EQUILATERAL WEIGHTS ON THE UNIT BALL OF $\mathbb{R}^n$

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Abstract. An equilateral set (or regular simplex) in a metric space $X$, is a set $A$ such that the distance between any pair of distinct members of $A$ is a constant. An equilateral set is standard if the distance between distinct members is equal to 1. Motivated by the notion of frame-functions, as introduced and characterized by Gleason in [6], we define an equilateral weight on a metric space $X$ to be a function $f : X \to \mathbb{R}$ such that $\sum_{i \in I} f(x_i) = W$, for every maximal standard equilateral set $\{x_i : i \in I\}$ in $X$, where $W \in \mathbb{R}$ is the weight of $f$. In this paper we characterize the equilateral weights associated with the unit ball $B^n$ of $\mathbb{R}^n$ as follows: For $n \geq 2$, every equilateral weight on $B^n$ is constant.

1. Introduction

Equilateral sets have been extensively studied in the literature for a number of metric spaces [2]. An equilateral set (or regular simplex) in a metric space $X$, is a set $A$ such that the distance between any pair of distinct members of $A$ is $\rho$, where $\rho \neq 0$ is a constant. The equilateral dimension of $X$ is defined to be $\sup \{|A| : A \text{ is an equilateral set in } X\}$.

Suppose that $\{x_1, \ldots, x_k\}$ is an equilateral set in $\mathbb{R}^n$ (equipped with the $\ell_2$-norm). Then the vectors $v_i := x_{i+1} - x_1$ for $i = 1, \ldots, k-1$ are linearly independent. Indeed, let $A$ be the $(k-1) \times (k-1)$ matrix $(a_{ij})$ defined by $a_{ij} := \langle v_i, v_j \rangle$. Then $a_{ij} = \frac{\rho^2}{2}(1 + \delta_{ij})$ where $\rho \neq 0$ is a constant and $\delta_{ij}$ is the Kronecker delta. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$ and let $B$ be the $n \times (k-1)$ matrix $(b_{ij})$ defined by $b_{ij} := \langle v_j, e_i \rangle$. Since $A = B^*B$ and $A$ is clearly non-singular, we deduce that $B$ is non-singular, i.e. the vectors $v_i := x_{i+1} - x_1$ for $i = 1, \ldots, k-1$ are linearly independent and therefore $k \leq n + 1$. To see that the equilateral dimension of $\mathbb{R}^n$ (equipped with the $\ell_2$-norm) is $n + 1$ observe that the set $\{x_1 - c, \ldots, x_k - c\}$ where $c := \frac{1}{k} \sum_{i=1}^k x_i$ has linear dimension $k-1$ and so if $k < n + 1$, there exists a unit vector $u \in \mathbb{R}^n$ such that $u \perp x_i - c$ for each $i = 1, \ldots, k$, and therefore the set $\{x_1, \ldots, x_k\}$ can be enlarged to a bigger equilateral set in $\mathbb{R}^n$. Let us only mention here that the situation is far more complicated for the other $\ell_p$-norms [11, 9, 1] (and others).

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An equilateral set in $\mathbb{R}^n$ is standard if the distance between distinct points is equal to 1. If $\{x_1, \ldots, x_k\}$ is a standard equilateral set in $\mathbb{R}^n$, its centre $\frac{1}{k} \sum_{i=1}^k x_i$ will be denoted by $c(x_1, \ldots, x_k)$. The radius of $\{x_1, \ldots, x_k\}$ is $\|x_i - c(x_1, \ldots, x_k)\|$ and is denoted by $\beta_k$. A simple calculation yields

$$\beta_k = \left\| x_i - c(x_1, \ldots, x_k) \right\| = \frac{1}{k} \sum_{1 \leq j \leq k, j \neq i} (x_j - x_i) = \frac{1}{k} \sqrt{k - 1 + \frac{(k - 1)(k - 2)}{2}} = \sqrt{\frac{k - 1}{2k}}.$$  

If $x_{k+1}$ is another point in $\mathbb{R}^n$ such that $\{x_1, \ldots, x_k, x_{k+1}\}$ is again a standard equilateral set, then $x_{k+1} - c(x_1, \ldots, x_k)$ is orthogonal to $x_i - c(x_1, \ldots, x_k)$ for every $i = 1, \ldots, k$, and thus

$$\left\| x_{k+1} - c(x_1, \ldots, x_k) \right\| = \sqrt{1 - \beta_k^2} = \sqrt{\frac{k + 1}{2k}}.$$  

We will call $\alpha_{k+1} := \sqrt{\frac{k + 1}{2k}}$ the perpendicular height of $\{x_1, \ldots, x_k, x_{k+1}\}$.

