Abstract. We consider the empirical eigenvalue distribution of an \( m \times m \) principal submatrix of an \( n \times n \) random unitary matrix distributed according to Haar measure. For \( n \) and \( m \) large with \( \frac{m}{n} = \alpha \), the empirical spectral measure is well-approximated by a deterministic measure \( \mu_{\alpha} \) supported on the unit disc. In earlier work, we showed that for fixed \( n \) and \( m \), the bounded-Lipschitz distance between the empirical spectral measure and the corresponding \( \mu_{\alpha} \) is typically of order \( \sqrt{\log(m)/m} \) or smaller. In this paper, we consider eigenvalues on a microscopic scale, proving concentration inequalities for the eigenvalue counting function and for individual bulk eigenvalues.

1. Introduction

Let \( U \) be an \( n \times n \) Haar-distributed unitary matrix and let \( U_m \) be the \( m \times m \) top-left block of \( U \), where \( m < n \). We refer to \( U_m \) as a truncation of \( U \). The eigenvalues of the truncation are all located within the unit disc and the asymptotic distribution of the eigenvalues can be described quite explicitly. Let \( \mu_m \) denote the empirical spectral measure of \( U_m \), that is,

\[
\mu_m = \frac{1}{m} \sum_{p=1}^{m} \delta_{\lambda_p},
\]

where \( \lambda_1, \ldots, \lambda_m \) are the eigenvalues of \( U_m \). Petz and Réffy [5] proved that if \( \frac{m}{n} \to \alpha \in (0, 1) \), then \( \mu_m \) converges almost surely to a limiting spectral measure \( \mu_{\alpha} \); it has radial density with respect to Lebesgue measure on \( \mathbb{C} \) given by

\[
f_{\alpha}(z) = \begin{cases} 
\frac{1-\alpha}{\pi \alpha (1-|z|^2)} & 0 < |z| < \sqrt{\alpha}; \\
0, & \text{otherwise.}
\end{cases}
\]

In [4], we proved the following non-asymptotic, quantitative version of this result. The rescaling was chosen so that the support of the limiting measure is the full unit disc, independent of \( \alpha \).

**Theorem 1** (E. Meckes and K. Stewart). Let \( n, m \in \mathbb{N} \) with \( 1 \leq m < n \). Let \( U \in \mathbb{U}(n) \) be distributed according to Haar measure, and let \( \lambda_1, \ldots, \lambda_m \) denote the eigenvalues of the top-left \( m \times m \) block of \( \sqrt{\frac{m}{n}} U \). The joint law of \( \lambda_1, \ldots, \lambda_m \) is denoted \( \mathbb{P}_{n,m} \). Let \( \mu_m \) be the random measure with mass \( \frac{1}{m} \) at each of the \( \lambda_p \), and let \( \alpha = \frac{m}{n} \). Let \( \mu_{\alpha} \) be the probability measure on the unit disc with the density \( g_{\alpha} \) defined by

\[
g_{\alpha}(z) = \begin{cases} 
\frac{(1-\alpha)}{\pi (1-|z|^2)} & 0 < |z| < 1; \\
0, & \text{otherwise.}
\end{cases}
\]
For any $r > 0$,
\[
\mathbb{P}_{n,m} \left[ d_{BL}(\mu_m, \mu_{\alpha}) \geq r \right] \leq e^{2 \exp \left\{-C_{\alpha} m^2 r^2 + 2m \log(m) + C'_{\alpha} m \right\}} + \frac{e}{2\pi} \sqrt{\frac{m}{1-\alpha}} e^{-m},
\]
where $C_{\alpha} = \frac{1}{128 \pi (1 + (3 + \log(\alpha^{-1}))^2)}$ and $C'_{\alpha} = 6 + 3 \log(\alpha^{-1})$.

The result above is essentially macroscopic; it says that with high probability, $d_{BL}(\mu_m, \mu_{\alpha})$ is of order $\sqrt{\frac{\log(m)}{m}}$. The purpose of this paper is to examine the microscopic level, by considering the eigenvalue counting function on small sets. Throughout the paper, we assume that $\alpha = \frac{m}{128 \pi}$ is bounded away from 0 and 1; i.e., that there is a fixed $\delta > 0$ such that $\alpha \in (\delta, 1 - \delta)$. Throughout the statements and proofs, there are constants $C_{\alpha}$ depending only on $\alpha$; their exact values may vary from one line to the next.

We begin by ordering the eigenvalues $\{\lambda_p\}_{p=1}^m$ in the spiral fashion introduced in [3]. Define a linear order $\prec$ on $\mathbb{C}$ by making 0 initial, and for nonzero $w, z \in \mathbb{C}$, declare $w \prec z$ if either of the following hold:

- $|w| < |z|
- |w| = |z|$ and $\arg w < \arg z$

We divide the disc of radius $\sqrt{\frac{m}{n}}$ (i.e., the support of the limiting eigenvalue density) into annuli with radii $r_i = \frac{i}{\sqrt{n-m+i}}$; it is verified below that the expected number of eigenvalues in the annulus from radius $r_{i-1}$ to $r_i$ is approximately $2i - 1$.

More generally, for $\theta \in (0, 2\pi]$, define
\[
A_{i, \theta} = \left\{ z \in \mathbb{C} \mid z \prec r_i e^{i\theta} \right\} = \left\{ z \in \mathbb{C} \mid |z| < r_i \right\} \cup \left\{ z \in \mathbb{C} \mid r_i \leq |z| < r_{i+1}, 0 < \arg z < \theta \right\},
\]
with $r_i = \frac{i}{\sqrt{n-m+i}}$ and $1 \leq i \leq \sqrt{m}$ (see Figure 1).

Our first main result is on the concentration of the eigenvalue counting function for the sets $A_{i, \theta}$.

**Theorem 2.** Let $N_{i, \theta}$ denote the number of eigenvalues of an $m \times m$ truncation of a Haar-distributed matrix in $\mathbb{U}(n)$ which lie in $A_{i, \theta}$. If $\epsilon_m = \sqrt{\frac{2 \log(m+1)}{m}}$, then for each $1 \leq i \leq \sqrt{m} \left(1 - \frac{\epsilon_m}{1-\alpha(1-\epsilon_m)}\right)^{\frac{1}{2}}, 0 \leq \theta \leq 2\pi$, and $t > 0$,
\[
\mathbb{P} \left[ \left| N_{i, \theta} - i^2 - \frac{\theta}{2\pi} (2i + 1) \right| \geq t \right] \leq 2e^{2 \exp \left\{- \min \left\{ \frac{t^2}{C_{\alpha} \sqrt{\log(i)}}, 1 \right\} \right\}}.
\]

If $t > \frac{12}{1-\alpha} \sqrt{2m \log(m + 1)}$, then this estimate is also valid for those $i$ with $\sqrt{m} \left(1 - \frac{\epsilon_m}{1-\alpha(1-\epsilon_m)}\right)^{\frac{1}{2}} \leq i \leq \sqrt{m}$.

We next define predicted locations $\{\tilde{\lambda}_p\}_{p=1}^m$ for the eigenvalues by choosing $2i - 1$ equally spaced points in the annulus with inner radius $r_{i-1}$ and outer radius $r_i$. The concentration
inequalities in Theorem 2 for the counting function lead to the following concentration inequality for bulk eigenvalues about their predicted locations.

