ON THE RELATIONSHIP BETWEEN IDEAL CLUSTER POINTS AND IDEAL LIMIT POINTS

MAREK BALCERZAK AND PAOLO LEONETTI

Abstract. Let $X$ be a first countable space which admits a non-trivial convergent sequence and let $I$ be an analytic P-ideal. First, it is shown that the sets of $I$-limit points of all sequences in $X$ are closed if and only if $I$ is also an $F_\sigma$-ideal.

Moreover, let $(x_n)$ be a sequence taking values in a Polish space without isolated points. It is known that the set $A$ of its statistical limit points is an $F_\sigma$-set, the set $B$ of its statistical cluster points is closed, and that the set $C$ of its ordinary limit points is closed, with $A \subseteq B \subseteq C$. It is proved the sets $A$ and $B$ own some additional relationship: indeed, the set $S$ of isolated points of $B$ is contained also in $A$.

Conversely, if $A$ is an $F_\sigma$-set, $B$ is a closed set with a subset $S$ of isolated points such that $B \setminus S \neq \emptyset$ is regular closed, and $C$ is a closed set with $S \subseteq A \subseteq B \subseteq C$, then there exists a sequence $(x_n)$ for which: $A$ is the set of its statistical limit points, $B$ is the set of its statistical cluster points, and $C$ is the set of its ordinary limit points.

Lastly, we discuss topological nature of the set of $I$-limit points when $I$ is neither $F_\sigma$- nor analytic P-ideal.

1. Introduction

The aim of this article is to establish some relationship between the set of ideal cluster points and the set of ideal limit points of a given sequence.

To this aim, let $I$ be an ideal on the positive integers $\mathbb{N}$, i.e., a collection of subsets of $\mathbb{N}$ closed under taking finite unions and subsets. It is assumed that $I$ contains the collection $\text{Fin}$ of finite subsets of $\mathbb{N}$ and it is different from the whole power set $\mathcal{P}(\mathbb{N})$.

Note that the family $I_0$ of subsets with zero asymptotic density, that is,

$$I_0 := \left\{ S \subseteq \mathbb{N} : \lim_{n \to \infty} \frac{|S \cap \{1, \ldots, n\}|}{n} = 0 \right\},$$

is an ideal. Let also $x = (x_n)$ be a sequence taking values in a topological space $X$, which will be always assumed hereafter to be Hausdorff. We denote by $\Lambda_x(I)$ the set of $I$-limit points of $x$, that is, the set of all $\ell \in X$ for which $\lim_{k \to \infty} x_{n_k} = \ell$, for some subsequence $(x_{n_k})$ such that $\{n_k : k \in \mathbb{N}\} \notin I$. In addition, let $\Gamma_x(I)$ be the set of $I$-cluster points of $x$, that is, the set of all $\ell \in X$ such that $\{n : x_n \in U\} \notin I$ for every neighborhood $U$ of $\ell$. Note that $L_x := \Lambda_x(\text{Fin}) = \Gamma_x(\text{Fin})$ is the set of ordinary limit points of $x$; we also shorten $\Lambda_x := \Lambda_x(I_0)$ and $\Gamma_x := \Gamma_x(I_0)$.

Statistical limit points and statistical cluster points (i.e., $I_0$-limit points and $I_0$-cluster points, resp.) of real sequences were introduced by Fridy [10], cf. also [2, 5, 11, 13, 15, 17].

2010 Mathematics Subject Classification. Primary: 40A35. Secondary: 54A20, 40A05, 11B05.

Key words and phrases. Ideal limit point, ideal cluster point, asymptotic density, analytic P-ideal, regular closed set, equidistribution, co-analytic ideal, maximal ideal.
We are going to provide in Section 2, under suitable assumptions on $X$ and $I$, a characterization of the set of $I$-limit points. Recall that $\Gamma_x(I)$ is closed and contains $\Lambda_x(I)$, see e.g. [4, Section 5]. Then, it is shown that:

(i) $\Lambda_x(I)$ is an $F_\sigma$-set, provided that $I$ is an analytic $P$-ideal (Theorem 2.2);

(ii) $\Lambda_x(I)$ is closed, provided that $I$ is an $F_\sigma$-ideal (Theorem 2.3);

(iii) $\Lambda_x(I)$ is closed for all $x$ if and only if $\Lambda_x(I) = \Gamma_x(I)$ for all $x$ if and only if $I$ is an $F_\sigma$-ideal, provided that $I$ is an analytic $P$-ideal (Theorem 2.5);

(iv) For every $F_\sigma$-set $A$, there exists a sequence $x$ such that $\Lambda_x(I) = A$, provided that $I$ is an analytic $P$-ideal which is not $F_\sigma$ (Theorem 2.6);

(v) Each of isolated point $I$-cluster point is also an $I$-limit point (Theorem 2.7).

In addition, we provide in Section 3 some joint converse results:

(vi) Given $A \subseteq B \subseteq C \subseteq \mathbb{R}$ where $A$ is an $F_\sigma$-set, $B$ is non-empty regular closed, and $C$ is closed, then there exists a real sequence $x$ such that $\Lambda_x = A$, $\Gamma_x = B$, and $L_x = C$ (Theorem 3.1 and Corollary 3.3);

(vii) Given non-empty closed sets $B \subseteq C \subseteq \mathbb{R}$, there exists a real sequence $x$ such that $\Lambda_x(I) = \Gamma_x(I) = B$ and $L_x = C$, provided $I$ is an $F_\sigma$-ideal different from $\text{Fin}$ (Theorem 3.4).

Lastly, it is shown in Section 4 that:

(viii) $\Lambda_x(I)$ is analytic, provided that $I$ is a co-analytic ideal (Proposition 4.1);

(ix) An ideal $I$ is maximal if and only if each real sequence $x$ admits at most one $I$-limit point (Proposition 4.2 and Corollary 4.3).

We conclude by showing that there exists an ideal $I$ and a real sequence $x$ such that $\Lambda_x(I)$ is not an $F_\sigma$-set (Example 4.4).

2. Topological structure of $I$-limit points

We recall that an ideal $I$ is said to be a $P$-ideal if it is $\sigma$-directed modulo finite, i.e., for every sequence $(A_n)$ of sets in $I$ there exists $A \in I$ such that $A_n \setminus A$ is finite for all $n$; equivalent definitions were given, e.g., in [1, Proposition 1].

By identifying sets of integers with their characteristic function, we equip $\mathcal{P}(\mathbb{N})$ with the Cantor-space topology and therefore we can assign the topological complexity to the ideals on $\mathbb{N}$. In particular, an ideal $I$ is analytic if it is a continuous image of a $G_\delta$-subset of the Cantor space. Moreover, a map $\varphi : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is a lower semicontinuous submeasure provided that: (i) $\varphi(\emptyset) = 0$; (ii) $\varphi(A) \leq \varphi(B)$ whenever $A \subseteq B$; (iii) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ for all $A, B$; and (iv) $\varphi(A) = \lim_n \varphi(A \cap \{1, \ldots, n\})$ for all $A$.

By a classical result of Solecki, an ideal $I$ is an analytic $P$-ideal if and only if there exists a lower semicontinuous submeasure $\varphi$ such that

$$I = I_\varphi := \{A \subseteq \mathbb{N} : \|A\|_\varphi = 0\}$$

and $\varphi(\mathbb{N}) < \infty$, where $\|A\|_\varphi := \lim_n \varphi(A \setminus \{1, \ldots, n\})$ for all $A \subseteq \mathbb{N}$, see [19, Theorem 3.1]. Note, in particular, that for every $n \in \mathbb{N}$ it holds

$$\|A\|_\varphi = \|A \setminus \{1, \ldots, n\}\|_\varphi.$$

Hereafter, unless otherwise stated, an analytic $P$-ideal will be always denoted by $I_\varphi$, where $\varphi$ stands for the associated lower semicontinuous submeasure as in (1).
Given a sequence \( x = (x_n) \) taking values in a first countable space \( X \) and an analytic P-ideal \( \mathcal{I}_\varphi \), define
\[
u(\ell) := \lim_{k \to \infty} \|\{n : x_n \in U_k\}\|_\varphi
\]for each \( \ell \in X \), where \( (U_k) \) is a decreasing local base of neighborhoods at \( \ell \). It is easy to see that the limit in (3) exists and its value is independent from the choice of \( (U_k) \).

