Diffusion Changes the Peak Value of Energy Loss Observed in a LArTPC

G. Putnam,\textsuperscript{a} D. W. Schmitz,\textsuperscript{a}
\textsuperscript{a}University of Chicago, Chicago, IL, 60637, USA

E-mail: grayputnam@uchicago.edu

ABSTRACT: Liquid Argon Time Projection Chamber (LArTPC) detectors observe ionization electrons to measure charged particle trajectories and energy. In a LArTPC, the long time (\sim ms) between when the ionization is produced and when it is collected means that diffusion can smear the charge by an amount comparable to the spatial resolution of the detector, given by the spacing between charge sensing channels (\sim mm). This smearing has an impact on the distribution of energy losses measured by each channel. In particular, the smearing acts to increase the effective “thickness” seen by each channel, and therefore the most-probable-value (MPV) of particle energy loss. We find, for example, that this effect shifts the MPV $dE/dx$ of a 1 GeV muon by \sim 4\% for a 2 ms drift time and 4.7 mm wire spacing, as in the DUNE-FD LArTPC. This has implications for the energy-scale calibration and electron lifetime measurements in a LArTPC, which both use the MPV of the muon energy loss Landau as a “standard candle”. The impact of diffusion on these calibrations is assessed.
1 Introduction

Liquid Argon Time Projection Chamber (LArTPC) detectors primarily leverage the ionization charge produced by charged particles traversing the medium to reconstruct their energy loss profile for use in particle identification and kinematic reconstruction. In a LArTPC, charge is drifted from the particle ionization column to a set of readout channels by the effect of a large electric field. The readout channels can be a plane of wires [1–7] or an array of pixels [8] depending on the detector. Between production and collection a number of effects perturbs the charge: impurities in the argon absorb electrons, distortions to the electric field (such as those from space-charge [9]) deflect the ionization from its path, and diffusion smears out the charge. These effects can cause non-uniformities in the detector response, and typically the calibration procedure in a LArTPC experiment will analyze and eliminate them to make the detector response uniform. This work focuses on the role of diffusion between ionization production and collection: we find that diffusion has a significant impact on the probability distribution of charge collected by each channel which...
has not been appreciated in previous LArTPC experiments. In contrast to other effects, which are treated as perturbations to the detector response, the role of diffusion should be understood as changing the underlying distribution of energy loss seen by each channel.

The probability distribution governing particle energy loss observed by a channel is the Landau-Vavilov distribution \[10\]. In the limit that the charged particle is relativistic and the slice of the particle that the channel is sensitive to is small (as is often the case in a LArTPC), this distribution is well-approximated by the Landau distribution \[11\]. For both distributions, the mean energy loss per length (given by the Bethe-Bloch theory \[12\]) is independent of the length of the charged particle observed by the channel. However, the shape of the distribution depends strongly on this length.

The Landau and Landau-Vavilov distributions are governed by the cross-section of charged particles incident on atomic electrons, which has a power law dependency on the energy transfer. This behavior means that the mean energy loss is influenced by a small number of large energy transfers to electrons well above the atomic excitation energy (\(\delta\)-rays). When the channel is only sensitive to a short enough length of the charged particle such that it will not see a \(\delta\)-ray most of the time, the bulk of the distribution is below the mean value of energy loss with a long tail extending out to high energy losses. The Landau distribution is an appropriate approximation in this case. As the channel-sensitive length increases, more of the \(\delta\) rays get absorbed into the bulk of the distribution and the peak increases as the variance drops, until the distribution is better modeled by a Landau-Vavilov.

