Abstract

We study the non-linear dynamics of self-gravitating irrotational dust in a general relativistic framework, using synchronous and comoving (i.e. Lagrangian) coordinates. All the equations are written in terms of a single tensor variable, the metric tensor of the spatial sections orthogonal to the fluid flow. This treatment allows an unambiguous expansion in inverse (even) powers of the speed of light. To lowest order, the Newtonian approximation – in Lagrangian form – is derived and written in a transparent way; the corresponding Lagrangian Newtonian metric is obtained. Post-Newtonian corrections are then derived and their physical meaning clarified. A number of results are obtained: i) the master equation of Lagrangian Newtonian dynamics, the Raychaudhuri equation, can be interpreted as an equation for the evolution of the Lagrangian–to–Eulerian Jacobian matrix, complemented by the irrotationality constraint; ii) the Lagrangian spatial metric reduces, in the Newtonian limit, to that of Euclidean 3-space written in time–dependent curvilinear coordinates, with non–vanishing Christoffel symbols, but vanishing spatial curvature (a particular example of it is given within the Zel’dovich approximation); iii) a Lagrangian version of the Bernoulli equation for the evolution of the “velocity potential” is obtained. iv) The Newtonian and post–Newtonian content of the electric and magnetic parts of the Weyl tensor is clarified. v) At the Post–Newtonian level, an exact and general formula is derived for gravitational–wave emission from non–linear cosmological perturbations; vi) a straightforward application to the anisotropic collapse of homogeneous ellipsoids shows that the ratio of these post–Newtonian terms to the Newtonian ones tends to diverge at least like the mass density. vii) It is argued that a stochastic gravitational–wave background is produced by non–linear cosmic structures, with present–day closure density $\Omega_{gw} \sim 10^{-5} - 10^{-6}$ on 1 – 10 Mpc scales.

Key words: gravitation – hydrodynamics – instabilities – cosmology: theory – large–scale structure of Universe.
1 Introduction

The gravitational instability of collisionless matter in a cosmological framework is usually studied within the Newtonian approximation, which basically consists in neglecting terms whose order is higher than the first in metric perturbations around a matter–dominated Friedmann–Robertson–Walker (FRW) background, while keeping non–linear density perturbations. This approximation is usually thought to produce accurate results in a wide spectrum of cosmological scales, namely on scales much larger than the Schwarzschild radius of collapsing bodies and much smaller than the Hubble horizon scale, where the peculiar gravitational potential \( \varphi_g \), divided by the square of the speed of light \( c^2 \) to obtain a dimensionless quantity, keeps much less than unity, while the peculiar matter flow never becomes relativistic. To be more specific, the Newtonian approximation consists in perturbing only the time–time component of the FRW metric tensor by an amount \( 2\varphi_g/c^2 \), where \( \varphi_g \) is related to the matter density fluctuation \( \delta \) via the cosmological Poisson equation,

\[
\nabla^2_x \varphi_g(x, t) = 4\pi G \rho_b(t) \delta(x, t),
\]

where \( \rho_b \) is the background matter density and \( a(t) \) the appropriate FRW scale–factor; the Laplacian operator \( \nabla^2_x \) has been used here with its standard meaning of Euclidean space. The fluid dynamics is then usually studied in Eulerian coordinates by accounting for mass conservation and using the cosmological version of the Euler equation for a self–gravitating pressureless fluid – as long as the flow is in the laminar regime – to close the system. To motivate the use of this “hybrid approximation”, which deals with perturbations of the matter and the geometry at a different perturbative order, one can either formally expand the correct equations of General Relativity (GR) in inverse powers of the speed of light (e.g. Weinberg 1972), or simply notice that the peculiar gravitational potential is strongly suppressed with respect to the matter perturbation by the square of the ratio of the perturbation scale \( \lambda \) to the Hubble radius \( r_H = cH^{-1} \) (\( H \) being the Hubble constant): \( \varphi_g/c^2 \sim \delta (\lambda/r_H)^2 \).

Such a simplified approach, however, already fails in producing an accurate description of the trajectories of relativistic particles, such as photons. Neglecting the relativistic perturbation of the space–space components of the metric, which in the so–called longitudinal gauge is just \(-2\varphi_g/c^2\), would imply a mistake by a factor of two in well–known effects such as the Rees–Sciama (1968) and gravitational lensing (e.g. Schneider, Ehlers & Falco 1992), as it would be easy to see, by looking at the solution of the eikonal equation. In other words, the level of accuracy not only depends on the peculiar velocity of the matter producing the spacetime curvature, but also on the nature of the particles carrying the signal to the observer. Said this way, it may appear that the only relativistic correction required to the usual Eulerian Newtonian picture is that of writing the metric tensor in the revised, “weak field”, form (e.g. Peebles 1993)

\[
\begin{align*}
\begin{cases}
\frac{dr}{dt} = &\frac{\varphi_g}{c^2} - \frac{2}{c^2} \varphi_g - \frac{2}{c^2} \varphi_g \\
\frac{d\theta}{dt} = &0
\end{cases}
\end{align*}
\]

However, as we are going to show, this is not the whole story. It is well–known in fact that the gravitational instability of aspherical perturbations (which is the generic case) leads to the
formation of very anisotropic structures whenever pressure gradients can be neglected (Lynden-Bell 1962; Lin, Mestel & Shu 1965; Zel’dovich 1970; Icke 1973; White & Silk 1979; Shandarin et al. 1995). Matter first flows in almost two–dimensional structures called pancakes, which then merge and fragment to eventually form one–dimensional filaments and point–like clumps. During the process of pancake formation the matter density, the shear and the tidal field formally become infinite along evanescent two–dimensional configurations corresponding to caustics; after this event a number of highly non–linear phenomena, such as vorticity generation by multi–streaming, merging, tidal disruption and fragmentation, occur. Most of the pathology of the caustic formation process, such as the local divergence of the density, shear and tide, and the formation of multi–stream regions, are just an artifact of extrapolating the pressureless fluid approximation beyond the point at which pressure gradients and viscosity become important. In spite of these limitations, however, it is generally believed that the general anisotropy of the collapse configurations, either pancakes or filaments, is a generic feature of cosmological structures originated through gravitational instability, which would survive even in the presence of a collisional component.

This simple observation already shows the inadequacy of the standard Newtonian paradigm. According to it, the lowest scale at which the approximation can be reasonably applied is set by the amplitude of the gravitational potential and is given by the Schwarzschild radius of the collapsing body, which is negligibly small for any relevant cosmological mass scale. What is completely missing in this criterion is the role of the shear which causes the presence of non–scalar contributions to the metric perturbations. A non–vanishing shear component is in fact an unavoidable feature of realistic cosmological perturbations and affects the dynamics in (at least) three ways, all related to non–local effects, i.e. to the interaction of a given fluid element with the environment.

First, at the lowest perturbative order the shear is related to the tidal field generated by the surrounding material by a simple proportionality law (because of this linear coincidence, in much of the literature “shear” and “tide” are used as synonyms). This sort of non–locality, however, is coded in the initial conditions of each fluid–element through a Coulomb–like interaction with arbitrarily distant matter. Because of its link with the initial data of each fluid element one can consider it as a local property. The later modification of these shear and tidal fields is one of the consequences of the non–linear evolution.

Second, it is related to a dynamical tidal induction: the modification of the environment forces the fluid element to modify its shape and density. In Newtonian gravity, this is an action–at–a–distance effect, which starts to manifest itself in second–order perturbation theory as an inverse–Laplacian contribution to the velocity potential (e.g. Catelan et al. 1995, and references therein).

Third, and most important here, a non–vanishing shear field leads to the generation of a traceless and divergenceless metric perturbation which can be understood as gravitational radiation emitted by non–linear perturbations. This contribution to the metric perturbations is statistically small on cosmologically interesting scales, but it becomes relevant whenever anisotropic (with the only exception of exactly one–dimensional) collapse takes place. In the Lagrangian picture considered here, such an effect already arises at the post–Newtonian (PN)
Note that the two latter effects are only detected if one allows for non–scalar perturbations in physical quantities. Contrary to a widespread belief, in fact, the choice of scalar perturbations in the initial conditions is not enough to prevent tensor modes to arise beyond the linear regime in a GR treatment. Truly tensor perturbations are dynamically generated by the gravitational instability of initially scalar perturbations, independently of the initial presence of gravitational waves.

This point is very clearly displayed in the GR Lagrangian second–order perturbative approach. The pioneering work in this field is by Tomita, who, back in 1967, calculated the gravitational waves emitted by non–linearly evolving scalar perturbations in an Einstein–de Sitter background, in the synchronous gauge (Tomita 1967). Matarrese, Pantano & Saez (1994a,b) obtained an equivalent result but with a different formalism in comoving and synchronous coordinates. According to these calculations, a traceless and divergenceless contribution to the spatial metric in the synchronous gauge,

$$\pi_{\alpha\beta}$$

[greek indices label Lagrangian spatial coordinates, while capital latin letters will label Eulerian space; lower–case latin indices will be used for spacetime coordinates], is produced, which, with growing mode initial conditions, obeys the inhomogeneous wave–equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} + \frac{4}{c^2}\frac{\partial}{\partial \tau} - \nabla_q^2\right)\pi_{\alpha\beta} = -\frac{\tau^4}{42}\nabla_q^2 S_{\alpha\beta},\quad (3)$$

where $\tau \propto t^{1/3}$ is the conformal time. The non–linear source tensor $S_{\alpha\beta}$ is given in terms of the initial peculiar gravitational potential, $\varphi_0 \equiv \varphi_g(t_0)$, by

$$S_{\alpha\beta} = \delta_{\beta}^{\alpha}\nabla_q^2\psi_0 + \psi_0,_{\alpha\beta} + 2(\varphi_0,_{\alpha\beta}\nabla_q^2\varphi_0 - \varphi_0,_{\alpha\mu}\varphi_0,_{\beta\mu}),\quad (4)$$

with

$$\nabla_q^2\psi_0 = -\frac{1}{2}\left((\nabla_q^2\varphi_0)^2 - \varphi_0,_{\mu}\varphi_0,_{\nu}\varphi_0,_{\mu}\right),\quad (5)$$

where spatial gradients, indicated by greek indices after a comma, are with respect to the Lagrangian coordinates $q^\alpha$ and indices are raised by the Kronecker symbol; finally $\nabla_q^2$ is just the standard (i.e. Euclidean) Laplacian in Lagrangian coordinates. To get from the above formula a form which can be compared with the standard Newtonian interpretation, we can expand $\pi_{\alpha\beta}$ in powers of $1/c^2$ (as we will see below, the absence of odd powers of the speed of light is a characteristic feature of the Lagrangian approach),

$$\pi_{\alpha\beta} = \pi^{(N)\alpha\beta} + \frac{1}{c^2}\pi^{(PN)\alpha\beta} + O(\frac{1}{c^4}).$$

To zeroth order one obtains the Newtonian term $\pi^{(N)\alpha\beta} = \frac{\tau^4}{42}S_{\alpha\beta}$ [which includes a non–local and non–causal contribution to the shear tensor through derivatives of the potential $\psi_0$ defined above. The meaning of this contribution has been discussed by Matarrese, Pantano & Saez (1994a,b), who obtained it by looking at perturbation scales much smaller than the Hubble radius; it represents the “relic” of a causal signal which, on sub–horizon scales, appears as an instantaneous Newtonian feature. To first order in $1/c^2$ one then gets a PN contribution,

\footnote{Actually, this expression is determined up to a harmonic divergenceless and traceless tensor, which can be set to zero if we require consistency with the standard Newtonian second–order results.}
\[ \nabla^2 \pi^{(PN)}_{\alpha\beta} = \frac{2\pi^2}{3} S^{\alpha\beta}, \] once again the causality of this gravitational–wave signal is lost because of the \(1/c^2\) expansion. In the formalism of (Matarrese, Pantano & Saez 1994a, b) this PN term would be detected as a sub–leading contribution for perturbation scales much smaller than the Hubble radius. The close relation between the two approximation schemes – inverse powers of the speed of light and powers of the ratio of the perturbation to the horizon scale – also helps in better understanding the actual physical meaning of the \(1/c^2\) expansion in the Lagrangian picture.

The latter PN effect will be recovered in Section 4 without any restriction to a second–order perturbation treatment. A heuristic estimate of the amplitude of this effect in the frame of current scenarios of cosmological structure formation is reported in Section 5. One can also speculate on the possibility to detect the resulting stochastic gravitational–wave background, e.g., through the secondary anisotropy it would induce on the Cosmic Microwave Background (CMB).

There is an intriguing aspect of the above expression for the tensor modes in the PN limit, namely \[ \nabla^2 \pi^{(PN)}_{\alpha\beta} = \frac{2\pi^2}{3} S^{\alpha\beta}. \] Our explicit PN result of Section 4, if further approximated to a perturbative second–order expansion around the FRW background, gives \[ \nabla^2 \pi^{(PN)}_{\alpha\beta} = \frac{\tau^2}{9} S^{\alpha\beta}, \] i.e. a factor of 6 smaller! Should one conclude that the \(1/c^2\) expansion and that in the amplitude of the perturbations around FRW do not commute? The explanation is actually much simpler: the splitting of the various perturbation modes into scalars, vectors ans tensors is obviously background–dependent; in particular, our PN tensor modes are defined with respect to a non–perturbative Newtonian background, while the tensor modes obtained in second–order perturbation theory are defined with respect to the usual FRW solution. When the PN expressions for the various geometric modes are further expanded to second order in perturbation theory and then compared to those obtained through a second–order followed by a \(1/c^2\) expansion the results do not generally coincide (because the Newtonian background itself contains perturbations of the FRW metric), but their sum, i.e. the overall metric perturbation, does. This completely accounts for the different numerical factor.

Finally, Eq.(3) allows to understand another important point: the complete insensitivity of the Newtonian approximation to the possible presence of free gravitational waves in the initial conditions, such as those produced by quantum effects in the early universe. These initial tensor modes, corresponding to solutions of the homogeneous equation associated to Eq.(3), would reduce to harmonic, transverse and traceless metric perturbations in the Newtonian limit, having no effect on physical quantities (they are gauge modes from the point of view of the Newtonian equations).

