Robust online joint state/input/parameter estimation of linear systems

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Abstract—This paper presents a method for jointly estimating the state, input, and parameters of linear systems in an online fashion. The method is specially designed for measurements that are corrupted with non-Gaussian noise or outliers, which are commonly found in engineering applications. In particular, it combines recursive, alternating, and iteratively-reweighted least squares into a single, one-step algorithm, which solves the estimation problem online and benefits from the robustness of least-deviation regression methods. The convergence of the iterative method is formally guaranteed. Numerical experiments show the good performance of the estimation algorithm in presence of outliers and in comparison to state-of-the-art methods.

I. INTRODUCTION

Reliable control of a dynamical system is often contingent upon an accurate system model and accurate state measurements. When the system dynamics change over time, system identification algorithms can be used to track the parameter variations in the system by using state and input samples. However, measurement noise and inaccurate actuation may strongly degrade the estimation accuracy. In this case, joint state/input and parameter estimation methods are required in order to estimate both the noiseless states/inputs and the unknown parameters. In addition, in order to be useful for control applications, the estimation algorithm must provide all estimates in real time. Online joint state/input and parameter estimation methods are used in power systems, battery management systems, and self-driving cars, among other applications [1, Section 3.3.3], [2], [3]. While the constantly growing range of sensing technologies often allows one to measure the complete state of a system, statistical outliers and non-Gaussian noise, introduced by complex sensors, faulty hardware, and cyberattacks, are often present [4], [5].

In these scenarios, very popular estimation algorithms built for input/output (rather than state/input) samples and assuming Gaussian noise distributions can be ineffective. The Extended Kalman Filter (EKF), which has been demonstrated to work well for moderate and Gaussian measurement noise, requires additional constraints and tuning to remain stable when the noise levels increase [6], [7], [8], [9], [10], [2]. Because the joint estimation problem for linear systems is bilinear, the Unscented Kalman Filter (UKF) can achieve higher accuracy as it relies on sigma points approximating Gaussian distributions rather than noise-sensitive linearizations [3]. However, the sigma-point approximations can still introduce significant inaccuracies for non-Gaussian noise. Recursive Alternating Least Squares and Dual Kalman Filters provide an alternative way to deal with the bilinearity by considering state/input and parameters separately [11], [12]. Error-in-variables (EIV) approaches, which include Total Least Squares (TLS) methods, can also be used for joint state and parameter estimation of linear systems, as EIV estimators implicitly reconstruct the inputs/states in order to find the parameters that best fit the data [13], [14]. Among EIV methods, Recursive Total Least Squares (RTLS) has been shown to outperform Kalman filtering for linear systems [14]. Similarly to EIV models, subspace identification uses noisy measurement to provide parameter estimates, which can then be used to find the most likely states and inputs of the system [15], [16].

The impact of outliers can be minimized using outlier detection techniques to pre-filter the data [17]. However, they usually amount to solve complex classification problems, which are prohibitive to solve in real time. A standard approach to improve robustness against outliers, and that can be adapted to online methods, is to consider tail-heavy noise distributions and/or to add regularization terms such as \( \ell_1 \) norm penalties [18], [19], [20], [21], [22], [23].

The non-smoothness of the \( \ell_1 \) norm creates complicated optimization problems when applied to the standard methods mentioned previously. ADMM, proximal, and sub-gradient methods have been used to solve optimization problems with \( \ell_1 \) costs [24], [25]. However, such approaches are tailored to batch estimation problems. Another method for dealing with \( \ell_1 \) terms is iterative reweighting, which consists of iteratively approximating a class of cost functions by quadratic ones, and by only relying on the current value of the cost [26]. This method is simpler, compared to the aforementioned alternatives, because it does not require a gradient or proximal operator computation. While convergence proofs exist for Iteratively-Reweighted Least Squares and its variants, they are not valid in a online (i.e. recursive) or alternating setting [26], [27].

In this paper, we propose a new Alternating and Iteratively-reweighed Recursive Least Squares algorithm (AIRLS) that addresses the three challenges of online, joint and robust estimation. A second contribution is the formal proof of convergence of the method. Finally, we present numerical experiments demonstrating the robustness of AIRLS to outliers. More in details, we show that, in presence of outliers, both EKF and a standard subspace identification method [28] can fail, even for a very simple system. In the same setting, RTLS shows greater robustness, but is outperformed by AIRLS.

