On Kerr black hole deformations admitting a Carter constant and an invariant criterion for the separability of the wave equation

Georgios O. Papadopoulos 1 · Kostas D. Kokkotas 2

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Abstract
In a previous work of ours, the most general family of Kerr deformations—admitting a Carter constant—has been presented. This time a simple, necessary and sufficient condition in order for the aforementioned family to have a separable Klein-Gordon equation is exhibited. In addition, we provide a solid theoretical foundation that the maximum number of free to be chosen radial functions is three.

Keywords Kerr deformations · Carter constant · Killing tensors · Klein-Gordon equation · Separability · Integrals of motion

Mathematics Subject Classification 83B05 · 83C20 · 83C57 · 83D05 · 83F05

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Georgios O. Papadopoulos
gopapado@phys.uoa.gr

Kostas D. Kokkotas
kostas.kokkotas@uni-tuebingen.de

1 Department of Physics, National and Kapodistrian University of Athens, Panepistimioupolis, Athens GR 157-71, Greece
2 Theoretical Astrophysics, IAAT, Eberhard Karls University of Tübingen, 72076 Tübingen, Germany
1 Introduction

During the last years an ever increasing interest on alternative variants of Black Holes (BHs) has been observed. The interest on BHs is due to not only their unique structure comprising the most extreme gravitational fields known to exist, but also their involvement in the evolution of small and large scale structures on the universe. Indeed, the extreme gravitational fields in their vicinity make them the ideal candidates for performing tests in strong field regime. One may test not only a given quantum gravity theory, but also the uniqueness of the solutions for BHs in General Relativity (GR). If, for instance, the uniqueness of the Kerr BH is compromised then the GR, as a whole, is under question. Thus alternative variants of BHs can lead to proposals for either alternative theories for gravity (cf. references in [1]) or variants of the well known general relativistic metrics such as the Kerr space—see e.g., [2]. These studies have been boosted by the detection of gravitational waves [3,4] and the HORIZON results [5,6] which are providing promising tests of the aforementioned claims.

The great progress made during the recent years in astronomy (electromagnetic, particle and gravitational) has transformed the study of BHs from (an initially) pure academic problem into a viable and promising (both at the observational and the experimental level) research. Various theoretical models, trying to phenomenologically interpret the observations by LIGO/VIRGO along with the expected ones from LISA, spring out massively in the literature [7–9]. In principle, there are two approaches in those works; the studies are made either within the framework of a new theory for gravity (like, e.g., the \( f(R) \) theory, Scalar-Tensor theories, etc) or in the context of a more pragmatic—but still theory agnostic (like e.g., a smooth deviation from the mathematical ideal of the Kerr solution)—approach to GR.

In both approaches, thus far, the current trend to be found in the literature is to begin with the assumption of a (mathematically convenient) phenomenological form for the metric suitable for modelling the astrophysical imprints of the source, but not (necessarily) always susceptible to some deeper analytical feature—like, for instance, the separability of a given family of equations like the wave equation etc.

In an earlier work [1] we introduced a simple and theory agnostic family of metrics which not only re-parameterises but also generalises many well known asymmetric metrics to be found in the literature. The novel feature of this metric is the admittance of an extra integral of motion for the corresponding geodesic equations; the well known Carter constant [10].

The metric tensor field introduced in [1] reads:

\[
g^{ab} = \frac{1}{A_1 + B_1} \begin{pmatrix} A_2 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & A_3 + B_3 & A_4 + B_4 \\ 0 & 0 & A_4 + B_4 & A_5 + B_5 \end{pmatrix}
\]

(1)

where the functions \( A_i \) depend on \( r \) (i.e., “radial functions”) and the functions \( B_i \) depend on \( x \equiv \cos(\theta) \) (i.e., “angular functions”), are such that the covariant metric be asymptotically flat. This metric, by construction, allows for the existence of a second
The acronyms and references are given in the first column. The second column presents the number of assumed free deforming functions by each one of them, while the third denotes the existence of a Killing tensor while the forth comments on the separability of the geodesic equations. The fifth column, is the most relevant for the current work i.e., the separability of the KG equation.

