Research Article

Topological Structure of Vague Soft Sets

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We introduce vague soft topological spaces which are defined over an initial universe with a fixed set of parameters. The notions of vague soft open sets, vague soft closed sets, vague soft interior, vague soft closure, and vague soft boundary are introduced and their basic properties and relations are investigated. Furthermore, with the help of examples they established that some properties of topological spaces and soft topological spaces do not hold in vague soft topological spaces. Vague soft connectedness and vague soft compactness are also studied.

1. Introduction

Researchers in economics, environmental science, social science, medical science, business, and many other fields deal daily with the complexities of modeling uncertain data. Classical methods are not always successful because the uncertainties appearing in these domains may be of various types. Fuzzy set theory [1], intuitionistic fuzzy set theory [2], vague set theory [3], interval mathematics [4], and other mathematical tools are well-known and often useful approaches to describing uncertainty. However, all of these theories have their own difficulties which have been pointed out in [5]. Molodtsov suggested that one reason for these difficulties may be due to the inadequacy of the parameterization tools of these theories. To overcome these difficulties, Molodtsov [5] introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Since then, many researches have investigated soft sets and have established some significant conclusions. For example, Jun and Park [6] proposed the notion of soft ideals and idealistic soft BCK/BCI-algebras and constructed several examples. Majumdar and Samanta [7] further generalized the concept of fuzzy soft sets and some of their properties were studied, and relations on generalized fuzzy soft sets were also discussed by them. Maji et al. [8] introduced the concept of fuzzy soft sets by combining fuzzy sets and soft sets. By combining the vague set and the soft set, Xu et al. [9] introduced the notion of vague soft sets, derived its basic properties, and illustrated its potential applications. Wang and Qu [10] introduced the definitions of entropy, similarity measure, and distance measure of vague soft sets, and the relations between these measures are discussed in detail. About soft topology, Shabir and Naz [11] defined several basic notions on soft topology and studied many properties. Hussain and Ahmad [12] continued investigating the properties of soft topological spaces and strengthened the foundations of the theory of soft topological spaces. Tanay and Kandemir [13] introduced the concept of fuzzy soft topology and some of its structural properties are studied.

By definition, a soft set is a parameterized family of subsets of the universal set. In other words, a soft set is a mapping from a set of parameters to the power set of an initial universe set. In the real world, the difficulty is that the objects in the universal set may not precisely satisfy the problem’s parameters, which usually represent some attributes, characteristics, or properties of the objects in the universal set. The concept of fuzzy soft sets proposed in [8] partially resolves this difficulty but falls short in dealing with additional complexity; that is, the mapping may be too vague. It is, therefore, desirable to extend soft set theory and fuzzy soft set theory using the concept of vague set theory. Vague set
theory is actually an extension of fuzzy set theory and vague sets are regarded as a special case of context-dependent fuzzy sets. The basic concepts of vague set theory and its extensions, as well as some interesting applications, can be found in [14–16]. Vague soft set theory makes descriptions of the object world more realistic, practical, and accurate, at least in some cases, making it a very promising tool. Since vague sets are equivalent to intuitionistic fuzzy sets [17], so vague soft sets are equivalent to intuitionistic fuzzy soft sets. Some scholars have studied intuitionistic fuzzy soft sets from different aspects. For example, Gunduz and Bayramov [18] introduced the concept of an intuitionistic fuzzy soft module and some operations on intuitionistic fuzzy soft sets were given; they also studied some of its basic properties. Jiang et al. [19] proposed the notion of the interval-valued intuitionistic fuzzy soft set, the complement, and/or union, intersection, and necessity, and possibility operations were defined on intuitionistic fuzzy soft sets. The basic properties of interval-valued intuitionistic fuzzy soft sets were given; they also studied some of its basic properties. Jiang et al. [19] proposed the notion of the interval-valued intuitionistic fuzzy soft set, the complement, and/or union, intersection, and necessity, and possibility operations were defined on intuitionistic fuzzy soft sets. The basic properties of interval-valued intuitionistic fuzzy soft sets were given; they also studied some of its basic properties.

In this section, we will recall several definitions and results which are necessary for our paper. They are stated as follows.

**Definition 1** (see [3]). A vague set $A$ in the universe $U = \{x_1, x_2, \ldots, x_n\}$ can be expressed by the following notion: $A = \{(x_i, [f_A(x_i), 1 - f_A(x_i)]) \mid x_i \in U\}$; that is, $A(x_i) = [f_A(x_i), 1 - f_A(x_i)]$, and the condition $0 \leq f_A(x_i) \leq 1 - f_A(x_i)$ should hold for any $x_i \in U$, where $f_A(x_i)$ is called the membership degree (true membership) of element $x_i$ to the vague set $A$, while $f_A(x_i)$ is the degree of nonmembership (false membership) of the element $x_i$ to the vague set $A$.

