Quasi-spin Model for Macroscopic Quantum Tunnelling between Two Coupled Bose-Einstein Condensates

Chaohong Lee

Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences, Wuhan 430071, P. R. China, and
Max Planck Institute for the Physics of Complex Systems, Noethnitzer Str. 38, D-01187 Dresden, Germany

Wenhua Hai

Department of Physics, Hunan Normal University, Changsha 410081, P. R. China

Xueli Luo, Lei Shi, and Kelin Gao

Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences, Wuhan 430071, P. R. China

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Abstract

The system of two coupled Bose-Einstein condensates is mapped onto a uniaxial spin with an applied magnetic field. The mean-field interaction, the coupling and the asymmetry or the detuning correspond to the anisotropy, the transverse field, and the longitudinal field, respectively. A generalized Bloch equation is derived. In the low barrier limit for the quasi-spin model, the tunneling rate is analyzed with an imaginary-time path-integral method. The dependence of the tunneling rate on the system parameters is obtained.

*Corresponding author: chlee@mpipks-dresden.mpg.de and chlee.cn@hotmail.com.
The crossover temperature $T_c$ from the thermal regime to the quantum regime is estimated. Below $T_c$ quantum tunnelling prevails, otherwise thermal activation dominates.
I. INTRODUCTION

The experimental realization of measuring the relative phase and the population oscillation between coupled Bose-Einstein condensates (BECs) stimulates great interest in investigating their macroscopic quantum tunnelling dynamics [1–3]. There are two different types of atomic tunnelling between coupled BECs, external tunnelling and internal tunnelling [2,3]. The former has different spatially separated single-particle states in a double-well or multi-well potential and the latter has different hyperfine internal states in a single-well potential. For external tunnelling, the phase interference between BECs confined in a multi-well potential has been observed [4,5]; the experimental observation of the tunnelling among BECs confined in multi-well potential has also been reported [6–8]. For internal tunnelling, JILA realized a two-component BEC coupled with Raman pulses [9], MIT observed the tunnelling across spin domains in BECs [10,11], and LENS reported the current-phase dynamics in two weakly coupled BECs trapped in different Zeeman states [12].

With the proceeding of the experimental exploration, a lot of theoretical investigation was performed simultaneously. Williams et al. demonstrated the existence of Josephson tunnelling in a driven two-state single-particle BEC in a single-well trap potential [13]. Kasamatsu et al. investigated theoretically the existence of a metastable state and the possibility of decay to the ground state through macroscopic quantum tunnelling in two-component BECs with repulsive interactions [14]. Smerzi et al. studied the coherent atomic tunneling and population oscillations between two zero-temperature BEC’s confined in a double-well potential [15–18]. Macroscopic quantum self-trapping (MQST), namely a self-maintained population imbalance with nonzero average value of the fractional population imbalance, and $\pi$—phase oscillations in which the time averaged value of the phase difference is equal to $\pi$ were detailed in Refs. [15,16]. The authors of Ref. [17] claim that interaction with a thermal cloud will damp all different oscillations to the zero-phase mode. In addition, macroscopic quantum fluctuations have also been discussed by using second-quantization approaches [18,19]. Within the time-dependent potential, chaotic population tunnelling
emerges. Abdullaev and Kraenkel analyzed the nonlinear resonances and chaotic oscillations of the fractional population imbalance between two coupled BEC’s in a double-well trap with a time-dependent tunneling amplitude for different damping [20]. They also considered the chaotic atomic population resonances and the possibility of stabilization of the unstable-mode regime in coupled BEC’s with oscillating atomic scattering length [21]. In a previous paper, we investigated the chaotic and frequency-locked population oscillation between two coupled BECs [22].

Although many papers appear in the field of the tunnelling between coupled BECs, because of the nonlinearity in the Gross-Pitaevskii equation (GPE), few of them address the question of calculating the tunnelling rate and the crossover temperature between different tunnelling regimes. However, the tunnelling rate and the crossover temperature of the spin systems have been studied systematically with the imaginary-time path-integral method, including models with applied magnetic field [24–30] and without [31–33]. For a two-state system described with linear Schrödinger equation, it is easy to visualize the effects of coupling between two states by introducing Bloch’s spin vector formalism [23]. Can we introduce a generalized Bloch vector for two coupled BECs described with the nonlinear Schrödinger equation to map it onto a spin system, and then calculate the tunneling rate and the crossover temperature with the imaginary-time path-integral method? If the coupled BECs is equivalent to a spin system, the tunnelling process is related to the decay of the metastable MQST state to the ground state. More interestingly, the crossover temperature corresponds to the transition from the classical or mean-field regime to the second quantization regime. In the next section, by introducing a generalized Bloch spin vector, the coupled BECs is mapped onto an uniaxial spin with an applied magnetic field. In section III, the tunneling rate is calculated with the imaginary-time path-integral method, and the crossover temperature is estimated. In the last section, a brief discussion and summary is given.
II. QUASI-SPIN MODEL FOR TWO COUPLED BOSE-EINSTEIN CONDENSATES

