EXTREMAL PROPERTIES OF THE FIRST EIGENVALUE OF SCHRÖDINGER-TYPE OPERATORS

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Abstract. Given a separable, locally compact Hausdorff space \( X \) and a positive Radon measure \( m(dx) \) on it, we study the problem of finding the potential \( V(x) \geq 0 \) that maximizes the first eigenvalue of the Schrödinger-type operator \( L + V(x) \); \( L \) is the generator of a local Dirichlet form \( (a, D[a]) \) on \( L^2(X, m(dx)) \).

Introduction

Let \( A > 0 \); for

\[
V \in B_A := \left\{ f : \left( \int_{\Omega} |f|^p \, m(dx) \right)^{1/p} \leq A \right\}, \quad 1 \leq p \leq \infty,
\]

we let \( \lambda_1(V) \) denote the first eigenvalue of \( L + V(x) \):

\[
\begin{align*}
Lu + Vu &= \lambda_1(V)u, \quad \text{in } X, \\
u &\in D[a].
\end{align*}
\]

(0.1)

In this paper we shall be concerned with the following

Problem: Determine whether

(1) the supremum \( \sup \{ \lambda_1(V) : V \in B_A \} \) is finite;
(2) there exists \( \tilde{V} \in B_A \) such that

\[
\sup \{ \lambda_1(V) : V \in B_A \} = \lambda_1(\tilde{V}).
\]

The main assumptions on the local Dirichlet form \( (a, D[a]) \) are (a1), (a2), (a3) in § 1 below. In particular we stress that \( (a, D[a]) \) need not be a regular Dirichlet form (according to the terminology in [3]).

The paper we mainly refer to, and which inspired the present work, is [2] by H. Egnell (cf. also the references therein for other contributions to
this problem). In $\mathbb{R}^d$, $d \geq 1$, $m(dx) = k^2 dx$, where $dx$ denotes the Lebesgue measure, $k \geq 0$ is a measurable function on $X$; the family $B_A$ is correspondingly defined as above; the Dirichlet form

$$a[u, u] := \int_X \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx,$$

with domain $D[a] = \{ u : a[u, u] < +\infty \}$; the matrix $(a_{ij})_{i,j}$ is symmetric, coercive, that is, there is a constant $\Lambda > 0$ such that

$$a_{ij}(x)\xi_i\xi_j \geq \Lambda |\xi|^2,$$

for all $x \in X$, $\xi \in \mathbb{R}^n$, and $a_{ij} \in L^1(X)$, $i, j = 1, \ldots, n$.

The paper is organized as follows. In the first section we fix the notation, introduce some definitions and preliminary results regarding the general theory of Dirichlet forms, and in the second section we present our solution to the problem considered. As in [2], the case $p = \infty$ is trivial (with maximal potential $V = A$), while the other two cases $1 < p < \infty$ and $p = 1$ are examined with different approaches. The case $1 < p < \infty$ is treated with a suitable use of standard methods in the Calculus of Variations. The remaining case $p = 1$ requires the form $(a, D[a])$ to be strongly local (cf. § 1) and this case is examined by the analysis of a related variational inequality (cf. Proposition 2.9); we have thus to generalize some results from the Theory of Variational Inequalities to this framework of Dirichlet forms (Theorem 3), which is done in the Appendix; we point out that the energy measure associated with the strongly local form $(a, D[a])$ (cf. § 1) plays an important role in this generalization.

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1. Preliminaries & notation

General notation. $X$ is a locally compact separable Hausdorff space. For any $E \subset X$, $\overline{E}$ denotes the closure of $E$ in $X$; also we let $\chi_E(x)$ be the function such that $\chi_E(x) = 1$ if $x \in E$, while $\chi_E(x) = 0$ otherwise in
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$X$. $C(X)$ denotes the space of all real-valued continuous functions $u$ on $X$. A Borel measure (on $X$) is an additive set function defined on the $\sigma$-algebra generated by the family of open sets of $X$; a Radon measure is a Borel measure which is finite on compact sets and different from zero on open non-empty sets. Unless otherwise specified, all the measures under consideration are non-negative. We let $m(dx)$ be a Radon measure whose support is the whole $X$, consider the real Hilbert space $L^2(X,m(dx))$ and for $u,v \in L^2(X,m(dx))$ we let $(u,v)$ denote their inner product. We also consider the Banach space $L^p(X,m(dx))$, $1 \leq p \leq \infty$, the norm of which is denoted by $\| \cdot \|_p$.

