ON THE VOLUME FUNCTIONAL OF COMPACT MANIFOLDS WITH BOUNDARY WITH CONSTANT SCALAR CURVATURE

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ABSTRACT. We study the volume functional on the space of constant scalar curvature metrics with a prescribed boundary metric. We derive a sufficient and necessary condition for a metric to be a critical point, and show that the only domains in space forms, on which the standard metrics are critical points, are geodesic balls. In the zero scalar curvature case, assuming the boundary can be isometrically embedded in the Euclidean space as a compact strictly convex hypersurface, we show that the volume of a critical point is always no less than the Euclidean volume bounded by the isometric embedding of the boundary, and the two volumes are equal if and only if the critical point is isometric to a standard Euclidean ball. We also derive a second variation formula and apply it to show that, on Euclidean balls and “small” hyperbolic and spherical balls in dimensions $3 \leq n \leq 5$, the standard space form metrics are indeed saddle points for the volume functional.

1. Introduction

Given a compact $n$-dimensional manifold $\Omega$ with a boundary $\Sigma$, we study variational properties of the volume functional on the space of constant scalar curvature metrics on $\Omega$ with a prescribed boundary metric on $\Sigma$. The dimension $n$ is assumed to be $\geq 3$. There are several motivations for us to consider this problem.

The first motivation comes from a recent result in [9]. There one considers an asymptotically flat 3-manifold $(M, g)$ with a given end. Let $\{x_i\}$ be a coordinate system at $\infty$ which defines the asymptotic structure of $(M, g)$. Let $S_r = \{ x \in M \mid |x| = r \}$ be the coordinate sphere, where $|x|$ denotes the coordinate length. Let $\gamma$ be the induced metric on $S_r$. When $r$ is large, $(S_r, \gamma)$ can be isometrically embedded in the Euclidean space $\mathbb{R}^3$ as a strictly convex hypersurface $S^0_r$. Let $V_0(r)$

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be the volume of the region enclosed by $S_r$ in $\mathbb{R}^3$ and $V_r$ be the volume of the region enclosed by $S_r$ in $(M^3, g)$. It was proved in [9] that, as $r \to \infty$,

\begin{equation}
V(r) - V_0(r) = 2m_{ADM}\pi r^2 + o(r^2)
\end{equation}

whenever $m_{ADM}$ is defined. Here $m_{ADM}$ is the ADM mass of $(M^3, g)$ [2]. Therefore if $R(g)$, the scalar curvature of $g$, is nonnegative, then by the Positive Mass Theorem [17, 22], one has

\begin{equation}
V(r) \geq V_0(r)
\end{equation}

for sufficiently large $r$. (Note that the case $m_{ADM} = 0$ would imply $(M, g)$ is isometric to $\mathbb{R}^3$, hence showing $V(r) = V_0(r)$.)

As a statement on volume comparison, (2) is rather intriguing. First, the curvature assumption it requires is only on scalar curvature. Second, it was formulated as a boundary value problem, i.e. the competitors involved have a same Dirichlet boundary geometry. It is natural to ask whether there exist related results on compact manifolds with boundary.

As a special case of Theorem 3.3 and Theorem 4.4 in this paper, we have:

**Theorem 1.1.** Let $\Omega$ be a 3-dimensional compact manifold with a connected boundary $\Sigma$. Let $\gamma$ be a given metric on $\Sigma$ such that $(\Sigma, \gamma)$ can be isometrically embedded in the Euclidean space $\mathbb{R}^3$ as a strictly convex hypersurface $\Sigma_0$. Let $\mathcal{M}_0^\gamma$ be the space of zero scalar curvature metrics $g$ on $\Omega$ such that the induced metric from $g$ on $\Sigma$ is $\gamma$.

(i) Suppose $g \in \mathcal{M}_0^\gamma$ is a critical point of the volume functional $V(\cdot)$ on $\mathcal{M}_0^\gamma$, then

\[ V(g) \geq V_0, \]

where $V_0$ is the Euclidean volume of the region enclosed by $\Sigma_0$ in $\mathbb{R}^3$. Equality holds if and only if $(\Omega, g)$ is isometric to a standard Euclidean ball.

(ii) There exists no element in $\mathcal{M}_0^\gamma$ that minimizes volume in $\mathcal{M}_0^\gamma$.

Our second motivation comes from a variational characterization of Einstein metrics on a closed manifold [5, 18]: Let $\mathcal{M}_{-1}$ be the space of metrics with constant scalar curvature $-1$ on a compact manifold $M$ without boundary. Then an element $g \in \mathcal{M}_{-1}$ is an Einstein metric if and only if $g$ is a critical point of the volume functional $V(\cdot)$ on $\mathcal{M}_{-1}$. This characterization follows from Proposition 4.47 in [5], or the argument preceding Lemma 1.2 in [18]. The proof used the fact
that Einstein metrics correspond to critical points of the total scalar curvature functional

\[ S(g) = V(g)^{\frac{2-n}{n}} \int_M R(g)dV_g, \]

while the restriction of \( S(\cdot) \) to the space \( \mathcal{M}_{-1} \) agrees with \(-V(\cdot)\frac{2}{n}\).

We want to establish a similar result on compact manifolds with boundary, with an aim to therefore find a proper concept of metrics that would sit between constant scalar curvature metrics and Einstein metrics.

As a special case of Theorem 2.1 in this paper, we have:

**Theorem 1.2.** Let \( \Omega \) be a compact \( n \)-dimensional manifold with smooth boundary \( \Sigma \). Let \( \gamma \) be a given metric on \( \Sigma \) and \( K = -n(n-1) \) or 0. Let \( \mathcal{M}_\gamma^K \) be the space of metrics on \( \Omega \) which have constant scalar curvature \( K \) and have induced metric on \( \Sigma \) given by \( \gamma \). Then \( g \in \mathcal{M}_\gamma^K \) is a critical point of the volume functional \( V(\cdot) \) on \( \mathcal{M}_\gamma^K \) if and only if there is a function \( \lambda \) on \( \Omega \) such that \( \lambda = 0 \) on \( \Sigma \) and

\[ -(\Delta_g \lambda)g + \nabla^2_g \lambda - \lambda \text{Ric}(g) = g, \]

where \( \Delta_g, \nabla^2_g \) are the Laplacian, Hessian operator with respect to \( g \) and \( \text{Ric}(g) \) is the Ricci curvature of \( g \).

It will be shown in Theorem 3.2 that if \( g \) is a metric, defined on some open set, which satisfies (3) with some function \( \lambda \), then \( g \) necessarily has constant scalar curvature. Furthermore, if \( \Omega \) is indeed a closed manifold and \( g \) is a metric on \( \Omega \) with negative scalar curvature for which (3) holds with some function \( \lambda \), then \( g \) is an Einstein metric.

Our method in deriving the first variation of the volume functional also leads to the following characterization of geodesic balls in the hyperbolic space \( \mathbb{H}^n \) and the sphere \( \mathbb{S}^n \) through the first variation of the total mean curvature integral (see Theorem 3.1 and Proposition 3.1).

**Theorem 1.3.** Let \( \Omega \) be a connected domain with compact closure in \( \mathbb{H}^n \) or \( \mathbb{S}^n \) and with a smooth (possibly disconnected) boundary \( \Sigma \). If \( \Omega \subset \mathbb{S}^n \), we also assume the volume of \( \Omega \) is less than half of the volume of \( \mathbb{S}^n \). Let \( g \) be the corresponding space form metric on \( \Omega \). Then \( \Omega \) is a geodesic ball if and only if

\[ \oint_{\Sigma} H'(0) = 0 \]

for any smooth variation \( \{g(t)\} \) of \( g \) on \( \Omega \) such that \( g(t) \) and \( g \) have the same scalar curvature and the same induced boundary metric. Here
$H'(0)$ is the variation of the mean curvature of $\Sigma$ in $(\Omega, g(t))$ with respect to the outward unit normal.

In particular, Theorem 1.3 implies that if $\Omega$ is not a geodesic ball in $\mathbb{H}^n$ or the half sphere $\mathbb{S}_n^+$, then there exists a deformation $\tilde{g}$ of the standard metric $g$ on $\Omega$ such that $\tilde{g}$ and $g$ have the same scalar curvature and the same induced boundary metric on $\partial \Omega$, and

$$\int_{\partial \Omega} H < \int_{\partial \Omega} \tilde{H},$$

where $H$, $\tilde{H}$ are the mean curvature of $\partial \Omega$ in $(\Omega, g)$, $(\Omega, \tilde{g})$. One wants to compare this with the results in [19] and [20] (see the remark after Proposition 3.1).

In [18], Schoen gave the following conjecture concerning the volume functional $V(\cdot)$ on a closed hyperbolic manifold.

**Conjecture** Let $(M^n, h)$ be a closed hyperbolic manifold. Let $g$ be another metric on $M$ with $R(g) \geq R(h)$, then $V(g) \geq V(h)$.

This conjecture remained widely open until recently its 3-dimensional case followed as a corollary of Perelman's work on geometrization [14, 15]. It is natural to wonder if there exists a similar conjecture or result on compact manifolds with boundary with a hyperbolic metric. In [1], related results were established by Agol, Storm and Thurston, on compact 3-manifolds whose boundary are minimal surfaces. In this paper, we note that the above conjecture on closed manifolds does not generalize directly to manifolds with boundary if only the Dirichlet boundary condition is imposed.

**Theorem 1.4.** Let $\mathbb{H}^3$ be the hyperbolic space and $g_{\mathbb{H}}$ be the standard hyperbolic metric on $\mathbb{H}^3$. There exists a small constant $\delta > 0$ such that if $B$ is a geodesic ball in $\mathbb{H}^3$ with geodesic radius less than $\delta$, then there is another metric $g$ on $B$ such that $g$ induces the same boundary metric on $\partial B$ as $g_{\mathbb{H}}$ does, $R(g) = R(g_{\mathbb{H}})$, and $V(g) < V(g_{\mathbb{H}})$ on $B$.

The paper is organized as follows. In Section 2 we first analyze the manifold structure of the space of constant scalar curvature metrics with a prescribed boundary metric, then we compute the first variation of the volume functional and derive the critical point equation. We also relate the first variation of the volume functional on domains in an Einstein manifold with non-zero scalar curvature to the first variation of the total boundary mean curvature integral of the associated metric variation. In Section 3 we show that the only domains in space forms, on which the standard metrics are critical points, are geodesic balls. We
also establish some general properties for metrics satisfying the critical point equation. Then we focus on the zero scalar curvature case to prove a theorem that compares the volume of a critical point with the corresponding Euclidean volume. In Section 4 we derive the second variational formula and apply it to geodesic balls in space forms. In particular, we show that, on Euclidean balls and “small” hyperbolic and spherical balls in dimensions $3 \leq n \leq 5$, the volume functional achieves a saddle point at the standard space form metrics. For completeness, we include an appendix in which we construct traceless and divergence free $(0, 2)$ symmetric tensors with prescribed compact support on space forms, which are needed in the second variation construction.

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2. First variational formula for the volume functional

Let $\Omega$ be an $n$-dimensional, connected, compact manifold with a smooth (possibly disconnected) boundary $\Sigma$. Let $\gamma$ be a smooth metric on $\Sigma$. Let $S^{k,2}$ be the space of $W^{k,2}$ symmetric $(0,2)$ tensors on $\Omega$. We will always assume $k > \frac{n}{2} + 2$ so that each $h \in S^{k,2}$ is $C^{2,\alpha}$ up to the boundary. Let $S_{0}^{k,2}$ be the subspace consisting those $h$ with $h|_{T(\Sigma)} = 0$. That is to say, $h(v,w) = 0$ for all $v,w$ tangent to $\Sigma$. Let $M_{\gamma}$ be the open set in $S^{k,2}$ consisting $g$ which is a Riemannian metric $g > 0$ such that $g|_{T(\Sigma)} = \gamma$. Let $W^{k-2,2}(\Omega)$ be the space of $W^{k-2,2}$ functions on $\Omega$. Let $R$ be the scalar curvature map which maps $g \in M_{\gamma}$ to its scalar curvature $R(g)$. The fact that $R$ is a smooth map was shown by Fischer and Marsden in [3]:

Lemma 2.1. The map $R : M_{\gamma} \to W^{k-2,2}(\Omega)$ is smooth.

Let $g_{0} \in M_{\gamma}$. Suppose the scalar curvature of $g_{0}$ is a constant $K$. We have

Lemma 2.2. Suppose $0$ is not one of the Dirichlet eigenvalues of the operator

$$(n-1)\Delta_{g_{0}} + K.$$ 

Then, near $g_{0}$, the set

$$(\mathcal{M}_{\gamma})^{K} = \{g \in M_{\gamma} | R(g) = K\}$$

is a submanifold of $M_{\gamma}$.

Proof. The linearization of $R$ at $g_{0}$ is

$$(4) \quad DR_{g_{0}}(h) = -\Delta_{g_{0}}(tr_{g_{0}}h) + div_{g_{0}}(div_{g_{0}}(h)) - \langle h, Ric(g_{0}) \rangle_{g_{0}}$$
for $h$ in the tangent space of $g_0$ in $\mathcal{M}_\gamma$, which is equal to $S_{g_0}^{k,2}$. Here $\text{tr}_{g_0}(h)$ and $\text{div}_{g_0}(h)$ denote the trace of $h$ and the divergence of $h$ with respect to the metric $g_0$. Let $f \in W^{k-2,2}(\Omega)$ which is the tangent space at an element in $W^{k-2,2}(\Omega)$. Consider the following boundary value problem:

\[
\begin{cases}
-(n-1)\Delta_{g_0} u - Ku = f & \text{in } \Omega \\
u = 0 & \text{on } \Sigma,
\end{cases}
\]

which has a unique solution by the assumption on the Dirichlet eigenvalues and the Fredholm alternative, see [10]. Let $h = ug_0$. We conclude that $D\mathcal{R}_{g_0}(h) = f$. Hence $D\mathcal{R}_{g_0}$ is surjective. As $S_{g_0}^{k,2}$ is a Hilbert space, the kernel of $D\mathcal{R}_{g_0}$ automatically splits. By the implicit function theorem, the lemma follows. □

We want to apply Lemma 2.2 to domains in space forms: the Euclidean space $\mathbb{R}^n$, the hyperbolic space $\mathbb{H}^n$ and the sphere $\mathbb{S}^n$.