**Definition 2** (see [3]). Let $A, B$ be two vague sets in the universe $U = \{x_1, x_2, \ldots, x_n\}$; then the union, intersection, and complement of vague sets are defined as follows:

$$A \cup B = \{(x_i, [f_A(x_i) \lor f_B(x_i)], (1 - f_A(x_i)) \lor (1 - f_B(x_i))) \mid x_i \in U\},$$

$$A \cap B = \{(x_i, [f_A(x_i) \land f_B(x_i)], (1 - f_A(x_i)) \land (1 - f_B(x_i))) \mid x_i \in U\},$$

$$A^c = \{(x_i, [f_A(x_i), 1 - f_A(x_i)]) \mid x_i \in U\}.$$

**Definition 3** (see [3]). Let $A, B$ be two vague sets in the universe $U = \{x_1, x_2, \ldots, x_n\}$. If $\forall x_i \in U, f_A(x_i) \leq f_B(x_i)$, then $A$ is called a vague subset of $B$, denoted by $A \subseteq B$, where $1 \leq i \leq n$.

**Definition 4** (see [5]). Let $U$ be an initial universe set, $P(U)$ the power set of $U$, $E$ a set of parameters, and $A \subseteq E$. A pair $(F, A)$ is called a soft set over $U$, where $F$ is a mapping given by $F : A \rightarrow P(U)$.

**Definition 5** (see [9]). Let $U$ be an initial universe set, $V(U)$ the set of all vague sets on $U$, $E$ a set of parameters, and $A \subseteq E$. A pair $(F, A)$ is called a vague soft set over $U$, where $F$ is a mapping given by $F : A \rightarrow V(U)$.

**Definition 6** (see [9]). Let $(F, A)$ and $(G, B)$ be two vague soft sets over a universe $U$. If $A \subseteq B$ and for all $e \in A, F(e)$ is a vague subset of $G(e)$, then $(F, A)$ is called a vague soft subset of $(G, B)$. This relation is denoted by $(F, A) \subseteq (G, B)$.

**Definition 7** (see [9]). Two vague soft sets $(F, A)$ and $(G, B)$ over a universe $U$ are said to be vague soft equal if $(F, A) = (G, B)$.

**Definition 8** (see [9]). Let $E = \{e_1, e_2, \ldots, e_n\}$ be a parameter set. The not set of $E$ denoted by $\neg E$ is defined by $\neg E = \{\neg e_1, \neg e_2, \ldots, \neg e_n\}$, where $\neg e_i = not e_i$.

**Definition 9** (see [9]). The complement of vague soft set $(F, A)$ is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, \neg A)$, where $F^c : \neg A \rightarrow V(U)$ is a mapping given by $F_{\neg A}(\alpha) = f_{\neg A}(\alpha), 1 - f_{\neg A}(\alpha) = 1 - t_{\neg A}(\alpha), \forall \alpha \in \neg A, x \in U$.

Clearly $(F^c)^c$ is the same as $F$ and $((F, A)^c)^c = (F, A)$.

**Definition 10** (see [9]). A vague soft set $(F, A)$ over $U$ is said to be a null vague soft set denoted by $\emptyset$, if $\forall e \in A, t_{\neg A}(\alpha) = 0, 1 - f_{\neg A}(\alpha) = 0, x \in U$.

**Definition 11** (see [9]). A vague soft set $(F, A)$ over $U$ is said to be an absolute vague soft set denoted by $\overline{A}$, if $\forall e \in E, t_{\neg A}(x) = 1, 1 - f_{\neg A}(x) = 1, x \in U$.
Definition 12 (see [9]). The union of two vague soft sets \((F, A)\) and \((G, B)\) over a universe \(U\) is a vague soft set \((H, C)\), where \(C = A \cup B\) and \(\forall e \in C\),

\[
t_{H(e)}(x) = \begin{cases} 
t_{F(e)}(x) & e \in A - B, x \in U, \\
t_{G(e)}(x) & e \in B - A, x \in U, \\
t_{F(e)}(x) \lor t_{G(e)}(x) & e \in A \cap B, x \in U,
\end{cases}
\]

\[1 - f_{H(e)}(x) = \begin{cases} 
1 - f_{F(e)}(x) & e \in A - B, x \in U, \\
1 - f_{G(e)}(x) & e \in B - A, x \in U, \\
(1 - f_{F(e)}(x)) \lor (1 - f_{G(e)}(x)) & e \in A \cap B, x \in U.
\end{cases}
\]  

We denote it by \((F, A) \cup (G, B) = (H, C)\).

Definition 13 (see [9]). The intersection of two vague soft sets \((F, A)\) and \((G, B)\) over a universe \(U\) is a vague soft set \((H, C)\), where \(C = A \cup B\) and \(\forall e \in C\),

\[
t_{H(e)}(x) = \begin{cases} 
t_{F(e)}(x) & e \in A - B, x \in U, \\
t_{G(e)}(x) & e \in B - A, x \in U, \\
t_{F(e)}(x) \land t_{G(e)}(x) & e \in A \cup B, x \in U,
\end{cases}
\]

\[1 - f_{H(e)}(x) = \begin{cases} 
1 - f_{F(e)}(x) & e \in A - B, x \in U, \\
1 - f_{G(e)}(x) & e \in B - A, x \in U, \\
(1 - f_{F(e)}(x)) \land (1 - f_{G(e)}(x)) & e \in A \cup B, x \in U.
\end{cases}
\]  

We denote it by \((F, A) \cap (G, B) = (H, C)\).