Given two functions $f,g$ on $X$, we denote by $\max\{f,g\}(x)$ (respectively $\min\{f,g\}(x)$) the pointwise maximum (respectively minimum) between $f(x)$ and $g(x)$, $x \in X$.

Dirichlet Forms (\cite{3}). A Dirichlet form $(a,D[a])$ on $L^2(X,m(dx))$ is a symmetric, non-negative bilinear form $a[u,v]$ defined on a dense subspace $D[a]$ of $L^2(X,m(dx))$; moreover $D[a]$ equipped with the intrinsic norm $(a[u,u]+(u,u))^{1/2}$ is itself a Hilbert space. Thus the embedding of Hilbert spaces $D[a] \hookrightarrow L^2(X,m(dx))$ is continuous.

The following result collects some standard properties of functions in $D[a]$ which will be used in the following.

**Proposition 1.1.** Let $(a,D[a])$ be a Dirichlet form. Then

1. If $u \in D[a]$ then the function $v := \min\{1, \max\{u,0\}\}$ belongs to $D[a]$ and $a[v,v] \leq a[u,u]$.
2. The sequence $(\max\{-n, \min\{u,n\}\})_n$ converges in $D[a]$ to $u$, as $n \to +\infty$.
3. $a[\|u\|,\|u\|] \leq a[u,u]$, for every $u \in D[a]$.
4. If $u,v \in D[a]$, then $\max\{u,v\}$, $\min\{u,v\} \in D[a]$.

The form is local if $a[u,v] = 0$ whenever $u,v \in D[a]$ have disjoint supports; the form is strongly local if $a[u,v] = 0$ whenever $u$ is constant on the support of $v$.

We shall consider in the rest of the paper the following conditions.

(a1) The embedding $D[a] \hookrightarrow L^2(X,m(dx))$ is compact.

(a2) “Urysohn-type Property”: For every compact set $K$ and each relatively compact open set $G \subset X$, with $K \subset G$, there exists a function
$u \in D[a]$ such that

\begin{align*}
u &= 1, \text{ on } K \\
u &= 0, \text{ on } X \setminus G.
\end{align*}

(a3) $C(X) \cap D[a]$ is a core of the form $(a, D[a])$, that is, $C(X) \cap D[a]$ is dense in both $D[a]$ (with respect to the intrinsic norm) and in $C(X)$ (with respect to the uniform convergence on compact sets).

If the form $a[u, v]$ is strongly local, then we write it as follows:

\begin{equation}
(1.2) \quad a[u, v] = \int_X \mu[u, v](dx), \quad u, v \in D[a].
\end{equation}

In (1.2), $\mu[u, v](dx)$ is a signed Borel measure (the energy measure associated with the form $a[u, v]$). The mapping $(u, v) \mapsto \mu[u, v](dx)$ is a symmetric non-negative bilinear form; moreover we assume that the energy measure satisfies the following localization property: If $A \subset X$ is any open set and $u = v$ m.a.e. on $A$, then

$$\chi_A(x)\mu[u, u](dx) = \chi_A(x)\mu[v, v](dx).$$

As a consequence of this property we have that

\begin{equation}
(1.3) \quad \chi_A(x)\mu[u, v](dx) = 0,
\end{equation}

whenever $u$ is constant on $A$, for every $v \in D[a]$.

Remark 1.4. The formula (1.2) is a particular case of a result by S. Albeverio, Z.M. Ma & M. Röckner [1, Theorem 1.1] concerning the representation of Dirichlet forms satisfying (a3) via an extension of the Beurling-Deny formula.

The generator of the form. The generator of a Dirichlet form $a[u, v]$ is the non-negative self-adjoint operator $L$ whose domain $D[L]$ is dense in the domain $D[a]$ of the form and satisfies

$$a[u, v] = (Lu, v),$$

for $u \in D[L]$ and $v \in D[a]$. 
2. Solution of the problem

In this section we present our solution of the problem stated in the Introduction. For the convenience of the reader, we rewrite the problem here.

