Direct Instanton Effects
in Current-Current Correlators

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Abstract

We compute the effect of small-size instantons on the coefficient function of the chiral condensate in the operator product expansion of current-current correlators. Furthermore, we also compute the instanton corrections associated with four-quark and six-quark operators in the factorization approximation. We discuss the phenomenological implications of our result.
1. Introduction

It is well known\(^{1}\) that instantons do spoil the operator product expansion by introducing corrections that are power-suppressed by 9 or more inverse powers of the momentum. Such corrections are fully calculable. In view of the large power suppression associated with them, in practical applications they will turn out to be either very large, or very small. They provide therefore a lower limit to the momentum scale at which perturbation theory can be applied. In a previous work\(^{2}\), the effect of small-size instantons on the correlator of two currents was computed, and the calculation was applied to the hadronic decays of the \(\tau\) lepton. In this case it was found that the effect was small, mainly because of a chiral suppression factor. In a subsequent work\(^{3}\) it was instead shown that in the sum rules that are usually employed to determine the mass of the light quarks, instantons effects are indeed very large.

The aim of the present work is to extend and complete the work of ref.\(^{2}\), with the inclusion of corrections to the coefficient function of quark condensates in the operator product expansion of two currents. Such corrections are less chiral suppressed than the one one computed in ref.\(^{2}\) (a discussion of this issue can be found in ref.\(^{??}\)). Results in this direction have already been obtained in ref.\(^{3}\), where the instanton correction of the coefficient function of a six-quark operator was computed in the factorization approximation.

Our paper is organized as follows: in section 2 we discuss the general framework for computing instanton corrections to the coefficients of the operator product expansion of a current-current correlator. In section 3 we describe in detail the computation of the instanton corrections to the coefficient function of the chiral condensate. In section 4 we extend our calculation to four- and six-quark operators. In section 5 we collect the final results. In section 6 we discuss some phenomenological applications, and in section 7 we give our conclusions.

2. Instanton Corrections to the

Current-Current Two-Point Functions

We begin by establishing a framework for the computation of instanton corrections to the coefficient functions of the OPE. Let us consider the time ordered product of
two currents. The operator product expansion reads (all indices were dropped for simplicity)

\[ J(x) J(y) \xrightarrow{x \to y} \sum_i C^{(i)}(\Delta) O_i(r) \]  

(2.1)

where

\[ r = \frac{x + y}{2}, \quad \Delta = x - y, \]

\[ O_1 = 1, \quad O_2, \ldots = \text{operators of higher dimension}. \]  

(2.2)

In order to compute the coefficients of the OPE in perturbation theory, one usually computes the expectation value of both sides of eq. (2.1) among suitably chosen perturbative states. Alternatively, one can compute in perturbation theory both sides of the equation

\[ \langle 0 | J(x) J(y) P | 0 \rangle \xrightarrow{x \to y} \sum_i C^{(i)}_0(\Delta) \langle 0 | O_i(r) P | 0 \rangle \]  

(2.3)

where \( P \) indicates a generic product of local fields at fixed positions away from the origin, and then solve for the \( C^{(i)}_0 \).

In order to determine the effect of instantons, we may now compute both sides of eq. (2.3) by including instanton effects. We must have

\[ \langle I | J(x) J(y) P | I \rangle \xrightarrow{x \to y} \sum_i C^{(i)}(\Delta) \langle I | O_i(r) P | I \rangle \]  

(2.4)

where now \( |I\rangle \) denotes the vacuum in the presence of instantons, and \( C^{(i)} = C^{(i)}_0 + \delta C^{(i)} \) represent the modified coefficient function in the presence of instantons. In the dilute gas approximation we have

\[ \langle I | J(x) J(y) P | I \rangle = \langle 0 | J(x) J(y) P | 0 \rangle + \int d\xi \ D(\xi) \left[ \langle \xi | J(x) J(y) P | \xi \rangle - \langle 0 | J(x) J(y) P | 0 \rangle \right] \]

(2.5)

where \( |\xi\rangle \) represents the one-instanton configuration. The variable \( \xi \) represents here all degrees of freedom (position, size, orientation, colour, signature) of the instanton

\[ \int d\xi = \int d\rho d^4 z dR \sum_x \]  

(2.6)
and $D(\xi)$ is the instanton density. For the operators we also have

$$\langle I | O_i(r) P | I \rangle = \langle 0 | O_i(r) P | 0 \rangle + \int d\xi \; D(\xi) \left[ \langle \xi | O_i(r) P | \xi \rangle - \langle 0 | O_i(r) P | 0 \rangle \right].$$

(2.7)

From eqs. (2.4), (2.5) and (2.7), treating the instanton correction as a small correction, we get

$$\sum_i \delta C^{(i)}(\Delta) \langle 0 | O_i(r) P | 0 \rangle = \int d\xi \; D(\xi) \left[ \langle \xi | J(x) J(y) P | \xi \rangle - \sum_i C^{(i)}_0(\Delta) \langle \xi | O_i(r) P | \xi \rangle \right].$$

(2.8)

The coefficient functions are then obtained by appropriately choosing $P$. If we choose $P = 1$ (the identity operator), then only the identity operator $O_\infty$ will survive on the left-hand side, and we will get

$$\delta C^{(1)}(\Delta) = \int d\xi \; D(\xi) \left[ \langle \xi | J(x) J(y) | \xi \rangle - \sum_{i=1}^{\infty} C^{(i)}_0(\Delta) \langle \xi | O_i | \xi \rangle \right].$$

(2.9)

The second term in the square bracket of eq. (2.9) subtracts the effect of large-size instantons from the first term. In fact, the expectation value of the product of two currents in a slowly varying classical field obeys the operator product expansion

$$\langle \xi | J(x) J(y) | \xi \rangle \xrightarrow{x \rightarrow y} \sum_i C^{(i)}_0(\Delta) \langle \xi | O_i | \xi \rangle$$

(2.10)

when the $x \rightarrow y$ limit is taken at fixed instanton size. Therefore, in eq. (2.9) the subtraction effectively cuts off the effects of large-size instantons. By looking back to the derivation of eq. (2.8), it is easy to see that large instantons affect the matrix elements of the OPE, but not the coefficient functions.

Equation (2.9) cannot be valid for operators of arbitrarily high dimension. In fact, after integrating over the instanton position, the expectation value of the operator $O_i$ in the instanton background is given by an integral over the instanton size, which behaves like $\int d\rho \rho^{6+n_f/3} \rho^{-d_i}$, where $d_i$ is the canonical dimension of the operator. Therefore, if $d_i \geq 7 + n_f/3$ the integral is ultraviolet divergent. One should then worry about subtractions. If we limit ourselves to operators with dimension smaller than $7 + n_f/3$, no UV divergences are present, and the instanton correction to the coefficient function is well defined, because all infrared divergences in eq. (2.9) are removed by the subtraction term. In the particular case in which $n_f$ is an odd multiple
of 3, not all the infrared divergences are removed in this way. In fact, operators of dimension $7 + n_f/3$ may exist, and their matrix elements are both infrared and ultraviolet divergent. One then has an ambiguity in deciding the scale of separation of short and long distance effects in the OPE. For our purposes, this problem will turn out to be irrelevant. In fact, we will see that the ambiguous term is a polynomial in $\Delta$, and therefore its Fourier transform is concentrated at $p = 0$, and it never contributes to physical quantities.

3. Instanton corrections to the coefficient function of the chiral condensate

In order to compute the correction to the coefficient function of the chiral condensate $O_{(ff)} = \bar{\psi}_f(t)\psi_f(t)$, we will choose $P = \bar{\psi}_f(t)\psi_f(t)$, where the point $t$ is chosen to be far away from the origin with respect to $x, y$. We get

$$\delta C^{(ff)}(\Delta) \langle 0|O_{(ff)}(r)\bar{\psi}_f(t)\psi_f(t)|0 \rangle = \int d\xi \ D(\xi) \left[ \langle \xi|J(x)J(y)\bar{\psi}_f(t)\psi_f(t)|\xi \rangle - \sum_i C^{(i)}_0(\Delta) \langle \xi|O_i(r)\bar{\psi}_f(t)\psi_f(t)|\xi \rangle \right].$$

It is easy to convince oneself that the only operator that can appear on the left-hand side is $O_{(ff)}$. In fact, since we are working at the tree level, in order to get a contribution the operator must be a fermion bilinear. Bilinears of the form $\bar{\psi}_f \gamma_\sigma \psi_f$, $\bar{\psi}_f \sigma_\sigma \delta \psi_f$, etc., give zero after one takes the fermion trace, while bilinears of the form $\bar{\psi}_f \partial_\sigma \psi_f$ vanish at a faster rate for large $t$.