**Theorem 3.** Let \( \{\lambda_p\}_{p=1}^m \) denote the eigenvalues of an \( m \times m \) truncation of a Haar-distributed matrix in \( U(n) \), ordered according to \( \prec \). Let \( l = \lceil \sqrt{p} \rceil \). There are constants \( c_\alpha, C_\alpha \) depending only on \( \alpha = \frac{m}{n} \) such that, if \( \epsilon_m = \sqrt{\frac{2 \log(m+1)}{m}} \), then for those \( p \) with

\[
2 \leq l \leq \sqrt{m} \left( 1 - \frac{\epsilon_m}{1 - \alpha(1 - \epsilon_m)} \right)^{\frac{1}{2}},
\]

when \( s \leq 2\pi(l - 1) \),

\[
P \left[ |\lambda_p - \tilde{\lambda}_p| \geq \frac{s}{\sqrt{n-m+(l-1)^2}} \right] \leq 2 \exp \left[ -\frac{s^2}{C_\alpha l \sqrt{\log(l)}} \right];
\]

when \( 2\pi(l - 1) < s \leq 2\sqrt{n - m + (l - 1)^2} \),

\[
P \left[ |\lambda_p - \tilde{\lambda}_p| \geq \frac{s}{\sqrt{n-m+(l-1)^2}} \right] \leq 2 \exp \left[ -c_\alpha s^2 \right];
\]

and when \( s > 2\sqrt{n - m + (l - 1)^2} \),

\[
P \left[ |\lambda_p - \tilde{\lambda}_p| \geq \frac{s}{\sqrt{n-m+(l-1)^2}} \right] = 0.
\]
By way of example, if $2\pi(l-1) \leq \sqrt{\frac{k}{c_\alpha} \log(n)}$, then

$$P\left[ |\lambda_p - \tilde{\lambda}_p| \geq \sqrt{\frac{k \log(n)}{c_\alpha (n-m + (l-1)^2)}} \right] \leq 2n^{-k},$$

whereas if, e.g., $\log(n) \leq \frac{4\pi^2(l-1)^2}{kC_\alpha l \sqrt{\log(l)}}$, then

$$P\left[ |\lambda_p - \tilde{\lambda}_p| \geq (\log(l))^{\frac{1}{4}} \sqrt{\frac{kC_\alpha l \log(n)}{n-m + (l-1)^2}} \right] \leq 2n^{-k},$$

For reference, spacing of predicted locations around $\tilde{\lambda}_p$ is about $\frac{1}{\sqrt{n-m+(l-1)^2}}$.

The concentration inequalities of Theorem 3 also easily imply the following variance bound for bulk eigenvalues.

**Corollary 4.** Let $\epsilon_m = \sqrt{\frac{2\log(m+1)}{m}}$ and $p$ be such that $2 \leq \lfloor \sqrt{p} \rfloor \leq \sqrt{m} \left(1 - \frac{\epsilon_m}{1-\alpha(1-\epsilon_m)}\right)^{\frac{1}{2}}$.

There is a constant $C_\alpha$ depending only on $\alpha = \frac{m}{n}$ such that

$$\text{Var}(\lambda_p) \leq C_\alpha \frac{p \log(p+1)}{n}.$$

2. MEANS AND VARIANCES

Throughout the proofs, we will make heavy use of the fact that the eigenvalues we consider are a determinantal point process on $\{|z| \leq 1\}$ with kernel (with respect to Lebesgue measure) given by

(1) $$K(z_1, z_2) = \sum_{j=1}^{m} \frac{1}{N_j} (z_1 \overline{z}_2)^{j-1} (1 - |z_1|^2)^{\frac{n-m-1}{2}} (1 - |z_2|^2)^{\frac{n-m-1}{2}},$$

with

$$N_j = \frac{\pi (j-1)! (n-m-1)!}{(n-m+j-1)!}.$$

See, e.g., [6] or [5].

Recall that for large $n$ and $\frac{m}{n} = \alpha \in (0, 1)$ the spectral measure of the truncation is approximately given by the measure $\mu_\alpha$, with density with respect to Lebesgue measure given by

$$f_\alpha(z) = \begin{cases} \frac{(1-\alpha)}{\pi \alpha (1-|z|^2)^2}, & 0 < |z| < \sqrt{\alpha}; \\ 0, & \text{otherwise}. \end{cases}$$

In particular, given a set $A \subseteq \{|z| \leq \sqrt{\alpha}\}$, the expected number $N_A$ of eigenvalues inside $A$ is approximately $m \mu_\alpha(A)$. We begin by giving explicit estimates quantifying this approximation.
Lemma 5. For any measurable \( A \subseteq \{ |z| \leq \sqrt{\alpha} \} \),
\[
m\mu_\alpha(A) - \frac{6\sqrt{2m \log(m+1)}}{1 - \alpha} \leq \mathbb{E}N_A \leq m\mu_\alpha.
\]
If additionally \( A \subseteq \{ |z|^2 \leq \alpha \left( 1 - \sqrt{\frac{2 \log(m+1)}{m}} \right) \} \), then
\[
m\mu_\alpha(A) - 4 \leq \mathbb{E}N_A \leq m\mu_\alpha(A).
\]

Proof. For a determinantal point process on \((\Lambda, \mu)\) with kernel \( K \), the expected number of points in a set \( A \) is given by
\[
\mathbb{E}N(A) = \int_A K(x, x) d\mu(x).
\]
From the formula for the kernel given in equation (1),
\[
K(z, z) = \frac{(1 - |z|^2)^{n-m-1}}{\pi} \sum_{j=1}^{m} \frac{(n-m+j-1)!}{(j-1)!(n-m-1)!} |z|^{2(j-1)}
\]
\[
= \frac{(n-m)(1 - |z|^2)^{n-m-1}}{\pi} \sum_{p=0}^{m-1} \frac{(n-m+1)_p}{p!} |z|^{2p},
\]
where \((a)_p = a(a+1) \cdots (a+p-1)\) is the rising Pochhammer symbol. Letting
\[
2F1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n
\]
denote the hypergeometric function with parameters \( a, b, c \),
\[
\sum_{p=0}^{\infty} \frac{(n-m+1)_p}{p!} |z|^{2p} = 2F1(n-m+1, 1; 1; |z|^2) = (1 - |z|^2)^{-(n+m-1)}.
\]
It follows that
\[
K(z, z) = \frac{n-m}{\pi} \left[ \frac{1}{(1 - |z|^2)^2} - (1 - |z|^2)^{n-m-1} \sum_{p=m}^{\infty} \frac{(n-m+1)_p}{p!} |z|^{2p} \right]
\]
\[
= mf_\alpha(z) \left[ 1 - (1 - |z|^2)^{n-m+1} \sum_{p=m}^{\infty} \frac{(n-m+1)_p}{p!} |z|^{2p} \right],
\]
and as an immediate consequence,
\[
\mathbb{E}N_A \leq \int_A mf_\alpha(z) d\lambda(z) = m\mu_\alpha(A).
\]
For the lower bound, we first treat the more restrictive case of
\[
A \subseteq \{ |z|^2 \leq \alpha \left( 1 - \sqrt{\frac{2 \log(m+1)}{m}} \right) \}.
\]
Consider the random variable $Y_k(x)$ on $\mathbb{N} \cup \{0\}$ with mass function
$$P[Y_k(x) = p] = \frac{(k)_p}{p!} (1 - x)^k x^p.$$ 

The moment generating function of $Y_k(x)$ is given by
$$E[e^{tY_k(x)}] = \sum_{p=0}^{\infty} \frac{(k)_p}{p!} (1 - x)^k (e^t x)^p = \left[ \frac{1 - x}{1 - xe^t} \right]^k.$$ 

Now,
$$(1 - |z|^2)^{n-m+1} \sum_{p=m}^{\infty} \frac{(n-m+1)_p}{p!} |z|^{2p}$$
$$= P[Y_{n-m+1}(|z|^2) \geq m] \leq e^{-tm} \left[ \frac{1 - |z|^2}{1 - |z|^2 e^t} \right]^{n-m+1},$$

for any $t > 0$. Since $|z|^2 < \alpha \left( 1 - \sqrt{\frac{2 \log(m+1)}{m}} \right) < \frac{\alpha n}{n+1}$, we may choose $t = \log \left( \frac{m}{|z|^2(n+1)} \right) > 0$. Then
$$(1 - |z|^2)^{n-m+1} \sum_{p=m}^{\infty} \frac{(n-m+1)_p}{p!} |z|^{2p}$$
$$\leq \left( \frac{|z|^2(n+1)}{\alpha n} \right)^{\alpha n} \left[ \frac{1 - |z|^2}{1 - \alpha \left( \frac{n}{n+1} \right)} \right]^{n(1-\alpha)+1}.$$ 

