Lagrange vs. Lyapunov stability of hierarchical triple systems: dependence on the mutual inclination between inner and outer orbits

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ABSTRACT

While there have been many studies examining the stability of hierarchical triple systems, the meaning of “stability” is somewhat vague and has been interpreted differently in previous literatures. The present paper focuses on “Lagrange stability”, which roughly refers to the stability against the escape of a body from the system, or “disruption” of the triple system, in contrast to “Lyapunov-like stability” that is related to the chaotic nature of the system dynamics. We compute the evolution of triple systems using direct N-body simulations up to $10^7 P_{\text{out}}$, which is significantly longer than previous studies (with $P_{\text{out}}$ being the initial orbital period of the outer body). We obtain the resulting disruption timescale $T_d$ as a function of the triple orbital parameters with particular attention to the dependence on the mutual inclination between the inner and outer orbits, $i_{\text{mut}}$. By doing so, we have clarified explicitly the difference between Lagrange and Lyapunov stabilities in astronomical triples. Furthermore, we find that the von Zeipel-Kozai-Lidov oscillations significantly destabilize inclined triples (roughly with $60^\circ < i_{\text{mut}} < 150^\circ$) relative to those with $i_{\text{mut}} = 0^\circ$. On the other hand, retrograde triples with $i_{\text{mut}} > 160^\circ$ become strongly stabilized with much longer disruption timescales. We show the sensitivity of the normalized disruption timescale $T_d/P_{\text{out}}$ to the orbital parameters of triple system. The resulting $T_d/P_{\text{out}}$ distribution is practically more useful in a broad range of astronomical applications than the stability criterion based on the Lyapunov divergence.

Keywords: celestial mechanics - (stars:) binaries (including multiple): close - stars: black holes
Hierarchical triple systems play a very basic and important role in a broad range of astronomical phenomena. Their dynamical stability has numerous implications for actual systems (e.g., Jha et al. 2000; Naoz et al. 2013; Antonini et al. 2014; Ransom et al. 2014; Perpinyà-Valles et al. 2019; Rivinius et al. 2020; Bodensteiner et al. 2020; Toonen et al. 2020; Tokovinin & Latham 2020; Hayashi et al. 2020; Hayashi & Suto 2020). Furthermore, it remains as one of the most long-standing questions in celestial mechanics (e.g. Eggleton & Kiseleva 1995; Mardling & Aarseth 1999, 2001; Georgakarakos 2013; Grishin et al. 2017; He & Petrovich 2018; Mylläri et al. 2018; Wei et al. 2021; Lalande & Trani 2022; Tory et al. 2022; Vynatheya et al. 2022).

Among all, the dynamical stability criterion for hierarchical triples proposed by Mardling & Aarseth (2001, hereafter MA01) is well-known and widely used in various studies of the triple dynamics. They examine the stability of triples as follows (R. Mardling, private communication). First, they derive the analytic expression for the stability criterion based on the chaotic evolution boundary (see also Mardling 1995a,b). In order to determine the numerical factor, they integrate two almost identical systems (the given system and its “ghost”) except for their different inner eccentricities $e_{in}$ of $\Delta e_{in} = 10^{-7}$, and monitor the difference in the inner semi-major axes $a_{in}$ at the time of outer apocenter passage. This variable is expected to be fairly insensitive to the initial difference for a stable system, while grow exponentially for chaotic and unstable systems. This consideration motivated MA01 to judge and classify the stable and unstable systems from the departure of $a_{in}$ evaluated at 100 times the outer orbital period, $P_{out}$ (see also Mardling 2008).

The Lyapunov stability is mathematically defined as the stability of solutions of dynamical systems near their fixed (equilibrium) points, but has been also interpreted as the stability around arbitrary solutions (or trajectories) of dynamical systems against the tiny changes of system parameters (see, for instance Lichtenberg & Lieberman 1983; Kandrup 1990; Suto 1991; Lichtenberg & Lieberman 1992). The methodology of MA01 is devised to identify the chaotic nature of triple systems in the spirit similar to the Lyapunov stability in the latter sense. We note that their methodology is close to the Lyapunov exponent method, which is widely used to characterize the chaoticity of systems. Therefore, we refer their stability condition as the Lyapunov stability below in a broader sense.