**Lemma 2.1.** The map \( \nu \) is upper semi-continuous. In particular, the set
\[
\Lambda_x(\mathcal{I}_\varphi, q) := \{\ell \in X : \nu(\ell) \geq q\}
\]is closed for every \( q > 0 \).

**Proof.** We need to prove that \( \mathcal{U}_y := \{\ell \in X : \nu(\ell) < y\} \) is open for all \( y \in \mathbb{R} \) (hence \( \mathcal{U}_y \) is open too). Clearly, \( \mathcal{U}_y = \emptyset \) if \( y \leq 0 \). Hence, let us suppose hereafter \( y > 0 \) and \( \mathcal{U}_y \neq \emptyset \). Fix \( \ell \in \mathcal{U}_y \) and let \( (U_k) \) be a decreasing local base of neighborhoods at \( \ell \). Then, there exists \( k_0 \in \mathbb{N} \) such that \( \|\{n : x_n \in U_k\}\|_\varphi < y \) for every \( k \geq k_0 \). Fix \( \ell' \in U_{k_0} \) and let \( (V_k) \) be a decreasing local base of neighborhoods at \( \ell' \). Fix also \( k_1 \in \mathbb{N} \) such that \( V_{k_1} \subseteq U_{k_0} \). It follows by the monotonicity of \( \varphi \) that
\[
\|\{n : x_n \in V_k\}\|_\varphi \leq \|\{n : x_n \in U_{k_0}\}\|_\varphi < y
\]
for every \( k \geq k_1 \). In particular, \( \nu(\ell') < y \) and, by the arbitrariness of \( \ell', U_{k_0} \subseteq \mathcal{U}_y \). \( \Box \)

At this point, we provide a useful characterization of the set \( \Lambda_x(\mathcal{I}_\varphi) \) (without using limits of subsequences) and we obtain, as a by-product, that it is an \( F_\sigma \)-set.

**Theorem 2.2.** Let \( x \) be a sequence taking values in a first countable space \( X \) and \( \mathcal{I}_\varphi \) be an analytic P-ideal. Then
\[
\Lambda_x(\mathcal{I}_\varphi) = \{\ell \in X : \nu(\ell) > 0\}
\]
In particular, \( \Lambda_x(\mathcal{I}_\varphi) \) is an \( F_\sigma \)-set.

**Proof.** Let us suppose that there exists \( \ell \in \Lambda_x(\mathcal{I}_\varphi) \) and let \( (U_k) \) be a decreasing local base of neighborhoods at \( \ell \). Then, there exists \( A \subseteq \mathbb{N} \) such that \( \lim_{n \to \infty, n \in A} x_n = \ell \) and \( \|A\|_\varphi > 0 \). At this point, note that, for each \( k \in \mathbb{N} \), the set \( \{n \in A : x_n \not\in U_k\} \) is finite, hence it follows by (2) that \( \nu(\ell) \geq \|A\|_\varphi > 0 \).

On the other hand, suppose that there exists \( \ell \in X \) such that \( \nu(\ell) > 0 \). Let \( (U_k) \) be a decreasing local base of neighborhoods at \( \ell \) and define \( A_k := \{n : x_n \in U_k\} \) for each \( k \in \mathbb{N} \); note that \( A_k \) is infinite since \( \|A_k\|_\varphi \downarrow \nu(\ell) > 0 \) implies \( A_k \not\in \mathcal{I}_\varphi \) for all \( k \). Set for convenience \( \theta_0 := 0 \) and define recursively the increasing sequence of integers \( (\theta_k) \) so that \( \theta_k \) is the smallest integer greater than both \( \theta_{k-1} \) and \( \min A_{k+1} \) such that
\[
\varphi(A_k \cap (\theta_{k-1}, \theta_k)) \geq \nu(\ell)(1 - 1/k).
\]
Finally, define \( A := \bigcup_{k} (A_k \cap (\theta_{k-1}, \theta_k)) \). Since \( \theta_k \geq k \) for all \( k \), we obtain
\[
\varphi(A \setminus \{1, \ldots, n\}) \geq \varphi(A_n+1 \cap (\theta_n, \theta_{n+1})) > \nu(\ell)(1 - 1/n)
\]
for all \( n \), hence \( \|A\|_\varphi \geq \nu(\ell) > 0 \). In addition, we have by construction \( \lim_{n \to \infty, n \in A} x_n = \ell \). Therefore \( \ell \) is an \( \mathcal{I}_\varphi \)-limit point of \( x \). To sum up, this proves (4).

Lastly, rewriting (4) as \( \Lambda_x(\mathcal{I}_\varphi) = \bigcup_n \Lambda_x(\mathcal{I}_\varphi, 1/n) \) and considering that each \( \Lambda_x(\mathcal{I}_\varphi, 1/n) \) is closed by Lemma 2.1, we conclude that \( \Lambda_x(\mathcal{I}_\varphi) \) is an \( F_\sigma \)-set. \( \Box \)
The fact that $\Lambda_x(I_x)$ is an $F_\sigma$-set already appeared in [3, Theorem 2], although with a different argument. The first result of this type was given in [13, Theorem 1.1] for the case $I_x = I_0$ and $X = \mathbb{R}$. Later, it was extended in [5, Theorem 2.6] for first countable spaces. However, in the proofs contained in [3, 5] it is unclear why the constructed subsequence $(x_n : n \in A)$ converges to $\ell$. Lastly, Theorem 2.2 generalizes, again with a different argument, [14, Theorem 3.1] for the case $X$ metrizable.

A stronger result holds in the case that the ideal is $F_\sigma$. We recall that, by a classical result of Mazur, an ideal $I$ is $F_\sigma$ if and only if there exists a lower semicontinuous submeasure $\varphi$ such that

$$I = \{ A \subseteq \mathbb{N} : \varphi(A) < \infty \},$$

with $\varphi(\mathbb{N}) = \infty$, see [16, Lemma 1.2].

**Theorem 2.3.** Let $x = (x_n)$ be a sequence taking values in a first countable space $X$ and let $I$ be an $F_\sigma$-ideal. Then $\Lambda_x(I) = \Gamma_x(I)$. In particular, $\Lambda_x(I)$ is closed.

**Proof.** Since it is known that $\Lambda_x(I) \subseteq \Gamma_x(I)$, the claim is clear if $\Gamma_x(I) = \emptyset$. Hence, let us suppose hereafter that $\Gamma_x(I)$ is non-empty. Fix $\ell \in \Gamma_x(I)$ and let $(U_k)$ be a decreasing local base of neighborhoods at $\ell$. Letting $\varphi$ be a lower semicontinuous submeasure associated with $I$ as in (5) and considering that $\ell$ is an $I$-cluster point, we have $\varphi(A_k) = \infty$ for all $k \in \mathbb{N}$, where $A_k := \{ n : x_n \in U_k \}$.

Then, set $a_0 := 0$ and define an increasing sequence of integers $(a_k)$ which satisfies

$$\varphi(A_k \cap [a_{k-1}, a_k]) \geq k$$

for all $k$ (note that this is possible since $\varphi(A_k \setminus S) = \infty$ whenever $S$ is finite). At this point, set $A := \bigcup_k A_k \cap [a_{k-1}, a_k]$. It follows by the monotonicity of $\varphi$ that $\varphi(A) = \infty$, hence $A \not\in I$. Moreover, for each $k \in \mathbb{N}$, we have that $\{ n \in A : x_n \notin U_k \}$ is finite: indeed, if $n \in A_j \cap [a_{j-1}, a_j]$ for some $j \geq k$, then by construction $x_n \in U_j$, which is contained in $U_k$. Therefore $\lim_{n \to \infty, n \in A} x_n = \ell$, that is, $\ell \in \Lambda_x(I)$. \hfill $\square$

Since summable ideals are $F_\sigma$ P-ideals, see e.g. [7, Example 1.2.3], we obtain the following corollary which was proved in [14, Theorem 3.4]:

**Corollary 2.4.** Let $x$ be a real sequence and let $I$ be a summable ideal. Then $\Lambda_x(I)$ is closed.

It turns out that, within the class of analytic P-ideals, the property that the set of $I$-limit points is always closed characterizes the subclass of $F_\sigma$-ideals:

**Theorem 2.5.** Let $X$ be a first countable space which admits a non-trivial convergent sequence. Let also $I_x$ be an analytic P-ideal. Then the following are equivalent:

(i) $I_x$ is also an $F_\sigma$-ideal;
(ii) $\Lambda_x(I_x) = \Gamma_x(I_x)$ for all sequences $x$;
(iii) $\Lambda_x(I_x)$ is closed for all sequences $x$;
(iv) there does not exist a partition $\{ A_n : n \in \mathbb{N} \}$ of $\mathbb{N}$ such that $\| A_n \|_\varphi > 0$ for all $n$ and $\lim_n \| \bigcup_{k>n} A_k \|_\varphi = 0$.

