A standard two-path interferometer fed into a linear N-port analyzer with coincidence detection of its output ports is analyzed. The N-port is assumed to be implemented as a discrete Fourier transformation $F$, i.e., to be balanced. For unbound bosons it allows us to detect N-particle interference patterns with an N-fold reduction of the observed de Broglie wavelength, perfect visibility and minimal noise. Because the scheme involves heavy filtering a lot of the signal is lost, yet, it is surprisingly robust against common experimental imperfections, and can be implemented with current technology.

42.50.Ar, 42.25.Hz, 03.65.Ta, 07.60.Ly, 42.79.-e

A quantum state containing $N$ particles can yield an $N$-fold reduction of the observed de Broglie wavelength in an interference experiment when compared with a single particle pattern [1]. This can lead to an $N$-fold increase of interferometric sensitivity, in the best case, reaching the quantum mechanical ‘Heisenberg-limit’ of particle-number–based enhancement in resolution [2]. For bound particles this is straightforward [3], as long as they are not split up by the interferometer, namely, their binding energy exceeds the interaction energies encountered when passing the interferometer’s beam splitters and mergers [1]. But, it is difficult to reach the Heisenberg-limit, or, more generally speaking, to detect N-particle patterns in imaging or interferometry using unbound particles [4–6] such as free photons.

Experimental results have so far only demonstrated a halving of the observed wavelength using two-photon states [7–9] or reported signatures of four-particle entanglement for photons [10,11] and ions [12]. Yet, weakly or unbound bosons such as photons or cold atoms are currently the most important particles for interferometry and imaging. Interest in several recent schemes to employ their multi-particle features in interferometers [1,4,5,7–9,13], high resolution imaging [14], and quantum lithography [15] has therefore been considerable [16].

Observing multi-particle quantum effects for unbound particles is hampered by several problems: special quantum states are needed that are hard to synthesize [1,2,5,14,15], the processing involves large non-linear particle-particle interactions [1] or joint detection of all particles with single particle resolution [5,7–9,13,15].

The linear scheme presented in this paper shows that equipping an interferometer’s output with a balanced N-port analyzer allows us to circumvent several of the above problems and detect multi-photon interference patterns with current technology. Note, that other linear schemes have recently been devised to circumvent similar problems in quantum information theory [17,18] and quantum state preparation [19,20]. Using the scheme presented here it should be straightforward to observe, for the first time, a more than two-fold reduction of the effective de Broglie wavelength of unbound photons [1,7–9,13]; the scheme is sketched in FIG. 1.

![FIG. 1. Sketch of the setup: dotted lines depict $N-2$ empty (vacuum) modes fed into the Fourier mixer $F_N$ together with the two active interferometer modes $a^{\dagger}_1$ and $a^{\dagger}_2$ resulting from mixing the modes $a^{\dagger}$ and $b^{\dagger}$ at a balanced beam splitter. Mode $a^{\dagger}_2$ suffers the interferometric phase shift $\phi$. The corresponding output modes of the Fourier mixer $F_N$ are $A^{\dagger}_1$ to $A^{\dagger}_N$; they are detected by $N$ single-photon detectors $D$ which are read out in coincidence.](image)

In order to model our system let us follow the evolution of a state of bosonic particles starting out in channel $a^{\dagger}_1$ of FIG. 1 and progressing from left to right. For the transformations describing the mixing of the interferometer’s input modes by a balanced beam-splitter in conjunction with the phase shift $\phi$ for mode $a^{\dagger}_2$ we choose the following operator equations

$$a^{\dagger}_1 = (\hat{a}^{\dagger} + \hat{b}^{\dagger})/\sqrt{2}, \quad a^{\dagger}_2 = \exp[-i\phi(\hat{a}^{\dagger} - \hat{b}^{\dagger})/\sqrt{2}].$$

