ON THE NONORIENTABLE GENUS OF THE GENERALIZED UNIT AND UNITARY CAYLEY GRAPHS OF A COMMUTATIVE RING

MAHDI REZA KHORSANDI∗ AND SEYED REZA MUSAWI

Faculty of Mathematical Sciences, Shahrood University of Technology, P.O. Box 36199-95161, Shahrood, Iran.

Abstract. Let $R$ be a commutative ring and let $U(R)$ be multiplicative group of unit elements of $R$. In 2012, Khashyarmanesh et al. defined generalized unit and unitary Cayley graph, $\Gamma(R, G, S)$, corresponding to a multiplicative subgroup $G$ of $U(R)$ and a non-empty subset $S$ of $G$ with $S^{-1} = \{s^{-1} | s \in S\} \subseteq S$, as the graph with vertex set $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if there exists $s \in S$ such that $x + sy \in G$. In this paper, we characterize all Artinian rings $R$ whose $\Gamma(R, U(R), S)$ is projective. This leads to determine all Artinian rings whose unit graphs, unitary Cayley graphs and co-maximal graphs are projective. Also, we prove that for an Artinian ring $R$ whose $\Gamma(R, U(R), S)$ has finite nonorientable genus, $R$ must be a finite ring. Finally, it is proved that for a given positive integer $k$, the number of finite rings $R$ whose $\Gamma(R, U(R), S)$ has nonorientable genus $k$ is finite.

1. Introduction

All rings considered in this paper are commutative rings with non-zero identity. We denote the ring of integers module $n$ by $\mathbb{Z}_n$ and the finite field with $q$ elements by $\mathbb{F}_q$. Let $R$ be a ring. We use $\mathbb{Z}(R)$, $U(R)$ and $J(R)$ to denote the set of zero-divisors of $R$, the set of units of $R$ and the Jacobson radical of $R$, respectively.

The idea of associating a graph to a commutative ring was introduced by Beck in [7]. The relationship between ring theory and graph theory has received significant attention in the literature. After introducing the zero-divisor graph by Beck, the authors assigned the other graphs to a commutative ring. Sharma and Bhatwadekar in [20], defined the co-maximal graph on $R$ as the graph whose vertex set is $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if $Rx + Ry = R$. Afterward, in [3] (resp., in [1]), the authors defined the unit (resp., unitary Cayley) graph, $G(R)$ (resp., Cay$(R, U(R))$), with vertex set $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in U(R)$ (resp., $x - y \in U(R)$). The unit and unitary Cayley graph were generalized in [15] as follows. The generalized unit and unitary Cayley graph, $\Gamma(R, G, S)$, corresponding to a multiplicative subgroup $G$ of $U(R)$...
and a non-empty subset $S$ of $G$ with $S^{-1} = \{s^{-1} \mid s \in S\} \subseteq S$, is the graph with vertex set $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if there exists $s \in S$ such that $x + sy \in G$. If we omit the word “distinct”, the corresponding graph is denoted by $\overline{\Gamma}(R, G, S)$. Note that the graph $\Gamma(R, G, S)$ is a subgraph of the co-maximal graph. For simplicity of notation, we denote $\Gamma(R, U(R), S)$ (resp., $\overline{\Gamma}(R, U(R), S)$) by $\Gamma(R, S)$ (resp., $\overline{\Gamma}(R, S)$).

The genus, $\gamma(\Gamma)$, of a finite simple graph $\Gamma$ is the minimum non-negative integer $g$ such that $\Gamma$ can be embedded in the sphere with $g$ handles. The crosscap number (nonorientable genus), $\tilde{\gamma}(\Gamma)$, of a finite simple graph $\Gamma$ is the minimum non-negative integer $k$ such that $\Gamma$ can be embedded in the sphere with $k$ crosscaps. The genus (resp., nonorientable genus) of an infinite graph $\Gamma$ is the supremum of genus (resp., nonorientable genus) of its finite subgraphs (see [18, 29]). The problem of finding the genus of a graph is NP-complete (see [27]). However, the genus of graphs that can be embedded in the projective plane can be computed in polynomial time (see [12]).

A genus 0 graph is called planar graph and a nonorientable genus 1 graph is called a projective graph. In [28], H.-J. Wang characterized all finite rings whose co-maximal graphs have genus at most one. Also, H.-J. Chiang-Hsieh in [9] determined all finite rings with projective zero-divisor graphs. Similar results are established for total graphs in [11,16,17,26]. Planar unit and unitary Cayley graphs were investigated in [1,3,23,24]. Also, Khashyarmanesh et al. in [15] characterized all finite rings $R$ in which $\Gamma(R, S)$ is planar. Recently, Asir et al. in [5,6], determined all finite rings $R$ whose $\Gamma(R, S)$ has genus at most two. Moreover, finite rings with higher genus unit and unitary Cayley graphs were investigated in [11,22] and [25], respectively. In this paper, we characterize all Artinian rings $R$ whose $\Gamma(R, S)$ is projective. This leads to determine all Artinian rings whose unit graphs, unitary Cayley graphs and co-maximal graphs are projective. Also, we prove that for an Artinian ring $R$ with $\tilde{\gamma}(\Gamma(R, S)) < \infty$, $R$ must be a finite ring. Finally, it is proved that for a given positive integer $k$, the number of finite rings $R$ such that $\tilde{\gamma}(\Gamma(R, S)) = k$ is finite.