**Proof.** (i) $\implies$ (ii) follows by Theorem 2.3 and (ii) $\implies$ (iii) is clear. (iii) $\implies$ (iv) By hypothesis, there exists a sequence $(\ell_n)$ converging to $\ell \in X$ such that $\ell_n \neq \ell$ for all $n$. Let us suppose that there exists a partition $\{ A_n : n \in \mathbb{N} \}$ of $\mathbb{N}$
such that \( \|A_n\|_\varphi > 0 \) for all \( n \) and \( \lim \| \bigcup_{n \geq k} A_n \|_\varphi = 0 \). Then, define the sequence \( x = (x_n) \) by \( x_n = \ell_i \) for all \( n \in A_i \). Then, we have that \( \{ \ell_n : n \in \mathbb{N} \} \subseteq \Lambda_x(\mathcal{I}_\varphi) \). On the other hand, since \( X \) is first countable Hausdorff, it follows that for all \( k \in \mathbb{N} \) there exists a neighborhood \( U_k \) of \( \ell \) such that
\[
\{ n : x_n \in U_k \} \subseteq \{ n : x_n = \ell_i \text{ for some } i \geq k \} = \bigcup_{n \geq k} A_n.
\]
Hence, by the monotonicity of \( \varphi \), we obtain \( 0 < \| \{ n : x_n \in U_k \} \|_\varphi \downarrow 0 \), i.e., \( u(\ell) = 0 \), which implies, thanks to Theorem 2.2, that \( \ell \notin \Lambda_x(\mathcal{I}_\varphi) \). In particular, \( \mathcal{I}_\varphi \) is not closed.

(iv) \( \implies \) (i) Lastly, assume that the ideal \( \mathcal{I}_\varphi \) is not an \( F_\sigma \)-ideal. According to the proof of [19, Theorem 3.4], cf. also [18, pp. 342–343], this is equivalent to the existence, for each given \( \varepsilon > 0 \), of some set \( M \subseteq \mathbb{N} \) such that \( 0 < \| M \|_\varphi \leq \varphi(M) < \varepsilon \). This allows to define recursively a sequence of sets \( (M_n) \) such that
\[
\| M_n \|_\varphi > \sum_{k \geq n+1} \varphi(M_k) > 0, \tag{6}
\]
for all \( n \) and, in addition, \( \sum_k \varphi(M_k) < \varphi(\mathbb{N}) \). Then, it is claimed that there exists a partition \( \{ A_n : n \in \mathbb{N} \} \) of \( \mathbb{N} \) such that \( \| A_n \|_\varphi > 0 \) for all \( n \) and \( \lim_n \| \bigcup_{k \geq n} A_k \|_\varphi = 0 \).

To this aim, set \( M_0 := \mathbb{N} \) and define \( A_n := M_{n-1} \setminus \bigcup_{k \geq n} M_k \) for all \( n \in \mathbb{N} \). It follows by the subadditivity and monotonicity of \( \varphi \) that
\[
\varphi(M_{n-1} \setminus \{ 1, \ldots, k \}) \leq \varphi(A_n \setminus \{ 1, \ldots, k \}) + \varphi \left( \bigcup_{k \geq n} M_k \right)
\]
for all \( n, k \in \mathbb{N} \); hence, by the lower semicontinuity of \( \varphi \) and (6),
\[
\| A_n \|_\varphi \geq \| M_{n-1} \|_\varphi - \varphi \left( \bigcup_{k \geq n} M_k \right) \geq \| M_{n-1} \|_\varphi - \sum_{k \geq n} \varphi(M_k) > 0
\]
for all \( n \in \mathbb{N} \). Finally, again by the lower semicontinuity of \( \varphi \), we get
\[
\| \bigcup_{k \geq n} A_k \|_\varphi = \| \bigcup_{k \geq n} M_k \|_\varphi \leq \varphi \left( \bigcup_{k \geq n} M_k \right) \leq \sum_{k \geq n} \varphi(M_k)
\]
which goes to 0 as \( n \to \infty \). This concludes the proof. \( \square \)

At this point, thanks to Theorem 2.2 and Theorem 2.5, observe that, if \( X \) is a first countable space which admits a non-trivial convergent sequence and \( \mathcal{I}_\varphi \) is an analytic \( P \)-ideal which is not \( F_\sigma \), then there exists a sequence \( \sigma \) such that \( \Lambda_x(\mathcal{I}_\varphi) \) is a non-closed \( F_\sigma \)-set. In this case, indeed, all the \( F_\sigma \)-sets can be obtained:

**Theorem 2.6.** Let \( X \) be a first countable space where all closed sets are separable and assume that there exists a non-trivial convergent sequence. Fix also an analytic \( P \)-ideal \( \mathcal{I}_\varphi \) which is not \( F_\sigma \) and let \( B \subseteq X \) be a non-empty \( F_\sigma \)-set. Then, there exists a sequence \( \sigma \) such that \( \Lambda_x(\mathcal{I}_\varphi) = B \).

**Proof.** Let \( (B_k) \) be a sequence of non-empty closed sets such that \( \bigcup_k B_k = B \). Let also \( \{ b_{k,n} : n \in \mathbb{N} \} \) be a countable dense subset of \( B_k \). Thanks to Theorem 2.5, there exists a partition \( \{ A_n : n \in \mathbb{N} \} \) of \( \mathbb{N} \) such that \( \| A_n \|_\varphi > 0 \) for all \( n \) and \( \lim_n \| \bigcup_{k \geq n} A_k \|_\varphi = 0 \).

Moreover, for each \( k \in \mathbb{N} \), set \( \theta_{k,0} := 0 \) and it is easily seen that there exists an increasing sequence of positive integers \( \{ \theta_{k,n} \} \) such that
\[
\varphi(A_k \cap (\theta_{k,n-1}, \theta_{k,n}]) \geq \frac{1}{2} \| A_k \setminus \{ 1, \ldots, \theta_{k,n-1} \} \|_\varphi = \frac{1}{2} \| A_k \|_\varphi
\]
for all $n$. Hence, setting $A_{k,n} := A_k \cap \bigcup_{m \in A_n} (\theta_{k,m-1}, \theta_{k,m}]$, we obtain that \{ $A_{k,n} : n \in \mathbb{N}$ \} is a partition of $A_k$ such that $\frac{1}{2} \|A_k\|_\varphi \leq \|A_{k,n}\|_\varphi \leq \|A_k\|_\varphi$ for all $n, k$.

At this point, let $x = (x_n)$ defined by $x_n = b_{k,m}$ whenever $n \in A_{k,m}$. Fix $\ell \in B$, then there exists $k \in \mathbb{N}$ such that $\ell \in B_k$. Let $(b_{k,m})_m$ be a sequence in $B_k$ converging to $\ell$. Thus, set $\tau_0 := 0$ and let $(\tau_n)_n$ be an increasing sequence of positive integers such that $\varphi(A_{k,m} \cap (\tau_{m-1}, \tau_m]) \geq \frac{1}{2m}\|A_{k,m}\|_\varphi$ for each $m$. Setting $A := \bigcup_{m} A_{k,m} \cap (\tau_{m-1}, \tau_m]$, it follows by construction that $\lim_{n \to \infty, n \in A} x_n = \ell$ and $\|A\|_\varphi \geq \frac{1}{4\|A_k\|_\varphi} > 0$. This shows that $B \subseteq \Lambda_\varphi(I_\varphi)$.