The reader at this point may be confused by the continuous interchange of Newtonian and PN concepts. However, this will appear unavoidable once one realizes that, as in any perturbative treatment (the perturbation parameter here being formally \(1/c^2\)), there are equations which mix different perturbation orders. So, the PN equations will have Newtonian sources, or read the other way around, there are Newtonian effects which are produced by PN sources. This point has been definitely clarified in a fundamental paper by Kofman & Pogosyan (1995), who showed how the Newtonian “electric” tidal field \(E_{\alpha\beta}^{\alpha}\) evolves in time according to a PN equation, so that the circulation of the PN “magnetic” Weyl tensor \(H_{\alpha\beta}^{\alpha}\), happens to be respon-
sible for the Newtonian non–local “tidal induction”. Bertschinger & Hamilton (1994) gave a different interpretation of the same effect.

Recently a number of different approaches to relativistic effects in the non–linear dynamics of cosmological perturbations have been proposed. Matarrese, Pantano & Saez (1993) proposed an algorithm based on neglecting the magnetic part of the Weyl tensor in the dynamics, obtaining strictly local fluid–flow evolution equations, i.e. the so–called “silent universe”. Using this formalism Bruni, Matarrese & Pantano (1995a) studied the asymptotic behaviour of the system, both for collapse and expansion, showing, in particular, that this kind of local dynamics generically leads to spindle singularities for collapsing fluid elements, thereby confirming the results of a previous analysis by Bertschinger & Jain (1994). This formalism, however, cannot be applied to cosmological structure formation inside the horizon, where the non–local tidal induction cannot be neglected, i.e. the magnetic Weyl tensor $H_{\alpha\beta}$ is non–zero, with the exception of highly specific initial configurations (Matarrese, Pantano & Saez 1994a; Bertschinger & Jain 1994). Rather, it is probably related to the non–linear dynamics of an irrotational fluid outside the (local) horizon (Matarrese, Pantano & Saez 1994a,b). One possible application (Bruni, Matarrese & Pantano 1995b), is in fact connected to the so–called Cosmic No–hair Theorem (e.g. Hawking & Moss 1982), i.e. to the conjecture that expanding patches of an initially inhomogeneous and anisotropic universe asymptotically tend to almost FRW solutions, thanks to the action of a cosmological constant–like term. The self–consistency of these “silent universe” models has been recently demonstrated by Lesame, Dunsby & Ellis (1995), extending an earlier analysis by Barnes & Rowlingson (1989). Lesame, Ellis & Dunsby (1996) discussed the role of the divergence of the magnetic Weyl tensor, whose presence reflects the fact that the shear and the electric tide generally have a different eigenframe. A local–tide approximation for the non–linear evolution of collisionless matter, which tries to overcome some limitations of the Zel’dovich approximation (Zel’dovich 1970), has been recently proposed by Hui & Bertschinger (1995).

In this work we will follow the more “conservative” approach of expanding the Einstein and continuity equations in inverse powers of the speed of light, which will then define a Newtonian limit and, at the next order, post–Newtonian corrections. The newer aspect of our approach is the choice of gauge: we use synchronous and comoving coordinates, because of which our approach can be legitimately called a Lagrangian one. Thanks to this choice, the dynamical variables involved are quite different to the standard ones; the gravitational potential, for instance, never appears explicitly in our expansion.

Various approaches have been proposed in the literature, which are somehow related to the present one. A PN approximation has been followed by Futamase (1988, 1989, 1991) to describe the dynamics of a clumpy universe; he however used non–comoving coordinates and focused his analysis on applications related to the so–called averaging problem in cosmology (e.g. Ellis 1984). Tomita (1988, 1991) also used non–comoving coordinates in a PN approach to cosmological perturbations. Shibata & Asada (1995) recently developed a PN approach to cosmological perturbations, but they also used non–comoving coordinates. Kasai (1995) [see

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2 Actually, a non–zero $\text{div}H$ only appears as a third–order effect in the amplitude of perturbations around FRW, unless the initial conditions contain a mixture of growing and decaying modes.
also (Kasai 1992, 1993) analyzed the non-linear dynamics of dust in the synchronous and comoving gauge; his approximation methods are however largely different. Finally, in a series of papers, based on the Hamilton–Jacobi approach (Croudace et al. 1994; Parry et al. 1994; Salopek, Stewart & Croudace 1994) a new approximation technique has been developed, which relies on an expansion in higher and higher gradients of an initial perturbation “seed”. In spite of its elegance and generality, however, this approximation scheme is by construction unable to reproduce the non-local aspects of the gravitational instability on sub–horizon scales; more specifically, terms containing the inverse of the Laplacian operator, which are unavoidable in the Newtonian limit, would formally require an infinite series of terms in a gradient expansion.

To help the reader from not being too much confused by the various perturbative techniques adopted in this work, we anticipate that our calculations contain, in different parts, three different kinds of expansion.

First, the entire paper is mostly based on an expansion in inverse even powers of the speed of light: the lowest order – or background – solution, in this case, describes the so-called Newtonian approximation in an expanding universe. Although in general we do not know the explicit form of the Newtonian background, we can safely assume it exists and use it to derive the next order terms. The result of first–order perturbation theory is then called post–Newtonian (PN); the second order would be the post–post–Newtonian (PPN) approximation. The range of application of this perturbative method has been already discussed above. Going to higher and higher orders would generally lead to a more accurate description of the system, account for some new relativistic effects, such as the generation of gravitational waves, and possibly allow for an extension of the range of scales to which the formalism can be applied.

Second, we will also use the most standard cosmological perturbation theory (see, e.g., Kodama & Sasaki 1984, and references therein; for a pressureless medium, see also Hwang 1994), which is basically an expansion in powers of the amplitude of the perturbations around a background, homogeneous and isotropic, FRW solution. The first–order, or “linear”, terms of the expansion are given in Section 2.4. No second–order calculations, in this sense, will be presented here, with the exception of Eqs.(3) – (5) reported above. The works by Tomita (1967), Matarrese, Pantano & Saez (1994a,b) and – in some respect – Pyne & Carroll (1995) follow precisely this perturbative approach up to second order and within GR. The range of applicability of this second perturbation technique is that of small fluctuations around a FRW background, but with no extra limitations on scale. Going to higher and higher orders here generally helps to follow the gravitational instability process on a longer time–scale and to account for new non–linear and non–local phenomena.

Third, there is another meaning of “perturbation theory” in Lagrangian coordinates, which is frequently used in the cosmological literature (e.g. Buchert 1995, and references therein). This refers to an expansion, within Newtonian gravity, in powers of the displacement vector from Lagrangian to Eulerian coordinates (Buchert 1989; Moutarde et al. 1991; Bouchet et al. 1992; Buchert 1992; Catelan 1995), the background being once more represented by the FRW models. The linear result is the so–called Zel’dovich approximation (see Section 3.2 below), while the second order terms are either called “second–order Lagrangian” or “post–Zel’dovich” (e.g. Munshi, Sahni & Starobinsky 1994). The peculiarity of this treatment,
at any order, is that, while the displacement vector is calculated from the equations at the required perturbative order, all the other dynamical variables, such as mass density, shear and so on, are calculated exactly from their non–perturbative definition. What comes out is a fully non–linear description of the system, which, though not being generally correct, “mimics” the true non–linear behaviour. This perturbation treatment basically exploits the advantages of the Lagrangian picture, leading, in particular, to a more accurate description of high density regions. Its limitations are generally set by the emerging of caustic singularities, besides those deriving from the underlying Newtonian approximation. A similar, Zel’dovich–like, approach can also be followed within GR [some progress in this direction has been recently made by Kasai (1995)]; this will be however the subject of a future investigation.

The plan of the paper is as follows. In Section 2 we introduce the GR Lagrangian formalism. Although we do not use the whole machinery of the ADM approach (Arnowitt, Deser & Misner 1962), some of the language is the same; in particular, the clear distinction between constraint and evolution equations plays a key role also in our work. Section 3 deals with the Newtonian limit of the GR equations in Lagrangian coordinates and gives a number of formal applications of the approach. Section 4 is instead devoted to the post–Newtonian limit of the GR equations. In particular, we discuss the dynamical role of gravitational waves generated by non–linear cosmic structures. It is shown that, during the collapse of a non–spherical (and non–planar) perturbation, PN tensor modes are produced, so as to give a dominant contribution near the collapse time. Conclusions are drawn in Section 5, which also contains a qualitative discussion on the amplitude of the PN gravitational–wave modes, as well as some speculations on their possible detectability.

2 Relativistic dynamics of irrotational dust in the Lagrangian picture

In this section we will derive the equations governing the evolution of an irrotational fluid of dust (i.e. \( p = \omega = 0 \)) in a synchronous and comoving system of coordinates (actually the possibility of making these two gauge choices simultaneously is a peculiarity of irrotational dust, which holds at any time, i.e. also beyond the linear regime). The starting point will be the Einstein equations \( R_{ab} - \frac{1}{2}g_{ab}R = \frac{8\pi G}{c^4}T_{ab} \), with \( R_{ab} \) the Ricci tensor, and the continuity equation \( T_{ab, a} = 0 \) for the matter stress–energy tensor \( T_{ab} = \rho c^2 u^a u^b \), where \( \rho \) is the mass density and \( u^a \) the fluid four–velocity (normalized to \( u^a u_a = -1 \)); a semicolon denotes covariant differentiation. The line–element reads

\[
\text{ds}^2 = -c^2 dt^2 + h_{\alpha\beta}(q,t) dq^\alpha dq^\beta .
\] (6)

The fluid four–velocity in comoving coordinates is \( u^a = (1, 0, 0, 0) \). A fundamental quantity of our analysis will be the velocity–gradient tensor, which is purely spatial,

\[
\Theta^a_{\beta} \equiv cu^a_{;\beta} = \frac{1}{2} h^{\alpha\gamma} h_{\alpha\beta} .
\] (7)
where a dot denotes partial differentiation with respect to the proper time $t$. The tensor $\Theta^\alpha_\beta$ represents the extrinsic curvature of the spatial hypersurfaces orthogonal to $u^a$.

Thanks to the spacetime splitting obtained in our frame, the 10 Einstein equations can be immediately divided into 4 constraints and 6 evolution equations. The time–time component of the Einstein equations is the so–called energy constraint of the ADM approach, which reads

$$\Theta^2 - \Theta^\alpha_\beta \Theta^\beta_\alpha + c^2 (^{(3)}R) = 16\pi G \rho,$$

where the volume–expansion scalar $\Theta$ is just the trace of the velocity–gradient tensor, $^{(3)}R$ is the trace of the three-dimensional Ricci curvature, $^{(3)}R^\alpha_\beta$, of the spatial hypersurfaces of constant time.

The space–time components give the momentum constraint,

$$\Theta^\alpha_\beta \Theta^\beta_\alpha = \Theta^\alpha_\alpha.$$

Finally, the space–space components represent the only truly evolution equations, i.e. those which contain second–order time derivatives of the metric tensor. They indeed govern the evolution of the extrinsic curvature tensor and read

$$\dot{\Theta}^\alpha_\beta + \Theta \Theta^\alpha_\beta + c^2 (^{(3)}R) = 4\pi G \rho \delta^\alpha_\beta.$$

Taking the trace of the last equation and combining it with the energy constraint, we obtain the Raychaudhuri equation (Raychaudhuri 1957),

$$\dot{\Theta} + \Theta^\alpha_\beta \Theta^\beta_\alpha + 4\pi G \rho = 0.$$

Mass conservation is provided by the equation

$$\dot{\rho} = -\Theta \rho.$$

Given that $\Theta = \frac{1}{2} h^{\alpha \gamma} \dot{h}_{\gamma \alpha} = \partial (\ln h^{1/2})/\partial t$, where $h \equiv \text{det} h_{\alpha \beta}$, we can write the solution of this equation in the form

$$\rho(q, t) = \rho_0(q) [h(q, t)/h_0(q)]^{-1/2}.$$

Here and in what follows quantities with a subscript 0 are meant to be evaluated at some initial time $t_0$.

Finally, let us introduce the so–called electric and magnetic parts of the Weyl tensor, which are both symmetric, flow–orthogonal and traceless. They read, respectively,

$$E^\alpha_\beta = \frac{1}{3} \delta^\alpha_\beta \left( \Theta^\mu_\nu \Theta^\nu_\mu - \Theta^2 \right) + \Theta \Theta^\alpha_\beta - \Theta^\gamma_\gamma \Theta^\gamma_\beta + c^2 \left( ^{(3)}R^\alpha_\beta - \frac{1}{3} \delta^\alpha_\beta ^{(3)}R \right)$$

and

$$H^\alpha_\beta = \frac{1}{2} h_{\gamma \nu} \left( \eta^{\mu \gamma \delta} \Theta^\alpha_\gamma \Theta^\delta_\beta + \eta^{\alpha \gamma \delta} \Theta^\mu_\gamma \Theta^\delta_\beta \right),$$

where $\eta^{\alpha \beta \gamma} = h^{-1/2} \epsilon^{\alpha \beta \gamma}$ is the three–dimensional, completely anti–symmetric Levi–Civita tensor relative to the spatial metric $h_{\alpha \beta}$ and $\epsilon^{\alpha \beta \gamma}$ is such that $\epsilon^{123} = 1$, etc...
Notice that, while the definition of the electric tide $E^\alpha_\beta$ is completely fixed, because of its well-known Newtonian limit, the magnetic tensor field has no straightforward Newtonian counterpart, and can be therefore defined up to arbitrary powers of the speed of light. The definition we are adopting here is the most straightforward one; it is such that no explicit powers of $c$ appear in Eq. (15), which means that its physical dimensions are $1/c$ those of $E^\alpha_\beta$. This choice can be motivated in analogy with electrodynamics, where the magnetic vector field is also scaled by $1/c$ with respect to the electric one. We will come back later, in Section 2.2 and Section 4.1, to the consequences of this choice.

2.1 Conformal rescaling and FRW background subtraction

With the purpose of studying gravitational instability in a FRW background, it is convenient to factor out the homogeneous and isotropic solutions of the above equations. To this aim we also perform a conformal rescaling of the metric with conformal factor $a(t)$, the scale–factor of FRW models, and change the time variable to the conformal time $\tau$, defined by $d\tau = dt/a(t)$.

