The analysis of income distribution (ID) has traditionally been of prime importance for economists and policymakers. However, the standard input–output (I–O) model is not particularly well equipped for studying current issues such as the consequences of decreasing access to primary inputs or the effects of specific redistributive policies. This paper addresses this gap in the existing literature. We propose that IDs can excellently be studied by restructuring the I–O relations. A new coefficients matrix is defined, the so-called augmented input coefficients matrix. This matrix is the sum of the intermediate input coefficients matrix and newly constructed matrices of sector-specific input coefficients that represent the existing distribution of income. We show that shifts in the distribution can be modelled by attributing weights to these matrices and vary these according to system-specific rules. Numerical illustrations based on the existing literature are given throughout the paper.

Keywords: Income distribution; Factor remuneration; Augmented input coefficients matrix; System numeraire

1. INTRODUCTION

The study of income distribution (ID) is generally understood as the study of which part of an economy’s national income (NI) goes to which party and why. Time and again, insight into the distribution and its dynamics has proven to be of prime importance to economic analysis and policy.

Here, Leontief-based input–output (I–O) analysis offers an excellent perspective for a detailed study. First, this is due to the level of detail that is normally incorporated into I–O tables. Second, the double-entry character of the tables means that the information on surplus and distribution is accounted for in two ways, that is, in the value-added rows and the final demand columns. Both registrations are interconnected in that total earnings are equal to total spending, while the links between individual rows and columns can be present as well. I–O models nowadays can be straightforwardly used to calculate the output and employment effects of, say, an increase in one or more final demand categories. Taking the outcomes for all factors together, we can see in which way absolute quantities will be affected, how distribution will change, etc., following shifts in exogenous final demand.

Typical ID modelling, nonetheless, emerged rather late. Here, we should mention the contributions reported in Miyazawa and Masegi (1963) and Miyazawa (1976), tracing back
their origins to Keynes–Kaldor types of multiplier analyses. Miyazawa proposed a way forward via a special extension of the real output model, closing it for household income groups represented by group-characteristic consumption bundles. In this way, several types of multipliers could be presented, showing the effects of shifts in demand on the selected groups. Kurz (1985) demonstrated how changes in ID will affect income–expenditure multipliers in a multi-commodity setting. Later models explored the effects of further disaggregation of the payment sectors into income classes (Rose et al., 1982). Schefold (1976) analysed the effects of forms of technical progress on ID, while Leontief (1985, 1986) studied the distributional effects of the arrival of new technologies in the USA.

However, by and large, much of ID analysis has been relegated to extensions and subfields built around special tools and instruments such as Social Accounting Matrices or SAMs.² To a large extent, this line of research dates back to pioneering work such as that done by Pyatt and Roe (1977) in studies focusing on poverty and income inequality. Many variants have been proposed. One line focuses on a fuller coverage of household characteristics, see Batey and Weeks (1989) or Duchin (1998); see also Pyatt (2001) for a recent perspective in terms of various types of multipliers.

As opposed to Leontief’s basic model, ID-determining mechanisms are very much part of the Sraffian basic model (Sraffa, 1960). The Sraffian model, essentially, consists of a price equation and a so-called actual system (basically equal to Leontief’s open, static model) to provide a physical measure of output. The model focuses on the distribution of income between two parties, labour and capital, and as such its orientation is quite different from that of the Leontief model. The model, most importantly, recognizes that price changes will accompany a shift in the distribution of income; here the necessary links are made by expressing the wage in terms of NI. The theoretical framework contains several normalizations and numeraires that are unique to the Sraffian approach, such as the so-called standard commodity. The differences with Leontief analysis are substantial, which may explain the lack of cross-fertilization between both these schools of thought.³,⁴

In fact, the standard I–O model as it is presented nowadays is not particularly well equipped for answering questions regarding ID-related issues. We may be interested, for example, in listing the alternatives that are opened up by a technological innovation or in the price implications of redistributive policies. In such cases, given the interdependence within the system, a shift in the position of one or more parties is very likely to have implications for the other ones, which, in effect, may be ‘crowded out’. Catching such effects employing the standard framework is not a straightforward procedure. In fact, so far, no satisfactory method has been found to trace such effects due to the lack of a transmission mechanism that links, say, shifts in factor remuneration to shifts in net output or vice versa. This means that it is not possible, in the present context, to systematically explore ID-determining mechanisms in a variety of relevant cases. This brings us directly to the main objective of this paper, that is, to show how the tools that are required for this can be incorporated into the standard framework.

² To illustrate, in Miller and Blair (2009), the term ‘income distribution’ appears only thrice, that is, on pages 7, 499, and 681, and then only in connection to SAMs.
³ To illustrate, and again referring to Miller and Blair (2009), only two references to Sraffa are provided, both in the final chapter.
⁴ See Steedman (2000) on differences and on further references to the intellectual history.
Below, we shall start with the well-known observation that in standard Leontief models factor prices are given exogenously, say in dollars or euros per unit of factor input. Changes in these prices have no influence on the real sphere. Commodity prices will reflect changes in factor prices, but this has no further consequences.

Adopting such a view of prices, however, ignores the fact that in many instances price changes do have consequences in the real sphere that we would like to track. This basically is the perspective that we explore in this paper. Our approach is based on the fact that factor remunerations can often be straightforwardly expressed in real terms, that is, in terms of commodity bundles of factor-specific composition. This immediately introduces a link between these ‘remuneration bundles’ and the final demand categories.

Accounting for remunerations in this way makes it possible to consider them in terms of inputs, that is, in terms of bundles of commodities analogous to the intermediate inputs into the sectoral production processes. By subsequently manipulating the size and composition of the remuneration bundles, shifts in NI can be generated, where precise outcomes will depend on the choices that are made.