The paper is structured as follows: Section II describes the joint estimation problem to be solved. A naive form of robust and online joint estimation is introduced in Section III and then used in Section IV to define the AIRLS method and prove its convergence. Section V presents the numerical
results. Section VI concludes the paper.

A. Preliminaries and Notation

The $n$-dimensional identity matrix is $I_n \in \mathbb{R}^{n \times n}$, and $0_n \in \mathbb{R}^n$ and $1_n \in \mathbb{R}^n$ are the vectors of all zeros and all ones. $[x_i^\top]^n_{i=1}$ is the matrix with rows equal to the vectors $x_i$. vec$(\cdot)$ is the column vectorization operator. A proportionality relation is denoted by $\propto$. The $\ell_1$ Frobenius norm is defined as $\|A\|_{\ell_1} = \|\text{vec}(A)\|_1 = \sum_i \|A[i,\cdot]\|_1$, where $[A[i,\cdot]]$ is the ith column of $A$. Similarly, the $\ell_2$ Frobenius norm is $\|A\|_F = \|\text{vec}(A)\|_2 = \sqrt{\sum_i \|A[i,\cdot]\|^2}$.

II. Problem Statement

A. System model

We want to reconstruct the matrices $A$ and $B$ in the model

$$x_{t+1} = Ax_t + Bu_t,$$

where $u_t \in \mathbb{R}^{n_u}$ are the inputs and $x_t \in \mathbb{R}^n$ are the states for the time steps $t = 1, \ldots, N$. We also want to provide a running estimate of $x_t$ and $u_t$ from measurements $\hat{x}_t$ and $\hat{u}_t$ that are corrupted by additive noise $\Delta x_t$ and $\Delta u_t$, i.e.

$$\hat{x}_t = x_t + \Delta x_t, \quad (2a)$$

$$\hat{u}_t = u_t + \Delta u_t. \quad (2b)$$

Note that we do not assume any specific probability distribution for the noise.

A recursive algorithm for solving this problem must be based on a fixed-size data matrix and a constant number of parameters. To this purpose, a common approach is to replace the observations $[x_{t+1}, x_t, u_t]$ by their discounted empirical correlation matrix

$$C_t = \sum_{i=0}^{t} \beta^{t-i} \Gamma_i : \Gamma_i = \begin{bmatrix} x_{i+1}^\top & x_i^\top & x_i^\top \\ x_i & u_i & u_i \end{bmatrix}, \quad (3)$$

where $1 - \beta \in (0, 1)$ is the forgetting factor. In presence of noise, one can only build $\hat{C}_t$ and $\hat{\Gamma}_t$ from measurements $\hat{x}_0, \ldots, \hat{x}_{t+1}$ and $\hat{u}_0, \ldots, \hat{u}_t$. We note that, usually, the matrix $\hat{C}_t$ is full rank due to the noise.

Next, we briefly review standard online estimation methods. The RTLS algorithms estimate the null space of $C_t$ by identifying the eigenvectors of $\hat{C}_t$ corresponding to its smallest eigenvalues. This can be done with the inverse power method [14]. The null space is formed by vectors $[x_{t+1}^\top, x_t^\top, u_t^\top]^\top$ verifying (1) and therefore allows one to retrieve the matrices $A$ and $B$.

Recursive subspace identification and the EKF recursively estimate propagator matrices, which have an expression similar to $\hat{C}_t$, and use their inverse for updating the estimates of $A$ and $B$ [16], [6]. Another similar method is the Frisch scheme, which also uses $\hat{C}_t$, but strongly differs in the way it filters the noise [29].

III. Online Robust Joint State/Input and Parameter Estimation

In this section, we provide a preliminary online estimation algorithm using $\hat{C}_t$ for estimating both $[A, B]$ and $\hat{C}_t$. Moreover, in Section IV, we show that the estimate of states and inputs at time $t$ can be recovered by projecting the noisy measurements $\hat{x}_t$ and $\hat{u}_t$ onto the null space of $[-I_n, A_t, B_t]$ (see Corollary 3 below).