2. Preliminaries

For a graph $\Gamma$, $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and edge set of $\Gamma$, respectively. The degree of a vertex $v$ in the graph $\Gamma$, denoted by $\deg(v)$, is the number of edges of $\Gamma$ incident with $v$, with each loop at $v$ counting as two edges. The minimum degree of $\Gamma$ is the minimum degree among the vertices of $\Gamma$ and is denoted by $\delta(\Gamma)$. A complete graph $\Gamma$ is a simple graph such that all vertices of $\Gamma$ are adjacent. In addition, $K_n$ denotes a complete graph with $n$ vertices. A graph $\Gamma$ is called bipartite if $V(\Gamma)$ admits a partition into two classes such that the vertices in the same partition class must not be adjacent. A simple bipartite graph in which every two vertices from different partition classes are adjacent is called a complete bipartite graph, denoted by $K_{m,n}$, where $m$ and $n$ are size of the partition classes. Two simple graphs $\Gamma$ and $\Delta$ are said to be isomorphic, and written by $\Gamma \cong \Delta$, if there exists a bijection $\varphi : V(\Gamma) \rightarrow V(\Delta)$ such that $xy \in E(\Gamma)$ if and only if $\varphi(x)\varphi(y) \in E(\Delta)$ for all $x, y \in V(\Gamma)$. A graph $\Gamma$ is called connected if any two of its vertices are linked by a path in $\Gamma$. A maximal connected subgraph of $\Gamma$ is called a component of $\Gamma$. 
A subdivision of a graph $\Gamma$ is a graph that can be obtained from $\Gamma$ by replacing (some or all) edges by paths. Two graphs are said to be homeomorphic if both can be obtained from the same graph by subdivision. Let $\Gamma_1$ and $\Gamma_2$ be two graphs without multiple edges. Recall that the tensor product $\Gamma = \Gamma_1 \otimes \Gamma_2$ is a graph with vertex set $V(\Gamma) = V(\Gamma_1) \times V(\Gamma_2)$ and two distinct vertices $(u_1, u_2)$ and $(v_1, v_2)$ of $\Gamma$ are adjacent if and only if $u_1v_1 \in E(\Gamma_1)$ and $u_2v_2 \in E(\Gamma_2)$. We refer the reader to [5] and [29] for general references on graph theory.

The following results give us some useful information about nonorientable genus of a graph.

**Lemma 2.1.** ([29] Chapter 11) The following statements hold:

1. Let $G$ be a graph. Then $\gamma(G) \leq 2\gamma(G) + 1$.
2. If $H$ is a subgraph of $G$, then $\gamma(H) \leq \gamma(G)$.
3. $\gamma(K_n) = \left\lceil \frac{n-3}{4} (n-4) \right\rceil$ if $n \geq 3$ and $n \neq 7$, $3$ if $n = 7$.

In particular, $\gamma(K_5) = 1$ if $n = 5, 6$.
4. $\gamma(K_{m,n}) = \left\lceil \frac{m-2}{4} (m-2) \right\rceil$ if $m, n \geq 2$.

In particular, $\gamma(K_{3,3}) = \gamma(K_{3,4}) = 1$ and $\gamma(K_{4,4}) = 2$.

**Lemma 2.2.** ([21] Theorem 1 and Corollary 3) Let $G$ be a graph with components $G_1, G_2, \ldots, G_n$. If for all $i = 1, \ldots, n$, $\gamma(G_i) > 2\gamma(G_i)$, then

$$\gamma(G) = 1 - n + \sum_{i=1}^{n} \gamma(G_i),$$

otherwise,

$$\gamma(G) = 2n - \sum_{i=1}^{n} \mu(G_i),$$

where $\mu(G_i) = \max\{2 - 2\gamma(G_i), 2 - \gamma(G_i)\}$.

If we combine Lemma 2.1a) and Lemma 2.2, we can conclude the following corollary:

**Corollary 2.3.** Let $G$ be a graph with components $G_1, G_2, \ldots, G_n$. Then

$$1 - n + \sum_{i=1}^{n} \gamma(G_i) \leq \gamma(G) \leq \sum_{i=1}^{n} \gamma(G_i).$$

**Lemma 2.4.** ([20] Corollaries 11.7 and 11.8) Let $G$ be a connected graph with $p \geq 3$ vertices and $q$ edges. Then $\gamma(G) \geq \frac{q}{3} - p + 2$. In particular, if $G$ has no triangle, then $\gamma(G) \geq \frac{q}{4} - p + 2$.

Now, from Corollary 2.3 together Lemma 2.4 we obtain the following corollary:

**Corollary 2.5.** Let $G$ be a graph with $n$ components, $p \geq 3$ vertices and $q$ edges. Then $\gamma(G) \geq \frac{q}{3} - p + n + 1$. In particular, if $G$ has no triangles, then $\gamma(G) \geq \frac{q}{4} - p + n + 1$.

The authors in [10], Lemma 2.2], obtained the following lemma (when the graph $G$ is connected), but they used Euler’s formula in their proof which is false in nonorientable case (see [29], p. 144)). Fortunately, the result is true and we prove it in general case. We remark here that the Euler’s formula also used in [9], which is false in nonorientable case and so the results in [9] must be checked again.
Lemma 2.6. Let $G$ be a graph with $n$ components and $p \geq 3$ vertices. Then

$$\delta(G) \leq 6 + \frac{6\gamma(G) - 6(n + 1)}{p}.$$ 

Proof. Since $\sum_{v \in V(G)} \deg(v) = 2q$, then $p\delta(G) \leq 2q$. Now, by Corollary 2.5, $2q \leq 6(p + \gamma(G) - (n + 1))$. This completes the proof. \qed

3. $\Gamma(R, S)$ with finite nonorientable genus

In this section, first we prove that for an Artinian ring $R$ with $\gamma(\Gamma(R, S)) = k < \infty$, for some non-negative integer $k$, $R$ must be a finite ring. Then, we prove that for a given positive integer $k$, the number of finite rings $R$ such that $\gamma(\Gamma(R, S)) = k$ is finite. We begin with some basic general properties of $\Gamma(R, G, S)$.

Lemma 3.1. (15, Remark 2.4) (a) For any vertex $x$ of $\Gamma(R, G, S)$, we have the inequalities

$$|G| - 1 \leq \deg(x) \leq |G||S|.$$ 

Furthermore, for any vertex $x$ of $\Gamma(R, G, S)$, $\deg(x) \geq |G|$.

