Y is defined by Tubular surfaces, vector space, and let X be two vectors in R3 be a vector space, respectively.

1. Preliminaries

Let R3 = \{(x1, x2, x3) | x1, x2, x3 ∈ R\} be a 3-dimensional vector space, and let X = (x1, x2, x3) and Y = (y1, y2, y3) be two vectors in R3. The Lorentz scalar product of X and Y is defined by

\[ \langle X, Y \rangle = x_1y_1 - x_2y_2 + x_3y_3. \]  

E1 \(= (R^3, \langle ., . \rangle)\) is called 3-dimensional Lorentzian space, Minkowski space or 3-dimensional semi-Euclidean space. Any X ∈ R3 is named

- spacelike if \(\langle X, X \rangle > 0\) or \(X = 0\),
- timelike if \(\langle X, X \rangle < 0\),
- null if \(\langle X, X \rangle = 0\) and \(X \neq 0\).

Let \(X, Y \in R_3\) and \(s \in I \subset R\).

- The norm of the vector X in R3 is defined as \(\|X\| = |\langle X, X \rangle|^{1/2}\).
- If \(\langle X, Y \rangle > 0\), then the vectors X and Y in R3 are said to be orthogonal.

If \(\|X\| = 1\), X is called a unit vector.

Similarly, if the velocity vector \(\alpha'(s) = T(s)\) at each point \(s\) is locally spacelike, timelike or null (lightlike), then \(\alpha\) is spacelike, timelike or null, respectively.

The Lorentzian vector product of X and Y is defined as

\[ X \wedge Y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1). \]  

Hyperbolic and Lorentzian spheres of center \(M = (m_1, m_2, m_3)\) with radius \(r\) in the space \(E_1^3\) can be written as

\[ S_0^3 = \{A = (a_1, a_2, a_3) \in E_1^3 | \langle A - M, A - M \rangle = -r^2\} \]  

and

\[ S_0^3 = \{A = (a_1, a_2, a_3) \in E_1^3 | \langle A - M, A - M \rangle = r^2\}, \]  

respectively.

If normal vectors at each point of \(M\) are timelike or spacelike vectors, then it is called as spacelike or timelike surface, respectively[6].

Let a curve \(\alpha = \alpha(s) : I \rightarrow E_1^3\) be given by arclength \(s\). We know that its velocity vector is \(T(s) = \alpha'(s) = \frac{d\alpha(s)}{ds}\).
Let as define the unit vector \( N = \frac{T'(s)}{||T'(s)||} \). Finally, define the vector \( B = N \wedge T \). The family \( \{ T, N, B \} \) is orthonormal triad. These three vectors are called the tangent, the principal normal and the binormal vectors, respectively. The family \( \{ T, N, B \} \) is called the Frenet frame.

For a non-lightlike curve \( \alpha \), the rate of change of the Frenet-Serret vector equations may be expressed as

\[
T' = \kappa N, \\
N' = \kappa T + \tau B, \\
B' = \tau N
\]

the coefficients \( \kappa \) and \( \tau \) are the first and the second curvatures of the \( \alpha \), respectively [7].

In \( \mathbb{E}^3 \) curvatures of an arbitrary curve \( X \) is derived as

\[
\kappa = \frac{||X'' \wedge X' ||}{||X'||^3}, \quad \tau = \frac{< X' \wedge X'' , X''' >}{||X' \wedge X''||^2}.
\]

where \( \wedge \) is cross product in \( \mathbb{E}^3 \) [3].

If \( \alpha' \) and \( \alpha'' \) are linearly independent in \( I \), then the curve \( \alpha \) is said to be good [8].

From now on, we will assume that the given curves are good curves.

Let

\[
f(s) = \frac{1}{2} \left( ||C_\alpha - \alpha||^2 - r^2 \right).
\]

If there exist infinitely close joint 4-points between the curve \( \alpha \) with its osculating sphere at \( s = s_0 \) then we have

\[
f(s_0) = f'(s_0) = f''(s_0) = f'''(s_0) = 0.
\]

The sphere, \( ||C_\alpha - \alpha||^2 = r^2 \), with the center \( C_\alpha \) obtained in this way is called the osculating Lorentzian sphere.