Before presenting the algorithm in Sections III-A and III-B, we need to introduce some notations. We partition $C_t = [Y_t^\top, Z_t^\top]^\top$, where $Y_t \in \mathbb{R}^{n \times m}$ and $Z_t \in \mathbb{R}^{(n + n_u) \times m}$ are two linearly dependent blocks. Using (1) and (3), this gives

$$Y_t = E_y C_t = [A, B] E_z C_t = [A, B] Z_t, \quad (4)$$

where $E_y = [I_n, 0_{n \times (n + n_u)}]$ and $E_z = [0_{(n + n_u) \times n}, I_{n + n_u}]$. Similar to $C_t$, the matrix $\hat{C}_t$ is partitioned into $\hat{Y}_t = E_y \hat{C}_t$ and $\hat{Z}_t = E_z \hat{C}_t$.

Because $Y_t$ depends on $A$, $B$, and $Z_t$, the estimation problem amounts to finding the estimates $\hat{A}_t$, $\hat{B}_t$, and $\hat{Z}_t$ from $C_t$. To improve accuracy, one can use the previous estimate $\hat{Z}_{t-1}$, rather than $Z_{t-1}$. This amounts to using $\hat{C}_t$ rather than $C_t$, which we define recursively as

$$\hat{C}_{t+1} = \beta \hat{C}_t + \hat{\Gamma}_{t+1} = \begin{bmatrix} \hat{Y}_t \\ \hat{Z}_t \end{bmatrix}, \quad (5)$$

where $I_m \propto I_m$ is arbitrarily chosen. Similar to $\hat{C}_t$, the matrix $\hat{C}_t$ is partitioned into $\hat{Y}_t = E_y \hat{C}_t = \hat{Y}_t$ and $\hat{Z}_t = E_z \hat{C}_t$.

Remark. EIV system identification often suffers from identifiability issues because (4) has more unknowns than equations [30]. This means that there are infinitely many possible estimates that would satisfy (4). Among these estimates, in this paper, we will always chose the system for which the predictions deviate the least from the data (in terms of $\ell_1$ norm, represented by the second term in the problem (6) that follows). Other solutions such as combining multiple experiments are proposed in [30] and can also be applied here.

A. A simple alternating joint estimation algorithm

The combined estimation of the state, input and parameter, based on least absolute deviations (i.e. the $\ell_1$ norm loss function) is given by [18]

$$\hat{A}_t, \hat{B}_t, \hat{Z}_t = \arg\min_{A, B, Z} \|\|A, B\|Z - Y_t\|_{F_1} + \|Z - Z_t\|_{F_1}.$$  \hspace{1cm} (6)

In addition to the robustness provided by least absolute deviations, one may want to include a regularization for $\Theta = [A, B]$. In this case, with $\hat{\Theta}_t = [\hat{A}_t, \hat{B}_t]$, (6) becomes

$$\hat{\Theta}_t, \hat{Z}_t = \arg\min_{\Theta, Z} \|\Theta Z - Y_t\|_1 + \|Z - Z_t\|_1 + \|\Theta - \Theta_0\|_1.$$  \hspace{1cm} (7)

where $\Psi$ and $\mu$ are a matrix and a vector that can be chosen to tune the regularization term. We note that the problem (7) is
(i) equivalent to (6) when $\Psi = 0^{T}_{n(n+n_u)}$, and $\mu = 0$,
(ii) an $\ell_\infty$ robust and $\ell_\infty$ distributionally robust formulation
of (6) according to [19, Theorem 3] and [31, Equation (4)],
when $\Psi = \epsilon I_{n(n+n_u)}$, and $\mu = 0^{n(n+n_u)}$.
Moreover, (7) is a maximum a posteriori estimation problem
with the prior belief that $\Psi \vec{v} \approx [32]$

