Moduli spaces of special Lagrangians and Kähler-Einstein structures

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Abstract. We shall construct a moduli space of pairs of Kähler-Einstein structures and special lagrangians and obtain smoothness of the moduli space of these pairs. Further we show that the moduli space of these pairs is locally embedded in a certain relative cohomology group.

§0. Introduction

Special lagrangians are calibrated submanifolds of Calabi-Yau manifolds with Kähler-Einstein structures, which have been extensively studied between differential geometry and mathematical physics [8],[10],[11],[16]. In particular, a deformation of special lagrangians is an intriguing topic and Mclean shows smoothness of the deformation space of special lagrangians, which is parameterized by an open set of the first cohomology group [14]. The moduli space of Kähler-Einstein structures are constructed by Fujiki-Schumacher [4] and Tian-Todorov show that the deformation space of Kähler-Einstein structures is also smooth [17],[18]. Special lagrangians are depending on a choice of Kähler-Einstein structures. Hence it is natural to study moduli spaces of special lagrangians.
under deformations of Kähler-Einstein structures. In this paper we consider a pair consisting of a Kähler-Einstein structure \( \Phi \) and a special lagrangian \( M \) with respect to \( \Phi \). These results of smoothness raise a question of whether we obtain a smooth moduli space of pairs of Kähler-Einstein structures and special lagrangians. The purpose of this paper is to show smoothness of the moduli space of such pairs, to obtain some further properties. The moduli space of such pairs is locally embedded into a certain relative cohomology group. In order to explain our results more precisely, we introduce our notations and definitions. A calabi-Yau manifold is a complex manifold with the trivial canonical line bundle. Let \( X \) be a compact Calabi-Yau manifold with a Kähler form. There exists a unique Kähler-Einstein form \( \omega \) on \( X \) with vanishing Ricci curvature in each Kähler class. We denote by \( \Omega \) a nonzero form of type \((n,0)\), where \( \text{dim}_\mathbb{C} X = n \). We call a pair \( \Phi = (\Omega, \omega) \) a Kähler-Einstein structure on \( X \). A special lagrangian \( M \) of \( X \) with respect to a Kähler-Einstein structure \( \Phi \) is, by definition, a real \( n \) dimensional submanifold of \( X \) satisfying equations,

\[
i_M^* \Omega^{Im} = 0, \quad i_M^* \omega = 0.
\]

where \( i_M : M \to X \) and \( \Omega^{Im} \) denotes the imaginary part of the complex form \( \Omega \). We assume that a special lagrangian \( M \) is compact. Then we have a moduli space \( \mathcal{P} \) of pairs consisting of Kähler-Einstein structures and special lagrangians (see definition 4-1-3 in section 4). At first we show that a connected component of the moduli space \( \mathcal{P} \) is regarded as a certain relative moduli space \( \mathcal{M}_{KE}(X,M) \), where \( M \) is a fixed real \( n \) dimensional submanifold. Then we obtain,

**Theorem.** The relative moduli space \( \mathcal{M}_{KE}(X,M) \) is a smooth manifold. In particular, \( \mathcal{M}_{KE}(X,M) \) is Hausdorff.

One of difficulties for a construction of the moduli space \( \mathcal{P} \) is that special lagrangians are real calibrated manifolds, in which we can not apply a general theory of deformations of complex geometry, such as Kodira-Spencer theory. In this paper we consider \( X \) as a real \( 2n \) dimensional
manifold and Kähler-Einstein structures are geometric structures defined by closed differential forms \( \Phi = (\Omega, \omega) \). (We refer to these closed differential forms as calibrations.) We construct the moduli space \( \mathcal{M}_{KE}(X) \) of these calibrations by using the implicit function theorem and obtain a smooth moduli space of Kähler-Einstein structures. Let \( \Phi^0 = (\Omega^0, \omega^0) \) be a Kähler-Einstein structure on \( X \). Then we obtain an elliptic complex \( \#_X \) with respect to \( \Phi^0 \):

\[
\begin{align*}
(\#_X) & \quad \Gamma_X(E^0_X) \xrightarrow{d_X} \Gamma_X(E^1_X) \xrightarrow{d_X} \Gamma_X(E^2_X).
\end{align*}
\]

We denote by \( H^i(\#_X) \) the cohomology group of the complex \( \#_X \). Then the infinitesimal deformation at \( \Phi^0 \) is given by the first cohomology group \( H^1(\#_X) \). Let \( i_M : M \to X \) be a special lagrangian with respect to \( \Phi^0 \). Then we also obtain an elliptic complex \( \#_M \) over \( M \):

\[
\begin{align*}
(\#_M) & \quad \Gamma_M(E^0_M) \xrightarrow{d_M} \Gamma_M(E^1_M) \xrightarrow{d_M} \Gamma_M(E^2_M).
\end{align*}
\]

By using the pull back \( i_M^* \), we have a surjective map of complexes, \( \kappa : \#_X \to \#_M \). Hence we have a short exact sequence of complexes:

\[
0 \to \#_{X,M} \to \#_X \xrightarrow{\kappa} \#_M \to 0,
\]

where \( \#_{X,M} \) is a complex,

\[
\begin{align*}
\Gamma_{X,M}(E^0_{X,M}) & \xrightarrow{d_{X,M}} \Gamma_{X,M}(E^1_{X,M}) \xrightarrow{d_{X,M}} \Gamma_{X,M}(E^2_{X,M}),
\end{align*}
\]

(see section 4 for definition). We denote by \( H^i(\#_{X,M}) \) the cohomology group of the complex \( \#_{X,M} \). Then we have

**Theorem.** Let \( \mathcal{M}_{KE}(X, M) \) be a relative moduli space as above. Then local coordinates at \( \Phi^0 \) is given by an open set of the cohomology group \( H^1(\#_{X,M}) \).

We also show that the cohomology group \( H^1(\#_{X,M}) \) is a subgroup of a relative de Rham cohomology group, which is topologically defined.
Then we show that the moduli space $\mathcal{M}_{KE}(X, M)$ is locally embedded in the relative de Rham cohomology group, which is called local Torelli type theorem (Theorem 4-2-8). It must be noted that the moduli space $\mathcal{M}_{KE}(X, M)$ is a total space of a fibre bundle,

$$\mathcal{M}_{KE}(X, M) \to \mathcal{M}_0(X, M),$$

where the base space $\mathcal{M}_0(X, M)$ is a submanifold of the moduli space of Kähler-Einstein structures $\mathcal{M}_{KE}(X)$, which corresponds to deformations preserving special lagrangians (Proposition 4-2-9) and each fibre is regarded as the moduli space of special lagrangians with respect to a fixed Kähler-Einstein structure. Our moduli space is defined as a certain quotient by the action of the identity component of the group of diffeomorphisms. It is interesting to ask what is the quotient by the action of whole diffeomorphisms. In theorem 4-2-11 we show that such a quotient is an orbifold (see theorem 1-9). In section 1, we discuss a general theory of geometric structures defined by closed differential forms and construct a moduli space of such closed differential forms. If a geometric structure is metrical, elliptic and topological, we obtain a smooth moduli space of them (see definition 1-1,2,3). Section 2 is devoted to prove theorems in section 1. We show in section 3 that the Kähler-Einstein structure is metrical, elliptic and topological. Hence we obtain a smooth moduli space of Kähler-Einstein structures. If we fix a class of Kähler forms, then we have a smooth moduli space of polarized Calabi-Yau structures. In section 4 we obtain the relative moduli space $\mathcal{M}_{KE}(X, M)$. In subsection 4-1 we study the cohomology groups $H^i(\#_X), H^i(\#_M)$ and $H^i(\#_{X,M})$. In subsection 4-2 we construct a slice $S_{\delta^0}(X, M)$, which is local coordinates of the moduli space $\mathcal{M}_{KE}(X, M)$, and prove our main theorem (theorem 4-1-5). In the case of hyperKähler structure, corresponding calibrated submanifolds are holomorphic lagrangians. Our construction also holds in this case and we obtain a smooth moduli space of pairs of hyperKähler structures and holomorphic lagrangians. We will discuss this in a forthcoming paper.
§1. Moduli spaces of calibrations

Let $V$ be a real vector space of dimension $n$. We denote by $\wedge^p V^*$ the vector space of $p$ forms on $V$. Let $\rho_p$ be the linear action of $G = \text{GL}(V)$ on $\wedge^p V^*$. Then we have the action $\rho$ of $G$ on the direct sum $\bigoplus_i \wedge^{p_i} V^*$ by

$$\rho: \text{GL}(V) \rightarrow \text{End}(\bigoplus_{i=1}^{l} \wedge^{p_i} V^*),$$

$$\rho = (\rho_{p_1}, \cdots, \rho_{p_l}).$$

We fix an element $\Phi^0 = (\phi_1, \phi_2, \cdots, \phi_l) \in \bigoplus_i \wedge^{p_i} V^*$ and consider the $G$-orbit $O = O_{\Phi^0}$ through $\Phi^0$:

$$O_{\Phi^0} = \{ \Phi = \rho_g \Phi^0 \in \bigoplus_i \wedge^{p_i} V^* \mid g \in G \}$$

The orbit $O_{\Phi^0}$ can be regarded as a homogeneous space,

$$O_{\Phi^0} = G/H,$$

where $H$ is the isotropy group

$$H = \{ g \in G \mid \rho_g \Phi^0 = \Phi^0 \}.$$  

We denote by $A(V)$ the orbit $O_{\Phi^0} = G/H$. The tangent space $E^1(V) = T_{\Phi^0} A(V)$ is given by

$$E^1(V) = T_{\Phi^0} A(V) = \{ \rho_\xi \Phi^0 \in \bigoplus_i \wedge^{p_i} V^* \mid \xi \in \mathfrak{g} \},$$

where $\rho$ denotes the differential representation of $\mathfrak{g}$. The vector space $E^1(V)$ is the quotient space $\mathfrak{g}/\mathfrak{h}$. We also define a vector space $E^0(V)$ by the interior product,

$$E^0(V) = \{ i_v \Phi^0 \in \bigoplus_i \wedge^{p_i-1} V^* \mid v \in V \}.$$  

$E^2(V)$ is define as a vector space spanned by the following set,

$$E^2(V) = \text{Span}\{ \theta \wedge \Phi \in \bigoplus_i \wedge^{p_i+1} V^* \mid \theta \in V^*, \Phi \in E^1(V) \}.$$  

Then we have the complex by the exterior product of a nonzero $u \in V^*$,

$$E^0(V) \xrightarrow{\wedge^u} E^1(V) \xrightarrow{\wedge^u} E^2(V).$$
Definition 1-1 (elliptic orbits). An orbit $O_{\Phi^0}$ is an elliptic orbit if the complex

$$E^0(V) \xrightarrow{\wedge u} E^1(V) \xrightarrow{\wedge u} E^2(V).$$

is exact for any nonzero $u \in V^*$. In other words, if $\alpha \wedge u = 0$ for $\alpha \in E^1(V)$, then there exists $\beta \in E^0(V)$ such that $\alpha = \beta \wedge u$.

Definition 1-2 (metrical orbits). Let $O_{\Phi^0}$ be an orbit as before. An orbit $O_{\Phi^0}$ is metrical if the isotropy group $H$ is a subgroup of $O(V)$ with respect to a metric $g_V$ on $V$.

Let $X$ be a compact real manifold of dimension $n$. Then we define the $G/H$-bundle $\mathcal{A}(X) = \mathcal{A}_\sigma(X)$ by

$$\mathcal{A}_\sigma(X) = \bigcup_{x \in X} \mathcal{A}(T_xX) \longrightarrow X.$$ 

We denote by $\mathcal{E}^1 = \mathcal{E}^1(X)$ the set of $C^\infty$ global sections of $\mathcal{A}(X)$,

$$\mathcal{E}^1(X) = \Gamma(X, \mathcal{A}(X)).$$

Let $\Phi^0$ be a closed element of $\mathcal{E}^1$. Then we have the vector spaces $E^i(T_xX)$ for each $x \in X$ and $i = 0, 1, 2$ respectively. We define the vector bundle $E^i_X = E^i$ over $X$ as

$$E^i_X = E^i := \bigcup_{x \in X} E^i(T_xX) \longrightarrow X.$$ 

for each $i = 0, 1, 2$. (Note that the fibre of $E^1$ is $g/\mathfrak{h}$.) Then we have a complex $\#_{\Phi^0}$

$$\begin{array}{ccc}
\Gamma(E^0) & \xrightarrow{d_0} & \Gamma(E^1) \\
\Gamma(E^1) & \xrightarrow{d_1} & \Gamma(E^2),
\end{array}$$

where $\Gamma(E^i)$ is the set of $C^\infty$ global sections for each vector bundle and $d_i = d|_{E^i}$ for each $i = 0, 1, 2$. The complex $\#_{\Phi^0}$ is a subcomplex of $\text{de}$.
Rham complex:
\[
\begin{array}{ccc}
\Gamma(E^0) & \xrightarrow{d_0} & \Gamma(E^1) & \xrightarrow{d_1} & \Gamma(E^2) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(\oplus_i \wedge^{p_i-1}) & \xrightarrow{d} & \Gamma(\oplus_i \wedge^{p_i}) & \xrightarrow{d} & \Gamma(\oplus_i \wedge^{p_i+1})
\end{array}
\]

Hence there is the map \( p \) from the cohomology group of the complex \( #\Phi_0 \) to de Rham cohomology group:
\[
p: H^1(#\Phi_0) \to \bigoplus_i H^{p_i}(X, \mathbb{R}),
\]
where
\[
H^1(#\Phi_0) = \{ \alpha \in \Gamma(E^1) | d_1 \alpha = 0 \} / \{ d\beta | \beta \in \Gamma(E^0) \}.
\]

**Definition 1-3 (Topological calibrations and topological orbits).** A closed element \( \Phi^0 \in \mathcal{E}^1(X) \) is a topological calibration if the map
\[
p: H^1(#\Phi_0) \to \bigoplus_i H^{p_i}(X, \mathbb{R})
\]
is injective. A manifold \( X \) is topological with respect to an orbit \( O \) in \( \oplus_i \wedge^{p_i} V^* \) if any closed element of \( \mathcal{E}^1(X) \) is a topological calibration. An orbit \( O \) is topological if \( p \) is injective for each closed form \( \Phi^0 \in \mathcal{E}^1(X) \) over any compact \( n \) dimensional manifold \( X \).