In any case, the Lyapunov stability or the local chaoticity may not be directly used in judging the global fate of the astronomical triple systems in reality. Even if triple systems are locally chaotic, their orbital configurations may be bounded, i.e., Lagrange stable. Indeed, the latter stability would correspond to a more relevant and practical definition in most applications for aircrafts and nuclear-plants (Gyftopoulos 1963), as well as for astronomical triple systems of our interest.

It seems that the stability criterion of triples derived by MA01 is sometimes misunderstood, and used as a criterion of triple disruption in the sense of Lagrange instability. Recently Hayashi et al. (2022, hereafter Paper I) computed the disruption timescales $T_d$ of triples by directly integrating the three-body dynamics up to $10^9 P_{in}$, where $P_{in}$ is the initial orbital period of the inner binary. The distribution of the normalized disruption timescale $T_d/P_{in}$ follows an approximate scaling similar to the MA01 criterion for coplanar-prograde triples in which the mutual inclination between inner and outer orbits $i_{mut}$ is $0^\circ$. The orthogonal and coplanar-retrograde triples, however, exhibit very different behavior, illustrating the clear difference between Lagrange and Lyapunov stabilities. In particular, Paper I finds that the von Zeipel-Kozai-Lidov (ZKL) oscillations (von Zeipel 1910; Kozai 1962; Lidov 1962) play a major role in disrupting significantly inclined triples. Because the characteristic ZKL
timescale $\tau_{ZKL}$ is comparable or even longer than $100 P_{\text{out}}$ in general, their Lagrange stability cannot be determined from the short-term local behavior that characterizes Lyapunov stability.

The relation between the chaoticity and Lagrange stability is subtle. Figure 6 of Paper I illustrated such examples; the lower panels in the figure show that a Lagrange-unstable triple with $T_d \approx 3000 P_{\text{in}}$ is robust against the change of the input initial conditions, unlike the other two cases (the upper and middle panels). Therefore, Paper I concluded that a Lagrange-unstable triple is not necessarily sensitive to the initial conditions. More strictly, a Lagrange-unstable triple may become chaotic eventually (just at the last moment of the disruption), but it is not clear how long it takes for them to exhibit noticeable chaoticity. Indeed, this is what we intend to examine quantitatively in Paper I and the present paper; we record the disruption timescales during the simulations, not merely the stable/unstable outcome alone.

In addition, we would like to emphasize that a chaotic triple does not always necessarily become Lagrange unstable. Thus, the Lagrange instability cannot be judged simply from the first few hundred (or fewer) orbits of a trajectory to its nearby ghost as MA01 assumed. This is exactly why we have performed a significantly longer time-integration (up to $10^9 P_{\text{in}}$ in Paper I and $10^7 P_{\text{out}}$ in the present paper) to check the Lagrange instability of various triples. Only with such direct confirmation of the disruption timescales, one can understand the relation between the Lagrange instability and the chaotic behavior of the triple systems.

This point is also discussed in a different approach recently by Gajdoš & Vaňko (2023), who found that several observed exoplanetary systems have relatively short Lyapunov times (calculated from the Lyapunov exponent), even down to $O(10)$ yrs. This fact implies that the Lyapunov time is not necessarily related to the Lagrange stability of the systems, in good agreement of our finding mentioned in the above.

This paper examines the disruption timescale distribution for hierarchical triples using the same methodology of Paper I, and addresses the following questions more specifically: (i) the relation between Lyapunov stability criterion and Lagrange stability as a function of disruption timescales of triples with different orbital configurations, (ii) the dependence of disruption timescales on the mutual inclination, $i_{\text{mut}}$, between the inner and outer orbits, and (iii) the effect of the relative orbital phases among the three bodies that has not been explicitly studied in most previous literatures.

2. METHOD

We consider hierarchical triple systems comprising an inner binary ($m_1$ and $m_2$), and a tertiary ($m_3$). Since one of the main purposes of the present paper is to identify the mutual inclination dependence on the disruption timescale, we simply fix the mass ratio as $m_1 = m_2 = 5m_3$, corresponding to $q_{21} \equiv m_2/m_1 = 1$ and $q_{23} \equiv m_2/m_3 = 5$ in Paper I. The numerical simulation employs the $N$-body integrator TSUNAMI (Trani et al. 2019, Trani et al., in prep.) that is specifically designed to accurately simulate few-body systems; further details of the code can be found in Trani et al. (2019); Trani & Spera (2022) and Paper I.

