Non-universal suppression of the excitation gap in chaotic Andreev billiards:
Superconducting terminals as sensitive probes for scarred states

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When a quantum-chaotic normal conductor is coupled to a superconductor, random-matrix theory predicts that a gap opens up in the excitation spectrum which is of universal size $E_g^{\text{RMT}} \approx 0.3h/t_D$, where $t_D$ is the mean scattering time between Andreev reflections. We show that a scarred state of long lifetime $t_S \gg t_D$ suppresses the excitation gap over a window $\Delta E \approx 2E_g^{\text{RMT}}$ which can be much larger than the narrow resonance width $\Gamma_S = h/t_S$ of the scar in the normal system. The minimal value of the excitation gap within this window is given by $\Gamma_S/2 \ll E_g^{\text{RMT}}$. Hence the scarred state can be detected over a much larger energy range than it is the case when the superconducting terminal is replaced by a normal one.

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Typical wavefunctions in classically chaotic quantum systems display a structure-less speckle pattern which is well captured by Berry’s random plain-wave model. These quantum systems obey a strong degree of universality which makes them amenable to a random-matrix theory (RMT) description. Recent advances of semiclassical techniques link the random-matrix universality to the condition of well-developed wave chaos. Two sources of deviations from this condition have been identified. The first source is rooted in the quasi-deterministic quantum-to-classical correspondence of the dynamics for times shorter than the Ehrenfest time $t_E$. The second source is the scarring of wavefunctions along the trajectories of periodic orbits. The complementary nature of both sources of non-universality becomes apparent when the system is opened up, so that typical states acquire a width of order $E_{\text{Th}} = h/t_D$ (the Thouless energy), where $t_D$ is the classical mean dwell time in the system. Quantum-to-classical correspondence then induces anomalously short-lived states with resonance width much larger than $E_{\text{Th}}$. Scarred states, on the other hand, can acquire lifetimes $t_S$ much larger than the typical lifetime $t_D$ when the openings are not visited by the periodic orbits which support the scar. The resulting narrow resonances of width $\Gamma_S = h/t_S$ have been observed experimentally and numerically in a large variety of physical systems, such as lateral quantum dots, microwave cavities, resonant tunnel diodes, micro-optical lasers, and surface waves.

In general, scarred states are a rare fraction of all states of a system and their influence can be ignored for almost all energies, with the exception of the resonance intervals of size $\Gamma_S$. The purposes of this paper is to demonstrate that scarred states can be detected over much larger energy ranges when the normal opening is replaced by a superconducting terminal so that the system forms a so-called Andreev billiard. Chaotic Andreev billiards have attracted much attention over the past years because they display a gap $E_g$ in the excitation spectrum which is induced by the superconductor due to the dynamical process of Andreev reflection (retro-reflections at the superconducting interface which convert electrons into holes or vice versa, at the expense of Cooper pairs which are absorbed or emitted from the superconductor). Random-matrix theory predicts that this gap is of size $E_g^{\text{RMT}} \approx 0.3 E_{\text{Th}}$, where the Thouless energy refers to the system with the normal opening (the associated dwell time $t_D$ is identical to the mean time between successive Andreev reflections in the Andreev billiard).

We show that a long-lived scarred state with lifetime $t_S \gg t_D$ reduces the excitation gap over an energy window $\Delta E \approx 2E_g^{\text{RMT}} \gg \Gamma_S$. The scar hence determines

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig1.png}
\caption{Left panel: Excitation spectrum as function of the Fermi energy $E_F$, obtained by numerical computations for the Andreev billiard shown in the right panel. The lowest excitation energy (triangles) determines the excitation gap. The second, third, forth and fifth excitation energy are shown as filled circles. The energy of the scarred eigenstate is highlighted by open squares. The dashed line shows the random-matrix prediction $E_g^{\text{RMT}}$ for the gap. Other lines are guides for the eye. Right panel: The Andreev billiard is a segment of a Sinai billiard with $W = 0.21a$, $h = 0.58a$, and $R = 0.8a$. The pair potential in the superconductor is $\Delta_0 = 0.015 E_F$. In the energy range of the left panel, there are $N = 25$ open channels in the superconducting lead.}
\end{figure}
FIG. 2: The square moduli $|u|^2$ of the electron and $|v|^2$ of the hole components of the wavefunction for $E_F/E_{\text{Th}} = 707.434 < E_{F,\text{min}}$ (a), for $E_F = E_{F,\text{min}}$ (b) and for $E_F/E_{\text{Th}} = 707.695 > E_{F,\text{min}}$ (c).

