Glueball and meson spectrum in large-$N$ massless $QCD$

Marco Bochicchio$^{a,b}$

$^a$Scuola Normale Superiore (SNS)
Piazza dei Cavalieri 7, Pisa, I-56100, Italy

$^b$INFN sez. Roma 1
Piazzale A. Moro 2, Roma, I-00185, Italy

E-mail: marco.bochicchio@roma1.infn.it

ABSTRACT: We provide outstanding numerical evidence that in large-$N$ massless $QCD$ the joint spectrum of the masses squared, for fixed integer spin $s$ and unspecified parity and charge conjugation, obeys exactly the following laws: $m_k^2 = (k + s/2)\Lambda_{QCD}^2$ for $s$ even, $m_k^2 = 2(k + s/2)\Lambda_{QCD}^2$ for $s$ odd, $k = 1, 2, \cdots$ for glueballs, and $m_n^2 = \frac{1}{2}(n + s/2)\Lambda_{QCD}^2$, $n = 0, 1, \cdots$ for mesons. One of the striking features of these laws is that they imply that the glueball and meson masses squared form exactly-linear Regge trajectories in the large-$N$ limit of massless $QCD$, all the way down to the low-lying states: A fact unsuspected so far.

The numerical evidence is based on lattice computations by Meyer-Teper in $SU(8)$ YM for glueballs, and by Bali et al. in $SU(17)$ quenched massless $QCD$ for mesons, that we analyze systematically. The aforementioned spectrum for spin-0 glueballs is implied by a Topological Field Theory underlying the large-$N$ limit of $YM$, whose glueball propagators satisfy as well fundamental universal constraints arising from the asymptotic freedom and the renormalization group. No other presently existing model meets both the infrared spectrum and the ultraviolet constraints. We argue that some features of the aforementioned spectrum of glueballs and mesons of any spin could be explained by the existence of a Topological String Theory dual to the Topological Field Theory.
1 Introduction and Conclusions

The main purpose of this paper is to provide outstanding numerical evidence that in ‘t Hooft large-N limit of massless QCD \(^1\) the glueball and meson spectrum for fixed integer spin \(s\) obeys the following laws, exactly at the leading large-N order.

For glueballs:

\[
m^{(s)2}_k = \left( k + \frac{s}{2} \right) \Lambda^2_{QCD}
\]

(1.1a)

for \(s\) even, and:

\[
m^{(s)2}_k = 2 \left( k + \frac{s}{2} \right) \Lambda^2_{QCD}
\]

(1.1b)

for \(s\) odd, with \(k = 1, 2, \cdots\) in both cases. For mesons:

\[
m^{(s)2}_n = \frac{1}{2} \left( n + \frac{s}{2} \right) \Lambda^2_{QCD}
\]

(1.1c)

with \(n = 0, 1, \cdots\).

\(m^{(s)}_k\) and \(m^{(s)}_n\) are the glueball and meson masses of spin \(s\) respectively, and unspecified charge conjugation and parity, labelled by the internal quantum numbers \(k\) and \(n\) respectively, and \(\Lambda_{QCD}\) is the QCD renormalization-group invariant scale in the scheme in which it coincides with the mass gap in the pure glue sector in ‘t Hooft large-N limit.

The numerical evidence is based on lattice computations by Meyer-Teper [1, 2] for glueballs and by Bali et al. [3, 4] for mesons, that we regard presently as the state of the art \(^2\) of the subject of large-N lattice QCD, and that we analyze systematically.

Phenomenologically, it is observed that the meson masses squared form Regge trajectories approximatively linear in the angular momentum.

Field theoretically, it is believed that the glueball and meson masses squared form Regge trajectories asymptotically linear in the angular momentum for large masses.

---

\(^1\)By massless QCD we mean QCD with massless quarks.

\(^2\)Meyer-Teper computation for glueballs is presently on the largest lattice with the smallest value of the coupling and with the largest group SU(8) in pure YM. Bali et al. computation for mesons involves presently the largest group SU(17) in a carefully defined large-N massless limit of quenched lattice QCD.
Figure 1. The glueball and meson spectrum of large-$N$ massless QCD: The points in black, and the straight trajectories labelled by the spin $s$, represent the spectrum implied by the laws Eqs. (1.1). The red and yellow points represent respectively glueballs and mesons actually found in the lattice computations [1–4].

