A new method to construct spacetimes with a spacelike circle action *

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Abstract

A new general procedure to construct realistic spacetimes is introduced. It is based on the null congruence on a time-oriented Lorentzian manifold associated to a certain timelike vector field. As an application, new examples of stably causal Petrov type D spacetimes which obey the timelike convergence condition and which admit an isometric spacelike circle action are obtained.

1 Introduction

The assumption of the existence of symmetries has been used extensively to obtain exact solutions of very complicated equations inPhysics. In particular, there are remarkable examples of exact solutions of the gravitational field equations which have been obtained by assuming the existence of an isometric action of a Lie group on the spacetime. On the other hand, if the orbit space is a (smooth) manifold, many physical and mathematical problems can be reduced to questions on the orbit space which has a lower dimension than the original spacetime. We recall for instance the spatial spherical symmetry for the classical Schwarzschild solution.

The simplest compact Lie group is the circle $S^1$ and so an isometric circle action could be the simplest symmetry assumption on a Lorentzian manifold. Classically, the case of $S^1$ acting by timelike isometries is well known, recall for example the timelike circle action admitted for the 3-dimensional anti-de Sitter spacetime. Note that a timelike circle action implies the existence of closed timelike curves and so the absence of the chronology condition. Recently the case of $S^1$ acting by spacelike isometries has received a deep and wide attention. In fact, Choquet-Bruhat and Moncrief [3, 4] have introduced a Lorentzian

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metric on each 4-dimensional manifold of the form $\Sigma \times \mathbb{R} \times S^1$, where $\Sigma$ is certain compact orientable 2-dimensional manifold, which is invariant under the action of the group $S^1$ and whose orbits are spacelike.

This paper is devoted to the introduction of a new general technique to construct 4-dimensional spacetimes which admit a spacelike circle action starting from certain 3-dimensional Lorentzian manifolds. Moreover, we will show that in several relevant cases this action is isometric. We would like to point out that every orientable 3-dimensional manifold can be endowed with a Lorentzian metric. Further, let us recall that the space of all Lorentzian metrics on a 3-dimensional orientable compact manifold, for a natural topology, possesses an infinity of connected components [17]. Therefore we think that a method to construct spacetimes from 3-dimensional Lorentzian manifolds would be interesting by itself and it will produce many new examples of 4-dimensional spacetimes.

The technique used here is based on the geometry of the null congruence associated to a timelike vector field on a time-oriented Lorentzian manifolds (see Section 2 for the definition and main properties of the null congruence). Given an $n(\geq 3)$-dimensional time-oriented Lorentzian manifold $(M, g)$ and a timelike vector field $K$ on $M$, the null congruence obtained from this manifold, $C_K M$, is a codimension-two oriented submanifold of the tangent bundle $TM$. It is constructed by taking a particular $(n-2)$-dimensional ellipse in the light cone above every point of $M$ and it can be viewed as the manifold of all null directions on the Lorentzian manifold $(M, g)$. The null congruence has been previously used to characterize Friedmann-Lemaître-Robertson-Walker spacetimes [14], to study infinitesimal null isotropy [16] and recently to analyze the behavior of conjugate points along null geodesics [8, 10], (see also [7, 9, 20] for more details). Moreover, in an appropriate sense, the null congruence allows one to study the null sectional curvature as a smooth function on the set of degenerate tangent planes [19]. We present now another application of a different nature. In fact, every null congruence, $C_K M$, inherits a natural Lorentzian metric, $\hat{g}$, from the well-known Sasaki one on $TM$ defined from $g$, see for instance [21], which is indefinite with index 2. The canonical projection $\pi : C_K M \to M$ becomes then a semi-Riemannian submersion and a fibre bundle with fibre-type a spacelike $(n-2)$-dimensional sphere [8]. In particular, starting from a time-oriented, 3-dimensional Lorentzian manifold $(M, g)$, we obtain a 4-dimensional spacetime $(C_K M, \hat{g})$ which is a fibre bundle with spacelike circle fibres. It should be noted that $\hat{g}$ depends on the metric $g$ and the choice of a timelike vector field $K$ on $M$. Therefore, this procedure provides a wide family of spacetimes and, as it will be shown, with a rich variety of geometric properties.

The content of this paper is organized as follows. Section 2 is devoted to introduce basic results and several definitions about null congruences. Then, the vertical and the horizontal subspaces for the semi-Riemannian submersion $\pi : C_K M \to M$ are found, and as a consequence the horizontal lift of a vector field on $M$ along this submersion is given in (5). It should be pointed out that the horizontal vectors for the null congruence are also horizontal for the tangent bundle only if the considered timelike vector field $K$ is parallel.

As an easy consequence, we can show that the null congruences inherits certain causality conditions imposed on the base spacetime. In fact, in causality theory, the stable causality condition is frequently used to describe suitable spacetimes. As shown in Remark 2, if
we assume that \((M, g)\) is stably causal, then every null congruence over \(M\) is also stably causal.

In Section 3, basic relations between the geometries of the Lorentzian manifold and the null congruence constructed over it are expressed from the O'Neill fundamental tensors \(T\) and \(A\) \[18\] of the semi-Riemannian submersion. The formulas obtained for \(T\) in Proposition \[3\] and for \(A\) in Proposition \[5\] show the strong dependence of the geometry of \(C_K M\) on the choice of the timelike vector field \(K\). In particular, the fibres of \(\pi|_{C_K M}\) are totally geodesic (i.e. \(T = 0\)) if and only if \(K\) is parallel, Corollary \[4\]. This contrasts with the well-known fact which says that the fibres of the tangent bundle are always totally geodesic. In our setting, \(T\) and \(A\) are different from zero. Recall that semi-Riemannian submersions with both O'Neill tensors non-zero are less studied in the literature. Thus, the Lorentzian metrics we obtain are neither products nor Kaluza-Klein, in general.

In Section 4, previous results are specialized for 3-dimensional time-oriented Lorentzian manifolds. In this case, each null congruence admits a natural action of \(S^1\) and it becomes a \(S^1\)-principal fibre bundle. A natural question arises in this setting: When is this action isometric? An answer will be found in Theorem \[7\] showing that this holds if and only if the normalized vector field \(1/\sqrt{-g(K, K)} K\) is parallel or equivalently, the horizontal distribution defines a connection on the principal circle bundle \((C_K M, \pi, M, S^1)\).

