Frozen Quantum Coherence

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We analyse under which dynamical conditions the coherence of an open quantum system is totally unaffected by noise. For a single qubit, specific measures of coherence are found to freeze under different conditions, with no general agreement between them. Conversely, for an N-qubit system with even N, we identify universal conditions in terms of initial states and local incoherent channels such that all bona fide distance-based coherence monotones are left invariant during the entire evolution. This finding also provides an insightful physical interpretation for the freezing phenomenon of quantum correlations beyond entanglement. We further obtain analytical results for distance-based measures of coherence in two-qubit states with maximally mixed marginals.

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Introduction. The coherent superposition of states stands as one of the characteristic features that mark the departure of quantum mechanics from the classical realm, if not the most essential one [1]. Quantum coherence is at the root of a number of intriguing phenomena of wide-ranging impact in quantum optics [2–5], quantum information [6, 7], solid state physics [8, 9], and thermodynamics [10–16]. It encapsulates the essence of entanglement [17] and by itself constitutes a powerful resource for quantum metrology [18]. In recent years, an invigorated interest in the role of quantum coherence for the optimal function of some biological systems has arisen considerably, de facto triggering the emergence of quantum biology as a new cross-disciplinary research field [19–33].

Despite the fundamental importance of quantum coherence, only very recently have important first steps been achieved towards developing a rigorous theory of coherence as a physical resource [34–36], and necessary constraints have been put forward to assess valid quantifiers of coherence [34, 37]. A number of coherence monotones have been proposed and investigated, such as the $L_1$-norm and relative entropy of coherence [34], and the skew information [38, 39] which admits an experimentally friendly lower bound [38]. Attempts to quantify coherence via a distance-based approach, which has been fruitfully adopted for entanglement and other correlations [17, 40–49], have revealed some subtleties [50]. More generally, operational interpretations of particular coherence monotones within the context of quantum technologies are still scarce. An interesting fact is that a coherent probe (e.g. laser light) can be used as a catalyst to drive quantum systems into a coherent superposition without any degradation of the coherence of the probe state [51]. This reveals that coherence can be preserved even when consumed as a resource.

Noise plays an integral part in any complex quantum system, and it is a natural question to ask under which conditions can coherence be invariant or even recovered in the presence of noise [52, 53]. This question is also relevant for practical applications, as one aims to protect the precious resource of quantum coherence and not to let it go away with losses. A lesson learned from biology is that noise is not always detrimental, and coherence-based effects can in fact flourish and persist at reasonable timescales under suitable exposure to de-cohering environments, as happens e.g. for light-harvesting complexes in the early stages of photosynthesis [27, 29, 30].

In this paper we investigate the dynamics of quantum coherence in open quantum systems under paradigmatic incoherent noisy channels. While coherence is generally nonincreasing under any incoherent channel [54], our goal is to identify initial states and dynamical conditions, here labelled freezing conditions, such that coherence will remain exactly constant (frozen) during the whole evolution, as illustrated in Fig. 1.

For a single qubit subject to a Markovian bit flip, bit-phase flip, phase flip, depolarising, amplitude damping, or phase damping channel [55], we study the evolution of the $L_1$-norm and relative entropy of coherence [34] with respect to the computational basis. We show that no nontrivial condition exists such that both measures are simultaneously frozen. We then turn our attention to two-qubit systems, for which we remarkably identify a set of initial states such that all bona fide distance-based measures of coherence are frozen forever when each qubit is independently experiencing a nondissipative flip channel. These results are extended to N-qubit systems with any even N, for which suitable conditions supporting the freezing of all distance-based measures of coherence are provided. Such a universal freezing of quantum coherence within the geometric approach is intimately related to the freezing of distance-based quantum correlations beyond entanglement [47, 49, 54, 55], shedding light on the latter from a physical perspective. Finally, some analytical results for the $L_1$-norm of coherence are obtained, and its freezing conditions in general one- and two-qubit states are identified.

FIG. 1: (color online) Frozen quantum coherence for an N-qubit system subject to incoherent noisy channels acting on each qubit.
Incoherent states and channels. Coherence embodies the capability of a quantum state to exhibit quantum interference phenomena. These effects are usually ascribed to the off-diagonal elements of a density matrix with respect to a reference basis. Here, for an $N$-qubit system associated to a Hilbert space $\mathbb{C}^{2^N}$, we fix the computational basis $|0\rangle, |1\rangle |^{\otimes N}$ as the reference basis, and define incoherent states as those whose density matrix $\rho$ is diagonal in such a basis,

$$\delta = \sum_{i_1, \ldots, i_N=0}^{1} d_{i_1, i_2, \ldots, i_N} |i_1, i_2, \ldots, i_N\rangle \langle i_1, i_2, \ldots, i_N|.$$  (1)

The dynamical evolution of an open quantum system is mathematically described by a completely positive trace-preserving (CPTP) map $\Lambda$, also known as a quantum channel, whose action on the state $\rho$ of the system can be characterised by a set of Kraus operators $\{K_j\}$ such that $\Lambda(\rho) = \sum_j K_j^\dagger \rho K_j$, where $\sum_j K_j^\dagger K_j = I$. Incoherent quantum channels (ICPTP maps) constitute a subset of quantum channels that satisfy the additional constraint $K_j^\dagger K_j \subset I$ for all $j$, where $I$ is the set of incoherent states [34]. This implies that ICPTP maps transform incoherent states into incoherent states, and no creation of coherence would be witnessed even if an observer had access to individual outcomes.

We will consider paradigmatic instances of incoherent channels which embody typical noise sources in quantum information processing [6, 34], and whose action on a single qubit is described as follows, in terms of a parameter $q \in [0, 1]$ which encodes the strength of the noise. The bit flip, bit-phase flip and phase flip channels are represented in Kraus form by

$$K_{0,0}^f = \sqrt{1 - q/2}, \quad K_{j,k}^f = 0, \quad K_{k}^p = \sqrt{q/2} \sigma_k,$$  (2)

with $k = 1, k = 2$ and $k = 3$, respectively, and $\{|\sigma_j\rangle\}$ being the $j$-th Pauli matrix. The depolarising channel is represented by

$$K_0^p = \sqrt{1-3q/4}, \quad K_j^p = \sqrt{q/4} \sigma_j,$$  (3)

with $j \in \{1, 2, 3\}$. Finally, the amplitude damping channel is represented by

$$K_0^d = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}, \quad K_1^d = \begin{pmatrix} 0 & \sqrt{q} \\ \sqrt{q} & 0 \end{pmatrix}.$$  (4)

and the phase damping channel by

$$K_0^b = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}, \quad K_1^b = \begin{pmatrix} 0 & \sqrt{q} \\ \sqrt{q} & 0 \end{pmatrix}.$$  (5)