**Corollary 2.1.** Suppose $\Omega$ is a connected domain with compact closure in $\mathbb{R}^n$, $\mathbb{H}^n$ or $\mathbb{S}^n$ and with a smooth (possibly disconnected) boundary $\Sigma$. Let $g$ be the standard metric on $\Omega$ and let $\gamma = g|\Sigma$. Then, near $g$, the set $\mathcal{M}_\gamma^K$ is a submanifold of $\mathcal{M}_\gamma$, where (i) $K = 0$ if $\Omega \subset \mathbb{R}^n$, (ii) $K = -n(n-1)$ if $\Omega \subset \mathbb{H}^n$, and (iii) $K = n(n-1)$ if $\Omega \subset \mathbb{S}^n$ and $V(\Omega) < \frac{1}{2}V(\mathbb{S}^n)$, where $V(\Omega)$ and $V(\mathbb{S}^n)$ denote the volume of $(\Omega, g)$ and $\mathbb{S}^n$.

**Proof.** Cases (i) and (ii) follow immediately from Lemma 2.2. In case (iii), since the volume of $\Omega \subset \mathbb{S}^n$ is less than the volume of a hemisphere, the first Dirichlet eigenvalue for the Laplacian is larger than $n$ by the Faber-Krahn inequality [21], the result follows from Lemma 2.2 again. □

Next we want to consider the volume functional

$$V : \mathcal{M}_\gamma \to \mathbb{R},$$

which is a smooth functional on $\mathcal{M}_\gamma$. For $g \in \mathcal{M}_\gamma$, the first variation of $V(\cdot)$ at $g$ is:

\[
DV_g(h) = \frac{1}{2} \int_\Omega \text{tr}_g(h) dV_g.
\]

We are interested in critical points of $V(\cdot)$ restricted to $\mathcal{M}_\gamma^K$, when $\mathcal{M}_\gamma^K$ is a submanifold. First, we will construct explicit deformations on $\mathcal{M}_\gamma^K$.

**Proposition 2.1.** Let $g_0 \in \mathcal{M}_\gamma^K$ be a smooth metric such that the first Dirichlet eigenvalue of $(n-1)\Delta_{g_0} + K$ is positive. Let $h$ be a smooth
symmetric $(0,2)$ tensor on $\Omega$ such that $h|_{\partial(\varOmega)} = 0$. Let $g(t) = g_0 + th$ which is a smooth metric provided $|t|$ is small enough. Then there is $t_0 > 0$ and $\epsilon > 0$ such that for all $|t| < t_0$ the following Dirichlet boundary value problem has a unique solution $u(t)$ such that $1 - \epsilon \leq u \leq 1 + \epsilon$:

\[
\begin{cases}
\alpha \Delta_{g(t)} u - R(t) u = -K u^a, & \text{in } \varOmega \\
u = 1, & \text{on } \Sigma,
\end{cases}
\]

where $\alpha = 4(n-1)/(n-2)$, $a = (n+2)/(n-2)$, and $R(t)$ is the scalar curvature of $g(t)$. Moreover, $v = \partial_u|_{t=0}$ exists and is a smooth function on $\overline{\varOmega}$ (the closure of $\varOmega$), which is the unique solution of:

\[
\begin{cases}
(n-1)\Delta_{g_0} v + Kv = \frac{n-2}{4} \partial R(0), & \text{in } \varOmega \\
v = 0, & \text{on } \Sigma.
\end{cases}
\]

**Proof.** Since the first eigenvalue of $(n-1)\Delta_{g_0} + K$ is positive, there is a smooth function $\phi$ on $\overline{\varOmega}$ and $\delta > 0$ such that

\[
(n-1)\Delta_{g_0} \phi + K \phi + \delta \phi = 0
\]

in $\varOmega$ and $\phi > 0$ on $\overline{\varOmega}$, see [10] for example. Let $b = \min_{\overline{\varOmega}} \phi$, then $b > 0$. Suppose $1 \geq t > 0$ is small enough such that $g(t)$ is a Riemannian metric on $\varOmega$. For each $t$, let $L(\cdot)$ denote the operator

\[
\alpha \Delta_{g(t)} (\cdot) - R(t)(\cdot) + K(\cdot)^a.
\]

Then

\[
L(1 + t\phi) = \alpha \Delta_{g(t)} (1 + t\phi) - R(t)(1 + t\phi) + K(1 + t\phi)^a
\]

\[
= \alpha \Delta_{g_0} (1 + t\phi) - K(1 + t\phi) + \alpha (\Delta_{g(t)} - \Delta_{g_0}) (1 + t\phi)
\]

\[
- (R(t) - K)(1 + t\phi) + K(1 + t\phi)^a
\]

\[
\leq -\frac{4}{n-2} \delta t \phi + C_1 t + C_2 t^2
\]

where $C_1$ is a positive constant depending only on $g_0$ and $h$, and $C_2 > 0$ depends also on $\phi$. Similarly,

\[
L(1 - t\phi) = \alpha \Delta_{g(t)} (1 - t\phi) - R(t)(1 - t\phi) + K(1 + t\phi)^a
\]

\[
= \alpha \Delta_{g_0} (1 - t\phi) - K(1 - t\phi) + \alpha (\Delta_{g(t)} - \Delta_{g_0}) (1 - t\phi)
\]

\[
- (R(t) - K)(1 - t\phi) + K(1 - t\phi)^a
\]

\[
\geq \frac{4}{n-2} \delta t \phi - C_3 t - C_4 t^2
\]
where $C_3$ is a positive constant depending only on $g_0$ and $h$, and $C_4 > 0$ depends also on $\phi$. By rescaling $\phi$, we may assume that $(4\delta b)/(n-2) \geq 2C_1$ and $(4\delta b)/(n-2) \geq 2C_3$. Then for $t > 0$ small enough $L(1+t\phi) \leq 0$ and $L(1-t\phi) \geq 0$. By [16, Theorem 2.3.1], (7) has a solution $u$ satisfying $1 - t\phi \leq u \leq 1 + t\phi$ provided $t > 0$ is small enough. The proof for $t < 0$ is similar.

To prove uniqueness, let $u_1$ and $u_2$ be two solutions of (7) such that $1 - \epsilon \leq u_1, u_2 \leq 1 + \epsilon$. Then there is $\delta_1 > 0$ depending only on $g_0$ and $h$ such that for $|t|$ small enough:

$$
\delta_1 \int_{\Omega} (u_1 - u_2)^2 dV_{g(t)} \leq -\frac{4}{n-2} \int_{\Omega} K(u_1 - u_2)^2 dV_{g(t)}
$$

$$
+ \int_{\Omega} [K(u_1^a - u_2^a) - R(t)(u_1 - u_2)] (u_1 - u_2) dV_{g(t)}
$$

$$
\leq (C_5|t| + C_6\epsilon) \int_{\Omega} (u_1 - u_2)^2 dV_{g(t)}
$$

where $C_5$ and $C_6$ are constants depending only on $g_0$ and $h$. Hence if $\epsilon > 0$ is small enough and $|t|$ is small enough, we must have $u_1 = u_2$.

To prove the last part of the proposition, let $u(t), t \neq 0$ be the solution of (7) obtained above, then $|u(t) - 1| \leq |t|\phi$. Let $w = (u - 1)/t$. Then $w$ satisfies:

$$
\alpha \Delta_{g(t)} w = R(t)w - \frac{K - R(t)}{t} - \frac{K(u^a - 1)}{t}, \text{ in } \Omega
$$

$$
w = 0, \text{ on } \Sigma.
$$

Since the right side of the equation is bounded by a constant independent of $t$ and $x \in \Omega$, there exist $\beta > 0$ and $C_7 > 0$ independent of $t$ such that the Hölder norm with respect to $g_0$ with exponent $\beta$ of $w$ in $\Omega$ is bounded by $C_7$, see [11, Theorem 8.29]. By the Schauder estimates [11, Theorem 6.19], one can conclude that for any $k \geq 2$, then $C^{k,\beta}$ norm of $w$ is uniformly bounded by a constant independent of $t$. Hence for any $t_i \to 0$, we can find a subsequence which converge to a solution $v$ of (8). Since $v$ is unique by the assumption on the first eigenvalue of $(n-1)\Delta_{g_0} + K$ and the Fredholm alternative, we conclude that $\frac{\partial v}{\partial t}|_{t=0} = v$. This completes the proof of the proposition.

**Theorem 2.1.** Let $g \in \mathcal{M}_K^\gamma$ be a smooth metric such that the first Dirichlet eigenvalue of $(n-1)\Delta_g + K$ is positive. Then $g$ is a critical point of the volume functional in $\mathcal{M}_K^\gamma$ if and only if there is a smooth
function \( \lambda \) on \( \overline{\Omega} \) such that

\[
\begin{align*}
-(\Delta_g \lambda)g + \nabla^2_g \lambda - \lambda \text{Ric}(g) &= g \text{ in } \Omega ; \\
\lambda &= 0 \text{ on } \Sigma.
\end{align*}
\]

Proof. By Lemma 2.2, there is a neighborhood \( U \) of \( g \) in \( M_\gamma \) such that \( U \cap M^K_\gamma \) is a submanifold. Suppose that \( g \) is a critical point of \( V(\cdot) \) in \( M^K_\gamma \). Since the first eigenvalue of \((n-1)\Delta_g + K\) is positive, by the Fredholm alternative we have a smooth function \( \lambda \) on \( \overline{\Omega} \) satisfying:

\[
\begin{align*}
\Delta \lambda &= -\frac{1}{n-1}(\lambda K + n), \text{ in } \Omega \\
\lambda &= 0, \text{ on } \Sigma.
\end{align*}
\]

We want to prove that \( \lambda \) satisfies the interior equation in (14). Let \( h \) be a smooth symmetric \((0,2)\) tensor with compact support in \( \Omega \). Let \( g(t) = g + th \), then \( g(t) \in M_\gamma \) for small \( t \). For each \( t \), consider the following Dirichlet boundary value problem:

\[
\begin{align*}
\alpha \Delta_{g(t)} u - R(t)u &= -Ku^a, \text{ in } \Omega \\
u &= 1, \text{ on } \Sigma,
\end{align*}
\]

where \( \alpha = 4(n-1)/(n-2) \), \( a = (n+2)/(n-2) \) and \( R(t) \) is the scalar curvature of \( g(t) \). For \( |t| \) small, the equation has a unique positive solution \( u(t) \) which is smooth up to the boundary by Proposition 2.1. Moreover, \( u^{4/(n-2)}(t)g(t) \) is in \( M^K_\gamma \) and is a \( C^1 \) curve in \( U \cap M^K_\gamma \). Since \( u \equiv 1 \) at \( t = 0 \),

\[
\left. \frac{d}{dt} (u^{4/(n-2)}(t)g(t)) \right|_{t=0} = \frac{4}{n-2}u'g + h,
\]

where \( u' = u'(0) \). Since \( g \) is a critical point of \( V(\cdot) \) in \( M^K_\gamma \), by (6), we have

\[
\int_{\Omega} \left( \frac{4n}{n-2}u' + \text{tr}_g(h) \right) dV_g = 0.
\]

Now by Proposition 2.1 again, \( u' \) satisfies:

\[
\begin{align*}
\alpha \Delta_g u' - R'(0) - Ku' &= -aKu', \text{ in } \Omega \\
u' &= 0, \text{ on } \Sigma.
\end{align*}
\]
Since $\lambda = 0$ on $\Sigma$, by (13) and (18), we have

$$
\frac{4n}{n-2} \int_{\Omega} u'dV_g = \frac{n\alpha}{n-1} \int_{\Omega} u'dV_g \\
= \alpha \int_{\Omega} \left( -\frac{\lambda K u'}{n-1} - u'\Delta_g \lambda \right) dV_g \\
= \alpha \int_{\Omega} \left( -\frac{\lambda K u'}{n-1} - \lambda \Delta_g u' \right) dV_g \\
= \int_{\Omega} \left( -\frac{\alpha \lambda K u'}{n-1} - \lambda R'(0) - (1-a) K \lambda u' \right) dV_g \\
= \int_{\Omega} \lambda (\Delta_g (\text{tr}_g h) - \text{div}_g (\text{div}_g (h)) + \langle h, \text{Ric}(g) \rangle_g) dV_g
$$

where we have used (3) in the last step. Combining this with (17) we have

$$
\int_{\Omega} \lambda (\Delta_g (\text{tr}_g h) - \text{div}_g (\text{div}_g (h)) + \langle h, \text{Ric}(g) \rangle_g) dV_g = 0.
$$

Let $f = \text{tr}_g (h)$. Let $\nu$ be the unit outward normal of $\Sigma$. Since $f$ has compact support in $\Omega$, we have

$$
\int_{\Omega} \lambda \Delta_g f dV_g = \int_{\Omega} f \Delta_g \lambda dV_g + \oint_{\Sigma} (\lambda f \nu - f \lambda \nu) = \int_{\Omega} f \Delta_g \lambda dV_g
$$

where we use $\psi_{\nu}$ to denote $\frac{\partial \psi}{\partial \nu}$ for a smooth function $\psi$ on $\bar{\Omega}$, and

$$
\int_{\Omega} \lambda \text{div}_g (\text{div}_g (h)) = \int_{\Omega} \lambda g^{ij} g^{kl} h_{ik, jl} dV_g \\
= -\int_{\Omega} \lambda g^{ij} g^{kl} h_{ik, jl} dV_g + \oint_{\Sigma} \lambda (\text{div} h)_k \nu^k \\
= \int_{\Omega} \langle \nabla_2^2 \lambda, h \rangle dV_g - \oint_{\Sigma} h (\nabla \lambda, \nu) + \oint_{\Sigma} \lambda (\text{div} h)_k \nu^k \\
= \int_{\Omega} \langle \nabla_2^2 \lambda, h \rangle dV_g
$$

where we have used the fact that $h$ has compact support. Combining (20)–(22), we have:

$$
0 = \int_{\Omega} \langle h, (\Delta_g \lambda) g - \nabla_2^2 \lambda + \lambda \text{Ric}(g) + g \rangle_g dV_g.
$$

Since $h$ is arbitrary, $\lambda$ must satisfy the interior equation in (14).
To prove sufficiency, suppose there is a smooth function \( \lambda \) satisfying (14), let \( h \in S^2_0 \) be in the tangent space of \( g \) in \( \mathcal{M}_\gamma^K \). Then \( h \) is in the kernel of \( DR_g \). Let \( f = \text{tr}_g(h) \) as before. By (4) and the computation in (21)–(22), we have

\[
0 = \int_{\Omega} \lambda \left( \Delta_g f - \text{div}_g(\text{div}_g(h)) + \langle h, \text{Ric}(g) \rangle_g \right) dV_g \\
= \int_{\Omega} \langle h, (\Delta_g \lambda) g - \nabla^2_g \lambda + \lambda \text{Ric}(g) + g \rangle_g dV_g - \int_{\Omega} f dV_g \\
- \oint_{\Sigma} f \lambda \nu + \oint_{\Sigma} h(\nabla \lambda, \nu) + \oint_{\Sigma} \lambda \left[ f_\nu - (\text{div}_g h)(\nu) \right] \\
= - \int_{\Omega} f dV_g \\
= -2DV(h),
\]

where we have used the fact that \( \lambda \) satisfies (14), \( h|_{T(\Sigma)} = 0 \), and \( \lambda = 0 \) on \( \Sigma \). Hence \( g \) is a critical point of \( V(\cdot) \) in \( \mathcal{M}_\gamma^K \).

\[\square\]

**Remark 2.1.**

(i) From the proof of Theorem 2.1, one can see that under the assumptions on \( g \), \( g \) is a critical point of the volume functional \( V(\cdot) \) in \( \mathcal{M}_\gamma^K \) if and only if \( V'(0) = \frac{d}{dt} V(g(t)) |_{t=0} = 0 \) for any smooth variation \( \{g(t)\} \) of \( g \) in \( \mathcal{M}_\gamma^K \).

