Selecting Penalty Parameters of High-Dimensional M-Estimators using Bootstrapping after Cross-Validation

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Abstract

We develop a new method for selecting the penalty parameter for $\ell_1$-penalized M-estimators in high dimensions, which we refer to as bootstrapping after cross-validation. We derive rates of convergence for the corresponding $\ell_1$-penalized M-estimator and also for the post-$\ell_1$-penalized M-estimator, which refits the non-zero entries of the former estimator without penalty in the criterion function. We demonstrate via simulations that our methods are not dominated by cross-validation in terms of estimation errors and can outperform cross-validation in terms of inference. As an empirical illustration, we revisit Fryer Jr (2019), who investigated racial differences in police use of force, and confirm his findings.

Keywords: Penalty parameter selection, penalized M-estimation, high-dimensional models, sparsity, cross-validation, bootstrap, inference, one-step debiasing.

1 Introduction

High-dimensional models have attracted substantial attention both in the econometrics and in the statistics/machine learning literature, see e.g. Belloni et al. (2018a) and Hastie et al.

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(2015), and $\ell_1$-penalized estimators have emerged among the most useful methods for learning parameters of such models. However, implementing these estimators requires a choice of the penalty parameter and with few notable exceptions, e.g. $\ell_1$-penalized linear mean, quantile and logit regression estimators, the choice of this penalty parameter in practice often remains unclear. In this paper, we develop a new method to choose the penalty parameter in the context of $\ell_1$-penalized M-estimation and show that our method leads to precise estimation and inference in a large variety of models.

We consider a model where the true value $\theta_0$ of some parameter $\theta$ is given by the solution to an optimization problem

$$
\theta_0 = \arg\min_{\theta \in \Theta} \mathbb{E}[m(X^T \theta, Y)],
$$

where $m: \mathbb{R} \times \mathcal{Y} \to \mathbb{R}$ is a known (potentially non-smooth) loss function that is convex in its first argument, $X = (X_1, \ldots, X_p)^\top \in \mathcal{X} \subseteq \mathbb{R}^p$ a vector of candidate regressors, $Y \in \mathcal{Y}$ one or more outcome variables, and $\Theta \subseteq \mathbb{R}^p$ a convex parameter space. Prototypical loss functions are square loss and negative log-likelihood, but the framework (1.1) also covers many other cross-sectional models and associated modern as well as classical estimation approaches including logit and probit models, logistic calibration (Tan, 2020), covariate balancing (Imai and Ratkovic, 2014), and expectile regression (Newey and Powell, 1987). It also subsumes approaches to estimation of panel-data models such as the fixed-effects/conditional logit for binary outcomes (Rasch, 1960), trimmed least-absolute-deviations and trimmed least-squares for censored outcomes (Honoré, 1992), and partial likelihood approaches to heterogeneous panel models for duration (Chamberlain, 1985). We detail some of these examples in Section 2.

For the purpose of estimation, we assume access to a sample $\{(X_i, Y_i)\}_{i=1}^n$ of $n$ independent observations from the distribution $P$ of the pair $(X, Y)$, where the number $p$ of candidate regressors in each $X_i = (X_{i,1}, \ldots, X_{i,p})^\top$ may be (potentially much) larger than the sample size $n$, meaning that we cover high-dimensional models. Following the literature on high-dimensional models, we assume that the vector $\theta_0 = (\theta_{0,1}, \ldots, \theta_{0,p})^\top$ is at least approximately (also known as “weakly”) sparse. While we postpone a formal definition to Section 3, approximate sparsity captures the idea that, even though the number of candidate regressors $p$ can be very large, the number of relevant regressors may be substantially smaller. In the simplest case, known as exact (or “strong”) sparsity, this assumption amounts to the number of non-zeros in $\theta_0$ being much smaller than $n$. Approximate sparsity relaxes this idea to allow possibly many—but typically small—non-zeros. With sparsity in mind,
we study the sparsity encouraging $\ell_1$-penalized M-estimator ($\ell_1$-ME)
\[ \hat{\theta}(\lambda) \in \hat{\Theta}(\lambda) := \arg\min_{\theta \in \Theta} \left\{ \frac{1}{n} \sum_{i=1}^{n} m(X_i^\top \theta, Y_i) + \lambda \|\theta\|_1 \right\}, \] (1.2)
where $\|\theta\|_1 = \sum_{j=1}^{p} |\theta_j|$ denotes the $\ell_1$ norm of $\theta$, and $\lambda \in [0, \infty)$ is a penalty parameter.\(^1\)
We also study the post-$\ell_1$-penalized M-estimator (post-$\ell_1$-ME), which refits the coefficients of the variables selected by $\ell_1$-ME without the penalty in the criterion function in (1.2).

Implementing the estimator $\hat{\theta}(\lambda)$ requires choosing $\lambda$. To do so, we first extend a probabilistic bound from Belloni and Chernozhukov (2011a), obtained for $\ell_1$-penalized quantile regression, to our general $\ell_1$-penalized M-estimation setting (1.2). (See also Negahban et al. (2012) for independently developed and closely related results.) The bound, which we state in Section 3, yields a general principle to choose $\lambda$. In particular, it suggests that, for an arbitrary choice of $c_0 \in (1, \infty)$, one should choose $\lambda$ as small as possible subject to the constraint that the event
\[ \lambda \geq c_0 \max_{1 \leq j \leq p} \left| \frac{1}{n} \sum_{i=1}^{n} m'_i(X_i^\top \theta_0, Y_i) X_{i,j} \right| \] (1.3)
occurs with probability approaching one, where $m'_i$ denotes the partial derivative of the loss function with respect to its first argument. We therefore wish to set $\lambda = c_0 q_n(1 - \alpha)$, where
\[ q_n(1 - \alpha) := (1 - \alpha)\text{-quantile of } \max_{1 \leq j \leq p} \left| \frac{1}{n} \sum_{i=1}^{n} m'_i(X_i^\top \theta_0, Y_i) X_{i,j} \right|, \] (1.4)
for some small user-specified probability tolerance level $\alpha = \alpha_n \to 0$ as $n \to \infty$. This choice, however, is typically infeasible since the random variable in (1.4) depends on the unknown $\theta_0$. We thus have a vicious circle: to choose $\lambda$, we need an estimator of $\theta_0$, but to estimate $\theta_0$, we need to choose $\lambda$. In this paper, we offer a solution to this problem, which constitutes our key contribution.

To obtain our solution, we show that even though (as we discuss below) the estimator $\hat{\theta}(\lambda)$ based on $\lambda$ chosen by cross-validation or its variants is generally difficult to analyze, it can be used to construct provably good, in a certain sense, estimators of the random vectors $m'_i(X_i^\top \theta_0, Y_i) X_i$. We are then able to derive an estimator, say $\tilde{q}(1 - \alpha)$, of $q_n(1 - \alpha)$ via bootstrapping, as discussed in Belloni et al. (2018a), and to set $\lambda = c_0 \tilde{q}(1 - \alpha)$, which we refer to as the bootstrap-after-cross-validation (BCV) method to choose $\lambda$. This method

\(^1\)Throughout the main text, we implicitly assume that an estimator exists. Simple conditions under which $\hat{\Theta}(\lambda)$ is non-empty (and related properties) are given in Appendix E.
is computationally rather straightforward, applicable in a wide variety of models, and non-conservative in the sense that it gives \( \lambda \) such that \( \lambda \approx c_0 q_n (1-\alpha) \) rather than \( \lambda \gg c_0 q_n (1-\alpha) \). We derive convergence rates of \( \ell_1 \)-ME and post-\( \ell_1 \)-ME based on this choice of \( \lambda \) in Section 4. In addition, we show in Section 5 that, upon debiasing via the double machine learning approach, these estimators yield simple inference procedures.

The main alternatives to our method are cross-validation and related sample-splitting techniques. One of the main complications with these methods is that they are difficult to analyze, at least in some important dimensions. Sample-splitting techniques yield bounds on the \( \ell_2 \) estimation error \( \| \hat{\theta}(\lambda) - \theta_0 \|_2 \), see e.g. Lecue and Mitchell (2012), but not on the \( \ell_1 \) estimation error \( \| \hat{\theta}(\lambda) - \theta_0 \|_1 \). In contrast, our method gives bounds on both \( \ell_2 \) and \( \ell_1 \) estimation errors. An \( \ell_1 \) error bound is crucial when we are interested in estimating dense functionals \( a^\top \theta_0 \) of \( \theta_0 \) with \( a \in \mathbb{R}^p \) being a vector of loadings with many non-zero components; see Belloni et al. (2018a) for details. Moreover, \( \ell_1 \) estimation error bounds are needed to perform inference on components of \( \theta_0 \) as in Section 5. When \( \lambda \) is selected by cross-validation, \( \ell_1 \) and \( \ell_2 \) estimation error bounds are typically both unknown. The only exception we are aware of is the linear mean regression model estimated by the LASSO. For this special case, bounds have been derived in Chetverikov et al. (2021) and Miolane and Montanari (2018), but the bounds appearing in those references are less sharp than those provided here. Moreover, and crucially, cross-validation may lead to rather poor inference results, in the sense of bad size control, even in relatively large samples, and does not dominate our method even in terms of estimation errors; see our simulation results in Section 6 for details.

Another alternative to our method is to base the penalty parameter choice on self-normalized moderate deviation (SNMD) theory, as proposed in Belloni et al. (2012) for the linear mean regression model and extended in Belloni et al. (2016) to the logit model. This method is slightly conservative, in the sense that it gives \( \lambda \) somewhat larger than \( c_0 q_n (1-\alpha) \), but yields estimation and inference results that are comparable in quality with those produced by the BCV method. The SNMD method can be further extended to cover

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2 Any two norms on a fixed and finite-dimensional space are equivalent. However, the equivalence constants generally depend on the dimension (here \( p \)), which makes translation of error bounds for one norm into another a non-trivial manner when the dimension is growing.

3 Dense functionals \( a^\top \theta_0 \) may appear in the analysis, for example, when the vector \( X \) consists of many dummy variables and we are interested in making comparisons between two cells, \( (x_2 - x_1)^\top \theta_0 \), where \( x_1 \) and \( x_2 \) represent the first and the second cell, respectively. In such examples, we can guarantee that \( (x_2 - x_1)^\top \hat{\theta}(\lambda) \) is close to \( (x_2 - x_1)^\top \theta_0 \) only when \( \| \theta(\lambda) - \theta_0 \|_1 \) is small.

4 It is possible to replace the requirement on \( \ell_1 \) estimation error by the requirement on \( \ell_2 \) estimation error via cross-fitting, as in Chernozhukov et al. (2018). However, the combination of sample-splitting and cross-fitting would require splitting the original sample into at least three subsamples, which may not lead to accurate inference in moderate samples.
any Lipschitz-continuous loss function, but it is not clear how to extend it to a non-Lipschitz setting. For example, the SNMD method can be applied to the logit model but not to the probit model. In contrast, our BCV method is nearly universally applicable, and does not require Lipschitz continuity. We provide several other important examples where the loss function is not Lipschitz-continuous in Section 2.

To showcase our method using real data, in Section 7 we revisit the setting of Fryer Jr (2019), who investigated racial differences in police use of force. We extend Fryer’s regression analysis in two ways. First, we change the model from a binary logit to a binary probit, keeping the regressors as in Fryer’s analysis, a relatively small list. Second, we add a large number of additional (technical) regressors resulting from interactions between the original regressors. The first change leads to a non-Lipschitz loss (the negative probit likelihood). The second change brings us into high-dimensional territory, causing classical methods to break down. Unlike existing methods, the methods developed in this paper can accommodate both challenges. Our analysis supports the conclusions of Fryer Jr (2019) in showing that they are robust to model specification and a much larger set of candidate controls than originally considered.

The literature on learning parameters of high-dimensional models via $\ell_1$-penalized M-estimation is large. Instead of listing all existing papers, we therefore refer the interested reader to the excellent textbook treatment in Wainwright (2019) and focus here on only a few key references. van de Geer (2008, 2016) derives bounds on the estimation errors of general $\ell_1$-penalized M-estimators (1.2) and provides some choices of the penalty parameter $\lambda$. However, her penalty formulas give values of $\lambda$ that are so large that the resulting estimators are typically trivial in moderate samples, with all coefficients being exactly zero. Recognizing this issue, van de Geer (2008, p. 621) remarks that her results should only be seen as an indication that her theory has something to say about finite sample sizes, and that other methods to choose $\lambda$ should be used in practice. Negahban et al. (2012) develop error guarantees in a very general setting, and when specialized to our setting (1.2), their results become quite similar to those in our Theorem 3.1. The same authors also note that a challenge to using these results in practice is that the random variable in (1.3) is usually impossible to compute because it depends on the unknown vector $\theta_0$ (ibid., p. 547). It is exactly this challenge that we overcome in this paper. Belloni and Chernozhukov (2011a) study the high-dimensional quantile regression model and note that the distribution of the random variable in (1.3) is in this case pivotal, making the choice of the penalty parameter simple. Similarly, Wang et al. (2020) study the high-dimensional mean regression model and show that one can obtain pivotality by replacing the square-loss function by Jaeckel’s dispersion function, again making the choice of the penalty parameter simple. However, these
are the only two settings we are aware of in which the distribution of the random variable in (1.3) is pivotal.\footnote{With a known censoring propensity, the Buchinsky and Hahn (1998) linear programming estimator for censored quantile regression boils down to a variant of quantile regression, thus leading to pivotality.} Finally, Ninomiya and Kawano (2016) consider information criteria for the choice of the penalty parameter $\lambda$ but focus on fixed-$p$ asymptotics, thus precluding high-dimensional models.

The rest of the paper is organized as follows. In Section 2 we provide a portfolio of examples that constitute possible applications of our method. In Section 3 we develop bounds on the estimation error of the $\ell_1$-ME, which motivate our method for choosing the penalty parameter. In Section 4, we introduce the BCV penalty method and derive convergence rates for the resulting $\ell_1$-ME and post-$\ell_1$-ME. In Section 5, we show how to perform inference on individual components of $\theta_0$ via debiasing. In Section 6, we present a simulation study shedding light on the finite-sample properties of our method and contrast it with cross-validation. Finally, in Section 7, we apply our method to the empirical setting of Fryer Jr (2019). All proofs are relegated to the Online Appendices.

**Notation**

The distribution $P$ of the pair $(X,Y)$ and features thereof, including the dimension $p$ of the vector $X$, may change with the sample size $n$ (that is, we consider triangular array sampling and asymptotics), but we suppress this potential dependence whenever this does not cause confusion in order to simplify notation. We use $E[f(X,Y)]$ (or $E_{X,Y}[f(X,Y)]$) to denote the expectation of a function $f$ of the pair $(X,Y)$ computed with respect to $P$, and we use $E_n[f(X_i,Y_i)] := n^{-1} \sum_{i=1}^n f(X_i,Y_i)$ to abbreviate the sample average. We use $\mathbb{R}$ and $\mathbb{N}$ to denote all real numbers and all positive integers $\{1,2,\ldots\}$, respectively. For $k \in \mathbb{N}$, we write $[k] := \{1,\ldots,k\}$ for all positive integers up to and including $k$. When only a non-empty subset $I \subset [n]$ is in use, we write $E_I[f(X_i,Y_i)] := |I|^{-1} \sum_{i \in I} f(X_i,Y_i)$ for the subsample average. For a set of indices $I \subset [n]$, we use $I^c$ to denote the elements of $\{n\}$ not in $I$. For $k \in \mathbb{N}$, we use $0_k$ to denote the vector in $\mathbb{R}^k$ whose components are all zero. Given a vector $\delta \in \mathbb{R}^k$, we denote its $\ell_r$ norms, $r \in [1,\infty]$, by $||\delta||_r$. We write $\text{supp}(\delta) := \{j \in [k]; \delta_j \neq 0\}$ for the support of $\delta$, and use the $\ell_0$ “norm” $||\delta||_0 := |\text{supp}(\delta)|$ to denote the number of non-zero elements of $\delta$, where $|J|$ denotes the cardinality of the set $J$. For any function $f: \mathbb{R} \times \mathcal{Z} \to \mathbb{R}$, whose first argument is a scalar, we use $f'_1$, $f''_1$ and $f'''_1$ to denote its partial derivatives with respect to the first argument of the first, second and third order, respectively. We abbreviate $a \vee b := \max\{a,b\}$ and $a \wedge b := \min\{a,b\}$. Unless explicitly stated otherwise, limits are understood as $n \to \infty$. For numbers $a_n$ and positive numbers $b_n$, $n \in \mathbb{N}$, we write $a_n = o(1)$ if $a_n \to 0$, and $a_n \preceq b_n$, if the sequence $a_n/b_n$ is bounded. For random variables $V_n$ and
positive numbers $b_n$, we write $V_n \lesssim_P b_n$, if the sequence $V_n/b_n$ is bounded in probability. We denote $\eta_n := \sqrt{\ln(p)}/n$. We use the word “constant” to refer to non-random quantities that do not depend on $n$. Finally, we take $n \geq 3$ and $p \geq 2$ throughout and introduce more notation as needed in the appendices.

2 Examples

In this section, we discuss a variety of models that fit into the M-estimation framework (1.1) with the loss function $m(t, y)$ being convex in its first argument. The following examples cover both discrete and continuous outcomes in likelihood and non-likelihood settings with smooth as well as kinked loss functions. Additional examples can be found in Appendix G.

**Example 1 (Binary Response Model).** A relatively simple model fitting our framework is the *binary response model*, i.e. a model for an outcome $Y \in \{0, 1\}$ with

$$P(Y = 1 \mid X) = F(X^\top \theta_0),$$

for a known cumulative distribution function (CDF) $F : \mathbb{R} \to (0, 1)$. The log-likelihood of this model yields the following loss function:

$$m(t, y) = -y \ln F(t) - (1 - y) \ln (1 - F(t)). \quad (2.1)$$

The *logit* model arises from setting $F(t) = 1/(1 + e^{-t}) =: \Lambda(t)$, the standard logistic CDF, and the loss function reduces in this case to

$$m(t, y) = \ln \left(1 + e^t\right) - yt. \quad (2.2)$$

The *probit* model arises from setting $F(t) = \int_{-\infty}^{t}(2\pi)^{-1/2}e^{-u^2/2}du =: \Phi(t)$, the standard normal CDF, and the loss function in this case becomes

$$m(t, y) = -y \ln \Phi(t) - (1 - y) \ln (1 - \Phi(t)). \quad (2.3)$$

Both loss functions (2.2) and (2.3) are convex in $t$.

More generally, any binary response model with both $F$ and complementary CDF $1 - F$ being log-concave leads to a loss (2.1) that is convex in $t$. For these log-concavities it suffices that $F$ admits a probability density function (PDF) $f = F'$, which is itself positive and log-concave (Pratt, 1981, Section 5). Both the standard logistic and standard normal PDFs are log-concave. Also, $\ln f$ is concave whenever $f$ is of the (Subbotin) form $f(t) \propto e^{-|t|^\alpha/\alpha}$ for
some $a \in [1, \infty)$, the extreme case being the Laplace distribution. See Pratt (1981, Section 6) for additional examples. We focus on the logit and probit cases for concreteness.

**Example 2 (Ordered Response Model).** Consider the ordered response model, i.e. a model for an outcome $Y \in \{0, 1, \ldots, V\}$ with

$$ P(Y = v \mid X) = F(\alpha_{v+1} - X^\top \theta_0) - F(\alpha_v - X^\top \theta_0), \quad v \in [V], $$

for a known CDF $F : \mathbb{R} \to (0, 1)$ and known cut-off points $-\infty = \alpha_0 < \alpha_1 < \cdots < \alpha_V < \alpha_{V+1} = +\infty$. (We interpret $F(-\infty)$ as zero and $F(+\infty)$ as one to subsume the end cases.) The log-likelihood of this model yields the loss function

$$ m(t, y) = -\sum_{v=0}^V 1(y = v) \ln \left( F(\alpha_{v+1} - t) - F(\alpha_v - t) \right), \quad (2.4) $$

which is convex in $t$ for any distribution $F$ admitting a positive and log-concave PDF $f = F'$ (Pratt, 1981, Section 3). See Example 1 for specific distributions satisfying this criterion. As for binary response, we focus on the logit and probit cases.

**Example 3 (Expectile Model).** Newey and Powell (1987) study the conditional ($\tau$th) expectile model $\mu_\tau(Y \mid X) = X^\top \theta_0$, where $\tau \in (0, 1)$ is a known number, and propose the asymmetric least squares (ALS) estimator of $\theta_0$ in this model. This estimator can be understood as an M-estimator with loss of the form

$$ m(t, y) = \rho_\tau(y - t), \quad (2.5) $$

with $\rho_\tau : \mathbb{R} \to \mathbb{R}$ being the “swoosh” function given by

$$ \rho_\tau(u) = |\tau - 1(u < 0)| u^2 = \begin{cases} (1 - \tau) u^2, & \text{if } u < 0, \\ \tau u^2, & \text{if } u \geq 0, \end{cases} $$

a piecewise quadratic and continuously differentiable analogue of the “check” function known from the quantile regression literature. The ALS estimator can be interpreted as a maximum likelihood estimator when model disturbances arise from a normal distribution with unequal weights placed on positive and negative disturbances (Aigner et al., 1976; Philipps, 2022). Note that $m(\cdot, y)$ in (2.5) is convex but not twice differentiable (at $y$) unless $\tau = 1/2$.

**Example 4 (Panel Censored Model).** Consider the panel censored model

$$ Y_\tau = \max \left( 0, \gamma + X_\tau^\top \theta_0 + \varepsilon_\tau \right), \quad \tau \in \{1, 2\}, $$

8
where \( Y = (Y_1, Y_2)^\top \in [0, \infty)^2 \) is a pair of outcome variables, \((X_1^\top, X_2^\top)^\top\) is a vector of regressors, \( \gamma \) is a unit-specific (possibly random) unobserved fixed effect, and \( \varepsilon_1 \) and \( \varepsilon_2 \) are unobserved error terms, which may or may not be centered. Honoré (1992) shows that under certain conditions, including exchangeability of \( \varepsilon_1 \) and \( \varepsilon_2 \) conditional on \((X_1, X_2, \gamma)\), \( \theta_0 \) in this model can be identified by

\[
\theta_0 = \arg\min_{\theta \in \mathbb{R}^p} \mathbb{E}[m(X^\top \theta, Y)],
\]

with \( X := X_1 - X_2 \) and \( m \) being the \textit{trimmed loss function}

\[
m(t, y) = \begin{cases} 
\Xi(y_1) - (y_2 + t) \xi(y_1), & \text{if } t \in (-\infty, -y_2], \\
\Xi(y_1 - y_2 - t), & \text{if } t \in (-y_2, y_1], \\
\Xi(-y_2) - (t - y_1) \xi(-y_2), & \text{if } t \in [y_1, \infty), 
\end{cases}
\]

and either \( \Xi = |\cdot| \) or \( \Xi = (\cdot)^2 \) and \( \xi \) its derivative (when defined).\(^6\) These choices lead to \textit{trimmed least absolute deviations} (trimmed LAD) and \textit{trimmed least squares} (trimmed LS) estimators, respectively, both of which are based on loss functions convex in \( t \). Note that trimmed LAD is based on a non-differentiable loss \( m(\cdot, y) \), while trimmed LS is based on a continuously differentiable but not twice differentiable loss.

\[\square\]

\section{Non-Asymptotic Bounds on Estimation Error}

In this section, we derive probabilistic bounds on the error of the \( \ell_1 \)-ME (1.2) in the \( \ell_1 \) and \( \ell_2 \) norms. The bounds reveal which quantities one needs to control in order to ensure good behavior of the estimator, motivating the choice of the penalty parameter \( \lambda \) in the next section.

Our bounds will be based on the following assumptions. Since Assumptions 3.3, 3.4, and 3.5 stated below are high level, we verify these assumptions under more low-level conditions in the familiar case of the linear model with square loss in Appendix A.1 and for all examples in Section 2 in Appendix A.2.

\begin{assumption}[Parameter Space] \( \text{Parameter space } \Theta \text{ is a non-empty convex subset of } \mathbb{R}^p \text{ for which } \theta_0 \text{ is interior.} \)
\end{assumption}

\begin{assumption}[Convexity] \( \text{The function } m(\cdot, y) \text{ is convex for all } y \in \mathcal{Y}. \)
\end{assumption}

\begin{assumption}[Differentiability and Integrability] \( \text{The derivative } m'_1(X^\top \theta, Y) \text{ exists almost surely for all } \theta \in \Theta, \text{ and } \mathbb{E}[|m(X^\top \theta, Y)|] < \infty \text{ for all } \theta \in \Theta. \)
\end{assumption}

\(^6\)When \( \Xi = |\cdot| \), we set \( \xi(0) := 0 \) to make (2.6) consistent with formulas in Honoré (1992).
Assumption 3.1 is a minor regularity condition. Both convexity and interiority follow trivially in the case of a full parameter space $\Theta = \mathbb{R}^p$. Assumption 3.2 is satisfied in all examples from the previous section, as discussed there. In the same examples, Assumption 3.3 imposes minor integrability conditions on the random vectors $X$ and $Y$. In addition, in the case of Example 4 with trimmed LAD loss function, this assumption requires that the conditional distribution of $Y_1 - Y_2$ given $(X, Y_1 + Y_2 > 0)$ is continuous; see Appendix A.2 for details.

Further, define the \textit{excess risk function} $\mathcal{E} : \Theta \to [0, \infty)$ by

\[
\mathcal{E}(\theta) := \mathbb{E} \left[ m(X^\top \theta, Y) - m(X^\top \theta_0, Y) \right], \quad \theta \in \Theta.
\]

By definition of $\theta_0$ in (1.1), this function is non-negative and takes value zero at $\theta = \theta_0$. The next assumption requires that it grows sufficiently fast as $\theta$ moves away from $\theta_0$.

**Assumption 3.4 (Margin).** There are constants $c_M \in (0, 1]$ and $c'_M \in (0, \infty]$ such that for all $\theta \in \Theta$ satisfying $\|\theta - \theta_0\|_2 \leq c'_M$, we have $\mathcal{E}(\theta) \geq c_M \|\theta - \theta_0\|_2^2$.

In addition to some technical regularity conditions, this assumption requires the matrix $\mathbb{E}[XX^\top]$ to be non-singular, which means that there should be no perfect regressor multicollinearity in the population. In the context of Example 4, it also requires $Y_1$ and $Y_2$ to be different with positive probability. Also, our formal analysis reveals that Assumption 3.4 could be relaxed by requiring the bound $\mathcal{E}(\theta) \geq c_M \|\theta - \theta_0\|_2^2$ to hold only for certain \textit{sparse} vectors $\theta$. We have opted for a less general statement to avoid additional technicalities.

The following assumption requires additional technical regularity of the loss function.

**Assumption 3.5 (Local Loss).** There are constants $c_L \in (0, \infty]$, $C_L \in [1, \infty)$ and $r \in (4, \infty)$, a non-random sequence $B_n$ in $[1, \infty)$, and a function $L : \mathcal{X} \times \mathcal{Y} \to [1, \infty)$ such that

1. for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$ and all $(t_1, t_2) \in \mathbb{R}^2$ satisfying $|t_1| + |t_2| \leq c_L$, 
   \[
   \left| m(x^\top \theta_0 + t_1, y) - m(x^\top \theta_0 + t_2, y) \right| \leq L(x, y)|t_1 - t_2|, \quad (3.1)
   \]
   with $\max_{1 \leq j \leq p} \mathbb{E}[|L(X, Y)X_j|^2] \leq C_L^2$ and $\mathbb{E}[|L(X, Y)||X||_\infty^r] \leq B_n^r$;

2. for all $\theta \in \Theta$ satisfying $\|\theta - \theta_0\|_2 \leq c_L$, we have 
   \[
   \mathbb{E} \left[ |m(X^\top \theta, Y) - m(X^\top \theta_0, Y)|^2 \right] \leq C_L^2 \|\theta - \theta_0\|_2^2;
   \]

3. for all $\theta \in \Theta$ satisfying $\|\theta - \theta_0\|_2 \leq c_L$, we have 
   \[
   \mathbb{E} \left[ |m'(X^\top \theta, Y) - m'(X^\top \theta_0, Y)|^2 \right] \leq C_L^2 \|\theta - \theta_0\|_2^2. \quad (3.2)
   \]
Assumption 3.5.1 states that the loss function is locally Lipschitz in the first argument with the Lipschitz “constant” $L(x, y)$ being sufficiently well-behaved. The local Lipschitzness required in (3.1) actually follows from the loss convexity in Assumption 3.2 (Rockafellar, 1970, Theorem 10.4), so Assumption 3.5.1 should be regarded as a mild moment condition. Assumptions 3.5.2 and 3.5.3 essentially state that, viewed as functions of $\theta$, both the loss and its derivative are mean-square continuous at $\theta_0$. When the loss $m(\cdot, y)$ is globally Lipschitz uniformly in $y$ (thus allowing the choice $c_L = \infty$), Assumption 3.5.1 boils down to the regressors having sufficiently many absolute moments, and Assumption 3.5.2 reduces to the requirement that the largest eigenvalue of $E[XX^\top]$ is bounded from above.\footnote{Boundedness of eigenvalues is a standard assumption in the semi- and non-parametric estimation literature. See e.g. Belloni et al. (2015, Condition A.2) and Sørensen (2024, Assumption 5).} Examples of globally Lipschitz losses are the logit likelihood loss in Example 1 and the trimmed LAD loss in Example 4.

Assumption 3.6 (Approximate Sparsity). There is a constant $q \in [0, 1]$ and a non-random sequence $s_q := s_{q,n}$ in $[1, \infty)$ such that $\sum_{j=1}^p |\theta_{0,j}|^q \leq s_q$.

This assumption is a sparsity condition, stating that $\theta_0$ lies in an $\ell_q$-“ball” of “radius” $s_q^{1/q}$. We interpret the $q = 0$ case in the limiting sense $\lim_{q \to 0^+} \sum_{j=1}^p |\theta_{0,j}|^q = \sum_{j=1}^p 1(\theta_{0,j} \neq 0)$ so as to nest the case of exact sparsity with (at most) $s_0$ non-zero entries. When $q > 0$, we have only approximate sparsity, allowing possibly many—but typically small—non-zero entries. Related notions of sparsity appear in many papers on estimation of high-dimensional models. See Remark 3.3 for further discussion.

Under Assumption 3.3, we can (almost surely) define $S_n \in \mathbb{R}^p$ by

$$S_n := \mathbb{E}_n \left[ \frac{\partial}{\partial \theta} m(X_i^\top \theta, Y_i) \bigg| \theta = \theta_0 \right] = \mathbb{E}_n \left[ m'_i(X_i^\top \theta_0, Y_i)X_i \right]. \quad (3.3)$$

In this paper we refer to $S_n$ as the score.

We are now ready to present a theorem that provides probabilistic guarantees for $\ell_1$ and $\ell_2$ estimation errors of the $\ell_1$-ME. The proof, given in Appendix B.1, builds on arguments of Belloni and Chernozhukov (2011a). Related statements appear also in van de Geer (2008), Bickel et al. (2009), and Negahban et al. (2012), among others. Although we could not find the exact same version of the theorem in the literature, we make no claims of originality for these bounds and include the theorem for expository purposes and in order to motivate our method for choosing the penalty parameter $\lambda$.

To state the theorem, recall that we denote $\eta_n = \sqrt{\ln(pn)/n}$. 
Theorem 3.1 (Non-Asymptotic Error Bounds for $\ell_1$-ME). Let Assumptions 3.1–3.6 hold, let $\lambda_n$ be a non-random sequence in $(0, \infty)$, let $c_0 \in (1, \infty)$, and define

$$u_n := \frac{4c_0 \sqrt{s_q \eta_n^{-q}}}{(c_0 - 1)c_M} \left(C_L \eta_n + \lambda_n\right).$$

Then there is a universal constant $C \in [1, \infty)$ such that for all $n \in \mathbb{N}$ and $t \in [1, \infty)$ satisfying

$$\eta_n \leq 1, \; Cu_n \leq c_M, \; \frac{B_n^2 \ln(pn)}{\sqrt{n}} \leq C_L^2 \text{ and } tn^{1/r}B_n \left(Cu_n \sqrt{s_q \eta_n^{-q}} + s_q \eta_n^{1-q}\right) \leq \frac{(c_0 - 1)c_L}{2c_0},$$

we have

$$\sup_{\hat{\theta} \in \hat{\Theta}()} \|\hat{\theta} - \theta_0\|_2 \leq Cu_n \quad \text{and} \quad \sup_{\hat{\theta} \in \hat{\Theta}()} \|\hat{\theta} - \theta_0\|_1 \leq \frac{2c_0}{c_0 - 1} \left(Cu_n \sqrt{s_q \eta_n^{-q}} + s_q \eta_n^{1-q}\right)$$

with probability at least $1 - P(\lambda < c_0 \|S_n\|_{\infty}) - P(\lambda > \lambda_n) - 4t^{-r} - C/\ln^2(pn) - n^{-1}$.

This theorem motivates our choice of the penalty parameter $\lambda$. Specifically, it demonstrates that we want a level of regularization sufficient to overrule the score ($\lambda \geq c_0 \|S_n\|_{\infty}$) with high probability, without making the penalty “too large” ($\lambda > \lambda_n$). An interested reader can also find an analogue of Theorem 3.1 for the post-$\ell_1$-ME in Appendix C, but the general principle for choosing $\lambda$ remains the same.

Remark 3.1 (Non-Uniqueness). Like similar statements appearing in the literature, Theorem 3.1 concerns the entire set $\hat{\Theta}(\lambda)$ of optimizers for the convex minimization problem (1.2). While the objective function is presumed convex, it need not be strictly convex, and the global minimum may be attained at more than one point. For example, no matter the choice of $\Xi$, the trimmed loss function (2.6) in Example 4 will have linear pieces and need therefore not produce a strictly convex objective function. The bounds stated here (and in what follows) hold for any of these optimizers. See also Appendix E for sufficient conditions for solution existence and uniqueness as well as related (sparsity) properties. Despite the possible multiplicity, we sometimes refer to any element $\hat{\theta} \in \hat{\Theta}(\lambda)$ as the $\ell_1$-ME.

Remark 3.2 (Margin). Our convexity, interiority and differentiability assumptions suffice to show that the excess risk function $\mathcal{E}(\theta)$ is differentiable at $\theta_0$, and so the estimand $\theta_0$ must satisfy the population first-order condition $\nabla \mathcal{E}(\theta_0) = 0$. Assumption 3.4 therefore amounts to assuming that $\mathcal{E}(\theta)$ admits a quadratic margin near $\theta_0$. The name margin condition appears to originate from Tsybakov (2004, Assumption A1), who invokes a similar
assumption in a classification context. van de Geer (2008, Assumption B) contains a more general formulation of margin behavior for estimation purposes. We consider the (focal) quadratic case for the sake of simplicity.

**Remark 3.3 (Sparsity Notions).** In Negahban et al. (2012, Section 4.3) the sparsity in Assumption 3.6 is referred to as strong for \( q = 0 \) and weak for \( q > 0 \). Wainwright (2019, Chapter 7) distinguishes between strong \( \ell_q \)-balls (like \( \{ \theta \in \mathbb{R}^p; \sum_{j=1}^p |\theta_j|^q \leq s_q \} \) implicitly considered here) and weak \( \ell_q \)-balls, which impose a polynomial decay in the non-increasing rearrangement of the absolute values of the coefficients. In Belloni et al. (2018a), restricting \( \theta_0 \) to a weak \( \ell_q \)-ball is referred to as approximate sparsity, and a \( \theta_0 \) having bounded \( \ell_1 \) norm (i.e. belonging to a strong \( \ell_1 \)-ball) is called dense. Both strong \( (q > 0) \) and weak ball restrictions formalize the idea of “weak” or “approximate” sparsity.

**Remark 3.4 (Free Parameter).** The free parameter \( c_0 \in (1, \infty) \) in Theorem 3.1 serves as a trade-off between the likelihood of score domination on the one hand and the bound quality on the other. A smaller \( c_0 \in (1, \infty) \) makes the event \( \lambda \geq c_0 \| S_n \|_\infty \) more probable but also worsens the bounds. Note that the free parameter \( c_0 \) appears, either explicitly or implicitly, in existing bounds as well.\(^8\) While asymptotic theory provides no guidance on the choice of \( c_0 \), our finite-sample experiments in Section 6 indicate that increasing \( c_0 \) away from one worsens performance but setting \( c_0 \) to any value near one, including one itself, does not impact the results by much (cf. Figures 6.2 and 6.3). Similar observations were made by Belloni et al. (2012, Footnote 7) in the context of the LASSO.\(^9\)

## 4 Bootstrapping after Cross-Validation

We next provide a method for choosing the penalty parameter which is broadly available yet amenable to theoretical analysis. We split the section into two parts. In Section 4.1, we discuss a generic bootstrap method that allows for choosing the penalty parameter \( \lambda \) under availability of some generic estimators \( \hat{U}_i \) of \( U_i := m'_1(X_i^\top \theta_0, Y_i), i \in [n] \). In Section 4.2, we show how to obtain suitable estimators \( \hat{U}_i \) via cross-validation. By analogy with linear mean regression, we refer to \( U := m'_1(X^\top \theta_0, Y) \) as the residual.\(^9\)

---

\(^8\)A free parameter is explicit in both Belloni and Chernozhukov (2011a) and van de Geer (2008). In deriving their bounds both Bickel et al. (2009) (for the LASSO) and Negahban et al. (2012) set \( c_0 = 2 \).

\(^9\)The linear mean model \( \mathbb{E}[Y|X] = X^\top \theta_0 \) and (half) square loss imply \( U = X^\top \theta_0 - Y \). The name “residual” stems from \( U \) agreeing with the deviation \( Y - \mathbb{E}[Y|X] \) from the mean up to a sign.
### 4.1 Bootstrapping the Penalty Level

Suppose for the moment that residuals $U_i = m'_{i} (X_i^\top \theta_0, Y_i)$ are observable. In this case, we can estimate the $(1 - \alpha)$-quantile $q_n(1 - \alpha)$ of $\|S_n\|_{\infty} = \|E[U_i X_i]\|_{\infty}$ via the Gaussian multiplier bootstrap.\(^{10}\) To this end, let $\{e_i\}_{i=1}^n$ be independent standard normal random variables that are independent of the data $\{(X_i, Y_i)\}_{i=1}^n$. Given that $E[U X] = 0$, under mild regularity conditions, the Gaussian multiplier bootstrap estimates $q_n(1 - \alpha)$ by

$$
\tilde{q}_n(1 - \alpha) := (1 - \alpha)\text{-quantile of } \max_{1 \leq j \leq p} |E_n[e_i U_i X_i,j]| \text{ given } \{(X_i, Y_i)\}_{i=1}^n.
$$

Under certain regularity conditions, $\tilde{q}_n(1 - \alpha)$ delivers a good approximation to $q_n(1 - \alpha)$, even if the dimension $p$ of the vectors $X_i$ is much larger than the sample size $n$. To see why this is the case, let $Z := (Z_1, \ldots, Z_p)^\top$ be a centered random vector in $\mathbb{R}^p$ and let $\{Z_i\}_{i=1}^n$ be independent copies of $Z$. As established in Chernozhukov et al. (2013, 2017), the random vectors $\{Z_i\}_{i=1}^n$ satisfy the following high-dimensional versions of the central limit and Gaussian multiplier bootstrap theorems: If for some constant $b \in (0, \infty)$ and a non-random sequence $\tilde{B}_n$ in $[1, \infty)$, possibly growing to infinity, one has

$$
\min_{1 \leq j \leq p} E[Z^2_j] \geq b, \quad \max_{k \in \{1, 2\}} \max_{1 \leq j \leq p} E[|Z^2_j|^{2 + k}] / \tilde{B}_n^k \leq 1 \quad \text{and} \quad E\left[\max_{1 \leq j \leq p} Z^4_j\right] \leq \tilde{B}_n^4,
$$

then there is a constant $C_b \in (0, \infty)$, depending only on $b$, such that

$$
\sup_{A \in A_p} \left| P \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i \in A \right) - P(N_n \in A) \right| \leq C_b \left( \frac{\tilde{B}_n^4 \ln^7 (pn)}{n} \right)^{1/6}, \quad (4.1)
$$

and, with probability approaching one,

$$
\sup_{A \in A_p} \left| P \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i Z_i \in A \right) - P(N_n \in A) \right| \leq C_b \left( \frac{\tilde{B}_n^4 \ln^7 (pn)}{n} \right)^{1/6}, \quad (4.2)
$$

where $A_p$ denotes the collection of all (hyper)rectangles in $\mathbb{R}^p$, and $N_n$ is a centered Gaussian random vector in $\mathbb{R}^p$ with covariance matrix $E[Z Z^\top]$. Provided $\tilde{B}_n^4 \ln^7 (pn)/n \to 0$, applying these two results with $Z_i = U_i X_i$ for all $i \in [n]$ and noting that sets of the form $A_t = \{u \in \mathbb{R}^p; \max_{1 \leq j \leq p} |u_j| \leq t\}$, $t \in [0, \infty)$, are indeed rectangles, suggest that the Gaussian multiplier bootstrap estimator $\tilde{q}_n(1 - \alpha)$ provides a good approximation to $q_n(1 - \alpha)$.

As we typically do not observe the residuals $U_i = m'_{i} (X_i^\top \theta_0, Y_i)$, the method described above is infeasible. Fortunately, the result (4.2) continues to hold upon replacing $\{Z_i\}_{i=1}^n$

\(^{10}\)Recall that $S_n$ is well-defined a.s. under Assumption 3.3. We omit the qualifier throughout this section.
with estimators $\{\hat{Z}_i\}_{i=1}^n$, provided these estimators are “sufficiently good,” in the sense to be defined below; see (4.5). Suppose therefore that residual estimators $\{\hat{U}_i\}_{i=1}^n$ are available. We then compute

$$
\hat{q}^{\text{bm}}(1 - \alpha) := (1 - \alpha)\text{-quantile of } \max_{1 \leq j \leq p} |\mathbb{E}_n[c_i \hat{U}_i X_{i,j}]| \text{ given } \{(X_i, Y_i, \hat{U}_i)\}_{i=1}^n,
$$

(4.3)

and a penalty level follows as

$$
\hat{\lambda}^{\text{bm}}_\alpha := c_0 \hat{q}^{\text{bm}}(1 - \alpha).
$$

(4.4)

We refer to this method for obtaining a penalty level as the bootstrap method (BM) and to $\hat{\lambda}^{\text{bm}}_\alpha$ itself as the bootstrap penalty level.

To ensure that $\hat{q}^{\text{bm}}(1 - \alpha)$ indeed delivers a good approximation to $q_n(1 - \alpha)$, we invoke the following assumption.

**Assumption 4.1 (Residuals).** There are constants $c_U \in (0, \infty)$ and $C_U \in [1, \infty)$ and a non-random sequence $\hat{B}_n$ in $[1, \infty)$, such that (1) $c^2_U \leq \mathbb{E}[|UX_j|^2] \leq C^2_U$ for all $j \in [p]$, (2) $\mathbb{E}[|UX_j|^4] \leq \hat{B}^2_n$ for all $j \in [p]$, and (3) $\mathbb{E}[\|UX\|_\infty^4] \leq \hat{B}^4_n$.

This assumption imposes a few minor regularity conditions. It requires, in particular, that all components of the vector $X$ are normalized to be on the same scale. Since this assumption is high level, we verify it under low-level conditions in Appendix A.2 for the examples in Section 2.

Our next result provides convergence rates for the $\ell_1$-ME based on the bootstrap method.

**Lemma 4.1 (Convergence Rates: Generic Bootstrap Method).** Let Assumptions 3.1–3.6 and 4.1 hold, let $\delta_n$ be a non-random sequence in $[0, \infty)$ such that

$$
P\left(\mathbb{E}_n[(\hat{U}_i - U_i)^2] > \delta_n^2 / \ln^2 (pn)\right) \to 0,
$$

(4.5)

let $\hat{\Theta}(\hat{\lambda}^{\text{bm}}_\alpha)$ be the solutions to the $\ell_1$-penalized M-estimation problem (1.2) with penalty level $\lambda = \hat{\lambda}^{\text{bm}}_\alpha$ given in (4.4) and $\alpha = \alpha_n \in (0, 1)$ satisfying $\alpha_n \to 0$ and $\ln(1/\alpha_n) \lesssim \ln(pn)$, and suppose that

$$
n^{1/r} B_n \left(\delta_n + s_q \eta_n^{1-q}\right) \to 0, \quad B_n^{4 \ln^2 (pn)} n \to 0 \quad \text{and} \quad \hat{B}_n^4 \ln^7 (pn) n \to 0.
$$

(4.6)

Then

$$
\sup_{\theta \in \hat{\Theta}(\hat{\lambda}^{\text{bm}}_\alpha)} \|\hat{\theta} - \theta_0\|_2 \lesssim_P \sqrt{s_q \eta_n^{2-q}} \quad \text{and} \quad \sup_{\theta \in \hat{\Theta}(\hat{\lambda}^{\text{bm}}_\alpha)} \|\hat{\theta} - \theta_0\|_1 \lesssim_P s_q \eta_n^{1-q}.
$$

The idea of using a bootstrap procedure to select the penalty level in high-dimensional estimation is in itself not new. Chernozhukov et al. (2013) use a Gaussian multiplier bootstrap
to tune the Dantzig selector (Candès and Tao, 2007) for the high-dimensional linear model allowing both non-Gaussian and heteroskedastic errors. Note, however, that Chernozhukov et al. (2013, Theorem 4.2) presumes access to a preliminary Dantzig selector, which is used to estimate residuals. The condition (4.5) is similarly high-level in the sense that it does not specify how one performs residual estimation in practice. Our primary contribution lies in providing a method for coming up with good residual estimators. We turn to this task in the next subsection, where we also compare the rates with those appearing in the literature and discuss the side conditions under which they are derived; see Remarks 4.1 and 4.2.

4.2 Cross-Validating Residuals

In this subsection, we explain how residual estimation can be performed via cross-validation (CV). To describe our CV residual estimator, fix any integer $K \geq 2$, and let $\{I_k\}_{k=1}^K$ partition the observation indices $[n]$. Provided $n$ is divisible by $K$, the even partition

$$I_k = \{(k-1)n/K + 1, \ldots, kn/K\}, \quad k \in [K],$$

(4.7)
is natural, but not necessary. For the formal results below, we only require that each $I_k$ specifies a “substantial” subsample; see Assumption 4.2 below.

Let $\Lambda_n$ denote a finite subset of $(0, \infty)$ composed by candidate penalty levels. In Assumption 4.3 below, we require $\Lambda_n$ to be “sufficiently rich.” Our CV procedure then goes as follows. First, estimate the vector of parameters $\theta_0$ by

$$\hat{\theta}_{I^c_k}(\lambda) \in \hat{\Theta}_{I^c_k}(\lambda) := \arg\min_{\theta \in \Theta} \left\{ \mathbb{E}_{I^c_k}[m(X_i^\top \theta, Y_i)] + \lambda \|\theta\|_1 \right\},$$

(4.8)

for each candidate penalty level $\lambda \in \Lambda_n$ and holding out each subsample $k \in [K]$ in turn. Second, determine the penalty level

$$\hat{\lambda}^{cv} \in \arg\min_{\lambda \in \Lambda_n} \sum_{k=1}^K \sum_{i \in I_k} m(X_i^\top \hat{\theta}_{I^c_k}(\lambda), Y_i)$$

(4.9)

by minimizing the out-of-sample loss over the set of candidate penalties. Third, estimate residuals $U_i = m'_1(X_i^\top \hat{\theta}_{I^c_k}(\hat{\lambda}^{cv}), Y_i), i \in [n]$, by predicting out of each estimation subsample, i.e.,

$$\hat{U}_i^{cv} := m'_1(X_i^\top \hat{\theta}_{I^c_k}(\hat{\lambda}^{cv}), Y_i), \quad i \in I_k, \quad k \in [K].$$

(4.10)

Note here that since $I_k$ and $I^c_k$ have no elements in common, the derivative $m'_1(X_i^\top \hat{\theta}_{I^c_k}(\lambda), Y_i)$ exists for all $i \in I_k$, $k \in [K]$, and $\lambda \in \Lambda_n$ almost surely by Assumption 3.3. The residual
estimates \( \{ \widehat{U}_i^{cv} \}_{i=1}^n \) are therefore almost-surely well-defined even though the function \( m(\cdot, y) \) is not necessarily differentiable.

Combining the bootstrap penalty level \( \widehat{\lambda}_n = c_0 q_b^{\text{bcv}} (1 - \alpha) \) from the previous subsection with the CV residual estimates \( \widehat{U}_i = \widehat{U}_i^{cv} \) from this subsection, we obtain the bootstrap-after-cross-validation (BCV) method for estimating the quantile \( q_n(1 - \alpha) \),

\[
q^{\text{bcv}} (1 - \alpha) := (1 - \alpha)\text{-quantile of } \max_{1 \leq j \leq p} \E_n [c_i \widehat{U}_i^{cv} X_{i,j}] \text{ given } \{(X_i, Y_i)\}_{i=1}^n,
\]

and the BCV penalty level follows as

\[
\widehat{\lambda}_n^{\text{bcv}} := c_0 q^{\text{bcv}} (1 - \alpha). \tag{4.12}
\]

To analyze the \( \ell_1 \)-ME implied by BCV, we invoke the following two assumptions.

**Assumption 4.2 (Data Partition).** The number of folds \( K \in \{2, 3, \ldots \} \) is constant. There is a constant \( c_D \in (0, 1) \) such that \( \min_{1 \leq k \leq K} |I_k| \geq c_D n \).

**Assumption 4.3 (Candidate Penalties).** There are constants \( c_\Lambda \) and \( C_\Lambda \) in \((0, \infty)\) and \( a \in (0, 1) \) such that

\[
\Lambda_n = \{ C_\Lambda a^\ell; a^\ell \geq c_\Lambda / n, \ell \in \{0, 1, 2, \ldots \} \}.
\]

Assumption 4.2 means that we rely upon classical \( K \)-fold cross-validation with fixed \( K \). This assumption does rule out leave-one-out cross-validation, since \( K = n \) and \( I_k = \{k\} \) imply \( |I_k|/n \to 0 \). Assumption 4.3 allows for a rather large candidate set \( \Lambda_n \) of penalty values. Note that the largest penalty value, \( C_\Lambda \), can be set arbitrarily large and the smallest value, \( c_\Lambda / n \), converges rapidly to zero. As a part of the proof of Theorem 4.1 below, we show that these properties ensure that the set \( \Lambda_n \) eventually contains a “good” penalty candidate, say \( \lambda_* \), in the sense of leading to a uniform bound on the excess risk of subsample estimators \( \widehat{\theta}_{I_k}^{\Lambda_\gamma}(\lambda_*), k \in [K] \) and, because of that, the CV residual estimators are reasonable inputs for the bootstrap method, in the sense of satisfying (4.5). Combining this finding with Lemma 4.1, we obtain convergence rates for the \( \ell_1 \)-ME implied by BCV.

**Theorem 4.1 (Convergence Rates: BCV Method, Penalized Estimator).** Let Assumptions 3.1–3.6 and 4.1–4.3 hold, let \( \widehat{\Theta}(\widehat{\lambda}_n^{\text{bcv}}) \) be the solutions to the \( \ell_1 \)-penalized M-estimation problem (1.2) with penalty level \( \lambda = \widehat{\lambda}_n^{\text{bcv}} \) given in (4.12) and \( \alpha = \alpha_n \in (0, 1) \) satisfying \( \alpha_n \to 0 \) and \( \ln(1/\alpha_n) \lesssim \ln(pn) \), and suppose that

\[
n^{1/r} B_n s_q n^{1 - q} \to 0, \quad \frac{B_n^4 s_q (\ln(pn))^{5 - q/2}(\ln n)^2}{n^{1 - q/2 - 4/r}} \to 0 \quad \text{and} \quad \frac{\overline{B}_n^4 \ln^7(pn)}{n} \to 0. \tag{4.13}
\]
Then
\[ \sup_{\hat{\theta} \in \hat{\Theta}(\lambda_{cv})} \|\hat{\theta} - \theta_0\|_2 \lesssim_P \sqrt{s_q \eta_n^{2-q}} \quad \text{and} \quad \sup_{\hat{\theta} \in \hat{\Theta}(\lambda_{cv})} \|\hat{\theta} - \theta_0\|_1 \lesssim_P s_q \eta_n^{1-q}. \] 

(4.14)

**Remark 4.1 (Convergence Rates).** The Theorem 4.1 (and Lemma 4.1) convergence rates are as one would expect in high-dimensional settings. For example, the \( \ell_2 \) rate in (4.14) coincides with that obtained for the LASSO in Negahban et al. (2012, Corollary 3) in the context of linear mean regression with \( \theta_0 \) belonging to an \( \ell_q \)-ball (Assumption 3.6). The rate is known to be minimax optimal in the context of sparse linear mean regression (Ye and Zhang, 2010; Raskutti et al., 2011). We expect it to remain optimal in the general high-dimensional M-estimation framework as well. In the special case of exact sparsity \( (q = 0) \), the \( \ell_2 \) and \( \ell_1 \) rates in (4.14) become \( \sqrt{s_0 \ln(pn)/n} \) and \( \sqrt{s_0^2 \ln(pn)/n} \), respectively.

**Remark 4.2 (Dense Case).** In the dense case \( (q = 1) \) \( \eta_n^{1-q} \) does not vanish, and so the side condition \( n^{1/r} B_n s_q \eta_n^{1-q} \to 0 \) in (4.13) fails. However, inspection of the proof reveals that we actually require \( n^{1/r} B_n s_q \eta_n^{1-q}/c_L \to 0 \) for \( c_L \in (0, \infty) \) in Assumption 3.5. The latter condition is trivially satisfied when \( c_L = \infty \), which is allowed when the loss \( m(t, y) \) is globally Lipschitz in its first argument uniformly in \( y \in \mathcal{Y} \). Hence, provided the loss is globally Lipschitz, even in the dense case Theorem 4.1 can produce the \( \ell_2 \) rate of convergence \( s_1^{1/2} (\ln(pn)/n)^{1/4} \). Examples of globally Lipschitz losses are the logit likelihood loss in Example 1 and the trimmed LAD loss in Example 4. The side condition \( n^{1/r} B_n s_q \eta_n^{1-q} \to 0 \) may also be relaxed in the special case of generalized linear models; see Negahban et al. (2012, Section 4.4) and Wainwright (2019, Section 9.5) for details.

**Remark 4.3 (Model Sparsity and Regressor Regularity).** The side condition
\[ \frac{B_n^4 s_q (\ln(pn))^{5-q/2} (\ln n)^2}{n^{1-q/2-4/\gamma}} \to 0 \]
in (4.13) necessitates \( r > 8/(2 - q) \), which reveals an interplay between the model sparsity as captured by the constant \( q \) in Assumption 3.6 and the regressor integrability as captured by the constant \( r \) in Assumption 3.5.1. In the special case of exact sparsity \( (q = 0) \) the regressors are required to have more than four finite moments.

We next consider the post-\( \ell_1 \)-penalized M-estimator (post-\( \ell_1 \)-ME). The main motivation for the post-\( \ell_1 \)-ME is that the \( \ell_1 \)-ME may be severely biased because it shrinks coefficients toward zero. By refitting the non-zero coefficients of the \( \ell_1 \)-ME without the penalty in the criterion function in (1.2), the post-\( \ell_1 \)-ME attempts to reduce this bias.
To define the post-$\ell_1$-ME, for any $\bar{\theta} \in \Theta$, we define the set $\tilde{\Theta}(\text{supp}(\bar{\theta})) \subseteq \Theta$ by

$$
\tilde{\Theta}(\text{supp}(\bar{\theta})) := \arg\min_{\theta \in \Theta, \text{supp}(\theta) \subseteq \text{supp}(\bar{\theta})} \mathbb{E}_n[m(X_i^T \theta, Y_i)].
$$

(4.15)

Then for any $\ell_1$-ME, i.e. a solution $\hat{\theta} \in \tilde{\Theta}(\lambda)$ to the optimization problem in (1.2), the corresponding post-$\ell_1$-ME is defined as any element $\tilde{\theta}$ of the set $\tilde{\Theta}(\text{supp}(\hat{\theta}))$. Note that there could be multiple post-$\ell_1$-MEs.\textsuperscript{11} Our treatment below covers the set $\tilde{\Theta}(\lambda)$ of all post-$\ell_1$-MEs, which we denote

$$
\tilde{\Theta}(\lambda) := \bigcup_{\tilde{\theta} \in \tilde{\Theta}(\lambda)} \tilde{\Theta}(\text{supp}(\tilde{\theta})).
$$

(4.16)

To analyze the post-$\ell_1$-ME, we will use the following two additional assumptions.

\textbf{Assumption 4.4 (Smoothness).} The function $m(\cdot, y)$ is differentiable for all $y \in \mathcal{Y}$ with its derivative being Lipschitz-continuous, i.e. $|m'_1(t_2, y) - m'_1(t_1, y)| \leq C_m|t_2 - t_1|$ for all $(t_1, t_2, y) \in \mathbb{R} \times \mathbb{R} \times \mathcal{Y}$ and some constant $C_m \in (0, \infty)$.

\textbf{Assumption 4.5 (Moments).} There is a constant $C_{\text{ev}} \in [1, \infty)$ such that $\mathbb{E}[(X^T \delta)^4] \leq C_{\text{ev}}\|\delta\|_2^4$ for all $\delta \in \mathbb{R}^p$.

Assumption 4.4 strengthens the almost-sure differentiability in Assumption 3.3. The stronger smoothness requirement precludes the trimmed LAD loss function in Example 4, but it can be easily verified under more low-level conditions for the trimmed LS loss function in the same example as well as for all other examples from Section 2; see Appendix A.2. Assumption 4.5 is satisfied if the entries of $X$ are independent standard Gaussian, for example. Related assumptions appear in the existing literature on high-dimensional estimation.

With these added assumptions, we can derive the convergence rates for the post-$\ell_1$-ME.

\textbf{Theorem 4.2 (Convergence Rates: BCV Method, Post-Penalized Estimator).} Let Assumptions 3.1–3.6 and 4.1–4.5 hold, let $\tilde{\Theta}(\lambda_{\text{bcv}})$ be the set of post-$\ell_1$-penalized M-estimators (4.16) with penalty level $\lambda = \lambda_{\text{bcv}}$ given in (4.12) and $\alpha = \alpha_n \in (0, 1)$ satisfying $\alpha_n \to 0$ and $\ln(1/\alpha_n) \lesssim \ln(n)$, and suppose that

$$
n^{1/r}B_n s_q n^{1-q} \ln(pn) \to 0, \quad \frac{B_n^4 s_q (\ln(pn))^{5-q/2} (\ln n)^2}{n^{1-q/2-4/r}} \to 0 \quad \text{and} \quad \frac{\tilde{B}_n^4 \ln^7 (pn)}{n} \to 0.
$$

(4.17)

\textsuperscript{11}As in our treatment of $\ell_1$-ME, we implicitly assume that a post-$\ell_1$-ME exists.
Then
\[
\sup_{\hat{\theta} \in \hat{\Theta}(\hat{\lambda}_{\text{cv}})} \|\hat{\theta} - \theta_0\|_2 \lesssim_P \sqrt{s_q n^{2-q} \ln(p n)} \quad \text{and} \quad \sup_{\hat{\theta} \in \hat{\Theta}(\hat{\lambda}_{\text{cv}})} \|\hat{\theta} - \theta_0\|_1 \lesssim_P s_q n^{1-q} \ln(p n). \tag{4.18}
\]

**Remark 4.4 (Comparison of Rates for \(\ell_1\)-ME and Post-\(\ell_1\)-ME).** The convergence rates for the post-\(\ell_1\)-ME we derive here are slightly slower than those we derived for the \(\ell_1\)-ME itself in Theorem 4.1. The technical reason for this difference is that the analysis of the post-\(\ell_1\)-ME requires not only that the penalty parameter \(\lambda\) is not too large but also that it is not too small, as we may end up with “too many” selected variables; see Appendix C for details.

In turn, our BCV method may yield low values of the penalty level if there is substantial correlation between regressors in the vector \(X\). A simple solution to this issue would be to censor the BCV penalty parameter \(\hat{\lambda}_{\text{cv}}\) from below so that it shrinks to zero no faster than \(\eta_n\), i.e. to replace \(\hat{\lambda}_{\text{cv}}\) by \(\max\{\hat{\lambda}_{\text{cv}}\, c \eta_n\}\) for some user-chosen constant \(c \in (0, \infty)\). In this case, the rates in (4.18) would coincide with the rates in (4.14). However, we prefer to state a slightly slower rate, as in (4.18), over the necessity to introduce another tuning parameter \(c\) (for which there is no obvious guiding principle). Moreover, under the additional assumption that the elements of the regressor vector \(X\) are not too correlated, it is possible to derive the same rates as in (4.14) for the post-\(\ell_1\)-ME even without censoring, in which case the rates for post-\(\ell_1\)-ME are as good as those for \(\ell_1\)-ME.

\[\square\]

## 5 Debiased Estimation and Inference

In this section, we describe how to construct \(\sqrt{n}\)-consistent and asymptotically normal estimators of individual components of the vector \(\theta_0\) defined in (1.1). Since these estimators are asymptotically unbiased and have easily estimable asymptotic variance, they lead to standard inference procedures for testing hypotheses about and building confidence intervals for individual components of \(\theta_0\). Our approach here is based on the concept of Neyman orthogonal equations and closely follows the literature on double/debiased machine learning (Chernozhukov et al., 2018). We note that the tools developed in this section rule out the trimmed LAD loss in Example 4, as this function is not sufficiently smooth to satisfy our Assumption 5.3.

Without loss of generality, suppose that we are interested in the first component of the vector \(\theta_0\), so that \(\theta_0 = (\beta_0, \gamma_0^\top)^\top\), where \(\beta_0 \in \mathbb{R}\) is the scalar parameter of interest and \(\gamma_0 \in \mathbb{R}^{p-1}\) is a vector of nuisance parameters. To derive a \(\sqrt{n}\)-consistent and asymptotically normal estimator of \(\beta_0\), write \(X = (D, W^\top)^\top\), so that \(X^\top \theta_0 = D\beta_0 + W^\top \gamma_0\), and let
\( \mu_0 \in \mathbb{R}^{p-1} \) be a vector that is defined as a solution to the following system of equations:

\[
E[m''_{11}(X^\top \theta_0, Y)(D - W^\top \mu_0)W] = 0_{p-1}.
\] (5.1)

Note that this system has a solution \( \mu_0 \) and this solution is unique as long as the matrix \( E[m''_{11}(X^\top \theta_0, Y)WW^\top] \) is non-singular, which is the case under our assumptions.\(^{12}\) With this definition in mind, by inspecting the first-order conditions associated with (1.1), we have

\[
E[m'_1(D\beta_0 + W^\top \gamma_0, Y)(D - W^\top \mu_0)] = 0.
\] (5.2)

We obtain an estimator of \( \beta_0 \) by solving an empirical version of this equation, where we replace the (high-dimensional) vectors \( \gamma_0 \) and \( \mu_0 \) by suitable estimators. Here, \( \sqrt{n} \)-consistent and asymptotically normal estimation of \( \beta_0 \) is possible due to (5.2) being Neyman orthogonal with respect to \( \gamma_0 \) and \( \mu_0 \), which means that this equation is first-order insensitive with respect to perturbations in \( \gamma_0 \) and \( \mu_0 \). Specifically, we have

\[
\frac{\partial}{\partial \gamma} E[m'_1(D\beta_0 + W^\top \gamma_0, Y)(D - W^\top \mu_0)] \bigg|_{\gamma=\gamma_0} = 0_{p-1} \quad \text{and}
\]

\[
\frac{\partial}{\partial \mu} E[m'_1(D\beta_0 + W^\top \gamma_0, Y)(D - W^\top \mu)] \bigg|_{\mu=\mu_0} = 0_{p-1},
\]

which follows from (5.1) and (1.1), respectively. Neyman orthogonality thus facilitates simple inference for the low-dimensional \( \beta_0 \) despite possibly complicated estimation of the high-dimensional \( \gamma_0 \) and \( \mu_0 \). Formally, we consider the following procedure:

**Algorithm 5.1 (Three-Step Debiasing).** Given rules for choosing penalty levels \( \lambda_1, \lambda_2 \in (0, \infty) \),\(^{13}\) follow the steps below to obtain a debiased estimator \( \hat{\beta} \) of \( \beta_0 \):

**Step 1 (Initiate):** a. Define the (preliminary) estimator \( \hat{\theta} = (\hat{\beta}, \hat{\gamma}^\top) \) of \( \theta_0 = (\beta_0, \gamma_0^\top) \) by

\[
\hat{\theta} \in \arg\min_{\theta \in \Theta} \left\{ E_n[m(X_i^\top \theta, Y_i)] + \lambda_1 \|	heta\|_1 \right\}.
\]

b. (Optional): Define \( \hat{T}_1 := \text{supp}(\hat{\theta}) \) and recast \( \hat{\theta} \) as the refitted estimator of \( \theta_0 \):

\[
\tilde{\theta} \in \arg\min_{\theta \in \Theta, \text{supp}(\theta) \subseteq \hat{T}_1} E_n[m(X_i^\top \theta, Y_i)].
\]

---

\(^{12}\)See Lemma B.22 in the appendix for the precise statement.

