A note on potential diagonalizability of crystalline representations

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Abstract Let \( K_0/\mathbb{Q}_p \) be a finite unramified extension, \( G_{K_0} \) the Galois group \( \text{Gal}(\mathbb{Q}_p/K_0) \). We show that all crystalline representations of \( G_{K_0} \) with Hodge-Tate weights \( \subseteq \{0, \ldots, p-1\} \) are potentially diagonalizable.

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1 Introduction

Let \( p \) be a prime, \( K \) a finite extension over \( \mathbb{Q}_p \) and \( G_K \) the absolute Galois group \( \text{Gal}(\overline{\mathbb{Q}}_p/K) \). In [1] §1.4, potential diagonalizability is defined for a potentially crys-

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talline representation of $G_K$. Since potential diagonalizability is the local condition at $p$ for a global Galois representation in the automorphy lifting theorems proved in [1] (cf. Theorem B, C), it is quite interesting to investigate what kind of potentially crystalline representations are indeed potentially diagonalizable. Let $K_0$ be a finite unramified extension of $\mathbb{Q}_p$. By using Fontaine–Laffaille’s theory, Lemma 1.4.3 (2) in [1] proved that any crystalline representation of $G_{K_0}$ with Hodge-Tate weights in $\{0, \ldots, p-2\}$ is potentially diagonalizable.

In this short note, we show that the idea in [1] can be extended to prove the potential diagonalizability of crystalline representations of $G_{K_0}$ with Hodge-Tate weights in $\{0, \ldots, p-1\}$. Let $\rho : G_{K_0} \to \text{GL}_d(\mathbb{Q}_p)$ be a crystalline representation with Hodge-Tate weights in $\{0, \ldots, p-1\}$. To prove the potential diagonalizability of $\rho$, we first reduce to the case that $\rho$ is irreducible. Then $\rho$ is nilpotent (see definition in Sect. 2.2). Note that Fontaine–Laffaille’s theory can be extended to nilpotent representations. Hence we can follow the similar idea in [1] to conclude the potential diagonalizability of $\rho$.

Notations

Throughout this note, $K$ is always a finite extension of $\mathbb{Q}_p$ with the absolute Galois group $G_K := \text{Gal}(\overline{\mathbb{Q}}_p / K)$. Let $K_0$ be a finite unramified extension of $\mathbb{Q}_p$ with residue field $k$. We denote $W(k)$ its ring of integers and $\text{Frob}_{W(k)}$ the arithmetic Frobenius on $W(k)$. If $E$ is a finite extension of $\mathbb{Q}_p$ then we write $\mathcal{O}$ the ring of integers, $\varpi$ its uniformizer and $E = \mathcal{O}/\varpi \mathcal{O}$ its residue field. If $A$ is a local ring, we denote $m_A$ the maximal ideal of $A$ and equip $A$ with the $m_A$-adic topology. Let $\rho : G_K \to \text{GL}_d(A)$ be a continuous representation with the ambient space $M = \bigoplus_{i=1}^d A$. We always denote $\rho^*$ the dual representation induced by $\text{Hom}_A(M, A)$. Let $\rho : G_K \to \text{GL}_d(A)$ be a de Rham representation of $G_K$. Then $D_{\text{dR}}(\rho^*)$ is a filtered $K \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$-module. For any embedding $\tau : K \to \overline{\mathbb{Q}}_p$, we define the set of $\tau$-Hodge-Tate weights

$$HT_{\tau}(\rho) := \{i \in \mathbb{Z} | \text{gr}^i(D_{\text{dR}}(\rho^*)) \otimes_{K \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p} (K \otimes_{K, \tau} \overline{\mathbb{Q}}_p) \neq 0\}.$$

In particular, if $\epsilon$ is the $p$-adic cyclotomic character then $HT_{\tau}(\epsilon) = \{1\}$ (here our convention is slightly different from that in [1]).