We consider an initially circular inner orbit with fixing the inner mean anomaly $M_{\text{in}}$ and the inner pericenter argument $\omega_{\text{in}}$. We specify their explicit values for each result below. In the barycentric reference frame whose $xy$-plane is chosen as the invariant plane perpendicular to its angular momentum vector, the inner and outer longitudes of the ascending node differ by $180^\circ$ (see also Fig. 1 of Paper I). Without a loss of generality, we set them to $\Omega_{\text{in}} = 180^\circ$ and $\Omega_{\text{out}} = 0^\circ$. Therefore, the phase information of the initial conditions is then fully specified by the values of the outer pericenter
argument and mean anomaly, $\omega_{\text{out}}$ and $M_{\text{out}}$. We fix these values first, and vary them randomly to check the initial phase dependence in section 3.3.

We adopt the definition of the disruption time $T_d$ following Paper I and similar to Manwadkar et al. (2020, 2021); at each timestep, the integrator evaluates the binding energy for each of the three pairs of bodies, i.e. $(m_1, m_2)$, $(m_1, m_3)$ and $(m_2, m_3)$. The pair with the highest (negative) binding energy is considered as the inner binary, and we call the pair of the inner binary and the remaining tertiary as the outer pair. When the binding energy of the outer pair becomes positive and the radial velocity of the tertiary becomes positive (i.e. is moving away from the inner binary), we record the system as a candidate of the disrupted system tentatively. We found that the disruption normally occurs very fast after one body is ejected from a system (see also Figure 14 in Paper I).

A fraction of them, however, just represent a transient unbound state, and become gravitationally bound again. Thus, in order to make sure the system is truly disrupted, we continue the run until its binary-single distance exceeds 20 times the binary semi-major axis. In that case, we stop the run, and the triple is classified as disrupted with recording the epoch as $T_d$. Otherwise, we reset the disruption candidate condition, and keep running the integration of the system.

Finally, we define a triple as (Lagrange) stable when $T_d > t_{\text{int}}$, i.e. when it does not disrupt and instead survives until our maximum integration time $t_{\text{int}}$. Nevertheless, we would like to emphasize again that the above definition is just for convenience, and the main purpose of the present paper is to study the disruption timescales related to the Lagrange stability. In the present paper, we set $t_{\text{int}} = 10^7 P_{\text{out}}$ unless otherwise specified, in contrast to Paper I that adopts $t_{\text{int}} = 10^9 P_{\text{in}}$, because the current paper expresses the normalized disruption time $T_d/P_{\text{out}}$ instead of $T_d/P_{\text{in}}$. In reality, however, the result is identical except for the boundary of stable and unstable triples. We also note that under Newtonian gravity alone, the distribution of $T_d/P_{\text{out}}$ is scale-free (Paper I). So, if we consider an outer orbit of $P_{\text{out}} = 10^3$ yr, for instance, the stopping time of our simulations corresponds to $t_{\text{int}} = 10$ Gyr, i.e., the present age of the universe. Even for $P_{\text{out}} = 10$ yr, $t_{\text{int}} = 10^8$ yr would be a practically reasonable timescale to consider the fate of astronomical triples.

We emphasize that the above definition of the disruption is expected to correspond to the Lagrange stability, instead of the Lyapunov stability studied by MA01. The Lyapunov instability may be a sufficient condition to the Lagrange instability (for infinite future). In practice, however, Lagrange instability is difficult to be identified with a limited computational time and its numerical accuracy. Thus we decided to examine the corresponding disruption timescales up to $10^7 P_{\text{out}}$, and not to repeat the Lyapunov-type analysis in the present paper. We simply clarify that the two stabilities are different concepts, and their relation is not trivial in general. As we show below, Lyapunov stability, at least the criterion by MA01 that involves an empirical extrapolation of the inclination dependence, is neither a necessary nor sufficient condition for Lagrange stability up to $10^7 P_{\text{out}}$.