For standard (homogeneous) labour, for example, the above would straightforwardly translate into the introduction of the wage in the form of a well-defined bundle of commodities consumed per unit of (labour) input. If (as will normally be the case) each sector employs labour, these (wage) bundles can be assembled in a square matrix, the columns of which then represent the real wage per unit of output of each sector. By expressing the remuneration of other factors in real terms as well, we can subsequently introduce a similar factor-specific remuneration matrix for each factor. This enables us to define an ‘initial situation’ which may serve as a benchmark for studying shifts in ID. As we will see, shifts in factor remuneration can be translated directly into shifts in ID and vice versa, by attaching weights to the newly defined factor remuneration matrices. A related problem is to find a rule (or set of rules) that allows us to consistently vary the distribution of income over the entire range of weights and parameters involved. We shall show that such a rule is provided by the mathematical rules that safeguard the I–O model’s internal consistency; in this context, specific ‘numeraire’ will have an important role.

We shall start, in Section 2, with the standard open I–O model with one primary factor. In Section 3, we discuss the Sraffian two-factor approach, and in Section 4, we return to a general class of I–O models. In Section 5, we conclude with a number of remarks.

2. AUGMENTED INPUT COEFFICIENTS MATRICES

We start our proposition with the standard I–O model with one primary factor (to be called labour), which consumes the entire net product. Three equations are sufficient to provide the essentials. We have

\[ x = Ax + f, \]  

where \( A \), the \((n, n)\) matrix of intermediate input coefficients, is square, nonnegative, indecomposable and of full rank with a Perron–Frobenius (PF) eigenvalue smaller than 1; \( f \) and \( x \) stand for, respectively, the \((n, 1)\) vectors of net product, here consumed in its entirety by the single primary factor, and total output. We also have

\[ L = \mathbf{1}'x, \]
where \( l' > 0 \) stands for the \((1, n)\) row vector of direct labour input coefficients and \( L \) for the size of the labour force employed.\(^5\) The corresponding price equation is

\[
p' = p'A + w'l',
\]

with \( p' \) standing for the \((1, n)\) row vector of commodity prices and \( w \) (a scalar) for the money wage per worker.

We observe that a shift in \( w \) will not have any influence on the real sphere as given by Equations 1 and 2. This is a fundamental property of the model; \( w \) is exogenously determined, and the model is constructed without any direct connection between \( w \) and the real sphere as expressed by Equations 1 and 2. The model clearly is transparent, but there is a price to pay; the signalled transparency deprives us from studying the interactions between shifts in the wage and the real sphere.

However, the above three equations can be reformulated to provide the basis for an extension that allows us to study distribution issues from a quite different perspective. In fact, we shall show that by ‘expressing the wage in terms of NI’, we obtain a direct link between price and output equations that can be further exploited. This implies, as we shall see, adopting specific ‘standards’ or ‘numeraires’ which reflect the real sphere.

The link can be established by expressing the wage in real terms, here an \((n, 1)\) commodity bundle. If we do this for all sectors combined, we can write these ‘real wage bundles’ as inputs, which provides the basic structure for our analysis. First, we have

\[
p'f = p'\left(\frac{f}{L}\right) L = p'\left[\left(\frac{f}{L}\right) l'\right] x.
\]

If we now adopt the rule

\[
w = p'\left(\frac{f}{L}\right),\]

we have expressed the wage in terms of NI (per head). We subsequently need a standardization to fix the value of \( w \), for example, by fixing the value of the bundle \( f/L \) at unity, in which case it becomes a numeraire. If all workers receive the same real wage, we can write Equation 3 as

\[
p' = p'A + p'\left(\frac{f}{L}\right) l'
\]

\[
= p'\left(A + \left(\frac{f}{L}\right) l'\right)
\]

from which we observe that the elements of the (standardized) final deliveries vector now appear as part of the economy’s cost structure, that is, as ‘inputs’. Correspondingly, we have for the real output system

\[
x = \left(A + \left(\frac{f}{L}\right) l'\right) x.
\]

\(^5\) The notation \( l' > 0 \) means that each element of the vector \( l' \) is positive.

\(^6\) Each element of the column vector \( f/L \) is obtained by dividing the corresponding element of \( f \) by \( L \), a scalar.
So, we have rewritten the model given by Equations 1–3 in terms of two eigenequations where \( \mathbf{p}' \) and \( \mathbf{x} \) stand for, respectively, the left- and right-hand PF eigenvectors of matrix \( (\mathbf{A} + (\mathbf{f}/\mathbf{L})') \). It is useful to have a compact notation for the wage part of this matrix. With

\[
\mathbf{B} = \left( \frac{\mathbf{f}}{\mathbf{L}} \right)' \tag{8}
\]

we have

\[
\mathbf{p}' = \mathbf{p}'(\mathbf{A} + \mathbf{B}). \tag{9}
\]

With \( \mathbf{f} > 0 \), we then have \( \mathbf{B} > 0 \) and \( r(\mathbf{B}) = 1 \), where \( r(\mathbf{B}) \) stands for the rank of matrix \( \mathbf{B} \). Correspondingly, we find

\[
\mathbf{x} = (\mathbf{A} + \mathbf{B})\mathbf{x} \tag{10}
\]

with \( \mathbf{B}\mathbf{x} = [(\mathbf{f}/\mathbf{L})']\mathbf{x} = \mathbf{f} \), using \( \mathbf{L} = \mathbf{l}'\mathbf{x} \). We shall call, following Seton (1977), matrix \( (\mathbf{A} + \mathbf{B}) \) an ‘augmented input coefficients matrix’. We continue by rewriting Equation 10 as

\[
[I - (\mathbf{A} + \mathbf{B})]\mathbf{x} = 0. \tag{11}
\]

From Equation 11, we have that the columns of matrix \( I - (\mathbf{A} + \mathbf{B}) \) are interdependent. This implies that its determinant must vanish, that is,

\[
|I - (\mathbf{A} + \mathbf{B})| = 0. \tag{12}
\]

At this point, we should note that matrix \( (\mathbf{A} + \mathbf{B}) \) contains what we may call ‘complete’ or ‘full’ information on the various segments of the economy. We have only one final demand category (labour), but it is fully represented in terms of its consumption bundle and value-added row. It is this ‘completeness’ that allows us to reformulate the available information in another form, that is, Equation 12. This equation and its variants provide all information that we need to study IDs. Later on, for example, we shall say that matrix \( \mathbf{B} \) has a weight ‘1’ in the interpretation of Equation 12 that we shall further explore.