For $|z|^2 \leq \frac{\alpha n}{n+1}$, this last quantity is increasing in $|z|$; if we further assume that $|z|^2 \leq \alpha(1 - \epsilon_n)$, we thus have that
$$(1 - |z|^2)^{n-m+1} \sum_{p=m}^{\infty} \frac{(n-m+1)_p}{p!} |z|^{2p}$$
$$\leq e^{\alpha} \left( 1 - \epsilon_n \right)^{\alpha n} \left[ \frac{1 - \alpha(1 - \epsilon_n)}{1 - \alpha \left( \frac{n}{n+1} \right)} \right]^{n(1-\alpha)+1}$$
$$= \exp \left\{ \alpha + \alpha n \log(1 - \epsilon_n) + (n(1 - \alpha) + 1) \log \left( 1 + \alpha \left( \frac{\epsilon_n - \frac{1}{n+1}}{1 - \alpha \left( \frac{n}{n+1} \right)} \right) \right\}$$
$$\leq \exp \left\{ \alpha - \alpha n \epsilon_n - \frac{\alpha n \epsilon_n^2}{2} + (n + 1) \alpha \left( \epsilon_n - \frac{1}{n+1} \right) \right\}$$
$$= \exp \left\{ \alpha \epsilon_n - \frac{\alpha n \epsilon_n^2}{2} \right\}.$$
The claimed estimate follows by taking $\epsilon_n = \sqrt{\frac{2 \log(m+1)}{m}} = \sqrt{\frac{2 \log(\alpha n + 1)}{\alpha n}}$ (the constant 4 in the statement is for concreteness; the actual estimate resulting from this choice of $\epsilon_n$ is $e^{\frac{2 \log(\alpha n + 1)}{n}}$).

Returning to the more general case, using the expression for $K(z, z)$ in (2)

$$\mathbb{E}N_A = m \mu_\alpha(A) - \int_A m f_\alpha(z)(1 - |z|^2)^{n-m+1} \sum_{p=m}^{\infty} \frac{(n - m + 1)p}{p!} |z|^{2p} d\lambda(z)$$

$$\geq m \mu_\alpha(A) - 4 \int_A \left\{ \alpha \left( 1 - \sqrt{\frac{2 \log(m+1)}{m}} \right) \leq |z|^2 \leq \alpha \right\} m f_\alpha(z)(1 - |z|^2)^{n-m+1}$$

$$\times \sum_{p=m}^{\infty} \frac{(n - m + 1)p}{p!} |z|^{2p} d\lambda(z),$$

making use of the analysis above. To estimate the remaining integral, we reconsider the quantity

$$\Pr[Y_{n-m+1}(|z|^2) \geq m],$$

this time simply estimating via Markov's inequality. Given $k$ and $x$,

$$\mathbb{E}Y_k(x) = \sum_{p=0}^{\infty} \frac{(k+1)^p}{p!} (1-x)^k x^p = \frac{kx}{1-x} \sum_{\ell=0}^{\infty} \frac{(k+1)^\ell}{\ell!} (1-x)^{k+1} x^\ell = \frac{xk}{1-x},$$

and so

$$(1 - |z|^2)^{n-m+1} \sum_{p=m}^{\infty} \frac{(n - m + 1)p}{p!} |z|^{2p}$$

$$= \Pr[Y_{n-m+1}(|z|^2) \geq m] \leq \frac{(n - m + 1)|z|^2}{m(1 - |z|^2)}.$$
for $|z|^2 \leq \alpha$, and so

$$
(n - m + 1) \left( \sup_{\alpha (1 - \sqrt{2\log(m + 1)/m}) \leq |z|^2 \leq \alpha} \frac{|z|^2 f_\alpha(z)}{(1 - |z|^2)} \right) \pi \alpha \sqrt{\frac{2 \log(m + 1)}{m}}
$$

$$
\leq (n - m + 1) \frac{\alpha}{(1 - \alpha)^2} \sqrt{\frac{2 \log(m + 1)}{m}}
$$

$$
= \frac{1}{1 - \alpha} \sqrt{2m \log(m + 1)} + \frac{\alpha}{(1 - \alpha)^2} \sqrt{\frac{2 \log(m + 1)}{m}}.
$$

It follows that

$$
EN_A \geq m\mu_\alpha(A) - 4 - \frac{1}{1 - \alpha} \sqrt{2m \log(m + 1)} - \frac{\alpha}{(1 - \alpha)^2} \sqrt{\frac{2 \log(m + 1)}{m}}.
$$

Observing that max $\left\{4, \frac{\alpha}{(1 - \alpha)^2} \sqrt{\frac{2 \log(m + 1)}{m}} \right\} \leq \frac{1}{1 - \alpha} \sqrt{2m \log(m + 1)}$ completes the proof.

The following is an immediate consequence of Lemma 5.

**Corollary 6.** Let $\theta \in (0, 2\pi]$ and let $r_i = \frac{i}{\sqrt{n - m + r^2}}$. Let $N_{i, \theta}$ be defined as above, and suppose that $m \geq 3$ and $1 \leq i \leq \sqrt{m} \left( 1 - \frac{\sqrt{2\log(m + 1)/m}}{1 - \alpha + \alpha \sqrt{2\log(m + 1)/m}} \right)$. Then

$$
\left| EN_{i, \theta} - i^2 - \frac{\theta}{2\pi} (2i + 1) \right| \leq 4.
$$

**Proof.** The condition on $i$ guarantees that $A_{i, \theta} \subseteq \left\{ |z|^2 \leq \alpha \left( 1 - \sqrt{2\log(m + 1)/m} \right) \right\}$, so that the sharper estimate from Lemma 5 applies.

For $A_{i, \theta}$ defined as above,

$$
\mu_\alpha(A_{i, \theta}) = \int_0^{2\pi} \int_0^{r_i} \frac{1 - \alpha}{\pi \alpha (1 - r^2)^2} r dr d\theta + \int_0^\theta \int_{r_i}^{r_{i+1}} \frac{1 - \alpha}{\pi \alpha (1 - r^2)^2} r dr d\theta
$$

$$
= \frac{1 - \alpha}{\alpha} \frac{r_i^2}{1 - r_i^2} + \frac{(1 - \alpha)\theta}{\alpha} \frac{r_{i+1}^2 - r_i^2}{(1 - r_i^2)(1 - r_{i+1}^2)},
$$

so that

$$
m\mu_\alpha(A_{i, \theta}) = (n - m) \frac{r_i^2}{1 - r_i^2} + \frac{(n - m)\theta}{2\pi} \frac{r_{i+1}^2 - r_i^2}{(1 - r_i^2)(1 - r_{i+1}^2)}
$$

$$
= i^2 + \frac{\theta}{2\pi} (2i + 1).
$$

We next estimate the variance of $N_{i, \theta}$.