Traditionally in LArTPC experiments, the length of the charged particle observed by each channel has been understood as being given by the spacing between channels. However, diffusion changes this length. Between production and collection, ionization electrons in a LArTPC diffuse in the direction of their drift velocity (“longitudinal” diffusion) and in the two perpendicular directions (“transverse” diffusion). The magnitude of the smearing is different between the longitudinal and transverse directions and is parameterized by two separate diffusion constants. In the transverse directions, the smearing is given by \[13, 14\]

\[
\sigma_T = \sqrt{2 \cdot D_T \cdot t_{\text{drift}}},
\]

where \(D_T\) is the transverse diffusion constant and \(t_{\text{drift}}\) is the time between production and collection for a cluster of ionization charge. The transverse diffusion constant has never been directly measured in LAr. It can in principle be extrapolated from measurements of the longitudinal diffusion constant \(D_L\) \[15, 16\], which have been performed \[17–19\]. However, existing longitudinal diffusion measurements are in tension with each other, so there is not a clear way to perform this extrapolation. From this picture there are two available predictions for the transverse diffusion constant: one from the Li et al. measurement \[18\] and another from Atrazhev-Timoshkin theory \[20\]. At a drift field of 500 V/cm (typical for existing LArTPCs), the Li et al. (Atrazhev-Timoshkin) parameterization
predicts that $D_T = 12.0 \text{ cm}^2/\text{s} (9.30 \text{ cm}^2/\text{s})$ and the transverse diffusion width in time is

$$\sigma_T = 1.55 \frac{1}{1 \text{ ms}} \sqrt{t_{\text{drift}}} \text{ mm}.$$  \hspace{1cm} (1.2)

Diffusion in the direction perpendicular to the drift orientation smears the positions of ionization electrons seen by each channel. Figure 1 illustrates the case of a plane of wires in a LArTPC. The diffusion in the perpendicular direction means that each channel sees a larger length of the charged particle than would just be given by the channel spacing: \textit{diffusion effectively makes the spacing a drift-dependent quantity}. This means that the Landau-Vavilov or Landau distribution of energy loss seen by each channel is influenced by diffusion. In particular, diffusion acts to increase the most-probable-value (MPV) of energy loss seen by each channel.

This effect is derived and discussed from a generalized perspective in section 2, with further details in appendix A. Then, in section 3, we apply the general results of section 2 to the LArTPC case, and in section 4 we discuss the outlook for LArTPC calibrations.

\section*{2 Distribution of Energy Loss Seen by a Generalized Channel}

The Landau-Vavilov and Landau distributions are derived assuming that a detector has a step-function sensitivity to the path of a charged particle: there is a region where the channel detects all of the deposited energy, and a region where it detects none. In the case of a LArTPC impacted by diffusion, this assumption does not hold. As is illustrated in figure 1, a channel in a LArTPC is sensitive to a fractional amount of the particle energy loss, dependent on its position. To parameterize this effect, we define the channel ionization weight function $w(x)$, which gives this fraction as a function of the particle position $x$. This weight function defines the probability distribution of energy that the channel measures.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Diagram of the effect of diffusion on the path length of a particle track seen by each channel in a LArTPC with a wire-plane readout. Diffusion in the direction transverse to the wire orientation smears charge in and out of the range each wire is sensitive to.}
\end{figure}
To obtain the distribution of energy loss for a general channel ionization weight function, we go through the usual derivation of the Landau-Vavilov distribution, keeping this weight function along the way. Section 2.1 introduces the principles in this derivation. Section 2.2 gives an analytic formula for the energy loss distribution. Section 2.3 discusses the relativistic/thin-film limit for a weight function. Details of the derivation are given in appendix A.

2.1 Principles of the Energy Loss Distribution

We construct the probability distribution of energy loss iteratively. We start from the simple base case of the distribution over an infinitesimal length, and then build it up into the distribution over finite length. Our tool in this construction is the convolution property: given the probability distribution of particle energy loss over a length $\ell$, $p_\ell(E)$, the probability of an energy loss $E$ over a distance $2\ell$ is equal to the probability of losing energy $E_0$ over $\ell$ ($p_\ell(E_0')$) times the probability of losing an energy $E - E'$ over $\ell$ ($p_\ell(E - E')$), for all $E'$. This combination represents all the ways that a particle can lose an amount of energy $E$ over a distance $2\ell$ in two steps of $\ell$. Thus the convolution property states that

$$p_{2\ell}(E) = \int p_\ell(E - E') p_\ell(E') dE'.$$

(2.1)

