GAUSS’S PROOF OF DESCARTES’S RULE OF SIGNS

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Abstract. For nonzero polynomials $f(x)$ with real coefficients, Descartes’s rule of signs gives an upper bound for the number of positive zeros. A variation of an argument of Gauss is used to prove Descartes’s theorem.

1. Variation of sign

The sign of a real number $\lambda$ is the function

$$\text{sign}(\lambda) = \begin{cases} 
  1 & \text{if } \lambda > 0 \\
  0 & \text{if } \lambda = 0 \\
  -1 & \text{if } \lambda < 0.
\end{cases}$$

Let $$A = (a_0, a_1, a_2, \ldots, a_n)$$ be a sequence of real numbers. The associated sign sequence is

$$\text{sign}(A) = (\text{sign}(a_0), \text{sign}(a_1), \text{sign}(a_2), \ldots, \text{sign}(a_n)).$$

Let $k + 1$ be the number of nonzero terms in this sequence. There is a strictly increasing sequence $0 \leq j_0 < j_1 < \cdots < j_k \leq n$ such that $(a_{j_0}, a_{j_1}, \ldots, a_{j_k})$ is the subsequence of nonzero terms in $A$. We have $\text{sign}(a_{j_i}) = \pm 1$ for all $i \in \{0, 1, \ldots, k\}$. The number of sign variations or sign changes in the sequence $A$, denoted $V(A)$, is the number of pairs $(a_{j_{i-1}}, a_{j_i})$ such that $\text{sign}(a_{j_{i-1}}) = -\text{sign}(a_{j_i})$. The number of sign changes in the sequence $(a_0, a_1, a_2, \ldots, a_n)$ equals the number of sign changes in the associated sign sequence sign($A$).

For example, if $A = (1, -1, 0, 1, -1)$, $A_1 = (1, -1, 0)$, and $A_2 = (0, 1, -1)$, then

$$V(A) = 3, \quad V(A_1) = 1, \quad \text{and} \quad V(A_2) = 1.$$ 

Note that

$$V(A) \neq V(A_1) + V(A_2).$$

Consider $n = 4$ and $\ell = 2$ in the following lemma.

Lemma 1. Let $A = (a_0, a_1, a_2, \ldots, a_n)$ be a sequence of real numbers. For $\ell \in \{1, 2, \ldots, n-1\}$, let

$$A_1 = (a_0, a_1, \ldots, a_\ell)$$

and

$$A_2 = (a_\ell, a_{\ell+1}, \ldots, a_n).$$
If \(a_\ell \neq 0\), then
\[
V(A) = V(A_1) + V(A_2).
\]

If \(a_i = 0\) for all \(i \in \{0, 1, \ldots, \ell\}\) or if \(a_j = 0\) for all \(j \in \{\ell, \ell + 1, \ldots, n\}\), then
\[
V(A) = V(A_1) + V(A_2).
\]

If \(a_\ell = 0\) and if \(a_i \neq 0\) for some \(i < \ell\) and \(a_j \neq 0\) for some \(j > \ell\) and if \(\ell_1\) is the largest integer such that \(\ell_1 < \ell\) and \(a_{\ell_1} \neq 0\), and \(\ell_2\) is the smallest integer such that \(\ell_2 > \ell\) and \(a_{\ell_2} \neq 0\), then
\[
V(A) = \begin{cases} 
V(A_1) + V(A_2) & \text{if sign}(a_{\ell_1}) = \text{sign}(a_{\ell_2}) \\
V(A_1) + V(A_2) + 1 & \text{if sign}(a_{\ell_1}) \neq \text{sign}(a_{\ell_2}).
\end{cases}
\]

**Proof.** This is a straightforward verification. \(\square\)

**Lemma 2.** Consider a sequence \(\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k)\) such that \(\varepsilon_i \in \{1, 0, -1\}\) for all \(i \in \{0, 1, \ldots, n\}\) and \(\varepsilon_0 \varepsilon_k \neq 0\).

(i) If \(\varepsilon_0 = \varepsilon_k\), then \(V(\varepsilon)\) is even.

(ii) If \(\varepsilon_0 = -\varepsilon_k\), then \(V(\varepsilon)\) is odd.

(iii) If \(\varepsilon_{i-1} \varepsilon_{i+1} = -1\) for some \(i \in \{1, \ldots, k-1\}\), then \(V(\varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}) = 1\) for all \(\varepsilon_i \in \{0, 1, -1\}\).

**Proof.** Statements (i) and (ii) follow from the simple but fundamental observation that changing signs an even number of times does not change the sign but changing signs an odd number of times does change the sign.

