot\|_2$-isometry. Then there exists a neighbourhood $W$ of $0 \in \mathbb{R}^d$, such that the map $u \mapsto \psi(x + E_x u)$ from $W$ to $M$ is open and, for all $f \in \text{Lip}_0(M)$ the function $\tilde{f}_x : W \to \mathbb{R},$

$$\tilde{f}_x(u) = S_n(f)(\psi(x + E_x u)),$$

is well-defined and $C^1$. Moreover, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left\|D\tilde{f}_x(u) - D\tilde{f}_x(u')\right\|_2 \leq \epsilon$$

whenever $f \in B_{\text{Lip}_0(M)}$ and $u, u' \in W$ satisfy $\|u - u'\|_2 \leq \sqrt{d}\delta$.

**Proof.** By Lemma 2.4, $D(\psi - I)(x)|_{T_x} = 0$. Because $\psi$ is $C^1$, there exists $V \ni x$ open in $(x + T_x) \cap M^\delta \cap \text{int}(B_{n^2 + \delta_n})$, such that $\text{Lip}_{1,2}((\psi - I)|_V) \leq \frac{1}{2}$. Therefore the inverse of $\psi|_V$ exists and is Lipschitz, as is shown after the proof of [13] Lemma 2.7. Let $\phi_x$ denote this inverse defined on some set $U \ni x$ that is open in $M \cap \text{int}(B_{n^2 + \delta_n})$ of $x$. We put $W = E_x^{-1}(\phi_x(U) - x)$.

For $f \in \text{Lip}_0(M)$ we have

$$\tilde{f}_x(u) = \hat{f}_{s_n}(\psi(x + E_x u)) - \hat{f}_{s_n}(0),$$

for $u \in W$. According to Lemma 2.3, $\hat{f}_{s_n}$ is $C^\infty$. Since $E_x$ is $C^\infty$ and $\psi$ is $C^1$ we have that $\tilde{f}_x$ is $C^1$. Now let $\epsilon > 0$ and by uniform continuity of $D\psi$ on compact sets choose

$$\delta \in \left(0, \frac{\epsilon s_n}{2GKL_n^3\sqrt{d}}\right) \quad (3.1)$$
such that
\[ \|D\psi(y) - D\psi(z)\|_{op} \leq \frac{\epsilon}{2KL_n} \] (3.2)
whenever \( y, z \in M^{\delta_n} \cap \text{int}(B_{n^2+\delta_n}) \) satisfy \( \|y - z\|_2 \leq \sqrt{d}\delta \). We note that \( \|D\psi(y)\|_2 \leq L_n \) when \( y \in M^{\delta_n} \cap B_{n^2+\delta_n} \).

Pick \( f \in B_{\text{Lip}_0(M)} \) and \( u, u' \in W \) satisfying \( \|u - u'\|_2 \leq \sqrt{d}\delta \). Note that \( x + E_x u, \psi(x + E_x u) \in M^{\delta_n} \cap \text{int}(B_{n^2+\delta_n}) \), and the same holds for \( u' \). By the chain rule,
\[ D\hat{f}_x(u) = (D\hat{f}_{s_n})(\psi(x + E_x u)) \circ (D\psi)(x + E_x u) \circ E_x. \]

Again by Lemma 2.9 together with the fact that \( E_x \) is a \( \|\cdot\|_2 \)-isometry, we obtain
\[
\begin{align*}
\|D\hat{f}_x(u) - D\hat{f}_x(u')\|_2 &\leq \|D\hat{f}_{s_n}(\psi(x + E_x u)) - (D\hat{f}_{s_n})(\psi(x + E_x u'))\|_2 \|D\psi)(x + E_x u)\|_2 \\
&\quad + \|D\hat{f}_{s_n}(\psi(x + E_x u'))\|_2 \|D\psi)(x + E_x u) - (D\psi)(x + E_x u')\|_2 \\
&\leq \frac{GKL_n}{s_n} \|\psi(x + E_x u) - \psi(x + E_x u')\|_2 L_n + KL_n \frac{\epsilon}{2KL_n} \\
&\leq \frac{GKL_n^3}{s_n} \|u - u'\|_2 + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{by (3.2) and (3.1)}. \quad \square
\end{align*}
\]

Now let \( x \in M \cap B_{n^2} \) and choose some \( \|\cdot\|_2 \)-isometry \( E_x : \mathbb{R}^d \to T_x \). Let \( W_x \) be the neighbourhood of \( 0 \in \mathbb{R}^d \) furnished by Theorem 3.2. Let \( C_x \) be some closed hypercube centered at \( 0 \) and contained in \( W_x \), and let \( U_x = \psi(x + E_x(\text{int}(C_x))) \). We have that \( U_x \) is open in \( M \). We form the cover \( \{U_x : x \in M\} \) of \( M \cap B_{n^2} \) and find a finite subcover \( U_1, \ldots, U_m \) with associated points \( x_1, \ldots, x_m \). Let \( J_n = \max\{\text{Lip}_{1,12}(\phi_{x_i} : x_i) : i = 1, \ldots, m\} \).