Equation 12 provides valuable insights into the system’s structure in that it provides an exact, quantitative statement on the interconnection between the coefficients of matrices \( \mathbf{A} \) and \( \mathbf{B} \). For example, if one or more coefficients of matrix \( (\mathbf{A} + \mathbf{B}) \) would change, this implies that also one or more of the remaining coefficients must change. We shall explore this connection in more detail further below. We can also observe that Equation 12, viewed as a statement on the system 1–3, is different from the well-known statements on the system’s productivity such as \( |I - \mathbf{A}| > 0.8 \) The latter statement is of a qualitative nature and provides information quite different from what Equation 12 tells us. The difference between both statements is a consequence of the fact that Equation 12 includes all segments of the economy, while a statement such as \( |I - \mathbf{A}| > 0 \) only concerns the intermediate part.

We may observe that Equations 9 and 10 allow us to distinguish between ‘structure’ and ‘scale’. We shall say that ‘structure’ is represented by the properties of matrix \( \mathbf{A} + \mathbf{B} \), while ‘scale’ is represented by vectors \( \mathbf{p}' \) and \( \mathbf{x} \). These vectors, being the left- and right-hand

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7 So, matrix \( \mathbf{B} \) is the outer product of the column vector \( \mathbf{f}/\mathbf{L} \) and the row vector \( \mathbf{l}' \).

8 This is one of a series of equivalent statements on an economy’s productivity (see Miller and Blair, 2009, apps. 2.1 and 2.2).
eigenvectors of matrix \((A + B)\), can be increased or decreased in size by a simple scalar multiplication without affecting the structure of the system; specific numeraires will be discussed in the following sections.\(^9\)

Equations 9 and 10 also enable us to ask a new type of question. In fact, we can straightforwardly determine what happens to NI if the coefficients of matrices \(A\) and \(B\) change. Suppose, for example, that an innovation reduces the use of one particular good in all production processes.\(^{10}\) This will have consequences for the economy’s net output, where the precise formulation of the model depends on the answers we are looking for. For example, if the proportions of the final demand bundle do not change, and if \(L\) also does not change, we can consider the alternative system

\[
x = (A + \beta B)x,
\]

where \(A\) is the post-innovation matrix of intermediate input coefficients and \(\beta\) a proportionality factor, where we would expect \(\beta > 1\). We can now analyse the relation between shifts in the original intermediate input coefficients matrix and the scalar \(\beta\) by considering

\[
|I - (A + \beta B)| = 0,
\]

a relation which must be satisfied. This, in broad lines, is the approach we shall pursue further below.

Equation 12 has additional properties that we can use. One such property concerns the case where \(r(B) = 1\), as above. We have

\textbf{Lemma 1} \hspace{1em} Let all symbols be as before, and let \(A^\#\) denote the adjoint of matrix \(A\). Then,

\[
det(A + \left(\frac{f}{L}\right)^\prime Y) = det(A) + \left[V^{\prime}\right]A^\# \left[\frac{f}{L}\right].
\]

In the next two sections, we shall explore the properties of Equation 12 and its variants in two ways. In Section 3, we shall explore the relation between NI and matrices \(A\) and \(B\) in a Sraffian context. In Section 4, we shall introduce further claims on net output in a Leontief type of context, with the claims on NI being modelled by introducing additional coefficient matrices, similar to matrix \(B\) in the single primary factor case. In both sections, ID-determining mechanisms will be studied by attaching weights to the individual matrices and by varying these weights. We thereby shall propose two additional lemmas, both based on Lemma 1.

Before continuing, we should mention that adding the ‘material’ and labour costs is quite well known in certain subfields. For example, Seton (1977, p. 17), in discussing Marxian production systems, points out that the total of these costs sometimes is referred to as the

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\(^9\) We should observe that the model of Equations 9 and 10 is fundamentally different from the so-called closed Leontief model (see, e.g. Pasinetti, 1977, Ch. IV, or Miller and Blair, 2009, Section 2.5). In that model, the production of a commodity called labour is interpreted just like the production of any other commodity, that is, in terms of fixed input coefficients. In our model, the final demand vector \(f\) stands for the household consumption bundle, which changes if preferences change.

\(^{10}\) Addressing this type of question was one of the motivations to develop the so-called RAS method; for origins, see Leontief (1951) or Stone and Brown (1962).

\(^{11}\) For proofs, see, for example, Rao and Bhimasankaram (2000, Section 6.7) or Trenkler (2000).
INCOME DISTRIBUTIONS IN I–O MODELS

The economy’s ‘augmented technology’, representing the total cost of each output unit to the capitalist producer. Pasinetti (1977, appendix to Ch. V) used a similar construct in discussing the structure of particular Marxian inspired models. However, the idea of combining the final demand and value-added categories was only put forward for the wage–labour input connection, without further discussion of systems where links between all categories are explored. As far as we know, such constructs have never been employed to study trade-offs in ID schemes of the type we shall be dealing with further below.

3. CAPITAL AND LABOUR AS COMPETING PARTIES

In this section, we shall see in which way augmented input coefficients matrices can be applied in a Sraffian analysis. As has been mentioned already in Section 1, the Sraffian approach is quite different from Leontief’s. Differences include a focus on distribution issues and the role of specific standardizations and numeraires therein in providing the necessary structure. The original model, as presented by Sraffa (1960) in his “Production of Commodities by Means of Commodities”, has initiated many comments, further propositions, and extensions, but interaction with the Leontief ‘school of thought’ has been minimal.12

Many aspects of Sraffa’s work, especially those concerning the validity of neo-classical theorizing and the revival of classical thought, are important in any contribution discussing differences and similarities with Leontief’s work. However, these are outside the scope of the present contribution. We refer the reader to Bharadwaj and Schefold (1990) for a recent set of essays on Sraffa’s life and works and to Kurz and Salvadori (2003) for a collection of original contributions.