Proposition 14. \((F, A) \cup (G, B)\) and \((F, G)\) over a universe \(U\) is a vague soft sets in \((H, C)\), where \(C = A \cup B\) and \(\forall e \in C\),

\[
t_{H(e)}(x) = \begin{cases} 
t_{F(e)}(x) & e \in A - B, x \in U, \\
t_{G(e)}(x) & e \in B - A, x \in U, \\
t_{F(e)}(x) \lor t_{G(e)}(x) & e \in A \cup B, x \in U,
\end{cases}
\]

\[1 - f_{H(e)}(x) = \begin{cases} 
1 - f_{F(e)}(x) & e \in A - B, x \in U, \\
1 - f_{G(e)}(x) & e \in B - A, x \in U, \\
(1 - f_{F(e)}(x)) \lor (1 - f_{G(e)}(x)) & e \in A \cup B, x \in U.
\end{cases}
\]  

3. Vague Soft Topological Spaces

Let \(X\) be an initial universe set, and let \(E\) be the nonempty set of parameters.

Definition 15. Let \(\tau\) be the collection of vague soft sets over \(X\); then \(\tau\) is said to be a vague soft topology on \(X\) if

1. \(\emptyset, X\) belong to \(\tau\),
2. the union of any number of vague soft sets in \(\tau\) belongs to \(\tau\),
3. the intersection of any two vague soft sets in \(\tau\) belongs to \(\tau\).

The triplet \((X, \tau, E)\) is called a vague soft topological space over \(X\).

Definition 16. Let \((X, \tau, E)\) be a vague soft topological space over \(X\); then the members of \(\tau\) are said to be vague soft open sets in \(X\).

Definition 17. Let \((X, \tau, E)\) be a vague soft topological space over \(X\). A vague soft set \((F, E)\) over \(X\) is said to be a vague soft closed set in \(X\), if its complement \((F, E)^c\) belongs to \(\tau\).

Proposition 18. Let \(\{\tau_k \mid k \in K\}\) be a family of vague soft topologies on \(X\); then \(\bigcap_{k \in K} \tau_k\) is a vague soft topology on \(X\).

Proof. (1) \(\emptyset, X\) belong to \(\bigcap_{k \in K} \tau_k\).

(2) Let \(\{F_i, E\} \mid i \in I\} \subseteq \bigcap_{k \in K} \tau_k\). Then for all \(i \in I\), \((F_i, E) \in \tau_k\), so \(\bigcup_{i \in I} F_i, E \in \tau_k\). Thus \(\bigcup_{k \in K} (F_i, E) \in \bigcap_{k \in K} \tau_k\).

(3) Let \((F, E), (G, E) \in \bigcap_{k \in K} \tau_k\). Then for all \(k \in K\), \((F, E), (G, E) \in \tau_k\). Since \(((F, E) \cap (G, E)) \in \tau_k\), so \(((F, E) \cap (G, E)) \in \bigcap_{k \in K} \tau_k\).

Definition 19. Let \((X, \tau_1, E)\) and \((X, \tau_2, E)\) be two vague soft topological spaces over the same universe \(X\). If each vague soft set \((F, E) \in \tau_1\) is \(\tau_2\), then \(\tau_2\) is called vague soft finer than \(\tau_1\), or \(\tau_1\) is vague soft coarser than \(\tau_2\).

Proposition 20. Let \(\{\tau_k \mid k \in K\}\) be a family of all vague soft topologies on \(X\); then \(\bigcap_{k \in K} \tau_k\) is the coarsest vague soft topology on \(X\).

Proof. \(\bigcap_{k \in K} \tau_k\) is a vague soft topology on \(X\) by Proposition 18. If each vague soft set \((F, E) \in \bigcap_{k \in K} \tau_k\), then \((F, E) \in \tau_k\) for all \(k \in K\). Hence \(\bigcap_{k \in K} \tau_k\) is the coarsest vague soft topology on \(X\).

Remark 21. Let \((X, \tau_1, E)\) and \((X, \tau_2, E)\) be two vague soft topological spaces over the same universe \(X\); then \((X, \tau_1 \cup \tau_2, E)\) may not be a vague soft topological space over \(X\).

Example 22. Let \(X = \{a, b, c\}\), \(E = \{e_1, e_2\}\) and \(\tau_1 = \{\emptyset, X, (F, E)\}\), \(\tau_2 = \{\emptyset, X, (G, E)\}\) be two vague soft topologies defined on \(X\) where \((F, E), (G, E)\) are vague soft sets over \(X\), defined as follows:

\[
F(e_1) = \{ [0.1, 0.8], [0.3, 0.9], [0.7, 0.8] \}, \\
F(e_2) = \{ [0.2, 0.5], [0.5, 0.6], [0.8, 1] \}, \\
G(e_1) = \{ [0.9], [0.4, 1], [0.0] \}, \\
G(e_2) = \{ [0.3, 0.6], [0.4, 0.7], [0.6, 0.8] \}.
\]  

Now, \(\tau_1 \cup \tau_2 = \{\emptyset, X, (F, E), (G, E)\}\).

If we take \((F, E) \cup (G, E) = (H, E), (F, E) \cap (G, E) = (P, E)\), then

\[
H(e_1) = \{ [0.1, 0.9], [0.4, 1], [0.7, 0.8] \}, \\
H(e_2) = \{ [0.3, 0.6], [0.5, 0.7], [0.8, 1] \}.
\]
\[ P(e_1) = \left\{ \frac{[0,0.8]}{a}, \frac{[0.3,0.9]}{b}, \frac{[0,0]}{c} \right\}, \]
\[ P(e_2) = \left\{ \frac{[0.2,0.5]}{a}, \frac{[0.4,0.6]}{b}, \frac{[0.6,0.8]}{c} \right\}. \]

\[ (H, E) \notin \tau_1 \cup \tau_2, (P, E) \notin \tau_1 \cup \tau_2, \text{ so } (X, \tau_1 \cup \tau_2, E) \text{ is not a vague soft topological space on } X. \]

**Definition 23.** Let \((X, \tau, E)\) be a vague soft topological space, and let \((F, E)\) be a vague soft set over \(X\). Then vague soft interior of \((F, E)\), denoted by \((F, E)^\circ\), is defined as the union of all vague soft open sets contained in \((F, E)\).

