Convergence of incremental adaptive systems

Mingxuan Sun

College of Information Engineering, Zhejiang University of Technology
Hangzhou 310023, CHINA

Abstract
In this paper, incremental adaptive mechanisms are presented and characterized, to provide design hints for the development of continuous-time adaptive systems. The comparison with the conventional integral adaptive systems indicates that the suggested design methodology will be a supplement to the existing ones. With the aid of a Barbalat-like lemma, convergence results of the incremental adaptive systems are established. It is shown that the proposed adaptive mechanisms are able to work well in handling parametric uncertainties in systems undertaken.

Keywords: convergence, incremental adaptation, incremental adaptive control, parametrization

1. Introduction
Consider the following uncertain system
\[
\dot{x} = \theta^0 t \varphi(t, x) + bu
\]
where \(x\) is the scalar state, \(u\) is the scalar input of the system, \(\theta^0\) is the vector of unknown parameters, \(\varphi(\cdot, \cdot)\) is the vector of known continuous nonlinearities, and \(b\) is the unknown control gain, but its sign is assumed to be known. Here, we do not specify whether \(b\) is positive or negative, implying that the adaptive systems to be developed in this paper are suitable to both situations.

It is seen that the parameter vector \(\theta^0\) appear linearly in (1), and this indicates that the main point of this paper is to handle the linear-in-the-parameters uncertainty. The problem is, for a given trajectory \(x_d(t), t \in \)
[0, +∞), to develop adaptive mechanisms for estimating the unknown parameters, and based on the estimates find the control $u(t), t \in [0, +∞)$, such that $x(t)$ follows $x_d(t)$ as close as possible, as $t \to +∞$.

Let us denote by $e = x - x_d$ the tracking error. The time derivative of the tracking error with respect to time can be expressed as

$$\dot{e} = b(\theta^T \varphi(t, x) + u)$$

with $\theta = \theta^0 / b$.

Let us begin with a discussion on the conventional integral adaptive systems. We refer the reader to literature [1, 2, 4], for design issues in continuous-time adaptive control, [3] for model reference adaptive methodologies, [6] for robustness of adaptive systems, [7, 8] for robust adaptive algorithms, [5] for adaptive backstepping designs, and [9] more recent immersion and invariance adaptive techniques. As is well known, it is difficult to establish the asymptotic stability of time-varying systems as it is not easy to find the Lyapunov function with a negative definite derivative. Fortunately, Barbalat’s Lemma is found to be useful in performance analysis of adaptive systems, which states that if the integral of a uniformly continuous function having a (finite) limit, then the function converges to zero asymptotically [5, 7]. Note that a simple alternative to Barbalat’s Lemma can be found in [8].

Consider the adaptive system consisting of system (1), the controller

$$u = -\text{sgn}(b)\kappa e - \hat{\theta}^T \varphi$$

and the adaptation law

$$\dot{\hat{\theta}} = -\text{sgn}(b)\gamma \varphi e$$

where $\kappa, \gamma > 0$ are design parameters, and $\text{sgn}(\cdot)$ is the sign function. Controller (3) is designed based on the certainty equivalent principle. To establish convergence of the adaptive system, we choose the positive definite function, $L = V + \frac{|b|}{2\gamma} \hat{\theta}^T \hat{\theta}$, where $V = \frac{1}{2} e^2$, and $\hat{\theta} = \hat{\theta} - \theta$. Applying (3) and (4), we have $\dot{L} = -c_1 V$, $c_1 = 2\kappa |b|$. This implies that $\dot{L}$ is negative semidefinite, which renders $L$ to be bounded. Due to the boundedness of $L$, it is easy to obtain the boundedness of $V$ and its derivative, $\dot{V}$, as well as $\int_0^t V(s)ds < +∞$, as $t \to +∞$. Invoking Barbalat’s Lemma shows that $\lim_{t\to +∞} V(t) = 0$. In turn, we conclude that $\lim_{t\to +∞} e(t) = 0$. 