Let \( A > 0 \), let \( p \in [1, \infty] \), and let us consider

\[
B_A := \{ f : \| f \|_p \leq A \} .
\]

Let \( (a,D[a]) \) be a Dirichlet form that satisfies (a1), and let \( L \) be the generator of the form. For \( V \in B_A \); we denote by \( \lambda_1(V) \) the first eigenvalue of the problem

\[
\begin{aligned}
Lu + Vu &= \lambda_1(V)u, \quad \text{in } X, \\
u &\in D[a],
\end{aligned}
\]

that is, \( u \in D[a] \) and

\[
a[u,w] + \int_X Vuw m(dx) = \lambda_1(V) \int_X uw m(dx),
\]

for every \( w \in D[a] \).

The problem is the following.

**Problem 2.2.** Determine whether:

1. the supremum \( \sup\{\lambda_1(V) : V \in B_A\} \) is finite;
2. there exists \( \tilde{V} \in B_A \) such that

\[
\sup\{\lambda_1(V) : V \in B_A\} = \lambda_1(\tilde{V}).
\]

If such a potential \( \tilde{V} \) exists, then we call the pair \( (\tilde{u}, \tilde{V}) \) composed by the solution \( \tilde{u} \) to (2.1) with potential \( \tilde{V} \) the extremal pair.

Let us associate to (2.1) above the corresponding Rayleigh quotient defined by

\[
R_V(u) := \frac{a[u,u] + \int_X V u^2 m(dx)}{\int_X u^2 m(dx)}, \quad u \in D[a], \ u \neq 0.
\]

Adapting the variational principle to our case, we have that the first eigenvalue \( \lambda_1(V) \) in (2.1) can be determined as the lower bound of the Rayleigh quotient:

\[
\lambda_1(V) = \inf\{R_V(u) : u \in D[a], \ u \neq 0\}.
\]
Remark 2.4. From the variational principle \([2.3]\) above, we see that the case \(p = \infty\) has the trivial solution \(V = A\).

Denoting by \(q\) the conjugate exponent of \(p\) \((p^{-1} + q^{-1} = 1)\), let us consider the following functional

\[
J(u) := \frac{a[u,u] + A\|u\|_2^2}{\|u\|_q^2}, \quad u \in D[a], \; u \neq 0.
\]

By standard properties of Dirichlet forms (cf. Proposition \([1.1]\) above) we have that

\[
J(|u|) \leq J(u).
\]

Notice that the functional \(J(u)\) is such that

\[
J(tu) = J(u), \quad t \in \mathbb{R}.
\]

By the Hölder inequality, we have

\[
RV(u) \leq J(u)
\]

for arbitrary \(u \in D[a], \; u \neq 0\). Thus from \([2.3]\) we get

\[
\lambda_1(V) \leq \inf \{J(u) : u \in D[a], \; u \neq 0\}.
\]

Thus we see that \(\sup_{V \in B_A} \lambda_1(V) < +\infty\) whenever the right-hand side in the above inequality is finite. The next result shows that this is indeed the case.

**Proposition 2.5.** Let \((a, D[a])\) be a Dirichlet form that satisfies \((a1)\) and \((a2)\) in § 1. Then the functional \(J(u)\) attains its minimum in \(D[a]\). If moreover \(V \geq 0\), then also \(RV(u)\) attains its minimum in \(D[a]\). Furthermore the minimizers for both \(J(u)\) and \(RV(u)\) are non-negative.

**Proof.** We shall prove the existence of minimizers for \(J(u)\), the other case being analogous. First of all we notice that the functional \(J(u)\) is not identically equal to \(+\infty\); this is a consequence of \((a2)\) and of \(m(dx)\) being a Radon measure. Let thus \((u_h)_h\) be a minimizing sequence normalized so that \(\|u_h\|_2 = 1\).

Let us consider first the case \(1 < p < \infty\). Then \((u_h)_h\) is bounded in \(D[a] \cap L^{2q}(X)\), and therefore a subsequence \((u_{h'})_{h'}\) of \((u_h)_h\) will converge to \(u \in D[a] \cap L^{2q}(X)\); as the embedding of \(D[a]\) into \(L^2(X, m(dx))\) is compact, then the whole sequence will converge to \(u\) in \(L^2(X, m(dx))\). Now a semicontinuity argument shows that

\[
J(u) \leq \liminf_{h \to +\infty} J(u_{h'}),
\]
hence $u$ is a minimizer. As $J(|u|) \leq J(u)$ (cf. Proposition 1.1), the minimizers are non-negative.