We will now focus on the computation of the first term inside the square bracket of eq. (3.1). We define

$$I_{\mu\nu} = \langle \xi|J_\mu(x)J_\nu(y)\bar{\psi}_f(t)\psi_f(t)|\xi \rangle$$

where our currents will be in general axial or vector currents, possibly flavour non-diagonal. The Euclidean fermionic propagator in the instanton background satisfies the equation

$$(i\gamma \cdot D_x - m) S^\pm(x, y) = \delta^4(x - y).$$

In the $m \to 0$ limit it has the expansion

$$S^\pm(x, y) = -\frac{\bar{\psi}_0(x)\psi^\dagger_0(y)}{m} + S_0^\pm(x, y) + m \int d^4 z \ S_0^\pm(x, z) S_0^\pm(z, y) + O(m^2);$$
$S_0^\pm$ satisfies the equation
\[
i\gamma \cdot D_x S_0^\pm(x, y) = \delta^4(x - y) - \psi_0(x) \psi_0^\dagger(y),
\]
where $D_\mu$ is the covariant derivative in the instanton background
\[
D_\mu = \partial_\mu - i t_a A_{\mu a}.
\]
In operator notation, $S_0^\pm$ has the explicit expression
\[
S_0^\pm = -i\gamma \cdot D \frac{1 \pm \gamma_5}{2} - i \frac{1}{-D^2} \gamma \cdot \frac{1 \mp \gamma_5}{2},
\]
where the meaning of the operator notation is specified as follows
\[
\int d^4x d^4y S_0^\pm(x, y) f_1(x) f_2(y) = \int d^4x d^4y f_1(x) S_0^\pm f_2(y).
\]
The null-mode projector
\[
P(x, y) = \sum_k \psi_0^k(x) \psi_0^{k\dagger}(y)
\]
is given in operator notation by the formula
\[
P = \left[1 - \gamma \cdot D \frac{1}{-D^2} \gamma \cdot D \right] \frac{1 \mp \gamma_5}{2}.
\]
Another useful identity is the following
\[
\left(S_0^\pm\right)^2 = -\gamma \cdot D \frac{1}{-D^2} \gamma \cdot \frac{1 \pm \gamma_5}{2} - \frac{1}{-D^2} \frac{1 \mp \gamma_5}{2}.
\]
Equations (3.7), (3.10) and (3.11) are easily verified, using the identity
\[
-\left(\gamma \cdot D\right)^2 \frac{1 \pm \gamma_5}{2} = D^2 \frac{1 \pm \gamma_5}{2}
\]
which is valid for any self-dual (anti-self-dual) field configuration (see ref. [7]).

In operator notation the fermion propagator is then
\[
S^\pm = -\frac{P}{m} + S_0^\pm + m \left(S_0^\pm\right)^2 + O(m^2).
\]
Let us assume now for concreteness that our currents are $ud$ currents, and that the scalar bilinear is a $dd$ current. We have
\[
I_{\mu\nu} = -\text{Tr} \left[\Gamma_\mu \langle x | S^\pm_u | y \rangle \Gamma_\nu \langle y | S^\pm_d | t \rangle \langle t | S^\pm_d | x \rangle + \text{Tr} \left[\Gamma_\mu \langle x | S^\pm_u | y \rangle \Gamma_\nu \langle y | S^\pm_d | x \rangle \right] \times \text{Tr} \left[\langle t | S^\pm_d | t \rangle\right]\right]
\]
(the minus sign in the first term is due to the fermion loop). Observe that we do not need to include tadpole diagrams involving the currents, since we are considering the flavour off-diagonal case. We want to isolate the least chiral suppressed contribution. We immediately find that the relevant contributions are given by

\[ I_{\mu\nu} = \frac{1}{m_d} I_{\mu\nu}^{(a)} + \frac{m_u}{m_d^2} I_{\mu\nu}^{(b)} + \frac{1}{m_u} I_{\mu\nu}^{(c)} + I'_{\mu\nu} \]

with

\[
I_{\mu\nu}^{(a)} = \text{Tr} \left[ \Gamma_\mu \langle x|S^+_0|y\rangle \Gamma_\nu \langle y|S^+_0|t\rangle \langle t|P|x\rangle \right] \\
+ \text{Tr} \left[ \Gamma_\mu \langle x|S^+_0|y\rangle \Gamma_\nu \langle y|P|t\rangle \langle t|S^+_0|x\rangle \right]
\]

\[
I_{\mu\nu}^{(b)} = -\text{Tr} \left[ \Gamma_\mu \langle x|(S^+_0)^2|y\rangle \Gamma_\nu \langle y|P|t\rangle \langle t|P|x\rangle \right]
\]

\[
I_{\mu\nu}^{(c)} = \text{Tr} \left[ \Gamma_\mu \langle x|P|y\rangle \Gamma_\nu \langle y|S^+_0|t\rangle \langle t|S^+_0|x\rangle \right].
\]

The remaining term is the tadpole

\[ I'_{\mu\nu} = \text{Tr} \left[ \Gamma_\mu \langle x|S^+_u|y\rangle \Gamma_\nu \langle y|S^+_d|x\rangle \right] \times \text{Tr} \left[ \langle t|S^+_d|t\rangle \right], \]

which is proportional to the vacuum term. All other terms are either more suppressed by powers of the fermion masses, or they vanish because of a wrong chiral structure.

The computation was carried out in the regular gauge. This is legitimate, since \( I_{\mu\nu} \) is a gauge-invariant quantity. The instanton field in the regular gauge has the expression

\[ A^\pm_\mu (x) \equiv A^{\pm}_{\mu b}(x) t^b = \frac{1}{g} \frac{\eta^\pm_{\mu\nu} x_\nu \sigma_a}{x^2 + \rho^2}, \]

where \( \sigma_a \) are the three Pauli matrices in the \( SU(2) \) subgroup of \( SU(3) \) in which the instanton lives. Equation (3.18) is for an instanton centred at the origin (we can go back to the general case by the replacement \( x \to x - z, y \to y - z \)). The \( \eta \) symbols are defined in ref. [6]. The scalar propagator has the simple expression

\[ \langle x| \frac{1}{-D^2} |y\rangle = \frac{\rho^2 + x \cdot y + i \eta^\pm_{\mu\nu} x_\mu y_\nu \sigma_a}{4\pi^2 (x - y)^2 (x^2 + \rho^2)^\frac{3}{2} (y^2 + \rho^2)^\frac{3}{2}}. \]

We introduce the \( \tau \) symbols

\[ \tau_\mu = (\sigma_1, \sigma_2, \sigma_3, i) \]

The \( \eta \) symbols satisfy the following equations

\[
\tau_\mu^\dagger \tau_\nu = \delta_{\mu\nu} + i \eta^+_{\mu\nu} \sigma_a
\]

\[
\tau_\mu \tau_\nu^\dagger = \delta_{\mu\nu} + i \eta^-_{\mu\nu} \sigma_a.
\]
In terms of the \( \tau \) symbols we have

\[
\langle x | \frac{1}{-D^2} | y \rangle = \frac{\rho^2 + (\tau^\dagger \cdot x) (\tau \cdot y)}{4\pi^2 (x - y)^2 (x^2 + \rho^2)^{\frac{3}{2}} (y^2 + \rho^2)^{\frac{3}{2}}},
\]

(3.22)

\[
t^b A^b_\mu(x) = \frac{i}{g} \frac{x_\mu - \tau^\dagger_\mu \tau \cdot x}{x^2 + \rho^2}
\]

(3.23)

\( (A^\mu_\mu(x) e \Delta^{-}(x,y) \) are obtained with the substitution \( \tau \leftrightarrow \tau^\dagger \)). It is also convenient to find a simpler expression for the null-mode projector \( P \). The zero modes in the regular gauge are given by

\[
[\psi^{\text{reg}}_0(x)]_{\alpha,i} = \frac{1}{\pi} \frac{\rho}{[x^2 + \rho^2]^{\frac{3}{2}}} \left[ i \gamma_4 \gamma_2 \frac{1 - \gamma_5}{2} \right]_{\alpha,i}
\]

(3.24)

where \( \alpha \) is a spinor and \( i \) is a colour index. The index \( i \) spans the two-dimensional space of left-handed spinor, and it corresponds in colour space to the \( SU(2) \) subspace in which the instanton lives. To be more specific, let us use the following representation for the gamma matrices

\[
\gamma^4 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad \tilde{\gamma} = \begin{bmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{bmatrix}, \quad \gamma^5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

(3.25)

so that

\[
\gamma^\mu = \begin{bmatrix} 0 & \tau^\mu \dagger \\ -\tau^\mu & 0 \end{bmatrix}, \quad \gamma^5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

(3.26)

We have

\[
\gamma^\mu \gamma^\nu = \begin{bmatrix} -\tau^\mu \dagger \tau^\nu & 0 \\ 0 & -\tau^\mu \tau^\nu \dagger \end{bmatrix}.
\]

(3.27)

Therefore

\[
\tau^\mu \tau^\nu = \frac{1 - \gamma_5}{2} \gamma^\mu \gamma^\nu
\]

(3.28)

where the left-hand side is naturally extended to four dimensions, with all the entries equal to zero, except for the upper-left two-dimensional block.