2
Now let us look into Eq. (4), by integrating its both sides as follows:

\[ \hat{\theta}(t) = \hat{\theta}(0) - \text{sgn}(b) \gamma \int_{0}^{t} \varphi(s, x(s)) e(s) ds \]  

which give the estimate for \( \theta \) through the indicated integration. The adaptation law (4) is usually referred to as an integral adaptive law. As for \( t > \tau \),

\[ \hat{\theta}(t - \tau) = \hat{\theta}(0) - \text{sgn}(b) \gamma \int_{0}^{t-\tau} \varphi(s, x(s)) e(s) ds \]  

Subtracting (6) from (5), we obtain

\[ \hat{\theta}(t) = \hat{\theta}(t - \tau) - \text{sgn}(b) \gamma \int_{t-\tau}^{t} \varphi(s, x(s)) e(s) ds, \quad t > \tau \]  

Then appealing to the integral mean-value theorem, an incremental form of (4) is obtained as follows:

\[ \hat{\theta}(t) = \hat{\theta}(t - \tau) - \text{sgn}(b) \tau \gamma \varphi(t_1, x(t_1)) e(t_1), \quad t > \tau \]  

where \( t_1 \) lies between \( t - \tau \) and \( t \), takes different values for different instants of time. To use \( x(t) \) or \( x(t - \tau) \) are two ways to approximate \( x(t_1) \) for implementation of the incremental adaptive mechanism.

Transient performance is always a major concern in an adaptive system design. Due to slow rate of parameter convergence, it may exhibit poor transient behavior together with ideal asymptotic performance. We are concerned about the second term of the right-hand side of (7), where \( \tau \) appears. The parameter estimates will become hard to adapt, as \( \tau \) is set to be small. One way is to reduce the sampling rate. However, the parameter estimates would not in time updated with the measured data, when setting \( \tau \) too large. As such, the expression of (7) suggests that we choose \( \gamma \) to be proportional to \( \tau \) as follows:

\[ \gamma = \frac{1}{\tau} \gamma' \]  

with \( \gamma' \) being a constant specified by designer. This discussion is a motivation for this paper to suggest the novel methodology of incremental adaptation. Unlike the conventional ones, the incremental adaptive mechanisms do not involve such a \( \tau \). We shall clarify in the next section how the incremental adaptive mechanisms are different from the conventional ones.
2. Analysis of incremental adaptive systems

Barbalat's lemma is a tool commonly employed for concluding the convergence results of integral adaptive systems. The following presents a Barbalat-like lemma, a slightly modified form of Lemma 1 in [11] and Lemma 1 in [12], which is specifically tailored for analysis of the incremental adaptive systems.

Lemma 1. Suppose that \( g(t) \), a time function positive on \([0, +\infty)\), satisfies
\[
\int_{t-\tau}^{t} \dot{g}^2(s)ds \leq M \tag{9}
\]
for \( t \in [\tau, +\infty) \), with \( \tau > 0 \) being a constant, and
\[
\lim_{t \to +\infty} \int_{t-\tau}^{t} g(s)ds = 0 \tag{10}
\]
Then \( \lim_{t \to +\infty} g(t) = 0 \).

Proof. See Appendix for the proof.

Corollary 1. Lemma 1 holds, if the condition (9) is replaced with that \( \dot{g}(t) \) is bounded.

Proof. This corollary follows immediately from the observation that Eq. (9) holds, whenever \( \dot{g}(t) \) is bounded.

We are now in a position to present the convergence result of an adaptive system, where the incremental adaptation mechanism is adopted.

Theorem 1. Consider the incremental adaptive system described by the system (1), the controller
\[
u = -\text{sgn}(b)\kappa e - \hat{\theta}^T \varphi \tag{11}
\]
and the adaptation law
\[
\hat{\theta}(t) = \begin{cases} 
\hat{\theta}(t - \tau) - \text{sgn}(b)\gamma \varphi(t, x(t))e(t) & \text{for } t \geq 0 \\
\hat{\theta}_0 & \text{for } t \in [-\tau, 0) \end{cases} \tag{12}
\]
where $\tau, \kappa, \gamma > 0$ are design parameters, and $\hat{\theta}_0$ is the initial setting for $\hat{\theta}$. Then the tracking error $e(t)$ will be guaranteed to converge to zero, as time increases, i.e.,

$$\lim_{t \to +\infty} e(t) = 0$$

while $e, x$, as well as $\int_{t-\tau}^{t} \| \hat{\theta}(s) \|^2 ds$ and $\int_{t-\tau}^{t} u^2(s) ds$, for $t \in [\tau, +\infty)$, are bounded.