We shall now introduce the notion of equilateral weights. The motivation behind this definition is the notion of frame functions. These were introduced and characterized by Gleason [6] in his famous theorem describing the measures on the closed subspaces of a Hilbert space. Gleason’s Theorem is of utmost importance in the laying down of the foundations of quantum mechanics [12, 10, 7, 4, 8] (and others). Let $S(0,1)$ denote the unit sphere of a Hilbert space $H$. A function $f : S(0,1) \to \mathbb{R}$ is called a frame function on $H$ if there is a number $w(f)$, called the weight of $f$, such that $\sum_{i \in I} f(u_i) = w(f)$ for every orthonormal basis $\{u_i : i \in I\}$ of $H$. We recall that a bounded operator $T$ on $H$ is of trace-class if the series $\sum_{i \in I} \langle Tu_i, u_i \rangle$ converges absolutely for any orthonormal basis $\{u_i : i \in I\}$ of $H$. (It is well-known that if the series converges for an orthonormal basis $\{u_i : i \in I\}$ then it converges for any orthonormal basis and the sum does not depend on the choice of the basis.) Clearly, if $T$ is self-adjoint and of trace-class the function $f_T(x) = \langle Tx, x \rangle$ ($x \in S(0,1)$) defines a continuous frame function on $H$. Gleason’s Theorem says that when $\dim H \geq 3$ every bounded frame function arises in this way. The heart of the proof of Gleason’s Theorem is the treatment of the case when $H$ is the real three-dimensional Hilbert space $\mathbb{R}^3$. In fact all the other cases can be reduced to this case. Thus, as a matter of fact, it can be said that the crux of this theorem can be rendered to the following statement: For every bounded frame function $f$ on $\mathbb{R}^3$ there exists a symmetric matrix $T$ on $\mathbb{R}^3$ such that $f(u) = \langle Tu, u \rangle$ for every unit vector $u \in \mathbb{R}^3$. The notion of frame functions and the fact that an orthonormal basis of $\mathbb{R}^3$...
is simply a maximal equilateral set on the unit sphere of \( \mathbb{R}^3 \), suggest the following definition:

**Definition 1.1.** Let \( X \) be a metric space and let \( W \in \mathbb{R} \). An equilateral weight on \( X \) with weight \( W \) is a function \( f : X \to \mathbb{R} \) such that

\[
\sum_{i \in I} f(x_i) = W
\]

whenever \( \{x_i : i \in I\} \) is a maximal standard equilateral set in \( X \).

Given a metric space, can one describe the equilateral weights associated with it?

**Example 1.2.** Every equilateral weight on \( \mathbb{R}^2 \) is constant. First observe that for every pair of points \( x \) and \( y \) in \( \mathbb{R}^2 \) there are points \( x_1, x_2, \ldots, x_n \) in \( \mathbb{R}^2 \) such that \( \|x_1 - x\| = \|x_{i+1} - x_i\| = \|y - x_n\| = 1 \) for every \( i = 1, \ldots, n - 1 \). Thus, it suffices to to show that \( f(x) = f(y) \) for all \( x, y \in \mathbb{R}^2 \) satisfying \( \|x - y\| = 1 \). Let \( x, y \in \mathbb{R}^2 \) such that \( \|x - y\| = 1 \). Observe that if \( \{a, b, c\} \) and \( \{d, b, c\} \) are the vertices of two unit equilateral triangles and \( f \) is an equilateral weight, then \( f(a) = f(d) \). Thus, \( f \) takes the constant value \( f(x) \) on the circle with centre \( x \) and radius \( \sqrt{3} \), and the constant value \( f(y) \) on the circle with centre \( y \) and radius \( \sqrt{3} \). Since these circles intersect, it follows that \( f(x) = f(y) \). Using a similar argument but replacing \( \sqrt{3} \) with \( 2\alpha_{n+1} \), one can easily show that every equilateral weight on \( \mathbb{R}^n \) is constant. The same cannot be said for \( \mathbb{R} \) – it is easy to find non-trivial equilateral weights on \( \mathbb{R} \).

**Example 1.3.** Let \( S \) be the sphere in a Hilbert space \( H \) with centre 0 and radius \( 1/\sqrt{2} \). Two vectors \( u \) and \( v \) in \( S \) satisfy \( \|u - v\| = 1 \) if, and only if, \( \langle u, v \rangle = 0 \). Thus, each maximal standard equilateral set in \( S \) corresponds to a rescaling of some orthonormal basis of \( H \) by a factor of \( 1/\sqrt{2} \). It is clear therefore that the equilateral weights on \( S \) correspond to the frame-functions on \( H \) (composite with a rescaling by a factor of \( \sqrt{2} \)). Thus, in view of Gleason’s Theorem if \( \dim H \geq 3 \) and \( f \) is a bounded equilateral weight on \( S \), there exists a self-adjoint, trace-class operator \( T \) such that

\[
f(u) = \langle Tu, u \rangle
\]

for all \( u \in S \). Let us emphasize that such a description does not hold when \( \dim H = 2 \) and that the assumption of boundedness is not redundant when \( \dim H \) is finite. It known that \( \mathbb{R}^n \) admits frame functions that are unbounded and that therefore cannot be described by such an equation (see [4, Proposition 3.2.4]).