\(^{13}\)Here, \( \lambda_1 \) can be chosen via the BCV method, and \( \lambda_2 \) can be chosen either via the BCV method or via the SNMD theory for weighted LASSO, as discussed in Belloni et al. (2016).
Step 2 (Orthogonalize): a. Based on $\tilde{\theta}$ from Step 1, define an estimator $\tilde{\mu}$ of $\mu_0$ by

$$\tilde{\mu} \in \arg\min_{\mu \in \mathbb{R}^{p-1}} \left\{ \mathbb{E}_n \left[ m''_{11}(X_i^T \tilde{\theta}, Y_i)(D_i - W_i^T \mu)^2 \right] + \lambda_2 \| \mu \|_1 \right\}. \tag{5.3}$$

b. (Optional): Define $\tilde{T}_2 := \text{supp}(\tilde{\mu})$ and recast $\tilde{\mu}$ as the refitted estimator of $\mu_0$:

$$\tilde{\mu} \in \arg\min_{\mu \in \mathbb{R}^{p-1}, \ \text{supp}(\mu) \subseteq \tilde{T}_2} \mathbb{E}_n \left[ m''_{11}(X_i^T \tilde{\theta}, Y_i)(D_i - W_i^T \mu)^2 \right]. \tag{5.4}$$

Step 3 (Update): Define the (debiased) estimator $\tilde{\beta}$ of $\beta_0$ as the one-step update of $\hat{\beta}$:

$$\tilde{\beta} := \hat{\beta} - \frac{\mathbb{E}_n \left[ m'_1(X_i^T \tilde{\theta}, Y_i)(D_i - W_i^T \hat{\mu}) \right]}{\mathbb{E}_n \left[ m''_{11}(X_i^T \tilde{\theta}, Y_i)(D_i - W_i^T \hat{\mu})D_i \right]}^{14} \tag{5.5}$$

Note that (even without refitting) this procedure gives two estimators of $\beta_0$: $\hat{\beta}$ on the first step and $\tilde{\beta}$ on the third step. As it turns out, the estimator $\tilde{\beta}$ is better, in the sense that it can be established as both asymptotically unbiased, $\sqrt{n}$-consistent and asymptotically normal. To derive these properties, we impose the following assumptions.

**Assumption 5.1 (Identifiability).** There exists a constant $c_1 \in (0, \infty)$ such that we have $\mathbb{E}[|m'_1(X^T \theta_0, Y)(D - W^T \mu_0)|^2] \geq c_1$.

**Assumption 5.2 (Integrability).** There are constants $C_M \in (0, \infty)$ and $\bar{r} \in (4, \infty)$ such that

$$\max_{1 \leq j \leq p}(\mathbb{E}[|X_j|^\bar{r}])^{1/\bar{r}} \leq C_M, \ (\mathbb{E}[|D - W^T \mu_0|^\bar{r}])^{1/\bar{r}} \leq C_M, \ \text{and} \ (\mathbb{E}[|m'_1(X^T \theta_0, Y)|^{\bar{r}}])^{1/\bar{r}} \leq C_M.$$

Assumption 5.1 essentially means that there is non-trivial variation in the variable of interest $D$ after partialling out the controls $W$. In the familiar case of the linear mean model with square loss, non-trivial variation follows from the usual rank condition for identification of $\theta_0$; see Appendix A.1 for details.\textsuperscript{15} Assumption 5.2 imposes minor regularity conditions requiring a certain amount of integrability of the random variables in the model and transformations thereof.

For inference purposes, we also invoke a stronger smoothness condition.

---

\textsuperscript{14}In Examples 3 and 4, the second derivative $m''_{11}(X_i^T \tilde{\theta}, Y_i)$ may not exist for some observations $i \in [n]$, in which case the estimator $\hat{\beta}$ may not be well-defined. To make it well-defined, we replace $m''_{11}(X_i^T \tilde{\theta}, Y_i)$ in (5.3), (5.4) and (5.5) for such observations by zero.

\textsuperscript{15}More generally, Assumption 5.1 is implied by the eigenvalues of the matrix $\mathbb{E}[U^2XX^T]$ being bounded away from zero, which is a non-degeneracy condition.
Assumption 5.3 (Smoothness). There are constants $C_m \in (0, \infty)$, $J \in \mathbb{N}$, and a possibly $y$-dependent partition $-\infty = t_{y,0} < t_{y,1} \leq \cdots \leq t_{y,J-1} < t_{y,J} = \infty$ of $\mathbb{R}$ such that for all $y \in \mathcal{Y}$, the function $m(\cdot, y)$ is continuously differentiable on $\mathbb{R}$ and three-times differentiable on each $(t_{y,j-1}, t_{y,j})$, $j \in [J]$, with second and third derivatives satisfying $|m''_{11}(t, y)| \leq C_m$ and $|m'''_{111}(t, y)| \leq C_m$. In addition, $m''_{11}(X^\top \theta_0, Y)$ exists almost surely.

This assumption strengthens Assumption 4.4 from Section 4 (which is why we reuse the symbol $C_m$ for the constant). Note that Assumption 5.3 does not hold for the trimmed LAD loss function in Example 4, which means that our inference approach does not apply for this loss function. In addition, Assumption 5.3 does not hold for the trimmed LS loss function in the same example whenever $\theta_0 = 0_p$. Although we believe it should be possible to perform inference in these cases using methods from Belloni et al. (2017) developed for the case of a high-dimensional linear quantile regression model, we leave this line of work for the future.

In Appendix A.2, we verify Assumption 5.3 for all other examples from Section 2 including Example 4 with the trimmed LS loss function whenever $\theta_0 \neq 0_p$.

The next assumption controls the impact of points of non-smoothness in the loss, if any.

Assumption 5.4 (Density). Provided $J \geq 2$, there is a constant $C_f \in (0, \infty)$ and a non-random sequence $\Delta_n$ in $(0, \infty)$ such that $P(X^\top \theta_0 - \Delta_n \leq t_{y,j} \leq X^\top \theta_0 + \Delta_n) \leq C_f \Delta_n$ for all $j \in [J-1]$.

Assumption 5.4 holds trivially in Examples 1 and 2, as for those examples Assumption 5.3 holds with $J = 1$. When combined with the requirement that $\Delta_n \to 0$ (sufficiently fast), Assumption 5.4 does impose quite a bit of structure in Examples 3 and 4, however. In Example 3, this assumption is satisfied if the conditional distribution of $Y$ given $X$ is absolutely continuous with bounded PDF. In Example 4 with the trimmed LS loss function, it is satisfied if the conditional distribution of $Y_1$ given $(X, Y_1 > 0)$, the conditional distribution of $Y_2$ given $(X, Y_2 > 0)$, and the (unconditional) distribution of $X^\top \theta_0$ are all absolutely continuous with bounded (uniformly over $n$) PDFs; see Appendix A.2 for details.\footnote{Note here that the requirement that the distribution of $X^\top \theta_0$ is absolutely continuous with bounded PDF implies that $\theta_0$ is sufficiently well separated from $0_p$.}

Assumption 5.5 (Convergence Rates). There is a non-random sequence $a_n$ in $(0, \infty)$ such that $a_n \to 0$ and $\|\tilde{\theta} - \theta_0\|_1 + \|\tilde{\mu} - \mu_0\|_1 \lesssim_P a_n$.

Assumption 5.5 is a high-level assumption placed on the estimators from Steps 1 and 2 in Algorithm 5.1. When $\tilde{\theta}$ is $\ell_1$-ME or post-$\ell_1$-ME based on BCV, we can lean on the bounds from Theorem 4.2. For the estimation error of $\tilde{\mu}$, however, we cannot use Theorem 4.2, as this estimator does not fit into our framework because of the presence of estimated

\[\]
weights in the optimization problems (5.3) and (5.4). However, these optimization problems correspond to LASSO and post-LASSO with estimated weights, and such estimators are well studied in the literature. See e.g. Belloni et al. (2016), where one can find the appropriate rates for the estimation error of $\hat{\beta}$ in terms of the sparsity of $\mu_0$.

We next present a theorem on the asymptotic distribution of the debiased estimator $\hat{\beta}$.

**Theorem 5.1 (Asymptotic Distribution).** Let Assumptions 3.1–3.6 and 5.1–5.5 hold, and suppose that $\sqrt{n}a_n^2 \to 0$, $a_n(n^{1/p}B_n + \sqrt{\ln(pn)}) \to 0$, and $B_n^2 \ln(pn) = o(n^{1-4/(r+5)})$. If $J \geq 2$, suppose also that $(1 + \sqrt{n}B_n a_n)(\Delta_n^{1/2} + (B_n a_n/\Delta_n)^{r/2}) \to 0$. Then

$$\frac{\sqrt{n}(\hat{\beta} - \beta_0)}{\sigma_0} \xrightarrow{D} N(0, 1), \quad \text{where} \quad \sigma_0^2 := \frac{E[(m'_i(X\top\theta_0, Y)(D - W\top\mu_0))^2]}{(E[m''_i(X\top\theta_0, Y)(D - W\top\mu_0)D])^2}. \quad (5.6)$$

This theorem shows that the estimator $\hat{\beta}$ is asymptotically unbiased and normal under plausible regularity conditions. The “asymptotic” variance $\sigma_0^2$ appearing in this theorem, which depends on $n$ in general via the distribution $P$ of $(X, Y)$, is easily estimable. For example, one can use a plug-in estimator

$$\hat{\sigma}^2 := \frac{E_n[(m'_i(D_i\hat{\beta} + W_i\top\tilde{\gamma}, Y_i)(D_i - W_i\top\tilde{\mu})^2]}{(E_n[m''_i(D_i\hat{\beta} + W_i\top\tilde{\gamma}, Y_i)(D_i - W_i\top\tilde{\mu})D_i])^2}. \quad (5.7)$$

with the estimators $\tilde{\beta}, \tilde{\gamma}$ and $\tilde{\mu}$ stemming from Steps 1 and 2 of Algorithm 5.1 (possibly with refitting) using BCV as the penalty rule in both steps. Alternatively, one can incorporate Step 3 of the same algorithm and use

$$\hat{\sigma}^2 := \frac{E_n[(m'_i(D_i\hat{\beta} + W_i\top\tilde{\gamma}, Y_i)(D_i - W_i\top\tilde{\mu})^2]}{(E_n[m''_i(D_i\hat{\beta} + W_i\top\tilde{\gamma}, Y_i)(D_i - W_i\top\tilde{\mu})D_i])^2}. \quad (5.8)$$

It is rather standard to derive consistency of these estimators. Also, because of asymptotic normality of $\hat{\beta}$, it is then straightforward to perform inference on $\beta_0$. For example, an asymptotically valid $(1 - \alpha) \times 100\%$ confidence interval for $\beta_0$ takes the standard form $[\hat{\beta} - z_{\alpha/2}\hat{\sigma}/\sqrt{n}, \hat{\beta} + z_{\alpha/2}\hat{\sigma}/\sqrt{n}]$, where $\hat{\sigma}$ is given either by (5.7) or by (5.8), and $z_{\alpha/2}$ is the $(1 - \alpha/2)$-quantile of the standard normal distribution.

**Remark 5.1 (Relation to Literature).** As discussed in the beginning of this section, our approach to inference in this section closely follows the developments in the literature. In particular, our estimator $\hat{\beta}$ is essentially the same as that proposed in van de Geer et al.\footnote{For both variance estimators (5.7) and (5.8), in case the second derivative $m''_i(X_i\top\theta, Y_i)$ fails to exist at $\theta = (\tilde{\beta}, \tilde{\gamma})\top$ or $\theta = (\tilde{\beta}, \tilde{\gamma})\top$ and some $i \in [n]$, we replace those second derivatives by zero.}
(2014), the only difference being that we allow refitting in the optional parts of Steps 1 and 2 of Algorithm 5.1. As we will see in the next section, this refitting can substantially improve inference, in terms of size control, even in approximately sparse models. More importantly, however, is that Theorem 5.1 is different from the corresponding theorem in van de Geer et al. (2014), as we tune the assumptions of our theorem toward the examples from Section 2. Specifically, we do not require the function \( m(\cdot, y) \) to be strictly convex (see ibid., p. 1179) or for it to be everywhere twice differentiable with a Lipschitz-continuous second derivative (see ibid., Condition (C1)). No matter the choice of “trimmer” \( \Xi \), the trimmed loss (2.6) in Example 4 has linear pieces and is therefore not strictly convex. Moreover, neither the asymmetric LS (Example 3 with \( \tau \neq 1/2 \)) nor trimmed LS loss functions have Lipschitz-continuous second derivatives. We also provide a detailed verification of our assumptions for all Section 2 examples in Appendix A.2. Related approaches to debiasing of high-dimensional estimators were also proposed in Javanmard and Montanari (2013) in a likelihood framework and in Belloni et al. (2016) for generalized linear models.

\[ \square \]

6 Simulations

In this section we investigate the finite-sample behavior of our estimators based on the bootstrap-after-cross-validation (BCV) method for obtaining penalty levels proposed in Section 4. We also compare our estimation and inference methods to \((K\)-fold\) cross-validation, which lacks general theoretical justification but is a popular method in practice.

6.1 Simulation Design

We consider a master data-generating process (DGP) of the form

\[
Y_i = 1 \left( \beta_0 D_i + \sum_{j=1}^{p-1} \gamma_{0j} W_{ij} + \varepsilon_i > 0 \right), \quad \varepsilon_i \mid D_i, W_i \sim N(0, 1), \quad i \in [n],
\]

thus implying a binary probit model as in Example 1. The regressors \( X = (D, W^\top)^\top \) are distributed jointly centered Gaussian \( X \sim N(0, \Sigma(\rho)) \) with covariances (and correlations)

\[
\Sigma_{j,k}(\rho) := \text{cov}(X_j, X_k) = E[X_j X_k] = \rho^{|j-k|}, \quad (j, k) \in [p]^2.
\]

Hence, the regressor covariance matrix \( \Sigma(\rho) \) takes a Toeplitz form with the overall correlation level being dictated by \( \rho \). We allow \( \rho \in \{0, .2, \ldots, .8\} \), thus running the gamut of (positive) correlation levels. Since \( \varepsilon_i \)'s are standard normal, the “noise” \( \text{var}(\varepsilon) \) in our DGP is fixed at
one. Hence, the signal-to-noise ratio (SNR) equals the “signal,”

$$\text{SNR} := \frac{\text{var}(X^\top \theta_0)}{\text{var}(\varepsilon)} = \theta_0^\top \Sigma (\rho) \theta_0,$$

which depends on both the correlation level and coefficient pattern. We consider the patterns:

- **Pattern 1**: \( \theta_0 = (1, 1, 0, \ldots, 0)^\top \), (Exactly Sparse)
- **Pattern 2**: \( \theta_{0,j} = (1/\sqrt{2})^{j-1}1 (j \leq 5), \quad j \in [p] \), (Intermediate)
- **Pattern 3**: \( \theta_{0,j} = (1/\sqrt{2})^{j-1} \), \( j \in [p] \). (Approximately Sparse)

The exactly sparse pattern has only non-zero coefficients for the first couple of regressors (\( s_0 = 2 \)), and both non-zeros are clearly separated from zero, thus allowing perfect variable selection. The implied signals (hence SNRs) are

$$\text{var} \left( X^\top \theta_0 \right) = 2 (1 + \rho) \in \{2, 2.4, 2.8, 3.2, 3.8\}. \quad (6.1)$$

Compared to existing simulation studies for high-dimensional binary response models, the SNRs considered here are relatively low.\(^{18}\)

Note that the SNR is increasing with the regressor correlation, such that sampling from a high-\( \rho \) DGP tends to produce an easier estimation problem compared to sampling from a low-\( \rho \) DGP, keeping all other things equal. When reporting results below for \( \rho = 0 \) (our baseline), we are thus considering the worst correlation scenario.\(^{19}\)

In contrast to the exactly sparse pattern, the approximately sparse pattern involves all non-zeros (\( s_q = p \)), which are not bounded away from zero, such that variable selection mistakes are bound to happen. To see that this pattern is in fact approximately sparse, note that for every \( q \in (0, 1] \) one has \( \sum_{j=1}^{p} |\theta_{0,j}|^q \leq \sum_{j=1}^{\infty} |\theta_{0,j}|^q = 1/(1 - 2^{-q/2}) \). Hence, for the purpose of Assumption 3.6, we can choose \( q \in (0, 1] \) freely and pair it with \( s_q = 1/(1 - 2^{-q/2}) \). The base \( 1/\sqrt{2} \) of the approximately sparse pattern was here chosen to (approximately) equate the signals arising from the approximately and exactly sparse coefficient patterns in the baseline case of uncorrelated regressors (\( \rho = 0 \)), which amounts to \( \|\theta_0\|_2^2 \). The relevance of a regressor, as measured by its coefficient, is rapidly decaying in the regressor index \( j \),

\(^{18}\)For example, the binary logit designs in Friedman et al. (2010, Section 5.2) and Ng (2004, Section 5) imply SNRs of three and over 30, respectively.

\(^{19}\)The same comments apply to the other coefficient patterns albeit with the more complicated signal

$$\text{var} \left( X^\top \theta_0 \right) = \sum_{j=1}^{p} \theta_{0,j}^2 + 2 \sum_{j=1}^{p-1} \sum_{k=j+1}^{p} \theta_{0,j} \theta_{0,k} \rho^{k-j}.$$
such that the vast majority of the signal is captured by a small fraction of the regressors. For example, in the baseline case of uncorrelated regressors ($\rho = 0$), the first 10 regressors account for 99.9 percent of the signal (two).

In between these two extremes lies the intermediate pattern. This pattern was created by cutting off the approximately sparse coefficient sequence at the smallest regressor index $j^*$, such that regressors $[j^*]$ account for at least 95 percent of the baseline signal. (Here: $j^* = 5$.) For this pattern, perfect variable selection is possible but unlikely.

We consider sample sizes $n \in \{100, 200, 400\}$ and limit attention to the high-dimensional regime by fixing $p = n$ throughout.

**Remark 6.1 (Sparsity of Debiasing Coefficient Vector).** One may wonder whether the above patterns for the structural coefficients $\theta_0 = (\beta_0, \gamma_0^\top)^\top$ agree or conflict with sparsity of the non-primitive debiasing coefficient vector $\mu_0$ in any sense of the word. While a thorough investigation of this question is beyond the scope of this paper, we can provide some insights for our concrete DGPs. Specifically, in Appendix H.1, we show that—in our collection of DGPs—the number of non-zeros in $\mu_0$ is bounded by the number of non-zeros in $\gamma_0$, $\|\mu_0\|_0 \leq \|\gamma_0\|_0$, i.e. the number of relevant controls. Hence, when $\gamma_0$ is (exactly) sparse, so is $\mu_0$. Moreover, we show via simulation that when $\gamma_0$ is only approximately sparse, the sorted absolute values of the elements of $\mu_0$ are rapidly decaying and approaching zero, cf. Figure H.2. Such a decay is in line with the notion of approximate sparsity.

### 6.2 Estimation and Implementation

We consider the following four estimators arising from $\ell_1$-ME (1.2) and post-$\ell_1$-ME (4.16) based on either the CV or BCV penalty levels in (4.9) and (4.12), respectively:

- $\ell_1$-ME based on bootstrapping after cross-validation ("BCV"),
- post-$\ell_1$-ME based on bootstrapping after cross-validation ("post-BCV"),
- $\ell_1$-ME based on cross-validation ("CV"), and
- post-$\ell_1$-ME based on cross-validation ("post-CV").

When discussing normal approximations based on three-step debiasing (Algorithm 5.1), we use the same method in both Steps 1 and 2. For example, the "post-BCV" inference procedure refers to post-BCV in the first step, followed by post-BCV in the second step (i.e. both optional steps are taken).

Our BCV and post-BCV estimation methods require us to specify a score markup $c_0 \in (1, \infty)$ and probability tolerance rule $\alpha = \alpha_n$. We here follow the recommendation in Belloni
et al. (2012, p. 2380) for the LASSO and post-LASSO and take $c_0 = 1.1$ and $\alpha_n = .1/\ln(p\lor n)$ as our benchmark. The latter function, slowly decaying in $p \lor n$, leads to $\alpha \approx 2.2\%, 1.9\%$ and $1.7\%$ for $n = 100, 200$ and $400$, respectively. We also look at the alternative score markups $\{1, 1.05\}$, the first one being excluded by the theory in Section 4. The alternative probability tolerance rule $\alpha_n = 10/n$ leads to qualitatively identical conclusions, cf. Appendix H.2. We stress that the benchmark choices of $c_0$ and $\alpha_n$ are only rules of thumb that tend to perform well in the simulation designs considered here. Other choices of score markups and probability tolerance rules may have better properties in other DGPs.

We have previously treated all coefficients in the same manner, in that they are all penalized and with equal weight. However, in an empirical application one is typically confident that an intercept belongs in the model. For this reason, the (intercept) coefficient on the constant regressor is usually not penalized during estimation. Moreover, to justify equal penalty weighting, prior to estimation one typically brings the (non-constant) regressors onto the same scale by dividing them by their respective sample standard deviations. To align our simulation study with these empirical practices, we include unpenalized intercepts in both Steps 1 and 2 of Algorithm 5.1 and rescale regressors. (The intercepts are still suppressed in our notation.) That is, we treat neither the zero (true) intercept nor equivariant regressors as information known to the researcher. In these aspects our simulations are therefore empirically calibrated.

For each sample size $n (= p)$, each correlation level $\rho$, and each coefficient pattern, we use 2,000 independent simulation draws and 1,000 independent standard Gaussian bootstrap draws per simulation draw and per estimation step (when relevant). We assign observations to $K$ approximately equally large folds $\{I_k\}_{k=1}^K$ for both the first and second steps, shuffling the assignments in between. We keep $K = 3$ throughout and use the same folds for all estimators to facilitate comparison.\(^{20}\)

All simulations are carried out in \texttt{R} with cross-validation done using \texttt{glmnet::cv.glmnet}, and refitting done using \texttt{stats::glm}.\(^{21}\) When constructing the candidate penalty set $\Lambda_n$, we use the \texttt{glmnet} default settings, which creates a log-scale equi-distant grid of a 100 candidate penalties from the threshold penalty level to essentially zero. The threshold is the (approximately) smallest level of penalization needed to set every coefficient to zero, thus resulting in a trivial (null) model.\(^{22}\)

\(^{20}\)Three folds is the minimum value allowed by \texttt{cv.glmnet}. Preliminary and unreported simulation experiments suggest that using 5-fold (instead of 3-fold) CV only affects the average errors reported below at the third decimal. Similarly, using 2,000 Gaussian bootstraps (instead of 1,000) appears to only affect these averages at the fourth decimal.

\(^{21}\)We use \texttt{R} version 4.2.2 and \texttt{glmnet} version 4.1-6.

\(^{22}\)Log-scale equi-distance from a “large” candidate value to essentially zero fits well with the form of $\Lambda_n$ in our Assumption 4.3 (interpreting $c_\Lambda/n \approx 0$). However, the threshold penalty is a function of the data.
Note that \texttt{cv.glmnet} calculates and stores the out-of-fold linear forms $X_i^\top \hat{\theta}_{I_k}(\lambda)$ (with an intercept, if relevant) for each $i \in I_k$, fold $k$ and candidate penalty $\lambda$, and allows for extraction of estimates for penalty levels off the regularization grid via linear interpolation. Hence, compared to CV, there is essentially zero added computational burden associated with using BCV.

6.3 Results

6.3.1 Non-Existence and Treatment of Missing Values

While the $\ell_1$-penalized probit estimators BCV and CV always exist (cf. Section E), refitting after variable selection based on either of these estimator can fail. For example, in our binary response setting, without any penalty one may encounter complete separation of the outcomes based on the fitted probabilities, in which case the refitted estimates fail to exist (as real numbers).\footnote{Strictly speaking, the $\ell_1$-penalized probit estimator fails to exist when all outcomes are of the same label and some coefficient (here: the intercept) goes unpenalized. In none of our simulated datasets did we encounter all zeros or all ones. See Appendix E and, in particular, Remark E.6 for more discussion.}

Across all simulation designs and draws, refitting after CV fails in nearly 15% of all cases. The fraction of such non-existent post-CV cases varies with the DGP and can be higher than 47%. Since post-CV estimation and debiasing procedures do not appear well-defined in our context, we drop them from further consideration.

In contrast, refitting after BCV fails to converge in only about 0.01% of all cases.\footnote{Specifically, convergence fails in 74 out of a total of 540,000 cases, where the total equals the product of the numbers of simulation draws (2,000), correlation levels (5), sample/problem sizes (3), coefficient patterns (3), score markups (3) and probability tolerance rules (2).} Since we find this fraction miniscule, when reporting results below we choose to simply omit the problematic cases from the relevant post-BCV statistics; see also the figure notes.

6.3.2 Estimation Error

Figure 6.1 shows the mean $\ell_2$ estimation error (for the slope coefficients, averaging over the 2,000 simulation draws) arising from BCV, post-BCV and CV, respectively, using benchmark tuning. Mean $\ell_2$ estimation error is here depicted as a function of the sample/problem size (the tile column), coefficient pattern (the tile row), and correlation level (the horizontal axis in each tile). The horizontal line at $\|\theta_0\|_2$ facilitates comparison with the trivial “estimator” $\hat{\theta} \equiv 0_p$.

and, thus, random. The resulting candidate penalty set used in our simulations is therefore also random, and thus, strictly speaking, not allowed by Assumption 4.3. Moreover, the number of candidate values $|\Lambda_n|$ is here held fixed. We believe these deviations from our theory to be only a minor issue.
One observation evident from this figure is that the error curves of BCV, post-BCV and CV can cross. Hence, these estimators cannot be ranked in terms of mean $\ell_2$ estimation error, in general. However, for the largest sample/problem size considered, post-BCV outperforms CV for small to medium levels of correlation, and CV outperforms BCV.\(^{25}\)

Increasing the sample size (moving from left to right) leads to a downward shift in mean estimation error for all three estimators, which is indicative of convergence. Convergence appears to take place no matter the coefficient pattern or regressor correlation level even though the number of candidate regressors matches the sample size. Increasing the number of non-zeros in $\theta_0$ (moving from top to bottom) leads to an upward shift in mean estimation error. This finding is consistent with convergence slowing down with $q$ and $s_q$ as predicted by Theorems 4.1 and 4.2.

We next investigate the impact of the choice of score markup $c_0$. Figures 6.2 and 6.3 plot the mean $\ell_2$ estimation error for $c_0 = 1, 1.05$ and (the previously used) 1.1, each sample/problem size and coefficient pattern, and for the BCV and post-BCV estimators, respectively. Figure 6.2 suggests that increasing $c_0$ away from one slightly worsens (mean

\(^{25}\)We reach qualitatively identical conclusions from inspecting the median $\ell_2$ estimation errors. Hence, these findings are not limited to one particular feature of the error distributions. We omit the corresponding median plots due to their similarity with the mean error plots. Figures are available upon request.
Figure 6.2: Mean $\ell_2$ BCV Estimation Error by Score Markup with $\alpha_n = .1/ \ln (p \lor n)$

Figure 6.3: Mean $\ell_2$ Post-BCV Estimation Error by Score Markup with $\alpha_n = .1/ \ln (p \lor n)$

Notes: The dotted black lines correspond to the (constant) error of the all-zeros estimator. Post-BCV cases dropped due to nonconvergence: 19 (0.007%).
estimation error) performance of BCV. While our theory takes \( c_0 \) strictly greater than one, any value near one—including the limit case of one itself—appears to lead to near identical results.\(^{26}\) For post-BCV (Figure 6.3), the findings are similar. In fact, at least for the largest sample/problem size, the exact value of \( c_0 \in \{1, 1.05, 1.1\} \) has little to no impact on mean error. Note that our findings for post-BCV apply even with the approximately sparse coefficient pattern, where variable selection mistakes are bound to occur.

To conclude this subsection, we note that it is a well-known puzzle in the LASSO literature that the theory typically requires that \( c_0 \) is strictly bigger than one, with the estimation error bounds deteriorating as \( c_0 \) approaches one, while simulation experience suggests that the estimation errors are insensitive with respect to \( c_0 \) when \( c_0 \) is close to one. We believe that solving this puzzle remains one of the key challenges in this literature.

6.3.3 Normal Approximation

We next assess the normal approximations resulting from three-step debiasing (Algorithm 5.1) using either BCV, post-BCV or CV. Instead of looking at the standardized estimate \( \sqrt{n}(\hat{\beta} - \beta_0)/\sigma_0 \) for the true asymptotic variance \( \sigma_0^2 \) given in (5.6), we form an estimate \( \hat{\sigma}^2 \) and consider the studentized estimate \( \sqrt{n}(\hat{\beta} - \beta_0)/\hat{\sigma} \). That is, we take into account the unknown nature of the \( \sigma_0^2 \), as required in an empirical application.

To construct the estimate \( \hat{\sigma}^2 \), we first leverage the binary response model to establish the (conditional information) equality \( E[\mathbf{m}_1'(\mathbf{X}^\top \mathbf{\theta}_0, \mathbf{Y})^2 | \mathbf{X}] = E[\mathbf{m}_1''(\mathbf{X}^\top \mathbf{\theta}_0, \mathbf{Y}) | \mathbf{X}] \). We then use the definition of \( \mu_0 \) to establish the (weighted projection) equality

\[
E[\mathbf{m}_1''(\mathbf{X}^\top \mathbf{\theta}_0, \mathbf{Y}) (D - \mathbf{W}^\top \mu_0)^2] = E[\mathbf{m}_1''(\mathbf{X}^\top \mathbf{\theta}_0, \mathbf{Y}) (D - \mathbf{W}^\top \mu_0) D] .
\]

Again using the binary response model, we can evaluate

\[
E[\mathbf{m}_1''(\mathbf{X}^\top \mathbf{\theta}_0, \mathbf{Y}) | \mathbf{X}^\top \mathbf{\theta}_0 = t] = \frac{f(t)^2}{F(t)(1 - F(t))} =: \omega_F(t) ,
\]

where \( f \) and \( F \) denote the PDF and CDF, respectively, associated with the binary response model.\(^{27}\) This allows us to simplify the expression for \( \sigma_0^2 \) to

\[
\sigma_0^2 = 1/E[\omega_F(\mathbf{X}^\top \mathbf{\theta}_0) (D - \mathbf{W}^\top \mu_0) D] .
\]

\(^{26}\)That mean BCV error is downward sloping for small to moderate \( \rho \) levels is due to the signal being increasing in \( \rho \) and need not translate to other correlation or coefficient patterns.

\(^{27}\)In our current binary probit setting, these functions are the standard normal PDF and CDF, respectively. In the empirical application in Section 7, we also use the logistic distribution, leading to the binary logit.
Figure 6.4: Densities of Studentized Estimates by $n(=p)$ with $\rho = 0$, $c_0 = 1.1$ and $\alpha_n = .1/\ln (p \lor n)$.

and leads to the following estimator of $\sigma_0^2$:

$$\hat{\sigma}^2 := 1/E_n [\omega_F (\bar{\beta}D_i + W_i^T \bar{\gamma}) (D_i - W_i^T \bar{\mu}) D_i],$$

with $\bar{\gamma}, \bar{\mu}$ and $\bar{\beta}$ given by Steps 1, 2 and 3, respectively, of Algorithm 5.1 given different rules for choosing the penalties $\lambda_1$ and $\lambda_2$.

Figure 6.4 shows the (kernel) densities of the studentized estimates using benchmark tuning and $\rho = 0$. The densities arising from BCV, post-BCV and CV, respectively, are here depicted as columns of tiles, where each tile row corresponds to a coefficient pattern and each graph within a tile a sample/problem size. Starting with the exactly sparse coefficient pattern (the top row), we see that both BCV and CV lead to considerable shrinkage bias even after debiasing the initial estimate of the focal parameter $\beta_0$. This feature is seen from the leftward shifts in the resulting densities compared to the standard normal density, here

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28Alternatively, one can use the “sandwich” estimators (5.7) and (5.8). Experimenting with these estimators, we obtained numerically similar results as reported below for the estimator $\hat{\sigma}^2$. We prefer $\hat{\sigma}^2$ since it leverages both the binary response and projection structure.

29All kernel densities are created using the R package ggplot2 with geom_density. In expectation of an approximately normal distribution, we use a Gaussian kernel and the Silverman (1986, Equation (3.31)) rule-of-thumb bandwidth (both geom_density defaults).
represented by the dotted line. These biases do not seem to disappear as \( n \) increases, holding \( n = p \). If anything, these distributions shift further left, which indicates that BCV requires a larger sample size. In contrast, the post-BCV density essentially collapses to the standard normal one, at least for \( n = p = 200 \) and 400.

As the coefficient pattern becomes less and less sparse (moving down), all approximations deteriorate, as is to be expected. While imperfect, the post-BCV densities are still decent approximations to the normal for both the intermediate and approximately sparse coefficient patterns. Moreover, only these densities appear to approach the standard normal as the sample/problem size increases.

While Figure 6.4 depicts the normal approximations for the worst-correlation case \( \rho = 0 \), in Figure 6.5 we display the normal approximations as a function of \( \rho \). We here focus on the largest sample size \( n(= p) = 400 \) and the (more challenging) approximately sparse coefficient pattern, again using benchmark tuning. Since post-BCV and CV appear to lead to better normal approximations than BCV, we display only results from the former two methods. Post-BCV leads to a relatively accurate normal approximation for every correlation level considered. Moreover, while the normal approximation stemming from CV appears to improve as \( \rho \) increases, at no correlation level considered does CV lead to a visually better
approximation than post-BCV.\footnote{We also investigated the robustness of the post-BCV-resulting normal approximations with respect to the markup $c_0 \in \{1, 1.05, 1.1\}$. Parallelling our findings for mean estimation error in Figure 6.3, the exact markup value appears to make little difference. Figures are available upon request.}

7 Revisiting Racial Differences in Police Use of Force

In this section we revisit the empirical setting in Fryer Jr (2019) (henceforth: Fryer), who explored racial differences in police use of force. We here focus on the part of Fryer’s regression analysis invoking the full Police-Public Contact Survey (PPCS) dataset with the outcome being an indicator for any use of force by the police (conditional on an encounter), thus leading to a binary response model as in our Example 1.\footnote{See Fryer Jr (2019) and the associated online appendix for alternative outcome variables and data sources as well as a detailed discussion of their relative merits and drawbacks.} Specifically, Fryer estimates models of the form

$$P(\text{Force} = 1 | \text{Race}, W) = F(\text{Race}^\top \alpha_0 + W^\top \gamma_0),$$

(7.1)

where Force indicates whether any force was used by the police when encountering a civilian, Race = (Black, Hisp, Other)$^\top$ indicate the race of the civilian (black, hispanic and other than white, with white being the reference race), and $W$ is a list of control variables (including a constant) capturing both civilian (e.g. gender and age), officer (e.g. majority race) and encounter characteristics (e.g. whether the civilian disobeyed, resisted or otherwise misbehaved).\footnote{See the Fryer Jr (2019, Table 2.B) notes for the full variable list and his online appendix A for descriptions.} Here $F$ is a placeholder for a strictly increasing known cumulative distribution function (CDF), which Fryer takes to be the logistic CDF $\Lambda$, thus leading to the binary logit model.

The PPCS logistic regression results reported in Fryer Jr (2019, Table 2.B) show that black and hispanic subjects are statistically significantly more likely to experience some form of force in interactions with the police, controlling for context and civilian behavior. We here look into the robustness of this finding by employing also (i) an alternative binary response model and (ii) large(r) sets of candidate regressors, in combination with $\ell_1$-penalization. For brevity, we here single out the black dummy (Black) and its coefficient $\beta_{\text{Black}}$ and group the other non-white dummies (Hisp and Other) with the controls (thus recasting $W$ and $\gamma_0$). We interpret the statement “there are no racial differences in police use of force” as there being no difference in the probability of force being used for black civilians relative to white
Table 1: \( t \)-Values for Testing a Zero Coefficient \( \beta_{\text{Black}} \) on Black

| Controls \ Loss | Unpenalized (ML) | Post-BCV |
|----------------|------------------|----------|
|                | Logit            | Probit   | Logit | Probit |
| Basic Controls | 8.8              | 8.7      | 10.5  | 9.6    |
| + Interactions | n.a.             | n.a.     | 20.7  | 18.9   |

subjects, holding everything else equal. That is,

\[
P \left( \text{Force} = 1 \mid \text{Black} = 1, \mathbf{W} = \mathbf{w} \right) = P \left( \text{Force} = 1 \mid \text{Black} = 0, \mathbf{W} = \mathbf{w} \right)
\]

for all realizations \( \mathbf{w} \) of the controls (with \text{Hisp} and \text{Other} both zero). Since the race dummies enter the strictly increasing \( F \) in (7.1) in an additive manner, using the model, such a zero probability difference is equivalent to a zero coefficient on the dummy for being black, i.e. \( \beta_{\text{Black}} = 0 \). We therefore take the latter as the hypothesis to be tested.

To this end, we first use the Fryer Jr (2019) supplementary files and descriptions in his online appendix to recollect and recreate the PPCS dataset. Using the same supplementary files, we then replicate the PPCS logistic regression results in Fryer Jr (2019, Table 2.B) to all reported digits, which leaves us confident that we are indeed considering the original dataset.

We next apply three-step debiasing (Algorithm 5.1) with the loss function being either the negative logit or probit log-likelihood. Our simulation findings indicate that post-BCV debiasing outperforms both the BCV and CV equivalents. We therefore only consider the former.\(^{33}\) We use two sets of regressors. The first set (Basic Controls) corresponds to that in Fryer Jr (2019, Table 2.B, Row l), and is Fryer’s largest set of controls. The only difference is that we include categorical regressors via dummies for their different levels, leaving one reference category for each. The second set of regressors (Basic Controls + Interactions) builds on the first by adding all first-order pairwise interactions between the controls (excluding the race dummies \text{Hisp} and \text{Other}). After eliminating variables with zero variance or perfect correlation, the two sets include 30 and 327 non-constant regressors, respectively, which should be compared to a sample size of \( n = 59,668 \) civilian–police encounters.\(^{34}\)

Table 1 displays the \( t \)-values associated with testing the null hypothesis using either unpenalized or \( \ell_1 \)-penalized methods. Using only basic controls (the logit case being covered in Fryer), the \( t \)-statistics take on similar values for both unpenalized maximum likelihood

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\(^{33}\)For both Steps 1 and 2, we here use 10-fold cross-validation, i.e. \( K = 10 \).

\(^{34}\)We note in passing that the 59,668 equals the total number of police-civilian encounters in the PPCS dataset covering the six surveys 1996, 1999, 2002, 2005, 2008 and 2011. The number of complete cases with respect to the regressors used is 9,930 and only leaves the years 2002 and 2011. For a clean comparison, we follow Fryer’s approach to missing values.
(ML) and post-BCV methods. Thus, with only 30 non-constant candidate regressors, regularization has little impact. In contrast, the specification including both basic controls and interactions thereof leads to complete separation in the data, such that the (unpenalized) maximum likelihood estimates do not exist (as real numbers). For this case, some regularization is necessary. Hence, although the numbers of regressors considered here may not appear overwhelmingly large when compared to the sample size, the set of regressors is of great importance. Even when we include all first-order interactions between the controls, the t-statistics resulting from our three-step debiasing procedure remain of the same order as before.\footnote{The increase in the t-values for post-BCV upon inclusion of interactions is for both the logit and probit loss due to both a somewhat larger point estimate and a somewhat smaller standard error. The values underlying Table 1 thus illustrate that more candidate regressors need not lead to larger standard error.} The t-tests based on our post-BCV debiasing lead us to reject the null hypothesis of no racial differences in police use of force at any reasonable significance level. This conclusion in Fryer Jr (2019) therefore appears robust to the choice of controls.

To gauge the economic impact of our change in estimation procedures, we estimate the average partial effect (APE) of changing the civilian race from white to black. Iterating expectations and using (7.1), the APE can be expressed as the average probability difference

\[
\text{APE}_{\text{Black}} := E \left[ P(\text{Force} = 1 \mid \text{Black} = 1, \text{Hisp} = 0, \text{Other} = 0, \mathbf{W}) - P(\text{Force} = 1 \mid \text{Black} = 0, \text{Hisp} = 0, \text{Other} = 0, \mathbf{W}) \right] 
= E \left[ F(\beta_{\text{Black}} + \mathbf{W}^\top \gamma_0) - F(\mathbf{W}^\top \gamma_0) \right], 
\]

(7.2)

where we bring back the other (non-white) civilian race dummies to clarify the comparison made. We estimate this APE by \( \hat{\text{APE}}_{\text{Black}} := \mathbb{E}_n[F(\hat{\beta}_{\text{Black}} + \mathbf{W}_i^\top \hat{\gamma}) - F(\mathbf{W}_i^\top \hat{\gamma})] \), for point estimates \( \hat{\beta}_{\text{Black}} \) and \( \hat{\gamma} \) of \( \beta_{\text{Black}} \) and \( \gamma_0 \), respectively. When results stem from (unpenalized) ML, we use the ML estimates. When results stem from three-step post-BCV debiasing, we use the debiased third-step estimate \( \hat{\beta}_{\text{Black}} \) and the (biased) first-step estimate \( \hat{\gamma} \). Table 2 reports the APE estimates (in percentage points) corresponding to these procedures including either basic controls or basic controls with interactions.

| Controls \ Loss | Unpenalized (ML) | Post-BCV |
|----------------|------------------|----------|
|                | Logit            | Probit   | Logit    | Probit |
| Basic Controls | 1.1              | 1.1      | 1.4      | 1.3    |
| + Interactions | n.a.             | n.a.     | 3.2      | 2.8    |

Using only basic controls, (unpenalized) ML and post-BCV lead to APE estimates in the range of 1.1–1.4 percentage points regardless of the CDF used. (For context, the unpenalized (ML) and post-BCV methods. Thus, with only 30 non-constant candidate regressors, regularization has little impact. In contrast, the specification including both basic controls and interactions thereof leads to complete separation in the data, such that the (unpenalized) maximum likelihood estimates do not exist (as real numbers). For this case, some regularization is necessary. Hence, although the numbers of regressors considered here may not appear overwhelmingly large when compared to the sample size, the set of regressors is of great importance. Even when we include all first-order interactions between the controls, the t-statistics resulting from our three-step debiasing procedure remain of the same order as before.\footnote{The increase in the t-values for post-BCV upon inclusion of interactions is for both the logit and probit loss due to both a somewhat larger point estimate and a somewhat smaller standard error. The values underlying Table 1 thus illustrate that more candidate regressors need not lead to larger standard error.} The t-tests based on our post-BCV debiasing lead us to reject the null hypothesis of no racial differences in police use of force at any reasonable significance level. This conclusion in Fryer Jr (2019) therefore appears robust to the choice of controls.

To gauge the economic impact of our change in estimation procedures, we estimate the average partial effect (APE) of changing the civilian race from white to black. Iterating expectations and using (7.1), the APE can be expressed as the average probability difference

\[
\text{APE}_{\text{Black}} := E \left[ P(\text{Force} = 1 \mid \text{Black} = 1, \text{Hisp} = 0, \text{Other} = 0, \mathbf{W}) 
- P(\text{Force} = 1 \mid \text{Black} = 0, \text{Hisp} = 0, \text{Other} = 0, \mathbf{W}) \right] 
= E \left[ F(\beta_{\text{Black}} + \mathbf{W}^\top \gamma_0) - F(\mathbf{W}^\top \gamma_0) \right], 
\]

(7.2)

where we bring back the other (non-white) civilian race dummies to clarify the comparison made. We estimate this APE by \( \hat{\text{APE}}_{\text{Black}} := \mathbb{E}_n[F(\hat{\beta}_{\text{Black}} + \mathbf{W}_i^\top \hat{\gamma}) - F(\mathbf{W}_i^\top \hat{\gamma})] \), for point estimates \( \hat{\beta}_{\text{Black}} \) and \( \hat{\gamma} \) of \( \beta_{\text{Black}} \) and \( \gamma_0 \), respectively. When results stem from (unpenalized) ML, we use the ML estimates. When results stem from three-step post-BCV debiasing, we use the debiased third-step estimate \( \hat{\beta}_{\text{Black}} \) and the (biased) first-step estimate \( \hat{\gamma} \). Table 2 reports the APE estimates (in percentage points) corresponding to these procedures including either basic controls or basic controls with interactions.

| Controls \ Loss | Unpenalized (ML) | Post-BCV |
|----------------|------------------|----------|
|                | Logit            | Probit   | Logit    | Probit |
| Basic Controls | 1.1              | 1.1      | 1.4      | 1.3    |
| + Interactions | n.a.             | n.a.     | 3.2      | 2.8    |

Using only basic controls, (unpenalized) ML and post-BCV lead to APE estimates in the range of 1.1–1.4 percentage points regardless of the CDF used. (For context, the unpenalized (ML) and post-BCV methods. Thus, with only 30 non-constant candidate regressors, regularization has little impact. In contrast, the specification including both basic controls and interactions thereof leads to complete separation in the data, such that the (unpenalized) maximum likelihood estimates do not exist (as real numbers). For this case, some regularization is necessary. Hence, although the numbers of regressors considered here may not appear overwhelmingly large when compared to the sample size, the set of regressors is of great importance. Even when we include all first-order interactions between the controls, the t-statistics resulting from our three-step debiasing procedure remain of the same order as before.\footnote{The increase in the t-values for post-BCV upon inclusion of interactions is for both the logit and probit loss due to both a somewhat larger point estimate and a somewhat smaller standard error. The values underlying Table 1 thus illustrate that more candidate regressors need not lead to larger standard error.} The t-tests based on our post-BCV debiasing lead us to reject the null hypothesis of no racial differences in police use of force at any reasonable significance level. This conclusion in Fryer Jr (2019) therefore appears robust to the choice of controls.

To gauge the economic impact of our change in estimation procedures, we estimate the average partial effect (APE) of changing the civilian race from white to black. Iterating expectations and using (7.1), the APE can be expressed as the average probability difference

\[
\text{APE}_{\text{Black}} := E \left[ P(\text{Force} = 1 \mid \text{Black} = 1, \text{Hisp} = 0, \text{Other} = 0, \mathbf{W}) 
- P(\text{Force} = 1 \mid \text{Black} = 0, \text{Hisp} = 0, \text{Other} = 0, \mathbf{W}) \right] 
= E \left[ F(\beta_{\text{Black}} + \mathbf{W}^\top \gamma_0) - F(\mathbf{W}^\top \gamma_0) \right], 
\]

(7.2)

where we bring back the other (non-white) civilian race dummies to clarify the comparison made. We estimate this APE by \( \hat{\text{APE}}_{\text{Black}} := \mathbb{E}_n[F(\hat{\beta}_{\text{Black}} + \mathbf{W}_i^\top \hat{\gamma}) - F(\mathbf{W}_i^\top \hat{\gamma})] \), for point estimates \( \hat{\beta}_{\text{Black}} \) and \( \hat{\gamma} \) of \( \beta_{\text{Black}} \) and \( \gamma_0 \), respectively. When results stem from (unpenalized) ML, we use the ML estimates. When results stem from three-step post-BCV debiasing, we use the debiased third-step estimate \( \hat{\beta}_{\text{Black}} \) and the (biased) first-step estimate \( \hat{\gamma} \). Table 2 reports the APE estimates (in percentage points) corresponding to these procedures including either basic controls or basic controls with interactions.

| Controls \ Loss | Unpenalized (ML) | Post-BCV |
|----------------|------------------|----------|
|                | Logit            | Probit   | Logit    | Probit |
| Basic Controls | 1.1              | 1.1      | 1.4      | 1.3    |
| + Interactions | n.a.             | n.a.     | 3.2      | 2.8    |
ditional average of contacts in which PPCS respondents reported any force being used for white civilians is .7 percent.) Including interactions of basic controls, the post-BCV APE estimates roughly double in size to about 3 percentage points. Of course, these relatively large APE estimates may come with relatively large estimation error. However, as the APE in (7.2) depends on many coefficients, it remains a non-trivial task to assign standard errors to these point estimates—a task falling outside the scope of this paper.

Finally, to get a feel for the computational burden associated with the methods proposed in this paper when applied to real data, in Table 3 we report the computing time used by the above-mentioned estimation routines. With only basic controls, the three-step post-BCV debiasing procedure takes at least ten times as long as (unpenalized) ML. This is not surprising, as the former method involves two rounds of (10-fold) CV, bootstrapping and refitting—no task of which is undertaken by ML. However, including also interactions, the ranking of the two approaches is reversed. The about tenfold increase in number of controls increases the computing time associated with post-BCV logit debiasing approximately linearly. For post-BCV probit debiasing, the corresponding increase is almost five fold. In contrast, as the ML estimates are not real numbers, without proper checks for solution existence, any (gradient-based) optimizer would iterate indefinitely in search of the ML estimates. We represent the non-existence of an ML estimate by infinite computing time.\footnote{While infinity may appear overly dramatic, we warn that \texttt{glmnet} does \textit{not} check for optimizer divergence (cf. Friedman et al., 2010, p. 9). We therefore opted for \texttt{stats::glm} for ML estimation and refitting.}

Of course, the Table 3 runtimes are only single observations arising from our particular \texttt{R} implementation of our procedures, using a specific dataset, and our specific computing environment. As such, they need not translate to other settings.

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**Table 3: Estimation Routine Timings (in Seconds)**

| Controls \ Loss | Unpenalized (ML) | Post-BCV |
|----------------|------------------|----------|
|                | Logit  | Probit | Logit  | Probit |
| Basic Controls | 1.5    | 1.5    | 21     | 40     |
| + Interactions | ∞      | ∞      | 210    | 199    |

Notes: All timings were carried out on an Intel Core i7-8700 3.20GHz CPU. When using \texttt{cv.glmnet}, we use the parallel computing option with all 12 virtual cores available.
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# Online Appendices

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A Verification of High-Level Assumptions

In this section, we discuss the high-level assumptions from the main text. We first consider the case of the linear model and square loss, and then turn to the examples from Section 2. In particular, Corollaries A.1 and A.2 state convergence rate and inference results under low-level conditions for each of the examples from Section 2.

A.1 Verification for Linear Model with Square Loss

It is instructive to illustrate the contents of Assumptions 3.1–3.5, 4.1, 4.4, 4.5, 5.1, 5.3, and 5.4 in the familiar case of the linear mean regression model. We here consider the following (strong) version of the linear model with independent Gaussian errors,

\[ Y = \mathbf{X}^\top \theta_0 + \varepsilon, \quad \varepsilon \mid \mathbf{X} \sim \mathcal{N}(0, \sigma_0^2), \]

where \( \theta_0 \in \mathbb{R}^p \) and \( \sigma_0^2 \in (0, \infty) \) are model parameters, and we use the (one-half) square loss \( m(t, y) = (1/2)(y - t)^2 \). Less restrictive dependence and distributional assumptions can be accommodated. We focus on the independent Gaussian case for simplicity.

We take \( \Theta \) to be the full space \( \mathbb{R}^p \), such that Assumption 3.1 is trivial. Convexity of the loss (Assumption 3.2) follows from differentiating twice with respect to \( t \) and observing that \( m''_{11}(t, y) = 1 > 0 \) no matter \( (t, y) \in \mathbb{R}^2 \). For Assumption 3.3, observe that for \( \theta \in \Theta \) arbitrary and denoting \( \delta := \theta - \theta_0 \), the derivative

\[ m'_1(X^\top \theta, Y) = X^\top \theta - Y = X^\top \delta - \varepsilon \]

surely exists. The finiteness of

\[ \mathbb{E} \left[ |m(X^\top \theta, Y)| \right] = \frac{1}{2} \mathbb{E} \left[ (\varepsilon - X^\top \delta)^2 \right] = \frac{1}{2} (\sigma_0^2 + \delta^\top \mathbb{E} [XX^\top] \delta) \]

boils down to finiteness of the matrix \( \mathbb{E} [XX^\top] \). Assumption 3.3 can then be guaranteed by finiteness of second moments \( \mathbb{E}[X^2_j] \) for all \( j \in [p] \), for example. Turning to Assumption 3.4, note that the excess risk at \( \theta \) is

\[ \mathcal{E}(\theta) = \frac{1}{2} \mathbb{E} \left[ (\varepsilon - X^\top \delta)^2 - \varepsilon^2 \right] = \frac{1}{2} \delta^\top \mathbb{E} [XX^\top] \delta \geq \frac{1}{2} \lambda_{\text{min}} (\mathbb{E} [XX^\top]) \| \delta \|_2^2, \]

with \( \lambda_{\text{min}}(\mathbb{E} [XX^\top]) \) denoting the smallest eigenvalue of \( \mathbb{E} [XX^\top] \). As long as the eigenvalues are bounded away from zero, Assumption 3.4 holds with \( c_M = 1 \wedge (1/2) \inf_{n \in \mathbb{N}} \lambda_{\text{min}}(\mathbb{E} [XX^\top]) \) and \( c'_M = \infty \). Positivity of \( \lambda_{\text{min}}(\mathbb{E} [XX^\top]) \) is precisely the rank condition for identification of
\( \theta_0 \). Without this condition \( \theta_0 \) does not uniquely minimize the expected loss. For Assumption 3.5.1, a calculation shows that for all \((x, y) \in \mathbb{R}^{p+1} \) and all \((t_1, t_2) \in \mathbb{R}^2\),

\[
\left| m(x^\top \theta_0 + t_1, y) - m(x^\top \theta_0 + t_2, y) \right| = \frac{1}{2} \left| t_1 + t_2 + 2 (x^\top \theta_0 - y) \right| |t_1 - t_2|
\]

\[
\leq \frac{1}{2} \left( |t_1| + |t_2| + 2 |y - x^\top \theta_0| \right) |t_1 - t_2|.
\]

Choosing \( c_L = 1 \) and \( L(x, y) = 1 + |y - x^\top \theta_0| \), we see that (3.1) holds for all \((x, y) \in \mathbb{R}^{p+1} \) and all \((t_1, t_2) \in \mathbb{R}^2 \) for which \(|t_1| \vee |t_2| \leq c_L\). For this \( L \), using independence and \((a+b)^2 \leq 2a^2 + 2b^2\), we get

\[
\max_{1 \leq j \leq p} \mathbb{E} \left[ L(X, Y) X_j \right]^2 = \mathbb{E} \left[ (1 + |\varepsilon|)^2 \right] \max_{1 \leq j \leq p} \mathbb{E} \left[ X_j^2 \right] \leq 2 \left( 1 + \sigma_0^2 \right) \max_{1 \leq j \leq p} \mathbb{E} \left[ X_j^2 \right].
\]

As long as the error has bounded variance and the regressors bounded second moments, the previous displays suggests

\[
C_{L, \text{Li}}^2 := 1 \vee 2 \sup_{n \in \mathbb{N}} \left\{ (1 + \sigma_0^2) \max_{1 \leq j \leq p} \mathbb{E} [X_j^2] \right\}.
\]

Also, picking \( r = 8 \), we get

\[
\mathbb{E} \left[ L(X, Y) \|X\|_\infty^8 \right] = \mathbb{E} \left[ (1 + |\varepsilon|)^8 \right] \mathbb{E} \left[ \|X\|_\infty^8 \right] \leq 2^7 \left( 1 + 105\sigma_0^8 \right) \mathbb{E} \left[ \|X\|_\infty^8 \right],
\]

where the last inequality uses normality to get \( \mathbb{E}[\varepsilon^8] = 105\sigma_0^8 \). Provided the right-hand side is finite, this calculation suggests

\[
B_\text{Li}^8 = 1 \vee 2^7 \left( 1 + 105\sigma_0^8 \right) \mathbb{E} \left[ \|X\|_\infty^8 \right].
\]

For Assumption 3.5.2, note that

\[
\mathbb{E} \left[ m(X^\top \theta, Y) - m(X^\top \theta_0, Y) \right]^2 = \frac{1}{4} \mathbb{E} \left[ (\varepsilon - X^\top \delta)^2 - \varepsilon^2 \right]^2
\]

\[
= \frac{1}{4} \mathbb{E} \left[ (X^\top \delta)^4 \right] + \mathbb{E} \left[ \varepsilon^2 (X^\top \delta)^2 \right] - \mathbb{E} \left[ \varepsilon (X^\top \delta)^3 \right]
\]

\[
= \frac{1}{4} \mathbb{E} \left[ (X^\top \delta)^4 \right] + \sigma_0^2 \mathbb{E} \left[ (X^\top \delta)^2 \right]
\]

\[
\leq \frac{1}{4} \mathbb{E} \left[ (X^\top \delta)^4 \right] + \sigma_0^2 \lambda_{\text{max}} \left( \mathbb{E} [XX^\top] \right) \|\delta\|_2^2,
\]

with \( \lambda_{\text{max}}(\mathbb{E} [XX^\top]) \) denoting the largest eigenvalue of \( \mathbb{E} [XX^\top] \). The term \((1/4) \mathbb{E} [(X^\top \delta)^4] \) is not generally of the form required by Assumption 3.5.2. However, if the regressors them-
Assumption 3.5 now follows from setting $C$ we see that $E[|X|^4] \leq 3\lambda_{\text{max}}(E[XX^\top])^2 \|\delta\|_2^4$. (A.1)

Hence, when $\|\delta\|_2 \leq c_L = 1$, $E[(X^\top \delta)^4] \leq 3\lambda_{\text{max}}(E[XX^\top])^2 \|\delta\|_2^2$, and, thus,

$$E\left[\left|m^{1}(X^\top \theta, Y) - m^{1}(X^\top \theta_0, Y)\right|^2\right] \leq \left(\frac{3}{4} \lambda_{\text{max}}(E[XX^\top])^2 + \sigma_0^2 \lambda_{\text{max}}(E[XX^\top])\right) \|\delta\|_2^2.$$

Provided also the eigenvalues of $E[XX^\top]$ are bounded from above, the previous display suggests

$$C_{L,2}^2 = 1 \lor \sup_{n \in \mathbb{N}} \left\{\frac{3}{4} \lambda_{\text{max}}(E[XX^\top])^2 + \sigma_0^2 \lambda_{\text{max}}(E[XX^\top])\right\}.$$

Finally, for Assumption 3.5.3, observe that no matter $\|\delta\|_2$,

$$E\left[\left|m^{1}(X^\top \theta, Y) - m^{1}(X^\top \theta_0, Y)\right|^2\right] = E\left[(X^\top \delta)^4\right] \leq \lambda_{\text{max}}(E[XX^\top]) \|\delta\|_2^4,$$

which suggests

$$C_{L,3}^2 = 1 \lor \sup_{n \in \mathbb{N}} \lambda_{\text{max}}(E[XX^\top]).$$

Assumption 3.5 now follows from setting $C_L = \max\{C_{L,1}, C_{L,2}, C_{L,3}\}$, which lies in $[1, \infty)$ under the previously stated assumptions.

With an eye on Assumption 4.1, from $m^{1}(X^\top \theta_0, Y) = -\varepsilon$ and independence of $\varepsilon$ and $X$, we see that $E[|UX_j|^2] = \sigma_0^2 E[X_j^2]$, and so

$$\sigma_0^2 \min_{1 \leq j \leq p} E[X_j^2] \leq E[|UX_j|^2] \leq \sigma_0^2 \max_{1 \leq j \leq p} E[X_j^2].$$

Hence, as long as the lower and upper bounds are bounded away from zero and infinity, respectively, for the purpose of Assumption 4.1.1, we can take

$$c_{\tilde{U}} = \inf_{n \in \mathbb{N}} \left\{\sigma_0^2 \min_{1 \leq j \leq p} E[X_j^2]\right\} \text{ and } C_{\tilde{U}}^2 = 1 \lor \sup_{n \in \mathbb{N}} \left\{\sigma_0^2 \max_{1 \leq j \leq p} E[X_j^2]\right\}.$$

Using also normality, $E[|UX_j|^4] = 3\sigma_0^4 E[X_j^4]$ and $E[|UX_j|^4] = 3\sigma_0^4 E[|UX_j|^4]$, which suggest

$$\tilde{B}_{n,1}^2 = 1 \lor 3\sigma_0^4 \max_{1 \leq j \leq p} E[X_j^4] \text{ and } \tilde{B}_{n,2}^4 = 1 \lor 3\sigma_0^4 E[|UX_j|^4],$$

respectively. The remainder of Assumption 4.1 now follows from setting $\tilde{B}_n = \tilde{B}_{n,1} \lor \tilde{B}_{n,2}$,
which lies in $[1, \infty)$ as long as $E[\|X\|_\infty] < \infty$, which holds when all regressors have finite fourth moments. Assumptions 4.2 and 4.3 have nothing to do with the model and loss function. Moving to Assumption 4.4, observing that $m'_1(t, y) = t - y$, setting $C_m := 1$ we have $|m'_1(t_1, y) - m'_1(t_2, y)| = C_m|t_1 - t_2|$ for all $t_1, t_2, y \in \mathbb{R}$. As shown in (A.1), if $X$ is Gaussian with $\lambda_{\max}(E[X X^\top])$ bounded, then Assumption 4.5 holds with $C_{ev} := 3 \sup_{n \in \mathbb{N}} \lambda_{\max}(E[X X^\top])^2$.

More generally, joint sub-Gaussianity with bounded (joint) sub-Gaussian norm here suffices.

Next, $\mu_0$ is uniquely given by $\mu_0 = (E[W W^\top])^{-1}E[W D]$ provided $\lambda_{\min}(E[W W^\top]) > 0$, which is implied by the previously discussed rank condition for identification of $\theta_0$. It follows that

$$E[|m'_1(X^\top \theta_0, Y)(D - W^\top \mu_0)|^2] = \sigma_0^2 E[(D - W^\top \mu_0)^2],$$

and Assumption 5.1 is satisfied if both right-hand side terms are bounded away from zero. To this end, note that

$$E[(D - W^\top \mu_0)^2] = \begin{bmatrix} 1 & -\mu_0^\top \end{bmatrix} E[X X^\top] \begin{bmatrix} 1 \\ -\mu_0 \end{bmatrix} \geq \lambda_{\min}(E[X X^\top]).$$

Our identifiability condition (Assumption 5.1) then essentially follows from the previously invoked $\inf_{n \in \mathbb{N}} \lambda_{\min}(E[X X^\top]) > 0$, which, again, is only slightly stronger than the rank condition for identification of $\theta_0$. The square loss is everywhere thrice differentiable. Hence, for the purpose of Assumption 5.3, we can take $J = 1$. The second derivative of the loss is $m''_{11}(\cdot, \cdot) \equiv 1$ and the third derivative is $m'''_{111}(\cdot, \cdot, \cdot) \equiv 0$, and, thus, the remainder of Assumption 5.3 follows. Since we can pick $J = 1$, Assumption 5.4 is trivially satisfied.

### A.2 Verication for Examples in Section 2

In this section, we state convergence rate and inference results for each of the examples from the main text under low-level assumptions. The main task of this section is to prove the following results.