Now let us examine the case $p = 1$. Then the sequence $(u_h)_h$ is bounded in $D[a] \cap L^\infty(X)$; passing to a subsequence $(u_{h'})_h$ we may assume that $u_{h'} \to u$ weakly in $D[a]$ (hence strongly in $L^2(X)$, and in particular $\|u\|_2 = 1$) and weak*-convex in $L^\infty(X)$; therefore $u \in D[a] \cap L^\infty(X)$, and

$$ J(u) \leq \liminf_{h' \to \infty} J(u_{h'}) ,$$

so that $u$ is a minimizer. The last inequality follows from the two inequalities

$$ a[u,u] \leq \liminf_{h' \to \infty} a[u_{h'},u_{h'}], $$

$$ \|u\|_\infty \leq \liminf_{h' \to \infty} \|u_{h'}\|_\infty .$$

Arguing as in the previous case, it is easily seen that the minimizers are non-negative. \qed 

The next proposition gives a necessary condition for the existence of a maximizing potential $\tilde{V}$.

**Proposition 2.6.** Let $(a,D[a])$ be a local Dirichlet form that satisfies (a1), (a2) in § 1; let $\tilde{u}$ be a minimizer of $J(u)$, and assume that there exists a function $\tilde{V} \in B_A$ with $\text{supp}(\tilde{V}) \subset \text{supp}(\tilde{u})$ such that

$$ L\tilde{u} + \tilde{V}\tilde{u} = \lambda \tilde{u} , $$

where $\lambda := J(\tilde{u})$ is the minimum value of $J(u)$. Then the minimum value of the Rayleigh quotient $R_{\tilde{V}}(u)$ is equal to the minimum value of the functional $J(u)$ and $\tilde{u}$ is a minimizer for $R_{\tilde{V}}(u)$:

$$ \inf\{R_{\tilde{V}}(u) : u \in D[a], \ u \neq 0\} = R_{\tilde{V}}(\tilde{u}) = \lambda = J(\tilde{u}) .$$

**Proof.** It is similar to the proof of Lemma 6 in [4] but for the convenience of the reader we present it as well. Without loss of generality we may assume that $\|\tilde{u}\|_{2q} = 1$. Let $v$ be a non-negative minimizer of $R_{\tilde{V}}$ and assume that $\lambda' := R_{\tilde{V}}(v) < \lambda$. Then

$$ L\tilde{u} + \tilde{V}\tilde{u} = \lambda \tilde{u} , $$

$$ Lv + \tilde{V}v = \lambda' v .$$
(As \( v \) is a minimizer of the Rayleigh quotient \( R_V(\cdot) \), \( v \) satisfies the latter equation, while \( \tilde{u} \) satisfies the former by hypothesis.) This implies that

\[(\lambda - \lambda') \int_X \tilde{u} v m(dx) = 0,\]

so that \( \tilde{u} v = 0, \text{ m-a.e. on } X \). As \( \text{supp}(\tilde{V}) \subset \text{supp}(\tilde{u}) \) and \( \tilde{u} v = 0, \text{ m-a.e. on } X \), we get \( \tilde{V} v = 0 \text{ m-a.e. in } X \). This implies in particular that

\[(\lambda - \lambda') \int_X \tilde{u} v m(dx) = 0,\]

so that \( \tilde{u} v = 0, \text{ m-a.e. on } X \). As \( \text{supp}(\tilde{V}) \subset \text{supp}(\tilde{u}) \) and \( \tilde{u} v = 0, \text{ m-a.e. on } X \), we get \( \tilde{V} v = 0 \text{ m-a.e. in } X \). This implies in particular that

\[\lambda' = \frac{a[v, v]}{\|v\|_2^2} \]

Let \( v_n := (1/n) \min \{v, n\} \), for \( n \in \mathbb{N} \), and let

\[\lambda_n := \frac{a[v_n, v_n]}{\|v_n\|_2^2}.\]

By Proposition \([1]\) \( nv_n \) is in \( D[a] \), and \( nv_n \) converges to \( v \) in \( D[a] \), hence in \( L^2(X, m(dx)) \), as \( n \to +\infty \) (recall that \( D[a] \) is compactly, hence continuously, embedded into \( L^2(X, m(dx)) \)); thus

\[\lim_{n \to +\infty} \lambda_n = \lambda' := \frac{a[v, v]}{\|v\|_2^2}.\]

Notice that, by definition, we get \( v_n \tilde{u} = 0 \text{ m-a.e. on } X \), hence by using a well-known result in the theory of Dirichlet forms (cf. e.g. [2, Lemma 3.1.4]) and by the local property of the form under consideration we have that \( a[\tilde{u}, v_n] = 0 \).