The plane spanned by the tangent vector and the principle normal vector of a curve is called the osculating plane. A point of a smooth curve in \( \mathbb{E}^3_1 \) for which the derivative of the curve of order 3 belongs to the osculating plane is called a flattening.

If there exist infinitely close 5-points in the neighbourhood of a point with the osculator sphere at \( s = s_0 \) of the curve \( \alpha \), it is called a vertex of the curve. Conversely, if there does not exist infinitely close 5-points in the neighbourhood of a point with the osculator sphere at \( s = s_0 \) of the curve \( \alpha \), it is called a non-vertex of the curve.

From now on, we assume that all points of the given curves are non-vertex.

2. Focal Curves in \( \mathbb{E}^3_1 \)

In this section, we will show that, in \( \mathbb{E}^3_1 \) it is possible to obtain a Lorentzian tubular surface around a spacelike focal curve.

**Definition 2.1.** [9] Let \( \alpha = \alpha(s) : I \rightarrow \mathbb{E}^3_1 \) be any curve. That the points of \( C_\alpha \) are the centres of the osculating spheres of \( \alpha \) is called the focal curve of \( \alpha \).

**Lemma 2.2.** Let \( \alpha \) be a spacelike curve with spacelike binormal in \( \mathbb{E}^3_1 \) and its Frenet frame be \( \{ T(s), N(s), B(s) \} \). Then the focal curve \( C_\alpha \) of \( \alpha \) is

\[
C_\alpha = \alpha + c_1 N + c_2 B
\]

and the focal coefficients of \( C_\alpha \) are given by

\[
c_1 = -\frac{1}{\kappa}, \quad c_2 = \frac{1}{\tau}
\]

where \( \kappa \neq 0 \) and \( \tau \neq 0 \) are the first and the second curvatures of the curve \( \alpha \).

**Proof.** We can always write the vector \( C_\alpha - \alpha \) with respect to the linear independence vectors \( \{ T(s), N(s), B(s) \} \). Namely

\[
C_\alpha - \alpha = c_0 T + c_1 N + c_2 B
\]

If we take the Lorentz scalar product with \( T, N \) and \( B \) both sides of equation (7), then

\[
< T, C_\alpha - \alpha > = c_0, \\
< N, C_\alpha - \alpha > = -c_1, \\
< B, C_\alpha - \alpha > = c_2.
\]

On the other hand by using equation (4), we may write

\[
f = 0 \Rightarrow < C_\alpha - \alpha, C_\alpha - \alpha > = r^2, \\
f' = 0 \Rightarrow < T, C_\alpha - \alpha > = 0, \\
f'' = 0 \Rightarrow < N, C_\alpha - \alpha > = \frac{1}{\kappa}, \\
f''' = 0 \Rightarrow < B, C_\alpha - \alpha > = \frac{1}{\tau}.
\]

Making use of the equations \( c_1 = \frac{1}{\kappa} \) and \( c_2 = \frac{1}{\tau} \). Finally, we may write the focal curve as

\[
C_\alpha(s) = \alpha(s) - \frac{1}{\kappa(s)} N(s) + \left( -\frac{1}{\kappa(s)} \right)' \frac{1}{\tau(s)} B(s).
\]

**Lemma 2.3.** Let \( \alpha = \alpha(s) : I \rightarrow \mathbb{E}^3_1 \) be a spacelike curve with spacelike binormal. If a non-flattening point of \( \alpha \) is a vertex, then

\[
c'_2 + c_1 \tau = 0.
\]

Converse is also true.

**Proof.** The equation of the Lorentzian spheres with center at \( C_\alpha \) is

\[
f(s) = \frac{1}{2} \left( ||C_\alpha - \alpha||^2 - r^2 \right).
\]

If there exist infinitely close 5-points between \( \alpha \) and its osculating sphere at \( s = s_0 \), then we have

\[
f(s_0) = f'(s_0) = f''(s_0) = f'''(s_0) = f^{(4)}(s_0) = 0.
\]

Calculating these derivatives we easily obtain the desired result \( c'_2 + c_1 \tau = 0 \).

