Deformation Quantization
of the Isotropic Rotator

A. Stern and I. Yakushin

Department of Physics, University of Alabama, Tuscaloosa, Al 35487, USA.

ABSTRACT
We perform a deformation quantization of the classical isotropic rigid rotator. The resulting quantum system is not invariant under the usual $SU(2) \times SU(2)$ chiral symmetry, but instead $SU_{q-1}(2) \times SU_q(2)$. 
The classical isotropic rotator is known to be invariant under $SU(2) \times SU(2)$ chiral transformations. In ref. [1] a new Hamiltonian formulation of the isotropic rotator was found where the left and right $SU(2)$ transformations are not canonical symmetries but rather Poisson Lie group symmetries.[2-8] The treatment given in ref. [1] further differs from the standard one because the classical Hamiltonian can not be expressed as the square of the angular momentum $J_i$, nor does $J_i$ satisfy an $SU(2)$ algebra. On the other hand, from this formulation one obtains the usual equations of motion for the isotropic rotator. They state that an $SU(2)$ matrix-valued degree of freedom $g$ denoting the orientation of the rigid body undergoes a uniform precession. This can be expressed as follows:

$$
\dot{g} g^\dagger = \frac{i}{2} J_i \sigma_i, \quad \dot{J}_i = 0,
$$

where the dot denotes a time derivative, $\sigma_i$ are Pauli matrices and we have set the moment of inertia equal to one. In the usual formalism a general chiral transformation is given by

$$
g \rightarrow w^{-1} g v, \quad J_i \sigma_i \rightarrow w^{-1} J_i \sigma_i w, \quad w, v \in SU(2),
$$

which leaves (1) invariant.

In this article we quantize the system of ref. [1] using the method of deformation quantization [9]. We show that the resulting system is invariant under $SU_{q^{-1}}(2) \times SU_q(2)$, and this is the quantum analogue of the classical $SU(2) \times SU(2)$ Poisson Lie group symmetry. The quantum mechanical observables for the system are associated with a pair of Hopf algebras [10] or equivalently a quantum double. We obtain dynamics on the quantum double which reduces to (1) when $\hbar \rightarrow 0$. Furthermore in analogy to (1) the quantum dynamics is such that the quantum operator corresponding to $g$ (now taking values in $SU_q(2)$) undergoes a “uniform precession”.

We first review the classical Hamiltonian formalism of ref. [1]. There it was shown that the six dimensional phase space describing a rigid body can be taken to be the group
\[ D = SL(2, C). \] This phase space which is known to be a “classical double”\[2-5,8\] can be parametrized by elements of the group \( G = SU(2) \) and its dual \( G^* \). The latter is the group of \( 2 \times 2 \) lower triangular matrices \( \{ g^* \} \),

\[ g^* = \begin{pmatrix} m & m \vphantom{-}^{-1} \\ x_+ & m \vphantom{-}^{-1} \end{pmatrix}, \]

where \( m \) is real and \( x_+ \) is complex. An element \( \gamma \) of \( D \) can be labeled by \( (g^*, g) \), \( g \in G \) and \( g^* \in G^* \), using the Iwasawa decomposition \( \gamma = g^* g \). The coordinates \( (g^*, g) \) do not globally cover \( D \) as, for instance, \((1, 1)\) and \((-1, -1)\) are both mapped to the identity in \( D \). Nevertheless, they serve as a useful parametrization of a finite region of \( D \).

Let \( e_i, i = 1, 2, 3 \), denote a basis for the Lie algebra \( \mathcal{G} \) associated with \( G \), and \( e^i \) denote a basis for the Lie algebra \( \mathcal{G}^* \) associated with \( G^* \). \( e_i \) and \( e^i \) together span the Lie algebra \( \mathcal{D} \) associated with \( D \). The Poisson brackets for \( g \) and \( g^* \) were expressed in terms of classical \( r \)-matrices, taking values in \( \mathcal{D} \otimes \mathcal{D} \). These matrices denoted by \( r \) and \( r^* \), were defined according to

\[ r = e^i \otimes e_i \quad \text{and} \quad r^* = - e_i \otimes e^i. \quad (2) \]

\( r \) and \( r^* \) satisfy

\[ r^* - r = \text{adjoint invariant}, \quad (3) \]

and the classical Yang-Baxter equations.[10]

In this article we will make use defining representation for \( D \). In this representation the generators \( e_i \) and \( e^i \) can be expressed in terms of Pauli matrices \( \sigma_i \) as follows:

\[ e_i = \frac{1}{2} \sigma_i, \quad e^i = \frac{1}{2} (i \sigma_i + \varepsilon_{ij3} \sigma_j). \quad (4) \]

From them, we obtain the following \( 4 \times 4 \) matrix representation for \( r \):

\[ r = i \frac{4}{4} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 4 & -1 & 1 \\ -1 & 4 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad (5) \]
with \( r^* \) being its hermitean conjugate \( r^* = r^\dagger \). Using (4), the right hand side of (3) is simply \(-\frac{i}{2} \sigma_i \otimes \sigma_i \).

