An Effective Bernstein-type Bound on Shannon Entropy over Countably Infinite Alphabets

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Abstract

We prove a Bernstein-type bound for the difference between the average of negative log-likelihoods of independent discrete random variables and the Shannon entropy, both defined on a countably infinite alphabet. The result holds for the class of discrete random variables with tails lighter than or on the same order of a discrete power-law distribution. Most commonly-used discrete distributions such as the Poisson distribution, the negative binomial distribution, and the power-law distribution itself belong to this class. The bound is effective in the sense that we provide a method to compute the constants in it.

Keywords: Concentration inequality; Bernstein-type bound; effective bound; Shannon entropy; countably infinite alphabet; moment generating function

1 Introduction

Concentration inequalities provide powerful tools for many subjects including information theory [5], algorithm analysis [4] and statistics [9, 8]. The goal of the present paper is to prove an exponential decay bound with computable constants for the difference between the negative log-likelihood of discrete random variables and the Shannon entropy, both defined on a countably infinite alphabet.

Let $X$ be a discrete random variable on a countably infinite alphabet $\mathcal{X} = \{x_1, \ldots, x_k, \ldots\}$. Let $p_k = \mathbb{P}(X = x_k)$ be the probability mass at $x_k$. Assume, without loss of generality, that $p_k > 0$ for each $k$; otherwise, simply remove $x_k$ with $p_k = 0$ from $\mathcal{X}$. Let $P(X)$ be the probability mass function, which is a random variable with $P(X) = p_k$ if $X = x_k$, $k \geq 1$. Then $\mathbb{E}[-\log P(X)] = -\sum_{k=1}^{\infty} p_k \log p_k$ is the Shannon entropy, which is a key concept in information theory [2, 2]. Note

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1Throughout the paper, “log” denotes the natural logarithm.
that neither $P(X)$ nor the entropy depends on the elements in $\mathcal{X}$. In fact, $\mathcal{X}$ is not necessarily a set of real numbers. The set can contain generic symbols such as letters, and is therefore named as alphabet.

Entropy on countably infinite alphabets does not always have finite values. We give a simple sufficient condition ensuring its finiteness at the beginning of Section 2, which is also the key assumption for the main result of the paper. The readers are referred to [1] for a more thorough discussion on conditions for finiteness of entropy on countably infinite alphabets.

Let $X_1, ..., X_n$ be independently and identically distributed (i.i.d.) copies of $X$. Then $\sum_{i=1}^{n} \log P(X_i)$ is the joint log-likelihood of $X_1, ..., X_n$. By the weak law of large numbers,

$$P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \log P(X_i) - \mathbb{E}[\log P(X)] \right| \geq \epsilon \right) \to 0,$$

provided that the entropy is finite. This result, particularly for the case of $|\mathcal{X}|$ being finite, is called the asymptotic equipartition property in the information theory literature, which is the foundation of many important results in this field [2, 3].

In this paper, we strengthen the above result by proving a Bernstein-type bound for the case of countably infinite alphabets:

$$P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \log P(X_i) - \mathbb{E}[\log P(X)] \right| \geq \epsilon \right) \leq 2 \exp \left( -\frac{ne^2}{c_1 + c_2 \epsilon} \right),$$

(1)

where $c_1$ and $c_2$ are computable constants that depend on $\{p_k\}_{k \geq 1}$.

Concentration inequalities for entropy have been studied recently. Zhao [10] proved a Bernstein-type inequality for entropy on finite alphabets with convergence rate $(K^2 \log K)/n = o(1)$, where $n$ is the sample size and $K$ is the size of the alphabet. Zhao [11] proved an exponential decay bound that improves the rate to $(\log K)^2/n = o(1)$ and showed that the new rate is optimal. Both papers studied inequalities for finite alphabets while we focus on countably infinite alphabets in this work. In Section 2 we prove (1) under a mild assumption. In Section 3 we show that this assumption holds if the tail of $\{p_k\}_{k \geq 1}$ drops faster or on the same order of a discrete power-law distribution; conversely, the assumption cannot be satisfied if the tail drops slower than any power-law distribution. Most commonly-used discrete distributions such as the Poisson distribution, the negative binomial distribution, and the power-law distribution itself satisfy this assumption. Furthermore, we propose a method to compute the constants in the bound (1).