The forthcoming theorem, lemmas and corollaries state the relations between \( \alpha \) and its focal curve \( C_\alpha \).

**Theorem 2.4.** Let \( \alpha : I \rightarrow \mathbb{E}^3_1 \) be a spacelike curve with spacelike binormal. Let \( \{ T, N, B \} \) be the Frenet frame to \( \alpha \) (resp. \( C_\alpha \)). Let \( \kappa \) and \( \tau \) be first and
second curvatures of \( \alpha \), respectively. Then we have the connections
\[
\begin{align*}
t &= \varepsilon_1 B, \quad (8) \\
n &= \varepsilon_1 \varepsilon_2 N, \quad (9) \\
b &= -\varepsilon_1 T, \quad (10)
\end{align*}
\]
between \( \{T,N,B\} \) and \( \{t,n,b\} \) where
\[
\varepsilon_1 = \frac{c_2 + c_1 \tau}{|c_2 + c_1 \tau|}, \quad \varepsilon_2 = \frac{\tau}{|\tau|}.
\]

**Proof.** Let \( \sigma \) be the arclength parameter of the focal curve \( C_\alpha \). If we take the derivative of both sides of (5) with respect to the arclength parameter \( \sigma \), we have
\[
\frac{dC_\alpha}{d\sigma} = \frac{dC_\alpha}{d\sigma} \frac{d\sigma}{ds} = \left[ c_2 + c_1 \tau \right] B, \quad (11)
\]
and if we take the norm of both sides of (11), we get
\[
ds = \frac{1}{|c_2 + c_1 \tau|},
\]
and if \( \varepsilon_1 = \frac{c_2 + c_1 \tau}{|c_2 + c_1 \tau|} \) then
\[
t = \varepsilon_1 B = \left( \frac{c_2 + c_1 \tau}{|c_2 + c_1 \tau|} \right) B = \frac{dC_\alpha}{d\sigma}. \quad (12)
\]

Now, differentiating both sides of (12) with respect to the arclength parameter \( s \), we obtain
\[
n = \varepsilon_1 \varepsilon_2 N, \quad (13)
\]
and
\[
\kappa_\tau = |\tau| \frac{c_2 + c_2 \tau}{|c_2 + c_1 \tau|}. \quad (14)
\]
On the other hand, we may write
\[
b = t \wedge n = (\varepsilon_1 B) \wedge (\varepsilon_1 \varepsilon_2 N)
\]
and
\[
b = -\varepsilon_1 T. \quad (15)
\]
Then, taking the derivative of (15) with respect to the arclength parameter \( s \), we obtain
\[
\kappa = |\tau| \frac{c_2 + c_1 \tau}{|c_2 + c_1 \tau|}. \quad (16)
\]

**Corollary 2.5.** Let \( \alpha = \alpha(s) : I \rightarrow \mathbb{E}^3 \) be a spacelike curve with spacelike binormal. If the curve \( \alpha \) is Lorentzian spherical, then
\[
r^2 = \|C_\alpha - \alpha\|^2 = \|c_1 N + c_2 B\|^2 = c_2^2 - c_1^2,
\]
where \( r \) is radius of the Lorentzian spherical and differentiating the last equation with respect to the arclength parameter \( s \) we get
\[
(r^2)' = 2c_2 (c_2' + c_1 \tau), \quad (17)
\]
Converse is also true. According to equation (17), if \( r \) is a constant, then
\[
c_2 = 0.
\]
Because the curve \( \alpha \) is a non-vertex curve, \( c_2' + c_1 \tau \neq 0 \).

**Corollary 2.6.** If we consider equations (17) and (17), the focal coefficients of \( c_1 \), \( c_2 \) of the curve \( \alpha \) satisfy the following matrix-vector equation
\[
\begin{bmatrix}
1 \\
\frac{c_1'}{c_2 - \frac{(r^2)'}{2c_2}}
\end{bmatrix} =
\begin{bmatrix}
0 & -\kappa & 0 \\
\kappa & 0 & \tau \\
0 & -\tau & 0
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
\]
If the curve \( \alpha \) is spherical, \( (r^2)' = 0 \).