We end this paper constructing spacetimes from the null congruence procedure when the base Lorentzian manifold is the 3-dimensional Minkowski spacetime \(\mathbb{L}^3\). According to Remark 2, \(C_K \mathbb{L}^3\) will be stably causal for any timelike vector field \(K\) on \(\mathbb{L}^3\). Even more, a suitable choice of such \(K\) will produce nice physically realistic spacetimes of type \(C_K \mathbb{L}^3\). Thus, Theorem \[11\] summarizes the main properties of the obtained examples as follows.

For any non-vanishing smooth function \(f = f(z), z \in \mathbb{R}\), the null congruence \(C_K \mathbb{L}^3\), with \(K = f \partial_z\), is stably causal, admits an isometric spacelike circle action and is semi-symmetric (i.e. \(R.R = 0\)). The spacetime \(C_K \mathbb{L}^3\) is of type D in Petrov’s classification and it is filled with an anisotropic perfect fluid. Moreover, \(C_K \mathbb{L}^3\) satisfies the timelike convergence condition if and only if \(2(f')^2 - f'' f \geq 0\).

It should be remarked that the semi-symmetric notion, \(R.R = 0\), can be geometrically interpreted in terms of the invariance, in first approximation, of the sectional curvature of every tangent plane after parallel transport along every coordinate parallelogram \[11\]. Clearly, it is strictly more weak than the locally symmetric condition \(\nabla R = 0\), and so it includes spaces of constant sectional curvature. Our technique also permits us to construct a family of semi-symmetric spacetimes \(C_K \mathbb{L}^3\), which are not locally symmetric, for suitable choices of the vector field \(K\), Theorem \[10\].

In view of these results and the general approach we show here, it seems that the construction of spacetimes from the null congruence is useful to provide new examples of spacetimes which admit a spacelike circle action.
2 Definition and properties of the null congruence

2.1 Preliminaries

Let $(M, g)$ be an $n(\geq 3)$-dimensional Lorentzian manifold, that is, a (connected) smooth manifold $M$ endowed with a nondegenerate metric $g$ with signature $(-, +, \ldots, +)$. We shall write $\nabla$ for its Levi-Civita connection, $R$ for its Riemann-Christoffel curvature tensor, and $\text{Ric}$ for its Ricci tensor. Remember that a tangent vector $u \in T_p M$ is said to be timelike if $g(u, u) < 0$, null if $g(u, u) = 0$ and $u \neq 0$, and spacelike if $g(u, u) > 0$ or $u = 0$. A vector field $K \in \mathfrak{X}(M)$ is said to be timelike if $K_p$ is timelike for all $p \in M$. From now on, the Lorentzian manifold $(M, g)$ is assumed to be time-oriented, that is, a global timelike vector field $K$ has been fixed and so a timelike or null tangent vector $v \in T_p M$ is said to be future (resp. past) with respect to $K$ if $g(v, K_p) < 0$ (resp. $g(v, K_p) > 0$).

The null congruence associated with $K$ is defined as the set

$$C_K M = \{ v \in TM \mid g(v, v) = 0 \text{ and } g(v, K_{\pi(v)}) = -1 \} ,$$

with $\pi : TM \to M$ the natural projection. For each $p \in M$ we put $(C_K M)_p = C_K M \cap T_p M$, and thus $C_K M = \bigcup_{p \in M} (C_K M)_p$. Note that we take the null congruence in the future null cone, in contrast to the definition used in [8, 14, 16], and that the number $-1$ is only a normalization and any other negative real could be used to define $C_K M$.

We next recall that $C_K M$ is an orientable submanifold of $TM$ with dimension $2(n-1)$ and $(C_K M, \pi, M, S^{n-2})$ is a spherical fibre bundle with structure group $O(n-1)$, proofs of these facts are given in [8].

Let $(M, g)$ be a time-orientable Lorentzian manifold. If $K$ and $T$ are arbitrary timelike vector fields on $M$, then we have the bundle diffeomorphism $\sigma : C_K M \to C_T M$ given as follows,

$$\sigma(v) = -\frac{v}{g(v, T_{\pi(v)})}. \tag{1}$$

Therefore, all null congruences on $(M, g)$ are diffeomorphic manifolds and they could be seen as its manifold of null directions.

Recall that the connection map or connector $c$ associated with the Levi-Civita connection $\nabla$ is defined by

$$c : TTM \to TM, \quad X \mapsto \frac{\nabla_\alpha}{dt} \mid_0 ,$$

where $\alpha$ is a curve in $TM$ with $\alpha'(0) = X$ and $\frac{\nabla_\alpha}{dt}$ is the covariant derivative of the vector field $\alpha$ along the curve $\pi \circ \alpha$ on $M$.

For each $v \in TM$, the canonical identification between the tangent spaces will be denoted by

$$(v) : T_{\pi(v)} M \to T_v T_{\pi(v)} M, \quad u \mapsto (u)_v,$$

and so $c((u)_v) = u$. If $X \in \mathfrak{X}(M)$, the vector field $\mathbb{X} \in \mathfrak{X}(TM)$, given by $\mathbb{X}_v = (X_{\pi(v)})_v$, is called the vertical lift of $X$ to $TM$. Recall further that the position vector field, or

1Our convention on the curvature tensor is $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\{X, Y\}} Z.$
Liouville field, \( A \in \mathfrak{X}(TM) \) is given by \( A_v = (v)_v \). These vector fields are characterized by the equalities,
\[
X\omega = \omega(X) \circ \pi, \quad A\omega = \omega,
\]
for every \( \omega \in \mathfrak{X}^*(M) \).

**Remark 1** We assume that \( M \) is oriented and \( \Omega_g \) is the volume form of the oriented Lorentzian manifold \((M, g)\). Let \( \Psi \) be the \((n-2)\)-form on \( C_K M \) given by,
\[
\Psi(\xi_1, \ldots, \xi_{n-2}) = \Omega_g(U_\pi(v), c(\xi_1), \ldots, c(\xi_{n-2}), v),
\]
where \( \xi_1, \ldots, \xi_{n-2} \in T_v C_K M \) and \( U = \frac{1}{\sqrt{-g(K, K)}} K \). It is not difficult to check that its restriction to every fibre \((C_K M)_p\) provides us an orientation and so the spherical fibre bundle \((C_K M, \pi, M, S^{n-2})\) is oriented in the sense of [6, Chapter 7]. Hence, when \( M \) is assumed to be compact, the de Rham cohomology groups of \( M \) and \( C_K M \) are related by the Gysin sequence [6, Chapter 8] as follows,
\[
\ldots \to H^k_{dR}(M) \to H^{k+n-1}_{dR}(M) \to H^{k+n-1}_{dR}(C_K M) \to H^{k+1}_{dR}(M) \to \ldots
\]
and, as usual, the Euler class \( \chi_{C_K M} \in H^{n-1}_{dR}(M) \) can be considered. Note that the vanishing of \( \chi_{C_K M} \) does not imply here the existence of a cross section (a null tangent vector field on \( M \)) for \((C_K M, \pi, M, S^{n-2})\). This fact is a remarkable difference with the Euler class of the spherical fibre bundle associated to a compact Riemannian manifold.