The line–element is then written in the form

$$ds^2 = a^2(\tau)[ - c^2d\tau^2 + \gamma_{\alpha\beta}(q, \tau)dq^\alpha dq^\beta ],$$

where $a^2(\tau)\gamma_{\alpha\beta}(q, \tau) \equiv h_{\alpha\beta}(q, t(\tau))$. For later convenience we fix the Lagrangian coordinates $q^\alpha$ to have physical dimension of length and the conformal time variable $\tau$ to have dimension of time. As a consequence the spatial metric $\gamma_{\alpha\beta}$ is dimensionless, as is the scale–factor $a(\tau)$ which must be determined by solving the Friedmann equations for a perfect fluid of dust

$$\frac{(a')^2}{a} = \frac{8\pi G}{3}\rho a^2 - \kappa c^2,$$

$$2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2 + \kappa c^2 = 0 .$$

Here primes denote differentiation with respect to the conformal time $\tau$ and $\kappa$ represents the curvature parameter of FRW models, which, because of our choice of dimensions, cannot be normalized as usual. So, for an Einstein–de Sitter universe $\kappa = 0$, but for a closed (open) model we simply have $\kappa > 0$ ($\kappa < 0$). Let us also note that the curvature parameter is related to a Newtonian squared time–scale $\kappa_N$ through $\kappa_N \equiv \kappa c^2$ (e.g. Peebles 1980; Coles & Lucchin 1995); in other words $\kappa$ is an intrinsically PN quantity.

By subtracting the isotropic Hubble–flow, we introduce a peculiar velocity–gradient tensor

$$\vartheta^\alpha_\beta \equiv a \, c u^\alpha_\beta - \frac{a'}{a} \delta^\alpha_\beta = \frac{1}{2} \gamma^{\alpha\gamma} \gamma_{\gamma\beta}' ,$$

where $\tilde{u}^\alpha = (1/a, 0, 0, 0)$.

Thanks to the introduction of this tensor we can rewrite the Einstein’s equations in a more cosmologically convenient form. The energy constraint becomes

$$\vartheta^2 - \vartheta^\nu_\nu \vartheta^\nu_\mu + 4\frac{a'}{a} \vartheta^\nu_\mu + c^2 (\mathcal{R} - 6\kappa) = 16\pi Ga^2 \rho = 0 .$$
where $R^\alpha_\beta(\gamma) = a^2 \, (3) R^\alpha_\beta(h)$ is the conformal Ricci curvature of the three–space, i.e. that corresponding to the metric $\gamma_{\alpha\beta}$; for the background FRW solution $\gamma_{\alpha\beta}^{\text{FRW}} = (1 + \kappa q^2)^{-2} \delta_{\alpha\beta}$, one has $R^\alpha_\beta(\gamma_{\alpha\beta}^{\text{FRW}}) = 2\kappa \delta^\alpha_\beta$. We also introduced the density contrast $\delta \equiv (\rho - \rho_b)/\rho_b$.

The momentum constraint reads

$$\vartheta^\alpha_{\beta||\alpha} = \vartheta_{\beta},$$

(21)

where the double vertical bars denote covariant derivatives in the three–space with metric $\gamma_{\alpha\beta}$.

Finally, after replacing the density from the energy constraint and subtracting the background contribution, the extrinsic curvature evolution equation becomes

$$\vartheta^\alpha_{\beta} + 2 \frac{a'}{a} \vartheta^\alpha_{\beta} + \vartheta \vartheta^\alpha_{\beta} + \frac{1}{4} \left( \vartheta^\mu_{\nu} \vartheta^\nu_{\mu} - \vartheta^2 \right) \delta^\alpha_{\beta} + \frac{\kappa^2}{4} \left[ 4 R^\alpha_{\beta} - (\mathcal{R} + 2\kappa) \delta^\alpha_{\beta} \right] = 0 \, .$$

(22)

The Raychaudhuri equation for the evolution of the peculiar volume–expansion scalar $\vartheta$ becomes

$$\vartheta + \frac{a'}{a} \vartheta + \vartheta \vartheta_{\mu} \vartheta^\mu + 4\pi G a^2 \varrho \delta = 0 \, .$$

(23)

The main advantage of this formalism is that there is only one dimensionless (tensor) variable in the equations, namely the spatial metric tensor $\gamma_{\alpha\beta}$, which is present with its partial time derivatives through $\vartheta^\alpha_{\beta}$ [Eq.(19) above], and with its spatial gradients through the spatial Ricci curvature $R^\alpha_{\beta}$. The only remaining variable is the density contrast which can be written in the form

$$\delta(\mathbf{q}, \tau) = (1 + \delta_0(\mathbf{q})) \left[ \gamma(\mathbf{q}, \tau)/\gamma_0(\mathbf{q}) \right]^{-1/2} - 1 \, ,$$

(24)

where $\gamma \equiv \text{det} \, \gamma_{\alpha\beta}$. A relevant advantage of having a single tensorial variable, for our purposes, is that there can be no extra powers of $c$ hidden in the definition of different quantities.

### 2.2 Fluid–flow approach

Following the fluid–flow approach, described in the classical review by Ellis (1971) [see also Ehlers (1993)], we can alternatively describe our system in terms of fluid properties, in our case matter density, volume–expansion scalar and shear tensor, and two geometric tensors, the electric and magnetic parts of the Weyl tensor defined above. The derivation of the equations reported below is thoroughly described by Ellis (1971) and will not be reported here.

For most cosmological purposes it is convenient to adopt the conformal rescaling and FRW background subtraction described in the previous sub–section. Therefore, we can start by writing the continuity equation directly in terms of the density contrast $\delta$,

$$\frac{D\delta}{D\tau} + (1 + \delta) \vartheta = 0 \, ,$$

(25)

with $\frac{D}{D\tau}$ denoting convective differentiation with respect to the conformal time $\tau$. In our Lagrangian frame, however, and for a scalar field, convective differentiation and partial differentiation coincide. The formal solution of this equation is given by Eq.(24) above. The
peculiar volume–expansion scalar $\dot{\vartheta}$ obeys the Raychaudhuri equation which we can rewrite in the form

$$\frac{D\vartheta}{D\tau} + \frac{a'}{a} \vartheta + \frac{1}{3} \vartheta^2 + \sigma^\alpha_\beta \sigma^\beta_\alpha + 4\pi Ga^2 \vartheta \delta = 0 , \quad (26)$$

where $\sigma^\alpha_\beta \equiv \dot{\vartheta}^\alpha_\beta - \frac{1}{3} \delta^\alpha_\beta \dot{\vartheta}$ is the shear tensor. The shear, in turn, evolves according to

$$\frac{D\sigma^\alpha_\beta}{D\tau} + \frac{a'}{a} \sigma^\alpha_\beta + \frac{2}{3} \sigma^\alpha_\beta + \sigma^{\alpha}_\gamma \sigma^\gamma_\beta - \frac{1}{3} \delta^\alpha_\beta \sigma^\gamma_\delta \sigma^\delta_\gamma + \mathcal{E}^\alpha_\beta = 0 , \quad (27)$$

where we have rescaled the electric tide as $\mathcal{E}^\alpha_\beta \equiv a^2 E^\alpha_\beta$, which can be written in terms of our new variables as

$$\mathcal{E}^\alpha_\beta = \frac{1}{3} \delta^\alpha_\beta \sigma^{\mu}_\nu \sigma^\mu_\nu + \frac{1}{3} \vartheta \sigma^\alpha_\beta + \frac{a'}{a} \sigma^\alpha_\beta - \sigma^\alpha_\gamma \sigma^\gamma_\beta + c^2 \left( \mathcal{R}^\alpha_\beta - \frac{1}{3} \delta^\alpha_\beta \mathcal{R} \right) . \quad (28)$$

Note that, for a generic second rank tensor $A^\alpha_\beta$, one has

$$\frac{D A^\alpha_\beta}{D\tau} = \frac{dA^\alpha_\beta}{d\tau} + \sigma^\gamma_\alpha A^\gamma_\beta - \sigma^\gamma_\beta A^\gamma_\alpha , \quad (29)$$

where $\frac{d}{d\tau}$ denotes the total derivative with respect to $\tau$, which in comoving coordinates coincides with the partial one. The two last terms in the r.h.s. come from writing the Christoffel symbols in our gauge. It is then clear that when the $\frac{D}{D\tau}$ operator acts on either the shear or the complete $\dot{\vartheta}^\alpha_\beta$ tensor, the second and third term in the r.h.s. cancel each other and the convective and total differentiation coincide. This cancellation also occurs for a generic $A^\alpha_\beta$ if either the relevant Christoffel symbols vanish (as it is the case for the Newtonian limit in Eulerian coordinates) or the convective derivative acts on the eigenvalues of $A^\alpha_\beta$ and such a tensor has the same eigenvectors of $\sigma^\alpha_\beta$ [as it is the case for the electric tide in the “silent universe” case (Barnes & Rowlingson 1989; Matarrese, Pantano & Saez 1993; Bruni, Matarrese & Pantano 1995b)].

The electric tidal tensor evolves according to

$$\frac{D\mathcal{E}^\alpha_\beta}{D\tau} + \frac{a'}{a} \mathcal{E}^\alpha_\beta + \dot{\mathcal{E}}^\alpha_\beta + \delta^\alpha_\beta \sigma^\gamma_\delta \mathcal{E}^\delta_\gamma - \frac{3}{2} \left( \mathcal{E}^\alpha_\gamma \sigma^\gamma_\beta + \sigma^\alpha_\gamma \mathcal{E}^\gamma_\beta \right) -$$

$$-\frac{c^2}{2} \gamma^{\gamma\delta} \left( \tilde{\eta}^{\gamma\delta} \mathcal{H}^\alpha_{\gamma|\delta} + \tilde{\eta}^{\alpha\gamma\delta} \mathcal{H}^\eta_{\gamma|\delta} \right) + 4\pi G a^2 \vartheta \delta (1 + \delta) \sigma^\alpha_\beta = 0 , \quad (30)$$

where we have rescaled the magnetic tide as $\mathcal{H}^\alpha_\beta \equiv a^2 H^\alpha_\beta$ and redefined the Levi–Civita tensor so that $\eta^{\alpha\beta\gamma} = \gamma^{-1/2} \epsilon^{\alpha\beta\gamma}$ (for simplicity we used the same symbol after rescaling).

Finally, the magnetic Weyl tensor evolves according to

$$\frac{D\mathcal{H}^\alpha_\beta}{D\tau} + \frac{a'}{a} \mathcal{H}^\alpha_\beta + \dot{\mathcal{H}}^\alpha_\beta + \delta^\alpha_\beta \sigma^\gamma_\delta \mathcal{H}^\delta_\gamma - \frac{3}{2} \left( \mathcal{H}^\alpha_\gamma \sigma^\gamma_\beta + \sigma^\alpha_\gamma \mathcal{H}^\gamma_\beta \right) +$$

$$+ \frac{1}{2} \gamma^{\gamma\delta} \left( \eta^{\gamma\delta} \mathcal{E}^\alpha_{\gamma|\delta} + \eta^{\alpha\gamma\delta} \mathcal{E}^\eta_{\gamma|\delta} \right) = 0 . \quad (31)$$
Note that, following the discussion above, in the last two equations the convective time derivative must include the two terms proportional to the shear, as in Eq.(29). Note that, apart from the cases listed after Eq.(29), these two terms cannot be disregarded even in the Newtonian limit.

In the fluid–flow approach, besides the evolution equations, one has to satisfy a number of constraint equations. One has: the momentum constraint, which we rewrite in the form

$$\sigma_{\alpha\beta\|\alpha} = \frac{2}{3} \partial_{\beta} \vartheta,$$

(32)

the $H-\sigma$ constraint (which we actually used in Section 2 to define the magnetic tide in terms of derivatives of the spatial metric),

$$H_{\alpha\beta} = \frac{1}{2} \gamma_{\beta\mu} \left( \eta^{\mu\gamma\delta} \sigma^{\alpha}_{\gamma\delta} + \eta^{\alpha\gamma\delta} \sigma^{\mu}_{\gamma\delta} \right),$$

(33)

the div $E$ constraint,

$$E^{\alpha}_{\beta\|\alpha} = -\gamma_{\beta\mu} \gamma_{\alpha\nu} \eta^{\mu\lambda\gamma} \sigma^{\alpha}_{\lambda\gamma} H_{\gamma} + \frac{8\pi G}{3} a^{2} \vartheta \delta_{\beta},$$

(34)

and the div $H$ constraint

$$c^2 H^{\alpha}_{\beta\|\alpha} = \gamma_{\beta\mu} \gamma_{\alpha\nu} \eta^{\mu\lambda\gamma} \sigma^{\alpha}_{\lambda\gamma} E^{\alpha}_{\gamma},$$

(35)

In the above equations one also needs to know the three–metric $\gamma_{\alpha\beta}$. This can be obtained from the evolution equation

$$\gamma_{\alpha\beta}' = 2 \gamma_{\alpha\gamma} \vartheta^\gamma_{\beta},$$

(36)

which is however only valid in our Lagrangian coordinates. In order to completely fix the spatial dependence of the metric one also needs to specify the energy constraint (the trace of the Gauss–Codacci relations), which we rewrite in the form

$$c^2 (\mathcal{R} - 6\kappa) = \sigma^{\alpha}_{\beta} \sigma^{\beta}_{\alpha} - \frac{2}{3} \vartheta^2 - 4 \frac{a'}{a} \vartheta + 16\pi G a^2 \vartheta \delta,$$

(37)

Although we will not use the fluid–flow approach in this paper it is interesting to have the complete form of the equations, with the correct powers of $c^2$ included, in order to understand the Newtonian meaning of the electric and magnetic tide. We will come back to this point in Section 4.1.