2 Main Result

Our result requires only one assumption on $\{p_k\}_{k \geq 1}$:

**Assumption 1.** There exists $0 < r < 1$ such that

$$\sum_{k=1}^{\infty} p_k^{1-r} \leq C_r < \infty.$$
Assumption 1 implies that the tail of \( \{ p_k \}_{k \geq 1} \) cannot be too heavy, and in Section 3 we will elaborate this assumption by showing that the assumption holds if the tail of \( \{ p_k \}_{k \geq 1} \) is lighter than or on the same order of a discrete power-law distribution; conversely, it cannot be satisfied if the tail is heavier than any power-law distribution.

First note that Assumption 1 ensures the finiteness of the entropy.

**Proposition 1.** Under Assumption 1, \( E[- \log P(X)] < \infty \).

*Proof.*

\[
E[- \log P(X)] = -\sum_{k=1}^{\infty} p_k \log p_k \leq \sum_{k=1}^{\infty} p_k^{1-r}(-p_k^r \log p_k) \leq \frac{1}{er} \sum_{k=1}^{\infty} p_k^{1-r}.
\]

The last inequality holds because \(-p_k^r \log p_k\) on \([0, 1]\) is maximized at \( p_k = e^{-1/r} \). This result can be easily verified by comparing the function value at the stationary point in \((0, 1)\), which is unique for this function, with the values on the boundaries. Here we use the convention \( q^{r \log q} = 0 \) at \( q = 0 \), which makes the function continuous on \([0, 1]\) since \( \lim_{q \to 0^+} q^{r \log q} = 0 \).

Let \( Y_i = \log P(X_i) - E[\log P(X)] \). The key ingredient of the proof is to bound the moment generating function (MGF) of \( Y_i \), which is defined as

\[
E[e^{\lambda Y_i}] = \left( \sum_{k=1}^{\infty} p_k^{\lambda+1} \right) \exp \left( -\lambda \sum_{k=1}^{\infty} p_k \log p_k \right).
\]

Denote the MGF of \( Y_i \) by \( M_{Y_i}(\lambda) \). Under Assumption 1, \( M_{Y_i}(\lambda) \) is finite for \( |\lambda| < r \) because

\[
\sum_{k=1}^{\infty} p_k^{\lambda+1} \leq \sum_{k=1}^{\infty} p_k^{1-r} < \infty.
\]

Conversely, if Assumption 1 does not hold then \( \sum_{k=1}^{\infty} p_k^{\lambda+1} \) diverges for all \( \lambda < 0 \), because if \( \sum_{k=1}^{\infty} p_k^{\lambda+1} \) converges for a certain negative \( \lambda \) then it must be in the interval \((-1, 0)\) and one can take \( r = -\lambda \).

We now give the main result.

**Theorem 1** (Main result). Under Assumption 1, that is, if there exists \( 0 < r < 1 \) such that

\[
\sum_{k=1}^{\infty} p_k^{1-r} \leq C_r < \infty,
\]

then for \( |\lambda| < r \),

\[
M_{Y_i}(\lambda) \leq \exp \left( C_r \frac{\lambda^2}{r^2} \frac{1}{1 - \frac{|\lambda|}{r}} \frac{1}{2\sqrt{\pi}} \right).
\]