Figure 2. The glueball and meson spectrum of large-$N$ massless QCD: The points in black, and the straight Regge trajectories labelled by the internal quantum numbers, $k$ for glueballs and $n$ for mesons, represent the spectrum implied by the laws Eqs. (1.1).
One of the striking features of Eqs. (1.1) is that they imply that the glueball and meson masses squared in large-\(N\) massless QCD form exactly-linear Regge trajectories, all the way down to the low-lying states: A fact unsuspected so far, for which we provide outstanding numerical evidence in Section 2.

Another striking feature is that Eqs. (1.1) refer to the joint spectrum for fixed spin, without specifying parity \(P\) and charge conjugation \(C\). Thus the preceding equations do not identify uniquely any particular one-particle state occurring in the large-\(N\) limit, but rather they describe the joint spectrum with respect to the other quantum numbers \(P\) and \(C\), without specifying the degeneracy for fixed spin in the glueball and meson sectors.

Yet, since no numerical evidence can substitute a theoretical understanding, we report here below the theoretical reasons that made us to suspect that Eqs. (1.1) hold exactly in the large-\(N\) limit.

Firstly, Eq. (1.1a) restricted to the glueball sector with \(s = 0\), positive \(C\) and unspecified \(P\), holds exactly at the leading large-\(N\) order in a Topological Field Theory (TFT) underlying the large-\(N\) limit of pure YM [5]. Obviously, the result extends to large-\(N\) QCD in ’t Hooft large-\(N\) limit in the pure glue sector.

Roughly speaking the TFT describes [5] the ground state of the large-\(N\) one-loop integrable sector of massless QCD of Ferretti-Heise-Zarembo [6]. The ground state in the pure glue sector is generated [6] by scalar operators constructed by the anti-selfdual (ASD) curvature \(F_{\alpha\beta}^{\ast} = F_{\alpha\beta} - \ast F_{\alpha\beta}\) with \(\ast F_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F_{\gamma\delta}\).

Hence the TFT computes [5] correlators that couple to scalar and pseudoscalar glue-balls of positive \(C\). The correlators in the TFT factorize exactly [5] on the spectrum in Eq. (1.1a) for \(s = 0\), without being able to resolve the parity of the states.

Besides, the chiral nature of the correlators in the TFT suggested us that the proper generalization of the \(s = 0\) case should not specify other quantum numbers but the spin, in order to fill all the points of the spectrum predicted by Eqs. (1.1) without holes. Morally, this would occur should the points of the spectrum implied by Eqs. (1.1) arise as poles in correlators in the whole sector of Ferretti-Heise-Zarembo, including chiral fermion bilinears, that has a chiral nature [6] and that therefore cannot resolve the parity of the states that couple to the operators therein.

A relevant feature of the TFT is that it agrees sharply [5] with Meyer-Teper numerical results for the lowest scalar and pseudoscalar states \[\frac{m_0^{++}}{m_0^{-+}} = 1.419; \frac{m_0^{++}}{m_0^{-+}} = 1.422\] since Eq. (1.1a) predicts for the same ratios \(\sqrt{2} = 1.4142 \cdots\). Besides, the glueball correlators of the TFT agree [5, 7] crucially with fundamental universal constraints arising by the asymptotic freedom and the renormalization group, as opposed to the correlators computed by means of the AdS String/Gauge Theory correspondence. Thus no presently existing model agrees both with the infrared and the ultraviolet of large-\(N\) QCD but the TFT [5]. For an analysis of all the implications of the TFT versus the AdS String/Gauge Theory correspondence see ref. [5]. For an analysis of the ultraviolet asymptotics of glueball propagators in massless QCD and in the TFT see ref. [7]. For theoretical foundations of the TFT see refs. [8–10]. For additional detailed computations see refs. [11, 12].

\[\text{We label glueballs and mesons by their } s^{PC}\text{ quantum numbers.}\]
Table 1. The glueball mass ratios implied by ref. [2] versus Eqs. (1.1) (theor.). The $s = 0$ case is actually a theoretical prediction of the $TFT$.

Secondly, were to exist a (Topological) String Theory ($TST$) dual to the $TFT$, some features of the whole spectrum in Eqs. (1.1) could be understood as well as follows.

By construction, i.e. by the exact duality, the $TST$ would reproduce the $s = 0$ case. Moreover, it is natural to conjecture that the slopes, differing by a factor of 2 between Eq. (1.1a) and Eq. (1.1b), would correspond to the open and closed sectors of strings in the adjoint representation respectively, and that the further factor of 2 for the slope in the meson sector Eq. (1.1c), with respect to the would-be open glueball sector in Eq. (1.1a), would correspond to an open string in the fundamental representation.