**Lemma 7.** Let $A_{i, \theta}$ be as above. There is a constant $C_\alpha$ depending only on $\alpha = \frac{m}{n}$ such that

$$
\text{Var}(N_{i, \theta}) \leq C_\alpha i \sqrt{\log(i)}.
$$
Proof. By an argument similar to the one in [1, Appendix B],

\[
\text{Var}(N_{i,\theta}) = \int_{\{\vert z \vert < r_i\}} \int_{\{|w| \geq r_i+1\}} |K(z, w)|^2 \, dw \, dz \\
+ \int_{\{\vert z \vert < r_i\}} \int_{\{r_i \leq |w| < r_i+1, \ \theta \leq \arg w \leq 2\pi\}} |K(z, w)|^2 \, dw \, dz \\
+ \int_{\{r_i \leq |z| < r_i+1, \ 0 < \arg z \leq \theta\}} \int_{\{|w| \geq r_i+1\}} |K(z, w)|^2 \, dw \, dz \\
+ \int_{\{r_i \leq |z| < r_i+1, \ 0 < \arg z \leq \theta\}} \int_{\{r_i \leq |w| < r_i+1, \ 0 < \arg w \geq \theta\}} |K(z, w)|^2 \, dw \, dz \\
= : V_1 + V_2 + V_3 + V_4
\]

(3)

Observe that for \(r_1, r_2 \leq 1\),

\[
\left|K(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2})\right|^2 = \frac{(n-m)^2}{\pi^2} (1 - r_1^2)^{n-m-1} (1 - r_2^2)^{n-m-1} \sum_{j,k=0}^{m-1} \binom{n-m+j}{j} \binom{n-m+k}{k} (r_1 r_2)^{j+k} e^{i(j-k)(\varphi_1 - \varphi_2)}.
\]

Integrating in polar coordinates gives that

\[
V_1 = 4(n-m)^2 \times \sum_{j=0}^{m-1} \binom{n-m+j}{j}^2 \int_0^{r_i} (1 - r^2)^{n-m-1} r^{2j+1} dr \int_0^{1} (1 - r^2)^{n-m-1} r^{2j+1} dr,
\]

since the angular integrals vanish unless \(j = k\). By repeated applications of integration by parts,

\[
2(n-m) \binom{n-m+j}{j} \int_0^{r_i} (1 - r^2)^{n-m-1} r^{2j+1} dr \\
= \sum_{\ell=j+1}^{n-m+j} \binom{n-m+j}{\ell} r_1^{2\ell} (1 - r_1^2)^{n-m+j-\ell},
\]

which is exactly \(P[Y_j > j]\) for \(Y_j \sim \text{Binom}(n-m+j, r_1^2)\). Similarly,

\[
2(n-m) \binom{n-m+j}{j} \int_{r_i+1}^1 (1 - r^2)^{n-m-1} r^{2j+1} dr \\
= \sum_{\ell=0}^{j} \binom{n-m+j}{\ell} r_1^{2\ell+1} (1 - r_1^2)^{n-m+j-\ell},
\]

which is \(P[X_j \leq j]\) for \(X_j \sim \text{Binom}(n-m+j, r_1^{2})\).
It thus follows from (4) that

$$V_1 = \sum_{j=0}^{m-1} \mathbb{P}[Y_j > j] \mathbb{P}[X_j \leq j] \leq \sum_{j=0}^{i^2-1} \mathbb{P}[X_j \leq j] + \sum_{j=i^2}^{m-1} \mathbb{P}[Y_j > j].$$

For the first sum, observe that $\mathbb{E}X_j = \frac{(n-m+j)(i+1)^2}{n-m+(i+1)^2} > j$ for $j \leq i^2 - 1$. By Bernstein’s inequality,

$$\mathbb{P}[X_j \leq j] = \mathbb{P} \left[ \mathbb{E}X_j - X_j \geq \frac{(n-m)((i+1)^2 - j)}{n-m+(i+1)^2} \right] \leq \exp \left\{ - \min \left\{ \frac{(n-m)((i+1)^2 - j)^2}{2(n-m+j)(i+1)^2}, \frac{(n-m)((i+1)^2 - j)}{2(n-m+(i+1)^2)} \right\} \right\}.$$

The first term of the minimum is smaller exactly when $j \geq j_0 := \frac{(i+1)^4}{n-m+2(i+1)^2}$. Note that $j_0 \leq \frac{(i+1)^2}{2}$, so that

$$\sum_{j=0}^{j_0} \exp \left\{ - \frac{(n-m)((i+1)^2 - j)}{2(n-m+(i+1)^2)} \right\} \leq \frac{(i+1)^2}{2} \exp \left\{ - \frac{(n-m)(i+1)^2}{4(n-m+(i+1)^2)} \right\} \leq \frac{(i+1)^2}{2} \exp \left\{ - \frac{(1-\alpha)(i+1)^2}{4} \right\},$$

which is bounded independent of $i$.

Now consider

$$\sum_{j=j_0}^{i^2-1} \mathbb{P}[X_j \leq j] \leq \sum_{j=j_0}^{(i+1)^2-(i+1)\sqrt{\frac{2\log(i+1)}{1-\alpha}}} \exp \left\{ - \frac{(n-m)((i+1)^2 - j)^2}{2(n-m+j)(i+1)^2} \right\} + (i+1) \sqrt{\frac{2\log(i+1)}{1-\alpha}},$$

where we have used the fact that the summand in the second line is increasing in $j$ and bounded by $\frac{1}{i+1}$ at the upper limit of the sum.

For the second sum of Equation (5), we again apply Bernstein’s inequality:

$$\mathbb{P}[Y_j > j] = \mathbb{P} \left[ Y_j - \mathbb{E}Y_j > j - (n-m+j)r_i^2 \right] \leq \exp \left\{ - \min \left\{ \frac{(j-(n-m+j)r_i^2)^2}{2(n-m+j)r_i^2(1-r_i^2)}, \frac{j-(n-m+j)r_i^2}{2} \right\} \right\}$$

$$= \exp \left\{ - \min \left\{ \frac{(n-m)(j-i^2)^2}{2(n-m+j)i^2}, \frac{(n-m)(j-i^2)}{2(n-m+j)i^2} \right\} \right\}.$$
The change in behavior of the bound is at \( j = j_1 := \frac{i^2 [2(n-m+i)]}{n-m} \). Note that \( j_1 \leq i^2 \left( 2 + \frac{\alpha}{1-\alpha} \right) \) since \( i^2 \leq m \). Decomposing as before,

\[
\sum_{j=j_1}^{m-1} \mathbb{P}[Y_j > j] \leq i \sqrt{2 \log(i) \frac{1}{1-\alpha} + \sum_{j=j_1}^{m-1} \exp \left\{ -\frac{(n-m)(j-i^2)^2}{2(n-m+i^2)} \right\}}
\]

\[
+ \sum_{j=j_1}^{m-1} \exp \left\{ -\frac{(n-m)(j-i^2)}{2(n-m+i^2)} \right\} \leq i \sqrt{2 \log(i) \frac{1}{1-\alpha} + \sum_{j=j_1}^{m-1} \exp \left\{ -\frac{(n-m)(j-i^2)}{2(n-m+i^2)} \right\}}.
\]

This last sum is

\[
e^{-\frac{(n-m)^2}{2(n-m+i^2)}} \sum_{j=j_1}^{m-1} e^{-\frac{(n-m)^2}{2(n-m+i^2)}} \leq e^{-\frac{(n-m)^2}{2(n-m+i^2)}} \left[ \frac{e^{-\frac{(n-m)}{2(n-m+i^2)}}}{1-e^{-\frac{(n-m)}{2(n-m+i^2)}}} \right] = \frac{e^{-\frac{(n-m)^2}{2(n-m+i^2)}}}{1-e^{-\frac{(n-m)}{2(n-m+i^2)}}},
\]

which is bounded independent of \( i \). Collecting terms, we have

\[
V_1 \leq C_\alpha i \sqrt{\log(i)}
\]

for a constant \( C_\alpha \) depending only on \( \alpha \).

The remaining terms of (3) are estimated similarly. For \( V_2 \), integrating in polar coordinates gives that

\[
V_2 = 4(n-m)^2 \left( 1 - \frac{\theta}{2\pi} \right) \sum_{j=0}^{m-1} \binom{n-m+j}{j}^2 \times \int_0^{r_1} (1-r^2)^{n-m-1} r^{2j+1} dr \int_{r_1}^{r_{j+1}} (1-r^2)^{n-m-1} r^{2j+1} dr
\]

\[
\leq 4(n-m)^2 \left( 1 - \frac{\theta}{2\pi} \right) \sum_{j=0}^{m-1} \binom{n-m+j}{j}^2 \times \int_0^{r_1} (1-r^2)^{n-m-1} r^{2j+1} dr \int_{r_1}^{1} (1-r^2)^{n-m-1} r^{2j+1} dr.
\]

Proceeding exactly as for \( V_1 \),

\[
V_2 \leq \left( 1 - \frac{\theta}{2\pi} \right) \sum_{j=0}^{m-1} \mathbb{P}[Y_j > j] \mathbb{P}[Y_j \leq j]
\]

\[
\leq \left( 1 - \frac{\theta}{2\pi} \right) \left( \sum_{j=0}^{i^2-1} \mathbb{P}[Y_j \leq j] + \sum_{j=i^2}^{m-1} \mathbb{P}[Y_j > j] \right)
\]

\[
\leq \left( 1 - \frac{\theta}{2\pi} \right) C_\alpha i \sqrt{\log(i)},
\]
where $Y_j \sim \text{Binom}(n - m + j, r_i^2)$.