(b) Suppose that $R_1$ and $R_2$ are rings and, for each $i$ with $i = 1, 2$, $G_i$ is a subgroup of $U(R_i)$. Also, assume that $S_i$ is a non-empty subset of $G_i$ with $S_i^{-1} \subseteq S_i$.

(i) Then $\Gamma(R_1 \times R_2, G_1 \times G_2, S_1 \times S_2) \cong \Gamma(R_1, G_1, S_1) \otimes \Gamma(R_2, G_2, S_2)$.

(ii) Furthermore, whenever $R_1 = R_2$, $G_1 \subseteq G_2$ and $S_1 \subseteq S_2$, then $\Gamma(R_1, G_1, S_1)$ is a subgraph of $\Gamma(R_2, G_2, S_2)$.

Lemma 3.2. (15, Theorem 2.7) The graph $\Gamma(R, G, S)$ is a complete graph if and only if the following statements hold.

(a) $R$ is a field.

(b) $G = U(R)$.

(c) $|S| \geq 2$ or $S = \{-1\}$.

Remark 3.3. (15, Remark 3.1) Suppose that $\{x_i + J(R)\}_{i \in I}$ is a complete set of coset representation of $J(R)$. Note that if $x \in U(R)$ and $j \in J(R)$, then $x + j \in U(R)$. Hence, whenever $x_1$ and $x_2$ are adjacent vertices in $\Gamma(R, S)$, then every element of $x_1 + J(R)$ is adjacent to every element of $x_j + J(R)$.

Lemma 3.4. (15, Proposition 3.2 and its proof) Let $m$ be a maximal ideal of $R$ such that $|\frac{R}{m}| = 2$. Then the graph $\Gamma(R, S)$ is bipartite. Furthermore, if $R$ is a local ring, then $\Gamma(R, S)$ is a complete bipartite graph with parts $m$ and $1 + m$.

Theorem 3.5. Let $R$ be an Artinian ring such that $\gamma(\Gamma(R, S)) = k < \infty$, for some non-negative integer $k$. Then $R$ is a finite ring.

Proof. First suppose that $|J(R)| = 1$. In this case, for some positive integer $n$, $R \cong F_1 \times \cdots \times F_n$, where $F_i$’s ($1 \leq i \leq n$) are fields. Suppose on the contrary $R$ is infinite. Hence, without loss of generality we can assume that $F_1$ is infinite. Let $s = (s_1, \ldots, s_n) \in S$ and $k' = \max\{3, 4k\}$. Since $F_1$ is infinite we can choose distinct elements $x_1, \ldots, x_{k'}, y_1, \ldots, y_{k'} \in F_1$ such that $-s_1y_1, \ldots, -s_1y_{k'} \notin \{x_1, \ldots, x_{k'}\}$. Now, every element of the form $(x_i, 1, \ldots, 1)$, $i = 1, \ldots, k'$, is adjacent to every element of the form $(y_j, 0, \ldots, 0)$, $j = 1, \ldots, k'$, in $\Gamma(R, S)$. Thus, $K_{k', k'}$ is a subgraph of $\Gamma(R, S)$ and so by parts (b) and (d) of Lemma 2.4, $k' \leq \sqrt{2k} + 2$ which is a contradiction.
Now, suppose that $|J(R)| > 1$. Since 0 is adjacent to 1 in $\Gamma(R, S)$, by Remark 3.3, every element of $0 + J(R)$ is adjacent to every element of $1 + J(R)$. Hence, $K_{|J(R)|, |J(R)|}$ is a subgraph of $\Gamma(R, S)$ and so by parts (b) and (d) of Lemma 2.1, $|J(R)| \leq \sqrt{|K_{|J(R)|, |J(R)|}} + 2$. Now, since $R$ is an Artinian ring, for some positive integer $n$, we can write $R \cong R_1 \times \cdots \times R_n$, where $R_i$’s ($1 \leq i \leq n$) are local rings. Thus, $|J(R)| = |J(R_1)| \times \cdots \times |J(R_n)|$ and so for all $i = 1, \ldots, n$, $|J(R_i)| < \infty$. On the other hand, for all $i = 1, \ldots, n$, $Z(R_i) = J(R_i)$ and so by [13, Theorem 1], $R_i$ is a finite ring. Hence, $R$ is a finite ring.

The following corollary is an immediate consequence from Lemma 2.1(a) and Theorem 3.5.

**Corollary 3.9.** Let $R$ be an Artinian ring such that $\gamma(\Gamma(R, S)) < \infty$. Then $R$ is a finite ring.

**Remark 3.7.** Let $R$ be an Artinian ring such that $\gamma(\Gamma(R, S)) < \infty$. Then $R$ is a finite ring.

**Theorem 3.8.** Let $R$ be a finite ring and $\tilde{\gamma}(\Gamma(R, S)) = k > 0$. Then either

$$|R| \leq 6k - 12 \quad \text{or} \quad R \cong (\mathbb{Z}_2)^\ell \times T,$$

where $0 \leq \ell \leq \log_2 k + 1$ and $T$ is a ring with $|T| \leq 16$.

**Proof.** By Lemma 2.8, $\delta(\Gamma(R, S)) \leq 6 + \frac{6k - 12}{|R|}$. If $|R| > 6k - 12$, then $\delta(\Gamma(R, S)) \leq 6$ and so by Lemma 3.1(a), $|U(R)| \leq 7$. Now, since $R$ is a finite ring, we can write $R \cong (\mathbb{Z}_2)^\ell \times T$, where $\ell \geq 0$ and $T$ is a finite ring. Since $|U(R)| \leq 7$, in view of [11, Theorem 3.8] and its proof, $|T| \leq 16$. It will suffice to prove that if $\ell > 0$, then $\ell \leq \log_2 k + 1$. Since $S = \{1\} \times \cdots \times \{1\} \times S'$, for some $S' \subseteq T$ and $\ell \geq 1$, by Remark 3.7, $\Gamma(\mathbb{Z}_2^\ell \times T, \{1\} \times \cdots \times \{1\} \times S') \cong 2^{\ell - 1} \Gamma(\mathbb{Z}_2^\ell \times T, \{1\} \times S')$. Set $t := \tilde{\gamma}(\Gamma(\mathbb{Z}_2^\ell \times T, \{1\} \times S'))$. If $t = 1$, then by Lemma 2.2, $k = 2^{\ell - 1}$ and so $\ell = \log_2 k + 1$. Now, suppose that $t > 1$. By Corollary 2.3, $k \geq 1 - 2^{\ell - 1} + 2^{\ell - 1}t$. Hence, $k \geq 2^{\ell - 1} + 1$ and so $\ell \leq \log_2 (k - 1) + 1$. This completes the proof.