**Corollary A.1 (Convergence Rates in Examples from Main Text).** Let $c_{de}$, $c_{ev}$, $c_f$, $c_{eps}$, $C_0$, $C_{ev}$ and $C_{pdf}$ be some constants in $(0, \infty)$, let $\bar{r}$ be a constant in $(4, \infty)$, and let $\bar{B}_n$ be a non-random sequence in $[1, \infty)$. Assume the setting of one of the examples from the main text: Example 1 (logit or probit), Example 2 (logit or probit), Example 3, or Example 4 (trimmed
LS or trimmed LAD). In all of these examples, take $\Theta = \mathbb{R}^p$ and assume that
\begin{equation}
E[|X^T\delta|^2] \geq c_{ev}\|\delta\|_2^2, \quad \text{and} \quad E[|X^T\delta|^4] \leq C_{ev}^2\|\delta\|_2^4, \quad \text{for all } \delta \in \mathbb{R}^p,
\end{equation}
(A.2)
\begin{equation}
\max \left\{ E[|X^T\theta_0|^8], E[|D - W^T\mu_0|^8], \max_{j \in [p]} E[|X_j|^8] \right\} \leq C_0^8,
\end{equation}
(A.3)
\begin{equation}
E \left[ (1 + |X^T\theta_0|)^p \|X\|_\infty^p \right] \leq B_n^p.
\end{equation}
(A.4)

For Example 3, assume in addition that $Y$ is continuously distributed given $X$, i.e., that
\begin{equation}
P(Y = t \mid X = x) = 0 \quad \text{for all } (t, x) \in \mathcal{Y} \times \mathcal{X},
\end{equation}
(A.5)
and that
\begin{align*}
& E[|Y|^8] \leq C_0^8, \quad \min_{j \in [p]} E[|Y - X^T\theta_0|^2X_j^2] \geq c_{eps}, \quad \text{and} \quad E[\|YX\|_\infty^p] \leq B_n^p.
\end{align*}
(A.6)

In Example 4 (trimmed LS), assume in addition that the conditional distributions of $Y_1$ and $Y_2$ given $X$ are continuous on $(0, \infty)$, i.e., that
\begin{equation}
P(Y_j = t \mid X = x) = 0 \quad \text{for all } (j, t, x) \in \{1, 2\} \times (0, \infty) \times \mathcal{X},
\end{equation}
(A.7)
and that
\begin{align*}
& E[|\bar{Y}|^8] \leq C_0^8, \quad E[\|\bar{Y}X\|_\infty^p] \leq B_n^p, \quad \text{and} \quad \min_{j \in [p]} E \left[ X_j^2 (Y_1^2 1\{Y_2 \leq -X^T\theta_0\} + Y_2^2 1\{Y_1 \leq X^T\theta_0\}) \right] \geq c_{eps},
\end{align*}
(A.8)
where $\bar{Y} := Y_1 \lor Y_2$, and
\begin{equation}
P \left( \min \left\{ P(Y_1 - Y_2 > X^T\theta_0 + c_{de} \mid X) \right\},
\right.
\end{equation}
\begin{equation}
\left. P(Y_1 - Y_2 < X^T\theta_0 - c_{de} \mid X) \right\} \geq 2c_f \geq 1 - \left( \frac{c_{ev}}{2\sqrt{2}C_{ev}} \right)^2.
\end{equation}
(A.9)

In Example 4 (trimmed LAD), assume in addition that
\begin{equation}
E[|\bar{Y}|^8] \leq C_0^8, \quad \min_{j \in [p]} E \left[ |X_j|^2 1\{Y_1 > 0, Y_2 > 0\} \right] \geq c_{eps},
\end{equation}
(A.10)
where $\bar{Y} := Y_1 \lor Y_2$, and that the conditional distribution of $Y_1 - Y_2$ given $(X, Y_1 > 0, Y_2 > 0)$ as well as the conditional distributions of $Y_1$ given $(X, Y_1 > 0, Y_2 = 0)$ and of $Y_2$ given
\((X, Y = 0, Y_2 > 0)\) are absolutely continuous with bounded PDFs, i.e., that
\[
f_{Y_1 \mid X, Y_1 > 0, Y_2 > 0}(t \mid x) \leq C_{pdf}, \quad \text{for all } (x, t) \in \mathcal{X} \times \mathbb{R}
\] (A.12)
and
\[
f_{Y_j \mid X, Y_j > 0, Y_{3-j} = 0}(t \mid x) \leq C_{pdf}, \quad \text{for all } (j, x, t) \in \{1, 2\} \times \mathcal{X} \times (0, \infty),
\] (A.13)
and that
\[
P\left(\inf_{|t| \leq c_{de}} f_{Y_1 \mid X, Y_1 > 0, Y_2 > 0}(X^\top \theta_0 + t \mid X)P(Y_1 > 0, Y_2 > 0 \mid X) \geq 2c_f\right)
\geq 1 - \left(\frac{c_{ev}}{2\sqrt{2C_{ev}}}\right)^2. \quad (A.14)
\]
Moreover, let Assumptions 3.6, 4.2 and 4.3 hold. Finally, let \(\hat{\Theta}(\hat{\lambda}_0^{bcv})\) be the \(\ell_1\)-MEs in (1.2) arising from the BCV penalty level \(\hat{\lambda}_0^{bcv}\) in (4.12) and \(\alpha = \alpha_n\) satisfying \(\alpha_n \to 0\) and \(\ln(1/\alpha_n) \leq \ln(pn)\), and suppose that
\[
n^{1/q} \bar{B}_n s_q \eta_n^{1-q} \to 0, \quad \frac{\bar{B}_n^4 s_q (\ln(pn))^{5-q/2}(\ln n)^2}{n^{1-q/2-4/p}} \to 0 \quad \text{and} \quad \frac{\bar{B}_n^4 \ln^7 (pn)}{n} \to 0.
\]
Then
\[
\sup_{\tilde{\theta} \in \hat{\Theta}(\hat{\lambda}_0^{bcv})} \|\tilde{\theta} - \theta_0\|_2 \overset{p}{\lesssim} \sqrt{s_q \eta_n^{2-q}} \quad \text{and} \quad \sup_{\tilde{\theta} \in \hat{\Theta}(\hat{\lambda}_0^{bcv})} \|\tilde{\theta} - \theta_0\|_1 \overset{p}{\lesssim} s_q \eta_n^{1-q}.
\]
Let \(\tilde{\Theta}(\tilde{\lambda}_0^{bcv})\) be the post-\(\ell_1\)-MEs in (4.15)–(4.16) resulting from \(\hat{\Theta}(\hat{\lambda}_0^{bcv})\). If, in addition, in all examples except for Example 4 (trimmed LAD), also \(n^{1/q} \bar{B}_n s_q \eta_n^{1-q} \ln(pn) \to 0\), then
\[
\sup_{\tilde{\theta} \in \tilde{\Theta}(\tilde{\lambda}_0^{bcv})} \|\tilde{\theta} - \theta_0\|_2 \overset{p}{\lesssim} \sqrt{s_q \eta_n^{2-q} \ln(pn)} \quad \text{and} \quad \sup_{\tilde{\theta} \in \tilde{\Theta}(\tilde{\lambda}_0^{bcv})} \|\tilde{\theta} - \theta_0\|_1 \overset{p}{\lesssim} s_q \eta_n^{1-q} \ln(pn).
\]

**Corollary A.2 (Inference in Examples from Main Text).** Assume the setting of one of the examples from the main text: Example 1 (logit or probit), Example 2 (logit or probit), Example 3, or Example 4 (trimmed LS), and let all assumptions of Corollary A.1 related to these examples hold. In addition, in Example 3, assume that the conditional distribution of \(Y \mid X\) is absolutely continuous with bounded PDF, i.e., that
\[
f_{Y \mid X}(y \mid x) \leq C_{pdf} \quad \text{for all } (y, x) \in \mathbb{R} \times \mathcal{X},
\] (A.15)
and in Example 4 (trimmed LS), assume that for all \(j \in \{1, 2\}\), the conditional distribution
of $Y_j$ given $(X, Y_j > 0)$ as well as the unconditional distribution of $X^\top \theta_0$ are absolutely continuous with bounded PDFs, i.e., that

$$f_{Y_j|X, Y_j > 0}(t \mid \mathbf{x}) \leq C_{pdf} \text{ for all } (t, \mathbf{x}) \in (0, \infty) \times \mathcal{X} \text{ and } f_{X^\top \theta_0}(t) \leq C_{pdf} \text{ for all } t \in \mathbb{R}. \quad (A.16)$$

Also, let Assumptions 5.1 and 5.5 hold and suppose that $\sqrt{n} a_n^2 \to 0$, $a_n (n^{1/6} \hat{B}_n + \sqrt{\ln(pn)}) \to 0$ and $\hat{B}_n^2 \ln(pn) = o(n^{1-4/(p+8)})$. Finally, in Examples 3 and 4 (trimmed LS), suppose also that $\sqrt{n}(\hat{B}_n a_n)^{(3p+2)/(2p+2)} \to 0$. Then the debiased estimator $\hat{\beta}$ given in (5.5) satisfies

$$\frac{\sqrt{n}(\hat{\beta} - \beta_0)}{\sigma_0} \xrightarrow{D} N(0, 1), \quad \text{where } \sigma_0^2 := \frac{\mathbb{E}[(m'_1(X^\top \theta_0, Y)(D - W^\top \mu_0))^2]}{(\mathbb{E}[m''_1(X^\top \theta_0, Y)(D - W^\top \mu_0)D])^2}.$$

Corollaries A.1 and A.2 follow immediately from Theorems 4.1, 4.2 and 5.1 as long as we can verify Assumptions 3.1, 3.2, 3.3, 3.4, 3.5, 4.1, 4.4, 4.5, 5.2, 5.3 and 5.4 from the main text under low-level example-specific assumptions of these corollaries with $B_n \lesssim \hat{B}_n$, $\hat{B}_n \lesssim \hat{B}_n$, $r = \tilde{r}$, and $\tilde{r} = 8$. Also, Assumption 3.1 holds trivially as we set $\Theta = \mathbb{R}^p$ and Assumption 3.2 holds trivially in all examples as we discussed in the main text. In addition, Assumptions 4.4 and 4.5 follow immediately from Assumption 5.3 and (A.2), respectively, so we do not have to verify them separately. In the rest of this section, we verify all the remaining assumptions.

Before starting the verification process, however, we note that there are various sets of sufficient low-level example-specific assumptions. In particular, throughout this section, we do not impose any restrictions on the set $\Theta$ and assume that $\Theta = \mathbb{R}^p$, which is convenient from the implementation point of view, but one could assume, for example, that the set $\Theta$ is $\ell_1$-bounded in the sense that $\sup_{\theta \in \Theta} \|\theta\|_1$ is finite but possibly growing with $n$, and relax some of the assumptions above. For brevity, we provide results only for assumptions that are listed in Corollaries A.1 and A.2.

Let $f : \mathbb{R} \times \mathcal{X} \to \mathbb{R}$ be the function defined by $f(t, \mathbf{x}) := \mathbb{E}[m(t, Y) \mid X = \mathbf{x}]$ for $(t, \mathbf{x}) \in \mathbb{R} \times \mathcal{X}$. We will show that it exists under our conditions. Also, in this section, to emphasize dependence between constants, we will use function arguments. For example, we will write $C = C(C_{ev}, C_0)$ when the constant $C \in (0, \infty)$ may depend on $C_{ev}$ and $C_0$.

To streamline the verification process, we first state five lemmas, whose proofs can be found at the end of this section.

**Lemma A.1.** Let Assumptions 3.1, 3.2 and 3.3 hold, and suppose that inequalities (A.2) and

\footnote{Our verification in the Examples 1, 2, 3 and 4 (trimmed LS) leaves $\Delta_n \in (0, \infty)$ unrestricted. The Corollary A.2 growth condition $\sqrt{n}(B_n a_n)^{(3p+2)/(2p+2)} \to 0$ results from optimizing this choice subject to the growth conditions stated in Theorem 5.1.}
(A.3) are satisfied. Let $c_f$ and $c_M'$ be some constants in $(0, \infty)$. In addition, suppose that for all $x \in \mathcal{X}$, $t \mapsto f(t, x)$ exists as a differentiable function from $\mathbb{R}$ to $\mathbb{R}$, with its first derivative $t \mapsto f'_1(t, x)$ being Lipschitz continuous on compacta, so that (per Rademacher’s theorem) there is a (possibly empty) Lebesgue null set $N(x) \subset \mathbb{R}$ for which also the second derivatives $f''_1(t, x)$, $t \in \mathbb{R} \setminus N(x)$, exist. Moreover, suppose that $E[|f'_1(X^\top \theta, X) X_j|] < \infty$ for all $\theta \in \Theta$ and $j \in [p]$, and that at least one of the following two conditions is satisfied:

$$P \left( \inf_{t \in [-C, C]} f''_1(t, X) \geq 4c_f \right) \geq 1 - \left( \frac{c_{ev}}{2\sqrt{2C_{ev}}} \right)^2 \text{ for } C := \frac{4C_{ev}}{c_{ev}} \sqrt{C_0^2 + C_{ev}(c_M')^2}, \quad (A.17)$$

$$P \left( \inf_{t = X^\top \theta_0 + u \in [-C, C] \setminus N(x)} f''_1(t, X) \geq 4c_f \right) \geq 1 - \left( \frac{c_{ev}}{2\sqrt{2C_{ev}}} \right)^2 \text{ for } C := \frac{2\sqrt{2c_M'C_{ev}^{3/2}}}{c_{ev}}. \quad (A.18)$$

Then Assumption 3.4 holds with the given $c_M'$ and $c_M := (c_{ev}c_f) \land 1$.

Lemma A.2. Suppose that inequalities (A.2), (A.3) and (A.4) are satisfied, and let $c_{m,1}$, $c_{m,2}$, and $c_{m,3}$ be some constants in $[0, \infty)$. Also, suppose that the function $m(\cdot, y)$ is continuously differentiable for all $y \in \mathcal{Y}$ with the first derivative $m(\cdot, y)$ satisfying

$$|m'_1(t, y)| \leq c_{m,1} + c_{m,2}|t|, \quad \text{for all } (t, y) \in \mathbb{R} \times \mathcal{Y} \quad (A.19)$$

and

$$|m'_1(t_2, y) - m'_1(t_1, y)| \leq c_{m,3}|t_2 - t_1|, \quad \text{for all } (t_1, t_2, y) \in \mathbb{R} \times \mathbb{R} \times \mathcal{Y}. \quad (A.20)$$

Then Assumption 3.5 is satisfied with $L(x, y) := (1 + c_{m,1} + c_{m,2})(1 + |x^\top \theta_0|)$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, $c_L := 1$, $r := \tilde{r}$, $B_n := (1 + c_{m,1} + c_{m,2})B_n$, and $C_L^2 := \max\{1, 2(1 + c_{m,1} + c_{m,2})^2C_{ev}(1 + C_0^2), 3(c_{m,1}C_{ev} + c_{m,2}^2C_{ev}^2 + c_{m,3}^2C_{ev}^2), c_{m,3}C_{ev}^2\}$.

Lemma A.3. Suppose that all conditions of Lemma A.2 are satisfied. In addition, suppose that there is a constant $c_U \in (0, \infty)$ such that $E[|m'_1(X^\top \theta_0, Y) X_j|^2] \geq c_U^2$ for all $j \in [p]$. Then Assumption 4.1 is satisfied with the given $c_U$, $C_U^2 := \max\{1, 2(c_{m,1}C_0^2 + c_{m,2}^2C_0^4)\}$ and $	ilde{B}_n^2 := \max\{1, 8(c_{m,1} + c_{m,2})^4C_0^4(1 + C_0^4), (c_{m,1} + c_{m,2})^2\bar{B}_n^2\}$.

Lemma A.4. Suppose that all conditions of Lemma A.2 are satisfied. Then Assumption 5.2 is satisfied with $\tilde{r} := 8$ and $C_M := \max\{C_0, (c_{m,1} + c_{m,2})(1 + C_0)\}$. 

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Lemma A.5. Let $Z$ be a random variable satisfying $E[|Z|] < \infty$. Then the function $f$ defined by $f(t) := E[(Z - t)1(Z \geq t)]$ or, equivalently, $f(t) := E[(Z - t)1(Z > t)]$, is a Lipschitz continuous mapping from $\mathbb{R}$ to $\mathbb{R}$. If, in addition, $Z$ is continuously distributed, i.e. $P(Z = t) = 0$ for all $t \in \mathbb{R}$, then $f$ is differentiable with derivative $f'(t) = -P(Z > t)$ for all $t \in \mathbb{R}$.

We are now ready to verify Assumptions 3.3, 3.4, 3.5, 4.1, 5.2, 5.3 and 5.4 in each of the examples from the main text.

Example 1 (Binary Response Model, Continued). We first consider the case of the logit loss function (2.2). In this case, the differentiability part of Assumption 3.3 is trivial. In addition, for all $\theta \in \Theta$, since $1 + e^t \leq 2e^t$ for $t \geq 0$, by (A.2) we have

$$E[|m(X^\top \theta, Y)|] \leq E[\ln(1 + e^{|X^\top \theta|})] + E[|X^\top \theta|] \leq \ln 2 + 2E[|X^\top \theta|] < \infty,$$

which gives the integrability part of Assumption 3.3.

Next, recalling that $P(Y = 1|X = x) = \Lambda(x^\top \theta_0)$ for the standard logistic CDF $\Lambda(t) = 1/(1 + e^{-t})$, we see that the function $f$ defined by

$$f(t, x) = E[m(t, Y) | X = x] = \ln(1 + e^t) - \Lambda(x^\top \theta_0)t, \quad (t, x) \in \mathbb{R} \times \mathcal{X},$$

is twice continuously differentiable in its first argument. Hence, for all $x \in \mathcal{X}$, $t \mapsto f'_1(t, x)$ is Lipschitz continuous on compacta. Moreover, $f$ satisfies

$$f'_1(t, x) = \Lambda(t) - \Lambda(x^\top \theta_0) \quad \text{and} \quad f''_1(t, x) = \Lambda'(t) = \Lambda(t)(1 - \Lambda(t)),$$

which, in particular, shows that $|f'_1(t, x)| \leq 1$, that $f''_1(t, x)$ is strictly positive for all $(t, x) \in \mathbb{R} \times \mathcal{X}$ and that $f''_1(t, x)$ does not actually depend on $x$. Thus, for all $\theta \in \Theta$ and $j \in [p]$, we have $E[|f''_1(X^\top \theta, X)X_j|] \leq E[|X_j|] < \infty$ by (A.3). Now, for arbitrary $C \in (0, \infty)$, defining $c_f(C) := (1/4)\inf_{|t| \leq C} \Lambda'(t) = (1/4)\Lambda'(C) \in (0, \infty)$, we see that $f''_1(t, x) \geq 4c_f(C)$ for all $(t, x) \in [-C, C] \times \mathcal{X}$. Choosing $c'_M := 1$ and the specific $C := C(C_0, c_{ev}, C_{ev}, c'_M)$ in (A.17), by Lemma A.1 with the implied $c_f := c_f(C)$, we see that Assumption 3.4 holds with $c'_M = 1$ and $c_M = c_M(C_0, c_{ev}, C_{ev})$. 


and so \(|m_1'(t, y)| \leq 1, |m_1''(t, y)| \leq 1, \) and \(|m_{111}(t, y)| \leq 1. \) Hence, (A.19) and (by way of the mean-value theorem) (A.20) are satisfied with \(c_{m,1} = 1, c_{m,2} = 0, \) and \(c_{m,3} = 1. \) Assumption 3.5 now follows from Lemma A.2 with \(Y \) is here Bernoulli distributed with \(P(Y = 1) = \frac{1}{2}. \) The remaining parts of Assumption 4.1 follow from Lemma A.3 with \(c \) and \(C \) as defined above.

Finally, letting \(C := C_0^2/\sqrt{c_{ev}/2}, \) noting that \(U = m_1'(X^\top \theta_0, Y) = \Lambda(X^\top \theta_0) - Y \) and letting \(c = c(C_0, c_{ev}) \) be the constant \(c := \inf_{|t| \leq C} \Lambda'(t) = \Lambda'(C) \in (0, \infty), \) since \(Y|X = x \) is here Bernoulli distributed with \(P(Y = 1|X = x) = \Lambda(x^\top \theta_0), \) iterating expectations, we have for all \(j \in [p] \) that

\[
E[|UX_j|^2] = E[|X_j|^2E[|Y - \Lambda(X^\top \theta_0)|^2|X]] = E[|X_j|^2\Lambda(X^\top \theta_0)(1 - \Lambda(X^\top \theta_0))] \\
\geq cE[|X_j|^21\{|X^\top \theta_0| \leq C\}] = c(E[|X_j|^2] - E[|X_j|^21\{|X^\top \theta_0| > C\}]) \\
\geq c(c_{ev} - E[|X_j|^21\{|X^\top \theta_0| \leq C\}]) \geq c\left(c_{ev} - \left(E[|X_j|^4]E[|X^\top \theta_0|^4]\right)^{1/2}/C^2\right) \\
\geq c(c_{ev} - C_0^4/C^2) = cc_{ev}/2, 
\]

by (A.2), (A.3), and the Cauchy-Schwarz and Hölder inequalities. The final inequality in the previous display shows that the lower bound in Assumption 4.1.1 is satisfied with \(c_U = c_U(C_0, c_{ev}). \) The remaining parts of Assumption 4.1 follow from Lemma A.3 with \(C_U = C_U(C_0) \) and \(\tilde{B}_n = \tilde{B}_n \vee C_1 \) for some \(C_1 = C_1(C_0). \) This observation completes the logit case.

Next, we consider the case of the probit loss function (2.3). Let \(\Phi \) and \(\phi \) denote the standard normal CDF and PDF, respectively. We start with deriving some basic inequalities. By Proposition 2.5(b) in Dudley (2014), for all \(t \in [1, \infty), \) we have \(\phi(t)/(1 - \Phi(t)) \leq 2t, \) so

\[
\frac{1}{1 - \Phi(t)} \leq 2\sqrt{2\pi}te^{t^2/2} \leq 2\sqrt{2\pi}e^{t^2/2}. 
\]
Also, for all $t \in (-\infty, 1)$, we have $1/(1 - \Phi(t)) \leq 1/(1 - \Phi(1))$. Hence,

$$|\ln(1 - \Phi(t))| \leq |\ln(1 - \Phi(1))| + \ln(2\sqrt{2\pi}) + |t| + \frac{t^2}{2},$$

for all $t \in \mathbb{R}$, \hspace{1cm} (A.21)

and, using the symmetry of the standard normal distribution to get $\Phi(t) = 1 - \Phi(-t)$,

$$|\ln(\Phi(t))| \leq |\ln(1 - \Phi(1))| + \ln(2\sqrt{2\pi}) + |t| + \frac{t^2}{2},$$

for all $t \in \mathbb{R}$. \hspace{1cm} (A.22)

In addition, for all $t \in (-\infty, 1)$, we have $\phi(t)/(1 - \Phi(t)) \leq \phi(0)/(1 - \Phi(1))$. Hence, both

$$\phi(t) \cdot \frac{1}{1 - \Phi(t)} \leq 2|t| \quad \text{and} \quad \frac{\phi(t)}{\Phi(t)} \leq \frac{\phi(0)}{1 - \Phi(1)} + 2|t|,$$

for all $t \in \mathbb{R}$, \hspace{1cm} (A.23)

where the second inequality follows from the first, $\Phi(t) = 1 - \Phi(-t)$ and $\phi(t) = \phi(-t)$. Moreover, by (1.2.2) in Adler and Taylor (2007), for all $t \in (0, \infty)$, we have $\phi(t)/(1 - \Phi(t)) > t$, and so

$$\frac{\phi(t)}{1 - \Phi(t)} - t > 0 \quad \text{and} \quad \frac{\phi(t)}{\Phi(t)} + t > 0,$$

for all $t \in \mathbb{R}$, \hspace{1cm} (A.24)

where the second inequality again follows from the first and symmetry. Again, by (1.2.2) in Adler and Taylor (2007), for all $t \in (1, \infty)$, we have

$$0 < \frac{\phi(t)}{1 - \Phi(t)} - t \leq \frac{1}{1/t - 1/t^3} - t = \frac{t}{t^2 - 1}.$$

The previous display shows that for $t \in (2, \infty),

$$\frac{\phi(t)}{1 - \Phi(t)} \cdot \left|\frac{\phi(t)}{1 - \Phi(t)} - t\right| \leq 2t \cdot \frac{t}{t^2 - 1} = \frac{2t^2}{t^2 - 1} \leq \frac{8}{3}.$$

The left-hand side (continuous) function is bound on compacta, including $[-2, 2]$. Finally, for $t \in (-\infty, 2)$, as $\phi$ is bounded, $1 - \Phi(\cdot)$ is bounded away from zero, and $\phi(t)$ decays more rapidly than $-t$ grows as $t \to -\infty$, the same left-hand side function remains bounded on $(-\infty, 2)$ as well. Conclude that there is a universal constant $C \in [1, \infty)$ such that

$$\frac{\phi(t)}{1 - \Phi(t)} \left|\frac{\phi(t)}{1 - \Phi(t)} - t\right| \leq C \quad \text{and} \quad \frac{\phi(t)}{\Phi(t)} \left|\frac{\phi(t)}{\Phi(t)} + t\right| \leq C,$$

for all $t \in \mathbb{R}$, \hspace{1cm} (A.25)

where the second inequality follows from symmetry and parallel reasoning. Finally, both

$$\lim_{t \to \infty} t^2 \left(\frac{\phi(t)}{1 - \Phi(t)} - t - \frac{1}{t}\right) = 0 \quad \text{and} \quad \lim_{t \to -\infty} t^2 \left(\frac{\phi(t)}{\Phi(t)} + t + \frac{1}{t}\right) = 0 \hspace{1cm} (A.26)$$

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which both follow from repeated application of L'Hôpital's rule.

With these inequalities in mind, we now verify the required assumptions. The differentiability part of Assumption 3.3 is trivial. In addition, for all \( \theta \in \Theta \), we have

\[
E[|m(X^\top \theta, Y)|] \leq E[|\ln(\Phi(X^\top \theta))| + |\ln(1 - \Phi(X^\top \theta))|] < \infty
\]

by (A.21), (A.22), the Cauchy-Schwarz inequality and (A.2), which yields also the integrability part of Assumption 3.3.

Next, viewed as a function of its first argument, the function \( f \) defined by

\[
f(t, x) = E[m(t, Y) | X = x] = -\Phi(x^\top \theta_0) \ln(\Phi(t)) - (1 - \Phi(x^\top \theta_0)) \ln(1 - \Phi(t)),
\]

for \((t, x) \in \mathbb{R} \times \mathcal{X}\), is seen to be twice continuously differentiable in its first argument, with

\[
f'_1(t, x) = -\Phi(x^\top \theta_0) \frac{\phi(t)}{\Phi(t)} + (1 - \Phi(x^\top \theta_0)) \frac{\phi(t)}{1 - \Phi(t)} \quad \text{and}
\]

\[
f''_{11}(t, x) = \Phi(x^\top \theta_0) \frac{\phi(t)}{\Phi(t)} \left(t + \frac{\phi(t)}{\Phi(t)}\right) + (1 - \Phi(x^\top \theta_0)) \frac{\phi(t)}{1 - \Phi(t)} \left(\frac{\phi(t)}{1 - \Phi(t)} - t\right) > 0.
\]

Thus, for all \( \theta \in \Theta \) and \( j \in [p] \), we have

\[
E[|f'_1(X^\top \theta, X)X_j|] \leq E \left[ \left( \frac{\phi(X^\top \theta)}{\Phi(X^\top \theta)} + \frac{\phi(X^\top \theta)}{1 - \Phi(X^\top \theta)} \right) |X_j| \right] < \infty
\]

by (A.23), (A.2), (A.3), and the Cauchy-Schwarz inequality. Also, by the positivities in (A.24), for arbitrary \( C \in (0, \infty) \), defining

\[
c_f(C) := \frac{1}{4} \min \left\{ \inf_{|t| \leq C} \frac{\phi(t)}{\Phi(t)} \left(t + \frac{\phi(t)}{\Phi(t)}\right), \inf_{|t| \leq C} \frac{\phi(t)}{1 - \Phi(t)} \left(\frac{\phi(t)}{1 - \Phi(t)} - t\right) \right\} \in (0, \infty)
\]

we see that \( f''_{11}(t, x) \geq 4c_f(C) \) for all \( t \in [-C, C] \times \mathcal{X} \). Choosing \( c'_M := 1 \) and the specific \( C := C(C_0, c_{ev}, C_{ev}, c'_M) \) in (A.17), by Lemma A.1 with the implied \( c_f := c_f(C) \), we see that Assumption 3.4 holds with \( c'_M = 1 \) and \( c_M = c_M(C_0, c_{ev}, C_{ev}) \).
Next, for all \((t, y) \in \mathbb{R} \times \mathcal{Y}\), differentiation shows that

\[
m'_1(t, y) = -y \frac{\phi(t)}{\Phi(t)} + (1 - y) \frac{\phi(t)}{1 - \Phi(t)},
\]

\[
m''_{11}(t, y) = y \frac{\phi(t)}{\Phi(t)} \left( t + \frac{\phi(t)}{\Phi(t)} \right) + (1 - y) \frac{\phi(t)}{1 - \Phi(t)} \left( \frac{\phi(t)}{1 - \Phi(t)} - t \right)
\]

and

\[
m'''_{111}(t, y) = y \frac{\phi(t)}{\Phi(t)} \left\{ 1 - \left( \frac{\phi(t)}{\Phi(t)} + t \right) \left( 2\phi(t) + t \right) \right\}
\]

\[
+ (1 - y) \frac{\phi(t)}{1 - \Phi(t)} \left\{ \left( \frac{\phi(t)}{1 - \Phi(t)} - t \right) \left( 2\phi(t) + t \right) - 1 \right\}.
\]

Therefore, for all \((t, y) \in \mathbb{R} \times \mathcal{Y}\), \(|m'_1(t, y)| \leq \phi(0)/(1 - \Phi(1)) + 2|t|\) by (A.23) and \(|m''_{11}(t, y)| \leq C\) for some universal constant \(C \in [1, \infty)\) by (A.25). Hence, both (A.19) and (via the mean-value theorem) (A.20) are satisfied with \(c_{m,1} = \phi(0)/(1 - \Phi(1))\), \(c_{m,2} = 2\), and \(c_{m,3} = C\), and so Assumption 3.5 follows from Lemma A.2 with \(L(x, y) = (3 + \phi(0)/(1 - \Phi(1)))(1 + |x^\top \theta_0|)\) for \((x, y) \in \mathcal{X} \times \mathcal{Y}\), \(c_L = 1\), \(r = \bar{r}\), \(B_n = (3 + \phi(0)/(1 - \Phi(1)))B_n\), and \(C_L = C_L(C_0, C_{ev})\).

Also, as \(t \to \infty\), uniformly over \(y \in \mathcal{Y}\),

\[
|m'''_{111}(t, y)| = o(1) + (t + o(t)) \left| (1/t + o(1/t^2))(t + 2/t + o(1/t^2)) - 1 \right| = o(1)
\]

and as \(t \to -\infty\), uniformly over \(y \in \mathcal{Y}\),

\[
|m'''_{111}(t, y)| = (-t + o(|t|)) \left| 1 - (-1/t + o(1/t^2))(-t - 2/t + o(1/t^2)) \right| + o(1) = o(1)
\]

using the limits in (A.26). Since each \(t \mapsto m'''_{111}(t, y), y \in \mathcal{Y}\), is continuous, from the previous two displays we deduce boundedness of \(t \mapsto m'''_{111}(t, y)\) uniformly in \(y \in \mathcal{Y}\). Hence, Assumption 5.3 is satisfied with \(J = 1\) and some universal constant \(C_m \in [1, \infty)\). Since \(J = 1\), Assumption 5.4 is trivially satisfied. Moreover, Assumption 5.2 follows from Lemma A.4 with \(\bar{r} = 8\) and \(C_M = C_M(C_0)\).

Finally, let \(C := C_0^2/\sqrt{c_{ev}/2}\) and let \(c = c(C_0, c_{ev})\) be the constant

\[
c := \inf_{|t| \leq C} \frac{\phi(t)^2}{\Phi(t)[1 - \Phi(t)]} \in (0, \infty).
\]

Since \(Y|X = x\) is here Bernoulli distributed with \(P(Y = 1|X = x) = \Phi(x^\top \theta_0)\), we have

\[
U = m'_1(X^\top \theta_0, Y) = \frac{\phi(X^\top \theta_0)}{\Phi(X^\top \theta_0)[1 - \Phi(X^\top \theta_0)]} \left[ \Phi(X^\top \theta_0) - Y \right].
\]
Iterating expectations, we see that for all $j \in [p],$

$$E[|UX_j|^2] = E\left[\frac{\phi(X^T\theta_0)^2}{\Phi(X^T\theta_0)^2[1 - \Phi(X^T\theta_0)]^2}|X_j|^2E\left[|Y - \Phi(X^T\theta_0)|^2|X\right]\right]$$
$$= E\left[\frac{\phi(X^T\theta_0)^2}{\Phi(X^T\theta_0)[1 - \Phi(X^T\theta_0)]}|X_j|^2\right]$$
$$\geq cE[|X_j|^2 1\{|X^T\theta_0| \leq C\}] = c\left(E[|X_j|^2] - E[|X_j|^2 1\{|X^T\theta_0| > C\}]\right)$$
$$\geq c\left(c_{ev} - E[|X_j|^2|X^T\theta_0|^2]/C^2\right) \geq c\left(c_{ev} - (E[|X_j|^4]E[|X^T\theta_0|^4])^{1/2}/C^2\right)$$
$$\geq c(c_{ev} - C_0^4/C^2) = cc_{ev}/2,$$

by (A.2), (A.3), the Cauchy-Schwarz and Hölder inequalities, and the choice of $C$. The final inequality in the previous display shows that the lower bound in Assumption 4.1.1 is satisfied with $c_{U'} = c_U(C_0, c_{ev})$. The remaining parts of Assumption 4.1 follow from Lemma A.3 with $C_U = C_U(C_0)$, and $\tilde{B}_n = C_1\tilde{B}_n \vee C_2$, where $C_1 \in [1, \infty)$ is a universal constant and $C_2 = C_2(C_0)$. This observation completes the probit case and, thus, the example. \qed

**Example 2 (Ordered Response Model, Continued).** We first consider the logit case. Here the CDF $F$ is of the logistic form, $F(t) = \Lambda(t) = 1/(1 + e^{-t})$, which has PDF $\Lambda'(t) = \Lambda(t)[1 - \Lambda(t)] = e^{-t}/(1 + e^{-t})^2$. In this case, the differentiability part of Assumption 3.3 is trivial. Also, for all $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$, by the Mean Value Theorem, there is a $\tau \in (t_1, t_2)$ such that

$$|\ln(\Lambda(t_2) - \Lambda(t_1))| = \ln\left(\frac{1}{\Lambda(t_2) - \Lambda(t_1)}\right) = \ln\left(\frac{1}{\Lambda'(\tau)(t_2 - t_1)}\right)$$
$$= \ln\left(\frac{1}{(1 + e^\tau)^2}\right) - \ln(t_2 - t_1)$$
$$\leq 2\ln(1 + e^{|\tau|}) + |\tau| - \ln(t_2 - t_1)$$
$$\leq 2\ln 2 + 3(|t_1| \vee |t_2|) - \ln(t_2 - t_1). \quad (A.27)$$

Moreover, since $1 + e^t \leq 2e^t$ for $t \geq 0$, for any $t \in \mathbb{R}$, we have

$$|\ln(\Lambda(t))| \vee |\ln(1 - \Lambda(t))| \leq \ln 2 + |t|. \quad (A.28)$$

Hence, for all $\theta \in \Theta$, using the bound (A.27) to control the terms with $v \in [V - 1]$ and (A.28) for the end cases $v \in \{0, V\}$, from (A.2), we see that

$$E\left[|m(X^T\theta, Y)|\right] \leq \sum_{v=0}^{V} E\left[|\ln(\alpha_{v+1} - X^T\theta) - \Lambda(\alpha_v - X^T\theta)|\right] < \infty.$$
(Recall that we interpret \( \Lambda(+\infty) \) as one and \( \Lambda(-\infty) \) as zero.) The previous display shows the integrability part of Assumption 3.3.

For each \( t \in \mathbb{R} \), \( m(t, Y) \) is integrable. Hence, for any \( x \in \mathcal{X} \), \( t \mapsto f(t, x) := E[m(t, Y)|X = x] \) is a well-defined function from \( \mathbb{R} \) to \( \mathbb{R} \), given by

\[
f(t, x) = -\sum_{v=0}^{V} [\Lambda(\alpha_{v+1} - x^\top \theta_0) - \Lambda(\alpha_v - x^\top \theta_0)] \ln (\Lambda(\alpha_{v+1} - t) - \Lambda(\alpha_v - t)).
\]

We see that \( t \mapsto f(t, x) \) is twice continuously differentiable, with derivatives

\[
f'_1(t, x) = \sum_{v=0}^{V} [\Lambda(\alpha_{v+1} - x^\top \theta_0) - \Lambda(\alpha_v - x^\top \theta_0)] [1 - \Lambda(\alpha_{v+1} - t) - \Lambda(\alpha_v - t)]
\]

and

\[
f''_{11}(t, x) = \sum_{v=0}^{V} [\Lambda(\alpha_{v+1} - x^\top \theta_0) - \Lambda(\alpha_v - x^\top \theta_0)] [\Lambda'(\alpha_{v+1} - t) + \Lambda'(\alpha_v - t)].
\]

Thus, for each \( \theta \in \Theta \) and \( j \in [p] \), we have \( E[|f'_1(X^\top \theta, X_j)|] \leq E[|X_j|] < \infty \) by (A.3). Also, for arbitrary \( C \in (0, \infty) \), gathering the cut-off points in \( \alpha := (\alpha_1, \ldots, \alpha_V)^\top \) and defining

\[
c_f(\alpha, C) := \frac{1}{4} \inf \min_{|t| \leq C} \{\Lambda'(\alpha_{v+1} - t) + \Lambda'(\alpha_v - t)\} \in (0, \infty),
\]

we see that, for all \( (t, x) \in [-C, C] \times \mathcal{X} \),

\[
f''_{11}(t, x) \geq \sum_{v=0}^{V} [\Lambda(\alpha_{v+1} - x^\top \theta_0) - \Lambda(\alpha_v - x^\top \theta_0)] [\Lambda'(\alpha_{v+1} - t) + \Lambda'(\alpha_v - t)]
\]

\[
\geq 4c_f(\alpha, C) \times \sum_{v=0}^{V} [\Lambda(\alpha_{v+1} - x^\top \theta_0) - \Lambda(\alpha_v - x^\top \theta_0)] = 4c_f(\alpha, C),
\]

where the equality follows from the probability differences summing to one. Choosing the specific \( C := C(C_0, c_{ev}, C_{ev}, c'_M) \) in (A.17) and \( c'_M := 1 \), by Lemma A.1 with the implied \( c_f := c_f(\alpha, C) \), we see that Assumption 3.4 holds with \( c'_M = 1 \) and \( c_M = c_M(\alpha, C_0, c_{ev}, C_{ev}) \).
Next, differentiating thrice shows that, for all $(t, y) \in \mathbb{R} \times \mathcal{Y}$,\[ m'_1(t, y) = \sum_{v=0}^{V} 1(y = v) \left[ 1 - \Lambda(\alpha_{v+1} - t) - \Lambda(\alpha_v - t) \right], \]
\[ m''_{11}(t, y) = \sum_{v=0}^{V} 1(y = v) \left[ \Lambda'(\alpha_{v+1} - t) + \Lambda'(\alpha_v - t) \right] \text{ and} \]
\[ m'''_{111}(t, y) = -\sum_{v=0}^{V} 1(y = v) \left( \Lambda'(\alpha_{v+1} - t) \left[ 1 - 2\Lambda(\alpha_{v+1} - t) \right] + \Lambda'(\alpha_v - t) \left[ 1 - 2\Lambda(\alpha_v - t) \right] \right), \]
from which we deduce that $|m'_1(t, y)| \leq 1$, $|m''_{11}(t, y)| \leq 2$, and $|m'''_{111}(t, y)| \leq 2$ for all $(t, y) \in \mathbb{R} \times \mathcal{Y}$. Hence, (A.19) and (A.20) are satisfied with $c_{m,1} = 1$, $c_{m,2} = 0$, and $c_{m,3} = 2$, and so Assumption 3.5 follows from Lemma A.2 with $(\alpha, \theta) \in \mathcal{X} \times \mathcal{Y}$, $c_L = 1$, $r = \bar{r}$, $B_a = 2B_n$, and $C_L = C_L(C_0, C_{ev})$. Also, Assumption 5.3 is satisfied with $J = 1$ and $C_m = 2$, and Assumption 5.4 holds trivially (because $J = 1$). Moreover, Assumption 5.2 follows from Lemma A.4 with $\bar{r} = 8$ and $C_M = C_M(C_0)$.

For arbitrary $C \in (0, \infty)$, defining \[ c(\alpha, C) := \inf_{|t| \leq C} \{ \Lambda(\alpha_1 - t) \left[ 1 - \Lambda(\alpha_1 - t) \right]^2 \} \in (0, \infty), \]
since all summands are non-negative, we must have \[ \inf_{|t| \leq C} \sum_{v=0}^{V} \left[ \Lambda(\alpha_{v+1} - t) - \Lambda(\alpha_v - t) \right] \left[ 1 - \Lambda(\alpha_{v+1} - t) - \Lambda(\alpha_v - t) \right]^2 \]
\[ \geq \inf_{|t| \leq C} \left[ \Lambda(\alpha_1 - t) - \Lambda(\alpha_0 - t) \right] \left[ 1 - \Lambda(\alpha_1 - t) - \Lambda(\alpha_0 - t) \right]^2 = c(\alpha, C), \]
where we have used that $\alpha_0 = -\infty$ implies $\Lambda(\alpha_0 - t) = 0$ for all $t \in \mathbb{R}$. For this example, \[ U = m'_1(X^\top \theta_0, Y) = \sum_{v=0}^{V} 1(Y = v) \left[ 1 - \Lambda(\alpha_{v+1} - X^\top \theta_0) - \Lambda(\alpha_v - X^\top \theta_0) \right], \]
such that its square \[ U^2 = \sum_{v=0}^{V} 1(Y = v) \left[ 1 - \Lambda(\alpha_{v+1} - X^\top \theta_0) - \Lambda(\alpha_v - X^\top \theta_0) \right]^2. \]

Hence, for the specific $C := C_0^2/\sqrt{c_{ev}}/2$ and the implied $c := c(\alpha, C) = c(\alpha, C_0, c_{ev})$, for all
By L'Hôpital's rule, we see that

\[ \phi(\cdot) \]

where we have inserted \( j \in [p] \), by iterated expectations, (A.2), (A.3), and the Cauchy-Schwarz inequality, we get

\[
E[|UX_j|^2] = E \left[ |X_j|^2 \sum_{v=0}^{\nu} \left[ \Lambda(\alpha_{v+1} - X^{\top} \theta_0) - \Lambda(\alpha_v - X^{\top} \theta_0) \right] \right]
\]

\[
\times \left[ 1 - \Lambda(\alpha_{v+1} - X^{\top} \theta_0) - \Lambda(\alpha_v - X^{\top} \theta_0) \right]^2 \]

\[
\geq c E[|X_j|^2 1 \{|X^{\top} \theta_0| \leq C \}] = c \left( E[|X_j|^2] - E \left[ |X_j|^2 1 \{|X^{\top} \theta_0| > C \} \right] \right)
\]

\[
\geq c \left( c_{ev} - E \left[ |X_j|^2 \right] E[|X^{\top} \theta_0|^2] / C^2 \right) \geq c \left( c_{ev} - (E[|X_j|^4]E[|X^{\top} \theta_0|^4])^{1/2} / C^2 \right)
\]

\[
\geq c \left( c_{ev} - C_0^4 / C^2 \right) = cc_{ev}/2.
\]

The final inequality provides the lower bound in Assumption 4.1.1 for \( c_U = c_U(\alpha, C_0, c_{ev}) \).

The remaining parts of Assumption 4.1 follow from Lemma A.3 with \( C_U = C_U(C_0) \) and \( \bar{B}_n = \bar{B}_n \lor C_1 \) for \( C_1 = C_1(C_0) \). This observation completes the logit case.

Next, we consider the probit case. Here the CDF \( F \) is of the standard normal form \( F(t) = \Phi(t) \). We first derive several basic inequalities for later reference. First, we show that

\[
|\ln (\Phi(t_2) - \Phi(t_1))| \leq \frac{(|t_1| \lor |t_2|)^2}{2} - \ln \left( \frac{t_2 - t_1}{\sqrt{2\pi}} \right) \quad \text{for } t_1, t_2 \in \mathbb{R} \text{ such that } t_1 < t_2. \quad (A.29)
\]

To this end, note that by the Mean Value Theorem, for some \( \tau \in (t_1, t_2) \) we have

\[
|\ln (\Phi(t_2) - \Phi(t_1))| = \ln \left( \frac{1}{\Phi(t_2) - \Phi(t_1)} \right) = \ln \left( \frac{1}{\phi(\tau)(t_2 - t_1)} \right)
\]

\[
= \frac{\tau^2}{2} - \ln \left( \frac{t_2 - t_1}{\sqrt{2\pi}} \right) \leq \frac{(|t_1| \lor |t_2|)^2}{2} - \ln \left( \frac{t_2 - t_1}{\sqrt{2\pi}} \right),
\]

where we have inserted \( \phi(t) = (2\pi)^{-1/2}e^{-t^2/2} \). The inequality (A.29) follows.

Second, we show that

\[
\frac{\phi(t_1) - \phi(t_2)}{\Phi(t_2) - \Phi(t_1)} \geq t_1 \quad \text{for } t_1, t_2 \in \mathbb{R} \text{ such that } t_1 < t_2. \quad (A.30)
\]

To do so, fix any \( t_1 \in \mathbb{R} \) and let \( g: (t_1, \infty) \to \mathbb{R} \) be the function defined by

\[
g(t) := \frac{\phi(t_1) - \phi(t)}{\Phi(t) - \Phi(t_1)}, \quad t \in (t_1, \infty). \quad (A.31)
\]

By L'Hôpital's rule, we see that

\[
\lim_{t \downarrow t_1} g(t) = t_1, \quad (A.32)
\]
which allows us to continuously extend the domain of $g$ to include $t_1$. Henceforth, we therefore interpret $g$ as the resulting function from $[t_1, \infty)$ to $\mathbb{R}$.

Suppose for the moment that $g'(t) \geq 0$ for all $t \in (t_1, \infty)$. By the Fundamental Theorem of Calculus, for $t, t' \in (t_1, \infty)$ satisfying $t \leq t'$, we must have

$$g(t) - g(t') = \int_t^{t'} g'(u) \, du \geq \int_t^{t'} 0 \, du = t - t' \geq 0.$$ 

Rearranging this inequality to get $g(t) \geq g(t')$ and taking limits as $t' \downarrow t_1$ gives $g(t) \geq g(t_1) = t_1$. To establish (A.30), it thus suffices to show that $g'(t) \geq 0$ for all $t \in (t_1, \infty)$. Differentiating $g$ shows

$$g'(t) = \frac{\phi(t)}{\Phi(t) - \Phi(t_1)} [t - g(t)] \text{ for } t \in (t_1, \infty),$$

so, on $(t_1, \infty)$, $g'(t) \geq 0$ if and only if $t \geq g(t)$. To show the latter claim, let $f : [t_1, \infty) \rightarrow \mathbb{R}$ be the function defined by $f(t) := g(t) - t$ for $t \in [t_1, \infty)$. Then $f$ is continuous, and $f(t_1) = 0$. Seeking a contradiction, suppose that there is a $t_2 \in (t_1, \infty)$ such that $f(t_2) > 0$. Let $T := \{t \in [t_1, t_2]: f(t) = 0\}$ be the roots of $f$ in $[t_1, t_2]$, which includes at least $t_1$. Per continuity of $f$, $T$ is closed, so $\bar{t} := \sup T$ lies in $T$. Since $f(t_2) > 0$ by supposition, per continuity of $f$ and definition of $\bar{t}$, we must have $f(t) > 0$ for all $t \in (\bar{t}, t_2)$. The Mean Value Theorem then implies that there is a $\tilde{t} \in (\bar{t}, t_2)$ such that

$$f'(\tilde{t}) = \frac{f(t_2) - f(\tilde{t})}{t_2 - \tilde{t}} = \frac{f(t_2)}{t_2 - \tilde{t}} > 0.$$

On the other hand, for any such mean value $\tilde{t}$, differentiation and $f(\tilde{t}) > 0$ imply that

$$f'(\tilde{t}) = g'(\tilde{t}) - 1 = \frac{\phi(\tilde{t})}{\Phi(\tilde{t}) - \Phi(t_1)} [-f(\tilde{t})] - 1 < 0,$$

a contradiction. Hence, it must be that $f(t) \leq 0$ for all $t \in (t_1, \infty)$, which means that $g(t) \leq t$ for all $t \in (t_1, \infty)$, as desired. This observation completes the proof of (A.30).

Third, we show that

$$\frac{t_2\phi(t_2) - t_1\phi(t_1)}{\Phi(t_2) - \Phi(t_1)} + \left( \frac{\phi(t_2) - \phi(t_1)}{\Phi(t_2) - \Phi(t_1)} \right)^2 > 0, \text{ for all } t_1, t_2 \in \mathbb{R} \text{ such that } t_1 < t_2. \quad (A.33)$$

To do so, we consider the three possible cases: (i) $t_2 > t_1 \geq 0$, (ii) $t_1 < t_2 \leq 0$, and (iii)
$t_1 < 0 < t_2$. In the first case, we have

$$\frac{t_2 \phi(t_2) - t_1 \phi(t_1)}{\Phi(t_2) - \Phi(t_1)} + \left( \frac{\phi(t_2) - \phi(t_1)}{\Phi(t_2) - \Phi(t_1)} \right)^2 > \frac{t_1 \phi(t_2) - t_1 \phi(t_1)}{\Phi(t_2) - \Phi(t_1)} + \left( \frac{\phi(t_2) - \phi(t_1)}{\Phi(t_2) - \Phi(t_1)} \right)^2$$

$$= \frac{\phi(t_1) - \phi(t_2)}{\Phi(t_2) - \Phi(t_1)} \left( \frac{\phi(t_1) - \phi(t_2)}{\Phi(t_2) - \Phi(t_1)} - t_1 \right) \geq 0$$

by (A.30). In the second case, we have

$$\frac{t_2 \phi(t_2) - t_1 \phi(t_1)}{\Phi(t_2) - \Phi(t_1)} + \left( \frac{\phi(t_2) - \phi(t_1)}{\Phi(t_2) - \Phi(t_1)} \right)^2 = \frac{|t_1| \phi(|t_1|) - |t_2| \phi(|t_2|)}{\Phi(|t_1|) - \Phi(|t_2|)} + \left( \frac{\phi(|t_1|) - \phi(|t_2|)}{\Phi(|t_1|) - \Phi(|t_2|)} \right)^2 > 0,$$

arguing as in the first case and using $|t_1| > |t_2| \geq 0$. In the third case, (A.33) is immediate from the first fraction being positive. Since we have considered all cases, (A.33) follows.

Fourth, we show that

$$\frac{|\phi(t_2) - \phi(t_1)|}{\Phi(t_2) - \Phi(t_1)} \leq \frac{\phi(0)}{1 - \Phi(1)} + 2 (|t_1| \wedge |t_2|), \quad \text{for all } t_1, t_2 \in \mathbb{R} \text{ such that } t_1 < t_2. \quad (A.34)$$

To do so, we again consider the three possible cases: (i) $t_2 > t_1 \geq 0$, (ii) $t_1 < t_2 \leq 0$, and (iii) $t_1 < 0 < t_2$. In the first case, we have

$$\frac{|\phi(t_2) - \phi(t_1)|}{\Phi(t_2) - \Phi(t_1)} = \frac{\phi(t_2) - \phi(t_1)}{\Phi(t_2) - \Phi(t_1)} \leq \frac{\phi(t_1)}{1 - \Phi(t_1)} \leq \frac{\phi(0)}{1 - \Phi(1)} + 2 (|t_1| \wedge |t_2|),$$

where the first inequality follows from $t_2 \mapsto g(t_2)$ in (A.31) being non-decreasing on $[t_1, \infty)$ and taking the limit as $t_2 \to \infty$, and the second from (A.23). In the second case, we have

$$\frac{|\phi(t_2) - \phi(t_1)|}{\Phi(t_2) - \Phi(t_1)} = \frac{\phi(|t_2|) - \phi(|t_1|)}{\Phi(|t_2|) - \Phi(|t_1|)} \leq \frac{\phi(|t_2|)}{1 - \Phi(|t_2|)} \leq \frac{\phi(0)}{1 - \Phi(1)} + 2 (|t_1| \wedge |t_2|)$$

by the same arguments. Consider now the third case, where $t_1 < 0 < t_2$. If $-t_1 = t_2$, then the left-hand side of (A.34) is zero per symmetry of the standard normal distribution about zero, and the bound is trivial. The remaining two subcases $-t_1 < t_2$ or $-t_1 > t_2$, can be handled exactly as in the above. Since we have considered all cases, (A.34) follows.

Fifth, we show that for any $\Delta \in (0, \infty)$,

$$\lim_{t \to \infty} t^2 \left( \frac{\phi(t) - \phi(t + \Delta)}{\Phi(t + \Delta) - \Phi(t)} - t - \frac{1}{t} \right) = 0. \quad (A.35)$$

To do so, fix any $\Delta \in (0, \infty)$ and observe that for all $t \in [1, \infty)$, we have $t \leq \phi(t)/[1 - \Phi(t)] \leq \phi(t)$.
2t by Proposition 2.5(b) in Dudley (2014). Hence, again for all \( t \in [1, \infty) \),

\[
\frac{1 - \Phi(t + \Delta)}{1 - \Phi(t)} = \frac{1 - \Phi(t + \Delta)}{\phi(t + \Delta)} \cdot \frac{\phi(t)}{1 - \Phi(t)} \cdot \frac{\phi(t + \Delta)}{\phi(t)} \leq \frac{1}{t + \Delta} \cdot 2t \cdot \frac{\phi(t + \Delta)}{\phi(t)},
\]

such that, given that \( \phi(t + \Delta)/\phi(t) = o(1/t^3) \) as \( t \to \infty \), we get

\[
\frac{1 - \Phi(t + \Delta)}{1 - \Phi(t)} = o(1/t^3) \quad \text{as} \quad t \to \infty.
\]

It follows that

\[
\left| \frac{\phi(t) - \phi(t + \Delta)}{\Phi(t + \Delta) - \Phi(t)} - \frac{\phi(t)}{1 - \Phi(t)} \right| = \frac{\phi(t)}{1 - \Phi(t)} \left| \frac{1 - \phi(t+\Delta)}{1 - \frac{\Phi(t + \Delta)}{\phi(t)}} - 1 \right| = o(1/t^2) \quad \text{as} \quad t \to \infty,
\]

which in combination with (A.26) gives (A.35).

Sixth, we show that for any \( \Delta \in (0, \infty) \),

\[
\frac{t \Delta \phi(t + \Delta)}{\Phi(t + \Delta) - \Phi(t)} \to 0 \quad \text{as} \quad t \to \infty.
\]  

(A.36)

To do so, observe that by a change of variables, for \( t \in (0, \infty) \) we have

\[
\frac{\Phi(t + \Delta) - \Phi(t)}{\phi(t + \Delta)} = \int_0^\Delta \frac{\phi(t + s)}{\phi(t + \Delta)} ds \geq \int_0^\Delta e^{t\Delta - ts} ds = \int_0^\Delta e^{ts} ds = \frac{e^{t\Delta} - 1}{t}.
\]

The claim (A.36) follows from rearranging, multiplying by \( t\Delta \), and taking the limit as \( t \to \infty \).

With these bounds in mind, we now verify the required assumptions. (Throughout, we continue to interpret \( \Phi(+\infty) \) as one and \( \Phi(-\infty), \phi(-\infty) \) and \( \phi(+\infty) \) as zero.) The differentiability part of Assumption 3.3 is immediate. In addition, for all \( \theta \in \Theta \), we have

\[
E \left[ m(X^\top \theta, Y) \right] \leq \sum_{v=0}^V E \left[ \ln \left( \Phi(\alpha_{v+1} - X^\top \theta) - \Phi(\alpha_v - X^\top \theta) \right) \right] < \infty
\]

where we use (A.21) and (A.22) for the cases \( v \in \{0, V\} \) and (A.29) for the cases \( v \in [V - 1] \), and (A.2). The integrability part of Assumption 3.3 follows.

Next, for all \( t \in \mathbb{R}, x \in X \) and \( v \in \{0, 1, \ldots, V\} \), denote

\[
\psi_v(t) := \frac{\phi(\alpha_v - t) - \phi(\alpha_{v+1} - t)}{\Phi(\alpha_{v+1} - t) - \Phi(\alpha_v - t)} \quad \text{and} \quad \kappa_v(x) := \Phi(\alpha_{v+1} - x^\top \theta_0) - \Phi(\alpha_v - x^\top \theta_0),
\]

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so that
\[
\lim_{t \to -\infty} (\alpha_v - t)^2 \left( \psi_v(t) - (\alpha_v - t) - \frac{1}{\alpha_v - t} \right) = 0
\] (A.37)
for all \( v \in [V - 1] \) by (A.35). For each \( t \in \mathbb{R}, m(t,Y) \) is integrable. Hence, for any \( x \in \mathcal{X}, t \mapsto f(t,x) := \mathbb{E}[m(t,Y) | X = x] \) is a well-defined function from \( \mathbb{R} \) to \( \mathbb{R} \), namely
\[
f(t,x) = -\sum_{v=0}^{V} \kappa_v(x) \ln \left( \Phi(\alpha_{v+1} - t) - \Phi(\alpha_v - t) \right).
\]
We see that \( t \mapsto f(t,x) \) is twice continuously differentiable, with derivatives
\[
f_1'(t,x) = \sum_{v=0}^{V} \kappa_v(x) [-\psi_v(t)] \quad \text{and} \quad f_1''(t,x) = \sum_{v=0}^{V} \kappa_v(x) \left\{ \frac{(\alpha_{v+1} - t)\phi(\alpha_{v+1} - t) - (\alpha_v - t)\phi(\alpha_v - t)}{\Phi(\alpha_{v+1} - t) - \Phi(\alpha_v - t)} + \psi_v(t)^2 \right\},
\]
where the terms \((\alpha_{v+1} - t)\phi(\alpha_{v+1} - t)\) and \((\alpha_v - t)\phi(\alpha_v - t)\) are understood to be zero for all \( t \in \mathbb{R} \). Thus, for all \( \theta \in \Theta \) and \( j \in [p] \), we have
\[
\mathbb{E} \left[ |f_1'(X^\top \theta, X)_j| \right] \leq \mathbb{E} \left[ |X_j| \sum_{v=0}^{V} \left| \psi_v(X^\top \theta_0) \right| \right] < \infty
\]
by (A.34), (A.2), (A.3), and the Cauchy-Schwarz inequality. Also, by (A.24) and (A.33), for arbitrary \( C \in (0, \infty) \), defining
\[
c_f(\alpha, C) := \frac{1}{4} \min_{v \in \{0,1,...,V\}} \inf_{|t| \leq C} \left\{ \frac{(\alpha_{v+1} - t)\phi(\alpha_{v+1} - t) - (\alpha_v - t)\phi(\alpha_v - t)}{\Phi(\alpha_{v+1} - t) - \Phi(\alpha_v - t)} + \psi_v(t)^2 \right\},
\]
we have \( c_f(\alpha, C) \in (0, \infty) \) and \( f_1''(t,x) \geq 4c_f(\alpha, C) \) for all \((t,x) \in [-C, C] \times \mathcal{X}\). Choosing the specific \( C := C(C_0, c_{ev}, C_{ev}, c'_M) \) in (A.17), with \( c'_M := 1 \) and the implied \( c_f := c_f(\alpha, C) = c_f(\alpha, C_0, c_{ev}, C_{ev}) \), Lemma A.1 now shows that Assumption 3.4 holds with \( c'_M = 1 \) and \( c_M = c_M(\alpha, C_0, c_{ev}, C_{ev}) \).
Next, differentiating thrice shows that for all \((t, y) \in \mathbb{R} \times \mathcal{Y}\),

\[
m'_1(t, y) = \sum_{v=0}^V 1(y = v) [-\psi_v(t)],
\]

\[
m''_{11}(t, y) = \sum_{v=0}^V 1(y = v) \left\{ \frac{(\alpha_{v+1} - t)\phi(\alpha_{v+1} - t) - (\alpha_v - t)\phi(\alpha_v - t)}{\Phi(\alpha_{v+1} - t) - \Phi(\alpha_v - t)} + \psi_v(t)^2 \right\}
\]

\[
m'''_{11}(t, y) = \sum_{v=0}^V 1(y = v) \left\{ \psi_v(t) \left[ \frac{(\alpha_{v+1} - t)\phi(\alpha_{v+1} - t) - (\alpha_v - t)\phi(\alpha_v - t)}{\Phi(\alpha_{v+1} - t) - \Phi(\alpha_v - t)} - \psi_v(t) \frac{(\alpha_{v+1} - t)\phi(\alpha_{v+1} - t) - (\alpha_v - t)\phi(\alpha_v - t)}{\Phi(\alpha_{v+1} - t) - \Phi(\alpha_v - t)} \right] \right\},
\]

where the terms \((\alpha_{V+1} - t)^2\phi(\alpha_{V+1} - t), (\alpha_{V+1} - t)\phi(\alpha_{V+1} - t), (\alpha_0 - t)^2\phi(\alpha_0 - t),\) and \((\alpha_0 - t)\phi(\alpha_0 - t)\) are all understood to be zero for all \(t \in \mathbb{R}\). Therefore, for all \((t, y) \in \mathbb{R} \times \mathcal{Y}\),

\[|m'_1(t, y)| \leq \phi(0)/(1 - \Phi(1)) + 2 \max_{v \in [V]} |\alpha_v| + 2|t|\]

by (A.23) and (A.34). Also, for all \(t \in (-\infty, \alpha_1]\), using the Mean Value Theorem for the denominator, we see that

\[
\frac{(\alpha_{v+1} - \alpha_v)\phi(\alpha_{v+1} - t)}{\Phi(\alpha_{v+1} - t) - \Phi(\alpha_v - t)} \leq 1, \quad \text{for all } v \in [V - 1],
\]

and applying this upper bound to the terms \(v \in [V - 1]\), for all \(t \in (-\infty, \alpha_1]\), we get

\[
|m''_{11}(t, y)| \leq 1(y = 0) \phi(\alpha_1 - t) \frac{\phi(\alpha_1 - t)}{\Phi(\alpha_1 - t)} + (\alpha_1 - t)\]

\[
+ \sum_{v=1}^{V-1} 1(y = v) \left( 1 + |\psi_v(t)| \cdot |\psi_v(t) - (\alpha_v - t)| \right)
\]

\[
+ 1(y = V) \phi(\alpha_V - t) \frac{\phi(\alpha_V - t)}{1 - \Phi(\alpha_V - t)} + (\alpha_V - t).
\]

Hence, we have \(\max_{y \in \mathcal{Y}} |m''_{11}(t, y)| = O(1)\) as \(t \to -\infty\) by (A.25) (for the cases \(v \in \{0, V\}\)) and (A.37) (for the cases \(v \in [V - 1]\)). Also, \(\max_{y \in \mathcal{Y}} |m''_{11}(t, y)| = O(1)\) as \(t \to \infty\) by a similar argument. Thus, there is a constant \(C(\alpha) \in (0, \infty)\) such that for all \((t, y) \in \mathbb{R} \times \mathcal{Y}\), we have

\[|m''_{11}(t, y)| \leq C(\alpha).\]

Hence, (A.19) and (A.20) are satisfied with \(c_{m,1} = \phi(0)/(1 - \Phi(1)) + 2 \max_{v \in [V]} |\alpha_v|, c_{m,2} = 2,\) and \(c_{m,3} = C(\alpha),\) and so Assumption 3.5 follows from Lemma A.2 with \(L(\mathbf{x}, y) = (3 + \phi(0)/(1 - \Phi(1)) + 2 \max_{v \in [V]} |\alpha_v|)(1 + |\mathbf{x}^\top \theta_0|)\) for \((\mathbf{x}, y) \in \mathcal{X} \times \mathcal{Y}, c_L = 1, r = \bar{r}, B_n = (3 + \phi(0)/(1 - \Phi(1)) + 2 \max_{v \in [V]} |\alpha_v|)\bar{B}_n,\) and \(C_L = C_L(\alpha, C_0, C_{ce}).\) Further,
for all \( v \in [V - 1] \), as \( t \to -\infty \),

\[
\frac{\phi(\alpha_{v+1} - t)}{\Phi(\alpha_{v+1} - t) - \Phi(\alpha_v - t)} \to 0 \quad \text{and} \quad \frac{(\alpha_v - t)\phi(\alpha_{v+1} - t)}{\Phi(\alpha_{v+1} - t) - \Phi(\alpha_v - t)} \to 0
\]

(A.38)

by (A.36),

\[
\frac{\psi_v(t)\phi(\alpha_{v+1} - t)}{\Phi(\alpha_{v+1} - t) - \Phi(\alpha_v - t)} \to 0
\]

(A.39)

by (A.36) and (A.37), Hence, uniformly over \( y \in \mathcal{Y} \),

\[
|m'''_{111}(t, y)| \leq o(1) + \sum_{v=1}^{V-1} |\psi_v(t)| \times |(\psi_v(t) - (\alpha_v - t))(2\psi_v(t) - (\alpha_v - t)) - 1|
\]

as \( t \to -\infty \), where the terms \( v \in \{0, V\} \) are treated by applying the same arguments as in the binary probit derivations from Example 2, and the terms \( v \in [V - 1] \) are handled by decomposing \( \alpha_{v+1} - t = \alpha_{v+1} - \alpha_v + \alpha_v - t \) and applying (A.38) and (A.39) to all the terms containing \( \alpha_{v+1} - \alpha_v \). Thus, \( \max_{y \in \mathcal{Y}} |m'''_{111}(t, y)| \to 0 \) as \( t \to -\infty \) by (A.37) and \( \max_{y \in \mathcal{Y}} |m'''_{111}(t, y)| \to 0 \) as \( t \to \infty \) by a similar argument. Therefore, Assumption 5.3 is satisfied with \( J = 1 \) and some \( C_m = C_m(\alpha) \), and Assumption 5.4 follows trivially (from \( J = 1 \)). Moreover, Assumption 5.2 holds from Lemma A.4 with \( \tilde{r} = 8 \) and \( C_M = C_M(\alpha, C_0) \).