Formula (3.24) was obtained via a gauge transformation of the standard singular gauge expression of ref. [14]:

\[
[\psi^{\text{sing}}_0(x)]_{\alpha,i} = \frac{1}{\pi} \frac{\rho}{[x^2 + \rho^2]^{\frac{3}{2}}} \left[ i \gamma \cdot x \frac{1 - \gamma_5}{2} \right]_{\alpha,i}.
\]

(3.29)

With the above normalization we get\(^2\)

\[
\int d^4x \ \text{Tr} \left[ \psi_0(x) \psi^\dagger_0(x) \right] = 1,
\]

(3.30)

\(^2\)The present work differs from ref. [14] in the normalization of the zero mode.
and the null-mode projector is given by

$$
\langle x|P|y \rangle = \psi_0(x) \psi_0^\dagger(y).
$$

(3.31)

Consider now a generic diagram containing a fermionic loop. If a null-mode projector is present, colour and spinor indices will mix, and Dirac and colour traces will generally have the form

$$
[\gamma_{a_1} \cdots \gamma_{a_n}]_{\beta_\alpha} \left[ \tau_{\beta_1}^\dagger \tau_{\beta_2} \cdots \tau_{\beta_{m-1}}^\dagger \tau_{\beta_m} \right]_{ji} \left[ \psi_0(x) \right]_{\alpha i} \left[ \psi_0(y) \right]_{\beta j}^* = \left[ \gamma_{a_1} \cdots \gamma_{a_n} \right]_{\beta_\alpha} \left[ \psi_0(x) \right]_{\alpha i} \left[ \tau_{\beta_1}^\dagger \tau_{\beta_2} \cdots \tau_{\beta_{m-1}}^\dagger \tau_{\beta_m} \right]^T \left[ \psi_0(y) \right]_{\beta j}^*,
$$

(3.32)

Using now eq. (3.28), together with the transposition rule

$$
\gamma^T_{\mu} = -\gamma_2 \gamma_4 \gamma_5 \gamma_1,
$$

(3.33)

we can transform our expression into a standard gamma matrix trace:

$$
(-1)^{\frac{\mu}{2}} \text{Tr} \left[ \frac{1}{2} - \gamma_5 \left( \gamma_{a_1} \cdots \gamma_{a_n} \right) \psi_0(x) \gamma_2 \gamma_4 \left( \gamma_{\beta_m} \cdots \gamma_{\beta_1} \right) \gamma_2 \psi_0^\dagger(y) \right]
$$

$$
= \frac{\rho^2}{\pi^2(x^2 + \rho^2)^{\frac{3}{2}}(y^2 + \rho^2)^{\frac{3}{2}}} (-1)^{\frac{\mu}{2}} \text{Tr} \left[ \frac{1}{2} - \gamma_5 \left( \gamma_{a_1} \cdots \gamma_{a_n} \right) \left( \gamma_{\beta_m} \cdots \gamma_{\beta_1} \right) \right].
$$

(3.34)

Notice that the factors $\gamma_2 \gamma_4$ in this formula cancel against the analogous factors appearing in the zero-modes, thus yielding a fully Lorentz invariant result. The spin and colour traces one encounters when computing the quantities $I_{\mu\nu}^{(a)}$ and $I_{\mu\nu}^{(c)}$ are precisely of the above form. In the case when we have two null-mode projectors in the trace, as in the $I_{\mu\nu}^{(b)}$ term, we get instead

$$
\langle \gamma_{a_1} \cdots \gamma_{a_n} \rangle_{\beta_\alpha} \left( \tau_{\beta_1}^\dagger \tau_{\beta_2} \cdots \tau_{\beta_{m-1}}^\dagger \tau_{\beta_m} \right)_{ji} \left[ \psi_0(x) \right]_{\alpha i} \left[ \psi_0(t) \right]_{\beta k} \left[ \psi_0(t) \right]_{\beta k} \left[ \psi_0(y) \right]_{\beta j}^*
$$

$$
= \frac{2 \rho^4 (-1)^{\frac{\mu}{2}}}{\pi^4(t^2 + \rho^2)^{\frac{3}{2}}(x^2 + \rho^2)^{\frac{3}{2}}(y^2 + \rho^2)^{\frac{3}{2}}} \text{Tr} \left[ \frac{1}{2} - \gamma_5 \left( \gamma_{a_1} \cdots \gamma_{a_n} \right) \left( \gamma_{\beta_m} \cdots \gamma_{\beta_1} \right) \right].
$$

(3.35)

With all the machinery developed so far we can perform the calculation using standard algebraic tools. We need to perform the spin and colour trace first. We choose for convenience a frame in which $x = -y = \Delta/2$. In the approximation $|t| \gg |x|, |y|, |z|, \rho$, neglecting terms that have opposite sign for the instanton and for the anti-instanton, we get

$$
I_{\mu\nu}^{(a)} = \frac{\rho^4}{2 \pi^6 \delta_{\mu\nu}} \left[ \left( \frac{1}{d_+^2 d_-^2} + \frac{1}{d_+^2 d_-^2} \right) \left( \Delta^2 \delta_{\mu\nu} - \Delta_\mu \Delta_\nu \right) - \frac{2 \Delta^2}{d_+^2 d_-^2} \Delta_\mu \Delta_\nu \right]
$$

(3.36)
$$I_{\mu\nu}^{(b)} = i_p \frac{\rho^4}{2 \pi^6 t^6 \Delta^2} \delta_{\mu\nu} \left[ - \left( \frac{1}{d_+ d_-^2} + \frac{1}{d_-^2 d_+} \right) + \frac{\Delta^2}{d_+^2 d_-^2} \right] \quad (3.37)$$

$$I_{\mu\nu}^{(c)} = -i_p \frac{\rho^2}{4 \pi^6 t^6} \delta_{\mu\nu} \left[ - \left( \frac{1}{d_+ d_-^2} + \frac{1}{d_-^2 d_+} \right) + \frac{\Delta^2 + 2 \rho^2}{d_+^2 d_-^2} \right], \quad (3.38)$$

where we have defined

$$d_{\pm} = \left( z \pm \frac{\Delta}{2} \right)^2 + \rho^2 \quad (3.39)$$

and $i_p$ is defined to be $1$ for vector-vector and $-1$ for axial-axial currents. We do not need to take into account the region of integration where $|z|$ is of the order of $|t|$, or where $|z - t|$ is of the order of $\rho$. According to our formula (B.1), such corrections will be subtracted away, since in fact they can be viewed as instanton corrections to the expectation value of the operator, and not to the coefficient functions. The integration over the instanton position can be performed as follows. We define the integrals

$$F = \int d^4 z \frac{1}{d_+ d_-}, \quad H = -2 \int d^4 z \frac{1}{d_+ d_-^2}, \quad G = \int d^4 z \frac{1}{d_+^2 d_-^2}. \quad (3.40)$$

The following identities are easily proved

$$H = \frac{\partial F}{\partial \rho^2}, \quad G = -\frac{1}{\rho^2} \frac{\partial}{\partial \Delta^2} \left( \Delta^2 \frac{\partial F}{\partial \Delta^2} \right). \quad (3.41)$$

Observe that $F$ is logarithmically infrared divergent, while $H$ and $G$ are finite. Since $F$ is dimensionless, it will always be possible to express it as a function of the ratio $\Delta/\rho$ plus a term proportional to $\log \frac{\Delta}{L}$, where $L$ is the infrared cutoff. On the other hand we can easily verify that both $H$ and $G$ remain invariant if we add to $F$ a term proportional to $\log \Delta$. Therefore we can always choose $F$ as a function of the ratio $\Delta/\rho$ alone. If we restrict ourselves to this choice we also have

$$H = \frac{\partial F}{\partial \rho^2} = -\frac{\Delta^2}{\rho^2} \frac{\partial F}{\partial \Delta^2}. \quad (3.42)$$