**Proof.** The convergence result of the closed-loop system composed of (11), (11) and (12) can be established by choosing the following Lyapunov-Krasovskii functional candidate

$$L(t) = V(t) + \frac{|b|}{2\gamma} \int_{t-\tau}^{t} \tilde{\theta}(s) \tilde{\theta}(s) ds$$

with $V = \frac{1}{2} e^2$. Its derivative with respect to time is

$$\dot{L}(t) = \dot{V}(t) + \frac{|b|}{2\gamma} \left[ \tilde{\theta}^T(t) \tilde{\theta}(t) - \tilde{\theta}^T(t-\tau) \tilde{\theta}(t-\tau) \right]$$

To proceed, the filtered error dynamics is expressed as, when applying (11),

$$\dot{e} = -\kappa |b| e + b\tilde{\theta}^T \varphi$$

The derivative of $V$ along the error trajectory (15) can be given as

$$\dot{V} = -\kappa |b| e^2 + b\tilde{\theta}^T \varphi e$$

The second term of the right-hand side of (14) satisfies

$$\tilde{\theta}^T(t-\tau) \tilde{\theta}(t-\tau)$$

$$= [\hat{\theta}(t-\tau) - \hat{\theta}(t) + \hat{\theta}(t) - \hat{\theta}(t)]^T [\hat{\theta}(t-\tau) - \hat{\theta}(t) + \hat{\theta}(t) - \hat{\theta}(t)]$$

$$= [\hat{\theta}(t-\tau) - \hat{\theta}(t)]^T [\hat{\theta}(t-\tau) - \hat{\theta}(t)]$$

$$+ 2[\hat{\theta}(t) - \hat{\theta}(t-\tau)]^T \hat{\theta}(t) + \tilde{\theta}^T(t) \tilde{\theta}(t)$$

Substituting (16) and (17) into (14), we obtain

$$\dot{L}(t) = -\kappa |b| e^2(t) + b\tilde{\theta}^T(t) \varphi(t, x(t)) e(t)$$

$$+ \frac{|b|}{\gamma} \tilde{\theta}^T(t) [\hat{\theta}(t) - \hat{\theta}(t-\tau)]$$

$$- \frac{|b|}{2\gamma} [\hat{\theta}(t) - \hat{\theta}(t-\tau)]^T [\hat{\theta}(t) - \hat{\theta}(t-\tau)]$$
Then applying (12) yields
\[ \dot{L}(t) = -\kappa |b| e^2(t) - \frac{|b|}{2\gamma} [\dot{\theta}(t) - \dot{\theta}(t-\tau)]^T [\dot{\theta}(t) - \dot{\theta}(t-\tau)] \]
\[ \leq -\kappa |b| e^2(t) \]
\[ = -c_1 V(t) \tag{18} \]
where \( c_1 = 2\kappa |b| \).

Eq. (18) makes \( \dot{L} \) negative semidefinite. The boundedness of \( L \) is ensured due to the boundedness of \( L(0) \). Hence, \( V \) is bounded, implying the boundedness of \( e \), and in turn that of \( x \). Furthermore, \( \int_{t-\tau}^t \|\hat{\theta}(s)\|^2 ds, t \in [\tau, +\infty) \), is bounded, by the definition of \( L \). It follows from (11) that
\[ u^2 \leq 2\kappa^2 e^2 + 2(\hat{\theta}^T \phi)^2 \]
\[ \leq c_2 + c_3 \|\hat{\theta}\|^2 \tag{19} \]
where \( c_2 = 2\kappa^2 \sup_{t \in [0, +\infty)} e^2 \), and \( c_3 = 2 \sup_{t \in [0, +\infty)} \|\phi\|^2 \). Hence, the boundedness of \( \int_{t-\tau}^t u^2(s) ds, t \in [\tau, +\infty) \), follows by noting that
\[ \int_{t-\tau}^t u^2(s) ds \leq c_2 \tau + c_3 \int_{t-\tau}^t \|\hat{\theta}(s)\|^2 ds < +\infty \tag{20} \]

The difference between instants of \( t \) and \( t - \tau \), \( L(t) - L(t-\tau) \), can be calculated by
\[ L(t) - L(t-\tau) = \int_{t-\tau}^t \dot{L}(s) ds \]
Again using (18),
\[ L(t) - L(t-\tau) \leq -c_1 \int_{t-\tau}^t V(s) ds \]
For each fixed instant \( t = t_i = i\tau + t_0, i = 1, 2, \ldots, t_0 \in [0, \tau) \).
\[ L(t_i) - L(t_{i-1}) \leq -c_1 \int_{t_{i-1}}^{t_i} V(s) ds \]
leading to
\[ L(t_i) - L(t_0) \leq -c_1 \sum_{j=1}^i \int_{t_{j-1}}^{t_j} V(s) ds \]
Consequently, by the finiteness of $L(t_0)$, the series $\sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} V(s)ds$ converges. Therefore,