To complete the proof, fix $\ell \notin B$ and let us suppose for the sake of contradiction that there exists $A \subseteq \mathbb{N}$ such that $\lim_{n \to \infty, n \in A} x_n = \ell$ and $\|A\|_\varphi > 0$. For each $m \in \mathbb{N}$, let $U_m$ be an open neighborhood of $\ell$ which is disjoint from the closed set $B_1 \cup \cdots \cup B_m$. It follows by the subadditivity and the monotonicity of $\varphi$ that there exists a finite set $Y$ such that

$$\|A\|_\varphi \leq \|Y\|_\varphi + \|\{ n \in A : x_n \notin B_1 \cup \cdots \cup B_m \}\|_\varphi \leq \|\bigcup_{k>m} A_k\|_\varphi.$$ 

The claim follows by the arbitrariness of $m$ and the fact that $\lim_{m} \|\bigcup_{k>m} A_k\|_\varphi = 0$. □

Note that every analytic P-ideal without the Bolzano-Weierstrass property cannot be $F_\sigma$, see [8, Theorem 4.2]. Hence Theorem 2.6 applies to this class of ideals.

It was shown in [5, Theorem 2.8 and Theorem 2.10] that if $X$ is a topological space where all closed sets are separable, then for each $F_\sigma$-set $A$ and closed set $B$ there exist sequences $a = (a_n)$ and $b = (b_n)$ with values in $X$ such that $A_a = A$ and $\Gamma_b = B$.

As an application of Theorem 2.2, we prove that, in general, its stronger version with $a = b$ fails (e.g., there are no real sequences $x$ such that $A_\varphi = \{0\}$ and $\Gamma_x = \{0, 1\}$).

Here, a topological space $X$ is said to be locally compact if for every $x \in X$ there exists a neighborhood $U$ of $x$ such that its closure $\overline{U}$ is compact, cf. [6, Section 3.3].

**Theorem 2.7.** Let $x = (x_n)$ be a sequence taking values in a locally compact first countable space and fix an analytic P-ideal $I_\varphi$. Then each isolated $I_\varphi$-cluster point is also an $I_\varphi$-limit point.

**Proof.** Let us suppose for the sake of contradiction that there exists an isolated $I_\varphi$-cluster point, let us say $\ell$, which is not an $I_\varphi$-limit point. Let $(U_k)$ be a decreasing local base of open neighborhoods at $\ell$ such that $\overline{U_1}$ is compact. Let also $m$ be a sufficiently large integer such that $U_m \cap \Gamma_x(I_\varphi) = \{\ell\}$. Thanks to [6, Theorem 3.3.1] the underlying space is, in particular, regular, hence there exists an integer $r > m$ such that $\overline{U_r}$ is a compact contained in $U_m$. In addition, since $\ell$ is an $I_\varphi$-cluster point and it is not an $I_\varphi$-limit point, it follows by Theorem 2.2 that

$$0 < \|\{ n : x_n \in U_k \}\|_\varphi \downarrow u(\ell) = 0.$$ 

In particular, there exists $s \in \mathbb{N}$ such that $0 < \|\{ n : x_n \in U_s \}\|_\varphi < \|\{ n : x_n \in U_r \}\|_\varphi$.

Observe that $K := U_r \setminus U_s$ is a closed set contained in $\overline{U_1}$, hence it is compact. By construction we have that $K \cap \Gamma_x(I_\varphi) = \emptyset$. Hence, for each $z \in K$, there exists an open neighborhood $V_z$ of $z$ such that $V_z \subseteq U_m$ and $\{ n : x_n \in V_z \} \in I_\varphi$, i.e., $\|\{ n : x_n \in V_z \}\|_\varphi = 0$. It follows that $\bigcup_{z \in K} V_z$ is an open cover of $K$ which is contained in $U_m$. Since $K$ is compact, there exists a finite set $\{ z_1, \ldots, z_l \} \subseteq K$ for which

$$K \subseteq V_{z_1} \cup \cdots \cup V_{z_l} \subseteq U_m.$$ 

(7)
At this point, by the subadditivity of \( \varphi \), it easily follows that \( \| A \cup B \|_\varphi \leq \| A \|_\varphi + \| B \|_\varphi \) for all \( A, B \subseteq \mathbb{N} \). Hence we have

\[
\| \{ n : x_n \in K \} \|_\varphi \geq \| \{ n : x_n \in \bigcup_f \} \|_\varphi - \| \{ n : x_n \in \bigcup_r \} \|_\varphi \\
\geq \| \{ n : x_n \in U_r \} \|_\varphi - \| \{ n : x_n \in U_s \} \|_\varphi > 0.
\]

On the other hand, it follows by (7) that

\[
\| \{ n : x_n \in K \} \|_\varphi \leq \| \{ n : x_n \in \bigcup^t_i V_i \} \|_\varphi \leq \sum^t_i \| \{ n : x_n \in V_i \} \|_\varphi = 0.
\]

This contradiction concludes the proof. \( \square \)

The following corollary is immediate (we omit details):

**Corollary 2.8.** Let \( x \) be a real sequence for which \( \Gamma_x \) is a discrete set. Then \( \Lambda_x = \Gamma_x \).

3. **Joint Converse results**

We provide now a kind of converse of Theorem 2.7, specializing to the case of the ideal \( \mathcal{I}_0 \): informally, if \( B \) is a sufficiently smooth closed set and \( A \) is an \( F_\sigma \)-set containing the isolated points of \( B \), then there exists a sequence \( x \) such that \( \Lambda_x = A \) and \( \Gamma_x = B \).

To this aim, we need some additional notation: let \( d^* \), \( d_\ast \), and \( d \) be the upper asymptotic density, lower asymptotic density, and asymptotic density on \( \mathbb{N} \), resp.; in particular, \( \mathcal{I}_0 = \{ S \subseteq \mathbb{N} : d^*(S) = 0 \} \).

Given a topological space \( X \), the interior and the closure of a subset \( S \subseteq X \) are denoted by \( S^o \) and \( \overline{S} \), respectively; \( S \) is said to be regular closed if \( S = \overline{S^o} \). We let the Borel \( \sigma \)-algebra on \( X \) be \( \mathcal{B}(X) \). A Borel probability measure \( \mu : \mathcal{B}(X) \to [0,1] \) is said to be strictly positive whenever \( \mu(U) > 0 \) for all non-empty open sets \( U \). Moreover, \( \mu \) is atomless if, for each measurable set \( A \) with \( \mu(A) > 0 \), there exists a measurable subset \( B \subseteq A \) such that \( 0 < \mu(B) < \mu(A) \). Then, a sequence \( (x_n) \) taking values in \( X \) is said to be \( \mu \)-uniformly distributed whenever

\[
\mu(F) \geq d^*(\{ n : x_n \in F \}) \tag{8}
\]

for all closed sets \( F \), cf. [9, Section 491B].

**Theorem 3.1.** Let \( X \) be a separable metric space and \( \mu : \mathcal{B}(X) \to [0,1] \) be an atomless strictly positive Borel probability measure. Fix also sets \( A \subseteq B \subseteq C \subseteq X \) such that \( A \) is an \( F_\sigma \)-set, and \( B, C \) are closed sets such that: (i) \( \mu(B) > 0 \), (ii) the set \( S \) of isolated points of \( B \) is contained in \( A \), and (iii) \( B \setminus S \) is regular closed. Then there exists a sequence \( x \) taking values in \( X \) such that

\[
\Lambda_x = A, \ \Gamma_x = B, \ \text{and} \ \Lambda_x = C. \tag{9}
\]

**Proof.** Set \( F := B \setminus S \) and note that, by the separability of \( X \), \( S \) is at most countable. In particular, \( \mu(S) = 0 \), hence \( \mu(F) = \mu(B) > 0 \).

Let us assume for now that \( A \) is non-empty. Since \( A \) is an \( F_\sigma \)-set, there exists a sequence \( (A_k) \) of non-empty closed sets such that \( \bigcup_k A_k = A \). Considering that \( X \) is (hereditarily) second countable, then every closed set is separable. Hence, for each \( k \in \mathbb{N} \), there exists a countable set \( \{ a_{k,n} : n \in \mathbb{N} \} \subseteq A_k \) with closure \( A_k \). Considering
that $F$ is a separable metric space on its own right and that the (normalized) restriction $\mu_F$ of $\mu$ on $F$, that is,

$$\mu_F : \mathcal{B}(F) \to [0, 1] : Y \mapsto \frac{1}{\mu(F)} \mu(Y)$$

is a Borel probability measure, it follows by [9, Exercise 491Xw] that there exists a $\mu_F$-uniformly distributed sequence $(b_n)$ which takes values in $F$ and satisfies (8). Lastly, let $\{c_n : n \in \mathbb{N}\}$ be a countable dense subset of $C$.

