Solitary wave solutions and global well-posedness for a coupled system of gKdV equations

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Abstract. In this work, we consider the initial-value problem associated with a coupled system of generalized Korteweg–de Vries equations. We present a relationship between the best constant for a Gagliardo–Nirenberg type inequality and a criterion for the existence of global solutions in the energy space. We prove that such a constant is directly related to the existence problem of solitary wave solutions with minimal mass, the so-called ground state solutions. A characterization of the ground states and the orbital instability of the solitary waves are also established.

1. Introduction

Nonlinear dispersive systems appear in many physical applications. They can be used, for instance, to model the propagation of waves in water surface or to describe the interaction of nonlinear internal waves. In the present paper, we are interested in systems having the Hamiltonian form

\[
\begin{align*}
\partial_t u + \partial_x^3 u + \mu \partial_x (H_u(u,v)) &= 0, \\
\partial_t v + \partial_x^3 v + \mu \partial_x (H_v(u,v)) &= 0,
\end{align*}
\]

where \(u = u(x,t)\) and \(v = v(x,t)\) are real-valued functions, \(H\) is a smooth function, \(H_u\) and \(H_v\) denote the derivative of \(H\) with respect to \(u\) and \(v\), respectively, and \(\mu\) is a real constant which we normalize to be \(\pm 1\).

Systems of the form (1.1) are said to be of KdV type and model important phenomena in the propagation on nonlinear waves. To cite a few examples, in the case \(\mu = 1\) and

\[
H(u, v) = Au^3 + Bv^3 + Cu^2v + Du^2v
\]

with \(A, B, C\) and \(D\) real constants, the system was derived by Gear and Grimshaw [16] to describe the strong interaction of two-dimensional long internal gravity waves propagating on neighboring pycnoclines in a stratified fluid. Also, in the case

\[
H(u, v) = u^2v,
\]

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system (1.1) is a particular case of the Majda–Biello system [26] (see also [4] and [3]), which models the nonlinear interaction of long-wavelength equatorial Rossby waves and barotropic Rossby waves.

The issue of local and global well-posedness for the initial-value problem (IVP) associated with (1.1) has become a major topic in the theory of dispersive equations in recent years. Let us briefly recall some results of our interest available in the current literature. The well-posedness problem associated with IVP (1.1) with function $H$ given by (1.2) was studied by many authors. For instance, Bona, Ponce, Saut and Tom [7] proved that, under some restrictions on the coefficients, the associated IVP is globally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1$. Also, Linares and Panthee in [21] obtained the sharp local result for Sobolev spaces with index $s > -\frac{3}{4}$. Besides, in [21] was also proved the global well-posedness for $s > -\frac{3}{10}$ under some restrictions on the coefficients $A, B, C$ and $D$. The well-posedness for the Majda–Biello system was studied, for instance, by Oh [29] where the author proved local well-posedness in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -\frac{1}{2}$ and in $H^{-\frac{1}{2}}(\mathbb{T}) \times H^{-\frac{1}{2}}(\mathbb{T})$. In [28], via the I-method, Oh established the global well-posedness $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 0$ and $H^s(\mathbb{T}) \times H^s(\mathbb{T})$, $s > -\frac{1}{2}$. Also, Guo et al. [18] considered the periodic problem and used a successive time-averaging method to prove the global well-posedness in the homogeneous Sobolev space $\dot{H}^s(\mathbb{T})$, $s \geq 0$.

Panthee and Scialom in [31] studied (1.1) with $H(u, v) = \frac{1}{3}u^3v^3$. In this case, the system contains a pair of “critical” generalized KdV equations. The authors showed local well-posedness in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 0$ utilizing the sharp smoothing estimates to the linear problem combined with the contraction mapping principle. Global well-posedness for data with small Sobolev norm was also established. In particular, they showed if $\|(u_0, v_0)\|_{L^2 \times L^2} < \|(S, S)\|_{L^2 \times L^2}$, where $S$ is an associated ground state solution, then the IVP is globally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s > \frac{3}{4}$.

Corcho and Panthee in [11] considered a coupled system of modified KdV equations. More precisely, they studied (1.1) with

$$H(u, v) = \frac{a_1 u^4}{4} + \frac{b_1}{4} v^4 + \frac{a_2}{2} (uv)^2 + \frac{a_3}{3} u^3v + \frac{a_4}{3} uv^3.$$ 

The authors used the second generations of the modified energy and almost conserved quantities introduced by Colliander, Keel, Staffilani, Takaoka, and Tao [9, 10] to obtain global well-posedness in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s > \frac{1}{4}$.

Alarcon, Angulo and Montenegro [1] studied (1.1) with $H(u, v) = \frac{u^{k+1}}{k+1}v^{k+1}$, where $k \geq 1$ is a natural number and obtained global well-posedness in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1$, under suitable conditions on $k$. Moreover, the authors also established sufficient conditions for the orbital stability and instability of the associated traveling waves.
Our main objective in this paper is to study the IVP associated with (1.1) when $H$ has the form
\[
H(u, v) = \frac{a}{2k+2} (u^{2k+2} + v^{2k+2}) + \frac{b}{k+1} (uv)^{k+1} + \frac{c}{k} u^{k+1}v^{k} + \frac{d}{k} u^{k}v^{k+2},
\]
(1.3)

with $k \geq 1$ a natural number and $a, b, c, d$ nonnegative real constants. More precisely, we are interested in the IVP
\[
\begin{aligned}
\partial_t u + \partial_x^3 u + \mu \partial_x (f(u, v)) &= 0, \\
\partial_t v + \partial_x^3 v + \mu \partial_x (g(u, v)) &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\
(u(x, 0), v(x, 0)) &= (u_0(x), v_0(x)),
\end{aligned}
\]
(1.4)

with
\[
\begin{aligned}
f(u, v) := H_u(u, v) &= a u^{2k+1} + b u^{k+1}v + \frac{k+2}{k} c u^{k+1}v^k + d u^{k-1}v^{k+2}, \\
g(u, v) := H_v(u, v) &= a v^{2k+1} + b v^{k+1}u + \frac{k+2}{k} d v^{k+1}u^k + c v^{k-1}u^{k+2}.
\end{aligned}
\]
(1.5)

From (1.3) and (1.5), it is easily seen that
\[
H(u, v) = \frac{1}{2k+2} \left[ f(u, v)u + g(u, v)v \right] = \frac{1}{2k+2} \left[ H_u(u, v)u + H_v(u, v)v \right].
\]
(1.6)

Following the standard nomenclature in the literature, for $\mu = 1$ the system (1.4) is said to be focusing, whereas for $\mu = -1$ it is called defocusing. Note that our function $H$ given by (1.3) generalizes the models in [1,5,7,11,21,31]. So our work may be seen as a natural extension of these works.

Let us now describe our results. First of all, the local well-posedness for IVP (1.4) can be established similarly to [1]. More specifically, combining smoothing effects with a contraction principle argument we obtain the following result.

**Theorem 1.1.** Let $k \geq 1$ and $s \geq 1$. Then, for any $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ there exist $T_0 = T((u_0, v_0)) > 0$ and a unique strong solution $(u, v)$ of the IVP (1.4) in the class
\[
(u, v) \in C([0, T_0]; H^s(\mathbb{R}) \times H^s(\mathbb{R})),
\]
\[
\begin{aligned}
\|u\|_{L^2_T L^\infty_x L^2_T L^\infty_x} &< \infty, \\
\|(u, v)\|_{L^4_T L^4_x L^4_T L^4_x} &< \infty, \\
\|\partial_x D^s_x (u, v)\|_{L^\infty_T L^\infty_x L^2_T L^2_x} &< \infty.
\end{aligned}
\]
(1.7)

Moreover, for any $T_0 \in (0, T)$ there exists a neighborhood $V_0$ of $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ such that the map $(u_0, v_0) \mapsto (u(t), v(t))$ from $V_0$ into the class defined in (1.7) (with $T_0$ instead of $T$) is Lipschitz.
By noting that \( f \) and \( g \) are homogeneous polynomials of degree \( 2k + 1 \), the proof of Theorem 1.1 is similar to that of Theorem 3.1 in [1]. So we will omit the details. Once we know the existence of local solutions, a natural question is about their extension to global ones. This question is partially answered for solutions in the energy space \( H^1(\mathbb{R}) \times H^1(\mathbb{R}) \) in view of the conservation laws. Indeed, it is not difficult to see that system (1.1) conserves the mass and the energy given, respectively, by

\[
M(u, v) = \frac{1}{2} \int |u^2 + v^2| dx
\]

(1.8)

and

\[
E(u, v) = \frac{1}{2} \int [(\partial_x u)^2 + (\partial_x v)^2 - 2\mu H(u, v)] dx.
\]

(1.9)

In addition, since the existence time in Theorem 1.1 depends on the norm of the initial data itself, in order to extend the solution globally in time, it suffices to establish an a priori bound on \( \|\partial_x (u, v)(t)\| \), where \( \|\cdot, \cdot\| \) denotes the \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \)-norm. Observe that (1.9) provides

\[
\|\partial_x (u, v)(t)\|^2 = 2E(u_0, v_0) + 2\mu \int H(u, v)(t) dx.
\]

(1.10)

As an immediate consequence, in the case \( \mu \int H(u, v) dx < 0 \), we have the following result.