Finally, for arbitrary \( C \in (0, \infty) \), define

\[
c(\alpha, C) := \inf_{|t| \leq C} \sum_{v=0}^{V} |\phi(\alpha_v - t) - \phi(\alpha_{v+1} - t)|^2 / \Phi(\alpha_{v+1} - t) - \Phi(\alpha_v - t) \in (0, \infty).
\]

In this example and case, the residual takes the form

\[
U = m'_{1}(X^\top \theta_0, Y) = \sum_{v=0}^{V} 1(Y = v) \left[-\psi_v(X^\top \theta_0)\right]
\]

so that its square

\[
U^2 = \sum_{v=0}^{V} 1(Y = v)\psi_v(X^\top \theta_0)^2,
\]

Consider now the specific \( C := C_0^2 / \sqrt{c_{ev}/2} \) and the implied \( c := c(\alpha, C) = c(\alpha, C_0, c_{ev}) \).
Then by iterated expectations, we get

\[
E\left[|UX_j|^2\right] = E\left[|X_j|^2 \sum_{v=0}^{V} \kappa_v(X)\psi_v(X^\top \theta_0)^2\right]
\]

\[
= E\left[|X_j|^2 \sum_{v=0}^{V} \frac{\phi(\alpha_v - X^\top \theta_0) - \phi(\alpha_{v+1} - X^\top \theta_0)}{\Phi(\alpha_{v+1} - X^\top \theta_0) - \Phi(\alpha_v - X^\top \theta_0)}\right]
\]

\[
\geq cE\left[|X_j|^2 1\{\|X^\top \theta_0\| \leq C\}\right] = c (E[|X_j|^2] - E\left[|X_j|^2 1\{\|X^\top \theta_0\| > C\}\right])
\]

\[
\geq c \left( c_{ev} - E\left[|X_j|^2|X^\top \theta_0|^2\right]/C^2\right) \geq c \left( c_{ev} - \left( E[|X_j|^4]E[|X^\top \theta_0|^4]\right)^{1/2}/C^2\right)
\]

\[
\geq (c_{ev} - C^4/C^2) = cc_{ev}/2,
\]

by (A.2), (A.3), and the Cauchy-Schwarz inequality. The previous display shows that the lower bound in Assumption 4.1.1 holds for \(c_U = c_U(\alpha, C_0, c_{ev})\). The remaining parts of Assumption 4.1 follow from Lemma A.3 with \(C = C_U(\alpha, C_0)\), and \(\tilde{B}_n = (C_1\tilde{B}_n) \lor C_2\), where \(C_1 = C_1(\alpha) \in [1, \infty)\) and \(C_2 = C_2(\alpha, C_0)\). This observation completes the probit case and, thus, the example.

\[\square\]

**Example 3 (Expectile Model, Continued).** The loss (2.5) is continuously differentiable in \(t\), thus implying the differentiability part of Assumption 3.3. In addition, for all \(\theta \in \Theta\), the basic inequality \((a + b)^2 \leq 2(a^2 + b^2)\), (A.6) and (A.2) combine to show that

\[
E[|m(X^\top \theta, Y)|] \leq E[|Y - X^\top \theta|^2] \leq 2E[Y^2] + 2E[|X^\top \theta|^2] < \infty,
\]

which shows also the integrability part of Assumption 3.3.

Next, by integrability of \(Y\) (implied by (A.6)), for all \(x \in \mathcal{X}\), \(t \mapsto f(t, x) = E[m(t, Y) \mid X = x]\) is a well-defined function from \(\mathbb{R}\) to \(\mathbb{R}\). Its partial derivative is

\[
f'_1(t, x) = 2E[(1 - \tau)(t - Y)1(Y < t) + \tau(t - Y)1(Y \geq t) \mid X = x], \quad (t, x) \in \mathbb{R} \times \mathcal{X}.
\]

By the conditional continuity (A.5), two applications of Lemma A.5 (conditional on \(X = x\), with \(Z\) there set to \(Y\) and, in turn, \(-Y\)) show that this partial derivative is itself differentiable in \(t\) with derivative

\[
f''_1(t, x) = 2(1 - \tau)P(Y < t \mid X = x) + 2\tau P(Y > t \mid X = x), \quad (t, x) \in \mathbb{R} \times \mathcal{X}.
\]

Thus, for all \(\theta \in \Theta\) and \(j \in [p]\), we have

\[
E[|f'_1(X^\top \theta, X)X_j|] \leq 2E[|X_jX^\top \theta|] + 2E[|YX_j|] < \infty
\]

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by the law of iterated expectations, the Jensen and Cauchy-Schwarz inequalities, (A.2), (A.3) and (A.6). Using the conditional continuity (A.5) again, we see that, for all \((t, x) \in \mathbb{R} \times X\),

\[
f''_{11}(t, x) \geq 2(\tau \wedge (1 - \tau)) [P(Y < t \mid X = x) + P(Y > t \mid X = x)] = 2(\tau \wedge (1 - \tau)).
\]

The previous display and Lemma A.1 with \(c_f := (1/2)(\tau \wedge (1 - \tau))\), combine to show that Assumption 3.4 holds for \(c_M = c_M(c_{ev}, \tau)\) and any \(c'_M \in (0, \infty)\).

Next, for all \((t, y) \in \mathbb{R} \times Y\), we have

\[
m''_1(t, y) = \begin{cases} 
2(1 - \tau)(t - y), & \text{if } y < t, \\
2\tau(t - y), & \text{if } y \geq t, 
\end{cases}
\]

which satisfies

\[|m'_1(t, y)| \leq 2|y| + 2|t|, \quad (t, y) \in \mathbb{R} \times Y.
\]

Moreover, going case by case, we see that no matter the ordering, for all

\[|m'_1(t, y) - m'_2(t, y)| \leq 2|t_1 - t_2|, \quad (t_1, t_2, y) \in \mathbb{R}^2 \times Y,
\]

i.e. \(t \mapsto m'_1(t, y)\) is Lipschitz continuous in \(t\) with Lipschitz constant 2 for all \(y \in Y\). The previous two displays show that (A.19) and (A.20) hold with \(c_{m,1}(y) = 2|y|, \ c_{m,2} = 2, \) and \(c_{m,3} = 2\). While \(c_{m,1}(y) = 2|y|\) is no constant, from minor modifications to the proof of Lemma A.2, leveraging now the added (A.6), we deduce that Assumption 3.5 is satisfied with \(L(x, y) = 2(1 + |y| + |x^\top \theta_0|)\) (now depending on \(y\)) for \((x, y) \in X \times Y, c_L = 1, r = \bar{r}, B_n = 4\bar{B}_n, \) and \(C_L = C_L(C_0, C_{ev})\).

Next, for all \((t, y) \in \mathbb{R} \times Y, \) such that \(t \neq y\), we have

\[
m'''_{11}(t, y) = \begin{cases} 
2(1 - \tau), & \text{if } y < t, \\
2\tau, & \text{if } y \geq t,
\end{cases}\]

and, thus, \(m'''_{11}(t, y) = \begin{cases} 
0, & \text{if } y < t, \\
0, & \text{if } y > t.
\end{cases}\)

The conditional continuity (A.5) now shows that Assumption 5.3 is satisfied with \(J = 2, t_{y,1} = y\) and \(C_m = 2\). Under the (stronger) bounded conditional density assumption (A.15), Assumption 5.4 is satisfied with any \(\overline{\Delta}_n \in (0, \infty)\) and \(C_f = 2C_{pdf}\). Moreover, Assumption 5.2 follows from (A.3) and (A.6) with \(\bar{r} = 8\) and \(C_M = C_M(C_0)\).
Finally, we have for all \( j \in [p] \) that, by (A.6),

\[
E[|UX_j|^2] = E[|m'_i(X^\top \theta_0, Y)X_j|^2]
\geq 4(\tau \wedge (1 - \tau))^2 E[(Y - X^\top \theta_0)^2 X_j^2] \geq 4(\tau \wedge (1 - \tau))^2 c_{\text{eps}},
\]

showing that the lower bound in Assumption 4.1.1 is satisfied with \( c_U = c_U(c_{\text{eps}}, \tau) \). While Lemma A.3 does not apply directly (due to the non-constant \( c_{m,1}(y) \)), from minor modifications to the arguments used in the proof of Lemma A.2, leveraging the added (A.6), we deduce that the rest of Assumption 4.1 is satisfied with \( C_U = C_U(C_0) \), and \( \tilde{B}_n = (C_1 B_n) \lor C_2 \), where \( C_1 = C_1(C_0) \in [1, \infty) \) and \( C_2 = C_2(C_0) \).

**Example 4 (Panel Censored Model, Continued).** We consider trimmed LS and trimmed LAD, in turn. For the former loss, (2.6) takes the form

\[
m(t, y) = \begin{cases} 
  y_1^2 - 2y_1(y_2 + t), & \text{if } t \in (-\infty, -y_2], \\
  (y_1 - y_2 - t)^2, & \text{if } t \in (-y_2, y_1), \\
  y_2^2 + 2y_2(t - y_1), & \text{if } t \in [y_1, \infty), 
\end{cases}
\]

which is continuously differentiable in \( t \) for each \( y \in \mathcal{Y} = [0, \infty)^2 \). The differentiability part of Assumption 3.3 follows. In addition, for all \( \theta \in \Theta \), we have

\[
E[|m(X^\top \theta, Y)|] \leq 9E[Y_1^2] + 9E[Y_2^2] + 9E[|X^\top \theta|^2] < \infty
\]

by (A.8) and (A.2), which gives the integrability part of Assumption 3.3.

A similar argument shows integrability of \( m(t, Y) \) for each \( t \in \mathbb{R} \). Hence, for all \( x \in \mathcal{X} \), \( t \mapsto f(t, x) = E[m(t, Y)|X = x] \) is well-defined as a function from \( \mathbb{R} \) to \( \mathbb{R} \). Direct calculation and case inspection show that it is differentiable with derivative given by

\[
f'_1(t, x) = \begin{cases} 
  2E[(Y_2 + t)1(Y_2 + t > 0) - Y_1 | X = x], & \text{if } t \in (-\infty, 0], \\
  2E[(t - Y_1)1(Y_1 - t > 0) + Y_2 | X = x], & \text{if } t \in (0, \infty).
\end{cases}
\]

From the continuities in (A.7) and two applications of Lemma A.5 conditional on \( X = x \), \( t \mapsto f'_1(t, x) \) is seen to be Lipschitz-continuous on \( \mathbb{R} \), and differentiable on \( \mathbb{R} \setminus \{0\} \) with

\[
f''_{11}(t, x) = \begin{cases} 
  2P(Y_2 > -t | X = x), & \text{if } t \in (-\infty, 0), \\
  2P(Y_1 > t | X = x), & \text{if } t \in (0, \infty).
\end{cases}
\]

Thus, for all \( \theta \in \Theta \) and \( j \in [p] \), we have \( E[|f'_1(X^\top \theta, X)X_j|] < \infty \) by the law of iterated
expectations, the Cauchy-Schwarz inequality, (A.2), (A.3), and (A.8). Also, for \( t \neq 0 \), we have
\[
f_{11}''(t, x) \geq \begin{cases} 2P(Y_1 - Y_2 < t \mid X = x), & \text{if } t \in (-\infty, 0), \\ 2P(Y_1 - Y_2 > t \mid X = x), & \text{if } t \in (0, \infty), \end{cases}
\]
which implies that
\[
\inf_{t \in [-c_{de}, c_{de}], x^\top \theta_0 + t \neq 0} f_{11}''(x^\top \theta_0 + t, x) \geq 2 \min \left\{ P \left( Y_1 - Y_2 < x^\top \theta_0 - c_{de} \mid X = x \right), P \left( Y_1 - Y_2 > x^\top \theta_0 + c_{de} \mid X = x \right) \right\}.
\]
Therefore, by (A.10), we have that the probability statement in (A.18) is satisfied with \( C = c_{de} \), the provided \( c_f \), and \( N(x) = \{0\} \) for all \( x \in \mathcal{X} \). Adjusting \( c_M \) (which is here free to vary) to match this \( C \), we see that Assumption 3.4 follows from Lemma A.1 with \( c_M' = c_M(c_{de}, c_{ev}, C_{ev}) \) and \( c_M = c_M(c_{ev}, c_f) \).

Next, for all \((t, y) \in \mathbb{R} \times \mathcal{Y}\), we have
\[
m_1'(t, y) = \begin{cases} -2y_1, & \text{if } t \in (-\infty, -y_2], \\ 2(t + y_2 - y_1), & \text{if } t \in (-y_2, y_2), \\ 2y_2, & \text{if } t \in [y_2, \infty), \end{cases}
\]
which implies both
\[
|m_1'(t, y)| \leq 2(y_1 \vee y_2) + 2|t|, \quad \text{for all } (t, y) \in \mathbb{R} \times \mathcal{Y}.
\]
and
\[
|m_1'(t_1, y) - m_1'(t_2, y)| \leq 2|t_1 - t_2|, \quad \text{for all } (t_1, t_2, y) \in \mathbb{R}^2 \times \mathcal{Y}.
\]
The previous two displays show that (A.19) and (A.20) hold with \( c_{m,1}(y) = 2(y_1 \vee y_2) \), \( c_{m,2} = 2 \), and \( c_{m,3} = 2 \). While \( c_{m,1}(y) \) is no constant, with minor modifications to the proof of Lemma A.2 and leveraging the added (A.8), it follows that Assumption 3.5 is satisfied with \( L(x, y) = 2(1 + (y_1 \vee y_2) + |x^\top \theta_0|) \) for \((x, y) \in \mathcal{X} \times \mathcal{Y}\), \( c_L = 1 \), \( r = \bar{r} \), \( B_n = CB_n \) for a universal constant \( C \in [1, \infty) \), and \( C_L = C_L(C_0, C_{ev}) \).

If \((y_1, y_2) = (0, 0)\), then the trimmed LS loss \( m(\cdot, 0, 0) \) is identically zero. If \((y_1, y_2) \neq
(0, 0), i.e. at least one element in \((y_1, y_2)\) is positive, then \((-y_2, y_1)\) is non-empty, and

\[
m''_{11}(t, y) = \begin{cases} 0, & \text{if } t \in (-\infty, -y_2), \\ 2, & \text{if } t \in (-y_2, y_1), \quad \text{and} \quad m''_{111}(t, y) = 0, & \text{if } t \notin \{-y_2, y_1\}. \\ 0, & \text{if } t \in (y_1, \infty), \end{cases}
\]

with the second (and thus the third) derivative being undefined at \(t \in \{-y_2, y_1\}\). This gives all conditions of Assumption 5.3 except for the last one with \(J = 3\), \(t_{y,1} = -y_2\), \(t_{y,2} = y_1\), and \(C_m = 2\). The last condition of Assumption 5.3 follows by noting that

\[
P(X^\top \theta_0 = Y_1) = P(X^\top \theta_0 = Y_1, Y_1 = 0) + P(X^\top \theta_0 = Y_1, Y_1 > 0) \\
\leq P(X^\top \theta_0 = 0) + P(X^\top \theta_0 = Y_1 \mid Y_1 > 0)P(Y_1 > 0) = 0
\]

and, similarly, \(P(X^\top \theta_0 = -Y_2) = 0\) under the continuities provided by (A.16). Also, by similar reasoning,

\[
P(X^\top \theta_0 - \Delta_n \leq Y_1 \leq X^\top \theta_0 + \Delta_n) \\
\leq P(X^\top \theta_0 - \Delta_n \leq 0 \leq X^\top \theta_0 + \Delta_n) \\
+ P(X^\top \theta_0 - \Delta_n \\leq Y_1 \leq X^\top \theta_0 + \Delta_n \mid Y_1 > 0) \cdot P(Y_1 > 0) \\
\leq 2C_{pdf} \Delta_n + 2C_{pdf} \Delta_n P(Y_1 > 0) \leq 4C_{pdf} \Delta_n
\]

and, analogously,

\[
P(X^\top \theta_0 - \Delta_n \leq -Y_2 \leq X^\top \theta_0 + \Delta_n) \leq 4C_{pdf} \Delta_n,
\]

showing that Assumption 5.4 is satisfied with any \(\Delta_n \in (0, \infty)\) and \(C_f = 4C_{pdf}\). Moreover, Assumption 5.2 follows from (A.3) and (A.8) with \(\bar{r} = 8\) and \(C_M = C_M(C_0)\).

Finally, by (A.9), we have for all \(j \in [p]\) that

\[
E[|UX_j|^2] = E \left[ |m'_1(X^\top \theta_0, Y)X_j|^2 \right] \\
\geq 4E \left[ X_j^2 \left( Y^2 I \{ Y_2 \leq -X^\top \theta_0 \} + Y_2^2 1 \{ Y_1 \leq X^\top \theta_0 \} \right) \right] \geq 4c_{eps},
\]

which shows that the lower bound in Assumption 4.1.1 is satisfied with \(c_U = c_U(c_{eps})\). While Lemma A.3 is not directly applicable (due to the non-constant \(c_{m,1}(y)\)), from minor modifications to the proof of that lemma, we see that the rest of Assumption 4.1 is satisfied with \(C_U = C_U(C_0)\) and \(\bar{B}_n = (C_1 \bar{B}_n) \lor C_2\), where \(C_1 = C_1(C_0) \in [1, \infty)\) and \(C_2 = C_2(C_0)\).
Next, we consider trimmed LAD, in which case the loss (2.6) can be written as

\[
m(t, y) = \begin{cases} 
|y_1 - y_2 - t|, & \text{if } y_1 > 0, y_2 > 0, \\
(y_2 + t)1(y_2 + t > 0), & \text{if } y_1 = 0, y_2 > 0, \\
(y_1 - t)1(y_1 - t > 0), & \text{if } y_1 > 0, y_2 = 0, \\
0, & \text{if } y_1 = 0, y_2 = 0.
\end{cases}
\]

The differentiability part of Assumption 3.3 follows from the absolute continuity provided by (A.12) and (A.13). In addition, for all \( \theta \in \Theta \), we have

\[
E[|m(X^\top \theta, Y)|] \leq E[Y_1] + E[Y_2] + E[|X^\top \theta|] < \infty
\]

by (A.11) and (A.2), which gives the integrability part of Assumption 3.3.

A similar argument shows that each \( m(t, Y) \) is integrable, which implies that, for all \( x \in \mathcal{X}, t \mapsto f(t, x) = E[m(t, Y)|X = x] \) is well-defined as a function from \( \mathbb{R} \) to \( \mathbb{R} \). By (A.12) and (A.13), \( t \mapsto f(t, x) \) is seen to be differentiable with derivative

\[
f'_1(t, x) = E[(1(Y_1 - Y_2 < t) - 1(Y_1 - Y_2 > t))1\{Y_1 > 0, Y_2 > 0\} | X = x] \\
+ E[1\{Y_2 > -t\}1\{Y_1 = 0, Y_2 > 0\} | X = x] - E[1\{Y_1 > t\}1\{Y_1 > 0, Y_2 = 0\} | X = x].
\]

Again by (A.12) and (A.13), we deduce that \( t \mapsto f'_1(t, x) \) is Lipschitz-continuous and differentiable. Inspection reveals that its derivative satisfies the lower bound

\[
f''_{11}(t, x) \geq 2f_{Y_1-Y_2|Y_1>0,Y_2>0}(t | x)P(Y_1 > 0, Y_2 > 0 | X = x).
\]

Thus, for all \( \theta \in \Theta \) and \( j \in [p] \), we have \( E[|f'_1(X^\top \theta, X)^j|] < \infty \) by (A.2). Also, by (A.14), we have that (A.18) is satisfied with \( C = c_{de} \) and the provided \( c_f \). Adjusting \( c_M' \) (which is here free to vary), Assumption 3.4 follows from Lemma A.1 with \( c_M' = c_M'(c_{de}, c_{ev}, C_{ev}) \) and \( c_M = c_M(c_{ev}, c_f) \).

Next, given that for all \( y \in \mathcal{Y} \), the function \( m(\cdot, y) \) is Lipschitz-continuous with Lipschitz constant one, using (A.2), (A.3) and (A.4), Assumptions 3.5.1 and 3.5.2 follow upon taking \( L(x, y) = 1 \) for \( (x, y) \in \mathcal{X} \times \mathcal{Y}, c_L = 1, r = \bar{r}, B_n = \bar{B}_n \) and \( C_L = C_L(C_0, C_{ev}) \).
such that Assumption 3.5 as a whole is satisfied with $C$, the previous display shows that Assumption 3.5.3 holds with $C$, the second from (A.12) and (A.13), and the third from Jensen’s inequality and (A.2). The previous display shows that Assumption 3.5.3 holds with $C = C_L(C_{ev}, C_{pdf})$, such that Assumption 3.5 as a whole is satisfied with $C_L = C_L(C_0, C_{ev}, C_{pdf})$.

Finally, we have for all $j \in [p]$ that

$$
E[|U X_j|^2] \geq E[|m_1'(X^\top \theta_0, Y) X_j|^2 1(Y_1 > 0, Y_2 > 0)] = E[|X_j|^2 1(Y_1 > 0, Y_2 > 0)] \geq c_{eps}
$$

by (A.12) and (A.11), which shows that the lower bound in Assumption 4.1.1 is satisfied with $c_U = \sqrt{c_{eps}}$. Given that the loss $m(\cdot, y)$ is Lipschitz-continuous with Lipschitz constant one for all $y \in \mathcal{Y}$, using (A.3) and (A.4), straightforward arguments show that the rest of Assumption 4.1 is satisfied with $C_U = C_0$, and $\bar{B}_n = \bar{B}_n \vee C_0$.

We end this section by providing the proofs for Lemmas A.1–A.5, in turn.

**Proof of Lemma A.1.** We first show that $E[f'(X^\top \theta_0, X)] = 0$. To do so, note that since $\theta_0$ is interior to $\Theta$ (Assumption 3.1), there is a radius $\tau_n \in (0, \infty)$ such that the ball $B_{\theta_0}(\tau_n) := \{\theta \in \mathbb{R}^p; \|\theta - \theta_0\|_2 \leq \tau_n\}$ is a subset of $\Theta$. Fix any $\theta \in B_{\theta_0}(\tau_n)$ and let

$$
g(\tau) := E[f(X^\top \theta_0 + \tau(X^\top \theta - X^\top \theta_0), X)], \quad \tau \in (-1, 1).
$$

Note that since $E[|m(X^\top \theta_0, Y)|] < \infty$ by Assumption 3.3, it follows from Dudley (2004, Theorem 10.1.1) that the conditional expectation $E[m(X^\top \theta_0, Y)|X]$ exists and hence is equal to $f(X^\top \theta_0, X)$ almost surely. Thus,

$$
E[|f(X^\top \theta_0, X)|] = E[|E[m(X^\top \theta_0 Y)|X]|] \leq E[|m(X^\top \theta_0, Y)|] < \infty.
$$

Also, since it follows from Assumption 3.2 that the function $t \mapsto f(t, x)$ is convex, we have
for any \( \tau \in (-1, 1) \setminus \{0\} \) that
\[
\tau^{-1} \left| f(X^\top \theta_0 + \tau (X^\top \theta - X^\top \theta_0), X) - f(X^\top \theta_0, X) \right| \\
\leq \left( |f'_1(X^\top \theta_0, X)| + |f'_1(X^\top \theta, X)| \right) \times |X^\top \theta - X^\top \theta_0|,
\]
where the right-hand side is integrable by assumption. Along with the differentiability and
integrability presumed in the statement of the lemma, the previous two displays suffice to
satisfy both Condition (A.2) and the difference quotient domination condition of Dudley
(2014, Corollary A.3) when applied to the function \( \tau \mapsto f(x^\top \theta_0 + \tau (x^\top \theta - x^\top \theta_0), x) \) in a
neighborhood of \( \tau_0 = 0 \). Hence, it follows from the same corollary that \( g \) is differentiable at
\( \tau_0 = 0 \) with derivative
\[
\left. g'(0) = E[f'_1(X^\top \theta_0, X)(X^\top \theta - X^\top \theta_0)] \right. 
\]
Therefore, by Taylor’s theorem (with Peano’s form of remainder),
\[
g(\tau) = g(0) + g'(0)\tau + h(\tau)\tau, \quad \tau \in (-1, 1),
\]
where \( h: (-1, 1) \to \mathbb{R} \) is a function such that \( h(\tau) \to 0 \) as \( \tau \to 0 \). On the other hand, by
the definition of \( \theta_0 \), we have
\[
g(\tau) = E[m(X^\top \theta_0 + \tau (X^\top \theta - X^\top \theta_0), Y)] \geq E[m(X^\top \theta_0, Y)] = g(0)
\]
for all \( \tau \in (-1, 1) \). Thus,
\[
g'(0) + h(\tau) \leq 0 \quad \text{for} \quad \tau \in (-1, 0)
\]
and
\[
g'(0) + h(\tau) \geq 0 \quad \text{for} \quad \tau \in (0, 1).
\]
Taking the limits as \( \tau \to 0_- \) and \( \tau \to 0_+ \), respectively, implies that \( g'(0) = 0 \). Since
\( \theta \in \mathcal{B}_{\theta_0}(\tau_n) \) used in defining \( f \) and, thus, \( g \) was arbitrarily chosen, varying \( \theta \in \mathcal{B}_{\theta_0}(\tau_n) \)
produces the desired \( E[f'_1(X^\top \theta_0, X)X] = 0_p \). (See the proof of Lemma B.10 for details.)

Let \( \theta \in \Theta \) satisfy \( \|\theta - \theta_0\|_2 \leq c'_M \), and introduce the abbreviations \( \delta := \theta - \theta_0 \) and
\[
g(r, x) := \begin{cases} 
    f''_{11}(x^\top (\theta_0 + r\delta), x), & \text{if } x^\top (\theta_0 + r\delta) \notin N(x), \\
    4c_f, & \text{if } x^\top (\theta_0 + r\delta) \in N(x), 
\end{cases}
\]

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for \( r \in [0, 1] \times \mathcal{X} \). Using 
\[ E[f_1'(X^\top \theta_0, X)X] = 0_p, \]
by a first-order Taylor expansion with remainder in the integral form (which is allowed per Lipschitz continuity of \( t \mapsto f_1'(t, x) \) on compacta), using that \( N(x) \) is a Lebesgue null set for all \( x \in \mathcal{X} \), we get

\[
\mathcal{E}(\theta) = E[f(X^\top \theta, X) - f(X^\top \theta_0, X)]
= E \left[ \int_{X^\top \theta_0} X^\top \theta' f_1''_1(t, X)(X^\top \theta - t) dt \right]
= E \left[ \int_{X^\top \theta_0} f_1''_1(X^\top (\theta_0 + r\delta), X) |X^\top \delta|^2 (1 - r) dr \right]
= E \left[ \int_{X^\top \theta_0} g(r, X) |X^\top \delta|^2 (1 - r) dr \right]
\geq 4c_f E \left[ \int_{X^\top \theta_0} |X^\top \delta|^2 1\{g(r, X) \geq 4c_f\} (1 - r) dr \right]. \tag{A.40}
\]

Splitting the expectation in (A.40) in two, we get

\[
E \left[ \int_{X^\top \theta_0} |X^\top \delta|^2 (1 - r) dr \right] - E \left[ \int_{X^\top \theta_0} |X^\top \delta|^2 1\{g(r, X) < 4c_f\} (1 - r) dr \right]
\geq \frac{c_{ev} \|\theta - \theta_0\|^2}{2} - \int_{X^\top \theta_0} \sqrt{E[|X^\top \delta|^4]} \sqrt{P(g(r, X) < 4c_f)(1 - r) dr}
\geq \frac{\|\theta - \theta_0\|^2}{2} \left( c_{ev} - C_{ev} \sup_{r \in [0, 1]} \sqrt{P(g(r, X) < 4c_f)} \right)
\]

where we have used Tonelli’s theorem, the Cauchy-Schwarz inequality and (A.2). Continuing with the right-hand side probability, for all \( r \in [0, 1] \), for \( C \) stated in (A.17), we see that

\[
P (g(r, X) < 4c_f) \leq P \left( \inf_{t \in [-C, C]} f_1''_1(t, X) < 4c_f \right) + P \left( |X^\top_0 (\theta_0 + r\delta)| > C \right)
\leq \left( \frac{c_{ev}}{2\sqrt{2C_{ev}}} \right)^2 + \frac{E[|X^\top_0 + rX^\top (\theta - \theta_0)|^2]}{C^2}
\leq \left( \frac{c_{ev}}{2\sqrt{2C_{ev}}} \right)^2 + \frac{2E[|X^\top_0|^2] + 2E[|X^\top (\theta - \theta_0)|^2]}{C^2}
\leq \left( \frac{c_{ev}}{2\sqrt{2C_{ev}}} \right)^2 + \frac{2(C_0^2 + C_{ev}(c_M')^2)}{C^2} = \left( \frac{c_{ev}}{2C_{ev}} \right)^2
\]

where we have used the union bound, (A.17), Markov’s inequality, the basic inequality \((a + b)^2 \leq 2(a^2 + b^2)\), Hölder’s inequality, (A.2) and (A.3).
Also, for all \( r \in [0, 1] \), with \( C \) stated in (A.18), we have
\[
P(g(r, X) < 4c_f) \leq P \left( \inf_{t \in [-C, C]} f''_{11}(X^\top \theta_0 + t, X) < 4c_f \right) + P \left( |X^\top (\theta - \theta_0)| > C \right)
\leq \left( \frac{c_{ev}}{2\sqrt{2C_{ev}}} \right)^2 + \frac{E[(X^\top (\theta - \theta_0))^2]}{C^2}
\leq \left( \frac{c_{ev}}{2\sqrt{2C_{ev}}} \right)^2 + \frac{C_{ev}(c_M')^2}{C^2} = \left( \frac{c_{ev}}{2C_{ev}} \right)^2
\]
by similar arguments. Combining these chains of inequalities gives the asserted claim.

**Proof of Lemma A.2.** First, for all \((x, y) \in X \times Y\) and all \((t_1, t_2) \in \mathbb{R}^2\) satisfying \(|t_1| \vee |t_2| \leq 1\), we have for some \( \tau \in [0, 1] \) that
\[
|m(x^\top \theta_0 + t_1, y) - m(x^\top \theta_0 + t_2, y)| = |m_1'(x^\top \theta_0 + t_1 + \tau(t_2 - t_1), y)(t_1 - t_2)|
\leq (c_{m,1} + c_{m,2}(|x^\top \theta_0 + t_1| \vee |x^\top \theta_0 + t_2|))|t_1 - t_2|
\leq (c_{m,1} + c_{m,2}(|x^\top \theta_0| + 1))|t_1 - t_2| \leq (1 + c_{m,1} + c_{m,2})(1 + |x^\top \theta_0|)|t_1 - t_2|,
\]
where the first line follows from the Mean Value Theorem, the second from (A.19) and the triangle inequality, and the third from \(|t_1| \vee |t_2| \leq 1\) and the triangle inequality. This gives (3.1) with the provided \( c_L \) and \( L \). Also, for this \( L \), by (A.4),
\[
E[\|L(X, Y)\| X \|_\infty|^r] = (1 + c_{m,1} + c_{m,2})^r E \left[ (1 + \|X^\top \theta_0\|_\infty)^r \|X\|_\infty \right] \leq (1 + c_{m,1} + c_{m,2})^r \bar{B}_n^r,
\]
which justifies the choices \( r = \bar{r} \) and \( B_n = (1 + c_{m,1} + c_{m,2}) \bar{B}_n \). In addition, for all \( j \in [p] \), continuing with the above \( L \), we have
\[
E[|L(X, Y)X_j|^2] = (1 + c_{m,1} + c_{m,2})^2 E[(1 + \|X^\top \theta_0\|_\infty)^2 X_j^2]
\leq 2(1 + c_{m,1} + c_{m,2})^2 \left( E[X_j^2] + \sqrt{E[\|X^\top \theta_0\|^4]} E[X_j^4] \right)
\leq 2(1 + c_{m,1} + c_{m,2})^2 C_{ev}(1 + C_0^2) \leq C_L^2,
\]
where the second line follows from the basic inequality \((a + b)^2 \leq 2(a^2 + b^2)\) and the Cauchy-Schwarz inequality, and the third from (A.2), (A.3) and the definition of \( C_L \). The previous display yields Assumption 3.5.1.
Second, for all \( \theta \in \Theta \) satisfying \( \| \theta - \theta_0 \|_2 \leq 1 \), we have for some \( \tau \in [0, 1] \) that

\[
E \left[ |m(X^T \theta, Y) - m(X^T \theta_0, Y)|^2 \right] \\
= E \left[ |m'(X^T \theta_0 + \tau X^T (\theta - \theta_0), Y)|^2 |X^T \theta - X^T \theta_0|^2 \right] \\
\leq E \left[ (c_{m,1} + c_{m,2}|X^T \theta_0| + |X^T (\theta - \theta_0)|)|^2 |X^T \theta - X^T \theta_0|^2 \right] \\
\leq 3E \left[ (c_{m,1}^2 + c_{m,2}^2|X^T \theta_0|^2 + c_{m,2}^2|X^T (\theta - \theta_0)|^2) |X^T \theta - X^T \theta_0|^2 \right] \\
\leq 3 (c_{m,1}^2 C_{ev} + c_{m,2}^2 C_0^2 C_{ev} + c_{m,2}^2 C_{ev}) \| \theta - \theta_0 \|_2^2 \leq C_L^2 \| \theta - \theta_0 \|_2^2,
\]

where the second line follows from the Mean Value Theorem, the third from (A.19) and the triangle inequality, the fourth from the basic inequality \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\), and the fifth from the Cauchy-Schwarz inequality, (A.2), (A.3) and the definition of \( C_L \). The previous display yields Assumption 3.5.2.

Third, for all \( \theta \in \Theta \) satisfying \( \| \theta - \theta_0 \|_2 \leq 1 \), we have

\[
E \left[ |m'(X^T \theta, Y) - m'(X^T \theta_0, Y)|^2 \right] \leq c_{m,3}^2 E \left[ |X^T \theta - X^T \theta_0|^2 \right] \\
\leq c_{m,3}^2 C_{ev} \| \theta - \theta_0 \|_2^2 \leq C_L^2 \| \theta - \theta_0 \|_2^2 \leq C_L^2 \| \theta - \theta_0 \|_2
\]

by (A.20), (A.2), and the definition of \( C_L \). The previous display yields Assumption 3.5.3 and completes the proof of the lemma. \( \Box \)

**Proof of Lemma A.3.** First, for all \( j \in [p] \), we have

\[
E[|m'(X^T \theta_0, Y)X_j|^2] \leq E [(c_{m,1} + c_{m,2}|X^T \theta_0|)^2 X_j^2] \\
\leq 2E [(c_{m,1}^2 + c_{m,2}^2|X^T \theta_0|^2)X_j^2] \leq 2(c_{m,1}^2 C_0^2 + c_{m,2}^2 C_0^4) \leq C_U^2
\]

by (A.19), the basic inequality \((a + b)^2 \leq 2(a^2 + b^2)\), (A.3), the Cauchy-Schwarz inequality, and the definition of \( C_U \). Together with the assumptions of the lemma, the previous display yields Assumption 4.1.1.

Second, for all \( j \in [p] \), we have

\[
E \left[ |m'(X^T \theta_0, Y)X_j|^4 \right] \leq E \left[ (c_{m,1} + c_{m,2}|X^T \theta_0|)^4 X_j^4 \right] \\
\leq (c_{m,1} + c_{m,2})^4 \left[ (1 + |X^T \theta_0|)^4 X_j^4 \right] \\
\leq 8(c_{m,1} + c_{m,2})^4 \left[ (1 + |X^T \theta_0|^4)X_j^4 \right] \\
\leq 8(c_{m,1} + c_{m,2})^4 C_0^4 (1 + C_0^4) \leq \tilde{B}_{n}^2.
\]
by (A.19), the basic inequality \((a + b)^2 \leq 2(a^2 + b^2)\), (A.3), the Cauchy-Schwarz inequality, and the definition of \(\tilde{B}_n\). The previous display yields Assumption 4.1.2.

Third, by (A.19), Hölder’s inequality and (A.4), and the definition of \(\tilde{B}_n\),

\[
\mathbb{E} \left[ \left\| m_1'(X^\top \theta_0, Y) \mathbf{X} \right\|_4^4 \right] \leq \mathbb{E} \left[ \left( c_{m,1} + c_{m,2} |X^\top \theta_0| \right)^4 \|X\|_\infty^4 \right]
\leq (c_{m,1} + c_{m,2})^4 \mathbb{E} \left[ \left( 1 + |X^\top \theta_0| \right)^4 \|X\|_\infty^4 \right]
\leq (c_{m,1} + c_{m,2})^4 \tilde{B}_n^4 \leq \tilde{B}_n^4.
\]

The previous display shows Assumption 4.1.3 and completes the proof of the lemma. \(\square\)

**Proof of Lemma A.4.** By (A.19), the triangle inequality, (A.3), and the definition of \(C_M\),

\[
\left( \mathbb{E} \left[ |m_1'(X^\top \theta_0, Y)|^8 \right] \right)^{1/8} \leq \left( \mathbb{E} \left[ \left( c_{m,1} + c_{m,2} |X^\top \theta_0| \right)^8 \right] \right)^{1/8}
\leq (c_{m,1} + c_{m,2}) \left( \mathbb{E} \left[ \left( 1 + |X^\top \theta_0| \right)^8 \right] \right)^{1/8}
\leq (c_{m,1} + c_{m,2}) \left( 1 + \left( \mathbb{E} \left[ |X^\top \theta_0|^8 \right] \right)^{1/8} \right)
\leq (c_{m,1} + c_{m,2}) (1 + C_0) \leq C_M.
\]

The previous display gives one part of Assumption 5.2 for \(\tilde{r} = 8\). Since the remaining parts follow trivially for the same \(\tilde{r}\) by (A.3), the asserted claim follows. \(\square\)

**Proof of Lemma A.5.** First, \(f\) is a well-defined and real-valued function on \(\mathbb{R}\) by integrability of \(Z\). For the alternative expression for \(f\), considering the three cases \(t < z\), \(t = z\) and \(t > z\), we see that the integrands \((z - t)1(z \geq t)\) and \((z - t)1(z > t)\) used in the two definitions are actually one and the same. Taking the expectation over \(Z\) gives the desired equivalence.

To argue Lipschitzness, let \(t_1, t_2 \in \mathbb{R}\). Consider the case \(t_1 < t_2\). Then

\[
f(t_2) - f(t_1) = \mathbb{E} [(Z - t_2)1(Z \geq t_2)] - \mathbb{E} [(Z - t_1)1(Z \geq t_1)]
= \mathbb{E} [(Z - t_1)(1(Z \geq t_2) - 1(Z \geq t_1))] - (t_2 - t_1)\mathbb{E} [1(Z \geq t_2)]
= -\left( \mathbb{E} [(Z - t_1)1(t_1 \leq Z < t_2)] + (t_2 - t_1)\mathbb{E} [1(Z \geq t_2)] \right),
\]

which implies

\[
|f(t_2) - f(t_1)| = \mathbb{E} [(Z - t_1)1(t_1 \leq Z < t_2)] + (t_2 - t_1)\mathbb{E} [1(Z \geq t_2)]
\leq (t_2 - t_1)\mathbb{E} [1(t_1 \leq Z < t_2)] + (t_2 - t_1)\mathbb{E} [1(Z \geq t_2)] \leq (t_2 - t_1) = |t_2 - t_1|.
\]

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The case \( t_1 > t_2 \) can be handled analogously, thus showing Lipschitzness of \( f \).

For the differentiability assertion, let \( t \in \mathbb{R} \) and \( \Delta \in (0, \infty) \) be arbitrary. Then (A.41) and the continuity of the distribution of \( Z \) combine to show that

\[
|f(t + \Delta) - f(t) - \Delta [-P(Z > t)]| \\
= \mathbb{E} \left[ (Z - t)1(t \leq Z < t + \Delta) \right] + \Delta [P(Z \geq t + \Delta) - P(Z > t)] \\
\leq \Delta [P(t \leq Z < t + \Delta) + P(Z \geq t + \Delta) - P(Z > t)] \\
= \Delta P(Z = t) = 0.
\]

It follows that

\[
\lim_{\Delta \to 0^+} \frac{f(t + \Delta) - f(t)}{\Delta} = -P(Z > t).
\]

The case \( \Delta \to 0^- \) can be handled analogously, thus completing the proof. \( \square \)

B Proofs for Statements in Main Text

B.1 Proofs for Section 3

For the arguments in this section, we introduce some additional notation. Let

\[
T_0 := \text{supp} (\theta_0) = \{j \in [p]; |\theta_{0,j}| > 0\}
\]

be the support of \( \theta_0 \), and let \( T(\eta) \subseteq T_0 \) be the \( \eta \)-thresholded version thereof, i.e.

\[
T(\eta) := \{j \in [p]; |\theta_{0,j}| > \eta\}, \quad \eta \in [0, \infty),
\]

such that \( T(0) = T_0 \). Given a vector \( \delta \in \mathbb{R}^p \) and a set of indices \( J \subseteq [p] \), we let \( \delta_J \) denote the vector in \( \mathbb{R}^p \) with coordinates given by \( \delta_{J,j} = \delta_j \) if \( j \in J \) and \( \delta_{J,j} = 0 \) otherwise. Also, for \( \bar{c}, \eta \in [0, \infty) \), let \( \mathcal{R}(\bar{c}, \eta) \) denote the restricted set

\[
\mathcal{R}(\bar{c}, \eta) := \{\delta \in \mathbb{R}^p; \left\| \delta_{T(\eta)'} \right\|_1 \leq \bar{c} \left\| \delta_{T(\eta)} \right\|_1 + (1 + \bar{c}) \left\| \theta_{0T(\eta)'} \right\|_1 \quad \text{and} \quad \theta_0 + \delta \in \Theta\}.
\]

In addition, for a constant \( c_0 \in (1, \infty) \), define the (random) empirical error function \( \epsilon_n: [0, \infty) \to [0, \infty) \) by

\[
\epsilon_n(u) := \sup_{\delta \in \mathcal{R}(c_0, \eta_n), \left\| \delta \right\|_2 \leq u} \left| \mathbb{E} \left[ m(X_i^T (\theta_0 + \delta), Y_i) - m(X_i^T \theta_0, Y_i) \right] \right|,
\]

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where $\bar{c}_0 = (c_0 + 1)/(c_0 - 1)$, and $\eta_n = \sqrt{\ln (pn)/n}$. Finally, for any $\lambda \in (0, \infty)$, any non-random sequence $\lambda_n$ in $(0, \infty)$, and non-random sequences $a_{\varepsilon,n}$ and $b_{\varepsilon,n}$ in $(0, \infty)$, to be specified later, define the events

$$\mathcal{S}_n := \{ \lambda \geq c_0 \| S_n \|_\infty \}, \quad \mathcal{L}_n := \{ \lambda \leq \lambda_n \}, \quad \mathcal{E}_n := \{ \varepsilon_n (\bar{u}_n) \leq a_{\varepsilon,n}\bar{u}_n + b_{\varepsilon,n} \}, \quad (B.1)$$

where

$$\bar{u}_n := \frac{2}{c_M} \left( a_{\varepsilon,n} + (1 + \bar{c}_0) \lambda_n \sqrt{s_q \eta_n^{1-q}} \right). \quad (B.2)$$

(That $S_n$ exists is left implicit in the definition of $\mathcal{S}_n$.) In proving Theorem 3.1, we rely on the following four lemmas, whose proofs can be found at the end of this section.

**Lemma B.1 (Strong Ball Consequences).** Let Assumption 3.6 hold. Then for any $\eta \in (0, \infty)$, we have $|T(\eta)| \leq s_q \eta^{-q}$ and $\| \theta_{T(\eta)} \|_1 \leq s_q \eta^{1-q}$.

**Lemma B.2 (Restricted Set Consequence).** Let Assumption 3.6 hold. Then for any $\bar{c}, \eta \in (0, \infty)$, $\delta \in \mathcal{R}(\bar{c}, \eta)$ implies $\| \delta \|_1 \leq (1 + \bar{c}) (s_q \eta^{-q} \| \delta \|_2 + s_q \eta^{1-q})$.

**Lemma B.3 (Non-Asymptotic Deterministic Bounds).** Let Assumptions 3.1–3.4 and 3.6 hold and suppose that $\bar{u}_n \leq c_M$ and

$$\left( a_{\varepsilon,n} + (1 + \bar{c}_0) \lambda_n \sqrt{s_q \eta_n^{-q}} \right)^2 \geq c_M \left( b_{\varepsilon,n} + (1 + \bar{c}_0) \lambda_n s_q \eta_n^{1-q} \right). \quad (B.3)$$

Then on the event $\mathcal{S}_n \cap \mathcal{L}_n \cap \mathcal{E}_n$, for all $\hat{\theta} \in \hat{\Theta}(\lambda)$, we have $\hat{\theta} - \theta_0 \in \mathcal{R}(\bar{c}_0, \eta_n)$,

$$\| \hat{\theta} - \theta_0 \|_2 \leq \bar{u}_n \text{ and } \| \hat{\theta} - \theta_0 \|_1 \leq (1 + \bar{c}_0) \left( \bar{u}_n \sqrt{s_q \eta_n^{-q}} + s_q \eta_n^{1-q} \right).$$

**Lemma B.4 (Empirical Error Bound).** Let Assumptions 3.3, 3.5 and 3.6 hold. Then there is a universal constant $C \in [1, \infty)$, such that for any $n \in \mathbb{N}$, $t \in [1, \infty)$ and $u \in (0, \infty)$ satisfying

$$\eta_n \leq 1, \quad \frac{B_n^2 \ln (pn)}{\sqrt{n}} \leq C_L^2 \quad \text{and} \quad t n^{1/r} B_n \left( u \sqrt{s_q \eta_n^{-q}} + s_q \eta_n^{1-q} \right) \leq \frac{c_L}{1 + \bar{c}_0}, \quad (B.4)$$

we have

$$\varepsilon_n (u) \leq C (1 + \bar{c}_0) C_L \left( u \sqrt{s_q \eta_n^{2-q}} + s_q \eta_n^{2-q} \right)$$

with probability at least $1 - 4t^{-r} - C/\ln^2 (pn) - n^{-1}$.
Remark B.1 (Alternative Non-Asymptotic Bounds). If the loss function $m$ is (globally) Lipschitz in its first argument with Lipschitz constant not depending on $y$, and the regressors are bounded, then symmetrization, contraction, and concentration arguments may be used to bound the modified empirical error

$$\tilde{\varepsilon}_n(u) := \sup_{\delta \in \mathbb{R}^p; \|\delta\|_1 \leq u} |(E_n - E) [m(X_i^T (\theta_0 + \delta), Y_i) - m(X_i^T \theta_0, Y_i)]|, \quad u \in (0, \infty),$$

now defined with respect to the $\ell_1$ norm and without the restricted set. This is the approach taken by van de Geer (2008), who shows that, under the above assumptions, there exists a constant $\tilde{C} \in (0, \infty)$ such that with probability approaching one,

$$\sup_{u \in (0, \infty)} \frac{\tilde{\varepsilon}_n(u)}{u} \leq \tilde{C} \left( \frac{\ln p}{n} + \frac{\ln p}{n} \right).$$

van de Geer (2008) demonstrates that bounds on the estimation error of the $\ell_1$-ME can be derived if $\lambda$ is chosen to exceed the right-hand side of this inequality, which motivates alternative methods to choose $\lambda$. Unfortunately, $\tilde{C}$ typically relies on design constants unknown to the researcher. Moreover, even if these constants were known, the resulting values of $\tilde{C}$ would typically be prohibitively large, yielding choices of $\lambda$ leading to trivial estimates of the vector $\theta_0$ in moderate samples. Our bounds therefore seem more suitable for devising methods to choose $\lambda$.

Proof of Theorem 3.1. We will prove the theorem with the universal constant $C \in [1, \infty)$ appearing in the statement of Lemma B.4. Let $\hat{\theta} \in \hat{\Theta}(\lambda)$ be arbitrary, fix any $t \in [1, \infty)$ satisfying the requirements of the theorem, and specify the sequences $a_{t,n} := C(1 + \tau_0)C_L \sqrt{\eta_n^{1-q}}$ and $b_{t,n} := C(1 + \tau_0)C_L s_n \eta_n^{1-q}$, so that (B.3) is satisfied (because $C(1 + \tau_0)C_L > 1 \geq c_M$) and $\tilde{a}_n \leq Cu_n$ with $\tilde{a}_n$ given in (B.2). Since $Cu_n \leq c'_M$ by assumption, we therefore obtain from Lemma B.3 that on the event $\mathcal{I}_n \cap \mathcal{L}_n \cap \mathcal{E}_n$, we have

$$\|\hat{\theta} - \theta_0\|_2 \leq Cu_n \quad \text{and} \quad \|\hat{\theta} - \theta_0\|_1 \leq \frac{2c_0}{c_0 - 1} \left( Cu_n \sqrt{s_q \eta_n^{-q} + s_q \eta_n^{1-q}} \right),$$

where we have used $1 + \tau_0 = 2c_0/(c_0 - 1)$. The asserted claim now follows from

$$P((\mathcal{I}_n \cap \mathcal{L}_n \cap \mathcal{E}_n)^c) \leq P(\mathcal{I}_n^c) + P(\mathcal{L}_n^c) + P(\mathcal{E}_n^c) \leq P(\mathcal{I}_n^c) + P(\mathcal{L}_n^c) + 4t^{-r} + C/\ln^2(pn) + n^{-1},$$

where the first inequality follows from the union bound and the second from Lemma B.4,
whose application is justified by the assumptions of the theorem.

We now turn to the proofs of the lemmas.

**Proof of Lemma B.1.** We proceed as in Negahban et al. (2012, p. 551). The first claim follows from

\[ s_q \geq \sum_{j=1}^{p} |\theta_{0,j}|^q \geq \sum_{j \in T(\eta)} |\theta_{0,j}|^q \geq |T(\eta)|\eta^q \]

upon rearrangement. The second claim follows from

\[ \|\theta_{0T(\eta)^c}\|_1 = \sum_{j \in T(\eta)^c} |\theta_{0,j}|^{1-q} |\theta_{0,j}|^q \leq \eta^{1-q} \sum_{j \in T(\eta)^c} |\theta_{0,j}|^q \leq s_q \eta^{1-q}. \]

**Proof of Lemma B.2.** The claim follows from

\[ \|\delta\|_1 = \|\delta_{T(\eta)}\|_1 + \|\delta_{T(\eta)^c}\|_1 \leq (1 + \tilde{c}) \left( \|\delta_{T(\eta)}\|_1 + \|\theta_{0T(\eta)^c}\|_1 \right) \]

\[ \leq (1 + \tilde{c}) \left( |T(\eta)|^{1/2} \|\delta_{T(\eta)}\|_2 + \|\theta_{0T(\eta)^c}\|_1 \right) \]

\[ \leq (1 + \tilde{c}) \left( \sqrt{s_q \eta^{-q}} \|\delta\|_2 + s_q \eta^{1-q} \right), \]

where the first inequality follows from \( \delta \in R(\tilde{c}, \eta) \), the second from the Cauchy-Schwarz inequality, and the third from Lemma B.1.

**Proof of Lemma B.3.** Let \( \hat{\Theta} \in \hat{\Theta}(\lambda) \) be arbitrary. We proceed in two steps. In the first step, we show that \( \hat{\Theta} - \theta_0 \in R(\tilde{c}_0, \eta_n) \) on the event \( \mathcal{S}_n \). In the second step, we derive bounds on \( \|\hat{\Theta} - \theta_0\|_2 \) and \( \|\hat{\Theta} - \theta_0\|_1 \) on the event \( \mathcal{S}_n \cap \mathcal{L}_n \cap \mathcal{E}_n \).

**Step 1:** Abbreviate \( \bar{\delta} := \hat{\Theta} - \theta_0 \). By minimization in (1.2),

\[ \mathbb{E}_n[m(X_i^\top \hat{\Theta}, Y_i) - m(X_i^\top \theta_0, Y_i)] \leq \lambda \left( \|\theta_0\|_1 - \|\hat{\Theta}\|_1 \right). \]

Let \( J \subseteq [p] \) be a for now arbitrary index set. By convexity in Assumption 3.2 followed by Hölder's inequality, score domination (\( \mathcal{S}_n \)), and the triangle inequality,

\[ \mathbb{E}_n[m(X_i^\top \hat{\Theta}, Y_i) - m(X_i^\top \theta_0, Y_i)] \geq S_n^\top (\hat{\Theta} - \theta_0) \geq -\|S_n\|_{\infty} \|\bar{\delta}\|_1 \geq -\frac{\lambda}{c_0} \left( \|\bar{\delta}_J\|_1 + \|\bar{\delta}_{J^c}\|_1 \right). \]
Moreover, since \( \hat{\theta} = \theta_0 + \hat{\delta} = \theta_{0,J} + \theta_{0,J^c} + \hat{\delta}_J + \hat{\delta}_{J^c} \), the triangle inequality shows that

\[
\|\hat{\theta}\|_1 - \|\theta_0\|_1 \geq \|\theta_{0,J} + \hat{\delta}_{J^c}\|_1 - \|\theta_{0,J^c} + \hat{\delta}_J\|_1 - \|\theta_0\|_1 \\
= \|\theta_{0,J}\|_1 + \|\hat{\delta}_{J^c}\|_1 - (\|\theta_{0,J}\|_1 + \|\hat{\delta}_J\|_1) - (\|\theta_{0,J^c}\|_1 + \|\hat{\delta}_{J^c}\|_1) \\
= \|\hat{\delta}_{J^c}\|_1 - \|\hat{\delta}_J\|_1 - 2\|\theta_{0,J^c}\|_1.
\]

Combining the three previous displays, we get

\[
\|\hat{\delta}_{J^c}\|_1 - \|\hat{\delta}_J\|_1 - 2\|\theta_{0,J^c}\|_1 \leq \|\hat{\theta}\|_1 - \|\theta_0\|_1 \leq \frac{1}{c_0} (\|\hat{\delta}_J\|_1 + \|\hat{\delta}_{J^c}\|_1),
\]

which implies that

\[
\|\hat{\delta}_J\|_1 \leq \frac{c_0 + 1}{c_0 - 1}\|\hat{\delta}_J\|_1 + \frac{2c_0}{c_0 - 1}\|\theta_{0,J^c}\|_1 = \tau_0\|\hat{\delta}_J\|_1 + (1 + \tau_0)\|\theta_{0,J^c}\|_1.
\]

Choosing \( J = T(\eta_n) \), we see that the event \( \mathcal{S}_n \) implies \( \hat{\delta} \in \mathcal{R}(\tau_0, \eta_n) \), as claimed.

**Step 2:** Define the (random) function \( \hat{F} : \mathbb{R}^p \to \mathbb{R} \) by

\[
\hat{F}(\delta) := \mathbb{E}_n \left[ m \left( X_i^\top (\theta_0 + \delta) , Y_i \right) - m \left( X_i^\top \theta_0 , Y_i \right) \right] + \lambda (\|\theta_0 + \delta\|_1 - \|\theta_0\|_1).
\]

Then \( \hat{F} \) is convex and \( \hat{F}(0_p) = 0 \). Moreover, since \( \Theta \) is convex (Assumption 3.1), \( \delta \in \mathcal{R}(\tau_0, \eta_n) \) and \( t \in [0, 1] \) imply \( \theta_0 + t\delta \in \Theta \) and

\[
\| (t\delta)_{T(\eta_n)^c} \|_1 = t \| \delta_{T(\eta_n)^c} \|_1 \leq t \left( \tau_0 \| \delta_{T(\eta_n)^c} \|_1 + (1 + \tau_0) \| \theta_0 \|_{T(\eta_n)^c} \|_1 \right) \\
= \tau_0 \| (t\delta)_{T(\eta_n)} \|_1 + (1 + \tau_0) t \| \theta_0 \|_{T(\eta_n)^c} \|_1 \\
\leq \tau_0 \| (t\delta)_{T(\eta_n)} \|_1 + (1 + \tau_0) \| \theta_0 \|_{T(\eta_n)^c} \|_1,
\]

which shows that \( t\delta \in \mathcal{R}(\tau_0, \eta_n) \). Hence, \( \mathcal{R}(\tau_0, \eta_n) \) is star-shaped with vantage point \( 0_p \).

Seeking a contradiction, suppose that we are on the event \( \mathcal{S}_n \cap \mathcal{L}_n \cap \mathcal{E}_n \), but \( \|\hat{\delta}\|_2 > \tilde{u}_n \).

Since \( \hat{\delta} \in \mathcal{R}(\tau_0, \eta_n) \) by Step 1, \( \mathcal{R}(\tau_0, \eta_n) \) being star-shaped implies \( (\tilde{u}_n/\|\hat{\delta}\|_2)\hat{\delta} \in \mathcal{R}(\tau_0, \eta_n) \).

By the definition (1.2) of \( \hat{\theta} \) as a minimizer, we have \( \hat{F}(\hat{\delta}) \leq 0 \). Convexity of \( \hat{F} \) and \( \hat{F}(0_p) = 0 \) then show that \( \hat{F}((\tilde{u}_n/\|\hat{\delta}\|_2)\hat{\delta}) \leq 0 \). Unpacking \( \hat{F} \), these findings imply

\[
0 \geq \inf_{\delta \in \mathcal{R}(\tau_0, \eta_n), \|\delta\|_2 = \tilde{u}_n} \left\{ \mathbb{E}_n \left[ m \left( X_i^\top (\theta_0 + \delta) , Y_i \right) - m \left( X_i^\top \theta_0 , Y_i \right) \right] + \lambda (\|\theta_0 + \delta\|_1 - \|\theta_0\|_1) \right\}.
\]

By superadditivity of infima and the definition of the empirical error function, on the event
\( L_n \), the right-hand side of the previous display is bounded from below by
\[
\begin{align*}
\inf_{\delta \in \mathcal{R}(\tau_0, \eta_n), \|\delta\|_2 = \bar{u}_n} & \ E \left[ m \left( X^\top (\theta_0 + \delta), Y \right) - m \left( X^\top \theta_0, Y \right) \right] \\
+ \inf_{\delta \in \mathcal{R}(\tau_0, \eta_n), \|\delta\|_2 = \bar{u}_n} & \left( E_n - E \right) \left[ m \left( X_i^\top (\theta_0 + \delta), Y_i \right) - m \left( X_i^\top \theta_0, Y_i \right) \right] \\
+ \lambda & \inf_{\delta \in \mathcal{R}(\tau_0, \eta_n), \|\delta\|_2 = \bar{u}_n} \left\{ \|\theta_0 + \delta\|_1 - \|\theta_0\|_1 \right\}
\end{align*}
\]
\[
\geq \inf_{\delta \in \mathcal{R}(\tau_0, \eta_n), \|\delta\|_2 = \bar{u}_n} \mathcal{E} (\theta_0 + \delta) - \epsilon_n (\bar{u}_n) - \bar{L}_n \sup_{\delta \in \mathcal{R}(\tau_0, \eta_n), \|\delta\|_2 = \bar{u}_n} \|\theta_0 + \delta\|_1 - \|\theta_0\|_1.
\]

Now, since \( \bar{u}_n \leq c_M' \) by hypothesis, Assumption 3.4 yields
\[
\inf_{\delta \in \mathcal{R}(\tau_0, \eta_n), \|\delta\|_2 = \bar{u}_n} \mathcal{E} (\theta_0 + \delta) \geq \inf_{\delta \in \mathcal{R}(\tau_0, \eta_n), \|\delta\|_2 = \bar{u}_n} \mathcal{E} (\theta_0 + \delta) \geq c_M \inf_{\delta \in \mathcal{R}(\tau_0, \eta_n), \|\delta\|_2 = \bar{u}_n} \|\delta\|_2^2 = c_M \bar{u}_n^2.
\]

Also, the triangle inequality followed by Lemma B.2 show that
\[
\sup_{\delta \in \mathcal{R}(\tau_0, \eta_n), \|\delta\|_2 = \bar{u}_n} \|\theta_0 + \delta\|_1 - \|\theta_0\|_1 \leq \sup_{\delta \in \mathcal{R}(\tau_0, \eta_n), \|\delta\|_2 = \bar{u}_n} \|\delta\|_1 \leq (1 + \bar{c}_0) \left( \bar{u}_n \sqrt{s_q \eta_n^{-q} + s_q \eta_n^{-q}} \right).
\]

In addition, \( \epsilon_n (\bar{u}_n) \leq a_{\epsilon,n} \bar{u}_n + b_{\epsilon,n} \) on the event \( \mathcal{E}_n \). Harvesting the results, it follows that
\[
0 \geq \inf_{\delta \in \mathcal{R}(\tau_0, \eta_n), \|\delta\|_2 = \bar{u}_n} \mathcal{E} (\theta_0 + \delta) - \epsilon_n (\bar{u}_n) - \bar{L}_n \sup_{\delta \in \mathcal{R}(\tau_0, \eta_n), \|\delta\|_2 = \bar{u}_n} \|\theta_0 + \delta\|_1 - \|\theta_0\|_1
\]
\[
\geq c_M \bar{u}_n^2 - (a_{\epsilon,n} \bar{u}_n + b_{\epsilon,n}) - (1 + \bar{c}_0) \bar{L}_n \left( \bar{u}_n \sqrt{s_q \eta_n^{-q} + s_q \eta_n^{-q}} \right)
\]
\[
= c_M \bar{u}_n^2 - \left( a_{\epsilon,n} + (1 + \bar{c}_0) \bar{L}_n \sqrt{s_q \eta_n^{-q}} \right) \bar{u}_n - \left( b_{\epsilon,n} + (1 + \bar{c}_0) \bar{L}_n s_q \eta_n^{-q} \right)
\]
\[
=: A_n \bar{u}_n^2 - B_n \bar{u}_n - C_n.
\]

The right-hand side quadratic in \( \bar{u}_n \) has \( A_n, B_n, C_n \in (0, \infty) \). Observing that \( \bar{u}_n \) is actually equal to \( 2B_n/A_n \) by the definition in (B.2), the right-hand side equals \( 2B_n^2/A_n - C_n \). Since \( B_n^2/A_n \geq C_n \) by (B.3) and \( C_n > 0 \), we arrive at the desired contradiction. We therefore conclude that provided \( \bar{u}_n \leq c_M' \), on the event \( \mathcal{L}_n \cap \mathcal{L}_n \cap \mathcal{E}_n \), we have \( \|\delta\|_2 \leq \bar{u}_n \), which establishes the \( \ell_2 \) bound. The \( \ell_1 \) bound then follows from \( \tilde{\delta} \) belonging to \( \mathcal{R}(\tau_0, \eta_n) \) and Lemma B.2. \( \Box \)
**Proof of Lemma B.4.** The claim will follow from an application of the maximal inequality in Theorem D.1. First, fix \( n \in \mathbb{N} \), \( t \in [1, \infty) \) and \( u \in (0, \infty) \) satisfying (B.4) and denote \( \Delta(u, \eta_n) := \mathcal{R}(\tau_0, \eta_n) \cap \{ \| \cdot \|_2 \leq u \} \). Lemma B.2 shows that

\[
\| \Delta(u, \eta_n) \|_1 := \sup_{\delta \in \Delta(u, \eta_n)} \| \delta \|_1 \leq (1 + \tau_0) \left( u \sqrt{s_q \eta_n^{-q} + s_q \eta_n^{1-q}} \right) =: \overline{\Delta}_n(u). \tag{B.5}
\]

Setting up for an application of Theorem D.1, define \( h : \mathbb{R} \times \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) by \( h(t, x, y) := m(x^\top \theta_0 + t, y) - m(x^\top \theta_0, y) \) for all \( t \in \mathbb{R} \) and \((x, y) \in \mathcal{X} \times \mathcal{Y} \). By construction, \( h(0, \cdot, \cdot) \equiv 0 \).

By Assumption 3.5.1, the restriction \( h : [-c_L, c_L] \times \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) is \( L(x, y) \)-Lipschitz in its first argument, thus verifying Condition 1 of Theorem D.1 with \( C_h = c_L \). Hölder’s inequality, Assumption 3.5.1, (B.5), and (B.4) imply that

\[
\max_{1 \leq i \leq n} \sup_{\delta \in \Delta(u, \eta_n)} \| X_i^\top \delta \| \leq \max_{1 \leq i \leq n} \| X_i \|_\infty \| \Delta(u, \eta_n) \|_1 \leq tn^{1/r} B_n \overline{\Delta}_n(u) \leq c_L
\]

with probability at least \( 1 - t^{-r} \), where the bound \( \mathbb{P}(\max_{1 \leq i \leq n} \| X_i \|_\infty > tn^{1/r} B_n) \leq t^{-r} \) follows from Markov’s inequality, since Assumption 3.5.1 implies that \( E[\| X \|_\infty] \leq B_n \). Condition 2 of Theorem D.1 therefore holds with \( C_h = c_L \) and \( \zeta_n = t^{-r} \). Given that \( s_q, t \in [1, \infty) \), and \( B_n \in [1, \infty) \), (B.4) implies that \( u \leq c_L \). Therefore, it follows from Assumption 3.5.2 that

\[
\sup_{\delta \in \Delta(u, \eta_n)} E[h(X^\top \delta, X, Y)^2] \\
\leq \sup_{\| \theta_0 + \delta \|_2 \leq u} E[\| m(X^\top (\theta_0 + \delta), Y) - m(X^\top \theta_0, Y) \|^2] \leq C_L^2 u^2,
\]

and so Condition 3 of Theorem D.1 holds for \( B_{1n} = C_L u \). For (the final) Condition 4 of Theorem D.1, we invoke Theorem D.3. Observe first that \( L(X, Y)^2 X_j^2 \geq 0 \) for all \( j \in [p] \). Assumption 3.5.1 entails \( \max_{1 \leq j \leq p} E[L(X, Y)^2 X_j^2] \leq C_L^2 \), so Condition 1 of Theorem D.3 holds with \( \mu_n = C_L^2 \), a constant. Assumption 3.5.1 also implies that \( E[\max_{1 \leq j \leq p} L(X, Y)^4 X_j^4] \leq B_n^4 \), so Condition 2 of Theorem D.3 holds with \( q = 2 \) and \( M_n = B_n^2 \). Equation (B.4) shows that \( B_n^2 \ln(pn)/n^{1/2} \leq C_L^2 \), which verifies Condition 3 of Theorem D.3. Theorem D.3 therefore shows that there is a universal constant \( C \in [1, \infty) \) such that

\[
\max_{1 \leq j \leq p} \mathbb{E}_n \left[ L(X_i, Y_i)^2 X_{i,j}^2 \right] \leq (CC_L)^2 \tag{B.6}
\]

with probability at least \( 1 - C/\ln^2(pn) \). Condition 4 of Theorem D.1 therefore holds with \( B_{2n} = CC_L \) and \( \gamma_n = C/\ln^2(pn) \). Theorem D.1 combined with the bound \( \overline{\Delta}_n(u) \) on
\[ \|\Delta(u, \eta_n)\|_1 \text{ from (B.5) and } \ln(8pn) \leq 4\ln(pn) \text{ (which follows from } p \geq 2) \text{ now show that} \]
\[ P\left(\sqrt{n}\epsilon_n(u) > \{4C_Lu\} \vee \left\{16\sqrt{2}CC_L\Delta_n(u)\sqrt{\ln(pn)}\right\}\right) \leq 4t^{-r} + 4C/\ln^2(pn) + n^{-1}. \quad \text{(B.7)} \]

Now, given that \(s, C \in [1, \infty), p \in [2, \infty), n \in [3, \infty)\) and \(\eta_n \in (0, 1]\), it follows that
\[ 4\sqrt{2}(1 + \bar{c}_0)[s\eta_n^q\ln(pn)]^{1/2} \geq 1, \]
and so \(16\sqrt{2}CC_L\Delta_n(u)\sqrt{\ln(pn)} \geq 4C_Lu\). Therefore, the asserted claim follows from (B.7) upon recognizing that
\[ \Delta_n(u)\sqrt{\frac{\ln(pn)}{n}} = (1 + \bar{c}_0)\left(u\sqrt{s_q\eta_n^{2-q}} + s_q\eta_n^{2-q}\right) \]
and redefining the universal constant \(C\) appropriately. \(\square\)

### B.2 Proofs for Section 4.1

For the arguments in this section, we first introduce some additional notation. Since \(\theta_0\) is interior to \(\Theta\) (Assumption 3.1), there is a radius \(\tau_n \in (0, \infty)\) such that \(\tau_n \leq \min(c_L, c'_M)\) and the ball \(B_{\theta_0}(\tau_n) := \{\theta \in \mathbb{R}^p; \|\theta - \theta_0\|_2 \leq \tau_n\}\) is a subset of \(\Theta\), with \(c_L, c'_M \in (0, \infty]\) provided by Assumptions 3.4 and 3.5, respectively. Fix \(\theta \in B_{\theta_0}(\tau_n)\), and define
\[ f(\tau, z) := m(x^\top\theta_\tau, y) - m(x^\top\theta_0, y) \text{ for each } (\tau, z) \in \mathbb{R} \times Z, \quad \text{(B.8)} \]
\[ g(\tau) := E[f(\tau, Z)] \text{ for each } \tau \in [-1, 1], \quad \text{(B.9)} \]

where we employ the shorthand notations, \(\theta_\tau := \theta_0 + \tau(\theta - \theta_0), \ z := (x, y), \ Z := (X, Y)\) and \(Z := X \times Y\). Below we show that \(g\) is well defined. Also, let \(f_1(\tau, z) := (\partial/\partial \tau)f(\tau, z)\) denote the partial derivative of \(f\) with respect to its first argument, when it exists.