The bilinear term $\Theta Z$ makes the optimization hard to solve.
A common approach to circumvent this issue is to
use a block coordinate descent method, which consists of
an iterative optimization procedure that alternates between
optimizing the estimate of $Z$ (for $\Theta$ fixed) and optimizing
the estimate of the parameters $\Theta$ (for $Z$ fixed). Hence, $\Theta Z$
is linear in each subproblem. More precisely, the iteration $k$
of the optimization subproblems using the data at time step $t$ are given by

\[
\hat{Z}_{t,k} = \arg \min_{Z} \left\| \begin{bmatrix} \hat{\Theta}_{t,k} \\ I_{n+n_u} \end{bmatrix} Z - \begin{bmatrix} Y_{i,t} \\ Z_{i,t} \end{bmatrix} \right\|_{F1}, \tag{8a}
\]

\[
\hat{\Theta}_{t,k+1} = \arg \min_{\Theta} \| \Theta \hat{Z}_{t,k} - Y_{i,t} \|_{F1} + \| \Psi \vec{v} \Theta \|_1. \tag{8b}
\]

The update (8a) estimates $\hat{Z}_{t,k}$, the $Z$ portion of the
correlation matrix at the iteration $k$ of the optimization using
the data at time $t$. It does so based on observations $Y_{i,t}$ and
$Z_{i,t}$, and the current parameter estimates $\hat{\Theta}_{t,k}$. The following
Lemma shows how (8a) can be decomposed into simpler
problems.

**Lemma 1.** The optimization problem (8a) can be split
into $n+n_u$ independent optimization problems, each only
depending on one column $Z_i$ of $Z$.

\[
\arg \min_{Z_i} \left\| \begin{bmatrix} \hat{\Theta}_{t,k} \\ I_{n+n_u} \end{bmatrix} Z_i - \begin{bmatrix} Y_{i,t} \\ Z_{i,t} \end{bmatrix} \right\|_{F1}, \tag{9}
\]

**Proof.** From the definition of the $\ell_1$ Frobenius norm, the cost in (8a) is composed of $n+n_u$ terms, each depending
only on one column of $\begin{bmatrix} \hat{\Theta}_{t,k} \\ I_{n+n_u} \end{bmatrix} Z_i$. This column is equal to

\[
\begin{bmatrix} \hat{\Theta}_{t,k} \\ I_{n+n_u} \end{bmatrix} Z_i. \tag{10}
\]

The proof is concluded by using the distribution
property of the arg min, i.e. $\arg \min_{x,y} af(x) + bg(y) = \arg \min_{x} f(x), \arg \min_{y} g(y). \tag{11}$

Using Lemma 1, we can update all columns $\hat{Z}_{i,t,k}$
independently using iterative reweighting [26]. Given an integer
$L_Z$, for an outer iteration $k$, iterative reweighting introduces the following $L_Z$ inner iterations indexed by $\ell = 1, \ldots, L_Z$
to approximate the $\ell_1$ norm:

\[
\hat{Z}_{i,t,k,\ell+1} = \arg \min_{Z_i} \left\| \begin{bmatrix} \hat{\Theta}_{t,k} \\ I_{n+n_u} \end{bmatrix} Z_i - \begin{bmatrix} Y_{i,t} \\ Z_{i,t} \end{bmatrix} \right\|_{W_{i,t,k,\ell}}^2, \tag{12}
\]

\[
W_{i,t,k,\ell}^{-1} = \sqrt{\text{diag} \left( \begin{bmatrix} \hat{\Theta}_{t,k} \hat{Z}_{i,t,k,\ell} - Y_{i,t} \\ \hat{Z}_{i,t,k,\ell} - Z_{i,t} \end{bmatrix} \right)^2 + \alpha I_{n+m}, \tag{13}
\]

where $0 < \alpha \ll 1$ is a small parameter introduced for numerical stability. After the last inner iteration, $\hat{Z}_{i,t,k,L_Z}$
is used as an approximate solution to (8a). Note that for each
time step $t$, we now have a double nested loop over $k$ and $\ell$,
which may be very slow in practice. A remedy for this issue
is described in Section IV.

Next, we analyze the parameter update (8b), which estimates $\hat{\Theta}_{t,k+1}$ based on $\hat{Z}_{t,k,L_Z}$. Unlike (8a), the problem
(8b) cannot easily be split into subproblems. However, one
can vectorize the parameters to simplify (8b) using iterative
reweighting.

\[
\theta = \vec{v}(\Theta), \quad \hat{\Theta}_{t,k+1} = \vec{v}(\hat{\Theta}_{t,k+1}), \quad \hat{Z}_{t,k} = \hat{Z}_{t,k}^T \otimes I_n \tag{14}
\]