We now give the symplectic structure on \( D \). The Poisson brackets between two \( SU(2) \) group elements \( g \) can be expressed according to:

\[
\{ g_1, g_2 \} = \left[ r, g_1 g_2 \right],
\]

where we use the usual tensor product notation with \( g_1 = g \otimes 1 \) and \( g_2 = 1 \otimes g \). The Jacobi identity is satisfied due to the classical Yang-Baxter equations. (We set the deformation parameter \( \lambda \) of ref. \[1\] equal to one for simplicity.) Poisson brackets involving group elements \( \ell(\pm) = g^* \in G^* \) and their conjugate inverses \( \ell(\mp) = g^{*\dagger-1} \) are given by:

\[
\{ \ell_1^{(\pm)}, \ell_2^{(\mp)} \} = -\left[ r^*, \ell_1^{(\pm)} \ell_2^{(\mp)} \right] \quad (7)
\]

\[
\{ \ell_1^{(\mp)}, \ell_2^{(\pm)} \} = -\left[ r^*, \ell_1^{(\mp)} \ell_2^{(\pm)} \right] \quad (8)
\]

The remaining Poisson brackets are between elements of \( G \) and elements of \( G^* \):

\[
\{ \ell_1^{(-)}, g_2 \} = -\ell_1^{(-)} r g_2 , \quad (9)
\]

\[
\{ \ell_1^{(+)}, g_2 \} = -\ell_1^{(+)} r^* g_2 . \quad (10)
\]

Concerning dynamics, the classical Hamiltonian describing a free isotropic rigid rotator on \( D \) was found to be

\[
\mathcal{H} = \frac{1}{2} Tr \ell^{(-)} \ell^{(+)-1} - 1 . \quad (11)
\]

It leads the following Hamilton’s equations of motion

\[
\dot{\ell}^{(\pm)} = 0 \quad \text{and} \quad (12)
\]

\[
\dot{g} g^\dagger = i \left( \ell^{(+)-1} \ell^{(-)} - \frac{1}{2} Tr(\ell^{(+)-1} \ell^{(-)}) \mathbf{1}_{2 \times 2} \right) . \quad (13)
\]

The right hand side of (13) is a traceless hermitean matrix. We can thus expand it in terms of Pauli matrices, identifying the coefficients with the classical angular momenta \( J_i \).
of eq. (1). This gives:

\[ J_i = \frac{1}{2} Tr \ell^{(-)} \sigma_i \ell^{(+)}^{-1} . \]  

(14)

Since \( \ell^{(-)} \) and \( \ell^{(+)} \) are constants of the motion, then so are \( J_i \). Therefore the variable \( g \in SU(2) \) undergoes a uniform precession, and we recover the equations of motion for an isotropic rigid body.

As previously stated, this Hamiltonian description differs from the usual one because the Hamiltonian (11) is not proportional to \( J_i J_I \), and \( J_i \) does not satisfy an \( SU(2) \) Poisson bracket algebra. Also different is the nature of the \( SU(2) \times SU(2) \) chiral symmetry which we lastly review.

Unlike in the standard formulation of the rigid rotator, the chiral transformations do not correspond to canonical symmetries, but rather to two Poisson Lie group symmetries. One of the Poisson Lie group symmetries is associated with the right action of \( SU(2) \) on \( G \). Elements of \( G^* \) are unchanged under such transformations. Thus

\[ g \rightarrow gv , \quad \ell^{(\pm)} \rightarrow \ell^{(\pm)} , \quad v \in SU(2) . \]  

(15)

The Poisson brackets (6-10) for the classical observables \( g \) and \( \ell^{(\pm)} \) were shown to be invariant under (15) upon insisting that \( v \) has the following Poisson bracket with itself

\[ \{ v_1, v_2 \} = [ r^* , v_1 v_2 ] , \]  

(16)

and zero Poisson bracket with \( g \) and \( \ell^{(\pm)} \). Then \( SU(2) \) right multiplication is a Poisson map and (15) is a Poisson Lie group transformation. Further, since \( \ell^{(\pm)} \) are unchanged by this transformation, the Hamiltonian (11) and hence the equations of motion (12) and (13) are also invariant under (15).