Clearly, \((F, E)^\circ\) is the largest vague soft open set contained in \((F, E)\).

**Theorem 24.** Let \((X, \tau, E)\) be a vague soft topological space over \(X\), and let \((F, E)\) and \((G, E)\) be two vague soft sets over \(X\). Then the following properties hold:

1. \(\hat{\emptyset} = \emptyset\) and \(\bar{X}^a = \bar{X}\);
2. \((F, E)^\circ \subseteq (F, E)\);
3. \((F, E)\) is a vague soft open set if and only if \((F, E)^\circ = (F, E)\);
4. \([(F, E)^\circ]^\circ = (F, E)^\circ\);
5. \((F, E) \subseteq (G, E)\) implies \((F, E)^\circ \subseteq (G, E)^\circ\);
6. \((F, E)^\circ \cap (G, E)^\circ = ((F, E) \cap (G, E))^\circ\).

**Proof.** (1) and (2) are obvious.

3. \(\Rightarrow\) If \((F, E)\) is a vague soft open set, then \((F, E)^\circ \subseteq (F, E)^\circ\). Since \((F, E)^\circ \subseteq (F, E)\) by part (2), so \((F, E) = (F, E)^\circ\).

\(\Leftarrow\) Suppose that \((F, E) = (F, E)^\circ\). Since \((F, E)^\circ\) is a vague soft open set, so \((F, E)\) is a vague soft open set.

4. Since \((F, E)^\circ\) is a vague soft open set, we have \([(F, E)^\circ]^{\circ} = (F, E)^\circ\) by part (3).

5. If \((F, E) \subseteq (G, E)\), then \((F, E)^\circ \subseteq (F, E)^\circ \subseteq (G, E)^\circ\).

\((F, E)^\circ\) is a vague soft open set contained in \((G, E)\), so \((F, E)^\circ \subseteq (G, E)^\circ\) by Definition 23.

6. Since \((F, E) \cap (G, E) \subseteq (G, E)\), \((F, E) \cap (G, E)^\circ \subseteq (G, E)^\circ\) by part (5). Thus \((F, E) \cap (G, E)^\circ \subseteq (F, E)^\circ \cap (G, E)^\circ\). Since \((F, E)^\circ \cap (F, E)^\circ \subseteq (F, E) \cap (G, E)^\circ\), so \((F, E)^\circ \cap (F, E)^\circ \subseteq (F, E) \cap (G, E)^\circ\) by part (5). \((F, E)^\circ \cap (G, E)^\circ \subseteq (F, E)^\circ \cap (G, E)^\circ\) is a vague soft open set; then \((F, E)^\circ \cap (G, E)^\circ = (F, E)^\circ \cap (G, E)^\circ\). Hence \((F, E)^\circ \cap (G, E)^\circ = ((F, E) \cap (G, E))^\circ\).

**Definition 25.** Let \((X, \tau, E)\) be a vague soft topological space, and let \((F, E)\) be a vague soft set over \(X\). Then vague soft closure of \((F, E)\), denoted by \((F, E)^\bar{c}\), is defined as the intersection of all vague soft closed sets containing \((F, E)\).

Clearly, \((F, E)^\bar{c}\) is the smallest vague soft closed set containing \((F, E)\).

**Theorem 26.** Let \((X, \tau, E)\) be a vague soft topological space over \(X\), and let \((F, E)\) and \((G, E)\) be two vague soft sets over \(X\). Then the following properties hold:

1. \(\hat{\emptyset} = \emptyset\) and \(\bar{X} = \bar{X}\);
2. \((F, E) \subseteq (F, E)^\circ\);
3. \((F, E)\) is a vague soft closed set if and only if \((F, E)^\circ = (F, E)\);
4. \((F, E)^\circ = (F, E)\);
5. \((F, E) \subseteq (G, E)\) implies \((F, E)^\circ \subseteq (G, E)^\circ\);
6. \((F, E)^\circ \cup (G, E) = (F, E)^\circ \cup (G, E)^\circ\).

**Proof.** The proof is similar to the proof of Theorem 24. \(\square\)

**Theorem 27.** Let \((X, \tau, E)\) be a vague soft topological space over \(X\), and let \((F, E)\) be a vague soft set over \(X\). Then the following properties hold:

1. \(((F, E)^\circ)^\circ = (F, E)^\circ\);
2. \(((F, E)^\circ)^\circ = (F, E)^\circ\).

**Proof.** (1) Suppose that \(\{ (F, E) : i \in I \} \) is the family of all vague soft open sets contained in \((F, E)\); that is, \((F, E)^\circ = \bigcup_{i \in I} (F, E)\). Then \(\{ (F, E)^\circ : i \in I \} \) is the family of all vague soft closed sets containing \((F, E)\); that is, \((F, E)^\circ = \bigcap_{i \in I} (F, E)^\circ\). Since \((F, E)^\circ = \bigcup_{i \in I} (F, E)^\circ = \bigcap_{i \in I} (F, E)^\circ\) by Proposition 14 (2), so \((F, E)^\circ = ((F, E)^\circ)^\circ\).