2 Definitions and preliminaries

2.1 Potential diagonalizability

We recall the definition of potential diagonalizability from [1]. Given two continuous representations $\rho_1, \rho_2 : G_K \to \text{GL}_d(O_{\overline{\mathbb{Q}}_p})$, we say that $\rho_1$ connects to $\rho_2$, denoted by $\rho_1 \sim \rho_2$, if:

- The two reductions $\tilde{\rho}_i := \rho_i \mod m_{\mathcal{O}_{\overline{\mathbb{Q}}_p}}$ are equivalent to each other;
- Both $\rho_1$ and $\rho_2$ are potentially crystalline;

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• For each embedding \( \tau : K \hookrightarrow \bar{Q}_p \), we have \( HT_{\tau}(\rho_1) = HT_{\tau}(\rho_2) \);
• \( \rho_1 \) and \( \rho_2 \) define points on the same irreducible component of the scheme \( Spec(R_{\rho_1,\mathbf{HT}}(1),K^{\prime}risc\{1\}) \) for some sufficiently large field extension \( K'/K \).

Here \( R_{\rho_1,\mathbf{HT}}(1),K^{\prime}risc \) is the quotient of the framed universal deformation ring \( R_{\rho_1} \) corresponding to liftings \( \rho \) with \( HT_{\tau}(\rho) = HT_{\tau}(\rho_1) \) for all \( \tau \) and with \( \rho |_{G_{K'}} \) crystalline. The existence of \( R_{\rho_1,\mathbf{HT}}(1),K^{\prime}risc \) is the main result of [7].

Clearly the relation \( \sim \) is an equivalence relation. A representation \( \rho : G_K \to GL_d(\mathcal{O}_{\bar{Q}_p}) \) is called diagonalizable if it is crystalline and connects to a sum of crystalline characters \( \chi_1 \oplus \cdots \oplus \chi_d \). It is called potentially diagonalizable if \( \rho |_{G_{K'}} \) is diagonalizable for some finite extension \( K'/K \).

**Remark 2.1.1** By Lemma 1.4.1 of [1], the potential diagonalizability is well defined for a representation \( \rho : G_K \to GL_d(\mathcal{O}_{\bar{Q}_p}) \) because for any two \( G_K \)-stable \( \mathcal{O}_{\bar{Q}_p} \)-lattices \( L \) and \( L' \), \( L \) is potentially diagonalizable if and only if \( L' \) is potentially diagonalizable.

**Lemma 2.1.2** Suppose \( \rho : G_K \to GL_d(\mathcal{O}_{\bar{Q}_p}) \) is potentially crystalline. Let \( Fil^i \) be a \( G_K \)-invariant filtration on \( \rho \). Then \( \rho \) is potentially diagonalizable if and only if \( \oplus_i \text{gr}^i \rho \) is potentially diagonalizable.

**Proof** We can always choose a \( G_K \)-stable \( \mathcal{O}_{\bar{Q}_p} \)-lattice \( M \) inside the ambient space of \( \rho \) such that \( Fil^i \rho \cap M \) is an \( \mathcal{O}_{\bar{Q}_p} \)-summand of \( M \) and the reduction \( \bar{M} \) is semi-simple. Then the lemma follows item (7) of the numbered list preceding Lemma 1.4.1 of [1]. \( \square \)

2.2 Nilpotency and Fontaine–Laffaille data

Let \( E \) be a finite extension of \( \mathbb{Q}_p \). Recall that we write \( \mathcal{O} \) the ring of integers, \( \sigma \) its uniformizer and \( \mathbb{F} = \mathcal{O}/\sigma \mathcal{O} \) its residue field. Write \( W(k) := W(k) \otimes_{\mathbb{Z}_p} \mathcal{O} \). By imitating [5] §7.7, let \( \mathcal{MF}_{\mathcal{O}} \) denote the category of finitely generated \( W(k) \mathcal{O} \)-modules \( M \) with

• A decreasing filtration \( Fil^i M \) by \( W(k) \mathcal{O} \)-submodules which are \( W(k) \)-direct summands, where \( Fil^0 M = M \) and \( Fil^p M = \{0\} \);
• \( \text{Frob}_{W(k)} \otimes 1 \)-semi-linear and \( 1 \otimes \mathcal{O} \)-linear maps \( \varphi_i : Fil^i M \to M \) with \( \varphi_i \big|_{Fil^{i+1} M} = p \varphi_{i+1} \) and \( \sum_{i=0}^{p-1} \varphi_i(Fil^i M) = M \).