The action of $N$ independent and identical local noisy channels (of a given type, say labelled by $\Xi = \{F_{ik}, D, A, P\}$) on each qubit of an $N$-qubit system, as depicted in Fig. [1] maps the system state $\rho$ into the evolved state

$$\Lambda^\Xi_{\otimes N}(\rho) = \sum_{j_1, j_2, \ldots j_N} K_{j_1}^f \otimes K_{j_2}^p \otimes \cdots \otimes K_{j_N}^p \rho K_{j_N}^b \otimes K_{j_{N-1}}^b \otimes \cdots \otimes K_{j_1}^b.$$  (6)

Coherence monotones. Baumgratz et al. [34] have formulated a set of physical requirements which should be satisfied by any valid measure of quantum coherence $C$, namely:

C1. $C(\delta) = 0$ for all incoherent states $\delta \in I$;
C2a. Contractivity under incoherent channels $\Lambda_{\text{ICPTP}}$, $C(\Lambda(\rho)) \leq C(\rho)$;
C2b. Contractivity under selective measurements on average, $C(\rho) \geq \sum_j p_j C(\rho_j)$, where $\rho_j = K_j^\dagger \rho K_j$ and $p_j = \text{Tr}(K_j^\dagger K_j)$, for any $\{|K_j\rangle\}$ such that $\sum_j K_j^\dagger K_j = I$ and $K_j I \subset I$ for all $j$;
C3. Convexity, $C(\rho_0 + (1-q)\rho_1) \leq q C(\rho_0) + (1-q) C(\rho_1)$ for any states $\rho_0$ and $\rho_1$.

We now recall known measures of coherence. The $l_1$-norm quantifies coherence in an intuitive way via the off-diagonal elements of a density matrix $\rho$ in the reference basis [34].

$$C_i(\rho) = \sum_{i \neq j} |\rho_{ij}|.$$  (8)

Alternatively, one can quantify coherence via a geometric approach. Given a distance $D$, a generic distance-based measure of coherence is defined as

$$C_D(\rho) = \min_{\delta \in I} D(\rho, \delta) = D(\rho, \delta_0),$$  (9)

where $\delta_0$ is one of the closest incoherent states to $\rho$ with respect to $D$. We refer to bona fide distances $D$ as those which satisfy natural properties [6] of contractivity under quantum channels, i.e. $D(\Lambda(\rho), \Lambda(\tau)) \leq D(\rho, \tau)$ for any states $\rho$ and $\tau$ and CPTP map $\Lambda$, and joint convexity, i.e. $D(q\rho + (1-q)\tau, q\sigma + (1-q)\varsigma) \leq q D(\rho, \tau) + (1-q) D(\sigma, \varsigma)$ for any states $\rho$, $\sigma$, $\tau$, $\varsigma$ and $q \in [0, 1]$. We then refer to bona fide distance-based measures of coherence $C_D$ as those defined by Eq. (9) using a bona fide distance $D$: all such measures will satisfy requirements C1, C2a, and C3 [34]. It is still unclear what additional property a distance $D$ needs to satisfy in order for the corresponding $C_D$ to obey C2b as well. For instance, the coherence quantifier defined via the squared Bures distance, which is contractive and jointly convex, is known not to obey C2b [50].

All our subsequent conclusions will apply to bona fide distance-based coherence quantifiers $C_D$, which include of course distance-based coherence monotones obeying all the resource-theory requirements recalled in (7). An example of distance-based full coherence monotone is the relative entropy of coherence [34], which takes the simple form

$$C_{\text{RE}}(\rho) = S(\rho_{\text{diag}}) - S(\rho),$$  (10)

for any state $\rho$, where $\rho_{\text{diag}}$ is the matrix containing only the leading diagonal elements of $\rho$ in the reference basis, and $S(\rho) = -\text{Tr}(\rho \log \rho)$ is the von Neumann entropy.

We can also define the trace distance of coherence $C_{\text{TR}}$, as in Eq. [1] using the bona fide trace distance $D_{\text{TR}}(\rho, \tau) = \frac{1}{2} \text{Tr}|\rho - \tau|$. For one-qubit states $\rho$, the trace distance of coherence coincides with the $l_1$-norm of coherence [45, 50], but this equivalence is not valid for higher dimensional systems, and it is still unknown whether $C_{\text{TR}}$ obeys requirement C2b in general.
Frozen coherence: one qubit. We now analyse conditions such that the \( l_1 \)-norm and relative entropy of coherence are invariant during the evolution of a single qubit (initially in a state \( \rho \)) under any of the noisy channels \( \Lambda^q_\eta \) described above. This is done by imposing a vanishing differential of the measures on the evolved state, \( \delta \rho C(\Lambda^q_\eta(\rho)) = 0 \) \( \forall q \in [0, 1] \), with respect to the noise parameter \( q \), which can also be interpreted as a dimensionless time \([56]\). We find that only the bit and bit-phase flip channels allow for nonzero frozen coherence (in the computational basis), while all the other considered incoherent channels leave coherence invariant only trivially when the initial state is already incoherent. We can then ask whether nontrivial common freezing conditions for \( C_l \) and \( C_{RE} \) exist.

Writing a single-qubit state in general as \( \rho = \frac{1}{2} (1 + \sum_j n_j \sigma_j) \) in terms of its Bloch vector \( \vec{n} = \{n_1, n_2, n_3\} \), the bit flip channel \( \Lambda^{B}_q \) maps an initial Bloch vector \( \vec{\rho}(0) \) to an evolved one \( \vec{\rho}(q) = \{n_1(0), (1-q)n_2(0), (1-q)n_3(0)\} \). As the \( l_1 \)-norm of coherence is independent of \( n_1 \), while \( n_1 \) is unaffected by the channel, we get that necessary and sufficient freezing conditions for \( C_l \) under a single-qubit bit flip channel amount to \( n_2(0) = 0 \) in the initial state. Similar conclusions apply to the bit-phase flip channel \( \Lambda^{F_\phi}_q \) by swapping the roles of \( n_1 \) and \( n_2 \).

Conversely, the relative entropy of coherence is also dependent on \( n_3 \). By analysing the \( q \)-derivative of \( C_{RE} \), we see that such a measure is frozen through the bit flip channel only when either \( n_1(0) = 0 \) and \( n_2(0) = 0 \) (trivial because the initial state is incoherent) or \( n_2(0) = 0 \) and \( n_3(0) = 0 \) (trivial because the initial state is invariant under the channel). Therefore, there is no nontrivial freezing of the relative entropy of coherence under the bit flip or bit-phase flip channel either.