We have therefore

$$I_{\mu\nu}^{(a)} = \frac{1}{2 \pi^6 t^6} \left[ \frac{\partial \rho^4}{\partial \Delta^2} \left( \delta_{\mu\nu} - \frac{\Delta_{\mu} \Delta_{\nu}}{\Delta^2} \right) + 2 \frac{\partial}{\partial \Delta^2} \left( \Delta^2 \frac{\partial \rho^2 F}{\partial \Delta^2} \right) \frac{\Delta_{\mu} \Delta_{\nu}}{\Delta^2} \right]$$

$$I_{\mu\nu}^{(b)} = i_p \frac{1}{2 \pi^6 t^6} \delta_{\mu\nu} \left[ -\frac{\partial \rho^2 F}{\partial \Delta^2} - \frac{\partial}{\partial \Delta^2} \left( \Delta^2 \frac{\partial \rho^2 F}{\partial \Delta^2} \right) \right] \quad (3.43)$$

$$I_{\mu\nu}^{(c)} = -i_p \frac{1}{4 \pi^6 t^6} \delta_{\mu\nu} \left[ -\Delta^2 \frac{\partial F}{\partial \Delta^2} - \Delta^2 \frac{\partial}{\partial \Delta^2} \left( \Delta^2 \frac{\partial F}{\partial \Delta^2} \right) - \frac{\partial}{\partial \Delta^2} \left( \Delta^2 \frac{\partial \rho^2 F}{\partial \Delta^2} \right) \right].$$
We only need to compute $F$. We get

$$ F = \pi^2 \left( - \log \frac{\rho^2}{\Delta^2} + \xi \log \frac{\xi - 1}{\xi + 1} \right) $$

with $\xi = \sqrt{1 + 4\rho^2/\Delta^2}$. The Mellin transform of $F$ is

$$ F_M = \int \rho^M F \, d\rho = -\Delta^{M+1} \frac{\pi^2}{2} \cos \left( \frac{\pi}{2} \left( \frac{M+1}{2} \right) \right) \Gamma^2 \left( \frac{M+1}{2} \right) \Gamma(-M-2) (M+1). $$

Using the formula

$$ \int d^4 \Delta e^{i\Delta \cdot p} \Delta^N = \rho^{-n-4} 4\pi \sin \frac{\pi(N+2)}{2} \frac{\Gamma \left( \frac{N+2}{2} \right) \Gamma(N+4)}{\Gamma \left( \frac{N+5}{2} \right)} $$

it is now easy to perform the Mellin transform and the Fourier transform of our result by a simple algebraic procedure. Defining

$$ \tilde{I}_{\mu\nu}^{(a,b,c)}(M) = \int \rho^M \int d\rho^M I_{\mu\nu}^{(a,b,c)} \quad (3.47) $$

we get

$$ \tilde{I}_{\mu\nu}^{(a)}(M) = -\frac{A(M)}{4 \pi^6} \rho^{-M-7} p_\mu p_\nu $$

$$ \tilde{I}_{\mu\nu}^{(b)}(M) = i \rho \frac{A(M)}{8 \pi^6} \rho^{-M-7} \delta_{\mu\nu} p^2 $$

$$ \tilde{I}_{\mu\nu}^{(c)}(M) = i \rho \frac{A(M)}{8 \pi^6} \rho^{-M-7} \delta_{\mu\nu} p^2 $$

where

$$ A(M) = \pi^4 \frac{\Gamma \left( \frac{3}{2} \right) \Gamma^3 \left( \frac{M+3}{2} \right)}{\Gamma \left( \frac{M+6}{2} \right)}.$$

The calculation of the disconnected term (eq. (3.17)) is quite simple. We have

$$ \text{Tr} \left[ \langle t|S_\alpha^\pm|t \rangle \right] = -\frac{1}{m_d} \text{Tr} \left[ \langle t|P^\pm|t \rangle \right] = -\frac{2\rho^2}{m_d \pi^2 t^6}. $$

The remaining factor $[\Gamma_\mu \langle x|S_\alpha^\pm|y \rangle \Gamma_\nu \langle y|S_\alpha^\pm|x \rangle]$ can be easily obtained by suitably adapting formula (2.28) of ref. 2. We get

$$ \tilde{I}_{\mu\nu}'(M) = -\frac{A(M)}{4 m_d \pi^6 t^6} \left\{ (M+5) \left[ i_F C \delta_{\mu\nu} p^2 - p_\mu p_\nu \right] + (i_F C - 1) \delta_{\mu\nu} p^2 \right\}, $$

(3.53)
where $C = (m_u/m_d + m_d/m_u)/2$. The relative minus sign with respect to ref. [2] is due to the fact that in our case the minus sign of the fermion loop has cancelled against the minus sign of the fermion loop of the tadpole, and the extra factor of $1/2$ accounts for the fact that our expression is not yet summed over the instanton and anti-instanton. We now need to include the instanton density in our result. We follow closely the notation of eq. (3.10) of ref. [2], defining

$$D(\rho) = H \left[ \log \frac{1}{\rho^2 \Lambda^2} \right]^c \rho^{6+\eta},$$

(3.54)

where $H$ and $c$ are given in ref. [2]. In leading logarithmic accuracy (which is the accuracy of our calculation) the logarithmic factor in the integration can be taken out of the integral sign, by simply replacing $\rho$ with $1/p$. We get

$$\int d\xi D(\xi) I_{\mu\nu} = \int d^4z d\rho \sum_{\pm} D(\rho) I_{\mu\nu} = H \left( \log \frac{p^2}{\Lambda^2} \right)^c \frac{A(7)}{4} \rho^{1-14/6} \left[ -\frac{2}{m_d} p_\mu p_\nu + i p \left( \frac{m_u}{m_d^2} + \frac{1}{m_u} \right) p^2 \delta_{\mu\nu} - \frac{2}{m_d} \left( \frac{12}{13} i p C \delta_{\mu\nu} p^2 - p_\mu p_\nu \right) + \frac{1}{13} (i p C - 1) \delta_{\mu\nu} p^2 \right].$$

(3.55)

Observe that in the case $m_u = m_d$ the result is transverse, which is an important check of our calculation. Another important observation has to do with the relative sign of the tadpole term with respect to the rest of the expression. Observe in fact that there is a term proportional to $m_u/m_d^2$ in the expression, which cancels after inclusion of the tadpole term. Since $H$ contains the factor $m_u m_d m_s$, such a term would otherwise give rise to corrections proportional to $m_s m_u/m_d$, so that the chiral limit $m_d, m_u \to 0$ (with $m_s$ fixed) would be undefined, with disastrous consequences for the usual interpretation of chiral symmetry in QCD.

Observe that in intermediate steps of the calculation there are infrared divergences that do not appear in the final answer. In fact, the Mellin transform of $F$, eq. (3.44) is divergent for odd integer $M$. In particular, for three light flavours $M$ turns out to be an odd integer. Our final answer is however finite, because the Fourier transform, eq. (3.46), vanishes when $N$ is an even integer (in fact, the Fourier transform of an even power of $\Delta$ is a derivative of the four-dimensional delta-function of $p$, and it vanishes when $p$ is away from zero). One may therefore doubt that our result may depend upon the regularization method that we implicitly used, which is to continue
our result for non-integer $M$. We should however remember that the IR divergent terms should be subtracted from $I_{\mu\nu}$, according to eq. (2.8). As already discussed in section 2, the subtractions are defined up to finite terms which are polynomials in $\Delta$. Once the subtractions are performed, the Mellin transform will turn out to be infrared finite, and it can therefore be regulated in any way we like. We can compute it for complex $M$, perform the Fourier transform, and then take the appropriate limit for $M$ integer. If we use this procedure the subtraction terms do not survive, because they are polynomials in $\Delta$, and therefore they have zero Fourier transform (for a more detailed discussion see ref. [2]).