Subsequent to the interferometer the balanced N-port $F_N$ acts as a discrete Fourier-transformer [21,22] and mixes the interferometric modes $a^{\dagger}_1$ and $a^{\dagger}_2$ with the $N-2$ vacuum modes $a^{\dagger}_3, ..., a^{\dagger}_N$. Using a vector notation for these input modes $a_k^{\dagger}$ and the output modes $A_k^{\dagger}$, i.e. $A_k^{\dagger} = F_N a_k^{\dagger}$, we specify the corresponding (Fourier-) transformation matrix elements as

$$[F_N]_{j,k} = \frac{1}{\sqrt{N}} \exp[i\frac{2\pi}{N}(j-1)(k-1)].$$
To make a connection with the textbook case of classical or single photon interference [23] let us look at the case $N = 2$, the conventional Mach-Zehnder interferometer. For the input state we assume a single photon is entering through channel $\hat{a}^\dagger$, namely $|\psi\rangle_{in} = \hat{a}^\dagger |0\rangle$. After the transformations (1) and (2) this state is converted into $|\psi\rangle_{in} = (1 + e^{-i\phi})/2 \left( \hat{A}_1^\dagger - \hat{A}_2^\dagger \right) |0\rangle$ yielding the customary classical interference pattern $\langle \hat{I}_1 \rangle = \langle \hat{A}_1^\dagger \hat{A}_1 \rangle = (1 + \cos \phi)/2$ and the antiphase pattern in the second channel $\langle \hat{I}_2 \rangle = (1 - \cos \phi)/2$. Of course, the coincidence pattern $\langle \hat{I}_1 \cdot \hat{I}_2 \rangle = 0$ vanishes, since only a single photon is present. The simplest non-trivial case is that of the Mach-Zehnder interferometer fed with a two-photon state $|\psi(2)\rangle_{in} = (\hat{a}_1^\dagger)^2 |0\rangle$: it yields the classical [23] first order interference patterns $\langle \hat{I}_{1/2} \rangle = \pm 1 \cos \phi$ and the coincidence signal $\langle \hat{I}_1 \cdot \hat{I}_2 \rangle = (1 - \cos 2\phi)/4$, which shows the desired halving of the effective de Broglie wavelength and full contrast [23] observed in experiments [7–9].

Generalizing the above discussion we now consider the $N$-channel coincidence count operator $\hat{I}_N$

$$\hat{I}_N = \prod_{j=1}^N \hat{I}_j = \prod_{j=1}^N \hat{A}_j^\dagger \prod_{k=1}^N \hat{A}_k ,$$

where we have used the fact that the output modes $A$ commute. Expressing $\prod_{k=1}^N \hat{A}_k$ in terms of the input operators $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ yields

$$\hat{I}_N = \frac{\hat{a}_2^\dagger N - (-\hat{a}_1^\dagger)^N \cdot \hat{a}_1^\dagger N - (-\hat{a}_2^\dagger)^N}{\sqrt{N^N}} \frac{\hat{a}_2^\dagger N + (-\hat{a}_1^\dagger)^N \cdot \hat{a}_1^\dagger N + (-\hat{a}_2^\dagger)^N}{\sqrt{N^N}},$$

where we have used (2) and assumed that $\hat{a}_1^\dagger = \hat{a}_2^\dagger = \ldots = \hat{a}_N^\dagger = 0$ in accordance with our scheme’s stipulation to leave these modes empty (tracing out the vacuum state).

Note, that the terms $\hat{a}_j^\dagger N \hat{a}_k^\dagger$ represent generalized $N$-particle intensity or $N$-particle dosage terms [15]. Correspondingly, the terms in square brackets represent $N$-particle cross-mode terms which generalize the well known single particle (or classical) interference terms [23] of the form $\hat{a}_1 \hat{a}_2^\dagger + \hat{a}_2 \hat{a}_1^\dagger$.

Eq. (5) contains a good and a bad message, the bad aspect is the emergence of the signal suppression factor $N^{-N}$ which is due to the fact that we require all output detectors to fire – a rare event: $\langle \hat{I}_N \rangle = 0$ if one of the output channel detectors fails to fire. Such losses are common whenever one tries to substitute non-linear by linear elements [17–20].