By contrast, the boundedness assumption is superfluous when the space is infinite dimensional. This surprising result is due to Dorofeev and Sherstnev [3] and allows us to describe the equilateral weights.
associated with the metric space $S$ of an infinite dimensional Hilbert space directly from Gleason’s Theorem.

**Proposition 1.4.** Let $H$ be an infinite dimensional Hilbert space and let $S$ be the sphere in $H$ with centre 0 and radius $1/\sqrt{2}$. If $f$ is an equi- lateral weight on $S$, then there exists a self-adjoint, trace-class operator $T$ on $H$ such that $f(u) = \langle Tu, u \rangle$ for every vector $u$ in $S$.

The aim of the present paper is to describe the equilateral weights associated with another bounded metric space; namely the unit ball of $\mathbb{R}^n$.

2. **Standard equilateral sets in the unit ball of $\mathbb{R}^n$**

In what follows we will be interested in standard equilateral sets contained in the (closed) unit ball of $\mathbb{R}^n$, denoted by $B^n$. It is clear that the equilateral dimension of $B^n$ is equal to that of $\mathbb{R}^n$. We start by exhibiting some properties of standard equilateral sets in $B^n$.

**Proposition 2.1.** Let $\{x_1, \ldots, x_k\}$ ($k \leq n+1$) be a standard equilateral set in $B^n$. Then $\|c(x_1, \ldots, x_k)\| \leq \alpha_{k+1}$.

**Proof.** First observe that

$$2\langle x_i, x_j \rangle = \|x_i\|^2 + \|x_j\|^2 - \|x_i - x_j\|^2 \leq 1,$$

and therefore

$$\|c(x_1, \ldots, x_k)\|^2 = k^{-2}\left(\sum_{i=1}^{k} x_i, \sum_{i=1}^{k} x_i\right) = k^{-2}\left[\sum_{i=1}^{k} \|x_i\|^2 + \sum_{1 \leq i, j \leq k, i \neq j} \langle x_i, x_j \rangle\right] \leq k^{-2}\left[k + \frac{k(k - 1)}{2}\right] = \alpha_{k+1}^2. \quad \square$$

In the extremal case $k = n + 1$ the bound obtained in Proposition 2.1 can be improved as shown in the next Proposition. This improvement is needed to prove Proposition 2.4. We first prove a lemma.

**Lemma 2.2.** Let $\{x_1, x_2, \ldots, x_{n+1}\}$ be a maximal standard equilateral set in $\mathbb{R}^n$ with centre at the origin and let $x \in \mathbb{R}^n$ satisfy $\langle x, x_i \rangle \geq 0$ for $i = 2, 3, \ldots, n + 1$. If $\|x\| \geq 1$, then $\langle x, x_2 + x_3 + \cdots + x_{n+1} \rangle \geq 1/2$.

**Proof.** Let $v := x_2 + x_3 + \cdots + x_{n+1}$ and let

$K := \{x \in \mathbb{R}^n : \langle x, v \rangle \leq 1/2, \langle x, x_i \rangle \geq 0 \text{ for each } i = 2, 3, \ldots, n + 1\}$. 

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$K$ is the intersection of half-spaces and therefore a point of $K$ is an extreme point if and only if it is the intersection of $n$ hyperplanes whose normals form a basis of $\mathbb{R}^n$. Using the fact that $\langle x_i, x_j \rangle$ is independent of $i, j$ (when $i \neq j$) it is easy to see that the extreme points of $K$ are $\{0, x_2 - x_1, x_3 - x_1, \ldots, x_{n+1} - x_1\}$. The norm, being a strictly convex function, i.e.

$$\|\lambda x + (1 - \lambda)y\| < \max(\|x\|, \|y\|), \quad x \neq y, \quad 0 < \lambda < 1 \quad (\star)$$

takes a maximum value at an extremal point and therefore, since $\|x_i - x_1\| = 1 \ (i = 2, 3, \ldots, n + 1)$, it follows that $\|x\| \leq 1$ for every $x \in K$.

From the strict inequality of $(\star)$ and from the fact that each of the vectors $x_i - x_1 \ (i = 2, 3, \ldots, n + 1)$ lies in the hyperplane $\langle x, v \rangle = 1/2$, it follows that if $x \in \mathbb{R}^n$ satisfies $\langle x, x_i \rangle \geq 0 \ (i = 2, 3, \ldots, n + 1)$ and $\langle x, v \rangle < 1/2$, then $\|x\| < 1$.

\begin{proposition}
Let $\{u_1, \ldots, u_{n+1}\}$ be a standard equilateral set in $B^n$. Then $\|c(u_1, \ldots, u_{n+1})\| \leq \beta_{n+1}$. \\
\begin{proof}
Let $\{u_1, u_2, \ldots, u_{n+1}\}$ be a maximal standard equilateral set in $B^n$. Then $\{0, u_2 - u_1, \ldots, u_{n+1} - u_1\}$ is again a maximal standard equilateral set in $B^n$. Let us denote its centre by $c$. Note that $\|c\| = \beta_{n+1}$.