The distribution of energy loss for an infinitesimal length is given by the limit that the particle travels a short enough distance such that it will scatter off at most one atomic electron. In this case, the probability of energy loss $T$ is equal to the probability of colliding with an atomic electron with precisely that amount of energy transfer: $\rho \ell d\sigma dT$, where $\rho$ is the number density of electrons and $d\sigma/dT$ is the differential cross section w.r.t. the energy transfer $T$. The probability of no energy loss is equal to the probability of not interacting with an atomic electron. Thus,

$$\lim_{\ell \to 0} p_\ell(T) = (1 - \rho \sigma \ell) \delta(T) + \rho \ell d\sigma dT,$$

(2.2)

where $\sigma = \int d\sigma dT$ and $\delta$ is the Dirac-Delta function.

In the case of heavy charged particles (such as muons and protons) elastically scattering on bare electrons, the cross section is given by the Rutherford formula [21]

$$d\sigma dT = \frac{2\pi r_e^2 m_e}{\beta^2 T_{\text{max}}} \left( 1 - \beta^2 T/T_{\text{max}} \right),$$

(2.3)

where $r_e$ is the classical electron radius, $m_e$ is the electron mass, $\beta$ is given by the particle velocity, and $T_{\text{max}}$ is the maximum energy transfer in a single collision, $T_{\text{max}} = \frac{2m_e \beta^2 r_e^2}{1 + 2ym_e/M^2}$, where $M$ is the mass of the charged particle. The cross section for atomic electrons is modified relative to the bare cross section at low energy transfer by atomic effects. A useful quantity related to the cross section is

$$\zeta = \rho \frac{2\pi r_e^2 m_e}{T_{\text{max}} \beta^2}.$$

(2.4)

In terms of $\zeta$, $\rho d\sigma dT = \zeta T_{\text{max}} \frac{1 - \beta^2 T/T_{\text{max}}}{T^2}$. Thus, $\zeta$ is a quantity with units of inverse length that encodes the rate of scattering.
For the probability distribution of energy loss over an infinitesimal length, the channel ionization weight function can always be assumed to be a constant $w$. In this case, for a channel to measure an energy deposition $E$ the particle must deposit an amount $E/w$. So in the presence of a weight function $w$, $p_{\ell}(E) \to \frac{1}{w} p_{\ell}(E/w)$ (the $1/w$ in front fixes the normalization) when $\ell$ is small.

2.2 Analytic Form of the Distribution

To build the particle energy loss distribution, we discretize $w(x)$ into weights $w_i$ over infinitesimal steps $dx_i$, and build up the probability distribution over the full weight function by performing a product of convolutions:

$$p_w(E) = \int dt_1 \int dt_2 \cdots \int dt_n p_{d_{0}} \left( \frac{E - T_n \cdots - T_1}{w_0} \right) \times \frac{p_{d_{1}} \left( \frac{T_n \cdots - T_1}{w_1} \right)}{w_1} \times \cdots \times \frac{p_{d_{n}} \left( \frac{T_n}{w_n} \right)}{w_n}.$$ (2.5)

Appendix A.1 goes through the process of simplifying this into an algebraic expression for the case of Rutherford scattering. In the process of this derivation, it is necessary to introduce the mean energy loss $\bar{E}$ as an external input to the distribution. This is because atomic effects modify the bare electron cross section at low $\bar{T}$ in a way that can be accounted for through the mean energy loss using Bethe-Bloch theory. We obtain:

$$p_w(E) = \frac{1}{2\pi \zeta T_{\text{max}}} \int_{-\infty}^{\infty} dz \exp \left\{ \frac{i z}{\zeta T_{\text{max}}} (E - \bar{E}) + \int dx \, \zeta (1 - e^{-iw(x)z/i\zeta}) - izw(x)(1 + \beta^2) + (\zeta \beta^2 + iw(x)z)(-\text{Ei}[-iw(x)z/i\zeta] + \log[iw(x)z/i\zeta] + \gamma_{\text{EM}}) \right\}.$$ (2.6)