At this point, let $\mathcal{C}$ be the set of non-zero integer squares and note that $d(\mathcal{C}) = 0$. For each $k \in \mathbb{N}$ define $\mathcal{A}_k := \{2^kn : n \in \mathbb{N} \setminus 2\mathbb{N}\} \setminus \mathcal{C}$ and $\mathcal{B} := \mathbb{N} \setminus (2\mathbb{N} \cup \mathcal{C})$. It follows by construction that $\{\mathcal{A}_k : k \in \mathbb{N}\} \cup \{\mathcal{B}, \mathcal{C}\}$ is a partition of $\mathbb{N}$. Moreover, each $\mathcal{A}_k$ admits asymptotic density and

$$\lim_{n \to \infty} d\left(\bigcup_{k \geq n} \mathcal{A}_k\right) = 0.$$  

Finally, for each positive integer $k$, let $\{\mathcal{A}_{k,m} : m \in \mathbb{N}\}$ be the partition of $\mathcal{A}_k$ defined by $\mathcal{A}_{k,1} := \mathcal{A}_k \cap \bigcup_{n \in \mathcal{A}_k \cup \mathcal{B},|n| \in 2, n \lesssim k}$ and $\mathcal{A}_{k,m} := \mathcal{A}_k \cap \bigcup_{n \in \mathcal{A}_m} \{n, (n + 1)!\}$ for all integers $m \geq 2$. Then, it is easy to check that

$$d^*(\mathcal{A}_{k,1}) = d^*(\mathcal{A}_{k,2}) = \ldots = d(\mathcal{A}_k) = 2^{-k-1}.$$  

Hence, define the sequence $x = (x_n)$ by

$$x_n = \begin{cases} 
  a_{k,m} & \text{if } n \in \mathcal{A}_{k,m}, \\
  b_m & \text{if } n \text{ is the } m\text{-th term of } \mathcal{B}, \\
  c_m & \text{if } n \text{ is the } m\text{-th term of } \mathcal{C}.
\end{cases}$$

To complete the proof, let us verify that (9) holds true:

**Claim (i):** $L_x = C$. Note that $x_n \in C$ for all $n \in \mathbb{N}$. Since $C$ is closed by hypothesis, then $L_x \subseteq C$. On the other hand, if $\ell \in C$, then there exists a sequence $(c_n)$ taking values in $C$ converging (in the ordinary sense) to $\ell$. It follows by the definition of $(x_n)$ that there exists a subsequence $(x_{n_k})$ converging to $\ell$, i.e., $C \subseteq L_x$.

**Claim (ii):** $\Gamma_x = B$. Fix $\ell \notin B$ and let $U$ be an open neighborhood of $\ell$ disjoint from $B$ (this is possible since, in the opposite, $\ell$ would belong to $\overline{B} = B$). Then, $\{n : x_n \in U\} \subseteq \mathcal{C}$, which implies that $\Gamma_x \subseteq B$.

Note that the Borel probability measure $\mu_F$ defined in (10) is clearly atomless. Moreover, given an open set $U \subseteq X$ with non-empty intersection with $F$, then $U \cap F^c \neq \emptyset$: indeed, in the opposite, we would have $F^c \subseteq U^c$, which is closed, hence $F = \overline{F^c} \subseteq U^c$, contradicting our hypothesis. This proves that every non-empty open set $V$ (relative to $F$) contains a non-empty open set of $X$. Therefore $\mu_F$ is also strictly positive. With these premises, fix $\ell \in F$ and let $V$ be a open neighborhood of $\ell$ (relative to $F$). Since $(b_n)$ is $\mu_F$-uniformly distributed and $\mu_F$ is strictly positive, it follows by (8) that

$$0 < \mu_F(V) = 1 - \mu_F(V^c) \leq 1 - d^*(\{n : b_n \in V^c\})$$

$$= d_x(\{n : b_n \in V\}) \leq d^*(\{n : b_n \in V\}).$$

Since $d(\mathcal{B}) = 1/2$, we obtain by standard properties of $d^*$ that

$$d^*(\{n : x_n \in V\}) \geq d^*(\{n \in \mathcal{B} : x_n \in V\}) = \frac{1}{2} d^*(\{n : b_n \in V\}) > 0.$$
We conclude by the arbitrariness of $V$ and $\ell$ that $F \subseteq \Gamma_x$.

Hence, we may only to show that $S \subseteq \Gamma_x$. To this aim, fix $\ell \in S$, thus $\ell$ is also an isolated point of $A$. Hence, there exist $k, m \in \mathbb{N}$ such that $a_{k,m} = \ell$. We conclude that $d^*\{\{n : x_n \in U\}\} \geq d^*\{\{n : x_n = \ell\}\} \geq d(\mathscr{A}_k) > 0$ for each neighborhood $U$ of $\ell$. Therefore $B = F \cup S \subseteq \Gamma_x$.

**Claim (iii):** $\Lambda_x = A$. Fix $\ell \in A$, hence there exists $k \in \mathbb{N}$ for which $\ell$ belongs to the (non-empty) closed set $A_k$. Since $\{a_{k,n} : n \in \mathbb{N}\}$ is dense in $A_k$, there exists a sequence $(a_{k,r_m} : m \in \mathbb{N})$ converging to $\ell$. Recall that $x_n = a_{k,r_m}$ whenever $n \in \mathscr{A}_{k,r_m}$ for each $m \in \mathbb{N}$. Set by convenience $\theta_0 := 0$ and define recursively an increasing sequence of positive integers $(\theta_m)$ such that $\theta_m$ is an integer greater than $\theta_{m-1}$ for which

$$d^*\left(\mathscr{A}_{k,r_m} \cap (\theta_{m-1}, \theta_m]\right) \geq \frac{d(\mathscr{A}_k)}{2} = 2^{-k-2}.$$ 

Then, setting $A := \bigcup_m \mathscr{A}_{k,r_m} \cap (\theta_{m-1}, \theta_m]$, we obtain that the subsequence $(x_n : n \in A)$ converges to $\ell$ and $d^*(A) > 0$. In particular, $A \subseteq \Lambda_x$.

On the other hand, it is known that $\Lambda_x \subseteq \Gamma_x$, see e.g. [10]. If $A = B$, it follows by Claim (ii) that $\Lambda_x \subseteq A$ and we are done. Otherwise, fix $\ell \in B \setminus A = F \setminus A$ and let us suppose for the sake of contradiction that there exists a subsequence $(x_{n_k})$ such that $\lim_k x_{n_k} = \ell$ and $d^*\{n_k : k \in \mathbb{N}\} > 0$. Fix a real $\varepsilon > 0$. Then, thanks to (11), there exists a sufficient large integer $n_0$ such that $d\left(\bigcup_{k>n_0} \mathscr{A}_k\right) \leq \varepsilon$. In addition, since $F$ is a metric space and $\mu_F$ is atomless and strictly positive (see Claim (ii)), we have

$$\lim_{n \to \infty} \mu_F(V_n) = \mu_F(\{\ell\}) = 0,$$

where $V_n$ is the open ball (relative to $F$) with center $\ell$ and radius $1/n$. Hence, there exists a sufficiently large integer $m'$ such that $0 < \mu_F(V_{m'}) \leq \varepsilon$. In addition, there exists an integer $m''$ such that $V_{m''}$ is disjoint from the closed set $A_1 \cup \cdots \cup A_{n_0}$. Then, set $V := V_{m''}$, where $m$ is an integer greater than $\max(m', m'')$ such that $\mu_F(V) < \mu_F(V_{\max(m', m'')})$. In particular, by the monotonicity of $\mu_F$, we have

$$0 < \mu_F(V) \leq \mu_F(V_{m'}) \leq \mu_F(V_{m''}) \leq \varepsilon. \quad (13)$$

At this point, observe there exists a finite set $Y$ such that

$$\{n_k : k \in \mathbb{N}\} = \{n_k : x_{n_k} \in V\} \cup Y$$

$$\subseteq \left(\bigcup_{k>n_0} \mathscr{A}_k\right) \cup \{n \in \mathscr{B} : x_n \in V\} \cup \mathscr{C} \cup Y.$$ 

Therefore, by the subadditivity of $d^*$, (8), and (13), we obtain

$$d^*\{n_k : k \in \mathbb{N}\} \leq \varepsilon + d^*\{n \in \mathscr{B} : x_n \in V\} \leq \varepsilon + d^*\{n \in \mathscr{B} : b_n \in V\} \leq \varepsilon + d^*\{n \in \mathscr{B} : b_n \in \bar{V}\} \leq \varepsilon + \mu_F(V) \leq 2\varepsilon.$$

It follows by the arbitrariness of $\varepsilon$ that $d\{n_k : k \in \mathbb{N}\} = 0$, i.e., $\Lambda_x \subseteq A$.