The specific form of Eqs. (1.1) would be a dynamical feature of the $TST$, but the integer or semi-integer character of the spectrum of the masses squared in units of a common scale squared should arise by its topological nature, since by construction the $TST$ would be the dual of the $TFT$ underlying $YM$.

We will work out elsewhere the spectrum of such $TST$, in order to check our stringy conjectures, along the lines foreseen in section 14 of ref. [11] and in [14].

Our conclusion, in addition to the main numerical evidence that is remarkable by itself,

---

4The $TST$ dual to the $TFT$ is essentially a theory of open strings, since the $TFT$ describes twistor Wilson loops supported on Lagrangian submanifolds of twistor space of non-commutative space-time [11]. Hence the dual topological string is the topological $A$-model on twistor space of non-commutative space-time, that describes open strings ending on Lagrangian submanifolds of twistor space of non-commutative space-time according to section 14 of ref. [11] and [14].
Table 2. The meson mass ratios implied by ref. [3] versus Eqs. (1.1) (theor.).

is that the aforementioned evidence favors the TFT of YM and indirectly the existence of a dual TST as well, since no other presently existing model agrees [5] with Eq. (1.1a) for $s = 0$ in the infrared and with asymptotic freedom for glueball propagators in the ultraviolet, and globally with the actual glueball and meson spectrum as reported in Fig. 1 and Fig. 2.

Moreover, we urge the lattice gauge theory community to further confirm or falsify Eqs. (1.1).

2 Numerical Analysis

We start our numerical analysis by comparing the lowest mass glueball $k = 1, s = 0$ with the mass of the $n = 2, s = 0$ meson, in order to set a common $\Lambda_{QCD}$ scale for glueballs and mesons, since on the basis of Eqs. (1.1a)–(1.1c) $m_{k=1}^{(0)} = m_{n=2}^{(0)} = \Lambda_{QCD}$.

The result of the lattice computations relevant for this paper were summarized by the authors of ref. [2] and of ref. [3] in Tab. 3 and Tab. 4 respectively, that we reproduce here.

In fact, we get numerically from Tab. 3 $\left[\frac{m_{n=1}}{\sqrt{\sigma}}\right]_{SU(8)} = 3.32$ [2] and from Tab. 4 $\left[\frac{m_{n=2}}{\sqrt{\sigma}}\right]_{SU(\infty)} = 3.39$ [3], where $\sigma$ is the string tension measured in the lattice gauge theory, with the first ratio computed in $SU(8)$ and the second ratio extrapolated to $SU(\infty)$.

We assume that the first ratio is accurately close to its large-$N$ limit, as it is generally believed [13]. Indeed, under this assumption, the ratio $\frac{m_{n=2}}{m_{k=1}} = 1.0211$ turns out to be 1 with 2% accuracy, consistently with Eqs. (1.1a)–(1.1c).
\[ J^{PC} \quad IR \quad m/\sqrt{\sigma} \quad \nu \quad \chi^2/(\nu - 2) \quad \text{Average } m/\sqrt{\sigma} \]

| \( J^{PC} \) | IR | \( m/\sqrt{\sigma} \) | \( \nu \) | \( \chi^2/(\nu - 2) \) | Average \( m/\sqrt{\sigma} \) |
|------------|----|----------------|-----|----------------|-------------------|
| 0^{++}     | \( A_1 \) | 3.32(15) | 4    | 0.41           | 3.32(15)          |
| 0^{+++}    | \( A_1 \) | 4.71(29) | 4    | 0.39           | 4.71(29)          |
| 2^{++}     | \( E \) | 4.74(21) | 4    | 0.20           | 4.65(19)          |
|            | \( T_2 \) | 4.57(19) | 4    | 0.45           |                   |
| 2^{+++}    | \( E \) | 6.47(50) | 4    | 1.0            | 6.47(50)          |
| 3^{++}     | \( A_2 \) | 7.2(1.3) | 3    | 0.08           | 7.2(1.3)          |
| 0^{--}     | \( A_1 \) | 4.72(32) | 4    | 1.1            | 4.72(32)          |
| \( \gamma^{--} \) | \( T_1 \) | 7.87(77) | 4    | 0.70           | 7.87(77)          |
| 2^{--}     | \( E \) | 6.21(53) | 4    | 0.28           | 5.67(40)          |
|            | \( T_2 \) | 5.36(40) | 4    | 0.22           |                   |
| 1^{++}     | \( T_1 \) | 5.70(29) | 4    | 0.85           | 5.70(29)          |
| 3^{+-}     | \( A_2 \) | 7.2(1.5) | 3    | 0.09           | 7.74(79)          |
|            | \( T_2 \) | 7.89(79) | 3    | 0.18           |                   |
| 1^{--}     | \( T_1 \) | 7.45(60) | 4    | 0.07           | 7.45(60)          |
| 2^{--}     | \( E \) | 7.4(1.4) | 3    | 0.87           | 7.3(1.4)          |
|            | \( T_2 \) | 7.2(1.5) | 3    | 0.01           |                   |
| 3^{--}     | \( T_1 \) | 7.1(1.2) | 3    | 0.001          | 7.5(1.1)          |
|            | \( T_2 \) | 7.9(1.1) | 3    | 0.004          |                   |