We next state and prove the following non-asymptotic version of Lemma 4.1.

**Theorem B.1 (Non-Asymptotic Error Bounds: Generic Bootstrap Method).** Let Assumptions 3.1–3.6 and 4.1 hold, let \(\beta_n\) and \(\delta_n\) be non-random sequences in \([0, 1]\) and \([0, \infty)\), respectively, such that
\[ P\left(\mathbb{E}_n[\hat{U}_i - U_i]^2 > \delta_n^2/\ln^2(pn)\right) \leq \beta_n, \quad \text{(B.10)} \]

let \(\hat{\Theta}(\hat{\lambda}_{bn})\) be the solutions to the \(\ell_1\)-penalized M-estimation problem (1.2) with penalty level.
\(\lambda = \widehat{\lambda}_b^m\) given in (4.4), and define

\[
u_{n,\alpha} := \frac{4c_0^2 \sqrt{s_q \eta_n^q}}{(c_0 - 1)c_M} \left( C_L \eta_n + c_0 C_U \sqrt{\frac{\ln(p/\alpha)}{n}} \right).
\] (B.11)

Then there is a constant \(C_1 \in [1, \infty)\), depending only on \(c_U\) and \(C_U\), and a universal constant \(C_2 \in [1, \infty)\) such that with

\[
\rho_n := C_1 \max \left\{ \beta_n + t^{-r}, t n^{1/r} B_n \delta_n, \left( \frac{\bar{B}_n^4 \ln^7 (pn)}{n} \right)^{1/6}, \frac{1}{\ln^2 (pn)} \right\},
\]

for \(n \in \mathbb{N}\) and \(t \in [1, \infty)\) satisfying

\[
\eta_n \leq 1, \quad C_2 \nu_{n,\alpha} \leq c_M', \quad \frac{B_n^2 \ln (pn)}{\sqrt{n}} \leq C_L^2, \quad \frac{\bar{B}_n^2 \ln (pn)}{\sqrt{n}} \leq C_U^2
\]

and

\[
t n^{1/r} B_n \left( C_2 \nu_{n,\alpha} \sqrt{s_q \eta_n^q + s_q \eta_1^n} \right) \leq \frac{(c_0 - 1) c_L}{2c_0},
\]

we have both

\[
\sup_{\hat{\theta} \in \hat{\Theta}(\lambda_b^m)} \| \hat{\theta} - \theta_0 \|_2 \leq C_2 \nu_{n,\alpha} \quad \text{and} \quad \sup_{\hat{\theta} \in \hat{\Theta}(\lambda_b^m)} \| \hat{\theta} - \theta_0 \|_1 \leq \frac{2c_0}{c_0 - 1} \left( C_2 \nu_{n,\alpha} \sqrt{s_q \eta_n^q + s_q \eta_1^n} \right)
\]

with probability at least \(1 - \alpha - \rho_n - \beta_n - 2C_2/\ln^2 (pn) - 5t^{-r} - n^{-1}\).

In proving Theorem B.1, we rely on the following six lemmas, whose proofs can be found at the end of this section. Recall that \(P\) denotes the distribution of \((X, Y) = Z\).

**Lemma B.5 (L1-Boundedness of Levels).** Let Assumptions 3.1, 3.4 and 3.5 hold. Then

\[
\sup_{\tau \in [-1, 1]} \mathbb{E} \| f(\tau, Z) \| \leq c_L C_L < \infty.
\]

**Lemma B.6 (Almost Sure Existence of Partials).** Let Assumptions 3.1, 3.3, 3.4 and 3.5 hold. Then for any \(\tau \in [-1, 1]\), the partial \(f_1'(\tau, z)\) exists for \(P\)-almost every (P-a.e.) \(z \in Z\), and then

\[
f_1'(\tau, z) = m_1' \left( x^\top \theta, y \right) x^\top (\theta - \theta_0).
\] (B.12)

**Lemma B.7 (L1-Boundedness of Partials).** Let Assumptions 3.1, 3.3, 3.4, 3.5 and 4.1 hold.
Then
\[ \sup_{\tau \in [-1,1]} \mathbb{E}[|f'_{\tau}(\tau, Z)|] \leq \tau_n \sqrt{p} \left( \tilde{B}_n + C_L B_n \sqrt{F_n} \right) < \infty. \]

**Lemma B.8 (Difference Quotient Domination).** Let Assumptions 3.1–3.5 and 4.1 hold. Then there is a \( P \)-integrable function \( \overline{f} : \mathbb{Z} \to [0, \infty] \) such that
\[ |f(\tau_0 + h, z) - f(\tau_0, z)| \leq |h| \overline{f}(z) \text{ for each } \tau_0 \in [-1,1], \ h \in [-d(\tau_0), d(\tau_0)], \ z \in \mathbb{Z}, \]
where \( d(\tau_0) := |\tau_0 + 1| \land |\tau_0 - 1| \) denotes the distance to the nearest interval endpoint.

**Lemma B.9 (Existence of Derivatives).** Let Assumptions 3.1–3.5 and 4.1 hold. Then \( g \) given in (B.9) is well defined as a mapping from \([-1,1]\) to \( \mathbb{R} \). This mapping is differentiable on \((-1,1)\) with derivative given by
\[ g'(\tau) = \mathbb{E}[f'_{\tau}(\tau, Z)], \quad \tau \in (-1,1). \]
In particular,
\[ g'(0) = \mathbb{E}[U X^\top (\theta - \theta_0)] = 0. \]

**Lemma B.10 (Zero Derivative).** Let Assumptions 3.1–3.5 and 4.1 hold. Then \( \mathbb{E}[U X] = 0_p \).

**Proof of Theorem B.1.** We first set up for an application of the multiplier bootstrap consistency result in Theorem D.9. To this end, note that Assumption 4.1 implies the moment conditions (D.1) for \( Z_{i,j} \), \( b \) and \( B_n \) there equal to \( U_i X_{i,j}, c_U^2 \) and \( C_U \tilde{B}_n \), respectively.

Fix \( t \in [1, \infty) \). Assumption 3.5.1 implies that \( \mathbb{E}[\|X\|_{\infty}] \leq B_n^* \), so \( P(\max_{1 \leq i \leq n} \|X_i\|_{\infty} > tn^{1/r} B_n) \leq t^{-r} \) by Markov’s inequality. It then follows from (B.10) that the estimation error condition (D.3) for \( \tilde{Z}_{i,j} = \tilde{U}_i X_{i,j} \) holds with \( \delta_n \) and \( \beta_n \) there replaced by \( tn^{1/r} B_n \delta_n \) and \( \beta_n + t^{-r} \), respectively. Since \( U X \) is centered (cf. Lemma B.10), Theorem D.9 therefore shows that there is a constant \( C_1 \in [1, \infty) \), depending only on \( c_U \) and \( C_U \), such that
\[ \sup_{\alpha \in (0,1)} \left| \mathbb{P}(\|S_n\|_{\infty} > \widehat{q}_{\alpha} \beta_n (1 - \alpha)) - \alpha \right| \leq C_1 \max \left\{ \beta_n + t^{-r}, tn^{1/r} B_n \delta_n, \left( \frac{B_n^4 \ln^7 (pn)}{n} \right)^{1/6}, \frac{1}{\ln^2 (pn)} \right\}. \]

Taking \( \rho_n \) to be this upper bound, it thus follows by construction of the bootstrap penalty level \( \lambda_n^{\text{bm}} = \rho_n \widehat{q}_{\alpha} (1 - \alpha) \) that \( \mathbb{P}(\lambda_n^{\text{bm}} < \rho_n \|S_n\|_{\infty}) \leq \alpha + \rho_n \). We proceed to establish the claimed bounds on the estimation error for this constant \( C_1 \).

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\(^{38}\) We here invoke the scaling property that \( q_V(\alpha) = t q_V(\alpha) \) for \( t \in (0, \infty) \) and \( \alpha \in (0,1) \) both non-random and \( q_V(\alpha) \) denoting the \( \alpha \) quantile of the random variable \( V \).
To this end, assume without loss of generality that $tn^{1/r}B_n\delta_n \leq 1$ (otherwise $\rho_n \geq 1$ and the probabilistic claim becomes vacuous) and use the observation that, conditional on $\{(X_i, Y_i, U_i)\}_{i=1}^n$, the random vector $E_n[\epsilon_i \tilde{U}_i X_i]$ is centered Gaussian in $\mathbb{R}^p$ with $j$th coordinate variance $n^{-1}E_n[\tilde{U}_i^2 X_{i,j}^2]$ in combination with Theorem D.4 to see that

$$\hat{q}^{bm}(1-\alpha) \leq (2 + \sqrt{2}) \sqrt{\frac{\ln(p/\alpha)}{n}} \max_{1 \leq j \leq p} E_n[\tilde{U}_i^2 X_{i,j}^2].$$

In addition, there is a universal constant $\tilde{C} \in [1, \infty)$ such that with probability at least $1 - \tilde{C}/\ln^2(pn) - t^{-r} - \beta_n$,

$$\max_{1 \leq j \leq p} E_n[\tilde{U}_i^2 X_{i,j}^2] \leq 2 \max_{1 \leq j \leq p} \left( E_n[U_i^2 X_{i,j}^2] + E_n[(\tilde{U}_i - U_i)^2 X_{i,j}^2] \right)$$

$$\leq 2(\tilde{C}C_U)^2 + 2tn^{1/r}B_n^2 E_n[(\tilde{U}_i - U_i)^2]$$

$$\leq 2(\tilde{C}C_U)^2 + 2tn^{1/r}B_n^2 \delta_n^2/\ln^2(pn)$$

$$\leq 4(\tilde{C}C_U)^2,$$

where the first inequality follows from the elementary inequality $(a + b)^2 \leq 2a^2 + 2b^2$, the second follows from the already established bound $P(\max_{1 \leq i \leq n} \|X_i\|_\infty > tn^{1/r}B_n) \leq t^{-r}$ and Theorem D.3 applied with $Z_j = U^2 X_j^2$, $\mu_n = C_U^2$, $q = 2$ and $M_n = \tilde{B}_n^2$ [which is justified by Assumptions 4.1.1 and 4.1.3 and the condition $\tilde{B}_n^2 \ln(pn)/\sqrt{n} \leq C_U^2$], the third from (B.10), and the fourth and final inequality follows from the inequalities $\tilde{C}C_U \geq 1 \geq tn^{1/r}B_n\delta_n$. Hence, with the same probability,

$$\hat{\lambda}_{\alpha}^{bm} \leq 2(2 + \sqrt{2})c_0 \tilde{C}C_U \sqrt{\frac{\ln(p/\alpha)}{n}} := \overline{\lambda}_n.$$

With $\overline{\lambda}_n$ defined as such the implied $u_n$ and the universal constant $C$ appearing in the statement of Theorem 3.1 satisfy $Cu_n \leq 2(2 + \sqrt{2})C\overline{\lambda}_n$. We therefore take $C_2 := 2(2 + \sqrt{2})C\tilde{C}$ as the universal constant. Up to this point, the choice of $t \in [1, \infty)$ has been arbitrary. With the restrictions placed on $n$ and $t$ in the statement of Theorem B.1 for this choice of $C_2$, the asserted probabilistic bounds on the estimation error follow from Theorem 3.1. \hfill \Box

**Proof of Lemma 4.1.** We set up for an application of Theorem B.1. First, to satisfy (B.10), for $\delta_n$ provided by (4.5), we set

$$\beta_n := P\left( E_n[(\tilde{U}_i - U_i)^2] > \delta_n^2/\ln^2(pn) \right),$$

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such that $\beta_n \to 0$. Next, from $n^{1/r}B_n s_q q_n^{1-q} \to 0$ in (4.6), we deduce that $s_q q_n^{-q} \to 0$ (recall that $B_n \geq 1$) and, thus, $\eta_n \to 0$ (recall that $s_q \geq 1$). It follows that $\eta_n \leq 1$ for sufficiently large $n$. Since $\alpha_n$ satisfies $\ln(1/\alpha_n) \leq (\ln(pn))$, we also have $u_{n,\alpha}^{bn} \leq \sqrt{s_q q_n^{2-q}} \to 0$. Letting $C_2$ be the universal constant from Theorem B.1, we eventually have $C_2 u_{n,\alpha,n}^{bn} \leq c'_M$. From $B_n^2 \ln(pn)/\sqrt{n} \to 0$ and $B_n^2 \ln^2(pn)/n \to 0$ in (4.6) we deduce that $B_n^2 \ln(pn)/\sqrt{n} \leq C_2^2$ eventually and $B_n^2 \ln(pn)/\sqrt{n} \leq C_2^2$ eventually, respectively. From $u_{n,\alpha}^{bn} \leq \sqrt{s_q q_n^{2-q}}$, we further deduce

$$u_{n,\alpha}^{bn} s_q q_n^{-q} + s_q q_n^{-q} \leq s_q q_n^{-q}.$$  

Choose now $t = t_n := 1 \vee [n^{1/r}B_n(\delta_n \vee s_q q_n^{-q})]^{-1/2}$. Then (4.6) guarantees $t_n = [n^{1/r}B_n(\delta_n \vee s_q q_n^{-q})]^{-1/2} \in [1, \infty)$ for sufficiently large $n$, and $t_n \to \infty$. By choice of $t_n$ and (4.6), we have

$$t_n n^{1/r} B_n \left( C_2 u_{n,\alpha}^{bn} \sqrt{s_q q_n^{-q} + s_q q_n^{-q}} \right) \to 0$$

implying that for sufficiently large $n$,

$$t_n n^{1/r} B_n \left( C_2 u_{n,\alpha}^{bn} \sqrt{s_q q_n^{-q} + s_q q_n^{-q}} \right) \leq \frac{(c_0 - 1) c_L}{2c_0}.$$  

The previous observations and (4.5) combine to show that the $\rho_n$ in Theorem B.1 implied by this choice of $t_n$ converges to zero. Since also $\alpha_n \to 0$, the estimation error bounds provided by Theorem B.1 hold with probability approaching one.

**Proof of Lemma B.5.** Fix $\tau \in [-1, 1]$. Since $\theta \in \mathcal{B}_{\theta_0}(r_n) \subset \Theta$ and $r_n \leq c_L$, we have $\|\theta - \theta_0\|_2 = |\tau|\|\theta - \theta_0\|_2 \leq c_L$, so using the Cauchy-Schwarz inequality followed by Assumption 3.5.2, we get

$$E[|f(\tau, Z)|] = E \left[ |m(X^\top \theta_\tau, Y) - m(X^\top \theta_0, Y)| \right]$$

$$\leq \sqrt{E \left[ |m(X^\top \theta_\tau, Y) - m(X^\top \theta_0, Y)|^2 \right]}$$

$$\leq \sqrt{C_L^2 \|\theta_\tau - \theta_0\|_2^2} \leq c_L C_L < \infty.$$  

This bound holds for all $\tau \in [-1, 1]$.  

**Proof of Lemma B.6.** By Assumption 3.3, $m'_1(x^\top \theta, y)$ exists for $P$-a.e. $z \in \mathcal{Z}$. Also, for any $\tau \in [-1, 1]$ we have $\|\theta_\tau - \theta_0\|_2 = |\tau|\|\theta - \theta_0\|_2 \leq r_n$, so that every $\theta_\tau, \tau \in [-1, 1]$, lies in $\mathcal{B}_{\theta_0}(r_n)$ and, thus, $\Theta$. Hence, for any $\tau \in [-1, 1]$ there is a set $A \subseteq \mathcal{Z}$ [possibly depending on $(\theta, \tau)$] such that $P(A) = 0$, and for all $z \in \mathcal{Z}\setminus A$, $m'_1(x^\top \theta_\tau, y)$ exists. In this case, the
The chain rule of differentiation shows that the partial $f'_1(\tau, z)$ exists and equals

$$f'_1(\tau, z) = \frac{\partial}{\partial \tau} m\left( x^\top [\theta_0 + \tilde{\tau}(\theta - \theta_0)], y \right) \bigg|_{\tilde{\tau} = \tau} = m'_1( x^\top \theta_\tau, y) x^\top (\theta - \theta_0)$$

which gives the asserted claim. \hfill \Box

**Proof of Lemma B.7.** From Lemma B.6 we know that for any $\tau \in [-1, 1]$, $f'_1(\tau, \mathbf{Z})$ exists a.s. and then takes the form in (B.12). Setting $\tau = 0$, we get

$$E \left[ |f'_1(0, \mathbf{Z})| \right] = E \left[ |m'_1( X^\top \theta_0, Y) X^\top (\theta - \theta_0)| \right]$$

$$= E \left[ |UX^\top (\theta - \theta_0)| \right]$$

$$\leq \|\theta - \theta_0\|_1 E[\|UX\|_\infty]$$

$$(\text{H"older})$$

$$\leq \tilde{B}_n \bar{r}_n \sqrt{p} < \infty,$$

where the last inequality stems from the Cauchy-Schwarz and Jensen inequalities and Assumption 4.1.3. Also, for any $\tau \in [-1, 1]$, $\|\theta_\tau - \theta_0\|_2 = |\tau|\|\theta - \theta_0\|_2 \leq \tau_n$, so that every $\theta_\tau, \tau \in [-1, 1]$, lies in $\mathcal{B}_{\theta_0}(\tau_n)$ and, thus, $\Theta$. Since $\tau_n \leq c_L$, from the Cauchy-Schwarz and Hölder inequalities and Assumptions 3.5.1 and 3.5.3 [recall that $L$ maps into $[1, \infty]$], we get

$$E \left[ |f'_1(\tau, \mathbf{Z}) - f'_1(0, \mathbf{Z})| \right] = E \left[ |m'_1( X^\top \theta_\tau, Y) - m'_1( X^\top \theta_0, Y)| X^\top (\theta - \theta_0)| \right]$$

$$\leq \sqrt{E \left[ |m'_1( X^\top \theta_\tau, Y) - m'_1( X^\top \theta_0, Y)|^2 \right]} \sqrt{E \left[ |X^\top (\theta - \theta_0)|^2 \right]}$$

$$\leq \sqrt{C'_L \tilde{B}_n \bar{r}_n \sqrt{p}}.$$  

The claim now follows from the triangle inequality and $\tau \in [-1, 1]$ being arbitrary. \hfill \Box

**Proof of Lemma B.8.** Since $m$ is (finite) convex in its first argument (Assumption 3.2) and $f(\tau, z) = m( x^\top \theta_\tau, y) - m( x^\top \theta_0, y)$, $f : \mathbb{R} \times \mathcal{Z} \to \mathbb{R}$ is (finite) convex and, thus, everywhere subdifferentiable in its first argument with compact-valued subdifferential (Rockafellar, 1970, Theorem 23.4). Letting $\partial_1 f(\tau, z)$ denote the subdifferential of $f$ with respect to its first argument evaluated at $(\tau, z)$, it follows that for any $(\tau_1, \tau_2) \in \mathbb{R}^2$ and any $(\tau'_1, \tau'_2) \in \partial_1 f(\tau_1, z) \times \partial_1 f(\tau_2, z)$,

$$f(\tau_1, z) - f(\tau_2, z) \geq \tau'_2 (\tau_1 - \tau_2) \geq - (|\tau'_1| + |\tau'_2|) |\tau_1 - \tau_2| \quad \text{and}$$

$$f(\tau_2, z) - f(\tau_1, z) \geq \tau'_1 (\tau_2 - \tau_1) \geq - (|\tau'_1| + |\tau'_2|) |\tau_1 - \tau_2|,$$
which combine to yield
\[
|f(\tau_1, z) - f(\tau_2, z)| \leq |\tau_1 - \tau_2| \left( \sup_{\tau_1^* \in \partial_1 f(\tau_1, z)} |\tau_1^*| + \sup_{\tau_2^* \in \partial_1 f(\tau_2, z)} |\tau_2^*| \right).
\]

Setting \(\tau_1 = \tau_0 + h\) and \(\tau_2 = \tau_0\), we see that
\[
|f(\tau_0 + h, z) - f(\tau_0, z)| \leq |h| F(z)
\]
for each \(\tau_0 \in [-1, 1], \ h \in [-d(\tau_0), d(\tau_0)], \ z \in Z\),

for \(F : Z \to [0, \infty]\) defined by
\[
F(z) := 2 \sup_{\tau \in [-1, 1]} \sup_{\tau^* \in \partial_1 f(\tau, z)} |\tau^*|, \ z \in Z.
\]

Note that \(F\) thus defined depends on neither \(\tau_0\) nor \(h\). It remains to show that \(F\) is \(P\)-integrable. To this end, note that for each \(\tau \in \mathbb{R}\) and \(z \in Z\), \(\partial_1 f(\tau, z)\) is the non-empty compact interval
\[
\partial_1 f(\tau, z) = [f_{1-}'(\tau, z), f_{1+}'(\tau, z)]
\]
(Rockafellar, 1970, p. 216), with \(f_{1-}'(\tau, z)\) and \(f_{1+}'(\tau, z)\) being the left and right (partial) derivatives
\[
f_{1-}'(\tau, z) := \lim_{\tau^* \uparrow \tau} \frac{f(\tau^*, z) - f(\tau, z)}{\tau^* - \tau},
f_{1+}'(\tau, z) := \lim_{\tau^* \downarrow \tau} \frac{f(\tau^*, z) - f(\tau, z)}{\tau^* - \tau},
\]
respectively, and both limits exist as real numbers per (finite) convexity of \(f(\cdot, z)\). It follows that the “inner” supremum in \(F(z)\) is attained at an interval endpoint, i.e. for all \((\tau, z) \in \mathbb{R} \times Z\),
\[
\sup_{\tau^* \in \partial_1 f(\tau, z)} |\tau^*| = |f_{1-}'(\tau, z)| \lor |f_{1+}'(\tau, z)|.
\]

Both the left and right derivatives are non-decreasing functions of \(\tau\), cf. Rockafellar (1970, Theorem 24.1) and \(f(\cdot, z)\) being finite convex (hence closed and proper). It follows that the “outer” supremum over \(\tau \in [-1, 1]\) is also attained at an interval endpoint, so
\[
\sup_{\tau \in [-1, 1]} \sup_{\tau^* \in \partial_1 f(\tau, z)} |\tau^*| = \sup_{\tau \in [-1, 1]} |f_{1-}'(\tau, z)| \lor \sup_{\tau \in [-1, 1]} |f_{1+}'(\tau, z)|
\]
\[
= \max \{ |f_{1-}'(-1, z)|, |f_{1-}'(1, z)|, |f_{1+}'(-1, z)|, |f_{1+}'(1, z)| \}.
\]
Since \(\partial_1 f(-1, Z) = \{f_1'(-1, Z)\}\) and \(\partial_1 f(1, Z) = \{f_1'(1, Z)\}\) a.s. (cf. Lemma B.6), we have
\[ f_1'_{-} (-1, Z) = f_1'_{+} (-1, Z) \text{ and } f_1'_{-} (1, Z) = f_1'_{+} (1, Z) \text{ a.s.}, \text{ and thus,} \]

\[
E \left[ \sup_{\tau \in [-1, 1]} \sup_{\tau^* \in \partial_1 f (\tau, Z)} |\tau^*| \right] = E \left[ \sup_{\tau \in [-1, 1]} |f_1'_{-} (-1, Z)| \vee |f_1'_{-} (1, Z)| \right] \\
\leq E \left[ |f_1'_{-} (-1, Z)| \right] + E \left[ |f_1'_{-} (1, Z)| \right] \\
\leq 2 \tau_n \sqrt{p \left( \tilde{B}_n + C_L B_n \sqrt{\tau_n} \right)} < \infty, \quad \text{(Lemma B.7)}
\]

implying that \( F \) is \( P \)-integrable.

**Proof of Lemma B.9.** Lemma B.5 shows that \( g \) in (B.9) is well defined as a map from \([-1, 1]\) to \( \mathbb{R} \). Lemmas B.5, B.6 and B.7 combine to verify Dudley (2014, Condition (A.2)) for our \( f \) in (B.8) for any \( \tau_0 \in (-1, 1) \) with \( \delta \) there being our \( d(\tau_0) \). Combining the difference quotient domination by a \( P \)-integrable function in Lemma B.8 with Dudley (2014, Corollary A.3) now show that \( g \) is differentiable at every \( \tau_0 \in (-1, 1) \) with

\[ g'(\tau_0) = E[f_1'(\tau_0, Z)] = E[m_1'(X^\top \theta_0, Y)X^\top (\theta - \theta_0)]. \]

It remains to show that \( g'(0) = 0 \). Since \( \theta \in B_{\theta_0}(\tau_n) \) and \( \tau_n \leq \epsilon'_M \), Assumption 3.4 tells us that for all \( h \in [-1, 1] \),

\[ g(h) - g(0) = E(\theta_0 + h(\theta - \theta_0)) \geq 0. \]

Seeking a contradiction, suppose first that \( g'(0) \in (0, \infty) \). Since the derivative exists, letting \( \{h_m\}_{m=1}^\infty \subset [-1, 0) \) be the strictly negative vanishing sequence \( h_m = -1/m \), we have

\[ g'(0) = \lim_{m \to \infty} \frac{g(h_m) - g(0)}{h_m} \in (0, \infty). \]

Hence, for all \( m \) sufficiently large,

\[ \frac{g(h_m) - g(0)}{h_m} \in (0, \infty), \]

which by \( h_m < 0 \) implies \( g(h_m) - g(0) \in (-\infty, 0) \), a contradiction. If we instead suppose that \( g'(0) \in (-\infty, 0) \), then letting \( \{h_m\}_{m=1}^\infty \subset (0, 1] \) be the strictly positive vanishing sequence \( h_m = 1/m \), we again reach a contradiction. It follows that \( g'(0) = 0 \).

**Proof of Lemma B.10.** From Lemma B.9 we know that

\[ E[U X^\top (\theta - \theta_0)] = 0 \]
for any $\theta \in B_{\theta_0}(\tau_n) \subseteq \Theta$. Seeking a contradiction, suppose that $E[UX]$ is non-zero. Then $E[UX] \in (0, \infty)$, so we may define

$$\theta := \theta_0 + \frac{\tau_n}{\|E[UX]\|_2} E[UX].$$

This $\theta$ belongs to $B_{\theta_0}(\tau_n)$, but

$$E[U^T (\theta - \theta_0)] = \tau_n \|E[UX]\|_2 \in (0, \infty),$$

a contradiction. Conclude that $E[UX] = 0_p$.

B.3 Proofs for Section 4.2

For the arguments in this section, we introduce some additional notation. For any non-empty $I \subseteq [n]$, define the subsample score by

$$S_I := E_I \left[ m'_i \left( X_i^T \theta_0, Y_i \right) X_i \right]$$

and the (random) subsample empirical error function $\epsilon_I : [0, \infty) \rightarrow [0, \infty)$ by

$$\epsilon_I(u) := \sup_{\delta \in R(\tau_0, \eta_n), \|\delta\|_2 \leq u} \left| (E_I - E) \left[ m \left( X_i^T (\theta_0 + \delta), Y_i \right) - m \left( X_i^T \theta_0, Y_i \right) \right] \right|.$$

Also, recall the notations $\eta_n = \sqrt{\ln(pn)/n}$ and $\tau_0 = (c_0 + 1)/(c_0 - 1)$ for the user-chosen constant $c_0 \in (1, \infty)$.

In proving Theorem 4.1, we will rely on the following eight lemmas, whose proofs can be found at the end of this section.

**Lemma B.11.** Let Assumption 4.3 hold. Then for any constant $C \in (0, \infty)$ satisfying $\eta_n \leq C_{\Lambda a}/C$ and $n\eta_n \geq c_{\Lambda}C_{\Lambda}/C$, the candidate penalty set $\Lambda_n$ and the interval $[C\eta_n, C\eta_n/a]$ have an element in common.

**Lemma B.12.** Let Assumptions 3.3, 3.5, 3.6 and 4.2 hold. Then there is a universal constant $C \in [1, \infty)$, such that for any $n \in \mathbb{N}$, $t \in [1, \infty)$ and $u \in (0, \infty)$ satisfying

$$\eta_n \leq 1, \quad \frac{B_n^2 \ln(pn)}{n^{1/2}} \leq C_L^2 \quad \text{and} \quad tn^{1/2} B_n \left( u \sqrt{s_q \eta_n^{-q}} + s_q \eta_n^{1-q} \right) \leq \frac{c_L}{1 + \tau_0}.$$

(B.13)
we have

\[
\max_{1 \leq k \leq K} \epsilon_{l_k}(u) \leq \frac{(1 + \bar{c}_0)CC_L}{(K - 1)c_D} \left( u \sqrt{s_q \eta_n^{2-q} + s_q \eta_n^{2-q}} \right)
\]

with probability at least \(1 - K \left( 4t^{-r} + C/\ln^2(pn) + [(K - 1)c_Dn]^{-1} \right)\).

**Lemma B.13.** Let Assumptions 3.1–3.5, 4.1 and 4.2 hold. Then there is a universal constant \(C \in [1, \infty)\), such that for any \(n \in \mathbb{N}\) satisfying

\[
\frac{\tilde{B}_n^2 \ln(pn)}{\sqrt{n}} \leq C_U \sqrt{(K - 1)c_D} \quad \text{and} \quad pn \geq \left( \frac{1}{(K - 1)c_D} \right)^2,
\]

we have

\[
\max_{1 \leq k \leq K} \| \mathbf{S}_{l_k} \|_\infty \leq \frac{CC_U \eta_n}{\sqrt{(K - 1)c_D}}
\]

with probability at least \(1 - CK/\ln^4(pn)\).

**Lemma B.14.** Let Assumptions 3.1–3.4 and 3.6 hold, and let \(a_{\epsilon,n}, b_{\epsilon,n}\) and \(\bar{\lambda}_n\) be non-random sequences in \((0, \infty)\). Define \(\tilde{u}_n\) as in (B.2) and suppose that \(\tilde{u}_n \leq c'_M\) and (B.3) are satisfied. Then for any \(k \in [K]\) and any (possibly random) \(\lambda \in \Lambda_n\), on the event

\[
\{ \lambda \geq c_0 \| \mathbf{S}_{l_k} \|_\infty \} \cap \{ \lambda \leq \bar{\lambda}_n \} \cap \{ \epsilon_{l_k}(\tilde{u}_n) \leq a_{\epsilon,n}{\tilde{u}_n} + b_{\epsilon,n} \},
\]

we have

\[
\sup_{\tilde{\theta} \in \hat{\Theta}_{l_k}(\lambda)} \mathcal{E}(\tilde{\theta}) \leq (1 + \bar{c}_0) \bar{\lambda}_n \left( \tilde{u}_n \sqrt{s_q \eta_n^{2-q} + s_q \eta_n^{2-q}} \right) + a_{\epsilon,n}{\tilde{u}_n} + b_{\epsilon,n}.
\]

**Lemma B.15.** Let Assumptions 3.1–3.6 and 4.1–4.3 hold, such that Lemmas B.12 and B.13 apply, and let \(C \in [1, \infty)\) be the largest universal constant appearing in these two lemmas. Define the constants

\[
C_1 := \frac{CC_U}{\sqrt{(K - 1)c_D}},
\]

\[
C_2 := \frac{CC_L}{(K - 1)c_D},
\]

\[
C_S := \frac{2(1 + \bar{c}_0)}{c_M} \left( \frac{c_0 C_1}{\epsilon} + C_2 \right) \quad \text{and}
\]

\[
C_\epsilon := c_M C_S (1 + C_S).
\]
Then for any $n \in \mathbb{N}$ and $t \in [1, \infty)$ satisfying both (B.14) and

$$\begin{align*}
&\left\{ \frac{c_c C_L}{c_0 C_1 n} \leq \eta_n \leq 1 \wedge \frac{C_A a}{c_0 C_1} , \quad \frac{B_c^2 \ln(pn)}{n^{1/2}} \leq C^2_L, \\
&\quad C_S \sqrt{s_0 \eta_n^{2-q}} \leq c'_M \quad \text{and} \quad (1 + C_S) t^{1/r} B_n s_q \eta_n^{1-q} \leq \frac{c_L}{1 + \eta_n} \right\} \quad (B.16)
\end{align*}$$

there is a non-random candidate penalty level $\lambda_* \in \Lambda_n$, such that

$$\max_{1 \leq k \leq K} \sup_{\theta \in \Theta_{k,\lambda}} \mathcal{E}(\theta) \leq C^*_n s_q \eta_n^{2-q}$$

with probability at least $1 - K(4t^{-r} + 2C/\ln^2(pn)) + [(K - 1)c_D n]^{-1}$.

**Lemma B.16.** Let Assumptions 3.1–3.4 hold. Then for all $\theta \in \Theta$ such that $\mathcal{E}(\theta) \leq c_M (c'_M)^2$, we have $\|\theta - \theta_0\|^2 \leq \mathcal{E}(\theta)/c_M$.

**Lemma B.17.** Let Assumptions 3.1–3.6 and 4.1–4.3 hold, let $C$, $C_S$, and $C_E$ be the constants defined in Lemma B.15, and define the non-random sequence $\mathcal{E}^*_n := C^*_n s_q \eta_n^{2-q}$. For each $(k, \lambda) \in [K] \times \Lambda_n$, fix a solution $\hat{\theta}_{k,\lambda} \in \Theta_{k,\lambda}$ to (4.8) and a solution $\hat{\lambda}^{cv}$ to (4.9) based on $\{\hat{\theta}_{k,\lambda}\}_{(k,\lambda) \in [K] \times \Lambda_n}$. Then for any $(n, t) \in \mathbb{N} \times [1, \infty)$ satisfying (B.14), (B.16) and

$$\left\{ n \geq \frac{1}{c_\Lambda \wedge a} \quad \text{and} \quad \frac{11 C_L}{4c_D c_M} \sqrt{\frac{3t \ln n}{\ln(1/a)n}} + \frac{3}{2} \sqrt{\mathcal{E}^*_n/c_D c_M} \leq c'_M \wedge c_L \right\}, \quad (B.17)$$

we have

$$\max_{1 \leq k \leq K} \|\hat{\theta}_{k,\lambda} - \theta_0\|^2 \leq \frac{11 C_L}{4c_D c_M} \sqrt{\frac{3t \ln n}{\ln(1/a)n}} + \frac{3}{2} \sqrt{\mathcal{E}^*_n/c_D c_M} \quad (B.18)$$

with probability at least $1 - K(4t^{-r} + 3t^{-1} + 2C/\ln^2(pn)) + [(K - 1)c_D n]^{-1}$.

**Lemma B.18.** Let Assumptions 3.1–3.6 and 4.1–4.3 hold, let $C$, $C_S$, and $C_E$ be the constants defined in Lemma B.15, and define the non-random sequence $\mathcal{E}^*_n = C^*_n s_q \eta_n^{2-q}$. For each $(k, \lambda) \in [K] \times \Lambda_n$, fix a solution $\hat{\theta}_{k,\lambda} \in \Theta_{k,\lambda}$ to (4.8) and a solution $\hat{\lambda}^{cv}$ to (4.9) based on $\{\hat{\theta}_{k,\lambda}\}_{(k,\lambda) \in [K] \times \Lambda_n}$, and define $\hat{U}^{cv}$ as in (4.10) based on $\{\hat{\theta}_{k,\lambda}^{\hat{\lambda}^{cv}}\}_{k \in [K]}$. Then for any $(n, t) \in \mathbb{N} \times [1, \infty)$ satisfying (B.14), (B.16) and (B.17), we have

$$\mathbb{E}[(\hat{U}^{cv} - U)^2] \leq \frac{3C^2_L t \ln n}{\ln(1/a)} \left( \frac{11 C_L}{4c_D c_M} \sqrt{\frac{3t \ln n}{\ln(1/a)n}} + \frac{3}{2} \sqrt{\mathcal{E}^*_n/c_D c_M} \right) \quad (B.19)$$

with probability at least $1 - K(4t^{-r} + 3t^{-1} + 2C/\ln^2(pn)) + [(K - 1)c_D n]^{-1}$.
Proof of Theorem 4.1. The proof will follow from Lemma 4.1, with (4.5) being verified via Lemma B.18. Observe that (4.13) ensures that there is a non-random sequence \( t_n \) in \([1, \infty)\) such that
\[
t_n \to \infty, \quad t_n n^{1/r} B_n s_q \eta_n^{1-q} \to 0 \quad \text{and} \quad \frac{t_n^3 B_n^4 s_q (\ln(pn))^{5-q/2}(\ln n)^2}{n^{1-q/2-4/r}} \to 0.
\]
Therefore, for \( \bar{E}_n^* \) appearing in the statement of Lemma B.18, setting
\[
\delta_n^2 := \frac{3C_L^2 t_n \ln n}{\ln(1/a)} \left( \frac{11C_L}{4cDcM} \sqrt{\frac{3t_n \ln n}{\ln(1/a)n}} + \frac{3}{2} \sqrt{\frac{\bar{E}_n^*}{cDcM}} \right) \ln^2(pn),
\]
a calculation shows that \( n^{1/r} B_n \delta_n \to 0 \). Together with (4.13), this implies that (4.6) is satisfied. Also, (4.13) and \( t_n n^{1/r} B_n s_q \eta_n^{1-q} \to 0 \) imply that, setting \( t = t_n \), (B.14) and (B.16) will hold for all \( n \) large enough. In addition, given that \( \delta_n \to 0 \) and \( t_n \to \infty \), it follows that, setting \( t = t_n \), (B.17) will hold for all \( n \) large enough as well. Thus, Lemma B.18 implies that, setting \( \hat{U}_i = \hat{U}_i^{cv} \) for all \( i \in [n] \), (4.5) is satisfied. The asserted claim now follows from applying Lemma 4.1.

Proof of Theorem 4.2. The asserted claim will follow from an application of Theorem C.1, which we state and prove in a separate appendix (see Section C), due to its length.

Observe that (4.17) ensures that there is a non-random sequence \( t_n \) in \([1, \infty)\) such that
\[
t_n \to \infty, \quad t_n n^{1/r} B_n s_q \eta_n^{1-q} \to 0 \quad \text{and} \quad \frac{t_n^3 B_n^4 s_q (\ln(pn))^{5-q/2}(\ln n)^2}{n^{1-q/2-4/r}} \to 0.
\]
Therefore, for \( \bar{E}_n^* \) appearing in the statement of Lemma B.18, setting
\[
\delta_n^2 := \frac{3C_L^2 t_n \ln n}{\ln(1/a)} \left( \frac{11C_L}{4cDcM} \sqrt{\frac{3t_n \ln n}{\ln(1/a)n}} + \frac{3}{2} \sqrt{\frac{\bar{E}_n^*}{cDcM}} \right) \ln^2(pn),
\]
we have \( n^{1/r} B_n \delta_n \to 0 \). Also, (4.17) and \( t_n n^{1/r} B_n s_q \eta_n^{1-q} \to 0 \) imply that (B.14) and (B.16) with \( t = t_n \) hold for all \( n \) large enough. In addition, given that \( \delta_n \to 0 \) and \( t_n \to \infty \), it follows that (B.17) with \( t = t_n \) holds for all \( n \) large enough as well. Thus, given that
\[
\text{E}[\max_{1 \leq i \leq n} \|X_i\|_\infty] \leq nB_n \quad \text{by Assumption 3.5.1, Lemma B.18 together with } n^{1/r} B_n \delta_n \to 0
\]
\[
\text{implies that}
\]
\[
P \left( \max_{1 \leq i \leq p} \mathbb{E}_n \left[ (\hat{U}_i^{cv} X_{i,j} - U_i X_{i,j})^2 \right] > \frac{\delta_n^2}{\ln^2(pn)} \right) \to 0 \quad \text{(B.20)}
\]
for some non-random sequence \( \tilde{\delta}_n \) in \( (0, 1) \) satisfying \( \tilde{\delta}_n \to 0 \). Let \( \mathcal{N} \) be a centered normal random vector in \( \mathbb{R}^p \) with covariance matrix \( \text{E}[U^2X'X] \) and for all \( \beta \in (0, 1) \), let \( q^\beta(\beta) \) be the \( \beta \)th quantile of \( ||\mathcal{N}||_\infty \). Theorem D.8 together with Assumption 4.1 and (4.17) then imply that with probability \( 1 - o(1) \),

\[
\frac{q^\beta(1 - \alpha_n - \rho_n)}{\sqrt{n}} \leq \tilde{q}^{\text{bev}}(1 - \alpha_n) \quad \text{and} \quad \tilde{q}^{\text{bev}}(1/2) \leq \frac{q^\beta(1/2 + \rho_n)}{\sqrt{n}}
\]

for some non-random sequence \( \rho_n \) in \( (0, \infty) \) satisfying \( \rho_n \to 0 \). Also, from (the median version of) Borell’s inequality (van der Vaart and Wellner, 1996, Proposition A.2.1), we get

\[
\tilde{q}^{\text{bev}}(1 - \alpha_n) \leq \tilde{q}^{\text{bev}}(1/2) + \sqrt{\max_{1 \leq j \leq p} \mathbb{E}_n[(\tilde{U}_i^{\text{bev}}X_{i,j})^2] \sqrt{\frac{2\ln (1/\alpha_n)}{n}}}
\]

Using (B.20) and the same arguments as those in the proof of Theorem B.1, we get

\[
\max_{1 \leq j \leq p} \mathbb{E}_n[(\tilde{U}_i^{\text{bev}}X_{i,j})^2] \leq 4(\tilde{C}C_V)^2
\]

with probability \( 1 - o(1) \) for some universal constant \( \tilde{C} \in [1, \infty) \). Since \( \ln(1/\alpha_n) \lesssim \ln(p n) \), we thus have

\[
\frac{q^\beta(1 - \alpha_n - \rho_n)}{\sqrt{n}} \leq \tilde{q}^{\text{bev}}(1 - \alpha_n) \leq \frac{q^\beta(1/2 + \rho_n)}{\sqrt{n}} + C\eta_n
\]

with probability \( 1 - o(1) \) for some constant \( C \in [1, \infty) \). Setting \( \underline{\lambda}_n \) and \( \overline{\lambda}_n \) to be \( c_0 \) times the left-hand side and the right-hand side of this chain of inequalities, respectively, we thus have \( \text{P}(\underline{\lambda}_n^{\text{bev}} > \overline{\lambda}_n) \to 0 \) and \( \text{P}(\overline{\lambda}_n^{\text{bev}} < \underline{\lambda}_n) \to 0 \). The same arguments as those used to invoke the multiplier bootstrap consistency result (Theorem D.9) in the proof Theorem B.1 show that \( \text{P}(\underline{\lambda}_n^{\text{bev}} < C_0 ||\mathcal{S}_n||_\infty) \leq \alpha_n + \rho_n \to 0 \). Since \( \rho_n \to 0 \), we must eventually have \( q^\beta(1/2 + \rho_n) \leq q^\beta(2/3) \). Using the Gaussian quantile bound (Lemma D.4) alongside Assumption 4.1, we therefore get \( \overline{\lambda}_n \lesssim \eta_n \). Similarly, because \( \alpha_n \to 0 \) and \( \rho_n \to 0 \), we must eventually have \( q^\beta(1 - \alpha_n - \rho_n) \geq q^\beta(1/2) \). Lower bounding the maximum \( ||\mathcal{N}_1||_\infty \) by the coordinate \( |\mathcal{N}_1| \) (for example), and observing that the quantile of a folded normal distribution scales linearly with its standard deviation, again using Assumption 4.1, we get \( n^{-1/2} \lesssim \underline{\lambda}_n \). It follows that \( \phi_n := ((\eta_n^2 + \overline{\lambda}_n^2)/\underline{\lambda}_n^2)^{1/2} \lesssim \sqrt{\ln(p n)} \). The asserted claim now follows from an application of Theorem C.1, which is justified by (4.17).

**Proof of Lemma B.11.** Fix any \( C \in (0, \infty) \) satisfying \( \eta_n \lesssim C\lambda a/C \) and \( n\eta_n \geq c\lambda C\lambda/C \). Denote \( b_n := C\eta_n \). We will show that there is an integer \( \ell_0 \in \{0, 1, 2, \ldots \} \) such that

\[
c\lambda C\lambda/n \leq b_n \leq C\lambda a^{\ell_0} \leq b_n/a.
\]

(B.21)
By Assumption 4.3, this will imply that $C\Lambda a^\ell_0$ belongs to both the candidate penalty set $\Lambda_n$ and the interval $[C\eta_n, C\eta_n/a]$. To prove (B.21), note that the condition $\eta_n \leq C\Lambda a/C$ implies that

$$0 \leq \frac{\ln (b_n/C\Lambda)}{\ln a} - 1.$$  

Thus, there exists an integer $\ell_0 \in \{0, 1, 2, \ldots\}$ such that

$$\frac{\ln (b_n/C\Lambda)}{\ln a} - 1 \leq \ell_0 \leq \frac{\ln (b_n/C\Lambda)}{\ln a}.$$  

In turn, the latter implies that $b_n \leq C\Lambda a^\ell_0 \leq b_n/a$. Moreover, the condition $n\eta_n \geq C\Lambda C\Lambda n/C$ means that $c\Lambda C\Lambda n/n \leq b_n$. Combining these inequalities gives (B.21) and completes the proof of the lemma.

**Proof of Lemma B.12.** The claim will follow from $K$ applications of the maximal inequality in Theorem D.1 in combination with the union bound. The proof is very similar to that of Lemma B.4. We include the steps for the sake of completeness. First, fix $n \in \mathbb{N}$, $t \in [1, \infty)$ and $u \in (0, \infty)$ satisfying (B.13) and denote $\Delta(u, \eta_n) := R(c_0, \eta_n) \cap \{\|\cdot\|_2 \leq u\}$.

Lemma B.2 shows that

$$\|\Delta(u, \eta_n)\|_1 := \sup_{\delta \in \Delta(u, \eta_n)} \|\delta\|_1 \leq (1 + c_0) \left( u\sqrt{s_q\eta_n^{-q} + s_q\eta_n^{1-q}} \right) =: \Delta_n(u).$$  

(B.22)

Setting up for an application of Theorem D.1, define $h : \mathbb{R} \times X \times Y \to \mathbb{R}$ by $h(t, x, y) := m(x^\top \theta_0 + t, y) - m(x^\top \theta_0, y)$ for all $t \in \mathbb{R}$ and $(x, y) \in X \times Y$. By construction, $h(0, \cdot, \cdot) \equiv 0$. By Assumption 3.5.1, the restriction $h : [-c_L, c_L] \times X \times Y \to \mathbb{R}$ is $L(x, y)$-Lipschitz in its first argument, thus verifying Condition 1 of Theorem D.1 with $C_h = c_L$. Hölder’s inequality, Assumption 3.5.1, (B.22), and (B.13) imply that

$$\max_{1 \leq k \leq K} \max_{i \in I_k} \sup_{\delta \in \Delta(u, \eta_n)} \|X_i^\top \delta\| \leq \max_{1 \leq i \leq n} \|X_i\|_\infty \|\Delta(u, \eta_n)\|_1 \leq tn^{1/r}B_n \Delta_n(u) \leq c_L$$

with probability at least $1 - t^{-r}$, where the bound $P(\max_{1 \leq i \leq n} \|X_i\|_\infty > tn^{1/r}B_n) \leq t^{-r}$ follows from Markov’s inequality since Assumption 3.5.1 implies that $E[\|X\|_\infty] \leq B_n^r$. Condition 2 of Theorem D.1 thus holds with $C_h = c_L$ and $\zeta_n = t^{-r}$. Further, given that $s_q, t, B_n \in [1, \infty)$, and $\eta_n \leq 1$ by assumption, (B.13) implies that $u \leq c_L$. Therefore, it
follows from Assumption 3.5.2 that
\[
\sup_{\delta \in \Delta(u, \eta_n)} \mathbb{E} \left[ h(\mathbf{X}^\top \mathbf{\delta}, \mathbf{X}, \mathbf{Y})^2 \right] \\
\leq \sup_{\mathbf{\theta}_0 + \delta \in \Theta} \mathbb{E} \left[ \left| m(\mathbf{X}^\top (\mathbf{\theta}_0 + \delta), \mathbf{Y}) - m(\mathbf{X}^\top \mathbf{\theta}_0, \mathbf{Y}) \right|^2 \right] \leq C_L^2 u^2,
\]
and so Condition 3 of Theorem D.1 holds for \( B_1 = C_L u \). For (the final) Condition 4 of Theorem D.1, note that for some universal constant \( C \in [1, \infty) \), we have
\[
\max_{1 \leq k \leq K} \max_{1 \leq j \leq p} \mathbb{E}_{I_k}[L(\mathbf{X}_i, \mathbf{Y}_i)^2 X_{i,j}^2] \leq \frac{1}{(K - 1)c_D} \max_{1 \leq j \leq p} \mathbb{E}_{n}[L(\mathbf{X}_i, \mathbf{Y}_i)^2 X_{i,j}^2] \leq \frac{(CC_L)^2}{(K - 1)c_D}
\]
with probability at least \( 1 - C/\ln^2(pn) \), where the first (deterministic) inequality follows from Assumption 4.2 and the second (probabilistic) inequality follows from the argument leading to (B.6) in the proof of Lemma B.4. Condition 4 of Theorem D.1 thus holds with \( \gamma_n = C/\ln^2(pn) \) and the now \((K, c_D)\)-dependent \( B_2 = DCC_L/\sqrt{(K - 1)c_D} \). Theorem D.1 combined with the bound \( \overline{\Delta}_n(u) \) on \( \| \Delta(u, \eta_n) \|_1 \) from (B.22) and \( \ln(8pn) \leq 4\ln(pn) \) (which follows from \( p \geq 2 \)) now show that for any given \( k \in [K] \), we have
\[
P \left( \sqrt{|I_k|} \epsilon_{I_k}(u) > \{4C_L u\} \cup \left\{ 16\sqrt{2}CC_L \overline{\Delta}_n(u) \sqrt{\frac{\ln(pn)}{(K - 1)c_D}} \right\} \right) \leq 4t^{-r} + 4C/\ln^2(pn) + \left( (K - 1)c_D n \right)^{-1}, \tag{B.23}
\]
where we also used Assumption 4.2 to bound \( |I_k|^{-1} \). Next, Assumption 4.2 also implies that \( (K - 1)c_D \leq Kc_D \leq 1 \). Now, given that \( s_q, C \in [1, \infty), p \in [2, \infty), n \in [3, \infty) \), and \( \eta_n \in (0, 1) \), it follows that \( 4\sqrt{2}C(1 + \bar{c}_0)\{s_q \bar{c}_n q \ln(pn)/[(K - 1)c_D]\}^{1/2} \geq 1 \), and so \( 16\sqrt{2}CC_L \overline{\Delta}_n(u) \{\ln(pn)/\left[ (K - 1)c_D \right]\}^{1/2} \geq 4C_L u \). It follows from (B.23) that
\[
\epsilon_{I_k}(u) \leq 16\sqrt{2}CC_L \overline{\Delta}_n(u) \sqrt{\frac{\ln(pn)}{(K - 1)c_D|I_k|}} \leq 16\sqrt{2}CC_L \overline{\Delta}_n(u) \eta_n
\]
with probability at least \( 1 - (4t^{-r} + 4C/\ln^2(pn) + \left( (K - 1)c_D n \right)^{-1}) \). The latter bound is uniform in \( k \in [K] \). Redefining the universal constant \( C \) appropriately, the union bound produces the desired result. \( \square \)

**Proof of Lemma B.13.** The claim will follow from \( K \) applications of Theorem D.2 in combination with the union bound. Fix \( k \in [K] \). We invoke Theorem D.2 with \( \mathbf{Z}_i = U_i \mathbf{X}_i \), and \( i \in I_k \), such that \( n \) there corresponds to \( |I_k| \). Lemma B.10 shows that these random
variables are centered. Assumption 4.1.1 involves \( \max_{1 \leq j \leq p} \mathbb{E}[|UX_j|^2] \leq C_U^2 \), so Condition 1 of Theorem D.2 is satisfied with \( \sigma_n = C_U \), a constant. Assumption 4.1.3 shows that Condition 2 of Theorem D.2 is satisfied with \( q = 4 \) and \( M_n = \bar{B}_n \). For (the final) Condition 3 of Theorem D.2, observe from (B.14) that

\[
\frac{\tilde{B}_n \sqrt{\ln(pn)}}{|I_k^c|^{1/4}} \leq \frac{\tilde{B}_n \sqrt{\ln(pn)}}{(K-1)c_D n^{1/4}} \leq C_U,
\]

where \( |I_k^c| \geq (K-1)c_D n \) follows from Assumption 4.2. Therefore, applying Theorem D.2, we obtain that there is a universal constant \( C \in [1, \infty) \), such that

\[
P\left( \sqrt{|I_k^c|} \| S_{I_k^c} \|_\infty > CC_U \sqrt{\ln(p|I_k^c|)} \right) \leq \frac{C}{\ln^4(p|I_k^c|)} \leq \frac{C}{\ln^4(p(K-1)c_D n)} \leq \frac{2^4 C}{\ln^4(pn)},
\]

where the last inequality follows from \( pn \geq 1/(K-1)c_D^2 \). Hence, with probability at least \( 1 - 2^4 C/\ln^4(pn) \),

\[
\| S_{I_k^c} \|_\infty \leq CC_U \sqrt{\frac{\ln(p|I_k^c|)}{|I_k^c|}} \leq \frac{CC_U}{\sqrt{(K-1)c_D} n} \sqrt{\frac{\ln(pn)}{n}} = \frac{CC_U \eta_n}{\sqrt{(K-1)c_D}}.
\]

The latter bound is uniform in \( k \in [K] \). The claim now follows from combining this inequality with the union bound, and redefining the universal constant \( C \) appropriately. \( \square \)

**Proof of Lemma B.14.** Fix \( k \in [K] \) and observe that Lemma B.3 still holds if we replace \( S_n, \epsilon_n(\bar{u}_n) \), and \( \mathcal{S}_n \cap \mathcal{L}_n \cap \mathcal{E}_n \) by \( S_{I_k^c}, \epsilon_{I_k^c}(\bar{u}_n) \), and (B.15), respectively. Fix \( \hat{\theta} \in \hat{\Theta}_{I_k^c}(\lambda) \) and denote \( \tilde{\theta}_i = \hat{\theta} - \theta_0 \). Then \( \tilde{\theta} \in \mathcal{R}(\bar{c}_0, \eta_n) \), \( \| \tilde{\theta} \|_2 \leq \bar{u}_n \) and \( \| \tilde{\theta} \|_1 \leq (1 + \bar{c}_0)(\bar{u}_n \sqrt{s_q \eta_n^q + s_q \eta_n^{1-q}}) \) on the event (B.15). It follows that

\[
\mathcal{E}(\hat{\theta}) \leq \mathbb{E}_{I_k^c} \left[ m(X_i^\top \hat{\theta}, Y_i) \right] - \mathbb{E}_{I_k^c} \left[ m(X_i^\top \theta_0, Y_i) \right] + \epsilon_{I_k^c}(\bar{u}_n)
\]

\[
\leq \lambda \| \theta_0 \|_1 - \| \hat{\theta} \|_1 + \epsilon_{I_k^c}(\bar{u}_n)
\]

\[
\leq \bar{c}_0 \| \tilde{\theta} \|_1 + \epsilon_{I_k^c}(\bar{u}_n)
\]

\[
\leq (1 + \bar{c}_0) \bar{c}_0 \left( \bar{u}_n \sqrt{s_q \eta_n^q + s_q \eta_n^{1-q}} \right) + a_{\epsilon,n} \bar{u}_n + b_{\epsilon,n},
\]

where the first inequality follows from the definition of \( \epsilon_{I_k^c}(\bar{u}_n) \), the second from the definition of \( \hat{\theta} \), the third from the triangle inequality and (B.15), and the fourth from (B.15) again. The asserted claim now follows from taking the supremum over \( \hat{\theta} \in \hat{\Theta}_{I_k^c}(\lambda) \). \( \square \)

**Proof of Lemma B.15.** Let \( (n, t) \in \mathbb{N} \times [1, \infty) \) satisfy (B.14) and (B.16). Then Lemma B.11 with the constant there equal to \( c_0 C_1 \) shows that \( [c_0 C_1 \eta_n, c_0 C_1 \eta_n/a] \cap \Lambda_n \neq \emptyset \), so we can
Proof of Lemma B.17. Fix any \( \lambda_s \in \Lambda_n \) satisfying \( c_0 C_1 \eta_n \leq \lambda_s \leq c_0 C_1 \eta_n / a \). Specify the non-random numbers

\[
\bar{\lambda}_n := \frac{c_0 C_1 \eta_n}{a}, \quad a_{\epsilon,n} := (1 + \bar{\epsilon}_0) C_2 \sqrt{s_q q^{2-q}} \quad \text{and} \quad b_{\epsilon,n} := (1 + \bar{\epsilon}_0) C_2 s_q q^{2-q}.
\]

The via (B.2) implies \( \bar{u}_n \) then reduces to \( C_S(s_q q^{2-q})^{1/2} =: \bar{u}_n \). Then \( \bar{u}_n \leq c'_M \) by (B.16), the requirement (B.3) reduces to \( C_S \geq 2 \), which holds true by construction. Define the events

\[
\mathcal{Z}_k := \{ \| S_{I_k} \|_{\infty} \leq C_1 \eta_n \} \quad \text{and} \quad \delta_k := \{ \epsilon_{I_k} (\bar{u}_n) \leq a_{\epsilon,n} \bar{u}_n + b_{\epsilon,n} \}, \quad k \in [K].
\]

Then Lemma B.14 and (B.16) imply that, for any \( k \in [K] \), on \( \mathcal{Z}_k \cap \delta_k \), the penalty level \( \lambda_s \) yields

\[
\sup_{\tilde{\theta} \in \Theta_{I_k} (\lambda_s)} \mathcal{E} (\tilde{\theta}) \leq (1 + \bar{\epsilon}_0) \bar{\lambda}_n \left( \bar{u}_n \sqrt{s_q q^{2-q} + s_q q^{1-q}} \right) + a_{\epsilon,n} \bar{u}_n + b_{\epsilon,n} = C_\varepsilon s_q q^{2-q}.
\] (B.24)

Further, Lemma B.12 (with \( u = \bar{u}_n \)) and (B.16) show that

\[
P \left( (\cap_{k=1}^K \delta_k)^c \right) \leq K \left( 4t^{-r} + C / \ln^2 (pn) + [(K - 1) c_D n]^{-1} \right).
\]

Finally, Lemma B.13 and (B.14) show that

\[
P \left( (\cap_{k=1}^K \mathcal{Z}_k)^c \right) \leq CK / \ln^4 (pn).
\]

It thus follows from the union bound and \( pn \geq e \) that (B.24) holds simultaneously for all \( k \in [K] \) with probability at least \( 1 - K (4t^{-r} + 2C / \ln^2 (pn) + [(K - 1) c_D n]^{-1}) \). \( \square \)

Proof of Lemma B.16. Fix any \( \theta \in \Theta \) such that \( \mathcal{E}(\theta) \leq c_M (c'_M)^2 \). If \( \| \theta - \theta_0 \|_2 \leq c'_M \), then \( \| \theta - \theta_0 \|_2 \leq \mathcal{E}(\theta) / c_M \) is immediate from Assumption 3.4. It thus suffices to prove that the case \( \| \theta - \theta_0 \|_2 > c'_M \) is not possible. Seeking a contradiction, suppose that \( \| \theta - \theta_0 \|_2 > c'_M \). Then we must have \( \| \theta - \theta_0 \|_2 > 0 \) and \( c'_M \in (0, \infty) \). It follows that \( u := c'_M / \| \theta - \theta_0 \|_2 \in (0, 1) \), and so defining \( \tilde{\theta} := \theta_0 + u (\theta - \theta_0) \), we have \( \| \tilde{\theta} - \theta_0 \|_2 = u \| \theta - \theta_0 \|_2 = c'_M \). Using Assumptions 3.2 and 3.4, we therefore see that

\[
c_M (c'_M)^2 = c_M \| \tilde{\theta} - \theta_0 \|_2 \leq \mathcal{E}(\tilde{\theta}) \leq (1 - u) \mathcal{E}(\theta_0) + u \mathcal{E}(\theta) = u \mathcal{E}(\theta) \leq \frac{c_M (c'_M)^3}{\| \theta - \theta_0 \|_2}.
\]

This implies that \( \| \theta - \theta_0 \|_2 \leq c'_M \), which contradicts \( \| \theta - \theta_0 \|_2 > c'_M \). Thus, the case \( \| \theta - \theta_0 \|_2 > c'_M \) is not possible, and the proof is complete. \( \square \)

Proof of Lemma B.17. Fix \( (n, t) \in \mathbb{N} \times [1, \infty) \) satisfying (B.14), (B.16) and (B.17) and
define the non-random sequences
\[ q_n := C_L \sqrt{\frac{tE^*}{c_Dc_Mn}} + \tilde{\varepsilon}_n^*, \quad \tilde{q}_n := C_L^2 \frac{3t \ln n}{c_D \ln(1/a)n} + C_L \frac{q_n}{c_M} \sqrt{\frac{3t \ln n}{c_D \ln(1/a)n}}, \]
\[ \tilde{u}_{1,n} := C_L \frac{3t \ln n}{c_D \ln(1/a)n} + \frac{q_n}{c_M}, \quad \text{and} \quad \tilde{u}_{2,n} := C_L \frac{3t \ln n}{c_D \ln(1/a)n} + \frac{q_n + \tilde{q}_n}{c_Dc_M}. \]

Note that since \( c_D \in (0, 1) \) by Assumption 4.2, we have \( \tilde{u}_{1,n} < \tilde{u}_{2,n} \) and, as we will show at the end of this proof via elementary inequalities, \( \tilde{u}_{2,n} \) is smaller than the right-hand side of (B.18). Therefore, \( \tilde{u}_{1,n} \lor \tilde{u}_{2,n} \leq c'_N \land c_L \) by (B.17). The latter inequality will be used below to justify applications of Assumptions 3.4 and 3.5.2.

