NON-SINGULAR INHOMOGENEOUS STIFF FLUID COSMOLOGY

L. Fernández-Jambrina
Theoretisch-Physikalisches Institut
Max-Wien-Platz 1
Friedrich-Schiller-Universität-Jena
07743-Jena, Germany

Abstract

In this talk we show a stiff fluid solution of the Einstein equations for a cylindrically symmetric spacetime. The main features of this metric are that it is non-separable in comoving coordinates for the congruence of the worldlines of the fluid and that it yields regular curvature invariants.

1 Introduction

The publication in 1990 of the first known cosmological perfect fluid solution of the Einstein equations with regular curvature invariants [1] has triggered the search of new solutions sharing this feature and the analysis of their properties. In a subsequent paper [2] it was proven that this solution not only has no curvature singularity but it is geodesically complete and singularity-free [3]. It has also been shown that it is part of a larger family which includes both singular and non-singular metrics [4]. Since then other non-singular solutions have been obtained, all of them describing spacetimes with an abelian $G_2$ group of isometries. The main aim of this field of research would be to establish whether there is an open set of metrics which are non-singular.

In this talk we shall show a new diagonal metric which has got regular curvature invariants and is causally stable. It also possess an abelian orthogonally transitive $G_2$ group of isometries. The matter source in this spacetime is a stiff perfect fluid. To our knowledge it is the first solution with cylindrical symmetry which is both non-separable in comoving coordinates and non-singular.

*Permanent address: Departamento de Geometría y Topología, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, E-28040-Madrid, Spain
2 The line element

For the line element constructed from the metric, $g$, of the solution we choose a set of coordinates \( \{ t, \phi, r, z \} \),

\[-\infty < t, z < \infty, \quad 0 < r < \infty, \quad 0 < \phi < 2\pi, \quad (1)\]

where the coordinates, $z$, and $\phi$, are adapted to the commuting Killing vectors, so that the metric functions depend only on the remaining coordinates, $r$, $t$, which are an isothermal parametrization for the two-dimensional submanifolds $z = \text{const.}$, $\phi = \text{const.}$ In this chart the metric takes the form,

\[ ds^2 = e^{K(t,r)} (-dt^2 + dr^2) + e^{-U(t,r)} dz^2 + e^{U(t,r)} r^2 d\phi^2, \quad (2)\]

where the functions $U(t,r)$ and $K(t,r)$ that have been introduced have the following expressions,

\[ K(t,r) = \frac{1}{2} \beta^2 r^4 + (\alpha + \beta) r^2 + 2t^2 \beta + 4t^2 \beta^2 r^2 \quad (3)\]
\[ U(t,r) = \beta (r^2 + 2t^2). \quad (4)\]

The metric satisfies the regularity conditions in the vicinity of the sub manifold $r = 0$ [5] and therefore it can be considered as an actual symmetry axis. A restriction needs be imposed on the values of the only free parameters $\alpha$ and $\beta$,

\[ \alpha > 0, \quad \beta > 0. \quad (5)\]

3 Curvature invariants

In order to write the expressions for the components of the Weyl tensor we shall make use of a complex null tetrad,

\[ l = \frac{\theta^0 + \theta^1}{\sqrt{2}}, \quad n = \frac{\theta^0 - \theta^1}{\sqrt{2}}, \quad m = \frac{\theta^2 + i\theta^3}{\sqrt{2}}, \quad \bar{m} = \frac{\theta^2 - i\theta^3}{\sqrt{2}}, \quad (6)\]

that can be constructed from the orthonormal coframe \( \{ \theta^0, \theta^1, \theta^2, \theta^3 \} \),

\[ \theta^0 = e^{\frac{1}{2} K(t,r)} dt, \quad \theta^1 = e^{\frac{1}{2} K(t,r)} dr, \quad \theta^2 = e^{-\frac{1}{2} U(t,r)} dz, \quad \theta^3 = e^{\frac{1}{2} U(t,r)} r d\phi. \quad (7)\]

After introducing two functions, $f_1$, $f_2$,

\[ f_1(t,r) = -3\beta + 2\beta^3 r^2 + 3\beta^2 r + \alpha (1 + 2\beta r^2) + 24\beta^3 t^2 + 12\beta^2 t^2 \quad (8)\]
\[ f_2(t,r) = 12\beta^3 r^3 t + 12\beta^2 r t + 16\beta^3 t^3 r + 4\alpha \beta t r. \quad (9)\]

the components of the Weyl tensor in the null tetrad can be written as follows,

\[ \Psi_0 = \frac{1}{2} (f_1(t,r) + f_2(t,r)) e^{-K(t,r)} \quad (10)\]
\[ \Psi_1 = 0 \] (11)

\[ \Psi_2 = \frac{1}{6} (3 \beta^2 r^2 + 3 \beta - \alpha - 12 \beta^2 t^2) e^{-K(t,r)} \] (12)

\[ \Psi_3 = 0 \] (13)

\[ \Psi_4 = \frac{1}{2} (f_1(t,r) - f_2(t,r)) e^{-K(t,r)}. \] (14)