To prove statement (iii), we observe that if \(\varepsilon_{i-1} = -\varepsilon_{i+1} = 1\), then \((\varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}) = (1, 1, -1)\) or \((1, 0, -1)\) or \((1, -1, -1)\) and each of these sequences has one sign change. Similarly, if \(\varepsilon_{i-1} = -\varepsilon_{i+1} = -1\), then \((\varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}) = (-1, 1, -1)\) or \((-1, 0, 1)\) or \((-1, -1, -1)\) and each of these sequences has one sign change. This completes the proof. \(\square\)

A \(C^n\) function is a function with \(n\) continuous derivatives. Let \(f(x)\) be a \(C^n\) function. For \(\lambda\) in the domain of \(f(x)\), we define the **derivative sequence**
\[
D_f^{(n)}(\lambda) = \left( f(\lambda), f'(\lambda), f''(\lambda), \ldots, f^{(n)}(\lambda) \right).
\]

The variation of sign function
\[
V_f^{(n)}(\lambda) = V \left( D_f^{(n)}(\lambda) \right) = V \left( f(\lambda), f'(\lambda), f''(\lambda), \ldots, f^{(n)}(\lambda) \right)
\]
counts the number of sign changes in the derivative sequence.

In this paper we consider only polynomials with real coefficients. Let \(f(x)\) be a polynomial of degree \(n\). For all \(\lambda \in \mathbb{R}\), there is the Taylor expansion
\[
f(x) = \sum_{j=0}^{n} a_j (x - \lambda)^j
\]
with Taylor coefficients
\[
a_j = \frac{f^{(j)}(\lambda)}{j!}
\]
It follows that
\[
\text{sign}(a_j) = \text{sign} \left( f^{(j)}(\lambda) \right)
\]
for all \( j \in \{0, 1, \ldots, n\} \) and so
\[
V^{(n)}_f(\lambda) = V(a_0, a_1, \ldots, a_n)
\]
also counts the number of sign changes in the sequence of Taylor coefficients of \( f(x) \).

For example, the polynomial
\[
f(x) = 10 + 8x - 3x^2 - 5x^3 + 2x^5 + 7x^8 + x^9
\]
has coefficient sequence
\[
A = (10, 8, -3, -5, 0, 2, 0, 0, 7, 1).
\]
The corresponding sign sequence is
\[
\text{sign}(A) = (1, 1, -1, -1, 0, 1, 0, 1, 1).
\]
There are two sign changes in these sequences and so \( V^{(n)}_f(0) = 2 \).

Let \( f(x) \) be a \( C^n \) function and let
\[
g(x) = f(x + a).
\]
For all \( j \in \{0, 1, \ldots, n\} \) we have \( g^{(j)}(x) = f^{(j)}(x + a) \) and so
\[
g^{(j)}(0) = f^{(j)}(a).
\]
Therefore,
\[
V^{(n)}_g(0) = V\left(g(0), g'(0), g''(0), \ldots, g^{(n)}(0)\right)
\]
\[
= V\left(f(a), f'(a), f''(a), \ldots, f^{(n)}(a)\right)
\]
\[
= V^{(n)}_f(a).
\]

2. Counting zeros

Let \( f(x) \) be a nonzero polynomial with real coefficients. Let \( \mu_f(\lambda) \) be the multiplicity of the real or complex number \( \lambda \) as a root of the polynomial \( f(x) \). We define the zero counting function
\[
Z_f(a, b) = \sum_{a < \lambda < b} \mu_f(\lambda).
\]
This is the number (counting multiplicity) of real numbers \( \lambda \) such that
\[
f(\lambda) = 0 \quad \text{and} \quad a < \lambda < b.
\]
In particular, \( Z_f(0, \infty) \) counts the number of positive roots of \( f(x) \) and \( Z_{f(-\infty, 0)} \) counts the number of negative roots of \( f(x) \).