From the Lorentzian metric \( g \) on \( M \), the Sasaki metric \( \widehat{g} \) on \( TM \) is defined as
\[
\widehat{g}(\zeta, \xi) = g(\zeta_*, \xi_*) + g(c(\zeta), c(\xi)),
\]
for all \( \zeta, \xi \in TTM \) with \( \zeta_* = d\pi(\zeta) \) and \( \xi_* = d\pi(\xi) \). Recall that \( \pi : (TM, \widehat{g}) \to (M, g) \) is a semi-Riemannian submersion in the terminology of Gray [5] and O’Neill [18]. Given \( X \in \mathfrak{X}(M) \), the horizontal lift \( \widehat{X} \in \mathfrak{X}(TM) \) of \( X \) along \( \pi \) is characterized by,
\[
\widehat{X}\omega = \nabla_X \omega, \tag{2}
\]
for each one-form \( \omega \in \mathfrak{X}^*(M) \) and satisfies \( d\pi(\widehat{X}) = X \circ \pi \) and \( c(\widehat{X}) = 0 \). Since \((M, g)\) is a Lorentzian manifold \((TM, \widehat{g})\) is a semi-Riemannian manifold with index 2.

### 2.2 Horizontal and vertical vectors for the null congruence

In the following, \( \widehat{g} \) will also represent the induced tensor on \( C_K M \) from the Sasaki one of \( TM \), unless otherwise stated. As it was shown in [8], \((C_K M, \widehat{g})\) is a Lorentzian manifold and \( \pi |_{C_K M} : (C_K M, \widehat{g}) \to (M, g) \) is a semi-Riemannian submersion with spacelike fibres, that is, for each \( p \in M \) the induced tensor \( \widehat{g}_p = \widehat{g} |_{(C_K M)_p} \) turns \(((C_K M)_p, \widehat{g}_p)\) into a Riemannian manifold.
As usual, the vectors of $TC_KM$ tangent to the fibres are called vertical while the vectors normal to the fibres are called horizontal. At each $v \in (C_KM)_p$, we denote by $\mathcal{V}_v$ the tangent space $T_v(C_KM)_p$ and $\mathcal{H}_v$ the orthogonal complement to $\mathcal{V}_v$ in $T_vC_KM$. We will write $\mathcal{V}$ and $\mathcal{H}$ for the corresponding distributions on $C_KM$ and also the orthogonal projections onto them.

Consider the differentiable map,

$$F : TM \to \mathbb{R}^2, \quad v \mapsto \left( \frac{1}{2}g(v,v), g(v, K_{\pi(v)}) \right).$$

Taking into account that $(0, -1)$ is a regular value of $F$ and $F^{-1}\{(0, -1)\} = C_KM$, a standard computation gives us, for $v \in C_KM$ with $\pi(v) = p$,

$$T_vC_KM = \left\{ \xi \in T_vTM : g(c(\xi), v) = g(c(\xi), K_p) + g(v, \nabla_{\xi} K) = 0 \right\},$$

$$\mathcal{V}_v = \left\{ (u)_v \in T_vT_pM : g(u, v) = g(u, K_p) = 0 \right\},$$

and

$$\mathcal{H}_v = \left\{ \xi \in T_vTM : c(\xi) = g(v, \nabla_{\xi} K)v \right\}.$$  \hfill (3)

As a direct consequence, the horizontal lift along the semi-Riemannian submersion $\pi |_{C_KM}$ of $X \in \mathfrak{X}(M)$ is the vector field $\tilde{X} \in \mathfrak{X}(C_KM)$, given for every $v \in C_KM$ by,

$$\tilde{X}_v = \hat{X}_v + g(v, \nabla X_{\pi(v)} K)A_v,$$  \hfill (5)

where $\hat{X}$ is the horizontal lift of $X$ to $TM$ given in (2).

The orthogonal subspace of $\mathcal{V}_v$ in $T_vT_pM$ is,

$$\mathcal{V}_v^\perp = \text{Span} \{ \xi_1(v), \xi_2(v) \} \subset T_vT_pM,$$  \hfill (6)

whereby

$$\xi_1 = \frac{K}{\sqrt{-\tilde{g}(K,K)}} \quad \text{and} \quad \xi_2 = \sqrt{-\tilde{g}(K,K)}A - \xi_1,$$  \hfill (7)

which satisfy $\tilde{g}(\xi_1, \xi_1) = -1 = -\tilde{g}(\xi_2, \xi_2)$ and $\tilde{g}(\xi_1, \xi_2) = 0$.

For every $X \in \mathfrak{X}(M)$, let $X^T$ be the tangent part to $C_KM$ of $X$. Since, $X_v \in T_vT_pM = \mathcal{V}_v \oplus \mathcal{V}_v^\perp$ for all $v \in C_KM$, it follows from (6) that,

$$X^T = X + \left[ g(K, K) \circ \pi \cdot X^\flat + K^\flat(X) \circ \pi \right] A + X^\flat K,$$  \hfill (8)

where $X^\flat$ is the one-form $g$-equivalent to the vector field $X$, i.e., $X^\flat(Y) = g(X, Y)$, for all $Y \in \mathfrak{X}(M)$.  

Remark 2 Two causality conditions which are frequently used in the study of Lorentzian manifolds are stable causality and the global hyperbolicity condition. A stably causal manifold \((M, g)\) is characterized by the existence of a function \(f\) on \(M\) whose gradient is everywhere timelike \([15, \text{Proposition 6.4.9}]\). A Cauchy hypersurface in \(M\) is a subset which is met exactly once by every inextendible causal curve in \(M\). Recently, the globally hyperbolic manifolds are characterized by the existence of a smooth Cauchy hypersurface \([1]\).

Every null congruence on a stably causal Lorentzian manifold inherits this condition. In fact, taking into account that \(\pi \mid_{\mathcal{C}K}\) is a semi-Riemannian submersion, it follows that if \(f \in \mathcal{C}^\infty(M)\) has a timelike gradient, then \(f \circ \pi \mid_{\mathcal{C}K}\) has also a timelike gradient.