Consider the experiments of JILA [9], two Bose-Einstein condensates in the $|F = 1, m_F = -1 >\rangle |1 >$ and $|F = 2, m_F = 1 >\rangle |2 >$ spin states of $^{87}Rb$ are coupled by a two-photon pulse with the two-photon Rabi-frequency $\Omega$ and a finite detuning $\delta = \omega_d - \omega_{hf}$. Where, $\omega_d = \omega_1 + \omega_2$ is the driven frequency of the two-photon pulses, $\omega_{hf}$ is the transition frequency between two hyperfine states. In the rotating frame, ignoring the damping and the finite-temperature effects, the coupled two-component BEC system can be described by a pair of coupled GPEs

\begin{align}
  i\hbar \frac{\partial \Psi_2(\vec{r},t)}{\partial t} &= (H_2^0 + H_2^{MF} - \frac{\hbar \delta}{2}) \Psi_2(\vec{r},t) + \frac{\hbar \Omega}{2} \Psi_1(\vec{r},t), \\
  i\hbar \frac{\partial \Psi_1(\vec{r},t)}{\partial t} &= (H_1^0 + H_1^{MF} + \frac{\hbar \delta}{2}) \Psi_1(\vec{r},t) + \frac{\hbar \Omega}{2} \Psi_2(\vec{r},t),
\end{align}

where, the free evolution Hamiltonians $H_i^0 = -\frac{\hbar^2 \nabla^2}{2m} + V_i(\vec{r})$ ($i = 1, 2$) and the mean-field interaction Hamiltonians $H_i^{MF} = \frac{4\pi \hbar^2}{m} (a_{ii} |\Psi_i(\vec{r},t)|^2 + a_{ij} |\Psi_j(\vec{r},t)|^2)$ ($i, j = 1, 2, i \neq j$). The coefficient $a_{ij}$ is the scattering length between states $i$ and $j$ and it satisfies $a_{ij} = a_{ji}$. Weak coupling is defined by the Rabi frequency satisfying $\Omega/(\omega_x \omega_y \omega_z)^{1/3} = \Omega/\overline{\omega} \ll 1$, where $\overline{\omega} = (\omega_x \omega_y \omega_z)^{1/3}$ is the geometric-averaged angular frequency for the trapping potential.

In this regime, we can write the macroscopic wavefunctions using the variational ansatz $\Psi_i(\vec{r},t) = \psi_i(t) \Phi_i(\vec{r})$ with $\psi_i(t) = \sqrt{N_i(t)} e^{i\alpha_i(t)}$ ($i = 1, 2$). In the ansatz, the functions $\Phi_i(\vec{r})$ describe the spatial distribution of the $i-$th component, the complex coefficient functions $\psi_i(t)$ are spatially uniform and contain all time-dependence in the macroscopic quantum wave-functions $\Psi_i(\vec{r},t)$. The symbols $N_i(t)$ and $\alpha_i(t)$ represent the populations and phases of the $i-$th condensate, respectively. Because the coupling is very weak, the spatial distributions vary slowly in time and are very close to the adiabatic solutions to the time-independent uncoupled case for GP equations (1), being slaved by the populations [13]. Thus, the complex coefficient functions $\psi_i(t)$ obey the nonlinear two-mode dynamical equations
\[
i\hbar \frac{d}{dt} \psi_2(t) = \left( E_2^0 - \frac{\hbar \delta}{2} + U_{22} |\psi_2(t)|^2 + U_{21} |\psi_1(t)|^2 \right) \psi_2(t) + \frac{K}{2} \psi_1(t),
\]
\[
i\hbar \frac{d}{dt} \psi_1(t) = \left( E_1^0 + \frac{\hbar \delta}{2} + U_{11} |\psi_1(t)|^2 + U_{12} |\psi_2(t)|^2 \right) \psi_1(t) + \frac{K}{2} \psi_2(t).
\]

The parameters satisfy
\[
E_i^0 = \int \Phi_i(\vec{r}) H_0^i \Phi_i(\vec{r}) d\vec{r}, \quad U_{ij} = \frac{4\pi \hbar^2 a_{ij}}{m} \int |\Phi_i(\vec{r})|^2 |\Phi_j(\vec{r})|^2 d\vec{r} = U_{ji} \quad (i,j = 1, 2).
\]

The terms in \( K \) describe population transfer (internal tunnelling) between two BEC states, whereas the terms in \( U_{ij} \), which depend on the numbers of atoms in each BEC state, describe the mean-field interaction between atoms. When \( U_{21} \) and \( \delta \) equal zero, these coupled equations can also describe the BECs in a double-well potential [15–18].

