Algebraic Linearization of Dynamics of Calogero Type for any Coxeter Group

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Abstract

Calogero-Moser systems can be generalized for any root system (including the non-crystallographic cases). The algebraic linearization of the generalized Calogero-Moser systems and of their quadratic (resp. quartic) perturbations are discussed.

I Introduction

The Calogero-Moser systems (\textsuperscript{1}) were extended to any semi-simple Lie algebras by Olshanetsky and Perelomov (\textsuperscript{2}) at the classical level. It was later generalized by Bordner, Corrigan and Sasaki (\textsuperscript{3}) to any root system (including the non-crystallographic case). The rational Calogero-Moser system displays a perturbation of special interest whose
solutions are all periodic of the same period. Such a type of perturbation was considered for any root system by Bordner, Corrigan and Sasaki and they proved the existence of a Lax pair which generalizes the one introduced by Olshanetsky-Perelomov (1) for the $A_{m-1}$ case. In this article, we follow the method introduced in Caseiro-Françoise (2) to prove an explicit algebraic linearization of several systems of Calogero-Moser or Ruijsenaars-Schneider type. This technique together with the general Lax matrix introduced by Bordner-Corrigan-Sasaki allows to show the full periodicity of all the orbits of the rational system perturbed by a confining quadratic potential. A proof of the involution of the eigenvalues of the Lax matrix is given by generalizing the original proof by Françoise (3) for the $A_{m-1}$ case. We show next that the flow associated to each eigenvalue has all orbits periodic of the same period.

In the third part of the article, we consider perturbations of quartic type which are still integrable as shown (in the $A_{m-1}$ case) by Françoise-Ragnisco (4) and we show the existence of a Lax pair for all Coxeter groups. In the fourth part, we extend the algebraic linearization of the trigonometric or hyperbolic systems (Sutherland systems) (5) to any root system having the minimal representation.

We postpone to further studies the semi-classical analysis of the associated quantum systems.

II Superintegrability of the rational Calogero-Moser system for any Coxeter group

The rational Calogero-Moser system for any Coxeter group was first considered by Bordner, Corrigan and Sasaki (5). They proved the classical integrability by constructing a universal Lax pair (see also 6, 7). The quantum case was discussed by Dunkl (8). In this article, we use the techniques of algebraic linearization discussed in (3) for the $A_{m-1}$ case. This yields the superintegrability of the rational Calogero-Moser system for any Coxeter group.

Let us denote by $\Delta$ a root system of rank $r$. The dynamical variables are (as before) the coordinates $q_i, i = 1, ..., r$ and their canonically conjugate momenta $p_i, i = 1, ..., r$. The Hamiltonian for the classical Calogero-Moser model is:
\[
\mathcal{H} = \frac{1}{2} p^2 + \frac{1}{2} \sum_{\rho \in \Delta_+} \frac{g^2_{|\rho|}}{(\rho \cdot q)^2} |\rho|^2,
\]

(2.1)
in which the coupling constants \( g_{|\rho|} \) are defined on orbits of the corresponding Coxeter group. That is, for the simple Lie algebra cases \( g_{|\rho|} = g \) for all roots in simply-laced models and \( g_{|\rho|} = g_L \) for long roots and \( g_{|\rho|} = g_S \) for short roots in non-simply laced models. Choose a representation \( \mathcal{D} \) of dimension \( D \) of the Coxeter group (see appendix), then define the \( D \times D \) matrix:

\[
p \cdot \hat{H} : \quad (p \cdot \hat{H})_{\alpha\beta} = (p \cdot \alpha) \delta_{\alpha\beta},
\]

(2.2)
where \( \alpha \) and \( \beta \) are vectors belonging to the representation.

