Supersymmetric completion of 
supersymmetric quantum mechanics

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Abstract

Via supersymmetry argument, we determine the effective action of the $SU(2)$ supersymmetric Yang-Mills quantum mechanics up to two constants, which results from the full supersymmetric completion of the $F^4$ term. The effective action, consisting of zero, two, four, six and eight fermion terms, agrees with the known perturbative one-loop calculations from the type II string theory and the matrix theory. Our derivation thus demonstrates its non-renormalization properties, namely, the one-loop exactness of the aforementioned action and the absence of the non-perturbative corrections. We briefly discuss generalizations to other branes and the comparison to the DLCQ supergravity analysis. In particular, our results show that the stringent constraints from the supersymmetry are responsible for the agreement between the matrix theory and supergravity with sixteen supercharges.

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1 Introduction and Summary

Supersymmetric matrix quantum mechanics (we alternatively call it matrix theory or supersymmetric quantum mechanics in this paper) is suggested to provide us with a quantum description of the eleven-dimensional supergravity in the large $N$ limit [1]. For the finite $N$ case, the eleven-dimensional supergravity formulated in terms of the discrete light-cone quantization (DLCQ) scheme of Susskind [2, 3, 4] is argued to be described by the supersymmetric quantum mechanics. The agreement of the effective action between the matrix theory and the DLCQ supergravity for particle (or other extended objects in $M$ theory that we do not consider in this paper) scatterings is by now well-reported in the literature [5]-[10].

The impressive agreement between these two radically different theories naturally lead us to wonder why they should agree in the first place. Intuitively, supersymmetries should play a key role; the scattering dynamics analyzed in, for example, Refs. [5] and [6] preserves sixteen supersymmetries. The matrix quantum mechanics, being the dimensional reduction to one-dimension from the ten-dimensional supersymmetric Yang-Mills (SYM) theory, describes the low energy dynamics of the D-particles in IIA string theory [11] and possesses sixteen supersymmetries along with the $SO(9)$ R-symmetry. Similarly, the supergravity space-time metric for $M$-momenta moving along the light-cone direction in the DLCQ supergravity has sixteen Killing spinors and $SO(9)$ transversal rotational isometry [10]. In the latter case, the detailed form of the metric is determined by specifying a nine-dimensional harmonic function, which is obtained by solving the BPS equations that are valid when there exist sixteen unbroken supersymmetries. Once the metric is determined in the supergravity side, the bosonic probe action that produces $v^4$ term in the small velocity expansion can be straightforwardly written down [10]. Given this purely bosonic $v^4$ term, the supersymmetrization uniquely determines all the other fermion terms via the superspace formalism [10].

A similar behavior to what happens in the supergravity case has been observed in the pioneering work of Paban, Sethi and Stern [12] in the matrix quantum mechanics. They show that, in the case of the supersymmetric quantum mechanics, the eight fermion terms, that result from the supersymmetric completion of the one-loop $F^4$ term, can be uniquely determined (up to an overall normalization) by the supersymmetry argument.
alone; the coefficient functions of the eight fermion terms satisfy an analog of the BPS equations from the supergravity \cite{12}. If the correspondence between the matrix theory and the supergravity holds up as presumed, one naturally hopes further that the eight fermion terms, once determined, should determine all the other remaining terms with zero, two, four, and six fermions. In this paper, we show that the constraints from the sixteen supersymmetries determine all fermion terms in the effective action belonging to the full supersymmetric completion of the $F^4$ terms of the matrix quantum mechanics. Our results demonstrate the formal similarity of the matrix quantum mechanics to the supergravity where the superspace formalism generates all fermion terms from the purely bosonic terms.

Main benefit of this line of approach is that the non-renormalization theorem \cite{12, 13} for all the terms that we calculate is guaranteed, since the sixteen supersymmetries are the exact symmetries in our context. Our effective action turns out to be identical to the one-loop perturbative terms reported in the literature including the bosonic term \cite{3}, two fermion terms \cite{8}, four fermion terms \cite{14}, and eight fermion terms \cite{15}. The six fermion terms have not been calculated in the perturbative supersymmetric quantum mechanics framework, but our results are identical to the ones obtained by the perturbative analysis in the IIA string theory framework \cite{19, 20, 21}. In view of these, our analysis demonstrates the non-renormalization properties, including the one-loop exactness and the lack of non-perturbative effects, for those terms that originate from the supersymmetric completion of the $F^4$ terms. Specifically, following the notations introduced in Sec. 2.1, our results are:

\[
I = \Gamma_{(2)} + \Gamma_{(4)} + F^6 \text{ terms} + \cdots ,
\]

where

\[
\Gamma_{(2)} = \int d\lambda \left( \frac{1}{2} v^2 + \frac{i}{2} \theta \dot{\theta} \right),
\]

and

\[
\Gamma_{(4)} = \int d\lambda \left\{ f^{(0)}(\phi)(v^2)^2 + v^2 v^j f^{(2)}_1(\phi) \phi^j (\theta \gamma^{ij} \theta) \\
+ v^i v^j \left[ f^{(4)}_2(\phi) \phi^k \phi^l + f^{(4)}_0(\phi) \delta^{kl} \right] (\theta \gamma^{ik} \theta)(\theta \gamma^{jl} \theta) \\
+ v^i \left[ f^{(6)}_3(\phi) \phi^j \phi^k \phi^l + 2 f^{(6)}_1(\phi) \delta^{jk} \phi^l \right] (\theta \gamma^{ij} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{ln} \theta) \\
+ \left[ f^{(8)}_4(\phi) \phi^i \phi^j \phi^k \phi^l + 4 f^{(8)}_2(\phi) \delta^{ik} \phi^j \phi^l + 2 f^{(8)}_0(\phi) \delta^{ik} \delta^{jl} \right] \\
\times (\theta \gamma^{im} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{kn} \theta)(\theta \gamma^{ln} \theta) \right\} ,
\]

(1)

where \( f^{(0)} \) is an \( SO(9) \) invariant nine-dimensional harmonic function

\[
f^{(0)} = k_1 + k_2 \frac{1}{\phi^9},
\]

the functions \( f^{(2p)}_q \) satisfy

\[
f^{(2p)}_q = C_p \left( \frac{d}{\phi d\phi} \right)^{(p+q)/2} f^{(0)},
\]

and the numbers \( C_p \) independent of \( q \) are given by

\[
C_1 = \frac{i}{2}, \quad C_2 = -\frac{1}{8}, \quad C_3 = -\frac{i}{144}, \quad C_4 = \frac{1}{8064}.
\]

In other words, the effective action \( \Gamma^{(4)} \) is fully determined up to two constants \( k_1 \) and \( k_2 \). These results are obtained purely on the basis of the existence of sixteen supersymmetries, \( SO(9) \) R-symmetry and the CPT invariance. Perturbative calculations of the bosonic effective action within the matrix theory framework \([6]\) show that \( k_1 = 0 \), while the supersymmetry allows it to be an arbitrary constant. To recover \( k_1 = 0 \) from the supergravity necessitates the use of the DLCQ framework where the asymptotic time direction of the background geometry is light-like \([4, 6, 10]\). The resulting background geometry is the non-asymptotically flat near-horizon D-particle geometry in ten dimensions \([19]\), or equivalently, the asymptotically flat Aichelberg-Sexl geometry in eleven dimensions \([4]\). The asymptotic time direction of the asymptotically flat D-particle background geometry is time-like, which implies \( k_1 > 0 \). Apart from the necessity of introducing the DLCQ framework, which is strictly speaking beyond supersymmetry argument, the complete effective action itself is determined by the supersymmetry with sixteen supercharges, up to an overall normalization. Therefore, the stringent constraints imposed by the maximal supersymmetries are responsible for the agreement for the two-body dynamics between the DLCQ supergravity and the matrix theory when there are sixteen supersymmetries. In view of this aspect, the crucial future tests to verify the (dis)agreements between the matrix theory and the supergravity should be directed to the cases when some of the supersymmetries are broken, as well as to the cases involving multi-body (especially large \( N \)) scatterings \([20]\).

The technical details are presented in Sec. 2, and we briefly discuss related issues, such as the extension of our analysis to the membrane case, in Sec. 3.
2 Supersymmetric completion of $F^4$ terms

This section is organized as follows. In Sec. 2.1, we classify the terms that can appear in the effective action obtained by the supersymmetric completion of the $F^4$ terms. The guiding principles here are the unbroken $SO(9)$ R-symmetry and the CPT invariance, both of which constrain the possible terms in the effective action. Appendix A and B contain some technical details such as a number of relevant Fierz identities. In Sec. 2.2, starting from the effective action constrained in Sec. 2.1, we explicitly work out the supersymmetry transformations to determine the full supersymmetrically completed effective action.