Similar to the coupled two-state system obeying the linear Schrödinger equation, we introduce a generalized Bloch spin vector \((u, v, w)\) with the components
\[
u = \psi_2^* \psi_1 + \psi_2 \psi_1^*, \quad v = -i (\psi_2 \psi_1^* - \psi_2^* \psi_1), \quad w = \psi_2^* \psi_2 - \psi_1^* \psi_1.
\]

Obviously, \( u^2 + v^2 + w^2 = (N_1 + N_2)^2 = N_T^2 \) is a conserved quantity when finite-temperature and damping effects can be ignored. Rescaling the time \( t/\hbar \) to \( t \), the Bloch spin vector satisfies
\[
\frac{du}{dt} = v(\gamma + \eta w), \quad \frac{dv}{dt} = Kw - u(\gamma + \eta w), \quad \frac{dw}{dt} = -Kv,
\]
where \( \gamma = E_2^0 - E_1^0 + N_T(U_{22} - U_{11})/2 - \hbar \delta \) and \( \eta = (U_{22} + U_{11} - 2U_{12})/2 \). Regarding the atom in one condensate as spin-up state and the atom in the other condensate as spin-down state, the coupled BECs can be described with the quasi-spin \( \vec{S} = u \hat{e}_x + v \hat{e}_y + w \hat{e}_z \). In this language, the longitudinal component \( w \) depicts the population difference, and the transverse components \( u \) and \( v \) characterize the coherence. Thus the effective Hamiltonian for the quasi-spin is
\[
E = -\frac{1}{2} \eta S_z^2 - KS_x - \gamma S_z.
\]

The above Hamiltonian is similar to the one of a uniaxial spin with an applied magnetic field [26–30], it indicates that the mean-field interaction brings the anisotropy \( \eta \), the coupling causes an effective transverse magnetic field \( K \) along axis-\( x \), and the asymmetry or the
detuning induces an effective longitudinal magnetic field $\gamma$. In the symmetric case ($E_2^0 = E_1^0$, $U_{22} = U_{11}$ and $\delta = 0$), it is consistent with the one derived from the second quantized Hamiltonian in [4].

III. TUNNELLING RATE AND CROSSOVER TEMPERATURE

In conventional spherical coordinates, the spin components can be written as $S_x = N_T \sin \theta \cos \phi$, $S_y = N_T \sin \theta \sin \phi$ and $S_z = N_T \cos \theta$ (see Fig. 1). Thus, the corresponding effective Hamiltonian is formulated as

$$E = -\eta N_T^2 \left( \frac{1}{2} \cos^2 \theta + \frac{K}{\eta N_T} \sin \theta \cos \phi + \frac{\gamma}{\eta N_T} \cos \theta \right). \tag{6}$$

Based upon the analysis of a spin in a uniaxial magnetic field [26–30], we know that there are stationary states if some angles ($\theta_0, \phi_0$) satisfy $\partial E/\partial \phi|_{\phi = \phi_0} = 0$ and $\partial E/\partial \theta|_{\phi = \phi_0} = 0$. The condition $\partial E/\partial \phi|_{\phi = \phi_0} = 0$ locates the stationary states in the XOZ plane ($\sin \phi_0 = 0$). The existence of multiple stationary states in this quasi-spin system is equivalent to the existence of multiple metastable MQST states in the coupled BECs. Near the metastable states the potential describes a "canyon" satisfying

$$E_\theta = E(\theta, \phi_0)/(\eta N_T^2) = -\frac{1}{2} \cos^2 \theta - P \cos(\theta - \theta_P). \tag{7}$$

The parameters obey $P = \sqrt{K^2 + \gamma^2}/|\eta N_T|$, $\sin \theta_P = K \cos \phi_0/\sqrt{K^2 + \gamma^2}$ and $\cos \theta_P = \gamma/\sqrt{K^2 + \gamma^2}$. As stated in the previous section, the parameter $K \propto \Omega > 0$, therefore $\sin \theta_P > 0$ and $\sin \theta_P < 0$ correspond to the equal-phase mode ($\phi_0 = 0$) and the anti-phase mode ($\phi_0 = \pi$) in the coupled two-component BECs, respectively. In the case of $E_2^0 - E_1^0 + N_T(U_{22} - U_{11})/2 = 0$, the parameter $\gamma$ is just the negative detuning $-\delta$, thus $\cos \theta_P > 0$ and $\cos \theta_P < 0$ correspond to the red detuning and the blue detuning of the coupling laser, respectively. The $\partial E/\partial \theta|_{\phi = \phi_0} = 0$ is equivalent to $\partial E_\theta/\partial \theta|_{\phi = \phi_0} = 0$, that is, $\sin 2\theta_0 + 2P \sin(\theta_0 - \theta_P) = 0$. For some critical points where both the first and the second derivatives of $E_\theta$ equal zero, an appreciable tunnelling rate appears. This gives
\[
\sin 2\theta_C + 2P_C \sin(\theta_C - \theta_P) = 0, \\
\cos 2\theta_C + P_C \cos(\theta_C - \theta_P) = 0, 
\]

(8)

here, \(\theta_C\) and \(P_C\) are critical values for \(\theta\) and \(P\), respectively. Solving the above equations, one can obtain \(\tan^3 \theta_C = -\tan \theta_P\) and \(P_C = (\sin^{2/3} \theta_P + \cos^{2/3} \theta_P)^{-3/2}\). The system has an instanton solution at the critical point \(P = P_C\), i.e., \((K/(\eta N_T))^{2/3} + (\gamma/(\eta N_T))^{2/3} = 1\). This critical point stands on the separatrix between the single stable regime and the multiple-stable regime. It separates the metastable multi-MQST behavior between the single-stable population oscillation in the coupled two-component BEC.