Introduce next the \( D \times D \) matrices \( X, L \) and \( M \):

\[
X = i \sum_{\rho \in \Delta_+} g_{|\rho|} (\rho \cdot \hat{H}) \frac{1}{(\rho \cdot q)} \hat{s}_\rho,
\]

(2.3)
\[
L = p \cdot \hat{H} + X,
\]

(2.4)
\[
M = \frac{i}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} |\rho|^2 \frac{1}{(\rho \cdot q)^2} \hat{s}_\rho,
\]

(2.5)
and a diagonal matrix:

\[
Q = q \cdot \hat{H}; \quad (Q)_{\alpha\beta} = (q \cdot \alpha) \delta_{\alpha\beta}.
\]

(2.6)
The time evolution of the matrix \( L \) along the flow of the Hamiltonian displays the following equations:

\[
\dot{L} = [L, M],
\]

(2.7a)
\[
\dot{Q} = [Q, M] + L.
\]

(2.7b)
Introduce now the functions

\[
F_k = \text{Tr}(L^k); \quad k = 1, \ldots, D,
\]

(2.8)
\[
G_k = \text{Tr}(QL^k); \quad k = 0, \ldots, D - 1,
\]

(2.9)
whose time-evolution displays:

\[
\dot{F}_k = 0,
\]

(2.10)
\[
\dot{G}_k = F_{k+1}.
\]

(2.11)
This provides the algebraic linearization.
Proposition II.1

The generalized Calogero-Moser system (2.1) is superintegrable for any Coxeter group.

Proof.

Introduce together with the $D$ first integrals $F_k$ the $D(D - 1)/2$ extra first integrals $H_{k,k'} = F_k G_{k'} - F_{k'+1} G_{k-1}$. Independent conserved quantities $F_k$ to be obtained from the Lax equation (2.7a) occur at such $k = 1 + \text{exponent}$ of the corresponding crystallographic root systems. For the non-crystallographic root systems, they arise at $(k = 2, m)$ for the dihedral group $I_2(m)$, $(k = 2, 6, 10)$ for $H_3$ and $(k = 2, 12, 20, 30)$ for $H_4$. These are the degrees at which Coxeter invariant polynomials exist ([18]).

III Full periodicity of the confining potential case

Integrable systems related to any Coxeter group first considered by Bordner, Corrigan and Sasaki ([8]) are obtained by adding to generalized rational Calogero-Moser systems a confining potential. We recall here the notations and results of this article concerned with the confining potential.

The Hamiltonian is now:

$$\mathcal{H}_\omega = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2 + \frac{1}{2} \sum_{\alpha \in \Delta} \frac{g^2_{\alpha^2} |\alpha|^2}{(\alpha \cdot q)^2}. \quad (3.1)$$

With the same matrices introduced above in the first paragraph, the time evolution displays:

$$\dot{L} = [L, M] - \omega^2 Q, \quad (3.2a)$$
$$\dot{Q} = [Q, M] + L. \quad (3.2b)$$

Introduce the matrices:

$$L^\pm = L \pm i \omega Q. \quad (3.3)$$

These matrices undergo the time evolution:

$$\dot{L}^\pm = \pm i \omega L^\pm + [L^\pm, M]. \quad (3.4)$$

It was then observed that the matrix $\mathcal{L} = L^+ L^-$ defines Lax matrix for the system:

$$\dot{\mathcal{L}} = [\mathcal{L}, M]. \quad (3.5)$$
Consider then the functions:

\[ F_k = \text{Tr}(L^+ L^k), \quad (3.6a) \]
\[ G_k = \text{Tr}(L^- L^k). \quad (3.6b) \]

The time evolution yields:

\[ \dot{F}_k = i\omega F_k, \quad (3.7a) \]
\[ \dot{G}_k = -i\omega G_k. \quad (3.7b) \]

Thus these functions provide the algebraic linearization of the system. The existence of this algebraic linearization relies on the fact that the Lax equation is supplemented by an extra-equation which provides the full dynamics. Another consequence of this extra-equation is the involution of the first integrals displayed by the Lax pair.

Indeed, the formal structure of the equations [(3.2), (3.5)] is the same for any Coxeter group. So the same line of arguments developed in (10) for the special case \( A_{m-1} \) shows the following: (which is also of interest in the rational case)

**Theorem III.1**

The Hamiltonian flows generated by the functions

\[ H_k = \text{Tr}(L^k), \quad k = 1, \ldots, D \]

Poisson commute.