Central to the Sraffian approach is the price model, traditionally written as

\[ p' = (1 + r)p'A + wl', \]

where \( r \) and \( w \) stand for, respectively, the rate of profit and the wage rate.13 The problem is to determine the ID when \( w \) and \( r \) shift, in relation to changing prices. Solving for \( p' \) results in

\[ p' = wl'[I - (1 + r)A]^{-1}. \]

Thus, price proportions (which do not depend on \( w \)) can be obtained straightforwardly. If \( w \) would be in money terms (say in $ or £ per unit of labour), prices are also in $ or £. In that case, there is no direct relation between \( r \) and \( w \) in the sense that knowledge of one enables us to calculate the other. Adopting a particular commodity as numeraire will establish a fixed relation, though. Following Kurz and Salvadori (1995, Ch. 4, Section 1), we may adopt a

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12 We should remark that Sraffa makes no assumption regarding the presence of constant returns to scale. For the argument in this section, no such assumption will be required, though. Sraffa (1960, p. v), however, adds that there also is no harm in adopting such an assumption as a temporary working hypothesis if such an assumption is found to be helpful (on this, see also Kurz and Salvadori, 1995, pp. 423–424, in a comparison of the analyses of von Neumann (1945) and Sraffa).

13 See, for example, Pasinetti (1977, Section V.3). We should add that the row vector \( l' \) is obtained via the aggregation of different labour types via wage rates. Sraffa’s assumption of labour to be of uniform quality is equivalent to assuming that any differences in quality have been reduced to equivalent differences in quantity such that each unit of labour receives the same wage (Sraffa, 1960, p. 10; see also Kurz and Salvadori, 1995, pp. 324–325); we are indebted to one referee for pointing this out.
particular commodity bundle, say \( d \), and assign the value 1 to that bundle, that is, \( p' d = 1 \). For given \( r \), this fixes the value of \( p' \) and, hence, \( w \). We can thus introduce a relation between \( r \) and \( w \) based on the bundle \( d \), provided the inverse of matrix \( [I - (1 + r) A] \) exists. In this way, a range of IDs can be obtained, each based on the particular commodity bundle chosen as the numeraire. If we decide to select a net output bundle as given by Equation 1, we obtain a distribution based on that particular output bundle. We may observe here that this method is not based on the notion that the economy’s final demand vector has an interpretation in terms of the real wage bundle, as put forward in Sections 1 and 2. Also, as we shall see in Section 4, this method cannot readily serve as a basis for analysing more complex cases where more than two parties are involved.

When the wage is expressed in terms of NI, \( w \in [0, 1] \) with \( w = 1 \) when \( r = 0 \). Similarly, \( r \in [0, R] \), where \( R \) is the maximum rate of profit, obtained when \( w = 0 \). A special role is played by the standard commodity. This is the net output bundle of an I–O system (the so-called standard system) with the special property that its net output vector has the same proportions as the vector of the aggregated means of production in the system as a whole (or, equivalently, its total output vector). Adopting the standard commodity as a numeraire for prices is Sraffa’s answer to the problem of finding a standard that is ‘invariable’ with respect to changes in ID. In fact, this composite commodity, used as a numeraire, enables us to formulate the distribution-determining relations independent of any price movements, because (as a unit of measurement for prices) it eliminates the influence of the characteristics of the individual production processes from the relation between \( r \) and \( w \). If this standard commodity is used as a numeraire, we will have, by construction, a case where the \( w–r \) relation is linear.\(^{14}\)

In the Sraffian literature – as far as we know – up to now, no general approach for characterizing the structural form of the ID over the entire range of \( r \) and \( w \) (as proposed in the previous section) has been presented. Below, therefore, we shall show how adopting the method described in our earlier sections, the \((r, w)\) relation can be derived for the general case.

We shall proceed following the approach described in the previous section. With given net output bundle \( f \), the economy’s NI is \( p' f \). The Sraffian approach contains two important standardizations, that is, \( p' f = 1 \) and \( L = 1 \). This means that also the income per head \( p'(f/L) = 1 \). We now may rewrite the price equation as

\[
p' = p'[(1 + r)A + w\left(\frac{f}{L}\right) l'],
\]

or

\[
p'\left(I - [(1 + r)A + w\left(\frac{f}{L}\right) l']\right) = 0
\]

again with \( p'(f/L) = 1 \) and \( w \in [0, 1] \). The sought-after wage–profit relation then is

\[
\varphi(r, w) = \left|I - \left[(1 + r)A + w\left(\frac{f}{L}\right) l'\right]\right| = 0.
\]

\(^{14}\) See further Pasinetti (1977, Ch. V), Abraham-Frois and Berrebi (1979, Ch. 2), Kurz and Salvadori (1995, 2007), or Steenge (1995) on this. Similarly, also in the case where \( l' \) in Equation 3 is proportional to \( p' \), we have a linear relation, see Pasinetti (1977, Ch. V). See also Kurz and Salvadori (2007) for a recent contribution on the origin of this particular line of research.
Equation 20 is the Sraffian equivalent of Equation 12. We see that both matrices \( A \) and \( B = (f/L)' \) have received weights \((1 + r)\) and \(w\), respectively. A closer look at Equation 20 gives us the relation between the weight factors and thus the sought-after ID. Starting from Equation 20 and defining

\[
M_{r,w} \equiv (1 + r)A + w \left( \frac{f}{L} \right)' ,
\]

(21)

we obtain the simplified form

\[
\varphi(r, w) = |I - M_{r,w}| = 0,
\]

(22)

which gives us the relation between \( r \) and \( w \), a polynomial in both variables. We should note that the analysis of the price equation does not require a simultaneous specification of the physical output side. In fact, we only need to possess knowledge of the final demand bundle \( f \) that is produced in the ‘actual system’. We have two additional lemmas here which will enable us to obtain analytical expressions for the \( r \)-\( w \) relation:

**Lemma 2** Let \( M_{\alpha, \beta} = [\alpha A + \beta (f/L)'] \), all symbols as before and \( \alpha \) and \( \beta \) scalars. Then,

\[
|I - M_{\alpha, \beta}| = |I - \alpha A| - \beta (I - \alpha A)\# \left( \frac{f}{L} \right).
\]

(23)

**Proof**

\[
|I - M_{\alpha, \beta}| = |I - \left[ \alpha A + \beta \left( \frac{f}{L} \right)' \right]| = |(I - \alpha A) - \beta (f/L)'| = |I - \alpha A| - \beta (f/L)'\#.
\]

\[\blacksquare\]

In our case, we then have

**Lemma 3** Let \( M_{\alpha, \beta} \) be as in Lemma 2 and let \( |I - M_{\alpha, \beta}| = 0 \). Then,

\[
\beta = |I - \alpha A| / Y (I - \alpha A)\# \left( \frac{f}{L} \right).
\]

(24)

That is, Lemma 3 informs us that \( \beta \) is a function of \( \alpha \).
3.1. Frontiers and Prices

Below, we shall present an example using Sraffa’s data on his ‘actual system’ (see also Pasinetti, 1977, Chs. 2 and 5 on this system). We have, rounding in three decimals,

\[ A = \begin{bmatrix} 0.413 & 2.571 & 0.500 \\ 0.027 & 0.286 & 0.050 \\ 0.020 & 0.286 & 0.250 \end{bmatrix} \]

and

\[ l' = [0.040 \ 0.571 \ 0.500]. \]

Final demand equals

\[ f = \begin{bmatrix} 180 \\ 0 \\ 30 \end{bmatrix}. \]

Straightforward calculation gives, after rounding,

\[ x = \begin{bmatrix} 450 \\ 21 \\ 60 \end{bmatrix} \]

and

\[ L = 60. \]

Substituting the values of \( f, l', \) and \( L \) in Equation 20, we obtain the expression

\[ \varphi(r, w) = 0.241 - 0.579r - 0.241w + 0.170r^2 + 0.119rw - 0.010r^3 - 0.010r^2w, \]

a polynomial of third degree in \( r \) and first degree in \( w \). Putting \( \varphi(r, w) = 0 \) and solving for \( w \), we find

\[ w(r) = \frac{0.241 - 0.579r + 0.170r^2 - 0.010r^3}{0.241 - 0.119r + 0.010r^2}. \]

Clearly, \( w(r) \) is nonlinear. To illustrate, the Taylor expansion of \( w(r) \) around the point \( r = 0 \) by a polynomial of degree 2 gives

\[ w(r) = 1.000 - 1.909r - 0.279r^2 + O(r^3). \]

Figure 1 shows the graph of \( w(r) \). The part of the left-most curve in the first quadrant is economically relevant, with \( r \) ranging from 0 to 0.482, its maximum value. Prices are obtained by substituting an admissible point \((r, w)\) in Equation 20 and solving for \( p'_r, w \), the left-hand PF eigenvector of matrix \( M_{r,w} \); standardization then gives absolute prices. For
example, for \( r = 0.2 \), we find \( w = 0.606 \). From Equation 20, we then have

\[
M_{0.2, 0.606} = \begin{bmatrix}
0.569 & 4.125 & 1.509 \\
0.032 & 0.343 & 0.060 \\
0.036 & 0.516 & 0.452
\end{bmatrix}
\]

with standardized left-hand PF eigenvector\(^{15}\)

\[
p_{0.2, 0.606}' = [0.205 \quad 1.891 \quad 0.771].
\]

### 3.2. Adopting the Standard Commodity as Numeraire

The standard commodity is defined as the net output vector (indicated by the symbol \( f^* \)) of an economic system (the so-called standard system, see above) that produces its net output in the same proportions as its aggregated means of production \( Ax^* \) or, equivalently, its total output vector \( x^* \). It can be shown that these proportions are those of the right-hand PF eigenvector of the matrix of intermediate input coefficients, that is, matrix \( A \).\(^{16}\) In this case, \( w(r) \) is linear, as mentioned. Keeping the size of the labour force unchanged (i.e. 60 units),

\(^{15}\) That is, \((p_{0.2,0.606}')f = 60.\)

\(^{16}\) See, for example, Pasinetti (1977, Section V.9).
we find that

\[ f^* = \begin{bmatrix} 141.77 \\ 11.58 \\ 14.47 \end{bmatrix} \]

and

\[ x^* = \begin{bmatrix} 435.57 \\ 35.59 \\ 44.48 \end{bmatrix}. \]

If we adopt \( f^* / L \) as our numeraire, we have, again using Sraffa’s data, the case of a linear trade-off between \( w \) and \( r \). Straightforward calculation gives

\[ w(r) = \frac{0.241 - 0.579r + 0.170r^2 - 0.010r^3}{0.241 - 0.080r - 0.005r^2}. \]

or, in terms of its Taylor expansion,\(^{17}\)

\[ w(r) = 1.000 - 2.073r + 0.000r^2 + O(r^3). \]

Figure 2 captures the changed situation in the same wider picture as the general case described in the previous sub-section. Again, the economically relevant part is given by the values of \( r \) in the closed interval \([0, R]\), with \( R = 0.482 \), the maximum rate of profit.