Furthermore, for all \( \epsilon > 0 \),

\[
\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} \log P(X_i) - E[\log P(X)] \geq \epsilon \right) \leq 2 \exp \left( -\frac{ne^2}{2C_r/(\sqrt{\pi}r^2) + 2\epsilon/r} \right). \quad (2)
\]
Proof. For $|\lambda| < r$,

$$\log M_Y(\lambda) = \log \left( \sum_{k=1}^{\infty} p_k^{\lambda+1} \right) - \lambda \sum_{k=1}^{\infty} p_k \log p_k$$

$$\leq \sum_{k=1}^{\infty} p_k^{\lambda+1} - 1 - \lambda \sum_{k=1}^{\infty} p_k \log p_k$$

$$= \sum_{k=1}^{\infty} p_k \exp(\lambda \log p_k) - 1 - \lambda \sum_{k=1}^{\infty} p_k \log p_k$$

$$= \sum_{k=1}^{\infty} \left( p_k + \lambda p_k \log p_k + \sum_{m=2}^{\infty} \frac{1}{m!} \lambda^m p_k (\log p_k)^m \right) - 1 - \lambda \sum_{k=1}^{\infty} p_k \log p_k, \quad (3)$$

where the inequality follows from $\log x \leq x - 1$ for $x > 0$.

For $m \geq 2$, it is easy to check that, the minimum of $p_k^r (\log p_k)^m$ on $[0, 1]$ when $m$ is an odd number, and the maximum when $m$ is an even number, are achieved at $e^{-m/r}$ by comparing the function value at the stationary point in $(0, 1)$, which is unique, with the values on the boundaries. Here we use the convention $q^r (\log q)^m = 0$ at $q = 0$ as before, which makes the function continuous on $[0, 1]$ since $\lim_{q \to 0^+} q^r (\log q)^m = 0$.

Therefore, for $m \geq 2$,

$$\left| \frac{1}{m!} \lambda^m p_k (\log p_k)^m \right|$$

$$\leq p_k^{1-r} \frac{1}{m!} |\lambda|^m |p_k^r (\log p_k)^m|$$

$$\leq p_k^{1-r} \frac{1}{m!} |\lambda|^m e^{-m} \left( \frac{m}{r} \right)^m$$

$$= p_k^{1-r} \frac{1}{m!} \left( \frac{|\lambda|}{r} \right)^m \left( \frac{m}{e} \right)^m$$

$$\leq p_k^{1-r} \frac{1}{m!} \left( \frac{|\lambda|}{r} \right)^m \frac{m!}{\sqrt{2\pi m}}$$

$$\leq p_k^{1-r} \left( \frac{|\lambda|}{r} \right)^m \frac{1}{2\sqrt{\pi}}, \quad (4)$$

where the second inequality is obtained by replacing $|p_k^r (\log p_k)^m|$ with its maximum and the third inequality follows from Stirling’s formula (see [6] for example):

$$m! \geq \sqrt{2\pi m} \left( \frac{m}{e} \right)^m, \text{ for } m \geq 1.$$ 

It follows that for $|\lambda| < r$,

$$\left| \sum_{m=2}^{\infty} \frac{1}{m!} \lambda^m p_k (\log p_k)^m \right| \leq \sum_{m=2}^{\infty} \left| \frac{1}{m!} \lambda^m p_k (\log p_k)^m \right| \leq p_k^{1-r} \sum_{m=2}^{\infty} \left( \frac{|\lambda|}{r} \right)^m \frac{1}{2\sqrt{\pi}} = p_k^{1-r} \frac{\lambda^2}{r^2} \frac{1}{1 - \frac{|\lambda|^2}{2r^2}}.$$
and
\[ \sum_{k=1}^{\infty} \sum_{m=2}^{\infty} \frac{1}{m!} \lambda^m p_k (\log p_k)^m \leq C_r \frac{\lambda^2}{r^2} \frac{1}{1 - \frac{|\lambda|}{r}} \frac{1}{2\sqrt{\pi}}. \]

Since the three terms under \( \sum_{k=1}^{\infty} \) in (3) all converge absolutely for \( |\lambda| < r \), one can take the sum term by term. Therefore, for \( |\lambda| < r \),
\[ \log M_{Y_1}(\lambda) \leq \sum_{k=1}^{\infty} \sum_{m=2}^{\infty} \frac{1}{m!} \lambda^m p_k (\log p_k)^m \leq C_r \frac{\lambda^2}{r^2} \frac{1}{1 - \frac{|\lambda|}{r}} \frac{1}{2\sqrt{\pi}}, \]
and
\[ M_{Y_1}(\lambda) \leq \exp \left( \frac{C_r \lambda^2}{r^2} \frac{1}{1 - \frac{|\lambda|}{r}} \frac{1}{2\sqrt{\pi}} \right). \quad (5) \]