(2) Suppose that \(\{ (F, E) : i \in I \} \) is the family of all vague soft closed sets containing \((F, E)\); that is, \((F, E)^\circ = \bigcap_{i \in I} (F, E)^\circ\). Then \((F, E)^\circ = \bigcup_{i \in I} (F, E)^\circ\) by Proposition 14 (2), so \((F, E)^\circ = (F, E)^\circ\).

**Definition 28.** Let \((X, \tau, E)\) be a vague soft topological space, and let \((F, E)\) be a vague soft set over \(X\). Then vague soft boundary of \((F, E)\), denoted by \(\partial (F, E)\), is defined as \(\partial (F, E) = (F, E)^\circ \cap (F, E)^\bar{c}\).

Clearly, the vague soft sets \((F, E)^\circ\) and \((F, E)^\bar{c}\) have same vague soft boundary; that is, \((F, E) = (F, E)^\circ\).

**Theorem 29.** Let \((X, \tau, E)\) be a vague soft topological space over \(X\), and let \((F, E)\) be a vague soft set over \(X\). Then the following properties hold:

1. \(\hat{\emptyset} = \emptyset\) and \(\bar{X} = \bar{X}\);
2. \((F, E)^\circ \subseteq (F, E)\);
3. \(((F, E)^\circ)^\circ = (F, E) \cup (F, E)^\circ\).
Proof. (1) is obvious.

(2) is as follows:

\[(F, E) = (F, E) \cap (F, E)^c = (F, E) \cap (F, E)^c \quad \text{(by Theorem 27)}\]

(3) is as follows:

\[(F, E)^c \cup ((F, E)^c)^c = ((F, E)^c \cap ((F, E)^c)^c)^c \quad \text{(by Proposition 14)}\]

Then \((F, E) = X\),

\[
(F, E) = \begin{cases} 
    e_1 = \left\{ \frac{[0.4,0.4]}{a}, \frac{[0.3,0.4]}{b} \right\}, \\
    e_2 = \left\{ \frac{[0.2,0.4]}{a}, \frac{[0.2,0.3]}{b} \right\} 
\end{cases}
\]

\[
(F, E)^c = \begin{cases} 
    e_1 = \left\{ \frac{[0.6,0.6]}{a}, \frac{[0.6,0.7]}{b} \right\}, \\
    e_2 = \left\{ \frac{[0.6,0.8]}{a}, \frac{[0.7,0.8]}{b} \right\} 
\end{cases}
\]

Hence \((F, E)^c \cup (F, E) \neq (F, E)\), \((F, E)^c \cap (F, E) \neq 0\).

**Theorem 32.** Let \((X, \tau, E)\) be a vague soft topological space over \(X\), and let \((F, E)\) be a vague soft set over \(X\).

- (1) If \((F, E) \cap (F, E) = 0\), then \((F, E)\) is a vague soft open set over \(X\).
- (2) If \((F, E)\) is a vague soft closed set over \(X\), then \((F, E) \subseteq (F, E)\).

**Proof.** (1) Let \((F, E) \cap (F, E) = 0\). Then \((F, E) \cap (F, E)^c = 0\), or \((F, E) \cap (F, E)^c = 0\), or \((F, E)^c \subseteq (F, E)^c\), which implies \((F, E)^c\) is a vague soft closed set. Hence \((F, E)\) is a vague soft open set.

(2) If \((F, E)\) is a vague soft closed set over \(X\), then \((F, E) = (F, E)\), since \((F, E) = (F, E) \cap (F, E)^c \subseteq (F, E)\) \(= (F, E)\); that is, \((F, E) \subseteq (F, E)\).

**Remark 33.** Let \((X, \tau, E)\) be a vague soft topological space over \(X\), and let \((F, E)\) be a vague soft set over \(X\).

- (1) If \((F, E)\) is a vague soft open set over \(X\), then \((F, E) \cap (F, E) = 0\) may not hold.
- (2) If \((F, E) \subseteq (F, E)\), then \((F, E)\) may not be a vague soft closed set over \(X\).
Example 34. Let us consider the vague soft topological space $(X,\tau,E)$ over $X$ in Example 31. We take $(F,E) = (F_1,E)$ is a vague soft open set over $X$. Then

$$(F,E) = \bar{X},$$

$$\bar{(F,E)} = \left\{ e_1 = \left\{ \frac{[0.4,0.5]}{a},\frac{[0.3,0.4]}{b} \right\}, \right.$$

$$\bar{(F,E)} = \left\{ e_2 = \left\{ \frac{[0.3,0.5]}{a},\frac{[0.2,0.3]}{b} \right\} \right\}.$$ (11)

Hence $(F,E) \cap (F,E) \neq \emptyset$. We have $(F,E) \subseteq (F,E)$, but $(F,E)$ is not a vague soft closed set over $X$.

Theorem 35. Let $(X,\tau,E)$ be a vague soft topological space over $X$, and let $(F,E)$ be a vague soft set over $X$. If $(F,E) = \emptyset$, then $(F,E)$ is a vague soft open set and vague soft closed set over $X$.