For each \((k, \lambda) \in [K] \times \Lambda_n\), fix a solution \( \tilde{\theta}_{i_{k}}^{c} (\lambda) \in \tilde{\Theta}_{i_{k}}^{c} (\lambda) \). For \( k \in [K], j \in \{1, 2\} \) and \( \lambda \in \Lambda_n \), denote
\[
\tilde{\theta}_{i_{k},j}^{c} (\lambda) := \begin{cases} 
\tilde{\theta}_{i_{k}}^{c} (\lambda) & \text{if } \| \tilde{\theta}_{i_{k}}^{c} (\lambda) - \theta_0 \|_2 \leq \tilde{u}_{j,n}, \\
\theta_0 + \frac{\tilde{u}_{j,n}}{\| \tilde{\theta}_{i_{k}}^{c} (\lambda) - \theta_0 \|_2} (\tilde{\theta}_{i_{k}}^{c} (\lambda) - \theta_0) & \text{if } \| \tilde{\theta}_{i_{k}}^{c} (\lambda) - \theta_0 \|_2 > \tilde{u}_{j,n}. 
\end{cases}
\]

(B.25)

In addition, for \( \theta \in \Theta \) and \( k \in [K] \), let
\[
f_k (\theta) := \langle I_k - E \rangle [m(\lambda X_i^\top \theta, Y_i) - m(X_i^\top \theta_0, Y_i)]
\]

let \( \lambda_s \in \Lambda_n \) be a penalty level satisfying the bound of Lemma B.15, and define events
\[
\mathcal{R}_n := \left\{ \max_{1 \leq k \leq K} \mathcal{E} (\tilde{\theta}_{i_{k}}^{c} (\lambda_s)) \leq \tilde{\varepsilon}_n^* \right\},
\]
\[
\mathcal{F}_n := \left\{ \max_{1 \leq k \leq K} |f_k (\tilde{\theta}_{i_{k}}^{c} (\lambda_s))| \leq C_L \sqrt{\frac{t\tilde{\varepsilon}_n^*}{c_Dc_Mn}} \right\} \quad \text{and}
\]
\[
\mathcal{C}_n := \left\{ |f_k (\tilde{\theta}_{i_{k},j}^{c} (\lambda))| \leq C_L \sqrt{\frac{3t \ln n}{c_D \ln(1/a)n}} \| \tilde{\theta}_{i_{k},j}^{c} (\lambda) - \theta_0 \|_2, \text{ all } k \in [K], j \in \{1, 2\}, \lambda \in \Lambda_n \right\}.
\]

We first derive a lower bound for \( P(\mathcal{R}_n \cap \mathcal{F}_n \cap \mathcal{C}_n) \) and then prove that (B.18) is satisfied on \( \mathcal{R}_n \cap \mathcal{F}_n \cap \mathcal{C}_n \).

To derive a lower bound for \( P(\mathcal{R}_n \cap \mathcal{F}_n \cap \mathcal{C}_n) \), first observe that for any \( k \in [K] \), the variance of the conditional distribution of
\[
\mathbb{E}_{I_k} \left[ m(X_i^\top \tilde{\theta}_{i_{k}}^{c} (\lambda_s), Y_i) - m(X_i^\top \theta_0, Y_i) \right]
\]

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given \( \{ (X_i, Y_i) \}_{i \in I_k^c} \) is bounded from above by

\[
|I_k|^{-1} E_{X,Y} \left[ |m(X^\top \hat{\theta}_{I_k^c} (\lambda_*), Y) - m(X^\top \theta_0, Y)|^2 \right],
\]

and so, by Assumption 4.2 and Chebyshev’s inequality applied conditional on \( \{ (X_i, Y_i) \}_{i \in I_k^c} \),

\[
P \left( |f_k(\hat{\theta}_{I_k^c}(\lambda_*))| > \frac{t}{c_D n} \sqrt{E_{X,Y} \left[ |m(X^\top \hat{\theta}_{I_k^c} (\lambda_*), Y) - m(X^\top \theta_0, Y)|^2 \right]} \right) \leq \frac{1}{t}.
\]

Also, by (B.17) and Lemma B.16, on the event \( \mathcal{A}_n \), we have \( \| \hat{\theta}_{I_k^c}(\lambda_*) - \theta_0 \|^2 \leq \frac{\overline{\mathcal{E}}^*}{c_M} \) and so by Assumption 3.5.2,

\[
E_{X,Y} \left[ |m(X^\top \hat{\theta}_{I_k^c} (\lambda_*), Y) - m(X^\top \theta_0, Y)|^2 \right] \leq \frac{C^2 \overline{\mathcal{E}}^*}{c_M}.
\]

Further, observe that for any \( \lambda \in \Lambda_n \) and \( j \in \{1, 2\} \), the variance of the conditional distribution of

\[
\mathbb{E}_{I_k} \left[ m(X_i^\top \hat{\theta}_{I_k^c,j} (\lambda), Y_i) - m(X_i^\top \theta_0, Y_i) \right]
\]

given \( \{ (X_i, Y_i) \}_{i \in I_k^c} \) is bounded from above by

\[
|I_k|^{-1} E_{X,Y} \left[ |m(X_i^\top \hat{\theta}_{I_k^c,j} (\lambda), Y_i) - m(X_i^\top \theta_0, Y_i)|^2 \right],
\]

and so, by Assumption 4.2 and Chebyshev’s inequality applied conditional on \( \{ (X_i, Y_i) \}_{i \in I_k^c} \),

\[
P \left( |f_k(\hat{\theta}_{I_k^c,j}(\lambda))| > \frac{3t \ln n}{c_D \ln(1/a) n} \sqrt{E_{X,Y} \left[ |m(X_i^\top \hat{\theta}_{I_k^c,j} (\lambda), Y_i) - m(X_i^\top \theta_0, Y_i)|^2 \right]} \right) \leq \left( \frac{1}{a} \right)/(3t \ln n).
\]

In addition, given that we have \( \max_{j \in \{1,2\}} \| \hat{\theta}_{I_k^c,j}(\lambda) - \theta_0 \|_2 \leq \hat{u}_{1,n} \vee \hat{u}_{2,n} \leq c_L \) it follows from Assumption 3.5.2 that

\[
E_{X,Y} \left[ |m(X^\top \hat{\theta}_{I_k^c,j} (\lambda), Y) - m(X^\top \theta_0, Y)|^2 \right] \leq C_L^2 \| \hat{\theta}_{I_k^c,j}(\lambda) - \theta_0 \|_2^2, \quad j \in \{1, 2\}.
\]

The qualifier \( a^\ell \geq c_\Lambda/n \) in the definition of \( \Lambda_n \) in Assumption 4.3 implies that

\[
\ell \leq \frac{\ln(1/c_\Lambda) + \ln n}{\ln(1/a)}.
\]
Since also $\ell \in \{0, 1, 2, \ldots\}$, from (B.17) and the previous display, one can deduce that

$$|\Lambda_n| \leq \frac{2 \ln n}{\ln(1/a)} + 1 \leq \frac{3 \ln n}{\ln(1/a)}.$$  

Combining the presented results with Lemma B.15 and the union bound, we obtain

$$\mathbb{P}(\mathcal{R}_n \cap \mathcal{F}_n \cap \mathcal{C}_n) \geq 1 - K \left( 4t^{-r} + 3t^{-1} + 2C/\ln^2(pn) + [(K - 1)c_Dn]^{-1} \right),$$

which is the desired bound.

We next prove that (B.18) holds on $\mathcal{R}_n \cap \mathcal{F}_n \cap \mathcal{C}_n$. For the rest of proof, we therefore fix a realization of the data $\{(X_i, Y_i)\}_{i=1}^n$ and assume that $\mathcal{R}_n \cap \mathcal{F}_n \cap \mathcal{C}_n$ is satisfied. Given

$$\hat{\lambda}^{cv} \in \arg\min_{\lambda \in \Lambda_n} \sum_{k=1}^K \sum_{i \in I_k} m(X_i^\top \hat{\theta}_{I_k}^c(\lambda), Y_i),$$

a problem for which $\lambda_*$ is feasible, we must have

$$\sum_{k=1}^K |I_k| \mathbb{E}_{I_k} \left[ m(X_i^\top \hat{\theta}_{I_k}^c(\hat{\lambda}^{cv}), Y_i) \right] \leq \sum_{k=1}^K |I_k| \mathbb{E}_{I_k} \left[ m(X_i^\top \hat{\theta}_{I_k}^c(\lambda_*), Y_i) \right].$$

Here, by $\mathcal{R}_n$ and $\mathcal{F}_n$, for each $k \in [K]$ we have

$$\mathbb{E}_{I_k} \left[ m(X_i^\top \hat{\theta}_{I_k}^c(\lambda_*), Y_i) \right] = \mathbb{E}_{I_k} \left[ m(X_i^\top \hat{\theta}_{0}, Y_i) \right] + \mathcal{E}(\hat{\theta}_{I_k}^c(\lambda_*)) \leq \mathbb{E}_{I_k} \left[ m(X_i^\top \theta_0, Y_i) \right] + q_n.$$

Therefore,

$$\sum_{k=1}^K \frac{|I_k|}{n} \mathbb{E}_{I_k} \left[ m(X_i^\top \hat{\theta}_{I_k}^c(\hat{\lambda}^{cv}), Y_i) \right] \leq \sum_{k=1}^K \frac{|I_k|}{n} \mathbb{E}_{I_k} \left[ m(X_i^\top \theta_0, Y_i) \right] + q_n. \quad (B.26)$$

Now, define $\hat{K}$ as

$$\hat{K} := \left\{ k \in [K] : \mathbb{E}_{I_k} \left[ m(X_i^\top \hat{\theta}_{I_k}^c(\hat{\lambda}^{cv}), Y_i) \right] \leq \mathbb{E}_{I_k} \left[ m(X_i^\top \theta_0, Y_i) \right] + q_n \right\} \quad (B.27)$$

and $\hat{K}^c := [K] \setminus \hat{K}$. We will prove that

$$\max_{k \in \hat{K}} \| \hat{\theta}_{I_k}^c(\hat{\lambda}^{cv}) - \theta_0 \|_2 \leq \hat{u}_{1,n} \quad \text{and} \quad \max_{k \in \hat{K}^c} \| \hat{\theta}_{I_k}^c(\hat{\lambda}^{cv}) - \theta_0 \|_2 \leq \hat{u}_{2,n} \quad (B.28)$$
To prove the first inequality in (B.28), seeking a contradiction, suppose that the inequality is not true, and fix any \( k \in \mathcal{K} \) such that \( \|\tilde{\theta}_{I_k}(\hat{\lambda}^{cv}) - \theta_0\|_2 > \tilde{u}_{1,n} \). Then \( \tilde{u}_{1,n} = \|\tilde{\theta}_{I_k}(\hat{\lambda}^{cv}) - \theta_0\|_2 < \|\tilde{\theta}_{I_k}(\hat{\lambda}^{cv}) - \theta_0\|_2 \), and so by (B.25), (B.27), and convexity (Assumption 3.2),

\[
\mathbb{E}_{I_k} \left[ m(X_i^\top \tilde{\theta}_{I_k}(\hat{\lambda}^{cv}), Y_i) \right] < \mathbb{E}_{I_k} \left[ m(X_i^\top \theta_0, Y_i) \right] + q_n.
\]

Therefore, by \( \mathcal{C}_n \) and \( k \in \mathcal{K} \),

\[
\mathcal{E}(\tilde{\theta}_{I_k,1}(\hat{\lambda}^{cv})) \leq |f_k(\tilde{\theta}_{I_k,1}(\hat{\lambda}^{cv}))| + \mathbb{E}_{I_k} \left[ m(X_i^\top \tilde{\theta}_{I_k,1}(\hat{\lambda}^{cv}), Y_i) \right] - \mathbb{E}_{I_k} \left[ m(X_i^\top \theta_0, Y_i) \right] < C_L \sqrt{\frac{3t \ln n}{c_D \ln(1/a)n}} \|\tilde{\theta}_{I_k,1}(\hat{\lambda}^{cv}) - \theta_0\|_2 + q_n,
\]

and so given that \( \tilde{u}_{1,n} \leq c_M \), by the margin condition (Assumption 3.4),

\[
\|\tilde{\theta}_{I_k,1}(\hat{\lambda}^{cv}) - \theta_0\|_2 < \frac{C_L}{c_M} \sqrt{\frac{3t \ln n}{c_D \ln(1/a)n}} \|\tilde{\theta}_{I_k,1}(\hat{\lambda}^{cv}) - \theta_0\|_2 + \frac{q_n}{c_M}.
\]

Using the elementary inequality that \( x^2 < Bx + C \) and \( B, C \geq 0 \) imply \( x < B + \sqrt{C} \), we see

\[
\tilde{u}_{1,n} = \|\tilde{\theta}_{I_k,1}(\hat{\lambda}^{cv}) - \theta_0\|_2 < \frac{C_L}{c_M} \sqrt{\frac{3t \ln n}{c_D \ln(1/a)n}} + \sqrt{\frac{q_n}{c_M}} = \tilde{u}_{1,n}.
\]

This contradiction proves the first inequality in (B.28).

To prove the second inequality in (B.28), observe that from the first inequality in (B.28) and the definition of \( \tilde{\theta}_{I_k,1}(\hat{\lambda}^{cv}) \) in (B.25), we know that for all \( k \in \mathcal{K} \), \( \tilde{\theta}_{I_k,1}(\hat{\lambda}^{cv}) = \tilde{\theta}_{I_k}(\hat{\lambda}^{cv}) \).

It therefore follows from \( \mathcal{C}_n \) that for all \( k \in \mathcal{K} \),

\[
\mathbb{E}_{I_k} \left[ m(X_i^\top \tilde{\theta}_{I_k}(\hat{\lambda}^{cv}), Y_i) \right] - \mathbb{E}_{I_k} \left[ m(X_i^\top \theta_0, Y_i) \right] = f_k(\tilde{\theta}_{I_k}(\hat{\lambda}^{cv})) + \mathcal{E}(\tilde{\theta}_{I_k}(\hat{\lambda}^{cv})) \geq f_k(\tilde{\theta}_{I_k,1}(\hat{\lambda}^{cv})) \geq -C_L \sqrt{\frac{3t \ln n}{c_D \ln(1/a)n}} \tilde{u}_{1,n}.
\]

Rearranging and using the definitions of \( \tilde{u}_{1,n} \) and \( \tilde{q}_n \), it follows from (B.26) that

\[
\sum_{k \in \mathcal{K}} \frac{|I_k|}{n} \mathbb{E}_{I_k} \left[ m(X_i^\top \tilde{\theta}_{I_k}(\hat{\lambda}^{cv}), Y_i) \right] \leq \sum_{k \in \mathcal{K}} \frac{|I_k|}{n} \mathbb{E}_{I_k} \left[ m(X_i^\top \theta_0, Y_i) \right] + q_n + \tilde{q}_n.
\]

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Hence, given that
\[
\min_{k \in \mathcal{K}} \left\{ \mathbb{E}_{I_k} \left[ m \left( X_i^\top \tilde{\theta}_{I_k}^* (\lambda^{cv}) , Y_i \right) \right] - \mathbb{E}_{I_k} \left[ m \left( X_i^\top \theta_0 , Y_i \right) \right] \right\} \geq 0
\]
by construction, it follows from Assumption 4.2 that for all \( k \in \mathcal{K} \),
\[
\mathbb{E}_{I_k} \left[ m \left( X_i^\top \tilde{\theta}_{I_k}^* (\lambda^{cv}) , Y_i \right) \right] \leq \mathbb{E}_{I_k} \left[ m \left( X_i^\top \theta_0 , Y_i \right) \right] + \frac{q_n + \tilde{q}_n}{c_D}.
\]

We now establish the second inequality in (B.28) using an argument parallel to that used to establish the first inequality with \( \tilde{\theta}_{I_k}^* (\lambda^{cv}) \), \( (q_n + \tilde{q}_n)/c_D \), and \( \tilde{u}_{2,n} \) playing the roles of \( \tilde{\theta}_{I_k}^* (\lambda^{cv}) \), \( q_n \) and \( \tilde{u}_{1,n} \), respectively. This observation finishes the proof of the inequalities in (B.28).

To complete the proof, note that since \( \tilde{u}_{1,n} < \tilde{u}_{2,n} \), the inequalities in (B.28) imply that
\[
\| \tilde{\theta}_{I_k}^* (\lambda^{cv}) - \theta_0 \|_2 \leq \tilde{u}_{2,n} \text{ for all } k \in [K].
\]
It thus remains to simplify the expression for \( \tilde{u}_{2,n} \). To do so, we denote \( T_n := (3t \ln n / [c_D \ln (1/a)]^{1/2} \) and use the following elementary inequalities, where the very first inequality uses \( n \geq 1/a \) in (B.17):
\[
q_n \leq \left( \sqrt{\mathcal{E}_n} + \frac{C_L T_n}{2 \sqrt{c_M}} \right)^2, \quad \tilde{q}_n \leq \left( \frac{\sqrt{q_n}}{2} + \frac{C_L T_n}{\sqrt{c_M}} \right)^2,
\]
\[
\sqrt{q_n + \tilde{q}_n} \leq \sqrt{q_n} + \sqrt{\tilde{q}_n} \leq \frac{3}{2} \sqrt{\mathcal{E}_n} + \frac{7 C_L T_n}{4 \sqrt{c_M}} \quad \text{and}
\]
\[
\tilde{u}_{2,n} \leq \frac{C_L T_n}{c_M} + \frac{7 C_L T_n}{4 c_M \sqrt{c_D}} + \frac{3}{2} \sqrt{\mathcal{E}_n} \leq \frac{11 C_L T_n}{4 c_M \sqrt{c_D}} + \frac{3}{2} \sqrt{\mathcal{E}_n}.
\]
Therefore, \( \tilde{u}_{2,n} \) is smaller than the right-hand side of (B.18), which completes the proof. \( \square \)

**Proof of Lemma B.18.** Let \((n, t) \in \mathbb{N} \times [1, \infty)\) satisfy (B.14), (B.16) and (B.17), and denote \( \Lambda_{n,k} := \{ \lambda \in \Lambda_n ; \| \tilde{\theta}_{I_k}^* (\lambda) - \theta_0 \|_2 \leq c_L \} \) for each \( k \in [K] \). Then for each \( k \in [K] \) and each \( \lambda \in \Lambda_{n,k} \), by Assumption 3.5.3 and Markov’s inequality applied conditional on \( \{ (X_i, Y_i) \}_{i \in I_k} \) we have
\[
P \left( \mathbb{E}_{I_k} \left[ m'_i \left( X_i^\top \tilde{\theta}_{I_k}^* (\lambda) , Y_i \right) - m'_i \left( X_i^\top \theta_0 , Y_i \right) \right] \right) > C_L^2 t \| \tilde{\theta}_{I_k}^* (\lambda) - \theta_0 \|_2 \leq t^{-1}.
\]
Also, since \( n \geq 1 / (c_A \wedge a) \) by (B.17), Assumption 4.3 implies that \( |\Lambda_n| \leq 3 (\ln n) / \ln (1/a) \). (See the proof of Lemma B.17 for more details.) Therefore, by the union bound, the proba-
bility that there exists \( k \in [K] \) and \( \lambda \in \Lambda_{n,k} \) such that

\[
E_{I_k}\left[|m'_i(X_i^\top \tilde{\theta}_{I_k}^c(\lambda), Y_i) - m'_i(X_i^\top \theta_0, Y_i)|^2\right] > \frac{3C^2_en}{\ln(1/a)} \|\tilde{\theta}_{I_k}^c(\lambda) - \theta_0\|_2
\]

is bounded from above by \( K/t \). In addition, by Lemma B.17 and (B.17),

\[
\max_{1 \leq k \leq K} \|\tilde{\theta}_{I_k}^c(\lambda) - \theta_0\|_2 \leq \frac{11C_L}{4c_Dc_M} \sqrt{\frac{3t \ln n}{\ln(1/a)n}} + \frac{3}{2} \sqrt{\frac{\mathcal{E}_n}{c_Dc_M}} \leq c_L
\]

with probability at least \( 1 - K \left( 4t^{-r} + 3t^{-1} + 2C/\ln^2(pn) + [(K - 1)c_Dn]^{-1} \right) \). Hence, with the same probability, all \( \Lambda_{n,k} \) are non-empty. It follows from the union bound that

\[
E_n[(\tilde{U}_i^{cv} - U_i)^2] = \sum_{k=1}^K \frac{|I_k|}{n} E_{I_k}\left[|m'_i(X_i^\top \tilde{\theta}_{I_k}^c(\lambda), Y_i) - m'_i(X_i^\top \theta_0, Y_i)|^2\right] 
\leq \frac{3C^2_en}{\ln(1/a)} \left( \frac{11C_L}{4c_Dc_M} \sqrt{\frac{3t \ln n}{\ln(1/a)n}} + \frac{3}{2} \sqrt{\frac{\mathcal{E}_n}{c_Dc_M}} \right)
\]

with probability at least \( 1 - K \left( 4t^{-r} + 4t^{-1} + 2C/\ln^2(pn) + [(K - 1)c_Dn]^{-1} \right) \), as desired. \( \square \)

### B.4 Proofs for Section 5

Note that according to Assumption 5.3, the second derivative \( m''_{11}(t, y) \) may not exist for some \((t, y) \in \mathbb{R} \times \mathcal{Y}\). With some abuse of notation, for such \( t \) and \( y \), throughout this section, we set \( m''_{11}(t, y) := 0 \), which is consistent with our convention in Algorithm 5.1. With this convention, the function \( m''_{11} \) is now defined on the entire set \( \mathbb{R} \times \mathcal{Y} \), although it may not have the interpretation as to the derivative of \( m'_1 \) with respect to its first argument.

In proving Theorem 5.1, we rely on the following eight lemmas, whose proofs can be found at the end of this section.

**Lemma B.19 (Second Derivative of Loss).** Let Assumption 5.3 hold. Then \(|m''_{11}(t, y)| \leq C_m\) for all \((t, y) \in \mathbb{R} \times \mathcal{Y}\).

**Lemma B.20 (Interpolation).** Let Assumption 5.3 hold. Then for any \((\tilde{\theta}_0, \tilde{\theta}_1) \in \Theta \times \Theta\) and \((x, y) \in \mathbb{R}^p \times \mathcal{Y}\), we have

\[
m'_1(x^\top \tilde{\theta}_1, y) - m'_1(x^\top \tilde{\theta}_0, y) = \int_0^1 m''_{11}(x^\top \tilde{\theta}_\tau, y)x^\top(\tilde{\theta}_1 - \tilde{\theta}_0)d\tau,
\]
where we denote \( \tilde{\theta}_r := \tilde{\theta}_0 + \tau(\tilde{\theta}_1 - \tilde{\theta}_0) \) for all \( \tau \in (0, 1) \). As a consequence,

\[
|m'_1(x^T \tilde{\theta}_1, y) - m'_1(x^T \tilde{\theta}_0, y)| \leq C_m |x^T(\tilde{\theta}_1 - \tilde{\theta}_0)|.
\]

Lemma B.21 (First- and Second-Order Conditions). Let Assumptions 3.1–3.5, 5.2, and 5.3 hold. Then

\[
E[m'_1(X^T \theta_0, Y)X] = 0_p,
\]

and

\[
E \left[ m''_{11}(X^T \theta_0, Y) |X^T(\theta - \theta_0)|^2 \right] \geq 2cM \|\theta - \theta_0\|^2 \quad \text{for all} \quad \theta \in \mathbb{R}^p.
\]

Lemma B.22 (Existence and Uniqueness of \( \mu_0 \)). Let Assumptions 3.1–3.5, 5.2, and 5.3 hold. Then there is a unique solution to (5.1), namely

\[
\mu_0 = (E[m''_{11}(X^T \theta_0, Y) W W^T])^{-1} E[m''_{11}(X^T \theta_0, Y) WD].
\]

Lemma B.23 (Variance Denominator Bound). Let Assumptions 3.1–3.5, 5.2, and 5.3 hold. Then

\[
E \left[ m''_{11}(X^T \theta_0, Y)(D - W^T \mu_0)D \right] \geq 2cM.
\]

Lemma B.24 (Remainder Term, I). Under the conditions of Theorem 5.1,

\[
R_{n,1} := E_n \left[ \left( m''_{11}(X_i^T \tilde{\theta}, Y_i) - m''_{11}(X_i^T \theta_0, Y_i) \right)(D_i - W_i^T \mu_0)D_i \right] = o_P(1).
\]

Lemma B.25 (Remainder Term, II). For all \( \tau \in [0, 1] \), denote \( \tilde{\theta}_r := (\beta_0 + \tau(\tilde{\beta} - \beta_0), \tilde{\gamma})^T \). Under the conditions of Theorem 5.1,

\[
R_{n,2} := \int_0^1 E_n \left[ \left( m''_{11}(X_i^T \tilde{\theta}_r, Y_i) - m''_{11}(X_i^T \theta_0, Y_i) \right)(D_i - W_i^T \mu_0)D_i \right] d\tau
\]

\[
\lesssim_p a_n + \frac{\Delta_n^{1/2}}{a_n} + (B_n a_n/\Delta_n)^{r/2}.
\]

Lemma B.26 (Remainder Term, III). For all \( \tau \in [0, 1] \), denote \( \tilde{\theta}_r := (\beta_0, \gamma_0^T + \tau(\tilde{\gamma} - \gamma_0)^T) \). Under the conditions of Theorem 5.1,

\[
R_{n,3} := \int_0^1 E_n \left[ \left( m''_{11}(X_i^T \tilde{\theta}_r, Y_i) - m''_{11}(X_i^T \theta_0, Y_i) \right)(D_i - W_i^T \mu_0)(W_i^T \tilde{\gamma} - W_i^T \gamma_0) \right] d\tau
\]

\[
\lesssim_p a_n^2 + B_n a_n \left( \frac{\Delta_n^{1/2}}{a_n} + (B_n a_n/\Delta_n)^{r/2} \right).
\]

Proof of Theorem 5.1. Denote the (normalized) denominator in the one-step update of
Algorithm 5.1 by
\[ D_n := \mathbb{E}_n \left[ m''_{11}(X_i^\top \theta, Y_i)(D_i - W_i^\top \mu)D_i \right]. \]

We decompose \( D_n \) as
\[ D_n = \mathbb{E} \left[ m''_{11}(X^\top \theta, Y)(D - W^\top \mu_0)D \right] + I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}, \]
with remainder terms
\[
\begin{align*}
I_{n,1} &= (\mathbb{E}_n - \mathbb{E}) \left[ m''_{11}(X_i^\top \theta, Y_i)(D_i - W_i^\top \mu_0)D_i \right], \\
I_{n,2} &= \mathbb{E}_n \left[ m''_{11}(X_i^\top \theta, Y_i)W_i^\top(\mu_0 - \bar{\mu})D_i \right], \\
I_{n,3} &= \mathbb{E}_n \left[ \left( m''_{11}(X_i^\top \theta, Y_i) - m''_{11}(X_i^\top \theta_0, Y_i) \right)(D_i - W_i^\top \mu_0)D_i \right], \\
I_{n,4} &= \mathbb{E}_n \left[ \left( m''_{11}(X_i^\top \theta, Y_i) - m''_{11}(X_i^\top \theta_0, Y_i) \right)W_i^\top(\mu_0 - \bar{\mu})D_i \right].
\end{align*}
\]

We handle each remainder term in turn. First, by Assumption 5.2, Lemma B.19, and the Chebyshev and Cauchy-Schwarz inequalities, we get \( I_{n,1} = o_p(1) \). Second, we have
\[
|I_{n,2}| \leq C_m \mathbb{E}_n \left[ \|W_i^\top(\mu_0 - \bar{\mu})D_i\|_1 \right] \leq C_m \|\bar{\mu} - \mu_0\| \sqrt{\mathbb{E}_n \left[ \|W_i\|_\infty^2 \right] \mathbb{E}_n \left[ D_i^2 \right]} \lesssim P a_n B_n, \tag{B.33}
\]
where the first line follows from the triangle inequality and Lemma B.19, and the second from the Hölder and Cauchy-Schwarz inequalities and Assumptions 3.5.1, 5.2 and 5.5. Given that \( a_n B_n \to 0 \) by assumption, it thus follows that \( I_{n,2} = o_p(1) \). Third, we have \( I_{n,3} = o_p(1) \) by Lemma B.24. Fourth, we have
\[
|I_{n,4}| \leq 2C_m \mathbb{E}_n \left[ \|W_i^\top(\bar{\mu} - \mu_0)D_i\| \right] \lesssim P a_n B_n,
\]
where the inequality follows from the triangle inequality and Lemma B.19, and the \( \lesssim P \) follows as in (B.33). Given that \( a_n B_n \to 0 \) by assumption, it thus follows that \( I_{n,4} = o_p(1) \). Conclude that
\[
D_n = \mathbb{E} \left[ m''_{11}(X^\top \theta_0, Y)(D - W^\top \mu_0)D \right] + o_p(1). \tag{B.34}
\]

In addition, Lemma B.23 shows that
\[
\mathbb{E}[m''_{11}(X^\top \theta_0, Y)(D - W^\top \mu_0)D] \geq 2c_M, \tag{B.35}
\]
and so \( D_n \) is bounded away from zero with probability tending to one, which implies in particular that the one-step update in Algorithm 5.1 is well-defined with probability tending
to one as well and that $1/D_n \lesssim P 1$.

Next, consider the (normalized) numerator in the one-step update of Algorithm 5.1. For all $\tau \in [0, 1]$, denote $\tilde{\theta}_\tau := (\tilde{\beta}_0 + \tau(\tilde{\beta} - \beta_0), \tilde{\gamma}^T)^T$ and

$$N_n := \int_0^1 \mathbb{E}_n \left[ m''(X_i^T \tilde{\theta}_\tau, Y_i)(D_i - W_i^T \mu)D_i \right] d\tau.$$

Then by Lemma B.20, we have

$$\mathbb{E}_n \left[ m_i'(D_i \tilde{\beta} + W_i^T \tilde{\gamma}, Y_i)(D_i - W_i^T \tilde{\mu}) \right] = \mathbb{E}_n \left[ m_i'(D_i \beta_0 + W_i^T \tilde{\gamma}, Y_i)(D_i - W_i^T \mu) \right] + N_n(\tilde{\beta} - \beta_0).$$

Hence, by definition of the one-step update in (5.5),

$$\tilde{\beta} - \beta_0 = (\tilde{\beta} - \beta_0) \left( 1 - \frac{N_n}{D_n} \right) - \frac{1}{D_n} \mathbb{E}_n \left[ m_i'(D_i \beta_0 + W_i^T \tilde{\gamma}, Y_i)(D_i - W_i^T \mu) \right].$$

By Lemma B.25, we know

$$|N_n - D_n| \lesssim P a_n + \bar{X}_n^{1/2} + (B_n a_n/\bar{X}_n)^r/2.$$

Recalling $1/D_n \lesssim P 1$, it follows that

$$\sqrt{n}(\tilde{\beta} - \beta_0) \left( 1 - \frac{N_n}{D_n} \right) \lesssim P \sqrt{n}a_n \left( a_n + \bar{X}_n^{1/2} + (B_n a_n/\bar{X}_n)^r/2 \right),$$

which $\to 0$ by Assumption 5.5 and by hypothesis of the theorem. It follows from (B.37) that

$$\sqrt{n}(\tilde{\beta} - \beta_0) = -\frac{1}{D_n} \sqrt{n} \mathbb{E}_n \left[ m_i'(D_i \beta_0 + W_i^T \tilde{\gamma}, Y_i)(D_i - W_i^T \mu) \right] + o_P(1).$$

We next further analyze the right-hand side numerator. To this end, denote

$$\tilde{I}_{n,1} := \sqrt{n} \mathbb{E}_n \left[ m_i'(X_i^T \theta_0, Y_i)(W_i^T \mu - W_i^T \mu_0) \right]$$

and observe that

$$|\tilde{I}_{n,1}| \leq \|\mu - \mu_0\|_1\|\sqrt{n} \mathbb{E}_n \left[ m_i'(X_i^T \theta_0, Y_i)W_i \right]\|_\infty.$$
addition, for \( q := (r \wedge \bar{r})/2 \in (2, \infty) \),

\[
\mathbb{E}[\|m'_i(X^\top \theta_0, Y)W\|_\infty^q] \leq \left( \mathbb{E}[\|m'_i(X^\top \theta_0, Y)\|^{2q}] \right)^{1/2} \left( \mathbb{E}[\|W\|_{\infty}^{2q}] \right)^{1/2} \leq C_M^q B_n^q
\]

by the Cauchy-Schwarz inequality and Assumptions 3.5.1 and 5.2. Therefore, given that \( B_n^2 \ln(pn) = o(n^{1-4/(r \wedge \bar{r})}) \) by hypothesis of the theorem, (the mean-zero part of) Lemma B.21 and Theorem D.2 combine to show that

\[
\|\sqrt{n}\mathbb{E}_n \left[ m'_i(X_i^\top \theta_0, Y_i)W_i \right]\|_\infty \lesssim_P \sqrt{\ln(pn)}
\]

Hence, given that \( \|\tilde{\mu} - \mu_0\|_1 \lesssim_P a_n \) by Assumption 5.5 and \( a_n \sqrt{\ln(pn)} \to 0 \) by assumption, it follows that \( \tilde{I}_{n,1} = o_P(1) \). Moreover, for all \( \tau \in [0, 1] \), denote \( \tilde{\theta}_\tau := (\beta_0, \gamma_0^\top + \tau(\tilde{\gamma} - \gamma_0)^\top)^\top \) and

\[
\tilde{I}_{n,2} := \sqrt{n}\mathbb{E}_n \left[ m'_i(D_i\beta_0 + W_i^\top \tilde{\gamma}, Y_i)(W_i^\top \tilde{\mu} - W_i^\top \mu_0) \right] - \tilde{I}_{n,1}.
\]

Then

\[
|\tilde{I}_{n,2}| \leq \sqrt{n} \int_0^1 \left| \mathbb{E}_n \left[ m''_{i1}(X_i^\top \tilde{\theta}_\tau, Y_i)(W_i^\top \tilde{\gamma} - W_i^\top \gamma_0)(W_i^\top \tilde{\mu} - W_i^\top \mu_0) \right] \right| d\tau
\]

\[
= \sqrt{n} \int_0^1 \left| (\tilde{\gamma} - \gamma_0)^\top \mathbb{E}_n \left[ m''_{i1}(X_i^\top \tilde{\theta}_\tau, Y_i)W_i W_i^\top \right] (\tilde{\mu} - \mu_0) \right| d\tau
\]

\[
\leq C_m \sqrt{n} \|\tilde{\gamma} - \gamma_0\|_1 \|\tilde{\mu} - \mu_0\|_1 \max_{1 \leq j, k \leq p-1} \mathbb{E}_n[|W_{i,j}W_{i,k}|], \quad (B.39)
\]

where the first line follows from Lemma B.20 and Jensen’s inequality, and the third from Lemma B.19 and the Hölder and triangle inequalities. Setting up for an application of Theorem D.3, observe that for all \( j, k \in [p - 1] \), we have \( \mathbb{E}[|W_j W_k|] \leq C_M^q \) by the Cauchy-Schwarz inequality and Assumption 5.2. In addition, for \( q := r/2 \in (1, \infty) \), we have \( \mathbb{E}[\|W\|_{\infty}^q \|W\|_{\infty}^q] \leq B_n^{2q} \) by Assumptions 3.5.1. Hence, given that \( B_n^{2} \ln(pn) = o(n^{1-2/r}) \) by hypothesis of the theorem, it follows from Theorem D.3 that

\[
\max_{1 \leq j, k \leq p-1} \mathbb{E}_n[|W_{i,j}W_{i,k}|] \lesssim_P 1.
\]

Therefore, given that \( \|\tilde{\gamma} - \gamma_0\|_1 \|\tilde{\mu} - \mu_0\|_1 \lesssim_P a_n^2 \) by Assumption 5.5, it follows that \( |\tilde{I}_{n,2}| \lesssim_P \sqrt{n}a_n^2 \), which \( \to 0 \) by assumption. We therefore have \( \tilde{I}_{n,1} + \tilde{I}_{n,2} = o_P(1) \), and it follows from (B.38) and \( 1/\mathcal{D}_n \lesssim_P 1 \) that

\[
\sqrt{n} (\tilde{\beta} - \beta_0) = -\frac{1}{\mathcal{D}_n} \sqrt{n}\mathbb{E}_n \left[ m'_i(D_i\beta_0 + W_i^\top \tilde{\gamma}, Y_i)(D_i - W_i^\top \mu_0) \right] + o_P(1). \quad (B.40)
\]
Further, by Lemma B.20, we have
\[
\sqrt{n} E_n[(m_i'(D_i \beta_0 + W_i^{\top} \tilde{\gamma}, Y_i) - m_i'(X_i^{\top} \theta_0, Y_i))(D_i - W_i^{\top} \mu_0)]
\]
\[
= \sqrt{n} \int_0^1 E_n[m''_{11}(X_i^{\top} \theta_0, Y_i)(D_i - W_i^{\top} \mu_0)(W_i^{\top} \tilde{\gamma} - W_i^{\top} \gamma_0)]d\tau =: \tilde{I}_{n,1}. \tag{B.41}
\]
In addition, denote
\[
\tilde{I}_{n,2} := \sqrt{n} E_n[m''_{11}(X_i^{\top} \theta_0, Y_i)(D_i - W_i^{\top} \mu_0)(W_i^{\top} \tilde{\gamma} - W_i^{\top} \gamma_0)].
\]
Then, by Lemma B.26 and hypothesis of the theorem, we have
\[
|\tilde{I}_{n,1} - \tilde{I}_{n,2}| \leq_P \sqrt{n} \left( a_n^2 + B_n a_n \left( \frac{\Delta^1_n}{n} + \frac{(B_n a_n/\Delta_n)^{r/2}}{} \right) \right) \rightarrow 0.
\]
Moreover,
\[
|\tilde{I}_{n,2}| \leq |\tilde{\gamma} - \gamma_0|_1 \sqrt{n} E_n[m''_{11}(X_i^{\top} \theta_0, Y_i)(D_i - W_i^{\top} \mu_0)W_i|_\infty.
\]
Setting up for an application of Theorem D.2, observe that for all \( j \in [p] \),
\[
E[|m''_{11}(X^{\top} \theta_0, Y)(D - W^{\top} \mu_0)W_j|^2] \leq C_m^2 C_M^4
\]
where we have used Lemma B.19, the Cauchy-Schwarz inequality, and Assumption 5.2. In addition, for \( q := (r \wedge \tilde{r})/2 \in (2, \infty) \),
\[
E[|m''_{11}(X^{\top} \theta_0, Y)(D - W^{\top} \mu_0)W||_\infty^q] \leq C_m^q \left( E[|D - W^{\top} \mu_0|^{2q}] \right)^{1/2} \left( E[||W||^{2q}] \right)^{1/2} \leq C_m^q C_M^q B_n^q
\]
by Lemma B.19, the Cauchy-Schwarz inequality, and Assumptions 3.5.1 and 5.2. Therefore, given that \( B_n^2 \ln(pn) = o(n^{1-4/(r \wedge \tilde{r})}) \) by assumption,
\[
\left\| \sqrt{n} E_n \left[ m''_{11}(X_i^{\top} \theta_0, Y_i)(D_i - W_i^{\top} \mu_0)W_i \right] \right\|_\infty \leq_P \sqrt{\ln(pn)}
\]
by (5.1) and Theorem D.2. Hence, given that \( |\tilde{\gamma} - \gamma_0|_1 \leq_P a_n \) by Assumption 5.5, it follows that \( |\tilde{I}_{n,2}| \leq_P a_n \sqrt{\ln(pn)} \), which \( \rightarrow 0 \) by hypothesis, and so \( \tilde{I}_{n,1} = o_P(1) \). Thus, it follows from (B.40) and \( 1/D_n \leq_P 1 \) that
\[
\sqrt{n}(\hat{\beta} - \beta_0) = -\frac{1}{D_n} \sqrt{n} E_n \left[ m'_{11}(X_i^{\top} \theta_0, Y_i)(D_i - W_i^{\top} \mu_0) \right] + o_P(1). \tag{B.42}
\]
Combining this bound with (B.34) and (B.35) in turn yields

\[
\sqrt{n}(\hat{\beta} - \beta_0) = -\frac{n^{-1/2} \sum_{i=1}^{n} m''_1(x_i^\top \theta_0, y_i)(D_i - W_i^\top \mu_0)}{E[m''_1(x_i^\top \theta_0, Y)(D - W^\top \mu_0)D]} + o_p(1).
\]

Using Assumptions 5.1 and 5.2 and Lemmas B.19 and B.23, we see that the asymptotic variance \( \sigma_0 \) is bounded from above and away from zero, and from the Cauchy-Schwarz inequality, we have

\[
E \left[ |m'_1(X^\top \theta_0, Y)(D - W^\top \mu_0)|^{2+(r-4)/2} \right] \leq C_M \in (0, \infty).
\]

It therefore follows that

\[
\frac{\sqrt{n}(\hat{\beta} - \beta_0)}{\sigma_0} = -\frac{n^{-1/2} \sum_{i=1}^{n} m''_1(x_i^\top \theta_0, y_i)(D_i - W_i^\top \mu_0)}{(E[m'_1(x_i^\top \theta_0, Y)(D - W^\top \mu_0)])^{1/2}} + o_p(1),
\]

and the asserted claim of the theorem now follows from Lyapunov’s version of the Central Limit Theorem in combination with Slutsky’s lemma. \( \square \)

**Proof of Lemma B.19.** Observe that for all \( t_1, t_2 \in \mathbb{R} \) and \( y \in \mathcal{Y} \), we have \( |m'_1(t_2, y) - m'_1(t_1, y)| \leq C_m |t_2 - t_1| \) by Assumption 5.3. Hence, for all \( t \in \mathbb{R} \) and \( y \in \mathcal{Y} \) such that the derivative of \( m'_1(t, y) \) with respect to the first argument exists, it is equal to \( m''_1(t, y) \) by definition and satisfies \( |m''_1(t, y)| \leq C_m \). Also, for all \( t \in \mathbb{R} \) and \( y \in \mathcal{Y} \) such that the derivative of \( m'_1(t, y) \) with respect to the first argument does not exist, we have \( m''_1(t, y) = 0 \) by convention, and so \( |m''_1(t, y)| \leq C_m \) as well. This gives the asserted claim. \( \square \)

**Proof of Lemma B.20.** Fix \( (\tilde{\theta}_0, \tilde{\theta}_1) \in \Theta \times \Theta \) and \( (x, y) \in \mathbb{R}^p \times \mathcal{Y} \). We provide the proof for the case where \( x^\top \tilde{\theta}_0 \leq x^\top \tilde{\theta}_1 \). The other case is analogous. Let \( J := \{ j \in [J]; x^\top \tilde{\theta}_0 \leq t_{y, j} \leq x^\top \tilde{\theta}_1 \} \). If \( J \) is empty (i.e. no threshold was encountered or crossed), then

\[
m'_1(x^\top \tilde{\theta}_1, y) - m'_1(x^\top \tilde{\theta}_0, y) = \int_{x^\top \tilde{\theta}_0}^{x^\top \tilde{\theta}_1} m''_1(t, y) dt = \int_{0}^{1} m''_1(x^\top \tilde{\theta}_r, y)x^\top (\tilde{\theta}_1 - \tilde{\theta}_0)d\tau, \quad (B.43)
\]

where the first equality follows from the fundamental theorem of calculus and Assumption 5.3 and the second from the change of variables \( t = x^\top (\tilde{\theta}_0 + \tau(\tilde{\theta}_1 - \tilde{\theta}_0)) \).

If the set \( J \) is non-empty, we let \( j_0 \) and \( j_1 \) be its smallest and largest element, respectively.
(These elements could coincide.) Then

\[ m'_1(x^\top \tilde{\theta}_1, y) - m'_1(x^\top \tilde{\theta}_0, y) = m'_1(x^\top \tilde{\theta}_1, y) - m'_1(t_{y,j_1}, y) + \sum_{j=j_0+1}^{j_1} m'_1(t_{y,j}, y) - m'_1(t_{y,j-1}, y) + m'_1(t_{y,j_0}, y) - m'_1(x^\top \tilde{\theta}_0, y), \tag{B.44} \]

where the sum is omitted if \( j_0 = j_1 \). Applying the same argument as that leading to (B.43) to each of the terms on the right-hand side of (B.44) shows that

\[
m'_1(x^\top \tilde{\theta}_1, y) - m'_1(x^\top \tilde{\theta}_0, y) = \int_{\tau_{j_1}}^{1} m''_{11}(x^\top \tilde{\theta}_\tau, y)x^\top(\tilde{\theta}_1 - \tilde{\theta}_0)d\tau + \sum_{j=j_0+1}^{j_1} \int_{\tau_{j-1}}^{\tau_j} m''_{11}(x^\top \tilde{\theta}_\tau, y)x^\top(\tilde{\theta}_1 - \tilde{\theta}_0)d\tau + \int_{0}^{\tau_j_0} m''_{11}(x^\top \tilde{\theta}_\tau, y)x^\top(\tilde{\theta}_1 - \tilde{\theta}_0)d\tau = \int_{0}^{1} m''_{11}(x^\top \tilde{\theta}_\tau, y)x^\top(\tilde{\theta}_1 - \tilde{\theta}_0)d\tau,
\]

where we denoted \( \tau_j := (t_{y,j} - x^\top \tilde{\theta}_0)/(x^\top(\tilde{\theta}_1 - \tilde{\theta}_0)) \) for all \( j \in \{j_0, \ldots, j_1\} \). The previous display yields the first claim. The second claim follows from the first and Lemma B.19. \[ \Box \]

**Proof of Lemma B.21.** Since \( \theta_0 \) is interior to \( \Theta \) (Assumption 3.1), there is a radius \( r_n \in (0, \infty) \) such that \( r_n \leq c'_M \) and the ball \( B_{\theta_0}(r_n) := \{ \theta \in \mathbb{R}^p; \| \theta - \theta_0 \|_2 \leq r_n \} \) is a subset of \( \Theta \), with \( c'_M \in (0, \infty] \) being provided in Assumption 3.4. Fix any \( \theta \in B_{\theta_0}(r) \) and define

\[ f(\tau, z) := m(x^\top \theta_\tau, y) - m(x^\top \theta_0, y) \text{ for each } (\tau, z) \in \mathbb{R} \times Z. \]

As in Section B.2, we here employ the shorthand notations \( \theta_\tau = \theta_0 + \tau(\theta - \theta_0) \), \( z = (x, y) \), \( Z = (X, Y) \) and \( Z = \mathcal{X} \times \mathcal{Y} \).

Now, for any \( \tau \in (-1, 1) \), we have \( \theta_\tau \in B_{\theta_0}(r_n) \subset \Theta \) by Assumption 3.1, and so \( E[\|f(\tau, Z)\|] < \infty \) by Assumption 3.3. Hence, \( g(\tau) := E[f(\tau, Z)] \), \( \tau \in (-1, 1) \), is a well-defined map from \((-1, 1)\) to \( \mathbb{R} \). Further, for any \( \tau \in (-1, 1) \) and \( z \in Z \), there is an \( \alpha \in [0, 1] \). \[ 116 \]
such that

$$
|f(\tau, z)| = |\tau| \cdot |m'_1(x^\top \theta_0 + \alpha \tau x^\top (\theta - \theta_0), y) \cdot |x^\top (\theta - \theta_0)|
\leq |\tau| \cdot \left( |m'_1(x^\top \theta_0 + x^\top (\theta - \theta_0), y)| + |m'_1(x^\top \theta_0 - x^\top (\theta - \theta_0), y)| \right) \cdot |x^\top (\theta - \theta_0)|
\leq 2|\tau| \cdot \left( |m'_1(x^\top \theta_0, y)| + C_m|x^\top (\theta - \theta_0)| \right) \cdot |x^\top (\theta - \theta_0)|,
$$

where the first line follows from the Mean Value Theorem and Assumption 5.3, the second from $m(\cdot, y)$ being convex and differentiable (Assumptions 3.2 and 5.3) and the fact that the inequality $a \leq b \leq c$ implies $|b| \leq |a| + |c|$, and the third follows from the triangle inequality (adding and subtracting $m'_1(x^\top \theta_0, y)$ twice) and two applications of the Lemma B.20 inequality. Hence, $E[\sup_{\tau \in (-1, 1)} |f(\tau, Z)|] < \infty$ by the Cauchy-Schwarz inequality and Assumption 5.2. Also, for any $\tau \in (-1, 1)$ and $z = (x, y) \in Z$, $f'_1(\tau, z) \exists$ by Assumption 5.3 and

$$
|f'_1(\tau, z)| = |m'_1(x^\top \theta_0 + \tau x^\top (\theta - \theta_0), y) \cdot |x^\top (\theta - \theta_0)|
\leq 2\left( |m'_1(x^\top \theta_0, y)| + C_m|x^\top (\theta - \theta_0)| \right) \cdot |x^\top (\theta - \theta_0)|
$$

by the same arguments as above, so $E[\sup_{\tau \in (-1, 1)} |f'_1(\tau, Z)|] < \infty$ as well. It then follows from [1] that $g$ is differentiable on $(-1, 1)$ with derivative given by $g'(\tau) = E[f'_1(\tau, Z)]$. In particular, $g'(0) = E[m'_1(X^\top \theta_0, Y)X^\top (\theta - \theta_0)]$.

Further, $f''_{11}(0, Z) \exists$ almost surely by Assumption 5.3. Also, for any $\tau \in (-1, 1)$ and $z = (x, y) \in Z$,

$$
|f'_1(\tau, z) - f'_1(0, z)| \leq |m'_1(x^\top \theta_0 + \tau x^\top (\theta - \theta_0), y) - m'_1(x^\top \theta_0, y)| \cdot |x^\top (\theta - \theta_0)|
\leq C_m|\tau| \cdot |x^\top (\theta - \theta_0)|^2 \tag{B.45}
$$

by Assumption 5.3 and the Lemma B.20 inequality as well. It follows that whenever $f''_{11}(0, Z) \exists$, it satisfies $|f''_{11}(0, Z)| \leq C_m|X^\top (\theta - \theta_0)|^2$, and so $E[|f''_{11}(0, Z)|] < \infty$ by Assumption 5.2. In addition, denoting $\overline{f}(z) := C_m|x^\top (\theta - \theta_0)|^2$, we have from (B.45) that $|f'_1(\tau, z) - f'_1(0, z)| \leq |\tau|\overline{f}(z)$, where $E[\overline{f}(Z)] < \infty$ by Assumption 5.2. It thus follows that $g'(\tau)$ is differentiable at $\tau = 0$ with derivative $g''(0) = E[f''_{11}(0, Z)]$ by [1], Corollary A.3 applied with $f'_1$ instead of $f$, and so $g(\tau)$ is twice differentiable at $\tau = 0$ with second derivative $g''(0) = E[f''_{11}(0, Z)] = E[m''_{11}(X^\top \theta_0, Y)X^\top (\theta - \theta_0)|^2]$.

Next, from Taylor’s theorem (with Peano’s form of remainder), we know that

$$
g(\tau) - g(0) = g'(0)\tau + \frac{1}{2}g''(0)\tau^2 + h(\tau)\tau^2, \quad \tau \in (-1, 1), \tag{B.46}
$$

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where the function \( h : (-1, 1) \to \mathbb{R} \) satisfies \( h(\tau) \to 0 \) as \( \tau \to 0 \). On the other hand, since \( \tau_n \leq c'_{M} \), Assumption 3.4 and \( \theta \in \mathcal{B}_{\theta_0}(\tau_n) \) imply \( \| \theta - \theta_0 \|_2 = |\tau| \| \theta - \theta_0 \|_2 \leq c'_{M} \), and thus

\[
g(\tau) - g(0) \geq c_M \tau^2 \| \theta - \theta_0 \|_2^2, \quad \tau \in (-1, 1).
\] (B.47)

We claim that (B.47) implies that \( g'(0) = 0 \) and \( g''(0) \geq 2c_M \| \theta - \theta_0 \|_2^2 \). To see the former, combine (B.46) and (B.47) to obtain

\[
g'(0) + \frac{1}{2} g''(0) \tau + h(\tau) \tau \geq c_M \tau \| \theta - \theta_0 \|_2^2, \quad \tau \in (0, 1),
\]

and

\[
g'(0) + \frac{1}{2} g''(0) \tau + h(\tau) \tau \leq c_M \tau \| \theta - \theta_0 \|_2^2, \quad \tau \in (-1, 0).
\]

Take the limits as \( \tau \to 0^+ \) and \( \tau \to 0^- \), respectively, to see that both \( g'(0) \geq 0 \) and \( g'(0) \leq 0 \), and so \( g'(0) = 0 \). To see the latter, combine (B.46), (B.47), and \( g'(0) = 0 \) to obtain

\[
\frac{1}{2} g''(0) + h(\tau) \geq c_M \| \theta - \theta_0 \|_2^2, \quad \tau \in (-1, 1) \setminus \{0\}.
\]

Using \( h(\tau) \to 0 \) as \( \tau \to 0 \), the claim follows from taking the limit as \( |\tau| \to 0^+ \). In turn, \( g'(0) = E[m'_1(X^\top \theta_0, Y)X^\top(\theta - \theta_0)] = 0 \) for all \( \theta \in \mathcal{B}_{\theta_0}(\tau_n) \) gives (B.29) by varying \( \theta \) over \( \mathcal{B}_{\theta_0}(\tau_n) \). (See the proof of Lemma B.10 for details.) Finally, \( g''(0) = E[m''_1(X^\top \theta_0, Y)|X^\top(\theta - \theta_0)|^2] \geq 2c_M \| \theta - \theta_0 \|_2^2 \) for all \( \theta \in \mathcal{B}_{\theta_0}(\tau_n) \) gives (B.30) by varying \( \theta \) over \( \mathcal{B}_{\theta_0}(\tau_n) \) and rescaling \( \theta - \theta_0 \).

**Proof of Lemma B.22.** Using Lemma B.19 followed by the triangle, Cauchy-Schwarz and Jensen inequalities along with Assumption 5.2, we see that \( E[m''_1(X^\top \theta_0, Y)|D - W^\top \mu||W_j| < \infty \) for any \( \mu \in \mathbb{R}^{p-1} \) and \( j \in [p-1] \), so the problem of solving the system of equations (5.1) is well-defined. Also, it follows from Lemma B.21 that for all \( \theta \in \mathbb{R}^p \),

\[
E \left[ m''_1(X^\top \theta_0, Y) |X^\top(\theta - \theta_0)|^2 \right] \geq 2c_M \| \theta - \theta_0 \|_2^2.
\]

For any \( \theta = (\beta, \gamma^\top)^\top \in \mathbb{R}^p \) such that \( \beta = \beta_0 \), we therefore obtain

\[
(\gamma - \gamma_0)^\top E \left[ m''_1(X^\top \theta_0, Y) W W^\top \right] (\gamma - \gamma_0) \geq 2c_M \| \gamma - \gamma_0 \|_2^2.
\]

The previous display implies that for any \( v \in \mathbb{R}^{p-1} \) such that \( \| v \|_2 = 1 \),

\[
v^\top E \left[ m''_1(X^\top \theta_0, Y) W W^\top \right] v \geq 2c_M > 0,
\]

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which further implies that $E[m''_{11}(X^\top \theta_0, Y)WW^\top]$ is positive definite. Hence, a solution $\mu_0$ to (5.1) exists, is unique, and is given by (B.31).

**Proof of Lemma B.23.** From Lemma B.21, we know that for all $\theta \in \mathbb{R}^p$,

$$E \left[ m''_{11} (X^\top \theta_0, Y) \mid X^\top (\theta - \theta_0) \right]^2 \geq 2c_M \|\theta - \theta_0\|_2^2.$$ 

Let $\Delta := 1/\sqrt{1 + \|\mu_0\|_2^2} \in (0, 1]$ with $\mu_0$ given by (B.31). Then $\theta' := (\beta_0 + \Delta, \gamma_0^\top - \Delta \mu_0^\top)^\top$ satisfies $\|\theta' - \theta_0\|_2^2 = (1 + \|\mu_0\|_2^2)^2 = 1$, and $X^\top (\theta' - \theta_0) = \Delta (D - W^\top \mu_0)$. With $\theta = \theta'$, the previous display produces

$$\Delta^2 E \left[ m''_{11} (X^\top \theta_0, Y) \right] (D - W^\top \mu_0)^2 \geq 2c_M,$$

and so, since $\Delta \in (0, 1]$, we get

$$E \left[ m''_{11} (X^\top \theta_0, Y) \right] (D - W^\top \mu_0)^2 \geq 2c_M.$$

On the other hand, it follows from (5.1) that

$$E \left[ m''_{11} (X^\top \theta_0, Y) \right] (D - W^\top \mu_0)^2 = E \left[ m''_{11} (X^\top \theta_0, Y) \right] D.$$

The asserted claim follows by combining the last two displays. □

**Proof of Lemma B.24.** Suppose first that $J = 1$, such that $m$ is everywhere twice continuously differentiable in its first argument. Then

$$|R_{n,1}| \leq C_m \mathbb{E}_n \left[ |(D_i - W_i^\top \mu_0)D_iX_i^\top (\tilde{\theta} - \theta_0)| \right]$$

$$\leq C_m \|\tilde{\theta} - \theta_0\|_1 \mathbb{E}_n \left[ |D_i - W_i^\top \mu_0||D_i||X_i|_\infty \right]$$

$$\leq C_m \|\tilde{\theta} - \theta_0\|_1 \left( \mathbb{E}_n \left[ |D_i - W_i^\top \mu_0|^4 \right] \mathbb{E}_n \left[ |D_i|^4 \right] \right)^{1/4} \sqrt{\mathbb{E}_n \|X_i\|_\infty^2}$$

$$\lesssim_P a_nB_n,$$ 

where the first inequality follows from the mean-value theorem and Assumption 5.3, the second from Hölder’s inequality, the third from the Cauchy-Schwarz inequality, and the $\lesssim_P$ from Markov’s inequality and Assumption 3.5.1 (recall that we take $L \geq 1$) in combination with Assumptions 5.2 and 5.5. Given that $a_nB_n \to 0$ by assumption, it thus follows that $R_{n,1} = o_P(1)$.  

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Suppose now that $J \geq 2$. For all $i \in [n]$, we denote $\Delta_i := |X_i^\top (\tilde{\theta} - \theta_0)|$,

\[
\mathcal{R}_{i,1} = \begin{cases} 
1, & \text{if } X_i^\top \theta_0 - \Delta_i \leq t_{Y,i,j} \leq X_i^\top \theta_0 + \Delta_i \text{ for some } j \in [J-1], \\
0, & \text{otherwise}, 
\end{cases} \tag{B.49}
\]

and

\[
\mathcal{R}_{i,2} = \begin{cases} 
1, & \text{if } X_i^\top \theta_0 - \Delta_n \leq t_{Y,i,j} \leq X_i^\top \theta_0 + \Delta_n \text{ for some } j \in [J-1], \\
0, & \text{otherwise}, 
\end{cases} \tag{B.50}
\]

Observe that for all $i \in [n]$, we have $\mathcal{R}_{i,1}^2 = \mathcal{R}_{i,1} \leq \mathcal{R}_{i,2} + 1(\Delta_i > \Delta_n)$ and

\[
1(\Delta_i > \Delta_n) \leq 1(\|X_i\|_\infty \|\tilde{\theta} - \theta_0\|_1 > \Delta_n) \leq \left(\|X_i\|_\infty \|\tilde{\theta} - \theta_0\|_1/\Delta_n\right)^r,
\]

with $r \in (4, \infty)$ provided by Assumption 3.5.1. Hence,

\[
\mathbb{E}_n[\mathcal{R}_{i,1}^2] \leq \mathbb{E}_n[\mathcal{R}_{i,2}] + \mathbb{E}_n[\|X_i\|_\infty^r] \left(\|\tilde{\theta} - \theta_0\|_1/\Delta_n\right)^r \lesssim \Delta_n + (B_n a_n/\Delta_n)^r, \tag{B.51}
\]

where the $\lesssim$ follows from Markov’s inequality and Assumptions 3.5.1, 5.4 and 5.5.

Next, decompose as $R_{n,1} = R_{n,1,1} + R_{n,1,2}$, where

\[
R_{n,1,1} := \mathbb{E}_n \left[ (1 - \mathcal{R}_{i,1}) \left( m''_{11}(X_i^\top \tilde{\theta}, Y_i) - m''_{11}(X_i^\top \theta_0, Y_i) \right) (D_i - W_i^\top \mu_0) D_i \right],
\]

\[
R_{n,1,2} := \mathbb{E}_n \left[ \mathcal{R}_{i,1} \left( m''_{11}(X_i^\top \tilde{\theta}, Y_i) - m''_{11}(X_i^\top \theta_0, Y_i) \right) (D_i - W_i^\top \mu_0) D_i \right].
\]

Then $R_{n,1,1} = o_P(1)$ as in the case of $J = 1$ treated previously. Also,

\[
|R_{n,1,2}| \leq 2C_m \mathbb{E}_n[\mathcal{R}_{i,1}](D_i - W_i^\top \mu_0) D_i]
\]

\[
\leq 2C_m \left( \mathbb{E}_n[\mathcal{R}_{i,1}^2] \mathbb{E}_n[(D_i - W_i^\top \mu_0)^2] \right)^{1/2} \tag{B.52}
\]

\[
\lesssim \Delta_n^{1/2} + (B_n a_n/\Delta_n)^{r/2} \tag{B.53}
\]

where the first inequality follows from the triangle inequality and Lemma B.19, the second from the Cauchy-Schwarz inequality, and the $\lesssim$ from (B.51), Assumption 5.2, and the Markov and Cauchy-Schwarz inequalities. Given that $\Delta_n \to 0$ and $B_n a_n/\Delta_n \to 0$ by assumption, it follows that $R_{n,1,2} = o_P(1)$, and so $R_{n,1} = R_{n,1,1} + R_{n,1,2} = o_P(1)$ as well. \qed
Proof of Lemma B.25. Suppose first that $J = 1$. Then
\[ |R_{n,2}| \lesssim C_m |\tilde{\beta} - \beta_0| \mathbb{E}_n [ |D_i - W_i^\top \tilde{\mu}|D_i^2] \]
by Assumption 5.3. The right-hand side average satisfies
\[
\mathbb{E}_n [ |D_i - W_i^\top \tilde{\mu}|D_i^2] \lesssim \mathbb{E}_n [ |D_i - W_i^\top \mu_0|D_i^2] + \mathbb{E}_n [ |W_i^\top (\tilde{\mu} - \mu_0)|D_i^2] \\
\lesssim \sqrt{\mathbb{E}_n [D_i^4]} \left( \sqrt{\mathbb{E}_n [ |D_i - W_i^\top \mu_0|^2]} + \|\tilde{\mu} - \mu_0\|_1 \sqrt{\mathbb{E}_n [\|W_i\|_\infty^2]} \right) \\
\lesssim P 1 + a_n B_n \lesssim 1,
\]
where the first inequality follows from the triangle inequality, the second from the Cauchy-Schwarz and Hölder inequalities, the $\lesssim P$ from Assumptions 3.5.1, 5.2 and 5.5, and the $\lesssim$ from $a_n B_n \rightarrow 0$, which holds by hypothesis. Since $|\tilde{\beta} - \beta_0| \lesssim P a_n$ by Assumption 5.5, it follows that $|R_{n,2}| \lesssim P a_n$.

Suppose now that $J \geq 2$. For all $i \in [n]$, denote $\Delta_i := |X_i^\top (\tilde{\theta} - \theta_0)| + |D_i (\tilde{\beta} - \beta_0)|$ and define $R_{i,1}$ and $R_{i,2}$ as in (B.49) and (B.50), respectively. Then the rate in (B.51) follows from the argument used in the proof of Lemma B.24. Next, decompose as $R_{n,2} = R_{n,2,1} + R_{n,2,2}$, where
\[
R_{n,2,1} := \int_0^1 \mathbb{E}_n \left[ (1 - R_{i,1}) \left( m''_{11}(X_i^\top \tilde{\theta}, Y_i) - m''_{11}(X_i^\top \bar{\theta}, Y_i) \right) (D_i - W_i^\top \tilde{\mu})D_i \right] d\tau, \\
R_{n,2,2} := \int_0^1 \mathbb{E}_n \left[ R_{i,1} \left( m''_{11}(X_i^\top \tilde{\theta}, Y_i) - m''_{11}(X_i^\top \bar{\theta}, Y_i) \right) (D_i - W_i^\top \tilde{\mu})D_i \right] d\tau.
\]
Then $|R_{n,2,1}| \lesssim P a_n$ as in the case $J = 1$. Also,
\[
|R_{n,2,2}| \lesssim 2C_m \mathbb{E}_n |R_{i,1}| (D_i - W_i^\top \tilde{\mu})D_i| \lesssim P \bar{\Delta}_n^{1/2} + (B_n a_n / \bar{\Delta}_n)^{r/2},
\]
where the inequality follows from triangle inequality and Lemma B.19 and the $\lesssim P$ follows from the Cauchy-Schwarz inequality, (B.51), and an argument similar to that leading to (B.54). Thus, $|R_{n,2}| \leq |R_{n,2,1}| + |R_{n,2,2}| \lesssim P a_n + \bar{\Delta}_n^{1/2} + (B_n a_n / \bar{\Delta}_n)^{r/2}$, as claimed. \hfill \Box

Proof of Lemma B.26. Suppose first that $J = 1$. Then
\[
|R_{n,3}| \leq C_m \mathbb{E}_n [ |D_i - W_i^\top \mu_0| (W_i^\top \tilde{\gamma} - W_i^\top \gamma_0)^2] \\
\leq C_m \|\tilde{\gamma} - \gamma_0\|^2 \max_{1 \leq j, k \leq p - 1} \mathbb{E}_n [(D_i - W_i^\top \mu_0)W_{i,j}W_{i,k}] \\
\]
by Assumption 5.3 followed by Hölder’s inequality. Setting up for an application of Theorem
D.3, note that for all \( j, k \in [p - 1] \), we have \( \mathbb{E}[|\langle D - W^\top \mu_0 \rangle W_j W_k |] \leq C_M^3 \) by the Cauchy-Schwarz inequality and Assumption 5.2. In addition, for \( q := (r \wedge \bar{r})/4 \in (1, \infty) \),

\[
\mathbb{E}[|D - W^\top \mu_0|^q \|W\|_\infty \|W\|_\infty] \leq C_M^q B_n^{2q}
\]

by the Cauchy-Schwarz inequality and Assumptions 3.5.1 and 5.2. Therefore, given that \( B_n^2 \ln(pn) = o(n^{1 - 4/(r \wedge \bar{r})}) \) by assumption, Theorem D.3 produces

\[
\max_{1 \leq j, k \leq p - 1} \mathbb{E}_n[|\langle D_i - W_i^\top \mu_0 \rangle W_{ij} W_{ik}|] \lesssim_P 1.
\]

Hence, given that \( \|\bar{\gamma} - \gamma_0\|_\infty \lesssim_P a_n \) by Assumption 5.5, it follows that \( |R_{n, 3}| \lesssim_P a_n^2 \).