The morphisms in \( \mathcal{MF}_{\mathcal{O}} \) are \( W(k) \mathcal{O} \)-linear morphisms that are compatible with \( \varphi_i \) and \( Fil^i \) structures. We denote \( \mathcal{MF}_{\mathcal{O},\text{tor}} \) the full sub-category of \( \mathcal{MF}_{\mathcal{O}} \) consisting of objects which are killed by some \( p \)-power, and denote \( \mathcal{MF}_{\mathcal{O},\text{fr}} \) the full category of \( \mathcal{MF}_{\mathcal{O}} \) whose objects are finite free over \( W(k) \mathcal{O} \). Obviously, if \( M \in \mathcal{MF}_{\mathcal{O},\text{fr}} \) then \( M/\sigma^m M \) is in \( \mathcal{MF}_{\mathcal{O},\text{tor}} \) for all \( m \).

It turns out that the category \( \mathcal{MF}_{\mathcal{O},\text{tor}} \) is abelian (see §1.10 in [5]). An object \( M \) in \( \mathcal{MF}_{\mathcal{O},\text{tor}} \) is called nilpotent if there is no nontrivial subobject \( M' \subset M \) such that \( Fil^1 M' = \{0\} \). Denote the full subcategory of nilpotent objects by \( \mathcal{MF}^n_{\mathcal{O},\text{tor}} \). An
object $M \in \mathcal{MF}_{O, \text{fr}}$ is called nilpotent if $M/\sigma^m M$ is nilpotent for all $m$. Denote by $\mathcal{MF}_{O, \text{fr}}^n$ the full subcategory of $\mathcal{MF}_{O, \text{fr}}$ whose objects are nilpotent.

We refer readers to [6] for the construction and details of the period ring $A_{\text{cris}}$ (and $A_{\text{cris}}$ is just $S$ in [5]). Here we just recall that $A_{\text{cris}}$ is a $W(k)$-algebra with a decreasing filtration of ideals $A_{\text{cris}} = \text{Fil}^0 A_{\text{cris}} \supset \text{Fil}^1 A_{\text{cris}} \supset \ldots$, a continuous ring endomorphism $\varphi$ which extends Frobenius on $W(k)$ and a continuous $G_{K_0}$-action which commutes with $\varphi$ and preserves $\text{Fil}^i A_{\text{cris}}$. It turns out that $\varphi(\text{Fil}^i A_{\text{cris}}) \subset p^i A_{\text{cris}}$ for $1 \leq i \leq p-1$ and we define maps $\varphi_i := \varphi/p^i : \text{Fil}^i A_{\text{cris}} \rightarrow A_{\text{cris}}$. Let $\text{Rep}_O(G_{K_0})$ be the category of finitely generated $O$-modules with continuous $O$-linear $G_{K_0}$-action. We define a functor $T_{\text{cris}}^*$ from the category $\mathcal{MF}_{O, \text{tor}}^n$ (resp. $\mathcal{MF}_{O, \text{fr}}^n$) to $\text{Rep}_O(G_{K_0})$:

$$T_{\text{cris}}^*(M) := \text{Hom}_{W(k), \varphi_i, \text{Fil}^i} (M, A_{\text{cris}} \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p)) \quad \text{if} \quad M \in \mathcal{MF}_{O, \text{tor}},$$

and

$$T_{\text{cris}}^*(M) := \text{Hom}_{W(k), \varphi_i, \text{Fil}^i} (M, A_{\text{cris}}) \quad \text{if} \quad M \in \mathcal{MF}_{O, \text{fr}}.$$

Let $\text{Rep}_{E, \text{cris}}^{[0, p-1]}(G_{K_0})$ denote the category of continuous $E$-linear $G_{K_0}$-representations on finite dimensional $E$-vector spaces $V$ which are crystalline with Hodge-Tate weights in $[0, \ldots, p-1]$. An object $V \in \text{Rep}_{E, \text{cris}}^{[0, p-1]}(G_{K_0})$ is called nilpotent if $V$ does not admit nontrivial unramified quotient (it is easy to check that $V$ admits a nontrivial unramified quotient as an $\mathbb{Q}_p$-representation if and only if $V$ admits a nontrivial unramified quotient as an $E$-representation. See the proof of Theorem 2.2.1 (4) below). We denote by $\text{Rep}_{O, \text{cris}}^{[0, p-1], n}(G_{K_0})$ the category of $G_{K_0}$-stable $O$-lattices in nilpotent representations in $\text{Rep}_{E, \text{cris}}^{[0, p-1]}(G_{K_0})$.