The question remains open if the global hyperbolicity condition is preserved by the null congruence construction.

3 The O’Neill fundamental tensors of the null congruence

In this section we study the relation between the geometries of a Lorentzian manifold and a null congruence over this manifold. This relation is expressed through the knowledge of the O’Neill fundamental \((1, 2)\)-tensors \(T\) and \(A\) of the semi-Riemannian submersion \(\pi \mid_{\mathcal{C}K}\). Recall that \(T\) and \(A\) are given by,

\[
T_{E_1 E_2} = \mathcal{H}\hat{\nabla}_{E_1} V E_2 + V \hat{\nabla}_{E_1} \mathcal{H} E_2 , \quad A_{E_1 E_2} = \mathcal{H}\hat{\nabla}_{HE_1} V E_2 + V \hat{\nabla}_{HE_1} \mathcal{H} E_2 ,
\]

whereby \(E_1, E_2 \in \mathfrak{X}(C_K M)\) and \(\hat{\nabla}\) is the Levi-Civita connection of \(\hat{g} \mid_{C_K M}\). Recall that if \(U, V \in \mathfrak{X}(C_K M)\) are vertical vector fields, there holds that \(T_U V\) is the second fundamental form of the fibres. Because \(T\) is vertical, i.e., \(T_E = T_{VE}\), the fibres of \(\pi\) are totally geodesic if and only if \(T = 0\). On the other hand, if \(X, Y \in \mathfrak{X}(C_K M)\) are horizontal, there holds that,

\[
A_X Y = \frac{1}{2} \mathcal{V} [X, Y].
\]

Because \(A\) is horizontal, i.e., \(A_E = A_{HE}\), the O’Neill fundamental tensor \(A\) characterizes the integrability of the horizontal distribution \(\mathcal{H}\), \([18]\).

For each \(z \in T_p M\) we can construct the following linear form on \(T_p M\), \(\Gamma_z(u) = g(u, \nabla_z K), u \in T_p M\). Then, if \(Z \in \mathfrak{X}(M)\), we can consider \(\Gamma_Z \in \mathfrak{X}^*(M)\) with \(\Gamma_Z(u) = g(u, \nabla_{Z(u)} K)\). In particular, from \((5)\), it follows that

\[
\tilde{Z} = \hat{Z} + \Gamma_Z A .
\]

We are now able to calculate the O’Neill fundamental tensor \(T\).

Proposition 3 Let \(v \in C_K M\) and \(\xi, \zeta \in \mathcal{V}_v\). The fundamental tensor \(T\) of the semi-Riemannian submersion \(\pi \mid_{C_K M}\) satisfies the condition
\[ \tilde{g}(T_z\xi, Z) = -\Gamma_{Z,v}(v) \tilde{g}(\xi, \zeta) , \] 

for every horizontal \( Z \in T_xC_KM. \)

**Proof:** Take \( x, y \in T_pM \) so that \( \xi = (x)_v, \zeta = (y)_v \) and extend the tangent vectors \( x \) and \( y \) to vector fields \( X, Y \in \mathfrak{X}(M) \) respectively. By using (8), we find that \( X_v = \xi \) and \( Y_v = \zeta \).

From the definition of \( T \), we have that,

\[ T_{X^T}Y^T = \mathcal{H}_{X^T} \mathcal{Y}^T. \]

Thus, for every \( Z \in \mathfrak{X}(M) \) we have to calculate \( \tilde{g}(\hat{\mathcal{V}}_{X^T} \mathcal{Y}^T, \tilde{Z})(v) \), where \( \tilde{Z} \) is the horizontal lift of \( Z \) along \( \pi \big|_{C_KM} \) given in (11).

From the Koszul formula we find,

\[
2 \tilde{g}(\hat{\mathcal{V}}_{X^T} \mathcal{Y}^T, \tilde{Z}) = X^T \tilde{g}(\mathcal{Y}^T, \tilde{Z}) + \mathcal{Y}^T \tilde{g}(X^T, \tilde{Z}) - \tilde{Z} \tilde{g}(X^T, \mathcal{Y}^T) \\
- \tilde{g} \left( X^T, \left[ \mathcal{Y}^T, \tilde{Z} \right] \right) + \tilde{g} \left( \mathcal{Y}^T, \left[ \tilde{Z}, X^T \right] \right) + \tilde{g} \left( \tilde{Z}, \left[ X^T, \mathcal{Y}^T \right] \right).
\]

The horizontal and vertical vector fields are \( \tilde{g} \)-orthogonal and the distribution formed by the vertical vector fields of every submersion is integrable. Thus, by using Frobenius’ theorem, the Koszul formula simplifies to,

\[
2 \tilde{g}(\hat{\mathcal{V}}_{X^T} \mathcal{Y}^T, \tilde{Z}) = -\tilde{Z} \tilde{g}(X^T, \mathcal{Y}^T) + \tilde{g} \left( X^T, \left[ \tilde{Z}, \mathcal{Y}^T \right] \right) + \tilde{g} \left( \mathcal{Y}^T, \left[ \tilde{Z}, X^T \right] \right). \tag{13}
\]

On the other hand, from (8) we have that

\[
\left[ \tilde{Z}, X^T \right] = \left[ \tilde{Z}, X \right] + \Theta_X \left[ \tilde{Z}, A \right] + \tilde{Z} \Theta_X \cdot A + X^b \left[ \tilde{Z}, K \right] + \tilde{Z} X^b \cdot K ,
\]

where, \( \Theta_X = g(K, K) \circ \pi \cdot (X^b + g(K, X) \circ \pi \in C^{\infty}(TM). \)

Taking into account that \( \mathcal{Y}^T \in \mathfrak{X}(C_KM) \) is vertical, we find

\[
\tilde{g} \left( \mathcal{Y}^T, \left[ \tilde{Z}, X^T \right] \right) = \tilde{g} \left( \mathcal{Y}^T, \left[ \tilde{Z}, X \right] + \Theta_X \left[ \tilde{Z}, A \right] + X^b \left[ \tilde{Z}, K \right] \right) . \tag{14}
\]