According to the dependence of \(E_\theta\) on \(\theta\), we obtain that the condition for the existence of multiple stationary states is \(P < P_C\), i.e., \((K/(\eta N_T))^{2/3} + (\gamma/(\eta N_T))^{2/3} < 1\). One can easily find the small oscillations around these stationary states with nonzero time-averaged values for \(S_z\) and \(\sqrt{S_x^2 + S_y^2}\). These oscillations correspond to the phase-locked MQST states with time-averaged relative phase 0 or \(\pi\) and multiple stationary states correspond to multiple metastable MQST states with fixed nonzero population difference and relative phase 0 or \(\pi\). The appearance of multiple stationary states indicates, only for some proper parameters, that multiple metastable MQST states exist. For simplicity we only consider the case where the parameter \(P\) is slightly lower than the fixed critical value \(P_C\), \(P = P_C(1 - \varepsilon)\), \(\varepsilon \ll 1\). This requires that the Rabi frequency, the detuning, the scattering lengths, and the total atomic number in the coupled BEC system must cooperate with each other to approach the critical values for the emergence of multiple metastable MQST states. One way to maintain the critical value \(P_C\) unchanged is fixing the values of the ratio \(\gamma/K\) and other correlated parameters (\(\eta\) and \(N_T\)), that is, keeping the angle \(\theta_P\) unchanged. By introducing a new positive variable \(\xi = \theta - \theta_0\), the potential (6) can be expanded into

\[
E_\theta(\theta) = E_\theta(\theta_0) + \frac{1}{4} \left[ \sqrt{\varepsilon} \xi^2 - \xi^3 + O(\xi^4) \right] \sin(2\theta_C). 
\]

(9)

With the definition in Refs. [26,27,29,30], the tunnelling rate \(\Gamma\) obeys \(N_p(t) = N_P(0) \exp(-\Gamma t)\) and it can be written as \(\Gamma = A \exp(-B)\) for the quantum tunnelling regime. Here, \(N_P(t)\) is the population occupying the metastable state at time \(t\) and the
tunnelling exponent $B \geq 0$ is determined by the imaginary time action of the instanton solution. Similar to Ref. [26], the tunnelling exponent follows from the path integral 
\[ \int D\{\phi(\tau)\} \int D\{\cos \theta(\tau)\} \exp(I/\hbar) \]
over the continuum of trajectories which start and end at $(\theta_0, \phi_0)$ and which are close to the instanton solution, where, $\tau$ is the imaginary time $it$, and $I$ is the imaginary time action $I = \int d\tau [iN_T (1 - \cos \theta) d\phi/d\tau + E(\theta, \phi)]$. Integrating the imaginary time action by parts, one can gain the tunnelling exponent
\begin{equation}
B = N_T \int_{-\infty}^{+\infty} d\tau \left\{ \frac{(d\xi/d\tau)^2 \sin \theta_C}{2\sin \theta_P} + \frac{1}{4} \sin(2\theta_C) \sqrt{6\varepsilon \xi^2 - \xi^3 + O(\xi^4)} \right\},
\end{equation}
\begin{equation}
= 16 \times 6^{1/4} N_T \varepsilon^{5/4} |\cot \theta_P|^{1/6}/5 = 16 \times 6^{1/4} N_T \varepsilon^{5/4} |\gamma/K|^{1/6}/5.
\end{equation}

From the definition of $\varepsilon$, one can obtain
\begin{equation}
\varepsilon = 1 - P/P_C = 1 - (1 + |\gamma/K|^2)(1 + |\gamma/K|^{2/3})^{-3/2}|K/(\eta N_T)|.
\end{equation}

Thus the tunnelling exponent can be expressed as
\begin{equation}
B = 16 \times 6^{1/4} N_T [1 - (1 + |\gamma/K|^2)(1 + |\gamma/K|^{2/3})^{-3/2}|K/(\eta N_T)|]^{5/4} |\gamma/K|^{1/6}/5.
\end{equation}