Actually the very existence of a Killing tensor is extremely important for it signals the formal integrability of the geodesic (or Hamilton-Jacobi) equations (cf. next Section). At this point if should be mentioned that the horizons are implicitly defined through the vanishing of the norms of the two Killing vector fields (KVF) (cf. the *The Mathematical Theory of Black Holes* by S. Chandrasekhar) so both the ergo-sphere and the horizon are given by the vanishing of the appropriate combinations of some of the metric components.

The effort for construction Kerr-like metrics has been initiated more than two decades ago, but the last ten years many variations have been presented [2,11–17], the properties of which are compared to those of the Kerr metric. Predominantly, the interest has been related to the Hamilton-Jacobi integrability and the Klein-Gordon (KG) equation separability.

The following table shows a timely current state of affairs in the literature (only the most general results are mentioned) regarding the various alternative deviations from the Kerr metric and the integrability (via separation of the independent variables) of two important families of equations: the geodesics and the KG equation (Table 1).

At this point three remarks are in order.

First, it should be made clear that all the aforementioned, metrics [2,15–17] are sub-cases of the PK [1] family of metrics; ditto for the corresponding Killing tensors.

\[
K_{ab} = \frac{1}{A_1 + B_1} \begin{pmatrix}
A_2 B_1 & 0 & 0 & 0 \\
0 & -B_2 A_1 & 0 & 0 \\
0 & 0 & B_1 A_3 - A_1 B_3 & B_1 A_4 - A_1 B_4 \\
0 & 0 & B_1 A_4 - A_1 B_4 & B_1 A_5 - A_1 B_5
\end{pmatrix}
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At this point three remarks are in order.

First, it should be made clear that all the aforementioned, metrics [2,15–17] are sub-cases of the PK [1] family of metrics; ditto for the corresponding Killing tensors.
Second, the well known Carter metric is more restrictive, for it contains 3 radial and 3 angular free functions in a particular position in the metric tensor field plus 1 constraint, while our result, in principle, contains 5 radial and 5 angular free functions plus 1 constraint.

Finally, barring the PK [1] metric, all the others—to some extent—are related to some theory (e.g., the J metric [2] is a pragmatic extension of GR, etc). Indeed, a deformed Kerr metric can be obtained by two ways:

- Either one implements the equivalent of a kinematical algorithm—something which does map solutions to the Einstein Field Equations (EFEs) to other solutions (to the EFEs); like the S metric [16] where the author starts with a non rotating solution to the EFEs, then exploits the Newman-Janis (complex) algorithm and ends up with another solution (to the same theory, i.e., GR)
- Or one considers a completely different metric, deviating smoothly from the Kerr solution, which has to satisfy some field equations (e.g., the GR plus some perturbation of the energy-momentum tensor see, e.g., [19], [17] and references therein)

On the contrary, the PK family of metrics [1] was constructed as a geometric object endowed with the maximum number of free functions along with some specific geometric features (like the existence of a Killing tensor, something very important for this leads to integrals of motion for the geodesic equations) without any reference to any theory. Of course there is a price for this generality; e.g., not all members of the family are—in principle—physically accepted; supplementary constraints like e.g., the energy conditions, have to be implemented when one considers any member of this family.

In the present work a further step is made and a new family of metrics, we will call it PK-I family, is presented. The new family is a subfamily of the PK one [1]. Indeed, the new family (PK-I) consists of the members of the previous one (PK) obeying a necessary and sufficient condition (upon the metric functions) such that the Klein-Gordon equation be separable.

It should be noted that classical references on the separability of the KG and its generalisation on the Kerr background are to be found in [10,20,21]. Also, in [22] a very interesting comparable analysis of the KG separability was presented for various alternative phenomenological families of metrics.