3. RESULT

3.1. Previous models of the disruption timescale and Lyapunov stability criteria

In order to illustrate the basic idea behind the present analysis, we show in Figure 1 an example of the distribution of the disruption timescale $T_d$ for the simulated orthogonal triples on $e_{\text{out}} - r_{\text{p, out}}/a_{\text{in}}$ plane. Here, $e_{\text{out}}$ is the eccentricity of outer orbit, and $r_{\text{p, out}}/a_{\text{in}}$ is the ratio of the pericenter distance of outer orbit and the semi-major axis of inner orbit.
Following Paper I, we sequentially perform the simulations at each $e_{\text{out}}$ from lower to higher values of $r_{p,\text{out}}/a_{\text{in}}$. Once the previous two realizations at lower $r_{p,\text{out}}/a_{\text{in}}$ do not disrupt before $t_{\text{int}}$, we stop further simulations for one sequence, and define the boundary to save computational cost. The detail of this procedure is described in Paper I. The left and right panels in Figure 1 plot $T_d/P_{\text{in}}$ and $T_d/P_{\text{out}}$, respectively. While the information context of the two panels is essentially the same, their comparison provides a useful insight into the purpose of the present paper.

For reference, we overlay the disruption timescale contours predicted from the random walk (RM) model by Mushkin & Katz (2020) in black solid lines:

\[
T_d/P_{\text{in}} = 2 \left( \frac{m_{123}m_{12}}{m_1m_2} \right)^2 \sqrt{1 - e_{\text{out}}} \left( \frac{r_{p,\text{out}}}{a_{\text{in}}} \right)^{-2} \exp \left[ \frac{4\sqrt{2}}{3} \sqrt{\frac{m_{12}}{m_{123}}} \left( \frac{r_{p,\text{out}}}{a_{\text{in}}} \right)^{3/2} \right],
\]

and

\[
T_d/P_{\text{out}} = 2 \left( \frac{m_{12}}{m_1m_2} \right)^2 \left( \frac{m_{123}}{m_{12}} \right)^{5/2} (1 - e_{\text{out}})^2 \left( \frac{r_{p,\text{out}}}{a_{\text{in}}} \right)^{-7/2} \exp \left[ \frac{4\sqrt{2}}{3} \sqrt{\frac{m_{12}}{m_{123}}} \left( \frac{r_{p,\text{out}}}{a_{\text{in}}} \right)^{3/2} \right],
\]

where $m_{123} \equiv m_1 + m_2 + m_3$ is the total mass of the system, and $m_{12} \equiv m_1 + m_2$ is the total mass of inner binary. Equations (1) and (2) serve as useful analytical estimates for the disruption timescale of Lagrange-unstable systems.

In Figure 1, we also include the dynamical stability criteria by MA01 and Vynatheya et al. (2022) as magenta solid and dashed lines, respectively. Vynatheya et al. (2022) defined the boundary from their simulations at $100P_{\text{out}}$ by selecting the triples whose inner and outer semi-major axes do not change by more than 10 percent of the initial values. This is not exactly the same condition adopted by MA01, but is expected to be similar from the viewpoint of Lyapunov stability.

According to them, those triples with $r_{p,\text{out}}/a_{\text{in}}$ exceeding the critical thresholds below are Lyapunov stable, respectively:

\[
\left( \frac{r_{p,\text{out}}}{a_{\text{in}}} \right)_{\text{MA}} \equiv 2.8 \left( 1 - 0.3 \frac{i_{\text{mut}}}{180^\circ} \right) \left( 1 + q_{\text{out}} \right) \frac{(1 + e_{\text{out}})}{\sqrt{1 - e_{\text{out}}}} \left[ \frac{1}{2} \right]^{2/5},
\]

and

\[
\left( \frac{r_{p,\text{out}}}{a_{\text{in}}} \right)_{V} \equiv 2.4 (1 + \tilde{e}_{\text{in}}) \left[ \frac{1 + q_{\text{out}}}{\sqrt{1 - e_{\text{out}}(1 + \tilde{e}_{\text{in}})}} \right]^{2/5} \times \left[ \frac{1 - 0.2 \tilde{e}_{\text{in}} + e_{\text{out}}}{8} \right] (\cos i_{\text{mut}} - 1) + 1,
\]

where $q_{\text{out}} \equiv m_3/m_{12}$, and $\tilde{e}_{\text{in}}$ is defined as

\[
\tilde{e}_{\text{in}} \equiv \max \left( e_{\text{in}}, 0.5 \left( 1 - \frac{5}{3} \cos^2 i_{\text{mut}} \right) \right).
\]