### 2.3 Local Eulerian coordinates

Our intuitive notion of Eulerian coordinates, involving a universal absolute time and globally flat spatial coordinates is intimately Newtonian; nevertheless it is possible to construct a local coordinates system which reproduces this picture for a suitable set of observers. This issue has been already addressed by Matarrese, Pantano & Saez (1994a,b), who introduced local
Eulerian – FRW comoving – coordinates $x^A$ which are related to the Lagrangian ones $q^a$ via the Jacobian matrix with elements

$$J^A_a(q, \tau) \equiv \frac{\partial x^A}{\partial q^a} \equiv \delta^A_a + D^A_a(q, \tau), \quad A = 1, 2, 3,$$

(38)

where $D^A_a(q, \tau)$ is called deformation tensor. Each matrix element $J^A_a$ labelled by the Eulerian index $A$ can be thought as a three–vector, namely a triad, defined on the hypersurfaces of constant conformal time. As shown in (Matarrese, Pantano & Saez 1994a,b), they evolve according to

$$J^A_a' = \vartheta^\gamma_{\alpha} J^A_{\gamma},$$

(39)

which also follows from the condition of parallel transport of the triads relative to $q$ along the world–line of the corresponding fluid element $D(aJ^A_a)/Dt = 0$ (see also Kasai 1995).

Our local Eulerian coordinates are such that the spatial metric takes the Euclidean form $\delta_{AB}$, i.e.

$$\gamma_{ab}(q, \tau) = \delta_{AB} J^A_a(q, \tau) J^B_b(q, \tau).$$

(40)

Correspondingly the matter density can be rewritten in the suggestive form

$$\varrho(q, \tau) = \varrho_b(\tau)(1 + \delta_0(q))[J(q, \tau)/J_0(q)]^{-1},$$

(41)

where $J \equiv \det J^A_a$. Note that, contrary to the Newtonian case, it is generally impossible in GR to fix $J_0 = 1$, as this would imply that the initial Lagrangian space is conformally flat, which is only possible if the initial perturbations vanish.

### 2.4 Linear perturbation theory in Lagrangian coordinates

In this subsection we will deal with the linearization of the equations obtained in Section 2.1. This will be done mostly for pedagogical purposes, in that it will allow us to obtain a number of results which will turn out to be useful for the $1/c^2$ expansion. Apart from this, it can be interesting to re–obtain the classical results of linear theory in the comoving and synchronous gauge only in terms of the spatial metric coefficients.

Let us then write the spatial metric tensor of the physical (i.e. perturbed) space–time in the form

$$\gamma_{\alpha\beta} = \bar{\gamma}_{\alpha\beta} + w_{\alpha\beta},$$

(42)

with $\bar{\gamma}_{\alpha\beta}$ the spatial metric of the background space – in our case the maximally symmetric FRW one, $\bar{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta}^{FRW}$ – and $w_{\alpha\beta}$ a small perturbation. Also, we assume that the only non–geometric quantity in our equations, namely the initial density contrast $\delta_0$, is everywhere much smaller than unity.

As usual, we can take advantage of the maximal symmetry of the background FRW spatial sections to classify metric perturbations as scalars, vectors and tensors (e.g. Bardeen 1980). We then write

$$w_{\alpha\beta} = \chi \bar{\gamma}_{\alpha\beta} + \zeta_{(\alpha\beta} + \frac{1}{2}(\xi_{a\beta} + \xi_{b\alpha}) + \pi_{\alpha\beta},$$

(43)
with
\[ \xi^\alpha |_\alpha = \pi^\alpha _\alpha = \pi^\alpha _\beta |_\alpha = 0 , \] (44)
where a single vertical bar is used for covariant differentiation in the background three–space with metric \( \gamma_{\alpha\beta} \). In the above decomposition \( \chi \) and \( \zeta \) represent scalar modes, \( \xi^\alpha \) vector modes and \( \pi^\alpha _\beta \) tensor modes (indices being raised by the contravariant background three–metric).

Before entering into the discussion of the equations for these perturbation modes, let us quote a result which will be also useful in the next sections. In the \( \vartheta^\alpha _\beta \) evolution equation and in the energy constraint the combination \( P^\alpha _\beta \equiv 4R^\alpha _\beta - (R + 2\kappa)\delta^\alpha _\beta \) and its trace appear. To first order in the metric perturbation one has
\[ P^\alpha _\beta (w) = -2\left[ (\nabla^2 - 2\kappa)\pi^\alpha _\beta + \chi^\alpha _\beta + \kappa\chi\delta^\alpha _\beta \right] , \] (45)
where \( \nabla^2 (\cdot) \equiv (\cdot) |^\gamma _\gamma \). Only the scalar mode \( \chi \) and the tensor modes contribute to the three–dimensional Ricci curvature.

As well known, in linear theory scalar, vector and tensor modes are independent. The equation of motion for the tensor modes is obtained by linearizing the traceless part of the \( \vartheta^\alpha _\beta \) evolution equation, Eq.(22). One has
\[ \pi_{\alpha\beta}'' + 2\frac{a'}{a}\pi_{\alpha\beta}' - c^2(\nabla^2 - 2\kappa)\pi_{\alpha\beta} = 0 , \] (46)
which is the equation for the free propagation of gravitational waves in a FRW background (compare with Eq.(3) in the Einstein–de Sitter case). The general solution of this equation is well known (e.g. Weinberg 1972) and will not be reported here.

At the linear level, in the irrotational case, the two vector modes represent gauge modes which can be set to zero, \( \xi^\alpha = 0 \).

The two scalar modes are linked together through the momentum constraint, which leads to the relation
\[ \chi = \chi_0 + \kappa(\zeta - \zeta_0) . \] (47)
The energy constraint gives
\[ (\nabla^2 + 3\kappa)\left[ \frac{a'}{a}\zeta' + (4\pi Ga^2_g - \kappa c^2)(\zeta - \zeta_0) - c^2\chi_0 \right] = 8\pi G a^2_g \delta_0 , \] (48)
while the evolution equation gives
\[ \zeta'' + 2\frac{a'}{a}\zeta' = c^2\chi . \] (49)

An evolution equation only for the scalar mode \( \zeta \) can be obtained by combining together the evolution equation and the energy constraint; it reads
\[ (\nabla^2 + 3\kappa)\left[ \zeta'' + \frac{a'}{a}\zeta' - 4\pi G a^2_g(\zeta - \zeta_0) \right] = -8\pi G a^2_g \delta_0 . \] (50)
On the other hand, linearizing the solution of the continuity equation, Eq.(24), gives
\[ \delta = \delta_0 - \frac{1}{2}(\nabla^2 + 3\kappa)(\zeta - \zeta_0), \] (51)
which replaced in the previous equation gives
\[ \delta'' + \frac{a'}{a}\delta' - 4\pi Ga^2 \varrho_0\delta = 0. \] (52)

This is the well-known equation for linear density fluctuations, whose general solution can be found in (Peebles 1980). Once \( \delta(\tau) \) is known, one can easily obtain \( \zeta \) and \( \chi \), which completely solves the linear problem.

Eq.(50) above has been obtained in whole generality; we could have used instead the well-known residual gauge ambiguity of the synchronous coordinates to simplify its form. In fact, \( \zeta \) is determined up to a space–dependent scalar, which would neither contribute to the spatial curvature, nor to the velocity–gradient tensor. For instance, we could fix \( \zeta_0 \) so that \((\nabla^2 + 3\kappa)\zeta_0 = -2\delta_0\), so that the \( \zeta \) evolution equation takes the same form as that for \( \delta \).

In order to better understand the physical meaning of the two scalar modes \( \chi \) and \( \zeta \), let us consider the simplest case of an Einstein–de Sitter background (\( \kappa = 0 \)), for which \( a(\tau) \propto \tau^2 \). By fixing the gauge so that \( \nabla^2 \zeta_0 = -2\delta_0 \) we obtain \( \chi(\tau) = \chi_0 \) and
\[ \zeta(\tau) = \frac{c^2}{10} \chi_0 \tau^2 + B_0 \tau^{-3}, \] (53)
where the amplitude \( B_0 \) of the decaying mode is an arbitrary function of the spatial coordinates. Consistency with the Newtonian limit suggests \( \chi_0 \equiv -\frac{10}{c^2} \varphi_0 \), with \( \varphi_0 \) the initial peculiar gravitational potential, related to \( \delta_0 \) through \( \nabla^2 \varphi_0 = 4\pi G\varrho_0\delta_0 \). We can then write
\[ \zeta(\tau) = -\frac{1}{3} \varphi_0 \tau^2 + B_0 \tau^{-3}. \] (54)

This result clearly shows that, at the Newtonian level, the linearized metric is
\[ \gamma_{\alpha\beta} = \delta_{\alpha\beta} + \zeta_{\alpha\beta}, \] (55)
while the perturbation mode \( \chi \) is already PN. Note that also the tensor modes are at least PN.

These results also confirm the above conclusion that in the general GR case the initial Lagrangian spatial metric cannot be flat, i.e. \( J_0 \neq 1 \), because of the initial “seed” PN metric perturbation \( \chi_0 \).

### 3 Newtonian approximation

The Newtonian equations in Lagrangian form can be obtained from the full GR equations of Section 2.1 by an expansion in inverse powers of the speed of light; as a consequence of our
gauge choice, however, no odd powers of \( c \) appear in the equations, which implies that the expansion parameter can be taken to be \( 1/c^2 \). The physical meaning of this expansion has been already outlined in Section 1.

Let us then expand the spatial metric in a form analogous to that used in our linear perturbation analysis of Section 2.4:

\[
\gamma_{\alpha\beta} = \bar{\gamma}_{\alpha\beta} + \frac{1}{c^2} w_{\alpha\beta}^{(PN)} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{56}
\]

where we made explicit the \( c \) dependence of the metric perturbation. The actual convergence of the series requires that the PN metric perturbation \( \frac{1}{c^2} w_{\alpha\beta}^{(PN)} \) is much smaller than the background Newtonian metric \( \bar{\gamma}_{\alpha\beta} \). Let us first concentrate on the Newtonian metric; the properties of \( w_{\alpha\beta} \) will be instead considered in Section 4.

To lowest order in our expansion, the extrinsic curvature evolution equation, Eq.(22), and the energy constraint, Eq.(20), imply that \( \bar{\mathcal{P}}_{\beta}^\alpha \equiv \mathcal{P}_{\beta}^\alpha(\bar{\gamma}) = 0 \), and recalling that \( \kappa = \kappa_N/c^2 \),

\[
\bar{\mathcal{R}}_{\alpha\beta} \equiv \mathcal{R}_{\alpha\beta}(\bar{\gamma}) = 0 : \tag{57}
\]

_in the Newtonian limit the spatial curvature identically vanishes_ (e.g. Ellis 1971). This important conclusion implies that \( \bar{\gamma}_{\alpha\beta} \) can be transformed to \( \delta_{AB} \) globally, i.e. that one can write

\[
\bar{\gamma}_{\alpha\beta} = \delta_{AB} \bar{J}_A^A \bar{J}_B^B, \tag{58}
\]

with integrable Jacobian matrix coefficients. In other words, at each time \( \tau \) there exist _global Eulerian coordinates_ \( x^A \) such that

\[
x(q, \tau) = q + S(q, \tau), \tag{59}
\]

where \( S(q, \tau) \) is called the _displacement vector_, and the deformation tensor becomes in this limit

\[
\bar{D}_A^\alpha = \frac{\partial S^A}{\partial q^\alpha}. \tag{60}
\]

The Newtonian Lagrangian metric can therefore be written in the form

\[
\bar{\gamma}_{\alpha\beta}(q, \tau) = \delta_{AB} \left( \delta^A_\alpha + \frac{\partial S^A(q, \tau)}{\partial q^\alpha} \right) \left( \delta^B_\beta + \frac{\partial S^B(q, \tau)}{\partial q^\beta} \right). \tag{61}
\]

We can rephrase the above result as follows: the Lagrangian spatial metric in the Newtonian limit is that of Euclidean three–space in time–dependent curvilinear coordinates \( q^\alpha \), defined at each time \( \tau \) in terms of the Eulerian ones \( x^A \) by inverting Eq.(59) above. As a consequence, the Christoffel symbols involved in spatial covariant derivatives (which we will indicate by a single bar or by a nabla operator followed by greek indices) do not vanish, but the vanishing of the spatial curvature implies that these covariant derivatives always commute.

Contrary to the evolution equation and the energy constraint, the Raychaudhuri equation, Eq.(23) and the momentum constraint, Eq.(21), contain no explicit powers of \( c \), and therefore preserve their form in going to the Newtonian limit. These equations therefore determine the
background Newtonian metric $\bar{\gamma}_{\alpha\beta}$, i.e. they govern the evolution of the displacement vector $\mathbf{S}$.

The Raychaudhuri equation becomes the master equation for the Newtonian evolution; it takes the form

$$\ddot{\bar{\gamma}} + \frac{a'}{a} \bar{\gamma} + \bar{\gamma}^{\mu}_{\nu} \bar{\gamma}^{\nu}_{\mu} + 4\pi Ga^2 \rho_0 (\bar{\gamma}^{-1/2} - 1) = 0 ,$$

(62)

where

$$\bar{\gamma}^\alpha_{\beta} \equiv \frac{1}{2} \bar{\gamma}^\alpha_\gamma \bar{\gamma}_\gamma^\beta ,$$

(63)

and, for simplicity, we assumed $\delta_0 = 0$ (a restriction which is, however, not at all mandatory). We also used the residual gauge freedom of our coordinate system to set $\bar{\gamma}_{\alpha\beta}(\tau_0) = \delta_{\alpha\beta}$, implying $\bar{J}_0 = 1$, i.e. to make Lagrangian and Eulerian coordinates coincide at the initial time. That this choice is indeed possible in the Newtonian limit can be understood from our previous linear analysis, where this is achieved by taking, e.g., $\zeta_0 = 0$.

The momentum constraint,

$$\bar{\gamma}^{\mu}_{\nu|\mu} = \bar{\gamma}_{\nu\mu} ,$$

(64)

is actually related to the irrotationality assumption. We will come back to this point in the next section.