Since $\vec{v}(XY) = (Y^T \otimes I) \vec{v}(Y), (11)$ gives

\[
\vec{v}(\Theta \hat{Z}_{t,k} - Y_{i,t}) = \hat{Z}_{t,k} \theta - \vec{v}(Y_{i,t}), \tag{15}
\]

and, therefore, (8b) can be written as

\[
\arg \min_{\theta} \left\| \begin{bmatrix} \hat{Z}_{t,k} \\ \Theta \end{bmatrix} \theta - \begin{bmatrix} \vec{v}(Y_{i,t}) \\ \mu \end{bmatrix} \right\|_1. \tag{16}
\]

Similar to (10), problem (13) can be solved for each
iteration $k$ using iteratively reweighted inner iterations $\ell = 1, \ldots, L_{\Theta}$ to approximate the $\ell_1$ norm [26], i.e.

\[
\hat{\Theta}_{t,k+1,\ell+1} = \arg \min_{\theta} \left\| \begin{bmatrix} \hat{Z}_{t,k} \\ \Theta \end{bmatrix} \theta - \begin{bmatrix} \vec{v}(Y_{i,t}) \\ \mu \end{bmatrix} \right\|_{V_{t,k+1,\ell}}^2, \tag{17}
\]

\[
V_{t,k+1,\ell}^{-1} = \sqrt{\text{diag} \left( \begin{bmatrix} \hat{Z}_{t,k} \hat{\theta}_{t,k+1,\ell} - \vec{v}(Y_{i,t}) \\ \Theta \hat{\theta}_{t,k+1,\ell} - \mu \end{bmatrix} \right)^2 + \alpha I_{n+m}, \tag{18}
\]

The quantity $\hat{\Theta}_{t,k,L_{\Theta}}$ obtained in the last iteration is an approximate solution to (8b).

**B. The overall algorithm**

The estimation procedure alternates between (8a) and (8b), solved iteratively using (10) and (14), respectively. The full
implementation of both outer and inner loops at each time
instant is provided in Algorithm 1.
Algorithm 1

\[
\begin{align*}
C_0 &= I_m \\
\text{for } t = 1, \ldots, N \text{ do} \\
\quad C_t &\leftarrow \beta C_{t-1} + [\tilde{x}_t, \tilde{x}_{t-1}, \tilde{u}_{t-1}]^T [\tilde{x}_t, \tilde{x}_{t-1}, \tilde{u}_{t-1}] \\
\quad \Theta_{t,k=0} &\leftarrow \Theta_{t-1,K} \\
\quad Z_{t,k=0} &\leftarrow Z_{t-1,K} \\
\text{for } k = 1, \ldots, K \text{ do} \\
\quad Z_{tk,t=0} &\leftarrow Z_{tk,k-1} \\
\quad \Theta_{tk,t=0} &\leftarrow \Theta_{tk,k-1} \\
\text{for } \ell = 1, \ldots, L_Z \text{ do} \\
\quad &\text{minimize (10) to obtain } \hat{Z}_{tk,\ell} \\
\text{end for} \\
\quad \hat{Z}_{tk} &\leftarrow \hat{Z}_{tk,L_Z} \\
\text{for } \ell = 1, \ldots, L_\Theta \text{ do} \\
\quad &\text{minimize (14) to obtain } \hat{\Theta}_{tk,\ell} \\
\text{end for} \\
\quad \hat{C}_t &\leftarrow \hat{E}_g \hat{E}_g C_t + E_\perp \hat{Z}_{tk,K} \\
\text{end for} \\
\end{align*}
\]

Algorithm 1 provides a robust solution for the joint state/input and parameter estimation problem using a fixed-size matrices, which suits online application. However, the nested loops are often too slow for real-time application. We will therefore not study the convergence of Algorithm 1. Instead, we will study the convergence of a more computationally efficient version, presented in the next section.

IV. ALTERNATING AND ITERATIVELY-WEIGHTED RECURSIVE LEAST SQUARES (AIRLS)

In this section, we will first show how to easily compute the optimizers of (10) and (14), and then prove that Algorithm 1 converges when \( K = L_Z = L_\Theta = 1 \). Problems (10) and (14), admit a closed-form solution, as discussed in the following.