where $\gamma_{\text{EM}} \approx 0.577$ is the Euler-Mascheroni constant and Ei is the exponential integral function, $\text{Ei}(x) = - \int_{-x}^{\infty} dt \, e^{-t}/t$. This is the probability distribution of particle energy loss for a channel with a general ionization weight function $w(x)$ as a function of the charged particle position. Appendix A.3 shows that this distribution reduces to a Landau-Vavilov precisely when the channel ionization weight function is given by a step function.

2.3 The Thin-Film/Relativistic Limit

To take the thin-film/relativistic limit, we take the case where the weight function is narrow compared to the size of the scattering length $1/\zeta$ (details in appendix A.2). In this limit, $p_w(E)$ converges to

$$p_w(E) = \frac{1}{2\pi i \zeta T_{\text{max}} \rho} \int_{-\infty}^{\infty} dz' \exp \left[ z' (\lambda + \log|z'|) \right],$$ (2.7)

where $\rho \equiv \int w(x)dx$ and $\lambda = \frac{E - \bar{E}}{T_{\text{max}}} - \log \zeta \rho + \gamma_{\text{EM}} - 1 - \beta^2 + \frac{\int dx \, \log[|w(x)|]}{\rho}$. $\rho$ is defined as the pitch. This equation can be recognized as the Landau distribution for a parameter $\lambda$, which has a maximum at $\lambda_{\text{MPV}} = \gamma_{\text{EM}} - 0.8 \approx -0.22278 \ldots$ [22]. Therefore:

$$E_{\text{MPV}} = \bar{E} + \zeta T_{\text{max}} \rho \left( \log \zeta \rho + 0.2 + \beta^2 - \frac{\int dx \, \log[|w(x)|]}{\rho} \right).$$
In the case where \( w(x) \) is a step function, then \( \int dx \, w(x)\log[w(x)] = 0 \) and \( E_{\text{MPV}} \) reduces to the usual result [21]. For a general \( w(x) \), \( \int dx \, w(x)\log[w(x)] \leq 0 \) (since \( 0 \leq w \leq 1 \)), so the \( \text{MPV} \) is in general larger than the result for a step function for a fixed pitch. We can recast this result by defining the thickness

\[
t = \rho e^{-\frac{\int dx \, w(x)\log[w(x)]}{\rho}}.
\]  

(2.8)

Then the formula for the \( \text{MPV} \) energy loss \( \frac{dE}{dx_{\text{MPV}}} \equiv \frac{E_{\text{MPV}}}{\rho} \) is

\[
\frac{dE}{dx_{\text{MPV}}} = \frac{dE}{dx} + \zeta_{\text{max}} \left( \log \zeta t + 0.2 + \beta^2 \right) .
\]  

(2.9)

This equation, with equation 2.8 as an input, is the main result of this derivation. In this formula, all the dependence on the weight function \( w(x) \) is encoded in the \( t \) parameter. This is precisely the usual formula for the energy loss \( \text{MPV} \) \( \frac{dE}{dx} \), just with the thickness given by equation 2.8. In general, the thickness \( t \) is larger than the pitch \( \rho \). Under scaling of \( w \), \( t \) is unchanged: \( w(x) \rightarrow a \cdot w(x), \rho \rightarrow a \cdot \rho, \ t \rightarrow t \). Under dilation of \( w \), \( t \) scales like the pitch: \( w(x) \rightarrow w(x/a), \rho \rightarrow a \cdot \rho, \ t \rightarrow a \cdot t \).