For each $i = 1, 2, \ldots, n+1$, let $x_i := u_i - u_1 - c$. Then $\{x_1, x_2, \ldots, x_{n+1}\}$ is a maximal standard equilateral set with centre at the origin. Note that

$$c(u_1, u_2, \ldots, u_{n+1}) = c(x_1, x_2, \ldots, x_{n+1}) + u_1 + c = u_1 + c.$$ 

Thus

$$\|c(u_1, u_2, \ldots, u_{n+1})\|^2 = \|u_1 + c\|^2 = \|u_1\|^2 + \|c\|^2 + 2\langle u_1, c \rangle,$$

and therefore for the proposition to hold we require

$$\left\langle \frac{-u_1}{\|u_1\|}, c \right\rangle \geq \frac{\|u_1\|}{2}. \quad (\star)$$

To this end we calculate

$$1 \geq \|u_1\|^2 = \|x_i + c\|^2 + \|u_1\|^2 + 2\langle u_1, x_i + c \rangle$$

$$= 1 + \|u_1\|^2 + 2\langle u_1, x_i \rangle + 2\langle u_1, c \rangle$$

which implies

$$\left\langle \frac{-u_1}{\|u_1\|}, x_i \right\rangle \geq \frac{\|u_1\|}{2} - \left\langle \frac{-u_1}{\|u_1\|}, c \right\rangle. \quad (\star\star)$$

for each $i = 2, 3, \ldots, n + 1$. Now, if the right hand side of $(\star\star)$ is $\leq 0$, then $(\star)$ is satisfied. On the other hand, if the right hand side of $(\star\star)$ is greater than $0$, then Lemma 2.2 can be applied to conclude

$$\frac{\|u_1\|}{2} \leq \frac{1}{2} \leq \left\langle \frac{-u_1}{\|u_1\|}, x_2 + x_3 + \cdots + x_{n+1} \right\rangle = \left\langle \frac{-u_1}{\|u_1\|}, -x_1 \right\rangle = \left\langle \frac{-u_1}{\|u_1\|}, c \right\rangle,$$

which completes the proof.
\end{proof}
\end{proposition}
Proposition 2.4. Every standard equilateral set in $B^n$ can be enlarged to one having size $n+1$ such that its members all lie in $B^n$.

Proof. Let $\{x_1, \ldots, x_k\}$ ($1 \leq k \leq n$) be a standard equilateral set in $B^n$. We show that there exists a vector $x_{k+1} \in B^n$ such that $\{x_1, \ldots, x_k, x_{k+1}\}$ is a standard equilateral set. The proof will then follow by induction.

Let $N := \text{span}\{x_i - c(x_1, \ldots, x_k) : 1 \leq i \leq k\}$ and set $a := (I - P_N)c(x_1, \ldots, x_k)$, where $P_N$ is the projection of $\mathbb{R}^n$ into $N$ and $I$ is the identity. The intersection of $B^n$ with the translation $a + N$ is a $(k-1)$-dimensional ball with centre $a$ and radius $\sqrt{1 - \|a\|^2}$. The set $\{x_1, \ldots, x_k\}$ is a standard equilateral set in $(a + N) \cap B^n$ and thus, in view of Proposition 2.3, it follows that $\|c(x_1, \ldots, x_k) - a\| \leq \beta_k$.

Set $u := -\alpha_{k+1}v$, where $v := a/\|a\|$ if $a \neq 0$ and any unit vector in $N^\perp$ if $a = 0$. Then $\|a + u\| \leq \|u\| = \alpha_{k+1}$ since $\alpha_{k+1} \geq \beta_k = \|c(x_1, \ldots, x_k)\| \geq \|a\|$. Put $x_{k+1} := c(x_1, \ldots, x_k) + u$. The set $\{x_1, \ldots, x_k, x_{k+1}\}$ is a standard equilateral set in $\mathbb{R}^n$. Moreover,

$$
\|x_{k+1}\|^2 = \|c(x_1, \ldots, x_k) + u\|^2 = \|c(x_1, \ldots, x_k) - a\|^2 + \|a + u\|^2 \\
\leq \beta_k^2 + \alpha_{k+1}^2 = 1.
$$

\[\square\]

3. Equilateral weights on $B^n$

In this section we shall prove that the only admissible equilateral weights on the unit ball of $\mathbb{R}^n$ are those that take a constant value.

For any linear subspace $M$ of $\mathbb{R}^n$, $a \in M$ and $r > 0$, we denote the closed ball in $M$ with centre $a$ and radius $r$ by $B^M(a, r)$, i.e. $B^M(a, r) = \{x \in M : \|x - a\| \leq r\}$. We will also denote by $S^M(a, r)$ the sphere in $M$ with centre $a$ and radius $r$, i.e. $S^M(a, r) = \{x \in M : \|x - a\| = r\}$. We will write $B(a, r)$ (resp. $S(a, r)$) instead of $B^{\mathbb{R}^n}(a, r)$ (resp. $S^{\mathbb{R}^n}(a, r)$). We will need the following definition.