The other Poisson Lie group symmetry is associated with rotations as it has a nontrivial action on \( \ell^{(\pm)} \) (as well as \( g \)) and therefore the angular momentum \( J_i \). It corresponds to the left action of \( SU(2) \). However unlike in the standard formalism, the left action is
not implemented directly on $G$, but rather the classical double $D$ (and its Hermitian conjugate). Under the left action of $SU(2)$, the variables $d^{(-)} = \gamma$ and $d^{(+)} = \gamma^{-1}$ transform according to

$$d^{(\pm)} \rightarrow w^{-1}d^{(\pm)}, \quad w \in SU(2). \quad (17)$$

The Poisson brackets for $d^{(\pm)}$ can be constructed from those of $g$ and $\ell^{(\pm)}$. One finds

$$\{d_1^{(\pm)}, d_2^{(\pm)}\} = -d_1^{(\pm)}d_2^{(\pm)} r - r^* d_1^{(\pm)}d_2^{(\pm)} . \quad (18)$$

$$\{d_1^{(-)}, d_2^{(+)}\} = -d_1^{(-)}d_2^{(+)} r - r d_1^{(-)}d_2^{(+)} . \quad (19)$$

These relations are invariant under (17) upon insisting that $w \in SU(2)$ has the following Poisson bracket with itself

$$\{w_1, w_2\} = [r^*, w_1w_2] , \quad (20)$$

and zero Poisson bracket with $d^{(\pm)}$. Then $SU(2)$ left multiplication is a Poisson map and (17) is a Poisson Lie group transformation. Further, since the classical Hamiltonian can be written

$$H = \frac{1}{2} Tr \ d^{(-)}d^{(+)^{-1}} - 1 , \quad (21)$$

we can use the cyclic property of trace to show that it is unchanged under (17).

We are now ready to consider the quantization of this system. In the spirit of deformation quantization [9], we do not identify the quantum mechanical commutation relations with $i\hbar$ times the corresponding classical Poisson brackets, but only demand that they agree in the $\hbar \rightarrow 0$ limit. Also, in the spirit of deformation quantization, we do not identify the quantum Hamiltonian $H$ with the classical Hamiltonian $H$. Rather we only require that $H$ reduces to $H$ (with classical variables replaced by quantum operators) in the limit $\hbar \rightarrow 0$. These requirements are of course not enough to completely determine the quantum mechanical system. For this purpose we shall in addition insist that the Heisenberg equations governing the quantum dynamics are the quantum analogs of the classical equations of motion (12) and (13), and that for each Poisson Lie group symmetry
present in the classical theory there is a corresponding quantum symmetry. Regarding the
former, we want that the quantum operators corresponding to \( \ell^{(\pm)} \) are constants of the
motion, and the quantum operator corresponding to \( g \) undergoes a “uniform precession”.
Concerning the latter, the resulting symmetry transformations are not associated with
groups, but Hopf algebras.\[10\]

We begin by writing down the quantum mechanical commutation relations. The Pois-
son bracket algebra (6) for \( g \) is known to be identical to the semiclassical limit of the
\( SU_q(2) \) Hopf algebra. \( SU_q(2) \) can be described in terms of \( 2 \times 2 \) matrices \( \{T\} \) whose
matrix elements are not c- numbers, but rather satisfy the commutation relations:

\[
RT_1T_2 = T_2T_1R
\]

with \( T_1 = T \otimes 1, T_2 = 1 \otimes T \) and \( R \) given by

\[
R = q^{-1/2} \begin{pmatrix}
q & 1 \\
q - q^{-1} & 1 \\
q & q
\end{pmatrix}.
\]

In addition, \( T^\dagger T = 1_{2\times2} \) and \( det_q T = T_{11}T_{22} - qT_{12}T_{21} = 1 \). \( R \) satisfies the quantum
Yang-Baxter equation, as well as

\[
[RR^T, T_2T_1] = 0
\]

which is the quantum analogue of the condition (3). The latter relation can be verified
using the \( 2 \times 2 \) matrix representation for \( T \). Here \( q = e^{\hbar/2} \). In the \( \hbar \to 0 \) limit \( R \) tends
to \( 1 - i\hbar r + O(\hbar^2) \), and consequently (22) reduces to \( [ T_1, T_2 ] = i\hbar [ r , T_1T_2 ] + O(\hbar^2) \).
We thereby recover the algebra given in (6).