We gather the following useful results from [5] and [8].

**Theorem 2.2.1**

1. The contravariant functor $T_{\text{cris}}^*$ from $\mathcal{MF}_{O, \text{tor}}^n$ to $\text{Rep}_O(G_{K_0})$ is exact and fully faithful.
2. An object $M \in \mathcal{MF}_{O, \text{fr}}^n$ is nilpotent if and only if $M/\sigma M$ is nilpotent.
3. The essential image of $T_{\text{cris}}^*$ : $\mathcal{MF}_{O, \text{tor}}^n \rightarrow \text{Rep}_O(G_{K_0})$ is closed under taking sub-objects and quotients.
4. Let $V$ be a crystalline representation of $G_{K_0}$ and $K’$ a finite unramified extension of $K_0$. Then $V$ is nilpotent if and only if $V|_{G_{K_0}}$ is nilpotent.
5. $T_{\text{cris}}^*$ induces an anti-equivalence between the category $\mathcal{MF}_{O, \text{fr}}^n$ and the category $\text{Rep}_{O, \text{cris}}^{[0, p-1], n}(G_{K_0})$.

**Proof**

(1) and (2) follow from Theorem 3.3 and Theorem 6.1 in [5]. Note that $U_\xi$ in [5] is just $T_{\text{cris}}^*$ here. To prove (3), we may assume that $O = \mathbb{Z}_p$ and it suffices to check that $T_{\text{cris}}^*$ sends simple objects in $\mathcal{MF}_{O, \text{tor}}^n$ to simple objects in $\text{Rep}_O(G_{K_0})$ (see Property 6.4.2 in [3]). And this is proved in [5], §6.13 (a). (4) is clear because $V$ is nilpotent if and only if $(V^*)^{|K_0|} = \{0\}$ where $I_{K_0}$ is the inertia subgroup of $G_{K_0}$.

(5) has been essentially proved in [5] and [8] but has not been recorded in literature. So we sketch the proof here. First, by §7.14 of [5], $T_{\text{cris}}^*(M)$ is a continuous $O$-linear $G_{K_0}$-representation on a finite free $O$-module $T$. By (1) and Theorem 0.6 in [5], we
have rank\(\mathcal{O}(T) = \text{rank}_{\mathcal{O}} W(k) \cdot M = d\). It is easy to see that \(M\) is a \(W(k)\)-lattice in \(D_{\text{cris}}(V^*)\) where \(V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T\). Hence \(V\) is crystalline with Hodge-Tate weights in \(\{0, \ldots, p - 1\}\). To see that \(V\) is nilpotent, note that \(V\) has an unramified quotient \(\tilde{V}\) is equivalent to that there exists an \(M' \subset M\) such that \(M' \cap \text{Fil}^1 M = \{0\}\) and \(M/M'\) has no \(p\)-torsion (just let \(M' := D_{\text{cris}}(V^*) \cap M\)). So \(M\) is nilpotent implies that \(V\) is nilpotent. Hence by (1), \(T_{\text{cris}}^*\) is an exact, fully faithful functor from \(\mathcal{M}_\mathcal{F}^{n}_{\mathcal{O}, \text{fr}}\) to \(\text{Rep}^{[0,p-1], n}_{\mathcal{O}, \text{cris}}(G_{K_0})\).