2.1 Constraints from the $SO(9)$ R-symmetry and CPT on the effective action

The effective action of the SYM quantum mechanics is described by nine scalars $\phi^i$ and their time derivatives $v^i = (d/d\lambda)(\phi^i) \equiv \dot{\phi}^i$ where $i = 1, \ldots, 9$ and $\lambda$ is a time coordinate. At the origin of the SYM moduli space, the $R$-symmetry is unbroken $SO(9)$. This is the situation that corresponds to the matrix theory description of the source-probe two-body dynamics where the probe $M$-momentum moves in the background geometry of the $N$ coincident source $M$-momenta. In this paper, we will start by computing the constraints imposed on the possible terms in the effective action resulting from the requirement of the unbroken $SO(9)$ $R$-symmetry. Furthermore, we will impose the CPT invariance that the matrix quantum mechanics inherits from the ten-dimensional IIA SYM theory. Our starting point will be the consideration of the $F^2$ terms, following the notation and set-up of Ref. [12]. Considering the $SO(9)$ invariance, the possible bosonic $F^2$ terms are $g_1(\phi)v^2$ and $g_2(\phi)(v^i\dot{\phi}^i)^2$. Here the $SO(9)$ vector indices are contracted by the Kronecker delta $\delta_{ij}$, for example, $v^2 = \delta_{ij}\dot{\phi}^i\dot{\phi}^j$, and $g_1$ and $g_2$ are $SO(9)$ invariant scalar functions, which depend only on an $SO(9)$ invariant $\phi = \sqrt{\phi^i\phi^i}$. By the inverse of the diffeomorphism of the form

\[ h_1(\phi)v^i \rightarrow h_2(\phi)v^i + h_3(\phi)(v^j\dot{\phi}^j)\dot{\phi}^i, \]

which squares to

\[ h_1^2v^2 \rightarrow h_2^2v^2 + (2h_2h_3 + h_3^2\phi^2)(v^j\dot{\phi}^j)^2 \equiv g_1v^2 + g_2(v^i\dot{\phi}^i)^2, \]
the general $SO(9)$ invariant moduli space metric consisting of the two terms $g_1 v^2 d\lambda^2$ and $g_2 (v^i \phi^i)^2 d\lambda^2$ can be reduced to a simple form as follows

$$ds^2 = g_1 d\phi^i d\phi^i + g_2 (v^i \phi^i)^2 d\lambda^2 = h^2 d\phi^i d\phi^i .$$  (9)

The key result of Ref. [12] is that this moduli space metric is constrained to be flat (corresponding to a free abelian theory) under the imposition of the supersymmetry with sixteen supercharges. For the $F^2$ terms, we thus choose a coordinate such that the quadratic effective action looks like the following form

$$\Gamma_2 = \int d\lambda \left( \frac{1}{2} v^2 + \frac{i}{2} \theta \dot{\theta} \right) ,$$  (10)

which is invariant under the supersymmetry transformation

$$\delta \phi^i = -i \epsilon \gamma^i \theta$$

$$\delta \theta = (\gamma^i v^i \epsilon).$$

Here $\theta$ is the sixteen component $SO(9)$ Majorana spinor and gamma matrices $\gamma^i$ are the $16 \times 16$ $SO(9)$ gamma matrices. For the following description, we introduce an ordering $O(v^i) = O(\theta \theta) = O(d/d\lambda) = 1$ and we assign $O(\epsilon) = -1/2$ for the $SO(9)$ Majorana spinor parameter $\epsilon$ for the supersymmetry transformation. The subscript of the effective action $\Gamma_2$ signifies the fact that the terms of (10) are of order two under the above ordering assignment.

What we are interested in in this paper is to determine the order of four terms, $\Gamma_4$, in the effective action when the spinor $\theta$ and the velocity $v^i$ are constant$^1$. In this case, we can preclude the possible acceleration and high order fermion derivative terms, which are, in general, present in the effective action. Due to the existence of the sixteen supersymmetries and the ordering assignment in the above, the structure of $\Gamma_4$ can be schematically written as [16, 17]

$$\Gamma_4 = \int d\lambda \left( [v^4] + [v^3 \theta^2] + [v^2 \theta^4] + [v \theta^6] + [\theta^8] \right) ,$$  (12)

$^1$These are consistent with the supersymmetry transformation, Eq. (11). As will be explained in Sec. 2.2, the inclusion of $\Gamma_4$ modifies the supersymmetry transformation, Eq. (11). The detailed consideration in Sec. 2.2 will show that using the leading order supersymmetry transformation, Eq. (11), is enough for $\Gamma_4$, while the correction plays an important role for the supersymmetric variation of $\Gamma_2$. 
where we suppress index structures; it consists of zero, two, four, six and eight fermion terms. The possible terms that can appear in Eq. (12) can be constrained by the requirement of the unbroken $SO(9)$ $R$-symmetry and the CPT theorem, as we will discuss now. Since $\theta$ is an $SO(9)$ Majorana spinor, the fermion bilinears satisfy $\theta \gamma^{i_1 i_2 \cdots i_k} \theta = 0$ for $k = 0, 1, 4, 5, 8, 9$. Here $\gamma^{i_1 i_2 \cdots i_k}$ is the totally anti-symmetrized $k$-product of gamma matrices normalized to unity. Therefore, the fermion structure of the $p$-fermion term shown in Eq. (12) is in general a $(p/2)$-product of fermion bilinears $J_{ij} \equiv \theta \gamma_{ij} \theta$ and $K_{ijk} \equiv \theta \gamma_{ijk} \theta$. The terms appearing in Eq. (12) can thus be classified as shown in Table 1.

| 0-fermion | $4v^i$ |  
| 2-fermion | $5v^i J$ | $6v^i K$ |
| 4-fermion | $6v^i J J$ | $7v^i J K$ | $8v^i K K$ |
| 6-fermion | $7v^i J J J$ | $8v^i J J K$ | $9v^i J K K$ | $10v^i K K K$ |
| 8-fermion | $8v^i J J J J$ | $9v^i J J J K$ | $10v^i J J K K$ | $11v^i J K K K$ | $12v^i K K K K$ |

Table 1. The classification of the possible terms in the effective action.

The superscript $v^m J^n K^p$ denotes the fact that the corresponding terms are composed of the products of $m$ velocity vector $v^i$’s, $n$ fermion bilinear $J^{ij}$’s and $p$ fermion bilinears $K^{ijk}$’s. The number $r$ ($r = 4, \cdots, 12$) in each entry denotes the total number of indices given by $r = m + 2n + 3p$. Since the terms in the effective action should be an $SO(9)$ scalar, an object with indices should be contracted with an appropriate number of $\phi^i$’s; an object with $r$ indices can however be self-contracted $s$ times ($0 \leq s \leq \lfloor r/2 \rfloor$) before the contraction with $(r-2s)$ $\phi^i$’s. Each term in Table 1 contains, in addition to the $v^m J^n K^p \phi^{r-2s}$ structure, an arbitrary coefficient function as an overall factor that depends only on an $SO(9)$ invariant $\phi = \sqrt{\phi^i \phi^i}$.

The CPT invariance dictates that the terms of the second ($6v^i K, 7v^i J K, 8v^i J J K, 9v^i J J J K$) and the fourth ($10v^i K K K, 11v^i J K K K$) columns of Table 1 should vanish; the terms in the first column, which are of the form of the perturbative terms reported in the literature, are all CPT invariant, and replacing one $J^{ij}$ with $K^{mnp}$ turns the aforementioned terms CPT violating. As proven in Ref. [12] for the fifth row and in Appendix A for the other rows, all the terms in the third and fifth columns can be Fierz-rearranged into the terms of the first column. Therefore, we can concentrate on the terms of the first column from now on, without losing generality. We note that the CPT violating terms, the terms of the second and the fourth columns, can not be turned into the terms of the first column via the Fierz
rearrangement. Noting a Fierz identity \((\theta\gamma^{ij}\theta)(\theta\gamma^{ij}\theta) = 0\), all the non-vanishing terms of Table 1 can be written down as follows, after working out all possible contractions:

0 – fermion : \((v^2)^2\),  \((v^2)^4\),  \((v^i\phi^j)^2v^2\),  \((v^i\phi^j)^4\),

2 – fermion : \(v^2v^i\phi^j(\theta\gamma^{ij}\theta)\),  \((v^i\phi^j)^2v^j\phi^k(\theta\gamma^{jk}\theta)\),

4 – fermion : \(v^2\phi^j\phi^k(\theta\gamma^{jk}\theta)(\theta\gamma^{ij}\theta)\),  \(v^jv^i\phi^k\phi^l(\theta\gamma^{jk}\theta)(\theta\gamma^{il}\theta)\),  \(v^jv^i(\theta\gamma^{jk}\theta)(\theta\gamma^{ij}\theta)\),

6 – fermion : \(v^j\phi^i\phi^k\phi^l(\theta\gamma^{ij}\theta)(\theta\gamma^{km}\theta)(\theta\gamma^{ln}\theta)\),  \(v^i\phi^j(\theta\gamma^{ij}\theta)(\theta\gamma^{jk}\theta)(\theta\gamma^{kl}\theta)\),

8 – fermion : \(\phi^i\phi^j\phi^k\phi^l(\theta\gamma^{lm}\theta)(\theta\gamma^{km}\theta)(\theta\gamma^{jn}\theta)(\theta\gamma^{ln}\theta)\),  \(\phi^i\phi^j\phi^k\phi^l(\theta\gamma^{jm}\theta)(\theta\gamma^{jn}\theta)(\theta\gamma^{im}\theta)\),  \(\phi^i\phi^j\phi^k(\theta\gamma^{ij}\theta)(\theta\gamma^{kl}\theta)(\theta\gamma^{jl}\theta)(\theta\gamma^{kl}\theta)(\theta\gamma^{il}\theta)\),

and

0 – fermion : \((v^i\phi^i)^2v^2\),  \((v^i\phi^i)^4\),

2 – fermion : \((v^i\phi^i)^2v^j\phi^k(\theta\gamma^{jk}\theta)\),

4 – fermion : \((v^i\phi^i)^2v^j\phi^k(\theta\gamma^{kl}\theta)(\theta\gamma^{jl}\theta)\),  \((v^i\phi^i)^2v^j\phi^k(\theta\gamma^{kl}\theta)(\theta\gamma^{jl}\theta)\),

6 – fermion : \((v^i\phi^i)^2\phi^j\phi^m(\theta\gamma^{jk}\theta)(\theta\gamma^{kl}\theta)(\theta\gamma^{jm}\theta)\).

