Isotropy Groups of Free Racks and Quandles

Jason Parker

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Abstract

In this article, we apply the methods developed in [5], [7] to characterize the (covariant) isotropy groups of free, finitely generated racks and quandles. As a consequence, we show that the usual inner automorphisms of such racks and quandles are precisely those automorphisms that are ‘coherently extendible’. We then use this result to compute the global isotropy groups of the categories of racks and quandles, i.e. the automorphism groups of the identity functors of these categories.

1 Introduction

In [3, Theorem 1], George Bergman proved that the usual inner automorphisms of a group $G$ (defined in terms of conjugation) are exactly those automorphisms of $G$ that can be coherently extended along morphisms out of $G$. More precisely, he showed that an automorphism $\alpha : G \to G$ is inner if and only if for any group homomorphism $f : G \to H$ with domain $G$, one can define a group automorphism $\pi_f : H \to H$ of the codomain in such a way that the resulting family of automorphisms $(\pi_f)_f$ is natural, meaning that if $f : G \to G'$ and $f' : G' \to G''$ are group homomorphisms, then the following square commutes:

\[
\begin{array}{c}
G' \\
\downarrow f' \\
G'' \\
\end{array}
\begin{array}{c}
\pi_f \\
\pi_{f' \circ f} \\
\end{array}
\begin{array}{c}
G' \\
\downarrow f' \\
G'' \\
\end{array}
\]

Such a family of automorphisms may be equivalently described as a natural automorphism of the projection functor $G/\text{Group} \to \text{Group}$, from the slice category under the group $G$ to the category $\text{Group}$. We refer to such a family of automorphisms as an extended inner automorphism of $G$. If $Z(G)$ denotes the group of all extended inner automorphisms of $G$, i.e. the group of all natural
automorphisms of the projection functor \( G/\text{Group} \to \text{Group} \), then Bergman also showed in \cite{Bergman80} Theorem 2 that \( Z(G) \) is isomorphic to the group \( G \) itself.

In \cite{Burke11}, the author and his collaborators took inspiration from this result of Bergman to analyze the extended inner automorphisms of the models of any (single-sorted) algebraic or equational theory whatsoever (as a special case, a group is a model of the algebraic theory of groups). In \cite{Burke13}, the author then extended this study from (single-sorted) algebraic theories to (multi-sorted) essentially algebraic theories (where operations are only required to be partially defined).

As for groups, there is a well-known notion of inner automorphism for a rack or quandle (which, along with the theories of racks and quandles, is defined and discussed in the subsequent section). Using techniques developed in \cite{Burke11} and \cite{Burke13}, we will prove a result for free racks and quandles that is analogous to the one that George Bergman proved for (arbitrary) groups: namely, we will show that an automorphism of a rack/quandle is inner (in the well-known sense) if and only if it can be ‘coherently extended’ along morphisms out of the rack/quandle, in the sense described for groups.

2 Background

We now review the relevant material from \cite{Burke11} and \cite{Burke13} Chapter 2 on the isotropy groups of free models of algebraic theories. For more details, see those references.

A single-sorted algebraic theory \( T \) is a set of equations between the terms of a single-sorted first-order signature \( \Sigma \) consisting of operation symbols. For example, the theories of (commutative) monoids, (abelian) groups, (commutative) rings with unit, and the theories of racks and quandles (to be defined explicitly below). A (set-based) model \( M \) of a single-sorted algebraic theory \( T \) is a set equipped with functions on \( M \) interpreting the function symbols of the signature, which satisfies the axioms of \( T \). One can then form the category \( T\text{mod} \) of (set-based) models of \( T \) and homomorphisms between them.

Given \( M \in T\text{mod} \), the (covariant) isotropy group of \( M \) is the group \( Z_T(M) \) of all natural automorphisms of the forgetful functor \( M/T\text{mod} \to T\text{mod} \). More concretely, an element of \( Z_T(M) \) is a family of automorphisms

\[
\pi = (\pi_h : \text{cod}(h) \xrightarrow{\sim} \text{cod}(h))_{\text{dom}(h) = M}
\]

in \( T\text{mod} \) indexed by morphisms \( h \in T\text{mod} \) with domain \( M \) that has the following naturality property: if \( h : M \to M' \) and \( h' : M' \to M'' \) are homomorphisms in \( T\text{mod} \), then the following diagram commutes:
We then say that an automorphism \( h : M \xrightarrow{\sim} M \) of \( M \in \mathcal{T}_{\text{mod}} \) is a categorical inner automorphism (or is coherently extendible) if there is some \( \pi \in Z_{\mathcal{T}}(M) \) with \( h = \pi_{\text{id}_M} : M \xrightarrow{\sim} M \); roughly, \( h \) is a categorical inner automorphism if it can be coherently extended along any morphism out of \( M \).

In [5] and [7] the author and his collaborators gave a logical characterization of the isotropy group of a model of an algebraic theory. We will only review the (simpler) characterization for the free, finitely generated models, as these are the only models that will concern us in this article. If \( \mathcal{T} \) is an algebraic theory and \( n \geq 0 \), then a model \( M_n \in \mathcal{T}_{\text{mod}} \) is free on \( n \) generators if it contains \( n \) distinct elements (the generators) \( m_1, \ldots, m_n \) and has the following universal property: for any \( N \in \mathcal{T}_{\text{mod}} \) and any elements \( a_1, \ldots, a_n \in N \), there is a unique homomorphism \( h_{a_1, \ldots, a_n} : M_n \to N \) with \( h_{a_1, \ldots, a_n}(m_i) = a_i \) for each \( 1 \leq i \leq n \).

For a fixed \( n \geq 0 \), the free \( \mathcal{T} \)-models on \( n \) generators are all isomorphic, so we may speak of the free \( \mathcal{T} \)-model on \( n \) generators (unique up to isomorphism).

The free \( \mathcal{T} \)-model on \( n \) generators can be given the following explicit description: let \( \{y_1, \ldots, y_n\} \) be a set of \( n \) distinct constants (not in \( \Sigma \)), and let \( \Sigma(y_1, \ldots, y_n) \) be the single-sorted signature obtained from \( \Sigma \) by adding the elements \( y_1, \ldots, y_n \) as new constant symbols. Let \( T(y_1, \ldots, y_n) \) be the algebraic theory with the same axioms as \( T \), but now regarded as an algebraic theory over the signature \( \Sigma(y_1, \ldots, y_n) \). Consider the set \( \text{Term}^e(\Sigma(y_1, \ldots, y_n)) \) of closed terms over the signature \( \Sigma(y_1, \ldots, y_n) \). We then define a relation \( \sim_{\mathcal{T}, n} = \sim_{\mathcal{T}} \) on \( \text{Term}^e(\Sigma(y_1, \ldots, y_n)) \) by setting \( s \sim_{\mathcal{T}} t \) iff

\[
T(y_1, \ldots, y_n) \vdash s = t
\]

for any \( s, t \in \text{Term}^e(\Sigma(y_1, \ldots, y_n)) \). Roughly, we have \( s \sim_{\mathcal{T}} t \) iff \( s \) can be proved equal to \( t \) using (only) the axioms of \( \mathcal{T} \). Then \( \sim_{\mathcal{T}} \) is a congruence relation on \( \text{Term}^e(\Sigma(y_1, \ldots, y_n)) \), i.e. an equivalence relation that is compatible with the operation symbols in \( \Sigma \). We can then form the quotient \( \mathcal{T} \)-model

\[
\text{Term}^e(\Sigma(y_1, \ldots, y_n))/\sim_{\mathcal{T}},
\]

whose objects are \( \sim_{\mathcal{T}} \)-congruence classes, which will have the desired universal property, with generators \( [y_1], \ldots, [y_n] \). So we can take

\[
M_n := \text{Term}^e(\Sigma(y_1, \ldots, y_n))/\sim_{\mathcal{T}}
\]

as an explicit construction of the free \( \mathcal{T} \)-model on \( n \) generators.
Now let $G_T(M_n)$ be the set of all elements
\[ [t] \in \text{Term}^c(\Sigma(x, y_1, \ldots, y_n))/\sim_T \]
(note the additional constant $x$) with the following properties:

- $[t]$ is invertible, meaning that there is some $t^{-1} \in \text{Term}^c(\Sigma(x, y_1, \ldots, y_n))$ such that
  \[ \mathcal{T}(x, y_1, \ldots, y_n) \vdash t[t^{-1}/x] = x = t^{-1}[t/x]. \]

- $[t]$ commutes generically with the operation symbols of $\Sigma$, meaning that if $f$ is an $m$-ary operation symbol of $\Sigma$, then
  \[ \mathcal{T}(x_1, \ldots, x_m, y_1, \ldots, y_n) \vdash t[f(x_1, \ldots, x_m)/x] = f(t[x_1/x], \ldots, t[x_m/x]). \]

Then this set $G_T(M_n)$ can be given the structure of a group (with unit element $[x]$ and multiplication given by substitution into $x$), and we then have (cf. [7, Corollary 2.4.15])
\[ Z_T(M_n) \cong G_T(M_n). \]

We refer to $G_T(M_n)$ as the **logical isotropy group** of $M_n$; thus, the (categorical) covariant isotropy group of $M_n$ is isomorphic to its logical isotropy group. Therefore, the extended inner automorphisms of $M$ can essentially be identified with those (congruence classes of) closed $\Sigma$-terms over the constant $x$ and the generating constants $y_1, \ldots, y_n$ that are invertible and commute generically with the operations of $\Sigma$.

Given any $[t] \in \text{Term}^c(\Sigma(x, y_1, \ldots, y_n))/\sim_T$ and any $T$-model $N$ with $n$ distinct elements $a_1, \ldots, a_n \in N$, the element $[t]$ induces a function
\[ [t]^{N,a_1,\ldots,a_n} : N \to N; \]
roughly, given any $a \in N$, the value $[t]^{N,a_1,\ldots,a_n}(a) \in N$ is the element of $N$ obtained by substituting $a_1, \ldots, a_n$ for $y_1, \ldots, y_n$ and $a$ for $x$ in $t$, and then interpreting/evaluating the result in $N$. The following results then follow from the definition of the isomorphism $Z_T(M_n) \cong G_T(M_n)$, cf. [7, Corollary 2.2.42]. For any homomorphism $h : M_n \to N$ in $\mathcal{T}\text{mod}$, let us write $h_1, \ldots, h_n \in N$ for the images $h([y_1]), \ldots, h([y_n]) \in N$ of the generators of $M_n$ under $h$.

- Given any (not necessarily natural) family
  \[ \pi = (\pi_h : \text{cod}(h) \to \text{cod}(h))_{\text{dom}(h)=M_n} \]
of endomorphisms in $\mathcal{T}\text{mod}$ indexed by morphisms with domain $M_n$, we have $\pi \in Z_T(M_n)$ iff there is some (uniquely determined) element $[t] \in G_T(M_n)$ with the property that
  \[ \pi_h = [t]^{N,h_1,\ldots,h_n} : N \to N \]
for each homomorphism $h : M_n \to N$ in $\mathcal{T}\text{mod}$ with domain $M_n$ (in particular, every such function $[t]^{N,h_1,\ldots,h_n}$ will be a $T$-model automorphism).
Given any endomorphism $h : M_n \to M_n$ in $\text{Tmod}$, we have that $h$ is a \textit{categorical inner automorphism} of $M_n$ iff there is some element $[t] \in G_T(M_n)$ with
\[ h = [t]^{id_1, \ldots, id_n} : M \to M \]
(where $id : M_n \to M_n$ is the identity morphism).

In this article, we will be computing the (logical) isotropy groups of the free, finitely generated models of the algebraic theories of \textit{racks} and \textit{quandles}. Our computations will depend heavily on the solutions of the word problems for free racks and quandles in terms of the solution of the word problem for free groups given in [4, Section 4.1]. We now review the definitions of these algebraic theories:

\textbf{Definition 2.1.}

1. Let $\Sigma$ be the single-sorted signature containing two binary function symbols $\lhd, \lhd^{-1}$, written in infix notation.

2. Let $\mathbb{T}_{\text{Rack}}$ be the algebraic theory over the signature $\Sigma$ with the following axioms (where $x, y, z$ are variables):
   - $x \lhd (y \lhd z) = (x \lhd y) \lhd (x \lhd z)$.
   - $x \lhd^{-1} (y \lhd^{-1} z) = (x \lhd^{-1} y) \lhd^{-1} (x \lhd^{-1} z)$.
   - $(x \lhd y) \lhd^{-1} y = x$.
   - $(x \lhd^{-1} y) \lhd y = x$.

3. Let $\mathbb{T}_{\text{Quandle}}$ be the algebraic theory over the signature $\Sigma$ whose axioms are those of $\mathbb{T}_{\text{Rack}}$ together with the following two additional axioms:
   - $x \lhd x = x$.
   - $x \lhd^{-1} x = x$.

Racks and quandles are algebraic structures that originally arose in the context of knot theory, to describe so-called \textit{Reidemeister moves}. Algebraically speaking, they axiomatize the notion of (group) \textit{conjugation} (without reference to multiplication or inverses). For example, a canonical quandle structure can be defined on (the underlying set of) any group $G$ by setting $g \lhd h := ghg^{-1}$ and $g \lhd^{-1} h := g^{-1}hg$. Examples of racks that are \textit{not} (necessarily) quandles include constant actions $x \lhd y := \sigma(x) := x \lhd^{-1} y$, where $\sigma$ is a permutation of a fixed set $X$. Any equation involving only conjugation that is provable in the theory of groups is also provable in the theory of quandles, so that quandles essentially axiomatize the concept of group-theoretic conjugation ([6, Theorem 4.2]).