Definition 3.1. Let $a, b \in B^n$, $a \neq b$ and $N := (b - a)^\perp$. For any subspace $M \neq \{0\}$ of $\mathbb{R}^n$ define

$$
\gamma^M(a, b) := \sup \left\{ r > 0 : \frac{a + b}{2} + B^{M \cap N}(0, r) \subseteq B^n \right\}.
$$

Note that the set involved in the definition of $\gamma^M(a, b)$ is not empty and bounded above by 1. Instead of $\gamma^{\mathbb{R}^n}(a, b)$ we will simply write $\gamma(a, b)$. It is easy to see that $\gamma^M(a, b)$ is in fact equal to the maximum of the set of its definition. In addition, if $M_1$ and $M_2$ are subspaces of $\mathbb{R}^n$ such that $M_1 \subseteq M_2$, then $\gamma^{M_2}(a, b) \leq \gamma^{M_1}(a, b)$. The motivation behind this definition lies in the following observation.
Lemma 3.2. Let \( a, b \in B^n \) such that \( \|b-a\| = 2\alpha_{n+1} \) and \( \gamma(a, b) \geq \beta_n \). Then \( f(a) = f(b) \) for every equilateral weight \( f \) on \( B^n \).

Proof. Let \( N := (b-a)^\perp \) and let \( \{x_1, \ldots, x_n\} \) be a standard equilateral set in 
\[
\frac{a + b}{2} + S^N(0, \beta_n) \subseteq B^n.
\]
Each \( x_i \) can be written as \( (a + b)/2 + n_i \), where \( n_i \in N \) and \( \|n_i\| = \beta_n \). Thus,
\[
\|x_i - a\|^2 = \left\| \frac{b-a}{2} + n_i \right\|^2 = \alpha_{n+1}^2 + \beta_n^2 = 1.
\]
Similarly, \( \|x_i - b\| = 1 \), i.e. \( \{a, x_1, \ldots, x_n\} \) and \( \{b, x_1, \ldots, x_n\} \) are maximal standard equilateral sets in \( B^n \), and therefore 
\[
f(a) + \sum_{i=1}^n f(x_i) = f(b) + \sum_{i=1}^n f(x_i),
\]
for every equilateral weight \( f \) on \( B^n \). \( \square \)

Lemma 3.3. Let \( a, b \in B^n \), \( a \neq b \) and let \( T \) be a two-dimensional subspace of \( \mathbb{R}^n \) containing \( a \) and \( b \). Then \( \gamma^T(a, b) = \gamma(a, b) \).

Proof. We show that \( \gamma(a, b) \geq \gamma^T(a, b) \). Let \( u \) be a unit vector in \( T \) such that \( \langle u, b-a \rangle = 0 \) and \( \langle u, b+a \rangle \geq 0 \). Set \( x_0 := (a+b)/2 \). Let \( r > 0 \) such that \( \|x_0 + ru\| \leq 1 \) and let \( x \in (b-a)^\perp \) such that \( \|x\| \leq r \). Then \( P_T x = \lambda u \) where \( |\lambda| \leq \|x\| \leq r \). Hence
\[
\|x_0 + x\|^2 = \|x_0\|^2 + \|x\|^2 + 2 \langle x_0, x \rangle \\
\leq \|x_0\|^2 + \|x\|^2 + 2 |\langle P_T x_0, x \rangle| \\
= \|x_0\|^2 + \|x\|^2 + 2 |\lambda| \langle x_0, u \rangle \\
\leq \|x_0\|^2 + r^2 + 2r \langle x_0, u \rangle \\
= \|x_0 + ru\|^2 \\
\leq 1,
\]
and therefore \( \gamma(a, b) \geq \gamma^T(a, b) \) as required. \( \square \)

Lemma 3.4. Let \( f \) be an equilateral weight on \( B^n \). There exists \( 0 \leq \lambda_n < 1 \) such that \( f \) is constant in \( \{ x \in B^n : \|x\| \geq \lambda_n \} \).

Proof. It suffices to show that there exists \( 0 \leq \lambda_n < 1 \) such that \( f \) is constant in \( \{ x \in B^n \cap T : \|x\| \geq \lambda_n \} \) for every two-dimensional subspace \( T \) of \( \mathbb{R}^n \).

Fix an arbitrary two-dimensional subspace \( T \) and let \( D \) denote the closed unit disc \( B^n \cap T \). To make calculations easier we fix a rectangular coordinate system in \( D \) with origin \( o \) at the centre of \( D \) (see Figure 1.). Consider the points \( w(0, -1), x(-1, 0), y(0, 1) \) and \( z(1, 0) \). Let \( C_w \) (resp. \( C_x, C_y, C_z \)) be the circular arc with centre \( w \) (resp. \( x, y, z \))
and radius $2\alpha_{n+1}$. The arcs $C_w$ and $C_z$ meet in $D$ at the point $a$ the coordinates of which can be easily calculated:

$$a\left(\frac{-1 + \sqrt{8\alpha_{n+1}^2 - 1}}{2}, -1 + \sqrt{8\alpha_{n+1}^2 - 1} \right).$$

Similarly, let $b, c, d \in D$ such that $C_x \cap C_y = \{b\}$, $C_y \cap C_z = \{c\}$ and $C_z \cap C_w = \{d\}$. Let $C_a$ (resp. $C_b$, $C_c$ and $C_d$) denote the circular arc in $D$ having centre $a$ and radius $2\alpha_{n+1}$ (see Figure 1).