Integrating in polar coordinates and proceeding as above,

$$V_3 = 4(n - m)^2 \frac{\theta}{2\pi} \sum_{j=0}^{m-1} \binom{n - m + j}{j} \binom{n - m + k}{k} \theta \int_{r_i}^{r_{i+1}} (1 - r^2)^{n - m - 1} r^{2j+1} dr \int_{r_i}^{r_{i+1}} (1 - r^2)^{n - m - 1} r^{2j+1} dr$$

$$\leq 4(n - m)^2 \frac{\theta}{2\pi} \sum_{j=0}^{m-1} \binom{n - m + j}{j} \theta \int_{r_i}^{r_{i+1}} (1 - r^2)^{n - m - 1} r^{2j+1} dr \int_{r_i}^{r_{i+1}} (1 - r^2)^{n - m - 1} r^{2j+1} dr$$

$$= \frac{\theta}{2\pi} \sum_{j=0}^{m-1} \mathbb{P}[X_j > j] \mathbb{P}[X_j \leq j]$$

$$\leq \frac{\theta}{2\pi} C_\alpha \sqrt{i \log(i)},$$

where $X_j \sim \text{Binom}(n - m + j, r_i^2)$. The final integral in (3) is

$$V_4 = \frac{(n - m)^2}{\pi^2} \sum_{j,k=0}^{m-1} \binom{n - m + j}{j} \binom{n - m + k}{k} \left( \int_{r_i}^{r_{i+1}} (1 - r^2)^{n - m - 1} r^{j+k+1} dr \right)^2 \int_0^\theta e^{i(j-k)\phi} d\phi \int_0^\theta e^{i(k-j)\phi} d\phi.$$

For $j \neq k$,

$$\int_\theta^{2\pi} e^{i(k-j)\phi} d\phi = - \int_0^\theta e^{i(k-j)\phi} d\phi = - \int_0^\theta e^{i(j-k)\phi} d\phi.$$

Therefore, if $j \neq k$ in the sum the term is negative. Thus

$$V_4 \leq 4(n - m)^2 \left( 1 - \frac{\theta}{2\pi} \right) \frac{\theta}{2\pi} \sum_{j=0}^{m-1} \binom{n - m + j}{j} \left( \int_{r_i}^{r_{i+1}} (1 - r^2)^{n - m - 1} r^{2j+1} dr \right)^2$$

$$\leq 4(n - m)^2 \left( 1 - \frac{\theta}{2\pi} \right) \frac{\theta}{2\pi} \sum_{j=0}^{m-1} \binom{n - m + j}{j} \left( \int_{r_i}^{r_{i+1}} (1 - r^2)^{n - m - 1} r^{2j+1} dr \right)^2$$

$$\times \int_0^\theta \mathbb{P}[X_j > j] \mathbb{P}[X_j \leq j]$$

$$\leq \left( 1 - \frac{\theta}{2\pi} \right) \frac{\theta}{2\pi} C_\alpha \sqrt{i \log(i)}.$$

All together then,

$$\text{Var}(N_i, \theta) = V_1 + V_2 + V_3 + V_4 \leq C_\alpha \sqrt{i \log(i)}$$

for a constant $C_\alpha$ depending only on $\alpha$. \qed
3. Concentration

We now move on to concentration for the counting functions \( N_{i,\theta} \). The key ingredient is the following general result on determinantal point processes.

**Theorem 8** (Hough–Krishnapur–Peres–Virág [2]). Let \( \Lambda \) be a locally compact Polish space and \( \mu \) a Radon measure on \( \Lambda \). Suppose that \( K : \Lambda \times \Lambda \to \mathbb{C} \) is the kernel of a determinantal point process, such that the corresponding integral operator \( \mathcal{K} : L^2(\mu) \to L^2(\mu) \) defined by

\[
[\mathcal{K} f](x) = \int_{\Lambda} K(x, y) f(y) d\mu(y)
\]

is self-adjoint, nonnegative, and locally trace-class. Let \( D \subseteq \Lambda \) be such that the restriction \( K_D(x, y) = 1_D(x)K(x, y)1_D(y) \) defines a trace-class operator \( \mathcal{K}_D \) on \( L^2(\mu) \). Then the number of points \( N_D \) lying in \( D \) of the process governed by \( K \) is distributed as \( \sum_k \xi_k \), where the \( \xi_k \) are independent Bernoulli random variables whose means are given by the eigenvalues of the operator \( \mathcal{K}_D \).

It is not hard to see that the kernel given in (1) has the properties required by Theorem 8 and so the random variable \( N_{i,\theta} \) is distributed as a sum of independent Bernoulli random variables. It is thus an immediate consequence of Bernstein’s inequality that

\[
\mathbb{P} \left[ |N_{i,\theta} - \mathbb{E} N_{i,\theta}| > t \right] \leq 2 \exp \left( -\frac{t^2}{2\sigma^2} \right).
\]

This is the key observation underlying the proof of Theorem 2.

**Proof of Theorem 2**. For the first claim, the assumption on \( i \) implies that

\[
A_{i,\theta} \subseteq \sqrt{\alpha} \left( 1 - \sqrt{\frac{2\log(m+1)}{m}} \right)^{1/2} D
\]

where \( D \) is the unit disc, so that by lemmas 5 and 7 together with Bernstein’s inequality, if \( t > 4 \), then

\[
\mathbb{P} \left[ |m\mu_{\alpha}(A_{i,\theta}) - N(A_{i,\theta})| \geq t \right] \leq \mathbb{P} \left[ |\mathbb{E} N(A_{i,\theta}) - N(A_{i,\theta})| \geq t - 4 \right] 
\]

\[
\leq \exp \left[ -\min \left\{ \frac{(t-4)^2}{C_{\alpha}i\sqrt{\log(i)}}, \frac{t-4}{2} \right\} \right].
\]

If \( t \geq 8 \), then \( t - 4 \geq t/2 \), so that

\[
\mathbb{P} \left[ |m\mu_{\alpha}(A_{i,\theta}) - N(A_{i,\theta})| \geq t \right] \leq \exp \left[ -\min \left\{ \frac{t^2}{C_{\alpha}i\sqrt{\log(i)}}, \frac{t}{4} \right\} \right];
\]

if \( t < 8 \), then

\[
\exp \left[ -\min \left\{ \frac{t^2}{C_{\alpha}i\sqrt{\log(i)}}, \frac{t}{4} \right\} \right] > e^{-2}
\]

and the first claim follows. The proof of the second claim is an immediate consequence of the second estimate of Lemma 5 together with Lemma 7.\[\Box\]
We now focus our attention on individual eigenvalues. Given $1 \leq p \leq m$, let $l = \lceil \sqrt{p} \rceil$ and $q = p - (l - 1)^2$, so that $p = (l - 1)^2 + q$ and $1 \leq q \leq 2l - 1$. Let

$$r_l = \frac{l}{\sqrt{n - m + l^2}}.$$ 

The predicted locations $\tilde{\lambda}_p$ for the eigenvalues are defined by

$$\tilde{\lambda}_p = r_{l-1}e^{2\pi i q/(2l-1)} = \frac{l - 1}{\sqrt{n - m + (l - 1)^2}} e^{2\pi i q/(2l-1)}.$$ 

To shed some light on this choice, consider the annulus $A_l$ with inner radius $r_{l-1}$ and outer radius $r_l$. Then

$$\mu_\alpha (A_l) = 2\pi \int_{r_{l-1}}^{r_l} \frac{1 - \alpha}{\pi \alpha (1 - r^2)^2} r dr$$

$$= \frac{1 - \alpha}{\alpha} \left[ \frac{r_l^2 - r_{l-1}^2}{(1 - r_l^2)(1 - r_{l-1}^2)} \right]$$

$$= \frac{1 - \alpha}{\alpha} \left[ \frac{n - (l-1)^2}{n - m + (l-1)^2} \frac{(l-1)^2}{n - m + l^2} \right]$$

$$= \frac{1 - \alpha}{\alpha} \left[ \frac{(n - m)(2l - 1)}{(n - m)^2} \right]$$

$$= \frac{2l - 1}{m}.$$ 

It follows from Lemma 5 that the expected number of eigenvalues in $A_l$ is approximately $2l - 1$.