We conclude that, although individual coherence monotones such as the \( l_1 \)-norm can be frozen for specific subsets of initial states under flip channels, nontrivial universal freezing of quantum coherence measures is impossible for the dynamics of a single qubit under paradigmatic incoherent maps.

Frozen coherence: two qubits. This is not true anymore when considering more than one qubit. We will now show that any bona fide distance-based measure of quantum coherence manifests freezing forever in the case of two qubits \( A \) and \( B \) undergoing local identical bit flip channels \([57]\) and starting from the initial conditions specified as follows. We consider two-qubit states with maximally mixed marginals (\( M^2_3 \) states), also known as Bell-diagonal states \([58]\), which are identified by a triple \( \vec{c} = \{c_1, c_2, c_3\} \) in their Bloch representation

\[
\rho = \frac{1}{4} \left( \mathbb{1}^A \otimes \mathbb{1}^B + \sum_{j=1}^{3} c_j \sigma_j^A \otimes \sigma_j^B \right).
\]

Local bit flip channels on each qubit map initial \( M^2_3 \) states with \( \vec{c}(0) = \{c_1(0), -c_1(0)c_3(0), c_3(0)\} \) to \( M^2_3 \) states with \( \vec{c}(q) = \{c_1(0), (1-q)c_2(0), (1-q)c_2(0)\} \). Then, the subset of \( M^2_3 \) states supporting frozen coherence for all bona fide distance-based measures is given by the initial condition \([47, 49, 54]\)

\[
c_2(0) = -c_1(0)c_3(0).
\]

To prove this claim, we first establish the following result, which simplifies the evaluation of distance-based coherence monotones \([9]\) for the relevant class of \( M^2_3 \) states.

**Lemma 1.** According to any contractive and convex distance \( D \), one of the closest incoherent states \( \delta_\rho \) to a \( M^2_3 \) state \( \rho \) is always a \( M^2_3 \) incoherent state, i.e. one of the form

\[
\delta_\rho = \frac{1}{4} \left( \mathbb{1}^A \otimes \mathbb{1}^B + s \sigma_3^A \otimes \sigma_3^B \right),
\]

for some coefficient \( s \in [-1, 1] \).

**Proof.** Let us consider an arbitrary two-qubit incoherent state \( \delta = \sum_{i,j=0}^3 d_i \hat{\sigma}_i \otimes \hat{\sigma}_j \), where \( \{|i\} \) denotes the standard basis. It can be characterised by the triple \( (\vec{x}_s, \vec{y}_s, T_\delta) \) which defines its Bloch representation,

\[
\vec{x}_s = \{0, 0, -1 + 2(d_1 + d_2)\}, \quad \vec{y}_s = \{0, 0, -1 + 2(d_1 + d_3)\}, \quad T_\delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - 2(d_2 + d_3) \end{pmatrix}.
\]

For any two-qubit incoherent state \( \delta \), there exist two corresponding incoherent states \( \delta_- = (\sigma_1 \otimes \sigma_1) \delta (\sigma_1 \otimes \sigma_1) \) and \( \delta_0 = \frac{1}{2}(\delta + \delta_-) \). Moreover, \( \delta_0 \) is always a \( M^2_3 \) incoherent state. We now observe that, for any incoherent state \( \delta \) and any \( M^2_3 \) state \( \rho \), one has

\[
D(\rho, \delta_0) = D(\rho, \frac{1}{2}(\delta + \delta_-)) \leq \frac{1}{2} D(\rho, \delta) + \frac{1}{2} D(\rho, \delta_-) = D(\rho, \delta),
\]

where we have exploited the convexity of \( D \), and its invariance under unitary operations as a consequence of contractivity, which implies that \( D(\rho, \delta) = D(\rho \otimes \sigma_1) \rho (\sigma_1 \otimes \sigma_1), (\sigma_1 \otimes \sigma_1) \delta (\sigma_1 \otimes \sigma_1)) = D(\rho, \delta_-) \). This proves that \( D \) is minimised on incoherent \( M^2_3 \) states.

We can then invoke some recent results on the dynamics of quantum correlations for \( M^2_3 \) states, specifically Lemma A1 in Ref.\([49]\), to obtain immediately the following.

**Lemma 2.** According to any contractive and convex distance \( D \), one of the closest incoherent states \( \delta_\rho \) to a \( M^2_3 \) state \( \rho \) with \( \{c_1, -c_1c_3, c_3\} \) is the \( M^2_3 \) state \( \delta_\rho \) with \( \{0, 0, c_3\} \).

Consequently, any bona fide distance-based measure of quantum coherence \( C_D \) for the \( M^2_3 \) states \( \rho(q) \), evolving from the initial freezing conditions \([12]\) under the action of local bit flip channels, is given by

\[
C_D(\rho(q)) = D(\{c_1(0), -(1-q)c_1(0)c_3(0), (1-q)c_2(0), (1-q)c_2(0)\}, \{0, 0, -1 + 2(d_1 + d_2)\})
\]

\[
= C_D(\rho(0)),
\]

which is frozen for any \( q \in [0, 1] \), or equivalently frozen forever for any \( t \) \([56]\), thanks to Theorem 1 of Ref.\([49]\); see also the Supplementary Material \([59]\) for a general proof. This shows that universal freezing of quantum coherence, measured within a bona-fide geometric approach, can occur in two-qubit systems subject to local decohering dynamics.

Coming back now to the two specific coherence monotones analysed here \([44]\), we know that the relative entropy of coherence \( C_{RE} \) is a bona fide distance-based measure, hence it
manifests freezing in the conditions of Eq. (12). Interestingly, we will now show that the \( l_1 \)-norm of coherence \( C_l \) coincides with the trace distance of coherence \( C_T \) for any \( M^2 \) state, which implies that \( C_l \) also freezes in the same dynamical conditions. To this aim we need to show that, with respect to the trace distance \( D_T \), one of the closest incoherent states \( \delta \rho \) to a \( M^2 \) state \( \rho \) is always its diagonal part \( \rho_{\text{diag}} \). The trace distance between a \( M^2 \) state \( \rho \) with \( |c_1, c_2, c_3 \rangle \) and one of its closest incoherent states \( \delta \rho \), which is itself a \( M^2 \) state of the form \( \rho + \sum_{j} \Lambda_j \rho \Lambda_j^* \), is attained by \( \delta = \rho_{\text{diag}} \) as claimed. Notice, however, that the equivalence between \( C_l \) and \( C_T \) does not extend to general two-qubit states, as can be confirmed numerically.