$$\lim_{i \to \infty} \int_{t_{i-1}}^{t_i} V(s)ds = 0$$

implying that

$$\lim_{t \to \infty} \int_{t-\tau}^{t} V(s)ds = 0 \quad (21)$$

Now, we consider the finiteness of $\int_{t-\tau}^{t} \dot{V}^2(s)ds$, $t \in [\tau, +\infty)$. It follows from (16) that

$$\dot{V}^2 = \left[ -\kappa |b|e^2 + b\dot{\theta}^T\varphi e \right]^2 \leq 2\kappa^2 b^2 e^4 + 2b^2 (\dot{\theta}^T \varphi)^2 e^2 \leq c_4 + c_5 \|\dot{\theta}\|^2$$

where $c_4 = 2\kappa^2 b^2 \sup_{t \in [0, +\infty)} e^4$ and $c_5 = 2b^2 \sup_{t \in [0, +\infty)} \|\varphi\|^2 \sup_{t \in [0, +\infty)} e^2$.

Integrating both sides and by the boundedness of $\hat{\theta}$ yield

$$\int_{t-\tau}^{t} \dot{V}^2(s)ds \leq c_4 \tau + c_5 \int_{t-\tau}^{t} \|\dot{\theta}(s)\|^2ds < +\infty \quad (22)$$

for $t \in [\tau, +\infty)$. In view of (21) and (22), and by Lemma 1, $\lim_{t \to +\infty} V(t) = 0$. In turn, we can conclude the convergence of $e(t)$, as $t \to +\infty$. ■

Remark 1. In comparison with (7), no $\tau$ appears in the second term of the right-hand side of the adaptation law (12), which indicates the main difference between the integral adaptation law and the incremental adaptation law.

Remark 2. The adaptation law given in Theorem 1 guarantees the boundedness of $\hat{\theta}$ in the sense as presented. In order to ensure the boundedness of $\hat{\theta}$ itself, the saturated learning is helpful [10]. In particular, for fully saturated learning, the entire right-hand side of the learning law is saturated, and the estimate is ensured to be within a pre-specified region. We apply the fully-saturated adaptation law as follows:

$$\begin{cases}
\hat{\theta}(t) = \text{sat}(\hat{\theta}^*(t)) \\
\hat{\theta}^*(t) = \text{sat}(\hat{\theta}^*(t-\tau)) - \text{sgn}(b)\gamma \varphi(t, x(t))e(t)
\end{cases} \quad (23)$$
for $t \geq 0$. By the boundedness of $e$, $x$ and $\hat{\theta}$, it is easy to obtain the boundedness of $u$ from (3), and that of $V$ from (16). By invoking Corollary 1, convergence of such an incremental adaptive system can be established.

The following theorem clarifies the flexibility of choice of incremental adaptive mechanisms.

**Theorem 2.** When the adaptive control law given by

$$u = -\text{sgn}(b)\kappa e - \hat{\theta}^T \varphi + u_1$$

$$u_1 = -\frac{1}{2}\text{sgn}(b)\gamma \varphi^2 e$$  \hspace{1cm} (24)

with the adaptation law

$$\hat{\theta}(t + \tau) = \begin{cases} 
\hat{\theta}(t) - \text{sgn}(b)\gamma \varphi(t, x(t))e(t) & \text{for } t \geq 0 \\
\hat{\theta}_0 & \text{for } t \in [0, \tau]
\end{cases}$$  \hspace{1cm} (26)

is applied to system (1), then the same results as in Theorem 1 are true.

**Proof.** In order to cope with the use of (26), we choose the following candidate Lyapunov-Krasovskii functional,

$$L(t) = V(t) + \frac{|b|}{2\gamma} \int_t^{t+\tau} \tilde{\theta}^T(s)\tilde{\theta}(s)ds$$

with the same $V(t)$ as that in (13). Employing (24), the error dynamics can be expressed as