**Theorem 3.3.** Let \( i \in \{1, \ldots, m\} \), and \( \epsilon > 0 \). There exists a finite-rank linear map \( P_i : \text{Lip}_0(M \cap B_{n^2+\delta_n}) \to C(U_i) \) such that
\[ \text{Lip}(P_i(S_n(f)) - S_n(f)|_{U_i}) \leq KJ_n(1 + \sqrt{d})\epsilon \]
and
\[ \|P_i(S_n(f)) - S_n(f)|_{U_i}\|_{\infty} \leq 2KL_n\sqrt{d}\epsilon \]
for all \( f \in B_{\text{Lip}_0(M)} \). Moreover, for any bounded sequence \( (f_k) \subseteq \text{Lip}_0(M \cap B_{n^2+\delta_n}) \) converging pointwise to \( f \) we have \( P_i(f_k) \to P_i(f) \) pointwise.

**Proof.** Fix \( i \in \{1, \ldots, m\} \) and \( \epsilon > 0 \). Choose \( \delta \in (0, \epsilon] \) furnished by Theorem 3.2 applied to \( x_i, E_{x_i} \) and \( \epsilon > 0 \).

Let \( e(C_{x_i}) \) be the edge length of \( C_{x_i} \), and let \( b \) be the least integer not smaller than \( \frac{e(C_{x_i})}{\delta} \). Put \( \xi = \frac{e(C_{x_i})}{b} \leq \delta \). Let \( C \) be the cover of \( C_{x_i} \), with hypercubes of edge length \( \xi \), determined by the mesh \( \xi \mathbb{Z}^d \). In other words,
\[ C = \{C \subseteq \mathbb{R}^d : C \text{ is a hypercube, } V_C = C \cap \xi \mathbb{Z}^d \text{ and } \text{int}(C \cap C_{x_i}) \neq \emptyset\}. \]
where \( V_C \) is the set of vertices of \( C \).
For any \( f \in \text{Lip}_1(M \cap B_{n^2+\delta n}) \), we define the function \( G(f) : W_{x_i} \to \mathbb{R} \) by \( G(f)(u) = f(\phi(x_i + E_{x_i}u)) \). Then we define \( \Lambda(G(f)) : C_{x_i} \to \mathbb{R} \), \( \Lambda(G(f))(u) = \Lambda(G(f), C)(u) \), where \( C \in \mathcal{C} \) is such that \( u \in C \). Note that if \( C, C' \in \mathcal{C} \) are different hypercubes with \( C \cap C' \neq \emptyset \), then \( \Lambda(G(f), C) \) and \( \Lambda(G(f), C') \) are equal on \( C \cap C' \). Therefore \( \Lambda(G(f)) \) is well-defined. Finally, we define

\[
P_i(f)(y) = \Lambda(G(f))(E_{x_i}^{-1}(\phi_{x_i}(y) - x_i))
\]

for \( y \in U_i \). We may and do extend \( P_i(f) \) to \( \overline{U_i} \), since \( P_i(f) \) is Lipschitz. We have that \( P_i \) is a finite-rank linear map.

Now fix \( f \in B_{\text{Lip}_0(M)} \) and consider the function \( g = G(S_n(f)) \). By the choice of \( \delta \) supplied by Theorem 3.2 the function \( g \) satisfies the assumptions of Lemma 2.10 for every \( C \in \mathcal{C} \). We obtain

\[
\text{Lip}_{1,2}((\Lambda(g) - g)|_{C_{x_i}}) \leq (1 + \sqrt{d})\epsilon \quad \text{and} \quad \|(\Lambda(g) - g)|_{C_{x_i}}\|_\infty \leq \sqrt{d}\delta \text{Lip}_{1,2}(g).
\]

Pick \( y, z \in U_i \). We have

\[
P_i(S_n(f))(y) = \Lambda(g)(E_{x_i}^{-1}(\phi_{x_i}(y) - x_i)),
\]

and

\[
S_n(f)(y) = G(S_n(f))(E_{x_i}^{-1}(\phi_{x_i}(y) - x_i)) = g(E_{x_i}^{-1}(\phi_{x_i}(y) - x_i))
\]