### 4 The energy momentum tensor

The energy momentum tensor corresponds to a stiff perfect fluid,

\[ T = \mu u \otimes u + p(g + u \otimes u), \] (15)

where the pressure, \( p \), is equal to the density, \( \mu \), of the fluid

\[ \mu = p = \alpha e^{-K(t,r)}, \] (16)

and happens to be non-singular. This metric can be generated from a vacuum spacetime making use of the Wainwright, Ince, Marshman algorithm [6]. The seed metric is obtained taking the zero value for \( \alpha \) in (2).

The four-velocity of the fluid has only projection on \( \partial_t \),

\[ u = e^{-\frac{1}{2} K(t,r)} \partial_t, \] (17)

since the coordinates that have been chosen are comoving. This implies that the acceleration of the fluid has only a radial component,

\[ a = r \left( \beta^2 r^2 + \alpha + \beta + 4 \beta^2 t^2 \right) \partial_r, \] (18)

due to the orthogonal transitivity requirement and to the fact that the velocity is orthogonal to the orbits of the group of isometries. The only fluid wordlines that are geodesic are those contained in the \( r = 0 \) submanifold.

The shear tensor for the fluid has the following expression,

\[ \sigma = \frac{4}{3} \beta t e^{-\frac{1}{2} K(t,r)} \left\{ (1 + 2 \beta r^2) \theta^1 \otimes \theta^1 - (2 + \beta r^2) \theta^2 \otimes \theta^2 + (1 - \beta r^2) \theta^3 \otimes \theta^3 \right\}. \] (19)

in the orthonormal coframe defined in (7). There is no vorticity since we are dealing with an orthogonally transitive \( G_2 \) group of isometries. Finally we write down the expansion, \( \Theta \), of the cosmological fluid,

\[ \Theta = 2 \beta t (1 + 2 \beta r^2) e^{-\frac{1}{2} K(t,r)}, \] (20)

which shows that the spacetime is contracting for negative values of the time coordinate and expanding for positive values of it.
5 Discussion

From the expressions for the Weyl tensor and the fluid density it is straightforward to check that there is no curvature singularity in this spacetime, since the curvature invariants are polynomial functions of them.

Besides the time coordinate, $t$, is a globally defined function whose gradient is always timelike, that is, it is a cosmic time [3] and therefore the spacetime is causally stable and the hierarchy of causality conditions under it (strong causality condition, chronology condition, ...) are satisfied.

The strong and dominant energy conditions [3] hold since the energy density of the fluid is positive everywhere and the equation of state corresponds to a stiff fluid. Also the generic condition on the Riemann tensor is fulfilled, since the fluid density does not vanish [7].

It remains to be shown whether the spacetime is bundle or geodesically complete. If the latter were true, the reason for avoiding the existence of singularities, according to the singularity theorems [3], would have to be looked for in the lack of closed trapped surfaces or of compact achronal sets without edge or of a null geodesic focalizing point in the spacetime, as it happens in other regular spacetimes [2]. There is work in progress in this direction. It would also be interesting to prove global hyperbolicity.

The curvature tends to zero for large values of the radial, $r$, and time, $t$, coordinates, as it can be inferred from the expressions for the components of the Weyl tensor and the density and the pressure of the fluid. The fluid is rather diluted in early times and becomes more and more dense until the time coordinate reaches the zero value. At this time the universe starts to expand. This feature is typical in other singularity-free cosmological models [1]. As in every other known non-singular cosmological model, the gradient of the transitivitiy surface area element of the metric is spacelike. There is a static limit for vanishing $\beta$ which happens to be same as for the non-singular metric in [8].

Further details on this spacetime will be given in a forthcoming paper [9].

Acknowledgements. The present work has been supported by the Direcccion General de Enseñanza Superior Project PB95-0371 and by a DAAD (Deutscher Akademischer Austausch-dienst) grant for foreign scientists. The author wishes to thank Prof. F. J. Chinea and Dr. L. M. Gonzalez-Romero for valuable discussions and Prof. Dietrich Kramer and the Theoretisch-Physikalisches Institut of the Friedrich-Schiller-Universitiit-Jena for their hospitality.

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