Before closing this section, let us notice a general property of our expression for the Lagrangian metric: at each time $\tau$ it can be diagonalized by going to the local and instantaneous principal axes of the deformation tensor. Calling $\bar{\gamma}_{\alpha}$ the eigenvalues of the metric tensor, $\bar{J}_{\alpha}$ those of the Jacobian and $\bar{d}_{\alpha}$ those of the deformation tensor, one has

$$\bar{\gamma}_{\alpha}(\mathbf{q}, \tau) = \bar{J}_{\alpha}^2(\mathbf{q}, \tau) = [1 + \bar{d}_{\alpha}(\mathbf{q}, \tau)]^2 .$$

(65)

In Section 3.2 below, the diagonal form of the metric tensor will be reconsidered in the frame of the Zel’dovich approximation. Beyond the mildly non–linear regime, where this approximation is consistently applied, diagonalizing the metric is in general, i.e. apart from specific initial configurations, of smaller practical use, because metric (and deformation) tensor, shear and tide generally have different eigenvectors.

From this expression it becomes evident that, at shell–crossing, where some of the Jacobian eigenvalues go to zero, the related covariant metric eigenvalues just vanish. On the other hand, other quantities, like the matter density, the peculiar volume expansion scalar and some eigenvalues of the shear and tidal tensor will generally diverge at the location of the caustics (see Bruni, Matarrese & Pantano 1995b, for a discussion). This diverging behaviour makes the description of the system extremely involved after this event. Although dealing with this problem is far outside the aim of the present paper, let us just mention that a number of ways out are available. One can convolve the various dynamical variables by a suitable low–pass filter, either at the initial time, in order to postpone the occurrence of shell–crossing singularities (e.g. Coles, Melott & Shandarin 1993; Kofman et al. 1994), or at the time when they form, in order to smooth out the singular behaviour (e.g. Nusser & Dekel 1992, and references therein); alternatively one can abandon the perfect fluid picture and resort to a discrete point–like particle set, which automatically eliminates the possible occurrence of
caustics, at least for generic initial data. At this level, anyway, we prefer to take a conservative point of view and assume that the actual range of validity of our formalism is up to shell-crossing.

### 3.1 Jacobian approach

A more direct way to deal with the Lagrangian Newtonian equations is to write them in terms of the Jacobian matrix $\mathcal{J}^A_\alpha$. This approach is obviously related to the more usual ones in terms of the displacement vector $\mathbf{S}$ or in terms of the deformation tensor $\mathcal{D}^A_\alpha$ (Buchert 1989; Moutarde et al. 1991; Bouchet et al. 1992; Buchert 1992; Catelan 1995). The evolution equation has been explicitly written directly in terms of the Jacobian matrix by Buchert & Götz (1987), Lachièze-Rey (1993) and Catelan (1995).

In order to rewrite the Raychaudhuri equation in terms of the Jacobian matrix, we notice that

$$\vartheta^\alpha_{\beta} = \mathcal{J}^\alpha_A \mathcal{J}^A_{\beta}'' - \vartheta^\alpha_{\mu} \vartheta^\mu_{\beta},$$

where we have introduced the inverse Jacobian matrix

$$\mathcal{J}^\alpha_A \equiv \frac{\partial q^{\alpha}}{\partial x^A}, \quad \mathcal{J}^\alpha_A \mathcal{J}^A_{\beta},$$

where Eulerian indices are raised and lowered by the Kronecker symbol. To make explicit our notation, we just stress that elements of $\frac{\partial q^A}{\partial q^\alpha}$ will be characterized by a greek (i.e. Lagrangian) index subscript, while elements of the inverse matrix $\frac{\partial q^{\alpha}}{\partial x^A}$ will be characterized by a greek index superscript.

Replacing the latter identity into the Newtonian expression Eq.(62) yields

$$\mathcal{J}^\alpha_A \mathcal{J}^A_{\beta}, + \frac{a'}{a} \mathcal{J}^{-1} \dot{\mathcal{J}}' = 4 \pi G a^2 \varrho_b (1 - \mathcal{J}^{-1}).$$

Note that this expression is, apart from the use of a different time variable, identical to Eq.(60), in Catelan (1995) [see also Appendix A in (Buchert 1989), and (Buchert 1992)].

We also notice that the parallel transport condition, Eq.(39), can be rewritten in the form

$$\vartheta^\alpha_{\beta} = \mathcal{J}^\alpha_A \mathcal{J}^A_{\beta}'.$$

This equation, together with $\vartheta^\alpha_{\beta} = \frac{1}{2} \gamma^\alpha_{\gamma \gamma} \gamma_{\gamma \beta}'$ gives the general relation

$$\mathcal{J}^\alpha_A \mathcal{J}^A_{\beta} = \mathcal{J}^\alpha_A \mathcal{J}^A_{\beta}'.$$

Replacing these relations in the momentum constraint we obtain in whole generality

$$\mathcal{J}^\alpha_A \mathcal{J}^A_{\beta}'' + \mathcal{J}^\alpha_A \mathcal{J}^A_{\beta}' = (\mathcal{J}^{-1} \mathcal{J}')_{\beta}.$$

On the other hand, in the Newtonian limit we have

$$\mathcal{J}^A_{\alpha, \beta} = \mathcal{J}^A_{\beta, \alpha}.$$
as it follows from the fact that $S^A,_{\alpha\beta} = S^A,_{\beta\alpha}$. Using this commutation property it is easy to verify that
\[ \bar{\Gamma}^\alpha_{\beta\gamma} = \bar{\mathcal{J}}_A^\alpha \bar{\mathcal{J}}^A_{\beta\gamma}. \] (73)

Thanks to the latter relation and to the well–known matrix identity $\text{Tr} \ln \mathbf{J} = \ln \det \mathbf{J}$, it is straightforward to verify that the momentum constraint in the Newtonian limit becomes an identity. It is then clear that Eq.(70) is more fundamental than the momentum constraint: it plays the role of an \textit{irrotationality condition} written in Lagrangian space. This is of course equivalent to the standard form [compare with Eq.(59) in (Catelan 1995)]
\[ \epsilon^{\alpha\beta\gamma} \bar{\mathcal{J}}_A^\alpha \bar{\mathcal{J}}^A_{\beta\gamma} = 0. \] (74)

This equation, together with the Raychaudhuri equation above, Eq.(68), completely determines the Newtonian problem, in terms of either the Jacobian matrix, the deformation tensor or the displacement vector.

The very fact that we have been able to recover the standard equations for the Newtonian approximation in the Lagrangian picture, by starting from the Lagrangian GR treatment and expanding in powers of $1/c^2$, should be considered as a further confirmation of the validity of our method.

### 3.2 Zel’dovich approximation

Having shown the equivalence of our method, in the Newtonian limit, with the standard one, it is now trivial to recover the Zel’dovich approximation (Zel’dovich 1970). This is obtained by expanding Eq.(68) and Eq.(70) to first order in the displacement vector. The result is
\[ x(q, \tau) = q + D(\tau) \nabla \Phi_0(q), \] (75)

where only the growing mode solution $D(\tau)$ of Eq.(52) has been considered, and we introduced the potential $\Phi_0(q)$, such that $\nabla^2 \Phi_0 = -\delta_0/D_0$, where $\nabla^2$ is the standard (i.e. Euclidean) Laplacian in Lagrangian coordinates; more in general, at this perturbative order covariant and partial derivatives with respect to the $q^a$ coincide. The potential $\Phi_0$ is easily related to the initial peculiar gravitational potential defined in Section 1, $\Phi_0 = -(4\pi G a_0^2 \delta_{0b} D_0)^{-1} \varphi_0$.

More interesting is to derive from the above expression the corresponding \textit{Zel’dovich metric}. It reads
\[ \gamma_{\alpha\beta}^{ZEL}(q, \tau) = \delta_{\gamma\delta} \left( \delta_{\alpha}^{\gamma} + D(\tau) \Phi_0,_{\alpha}(q) \right) \left( \delta_{\beta}^{\delta} + D(\tau) \Phi_0,_{\beta}(q) \right). \] (76)

One can of course diagonalize this expression by going to the principal axes of the deformation tensor. Calling $\lambda_\alpha$ the eigenvalues of the matrix $\Phi_0,_{\alpha\beta}$, one finds
\[ \gamma_{\alpha}^{ZEL}(q, \tau) = [1 + D(\tau) \lambda_\alpha(q)]^2. \] (77)

Note that, contrary to what has been commonly done so far in the literature, the metric tensor must be evaluated at second order in the displacement vector, in order to obtain back the correct Zel’dovich expressions for the dynamical variables (density, shear, etc ...).
The above diagonal form of the metric allows a straightforward calculation of all the relevant quantities. The well-known expression for the mass density is consistently recovered,

$$\rho^{ZEL} = \rho_b \prod_\alpha (1 + D\lambda_\alpha)^{-1}. \quad (78)$$

The peculiar velocity-gradient tensor has the same eigenframe of the metric; its eigenvalues read

$$\vartheta^{ZEL}_\alpha = \frac{D'\lambda_\alpha}{1 + D\lambda_\alpha}. \quad (79)$$

By summing over $\alpha$ the latter expression we can obtain the peculiar volume-expansion scalar

$$\vartheta^{ZEL} = \sum_\alpha \frac{D'\lambda_\alpha}{1 + D\lambda_\alpha} \quad (80)$$

and then the shear eigenvalues

$$\sigma^{ZEL}_\alpha = \frac{D'\lambda_\alpha}{1 + D\lambda_\alpha} - \frac{1}{3} \sum_\alpha \frac{D'\lambda_\alpha}{1 + D\lambda_\alpha} \quad (81)$$

The electric tide comes out just proportional to the shear. Its eigenvalues read

$$\mathcal{E}^{ZEL}_\alpha = -4\pi G a^2 \varrho_b \frac{D}{D\mu} \sigma^{ZEL}_\alpha. \quad (82)$$

These expressions for the shear and the tide completely agree with those obtained by Kofman & Pogosyan (1995) and Hui & Bertschinger (1995). The fact that metric, shear and tide have simultaneous eigenvectors shows that fluid elements in the Zel’dovich approximation actually evolve as in a “silent universe” (Matarrese, Pantano & Saez 1994a; Bruni Matarrese & Pantano 1995b), with no influence from the environment, except for that implicit in the self-consistency of the initial conditions.

So far the Zel’dovich approximation has been obtained by first taking the Newtonian limit ($c \to \infty$) of the GR equations and then linearizing them with respect to the Newtonian displacement vector. One could also drop the first step and linearize the GR equations of Section 2.1 with respect to the local deformation tensor as introduced in Section 2.3; in such a case one would get a fully relativistic version of the Zel’dovich approximation.

The latter problem has been already discussed a number of times by various authors. Unfortunately, there has been a lot of misunderstanding on what the “relativistic Zel’dovich approximation” should actually be. Most authors just deal with the GR version of the Zel’dovich solution, i.e. with the non-linear evolution of planar perturbations, which is a sub-case of the well-known exact solutions obtained by Szekeres (1975). Such an approach, however, does not allow to deal with the approximate non-linear behaviour of generic perturbations in a relativistic framework.
3.3 Lagrangian Bernoulli equation

As we have demonstrated above, it is always possible, in the frame of the Newtonian approximation, to define a global Eulerian picture. This will be the picture of the fluid evolution as given by an observer that, at the point \( \mathbf{x} = \mathbf{q} + \mathbf{S}(\mathbf{q}, \tau) \) and at the time \( \tau \) observes the fluid moving with physical peculiar three-velocity \( \mathbf{v} = d\mathbf{S}/d\tau \). From the point of view of a Lagrangian observer, who is comoving with the fluid, the Eulerian observer, which is located at constant \( \mathbf{x} \), is moving with three-velocity \( d\mathbf{q}(\mathbf{x}, \tau)/d\tau = -\mathbf{v} \).

The line-element characterizing the Newtonian approximation in the Eulerian frame is well known (e.g. Peebles 1980)

\[
ds^2 = a^2(\tau) \left[ -\left(1 + \frac{2\varphi_g(\mathbf{x}, \tau)}{c^2}\right)c^2d\tau^2 + \delta_{AB}dx^Adx^B \right],
\]

(83)

with \( \varphi_g \) the peculiar gravitational potential, determined by the mass distribution through the Eulerian Poisson equation,

\[
\nabla^2_x \varphi_g(\mathbf{x}, \tau) = 4\pi G a^2(\tau) \varrho_b(\tau) \delta(\mathbf{x}, \tau),
\]

(84)

where the Laplacian \( \nabla^2_x \), as well as the nabla operator \( \nabla \), have their standard Euclidean meaning. The perturbation in the time–time component of the metric tensor here comes from the different proper time of the Eulerian and Lagrangian observers. As already noticed in the Introduction, this Newtonian line–element is different to that of Eq.(2), describing the so-called weak-field limit; the extra term \( -2\varphi_g/c^2 \), multiplying the spatial line–element, would in fact give rise to PN corrections in the equations of motion for the matter.

It is now crucial to realize that all the dynamical equations obtained so far, being entirely expressed in terms of three–tensors, keep their form in going to the Eulerian picture, only provided the convective time derivatives of tensors of any rank (scalars, vectors and tensors) are modified as follows:

\[
\frac{D}{D\tau} \rightarrow \frac{\partial}{\partial \tau} + \mathbf{v} \cdot \nabla, \quad \mathbf{v} \equiv \frac{d\mathbf{S}}{d\tau}.
\]

(85)

This follows from the fact that, for the metric above, \( \bar{\Gamma}^0_{AB} = \bar{\Gamma}^A_{0B} = \bar{\Gamma}^A_{BC} = 0 \), which also obviously implies that covariant derivatives with respect to \( x^A \) reduce to partial ones.