According to the perturbative calculations for zero, two, four and eight fermion terms and the type II side calculations [7, 8] [14]-[18], the terms of the form \((23)-(27)\) do not appear in the effective action under the choice of \(\Gamma^{(2)}\) in Eq. (10). Considering the non-renormalization theorem of Refs. [12] and [13], we can set the coefficient functions of \((23)-(27)\) as zero. In fact, consistent with the analysis of Ref. [12], the diffeomorphism of the form Eq. (7) generates all the terms of \((23)-(27)\) from \((13)-(19)\), just like the same diffeomorphism generates the \((v^i\phi^i)^2\) term from the bosonic kinetic term \(v^2\) in Eq. (10) (see Eq. (8)). Specifically, under the diffeomorphism Eq. (7), the terms of \((13)-(15)\) generate the following terms:

\[(13) \rightarrow (13) + (23),\]
\[(14) \rightarrow (14) + (24) ,
(15) \rightarrow (15) + (26) ,
(16) \rightarrow (16) ,
(17) \rightarrow (17) + (25) + (26) ,
(18) \rightarrow (18) ,
(19) \rightarrow (19) + (27) ,
(13) \rightarrow (18) ,
\]

where we use the identity \(\phi^i \phi^j \theta \gamma^{ij} \theta = 0\). It is instructive to observe the same situation in the eleven-dimensional DLCQ supergravity. In Ref. [10], it is shown that (10), (13) and (14) terms are correctly reproduced from the probe action of a massless eleven-dimensional superparticle moving in the background geometry produced by \(N\) source \(M\)-momenta.

In the same reference, choosing a static gauge \((dX^0/d\lambda) = 1\) renders the kinetic terms be of the form of (10) and the order of four terms be of the form (13) + (14), while the terms of the form \((v^i \phi^i)^2\), (23) and (24), are absent.

We are now left to consider the terms of (13)-(22). We note the following property for the terms of (16)-(22); replacing \(\phi^i \phi^j\) with \(\delta^{ij}\) reduces (16) into (17), (18) into (19), (20) into (21), and (21) into (22), again noting a Fierz identity \((\theta \gamma^{ij} \theta)(\theta \gamma^{ij} \theta) = 0\). The same replacement, when applied to (15), makes it vanish. Utilizing this property, the terms of \(\Gamma_4\) can in general be written as

\[
[v^4] = f^{(0)}(\phi)(v^2)^2 ,
[v^3 \theta^2] = v^2 v^j f^{(2)}_1(\phi) \phi^j (\theta \gamma^{ij} \theta) ,
[v^2 \theta^4] = v^2 g^{(4)}_2(\phi) \phi^i \phi^j (\theta \gamma^{ik} \theta)(\theta \gamma^{jk} \theta)
+ v^i v^j \left[ f^{(4)}_2(\phi) \phi^k \phi^l + f^{(4)}_0(\phi) \delta^{kl} \right] (\theta \gamma^{ik} \theta)(\theta \gamma^{jl} \theta) ,
[v \theta^6] = v^i \left[ f^{(6)}_3(\phi) \phi^j \phi^k \phi^l + 2 f^{(6)}_1(\phi) \delta^{jk} \phi^l \right] (\theta \gamma^{ij} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) ,
[\theta^8] = \left[ f^{(8)}_4(\phi) \phi^i \phi^j \phi^k \phi^l + 4 f^{(8)}_2(\phi) \delta^{ik} \phi^j \phi^l + 2 f^{(8)}_0(\phi) \delta^{ij} \delta^{kl} \right]
\times (\theta \gamma^{im} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{kn} \theta)(\theta \gamma^{ln} \theta) .
\]

The scalar function \(f_q^{(2p)}\) represents the coefficient function of the \(q\)-scalar term \(^\square\) among the \(2p\)-fermion terms. Among \(2p\)-fermion terms, the maximum scalar number is \(p\), as can be seen from (28)-(32). In Sec. 2.2, we will determine the ten coefficient functions \(f^{(0)}\), \(f^{(2)}_1\), \(f^{(4)}_2\), \(f^{(6)}_3\), \(f^{(6)}_1\), \(f^{(8)}_4\), \(f^{(8)}_2\), \(f^{(8)}_0\), and \(g^{(4)}_2\) by the supersymmetry argument, and it turns out that \(g^{(4)}_2 = 0\).

\(^2\)The calculations for the higher fermion terms from the supergravity side are not yet available in the literature, except for the four fermion terms of Ref. [3].

\(^3\)Throughout this paper, the scalar number refers to the number of scalars \(\phi^i\) contracted to the indices of the fermion bilinears. Thus, \(v^i \phi^j\), for example, has the scalar number zero.
We make the following formal observation; for a function \( f \) depending only on an \( SO(9) \) invariant \( \phi \), the derivatives respect to \( \phi \) can be computed as follows via the chain rule:

\[
\partial_i f = \phi^j \left( \frac{d}{\partial \phi} \right) f ,
\]

\[
\partial_i \partial_j f = \phi^j \phi^l \left( \frac{d}{\partial \phi} \right)^2 f + \delta^{ij} \left( \frac{d}{\partial \phi} \right) f ,
\]

\[
\partial_i \partial_j \partial_k f = \phi^j \phi^k \left( \frac{d}{\partial \phi} \right)^3 f + (\delta^{ij} \phi^k + \delta^{ik} \phi^j + \delta^{jk} \phi^i) \left( \frac{d}{\partial \phi} \right)^2 f ,
\]

\[
\partial_i \partial_j \partial_k \partial_l f = \phi^j \phi^k \phi^l \left( \frac{d}{\partial \phi} \right)^4 f
\]

\[
+ (\delta^{ij} \phi^k \phi^l + \delta^{ik} \phi^j \phi^l + \delta^{jk} \phi^i \phi^l + \delta^{jl} \phi^i \phi^k + \delta^{il} \phi^j \phi^k) \left( \frac{d}{\partial \phi} \right)^3 f
\]

\[
+ (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) \left( \frac{d}{\partial \phi} \right)^2 f
\].

Therefore, we have

\[
\partial_j f(\theta \gamma^{ij} \theta) = \phi^i \left( \frac{d}{\partial \phi} \right) f(\theta \gamma^{ij} \theta) ,
\]

\[
\partial_k \partial_j f(\theta \gamma^{ik} \theta)(\theta \gamma^{jl} \theta) = \left[ \phi^k \phi^l \left( \frac{d}{\partial \phi} \right)^2 f + \delta^{kl} \left( \frac{d}{\partial \phi} \right) f \right] (\theta \gamma^{ik} \theta)(\theta \gamma^{jl} \theta) ,
\]

\[
\partial_j \partial_k \partial_l f(\theta \gamma^{ij} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) =
\]

\[
\left[ \phi^j \phi^k \phi^l \left( \frac{d}{\partial \phi} \right)^3 f + 2\delta^{ij} \phi^l \left( \frac{d}{\partial \phi} \right)^2 f \right] (\theta \gamma^{ij} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) ,
\]

\[
\partial_i \partial_j \partial_k \partial_l f(\theta \gamma^{im} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{kn} \theta)(\theta \gamma^{ln} \theta) =
\]

\[
\left[ \phi^i \phi^j \phi^k \phi^l \left( \frac{d}{\partial \phi} \right)^4 f + 4\delta^{ik} \phi^j \phi^l \left( \frac{d}{\partial \phi} \right)^3 f + 2\delta^{ik} \delta^{jl} \left( \frac{d}{\partial \phi} \right)^2 f \right]
\]

\[
\times (\theta \gamma^{im} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{kn} \theta)(\theta \gamma^{ln} \theta) ,
\]

where we note that Eqs. (33)-(36) are identical to Eqs. (29)-(32) if we choose

\[
\begin{align*}
\text{Eq. } (29) & = C_p \left( \frac{d}{\partial \phi} \right)^{\frac{(p+q)}{2}} f ,
\end{align*}
\]

\[
\text{Eq. } (32) = 0 ,
\]

and \( C_p \) are constants.
2.2 Supersymmetry transformation and the determination of the coefficient functions

From Sec. 2.1, we have explicit form of the possible terms of \( \Gamma_4 \). Upon adding \( \Gamma_4 \) to the quadratic terms of \( \Gamma_2 \), the supersymmetry transformation law in Eq. (11) should be modified. We denote the \( \Gamma_4 \)-corrected supersymmetry transformation as

\[
\delta \phi^i = -i\epsilon \gamma^i \theta + \epsilon N^i \theta, \tag{37}
\]
\[
\delta \theta = \gamma^i v^i \epsilon + M \epsilon.
\]

We note that \( O(N) = 2 \) and \( O(M) = 3 \), which let us schematically write

\[
N^i = \left[ v^2 N^{i(0)} \right] + \left[ v N^{i(2)} \theta^2 \right] + \left[ N^{i(4)} \theta^4 \right] \tag{38}
\]

and

\[
M = \left[ v^3 M^{(0)} \right] + \left[ v^2 M^{(2)} \theta^2 \right] + \left[ v M^{(4)} \theta^4 \right] + \left[ M^{(6)} \theta^6 \right]. \tag{39}
\]

When we take the supersymmetry variation of \( \Gamma_2 + \Gamma_4 \), the supersymmetry transformation Eq. (11) leaves \( \Gamma_2 \) terms invariant. However, the correction terms in Eq. (37) generate fourth order terms from \( \Gamma_2 \). Up to an order of four terms, when it comes to \( \Gamma_4 \) part, considering the variation of \( \Gamma_4 \) under Eq. (11) is enough. The correction terms in Eq. (37) when acting on \( \Gamma_4 \) produce terms of order six.