Let us examine first the case \( p \in (1, \infty) \). Choose \( n \) large so that \( \lambda_n < \lambda \) and consider \( J(\tilde{u} + \varepsilon v_n) \); we have

\[J(\tilde{u} + \varepsilon v_n) = \lambda - \varepsilon^2 (\lambda - \lambda_n) \frac{\int_X \tilde{u} v_n m(dx)}{\int_X \tilde{u}^2 m(dx)} + o(\varepsilon^2) < \lambda,\]

for \( \varepsilon > 0 \) small enough, hence a contradiction. As in general \( \lambda \geq \lambda' \), we have thus \( \lambda = \lambda' \), hence the result is proved for \( p \in (1, \infty) \).

Let us consider the remaining case \( p = 1 \). Using \( \|\tilde{u} + v_n\|_\infty = \|\tilde{u}\|_\infty = 1 \), we get that

\[J(\tilde{u} + v_n) = \frac{\lambda \|\tilde{u}\|_2^2 + \lambda_n \|v_n\|_2^2}{\|\tilde{u}\|_2^2 + \|v_n\|_2^2} < \lambda,\]

if \( n \) is large enough. This is a contradiction, hence \( \lambda' = \lambda \) also in the case when \( p = 1 \) and the proof is thus concluded. \( \square \)
2.1. The case $1 < p < \infty$. Now we are in a position to state and prove the existence of the extremal pair for this case.

**Theorem 1.** Let $(a, D[a])$ be a local Dirichlet form that satisfies (a1), (a2) in § 1, and let $L$ be the associated non-negative self-adjoint operator. Let $1 < p < \infty$ and let $q = \frac{p}{p-1}$ be its conjugate exponent. Then there exists an extremal pair $(\tilde{u}, \tilde{V})$ that solves Problem (2.2); the potential $\tilde{V}$ is the unique maximizer of the first eigenvalue of Problem (2.2) in $\{f \in L^p(X, m(dx)) : \|f\|_p \leq A\}$; the function $\tilde{u}$ is a non-negative minimizer of the functional $J(\cdot)$ and is also the first eigenfunction of (2.1):

$$\begin{aligned}
L \tilde{u} + \tilde{V} \tilde{u} &= \lambda_1(\tilde{V}) \tilde{u}, \text{ in } X, \\
\tilde{u} &\in D[a],
\end{aligned}$$

where

$$\tilde{V} = \left(A \|\tilde{u}\|^2_1 q^{-q} \right) \tilde{u}^{2(q-1)},$$

and $\lambda_1(\tilde{V})$ is the maximal first eigenvalue with

$$\lambda_1(\tilde{V}) = J(\tilde{u}) = R_{\tilde{V}}(\tilde{u}).$$

**Proof.** The functional $J(\cdot)$ is Gateaux-differentiable; for $\phi \in D[a] \cap L^{2q}(X) \neq \emptyset$, by (a2), and being $m(dx)$ a Radon measure we have

$$J'_\phi(u) = \frac{2}{\|u\|^2_2} \left(a[u, \phi] + A\|u\|^2_1 q^{-q} \int_X |u|^{2(q-1)}u\phi \, m(dx) - J(u) \int_X u\phi \, m(dx) \right).$$

By Proposition 2.5, $J(\cdot)$ has (non-trivial) non-negative minimizers; thus a minimizer $\tilde{u} \geq 0$ of $J(\cdot)$ solves the equation

$$\begin{aligned}
L \tilde{u} + \tilde{V} \tilde{u} &= \lambda_1(\tilde{V}) \tilde{u}, \\
\tilde{u} &\in D[a],
\end{aligned}$$

that is,

$$\begin{aligned}
a[\tilde{u}, w] + \int_X \tilde{V} \tilde{u} w \, m(dx) &= \lambda_1(\tilde{V}) \int_X \tilde{u} w \, m(dx), \text{ for every } w \in D[a] \\
\tilde{u} &\in D[a],
\end{aligned}$$

where

$$\lambda_1(\tilde{V}) := J(\tilde{u}) \text{, and } \tilde{V} := \left(A \|\tilde{u}\|^2_1 q^{-q} \right) |u|^{2(q-1)}.$$
A direct computation shows that $\|\tilde{V}\|_p = A$, hence $\tilde{V} \in B_A$, and, by its definition, supp($\tilde{V}$) $\subset$ supp($\tilde{u}$); thus by Proposition 2.6 ($\tilde{u}, \tilde{V}$) is the extremal couple and $\lambda_1(\tilde{V}) = J(\tilde{u})$ is the maximal first eigenvalue. Notice that $\tilde{u}$ is the first eigenfunction corresponding to the eigenvalue $\lambda_1(\tilde{V})$. As for the uniqueness of the maximizing potential, it is proven similarly as in [2, Theorem 16].