\(^{17}\) The curve \( w(r) \) is not completely linear due to rounding.
The price proportions are given by the left-hand PF eigenvector of matrix:

\[
M_{0.2,0.585} = \begin{bmatrix}
0.551 & 3.876 & 1.292 \\
0.037 & 0.407 & 0.116 \\
0.030 & 0.424 & 0.371
\end{bmatrix}.
\]

The standardized price vector is

\[
p'_{0.2,0.585} = [0.198 \quad 1.827 \quad 0.744],
\]

with standardization to \((p'_{0.2,0.585})f^* = 60\). We observe that the price proportions have changed in correspondence to the new numeraire (i.e. \(f^*\)). In this way, we find, over the entire range of \(r\) and \(w\), the corresponding price adaptations.\(^{18}\)

4. INCOME DISTRIBUTION IN THE LEONTIEF MODEL

In Section 2, we constructed matrix \(B\) as a matrix of ‘imputed input coefficients’. By construction, \(r(B) = 1\), with \(B\) being the outer product of the column vector \(f/L\) and the row vector \(l'\). The columns of matrix \(B\) thus stood for the real wage per unit of output of the relevant sector and its rows for the imputed quantities of the relevant commodity.

In the above case, the imputed coefficients were obtained by combining information from two sources, that is, from the net output and the primary input categories. However, this need not standardly be the case; often the necessary information – to determine IDs – is available in another form. An example is matrix \(rA\) given in the previous section. There the imputed input coefficients were readily available as the elements of the intermediate input coefficients matrix \(A\) multiplied by the rate of profit \(r\). This procedure, that is, obtaining the required coefficients from other sources, can also be used in the Leontief-based distribution problems. We provide an illustration in Section 4.1. In Section 4.2, the case of two primary factors is discussed, while in Section 4.3 the general case is discussed.

Again, the proportions of the relevant output and price vectors are obtained as, respectively, the left- and right-hand PF eigenvectors of the system’s augmented input coefficients matrix, after which standardizing NI results in absolute levels. Subsequent varying of the weights of the sector-specific input coefficients matrices allows us to trace the ID patterns.

4.1. New Claims on the NI

Suppose that a country is confronted, in a relatively short time, with challenges of an entirely new nature. Suppose also that these challenges are of a scale that is significant in terms of its NI. We may think here of ambitious climate change-oriented initiatives, issues related to shifting patterns in international trade, or problems related to population dynamics.\(^{19}\) In

\(^{18}\) Leontief (1985, 1986), in exploring the effects of technological change on ID, employed a price equation quite similar to Sraffa’s. However, we may observe that getting insight into the general shape of the wage rate–rate of profit relation is not an easy matter and that the method itself is difficult to generalize.

\(^{19}\) An interesting example is provided by present-day water policy in the Netherlands. Investment expenditures in water management by central and local governments, water boards, and industry amount to about 5 billion euros annually (Brouwer and van der Veeren, 2009). New findings, possibly related to climate change policies, may easily ask for a rapid increase of this figure; see further Barro (2006) on the sheer scale of the potential welfare loss in advanced economies due to infrequently occurring disasters.
such a situation, many issues – often up to then dormant – may become critical. One of
these is the supply of primary resources. If this supply is not sufficient to meet the new
demands, their entry will be at the cost of the more traditional claims on NI. The ‘older’
claims can be ‘crowded out’, and the question then becomes what precisely will happen
in terms of which parties will benefit and which will not, at what rate, which prices will
be realized, etc.\textsuperscript{20} Stated otherwise, are there also here ‘laws’ of the type presented in the
previous section? And if so, what do they tell us?

Below, we shall, in analogy with the model described in the previous section, show that
the problems of the above sketched nature can be addressed by employing a framework
which is able to ‘weigh’ all economic activities within the structure of an augmented input
coefficients matrix. Suppose we have a matrix of intermediate input coefficients,

\[
A = \begin{bmatrix}
0.25 & 0.05 & 0.15 \\
0.09 & 0.19 & 0.06 \\
0.07 & 0.09 & 0.39
\end{bmatrix}
\]

and a final demand vector,

\[
f = \begin{bmatrix}
100 \\
250 \\
150
\end{bmatrix},
\]

with the corresponding labour input coefficients vector,

\[
l' = [0.20 \ 0.30 \ 0.15].
\]

We shall suppose furthermore that the ‘new challenges’ manifest themselves in the form of
extra production being required to fulfil the various needs, old and new, and that the costs of
producing the additional goods and services will be borne by the industrial sectors according
to some earlier agreed-upon distribution scheme, say on the basis of sectoral output. If the
additionally required production is known in its totality, we may proceed analogously to
the previous section. Suppose that the estimated extra production over time is given by the
vector

\[
h = z \begin{bmatrix}
10 \\
20 \\
30
\end{bmatrix},
\]

where \( z \) is a scale parameter, and that the programme specifies that in each specific period
a certain part of \( h \) must be produced and put in place. An arrangement such as this means
that each commodity price now must reflect the new commitment in the form of a separate
cost category to be dealt with. If costs are allocated strictly proportional according to the

\textsuperscript{20} Regarding this, we should recall that quite a number of studies have appeared in the last decade based on
forecasting, scenarios, or other types of analysis. Hirsch et al. (2005), for example, predict that price volatility will
increase dramatically, with enormous associated economic, social, and political costs. In addition, the owners of
the relevant resources may be expected to claim a much greater share of total income. Friedrichs (2010) predicts
three possible scenarios, that is, predatory militarism, totalitarian retrenchment, and socio-economic adaptation,
also against a background of a redistribution of power and wealth from resource importers to exporters.
following matrix $D$, where

$$
D = \begin{bmatrix}
0.05 & 0.05 & 0.05 \\
0.10 & 0.10 & 0.10 \\
0.15 & 0.15 & 0.15 \\
\end{bmatrix},
$$

only a proportionality constant has to be determined to obtain the consequences per period.