fundamental system properties over a much larger range of energies than in the open normal system. The smallest value of the excitation gap within the window is given by $E_{g,\text{min}} \approx \Gamma_S/2$. The minimal size of the gap is therefore much smaller than the RMT prediction $E_g^{\text{RMT}}$. In particular, the reduction of the excitation gap can be far stronger than the reduction due to both the quantum-to-classical correspondence for times smaller than the Ehrenfest time, observed in Refs. 24,25, and the mesoscopic fluctuations, studied in Ref. 26.

We first present numerical evidence for these results and then develop a theoretical description which relates the excitation energy of scarred states in the Andreev billiard to properties of the scar in the normal billiard.

The excitation energies $\varepsilon$ of an Andreev billiard are given by the eigenvalues of the Bogoliubov-de Gennes (BdG) equation

$$
\begin{pmatrix}
\hat{H}_0(r) & \Delta(r) \\
\Delta^*(r) & -\hat{H}_0(r)
\end{pmatrix}
\begin{pmatrix}
u(r) \\
u^*(r)
\end{pmatrix}
= \varepsilon
\begin{pmatrix}
u(r) \\
u^*(r)
\end{pmatrix}. \tag{1}
$$

Here $\hat{H}_0 = -\hbar^2/2m \nabla^2 - E_F$ is the single-particle Hamiltonian with the chemical potential in the superconductor set equal to the Fermi energy $E_F$ in the normal part of the system. The wavefunction is composed of an electronic part $u(r)$ and a hole part $v(r)$ which are coupled via the superconducting pair potential $\Delta(\mathbf{r})$ with value $\Delta_0$ in the bulk of the superconductor. Typical Andreev billiards satisfy the separation of energy scales $E_{\text{Th}} \ll \Delta_0 \ll E_F$. One can then assume that $\Delta(\mathbf{r}) = \Delta_0$ is constant in the superconducting region and vanishes in the normal part of the system 22. Solutions of Eq. (1) come in pairs of excitation energies $\pm \varepsilon$, where one solution is of the form $(u, v)$ and the other solution is of the form $(v, u)$.

In order to convince ourselves of the visibility of scars in the excitation spectrum of an Andreev billiard we solved the Bogoliubov-de Gennes equation numerically for the chaotic billiard shown in the right panel of Fig. 1.22,23. The normal part is a de-symmetrized segment of a Sinai billiard. The superconducting terminal is attached to one of the straight walls. The left panel of Fig. 1 shows the positive excitation energies as a function of the Fermi energy $E_F$, scaled in units of the Thouless energy $E_{\text{Th}}$ (the dependence of the Thouless energy on the Fermi energy is taken into account in this rescaling). The second, third etc. excitation energies show the characteristic level dynamics of chaotic systems when one parameter is changing, and avoided crossings can also be observed. Random matrix theory predicts that the excitation gap (set by the the lowest positive excitation energy) fluctuates around the value $E_g^{\text{RMT}} = 0.3 E_{\text{Th}}$ (dashed line) 22,23. However, there is an energy interval of width $\Delta E/E_{\text{Th}} \approx 0.26$ in which the excitation gap drops distinctively below this value. The smallest gap $E_{g,\text{min}} = 0.0654 E_{\text{Th}}$ is attained at Fermi energy $E_{F,\text{min}} = 707.567 E_{\text{Th}}$.