In the same assumptions and notation of the above theorem we have the following result.

**Proposition 2.7.** The extremal pair ($\tilde{u}, \tilde{V}$) satisfies

$$\|\tilde{u}\|_\infty \leq \left( \frac{\lambda_1(\tilde{V})}{A} \right)^{\frac{p-1}{2}} \|\tilde{u}\|_2 < +\infty,$$

$$\|\tilde{V}\|_\infty \leq \lambda_1(\tilde{V}),$$

Proof. As $\tilde{u}$ is a minimizer of $J(\cdot)$, by Proposition 2.5 $\tilde{u} \geq 0$, and without loss of generality we may as well assume that $\|\tilde{u}\|_2 = 1$; thus $\|\tilde{u}\|_q^{1-q} = 1$. Let $c := \left( \frac{\lambda_1(\tilde{V})}{A} \right)^{\frac{p-1}{2}}$ and define $\xi := \tilde{u} - \min\{\tilde{u}, c\}$. Notice that $\xi \geq 0$, $\xi \in D[a]$ (by Proposition 1.1) and

$$a[\xi, \xi] = a[\xi, \tilde{u}] = \int_X \left( \lambda_1(\tilde{V}) - A|\tilde{u}|^{2(q-1)} \right) \tilde{u} \xi \, m(dx);$$

observe that, with our choice of $c$, the integrand is negative when $\xi > 0$; thus $\xi = 0$ and this gives $\tilde{u} \leq c$. The estimate on $\tilde{V}$ follows with a direct computation by using the estimate on $\tilde{u}$.  

2.2. **The case** $p = 1$. Let us consider

$$K := \{ v \in D[a] : |v| \leq 1 \text{ m.a.e. on } X \}.$$  

Using the properties of the Dirichlet form $a[u, v]$ in Proposition 1.1 it is not difficult to see that $K$ is a non-empty, closed, convex set in $D[a]$.

Let us also consider the functional

$$T(v) := \frac{a[v, v] + A}{\|v\|_2^2}, \quad v \in K.$$  

Let $\tilde{u}$ be a minimizer of $J(u)$; as $J(tu) = J(u)$, $t \in R$, we can assume that $\|\tilde{u}\|_\infty = 1$; thus $\tilde{u}$ is also a minimizer of $T(\cdot)$ and $J(\tilde{u}) = T(\tilde{u})$.

We have the following result.
Proposition 2.9. Let \((a, D[a])\) be a local Dirichlet form on \(L^2(X, m(dx))\) that satisfies (a1), (a2) and (a3) in \(\S\ 1\). Then \(\bar{u}\) is a solution of the variational inequality

\[
\begin{cases}
  a[v, v - u] \geq J(\bar{u}) \int_X u(v - u) \, m(dx), \ \forall v \in K, \\
  u \in K.
\end{cases}
\]

(2.10)

Proof. Let \(t \in (0, 1), \ v \in K\); then \(\bar{u} + t(v - \bar{u}) \in K, \ J(\bar{u}) = T(\bar{u})\) and

\[T(\bar{u}) \leq T(\bar{u} + t(v - \bar{u})).\]

A direct computation, similarly as in the proof of [3, Proposition 12], shows that \(\bar{u}\) satisfies (2.10). \(\square\)

Now we are in a position to state and prove the main result for the case \(p = 1\).

Theorem 2. Let \((a, D[a])\) be a strongly local Dirichlet form that satisfies (a1), (a2), (a3) in \(\S\ 1\). Then there exists an extremal pair \((\bar{u}, \bar{V})\) that solves Problem 2.2 and has the following properties:

(i) \(\bar{u}\) is a minimizer of \(J(\cdot)\).

(ii) \(\bar{u} \geq 0, \ \|\bar{u}\|_\infty = 1\).

(iii) Let \(I := \{x \in X : \bar{u}(x) = 1\}\); then \(m(I) > 0\).