First we show that $\gamma^T(a, w) \geq \beta_n$. Let $g$ be the point $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$. Since $2\alpha_{n+1} \leq \sqrt{3}$, it easy to see that the circular arc in $D$ having centre $g$ and radius 1 intersects $C_w$, say at $h$. Observe that if $l$ is the midpoint of the line segment $wh$, then $|lg| = \beta_n$. So to show that $\gamma^T(w, a) \geq \beta_n$, it suffices to show that the angle $\widehat{oaw}$ is less than or equal to the angle $\widehat{owh}$. To this end, it is enough to show that $\sin \widehat{oaw} \leq \sin \widehat{owh}$. Since $\widehat{doa} = \frac{\pi}{2}$, we have

$$\sin \widehat{oaw} = \sin(\pi/4 - \widehat{ow})$$

$$= \frac{1}{\sqrt{2}}(\cos \widehat{ow} - \sin \widehat{ow}).$$

Applying the sine rule for triangle $oaw$ we deduce that

$$\sin \widehat{ow} = \frac{\sin 3\pi/4}{2\alpha_{n+1}} = \frac{1}{2}\sqrt{\frac{n}{n+1}}$$

and

$$\cos \widehat{ow} = \frac{1}{2}\sqrt{\frac{3n+4}{n+1}}.$$
Thus, 
\[ \sin \hat{ow} = \frac{1}{2\sqrt{2}} \left( \sqrt{3 + \frac{1}{n+1}} - \sqrt{1 - \frac{1}{n+1}} \right). \]

On the other-hand 
\[ \sin \hat{oh} = \sin(\pi/3 - \hat{wg}) \]
\[ = \frac{1}{2}(\sqrt{3}\cos \hat{wg} - \sin \hat{wg}) \]
\[ = \frac{1}{2}(\sqrt{3}\alpha_{n+1} - \beta_n) \]
\[ = \frac{1}{2\sqrt{2}} \left( \sqrt{3 + \frac{3}{n}} - \sqrt{1 - \frac{1}{n}} \right). \]

Thus, \( \sin \hat{ow} \leq \sin \hat{oh} \) and therefore \( \gamma^T(w,a) \geq \beta_n. \)

It is clear (see Figure 1.) that \( \gamma^T(u,a) \geq \gamma^T(w,a) \) for every \( u \in C_a \). Thus, in view of Lemma 3.2 and Lemma 3.3, it follows that \( f \) is constant on \( C_n \). By symmetry, it follows that \( f \) is constant on the circuit \( C_a \cup C_b \cup C_c \cup C_d \). If \( \{w', x', y', z'\} \) is another quadruple of points on the circumference of \( D \) such that \( w'y' \) and \( x'z' \) are perpendicular, then we can repeat the same as above to deduce that \( f \) is constant on the corresponding circuit joining the points \( w', x', y' \) and \( z' \). Moreover, since any two such circuits intersect, it follows that \( f \) is constant in the annulus \( \{u \in D : |ou| \geq 2\alpha_{n+1} - |oa|\} \). Let \( \lambda_n := 2\alpha_{n+1} - |oa| \). From the coordinates of \( a \) one can calculate
\[ \lambda_n = \frac{1}{\sqrt{2}} \left( 1 + \sqrt{4 + \frac{4}{n}} - \sqrt{3 + \frac{4}{n}} \right). \]

For each \( \rho \in [\beta_n, 1] \) define \( \eta_n(\rho) := \alpha_{n+1} - \sqrt{\rho^2 - \beta_n^2} \). Observe that the value \( \eta_n(\rho) \) decreases strictly from \( \alpha_{n+1} \) (when \( \rho = \beta_n \)) to 0 (when \( \rho = 1 \)) and \( \eta_n(\rho) = 0 \) if, and only if, \( \rho = \beta_n \). Thus, \( \eta_n(\rho) \geq \rho \) for every \( \rho \in [\beta_n, \beta_{n+1}] \) and \( \eta_n(\rho) < \rho \) when \( \rho \in (\beta_{n+1}, 1] \). The geometric meaning of \( \eta_n(\rho) \) becomes apparent from the following Lemma.

**Lemma 3.5.** (a) Let \( 1 \geq \rho \geq \beta_n \) and let \( x \in B^n \) such that \( \|x\| = \eta_n(\rho) \). Then there exists a standard equilateral set \( \{x_1, x_2, \ldots, x_n\} \) such that \( \|x_i\| = \rho \) and \( \|x_i - x\| = 1 \) for every \( i = 1, 2, \ldots, n \).