**Lemma 1-4.** Let \( O \) be a metrical orbit and \( \Phi^0 \) an element of \( \mathcal{E}^1 = \Gamma(X, \mathcal{A}_O(X)) \). Then there is a canonical metric \( g_{\Phi^0} \) on \( X \) corresponding to each \( \Phi^0 \).

**Proof.** The orbit \( O \) is defined in terms of \( \Phi^0 = \Phi^0_V \in \oplus_i \wedge^{p_i} V^* \) on \( V \). We also have \( \Phi^0(x) \in \mathcal{A}(T_x X) \) on each tangent space \( T_x X \). Let \( \text{Isom}(V, T_x X) \) be the set of isomorphisms between \( V \) and \( T_x X \). Then define \( H_x \) by
\[
H_x = \{ h \in \text{Isom}(V, T_x X) | \Phi^0_V = h^* \Phi^0(x) \}.
\]
Then we see that \( H_x \) is isomorphic to the isotropy group \( H \). \( h^* g_V \) defines the metric on the tangent space \( T_x X \) for \( h \in H_x \). Since \( H \) is a subgroup of \( O(V) \), the metric \( h^* g_V \) does not depend on a choice of \( h \in H_x \). \( \square \)

At first we shall show the following:
**Proposition 1-5.** Let \( O \) be a metrical and elliptic orbit and \( \Phi \) a topological element of \( \mathcal{E}^1 \). Then the dimension of \( H^1(\# \Phi) \) is invariant under deformations of \( \Phi \in \mathcal{E}^1 \).

**Proof.** The complex \( \{ \Gamma(E^i), d_i \} \) is a subcomplex of de Rham complex. Hence we have the commutative diagram:

\[
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Gamma(E^0) & \xrightarrow{d_0} & \Gamma(E^1) \\
\downarrow & & \downarrow \\
\oplus_i \Gamma(\wedge_{p_i-1}) & \xrightarrow{d} & \oplus_i \Gamma(\wedge_{p_i}) \\
\downarrow & & \downarrow \\
\oplus_i \Gamma(\wedge_{p_i-1})/\Gamma(E^0) & \rightarrow & \oplus_i \Gamma(\wedge_{p_i})/\Gamma(E^1) \\
\downarrow & & \downarrow \\
0 & & 0 \\
\end{array}
\]

Let \( g_\Phi \) be the metric corresponding to the metrical calibration \( \Phi \). We denote by \( E^i_\perp \) the orthogonal compliment of each vector bundle \( E^i \) for \( i = 0, 1, 2 \). Then we have the identification:

\[
E^0_\perp \cong \oplus_i \wedge_{p_i-1} / E^0 \\
E^1_\perp \cong \oplus_i \wedge_{p_i} / E^1
\]

We denote by \( \hat{E}^2 \) by the image

\[
\hat{E}^2 = d_1 \Gamma(E^1).
\]
Then we have
\[
\begin{array}{c}
\Gamma(E^0) \xrightarrow{d_0} \Gamma(E^1) \xrightarrow{d_1} \hat{E}^2 \\
\downarrow \\
\oplus_i \Gamma(\wedge^{p_i-1}) \xrightarrow{d} \oplus_i \Gamma(\wedge^{p_i}) \xrightarrow{d} \oplus_i \Gamma(\wedge^{p_i+1}) \\
\downarrow \\
\Gamma(E^0_\perp) \xrightarrow{} \Gamma(E^1_\perp) \xrightarrow{} \Gamma(E^2_\perp).
\end{array}
\]

Hence we have the long exact sequence. Since \(\Gamma(E^1) \rightarrow \hat{E}^2\) is surjective and \(\Phi\) is topological, we have the exact sequence:

\[
(1) 0 \rightarrow H^1(#) \rightarrow \oplus_i H^p_{DR}(X) \rightarrow H^1(\#_\perp) \rightarrow 0,
\]

When we consider symbols of differential operators in the diagram, we see that the complex \(\#_\perp\)
\[
\Gamma(E^0_\perp) \xrightarrow{} \Gamma(E^1_\perp) \xrightarrow{} \Gamma(E^2_\perp)
\]
is an elliptic complex. Hence \(H^1(\#_\perp)\) is Kernel of the elliptic operator. Since \(\#\) is elliptic, \(H^1(#)\) is also kernel of the elliptic operator. So the dimension of each cohomology group is an upper semi continuous. Hence from (1) we see that the dimension of \(H^1(#)\) is invariant under deformations. □

Let \(O\) be an orbit in \(\oplus_i \wedge^{p_i} V^*\). Then we define the moduli space \(\mathcal{M}_\circ(X)\) by
\[
\mathcal{M}_\circ(X) = \{ \Phi \in \mathcal{E}^1 \mid d\Phi = 0 \}/\text{Diff}_0(X),
\]
where \(\text{Diff}_0(X)\) is the identity component of the group of diffeomorphisms for \(X\). We denote by \(\widetilde{\mathcal{M}}_\circ(X)\) the set of closed elements in \(\mathcal{E}^1\). We have the natural projection \(\pi: \widetilde{\mathcal{M}}_\circ(X) \rightarrow \mathcal{M}_\circ(X)\). We shall prove the following theorems in section 2.
Theorem 1-6. If an orbit $O$ is metrical, elliptic and topological, then the corresponding moduli space $\mathcal{M}_O(X)$ is a smooth manifold. (In particular $\mathcal{M}_O(X)$ is Hausdorff.) Further each coordinates of $\mathcal{M}_O(X)$ around $\pi(\Phi)$ is canonically given by an open ball of the cohomology group $H^1(\#\Phi)$ for each $\Phi \in \widetilde{\mathcal{M}}_O(X)$.

Since de Rham cohomology group is invariant under the action of $\text{Diff}_0$, we have the map

$$P: \mathcal{M}_O(X) \longrightarrow \bigoplus_i H^p_{dR}(X).$$

Then we have

Theorem 1-7. If an orbit $O$ is metrical, elliptic and topological, then the map $P$ is locally injective.

Theorem 1-8. Let $I(\Phi)$ be the isotropy group,

$$I(\Phi) = \{ f \in \text{Diff}_0(X) \mid f^*\Phi = \Phi \}.$$

Then there is a sufficiently small slice $S_{\Phi^0}$ at $\Phi^0$ such that the isotropy group $I(\Phi^0)$ is a subgroup of $I(\Phi)$ for each $\Phi \in S_{\Phi^0}$, i.e.,

$$I(\Phi^0) \subset I(\Phi).$$

(Our definition of the slice will be given in section 2)

Theorem 1-9. Let $\widetilde{\mathcal{M}}_O(X)$ be the set of closed elements of $\mathcal{E}^1$. We denote by $\text{Diff}(X)$ the group of diffeomorphism of $X$. There is the action of $\text{Diff}(X)$ on $\widetilde{\mathcal{M}}_O(X)$. Then the quotient $\widetilde{\mathcal{M}}_O(X)/\text{Diff}(X)$ is an orbifold.

§2. PROOF OF THEOREMS

Let $X$ be a real $n$ dimensional compact manifold. We denote by $C^\infty(X, \wedge^p)$ the set of smooth $p$ forms on $X$. Let $L^2_s(X, \wedge^p)$ be the Sobolev space and suppose that $s > k + \frac{n}{2}$., i.e., the completion of $C^\infty(X, \wedge^P)$ with respect to the Sobolev norm $\| \cdot \|_s$, where $k$ is sufficiently large. Then we have the inclusion $L^2_s(X, \wedge^p) \longrightarrow C^k(X, \wedge^n)$. We define $\mathcal{E}^1_s$ by

$$\mathcal{E}^1_s = C^k(X, \mathcal{A}_O(X)) \cap L^2_s(X, \bigoplus_{i=1}^l \wedge^{p_i}).$$

Then we have
Lemma 2-1. $\mathcal{E}_s^1$ is a Hilbert manifold. The tangent space $T_{\Phi_0}\mathcal{E}_s^1$ at $\Phi^0$ is given by

$$T_{\Phi_0}\mathcal{E}_s^1 = L^2_s(X, E^1).$$

Proof. We denote by $\exp$ the exponential map of Lie group $G = \text{GL}(n, \mathbb{R})$. Then we have the map $k_x$

$$k_x : E^1(T_x X) \rightarrow A(T_x X),$$

by

$$k_x(\rho \xi \Phi^0(x)) = \rho \exp \xi \Phi^0(x).$$

for each tangent space $T_x X$. From 2-3 we have the map $k$

$$k : L^2_s(E^1) \rightarrow \mathcal{E}_s^1,$$

by

$$k|_{E^1(T_x X)} = k_x.$$

The map $k$ defines local coordinates of $\mathcal{E}_s^1$. □

For each $\Phi$ we have the vector bundles as in section one. We denote them by $E^0_s, E^1_s$ and $E^2_s$. We define $\mathcal{Z}$ to be the set of closed forms in $\mathcal{E}_s^1$,

$$\mathcal{Z} = \{ \Phi \in \mathcal{E}_s^1 | d\Phi = 0 \}.$$

We also denote by $Z_\Phi$ the Hilbert space of closed forms of $T_\Phi \mathcal{E}_s^1 = L^2_s(X, E^1_\Phi)$,

$$Z_\Phi = \{ a \in L^2_s(X, E^1_\Phi) | da = 0 \}.$$

We then have
Lemma 2-2. \( Z \) is a Hilbert manifold with \( T_\Phi Z = Z_\Phi \).

Proof. The tangent space of the Hilbert manifold \( \mathcal{E}_s^1 \) at \( \Phi \) is the Hilbert space \( L_s^2(X, E_s^1) \). We then have a distribution \( Z \) defined by the closed subspace \( Z_\Phi \) of the tangent space \( T_\Phi \mathcal{E}_s^1 \). We shall show that the distribution \( Z \) is integrable. Let \( \mathcal{V}^1 \) (resp. \( \mathcal{V}^2 \)) denote the Sobolev space \( L_s^2(X, \oplus_i \wedge p_i) \) (resp. \( L_{s-1}^2(X, \oplus_i \wedge p_i + 1) \)). We define a map \( \eta: \mathcal{V}^1 \to \mathcal{V}^2 \) by the exterior derivative \( a \mapsto \partial a \), where \( a \in \mathcal{V}^1 \). Then \( \eta \) is regarded as a \( \mathcal{V}^2 \) valued one from on the Hilbert manifold \( \mathcal{V}^1 \). We immediately see that \( \eta \) is a closed form. Hence the pull back \( i^{\#} \) is also closed where \( i: \mathcal{E}_s^1 \to \mathcal{V}^1 \) denotes the inclusion.

The distribution \( Z \) is defined by the closed one form \( i^{\#} \eta \),

\[
Z_\Phi = \{ a \in \mathcal{E}_s^1 \mid i^*_\mathcal{E} \eta(a) = 0 \}.
\]

Hence we see that the distribution \( Z \) is integrable. From the Frobenius theorem for Hilbert manifolds, there exists the integrable manifold \( Z' \) such that \( T_\Phi Z' = Z_\Phi \). We then see that \( Z \) coincides with the integral manifold \( Z' \). \( \square \)

Let \( \Phi^0 \) be a smooth closed element of \( \mathcal{E}^1 \). Since the orbit \( \mathcal{O} \) is metrical, we have the corresponding smooth metric \( g_{\Phi^0} \). The vector bundle \( E_s^i \) is defined from \( \Phi^0 \) as in section one for each \( i = 0, 1, 2 \). Then we have the orthogonal projection \( \pi_{E_s^i} \),

\[
\pi_{E_s^i}: L_{s-1}^2(X, \oplus_i \wedge p_i + 1) \to L_{s-1}^2(E_s^i),
\]

for \( i = 0, 1 \). We also denote by \( d^* \) the adjoint operator with respect to the metric \( g_{\Phi^0} \). We consider the complex:

\[
\begin{array}{c}
L_{s+1}^2(E^0) \xrightarrow{d_0} L_s^2(E^1) \xrightarrow{d_1} L_{s-1}^2(E^2).
\end{array}
\]

(\#_{\Phi^0})

Since \( \mathcal{O} \) is elliptic, the complex \( \#_{\Phi^0} \) is an elliptic complex. We then have the Laplacian \( \Delta_\# = d_0 d_0^* + d_1^* d_1 \) of the complex \( \#_{\Phi^0} \) and the Hodge decomposition:

\[
L_s^2(E^1) = \mathbb{H}^1(\#_{\Phi^0}) + \Delta_\# G_\# L_s^2(E^1),
\]

(2-7)
where $\mathbb{H}^1(\#_{\Phi^0}) = \text{Ker} \triangle_{\#}$ and $G_\#$ denotes the Green operator. We also see the the adjoint operator $d^*_0(\text{resp. } d^*_1)$ is given by $\pi_{E^0} \circ d^*$ (resp. $\pi_{E^1} \circ d^*$).