(iv) \(\bar{u}\) is the first eigenfunction of Problem 2.1, \(\bar{V} = \frac{A}{m(I)} \chi_I\) and the maximal first eigenvalue \(\lambda_1(\bar{V}) = \frac{A}{m(I)}\).

(v) \(R_{\bar{V}}(\bar{u}) = J(\bar{u})\).

(vi) The potential \(\bar{V}\) is the unique maximizer of the first eigenvalue of Problem 2.1.

Proof. By Proposition 2.5 the functional \(J(u)\) attains its minimum in \(D[a] \cap L^\infty(X)\), and its minimizers are non-negative. Let \(\bar{u}\) be a minimizer of \(J(u)\) and without loss of generality we may assume that \(\|\bar{u}\|_\infty = 1\) (recall that \(J(t\cdot) = J(\cdot), \ for \ t \in \mathbb{R}\) so that \(0 \leq \bar{u} \leq 1\). By Proposition 2.3 \(\bar{u}\) is a solution of the variational inequality (2.9). Letting \(\lambda = J(\bar{u})\) and considering the “obstacle” equal to the constant function \(\psi = 1\), by Theorem 3 in the Appendix we have that \(L\bar{u} = \lambda\bar{u}\), on \(X \setminus I\), and \(L\bar{u} = 0\) on
so that $L\tilde{u} + \chi_I(x)\lambda \tilde{u} = \lambda \tilde{u}$, that is,

$$a[\tilde{u}, v] + \lambda \int_X \chi_I(x)\tilde{u}(x)v(x) \, m(dx) = \lambda \int_X \tilde{u}(x)v(x) \, m(dx),$$

for every $v \in D[a]$; in particular for $v = \tilde{u}$, and recalling that $\tilde{u} = 1$ on $I$, we have

$$a[\tilde{u}, \tilde{u}] + \lambda m(I) = \lambda \int_X |\tilde{u}|^2 \, m(dx).$$

If $m(I) = 0$ then from the relation above we get that $\lambda$ is the first eigenvalue of the problem (2.1) with $V = 0$, $\tilde{u}$ being the corresponding eigenfunction; thus from the variational principle we have

$$(2.11) \quad \frac{a[\tilde{u}, \tilde{u}]}{\|\tilde{u}\|_2^2} = \lambda;$$

as $\tilde{u}$ is also a minimizer for $J(u)$ with $\|u\|_\infty = 1$, $\tilde{u}$ is also a minimizer for $T(u)$ and

$$\lambda = T(\tilde{u}) = \frac{a[\tilde{u}, \tilde{u}]}{\|\tilde{u}\|_2^2} + A$$

but from (2.11) we get a contradiction, since $A > 0$. Therefore $m(I) > 0$ and we have that

$$\lambda = \frac{a[\tilde{u}, \tilde{u}] + \lambda m(I)}{\|\tilde{u}\|_2^2} = J(\tilde{u}),$$

and this implies $A = \lambda m(I)$. Therefore if we define $\tilde{V} := \frac{A}{m(I)} \chi_I$, then by Proposition 2.6 we have that $(\tilde{u}, \tilde{V})$ is an extremal pair, $\lambda_1(\tilde{V}) := A/m(I)$ is the extremal eigenvalue and $R_{\tilde{V}}(\tilde{u}) = J(\tilde{u})$; this proves the first five statements in the theorem. As for the uniqueness of $\tilde{V}$, we can argue similarly as in [2, Theorem 16] and conclude the proof. \hfill \square

3. Appendix

In this section we deal with a strongly local Dirichlet form $(a, D[a])$ that satisfies (a1), (a2), (a3) in §1. With these assumptions we can, similarly as in [3, Chapter 3], introduce the notions of capacity (associated with $(a, D[a])$) and quasi-continuity; in particular we can associate to each $u \in D[a]$ a sequence of closed sets $(F_k)_k$ (a “nest”) such that the union $\bigcup_k F_k$ is equal to $X$ (with the exception perhaps of a set of capacity zero) and the restriction of $u$ to $F_k$ is continuous on $F_k$, $k \in \mathbb{N}$ (cf. Theorems 3.1.2, 3.1.3 in [3]).
Denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $D[a]$ and its dual $(D[a])'$. As the form $a[\cdot, \cdot]$ continuous on $D[a]$, we have that $L\phi$, defined on $D[a]$ by

$$L\phi : v \mapsto a[\phi, v],$$

is well-defined as an element in $(D[a])'$ and $\langle L\phi, v \rangle = a[\phi, v]$.