The positive aspect is the result that the form of $\hat{I}_N$ in Eq. (5) suggests that for those subensembles of events that trigger firing of all $N$ detectors a ‘perfect’ $N$th order interference pattern measurement can be performed. Only linear elements and an array of conventional detectors is needed for its implementation, it will turn out that the scheme is rather robust with regards to detector imperfections. Current technology suffices, Eq. (5) therefore is important, it is the main result of this paper.

In order to prove that $\hat{I}_N$ does indeed describe a ‘perfect’ $N$th order interference pattern measurement let us check its signal and noise properties. The combination of operators suggests that $\hat{I}_N$ probes for the presence of the number-entangled state

$$|\psi\rangle_{inside} = |N\rangle a_1^\dagger a_2 + e^{iN\phi} |0\rangle a_1 |N\rangle a_2 ^\dagger \sqrt{2} .$$

(6)

This is the superposition-state needed when trying to reach the Heisenberg limit in interferometry [2] using single photon count techniques. Note, that to synthesize this kind of state one typically needs correlated input in both modes $\hat{a}^\dagger$ and $\hat{b}^\dagger$, see e.g. [20], this case is obviously not covered by our sketch of the setup in FIG. 1 where the source is only seen to feed into mode $\hat{a}^\dagger$.

Its signal is $\langle \hat{I}_N \rangle = N! / N^N \times (1 - (-1)^N \cos (N \phi))$ and does indeed display the term $\cos (N \phi)$ due to an $N$-fold decreased effective de Broglie wavelength [1,7–9,13] and an interference pattern with perfect visibility.

The second moment is $\langle \hat{I}_N^2 \rangle = N^2 / N^{2N} \sin^2 (N \phi)$. To study the scheme’s noise features, we will estimate the noise induced phase spread $\Delta \phi$ from the quotient of the signal’s fluctuations and the signal’s phase gradient (see a good textbook such as [24])

$$\Delta \phi = \Delta \langle \hat{I}_N \rangle / |\partial \langle \hat{I}_N \rangle / \partial \phi|,$$

(7)

with $\Delta \langle \hat{I}_N \rangle = \sqrt{\langle \hat{I}_N^2 \rangle - \langle \hat{I}_N \rangle^2}$. For the number-entangled state (6) the phase spread therefore is

$$\Delta \phi = 1/N .$$

(8)

The subensemble of registered events (all detectors fire) reaches the Heisenberg limit [24] which proves that our scheme has the least possible noise [2].

According to Ou’s analysis [2] this limit cannot be reached using arbitrary states, i.e. in our case, states other than (6).

Let us therefore study the signal and noise properties of some representative states that are fed through one channel, e.g. $\hat{a}^\dagger$, only. To start out, let us consider $N$-photon Fock states $|\psi(N)\rangle_{in} = \hat{a}^N |0\rangle / \sqrt{N^N}$, which can in principle be synthesized using projection measurements [19]; the signal $\langle \hat{I}_N \rangle$ has the form

$$\langle \hat{I}_N \rangle \doteq \left( \prod_{j=1}^N \hat{I}_j \right) = \frac{N!}{2(N-1) N^N} (1 - (-1)^N \cos (N \phi)) .$$

(9)

We see that an interference pattern (9) with $N$-fold reduction and perfect contrast is observable, yet we will find that the imperfect matching to the ideal state (6) leads to large noise.
According to Stirling’s formula, this signal scales like
\[ \frac{N!}{2^{N-1}N^N} \approx \frac{\sqrt{\pi N}}{(2e)^N}, \]
with the corresponding numerical values \(1/4, 1/18, 3/256, 3/1250, 5/10368\) for \(N = 2, 3, \ldots, 6\). Evidently, the requirement for simultaneous firing of all detectors reduces the signal strength considerably.

The joint-detection operator \(\hat{I}_N\) extracts suitable components to show \(N\)-photon interference patterns out of every state with sufficiently large photon numbers. In order to prove this for general input into channel \(\alpha\) let us now consider an excess \(E\) of particles \(N_{\text{state}} = N + E\) over the number of ports \(N\). Trivially, a negative excess, i.e., a number of particles fewer than expected detector clicks will give a zero signal. For a Fock-state \(\psi_{(N+E)}\) let us now consider an excess \(E\) of particles \(N_{\text{state}} = N + E\) over the number of ports \(N\). Moreover, large photon number excess also reduces the noise, see below.