**Corollary 3.9.** Let $R$ be a finite ring such that $\tilde{\gamma}(\Gamma(R, S)) = k > 0$. Then $|R| \leq 32k$. In particular, for any positive integer $k$, the number of finite rings $R$ such that $\tilde{\gamma}(\Gamma(R, S)) = k$ is finite.

**Proof.** If $|R| > 6k - 12$, then by Theorem 3.8, $R \cong (\mathbb{Z}_2)^\ell \times T$, where $0 \leq \ell \leq \log_2 k + 1$ and $T$ is a ring with $|T| \leq 16$. In this case, $|R| = 2^{\ell} \times |T| \leq 2^{\ell} \times 16 \leq 32k$. Thus, $|R| \leq \max\{6k - 12, 32k\} = 32k$.

The following Corollary is an immediate consequence from Corollary 3.9 and Lemma 2.1(a).

**Corollary 3.10.** For a given positive integer $g$, the number of finite rings $R$ such that $\gamma(\Gamma(R, S)) = g$ is finite.
4. \( \Gamma(R, S) \) WITH NONORIENTABLE GENUS ONE

A graph \( G \) is irreducible for a surface \( S \) if \( G \) does not embed in \( S \), but any proper subgraph of \( G \) does embed in \( S \). Kuratowski’s theorem states that any graph which is irreducible for the sphere is homeomorphic to either \( K_5 \) or \( K_{3,3} \). Glover, Huneke, and Wang in [13] have constructed a list of 103 graphs which are irreducible for projective plane. Afterward, Archdeacon [2] showed that their list is complete. Hence a graph embeds in the projective plane if and only if it contains no subgraph homeomorphic to one of the graphs in the list of 103 graphs in [13].

In this section we characterize all finite rings \( R \) whose \( \Gamma(R, S) \) is projective. First, we focus in the case that \( R \) is local.

Lemma 4.1. Let \( R \) be a finite ring such that \( \bar{\gamma}(\Gamma(R, S)) = 1 \). Then \( |U(R)| \leq 6 \) and \( |J(R)| \leq 3 \).

Proof. By Lemma 3.1(a), \( |U(R)| \leq \bar{\delta}(\Gamma(R, S)) \) and by Lemma 2.6, \( \bar{\delta}(\Gamma(R, S)) \leq 6 \). Thus, \( |U(R)| \leq 6 \). Now, it is sufficient to prove that \( |J(R)| \leq 3 \). From the proof of Theorem 3.5 follows that either \( |J(R)| = 1 \) or \( |J(R)| \leq \sqrt{2} + 2 \). This completes the proof.

Corollary 4.2. Let \( R \) be a finite local ring such that \( \bar{\gamma}(\Gamma(R, S)) = 1 \). Then \( |R| \leq 9 \).

Proof. Let \( m \) be the unique maximal ideal of \( R \). By Lemma 4.1, \( |U(R)| \leq 6 \) and \( |m| \leq 3 \). This implies that \( |R| = |U(R)| + |m| \leq 6 + 3 = 9 \). In addition, if \( R \) is a field, then \( |R| = 1 \).

Lemma 4.3. Let \( R \) be a finite local ring which is not a field.

(a) If \( |R| = 8 \), then \( \bar{\gamma}(\Gamma(R, S)) = 2 \).

(b) If \( |R| = 9 \), then \( \bar{\gamma}(\Gamma(R, S)) \geq 2 \).

Proof. Let \( m \) be the unique maximal ideal of \( R \).

(a) Since \( R \) is not a field, in view of [10] p. 687, \( |m| = 4 \). Hence, \( |R/m| = 2 \) and by Lemma 3.4, \( \Gamma(R, S) \) is a complete bipartite graph with parts \( m \) and \( 1 + m \). Thus, \( \Gamma(R, S) \cong K_{4,4} \) and so by Lemma 2.1(d), \( \bar{\gamma}(\Gamma(R, S)) = 2 \).

(b) Since \( |R| \) is odd, by [15] Corollary 2.3, \( 2 \notin U(R) \). It follows that 0 is adjacent to 2. Now, by Remark 3.3, every element of \( m \) is adjacent to every element of \( 1 + m \) and \( 2 + m \). On the other hand, since \( R \) is not a field, \( |m| = 3 \). Thus, \( K_{3,6} \) is a subgraph of \( \Gamma(R, S) \) and so by parts (b) and (d) of Lemma 2.1, \( \bar{\gamma}(\Gamma(R, S)) \geq 2 \).

Lemma 4.4. ([15] Theorem 3.7) Let \( R \) be a finite ring. Then \( \Gamma(R, S) \) is planar if and only if one of the following conditions holds.

(a) \( R \cong (\mathbb{Z}_2)^\ell \times T \), where \( \ell \geq 0 \) and \( T \) is isomorphic to one of the following rings:

\[ \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4 \text{ or } \frac{\mathbb{Z}_2[x]}{(x^2)} \]

(b) \( R \cong \mathbb{F}_4 \).

(c) \( R \cong (\mathbb{Z}_2)^\ell \times \mathbb{F}_4 \), where \( \ell \geq 0 \) with \( S = \{1\} \).