Proof. First we prove that $(F,E)$ is a vague soft open set over $X$. Consider $(F,E) = \emptyset \Rightarrow (F,E) \cap \bar{(F,E)} = \emptyset \Rightarrow (F,E) \cap ((F,E)^c)^c = \emptyset \Rightarrow (F,E)^c \subseteq (F,E)^c \Rightarrow (F,E) \subseteq (F,E)^c$. This implies that $(F,E)$ is a vague soft open set.

We now prove that $(F,E)$ is a vague soft closed set over $X$. Consider $(F,E) = \emptyset \Rightarrow (F,E) \cap \bar{(F,E)} = \emptyset \Rightarrow (F,E) \subseteq ((F,E)^c)^c = (F,E)^c \Rightarrow (F,E) \subseteq (F,E)$. This implies that $(F,E)$ is a vague soft closed set.

Remark 36. If $(F,E)$ is a vague soft open set and vague soft closed set over $X$, then $(F,E)$ may not be $\emptyset$.

Example 37. Let $X = \{a,b\}$, $E = \{e_1,e_2\}$, and $\tau = \{\emptyset,\bar{X},(F,E),(G,E)\}$ where $(F,E)$, $(G,E)$ are vague soft sets over $X$, defined as follows:

$$F(e_1) = \left\{ \frac{[0.2,0.8]}{a},\frac{[0.4,0.6]}{b} \right\},$$

$$F(e_2) = \left\{ \frac{[0.3,0.7]}{a},\frac{[0.5,0.5]}{b} \right\},$$

$$G(e_1) = \left\{ \frac{[0.3,0.9]}{a},\frac{[0.5,0.7]}{b} \right\},$$

$$G(e_2) = \left\{ \frac{[0.4,0.8]}{a},\frac{[0.6,0.7]}{b} \right\}. $$ (12)

Then $\tau$ defines a vague soft topology on $X$ and $(X,\tau,E)$ is a vague soft topological space over $X$. We have $(F,E)$ is a vague soft open set and vague soft closed set over $X$, but $(\overline{F,E}) = (F,E) \neq \emptyset$.

Definition 38. A vague soft topological space $(X,\tau,E)$ is called strongly disconnected, if there exist vague soft open sets $(F,E) \neq \emptyset$ and $(G,E) \neq \emptyset$ such that $(F,E) \cup (G,E) = \bar{X}$ and $(F,E) \cap (G,E) = \emptyset$.

Definition 39. A vague soft topological space $(X,\tau,E)$ is called strongly connected, if it is not vague soft strongly disconnected.

Definition 40. A vague soft topological space $(X,\tau,E)$ is called weakly disconnected, if there exists vague soft set $(F,E)$ which is both vague soft open set and vague soft closed set such that $\emptyset \neq (F,E) \neq \bar{X}$.

Definition 41. A vague soft topological space $(X,\tau,E)$ is called weakly connected, if it is not vague soft weakly disconnected.

Theorem 42. Let $(X,\tau,E)$ be a vague soft topological space over $X$. If $(X,\tau,E)$ is strongly disconnected, then it is also weakly disconnected.

Proof. If $(X,\tau,E)$ is strongly disconnected, then there exist vague soft open sets $(F,E) \neq \emptyset$ and $(G,E) \neq \emptyset$ such that $(F,E) \cup (G,E) = \bar{X}$ and $(F,E) \cap (G,E) = \emptyset$. For any $e \in E$, we have $t_{F(e)}(x) \vee t_{G(e)}(x) = 1$, $(1 - t_{F(e)}(x)) \vee (1 - t_{G(e)}(x)) = 1$ and $t_{F(e)}(x) \wedge t_{G(e)}(x) = 0$, $(1 - t_{F(e)}(x)) \wedge (1 - t_{G(e)}(x)) = 0$, $x \in X$. When $t_{F(e)}(x) = 1$, then $1 - t_{F(e)}(x) = 1$ and $t_{G(e)}(x) = 0$, $1 - t_{G(e)}(x) = 0$. Then $t_{G(e)}(x) = 1$, then $1 - t_{G(e)}(x) = 1$ and $t_{F(e)}(x) = 0$, $1 - t_{F(e)}(x) = 0$. Hence $(F,E) = (G,E)^c$; that is, $(F,E)$ is a vague soft set which is both vague soft open set and vague soft closed set such that $\emptyset \neq (F,E) \neq \bar{X}$. Hence $(X,\tau,E)$ is weakly disconnected.

Remark 43. If $(X,\tau,E)$ is weakly disconnected, then it may not be strongly disconnected.

Example 44. Let $X = \{a,b\}$, $E = \{e_1,e_2\}$, and $\tau = \{\emptyset,\bar{X},(F_1,E),(F_2,E)\}$ where $(F_1,E)$, $(F_2,E)$ are two vague soft sets over $X$, defined as follows:

$$F_1(e_1) = \left\{ \frac{[0.1,0.9]}{a},\frac{[0.3,0.7]}{b} \right\},$$

$$F_1(e_2) = \left\{ \frac{[0.4,0.6]}{a},\frac{[0.2,0.8]}{b} \right\},$$

$$F_2(e_1) = \left\{ \frac{[0.1,0.8]}{a},\frac{[0.2,0.6]}{b} \right\},$$

$$F_2(e_2) = \left\{ \frac{[0.3,0.6]}{a},\frac{[0.2,0.7]}{b} \right\}. $$ (13)

Then $\tau$ defines a vague soft topology on $X$ and $(X,\tau,E)$ is a vague soft topological space over $X$. Because $(F_1,E)$ is a vague soft set which is both vague soft open set and vague soft closed
set such that $\emptyset \neq (F, E) \neq \overline{X}$, $(X, \tau, E)$ is weakly disconnected, but it is not strongly disconnected.