To complete the proof, assume now that $A = \emptyset$. In this case, note that necessarily $S = \emptyset$, and it is enough to replace (12) with

$$x_n = \begin{cases} \frac{b_n - \lfloor \sqrt{n} \rfloor}{\sqrt{n}} & \text{if } n \notin \mathscr{C}, \\ \frac{c\sqrt{n}}{n} & \text{if } n \notin \mathscr{C}. \end{cases}$$

Then, it can be shown with a similar argument that $\Lambda_x = \emptyset$, $\Gamma_x = B$, and $L_x = C$. \qed
It is worth noting that Theorem 3.1 cannot be extended to the whole class of analytic P-ideals. Indeed, it follows by Theorem 2.3 that if \( I \) is an \( F_\sigma \) ideal on \( \mathbb{N} \) then the set of \( I \)-limit points is closed set, cf. also Theorem 3.4 below.

In addition, the regular closedness of \( B \setminus S \) is essential in the proof of Theorem 3.1. On the other hand, there exist real sequences \( x \) such that \( \Gamma_x \) is the Cantor set \( C \) (which is a perfect set but not regular closed):

**Example 3.2.** Given a real \( r \in [0, 1) \) and an integer \( b \geq 2 \), we write \( r \) in base \( b \) as \( \sum_n a_n/b^n \), where each \( a_n \) belongs to \( \{0, 1, \ldots, b-1\} \) and \( a_n = \zeta \) for all sufficiently large \( n \) only if \( \zeta = 0 \). This representation is unique.

Let \( x = (x_n) \) be the sequence \((0, 0, 1, 0, \frac{1}{3}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, \ldots)\). This sequence is uniformly distributed in \([0, 1]\), i.e., \( d(\{n : x_n \in [a, b]\}) = b-a \) for all \( 0 \leq a < b \leq 1 \), and \( \Gamma_x = [0, 1] \), see e.g. [10, Example 4]. Let also \( T : [0, 1] \to C \) be the injection defined by \( r \mapsto T(r) \), where if \( r = \sum_n a_n/2^n \in [0, 1) \) in base \( 2 \) then \( T(r) = \sum_n 2a_n/3^n \) in base \( 3 \), and \( 1 \mapsto 1 \).

Observe that \( C \setminus T([0, 1)) \) is the set of points of the type \( 2(1/3^n + \cdots + 1/3^{n_k-1}) + 1/3^{n_k} \), for some non-negative integers \( n_1 < \cdots < n_k \); in particular, \( T([0, 1]) = C \).

Since the sequence \( T(x) := (T(x_n)) \) takes values in the closed set \( C \), it is clear that \( \Gamma_{T(x)} \subseteq C \). On the other hand, fix \( \ell \in T([0, 1)) \) with representation \( \sum_n 2a_n/3^n \) in base \( 3 \), where \( a_n \in \{0, 1\} \) for all \( n \). For each \( k \), let \( U_k \) be the open ball with center \( \ell \) and radius \( 1/3^k \). It follows that

\[
\{n : T(x_n) \in U_k\} \geq \left\{ n : T(x_n) \in \left[ \frac{2a_1}{3} + \cdots + \frac{2a_k}{3^k}, \frac{2a_1}{3} + \cdots + \frac{2a_k}{3^k} + \frac{1}{3^k} \right) \right\}.
\]

Since \((x_n)\) is equidistributed, then \( d^*(\{n : T(x_n) \in U_k\}) \geq 1/2^k \) for all \( k \). In particular, \( \Gamma_{T(x)} \) is a closed set containing \( T([0, 1]) \), therefore \( \Gamma_{T(x)} = C \).

Finally, we provide a sufficient condition for the existence of an atomless strictly positive Borel probability measure:

**Corollary 3.3.** Let \( X \) be a Polish space without isolated points and fix sets \( A \subseteq B \subseteq C \subseteq X \) such that \( A \) is an \( F_\sigma \)-set, \( B \neq \emptyset \) is regular closed, and \( C \) is closed. Then, there exists a sequence \( x \) taking values in \( X \) which satisfies (9).

**Proof.** First, observe that the restriction \( \lambda \) of the Lebesgue measure \( \lambda \) on the set \( \mathcal{F} := (0, 1) \setminus \mathbb{Q} \) is an atomless strictly positive Borel probability measure. Thanks to [6, Exercise 6.2.A(e)], \( X \) contains a dense subspace \( D \) which is homeomorphic to \( \mathbb{R} \setminus \mathbb{Q} \), which is turn is homeomorphic to \( \mathcal{F} \), let us say through \( \eta : D \to \mathcal{F} \). This embedding can be used to transfer the measure \( \lambda \) to the target space by setting

\[
\mu : B(X) \to [0, 1] : Y \mapsto \lambda(\eta(Y \cap D)).
\]

Lastly, since \( B \) is non-empty closed regular, then it has no isolated points and contains an open set \( U \) of \( X \). In particular, considering that \( \eta \) is an open map, we get by (14) that \( \mu(U) = \lambda(\eta(U \cap D)) > 0 \). The claim follows by Theorem 3.1. \( \Box \)

Note that, in general, the condition \( B \neq \emptyset \) cannot be dropped: indeed, it follows by [5, Theorem 2.14] that, if \( X \) is compact, then every sequence \((x_n)\) admits at least one statistical cluster point.
We conclude with another converse result related to ideals $\mathcal{I}$ of the type $F_\sigma$ (recall that, thanks to Theorem 2.3, every $\mathcal{I}$-limit point is also an $\mathcal{I}$-cluster point):

**Theorem 3.4.** Let $X$ be a first countable space where all closed sets are separable and let $\mathcal{I} \neq \Fin$ be an $F_\sigma$-ideal. Fix also closed sets $B, C \subseteq X$ such that $\emptyset \neq B \subseteq C$. Then there exists a sequence $x$ such that $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I}) = B$ and $\mathcal{I}_x = C$.

**Proof.** By hypothesis, there exists an infinite set $I \in \mathcal{I}$. Let $\varphi$ be a lower semicontinuous submeasures associated to $\mathcal{I}$ as in (5). Let $\{b_n : n \in \mathbb{N}\}$ and $\{c_n : n \in \mathbb{N}\}$ be countable dense subsets of $B$ and $C$, respectively. In addition, set $m_0 := 0$ and let $(m_k)$ be an increasing sequence of positive integers such that $\varphi((\mathbb{N} \setminus I) \cap (m_{k-1}, m_k]) \geq k$ for all $k$ (note that this is possible since $\varphi(\mathbb{N} \setminus I) = \infty$ and $\varphi$ is a lower semicontinuous submeasure). At this point, given a partition $\{H_n : n \in \mathbb{N}\}$ of $\mathbb{N} \setminus I$, where each $H_n$ is infinite, we set

$$M_k := (\mathbb{N} \setminus I) \cap \bigcup_{n \in H_k} (m_{n-1}, m_n]$$

for all $k \in \mathbb{N}$. Then, it is easily checked that $\{M_k : k \in \mathbb{N}\}$ is a partition of $\mathbb{N} \setminus I$ with $M_k \notin \mathcal{I}$ for all $k$, and that the sequence $(x_n)$ defined by

$$x_n = \begin{cases} b_k & \text{if } n \in M_k, \\ c_k & \text{if } n \text{ is the } k\text{-th term of } I. \end{cases}$$

satisfies the claimed conditions. \hfill \square

In particular, Theorem 2.6 and Theorem 3.4 fix a gap in a result of Das [3, Theorem 3] and provide its correct version.