The left panel of Figure 1 is adapted from the simulation results with $t_{\text{int}} = 10^9 P_{\text{in}}$ in Paper I. Due to the chaotic nature of three-body dynamics, the distribution of $T_d$ exhibits relatively large scatters; see Figures 6, 7, 16, 17 and 18 in Paper I. Nevertheless, the values of $T_d/P_{\text{in}}$ are basically
Figure 1. Disruption timescale distribution for orthogonal triples ($i_{\text{mut}} = 90^\circ$) with an equal-mass inner binary ($m_1 = m_2$) and the tertiary of $m_3 = m_1/5$. We adopt ($\omega_{\text{in}}, M_{\text{in}}, \omega_{\text{out}}, M_{\text{out}}$) = ($180^\circ, 30^\circ, 0^\circ, 45^\circ$). The disruption timescales $T_d$ are normalized in units of $P_{\text{in}}$ (left) and $P_{\text{out}}$ (right). The magenta solid and dashed curves represent the dynamical stability criteria, Mardling & Aarseth (2001) and Vynatheya et al. (2022) (see equations (3) and (4)), respectively. The black counters show the Random Walk model estimations of disruption times from Mushkin & Katz (2020) (see equations (1) and (2)). Cross symbols indicate the triples that have not been broken until $t_{\text{int}}$. Left and right plots adopt $t_{\text{int}} = 10^9 P_{\text{in}}$ and $t_{\text{int}} = 10^7 P_{\text{out}}$, respectively. We note that the left plot is adapted from Paper I.

determined by the value of $r_{p,\text{out}}/a_{\text{in}}$. This is consistent with the RW model prediction, equation (1), which suggests that $T_d/P_{\text{in}}$ depends on $e_{\text{out}}$ very weakly ($\propto \sqrt{1 - e_{\text{out}}}$). The right panel shows $T_d/P_{\text{out}}$ from our new simulations with $t_{\text{int}} = 10^7 P_{\text{out}}$.

Since equations (3) and (4) represent the thresholds for Lyapunov stability, there is no reason why their criteria are directly related to the disruption timescale characterizing Lagrange stability. Figure 1 indicates that their Lyapunov stability boundaries indeed correspond to a broad range of disruption timescales, $10^2 < T_d/P_{\text{out}} < 10^5$ in our simulation runs. In other words, their criteria should not be interpreted as a black-and-white boundary of the disruption of triples. This seems to be a common misinterpretation in works studying triple stability (R. Mardling, private communication). This is why we have repeatedly stressed the conceptual difference between Lagrange and Lyapunov stabilities.

The comparison of the two panels in Figure 1 motivates us to study more systematically the dependence of $T_d/P_{\text{out}}$ on the mutual inclination $i_{\text{mut}}$ of the inner and outer orbits. We note again that Paper I adopted the normalized disruption timescales of $T_d/P_{\text{in}}$, while the present paper uses $T_d/P_{\text{out}}$. This is not essential and just a matter of definition, but the different normalization may be useful in providing a complementary view to the problem relative to Paper I. Furthermore, if the outer binary is highly eccentric, most of the energy transfer between the two orbits, which is responsible for triggering the instability, should occur once every $P_{\text{out}}$ when the outer body is at pericenter. Therefore, the scaling of $T_d/P_{\text{out}}$ may be more relevant from the physical perspective.