Following eq. (3.1), in order to give our final expression for the correction to $\delta C^{(\bar{d}d)}$ we still need to divide by

$$
\langle 0|O_{(\bar{d}d)}(0)\bar{\psi}_d(t)\psi_d(t)|0\rangle = -3 \operatorname{Tr} \left[ \frac{it \cdot \gamma}{2\pi^2 t^4} \frac{-it \cdot \gamma}{2\pi^2 t^4} \right] = \frac{3}{\pi^4 t^6} \quad (3.56)
$$

(the $-3$ comes from the fermion loop and the colour sum). Our final result is then

$$
\delta C^{(\bar{d}d)}_{\mu\nu} = \frac{HA(7)}{78\pi^2} \left( \log \frac{p^2}{\Lambda^2} \right)^c p^{-14} \frac{1}{m_d} \left( \delta_{\mu\nu} p^2 - p_\mu p_\nu \right) . \quad (3.57)
$$

The corresponding expression for $\delta C^{(\bar{u}u)}$ is obtained with the obvious replacement $m_d \rightarrow m_u$. The coefficient $\delta C^{(\bar{s}s)}$ instead receives contributions only from the tadpole term. We have

$$
\delta C^{(\bar{s}s)}_{\mu\nu} = -\frac{HA(7)}{78\pi^2} \left( \log \frac{p^2}{\Lambda^2} \right)^c p^{-14} \frac{1}{m_s} \left[ 12 \left( i p C \delta_{\mu\nu} p^2 - p_\mu p_\nu \right) + \left( i p C - 1 \right) \delta_{\mu\nu} p^2 \right] . \quad (3.58)
$$

The case of flavour diagonal currents can be treated similarly. For a $\bar{d}d$ vector current we get

$$
\delta C^{(\bar{d}d)}_{\mu\nu,dd} = \frac{HA(7)}{78\pi^2} \left( \log \frac{p^2}{\Lambda^2} \right)^c p^{-14} \frac{1}{m_d} \left[ \delta_{\mu\nu} p^2 - p_\mu p_\nu + 13 \left( i p p^2 \delta_{\mu\nu} - p_\mu p_\nu \right) \right] \quad (3.59)
$$

and a similar one for the $\bar{u}u$ contributions. Observe that in the case of axial currents eqs. (3.59) and (3.60) are incomplete, and other contributions must be added. This is because in the flavour diagonal case we could have extra diagrams, in which there
is a tadpole attached to one or both of the currents. It is easy to show that such contributions vanish for the vector currents. In fact, given the tadpole expression

\[ T_\mu = \text{Tr}[\Gamma_\mu \langle x | S_\alpha^+ | x \rangle] \] (3.61)

rotational invariance implies that it must have the form (taking the instanton centre at the origin)

\[ T_\mu = x_\mu \, g(x^2). \] (3.62)

In the case of a vector current we must also have \( \partial_\mu T_\mu = 0 \), which implies immediately \( T_\mu = 0 \), since we cannot have a radial field with no sources. Therefore there are no extra tadpole contributions in the case of vector currents. The divergence of the axial current is instead non-zero, and its value is determined by the anomaly. Since we have no application in mind for the flavour-diagonal axial-current correlator, we will not extend our calculation in order to cover this case.

4. Instanton corrections for higher-dimension condensates

The coefficient functions of operators with four and six quark fields also receive corrections from the instanton. In particular, the coefficient functions of four quark operators can receive corrections of the order of \( m_s \), while the coefficient functions of six-quark operators can receive corrections with no chiral suppression factors at all. Since there are many such operators, and since their expectation values are not known, it would be useless to determine the corresponding correction to each coefficient function. In ref. [5] the simplifying assumption was made that the expectation value of six quark operators factorizes in terms of the \( \sum \bar{\psi}_f \psi_f \) quark condensate. Under this assumption, we do not need to determine the corrections to each coefficient function independently. In general, in order to compute the effect of \( 2n_f \) quark condensates in the factorization approximation it is enough to compute eq. (2.8) with \( P = (\sum \bar{\psi}_f \psi_f)^{n_f} \) in order to generate on the left-hand side of the equation the combination of operators appropriate to the factorization hypothesis. In other words, we use the operator \( P \) as the source of a factorized expectation value for \( 2n_f \) quark operators. We then observe that

\[ \langle 0 | (\prod_f \bar{\psi}_f(0) \psi_f(0)) (\sum_f \bar{\psi}_f(t) \psi_f(t))^{n_f} | 0 \rangle = n_f! \left( \frac{3}{4 \pi \alpha} \right)^{n_f} \] (4.1)

while on the physical vacuum, according to the factorization hypothesis

\[ \langle \text{vac} | \prod_f \bar{\psi}_f(0) \psi_f(0) | \text{vac} \rangle = (\bar{q}q)^{n_f}, \] (4.2)
where we have assumed a flavour-symmetric vacuum expectation value

$$\langle \text{vac}|\bar{\psi}_f(0)\psi_f(0)|\text{vac} \rangle = \langle \bar{q}q \rangle$$  \hspace{1cm} (4.3)$$

independent of $f$. In order to get the correction to the correlator coming from $n_f$ quark condensates in the factorization approximation we should therefore multiply our result by

$$\frac{1}{n_f!} \left( \frac{\langle \bar{q}q \rangle \pi^4 t^6}{3} \right)^{n_f}.$$  \hspace{1cm} (4.4)

In order to simplify the calculation, one should keep in mind that terms proportional to the inverse of a quark mass cannot appear with a power greater than 1, since they must cancel against the factor of a quark mass in the instanton density. This fact is rather general, and it has to do with the fermionic nature of the zero modes. The contributing terms are given by the tadpole-type term

$$K^{(a)}_{\mu\nu} = - \text{Tr}[\Gamma_\mu \langle x | S_0^+ | y \rangle \Gamma_\nu \langle y | S_0^+ | x \rangle] \times \frac{6}{m_u m_d m_s} \text{Tr}[\langle t | P | t \rangle_u] \text{Tr}[\langle t | P | t \rangle_d] \text{Tr}[\langle t | P | t \rangle_s]$$  \hspace{1cm} (4.5)

(where the 6 is a combinatoric factor) and the following connected contributions

$$K^{(b)}_{\mu\nu} = \frac{1}{m_d} \text{Tr}[\langle t | P | x \rangle \Gamma_\mu \langle x | S_0^+ | y \rangle \Gamma_\nu \langle y | S_0^+ | t \rangle] \times \frac{6}{m_u m_s} \text{Tr}[\langle t | P | t \rangle_u] \text{Tr}[\langle t | P | t \rangle_s]$$

$$+ \frac{1}{m_d} \text{Tr}[\langle t | S_0^+ | x \rangle \Gamma_\mu \langle x | S_0^+ | y \rangle \Gamma_\nu \langle y | P | t \rangle] \times \frac{6}{m_u m_s} \text{Tr}[\langle t | P | t \rangle_u] \text{Tr}[\langle t | P | t \rangle_s]$$

$$+ \frac{1}{m_u} \text{Tr}[\langle t | S_0^+ | y \rangle \Gamma_\nu \langle y | S_0^+ | x \rangle \Gamma_\mu \langle x | S_0^+ | t \rangle] \times \frac{6}{m_d m_s} \text{Tr}[\langle t | P | t \rangle_d] \text{Tr}[\langle t | P | t \rangle_s]$$

$$+ \frac{1}{m_u} \text{Tr}[\langle t | S_0^+ | y \rangle \Gamma_\nu \langle y | S_0^+ | x \rangle \Gamma_\mu \langle x | P | t \rangle] \times \frac{6}{m_d m_s} \text{Tr}[\langle t | P | t \rangle_d] \text{Tr}[\langle t | P | t \rangle_s]$$

$$K^{(c)}_{\mu\nu} = \frac{1}{m_u} \text{Tr}[\langle t | S_0^+ | y \rangle \Gamma_\nu \langle y | S_0^+ | P | x \rangle \Gamma_\mu \langle x | S_0^+ | t \rangle] \times \frac{6}{m_d m_s} \text{Tr}[\langle t | P | t \rangle_d] \text{Tr}[\langle t | P | t \rangle_s]$$

$$+ \frac{1}{m_d} \text{Tr}[\langle t | S_0^+ | y \rangle \Gamma_\nu \langle y | P | x \rangle \Gamma_\mu \langle x | S_0^+ | t \rangle] \times \frac{6}{m_u m_s} \text{Tr}[\langle t | P | t \rangle_u] \text{Tr}[\langle t | P | t \rangle_s]$$

$$K^{(d)}_{\mu\nu} = - \frac{2}{m_u m_d} \text{Tr}[\langle t | S_0^+ | x \rangle \Gamma_\mu \langle x | P | y \rangle \Gamma_\nu \langle y | S_0^+ | t \rangle \langle t | P | t \rangle_d] \times \frac{3}{m_s} \text{Tr}[\langle t | P | t \rangle_s]$$

$$- \frac{2}{m_u m_d} \text{Tr}[\langle t | S_0^+ | y \rangle \Gamma_\nu \langle y | P | x \rangle \Gamma_\mu \langle x | S_0^+ | t \rangle \langle t | P | t \rangle_u] \times \frac{3}{m_s} \text{Tr}[\langle t | P | t \rangle_s]$$

$$K^{(e)}_{\mu\nu} = - \frac{2}{m_u m_d} \text{Tr}[\langle t | S_0^+ | x \rangle \Gamma_\mu \langle x | P | t \rangle_u \langle t | S_0^+ | y \rangle \Gamma_\nu \langle y | P | t \rangle_d] \times \frac{3}{m_s} \text{Tr}[\langle t | P | t \rangle_s]$$
\[- \frac{2}{m_u m_d} \text{Tr}[\langle t | P | x \rangle \Gamma_\mu(x) S_0^\pm(t) \langle t | P | y \rangle \Gamma_\nu(y) S_0^\pm(t) \rangle] \times \frac{3}{m_s} \text{Tr}[\langle t | P | t \rangle_s] \]