The variation \( \delta(\Gamma_2 + \Gamma_4) \) contains one, three, five, seven and nine \( \theta \) terms, and they have to separately vanish (up to total derivatives) for the invariance of the effective action under supersymmetry transformations. Specifically, we have:

\[
\delta_F \left( \left[ v^3 \theta^2 \right] \right) + \delta_B \left( \left[ v^4 \right] \right) + v^i \epsilon \left[ v^2 \hat{N}^{i(0)} \right] \theta + \frac{i}{2} \theta \left[ v^3 \hat{M}^{(0)} \right] \epsilon \simeq 0, \tag{40}
\]
\[
\delta_F \left( \left[ v^2 \theta^4 \right] \right) + \delta_B \left( \left[ v^3 \theta^2 \right] \right) + v^i \epsilon \left[ v \hat{N}^{i(2)} \theta^2 \right] \theta + \frac{i}{2} \theta \left[ v^2 \hat{M}^{(2)} \theta^2 \right] \epsilon \simeq 0, \tag{41}
\]
\[
\delta_F \left( \left[ v \theta^6 \right] \right) + \delta_B \left( \left[ v^2 \theta^4 \right] \right) + v^i \epsilon \left[ \hat{N}^{i(4)} \theta^4 \right] \theta + \frac{i}{2} \theta \left[ v \hat{M}^{(4)} \theta^4 \right] \epsilon \simeq 0, \tag{42}
\]
\[
\delta_F \left( \left[ \theta^8 \right] \right) + \delta_B \left( \left[ v \theta^6 \right] \right) + \frac{i}{2} \theta \left[ \hat{M}^{(6)} \theta^6 \right] \epsilon \simeq 0, \tag{43}
\]
\[
\delta_B \left( \left[ \theta^8 \right] \right) \simeq 0, \tag{44}
\]

where \( \delta_B \) and \( \delta_F \) represent the supersymmetric variation of the bosonic fields and the fermionic fields, respectively. The symbol \( \simeq \) denotes the fact that the equality holds up
to a total derivative. We can rewrite Eqs. (40)-(43) for an easier tractability by introducing 16 × 16 matrices
\[
L(p) = -\frac{i}{2} \left[ v^{3-p/2} M^T(p) \theta^2 \right] + v^i \left[ v^{2-p/2} N^i(p) \right],
\]
where \( p = 0, 2, 4, 6 \). In terms of \( L(p) \), Eqs. (40)-(43) become:
\[
\begin{align*}
\delta F \left( \left[ v^3 \theta^2 \right] \right) &+ \delta B \left( \left[ v^4 \right] \right) + v^i \frac{\partial}{\partial \phi^i} (\epsilon L(0) \theta) = 0, \\
\delta F \left( \left[ v^2 \theta^4 \right] \right) &+ \delta B \left( \left[ v^3 \theta^2 \right] \right) + v^i \frac{\partial}{\partial \phi^i} (\epsilon L(2) \theta) = 0, \\
\delta F \left( \left[ v \theta^6 \right] \right) &+ \delta B \left( \left[ v^2 \theta^4 \right] \right) + v^i \frac{\partial}{\partial \phi^i} (\epsilon L(4) \theta) = 0, \\
\delta F \left( \left[ \theta^8 \right] \right) &+ \delta B \left( \left[ v \theta^6 \right] \right) + v^i \frac{\partial}{\partial \phi^i} (\epsilon L(6) \theta) = 0,
\end{align*}
\]
modulo acceleration terms and higher fermion derivative terms. Modulo the same terms, the time derivative \( d/d\lambda \) has been replaced as
\[
\frac{d}{d\lambda} \to v^i \frac{\partial}{\partial \phi^i}.
\]
We also absorbed the possible total derivative terms (if any) into \( L(p) \). In general, considering the fact that \( \theta \) is multiplied from the right side of \( L(p) \) in Eqs. (46)-(49), the part of matrices \( L(p) \) that can give non-trivial contributions to Eqs. (46)-(49) can be expanded as
\[
L(p) = a^{(p)} I_{16} + a^{(p)} i \gamma^i + a^{(p)} i j \gamma^{ij} + a^{(p)} i j k \gamma^{ijk} + a^{(p)} i j k l \gamma^{ijkl},
\]
where \( a^{(p)} \)'s are \( SO(9) \) totally anti-symmetric tensors made of \( p \)-fermions \( \theta \), \((3-p/2)\)-vectors \( v^i \) and an appropriate number of scalars \( \phi^i \). Each term in Eq. (51) has a coefficient function depending only on an \( SO(9) \) invariant \( \phi \). Our goal is to solve Eqs. (46)-(49) to determine the coefficient functions of the effective action.

We first compute \( \delta F \left( \left[ v^3 \theta^2 \right] \right) \) to get
\[
\delta F \left( \left[ v^3 \theta^2 \right] \right) = 2 f_1^{(2)} (v^2)^2 \phi^i (\epsilon \gamma^i \theta) - 2 f_1^{(2)} v^2 v^i (v^j \phi^j) (\epsilon \gamma^i \theta),
\]
and we have
\[
\delta B \left( \left[ v^4 \right] \right) = -i \left( \frac{d}{d\phi} \right) f^{(0)} (v^2)^2 \phi^i (\epsilon \gamma^i \theta) .
\]
We plug Eqs. (52) and (53) into Eq. (46). We note that the two terms of Eq. (52) can not cancel with each other. Since Eqs. (52) and (53) contain only \( (\epsilon \gamma^i \theta) \), we can set \( a^{(0)} \),
$a^{(0)ij}$, $a^{(0)ijk}$ and $a^{(0)ijkl}$ to zero in Eq. (51). Furthermore, the possible terms of $a^{(0)i}$ can
not contain factors like $(v^i\phi^i)^n$ ($n > 0$), for the partial derivative $v^i\partial/(\partial \phi^i)$ then produces
$(v^i\phi^i)^{n+1}$ terms when acting on their coefficient functions. The resulting terms are not
present in Eqs. (52) and (53). This leaves us with a unique possibility

$$\epsilon L^{(0)}\theta = h^{(0)}v^2v^i(\epsilon\gamma^i\theta).$$

Upon inserting Eq. (54) into Eq. (46), the second term of Eq. (52) should cancel the term
from Eq. (54) resulting

$$\left(\frac{d}{\phi d\phi}\right) h^{(0)} = 2f^{(2)}_1.$$ (55)

The first term of Eq. (52) should cancel Eq. (53) to yield

$$f^{(2)}_1 = \frac{i}{2} \left(\frac{d}{\phi d\phi}\right) f^{(0)}.$$ (56)

The spin-orbit coupling term $f^{(2)}_1$ is now determined in terms of the bosonic coefficient
function $f^{(0)}$. It is identical to the one-loop result computed in Ref. [8] using the pertur-
native matrix theory framework.

Going to Eq. (47), we compute

$$\delta_F\left[v^2\theta^4\right] = 4v^2(f^{(4)}_2v^i\phi^i\phi^k + f^{(4)}_0v^i\delta^{ik})(\epsilon\gamma^k)(\theta\gamma^{ij}\theta)$$

$$-4f^{(4)}_2(v^i\phi^i)\phi^k(\epsilon\gamma^k)(\theta\gamma^{ij}\theta)$$

$$+2g^{(4)}_2v^2 \left(\phi^j\phi^k v^i(\epsilon\gamma^{ij}\theta)(\theta\gamma^{kl}\theta) - \phi^i\phi^j v^k(\epsilon\gamma^k)(\theta\gamma^{ij}\theta)$$

$$+ (v^k\phi^k)(\phi^i(\epsilon\gamma^j)(\theta\gamma^{ij}\theta)) \right),$$ (57)

and

$$\delta_B\left[v^3\theta^2]\right] = -iv^2 \left(\frac{d}{\phi d\phi}\right) f^{(2)}_1 v^i\phi^i\phi^k + f^{(2)}_1 v^i\delta^{ik} \right) (\epsilon\gamma^k)(\theta\gamma^{ij}\theta).$$ (58)

We note that the first term of the $g^{(4)}_2$-dependent terms of Eq. (57) can not be canceled
with any other terms of Eqs. (57) and (58). For the same reason as before, we set $a^{(2)} = 0$
and consider terms of $\epsilon L^{(2)}\theta$ that do not contain $(v^i\phi^i)^n$ ($n > 0$) terms. All possible
candidates from $\epsilon L^{(2)}\theta$ are as follows:

$$(\epsilon\gamma^i)(\theta\gamma^{ij}\theta)v^2\phi^j$$

$$(\epsilon\gamma^i)(\theta\gamma^{ik}\theta)v^i v^j\phi^k$$
recalling the Fierz identities in Appendix B. The terms of Eqs. (61)-(69) look as if they can possibly cancel the $g_2^{(4)}$-dependent terms of Eq. (57). However, once the derivative $v^i \partial / \partial \phi^i$ is taken, the maximum scalar number of the terms resulting from Eqs. (61)-(69) that do not contain the $(v^i \phi^i)$ factor is one, while the $g_2^{(4)}$-dependent terms in Eq. (57) has the maximum scalar number of two. Therefore, there are no other terms to cancel the $g_2^{(4)}$-dependent terms in Eq. (70) and this gives a non-trivial result

$$g_2^{(4)} = 0.$$  \hspace{1cm} (70)