**Definition 1.** For any pair of matrices \( X \) and \( W \) such that \( WX \) exists and has full column rank, the weighted pseudo-inverse is \( X^{+}_w = (X^T WX)^{-1}X^T W \).

Note that, by construction \( X^T W X = X \).

The problem (14) is quadratic and solved by

\[
\hat{\Theta}_{tk+1,\ell+1} = \hat{Z}_{tk}^\dagger \left[ \begin{array}{c} \text{vec}(Y_{t}) \\ \mu \end{array} \right]_{V_{tk+1,\ell}}.
\]

Computing (15) amounts to solve a linear system with as many equations as the number of parameters in \( \Theta \). The problem (10) is composed of \( N \) multivariate optimization problems, which can all be solved by an oblique projection of the \( i^{th} \) column of \( C_t \) on the null space of \([-I_n, \Theta_{tk}]\), weighted by \( W_{t,k\ell} \).

**Theorem 2.** The problem (10) is solved by

\[
\hat{Z}_{t,tk+1,\ell+1} = E_\perp P_{\ell,t} C_{t,i^t},
\]

where

\[
P_{\ell,t} = I_m - ((-I_n, \hat{\Theta}_{tk})^T W_{t,k\ell})^T [-I_n, \hat{\Theta}_{tk}].
\]

**Proof.** For a basis \( B \) such that \( \text{range}(B) = \text{null}([-I_n, \hat{\Theta}_{tk}]) \), \( P_{\ell,t} \) provides the weighted least squares solution [33]

\[
P_{\ell,t} [C_t]_i = \arg \min_d \| d - [C_t]_i \|_{W_{t,k\ell}}^2, \quad \text{s.t. } \ d \in \text{range}(B).
\]

Choosing \( d = [Y^T, Z^T]^T \), the problem (18) becomes

\[
P_{\ell,t} [Y^T, Z^T] = \arg \min \| Y_i^T Z_i^T - [Y^T, Z^T]_i^T \|_{W_{t,k\ell}}^2, \quad \text{s.t. } [-I_n, \hat{\Theta}_{tk}] [Y_i^T Z_i^T] = 0_n.
\]

Plugging the constraint to replace \( Y_i \) in (19) yields exactly (10) with \( t + 1 \) instead of \( t \).

**Corollary 3.** The estimate of the state and input at a particular time step \( t \) is given by

\[
\hat{x}_{t+1} = P_{x,t} k [\hat{x}_{t+1}, \hat{x}_t, \hat{u}_t]^T,
\]

where \( P_{x,t} \) is defined by (17) with

\[
W_{x,t}^{-1} = \sqrt{\text{diag}(\left[ \hat{\Theta}_{tk}, [\hat{x}_t^T, \hat{u}_t]^T \right]^T [-\hat{x}_{t+1}, \hat{u}_t^T])^2} + \alpha I_m.
\]

**Proof.** The proof is given by replacing \( [C_t]_i \) by \( [\hat{x}_t^T, \hat{u}_t]^T \) and \( \hat{Z}_{tk,\ell} \) by \( [\hat{x}_{tk,\ell}, \hat{u}_{tk,\ell}]^T \) in (10) and in the proof of Theorem 2.

Computing the projector \( P_{x,t} \) may be expensive due to the pseudo-inverse (17). However, (16) only requires the projection of the correlation matrix \( C_t \) (i.e. \( n_a + 2n \) vectors), which is much faster to compute.

**A. AIRLS estimator**

The AIRLS algorithm is defined as Algorithm 1 with \( K = L_Z = L_\Theta = 1 \) and where (16) and (15) are used for computing the optimizers of (10) and (14), respectively. It has the computational advantage of replacing the nested loops in Algorithm 1 with one-step updates. In a sense, the robustness provided by the \( \ell_1 \) cost and the regularization term in (7) help compensate for the unfinished loops. In the sequel, for simplicity, we will drop the subscripts \( k = 1 \) and \( \ell = 1 \).

**Definition 2.** Let \( \Gamma_t \) represent an average measurement of the system such that the corresponding asymptotic correlation matrix \( \sum_{i=0}^{\infty} \beta^i \Gamma_t = C_t \). Similarly to the inverse power method [14], AIRLS needs \( C_t \) to be full rank to ensure convergence. Numerical experiments in Section V show that it also converges when \( C_t \) has rank \( n + n_u \), i.e. in the noiseless case.