The irrotationality assumption now has the obvious consequence that we can define an Eulerian velocity potential \( \Phi_v \) through

\[
\mathbf{v}(\mathbf{x}, \tau) = \nabla \Phi_v(\mathbf{x}, \tau).
\]

(86)

The Newtonian peculiar velocity–gradient tensor then becomes

\[
\bar{\bar{\sigma}}_{AB} = \frac{\partial^2 \Phi_v}{\partial x^A \partial x^B},
\]

(87)

because of which the momentum constraint gets trivially satisfied and the magnetic Weyl tensor becomes identically zero in the Newtonian limit.
We can now write the Raychaudhuri equation for the Eulerian peculiar volume–expansion scalar $\bar{\vartheta}$, and use the Poisson equation to get, as a first spatial integral, the Euler equation

$$\mathbf{v}' + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{a'}{a} \mathbf{v} = -\nabla \varphi_g .$$

(88)

This can be further integrated to give the Bernoulli equation

$$\Phi' + \frac{a'}{a} \Phi + \frac{1}{2} (\nabla \Phi)^2 = -\varphi_g .$$

(89)

On the other hand, by taking gradients of the Euler equation we can obtain an Eulerian evolution equation for the tensor $\bar{\vartheta}_{\alpha\beta}$. More interesting is that this equation can be transported back to the Lagrangian frame to get

$$\bar{\vartheta}'_{\alpha\beta} + \frac{a'}{a} \bar{\vartheta}_{\alpha\beta} + \bar{\vartheta}_{\alpha\gamma} \bar{\vartheta}^{\gamma}_{\beta} = -\varphi^{(L)}_{g|\beta} .$$

(90)

where $\varphi^{(L)}_g$ must be thought as a Lagrangian peculiar gravitational potential to be determined through the Lagrangian Poisson equation

$$\varphi^{(L)}_{g|\alpha} = 4\pi G a^2 \mathcal{Q}_0 (\gamma^{-1/2} - 1) .$$

(91)

These two Lagrangian expressions will turn out to be very useful for the PN calculations of Section 4.

There is, however, another consequence of these equations that we can easily derive. While in the Lagrangian frame the three–velocity field does not exist, the tensor $\bar{\vartheta}_{\alpha\beta}$ is well–defined, so that we can rewrite Eq.(87) in Lagrangian coordinates to obtain a Lagrangian velocity potential $\Phi^{(L)}_v(q, \tau)$, through

$$\bar{\vartheta}^{\alpha}_{\beta} \equiv \Phi^{(L)}_{v|\alpha} .$$

(92)

This potential obeys what we can name the Lagrangian Bernoulli equation, which is easily obtained from the Bernoulli equation above, provided we recollect the convective time derivative and express it in Lagrangian form. We get

$$\Phi^{(L)}_v' + \frac{a'}{a} \Phi^{(L)}_v - \frac{1}{2} \gamma^{\alpha\beta} \Phi^{(L)\alpha}_v \Phi^{(L)\beta}_v = -\varphi^{(L)}_g .$$

(93)

The most astonishing difference between the Eulerian and Lagrangian versions of the Bernoulli equation is the relative sign of the temporal and spatial derivatives. We could obtain more similar forms by reversing the arrow of time and the sign of the gravitational interaction. In this sense, therefore, the Lagrangian Bernoulli equation acts as a sort of time machine (cf. Nusser & Dekel 1992). This fact becomes more clear if we think to the fact that, by solving it, we are indeed asking how the Lagrangian (i.e. initial) geometry at $q$ should modify itself in order to reproduce the Eulerian (i.e. evolved) properties of the velocity and density fields at the point $x(q, \tau)$ as time goes on.
This equation could be used in principle as an alternative Lagrangian formulation of Newtonian theory, whose fundamental variables would be the velocity potential $\Phi^{(L)}_v$, the gravitational potential $\varphi^{(L)}_g$ and the metric tensor $\bar{\gamma}_{\alpha\beta}$. This approach could be useful, in particular, in order to obtain new self-consistent approximation schemes to the non-linear evolution of dust in the Lagrangian frame. To this aim, however, we need two more equations to close the system. These can be provided by the Lagrangian Poisson equation above, Eq.(91), and by the very definition of $\bar{\vartheta}_{\alpha\beta} = \bar{\gamma}_{\alpha\gamma} \bar{\vartheta}_{\gamma\beta}$, which implies

$$\Phi^{(L)}_{v|\alpha\beta} = \frac{1}{2} \bar{\gamma}'_{\alpha\beta}.$$  \hspace{1cm} (94)

Of course, the Lagrangian scalars $\Phi^{(L)}_v$ and $\varphi^{(L)}_g$ are related to their Eulerian counterparts by a simple coordinate transformation, namely $\Phi^{(L)}_v(q, \tau) = \Phi^{(E)}_v(x(q, \tau), \tau)$ and $\varphi^{(L)}_g(q, \tau) = \varphi^{(E)}_g(x(q, \tau), \tau)$.

### 4 Post–Newtonian approximation

Having examined all the aspects of our formalism in the Newtonian limit, we are now ready to proceed to the next perturbative order in $1/c^2$. The PN terms $\frac{1}{c^2} w^{(PN)}_{\alpha\beta}$ in Eq.(56) should be thought as small perturbations superposed on a Newtonian background $\bar{\gamma}_{\alpha\beta}$. The fact that the three–metric in the Newtonian limit is that of Euclidean space in time–dependent curvilinear coordinates $q^\alpha$, implies that we can apply most of the standard tools of linear perturbation theory in a flat spatial background. In particular, we can classify our PN metric perturbations as scalar, vector and tensor modes, as usual.

We then write

$$w^{(PN)}_{\alpha\beta} = \chi^{(PN)}_{\alpha\beta} + \zeta^{(PN)}_{|\alpha\beta} + \frac{1}{2} (\xi^{(PN)}_{|\alpha\beta} + \eta^{(PN)}_{|\alpha\beta}) + \pi^{(PN)}_{\alpha\beta},$$  \hspace{1cm} (95)

with

$$\xi^{(PN)}_{|\alpha} = \pi^{(PN)}_{\alpha} = \pi^{(PN)}_{\beta|\alpha} = 0,$$  \hspace{1cm} (96)

where greek indices after a single vertical bar, or nabla operators with a greek index, denote covariant differentiation in the Newtonian background three–space with metric $\bar{\gamma}_{\alpha\beta}$. In the above decomposition $\chi^{(PN)}$ and $\xi^{(PN)}$ represent PN scalar modes, $\xi^{(PN)}_{\alpha}$ PN vector modes and $\pi^{(PN)}_{\alpha\beta}$ PN tensor ones (indices being raised by the contravariant background three–metric). We deliberately used the same symbols as in Section 2.4, in order to emphasize the analogy with the linear problem. Some of these PN modes, namely $\chi^{(PN)}$ and $\pi^{(PN)}_{\alpha\beta}$, also have a non-vanishing linear counterpart, as noticed in Section 2.4 (actually the linear part of $\pi^{(PN)}_{\alpha\beta}$ appears as a gauge mode in the equations), while others, namely $\zeta^{(PN)}$ and $\xi^{(PN)}_{\alpha}$ are intrinsically non–linear. Unlike linear perturbation theory in a FRW background, metric perturbations of different rank do not decouple: this is because our time–dependent Newtonian background enters the equations not only through the metric $\bar{\gamma}_{\alpha\beta}$, but also through the peculiar velocity–gradient tensor $\bar{\vartheta}_{\alpha\beta}$, which also contains scalar, vector and tensor modes. This fact leads to
non–linear scalar–vector, scalar–tensor and vector–tensor mode mixing, which also explains why we had to account for the vector modes $\xi(PN)_\alpha^{\beta}$ in the expansion of $w^{(PN)}_{\alpha\beta}$, in spite of the irrotational character of our fluid motions. Actually, that vector modes appear in the non–linear evolution of an irrotational fluid in Lagrangian coordinates is well known also in the Newtonian framework (see Buchert 1994; Catelan 1995).

As in every perturbative calculation, some of the equations have the property to mix different perturbative orders. This is of course necessary in order to make the $n$–th order coefficients of the expansion calculable in terms of those of order $n - 1$. In our case the energy constraint and the extrinsic curvature evolution equation (which at the Newtonian level implies $R^\alpha_\beta(\bar{\gamma}) = 0$) play this role. Therefore we assume that the Newtonian metric and its derivatives are known by solving the Raychaudhuri equation and the momentum constraint, and we calculate the PN metric perturbations in terms of them.

Let us first compute the tensor $P^\alpha_\beta \equiv 4\mathcal{R}^\alpha_\beta - (\mathcal{R} + 2\kappa)\delta^\alpha_\beta$ to first order in $1/c^2$. We obtain

\[ c^2P^\alpha_\beta((w^{(PN)})) = -2\left(\nabla^2\pi^{(PN)}_{\alpha\beta} + \chi^{(PN)}_{\alpha\beta}\right), \tag{97} \]

where now $\nabla^2(\cdot) \equiv (\cdot)_{\alpha\beta}^\alpha$.

In the energy constraint only the scalar $\chi$ enters,

\[ \nabla^2\chi = \frac{1}{2}\left(\ddot{v} - \ddot{\bar{v}}\bar{v} - \ddot{\bar{v}}\right) + 2\frac{a'}{a} \ddot{v} - 8\pi G a^2 \bar{\rho}(\bar{\gamma}^{-1/2} - 1), \tag{98} \]

where, here and from now on, we have dropped the superscript (PN) on PN terms. One can also obtain an equation for $\chi$ from the trace of the evolution equation, Eq.(22), which is however equivalent to the latter, thanks to the Newtonian Raychaudhuri equation.

The tensor perturbations $\pi^\alpha_\beta$ are instead determined via the evolution equation, Eq.(22) (actually from its trace–free part),

\[ \nabla^2\pi^\alpha_\beta = 2\ddot{\bar{v}}\ddot{v} + 2\left(2\frac{a'}{a} + \ddot{v}\right)\ddot{v} - \frac{1}{2}\delta^\alpha_\beta \left(\ddot{\bar{v}} - \ddot{\bar{v}}\bar{v} - \ddot{\bar{v}}\bar{v}\right) - \chi^\alpha_\beta. \tag{99} \]

A by–product of the latter equation is that linear tensor modes, which in the $c \to \infty$ limit appear as harmonic functions (i.e. pure gauge modes), do not contribute to the r.h.s., i.e. to the Newtonian evolution of the system, as expected.

In order to get an equation for the tensor modes decoupled from the scalar mode $\chi$ we can resort to the equations obtained in Section 3.3 above. To this aim we define the auxiliary Newtonian potential $\Psi^{(L)}_v$, through

\[ \nabla^2\Psi^{(L)}_v = -\frac{1}{2}\left(\ddot{v} - \ddot{\bar{v}}\bar{v}\right). \tag{100} \]

Using this definition in Eq.(98), we obtain

\[ \chi = -\Psi^{(L)}_v + 2\frac{a'}{a} \Phi^{(L)} - 2\varphi^{(L)} \tag{101} \]
By replacing this expression into the momentum constraint, Eq.(21), we obtain
\[ \nabla^2 \Pi^\alpha_\beta = \left( \nabla^\alpha \nabla_\beta + \delta^\alpha_\beta \nabla^2 \right) \Psi^{(L)} + 2 \left( \bar{\vartheta} \delta^\alpha_\beta - \bar{\vartheta}_\gamma^\alpha \bar{\vartheta}^\gamma_\beta \right), \]  
which has the significant advantage of being explicitly second order (in any possible perturbative approach). This equation is one of the most important results of this paper: it gives (in the so-called near zone) the amount of gravitational waves emitted by non-linear cosmological perturbations, evolved within Newtonian gravity. In other terms, this equation, which is only applicable on scales well inside the horizon, describes gravitational waves produced by an inhomogeneous Newtonian background. At first sight it may appear surprising that gravitational waves already appear at the zeroth order, whereas in the longitudinal gauge the gravitational potential carries a \(1/c^2\) factor.

This formula can be compared with the PN limit of Eq.(3) in Section 1, to which it actually reduces (up to an already mentioned numerical factor) if the \(\vartheta^\alpha_\beta\) are calculated from linear theory and in an Einstein–de Sitter model. We have \(\bar{\vartheta}^\alpha_\beta(q, \tau) = D'(\tau)\Phi^\alpha_\beta(q)\), with \(D(\tau)\) the growing mode solution of Eq.(52) \((D(\tau) \propto a(\tau) \propto \tau^2\) in the Einstein–de Sitter case) and \(\Phi_0\) defined in Section 3.2. Therefore
\[ \nabla_q^2 \alpha^\alpha_\beta = D^2 \left[ \Psi_0,^\alpha_\beta + \delta^\alpha_\beta \nabla_q^2 \Psi_0 + 2 \left( \Phi_0,^\alpha_\beta \nabla_q^2 \Phi_0 - \Phi_0,^\alpha_\gamma \Phi_0,^\gamma_\beta \right) \right], \]  
where the symbol \(\nabla_q^2\) indicates the standard (Euclidean) form of the Laplacian in Lagrangian coordinates and \(\Psi_0 = \Psi_0(\tau_0)\) and indices are raised by the Kronecker symbol.

To completely determine the PN metric perturbations we still need the scalar mode \(\zeta\) and the vector modes \(\xi_\alpha\), which can be computed through the momentum constraint and the Raychaudhuri equation. We then need \(\vartheta^\alpha_\beta\) to PN order; it reads
\[ \bar{\vartheta}^\alpha_\beta = \tilde{\vartheta}^\alpha_\beta + \frac{1}{c^2} \left( \tilde{\vartheta}_\gamma^\alpha w^\gamma_\beta - \tilde{\vartheta}_\beta^\gamma w^\alpha_\gamma + \frac{1}{2} w^\alpha_\beta \right). \]  
By replacing this expression into the momentum constraint, Eq.(21), we obtain
\[ (w^\alpha_\beta)_{|\alpha} - w_{|\beta} + 2 \tilde{\vartheta}_\alpha w^\alpha_\beta - 2 \tilde{\vartheta}_\beta w^\alpha_\gamma + 2 \tilde{\vartheta}_\gamma w^\alpha_\beta - 2 \tilde{\vartheta}_\beta w^\alpha_\gamma - \tilde{\vartheta}_\gamma w^\alpha_\beta + \bar{\vartheta}_{\alpha\beta} w_{|\alpha} = -4\kappa_N \Phi^{(L)}_{v\beta}, \]  
where the term on the r.h.s. comes from the relation \(\vartheta^\alpha_\beta_{|\alpha} - \vartheta_{|\beta} = 2\kappa \Phi^{(L)}_{v\beta}\), which can be easily derived by expanding the Jacobi identity \(\Phi^{(L)}_{v|\alpha\beta} - \Phi^{(L)}_{v|\beta\alpha} = \Phi^{(L)}_{v\beta} R^\alpha_\beta\), in powers of \(1/c^2\).