In order to appreciate the very considerable intensity gain attributable to such a photon-number excess, note, that at the rather modest cost of doubling the number of photons, one can roughly compensate the losses due to an increased number of ports. For example, the losses due to moving from \(N = 2\) photons and channels to 10 photons and channels is offset by moving to 10 channels and 20 photons instead. Moreover, large photon number excess also reduces the noise, see below.

In order to generalize our results to states other than Fock-states let us maintain that the \(\beta\)-channel is empty and let us trace it out: \(\beta = \beta^\dagger = 0\). In this case the coincidence count operator \(\hat{I}_N\) commutes with the input photon number operator \(\hat{a}\dagger\hat{a}\) implying that any Fock-state component exceeding the port number \(N\) contributes separately to the overall interference pattern and its noise features. Hence, for a general \(\alpha\)-channel input state \(\psi_{\alpha} = \sum_{j=0}^{\infty} c_j |\alpha_j\rangle \langle \alpha_j| 0\rangle\), the coincidence pattern has the form
\[ \langle \hat{I}_N \rangle = \sum_{E=0}^{\infty} |c_{N+E}|^2 \frac{(N + E)!}{E!} \cdot \frac{1 - (-1)^N \cos(N\phi)}{2(N-1)N^N}. \]

A corresponding generalization applies to mixed input states; this shows that our scheme allows for a robust formation of an \(N\)-fold reduced interference pattern with perfect visibility in the general case of states fed through channel \(\alpha^\dagger\) only.

Let us now discuss the noise properties of the \(N\)-photon state \(\psi_{(N)} = \frac{(\hat{\alpha}^\dagger)^N}{\sqrt{N!}} |0\rangle\). The second moment is \(\langle \hat{I}_N^2 \rangle = N^{2N} / (N^2 N^{N-2}) \times (1 - (-1)^N \cos(N\phi))\) yielding the minimum phase estimate (at positions \(\phi = 0\) for even \(N\) and \(\phi = \pi/N\) for odd \(N\))
\[ \Delta\phi = \frac{\sqrt{2N-1}}{N}. \]

A Fock state fed into the \(\alpha^\dagger\)-channel has little overlap with the number-entangled state (6). This leads to extra noise increasing the phase spread by the factor \(\sqrt{2N-1}\).

We have seen that the use of Fock states with a photon number excess \(N + E\) over the port number \(N\) leads to a considerable signal increase (10). Also the noise performance in this case is better. The corresponding expression for the noise is a bit involved (it was determined using the ‘Maple’ symbolic manipulation program). For large photon number excess \(E\) we find that the noise falls to the shot-noise level, but not below; see FIG. 2.

Note, that we restricted most of the above analysis to the case of states fed through the input channel \(\alpha^\dagger\) only, this is because there are few transparent analytical expressions in the general case. However, in order to go below the shot noise level quantum correlated input, using channel \(\beta^\dagger\) as well, is needed (remember Eq. (8)); this can either reduce or enhance the interferometric signal [23]. Our discussion shows that currently available quantum states can be used and that moving towards the two-mode entangled number state [20] allows us not only to observe an \(N\)-fold reduced de Broglie wavelength but also noise below the shot-noise level (for the subensembles selected by the coincidence clicks).

Our scheme is not only versatile with regards to the states that can be used, it also is quite insensitive to detector imperfections (other than dark counts) which often lead to serious degradation for quantum state reconstruction schemes [25]. Let us first consider detector losses, described by the admixture of vacuum modes \(V^\dagger_j\) into every detector mode \(\hat{A}^\dagger_j \rightarrow \tau_j^\dagger \hat{A}^\dagger_j + \rho_j V^\dagger_j\), where \(|\tau_j|^2 + |\rho_j|^2 = 1\). We find that the multi-channel coincidence signal operator for lossy detectors \(\hat{I}_{N,\text{lossy}}\) acquires the form