Similarly to the single-qubit case, we can derive a larger set of necessary and sufficient freezing conditions valid specifically for the \( l_1 \)-norm of coherence. Every two-qubit state \( \rho \) can be transformed, by local unitaries, into a standard form \( 60 \) with Bloch representation \( \rho = \frac{1}{4} \left( |I^A \rangle \langle I^B| + |J^A \rangle \langle J^B| \right) \) and one of its closest incoherent states \( \delta \rho \). We have then that initial states of this form, with \( |c_1, c_2, c_3 \rangle \) in the computational basis, respect the freezing condition \( C \rho = 0 \), and one of its closest incoherent states with \( |c_1, c_2, c_3 \rangle \) in the computational basis is the product basis which minimises coherence (according to suitable bona fide measures) for particular \( M^2 \) states undergoing local bit flip noise \( \Lambda_{F_1} \), up to \( t < t^* \), while coherence is afterwards minimised in the eigenbasis of \( \sigma_1 \), which is the pointer basis towards which the system eventually converges due to the local decoherence \( 69 \); similar conclusions can be drawn for the other \( k \)-flip channels \( 57 \).

\textbf{Conclusions.} We have determined exact conditions such that any bona fide distance-based measure of quantum coherence \( 33 \) is dynamically frozen: this occurs for an even number of qubits, initialised in a particular class of states with maximally mixed marginals, and undergoing local independent and identical nondissipative flip channels (Fig. 1). We have also shown that there is no general agreement on freezing conditions between specific coherence monotones when considering either the one-qubit case or more general \( N \)-qubit initial states, thus highlighting the prominent role played by the aforementioned universal freezing conditions in ensuring a durable physical exploitation of coherence, regardless of how it is quantified. It will be interesting to explore practical realisations of such dynamical conditions \( 69 \), or variations thereof such that coherence remains effectively, even if not exactly, frozen \( 55 \).

Complex and living systems are inevitably subject to noise, hence it is natural and technologically crucial to question under what conditions the quantum resources that we can extract from them are not deteriorated during open evolutions. In addressing this problem by focusing on coherence, we have also provided an intrinsic physical explanation for the freezing of discord-type correlations, by exposing and exploiting the intimate link between these two nonclassical signatures. Providing unified quantitative resource-theory frameworks for coherence, entanglement, and other quantum correlations is certainly a task worthy of further investigation.

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Appendix A: Supplementary Material

Consider the $N$-qubit states with the following representation in the computational basis:

$$
\rho = \frac{1}{2^N} \left( I^{\otimes N} + \sum_{i=1}^{3} c_i \sigma_i^{\otimes N} \right),
$$

where $I$ is the $2 \times 2$ identity matrix, $\sigma_i$ is the $i$-th Pauli matrix and $c_i = \text{Tr} \left[ \rho \sigma_i^{\otimes N} \right] \in [-1, 1]$. These states will be referred to as $M_N^3$ states, as they have maximally mixed marginals (by tracing out any $K < N$ qubits), and will be uniquely identified by the triple $(c_1, c_2, c_3)$.

In this appendix we will show that, for an even number $N$ of qubits, all bona fide distance-based measures of quantum coherence will exhibit the freezing phenomenon when each qubit is subject to local independent bit flip noise, for an initial $M_N^3$ state specified by $(c_1, (-1)^{N/2}c_3, c_3)$.

The evolution of an $N$-qubit state $\rho$ under local independent identical $k$-flip channels, where the index $k \in \{1, 2, 3\}$ respectively identifies the bit flip ($k = 1$), bit-phase flip ($k = 2$), and phase flip ($k = 3$) channel, can be characterised in the operator-sum representation by the map

$$
A_q^{F_i \otimes N}(\rho) = \sum_{j_1, j_2, \ldots, j_N} K_{j_1}^{F_i} \otimes K_{j_2}^{F_i} \cdots \otimes K_{j_N}^{F_i} \rho K_{j_1}^{F_i \dagger} \otimes K_{j_2}^{F_i \dagger} \cdots \otimes K_{j_N}^{F_i \dagger},
$$

where the single-qubit Kraus operators $K_j^{F_i}$ are reported in the main text in terms of the strength of the noise $q \in [0, 1]$, which in dynamical terms can be expressed as $q(t) = 1 - \exp(-yt)$ with $t$ representing time and $y$ being the decoherence rate. From Eqs. (A1) and (A2), one can easily see that $N$ non-interacting qubits initially in a $M_N^3$ state, undergoing local identical flip channels, evolve preserving the $M_N^3$ structure during the entire dynamics (i.e., for all $q \in [0, 1]$, or equivalently for all $t \geq 0$). More precisely, the triple $(c_1(q), c_2(q), c_3(q))$ characterising the $M_N^3$ evolved state $\rho(q)$ can be written as follows

$$
c_{1,2,3}(q) = (1 - q)^N c_{1,2,3}(0), \quad c_k(q) = c_k(0),
$$

where $(c_1(0), c_2(0), c_3(0))$ is the triple characterising the initial $M_N^3$ state $\rho$.

We start by showing that, for even $N$, the eigenvectors and eigenvalues of an arbitrary $M_N^3$ state $\rho$ are given by, respectively

$$
|\beta_1^+\rangle = \frac{1}{\sqrt{2}} (000 \ldots 000) \pm |111 \ldots 111\rangle),
$$

$$
|\beta_2^+\rangle = \frac{1}{\sqrt{2}} (000 \ldots 001) \pm |111 \ldots 110\rangle),
$$

$$
|\beta_3^+\rangle = \frac{1}{\sqrt{2}} (000 \ldots 010) \pm |111 \ldots 101\rangle),
$$

$$
|\beta_4^+\rangle = \frac{1}{\sqrt{2}} (000 \ldots 001) \pm |111 \ldots 110\rangle),
$$

$$
|\beta_5^+\rangle = \frac{1}{\sqrt{2}} (000 \ldots 111) \pm |111 \ldots 110\rangle),
$$

... 

$$
|\beta_{2^{2N-1}+1}^+\rangle = \frac{1}{\sqrt{2}} (011 \ldots 110) \pm |100 \ldots 001\rangle),
$$

and

$$
\rho_{\beta^+} = \frac{1}{2^N} \left[ 1 \pm c_1 \pm (-1)^{N/2}(-1)^p c_2 + (-1)^p c_3 \right],
$$

where $p$ is the parity of $|\beta^+\rangle$ with respect to the parity operator along the $z$-axis $\Pi_3 \equiv \sigma_3^{\otimes N}$, i.e.