(d) \( R \cong \mathbb{Z}_5 \) with \( S = \{1\} \).
Lemma 4.6. Let \( R \) be a finite local ring. Then \( \Gamma(R, S) \) is projective if and only if \( R \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \) with \( S = \{(1,1), (1,-1), (-1,1)\} \).

Proof. Suppose that \( \gamma(\Gamma(R, S)) = 1 \). By Corollary 4.2 and Lemma 4.3, \(|R| \leq 7\).

If either \(|R| \leq 4\) or \( R \cong \mathbb{Z}_5 \) with \( S = \{1\} \), then by Lemma 4.4, \( \Gamma(R, S) \) is planar which is not projective. On the other hand, since \( R \) is a finite local ring, the order of \( R \) is a power of a prime number. Thus, either \( R \cong \mathbb{Z}_5 \) with \( S \neq \{1\} \) or \( R \cong \mathbb{Z}_7 \). If \( R \cong \mathbb{Z}_5 \) with \( S \neq \{1\} \), then by Lemma 3.2, \( \Gamma(R, S) \cong K_5 \) and so by Lemma 2.1(c), \( \gamma(\Gamma(R, S)) = 1 \). Now, suppose that \( R \cong \mathbb{Z}_7 \). If either \(|S| \geq 2\) or \( S = \{-1\} \), then by Lemma 3.2, \( \Gamma(R, S) \cong K_7 \) and in this case by Lemma 2.1(c), \( \gamma(\Gamma(R, S)) = 3 \).

If \( S = \{1\} \), then \( \Gamma(\mathbb{Z}_7, \{1\}) \), as shown in Figure 1, is isomorphic to the graph \( A_2 \) which is one of the 103 graphs listed in [14]. Thus, \( \Gamma(\mathbb{Z}_7, \{1\}) \) is not projective. □

![Figure 1. The graph \( \Gamma(\mathbb{Z}_7, \{1\}) \).](image)

Now, we determine all finite non-local rings \( R \) whose \( \Gamma(R, S) \) is projective. First, we state some especial cases.

Lemma 4.6. Let \( R \cong \mathbb{Z}_2 \times T \) and \( T \in \{\mathbb{Z}_5, \mathbb{Z}_7, \mathbb{Z}_9, \frac{\mathbb{Z}_9}{(x^2)}\} \). Then \( \gamma(\Gamma(R, S)) \geq 2 \).

Proof. Since \( \Gamma(\mathbb{Z}_2, \{1\}) = \Gamma(\mathbb{Z}_2, \{1\}) \), then \( \Gamma(R, S) = \Gamma(R, S) \) and so by Lemma 3.1(a), for any vertex \( x \) of \( \Gamma(R, S) \), \( \deg(x) \geq |U(R)| \). On the other hand, since \( m = \{0\} \times T \) is a maximal ideal of \( R \) such that \(|R_m| = 2\), then by Lemma 3.4, \( \Gamma(R, S) \) is a bipartite graph. Hence, \( \Gamma(R, S) \) has no triangles. Now, consider the following cases:

**Case 1:** \( T = \mathbb{Z}_5 \). In this case \(|R| = 10\) and \(|U(R)| = 4\). It follows that \( \Gamma(R, S) \) has at least 20 edges. Thus, by second part of Corollary 2.5, \( \gamma(\Gamma(R, S)) \geq 2 \).

**Case 2:** \( T = \mathbb{Z}_7 \). In this case \(|R| = 14\) and \(|U(R)| = 6\). Hence, \( \Gamma(R, S) \) has at least 42 edges and so by second part of Corollary 2.5, \( \gamma(\Gamma(R, S)) \geq 9 \).

**Case 3:** \( T \in \{\mathbb{Z}_9, \frac{\mathbb{Z}_9}{(x^2)}\} \). In this case \(|R| = 18\) and \(|U(R)| = 6\). Thus, \( \Gamma(R, S) \) has at least 54 edges and so by second part of Corollary 2.5, \( \gamma(\Gamma(R, S)) \geq 11 \). □

Lemma 4.7. Let \( R \equiv R_1 \times R_2 \), \( R_1 \in \{\mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_3}{(x^2)}, \mathbb{Z}_4\} \) and \( R_2 \in \{\mathbb{Z}_4, \frac{\mathbb{Z}_4}{(x^2)}\} \). Then \( \gamma(\Gamma(R, S)) \geq 2 \).

Proof. Since for every \( s \in U(R_2) \), \( 1+s \notin U(R_2) \), we have \( \Gamma(R, S) = \Gamma(R, S) \) and so by Lemma 3.1(a), for any vertex \( x \) of \( \Gamma(R, S) \), \( \deg(x) \geq |U(R)| \). On the other hand, by Lemma 3.4, \( \Gamma(R, S) \) is a bipartite graph. Indeed, if \( n \) be the unique maximal
ideal of \( R_2 \), then \( m = R_1 \times n \) is a maximal ideal of \( R \) such that \(|\frac{R_1}{m}| = 2\). Hence, \( \Gamma(R, S) \) has no triangles. Now, consider the following cases:

**Case 1:** \( R_1 = \mathbb{Z}_3 \). In this case \(|R| = 12\) and \(|U(R)| = 4\). It follows that \( \Gamma(R, S) \) has at least 24 edges. Thus, by second part of Corollary 2.5, \( \gamma(\Gamma(R, S)) \geq 2 \).

**Case 2:** \( R_1 \in \{ \mathbb{Z}_4, \mathbb{Z}_2[\{x\}] / (x^2) \} \). In this case \(|R| = 16\) and \(|U(R)| = 4\). Hence, \( \Gamma(R, S) \) has at least 32 edges and so by second part of Corollary 2.5, \( \gamma(\Gamma(R, S)) \geq 2 \).

**Case 3:** \( R_1 = \mathbb{F}_4 \). In this case \(|R| = 16\) and \(|U(R)| = 6\). Thus, \( \Gamma(R, S) \) has at least 48 edges and so by second part of Corollary 2.5, \( \gamma(\Gamma(R, S)) \geq 10 \).