**Proof.**

Consider the symplectic form:

\[ \Omega = \text{Tr}(dQ \wedge dL) = C_D \sum_{j=1}^{r} dq_j \wedge dp_j, \quad (3.8) \]

defined on the product of two copies of the representation. The constant \( C_D \) depends actually on the representation. We can in the following forget about this factorizing constant and simply consider the Hamiltonian system defined by:

\[ \Omega = \text{Tr}(dQ \wedge dL), \quad (3.9a) \]
and the Hamiltonian:
\[ H_\omega = (1/2C_D) \text{Tr}(L). \] (3.9b)

Let \( \Lambda \) be an eigenvalue of the matrix \( L \) and let \( T \) be the matrix of the projection onto the eigenspace corresponding to this eigenvalue. Classical result of linear perturbation theory yields:
\[ d\Lambda = \text{Tr}(dT^T). \] (3.10)

The Hamiltonian flow generated by the function \( \Lambda \) and the symplectic form \( \Omega \) displays:
\[ \text{Tr}(QdL - LdQ) = \text{Tr}(dT). \] (3.11)

This yields:
\[ \dot{Q} = [Q, M] + i\omega[T, Q] + (LT + TL), \] (3.12a)
\[ \dot{L} = [L, M] + i\omega[T, L] - \omega^2(QT + TQ), \] (3.12b)

and thus:
\[ \dot{L} = [L, M] + 2i\omega[T, L] = [L, M]. \] (3.13)

This shows that the eigenvalues of the Lax matrix \( L \) are constants of motion for the Hamiltonian flow generated by any of its eigenvalues. In particular this proves that the Hamiltonian flows generated by the eigenvalues of the Lax matrix \( L \) Poisson commute.

**Proposition III.2**

The Hamiltonian flows generated by any eigenvalues of the Lax matrix \( L \) have all orbits periodic of the same period \( \pi/\omega \).

**Proof.**

The time evolution along the Hamiltonian flow displays:
\[ \dot{L}^+ = [L^+, M] + 2i\omega T^+, \] (3.14a)
\[ \dot{L}^- = [L^-, M] - 2i\omega L^- T. \] (3.14b)

Introduce \( U \) the (time dependent) matrix solution of the Cauchy problem:
\[ \dot{U} = UM, \quad U(0) = 1. \] (3.15)
The conjugated matrix $ULU^{-1}$ is then a constant of motion. Denote $V$ a time-independent matrix which diagonalizes this matrix. Conjugate all the matrices $UL^\pm U^{-1}UTU^{-1}$ by the matrix $V$ yields:

\begin{align}
\dot{L}^+ &= 2i\omega \tau L^+,
\dot{L}^- &= -2i\omega L^-\tau,
\end{align}

where $L^\pm = VUL^\pm U^{-1}V^{-1}$ and $\tau$ is the constant diagonal matrix whose entries are equal to zero except the diagonal term equals to 1 in the position corresponding to the eigenvalue. Equations (3.16) can be easily integrated and they yield the periodicity of the eigenvalues of the matrix $Q = (1/2i\omega)(L^+ - L^-)$ (the conservation of the Hamiltonian prevents collisions). This clearly implies that the positions $q$ are periodic in time of period $\pi/\omega$.

### IV The generalized rational Calogero-Moser system with an external quartic potential

The rational Calogero-Moser can be deformed into an integrable system by adding a quartic potential (cf. ([1]))). We include now a proof of the existence of a Lax matrix for the generalized rational Calogero-Moser system with an external quartic potential. Define again the same matrices $L, Q, X$ and $M$. Let $h(Q) = aQ + bQ^2$ be a matrix quadratic in $Q; (a, b)$ are just two new independent parameters. The perturbed Hamiltonian is now (up to the normalization constant $2C_D$):

$$
\mathcal{H}_h \propto \text{Tr}(L^2 + h(Q)^2),
$$

**Theorem IV.1**

The time evolution of the quartic type Hamiltonian system (4.1) can be cast into a Lax pair.

