On Sliced Spaces: Global Hyperbolicity revisited

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Abstract

We give a topological condition for a generic sliced space to be globally hyperbolic, without any hypothesis on the lapse function, shift function and spatial metric.

1. Preliminaries.

We begin with the definition of a sliced space, that one can read in [3], as a continuation of a study in [1] and [2] on systems of Einstein equations.

Let \( V = M \times I \), where \( M \) is an \( n \)-dimensional smooth manifold and \( I \) is an interval of the real line, \( \mathbb{R} \). We equip \( V \) with a \( n + 1 \)-dimensional Lorentz metric \( g \), which splits in the following way:

\[
g = -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j,
\]

where \( \theta^0 = dt \), \( \theta^i = dx^i + \beta^i dt \), \( N = N(t, x^i) \) is the lapse function, \( \beta^i(t, x^j) \) is the shift function and \( M_t = M \times \{t\} \), spatial slices of \( V \), are spacelike submanifolds equipped with the time-dependent spatial metric \( g_t = g_{ij}dx^i dx^j \). Such a product space \( V \) is called a sliced space.
Throughout the paper, we will consider $I = \mathbb{R}$.

The author in [3] considered sliced spaces with uniformly bounded lapse, shift and spatial metric; by this hypothesis, it is ensured that parameter $t$ measures up to a positive factor bounded (below and above) the time along the normals to spacelike slices $M_t$, the $g_t$ norm of the shift vector $\beta$ is uniformly bounded by a number and the time-dependent metric $g_{ij}dx^idx^j$ is uniformly bounded (below and above) for all $t \in I (= \mathbb{R})$, respectively.

Given the above hypothesis, in the same article the following theorem is proved.

**Theorem 1.1** (Cotsakis). Let $(V, g)$ be a sliced space with uniformly bounded lapse $N$, shift $\beta$ and spatial metric $g_t$. Then, the following are equivalent:

1. $(M_0, \gamma)$ a complete Riemannian manifold.
2. The spacetime $(V, g)$ is globally hyperbolic.

In this article we review global hyperbolicity of sliced spaces, in terms of the product topology defined on the space $M \times \mathbb{R}$, for some finite dimensional smooth manifold $M$.

## 2 Strong Causality of Sliced Spaces.

Let $(V = M \times \mathbb{R}, g)$ be a sliced space. Consider the product topology $T_P$, on $V$. Since $M$ is finite-dimensional, a base for $T_P$ consists of all sets of the form $A \times B$, where $A \in T_M$ and $B \in T_\mathbb{R}$. Here $T_M$ denotes the natural topology of the manifold $M$ where, for an appropriate Riemann metric $h$, it has a base consisting of open balls $B^h_r(x)$ and $T_\mathbb{R}$ is the usual topology on the real line, with a base consisting of open intervals $(a, b)$. For trivial topological reasons, we can restrict our discussion on $T_P$ to basic-open sets $B^h_r(x) \times (a, b)$, which can be intuitively called as “open cylinders” in $V$.

We remind the Alexandrov topology $T_A$ (see [4]) has a base consisting of open sets of the form $<x, y> = I^+(x) \cap I^-(y)$, where $I^+(x) = \{z \in V : x \ll z\}$ and $I^-(y) = \{z \in V : z \ll y\}$, where $\ll$ is the chronological order defined as $x \ll y$ iff there exists a future oriented timelike curve, joining $x$ with $y$. By $J^+(x)$ one denotes the topological closure of $I^+(x)$ and by $J^-(y)$ that one of $I^-(y)$.

We use the definition of global hyperbolicity from [4] where the reader can read about global causality conditions in more detail as well as characterisations for strong causality. In
particular, a spacetime is strongly causal, iff it possesses no closed timelike curves and global hyperbolicity is an important causal condition in a spacetime related to major problems such as spacetime singularities, cosmic censorship etc.

**Definition 2.1.** A spacetime is globally hyperbolic, iff it is strongly causal and the “causal diamonds” \( J^+(x) \cap J^-(y) \) are compact.

We prove the following theorem.

**Theorem 2.1.** Let \((V, g)\) be a Hausdorff sliced space. Then, the following are equivalent.

1. \(V\) is strongly causal.
2. \(T_A \equiv T_P\).
3. \(T_A\) is Hausdorff.

**Proof.** 2. implies 3. is obvious and that 3. implies 1. can be found in \([4]\).

For 1. implies 2. we consider two events \(X, Y \in V\), such that \(X \neq Y\); we note that each \(X \in V\) has two coordinates, say \((x_1, x_2)\), where \(x_1 \in M\) and \(x_2 \in \mathbb{R}\). Obviously, \(X \in M_x = M \times \{x\}\) and \(Y \in M_y = M \times \{y\}\). Then, \(\langle X, Y \rangle = I^+(X) \cap I^-(Y) \in T_A\). Let also \(A \in M_a = M \times \{a\}\), where \(a < x\) (\(<\) is the natural order on \(\mathbb{R}\)) and \(B \in M_b = M \times \{b\}\), where \(y < b\). Consider some \(\epsilon > 0\), such that \(B^h(\epsilon)(A) \in M\). Obviously, \(B^h(\epsilon)(A) \times (a, b) \in T_P\) and, for \(\epsilon > 0\) sufficiently large enough, \(\langle X, Y \rangle \subset B^h(\epsilon)(A) \times (a, b)\). Thus, \(\langle X, Y \rangle \in T_P\).

For 2. implies 1. we consider \(\epsilon > 0\), such that \(B^h(\epsilon)(A) \in T_M\), so that \(B^h(\epsilon)(A) \times (a, b) = B \in T_P\). We let strong causality hold at an event \(P\) and consider \(P \in B \in T_P\). We show that there exists \(\langle X, Y \rangle \in T_A\), such that \(P \in \langle X, Y \rangle \subset B\). Now, consider a simple region \(R\) in \(\langle X, Y \rangle\) which contains \(P\) and \(P \in Q\), where \(Q\) is a causally convex-open subset of \(R\). Thus, we have \(U, V \in Q\), such that \(P \in \langle U, V \rangle \subset Q\). Finally, \(P \in \langle U, V \rangle \subset Q \subset B\) and this completes the proof. 

### 3 Global Hyperbolicity of Sliced Spaces, Revisited.

For the following theorem, we use Nash’s result, that refers to finite-dimensional manifolds (see [5]).
Theorem 3.1. Let $(V, g)$ be a Hausdorff sliced space, where $V = M \times R$, $M$ is an $n$-dimensional manifold and $g$ the $n+1$ Lorentz metric in $V$. Then, $(V, g)$ is globally hyperbolic, iff $T_P = T_A$ in $V$.

Proof. Given the proof of Theorem 2.1, strong causality in $V$ holds iff $T_P = T_A$ and given Nash’s theorem, the closure of $B^h_t (x) \times (a, b)$ will be compact. $\square$

We note that neither in Theorem 2.1 nor in Theorem 3.1 we made any hypothesis on the lapse function, shift function or on the spatial metric.

References

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