**Lemma 2-3.** Let $\Phi^0$ be a smooth closed element of $E^1_s$. We define a slice $S_{\Phi^0}$ by

$$S_{\Phi^0} = \{ \Phi \in E^1_s \mid d\Phi = 0, \pi_{E^0} \circ d^*\Phi = 0 \}.$$  

Then $S_{\Phi^0} \cap U$ is a Hilbert manifold with $T_{\Phi^0}S_{\Phi^0} = \mathbb{H}^1(\#_{\Phi^0})$, where $U$ is a sufficiently small neighborhood of $E^1_s$ at $\Phi^0$.

**Proof.** The slice $S_{\Phi^0}$ is written as

$$S_{\Phi^0} = \{ \Phi \in Z \mid d_0d^*_0\Phi = 0 \},$$

where $d^*_0$ denotes $\pi_{E^0} \circ d^*$. We define the map $F$ by

$$F: Z \longrightarrow L^2_s(X, E^1),$$

$$\Phi \longrightarrow d_0d^*_0\Phi.$$  

The map $F$ is the map from the Hilbert manifold $Z$ to the closed sub space of $d_0$ exact forms $d_0L^2_{s-1}(E^0)$. The differential of $F$ at $\Phi^0$ is given by

$$dF_{\Phi^0}: T_{\Phi^0}Z \longrightarrow d_0L^2_{s-1}(E^0)$$

$$a \longrightarrow d_0d^*_0a,$$

where $a \in T_{\Phi^0}Z = Z_{\Phi^0}$. From the Hodge decomposition of the complex $\#_{\Phi^0}$, we see that $dF_{\Phi^0}$ is surjective. Hence from the implicit function theorem, the slice $S_{\Phi^0} \cap U$ is a Hilbert manifold for a neighborhood $U$ at $\Phi^0$. The tangent space of $T_{\Phi^0}S_{\Phi^0}$ is given by the kernel of the map $dF_{\Phi^0}$. From lemma 2-2 we see that the kernel of $dF_{\Phi^0}$ is $\mathbb{H}^1(\#_{\Phi^0})$. □
Lemma 2-4. We define a map $P_{S_{\Phi^0}}$ by

$$P_{S_{\Phi^0}} : S_{\Phi^0} \rightarrow \bigoplus_i H^{p_i}_{dR}(X),$$

$$P_{S_{\Phi^0}}(\Phi) = ([\phi_1]_{dR}, \cdots, [\phi_l]_{dR}),$$

where $\Phi = (\phi_1, \cdots, \phi_l) \in S_{\Phi^0}$ and $[\phi_i]_{dR}$ is a class of de Rham cohomology group represented by $\phi_i$. Then $P_{S_{\Phi^0}}$ is injective on a small neighborhood at $\Phi^0$.

Proof. An open set of $H^1(\#_{\Phi^0})$ is a local coordinates of the slice $S_{\Phi^0}$. Hence the result follows since the orbit $\mathcal{O}$ is topological. $\square$

Let $C^1\text{Diff}_0$ be the identity component of $C^1$–diffeomorphisms from $X$ to $X$. We define $\text{Diff}_0^{s+1}$ by

$$\text{Diff}_0^{s+1} = C^1\text{Diff}_0 \cap L^2_{s+1}(\text{Diff}(X)).$$

Then $\text{Diff}_0^{s+1}$ is the Hilbert Lie group and the action

$$\mathcal{E}_s^1 \times \text{Diff}_0^{s+1} \rightarrow \mathcal{E}_s^1$$

is well defined (see [3]). We define $\overline{\mathcal{M}}_\circ^s(X)$ by

$$\overline{\mathcal{M}}_\circ^s(X) = \overline{\mathcal{M}}_\circ^s(X)/\text{Diff}_0^{s+1},$$

where

$$\overline{\mathcal{M}}_\circ^s(X) = \{ \Phi \in \mathcal{E}_s^1 | d\Phi = 0 \}.$$

Let $\pi$ be the natural projection

$$\pi : \overline{\mathcal{M}}_\circ^s(X) \rightarrow \mathcal{M}_\circ^s(X).$$
Lemma 2-5. Let $S_{\Phi^0}$ be the slice through $\Phi^0$. Then image $\pi(S_{\Phi^0})$ is an open set in $\mathcal{M}_\circ(X)$.

Proof. As in proof of lemma 2-3 it follows from implicit function theorem that a neighborhood of $\Phi^0$ is homeomorphic to $S_{\Phi^0} \times V$, where $V$ is an neighborhood of $\text{Diff}_0$ at the identity. Hence we have the result. \[\square\]

Lemma 2-6. Each element $\Phi$ of the slice $S_{\Phi^0}$ consists of smooth forms, i.e.,

$$\Phi = (\phi_1, \cdots, \phi_l), \quad \phi_i \in C^\infty(X, \wedge^{p_i}).$$

So that is,

$$S_{\Phi^0} \subset \mathcal{M}_\circ(X).$$

Proof. The tangent space $T_{\Phi^0}S_{\Phi^0}$ is the Kernel of Laplacian $\triangle_\#$. Hence from elliptic regularity we have

$$T_{\Phi^0}S_{\Phi^0} \subset C^\infty(X, \oplus_i \wedge^{p_i}).$$

Hence from implicit function theorem we see that

$$S_{\Phi^0} \subset C^\infty(X, \oplus_i \wedge^{p_i}).$$

\[\square\]

Proposition 2-7. Let $\pi$ be the natural projection $\pi: \mathcal{M}_\circ(X) \to \mathcal{M}_\circ(X)$. We restrict the map $\pi$ to a slice $S_{\Phi^0}$. Then the restricted map $\pi|_{S_{\Phi^0}}$ to the image

$$\pi|_{S_{\Phi^0}}: S_{\Phi^0} \to \pi(S_{\Phi^0})$$

is a homeomorphism.

Proof. It is sufficient to show that $\pi|_{S_{\Phi^0}}$ is injective. We assume that $\pi(\Phi) = \pi(\Phi')$ for $\Phi, \Phi' \in S_{\Phi^0}$. It implies that there exists $f \in \text{Diff}_0$ such that $\Phi' = f^* \Phi$. Since each class of de Rham cohomology group is invariant under the action of $\text{Diff}_0$, we have

$$[\Phi]_{dR} = [\Phi']_{dR} \in \oplus_i H^{p_i}(X).$$

From lemma 2-4, the map $P_{S_{\Phi^0}}$ is injective for sufficiently small $S_{\Phi^0}$. Hence we have $\Phi = \Phi'$. \[\square\]
Proposition 2-8. The quotient $\mathcal{M}_\varnothing(X)$ is Hausdorff.

Proof. We define $\tilde{\mathcal{M}}_s(X)$ by

$$\tilde{\mathcal{M}}_s(X) = \{ \Phi \in \mathcal{E}_s^1 | d\Phi = 0 \}.$$ 

Since $\mathcal{O}$ is metrical, we have the metric $g_\Phi$. Hence each tangent space $T_\Phi \tilde{\mathcal{M}}(X)$ has the metric and $\tilde{\mathcal{M}}_s(X)$ is a Riemannian manifold. We also see that the action of $\text{Diff}_0$ on $\tilde{\mathcal{M}}_s(X)$ is isometric (see [3]). Let $d$ be the distance of the Riemannian manifold $\tilde{\mathcal{M}}_s(X)$ and $\pi$ the natural projection $\pi: \tilde{\mathcal{M}}_s(X) \to \mathcal{M}_\varnothing(X)$. Then we define $d(\pi(\Phi^1), \pi(\Phi^2))$ by

$$d(\pi(\Phi^1), \pi(\Phi^2)) = \inf_{f, g \in \text{Diff}_0} d(f^* \Phi^1, g^* \Phi^2),$$

where $\Phi, \Phi' \in \tilde{\mathcal{M}}_s(X)$. Since the action of $\text{Diff}_0$ preserves the distance $d$,

$$d(\pi(\Phi^1), \pi(\Phi^2)) = \inf_{f \in \text{Diff}_0} d(f^* \Phi^1, \Phi^2).$$

Hence we have triangle inequality,

$$d(\pi(\Phi^1), \pi(\Phi^2)) + d(\pi(\Phi^2), \pi(\Phi^3)) = \inf_{f \in \text{Diff}_0} d(f^* \Phi^1, \Phi^2) + \inf_{g \in \text{Diff}_0} d(\Phi^2, g^* \Phi^3) \\
\leq \inf_{f, g \in \text{Diff}_0} d(f^* \Phi^1, g^* \Phi^3) = d(\pi(\Phi^1), \pi(\Phi^3)).$$

We shall that $d$ is a distance of $\mathcal{M}_\varnothing(X)$. We assume that the distance $d(\pi(\Phi^0), \pi(\Phi)) = 0$. Then $\inf_{f \in \text{Diff}_0} d(\Phi^0, f^* \Phi) = 0$. Hence $f^* \Phi$ is in a small neighborhood $U$ at $\Phi^0$. As in lemma 2-1 a neighborhood $U$ of $\tilde{\mathcal{M}}(X)$ at $\Phi^0$ is homeomorphic to a product $V \times S_{\Phi^0}$, where $V$ is a neighborhood of $\text{Diff}_0$ at the identity. We define a distance $d_{\Phi^0}$ on the cohomology group $\bigoplus_i H^p_{dR}(X)$ by using the harmonic representation with respect to the metric $g_{\Phi^0}$. From lemma 2-4, we have the injective map

$$P: S_{\Phi^0} \to \bigoplus_i H^p_{dR}(X).$$
Then we see that
\[
\inf_{f \in \text{Diff}_0, \ f^* \Phi \in U} d(\Phi^0, f^* \Phi) = C \ d_{\Phi^0}(P(\Phi^0), P(\Phi)),
\]
where $C$ is a constant. (Note that the action of $\text{Diff}_0$ preserves a class of $\bigoplus_i H^p_{dR}(X)$.) Hence from our assumption we have $P(\Phi) = P(\Phi^0)$. since $P$ is injective, $\pi(\Phi) = \pi(\Phi^0)$. Hence $d$ is a distance on $\mathcal{M}_\mathcal{O}(X)$. □

**Proof of theorem 1-6.** Since $\mathcal{O}$ is elliptic, each slice $S_\Phi$ is smooth from lemma 2-3. Each slice $S_\Phi$ is coordinates of $\mathcal{M}_\mathcal{O}(X)$ since $\mathcal{O}$ is topological from lemma 2-5. Each slice is homeomorphic to an open set of the cohomology group $H^1(\Phi)$. Since $\mathcal{O}$ is metrical, $\mathcal{M}_\mathcal{O}(X)$ is Hausdorff and each slice is canonically constructed. □

**Proof of theorem 1-7.** Each slice is a local coordinates of $\mathcal{M}_\mathcal{O}(X)$. Then the result follows from lemma 2-4. □

Since $\mathcal{O}$ is metrical, we have the metric $g_\Phi$ for each $\Phi \in \mathcal{E}^1$. Hence the metric $g_\Phi$ defines the metric on each tangent space $E^1 = T_\Phi \mathcal{E}^1$. So $\mathcal{E}^1$ can be considered as a Riemannian manifold. Then we see that the action of $\text{Diff}_0$ on $\mathcal{E}^1$ is isometry. Let $I(\Phi)$ be the isotropy group of $\text{Diff}_0$ at $\Phi$,

\[I(\Phi) = \{ f \in \text{Diff}_0(X) | f^* \Phi = \Phi \}.\]

Let $S_{\Phi^0}$ be a slice at $\Phi^0$. Then we shall compare $I_{\Phi^0}$ to other isotropy group $I_\Phi$ for $\Phi \in S_{\Phi^0}$.

**Theorem 1-8.** Let $I(\Phi^0)$ be the isotropy group of $\text{Diff}_0(X)$ at $\Phi^0$ and $S_{\Phi^0}$ the slice at $\Phi^0$. Then $I_{\Phi^0}$ is a subgroup of the isotropy group $I_\Phi$ for each $\Phi \in S_{\Phi^0}$. (We take $S_{\Phi^0}$ sufficiently small for necessary.)

**Proof of theorem 1-8.** From definition of $S_{\Phi^0}$, the slice $S_{\Phi^0}$ is invariant under the action of $I_{\Phi^0}$. The map $P|_{S_{\Phi^0}} : S_{\Phi^0} \rightarrow \bigoplus_i H^p_i(X)$ is locally injective. Since the action of $\text{Diff}_0$ preserves each class of de Rham cohomology group, we can take a sufficiently small $S_{\Phi^0}$ such that the
action of $I_{\Phi^0}$ is trivial on the slice $S_{\Phi^0}$. Hence $I_{\Phi^0}$ is a subgroup of the isotropy group $I_\Phi$ for each $\Phi \in S_{\Phi^0}$. □

Proof of theorem 1-9. The slice $S_{\Phi^0}$ is local coordinates of $\mathcal{M}_{KE}(X)$ and invariant under the action of $\text{Diff}(X)$. Hence the moduli space $\tilde{\mathcal{M}}_{KE}(X)/\text{Diff}(X)$ is locally homeomorphic to the quotient space $S_{\Phi^0}/I$, where $I$ is the isotropy,

$$I = \{ f \in \text{Diff}(X) | f^* \Phi^0 = \Phi^0 \}.$$

As in proof of proposition 2-8, $\text{Diff}(X)$ acts on $S_{\Phi^0}$ isometrically. Hence we see that there is an open set $V$ of $T_{\Phi^0}S_{\Phi^0}$ with the action of $I$ such that the quotient $V/I$ is homeomorphic to $S_{\Phi^0}/I$. Since $T_{\Phi^0}S_{\Phi^0}$ is isomorphic to $H^1(#_{\Phi^0})$ and the action of $I$ on $H^1(#_{\Phi^0})$ is a isometry with respect to $g_{\Phi^0}$. The action of $I$ preserves integral cohomology class. Hence from lemma 2-4 we see that $V/I$ is the quotient by a finite group. □

§3-1. CALABI-YAU STRUCTURES

Let $V$ be a real vector space of $2n$ dimensional. We consider the complex vector space $V \otimes \mathbb{C}$ and a complex form $\Omega \in \wedge^n V^* \otimes \mathbb{C}$. The vector space $\text{ker} \, \Omega$ is defined as

$$\text{Ker} \, \Omega = \{ v \in V \otimes \mathbb{C} | i_v \Omega = 0 \},$$

where $i_v$ denotes the interior product.