2.4 Derivation Summary

In the nominal case (a step-function weight function), in the thin-film/relativistic approximation, the distribution of particle energy loss is a Landau distribution. For a general weight function in the relativistic limit, the shape of the distribution is unchanged relative to the nominal case: it is still a Landau. However, the \( \text{MPV} \) of the distribution shifts in a way parameterized by the channel thickness defined in equation 2.8. Outside of the relativistic/thin-film limit, in the nominal case of the weight function the distribution of particle energy loss is a Landau-Vavilov. In general, for any weight function outside of the nominal case, this distribution is not equivalent to a Landau-Vavilov. The form of the probability distribution of particle energy loss for a general weight function is given by equation 2.6.

These results can be understood in terms of the stability of the input Landau and Landau-Vavilov distributions. The probability distribution for a weight function \( w(x) \) is given by the sum of probability distributions for an infinitesimal length \( dx \). At each \( dx \), \( w(x) \) is a constant and the infinitesimal distribution is a Landau (in the relativistic limit) or a Landau-Vavilov (in the general case). Since the Landau distribution is stable, the sum of all these infinitesimal distributions must also be a Landau. In the general case, each infinitesimal probability distribution is a Landau-Vavilov, which is not stable, and their sum will in general converge to a different probability distribution.

3 The LArTPC Channel Ionization Weight Function

We now apply these results to the case of a LArTPC with a wire-plane readout. As illustrated in figure 1, in the LArTPC case the channel ionization weight function is given by the step function of the wire pitch convolved with the Gaussian effect of smearing induced by transverse diffusion. In addition, as shown in figure 2, a track will in general be at some angle \( \gamma \) to the wire orientation, which dilates the channel weight function \( w \). Putting this together, we obtain

\[
w_{\text{LArTPC}}(x) = \int_0^a \frac{dx'}{\sqrt{2\pi}\sigma_T} e^{-(x-x')^2/(2\sigma_T^2)},
\]  

(3.1)
for a wire separation $a$ and $\sigma_T$ given by transverse diffusion (equation 1.1).

Figure 2. Diagram of the relationship between track orientation and track pitch in a LArTPC.

This distribution has a pitch $\rho = a / \cos \gamma$ (note that the formula for the pitch $\rho$ here coincides with the usual definition of the track pitch in a LArTPC). In the limit of a small wire separation relative to the Gaussian width, $w(x) = \sqrt{2\pi} \sigma_T \exp[-\frac{x^2 \cos^2 \gamma}{2 \sigma_T^2}]$ and $t = \sigma_T \sqrt{2\pi e / \cos \gamma}$. In general though, we have not found a useful closed form of the thickness for this weight function. A plot of the thickness $t$ as a function of $\sigma_T / a$ for $\cos \gamma = 1$ is shown in figure 3. By the scaling property of the thickness under dilation, $t(\cos \gamma) = t(1) / \cos \gamma$.

Figure 3. (Left) Smearing widths from transverse diffusion (using the Li et al. parameterization for $D_T$ [18]) by drift time for the DUNE (4.7mm) and SBN (3mm) wire pitches. (Right) Thickness observed by a channel with a weight function given by a step function of width $a$ convolved with a Gaussian of width $\sigma_T$, as a function of $\sigma_T / a$.

Figure 4 plots the most-probable-value of energy loss obtained from a Monte-Carlo simulation of muon energy loss observed by a LArTPC-like weight function, as a function of the smearing width $\sigma_T$. (Implementation details are in appendix B.) The value is compared to the energy loss estimate from the thin-film approximation (i.e., using the value of the thickness in the Landau MPV formula, equation 2.9). The Landau approximation works well at small thickness and large muon energy. Outside of this region, numerically obtaining the peak of the general distribution with the weight function (equation 2.6) would be required. Since the LArTPC weight function is not a step function, in this region the distribution is also not a Landau-Vavilov distribution. In the usual
Vavilov case, one defines this region of phase space using the parameter $\zeta \cdot p$, which is a unit-less measure of the “film-thickness” of a channel that is small for large particle energy and small channel thicknesses. Typically, the Landau distribution is taken to be valid for $\zeta \cdot p < 0.01$ [21]. For our purposes, it is natural to leverage $\zeta \cdot t$. This value is plotted for the numerically computed energies and thicknesses in figure 5.