To control the tunnelling, one has to select proper values for parameters $\gamma$, $K$ and $\eta$. In the experiments performed in a double-well potential [4,5], it can be realized by modifying the barrier position, the barrier height and the magnetic field (using Feshbach resonances to adjust the scattering lengths, [35]), respectively. In the experiments with two-component BECs in a single-well potential [9], it can be realized by adjusting the laser detuning, the laser intensity, and the magnetic field, respectively. For fixed value of $\eta$ and $\gamma/K$, the tunnelling exponent $B$ decreases with the increasing of the intensity of the coupling laser. In Fig. 2, we show how the tunnelling exponent $B$ depends on the angle $\theta_P$. In the region between $0$ and $\pi$, the ratio $B(\theta_P)/B(\pi/4)$ decreases from positive infinity to zero when the angle $\theta_P$ equals $\pi/2$, which corresponds to the symmetric case ($\gamma = 0$), and then increases to positive infinity when the angle $\theta_P$ is close to $\pi$. It is almost flat when the angle $\theta_P$ is not close to $0$, $\pi/2$ and $\pi$. This angular dependence indicates, in the case of fixed value of $\varepsilon$, that the tunnelling exponent increases with increasing $|\gamma/K|$.
The result for the angle $\theta_P$ close to $\pi/2$, which corresponds to the symmetric case $\gamma = 0$, should be taken with great caution because the coefficient $\sin(2\theta_C)$ in the Taylor expansion series (9) is equal to zero. In this case, the problem corresponds to the tunnelling between two equivalent minima which correspond to the angle $\theta_P$ equal 0 and $\pi$. Thus the potential can be expanded into the form of $\xi^2 - \xi^4$ and the tunnelling exponent $B$ is expressed as $B = 4S\varepsilon^{3/2} = 4N_T\varepsilon^{3/2}$. Therefore, the tunnelling exponents (10) and (12) only hold for the asymmetric case where $\gamma \neq 0$.

To confirm our prediction from the quasi-spin model, we perform a numerical simulation of the equation (2). A qualitative change in the stationary-state behavior occurs at $|K/(\eta N_T)| = 1$. When $|K/(\eta N_T)| > 1$, there are no metastable states for any effective detuning $\gamma$. However, when $|K/(\eta N_T)| < 1$, metastable states exist in the region $[-\gamma_c, +\gamma_c]$ for proper relative phase, where $\gamma_c$ satisfies $(K/(\eta N_T))^{2/3} + (\gamma_c/(\eta N_T))^{2/3} = 1$. See the left column of Fig. 3. Two stationary states, indicated as $S_1$ and $S_2$ in the figure, are stable and the other one ($U$) is unstable. Adiabatically changing the effective detuning $\gamma$ from $\gamma_c - \varepsilon$ to $\gamma_c + \varepsilon$ ($\varepsilon$ is a very small positive number), in the space of the fractional population difference $z = (N_2 - N_1)/N_T$ and the relative phase $\phi = \alpha_2 - \alpha_1$, a trajectory in the vicinity of $S_2$ becomes a large orbit $C$ encircling $S_1$. From the views of instanton method, the tunnelling exponent is determined by the canonical action of the orbit, i.e., $B$ follows from the path integral $\int D\{z(\tau)\} \int D\{\phi(\tau)\} \exp(I_c/h)$ over the continuum of trajectories which are close to the instanton solution. At different bifurcation points $\gamma_c$, the numerical results show $B(|\gamma/K|)/B(|\gamma/K| = 1) \propto |\gamma/K|^{0.163\pm0.002} \approx |\gamma/K|^{1/6}$, this confirms our previous prediction from the quasi-spin model (see the right column of Fig. 3).

There are two important aspects which must be noted. The one is that these results for tunnelling are only valid in the low barrier limit for the quasi-spin model, i.e., $\varepsilon << 1$. This means that the above results only hold in the region which approaches the critical point of emergence of multiple metastable MQST states. The parametric dependence of the general case is still an open problem. The other is the validity of the Wentzel-Kramers-Brillouin (WKB) semiclassical approximation. The semiclassical approach can only be used in the
case of small tunnelling probability, that is, $B \gg 1$. In this low barrier limit, from the
Taylor expansion series (9) one can obtain the following tunnelling amplitude by using the
theory developed by Caldeira and Leggett [34],

$$A = (15B/8\pi)^{1/2}\omega, \quad (13)$$

$$= \eta N_T\left(\frac{15B}{2\pi}\right)^{1/2}\left(\frac{3\epsilon}{8}\right)^{3/4}\cot \theta_P^{1/6}/(1 + \cot^{2/3}\theta_P),$$

$$= \eta N_T\left(\frac{15B}{2\pi}\right)^{1/2}\left(\frac{3\epsilon}{8}\right)^{3/4}\frac{1}{\left(1 + |\gamma/K|^{2/3}\right)}. \quad (13)$$

Here, $\omega$ is the angular frequency of small oscillations near the bottom of the inverse potential.

Apparently, when the angle $\theta_P$ is close to the $k\pi$ ($k = 0, 1$), which corresponds to small
Rabi frequency or large detuning of the coupling laser between two BECs, the tunnelling amplitude $A$ approaches to zero, see Fig. 4. As presented in above, in the case of $\theta_P$ close to
$\pi/2$ which corresponds to the symmetric case $\gamma = 0$, the potential is not in form of $\xi^2 - \xi^3$
but in form of $\xi^2 - \xi^4$ because the coefficient $\sin(2\theta_C)$ in the Taylor expansion series (9)
equals zero. Therefore, the above formula for the tunnelling amplitude only holds for the
asymmetric case $\gamma \neq 0$. Generally, contrary to the tunnelling exponent $B$, the tunnelling
amplitude $A$ is sensitive to the structure of quantum levels in the potential. Therefore, for
the case of the full potential (6) and (7), the estimation of $A$ is still an open problem.