(ii) The differential equation in (14) can be equivalently written as \( DR^*_g(\lambda) = g \), where \( DR^*_g \) is the formal \( L^2 \)-adjoint of \( DR_g \), and a weak form of (14) can also be derived using the infinite dimensional Lagrangian multiplier method employed by Bartnik in \([3]\).

Theorem 2.1 shows that, for a constant scalar curvature metric \( g \) to be a critical point for \( V(\cdot) \) in \( \mathcal{M}_\gamma^K \), there need to exist a function \( \lambda \) which satisfies both the interior equation

\[
- (\Delta_g \lambda) g + \nabla^2_g \lambda - \lambda \text{Ric}(g) = g \quad \text{on} \ \Omega
\]

and the boundary condition \( \lambda|_{\Sigma} = 0 \). (Later in Theorem 3.2 one will see that (25) alone implies that \( g \) has constant scalar curvature.) It remains interesting to know what the first variation of \( V(\cdot) \) in \( \mathcal{M}_\gamma^K \) would be if only (25) is satisfied but \( \lambda|_{\Sigma} \) is not necessarily zero.

**Proposition 2.2.** Let \( g \in \mathcal{M}_\gamma^K \) be a smooth metric. Suppose there exists a smooth function \( \lambda \) on \( \Omega \) such that

\[
- (\Delta_g \lambda) g + \nabla^2_g \lambda - \lambda \text{Ric}(g) = g \quad \text{on} \ \Omega.
\]
Let \( \{g(t)\} \) be a smooth path of metrics in \( \mathcal{M}_\gamma \) such that \( g(0) = g \) and \( g(t) \in \mathcal{M}^K_\gamma \). Then

\[
\frac{d}{dt} V(g(t))|_{t=0} = \oint_\Sigma \lambda H'(0),
\]

where \( H = H(t) \) is the mean curvature of \( \Sigma \) in \( (\Omega, g(t)) \) with respect to the unit outward pointing normal vector \( \nu \).

**Proof.** Similar to (24), we have

\[
d(27) \quad \frac{d}{dt} V(g(t))|_{t=0} = \oint_\Sigma \lambda H'(0),
\]

where we let

\[
(28) \quad \frac{d}{dt} V(g(t)) = \oint_\Sigma \lambda H'(0),
\]

we have

\[
(29) \quad \frac{d}{dt} V(g(t)) = \oint_\Sigma \lambda H'(0),
\]

where we let \( h = g'(0) \), \( f = \text{tr}_g h \) and used the fact \( D\mathcal{R}_g(h) = 0 \). By (26), we have

\[
(29) \quad \oint_\Omega f dV_g = - \oint_\Sigma f \lambda \nu + \oint_\Sigma h(\nabla \lambda, \nu) + \oint_\Sigma \lambda [f \nu - (\text{div}_g h)(\nu)],
\]

where \( \nabla \Sigma \lambda \) be the gradient of \( \lambda \) on \((\Sigma, \gamma)\) and integrate by parts

\[
(30) \quad \oint_\Sigma h(\nabla \lambda, \nu) = \oint_\Sigma h(\nabla \Sigma \lambda, \nu) + \oint_\Sigma h(\nu, \nu) \lambda \nu
\]

where \( X \) is the vector field on \( \Sigma \) that is dual to the one form \( h(\nu, \cdot)|_{\Gamma(\Sigma)} \) on \((\Sigma, \gamma)\) and \( \text{div}_\gamma X \) denotes the divergence of \( X \) on \((\Sigma, \gamma)\). Plug (30) in (29), we have

\[
(31) \quad \oint_\Omega f dV_g = - \oint_\Sigma \langle \gamma, h \rangle \gamma \lambda \nu + \oint_\Sigma \lambda [-\text{div}_\gamma X + f \nu - (\text{div}_g h)(\nu)].
\]

Now, let \( p \in \Sigma \) and let \( \{x^i \mid i = 1, \ldots, n\} \) be a coordinate chart around \( p \) in \( \Omega \) such that \( \{x^A \mid A = 1, \ldots, n-1\} \) gives a coordinate chart on \( \Sigma \) and \( \partial_n = \nu \). Direct calculation shows

\[
(32) \quad (\text{div}_g h)_n = h_{mn,n} + \gamma^{AB} h_{An;B},
\]

\[
h_{nA;B} = \partial_B(h_{nA}) - \Gamma^C_{AB} h_{nC} - \Gamma^n_{AB} h_{nn} - \Gamma^i_{nB} h_{iA}
\]

\[
= X_{A;B} + \mathbb{II}_{AB} h_{nn} - \mathbb{II}_C h_{CA},
\]

where \( \mathbb{II}_{AB} = \langle \nabla_{\partial_n} \nu, \partial_B \rangle \gamma \) is the second fundamental form of \( \Sigma \). Thus,

\[
(32) \quad (\text{div}_g h)_n = h_{mn,n} + \text{div}_\gamma X + H h_{nn} - \langle \mathbb{II}, h \rangle_\gamma.
\]
On the other hand, we have the following formula of the linearization of the mean curvature, see equation (42) in [13] for example:

\begin{equation}
H'(0) = \frac{1}{2} h_{nn;n} + \frac{1}{2} H h_{nn} - \langle \Pi, h \rangle_g - \left[ \text{div}_g h - \frac{1}{2} d(\tr_g h) \right]_n.
\end{equation}

(Note that the sign convention of $\Pi$ in [13] is opposite to the one used here.) Hence (32) and (33) imply that

\begin{equation}
2H'(0) = [d(\tr_g h) - \text{div}_g h]_n - \text{div}_X - \langle II, h \rangle_\gamma.
\end{equation}

Therefore, (31) becomes

\begin{equation}
\int_\Omega fdV_g = -\oint_\Sigma \langle \gamma, h \rangle_\gamma \lambda_\nu + \oint_\Sigma \lambda [2H'(0) + \langle \Pi, h \rangle_\gamma].
\end{equation}

Finally, by (4) and the boundary condition $h|_{\Gamma(\Sigma)} = 0$, we have

\begin{equation}
\frac{d}{dt} V(g(t))|_{t=0} = \oint_\Sigma \lambda H'(0).
\end{equation}

\[\square\]

As an application, we have the following:

**Corollary 2.2.** Suppose $g \in \mathcal{M}_K^K$ is a smooth Einstein metric with $\text{Ric}(g) = \kappa g$ (so $K = n\kappa$). Suppose $\kappa \neq 0$. Let $\{g(t)\}$ be a smooth path of metrics in $\mathcal{M}_\gamma^K$ such that $g(0) = g$ and $g(t) \in \mathcal{M}_\gamma^K$. Then

\begin{equation}
\frac{d}{dt} V(g(t))|_{t=0} = -\frac{1}{\kappa} \oint_\Sigma H'(0).
\end{equation}

In particular, if the first Dirichlet eigenvalue of $(n-1)\Delta_g + K$ is positive, then $g$ is a critical point of the volume functional $V(\cdot)$ in $\mathcal{M}_\gamma^K$ if and only if $\oint_\Sigma H'(0) = 0$ for any smooth variation $\{g(t)\}$ of $g$ in $\mathcal{M}_\gamma^K$.

**Proof.** If $g$ satisfies $\text{Ric}(g) = \kappa g$, then the constant function $\lambda = -1/\kappa$ satisfies (26). By Proposition 2.2, (37) is true. The last part of the corollary follows from (37) and Theorem 2.1, see Remark 2.1(i). \[\square\]

For example, if $\Omega$ is a domain in $\mathbb{S}^n$, then

\[\frac{d}{dt} V(g(t))|_{t=0} = -\frac{1}{n-1} \oint_\Sigma H'(0)\]

for any smooth variation $\{g(t)\}$ of the standard metric on $\Omega$ which keeps the induced boundary metric fixed.
3. Critical points in space forms

In this section, we shall discuss the volume functional on domains in space forms.

**Theorem 3.1.** Let \( \Omega \) be a connected domain with compact closure in \( \mathbb{R}^n, \mathbb{H}^n \) or \( \mathbb{S}^n \) and with a smooth (possibly disconnected) boundary \( \Sigma \). If \( \Omega \subset \mathbb{S}^n \), we also assume that \( V(\Omega) < \frac{1}{2} V(\mathbb{S}^n) \). Let \( g \) be the standard metric on \( \Omega \) and let \( \gamma = g|_{\mathcal{T}(\Sigma)} \). Suppose \( g \) is a critical point of the volume functional \( V(\cdot) \) in \( \mathcal{M}^K \), where (i) \( K = 0 \) if \( \Omega \subset \mathbb{R}^n \), (ii) \( K = -n(n-1) \) if \( \Omega \subset \mathbb{H}^n \), and (iii) \( K = n(n-1) \) if \( \Omega \subset \mathbb{S}^n \), then \( \Omega \) is a geodesic ball. Conversely, if \( \Omega \) is a geodesic ball, then the standard metric \( g \) is a critical point of the volume functional \( V(\cdot) \) in \( \mathcal{M}^K \).

**Proof.** By Theorem 2.1, \((\Omega, g)\) is a critical point if and only if there exists a smooth function \( \lambda \) such that
\[
\begin{align*}
-(\Delta g \lambda) + \nabla^2 g \lambda - \lambda \text{Ric}(g) &= g \text{ in } \Omega; \\
\lambda &= 0 \text{ on } \Sigma.
\end{align*}
\]

Taking trace of the equation, we have:
\[
(38) \quad \Delta g \lambda = -\frac{1}{n-1} (\lambda K + n) = \begin{cases} 
-\frac{n}{n-1}, & \text{if } K = 0; \\
\lambda n - \frac{1}{n-1}, & \text{if } K = -n(n-1); \\
-\lambda n - \frac{1}{n-1}, & \text{if } K = n(n-1).
\end{cases}
\]

Hence \( \lambda \) satisfies
\[
(39) \quad \nabla^2 g \lambda = \lambda \text{Ric}(g) - \frac{\lambda K + 1}{n-1} g = \begin{cases} 
-\frac{1}{n-1} g, & \text{if } K = 0; \\
(\lambda - \frac{1}{n-1}) g, & \text{if } K = -n(n-1); \\
(-\lambda - \frac{1}{n-1}) g, & \text{if } K = n(n-1).
\end{cases}
\]

In case (i), where \( \Omega \) is a domain in \( \mathbb{R}^n \), (39) directly implies
\[
(40) \quad \lambda = -\frac{1}{2(n-1)} |x|^2 + \sum_{i=1}^n b_i x^i + c,
\]
where \( x^1, \ldots, x^n \) are the standard coordinates on \( \mathbb{R}^n \) and \( b_i, c \) are constants. By translating the origin, we may assume that \( \lambda = -\frac{1}{2(n-1)} |x|^2 + c \) for a possibly different \( c \). Since \( \lambda \) is zero at the boundary of \( \Omega \), \( \Omega \) must be a Euclidean ball and \( c = \frac{1}{2(n-1)} R^2 \), where \( R \) is the radius of the ball. Conversely, if \( \Omega \) is a Euclidean ball of radius \( R \), then it is easy to see that \( \lambda \) given above satisfies the conditions in Theorem 2.1, therefore the standard metric is a critical point.

Next we consider case (ii), where \( \Omega \) is a domain in \( \mathbb{H}^n \). Suppose the standard metric is a critical point of the volume functional. Let
\( \lambda \) satisfy (39), then \( \lambda \) is not identically zero. Since \( \lambda = 0 \) on \( \Sigma \), there must be an interior point \( p \in \Omega \) such that \( \nabla \lambda(p) = 0 \). Henceforth, we use \( \nabla \) to denote the covariant derivative with respect to \( g \).

Embed \( \mathbb{H}^n \) in \( \mathbb{R}^{n,1} \), the Minkowski space with metric \( dx_1^2 + \cdots + dx_n^2 - dt^2 \) such that
\[
\mathbb{H}^n = \{(x_1, \ldots, x_n, t) \mid x_1^2 + \cdots + x_n^2 - t^2 = -1, \ t > 0 \}
\]
and such that \( p \) is mapped to the point \((0, \cdots, 0, 1)\). Then \( \nu = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + t \frac{\partial}{\partial t} \) is normal to \( \mathbb{H}^n \) with \( \langle \nu, \nu \rangle = -1 \) in \( \mathbb{R}^{n,1} \). Let \( \nabla \) and \( D \) be the covariant derivatives of \( \mathbb{H}^n \) and \( \mathbb{R}^{n,1} \) respectively. Consider a point in \( \mathbb{H}^n \) and a function \( f \) defined near that point in \( \mathbb{R}^{n,1} \). Let \( \{e_i\} \) be a basis for the tangent space of \( \mathbb{H}^n \), we have
\[
\nabla_i \nabla_j f = D_i D_j f + \Pi_{ij} \langle \text{grad} f, \nu \rangle
\]
where \( \text{grad} f \) is the gradient of \( f \) in \( \mathbb{R}^{n,1} \), \( \Pi_{ij} = -\langle D_i e_j, \nu \rangle \) is the second fundamental form of \( \mathbb{H}^n \) and is equal to the induced metric \( g_{ij} \) on \( \mathbb{H}^n \).