![FIG. 2. Logarithmic plot of minimum phase spread \(\Delta\phi\) (logarithm ‘ld’ to basis 10) for setups with \(N\) ports. The poor performance (12) of a Fock state with \(N\) photons fed into the \(\alpha^\dagger\)-channel (top sheet) is compared to shot noise \(\Delta\phi = 1/\sqrt{N + E}\) (bottom sheet). Shot noise can be reached employing coherent states \(|\sqrt{N + E}\rangle\). For a Fock-state \(\psi_{(N+E)}\) with sufficiently large photon excess \(E\) over the number of ports \(N\) (middle sheet) the noise is reduced, reaching (but not falling below) shot noise.](image-url)
\[ \hat{I}_{N,\text{lossy}} = \prod_{j=1}^{N} (\tau_j^* \hat{A}_j^\dagger + \rho_j^* \hat{V}_j^\dagger)(\tau_j \hat{A}_j + \rho_j \hat{V}_j) \]

\[ = \prod_{k=1}^{N} |\tau_k|^2 \prod_{j=1}^{N} \hat{A}_j^\dagger \hat{A}_j \doteq T \prod_{j=1}^{N} \hat{I}_j = T \hat{I}_N. \]  

Again, the fact that all modes \( V_j^\dagger \) are vacuum modes and can be traced out by setting \( \hat{V}_j = 0 \) has been used. The only net-effect of detector losses is the rescaling of the signal strength by the transmission loss factor \( T = \prod_{k=1}^{N} |\tau_k|^2 \). Since this factor cancels when forming the appropriate quotients of Eq. (7) it does not even affect the phase-resolution properties directly.

Most detectors are unable to distinguish one from two and more photons, see [26] though. This inability often poses problems [5,24,25] which our present scheme is also quite insensitive to. This can be seen from the following argument: if only one photon exits per channel \( A_j^\dagger \) there obviously is no difference in detection by number-sensitive detectors and those that only discriminate between presence and absence of photons. For the later case let us, in analogy to expression (3) define the N-channel photon-presence operator \( \hat{P}_N = \prod_{j=1}^{N} (\hat{1}_j - |0\rangle_j \langle 0|) \) where the projectors ‘unity minus vacuum’ \( (\hat{1}_j - |0\rangle_j \langle 0|) \) project out the vacuum modes in the exit channels \( A_j^\dagger \). This is equivalent to saying that we determine the signal of the N-channel coincidence count operator \( \hat{I}_N \) divided by the respective photon number in the output \( \hat{P}_N = \hat{P}_N \hat{I}_N (\prod_{j=1}^{N} A_j^\dagger A_j)^{-1} \hat{P}_N \).

Consequently \( \langle \hat{P}_N(\phi) \rangle \leq \langle \hat{I}_N(\phi) \rangle \) and \( \langle \hat{P}_N(\phi) \rangle > 0 \) iff \( \langle \hat{I}_N(\phi) \rangle > 0 \). In other words \( \langle P_N(\phi) \rangle \) and \( \langle I_N(\phi) \rangle \) have the same zeros and similar global behavior: the same period in interference patterns can be seen. It is not difficult to show that the diminished signal \( \langle P_N(\phi) \rangle \) encounters increased noise though.

Note the superficial similarity of our setup to imaging in the Fraunhofer-limit, see FIG. 1. The ‘object’ (beam splitter plus phase shifter) is being mapped by a ‘lens’ (N-port) into the ‘far-field’ (detector array \( A_j^\dagger \)). It remains to be seen whether the discussion presented here can be extended to quantum enhanced imaging [14].

It should also be interesting to investigate our setup for correlated input states feeding several of the input modes.

To conclude: we have studied an interferometric setup using an N-port readout containing only linear elements. The N output detectors have to fire in coincidence, this way a lot of signal and noise are filtered away. The selected subensemble can reach the Heisenberg limit which proves that the scheme is noise-optimized. The scheme is versatile with respect to the type of quantum states that can be employed, allows us to see an N-fold reduction of the measured de Broglie wavelength with perfect visibility, is surprisingly insensitive to common detector imperfections, and readily implementable using present technology.

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