The Poisson bracket algebra for \( \ell^{(\pm)} \) given in (7) and (8) is known to be identical to
the semiclassical limit of the \( U_q(sl(2)) \) Hopf algebra. \( U_q(sl(2)) \) can be described by the
set of \( 2 \times 2 \) lower triangular matrices \( \{L^{-}\} \) and upper triangular matrices \( \{L^{(+)}\} \) given
by

\[
L^{-} = \begin{pmatrix}
q^{-H/2} & q^{H/2} \\
-(q - q^{-1})X_+ & q^{-H/2}
\end{pmatrix} \quad \text{and} \quad L^{(+)} = \begin{pmatrix}
q^{H/2} & (q - q^{-1})X_- \\
q^{-H/2} & X_-
\end{pmatrix}.
\]
The commutation relations for $H$, $X_+$ and $X_- = X_+$ are
\[ [H, X_\pm] = \pm 2X_\pm, \quad \text{and} \quad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad (26) \]
or equivalently,
\[ R^{(+)} L_1^{(\pm)} L_2^{(\pm)} = L_2^{(\pm)} L_1^{(\pm)} R^{(+)} \quad (27) \]
\[ R^{(+)} L_1^{(+)} L_2^{(-)} = L_2^{(-)} L_1^{(+)} R^{(+)} \quad (28) \]
with $L_1^{(\pm)} = L^{(\pm)} \otimes 1$, $L_2^{(\pm)} = 1 \otimes L^{(\pm)}$ and $R^{(+)} = R^T$. In addition to the identity (24) we have $[RR^T, L_1^{(\pm)} L_2^{(\pm)}] = [RR^T, L_1^{(+)} L_2^{(-)}] = 0$. In the $\hbar \to 0$ limit $R^{(+)}$ tends to $1 + i\hbar r^* + O(\hbar^2)$ and consequently we recover the classical algebra (7) and (8) from (27) and (28).

Since the Poisson brackets between $g$ and $\ell^{(\pm)}$ do not vanish, it follows that the elements $T \in SU_q(2)$ do not commute with the elements $L^{(\pm)} \in U_q(sl(2))$ in the quantum theory. The quantum mechanical commutation rules for $T$ with $L^{(\pm)}$ must correspond to the Poisson brackets (9) and (10) in the limit $\hbar \to 0$. A suitable set of commutation relations consistent with this requirement is
\[ L_1^{(\pm)} R^{(\pm)} T_2 = T_2 L_1^{(\pm)}, \quad (29) \]
where $R^{(-)} = R^{-1}$. In the limit $\hbar \to 0$, $R^{(+)}$ and $R^{(-)}$ tend to $1 + i\hbar r^* + O(\hbar^2)$ and $1 + i\hbar r + O(\hbar^2)$, respectively, and the Poisson brackets (9) and (10) are recovered from (29).

The commutation relations for $L^{(\pm)}$ and $T$ are completely determined by eqs. (22,27-29). With the use of the quantum Yang-Baxter relations it can be checked that no further conditions on $L^{(\pm)}$ and $T$ result from commuting $L^{(\pm)}$ and $T$ through (22,27-29).

We next determine the quantum Hamiltonian $H$. Since we insist that the Heisenberg equations governing the quantum dynamics correspond with the classical equations of motion (12) and (13), $L^{(\pm)}$ should be constants of the motion, and $T$ should undergo a
“uniform precession”. For the former we have that \( L^{(\pm)} \) commutes with \( \mathbf{H} \), and thus \( \mathbf{H} \) must be a function of only the Casimir \( \mathbf{C} \) for \( U_q(sl(2)) \). The latter is known to be

\[
\mathbf{C} = \text{Tr}_q L^{-}S(L^{(+)} = q^{H+1} + q^{-H-1} + (q - q^{-1})^2X_+X_- , \tag{30}
\]

where \( S(A) \) denotes the antipode of \( A \). For both Hopf algebras \( SU_q(2) \) and \( U_q(sl(2)) \) the antipode is known to behave like a matrix inverse, ie. \( S(A)A = AS(A) = I_{2x2} \). \( S(L^{(\pm)}) \) are given in terms of \( 2 \times 2 \) matrices according to

\[
S(L^{-}) = \begin{pmatrix}
q^{H/2} & (1 - q^{-2})X_+ \\
1 - q^{-2} & q^{-H/2}
\end{pmatrix}
\quad \text{and} \quad
S(L^{(+)}) = \begin{pmatrix}
q^{-H/2} & -(q^2 - 1)X_- \\
q^H & q^{H/2}
\end{pmatrix} . \tag{31}
\]