Suppose now that \( J \geq 2 \). For all \( i \in [n] \), denote \( \Delta_i := |\langle W_i^\top (\bar{\gamma} - \gamma_0) \rangle| \) and define \( \mathcal{R}_{i, 1} \) and \( \mathcal{R}_{i, 2} \) as in (B.49) and (B.50), respectively. Then the rate in (B.51) follows as in the proof of Lemma B.24. Next, decompose as \( R_{n, 3} = R_{n, 3, 1} + R_{n, 3, 2} \), where

\[
R_{n, 3, 1} := \int_0^1 \mathbb{E}_n \left[ 1 - \mathcal{R}_{i, 1} \right] \left( m''_{11}(X_i^\top \theta_\tau, Y_i) - m''_{11}(X_i^\top \theta_0, Y_i) \right) \langle D_i - W_i^\top \mu_0 \rangle W_i^\top (\bar{\gamma} - \gamma_0) d\tau,
\]

\[
R_{n, 3, 2} := \int_0^1 \mathbb{E}_n \left[ \mathcal{R}_{i, 1} \right] \left( m''_{11}(X_i^\top \theta_\tau, Y_i) - m''_{11}(X_i^\top \theta_0, Y_i) \right) \langle D_i - W_i^\top \mu_0 \rangle W_i^\top (\bar{\gamma} - \gamma_0) d\tau.
\]

Then \( |R_{n, 3, 1}| \lesssim_P a_n^2 \) follows from the argument in the case \( J = 1 \). Also,

\[
|R_{n, 3, 2}| \leq 2C_m \mathbb{E}_n \left[ \mathcal{R}_{i, 1} \right] \langle D_i - W_i^\top \mu_0 \rangle W_i^\top (\bar{\gamma} - \gamma_0) \right] \lesssim_P B_n a_n \left( \overline{\Delta}_n^{1/2} + (B_n a_n / \overline{\Delta}_n)^{r/2} \right).
\]

where the inequality follows from the triangle inequality and Lemma B.19, and the \( \lesssim_P \) follows from the Cauchy-Schwarz and Hölder inequalities, (B.51), and Assumptions 3.5.1, 5.2, and 5.5. Hence, \( |R_{n, 3}| \leq |R_{n, 2, 1}| + |R_{n, 3, 2}| \lesssim_P a_n^2 + B_n a_n (\overline{\Delta}_n^{1/2} + (B_n a_n / \overline{\Delta}_n)^{r/2}) \), as claimed. \( \square \)

## C Analysis of Post-Penalized M-Estimation

In this section, we derive an analog of Theorem 3.1 for the post-\( \ell_1 \)-penalized M-estimator (post-\( \ell_1 \)-ME). The main message of this section is similar to that of Section 3: like in the case of the \( \ell_1 \)-ME, in order to obtain the post-\( \ell_1 \)-ME with small estimation errors, we should choose the penalty parameter \( \lambda \) such that it is as small as possible but larger than the (slightly inflated) maximum of the score \( c_0 \|S_n\|_\infty \) with high probability, with \( S_n \) defined in (3.3). Theorem C.1, which is the main result of this section, serves as the key building block for the proof of Theorem 4.2 in the main text.
Recall the definitions of \( \tilde{\Theta}(\supp(\tilde{\theta})) \) and \( \tilde{\Theta}(\lambda) \) in (4.15) and (4.16), respectively, and that
\[
\eta_n = \sqrt{\ln(pn)/n}.
\]
The following theorem yields estimation error bounds for the post-\( \ell_1 \)-ME.

**Theorem C.1 (Non-Asymptotic Error Bounds for Post-\( \ell_1 \)-ME).** Let Assumptions 3.1–3.6, 4.4, and 4.5 hold, let \( \bar{\lambda}_n \) and \( \underline{\lambda}_n \) be non-random sequences in \((0, \infty)\) such that \( \bar{\lambda}_n \geq \underline{\lambda}_n \), and let \( \phi_n := \sqrt{(\eta_n^2 + \bar{\lambda}_n^2)/\underline{\lambda}_n^2} \). In addition, suppose that
\[
\frac{B_n^2 \ln(pn)}{\sqrt{n}} \to 0 \quad \text{and} \quad n^{1/r} B_n s_q \eta_n^{-q} \left( \bar{\lambda}_n \phi_n + \eta_n \phi_n^2 \right) \to 0.
\]
Then there is a constant \( C \in [1, \infty) \), depending only on \( c_0, C_{ev}, C_L, c_M \) and \( C_m \), such that
\[
\sup_{\tilde{\theta} \in \tilde{\Theta}(\lambda)} \| \tilde{\theta} - \theta_0 \|_2 \leq C \sqrt{s_q \eta_n^{-q} \left( \bar{\lambda}_n \phi_n + \eta_n \phi_n^2 \right)}
\]
and
\[
\sup_{\tilde{\theta} \in \tilde{\Theta}(\lambda)} \| \tilde{\theta} - \theta_0 \|_1 \leq C s_q \eta_n^{-q} \left( \bar{\lambda}_n \phi_n + \eta_n \phi_n^2 \right)
\]
with probability at least \( 1 - C(\mathbb{P}(\lambda < c_0 \| S_n \|_\infty) + \mathbb{P}(\lambda > \bar{\lambda}_n) + \mathbb{P}(\lambda < \underline{\lambda}_n)) - o(1) \).

Before we prove this theorem, we introduce some extra notation. For \( k \in \mathbb{N} \), define the \( (\ell_0) \)-restricted set
\[
\tilde{\mathcal{R}}(k) := \{ \delta \in \mathbb{R}^p; \| \theta_0 + \delta \|_0 \leq k \text{ and } \theta_0 + \delta \in \Theta \},
\]
and the associated (random) empirical error function \( \tilde{\epsilon} : [0, \infty) \times \mathbb{N} \to [0, \infty) \) by
\[
\tilde{\epsilon}_n(u, k) := \sup_{\delta \in \tilde{\mathcal{R}}(k), \| \delta \|_2 \leq u} \left| \left( (\mathbb{E}_n - E) \left[ m \left( X_i^T (\theta_0 + \delta), Y_i \right) \right] - m \left( X_i^T \theta_0, Y_i \right) \right) \right|.
\]
Also, let \( a_{\epsilon,n}, b_{\epsilon,n}, \tilde{a}_{\epsilon,n}, \tilde{b}_{\epsilon,n} \) and \( \bar{\lambda}_n \) be non-random sequences in \((0, \infty)\), and let \( \tilde{K}_n \) be a non-random sequence in \( \mathbb{N} \), all to be specified later. Based on \( a_{\epsilon,n}, b_{\epsilon,n} \) and \( \bar{\lambda}_n \), define the non-random sequence \( \tilde{u}_n \) in \((0, \infty)\) as in (B.2) and the events \( \mathcal{I}_n, \mathcal{L}_n, \) and \( \mathcal{E}_n \) as in (B.1). Moreover, define the events
\[
\tilde{\mathcal{E}}_n := \left\{ \tilde{\epsilon}_n(\tilde{u}_n + \tilde{u}_n, \tilde{K}_n) \leq \tilde{a}_{\epsilon,n}(\tilde{u}_n + \tilde{u}_n) + \tilde{b}_{\epsilon,n} \right\} \quad \text{and} \quad \mathcal{K}_n := \left\{ \sup_{\tilde{\theta} \in \tilde{\Theta}(\lambda)} \| \tilde{\theta} \|_0 \leq \tilde{K}_n \right\},
\]
where
\[
\tilde{u}_n := 4\tilde{u}_n + 2\tilde{a}_{\epsilon,n}/c_M.
\]
The proof of Theorem C.1 will be based on the following four lemmas, whose proofs can be found at the end of this section.

**Lemma C.1** (Restricted Set Consequence, II). For any \( \eta \in (0, \infty) \) and \( k \in \mathbb{N} \), \( \delta \in \widetilde{R}(k) \) implies \( \|\delta\|_1 \leq \|\delta\|_2 \sqrt{k + s_q \eta^{-q}} + s_q \eta^{1-q} \).

**Lemma C.2** (Non-Asymptotic Deterministic Bounds for Post-\( \ell_1 \)-ME). Let Assumptions 3.1–3.6 hold and suppose that \( \tilde{u}_n + \tilde{v}_n \leq c'_M \),

\[
(a_{e,n} + (1 + \bar{c}_0)\lambda_n \sqrt{s_q \eta^{-q}})^2 \geq c_M \left(b_{e,n} + (1 + \bar{c}_0) \lambda_n \sqrt{s_q \eta^{-q} + \lambda_n s_q \eta^{1-q}}\right)
\]

and

\[
(2c_M \tilde{u}_n + \tilde{v}_n)^2 \geq c_M \left(\tilde{a}_{e,n} \tilde{u}_n + \tilde{b}_{e,n} + \tilde{\lambda}_n (1 + \bar{c}_0) \tilde{u}_n \sqrt{s_q \eta^{-q} + \tilde{\lambda}_n s_q \eta^{1-q}}\right)
\]

Then on the event \( \mathcal{S}_n \cap \mathcal{L}_n \cap \mathcal{E}_n \cap \tilde{\mathcal{E}}_n \cap \mathcal{K}_n \), for all \( \tilde{\theta} \in \tilde{\Theta}(\lambda) \), we have \( \tilde{\theta} - \theta_0 \in \tilde{R}(\kappa_n) \),

\[
\|\tilde{\theta} - \theta_0\|_2 \leq \tilde{u}_n + \tilde{v}_n \quad \text{and} \quad \|\tilde{\theta} - \theta_0\|_1 \leq (\tilde{u}_n + \tilde{v}_n) \sqrt{\tilde{\kappa}_n + s_q \eta^{-q} + s_q \eta^{1-q}}
\]

**Lemma C.3** (Empirical Error Bound, II). Let Assumptions 3.5 and 3.6 hold. Then there is a universal constant \( C \in [1, \infty) \), such that for any \( k \in \mathbb{N} \), \( n \in \mathbb{N} \), \( t \in [1, \infty) \) and \( u \in (0, \infty) \) satisfying

\[
\eta_n \leq 1, \quad \frac{B_n^2 \ln(pn)}{\sqrt{n}} \leq C_L^2 \quad \text{and} \quad tn^{1/r} \left(u \sqrt{k + s_q \eta^{-q} + s_q \eta^{1-q}}\right) \leq c_L,
\]

we have

\[
\bar{c}_n(u,k) \leq CC_L \left(u \eta_n \sqrt{k + s_q \eta^{-q} + s_q \eta^{2-q}}\right)
\]

with probability at least \( 1 - 4t^{-r} - C/\ln^2(pn) - n^{-1} \).

**Lemma C.4** (Sparsity Bound). Let Assumptions 3.1–3.6, 4.4 and 4.5 hold and let \( \lambda_n \) and \( \tilde{\lambda}_n \) be non-random sequences in \((0, \infty)\) such that \( \lambda_n \geq \lambda_n \) and set \( \phi_n := \sqrt{\eta_n^2 + \lambda_n^2} / \lambda_n^2 \). In addition, assume that

\[
\frac{B_n^2 \ln(pn)}{\sqrt{n}} \to 0 \quad \text{and} \quad n^{1/r} B_n s_q \eta^{-q} (\lambda_n + \eta_n \phi_n^2) \to 0.
\]
Then there is a constant $C \in [1, \infty)$, depending only on $c_0, C_{cv}, C_L, c_M$ and $C_m$, such that
\[
\sup_{\hat{\theta} \in \hat{\Theta}(\lambda)} \| \hat{\theta} \|_0 \leq C s_q \eta_n^{-q} \phi_n^2
\]
with probability at least
\[
1 - 2P(\lambda < c_0 \| S_n \|_\infty) - P(\lambda > \bar{\lambda}_n) - P(\lambda < \underline{\lambda}_n) - o(1).
\]

**Proof of Theorem C.1.** Let $\bar{C}$ and $\underline{C}$ be universal constants $C$ from Lemmas B.4 and C.3, respectively, and let $\bar{C}$ be the constant from Lemma C.4. Define
\[
a_{\epsilon,n} := \bar{C}(1 + \epsilon_0) C_L \sqrt{s_q \eta_n^{-q}}, \quad b_{\epsilon,n} := \bar{C}(1 + \epsilon_0) C_L s_q \eta_n^{-q}, \quad \bar{k}_n := [\bar{C} s_q \eta_n^{-q} \phi_n^2],
\]
\[
\bar{a}_{\epsilon,n} := \underline{C} C_L \eta_n \sqrt{\bar{k}_n + s_q \eta_n^{-q}}, \quad \text{and} \quad \bar{b}_{\epsilon,n} := \underline{C} C_L s_q \eta_n^{-q}.
\]
Then, under (C.1), Lemmas B.4 and C.3 imply (via appropriate choices of $t_n$ sequences) that $P(\epsilon_n^c) \to 0$ and $P(\tilde{\epsilon}_n^c) \to 0$, respectively, and Lemma C.4 implies that $P(\mathcal{K}_n^c) \leq 2P(\lambda < c_0 \| S_n \|_\infty) + P(\lambda > \bar{\lambda}_n) + P(\lambda < \underline{\lambda}_n) + o(1)$. In addition, again under (C.1), we have that $\tilde{u}_n + \tilde{a}_n \to 0$, and using the above definitions and rearranging shows that both (C.6) and (C.7) are satisfied since $\bar{C}, \underline{C}, C_L > 0$ and $c_M \leq 1$. Hence, for a sufficiently large constant $C \in [1, \infty)$, that can be chosen to depend only on $c_0, C_{cv}, C_L$ and $C_m$, Lemma C.2 implies that the bounds (C.2) and (C.3) hold with probability at least $P(\mathcal{S}_n \cap \mathcal{L}_n \cap \mathcal{E}_n \cap \tilde{\mathcal{E}}_n \cap \mathcal{K}_n^c$) for all $n$ large enough (so that $\tilde{u}_n + \tilde{a}_n \leq c_M^t$). Combining these bounds gives the asserted claim. \[\square\]

**Proof of Lemma C.1.** Recall that, given a vector $\delta \in \mathbb{R}^p$ and a set of indices $J \subseteq [p]$, we let $\delta_J$ denote the vector in $\mathbb{R}^p$ with coordinates given by $\delta_{J,j} = \delta_j$ if $j \in J$ and $\delta_{J,j} = 0$ otherwise. Now, fix $\eta \in (0, \infty)$ and $k \in \mathbb{N}$, let $\delta \in \hat{R}(k)$, and denote $\theta := \theta(\delta) := \theta_0 + \delta$ and $J := \text{supp}(\theta)$. Then we may decompose as follows
\[
\| \delta \|_1 = \| \delta_{J \cap T(\eta)} \|_1 + \| \delta_{J^c \cap T(\eta)}^c \|_1,
\]
with $T(\eta) = \{ j \in [p]; |\theta_{0,j}| > \eta \}$. The first term on the right-hand side satisfies
\[
\| \delta_{J \cap T(\eta)} \|_1 \leq \| \delta_{J \cap T(\eta)} \|_2 \sqrt{|J \cup T(\eta)|} \quad \text{(Cauchy-Schwarz)}
\]
\[
\leq \| \delta \|_2 \sqrt{|J| + |T(\eta)|} \leq \| \delta \|_2 \sqrt{k + s_q \eta^{-q}}. \quad \text{([|J| = \| \theta_0 + \delta \|_0 \text{ and Lemma B.1])}
\]

\[39\text{To verify (C.7), for example, note that } (2c_M \bar{u}_n + \bar{a}_{\epsilon,n})^2 = 4c_M^2 \bar{u}_n^2 + 4c_M \bar{u}_n a_{\epsilon,n} + \bar{a}_{\epsilon,n}^2 \text{ and then } 4c_M^2 \bar{u}_n^2 \geq c_M \bar{\lambda}_n (1 + \epsilon_0) \bar{u}_n \sqrt{s_q \eta_n^{-q}}, c_M \bar{u}_n a_{\epsilon,n} \geq c_M \bar{b}_{\epsilon,n} \text{ and } c_M \bar{u}_n a_{\epsilon,n} \geq c_M \bar{\lambda}_n s_q \eta_n^{-q}. \text{ Condition (C.6) follows similarly.}
\]
Since \( \text{supp}(\theta) = J \), the second term on the right-hand side satisfies

\[
\|\delta_{J \cap T(\eta)}\|_1 = \|\theta - \theta_0_{J \cap T(\eta)}\|_1 = \|\theta_0_{J \cap T(\eta)}\|_1 \leq \|\theta_0 T(\eta)\|_1 \leq s_q \eta^{1-q}. \quad \text{(Lemma B.1)}
\]

The claimed \( \ell_1 \) bound now arises from combining the previous three displays. \( \square \)

**Proof of Lemma C.2.** Let \( \tilde{\theta} \in \tilde{\Theta}(\lambda) \) and \( \tilde{\theta} \in \tilde{\Theta}(\text{supp}(\tilde{\theta})) \) be arbitrary, and let the event \( \mathcal{S}_n \cap \mathcal{L}_n \cap \mathcal{E}_n \cap \mathcal{E}_n \cap \mathcal{X}_n \) hold. Abbreviate \( \tilde{\delta} := \tilde{\theta} - \theta_0 \). Then \( \text{supp}(\tilde{\theta}) \subseteq \text{supp}(\tilde{\theta}) \) and \( \mathcal{X}_n \) imply

\[
\|\theta_0 + \tilde{\delta}\|_0 = \|\tilde{\theta}\|_0 \leq \|\theta_0\|_0 \leq \overline{\kappa}_n
\]

and, thus, \( \tilde{\delta} \in \tilde{\mathcal{R}}(\overline{\kappa}_n) \), as claimed. The stated \( \ell_1 \) bound will therefore follow from Lemma C.1 and the \( \ell_2 \) bound. It remains to show the latter bound.

Suppose to the contrary of the asserted claim that \( \|\tilde{\theta} - \theta_0\|_2 > \tilde{u}_n + \check{u}_n \). Then we must have \( \|\tilde{\theta} - \hat{\theta}\|_2 > \check{u}_n \), since \( \|\tilde{\theta} - \theta_0\|_2 \leq \check{u}_n \) by Lemma B.3. By definition of \( \tilde{\Theta}(\text{supp}(\tilde{\theta})) \) in (4.15),

\[
\mathbb{E}_n[m(X_i^T\tilde{\theta}, Y_i)] \leq \mathbb{E}_n[m(X_i^T\hat{\theta}, Y_i)].
\]

For \( t := \check{u}_n/\|\tilde{\theta} - \hat{\theta}\|_2 \in (0, 1) \), it then follows from convexity (Assumption 3.2) that

\[
\mathbb{E}_n[m(X_i^T(\hat{\theta} + t(\tilde{\theta} - \hat{\theta})), Y_i)] \leq \mathbb{E}_n[m(X_i^T\tilde{\theta}, Y_i)].
\]

In addition, by definition of \( \tilde{\Theta}(\lambda) \) in (1.2), the triangle inequality, and the \( \ell_1 \)-bound of Lemma B.3, we have

\[
\mathbb{E}_n[m(X_i^T\hat{\theta}, Y_i)] - \mathbb{E}_n[m(X_i^T\theta_0, Y_i)] \leq \lambda(\|\theta_0\|_1 - \|\hat{\theta}\|_1) \leq \overline{\lambda}_n \Delta_n,
\]

where \( \Delta_n := (1 + \bar{c}_0)(\tilde{u}_n \sqrt{s_q \eta^{1-q}} + s_q \eta^{1-q}) \). Combining these bounds, we obtain

\[
\overline{\lambda}_n \Delta_n \geq \mathbb{E}_n[m(X_i^T(\hat{\theta} + t(\tilde{\theta} - \hat{\theta})), Y_i)] - \mathbb{E}_n[m(X_i^T\theta_0, Y_i)].
\]

From the triangle inequality, we see that

\[
\check{u}_n - \tilde{u}_n \leq \|\hat{\theta} + t(\tilde{\theta} - \hat{\theta}) - \theta_0\|_2 \leq \check{u}_n + \tilde{u}_n.
\]

Convexity of the parameter space (Assumption 3.1) shows that \( \hat{\theta} + t(\tilde{\theta} - \hat{\theta}) \in \Theta \), and \( \text{supp}(\hat{\theta}) \subseteq \text{supp}(\tilde{\theta}) \) and the event \( \mathcal{X}_n \) combine to show that

\[
\|\hat{\theta} + t(\tilde{\theta} - \hat{\theta})\|_0 \leq \|\hat{\theta}\|_0 \leq \overline{\kappa}_n.
\]

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Deduce that \( \hat{\theta} + t(\tilde{\theta} - \hat{\theta}) - \theta_0 \in \tilde{R}(\tilde{k}_n) \). It follows from superadditivity of infima and the definition of the empirical error function \( \tilde{e}_n \) in (C.4) that

\[
\tilde{\lambda}_n \tilde{\Delta}_n \geq \mathbb{E}_n[m(X_i^\top(\hat{\theta} + t(\tilde{\theta} - \hat{\theta})), Y_i)] - \mathbb{E}_n[m(X_i^\top\theta_0, Y_i)]
\geq \inf_{\delta \in \tilde{R}(\tilde{k}_n), \tilde{u}_n - \tilde{u}_n \leq \|\delta\|_2 \leq \tilde{u}_n + \tilde{u}_n} \left\{ \mathbb{E}_n[m(X_i^\top(\theta_0 + \delta), Y_i)] - \mathbb{E}_n[m(X_i^\top\theta_0, Y_i)] \right\}
\geq \inf_{\delta \in \tilde{R}(\tilde{k}_n), \tilde{u}_n - \tilde{u}_n \leq \|\delta\|_2 \leq \tilde{u}_n + \tilde{u}_n} \mathcal{E}(\theta_0 + \delta) - \tilde{e}_n(\tilde{u}_n + \tilde{u}_n, \tilde{k}_n)
\geq c_M(\tilde{u}_n - \tilde{u}_n)^2 - \tilde{a}_{\epsilon,n}(\tilde{u}_n + \tilde{u}_n) - \tilde{b}_{\epsilon,n},
\]

where the final inequality uses Assumption 3.4 and the event \( \tilde{\mathcal{E}}_n \). Expanding the square, rearranging terms, and using \( c_M \tilde{u}_n^2 \geq 0 \), it further follows that

\[
A_n \tilde{u}_n^2 - B_n \tilde{u}_n - C_n \leq 0,
\]

where \( A_n := c_M, B_n := 2c_M \tilde{u}_n + \tilde{a}_{\epsilon,n}, \) and \( C_n := \tilde{a}_{\epsilon,n} \tilde{u}_n + \tilde{b}_{\epsilon,n} + \tilde{\lambda}_n \tilde{\Delta}_n \). The definition of \( \tilde{u}_n \) in (C.5) means that \( \tilde{u}_n = 2B_n/A_n \), so the displayed inequality can be written as \( 2B_n^2 \leq A_n C_n \). On the other hand, (C.7) can be rewritten as \( B_n^2 \geq A_n C_n \), yielding the desired contradiction. Conclude that \( \|\tilde{\theta} - \theta_0\|_2 \leq \tilde{u}_n + \tilde{u}_n \). \( \square \)

**Proof of Lemma C.3.** The proof of this lemma is closely related to that of Lemma B.4. In particular, as in the case of Lemma B.4, the proof will follow from an application of the maximal inequality in Theorem D.1. First, fix any \( k \in \mathbb{N}, n \in \mathbb{N}, t \in [1, \infty) \) and \( u \in (0, \infty) \) satisfying (C.8) and denote \( \Delta(u, k) := \tilde{R}(k) \cap \{\|\cdot\|_2 \leq u\} \). If \( \Delta(u, k) = \emptyset \), then the postulated bound holds with probability one (interpreting the supremum over an empty set as \(-\infty\)). We therefore assume that \( \Delta(u, k) \neq \emptyset \). Lemma C.1 shows that

\[
\|\Delta(u, k)\|_1 := \sup_{\delta \in \Delta(u, k)} \|\delta\|_1 \leq u \sqrt{k + s_q \eta_n^{-q} + s_q \eta_n^{-q}} =: \tilde{\lambda}_n(u, k).
\]

Setting up for an application of Theorem D.1, define \( h : \mathbb{R} \times X \times Y \to \mathbb{R} \) by \( h(t, x, y) := m(x^\top\theta_0 + t, y) - m(x^\top\theta_0, y) \) for all \( t \in \mathbb{R} \) and \( (x, y) \in X \times Y \). By construction, \( h(0, \cdot, \cdot) \equiv 0 \). By Assumption 3.5.1, the restriction \( h : [-c_L, c_L] \times X \times Y \to \mathbb{R} \) is \( L(x, y) \)-Lipschitz in its first argument, thus verifying Condition 1 of Theorem D.1 with \( C_h = c_L \). Hölder’s inequality, Assumption 3.5.1, (C.10), and (C.8) imply that

\[
\max_{1 \leq i \leq n} \sup_{\delta \in \Delta(u, k)} \|X_i^\top \delta\| \leq \max_{1 \leq i \leq n} \|X_i\|_\infty \|\Delta(u, k)\|_1 \leq tn^{1/r} B_n \tilde{\lambda}_n(u, k) \leq c_L
\]
with probability at least $1 - t^{-r}$, where the bound $P(\max_{1 \leq i \leq n} \|X_i\|_\infty > tn^{1/r}B_n) \leq t^{-r}$ follows from Markov’s inequality, since Assumption 3.5.1 implies that $E[\|X\|_\infty] \leq B_n^\alpha$. Condition 2 of Theorem D.1 therefore holds with $C_h = c_L$ and $\zeta_n = t^{-r}$. Further, swapping $\Delta(u, \eta_n)$ for the $\Delta(u, k)$ in the proof of Lemma B.4, we verify Conditions 3 and 4 of Theorem D.1 in exactly the same way, and with the same constants as those appearing in the proof of Lemma B.4. Therefore, Theorem D.1 combined with the bound $\Delta_n(u, k)$ on $\|\Delta(u, k)\|_1$ from (C.10) and $\ln(8pn) \leq 4\ln(pn)$ (which follows from $p \geq 2$) now show that

$$P\left(\sqrt{n}\epsilon_n(u, k) > \{4C_Lu\} \cup \{16\sqrt{2}CC_n(u, k)\sqrt{\ln (pn)}\}\right) \leq 4t^{-r} + 4C/\ln(pn)^2 + n^{-1}$$

for some universal constant $C \in [1, \infty)$. Now, given that $k, s, q \in [1, \infty)$, $p \in [2, \infty)$, $n \in [3, \infty)$, and $\eta_n \in (0, 1]$, it follows that $4\sqrt{2C}/\sqrt{\ln(pn)} \geq 1$ and $\Delta_n(u, k) \geq u$. Hence, $16\sqrt{2}CC_n(u, k)\sqrt{\ln(pn)} \geq 4C_Lu$. Using this bound in the previous display and redefining the universal constant $C$ appropriately, we arrive at the asserted claim. \hfill \Box

**Proof of Lemma C.4.** The proof of this lemma adapts arguments developed by Belloni et al. (2012) for post-LASSO to our setting with a more general loss function. For all $k \in \mathbb{N}$, let

$$S_k^p := \{\delta \in \mathbb{R}^p; \|\delta\|_2 \leq 1, \|\delta\|_0 \leq k\} \quad \text{and} \quad \hat{\phi}(k) := \sup_{\delta \in S_k^p} E_n[(X_i^\top \delta)^2].$$

We proceed in five steps.

**Step 1:** Let $a_{1,n}$ and $a_{2,n}$ be non-random sequences in $(0, \infty)$ for which $n^{1/r}B_na_{1,n} \to 0$, and let

$$\Delta_n := \{\delta \in \mathbb{R}^p; \|\delta\|_1 \leq a_{1,n}, \|\delta\|_2 \leq a_{2,n}\}.$$ 

In this step, we show that there is a constant $C \in [1, \infty)$, depending only on $C_{ev}$ and $C_L$, such that, with probability $1 - o(1)$,

$$\sup_{\delta \in \Delta_n} E_n[(X_i^\top \delta)^2] \leq C(a_{2,n}^2 + a_{1,n}\eta_n).$$

To do so, we set up for an application of Theorem D.1 with $h(t, x, y) = t^2$ and $\Delta = \Delta_n$. By construction, we have $\|\Delta_n\|_1 := \sup_{\delta \in \Delta_n} \|\delta\|_1 \leq a_{1,n}$. Since $h(t_1, x, y) - h(t_2, x, y) = (t_1 + t_2)(t_1 - t_2)$, Condition 1 of Theorem D.1 holds with $C_h = 1/2$ and $L(\cdot, \cdot) \equiv 1$. Condition 2 of Theorem D.1 holds for some $\zeta_n = o(1)$, since

$$\max_{1 \leq i \leq n} \sup_{\delta \in \Delta_n} |X_i^\top \delta| \leq \max_{1 \leq i \leq n} \|X_i\|_\infty \|\Delta_n\|_1 \leq n^{1/r}B_na_{1,n} \to 0$$

which can be argued from Hölder’s inequality, Assumption 3.5.1 and Markov’s inequality,
and the $\ell_1$-restriction in $\Delta_n$. Condition 3 of Theorem D.1 holds with $B_{1n} = \sqrt{C_{ev}a_{2, n}}$ by Assumption 4.5. To verify Condition 4 of Theorem D.1, we set up for an application of Theorem D.3 with $Z_{i,j} = X_{i,j}^2 \geq 0$. Assumption 3.5.1 shows that Conditions 1 and 2 of Theorem D.3 hold with $\mu_n = C_L^2$, $q = 2$, and $M_n = B_{2n}^2$. Condition 3 of Theorem D.3 then translates to $B_{2n}^2 \log(pn)/\sqrt{n} \leq C_L^2$, which holds eventually, cf. (C.9). Theorem D.3 therefore implies that $\max_{1 \leq i \leq p} E_n[X_{i,j}^2] \leq (\tilde{C} C_L)^2$ with probability $1 - o(1)$ for some universal constant $\tilde{C} \in (0, \infty)$. It follows that Condition 4 of Theorem D.1 holds for some $\gamma_n = o(1)$ and $B_{2n} = \tilde{C} C_L$. Applying Theorem D.1, it follows that for some universal constant $C \in [1, \infty)$, with probability $1 - o(1)$,

$$\sup_{\delta \in \Delta_n} \left| E_n[(X_i^\top \delta)^2] - E[(X^\top \delta)^2] \right| \leq C \left( \sqrt{C_{ev}a_{2, n}^2}/\sqrt{n} + C_L a_{1, n} \eta_n \right). \tag{C.11}$$

In addition, from Assumption 4.5, we get

$$\sup_{\delta \in \Delta_n} E[(X^\top \delta)^2] \leq \sup_{\delta \in \Delta_n} (E[(X^\top \delta)^4])^{1/2} \leq \sqrt{C_{ev}a_{2, n}^2}.$$

Combining these bounds, the asserted claim of this step follows from the triangle inequality.

**Step 2:** Let $k_n$ be a non-random sequence in $\mathbb{N}$ satisfying $k_n n^{1/r} B_n \eta_n \to 0$, whose existence follows from (C.9). In this step, we show that for such a sequence, with probability $1 - o(1)$,

$$\hat{\phi}(k_n) \leq \sqrt{C_{ev}} + o(1).$$

To this end, let $t_n$ be a non-random sequence in $(0, \infty)$ such that $\sqrt{k_n} n^{1/r} B_n / t_n \to 0$ and $t_n \sqrt{k_n} \eta_n \to 0$, whose existence follows from $k_n n^{1/r} B_n \eta_n \to 0$. Since $\|\delta\|_1 \leq \sqrt{k_n}\|\delta\|_2$ for any $\delta \in S_{k_n}^p$ (per Cauchy-Schwarz inequality), it follows that the set $S_{k_n}^p / t_n$ is contained in $\Delta_n$ introduced in Step 1 provided we set $a_{1, n} = \sqrt{k_n}/t_n$ and $a_{2, n} = 1/t_n$. In this case, the premise $n^{1/r} B_n a_{1, n} \to 0$ of Step 1 is satisfied, and it follows as in (C.11) that

$$\sup_{\delta \in S_{k_n}^p / t_n} \left| E_n[(X_i^\top \delta)^2] - E[(X^\top \delta)^2] \right| \leq C \left( \sqrt{C_{ev}/(t_n^2 \sqrt{n})} + C_L \sqrt{k_n} \eta_n / t_n \right)$$

with probability $1 - o(1)$ for some universal constant $C \in [1, \infty)$. Hence, with the same constant $C$,

$$\sup_{\delta \in S_{k_n}^p} \left| E_n[(X_i^\top \delta)^2] - E[(X^\top \delta)^2] \right| \leq C \left( C_{ev}/\sqrt{n} + C_L t_n \sqrt{k_n} \eta_n \right) \to 0$$

with probability $1 - o(1)$. In addition, using the Cauchy-Schwarz inequality and Assumption
4.5, we get
\[
\sup_{\delta \in \mathcal{S}_{kn}} \mathbb{E}[(\mathbf{X}^\top \delta)^2] \leq \sup_{\delta \in \mathcal{S}_{kn}} (\mathbb{E}[(\mathbf{X}^\top \delta)^4])^{1/2} \leq \sqrt{C_{cv}}.
\]
Combining these bounds gives the asserted claim of this step.

**Step 3:** In this step, we show that there exists a constant \( \bar{C} \in (0, \infty) \), depending only on \( c_0, C_{cv}, C_L, c_M \) and \( C_m \), such that
\[
\|\hat{\theta}\|_0 \leq \bar{C} \hat{\delta}(\|\hat{\theta}\|_0)s_q \eta_n^{-q} \phi_n^2 \text{ for all } \hat{\theta} \in \tilde{\Theta}(\lambda),
\]
with probability at least \( 1 - 2P(\lambda < c_0 \|S_n\|_{\infty}) - P(\lambda > \bar{\lambda}_n) - P(\lambda < \underline{\lambda}_n) - o(1) \). To do so, fix any \( \hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_p)^\top \in \tilde{\Theta}(\lambda) \) and observe that the first-order conditions for the optimization problem (1.2) imply that
\[
\mathbb{E}_n[m'_1(\mathbf{X}_i^\top \hat{\theta}, \mathbf{Y}_i)X_{i,j}] + \lambda Z_j = 0
\]
for all \( j \in [p] \) and some \((Z_1, \ldots, Z_p)^\top \in [-1, 1]^p \) satisfying \( Z_j = 1 \) if \( \hat{\theta}_j > 0 \) and \( Z_j = -1 \) if \( \hat{\theta}_j < 0 \). Therefore, denoting \( \hat{T} := \text{supp}(\hat{\theta}) \), it follows that for all \( j \in \hat{T} \),
\[
|\mathbb{E}_n[m'_1(\mathbf{X}_i^\top \hat{\theta}, \mathbf{Y}_i)X_{i,j}]|^2 = \lambda^2,
\]
and so
\[
\lambda^2 |\hat{T}| = \sum_{j \in \hat{T}} |\mathbb{E}_n[m'_1(\mathbf{X}_i^\top \hat{\theta}, \mathbf{Y}_i)X_{i,j}]|^2.
\]
Hence, by the triangle inequality,
\[
\lambda \sqrt{|\hat{T}|} \leq \left( \sum_{j \in \hat{T}} |\mathbb{E}_n[m'_1(\mathbf{X}_i^\top \hat{\theta}_0, \mathbf{Y}_i)X_{i,j}]|^2 \right)^{1/2} + \left( \sum_{j \in \hat{T}} |\mathbb{E}_n[(m'_1(\mathbf{X}_i^\top \hat{\theta}, \mathbf{Y}_i) - m'_1(\mathbf{X}_i^\top \hat{\theta}_0, \mathbf{Y}_i))X_{i,j}]|^2 \right)^{1/2}.
\] (C.13)

The first term on the right-hand side is bounded from above by \( \sqrt{|\hat{T}|} \|S_n\|_{\infty} \leq \sqrt{|\hat{T}|} \lambda/c_0 \) with probability at least \( 1 - P(\lambda < c_0 \|S_n\|_{\infty}) \). By the dual norm inequality, the second term
is bounded from above by

\[
\sup_{\delta \in \mathcal{S}_i} \mathbb{E}_n[(m_1'(X_i^\top \tilde{\theta}, Y_i) - m_1'(X_i^\top \theta_0, Y_i))X_i^\top \delta] \\
\leq \sup_{\delta \in \mathcal{S}_i} \left( \mathbb{E}_n[(m_1'(X_i^\top \tilde{\theta}, Y_i) - m_1'(X_i^\top \theta_0, Y_i))^2] \right)^{1/2} \left( \mathbb{E}_n[(X_i^\top \delta)^2] \right)^{1/2} \\
\leq C_m \sqrt{\tilde{\phi}(|\tilde{T}|)} \left( \mathbb{E}_n[(X_i^\top \tilde{\theta} - X_i^\top \theta_0)^2] \right)^{1/2},
\]

where the first inequality follows from the Cauchy-Schwarz inequality, and the second from the definition of \(\tilde{\phi}(|\tilde{T}|)\) and Assumption 4.4. To control the right-hand side average, observe that condition (C.9) allows us to pick a \(t = t_n \to \infty\) so as to eventually satisfy all side conditions of Theorem 3.1. Using Theorem 3.1, a calculation then shows that

\[
\|\tilde{\theta} - \theta_0\|_2 \leq C_2 \sqrt{s q \tilde{\eta}^{-q}} (\eta_n + \tilde{\lambda}_n) \quad \text{and} \quad \|\tilde{\theta} - \theta_0\|_1 \leq C_1 s q \tilde{\eta}^{-q} (\eta_n + \tilde{\lambda}_n),
\]

with probability at least \(1 - P(\lambda < c_0 \|S_n\|_\infty) - P(\lambda > \tilde{\lambda}_n) - o(1)\) for some constants \(C_1, C_2 \in (0, \infty)\), depending only on \(c_0, C_L, c_M\). Call the two right-hand sides of the previous display \(a_{1,n}\) and \(a_{2,n}\), respectively, and consider \(\Delta_n\) as defined in Step 1, but now based on these specific sequences. From (C.9) we see that \(n^{1/r} B_n a_{1,n} \to 0\), so Step 1 implies

\[
\mathbb{E}_n[|X_i^\top (\tilde{\theta} - \theta_0)|^2] \leq \sup_{\delta \in \Delta_n} \mathbb{E}_n[(X_i^\top \delta)^2] \leq C (a_{2,n}^2 + a_{1,n} \eta_n) \leq C' s q \eta_n^{-q} (\eta_n^2 + \tilde{\lambda}_n^2),
\]

with probability at least \(1 - P(\lambda < c_0 \|S_n\|_\infty) - P(\lambda > \tilde{\lambda}_n) - o(1)\), where \(C\) is the constant from Step 1, and the constant \(C'\) depends only on \(c_0, C_{ev}, C_L\) and \(c_M\).

Continuing the inequality (C.13) with the upper bounds thus developed, we see that

\[
\lambda \sqrt{|\tilde{T}|} \leq \lambda \sqrt{|\tilde{T}|/c_0 + \tilde{C} \sqrt{\tilde{\phi}(|\tilde{T}|)} s q \eta_n^{-q} (\eta_n^2 + \tilde{\lambda}_n^2)}
\]

with probability at least \(1 - 2P(\lambda < c_0 \|S_n\|_\infty) - P(\lambda > \tilde{\lambda}_n) - o(1)\), where \(\tilde{C}\) is a constant depending only on \(c_0, C_{ev}, C_L, c_M,\) and \(C_m\). Observe that all probabilistic bounds in this step hold simultaneously for all \(\tilde{\theta} \in \tilde{\Theta}(\lambda)\). Noting that \(c_0 > 1\) and \(|\tilde{T}| = \|\tilde{\theta}\|_0\), the desired inequality follows from rearranging the previous display, bounding \(\lambda\) from below by \(\tilde{\lambda}_n\) and recasting the constant \(\tilde{C} \in (0, \infty)\).

**Step 4:** For the same constant \(\tilde{C}\) as that in the Step 3, let

\[
\tilde{k}_n := \min \left\{ k \in \mathbb{N}; \; k > 2\tilde{C} \phi(k) s q \eta_n^{-q} \phi_n^2 \right\}. \quad (C.14)
\]

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Such a \( \tilde{k}_n \) exists in \( \mathbb{N} \), since \( \widehat{\phi}(k) = \widehat{\phi}(p) \) for \( k \geq p \). In this step, we show that

\[
\sup_{\hat{\theta} \in \hat{\Theta}(\lambda)} \| \hat{\theta} \|_0 \leq \tilde{k}_n
\]  

(C.15)

with probability at least \( 1 - 2P(\lambda < c_0 \| S_n \|_\infty) \) – \( P(\lambda > \bar{\lambda}_n) \) – \( P(\lambda < \underline{\lambda}_n) \) – \( o(1) \). To this end, let \( \bar{s}_n := s_q \eta_n^{-q} \phi_n^2 \), and let the event \( (\text{C.12}) \) hold. Seeking a contradiction, note that if \( \tilde{k}_n < \| \hat{\theta} \|_0 \) for some \( \hat{\theta} \in \hat{\Theta}(\lambda) \), then for such \( \hat{\theta} \),

\[
\| \hat{\theta} \|_0 \leq C \widehat{\phi}(\| \hat{\theta} \|_0) \bar{s}_n = C \widehat{\phi}(\| \hat{\theta} \|_0 / \tilde{k}_n) \tilde{k}_n \bar{s}_n \leq C \| \hat{\theta} \|_0 / \tilde{k}_n \tilde{\phi}(k_n) \tilde{s}_n \leq 2C'(\| \hat{\theta} \|_0 / k_n) \tilde{\phi}(k_n) \tilde{s}_n
\]

where the second inequality in the second line follows from Lemma 9 in Belloni et al. (2012). However, this chain of inequalities implies that

\[
\tilde{k}_n \leq 2C \widehat{\phi}(k_n) \tilde{s}_n.
\]

which contradicts the definition of \( \tilde{k}_n \) in (C.14). Hence, on the event \( (\text{C.12}) \), we have (C.15). Combining this result with the previous step yields the asserted claim of this step.

**Step 5:** We now complete the proof. To this end, for the constant \( \tilde{C} \) from Step 3, let

\[
k_n := \lceil 4\tilde{C} C_{ev} s_q \eta_n^{-q} \phi_n^2 \rceil.
\]

By (C.9), we have \( k_n n^{1/r} B_n \eta_n \to 0 \), and so it follows from Step 2 that

\[
k_n > 2 \tilde{C} \phi(k_n) s_q \eta_n^{-q} \phi_n^2
\]

with probability \( 1 - o(1) \) since \( C_{ev} \geq 1 \). Hence, \( \tilde{k}_n \) defined in (C.14) satisfies \( \tilde{k}_n \leq k_n \) with probability \( 1 - o(1) \). Combining this result with Step 4 yields the asserted probabilistic claim of the lemma.

\[\square\]

**D Fundamental Tools**

**D.1 Maximal Inequalities**

Let \( \mathbb{G}_n[f(Z_i)] := \sqrt{n}(\mathbb{E}_n - E)[f(Z_i)] \) abbreviate the centered and scaled empirical average of the \( \{ f(Z_i) \}_{i=1}^n \).
Theorem D.1 (Maximal Inequality, I). Let \( \{Z_i\}_{i=1}^n \) be independent copies of a random vector \( Z \), with support \( Z \), of which \( X \) is a \( p \)-dimensional subvector, let \( \Delta \) be a non-empty subset of \( \mathbb{R}^p \), and let \( h : \mathbb{R} \times Z \rightarrow \mathbb{R} \) be a measurable function satisfying \( h(0, \cdot) \equiv 0 \). Suppose that there are non-random sequences \( B_{1n}, B_{2n} \in [0, \infty), \zeta_n, \gamma_n \in (0, 1) \), a constant \( C_h \in (0, 1) \), and a measurable function \( L : Z \rightarrow [0, \infty) \) such that

1. for all \( z \in \mathbb{Z} \) and all \( t_1, t_2 \in \mathbb{R} \) satisfying \( |t_1| \vee |t_2| \leq C_h \),
   \[ |h(t_1, z) - h(t_2, z)| \leq L(z)|t_1 - t_2|; \]

2. \( \max_{1 \leq i \leq n} \sup_{\delta \in \Delta} |X_i^\top \delta| \leq C_h \) with probability at least \( 1 - \zeta_n \);

3. \( \sup_{\delta \in \Delta} E[h(X^\top \delta, Z)^2] \leq B_{1n}^2 \); and,

4. \( \max_{1 \leq i \leq p} E_n[L(Z_i)^2 X_{i,j}^2] \leq B_{2n}^2 \) with probability at least \( 1 - \gamma_n \).

Then, denoting \( \|\Delta\|_1 := \sup_{\delta \in \Delta} \|\delta\|_1 \), we have

\[
P \left( \max_{\delta \in \Delta} \left| \mathbb{E}[n|h(X_i^\top \delta, Z_i)|] \right| > u \right) \leq 4\zeta_n + 4\gamma_n + n^{-1},
\]

provided \( u \geq \{4B_{1n}\} \vee \{8\sqrt{2}B_{2n} \|\Delta\|_1 \sqrt{\ln(8pn)}\} \).

Proof. The claim follows from the proof of Belloni et al. (2018a, Lemma D.3), where in Step 1 we replace the set \( \Omega \) by the intersection of \( \Omega \) and \( \{\max_{1 \leq i \leq n} \sup_{\delta \in \Delta} |X_i^\top \delta| \leq C_h\} \).

Theorem D.2 (Maximal Inequality, II). Let \( \{Z_i\}_{i=1}^n \) be independent copies of a random vector \( Z = (Z_1, \ldots, Z_p)\top \) in \( \mathbb{R}^p \) with \( p \geq 2 \), and assume that for non-random sequences \( M_n, \sigma_n \in (0, \infty) \) and a constant \( q \in (2, \infty) \), we have (1) \( \max_{1 \leq j \leq p} \mathbb{E}[Z_j^2] \leq \sigma_n^2 \), (2) \( \mathbb{E}\|Z\|^q_n \leq M_n^q \), and (3) \( M_n \leq \sigma_n n^{1/2 - 1/q} / \sqrt{\ln(pn)} \). Then

\[
P \left( \max_{1 \leq j \leq p} |\mathbb{G}_n(Z_{i,j})| > C\sigma_n \sqrt{\ln(pn)} \right) \leq \frac{c_q}{\ln^q(pn)},
\]

where \( C \in (0, \infty) \) is a universal constant and \( c_q \in (0, \infty) \) is a constant depending only on \( q \).

Proof. Belloni et al. (2018a, Lemma A.3) implies that, for a universal constant \( K \in (0, \infty) \),

\[
\mathbb{E} \left[ \max_{1 \leq j \leq p} |\mathbb{G}_n(Z_{i,j})| \right] \leq K \left( \sqrt{\max_{1 \leq j \leq p} \mathbb{E}[Z_j^2]} \ln p + \sqrt{\frac{\mathbb{E}[\max_{1 \leq i \leq n} \|Z_i\|_\infty^2]}{n}} \ln p \right).
\]
By Jensen’s inequality, Condition 2 shows

\[
\mathbb{E} \left[ \max_{1 \leq i \leq n} \| Z_i \|_\infty^2 \right] \leq \left( \mathbb{E} \left[ \max_{1 \leq i \leq n} \| Z_i \|_\infty^q \right] \right)^{2/q} \leq \left( \mathbb{E} \left[ \sum_{i=1}^n \| Z_i \|_\infty^q \right] \right)^{2/q} \leq n^{2/q} M_n^2.
\]

Using also Conditions 1 and 3, we thus arrive at

\[
\mathbb{E} \left[ \max_{1 \leq j \leq p} \left| G_n(Z_{i,j}) \right| \right] \leq K \left( \sigma_n \sqrt{\ln p} + n^{1/q-1/2} M_n \ln p \right) \leq 2K \sigma_n \sqrt{\ln p}.
\]

Using Belloni et al. (2018a, Lemma A.2) with \( s = q \) and \( t = \sigma_n \sqrt{3n \ln (pn)} \), we see that

\[
\mathbb{P} \left( \max_{1 \leq j \leq p} \left| G_n(Z_{i,j}) \right| > (4K + \sqrt{3}) \sigma_n \sqrt{\ln (pn)} \right) \leq \frac{1}{pn} + \frac{C_q n \mathbb{E} \left[ \| Z - E[Z] \|_\infty^q \right]}{\sigma_n^q \left( n \ln (pn) \right)^{q/2}},
\]

where \( C_q \in (0, \infty) \) is a constant depending only on \( q \). The triangle and Jensen inequalities yield

\[
\mathbb{E} \left[ \| Z - E[Z] \|_\infty^q \right] \leq \mathbb{E} \left[ (\| Z \|_\infty + \| E[Z] \|_\infty)^q \right] \leq \mathbb{E} \left[ (\| Z \|_\infty + (E[\| Z \|_\infty^q])^{1/q})^q \right].
\]

Applying the inequality \( (a + b)^q \leq 2^q - 1(a^q + b^q) \), which is valid for any \( a, b \in [0, \infty) \) and any \( q \in [1, \infty) \), using Condition 2 we see that

\[
\mathbb{E} \left[ (\| Z \|_\infty + E[\| Z \|_\infty])^q \right] \leq 2^q - 1(\| Z \|_\infty^q + E[\| Z \|_\infty^q]) = 2^q E[\| Z \|_\infty^q] \leq 2^q M_n^q.
\]

Gathering these bounds, and using Condition 3, we arrive at

\[
\mathbb{P} \left( \max_{1 \leq j \leq p} \left| G_n(Z_{i,j}) \right| > (4K + \sqrt{3}) \sigma_n \sqrt{\ln (pn)} \right) \leq \frac{1}{pn} + \frac{2^q C_q}{\ln^q(pn)}.
\]

Since the polynomial \( x^{1/q} \) dominates the logarithm \( \ln x \) as \( x \to \infty \), there is a constant \( K_q \in (0, \infty) \) depending only on \( q \) such that \( \ln x \leq K_q x^{1/q} \) for all \( x \in [1, \infty) \). The claim now follows from specifying \( C = 4K + \sqrt{3} \) and \( c_q = K_q^q + 2^q C_q. \)

**Theorem D.3 (Maximal Inequality, III).** Let \( \{Z_i\}_{i=1}^n \) be independent copies of a random vector \( Z = (Z_1, \ldots, Z_p)^\top \) in \( \mathbb{R}^p \) with \( p \geq 2 \) such that \( Z_j \geq 0 \) for all \( j \in [p] \) and assume that for non-random sequences \( M_n, \mu_n \in (0, \infty) \) and a constant \( q \in (1, \infty) \), we have

1. \( \max_{1 \leq j \leq p} E[Z_j] \leq \mu_n \),
2. \( E[\| Z \|_\infty^q] \leq M_n^q \),
3. \( M_n \leq \mu_n n^{-1/q} / \ln(pn) \).

Then

\[
\mathbb{P} \left( \max_{1 \leq j \leq p} E_n[Z_{i,j}] > C \mu_n \right) \leq \frac{c_q}{\ln^q(pn)}.
\]
where $C \in (0, \infty)$ is a universal constant and $c_q \in (0, \infty)$ is a constant depending only on $q$.

**Proof.** Belloni et al. (2018a, Lemma A.5) shows that, for a universal constant $K \in (0, \infty)$,

$$
E \left[ \max_{1 \leq j \leq p} \mathbb{E}_n [Z_{i,j}] \right] \leq K \left( \max_{1 \leq j \leq p} E[Z_j] + \frac{E[\max_{1 \leq i \leq n} \|Z_i\|_\infty] \ln p}{n} \right).
$$

Jensen’s inequality and Condition 2 imply

$$
E \left[ \max_{1 \leq i \leq n} \|Z_i\|_\infty \right] \leq \left( E \left[ \max_{1 \leq i \leq n} \|Z_i\|^q \right] \right)^{1/q} \leq \left( E \left[ \sum_{i=1}^n \|Z_i\|^q \right] \right)^{1/q} \leq n^{1/q} M_n.
$$

Using also Conditions 1 and 3, we thus arrive at

$$
E \left[ \max_{1 \leq j \leq p} \mathbb{E}_n [Z_{i,j}] \right] \leq K (\mu_n + n^{1/q-1} M_n \ln p) \leq 2K \mu_n.
$$

Using Belloni et al. (2018a, Lemma A.4) with $s = q$ and $t = n \mu_n$, we see that

$$
P \left( \max_{1 \leq j \leq p} \mathbb{E}_n [Z_{i,j}] > (4K + 1) \mu_n \right) \leq \frac{c_q E[\max_{1 \leq i \leq n} \|Z_i\|^q]}{n^q \mu_n^q},
$$

where $c_q \in (0, \infty)$ is a constant depending only on $q$. Using Condition 2, we see that

$$
E \left[ \max_{1 \leq i \leq n} \|Z_i\|^q \right] \leq E \left[ \sum_{i=1}^n \|Z_i\|^q \right] \leq n M_n^q,
$$

Hence, by Condition 3,

$$
P \left( \max_{1 \leq j \leq p} \mathbb{E}_n [Z_{i,j}] > (4K + 1) \mu_n \right) \leq \frac{c_q n^{1-q} M_n^q}{\mu_n^q} \leq \frac{c_q}{\ln^q (p/\alpha)},
$$

which gives the asserted claim. \qed

**D.2 Gaussian Inequality**

**Theorem D.4 (Gaussian Quantile Bound).** Let $(Y_1, \ldots, Y_p)$ be centered Gaussian in $\mathbb{R}^p$ with $\sigma^2 := \max_{1 \leq j \leq p} E[Y_j^2]$ and $p \geq 2$. Let $q^Y (1 - \alpha)$ denote the $(1 - \alpha)$-quantile of $\max_{1 \leq j \leq p} |Y_j|$ for $\alpha \in (0, 1)$. Then $q^Y (1 - \alpha) \leq (2 + \sqrt{2}) \sigma \sqrt{\ln (p/\alpha)}$.

**Proof.** By the Borell-TIS (Tsirelson-Ibragimov-Sudakov) inequality (Adler and Taylor, 2007,
Theorem 2.1.1), for any \( t \in (0, \infty) \) we have

\[
P\left( \max_{1 \leq j \leq p} |Y_j| > E\left[ \max_{1 \leq j \leq p} |Y_j| \right] + \sigma t \right) \leq e^{-t^2/2}.
\]

This inequality translates to the quantile bound

\[
q^Y(1 - \alpha) \leq E\left[ \max_{1 \leq j \leq p} |Y_j| \right] + \sigma \sqrt{2 \ln (1/\alpha)}.
\]

Talagrand (2010, Proposition A.3.1) shows that

\[
E\left[ \max_{1 \leq j \leq p} |Y_j| \right] \leq \sigma \sqrt{2 \ln (2p)},
\]

thus implying

\[
q^Y(1 - \alpha) \leq \sigma \left( \sqrt{2 \ln (2p)} + \sqrt{2 \ln (1/\alpha)} \right).
\]

The claim now follows from \( p \geq 2 \). \( \square \)

D.3 Central Limit Theorem and Bootstrap in High Dimensions

Throughout this section we let \( \{ Z_i \}_{i=1}^n \) be independent copies of a centered random vector \( Z \) in \( \mathbb{R}^p \) and denote their scaled average and (common) variance by

\[
S_n := \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i \quad \text{and} \quad \Sigma := E[ZZ^\top],
\]

respectively. (The existence of \( \Sigma \) in \( \mathbb{R}^{p \times p} \) is guaranteed by our assumptions below.) Write \( \mathcal{N}_n \) for a centered \( p \)-dimensional Gaussian vector with variance \( \Sigma \). For \( \mathbb{R}^p \)-valued random variables \( U \) and \( V \), define the distributional measure of distance

\[
\rho (U, V) := \sup_{A \in \mathcal{A}_p} |P (U \in A) - P (V \in A)|,
\]

where \( \mathcal{A}_p \) denotes the collection of all hyperrectangles in \( \mathbb{R}^p \).

**Theorem D.5 (High-Dimensional CLT).** If for some constant \( b \in (0, \infty) \) and a non-random sequence \( B_n \) in \( [1, \infty) \),

\[
\min_{1 \leq j \leq p} E\left[ Z_j^2 \right] \geq b, \quad \max_{k \in \{1, 2\}} \max_{1 \leq j \leq p} E\left[ |Z_j|^{2+k} \right] / B_n^k \leq 1 \quad \text{and} \quad E\left[ \max_{1 \leq j \leq p} Z_j^4 \right] \leq B_n^4,
\]

(D.1)
then there is a constant $C_b \in (0, \infty)$, depending only on $b$, such that

$$\rho(S_n, N_n) \leq C_b \left( \frac{B_n^4 \ln^7 (pn)}{n} \right)^{1/6}. \quad (D.2)$$

**Proof.** The claim follows from Chernozhukov et al. (2017, Proposition 2.1) with $q = 4$. \qed

Let $\{\tilde{Z}_i\}_{i=1}^n$ be random elements of $\mathbb{R}^p$, and let $\{e_i\}_{i=1}^n$ be i.i.d. standard Gaussians independent of $\{(Z_i, bZ_i)\}_{i=1}^n$. Define

$$\tilde{S}_n^e := \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \tilde{Z}_i,$$

and let $P_e$ denote the (conditional) probability measure computed with respect to $\{e_i\}_{i=1}^n$ for fixed $\{(Z_i, bZ_i)\}_{i=1}^n$. Also, abbreviate

$$\tilde{\rho}(\tilde{S}_n^e, N_n) := \sup_{A \in \mathcal{A}_p} \left| P_e(\tilde{S}_n^e \in A) - P(N_n \in A) \right|,$$

with the tilde stressing that $\tilde{\rho}(\tilde{S}_n^e, N_n)$ is a random quantity.

**Theorem D.6 (Multiplier Bootstrap for Many Approximate Means).** Let (D.1) hold for some constant $b \in (0, \infty)$ and a non-random sequence $B_n$ in $[1, \infty)$, and let $\beta_n$ and $\delta_n$ be non-random sequences in $[0, \infty)$ such that

$$P \left( \max_{1 \leq j \leq p} \mathbb{E}_{n}[(\tilde{Z}_{i,j} - Z_{i,j})^2] > \frac{\delta_n^2}{\ln^2 (pn)} \right) \leq \beta_n. \quad (D.3)$$

Then there is a constant $C_b \in (0, \infty)$, depending only on $b$, such that with probability at least $1 - \beta_n - 1/\ln^2 (pn)$,

$$\tilde{\rho}(\tilde{S}_n^e, N_n) \leq C_b \left( \delta_n \vee \left( \frac{B_n^4 \ln^6 (pn)}{n} \right)^{1/6} \right). \quad (D.4)$$

**Proof.** The claim essentially follows from the proof of Belloni et al. (2018a, Theorem 2.2). We include the argument for completeness and to clarify the dependence on $b$.

First, denote $\Delta_n := \delta_n / \sqrt{\ln(pn)}$ and consider the random element $S_n^e$ of $\mathbb{R}^p$ defined by

$$S_n^e := \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i Z_i.$$

Observe that, conditional on $\{(Z_i, \tilde{Z}_i)\}_{i=1}^n$, the elements $\pm(\tilde{S}_{n,j}^e - S_{n,j}^e)_{j=1}^p$ are jointly
centered Gaussian with largest (conditional) variance

\[ \tilde{\sigma}_e^2 := \max_{1 \leq j \leq p} \mathbb{E}_n[(\tilde{Z}_{i,j} - Z_{i,j})^2]. \]

Applying Lemma D.4 conditional on \( \{(Z_i, \tilde{Z}_i)\}_{i=1}^n \) and with \( \alpha = 1/n \) shows that

\[ P_e \left( \| \tilde{S}_n^e - S_n^e \|_\infty > K_1 \tilde{\sigma}_e \sqrt{\ln(pn)} \right) \leq n^{-1} \]

for the absolute constant \( K_1 := 2 + \sqrt{2} \). Since (D.3) means that \( \tilde{\sigma}_p^2 \leq \Delta_p^2 / \ln(pn) \) with probability at least \( 1 - \beta_n \), with the same probability we have

\[ P_e \left( \| \tilde{S}_n^e - S_n^e \|_\infty > K_1 \Delta_n \right) \leq n^{-1}. \]

(D.5)

Next, consider any (hyper)rectangle \( A \in \mathcal{A}_p \). Then there are \( p \)-dimensional vectors \( \boldsymbol{w}^l = (w^l_1, \ldots, w^l_p)^\top \) and \( \boldsymbol{w}^u = (w^u_1, \ldots, w^u_p)^\top \) of (possibly extended) reals, for which

\[ A = \{ \boldsymbol{w} \in \mathbb{R}^p; w^l_j \leq w_j \leq w^u_j \text{ for all } j \in [p] \}. \]

Based on this representation, define the (expanded) set

\[ A^+ := \{ \boldsymbol{w} \in \mathbb{R}^p; w^l_j - K_1 \Delta_n \leq w^l_j \leq w^u_j + K_1 \Delta_n \text{ for all } j \in [p] \}. \]

Then also \( A^+ \in \mathcal{A}_p \) and \( A \subseteq A^+ \). It follows that, on the event (D.5),

\[ P_e(\tilde{S}_n^e \in A) \leq P_e(\mathbf{S}_n^e \in A^+) + n^{-1} \]

\[ \leq P(\mathbf{N}_n \in A^+) + \tilde{\rho}(\mathbf{S}_n^e, \mathbf{N}_n) + n^{-1}, \]

where \( \tilde{\rho}(\mathbf{S}_n^e, \mathbf{N}_n) \) is the random variable given by

\[ \tilde{\rho}(\mathbf{S}_n^e, \mathbf{N}_n) := \sup_{A \in \mathcal{A}_p} |P_e(\mathbf{S}_n^e \in A) - P(\mathbf{N}_n \in A)|. \]

Repeating the anti-concentration argument on Belloni et al. (2018a, p. 69), we see that there is a constant \( K_2 \in (0, \infty) \), depending only on \( b \), such that

\[ P(\mathbf{N}_n \in A^+) \leq P(\mathbf{N}_n \in A) + 2K_2 \delta_n. \]
Hence, on the event (D.5), we have the upper bound

$$P_e(\hat{S}_n^e \in A) \leq P(N_n \in A) + \tilde{\rho}(S_n^e, N_n) + K_2 \delta_n + n^{-1}.$$ 

To get a lower bound, we instead consider the set

$$A^- := \{ w \in \mathbb{R}^p; w_j^l + K_1 \Delta_n \leq w_j^u \leq w_j^u - K_1 \Delta_n \text{ for all } j \in [p] \},$$

which satisfies $A^- \in A_p$ and $A^- \subseteq A$. A parallel argument shows that on the event (D.5),

$$P_e(\hat{S}_n^e \in A) \geq P(N_n \in A) - \tilde{\rho}(S_n^e, N_n) - K_2 \delta_n - n^{-1}.$$

Combine the upper and lower bounds and take the supremum over $A \in A_p$ to see that on the event (D.5),

$$\tilde{\rho}(S_n^e, N_n) \leq \tilde{\rho}(S_n^e, N_n) + K_2 \delta_n + n^{-1}.$$ 

Condition (D.1) and Belloni et al. (2018a, Theorem A.2(ii)) [with their $q = 4$ and their $\beta = 1/\ln^2(pn)$, the latter choice being justified by $p \geq 2$ and $n \geq 3$] combine to show that there is a constant $K_3 \in (0, \infty)$, depending only on $b$, such that with probability at least $1 - 1/\ln^2(pn)$,

$$\tilde{\rho}(S_n^e, N_n) \leq K_3 \left( \frac{B_n^4 \ln^6(pn)}{n} \right)^{1/6}.$$ 

The previous two displays combine to show that with probability at least $1 - \beta - 1/\ln^2(pn)$,

$$\tilde{\rho}(S_n^e, N_n) \leq K_3 \left( \frac{B_n^4 \ln^6(pn)}{n} \right)^{1/6} + K_2 \delta_n + n^{-1}.$$ 

The claim now follows from taking $C_b := K_3 + K_2 + 1$, which depends only on $b$. \hfill \Box

Let $N_M$ be a $p$-dimensional centered Gaussian vector with variance matrix $M \in \mathbb{R}^{p \times p}$. Define $q_{N_M}^\mathcal{N}: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ as the (extended) quantile function of $\|N_M\|_\infty$, i.e.

$$q_{N_M}^\mathcal{N}(\alpha) := \inf \{ t \in \mathbb{R}; P(\|N_M\|_\infty \leq t) \geq \alpha \}, \quad \alpha \in \mathbb{R}.$$ 

We here interpret $q_{N_M}^\mathcal{N}(\alpha)$ as $+\infty (= \inf \emptyset)$ if $\alpha \geq 1$, and $-\infty (= \inf \mathbb{R})$ if $\alpha \leq 0$, such that $q_{N_M}^\mathcal{N}$ is monotone increasing.