$$
\Pi_3|\beta^\rangle = (-1)^p |\beta^\rangle.
$$

It will suffice to prove the following equation:

$$
\rho_{\beta^+} = \lambda_p^\rho |\beta^+\rangle < (p, N - p) \pm (|N - p, p )>,
$$

where $|n_0, n_1\rangle$ denotes any element of the $N$-qubit computational basis whose number of 0’s (1’s) is equal to $n_0 (n_1)$, one can easily see that

$$
\rho_{\beta^+}^{\otimes N} = \frac{1}{2^N} (|N - p, p \rangle \pm |p, N - p \rangle),
$$

$$
\rho_{\beta^+}^{\otimes N} = \frac{1}{2^N} (-1)^{N/2} (|-1|^p|N - p, p \rangle \pm (-1)^{N-p}|N - p, p \rangle),
$$

$$
\rho_{\beta^+}^{\otimes N} = \frac{1}{2^N} (-1)^{N-p} |p, N - p \rangle \pm (-1)^p|N - p, p \rangle).
$$

Eventually, by using the above three equations, Eq. (A1) and the fact that $N$ is even, so that $(-1)^{N-p} = (-1)^p$, one can easily verify that $\rho_{\beta^+}^{\otimes N}$ is equal to $\lambda_p^\rho |\beta^+\rangle$, i.e. that Eq. (A7) holds.

Now we are ready to show the three essential pieces which will lead us to prove the main result on the universal freezing phenomenon of bona fide distance-based measures of quantum coherence in the $N$-qubit setting (with even $N$).

Lemma A.1. For all even $N$, any contractive distance satisfies the following translational invariance properties within the space of $N$-qubit $M_N^3$ states:

$$
D(c_1, (-1)^{N/2}c_3, c_3) = D([0, 0, c_3], [0, 0, 0])
$$

and

$$
D(c_1, (-1)^{N/2}c_3, c_3) = D([0, 0, c_3], [0, 0, 0])
$$

for all $c_1$ and $c_3$, where $c_1, (-1)^{N/2}c_3, c_3$ denotes a $M_N^3$ state in Eq. (A1) with $c_2 = (-1)^{N/2}c_1c_3$.

Proof. Let us start by proving Eq. (A9). First of all, by considering the channel $\Lambda^{F_i \otimes N}$ representing the local independent phase flip noise expressed by Eq. (A2), when $k = 3$ and $q = 1$ (i.e. $t \to \infty$), we have the following inequality

$$
D([0, 0, c_3], [0, 0, 0]) \leq D(c_1, (-1)^{N/2}c_3, c_3).
$$

for all $c_1$ and $c_3$
where the first equality is due to the fact that
\[ \{0, 0, c_3\} = \Lambda^{F \otimes N}_{c_3}(c_1, (-1)^{N/2}c_3, c_3), \quad (A12) \]
\[ \{0, 0, 0\} = \Lambda^{F \otimes N}_{0}(c_1, 0, 0), \quad (A13) \]
while the final inequality in (A11) is due to the contractivity of the distance \( D \).

In order to prove the opposite inequality and thus Eq. (A9), we now introduce a \( N \)-qubit global rephasing channel \( \Lambda^r \), which is defined in the operator-sum representation as
\[
\Lambda^r_{R^3}(\rho) = \sum_{i, \pm} K^r_{i, \pm} \rho K^r_{i, \pm}^\dagger, \quad (A14)
\]
with
\[
K^r_{1, \pm} = \sqrt{\frac{1 \pm r}{2}} \ketbra{\beta^{r}_{+}}{000 \ldots 000}, \quad (A15)
K^r_{2, \pm} = \sqrt{\frac{1 \pm r}{2}} \ketbra{\beta^{r}_{\pm}}{000 \ldots 001},
K^r_{3, \pm} = \sqrt{\frac{1 \pm r}{2}} \ketbra{\beta^{r}_{\pm}}{000 \ldots 010},
K^r_{4, \pm} = \sqrt{\frac{1 \pm r}{2}} \ketbra{\beta^{r}_{\pm}}{000 \ldots 011},
\]
\[ \ldots \]
\[
K^r_{2^{N-1} - 1, \pm} = \sqrt{\frac{1 \pm r}{2}} \ketbra{\beta^{r}_{\pm}}{011 \ldots 110},
K^r_{2^{N-1}} = \sqrt{\frac{1 \pm r}{2}} \ketbra{\beta^{r}_{\pm}}{011 \ldots 111},
K^r_{2^{N-1} + 1, \pm} = \sqrt{\frac{1 \pm r}{2}} \ketbra{\beta^{r}_{\pm}}{100 \ldots 000},
K^r_{2^{N-1} + 2, \pm} = \sqrt{\frac{1 \pm r}{2}} \ketbra{\beta^{r}_{\pm}}{100 \ldots 001},
\]
\[ \ldots \]
\[
K^r_{2^{N-3}, \pm} = \sqrt{\frac{1 \pm r}{2}} \ketbra{\beta^{r}_{\pm}}{111 \ldots 100},
K^r_{2^{N-2}, \pm} = \sqrt{\frac{1 \pm r}{2}} \ketbra{\beta^{r}_{\pm}}{111 \ldots 101},
K^r_{2^{N-1}, \pm} = \sqrt{\frac{1 \pm r}{2}} \ketbra{\beta^{r}_{\pm}}{111 \ldots 110},
K^r_{2^{N}, \pm} = \sqrt{\frac{1 \pm r}{2}} \ketbra{\beta^{r}_{\pm}}{111 \ldots 111},
\]
where \( r \in [0, 1] \) is a parameter denoting the rephasing strength, \( \ket{\beta^{r}_{\pm}} \) is the \( N \)-qubit basis defined in Eq. (A4), and the \( 2^{N+1} \) Kraus operators satisfy \( \sum_{i, \pm} K^r_{i, \pm} K^r_{i, \pm}^\dagger = \mathbb{1}_{\otimes N} \), thus ensuring that \( \Lambda^r \) is a CPTP map.