Definition 3-1-1 (Calabi-Yau structures). A complex $n$ form $\Omega$ is a Calabi-Yau structure on $V$ if $\dim_{\mathbb{C}} \text{Ker} \, \Omega = n$ and $\text{Ker} \, \Omega \cap \overline{\text{Ker} \, \Omega} = \{0\}$, where $\overline{\text{Ker} \, \Omega}$ is the conjugate vector space.

We denote by $\mathcal{A}_{\text{CY}}(V)$ the set of Calabi-Yau structures on $V$. We define the almost complex structure $I_\Omega$ on $V$ by

$$I_\Omega(v) = \begin{cases} \sqrt{-1}v & \text{if } v \in \text{Ker} \, \Omega, \\ -\sqrt{-1}v & \text{if } v \in \overline{\text{Ker} \, \Omega}. \end{cases}$$
So that is, Ker $\Omega = T^{1,0}V$ and $\overline{\text{Ker} \Omega} = T^{0,1}V$ and $\Omega$ is a non-zero $(n,0)$ form on $V$ with respect to $I_\Omega$. Let $\mathcal{J}$ be the set of almost complex structures on $V$. Then $\mathcal{A}_{CY}(V)$ is the $\mathbb{C}^*-\text{bundle}$ over $\mathcal{J}$. We denote by $\rho$ the action of the real general linear group $G = GL(V)$ on the complex $n$ forms,

$$\rho : GL(V) \rightarrow \text{End}(\wedge^n(V \otimes \mathbb{C})^*).$$

Since $G$ is a real group, $\mathcal{A}_{CY}(V)$ is invariant under the action of $G$. Then we see that the action of $G$ on $\mathcal{A}_{CY}(V)$ is transitive. The isotropy group $H$ is defined as

$$H = \{ g \in G \mid \rho_g \Omega = \Omega \}.$$ 

Then we see $H = \text{SL}(n, \mathbb{C})$. Hence the set of Calabi-Yau structures $\mathcal{A}_{CY}(V)$ is the homogeneous space,

$$\mathcal{A}_{CY}(V) = G/H = GL(2n, \mathbb{R})/\text{SL}(n, \mathbb{C}).$$

(Note that the set of almost complex structures $\mathcal{J} = \text{GL}(2n, \mathbb{R})/\text{GL}(n, \mathbb{C})$. ) An almost complex structure $I$ defines a complex subspace $T^{1,0}$ of dimension $n$. Hence we have the map $\mathcal{J} \rightarrow \text{Gr}(n, C^{2n})$. We also have the map from $\mathcal{A}_{CY}(V)$ to the tautological line bundle $L$ over the Grassmannian $\text{Gr}(n, C^{2n})$ removed $0-$section. Then we have the diagram:

$$\begin{array}{ccc}
\mathcal{A}_{CY}(V) & \longrightarrow & L\backslash 0 \\
\downarrow \mathbb{C}^* & & \downarrow \\
\mathcal{J} & \longrightarrow & \text{Gr}(n, C^{2n})
\end{array}$$

$\mathcal{A}_{CY}(V)$ is embedded as a smooth submanifold in $n-$forms $\wedge^n$. This is Plücker embedding described as follows,

$$\begin{array}{ccc}
\mathcal{A}_{CY}(V) & \longrightarrow & L\backslash 0 \\
\downarrow \mathbb{C}^* & & \downarrow \\
\mathcal{J} & \longrightarrow & \text{Gr}(n, C^{2n})
\end{array} \longrightarrow \mathbb{C}\text{P}^n.$$
Hence the orbit $\mathcal{O}_{CY} = \mathcal{A}_{CY}(V)$ is a submanifold in $\wedge^n$ defined by Plücker relations. Let $X$ be a real compact manifold of $n$ dimensional. Then we have the $G/H$ bundle $\mathcal{A}_{CY}(X)$ over $X$ as in section 1. We denote by $\mathcal{E} = \mathcal{E}_{CY}^1$ the set of smooth global section of $\mathcal{A}_{CY}(X)$. Then we have the almost complex structure $I_\Omega$ corresponding to $\Omega \in \mathcal{E}^1$. Then we have

**Lemma 3-1-2.** If $\Omega \in \mathcal{E}^1$ is closed, then the almost complex structure $I_\Omega$ is integrable.

**Proof.** Let $\{\theta_i\}_{i=1}^n$ be a local basis of $\Gamma(\wedge^{1,0})$ with respect to $\Omega$. From Newlander-Nirenberg’s theorem it is sufficient to show that $d\theta_i \in \Gamma(\wedge^{2,0} \oplus \wedge^{1,1})$ for each $\theta_i$. Since $\Omega$ is of type $\wedge^{n,0}$,

$$\theta_i \wedge \Omega = 0.$$ 

Since $d\Omega = 0$, we have

$$d\theta_i \wedge \Omega = 0.$$ 

Hence $d\theta_i \in \Gamma(\wedge^{2,0} \oplus \wedge^{1,1})$. □

Then we have the moduli space of Calabi-Yau structures on $X$,

$$\mathcal{M}_{CY}(X) = \{ \Omega \in \mathcal{E}_{CY}^1 | d\Omega = 0 \}/\text{Diff}_0(X).$$

From lemma 3-1-2 we see that $\mathcal{M}_{CY}(X)$ is the $\mathbb{C}^*$—bundle over the moduli space of integrable complex structures on $X$ with trivial canonical line bundles.

**Proposition 3-1-3.** The orbit $\mathcal{O}_{CY}$ is elliptic.

**Proof.** Let $\wedge^{p,q}$ be $(p,q)$—forms on $V$ with respect to a closed form $\Omega^0 \in \mathcal{A}_{CY}(V)$. In this case we see that

$$E^0 = \wedge^{n-1,0}$$

$$E^1 = \wedge^{n,0} \oplus \wedge^{n-1,1}$$

$$E^2 = \wedge^{n,1} \oplus \wedge^{n-1,2}.$$
Hence we have the complex:

\[ \wedge^{n-1,0} \xrightarrow{\wedge u} \wedge^{n,0} \oplus \wedge^{n-1,1} \xrightarrow{\wedge u} \wedge^{n,1} \oplus \wedge^{n-1,2}, \]

for \( u \in V \). Let \( a = (x,y) \) be an element of \( E^1 \). We assume that \( u \wedge a = 0 \). Then since de Rham complex is elliptic, there is an element \( b = (z,w) \in \wedge^{n-1,0} \oplus \wedge^{n-2,1} \) such that \( a = u \wedge b \). Hence we have

(1) \[ x = u^{1,0} \wedge z, \]

(2) \[ y = u^{0,1} \wedge z + u^{1,0} \wedge w, \]

(3) \[ 0 = u^{0,1} \wedge w \]

From the equation (3) and ellipticity of Dolbeault complex, there is an element \( \gamma \in \wedge^{n-2,0} \) such that \( w = u^{0,1} \wedge \gamma \). Hence \( y = u^{0,1} \wedge z + u^{1,0} \wedge (u^{0,1} \wedge \gamma) = u^{0,1} \wedge \hat{z} \), where \( \hat{z} = z - u^{1,0} \wedge \gamma \in \wedge^{n-1,0} \). Hence \( a = (u^{1,0} \wedge \hat{z}, u^{0,1} \wedge \hat{z}) \) for \( \hat{z} \subset \wedge^{n-1,0} \). □

**Proposition 3-1-4.** Let \( I_\Omega \) be the complex structure corresponding to \( \Omega \in \mathcal{E}^1 \). If \( \partial \bar{\partial} \) lemma holds for the complex manifold \((X, I_\Omega)\), \( \Omega \) is topological. In particular, \((X, I_\Omega)\) is Kählerian, \( \Omega \) is topological.

*Proof.* As in proof of proposition 3-1-2 the complex \( \#_\Omega \) is given as

\[ \Gamma(\wedge^{n-1,0}) \xrightarrow{d} \Gamma(\wedge^{n,0} \oplus \wedge^{n-1,1}) \xrightarrow{d} \Gamma(\wedge^{n,1} \oplus \wedge^{n-1,2}). \]

Then we have the following double complex:

\[ \begin{array}{cccc}
\Gamma(\wedge^{n,0}) & \xrightarrow{\bar{\partial}} & \Gamma(\wedge^{n,1}) & \xrightarrow{\bar{\partial}} \\
\partial & \uparrow & \partial & \uparrow \\
\Gamma(\wedge^{n-1,0}) & \xrightarrow{\bar{\partial}} & \Gamma(\wedge^{n-1,1}) & \xrightarrow{\bar{\partial}} \\
\partial & \uparrow & \partial & \uparrow \\
\Gamma(\wedge^{n-2,0}) & \xrightarrow{\bar{\partial}} & \Gamma(\wedge^{n-2,1}) & \xrightarrow{\bar{\partial}} \\
\end{array} \]
Let \( a = (x, y) \) be a closed element of \( \Gamma(\wedge^n, 0) \oplus \Gamma(\wedge^{n-1}, 1) \). We assume that \( a = db \). Then \( b = (z, w) \in \Gamma(\wedge^{n-1}, 0) \oplus \Gamma(\wedge^{n-2}, 1) \), satisfying
\[
\begin{align*}
x &= \partial z \\
y &= \overline{\partial} z + \partial w \\
0 &= \overline{\partial} w.
\end{align*}
\]
Since \( \partial w \in \Gamma(\wedge^{n-1}, 1) \) is \( \overline{\partial} \) closed, We apply \( \partial \overline{\partial} \) lemma to \( \partial w \). Then there is an element \( \gamma \in \Gamma(\wedge^{n-2}, 0) \) such that
\[
\partial w = \partial \overline{\partial} \gamma = -\overline{\partial} \partial \gamma.
\]
Hence \( a \) is written as
\[
a = d(z - \partial \gamma).
\]
It implies that the map \( p: H^1(\#) \to H^n(X) \) is injective. □

Remark. We define \( F^{n-1} \wedge^* \)
\[
F^{n-1} \wedge^m = \bigoplus_{p+q=m} \Gamma(\wedge^{p,q}).
\]
Then the complex \( \#_\Omega \) is
\[
F^{n-1} \wedge^{n-1} \xrightarrow{d} F^{n-1} \wedge^n \xrightarrow{d} F^{n-1} \wedge^{n+1}.
\]
Hence \( \Omega \) is topological if and only if we have the following:
\[
F^{n-1} H^n(X) = H^n(F^{n-1} \wedge^*).
\]
Hence from proposition 3-1-3, we have the smooth slice \( S_\Omega \) corresponding to each \( \Omega \) and \( S_\Omega \) is the space of deformations of \( \Omega \). However \( \mathcal{O}_{CY} \) is not metrical and the moduli space \( \mathcal{M}_{CY}(X) \) is not Hausdorff in general. In fact, It is known that \( \mathcal{M}_{CY}(X) \) is not Hausdorff for a K3 surface. Hence in order to obtain a Hausdorff moduli space, we must introduce extra geometric structures. The most natural structure is Kähler-Einstein structure on a Calabi-Yau manifold.
§3-2. Kähler-Einstein structures

Let $V$ be a real vector space of $2n$ dimensional. We consider a pair $\Phi = (\Omega, \omega)$ of a Calabi-Yau structure $\Omega$ and a real symplectic structure $\omega$ on $V$,

$$\Omega \in \mathcal{A}_{CY}(V),$$

$$\omega \in \wedge^2 V^*, \quad \omega \wedge \cdots \wedge \omega \neq 0.$$

We define $g_{\Omega, \omega}$ by

$$g_{\Omega, \omega}(u, v) = \omega(I_{\Omega} u, v),$$

for $u, v \in V$.

**Definition 3-2-1 (Kähler-Einstein structures ).** A Kähler-Einstein structure on $V$ is a pair $\Phi = (\Omega, \omega)$ such that

1. $\Omega \wedge \omega = 0, \quad \overline{\Omega} \wedge \omega = 0$
2. $\Omega \wedge \overline{\Omega} = c_n \omega \wedge \cdots \wedge \omega$
3. $g_{\Omega, \omega}$ is positive definite.

where $c_n$ is a constant depending only on $n$, i.e,

$$c_n = (-1)^{\frac{n(n-1)}{2}} \frac{2^n}{i^n n!}.$$

From the equation (1) we see that $\omega$ is of type $\wedge^{1,1}$ with respect to the almost complex structure $I_{\Omega}$. The equation (2) is called Monge-Ampère equation.