Inspection of the wavefunction of the lowest eigenstate in this energy interval reveals an interesting feature: for $E_F$ smaller than $E_{F,\text{min}}$ [see Fig. 2(a)], the electronic component $u$ is nonuniform and shows clear signatures of scarring along a family of marginally stable periodic orbits which bounce between the straight segments of the billiard. Calculating the probability $P_e = \int |u|^2 \mathrm{d} \mathbf{r}$ ($P_h = \int |v|^2 \mathrm{d} \mathbf{r}$) of finding the quasiparticle in the electron (hole) state in the normal part of the system, one finds $P_e = 0.915$ and $P_h = 0.078$, i.e. the quasiparticle is dominantly electron-like (the remaining weight is in the superconductor). The next panel [Fig. 2(b)] corresponds to the eigenstate at the minimum of the excitation gap in Fig. 1(a), showing that in this case both $u$ and $v$ are nonuniform, and the probabilities are $P_e = 0.520$ and $P_h = 0.474$. In Fig. 2(c) we show an eigenstate at a Fermi energy which is larger than $E_{F,\text{min}}$. Here the hole component is scarred by the same family of periodic orbits and the quasiparticle becomes dominantly hole-like with $P_e = 0.064$ and $P_h = 0.929$.

These observations suggest that the suppression of the excitation gap and the scarring of the wavefunction stems from a long-lived scarred quasibound state of the normal open billiard. When the superconducting terminal is replaced by a normal lead, a scarred state can indeed be detected for energies very close to $E_{F,\text{min}}$ – see the left panel of Fig. 3 which shows a scattering state computed at the energy $E = E_{F,\text{min}}$. For a slightly different energy $E = 707.657 E_{\text{Th}}$, however, this scar is already very much diminished, as is shown in the right panel of Fig. 3. At this energy, the gap in the excitation spectrum of the Andreev billiard is still strongly suppressed. These numerical results demonstrate that a long-lived scarred state can be detected over an energy window much larger than its resonance width $\Gamma_S$ in the normal billiard when the system is coupled to a superconducting terminal.
We now present a theory which explains the numerical observations. In particular, we relate the Fermi-energy dependence of the scarred excitation in the Andreev billiard to the properties of the scarred state in the open normal billiard. Such relations can be derived because the excitation spectrum of Andreev billiards can be calculated from the scattering matrix $S(E)$ of the normal system using the quantization condition

$$\det[1 + S^*(E_F - \varepsilon)S(E_F + \varepsilon)] = 0,$$

which is valid in the limit $\varepsilon \ll \Delta_0 \ll E_F$. The scattering matrix is a unitary (and in absence of a magnetic field also symmetric) matrix of dimensions $N \times N$, given by the number of open channels $N$ in the opening. In Eq. (2), $S(E_F + \varepsilon)\ [S^*(E_F - \varepsilon)]$ describes the propagation of electrons [holes] and $*$ denotes complex conjugation. Equation (2) embodies the condition of total constructive interference after two Andreev reflections, so that an electron is converted first to a hole and then back to an electron. (Each Andreev reflection contributes a phase factor of $i$.)

Narrow resonances in the normal system are associated to poles of the scattering matrix which are situated close to the real axis in the lower half of the complex energy plane. In what follows we will assume that there is only one such narrow resonance in the energy range of interest, namely that associated to the scarred state, located at a complex energy $\epsilon = E_S - i\Gamma_S/2$. In order to isolate the contribution of this narrow resonance to the scattering matrix we first diagonalize $S(E) = O(E)\Lambda(E)O^T(E)$ where $O(E)$ is an $N \times N$ dimensional orthogonal matrix and $\Lambda(E)$ is a diagonal matrix containing the eigenvalues $\lambda_i(E)$, $i = 1 \ldots N$. In the vicinity of the narrow resonance the energy dependence of $O(E)$ can be neglected. Inserting the decomposition $S(E) = O\Lambda(E)O^T$ into Eq. (2) one then finds that the quantization condition can be written as

$$\prod_i [1 + \lambda_i^*(E_F - \varepsilon)\lambda_i(E_F + \varepsilon)] = 0.$$  \hspace{0.5cm} (3)

One of the eigenvalues, $\lambda_S(E)$, is associated to the narrow resonance, while the other eigenvalues are associated to the remaining non-resonant states in the system. The resonant eigenvalue can be approximated as $\lambda_S(E) = e^{2i\phi_S(E)}\frac{E - E_S}{\Gamma_S/2}$, where the energy dependence of $\phi_S(E)$ can again be neglected over the energy range of interest. Demanding that the term involving $\lambda_S$ in Eq. (3) be zero,

$$1 + \frac{E_F - \varepsilon_S - E_S - i\Gamma_S/2}{E_F - \varepsilon_S - i\Gamma_S/2} + \frac{\varepsilon_S - E_S + i\Gamma_S/2}{E_F - \varepsilon_S - E_S + i\Gamma_S/2} = 0,$$

one can find the excitation energies

$$\varepsilon_S = \pm \sqrt{(E_F - E_S)^2 + \Gamma_S^2/4}$$

associated to the scarred eigenstate of the Andreev billiard in terms of the energy and width of the scarred resonance in the open normal system. The excitation energies come in pair as required by the particle-hole symmetry of the BdG equation.