**Proposition 1.2.** Let \((u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})\). If \( \mu \int H(u, v) dx < 0 \), then there exists a unique solution \((u, v)\) of IVP (1.4) satisfying

\[
(u, v) \in C(\mathbb{R} : H^1(\mathbb{R}) \times H^1(\mathbb{R})) \cap L^\infty(\mathbb{R} : H^1(\mathbb{R}) \times H^1(\mathbb{R})).
\]

**Remark 1.3.** Assume \( \mu = -1 \). When \( k \) is an even number and \( b \) is null or when \( k \) is odd and \( c = d = 0 \), we have \( \int H(u, v) dx > 0 \). In particular, in these cases we see that assumption in Proposition 1.2 is fulfilled.

On the other hand, from Sobolev’s embedding and Cauchy–Schwarz’s inequality, the following estimate holds

\[
2 \int H(|u|, |v|) dx \leq C \|(u, v)\|^{k+2} \|\partial_x (u, v)\|^k,
\]

(1.11)

where \( C \) is a positive constant. So, in view of (1.10),

\[
\|\partial_x (u, v)\|^2 \leq 2E(u_0, v_0) + C \|(u, v)\|^{k+2} \|\partial_x (u, v)\|^k.
\]

Hence, by using a standard argument (see, for instance, [22, Chapter 6]) we can establish the existence of global solutions for (1.4) under certain conditions. More precisely,
Proposition 1.4. Let \((u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})\). Then, the solution \((u, v)\) given by Theorem 1.1 can be extended to any interval \([0, T]\), \(T > 0\), under one of the following assumptions:

(i) \(k = 1\) and no restrictions on the initial data.
(ii) \(k = 2\) and \(||(u_0, v_0)||\) small enough.
(iii) \(k > 2\) and \(||(u_0, v_0)||_{H^1 \times H^1}\) small enough.

Proposition 1.4 is in agreement with the result in [1, Theorem 4.1]. Note that in the case \(k \geq 2\) we always need a smallness assumption on the initial data. In particular, the case \(k = 2\) is the \(L^2\) critical one and must be treated as the critical KdV equation.

In this paper, our main contribution is to give a more precise description of how small the initial data must be.

Our main result reads as follows (for the precise definition of ground states see Definition 2.3)

Theorem 1.5. (Global well-posedness in \(H^1(\mathbb{R}) \times H^1(\mathbb{R})\)) Assume \(\mu = 1\) and \(k \geq 2\). Let \((u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})\) and suppose that

\[
M(u_0, v_0)^{k+2} E(u_0, v_0)^{k-2} < M(\Phi, \Psi)^{k+2} E(\Phi, \Psi)^{k-2},
\]

where \((\Phi, \Psi)\) is a ground-state solution of the elliptic system

\[
\begin{align*}
\phi'' - \phi + f(\phi, \psi) &= 0, \\
\psi'' - \psi + g(\phi, \psi) &= 0.
\end{align*}
\]

If

\[
||\partial_x (u_0, v_0)||^{k-2} ||(u_0, v_0)||^{k+2} < ||\partial_x (\Phi, \Psi)||^{k-2} ||(\Phi, \Psi)||^{k+2},
\]

then as long as the local solution given in Theorem 1.1 exists, it satisfies

\[
||\partial_x (u(t), v(t))||^{k-2} ||(u_0, v_0)||^{k+2} < ||(\partial_x (\Phi, \Psi)||^{k-2} ||(\Phi, \Psi)||^{k+2}.
\]

In particular, the solution exists globally in time in \(H^1(\mathbb{R}) \times H^1(\mathbb{R})\).

Remark 1.6. Note that in the case \(k = 2\) assumptions (1.12) and (1.14) reduce to the same one and are equivalent to \(||(u_0, v_0)|| < ||(\Phi, \Psi)||\).

To prove Theorem 1.5, we first relate the best constant one can place in inequality (1.11) with the problem of existence of ground state solutions associated with (1.13) (see Corollary 2.8). The main idea is to see the ground states as minima of a Weinstein-type functional. Nowadays, many strategies can be used to obtain the minima of such functional; for instance, one can use the mountain pass theorem, the concentration-compactness method, the Lieb’s translation lemma, and others (see [13, 14, 19, 20, 23–25, 27, 30, 32, 33] and references therein).
By setting \( v = 0 \) in (1.4), we see that the system reduces to the generalized KdV equation

\[
\partial_t u + \partial_x^3 u + \mu a \partial_x (u^{2k+1}) = 0.
\] (1.16)

Equation (1.16) together with the Schrödinger equation is the most studied dispersive equations. Many results concerning local and global well-posedness, asymptotic behavior, and several other properties of the solutions can be found in the current literature, which we refrain from listing them at this stage. However, a similar result for (1.16) as the one in Theorem 1.5 was established in [15]. So, Theorem 1.5 may also be seen as an extension to that result for system (1.4).

As our second main result, we give a suitable characterization of the ground states. Indeed, as we see in Sect. 2, if \((\Phi, \Psi)\) is a ground state of (1.13), then it is nonnegative, that is, \(\Phi \geq 0, \Psi \geq 0\). In addition, since the coefficients of \(f\) and \(g\) are nonnegative,

\[
\begin{align*}
\Phi'' - \Phi &= -f(\Phi, \Psi) \leq 0, \\
\Psi'' - \Psi &= -g(\Phi, \Psi) \leq 0.
\end{align*}
\]

By the maximum principle (see [17, Theorem 3.5]), it follows that \(\Phi\) is strictly positive or vanishes everywhere. A similar statement holds for \(\Psi\). If \(\Psi \equiv 0\), for instance, then \(\Phi\) is a solution of the scalar equation \(\Phi'' - \Phi + f(\Phi, 0) = 0\). A natural and interesting question is when the ground states are of the form \((\Phi, 0)\) or \((0, \Psi)\), which we will pay particular attention below.

It is well known that equation

\[
- Q'' + Q - Q^{2k+1} = 0,
\] (1.17)

has a unique ground state (up to translations), which is positive, radially symmetric and has an exponential decay at infinity (see, for instance, [8, Chapter 8]).

Our main result here is the following.

**Theorem 1.7.** Let \(F(x, y) = (2k + 2)H(x, y)\), where \(H\) is given in (1.3). Let \(Y\) be the set of all points \((x_0, y_0)\) satisfying

\[
F(x_0, y_0) = F_{\text{max}} := \max\{F(x, y) : x^2 + y^2 = 1, x \geq 0, y \geq 0\}.
\]

A pair \((u, v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})\) is a nonnegative ground state of (1.13) if and only if there exist \(\alpha, \beta \geq 0\) such that \((F_{\text{max}})^{\frac{1}{2}}(\alpha, \beta) \in Y\) and

\[
(u, v) = (\alpha Q, \beta Q),
\]

where \(Q\) is the ground state of (1.17).

In particular, uniqueness (up to translations) of the ground states holds provided \(Y\) has only one point.
The proof Theorem 1.7 relies on an extension of the arguments in [12]. In particular it heavily depends on the fact that $H$ is an homogeneous function. Thus, the ground states can also be viewed as minima of another suitable minimization problem.

Observe that system (1.13) appears when we look for solitary waves (with velocity one) of system in (1.4) with $\mu = 1$. Indeed, a solitary wave solution of (1.4) is a solution having the form $u(x, t) = \phi(x - \omega t)$, $v(x, t) = \psi(x - \omega t)$, where $\omega$ is a real constant representing the velocity of the traveling wave. By substituting this form in (1.4), with $\omega = 1$, we promptly see that $(\phi, \psi)$ must satisfy (1.13). Another question of our interest here concerns the orbital stability/instability of the solitary waves. As we see in Sect. 5 at least under some restrictions on the coefficients appearing in the definition of $H$ we are able to establish the orbital instability.

The paper is organized as follows. In Sect. 2, we recall the main steps to prove the existence of ground state solutions associated with system (1.13). As a consequence, we also obtain a sharp Gagliardo–Nirenberg inequality. Section 3 is devoted to prove Theorem 1.5. In Sect. 4, we give our characterization of the ground states by proving Theorem 1.7. Finally, in Sect. 5 we establish our instability result.

2. Gagliardo–Nirenberg type inequality and ground states

As we pointed out above, the proof of Proposition 1.4 is an immediate consequence of the Gagliardo–Nirenberg type inequality (1.11) and a standard argument. In addition, it is clear that the smallness assumption in Proposition 1.4 is related to the constant appearing in (1.11). Hence, the main goal of this section is to study the best constant one can place in (1.11). From now on, we assume $\mu = 1$.