The second part of the theorem follows from a standard argument using the Chernoff bound, which can be found in Chapter 2 of [9]. We give the details for completeness. For \( t > 0 \) and \( 0 < \lambda < r \),
\[ P \left( \sum_{i=1}^{n} Y_i \geq t \right) = P \left( e^{\lambda \sum_{i=1}^{n} Y_i} \geq e^{\lambda t} \right) \leq \prod_{i=1}^{n} \frac{M_{Y_i}(\lambda)}{e^{\lambda t}} \leq \exp \left\{ \frac{nC_r \lambda^2}{r^2} \frac{1}{1 - \frac{|\lambda|}{r}} \frac{1}{2\sqrt{\pi}} \lambda t \right\}, \]
where the first inequality is Markov’s inequality and the second inequality follows from (5). By setting
\[ \lambda = \frac{t}{nC_r/(\sqrt{\pi} r^2) + t/r} \in (0, r), \]
we obtain
\[ P \left( \sum_{i=1}^{n} Y_i \geq t \right) \leq \exp \left( -\frac{t^2}{2nC_r/(\sqrt{\pi} r^2) + 2t/r} \right). \]

The left tail bound can be obtained similarly by setting \( \lambda = -\frac{t}{nC_r/(\sqrt{\pi} r^2) + t/r} \). Therefore,
\[ P \left( \left| \sum_{i=1}^{n} Y_i \right| \geq t \right) \leq 2 \exp \left( -\frac{t^2}{2nC_r/(\sqrt{\pi} r^2) + 2t/r} \right). \]

Finally, letting \( t = n\epsilon \),
\[ P \left( \left| \frac{1}{n} \sum_{i=1}^{n} Y_i \right| \geq \epsilon \right) \leq 2 \exp \left( -\frac{n\epsilon^2}{2C_r/(\sqrt{\pi} r^2) + 2\epsilon/r} \right). \]
Theorem 1 can be generalized to \( \{X_i\}_{i=1,...,n} \) with independent but non-identical distributions. Let \( p_{ik} = P(X_i = x_k) \) be the probability mass of \( X_i \) at \( x_k \) and \( \mathbb{E}[-\log P(X_i)] = -\sum_{k=1}^{\infty} p_{ik} \log p_{ik} \) be the entropy of \( X_i \). Furthermore, redefine \( Y_i \) and \( M_{Y_i}(\lambda) \) accordingly. We have the following result for non-identical distributions:

**Corollary 1.** If there exists \( 0 < r < 1 \) such that

\[
\sum_{k=1}^{\infty} p_{1k}^{1-r} \leq C_{r,i} < \infty, \ i = 1, ..., n,
\]

then for \( |\lambda| < r \),

\[
M_{Y_i}(\lambda) \leq \exp \left( \frac{C_{r,i} \lambda^2}{r^2} \frac{1}{1 - |\lambda|^2} \frac{1}{2} \pi \frac{1}{\pi r^2} \right).
\]

Furthermore, for all \( \epsilon > 0 \),

\[
P \left( \left| \frac{1}{n} \sum_{i=1}^{n} (\log P(X_i) - \mathbb{E}[\log P(X_i)]) \right| \geq \epsilon \right) \leq 2 \exp \left( -\frac{n \epsilon^2}{2 \sum_{i=1}^{\infty} C_{r,i}/(n \pi r^2) + 2 \epsilon/r} \right).
\]

The proof is the same as of Theorem 1.

### 3 Determining the Constants in the Bound

The radius of convergence \( r \) in Theorem 1 and the upper bound \( C_r \) for \( \sum_{k=1}^{\infty} p_k^{1-r} \) are the only constants to be determined if one wants to use (2) as an effective upper bound for a given distribution \( \{p_k\}_{k \geq 1} \).