**Proof.**

Define the matrices:

$$
L^\pm = L \pm ih(Q),
$$

7
and
\[ \mathcal{L} = L^+ L^- \] (4.3)
The time evolution equations (3.4) of the matrices \( L^\pm \) get modified as follows:
\[
\dot{L}^+ = [L^+, M - i h'(Q)/2] + i L^+ h'(Q),
\] (4.4a)
\[
\dot{L}^- = [L^-, M - i h'(Q)/2] - i h'(Q) L^-.
\] (4.4b)
This yields the Lax pair equation:
\[ \dot{\mathcal{L}} = [\mathcal{L}, M - i h'(Q)/2]. \] (4.5)
In the special limit \( b = 0, a = \omega \), the quartic system reduces to the confining quadratic potential considered in the paragraph III.

V The trigonometric (hyperbolic) Calogero-Sutherland system

The Hamiltonian of the trigonometric Calogero-Sutherland model writes:
\[ \mathcal{H} = \frac{1}{2} \rho^2 + \frac{1}{2} \sum_{\alpha \in \Delta^+} \frac{g_{\alpha}^2 |\alpha|^2}{\sin^2(\alpha \cdot q)}. \] (5.1)
In order to get the hyperbolic case it suffices to change \( \sin \) into \( \sinh \). In the following, we only demonstrate the algebraic linearization of the trigonometric case. The hyperbolic case can be deduced easily by the above replacement. We consider the matrices:
\[
L = p \cdot \hat{H} + X, \] (5.2a)
\[
X = i \sum_{\rho \in \Delta^+} g_{|\rho|} (\rho \cdot \hat{H}) \frac{1}{\sin(\rho \cdot q)} \hat{s}_\rho, \] (5.2b)
\[
M = -\frac{i}{2} \sum_{\rho \in \Delta^+} g_{|\rho|} |\rho|^2 \cos(\rho \cdot q) \frac{1}{\sin^2(\rho \cdot q)} \hat{s}_\rho, \] (5.2c)
and diagonal matrices:
\[
Q = q \cdot \hat{H}; \quad (Q)_{\alpha \beta} = (q \cdot \alpha) \delta_{\alpha \beta}, \] (5.2d)
\[
R = e^{iQ}. \] (5.2e)
**Theorem V.1**

In the case when the root system admits a minimal representation, the time evolution along the flow of the Hamiltonian (5.1) displays:

\[
\dot{L} = [L, M], \quad (5.3a)
\]
\[
\dot{R} = [R, M] + i(RL + LR). \quad (5.3b)
\]

The algebraic linearization of the system (5.1) follows with the functions:

\[
a_k = \text{Tr}(L^k), \quad k = 1, \ldots, D, \quad (5.4a)
\]
and

\[
b_k = \text{Tr}(RL^k), \quad k = 1, \ldots, D, \quad (5.4b)
\]

whose time evolution reads:

\[
\dot{a}_k = 0, \quad (5.5a)
\]
\[
\dot{b}_k = 2ib_{k+1}. \quad (5.5b)
\]

**Proof.**

It can be easily checked that:

\[
\dot{R} = 2ip \cdot \hat{H}R = i(p \cdot \hat{H}R + Rp \cdot \hat{H}), \quad (5.6)
\]
\[
\dot{R} = i[(L - X)R + R(L - X)] = i(LR + RL) - i(XR + RX). \quad (5.7)
\]

We only need to show that:

\[
[R, M] = -i(XR + RX). \quad (5.8)
\]

Let us first evaluate the bracket $[R, M]$:

\[
[R, M] = -\frac{i}{2} \sum_{\rho \in \Delta^+} g_{[\rho]} \left\{ \frac{\rho \cdot q}{\sin^2(\rho \cdot q)} \right\} e^{2iqR} \hat{s}_\rho. \quad (5.9)
\]