**Table 3.** The \( SU(8) \) glueball spectrum from ref. [2].

Besides, we fit by means of a global look at Fig. 1 and Fig. 2 \( \Lambda_{QCD} \equiv (3.32)^2 \sqrt{\sigma} \) and we compute all the mass ratios in units of \( \Lambda_{QCD} \). We report the result for the ratios of glueball masses in Tab. 1, and for the ratios of meson masses in Tab. 2. The plots of the linear trajectories versus the observed states are displayed in Fig. 1 and Fig. 2.

For 10 of the 12 glueball states whose spin was identified in ref. [2] the actual difference between \( \frac{m}{\Lambda_{QCD}} \) and the ratio implied by the laws in Eqs. (1.1) is within about 1.5% or better, for the \( 3^{++} \) state it is within about 2% and for the remaining \( 3^{+-} \) state it is within 5%, but the \( 3^{+-} \) state has the largest estimated variance in Tab. 1, so that a 5% accuracy is still perfectly compatible.

In fact, the central values of the glueball mass ratios must be correlated much more than the estimate of their variance in ref. [2] implies, in order to explain such a good agreement with Eqs. (1.1).

For the mesons the agreement with Eqs. (1.1) is slightly worse: Only 3% or better for
Particle | $J^{PC}$ | $m_q = 0$ | $m_q = m_{ud}$ | $m_q = m_s$ | $m_q = 0$ | $m_q = m_{ud}$ | $m_q = m_s$
--- | --- | --- | --- | --- | --- | --- | ---
$\pi$ | $0^{-+}$ | 0 | 0.417(100) | 1.62(10) | 0 | 1.92(46) | 7.46(48)
$\rho$ | $1^{--}$ | 1.5382(65) | 1.6382(66) | 1.9130(79) | 7.08(10) | 7.54(11) | 8.80(13)
$a_0$ | $0^{++}$ | 2.401(31) | 2.493(31) | 2.755(32) | 11.04(21) | 11.47(22) | 12.67(23)
$a_1$ | $1^{++}$ | 2.860(21) | 2.938(21) | 3.158(22) | 13.16(21) | 13.51(21) | 14.53(23)
$b_1$ | $1^{+-}$ | 2.901(23) | 2.978(23) | 3.197(23) | 13.35(21) | 13.70(22) | 14.71(23)
$\pi^*$ | $0^{-+}$ | 3.392(57) | 3.462(57) | 3.659(58) | 15.61(34) | 15.93(35) | 16.83(36)
$\rho^*$ | $1^{--}$ | 3.696(54) | 3.756(54) | 3.928(54) | 17.00(34) | 17.28(35) | 18.07(36)
$a_0^*$ | $0^{++}$ | 4.356(65) | 4.420(65) | 4.603(66) | 20.04(41) | 20.33(41) | 21.18(42)
$a_1^*$ | $1^{++}$ | 4.587(75) | 4.646(75) | 4.816(77) | 21.10(46) | 21.38(46) | 22.15(47)
$b_1^*$ | $1^{+-}$ | 4.609(99) | 4.673(99) | 4.85(10) | 21.20(54) | 21.50(55) | 22.33(56)
$f_{\pi}^*$ | — | 0.3074(43) | 0.3271(44) | 0.3784(56) | $\sqrt{2}$ | 1.505(29) | 1.741(36)
$f_{\rho}^*$ | — | 0.5721(49) | 0.5855(50) | 0.6196(64) | 2.632(43) | 2.694(44) | 2.850(50)

Table 4. The $N = \infty$ meson spectrum in quenched massless QCD from ref. [3].