\( \Box \)

**Lemma 4.8.** Let \( R \cong \mathbb{Z}_3 \times \mathbb{F}_4 \). Then \( \gamma(\Gamma(R, S)) \geq 2 \).

**Proof.** Since \( S \) is a non-empty subset of \( U(R) \) such that \( S^{-1} \subseteq S \), there exists an element \((s_1, s_2) \in S \) where \( s_1 \in \{1, -1\} \) and \((s_1, s_2)^{-1} \in S \). Set \( S_2 := \{s_2, s_2^{-1}\}, S' := \{s_1\} \times S_2 \) and \( G := \overline{\Gamma}(\mathbb{Z}_3, \{s_1\}) \). Since \( \text{Char}(\mathbb{F}_4) = 2 \), then \(|S_2| = 1\) if and only if \( s_2 = -1 \) and so by Lemma 3.2, \( \Gamma(\mathbb{F}_4, S_2) \cong K_4 \). On the other hand, by Lemma 3.1(b)(i),

\[
\Gamma(R, S') \cong \overline{\Gamma}(\mathbb{Z}_3, \{s_1\}) \odot \overline{\Gamma}(\mathbb{F}_4, S_2).
\]

Hence, \( G \odot K_4 \) is a subgraph of \( \Gamma(R, S') \). Note that by Lemma 3.1(b)(i) and Lemma 3.2,

\[
\Gamma(\mathbb{Z}_3 \times \mathbb{F}_4, \{(s_1, -1)\}) = \overline{\Gamma}(\mathbb{Z}_3 \times \mathbb{F}_4, \{(s_1, 1)\}) \cong G \odot K_4.
\]

Thus, by Lemma 3.1(a), \( G \odot K_4 \) is a 6-regular graph and so by Corollary 2.5, \( \gamma(G \odot K_4) \geq 2 \). Now, by Lemma 2.1(b) and Lemma 3.1(b)(ii), we have the following inequalities:

\[
\gamma(\Gamma(R, S)) \geq \gamma(\Gamma(R, S')) \\
\geq \gamma(G \odot K_4) \\
\geq 2.
\]

\( \Box \)

**Lemma 4.9.** Let \( R \cong \mathbb{Z}_2 \times R_1 \times R_2 \) where \( R_1 \) and \( R_2 \) are local rings of order 3 or 4. Then \( \gamma(\Gamma(R, S)) \geq 2 \).

**Proof.** Without loss of generality we can assume that \(|R_1| \leq |R_2|\). Since \( \Gamma(\mathbb{Z}_2, \{1\}) = \overline{\Gamma}(\mathbb{Z}_2, \{1\}) \), then \( \Gamma(R, S) = \overline{\Gamma}(R, S) \) and so by Lemma 3.1(a), for any vertex \( x \) of \( \Gamma(R, S) \), \( \deg(x) \geq |U(R)| \). On the other hand, since \( m = \{0\} \times R_1 \times R_2 \) is a maximal ideal of \( R \) such that \(|\frac{R}{m}| = 2\), then by Lemma 3.1, \( \Gamma(R, S) \) is a bipartite graph. Hence, \( \Gamma(R, S) \) has no triangles. Now, consider the following cases:

**Case 1:** \(|R_1| = |R_2| = 3\). In this case \(|R| = 18\) and \(|U(R)| = 4\). It follows that \( \Gamma(R, S) \) has at least 36 edges. Thus, by second part of Corollary 2.5, \( \gamma(\Gamma(R, S)) \geq 2 \).

**Case 2:** \(|R_1| = 3\) and \(|R_2| = 4\). In this case \(|R| = 24\) and \(|U(R)| \geq 4\). Hence, \( \Gamma(R, S) \) has at least 48 edges and so by second part of Corollary 2.5, \( \gamma(\Gamma(R, S)) \geq 2 \).

**Case 3:** \(|R_1| = |R_2| = 4\). In this case \(|R| = 32\) and \(|U(R)| \geq 4\). Thus, \( \Gamma(R, S) \) has at least 64 edges and so by second part of Corollary 2.5, \( \gamma(\Gamma(R, S)) \geq 2 \).

\( \Box \)

**Theorem 4.10.** Let \( R \) be a non-local finite ring. Then \( \Gamma(R, S) \) is projective if and only if \( R \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) with \( S = \{(-1, -1)\}, S = \{(1, -1), (-1, -1)\} \) or \( S = \{(-1, 1), (-1, -1)\} \).
Proof. Suppose that \( \tilde{\gamma}(\Gamma(R, S)) = 1 \). Then by Lemma 4.1, \(|U(R)| \leq 6 \) and \(|J(R)| \leq 3 \). On the other hand, by Theorem 3.8, \( R \cong (\mathbb{Z}_2)^\ell \times T \), where \( 0 \leq \ell \leq 1 \) and \( T \) is a ring with \(|T| \leq 16 \). Since \( T \) is a finite ring, it is a finite direct product of finite local rings. Now, by Lemmas 4.4, 4.6, 4.7, 4.8 and 4.9, either \( \Gamma(R, S) \sim \mathbb{Z}_3 \) or \( R \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) with \( S \neq \{(1,1)\} \) or \( R \cong \mathbb{Z}_3 \times S \) with \( S \neq \{(1,1)\}, S \neq \{(1,-1)\} \) and \( S \neq \{(-1,1)\} \).