In the following section a formal description of the PK-I (sub)family of metric spaces will be presented, based solely on an implicit yet invariant criterion. An immediate yet fundamental application of this will be exhibited.

2 PK-I metrics: formal existence, integrals of motion, and an application

In general, separability structures are closely related to the existence of the so called (possibly hidden) symmetries; a kind of physical degeneracy (e.g., cf. Nöther’s theorem, and [23]). When it comes to gravitation the studies focus mainly on the geodesic
equations. Never the less, there are other equations of physical interest as well. Such an example is the KG equation, which is also a very useful tool in observational physics. So it would be interesting if one could somehow enhance the previous considerations—on symmetries and integrability—to the case of KG equation.

Let assume the metric $g^{ab}$ as it is given in (15). Then this metric has the fundamental property of admitting a Carter constant. Indeed, let construct the scalar functional

$$ I = K_{ab} \dot{x}^a \dot{x}^b \quad (3) $$

with the Killing tensor obeying, by definition, the condition

$$ \nabla_a (K_{bc}) = 0 \quad (4) $$

then $I$ is a constant of motion of the geodesics equations

$$ \ddot{x}^c + \Gamma^c_{ab} \dot{x}^a \dot{x}^b = 0 \quad (5) $$

since the combination of (4) and (5) leads to

$$ \dot{I} = \nabla_c K_{ab} \dot{x}^a \dot{x}^b \dot{x}^c = 0 \quad (6) $$

i.e., the last quantity vanishes by virtue of the geodesics and the Killing tensor definition.

Now the focus is on the Klein-Gordon equation for a scalar field $\Psi$ on the background of the aforementioned metric:

$$ \nabla^a \nabla_a \Psi = 0 \quad (7) $$

A tedious yet straightforward calculation shows that this equation is susceptible to separation of variables if under the Ansatz

$$ \Psi(t, r, x, \phi) = e^{i(m\phi - \omega t)} \mathcal{X}(r) \mathcal{Y}(x) \quad (8) $$

it holds that

$$ \nabla^a \nabla_a \Psi = 0 \Rightarrow \text{Radial Part} + \text{Angular Part} = 0 \quad (9) $$

where (modulo overall factors)

$$ \text{Radial Part} : A_2 \partial_r \mathcal{X} + \left( \partial_r A_2 + A_2 \partial_r \Omega \right) \partial_r \mathcal{X} - \left( m^2 A_3 - 2m \omega A_4 + \omega^2 A_5 + \lambda \right) \mathcal{X} = 0 \quad (10) $$

$$ \text{Angular Part} : B_2 \partial_{\phi \phi} \mathcal{Y} + \left( \partial_{\phi} B_2 + B_2 \partial_{\phi} \Omega \right) \partial_{\phi} \mathcal{Y} - \left( m^2 B_3 - 2m \omega B_4 + \omega^2 B_5 - \lambda \right) \mathcal{Y} = 0 \quad (11) $$
with \( \lambda \) denoting the separation constant, \( g \) the determinant of the metric tensor, while the quantity \( \Omega \) is identified as

\[
\Omega = \ln \left( \frac{\sqrt{-g}}{(A_1(r) + B_1(x))} \right).
\]  

(12)

Now, it is obvious that the Radial and the Angular parts separate if and only if:

\[
\Omega = F_1(r) + F_2(x)
\]

(13)

where \( F_1, F_2 \) are supposed to be well behaved functions of their designated arguments. Just for reference, in the case of the Kerr metric itself it is \( \Omega_{\text{Kerr}} = 0 \) while in the case of the Kerr-Sen metric \([18]\) it is \( \Omega_{\text{Kerr-Sen}} = 1 \). Also, it should be noted that the resulted metric is just a member of the PK metric. This calls the fact that the Johannsen’s metric is even more restrictive than the PK family.