6 of the 10 states, including the pion that is exactly massless according to Eqs. (1.1), as it should be.

However, the meson masses are obtained numerically by extrapolating to the large-$N$ limit and to the massless limit but not to the continuum limit, that according to ref. [3] may introduce additional systematic errors on the order of 5%.

The $\rho$ mass ratio agrees with Eqs. (1.1) only within 7%, but it may be more sensitive to the extrapolation to the massless limit, since the $\rho$ meson is the lightest massive meson. By means of Fig. 1 and Tab. 4 we observe that the 3 most massive meson states would fit perfectly the Regge trajectory implied by Eqs. (1.1), were their spin shifted by 1 with respect to the spin attributed in ref. [3]. Otherwise, the difference of $\frac{m}{\Lambda_{QCD}}$ for the aforementioned 3 meson states with respect to Eqs. (1.1) is on the order of 6% as it is seen in Fig 2, a fact still perfectly compatible with the 5% possible estimated additional systematic error in [3]. In any case their masses squared are semi-integer valued in units of $\frac{1}{2}\Lambda_{QCD}^2$ within 2% accuracy or better, a fact quite remarkable.

Taking the square doubles the errors, but the global agreement of $(\frac{m}{\Lambda_{QCD}})^2$ with Eqs. (1.1) is still impressive, within about 3% for 10 of the 12 glueball states and within about 6% for 6 of the 10 meson states. Above all our numerical conclusions follow simply looking at these plots.
3 Acknowledgements

We would like to thank Barbara Mele and Giulio D’Agostini for a clarifying discussion on the variance of the mass ratios. We would like to thank Alessandro Pilloni for working out the wonderful colorful plots in Fig. 1 and Fig. 2.

References

[1] H. B. Meyer, M. J. Teper, Glueball Regge trajectories and the Pomeron – a lattice study –, Phys. Lett. B 605 344 (2005) [hep-ph/0409183].

[2] H. B. Meyer, Glueball Regge Trajectories, hep-lat/0508002.

[3] G. S. Bali, F. Bursa, L. Castagnini, S. Collins, L. Del Debbio, B. Lucini, M. Panero, The meson spectrum in large-N QCD, PoS (Confinement X) 278 (2013) [arXiv:1302.1502].

[4] G. S. Bali, F. Bursa, L. Castagnini, S. Collins, L. Del Debbio, B. Lucini, M. Panero, Mesons in large-N QCD, JHEP 06 (2013) 071 [hep-th/1304.4437].

[5] M. Bochicchio, Glueball and meson propagators of any spin in large-N QCD, Nucl. Phys. B 875 (2013) 621, [hep-th/1305.0273].

[6] M. Bochicchio, S. P. Muscinelli, Asymptotics of glueball propagators, JHEP 08 (2013) 064 [hep-th/1304.6409].

[7] M. Bochicchio, Quasi BPS Wilson loops, localization of loop equation by homology and exact beta function in the large-N limit of SU(N) Yang-Mills theory, JHEP 0905 (2009) 116 [hep-th/0809.4662].

[8] M. Bochicchio, Exact beta function and glueball spectrum in large-N Yang-Mills theory, PoS EPS-HEP2009: 075(2009) [hep-th/0910.0776].

[9] M. Bochicchio, Yang-Mills mass gap at large-N, topological quantum field theory and hyperfiniteness, hep-th/1202.4476, a byproduct of the Simons Center workshop "Mathematical Foundations of Quantum Field Theory", Stony Brook, USA, Jan 16-20 (2012).

[10] M. Bochicchio, Glueballs in large-N YM by localization on critical points, hep-th/1107.4320, extended version of the talk at the Galileo Galilei Institute Conference "Large-N Gauge Theories", Florence, Italy, May 2011.

[11] M. Bochicchio, Glueball propagators in large-N YM, hep-th/1111.6073.

[12] B. Lucini, M. Panero, SU(N) gauge theories at large N, Physics Reports 526 (2013) 93 [hep-th/1210.4997].

[13] M. Bochicchio, The Yang-Mills String as the A-Model on the Twistor Space of the Complex Two-Dimensional Projective Space with Fluxes and Wilson Loops: the Beta Function, hep-th/0811.2547.