We define similarly
\[
Z_f(a, b] = \sum_{a < \lambda \leq b} \mu_f(\lambda), \quad \text{and} \quad Z_{f[a, b)} = \sum_{a \leq \lambda < b} \mu_f(\lambda).
\]
Let \( f(x) \) be a nonzero polynomial and let \( g(x) = f(-x) \). For \( \lambda > 0 \) we have \( f(-\lambda) = 0 \) if and only if \( g(\lambda) = 0 \), and so
\[
Z_f(-\infty, 0) = Z_g(0, \infty).
\]
For \( a, b \in \mathbb{R} \) with \( a < b \), we have
\[
Z_f(a, b] = Z_f(a, \infty) - Z_f(b, \infty).
\]
Let \( g(x) = f(x + a) \). Equivalently, \( f(x) = g(x - a) \). We have \( f(\lambda) = 0 \) if and only if \( g(\lambda - a) = 0 \). Because \( \lambda > a \) if and only if \( \lambda - a > 0 \), it follows that
\[
Z_f(a, \infty) = Z_g(0, \infty).
\]
Similarly, if \( h(x) = f(x + b) \), then
\[
Z_f(b, \infty) = Z_h(0, \infty).
\]
Relation (2) implies that if \( f(b) \neq 0 \), then
\[
Z_f(a, b) = Z_g(0, \infty) - Z_h(0, \infty).
\]
We may use Descartes’s rule of signs (Theorem I) to compute zero counting functions of the form \( Z_f(a, b) \).

If \( \mu = \mu_f(0) \), then \( f(x) = x^n \ell(x) \), where \( \ell(x) \) is a polynomial of degree \( n - \mu \) such that \( \ell(0) \neq 0 \). For \( \lambda \neq 0 \), we have \( f(\lambda) = 0 \) if and only if \( \ell(\lambda) = 0 \), and so the nonzero roots of \( f(x) \) are the nonzero roots of \( \ell(x) \). Moreover, \( \mu_f(\lambda) = \mu(\lambda) \) for all \( \lambda \neq 0 \). Thus, to count the number of nonzero roots of polynomials, it suffices to consider only polynomials \( f(x) \) with \( f(0) \neq 0 \), that is, polynomials with nonzero constant terms.

**Theorem 1** (Descartes). Let \( f(x) \) be a polynomial with \( f(0) \neq 0 \). There exists a nonnegative even integer \( \nu \) such that
\[
Z_f(0, \infty) = V_f^{(n)}(0) - \nu.
\]

**Corollary 1.** If \( V_f^{(n)}(0) = 1 \), then \( f \) has exactly one positive root \( \lambda \) and \( \lambda \) is a simple root.

**Proof.** Because \( \nu \) is nonnegative and even, if \( \nu \neq 0 \), then \( \nu \geq 2 \) and
\[
0 \leq Z_f(0, \infty) = V_f^{(n)}(0) - \nu = 1 - \nu \leq -1
\]
which is absurd. Therefore, \( \nu = 0 \) and \( Z_f(0, \infty) = V_f^{(n)}(0) - \nu = 1 \). This completes the proof. \( \square \)

Let \( a, b \in \mathbb{R} \) with \( a < b \). Let \( f(x) \) be a polynomial of degree \( n \) with \( f(b) \neq 0 \). Consider the polynomials \( g(x) = f(x + a) \) and \( h(x) = f(x + b) \). Descartes’s rule of signs and relation (3) give nonnegative even integers \( \nu_g \) and \( \nu_h \) such that
\[
Z_f(a, \infty) = Z_g(0, \infty) = V_g^{(n)}(0) - \nu_g = V_f^{(n)}(a) - \nu_g
\]
\[
Z_f(b, \infty) = Z_h(0, \infty) = V_h^{(n)}(0) - \nu_h = V_f^{(n)}(b) - \nu_h
\]
From equation (3) we obtain
\[
Z_f(a, b) = V_f^{(n)}(a) - V_f^{(n)}(b) - \nu
\]
where
\[
\nu = \nu_g - \nu_h.
\]

This argument does not prove the nonnegativity of the integer \( \nu \). The nonnegativity of \( \nu \) would immediately imply the following result, which is the Budan-Fourier theorem.

**Theorem 2** (Budan-Fourier). Let \( f(x) \) be a polynomial of degree \( n \) and let \( a, b \in \mathbb{R} \) with \( a < b \). There exists a nonnegative even integer \( \nu \) such that
\[
Z_f(a, b) = V_f^{(n)}(a) - V_f^{(n)}(b) - \nu.
\]
The Budan-Fourier theorem implies Descartes’s rule of signs.
For the history and alternate proofs of Descartes’s theorem, see [1, 2, 3, 4, 6, 7].
A recent proof of the Budan-Fourier theorem is Nathanson [8].

3. GAUSS’S PROOF OF DESCARTES’S THEOREM

The following result is the essential part of Gauss’s beautiful proof [5] of Descartes’s theorem.