It is now essential to see that the effect of \( \Lambda^r_{R^3} \) on a \( M^3_N \) state of the form \( \{0, 0, c_3\} \) is given by
\[
\Lambda^r_{R^3}(\{0, 0, c_3\}) = \{r, (-1)^{N/2}r, c_3\}, \quad (A16)
\]
for any even \( N \). To prove Eq. (A16), it will be useful to split the \( N \)-qubit states \( \ket{\beta^{r}_{\pm}} \) into the states \( \ket{\Phi^{r}_{\pm}} \) and \( \ket{\Psi^{r}_{\pm}} \) with even and odd parity, respectively, i.e. such that
\[
\Pi_3 \ket{\Phi^{r}_{\pm}} = \ket{\Phi^{r}_{\pm}}, \quad (A17)
\]
\[
\Pi_3 \ket{\Psi^{r}_{\pm}} = -\ket{\Psi^{r}_{\pm}}, \quad (A18)
\]
where \( i \in \{1, \ldots, 2^{N-2}\} \). Thanks to Eqs. (A5), (A6), (A7) and (A17), one gets that the spectral decomposition of a \( M^3_N \) state \( \rho_{\{c_1, c_2, c_3\}} \) with generic triple \( \{c_1, c_2, c_3\} \) can be written as follows,
\[
\rho_{\{c_1, c_2, c_3\}} = \frac{1}{2^N} \left[ 1 + c_1 + (-1)^{N/2}c_2 + c_3 \right] \sum_i \ketbra{\Phi^{r}_{i}}{\Phi^{r}_{i}}
+ \frac{1}{2^N} \left[ 1 - c_1 + (-1)^{N/2}c_2 + c_3 \right] \sum_i \ketbra{\Phi^{r}_{i}}{\Phi^{r}_{i}}
+ \frac{1}{2^N} \left[ 1 + c_1 - (-1)^{N/2}c_2 - c_3 \right] \sum_i \ketbra{\Psi^{r}_{i}}{\Psi^{r}_{i}}
+ \frac{1}{2^N} \left[ 1 - c_1 - (-1)^{N/2}c_2 - c_3 \right] \sum_i \ketbra{\Psi^{r}_{i}}{\Psi^{r}_{i}}.
\]
As a consequence
\[
\rho_{\{0,0,c_3\}} = \frac{1}{2^N} (1 + c_3) \sum_i \ketbra{\Phi^{r}_{i}}{\Phi^{r}_{i}}
+ \frac{1}{2^N} (1 - c_3) \sum_i \ketbra{\Psi^{r}_{i}}{\Psi^{r}_{i}}, \quad (A19)
\]
while
\[
\rho_{\{-1,N/2,r,c_3\}} = \frac{1}{2^N} (1 + r)(1 + c_3) \sum_i \ketbra{\Phi^{r}_{i}}{\Phi^{r}_{i}}
+ \frac{1}{2^N} (1 - r)(1 + c_3) \sum_i \ketbra{\Phi^{r}_{i}}{\Phi^{r}_{i}}
+ \frac{1}{2^N} (1 + r)(1 - c_3) \sum_i \ketbra{\Psi^{r}_{i}}{\Psi^{r}_{i}}
+ \frac{1}{2^N} (1 - r)(1 - c_3) \sum_i \ketbra{\Psi^{r}_{i}}{\Psi^{r}_{i}}. \quad (A20)
\]
By exploiting the following equalities
\[
\Lambda^r_{R^3}(\ketbra{\Phi^{r}_{i}}{\Phi^{r}_{i}}) = \frac{1 + r}{2} \ketbra{\Phi^{r}_{i}}{\Phi^{r}_{i}} + \frac{1 - r}{2} \ketbra{\Phi^{r}_{i}}{\Phi^{r}_{i}}, \quad (A21)
\]
\[
\Lambda^r_{R^3}(\ketbra{\Phi^{r}_{i}}{\Phi^{r}_{i}}) = \frac{1 + r}{2} \ketbra{\Phi^{r}_{i}}{\Phi^{r}_{i}} + \frac{1 - r}{2} \ketbra{\Phi^{r}_{i}}{\Phi^{r}_{i}},
\]
\[
\Lambda^r_{R^3}(\ketbra{\Psi^{r}_{i}}{\Psi^{r}_{i}}) = \frac{1 + r}{2} \ketbra{\Psi^{r}_{i}}{\Psi^{r}_{i}} + \frac{1 - r}{2} \ketbra{\Psi^{r}_{i}}{\Psi^{r}_{i}},
\]
\[
\Lambda^r_{R^3}(\ketbra{\Psi^{r}_{i}}{\Psi^{r}_{i}}) = \frac{1 + r}{2} \ketbra{\Psi^{r}_{i}}{\Psi^{r}_{i}} + \frac{1 - r}{2} \ketbra{\Psi^{r}_{i}}{\Psi^{r}_{i}},
\]
and the linearity of the global rephasing channel, we get

\[ \Lambda^{R_i}_i([0, 0, c_3]) = \frac{1}{2^N} (1 + c_3) \sum_i \Lambda^{R_i}([\Psi_i^+]) (\Phi_i^+) \]  
(A22)

\[ + \frac{1}{2^N} (1 + c_3) \sum_i \Lambda^{R_i}([\Phi_i^-]) (\Phi_i^-) \]

\[ + \frac{1}{2^N} (1 - c_3) \sum_i \Lambda^{R_i}([\Psi_i^-]) (\Psi_i^-) \]

\[ + \frac{1}{2^N} (1 - c_3) \sum_i \Lambda^{R_i}([\Psi_i^-]) (\Psi_i^-) \]

\[ = [r, (-1)^{3/2} r, c_1, c_3]. \]

We then have the inequality

\[ D([c_1, (-1)^{3/2} c_1, c_3], [c_1, 0, 0]) \]

\[ = D(\Lambda^{R_i}_i([0, 0, c_3], \Lambda^{R_i}_i([0, 0, 0])) \]

\[ \leq D([0, 0, c_3], [0, 0, 0]), \]

where the first equality is due to the fact that

\[ [c_1, (-1)^{3/2} c_1, c_3] = \Lambda^{R_i}_i([0, 0, c_3], \Lambda^{R_i}_i([0, 0, 0]), \]

\[ [c_1, 0, 0] = \Lambda^{R_i}_i([0, 0, 0]), \]

while the final inequality in (A23) is again due to the contractivity of the distance \( D \). By putting together the two inequalities (A11) and (A23), we immediately get the invariance of Eq. (A9) for any contractive distance.