4. **Concluding remarks**

In this last section, we are interested in the topological nature of the set of $\mathcal{I}$-limit points when $\mathcal{I}$ is neither $F_\sigma$- nor analytic $\mathcal{P}$-ideal.

Let $\mathcal{N}$ be the set of strictly increasing sequences of positive integers. Then $\mathcal{N}$ is a Polish space, since it is closed subspace of the Polish space $\mathbb{N}^{\mathbb{N}}$ (equipped with the product topology of the discrete topology on $\mathbb{N}$). Let also $x = (x_n)$ be a sequence taking values in a first countable regular space $X$ and fix an arbitrary ideal $\mathcal{I}$ on $\mathbb{N}$. For each $\ell \in X$, let $(U_{\ell,m})$ be a decreasing local base of open neighborhoods at $\ell$. Then, $\ell$ is an $\mathcal{I}$-limit point of $x$ if and only if there exists a sequence $(n_k) \in \mathcal{N}$ such that

$$\{n_k : k \in \mathbb{N}\} \notin \mathcal{I} \quad \text{and} \quad \{n : x_n \notin U_{\ell,m}\} \in \Fin \text{ for all } m. \quad (15)$$

Set $\mathcal{I}^c := \mathcal{P}(\mathbb{N}) \setminus \mathcal{I}$ and define the continuous function $\psi : \mathcal{N} \to \{0,1\}^{\mathbb{N}} : (n_k) \mapsto \chi_{\{n_k : k \in \mathbb{N}\}}$, where $\chi_S$ is the characteristic function of a set $S \subseteq \mathbb{N}$. Moreover, define

$$\zeta_m : \mathcal{N} \times X \to \{0,1\}^{\mathbb{N}} : (n_k) \times \ell \mapsto \chi_{\{n : x_n \notin U_{\ell,m}\}}$$

for each $m$. Hence, it easily follows by (15) that

$$\Lambda_x(\mathcal{I}) = \pi_X \left( \bigcap_m (\psi^{-1}(I^c) \times X \cap \zeta_m^{-1}(\Fin)) \right),$$

where $\pi_X : \mathcal{N} \times X \to X$ stands for the projection on $X$.

**Proposition 4.1.** Let $x = (x_n)$ be a sequence taking values in a first countable regular space $X$ and let $\mathcal{I}$ be a co-analytic ideal. Then $\Lambda_x(\mathcal{I})$ is analytic.
Proof. For each \((n_k) \in \mathcal{N}\) and \(\ell \in X\), the sections \(\zeta_m((n_k), \cdot)\) and \(\zeta_m(\cdot, \ell)\) are continuous. Hence, thanks to [20, Theorem 3.1.30], each function \(\zeta_m\) is Borel measurable. Since \(\text{Fin}\) is an \(F_\sigma\)-set, we obtain that each \(\zeta_m^{-1}(\text{Fin})\) is Borel. Since \(\text{Fin}\) is a co-analytic ideal and \(\psi\) is continuous, it follows that \(\psi^{-1}(I) \times X\) is an analytic subset of \(\mathcal{N} \times X\). Therefore \(\Lambda_x(I)\) is the projection on \(X\) of the analytic set \(\bigcap_m (\psi^{-1}(I^c) \times X \cap \zeta_m^{-1}(\text{Fin}))\), which proves the claim. \(\Box\)

The situation is much different for maximal ideals, i.e., ideals which are maximal with respect to inclusion. In this regard, we recall if \(I\) is a maximal ideal then every bounded real sequence \(x\) is \(I\)-convergent, i.e., there exists \(\ell \in \mathbb{R}\) such that \(\{n : |x_n - \ell| \geq \varepsilon\} \in I\) for every \(\varepsilon > 0\), cf. [12, Theorem 2.2].

Let \(B(a, r)\) the open ball with center \(a\) and radius \(r\) in a given metric space \((X, d)\), and denote by \(\text{diam} S\) the diameter of a non-empty set \(S \subseteq X\), namely, \(\sup_{a,b \in S} d(a, b)\). Then, the metric space is said to be smooth if

\[
\lim_{k \to \infty} \sup_{a \in X} \text{diam}\left(B(a, 1/k)\right) = 0. \tag{16}
\]

Note that (16) holds if, e.g., the closure of each open ball \(B(a, r)\) coincides with the corresponding closed ball \(\{b \in X : d(a, b) \leq r\}\).

**Proposition 4.2.** Let \(x\) be a sequence taking values in a smooth compact metric space \(X\) and let \(I\) be a maximal ideal. Then \(x\) has exactly one \(I\)-cluster point. In particular, \(\Lambda_x(I)\) is closed.

**Proof.** Since \(X\) is a compact metric space, then \(X\) is totally bounded, i.e., for each \(\varepsilon > 0\) there exist finitely many open balls with radius \(\varepsilon\) covering \(X\). Moreover, it is well known that an ideal \(I\) is maximal if and only if either \(A \in I\) or \(A^c \in I\) for every \(A \subseteq \mathbb{N}\).

Hence, fix \(k \in \mathbb{N}\), let \(\{B_{k,1}, \ldots, B_{k,m_k}\}\) be a cover of \(X\) of open balls with radius \(1/k\), and define \(\mathcal{G}_{k,i} := \{n : x_n \in C_{k,i}\}\) for each \(i \leq m_k\), where \(C_{k,i} := B_{k,i} \setminus \bigcup_{j=1}^{i-1} B_{k,j}\) and \(B_{k,0} := \emptyset\). Considering that \(\{\mathcal{G}_{k,1}, \ldots, \mathcal{G}_{k,m_k}\}\) is a partition of \(\mathbb{N}\), it follows by the above observations that there exists a unique \(i_k \in \{1, \ldots, m_k\}\) for which \(\mathcal{G}_{k,i_k} \notin I\).

At this point, let \((G_k)\) be the decreasing sequence of closed sets defined by

\[
G_k := \overline{C_{1,i_1} \cap \cdots \cap C_{k,i_k}}
\]

for all \(k\). Note that each \(G_k\) is non-empty, the diameter of \(G_k\) (which is contained in \(B_{k,i_k}\)) goes to \(0\) as \(k \to \infty\), and \(\{n : x_n \in G_k\} \notin I\) for all \(k\). Since \(X\) is a compact metric space, then \(\bigcap_k G_k\) is a singleton \(\{\ell\}\). Considering that every open ball with center \(\ell\) contains some \(G_k\) with \(k\) sufficiently large, it easily follows that \(\Gamma_x(I) = \{\ell\}\). In particular, since each \(I\)-limit point is also an \(I\)-cluster point, we conclude that \(\Lambda_x(I)\) is either empty or the singleton \(\{\ell\}\). \(\Box\)

**Corollary 4.3.** An ideal \(I\) is maximal if and only if every real sequence \(x\) has at most one \(I\)-limit point.

**Proof.** First, let us assume that \(I\) is a maximal ideal. Let us suppose that there exists \(k > 0\) such that \(A_k := \{n : |x_n| > k\} \in I\) and define a sequence \(y = (y_n)\) by \(y_n = k\) if \(n \in A_k\) and \(y_n = x_n\) otherwise. Then, it follows by [3, Theorem 4] and Proposition 4.2 that there exists \(\ell \in \mathbb{R}\) such that \(\Lambda_x(I) = \Lambda_y(I) \subseteq \Gamma_y(I) = \{\ell\}\).
Now, assume that $A_k^n \in \mathcal{I}$ for all $k \in \mathbb{N}$. Hence, letting $z = (z_n)$ be the sequence defined by $z_n = x_n$ if $n \in A_k$ and $z_n = k$ otherwise, we obtain

$$\Lambda_x(\mathcal{I}) = \Lambda_x(\mathcal{I}) \subseteq \mathbb{L}_z \subseteq \mathbb{R} \setminus (-k, k).$$

Therefore, it follows by the arbitrariness of $z$ defined by $\mathcal{I}$.

Conversely, let us assume that $\mathcal{I}$ is not a maximal ideal. Then there exists $A \subseteq \mathbb{N}$ such that $A \notin \mathcal{I}$ and $A^c \notin \mathcal{I}$. Then, the sequence $(x_n)$ defined by $x_n = \chi_{A(n)}$ for each $n$ has two $\mathcal{I}$-limit points.