3.2. Mutual inclination dependence of the disruption timescales
Figure 2 plots $T_d/P_{\text{out}}$ from our simulations on the $e_{\text{out}} - r_{\text{p, out}}/a_{\text{in}}$ plane for different values of $i_{\text{mut}}$. This generalizes Paper I that focused on three representative mutual inclinations ($i_{\text{mut}} = 0^\circ$, $90^\circ$ and $180^\circ$) alone, and quantitatively estimates the disruption timescales associated with Lagrange stability by systematically varying the initial inclinations $i_{\text{mut}}$ in an interval of $15^\circ$. We adopt the same initial parameter sets in Paper I and Figure 1, and fix the initial phases as $(\omega_{\text{in}}, M_{\text{in}}, \omega_{\text{out}}, M_{\text{out}}) = (180^\circ, 30^\circ, 0^\circ, 45^\circ)$. The estimated timescales are supposed to vary by one or two orders of magnitude when the initial phases are randomly assigned, in addition to the similar amount of scatters due to the intrinsic chaotic nature of the triple dynamics (Paper I). Thus, the distribution of $T_d/P_{\text{out}}$ shown in Figure 2 should be understood to have a scatter of one or two orders of magnitude in general. The variation due to the different initial phases will be examined in Figure 3 below.

For reference, we plot the stability boundary proposed by Vynatheya et al. (2022) alone in Figure 2; as Figure 1 indicates, the MA01 boundary is roughly identical. The top three panels indicate that triples with $r_{\text{p, out}}/a_{\text{in}} < (r_{\text{p, out}}/a_{\text{in}})_{\text{V}}$ are disrupted approximately within $T_d < 10^3 P_{\text{out}}$, while those with $r_{\text{p, out}}/a_{\text{in}} > (r_{\text{p, out}}/a_{\text{in}})_{\text{V}}$ mostly survive more than $10^7 P_{\text{out}}$. In that sense, Lyapunov stable triples with $i_{\text{out}} < 45^\circ$ (nearly coplanar prograde) are Lagrange stable as well.

However, it is not the case in general. In particular, strong ZKL oscillations for $i_{\text{mut}} > 60^\circ$ tend to destabilize the triple, and the fate of those triples can be revealed only by integrating them much longer than the corresponding quadrupole ZKL timescale (e.g. Antognini 2015).

$$\frac{T_{\text{ZKL}}}{P_{\text{out}}} = \frac{m_{123} P_{\text{out}}}{m_3 P_{\text{in}}} (1 - e_{\text{out}}^2)^{3/2} = \frac{m_{12}}{m_3} \sqrt{\frac{m_{12}}{m_3}} \left( \frac{r_{\text{p, out}}}{a_{\text{in}}} \right)^{3/2} (1 + e_{\text{out}})^{3/2}. \quad (6)$$

Indeed, the disruption timescales for triples with $60^\circ < i_{\text{mut}} < 150^\circ$ monotonically increase as $r_{\text{p, out}}/a_{\text{in}}$ without any discontinuous change around $r_{\text{p, out}}/a_{\text{in}} = (r_{\text{p, out}}/a_{\text{in}})_{\text{V}}$. This fact implies that the Lagrange instability is mainly triggered via the repeated ZKL oscillations over long timescales.

Furthermore, the significant suppression of the energy transfer between the inner and outer orbits for $i_{\text{mut}} \approx 180^\circ$ strongly stabilize those triples in coplanar retrograde orbits (Paper I) from the viewpoint of disruption timescales. Thus an accurate prediction of their fate requires a much longer-time integration as well.

Figure 2 clearly illustrates the impact of $i_{\text{mut}}$ on the disruption timescale of triples. In particular, it is worth noting that the Lagrange stability boundary appears to be highly asymmetric with respect to $i_{\text{mut}} = 90^\circ$. This is not expected from the quadrupole ZKL interaction mechanism alone, because it should be symmetric around $90^\circ$. The broken symmetry might arise due to semi-secular effects that excite the eccentricity of the inner binary more efficiently than the simple quadrupole interaction mechanism (Grishin et al. 2018; Mangipudi et al. 2022), and thus increase the energy diffusion between inner and outer orbits (Paper I).