If $R$ is any rack with $r \in R$, then it can be shown that the function $(-) \lhd r : R \to R$ is a rack automorphism, with inverse $(-) \lhd^{-1} r : R \to R$. Let $\text{Aut}(R)$ be
the group of all rack automorphisms of $R$. Then the group $\text{Inn}(R)$ of \textit{algebraic inner automorphisms} of $R$ is defined to be the subgroup of $\text{Aut}(R)$ generated by all such rack automorphisms ($\cdot$) $\triangleleft r$. If $Q$ is a quandle, then we define $\text{Inn}(Q)$ analogously. Explicitly, an automorphism $f : R \to R$ of a rack $R$ is an \textit{algebraic inner} automorphism iff there are $p, n \geq 0$ and $r_1, \ldots, r_n \in R$ and $\delta_1, \ldots, \delta_p, \epsilon_1, \ldots, \epsilon_n = \pm 1$ such that

$$f(r) = (\ldots(((r \triangleleft _{\delta_1} r) \triangleleft _{\delta_2} r) \ldots) \triangleleft _{\epsilon_1} r_1) \triangleleft _{\epsilon_2} r_2) \ldots) \triangleleft _{\epsilon_n} r_n$$

for all $r \in R$. Similarly, an automorphism $f : Q \to Q$ of a quandle $Q$ is an \textit{algebraic inner} automorphism iff there are $n \geq 0$ and $q_1, \ldots, q_n \in Q$ and $\epsilon_1, \ldots, \epsilon_n = \pm 1$ such that

$$f(q) = (\ldots((q \triangleleft _{\epsilon_1} q_1) \triangleleft _{\epsilon_2} q_2) \ldots) \triangleleft _{\epsilon_n} q_n$$

for all $q \in Q$. It is not difficult to show that any algebraic inner automorphism of an arbitrary rack or quandle is also a \textit{categorical} inner automorphism. As a consequence of our results in the next two sections, we will show that the converse is true for \textit{free, finitely generated} racks and quandles; i.e. we will show that any \textit{categorical} inner automorphism of a free, finitely generated rack or quandle must be an \textit{algebraic} inner automorphism as well.

\section{Isotropy Groups of Free Quandles}

We will first characterize the (logical) isotropy groups of free, finitely generated quandles, because this turns out to be the simpler task. First, we give an explicit description of the free quandle on $n$ generators, as given in e.g. \cite[Proposition 4.2]{4}.

Let $\Sigma_{\text{Grp}}$ be the single-sorted signature of the algebraic theory $T_{\text{Grp}}$ of groups, with three function symbols $\cdot$ (binary), $-1$ (unary), and $e$ (constant), the first two written in infix notation. For any (finite) set $X$, let $\Sigma_{\text{Grp}}(X)$ be the signature that extends the signature $\Sigma_{\text{Grp}}$ by adding the elements of $X$ as new constants, and let $\text{Term}^e(\Sigma_{\text{Grp}}(X))$ be the set of \textit{closed} terms over the signature $\Sigma_{\text{Grp}}(X)$. Also let $T_{\text{Grp}}(X)$ be the theory with the same axioms as $T_{\text{Grp}}$, but now regarded as an algebraic theory over the signature $\Sigma_{\text{Grp}}(X)$.

Given a (finite) set $X$, it has been shown (cf. \cite[Proposition 4.2]{4}) that the free quandle on $X$ has the following presentation, which we denote by $\text{Conj}(F_X)$, where $F_X$ is the free group on $X$ (with the presentation given in Section 1 for $T = T_{\text{Grp}}$). The underlying set of $\text{Conj}(F_X)$ is just the underlying set of $F_X$, and for any $[s], [t] \in F_X$ (so that $s, t \in \text{Term}^e(\Sigma_{\text{Grp}}(X))$) we have

$$[s] \triangleleft [t] := [t^{-1} \cdot s \cdot t] \in F_X$$

and

$$[s] \triangleleft ^{-1} [t] := [t \cdot s \cdot t^{-1}] \in F_X.$$  

(When brackets are omitted when writing group multiplication, we will assume that they associate to the left).
Let $\Sigma(X)$ be the signature that extends $\Sigma$ (the signature for racks and quandles) by adding the elements of $X$ as constants, and let $\text{Term}^c(\Sigma(X))$ be the set of closed terms over the signature $\Sigma(X)$. Then there is a function

$$E_X : \text{Term}^c(\Sigma(X)) \to \text{Term}^c(\Sigma_{\text{Grp}}(X))$$

defined by induction on the structure of closed terms by

$$E_X(x) := x \quad (x \in X)$$

$$E_X(s \cdot t) := E(t)^{-1} \cdot E(s) \cdot E(t)$$

$$E_X(s \cdot t^{-1}) := E(t) \cdot E(s) \cdot E(t)^{-1}$$

for $s, t \in \text{Term}^c(\Sigma(X))$. Technically the map $E_X$ depends on $X$, but we will omit the subscript when confusion will not arise. We then have the following substitution lemma:

**Lemma 3.1.** Let $X$ be an arbitrary (finite) set with designated element $x \in X$, and let $X'$ be another (finite) set with $X \subseteq X'$. Then for any $t \in \text{Term}^c(\Sigma(X))$ and $s \in \text{Term}^c(\Sigma(X'))$, we have

$$E(t[s/x]) \sim_{\text{Grp}} E(t)[E(s)/x]$$

in $F_{X'}$.

**Proof.** For a fixed $s \in \text{Term}^c(\Sigma(X'))$, we prove the claim by induction on $t \in \text{Term}^c(\Sigma(X))$:

- If $t \equiv x$, then we have
  $$E(t[s/x]) = E(x[s/x])$$
  $$= E(s)$$
  $$= x[E(s)/x]$$
  $$= E(x)[E(s)/x]$$
  $$= E(t)[E(s)/x].$$

- If $t \equiv y$ for some $y \in X$ with $x \neq y$, then we have
  $$E(t[s/x]) = E(y[s/x])$$
  $$= E(y)$$
  $$= y$$
  $$= y[E(s)/x]$$
  $$= E(y)[E(s)/x]$$
  $$= E(t)[E(s)/x].$$
• Suppose that $t \equiv t_1 < t_2$ for some $t_1, t_2 \in \textbf{Term}^c(\Sigma(X))$ with

$$E(t_1[s/x]) \sim E(t_1)[E(s)/x]$$

and

$$E(t_2[s/x]) \sim E(t_2)[E(s)/x].$$

Then we have

$$E(t[s/x]) = E((t_1 < t_2)[s/x])$$
$$= E(t_1[s/x] < t_2[s/x])$$
$$= E(t_2[s/x])^{-1} \cdot E(t_1[s/x]) \cdot E(t_2[s/x])$$
$$\sim (E(t_2)[E(s)/x])^{-1} \cdot E(t_1)[E(s)/x] \cdot E(t_2)[E(s)/x]$$
$$\sim E(t_2)^{-1}[E(s)/x] \cdot E(t_1)[E(s)/x] \cdot E(t_2)[E(s)/x]$$
$$= (E(t_2)^{-1} \cdot E(t_1) \cdot E(t_2))[E(s)/x]$$
$$= E(t_1 < t_2)[E(s)/x]$$
$$= E(t)[E(s)/x];$$

note that the fifth equality holds because for any $u, v \in \textbf{Term}^c(\Sigma_{\text{Grp}}(X'))$ we have $u[v/x]^{-1} \sim u^{-1}[v/x]$, as one can easily prove by induction on $u$ for a fixed $v$. The case where $t \equiv t_1 < t_2$ is exactly analogous.

\[ \square \]

In [4, Section 4.1], the following result was proven:

**Theorem 3.2** (Dehornoy [4]). *For any (finite) set $X$ and $s, t \in \textbf{Term}^c(\Sigma(X))$, we have

$$T_{\text{Quandle}}(X) \vdash s = t$$

iff

$$E(s) \sim E(t).$$

\[ \square \]

We will also require the following lemma. Recall that a closed term $s \in \textbf{Term}^c(\Sigma_{\text{Grp}}(X))$ is said to be reduced if it is either $e$ or $x \in X$, or else has the form $t_1^i \cdot \ldots \cdot t_m^i$ with $m \geq 2$ and $t_1, \ldots, t_m \in X$ and $e_1, \ldots, e_m \in \{1, -1\}$ and $t_i = t_{i+1}$ implies $e_i = e_{i+1}$ for each $1 \leq i < m$. We will sometimes refer to such reduced terms as reduced group words (over $X$). It is a standard fact about free groups that if $s, t$ are reduced group words over $X$ with $s \sim_{\text{Grp}} t$, then $s \equiv t$ (i.e. $s$ and $t$ must be the same word).

**Lemma 3.3.** Let $X := \{x_0, x_1, y_1, \ldots, y_n\}$ for some $n \geq 0$. If $s \in \textbf{Term}^c(\Sigma_{\text{Grp}}(x, y_1, \ldots, y_n))$ is reduced and

$$s[x_1/x] \cdot s[x_1^{-1}x_0x_1/x] \sim s[x_0/x] \cdot s[x_1/x],$$

then $s$ has at most one occurrence of $x$ (which must then have exponent 1).
Proof. Assume the hypothesis. Then it is not difficult to see that if \( s \) had at least two occurrences of \( x \), then the reduction of the term on the left side would have an occurrence of \( x_1 \) to the left of an occurrence of \( x_0 \) while the reduction of the term on the right side would have all occurrences of \( x_1 \) to the right of all occurrences of \( x_0 \). But if these terms are congruent (modulo \( \sim_{\text{Grp}} \)), then their reductions must be equal, which is impossible, as just shown. So \( s \) has at most one occurrence of \( x \).

Lastly, if the reduced term \( s \) contains an occurrence of \( x \), then this unique occurrence must have exponent 1. For suppose otherwise; then \( s \equiv t_1 \cdot x^{-1} \cdot t_2 \) for some reduced (possibly empty) words \( t_1, t_2 \in \text{Term}^\ast(\Sigma_{\text{Grp}}(y_1, \ldots, y_n)) \). By the assumed congruence, we then have

\[
t_1 \cdot x_1^{-1} \cdot t_2 \cdot t_1 \cdot x_1^{-1} \cdot x_0^{-1} \cdot (x_1^{-1})^{-1} \cdot t_2 \sim t_1 \cdot x_0^{-1} \cdot t_2 \cdot t_1 \cdot x_1^{-1} \cdot t_2.
\]

However, even if \( t_2 \cdot t_1 \sim e \), it is easy to see that the reductions of these terms will not be identical, and hence these terms cannot be congruent (modulo \( \sim \)). Therefore, if the reduced term \( s \) contains an occurrence of \( x \), then this occurrence must have exponent 1.

We can now characterize the isotropy group of the free quandle on \( n \) generators \( y_1, \ldots, y_n \). Recall from Section 1 that if \( Q_n \) is the free quandle on \( n \) generators \( y_1, \ldots, y_n \), then the logical isotropy group \( G_{\text{Quandle}}(Q_n) \) of \( Q_n \) is (isomorphic to) the group of all elements

\[
[t] \in \text{Term}^\ast(\Sigma(x, y_1, \ldots, y_n)) / \sim_{\text{Quandle}}
\]

that are invertible and commute generically with the function symbols \( \triangleleft, \triangleleft^{-1} \), in the sense that there is some \( s \in \text{Term}^\ast(\Sigma(x, y_1, \ldots, y_n)) \) with

\[
T_{\text{Quandle}}(x, y_1, \ldots, y_n) \vdash t[s/x] = x = s[t/x],
\]

\[
T_{\text{Quandle}}(x_0, x_1, y_1, \ldots, y_n) \vdash t[x_0 \triangleleft x_1/x] = t[x_0/x] \triangleleft t[x_1/x],
\]

and

\[
T_{\text{Quandle}}(x_0, x_1, y_1, \ldots, y_n) \vdash t[x_0 \triangleleft^{-1} x_1/x] = t[x_0/x] \triangleleft^{-1} t[x_1/x].
\]

Theorem 3.4 (Isotropy Group of a Free Quandle). Let \( Q_n \) be the free quandle on \( n \) generators \( y_1, \ldots, y_n \). Then for any \( t \in \text{Term}^\ast(\Sigma(x, y_1, \ldots, y_n)) \), we have

\[
[t] \in G_{\text{Quandle}}(Q_n)
\]

iff there are \( m \geq 0 \) and \( 1 \leq i_1, \ldots, i_m \leq n \) and \( \epsilon_1, \ldots, \epsilon_m = \pm 1 \) such that

\[
[t] = [x \triangleleft^{i_1} y_{i_1} \triangleleft^{i_2} \ldots \triangleleft^{i_m} y_{i_m}]
\]

and the corresponding group word \( y_{i_1}^{\epsilon_1} \ldots y_{i_m}^{\epsilon_m} \) is reduced (we have written \( x \triangleleft^{i_1} y_{i_1} \triangleleft^{i_2} \ldots \triangleleft^{i_m} y_{i_m} \) instead of the more cumbersome \( \ldots ((x \triangleleft^{i_1} y_{i_1}) \triangleleft^{i_2} y_{i_2}) \ldots) \triangleleft^{i_m} y_{i_m} \), i.e. bracketing of quandle terms is assumed to associate to the left).
Proof. We first show that the $\sim_{\text{Quandle}}$-class of any term of the described form belongs to $G_{\text{Quandle}}(Q_n)$. So let

\[ t := x^{-1} y_{i_1}^{-1} \cdots y_{i_m}^{-1} \]

for some $m \geq 0$ and $1 \leq i_1, \ldots, i_m \leq n$ and $\epsilon_1, \ldots, \epsilon_m = \pm 1$. We must show that $[t]$ is invertible and commutes generically with the quandle operations. For invertibility, we show that the $\sim_{\text{Quandle}}$-class of

\[ t^{-1} := x^{-1} y_{i_m}^{-1} \cdots y_1^{-1} \]

is the inverse of $[t]$. So we must show that $t[t^{-1}/x] \sim_{\text{Quandle}} x$ and $t^{-1}[t/x] \sim_{\text{Quandle}} x$. Since the two claims have analogous proofs, we will only prove the first.

By Theorem 1, it suffices to show that $E(t[t^{-1}/x]) \sim E(x) \equiv x$ in the free group on $x, y_1, \ldots, y_n$. Using Lemma 1, we have

\[
E(t[t^{-1}/x]) = E(t)E(t^{-1})/x \equiv (y_{i_m}^{-\epsilon_m} \cdots y_{i_1}^{-\epsilon_1} x_0^{-1} x_1 y_{i_1}^{-1} \cdots y_{i_m}^{-1}) \big[ x_1^{-1} x_0^{-1} x_1 y_{i_1}^{-1} \cdots y_{i_m}^{-1} \big] \equiv E(x),
\]

as desired.