Case 1: \( R \cong \mathbb{Z}_2 \times \mathbb{F}_4 \) with \( S \neq \{(1,1)\} \). In this case \( S = \{1\} \times S' \), where \(|S'| \geq 2 \). By Lemma 3.1(b)(i),

\[
\Gamma(R, S) \cong \Gamma(\mathbb{Z}_2, \{1\}) \otimes \Gamma(\mathbb{F}_4, S')
\]

On the other hand, since \( \mathbb{F}_4 \) is a field and \(|S'| \geq 2 \), in view of the proof of Lemma 3.2, all vertices in \( \Gamma(\mathbb{F}_4, S') \) are adjacent with the exception that \( 0 \) is not adjacent to \( 0 \). Hence, \( \Gamma(R, S) \), as shown in Figure 2, is isomorphic to the graph \( E_{18} \) which is one of the 103 graphs listed in [14]. Thus, in this case \( \Gamma(R, S) \) is not projective.

Figure 2. The graph \( \Gamma(\mathbb{Z}_2 \times \mathbb{F}_4, S) \) with \( S \neq \{(1,1)\} \).

Case 2: \( R \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) with \( S \neq \{(1,1)\}, S \neq \{(1,-1)\} \) and \( S \neq \{(-1,1)\} \). In this case by Lemma 4.1, \( \Gamma(R, S) \) is not planar and so \( \tilde{\gamma}(\Gamma(R, S)) \geq 1 \). If \( S = \{(-1,1)\} \), then Figure 3 shows that \( \Gamma(R, S) \) can be embedded in the projective plane. Thus, \( \Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(-1,1)\}) \) is projective.

If either \( S = \{(1,1),(-1,1)\} \) or \( S = \{(1,-1),(-1,1)\} \), then every vertex in \( \{(0,0),(0,1),(0,2),(1,0)\} \) is adjacent to every vertex in \( \{(1,1),(1,2),(2,1),(2,2)\} \). Hence, \( K_{4,4} \) is a subgraph of \( \Gamma(R, S) \) and so by parts (b) and (d) of Lemma 2.1, \( \tilde{\gamma}(\Gamma(R, S)) \geq 2 \). If \(|S| \geq 3 \), then either \( \{(1,1),(-1,1)\} \) or \( \{(1,1),(-1,1)\} \) is a subset of \( S \) and so by Lemma 3.1(b)(ii) and Lemma 2.1(b), \( \tilde{\gamma}(\Gamma(R, S)) \geq 2 \).

If \( S = \{(1,1),(-1,1)\} \), then Figure 4 shows that \( \Gamma(R, S) \) contains a subgraph isomorphic to \( B_3 \) which is one of the 103 graphs listed in [14]. Hence, by Lemma 2.1(b), \( \Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,1),(-1,1)\}) \) is not projective.

If \( S = \{(1,1),(-1,1)\} \), then by Lemma 3.1(b)(i),

\[
\Gamma(R, S) = \Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,1),(-1,1)\})
\]

\[
\cong \Gamma(\mathbb{Z}_3, \{1\}) \otimes \Gamma(\mathbb{Z}_3, \{1\})
\]

\[
\cong \Gamma(\mathbb{Z}_3, \{1\}) \otimes \Gamma(\mathbb{Z}_3, \{-1\})
\]

\[
\cong \Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{1\} \times \{-1\})
\]

\[
= \Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,1),(-1,1)\}).
\]

This implies that \( \tilde{\gamma}(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,1),(-1,1)\})) \geq 2 \).
Figure 3. Embedding of $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(−1, −1)\})$ in the projective plane.

Figure 4. A sugraph of $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1, 1), (1, −1)\})$.

If $S = \{(1, −1), (−1, −1)\}$, then by Figure 3, $\Gamma(R, S)$ can be embedded in the projective plane. Hence, $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1, −1), (−1, −1)\})$ is projective.

Finally, if $S = \{(−1, 1), (−1, −1)\}$, then similarly by Lemma 3.1(b)(i),

$\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1, −1), (−1, −1)\})$.

It follows that $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(−1, 1), (−1, −1)\})$ is also projective.

Corollary 4.11. Let $R$ be a finite ring. Then $\Gamma(R, S)$ is projective if and only if one of the following conditions holds:

(a) $R \cong \mathbb{Z}_5$ with $S \neq \{1\}$.
(b) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ with $S = \{(−1, −1)\}$, $S = \{(1, −1), (−1, −1)\}$ or $S = \{(−1, 1), (−1, −1)\}$.

Proof. It is an immediate consequence from Theorems 4.5 and 4.10.

Corollary 4.12. There is no finite ring $R$ such that the unit graph $G(R)$ is projective.
Figure 5. Embedding of $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1, -1), (-1, -1)\})$ in the projective plane.

Proof. It follows from the fact that $G(R) = \Gamma(R, \{1\})$ and Corollary 4.11.

Corollary 4.13. Let $R$ be a finite ring. Then the unitary Cayley graph $\text{Cay}(R, U(R))$ is projective if and only if $R$ is isomorphic to $\mathbb{Z}_5$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Proof. Since $\text{Cay}(R, U(R)) = \Gamma(R, \{-1\})$, by Corollary 4.11 there is nothing to prove.

Some properties of the special case of the graph $\Gamma(R, S)$ in the case that $S = U(R)$ were studied by Naghipour et al. in [19]. As a consequence of Corollary 4.11 we have the following corollary.

Corollary 4.14. Let $R$ be a finite ring. Then $\Gamma(R, U(R))$ is projective if and only if $R$ is isomorphic to $\mathbb{Z}_5$.

5. Projective Co-maximal Graphs

Let $R$ be a ring. As in [6], we denote the co-maximal graph of $R$ by $C_T(R)$. For every $S \subseteq U(R)$ with $S^{-1} \subseteq S$, $\Gamma(R, S)$ is a subgraph of $C_T(R)$. The authors in [28] and [6] determined all finite rings whose co-maximal graph has genus at most one and genus two, respectively. The question first posed in [16] was “which rings have projective co-maximal graphs?” In this section, we characterize all Artinian rings $R$ whose co-maximal graphs $C_T(R)$ are projective.

Theorem 5.1. Let $R$ be an Artinian ring such that $\tilde{\gamma}(C_T(R)) < \infty$. Then $R$ is a finite ring.