Let us start by introducing the set
\[
P = \{ (u, v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}) \setminus \{(0, 0)\} : P(u, v) := \int H(u, v) dx > 0 \}
\] (2.1)
and the functional
\[
J(u, v) = \frac{\| (u, v) \|^{k+2} \| \partial_x (u, v) \|^{k}}{2 \int H(u, v) dx}.
\] (2.2)

**Remark 2.1.** We always have $P \neq \emptyset$. Indeed, for any $u \in H^1(\mathbb{R}) \setminus \{0\}$ we obtain
\[
\int H(u, u) = \left( \frac{a + b}{k + 1} + \frac{c + d}{k} \right) \int u^{2k+2} dx > 0,
\]
which means that $(u, u) \in P$.

From (1.11), we immediately see that, on $P$, functional $J$ is bounded from below by a positive constant. As a consequence, the best constant we can place in (1.11) is $K_{opt}$ given by
\[
K_{opt}^{-1} = \inf \{ J(u, v) : (u, v) \in P \}.
\] (2.3)
So, our task is to understand the infimum of $J$ on the set $\mathcal{P}$. As we will see below, such an infimum is attained in a special solution of (1.13).

**Definition 2.2.** The pair $(\phi, \psi) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ is said to be a (weak) solution of (1.13) if

\[
\begin{cases}
\int \phi w \, dx + \int \phi' w' \, dx - \int f(\phi, \psi) w \, dx = 0, \\
\int \psi z \, dx + \int \psi' z' \, dx - \int g(\phi, \psi) z \, dx = 0,
\end{cases}
\tag{2.4}
\]

for any $(w, z) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$.

It is not difficult to see that $(\phi, \psi)$ is a solution of (1.13) if and only if it is a critical point of the action functional

\[
I(u, v) := M(u, v) + E(u, v) = \frac{1}{2} \|(u, v)\|^2 + \frac{1}{2} \|\partial_x (u, v)\|^2 - P(u, v). \tag{2.5}
\]

In addition, by the standard elliptic regularity theory any weak solution is indeed smooth and can be regarded as a solution in the strong sense (see, for instance, [8, Chapter 8]). Among all critical points of (2.5), the minima play a distinguished role in several aspects of (1.13); they are called ground states.

**Definition 2.3.** A pair of real-valued functions $(\phi, \psi) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ is called a ground-state solution of (1.13) if

\[
I(\phi, \psi) = \inf \{ I(u, v) : (u, v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}) \setminus (0, 0) \text{ and } I'(u, v) = 0 \}.
\]

Next we give some properties of the solutions of (1.13).

**Proposition 2.4.** (Pohozaev type identities) Let $(\phi, \psi)$ be a solution of (1.13). The following identities hold.

\[
\begin{align*}
(i) \quad & \|\phi, \psi\|^2 + \|\partial_x (\phi, \psi)\|^2 = (2k + 2)P(\phi, \psi); \tag{2.6} \\
(ii) \quad & \|\phi, \psi\|^2 - \|\partial_x (\phi, \psi)\|^2 = 2P(\phi, \psi); \tag{2.7} \\
(iii) \quad & \|\partial_x (\phi, \psi)\|^2 = \frac{k}{k + 2} \|\phi, \psi\|^2; \tag{2.8} \\
(iv) \quad & P(\phi, \psi) = \frac{1}{k + 2} \|\phi, \psi\|^2; \tag{2.9} \\
(v) \quad & P(\phi, \psi) = \frac{1}{k} \|\partial_x (\phi, \psi)\|^2. \tag{2.10}
\end{align*}
\]

In particular, any nontrivial solution of (1.13) belongs to $\mathcal{P}$.

**Proof.** By taking $(w, z) = (\phi, \psi)$ in (2.4), we obtain

\[
\|\phi, \psi\|^2 + \|\partial_x (\phi, \psi)\|^2 = \int_{\mathbb{R}} [f(\phi, \psi) \phi + g(\phi, \psi) \psi] \, dx. \tag{2.11}
\]
From (1.6), we conclude the prove of (2.6). On the other hand, we show (2.7) by multiplying the equations in (1.13) by \(x\phi'\) and \(x\psi'\), respectively, integrating on the spatial variable and applying integration by parts. The identity (2.8) results from multiplying (2.7) by \(-k+1\) and adding to (2.6). Finally, identities (2.9) and (2.10) are obtained by adding and subtracting, respectively, equations (2.6) and (2.7). □

The Pohozaev identities allow us to prove the equivalence between minimizing the functionals \(J\) and \(I\).

**Proposition 2.5.** If \((\phi, \psi)\) is a solution of (1.13), then

\[
J(\phi, \psi) = \frac{k}{2} \left( \frac{k}{k+2} \right)^{\frac{k}{2}} \| (\phi, \psi) \|^{2k}. \tag{2.12}
\]

and

\[
I(\phi, \psi) = \frac{k}{k+2} \| (\phi, \psi) \|^2. \tag{2.13}
\]

In particular, a nontrivial solution \((\phi, \psi) \in P\) of (1.13) is a minimizer of \(J\) if and only if it is a ground state.

**Proof.** From Proposition 2.4, we obtain

\[
J(\phi, \psi) = \frac{\| (\phi, \psi) \|^{k+2}}{2} \left( \frac{k}{k+2} \right)^{\frac{k}{2}} \| (\phi, \psi) \|^{2k} = \frac{k+2}{2} \left( \frac{k}{k+2} \right)^{\frac{k}{2}} \| (\phi, \psi) \|^{2k}
\]

and

\[
I(u, v) = \frac{1}{2} \| (u, v) \|^2 + \frac{1}{2} \left( \frac{k}{k+2} \right) \| (u, v) \|^2 - \frac{1}{k+2} \| (u, v) \|^2 = \frac{k}{k+2} \| (u, v) \|^2,
\]

which is the desired. □

**Remark 2.6.** Proposition 2.5 ensures that the ground-state solutions of (1.13) are solutions which minimize the \(L^2(\mathbb{R}) \times L^2(\mathbb{R})\) norm.

Next we present our result concerning of existence of ground states.

**Theorem 2.7.** (Existence of ground states) There exists at least one ground state solution for the elliptic system (1.13).

As we already said in introduction, there are many ways of proving Theorem 2.7. So we will not give the details. The main idea is to obtain a sequence (using, for instance, the mountain pass theorem) that, up to a subsequence, converges to a ground state solution (see [14, 19, 20, 25, 27, 32, 33] and references therein). Here, for future references, we just recall that solutions of (1.13) must belong to the Nehari manifold

\[
\mathcal{N} = \{(u, v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}) \setminus \{(0, 0)\} : I'(u, v)(u, v) = 0\}.
\]
and any solution of the minimization problem
\[
\inf_{(u,v) \in N} I(u,v)
\]
is a ground state. In addition, from (1.6) it is easily seen that
\[
I'(u,v)(u,v) = \| (u,v) \|^2_{H^1 \times H^1} - (2k + 2) P(u,v),
\]
which immediately gives that \( N \subset P \). In addition, if \( (u,v) \in N \), then
\[
\| (u,v) \|^2_{H^1 \times H^1} = (2k + 2) P(u,v).
\]
(2.15)

As an immediate consequence of the existence of ground states, we may establish the following.

Corollary 2.8. For any \( (u,v) \in P \), we have
\[
2P(u,v) \leq K_{opt} \| (u,v) \|^k \| \partial_x (u,v) \|^k,
\]
with the sharp constant \( K_{opt} > 0 \) given by
\[
K_{opt} = \frac{2}{k + 2} \left( \frac{k + 2}{k} \right)^{\frac{k}{2}} \frac{1}{\| (\Phi, \Psi) \|^{2k}},
\]
(2.16)
where \((\Phi, \Psi)\) is any ground state solution of (1.13).

Proof. This follows from (2.3) and Proposition 2.5. \( \square \)

3. Global well-posedness: proof of Theorem 1.5

This section is devoted to prove Theorem 1.5. The main tool here is the sharp Gagliardo–Nirenberg inequality obtained in Corollary 2.8. Before proceeding, we recall the following continuity lemma.

Lemma 3.1. Let \( \mathcal{I} \subset \mathbb{R} \) an open interval containing 0. Let \( m > 1 \), \( B > 0 \) and \( A \) be real constants. Define \( \gamma = (Bm)^{-\frac{1}{m-1}} \) and \( f(r) = A - r + Br^m \), for \( r \geq 0 \). Let \( G(t) \) be a continuous nonnegative function on \( \mathcal{I} \). Assume that \( A < \left( 1 - \frac{1}{m} \right) \gamma \) and \( f \circ G \geq 0 \).

(i) If \( G(0) < \gamma \), then \( G(t) < \gamma \), for any \( t \in \mathcal{I} \).

(ii) If \( G(0) > \gamma \), then \( G(t) > \gamma \), for any \( t \in \mathcal{I} \).