(and similarly for \( z \)). Therefore

\[
|P_i(S_n(f))(y) - S_n(f)(y) - (P_i(S_n(f))(z) - S_n(f)(z))| \\
= |(\Lambda(g) - g)(E_{x_i}^{-1}(\phi_{x_i}(y) - x_i)) - ((\Lambda(g) - g)(E_{x_i}^{-1}(\phi_{x_i}(z) - x_i)))| \\
\leq \text{Lip}_{1,2}((\Lambda(g) - g)|_{C_{x_i}}) \|(E_{x_i}^{-1}(\phi_{x_i}(y) - x_i)) - (E_{x_i}^{-1}(\phi_{x_i}(z) - x_i))\|_2 \\
\leq (1 + \sqrt{d})\epsilon \|\phi_{x_i}(y) - \phi_{x_i}(z)\|_2 \leq KJ_n(1 + \sqrt{d})\epsilon \|y - z\|.
\]

Therefore \( \text{Lip}_{1,2}(P_i(S_n(f)) - S_n(f)|_{U_i}) \leq KJ_n(1 + \sqrt{d})\epsilon \).

For the uniform norm, we have

\[
\|P_i(S_n(f)) - S_n(f)|_{U_i}\|_\infty \leq \|(\Lambda(g) - g)|_{C_{x_i}}\|_\infty \leq \sqrt{d}\delta \text{Lip}_{1,2}(g).
\]

If \( u, v \in C_{x_i} \), then

\[
|g(u) - g(v)| = |S_n(f)(\psi(x_i + E_{x_i}u)) - S_n(f)(\psi(x_i + E_{x_i}v))| \\
\leq \text{Lip}(S_n(f)) \|\psi(x_i + E_{x_i}u) - \psi(x_i + E_{x_i}v)\| \\
\leq \left(1 + \frac{1}{n}\right) K \|\psi(x_i + E_{x_i}u) - \psi(x_i + E_{x_i}v)\|_2 \leq 2KL_n \|u - v\|_2.
\]

Therefore \( \text{Lip}_{1,2}(g) \leq 2KL_n \) and

\[
\|P_i(S_n(f)) - S_n(f)|_{U_i}\|_\infty \leq 2KL_n \sqrt{d}\delta \leq 2KL_n \sqrt{d}\epsilon.
\]

Finally, let \( (f_k)_k \) be a bounded sequence in \( \text{Lip}_0(M \cap B_{n^2+\delta n}) \) converging pointwise to \( f \) and let \( y \in U_i \). We have that \( G(f_k) \to G(f) \) pointwise. Let \( C \in \mathcal{C} \) be a hypercube containing the point \( u = E_{x_i}^{-1}(\phi_{x_i}(y) - x_i) \). Then \( P_i(f_k)(y) = \Lambda(G(f_k))(u) = \).
Moreover, for a fixed $x \in U_i$ was arbitrary we have that $P_i(f_k) \to P_i(f)$ pointwise on $U_i$, and convergence on $\overline{U_i}$ follows easily. \hfill \Box

Finally we have the ingredients needed to prove Theorem 1.1.

Proof of Theorem 1.1. As described in [19, p. 44], it suffices to construct a sequence $(\Gamma_n)_n$ of finite-rank dual operators on $\text{Lip}_0(M)$ such that $\|\Gamma_n\| \to 1$ and for every $x \in M,$ $\Gamma_n(f)(x) \to f(x)$ uniformly in $f \in B_{\text{Lip}_0(M)}.

For a fixed $n \in \mathbb{N}$, consider the cover $U_1, \ldots, U_m$ of $M \cap B_{n^2}$ defined before Theorem 3.3 and let $\alpha_i: M \to [0, 1], i = 1, \ldots, m$ be $H$-Lipschitz functions, where $H \geq 1$, forming a partition of unity subordinate to $U_1, \ldots, U_m$. Apply Theorem 3.3 to $\epsilon := (2mnHK(1 + \sqrt{d})\max(L_n, J_n))^{-1}$ to obtain corresponding operators $P_i, i = 1, \ldots, m$. Then

$$
\|P_i(S_n(f)) - S_n(f)|_{U_i}\|_\infty \text{Lip}(P_i(S_n(f)) - S_n(f)|_{U_i}) \leq \frac{1}{mnH}
$$

for all $f \in B_{\text{Lip}_0(M)}$. Define the operator $Q'_n: \text{Lip}_0(M) \to C(M \cap B_{n^2})$ by

$$
Q'_n(f)(x) = \sum_{i=1}^m \alpha_i(x)P_i(S_n(f))(x).
$$

By Theorem 3.3, $Q'_n$ has finite rank. Then using Lemma 2.11 we obtain

$$
\|Q'_n(f) - S_n(f)|_{M \cap B_{n^2}}\|_\infty \leq \frac{1}{n} \text{ and Lip}(Q'_n(f) - S_n(f)|_{M \cap B_{n^2}}) < \frac{1 + mH}{mnH} \leq \frac{2}{n},
$$