For every \( \omega \in \mathfrak{X}^*(M) \), using (11), we have

\[
\left[ \tilde{Z}, X \right] \omega = Z(\omega(X)) \circ \pi - X(\nabla_Z \omega + \Gamma_Z \cdot \omega) \\
= \omega(\nabla_Z X) \circ \pi - (\Gamma_Z(X) \circ \pi) \cdot \omega - \Gamma_Z \cdot (\omega(X) \circ \pi) ,
\]

or

\[
\left[ \tilde{Z}, X \right] = \mathbb{D}(Z, X) - (\Gamma_Z(X) \circ \pi)A - \Gamma_Z X ,
\]

where \( \mathbb{D}(Z, X) \) is the vertical lift of \( \nabla_Z X \) to \( TM \). Therefore,
\[ \hat{g}(Y^T, [\tilde{Z}, X]) = \hat{g}(Y^T, D(Z, X) - \Gamma_Z X), \]

and analogously,
\[ \hat{g}(Y^T, [\tilde{Z}, K]) = \hat{g}(Y^T, D(Z, K)). \]

Further, using the fact that \([\tilde{Z}, A] = -\Gamma_Z A\), we find from (14) that,
\[ \hat{g}(Y^T, [\tilde{Z}, X^T]) = \hat{g}(Y^T, D(Z, X) - \Gamma_Z X) + X^p \hat{g}(Y^T, D(Z, K)), \]

and thus, taking into account that \(g(X_p, v) = 0\), we get,
\[ \hat{g}(Y^T, [\tilde{Z}, X^T])(v) = g(Y_p, \nabla Z_p X) - \Gamma_Z(v) g(X_p, Y_p). \tag{15} \]

Since \(\Theta_X, \Theta_Y, \hat{g}(X, A), \hat{g}(Y, A), \hat{g}(X, K), \hat{g}(Y, K), X^p, \) and \(Y^p\) vanish at \(v\), it is not difficult to check that,
\[ \tilde{Z} \hat{g}(X^T, Y^T)(v) = Z_p g(X, Y). \tag{16} \]

Finally, from (14), (15) and (16), and using the fact that the Levi-Civita connection is metric compatible, the result follows from (13).

As a direct consequence we find the following result.

**Corollary 4** *The fibres of the semi-Riemannian submersion \(\pi|_{C_{K\mathcal{M}}}\) are totally geodesic if and only if the timelike vector field \(K\) is parallel.*

We will now calculate the O’Neill fundamental tensor \(A\) of \(\pi|_{C_{K\mathcal{M}}}\).

**Proposition 5** *For every \(X, Y \in \mathfrak{X}(M)\), the fundamental tensor \(A\) of the semi-Riemannian submersion \(\pi|_{C_{K\mathcal{M}}}\) satisfies*
\[ 2A_X \tilde{Y} = -(R_{XY})^- - \hat{g}((R_{XY})^-, K) A, \]

*where \((R_{XY})^- \in \mathfrak{X}(TM)\) is given by \((R_{XY})^-_v = (R_{X_p Y_p} v)_v\) with \(p = \pi(v)\).*

**Proof:** From (11) we know that \(A_X \tilde{Y} = \frac{1}{2} \nabla [\tilde{X}, \tilde{Y}].\) Since \([\tilde{X}, A] = [\tilde{Y}, A] = 0\), using (11), we find,
\[ [\tilde{X}, \tilde{Y}] = [\tilde{X}, \tilde{Y}] + (\tilde{X} \Gamma_Y - \tilde{Y} \Gamma_X) A. \]
Let $V^\ast$ be the vertical projection of the semi-Riemannian submersion $\pi : (TM, \hat{g}) \to (M, g)$. It is well-known from [5] that $V^\ast [\hat{X}, \hat{Y}] = -(R_{XY})^-$ and therefore,

$$V^\ast [\tilde{X}, \tilde{Y}] = -(R_{XY})^- + (\tilde{X}\Gamma_Y - \tilde{Y}\Gamma_X)A.$$

Since $V^\ast [\tilde{X}, \tilde{Y}] (v) \in T_vT_pM = V_v \oplus V_v^\perp$ for all $v \in C_KM$, its vertical part with respect to the null congruence is given by,

$$V [\tilde{X}, \tilde{Y}] = V^\ast [\tilde{X}, \tilde{Y}] + \hat{g}(V^\ast [\tilde{X}, \tilde{Y}], \xi_1)\xi_1 - \hat{g}(V^\ast [\tilde{X}, \tilde{Y}], \xi_2)\xi_2. \tag{17}$$

If we take into account the symmetries of the curvature tensor, we find that

$$\hat{g}((R_{XY})^-, A) = 0,$$

and so a straightforward computation from (17) gives us the result.

\textbf{Corollary 6} The fundamental tensor $A$ of $\pi |_{C_KM}$ vanishes if and only if $(M, g)$ is flat.

\textbf{Proof:} From Proposition 5 it follows that if $R = 0$, $A$ vanishes on the horizontal vectors, and hence identically. Conversely, suppose that $A = 0$ and fix $v \in C_KM$. From Proposition 5 it follows that,

$$R_{xy}v = -g(R_{xy}v, K_{\pi(v)}) v , \tag{18}$$

for all $x, y \in T_{\pi(v)}M$. Let $\Pi$ be a degenerate plane containing $v$, and suppose that $\Pi = \text{Span}\{v, z\}$. From (18) it follows that the null sectional curvature of the plane $\Pi$ with respect to $v$ vanishes. Thus, all null sectional curvatures vanish and therefore the sectional curvature of $(M, g)$ must be a constant $k$ [13]. Suppose now that $x, y \in (C_KM)_{\pi(v)}$, with $x, y, v$ linearly independent. We can then write (18) as follows,

$$k \{g(v, y)x - g(v, x)y\} = k \{g(v, y) - g(v, x)\} v .$$

Thus, $kg(v, x) = kg(v, y) = 0$. But, two orthogonal null vectors must be collinear, hence $k = 0$.  

\section{The 3-dimensional case}

\subsection{General facts}

Let $(M, g)$ be an oriented 3-dimensional Lorentzian manifold which admits a timelike vector field $K \in \mathfrak{X}(M)$. We normalize $K$ in order to obtain a unitary timelike vector field $T = fK$, where $f = \frac{1}{\sqrt{-g(K, K)}}$. 
We can introduce an action of the Lie group $S^1$ on $C_T M$ as follows,

$$S^1 \times C_T M \longrightarrow C_T M, \quad (e^{i\theta}, w) \mapsto e^{i\theta} \cdot w = T_{\pi(w)} \cos \theta \bar{w} + \sin \theta \bar{u},$$

where $w = T_{\pi(w)} \bar{w}$, with $\bar{w} \in T^\perp$ and $B = \{ T_{\pi(w)}, \bar{w}, \bar{u} \}$ is an oriented $g$-orthonormal basis for $T_{\pi(w)} M$. It is not difficult to show that $(C_T M, \pi, M, S^1)$ is a principal fibre bundle.