At the same time, we are now forced to set all the terms of Eqs. (61)-(69) to zero. From the perturbative one-loop four fermion terms calculated in Ref. [14], we know that, perturbatively, the spin-spin terms are absent among the four fermion terms. Eq. (70) is the non-perturbative version of the same statement. To solve the remaining equations, we set

$$\epsilon L^{(2)} \theta = \tilde{h}_1^{(2)} v^2 \phi^j (\epsilon \gamma^j \theta) (\theta \gamma^i \theta) + h_1^{(2)} v^i v^j \phi^k (\epsilon \gamma^j \theta) (\theta \gamma^i \theta), \hspace{1cm} (71)$$

from Eqs. (59) and (60), which yields

$$\epsilon (v^k \phi^k) \left( \frac{d}{d \phi \phi} \right) \tilde{h}_1^{(2)} v^2 \phi^j (\epsilon \gamma^j \theta) (\theta \gamma^i \theta) + (v^i \phi^l) \left( \frac{d}{d \phi \phi} \right) h_1^{(2)} v^i v^j \phi^k (\epsilon \gamma^j \theta) (\theta \gamma^i \theta) + \tilde{h}_1^{(2)} v^2 v^j (\epsilon \gamma^j \theta) (\theta \gamma^i \theta), \hspace{1cm} (72)$$
upon the differentiation in Eq. (47). From Eqs. (57), (58), (70) and (72), we immediately find
\[ \tilde{h}^{(2)}_1 = 0, \quad \left( \frac{d}{d\theta \phi} \right) h^{(2)}_1 = 4f^{(4)}_2, \] (73)
\[ f^{(4)}_2 = \frac{i}{4} \left( \frac{d}{d\theta \phi} \right) f^{(2)}_1 = - \frac{1}{8} \left( \frac{d}{d\theta \phi} \right)^2 f^{(0)}, \] (74)
and
\[ f^{(4)}_0 = \frac{i}{4} f^{(2)}_1 = - \frac{1}{8} \left( \frac{d}{d\theta \phi} \right) f^{(0)}. \] (75)

These are precisely the same as the perturbatively calculated one-loop spin-orbit four fermion terms [14].

We now consider Eq. (48). We compute
\[ f^{(6)}_3 = f^{(6)}_3 (v\theta^6) = f^{(6)}_3 (v\phi^i \phi^j \phi^k \phi^l \delta F) \left[ (\theta \gamma^{ij} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) \right] + 2f^{(6)}_1 (v\phi^i \phi^j \phi^k \phi^l \delta F) \left[ (\theta \gamma^{ij} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{km} \theta) \right], \] (76)
and
\[ f^{(4)}_2 = f^{(4)}_2 (v\theta^4) = - \frac{i}{4} f^{(4)}_2 (v\phi^i \phi^j \phi^k \phi^l \delta F) \left[ (\theta \gamma^{ij} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) \right] - f^{(4)}_2 (v\phi^i \phi^j \phi^k \phi^l \delta F) \left[ (\theta \gamma^{ij} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{km} \theta) \right] \] (77)

where the (apparent) three scalar term of Eq. (76) is given by
\[ f^{(4)}_2 (v\theta^4) = f^{(4)}_2 (v\phi^i \phi^j \phi^k \phi^l \delta F) \left[ (\theta \gamma^{ij} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) \right] = 2v^i \phi^j \phi^k \phi^l \left[ v^i (\varepsilon \gamma^j \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) - v^j (\varepsilon \gamma^i \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) \right] + 2v^p (\varepsilon \gamma^{m} \theta)(\theta \gamma^{lm} \theta)(\theta \gamma^{jm} \theta) + 2v^k (\varepsilon \gamma^{m} \theta)(\theta \gamma^{lm} \theta)(\theta \gamma^{jm} \theta) - 2v^m (\varepsilon \gamma^{k} \theta)(\theta \gamma^{lm} \theta)(\theta \gamma^{jm} \theta), \] (78)

and the (apparent) one scalar term of Eq. (76) is
\[ f^{(4)}_2 (v\theta^4) = f^{(4)}_2 (v\phi^i \phi^j \phi^k \phi^l \delta F) \left[ (\theta \gamma^{ij} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{km} \theta) \right] = 2\phi^k \left[ v^i (\varepsilon \gamma^j \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{km} \theta) - v^i v^p (\varepsilon \gamma^j \theta)(\theta \gamma^{pm} \theta)(\theta \gamma^{km} \theta) \right] + v^{i} v^{p} (\varepsilon \gamma^{mp} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{ij} \theta) + v^{i} v^{p} (\varepsilon \gamma^{pm} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{ij} \theta) + v^{i} v^{k} (\varepsilon \gamma^{m} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{ij} \theta) + v^{i} v^{k} (\varepsilon \gamma^{m} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{ij} \theta) - v^{i} v^{p} (\varepsilon \gamma^{ij} \theta)(\theta \gamma^{kp} \theta)(\theta \gamma^{ij} \theta) \right], \] (79)
Among the terms of Eqs. (78) and (79), there are terms with a single \((v^i \phi^i)\) factor and terms of the form \((\epsilon \gamma^{mpk} \theta)(\theta \gamma^{lm} \theta)(\theta \gamma^{ij} \theta)\). Via Fierz identities, the second group of terms can be reduced to the simpler terms like the ones in Eq. (77). In this process, some of the indices appearing in the fermion bilinears are contracted, resulting the contractions among the \(v’s\) and \(\phi’s\). Closer inspection of the Fierz identities and the structure of the terms of Eqs. (78) and (79) show that the double contractions of \(v’s\) and \(\phi’s\) vanish and only a single contraction is allowed for the terms of Eqs. (78) and (79). Therefore, the terms of Eqs. (76) and (77) are classified as shown in Table 2.

| \((r, s)\) | \(\epsilon \gamma^{ij} \theta \gamma^{kl} \theta\) | \(\epsilon \gamma^{ij} \theta \gamma^{kl} \theta\) | \(h^{(4)}_2\) | \(h^{(4)}_0\) |
|---------|------------------|------------------|----------|----------|
| \((a)\) | \(3, 0\) | \(\frac{d}{\partial \phi^i}\) \(f_2^{(4)}\) | \(f_3^{(6)}\) | \(h^{(4)}_2\) | \(h^{(4)}_0\) |
| \((b)\) | \(2, 1\) | \((v^i \phi^i)\) \(f_3^{(6)}\) | \((v^i \phi^i)\) \(\frac{d}{\partial \phi^i}\) \(h^{(4)}_2\) |
| \((c)\) | \(1, 0\) | \(f_2^{(4)}\) | \(\phi^2 \phi^i\) \(f_3^{(6)}\) | \(h^{(4)}_2\) |
| \((d)\) | \(0, 1\) | \((v^i \phi^i)\) \(f_1^{(6)}\) | \((v^i \phi^i)\) \(\frac{d}{\partial \phi^i}\) \(h^{(4)}_0\) |

Table 2. The classification of the terms appearing in Eq. (48). Pairs of numbers \((r, s)\) shown in the second column denote the scalar number and the number of \((v^i \phi^i)\), respectively. The terms with different \((r, s)\) can not cancel with each other. Each entry in the table shows the \(SO(9)\) scalar factor of each term, suppressing the degeneracy and the tensor structures.

Also shown in Table 2 is the classification of the terms from \(\epsilon L^{(4)} \theta\) that show up in Eq. (48). Generally, the possible terms of \(\epsilon L^{(4)} \theta\) include zero, one, two and three scalar structure terms whose scalar coefficient functions we denote as \(h^{(4)}_q\) where \(q = 0, 1, 2, 3\) is the number of scalars, since \(\epsilon L^{(4)} \theta \propto (\epsilon \gamma^{ij} \theta)(\theta \gamma^{kl} \theta)(\theta \gamma^{ij} \theta)\). The three scalar structure terms of \(\epsilon L^{(4)} \theta\) produce \((3, 1)\) and \((2, 0)\) terms that do not appear elsewhere in Eq. (48), upon taking the derivative \(v^i \partial^i/\partial \phi^i\). Likewise, the one scalar structure terms of \(\epsilon L^{(4)} \theta\) yield \((1, 1)\) and \((0, 0)\) terms in Eq. (48), which are also absent in Table 2. As such, the coefficient functions \(h^{(4)}_3 = h^{(4)}_1 = 0\). Therefore, only the two scalar and zero scalar terms of \(\epsilon L^{(4)} \theta\) are non-vanishing and they correspond to the \(h^{(4)}_2\) and \(h^{(4)}_0\) columns of Table 2.

It is seen to be clear how to determine the scalar coefficient functions. We notice from the row \((a)\) of Table 2 that there are no contributions from \(\epsilon L^{(4)} \theta\) for the maximum scalar structure terms of type \((3, 0)\). The \((3, 0)\) terms from Eq. (78) should directly cancel the

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\(^4\)Precisely this observation was used to simplify the membrane spin-orbit coupling calculations of Ref. [23].
maximum scalar number terms of Eq. (77), yielding \( f_3^{(6)} \propto (d/(\phi d\phi)) f_2^{(4)} \). Next, from the row (b) of Table 2, \( h_2^{(4)} \) is determined to give \( h_2^{(4)} \propto f_2^{(4)} \) via \( (d/(\phi d\phi)) h_2^{(4)} \propto f_3^{(6)} \). The third row (c) determines \( f_1^{(6)} \) in terms of \( f_2^{(4)} \), \( \phi^2 f_3^{(6)} \) and \( h_2^{(4)} \) to be \( f_1^{(6)} \propto f_2^{(4)} \). Finally, from the row (d), the functions \( h_0^{(4)} \) are obtained as \( h_0^{(4)} \propto f_0^{(4)} \). Hereafter, we present the detailed derivation of \( f_3^{(6)} \) from the row (a). One technical comment should be in order; as the number of fermions increases, we need progressively more complicated Fierz identities. Especially when the Fierz identities involve two different constant spinors \( \epsilon \) and \( \theta \), they become even more complicated. For the simplification of the computations, we note that an arbitrary \( \epsilon \) can be obtained by multiplying \( \theta \) with an appropriate \( 16 \times 16 \) matrix. Recalling a complete expansion of the form (51), it is thus equivalent to consider \( \theta \gamma^i \), \( \theta \gamma^{ij} \), \( \theta \gamma^{ijk} \) and \( \theta \gamma^{ijkl} \) in place of \( \epsilon \). From now on, we will replace \( \epsilon \) with \( \theta \gamma^i \). The cases for \( \theta \gamma^{ij} \), \( \theta \gamma^{ijk} \) and \( \theta \gamma^{ijkl} \) can be analyzed in a similar fashion to show the complete consistency.