We first determine the types of distributions and the range of \( r \) that can make \( \sum_{k=1}^{\infty} p_k^{1-r} \) converge. Intuitively speaking, for distributions that satisfy Assumption 1, the tail of \( \{p_k\}_{k \geq 1} \) cannot be too heavy. We make the above statement precise in the following theorem.

**Theorem 2.** The distribution \( \{p_k\}_{k \geq 1} \) satisfies Assumption 1 if the tail of \( \{p_k\}_{k \geq 1} \) is lighter than or on the same order of a discrete power-law distribution; conversely, Assumption 1 cannot be satisfied if the tail is heavier than any power-law distribution. Specifically,

(i) If

\[
\lim_{k \to \infty} \frac{p_k}{k^{-\alpha}} = 0, \text{ for all } \alpha > 1,
\]

then

\[
\sum_{k=1}^{\infty} p_k^{1-r} < \infty, \text{ for all } 0 < r < 1.
\]
(ii) If
\[ 0 < \liminf_{k \to \infty} \frac{p_k}{k^{-\alpha}} \leq \limsup_{k \to \infty} \frac{p_k}{k^{-\alpha}} < \infty, \text{ for some } \alpha > 1, \]
then
\[ \sum_{k=1}^{\infty} p_k^{1-r} < \infty, \text{ if and only if } 0 < r < \frac{\alpha - 1}{\alpha}. \]

(iii) If
\[ \lim_{k \to \infty} \frac{p_k}{k^{-\alpha}} = \infty, \text{ for all } \alpha > 1, \]
then
\[ \sum_{k=1}^{\infty} p_k^{1-r} = \infty, \text{ for all } 0 < r < 1. \]

Proof. Recall that \( \sum_{k=1}^{\infty} k^{-\beta} \) converges for \( \beta > 1 \), and diverges for \( \beta \leq 1 \). Statement (i) is obvious by taking \( \alpha > 1/(1-r) \). Statement (ii) is also obvious by noticing that the assumption implies that there exist positive constants \( a_1, a_2 \) such that \( a_1 k^{-\alpha} \leq p_k \leq a_2 k^{-\alpha} \) for sufficiently large \( k \). We prove (iii) by contradiction. If there exists \( 0 < r < 1 \) such that \( \sum_{k=1}^{\infty} p_k^{1-r} < \infty \), then
\[ \liminf_{k \to \infty} \frac{p_k^{1-r}}{k^{-1}} = 0. \]
It implies
\[ \liminf_{k \to \infty} \frac{p_k}{k^{-1/(1-r)}} = 0, \]
which contradicts the assumption since \( 1/(1-r) > 1 \).

Theorem 2 implies that there are a wide class of discrete distributions satisfying Assumption 1, including the most commonly-used ones such as the Poisson distribution, the negative binomial distribution, and the power-law distribution itself. The class even contains certain discrete random variables that do not have finite expectations. In fact, if \( X \) follows a discrete power-law distribution with \( 1 < \alpha \leq 2 \) then \( E[X] = \infty \) since \( \sum_{k=1}^{\infty} k^{-(\alpha-1)} \) diverges. But such distributions satisfy Assumption 1 by Theorem 2 (ii).

Remark. It may be surprising, at first glance, to get an exponential decay bound for a power-law distribution, which itself is heavy-tailed. But note that (ii) is a concentration bound for \( \log P(X) \), not for \( X \). The log-likelihood \( \log P(X) \) is typically better-behaved than \( X \) that takes values on non-negative integers and follows a heavy-tailed distribution. For example, for a power-law distribution with \( 1 < \alpha \leq 2 \), \( E[X] = \infty \); on the contrary, the entropy \( E[-\log P(X)] \) is finite by Proposition 1 and Theorem 2 (ii). This phenomenon can be explained by noticing that \( -\log(k^{-\alpha}) \) grows much slower than \( k \). Moreover, the MGF of \( X \) is infinite if \( X \) follows a power-law distribution while
the MGF of \( \log P(X) \) is finite. The tail of \( \log P(X) \) is not heavy in this sense, which makes (2) possible.