**Proof of Theorem** The essential idea of the proof is that if $\lambda_p$ is far from its predicted location $\tilde{\lambda}_p$, then there is either a set of the form $A_\ell, \theta$ with substantially more eigenvalues than predicted by the mean (if $\lambda_p$ comes early) or a set of the form $A_\ell, \theta$ with substantially fewer eigenvalues than predicted by the mean (if $\lambda_p$ comes late). Theorem then gives control on the probabilities of such events.

To implement this strategy, several cases must be considered, which we first outline here.

1. $\lambda_p < \tilde{\lambda}_p$
   - (A) $\frac{s}{2l - 1} < \frac{2\pi q}{2l - 1}$
   - (B) $\frac{2\pi q}{2l - 1} \leq \frac{s}{2l - 1}$

2. $\lambda_p > \tilde{\lambda}_p$
   - (A) $\frac{s}{2l - 1} < 2\pi - \frac{2\pi q}{2l - 1}$
   - (B) $2\pi - \frac{2\pi q}{2l - 1} \leq \frac{s}{2l - 1} \leq \pi$
   - (C) $\pi < \frac{s}{2l - 1} \leq \frac{\sqrt{m(1 - \alpha(1 - \epsilon_m))^{1/2}} + l - 1}{2(l - 1)}$
   - (D) $\frac{\sqrt{m(1 - \alpha(1 - \epsilon_m))^{1/2}} + l - 1}{2(l - 1)} < \frac{s}{2l - 1}$
Combining cases (A) and (B) from both I and II yields the first part of the lemma (small $s$) and combining (C) and (D) of II gives the second part of the lemma (large $s$).

In most of the cases we will make use of the fact that

$$|Re^{i\theta} - re^{i\phi}| \leq ra(\theta, \phi) + |R - r|,$$

where $a(\theta, \phi)$ denotes the length of the shorter arc on the unit circle between $e^{i\theta}$ and $e^{i\phi}$.

**Case (I, A)** Suppose that $|\lambda_p - \tilde{\lambda}_p| \geq \frac{s}{\sqrt{n - m + (l - 1)^2}}$, that $\lambda_p \prec \tilde{\lambda}_p$, and that $\frac{s}{2(l - 1)} < \frac{2\pi q}{2l - 1}$.

We claim that

$$\lambda_p \prec r_{l-1} \exp \left[ i \left( \frac{2\pi q}{2l - 1} - \frac{s}{2(l - 1)} \right) \right].$$

Indeed, since $\lambda_p \prec \tilde{\lambda}_p$, either

(i) $r_{l-1} \leq |\lambda_p| < r_l$ and $\arg \lambda_p < \arg \tilde{\lambda}_p = \frac{2\pi q}{2l - 1}$ or

(ii) $|\lambda_p| < |\tilde{\lambda}_p| = r_{l-1}$.
If $|\lambda_p| < r_{l-1}$ holds, then the claim holds trivially. Otherwise, the estimate in (9) implies

$$|\lambda_p - \tilde{\lambda}_p| \leq r_{l-1} a \left( \arg \lambda_p, \frac{2\pi q}{2l-1} \right) + ||\lambda_p| - r_{l-1}|$$

$$\leq \frac{l-1}{\sqrt{n-m+(l-1)^2}} a \left( \arg \lambda_p, \frac{2\pi q}{2l-1} \right) + \frac{s}{2\sqrt{n-m+(l-1)^2}}.$$  

Therefore when condition (i) holds and $|\lambda_p - \tilde{\lambda}_p| \geq \frac{s}{\sqrt{n-m+(l-1)^2}}$, then

$$(10) \quad a \left( \arg \lambda_p, \frac{2\pi q}{2l-1} \right) \geq \frac{s}{2(l-1)}$$

and so $\arg \lambda_p < \frac{2\pi q}{2l-1} - \frac{s}{2(l-1)}$. In this case as well, then,

$$\lambda_p < r_{l-1} \exp \left[ i \left( \frac{2\pi q}{2l-1} - \frac{s}{2(l-1)} \right) \right].$$

It follows from the claim that

$$N_{l-1, \frac{2\pi q}{2l-1}, \frac{s}{2(l-1)}} \geq p.$$  

Now, the computation of $m\mu_\alpha (A_{l,\theta})$ in the proof of Corollary 6 gives that

$$m\mu_\alpha \left( A_{l-1, \frac{2\pi q}{2l-1} - \frac{s}{2(l-1)}} \right) = (l-1)^2 + q - \frac{s(2l-1)}{4\pi(l-1)} = p - \frac{s(2l-1)}{4\pi(l-1)} \leq p - \frac{s}{2\pi}.$$  

Then Theorem 2 implies that

$$\mathbb{P} \left[ N_{l-1, \frac{2\pi q}{2l-1}, \frac{s}{2(l-1)}} \geq p \right]$$

$$\leq \mathbb{P} \left[ N_{l-1, \frac{2\pi q}{2l-1}, \frac{s}{2(l-1)}} - m\mu_\alpha \left( A_{l-1, \frac{2\pi q}{2l-1} - \frac{s}{2(l-1)}} \right) \geq \frac{s}{2\pi} \right]$$

$$\leq 2 \exp \left[ - \min \left\{ \frac{s^2}{C_\alpha(l-1)\sqrt{\log(l-1)}}, \frac{s}{8\pi} \right\} \right]$$

$$= 2 \exp \left\{ - \frac{s^2}{C_\alpha l\sqrt{\log(l)}} \right\},$$

since $s \leq 2\pi(l-1)$.

(I, B) Suppose that $|\lambda_p - \tilde{\lambda}_p| \geq \frac{s}{\sqrt{n-m+(l-1)^2}}$, $\lambda_p < \tilde{\lambda}_p$ and $\frac{2\pi q}{2l-1} \leq \frac{s}{2(l-1)}$. We claim that

$$\lambda_p < \frac{l-2}{\sqrt{n-m+(l-2)^2}} \exp \left[ i \left( \frac{2\pi q}{2l-1} - \frac{s}{2(l-1)} \right) \right].$$

The estimate (9) implies condition (ii) above must hold; that is, $|\lambda_p| < r_{l-1}$. If $|\lambda_p| \geq r_{l-1} - \frac{s}{2\sqrt{n-m+(l-1)^2}}$, then the estimate (9) again implies that

$$\pi \geq a \left( \arg \lambda_p, \frac{2\pi q}{2l-1} \right) \geq \frac{s}{2(l-1)}.$$
In particular, when \( r_{l-2} \leq |\lambda_p| < r_{l-1}, \) \( \arg \lambda_p < 2\pi + \frac{2\pi q}{2l-1} - \frac{s}{2l-1}, \) If \( |\lambda_p| < r_{l-1} - \frac{s}{2\sqrt{n-m+(l-1)^2}} < r_{l-2}, \) then the estimate (9) implies

\[
|\lambda_p - \tilde{\lambda}_p| \leq r_{l-1}a \left( \arg \lambda_p, \frac{2\pi q}{2l-1} \right) + |\lambda_p| - r_{l-1} \leq \frac{l - 1}{\sqrt{n - m + (l - 1)^2}} a \left( \arg \lambda_p, \frac{2\pi q}{2l-1} \right) + \frac{4(l - 1) - s}{2\sqrt{n - m + (l - 1)^2}}.
\]