$$\dot{e} = -\kappa |b| e + b\tilde{\theta}^T \varphi + bu_1$$

The derivative of $V$ along trajectories of the error dynamics is given by

$$\dot{V} = ee$$

$$= -\kappa |b| e^2 + b\tilde{\varphi}e + bu_1 e$$

Hence, the derivative of $L$ can be calculated as

$$\dot{L}(t) = \dot{V}(t) + \frac{|b|}{2\gamma} \left[ \tilde{\theta}^T(t + \tau)\tilde{\theta}(t + \tau) - \tilde{\theta}^T(t)\tilde{\theta}(t) \right]$$
The second term of the right-hand side of the above equation satisfies
\[ \tilde{\theta}^T(t + \tau)\tilde{\theta}(t + \tau) \]
\[ = [\hat{\theta}(t + \tau) - \hat{\theta}(t)]^T[\hat{\theta}(t + \tau) - \hat{\theta}(t)] \]
\[ + 2[\hat{\theta}(t + \tau) - \hat{\theta}(t)]^T\hat{\theta}(t) + \tilde{\theta}^T(t)\hat{\theta}(t) \]
It follows that
\[ \dot{L}(t) = -\kappa|b|e^2(t) + b\tilde{\theta}^T(t)\varphi(t, x(t))e(t) + bu_1(t)e(t) \]
\[ + \frac{|b|}{\gamma}\tilde{\theta}^T(t)[\tilde{\theta}(t + \tau) - \tilde{\theta}(t)] \]
\[ + \frac{|b|}{2\gamma}[\tilde{\theta}(t + \tau) - \tilde{\theta}(t)]^T[\tilde{\theta}(t + \tau) - \tilde{\theta}(t)] \]
Applying (25) and (26) yields
\[ \dot{L} = -\kappa|b|e^2 \]
The proof can be carried out by evaluating the term \( L(t) - L(t - \tau) \), with similar lines to those of the proof for Theorem 1.

Remark 3. It is seen in (24) that an additional component, \( u_1 \), is added into (11), for canceling the term appeared when applying (26).

3. Robust treatments

In this section, we shall provide an approach for analysis of the adaptive system to be developed, in the presence of bounded uncertainty, by considering the class of single-input single-output continuous-time systems
\[ y(n) + \sum_{i=1}^{n_a} a_i Y_i(t, y, \dot{y}, \cdots, y^{(n-1)}) = bu + w \quad \text{(27)} \]
where \( u \) and \( y \) are the scalar input and output of the system, respectively, and \( w \) represents the lumped non-parametric uncertain term; \( a_i, i = 1, 2, \cdots, n_a \), are unknown coefficients, and \( Y_i, i = 1, 2, \cdots, n_a \), represent known nonlinearities, being bounded as \( y, \dot{y}, \cdots, y^{(n-1)} \) are bounded; and \( b \) is the unknown control gain.
By introducing the state vector \( x = [x_1, \cdots, x_n]^T \), and the state space representation for system \((27)\) can be given as follows:

\[
\begin{align*}
\dot{x}_i &= x_{i+1}, \quad i = 1, 2, \cdots, n-1 \\
\dot{x}_n &= -\sum_{i=1}^{n} a_iY_i(t, x) + bu + w \\
y &= x_1 
\end{align*}
\]  

(28)

Let us denote by \( e = x - x_d = [e_1, e_2, \cdots, e_n]^T \) the tracking error, where \( x_d = [y_d, \dot{y}_d, \cdots, y_d^{(n-1)}]^T \), and for \( \lambda > 0 \), \( e_f = (\frac{d}{dt} + \lambda)^{n-1} e_1 \) the filtered error, where \( y_d(t), t \in [0, +\infty) \), is the desired trajectory. The time derivative of \( e_f \) with respect to time is of the form

\[
\dot{e}_f = -\sum_{i=1}^{n} a_iY_i(t, x) + bu + w + \nu \tag{29}
\]

with \( \nu = [0 \ \Lambda^T]e - y_d^{(n)} \) and \( \Lambda = [\lambda^{n-1}, (n-1)\lambda^{n-2}, \cdots, (n-1)\lambda]^T \).

**Assumption 1.** The sign of the control gain \( b \) is known.

As discussed before, we again does not specify whether \( b \) is positive or negative. Define \( \theta = [a_1/b, \cdots, a_n/b, 1/b]^T \) and \( \varphi(t, x) = [-Y_1(t, x), \cdots, -Y_n(t, x), \nu]^T \). Eq. (29) can be rewritten as

\[
\dot{e}_f = b(\theta^T \varphi(t, x) + u + w_b) \tag{30}
\]

where \( w_b = w/b \).

**Assumption 2.** The uncertain term \( w_b \) is assumed to be bounded, satisfying

\[
|w_b| \leq \bar{w}_b \tag{31}
\]

where \( \bar{w}_b = \frac{\bar{w}}{|b|} \), and \( |w| \leq \bar{w} \).