For Fermi energies far less than the energy $E_S$ of the scar in the normal system, the positive excitation energy is given by $\varepsilon_S \approx E_S - E_F$, which is then far larger than the RMT gap and hence corresponds to a highly excited state. As $E_F$ is increased, the excitation drifts down through the spectrum, and assuming $\Gamma_S \ll \Gamma^\text{RMT}$ the scarred state eventually becomes the lowest excited state at an energy $E_F \approx E_S - \Gamma^\text{RMT}$. This defines the beginning of the energy interval in which the scar can be detected by measuring the excitation gap of the Andreev billiard. The minimal value $E_{g,\text{min}} = \min_{E_F} |\varepsilon_S| = \Gamma_S/2$ of the excitation gap is obtained for $E_{F,\text{min}} = E_S$. When the Fermi energy is further increased the excitation energy increases as well, and for values $E_F \gtrsim E_S + \Gamma^\text{RMT}$ the scarred state becomes again a higher excited state, with energy $\varepsilon_S \approx E_F - E_S$. This explains the qualitative features observed in Fig. (a).

The microscopic parameters $E_S$ and $\Gamma_S/2$ entering Eq. (3) can be determined directly from properties of the open normal billiard. The left panel of Fig. (a) shows the phase $\Theta(E) = \text{Im} \ln |S(E)|$ of the scattering matrix over the energy range in which the scar determines the excitation gap. The energy dependence due to the resonant eigenvalue $\lambda_S$ is given by $\text{Im} \ln \lambda_S \approx 2 \arctan \left( \frac{\Gamma_S}{2(E - E_S)} \right)$, while the non-resonant states in the system contribute a linear background $CE + \Theta_0$ with $C \approx 2\pi/\delta_N$, where $\delta_N$ is the mean level spacing of the normal system. A fit of the curve $2 \arctan \left( \frac{\Gamma_S}{2(E - E_S)} \right) + CE + \Theta_0$ to the numerical data delivers the values $E_S = 707.567E_{\text{Th}}$ and $\Gamma_S/2 = 0.0745E_{\text{Th}}$. The width is in good agreement with the smallest value $E_{g,\text{min}}$ of the excitation gap given above, and the energy of the scar coincides exactly with the Fermi energy $E_{F,\text{min}}$ at which this minimal value is attained. The right panel of Fig. (a) shows that substituting the obtained parameters $E_S$ and $\Gamma_S$ into Eq. (3), a quantitative agreement can be found between the theoretical prediction and the computed values of the excitation energies of the scarred states.
The weight of the scar in each component can be estimated by tunnel spectroscopy [29]. The Fermi energy can be related to the excitation gap as shown in the observations in Fig. 2. Right panel: Quantitative comparison of the excitation energies of the scarred Andreev bound state (squares) with Eq. (5) (solid line), using the parameters $E_S$ and $\Gamma_S$ of the scar in the normal system.

Finally, we turn to the observation that the scattered character of the state in the Andreev billiard shifts from the electronic part of the wavefunction, while the hole part is scattered for $E_F \gg E_S$. For $E_F = E_S$ the weights in both components are predicted to be equal. This explains the observations in Fig. 2.

In summary, we have demonstrated theoretically and numerically that long-lived scarred wavefunctions of quantum-chaotic system can be observed over large ranges of the Fermi energy when the system is connected to a superconducting terminal. The resulting suppression of the excitation gap should be accessible in experiments by tunnel spectroscopy [29]. The Fermi energy can be controlled by side gates [30].

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[1] F. Haake, Quantum signature of Chaos, 2nd ed. (Springer, Berlin, 2001).