Now we show that $[t]$ commutes generically with the quandle operations. Since the proofs of both claims are analogous, we only prove that $[t]$ commutes generically with $\prec$. By Theorem 1, it suffices to show that

\[ E(t[x_0 \prec x_1/x]) \sim E(t[x_0/x] \prec t[x_1/x]) \]

in the free group on $x_0, x_1, y_1, \ldots, y_n$. Starting from the right side, we have

\[
E(t[x_0/x] \prec t[x_1/x]) = E(t[x_1/x])^{-1} \cdot E(t[x_0/x]) \cdot E(t[x_1/x]) \\
\sim E(t^{-1}[x_1/x] \cdot E(t[x_0/x] \cdot E(t[x_1/x]) \\
\sim y_{i_m}^{-\epsilon_m} \cdots y_{i_1}^{-\epsilon_1} x_0 y_{i_1}^{-1} \cdots y_{i_m}^{-1} x_1 y_{i_1}^{-1} \cdots y_{i_m}^{-1} \\
\sim y_{i_m}^{-\epsilon_m} \cdots y_{i_1}^{-\epsilon_1} x_0 \cdot E(x_1^{-1} x_0 x_1 y_{i_1}^{-1} \cdots y_{i_m}^{-1}) \\
\sim E(t)E(x_0 \prec x_1/x) \\
\sim E(t[x_0 \prec x_1/x]),
\]

as desired (where we applied Lemma 1 to obtain the final congruence). This completes the proof that $[t] \in G_{\pi_{\text{Quandle}}}(Q_n)$.
Now suppose that \( t \in \text{Term}^c(\Sigma(x_1, \ldots, y_n)) \) and \([t] \in G_{\mathbb{T}_\text{Quandle}}(Q_n)\). We show that \( t \) can be assumed to have the form described in the statement of the theorem. Since \([t] \in G_{\mathbb{T}_\text{Quandle}}(Q_n)\), we know that \([t]\) commutes generically with the quandle operations. In particular, we have

\[
\mathbb{T}_\text{Quandle}(x_0, x_1, \ldots, y_n) \vdash t[x_0 < x_1/x] = t[x_0/x] < t[x_1/x].
\]

By Theorem 1, it then follows that the following relation holds in the free group on the same generators:

\[
E(t[x_0 < x_1/x]) \sim E(t[x_0/x] < t[x_1/x]).
\]

Then since we have

\[
E(t[x_0/x] < t[x_1/x]) = E(t[x_1/x])^{-1} \cdot E(t[x_0/x]) \cdot E(t[x_1/x])
\approx E(t)^{-1}[x_1/x] \cdot E(t)[x_0/x] \cdot E(t)[x_1/x],
\]

and since (by Lemma 1) we have

\[
E(t[x_0 < x_1/x]) \sim E(t)[E(x_0 < x_1)/x]
= E(t)[x_1^{-1}x_0x_1/x],
\]

it follows that we have

\[
E(t)[x_1^{-1}x_0x_1/x] \sim E(t)^{-1}[x_1/x] \cdot E(t)[x_0/x] \cdot E(t)[x_1/x].
\]

Now let \( s \) be the unique reduced word congruent to \( E(t) \in \text{Term}^c(\Sigma_{\text{Grp}}(x_1, \ldots, y_n)) \), so that \( E(t) \sim s \). Then we obtain

\[
s[x_1^{-1}x_0x_1/x] \sim s^{-1}[x_1/x] \cdot s[x_0/x] \cdot s[x_1/x],
\]

which implies

\[
s[x_1/x] \cdot s[x_1^{-1}x_0x_1/x] \sim s[x_0/x] \cdot s[x_1/x].
\]

Then by Lemma 2, since \( s \) is reduced, it follows that \( s \) has at most one occurrence of \( x \), which will have exponent 1. Now we show that \( s \) has at least one, and hence exactly one, occurrence of \( x \). Since \([t] \in G_{\mathbb{T}_\text{Quandle}}(Q_n)\), it is invertible, and hence there is some \( t^{-1} \in \text{Term}^c(\Sigma(x, y_1, \ldots, y_n)) \) such that

\[
\mathbb{T}_\text{Quandle}(x_1, y_1, \ldots, y_n) \vdash t[t^{-1}/x] = x = t^{-1}[t/x].
\]

Then by Theorem 1 and Lemma 1, it follows that

\[
x \equiv E(x) \sim E(t[t^{-1}/x]) \sim E(t)[E(t^{-1})/x].
\]

Since \( E(t) \sim s \), it then follows that

\[
s[E(t^{-1})/x] \sim x.
\]
If \( s \) did not have at least one occurrence of \( x \), then we would have \( s|E(t^{-1})/x| \equiv s \), so that we could deduce \( s \sim x \). But then since \( s \) and \( x \) are reduced, this would imply that \( s \equiv x \), contradicting the assumption that \( s \) has no occurrence of \( x \). So it follows that \( s \) has at least one, and hence exactly one, occurrence of \( x \). So then \( s \equiv t_1 \cdot x \cdot t_2 \) for some reduced (possibly empty) words \( t_1, t_2 \in \operatorname{Term}^e(\Sigma_{\operatorname{Grp}}(y_1, \ldots, y_n)) \). From

\[
 s[x_1/x] \cdot s[x_1^{-1}x_0x_1/x] \sim s[x_0/x] \cdot s[x_1/x]
\]

we then infer

\[
t_1 \cdot x_1 \cdot t_2 \cdot t_1 \cdot x_1^{-1}x_0x_1 \cdot t_2 \sim t_1 \cdot x_0 \cdot t_2 \cdot t_1 \cdot x_1 \cdot t_2.
\]

So the reductions of both words are identical, which implies that \( t_2 \cdot t_1 \sim e \), so that \( t_1 \sim t_2^{-1} \cdot t_1^{-1} \), and hence \( s \equiv t_1 \cdot x \cdot t_2 \sim t_1 \cdot x \cdot t_2^{-1} \). So we now have \( E(t) \sim s \sim t_1 \cdot x \cdot t_2^{-1} \) for some reduced word \( t_1 \in \operatorname{Term}^e(\Sigma_{\operatorname{Grp}}(y_1, \ldots, y_n)) \).

Now let \( t_1 \equiv y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m} \) for some \( m \geq 0 \), with \( \epsilon_j = \pm 1 \) and \( 1 \leq i_j \leq n \) for each \( 1 \leq j \leq m \). Then we have \( t_1^{-1} \sim y_{i_1}^{-\epsilon_1} \cdots y_{i_m}^{-\epsilon_m} \), so that

\[
 E(t) \sim y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m} x y_{i_1}^{-\epsilon_1} \cdots y_{i_m}^{-\epsilon_m}.
\]

Then we have

\[
 E(t) \sim y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m} x y_{i_1}^{-\epsilon_1} \cdots y_{i_m}^{-\epsilon_1}
 \sim E(x <^{\epsilon_m} y_{i_m} <^{\epsilon_{m-1}} \cdots <^{\epsilon_1} y_{i_1}).
\]

By Theorem 1, we then deduce that

\[
t \sim_{\text{Quandle}} x <^{\epsilon_m} y_{i_m} <^{\epsilon_{m-1}} \cdots <^{\epsilon_1} y_{i_1},
\]

so that \( t \) is \( \sim_{\text{Quandle}} \)-congruent to a term of the desired form (since \( t_1 \equiv y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m} \) is reduced, which implies that \( y_{i_m}^{-\epsilon_m} \cdots y_{i_1}^{-\epsilon_1} \) is reduced). This completes the proof of the theorem.

Given this logical description of the isotropy group of the free quandle on \( n \) generators, we now give a more algebraic description of this isotropy group:

**Corollary 3.5.** Let \( F_n \) be the free group on \( n \) generators \( y_1, \ldots, y_n \). Then

\[
 G_{\text{Quandle}}(Q_n) \cong F_n.
\]

That is, the logical isotropy group of the free quandle on \( n \) generators is isomorphic to the free group on \( n \) generators.

**Proof.** Since \( F_n \) is the free group on \( n \) generators and \( G_{\text{Quandle}}(Q_n) \) is a group, there is a unique group homomorphism

\[
 \phi : F_n \rightarrow G_{\text{Quandle}}(Q_n)
\]
with
\[ \phi(y_i) = [x \triangleleft y_i] \]
for each \( 1 \leq i \leq n \), since \([x \triangleleft y_i] \in G_{\text{Quandle}}(Q_n)\) by Theorem 2. So it remains to show that \( \phi \) is a bijection. Note first that for any \( 1 \leq i \leq n \) we have
\[
\phi([y_i^{-1}]) = \phi([y_i])^{-1} = [x \triangleleft y_i]^{-1} = [x \triangleleft^{-1} y_i]
\]
(cf. the proof of Theorem 2 for the last equality), and hence for any \( \epsilon_i \) with \( 1 \leq i_j \leq n \) for each \( 1 \leq j \leq m \), we have (since the product in \( G_{\text{Quandle}}(Q_n) \) is given by substitution into \( x \))
\[
\phi([y_{i_1}^{e_1} \cdots y_{i_m}^{e_m}]) = \phi([y_{i_1}^{e_1}]) \cdots \phi([y_{i_m}^{e_m}]) = [x \triangleleft e_1 y_{i_1}] \cdots [x \triangleleft^{e_m} y_{i_m}] = [x \triangleleft^{e_m} y_{i_m} \triangleleft^{e_{m-1}} \cdots \triangleleft^{e_1} y_{i_1}].
\]
Now, that \( \phi \) is surjective is obvious, because by Theorem 2, if \([t] \in G_{\text{Quandle}}(Q_n)\), then either \([t] = [x]\), in which case we have \( \phi([e]) = [x] = [t] \) (because \( \phi \) is a group homomorphism and \([x] \) is the identity element of \( G_{\text{Quandle}}(Q_n) \)), or otherwise there is some \( m \geq 1 \) such that
\[
[t] = [x \triangleleft^{e_1} y_{i_1} \triangleleft^{e_2} \cdots \triangleleft^{e_m} y_{i_m}],
\]
with \( e_j = \pm 1 \) and \( 1 \leq i_j \leq n \) for all \( 1 \leq j \leq m \). But then we have
\[
\phi([y_{i_m}^{e_m} \cdots y_{i_1}^{e_1}]) = [t],
\]
as desired.

To prove that \( \phi \) is injective, let \( y_{i_1}^{e_1} \cdots y_{i_m}^{e_m} \) and \( y_{j_1}^{\delta_1} \cdots y_{j_p}^{\delta_p} \) be reduced group words over the generators \( y_1, \ldots, y_n \) with \( m, p \geq 1 \) (if one of the words is just \( e \), then the argument that follows is even easier), and suppose that
\[
\phi([y_{i_1}^{e_1} \cdots y_{i_m}^{e_m}]) = \phi([y_{j_1}^{\delta_1} \cdots y_{j_p}^{\delta_p}]),
\]
in order to show that
\[
y_{i_1}^{e_1} \cdots y_{i_m}^{e_m} \sim y_{j_1}^{\delta_1} \cdots y_{j_p}^{\delta_p}
\]
in the free group on \( y_1, \ldots, y_n \). The assumption implies that
\[
x \triangleleft^{e_m} y_{i_m} \triangleleft^{e_{m-1}} \cdots \triangleleft^{e_1} y_{i_1} \sim_{\text{Quandle}} x \triangleleft^{\delta_p} y_{j_p} \triangleleft^{\delta_{p-1}} \cdots \triangleleft^{\delta_1} y_{j_1}
\]
in the free quandle on \( x, y_1, \ldots, y_n \). By Theorem 1, this in turn implies that
\[
E(x \triangleleft^{e_m} y_{i_m} \triangleleft^{e_{m-1}} \cdots \triangleleft^{e_1} y_{i_1}) \sim E(x \triangleleft^{\delta_p} y_{j_p} \triangleleft^{\delta_{p-1}} \cdots \triangleleft^{\delta_1} y_{j_1})
\]
in the free group on \( x, y_1, \ldots, y_n \), i.e. that
\[
y_{i_1}^{e_1} \cdots y_{i_m}^{e_m} xy_{i_m} \cdots y_{i_1}^{e_1} \sim y_{j_1}^{-\delta_1} \cdots y_{j_p}^{-\delta_p} xy_{j_p} \cdots y_{j_1}^{\delta_1}
\]
in the free group on \(x, y_1, \ldots, y_n\). This implies that
\[
x \sim y_{i_1}^{\epsilon_1} y_{j_1}^{-\delta_1} \cdots y_{i_p}^{\epsilon_p} y_{j_p}^{-\delta_p} y_{j_1}^{\delta_1} \cdots y_{i_m}^{-\delta_m},
\]
which then implies that
\[
y_{i_1}^{\epsilon_1} y_{j_1}^{-\delta_1} \cdots y_{i_p}^{\epsilon_p} \sim e
\]
and
\[
y_{j_p}^{\delta_p} y_{j_1}^{\delta_1} \cdots y_{i_m}^{-\delta_m} \sim e,
\]
which finally imply that
\[
y_{i_1}^{\epsilon_1} y_{j_1}^{\delta_1} \cdots y_{i_m}^{\epsilon_m} \sim y_{j_p}^{\delta_p} y_{j_1}^{\delta_1} \cdots y_{i_m}^{-\delta_m}.
\]
Since \(y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m}\) and \(y_{j_1}^{\delta_1} \cdots y_{j_p}^{\delta_p}\) are reduced words by assumption, this entails that \(y_{i_1}^{\epsilon_1} y_{j_1}^{-\delta_1} \cdots y_{i_p}^{\epsilon_p} y_{j_p}^{-\delta_p} y_{j_1}^{\delta_1} \cdots y_{i_m}^{-\delta_m}\) and \(y_{j_1}^{\delta_1} \cdots y_{j_p}^{\delta_p}\) are also reduced words. Therefore, since we are working in the free group on \(y_1, \ldots, y_n\), this implies that \(m = p\) and \(y_{i_k} = y_{j_k}\) and \(\epsilon_k = \delta_k\) for all \(1 \leq k \leq m = p\). This proves that \(y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m} \sim y_{j_1}^{\delta_1} \cdots y_{j_p}^{\delta_p}\), as desired. \(\square\)

From our characterization(s) of the logical isotropy groups of the free, finitely generated quandles, we can now deduce characterizations of the categorical isotropy groups of these quandles. The proof of the following corollary invokes the two bullet points preceding Definition 1 in Section 1, the characterization given in Theorem 3.4, and the fact that reduced group words congruent modulo \(\sim_{\text{Grp}}\) must be identical.