According to this, we can express the following corollary.

**Corollary 2.7.** Let \( \kappa \) and \( \tau \) (resp. \( \kappa_\tau \) and \( \tau_\tau \)) be the first and the second curvatures of \( \alpha \) (resp. the first and the second curvatures of the focal curve \( C_\alpha \)). If we consider equations (14) and (16), then
\[
\kappa_\tau = |\tau| \frac{c_2}{|c_2 + c_1 \tau|} = \frac{2|c_2|}{|c_2' + c_1 \tau|}. \quad (18)
\]

**Corollary 2.8.** Because \( \det(t,n,b) = 1 \), the focal curve \( C_\alpha \) is a right-handed curve.

From now on, we assume that the ranking of \( \{t,n,b\} \) will be \{space, time, space\} or \{space, space, time\} type.

**Lemma 2.9.** Let \( r \) be the radius of Lorentzian osculating sphere. If \( r \) is constant, then \( \kappa \) is constant and
\[
r = |c_1| = \frac{1}{\kappa},
\]
where \( \kappa \) and \( c_1 \) are first curvature of the curve \( \alpha \) and the first focal coefficient of the focal curve \( C_\alpha \), respectively.

**Proof.** Since \( r \) is constant equation (17) implies either \( c_2 = 0 \) or \( c_2' + c_1 \tau = 0 \). If \( c_2' + c_1 \tau = 0 \), then the curve is spherical. If \( c_2 = 0 \), \( c_1 = 0 \). This means that \( c_1 = -\frac{1}{\kappa} \) is constant.

**Lemma 2.10.** If we take the derivative of the Frenet frame \( \{t,n,b\} \) of the focal curve \( C_\alpha \) with respect to the arclength parameter \( s \), we have
\[
\begin{bmatrix}
t' \\
n' \\
b'
\end{bmatrix} =
\begin{bmatrix}
0 & \nu \kappa & 0 \\
\nu \kappa & 0 & \nu \tau \\
0 & \nu \tau & 0
\end{bmatrix}
\begin{bmatrix}
t \\
n \\
b
\end{bmatrix},
\]
where \( \nu = \frac{d\sigma}{ds} = |c_2' + c_1 \tau| \). If the radius of the osculating sphere \( r \) is constant, then
\[
\nu = \frac{d\sigma}{ds} = r |\tau|,
\]
where \( s \) and \( \sigma \) are the arclength parameters of the curve \( \alpha \) and the focal curve \( C_\alpha \), respectively.

Now, let us state the equations for canal and tubular surfaces around any good curve in \( \mathbb{E}^3 \).
3. Canal Surfaces in $\mathbb{E}_3^1$

Let us recall the definitions and the results of [1, 9]. A canal surface is named as the envelope of a family of 1-parameter spheres. In other words, it is the envelope of a moving sphere with varying radius, defined by the trajectory with center $\alpha(t)$ and a radius function $r(t)$. This moving sphere $S(t)$ touches it at a characteristic circle $K(t)$. If the radius function $r(t) = r$ is a constant, then it is called a tubular or pipe surface. Let $\{T, N, B\}$ be the Frenet vector fields of $\alpha$, where $T, N$ and $B$ are tangent, principal normal and binormal vectors to $\alpha$, respectively. Since the canal surface $K(t, \theta)$ is the envelope of a family of one parameter spheres with the center $\alpha$ and radius function $r$, it is parametrized as

$$K(t, \theta) = \alpha(t) - r(t)r'(t) \frac{\alpha'(t)}{\|\alpha'(t)\|} \pm \cos \theta r(t) \sqrt{\|\alpha'(t)\|^2 - r'(t)^2} N(t)$$
$$\pm \sin \theta r(t) \sqrt{\|\alpha'(t)\|^2 - r'(t)^2} B(t).$$