Population transfer between two states in a bistable system can occur either due to
classical thermal activation which depends on the system temperature or due to quantum
tunnelling which does not depend on the system temperature. There exists a phase transition
from the thermal regime to the quantum regime which occurs at the crossover temperature
$T_C$. Above $T_C$, quantum effects are very small and the population transfer rate follows the
Arrhenius law,

$$\Gamma_{thermal} = \Gamma_0 \exp\left(-\frac{U_B}{k_BT}\right).$$

(14)

Here, $U_B$ is the height of the energy barrier between two states and $k_B$ is the Boltzmann
constant. Below $T_C$, the population transfer is purely quantum,

$$\Gamma_{quantum} = A \exp\left(-B\right).$$

(15)
with $B$ independent of the system temperature. Thus the transition occurs when $\Gamma_{\text{thermal}} = \Gamma_{\text{quantum}}$. Neglecting the prefactors and equating the exponents, the crossover temperature can be estimated as

$$T_C = U_B/(k_B B).$$

(16)

The transition region is approximately the temperature interval $[T_C(1-B^{-1}), T_C(1+B^{-1})]$. This crossover resembles a first-order phase transition of the tunnelling rate $\Gamma$ because it is accompanied with the discontinuity of $d\Gamma/dT$ at $T_C$ [29].

There is another regime for tunnelling, the thermally assisted tunnelling (TAT), in which the particle strides over the barrier to the bottom of the potential with lowering temperature [29,30]. The transition from the classical regime to the TAT regime resembles a second-order classical-quantum phase transition of the tunnelling rate $\Gamma$ because it is accompanied with a discontinuity of $d^2\Gamma/dT^2$ and no discontinuity of $d\Gamma/dT$ at the crossover temperature. The corresponding transition temperature can be estimated as

$$T_C^{\text{T}} = \hbar/(\tau_0 k_B) = \hbar \omega/(2\pi k_B),$$

(17)

where $\tau_0$ and $\omega$ are the period and the angular frequency of small oscillations near the bottom of the inverse potential, respectively [25–27,29,30]. In the low barrier limit ($\varepsilon << 1$), from the Taylor expansion series (9), one can obtain the barrier height

$$U_B = \eta N_T^2 (2\varepsilon/3)^{3/2} |\sin(2\theta_C)| = \frac{5\pi}{18} (\hbar/\tau_0) B,$$

(18)

where,

$$|\sin(2\theta_C)| = 2|\gamma/K|_C/(1 + |\gamma/K|_C^2).$$

(19)

Comparing both crossover temperatures, one can easily find that they differ by a factor $T_C/T_C^{\text{T}} = 5\pi/18 = 1/1.15$, which means that they are of the same order of magnitude and can both be used to estimate the crossover temperature.

Below, from the experimental parameters in the experiments of JILA [9], we will give a quantitative estimation for the tunnelling rate and the crossover temperature. In those
experiments, the atomic mass \( m_1 = m_2 = m_{\text{Rb}} = 1.45 \times 10^{-25} \text{ kg} \), the time-averaged orbiting potential (TOP) magnetic trap has an axial frequency \( v_z = 59 \text{ Hz} \) and a radial frequency \( v_{x,y} = v_r = v_z / \sqrt{8} = 21 \text{ Hz} \), the s-wave scatter lengths \( a_{11} = 5.36 \text{ nm} \), \( a_{12} = a_{21} = 5.53 \text{ nm} \) and \( a_{22} = 5.70 \text{ nm} \), and the total atomic number \( N_T \approx 5 \times 10^5 \). To obtain the numerical values conveniently, we choose the natural units of the problem, in which, time is in units of \( 1 / (\omega_x \omega_y \omega_z)^{1/3} = 1 / \overline{\omega} \), length is in units of the geometric-averaged harmonic-oscillator length \( \overline{d} = \sqrt{\hbar / (\omega_x \omega_y \omega_z)^{1/3} m_{\text{Rb}}} = \sqrt{\hbar / (\overline{\omega} m_{\text{Rb}})} \), energy is in units of the geometric-averaged trap level spacing \( \overline{\hbar (\omega_x \omega_y \omega_z)}^{1/3} = \hbar \overline{\omega} \), and mass is in units of Rb atomic mass \( m_{\text{Rb}} \).