Then

\[
\frac{s}{2(l - 1)} \leq \frac{l - 2}{\sqrt{n - m + (l - 2)^2}} \exp \left[ i \left( 2\pi + \frac{2\pi q}{2l-1} - \frac{s}{2l-1} \right) \right],
\]

and

\[
N_{l-2,2\pi+\frac{2\pi q}{2l-1} - \frac{s}{2l-1}} \geq p.
\]

Now the computation in the proof of Corollary 6 yields

\[
m\mu_\alpha \left( A_{l-2,2\pi+\frac{2\pi q}{2l-1} - \frac{s}{2l-1}} \right) = (l - 1)^2 + \frac{q(2l - 3)}{2l - 1} - \frac{s(2l - 3)}{4\pi(l - 1)} \leq p - \frac{s}{4\pi}
\]

for \( l \geq 2. \) Therefore in this range of \( s, \) Theorem 2 implies that

\[
P \left[ N_{l-2,2\pi+\frac{2\pi q}{2l-1} - \frac{s}{2l-1}} \geq p \right] \leq P \left[ N_{l-2,2\pi+\frac{2\pi q}{2l-1} - \frac{s}{2l-1}} - m\mu_\alpha \left( A_{l-2,2\pi+\frac{2\pi q}{2l-1} - \frac{s}{2l-1}} \right) \geq \frac{s}{4\pi} \right]
\]

\[
\leq 2 \exp \left[ -\min \left\{ \frac{s^2}{C_\alpha(l - 2)\sqrt{\log(l - 2)}}, \frac{s}{16\pi} \right\} \right]
\]

\[
= 2 \exp \left[ -\frac{s^2}{C_\alpha l\sqrt{\log(l)}} \right].
\]

The estimates above cover the entire range of \( s \) when \( \lambda_p < \tilde{\lambda}_p \) and so

\[
P \left[ |\lambda_p - \tilde{\lambda}_p| \geq \frac{s}{\sqrt{n - m + (l - 1)^2}} : \lambda_p < \tilde{\lambda}_p \right] \leq 2 \exp \left\{ -\frac{s^2}{C_\alpha l\sqrt{\log(l)}} \right\}
\]

for all \( s > 0. \)

(II, A) Suppose that \( |\lambda_p - \tilde{\lambda}_p| \geq \frac{s}{\sqrt{n - m + (l - 1)^2}}, \) that \( \lambda_p > \tilde{\lambda}_p, \) and that \( \frac{s}{2l-1} < 2\pi - \frac{2\pi q}{2l-1}. \)

We claim that

\[
\lambda_p \geq \frac{l - 1}{\sqrt{n - m + (l - 1)^2}} \exp \left[ i \left( \frac{2\pi q}{2l - 1} + \frac{s}{2(2l-1)} \right) \right].
\]
Indeed, since $\lambda_p \succ \check{\lambda}_p$, either

(i) $r_{l-1} \leq |\lambda_p| < r_l$ and $\arg \lambda_p > \arg \check{\lambda}_p$ or

(ii) $|\lambda_p| \geq r_l = \frac{l}{\sqrt{n-m+l}}$.

If $|\lambda_p| \geq r_l$ holds, then the claim is trivially true. Suppose that condition (i) holds. Then for $s \geq 2$, $|\lambda_p| < r_l < r_{l-1} + \frac{s}{2\sqrt{n-m+(l-1)^2}}$. As above, combining this observation with the fact that $|\lambda_p - \check{\lambda}_p| \geq \frac{s}{\sqrt{n-m+(l-1)^2}}$ and the estimate (9) implies $a\left(\arg \lambda_p, \frac{2\pi q}{2l-1}\right) \geq \frac{s}{2(l-1)}$. It follows that if condition (i) holds, then $\arg \lambda_p > \frac{2\pi q}{2l-1} + \frac{s}{2(l-1)}$, and so

$$\lambda_p > \frac{l-1}{\sqrt{n-m+(l-1)^2}} \exp\left[i\left(\frac{2\pi q}{2l-1} + \frac{s}{2(l-1)}\right)\right].$$

It follows from the claim that

$$N_{l-1, \frac{2\pi q}{2l-1} + \frac{s}{2(l-1)}} < p.$$
By the proof of Corollary (6),

$$m\mu_\alpha \left( A_{l-1, \frac{2\pi q}{2l-1} + \frac{s}{2(l-1)}} \right) = p + \frac{s}{2\pi} \frac{2l-1}{2l-2} \geq p + \frac{s}{2\pi},$$

and so Theorem 2 implies that

$$\Pr \left[ N_{l-1, \frac{2\pi q}{2l-1} + \frac{s}{2(l-1)}} < p \right] \leq \Pr \left[ m\mu_\alpha \left( A_{l-1, \frac{2\pi q}{2l-1} + \frac{s}{2(l-1)}} \right) - N_{l-1, \frac{2\pi q}{2l-1} + \frac{s}{2(l-1)}} > \frac{s}{2\pi} \right]$$

$$\leq 2 \exp \left[ - \min \left\{ \frac{s^2}{C_\alpha (l-1) \sqrt{\log(l-1)}}, \frac{s}{8\pi} \right\} \right]$$

$$= 2 \exp \left[ - \frac{s^2}{C_\alpha (l-1) \sqrt{\log(l-1)}} \right],$$

since $s \leq 2\pi(l-1)$.

(II, B) Suppose that $|\lambda_p - \tilde{\lambda}_p| \geq \frac{s}{\sqrt{n-m+(l-1)^2}}$, that $\lambda_p > \tilde{\lambda}_p$ and that $2\pi - \frac{2\pi q}{2l-1} \leq \frac{s}{2(l-1)} \leq \pi$. We claim that

$$\lambda_p > \frac{l}{\sqrt{n-m+l^2}} \exp \left[ i \left( \frac{2\pi q}{2l-1} + \frac{s}{2(l-1)} - 2\pi \right) \right].$$

By the estimate (9) again, it must be the case that condition (ii) holds and $|\lambda_p| \geq r_l$. If $s \geq 2$ and $|\lambda_p| \geq r_{l-1} + \frac{s}{2\sqrt{n-m+(l-1)^2}} > r_{l+1}$, then the claim holds trivially. If $|\lambda_p| < r_{l-1} + \frac{s}{2\sqrt{n-m+(l-1)^2}}$, then the estimate (9) yields $\arg \lambda_p > \frac{2\pi q}{2l-1} + \frac{s}{2(l-1)} - 2\pi$. In particular, if $r_l \leq |\lambda_p| < r_{l+1}$, then $\arg \lambda_p > \frac{2\pi q}{2l-1} + \frac{s}{2(l-1)} - 2\pi$ and so

$$\lambda_p > \frac{l}{\sqrt{n-m+l^2}} \exp \left[ i \left( \frac{2\pi q}{2l-1} + \frac{s}{2(l-1)} - 2\pi \right) \right].$$

It follows from the claim that

$$N_{l, \frac{2\pi q}{2l-1} + \frac{s}{2(l-1)} - 2\pi} < p.$$

Since

$$m\mu_\alpha \left( A_{l, \frac{2\pi q}{2l-1} + \frac{s}{2l-1} - 2\pi} \right) = l^2 + \frac{2\pi q}{2l-1} + \frac{s}{2(l-1)} - \frac{2\pi}{2\pi} (2l+1) \geq p + \frac{s}{2\pi},$$
Theorem 2 implies that in this regime,

\[
P\left[\mathcal{N}_{t,2\pi} < 2\pi \right]
\leq P\left[ \mu_{\alpha} \left( A_{t,2\pi} \right) - \mathcal{N}_{t,2\pi} > \frac{s}{2\pi} \right]
\leq 2 \exp \left[ -\min\left\{ \frac{s^2}{C \alpha \sqrt{\log(l)}}, \frac{s}{8\pi} \right\} \right]
\]

Combining cases (I, A), (I, B), (II, A), and (II, B) thus yields

\[
P\left[ |\lambda_p - \tilde{\lambda}_p| \geq \frac{s}{\sqrt{n-m+(l-1)^2}} \right] \leq 2 \exp \left[ -\frac{s^2}{C \alpha \sqrt{\log(l)}} \right].
\]

when \( s \leq 2\pi(l-1) \). This proves the first part of the Theorem (small \( s \)).