Concentrating on a specific period, say 1 year, and with $x$ standing for the total output vector for that year, $Dx$ then is the vector of the additionally required production. Let further $\delta \geq 0$ be the proportionality constant we referred to. The vector $\delta Dx$ then indicates which part of $Dx$ must be produced during the year under consideration. Assuming that labour is still the only primary factor and that the real wage part again is given by matrix $B$ as defined in Section 2, the new price equation reads

$$
p' = p'(A + \delta D + \beta B).
$$

From Equation 25, we see that we have here a variant of the Sraffian price equation, now with matrix $\delta D$ performing the role of matrix $rA$ in Equation 18. For the real outputs, we have correspondingly

$$
x = (A + \delta D + \beta B)x.
$$

Both equations directly lead to a determinantal equation of the type we discussed earlier, that is,

$$
\phi(\beta, \delta) = 0.354 - 0.354\beta + 0.017\beta\delta - 0.205\delta = 0,
$$
a hyperbola. Solving for $\delta$, we have

$$
\delta(\beta) = \frac{0.354 - 0.354\beta}{0.205 - 0.017\beta}.
$$

It is useful to consider the Taylor expansion around the point $\beta = 0$. This gives

$$
\delta(\beta) = 1.727 - 1.584\beta - 0.131\beta^2 + O(\beta^3),
$$

which shows that $\delta$ is nearly linear in $\beta$ in the economically relevant part of the curve, see Figure 3.

We observe that price and output proportions can be obtained by substituting admissible values for $\beta$ and $\delta$ (see Figure 3) in Equation 25 or 26 and subsequent calibrating by selecting an appropriate standard, say by keeping total employment constant. For example, for $\beta = 1$

---

21 That is, industry 1 faces additional costs per unit of production as given by the first column of matrix $D$ multiplied by $\delta$, etc. These additional costs can consist of specific taxes to be paid, claims by capital owners, costs associated with R&D expenses, etc. We may wish to compare this with the role of profit in the price equation of Section 3, where profits are an integral part of each commodity price.
and $\delta = 0$, we find that the total employment $L$ equals 200.202 and

$$X_{1.0} = \begin{bmatrix} 221.973 \\ 357.312 \\ 324.092 \end{bmatrix}$$

with the corresponding standardized price vector

$$p_{1.0} = [0.354 \ 0.434 \ 0.376].$$

We may straightforwardly vary $\beta$ and $\delta$ over their entire admissible range. For example, for $\beta = 0.5$ and $\delta = 0.902$, we find, calibrating at $L = 200.202$,

$$X_{0.5,0.902} = \begin{bmatrix} 225.445 \\ 314.608 \\ 404.871 \end{bmatrix}$$

with the standardized price vector

$$p_{0.5,0.902} = [0.369 \ 0.393 \ 0.434].$$

The above example shows that it is not necessary to introduce a new and clearly identifiable primary factor in the context of the new programme. The introduction of the 'new challenges'
is at the expense of labour’s real wage at a rate dictated by the above equations. In this sense, the model described in this sub-section has a lot in common with the Sraffian model described in the previous section.

### 4.2. Several Factors and Final Demand Categories

In this sub-section, we shall discuss the case where two factors and two final demand categories are distinguished. We shall assume that for both factors a direct relation exists between their demand for final products and the amount of services they provide in return. We can then construct coefficient matrices analogous to matrix \( B \) given in Section 2. For convenience, the two factors shall be called labour and capital again; regarding capital, we shall assume that it consists of a well-specified bundle of capital goods that will be distributed over the sectors according to their needs.

We shall use the same intermediate input coefficients matrix as in Section 4.1. The final demand bundle now consists of two parts: one part \( (f_1) \) going to households in the form of real wage for the labour contributed, while the other part \( (f_2) \) goes to capital owners for investment, depreciation allowances, replacement, and other capital-related purposes. Correspondingly, we have, next to the \((1,n)\) row vector of direct labour input coefficients \( l' \) (as in Section 4.1), a second \((1,n)\) row vector, \( k' \), standing for the required capital-related inputs into production.\(^{22}\) Let us assume

\[
\begin{align*}
 f_1 &= \begin{bmatrix} 100 \\ 70 \\ 80 \end{bmatrix} \\
 f_2 &= \begin{bmatrix} 0 \\ 180 \\ 70 \end{bmatrix}
\end{align*}
\]

(so \( f_1 + f_2 = f \) with \( f \) as in Section 4.1). For the labour and capital-related vectors of input coefficients, we have, respectively,

\[
\begin{align*}
 l' &= \begin{bmatrix} 0.20 \\ 0.30 \\ 0.15 \end{bmatrix} \\
 k' &= \begin{bmatrix} 0.05 \\ 0.02 \\ 0.03 \end{bmatrix}
\end{align*}
\]

Calculation gives

\[
\begin{bmatrix} 221.973 \\ 357.312 \\ 324.092 \end{bmatrix}
\]

Total employment, \( L \), is 200.202 (with \( L = l'x \)). With \( K \) denoting the total supply of capital-related input contributions, we find \( K = 27.968 \) (via \( K = k'x \)). The input coefficients

\(^{22}\) Questions concerning the way in which the bundle \( f_2 \) will be allocated between the sectors for replacement and other purposes fall outside the scope of this paper. If necessary, appropriate mechanisms can be built in without undue effort.
matrices now can be simply obtained. With

\[ \mathbf{M} = \mathbf{A} + \mathbf{B} + \mathbf{H}, \]  

(27)

we have

\[ \mathbf{B} = \begin{pmatrix} f_1 \\ L \end{pmatrix} l', \]  

(28)

and

\[ \mathbf{H} = \begin{pmatrix} f_2 \\ K \end{pmatrix} k'. \]  

(29)

Substitution gives

\[ \mathbf{M} = \begin{bmatrix} 0.350 & 0.200 & 0.225 \\ 0.482 & 0.424 & 0.306 \\ 0.275 & 0.260 & 0.525 \end{bmatrix}. \]

The corresponding price proportions are given by the left-hand PF eigenvector of \( \mathbf{M} \). Calibrating, as in Section 4.1, at \( p' f = 200.202 \), we find

\[ p' = [0.450 \ 0.355 \ 0.442]. \]

The shares in NI are, respectively, 105.280 and 94.922.