The commutation relations:

\[
[\hat{H}_j, \hat{s}_\alpha] = \alpha_j (\alpha^\vee \cdot \hat{H}) \hat{s}_\alpha. \quad (5.10)
\]
yield
\[ [R, M] = -\frac{i}{2} \sum_{\rho \in \Delta_+} g_{[\rho]} |\rho|^2 \cos(\rho \cdot q) \frac{e^{2iq \cdot \hat{H}}}{\sin^2(\rho \cdot q)} \left( 1 - e^{-2i\rho \cdot q^\vee \cdot \hat{H}} \right) \hat{s}_\rho. \] (5.11)

The minimal representation (cf. [8]) is such that for all roots \( \rho \):
\[ \rho^\vee \cdot \hat{H} = 0, \pm 1. \] (5.12)

For each fixed positive root \( \rho \), three different cases have to be considered:

i) In case \( \rho^\vee \cdot \hat{H} = 0 \), then the right hand side of the equation (5.11) is zero. In this case, the contribution from \( XR + RX \) is zero as well because \( X \) itself is zero.

ii) In case \( \rho^\vee \cdot \hat{H} = 1 \), the contribution of \( \rho \) to the sum in (5.11) reads:
\[ -\frac{1}{2} g_{[\rho]} \frac{|\rho|^2}{\sin(\rho \cdot q)} (e^{2iq \cdot \hat{H}} + e^{2iq \cdot \hat{H} - 2i\rho \cdot q^\vee \cdot \hat{H}}) \hat{s}_\rho. \] (5.13)

Since \( \rho^\vee \cdot \hat{H} = 1 \), we have:
\[ (1/2)|\rho|^2 = \rho \cdot \hat{H}. \] (5.14)

Then the above expression (5.13) reads:
\[ g_{[\rho]} \rho \cdot \hat{H} \frac{1}{\sin(\rho \cdot q)} (e^{2iq \cdot \hat{H}} \hat{s}_\rho + \hat{s}_\rho e^{2iq \cdot \hat{H}}). \] (5.15)

which is exactly the same as the contribution of \( \rho \) to the expression of \(-i(XR + RX)\).

iii) The third case \( \rho^\vee \cdot \hat{H} = -1 \) can be treated analogously.

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10
Appendix. Root systems and finite reflection groups

We now review some facts about root systems and their reflection groups in order to introduce notation (cf. [8]). We consider only reflections in Euclidean space. A root system $\Delta$ of rank $r$ is a set of vectors in $\mathbb{R}^r$ which is invariant under reflections in the hyperplane perpendicular to each vector in $\Delta$:

$$\Delta \ni s_\alpha(\beta) = \beta - (\beta^\vee \cdot \alpha)\alpha, \quad \alpha^\vee = \frac{2\alpha}{|\alpha|^2}, \quad \alpha, \beta \in \Delta.$$  

Once chosen a representation $\mathcal{D}$, the reflection is represented by the operator $\hat{s}_\alpha$.

The set of positive roots $\Delta_+$ may be defined in terms of a vector $V \in \mathbb{R}^r$, with $V \cdot \alpha \neq 0$, $\forall \alpha \in \Delta$, as those roots $\alpha \in \Delta$ such that $\alpha \cdot V > 0$.

The set of reflections $\{s_\alpha, \alpha \in \Delta\}$ generates a group, known as a Coxeter group.

The root systems for finite reflection groups may be divided into two types: crystallographic and non-crystallographic root systems. Crystallographic root systems satisfy the additional condition

$$\alpha^\vee \cdot \beta \in \mathbb{Z}, \quad \forall \alpha, \beta \in \Delta.$$  

These root systems are associated with simple Lie algebras: $(A_r, r \geq 1)$, $(B_r, r \geq 2)$, $(C_r, r \geq 2)$, $(D_r, r \geq 4)$, $E_6$, $E_7$, $E_8$, $F_4$, and $G_2$ and $(BC_r, r \geq 2)$. The Coxeter groups for these root systems are called Weyl groups. The remaining non-crystallographic root systems are $H_3$, $H_4$, and the dihedral group of order $2m$, $(I_2(m), m \geq 4)$.

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