Now an action of the Lie group $S^1$ on $C_K M$ can be introduced, by using the bundle diffeomorphism $\sigma$ given in (1), as follows,

$$\Phi : S^1 \times C_K M \longrightarrow C_K M, \quad \Phi(e^{i\theta}, v) = \sigma^{-1}(e^{i\theta} \cdot \sigma(v)) = f(\pi(v))(e^{i\theta} \cdot \sigma(v)).$$

Hence $(C_K M, \pi, M, S^1)$ becomes a principal fibre bundle for every timelike vector field $K$. As usual, we denote $\Phi(e^{i\theta}, v) = \Phi_\theta(v)$.

**Theorem 7** For the action $\Phi$ defined above, the following are equivalent:

1. $T$ is a parallel vector field,
2. The horizontal distribution $H$ of the semi-Riemannian submersion,

$$\pi : (C_K M, \tilde{g}) \rightarrow (M, g),$$

defines a connection on the principal fibre bundle $(C_K M, \pi, M, S^1)$,
3. The action $\Phi$ is isometric for $\tilde{g} |_{C_K M}$. 

**Proof:** Let $v \in (C_K M)_p$, the horizontal distribution $H$ defines a connection if for each $e^{i\theta} \in S^1$,

$$d\Phi_\theta(X) \in H_{\Phi_\theta(v)},$$

with $X \in H_v$. Let $\lambda$ be a curve in $C_K M$ with $\lambda(0) = v$ and $\lambda'(0) = X$. A direct computation yields that,

$$c \left( d\Phi_\theta(X) \right) = (X \ast f)(e^{i\theta} \cdot \sigma(v)) + f(p) \frac{\nabla(e^{i\theta} \cdot (\sigma \circ \lambda))}{dt} \bigg|_0.$$

From (20) and (11) it follows immediately that (19) is equivalent to,

$$g \left( e^{i\theta} \cdot (\sigma \circ \lambda), \nabla_X T \right) (e^{i\theta} \cdot \sigma(v)) = \frac{\nabla(e^{i\theta} \cdot (\sigma \circ \lambda))}{dt} \bigg|_0.$$

Take the vector fields $\bar{V}, \bar{U} \in \mathfrak{X}(\pi \circ \lambda)$ such that $\sigma \circ \lambda = T + \bar{V}$, with $\bar{V} \in T^\perp$, and $\{ T, \bar{V}, \bar{U} \}$ is an oriented orthonormal basis along $\pi \circ \lambda$. Then,

$$e^{i\theta} \cdot (\sigma \circ \lambda) = T \cos \theta \bar{V} \sin \theta \bar{U}.$$
Let us fix an oriented orthonormal basis \{T_v, T_u\}. Using (24) and (25) one can show that, equation (21) is now equivalent to,

\[ \nabla_X (\sigma + \cos \theta \tilde{V} + \sin \theta \tilde{U}), \]

where \( \tilde{V}(0) = \tilde{v} \) and \( \tilde{U}(0) = \tilde{u} \). Taking into account that \( X \) is a horizontal tangent vector, we obtain that,

\[ g(v, \nabla_X(\sigma + \tilde{V}))v = f(p)^2 \left( \nabla_X(\sigma + \tilde{V}) \right), \]

Assume that \( \sigma + \tilde{V} \) is a parallel vector field, from (23) it is easily deduced that \( \nabla_X(\sigma + \tilde{V}) = 0 \) and hence also \( \nabla_X(\sigma + \tilde{U}) = 0 \). Using (22) we complete the proof of (1) \( \Rightarrow \) (2).

Conversely, if the equality holds in (22) for every \( e^{i\theta} \in S^1 \), by taking the product with \( \tilde{v} \), we find,

\[ \cos^2 \theta g(\nabla_X(\sigma + \tilde{V}), \tilde{v}) + \sin \theta \cos \theta g(\nabla_X(\sigma + \tilde{V}), \tilde{u}) = g(\sigma + \tilde{V}, \tilde{v}) + \sin \theta g(\nabla_X(\sigma + \tilde{U}), \tilde{v}) , \]

and from (23) we have,

\[ -\sin^2 \theta g(\nabla_X(\sigma + \tilde{V}), \tilde{v}) - \sin \theta (1 - \cos \theta) g(\nabla_X(\sigma + \tilde{U}), \tilde{v}) = 0 . \]

Therefore, \( g(\tilde{v}, \nabla_X(\sigma + \tilde{V})) = g(\tilde{u}, \nabla_X(\sigma + \tilde{U})) = 0 \) and \( \sigma + \tilde{V} \) is a parallel vector field.

Clearly, if \( \Phi \) is an isometric action, then \( d\Phi_\theta(\mathcal{H}_\alpha) = \mathcal{H}_{\Phi_\theta(\alpha)} \) and so (3) \( \Rightarrow \) (2).

Finally, if we assume that \( \sigma + \tilde{V} \) is a parallel vector field, the horizontal distribution is a connection and therefore it can be easily deduced from (19) that the action \( \Phi \) is isometric on horizontal tangent vectors.

On the other hand, if \( \xi \in \mathcal{V}_v \) and \( \alpha \) is a curve in \((C_K^i)\mathcal{M}_p\), with \( \alpha(0) = v \) and \( \alpha'(0) = \xi \), then, in a similar way as in (20) we can show that,

\[ g(c(d\Phi_\theta(\xi)), c(d\Phi_\theta(\xi))) = f(p)^2 g\left( \frac{\nabla(c(e^{i\theta} \cdot (\sigma \circ \alpha)))}{dt} \bigg|_0, \frac{\nabla(e^{i\theta} \cdot (\sigma \circ \alpha))}{dt} \bigg|_0 \right). \]

(24)

Let us fix an oriented orthonormal basis \( \{\mathcal{T}_p, e_1, e_2\} \) for \( T_p\mathcal{M} \). We can then decompose \( \sigma \circ \alpha = \mathcal{T}_p + \tilde{V}^\alpha \), where \( \tilde{V}^\alpha = \tilde{V}_1^\alpha e_1 + \tilde{V}_2^\alpha e_2 \in T_p\mathcal{M} \). We can further construct a vector \( \bar{U}^\alpha = -\tilde{V}_2^\alpha e_1 + \tilde{V}_1^\alpha e_2 \) such that \( \{\mathcal{T}_p, \tilde{V}^\alpha, \bar{U}^\alpha\} \) is an orthonormal basis of \( T_p\mathcal{M} \). Then,

\[ c(\xi) = f(p) \left( \frac{d\tilde{V}_1^\alpha}{dt} \bigg|_0 e_1 + \frac{d\tilde{V}_2^\alpha}{dt} \bigg|_0 e_2 \right). \]

(25)

Using (24) and (25) one can show that,

\[ g(c(d\Phi^K_\theta(\xi)), c(d\Phi^K_\theta(\xi))) = g(c(\xi), c(\xi)) , \]

which completes the proof of (1) \( \Rightarrow \) (3).