Proof. See Lemma 3.1 in [32]. \( \square \)

Next, let \( (u(t), v(t)) \) be the solution of (1.4) with initial data \( (u_0, v_0) \). As in (1.10), we use the conservation laws (1.8) and (1.9) and Corollary 2.8 to write
\[
\| \partial_x (u(t), v(t)) \|^2 = 2E(u_0, v_0) + 2\mu P(u(t), v(t))
\]
\[ \leq 2E(u_0, v_0) + 2P(|u(t)|, |v(t)|) \]
\[ \leq 2E(u_0, v_0) + K_{opt} \|(u_0, v_0)\|^{k+2}\|\partial_x(u(t), v(t))\|^k. \] (3.1)

Now we split the proof into the cases \( k > 2 \) and \( k = 2 \).

**Case** \( k > 2 \).

First, we note that under condition (1.14) we have \( E(u_0, v_0) > 0 \). In fact, since (3.1) holds as long as the solution exists, by taking \( t = 0 \) and using (1.14), we obtain
\[ \|\partial_x(u_0, v_0)\|^2 \leq 2E(u_0, v_0) + K_{opt} \|(u_0, v_0)\|^{k+2}\|\partial_x(u_0, v_0)\|^k \]
\[ < 2E(u_0, v_0) + K_{opt} \|(\Phi, \Psi)\|^{k+2}\|\partial_x(\Phi, \Psi)\|^{k-2}\|\partial_x(u, v)\|^2. \] (3.2)

On the other hand, combining (2.8) with (2.16) it follows that
\[ K_{opt} \|(\Phi, \Psi)\|^{k+2}\|\partial_x(\Phi, \Psi)\|^{k-2} = \frac{2}{k}. \]

Since \( k > 2 \), (3.2) then yields
\[ E(u_0, v_0) > \frac{k - 2}{2k} \|\partial_x(u, v)\|^2 > 0. \]

The idea now is to apply Lemma 3.1. For this, we set
\[ A = 2E(u_0, v_0) > 0, \quad B = K_{opt} \|(u_0, v_0)\|^{k+2}, \quad \text{and} \quad G(t) = \|\partial_x(u(t), v(t))\|^2. \]

Thus, we can write (3.1) as
\[ A - G(t) + BG^\frac{k}{2}(t) \geq 0, \quad \text{for} \ t \in [0, T], \]
with \( T \) given by Theorem 1.1. Thus, by defining \( f(r) = A - r + Br^m, \ m = \frac{k}{2} \), we promptly see that \( f(G(t)) \geq 0, \ t \in [0, T] \). Moreover, using (2.16), in the notation of Lemma 3.1,
\[ \gamma = \left(\frac{k}{2B}\right)^{-\frac{2}{k+2}} \]
\[ = \left[ \left(\frac{k+2}{k}\right)^{\frac{k}{2}-1}\frac{\|(u_0, v_0)\|^{k+2}}{\|(\Phi, \Psi)\|^{2k}} \right]^{-\frac{2}{k+2}} \]
\[ = \frac{k}{k + 2} \frac{\|(\Phi, \Psi)\|^{\frac{4k}{2(k+2)}}}{\|(u_0, v_0)\|^{\frac{2(k+2)}{2(k+2)}}}. \]

Hence,
where in the last inequality we used (2.8). Thus, we see that $G(0) < \gamma$ is equivalent to (1.14). Also, from (2.8) and (2.9) it is easily checked that

$$E(\Phi_1, \Psi_1) = \frac{k-2}{2} \|\Phi_1, \Psi_1\|_2 \|\Phi_1, \Psi_1\|_2,$$

which means that $A < \left(1 - \frac{1}{m}\right) \gamma$ is equivalent to (1.12). As an application of Lemma 3.1 we deduce that $G(t) < \gamma$ which in turn is equivalent to (1.15). This completes the proof in the case $k > 2$.

Case $k = 2$. In this case, from (3.1),

$$\|\partial_x(u(t), v(t))\|^2 \leq 2E(u_0, v_0) + K_{opt} \|u_0, v_0\|^4 \|\partial_x(u(t), v(t))\|^2.$$

Thus, it suffices to require

$$K_{opt} \|u_0, v_0\|^4 < 1.$$

But from (2.16) with $k = 2$,

$$K_{opt} \|u_0, v_0\|^4 < 1 \Leftrightarrow \frac{2}{4} \left(\frac{4}{3}\right)^{\frac{1}{2}} \frac{1}{\|\Phi, \Psi\|^2} \|u_0, v_0\|^4 < 1 \Leftrightarrow \|u_0, v_0\|^4 < \|\Phi, \Psi\|^2,$$

which is the desired.

In both cases, we obtain a uniform bound for $\|\partial_x(u(t), v(t))\|$ and the proof of the theorem is completed.

4. Characterization of the ground states: proof of Theorem 1.7

In this section, we will give a characterization of the ground states. Introduce the functionals

$$S(u, v) := \int (u_x^2 + v_x^2 + u^2 + v^2) dx = \|(u, v)\|^2_{H^1 \times H^1} \quad (4.1)$$
\[ \tilde{P}(u,v) = (2k+2)P(u,v) = (2k+2) \int H(u,v)dx. \]

For \( \lambda > 0 \), consider the following minimization problem

\[ S_\lambda = \inf \{ S(u,v) : (u,v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}) \text{ with } \tilde{P}(u,v) = \lambda \}. \quad (4.2) \]

We will show that for a specific value of \( \lambda \) the ground states of (1.13) are also solutions of (4.2). We start by noting that from (1.11) we have \( \tilde{P}(u,v) \leq CS(u,v)^{k+1} \), which implies that \( S_\lambda \) must be positive for any \( \lambda > 0 \). In addition, the homogeneity of \( S \) and \( \tilde{P} \) gives that

\[ S_\lambda = \frac{1}{\lambda} S_1. \quad (4.3) \]

In what follows, we set

\[ \lambda_1 := (S_1)^{\frac{k+1}{k}}. \]

Our first result is the following.

**Proposition 4.1.** For every \((u,v) \in \mathcal{P}\) there exists a unique number \( \ell > 0 \) such that \((\ell u, \ell v) \in \mathcal{N}\) and \( \max_{t \geq 0} I(tu, tv) = I(\ell u, \ell v) \), where \( \mathcal{N} \) is the Nehari manifold introduced in (2.14).

**Proof.** Let \( h : [0, \infty) \to \mathbb{R} \) be defined as \( h(\ell) = I(\ell u, \ell v) \). We have

\[ I'(\ell u, \ell v)(\ell u, \ell v) = \ell \left( \frac{\| (u,v) \|^2_{H^1 \times H^1}}{(2k+2)\ell P(u,v)} \right)^{\frac{1}{k}} = \ell h'(\ell). \]

It is a simple matter to check that

\[ \ell = \left( \frac{\| (u,v) \|^2_{H^1 \times H^1}}{(2k+2)P(u,v)} \right)^{\frac{1}{k}} \quad (4.4) \]

is the unique positive critical point of \( h \). In addition, since \( \ell \) is clearly a maximum point, we obtain the desired. \( \square \)

**Lemma 4.2.** Let \( \omega_\mathcal{N} \) be the Nehari level introduced by

\[ \omega_\mathcal{N} = \inf_{\mathcal{N}} I(u,v). \quad (4.5) \]

Then,

\[ \lambda_1 = \frac{2k+2}{k} \omega_\mathcal{N}. \]
Proof. Let \((u, v)\) be a ground state solution of (1.13). In particular, we have \(\omega_{\mathcal{N}} = I(u, v)\). Recall from (2.15), we must have \(S(u, v) = \tilde{P}(u, v)\). Hence,

\[
\omega_{\mathcal{N}} = I(u, v) = \frac{1}{2} S(u, v) - \frac{1}{2k + 2} \tilde{P}(u, v) = \frac{k}{2k + 2} S(u, v). \tag{4.6}
\]

Define \((U, V) = \left(\frac{k}{\omega_{\mathcal{N}}(2k + 2)}\right)^{\frac{1}{k+1}} (u, v)\). From (4.6), we deduce

\[
\tilde{P}(U, V) = \frac{k}{\omega_{\mathcal{N}}(2k + 2)} \tilde{P}(u, v) = \frac{k}{\omega_{\mathcal{N}}(2k + 2)} S(u, v) = 1
\]

and

\[
S_1 \leq S(U, V) = \left(\frac{k}{\omega_{\mathcal{N}}(2k + 2)}\right)^{\frac{1}{k+1}} S(u, v) = \left(\frac{k}{\omega_{\mathcal{N}}(2k + 2)}\right)^{-\frac{k}{k+1}}.
\]

This last inequality yields \(\lambda_1 \leq \frac{2k + 2}{k} \omega_{\mathcal{N}}\). To show the opposite inequality, it suffices to prove that \(\left(\frac{\omega_{\mathcal{N}}(2k + 2)}{k}\right)^{\frac{1}{k+1}} \leq S(z, w)\), for any \((z, w)\) satisfying \(\tilde{P}(z, w) = 1\). To do so, from Proposition 4.1, if we set \(\tilde{\ell}^{2k} = S(z, w)\) (see (4.4)), then \((Z, W) = \tilde{\ell}(z, w) \in \mathcal{N}\). Consequently,

\[
\omega_{\mathcal{N}} \leq I(Z, W) = \frac{1}{2} \tilde{\ell}^2 S(z, w) - \frac{1}{2k + 2} \tilde{\ell}^{2k+2} = \frac{k}{2k + 2} S(z, w)^{\frac{k}{k+1}},
\]

which immediately gives the desired inequality. \(\square\)

With the above lemma in hand, we are able to give the following characterization of the ground states.