In order now to prove Eq. (A10), we introduce the local unitary \( V^{oN} \) with \( V = \frac{1}{\sqrt{2}} (I + i\sigma_2) \). The effect of \( V^{oN} \) on a general \( M^3 \) state is given by

\[ V^{oN}[c_1, c_2, c_3] V^{oN}\dagger = [c_3, c_2, c_1], \]

(A24)

where this can be easily seen by utilising the fact that \( N \) is even and the following single-qubit identities:

\[ V\sigma_1 V\dagger = \sigma_3, \]

\[ V\sigma_2 V\dagger = \sigma_2, \]

\[ V\sigma_3 V\dagger = -\sigma_1, \]

Thanks to the invariance under unitaries of any contractive distance \( D \), the effect of the unitary \( V^{oN} \) expressed by Eq. (A24), and the just proven invariance expressed by Eq. (A9), we eventually have

\[ D([c_1, (-1)^{3/2} c_1, c_3], [c_1, 0, 0]) \]

\[ = D(V^{oN}[c_1, (-1)^{3/2} c_1, c_3] V^{oN}\dagger, V^{oN}[0, 0, c_3] V^{oN}\dagger) \]

\[ = D([c_3, (-1)^{3/2} c_1, c_3], [c_3, 0, 0]) \]

\[ = D([0, 0, c_1], [0, 0, 0]) \]

\[ = D(V^{oN}[0, 0, c_1] V^{oN}\dagger, V^{oN}[0, 0, 0] V^{oN}\dagger) \]

\[ = D([c_1, 0, 0], [0, 0, 0]), \]

that is Eq. (A10).

\[ \textbf{Lemma A.2.} \text{ For all even } N, \text{ according to any contractive and convex distance } D, \text{ one of the closest incoherent states } \delta_{\rho} \text{ to a } M^3 \text{ state } \rho \text{ is always a } M^3 \text{ incoherent state, i.e. one of the form} \]

\[ \delta_{\rho} = \frac{1}{2^N} (I^{\otimes N} + s \sigma_3^{oN}) \]

(A26)

for some coefficient \( s \in [-1, 1] \).

\[ \text{Proof.} \text{ Consider an arbitrary } N \text{-qubit state } \rho, \text{ which can be represented as} \]

\[ \rho = \sum_{i_1, i_2, \ldots, i_N=0}^{3} \tau_{i_1 i_2 \ldots i_N} \sigma_{i_1} \otimes \sigma_{i_2} \ldots \otimes \sigma_{i_N}, \]

(A27)

where the coefficients \( \tau_{i_1 i_2 \ldots i_N} = \text{Tr}[\rho \sigma_{i_1} \otimes \sigma_{i_2} \ldots \otimes \sigma_{i_N}] \in [-1, 1] \) are the correlation tensor elements of \( \rho \), and \( \sigma_0 \equiv I \). Any term involving \( \sigma_1 \) or \( \sigma_2 \) in the tensorial sum (A27) introduces off-diagonal elements, therefore we can write a general \( N \)-qubit incoherent state, with respect to the computational basis, as

\[ \delta = \sum_{i_1, i_2, \ldots, i_N=0}^{3} \tau_{i_1 i_2 \ldots i_N} \sigma_{i_1} \otimes \sigma_{i_2} \ldots \otimes \sigma_{i_N}, \]

(A28)

where each index \( i_j \) can now take either 0 or 3 as the only values. For any \( N \)-qubit incoherent state \( \delta \), we can define a corresponding incoherent \( M^3 \) state \( \delta_{\rho} \), whose \( \tau \) tensor is obtained from the one of \( \delta \) by setting all the \( \tau_{i_1 i_2 \ldots i_N} \) equal to zero, but for the two entries \( \tau_{0000} \) and \( \tau_{3333} \). We want to show that \( D(\rho, \delta_{\rho}) \leq D(\rho, \delta) \) for any \( M^3 \) state \( \rho \) and \( N \)-qubit incoherent state \( \delta \), which readily implies that one of the closest incoherent states \( \delta_{\rho} \), to a \( M^3 \) state \( \rho \) is indeed a \( M^3 \) state.

To begin, first consider the family of \( N-1 \) unitaries

\[ \{U\}_{j=1}^{N-1} = \{([\sigma_1 \otimes \sigma_1 \otimes I^{\otimes N-2}], (I \otimes \sigma_1 \otimes \sigma_1 \otimes I^{\otimes N-2}) \}

(A29)

\[ \{I^{\otimes 2} \otimes \sigma_1 \otimes \sigma_1 \otimes I^{\otimes N-4} \}, (I^{\otimes 3} \otimes \sigma_1 \otimes \sigma_1 \otimes I^{\otimes N-5} \}, \]

\[ \ldots, \{I^{\otimes N-3} \otimes \sigma_1 \otimes \sigma_1 \otimes I \}, (I^{\otimes N-2} \otimes \sigma_1 \otimes \sigma_1) \} \].

We note that every \( M^3 \) state \( \rho \) is invariant under the action of any \( U_j \). This can be seen as follows

\[ U_j \rho U_j^\dagger = \frac{1}{2^N} \left( U_j \rho^{oN} U_j^\dagger + \sum_{i=1}^{3} c_i U_j \sigma_i^{oN} U_j^\dagger \right) \]

(A30)

\[ = \frac{1}{2^N} \left( I^{\otimes N} + \sum_{i=1}^{3} c_i \sigma_i^{oN} \right) = \rho, \]

where in the second equality we use \( U_j I^{\otimes N} U_j^\dagger = I^{\otimes N} \) and \( U_j \sigma_i^{oN} U_j^\dagger = \sigma_i^{oN} \) which arises simply by recalling \( \sigma_1 \sigma_1 \sigma_1 = \sigma_1 \), \( \sigma_1 \sigma_2 \sigma_1 = -\sigma_2 \) and \( \sigma_1 \sigma_3 \sigma_1 = -\sigma_3 \) and noting that there are always two \( \sigma_1 \)’s in each unitary.