Now we present the robust treatments in forming an incremental adaptation mechanism in the presence of \( w_b \). Let us introduce functions \( \iota_\epsilon(\cdot) \) and \( \varsigma_\epsilon(\cdot) \) as follows:

\[
\iota_\epsilon(\cdot) = \begin{cases}
1 & \text{if } |\cdot| > \epsilon \\
0 & \text{if } |\cdot| \leq \epsilon
\end{cases}
\]  

(32)

and

\[
\varsigma_\epsilon(\cdot) = \begin{cases}
\text{sgn(\cdot)} & \text{if } |\cdot| > \epsilon \\
0 & \text{if } |\cdot| \leq \epsilon
\end{cases}
\]  

(33)

and define the error variable \( e_\epsilon(t) = (|e_f(t)| - \epsilon)\iota_\epsilon(t) \).
Theorem 3. Consider the incremental adaptive system described by the system (27), the controller
\[ u = -\text{sgn}(b)\kappa e\varsigma_e - \text{sgn}(b)\bar{w}_b - \hat{\theta}^T \varphi_e \] (34)
and the adaptation law
\[ \hat{\theta}(t) = \begin{cases} \hat{\theta}(t - \tau) - \text{sgn}(b)\gamma \varphi(t, x(t))e_e(t)e_e(t) & \text{for } t \geq 0 \\ \hat{\theta}_0 & \text{for } t \in [-\tau, 0) \end{cases} \] (35)
where \( \tau, \kappa, \gamma > 0 \) are parameters to be specified by designer, and \( \hat{\theta}_0 \) is the initial setting for \( \hat{\theta} \). Then the error variable \( e_e(t) \) can be made to converge to zero, as time increases, i.e.,
\[ \lim_{t \to +\infty} e_e(t) = 0 \]
while \( e_e, e_f, e, x, \) as well as \( \int_{t-\tau}^t \|\hat{\theta}(s)\|^2 ds \) and \( \int_{t-\tau}^t u^2(s) ds, t \in [\tau, +\infty) \), are all bounded.

Proof. The proof follows similar lines to those of the proof of Theorem 1 with the positive definite function
\[ V = \frac{1}{2} e^2_e \] (36)
By (32) and (33), the derivative of \( V \) with respect to time is calculated as
\[ \dot{V} = e_e \hat{e}_e \] (37)
We take the same positive definite function (13) as a Lyapunov-Krasovskii functional candidate, with the defined \( V(t) \) in (36). The derivative of \( L(t) \) can be calculated as
\[ \dot{L}(t) = V(t) + \frac{|b|}{2\gamma} \left( \hat{\theta}^T(t) \hat{\theta}(t) - \hat{\theta}^T(t - \tau) \hat{\theta}(t - \tau) \right) \] (38)
Using (17) and (37), \( \dot{L}(t) \) given by Eq. (38) satisfies
\[ \dot{L}(t) \leq -|b| \kappa e^2_e(t) - b \hat{\theta}^T(t) \varphi(x(t), t)e_e(t)\varsigma_e(t) - \frac{|b|}{\gamma} \hat{\theta}^T(t)[\hat{\theta}(t) - \hat{\theta}(t - \tau)] \\
- \frac{|b|}{2\gamma} [\hat{\theta}(t) - \hat{\theta}(t - \tau)]^T[\hat{\theta}(t) - \hat{\theta}(t - \tau)] \]
Applying the adaptation law (35), we obtain
\[
\dot{L}(t) \leq -|b| \kappa e_{\epsilon}^2(t) - \frac{1}{2\gamma} [\dot{\theta}(t) - \dot{\theta}(t - \tau)]^T [\dot{\theta}(t) - \dot{\theta}(t - \tau)]
\]
\[
\leq -|b| \kappa e_{\epsilon}^2(t)
\]
\[
= -c_1 V(t)
\]
(39)

where \( c_1 = 2|b|\kappa \).

By (39), the boundedness of \( L \) is ensured, as \( \dot{L} \) is negative semidefinite and \( L(0) \) is bounded. Thus, by the definition of \( L \), \( V \) is bounded, implying the boundedness of \( e_{\epsilon}, e_f, e \), and in turn that of \( x \). Moreover, by the definition of \( L \), \( \int_{t-\tau}^{t} \|\dot{\theta}(s)\|^2 ds \) is bounded for \( t \in [\tau, \infty) \). With the similar derivations to those of the proof for Theorem 1, we can conclude the boundedness of \( \int_{t-\tau}^{t} u^2(s) ds \) for \( t \in [\tau, \infty) \).