\( \text{Tr}_q \) in eq. (30) denotes a “quantum” trace. The \( \text{Tr}_q \) of a \( 2 \times 2 \) matrix \( M = [M_{ij}] \) is defined according to:

\[
\text{Tr}_q M = qM_{11} + q^{-1}M_{22} . \tag{32}
\]

Unlike the usual trace, \( \text{Tr}_q \) does not have the general property of invariance under cyclic permutations. It does however serve as an “adjoint invariant” with respect to both Hopf algebras \( SU_q(2) \) and \( U_q(sl(2)) \). By this we mean the following:

\[
\text{Tr}_q L^{(\pm)}M S(L^{(\pm)}) = \text{Tr}_q M , \quad [L_1^{(\pm)} , M_2] = 0 , \tag{33}
\]

\[
\text{Tr}_q S(T)MT = \text{Tr}_q M , \quad [T_1 , M_2] = 0 . \tag{34}
\]

These relations can be explicitly verified using the \( 2 \times 2 \) representations for \( L^{(\pm)} \) and \( T \). From the requirement that the quantum Hamiltonian \( \mathbf{H} \) reduces to \( \mathbf{H} \) in the limit \( \hbar \to 0 \), we can choose

\[
\mathbf{H} = \frac{1}{2}\mathbf{C} - 1 . \tag{35}
\]

(More generally we can add terms to (35) that are of order \( \hbar \). We shall not consider that possibility here.)

To compute the equation of motion for \( T \in SU_q(2) \) we take its commutator with \( \mathbf{C} \). Using (29), we find

\[
[C,T_2] = \text{Tr}_q L^{-}_1 \left(1 - \left(R^T R\right)^{-1}\right) S(L^{(+)})_1T_2 , \tag{36}
\]
where the “1” index in $Tr^1_q$ indicates that the “trace” is performed only on the first space in the tensor product. So from the Heisenberg equation of motion $\dot{T}_2 = i\hbar[H, T_2]$, we get

$$
\dot{T}_2 S(T)_2 = \frac{i}{2\hbar} Tr^1_q L_1^{(-)} \left(1 - (R^T R)^{-1}\right) S(L^{(+)})_1 .
$$

(37)

Some work shows that this equation can be rewritten according to

$$
-2i\hbar \dot{T} S(T) = (1 - q^{-2})S(L^{(+)})L^{(-)} + (1 - q)C_{12} \times 2 .
$$

(38)

Since the right hand side of the Heisenberg equation of motion (38) is a function of $L^{(\pm)}$ only it is a constant of the motion, just as is the right hand side of the classical equation of motion (13). We therefore conclude that in analogy to $g$, $T$ undergoes a “uniform precession”. In the $\hbar \rightarrow 0$ limit, eq. (38) reduces to

$$
-2i \dot{T} S(T) \rightarrow S(L^{(+)})L^{(-)} - \frac{1}{2} Tr(L^{(-)} S(L^{(+)}) \times 2
$$

(39)

which agrees with the classical equation of motion (13).

We finally show that the above system is invariant under $SU_{q^{-1}}(2) \times SU_q(2)$. This is the quantum analog of the $SU(2) \times SU(2)$ Poisson Lie group symmetries of the classical theory. One of the Poisson Lie group symmetries is associated with the right action of $SU(2)$ on itself given in (15). The corresponding symmetry transformation in the quantum theory is the right action of $SU_q(2)$ on itself. Elements of $U_q(sl(2))$ are unchanged under these transformations. Thus

$$
T \rightarrow TV , \quad L^{(\pm)} \rightarrow L^{(\pm)} , \quad V \in SU_q(2) .
$$

(40)