\( \diamond \)
4.2 Null congruences on $\mathbb{L}^3$

In order to obtain a family of examples, we consider the case when the base manifold $M$ is the 3-dimensional Minkowski space $\mathbb{L}^3$, where the natural coordinate vector field $\partial_z$ is unit and timelike. From Remark 2 we find that all the null congruences $C_K\mathbb{L}^3$ are 4-dimensional, stably causal, Lorentzian manifolds and from Corollary 6 it follows that the fundamental tensor of O’Neill $A$ of every null congruence on $\mathbb{L}^3$ vanishes identically. Therefore, from [2, Theorem 9.28] it follows that the only non-vanishing components of the curvature tensor $\hat{R}$ of $C_K\mathbb{L}^3$ have the form,

$$\hat{g}(\hat{R}(X, U)Y, U) = \hat{g}(T_U X, T_U Y) - \hat{g}((\hat{\nabla}_X T)U, Y),$$  \hspace{1cm} (26)

where $X, Y$ are horizontal vector fields and $U$ is the spacelike unitary vector field which spans the vertical distribution. Remark that our choice of sign of the curvature tensor is opposite as in [2].

Let $\tilde{X}$ be the horizontal lift of $\partial_z$ to $C_K\mathbb{L}^3$, and analogously for $\tilde{Y}$ and $\tilde{Z}$. It can be deduced from (26) that the Ricci tensor $\hat{\text{Ric}}$ of every null congruence $C_K\mathbb{L}^3$ satisfies the following properties,

$$\hat{\text{Ric}}(U, U) = \hat{\text{Ric}}(\tilde{X}, \tilde{X}) + \hat{\text{Ric}}(\tilde{Y}, \tilde{Y}) - \hat{\text{Ric}}(\tilde{Z}, \tilde{Z})$$  \hspace{1cm} (27)

$$\hat{\text{Ric}}(\tilde{X}, U) = \hat{\text{Ric}}(\tilde{Y}, U) = \hat{\text{Ric}}(\tilde{Z}, U) = 0.$$

**Remark 8** There exists no non-flat Einstein null congruence $C_K\mathbb{L}^3$. In fact, from (27), every Einstein null congruence $C_K\mathbb{L}^3$ is Ricci flat and it can be deduced that the sectional curvature $\hat{K}$ of non-degenerate planes in a Ricci flat null congruence $C_K\mathbb{L}^3$ satisfies $\hat{K} = 0$.

**Lemma 9** The only non-vanishing components of the Riemann-Christoffel curvature tensor of the null congruence $C_K\mathbb{L}^3$, at a point $v \in C_K\mathbb{L}^3$ with $\pi(v) = p$, are given as follows,

$$\hat{g}(\hat{R}(X, U)Y, U)_v = g\left(v, \nabla_{X_{\mid p}} \nabla_{Y_{\mid p}} K\right)$$

$$-g\left(v, \nabla_{\nabla_{X_{\mid p}} Y_{\mid p}} K\right) + 2g\left(v, \nabla_{X_{\mid p}} K\right)g\left(v, \nabla_{Y_{\mid p}} K\right),$$  \hspace{1cm} (28)

where $X, Y$ are horizontal vector fields and $U$ is the spacelike unitary vector field which spans the vertical distribution.

**Proof:** From [2, Theorem 9.18] it is well known that $T$ satisfies $\hat{g}(T_U X, T_U Y) = -\hat{g}(T_U T_U X, Y)$. Therefore, Proposition 3 implies that,

$$\hat{g}(T_U X, T_U Y)_v = g\left(v, \nabla_{X_{\mid p}} K\right)g\left(v, \nabla_{Y_{\mid p}} K\right).$$  \hspace{1cm} (29)

On the other hand, there holds that,

$$(\hat{\nabla}_X T)_U U = \hat{\nabla}_X (T_U U) - T_{\hat{\nabla}_X U} U - T_U (\hat{\nabla}_X U),$$
and from [18, Lemma 3] and \( A = 0 \), it follows that \( \nabla_X U = \mathcal{V} \nabla_X U \). Taking into account that \( X \hat{g}(U, U) = 2 \hat{g}(\nabla_X U, U) = 0 \), one can deduced that \( \nabla_X U = 0 \). Thus,

\[
\hat{g}((\nabla_X T)_v U, Y) = X(\hat{g}(T_v U, Y)) - \hat{g}(T_v U, \nabla_X Y).
\]

Assume that \( X, Y \in \mathfrak{X}(\mathbb{L}^3) \) are basic horizontal vector fields, i.e., they are \( \pi \)-related to the vector fields \( X_\ast, Y_\ast \in \mathfrak{X}(M) \). There holds that \( \nabla_X Y = \mathcal{H} \nabla_X Y \) and thus \( d\pi(\nabla_X Y) = \left( \nabla_X Y_\ast \right) \circ \pi, [18, \text{Lemma 1}] \). From Proposition 3 we then find that,

\[
\hat{g}(T_v U, \nabla_X Y)_v = -g(v, \nabla_{v_X, Y_\ast})_v K.
\]

Finally, from (13) and (12), we get,

\[
X(\hat{g}(T_v U, Y)) = -\nabla_{X_\ast} Y_\ast - \Gamma_{X_\ast} Y_\ast,
\]

which completes the proof.

Recall that a semi-Riemannian manifold \((M, g)\) is called locally symmetric if its Riemann-Christoffel curvature tensor satisfies the condition \( \nabla R = 0 \). Manifolds which satisfy this condition, also satisfy the integrability condition \( R \cdot R = 0 \), i.e.,

\[
(R \cdot R)(X_1, X_2, X_3, X_4; X, Y) = 0 \tag{30}
\]

\[
= -g(R(R(X, Y)X_1, X_2)X_3, X_4) - g(R(X_1, R(X, Y)X_2)X_3, X_4) - g(R(X_1, X_2)X_3, R(X, Y)X_4) = 0,
\]

with \( X_1, X_2, X_3, X_4, Y \in \mathfrak{X}(M) \). However, the converse is not true in general. A manifold which satisfies the condition \( R \cdot R = 0 \) is called semi-symmetric. A geometric interpretation of this class of manifolds is given in [11].