**Lemma 3-2-2.** Let $\mathcal{A}_{KE}(V)$ be the set of Kähler-Einstein structures on $V$. Then There is the transitive action of $G = GL(2n, \mathbb{R})$ on $\mathcal{A}_{KE}(V)$ and $\mathcal{A}_{KE}(V)$ is the homogeneous space

$$\mathcal{A}_{KE}(V) = GL(2n, \mathbb{R})/SU(n).$$
Proof. Let $g_{\Omega,\omega}$ be the Kähler metric. Then we have a unitary basis of $TX$. Then the result follows from (1),(2). □

Hence the set of Kähler-Einstein structures on $V$ is the orbit $O_{KE}$,

$$O_{KE} \subset \wedge^n(V \otimes \mathbb{C})^* \oplus \wedge^2 V^*.$$  

**Theorem 3-2-3.** The orbit $O_{KE}$ is metrical, elliptic and topological.

Proof. From lemma 3-2-2 the isotropy group is $SU(n)$. Hence $O_{KE}$ is metrical. At first we shall show that $O_{KE}$ is elliptic. Let $(\Omega^0,\omega^0)$ be an element of $A_{KE}(V)$. Then we have the vector space $E^0(V) = E^0_{KE}(V)$ by

$$E^0_{KE}(V) = \{ (i_v\Omega^0, i_v\omega^0) \in \wedge^{n-1}_C \oplus \wedge^{n-1} | v \in V \}$$

The vector space $E^1(V) = E^1_{KE}(V)$ is the set of $(\alpha,\beta) \in \wedge^n_C \oplus \wedge^2$ satisfying equations

$$\alpha \wedge \omega^0 + \Omega^0 \wedge \beta = 0,$$

$$\alpha \wedge \bar{\Omega}^0 + \bar{\Omega}^0 \wedge \bar{\alpha} = n c_n \beta \wedge (\omega^0)^{n-1} \quad (4)$$

We assume that $u \wedge \alpha = 0, u \wedge \beta = 0$ for some non zero vector $u \in V$. Then since the orbit $O_{CY}$ is elliptic, $(\alpha,\beta)$ is given as

$$\alpha = u \wedge s, \quad \beta = u \wedge t, \quad (5)$$

form some $s \in \wedge^{n-1,0}_{\mathcal{H}^*}$ and $t \in \wedge^1$. Hence $s, t$ are written as

$$s = i_{v_1}\Omega^0, \quad t = i_{v_2}\omega^0, \quad (6)$$

for some $v_1, v_2 \in V$. Set $v = v_1 - v_2$. Then from (4),(5) and (6) using (1),(2), we have

$$u \wedge (i_v\omega^0) \wedge \Omega^0 = 0 \quad (7)$$

$$u \wedge (i_v\omega^0) \wedge (\omega^0)^{n-1} = 0. \quad (8)$$
Form (7) we have

\( u \wedge i_v \omega^0 \in \wedge^{2,0} \oplus \wedge^{1,1}. \)

We also have from (8)

\( u \wedge i_v \omega^0 \in \wedge^{2,0} \oplus \wedge^{0,2}. \)

Since \( u \wedge i_v \omega^0 \) is a real form, we see from (9),(10) that

\( u \wedge i_v \omega^0 = 0. \)

Hence \((\alpha, \beta)\) is given as

\[ \alpha = u \wedge i_{v_1} \Omega^0, \]
\[ \beta = u \wedge i_{v_2} \omega^0 = u \wedge i_{v_1} \omega^0 - u \wedge i_v \omega^0 = u \wedge i_{v_1} \omega^0. \]

From (12) we see that the complex

\[ E^0_{KE}(V) \xrightarrow{\wedge u} E^1_{KE}(V) \xrightarrow{\wedge u} E^2_{KE}(V) \]

is elliptic. Next we shall show that \( \mathcal{O}_{KE} \) is topological. Let \((\alpha, \beta)\) be an element of \( \Gamma(E^1_{KE}) \). We assume that \( \alpha \) and \( \beta \) are exact forms respectively. Then since \( \mathcal{O}_{CY} \) is topological, we have \( \alpha = ds, \beta = dt \) for some \( s \in \Gamma(\wedge_{\mathbb{C}}^{n-1}), \) \( t \in \Gamma(\wedge^2). \) Hence \( s, t \) are written as

\( s = i_{v_1} \Omega^0, \) \( t = i_{v_2} \omega^0, \)

for some \( v_1, v_2 \in \Gamma(TX) \). Then from equations (4),(13) using (1),(2) we have

\( d(i_v \omega^0) \wedge \Omega^0 = 0 \)
\( d(i_v \omega^0) \wedge (\omega^0)^{n-1} = 0, \)
where \( v = v_1 - v_2 \). From (14), (15) we have

\[
\begin{align*}
(16) & \quad di_v \omega^0 \in \Gamma(\wedge^{2,0} \oplus \wedge^{1,1}) \\
(17) & \quad d_v \omega^0 \in \Gamma(\wedge^{2,0} \oplus \wedge^{0,2}).
\end{align*}
\]

Since \( di_v \omega^0 \) is real, we have from (16), (17)

\[
(18) \quad di_v \omega^0 = 0.
\]

Hence \( \beta = di_{v_2} \omega^0 = di_{v_1} \omega^0 - di_v \omega^0 \). Then from (18) we see that

\[
\alpha = di_{v_1} \Omega^0, \quad \beta = di_{v_1} \omega^0.
\]

Hence the map \( p: H^1(\#) \longrightarrow H^n(X, \mathbb{C}) \oplus H^2(X, \mathbb{R}) \) is injective. \( \square \)

Hence from theorem 1-6 in section 1 we have the following:

**Theorem 3-2-4.** Let \( \mathcal{M}_{KE}(X) \) be the moduli space of Kähler-Einstein structures over \( X \),

\[
\mathcal{M}_{KE}(X) = \tilde{\mathcal{M}}_{KE}(X)/\text{Diff}_0(X),
\]

where

\[
\tilde{\mathcal{M}}_{KE}(X) = \{ (\Omega, \omega) \in \mathcal{E}_{KE}^1 \mid d\Omega = 0, d\omega = 0 \}.
\]

Then \( \mathcal{M}_{KE}(X) \) is a smooth manifold. Let \( \pi \) be the natural projection

\[
\pi: \tilde{\mathcal{M}}_{KE}(X) \longrightarrow \mathcal{M}_{KE}(X).
\]

Then coordinates of \( \mathcal{M}_{KE}(X) \) at each \( (\Omega, \omega) \in \tilde{\mathcal{M}}_{KE}(X) \) is canonically given by an open ball of the cohomology group \( H^1(\#) \).

We have the Dolbeault cohomology group \( H^{p,q}(X) \) with respect to each \( \Omega \). Then we have
Theorem 3-2-5. The cohomology group \( H^1(\#) \) is the subspace of \( H^n(X, \mathbb{C}) \oplus H^2(X, \mathbb{R}) \) which is defined by equations

\[
\alpha \wedge \omega + \Omega \wedge \beta = 0,
\]

\[
\alpha \wedge \overline{\Omega} + \Omega \wedge \overline{\alpha} = n c_n \beta \wedge \omega^{n-1},
\]

where \( \alpha \in H^n(X, \mathbb{C}), \beta \in H^2(X, \mathbb{R}) \).

Let \( P^{p,q}(X) \) be the primitive cohomology group with respect to \( \omega \). Then we have Lefschetz decomposition,

\[
\alpha = \alpha^{n,0} + \alpha^{n-1,1} + \alpha^{n-2,0} \wedge \omega \in P^{n,0}(X) \oplus P^{n-1,0}(X) \oplus P^{n-2,0}(X) \wedge \omega.
\]

\[
\beta = \beta^{2,0} + \beta^{1,1} + \beta^{0,0} \wedge \omega + \beta^{0,2} \in P^{2,0}(X) \oplus P^{1,1}(X) \oplus P^{0,0}(X) \wedge \omega \oplus P^{0,2}(X).
\]

Then equation (19) is written as

\[
\alpha^{n-2,0} \wedge \omega \wedge \omega + \Omega \wedge \beta^{0,2} = 0,
\]

\[
\alpha^{n,0} \wedge \overline{\Omega} = n c_n \beta^{0,0} \omega^n.
\]

We see that \( \alpha^{n,0} \in P^{n,0}(X) \) and \( \beta^{0,0} \in P^{0,0}(X) \) are corresponding to the deformation in terms of constant multiplication:

\[
\Omega \rightarrow t\Omega, \quad \omega \rightarrow s\omega
\]

If a Kähler class [\( \omega \)] is not invariant under a deformation, such a deformation corresponds to an element of \( \beta^{2,0} \) and \( \alpha^{n-2,0} \). This is in the case of Calabi family of hyperKähler manifolds, i.e., Twistor space gives such a deformation. It must be noted that there is no relation between \( \alpha^{n-1,1} \in P^{n-1,1}(X) \) and \( \beta^{1,1} \in P^{1,1}(X) \). We have from theorem 1-7 in section 1,

Theorem 3-2-6. The map \( P \) is locally injective,

\[
P : \mathcal{M}_{KE}(X) \rightarrow H^n(X, \mathbb{C}) \oplus H^2(X, \mathbb{R}).
\]

We also have from theorem 1-8 in section 1,
**Theorem 3-2-7.** Let $I(\Omega, \omega)$ be the isotropy group of $(\Omega, \omega)$,

$$I(\Omega, \omega) = \{ f \in \text{Diff}_0(X) \mid f^*\Omega = \Omega, \ f^*\omega = \omega \}.$$

We consider the slice $S_0$ at $\Phi^0 = (\Omega^0, \omega^0)$. Then the isotropy group $I(\Omega^0, \omega^0)$ is a subgroup of $I(\Omega, \omega)$ for each $(\Omega, \omega) \in S_0$.

We define the map $P_{H^2}$ by

$$P_{H^2} : \mathcal{M}_{KE}(X) \longrightarrow \mathbb{P}(H^2(X)),$$

where

$$P_{H^2}([\Omega, \omega]) \longrightarrow [\omega]_{dR} \in \mathbb{P}(H^2(X)),$$

$\mathbb{P}(H^2(X))$ denoted the projective space $(H^2(X) - \{0\})/\mathbb{R}^*$. Then we have

**Theorem 3-2-8.** The inverse image $P_{H^2}^{-1}([\omega]_{dR})$ is a smooth manifold.

**Proof.** From theorem 3-2-5 and theorem 3-2-6 the differential of the map $P_{H^2}$ is surjective. Hence from the implicit function theorem $P_{H^2}^{-1}([\omega]_{dR})$ is a smooth manifold.

**Remark.** $P_{H^2}^{-1}([\omega]_{dR})$ is the $\mathbb{C}^*$ bundle over the moduli space of polarized Calabi-Yau manifolds [4].

§4. Special lagrangians and Kähler-Einstein structures

§4-1. Let $\Phi^0 = (\Omega^0, \omega^0)$ be a Kähler-Einstein structure on a compact Calabi-Yau manifold $X$ and $i_M : M \subset X$ a special lagrangian submanifold of $X$ with respect to $(\Omega^0, \omega^0)$, i.e,

$$i_M^*(\Omega^0)^{Im} = 0, \quad i_M^*\omega^0 = 0,$$

where $(\Omega)^{Im}$ is the imaginary part of the complex form $\Omega^0$. We assume that $M$ is a compact $n$ dimensional real manifold. We denote by $\text{Diff}_0(X, M)$ the subgroup of $\text{Diff}_0(X)$ preserving the submanifold $M$,

$$\text{Diff}_0(X, M) = \{ f \in \text{Diff}_0(X) \mid f(M) = M \},$$

where $\text{Diff}_0(X)$ is the identity component of the group of diffeomorphisms of $X$. 
**Definition 4-1-1.** We define the relative moduli space $\mathcal{M}_{KE}(X, M)$ by

$$\mathcal{M}_{KE}(X, M) = \widetilde{\mathcal{M}}_{KE}(X, M)/\text{Diff}_0(X, M),$$

where $\widetilde{\mathcal{M}}_{KE}(X, M)$ is given as

$$\widetilde{\mathcal{M}}_{KE}(X, M) = \{ \Phi = (\Omega, \omega) \in \mathcal{E}_{KE}^1(X) | d\Phi = 0, i^*_M \Omega^m = 0, i^*_M \omega = 0 \}.$$

For simplicity we use the following notation:

**Notation 4-1-2.** Let $\Phi = (\Omega, \omega)$ be a Kähler-Einstein structure. Then a pair $(\phi, \psi)$ denotes $\Phi$, where

$$\phi = \Omega^R, \quad \psi = (\Omega^I, \omega)$$

where $\Omega^R$ is the real part of $\Omega$.

Then $\mathcal{M}_{KE}(X, M) = \mathcal{M}(X, M)$ is rewritten as

$$\mathcal{M}_{KE}(X, M) = \{ \Phi = (\phi, \psi) \in \mathcal{E}_{KE}^1(X) | d\Phi = 0, i^*_M \psi = 0 \}/\text{Diff}_0(X, M)$$

We consider a special lagrangian $M'$ with respect to a Kähler-Einstein structure $\Phi = (\Omega, \omega)$ on $X$. Then there is the action of $\text{Diff}_0(X)$ on the set of pairs $(M', \Phi)$ by

$$f(M', \Phi) = (f^{-1}(M), f^*\Phi),$$

for $f \in \text{Diff}_0$. Hence we have the moduli space of pairs of special lagrangian submanifolds and Kähler-Einstein structures on $X$:

**Definition 4-1-3.**

$$\mathcal{P} = \{ (\Phi, M) | \Phi \in \widetilde{\mathcal{M}}_{KE}(X), M \subset X, \text{ a special lagrangian } \} /\text{Diff}_0(X),$$

where

$$\widetilde{\mathcal{M}}_{KE}(X) = \{ \Phi = (\Omega, \omega) \in \mathcal{E}_{KE}^1(X) | d\Phi = 0 \}.$$
Lemma 4-1-4. $\mathcal{M}_{KE}(X, M)$ is a connected component of the moduli space $\mathcal{P}$.