for all $f \in B_{\text{Lip}_0(M)}$. According to Theorem 3.1 we have $\|Q'_n(f) - f|_{M \cap B_{n^2}}\|_\infty < \frac{2}{n}$ and $\text{Lip}(Q'_n(f)) \leq \frac{2}{n} + \text{Lip}(S_n(f)) \leq 1 + \frac{2}{n}$ for all $f \in B_{\text{Lip}_0(M)}$. Define $Q_n: \text{Lip}_0(M) \to \text{Lip}_0(M \cap B_{n^2}),

$$
Q_n(f)(x) = Q'_n(f)(x) - Q'_n(f)(0).
$$

Then $Q_n(f)(0) = 0$ for all $f \in \text{Lip}_0(M)$. Since for $f \in B_{\text{Lip}_0(M)}, |Q'_n(f)(0) - f(0)| = |Q'_n(f)(0) - f(0)| \leq \frac{2}{n}$, we have $\|Q_n(f) - f|_{M \cap B_{n^2}}\|_\infty < \frac{4}{n}$ Moreover, $\text{Lip}(Q_n(f)) = \text{Lip}(Q'_n(f)) \leq 1 + \frac{2}{n}.

Finally, by Proposition 2.12 there exists an operator $\Phi_n: \text{Lip}_0(M \cap B_{n^2}) \to \text{Lip}_0(M)$, such that $\|\Phi_n\| \leq 1 + \frac{K^2}{\log n}$ and $\Phi_n(f)$ agrees with $f$ on $M \cap B_n$ and equals 0 outside $B_{n^2}$ for all $f \in \text{Lip}_0(M)$. Define $\Gamma_n: \text{Lip}_0(M) \to \text{Lip}_0(M)$ by $\Gamma_n(f) = \Phi_n(Q_n(f)).$

We have

$$
\|\Gamma_n\| \leq \left(1 + \frac{K^2}{\log n}\right) \left(1 + \frac{3}{n}\right) \to 1 \text{ as } n \to \infty.
$$

Moreover, for a fixed $x \in M$ and all $n \geq \|x\|_2$ we have

$$
|\Gamma_n(f)(x) - f(x)| = |Q_n(f)(x) - f(x)| \leq \frac{4}{n},
$$

for all $f \in B_{\text{Lip}_0(M)}$.

To show that $\Gamma_n$ is a dual operator it suffices to show that it is $w^* - w^*$ continuous. By the Banach-Dieudonné Theorem it suffices to show that the restriction of
Theorem 4.1. The space $F(M)$ has the MAP.

Proof. Denote by $B$ be closed unit ball of $\|\cdot\|$. Let $D = 2B \setminus \text{int}(B)$, where $2B := \{2x : x \in B\}$. Equip $D$ with the metric $d$ and let $x_0$ be the distinguished point for $D$ as well. By [2, Theorem 5], there exists a map $h : \mathbb{R}^2 \to B$, such that $h(x) = x$ for all $x \in B$ and $h$ is contractive, i.e. $d(h(x), h(y)) \leq d(x, y)$ for all $x, y \in \mathbb{R}^2$. Let $h|_D$ be the restriction of $h$ to $D$ and note the the range of $h|_D$ is $M$. By the lifting property for Lipschitz maps, $h|_D$ induces a norm-one projection $H : F(D) \to F(M)$. Therefore it suffices to show that $F(D)$ has the MAP. By [19, Corollary 2.4] it suffices to show that $D$ is locally downwards closed, i.e. given $x \in \partial D$, there exists open $U \ni x$ and $v \neq 0$ such that $y - tv \in \text{int}(D)$ whenever $y \in U \cap D$, $y - tv \in U$ and $t > 0$. This is straightforward to see: given $x \in \partial D$, set $v = x$ if $x \in \partial (2B)$ and $v = -x$ if $x \in \partial B$, and let $U$ be a sufficiently small open ball having centre $x$. \hfill $\Box$

4. Open problems

We conclude with a few remarks about open problems. We don’t know if Theorem 1.1 can be generalised to include non-$C^1$-submanifolds. In particular, we don’t know the answer in the case where $M$ is the unit sphere of an arbitrary norm on $\mathbb{R}^N$, $N \geq 3$. We can answer this in the case $N = 2$ by appealing to existing results.

Let $\|\cdot\|$, $\|\cdot\|$ be two norms on $\mathbb{R}^2$ (neither necessarily differentiable), let $M$ be the unit sphere of $\|\cdot\|$ endowed with the metric $d$ induced by $\|\cdot\|$ and choose a distinguished point $x_0 \in M$. We will denote by $\partial D$ the boundary of a subset $D \subseteq \mathbb{R}^2$.

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