In order to write the last equation in terms of the various PN perturbation modes, the Newtonian identity \(\tilde{\Gamma}^\mu_{\nu\rho} = \bar{\vartheta}^\mu_{\nu\rho}\) is also useful, implying
\[ (w^\alpha_\beta)'_{|\alpha} = (w^\alpha_\beta_{|\alpha})' - \bar{\vartheta}_\alpha w^\alpha_\beta + \bar{\vartheta}^\gamma_{|\beta} w^\alpha_\gamma. \]  
25
By replacing the expansion of $w^{\alpha\beta}$ into this equation we finally get

$$2\chi,\beta + \bar{\theta} \chi,\beta - 3\bar{\theta}^{\alpha} \chi,\alpha - \bar{\theta}_{\alpha} \pi_{\beta} + \bar{\theta}_{\gamma} \pi_{\alpha} - 2\bar{\theta}^{\alpha} \pi_{\beta} - 2\bar{\theta}_{\beta} \pi_{\alpha} - \bar{\theta}^{\gamma} \pi_{\beta} + \bar{\theta}_{\gamma} \pi_{\alpha} - \frac{1}{2}(\nabla^2 \xi_{\beta})' - \frac{1}{2}\bar{\theta}_{\alpha} \xi_{\beta} - \frac{1}{2}\bar{\theta}_{\alpha} \xi_{\beta} -$$

$$= - \frac{1}{2}\bar{\theta}_{\alpha} \xi_{\beta} + \bar{\theta}^{\gamma} \xi_{\alpha} - \bar{\theta}^{\gamma} \xi_{\beta} - \bar{\theta}^{\alpha} \xi_{\gamma} + \bar{\theta}_{\beta} \xi_{\gamma} - \frac{1}{2}(\nabla^2 \xi_{\beta})' - \frac{1}{2}\bar{\theta}_{\alpha} \xi_{\beta} -$$

$$= - \frac{1}{2}\bar{\theta}_{\alpha} \xi_{\beta} + \bar{\theta}^{\gamma} \xi_{\alpha} - \bar{\theta}^{\gamma} \xi_{\beta} - \bar{\theta}^{\alpha} \xi_{\gamma} + \bar{\theta}_{\beta} \xi_{\gamma} - \frac{1}{2}(\nabla^2 \xi_{\beta})' - \frac{1}{2}\bar{\theta}_{\alpha} \xi_{\beta} -$$

which, at the linear level, reduces to Eq.(47) above.

One can verify that the three–divergence of the last equation reduces to an identity, therefore, in order to completely determine the three remaining PN modes $\zeta$ and $\xi^\alpha$ we need one more equation. This is in fact provided by the PN Raychaudhuri equation, which reads

$$w'' + \frac{d'}{a}w' + 2\bar{\theta}_{\nu} w_{\nu}' = 4\pi G a^2 \bar{\theta}_b (1 + \bar{\delta})(w - w_0),$$

having assumed $\delta_0^{(PN)} = 0$, as suggested by linear theory. By replacing the expansion of $w^{\alpha\beta}$ into this equation we get

$$(3\chi + \nabla^2 \zeta)' + \frac{d'}{a}(3\chi + \nabla^2 \zeta)' + 2\bar{\theta}_\chi' + 2\bar{\theta}_{\nu}(\zeta_{\nu}' + \xi_{\nu}^{\mu} \pi_{\mu}')' =$$

$$= 4\pi G a^2 \bar{\theta}_b (1 + \bar{\delta}) \left[3(\chi - \chi_0) + \nabla^2 \zeta \right],$$

where we have also set $w_0 = 3\chi_0$, in agreement with linear theory.

Unfortunately, we have not been able to further simplify these equations, which nevertheless show that $\zeta$ and $\xi^\alpha$ are implicitly determined by the Newtonian quantities, once $\chi$ and $\pi_{\alpha\beta}$ have been computed. There is a caveat concerning second–order perturbation theory, where $\zeta$ is left undetermined by the above equations. Nevertheless, such an ambiguity is completely removed by going to the PPN energy constraint, which can be easily shown to fix $w$ and then $\zeta$ without explicit knowledge of the truly PPN coefficients. Let us complete this discussion by reporting the PPN energy constraint. Defining a PPN metric perturbation $w_{(PPN)}^{(PPN)}/c^4$, one gets

$$(\bar{\theta} + \frac{d'}{a})w - \bar{\theta}_{\nu} w_{\nu}' + \mathcal{R}^{(PPN)} = -8\pi G a^2 \bar{\theta}_b (1 + \bar{\delta})(w - w_0),$$

with $\mathcal{R}^{(PPN)}$ the PPN conformal Ricci scalar,

$$\mathcal{R}^{(PPN)} = -2\kappa_N w + w^{(PPN)\mu\nu}_| |_{\mu\nu} - \nabla^2 w^{(PPN)} + w^{\mu}_|(w^{\nu}_| + \nabla^2 w^{\nu}_| - 2w^{\nu\gamma}_{|\mu\gamma} +$$

$$+ \frac{3}{4} w^{\mu\gamma}_{|\mu\gamma} - \frac{1}{2} w^{\mu}_| w^{\gamma}_| - \frac{1}{2} w^{\mu}_| w^{\gamma}_| - \frac{1}{4}(w^{\mu}_| - 2w^{\mu}_{\mu\gamma})(w^{\nu}_| - 2w^{\nu}_{\nu\gamma}).$$

(111)
4.1 Fluid–flow approach in the Newtonian limit

We are now ready to discuss the fluid–flow approach presented in Section 2.2, within the Newtonian approximation. The reason why this discussion has been included in this Section is that, as we shall see, some of the relevant tensors must be computed at the PN order in order to provide the correct Newtonian evolution of the system.

We just have to discuss the order in our $1/c^2$ expansion at which the various tensors enter the equations of Section 2.2. It is immediately clear that the mass continuity equation, Eq.(25), the Raychaudhuri equation, Eq.(26), and the shear evolution equation, Eq.(27), where no explicit powers of $c$ appear, just keep their form, once the various tensors are replaced by their Newtonian counterparts. So, we have

$$\bar{\delta}' + (1 + \bar{\delta}) \bar{\rho} = 0 ,$$  \hspace{1cm} (112)

$$\bar{\rho}' + \frac{d}{a} \bar{\rho} + \frac{1}{3} \bar{\rho}^2 + \bar{\sigma}_\alpha^\beta \bar{\sigma}_\beta^\alpha + 4\pi G a^2 \bar{\rho}_\beta \bar{\delta} = 0 ,$$  \hspace{1cm} (113)

and

$$\bar{\sigma}_\beta^\alpha + \frac{a'}{a} \bar{\sigma}_\beta^\alpha + \frac{2}{3} \bar{\rho} \bar{\sigma}_\beta^\alpha + \bar{\sigma}_\gamma^\alpha \bar{\sigma}_\beta^\gamma - \frac{1}{3} \bar{\sigma}_\beta^\alpha \bar{\sigma}_\beta^\alpha \bar{\rho}^\gamma + \mathcal{E}_\beta^\alpha = 0 .$$  \hspace{1cm} (114)

On the other hand, by its very definition, Eq.(28), the electric tide contains a contribution coming from the PN terms $\chi$ and $\pi_\alpha^\beta$ because of the spatial curvature terms,

$$\bar{\mathcal{E}}_\beta^\alpha = \frac{1}{3} \delta_\beta^\alpha \bar{\sigma}_\mu^\nu \bar{\sigma}_\mu^\nu + \frac{1}{3} \bar{\rho} \bar{\sigma}_\beta^\alpha + \frac{a'}{a} \bar{\sigma}_\beta^\alpha - \bar{\sigma}_\gamma^\alpha \bar{\sigma}_\beta^\gamma - \frac{1}{2} \left[ \nabla^2 \pi_\beta^\alpha + \left( \nabla^\alpha \nabla_\beta - \frac{1}{3} \delta_\beta^\alpha \nabla^2 \right) \chi \right] .$$  \hspace{1cm} (115)

It is however immediate to realize that, once the expressions of this section for the PN tensors $\chi$ and $\pi_\alpha^\beta$ are used, one recovers the simpler form,

$$\bar{\mathcal{E}}_\beta^\alpha \left( \nabla^\alpha \nabla_\beta - \frac{1}{3} \delta_\beta^\alpha \nabla^2 \right) \varphi^{(L)}_g ,$$  \hspace{1cm} (116)

which, in Eulerian coordinates reduces to the standard form

$$\bar{\mathcal{E}}_{AB} = \varphi_{g,AB} - \frac{1}{3} \delta_{AB} \nabla^2 \varphi_g .$$  \hspace{1cm} (117)

On the other hand, if we replace in Eq.(33), the Newtonian peculiar velocity–gradient tensor, we obtain the well–known result (e.g. Ellis 1971) that the magnetic tensor identically vanishes in the Newtonian limit. This can be very easily shown by either applying the formalism of Section 3.3, i.e. writing $\bar{\rho}_\beta^\alpha$ through covariant derivatives of the Lagrangian velocity potential, or by writing the same tensor in terms of the Jacobian matrix of Section 3.1. The physics underlying this result is the conformal flatness of the Newtonian spatial sections, implying the commutation of spatial covariant derivatives. A simple consequence of this fact is that, at the Newtonian level, the div $\mathcal{E}$ constraint, Eq.(34), reduces to

$$\bar{\mathcal{E}}_{\beta|\alpha} = \frac{8\pi G}{3} \frac{a^2 \bar{\rho}_\beta \bar{\delta}_\alpha}{a} ,$$  \hspace{1cm} (118)
which, owing to our expression for $\bar{E}$, turns out to be just the gradient of the Lagrangian Poisson equation, Eq.(91), namely

$$\nabla^2 \varphi_g^{(L)} = 4\pi G a^2 \varrho_0 \delta .$$

(119)

Let us now come to the tide evolution equation, Eq.(30). In that evolution equation the circulation of the magnetic tensor is multiplied by $c^2$, which means that the PN part of curl $\mathcal{H}$ is the source of non–locality in the Newtonian electric tide evolution equation. On the other hand, if we look at the magnetic tide evolution equation, which starts to be non–trivial at the PN order, we see that curl $\mathcal{E}$ is consistently a PN quantity.

The Newtonian meaning of the momentum constraint, Eq.(32), has been already discussed in Section 3. Also interesting is the div $\mathcal{H}$ constraint, Eq.(35), telling us that the general non–vanishing of the PN magnetic tensor (see also Lesame, Ellis & Dunsby 1996), implies that the Newtonian shear and electric tide do not commute, i.e. they have different eigenvectors (viceversa, their non–alinement causes a non–zero div $\mathcal{H}$). Another possible version of this result is that the ratio of the velocity potential to the gravitational potential beyond the linear regime becomes space–dependent.

To summarize our results, we can say that within the Newtonian approximation the fluid–flow approach in Lagrangian coordinates can be formulated in terms of mass continuity, Raychaudhuri and shear evolution equations plus the Newtonian div $\mathcal{E}$ constraint, which closes the system, provided we remind the circulation–free character of the electric tide in this limit. Of course the direct use of a constraint to close the system of evolution equations, has the disadvantage of breaking the intrinsic hyperbolicity of the GR set of evolution equations, so that the entire method looses its basic feature. No ways out: this is the price to pay to the intrinsic non–causality of the Newtonian theory [see also Ellis (1990)].

The above discussion on the role of the PN magnetic tidal tensor, as causing non–locality in the Newtonian fluid–flow evolution equations, completely agrees with a similar analysis by Kofman & Pogosyan (1995). The only variant is that we obtained our results directly in Lagrangian space, while they worked in non–comoving (i.e. Eulerian) coordinates. A different point of view on the subject is expressed by Bertschinger & Hamilton (1994), according to which the magnetic part of the Weyl tensor is non–vanishing already at the Newtonian level. According to Kofman & Pogosyan (1995) the difference might be “semantic”; most important, there is general agreement on the fundamental fact that the Newtonian tide evolution is affected by non–local terms.

### 4.2 Post–Newtonian tensor modes within a collapsing homogeneous ellipsoid

The PN expression for $\pi_{\alpha\beta}$, Eq.(102), has the relevant feature of being non–local, through the presence of the scalar $\Psi_v$. A simpler way to deal with this problem is to transform the equation in Eulerian form, where it is easier to deal with the Laplacian operator $\nabla^2_x$ (which has there the standard Euclidean form), obtain the Eulerian gravitational–wave tensor $\pi_{AB}$ and then go back to the Lagrangian expression through $\pi_{\alpha\beta}(q, \tau) = \mathcal{J}^A_{\alpha} \mathcal{J}^B_{\beta} \pi_{AB}(x(q, \tau), \tau)$. 