This surface is called the canal surface around the curve $\alpha$. Clearly, $N(t)$ and $B(t)$ are spanning the plane that contains the characteristic circle. If the spine curve $\alpha(s)$ has an arclength parametrization $\left(\|\alpha'(s)\| = 1\right)$, then the canal surface is reparametrized as

$$K(s, \theta) = \alpha(s) - r(s)r'(s)T(s) \pm \cos \theta r(s) \sqrt{1 - r'(s)^2} N(s)$$
$$\pm \sin \theta r(s) \sqrt{1 - r'(s)^2} B(s).$$

For the constant radius case $r(s) = r$, the canal surface is called a tubular (pipe) surface and in this case the equation takes the form

$$K(s, \theta) = \alpha(s) + r(\cos \theta N(s) + \sin \theta B(s)),$$

where $0 \leq \theta \leq 2\pi$.

Let a regular curve $\alpha : I \longrightarrow M$ be parametrized so that $\|\alpha'(s)\| = 1$. Then we have

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix},$$

where $\kappa$ and $\tau$ are the curvature and the torsion of the curve $\alpha(s)$, respectively.

Now, let us see what happens if we take the focal curve $C_\alpha$ of $\alpha$ instead of the curve $\alpha$ itself in $\mathbb{E}_3^1$.

4. Canal Surfaces in $\mathbb{E}_3^1$

Now, we state and prove an important theorem related to our present study. However, first we need the following definition.

**Definition 4.1.** A canal surface in $\mathbb{E}_3^1$ is named as the envelope of a family of 1-parameter Lorentzian spheres. In other words, it is the envelope of a moving Lorentzian sphere with varying radius, defined by the trajectory with center $C_\alpha(s)$ and a radius function $r(t)$. This moving sphere $S(t)$ touches it at a Lorentzian characteristic circle $K(t)$. If the radius function $r(t) = r$ is a constant, then it is called a Lorentzian tubular or pipe surface in $\mathbb{E}_3^1$.

**Theorem 4.2.** Let $\alpha = \alpha(s) : I \longrightarrow \mathbb{E}_3^1$ be a spacelike curve with spacelike binormal. Then, the canal surface around its spacelike focal curve $C_\alpha(s)$ can be parametrized as follows

$$K(s, t) = C_\alpha(s) - \frac{r(s)r'(s)}{\nu} B(s)$$
$$\mp \nu(t) r(s) \sqrt{1 - \left(\frac{r'(s)}{\nu}\right)^2} T(s)$$
$$\pm \nu(t) r(s) \sqrt{1 - \left(\frac{r'(s)}{\nu}\right)^2} N(s).$$

**Proof.** Let $K$ be any point of the canal surface and $C_\alpha$ be the center of a Lorentzian spheres $S_\alpha^2(s)$. Then the difference $K(s, t) - C_\alpha(s)$ can be written in terms of the orthogonal vectors $\{t, n, b\}$ as

$$K(s, t) - C_\alpha(s) = c(s, t) t(s) + b(s, t) n(s) + a(s, t) b(s).$$

By using the connections in (8), the last equation can be rewritten as

$$K(s, t) - C_\alpha(s) = -a(s, t) e_n T(s) - b(s, t) e_t e_n N(s)$$
$$+ c(s, t) e_b B(s).$$

where $a, b$ and $c$ have partial derivatives with respect to the variables $s$ and $t$ on $I$. On the other hand, taking the norm of both sides of equation (18) we obtain

$$\|K(s, t) - C_\alpha(s)\|^2 = r^2(s).$$

The equation (19) expresses that $K(s, t)$ lies on a Lorentzian sphere $S_\alpha^2(s)$. Additionally, $K(s, t) - C_\alpha(s)$ is an orthogonal vector to the canal surface which means that

$$<K(s, t) - C_\alpha(s), K_t> = 0,$$
$$<K(s, t) - C_\alpha(s), K_r> = 0.$$

The equations in (20) and (21) indicate that velocity vector of parameter curves $K_t$ and $K_r$ of the canal surface are
tangent to $S_t^2(s)$. By making use of (18) and (19), we immediately obtain the equations
\[
\begin{align*}
    a^2 - b^2 + c^2 &= r^2, \\
aa_s - bb_s + cc_s &= rr'.
\end{align*}
\] (22)