**Corollary 3.6.** Let \(n \geq 0\).

1. Let
\[
\pi = (\pi_h : \text{cod}(h) \rightarrow \text{cod}(h))_{\text{dom}(h) = Q_n}
\]
be a (not necessarily natural) family of endomorphisms of quandles, indexed by quandle morphisms \(h\) with domain \(Q_n\). Then \(\pi \in \mathcal{Z}_{\text{Quandle}}(Q_n)\) iff there is a unique reduced word
\[
y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m} \in \text{Term}^c(\Sigma_{\text{Grp}}(y_1, \ldots, y_n))
\]
with the property that for any quandle morphism \(h : Q_n \rightarrow Q\) we have
\[
\pi_h(q) = q \triangleleft \epsilon_1 \ h_{i_1} \triangleleft \epsilon_2 \ \cdots \triangleleft \epsilon_m \ h_{i_m} \in Q.
\]

2. Let \(h : Q_n \rightarrow Q_n\) be a quandle endomorphism. Then \(h\) is a categorical inner automorphism iff there is a unique reduced word \(y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m} \in \text{Term}^c(\Sigma_{\text{Grp}}(y_1, \ldots, y_n))\) such that
\[
h([s]) = [s \triangleleft \epsilon_1 y_{i_1} \triangleleft \epsilon_2 \ \cdots \triangleleft \epsilon_m y_{i_m}] \in Q_n
\]
for any \([s] \in Q_n\) (so \(s \in \text{Term}^c(\Sigma(y_1, \ldots, y_n))\)).
3. Let \( h : Q_n \to Q_n \) be a quandle endomorphism. Then \( h \) is a categorical inner automorphism iff \( h \) is an algebraic inner automorphism.

Finally, we can deduce a characterization of the global isotropy group of the category \( \text{Quandle} \) of quandles and their homomorphisms, i.e. the group \( \text{Aut}(\text{Id}_{\text{Quandle}}) \) of automorphisms of the identity functor \( \text{Id}_{\text{Quandle}} : \text{Quandle} \to \text{Quandle} \) (which is also the group of invertible elements of the centre of the category \( \text{Quandle} \), which is the monoid \( \text{End}(\text{Id}_{\text{Quandle}}) \) of natural endomorphisms of the identity functor). Since the category \( \text{Quandle} \) has an initial object, namely the absolutely free quandle \( Q_0 \) (whose carrier is just the empty set), it is easy to see that the global isotropy group of \( \text{Quandle} \) is exactly the (covariant) categorical isotropy group of the initial object \( Q_0 \), i.e.

\[
\text{Aut}(\text{Id}_{\text{Quandle}}) = Z_{\text{Quandle}}(Q_0).
\]

Since

\[
Z_{\text{Quandle}}(Q_0) \cong C_{\text{Quandle}}(Q_0) \cong F_0
\]

by Corollary 3.5 and \( F_0 \) is the trivial group (being the free group on 0 generators), we thus obtain:

**Corollary 3.7.** The global isotropy group of the category \( \text{Quandle} \) is the trivial group, i.e. the only automorphism of the identity functor \( \text{Id}_{\text{Quandle}} \) is the identity natural transformation.

We also note in connection with Corollary 3.7 that M. Szymik independently proved in [8, Theorem 5.5] that the center \( \text{End}(\text{Id}_{\text{Quandle}}) \) of the category \( \text{Quandle} \) is trivial as well. Thus, we obtain the following further corollary:

**Corollary 3.8.** The global isotropy group of the category \( \text{Quandle} \) is equal to its center, and both are trivial.

## 4 Isotropy Groups of Free Racks

In this section, we will proceed to characterize the isotropy groups of free, finitely generated racks, which is a slightly more involved task than the characterization for quandles (due to the increased complexity of the word problem for free racks). Given a (finite) set \( X \), it has been shown (cf. [4, Proposition 4.2]) that the free rack on \( X \) has the following presentation, which we denote by \( \text{HalfConj}(X, F_X) \), where \( F_X \) is once again the free group on \( X \). The underlying set of \( \text{HalfConj}(X, F_X) \) is the set \( X \times F_X \), and the rack operations on this set are defined as follows, for any \( x, y \in X \) and \( [s], [t] \in F_X \):

\[
(x, [s]) \triangleleft (y, [t]) := (x, [s \cdot t^{-1} \cdot y \cdot t]),
\]

\[
(x, [s]) \triangleleft^{-1} (y, [t]) := (x, [s \cdot t^{-1} \cdot y^{-1} \cdot t]).
\]
There is now a function

\[ E_X : \text{Term}^c(\Sigma(X)) \to X \times \text{Term}^c(\Sigma_{\text{Grp}}(X)) \]

with

\[ E_X(x) := (x, e) \quad (x \in X) \]

and

\[ E_X(s <^\epsilon t) := (\pi_1(E_X(s)), \pi_2(E_X(s)) \cdot \pi_2(E_X(t))^{-1} \cdot \pi_1(E_X(t))^{\epsilon} \cdot \pi_2(E_X(t))) \]

for \( \epsilon = \pm 1 \) and \( s, t \in \text{Term}^c(\Sigma(X)) \). As before, we will omit the subscript on \( E \) to increase readability.

First, we have the following definition and lemma concerning the relationship between \( E \) and the first projection function \( \pi_1 : \text{HalfConj}(X, F_X) \to X \).

**Definition 4.1.** Let \( t \in \text{Term}^c(\Sigma(X)) \) for a (finite) set \( X \). We define \( \text{Left}(t) \in X \) (intuitively, the ‘leftmost’ element of \( X \) occurring in \( t \)) by induction on \( t \):

- If \( t \equiv x \) for some \( x \in X \), then \( \text{Left}(t) := x \).
- If \( t \equiv t_1 <^\epsilon t_2 \) for \( t_1, t_2 \in \text{Term}^c(\Sigma(X)) \) and \( \epsilon = \pm 1 \) such that \( E(t_1) = \text{Left}(t_1) \), then \( \text{Left}(t) := \text{Left}(t_1) \).

To increase readability, we will now write \( \pi_i(x, s) \) as \( (x, s)_i \) for \( i \in \{1, 2\} \) and \((x, s) \in X \times \text{Term}^c(\Sigma_{\text{Grp}}(X))\).

**Lemma 4.2.** Let \( t \in \text{Term}^c(\Sigma(X)) \) for a (finite) set \( X \). Then

\[ E(t)_1 = \text{Left}(t) \in X. \]

**Proof.** We prove this by induction on \( t \).

- If \( t \equiv x \) for some \( x \in X \), then we have

  \[ E(t)_1 = E(x)_1 = (x, e)_1 = x = \text{Left}(x) = \text{Left}(t), \]

  as desired.

- Let \( t \equiv t_1 <^\epsilon t_2 \) for some \( t_1, t_2 \in \text{Term}^c(\Sigma(X)) \) and \( \epsilon = \pm 1 \) such that \( E(t_1)_1 = \text{Left}(t_1) \). Then by definition of \( E \) we have

  \[ E(t_1 <^\epsilon t_2)_1 = E(t_1)_1 = \text{Left}(t_1) = \text{Left}(t_1 <^\epsilon t_2), \]

  as desired.

\[ \square \]

We also have the following substitution lemma:
Lemma 4.3. Let $X$ be an arbitrary (finite) set with designated element $x \in X$, and let $X'$ be another (finite) set with $X \subseteq X'$ and $x_0, x_1 \in X' \setminus X$. Then for any $t \in \text{Term}^c(\Sigma(X))$ we have:

- If $E(t)_1 = x$, then
  \[ E(t[x_0 \lhd x_1])_1 = x_0 \]
  for $\epsilon = \pm 1$ and
  \[ E(t[x_0 \lhd x_1])_2 \sim x_1 \cdot E(t[x_1^{-1}x_0x_1/x]), \]
  \[ E(t[x_0 \lhd x_1])_2 \sim x_1^{-1} \cdot E(t[x_1x_0x_1^{-1}/x]). \]

- If $E(t)_1 \neq x$, then
  \[ E(t[x_0 \lhd x_1])_1 = E(t)_1 \]
  for $\epsilon = \pm 1$ and
  \[ E(t[x_0 \lhd x_1])_2 \sim E(t[x_1^{-1}x_0x_1/x]), \]
  \[ E(t[x_0 \lhd x_1])_2 \sim E(t[x_1x_0x_1^{-1}/x]). \]

Proof. We prove this by induction on $t \in \text{Term}^c(\Sigma(X))$. We will only consider the claims for $\lhd$, since the claims for $\lhd^{-1}$ have analogous proofs.

- If $t \equiv x$, then $E(t)_1 = (x, e)_1 = x$, so that we must prove
  \[ E(x_0 \lhd x_1)_1 = x_0 \]
  and
  \[ E(x_0 \lhd x_1)_2 \sim x_1 \cdot e[x_1^{-1}x_0x_1/x] \equiv x_1 \cdot e. \]
  Since $E(x_0) = (x_0, e)$ and $E(x_1) = (x_1, e)$, we have by definition of $E$
  \[ E(x_0 \lhd x_1) = (x_0, e \cdot e^{-1} \cdot x_1 \cdot e), \]
  which clearly yields the desired result.

- If $t \equiv y$ for some $y \in X$ with $y \neq x$, then $E(t)_1 = (y, e)_1 = y$ and hence we must show
  \[ E(y)_1 = y \]
  and
  \[ E(y)_2 \sim e[x_1^{-1}x_0x_1/x]) \equiv e, \]
  which clearly follows by definition of $E(y)$.

- For the induction step, suppose that $t \equiv t_1 \lhd t_2$ for some $t_1, t_2 \in \text{Term}^c(\Sigma(X))$ for which the result holds.
– Suppose first that $E(t)_1 = x$. Then by Lemma 4.2 it follows that
\[
x = \text{Left}(t) = \text{Left}(t_1) = E(t_1)_1.
\]
Then by the induction hypothesis for $t_1$ we have
\[
E(t_1[x_0 < x_1/x])_1 = x_0
\]
and
\[
E(t_1[x_0 < x_1/x])_2 \sim x_1 \cdot E(t_1)_2[x_1^{-1}x_0x_1/x],
\]
so that by definition of $E$ we have
\[
E(t[x_0 < x_1/x])_1 = E(t_1[x_0 < x_1/x])_1 = x_0,
\]
as required. To compute $E(t[x_0 < x_1/x])_2$, suppose in addition that $E(t_2)_1 = x$, so that the induction hypothesis for $t_2$ gives
\[
E(t_2[x_0 < x_1/x])_1 = x_0
\]
and
\[
E(t_2[x_0 < x_1/x])_2 \sim x_1 \cdot E(t_2)_2[x_1^{-1}x_0x_1/x].
\]
Using the definition of $E$, the induction hypotheses, and the current assumption that $E(t_2)_1 = x$, we then have:
\[
E(t[x_0 < x_1/x])_2
= E(t_1[x_0 < x_1/x] < t_2[x_0 < x_1/x])_2
\sim x_1 \cdot E(t_1)_2[x_1^{-1}x_0x_1/x] \cdot \left( E(t_2)_2[x_1^{-1}x_0x_1/x] \right)^{-1} \cdot x_1^{-1} \cdot x_0 \cdot x_1 \cdot E(t_2)_2[x_1^{-1}x_0x_1/x]
\sim x_1 \cdot E(t_1)_2[x_1^{-1}x_0x_1/x] \cdot E(t_2)_2^{-1} \cdot E(t_1)_2[x_1^{-1}x_0x_1/x] \cdot x_1^{-1} \cdot x_0 \cdot x_1 \cdot E(t_2)_2[x_1^{-1}x_0x_1/x]
\equiv x_1 \cdot (E(t_1)_2 \cdot E(t_2)_2^{-1} \cdot x \cdot E(t_2)_2) [x_1^{-1}x_0x_1/x]
= x_1 \cdot (E(t_1)_2 \cdot E(t_2)_2^{-1} \cdot E(t_2)_2) [x_1^{-1}x_0x_1/x]
= x_1 \cdot E(t_1 < t_2)_2[x_1^{-1}x_0x_1/x]
= x_1 \cdot E(t)_2[x_1^{-1}x_0x_1/x],
\]
as desired.
Now suppose that $E(t_2)_1 = y$ for some $y \in X$ with $y \neq x$. Then by the induction hypothesis for $t_2$, it follows that
\[
E(t_2[x_0 < x_1/x])_1 = y
\]
and
\[
E(t_2[x_0 < x_1/x])_2 \sim E(t_2)_2[x_1^{-1}x_0x_1/x].
\]
Then we calculate as follows:

\[ E(t[x_0 < x_1/\mathbf{x}])_2 \]
\[ = E(t_1[x_0 < x_1/\mathbf{x}] < t_2[x_0 < x_1/\mathbf{x}])_2 \]
\[ \sim x_1 \cdot E(t_1)_2[x_1^{-1}x_0x_1/\mathbf{x}] \cdot (E(t_2)_2[x_1^{-1}x_0x_1/\mathbf{x}])^{-1} \cdot y \cdot E(t_2)_2[x_1^{-1}x_0x_1/\mathbf{x}] \]
\[ = x_1 \cdot E(t_1)_2 \cdot E(t_2)_2^{-1} \cdot y \cdot E(t_2)_2 [x_1^{-1}x_0x_1/\mathbf{x}] \]
\[ = x_1 \cdot E(t_1 < t_2)_2 [x_1^{-1}x_0x_1/\mathbf{x}] \]
\[ = x_1 \cdot E(t)_2 [x_1^{-1}x_0x_1/\mathbf{x}] \]

as desired. This completes the proof for the case where \( E(t)_1 = x \).