Finally, we discuss how to compute \( C_r \) after \( r \) is selected by Theorem 2. In practice, one can compute the partial sum of \( \sum_{k=1}^{\infty} p_k^{1-r} \) until the increment is negligible. The value obtained in this way, however, is a lower bound for \( \sum_{k=1}^{\infty} p_k^{1-r} \) and a generic truncation error bound does not exist for positive infinite series because in principle, the tail behavior cannot be predicted by a finite number of terms.\(^2\)

If the tail of \( \{p_k\}_{k \geq 1} \) is dominated by a power-law distribution, we propose a method that can compute an upper bound for \( \sum_{k=1}^{\infty} p_k^{1-r} \) at any tolerance level. Specifically, the next theorem shows how to compute an upper bound \( C_r \) for \( \sum_{k=1}^{\infty} p_k^{1-r} \) with \( |\sum_{k=1}^{\infty} p_k^{1-r} - C_r| \) smaller than a pre-specified tolerance level if we find \( k_0 \) such that \( p_k \leq c_0 k^{-\alpha} \) for \( k > k_0 \). Note that such \( k_0 \) exists if \( \{p_k\}_{k \geq 1} \) satisfies the condition in (i) or (ii) in Theorem 2.

**Theorem 3.** Suppose \( k_0 \) is a positive integer such that \( p_k \leq c_0 k^{-\alpha} \) for certain \( \alpha > 1 \) and all \( k > k_0 \), where \( c_0 > 0 \). Pick \( r \) such that \( 0 < r < (\alpha - 1)/\alpha \). For all \( \epsilon > 0 \), let

\[
k_1 = \max \left\{ k_0, \left\lceil \left( \frac{\epsilon(\alpha(1-r) - 1)}{c_0} \right)^{-\frac{1}{\alpha(1-r)-1}} \right\rceil \right\},
\]

where \( \lceil \cdot \rceil \) means rounding up to the next integer. Then

\[
C_r = \sum_{k=1}^{k_1} p_k^{1-r} + \epsilon
\]

satisfies

\[
0 \leq C_r - \sum_{k=1}^{\infty} p_k^{1-r} \leq \epsilon.
\]

**Proof.** We only need to bound the tail probability for \( k > k_1 \).

\[
\sum_{k=k_1+1}^{\infty} p_k^{1-r} \leq c_0 \sum_{k=k_1+1}^{\infty} k^{-\alpha(1-r)}
\]

\[
= c_0 \sum_{k=k_1}^{\infty} \int_k^{k+1} (k + 1)^{-\alpha(1-r)} \, dx
\]

\[
\leq c_0 \int_{k_1}^{\infty} x^{-\alpha(1-r)} \, dx
\]

\[
= \frac{c_0}{\alpha(1-r) - 1} k_1^{-(\alpha(1-r)-1)} \leq \epsilon,
\]

\(^2\)This issue is minor in practice especially when \( p_k \) drops exponentially. The series \( \sum_{k=1}^{\infty} p_k^{1-r} \) usually converges very fast in this case. It is nothing wrong to take the partial sum until the increment is negligible. The method in Theorem 3 is useful to someone who needs a rigorous upper bound.
where the first inequality holds because $p_k \leq c_0 k^{-\alpha}$ for all $k > k_0$ and the last inequality holds because $k_1 \geq \left\lceil \left( \frac{(\alpha(1-r)-1)}{c_0} \right)^{-\frac{1}{\alpha(1-r)-1}} \right\rceil$.

Therefore,

$$\sum_{k=1}^{\infty} p_k^{1-r} = \sum_{k=1}^{k_1} p_k^{1-r} + \sum_{k=k_1+1}^{\infty} p_k^{1-r} \leq \sum_{k=1}^{k_1} p_k^{1-r} + \epsilon.$$

$\square$

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