The commutation relations for the quantum mechanical observables $T$ and $L^{(\pm)}$ are invariant under (40) if we insist that $V$ satisfies the $SU_q(2)$ commutativity relation: $RV_1 V_2 = V_2 V_1 R$. For this we also assume that $V$ commutes with $T$ and $L^{(\pm)}$. Since $L^{(\pm)}$ are unaffected by the action of $SU_q(2)$, the Hamiltonian and hence the Heisenberg equation of motion are unchanged under (40).
The other Poisson Lie group symmetry was associated with the left action of $SU(2)$ on $D$ given in (17). In order to find the corresponding symmetry transformation in the quantum theory we first need the analogue of the classical observables $d^{(\pm)}$. For this we define

$$D^{(\pm)} = L^{(\pm)} T . \quad (41)$$

From the commutation relations for $L^{(\pm)}$ and $T$ we obtain:

$$\begin{align*}
R^{(+)} D^{(\pm)}_1 & = D^{(\pm)}_2 R , \\
R^{(-)} D^{(-)}_1 & = D^{(+)}_2 R .\quad (42) \quad (43)
\end{align*}$$

The symmetry transformation in the quantum theory associated with (17) is

$$D^{(\pm)} \rightarrow S(W) D^{(\pm)} , \quad W \in SU_q(2) , \quad (44)$$

where $RW_1 W_2 = W_2 W_1 R$ , and we also assume that $W$ commutes with $D^{(\pm)}$. Since $S(W)$ satisfies the $SU_{q^{-1}}(2)$ commutation relations, we say that the transformation (44) is the left action of $SU_{q^{-1}}(2)$. To show that the commutation relations (42) and (43) for $D^{(\pm)}$ are unchanged under (44) we can use $[R^{-1} R^{-T} , S(W)_2 S(W)_1 ] = 0$ which follows from (24) (with $T$ replaced by $W$). The quantum Hamiltonian and the Heisenberg equation of motion can be written in terms of $D^{(\pm)}$ according to

$$\begin{align*}
\mathbf{H} & = \frac{1}{2} \text{Tr}_q \Gamma - 1 , \\
\Gamma & = D^{(-)} S(D^{(+)}) , \quad (45)
\end{align*}$$

and

$$\begin{align*}
-2i\hbar \dot{D}^{(-)} S(D^{(-)}) & = (1 - q^{-2}) \Gamma + (1 - q) \text{Tr}_q \Gamma \ 1_{2 \times 2} , \\
\dot{\Gamma} & = 0 ,\quad (46)
\end{align*}$$

respectively, where we use $S(D^{(\pm)}) = S(T) S(L^{(\pm)})$. They are unchanged under (44) due to the property (34) of the deformed trace.

**Acknowledgements**
We were supported in part by the Department of Energy, USA under contract number DE-FG-05-84ER-40141. A. S. wishes to thank the group in Naples for their hospitality where this work was initiated. We are grateful for discussions with S. Rajeev.

References

[1] G. Marmo, A. Simoni and A. Stern, Alabama preprint UAHEP 9312.

[2] V.G. Drinfel’d, Sov. Math. Doklady 27 (1983) 68; in Proc. Int. Congr. Math. (Berkeley), vol. 1 Academic Press, New York, 1986.

[3] M.A. Semenov-Tian-Shansky, Publ. RIMS, Kyoto University 21, no. 6 (1985) 1237; Theor. Math. Phys. 93 (1992) 302 (in Russian).

[4] S. Majid, Pacific J. Math 141 (1990) 311.

[5] A. Yu. Alekseev and A. Z. Malkin, Paris preprint PAR-LPTHE 93-08.

[6] A. Yu. Alekseev and I. T. Todorov, Vienna preprint. The undeformed version of the system discussed in this paper was examined in A. P. Balachandran, S. Borchardt and A. Stern, Phys. Rev. D17 (1978) 3247.

[7] J. Avan and M. Bellon, Phys. Lett. B213 (1988) 459.

[8] O. Babelon and D. Bernard, Phys. Lett. B260 (1991) 81; Commun. Math. Phys. 149 (1992) 279; Int. J. of Mod. Phys. A8 (1993) 507.

[9] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Ann. of Phys. 111 (1978) 61; 111.

[10] For reviews see, L. Takhtajan in Introduction to Quantum Group and Integrable Massive Models of Quantum Field Theory, M-L. Ge and B-H. Zhao (eds.) (World
[11] L. D. Faddeev, N. Reshetikhin and L. Takhtajan, Alg. Anal. 1 (1989) 178; N. Reshetikhin Alg. Anal. 1 No.2 (1989) 169; B. Zumino, in Proc. of X-th IAMP Conf., Springer-Verlag (1992); P. Schupp, P. Watts and B. Zumino, Lett. Math. Phys. 25 (1992) 139; A. P. Isaev and R. P. Malik, Phys. Lett. B280 (1992) 219; A. P. Isaev and Z. Popowicz, Phys. Lett. B281 (1992) 271.