- Now suppose that \( E(t)_1 = y \) for some \( y \neq x \). As before, this implies that \( E(t)_1 = y \) as well. Then by the induction hypothesis for \( t_1 \), we have

\[ E(t_1[x_0 < x_1/\mathbf{x}])_1 = y \]

and

\[ E(t_1[x_0 < x_1/\mathbf{x}])_2 \sim E(t_1)_2[x_1^{-1}x_0x_1/\mathbf{x}] \]

which implies as before that

\[ E(t[x_0 < x_1/\mathbf{x}])_1 = y. \]

To compute \( E(t[x_0 < x_1/\mathbf{x}])_2 \), suppose first that \( E(t)_2 = x \). Then by the induction hypothesis for \( t_2 \), we have

\[ E(t_2[x_0 < x_1/\mathbf{x}])_1 = x_0 \]

and

\[ E(t_2[x_0 < x_1/\mathbf{x}])_2 \sim x_1 \cdot E(t)_2[x_1^{-1}x_0x_1/\mathbf{x}] \]

Then we calculate as follows:

\[ E(t[x_0 < x_1/\mathbf{x}])_2 \]
\[ = E(t_1[x_0 < x_1/\mathbf{x}] < t_2[x_0 < x_1/\mathbf{x}])_2 \]
\[ \sim E(t_1)_2[x_1^{-1}x_0x_1/\mathbf{x}] \cdot (E(t_2)_2[x_1^{-1}x_0x_1/\mathbf{x}])^{-1} \cdot x_1^{-1} \cdot x_0 \cdot x_1 \cdot E(t_2)_2[x_1^{-1}x_0x_1/\mathbf{x}] \]
\[ = E(t_1)_2[x_1^{-1}x_0x_1/\mathbf{x}] \cdot E(t_2)_2^{-1} \cdot x_1^{-1} \cdot x_0 \cdot E(t_2)_2[x_1^{-1}x_0x_1/\mathbf{x}] \]
\[ = E(t_1)_2 \cdot E(t_2)_2^{-1} \cdot x \cdot E(t_2)_2 [x_1^{-1}x_0x_1/\mathbf{x}] \]
\[ = E(t_1 < t_2)_2 [x_1^{-1}x_0x_1/\mathbf{x}] \]
\[ = E(t)_2 [x_1^{-1}x_0x_1/\mathbf{x}] \]

as desired.
Finally, suppose that \( E(t_2)_1 = z \) for some \( z \in X \) with \( z \neq x \). Then by the induction hypothesis for \( t_2 \), we have
\[
E(t_2[x_0 \triangleleft x_1/x])_1 = z
\]
and
\[
E(t_2[x_0 \triangleleft x_1/x])_2 \sim E(t_2)[x_1^{-1}x_0x_1/x]).
\]
Then we calculate as follows:
\[
\begin{align*}
E(t[x_0 \triangleleft x_1/x])_2 & = E(t_1[x_0 \triangleleft x_1/x] \triangleleft t_2[x_0 \triangleleft x_1/x])_2 \\
& \sim E(t_1)[x_1^{-1}x_0x_1/x] \cdot (E(t_2)[x_1^{-1}x_0x_1/x])^{-1} \cdot z \cdot E(t_2)[x_1^{-1}x_0x_1/x] \\
& \equiv (E(t_1) \cdot E(t_2^{-1}) \cdot z \cdot E(t_2))[x_1^{-1}x_0x_1/x] \\
& = E(t_1 \triangleleft t_2)[x_1^{-1}x_0x_1/x] \\
& = E(t_2)[x_1^{-1}x_0x_1/x],
\end{align*}
\]
as desired.

This completes the proof for the case \( t = t_1 \triangleleft t_2 \), which completes the induction and hence the proof.

\[\square\]

We now require the following lemma, definition, and lemma.

**Lemma 4.4.** Let \( t \in \text{Term}^e(\Sigma(X)) \) for a (finite) set \( X \), and assume that \( t \) has the form
\[
t = z_0 \triangleleft^e \cdots \triangleleft^e z_m,
\]
with \( \epsilon_j = \pm 1 \) and \( z_k \in X \) for all \( 1 \leq j \leq m \) and \( 0 \leq k \leq m \). Then
\[
E(t)_1 = z_0
\]
and
\[
E(t)_2 \sim z_1^e \cdots z_m^e.
\]

**Proof.** We prove this by induction on \( m \geq 0 \). If \( m = 0 \), then \( t = z_0 \) for some \( z_0 \in X \), and we must show that \( E(z_0)_1 = z_0 \) and \( E(z_0)_2 \sim \epsilon \), which is true by definition of \( E \).

Now suppose that the result holds for some \( m \geq 0 \), and let
\[
t = z_0 \triangleleft^e \cdots \triangleleft^e z_m \triangleleft^e z_{m+1},
\]
and let
\[
s := z_0 \triangleleft^e \cdots \triangleleft^e z_m.
\]

Then
\[
E(t)_1 = E(s)_1 = z_0
\]
and
\[
E(t)_2 \sim E(s)_2 \cdot z_{m+1} \sim z_1^e \cdots z_m^e \cdot z_{m+1}.
\]
Then by the induction hypothesis for \( s \), we have
\[
E(s)_1 = z_0
\]
and
\[
E(s)_2 \sim z_1^1 \cdot \ldots \cdot z_m^m.
\]
By definition of \( E \) and the induction hypothesis (and the fact that bracketing associates to the left), we then have
\[
E(t)_1 = E(z_0 \triangleleft^{t_1} \ldots \triangleleft^{t_m} z_m \triangleleft^{t_{m+1}} z_{m+1})_1 = E(s)_1 = z_0,
\]
as well as (recalling that \( E(z_{m+1}) = (z_{m+1}, e) \))
\[
E(t)_2 = E(z_0 \triangleleft^{t_1} \ldots \triangleleft^{t_m} z_m \triangleleft^{t_{m+1}} z_{m+1})_2
= E(s \triangleleft^{t_{m+1}} z_{m+1})_2
\sim z_1^1 \cdot \ldots \cdot z_m^m \cdot e^{-1} \cdot z_{m+1}^{m+1} \cdot e
\sim z_1^1 \cdot \ldots \cdot z_m^m \cdot z_{m+1}^{m+1},
\]
as desired. This completes the induction and hence the proof.

\[\square\]

**Definition 4.5.** Let \( t \in \text{Term}^c(\Sigma(X)) \) for a (finite) set \( X \). Then we define \( \text{W}(t) \in \text{Term}^c(\Sigma_{\text{Grp}}(X)) \) to be
\[
\text{W}(t) := E(t)^{-1} \cdot E(t)_1 \cdot E(t)_2.
\]

\[\square\]

**Lemma 4.6.** Let \( t, t' \in \text{Term}^c(\Sigma(X)) \) for a (finite) set \( X \), where \( x \in X \) is a distinguished element and \( E(t')_1 = x \) and \( t \) has the form
\[
t \equiv x \triangleleft^{t_1} z_1 \triangleleft^{t_2} \ldots \triangleleft^{t_m} z_m,
\]
with \( \epsilon_j = \pm 1 \) and \( z_j \in X \) for all \( 1 \leq j \leq m \). Then
\[
E(t[t'/x])_1 = x
\]
and
\[
E(t[t'/x])_2 \sim E(t')_2 \cdot E(t)_2 \cdot \text{W}(t')/x.
\]

**Proof.** We prove this by induction on the length of \( t \).
- For the base case, let \( t \equiv x \), so that \( E(t)_2 = E(x)_2 = \pi_2(x, e) = e \). Then
  \[
  E(t[t'/x])_1 = E(x[t'/x])_1 = E(t')_1 = x
  \]
by hypothesis on \( t' \), and we have
  \[
  E(t[t'/x])_2 = E(x[t'/x])_2
  = E(t')_2
  \sim E(t')_2 \cdot e
  \equiv E(t')_2 \cdot e \cdot \text{W}(t')/x
  = E(t')_2 \cdot E(t)_2 \cdot \text{W}(t')/x,
  \]
as required.

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• Now suppose that the result holds for all terms $t$ of the described form of some length $n \geq 1$, and consider

$$t \equiv \mathbf{x} \triangleq \mathbf{t}_1 \triangleq \mathbf{t}_2 \ldots \triangleq \mathbf{t}_{m+1},$$

with $m \geq 0$ and $\epsilon_j = \pm 1$ and $z_j \in X$ for all $1 \leq j \leq m + 1$. If we set

$$s := \mathbf{x} \triangleq \mathbf{t}_1 \triangleq \mathbf{t}_2 \ldots \triangleq \mathbf{t}_{m},$$

then by the induction hypothesis we have

$$E(s[t'/x])_1 = x$$

and

$$E(s[t'/x])_2 \sim E(t')_2 \cdot E(s)_2 \cdot W(t')/x.$$ 

So $t \equiv s \triangleq \mathbf{t}_{m+1}$, and hence we have

$$E(t[t'/x])_1 = E(s[t'/x]) \triangleq \mathbf{t}_{m+1}[t'/x])_1 = E(s[t'/x])_1 = x$$

by definition of $E$ and the induction hypothesis, as well as

$$E(t[t'/x])_2$$

$$= E(s[t'/x]) \triangleq \mathbf{t}_{m+1}[t'/x])_2$$

$$\sim E(t')_2 \cdot E(s)_2 \cdot W(t')/x \cdot E(z_{m+1}[t'/x])_2^{-1} \cdot E(z_{m+1}[t'/x])_1 \cdot E(z_{m+1}[t'/x])_2$$

$$\sim E(t')_2 \cdot (x \cdot z_1 \ldots z_m) \cdot W(t')/x \cdot E(z_{m+1}[t'/x])_2^{-1} \cdot E(z_{m+1}[t'/x])_1 \cdot E(z_{m+1}[t'/x])_2$$

with the last congruence justified by Lemma 4.4.

Suppose first that $z_{m+1} \neq x$. Then we have $E(z_{m+1}[t'/x]) = E(z_{m+1}) = (z_{m+1}, e)$, so it follows that

$$E(t[t'/x])_2$$

$$\sim E(t')_2 \cdot (x \cdot z_1 \ldots z_m) \cdot W(t')/x \cdot E(z_{m+1}[t'/x])_2^{-1} \cdot E(z_{m+1}[t'/x])_1 \cdot E(z_{m+1}[t'/x])_2$$

$$= E(t')_2 \cdot (x \cdot z_1 \ldots z_m) \cdot W(t')/x \cdot e^{-1} \cdot z_{m+1} \cdot e$$

$$\sim E(t')_2 \cdot (x \cdot z_1 \ldots z_m) \cdot W(t')/x \cdot z_{m+1}$$

$$= E(t')_2 \cdot (x \cdot z_1 \ldots z_{m+1}) \cdot W(t')/x$$

$$\sim E(t')_2 \cdot E(t)_2 \cdot W(t')/x$$

as desired, with the last equality justified by Lemma 4.4 and the assumption that $z_{m+1} \neq x$.

Now suppose that $z_{m+1} = x$. Then we have $E(z_{m+1}[t'/x]) = E(x[t'/x]) =$
Proof. We prove this by induction on the length of $s$

Lemma 4.7. We finally require the following two technical lemmas about reduced words in free groups.

**Lemma 4.7.** Let $s \in \operatorname{Term}^c(\Sigma_{\operatorname{Grp}}(x_1, \ldots, y_n))$ be reduced. Then in the free group on the set $\{x_0, x_1, y_1, \ldots, y_n\}$, the following claims hold:

(i) If $s \equiv e$, then the unique reduced word obtained from $x_1 \cdot s[x_1^{-1} x_0 x_1 / x]$ ends in $x_1$.

(ii) If $s$ ends in $x$, then the unique reduced word obtained from $x_1 \cdot s[x_1^{-1} x_0 x_1 / x]$ ends in $x_0 x_1$.

(iii) If $s$ ends in $x^{-1}$, then the unique reduced word obtained from $x_1 \cdot s[x_1^{-1} x_0 x_1 / x]$ ends in $x_0^{-1} x_1$.

(iv) If $s$ ends in $y_i^\epsilon$ for some $1 \leq i \leq n$ and $\epsilon = \pm 1$, then the unique reduced word obtained from $x_1 \cdot s[x_1^{-1} x_0 x_1 / x]$ ends in $x_1 \cdot t$ for some reduced word $t \in \operatorname{Term}^c(\Sigma_{\operatorname{Grp}}(y_1, \ldots, y_n))$ that ends in $y_i^\epsilon$.

**Proof.** We prove this by induction on the length of $s$.

- If $s \equiv e$, then (i) clearly holds.

- If $s \equiv x$, then we have

$$x_1 \cdot s[x_1^{-1} x_0 x_1 / x] \equiv x_1 \cdot x_1^{-1} x_0 x_1 \sim x_0 x_1,$$

as desired for (ii).
• If \( s \equiv x^{-1} \), then we have
\[
x_1 \cdot s[x_1^{-1}x_0x_1/x] \equiv x_1 \cdot (x_1^{-1}x_0x_1)^{-1} \sim x_1x_1^{-1}x_0^{-1}x_1 \sim x_0^{-1}x_1,
\]
as required for (iii).

• If \( s \equiv y_i^\epsilon \) for some \( 1 \leq i \leq n \) and \( \epsilon = \pm 1 \), then we have
\[
x_1 \cdot s[x_1^{-1}x_0x_1/x] \equiv x_1 \cdot y_i^\epsilon,
\]
as desired for (iv).

• Now let \( s \in \text{Term}^\sim(\Sigma_{\text{Grp}}(x, y_1, \ldots, y_n)) \) be reduced of length \( n \) for some \( n \geq 1 \), and assume that the result holds for \( s \). We show that the result holds for \( s \cdot x^\pm 1 \) and \( s \cdot y_i^\pm 1 \) (for any \( 1 \leq i \leq n \)), assuming that these words are reduced.

  - First we consider \( s \cdot x \). If this word is reduced, then \( s \) does not end with \( x^{-1} \). So then \( s \) ends with either \( x \) or \( y_i^\epsilon \) for some \( 1 \leq i \leq n \) and \( \epsilon = \pm 1 \).

    Suppose first that \( s \) ends with \( x \). Then by the induction hypothesis, there is some reduced (possibly empty) word \( t \in \text{Term}^\sim(\Sigma_{\text{Grp}}(x, y_1, \ldots, y_n)) \) such that
\[
x_1 \cdot s[x_1^{-1}x_0x_1/x] \sim t \cdot x_0x_1.
\]
So then we have
\[
x_1 \cdot (s \cdot x)[x_1^{-1}x_0x_1/x] \sim x_1 \cdot s[x_1^{-1}x_0x_1/x] \cdot x_1^{-1}x_0x_1 \\
\sim t \cdot x_0x_1 \cdot x_1^{-1}x_0x_1 \\
\sim t \cdot x_0x_0x_1,
\]
so that the reduced word obtained from \( x_1 \cdot (s \cdot x)[(x_1^{-1}x_0x_1)/x] \) ends in \( x_0x_1 \), as desired for (ii).