Using the partial derivative
\[
K_s = \left( -a_t\epsilon_n + b_t\epsilon_n\nu \right) T \\
+ \left( -a_t\epsilon_n + b_t\epsilon_n + c_t\nu \right) N \\
+ \left( a_t\epsilon_n + b_t\epsilon_n\nu + c_t\epsilon_n \right) B
\] (23)
of (18) with respect to $s$, we may rewrite equation (20) as
\[
<K(s,t) - C_\alpha(s), K_s> = aa_t - bb_s + cc_s + cv = 0. \tag{24}
\]
Then equation (22) together with (24), lead to the equalities
\[-cv = rr'
\]
and
\[
a^2 - b^2 = r^2 \left[ 1 - \left( \frac{r'}{v} \right)^2 \right]. \tag{25}
\]

from which we obtain
\[
a = \mp r \sqrt{1 - \left( \frac{r'}{v} \right)^2} \cosh t,
\]
\[
b = \mp r \sqrt{1 - \left( \frac{r'}{v} \right)^2} \sinh t.
\]

If we substitute these values of $a$ and $b$ in (18), we obtain the equation
\[
K(s,t) = C_\alpha(s) - \frac{\epsilon_n r'(s)r'(s)}{v} B(s)
\]
\[
\mp \epsilon_n (\cosh t)r(s) \sqrt{1 - \left( \frac{r'(s)}{v} \right)^2} T(s)
\]
\[
\pm \epsilon_t \epsilon_n (\sinh t)r(s) \sqrt{1 - \left( \frac{r'(s)}{v} \right)^2} N(s). \tag{26}
\]

If the radius $r$ is constant, the equation (26) takes the form
\[
L(s,t) = C_\alpha(s) + \epsilon_n \cosh t T(s) - \epsilon_t \epsilon_n \sinh t N(s). \tag{27}
\]
This means that equation (27) is the Lorentzian tubular surface with parameters $s$ and $t$. Without loss of generality, in (27) we can take $\epsilon_t = \epsilon_n = 1$. With this choice, (27) reads as
\[
L(s,t) = C_\alpha(s) + r \cosh t T(s) - r \sinh t N(s). \tag{28}
\]

In the next section, we give the fundamental forms which are crucial for the characterization of the Lorentzian tubular surfaces.

5. Fundamental Forms

Let $\alpha = \alpha(s) : I \rightarrow E^3$ be any unit speed spacelike curve with spacelike binormal. A parametrization $L(s,t)$ of the Lorentzian tubular surface around its spacelike focal curve $C_\alpha(s)$ has given in (28). The partial derivatives of $L$ with respect to the surface parameters $s$ and $t$ can be expressed in terms of Frenet vector fields of $\alpha$ as
\[
L_s = - \sinh t T + \cosh t N + r t (1 - \sinh t) B, \\
L_t = r \cosh t T - r \sinh t N.
\]
We can also choose a unit normal vector field $U$ as
\[
U = \frac{L_s \wedge L_t}{||L_s \wedge L_t||} = \cosh t T - \sinh t N,
\]
where we know that
\[
||L_s \wedge L_t||^2 = EG - F^2 = r^4 t^2 (1 - \sinh t)^2. \tag{29}
\]
The first fundamental form $I$ of $L$ is defined as
\[
I = Edx^2 + 2F dx dy + Gdy^2
\]
in which
\[
E = <L_s, L_s> = 1 + r^2 \tau^2 (1 - \sinh t)^2, \\
F = <L_s, L_t> = r, \\
G = <L_t, L_t> = r^2.
\]
On the other hand, the second fundamental form $II$ of $L$ is defined as
\[
II = edx^2 + 2fdx dy + gdy^2
\]
in which
\[
e = <U, L_s> = \kappa + r \tau^2 \sinh t (1 - \sinh t), \\
f = <U, L_t> = -1, \\
g = <U, L_t> = r.
\]

Corollary 5.1. The tubular surface in (28) is a timelike surface.