**Proposition 4.3.** A pair \((u, v)\) \(\in H^1(\mathbb{R}) \times H^1(\mathbb{R})\) is a ground state solution of (1.13) if and only if \(S(u, v) = S_{\lambda_1}\) and \(\tilde{P}(u, v) = \lambda_1\).

In particular, the minimization problem (4.2) with \(\lambda = \lambda_1\) has at least one solution.

**Proof.** Let \((u, v)\) be a ground state solution of (1.13). From (4.6), Lemma 4.2, and (4.3),

\[
S(u, v) = \frac{2k + 2}{k} \omega_{\mathcal{N}} = \lambda_1 = \lambda_1^{\frac{1}{k+1}} S_1 = S_{\lambda_1} \tag{4.7}
\]

and

\[
\tilde{P}(u, v) = S(u, v) = \lambda_1.
\]

Now, take \((u, v)\) \(\in H^1(\mathbb{R}) \times H^1(\mathbb{R})\) satisfying \(S(u, v) = S_{\lambda_1}\) and \(\tilde{P}(u, v) = \lambda_1\). Let us first show that \((u, v)\) is indeed a solution of (1.13). From Lagrange’s multiplier theorem, there is a constant \(\theta\) such that

\[
\int (u'w' + uw)dx = (k + 1)\theta \int f(u, v)w \, dx,
\]

\[
\int (v'z' + vz)dx = (k + 1)\theta \int g(u, v)z \, dx,
\]

for any \((w, z)\) \(\in H^1(\mathbb{R}) \times H^1(\mathbb{R})\). Thus, we must show that \((k + 1)\theta = 1\). By taking \(w = u, z = v\), and adding the above identities, we deduce
\[ S(u, v) = (k + 1)\theta \int [f(u, v)u + g(u, v)v]dx = (k + 1)\theta (2k + 2) \]
\[ \int H(u, v)dx = (k + 1)\theta \tilde{P}(u, v), \]
that is, \( S_{\lambda_1} = (k + 1)\theta \lambda_1. \) Since \( \lambda_1 = S_{\lambda_1} \) (see (4.7)), we obtain the desired equality.

It remains to show that \( (u, v) \) minimizes the functional \( I. \) Let \( (z, w) \) be any solution of (1.13) and set \( \xi = \tilde{P}(z, w). \) Since \( S(z, w) = \tilde{P}(z, w) \) (see (2.6)), we obtain
\[ I(z, w) = \frac{1}{2} S(z, w) - \frac{1}{2k + 2} \tilde{P}(z, w) = \frac{k}{2k + 2} \xi. \]  
(4.8)

Next define \( (Z, W) = \left( \frac{\lambda_1}{\xi} \right)^{\frac{1}{2k+2}} (z, w). \) We have \( \tilde{P}(Z, W) = \lambda_1 \) and
\[ \lambda_1^{\frac{1}{2k+2}} S_1 = S_{\lambda_1} = S(u, v) \leq S(Z, W) = \left( \frac{\lambda_1}{\xi} \right)^{\frac{1}{2k+2}} S(z, w) = \lambda_1^{\frac{1}{2k+2}} S_1 \xi \frac{k}{2k+2}, \]
implying that \( \xi \geq (S_1)^{\frac{k+1}{k}} = \lambda_1. \) Therefore, from (4.8),
\[ I(u, v) = \frac{1}{2} \tilde{P}(u, v) - \frac{1}{2k + 2} \tilde{P}(u, v) = \frac{k}{2k + 2} \lambda_1 \leq \frac{k}{2k + 2} \xi = I(z, w). \]
This completes the proof of the proposition. \( \square \)

Finally, we are in a position to prove Theorem 1.7

**Proof of Theorem 1.7.** \( (\Rightarrow) \) Assume first that \( (u, v) \) is a nonnegative ground state. Let \( U(x) = \sqrt{u(x)^2 + v(x)^2} \) and fix \( (x_0, y_0) \in Y. \) From Proposition 4.3 and the homogeneity of \( F, \)
\[ \lambda_1 = \tilde{P}(u, v) = \int F(u(x), v(x))dx = \int F \left( \frac{1}{U(x)}(u(x), v(x)) \right) U(x)^{2k+2}dx \]
\[ \leq \int F(x_0, y_0)U(x)^{2k+2}dx = \tilde{P}(U(x)x_0, U(x)y_0). \]  
(4.9)

In addition, since \( |U'|^2 \leq |u'|^2 + |v'|^2 \) and \( x_0^2 + y_0^2 = 1, \) we derive
\[ S(U(x)x_0, U(x)y_0) = \int (|U'(x)|^2 + |U(x)|^2)dx \leq S(u, v). \]  
(4.10)

From the homogeneity of \( \tilde{P} \) and (4.9), there is \( t \in (0, 1] \) such that \( \lambda_1 = \tilde{P}(tU(x)x_0, tU(x)y_0). \) However, from (4.10),
\[ S(tU(x)x_0, tU(x)y_0) = t^2 S(U(x)x_0, U(x)y_0) \leq t^2 S(u, v). \]

Since \( (u, v) \) is a minimum of \( S \) restricted to \( \tilde{P} = \lambda_1 \) (see Proposition 4.3), we must have \( t = 1, \) implying that
\[ \lambda_1 = \tilde{P}(U(x)x_0, U(x)y_0) \quad \text{and} \quad S(U(x)x_0, U(x)y_0) = S(u, v) = S_{\lambda_1}. \]
Another application of Proposition 4.3 yields that \((U(x)x_0, U(x)y_0)\) is a ground state of (1.13). Thus,

\[
\begin{align*}
(U'' - U)x_0 + f(x_0, y_0)U^{2k+1} &= 0, \\
(U'' - U)y_0 + g(x_0, y_0)U^{2k+1} &= 0.
\end{align*}
\]

(4.11)

By multiplying the first and second equations in (4.11) by \(x_0\) and \(y_0\), respectively, and adding the obtained equations, we get

\[U'' - U + [f(x_0, y_0)x_0 + g(x_0, y_0)y_0]U^{2k+1} = 0.\]

Recalling (1.6) and the definition of \(F\), we finally deduce that \(U\) must be a solution of

\[U'' - U + F_{\text{max}}U^{2k+1} = 0.\]

(4.12)

Moreover, since

\[I(U(x)x_0, U(x)y_0) = \frac{1}{2} \int [U'(x)^2 + U(x)^2]dx - \frac{F(x_0, y_0)}{2k + 2} \int U(x)^{2k+2}dx\]

and \((U(x)x_0, U(x)y_0)\) is a ground state, it follows that \(U\) is a ground state of (4.12). Consequently, \((F_{\text{max}})^{\frac{1}{2k}}U\) is a ground state of (1.17). Recall that a ground state solution of

\[h'' - h + ah^{2k+1} = 0, \quad (a > 0),\]

is a solution that minimizes the action

\[I_a(h) := \frac{1}{2} \int [h'^2 + h^2]dx - \frac{a}{2k + 2} \int h^{2k+2}dx.\]

(4.13)

From the uniqueness of the ground state of (1.17), we deduce (up to a translation)

\[(F_{\text{max}})^{\frac{1}{2k}}U = Q.\]

(4.14)

Next, since \(u(x) \geq 0\) and \(v(x) \geq 0\), we may write \((u(x), v(x)) = U(x)(z(x), w(x))\) with \(z(x)^2 + w(x)^2 = 1\) and \(z(x) \geq 0, w(x) \geq 0\). Thus,

\[
\int F_{\text{max}}U(x)^{2k+2}dx = \int F(x_0, y_0)U(x)^{2k+2}dx = \tilde{P}(U(x)x_0, U(x)y_0) = \tilde{P}(u,v) = \int F(z(x), w(x))U(x)^{2k+2}dx,
\]

from which follows

\[
\int (F_{\text{max}} - F(z(x), w(x)))U(x)^{2k+2}dx = 0,
\]
and, consequently, \( F(z(x), w(x)) = F_{\max} \) almost everywhere. This implies that 
\( (z(x), w(x)) = (F_{\max})^{\frac{1}{\alpha}}(\alpha, \beta) \), where the point \((F_{\max})^{\frac{1}{\alpha}}(\alpha, \beta)\) belongs to \( Y \). Therefore, from (4.14),
\[
(u, v) = U(z, w) = (F_{\max})^{\frac{1}{\alpha}}(U\alpha, U\beta) = (\alpha Q, \beta Q),
\]
as desired.