To proceed, we recall the expression of the difference of \( L(t) - L(t - \tau) \), given by
\[
L(t) - L(t - \tau) = V(t) - V(t - \tau) + \frac{|b|}{2\gamma} \int_{t-\tau}^{t} (\dot{\theta}^T(s)\dot{\theta}(s) - \dot{\theta}^T(s - \tau)\dot{\theta}(s - \tau)) ds
\]

It follows from (37) that
\[
V(t) - V(t - \tau) \leq -|b|\kappa \int_{t-\tau}^{t} e_{\epsilon}^2(s) ds - b \int_{t-\tau}^{t} \dot{\theta}^T(s)\varphi(x(s), s)e_\epsilon(s)e_\epsilon(s) ds
\]
and from (39),
\[
L(t) - L(t - \tau) \leq -|b|\kappa \int_{t-\tau}^{t} e_{\epsilon}^2(s) ds
\]
\[
- \frac{1}{2\gamma} \int_{t-\tau}^{t} [\dot{\theta}(s) - \dot{\theta}(s - \tau)]^T [\dot{\theta}(s) - \dot{\theta}(s - \tau)] ds
\]
\[
\leq -|b|\kappa \int_{t-\tau}^{t} e_{\epsilon}^2(s) ds
\]
\[
= -c_1 \int_{t-\tau}^{t} V(s) ds
\]

where \( c_1 = 2|b|\kappa \). It follows that
\[
L(t_i) - L(t_0) \leq -c_1 \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} V(s) ds
\]
12
for each fixed instant $t = t_i = i\tau + t_0$, $i = 1, 2, \ldots, t_0 \in [0, \tau)$. Consequently, by the finiteness of $L(t_0)$,

$$\lim_{i \to \infty} \int_{t_{i-1}}^{t_i} V(s)ds = 0$$

implying that

$$\lim_{t \to \infty} \int_{t-\tau}^{t} V(s)ds = 0 \quad (40)$$

Using (37) again,

$$\dot{V}^2 \leq 2[b^2\kappa^2 e^2 + [\tilde{\theta}^T \varphi e_i] e_i^2] \leq 2b^2[\kappa^2 + \|\tilde{\theta}\|^2 \|\varphi\|^2] e^2_i$$

leading to

$$\int_{t-\tau}^{t} \dot{V}^2(s)ds < +\infty \quad (41)$$

for $t \in [\tau, +\infty)$, which holds due to the boundedness of $\int_{t-\tau}^{t} \|\tilde{\theta}(s)\|^2 ds$ for $t \in [\tau, +\infty)$. In view of (40) and (41), by using Lemma 1 we conclude that $\lim_{t \to \infty} V(t) = 0$, and in turn $\lim_{t \to \infty} e(t) = 0$. This completes the proof. ■

**Remark 4.** Theorem 3 indicates the convergence of the error variable $e(t)$ of the incremental adaptive system, as time increases. In addition, whenever $e(t)$ converging to zero, $e(t)$ converges to the interval $(-\epsilon, \epsilon)$, as $t \to \infty$.

4. Concluding remarks

We suggest incremental adaptive mechanisms, in this paper, applicable to develop continuous-time adaptive systems, and illustrate design hints for the development. A comparison between the integral and incremental adaptive systems is made to clarify why our approach makes sense. It is interesting to note that the update term of an incremental adaptation law looks the same as the right-hand side term of the integral adaptation law, and the integral adaptation law can be considered as a kind of incremental adaptive one, where $\tau$, the duration of adaptation, appears in the update term. With
the aid of the Barbalat-like lemma, a unified approach for the analysis of incremental adaptive systems has been presented, by which the convergence has been established in the absence or presence of the disturbance term. In this paper, we present our preliminary results on the incremental adaptation. For future work, we would like to extend it to wide range of situations where the conventional integral adaptive mechanisms are applicable.