Finally, we consider cases for the larger values of \( s \) based on which part of Theorem 2 applies.

(C) Let \( \epsilon_m = \sqrt{\frac{2\log(m+1)}{m}} \) and suppose that

\[
|\lambda_p - \tilde{\lambda}_p| \geq \frac{s}{\sqrt{n-m+(l-1)^2}},
\]

that \( \lambda_p > \tilde{\lambda}_p \), and that

\[
2\pi(l-1) \leq s \leq \sqrt{m} \left( 1 - \frac{\epsilon_m}{1 - \alpha(1-\epsilon_m)} \right)^{\frac{1}{2}} + l - 1.
\]

(That is, \( s - l + 1 \leq \sqrt{m} \left( 1 - \frac{\epsilon_m}{1 - \alpha(1-\epsilon_m)} \right)^{\frac{1}{2}} \).) By the triangle inequality,

\[
|\lambda_p| \geq \frac{s}{\sqrt{n-m+(l-1)^2}} - |\tilde{\lambda}_p| = \frac{s-l+1}{\sqrt{n-m+(l-1)^2}}.
\]

Now, \( \frac{s}{2(l-1)} > \pi \) implies \((l-1)^2 \leq (s-l+1)^2\). It follows that

\[
|\lambda_p| \geq \frac{s-l+1}{\sqrt{n-m+(s-l+1)^2}};
\]

so

\[
\mathcal{N}_{s-l+1,2\pi} < p.
\]

Since \( p = (l-1)^2 + q, 1 \leq q \leq 2l - 1 \), and \( l < \frac{s}{2\pi} + 1 \),

\[
m\mu_{\alpha} \left( A_{s-l+1,2\pi} \right) = \left( [s-l+1]^2 + 2[s-l+1] + 1 \right)
\geq s^2 - 2sl + l^2 + 2s - 2l + 1
\geq s^2 - 2s(l-1) + p - q \geq cs^2 + p.
\]
since $s \geq 2\pi$. It follows from Theorem 2
$$\mathbb{P} \left[ N_{[s-l+1,2\pi]} < p \right] \leq \mathbb{P} \left[ m\mu_\alpha \left( A_{[s-l+1,2\pi]} \right) - N_{[s-l+1,2\pi]} > cs^2 \right]$$
$$\leq 2 \exp \left[ - \min \left\{ \frac{c^2 s^4}{C_\alpha([s-l+1]) \sqrt{\log([s-l+1])}}, \frac{cs^2}{2} \right\} \right]$$
$$\leq 2 \exp \left[ -c_\alpha s^2 \right],$$
since $s \geq 2\pi(l-1)$.

(D) Suppose that $|\lambda_p - \tilde{\lambda}_p| \geq \frac{s}{\sqrt{n-m+(l-1)^2}}$, that $\lambda_p \succ \tilde{\lambda}_p$, and that
$$\frac{s}{\sqrt{n-m+(l-1)^2}} \geq \frac{s}{2(l-1)};$$
that is,
$$s \geq \frac{s}{2(l-1)}.$$

As in the previous case,
$$N(A_{[s-l+1,2\pi]} < p).$$

If $\frac{s-l+1}{\sqrt{n-m+(s-l+1)^2}} \geq 2$, then $\mathbb{P} \left[ N(A_{[s-l+1,2\pi]} < p) = 0 \right]$. Otherwise, by the second estimate in Lemma 5
$$\mathbb{P} \left[ N(A_{[s-l+1,2\pi]} < p) \right]$$
$$\leq \mathbb{P} \left[ \mathbb{E} N(A_{[s-l+1,2\pi]} - N(A_{[s-l+1,2\pi]} > m\mu_\alpha \left( A_{[s-l+1,2\pi]} \right) - \frac{6\sqrt{2m \log(m+1)}}{1-\alpha} - p \right]$$
$$\leq \mathbb{P} \left[ \mathbb{E} N(A_{[s-l+1,2\pi]} - N(A_{[s-l+1,2\pi]} > cs^2 - \frac{6\sqrt{2m \log(m+1)}}{1-\alpha} \right].$$

In this range, $s^2 \geq m \left( 1 - \frac{\epsilon_m}{1-\alpha(1-\epsilon_m)} \right)$, so the lower bound can be replaced, for large enough $m$, by $cs^2$ by slightly reducing the value of $c$. Theorem 2 applied with $t = cs^2 \geq \frac{12\sqrt{2m \log(m)}}{1-\alpha}$ (again for $m$ large enough) then yields
$$\mathbb{P} \left[ N(A_{[s-l+1,2\pi]} < p) \right] \leq 2 \exp \left[ - \min \left\{ \frac{c^2 s^4}{C_\alpha([s-l+1]) \sqrt{\log([s-l+1])}}, \frac{cs^2}{2} \right\} \right]$$
$$\leq 2 \exp \left[ -c_\alpha s^2 \right].$$

Cases (C) and (D) thus yield
$$\mathbb{P} \left[ |\lambda_p - \tilde{\lambda}_p| \geq \frac{s}{\sqrt{n-m+(l-1)^2}} \right] \leq 2 \exp \left[ -c_\alpha s^2 \right].$$
for $s \geq 2\pi(l-1)$. Finally, the empirical spectral measure is supported on the disc of radius 1. It follows that if $s \geq 2\sqrt{n-m+(l-1)^2}$, then
\[ \mathbb{P}\left[ |\lambda_p - \tilde{\lambda}_p| \geq \frac{s}{\sqrt{n-m+(l-1)^2}} \right] = 0. \]
This completes the proof. \[ \square \]

Proof of Corollary 4. Let $\epsilon_m = \sqrt{\frac{2\log(m+1)}{m}}$ and $p$ be such that
\[ 2 \leq l = \left\lfloor \sqrt{p} \right\rfloor \leq \sqrt{m} (1 - \epsilon_m) \frac{1}{1 - \alpha(1 - \epsilon_m)} \]
Then by Fubini’s theorem and Theorem 3,
\[ \begin{align*}
\text{Var}(\lambda_p) &\leq \mathbb{E} \left| \lambda_p - \tilde{\lambda}_p \right|^2 \\
&= \int_0^\infty 2t \mathbb{P}\left[ \left| \lambda_p - \tilde{\lambda}_p \right| > t \right] dt \\
&= \frac{2}{n-m+(l-1)^2} \int_0^\infty s \mathbb{P}\left[ \left| \lambda_p - \tilde{\lambda}_p \right| > \frac{s}{\sqrt{n-m+(l-1)^2}} \right] ds \\
&\leq \frac{2}{n-m+(l-1)^2} \left[ \int_0^{2\pi(l-1)} 2se^{-\frac{s^2}{c_\alpha\sqrt{\log(l+1)}}} ds + \int_{2\pi(l-1)}^{2\sqrt{n-m+(l-1)^2}} 2se^{-c_\alpha s^2} ds \right] \\
&\leq \frac{2}{n-m+(l-1)^2} \left[ \int_0^\infty 2se^{-\frac{s^2}{c_\alpha\sqrt{\log(l+1)}}} ds + \int_{2\pi(l-1)}^\infty 2se^{-c_\alpha s^2} ds \right] \\
&\leq \frac{C_\alpha}{n-m+(l-1)^2} \left[ \frac{l\sqrt{\log(l+1)}}{n} \right] \\
&\leq C_\alpha \frac{l\sqrt{\log(l+1)}}{n},
\end{align*} \]
since $(l-1)^2 \leq m$. \[ \square \]

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