Let us now again introduce distribution parameters, \( \beta \) for the wage part and \( \gamma \) for the capital-related part. Employing the symbol \( \mathbf{M}_{\beta,\gamma} \) for the corresponding augmented input coefficients matrix, we then have

\[ \mathbf{M}_{\beta,\gamma} = \mathbf{A} + \beta \mathbf{B} + \gamma \mathbf{H}. \]

(30)

First, let us consider the relation between \( \beta \) and \( \gamma \) for the case at hand. We have

\[ |\mathbf{I} - \mathbf{M}_{\beta,\gamma}| = 0. \]

(31)

Written out, we have

\[ \varphi(\beta, \gamma) = 0.354 - 0.169\beta - 0.149\gamma - 0.035\beta\gamma = 0. \]

Figure 4 shows, for the interval \( -1 \leq \beta \leq 3 \), the relation between \( \beta \) and \( \gamma \).

We see that \( \gamma \) is a function of \( \beta \) and is convex in its economically relevant part. In this respect, we should realize that its functional form is completely determined by the characteristics of the technological and behavioural parameters of the economy in case.

---

23 We can verify that \( \mathbf{x} \) satisfies \( p' f = p'\mathbf{x} - p'\mathbf{Ax} \).

24 So, matrix \( \mathbf{M} \) reflects the case where both \( \beta \) and \( \gamma \) have unit value, with \( L = l'\mathbf{x} = 200.202 \) and \( K = k'\mathbf{x} = 27.968 \).
To illustrate, let us assume $\beta = 1.25$. Calculation gives $\gamma = 0.735$. This results in the following augmented input coefficients matrix:

$$
M_{1.25,0.735} = \begin{bmatrix}
0.375 & 0.237 & 0.244 \\
0.414 & 0.416 & 0.268 \\
0.262 & 0.277 & 0.520
\end{bmatrix}.
$$

The price and output proportions again are given by the left- and right-hand PF eigenvectors of $M_{1.25,0.735}$. Calibrating at $NI = 200.202$, we find

$$
p'_{1.25,0.735} = [0.419 \ 0.369 \ 0.418],
$$

and, assuming that the supply of capital will be sufficient,

$$
x_{1.25,0.735} = \begin{bmatrix}
254.964 \\
332.049 \\
330.629
\end{bmatrix},
$$

while shares in NI are 126.516 and 73.688, respectively.

### 4.3. The General Case

The general case is now straightforward. Again, we may include the additional factor-related terms in a new and extended system matrix. Assembling the new terms, we then obtain

$$
x = (A + B + H_1 + \ldots + H_k)x,
$$

(32)
where $k$ reflects the number of additional final demand/value-added categories we wish to distinguish. This gives, correspondingly, the following equation, which contains the basic information we need for identifying IDs via the process of attaching weights to the individual constituting (sub)matrices:

$$|I - (A + \beta B + \gamma_1 H_1 + \cdots + \gamma_k H_k)| = 0. \tag{33}$$

Hereafter, we may proceed by selecting values for the distribution parameters, thereby – as before – taking care of the role of possible critical values for one or more of the primary factors.

At this point, we should be precise about the results we wish to obtain. In the Leontief model, a choice is offered: we may start from the classification as given by the final demand categories or from the classification of the value-added categories. Our ultimate choices should reflect the research issues we are interested in.

A different point is that finding a satisfactory ‘match’ between the final demand and value-added categories may not be straightforward. We may, for example, wish to start with relatively less complex matchings such as those between forms of personal consumption and wages and salaries. Investment (gross or net) may be coupled to capital returns, depreciation, and other capital-related categories, taxes to government-based expenditure, and so on. In carrying out such exercises, we should recall that the total of all final demand categories is equal, by construction, to the total of all value-added categories. Starting with the less complex categories, the remaining categories may be determined successively.

A separate problem is the fact that the I–O model’s primary resources are not the standard ones as distinguished traditionally in other areas of economics. For example, there is no clear classification in terms of the rents of primary factors such as land, labour, and capital, including their sub-divisions. The lack of harmonization with the standard approaches in other fields of economics may mean that additional decisions regarding classification and possible sub-classification will have to be made.

5. FINAL REMARKS

In this paper, we have shown that ID mechanisms can fruitfully be introduced in a standard I–O framework. An interpretation of factor remunerations in real terms appeared to be central, that is, as commodity bundles exchanged for factor inputs. Assembling the available information for each factor resulted in factor-specific ‘remuneration matrices’. This procedure allowed us to interpret the remunerations as inputs on the same footing as intermediate inputs. The subsequent introduction of the so-called augmented input coefficients matrices enabled us to find a mathematical expression for the relation between shifts in prices, quantities, and income shares.

In its essence, the concept of an augmented input coefficient constitutes an extension of the concept of an intermediate input coefficient. Such an extension has been proposed before, but only confined to labour input. We have shown that the principle can be extended to include all factors that claim a part of NI, where the associated augmented coefficient matrices give a complete picture of all such claims. IDs subsequently are obtained by attributing weights to these matrices. We have proposed three lemmas that allow us to trace the relations between these weights, subject to the constraints imposed by the I–O
framework. We have also shown that prices and quantities (which express the size or scale of the system) are straightforwardly obtained as the standardized left- and right-hand PF eigenvectors of the relevant augmented input coefficients matrix.

The above will definitely not be the last word. Very likely, additional connections will be found and explored as well. In this respect, the methodology can be refined in a number of ways. For example, we may try to establish a link between the weights of the coefficient matrices we have introduced and other concepts such as substitution or other elasticities, possibly in relation to behavioural shifts motivated by changed circumstances. Also the role of government-directed tax-subsidy programmes can be investigated, including the distributive effects of specific policy initiatives. Finally, we may also wish to start the other way around, that is, with changes in the prevailing prices, and see which shifts in distribution these may trigger.

We have remarked that in Leontief modelling, we may find the existing categorization in the underlying tables sometimes appropriate, sometimes not. In fact, depending on the way the final demand and value-added categories have been defined, proceeding as proposed by us may involve some rearranging of the data in the value-added rows. For example, household consumption may be paid for out of wages and salaries and capital income. If the value-added classification is not conforming, the researcher will have to go back to the original underlying data and construct data according to the appropriate classification. If the original data are not available or of low quality, additional research may be required.

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