**Theorem 45.** $(X, \tau, E)$ is strongly disconnected if and only if there exist vague soft closed sets $(F, E) \neq \emptyset$ and $(G, E) \neq \emptyset$ such that $(F, E) \cup (G, E) = \overline{X}$ and $(F, E) \cap (G, E) = \emptyset$.

**Proof.** $\Rightarrow$ If $(X, \tau, E)$ is strongly disconnected, then there exist vague soft open sets $(F, E) \neq \emptyset$ and $(G, E) \neq \emptyset$ such that $(F, E) \cup (G, E) = \overline{X}$ and $(F, E) \cap (G, E) = \emptyset$. Then $(F, E)^c$ and $(G, E)^c$ are vague soft closed sets such that $(F, E)^c \cup (G, E)^c = \overline{X}$ and $(F, E)^c \cap (G, E)^c = \emptyset$.

$\Leftarrow$ The proof is similar to the above. $\blacksquare$

**Theorem 46.** $(X, \tau, E)$ is weakly disconnected if and only if there exist vague soft open sets $(F, E) \neq \emptyset$ and $(G, E) \neq \emptyset$ such that $(F, E) = (G, E)^c$.

**Proof.** $\Rightarrow$ If $(X, \tau, E)$ is weakly disconnected, then there exists vague soft set $(G, E)$ which is both vague soft open set and vague soft closed set such that $\emptyset \neq (G, E) \neq \overline{X}$. Take $(F, E) = (G, E)^c$; thus $(F, E)$ is a vague soft open set and $(F, E) \neq \emptyset$.

$\Leftarrow$ If there exist vague soft open sets $(F, E) \neq \emptyset$ and $(G, E) \neq \emptyset$ such that $(F, E) = (G, E)^c$, then $(F, E)$ is both vague soft open set and vague soft closed set. Since $(F, E) \neq \emptyset$, $(F, E) = (G, E)^c \neq \overline{X}$, so $(X, \tau, E)$ is weakly disconnected. $\blacksquare$

**Theorem 47.** $(X, \tau, E)$ is weakly disconnected if and only if there exist vague soft closed sets $(F, E) \neq \emptyset$ and $(G, E) \neq \emptyset$ such that $(F, E) = (G, E)^c = ((G, E)^c)^c = (F, E)^c$.

**Proof.** $\Rightarrow$ If $(X, \tau, E)$ is weakly disconnected, then there exists vague soft set $(F, E)$ which is both vague soft open set and vague soft closed set such that $\emptyset \neq (F, E) \neq \overline{X}$. Take $(G, E) = (F, E)^c$; thus $(G, E)$ is both vague soft open set and vague soft closed set. Since $(F, E), (G, E)$ are vague open sets, $(G, E) = (G, E)^c, (F, E) = (F, E)^c$. Hence, $(G, E) = ((F, E)^c)^c = (F, E)^c$.

$\Leftarrow$ Suppose there exist vague soft set $(F, E) \neq \emptyset$ and $(G, E) \neq \emptyset$ such that $(F, E) = (G, E)^c = ((F, E)^c)^c = (F, E)^c$ and $(F, E) = (G, E)^c$. Since $(F, E)^c$ and $(G, E)^c$ are vague soft closed sets, $(G, E) = (F, E)^c$ and $(F, E) = (G, E)^c$ are vague soft closed sets. Since $(G, E) = (F, E)^c$, so $(G, E)$ is a vague soft open set. Thus, vague soft set $(G, E)$ is both vague soft open set and vague soft closed set such that $\emptyset \neq (G, E) \neq \overline{X}$. Hence, $(X, \tau, E)$ is weakly disconnected. $\blacksquare$

**Theorem 48.** $(X, \tau, E)$ is weakly disconnected if and only if there exist vague soft closed sets $(F, E) \neq \emptyset$ and $(G, E) \neq \emptyset$ such that $(F, E) = (G, E)^c = ((G, E)^c)^c = (F, E)^c$.

**Proof.** The proof is similar to the proof of Theorem 47. $\blacksquare$

**Definition 49.** Let $(X, \tau, E)$ be a vague soft topological space over $X$. If a family $\{F_i, E\} | i \in I \}$ of vague soft open sets satisfies the condition $\bigcup_{i \in I} (F_i, E) = \overline{X}$, then it is called a vague soft open cover of $X$. If a finite subfamily of a vague soft open cover is also a vague soft open cover of $X$, then it is called a finite subcover of $\{(F_i, E) | i \in I \}$. 

**Definition 50.** A vague soft topological space $(X, \tau, E)$ is called vague soft compact if every vague soft open cover of $X$ has a finite subcover.