Proof. There is the natural map $\tilde{\mathcal{M}}_{KE}(X, M) \to \mathcal{P}$. From Definition 4-1-1 and 2, we see this map is injective. Let $M'$ be a special lagrangian submanifold with respect to a Kähler-Einstein structure $\Phi'$. If the class $[\Phi', M'] \in \mathcal{P}$ belongs to the same connected component as to the one of the class $[M, \Phi]$, we have

$$M' = f(M),$$

for some $f \in \text{Diff}_0(X)$. Hence the map $\tilde{\mathcal{M}}_{KE}(X, M) \to \mathcal{P}$ is bijective map to the connected component of $\mathcal{P}$. \qed

As in section 3, we have vector bundles $E^0_X, E^1_X$ and $E^2_X$ for each Kähler-Einstein structure $\Phi^0 = (\Omega^0, \omega^0)$ by

$$\Gamma_X(E^0_X) = \{(i_v \Omega^0, i_v \omega^0) \in \Gamma_X(\wedge^{n-1}_C \oplus \wedge^1) \mid v \in \Gamma_X(TX)\}$$

$$\Gamma_X(E^1_X) = \{(\theta \wedge i_v \Omega^0, \theta \wedge i_v \omega^0) \in \Gamma_X(\wedge^n C \oplus \wedge^2) \mid \theta \in \Gamma_X(\wedge^1), v \in \Gamma_X(TX)\}$$

$$\Gamma_X(E^2_X) = \{(\theta \wedge \alpha, \theta \wedge \beta) \in \Gamma_X(\wedge^{n+1}_C \oplus \wedge^3) \mid \theta \in \Gamma_X(\wedge^1), (\alpha, \beta) \in \Gamma_X(E^1_X)\},$$

we have the complex $\#_X = \#_{\Phi^0}$,

$$\begin{array}{c}
\Gamma_X(E^0_X) \xrightarrow{d_{X,0}} \Gamma_X(E^1_X) \xrightarrow{d_{X,1}} \Gamma_X(E^2_X)
\end{array}$$

By using Notation $\Phi^0 = (\phi^0, \psi^0)$, we have

$$\Gamma_X(E^0_X) = \{(i_v \phi^0, i_v \psi^0) \mid v \in \Gamma_X(TX)\}$$

$$\Gamma_X(E^1_X) = \{(\theta \wedge i_v \phi^0, \theta \wedge i_v \psi^0) \mid \theta \in \Gamma_X(\wedge^1), v \in \Gamma_X(TX)\}$$

$$\Gamma_X(E^2_X) = \{(\theta \wedge \alpha, \theta \wedge \beta) \mid \theta \in \Gamma_X(\wedge^1), (\alpha, \beta) \in \Gamma_X(E^1_X)\}$$

Let $i_M$ be the inclusion $M \to X$. Then by using the pull back $i_M^*$ of differential forms in the second component of each $E^i_X$, we obtain vector bundles $E^i_M$ over $M$ $(i = 0, 1, 2)$,

$$\Gamma_M(E^0_M) = \{i_M^*(i_v \psi^0) \mid v \in \Gamma_X(TX)\}$$

$$\Gamma_M(E^1_M) = \{i_M^*(\theta \wedge i_v \psi^0) \mid \theta \in \Gamma_M(\wedge^1), v \in \Gamma_X(TX)\}$$

$$\Gamma_M(E^2_M) = \{i_M^*(\theta \wedge \beta) \mid \theta \in \Gamma_M(\wedge^1), (\alpha, \beta) \in \Gamma_X(E^1_X)\}.$$
We denote by $\#_M$ the complex on $M$,

\[
\begin{align*}
\left(\#_M\right) \quad & \Gamma_M(E^0_M) \xrightarrow{d_{M,0}} \Gamma_M(E^1_M) \xrightarrow{d_{M,1}} \Gamma_M(E^2_M) \\
\end{align*}
\]

From our construction of $E^i_M$, we have the map $\kappa$ in terms of the pull back $i^*_M$,

\[
\kappa: \Gamma_X(E^i_X) \longrightarrow \Gamma_M(E^i_M).
\]

Then the map $\kappa$ is the map from the complex $\#_X$ to the complex $\#_M$. Hence we have the short exact sequence of complexes,

\[
\begin{align*}
0 & \longrightarrow 0 \longrightarrow 0 \\
\downarrow & \downarrow \downarrow \downarrow \\
\Gamma_{X,M}(E^0_X) & \longrightarrow \Gamma_{X,M}(E^1_X) \longrightarrow \Gamma_{X,M}(E^2_X) \\
\downarrow & \downarrow \downarrow \downarrow \\
\Gamma_X(E^0_X) & \longrightarrow \Gamma_X(E^1_X) \longrightarrow \Gamma_X(E^2_X) \\
\downarrow & \downarrow \downarrow \downarrow \\
\Gamma_M(E^0_M) & \longrightarrow \Gamma_M(E^1_M) \longrightarrow \Gamma_M(E^2_M) \\
\downarrow & \downarrow \downarrow \downarrow \\
0 & \longrightarrow 0 \longrightarrow 0
\end{align*}
\]

where $\Gamma_{X,M}(E^*_X)$ is given as

\[
\begin{align*}
\Gamma_{X,M}(E^0_X) & = \{(i_v \phi^0, i_v \psi^0) \in \Gamma_X(E^0_X) \mid i^*_M (i_v \psi^0) = 0\} \\
\Gamma_{X,M}(E^1_X) & = \{(\theta \wedge i_v \phi^0, \theta \wedge i_v \psi^0) \in \Gamma_X(E^1_X) \mid i^*_M (\theta \wedge i_v \psi^0) = 0\} \\
\Gamma_{X,M}(E^2_X) & = \{(\theta \wedge \alpha, \theta \wedge \beta) \in \Gamma_X(E^2_X) \mid i^*_M (\theta \wedge \beta) = 0\},
\end{align*}
\]

We denote by $\#_{X,M}$ the complex $(\Gamma_{X,M}(E^*_X), d_{X,M})$,

\[
\begin{align*}
\left(\#_{X,M}\right) \quad & \Gamma_{X,M}(E^0_X) \xrightarrow{d_{X,M}} \Gamma_{X,M}(E^1_X) \xrightarrow{d_{X,M}} \Gamma_{X,M}(E^2_X) \\
\end{align*}
\]
Theorem 4-1-5. Let \( \mathcal{M}_{KE}(X, M) \) be the relative moduli space. Then \( \mathcal{M}_{KE}(X, M) \) is a smooth manifold (in particular, \( \mathcal{M}_{KE}(X, M) \) is Hausdorff). Further local coordinates of \( \mathcal{M}_{KE}(X, M) \) is given by an open ball of the cohomology group \( H^1(#_{X,M}) \), where \( H^1(#_{X,M}) \) is the first cohomology group of the complex \( #_{X,M} \).

We will prove theorem 4-1-5 in the rest of this section.

Lemma 4-1-6. Let \( H^i(#_X) \) be the cohomology group of the complex \( #_X \). Then we have \( H^0(#_X) = H^1(X) \).

Proof. Let \( i_v \Phi^0 = (i_v \Omega^0, i_v \omega^0) \) be an element of \( \Gamma_X(E^0_X) \), for some real vector \( v \in \Gamma_X(TX) \). Let \( g_{\Phi^0} \) be the metric of \( X \) corresponding to \( \Phi^0 = (\Omega^0, \omega^0) \) and \( * \) the Hodge star operator with respect to \( g_{\Phi^0} \). Then we see that

\[
* i_v \Omega^0 = c_1 i_v \omega^0 \wedge \overline{\Omega^0} \\
* i_v \omega^0 = c_2 i_v \Omega^0 \wedge \overline{\Omega^0} + c_2 i_v \overline{\Omega^0} \wedge \Omega^0,
\]

where \( c_1, c_2 \) are constants depending only on the dimension \( n \). Hence if \( di_v \omega^0 = 0 \) and \( dI_v \Omega^0 = 0 \), the \( i_v \omega^0 \) is a harmonic 1 form with respect to the metric \( g_{\Phi^0} \) from (1). We define the complex Hodge star operator \( *_c \) by

\[
*_c : \wedge^p X \to \wedge^{n-p} X,
\]

\[
c_2 (*_c \alpha) \wedge \overline{\Omega^0} = * \alpha.
\]

Then we have

\[
*_c i_v \omega^0 = i_v \Omega^0.
\]

Conversely any 1 form is written as \( i_v \omega^0 \) since \( \omega^0 \) is a symplectic form. If \( i_v \omega^0 \) is a harmonic one form, then from (3) \( i_v \Omega^0 \) is also harmonic. Hence we have the result. \( \square \)

Then from lemma 4-1-6 we have
Lemma 4-1-7. 

\[ H^0(\#_X) = \{ (a, b) \in \mathbb{H}^{n-1}(X, \mathbb{C}) \oplus \mathbb{H}^1(X) \mid *_c a = (b)^{1,0} \}, \]

where \( \mathbb{H}^i(X) \) denotes harmonic forms on \( X \) with respect to \( g_{\Phi^0} \).

**Proof.** This directly follows from lemma 6.

Lemma 4-1-8. Let \( N_M \) be the normal bundle on the special lagrangian \( M \) with respect to a Kähler-Einstein structure \( \Phi^0 = (\Omega^0, \omega^0) \). Then we have the canonical identification,

\[ N_M \cong E^0_M, \]

**Proof.** We consider a splitting \( \Gamma_M(N_M) \to \Gamma_X(TX) \) of the exact sequence,

\[ 0 \to \Gamma_{X,M}(TX) \to \Gamma_X(TX) \to \Gamma_M(N_M) \to 0. \]

Since \( M \) is a special lagrangian, \( i_M^* \psi^0 = 0 \) for \( \Phi^0 = (\phi^0, \psi^0) \), where \( \psi = ((\Omega^0)^{Im}, \omega^0) \). Using the splitting we have the identification:

\[ N_M \cong E^0_M, \]

\[ v \mapsto i_M^* v \psi^0. \]

The identification does not depend on a choice of a splitting. \( \square \)

Lemma 4-1-9. \( E^0_M \) is the set of self dual forms of \( \wedge^1_M \oplus \wedge^{n-1}_M \),

\[ E^0_M = \{ (a^1, a^{n-1}) \in \wedge^1_M \oplus \wedge^{n-1}_M \mid *_M a^1 = a^{n-1} \}, \]

where \( *_M \) is the Hodge star operator with respect to the pull back metric \( i_M^* g_{\Phi^0} \) on \( M \).

**Proof.** Any element of \( E^0_M \) is given as \( i_M^* (i_v \psi^0) \). Then we see that

\[ (4) \quad i_M^* (i_v \Omega)^{Im} = *_M i_M^* (i_v \omega^0). \]

Hence \( i_M^* (i_v \psi^0) \) is a self-dual form. Conversely any self dual form is written as \( i_M^* (i_v \psi^0) \) for some \( v \in N_M \). \( \square \)
Lemma 4-1-10. Let $H^0(\#_M)$ be the cohomology group of the complex $\#_M$. Then we have

$$H^0(\#_M) = H^1(M).$$

Proof. From lemma 4-1-9, $H^0(\#_M)$ is the set of self dual harmonic forms $(H^1_1(M) \oplus H^{n-1}(X))^\bot$. Hence we have the result. □

Lemma 4-1-11. The image $d_{M,0}\Gamma_M(E^0_M)$ coincides with the closed subset $d\Gamma_M(\wedge^1_M \oplus \wedge^{n-1}_M)$.

Proof. This follows from lemma 4-1-9. □

Let $\alpha$ be a real harmonic one form on $X$. Then we see that $\ast\Omega(\alpha)^{1,0} \in \Gamma_X(\wedge^{n-1,0})$ is also Harmonic. Hence by using the pull back $i^*_M$, from (3) and (4) we have

$$i^*_M(\ast\Omega(\alpha)^{1,0}) \in H^1(M).$$

Hence we have the commutative diagram:

Lemma 4-1-12.

$$\begin{array}{ccc}
H^1(X) & \overset{\gamma_{H^1}}{\longrightarrow} & H^1(M) \\
\downarrow \ast\Omega & & \downarrow \ast_M \\
H^{n-1}(X, \mathbb{C}) & \longrightarrow & H^{n-1}(M, \mathbb{R}),
\end{array}$$

where $\gamma_{H^1}$ is the induced map from the pull back $i^*_M$.