(4.6)

The suffixes \( u, d, s \) are shown as a reminder of the flavour flowing in the line. When a projector \( P \) is present in the spin-colour trace, the trace can be split into expectation values over zero modes, by replacing \( P \to \psi \psi^\dagger \). Defining as usual

\[ \tilde{K}_{\mu\nu}^{(a...c)}(M) = \int d^4 \Delta e^{i\Delta p} \int d\rho \, \rho^M K_{\mu\nu}^{(a...c)} \]  

(4.7)

we get

\[
\begin{align*}
K_{\mu\nu}^{(a)} &= -\frac{A(M')}{4 \pi^6 t^6 (M' + 6)} \{ -(M' + 5)p_\mu p_\nu - \delta_{\mu\nu} p^2 \} \times \frac{6}{m_u m_d m_s} \left( \frac{2}{\pi^2 t^6} \right)^2 \\
K_{\mu\nu}^{(b)} &= -\frac{A(M')}{4 t^6 \pi^6} p^{-M' - 7} p_\mu p_\nu \times \frac{12}{m_u m_d m_s} \left( \frac{2}{\pi^2 t^6} \right)^2 \\
K_{\mu\nu}^{(c)} &= i_p \frac{A(M')}{8 t^6 \pi^6} p^{-M' - 7} \delta_{\mu\nu} p^2 \times \frac{12}{m_u m_d m_s} \left( \frac{2}{\pi^2 t^6} \right)^2 \\
K_{\mu\nu}^{(d)} &= -i_p \frac{A(M')}{8 t^6 \pi^6} p^{-M' - 7} p_\mu p_\nu \times \frac{12}{m_u m_d m_s} \left( \frac{2}{\pi^2 t^6} \right)^2 \\
K_{\mu\nu}^{(e)} &= \frac{A(M')}{8 t^6 \pi^6} p^{-M' - 7} p_\mu p_\nu \times \frac{12}{m_u m_d m_s} \left( \frac{2}{\pi^2 t^6} \right)^2,
\end{align*}
\]

(4.8)

where now \( M' = M + 4 \) (because of the four extra powers of \( \rho \) coming from the tadpole terms). The term \( K_{\mu\nu}^{(a)} \) was obtained from eq. (3.53) suppressing the term proportional to \( C \), while \( K_{\mu\nu}^{(b)} \) and \( K_{\mu\nu}^{(c)} \) are taken from eqs. (3.48) and (3.50). Only the terms \( \tilde{K}_{\mu\nu}^{(d)} \) and \( \tilde{K}_{\mu\nu}^{(e)} \) have a form different from that of the previously computed terms. Their calculation can however be easily performed using the techniques previously described. Adding up all the contributions we get

\[
\sum_{x=a...e} \tilde{K}^{(x)} = -\frac{A(M')}{(m_u m_d m_s) \pi^{10} t^{18}} \frac{6(M' + 7)}{M' + 6} \times \left[ 1 + \frac{M' + 6}{M' + 7} (i_p - 1) \right] (p_\mu p_\nu - \delta_{\mu\nu} p^2).
\]

(4.9)

Taking now into account the normalization factor eq. (4.4) and the instanton density, including a factor of 2 for the instanton anti-instanton contributions, we get the correction to the vacuum polarization coming from six quark operators in the
factorization approximation

\[ \delta I^{(\bar{q}q)^3}_{\mu\nu} = -H \left( \log \frac{p^2}{\Lambda^2} \right)^c \frac{4 \pi^2 A(11) p^{-18} \langle \bar{q}q \rangle^3}{51} \frac{m_u m_d m_s}{m_{\bar{q}} m_{\bar{q}}} \times \left[ 1 + \frac{45}{16}(i_p - 1) \right] \left( p_\mu p_\nu - \delta_{\mu\nu} p^2 \right). \]  

(4.10)

We observe that with no further effort we can also write down at this point the correction to the vacuum polarization coming from four-quark operators involving \( u \) and \( d \) quarks only in the factorization approximation. This is a sensible thing to consider, since the strange quark mass is not so small, and one may doubt that a factor of \( 1/m_s \) can provide a sensible chiral enhancement. We get

\[ \delta I^{(\bar{q}q)^2}_{\mu\nu} = -H \left( \log \frac{p^2}{\Lambda^2} \right)^c \frac{16 A(9) p^{-16} \langle \bar{q}q \rangle^2}{135} \frac{m_u m_d}{m_{\bar{q}} m_{\bar{q}}} \times \left[ 1 + \frac{45}{16}(i_p - 1) \right] \left( p_\mu p_\nu - \delta_{\mu\nu} p^2 \right). \]  

(4.11)

5. Final results

It is now time to collect all our results. We should remember that the quark masses appearing in our formula are running masses, given by

\[ m(\mu) = \hat{m} \left( \log \frac{\mu}{\Lambda} \right)^{-\frac{12}{33-2n_f}}. \]  

(5.1)

The condensate is also renormalization-group dependent. We have

\[ \langle \bar{q}q \rangle = -\hat{\mu}^3 \left( \log \frac{\mu}{\Lambda} \right)^{\frac{12}{33-2n_f}}. \]  

(5.2)

In terms of the renormalization-group-invariant quantities, we get

\[ \delta \Pi^{(\bar{q}q)}_{\mu\nu} = -\frac{H_0 A(7)}{78 \pi^2} 2^{-\frac{24}{33-2n_f}} \left( \log \frac{p^2}{\Lambda^2} \right)^{c+\frac{24}{33-2n_f}} \left( \frac{\Lambda^9 \hat{m}_u \hat{m}_d \hat{m}_s \hat{\mu}^3}{p^9} \right) \times \left\{ \left( \frac{1}{\hat{m}_u} + \frac{1}{\hat{m}_d} + \frac{1}{\hat{m}_s} \right) \left( \delta_{\mu\nu} p^2 - p_\mu p_\nu \right) - \frac{13}{\hat{m}_s} \left( i_p C \delta_{\mu\nu} p^2 - p_\mu p_\nu \right) \right\} \]  

(5.3)

\[ \delta \Pi^{(\bar{q}q)^2}_{\mu\nu} = -\frac{16 H_0 A(9)}{135} 2^{-\frac{48}{33-2n_f}} \left( \log \frac{p^2}{\Lambda^2} \right)^{c+\frac{48}{33-2n_f}} \left( \frac{\Lambda^9 \hat{m}_s \hat{\mu}^6}{p^9} \right) \times \left[ 1 + \frac{45}{16}(i_p - 1) \right] \left( p_\mu p_\nu - \delta_{\mu\nu} p^2 \right). \]  

(5.4)
\[ \delta \Pi^{(qq)}_{\mu\nu} = \frac{4\pi^2 H_0 A(11)}{51} 2^{-\frac{72}{33-2n_f}} \left( \log \frac{p^2}{\Lambda^2} \right)^{c+\frac{72}{33-2n_f}} \left( \frac{\Lambda_9^9}{p^9} \frac{\hat{\mu}^9}{p^9} \right) \times \left[ 1 + \frac{17}{19}(i_p - 1) \right] \left( p_{\mu} p_{\nu} - \delta_{\mu\nu} p^2 \right), \]  