Due to gravity acting besides the TOP, the centers of two condensates will displace along the vertical direction and the two equilibrium displacements are generally not the same. Thus, if the interparticle interaction is absent, the lowest single-particle state has the familiar wave function,

\[
\Phi_{0i}(\vec{r}) = \frac{1}{\pi^{3/4} (d_x d_y d_z)^{1/2}} \exp(-\frac{x^2}{2d_x^2} - \frac{y^2}{2d_y^2} - \frac{(z - F_i z_0)^2}{2d_z^2}).
\]

(20)

Where, \( F_1 = +1, F_2 = -1 \), \( 2z_0 \) is the offset between two potential centers along the vertical axis, \( d_k = \sqrt{\hbar / (\omega_k m_{\text{Rb}})} \) \( (k = x, y, z) \) are the oscillator lengths. The offset \( 2z_0 \) between two condensates can be varied by adjusting the magnitude of the rotating magnetic field. In the presence of interatomic interaction, the dimensions of the condensates are changed. The spatial parts of the macroscopic quantum wave functions are in the shape of

\[
\Phi_i(\vec{r}) = \frac{1}{\pi^{3/4} (b_{ix} b_{iy} b_{iz})^{1/2}} \exp(-\frac{x^2}{2b_{ix}^2} - \frac{y^2}{2b_{iy}^2} - \frac{(z - F_i z_0)^2}{2b_{iz}^2}).
\]

(21)

The variational parameters \( b_{ik} \) \( (k = x, y, z; i = 1, 2) \) depend on the scattering length, the total atom number, and the trapping potential and they have almost the same numerical values as \( d_k \). For proper values of the offset \( 2z_0 \), the numerical results of [13] show that the spatial distributions \( \Phi_i(\vec{r}) \) and their overlap only weakly depend on the total atom numbers in each condensate. For simplicity, in the following calculations, the variational parameters \( b_{ik} \) are replaced by the oscillator lengths \( d_k \). Therefore, the parameters \( E_{0i}^0 \), \( U_{ij} \) and \( K \) are determined by
\[ E_1^0 = E_2^0 = \hbar(\omega_x + \omega_y + \omega_z)/2, \]
\[ U_{ii} = 4\pi\hbar^2a_{ii}/[(\sqrt{2\pi})^3d_xd_yd_zm_{Rb}], (i = 1, 2), \]
\[ U_{12} = 4\pi\hbar^2a_{12}\exp(-2z_0^2/d_z^2)/[(\sqrt{2\pi})^3d_xd_yd_zm_{Rb}], \]
\[ K = \hbar\Omega\exp(-z_0^2/d_z^2). \]

So the corresponding parameters in the quasi-spin model (5) can be written as \( \gamma = \hbar^2N_T(a_{22} - a_{11})/(\sqrt{2\pi}d_xd_yd_zm_{Rb}) - \hbar\delta \) and \( \eta = \hbar^2[a_{22} + a_{11} - 2a_{12}\exp(-2z_0^2/d_z^2)]/(\sqrt{2\pi}d_xd_yd_zm_{Rb}) \). In the case of complete overlap \( (2z_0 = 0) \), the anisotropy parameter \( \eta \) equals zero, thus the metastable multi-MQST behavior will never appear, but some running-phase MQST states may still exist. This indicates that, to insure the existence of multiple metastable MQST states, a finite offset must be kept between two condensates. Furthermore, the appearance of this kind of MQST requires \( K^{2/3} + \gamma^{2/3} < (\eta N_T)^{2/3} \). Because \( K \propto \Omega \) and \( \gamma \propto \delta \), this inequality indicates that the Rabi frequency and the detuning of the coupling pulses must be relatively small. Choosing the total atom number \( N_T = 2.0 \times 10^4 \), the half offset \( z_0 = 0.20d_z \), the Rabi frequency \( \Omega = 2\pi \times 10 \) Hz, and the detuning \( \delta = -179 \) Hz, one can get \( \eta N_T = 6.70 \times 10^{-32} \), \( \gamma = 4.57 \times 10^{-32} \), \( K = 6.90 \times 10^{-33} \) and \( \epsilon = 9.78 \times 10^{-3} \). Thus, the corresponding tunnelling exponent \( B \) and crossover temperature \( T_C \) are around \( 4.22 \times 10^2 \) and \( 3.54 \times 10^{-2} \) nK, respectively. Obviously, the crossover temperature \( T_C \), which corresponds to a phase transition from classical tunnelling to quantum tunnelling, is far below than the critical temperature \( T_0 \approx 150 \) nK for Bose-Einstein condensation in a dilute gas of \(^{87}\text{Rb} \).

**IV. DISCUSSION AND SUMMARY**

The generalized Bloch equation (4) and its stability analysis will help to control the population transfer and realize the single-qubit operation with BECs qubit. Theoretically, any two-state quantum system can serve as a qubit, many of them have been realized experimentally. To make use of two quantum states, the coherence and superposition between them is the most essential qualification. The experimental observation of coherence and superposi-
tion between two BECs indicates the possibility of encoding two coupled BECs as a qubit. However, because of the mean-field interaction among Bosonic condensed atoms, the qubit operations become very difficult to perform. To accomplish a single-qubit operation, it must be possible rotated arbitrarily in the Hilbert space. This requires the atomic populations can be transferred arbitrarily. From the Bloch equations (4), we find that MQST prevents the arbitrary rotation of the state vector. And even if there no MQST, when $\eta \neq 0$, the complete population inversion can not be accomplished with linear operations. Thus, to accomplish a linear qubit operation, one has to adjust the parameter $\eta$ to zero by varying the atomic scattering length with a Feshbach resonance [35]. In this case, the mean-field interaction gives a density-shift to the original energy levels and, according to Rabi’s theory, the arbitrary rotation of the state vector can be performed easily. Thus, if one encodes the qubit states $|0\rangle$ and $|1\rangle$ as the condensate wavefunctions for two condensates in a double-well potential or two hyperfine-state condensates coupled with Raman pulses [36], an arbitrary one-bit linear operation can be realized when the anisotropy is absent ($\eta = 0$) and an arbitrary one-bit nonlinear operation can be realized when the metastable multi-MQST behavior is absent ($|\mathcal{K}| > |\eta \mathcal{N}_T|$). This means that, to perform an arbitrary one-bit transformation, it at least needs choosing proper parameters to avoid the emergence of the metastable multi-MQST behavior.