To prove the essential surjectivity of \(T_{\text{cris}}^*\), it suffices to assume that \(\mathcal{O} = \mathbb{Z}_p\). Indeed, suppose that \(T\) is an object in \(\text{Rep}^{[0,p-1], n}_{\mathcal{O}, \text{cris}}(G_{K_0})\) with \(d = \text{rank}_{\mathcal{O}} T\). Let \(V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T\) and \(D = D_{\text{cris}}(V^*)\). It is well-known that \(D\) is a finite free \(E \otimes_{\mathbb{Q}_p} K_0\)-module with rank \(d\). If there exists an \(M \in \mathcal{M}_\mathcal{F}^{n}_{\mathcal{O}, \text{fr}}\) such that \(T_{\text{cris}}^*(M) \simeq T\) as \(\mathbb{Z}_p[G]\)-modules. By the full faithfulness of \(T_{\text{cris}}^*\), \(M\) is naturally a \(W(k)\)-module. Since \(D\) is \(E \otimes_{\mathbb{Q}_p} K_0\)-free, it is standard to show that \(M\) is a \(W(k)\)-free by computing \(\mathcal{O}_i\)-rank of \(M_i\), where \(M_i := M \otimes_{W(k)} \mathcal{O}_i\) and \(W(k) \simeq \prod_i \mathcal{O}_i\).

Now suppose that \(V \in \text{Rep}^{[0,p-1], n}_{\mathcal{O}, \text{cris}}(G_{K_0})\) is nilpotent and \(D = D_{\text{cris}}(V^*)\). By [8] §3.2, there always exists a \(W(k)\)-lattice \(M \in \mathcal{M}_\mathcal{F}^{n}_{\mathcal{Z}_p, \text{fr}}\) inside \(D\). We claim that \(M\) is nilpotent. Suppose otherwise, then \(\tilde{M} := M/pM\) is not nilpotent, and there exists \(N \subset \tilde{M}\) such that \(\text{Fil}^1 N = \{0\}\). Consequently \(\varphi_0(\text{Fil}^0 N) = \varphi_0(N) = N\). Thus \(\bigcap_m (\varphi_0)^m(M) \neq \{0\}\). By Fitting Decomposition Theorem, we see that \(M^{\text{mult}} := \bigcap_m (\varphi_0)^m(M) \neq \{0\}\) is in fact a direct summand of \(M\). Let \(D^{\text{mult}} = M^{\text{mult}} \otimes_{W(k)} K_0\), it is an \(\varphi\)-submodule of \(D\). Since \(D\) is weakly admissible, \(I_H(D^{\text{mult}}) \leq I_N(D^{\text{mult}}) = 0\). Thus we must have \(I_H(D^{\text{mult}}) = I_N(D^{\text{mult}}) = 0\), and \(D^{\text{mult}}\) is weakly admissible. It is clear that \(V_{\text{cris}}^*(D^{\text{mult}})\) is an unramified quotient of \(V\), contradicting that \(V\) is nilpotent. Thus, \(M\) is nilpotent.

It remains to show that any \(G_{K_0}\)-stable \(\mathbb{Z}_p\)-lattices \(L' \subset V\) is given by an object \(M' \in \mathcal{M}_\mathcal{F}^{n}_{\mathcal{Z}_p, \text{fr}}\). Let \(L := T_{\text{cris}}^*(M)\). Without loss of generality, we can assume that \(L' \subset L\). For sufficiently large \(m\), \(p^m L' \subset L\), so \(L'/p^m L' \subset L/p^m L\). Since \(L/p^m L \simeq T_{\text{cris}}^*(M/p^m M)\). By (3), there exists an object \(M'_m \in \mathcal{M}_\mathcal{F}^{n}_{\mathcal{O}, \text{tor}}\) such that \(T_{\text{cris}}^*(M'_m) \simeq L'/p^m L\). Finally \(M' = \lim_m M'_m\) is the desired object in \(\mathcal{M}_\mathcal{F}^{n}_{\mathcal{O}, \text{fr}}\). \(\square\)