    Now suppose that \( s \) ends with \( y_i^\epsilon \) for some \( 1 \leq i \leq n \) and \( \epsilon = \pm 1 \). Then by the induction hypothesis, there is some reduced (possibly empty) word \( t \in \text{Term}^\sim(\Sigma_{\text{Grp}}(x, y_1, \ldots, y_n)) \) with
\[
x_1 \cdot s[x_1^{-1}x_0x_1/x] \sim t \cdot x_1 \cdot t'
\]
for some reduced \( t' \in \text{Term}^\sim(\Sigma_{\text{Grp}}(y_1, \ldots, y_n)) \) that ends in \( y_i^\epsilon \). So then we have
\[
x_1 \cdot (s \cdot x)[x_1^{-1}x_0x_1/x] \sim x_1 \cdot s[x_1^{-1}x_0x_1/x] \cdot x_1^{-1}x_0x_1 \\
\sim t \cdot x_1 \cdot t' \cdot x_1^{-1}x_0x_1,
\]
so that the reduced word obtained from \( x_1 \cdot (s \cdot x)[(x_1^{-1}x_0x_1)/x] \) again ends in \( x_0x_1 \), as desired for (ii). This completes the case for \( s \cdot x \).
Now we consider $s \cdot x^{-1}$. If this word is reduced, then $s$ does not end with $x$. So then $s$ ends with either $x^{-1}$ or $y'_i$ for some $1 \leq i \leq n$ and $\epsilon = \pm 1$.

Suppose first that $s$ ends with $x^{-1}$. Then by the induction hypothesis, there is some reduced (possibly empty) word $t \in \text{Term}^c(\Sigma_{\text{Grp}}(x, y_1, \ldots, y_n))$ with

$$x_1 \cdot s[x_1^{-1} x_0 x_1/x] \sim t \cdot x_0^{-1} x_1.$$ 

So then we have

$$x_1 \cdot (s \cdot x^{-1})[x_1^{-1} x_0 x_1/x] \sim x_1 \cdot s[x_1^{-1} x_0 x_1/x] \cdot x_1^{-1} x_0^{-1} x_1$$

$$\sim t \cdot x_0^{-1} x_1 \cdot x_1 x_1^{-1} x_0^{-1} x_1$$

$$\sim t \cdot x_0^{-1} x_0^{-1} x_1,$$

so that the reduced word obtained from $x_1 \cdot (s \cdot x^{-1})[x_1^{-1} x_0 x_1/x]$ ends in $x_0^{-1} x_1$, as desired for (iii).

Now suppose that $s$ ends with $y_i^\epsilon$ for some $1 \leq i \leq n$ and $\epsilon = \pm 1$. Then by the induction hypothesis, there is some reduced (possibly empty) word $t \in \text{Term}^c(\Sigma_{\text{Grp}}(x, y_1, \ldots, y_n))$ with

$$x_1 \cdot s[x_1^{-1} x_0 x_1/x] \sim t \cdot x_1 \cdot t'$$

for some reduced $t' \in \text{Term}^c(\Sigma_{\text{Grp}}(y_1, \ldots, y_n))$ that ends with $y_i^\epsilon$. Then we have

$$x_1 \cdot (s \cdot x^{-1})[x_1^{-1} x_0 x_1/x] \sim x_1 \cdot s[x_1^{-1} x_0 x_1/x] \cdot x_1^{-1} x_0^{-1} x_1$$

$$\sim t \cdot x_1 \cdot t' \cdot x_1^{-1} x_0^{-1} x_1,$$

so that the reduced word obtained from $x_1 \cdot (s \cdot x^{-1})[x_1^{-1} x_0 x_1/x]$ again ends in $x_0^{-1} x_1$, as desired for (iii). This completes the case for $s \cdot x^{-1}$.

Lastly we consider $s \cdot y_i^\epsilon$ for any $1 \leq i \leq n$ and $\epsilon = \pm 1$. If this word is reduced, then $s$ does not end with $y_i^\pm$ (equating $-(1)$ with $1$). So then $s$ ends with $x$, with $x^{-1}$, with $y_i^\epsilon$, or with $y_j^\delta$ for any $1 \leq j \neq i \leq n$ and $\delta = \pm 1$.

If $s$ ends with $x$, then by the induction hypothesis, there is some reduced (possibly empty) word $t \in \text{Term}^c(\Sigma_{\text{Grp}}(x, y_1, \ldots, y_n))$ with

$$x_1 \cdot s[x_1^{-1} x_0 x_1/x] \sim t \cdot x_0 x_1.$$ 

So then we have

$$x_1 \cdot (s \cdot y_i^\epsilon)[x_1^{-1} x_0 x_1/x] \sim x_1 \cdot s[x_1^{-1} x_0 x_1/x] \cdot y_i^\epsilon$$

$$\sim t \cdot x_0 x_1 \cdot y_i^\epsilon,$$

so that the reduced word obtained from $x_1 \cdot (s \cdot y_i^\epsilon)[x_1^{-1} x_0 x_1/x]$ ends in $x_1 y_i^\epsilon$, as desired for (iv). Exactly similar reasoning works for the case where $s$ ends with $x^{-1}$.

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Now suppose that \( s \) ends with \( y_i^\epsilon \). Then by the induction hypothesis, there is some reduced (possibly empty) word \( t \in \text{Term}^c(\Sigma_{\text{Grp}}(x,y_1,\ldots,y_n)) \) with
\[
x_1 \cdot s[x_1^{-1}x_0x_1/x] \sim t \cdot x_1 \cdot t'
\]
for some reduced \( t' \in \text{Term}^c(\Sigma_{\text{Grp}}(y_1,\ldots,y_n)) \) that ends with \( y_i^\epsilon \). Then we have
\[
x_1 \cdot (s \cdot y_i^\epsilon)[x_1^{-1}x_0x_1/x] \sim x_1 \cdot s[x_1^{-1}x_0x_1/x] \cdot y_i^\epsilon \sim t \cdot x_1 \cdot t' \cdot y_i^\epsilon,
\]
so that the reduced word obtained from \( x_1 \cdot (s \cdot y_i^\epsilon)[x_1^{-1}x_0x_1/x] \) has the form required for (iv).

Finally, suppose that \( s \) ends with \( y_j^\delta \) for some \( 1 \leq j \neq i \leq n \) and \( \delta = \pm 1 \). Then by the induction hypothesis, there is some reduced (possibly empty) word \( t \in \text{Term}^c(\Sigma_{\text{Grp}}(x,y_1,\ldots,y_n)) \) with
\[
x_1 \cdot s[x_1^{-1}x_0x_1/x] \sim t \cdot x_1 \cdot t'
\]
for some reduced \( t' \in \text{Term}^c(\Sigma_{\text{Grp}}(y_1,\ldots,y_n)) \) that ends with \( y_j^\delta \). So we have
\[
x_1 \cdot (s \cdot y_j^\delta)[x_1^{-1}x_0x_1/x] \sim x_1 \cdot s[x_1^{-1}x_0x_1/x] \cdot y_j^\delta \sim t \cdot x_1 \cdot t' \cdot y_j^\delta,
\]
so that the reduced word obtained from \( x_1 \cdot (s \cdot y_j^\delta)[x_1^{-1}x_0x_1/x] \) again has the form required for (iv), because \( j \neq i \) and hence \( t \cdot x_1 \cdot t' \cdot y_j^\delta \) is reduced and ends with \( x_1 \cdot t'' \) for some \( t'' \in \text{Term}^c(\Sigma_{\text{Grp}}(y_1,\ldots,y_n)) \) that ends with \( y_j^\delta \).

This completes the induction and hence the proof of the lemma.

\[\square\]

**Lemma 4.8.** Let \( s \in \text{Term}^c(\Sigma_{\text{Grp}}(x,y_1,\ldots,y_n)) \) be reduced, and assume that the congruence
\[
x_1 \cdot s[x_1^{-1}x_0x_1/x] \sim s[x_0/x] \cdot s[x_1/x]^{-1} \cdot x_1 \cdot s[x_1/x]
\]
holds in the free group on the set \( \{x_0,x_1,y_1,\ldots,y_n\} \). Then all occurrences of \( x \) in \( s \) must precede all occurrences of \( y_1,\ldots,y_n \) in \( s \).

**Proof.** Suppose towards a contradiction that \( s \) satisfies the assumptions but contains an occurrence of \( y_i \) (for some \( 1 \leq i \leq n \)) to the left of some occurrence of \( x \). Then there are \( \epsilon = \pm 1 \) and reduced (possibly empty) words \( s_1,s_2,s_3 \in \text{Term}^c(\Sigma_{\text{Grp}}(x,y_1,\ldots,y_n)) \) with
\[
s \equiv s_1y_i^\epsilon s_2xs_3
\]
Suppose first that \( s \equiv s_1 y_i' s_2 x^{-1} s_3 \). Since \( s \) is reduced, it follows that \( s_1 \) does not end in \( y_i'^{-} \), that \( s_2 \) does not start with \( y_i'^{+} \) or end with \( x^{-1} \), and that \( s_3 \) does not start with \( x^{-1} \). The assumption on \( s \) then implies that
\[
x_1 \cdot s_1 \left[ x_1^{-1} x_0 x_1 / x \right] \cdot y_i' \cdot s_2 \left[ x_1^{-1} x_0 x_1 / x \right] \cdot x_1^{-1} x_0 x_1 \cdot s_3 \left[ x_1^{-1} x_0 x_1 / x \right]
\equiv
s_1 \left[ x_0 / x \right] y_i' s_2 \left[ x_0 / x \right] x_0 s_3 \left[ x_0 / x \right] \cdot s_3^{-1} \left[ x_1 / x \right] x_1^{-1} s_2^{-1} \left[ x_1 / x \right] y_i'^{-} s_1^{-1} \left[ x_1 / x \right] \\
\cdot x_1 \cdot s_1 \left[ x_1 / x \right] y_i' s_2 \left[ x_1 / x \right] x_1 s_3 \left[ x_1 / x \right].
\]

If \( s_1 \) is the empty word, then the reduction of the top word will begin with \( x_1 y_i' \), while the reduction of the bottom word will begin with just \( y_i' \), which is impossible, since the reductions of these words are congruent in the free group on \( \{x_0, x_1, y_1, \ldots, y_n\} \) and hence must be identical.

If \( s_1 \) is non-empty and ends with \( x \), then since \( s_1 \) is reduced, it will follow from Lemma 4.7 that the reduction of the top word will begin with \( t \cdot x_0 x_1 y_i' \) for some reduced \( t \in \text{Term}^\sim(\Sigma_{\text{Grp}}(x_0, x_1, y_1, \ldots, y_n)) \). In particular, the reduced word obtained from the top word will have an occurrence of \( x_1 \) before the first occurrence of \( y_i' \). However, the reduced word obtained from the bottom word will not have any occurrences of \( x_1 \) before the first occurrence of \( y_i' \), which is impossible for the reason given in the last paragraph. If \( s_1 \) is non-empty and ends with \( x^{-1} \), or with \( y_i' \), or with \( y_j' \) for some \( 1 \leq j \neq i \leq n \) and \( \delta = \pm 1 \), then exactly similar reasoning (with the use of Lemma 4.7) leads to a contradiction.

This proves that we cannot have \( s \equiv s_1 y_i' s_2 x^{-1} s_3 \), and parallel reasoning also shows that we cannot have \( s \equiv s_1 y_i' s_2 x^{-1} s_3 \) either, which contradicts the original assumption. So it follows that all occurrences of \( x \) in \( s \) must precede all occurrences of \( y_1, \ldots, y_n \) in \( s \), as desired.

We can finally give a characterization of the logical isotropy groups of the free, finitely generated racks. First, the following result was proven in [4, Section 4.1]:

**Theorem 4.9 (Dehornoy [4]).** For any (finite) set \( X \) and \( s, t \in \text{Term}^\sim(\Sigma(X)) \), let \( E(s) = (x, \omega) \) and \( E(t) = (x', \omega') \) for some \( x, x' \in X \) and \( \omega, \omega' \in \text{Term}^\sim(\Sigma_{\text{Grp}}(X)) \). Then
\[
\mathcal{T}_{\text{Rack}}(X) + s = t
\]
iff
\[
x = x' \quad \text{and} \quad \omega \sim \omega.
\]

**Theorem 4.10 (Isotropy Group of a Free Rack).** Let \( R_n \) be the free rack on \( n \) generators \( y_1, \ldots, y_n \). For any \( t \in \text{Term}^\sim(\Sigma_{\text{Grp}}(x, y_1, \ldots, y_n)) \), we have
\[
[t] \in \mathcal{G}_{\mathcal{T}_{\text{Rack}}(R_n)}
\]

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iff there are \( p, m \geq 0 \) and \( 1 \leq i_1, \ldots, i_m \leq n \) such that

\[
[t] = [x^{\delta_1} \ldots x^{\delta_p} x^{\epsilon_1} y_{i_1}^{\epsilon_2} \ldots y_{i_m}^{\epsilon_m}],
\]

with \( \delta_j = \pm 1 \) for all \( 1 \leq j \leq p \) and \( \epsilon_k = \pm 1 \) for all \( 1 \leq k \leq m \), and the corresponding term \( x^{\delta_1} \ldots x^{\delta_p} y_{i_1}^{\epsilon_2} \ldots y_{i_m}^{\epsilon_m} \in \text{Term}^r(\Sigma_{\text{Grp}}(x, y_1, \ldots, y_n)) \) is reduced.

**Proof.** First we prove that if \( t \) has the stated form, then \( [t] \in G_{T_{\text{Rack}}} (R_n) \). So let \( t \) have the form described in the statement of the theorem. We must show that \( [t] \) is invertible and commutes generically with the rack operations.