Let \( f \) be the function \( at + \frac{1}{n-1} \) where \( a \) is chosen so that \( a + \frac{1}{n-1} = \lambda(p) \). Hence \( f(p) = \lambda(p) \) and \( \nabla f(p) = 0 \). Since \( \langle \text{grad} f, \nu \rangle = at \), one can check that \( f \) satisfies (39). So \( \nabla^2 (\lambda - f) = (\lambda - f) g \). Consider a geodesic \( \sigma(s) \) on \( \mathbb{H}^n \) emanating from \( p \). Restricted to \( \sigma(s) \), the function \( \lambda - f \) satisfies the ODE
\[
(\lambda - f)'' = \lambda - f,
\]
where "" is taken with respect to \( s \). Since initially \( \lambda - f = (\lambda - f)' = 0 \), \( \lambda - f \) must be identically zero. So \( \lambda = f \). But \( f = \lambda = 0 \) at the boundary of \( \Omega \). Hence \( \Omega \) is a geodesic ball. When restricted on \( \mathbb{H}^n \), \( t = \cosh r \) where \( r \) is the geodesic distance from the point \((0, \cdots, 0, 1)\). Hence \( a = -((n - 1) \cosh R)^{-1} \) where \( R \) is the radius of the geodesic ball \( \Omega \).

Conversely, if \( \Omega \) is a geodesic ball with center at \((0, \cdots, 0, 1)\) with geodesic radius \( R \). Then
\begin{equation}
\lambda = \frac{1}{n-1} \left( 1 - \frac{\cosh r}{\cosh R} \right)
\end{equation}
satisfies the conditions in Theorem 2.1 which implies the standard metric is a critical point.

Finally we consider case (iii), where \( \Omega \) is a domain with compact closure in \( \mathbb{S}^n \). Suppose the standard metric is a critical point of the volume functional. Let \( \lambda \) satisfy (39), then \( \lambda \) is not identically zero. Since \( \lambda = 0 \) on \( \Sigma \), there must be a point \( p \in \Omega \) such that \( \nabla \lambda(p) = 0 \). Embed \( \mathbb{S}^n \) in \( \mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n)\} \) as the unit sphere centered at the origin such that \( p \) is mapped to the point \((0, \cdots, 0, 1)\). Let \( f = ax_n - \frac{1}{n-1} \), where \( a \) is chosen so that \( a - \frac{1}{n-1} = \lambda(p) \). Hence
\( f(p) = \lambda(p) \) and \( \nabla f(p) = 0 \). As in case (ii), one can check that \( f \) satisfies (39). Then one can prove as before that \( \lambda = f \). Since \( f = \lambda = 0 \) on the boundary of \( \Omega \), \( \Omega \) must be a geodesic ball centered at \((0, \ldots, 0, 1)\) whose boundary can not be the equator \( \{x_n = 0\} \). As we assume \( V(\Omega) < \frac{1}{2} V(S^n) \), \( \Omega \) must be strictly contained in the upper hemisphere. When restricted to \( S^n \), \( x_n = \cos r \) where \( r \) is the geodesic distance from the point \((0, \cdots, 0, 1)\). Hence \( a = ((n-1) \cos R)^{-1} \) where \( R \) is the radius of the geodesic ball \( \Omega \).

Conversely, if \( \Omega \) is a geodesic ball in \( S^n \) with radius \( R \neq \frac{\pi}{2} \), then
\[
(42) \quad \lambda = \frac{1}{n-1} \left( \frac{\cos r}{\cos R} - 1 \right)
\]
satisfies (14) in Theorem 2.1. If furthermore \( \Omega \) is contained in a hemisphere, then Theorem 2.1 shows that the standard metric is a critical point. \( \square \)

As a direct application of Theorem 3.1 and Corollary 2.2, we have

**Proposition 3.1.** Let \( \Omega \) be a connected domain with compact closure in \( \mathbb{H}^n \) or \( S^n \) and with a smooth (possibly disconnected) boundary \( \Sigma \). If \( \Omega \subset S^n \), we also assume that \( V(\Omega) < \frac{1}{2} V(S^n) \). Let \( g \) be the standard metric on \( \Omega \), let \( K \) be the constant that is equal to the scalar curvature of \( g \) and let \( \gamma = g|_{T(\Sigma)} \). Then \( \Omega \) is a geodesic ball if and only if
\[
\oint_{\Sigma} H'(0) = 0
\]
for any smooth variation \( \{g(t)\} \) of \( g \) in \( M_K^\gamma \). Here \( H'(0) \) is the variation of the mean curvature of \( \Sigma \) in \((\Omega, g(t))\) with respect to the outward unit normal.

**Proof.** By Theorem 3.1, \( \Omega \) is a geodesic ball if and only if the standard metric \( g \) is a critical point of the volume functional \( V(\cdot) \) in \( M_K^\gamma \). On the other hand, as \( g \) is an Einstein metric with non-zero scalar curvature, Corollary 2.2 shows \( g \) is a critical point of \( V(\cdot) \) in \( M_K^\gamma \) if and only if
\[
\oint_{\Sigma} H'(0) = 0
\]
for any smooth variation \( \{g(t)\} \) of \( g \) in \( M_K^\gamma \). The proposition follows. \( \square \)

Before we proceed to discuss properties of general critical metrics of \( V(\cdot) \), we want to relate the result in Proposition 3.1 to the results in [19] and [20]. Let \( \Sigma \) be any given compact strictly convex hypersurface in \( \mathbb{R}^3 \). If \((\Omega, g)\) is a compact Riemannian 3-manifold with nonnegative
scalar curvature whose boundary is isometric to $\Sigma$ and has positive mean curvature $H$, then it was proved in [19] that

\begin{equation}
\int_{\Sigma} H_0 \geq \oint_{\partial \Omega} H,
\end{equation}

where $H_0$ is the mean curvature of $\Sigma$ in $\mathbb{R}^3$. In particular, if $\Omega$ is the domain enclosed by $\Sigma$ in $\mathbb{R}^3$, (43) then implies that

\[ \int_{\Sigma} H'(0) = 0 \]

for any smooth variation $\{g(t)\}$ of $g$ in $\mathcal{M}_\gamma^0$. This contrasts sharply with Proposition 3.1 by which we know the unique compact convex surfaces in $\mathbb{H}^3$ or $S^3_+$ with that property are geodesic spheres. As a result, Proposition 3.1 implies that (43) does not generalize directly to an arbitrary compact convex surface in $\mathbb{H}^3$ or $\Sigma^3_+$. On the other hand, it was proved in [20] that (43) does hold if $\Sigma$ is a geodesic sphere in $\mathbb{H}^3$ and $(\Omega, g)$ has scalar curvature no less than $-6$, which is consistent with Proposition 3.1. It remains an interesting question to know whether (43) is true for a compact 3-manifold $(\Omega, g)$ whose boundary is isometric to a geodesic sphere in $\Sigma^3_+$ and whose scalar curvature is greater than or equal to $+6$.

Next we discuss some general properties of a Riemannian metric $g$ for which there exists a function $\lambda$ satisfying the differential equation in (44) in Theorem 2.1

**Theorem 3.2.** Let $(\Omega, g)$ be a connected, smooth Riemannian manifold. Suppose there is a smooth function $\lambda$ on $\Omega$ such that

\begin{equation}
-(\Delta_g \lambda)g + \nabla^2 \lambda - \lambda \text{Ric}(g) = g.
\end{equation}

Then

(i) $g$ has constant scalar curvature.

(ii) If $\Omega$ is compact without boundary and $g$ has negative scalar curvature, then $g$ is an Einstein metric.

(iii) If $\Omega$ is compact with a smooth (possibly disconnected) boundary $\Sigma$ such that $\lambda = 0$ at $\Sigma$, and if the first Dirichlet eigenvalue of $(n - 1)\Delta_g + K$ is nonnegative where $K$ is the scalar curvature of $g$, then along each connected component $\Sigma_\alpha$ of $\Sigma$, the mean curvature of $\Sigma_\alpha$ with respect to the outward unit normal $\nu$ is a positive constant. In fact, $\Sigma_\alpha$ is umbilic and its second fundamental form $\mathbb{II}_\alpha$ satisfies $\mathbb{II}_\alpha = a_\alpha g|_{T(\Sigma_\alpha)}$ for some constant $a_\alpha > 0$. 
(iv) Under the same assumptions as in (iii), at each point in $\Sigma$,

$$2\text{Ric}(\nu,\nu) + R_{\Sigma} = K + \frac{n-2}{n-1}H^2,$$

where $R_{\Sigma}$ is the scalar curvature of $\Sigma$ and $H$ is mean curvature of $\Sigma$.

Proof. (i) can be proved as in [8]. By taking the divergence of (44), and using the Bianchi identity, we conclude that $\lambda dR = 0$ where $R$ is the scalar curvature of $g$. At a point $p$ where $\lambda(p) \neq 0$, we have $dR = 0$. Suppose $p \in \Omega$ is an interior point where $\lambda(p) = 0$. The equation

$$\Delta_g \lambda = -\frac{n}{n-1} - \frac{R}{n-1} \lambda$$

then implies $\Delta \lambda(p) < 0$. Thus, either $\nabla \lambda(p) \neq 0$ or $p$ is a strict local maximal point for $\lambda$. In either case, we would have $dR = 0$ in a neighborhood around $p$. Hence, $dR = 0$ in $\Omega$.

To prove (ii), we know that $R$ is a negative constant by (i) and the assumption. As $\Omega$ is compact without boundary, it follows from (46) and the maximum principle that $\lambda$ must be a constant. Hence, $g$ is an Einstein metric.

To prove (iii), we note that the boundary condition $\lambda = 0$ at $\Sigma$, together with (44), implies

$$\nabla^2_g \lambda = -\frac{1}{n-1}g$$

at $\Sigma$. Now choose a local orthonormal frames $\{e_i\}$ at the boundary so that $e_i$ is tangential for $1 \leq i \leq n-1$ and $e_n = \nu$ is the unit outward normal. For $1 \leq i \leq n-1$, (47) implies

$$-\frac{1}{n-1} = \nabla^2_g \lambda(e_i, e_i) = e_i e_i(\lambda) - \nabla_{e_i} e_i(\lambda) = -\langle \nabla_{e_i} e_i, e_n \rangle e_n(\lambda).$$

Summing over $1 \leq i \leq n-1$, we have

$$-1 = H \frac{\partial \lambda}{\partial \nu},$$

where $H$ is the mean curvature of $\Sigma$ with respect to $\nu$. If $\lambda < 0$ somewhere, then the set $U = \{x \in \Omega | \lambda < 0\}$ is a nonempty open set contained in $\Omega$ such that $\lambda = 0$ at $\partial U$ because $\lambda = 0$ at $\Sigma$. Since $(n-1)\Delta \lambda + K \lambda = -n < 0$, we have

$$\int_U ((n-1)|\nabla \lambda|^2 - K\lambda^2) \, dV < 0,$$
contradicting the fact that the first Dirichlet eigenvalue of \((n-1)\Delta + K\) is nonnegative. Therefore \(\lambda > 0\) in \(\Omega\). This together with (48) implies \(\frac{\partial \lambda}{\partial \nu} < 0\) at \(\Sigma\). Hence \(H > 0\) and \(e_n = \nu = -\nabla \lambda / |\nabla \lambda|\).

Next, let \(X\) and \(Y\) be two arbitrary vector fields on \(\Sigma\) which are tangential to \(\Sigma\). At \(\Sigma\), (47) implies
\[
0 = \nabla^2_g \lambda (X, \nu) = X \nu(\lambda) - \nabla_X \nu(\lambda) = -X(|\nabla \lambda|)
\]
where we used \(\lambda = 0\) at \(\Sigma\), and
\[
-\frac{1}{n-1} g(X, Y) = \nabla^2_g \lambda (X, Y) = XY(\lambda) - \nabla_X Y(\lambda) = \langle \nabla_X Y, |\nabla \lambda| \nu \rangle_g = -|\nabla \lambda| \mathbb{II},
\]
where \(\mathbb{II}\) denotes the second fundamental form of \(\Sigma\) with respect to \(\nu\). Hence, along each connected component \(\Sigma_\alpha\) of \(\Sigma\), we conclude that \(|\nabla \lambda|\) is a positive constant and \(\mathbb{II}\) equals a positive constant multiple of the induced metric on \(\Sigma\). (The constants may depend on \(\alpha\).)

(iv) follows directly from the Gauss equation and (iii). This completes the proof of the theorem. \(\square\)

Theorem 3.1 shows that the standard metrics on geodesic balls (which are contained in a hemisphere for the case of \(\mathbb{S}^n\)) in space forms are critical points of the volume functional in the corresponding spaces of metrics. It is natural to ask whether they are the only critical points with that boundary condition. Using Theorem 3.2 we give some partial answer to this question.

**Corollary 3.1.** Let \(\Omega\) be an \(n\)-dimensional compact manifold with a smooth connected boundary \(\Sigma\). Let \(\gamma\) be a given metric on \(\Sigma\) and let \(K = 0\) or \(-n(n-1)\). Suppose \(g \in \mathcal{M}^K_\gamma\) is a smooth metric and \(g\) is a critical point of the volume functional \(V(\cdot)\) in \(\mathcal{M}^K_\gamma\). Let \(\nu\) be the outward unit normal vector to \(\Sigma\) in \((\Omega, g)\).

(i) If \(K = 0\), \((\Sigma, \gamma)\) is isometric to a geodesic sphere in \(\mathbb{R}^n\) and \(\Omega\) is spin if \(n \geq 8\), then \(\text{Ric}(\nu, \nu)\) is a non-positive constant along \(\Sigma\), and \(\text{Ric}(\nu, \nu) = 0\) if and only if \((\Omega, g)\) is isometric to a standard ball in \(\mathbb{R}^n\).

(ii) If \(n = 3\), \(K = 0\), \(\Omega\) is oriented and \(\text{Ric}(\nu, \nu) = 0\) along \(\Sigma\), then \((\Omega, g)\) is isometric to a standard ball in \(\mathbb{R}^3\).
(iii) If \( n = 3 \), \( K = -6 \), and \((\Sigma, \gamma)\) is isometric to a geodesic sphere in \( \mathbb{H}^3 \), then \( \text{Ric}(\nu, \nu) \) is a constant satisfying \( \text{Ric}(\nu, \nu) \leq -2 \) along \( \Sigma \), and \( \text{Ric}(\nu, \nu) = -2 \) if and only if \((\Omega, g)\) is isometric to a geodesic ball in \( \mathbb{H}^3 \).