Proof. Since $\Gamma(R, S)$ is a subgraph of $C_T(R)$, by Lemma 2.1(b), $\tilde{\gamma}(\Gamma(R, S)) \leq \tilde{\gamma}(C_T(R))$. Hence, $\tilde{\gamma}(\Gamma(R, S) < \infty$ and so by Theorem 3.5 $R$ is a finite ring.

The following corollary is an immediate consequence from Lemma 2.1(a) and Theorem 5.1.
Corollary 5.2. Let \( R \) be an Artinian ring such that \( \gamma(C_\Gamma(R)) < \infty \). Then \( R \) is a finite ring.

Remark 5.3. Let \( R \) be a finite ring such that \( \tilde{\gamma}(C_\Gamma(R)) = k > 0 \). Since \( \Gamma(R, S) \) is a subgraph of \( C_\Gamma(R) \), by Lemma 2.1(b) and Corollary 3.9 \( |R| \leq 32k \). In particular, for any positive integer \( k \), the number of finite rings \( R \) such that \( \tilde{\gamma}(\Gamma(R, S)) = k \) is finite. Also, by Lemma 2.1(a), for a given positive integer \( g \), the number of finite rings \( R \) such that \( \gamma(C_\Gamma(R)) = g \) is finite.

Lemma 5.4. ([28 Corollary 5.3]) Let \( R \) be a finite ring. Then \( C_\Gamma(R) \) is planar if and only if \( R \) is isomorphic to one of the following rings:

\[
\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2[x]/(x^2), \quad \mathbb{F}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_3 \quad \text{or} \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.
\]

Theorem 5.5. Let \( R \) be a finite ring. Then \( C_\Gamma(R) \) is projective if and only if \( R \) is isomorphic to one of the following rings:

\[
\mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2) \quad \text{or} \quad \mathbb{Z}_5.
\]

Proof. Suppose that \( \tilde{\gamma}(C_\Gamma(R)) = 1 \). Since \( \Gamma(R, S) \) is a subgraph of \( C_\Gamma(R) \), by Lemma 2.1(b), \( \tilde{\gamma}(\Gamma(R, S)) \leq 1 \), for every \( S \subseteq U(R) \) with \( S^{-1} \subseteq S \). Thus, by Lemma 4.4 Corollary 4.11 and Lemma 5.4, we have the following candidates for \( R \):

(a) \( \mathbb{Z}_\ell^2 \) where \( \ell \geq 4 \).

(b) \( (\mathbb{Z}_2)^\ell \times \mathbb{Z}_3 \) where \( \ell \geq 2 \).

(c) \( (\mathbb{Z}_2)^\ell \times T \) where \( \ell \geq 1 \) and \( T \) is isomorphic to \( \mathbb{Z}_4 \) or \( \mathbb{Z}_2[x]/(x^2) \).

(d) \( \mathbb{Z}_5 \).

Case 1: \( R \cong \mathbb{Z}_\ell^2 \) where \( \ell \geq 4 \). If \( \ell = 4 \), then Figure 6 shows that \( C_\Gamma(R) \) contains a subdivision of \( K_{4,4} \) and so by parts (b) and (d) of Lemma 2.1, \( \tilde{\gamma}(C_\Gamma(R)) \geq 2 \). If \( \ell \geq 5 \), then \( C_\Gamma(\mathbb{Z}_\ell^2) \) is a subgraph of \( C_\Gamma(R) \) and so by Lemma 2.1(b), \( \tilde{\gamma}(C_\Gamma(R)) \geq 2 \).

Case 2: \( R \cong (\mathbb{Z}_2)^\ell \times \mathbb{Z}_3 \) where \( \ell \geq 2 \). If \( \ell = 2 \), then it is easy to check that the graph \( C_\Gamma(R) \) has 35 edges (see also [6 Figure 8]) and so by Corollary 2.5 \( \tilde{\gamma}(C_\Gamma(R)) \geq 2 \). If \( \ell \geq 3 \), then \( C_\Gamma(\mathbb{Z}_2^\ell) \) is a subgraph of \( C_\Gamma(R) \) and so by Lemma 2.1(b), \( \tilde{\gamma}(C_\Gamma(R)) \geq 2 \).

![Figure 6](image-url)
**Case 3:** $R \cong (\mathbb{Z}_2)^\ell \times T$ where $\ell \geq 1$ and $T$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. Note that $C_1((\mathbb{Z}_2)^\ell \times \mathbb{Z}_4) \cong C_1((\mathbb{Z}_2)^\ell \times \mathbb{Z}_2[x]/(x^2))$. Hence, it is enough to consider the case $R \cong (\mathbb{Z}_2)^\ell \times \mathbb{Z}_4$. If $\ell = 1$, then Figure 7 shows that the graph $C_\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$ can be embedded in the projective plane. If $\ell = 2$, then all the vertices $(1,1,0), (1,1,1), (1,1,2)$ and $(1,1,3)$ are adjacent to the vertices $(0,0,1), (0,1,1), (1,0,1)$ and $(1,0,3)$ in $C_\Gamma(R)$. Thus, $K_{4,4}$ is a subgraph of $C_\Gamma(R)$ and so by parts (b) and (d) of Lemma 2.1, $\bar{\gamma}(C_\Gamma(R)) \geq 2$. Finally, if $\ell \geq 3$, then $C_\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)$ is a subgraph of $C_\Gamma(R)$ and so by Lemma 2.1(b), $\bar{\gamma}(C_\Gamma(R)) \geq 2$.

**Figure 7.** Embedding of $C_\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$ in the projective plane.

**Case 4:** $R \cong \mathbb{Z}_5$. Note that $\Gamma(R, \{-1\})$ is a subgraph of $C_\Gamma(R)$ and by Lemma 3.2, $\Gamma(R, \{-1\}) \cong K_5$. Thus, $C_\Gamma(R) \cong K_5$ and so by Lemma 2.1(c), $C_\Gamma(R)$ is projective. \hfill $\square$

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