\((\Leftarrow)\) Assume now \((F_{\max})^{\frac{1}{\alpha}}(\alpha, \beta) =: (x_0, y_0) \in Y\) and let 
\((u, v) = (\alpha Q, \beta Q)\), where \( Q \) is the ground state of (1.17). We first claim that \((u, v) = (\alpha Q, \beta Q)\). Indeed, if either \( \alpha = 0 \) or \( \beta = 0 \) (equivalently \( x_0 = 0 \) or \( y_0 = 0 \)), this is trivial because \( F \) assumes the maximum value \( a \) at the points \((1,0)\) and \((0,1)\). So, we may assume \( x_0 \neq 0 \) and \( y_0 \neq 0 \). In this case, from Lagrange’s multiplier theorem there exists a constant \( \theta \) such that

\[
\begin{align*}
(k + 1)f(x_0, y_0) & = \theta x_0, \\
(k + 1)g(x_0, y_0) & = \theta y_0. 
\end{align*}
\]

By multiplying the first equation in (4.15) by \( x_0 \), the second by \( y_0 \) and adding the obtained equations we deduce that
\[
\theta = (k + 1)F(x_0, y_0) = (k + 1)F_{\max}. \tag{4.16}
\]

Hence, (4.15) and (4.16) imply
\[
\frac{f(\alpha, \beta)}{\alpha} = (F_{\max})^{-1}\frac{f(x_0, y_0)}{x_0} = (F_{\max})^{-1}\frac{\theta}{k + 1} = 1
\]
and
\[
\frac{g(\alpha, \beta)}{\beta} = (F_{\max})^{-1}\frac{g(x_0, y_0)}{y_0} = (F_{\max})^{-1}\frac{\theta}{k + 1} = 1.
\]
Consequently,
\[
\begin{align*}
(\alpha Q)^{\prime\prime} - \alpha Q + f(\alpha Q, \beta Q) & = \alpha \left( Q^{\prime\prime} - Q + \frac{f(\alpha, \beta)}{\alpha} Q^{2k+1} \right) = 0, \\
(\beta Q)^{\prime\prime} - \beta Q + g(\alpha Q, \beta Q) & = \beta \left( Q^{\prime\prime} - Q + \frac{g(\alpha, \beta)}{\beta} Q^{2k+1} \right) = 0,
\end{align*}
\]
which gives that \((\alpha Q, \beta Q)\) is a solution of (1.13).

It remains to show that \((\alpha Q, \beta Q)\) is in fact a ground state. To prove this, first note that
\[
I(\alpha Q, \beta Q) = (F_{\max})^{-\frac{1}{\alpha}}I_1(Q), \tag{4.17}
\]
where \( I_1 \) is given in (4.13). On the other hand, if \((z, w)\) is any ground state, by using we have already proved, we must have \((z, w) = (\tilde{\alpha} Q, \tilde{\beta} Q)\) for some \((\tilde{\alpha}, \tilde{\beta})\) satisfying \((F_{\max})^{\frac{1}{\alpha}}(\tilde{\alpha}, \tilde{\beta}) \in Y\). Hence,
\[
I(z, w) = I(\tilde{\alpha} Q, \tilde{\beta} Q) = (F_{\max})^{-\frac{1}{\alpha}}I_1(Q). \tag{4.18}
\]
By comparing (4.17) and (4.18), we then see that \((\alpha Q, \beta Q)\) is in indeed a ground state. \(\square\)
An immediate consequence of the characterization of the ground states in Theorem 1.7 is the following.

**Corollary 4.4.** The ground states of (1.13) are, up to translations, radially symmetric with an exponential decay at infinity.

To illustrate an application of Theorem 1.7, we will consider two examples: one in the case $k = 2$ and another one for $k > 2$.

**Corollary 4.5.** Assume $k = 2$ and suppose $a = c = d > 0$ and $b > 0$. Then, (1.13) has a unique ground state solution, up to translations, which is given by

$$(\alpha Q, \alpha Q), \quad \alpha = \frac{1}{(4a + b)^{\frac{1}{4}}},$$

where $Q$ is the ground state solution of

$$Q'' - Q + Q^5 = 0.$$

**Proof.** According to Theorem 1.7, we need to find the maximum points of

$$F(x, y) = a\left(x^6 + y^6 + 3(x^4y^2 + x^2y^4)\right) + 2bx^3y^3,$$

restricted to the set $Z = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1, x \geq 0, y \geq 0\}$. To obtain the critical points, from Lagrange’s multiplier theorem, we must find all points $(x, y, \theta)$ satisfying

$$\begin{aligned}
3(ax^5 + bx^2y^3 + 2ax^3y^2 + axy^4) &= \theta x, \\
3(ay^5 + bx^3y^2 + ax^4y + 2ax^2y^3) &= \theta y, \\
x^2 + y^2 &= 1, \quad x \geq 0, \quad y \geq 0.
\end{aligned}$$

(4.19)

Note that $(1, 0, 3a)$ and $(0, 1, 3a)$ are always solutions of (4.19). Moreover, $F(0, 1) = F(1, 0) = a$. To find other possible critical points, we then may assume $x \neq 0, y \neq 0$. Now, dividing the first equation in (4.19) by $x$, the second one by $y$ we see that

$$ax^4 + bxy^3 + 2ax^2y^2 + ay^4 = ay^4 + bx^3y + ax^4 + 2ax^2y^2,$$

or equivalently,

$$bxy(y^2 - x^2) = 0.$$

This implies that $x = y = \frac{1}{\sqrt{2}}$. As a consequence, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{3}{4}(4a + b)\right)$ is also a solution of (4.19) with $F\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = a + \frac{b}{4}$. Since $F\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) > a = F(1, 0)$, we are done.

□
Corollary 4.6. Assume $k > 2$, $a > 0$, $b > 0$, and $c = d = \gamma a$, with $\gamma \geq 0$ satisfying

$$1 + \frac{2\gamma}{k} < \begin{cases} \frac{k!}{(\frac{k}{2})!^2}, & k \text{ even}, \\ \frac{k!}{(\frac{k+1}{2})(\frac{k-1}{2})!}, & k \text{ odd}. \end{cases}$$

Let $Q$ be the ground state solution of (1.17). Then, we have the following.

(i) If $a \left(2^{k} - 1 - \frac{2\gamma (k+1)}{k}\right) > b$ then the ground states of (1.13) are of the form

$$(\alpha Q, 0) \text{ and } (0, \alpha Q), \quad \alpha = a^{-\frac{1}{\pi}}.$$

(ii) If $a \left(2^{k} - 1 - \frac{2\gamma (k+1)}{k}\right) = b$, then the ground states of (1.13) are of the form

$$(\alpha Q, 0), \quad (0, \alpha Q), \quad \alpha = a^{-\frac{1}{\pi}},$$

or

$$(\alpha Q, \alpha Q), \quad \alpha = \frac{a^{-\frac{1}{\pi}}}{\sqrt{2}}.$$

(iii) If $a \left(2^{k} - 1 - \frac{2\gamma (k+1)}{k}\right) < b$, then the ground states of (1.13) are of the form

$$(\alpha Q, \alpha Q) \quad \alpha = \left(a + b + \frac{2a\gamma (k+1)}{k}\right)^{-\frac{1}{\pi}}.$$

In particular, the ground state is unique in this case.

Proof. Following the ideas in Corollary 4.5, we need to find the critical points of

$$F(x, y) = a(x^{2k+2} + y^{2k+2}) + 2bx^{k+1}y^{k+1} + \frac{2a\gamma (k+1)}{k}(x^{k+2}y^k + x^k y^{k+2})$$

restricted to the set $Z = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1, x \geq 0, y \geq 0\}$. The Lagrange multiplier theorem implies we must solve the system

$$\begin{cases} (k + 1)[a x^{2k+1} + b x^k y^{k+1} + \frac{k+2}{k}a\gamma x^{k+1}y^k + a\gamma x^{k-1}y^{k+1}] = \theta x, \\ (k + 1)[a y^{2k+1} + b x^{k+1}y^k + \frac{k+2}{k}a\gamma x^k y^{k+1} + a\gamma x^{k+2}y^{k-1}] = \theta y, \quad (4.20) \\ x^2 + y^2 = 1, \ x \geq 0, \ y \geq 0. \end{cases}$$

We first observe that $(1, 0, a(k+1))$ and $(0, 1, a(k+1))$ are solutions of (4.20) with $F(0, 1) = F(1, 0) = a$. To find the other solutions, we may assume $x > 0$ and $y > 0$. As in (4.16), we deduce that at any critical point $(x_0, y_0)$ we must have

$$F(x_0, y_0) = \frac{\theta}{k + 1}, \quad (4.21)$$
On the other hand, dividing the first equation in (4.20) by $x$ and comparing the result with (4.21) we deduce that at any critical point $(x_0, y_0)$ of $F$ restricted to $Z$,

$$F(x_0, y_0) = ax^{2k}_0 + bx^{k-1}_0 y^{k+1}_0 + \frac{a\gamma(k+2)}{k} x^k_0 y^k_0 + a\gamma x^{k-2}_0 y^{k+2}_0. \quad (4.22)$$