Let $\psi : X \to \mathbb{R}$ be a quasi-continuous function and consider

$$K_\psi := \{ v \in D[a] : v \geq \psi \text{ m.a.e. on } X \};$$

$K_\psi$ is a closed convex set which we assume to be non-empty. Let us consider the following obstacle problem: Given $f \in (D[a])'$, find

$$\begin{cases} u \in K_\psi, \\ a[u, v - u] \geq \langle f, v - u \rangle, \forall v \in K_\psi, \end{cases}$$

(3.1)

**Theorem 3.** Assume that there exists a unique solution $\tilde{u}$ to the obstacle problem (3.1), and let $I := \{ x \in X : \tilde{u}(x) = \psi(x) \}$. Then

$$L\tilde{u} = f, \text{ on } X \setminus I.$$ 

Furthermore, if the obstacle function $\psi$ is equal to a constant function, then

$$L\tilde{u} = 0, \text{ on } I,$$

i.e., $\int_X \chi_I(x) \mu[\tilde{u}, v](dx) = 0$, for every $v \in D[a]$.

**Proof.** Adapting some arguments in [4] (cf. in particular Definition 6.7 in [4, Chapter II]), it can be shown that the set $X \setminus I$ is open; thus for $x_o \in X \setminus I$, there are two neighborhoods $U, G$ of $x_o$ such that $U \subset \overline{U} \subset G \subset X \setminus I$, and without loss of generality we can assume that $G$ is a relatively compact open set. By (a2), with $K = \overline{U}$, there exists a function $\phi$ contained in the domain of the form such that $\tilde{u} > \psi + \phi$; moreover for any $\zeta \in D[a]$ with support in $U$ there is $\varepsilon > 0$ such that

$$\tilde{u} + \varepsilon \zeta \geq \psi + \frac{1}{2} \phi.$$

Thus $v = \tilde{u} + \varepsilon \zeta \in K_\psi$; substituting this $v$ in (3.1) and dividing by $\varepsilon$ we get

$$\int_U \mu[\tilde{u}, \zeta](dx) \geq \langle f, \zeta \rangle,$$
for every $\zeta \in D[a]$, with support in $U$. We can argue similarly with $v = \tilde{u} - \varepsilon \zeta$ and get
\[
\int_U \mu[\tilde{u}, \zeta](dx) \geq \langle f, \zeta \rangle;
\]
hence $L\tilde{u} = f$, in $X \setminus I$.

Now assume that the obstacle $\psi$ is a constant function, and without affecting the generality of the argument that follows we can assume that $\psi = 0$; moreover we can also assume that $I$ is contained in some relatively compact open set $\Omega \subset X$. Let $(F_k)_k$ be the nest associated with $\tilde{u}$. Thus, except perhaps for a set of arbitrarily small capacity, we can assume that the function $\tilde{u}$ is continuous on $\Omega'$ with $\Omega' \subset \overline{\Omega'} \subset \Omega$, hence uniformly continuous on $\overline{\Omega'}$. Due to the uniform continuity of $\tilde{u}$ on $\overline{\Omega'}$, for every $\varepsilon > 0$ we can find an open neighborhood $U_\varepsilon$ of $I \cap \Omega'$ such that $U_\varepsilon \subset \{\tilde{u} \leq \varepsilon\}$. By the Urysohn-type property (a2), there exists $w_\varepsilon \in D[a]$ with compact support in $\Omega'$ such that $w_\varepsilon = \varepsilon$ on $\{\tilde{u} \leq \varepsilon\}$, hence on $U_\varepsilon$; we define $u_\varepsilon := \max\{\tilde{u}, w_\varepsilon\}$ so that $u_\varepsilon = \varepsilon$ on $U_\varepsilon$, $u_\varepsilon \in D[a]$ (cf. Proposition 1.1) and $u_\varepsilon$ converges to $\tilde{u}$ in $D[a]$. By the local character of the energy measure (cf. (1.3)) the restriction of the energy measure to $U_\varepsilon$, $\chi_{U_\varepsilon}(x)\mu[u_\varepsilon, v](dx)$, is equal to zero, for every $\varepsilon > 0$. Letting $\varepsilon \to 0$ we can conclude the proof. 

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