Appendix

For purpose of analysis, let us denote \( t = k\tau + \sigma, \sigma \in [0, \tau) \), and \( k = 0, 1, \ldots \). We prove by contradiction. Suppose that we can find \( \sigma_0 \in [0, \tau) \), such that \( g(k\tau + \sigma_0) \) does not converge to zero as \( k \to \infty \). Then we know that there exist a subsequence \( k_i \) and an \( \varepsilon > 0 \) such that

\[
g(k_i\tau + \sigma_0) \geq \sqrt{\varepsilon} + \frac{1}{2}\sqrt{\varepsilon M} \tag{42}
\]

and

\[
\sigma_0 - \frac{\sqrt{\varepsilon}}{2} \geq 0, \quad \sigma_0 + \frac{\sqrt{\varepsilon}}{2} < \tau \tag{43}
\]

Select \( \sigma_1 \in [\sigma_0 - \frac{\sqrt{\varepsilon}}{2}, \sigma_0 + \frac{\sqrt{\varepsilon}}{2}] \). By (9) and Schwarz’s inequality, we obtain

\[
|g(k_i\tau + \sigma_1) - g(k_i\tau + \sigma_0)| = \left| \int_{k_i\tau + \sigma_0}^{k_i\tau + \sigma_1} \dot{g}(s) ds \right| \\
\leq \frac{1}{2} \int_{k_i\tau + \sigma_0 - \sqrt{\varepsilon}/2}^{k_i\tau + \sigma_0 + \sqrt{\varepsilon}/2} |\dot{g}(s)| ds \\
\leq \frac{1}{2} \left( \int_{k_i\tau + \sigma_0 - \sqrt{\varepsilon}/2}^{k_i\tau + \sigma_0 + \sqrt{\varepsilon}/2} ds \int_{k_i\tau + \sigma_0 - \sqrt{\varepsilon}/2}^{k_i\tau + \sigma_0 + \sqrt{\varepsilon}/2} |\dot{g}(s)|^2 ds \right)^{1/2} \\
\leq \frac{1}{2} \sqrt{\varepsilon M} \tag{44}
\]

Combining (42) and (44) yields

\[
|g(k_i\tau + \sigma_1)| \geq |g(k_i\tau + \sigma_0)| - |g(k_i\tau + \sigma_1) - g(k_i\tau + \sigma_0)| \\
\geq \sqrt{\varepsilon} \tag{45}
\]
It follows from (45) that

$$\int_{k_1 \tau}^{(k_i+1)\tau} g(s)ds \geq \int_{k_1 \tau+\sigma_0+\sqrt{\varepsilon}/2}^{k_1 \tau+\sigma_0-\sqrt{\varepsilon}/2} g(s)ds \geq \sqrt{\varepsilon} \sqrt{\varepsilon} = \varepsilon$$

which contradicts to (10). Therefore, for each fixed $\sigma \in [0, \tau)$, $g(k\tau + \sigma)$ converges to zero as $k \to \infty$. This completes the proof.

References

[1] Goodwin G. C., Mayne D. Q.: A parameter perspective of continuous time model reference adaptive control, Automatica, 23 (1987) 57-70.

[2] Middleton R. H., Goodwin G. C., Hill D. J., Mayne D. Q.: Design issues in adaptive control, IEEE Transactions on Automatic Control, 33 (1988) 50-58.

[3] Narendra K. S., Annaswamy A. M.: Stable Adaptive Systems, Englewood Cliffs, NJ: Prentice Hall, 1989.

[4] Sastry S., Bodson M.: Adaptive Control: Stability, Convergence, and Robustness, Englewood Cliffs, NJ: Prentice-Hall, 1989.

[5] Krstic M., Kanellakopoulos I., Kokotovic P. V.: Nonlinear and Adaptive Control Design, NY: Wiley, 1995.

[6] Wen C.: An indirect robust continuous-time adaptive controller with minimal modifications, Automatica, 31 (1995) 293-296.

[7] Ioannou P. A., Sun J.: Robust Adaptive Control, Englewood Cliffs, NJ: Prentice-Hall, 1996.

[8] Tao G.: Adaptive Control Design and Analysis, NY: Wiley, 2003.

[9] Astolfi A., Karagiannidis, D., Ortega, R.: Nonlinear and Adaptive Control with Applications, Springer, Berlin, 2007.

[10] Sun, M., Ge, S. S.: Adaptive repetitive control for a class of nonlinearly parametrized systems. IEEE Transactions on Automatic Control, 51 (2006), 1684-1688.
[11] Sun M.: A Barbalat-like lemma with its application to learning control, IEEE Transactions on Automatic Control, 54 (2009) 2222-2225.

[12] Sun M.: Partial-period adaptive repetitive control by symmetry, Automatica, 48 (2012) 2137-2144.