Next, by dividing the first equation in (4.20) by $x$, the second one by $y$, we see that any critical point of $F$ must satisfy

$$ax^{2k} + bx^{k-1} y^{k+1} + a\gamma x^{k-2} y^{k+2} = ay^{2k} + bx^{k+1} y^{k-1} + a\gamma x^{k+2} y^{k-2},$$

or, which is the same,

$$a(x^{2k} - y^{2k}) + b(x^{k-1} y^{k+1} - x^{k+1} y^{k-1}) + a\gamma(x^{k-2} y^{k+2} - x^{k+2} y^{k-2}) = 0.$$  

Since $y \neq 0$, we may introduce the variable $r = \frac{x}{y}$. Thus, the last identity reads as

$$a(r^{2k} - 1) - br^{k-1}(r^2 - 1) - a\gamma r^{k-2}(r^4 - 1) = 0.$$  

Observing that $r^{2k} - 1 = (r^2 - 1)p(r)$, where $p(r) = r^{2k-2} + r^{2k-4} + \ldots + r^2 + 1$, we see that our task reduces to finding all positive solutions of

$$a(r^2 - 1)\left(p(r) - \gamma(r^2 + 1)r^{k-2}\right) - br^{k-1}(r^2 - 1) = 0. \quad (4.23)$$

It is clear that $r = 1$ is a solution of (4.23). This means that \(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)\) is a critical point with $F\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right) = \left(a + b + \frac{2a\gamma(k+1)}{k}\right)\frac{1}{2\pi}$. The other solutions of (4.23) (if they exist) must satisfy

$$ap(r) - br^{k-1} - a\gamma(r^2 + 1)r^{k-2} = 0, \quad (r > 0). \quad (4.24)$$

Now, let $r_0$ be any solution of (4.24). This means that

$$(x_0, y_0) = \left(\frac{r_0}{\sqrt{1 + r^2_0}}, \frac{1}{\sqrt{1 + r^2_0}}\right)$$

is a critical point of $F$ restricted to $Z$. We claim that $F(x_0, y_0) < a = F(1, 0)$. Indeed, using (4.22) and (4.24) we have

$$F(x_0, y_0) = \frac{ar^{2k}_0}{(1 + r^2_0)^k} + \frac{br^{k-1}_0}{(1 + r^2_0)^k} + \left(\frac{k + 2}{k}\right) \frac{a\gamma r^k_0}{(1 + r^2_0)^k} + \frac{a\gamma r^{k-2}_0}{(1 + r^2_0)^k}$$

$$= \frac{ar^{2k}_0 + p(r_0)}{(1 + r^2_0)^k} + \left(\frac{2}{k}\right) \frac{a\gamma r^k_0}{(1 + r^2_0)^k},$$

where in the last equality we have used that $r_0$ is a solution of (4.24) to write $br^{k-1}_0 = ap(r_0) - a\gamma r^{k-2}_0(r^2_0 + 1)$. If $\gamma = 0$, we have $r^{2k}_0 + p(r_0) < (1 + r^2_0)^k$ for any $r_0 > 0$ and $k > 2$; so $F(x_0, y_0) < a$, as claimed. On the other hand, if $\gamma > 0$, it suffices that

$$r^{2k}_0 + p(r_0) + \frac{2a\gamma r^k_0}{k} < (1 + r^2_0)^k. \quad (4.25)$$
Assume first that $k$ is even. Using the definition of $p$, expanding the right-hand side of (4.25) and using that $k$ is even, we see that it suffices to assume that

$$1 + \frac{2\gamma}{k} < \frac{k!}{((k/2)!)^2}. \quad (4.26)$$

which is our assumption.

Assume now that $k$ is odd. Using the definition of $p$, we can rewrite (4.25) as

$$\frac{2\gamma r_0^k}{k} + r_0^{k-1} + r_0^{k+1} + \sum_{j=0, j \neq \frac{k-1}{2}}^{j = k} C_j r_0^{2j},$$

where $C_j := \binom{k}{k-j}$ is the Newton coefficient. Since $C_j \geq 1$ for all $j$, it is suffices that

$$\frac{2\gamma r_0^k}{k} + r_0^{k-1} + r_0^{k+1} < C_{\frac{k-1}{2}} r_0^{k-1} + C_{\frac{k+1}{2}} r_0^{k+1}. \quad (4.27)$$

Observe that for $r_0 \geq 1$ we have $r_0^k < r_0^{k+1}$ and for $r_0 < 1$, we have $r_0^k < r_0^{k-1}$. So, for (4.27) it is sufficient to assume

$$\left(1 + \frac{2\gamma}{k}\right) < \frac{k!}{((k+1)/2)!(k-1/2)!},$$

which is our assumption. The claim is thus proved.

As a consequence, the maximum of $F$ restricted to $Z$ may occur only at the points $(0, 1), (1, 0)$, or $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. By comparing the maximum value of $F$ at these points and using Theorem 1.7, we complete the proof. \(\square\)

5. Instability of ground states for $k > 2$

In this section, we assume $k > 2$ and restrict our attention to the system

$$\begin{align*}
\partial_t u + \partial_3^3 u + \partial_x (H_u(u, v)) &= 0, \\
\partial_t v + \partial_3^3 v + \partial_x (H_v(u, v)) &= 0,
\end{align*} \quad (5.1)$$

with $H$ given according to Corollary 4.6. Recall that a solitary wave for (5.1) is a solution of the form $(\phi(x - \omega t), \psi(x - \omega t))$ with $\phi$ and $\psi$ having a suitable decay at infinity. By replacing this ansatz in (5.1) and integrating once, we obtain

$$\begin{align*}
\phi'' - \omega \phi + H_u(\phi, \psi) &= 0, \\
\psi'' - \omega \psi + H_v(\phi, \psi) &= 0,
\end{align*} \quad (5.2)$$

which reduces to system (1.13), if $\omega = 1$.

Our first result concerns the existence of solutions for (5.2) for any $\omega > 0$. 
Lemma 5.1. Let \((\phi_1, \psi_1)\) be any ground state of (1.13) satisfying the assumptions of Corollary 4.6. Then, system (5.2) has a smooth curve of ground state solutions,

\[
\omega \in (0, +\infty) \mapsto (\phi_\omega, \psi_\omega) \in H^\infty(\mathbb{R}) \times H^\infty(\mathbb{R}),
\]

which agrees with \((\phi_1, \psi_1)\) at \(\omega = 1\).

Proof. This follows immediately by setting

\[
\phi_\omega(x) = \omega^{\frac{1}{2k}} \phi_1(\sqrt{\omega}x), \quad \psi_\omega(x) = \omega^{\frac{1}{2k}} \psi_1(\sqrt{\omega}x), \quad \omega > 0,
\]

and using that \(H_u\) and \(H_v\) are homogeneous functions of degree \(2k + 1\).

Next let us recall the definition of orbital stability/instability.

Definition 5.2. We say that the solitary wave \((\phi_\omega, \psi_\omega)\) is stable in \(H^1(\mathbb{R}) \times H^1(\mathbb{R})\) if for any \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \((u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})\) satisfies

\[
\| (u_0, v_0) - (\phi_\omega, \psi_\omega) \|_{H^1 \times H^1} < \delta
\]

then the corresponding solution of (5.1) with initial data \((u_0, v_0)\) exists globally and satisfies

\[
\inf_{r \in \mathbb{R}} \| (u(t), v(t)) - (\phi_\omega(\cdot + r), \psi_\omega(\cdot + r)) \|_{H^1 \times H^1} < \varepsilon
\]

for any \(t \geq 0\). Otherwise, we say that \((\phi_\omega, \psi_\omega)\) is unstable in \(H^1(\mathbb{R}) \times H^1(\mathbb{R})\).

We will show that the solitary waves in Lemma 5.1 are unstable in \(H^1(\mathbb{R}) \times H^1(\mathbb{R})\).

To do so, we define \(G(u, v) = E(u, v) + \omega M(u, v)\) with \(E\) and \(M\) given in (1.8) and (1.9). From (5.2), we immediately see that \((\phi_\omega, \psi_\omega)\) is a critical point of \(G\), that is,

\[
G'(\phi_\omega, \psi_\omega) = (0, 0). \tag{5.3}
\]

In addition, the linearization of \(G\) around \((\phi_\omega, \psi_\omega)\) is the operator

\[
\mathcal{L} := G''(\phi_\omega, \psi_\omega) = \begin{pmatrix}
-\partial_x^2 + \omega & 0 \\
0 & -\partial_x^2 + \omega
\end{pmatrix} - \begin{pmatrix}
H_{uu}(\phi_\omega, \psi_\omega) & H_{uv}(\phi_\omega, \psi_\omega) \\
H_{vu}(\phi_\omega, \psi_\omega) & H_{vv}(\phi_\omega, \psi_\omega)
\end{pmatrix} \tag{5.4}
\]

From Corollary 4.6, we know that \((\phi_\omega, \psi_\omega) = (\alpha Q_\omega, \beta Q_\omega)\), where \(Q_\omega\) is the (unique) ground state solution of

\[
- Q'' + \omega Q - Q^{2k+1} = 0. \tag{5.5}
\]

Thus, recalling that \(F(x, y) = (2k + 2)H(x, y)\) we obtain

\[
\mathcal{L} = \begin{pmatrix}
-\partial_x^2 + \omega & 0 \\
0 & -\partial_x^2 + \omega
\end{pmatrix} - \frac{(F_{\text{max}})}{2k+2} \begin{pmatrix}
F_{xx}(x_0, y_0) & F_{xy}(x_0, y_0) \\
F_{yx}(x_0, y_0) & F_{yy}(x_0, y_0)
\end{pmatrix} Q_\omega^{2k}, \tag{5.6}
\]

where \((x_0, y_0)\) is a maximum point of \(F\), that is, according to Corollary 4.6, \((x_0, y_0)\) is either \((1, 0)\), \((0, 1)\), or \(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\).