Definition 5.2. [1] Let $M$ be any surface and the set $\{E, F, G\}$ be the coefficients of its first fundamental form. $M$ is called a regular surface if $EG - F^2 \neq 0$.

Lemma 5.3. $L(s,t)$ is a regular tube, iff $\sinh t \neq 1$.

Proof. It can easily be proved by using equation (29) and definition 5.2. \qed

Theorem 5.4. The mean and the Gaussian curvatures of a regular surface $L(s,t)$ are
\[
H = \frac{eG - f^2 F + gE}{2(EG - F^2)} = \frac{1}{2} \left( r \kappa - \frac{1}{r} \right) \tag{30}
\]
and
\[
K = \frac{eg - f^2}{EG - F^2} = -\frac{\sinh t}{r^2 (1 - \sinh t)} \tag{31}
\]
respectively.
6. Some Special Parameter Curves on The Lorentzian Tubular Surfaces in $\mathbb{E}_1^3$

**Theorem 6.1.** [5] Let the curve $\gamma$ lie on a surface. If $\gamma$ is an asymptotic curve, then the acceleration vector is orthogonal to the normal vector of the surface.

**Theorem 6.2.** Let $L(s, t)$ be a Lorentzian tubular surface around spacelike focal curve of $\alpha(s)$, then the curves $L_s$ and $L_t$ cannot be asymptotic.

Proof. For the $s-$parameter curves we obtain the first coefficient $e$ of second fundamental form as

$$e = \langle U, L_{ss} \rangle = (\kappa + r^2 \sinh t) (1 - \sinh t) \neq 0$$

showing that they cannot be asymptotic. Similarly, for the $t-$parameter curves we obtain the third coefficient $g$ of second fundamental form as

$$g = \langle U, L_{tt} \rangle = r \neq 0$$

which implies that they cannot be asymptotic. \qed

**Theorem 6.3.** [2] Let the curve $\gamma$ lie on a surface. If $\gamma$ is a geodesic curve, then the acceleration vector $\gamma''$ and the normal vector $U$ of the surface are linearly dependent. That is, $U \wedge \gamma' = 0$.

**Theorem 6.4.** Let $L(s, t)$ be a Lorentzian tubular surface around a spacelike focal curve of $\alpha(s)$, then

1. The $L_s$ curves can not be geodesic.
2. The $L_t$ curves are geodesic curves.

Proof. For the $s-$parameter curves, we have

$$U \wedge L_{ss} = - \sinh t \left[ \tau \cosh t + r \tau' (1 - \sinh t) \right] T + \cosh t \left[ \tau \cosh t + r \tau' (1 - \sinh t) \right] N - r \tau^2 (1 - \sinh t) \cosh t B.$$  \hspace{1cm} (32)

If the last equation were zero, i.e., $U \wedge L_{ss} = 0$, we would have

$$\sinh t \left[ \tau \cosh t + r \tau' (1 - \sinh t) \right] = 0, \hspace{1cm} \cosh t \left[ \tau \cosh t + r \tau' (1 - \sinh t) \right] = 0, \hspace{1cm} r \tau^2 (1 - \sinh t) \cosh t = 0 \hspace{1cm} (32)$$

since the vectors $\{T, N, B\}$ are linearly independent. However, since $L(s, t)$ is a regular surface, equation (32) cannot be zero. Therefore $U \wedge L_{ss} \neq 0$ which shows that $L_s$ curves cannot be geodesics. On the other hand, since

$$U \wedge L_{tt} = U \wedge rU = 0 \hspace{1cm} (33)$$

the $t-$parameter curves $L_t$ are geodesics. Converse is also true and it is trivial. \qed

**Example 6.5.** Let $\gamma$ be a spacelike curve in $\mathbb{E}_1^3$ defined by

$$\gamma: \mathbb{E}_1^3 \ni s \mapsto \gamma(s) = \left( \sinh s \frac{s}{\sqrt{2}}, \cosh s \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right),$$

where $-4 \leq s \leq 4$. Figure 2 includes the graph of the curve.