**Theorem 10** Let \( C_K \mathbb{L}^3 \) be the null congruence associated with a timelike vector field \( K \in \mathfrak{X}(\mathbb{L}^3) \) which satisfies the condition \( \nabla K = \alpha K \otimes K \), with \( \alpha \in C^\infty(\mathbb{L}^3) \). Then, \( C_K \mathbb{L}^3 \) is a semi-symmetric Lorentzian manifold. Moreover, if \( K(K(\alpha)) \neq 0 \), the null congruence \( C_K \mathbb{L}^3 \) is not locally symmetric.

**Proof**: From Lemma 9 we find,

\[
\hat{g}\left( \tilde{R}(X, U)Y, U \right)_v = g(X_\ast \mid_p, \nabla \alpha) g(Y_\ast \mid_p, K),
\]

where \( v \in C_K \mathbb{L}^3 \) with \( \pi(v) = p \). Let \( E_3 \in \mathfrak{X}(C_K \mathbb{L}^3) \) the horizontal lift of \( \frac{K}{\sqrt{-g(K,K)}} \) and \( \{e_1, e_2, e_3\} \) an orthonormal basis of \( T_p \mathbb{L}^3 \) adapted to our choice of \( K \), i.e., take as timelike...
direction $e_3 = E_3(v)$. Let $\{E_1, E_2, E_3(v)\}$ be the horizontal lift of $\{e_1, e_2, e_3\}$ to $T_v C_K \mathbb{L}^3$. With respect to the $\widehat{g}$-orthonormal basis $\{E_1, E_2, E_3(v), U_v\}$, the only component of the curvature tensor $\widehat{R}$ which can be non-vanishing is,

$$\widehat{g}(\widehat{R}(E_3, U)E_3, U)_v = -K_p(\alpha).$$

We can introduce a Newman-Penrose null basis as follows,

$$\widehat{L} = \frac{1}{\sqrt{2}}(U_v + E_3(v)) \ , \ \widehat{N} = \frac{1}{\sqrt{2}}(U_v - E_3(v)) \ , \ \widehat{M} = \frac{1}{\sqrt{2}}(E_1 + i E_2).$$

The scalars which determine the curvature in the Newman-Penrose formalism are \[22\],

$$\widehat{\Psi}_0 = \widehat{\Psi}_1 = \widehat{\Psi}_3 = \widehat{\Psi}_4 = 0 \text{ and } \widehat{\Psi}_2 = \overline{\widehat{\Psi}}_2 = -2\Lambda = -\frac{1}{6}K(\alpha), \quad (32)$$

$$\widehat{\Phi}_{0i} = \widehat{\Phi}_{2i} = 0 \ , \ i = 0, 1, 2 \text{ and } \widehat{\Phi}_{11} = -\frac{1}{4}K(\alpha). \quad (33)$$

From the classification given in \[12\] it follows that $(C_K \mathbb{L}^3, \widehat{g})$ is a semi-symmetric manifold.

In order to show that the null congruence $C_K \mathbb{L}^3$ is not locally symmetric, we point out that from $\widehat{\nabla}_E U = \nabla\widehat{\nabla}_E U$ and $E_3 \widehat{g}(U, U) = 0$, we deduce that $\widehat{\nabla}_E U = 0$. Further, since $\widehat{\nabla}_E(U_3 + H\nabla) = H\nabla E_3$ and $E_3 \widehat{g}(E_3, E_3) = 0$, we have that $(\widehat{\nabla}_E E_3)_v \in \text{Span}\{E_1, E_2\}$. Therefore, from \[20\], a straightforward computation gives us,

$$\widehat{g}\left((\widehat{\nabla}_E \widehat{R})(E_3, U)E_3, U\right)_v = E_3(v)\left(\widehat{g}(\widehat{R}(E_3, U)E_3, U)\right) = -\frac{1}{\sqrt{-g(K_p, K_p)}}K_p(K(\alpha)),$$

which completes the proof.

Recall that a spacetime satisfies the timelike convergence condition if and only if $\text{Ric}(W, W) \geq 0$, for every timelike vector field $W$. This is the mathematical translation of the fact that on average gravity attracts.

**Theorem 11** Let $f = f(z) \in C^\infty(\mathbb{L}^3)$ be a non-vanishing function. Then, the 4-dimensional, semi-symmetric, Lorentzian manifold $C_K \mathbb{L}^3$, with $K = f \partial_z$, is stably causal and admits an isometric spacelike circle action. This spacetime is further of type D in Petrov’s algebraic classification of the Weyl tensor and is filled with an anisotropic perfect fluid. The null congruence $C_K \mathbb{L}^3$ satisfies the timelike convergence condition if and only if $2(f')^2 - f'' f \geq 0$.

**Proof:** Note that $K$ satisfies the condition $\nabla K = -f'f^{-2} K^\flat \otimes K$. From \[32\] and \[33\] it follows immediately that $C_K \mathbb{L}^3$ is an anisotropic perfect fluid of Petrov type D. Recall that
\( \tilde{X} \) denotes the horizontal lift of \( \partial_z \) to \( C_K \mathbb{L}^3 \), and analogously for \( \tilde{Y} \) and \( \tilde{Z} \). An unitary timelike vector field \( W \in \mathfrak{X}(C_K \mathbb{L}^3) \) can be written as follows,

\[
W = \alpha \tilde{X} + \beta \tilde{Y} + \gamma \tilde{Z} + \delta U .
\]

From (31) it deduces that \( \hat{\text{Ric}}(\tilde{X}, .) = \hat{\text{Ric}}(\tilde{Y}, .) = 0 \) and therefore from (27),

\[
\hat{\text{Ric}}(W, W) = (\gamma^2 - \delta^2) \hat{\text{Ric}}(\tilde{Z}, \tilde{Z}) = (\gamma^2 - \delta^2) \left\{ 2 \left( \frac{f'}{f} \right)^2 - \frac{f''}{f} \right\} ,
\]

which completes the proof.

\( \diamond \)

**Remark 12** If \( f(z) = e^{az} \), with \( a \in \mathbb{R} - \{0\} \), the null congruence \( C_K \mathbb{L}^3 \), with \( K = f \partial_z \), satisfies the timelike convergence condition and is not locally symmetric.

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