(b) Conversely, if \( \{x_1, x_2, \ldots, x_{n+1}\} \) is a maximal standard equilateral set in \( B^n \) and \( \|x_i\| = \rho \) for every \( i = 1, 2, \ldots, n \), then \( \rho \geq \beta_n \) and if \( \text{conv}(x_1, \ldots, x_{n+1}) \) contains 0, then \( \|x_{n+1}\| = \eta_n(\rho) \).

**Proof.** (a) First note that if \( \rho = 1 \), then \( 0 = \eta_n(\rho) = \|x\| \) and therefore the statement is true in this case. Suppose that \( \beta_n \leq \rho < 1 \). Let \( \{u_1, u_2, \ldots, u_n\} \) be a maximal standard equilateral set in \( x^+ \) with centre
0. Then \( \|u_i\| = \beta_n \). It is easy to check that the vectors
\[
x_i := u_i - \sqrt{\rho^2 - \beta_n^2} \frac{x}{\|x\|} \quad (i = 1, 2, \ldots, n)
\]
satisfy the required conditions.

(b) The locus of points in \( \mathbb{R}^n \) equidistant from each of the \( x_i \)'s \((i = 1, \ldots, n)\) is the line passing through 0 and parallel to \( x_{n+1} - c(x_1, \ldots, x_n) \). The point on this line with shortest distance from any (and therefore from each) of the \( x_i \)'s \((i = 1, \ldots, n)\) is that with position vector \( c(x_1, \ldots, x_n) \). Thus
\[
\beta_n = \|c(x_1, \ldots, x_n) - x_i\| \leq \|x_i\| = \rho \quad (i = 1, 2, \ldots, n).
\]

If \( 0 \in \text{conv}(x_1, \ldots, x_{n+1}) \), then \( 0 = \lambda x_{n+1} + (1 - \lambda)c(x_1, \ldots, x_n) \) for some \( \lambda \in [0, 1] \). Thus
\[
\alpha_{n+1} = \|x_{n+1} - c(x_1, \ldots, x_n)\| = \|x_{n+1}\| + |c(x_1, \ldots, x_n)|
= \|x_{n+1}\| + \sqrt{\rho^2 - \beta_n^2}.
\]

\[\square\]

**Lemma 3.6.** Let \( f \) be an equilateral weight on \( B^n \) taking the constant value \( \delta \) in \( \{ x \in B^n : \|x\| \geq \rho_0 \} \), where \( \rho_0 \in [\beta_n, 1] \). Then \( f \) takes the constant value \( W - n\delta \) in \( B(0, \eta_n(\rho_0)) \) where \( W \) is the weight of \( f \). If \( \rho_0 \leq \beta_{n+1} \), then \( f \) takes the constant value \( \frac{W}{n+1} \) in \( B^n \).

**Proof.** Let \( x \in B(0, \eta_n(\rho_0)) \). The inequality \( 0 \leq \|x\| \leq \eta_n(\rho_0) \) implies that there exists \( 1 \geq \rho \geq \rho_0 \) such that \( \eta_n(\rho) = \|x\| \). Thus, by Lemma 3.5 there are vectors \( \{x_1, x_2, \ldots, x_n\} \) such that \( \|x_i\| = \rho \) for \( 1 \leq i \leq n \) and such that \( \{x, x_1, x_2, \ldots, x_n\} \) is a maximal standard equilateral set in \( B^n \). So, \( f(x) + n\delta = W \).

If \( \rho_0 \leq \beta_{n+1} \), then \( \eta_n(\rho_0) \geq \rho_0 \), i.e.
\[
\{ x \in B^n : \|x\| \geq \rho_0 \} \cap B(0, \eta_n(\rho_0)) \neq \emptyset,
\]
and thus \( W - n\delta = \delta \). \[\square\]

We are now ready to prove the result announced in the abstract.

**Theorem 3.7.** Every equilateral weight on \( B^n \) is constant.

**Proof.** Set \( \mu_n(\rho) := 1 - \eta_n(\rho) \) and \( \nu_n(\rho) := \rho - \mu_n(\rho) \) when \( \rho \in [\beta_n, 1] \). Observe that \( \mu_n \) is strictly increasing with range \([1 - \alpha_{n+1}, 1]\). It is easy to check that \( \nu_n \) is strictly decreasing and that \( \nu_n(1) = 0 \). Thus, \( \mu_n(\rho) < \rho \) for all \( \rho \in [\beta_n, 1] \).