(5.5)

where

\[ H_0 = \frac{2}{\pi^2} \left( \frac{33 - 2n_f}{12} \right) \frac{2^{\frac{12n_f}{33-2n_f}}}{2^{\frac{12n_f}{33-2n_f}}} \exp \left[ -\alpha(1) + \frac{1}{2} + (2n_f - 2)\alpha \left( \frac{1}{2} \right) \right] \]

\[ c = \frac{45 - 5n_f}{33 - 2n_f} \]

\[ C = \frac{1}{2} \left( \frac{m_u}{m_d} + \frac{m_d}{m_u} \right) \]

\[ i_p = 1 \text{ for vector currents, } -1 \text{ for axial currents.} \]  

(5.6)

For \( n_f = 3 \), performing the replacements \( p^2 \rightarrow -p^2 \) and \( \delta_{\mu\nu} \rightarrow -g_{\mu\nu} \) in order to go to the Minkowski space, our formula for the current-current correlator, including instanton corrections is then

\[ \Pi_{\mu\nu}(p^2) = \frac{1}{4\pi^2} \left\{ - (p_{\mu} p_{\nu} - p^2 g_{\mu\nu}) \log(-p^2) - \frac{\hat{m}_u \hat{m}_d \hat{m}_s}{p^3} \left( \frac{5.1701 \Lambda}{p} \right)^9 \left[ \log \frac{-p^2}{\Lambda^2} \right]^\frac{10}{9} \times \left[ \left( \frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s} \right) \left( p_{\mu} p_{\nu} - g_{\mu\nu} p^2 \right) + \frac{13}{m_s} (i_p C g_{\mu\nu} p^2 - p_{\mu} p_{\nu}) \right] \right. \]

\[ \left. - \frac{\hat{m}_s \hat{\mu}^6}{p^7} \left( \frac{14.446 \Lambda}{p} \right)^9 \left[ \log \frac{-p^2}{\Lambda^2} \right]^\frac{26}{9} (1 + \frac{17}{19}(i_p - 1)) \left( p_{\mu} p_{\nu} - g_{\mu\nu} p^2 \right) \right. \]

\[ \left. - \frac{\hat{\mu}^9}{p^9} \left( \frac{25.370 \Lambda}{p} \right)^9 \left[ \log \frac{-p^2}{\Lambda^2} \right]^\frac{34}{9} (1 + \frac{17}{19}(i_p - 1)) \left( p_{\mu} p_{\nu} - g_{\mu\nu} p^2 \right) \right\}, \]  

(5.7)

where we have also included for completeness the term computed in ref. \[4\]. The last term of our result agrees with ref. \[3\], except that the expression they use for the instanton density, taken from ref. \[9\], does not agree with ours. The instanton density for \( SU(N) \) was computed in ref. \[10\] in the Pauli-Villaars scheme, and the conversion to the \( \bar{M}S \) scheme was performed in refs. \[3\] and \[11\] (for a more detailed discussion see also ref. \[2\]).
In order to estimate the instanton corrections, we choose the following values for the various parameters

\[ \hat{m}_u = 8.7 \text{ MeV}, \quad \hat{m}_d = 15.4 \text{ MeV}, \quad \hat{m}_s = 283 \text{ MeV}, \]
\[ \hat{\mu} = 180 \text{ MeV}, \quad \Lambda_3 = 400 \text{ MeV}. \] (5.8)

We find then

\[ \Pi_{\mu\nu}(p^2) = \frac{1}{4\pi^2} \left\{ - (p_\mu p_\nu - p^2 g_{\mu\nu}) \log(-p^2) - \left( \frac{0.738}{p} \right)^{12} \left[ \log \left( \frac{-p^2}{\Lambda^2} \right) \right]^{10} \right. \]
\[ \times \left[ \left( \frac{15}{16} i_p C - \frac{1}{16} \right) (p^2 g_{\mu\nu} - p_\mu p_\nu) + \frac{15}{16} (i_p C - 1) p_\mu p_\nu \right] \]
\[ - \left( \frac{0.8861}{p} \right)^{14} \left[ \log \left( \frac{-p^2}{\Lambda^2} \right) \right]^{26} \]
\[ \times \left[ (p_\mu p_\nu - g_{\mu\nu} p^2) + 0.25 \left( i_p C g_{\mu\nu} p^2 - p_\mu p_\nu \right) \right] \]
\[ - \left( \frac{1.303}{p} \right)^{16} \left[ \log \left( \frac{-p^2}{\Lambda^2} \right) \right]^{26} \left( 1 + \frac{15}{16} (i_p - 1) \right) (p_\mu p_\nu - g_{\mu\nu} p^2) \]
\[ - \left( \frac{1.352}{p} \right)^{18} \left[ \log \left( \frac{-p^2}{\Lambda^2} \right) \right]^{34} \left( 1 + \frac{15}{16} (i_p - 1) \right) (p_\mu p_\nu - g_{\mu\nu} p^2) \right\}. \] (5.9)

This result deserves a few comments. First of all, we see that the six-quark correction is of the same order of magnitude as the four-quark correction, they both become of order 1 at a scale of 1.3 GeV. The two-quark correction becomes of order 1 at a scale of 0.9 GeV, while the vacuum correction is important at a lower scale of 0.74. This is consistent with the assumption that QCD is very near the \( SU(2) \) chiral limit, but only marginally near the \( SU(3) \) chiral limit. Therefore corrections suppressed by powers of \( m_u \) or \( m_d \) are indeed small, while those suppressed by powers of \( m_s \) are not necessarily small. We also observe that although the four- and six-quark corrections are larger than the vacuum and two-quark corrections, they do not always dominate. In fact they are purely transverse, and they do not enter in sum rules involving the divergence of the current.

The four- and six-quark corrections have been obtained by assuming factorization, while this assumption was not needed for the two-quark correction. We should therefore consider the latter as being on more solid ground than the former. Nevertheless, it is fair to assume that the corrections computed in the factorization approximation
should at least give an order-of-magnitude estimate of the effect. We also notice that
the six quark correction has the same power behaviour of the two instanton correction
(see ref. \cite{ref}).

In ref. \cite{ref} the integration over the instanton density is performed with a method
that includes also effects subleading by powers of logarithms of \( p^2 / \Lambda^2 \). In practice
these effects lower the effective scale at which the logarithms are computed. We
preferred instead to use only the leading logarithmic expression. In fact, if \( p \gg \Lambda \) one
may indeed get a better estimate of the effect by using the method of ref. \cite{ref}. However,
in the case when the effective scale is too close to \( \Lambda \), the logarithm approaches zero,
and it therefore yields a suppression that does not have any physical basis. In other
words, we should regard our logarithmic factors as expression that do in fact approach
a logarithm for large values of their argument, but become of order 1 when the
argument is of order 1.

In the case of \( V - A \) currents, the terms proportional to \( i_\rho \) disappear, and in the
four-quark and six-quark corrections a further suppression arises, which can be viewed
as a cancellation between the axial and vector current effects. As noticed in ref. \cite{ref}
such cancellation could be an artefact of the factorization approximation. In the
present work we prefer to take the results in the factorization approximation at their
face value, in order to avoid making too many uncontrolled assumptions. Similarly,
we completely neglect the fact that the factorization hypothesis is inconsistent with
the renormalization group equation.

6. Phenomenological results

In this section we will discuss some phenomenological applications of our calcu-
lation. First of all, we would like to stress the basic difference between the formulae
for instanton corrections and standard perturbative formulae. Instanton corrections
are proportional to a power of \( \Lambda \). Therefore, at the leading logarithmic level they
are defined up to a multiplicative constant of order 1. At the next-to-leading or-
der level, the prefactor is defined only in a leading logarithmic sense. The powers
of \( \log \frac{q^2}{\Lambda^2} \) appearing in the prefactor can therefore be written in various ways, differing
by subleading logarithmic terms. For example, the logarithm could be replaced
by \( 1 / (b_0 \alpha_s(q^2)) \), and then the two-loop form of the running coupling could be used.
These kind of changes should only produce small variations of the answer, if the scale
\( q \) is large enough for the leading logarithmic approximation to be working. If a large variation is found, this should be taken as an indication that subleading logarithmic corrections are large, and the result of the calculation should then only be taken as an estimate of the effect.