Upon replacing \( \epsilon \rightarrow \theta \gamma^n \) and retaining the maximum three scalar terms of Eq. (77), we have

\[
-i \left( \frac{d}{\phi d\phi} \right) f_2^{(4)} v^i v^p \phi^j \phi^k \phi^l (\theta \gamma^{nj} \theta)(\theta \gamma^{mk} \theta)(\theta \gamma^{pl} \theta) ,
\]

and

\[
2 f_3^{(6)} v^i v^p \phi^j \phi^k \phi^l \left[ \delta^{ip} (\theta \gamma^{nj} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) + 2 \delta^{np} (\theta \gamma^{ij} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) 
- 2 \delta^{nk} (\theta \gamma^{ij} \theta)(\theta \gamma^{pm} \theta)(\theta \gamma^{lm} \theta) \right] ,
\]

from Eq. (78) (apparent) three scalar terms using the Wick theorem. We have to check whether the terms of Eq. (81) turn into the terms of Eq. (80) up to \( (v^i \phi^j) \) terms and the lower scalar number terms. Using the Fierz identities Eqs. (115)-(119), we can show that:

\[
v^i v^p \phi^j \phi^k \phi^l \left[ \delta^{ip} (\theta \gamma^{nj} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) + 2 \delta^{np} (\theta \gamma^{ij} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) 
- 2 \delta^{nk} (\theta \gamma^{ij} \theta)(\theta \gamma^{pm} \theta)(\theta \gamma^{lm} \theta) \right]
= v^i v^p \phi^j \phi^k \phi^l \left[ 45 (\theta \gamma^{nj} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{pl} \theta) 
- 4 \delta^{ip} (\theta \gamma^{nj} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) 
- 8 \delta^{np} (\theta \gamma^{ij} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) 
+ 8 \delta^{nk} (\theta \gamma^{ij} \theta)(\theta \gamma^{pm} \theta)(\theta \gamma^{lm} \theta) \right]
+ (v^i \phi^j) \text{ terms and lower scalar terms} \,.
\]
We start from using Fierz identity Eq. (116). We sequentially use the Fierz identity Eq. (115) and the five free index Fierz identities, Eqs. (117)-(119) to reduce the terms of the left hand side into the form where we can use the Fierz identity Eq. (116). Then, using Eq. (116) again further reduces all the terms into the three products of $(\theta\gamma^{ij}\theta)$'s, producing the identity Eq. (82). In this process, noticing the permutation symmetry of $\phi^j\phi^k\phi^l$ simplifies the calculations. Eq. (82) can be rearranged to yield:

$$v^i v^p \phi^j \phi^k \phi^l \left[ \phi^{ij} \phi^{km} \phi^{nl} + 2 \phi^{ip} \phi^{km} \phi^{nl} - 2 \phi^{nk} \phi^{ij} \phi^{pm} \phi^{nl} \right] = 9 v^i v^p \phi^j \phi^k \phi^l \left[ \phi^{ij} \phi^{km} \phi^{nl} + (v^i \phi^i) \text{terms and lower scalar terms} \right]. \quad (83)$$

Now Eq. (83) immediately implies that Eq. (81) becomes

$$18 f_3^{(6)} v^i v^p \phi^j \phi^k \phi^l \left[ \phi^{ij} \phi^{km} \phi^{nl} + (v^i \phi^i) \text{terms and lower scalar terms} \right], \quad (84)$$

and, via Eq. (48), yields

$$f_3^{(6)} = \frac{i}{18} \left( \frac{d}{d\phi} \right) f_2^{(4)} = - \frac{i}{144} \left( \frac{d}{d\phi} \right)^3 f^{(0)}. \quad (85)$$

Now that we determined $f_3^{(6)}$, we can recursively solve the row (b) and (c) of Table 2 to determine the function $f_1^{(6)}$. The result is:

$$f_1^{(6)} = \frac{i}{18} f_2^{(4)} = - \frac{i}{144} \left( \frac{d}{d\phi} \right)^2 f^{(0)}. \quad (86)$$

We remark that we again need to use the five free index Fierz identities shown in Appendix B. The terms of the effective action, Eqs. (83) and (86), for six fermion mixed spin-orbit spin-spin couplings have not yet been calculated within the perturbative matrix theory framework. However, Eqs. (83) and (86) are identical to the corresponding terms computed from the perturbative IIA string theory framework [17, 18].

Before proceeding to Eq. (49), we recall that the eight fermion terms in the effective action can be completely determined up to an overall normalization by solving Eq. (44) [12]. Thus, the remaining task is only to consider the maximum scalar number terms of Eq. (49), which links the bosonic variation of the three scalar term of the six fermion terms to the fermionic variation of the four scalar term of the eight fermion terms. As noted in the computation of the six fermion terms, the knowledge of $\epsilon L^{(6)}\theta$ terms is not
necessary for this purpose. Furthermore, replacing $\epsilon$ with $\theta \gamma^i$ is enough to get the desired result. The consideration of $\theta, \theta \gamma^i, \theta \gamma^{jk}$ and $\theta \gamma^{ijkl}$ in place of $\epsilon$ can be straightforwardly performed to show the complete consistency, although the computations are quite lengthy in these cases. We compute the fermionic variation of the eight fermion terms

$$
\delta_F[\theta^8] = f_4^{(8)} \phi^i \phi^j \phi^k \phi^l \delta_F \left[ (\theta \gamma^{im} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{kn} \theta)(\theta \gamma^{ln} \theta) \right]
+ 4f_2^{(8)} \delta^i_k \delta^j_l \delta_F \left[ (\theta \gamma^{im} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{kn} \theta)(\theta \gamma^{ln} \theta) \right]
+ 2f_0^{(8)} \delta^i_k \delta^j_l \delta_F \left[ (\theta \gamma^{im} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{kn} \theta)(\theta \gamma^{ln} \theta) \right]
$$

and the bosonic variation of the three scalar term of the six fermion terms

$$
\delta_B[v^6] = -i \left( \frac{d}{\phi d\phi} \right) f_3^{(6)} v^p \phi^i \phi^j \phi^k \phi^l (\epsilon^i \theta)(\theta \gamma^m \theta)(\theta \gamma^k \theta)(\theta \gamma^l \theta) + \text{lower scalar terms},
$$

both of which appear in Eq. \([49]\). Written explicitly, the (apparent) four scalar term of Eq. \([57]\) is given by

$$
f_4^{(8)} \phi^i \phi^j \phi^k \phi^l \delta_F \left[ (\theta \gamma^{im} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{kn} \theta)(\theta \gamma^{ln} \theta) \right] = 8f_4^{(8)} \phi^i \phi^j \phi^k \phi^l \left[ v^p (\epsilon^i \theta)(\theta \gamma^m \theta)(\theta \gamma^k \theta)(\theta \gamma^l \theta) - v^m (\epsilon^i \theta)(\theta \gamma^m \theta)(\theta \gamma^k \theta)(\theta \gamma^l \theta) \right].
$$

Upon the replacement $\epsilon \to \theta \gamma^s$, the four scalar terms of Eq. \([88]\) become

$$
- i \left( \frac{d}{\phi d\phi} \right) f_3^{(6)} v^p \phi^i \phi^j \phi^k \phi^l (\theta \gamma^{si} \theta)(\theta \gamma^{pj} \theta)(\theta \gamma^{kn} \theta)(\theta \gamma^{ln} \theta),
$$

and Eq. \([89]\) becomes

$$
8f_4^{(8)} \phi^i \phi^j \phi^k \phi^l v^p \left[ \delta^{si}(\theta \gamma^{pm} \theta)(\theta \gamma^{mj} \theta)(\theta \gamma^{kn} \theta)(\theta \gamma^{ln} \theta) + \delta^{sp}(\theta \gamma^{mi} \theta)(\theta \gamma^{mj} \theta)(\theta \gamma^{kn} \theta)(\theta \gamma^{ln} \theta) \right],
$$

using the Wick theorem. We have to check whether the terms of Eq. \([11]\) become the form of Eq. \([10]\) up to $(v^i \phi^j)$ terms and the lower scalar number terms. Using the Fierz identities in Appendix B, we can show the following identity:

$$
\phi^i \phi^j \phi^k \phi^l v^p \left[ \delta^{si}((\theta \gamma^{pm} \theta)(\theta \gamma^{mj} \theta)(\theta \gamma^{ln} \theta) + \delta^{sp}(\theta \gamma^{mi} \theta)(\theta \gamma^{mj} \theta)(\theta \gamma^{ln} \theta) \right]
= \phi^i \phi^j \phi^k \phi^l v^p \left[ -35(\theta \gamma^{si} \theta)(\theta \gamma^{pj} \theta)(\theta \gamma^{kn} \theta)(\theta \gamma^{ln} \theta)
+ 6\delta^{si}(\theta \gamma^{pm} \theta)(\theta \gamma^{mj} \theta)(\theta \gamma^{ln} \theta)
+ 6\delta^{sp}(\theta \gamma^{mi} \theta)(\theta \gamma^{mj} \theta)(\theta \gamma^{kn} \theta)(\theta \gamma^{ln} \theta) \right]
+ (v^i \phi^j) \text{ terms and lower scalar terms}.
$$