To show that \( [t] \) is invertible, consider the term

\[
t^{-1} := x^{-\delta_p} \ldots x^{-\delta_1} x^{-\epsilon_m} y_{i_m}^{-\epsilon_{m-1}} \ldots x^{-\epsilon_1} y_{i_1}.
\]

To show that \( [t^{-1}/x] \sim_{\text{Rack}} x \) and \( t^{-1} [t/x] \sim_{\text{Rack}} x \) in the free rack on \( \{x, y_1, \ldots, y_n\} \), it suffices by Theorem 4.4 to show that

\[
E(t [t^{-1}/x])_1 = E(x)_1 = E(t^{-1} [t/x])_1
\]

and

\[
E(t [t^{-1}/x])_2 \sim E(x)_2 \sim E(t^{-1} [t/x])_2.
\]

We have \( E(x) = (x, e) \), and by Lemma 4.4 we have

\[
E(t)_1 = x = E(t^{-1})_1
\]

as well as

\[
E(t)_2 \sim x^{\delta_1} \ldots x^{\delta_p} y_{i_1}^{\epsilon_2} \ldots y_{i_m}^{\epsilon_m}
\]

and

\[
E(t^{-1})_2 \sim x^{-\delta_p} \ldots x^{-\delta_1} y_{i_m}^{-\epsilon_{m-1}} \ldots y_{i_1}^{-\epsilon_1}.
\]

Note also that

\[
W(t^{-1}) := E(t^{-1})_2^{-1} \cdot E(t^{-1})_1 \cdot E(t^{-1})_2
\]

\[
\sim y_{i_1}^{\epsilon_2} \ldots y_{i_m}^{\epsilon_m} x^{\delta_1} \ldots x^{\delta_p} x^{\delta_1} \ldots x^{-\delta_1} y_{i_m}^{-\epsilon_{m-1}} \ldots y_{i_1}^{-\epsilon_1}
\]

\[
\sim y_{i_1}^{\epsilon_2} \ldots y_{i_m}^{\epsilon_m} x^{\delta_1} \ldots x^{\delta_p} y_{i_m} \ldots y_{i_1}^{\epsilon_1}
\]

\[
\sim y_{i_1}^{\epsilon_2} \ldots y_{i_m}^{\epsilon_m} x y_{i_m}^{-\epsilon_{m-1}} \ldots y_{i_1}^{-\epsilon_1}.
\]

Then by Lemma 4.6 we have

\[
E(t [t^{-1}/x])_1 = x
\]
and
\[ E(t[t^{-1}/x])_2 \]
\[ \sim E(t^{-1})_2 \cdot E(t)_2 [W(t^{-1})/x] \]
\[ \sim x^{-\delta_p} \cdots x^{-\delta_1} y^{-\epsilon_m} \cdots y^{-\epsilon_1} \cdot \left( x^{\delta_1} + \cdots + \delta_p y^{\epsilon_1} \right) [W(t^{-1})/x] \]
\[ \sim x^{-\delta_p} \cdots x^{-\delta_1} y^{-\epsilon_m} \cdots y^{-\epsilon_1} \cdot \left( x^{\delta_1} + \cdots + \delta_p y^{\epsilon_1} y^{\epsilon_m} \right) [W(t^{-1})/x] \]
\[ \sim x^{-\delta_p} \cdots x^{-\delta_1} y^{-\epsilon_m} \cdots y^{-\epsilon_1} \cdot \left( y^{\epsilon_1} y^{\epsilon_m} y^{-\epsilon_m} y^{-\epsilon_1} \right) [W(t^{-1})/x] \]
\[ \sim x^{-\delta_p} \cdots x^{-\delta_1} y^{-\epsilon_m} \cdots y^{-\epsilon_1} \cdot \left( y^{\epsilon_1} y^{\epsilon_m} y^{-\epsilon_m} y^{-\epsilon_1} \right) [W(t^{-1})/x] \]
\[ \sim x^{-\delta_p} \cdots x^{-\delta_1} y^{-\epsilon_m} \cdots y^{-\epsilon_1} \cdot \left( y^{\epsilon_1} y^{\epsilon_m} y^{-\epsilon_m} y^{-\epsilon_1} \right) [W(t^{-1})/x] \]
\[ \sim e, \]
as desired, where the sixth congruence holds because in any group \( G \) we have \((ghg^{-1})^n = gh^n g^{-1}\) for any \( g, h \in G \) and \( n \in \mathbb{Z} \). The proof that \( E(t^{-1}[t/x])_2 \sim e \) is similar. It follows that \([t] \) is invertible, as desired.

Now we show that \([t] \) commutes generically with the rack operations. So we must show that the following congruences hold in the free rack on \( \{x_0, x_1, y_1, \ldots, y_m \} \):

\[ t[x_0 < x_1/x] \sim_{\text{Rack}} t[x_0/x] < t[x_1/x] \]

and

\[ t[x_0 < t^{-1} x_1/x] \sim_{\text{Rack}} t[x_0/x] < t^{-1} t[x_1/x]. \]

Since the proofs are similar, we only show the first. By Theorem \ref{thm:commute}, it suffices to show that

\[ E(t[x_0 < x_1/x])_1 = E(t[x_0/x] < t[x_1/x])_1 \]

and

\[ E(t[x_0 < x_1/x])_2 \sim E(t[x_0/x] < t[x_1/x])_2. \]

By Lemma \ref{lem:invertible}, since \( E(t) = x \), we have

\[ E(t[x_0 < x_1/x])_1 = x_0 \]

and

\[ E(t[x_0 < x_1/x])_2 \sim x_1 \cdot E(t)_2 [x_0 x_1/x] \]

\[ \sim x_1 \cdot \left( x^{\delta_1} + \cdots + \delta_p y^{\epsilon_1} \right) [x_0 x_1/x] \]

\[ \sim x_1 \cdot x_0 \left( x^{\delta_1} + \cdots + \delta_p y^{\epsilon_1} \right) y^{\epsilon_m} \]

\[ \sim x_0 \left( x^{\delta_1} + \cdots + \delta_p y^{\epsilon_1} \right) y^{\epsilon_m}, \]

where the third congruence again holds because of the previously mentioned group-theoretic fact. Then, given that

\[ E(t[x_0/x])_1 = x_0 \]
We prove this by induction on \( \text{Proof.} \)

\[
E(t[x_0/x])_2 \sim x_0^\delta_1 \cdots \delta^m \gamma_1^m \gamma_{t,m}^m
\]

and

\[
E(t[x_1/x])_1 = x_1
\]

and

\[
E(t[x_1/x])_2 \sim x_1^\delta_1 \cdots \delta^m \gamma_1^m \gamma_{t,m}^m, \]

we obtain

\[
E(t[x_0/x] \triangleleft t[x_1/x])_1 = E(t[x_0/x])_1 = x_0 = E(t[x_0 \triangleleft x_1/x])_1\]

and

\[
E(t[x_0/x] \triangleleft t[x_1/x])_2
= x_0^\delta_1 \cdots \delta^m \gamma_1^m \gamma_{t,m}^m - \delta_1 \cdots - \delta^m \gamma_1^m \gamma_{t,m}^m + x_1 \cdot x_1^\delta_1 \cdots \delta^m \gamma_1^m \gamma_{t,m}^m
\sim x_0^\delta_1 \cdots \delta^m \gamma_1^m \gamma_{t,m}^m
\sim E(t[x_0 \triangleleft x_1/x])_2,\]

as required. This proves that \([t]\) commutes generically with the rack operations, which completes the proof that \([t] \in G_{\text{Rack}}(R_n)\).

Now let \( t \in \text{Term}^\circ(\Sigma(x,y_1,\ldots,y_n))\) with \([t] \in G_{\text{Rack}}(R_n)\). We show that \( t \) can be assumed to have the form in the statement of the theorem.

First we show that \( E(t)_1 = x \). Since \([t] \in G_{\text{Rack}}(R_n)\), it follows that \([t]\) is invertible, and so there is some \( s \in \text{Term}^\circ(\Sigma(x,y_1,\ldots,y_n))\) such that

\[
t[s/x] \sim \text{Rack} \times \text{Rack}_s [t/x]\]

in the free rack on \( \{x,y_1,\ldots,y_n\} \). By Theorem \[4.9\] it then follows that \( E(t[s/x])_1 = E(x)_1 = x \). To show that \( E(t)_1 = x \) follows from this, we first prove the following claim:

**Claim.** Let \( u,v \in \text{Term}^\circ(\Sigma(x,y_1,\ldots,y_n))\).

- If \( \text{Left}(u) = x \), then \( \text{Left}(u[v/x]) = \text{Left}(v) \).
- If \( \text{Left}(u) = y_i \) for some \( 1 \leq i \leq n \), then \( \text{Left}(u[v/x]) = y_i \).

**Proof.** We prove this by induction on \( u \) (for a fixed \( v \)).

- If \( u \equiv x \), then we have \( \text{Left}(u) = x \) and
  
  \[
  \text{Left}(u[v/x]) = \text{Left}(x[v/x]) = \text{Left}(v),
  \]
  
as desired.

- If \( u \equiv y_i \) for some \( 1 \leq i \leq n \), then we have \( \text{Left}(u) = y_i \) and
  
  \[
  \text{Left}(u[v/x]) = \text{Left}(y_i[v/x]) = \text{Left}(y_i) = y_i,
  \]
  
as desired.
• Suppose that \( u \equiv u_1 \triangleleft u_2 \) for some \( u_1, u_2 \in \text{Term}^c(\Sigma(x,y_1,\ldots,y_n)) \) for which the induction hypothesis holds. If \( \text{Left}(u_1 \triangleleft u_2) = x \), then \( \text{Left}(u_1) = x \) by definition of \( \text{Left} \). So by the induction hypothesis for \( u_1 \), we have \( \text{Left}(u_1[v/x]) = \text{Left}(v) \). Then we have

\[
\text{Left}((u_1 \triangleleft u_2)[v/x]) = \text{Left}(u_1[v/x] \triangleleft u_2[v/x]) = \text{Left}(u_1[v/x]) = \text{Left}(v),
\]
as desired.

If \( \text{Left}(u_1 \triangleleft u_2) = y_i \) for some \( 1 \leq i \leq n \), then \( \text{Left}(u_1) = y_i \) by definition of \( \text{Left} \). So by the induction hypothesis for \( u_1 \), we have \( \text{Left}(u_1[v/x]) = y_i \). Then we have

\[
\text{Left}((u_1 \triangleleft u_2)[v/x]) = \text{Left}(u_1[v/x] \triangleleft u_2[v/x]) = \text{Left}(u_1[v/x]) = y_i,
\]
as desired.

Recall from Lemma 4.2 that \( E(u_1) = \text{Left}(u) \) for any \( u \in \text{Term}^c(\Sigma(x,y_1,\ldots,y_n)) \). So, given that \( E(t[\Sigma/x])_1 = x \), we then have \( \text{Left}(t[\Sigma/x]) = x \), and we want to show that \( \text{Left}(t) = x \). But if we had \( \text{Left}(t) = y_i \) for some \( 1 \leq i \leq n \) instead, then from the Claim it would follow that \( \text{Left}(t[\Sigma/x]) = y_i \), as desired, contrary to assumption. So we must have \( \text{Left}(t) = E(t)_1 = x \), as desired.

Since \( [t] \in G_{\text{Poly}}(R_n) \), we know that \( [t] \) commutes generically with the rack operations, and so it follows that \( t[x_0 \triangleleft x_1/x] \sim_{\text{Rack}} t[x_0/x] \triangleleft t[x_1/x] \) holds in the free rack on \( \{x_0,x_1,y_1,\ldots,y_n\} \), which implies by Theorem 4.9 that

\[
E(t[x_0 \triangleleft x_1/x])_1 = E(t[x_0/x] \triangleleft t[x_1/x])_1
\]
and

\[
E(t[x_0 \triangleleft x_1/x])_2 \sim E(t[x_0/x] \triangleleft t[x_1/x])_2.
\]

Then since \( E(t)_1 = x \), we can use Lemma 4.3 and the definition of \( E \) to reason as follows:

\[
x_1 \cdot E(t)_2[x_1^{-1}x_0x_1/x]\sim E(t[x_0 \triangleleft x_1/x])_2
\sim E(t[x_0/x] \triangleleft t[x_1/x])_2
\sim E(t[x_0/x])_2 \cdot E(t[x_1/x])_2^{-1} \cdot x_1 \cdot E(t[x_1/x])_2
\sim E(t)_2[x_0/x] \cdot E(t)_2^{-1} \cdot x_1 \cdot E(t)_2[x_1/x] = E(t)_2[x_1/x].
\]

Now let \( s := E(t)_2 \). Then the above congruence becomes

\[
x_1 \cdot s[x_1^{-1}x_0x_1/x] \sim s[x_0/x] \cdot s^{-1}[x_1/x] \cdot x_1 \cdot s[x_1/x],
\]
Now let \( s_r \) be the unique reduced word obtained from \( s \), so that \( s \sim s_r \) and we have

\[
x_1 \cdot s_r[x_1^{-1}x_0x_1/x] \sim s_r[x_0/x] \cdot s_r^{-1}[x_1/x] \cdot x_1 \cdot s_r[x_1/x].
\]
Then by Lemma 4.8, it follows that all occurrences of \( x \) in \( s_r \) precede all occurrences of \( y_1, \ldots, y_n \) in \( s_r \). So then \( s_r \) must have the form \( s_r \equiv x^{\delta_1} \cdots x^{\delta_p} y_1^{\epsilon_1} \cdots y_n^{\epsilon_n} \) for some \( p, m \geq 0 \) and \( 1 \leq i_1, \ldots, i_m \leq n \), with \( \delta_i = \pm 1 \) for all \( 1 \leq i \leq p \) and \( \epsilon_j = \pm 1 \) for all \( 1 \leq j \leq m \) and \( \delta_i + \delta_{i+1} \neq 0 \) for all \( 1 \leq i < p \) (because \( s_r \) is reduced). Then we have

\[
E(t)_2 = s \\
\sim s_r \\
\equiv x^{\delta_1} \cdots x^{\delta_p} y_1^{\epsilon_1} \cdots y_n^{\epsilon_n} \\
= (x, x^{\delta_1} \cdots x^{\delta_p} y_1^{\epsilon_1} \cdots y_n^{\epsilon_n})_2 \\
= E (x \prec^{\delta_1} \cdots \prec^{\delta_p} x \prec^{\epsilon_1} y_1 \prec^{\epsilon_2} \cdots \prec^{\epsilon_n} y_n)_2,
\]

with the last equality justified by Lemma 4.4. Since we also have

\[
E(t)_1 = x = E (x \prec^{\delta_1} \cdots \prec^{\delta_p} x \prec^{\epsilon_1} y_1 \prec^{\epsilon_2} \cdots \prec^{\epsilon_m} y_m)_1,
\]

it follows that

\[
E(t) = E (x \prec^{\delta_1} \cdots \prec^{\delta_p} x \prec^{\epsilon_1} y_1 \prec^{\epsilon_2} \cdots \prec^{\epsilon_n} y_n)_2.
\]

Then by Theorem 4, this entails that

\[
t \sim_{\text{Rack}} x \prec^{\delta_1} \cdots \prec^{\delta_p} x \prec^{\epsilon_1} y_1 \prec^{\epsilon_2} \cdots \prec^{\epsilon_n} y_n,
\]

so that \( t \) is congruent (in the free rack on \( \{x, y_1, \ldots, y_n\} \)) to a term of the form described in the statement of the theorem, as desired. \( \square \)