3.3. Dependence of disruption timescales on $r_{\text{p, out}}/a_{\text{in}}$, $e_{\text{out}}$ and $i_{\text{mut}}$ with randomized initial phases

Due to the chaotic nature of the triple dynamics, the result of Figure 2 may change significantly by the difference of the initial phases (see Paper I). Therefore, we examine the sensitivity of the disruption timescales to the initial phases by computing sections of Figure 2 along the constant $e_{\text{out}}$ direction. To be specific, we fix $M_{\text{in}} = 0^\circ$ and $\omega_{\text{in}} = 0^\circ$, and change $r_{\text{p, out}}/a_{\text{in}}$ by a step of 0.05, which is four times denser sampling than those adopted in Figure 2. Furthermore, we compute 10 different realizations for the same set of parameters, $(e_{\text{out}}, r_{\text{p, out}}/a_{\text{in}}, i_{\text{mut}})$ using randomly selected values of
Figure 2. The normalized disruption timescales $T_d/P_{out}$ on $e_{out} - r_{p, out}/a_{in}$ plane for different values of $i_{mut}$. The disruption timescales evaluated from simulations are indicated according to the side color scales. Magenta curves represent the dynamical stability criterion from Vynatheya et al. (2022) (see equation (4)). Cross symbols indicate the triples that have not been broken until $t_{int} = 10^7 P_{out}$. While the result for $i_{mut} = 30^\circ$ is not shown here, it is very similar to a simple interpolation of those for $i_{mut} = 15^\circ$ and $45^\circ$. 
\(\omega_{\text{out}}\) and \(M_{\text{out}}\). We emphasize that the tiny difference of other orbital parameters induces the scatters of \(T_d/P_{\text{out}}\) by a similar amount (see Paper I).

Figure 3 shows the resulting \(T_d/P_{\text{out}}\) distributions for coplanar prograde (top), orthogonal (middle), and coplanar retrograde (bottom) triples. Just for reference, we plot the expected stability boundary, equation (4), in dashed lines. The top panel shows that \(T_d/P_{\text{out}}\) is a very sensitive function of \(r_{\text{p, out}}/a_{\text{in}}\) for low-inclined systems, even taking into account the scatters due to initial phase differences; \(T_d/P_{\text{out}}\) changes significantly around the transition region corresponding to the value of \(10^4-10^5\). The top panel have very few data points between \(10^5-10^7\), while a majority of triples are simply located at the upper limit of integration time \((T_d/P_{\text{out}} = 10^7)\). This result agrees qualitatively with that of Vynatheya et al. (2022) except for highly eccentric cases. For a highly eccentric case of \(e_{\text{out}} = 0.9\), the transition becomes less abrupt; \(T_d/P_{\text{out}}\) increases gradually with \(r_{\text{p, out}}/a_{\text{in}}\) up to 4.5, and then shows significant jump.

The bottom panel, for coplanar retrograde triples, shows a similar behavior, although the systems are significantly more stable than prograde cases, and the transition looks less abrupt.

In contrast, for orthogonal triples (middle panel); \(T_d/P_{\text{out}}\) increases gradually with \(r_{\text{p, out}}/a_{\text{in}}\) until \(r_{\text{p, out}}/a_{\text{in}} \approx 3\) and 4 for \(e_{\text{out}} = 0.1\) and 0.5, respectively, and then exhibits significant increase only when \(r_{\text{p, out}}/a_{\text{in}}\) exceeds those values. Even after exceeding those values, the transition is still not so abrupt for orthogonal triples, unlike the case for coplanar prograde triples. For \(e_{\text{out}} = 0.9\), \(T_d/P_{\text{out}}\) continues to increase gradually with \(r_{\text{p, out}}/a_{\text{in}}\) up to \(r_{\text{p, out}}/a_{\text{in}} = 6\) where \(T_d/P_{\text{out}}\) reaches our upper limit of the integration time \((t_{\text{int}} = 10^7P_{\text{out}})\). Those trends are statistically robust against the different choices of the initial phases as illustrated by the scatters of the data points.

4. SUMMARY AND CONCLUSION

We have computed the distribution of disruption timescales for hierarchical triples with an equal-mass and initially circular inner binary, using a series of N-body simulations. We extend Hayashi et al. (2022) (Paper I), and systematically examine the dependence on the mutual inclination between the inner and outer orbits in particular. By integrating the triple systems much longer than the previous studies, we are able to reveal the fate of those triples more reliably than before. Our main findings are summarized as follows.

(1) In studying the dynamical stability of hierarchical triple systems, it is important to distinguish between Lyapunov and Lagrange stabilities. A widely used stability criterion by Mardling & Aarseth (1999, 2001) and Mardling (2008) is derived from the local divergence of the orbits, and thus corresponds to the former. Paper I and the present paper compute the disruption timescale of triple that is related to the Lagrange stability. Those concepts are complementary but very different, which should be kept in mind when applying those results for specific purposes.