**Theorem 4 (convergence).** With bounded measurements \( Y_t \leq \gamma_{\text{max}} I_m \) for all \( t \), and with \( \sum_{i=0}^{\infty} \beta^i \gamma_{\text{min}} I_m, a \) forgetting factor satisfying \( 1 - \beta \leq \gamma_{\text{max}}^2 \gamma_{\text{min}} \) guarantees that the
AIRLS update converges and can only decrease unweighted residuals of (14), i.e.
\[ R(\hat{\Theta}_t, C_t) = \hat{Z}_t \Psi \text{vec}(\hat{\Theta}_t) - Y \] 
\[ \hat{Z}_t \Psi \text{vec}(\hat{\Theta}_t) - Y \]

Proof. Using (12) and (15) to write \( R(\hat{\Theta}_{t+1}, C_{t+1}) \) depending on \( C_t \) and \( \hat{\Theta}_t \) yields
\[ \left[ \hat{Z}_t \Psi \text{vec}(\hat{\Theta}_{t+1}) - \left[ \begin{array}{c} Y_t \\ \mu_t \end{array} \right] \right] = \left( I_{n(n+n_u)+m} - \left[ \begin{array}{cc} \hat{Z}_t \Psi \text{vec}(\hat{\Theta}_t) - Y_t \\ \mu_t \end{array} \right] \right) \left[ \hat{Z}_t \Psi \text{vec}(\hat{\Theta}_t) - Y_t \right] \]

By definition, \( X \Psi_i \) is an oblique projection for any \( X \) and \( W \), which means that \( \| R(\hat{\Theta}_{t+1}, C_{t+1}) \|_2^2 \leq \| R(\hat{\Theta}_t, C_t) \|_2^2 \).

We now compare \( \| R(\hat{\Theta}_t, C_t) \|_2^2 \) to \( \| R(\hat{\Theta}_t, C_{t+1}) \|_2^2 \).

First, note that the second block in (22) does not contain \( C_t \). This means that \( \| R(\hat{\Theta}_t, C_t) \|_2^2 \) is equal to \( \| \text{vec}(\hat{Z}_t \Psi \text{vec}(\hat{\Theta}_t) - Y_t) \|_2^2 \) when \( C_t \) is dropped.

Second, we write the following decomposition:
\[ \| \text{vec}(\hat{Z}_t \Psi \text{vec}(\hat{\Theta}_t) - Y_t) \|_2^2 = \sum_{i=1}^{m} \| \text{vec}(\hat{Z}_t \Psi \text{vec}(\hat{\Theta}_t) - Y_t) \|_2^2 \].

Third, we write the closed form solution of (10) (which is equal to (16)) as
\[ \hat{Z}_{t+1} = \left( I_{n(n+n_u)+m} - \left[ \begin{array}{cc} \hat{Z}_t \Psi \text{vec}(\hat{\Theta}_t) - Y_t \\ \mu_t \end{array} \right] \right) \left[ \hat{Z}_t \Psi \text{vec}(\hat{\Theta}_t) - Y_t \right] \]

Similar to (23), we can construct a projection with \( X = \left[ \hat{\Theta}_t^T, I_{n(n+n_u)+m} \right]^T \) instead of \( \left[ \hat{Z}_t \Psi, \Psi^T \right]^T \):
\[ X \hat{Z}_{t+1} = \left( \hat{Z}_t \Psi \text{vec}(\hat{\Theta}_t) - Y_t \right) \]

Moreover, because of the last step of Algorithm 1,
\[ \| [-I_n, \hat{\Theta}_t] C_{t+1} \|_2^2 = \left\| \hat{\Theta}_t \hat{Z}_{t+1} - E_g (\beta C_{t+1} + \Gamma_{t+1}) \right\|_2^2 \]
\[ \leq \left\| \hat{\Theta}_t \hat{Z}_{t+1} - E_g (\beta C_{t+1} + \Gamma_{t+1}) \right\|_2^2 \]
\[ \leq \left\| \hat{\Theta}_t \hat{Z}_{t+1} - E_g (\beta C_{t+1} + \Gamma_{t+1}) \right\|_2^2 \]