Lemma 4-1-13. Let $H^1(\#_M)$ be the first cohomology group of the complex $\#_M$. Then we have

$$H^1(\#_M) \cong H^n(M, \mathbb{R}) \oplus H^2(M, \mathbb{R}).$$

Proof. Let $I = I_{\Omega^0}$ be the complex structure on $X$ corresponding to $\Omega^0$. Since $I$ is an element of $\text{End}(TX)$, we have $\rho_I(\Omega^0) \in \Gamma_X(E^1_X)$ as
in section one. We see that $\rho_I \Omega^0 = i \Omega^0$ since $\Omega^0 \in \wedge^{n,0}$. Since $M$ is a special lagrangian, $i_M^* (\Omega^0)^{Re} = vol_M$. Hence

\begin{equation}
(5) \quad i_M^* ((\rho_I \Omega^0)^{Im}) = i_M^* (-\Omega^{Re}) = -vol_M.
\end{equation}

Since $\omega$ is a Kähler form, $\rho_I \omega^0 = \omega^0$. Hence $(vol_M, i_M^* \omega) = (vol_M, 0)$ is an element of $\Gamma_M(E^0_M)$. We also see that $\{ i_v \omega \mid v \in \Gamma_X(TX) \} = \Gamma_X(\wedge^2 X)$. Hence $\{ i_M^* (i_v \omega) \mid v \in \Gamma_X(TX) \} = \Gamma_M(\wedge^2 M)$. Then from (5) we see that $\Gamma_M(E^1_M) = \Gamma_M(\wedge^2_M \oplus \wedge^2 M)$. Then from lemma 11 we have the result. \hfill \Box

Let $\gamma_{H^2} : H^2(X, \mathbb{R}) \to H^2(M, \mathbb{R})$ be the induced map from the pull back $i_M^*$. We denote by Image $\gamma_H^2$ the image of the map $\gamma_H^2$. As in (1) we have the map from the complex $\#_X$ to the complex $\#_M$. Then we have the map of cohomologies,

$$\gamma^1_{\#} : H^1(\#_X) \to H^1(\#_M).$$

Lemma 4-1-14. The image of the map $H^1(\#_X) \to H^1(\#_M)$ is the direct sum $H^n(M, \mathbb{R}) \oplus \text{Image } \gamma_{H^2}$, so that is, we have the commutative diagram:

$$
\begin{array}{ccc}
H^1(\#_X) & \longrightarrow & H^n(M, \mathbb{R}) \oplus \text{Image } \gamma_{H^2} \\
\gamma^1_{\#} & \downarrow & \downarrow \\
H^1(\#_M)
\end{array}
$$

Proof. Since $\rho_I(\Omega^0) = i \Omega^0$ and $\rho_I \omega^0 = \omega$, $(i \Omega^0, \omega)$ is an element of $\Gamma_X(E^1_X)$. Hence $i_M^* ((i \Omega^0)^{Im}, \omega) = (-vol_M, 0)$ is an element of $\Gamma_M(E^1_M)$. From proposition 3-2-8 in section 3, $P_{H^2} : H^1(\#_X) \to H^2(X, \mathbb{R})$ is surjective. Hence we have the result. \hfill \Box
We also denote by $\gamma_{H^1}$ the induced map from $i_*^*: H^1(X, \mathbb{R}) \to H^1(M, \mathbb{R})$. From (1) we have the long exact sequence:

$$
\begin{array}{cccccccc}
H^0(\#_X) & \rightarrow & H^0(\#_M) & \rightarrow & H^1(\#_{X,M}) & \rightarrow & H^1(\#_X) & \xrightarrow{\gamma^1_{\#}} & H^1(\#_M) \\
\| & & \| & & \| & & \| & & \\
H^1(X) & \xrightarrow{\gamma_{H^1}} & H^1(M) & & H^n(M) \oplus H^2(M)
\end{array}
$$

Hence we have an exact sequence:

$$(6) \quad 0 \rightarrow \text{Coker } \gamma_{H^1} \rightarrow H^1(\#_{X,M}) \rightarrow \text{Ker } \gamma^1_{\#} \rightarrow 0.$$

**Proposition 4-1-15.** The dimension of $H^1(\#_{X,M})$ is invariant under local deformations of $\Phi^0 \in \mathbb{M}_{KE}(X, M)$.

**Proof.** From the exact sequence (6),

$$\dim H^1(\#_{X,M}) = \dim \text{Coker } \gamma_{H^1} + \dim \text{Ker } \gamma^1_{\#}.$$

From lemma 14 ker $\gamma^1_{\#}$ is the ker of the map $H^1(\#_X) \to H^n(M) \oplus \text{Image } \gamma_{H^2}$, where Image $\gamma_{H^2}$ does not depend on $\Phi^0$. Hence $\dim \text{Ker } \gamma^1_{\#}$ is invariant since $H^1(\#_X)$ is invariant from proposition 1-5 in section one. Hence we see that $H^1(\#_{X,M})$ is also invariant since image $\gamma_{H^1}$ is topological. □

As in section 3, the complex $\#_X$ is a subcomplex of the following de Rham complex $dR_X$:

$$\cdots \rightarrow \Gamma_X(\wedge^{n-1}_c \oplus \wedge^1) \xrightarrow{d_X} \Gamma_X(\wedge^n_c \oplus \wedge^2) \xrightarrow{d_X} \Gamma_X(\wedge^{n+1}_c \oplus \wedge^3) \rightarrow \cdots.$$

We also see that the complex $\#_M$ is a subcomplex of de Rham complex $dR_M$:

$$\cdots \rightarrow \Gamma_M(\wedge^{n-1} \oplus \wedge^1) \xrightarrow{d_M} \Gamma_M(\wedge^n \oplus \wedge^2) \xrightarrow{d_M} \Gamma_M(\wedge^{n+1} \oplus \wedge^3) \rightarrow \cdots.$$
Hence we see that the complex $\#_{X,M}$ is a subcomplex of the following relative de Rham complex $dR_{X,M}$:

$$
\cdots \to \Gamma_{X,M}(\wedge^n_{\mathbb{C}} \oplus \wedge^1) \xrightarrow{d} \Gamma_{X,M}(\wedge^n_{\mathbb{C}} \oplus \wedge^2) \xrightarrow{d} \Gamma_{X,M}(\wedge^n_{\mathbb{C}} \oplus \wedge^3) \to \cdots,
$$

where $\Gamma_{X,M}(\wedge^p_{\mathbb{C}} \oplus \wedge^q)$ is given as

$$\Gamma_{X,M}(\wedge^p_{\mathbb{C}} \oplus \wedge^q) = \{ (a,b) \in \Gamma_X(\wedge^p_{\mathbb{C}} \oplus \wedge^q) \mid i^*_M a = 0, i^*_M b = 0 \}.$$

Then we have the map between short exact sequences of complexes:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \#_{X,M} & \longrightarrow & \#_X & \longrightarrow & \#_M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & dR_{X,M} & \longrightarrow & dR_X & \longrightarrow & dR_M & \longrightarrow & 0
\end{array}
$$

Hence we have the map between long exact sequences:

$$
\begin{array}{cccccc}
H^0(\#_X) \xrightarrow{\gamma^0} H^0(\#_M) & \longrightarrow & H^1(\#_{X,M}) & \longrightarrow & H^1(\#_X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow
\end{array}
\begin{array}{cccccc}
H^0(dR_X) \xrightarrow{\gamma_{dR}} H^0(dR_M) & \longrightarrow & H^1(dR_{X,M}) & \longrightarrow & H^1(dR_X)
\end{array}
$$

**Proposition 4-1-16.** The map $p_{X,M} : H^1(\#_{X,M}) \to H^1(dR_{X,M})$ is injective.

**Proof.** We consider the induced map between quotient vector spaces in the diagram:

$$
(7) \quad H^0(\#_M)/\gamma^0_# H^0(\#_X) \to H^0(dR_M)/\gamma_{dR}H^0(dR_X).
$$

From lemma 4-1-12 we see the map (7) is injective. From proposition 3-2-3 of section 3, $p_X : H^1(\#_X) \to H^1(dR_X)$ is also injective. Hence we have the result. $\square$
§4-2. We shall construct a slice on $\mathfrak{M}(X, M)$ for the action of $\text{Diff}_0(X)$. At first we consider the slice $S_{\Phi^0} = S_{\Phi^0}(X)$ in section 2 and define the map $\lambda$:

$$
\lambda: S_{\Phi^0}(X) \longrightarrow H^n(M, \mathbb{R}) \oplus H^2(M, \mathbb{R}),
$$

$$
\Phi = (\phi, \psi) \mapsto [i_M^* \psi] dR,
$$

where $\Phi \in S_{\Phi^0}$.

**Lemma 4-2-1.** We define $\hat{S}_{\Phi^0}(X)$ by

$$
\hat{S}_{\Phi^0}(X) = \{ \Phi \in S_{\Phi^0}(X) | \lambda(\Phi) = 0 \}.
$$

Then a neighborhood of $\hat{S}_{\Phi^0}(X)$ at $\Phi^0$ is homeomorphic to an open set of $\text{Ker } \gamma_1^\#$ in $H^1(\# X)$, so that is, $\hat{S}_{\Phi^0}(X)$ is a manifold around $\Phi^0$ with,

$$
T_{\Phi^0} \hat{S}_{\Phi^0}(X) = \text{Ker } \gamma_1^\#.
$$

**Proof.** From lemma 4-1-13 the image of the map $\gamma_1^\#: H^1(\# X) \to H^n(M) \oplus H^2(M)$ is given by $H^n(M) \oplus \text{Image } \gamma_{H^2}$. Hence $\text{Image } \gamma_1^\#$ does not depend on $\Phi^0 \in \mathfrak{M}_{KE}(X, M)$. Let $\Phi$ be an element of the slice $S_{\Phi^0}(X)$.

Then we have the map from the tangent space $T_{\Phi} S_{\Phi^0}(X)$ to $H^n(M) \oplus H^2(M)$ by taking cohomology classes. Since $\Phi$ is a topological calibration, the image of this map coincides $H^n(M) \oplus \text{Image } \gamma_{H^2}$. (Note that the difference between $T_{\Phi} S_{\Phi}$ and $T_{\Phi} S_{\Phi^0}$ is given by exact forms.) Hence we see that the image $\lambda(S_{\Phi^0})$ is $H^n(M) \oplus \text{Image } \gamma_{H^2}$. By using implicit function theorem we see that the inverse image $\lambda^{-1}(0)$ is a manifold around $\Phi^0$. □

As in section 2, we define the Sobolev space $L^2_s(M, \wedge^p)$, for $s > k + \frac{2n}{2}$, where $s$ is sufficiently large. Then we denote by $L^2_s(E^i_M) = L^2_s(M, E^i_M)$ the Sobolev space $L^2_s(M, \wedge^{n-1+i} \oplus \wedge^{i+1}) \cap C^k(M, E^i_M)$ for $i = 0, 1, 2$. Since the complex $\#^i_M$ is an elliptic complex, we have the Hodge decomposition,

$$
L^2_s(M, E^0_M) = H^0(\# M) \oplus d_{M, 0}^* L^2_{s+1}(M, E^1_M),
$$
where $\mathbb{H}^1(#_M)$ is the Ker $d_{M,0}$. We have the commutative diagram:

$$
\begin{array}{lcr}
L^2_s(X, E^0_X) & \xrightarrow{d_{X,0}} & L^2_{s-1}(X, E^1_X) \\
\kappa^0 & & \kappa^1 \\
L^2_s(M, E^0_M) & \xrightarrow{d_{M,0}} & L^2_{s-1}(X, E^1_M)
\end{array}
$$

Then we denote by $\gamma^0 = \gamma^0_#$ the induced map on the cohomologies from $\kappa^0$,

$$
\gamma^0 : H^0(#_X) \longrightarrow H^0(#_M).
$$

Let $\gamma^0(Ker d_{X,0}) = \gamma^0_0(\mathbb{H}^0(#_X))$ be the image of $\mathbb{H}^0(#_X)$ in $L^2_s(E^0_M)$. We denote by $(\text{Image } \gamma^0)^\perp$ the orthogonal complement of the image $\gamma^0_0(\mathbb{H}^0(#_X))$ in $L^2_s(M, E^0_M)$. Since $\text{Image } \gamma^0_0$ is the subspace of $\mathbb{H}^0(#_M)$, we have the decomposition:

$$
L^2_s(M, E^0_M) = \text{Image } \gamma^0 \oplus (\text{Image } \gamma^0)^\perp = \text{Image } \gamma^0 \oplus \mathbb{H}^0(#_X, #_M) \oplus d^*_{M,0} L^2_{s+1}(M, E^1_M),
$$

where $H^0(#_X, #_M)$ denotes the orthogonal complement of Image $\gamma^0$ in $H^0(#_M)$. We recall the identification $\Gamma(E^0_M) \cong \Gamma(N_M)$ in lemma 8 and fix a splitting $\Gamma_M(N_M) \to \Gamma_X(TX)$ of the exact sequence

$$
0 \longrightarrow \Gamma_{X,M}(TX) \longrightarrow \Gamma_X(TX) \longrightarrow \Gamma_M(N_M) \longrightarrow 0.
$$

Under the identification $\Gamma(E^0_M) \cong \Gamma(N_M)$, $v \in L^2_s(E^0_M) \cong L^2_s(N_M)$ is regarded with an element of $L^2_s(TX)$ by the splitting map. Let $\text{exp}$ be the exponential map with respect to the metric $g_{\Phi^0}$. We define the map $F_{X,M}$ by

$$
F_{X,M} : (\text{Image } \gamma^0)^\perp \times \hat{S}_{\Phi^0} \longrightarrow dL^2_s(M, E^0_M),
$$

$$( v , \Phi ) \longrightarrow i^*_M \text{exp}_v \psi,
$$

where $\Phi = (\phi, \psi) \in \hat{S}_{\Phi^0}$ and $\text{exp}_v^* \psi = (\text{exp}_v^*(\Omega)^M, \text{exp}_v^* \omega)$ is the pull back of $\psi$ by $\text{exp}_v \in \text{Diff}_0(X)$. From Lemma 4-2-1 $i^*_M \text{exp}_v \psi$ is an exact form and we see that the image of the map $F_{X,M}$ is $dL^2_s(M, E^0_M)$ from lemma 4-1-1.
Proposition 4-2-2. The inverse image $F_{X,M}^{-1}(0)$ is a manifold around $(0, \Phi^0)$.