**Proof.** (i) As \((\Sigma, \gamma)\) is isometric to a geodesic sphere, say \( \Sigma_0 \), in \( \mathbb{R}^n \), \( \text{Ric} \) is a constant along \( \Sigma_0 \). By (iii) and (iv) in Theorem 3.2, \( \text{Ric}(\nu, \nu) \) is a constant along \( \Sigma \). On the other hand, applying the Gauss equation to \( \Sigma_0 \) in \( \mathbb{R}^n \), we have

\[
R_{\Sigma} = \frac{n - 2}{n - 1} H_0^2,
\]

where \( H_0 \) is the mean curvature of \( \Sigma_0 \) in \( \mathbb{R}^n \) (which is a constant). Hence, it follows from (iv) in Theorem 3.2 and (49) that

\[
2 \text{Ric}(\nu, \nu) = \frac{n - 2}{n - 1} [H^2 - H_0^2].
\]

By the results in [19, 12], which are generalizations of the positive mass theorem in [17, 22, 18], we have \( H \leq H_0 \), and \( H = H_0 \) if and only if \((\Omega, g)\) is isometric to a standard ball in \( \mathbb{R}^n \). From these, (i) follows.

(ii) By the assumption, \( \Sigma \) is a connected orientable 2-surface. Let \( K_{\Sigma} \) be the Gaussian curvature of \((\Sigma, \gamma)\). It follows from (iii), (iv) in Theorem 3.2 and the facts \( K = 0 \), \( \text{Ric}(\nu, \nu) = 0 \) that

\[
K_{\Sigma} = \frac{1}{4} H^2,
\]

which is a positive constant. Therefore, \((\Sigma, \gamma)\) is isometric to a round sphere in \( \mathbb{R}^3 \). Now (ii) follows from (i).

(iii) The proof is similar to the proof of (i). As \((\Sigma, \gamma)\) is isometric to a geodesic sphere, say \( \Sigma_1 \), in \( \mathbb{H}^3 \), \( R_{\Sigma} \) is a constant. Hence, \( \text{Ric}(\nu, \nu) \) is a constant. On the other hand, applying the Gauss equation to \( \Sigma_1 \) in \( \mathbb{H}^3 \), we have

\[
2(-2) + R_{\Sigma} = -6 + \frac{n - 2}{n - 1} H_1^2,
\]

where \( H_1 \) is the mean curvature of \( \Sigma_1 \) in \( \mathbb{H}^3 \) (which is a constant). Hence, it follows from (iv) in Theorem 3.2 and (51) that

\[
2[\text{Ric}(\nu, \nu) - (-2)] = \frac{n - 2}{n - 1} [H^2 - H_1^2].
\]

By Theorem 3.8 in [20], we have \( H \leq H_1 \), and \( H = H_1 \) if and only if \((\Omega, g)\) is isometric to a geodesic ball in \( \mathbb{H}^3 \). From these, (iii) follows. □
As another application of Theorem 3.2, assuming $(\Sigma, \gamma)$ can be isometrically embedded as a convex hypersurface in $\mathbb{R}^n$, we can compare the volume of any critical point in $\mathcal{M}_\gamma^0$ with the Euclidean volume enclosed by $(\Sigma, \gamma)$ in $\mathbb{R}^n$.

**Theorem 3.3.** Let $(\Omega, g)$ be an $n$-dimensional smooth compact Riemannian manifold with zero scalar curvature, with a smooth connected boundary $\Sigma$, such that $g$ is a critical point of the volume functional in $\mathcal{M}_\gamma^0$ where $\gamma = g|_{\Sigma}$. Suppose $(\Sigma, \gamma)$ can be isometrically embedded in $\mathbb{R}^n$ as a compact strictly convex hypersurface $\Sigma_0$. If the dimension $n \geq 8$, $\Omega$ is also assumed to be spin. Then

$$V(g) \geq V_0$$

where $V(g)$ is the volume of $(\Omega, g)$ and $V_0$ is the Euclidean volume of the domain bounded by $\Sigma_0$ in $\mathbb{R}^n$. Moreover, $V(g) = V_0$ if and only if $(\Omega, g)$ is isometric to a standard ball in $\mathbb{R}^n$.

**Proof.** As $g$ is a smooth critical point of the volume functional in $\mathcal{M}_\gamma^0$, by Theorem 2.1 there is a smooth function $\lambda$ on $\bar{\Omega}$ such that $\lambda = 0$ at $\Sigma$ and

$$-(\Delta_g \lambda)g + \nabla^2_g \lambda - \lambda \text{Ric}(g) = g$$

in $\Omega$. As $g$ has zero scalar curvature, the condition in (iii) in Theorem 3.2 is satisfied, therefore the mean curvature $H$ of $\Sigma$ in $(\Omega, g)$ with respect to the unit outward normal $\nu$ is a positive constant. Moreover, by (48), $H$ and $\frac{\partial \lambda}{\partial \nu}$ satisfies

$$-1 = H \frac{\partial \lambda}{\partial \nu}$$

at $\Sigma$. Integrating on $\Sigma$, we have

$$|\Sigma| = -\oint_{\Sigma} H \frac{\partial \lambda}{\partial \nu}$$

$$= -H \int_{\Omega} \Delta_g \lambda dV_g$$

$$= \frac{n}{n-1} HV(g),$$

where the last step follows from the fact that

$$\Delta_g \lambda = -\frac{n}{n-1}.$$ 

On the other hand, by the results in [19],

$$\oint_{\Sigma_0} H_0 \geq \oint_{\Sigma} H = |\Sigma| H$$

(54)
where $H_0$ is the mean curvature of $\Sigma_0$ with respect to the outward normal in $\mathbb{R}^n$. By a Minkowski inequality [6]:

\begin{equation}
|\Sigma|^2 \geq \frac{n}{n-1} V_0 \int_{\Sigma_0} H_0.
\end{equation}

It follows from (53), (54) and (55) that

\begin{equation}
\frac{|\Sigma|}{V(g)} = \frac{n}{n-1} H 
\end{equation}

\begin{equation}
\leq \frac{n}{(n-1)|\Sigma|} \int_{\Sigma_0} H_0
\end{equation}

\begin{equation}
\leq \frac{|\Sigma|}{V_0}.
\end{equation}

Hence $V(g) \geq V_0$. If $V(g) = V_0$, then (54) becomes an equality. By the results in [19] again (see also [12]), we know that $(\Omega, g)$ is isometric to a domain in $\mathbb{R}^n$. Finally by Theorem 3.1, we conclude that $(\Omega, g)$ is isometric to a standard ball in $\mathbb{R}^n$. 

\[\square\]

4. Second variational formula for the volume functional

In this section, we will compute the second variation of the volume functional $V(\cdot)$ at critical points in $\mathcal{M}^K_\gamma$. First, we give a formula for the second derivative of the scalar curvature. We remark that our notation convention for the curvature tensor gives

\[R_{ijkl} = \kappa (g_{ik} g_{jl} - g_{il} g_{jk})\]

in the case that $g$ has constant sectional curvature $\kappa$.

**Lemma 4.1.** Let \{g(t)\} be a smooth path of $C^2$ metrics with $g(0) = g$. Let $R(t)$ be the scalar curvature of $g(t)$. Then

\begin{equation}
R''(0) = \Delta_g (|h|_g^2) + 2\langle h, \nabla^2_g (\text{tr}_g h) \rangle_g - 4 \langle \nabla_g (\text{div}_g h), h \rangle_g
\end{equation}

\begin{equation}
- 2 \langle \text{div}_g h - \frac{1}{2} \text{d} (\text{tr}_g h) |g \rangle_g^2 - \frac{1}{2} |\nabla_g h|_g^2 - g^{pq} h^{lk}_{,q} h_{lk}\)
\end{equation}

\begin{equation}
+ 2 h^{lp} R_{ikps} h^{sk} + D R_g (h'),
\end{equation}

where $h = g'(0)$, $h' = g''(0)$, $\nabla_g (\cdot)$ and $' ; '$ denote covariant derivative with respect to $g$, $R_{ijkl}$ is the curvature tensor of $g$, and $D R_g (\cdot)$ is the linearization of the scalar curvature map $\mathcal{R}$ at $g$.

**Proof.** The first derivative of the scalar curvature is given by

\begin{equation}
R'(t) = -\Delta_g (\text{tr}_g g') + \text{div}_g (\text{div}_g (g')) - \langle g', \text{Ric} \rangle_g.
\end{equation}
For any fixed point $p$, let $\{x_i\}$ be a normal coordinate chart at $p$ with respect to $g(0) = g$. We will use '$;$' to denote partial derivative and '$,'$ to denote covariant derivative. At $t = 0$, we have

$$-|\Delta_g (\text{tr}_g g')'| = - \left[ g^{ij} (\text{tr}_g g')_{,ij} - g^{ij} \Gamma_{ij}^k (\text{tr}_g g')_{,k} \right]'$$

$$= - \left[ -k^p g_{ip} (\text{tr}_g h)_{,ij} - \Delta_h |h|^2 \right] - g_{ij} \Gamma_{ij}^{kl} (\text{tr}_g h)_{,kl}$$

$$= (59)$$

and

$$\frac{1}{2} \{ \text{div}_g h, d(\text{tr}_g h) \}_g - \frac{1}{2} |d(\text{tr}_g h)|_g^2,$$

$$= (60)$$

$$\left[ \text{div}_g (\text{div}_g g') \right]' = (g^{ij} g^{kl})' h_{jlt;ki} + g^{ij} g^{kl} (g'_{jlt;ki})'$$

$$= - \left[ \nabla_g \text{div}_g h, d(\text{tr}_g h) \right]_g - g^{ij} g^{kp} g^{q} h_{pq} h'_{jlt;ki} + g^{ij} g^{kl} (g'_{jlt;ki})'$$

$$= -2 \left[ \nabla_g \text{div}_g h, d(\text{tr}_g h) \right]_g - h^{ip} R_{ik} p_k h_{s} - h^{ij} g^{kl} R_{ijkl}$$

$$+ g^{ij} g^{kl} (g'_{jlt;ki})',$$

where we have used the Ricci identities. Now

$$g^{ij} g^{kl} (g'_{jlt;ki})' = g^{ij} g^{kl} (g'_{jlt;ki} - \Gamma_{ij}^s g'_{s;lt;ki} - \Gamma_{ik}^s g'_{jlt;s} - \Gamma_{ik}^s g'_{jlt;s})$$

$$= g^{ij} g^{kl} \left[ \left( h'_{jlt;ki} - (\Gamma_{ij}^s)' h_{st;ik} - (\Gamma_{ik}^s)' h_{jlt;ik} \right) \right]$$

and

$$g^{ij} g^{kl} (h'_{jlt;ik})_i = \text{div}_g (\text{div}_g h'),$$

$$-g^{ij} g^{kl} \left( (\Gamma_{kj}^s)' h_{st} \right)_i = - \frac{1}{2} g^{ij} g^{kl} (g^{sm} (h_{km;j} + h_{mj;k} - h_{kjm}) h_{st})_i$$

$$= - \frac{1}{2} g^{ij} g^{kl} (g^{sm} h_{km;j} h_{st})_i$$

$$= - \frac{1}{4} \Delta_g |h|^2,$$

$$-g^{ij} g^{kl} \left( (\Gamma_{kl}^s)' h_{js} \right)_s = - \frac{1}{2} g^{ij} g^{kl} (g^{sm} (h_{km;l} + h_{ml;k} - h_{kl;m}) h_{js})_s$$

$$= - \left[ \nabla_g \text{div}_g h, d(\text{tr}_g h) \right]_g + \frac{1}{2} \left[ \nabla_g^2 (\text{tr}_g h), h \right]_g$$

$$- |\text{div}_g h|^2 + \frac{1}{2} \left[ \text{div}_g h, d(\text{tr}_g h) \right]_g,$$
Hence,

\[ -g^{ij} g^{kl} \left( \Gamma^s_{ij} \right)' h_{sl;k} = -\frac{1}{2} g^{ij} g^{kl} g^{sm} \left( h_{im;j} + h_{mj;i} - h_{ij;m} \right) h_{sl;k} \]
\[ = -\left| \text{div}_g h \right|_g^2 + \frac{1}{2} \langle \text{div}_g h, d(\text{tr}_g h) \rangle_g, \]
\[ -g^{ij} g^{kl} \left( \Gamma^s_{il} \right)' h_{js;k} = -\frac{1}{2} g^{ij} g^{kl} g^{sm} \left( h_{im;l} + h_{ml;i} - h_{il;m} \right) h_{js;k} \]
\[ = -\frac{1}{2} \left| \nabla_g h \right|_g^2, \]
\[ -g^{ij} g^{kl} \left( \Gamma^s_{ik} \right)' h_{jt;s} = -\frac{1}{2} g^{ij} g^{kl} g^{sm} \left( h_{im;k} + h_{mk;i} - h_{ik;m} \right) h_{jt;s} \]
\[ = -g^{pq} h^{lk}_{;p} h_{tq;k} + \frac{1}{2} \left| \nabla_g h \right|_g^2. \]

Hence,

\[ (\text{div}_g (\text{div}_g g'))' = -3 \langle \nabla_g \text{div}_g h, h \rangle_g + h^{lp} R_{lkps} h^{sk} - h^{ij} g^{kl} R_{jk} h_{il} \]
\[ + \text{div}_g (\text{div}_g h') - \frac{1}{4} \Delta_g |h|_g^2 + \frac{1}{2} \langle \nabla_g^2 (\text{tr}_g h), h \rangle_g \]
\[ + \langle \text{div}_g h, d(\text{tr}_g h) \rangle_g - 2 |\text{div}_g h|_g^2 - g^{pq} h^{lk}_{;p} h_{tq;k}. \] (61)