**Example 51.** Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, and $\tau = \{\emptyset, \overline{X}, (F_n, E)\}$ where $(F_n, E)$ is vague soft sets over $X$, defined as follows:

$$F_n(e_1) = \left\{ \frac{1 - 1/n, 1 - 1/((n + 1))}{a}, \frac{1 - 1/n, 1 - 1/((n + 1))}{b} \right\},$$

$$F_n(e_2) = \left\{ \frac{[1, 1]}{a}, \frac{[1 - 1/n, 1 - 1/((n + 1))]}{b} \right\}. \quad (14)$$

Then $\tau$ defines a vague soft topology on $X$ and $(X, \tau, E)$ is a vague soft topological space over $X$.

$(X, \tau, E)$ is not vague soft compact, because the vague soft open cover $\{(F_n, E) | n \in N \}$ has no finite subcover.

**Definition 52.** Let $(X, \tau, E)$ be a vague soft topological space, and let $\{F_i, E\} | i \in I \}$ be a family of vague soft sets over $X$. $(F_i, E) | i \in I \}$ is said to satisfy the finite intersection property, if every finite subfamily $\{F_i, E\} | i = 1, 2, \ldots, n \}$ satisfies the condition $\bigcap_{i=1}^n (F_i, E) \neq \emptyset$.

**Theorem 53.** A vague soft topological space $(X, \tau, E)$ is vague soft compact if and only if every family $\{F_i, E\} | i \in I \}$ of vague soft closed sets which satisfies the finite intersection property has a nonempty intersection.

**Proof.** $\Rightarrow$ Let $\{F_i, E\} | i \in I \}$ be a family of vague soft closed sets which satisfies the finite intersection property. Suppose that $\bigcap_{i \in I} (F_i, E) = \emptyset$; then $\bigcup_{i \in I} (F_i, E)^c = \left( \bigcap_{i \in I} (F_i, E)^c \right)^c = \overline{X}$, so $\{F_i, E\} | i \in I \}$ is a vague soft open cover of $X$. Since $(X, \tau, E)$ is vague soft compact, $\{F_i, E\} | i \in I \}$ has a finite subcover, denoted by $\{F_i, E\} | i = 1, 2, \ldots, n \}$. Thus, $\bigcap_{i=1}^n (F_i, E) = \left( \bigcup_{i=1}^n (F_i, E)^c \right)^c = \emptyset$, which is a contradiction. Hence, $\{F_i, E\} | i \in I \}$ has a nonempty intersection.

$\Leftarrow$ Let $\{G_i, E\} | i \in I \}$ is a vague soft open cover of $X$. Then $\{G_i, E\} | i \in I \}$ is a family of vague soft closed sets and $\bigcap_{i \in I} (G_i, E) = (\bigcup_{i \in I} (G_i, E))^c = \emptyset$. Since every family of vague soft closed sets which satisfies the finite intersection property has a nonempty intersection, so $\{G_i, E\} | i \in I \}$ does not have the finite intersection property; that is, a subfamily $\{G_i, E\} | i = 1, 2, \ldots, n \} \subset \{G_i, E\} | i \in I \}$ satisfies $\bigcap_{i=1}^n (G_i, E) = \emptyset$. Thus, $\bigcup_{i=1}^n (G_i, E) = \left( \bigcap_{i=1}^n (G_i, E)^c \right)^c = \overline{X}$; that is, $\{G_i, E\} | i = 1, 2, \ldots, n \}$ is a finite subcover of $\{G_i, E\} | i \in I \}$. Hence, $(X, \tau, E)$ is vague soft compact. $\blacksquare$

**Definition 54.** Let $(X, \tau, E)$ be a vague soft topological space, and let $(F, E)$ be a vague soft set over $X$. If a family $\{G_i, E\} | i \in I \}$ of vague soft open sets satisfies the condition $(F, E) \subset \bigcup_{i \in I} (G_i, E)$, then it is called a vague soft open cover of $(F, E)$. If a finite subfamily of a vague soft open cover is also a vague soft open cover of $(F, E)$, then it is called a vague soft finite subcover of $(F, E)$.
soft open cover of \((F, E)\), then it is called a finite subcover of \(\{(G_i, E) \mid i \in I\}\).

**Definition 55.** A vague soft set \((F, E)\) is called vague soft compact if every vague soft open cover of \((F, E)\) has a finite subcover.

**Theorem 56.** A vague soft set \((F, E)\) is vague soft compact if and only if every family \(\{(G_i, E) \mid i \in I\}\) of vague soft open sets with properties that, for any \(e \in E\), \(t_{F(e)}(x) \leq \bigvee_{i \in I} t_{G_i(e)}(x)\) and \(1 - f_{F(e)}(x) \leq \bigvee_{i \in I} (1 - f_{G_i(e)}(x))\), \(x \in X\), has a finite subfamily \(\{(G_i, E) \mid i = 1, 2, \ldots, n\}\) such that \(t_{F(e)}(x) \leq \bigvee_{i=1}^{n} t_{G_i(e)}(x)\) and \(1 - f_{F(e)}(x) \leq \bigvee_{i=1}^{n} (1 - f_{G_i(e)}(x))\), \(x \in X\).

**Proof.** It follows from Definitions 54 and 55.

### 4. Conclusion

Topology is an important and major area of mathematics. In this paper, we have introduced vague soft topological spaces which are defined over an initial universe with a fixed set of parameters. The notions and properties of vague soft open set, vague soft closed set, vague soft interior, vague soft closure, vague soft boundary, vague soft connectedness, and vague soft compactness are introduced and investigated. The findings in this paper will enhance and promote the further study on vague soft topological space to establish a general framework for practical application.

### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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