**Theorem D.7.** Let $M \in \mathbb{R}^{p \times p}$ be a symmetric positive semi-definite matrix with strictly positive diagonal, i.e. $M_{j,j} \in (0, \infty)$ for all $j \in [p]$, let $U$ be an $\mathbb{R}^p$-valued random variable,
and let $q$ denote the quantile function of $\|U\|_\infty$. Then

$$q_M^N(\alpha - 2\rho(U, \mathcal{N}_M)) \leq q(\alpha) \leq q_M^N(\alpha + \rho(U, \mathcal{N}_M)) \text{ for all } \alpha \in (0, 1).$$

Proof. If $\rho(U, \mathcal{N}_M) = 0$, then the distributions of $U$ and $\mathcal{N}_M$ agree on all hyperrectangles. In particular, these distributions agree on all cubes $[-t, t]^p$, $t \in [0, \infty)$, implying equality of quantile functions, as claimed. For the remainder of the proof, we therefore take $\rho(U, \mathcal{N}_M) \in (0, \infty)$. Since $M$ has a strictly positive diagonal, by the union bound, for any $t \in \mathbb{R}$ we have

$$P(\|\mathcal{N}_M\|_\infty = t) \leq \sum_{j=1}^p P(|\mathcal{N}(0, 1)| = t/\sqrt{M_{j,j}}) = 0.$$  

It follows that for each $\alpha \in (0, 1)$, $q_M^N(\alpha) \in \mathbb{R}$ is uniquely defined by

$$P(\|\mathcal{N}_M\|_\infty \leq q_M^N(\alpha)) = \alpha.$$  

Fix $\alpha \in (0, 1)$. In establishing the lower bound we may take $2\rho(U, \mathcal{N}_M) < \alpha$. (Otherwise $q_M^N(\alpha - 2\rho(U, \mathcal{N}_M)) = -\infty$ and there is nothing to show.) Then

$$[-q_M^N(\alpha - 2\rho(U, \mathcal{N}_M)), q_M^N(\alpha - 2\rho(U, \mathcal{N}_M))]^p$$

is a rectangle and, thus,

$$P(\|U\|_\infty \leq q_M^N(\alpha - 2\rho(U, \mathcal{N}_M))) \leq P(\|\mathcal{N}_M\|_\infty \leq q_M^N(\alpha - 2\rho(U, \mathcal{N}_M))) + \rho(U, \mathcal{N}_M) < \alpha,$$

which implies the lower bound. In establishing the upper bound we may assume $\rho(U, \mathcal{N}_M) < 1 - \alpha$. (Otherwise $q_M^N(\alpha + \rho(U, \mathcal{N}_M)) = +\infty$ and there is nothing to show.) Then from the rectangle

$$[-q_M^N(\alpha + \rho(U, \mathcal{N}_M)), q_M^N(\alpha + \rho(U, \mathcal{N}_M))]^p,$$

a similar calculation shows

$$P(\|U\|_\infty \leq q_M^N(\alpha + \rho(U, \mathcal{N}_M))) \geq \alpha,$$

which by definition of quantiles implies the upper bound. \qed
Now, define $q_n(\alpha)$ as the $\alpha$-quantile of $\|S_n\|_{\infty}$

$$q_n(\alpha) := \inf \{ t \in \mathbb{R}; P(\|S_n\|_{\infty} \leq t) \geq \alpha \}, \quad \alpha \in (0, 1),$$

and let $\tilde{q}_n(\alpha)$ be the $\alpha$-quantile of $\|\tilde{S}_n^e\|_{\infty}$ computed conditional on $\{(Z_i, \tilde{Z}_i)\}_{i=1}^n$, i.e.

$$\tilde{q}_n(\alpha) := \inf \{ t \in \mathbb{R}; P_e(\|\tilde{S}_n^e\|_{\infty} \leq t) \geq \alpha \}, \quad \alpha \in (0, 1).$$

**Theorem D.8 (Quantile Comparison).** If (D.1) holds for some constant $b \in (0, \infty)$ and a non-random sequence $B_n$ in $[1, \infty)$, and

$$\rho_n := 2C_b \left( \frac{B^4_n \ln^7 (pn)}{n} \right)^{1/6}$$

denotes two times the upper bound (D.2) in Theorem D.5, then

$$q_N(1 - \alpha - \rho_n) \leq q_n(1 - \alpha) \leq q_N(1 - \alpha + \rho_n) \text{ for all } \alpha \in (0, 1).$$

If, in addition, (D.3) holds for some non-random sequences $\beta_n$ and $\delta_n$ in $[0, \infty)$, and

$$\rho'_n := 2C'_b \left( \delta_n \vee \left( \frac{B^4_n \ln^6 (pn)}{n} \right)^{1/6} \right)$$

denotes two times the upper bound (D.4) in Theorem D.6, then with probability at least $1 - \beta_n - 1/\ln^2(pn)$,

$$q_N(1 - \alpha - \rho'_n) \leq \tilde{q}_n(1 - \alpha) \leq q_N(1 - \alpha + \rho'_n) \text{ for all } \alpha \in (0, 1).$$

**Proof.** Observe that (D.1) implies that $\Sigma$ is a symmetric positive semi-definite matrix with strictly positive diagonal. Apply Theorem D.7 with $U = S_n$ to obtain

$$q_N(1 - \alpha - 2\rho(S_n, N_n)) \leq q_n(1 - \alpha) \leq q_N(1 - \alpha + \rho(S_n, N_n)) \text{ for all } \alpha \in (0, 1).$$

The first pair of inequalities then follows from $2\rho(S_n, N_n) \leq \rho_n$, cf. Theorem D.5. To establish the second claim, apply Theorem D.7 with $U = \tilde{S}_n^e$ and conditional on the $\{(Z_i, \tilde{Z}_i)\}_{i=1}^n$ to obtain

$$q_N(1 - \alpha - 2\tilde{\rho}(\tilde{S}_n^e, N_n)) \leq \tilde{q}_n(1 - \alpha) \leq q_N(1 - \alpha + \tilde{\rho}(\tilde{S}_n^e, N_n)) \text{ for all } \alpha \in (0, 1).$$
The second pair of inequalities then follows on the event \(2\rho(\hat{S}_n^6, \mathcal{N}_n) \leq \rho'_n\), which by Theorem D.6 occurs with probability at least \(1 - \beta_n - 1/\ln^2(pn)\).

**Theorem D.9 (Multiplier Bootstrap Consistency).** Let (D.1) and (D.3) hold for some constant \(b \in (0, \infty)\) and non-random sequences \(B_n\) in \([1, \infty)\) and \(\delta_n\) and \(\beta_n\) both in \([0, \infty)\). Then there is a constant \(C_b \in (0, \infty)\), depending only on \(b\), such that

\[
\sup_{\alpha \in (0,1)} |P(\|S_n\|_\infty > \hat{q}_n(1 - \alpha)) - \alpha| \leq C_b \max \left\{ \beta_n, \delta_n, \left( \frac{B_n^4 \ln^7(pn)}{n} \right)^{1/6}, \frac{1}{\ln^2(pn)} \right\}.
\]

**Proof.** Fix \(\alpha \in (0,1)\). By Theorems D.5 and D.8,

\[
P \left( \|S_n\|_\infty \leq \hat{q}_n(1 - \alpha) \right) \leq P \left( \|S_n\|_\infty \leq q_n^N(1 - \alpha + \rho'_n) \right) + \beta_n + \frac{1}{\ln^2(pn)}
\]

\[
\leq P \left( \|N_n\|_\infty \leq q_n^N(1 - \alpha + \rho'_n) \right) + \rho_n + \beta_n + \frac{1}{\ln^2(pn)},
\]

with \(\rho_n\) and \(\rho'_n\) defined in Theorem D.8. If \(\rho'_n \geq \alpha\), then \(q_n^N(1 - \alpha + \rho'_n) = +\infty\) and, thus,

\[
P \left( \|N_n\|_\infty \leq q_n^N(1 - \alpha + \rho'_n) \right) = 1 \leq 1 - \alpha + \rho'_n.
\]

If \(\rho'_n < \alpha\), then

\[
P \left( \|N_n\|_\infty \leq q_n^N(1 - \alpha + \rho'_n) \right) = 1 - \alpha + \rho'_n.
\]

Continuing the initial string of inequalities, in either case we arrive at

\[
P \left( \|S_n\|_\infty \leq \hat{q}_n(1 - \alpha) \right) \leq 1 - \alpha + \rho'_n + \rho_n + \beta_n + \frac{1}{\ln^2(pn)}.
\]

Parallel reasoning shows

\[
P \left( \|S_n\|_\infty \leq \hat{q}_n(1 - \alpha) \right) \geq 1 - \alpha - \left( \rho'_n + \rho_n + \beta_n + \frac{1}{\ln^2(pn)} \right).
\]

The claim now follows from combining and rearranging the previous two displays, taking the supremum over \(\alpha \in (0, 1)\), and recasting the constant \(C_b \in (0, \infty)\), which can be chosen to depend only on \(b\). \(\square\)
E  Solution Existence, Sparsity and Uniqueness

The relation $\hat{\theta}(\lambda) \in \Theta(\lambda)$ in (1.2) hinges on the fundamental property of existence of a minimizer, i.e. the non-emptiness of $\Theta(\lambda)$. The non-emptiness, cardinality, and related properties of the solution set (a subset of $\mathbb{R}^p$) generally depend on the data, penalty level, and parameter space. While we in the main sections of the paper presume the existence of a minimizer, for the sake of completeness, in this section we provide criteria for existence of a solution as well as related properties. To keep matters interesting yet statements simple, we here consider the full (hence unbounded) parameter space $\Theta = \mathbb{R}^p$.

Denote the $n \times p$ regressor matrix $X := [X_1 : \cdots : X_n]^\top$ and use $X_J$ to denote the $n \times |J|$ submatrix of $X$ with columns indexed by $J \subseteq [p]$. (The rank of $X_\emptyset$ is interpreted as zero.)

**Theorem E.1 ($\ell_1$-ME Existence and Sparsity).** Let the loss function $m : \mathbb{R} \times \mathcal{Y} \to \mathbb{R}$ be non-negative, $m(\cdot, y)$ convex for all $y \in \mathcal{Y}$, and $\Theta = \mathbb{R}^p$. Then for any $\lambda \in (0, \infty)$ and any realization of $\{(Y_i, X_i)\}_{i=1}^n$, the following properties hold:

1. The set of minimizers $\hat{\Theta}(\lambda)$ in (1.2) is non-empty, convex and compact.

2. For at least one minimizer $\hat{\theta} \in \hat{\Theta}(\lambda)$, the columns $\{X_j : j \in \text{supp}(\hat{\theta})\}$ of $X$ corresponding to the non-zero entries of $\hat{\theta}$ are linearly independent, i.e. $\text{rank}(X_{\text{supp}(\hat{\theta})}) = \|\hat{\theta}\|_0$.

In particular, for such a minimizer, $\|\hat{\theta}\|_0 \leq n \wedge p$.

As stated in the theorem, non-negativity establishes not only existence of a solution, but also existence of a solution for which the “active” columns of $X$ are linearly independent. Such a solution is therefore sparse in the sense of having no more than $n \wedge p$ non-zeros, with the interesting part of the bound being $\|\hat{\theta}\|_0 \leq n$. All examples in Section 2 concern non-negative loss functions, thus guaranteeing the existence of a (sparse) solution.

Theorem E.1 certainly has precursors in the literature. Existence results for $\ell_1$-penalized M-estimators are mentioned in Tibshirani (2013, Section 2.3) for differentiable and strictly convex loss functions. As shown in Theorem E.1, neither differentiability nor strict convexity is necessary for this conclusion. The existence of a solution associated with linearly independent active columns dates back to Osborne et al. (2000) for the LASSO and Rosset et al. (2004, Theorem B.3) for differentiable non-negative convex loss functions. See also Tibshirani (2013, Section 5.2) for discussion. Again, the differentiability is not necessary. In

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40 Of course, since the criterion is presumed convex (hence continuous), non-emptiness and compactness of the parameter space suffice for the existence of a solution, cf. Weierstrass’ extreme value theorem.

41 Moreover, one can find a differentiable and strictly convex function and a positive penalty level $\lambda$ such that the criterion function in (1.2) can be made arbitrarily small and, thus, no minimizer exists. A condition (such as non-negativity) therefore appears to be missing or implicit in the treatment in Tibshirani (2013).
particular, Theorem E.1 applies to both quantile regression and trimmed LAD loss functions, both of which are non-differentiable.

**Remark E.1 (Non-Negativity).** The non-negativity of the loss function used in establishing the existence of an ℓ₁-ME is actually a bit of a red herring as adding or subtracting a constant from the loss bears no impact on the solution set. The crucial element is that the loss is bounded from below, and non-negativity is a simple way to ensure this. Inspecting the proof of Theorem E.1, we see that this property is only used to ensure positivity of the “asymptotic slope” of the ℓ₁-penalized M-estimation criterion [assuming \( \lambda \in (0, \infty) \)]. A positive asymptotic slope follows from the condition

\[
\lim_{\tau \to \infty} \frac{m(\tau v, y)}{\tau} \in [0, \infty] \quad \text{for both } v \in \{-1, 1\} \quad \text{and all } y \in \mathcal{Y}.
\]

A *positive* asymptotic slope means that the loss eventually increases, and a *zero* asymptotic slope means that the loss eventually flattens and, hence, the penalty eventually dominates. In the previous display, existence of the limits (as possibly extended real numbers) is guaranteed by convexity. Non-negativity of the displayed limits follows trivially from non-negativity of the loss itself.

**Remark E.2 (Post-LASSO Existence).** The existence of a solution for which the active columns are linearly independent is particularly interesting in the context of least squares post variable selection based on the LASSO, also known as *post-LASSO* (Belloni et al., 2012; Belloni and Chernozhukov, 2011b, 2013). Specifically, the linear independence implied by such a LASSO solution \( \widehat{\theta} \) ensures that least squares based on the regressors selected by \( \widehat{\theta} \) has a unique solution, namely \( (X^\top_{\text{supp}(\widehat{\theta})} X_{\text{supp}(\widehat{\theta})})^{-1} X^\top_{\text{supp}(\widehat{\theta})} Y \), where \( Y \) denotes the \( n \times 1 \) vector of outcomes. Hence, while the post-LASSO need not exist uniquely for every LASSO solution, as long as \( \lambda \in (0, \infty) \), there is a solution for which it does.

**Remark E.3 (Non-Sparse Solutions).** Not all minimizers are necessarily sparse. In fact, one can construct examples with \( p > n \) and some solution having all \( p \) non-zeros. For a simple numerical example, take the LAD loss \( m(t, y) = |y - t| \) from median regression with \( n = 1 \) observation and \( p = 2 \) parameters, data \( Y = X_1 = X_2 = 1 \), and the penalty level \( \lambda \in (0, 1) \). From the objective \( |1 - (\theta_1 + \theta_2)| + \lambda(|\theta_1| + |\theta_2|) \) and \( \lambda \in (0, 1) \) we see that it is more costly to move \( \theta_1 + \theta_2 \) away from one than to move \( \theta_1 \) or \( \theta_2 \) away from zero. Hence, any solution sets \( \theta_2 = 1 - \theta_1 \). The function \( |\theta_1| + |1 - \theta_1| \) equals one for all \( \theta_1 \in [0, 1] \) and is strictly greater otherwise, so the set of solutions is the closed line segment \( \{(u, 1-u) \in \mathbb{R}^2; u \in [0, 1]\} \).

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\(^{42}\)Post-LASSO is sometimes referred to as *Gauss LASSO*. The method coincides with the Meinshausen (2007) *relaxed LASSO*, when the relaxation parameter is set to zero.
that the solution set is both closed and bounded and involves both sparse solutions (the two end points) and non-sparse solutions (everything in between).

We next turn to the question of uniqueness. We say that the columns of \( X \) are in general position if for any integer \( k \in \{0, 1, \ldots, (n \wedge p) - 1\} \), any \( k + 1 \) column indices \( j_1, \ldots, j_{k+1} \in [p] \), and any signs \( \sigma_1, \ldots, \sigma_{k+1} \in \{-1, 1\} \), the affine span of the \( k + 1 \) signed columns \( \{\sigma_1 X_{j_1}, \ldots, \sigma_{k+1} X_{j_{k+1}}\} \) does not contain any element of \( \{\pm X_j; j \in [p] \setminus \{j_1, \ldots, j_{k+1}\}\} \).

**Theorem E.2 \((\ell_1\text{-ME Uniqueness via General Position})\).** Let the loss function \( m : \mathbb{R} \times \mathcal{Y} \to \mathbb{R} \) be non-negative, \( m(\cdot, y) \) strictly convex for all \( y \in \mathcal{Y} \), and \( \Theta = \mathbb{R}^p \). Then for any \( \lambda \in (0, \infty) \) and any realization of \( \{(Y_i, X_i)\}_{i=1}^n \) for which the columns of \( X \) are in general position, there is a unique minimizer \( \hat{\Theta}(\lambda) = \{\hat{\theta}(\lambda)\} \) with \( \|\hat{\theta}(\lambda)\|_0 \leq n \wedge p \).

Sparsity of the solution follows from uniqueness, cf. Theorem E.1. Like the existence result, the uniqueness theorem have precursors in earlier literature, including Osborne et al. (2000) for the LASSO and Rosset et al. (2004, Theorem B.5) for differentiable non-negative convex loss functions. See Tibshirani (2013) for discussion and additional references. Our modest contribution here lies in showing that the crucial parts of the argument can accommodate non-differentiability by appealing to the Karush-Kuhn-Tucker (KKT) conditions for optimality.

While general position is an abstract condition, it is satisfied with probability one when the regressors are drawn from an absolutely continuous distribution on \( \mathbb{R}^{p \times n} \). This observation leads us to the following corollary.

**Corollary E.1 \((\ell_1\text{-ME Uniqueness via Absolute Continuity})\).** Let the loss function \( m : \mathbb{R} \times \mathcal{Y} \to \mathbb{R} \) be non-negative, \( m(\cdot, y) \) strictly convex for all \( y \in \mathcal{Y} \), \( \Theta = \mathbb{R}^p \), and the elements of \( X \) absolutely continuously distributed with respect to Lebesgue measure on \( \mathbb{R}^{np} \). Then for any \( \lambda \in (0, \infty) \) and no matter the distribution of \( \{Y_i\}_{i=1}^n \), with probability one, there is a unique \( \ell_1\text{-ME} \), which then has at most \( n \wedge p \) non-zero components.

See Tibshirani (2013, Lemma 5) for a similar statement for differentiable and strict convex (non-negative) loss functions and *ibid.* (p. 1463) for the argument of almost surely sufficiency of absolute continuity for general position of \( X \).

**Remark E.4 (Necessity of General Position).** One cannot in general achieve uniqueness without the columns of \( X \) being in general position. For a simple numerical example, take the square loss \( m(t, y) = (1/2)(y - t)^2 \) from mean regression with \( n = 1 \) observation and \( p = 2 \) parameters, data \( Y = X_1 = X_2 = 1 \), and penalty level \( \lambda = \frac{1}{2} \). While the square loss is strictly convex, since \( X_1 = X_2 \), the columns of \( X \) are not in general position. The KKT
conditions associated with minimizing $\frac{1}{2}(1-\theta_1-\theta_2)^2 + \frac{1}{2}(|\theta_1| + |\theta_2|)$ are satisfied by any pair $(u, \frac{1}{2} - u)$ with $u \in (0, \frac{1}{2})$, so the solution is not unique.

Remark E.5 (Necessity of Strict Convexity). One cannot in general achieve uniqueness without strict convexity of the loss function. For a simple numerical example, take the (not strictly) convex LAD loss $m(t, y) = |y - t|$ from median regression with $n = 1$ observation and $p = 2$ parameters, data $Y = X_1 = 1$ and $X_2 = 0$, and penalty level $\lambda = 1$. Since $X_1$ and $X_2$ differ in terms of more than just their signs, the columns of $X$ are in general position. (Recall that the affine span of a singleton is the singleton itself.) However, the objective $|1 - (\theta_1 + 0 \cdot \theta_2)| + |\theta_1| + |\theta_2|$ is minimized at any $\theta_1 \in [0, 1]$ with $\theta_2 = 0$, so the solution is not unique.

Remark E.6 (Unpenalized Coefficients). In cases where one leaves one or more coefficients out of the penalty, the existence of solution can no longer be guaranteed independently of the data. For example, consider including an unpenalized intercept in a binary response setting with negative log-likelihood loss. In the event that all outcomes are of the same label (all zeros or all ones), one can achieve complete separation based on the constant regressor alone. Statements such as "\(\ell_1\)-penalized logistic regression always has a solution" appearing in the literature must therefore explicitly or implicitly penalize even the intercept (provided one is present). For a treatment of existence in the so-called generalized LASSO problem (with possibly non-square loss), where the penalty is the \(\ell_1\) norm of a (not necessarily identity) matrix times the coefficient vector, see Ali and Tibshirani (2019).

F Proofs for Appendix E

For this section, we let $\hat{M} : \Theta \rightarrow \mathbb{R}$ abbreviate the average loss, defined by

$$
\hat{M}(\theta) := \mathbb{E}_n[m(X_i^\top \theta, Y_i)], \quad \theta \in \Theta = \mathbb{R}^p.
$$

Proof of Theorem E.1. First, we consider Part 1 of the theorem. To show existence, consider the objective function $f$ on $\mathbb{R}^p$ defined by

$$
f(\theta) := \hat{M}(\theta) + \lambda \|\theta\|_1.
$$

Then $f$ is convex and finite (i.e. real-valued). The recession cone $R_f := \{\theta \in \mathbb{R}^p; f^\infty(\theta) \leq 0\}$ of $f$ consists of the vectors $\theta \in \mathbb{R}^p$ such that the recession function $f^\infty$ at $\theta$ is non-positive.
To see that $R_f = \{0_p\}$, invoke Rockafellar (1970, Corollary 8.5.2) to evaluate $f^\infty$ as

$$f^\infty(\theta) = \lim_{\tau \to \infty} \frac{f(\tau \theta)}{\tau} = \lim_{\tau \to \infty} \frac{M(\tau \theta) + \lambda \|\tau \theta\|_1}{\tau} = \lim_{\tau \to \infty} \frac{M(\tau \theta)}{\tau} + \lambda \|\theta\|_1.$$

Loss non-negativity implies $f^\infty(\theta) \geq \lambda \|\theta\|_1$, so from $\lambda \in (0, \infty)$ we conclude that $f^\infty(\theta) > 0$ for all $\theta \in \mathbb{R}^p \setminus \{0\}$. Rockafellar (1970, Theorem 27.1(d)) now shows that the set $\hat{\Theta}(\lambda)$ of minimizers of $f$ is non-empty and bounded. Convexity of $\hat{\Theta}(\lambda)$ follows from convexity of $f$. That the level set $\hat{\Theta}(\lambda)$ is closed (hence compact) follows from continuity of $f$, which is a consequence of its convexity and finiteness on $\mathbb{R}^p$ (Rockafellar, 1970, Corollary 10.1.1).

Now we consider Part 2 of the theorem. To show linear independence, let $\hat{\theta} \in \hat{\Theta}(\lambda) (\neq \emptyset)$ be any solution. We know that $\text{rank}(X_{\text{supp}(\hat{\theta})}) \leq \|\hat{\theta}\|_0$. If equality holds, then we are done, so suppose that $\text{rank}(X_{\text{supp}(\hat{\theta})}) < \|\hat{\theta}\|_0$. Let $\hat{T} := \text{supp}(\hat{\theta})$ abbreviate the support of $\hat{\theta}$. Then there is a $v \in \mathbb{R}^{\|\hat{\theta}\|_0 \setminus \{0\}}$ such that $X_{\hat{T}}v = 0$. Since $v$ is non-zero, there is an index $j \in \hat{T}$ such that $v_j \neq 0$. Fix such an index $j$. It then follows that

$$X_j = \sum_{k \in \hat{T} \setminus \{j\}} c_k X_k, \quad \text{where} \quad c_k := -\frac{v_k}{v_j}, \quad k \in \hat{T} \setminus \{j\}. \quad (F.1)$$

Per convexity, optimality of $\hat{\theta}$ is equivalent to $0 \in \partial f(\hat{\theta})$, where $\partial f(\hat{\theta})$ denotes the subdifferential of $f$ at $\hat{\theta}$. Since all functions involved are finite convex, Rockafellar (1970, Theorems 23.8 and 23.9) combine to show

$$\partial f(\hat{\theta}) = \partial \hat{M}(\hat{\theta}) + \lambda \partial \|\hat{\theta}\|_1 = \mathbb{E}_n \left[ X_i \partial_1 m(X_i^\top \hat{\theta}, Y_i) \right] + \lambda \sum_{k=1}^p \partial |\hat{\theta}_k|,$$

where summation is understood in the Minkowski (i.e. set) sense, and $\partial_1 m(t, y)$ denotes the subdifferential of $m(\cdot, y)$ at $t$. Since $\hat{\theta}$ is a solution, we can find $z_i \in -\partial_1 m(X_i^\top \hat{\theta}, Y_i), i \in [n]$, and $\gamma_k \in \partial |\hat{\theta}_k|, k \in [p]$, such that

$$\mathbb{E}_n [z_i X_{i,k}] = \lambda \gamma_k, \quad k \in [p].$$

Fix such $\{z_i\}_{i=1}^n$ and $\{\gamma_k\}_{k=1}^p$. Note that $\gamma_k = \text{sgn}(\hat{\theta}_k)$ for $\hat{\theta}_k \neq 0$. From (F.1) we know that $X_{i,j} = \sum_{k \in \hat{T} \setminus \{j\}} c_k X_{i,k}$ for all $i \in [n]$, so multiplying each side of (F.1) by $\gamma_j$ and using $\gamma_k^2 = 1$ for all $k \in \hat{T}$, we arrive at

$$\gamma_j X_{i,j} = \sum_{k \in \hat{T} \setminus \{j\}} a_k \gamma_k X_{i,k}, \quad i \in [n], \quad \text{where} \quad a_k := c_k \gamma_j \gamma_k, \quad k \in \hat{T} \setminus \{j\}. \quad (F.2)$$
Multiplying each side of (F.2) by $z_i$ and then averaging over $i \in [n]$, we therefore arrive at

$$\gamma_j \mathbb{E}_n [z_i X_{i,j}] = \sum_{k \in \hat{T} \setminus \{j\}} a_k \gamma_k \mathbb{E}_n [z_i X_{i,k}] = \lambda.$$  

The previous display and $\lambda \in (0, \infty)$ imply $\sum_{k \in \hat{T} \setminus \{j\}} a_k = 1$. We now follow the argument on Rosset et al. (2004, p. 969), included here for completeness. Define $\vartheta \in \mathbb{R}^p$ by $\vartheta_j := -\gamma_j$, $\vartheta_k := a_k \gamma_k$, $k \in \hat{T} \setminus \{j\}$, and $\vartheta_k = 0$, $k \notin \hat{T}$. We construct $\tilde{\vartheta} \in \mathbb{R}^p$ by moving $\tilde{\vartheta}$ in the direction $\vartheta$ until we hit a new zero. That is, we let

$$\tilde{\vartheta} := \hat{\vartheta} + \tau_0 \vartheta \quad \text{where} \quad \tau_0 := \inf \left\{ \tau \geq 0; \hat{\vartheta}_k + \tau \vartheta_k = 0 \text{ for some } k \in \hat{T} \right\}.$$  

Note that $\tau_0$ is finite, since $\hat{\vartheta} + \tau \vartheta = 0$ is solved by $\tau = |\hat{\vartheta}_j|$. For indices $J \subseteq [p]$, we let $\delta_{(J)}$ denote the $|J|$-dimensional subvector of $\delta \in \mathbb{R}^p$ with indices indexed by $J$. Then

$$X \tilde{\vartheta} = X^T \tilde{\vartheta}_{(\hat{T})} = X^T \tilde{\vartheta}_{(\hat{T})} + \tau_0 X^T \vartheta_{(\hat{T})}$$

$$= X^T \tilde{\vartheta}_{(\hat{T})} + \tau_0 \left( -\gamma_j X_j + \sum_{k \in \hat{T} \setminus \{j\}} a_k \gamma_k X_k \right) = X \hat{\vartheta}.$$  

It follows that $\hat{M}(\tilde{\vartheta}) = \hat{M}(\hat{\vartheta})$, so $\tilde{\vartheta}$ achieves the same loss as $\hat{\vartheta}$.

Moreover, from

$$\|\tilde{\vartheta}\|_1 = |\hat{\vartheta}_j| + \sum_{k \in \hat{T} \setminus \{j\}} |\hat{\vartheta}_k|$$

$$= (\hat{\vartheta}_j + \tau_0 \vartheta_j) \text{sgn}(\hat{\vartheta}_j) + \sum_{k \in \hat{T} \setminus \{j\}} (\hat{\vartheta}_k + \tau_0 \vartheta_k) \text{sgn}(\hat{\vartheta}_k)$$

$$= (\hat{\vartheta}_j - \tau_0 \gamma_j) \text{sgn}(\hat{\vartheta}_j) + \sum_{k \in \hat{T} \setminus \{j\}} (\hat{\vartheta}_k + \tau_0 a_k \gamma_k) \text{sgn}(\hat{\vartheta}_k)$$

$$= \hat{\vartheta}_j \text{sgn}(\hat{\vartheta}_j) - \tau_0 + \sum_{k \in \hat{T} \setminus \{j\}} \hat{\vartheta}_k \text{sgn}(\hat{\vartheta}_k) + \tau_0 \sum_{k \in \hat{T} \setminus \{j\}} a_k$$

$$= |\hat{\vartheta}_j| + \sum_{k \in \hat{T} \setminus \{j\}} |\hat{\vartheta}_k|$$

$$= \|\hat{\vartheta}\|_1$$

we see that $\tilde{\vartheta}$ achieves the same $\ell_1$ norm as well. It follows that $\tilde{\vartheta}$ is also a solution and that $\tilde{\vartheta}$ has (at least) one more zero than $\hat{\vartheta}$. We can repeat the above argument until we arrive at
a solution for which the columns indexed by its support are linearly independent.

Finally, letting \( \hat{\theta} \) be a solution for which \( \text{rank}(X_{\text{supp}(\hat{\theta})}) = \|\hat{\theta}\|_0 \), sparsity then follows from \( \text{rank}(X_{\text{supp}(\hat{\theta})}) \leq n \wedge p \).

\[\square\]

**Proof of Theorem E.2.** Existence follows from Theorem E.1. We argue uniqueness in four steps. In Step 1, we show that strict convexity of \( m(\cdot, y) \) implies that every solution \( \hat{\theta} \in \hat{\Theta}(\lambda) \) leads to the same linear forms \( X\hat{\theta} \). In Step 2, we use this observation in combination with the optimality conditions to define the so-called equicorrelation set \( \mathcal{E} \), which (as we establish) contains the supports of all solutions \( \hat{\theta} \in \hat{\Theta}(\lambda) \). In Step 3, we leverage the columns of \( X \) being in general position to show that the columns picked out by the equicorrelation set are linearly independent. In Step 4, we show that every solution can be characterized as the solution to the same strictly convex programming problem and must therefore coincide.

**Step 1.** To establish equality of linear forms, seeking a contradiction, suppose that we can find solutions \( \hat{\theta}^{(0)} \) and \( \hat{\theta}^{(1)} \) for which \( X\hat{\theta}^{(0)} \neq X\hat{\theta}^{(1)} \). Then \( X_i^\top \hat{\theta}^{(0)} \neq X_i^\top \hat{\theta}^{(1)} \) for some \( i \in [n] \). Consider the objective function \( f \) on \( \mathbb{R}^p \) defined by \( f(\theta) := M(\theta) + \lambda \|\theta\|_1 \). Per strict convexity of \( m(\cdot, y) \), \( y \in \mathcal{Y} \), and convexity of \( \|\cdot\|_1 \), for any \( \tau \in (0, 1) \) and \( \hat{\theta}^{(\tau)} := (1 - \tau)\hat{\theta}^{(0)} + \tau\hat{\theta}^{(1)} \) we have

\[
f(\hat{\theta}^{(\tau)}) = \mathbb{E}_n[m((1 - \tau)X_i^\top \hat{\theta}^{(0)} + \tau X_i^\top \hat{\theta}^{(1)}, Y_i)] + \lambda\|(1 - \tau)\hat{\theta}^{(0)} + \tau\hat{\theta}^{(1)}\|_1
\]

\[
< (1 - \tau)\mathbb{E}_n[m(X_i^\top \hat{\theta}^{(0)}, Y_i)] + \tau\mathbb{E}_n[m(X_i^\top \hat{\theta}^{(1)}, Y_i)] + (1 - \tau)\lambda\|\hat{\theta}^{(0)}\|_1 + \tau\lambda\|\hat{\theta}^{(1)}\|_1
\]

\[
= (1 - \tau)\left[M(\hat{\theta}^{(0)}) + \lambda\|\hat{\theta}^{(0)}\|_1\right] + \tau\left[M(\hat{\theta}^{(1)}) + \lambda\|\hat{\theta}^{(1)}\|_1\right] = \inf_{\mathbb{R}^p} f,
\]

which contradicts optimality of \( \hat{\theta}^{(0)} \) and \( \hat{\theta}^{(1)} \). Hence, \( X\hat{\theta} \) is constant across solutions \( \hat{\theta} \in \hat{\Theta}(\lambda) \).

**Step 2.** As described in the proof of Theorem E.1, optimality of \( \hat{\theta} \) is equivalent to there being \( z_i \in -\partial_t m(X_i^\top \hat{\theta}, Y_i), i \in [n] \), and \( \gamma_j \in \partial|\hat{\theta}_j|, j \in [p] \), such that

\[
\mathbb{E}_n[z_iX_{i,j}] = \lambda\gamma_j \quad j \in [p],
\]

where \( \partial_t m(t, y) \) denotes the subdifferential of \( m(\cdot, y) \) at \( t \). Note that \( \gamma_j = \text{sgn}(\hat{\theta}_j) \) for \( \hat{\theta}_j \neq 0 \).

From Step 1 we know that every solution \( \hat{\theta} \in \hat{\Theta}(\lambda) \) leads to the same linear forms \( X\hat{\theta} \). We can therefore define the **equicorrelation set** \( \mathcal{E} \) independently of the solution by

\[
\mathcal{E} := \left\{ j \in [p] : \mathbb{E}_n[z_iX_{i,j}] = \lambda \text{ for some } z_i \in -\partial_t m(X_i^\top \hat{\theta}, Y_i), i \in [n] \right\}.
\]

We claim that \( \mathcal{E} \) contains the support of every solution. Indeed, let \( \hat{\theta} \in \hat{\Theta}(\lambda) \) and \( j \in \text{supp}(\hat{\theta}) \). Then there are \( z_i \in -\partial_t m(X_i^\top \hat{\theta}, Y_i), i \in [n] \), and \( \gamma_j \in \partial|\hat{\theta}_j|, j \in [p] \), such that
\[ E_n [z_i X_{i,j}] = \lambda \gamma_j \] with \( \gamma_j = \text{sgn}(\hat{\theta}_j) \) being either plus or minus one. Hence \( j \in \mathcal{E} \).

**Step 3.** This step is similar to the linear independence argument in the proof of Theorem E.1. We here show that the columns \( X \) being in general position implies that \( \text{rank}(X_{\mathcal{E}}) = |\mathcal{E}| \). We prove the contra-positive statement, which is that \( \text{rank}(X_{\mathcal{E}}) < |\mathcal{E}| \) implies that the columns of \( X \) are not in general position. To this end, we may without loss of generality assume that \( |\mathcal{E}| \leq (n \land p) + 1 \). (Indeed, if \( |\mathcal{E}| > (n \land p) + 1 \), then taking any subset \( \mathcal{E}' \) of \( \mathcal{E} \) with \( |\mathcal{E}'| = (n \land p) + 1 \), we have \( \text{rank}(X_{\mathcal{E}'}) \leq n \land p < |\mathcal{E}'| \), and we continue the argument with \( \mathcal{E}' \) in place of \( \mathcal{E} \).) Now, since \( \text{rank}(X_{\mathcal{E}}) < |\mathcal{E}| \), there is a \( v \in \mathbb{R}^{|\mathcal{E}|} \setminus \{0\} \) such that \( X_{\mathcal{E}} v = 0 \). Since \( v \) is non-zero, there is a \( j \in \mathcal{E} \) such that \( v_j \neq 0 \). Fix such a \( j \). It then follows that

\[
X_j = \sum_{k \in \mathcal{E} \setminus \{j\}} c_k X_k, \quad \text{where} \quad c_k := -\frac{v_k}{v_j}, \quad k \in \mathcal{E} \setminus \{j\}.
\]

From \( k \in \mathcal{E} \), we know there are \( z_i \in -\partial_1 m(X_i^\top \hat{\theta}, Y_i), i \in [n] \) such that \( |E_n [z_i X_{i,k}]| = \lambda \). Fix such \( \{z_i\}_{i=1}^n \), and abbreviate \( \xi_k := \text{sgn}(E_n [z_i X_{i,k}]) \) for \( k \in \mathcal{E} \). Since \( \lambda \in (0, \infty) \), we have \( \xi_k \in \{-1, 1\} \) and, thus, \( \xi_k^2 = 1 \) for each \( k \in \mathcal{E} \). It therefore follows that

\[
\xi_j X_j = \sum_{k \in \mathcal{E} \setminus \{j\}} a_k \xi_k X_k, \quad a_k := \xi_j\xi_k c_k, \quad k \in \mathcal{E} \setminus \{j\}.
\]

Multiplying the \( i \)th equation by \( z_i \) and averaging over \( i \in [n] \), we see that

\[
\frac{\xi_j E_n [z_i X_{i,j}]}{\lambda} = \sum_{k \in \mathcal{E} \setminus \{j\}} a_k \frac{\xi_k E_n [z_i X_{i,k}]}{\lambda},
\]

which via \( \lambda \in (0, \infty) \) further implies that \( \sum_{k \in \mathcal{E} \setminus \{j\}} a_k = 1 \). Deduce that \( \xi_j X_j \) lies in the affine span of \( \{\xi_k X_k; k \in \mathcal{E} \setminus \{j\}\} \). Since \( |\mathcal{E}| \leq (n \land p) + 1 \), we have \( |\mathcal{E} \setminus \{j\}| \leq n \land p \), which shows that the columns of \( X \) are not in general position.

**Step 4.** Since \( \mathcal{E} \) contains the support of every solution, we can characterize any solution \( \hat{\theta} \in \hat{\Theta}(\lambda) \) by \( \hat{\theta}_{(\mathcal{E})} = 0 \) and

\[
\hat{\theta}_{(\mathcal{E})} \in \arg\min_{\theta \in \mathbb{R}^{|\mathcal{E}|}} \{E_n [m(X_{i\theta}^\top \theta, Y_i)] + \lambda \|\theta\|_1 \}.
\]

Strict convexity of \( m(\cdot, y), y \in \mathcal{Y} \), and \( \text{rank}(X_{\mathcal{E}}) = |\mathcal{E}| \) show that the function \( \theta \mapsto E_n [m(X_{i\theta}^\top \theta, Y_i)] \) defined on \( \mathbb{R}^{|\mathcal{E}|} \) is strictly convex. The right-hand side problem therefore has a unique solution, so \( \hat{\Theta}(\lambda) \) must be singleton. \( \square \)
G  Additional Examples

In this section, we provide additional examples of loss functions that fit the M-estimation framework (1.2) with the loss function $m(t, y)$ being convex in its first argument.

**Example 5 (Logistic Calibration).** In the context of average treatment effect estimation under a conditional independence assumption with a high-dimensional vector of controls, consider the logit propensity score model

$$P(Y = 1|X) = \Lambda(X^\top \theta_0), \quad (G.1)$$

where $Y \in \{0, 1\}$ is a treatment indicator, $X$ a vector of controls, and $\Lambda$ the logistic CDF. Using (1.1), $\theta_0$ can be identified with the logistic loss function in (2.2). However, as shown by Tan (2020), $\theta_0$ can also be identified using (1.1) with the logistic calibration loss

$$m(t, y) = ye^{-t} + (1 - y)t, \quad (G.2)$$

which is convex in $t$ as well. As demonstrated by Tan (2020), use of this alternative loss function leads to average treatment effect estimators with certain robustness properties. Specifically, under Tan’s conditions, these treatment effect estimators remain root-$n$ consistent and asymptotically normal even if the model for the outcome regression function is misspecified (ibid.).

**Example 6 (Logistic Balancing).** In the same setting as that of the previous example, the covariate balancing approach of Imai and Ratkovic (2014) amounts to specifying a parametric model for the treatment indicator $Y \in \{0, 1\}$,

$$P(Y = 1|X) = F(X^\top \theta_0)$$

and ensuring covariate balance in the sense that

$$E \left[ \left\{ \frac{Y}{F(X^\top \theta_0)} - \frac{1 - Y}{1 - F(X^\top \theta_0)} \right\} X \right] = 0_p.$$  

Balancing here amounts to enforcing a collection of moment conditions and is therefore naturally studied in a generalized method of moments (GMM) framework. However, specifying $F$ to be the logistic CDF $\Lambda$, covariate balancing can be achieved via M-estimation of $\theta_0$ based on the loss function

$$m(t, y) = (1 - y)e^t + ye^{-t} + (1 - 2y)t,$$  

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which is also convex in $t$. See Tan (2020) for details. \hfill \square

**Example 7 (Panel Logit Model).** Consider the *panel logit model*

$$ P(Y_{\tau} = 1|X, \gamma, Y_0, \ldots, Y_{\tau-1}) = \Lambda(\gamma + X_{\tau}^T \theta_0), \quad \tau \in \{1, 2\}, $$

where $Y = (Y_1, Y_2)^T \in \{0, 1\}^2$ is a pair of binary outcome variables, $X = (X_1^T, X_2^T)^T$ is a vector of regressors, and $\gamma$ is a unit-specific unobserved fixed effect. Rasch (1960) provides conditions under which $\theta_0$ can be identified by $\theta_0 = \underset{\theta \in \mathbb{R}^p}{\text{argmin}} \mathbb{E}[m((X_1 - X_2)^T \theta, Y)]$, where

$$ m(t, y) = 1(y_1 \neq y_2) \left[\ln \left(1 + e^t\right) - y_1 t\right], \quad (G.3) $$

which is convex in $t$.43 \hfill \square

**Example 8 (Panel Duration Model).** Consider the *panel duration model* with the log-linear hazard specification

$$ \ln h_{\tau}(y) = X_{\tau}^T \theta_0 + \ln h_0(y), \quad \tau \in \{1, 2\}. $$

Here $h_\tau$ denotes the (conditional) hazard for spell $\tau$, and both $h_0$ and $h_\tau$ are allowed to be unit-specific. This model is a special case of the duration models studied in Chamberlain (1985, Section 3.1). Chamberlain presumes that the spells $Y_1$ and $Y_2$ are (conditionally) independent of each other and shows that the partial log-likelihood contribution is44

$$ \theta \mapsto 1(Y_1 < Y_2) \ln \Lambda((X_1 - X_2)^T \theta) + 1(Y_1 \geq Y_2) \ln \left(1 - \Lambda((X_1 - X_2)^T \theta)\right). $$

The implied loss function

$$ m(t, y) = \ln \left(1 + e^t\right) - 1(y_1 < y_2) t \quad (G.4) $$

is of the logit form (see Example 1), hence convex in $t$. With more than two completed spells, the partial log-likelihood takes a conditional-logit form (ibid.), and the resulting loss is therefore still a convex function (albeit involving multiple indices). \hfill \square

**H Additional Simulations**

In this section we present additional results for the simulation setting in Section 6. In Appendix H.1, we relate the $\ell_0$ norm of the debiasing coefficient vector $\mu_0$ to that of the

43See also Chamberlain (1984, Section 3.2) and Wooldridge (2010, Section 15.8.3).

44See also Lancaster (1992, Chapter 9, Section 2.10.2).
structural coefficients $\gamma_0$ attached to the controls. We then present additional simulation results in Appendix H.2, which stem from a different choice of probability tolerance $\alpha = \alpha_n$.

**H.1 Sparsity of Debiasing Coefficient Vector**

In this section we argue that, at least in our simulation setting, the $\ell_0$ norm of the non-primitive debiasing vector $\mu_0$ is bounded by that of the structural coefficients $\gamma_0$. We consider a master data-generating processes (DGP) akin to the one in the main text, where

$$X = \begin{pmatrix} D \\ W \end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0_{(p-1)\times 1} \end{pmatrix}, \begin{pmatrix} \Sigma_{DD} & \Sigma_{WD} \\ \Sigma_{WD} & \Sigma_{WW} \end{pmatrix} \right)$$

$$Y \mid X \sim \text{Ber} \left( F \left( X^\top \theta_0 \right) \right) \overset{d}{=} \text{Ber} \left( F \left( \beta_0 D + W^\top \gamma_0 \right) \right) \, ,$$

$$(\Sigma_{XX})_{j,k} = \rho^{j-k}, \quad |\rho| < 1,$$

and $F : \mathbb{R} \rightarrow [0, 1]$ is a twice differentiable cumulative distribution function (CDF) with everywhere positive derivative $f = F'$ and satisfying the technical condition

$$\mathbb{E} \left[ \left| \frac{d}{dt} F(t) \frac{f(t)}{F(t)(1-F(t))} \right|_{t=X^\top \theta_0} \right] < \infty \, ;$$

which we use to guarantee the finiteness of $\mathbb{E}[m''_{11}(X^\top \theta_0, Y)]$. For the logit model $F = \Lambda$, one has $\Lambda' = \Lambda(1 - \Lambda)$, and the latter condition is trivial. For the probit model $F = \Phi$ considered in the main text, the absolute integrability in the previous display is less obvious, although it can be shown to follow. (See also Appendix A.2 for detailed derivations for the examples in Section 2.)

Suppose that for some integer $C \in [p-1]$ the first $C$ controls $(W_1, \ldots, W_C)^\top =: W[C]$ are relevant in the sense that we have $\gamma_{0,j} \neq 0, j \in [C], and the remaining controls $(W_{C+1}, \ldots, W_{p-1})^\top =: W_{[p-1]\setminus[C]}$ are irrelevant, in that $\gamma_{0,j} = 0, j \in [p-1] \setminus [C]$. This structure fits the exactly sparse and intermediate coefficient patterns presented in Section 6 for which we have $C = 1$ and $4$, respectively.$^{45}$

We wish to quantify the sparsity of

$$\mu_0 = \left( \mathbb{E} \left[ m''_{11}(X^\top \theta_0, Y) WW^\top \right] \right)^{-1} \mathbb{E} \left[ m''_{11}(X^\top \theta_0, Y) WD \right] ,$$

$^{45}$The structure also fits with the approximately sparse pattern of Section 6, but there $\|\mu_0\|_0 \leq p - 1 (=\|\gamma_0\|_0)$ holds trivially.
which is well-defined under the assumptions of the main text. For binary response we have

\[ m(t, y) = -y \ln F(t) - (1 - y) \ln (1 - F(t)). \]

Differentiating once and simplifying, we get

\[ m'_1(t, y) = \frac{f(t)}{F(t)(1 - F(t))} (F(t) - y). \]

Differentiating once more, we see that

\[ m''_{11}(t, y) = \frac{f(t)^2}{F(t)(1 - F(t))} + \left( \frac{f(t)}{F(t)(1 - F(t))} \right)' (F(t) - y), \]

and, thus, using finiteness of \( E[m''_{11}(X^\top \theta_0, Y)] \),

\[ E[m''_{11}(X^\top \theta_0, Y) | X^\top \theta = t] = \frac{f(t)^2}{F(t)(1 - F(t))} =: \omega_F(t). \]

It follows that we can express \( \mu_0 \) as

\[ \mu_0 = \left( E[\omega_F(X^\top \theta_0) W W^\top] \right)^{-1} E[\omega_F(X^\top \theta_0) W D]. \]

We claim that \( \mu_0 \) coincides with \( \mu^* \) defined as

\[ \mu^* := \left( \left( E[\omega_F(X^\top \theta_0) W[C] W[C]^\top] \right)^{-1} E[\omega_F(X^\top \theta_0) W[C] D] \right)_{(p-1-C)\times 1}, \]

which, in particular, implies that \( \mu_0 \) has no more than \( C \) non-zero entries, since \( \|\mu_0\|_0 \leq C = \|\gamma_0\|_0 \). To establish this claim, we show that \( \mu^* \) solves the linear system of equations

\[ E[\omega_F(X^\top \theta_0) W W^\top] \mu = E[\omega_F(X^\top \theta_0) W D] \quad \text{(H.1)} \]

in \( \mu \in \mathbb{R}^{p-1} \), in which case it must coincide with the (previously established) unique solution \( \mu_0 \). To see that \( \mu^* \) is indeed a solution, first note that by zero means and Gaussianity

\[ E \left[ W[p-1\setminus[C] | D, W[C]] \right] = \Sigma_{W[p-1\setminus[C],DW[C]]} \Sigma_{DW[C]}^{-1} D \left( \begin{array}{c} D \\ W[C] \end{array} \right). \]

\footnote{To be more precise, Lemma B.22 provides one set of sufficient conditions for the existence and uniqueness a solution to (5.1). Other sets of sufficient conditions are possible.}
Using the Toeplitz correlation structure, we get
\[
\Sigma_{W[p-1]\setminus[C],DW[C]} \Sigma^{-1}_{DW[C],DW[C]} = \left( \begin{array}{ccc} 0_{(p-(1+C))\times 1} & \cdots & 0_{(p-(1+C))\times 1} \\ \Sigma_{W[p-1]\setminus[C],W_C} \end{array} \right).
\]

Indeed, observe that
\[
\left( \begin{array}{ccc} 0_{(p-1-C)\times 1} & \cdots & 0_{(p-1-C)\times 1} \\ \Sigma_{W[p-1]\setminus[C],W_C} \end{array} \right) \Sigma_{DW[C],DW[C]} = \left( \begin{array}{cccc} 1 & \rho & \rho^2 & \cdots & \rho^C \\ \rho & 1 & \rho & \cdots & \rho \\ \rho^2 & \rho & \ddots & \ddots & \rho \\ \vdots & \vdots & \ddots & \ddots & \rho \\ \rho^C & \cdots & \rho & 1 \end{array} \right) = \Sigma_{W[p-1]\setminus[C],DW[C]},
\]
as desired. We therefore get the simplification
\[
E[W[p-1]\setminus[C] | D, W[C]] = W_C \Sigma_{W[p-1]\setminus[C],W_C}.
\]

Since \(X^\top \theta_0\) does not depend on \(W[p-1]\setminus[C]\), the right-hand side (RHS) vector of (H.1) is
\[
E\left[\omega_F \left(X^\top \theta_0\right) WD\right] = E\left[\omega_F \left(X^\top \theta_0\right) \left( \begin{array}{c} W[C] \\ W[p-1]\setminus[C] \end{array} \right) D\right] 
= E\left[\omega_F \left(X^\top \theta_0\right) \left( \begin{array}{c} W[C] \\ E[W[p-1]\setminus[C] | D, W[C]] \end{array} \right) D\right] 
= E\left[\omega_F \left(X^\top \theta_0\right) \left( \begin{array}{c} W[C] \\ W_C \Sigma_{W[p-1]\setminus[C],W_C} \end{array} \right) D\right] 
= \left( E\left[\omega_F \left(X^\top \theta_0\right) W_C D\right] \right) 
\left( E\left[\omega_F \left(X^\top \theta_0\right) W_C D \Sigma_{W[p-1]\setminus[C],W_C} \right] \right) .
\]
Similarly, the left-hand side (LHS) matrix of (H.1) is

\[
\begin{align*}
E \left[ \omega_F \left( X^\top \theta_0 \right) WW^\top \right] &= \begin{pmatrix} 
E \left[ \omega_F \left( X^\top \theta_0 \right) W_{[C]} W_{[C]}^\top \right] & E \left[ \omega_F \left( X^\top \theta_0 \right) W_{[p-1]\{C\}} W_{[p-1]\{C\}}^\top \right] \\
E \left[ \omega_F \left( X^\top \theta_0 \right) W_{[p-1]\{C\}} W_{[p-1]\{C\}}^\top \right] & E \left[ \omega_F \left( X^\top \theta_0 \right) W_{[p-1]\{C\}} W_{[p-1]\{C\}}^\top \right]
\end{pmatrix} \\
&= \begin{pmatrix} 
E \left[ \omega_F \left( X^\top \theta_0 \right) W_{[C]} W_{[C]}^\top \right] & E \left[ \omega_F \left( X^\top \theta_0 \right) W_{[C]} W_{[C]}^\top \right] \\
E \left[ \omega_F \left( X^\top \theta_0 \right) W_{[C-1]} W_{[C-1]}^\top \right] & E \left[ \omega_F \left( X^\top \theta_0 \right) W_{[C-1]} W_{[C-1]}^\top \right]
\end{pmatrix}^{-1} \\
&= \begin{pmatrix} 
A & b \\
c^\top & d
\end{pmatrix}^{-1}.
\end{align*}
\]

The bottom left block of the LHS matrix of (H.1) is \( \Sigma_{W_{[p-1]\{C\]},W_C} \) times

\[
E \left[ \omega_F \left( X^\top \theta_0 \right) W_C W_{[C]}^\top \right] = \begin{pmatrix} 
E \left[ \omega_F \left( X^\top \theta_0 \right) W_C W_{[C-1]}^\top \right] \\
E \left[ \omega_F \left( X^\top \theta_0 \right) W_C W_{[C-1]}^\top \right]
\end{pmatrix}
= \begin{pmatrix} 
c^\top \\
d
\end{pmatrix}.
\]

Inserting \( \mu^* \) to check and verify the bottom part of the system, we get

\[
\begin{align*}
\left( \Sigma_{W_{[p-1]\{C\]},W_C} E \left[ \omega_F \left( X^\top \theta_0 \right) W_C W_{[C]}^\top \right] \right) \mu^* \\
= \Sigma_{W_{[p-1]\{C\]},W_C} E \left[ \omega_F \left( X^\top \theta_0 \right) W_C W_{[C]}^\top \right] \times \left( E \left[ \omega_F \left( X^\top \theta_0 \right) W_{[C]} W_{[C]}^\top \right] \right)^{-1} E \left[ \omega_F \left( X^\top \theta_0 \right) W_{[C]} D \right]
\end{align*}
\]

\[
= \Sigma_{W_{[p-1]\{C\]},W_C} \left( c^\top \ d \right) \left( \begin{pmatrix} A & b \\
c^\top & d \end{pmatrix} \right)^{-1} E \left[ \omega_F \left( X^\top \theta_0 \right) W_{[C]} D \right].
\]

Since

\[
\begin{pmatrix} 0_{1 \times (C-1)} & 1 \end{pmatrix} \begin{pmatrix} A & b \\
c^\top & d \end{pmatrix} = \begin{pmatrix} c^\top & d \end{pmatrix},
\]

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we have
\[
\begin{pmatrix}
\begin{bmatrix} c \end{bmatrix} & d
\end{bmatrix}
\begin{pmatrix}
\begin{bmatrix} A \\ c^\top \\
\end{bmatrix} & b
\end{pmatrix}
\end{pmatrix}^{-1}
= \begin{pmatrix}
0_{1 \times (C-1)} & 1
\end{pmatrix}.
\]

Hence,
\[
\Sigma_{W[p-1] \setminus [C],W_C} \begin{pmatrix}
\begin{bmatrix} c \end{bmatrix} & d
\end{bmatrix}
\begin{pmatrix}
\begin{bmatrix} A \\ c^\top \\
\end{bmatrix} & b
\end{pmatrix}
\end{pmatrix}^{-1}
E \left[ \omega_F \left( X^\top \theta_0 \right) W_{[C]} D \right]
= \Sigma_{W[p-1] \setminus [C],W_C} \begin{pmatrix}
0_{1 \times (C-1)} & 1
\end{pmatrix}
\begin{pmatrix}
E \left[ \omega_F \left( X^\top \theta_0 \right) W_{[C]} D \right] &
E \left[ \omega_F \left( X^\top \theta_0 \right) W_{C} D \right]
\end{pmatrix}
= E \left[ \omega_F \left( X^\top \theta_0 \right) W_{C} D \right] \Sigma_{W[p-1] \setminus [C],W_C},
\]
as desired.

To verify the above calculation, we return to the binary probit model \((F = \Phi)\) in our main simulations section (Section 6). Figures H.1 and H.2 display the raw and sorted absolute values, respectively, of the debiasing coefficients \(\mu_0\) for the different coefficient patterns and correlation levels considered in the main text. These values are obtained via simulation for \(p = 100\) and a sample size of 100,000. For clarity of plots, we only display the first and largest 10 coefficients, respectively. The figures show that, when the structural coefficients \(\gamma_0\) are \(\ell_0\)-sparse, this sparsity is indeed inherited by the debiasing coefficients coefficients \(\mu_0\). In addition, when the structural coefficients are only approximately sparse, the sorted absolute values of the debiasing coefficients still decay polynomially fast in the index, which is a form of approximate sparsity.

**H.2 Additional Results**

In this section, we give additional simulation results based on the simulation design in the main text (Section 6). The only difference is that we use the ad hoc probability tolerance rule \(\alpha_n = 10/n\) instead of the rule \(\alpha_n = .1/\ln(p \lor n)\) from Belloni et al. (2012).\(^{47}\) The ad hoc rule was reverse engineered to yield \(\alpha = 10\%, 5\%\) and \(2.5\%\) for \(n = 100, 200\) and \(400\), respectively, which can be compared with \(\alpha \approx 2.2\%, 1.9\%\) and \(1.7\%\) appearing in Section 6.

Following the progression of Section 6, we display the estimation errors stemming from the ad hoc rule and then provide the resulting normal approximations. Specifically, Figures H.3, H.4, H.5, H.6 and H.7 below should be compared to Figures 6.1, 6.2, 6.3, 6.4 and 6.5, respectively.

\(^{47}\)Strictly speaking, the results in Section 6 and this section also differ in terms of their random number generator seeds and, hence, the resulting datasets. Thus, even though CV does not depend on \(\alpha_n\), one may also see small numerical differences when contrasting figures for CV across the two sections.
Figure H.1: Raw Debiasing Coefficients $\mu_0, p = 100$, First 10 Coefficients

Figure H.2: Sorted Absolute Debiasing Coefficients $\mu_0, p = 100$, Largest 10 Coefficients
Figure H.3: Mean $\ell_2$ Estimation Error by Method with $c_0 = 1.1$ and $\alpha_n = 10/n$

![Graph showing $\ell_2$ estimation error for different methods and sparsity levels.](image)

Notes: The dotted black lines correspond to the (constant) error of the all-zeros estimator. Post-BCV cases dropped due to nonconvergence: 19 (0.02%).

Figure H.4: Mean $\ell_2$ BCV Estimation Error by Score Markup with $\alpha_n = 10/n$

![Graph showing $\ell_2$ BCV estimation error for different $c_0$ values and sparsity levels.](image)

Notes: The dotted black lines correspond to the (constant) error of the all-zeros estimator.
Figure H.5: Mean $\ell_2$ Post-BCV Estimation Error by Score Markup with $\alpha_n = 10/n$

![Graph showing mean $\ell_2$ Post-BCV estimation error by score markup with $\alpha_n = 10/n$ for different sample sizes ($n, p = 100, 200, 400$).](image)

Notes: The dotted black lines correspond to the (constant) error of the all-zeros estimator. Post-BCV cases dropped due to nonconvergence: 55 (0.02%).

Figure H.6: Densities of Studentized Estimates by $n(=p)$ with $\rho = 0$, $c_0 = 1.1$ and $\alpha_n = 10/n$

![Graph showing densities of studentized estimates for different models (BCV, Post-BCV, CV) for exactly sparse, intermediate, and approximately sparse cases.](image)

Notes: The dotted black lines depict the standard normal density.
For both estimation and inference purposes, the BCV and Post-BCV estimators resulting from the ad hoc rule perform similarly to those resulting from the rule $\alpha_n = 1/\ln(p \lor n)$. The rankings and conclusions of the main text therefore appear robust to the choice of $\alpha_n$.

I Comparison with Existing Penalty Methods

For specific models and loss functions, existing methods for choosing the penalty can be used to estimate $\theta_0$. One theoretically justifiable method is based on moderate deviation theory for self-normalized sums as detailed in Jing et al. (2003) and de la Peña et al. (2009). Building on various structures (e.g. mean-square projection or Lipschitz loss), data-driven self-normalization is used in both Belloni et al. (2012) and Belloni et al. (2016) for the LASSO and $\ell_1$-penalized logistic regression, respectively. We next compare the performance of our BCV penalty method with the self-normalized penalty methods in each of these papers. Our main finding is that whenever the self-normalized penalty methods apply, they provide results which are rather similar to those obtained from our BCV method.
I.1 Comparing with Belloni et al. (2012) Penalty Method

We here consider data-generating processes identical to those in Section 6 except that the outcome is generated as

\[ Y_i = X_i^\top \theta_0 + \varepsilon_i, \quad \varepsilon_i \mid X_i \sim N(0,1), \quad i \in [n], \]

thus implying a linear model with Gaussian errors. Taking \( m \) to be the (one-half) square loss \( \frac{1}{2}(y-t)^2 \), the \( \ell_1 \)-ME in (1.2) is the LASSO (Tibshirani, 1996).

Belloni et al. (2012, Algorithm A.1) provides a data-driven penalization scheme allowing for non-Gaussianity, conditionally heteroskedastic errors and regressors measured on different scales. We here compare with a slightly simplified two-step version of their algorithm, presuming that the researchers knows that (i) the regressors are measured on the same scale, and (ii) the errors are conditionally homoskedastic (with variance unknown). These simplifications are only made to ease exposition and reduce the computational burden involved in the comparison.\(^{48}\)

With conditionally homoskedastic errors and equivariant regressors, a two-step version of Belloni et al. (2012, Algorithm A.1) goes as follows:

1. **Initial:** Calculate an initial penalty level

\[ \lambda_{b\ell} := \frac{c_0 \hat{\sigma}_y \Phi^{-1}(1 - \alpha(2p))}{\sqrt{n}}, \]

where \( \hat{\sigma}_y^2 \) is the outcome sample variance, and store the residuals \( \hat{\varepsilon}_i := Y_i - X_i^\top \hat{\theta}, i \in [n], \) from any LASSO solution \( \hat{\theta} \in \hat{\Theta}(\lambda_{b\ell}) \).

2. **Refined:** Calculate a refined penalty level

\[ \lambda_{bc\ell} := \frac{c_0 \hat{\sigma}_\varepsilon \Phi^{-1}(1 - \alpha(2p))}{\sqrt{n}}, \]

where \( \hat{\sigma}_\varepsilon^2 \) is the residual sample variance \( \{\hat{\varepsilon}_i\}_{i=1}^n \). The Belloni et al. (2012) estimator is then any LASSO solution \( \hat{\theta} \in \hat{\Theta}(\lambda_{bc\ell}) \).

As in Belloni et al. (2012), in the above we use \( c_0 = 1.1 \) and \( \alpha = \alpha_n = .1/\ln(n \vee p) \). Note that the (simplified) initial step here corresponds to the penalty level \( \lambda_{b\ell} \) and resulting estimator suggested in Bickel et al. (2009).

\(^{48}\)We also presume that the researcher knows that the true intercept is zero. Hence, we do not include a constant regressor and penalize all regressor coefficients.
Figure I.1 plots the mean $\ell_2$ estimation errors resulting from the BCV, Bickel et al. (2009) (BRT09) and in Belloni et al. (2012) (BCCH12) penalty methods, respectively, considering the sample and problem sizes ($n$ and $p$), correlation levels ($\rho$) and coefficient patterns considered in Section 6 (for the binary probit). The figure shows that BCV performs at least as well as the other methods. However, compared to the here theoretically justifiable BCCH12 method, the BCV improvement is modest.

### I.2 Comparing with Belloni et al. (2016) Penalty Method

We here consider data-generating processes identical to those in Section 6 except that the outcome is generated as

$$Y_i = 1 \left( X_i^T \theta_0 + \varepsilon_i > 0 \right), \quad \varepsilon_i \mid X_i \sim \text{Logistic}(0, 1), \quad i \in [n],$$

thus implying a binary logit model as in Example 1. We take $m$ to be the negative logit likelihood loss in (2.2).
Figure I.2: Comparing with Belloni et al. (2016) Penalty Method (for $\ell_1$-Penalized Logit)

The notes of Belloni et al. (2016, Table 1) suggest the penalty level

$$\lambda_{\alpha,n}^{bcw} := \frac{c_0\Phi^{-1}(1 - \alpha/(2p))}{2\sqrt{n}},$$

where we continue to use $c_0 = 1.1$ and $\alpha = \alpha_n = .1/\ln(n \lor p)$.$^{49}$ As in Belloni et al. (2016), we (correctly) presume that the regressors are equivariant, and that the true intercept is zero. Hence, we do not include a constant regressor and penalize all regressor coefficients.

Figure I.2 plots the mean $\ell_2$ estimation errors resulting from the BCV and Belloni et al. (2016) (BCW16) penalty methods, respectively, considering the sample and problem sizes ($n$ and $p$), correlation levels ($\rho$) and coefficient patterns considered in Section 6 (for the binary probit). The figure shows that BCV performs at least as well as BCW16. However, compared to the here theoretically justifiable BCW16 method, the BCV improvement is again modest.

$^{49}$See Belloni et al. (2018b, Algorithm 3) for details on how to handle regressors measured on different scales.