Contravariant functors like \(T_{\text{cris}}^*\) are not convenient for deformation theory. So we define a covariant variant for \(T_{\text{cris}}^*\). Define \(T_{\text{cris}}(M) := (T_{\text{cris}}^*(M))^*(p - 1)\), more precisely,

\[
T_{\text{cris}}(M) := \text{Hom}_{\mathcal{O}}(T_{\text{cris}}^*(M), E/\mathcal{O})(p - 1) \quad \text{if} \ M \in \mathcal{M}_\mathcal{F}^{n}_{\mathcal{O}, \text{tor}},
\]

and

\[
T_{\text{cris}}(M) := \text{Hom}_{\mathcal{O}}(T_{\text{cris}}^*(M), \mathcal{O})(p - 1) \quad \text{if} \ M \in \mathcal{M}_\mathcal{F}^{n}_{\mathcal{O}, \text{fr}}.
\]

Let \(\rho : G_{K_0} \to \text{GL}_d(\mathcal{O})\) be a continuous representation such that there exists an \(M \in \mathcal{M}_\mathcal{F}^{n}_{\mathcal{O}, \text{fr}}\) satisfying \(T_{\text{cris}}(M) = \rho\). Then \(T_{\text{cris}}(\tilde{M}) = \tilde{\rho} := \rho \mod \varpi \mathcal{O}\) where \(\tilde{M} := M/\varpi M\). Let \(\mathcal{C}_\mathcal{O}^f\) denote the category of Artinian local \(\mathcal{O}\)-algebras for which the structure map \(\mathcal{O} \to R\) induces an isomorphism on residue fields. The morphisms in

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the category are local homomorphisms inducing isomorphisms on the residue fields. Define a deformation functor

\[ D_{cris}^n(R) := \{ \text{lifts } \tilde{\rho} : G_{K_0} \to \text{GL}_d(R) \text{ of } \tilde{\rho} \exists M \in \mathcal{MF}_{O,tor}^n \text{ satisfying } T_{cris}(M) \simeq \tilde{\rho} \}. \]

Here \( T_{cris}(M) \simeq \tilde{\rho} \) as \( \mathcal{O}[G_{K_0}] \)-modules. To recapture the \( R \)-structure, let \( \mathcal{MF}_R \) be the category similarly defined as \( \mathcal{MF}_O \) by changing \( O \) to \( R \) everywhere (morphisms in \( \mathcal{MF}_R \) are \( W(k)_R \)-morphisms). It is clear that \( \mathcal{MF}_R \) is a subcategory of \( \mathcal{MF}_{O,tor} \) (note that \( R \) is a \( p \)-power torsion ring). Let \( \mathcal{MF}_R^n, fr \) be the full subcategory of \( \mathcal{MF}_R \) whose objects are nilpotent (as objects in \( \mathcal{MF}_{O,tor} \)) and finite \( W(k)_R \)-free, and \( \text{Rep}_{R,fr}(G_{K_0}) \) the category of \( R \)-linear continuous representations of \( G_{K_0} \) on finite free \( R \)-modules. It is easy to show that \( T_{cris} \) restricted to \( \mathcal{MF}_R^n, fr \) is an exact fully faithful functor from \( \mathcal{MF}_R^n, fr \) to \( \text{Rep}_{R,fr}(G_{K_0}) \). Thus, the \( R \)-structure on \( \tilde{\rho} \) guarantees an \( R \)-structure on \( M \) in the definition of \( D_{cris}^n(R) \), i.e., if \( T_{cris}(M) \simeq \tilde{\rho} \), then \( M \in \mathcal{MF}_R^n, fr \).

**Proposition 2.2.2** Assume that \( K_0 \subset E \). Then \( D_{cris}^n \) is pro-represented by a formally smooth \( O \)-algebra \( R_{\tilde{\rho}, cris}^n \).