Now we have the following result.
Theorem 5.3. Assume that $\mathcal{L}$ has a unique negative eigenvalue which is simple. Assume also that zero is a simple eigenvalue and the rest of the spectrum is positive and bounded away from zero. Then, the solitary wave $(\phi_\omega, \psi_\omega)$ is unstable in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ provided that $\Lambda''(\omega) < 0$, where $\Lambda(\omega) = E(\phi_\omega, \psi_\omega) + \omega M(\phi_\omega, \psi_\omega)$.

Proof. The proof follows the same ideas as in Theorem 6.2 in [1], which in turn is an extension to systems of the results in [6]. So we will omit the details. We just highlight that usually we need some strong decay at infinity of the solitary waves. Here this is not an issue because from Corollary 4.4 our solitary waves has an exponential decay.

With Theorem 5.3 in hand, we are able to prove the following.

Theorem 5.4. Assume $k > 2$. The solitary waves in Lemma 5.1 are unstable in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ for any $\omega > 0$.

Proof. We will use Theorem 5.3. First let us study the spectrum of the operator $\mathcal{L}$. We will consider only the case when $(x_0, y_0) = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$. The cases $(x_0, y_0) = (1, 0)$ or $(x_0, y_0) = (0, 1)$ are simpler.

By taking the derivative with respect to $x$ in (5.2), we promptly obtain that $(Q'_\omega, Q'_\omega)$ belongs to the kernel of $\mathcal{L}$. Assume now $(u, v)$ is an eigenfunction of $\mathcal{L}$ associated with the eigenvalue $\lambda$. Thus,

$$
\begin{align*}
-u'' + \omega u - \frac{(F_{\max})^{-1}}{2k + 2} (F_{xx}(x_0, y_0)u + F_{xy}(x_0, y_0)v) Q_{\omega}^{2k} &= \lambda u, \\
-v'' + \omega v - \frac{(F_{\max})^{-1}}{2k + 2} (F_{xy}(x_0, y_0)u + F_{yy}(x_0, y_0)v) Q_{\omega}^{2k} &= \lambda v.
\end{align*}
$$

(5.7)

Subtracting and adding the equations in (5.7), we obtain

$$
\begin{align*}
\begin{cases}
-u'' + \omega (u - v) - \frac{(F_{\max})^{-1}}{2k + 2} ((F_{xx}(x_0, y_0) - F_{xy}(x_0, y_0))u \\
+ (F_{xy}(x_0, y_0) - F_{yy}(x_0, y_0))v) Q_{\omega}^{2k} = \lambda (u - v), \\
-u'' + \omega (u + v) - \frac{(F_{\max})^{-1}}{2k + 2} ((F_{xx}(x_0, y_0) + F_{xy}(x_0, y_0))u \\
+ (F_{xy}(x_0, y_0) + F_{yy}(x_0, y_0))v) Q_{\omega}^{2k} = \lambda (u + v).
\end{cases}
\end{align*}
$$

(5.8)

Using the definition of $F$, we deduce that

$$
\frac{(F_{\max})^{-1}}{2k + 2} (F_{xx}(x_0, y_0) - F_{xy}(x_0, y_0)) = -\frac{(F_{\max})^{-1}}{2k + 2} (F_{xy}(x_0, y_0) - F_{yy}(x_0, y_0)) = \left( a + b + \frac{2a\gamma(k + 1)}{k} \right)^{-1}
\times \left( (2k + 1)a - b - \frac{2a\gamma(k - 1)}{k} \right)
$$
\[
\frac{(F_{\text{max}})^{-1}}{2k + 2} (F_{xx}(x_0, y_0) + F_{xy}(x_0, y_0)) = \frac{(F_{\text{max}})^{-1}}{2k + 2} (F_{xy}(x_0, y_0) + F_{yy}(x_0, y_0)) = 2k + 1
\]

Thus, (5.8) reduces to

\[
\begin{cases}
-(u - v)'' + \omega(u - v) - \left(a + b + \frac{2a\gamma(k+1)}{k}\right)^{-1} \\
\left((2k + 1)a - b - \frac{2a\gamma(k-1)}{k}\right) Q_{\omega}^{2k}(u - v) = \lambda(u - v), \\
-(u + v)'' + \omega(u + v) - (2k + 1)Q_{\omega}^{2k}(u + v) = \lambda(u + v).
\end{cases}
\tag{5.9}
\]

Now we introduce the operators \(L_1 = -\partial^2_x + \omega - (2k+1)Q_{\omega}^{2k}\) and \(L_2 = -\partial^2_x + \omega - Q_{\omega}^{2k}\). From (5.5) we see that \(L_1(Q_{\omega}') = 0\) and \(L_2(Q_{\omega}) = 0\). It is well known that \(Q_{\omega}\) is given by

\[Q_{\omega}(x) = \left((k + 1)\omega \text{sech}^2(k\sqrt{\omega}x)\right)^{\frac{1}{2k}}\]

from which we obtain that \(Q_{\omega}'\) has only one zero on the whole line. In particular, it follows from Sturm–Liouville theory that \(L_1\) has a unique negative eigenvalue, zero is a simple eigenvalue and the rest of the spectrum is positive and bounded away from zero (see, for instance, [2, Theorem B.61]). Also, since \(Q_{\omega}\) has no zeros on the whole line, it follows that zero is the first eigenvalue of \(L_2\) (see, for instance, [2, Theorem B.59]) and the rest of the spectrum is bounded away from zero. In addition, recalling the relations between \(a\) and \(b\) in Corollary 4.6 and that we are assuming \((x_0, y_0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\), we get

\[ (2k + 1)a - b - \frac{2a\gamma(k-1)}{k} < a + b + \frac{2a\gamma(k+1)}{k}. \]

Thus, comparing with \(L_2\) we infer that the first eigenvalue of the operator

\[-\partial^2_x + \omega - \left(a + b + \frac{2a\gamma(k+1)}{k}\right)^{-1} \left((2k + 1)a - b - \frac{2a\gamma(k-1)}{k}\right) Q_{\omega}^{2k}\]

must be positive and the rest of the spectrum is positive and bounded away from zero. Thus, the negative and null eigenvalues come only from \(L_1\). Putting all these information together and using (5.9), we finally deduce that \(L\) has a unique negative eigenvalue, its kernel is one-dimensional, and the rest of the spectrum is positive and bounded away from zero.
In order to conclude the proof of the theorem, it remains to establish that \( \Lambda''(\omega) < 0 \). But from (5.3) we obtain \( \Lambda'(\omega) = M(\phi_\omega, \psi_\omega) \). Since

\[
M(\phi_\omega, \psi_\omega) = M(\alpha Q_\omega, \alpha Q_\omega)
\]

\[
= \left( F_{\text{max}} \right)^{-\frac{1}{2}} \frac{k}{2} M(Q_\omega, Q_\omega)
\]

\[
= \left( F_{\text{max}} \right)^{-\frac{1}{2}} \frac{k}{2} \left( k + 1 \right)^{\frac{1}{2}} \omega^{\frac{1}{2}} \frac{1}{k} \int_{\mathbb{R}} \text{sech}^2 \left( x \right) dx.
\]

Consequently,

\[
\Lambda''(\omega) = \frac{\left( F_{\text{max}} \right)^{-\frac{1}{2}}}{2} \frac{k + 1}{k + 2} \left( \frac{1}{k} - \frac{1}{2} \right) \omega^{\frac{1}{2}} \frac{1}{k} \int_{\mathbb{R}} \text{sech}^2 \left( x \right) dx < 0,
\]

because \( k > 2 \). The proof of the theorem is thus completed. \( \square \)

**Remark 5.5.** The main difficulty in proving the instability in the general case (with \( k > 2 \)) consists in establishing the spectral properties of the linearized operator in (5.6). Indeed, note that our calculations in (5.10) depend only on the characterization of the ground states in Theorem 1.7. So, once we have the spectral properties according to Theorem 5.3, an instability result as in Theorem 5.4 follows.

Note also that in the case \( k = 2 \) (critical case) we obtain \( \Lambda''(\omega) = 0 \). So that we are unable to conclude the stability/instability of the traveling waves.

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