18
In deriving Eq. (92), we first use the Fierz identity Eq. (116). Next, Eq. (115) and the five free index Fierz identities, Eqs. (118) and (119), are utilized to transform the terms on the left hand side into the form where we can use the Fierz identity Eq. (116) again. Then, upon using Eq. (116), we reduce all the terms into four products of \((\theta \gamma^{ij})\)’s. Using the identity Eq. (92), we rewrite Eq. (91) as

\[
56f^{(8)}_4 \phi^i \phi^j \phi^k \phi^l v^p (\theta \gamma^{si} \theta) (\theta \gamma^{pj} \theta) (\theta \gamma^{kn} \theta) (\theta \gamma^{ln} \theta) + (v^i \phi^j) \text{ terms and lower scalar terms,}
\]

which, via Eq. (94), implies

\[
f^{(8)}_4 = \frac{i}{56} \left( \frac{d}{\phi d\phi} \right) f^{(6)}_3 = \frac{1}{8064} \left( \frac{d}{\phi d\phi} \right)^4 f^{(0)}.
\]  

By solving Eq. (44), the coefficient functions \(f^{(8)}_4, f^{(8)}_2\) and \(f^{(8)}_0\) are computed to be proportional to \(\phi^{-15}, \phi^{-13}\) and \(\phi^{-11}\), respectively, with known relative coefficients [12]:

\[
f^{(8)}_4 = c \frac{1}{\phi^{15}}, \quad f^{(8)}_2 = -c \frac{1}{13 \phi^{13}}, \quad f^{(8)}_0 = \frac{c}{143 \phi^{11}}
\]

where \(c\) is an overall constant. We can immediately integrate Eq. (94) to obtain

\[
f^{(0)} = k_1 + k_2 \frac{1}{\phi^5} + k_3 \phi^2 + k_4 \phi^4 + k_5 \phi^6,
\]

where \(k_1, k_3, k_4, k_5\) are constants of integration and \(k_2 = (128/143)c\). The \(\phi \to \infty\) limit of the \(F^4\) terms should be finite from the physical point of view and we thus set \(k_3 = k_4 = k_5 = 0\). We therefore find that

\[
f^{(8)}_2 = \frac{1}{8064} \left( \frac{d}{\phi d\phi} \right)^3 f^{(0)}, \quad f^{(8)}_0 = \frac{1}{8064} \left( \frac{d}{\phi d\phi} \right)^2 f^{(0)}.
\]

Eqs. (94) and (97) are identical to the one-loop spin-spin terms calculated in Ref. [15]. This completes our analysis and we derived the results shown in Sec. 1, Eqs. (1)-(6), by collecting Eqs. (10), (96), (56), (70), (74), (75), (85), (86), (94) and (97).

3 Discussions

The consideration in this paper has been restricted to the case of the \(M\)-momentum scatterings. However, for the extended objects, we can also use the results of the analysis.
presented in this paper as far as the ‘center of mass’ dynamics is concerned. For example, for the membrane dynamics considered in Ref. [22] (see also [21]), the four scalar term of the eight fermion terms in $n$-instanton sector has been computed to be

$$f_4^{(8)} = k_1 n^{13/2} \phi^{-13/2} K_{13/2}(n\phi/g)e^{in\phi/g},$$  \hfill (98)

and the two scalar term and the zero scalar term coefficient functions $4f_2^{(8)}$ and $2f_0^{(8)}$ are the inhomogeneous solutions of

$$\left( \frac{d^2}{d\phi^2} + \frac{10}{\phi} \frac{d}{d\phi} - \frac{n^2}{g^4} \right) 4f_2^{(8)} = -8f_4^{(8)}$$  \hfill (99)

and

$$\left( \frac{d^2}{d\phi^2} + \frac{6}{\phi} \frac{d}{d\phi} - \frac{n^2}{g^4} \right) 2f_0^{(8)} = -2(4f_2^{(8)}),$$  \hfill (100)

respectively. Here $k_1$ and $g$ are dimensionful constants and $K_\nu$ is the modified Bessel function with a half-integer coefficient $\nu$. The $SO(7)$ vectors $\phi^i$ ($i = 1, \cdots, 7$) combine to give an $SO(7)$ invariant $\phi^2 = \phi^i \phi^i$, and $\phi^8$ is the dual magnetic scalar. It can be easily shown that the function $f_\nu = \phi^{-\nu} K_\nu(n\phi/g^2)$ satisfies the differential equation

$$\left( \frac{d^2}{d\phi^2} + \frac{2\nu + 1}{\phi} \frac{d}{d\phi} - \frac{n^2}{g^4} \right) f_\nu = 0.$$  \hfill (101)

From the recursion relation

$$\left( \frac{d}{zd\zeta} \right)^a (z^{-\nu} K_\nu(z)) = (-1)^a z^{-\nu-a} K_{\nu+a}(z),$$  \hfill (102)

we can immediately write down the inhomogeneous solutions of Eqs. (99) and (100) as

$$4f_2^{(8)} = -4k_1 gn^{11/2} \phi^{-11/2} K_{11/2}(n\phi/g)e^{in\phi/g}$$  \hfill (103)

and

$$2f_0^{(8)} = 2k_1 g^2 n^{9/2} \phi^{-9/2} K_{9/2}(n\phi/g)e^{in\phi/g},$$  \hfill (104)

where we simultaneously use Eq. (101) to delete the second derivative terms and the zero derivative terms of Eqs. (99) and (100). Noting that

$$f^{(0)} = k_2 n^{5/2} \phi^{-5/2} K_{5/2}(n\phi/g)e^{in\phi/g},$$  \hfill (105)

where $k_2$ is a constant, from Ref. [21] and recalling Eq. (102), we conclude that Eqs. (103) and (104) are completely consistent with Eq. (5) for $p = 4$ and $q = 0, 2, 4$. It will be interesting to apply this type of arguments to the higher brane two-body dynamics.
The derivation presented in Sec. 2.2 does not appear to sensitively depend on the existence of the unbroken \(SO(9)\) \(R\)-symmetry, even if the classification of the possible terms in Sec. 2.1 does. Consequently, for an arbitrary point in the moduli space, that generally breaks \(SO(9)\) to its subgroup and represents an arbitrarily separated source \(M\)-momenta, we write down the effective action (by the linear superposition of the source \(M\)-momenta, which is valid when there are full supersymmetries):

\[
\Gamma^{(4)} = \int \mathcal{D} \lambda \left[ f(v^2)^2 + \frac{i}{2} v^i v^j \partial_j f(\theta \gamma^{ij} \theta) - \frac{1}{8} v^i v^j \partial_k \partial_l f(\theta \gamma^{ik} \theta)(\theta \gamma^{jl} \theta) \right.
\]

\[
- \frac{i}{144} v^i \partial_j \partial_k \partial_l f(\theta \gamma^{ij} \theta)(\theta \gamma^{km} \theta)(\theta \gamma^{lm} \theta) + \frac{1}{8064} \partial_i \partial_j \partial_k \partial_l f(\theta \gamma^{im} \theta)(\theta \gamma^{jm} \theta)(\theta \gamma^{kn} \theta)(\theta \gamma^{ln} \theta) \biggr],
\]

(106)

where \(f\) is an arbitrary nine-dimensional harmonic function that vanishes as \(\phi^i \to \infty\) and we use the normalization convention of the quadratic terms of Eq. (10). The formal observation at the end of Sec. 2.1 is used to write down the action (106). Up to two fermion terms, Eq. (106) agrees with the probe dynamics calculations in the DLCQ supergravity framework using the multi-center \(M\)-momenta solutions as the background geometry [10]. The multi-center background geometry solutions of the DLCQ supergravity preserve the sixteen supersymmetries as in the case of the single center solutions. Furthermore, the BPS solution space for the \(N\) source \(M\)-momenta from the supergravity is the \(N\)-symmetric product of \(R^9\), just like the SYM quantum mechanics moduli space.

Beyond our analysis presented in this paper, a possible next step is to repeat the same type of analysis to the \(F^6\) terms. The supersymmetric completion of these terms, once determined, can be used to prove the two-loop exactness of the \(v^6\) term, which is a necessary element in firmly establishing the matrix theory/DLCQ supergravity correspondence. Furthermore, at this order, we expect that the matrix theory produces genuine quantum gravity corrections to the eleven-dimensional supergravity. It will be interesting to explicitly compute these terms and compare them to the quantum corrected DLCQ supergravity and to the type II stringy corrections. Another very interesting issue, as mentioned in Sec. 1, is to understand how much of the constructions presented here can survive under the supersymmetry breaking.
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Appendix

A Reduction of $\theta\gamma^{ijk}\theta$ products to $\theta\gamma^{ij}\theta$ products

We start from writing all possible terms of $8v^2KK$. There are seven possible terms:

\begin{align*}
(v^i \phi^j) (v^j \phi^k (\theta\gamma^{jlm}\theta)(\theta\gamma^{klm}\theta)) , \\
v^2 \phi^i \phi^j (\theta\gamma^{ijkl}\theta)(\theta\gamma^{jkl}\theta) , \\
v^i v^j \phi^k (\theta\gamma^{ikm}\theta)(\theta\gamma^{jlm}\theta) , \\
(v^i \phi^j)(v^j \phi^k (\theta\gamma^{jlm}\theta)(\theta\gamma^{klm}\theta)) , \\
(v^i \phi^j) (v^j \phi^k (\theta\gamma^{jkl}\theta)(\theta\gamma^{jkl}\theta)) , \\
v^2 (\theta\gamma^{ijk}\theta)(\theta\gamma^{ijk}\theta) , \\
v^i v^j (\theta\gamma^{ikl}\theta)(\theta\gamma^{jkl}\theta) .
\end{align*}

Using the Fierz identities (108), (113) and (116), we can show that all the terms in the above reduce to the terms of $6v^2JJ$ or vanish. In the case of the terms of $7v^2JK$, there are terms like

\begin{align*}
v^i v^j \phi^k (\theta\gamma^{ijk}\theta)(\theta\gamma^{ijk}\theta),
\end{align*}

which can not be reduced to the terms of $6v^2JJ$ even if we use Fierz identities; the number of scalars do not match.