**Proof** By (1) and (3) in Theorem 2.2.1, and by §1 in [9], \( D_{cris}^n \) is a sub-functor of the framed Galois deformation functor of \( \tilde{\rho} \) and pro-represented by an \( O \)-algebra \( R_{\tilde{\rho}, cris}^n \). The formal smoothness of \( R_{\tilde{\rho}, cris}^n \) can be proved similarly as in Lemma 2.4.1 in [4]. Indeed, suppose that \( R \) is an object of \( C^f_O \) and \( I \) is an ideal of \( R \) with \( m_R I = (0) \). To prove the formal smoothness of \( R_{\tilde{\rho}, cris}^n \), we have to show that any lift in \( D_{cris}^n(R/I) \) admits a lift in \( D_{cris}^n(R) \). Then this is equivalent to lift the corresponding \( N \in \mathcal{MF}_{R/I,fr} \) to \( \tilde{N} \in \mathcal{MF}_{R,fr} \) [note that any lift \( N \) of \( \tilde{M} \) will be automatically in \( \mathcal{MF}_{R,fr} \) by Theorem 6.1 (i) of [5] or Theorem 2.2.1 (3)]. The proof is verbatim as in Lemma 2.4.1 in [4]. Note that the proof did not use the restrictions (assumed for §2.4.1 in loc. cit.) that \( \text{Fil}^p M = \{0\} \) and \( \dim_k(\text{gr}_{\tilde{\tau}}^{-1}(\tilde{\bar{r}}(G_{F_{\tilde{\tau}}})) \otimes O_{F_{\tilde{\tau}}, \tilde{\bar{r}}} \leq 1 \). □

### 3 The main theorem and its proof

**Theorem 3.0.3** (Main Theorem) Suppose \( \rho : G_{K_0} \to \text{GL}_d(\overline{Q}_p) \) is a crystalline representation, and for each \( \tau : K_0 \hookrightarrow \overline{Q}_p \), the Hodge-Tate weights \( \text{HT}_\tau(\rho) \subseteq \{ a_\tau, \ldots, a_\tau + p - 1 \} \) for some \( a_\tau \), then \( \rho \) is potentially diagonalizable.

**Proof** By Lemma 2.2.1.1 of [2], we may assume that \( \rho \) factors through \( \text{GL}_d(\mathcal{O}) \) for a sufficiently large \( \mathcal{O} \). By Lemma 2.1.2, we can assume that \( \rho \) is irreducible and hence \( \rho^*(p - 1) \) is nilpotent. As in the proof of Lemma 1.4.3 of [1], twisting by a suitable crystalline character, we can assume \( a_\tau = 0 \) for all \( \tau \). Then we can choose an unramified extension \( K' \), such that \( \overline{\rho} |_{G_{K'}} \) has a \( G_{K'} \)-invariant filtration with 1-dimensional graded pieces. By Theorem 2.2.1 (4), \( \rho^*(p - 1) \) is still nilpotent when restricted to \( G_{K'} \). Without loss of generality, we can assume that \( K_0 = K' \). Now by Theorem 2.2.1 (5), there exists an \( M \in \mathcal{MF}_{O,fr}^n \) such that \( T_{cris}(M) \simeq \rho \). Then \( \tilde{M} := M/\sigma M \) is nilpotent and \( T_{cris}(\tilde{M}) \simeq \tilde{\rho} \). Note that \( \tilde{M} \) has a filtration with rank-1...
$k \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathbb{F}$-graded pieces to correspond to the filtration of $\tilde{\rho}$. Now by Lemma 1.4.2 of [1], we can lift $\tilde{M}$ to $M' \in \mathcal{MF}_{\mathcal{O},fr}$ which has filtration with rank-1 $W(k)_{\mathcal{O}}$-graded pieces (note that the proof of Lemma 1.4.2 of [1] did not use the restriction that $HT_\tau(\rho) \subseteq \{0, \ldots, p - 2\}$). Hence $M'$ is nilpotent by Theorem 2.2.1 (2). Then $\rho' = T_{\text{cris}}(M')$ is crystalline and has a $G_{K_0}$-invariant filtration with 1-dimensional graded pieces by Theorem 2.2.1 (5). Then part 1 of Lemma 1.4.3 of [1] implies that $\rho'$ is potentially diagonalizable. Now it suffices to show that $\rho$ connects to $\rho'$. But it is obvious that $R_{\rho, \text{cris}}^{\square}$ is a quotient of $R_{\rho, [HT_\tau(\rho), K-\text{cris}}^{\square}$. By Proposition 2.2.2, we see that $\rho$ and $\rho'$ must be in the same connected component of $\text{Spec}(R_{\rho, [HT_\tau(\rho), K-\text{cris}}^{\square}(\frac{1}{p}))$. Hence $\rho \sim \rho'$ and $\rho$ is potentially diagonalizable. \hfill \Box

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