Next,

\[ -[\langle g', \text{Ric} \rangle_g]' = -\left( g^{ij} g^{kl} g^{ij}_{;ik} R_{jl} \right)' \]
\[ = 2 h^{ij} g^{kl} R_{jk} h_{il} - \langle h', \text{Ric} \rangle_g - g^{ij} g^{kl} h_{ik} R_{jl}'. \]
\[ g^{ij} g^{kl} h_{ik} R_{jl}' = \frac{1}{2} g^{ij} g^{kl} h_{ik} g^{pq} \left( h_{lp;jq} + h_{jp;lq} - h_{jip;q} - h_{pq;jl} \right) \]
\[ = -\frac{1}{4} \Delta_g (|h|_g^2) + \frac{1}{2} \left| \nabla_g h \right|_g^2 - \frac{1}{2} \langle \nabla_g^2 (\text{tr}_g h), h \rangle_g \]
\[ + g^{ij} g^{kl} g^{pq} h_{ik} h_{lp;jq} \]
\[ = -\frac{1}{4} \Delta_g (|h|_g^2) + \frac{1}{2} \left| \nabla_g h \right|_g^2 - \frac{1}{2} \langle \nabla_g^2 (\text{tr}_g h), h \rangle_g \]
\[ + \langle \nabla_g \text{div}_g h, h \rangle_g + h^{ij} g^{kl} R_{jk} h_{il} - h^{lp} R_{lkps} h^{sk}, \]

where we have used the Ricci identity in the last step. Hence

\[ -[\langle g', \text{Ric} \rangle_g]' = h^{ij} g^{kl} R_{jk} h_{il} - \langle h', \text{Ric} \rangle_g + \frac{1}{4} \Delta_g |h|_g^2 - \frac{1}{2} |\nabla_g h|_g^2 \]
\[ + \frac{1}{2} \langle \nabla_g^2 (\text{tr}_g h), h \rangle_g - \langle \nabla_g \text{div}_g h, h \rangle_g + h^{lp} R_{lkps} h^{sk}. \] (63)

So (57) follows from (58), (59), (61) and (63). \qed

Now we are in a position to compute the second variation of \( V(\cdot) \) in \( \mathcal{M}_g^K \). We state the formula in a general setting which does not require the manifold structure of \( \mathcal{M}_g^K \).
Theorem 4.1. Let \( \Omega \) be an \( n \)-dimensional connected compact manifold with a smooth boundary \( \Sigma \). Suppose \( g \) is a smooth metric on \( \Omega \) such that there is a smooth function \( \lambda \) on \( \Omega \) satisfying

\[
\begin{align*}
- (\Delta_g \lambda) g + \nabla^2_g \lambda - \lambda \text{Ric}(g) &= g \text{ in } \Omega; \\
\lambda &= 0 \text{ on } \Sigma.
\end{align*}
\]

Let \( \gamma = g|_{T(\Sigma)} \) and let \( K \) be the constant that equals the scalar curvature of \( g \). Suppose \( \{g(t)\} \) is a smooth path of metrics in \( \mathcal{M}_\gamma \) with \( g(0) = g \) and \( g(t) \in \mathcal{M}_{\gamma}^K \). Let \( V(t) \) be the volume of \( (\Omega, g(t)) \), then

\[
V''(0) = \int_\Omega \left\{ \frac{1}{4} (\text{tr}_g h)^2 + \lambda \left[ \text{div}_g(h) - \frac{1}{2} d(\text{tr}_g h)_g + \frac{1}{4} |\nabla_g h|_g^2 \right] \right\} dV_g \\
+ \int_\Omega \lambda \left[ \langle \nabla_g \text{div}_g h, h \rangle - \langle h, \nabla^2_g (\text{tr}_g h) \rangle \right] dV_g \\
- \int_\Omega \frac{1}{2} \lambda \left[ |\text{div}_g h|_g^2 + h^{sp} R_{g,kpl,s} h^{lk} \right] dV_g,
\]

where \( h = g'(0) \).

Proof. We first note that \( V'(0) = 0 \) by (64) and the proof of Theorem 2.1. To compute \( V''(0) \), we start with

\[
V''(0) = \int_\Omega \left[ \frac{1}{4} (\text{tr}_g h)^2 - \frac{1}{2} |h|_g^2 + \frac{1}{2} (\text{tr}_g h') \right] dV_g,
\]

where \( h = g'(0) \) and \( h' = g''(0) \). Our aim is to express the last integral in terms of \( h \).

Recall that (64) implies

\[
\Delta_g \lambda = -\frac{1}{n-1} (K \lambda + n)
\]

and

\[
\nabla^2_g \lambda = \lambda \text{Ric} - \frac{1}{n-1} (\lambda K + 1) g.
\]

Applying the fact that \( g(t) \) has constant scalar curvature \( K \) and Lemma 4.1, we have

\[
0 = R''(0)
= I - \Delta_g (\text{tr}_g h') + \text{div}_g (\text{div}_g h') - \langle h', \text{Ric} \rangle_g,
\]
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where

\[
I = \Delta_g(|h|^2_g) + 2\langle h, \nabla^2_g (\text{tr}_g h) \rangle_g - 4\langle \nabla_g \text{div}_g h, h \rangle_g \\
- 2|\text{div}_g(h) - \frac{1}{2}d(\text{tr}_g h)|^2_g - \frac{1}{2}|\nabla_g h|^2_g - g^{pq} h^{lk}_{\cdot p} h_{iq;lk} \\
+ 2h^{lp} R_{lkps} h^{sk}.
\]

(70)

Note that \(I\) involves only \(h\) and its derivatives. In what follows, we will omit the volume form and the area form in integrals for convenience. All integrals are taken with respect to the metric \(g = g(0)\). Integrating by parts, we have

\[
\int_\Omega \left[ -\lambda \Delta_g (\text{tr}_g (h')) + \lambda \text{div}_g (\text{div}_g h') - \lambda \langle h', \text{Ric} \rangle_g \right] \\
= \int_\Omega \left[ -\langle \text{tr}_g h' \rangle (\Delta_g \lambda) + \langle \nabla^2_g \lambda, h' \rangle_g - \lambda \langle h', \text{Ric} \rangle_g \right] \\
+ \oint_\Sigma \left[ \lambda \nu (\text{tr}_g h') - h'(\nu, \nabla \lambda) \right] \\
= \int_\Omega \nu (tr_g h') + \oint_\Sigma \lambda \nu (tr_g h') - \oint_\Sigma h'(\nu, \nabla \lambda)
\]

(71)

where we have used \(\lambda |_\Sigma = 0\), (67) and (68). The condition \(\lambda |_\Sigma = 0\), together with \(h'|_{T(\Sigma)} = 0\), also implies

\[-\lambda \nu (\text{tr}_g h') + h'(\nu, \nabla \lambda) = 0 \text{ on } \Sigma.\]

Combining this with (69) and (71), we have,

\[
(72) \quad \int_\Omega (\text{tr}_g h') = - \int_\Omega \lambda I.
\]

Next we compute the integral of \(\lambda I\). Integrating by parts and applying (67) and (68), we have

\[
\int_\Omega \lambda \Delta_g (|h|^2_g) = \int_\Omega (\Delta_g \lambda) |h|^2_g - \oint_\Sigma \lambda \nu |h|^2_g \\
= - \int_\Omega \frac{1}{n-1} (K \lambda + n)|h|^2_g - \oint_\Sigma \lambda \nu |h|^2_g.
\]

(73)

Integrating by parts, we also have

\[
\int \lambda g^{pq} h^{lk}_{\cdot p} h_{iq;lk} = \int (\lambda g^{pq} h^{lk}_{\cdot p} h_{iq;lk})_{\cdot p} - g^{pq} \lambda \nu p h^{lk}_{\cdot p} h_{iq;lk} - \lambda g^{pq} h^{lk}_{\cdot p} h_{iq;lk} \\
= \int -g^{pq} \lambda \nu p h^{lk}_{\cdot p} h_{iq;lk} - \lambda g^{pq} h^{lk}_{\cdot p} h_{iq;lk}.
\]

(74)
Now

\[(75)\]
\[
\int \Omega \int g^{pq} \lambda_p h^{lk} h_{tq;k} = 
\int \Omega \int (g^{pq} \lambda_p h^{lk} h_{tq})_{;k} - g^{pq} \lambda_p h^{lk} h_{tq} - g^{pq} \lambda_p h^{lk} h_{tq} \]

\[
= \int \Omega \int g^{pq} \lambda_p h^{lk} h_{tq} + \int \Omega \frac{1}{n-1} (K \lambda + 1) |h|^2_g - \int \Omega \lambda g^{pq} R_{pk} h^{lk} h_{tq} - \int \Omega \lambda_p (\text{div}_g h) h^{lp}
\]

\[
= \int \Sigma \lambda_\nu |h|^2_g + \int \Omega \frac{1}{n-1} (K \lambda + 1) |h|^2_g - \int \Omega \lambda g^{pq} R_{pk} h^{lk} h_{tq} - \int \Omega \lambda_p (\text{div}_g h) h^{lp}
\]

where we have used the fact that \( h|_{\Gamma(S)} = 0 \). Also

\[(76)\]
\[
\int \Omega \lambda_p (\text{div}_g h) h^{lp} = \int \Omega \int [\lambda(\text{div}_g h)_{;lp}]_{;p} - \lambda(\text{div}_g h)_{;lp} - \lambda |\text{div}_g h|^2_g
\]

\[
= - \int \Omega \lambda (\text{div}_g h, h)_g - \int \Omega \lambda |\text{div}_g h|^2_g.
\]

By the Ricci identity,

\[(77)\]
\[
\int \Omega \lambda g^{pq} h^{lk} h_{tq;kp} = \int \Omega \lambda (\text{div}_g h, h)_g + \lambda g^{pq} R_{pk} h^{lk} h_{tq} - \lambda h^{lp} R_{kp} h^{lk}.
\]

It follows from (73) – (77) that

\[(78)\]
\[
\int \Omega \lambda g^{pq} h^{lk} h_{tq;kp} = - \int \Omega \lambda_\nu |h|^2_g - \int \Omega \frac{1}{n-1} (K \lambda + 1) |h|^2_g

\[
- \int \Omega 2\lambda (\text{div}_g h, h)_g - \int \Omega \lambda |\text{div}_g h|^2_g

\[
+ \int \Omega \lambda h^{lp} R_{kp} h^{lk}.
\]

Combining (70), (73) and (78) we have

\[(79)\]
\[
\int \Omega \lambda I = \int \Omega - |h|^2_g + 2\lambda (h, \text{div}_g (\text{tr}_g (h)))_g - 2\lambda (\text{div}_g (h), h)_g

\[
- \int \Omega \left[2\lambda |\text{div}_g (h)|^2 - \frac{1}{2} |\text{tr}_g (h)|^2_g + \frac{1}{2} \lambda |\text{div}_g (h)|^2_g\right]

\[
+ \int \Omega \lambda |\text{div}_g (h)|^2_g + \lambda h^{lp} R_{kp} h^{lk}.
\]

Now (65) follows from (66), (72) and (79). \(\square\)
Next we focus on domains $\Omega$ in space forms on which the standard metrics are critical points of $V(\cdot)$. By Theorem 3.1, those domains are precisely geodesic balls.

**Theorem 4.2.** Let $\Omega$ be a geodesic ball with compact closure in $\mathbb{R}^n$, $\mathbb{H}^n$ or $\mathbb{S}^n_+$. Let $\Sigma$ be its boundary and $g$ be the standard metric on $\Omega$. Let $\gamma = g|_{\Gamma(\Sigma)}$ and $K$ be the constant that equals the scalar curvature of $g$. Let \( \{g(t)\} \) be a smooth path of metrics in $\mathcal{M}_\gamma^K$ with $g(0) = g$. Let $h$ denote $g'(0)$ which is assumed to be nonzero.

(i) If $\Omega \subset \mathbb{R}^n$ or $\Omega \subset \mathbb{S}^n_+$, then

\[
\frac{d^2}{dt^2} V(g(t))|_{t=0} > 0
\]

for any $h$ satisfying

\[
\text{div}_g h = 0 \quad \text{and} \quad \text{tr}_g h = 0.
\]

(ii) If $\Omega \subset \mathbb{H}^n$, then for any $p \in \Omega$, there is a constant $\delta > 0$, depending on $p$ and $(\Omega, g)$, such that

\[
\frac{d^2}{dt^2} V(g(t))|_{t=0} > 0
\]

for any $h$ which has compact support in $B_\delta(p) \subset \Omega$ and satisfies

\[
\text{div}_g h = 0 \quad \text{and} \quad \text{tr}_g h = 0.
\]

Here $B_\delta(p)$ denotes the geodesic ball with radius $\delta$ centered at $p$ in $\mathbb{H}^n$.

**Proof.** From the proof of Theorem 3.1, we know there exists a smooth positive function $\lambda$ on $\bar{\Omega}$ satisfying (64). (The explicit expression of $\lambda$ is given by (40), (41) and (42)). Applying Theorem 4.1 and the assumption $\text{div}_g h = 0$ and $\text{tr}_g h = 0$, we have

\[
(80) \quad \frac{d^2}{dt^2} V(t)|_{t=0} = \frac{1}{4} \int_{\Omega} \lambda |\nabla_g h|^2 - \frac{1}{2} \int_{\Omega} \lambda h^{sp} R_{kpls} h^{lk}.
\]

As $g$ has constant sectional curvature $\frac{K}{n(n-1)}$, the integrant in the last integral above reduces to

\[
\lambda h^{sp} R_{kpls} h^{lk} = \frac{K\lambda}{n(n-1)} [(\text{tr}_g h)^2 - |h|^2_g].
\]

Therefore (80) becomes

\[
(81) \quad \frac{d^2}{dt^2} V(t)|_{t=0} = \frac{1}{4} \int_{\Omega} \lambda |\nabla_g h|^2 - \frac{K}{2n(n-1)} \int_{\Omega} \lambda |h|^2_g.
\]
where we again used the assumption $\text{tr}_g h = 0$. As $\lambda > 0$ on $\Omega$, (i) follows directly from (81).

Now suppose $\Omega \subset \mathbb{H}^n$, we then have $K = -n(n - 1)$ and
\[
\frac{d^2}{dt^2} V(t)|_{t=0} = \frac{1}{4} \int_{\Omega} \lambda |\nabla g h|_g^2 - \frac{1}{2} \int_{\Omega} \lambda |h|_g^2.
\]

For any $p \in \Omega$, first choose $\delta > 0$ such that $B_{\delta}(p) \subset \Omega$ and
\[
\min_{B_{\delta}(p)} \lambda \geq \frac{1}{2} \lambda(p), \quad \max_{B_{\delta}(p)} \lambda \leq 2 \lambda(p).
\]

If $h$ has compact support contained in $B_{\delta} = B_{\delta}(p)$, then
\[
\frac{d^2}{dt^2} V(t)|_{t=0} \geq \lambda(p) \left\{ \frac{1}{8} \int_{B_{\delta}} |\nabla g h|_g^2 - \int_{B_{\delta}} |h|_g^2 \right\} \geq \lambda(p) \left\{ \frac{1}{8} \int_{B_{\delta}} |\nabla h|_g^2 - \int_{B_{\delta}} |h|_g^2 \right\},
\]
where we used $|\nabla g h|_g \geq |\nabla h|_g$ in the second step. Let $\lambda_1(B)$ be the first eigenvalue of $\Delta_g$ on $B_{\delta}$, then
\[
\frac{d^2}{dt^2} V(t)|_{t=0} \geq \lambda(p) \left\{ \frac{1}{8} \lambda_1(B_{\delta}) - 1 \right\} \int_{B_{\delta}} |h|_g^2.
\]
As $\lim_{\delta \to 0} \lambda_1(B_{\delta}) = +\infty$, we conclude that $\frac{d^2}{dt^2} V(t)|_{t=0} > 0$ if $\delta$ is smaller than some constant $\delta_0$ depending only on $p$ and $(\Omega, g)$. Therefore, (ii) is proved.