Proof. The differential of $F_{X,M}$ at $(0, \Phi^0)$ is given by

$$dF_{X,M}(\dot{v}, \dot{\Phi}) \mapsto i^*_M d\dot{v} \psi^0 + i^*_M \dot{\psi},$$

where $(\dot{v}, \dot{\Phi}) \in (\text{Image } \gamma^0)^\perp \oplus T_{\Phi^0} \hat{\mathcal{S}}_{\Phi^0}$ and $\dot{\Phi} = (\dot{\phi}, \dot{\psi})$. Hence we see that $dF_{X,M}$ is surjective. From the implicit function theorem, we have the result. □

Definition 4-2-3. Let $\exp$ be the exponential map corresponding to the metric $g_{\Phi^0}$. We define the slice $S_{\Phi^0}(X, M)$ by

$$S_{\Phi^0}(X, M) = \{ \exp^*_v \Phi \in \tilde{\mathcal{M}}(X) \cap U \mid (v, \Phi) \in F_{X,M}^{-1}(0) \},$$

where $U$ is a sufficiently small neighborhood of $\tilde{\mathcal{M}}(X)$ at $\Phi^0$.

Then we have

Proposition 4-2-4. The slice $S_{\Phi^0}(X, M)$ is a submanifold of $\tilde{\mathcal{M}}(X, M)$.

Proof. From elliptic regularity we see that $(v, \Phi) \in F_{X,M}^{-1}(0)$ is smooth. From definition, $i^*_M \exp^*_v \Phi = 0$. Hence $S_{\Phi^0}(X, M)$ is a subset of $\tilde{\mathcal{M}}(X, M)$. As in section 2, $U \cong V \times S_{\Phi^0}$, where $V$ is a neighborhood of $\text{Diff}_0(X)$ at the identity. Hence $S_{\Phi^0}(X, M)$ is homeomorphic to a neighborhood of $F_{X,M}^{-1}(0)$. From Proposition 4-2-2 we have the result. □

We recall the exact sequence (1):

$$0 \longrightarrow \#_{X,M} \longrightarrow \#_X \xrightarrow{\kappa} \#_M \longrightarrow 0.$$

From the map $\kappa: \#_X \to \#_M$, we define the following relative complex:

$$C^p(\#_X \to \#_M) = \Gamma_X(E^p_X) \oplus \Gamma_M(E^{p-1}_M),$$

where the coboundary map $\delta_{X,M}$ is

$$\delta_{X,M}(a^p_x, b^{p-1}_M) = (da^p_x, (-1)^{p-1}\kappa(a) + db^{p-1}_M),$$

where $(a^p_x, b^{p-1}_M) \in \Gamma_X(E^p_X) \oplus \Gamma_M(E^{p-1}_M)$. (Note that $C^0(\#_X \to \#_M) = \Gamma_X(E^0)$. ) Then we have a cohomology group $H^p(\#_X \to \#_M)$ of the complex $(C^p(\#_X \to \#_M), \delta_{X,M})$. 
Lemma 4-2-5. Let $H^p(\#_{X,M})$ be the cohomology group of the complex $\#_{X,M}$. Then we have

$$H^p(\#_{X,M}) \cong H^p(\#_X \to \#_M).$$

Proof. Let $a^p_X$ be an element of $\Gamma_{X,M}(E^p_X)$. Then we define the map from $\Gamma_{X,M}(E^p_X)$ to $C^p(\#_X \to \#_M)$ by

$$a^p_X \mapsto (a^p_X, 0).$$

We see that this map is quasi isomorphism between complexes. □

Proposition 4-2-6. Let $S_{\Phi^0}(X, M)$ be the slice as in definition 4-2-3. Then the tangent space of the slice is given by the relative cohomology group:

$$T_{\Phi^0}S_{\Phi^0}(X, M) = H^1(\#_{X,M}).$$

Proof. From Proposition 4-2-4 the tangent space $T_{\Phi^0}S_{\Phi^0}(X, M)$ is ker $dF_{X,M}$. From Proposition 4-2-2 Ker $dF_{X,M}$ is given as

$$\text{Ker } dF_{X,M} = \left\{ (\dot{\Phi}, i_{\dot{\psi}}\psi^0) \in (\text{Image } \gamma^0)^\perp \oplus T_{\Phi^0}\tilde{S}_{\Phi^0} \mid i_M^*d i_{\dot{\psi}}\psi^0 + i_M^*\dot{\psi} = 0 \right\}.$$

Then $(\dot{\Phi}, i_{\dot{\psi}}\psi^0) \in \text{Ker } dF_{X,M}$ is a representative of $H^1(\#_X \to \#_M)$. Hence we have the map Ker $dF_{X,M} \to H^1(\#_X \to \#_M)$. From Hodge decomposition, we see that this map is bijective. □

Proposition 4-2-7. Let $\pi_{X,M}: \tilde{\mathcal{M}}(X, M) \to \mathcal{M}(X, M)$ be the natural projection. We restrict $\pi_{X,M}$ to the slice $S_{\Phi^0}(X, M)$. Then the restricted map is injective and its image is an open set of $\mathcal{M}(X, M)$.

Proof. Let $\Phi$ be an element of the slice $S_{\Phi^0}(X, M)$. Then $\Phi$ is a closed form of $\Gamma_{X,M}(E^1_X)$. Let $P$ be the map from the slice $S_{\Phi^0}(X, M)$ to the relative de Rham cohomology group $H^1(dR_{X,M})$:

$$P: S_{\Phi^0}(X, M) \to H^1(dR_{X,M}),$$
\[ \Phi \mapsto [\Phi]_{dR_{X,M}}. \]

From proposition 4-2-6, the differential of the map \( P \) at \( \Phi^0 \) is an isomorphism

\[ T_{\Phi^0}S_{\Phi^0}(X, M) \cong H^1(#_{X,M}). \]

From proposition 4-1-16 \( H^1(#_{X,M}) \rightarrow H^1(dR_{X,M}) \) is injective. Hence the map \( P \) is injective if we take \( S_{\Phi^0}(X, M) \) sufficiently small. A class of \( H^1(dR_{X,M}) \) is invariant under the action of \( \text{Diff}_0(X, M) \). Hence the restricted map \( \pi_{X,M}|_{S_{\Phi^0}(X, M)} \) is injective. Let \( U \) be a small open set of \( \tilde{\mathcal{M}}(X) \) at \( \Phi^0 \). Then as in section 2, \( U \cong V \times S_{\Phi^0} \), where \( V \) is a neighborhood of \( \text{Diff}_0(X) \) at the identity. Let \( \Phi \) be an element of \( \tilde{\mathcal{M}}(X, M) \cap U \). Then \( \Phi \) is written as \( \Phi = f^* \hat{\Phi} \), where \( f \in V, \hat{\Phi} \in S_{\Phi^0} \). Since \( i^*_M \Phi = 0 \), we have \([i^*_M \hat{\Phi}]_{dR} = 0\). Hence \( \hat{\Phi} \in \hat{S}_{\Phi^0} \). The open set \( V \) of \( \text{Diff}_0(X) \) at the identity is identified with an open set of \( \Gamma_X(TX) \) at zero section. By using a splitting, we have the decomposition

\[ \Gamma_X(TX) = \Gamma_{X,M}(TX) \oplus \Gamma(N_M). \]

Let \( u \) be an element of \( \gamma^0(H^0(#_X)) \). Then we see \( di_u \Phi^0 = 0 \). Since \( U \) is sufficiently small, \( \Phi \) is written as

\[ \Phi = f_1^* \exp_u^* \hat{\Phi}, \]

where \( f_1 \in \text{Diff}_0(X, M) \) and \( u \in (\text{Image } \gamma)^\perp \subset H^0(#_M) \). Hence \( \text{Image } \pi_{X,M}(S_{\Phi^0}(X, M)) = \pi(U) \) is an open set.

We define the map \( P_{X,M} \) by

\[ P_{X,M} : \mathcal{M}(X, M) \rightarrow H^1(dR_{X,M}), \]

\[ \Phi \mapsto [\Phi]_{dR}. \]
Theorem 4-2-8. The map \( P \) is locally injective.

Proof. This follows from proposition 4-1-16. □

Proof of theorem 4-1-5. Let \( \tilde{M}_{KE}(X,M) \) be as in definition 1. Since \( \tilde{M}_{KE}(X,M) \) is a submanifold of \( \tilde{M}_{KE}(X) \), we have the distance \( d \) on \( \tilde{M}_{KE}(X,M) \) as in proposition 2-9. Then from theorem 4-2-8 we see that \( d \) gives a distance on the moduli space \( M_{KE}(X,M) \) as in a proof of proposition 2-9. Hence \( M_{KE}(X,M) \) is Hausdorff. Let \( f \) be an element of \( \text{Diff}_0(X,M) \). Then \( f \) gives a diffeomorphism from the slice \( S_{\Phi^0}(X,M) \) to \( S_{f^*\Phi^0}(X,M) \). Hence from proposition 4-2-7, we see that each slice is local coordinates of \( M(X,M) \). From Proposition 4-2-4 and 4-2-6, we have the result. □

We define \( \hat{M}(X,M) \) by

\[
\hat{M}(X,M) = \left\{ \Phi \in \mathcal{E}^1_{KE}(X) \mid d\Phi = 0, \ [i^*_M \psi] d\gamma = 0 \right\} / \text{Diff}_0(X).
\]

Then we have the natural map

\[
M(X,M) \longrightarrow \hat{M}(X,M),
\]

\[ [\Phi]_{\text{Diff}_0(X,M)} \mapsto [\Phi]_{\text{Diff}_0(X)}, \]

where \([\Phi]_{\text{Diff}_0(X,M)} \) ( resp. \([\Phi]_{\text{Diff}_0(X)} \) ) denotes a class of \( M(X,M) \) (resp. \( \hat{M}(X,M) \) ). We denote by \( \hat{M}_0(X,M) \) the image of this map.

proposition 4-2-9. \( \hat{M}_0(X,M) \) is a smooth manifold and local coordinates are given by an open ball of \( \text{Ker} \gamma^1_{\#} \subset H^1(\#X) \)

Proof. Local coordinates of \( \hat{M}_0(X,M) \) is given by \( \hat{S}_{\Phi^0} \) as in Lemma 4-2-1.

Theorem 4-2-10. \( M(X,M) \longrightarrow \hat{M}_0(X,M) \) is a fibre bundle and local coordinates of each fibre is given by an open ball of \( \text{Coker} \gamma_{H^1} \).

Proof. Let \( \Phi^0 \) be an element of \( \hat{M}(X,M) \). Then the fibre through \( \Phi^0 \) is written as the quotient,

\[
\left\{ f^*\Phi^0 \mid i^*_M f^*\psi^0 = 0, f \in \text{Diff}_0(X) \right\} / \left\{ g^*\Phi^0 \mid g \in \text{Diff}_0(X,M) \right\}.
\]
An neighborhood of $\text{Diff}_0(X)/\text{Diff}_0(X, M)$ is given by an open set of $\Gamma_M(N_M)$. Under the identification $\Gamma_M(N_M) \cong \Gamma_M(E^0_M)$, we see that each fibre is parameterized by an open ball of $\text{coker } \gamma_{H^1} \subset H^0(#_M)$. □

Each fibre can be regarded as the moduli space of special lagrangian submanifolds with respect to a fixed Kähler-Einstein structure $\Phi$.

**Theorem 4-2-11.** Let $\tilde{\mathcal{M}}_{KE}(X, M)$ be as in definition 4-1-1. We denote by $\text{Diff}(X, M)$ the group of diffeomorphisms of $X$ preserving $M$. There is the action of $\text{Diff}(X, M)$ on $\tilde{\mathcal{M}}_{KE}(X, M)$. Then the quotient $\tilde{\mathcal{M}}_{KE}(X, M)/\text{Diff}(X, M)$ is an orbifold.

*Proof.* The slice $S_{\Phi^0}(X, M)$ is local coordinates of $\mathcal{M}_{KE}(X, M)$ and invariant under the action of $\text{Diff}(X, M)$. Hence we see that the moduli space $\tilde{\mathcal{M}}_{KE}(X, M)/\text{Diff}(X, M)$ is locally homeomorphic to the quotient space $S_{\Phi^0}(X, M)/I_{X,M}$, where $I_{X,M}$ is the isotropy,

$$I_{X,M} = \{ f \in \text{Diff}(X, M) | f^*\Phi^0 = \Phi^0 \}.$$

As in proof of proposition 2-9, $\text{Diff}(X, M)$ acts on $S_{\Phi^0}(X, M)$ isometrically. Hence we see that there is an open set $V$ of $T_{\Phi^0}S_{\Phi^0}(X, M)$ with the action of $I_{X,M}$ such that the quotient $V/I_{X,M}$ is homeomorphic to $S_{\Phi^0}(X, M)/I_{X,M}$. Since $T_{\Phi^0}S_{\Phi^0}(X, M)$ is isomorphic to $H^1(#_{X,M})$ and the action of $I_{X,M}$ on $H^1(#_{X,M})$ is a isometry with respect to the metric $g_{\Phi^0}$. The action of $I_{X,M}$ preserves integral cohomology class. Hence from proposition 4-1-16 we see that $V/I_{X,M}$ is the quotient by a finite group. □

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