Let \( f \) be an equilateral weight on \( B^n \). In view of Lemma 3.4 we can define \( \theta := \inf\{\rho : f \text{ is constant in } B^n \setminus B(0, \rho)\} \) and note that \( \theta \leq \lambda_n \). In view of Lemma 3.6 the proof would be complete if we could show that \( \theta < \beta_{n+1} \). So we suppose that \( \theta \geq \beta_{n+1} \) and seek a contradiction. Let \( \epsilon \) be a positive real number satisfying
\[
\epsilon < \min\{\nu_n(\lambda_n), \beta_{n+1} - \beta_n\}.
\]
Then \( \theta - \epsilon > \beta_n > 1 - \alpha_{n+1} \) and thus \( \mu_n^{-1}(\theta - \epsilon) \) is defined. In addition, it follows that \( \mu_n^{-1}(\theta - \epsilon) > \theta \), for if \( \mu_n^{-1}(\theta - \epsilon) \leq \theta \), then (since \( \mu_n \) is strictly increasing) we would have \( \theta - \epsilon \leq \mu_n(\theta) \) and this would lead to \( \epsilon \geq \nu_n(\theta) \geq \nu_n(\lambda_n) \), which contradicts our choice of \( \epsilon \).

Fix \( \rho_0 := \mu_n^{-1}(\theta - \epsilon) \). Then, since \( \mu_n^{-1}(\theta - \epsilon) > \theta \), \( f \) takes a constant value, say \( \delta \), in the annulus \( \{ x \in B^n : \|x\| \geq \rho_0 \} \) and therefore, by virtue of Lemma 3.6, \( f \) takes the constant value \( W - n\delta \) in \( B(0, \eta_n(\rho_0)) \), where \( W \) is the weight of \( f \). We show that \( f \) then must take the constant value \( \delta \) in the annulus \( \{ x \in B^n : \|x\| \geq \mu(\rho_0) \} \). This would contradict the definition of \( \theta \) and thus conclude the proof.

To this end, fix and arbitrary vector \( u \in B^n \) such that

\[
1 - \eta_n(\rho_0) = \mu_n(\rho_0) \leq \|u\| \leq \rho_0,
\]

and let \( v = -\frac{1-\|u\|}{\|u\|}u \). Then \( v \in B^n \) and \( 1 = \|u - v\| = \|u\| + \|v\| \).

From the inequalities

\[
1 - \eta_n(\rho_0) + \|v\| \leq \|u\| + \|v\| = 1 \leq \rho_0 + \|v\|
\]

we obtain \( 1 - \rho_0 \leq \|v\| \leq \eta_n(\rho_0) \) and therefore, in virtue of Lemma 3.6, we obtain \( f(v) = W - n\delta \). We can now apply Proposition 2.4 to obtain an enlargement \( \{ x_1, \ldots, x_{n-1}, u, v \} \) of \( \{u, v\} \) to a maximal standard equilateral set in \( B^n \). Let \( w := (u + v)/2 \). For each \( i = 1, 2, \ldots, n - 1 \) we have

\[
\|x_i\|^2 = \|x_i - w\|^2 + \|w\|^2 = \frac{3}{4} + \left( \|u\| - \frac{1}{2} \right)^2.
\]

If \( \eta_n(\rho_0) > \frac{1}{2} \), then \( \rho_0^2 < 5/4 - \alpha_{n+1} \) and thus

\[
\|x_i\|^2 \geq \frac{3}{4} + \frac{5}{4} - \frac{1}{\sqrt{2}} > \frac{5}{4} - \alpha_{n+1} > \rho_0^2.
\]

On the other-hand, if \( \eta_n(\rho_0) \leq \frac{1}{2} \), then \((*)\) implies

\[
\frac{1}{2} \leq 1 - \eta_n(\rho_0) \leq \|u\|
\]

and therefore

\[
\|x_i\|^2 = \frac{3}{4} + \left( \|u\| - \frac{1}{2} \right)^2
\]

\[
\geq \frac{3}{4} + \left( \frac{1}{2} - \eta_n(\rho_0) \right)^2
\]

\[
= 1 - \eta_n(\rho_0) + \eta_n(\rho_0)^2
\]

\[
= (1 - 2\alpha_{n+1}) \left( \sqrt{\rho_0^2 - \beta_n^2} - \alpha_{n+1} \right) + \rho_0^2
\]

\[
\geq \rho_0^2.
\]

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So in both cases we conclude that \( f(x_i) = \delta \) for each \( i = 1, 2, \ldots, n - 1 \) and therefore

\[
    f(u) = W - f(v) - \sum_{i=1}^{n-1} f(x_i)
    = W - (W - n\delta) - (n - 1)\delta = \delta,
\]
as required. This completes the proof. \( \square \)

**Remark 3.8.**

(i) It follows immediately from the theorem proved here that an equilateral weight on a connected subset of \( \mathbb{R}^n \) that is the union of unit balls, is constant.

(ii) Our method of the proof should work also to show that an equilateral weight on an \( n \)-dimensional (closed) ball with radius greater than \( \alpha_{n+1} \) is constant. What is not completely clear to us is the case when the radius lies in the interval \( (\beta_{n+1}, \alpha_{n+1}] \).

(iii) Although we have defined equilateral weights as real-valued functions, it is apparent from the proof that the same conclusion can be drawn if one considers group-valued equilateral weights on the unit ball of \( \mathbb{R}^n \).

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