**Lemma 4.2.** Suppose $g \in \mathcal{M}_g^K$ is a smooth Einstein metric satisfying the property that the first eigenvalue of $(n - 1) \Delta_g + K$ is positive. Let $h$ be a symmetric $(0,2)$ tensor on $\Omega$ such that $h|_{\partial \Omega} = 0$, $\text{tr}_g h = 0$ and $\text{div}_g \text{div}_g h = 0$. Then there is a variation $\{g(t)\} \subset \mathcal{M}_g^K$ of $g$ such that $g'(0) = h$.

**Proof.** As in the proof of Theorem 2.1, we can find $g(t) \in \mathcal{M}_g^K$ such that $g(t)$ is of the form $u(t)^{\frac{4}{n-2}} (g + th)$ with $u(0) = 1$. Moreover $v = u'(0)$ satisfies:
\[
\begin{cases}
(n - 1) \Delta_g v + K v = \left( \frac{n-2}{4} \right) D\mathcal{R}_g(h), & \text{in } \Omega \\
v = 0, & \text{on } \partial \Omega.
\end{cases}
\]
As $\text{tr}_g(h) = 0$, $\text{div}_g(\text{div}_g(h)) = 0$ and $g$ is Einstein, we know $D\mathcal{R}_g(h) = 0$ by (58). As the first eigenvalue of $(n - 1) \Delta_g + K$ is positive, we have $v \equiv 0$, hence $g'(0) = h$. □
In the appendix following this section, we will explicitly construct trace free and divergence free \((0,2)\) symmetric tensors with prescribed compact support on space forms (see also [4, 7]). Therefore, by the existence of such tensors and by Theorem 4.2 and Lemma 4.2, we have:

**Corollary 4.1.** Let \((\Omega, g)\) be given as in Theorem 4.2. There exists a variation \(\{g(t)\} \subset \mathcal{M}_\gamma^K\) of \(g\) such that \(\frac{d}{dt} V(g(t))|_{t=0} = 0\) and
\[
\frac{d^2}{dt^2} V(g(t))|_{t=0} > 0.
\]
In particular, the volume of the standard metric \(g\) is a strict local minimum along such a variation.

Next, we show that there indeed exist deformations along which the volume of the standard metric is a strict local maximum.

**Theorem 4.3.** Let \(\Omega\) be a geodesic ball with compact closure in \(\mathbb{R}^n\), \(\mathbb{H}^n_+\) or \(\mathbb{S}^n_+\). Let \(\Sigma\) be its boundary and \(g\) be the standard metric. Let \(\gamma = g|_{T(\Sigma)}\) and \(K\) be the constant that equals the scalar curvature of \(g\). Suppose the dimension \(n\) satisfies \(3 \leq n \leq 5\).

(i) If \(\Omega \subset \mathbb{R}^n\), there exists a variation \(\{g(t)\} \subset \mathcal{M}_\gamma^K\) of \(g\) such that \(\frac{d}{dt} V(g(t))|_{t=0} = 0\) and
\[
\frac{d^2}{dt^2} V(g(t))|_{t=0} < 0.
\]

(ii) If \(\Omega \subset \mathbb{H}^n_+\) or \(\mathbb{S}^n_+\), there exists a small positive constant \(\delta\) such that, if the geodesic radius of \(\Omega\) is less than \(\delta\), then there exists a variation \(\{g(t)\} \subset \mathcal{M}_\gamma^K\) of \(g\) such that \(\frac{d}{dt} V(g(t))|_{t=0} = 0\) and
\[
\frac{d^2}{dt^2} V(g(t))|_{t=0} < 0.
\]

**Proof.** We first consider the case \(\Omega \subset \mathbb{R}^n\). We use \(g_0\) to denote the standard Euclidean metric on \(\mathbb{R}^n\) and \(\nabla_0(\cdot)\) to denote the covariant derivative taken with respect to \(g_0\). Let \(\hat{h}\) be an arbitrary, nonzero, symmetric \((0,2)\) tensor that is parallel on \((\Omega, g_0)\) and satisfies \(\text{tr}_{g_0} \hat{h} = 0\). Define \(h = \lambda \hat{h}\). Then \(h|_{T(\Sigma)} = 0\) and satisfies
\[
\text{tr}_{g_0} h = 0
\]
\[
(\text{div}_{g_0} h)_i = g^{ik} \lambda_k \hat{h}_{ij},
\]
\[
\text{div}_{g_0} (\text{div}_{g_0} h) = \langle \nabla^2_{g_0} \lambda, \hat{h}\rangle_{g_0} = 0,
\]
where we have used the equation
\begin{equation}
\nabla^2 g_0 \lambda = -\frac{1}{n-1} g_0.
\end{equation}

By Lemma 4.2, \( h \in T_{g_0} \mathcal{M}_g^K \). Plug this \( h \) into (65) in Theorem 4.1 and use the fact \( g_0 \) has zero curvature, we have
\[
\frac{d^2}{dt^2} V(g(t))|_{t=0} = \int_{\Omega} \lambda \left[ \frac{1}{2} |\text{div}_{g_0}(h)|^2_g + \frac{1}{4} |\nabla_0 h|^2_{g_0} + \langle \nabla_0 (\text{div}_{g_0} h), h \rangle_{g_0} \right].
\]

Applying (87) and (89), integrating by parts and using the fact that \( \hat{h} \) is parallel, we have
\begin{align}
\int_{\Omega} \lambda |\text{div}_{g_0} h|^2_{g_0} &= \frac{1}{2(n-1)} \int_{\Omega} \lambda^2 |\hat{h}|^2_{g_0}, \\
\int_{\Omega} \lambda |\nabla h|^2_{g_0} &= \frac{n}{2(n-1)} \int_{\Omega} \lambda^2 |\hat{h}|^2_{g_0}, \\
\int_{\Omega} \lambda (\text{div}_{g_0} h)_{\text{tr}} h^{lp} &= -\frac{1}{n-1} \int_{\Omega} \lambda^2 |\hat{h}|^2_{g_0}.
\end{align}

Thus,
\begin{equation}
V''(0) = \frac{1}{8} \left( \frac{n-6}{n-1} \right) \int_{\Omega} \lambda^2 |\hat{h}|^2_{g_0} < 0,
\end{equation}
for \( n = 3, 4, 5 \).

Next, we consider the case \( \Omega \subset \mathbb{H}^n \). For any \( \kappa > 0 \), consider the metric \( g_\kappa = \left( 1 - \frac{\kappa^2}{4} |x|^2 \right)^{-2} g_0 \), which is defined on \( \{|x| < \frac{2}{\kappa} \} \) and has constant sectional curvature \(-\kappa^2\). Let \( B = \{|x| < 1\} \) and \( \Sigma \) be its boundary. Let \( \lambda_k \) be the smooth function on \( B \) defined by
\begin{equation}
\lambda_k = \frac{1}{(n-1)\kappa^2} \left( 1 - \frac{4 - \kappa^2}{4 + \kappa^2} \cosh \kappa r \right),
\end{equation}
where \( r \) is the geodesic distance from the origin in \((B, g_\kappa)\). We also define
\begin{equation}
\lambda_0 = \frac{1}{2(n-1)} \left( 1 - |x|^2 \right).
\end{equation}
For \( \kappa \geq 0 \), the function \( \lambda_\kappa \) satisfies
\begin{equation}
\begin{cases}
-(\Delta_{g_\kappa} \lambda_\kappa) g_\kappa + \nabla^2_{g_\kappa} \lambda_\kappa - \lambda_\kappa \text{Ric}(g_\kappa) &= g_\kappa \\
\lambda_\kappa|_{\Sigma} &= 0,
\end{cases}
\end{equation}
and the metric \( g_\kappa \) is a critical point of the volume functional \( V(\cdot) \) on the manifold \( \mathcal{M}_{\gamma_\kappa}^{-n(n-1)\kappa^2} \), where \( \gamma_\kappa = g_\kappa|_{T(\Sigma)} \). For any \( h \in T_{g_\kappa} \mathcal{M}_{\gamma_\kappa}^{-n(n-1)\kappa^2} \), define the second variational functional of the volume

\[
\mathcal{F}_\kappa(h) = \int_B \left\{ \frac{1}{4}(\text{tr}_{g_\kappa} h)^2 + \lambda_\kappa \left[ \text{div}_{g_\kappa}(h) - \frac{1}{2} d(\text{tr}_{g_\kappa} h)_{g_\kappa} + \frac{1}{4} \nabla h |_{g_\kappa}^2 \right] \right\} dV_{g_\kappa}
+ \int_B \lambda_\kappa \left[ (\text{div}_{g_\kappa} h)_{\text{tr}^p} - \langle h, \nabla_\kappa^2 (\text{tr}_{g_\kappa} h) \rangle_{g_\kappa} \right] dV_{g_\kappa}
- \int_B \frac{1}{2} \lambda_\kappa |\text{div}_{g_\kappa} h|_{g_\kappa}^2 dV_{g_\kappa} + \int_B \frac{\kappa^2}{2} \lambda_\kappa \left[ (\text{tr}_{g_\kappa} h)^2 - |h|_{g_\kappa}^2 \right] dV_{g_\kappa}
\]

according to (65) in Proposition 4.1 where \( \nabla_\kappa(\cdot) \) denotes the covariant derivative with respect to \( g_\kappa \) and \( dV_{g_\kappa} \) denotes the volume form of \( g_\kappa \).

Now, for \( \kappa = 0 \), choose an \( h_0 \in T_{g_0} \mathcal{M}_{\gamma_0}^0 \) such that \( \mathcal{F}_0(h_0) < 0 \). The existence of such an \( h_0 \) was proved in (i). For any \( \kappa \in (0, 1] \), by Proposition 2.1, there is a \( t_\kappa > 0 \) and \( \epsilon_\kappa > 0 \) such that for all \( |t| < t_\kappa \), \( \tilde{g}_\kappa(t) = g_\kappa + th_0 \) is a smooth metric on \( B \) and the following Dirichlet boundary value problem has a unique solution \( u_\kappa(t) \) such that \( 1 - \epsilon_\kappa \leq u_\kappa(t) \leq 1 + \epsilon_\kappa \):

\[
\begin{aligned}
\alpha \Delta_{\tilde{g}_\kappa(t)} u - R_\kappa(t) u &= -K u_\kappa^a, \quad \text{in } B \\
u_\kappa &= 1, \quad \text{on } \Sigma.
\end{aligned}
\]

where \( \alpha = 4(n - 1)/(n - 2), \quad a = (n + 2)/(n - 2) \), and \( R_\kappa(t) \) is the scalar curvature of \( \tilde{g}_\kappa(t) \). Moreover, \( v_\kappa = u_\kappa(0) \) exists and is a smooth function on \( \overline{B} \) which is the unique solution of:

\[
\begin{aligned}
(n - 1)\Delta_{g_\kappa} v_\kappa - n(n - 1)\kappa^2 v_\kappa &= \frac{n - 1}{4} R'_\kappa(0), \quad \text{in } B \\
v_\kappa &= 0, \quad \text{on } \Sigma,
\end{aligned}
\]

where

\[
R'_\kappa(0) = DR_{g_\kappa}(h_0) = -\Delta_{g_\kappa} (\text{tr}_{g_\kappa} h_0) + \text{div}_{g_\kappa} (\text{div}_{g_\kappa} h_0) - \langle h_0, \text{Ric}(g_\kappa) \rangle_{g_\kappa}.
\]

Since \( g_\kappa \to g_0 \) in \( C^\infty(\overline{B}) \) as \( \kappa \to 0 \), we have \( DR_{g_\kappa}(h_0) \to 0 \) in \( C^\infty(\overline{B}) \) and, by (99), \( v_\kappa \to 0 \) in \( C^\infty(\overline{B}) \). Now define \( g_\kappa(t) = u_\kappa^{\frac{4}{n - 2}}(t) \tilde{g}_\kappa(t) \), \( \{g_\kappa(t)\}_{|t| < t_\kappa} \) is a smooth path in \( \mathcal{M}_{\gamma_\kappa}^{-n(n-1)\kappa^2} \) with \( g_\kappa(0) = g_\kappa \). Let \( h_\kappa = g_\kappa'(0) \), then

\[
h_\kappa = \frac{4}{n - 2} v_\kappa g_0 + h_0.
\]
Let $\kappa \to 0$, we have $h_\kappa \to h_0$ in $C^\infty(B)$ and $\mathcal{F}_\kappa(h_\kappa) \to \mathcal{F}_0(h_0)$. As $h_0$ is chosen so that $\mathcal{F}_0(h_0) < 0$, we see that there is a small $\kappa_0 > 0$, depending only on $B$, $g_0$ and $h_0$, such that $\mathcal{F}_\kappa(h_\kappa) < 0$ for $\kappa < \kappa_0$.

Recall $d^2 dt^2 V(g_\kappa(t))|_{t=0} = \mathcal{F}_\kappa(h_\kappa)$, we conclude that the case $\Omega \subset \mathbb{H}^n$ is proved by scaling the metric $g_\kappa$ to a metric with constant sectional curvature $-1$.

The case $\Omega \subset \mathbb{S}^n_+$ can be proved in a similar way by replacing $g_\kappa$ with

$$g_\kappa = \left(1 + \frac{\kappa^2}{4} |x|^2\right)^{-2} g_0$$

and by replacing $\lambda_\kappa$ with

$$\lambda_\kappa = \frac{1}{(n-1)\kappa^2} \left(-1 + \frac{4 + \kappa^2}{4 - \kappa^2} \cos \kappa r\right),$$

where $r$ is the geodesic distance from the origin in $(B, g_\kappa)$. This completes the proof of the theorem. $\square$

As a corollary of Theorem 3.1, Theorem 3.3 and Theorem 4.3, we have the following nonexistence result on the global volume minimizer in $\mathcal{M}_\gamma^0$.