**Lemma 3.** Let \( h(x) \) be a nonzero polynomial with \( h(0) \neq 0 \) and let \( \lambda > 0 \). If
\[
\ell(x) = (x - \lambda)h(x)
\]
then
\[
V^{(n)}_\ell(0) = V^{(n)}_h(0) + 1 + \nu
\]
for some nonnegative even integer \( \nu \).

**Proof.** Let
\[
h(x) = \sum_{j=0}^{m} c_j x^j
\]
be a polynomial of degree \( m \) with \( h(0) = c_0 \neq 0 \) and let
\[
V^{(m)}_h(0) = V(c_0, c_1, \ldots, c_m) = v
\]
be the number of sign changes in the coefficient sequence of \( h(x) \). Note that
\[
v \in \{0, 1, \ldots, m\}.
\]
We shall construct an integer sequence \((j_i)_{i=0}^v\) with
\[
0 = j_0 < j_1 < j_2 < \cdots < j_v \leq m
\]
such that
\[
V(c_{j_{i-1}}, \ldots, c_{j_i-1}) = 0 \quad \text{and} \quad V(c_{j_{i-1}}, \ldots, c_{j_i}) = 1
\]
for all \( i = 1, 2, \ldots, v \), and
\[
V(c_{j_v}, \ldots, c_m) = 0.
\]
The construction is by induction. Let \( j_0 = 0 \). Let \( i \in \{1, \ldots, v\} \). If \( j_{i-1} \in \{0, 1, \ldots, m\} \) satisfies
\[
V(c_0, \ldots, c_{j_{i-1}}) = i - 1 \leq v - 1 < V^{(m)}_h(0)
\]
then there is a smallest integer \( j_i > j_{i-1} \) such that
\[
\text{sign}(c_{j_i}) = -\text{sign}(c_{j_{i-1}})
\]
and so
\[
V(c_0, \ldots, c_{j_{i-1}}, c_{j_i}) = i.
\]
The sequence \((j_i)_{i=0}^v\) satisfies conditions [4] – [7]. Moreover, if \( j_v < m \), then
\[
\text{sign}(c_{j_v}) = \text{sign}(c_m).
\]
Consider the polynomial
\[
\ell(x) = (x - \lambda)h(x) = \sum_{j=0}^{m} c_j x^{j+1} - \sum_{j=0}^{m} \lambda c_j x^j
\]
\[
= -\lambda c_0 + \sum_{j=1}^{m} (c_{j-1} - \lambda c_j) x^j + c_m x^{m+1}
\]
\[
= \sum_{j=0}^{m+1} b_j x^j
\]
with
\[
b_j = \begin{cases} 
-\lambda c_0 & \text{for } j = 0 \\
c_{j-1} - \lambda c_j & \text{for } j = 1, \ldots, m \\
c_m & \text{for } j = m + 1.
\end{cases}
\]
For all \(i \in \{1, \ldots, v\}\) we have
\[
b_{j_i} = c_{j_i - 1} - \lambda c_{j_i}.
\]
It follows from the construction of the sequence \((j_i)_{i=0}^{v}\) that either
\[
c_{j_i - 1} = 0
\]
or
\[
\text{sign}(c_{j_i - 1}) = \text{sign}(c_{j_i - 1}) = -\text{sign}(c_{j_i}).
\]
Because \(\lambda > 0\), we have
\[
\text{sign}(b_{j_i}) = \text{sign}(c_{j_i - 1} - \lambda c_{j_i}) = -\text{sign}(c_{j_i})
\]
and so
\[
\text{sign}(b_{j_i}) = -\text{sign}(c_{j_i}) = \text{sign}(c_{j_i - 1}) = -\text{sign}(b_{j_i - 1}).
\]
We have \(j_v \leq m\). It follows from (7) that
\[
\text{sign}(b_{j_v}) = -\text{sign}(c_{j_v}) = -\text{sign}(c_m) = -\text{sign}(b_{m+1}).
\]
By Lemma 2 there exist nonnegative even integers \(\nu_0, \nu_1, \ldots, \nu_v\) such that
\[
V(b_{j_i - 1}, \ldots, b_{j_i}) = 1 + \nu_{i-1}
\]
for all \(i \in \{1, \ldots, v\}\) and
\[
V(b_{j_v}, \ldots, b_{m+1}) = 1 + \nu_v.
\]
Because \(b_{j_i} \neq 0\) for all \(i \in \{1, \ldots, v\}\), the addition formula in Lemma 3 gives
\[
V_{\ell}^{(m+1)}(0) = V(b_0, \ldots, b_{m+1}) = \sum_{i=1}^{v} V(b_{j_i - 1}, \ldots, b_{j_i}) + V(b_{j_v}, \ldots, b_{m+1})
\]
\[
= v + 1 + \sum_{i=0}^{v} \nu_i
\]
\[
= V_{h}^{(m)}(0) + 1 + \nu
\]
where \(\nu = \sum_{i=0}^{v} \nu_i\) is a nonnegative even integer. This completes the proof. \(\square\)
Lemma 4. Let $h(x)$ be a nonzero polynomial with $h(0) \neq 0$. Let $(\lambda_i)_{i=1}^k$ be a sequence of not necessarily distinct positive numbers. If