Now consider the action of \( U_1 \) on a generic incoherent state \( \delta \). The state transforms as

\[ U_1 \delta U_1^\dagger = \sum_{i_1, i_2, \ldots, i_N=0}^{3} \tau_{i_1 i_2 \ldots i_N} \sigma_{i_1} \sigma_1 \sigma_1 \sigma_1 \sigma_1 \sigma_1 \sigma_1 \sigma_1 \sigma_1 \sigma_1 \otimes \ldots \otimes \sigma_{i_N}. \]

(A31)
We have $\sigma_1^1\sigma_0^1 = \sigma_0^1$ and $\sigma_1^3\sigma_1^3 = -\sigma_3^3$, hence the coefficients $\tau_{i_1i_2i_3}^{U_j}$ of $U_j\delta U_j$ are $\tau_{i_1i_2i_3}^{U_j} = \tau_{000}^{U_j} = \tau_{333}^{U_j} = \tau_{333}^{U_j} = -\tau_{000}^{U_j}$ and $\tau_{303}^{U_j} = -\tau_{300}^{U_j}$. In other words, $U_j$ flips the sign of any element $\tau_{i_1i_2i_3}$ for which $i_1 \neq i_2$. We can further define a state that is a linear combination of $\delta$ and $U_j\delta U_j$,

$$\delta^1 = \frac{1}{2}(\delta + U_1\delta U_1^\dagger).$$

(A32)

The coefficients $\tau_{i_1i_2i_3}^{U_j}$ of $\delta^1$ can be found simply as

$$\tau_{i_1i_2i_3}^{U_j} = \frac{1}{2}(\tau_{i_1i_2i_3} + \tau_{i_1i_2i_3}).$$

(A33)

We see therefore that $\tau_{i_1i_2i_3}^{U_j} = \tau_{000}^{U_j} = \tau_{333}^{U_j} = \tau_{303}^{U_j} = 0$ and $\tau_{300}^{U_j}$.

Returning to the action of $U_j$ on $\delta$, it is a simple extension of the previous argument for $U_1$ to see that $U_j$ flips the value of $\tau_{i_1i_2i_3}$ when $i_j \neq i_j+1$. We are now in a position to define a set of incoherent states $\{\rho^0, \delta^1, \delta^2 \ldots \delta^{N-1}\}$ in an iterative way

$$\delta^j = \frac{1}{2}(\delta^{j-1} + U_j\delta^{j-1} U_j^\dagger),$$

where we use, in order, the definition of $\delta^j$ for $j \in [1, N - 1]$, the convexity of $D$, the invariance of $\rho$ through any $U_j$, i.e., $U_j\rho U_j^\dagger = \rho$, and the invariance of $D$ through unitaries, $D(\rho, \delta^j) \leq D(\rho, \delta^{j-1})$, which chained together imply $D(\rho, \delta^0) \leq D(\rho, \delta^{N-1})$. We know that $\delta^0 \equiv \delta$ and $\delta^{N-1} \equiv \delta_{M_3}^\delta$, hence we have shown that

$$D(\rho, \delta_{M_3}^\delta) \leq D(\rho, \delta) \forall \delta. \quad (A37)$$

Lemma A.3. For all even $N$, according to any contractive distance $D$, it holds that one of the closest incoherent $M_3^N$ states $\delta$ with triple $\{0, 0, s\}$ to a $M_3^N$ state $\rho$ with triple $\{c_1, (-1)^{N/2}c_1c_3, c_3\}$ is specified by $s = c_3$.

Proof. We need to prove that, for any $z$, it holds that

$$D((c_1, (-1)^{N/2}c_1c_3, c_3), \{0, 0, c_3\}) \leq D((c_1, (-1)^{N/2}c_1c_3, c_3), \{0, 0, c_3 + z\}).$$

In fact

$$D((c_1, (-1)^{N/2}c_1c_3, c_3), \{0, 0, c_3\}) = D((c_1, 0, 0), \{0, 0, 0\}) = D(A^1_{\delta^{N/2}}(c_1, (-1)^{N/2}c_1c_3, c_3), A^1_{\delta^{N/2}}(0, 0, c_3 + z)) \leq D((c_1, (-1)^{N/2}c_1c_3, c_3), \{0, 0, c_3 + z\}),$$

where the first equality is due to Lemma A.1 which holds for any contractive distance $D$ and any even $N$, the second equality is due to the fact that

$$\{c_1, 0, 0\} = A^1_{\delta^{N/2}}(c_1, (-1)^{N/2}c_1c_3, c_3), \quad \{0, 0, 0\} = A^1_{\delta^{N/2}}(0, 0, c_3 + z),$$

and $A^1_{\delta^{N/2}}$ representing the action of $N$ local independent bit flip noisy channels expressed by Eq. (A2), when $k = 1$ and $q = 1$ (i.e., $t \to \infty$), and finally the inequality is due to contractivity of the distance $D$.

Due to Lemma A.2 and Lemma A.3 we conclude that any bona fide distance-based measure of quantum coherence $C_D$ of the evolved $M_3^N$ state $\rho(q)$, given in Eq. (A3), is equal to the following distance

$$C_D(\rho(q)) = D((c_1, (-1)^{N/2}(1 - q)^Nc_1c_3, (1 - q)^Nc_3), \{0, 0, (1 - q)^Nc_3\}),$$

which is frozen for any $q$ (equivalently, for any time $t$) thanks to Lemma A.1, Eq. (A10). This concludes the proof of the central result in the main text.
rino, Phys. Rev. Lett. 112, 140501 (2014).
[67] A. Streltsov, H. Kampermann, and D. Bruß, Phys. Rev. Lett. 106, 160401 (2011).
[68] D. Girolami, T. Tufarelli, and G. Adesso, Phys. Rev. Lett. 110, 240402 (2013).
[69] M. F. Cornelio, O. J. Farias, F. F. Fanchini, I. Frerot, G. H. Aguilar, M. O. Hor-Meyll, M. C. de Oliveira, S. P. Walborn, A. O. Caldeira, and P. H. S. Ribeiro, Phys. Rev. Lett. 109, 190402 (2012).
[70] J.-S. Xu, X.-Y. Xu, C.-F. Li, C.-J. Zhang, X.-B. Zou, and G.-C. Guo, Nat. Commun. 1, 7 (2010).
[71] R. Auccaise, L. C. Céleri, D. O. Soares-Pinto, E. R. deAzevedo, J. Maziero, A. M. Souza, T. J. Bonagamba, R. S. Sarthour, I. S. Oliveira, and R. M. Serra, Phys. Rev. Lett. 107, 140403 (2011).
[72] J. S. Xu, K. Sun, C. F. Li, X. Y. Xu, G. C. Guo, E. Andersson, R. Lo Franco, and G. Compagno, Nat. Commun. 4, 2851 (2013).
[73] I. A. Silva, D. Girolami, R. Auccaise, R. S. Sarthour, I. S. Oliveira, T. J. Bonagamba, E. R. deAzevedo, D. O. Soares-Pinto, and G. Adesso, Phys. Rev. Lett. 110, 140501 (2013).
[74] F. M. Paula, I. A. Silva, J. D. Montealegre, A. M. Souza, E. R. deAzevedo, R. S. Sarthour, A. Saguia, I. S. Oliveira, D. O. Soares-Pinto, G. Adesso, et al., Phys. Rev. Lett. 111, 250401 (2013).