Combining (27) with the projection (28) yields
\[ \| [-I_n, \hat{\Theta}_t] C_{t+1} \|_2^2 \leq \| [-I_n, \hat{\Theta}_t] (\beta C_{t+1} + \Gamma_{t+1}) \|_2^2 \]

because \( \| X E_g - I_m \| = \left[ \begin{array}{c} [-I_n, \hat{\Theta}_t] \end{array} \right] \). Hence,
\[ \| R(\hat{\Theta}_t, C_{t+1}) \|_2^2 \leq \| R(\hat{\Theta}_t, \beta C_{t+1} + \Gamma_{t+1}) \|_2^2 \]

Finally, to prove that \( \| R(\hat{\Theta}_{t+1}, C_{t+1}) \|_2^2 \leq \| R(\hat{\Theta}_t, C_{t+1}) \|_2^2 \)
we need \( \| [-I_n, \hat{\Theta}_t] (\beta C_{t+1} + \Gamma_{t+1}) \|_2^2 \leq \| [-I_n, \hat{\Theta}_t] C_{t+1} \|_2^2 \)
which is true if
\[ \| [-I_n, \hat{\Theta}_t] \Gamma_{t+1} \|_2^2 \leq (1 - \beta^2) \| [-I_n, \hat{\Theta}_t] C_{t+1} \|_2^2 \].

The assumption that \( \tilde{\gamma}_t \leq \gamma_{\max} I_m \) and that \( \tilde{\gamma}_t \geq \gamma_{\min} I_m \)
ensure (29) if
\[ \frac{\gamma_{\max}^2}{\gamma_{\min}^2} \leq \frac{1 - \beta^2}{(1 - \beta^2)^2} \]

which is guaranteed if \( 1 - \beta \leq \gamma_{\max}^{-2} \gamma_{\min} \). The function \( \| R(\hat{\Theta}_t, C_{t+1}) \|_2^2 \) is therefore decreasing and lower bounded, which proves the theorem.

V. NUMERICAL EXPERIMENTS

In this section, we will compare the parameter estimates and state predictions in the asymptotic regime using AIRLS with \( \Psi = 10^{-3} I_8 \) and \( \mu = 0 \), the EKF from [6], the RTLS from [14] and subspace identification. For the latter, we use the batch method provided by the function n4sid in MATLAB [28]. We use the system
\[ x_{t+1} = \begin{bmatrix} 0.8 & -0.25 \\ -0.25 & 0.25 \end{bmatrix} x_t + \begin{bmatrix} 10 & 2 \\ 2 & 10 \end{bmatrix} u_t, \]

with a random persistent excitation \( u \sim N(0, 0.01I_2) \). We add weak Gaussian measurement noise with a signal to noise ratio of 100 to all samples, and much stronger noise (uniformly distributed in \([-0.2, 0.2]\)) for a small portion of randomly chosen samples, varying between 0.02% and 5% of all samples. The points affected by the strong noise are outliers. For each proportion of outliers, we average the estimates of 10 different experiments.

Fig. 1 shows the relative Frobenius error
\[ \epsilon_F = \frac{\| [A, B] - [\hat{A}_N, \hat{B}_N] \|_F}{\| [A, B] \|_F} \]

for all 4 methods and various proportions of outliers. We observe that both subspace identification and EKF get large errors with as low as 0.1% outliers. This means that they may perform poorly even with the help of an outlier detection system that is not 100% accurate. The RTLS is much more robust, but still performs much worse than AIRLS.

Fig. 2 shows the state estimation for \( t > 50000 \), i.e. when the parameters have converged. The error on parameters manifests as excessive smoothing of the state estimation.

1Batch estimation is expected to outperform any recursive implementation for the same sample size.
We conclude by highlighting that with standard Gaussian noise and no outliers, all methods achieve similar performance, and that without any noise, all methods have 100% accuracy.

VI. CONCLUSIONS

We show that AIRLS, an algorithm that combines recursive, alternating, and iteratively-reweighted least squares, converges and allows one to perform robust and online joint state/input and parameter estimation for linear systems. Numerical experiments show that the accuracy of the AIRLS estimates is higher than state-of-the-art methods in the presence of outliers.

Future work includes extending AIRLS to more general loss functions and noise distributions. Practical applications, including power systems and self driving cars will also be addressed.

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