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We immediately obtain the Eulerian expression
\[
\nabla^2 x \pi_{AB} = \Psi^{(E)}_{v,AB} + \delta_{AB} \nabla^2 x \Psi^{(E)}_v + 2 \left( \partial \partial_{AB} - \partial_{AC} \partial^C_B \right),
\]
with
\[
\nabla^2 x \Psi^{(E)}_v = - \frac{1}{2} \left( \partial^2 - \partial^A_B \partial_B^A \right),
\]
which generally allows a simpler derivation of \(\pi_{AB}\), given the (gradients of the) velocity potential. For a general homogeneous and isotropic random field \(\Phi_v\), for instance, \(\pi_{AB}\) can be obtained by a simple convolution in Fourier space. Nevertheless, we would like to obtain here an analytic estimate of this tensor, in some simple cases. What we need is a model for non–spherical and non–planar collapse. The simplest model we can figure out is that of a homogeneous ellipsoid with uniform internal overdensity \(\delta(\tau)\) with respect to a FRW background, of density \(\varrho_b(\tau)\) and scale factor \(a(\tau)\), in which it is embedded (White & Silk 1979; Peebles 1980). Calling \(R_A(\tau) = a(\tau)X_A(\tau), A = 1, 2, 3\), the physical length of the three axes, the peculiar gravitational potential within the ellipsoid is given by the simple expression
\[
\phi_g(x, \tau) = \pi G a^2 \varrho_b \delta \sum_A \alpha_A x_A^2,
\]
The dimensionless structure constants \(\alpha_A\) are defined in terms of the three \(X_A\) through
\[
\alpha_A = X_1 X_2 X_3 \int_0^{\infty} ds (X_A^2 + s)^{-1} \prod_B (X_B^2 + s)^{-1/2}
\]
amd are normalized so that \(\sum_A \alpha_A = 2\). In the particular case of oblate or prolate spheroids one can get explicit expressions for the \(\alpha_A\) in terms of an eccentricity parameter (e.g. Kellogg 1953). The simplest non–trivial case, however, is that of an infinite cylinder, for which the two non–vanishing structure constants have the value 1 at any time. The intrinsic self–similarity of the equations of motion for fluid elements within the object implies that the overall shape and homogeneity are preserved at all times. One then usually makes the reasonable approximation that also the universe outside the ellipsoid stays uniform. In such a case, the ellipsoid axes obey the equation
\[
X_A'' + \frac{a'}{a} X_A' = -2 \pi G a^2 \varrho_b \delta \alpha_A X_A,
\]
while mass conservation implies
\[
(1 + \delta) X_1 X_2 X_3 = \text{const}.
\]
The peculiar velocity–gradient tensor has eigenvalues \(\vartheta_A = X_A' / X_A\) (which remain unchanged in Lagrangian coordinates). These determine the Eulerian potential \(\Psi^{(E)}_v\), through the equation
\[
\nabla^2 x \Psi^{(E)}_v = - \sum_A \vartheta_A \vartheta_{A-1},
\]
where we adopt a notation such that $A-1 = 3$ if $A = 1$ and $A+1 = 1$ if $A = 3$. Accounting for the ellipsoidal symmetry of the problem, this equation is solved by

$$\Psi^{(E)}_v = -\frac{1}{4} \sum_A \vartheta_A \vartheta_{A-1} \sum_B \alpha_B x_B^2 .$$

(127)

Replacing this solution in Eq.(120) gives

$$\nabla^2 x \pi_A = \mu_A(\tau) \equiv -\frac{1}{2} (\vartheta_A \vartheta_{A-1} + \vartheta_A \vartheta_{A+1} + \vartheta_{A-1} \vartheta_{A+1}) \alpha_A + (\vartheta_A \vartheta_{A-1} + \vartheta_A \vartheta_{A+1} - \vartheta_{A-1} \vartheta_{A+1}) ,$$

(128)

where $\pi_A \equiv \pi_{AA}$ (no summation over repeated indices is understood) indicates a diagonal component, and $\sum_A \mu_A = 0$. This equation is solved by

$$\pi_A = \frac{1}{4} \mu_A \sum_B \alpha_B x_B^2 .$$

(129)

The off–diagonal components are instead harmonic functions,

$$\nabla^2 x \pi_{AB} = 0 , \quad A \neq B .$$

(130)

These equations must be solved accounting for the transversality condition $\sum_A \pi_{AB,A} = 0$, which gives

$$\pi_{AB} = \frac{1}{4} \nu_{AB} x_A x_B , \quad A \neq B$$

(131)

(no summation over repeated indices), with $\nu_{AB} = \nu_{BA}$ and

$$\nu_{12} = \mu_3 \alpha_3 - \mu_1 \alpha_1 - \mu_2 \alpha_2 ,$$

$$\nu_{13} = \mu_2 \alpha_2 - \mu_1 \alpha_1 - \mu_3 \alpha_3 ,$$

$$\nu_{23} = \mu_1 \alpha_1 - \mu_2 \alpha_2 - \mu_3 \alpha_3 .$$

(132)

These formulae are completely general and do not contain approximations, (apart from those implicit in the homogeneous ellipsoid model). The results of a numerical integration of Eqs.(123), (124) and (125) are shown in Figure 1, for the collapse of a triaxial homogeneous ellipsoid.

One might also use them to get the gravitational–wave emission outside the object (i.e. in the wave zone), by suitable matching with the interior solution. Far away from the body, one could, of course, also obtain the emitted gravitational radiation in terms of the quadrupole moments of our homogeneous ellipsoid (e.g. Landau & Lifshitz 1980, Sects. 41 and 105) and recover the usual $1/c^5$ dependence of the radiated energy. In the wave zone, moreover, the transverse and traceless character of the $\pi_{AB}$ would allow to define the two polarization states of the graviton. These problems will not be considered here; it is nevertheless important to realize that our formalism is completely consistent with the qualitative expectations of
the quadrupole approximation, as it implies identically vanishing tensor modes in the case of spherical ($\alpha_1 = \alpha_2 = \alpha_3 = 2/3$) and plane–parallel ($\alpha_1 = \alpha_2 = 0, \alpha_3 = 2$) collapse.

What we are interested in here is the behaviour of the PN tensor modes when the object is close to collapse. Of course, the set formed by Eq.(123), Eq.(124) and Eq.(125) could be integrated numerically to get the time evolution for the axes $X_A$, the eigenvalues $\vartheta_A$, and the structure constants $\alpha_A$ (e.g. White & Silk 1979). To catch the qualitative behaviour close to collapse, however, we can safely apply the Zel’dovich approximation, which, for the evolution of the axes, yields

$$X_A(\tau) = X_A(\tau_0)(1 + D(\tau)\lambda_A) ,$$

with $\lambda_A = -\frac{6}{2D_0}\alpha_A(\tau_0)$. These expressions should then be replaced into the definition of the $\alpha_A$ to get them self–consistently. Nevertheless, according to White & Silk (1979) a rough estimate is obtained by simply neglecting the time–dependence of the $\alpha_A$. One can immediately derive the Jacobian eigenvalues $\bar{J}_\alpha = 1 + D\lambda_a$, with $A = \alpha$, and those of the peculiar velocity–gradient tensor $\bar{\vartheta}_a = D'\lambda_a/(1 + D\lambda_a)$.

These expressions can then be replaced into the previous equations to get the Lagrangian relations

$$\pi_\alpha \equiv \pi_{\alpha\alpha}(q, \tau) = \frac{1}{4}\mu_\alpha \bar{J}_\alpha^2 \sum_\beta \alpha_\beta \bar{J}_\beta^2 q_\beta^2 ,$$

for the diagonal components, and

$$\pi_{\alpha\beta}(q, \tau) = \frac{1}{4}\nu_{\alpha\beta} \bar{J}_\alpha^2 \bar{J}_\beta^2 q_\alpha q_\beta , \quad \alpha \neq \beta ,$$

for the off–diagonal ones, with

$$\mu_\alpha = \frac{D'^2}{\bar{J}} \left[ \frac{1}{2} \left( \lambda_{a-1}\lambda_{a+1} + \lambda_a\lambda_{a-1} + \lambda_a\lambda_{a+1} + 3D\lambda_a\lambda_{a-1}\lambda_{a+1} \right) \alpha_a + \right.$$

$$\left. + \left( \lambda_a\lambda_{a-1} + \lambda_a\lambda_{a+1} - \lambda_{a-1}\lambda_{a+1} + D\lambda_a\lambda_{a-1}\lambda_{a+1} \right) \right]$$

(136)

and $\nu_{\alpha\beta}$ calculated from these $\mu_\alpha$ according to Eq.(132). For the most typical case of pancake collapse, where one Jacobian eigenvalue goes to zero first, these expressions also go to zero, like $\bar{J}$.

At this point we are able to compare the behaviour of these PN tensor modes to that of the Newtonian part of the metric, which is diagonal with eigenvalues $\bar{\gamma}_a = \bar{J}_a^2$. It is then clear that these PN modes vanish more slowly than the Newtonian part; their ratio diverges like $\bar{J}^{-1}$, i.e. like the mass density at collapse. Using a more refined approximation for the axes evolution, such as the one proposed by White & Silk (1979) or that recently proposed by Hui & Bertschinger (1995), would not change this qualitative result. Indeed, the numerical integration we have performed shows that such a divergence can be even stronger, as clearly displayed by the last panel in Figure 1.

The homogeneous ellipsoid model we have worked out does not allow, unfortunately, to distinguish the global collapse from a shell–crossing singularity, but we may argue that this qualitative behaviour would generally apply even at shell–crossing.
5 Conclusions

In this paper we have considered a Lagrangian approach to the evolution of an irrotational and collisionless fluid in general relativity. The use of a synchronous and comoving gauge allowed to reduce the fundamental variables of the system to the six metric tensor components of the spatial hypersurface orthogonal to the flow lines. Our method was based on a standard $1/c$ expansion of the Einstein and continuity equations which led to a new, purely Lagrangian, derivation of the Newtonian approximation. One of the most important results in this respect is that we obtained a simple and transparent expression for the Lagrangian metric; exploiting the vanishing of the spatial curvature in the Newtonian limit we were able to write it in terms of the displacement vector $S(q, \tau) = x(q, \tau) - q$, from the Lagrangian coordinate $q$ to the Eulerian one $x$ of each fluid element, namely

$$ds^2 = a^2(\tau) \left[ -c^2 d\tau^2 + \delta_{AB} \left( \delta^A_\alpha + \frac{\partial S^A(q, \tau)}{\partial q^\alpha} \right) \left( \delta^B_\beta + \frac{\partial S^B(q, \tau)}{\partial q^\beta} \right) \right].$$

(137)

The spatial metric is that of Euclidean space in time–dependent curvilinear coordinates, consistently with the intuitive notion of Lagrangian picture in the Newtonian limit. Read this way, the complicated equations of Newtonian gravity in the Lagrangian picture become much easier: one just has to deal with the spatial metric tensor and its derivatives. The involved matrices appearing in the standard formulation are nothing else than the covariant and contravariant metric tensor and the spatial Christoffel symbols, appearing in covariant derivatives. Moreover, the fact that the spatial Ricci curvature vanishes in this limit has the great practical advantage that spatial covariant derivatives commute.

Next, we considered the post–Newtonian corrections to the metric and wrote equations for them. In particular, we were able to derive a simple and general equation for gravitational–wave emission from non–linear structures described through Newtonian gravity. The result is expressed in Lagrangian coordinates by Eq.(102), but it can also be given the Eulerian form of Eq.(120). These formulae allow to calculate the amplitude of the gravitational–wave modes in terms of the velocity potential $\Phi_v$, which in turn can be deduced from observational data on radial peculiar velocities of galaxies, applying the POTENT technique (Bertschinger & Dekel 1989).

In the standard case, where the cosmological perturbations form a homogeneous and isotropic random field, we can obtain a heuristic perturbative estimate of their amplitude in terms of the $rms$ density contrast and of the ratio of the typical perturbation scale $\lambda$ to the Hubble radius $r_H = cH^{-1}$. One simply has

$$\frac{\pi_{rms}}{c^2} \sim \delta_{rms}^2 \left( \frac{\lambda}{r_H} \right)^2,$$

(138)
as it can be easily deduced from Eq.(103), specialized to an Einstein–de Sitter model. This effect gives rise to a stochastic background of gravitational waves which gets a non–negligible amplitude in the so–called extremely–low–frequency band (e.g. Thorne 1995), around $10^{-14}$ –
$10^{-15}$ Hz. We can roughly estimate that the present–day closure density of this gravitational–wave background is

$$\Omega_{gw}(\lambda) \sim \delta_{\mathrm{rms}}^4 \left(\frac{\lambda}{r_H}\right)^2.$$  \hspace{1cm} (139)

Note that this background is mostly produced here and now, so its energy is not affected by the usual $a^{-4}$ dilution of gravitational radiation within the Hubble radius. In standard scenarios for the formation of structure in the universe, the typical density contrast on scales 1 – 10 Mpc implies that $\Omega_{gw}$ is about $10^{-5} - 10^{-6}$. We might speculate that such a background would give rise to secondary CMB anisotropies on intermediate angular scales: a sort of tensor Rees–Sciama effect. This issue will be considered in more detail elsewhere.

On much smaller scales, where the effect might be even more relevant, pressure gradients and viscosity cannot be disregarded anymore and the entire formalism needs to be largely modified.

However, our PN formula also applies to isolated structures, where the density contrast can be much higher than the $\text{rms}$ value, and, what is most important here, shear anisotropies play a fundamental role, as it happens in the formation of pancakes. A calculation of $\pi_{\alpha\beta}$ in the simple case of a homogeneous ellipsoid showed that the PN tensor modes become dominant, compared to the Newtonian contributions to the metric tensor, during the late stages of collapse, and possibly even in the case of a shell–crossing singularity. There are a number of important limitations of this result, the most important of which is the role that pressure would certainly play during the highly non–linear stages. A possible consequence could be that pressure gradients halt the growth of anisotropy before our relativistic effects come into play. It is nevertheless important to stress that our effect generally contradicts the standard paradigm, according to which the smallest scale for the applicability of the Newtonian approximation is set by the Schwarzschild radius of the object. Such a critical scale is indeed only relevant for nearly spherical collapse, whereas our effect becomes important precisely if the collapsing structure strongly deviates from sphericity. On the other hand, if we consider the dynamics of a collisionless fluid as a formal problem on itself, the fact that PN terms dominate over Newtonian ones implies that in such a regime the perturbative $1/c$ expansion breaks down and one should resort to a fully relativistic approach.

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Figure 1. Evolution of the PN tensor modes during the collapse of a triaxial homogeneous ellipsoid embedded in an Einstein–de Sitter background; as well known the generic triaxial case exhibits pancake collapse. The top left panel shows the behaviour of the physical axes of the ellipsoid, $R_A(\tau) = a(\tau)X_A(\tau)$ ($A = 1, 2, 3$), vs. the FRW scale–factor $a(\tau)$; the lengths of the three axes are initially scaled as $1 : 1.25 : 1.5$; the initial density contrast is $\delta_0 = 0.05$; the evolution of the structure constant $\alpha_A$ is also shown (top right panel). The bottom left panel shows the evolution of the quantities $\mu_A$ [defined in Eq.(128)], in terms of which the PN tensor modes $\pi_{AB}$ are obtained; note that two of the $\mu_A$ tend to diverge near the pancake collapse of the ellipsoid; as shown in the bottom right panel, such a divergence is even stronger than that of the comoving mass density, $1 + \delta(\tau)$. 