The tunnelling of the quasi-spin model described by the Hamiltonian (5) has also been investigated by mapping it onto a particle moving in an asymmetric double-well potential [27–29]. Using this approach, Garanin et. al. have explored some new fascinating feature of this uniaxial spin model in the strongly biased limit [29]. They find that there exist two different regimes for the classical-quantum transition of the tunnelling rate and the kind of transition depends on both the strength and the direction of the magnetic field. In this article, we directly analyze the tunnelling in the low barrier limit for the quasi-spin model, which corresponds to the effective magnetic fields near their critical values for appearance of metastable states. This requires that all physical parameters of the coupled BECs collaborate with each other to approach the critical point of appearance of multiple

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metastable MQST states. The symmetric case ($\gamma = 0$) of the coupled BECs corresponds to the unbiased case ($H_z = 0$) of the anisotropic spin model, which has been investigated in details by mapping it onto a particle moving in a symmetric double-well potential [25].

The macroscopic quantum tunnelling of two-component BECs has also been investigated by the Kasamatsu group. Using a numerical approach, they have analyzed the tunnelling between two kinds of metastable stationary states, a symmetry-breaking state (SBS) and a symmetry-preserving state (SPS), in uncoupled two-component BECs [14]. To improve the usual Gaussian variational method, they have introduced a collective coordinate approach and then calculated the tunnelling rate within the WKB approximation. In that system, the populations of the two components can not be converted into each other because of the absence of coupling. This means, the tunneling does not occur between two components but between stationary states with different spatial configurations. Thus, this kind of tunnelling originates from the quantized spatial structure of the Hamiltonian. In our model, due to the coupling, the population can be transferred from one component to the other. Furthermore, we assume the coupling is very weak, thus both components stay in their ground stationary states through the full process. The metastability (metastable MQST) is the result of the cooperation between the coupling and the mean-field interaction (including both the intra-component and the inter-component interaction). Correspondingly, the tunneling from the metastable self-trapped state to its ground state of the coupled two-component BECs is caused by the quantized structure of their field operators.

In conclusion a system of coupled BECs (two BECs in a double-well potential or two internal state BECs coupled with laser pulses) has been mapped to a spin in a magnetic field by introducing a generalized Bloch vector. The mean-field interaction, the coupling and the asymmetry or the detuning are relevant to the anisotropy, the transverse magnetic field and the longitudinal magnetic field, respectively. The corresponding generalized Bloch equation is obtained. The analysis of this generalized Bloch equation will be propitious to control the population transfer and realize the quantum computation with coupled BECs. Based upon experience from the well-studied tunneling of spin systems, the detailed information about
the tunnelling between two metastable MQST states in coupled two-component BECs can be obtained with the imaginary-time path-integral method. The crossover temperature $T_C$ at the critical point for a transition from the classical thermal regime to the quantum regime was obtained. When the system temperature decreases through $T_C$, the population conversion goes from classical thermal activation regime to purely quantum tunnelling regime. This means, below the crossover temperature $T_C$, the quantum fluctuations in the atomic fields take the dominant position. We also find that the tunnelling rate can be adjusted by varying the coupling and the trapping magnetic field.

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Figure caption

Fig. 1 The quasi-spin $\vec{S}$ and it’s components $(u, v, w)$ in conventional spherical coordinates.

Fig. 2 The tunnelling exponent ratio $B(\theta_P)/B(\theta_P = \pi/4)$ versus different $\theta_P$. Where, the angle $\theta_P$ characterizes the angle between the effective magnetic field $\vec{B}_{\text{eff}} = K \hat{e}_x + \gamma \hat{e}_z$ and axis-$z$.

Fig. 3 In the left column, the stationary states for $|K| < |\eta N_T|$ are shown. There are two metastable states $S_1$, $S_2$ and one unstable state $U$. In the right column, the tunnelling exponent ratio $B/B_0$ versus different $|\gamma/K|$ is presented, where $B_0 = B(|\gamma/K| = 1)$. The black dots show the numerical data and the straight line represents the linear fit for the logarithmic data.

Fig. 4 The tunnelling amplitude ratio $A(\theta_P)/A(\theta_P = \pi/4)$ versus different $\theta_P$, where the angle $\theta_P$ characterizes the angle between the effective magnetic field $\vec{B}_{\text{eff}} = K \hat{e}_x + \gamma \hat{e}_z$ and axis-$z$.
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Fig. 1