(2) Dynamical disruption timescales of triples are very sensitive to the mutual inclination \(i_{\text{mut}}\) of the inner and outer orbits. They change drastically around the Lyapunov stability boundary if \(i_{\text{mut}} < 45^\circ\). Thus, the Lyapunov stability condition also guarantees the long-term Lagrange stability for moderately inclined triples. In contrast, for triples of significantly inclined inner and outer orbits \((60^\circ < i_{\text{mut}} < 150^\circ)\), the disruption timescales vary smoothly against \(r_{\text{p, out}}/a_{\text{in}}\) due to the ZKL oscillations. The difference of the initial phases adds one or two orders of magnitudes scatters to the disruption timescales, but does not change the behavior in a statistical sense.
Figure 3. Behavior of the stable and unstable transition for triples with $i_{\text{mut}} = 0^\circ$, $90^\circ$, and $180^\circ$. In each panel, the different colored symbols plot $T_d/P_{\text{out}}$ against $r_{p,\text{out}}/a_{\text{in}}$ for $e_{\text{out}} = 0.1$ (red), 0.5 (blue), and 0.9 (black). For each set of $(e_{\text{out}}, r_{p,\text{out}}/a_{\text{in}}, i_{\text{mut}})$, we run 10 realizations with different initial phases ($\omega_{\text{out}}, M_{\text{out}}$) randomly sampled from uniform distribution. The value of $N$ in the panels denotes the total number of the corresponding runs (data points).
(3) The Lagrange stability of triples does not change monotonically with the mutual inclination. For triples with moderate inclinations \( (i_{\text{mut}} < 45^\circ) \), the stability is not so sensitive to \( i_{\text{mut}} \). By increasing \( i_{\text{mut}} \) up to \( \sim 100^\circ \), those triples become less stable due to the ZKL oscillations. Retrograde triples with \( i_{\text{mut}} > 120^\circ \) become stabilized again as \( i_{\text{mut}} \) increases, and the coplanar retrograde triple \( (i_{\text{mut}} = 180^\circ) \) is the most stable configuration. Grishin et al. (2017) found the similar behavior about the mutual inclination dependence in the Hill stability, and pointed out that the ZKL oscillations affect the stability. Interestingly, the above behavior is not symmetric around \( i_{\text{mut}} = 90^\circ \), in contrast to a naive expectation from the quadrupole ZKL oscillations, indicating the importance of quasi-secular effects (e.g. Grishin et al. 2018; Mangipudi et al. 2022) beyond the simple ZKL quadrupole interaction.

The present paper has adopted a range of parameters in which the Newtonian gravity is dominant and the general relativity (GR) corrections are negligible. However, it is known that the GR precessions reduce the ZKL effects for specific parameter ranges (e.g. Liu et al. 2015). In this case, the Lagrange stability for inclined cases may be highly affected by GR corrections. In addition, gravitational wave emissions would reduce the semi-major axis of the inner binary, and thus affect the stability of triples with inner compact objects.

In this paper, we also fix the mass ratios, \( q_{21} \equiv m_2/m_1 = 1 \) and \( q_{23} \equiv m_2/m_3 = 5 \). Different choices of those parameters are important in applying to a variety of interesting astronomical objects. For instance, small \( q_{21} \) and \( q_{23} \) are relevant for planetary systems. In addition, octupole ZKL oscillations play an important role if \( q_{21} \) deviates significantly from unity, i.e. for a very unequal mass inner binary. The strong enhancement of the eccentricity of the inner binary by the octupole-level ZKL interactions would affect the stability.

Many scenarios are proposed recently to explain a variety of astronomical phenomena via the ZKL oscillations, including the formation of extrasolar hot jupiters (e.g. Wu & Murray 2003; Naoz et al. 2012), and the acceleration of compact mergers (e.g. Liu & Lai 2018; Trani et al. 2021). We show that the concept of the disruption timescales is particularly more important for triples with strong ZKL oscillations. Therefore, quantitative implications of these scenarios have to be discussed with taking account the disruption timescales.

Generalizing the current stability analysis of triples by taking account of the problems above-mentioned is an important area of research, which we plan to address in due course.

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