The hadronic width of the \( \tau \) lepton has been used extensively for a determination of the strong coupling constant \( \alpha_S \). Since the \( \tau \) mass is only marginally large compared to typical hadronic scales, one should make sure that non-perturbative effects in \( \tau \) hadronic decays are under control. The \( \tau \) hadronic width is easily expressed in terms of the current-current correlators of axial and vector currents

\[
R_\tau = 6\pi i \oint_{|z|=1} dz (1 - z)^2 \left[ (1 + 2z)\Pi_{A+V}^{(T)}(p^2) + \Pi_{A+V}^{(L)}(p^2) \right],
\]

(6.1)

where \( z = p^2/M_\tau^2 \), and

\[
\Pi_{A+V}^{\mu\nu} = \Pi_{A+V}^{(T)}(p^2)(p^\mu p^\nu - p^2 g^{\mu\nu}) + \Pi_{A+V}^{(L)}(p^2)p^\mu p^\nu.
\]

(6.2)

For the purpose of illustration we will now fix our reference values for the quark masses and condensates to those given in eq. (5.8). Defining

\[
R_\tau = R_\tau^{(0)} \left( 1 + \delta_I + \delta_I^{(\bar{q}q)} + \delta_I^{(\bar{q}q)^2} + \delta_I^{(\bar{q}q)^3} + \ldots \right)
\]

(6.3)

(we do not include other QCD corrections in this formula) from formula (6.1) and eq. (5.7) we obtain the following parametrization of the instanton corrections to \( R_\tau \)

\[
\delta_I = \frac{1}{(b_0\alpha_S(m_\tau))^{1/4}} \left( \frac{0.977^{+0.005}_{-0.006} \Lambda_3}{m_\tau} \right)^9 \delta_I^{(\bar{q}q)} = -\frac{1}{(b_0\alpha_S(m_\tau))^{1/4}} \left( \frac{1.39^{+0.01}_{-0.00} \Lambda_3}{m_\tau} \right)^9
\]

\[
\delta_I^{(\bar{q}q)^2} = -\frac{1}{(b_0\alpha_S(m_\tau))^{1/4}} \left( \frac{1.70^{+0.02}_{-0.02} \Lambda_3}{m_\tau} \right)^9 \delta_I^{(\bar{q}q)^3} = -\frac{1}{(b_0\alpha_S(m_\tau))^{1/4}} \left( \frac{1.59^{+0.00}_{-0.00} \Lambda_3}{m_\tau} \right)^9
\]

(6.4)

where the numbers in parenthesis are only slightly sensitive to \( \Lambda_3 \). Their central value was computed for \( \Lambda_3 = 400 \text{ MeV} \), while the upper (lower) variations correspond to \( \Lambda_3 = 500 \text{ MeV} \) (\( \Lambda_3 = 300 \text{ MeV} \)). The range chosen for \( \Lambda_3 \) corresponds to \( \alpha_S(M_Z) \) values of 0.113, 0.119 and 0.125, which is a reasonable representation of the present uncertainties. These results were obtained by replacing \( \log(q^2/\Lambda^2) \) with \( 1/(b_0\alpha_S(q^2)) \), and then using the two-loop formula for the strong coupling constant in the complex plane integration. Numerical values for the corrections are given in table I, where the
values of the corrections obtained by using the one-loop form of the strong coupling constant are also shown in parenthesis for comparison.

The corrections coming from the four- and six-quark condensates are by far the largest. They can reach the 2% level for the high end of the range of $\Lambda_3$. This result is in qualitative agreement with ref. [5] (we cannot expect exact agreement since we differ by subleading effects in our estimates) up to a sign, for which we have no explanation, since our analytical results do agree in sign. The four-quark condensate correction is of the same order as the six-quark correction. From these results we may conclude that instanton effects are not very important for the hadronic $\tau$ decay, and that they are smaller than other sources of uncertainty.

In ref. [3] it was shown that instanton effects in the divergence of the correlator of two axial currents are large enough to spoil the standard methods to determine light quark masses from QCD some rules[13]. The newly computed correction to the coefficient of the chiral condensate also contributes to these sum rules. The relevant quantity is

$$\Psi_5(p^2) = p_\mu p_\nu \Pi^{\mu\nu}. \quad (6.5)$$

With a straightforward generalization of the formalism of ref. [3], defining

$$m = \frac{m_u + m_d}{2} = \frac{\hat{m}_u + \hat{m}_d}{2} \left( \log \frac{p}{\Lambda} \right)^{-\frac{12}{3n_f-2n_t}} \quad (6.6)$$

| $\Lambda_3$ (MeV) | 300          | 400          | 500          |
|-------------------|-------------|-------------|-------------|
| $\delta_I$        | $0.105 \times 10^{-6}$ ($0.87 \times 10^{-7}$) | $0.141 \times 10^{-5}$ ($0.11 \times 10^{-5}$) | $0.84 \times 10^{-5}$ ($0.106 \times 10^{-4}$) |
| $\delta_I^{(qq)}$ | $-0.99 \times 10^{-3}$ ($-0.60 \times 10^{-5}$) | $-0.121 \times 10^{-3}$ ($-0.70 \times 10^{-4}$) | $-0.839 \times 10^{-3}$ ($-0.46 \times 10^{-3}$) |
| $\delta_I^{(qq)^2}$ | $-0.247 \times 10^{-3}$ ($-0.10 \times 10^{-3}$) | $-0.263 \times 10^{-2}$ ($-0.95 \times 10^{-3}$) | $-0.161 \times 10^{-1}$ ($-0.51 \times 10^{-2}$) |
| $\delta_I^{(qq)^3}$ | $-0.615 \times 10^{-3}$ ($-0.14 \times 10^{-3}$) | $-0.524 \times 10^{-2}$ ($-0.82 \times 10^{-3}$) | $-0.249 \times 10^{-1}$ ($-0.16 \times 10^{-2}$) |

Table 1: Instanton corrections to the $\tau$ hadronic width
we obtain:

\[
\frac{1}{\pi} \text{Im}\Psi_5(p^2) = \frac{3}{2\pi^2} p^2 m^2 \left\{ 1 - \frac{11}{30} \left( \log \frac{p}{\Lambda} \right)^{\frac{9}{2}} \frac{\tilde{m}_q}{q} \frac{1}{\pi} \text{Im} \left[ \left( \log \frac{-p^2}{\lambda^2} \right)^{\frac{11}{2}} \right] 
\right.
- \frac{52}{3} \left( \frac{6.371 \Lambda}{p} \right)^{9} \frac{\tilde{\mu}^3}{p^3} \left( \log \frac{p}{\Lambda} \right)^{\frac{17}{2}} \right\},
\]

(6.7)

which extends the result of ref. [3]. The contributions of the various terms of eq. (6.7) to the finite energy sum rule is given by

\[
\frac{1}{2\pi i} \oint_{|s|=s_0} ds \text{Im}\Psi_5(s) = \frac{3}{4\pi^2} s_0^2 m^2 \left\{ 1 + R(s_0) + T(s_0) \right\}
\]

(6.8)

where \( R(s_0) \) is given in ref. [3], and

\[
T(s_0) = \frac{26}{3} \left( \frac{5.899 \Lambda}{\sqrt{s_0}} \right)^{9} \frac{\tilde{\mu}^3}{\sqrt{s_0}} \left( \log \frac{\sqrt{s_0}}{\Lambda} \right)^{\frac{17}{2}}.
\]

(6.9)

It is easy to see that with \( \tilde{\mu} = 180 \text{ MeV} \), \( T(s_0) \) is of order 1 for \( \sqrt{s_0} \simeq 4.1 \Lambda \). For \( \Lambda = 400 \text{ MeV} \) the instanton correction equals 1 already for \( s_0 \simeq 2.7 \text{ GeV}^2 \) (while \( R(s_0) \) becomes 1 for \( s_0 \simeq 2.2 \text{ GeV}^2 \)).

7. Conclusions

In this paper we have presented a calculation of instanton corrections to the coefficient function of chiral condensate operators in the operator product expansion of the correlator of flavour non-diagonal axial or vector currents. The correction to the coefficient of the two-quark chiral condensate, together with the instanton correction due to four-quark condensates in the factorization approximation is a new result of this work. We also repeated the calculation of the instanton correction due to six-quark condensates in the factorization approximation, and found agreement with ref. [4]. Our formula (5.7) contains all the leading one instanton corrections to the correlator of off-diagonal vector or axial currents.

The pattern of the corrections we found confirms the chiral nature of QCD. Thus, corrections proportional to the quark masses are smaller than corrections proportional to condensates, with the possible exception of corrections proportional to the strange quark mass and to the square of the chiral condensate, which may become larger than corrections proportional to the third power of the chiral condensate.
We applied our result to the calculation of the $\tau$ hadronic width, where we found negligible corrections, and to the finite energy sum rules used to determine the light quark masses, where we found instead large corrections. **Acknowledgements**

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