An extended yet straightforward calculation exhibits the following conserved current:

\[
J^a = Q^{ab} \nabla_b \Psi, \quad \nabla_a J^a \bigg|_{\text{Radial Part}=0, \text{Angular Part}=0} = 0
\]

(14)

with

\[
Q^{ab} = \frac{1}{A_1 + B_1} \begin{pmatrix}
0 & 0 & \omega \Sigma_1 & m \Sigma_1 \\
0 & B_2 & \omega \Sigma_2 & m \Sigma_2 \\
\omega \Sigma_1 & \omega \Sigma_2 & 0 & 0 \\
m \Sigma_1 & m \Sigma_2 & 0 & 0
\end{pmatrix}
\]

(15)

where the \( \Sigma_1 \) and \( \Sigma_2 \) quantities depend on both \( r \) and \( x \) coordinates. To the best of the authors’ knowledge this, two-parametric, integral is new to the literature and its nature is not very clear.

An interesting application of the previous consideration would be the counting of the maximum number of free radial functions, when the angular functions are taken to be the Kerrian ones:

\[
B_1 = a^2 x^2
\]

(16a)

\[
B_2 = 1 - x^2
\]

(16b)

\[
B_3 = \frac{1}{1 - x^2}
\]

(16c)

\[
B_4 = a
\]

(16d)

\[
B_5 = a^2 (1 - x^2).
\]

(16e)

The necessary and sufficient condition, regarding \( \Omega \) becomes a polynomial constraint, in the angular variable \( x \). The vanishing of the coefficients results in three free radial
functions: $A_2$, $A_3$ and any of rest three (i.e., any one of $A_1$, $A_4$, $A_5$), as the desired maximum number. Actually, the three free functions are completely at one’s disposal, while the rest two are related to those three. This result is in full agreement with the existing relevant attempts to be found in the recent literature—cf. [15] and [16]. Indeed, for reference [15] it is:

\begin{align*}
A_1 &= r^2 R_\Sigma(r) \\
A_2 &= \frac{a^2 - r R_M(r) + r^2 R_\Sigma(r)}{R_B(r)^2} \\
A_3 &= -\frac{a^2}{a^2 - r R_M(r) + r^2 R_\Sigma(r)} \\
A_4 &= -\frac{a^3 + ar^2 R_\Sigma(r)}{a^2 - r R_M(r) + r^2 R_\Sigma(r)} \\
A_5 &= -\frac{(a^2 + r^2 R_\Sigma(r))^2}{a^2 - r R_M(r) + r^2 R_\Sigma(r)} \\
\Omega &= \ln \left( \frac{R_B(r)}{R_B(r)} \right)
\end{align*}

while for reference [16] it is:

\begin{align*}
A_2 &= \Delta(r) \\
A_3 &= -\frac{a^2}{\Delta(r)} \\
A_4 &= -\frac{aX(r)}{\Delta(r)} \\
A_5 &= -\frac{X(r)^2}{\Delta(r)} \\
\Omega &= \ln \left( \frac{a^2 x^2 + A_1(r)}{-a^2(1 - x^2) + X(r)} \right)
\end{align*}

and this time, $A_1(r)$ is a proper function such that the corresponding $\Omega$ be separable as a function.

### 3 Discussion

In this short work, we have reported on those (sub)families of space times which constitute (smooth) Kerr deformations endowed with the properties of

- Admitting a Carter constant—something signaling the separability and the complete integration of the geodesic equations
- Allowing for the separability of the KG equation.

This (sub)family is given rather indirectly, through an invariant criterion, never the less the result itself is important per se for, two simple reasons:
1. It confirms that for the Kerrian choice of radial functions, the maximum freedom of the radial functions is three—something which not only confirms other, less general attempts in the recent literature but it inherits to them a solid theoretical foundation.

2. It provides, the most general subfamily of those space-time smooth Kerr deformations which on one hand admit a Carter constant and on the other hand the KG equation is separable with the freedom of (at most) ten free functions obeying a single constraint.

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