$$f(x) = h(x) \prod_{i=1}^k (x - \lambda_i)$$

then

$$V_f^{(n)}(0) = k + V_h^{(n)}(0) + \nu$$

for some nonnegative even integer $\nu$.

Proof. This follows from Lemma 3 by induction on $k$. □

We now prove Descartes’s rule of signs (Theorem 1).

Proof. Let $f(x)$ be a monic polynomial of degree $n$ with $f(0) \neq 0$ and let $Z_f(0, \infty) = k$.

Let $\Re(z)$ denote the real part of the complex number $z$.

Let $(\lambda_i)_{i=1}^n$ be the sequence of $n$ not necessarily distinct real and complex roots of $f(x)$. Note that $\lambda_i \neq 0$ for all $i \in \{1, \ldots, n\}$. We order the roots so that, for nonnegative integers $k, \ell, m$ with $k + \ell + 2m = n$

we have

(i) $(\lambda_i)_{i=1}^k$ is the sequence of positive roots of $f(x)$,

(ii) $(\lambda_i)_{i=k+1}^{k+\ell}$ is the sequence of negative roots of $f(x)$,

(iii) $(\lambda_i)_{i=k+\ell+1}^{k+\ell+2m}$ is the sequence of nonreal complex roots of $f(x)$ with

$$\overline{\lambda_{k+\ell+r}} = \lambda_{k+\ell+r+m}, \quad \text{for } r \in \{1, \ldots, m\}.$$ 

For every nonreal complex number $\lambda_i$, we have

$$(x - \lambda_i)(x - \overline{\lambda_i}) = x^2 - 2\Re(\lambda_i)x + |\lambda_i|^2.$$ 

The constant term of this quadratic polynomial is positive and so the constant term of the polynomial

$$\prod_{i=k+\ell+1}^{k+\ell+2m} (x - \lambda_i) = \prod_{i=k+\ell+1}^{k+\ell+m} (x - \lambda_i)(x - \overline{\lambda_i})$$

$$= \prod_{i=k+\ell+1}^{k+\ell+m} (x^2 - 2\Re(\lambda_i)x + |\lambda_i|^2)$$

is positive:

$$\prod_{i=k+\ell+1}^{k+\ell+m} |\lambda_i|^2 > 0$$

Because $\lambda_i < 0$ for all $i \in \{k + 1, \ldots, k + \ell\}$, the constant term of the polynomial

$$\prod_{i=k+1}^{k+\ell} (x - \lambda_i)$$

is also positive. Thus, the constant term of $f(x)$ is positive, which implies that $V_f^{(n)}(0) = k + \nu$ for some nonnegative even integer $\nu$. □
is also positive:
\[ \prod_{i=k+1}^{k+\ell} |\lambda_i| > 0. \]

Therefore, the constant term of the monic polynomial
\[ h(x) = \prod_{i=k+1}^{n} (x - \lambda_i) \]
is positive:
\[ h(0) = \prod_{i=k+1}^{k+\ell} |\lambda_i| \prod_{i=k+\ell+1}^{k+\ell+m} |\lambda_i|^2 > 0. \]
The leading coefficient and the constant term of \( h(x) \) are positive. By Lemma 2, the number of sign changes in the coefficient sequence of \( h(x) \) is the nonnegative even integer \( V_h^{(n-k)}(0) = \nu_h \).

We have
\[ f(x) = h(x) \prod_{i=1}^{k} (x - \lambda_i). \]

Because \( (\lambda_i)_{i=1}^{k} \) is a sequence of positive real numbers, Lemma 3 implies that there is a nonnegative even integer \( \nu_f \) such that
\[ V_f^{(n)}(0) = V_h^{(n)}(0) + k + \nu_f = k + \nu_h + \nu_f = k + \nu \]
for \( \nu = \nu_h + \nu_f \), and so
\[ Z_f(0, \infty) = k = V_f^{(n)}(0) - \nu. \]
This completes the proof. \( \square \)

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