**Theorem 4.4.** Let $\Omega$ be a domain in $\mathbb{R}^n$ bounded by a smooth, compact, strictly convex hypersurface $\Sigma$. Let $g_0$ be the standard Euclidean metric on $\mathbb{R}^n$ and let $\gamma = g_0|_T(\Sigma)$. If $\Sigma$ is a round sphere in $\mathbb{R}^n$, the dimension $n$ is assumed to satisfy $3 \leq n \leq 5$. Define $\beta = \inf\{V(g) \mid g \in \mathcal{M}_\gamma^0\}$, then there does not exist a smooth metric $g$ in $\mathcal{M}_\gamma^0$ such that $V(g) = \beta$.

**Proof.** Suppose there exists a smooth metric $g \in \mathcal{M}_\gamma^0$ with $V(g) = \beta$, then $g$ is a critical point of $V(\cdot)$ in $\mathcal{M}_\gamma^0$. By Theorem 3.3, $V(g) \geq V_0$, where $V_0$ is the Euclidean volume of $\Omega$. Therefore, $\beta \geq V_0$.

Suppose $\Sigma$ is not a round sphere in $\mathbb{R}^n$. By Theorem 3.1, $g_0$ is not a critical point of $V(\cdot)$ in $\mathcal{M}_\gamma^0$. In particular, there is a path of metrics $\{g(t)\}$ in $\mathcal{M}_\gamma^0$ with $g(0) = g_0$ such that $\frac{d}{dt} V(g(t))|_{t=0} < 0$. Hence, $V(g(t)) < V_0$ for small positive $t$, contradicting $\beta \geq V_0$.

If $\Sigma$ is a round sphere in $\mathbb{R}^n$, by Theorem 4.3, there is a path of metrics $\{g(t)\}$ in $\mathcal{M}_\gamma^0$ with $g(0) = g_0$ such that $\frac{d^2}{dt^2} V(g(t))|_{t=0} = 0$ and $\frac{d^2}{dt^2} V(g(t))|_{t=0} < 0$. Hence, $V(g(t)) < V_0$ for small $t$, again contradicting $\beta \geq V_0$. The theorem is proved. $\square$

Before we end this section, we give a discussion of “large” geodesic balls in $\mathbb{S}^n$ which strictly contains a hemisphere.
Proposition 4.1. Let $\Omega$ be a geodesic ball in $\mathbb{S}^n$ with geodesic radius $R$ satisfying $\frac{\pi}{2} < R < \pi$. Let $g$ be the standard metric on $\mathbb{S}^n$. Let $\Sigma$ be the boundary of $\Omega$ and $\gamma = g|_{\Gamma(\Sigma)}$. Suppose $\{g(t)\}$ is a smooth path of metrics in $\mathcal{M}_\gamma$ with $g(0) = g$ and $g(t) \in \mathcal{M}^{n(n-1)}$. Then

$$\frac{d}{dt} V(g(t))|_{t=0} = 0.$$ 

If $h = g'(0)$ is nonzero and satisfies

$$\text{div}_g h = 0 \quad \text{and} \quad \text{tr}_g h = 0,$$

then

$$\frac{d^2}{dt^2} V(g(t))|_{t=0} < 0.$$ 

Proof. We embed $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ as the unit sphere

$$\mathbb{S}^n = \{(x_0, x_1, \ldots, x_n) \mid x_0^2 + x_1^2 + \ldots + x_n^2 = 1\}.$$ 

Suppose $\Omega$ is given by the set

$$\Omega = \mathbb{S}^n \cap \{x_n > \frac{1}{a}\}$$

for some constant $a < -1$. Consider the function

$$\lambda = \frac{1}{n-1} (ax_n - 1).$$

Then $g$ and $\lambda$ satisfy

\begin{equation}
\begin{cases}
- (\Delta_g \lambda) g + \nabla^2_g \lambda - \lambda \text{Ric}(g) = g & \text{in } \Omega; \\
\lambda = 0 & \text{on } \Sigma.
\end{cases}
\end{equation}

(103)

From the proof of Theorem 2.1, we know $\frac{d}{dt} V(g(t))|_{t=0} = 0$. By Theorem 4.1 and the assumption $\text{div}_g h = 0$ and $\text{tr}_g h = 0$, we have

\begin{equation}
\frac{d^2}{dt^2} V(g(t))|_{t=0} = \frac{1}{4} \int_{\Omega} \lambda |\nabla h|^2_g + \frac{1}{2} \int_{\Omega} \lambda |h|^2_g.
\end{equation}

(104)

Note that, unlike the case $\Omega \subset \mathbb{S}^n_+$ in which $\lambda$ is positive, we have $\lambda < 0$ in this case. Therefore, $\frac{d^2}{dt^2} V(g(t))|_{t=0} < 0$. \hfill \Box

Comparing Proposition 4.1 with the case $\Omega \subset \mathbb{S}^n_+$ in Theorem 4.2, we find there exists a dichotomy between the variational properties of $V(\cdot)$ on “small” geodesic balls in $\mathbb{S}^n$ strictly contained in a hemisphere and “large” geodesic balls in $\mathbb{S}^n$ strictly containing a hemisphere.
5. Appendix: TT Tensors with Prescribed Compact Support

In this appendix, we give a construction of trace free and divergence free \((0,2)\) symmetric tensors with prescribed compact support on a rotationally symmetric manifold. As mentioned earlier, such tensors are used to construct special metric variations in Section 4.

Let \(m \geq 2\) be an integer. Let \(S^m\) be the \(m\)-dimensional sphere with the standard differential structure. Let \(g_{S^m}\) be the standard metric on \(S^m\) with constant sectional curvature +1. For a given \(R > 0\), consider the product manifold

\[
M = (0, R) \times S^m
\]  

with a rotationally symmetric metric

\[
g = \frac{1}{N^2} dr^2 + r^2 g_{S^m},
\]

where \(r\) denotes the usual coordinate on \((0, R)\) and \(N = N(r)\) is a given smooth positive function on \((0, R)\).

In what follows, we let \(\omega\) denote points in \(S^m\) and let \(Y = Y(\omega)\) be a fixed eigenfunction of \(S^m = (S^m, g_{S^m})\) whose eigenvalue \(\kappa\) is not the first eigenvalue of \(S^m\), i.e.

\[
\Delta g_{S^m} Y + \kappa Y = 0, \quad \kappa > m.
\]

**Theorem 5.1.** For any \(0 < r_1 < r_2 < R\), and any given smooth function \(a(r)\) with support in \((r_1, r_2)\), there exists a smooth divergence free and trace free \((0,2)\) symmetric tensor \(h\) on \((M, g)\) such that the support of \(h\) is contained in \((r_1, r_2) \times S^m\) and \(h(\partial_r, \partial_r) = a(r) Y\).

**Proof.** We first assume that \(a(r), b(r), c(r), d(r)\) are some smooth functions of the variable \(r\) alone, which are to be determined later. For each \(r\), let \(\Sigma_r\) be the leaf \(\{r\} \times S^m\) in \(M\). We define a \((0,2)\) symmetric tensor \(h\) on \(M\) as follows,

\[
h(\partial_r, \partial_r) = a(r) Y,
\]

\[
h(\partial_r, \cdot)|_{T(\Sigma_r)} = b(r) dY,
\]

\[
h(\cdot, \cdot)|_{T(\Sigma_r)} = r^2 [c(r) \nabla_{g_{S^m}}^2 Y + d(r) Y g_{S^m}],
\]

where \(h(\partial_r, \cdot)|_{T(\Sigma_r)}, h(\cdot, \cdot)|_{T(\Sigma_r)}\) are the restriction of \(h(\partial_r, \cdot), h(\cdot, \cdot)\) to the tangent space of \(\Sigma_r\), \(dY\) is the differential of \(Y\) on \(S^m\) and \(\nabla_{g_{S^m}}^2 Y\) is the Hessian of \(Y\) on \(S^m\).
Let \( \{ \omega_A \mid A = 1, \ldots, m \} \) be a local coordinate chart on \( S^m \) and \( \partial_A = \frac{\partial}{\partial \omega_A} \) be the associated tangent vector. Then
\[
\text{tr}_g h = g^{rr} h_{rr} + g^{AB} h_{AB}
\]
\[
= N^2 a(r) Y + c(r) \Delta_{g_{\Sigma r}} Y + md(r) Y
\]
\[
= [N^2 a(r) - \kappa c(r) + md(r)] Y.
\]

To compute \( \text{div}_g h \), we let \( \partial_n = N \partial_r \) be the unit normal vector to \( \Sigma_r \). Let \( \{ \Gamma^i_{jk} \} \) be the Christoffel symbol of \( g \) with respect to the frame field \( \{ \partial_A, \partial_n \} \), where \( i, j, k \in \{ 1, \ldots, m, n \} \). Let ‘;’ denote the covariant differentiation with respect to \( g \) and ‘,’ denote the partial differentiation. Direct computations shows
\[
(\text{div}_g h)_{n} = h_{nn;n} + g^{AB} h_{nA;B}
\]
\[
= h_{nn;n} + g^{AB} (h_{nA,B} - h_{iA} \Gamma^i_{nB} - h_{ni} \Gamma^i_{AB})
\]
\[
= h_{nn;n} + g^{AB} (h_{nA,B} - h_c A \Gamma^C_{nB} - h_{nc} \Gamma^C_{AB} - h_{nn} \Gamma^n_{AB})
\]
\[
= h_{nn;n} + g^{AB} (h_{nA,B} - h_{cA} \Gamma^C_{AB} - h_{cA} \Gamma^C_{AB} + h_{nn} \Gamma^n_{AB})
\]
\[
= h_{nn;n} + \text{div}_\Sigma [h(\partial_n, \cdot)|T(\Sigma_r)] - \langle h|T(\Sigma_r), \Pi \rangle + h_n H,
\]
where \( \text{div}_\Sigma(\cdot) \) denotes the divergence operator on \( (\Sigma_r, g|T(\Sigma_r)) \), \( \Pi_{AB} = \langle \nabla_A \partial_n, \partial_B \rangle \) is the second fundamental form of \( \Sigma_r \) and \( H \) is the mean curvature of \( \Sigma_r \). Similarly,
\[
(\text{div}_g h)_A = h_{nA;n} + g^{BC} h_{AB;C}
\]
\[
= h_{nA;n} + g^{BC} (h_{AB,C} - h_{iB} \Gamma^i_{AC} - h_{Ai} \Gamma^i_{BC})
\]
\[
= h_{nA;n} + (\text{div}_\Sigma[h|T(\Sigma_r)])_A + g^{BC} h_{nB} \Pi_{AC} + H h_{nA}.
\]

Meanwhile, by (106), (107), (108), (109) and (110), we have
- \( H = N m r^{-1} \), \( \Pi_{AB} = N r^{-1} g_{AB} \)
- \( h_{nA} = N h(\partial_r, \partial_A) = Nb(r) Y_A \)
- \( h_{nn} = N^2 h(\partial_r, \partial_r) = N^2 a(r) Y \)
- \( h_{nn,n} = h_{nn,n} = N \partial_r [N^2 a(r) Y] = N \partial_r [N^2 a(r)] Y \)
- \( h_{nA;n} = h_{nA,n} - h_{ni} \Gamma^i_{An} - h_{Ai} \Gamma^i_{nn} = N \partial_r [Nb(r) Y_A] - N^2 r^{-1} b(r) Y_A \)
- \( h_{n|T(\Sigma_r), \Pi} = N r^{-1} [-\kappa c(r) + md(r)] Y \)
- \( g^{BC} h_{nB} \Pi_{AC} = N^2 r^{-1} b(r) Y_A \)
and
\[
\text{div}_\Sigma[h(\partial_n, \cdot)|T(\Sigma_r)] = r^{-2} \text{div}_{g_{\Sigma r}} [Nb(r) dY] = -r^{-2} Nb(r) \kappa Y,
\]
\[
\text{div}_\Sigma[h|T(\Sigma_r)] = \text{div}_{g_{\Sigma r}} [c(r) \nabla^2_{g_{\Sigma r}} Y + d(r) Y g_{\Sigma r}]
\]
\[
= c(r) [d(\Delta_{g_{\Sigma r}} Y) + (m - 1) dY] + d(r) dY
\]
\[
= \{c(r)[-\kappa + (m - 1)] + d(r)\} dY,
\]
\[
\text{div}_\Sigma[h|T(\Sigma_r)] = \text{div}_{g_{\Sigma r}} [c(r) \nabla^2_{g_{\Sigma r}} Y + d(r) Y g_{\Sigma r}]
\]
\[
= c(r) [d(\Delta_{g_{\Sigma r}} Y) + (m - 1) dY] + d(r) dY
\]
\[
= \{c(r)[-\kappa + (m - 1)] + d(r)\} dY,
\]
where in (115) we have also used the fact \( \text{Ric}(g^m) = (m - 1)g^m \). Therefore, it follows from (111), (112) and (113) that \( h \) is trace free and divergence free if and only if the following system holds

\[
\left\{
\begin{array}{c}
N^2 a - \kappa c + md = 0 \\
\partial_r (N^2 a) - r^{-2} b \kappa - r^{-1} [-\kappa c + md] + mr^{-1} N^2 a = 0 \\
N \partial_r (Nb) + c [-\kappa + (m - 1)] + d + mr^{-1} N^2 b = 0
\end{array}
\right.
\]

or equivalently

\[
\left\{
\begin{array}{c}
-\kappa c + md = -N^2 a \\
r^{-2} b \kappa = \partial_r (N^2 a) + (m + 1) r^{-1} N^2 a \\
c [-\kappa + (m - 1)] + d = -N \partial_r (Nb) - mr^{-1} N^2 b.
\end{array}
\right.
\]

Now, let \( a = a(r) \) be given as in the Theorem. By the second equation in (117), \( b = b(r) \) is then determined accordingly, furthermore \( b(r) \) is smooth and has compact support in \((r_1, r_2)\). With \( a(r) \) and \( b(r) \) given, the first and the third equations in (117) become a linear system for \( c = c(r) \) and \( d = d(r) \). As

\[
\left|\begin{array}{cc}
-\kappa & m \\
-\kappa + (m - 1) & 1
\end{array}\right| = (m - 1)(\kappa - m)
\]

and \( \kappa \) is chosen so that \( \kappa > m \), \( c(r) \) and \( d(r) \) are uniquely determined by the first and the third equations in (117) and they both are smooth and have compact support in \((r_1, r_2)\). With such a choice of \( a(r), b(r), c(r), d(r) \), we conclude that the \((0, 2)\) symmetric tensor \( h \), defined by (108)–(110), satisfies all the conditions in the Theorem. □

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