Using this characterization of the logical isotropy group of the free rack \( R_n \) on \( n \) generators, we now deduce the following more algebraic characterization:

**Corollary 4.11.** Let \( F_n \) and \( R_n \) be the free group and free rack on \( n \) generators \( y_1, \ldots, y_n \), respectively. Then

\[
G_{\text{Track}}(R_n) \cong \mathbb{Z} \times F_n.
\]

**Proof.** We define a function

\[
\phi : \mathbb{Z} \times F_n \to G_{\text{Track}}(R_n)
\]

as follows. Let \((z, [t]) \in \mathbb{Z} \times F_n \), with \( t \in \text{Term}^r(\Sigma_{\text{Grp}}(y_1, \ldots, y_n)) \). We may suppose without loss of generality that \( t \) is reduced (since \( t \sim t_r \), where \( t_r \) is the unique reduced word congruent to \( t \)). So \( t = y_1^{\epsilon_1} \cdots y_n^{\epsilon_n} \) for some \( m \geq 0 \) and \( 1 \leq i_1, \ldots, i_m \leq n \) and \( \epsilon_j = \pm 1 \) for each \( 1 \leq j \leq m \) (if \( m = 0 \), then \( t \equiv e \)). If \( z = 0 \), then we set

\[
\phi \left( z, [y_1^{\epsilon_1} \cdots y_n^{\epsilon_n}] \right) := [x \prec^{\epsilon_1} y_1 \prec^{\epsilon_2} \cdots \prec^{\epsilon_n} y_n] \in G_{\text{Track}}(R_n).
\]

Otherwise, we set

\[
\phi \left( z, [y_1^{\epsilon_1} \cdots y_n^{\epsilon_n}] \right) := [x \prec^{\delta_1} \cdots \prec^{\delta_p} x \prec^{\epsilon_1} y_1 \prec^{\epsilon_2} \cdots \prec^{\epsilon_n} y_n] \in G_{\text{Track}}(R_n),
\]

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where $\delta_1, \ldots, \delta_z = 1$ if $z > 0$ and $\delta_1, \ldots, \delta_z = -1$ if $z < 0$.

To see that $\phi$ is well-defined, note that if we have $s, t \in \text{Term}^c(\Sigma_{\text{Grp}}(y_1, \ldots, y_n))$ with $[s] = [t]$, i.e. $s \sim t$, then $s$ and $t$ will have the same (unique) reduction, and hence we will indeed have $\phi(z, [s]) = \phi(z, [t])$ for any $z \in \mathbb{Z}$.

Now we show that $\phi$ is actually a group anti-homomorphism. So let

$$
(z, [y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m}]), (z', [y_{j_1}^{\delta_1} \cdots y_{j_p}^{\delta_p}]) \in \mathbb{Z} \times F_n,
$$

with $y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m}, y_{j_1}^{\delta_1} \cdots y_{j_p}^{\delta_p} \in \text{Term}^c(\Sigma_{\text{Grp}}(y_1, \ldots, y_n))$ reduced. We want to show that

$$
\phi (z + z', [y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m} y_{j_1}^{\delta_1} \cdots y_{j_p}^{\delta_p}]) = \phi (z', [y_{j_1}^{\delta_1} \cdots y_{j_p}^{\delta_p}]) \cdot \phi (z, [y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m}]).
$$

Since the group multiplication in $G_{\text{Rack}}(R_n)$ is given by substitution into $x$, this means showing that if

$$
\phi (z + z', [y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m} y_{j_1}^{\delta_1} \cdots y_{j_p}^{\delta_p}]) = [t],
$$

$$
\phi (z', [y_{j_1}^{\delta_1} \cdots y_{j_p}^{\delta_p}]) = [t_1],
$$

and

$$
\phi (z, [y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m}]) = [t_2],
$$

then

$$
t \sim_{\text{Rack}} t_1 [t_2/x]
$$

holds in the free rack on $\{x, y_1, \ldots, y_n\}$. By Theorem 4.9, it suffices to show that $E(t_1) = E(t_1 [t_2/x])_1 \in \{x, y_1, \ldots, y_n\}$ and $E(t_2) \sim E(t_1 [t_2/x])_2$ holds in the free group on $\{x, y_1, \ldots, y_n\}$. By Lemma 4.4 and the definition of $\phi$, we have

$$
E(t_1) = x = E(t_1)_1 = E(t_2)_1
$$

and

$$
E(t_2) \sim x^{z + z'} y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m} y_{j_1}^{\delta_1} \cdots y_{j_p}^{\delta_p},
$$

$$
E(t_1)_2 \sim x^{-z} y_{i_1}^{\epsilon_1} \cdots y_{j_1}^{\delta_1} \cdots y_{j_p}^{\delta_p},
$$

$$
E(t_2)_2 \sim x^{-z} y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m}.
$$

Now note that

$$
W(t_2) := E(t_2)_2^{-1} \cdot E(t_2)_1 \cdot E(t_2)_2
$$

$$
\sim y_{i_1}^{-\epsilon_1} \cdots y_{i_m}^{-\epsilon_m} \cdot x^{-z} \cdot x \cdot x^{-z} y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m}
$$

$$
\sim y_{i_1}^{-\epsilon_1} \cdots y_{i_m}^{-\epsilon_m} \cdot y_{i_1} y_{i_1}^{\epsilon_1} \cdots y_{i_m}^{\epsilon_m}.
$$
So then by Lemma 4.6, we have

\[ E(t_1|t_2/x) \sim E(t_2) \cdot E(t_1|W(t_2)/x) \]

\[ \sim x^z y_{i_1}^\varepsilon \cdots y_{l_m}^\varepsilon \cdot \left( x^{z'} y_{j_1}^\delta \cdots y_{j_p}^\delta \right) |W(t_2)/x| \]

\[ \sim x^z y_{i_1}^\varepsilon \cdots y_{l_m}^\varepsilon \cdot y_{i_1}^{-\varepsilon} \cdots y_{j_1}^{-\varepsilon} \cdot x^{z'} y_{j_1}^\delta \cdots y_{j_p}^\delta \]

\[ \sim x^{z+z'} y_{i_1}^\varepsilon \cdots y_{l_m}^\varepsilon \cdot y_{j_1}^\delta \cdots y_{j_p}^\delta \]

\[ \sim E(t_1)\cdot x^z \]

as desired. This completes the proof that \( \phi \) is an anti-homomorphism.

Now we show that \( \phi \) is bijective. That \( \phi \) is surjective follows almost immediately from Theorem 4.10. To show that \( \phi \) is injective, let \((z, [y_{i_1}^\varepsilon \cdots y_{l_m}^\varepsilon]) \in \mathbb{Z} \times F_n\) with \(y_{i_1}^\varepsilon \cdots y_{l_m}^\varepsilon\) reduced, and suppose that

\[ \phi (z, [y_{i_1}^\varepsilon \cdots y_{l_m}^\varepsilon]) = [x], \]

the unit element of the group \(G_{T_{\text{Rack}}}(R_n)\). We must show that

\[ (z, [y_{i_1}^\varepsilon \cdots y_{l_m}^\varepsilon]) = (0, [e]) \in \mathbb{Z} \times F_n. \]

By definition of \( \phi \) and Theorem 4.9 and Lemma 4.4, the assumption implies that

\[ x^z \cdot y_{i_1}^\varepsilon \cdots y_{l_m}^\varepsilon \sim e, \]

which forces \( z = 0 \) and \( y_{i_1}^\varepsilon \cdots y_{l_m}^\varepsilon \sim e \), so that \([y_{i_1}^\varepsilon \cdots y_{l_m}^\varepsilon] = [e]\), as desired. This proves that \( \phi \) is bijective, which means that

\[ \phi : \mathbb{Z} \times F_n \to G_{T_{\text{Rack}}}(R_n) \]

is a group anti-isomorphism. However, it is a simple fact of group theory that any two anti-isomorphic groups are isomorphic, and so it follows that

\[ \mathbb{Z} \times F_n \cong G_{T_{\text{Rack}}}(R_n), \]

as desired.

We can now use our characterization(s) of the logical isotropy groups of free, finitely generated racks to deduce the following characterizations of the categorical isotropy groups of these racks, whose proofs are similar to those of Corollary 3.6.

**Corollary 4.12.** Let \( n \geq 0 \).

1. Let

\[ \pi = (\pi_h : \text{cod}(h) \to \text{cod}(h))_{\text{dom}(h) = R_n}, \]
be a (not necessarily natural) family of endomorphisms of racks, indexed by rack morphisms \( h \) with domain \( R_n \). Then \( \pi \in \mathbb{Z}_{\text{Rack}}(R_n) \) iff there is a unique integer \( z \in \mathbb{Z} \) and a unique reduced word \( y_i^1 \cdots y_i^{m} \in \text{Term}^{c}(\Sigma_{\text{Grp}}(y_1, \ldots, y_n)) \) with the property that for any rack morphism \( h : R_n \to R \) we have
\[
\pi h(r) = r \triangleleft^{\delta_1} \cdots \triangleleft^{\delta_z} r \triangleleft^{\epsilon_1} h_{i_1} \triangleleft^{\epsilon_2} \cdots \triangleleft^{\epsilon_m} h_{i_m} \in R,
\]
where \( \delta_1, \ldots, \delta_z = 1 \) if \( z > 0 \) and \( \delta_1, \ldots, \delta_z = -1 \) if \( z < 0 \).

2. Let \( h : R_n \to R_n \) be a rack endomorphism. Then \( h \) is a categorical inner automorphism iff there is a unique integer \( z \in \mathbb{Z} \) and a unique reduced word \( y_i^1 \cdots y_i^{m} \in \text{Term}^{c}(\Sigma_{\text{Grp}}(y_1, \ldots, y_n)) \) such that
\[
h([s]) = [s \triangleleft^{\delta_1} \cdots \triangleleft^{\delta_z} s \triangleleft^{\epsilon_1} y_{i_1} \triangleleft^{\epsilon_2} \cdots \triangleleft^{\epsilon_m} y_{i_m}] \in R_n
\]
for any \( [s] \in R_n \) (so \( s \in \text{Term}^{c}(\Sigma(y_1, \ldots, y_n)) \) and \( \delta_1, \ldots, \delta_z \) are as above).

3. Let \( h : R_n \to R_n \) be a rack automorphism. Then \( h \) is a categorical inner automorphism iff \( h \) is an algebraic inner automorphism.

As for quandles, we can deduce a characterization of the global isotropy group of the category \( \text{Rack} \) of racks and their homomorphisms, i.e. the group \( \text{Aut}(\text{Id}_{\text{Rack}}) \) of automorphisms of the identity functor \( \text{Id}_{\text{Rack}} : \text{Rack} \to \text{Rack} \) (which is also the group of invertible elements of the centre of the category \( \text{Rack} \), which is the monoid \( \text{End}(\text{Id}_{\text{Rack}}) \) of natural endomorphisms of the identity functor). Since the category \( \text{Rack} \) has an initial object, namely the absolutely free rack \( R_0 \) (whose carrier is just the empty set), it is easy to see that the global isotropy group of \( \text{Rack} \) is exactly the (covariant) categorical isotropy group of the initial object \( R_0 \), i.e.
\[
\text{Aut}(\text{Id}_{\text{Rack}}) = \mathbb{Z}_{\text{Rack}}(R_0).
\]

Since
\[
\mathbb{Z}_{\text{Rack}}(R_0) \cong G_{\text{Rack}}(R_0) \cong \mathbb{Z} \times F_0 \cong \mathbb{Z}
\]
by Corollary 4.11 (since \( F_0 \) is the trivial group), we thus obtain:

**Corollary 4.13.** The global isotropy group of the category \( \text{Rack} \) is isomorphic to the group \( \mathbb{Z} \):
\[
\text{Aut}(\text{Id}_{\text{Rack}}) \cong \mathbb{Z}.
\]

Explicitly, the natural automorphisms of \( \text{Id}_{\text{Rack}} \) are exactly the natural transformations \( \psi : \text{Id}_{\text{Rack}} \to \text{Id}_{\text{Rack}} \) with
\[
\psi_R(r) = r \triangleleft^{\delta_1} \cdots \triangleleft^{\delta_z} r \quad (r \in R)
\]
for any \( z \in \mathbb{Z} \) and rack \( R \) (with \( \delta_1, \ldots, \delta_z = \pm 1 \) as in Corollary 4.12).
We also note in connection with Corollary 4.13 that M. Szymik independently proved in [8, Theorem 5.4] that the center \( \text{End} (\text{Id}_{\text{Rack}}) \) of the category Rack is also isomorphic to \( \mathbb{Z} \), the free group on one generator. So we obtain as a further corollary:

**Corollary 4.14.** The global isotropy group of the category Rack is equal to its center, and both are isomorphic to \( \mathbb{Z} \).

## 5 Conclusions

We have characterized the isotropy groups of the free, finitely generated racks and quandles both logically and categorically, and shown as a consequence that the notions of categorical and algebraic inner automorphism coincide for such racks and quandles. One would hope to be able to extend the results herein to arbitrary (or at least finitely presented) racks and quandles, but it is not clear how one could accomplish this, since the results herein relied heavily on the solutions of the word problems for free racks and quandles given in terms of the solution of the word problem for free groups ([4, Section 4.1]), whereas it is known that the general word problems for finitely presented racks and quandles are undecidable ([2]).

However, as shown in [5] and [7], in order to compute the isotropy group of a quandle \( Q \), one ideally only needs an effective description of the quandle \( Q(x) \) obtained from \( Q \) by freely adjoining a new element \( x \), which is just the free product (or coproduct) of the quandle \( Q \) with the free quandle \( \langle x \rangle \) on one generator \( x \). In the recent work [1], it is shown for quandles \( Q_1 \) and \( Q_2 \) that if the canonical maps of \( Q_1 \) and \( Q_2 \) into their associated groups are injective, then the free product (i.e. coproduct) of \( Q_1 \) and \( Q_2 \) has an explicit presentation. However, despite the significant improvement of this presentation over the canonical presentation of the coproduct of models of a general equational theory, this presentation is still much more difficult to work with than the presentations of free quandles, even in the specific case where \( Q_2 \) is the free quandle on one generator. We have thus far not been able to extend the results herein to quandles \( Q \) for which the canonical map into its associated group is injective.

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