We conclude by showing that there exists an ideal $\mathcal{I}$ and a real sequence $x$ such that $\Lambda_x(\mathcal{I})$ is not an $F_n$-set.

**Example 4.4.** Fix a partition $\{P_m : m \in \mathbb{N}\}$ of $\mathbb{N}$ such that each $P_m$ is infinite. Then, define the ideal

$$\mathcal{I} := \{A \subseteq \mathbb{N} : \{m : A \cap 2^m \notin \text{Fin} \} \in \text{Fin}\},$$

which corresponds to the Fubini product $\text{Fin} \times \text{Fin}$ on $\mathbb{N}^2$ (it is known that $\mathcal{I}$ is a $\sigma_{\delta\sigma}$-ideal and it is not a $\sigma$-ideal). Given a real sequence $x = (x_n)$, let us denote by $x \upharpoonright P_m$ the subsequence $(x_n : n \in P_m)$. Hence, a real $\ell$ is an $\mathcal{I}$-limit point of $x$ if and only if there exists a subsequence $(x_{n_k})$ converging to $\ell$ such that $\{n_k : k \in \mathbb{N}\} \cap P_m$ is infinite for infinitely many $m$. Moreover, for each $m$ of this type, the subsequence $(x_{n_k}) \upharpoonright P_m$ converges to $\ell$. It easily follows that

$$\Lambda_x(\mathcal{I}) = \bigcap_{k \geq m} \mathbb{L}_x \upharpoonright P_m,$$

(17)

(In particular, since each $\mathbb{L}_x \upharpoonright P_m$ is closed, then $\Lambda_x(\mathcal{I})$ is an $F_{\sigma\delta}$-set.)

At this point, let $(q_t : t \in \mathbb{N})$ be the sequence $(9/1, 1/1, 9/2, 1/2, 2/1, 9/3, 1/3, 2/3, \ldots)$, where $q_t := a_t/b_t$ for each $t$, and note that $\{q_t : t \in \mathbb{N}\} = \mathbb{Q} \cap [0, 1]$. It follows by construction that $a_t \leq b_t$ for all $t$ and $b_t = \sqrt{2t(1 + o(1))}$ as $t \to \infty$. In particular, if $m$ is a sufficiently large integer, then

$$\min_{t \leq m} |q_t - q_m| \geq \left(\frac{1}{\sqrt{2m(1 + o(1))}}\right)^2 > \frac{1}{3m}.$$  

(18)

Lastly, for each $m \in \mathbb{N}$, define the closed set

$$C_m := [0, 1] \cap \bigcap_{t \leq m} \left(q_t - \frac{1}{2m}q_t + \frac{1}{2m}\right)^c.$$

We obtain by (18) that, if $m$ is sufficiently large, let us say $\geq k_0$, then

$$C_m \cup C_{m+1} = [0, 1] \cap \bigcap_{t \leq m} \left(q_t - \frac{1}{2m+1}q_t + \frac{1}{2m+1}\right)^c.$$

It follows by induction that

$$C_m \cup C_{m+1} \cup \cdots \cup C_{m+n} = [0, 1] \cap \bigcap_{t \leq m} \left(q_t - \frac{1}{2m+n}q_t + \frac{1}{2m+n}\right)^c.$$

for all $n \in \mathbb{N}$. In particular, $\bigcup_{m \geq k} C_m = [0, 1] \setminus \{q_1, \ldots, q_k\}$ whenever $k \geq k_0$. 
Let $x$ be a real sequence such that each $\{x_n : n \in P_m\}$ is a dense subset of $C_m$. Therefore, it follows by (17) that

$$\Lambda_x(I) = \bigcap_{k \geq k_0} \bigcup_{m \geq k} C_m = \bigcap_{k \geq k_0} [0,1] \setminus \{q_1, \ldots, q_k\} = [0,1] \setminus \mathbb{Q}.$$ 

On the other hand, if a rational $q_t$ belongs to $\Lambda_x(I)$, then $q_t \in \bigcup_{m \geq k} C_m$ for all $k \in \mathbb{N}$, which is impossible whenever $k \geq t$. This proves that $\Lambda_x(I) = [0,1] \setminus \mathbb{Q}$, which is not an $F_\sigma$-set.

We leave as an open question to determine whether there exists a real sequence $x$ and an ideal $I$ such that $\Lambda_x(I)$ is not Borel measurable.

Acknowledgments

The authors are grateful to Szymon Głąb (Łódź University of Technology, PL) for suggesting to investigate the ideal $\text{Fin} \times \text{Fin}$ in Example 4.4.

References

[1] M. Balcerzak, K. Dems, and A. Komisarski, Statistical convergence and ideal convergence for sequences of functions, J. Math. Anal. Appl. 328 (2007), no. 1, 715–729.
[2] J. Connor and J. Kline, On statistical limit points and the consistency of statistical convergence, J. Math. Anal. Appl. 197 (1996), no. 2, 392–399.
[3] P. Das, Some further results on ideal convergence in topological spaces, Topology Appl. 159 (2012), no. 10-11, 2621–2626.
[4] P. Das and B. K. Lahiri, $I$ and $I^*$-convergence in topological spaces, Math. Bohem. 130 (2005), no. 2, 153–160.
[5] G. Di Maio and L. D. R. Kočinac, Statistical convergence in topology, Topology Appl. 156 (2008), no. 1, 28–45.
[6] R. Engelking, General topology, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989. Translated from the Polish by the author.
[7] I. Farah, Analytic quotients: theory of liftings for quotients over analytic ideals on the integers, Mem. Amer. Math. Soc. 148 (2000), no. 702, xvi+177.
[8] R. Filipów, N. Mrożek, I. Reclaw, and P. Szuca, Ideal convergence of bounded sequences, J. Symbolic Logic 72 (2007), no. 2, 501–512.
[9] D. H. Fremlin, Measure theory. Vol. 4, Torres Fremlin, Colchester, 2006. Topological measure spaces. Part I, II, Corrected second printing of the 2003 original.
[10] J. A. Fridy, Statistical limit points, Proc. Amer. Math. Soc. 118 (1993), no. 4, 1187–1192.
[11] J. A. Fridy and C. Orhan, Statistical limit superior and limit inferior, Proc. Amer. Math. Soc. 125 (1997), no. 12, 3625–3631.
[12] P. Kostyrko, M. Mačaj, T. Šalát, and M. Sleziak, $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points, Math. Slovaca 55 (2005), no. 4, 443–464.
[13] P. Kostyrko, M. Mačaj, T. Šalát, and O. Strauch, On statistical limit points, Proc. Amer. Math. Soc. 129 (2001), no. 9, 2647–2654.
[14] P. Leonetti, Invariance of ideal limit points, preprint, last updated: Jul 24, 2017 (arXiv:1707.07475).
[15] M. A. Mamedov and S. Pehlivan, Statistical cluster points and turnpike theorem in nonconvex problems, J. Math. Anal. Appl. 256 (2001), no. 2, 686–693.
[16] K. Mazur, $F_\sigma$-ideals and $\omega_1\omega_1$-gaps in the Boolean algebras $P(\omega)/I$, Fund. Math. 138 (1991), no. 2, 103–111.
[17] H. I. Miller, A measure theoretical subsequence characterization of statistical convergence, Trans. Amer. Math. Soc. 347 (1995), no. 5, 1811–1819.
[18] S. Solecki, Analytic ideals, Bull. Symbolic Logic 2 (1996), no. 3, 339–348.
[19] ______, Analytic ideals and their applications, Ann. Pure Appl. Logic 99 (1999), no. 1-3, 51–72.
[20] S. M. Srivastava, A course on Borel sets, Graduate Texts in Mathematics, vol. 180, Springer-Verlag, New York, 1998.

Institute of Mathematics, Łódź University of Technology, ul. Wólczańska 215, 93-005 Łódź, Poland
E-mail address: marek.balcerzak@p.lodz.pl

Department of Statistics, Università “L. Bocconi”, via Roentgen 1, 20136 Milan, Italy
E-mail address: leonetti.paolo@gmail.com
URL: https://sites.google.com/site/leonettipaolo/