![Figure 2. The curve $\gamma$ of Example 6.5.](image)

Its velocity vector of the curve is

$$\gamma'(s) = \left( \frac{1}{\sqrt{2}} \cosh s \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \sinh s \frac{1}{\sqrt{2}}, 1 \right).$$

In this example, we will consider the Lorentz scalar product in (1) and the Lorentzian vectorial product in (2). The Frenet vectors $\{T, N, B\}$ of the curve $\gamma$ are

$$T = \left( \frac{1}{\sqrt{2}} \cosh s \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \sinh s \frac{1}{\sqrt{2}}, 1 \right),$$

$$N = \left( \sinh s \frac{s}{\sqrt{2}}, \cosh s \frac{s}{\sqrt{2}}, 0 \right),$$

$$B = \left( - \frac{1}{\sqrt{2}} \cosh s \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \sinh s \frac{s}{\sqrt{2}}, 1 \right).$$

![Figure 3. The focal curve $C_\gamma$ of the curve $\gamma$ in Example 6.5.](image)

The curvatures of $\gamma$ are found to be $\kappa = \frac{1}{2}$ and $\tau = -\frac{1}{4}$ by making use of the equation in (3). Hence, $\tau/\kappa$ is constant. Therefore, the curve $\gamma$ is the Lorentz circular helix in $\mathbb{E}_1^3$. The focal coefficients of $\gamma$ can be computed from (6) as $C_1 = -2$ and $C_2 = 0$. For this specific example, by using (5), the focal curve $C_\gamma$ of $\gamma$ may be computed as

$$C_\gamma = \gamma - 2N.$$

The last equation and equation (28) with $r = 2$ lead to the components

$$C_\gamma = \gamma - 2N.$$
\[ x(s,t) = -\sinh \frac{s}{\sqrt{2}} + \frac{2}{\sqrt{2}} \cosh t \cosh \frac{s}{\sqrt{2}} - 2\sinh t \sinh \frac{s}{\sqrt{2}} \]

\[ y(s,t) = -\cosh \frac{s}{\sqrt{2}} + \frac{2}{\sqrt{2}} \cosh t \sinh \frac{s}{\sqrt{2}} - 2\sinh t \cosh \frac{s}{\sqrt{2}} \]

\[ z(s,t) = \frac{s}{\sqrt{2}} + \frac{2}{\sqrt{2}} \cosh t. \]

of the tubular surface \( L(s,t) = (x(s,t), y(s,t), z(s,t)) \).

Tubular surface around the focal curve \( C_\gamma \) is shown in Figure 3.

References

[1] Do˘gan, F., Yaylı, Y. 2011. On the curvatures of tubular surface with Bishop frame. Commun. Fac. Sci. Univ. Ank. Series A1 Volume 60 (1) (2011), 59-69.

[2] Hacısılaıhoğlu, H. H. 2000. Diferensiyel Geometri II. Hacısılaıhoğlu Yayınları, 45-46.

[3] Karacan, M. K., Bukcu, B. 2007. An alternative moving frame for tubular surfaces around spacelike curves with a spacelike Binormal in the Minkowski 3-space. Mathematica Moravica, Volume 11, 47-54.

[4] Kühnel, W. 2003. Differential Geometry Curves-Surfaces-Manifolds, Second ed. Friedr. Vieweg & Sohn Verlag, 380 p.

[5] Oprea, J. 1997. Differential Geometry and Its Applications. Prentice-Hall Inc, 78-80.

[6] Özdemiı, M., Ergin, A. A. 2007. Spacelike Darboux curves in Minkowski 3-space. Differential Geometry-Dynamical Systems, vol. 9, 131-137.

[7] Petrovic, M., Sucurovic, E. 2000. Some characterizations of the spacelike, the timelike and the null curves on the pseudohyperbolic space \( H^2_0 \) in \( E^3_1 \). Kragujevac J.Math, vol 22, 71-82.

[8] Uribe-Vargas, R. 2005. On vertices focal curvatures and differential geometry of space curves. Bull. Brazilian Math. Soc. 36 (3), 285–307.

[9] Yıldırım, A. 2016. Tubular surface around a Legendre curve in BCV spaces. New Trends in Mathematical Sciences, 4(2), 61-71. (04.10.2017)