The possible $9v^2KK$ terms are as follows.

\begin{align*}
v^i \phi^j \phi^k \phi^l (\theta\gamma^{ij}\theta)(\theta\gamma^{klmn}\theta)(\theta\gamma^{lmm}\theta) , \\
v^i \phi^j (\theta\gamma^{ik}\theta)(\theta\gamma^{jkmn}\theta)(\theta\gamma^{jmn}\theta) , \\
v^i \phi^j (\theta\gamma^{ij}\theta)(\theta\gamma^{klm}\theta)(\theta\gamma^{klm}\theta) , \\
v^i \phi^j (\theta\gamma^{ijkl}\theta)(\theta\gamma^{jkl}\theta) .
\end{align*}
An efficient algorithm for generating Fierz identities has recently been given in Ref. [7]. By

\[ v^i \phi^j \phi^k \phi^l (\theta \gamma^{jm} \theta) (\theta \gamma^{kn} \theta) (\theta \gamma^{ln} \theta) \],

\[ v^i \phi^m (\theta \gamma^{jl} \theta) (\theta \gamma^{ik} \theta) (\theta \gamma^{lm} \theta) \],

\[ v^i \phi^m (\theta \gamma^{km} \theta) (\theta \gamma^{ij} \theta) (\theta \gamma^{kl} \theta) \],

\[ (v^i \phi^j) (\theta \gamma^{kl} \theta) (\theta \gamma^{jm} \theta) (\theta \gamma^{kn} \theta) \],

\[ (v^i \phi^j) (\theta \gamma^{jl} \theta) (\theta \gamma^{km} \theta) (\theta \gamma^{im} \theta) \],

\[ (v^i \phi^j) (\theta \gamma^{jk} \theta) (\theta \gamma^{im} \theta) (\theta \gamma^{kn} \theta) = 0 \].

Again, these terms either vanish or reduce to the terms of \( \tilde{T}^{vJJ} \) upon using the Fierz
identities Eqs. (108), (113) and (116).

**B Fierz Identities**

An efficient algorithm for generating Fierz identities has recently been given in Ref. [7]. By
implementing that algorithm using Mathematica, we obtain the following Fierz identities.

\[ (\epsilon \gamma^{a_1 a_2} \theta) (\theta \gamma^{a_1 a_2} \theta) = 0 \] (107)

\[ (\epsilon \gamma^{a_1 a_2 a_3} \theta) (\theta \gamma^{a_1 a_2 a_3} \theta) = 0 \] (108)

\[ (\epsilon \gamma^{a_1 a_2} \theta) (\theta \gamma^{a_1 a_2} \theta) = 2(\epsilon \gamma^{a_1} \theta) (\theta \gamma^{a_1} \theta) \] (109)

\[ (\epsilon \gamma^{a_1 a_2} \theta) (\theta \gamma^{a_1 a_2} \theta) = -2(\epsilon \gamma^{a_1} \theta) (\theta \gamma^{a_1} \theta) \] (110)

\[ (\epsilon \gamma^{a_1 a_2 a_3} \theta) (\theta \gamma^{a_1 a_2 a_3} \theta) = -6(\epsilon \gamma^{a_1} \theta) (\theta \gamma^{a_1} \theta) \] (111)

\[ (\theta \gamma^{a_1 a_2} \theta) (\theta \gamma^{a_1 a_2} \theta) = 0 \] (112)

\[ (\theta \gamma^{a_1 a_2} \theta) (\theta \gamma^{a_1 a_2} \theta) = 2(\theta \gamma^{a_1} \theta) (\theta \gamma^{a_1} \theta) \] (113)

\[ (\theta \gamma^{a_1 a_2} \theta) (\theta \gamma^{a_1 a_2} \theta) = 0 \] (114)

\[ (\theta \gamma^{a_1 a_2} \theta) (\theta \gamma^{a_1 a_2} \theta) + (\theta \gamma^{a_1} \theta) (\theta \gamma^{a_1} \theta) + (\theta \gamma^{a_1} \theta) (\theta \gamma^{a_1} \theta) = 0 \] (115)

\[ (\theta \gamma^{a_1 ij} \theta) (\theta \gamma^{a_1 kl} \theta) = -3(\theta \gamma^{ij} \theta) (\theta \gamma^{kl} \theta) - 2(\theta \gamma^{ik} \theta) (\theta \gamma^{jl} \theta) + 2(\theta \gamma^{il} \theta) (\theta \gamma^{jk} \theta) \]

\[ - \delta^{jk} (\theta \gamma^{a_1 l} \theta) (\theta \gamma^{a_1 l} \theta) - \delta^{ij} (\theta \gamma^{a_1 j} \theta) (\theta \gamma^{a_1 k} \theta) + \delta^{ik} (\theta \gamma^{a_1 j} \theta) (\theta \gamma^{a_1 l} \theta) \]

\[ + \delta^{il} (\theta \gamma^{a_1 j} \theta) (\theta \gamma^{a_1 k} \theta) \] (116)
\[
0 = (\theta \gamma^{lm}) (\theta \gamma^{ijk}) - (\theta \gamma^{km}) (\theta \gamma^{jil}) + (\theta \gamma^{kl}) (\theta \gamma^{ijm}) \\
- (\theta \gamma^{jm}) (\theta \gamma^{ikl}) + (\theta \gamma^{il}) (\theta \gamma^{jkm}) - (\theta \gamma^{jk}) (\theta \gamma^{ilm}) \\
+ (\theta \gamma^{im}) (\theta \gamma^{jkl}) - (\theta \gamma^{i}) (\theta \gamma^{jkm}) + (\theta \gamma^{ik}) (\theta \gamma^{jlm}) + 3(\theta \gamma^{ij}) (\theta \gamma^{klm}) \\
- \delta^{ik}(\theta \gamma^{aij}) (\theta \gamma^{a1lm}) + \delta^{il}(\theta \gamma^{a1j}) (\theta \gamma^{a1km}) - \delta^{im}(\theta \gamma^{a1j}) (\theta \gamma^{a1kl}) \\
+ \delta^{jk}(\theta \gamma^{a1i}) (\theta \gamma^{a1lm}) - \delta^{jl}(\theta \gamma^{a1i}) (\theta \gamma^{a1km}) + \delta^{jm}(\theta \gamma^{a1i}) (\theta \gamma^{a1kl}) \quad (117)
\]

\[
(\theta \gamma^{a1a2i})(\theta \gamma^{a1a2jklm}) = 2(\theta \gamma^{lm})(\theta \gamma^{ijk}) - 2(\theta \gamma^{km})(\theta \gamma^{jil}) + 2(\theta \gamma^{kl})(\theta \gamma^{ijm}) \\
+ 2(\theta \gamma^{jm})(\theta \gamma^{ikl}) - 2(\theta \gamma^{il})(\theta \gamma^{jkm}) + 2(\theta \gamma^{ik})(\theta \gamma^{jlm}) \\
+ 2(\theta \gamma^{im})(\theta \gamma^{jkl}) - 2(\theta \gamma^{il})(\theta \gamma^{jkm}) + 2(\theta \gamma^{jk})(\theta \gamma^{jlm}) \\
- 2(\theta \gamma^{ij})(\theta \gamma^{klm}) \quad (118)
\]

\[
0 = -2(\theta \gamma^{lm})(\theta \gamma^{ijk}) + 2(\theta \gamma^{km})(\theta \gamma^{jil}) - 2(\theta \gamma^{kl})(\theta \gamma^{ijm}) \\
- 2(\theta \gamma^{jm})(\theta \gamma^{ikl}) + 2(\theta \gamma^{il})(\theta \gamma^{jkm}) - 2(\theta \gamma^{jk})(\theta \gamma^{ilm}) \\
+ 2(\theta \gamma^{im})(\theta \gamma^{jkl}) - 2(\theta \gamma^{il})(\theta \gamma^{jkm}) + 2(\theta \gamma^{ik})(\theta \gamma^{jlm}) \\
+ 10(\theta \gamma^{ij})(\theta \gamma^{klm}) - (\theta \gamma^{a1a2j})(\theta \gamma^{a1a2jklm}) + (\theta \gamma^{a1a2i})(\theta \gamma^{a2a1klm}) \\
+ 2\delta^{ik}(\theta \gamma^{a1m})(\theta \gamma^{a1j}) - 2\delta^{ik}(\theta \gamma^{a1j})(\theta \gamma^{a1m}) - 2\delta^{il}(\theta \gamma^{a1m})(\theta \gamma^{a1k}) \\
+ 2\delta^{il}(\theta \gamma^{a1k})(\theta \gamma^{a1m}) + 2\delta^{im}(\theta \gamma^{a1i})(\theta \gamma^{a1k}) + 2\delta^{im}(\theta \gamma^{a1k})(\theta \gamma^{a1j}) \\
- 2\delta^{jk}(\theta \gamma^{a1m})(\theta \gamma^{a1il}) + 2\delta^{jk}(\theta \gamma^{a1l})(\theta \gamma^{a1im}) + 2\delta^{jl}(\theta \gamma^{a1m})(\theta \gamma^{a1ik}) \\
- 2\delta^{jl}(\theta \gamma^{a1k})(\theta \gamma^{a1im}) - 2\delta^{jm}(\theta \gamma^{a1l})(\theta \gamma^{a1ik}) + 2\delta^{jm}(\theta \gamma^{a1k})(\theta \gamma^{a1il}) \quad (119)
\]

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