These same principles also apply to a LArTPC with a readout from an array of charge sensing pixels. In this case, diffusion along both transverse directions is important. For a particle moving in the ($\hat{y}$, $\hat{z}$) direction in the transverse plane that hits the pixel at the location $(y_0, z_0)$, the weight function of a square pixel with a side length $a$ as a function of the particle position $r$ is

$$w_{\text{pixel-LArTPC}}(r) = \int_0^a dy' \int_0^a dz' \frac{1}{2\pi \sigma_T^2} e^{-\left[(r - \hat{y} y_0 - y')^2/(2\sigma_T^2)\right]} e^{-\left[(r - \hat{z} z_0 - z')^2/(2\sigma_T^2)\right]}.$$  \hspace{1cm} (3.2)

The location that the track hits the pixel at is likely challenging to reconstruct. This is not just a challenge for computing the weight function; the pitch of the track across the pixel also depends on this location. One workaround would be to sum the reconstructed charge across the pixels in either the $\hat{y}$ or $\hat{z}$ directions. In this case, the weight function reduces to the wire-plane readout case.

**Figure 4.** Comparison of the MPV energy loss from a Monte-Carlo simulation of the probability distribution in equation 2.6 (details in appendix B) and the Landau MPV approximation (equation 2.9) for a channel ionization weight function given by a step function of width 3mm convolved with a Gaussian of varying widths (equation 3.1). The comparison is made at various values of the muon energy. At low muon energy and large channel thickness, the thin-film/relativistic approximation underlying the Landau MPV prediction begins to breakdown, up to the percent level at 0.2 GeV.
4 Impact on LArTPC Calibration

These findings impact LArTPC detector calibrations, which use particle energy loss (especially from cosmic muons) as a “standard candle”.

4.1 Energy Scale Calibration

LArTPC detectors use muons to calibrate the map from digitized charge to energy. This is done by calibrating the gain of the readout electronics (from ADC to number of electrons), applying any detector uniformity factors, and then using external measurements of the ionization work function [23] and recombination [24, 25] to map from electrons to energy. The measurement of the gain is done by measuring the \( dQ/dx \) distribution from muon energy loss binned in muon kinematics and orientation to select for a single Landau distribution. A Landau convolved with a Gaussian is fit to this distribution, and the MPV of the fit is compared to the prediction from Bethe-Bloch theory with the recombination model to extract the detector gain (see prior calibrations: [26–29]). Since these calibrations rely on the MPV, the energy loss they calibrate to depends on diffusion. The fact that prior experiments neglected the role of diffusion in this calibration means that they were systematically biased by an effect of a few percent.

In the MicroBooNE detector (where this effect is large due to the large drift time), for a muon at an energy \( E = 1 \) GeV at the cathode (with a maximal drift time), the \( dE/dx \) MPV is 1.79 MeV/cm.
accounting for the thickness\(^2\), whereas it would be 1.69 MeV/cm computed (incorrectly) using the track pitch. For further reference, table 1 lists the MPV \(dE/dx\) for a 1 GeV muon for a few LArTPC detector configurations.

In order to get the energy scale calibration correct, the effect of diffusion must be included. In the relativistic limit, this can be done by using equation 2.8 to compute the thickness input to the Landau MPV \(dE/dx\) equation. The thickness equation has four inputs: the channel spacing, the transverse diffusion constant, the drift time, and the muon track orientation. The channel spacing is a constant of each detector. The transverse diffusion constant is a LAr property which can be measured. And the drift time and track orientation can be reconstructed for each individual energy deposition. Where the relativistic limit breaks down (i.e. where \(\zeta \cdot t > 0.01\)), the general distribution of particle energy loss must be used (equation 2.6). Since the LArTPC channel weight function is not a step function, in this region this distribution is not a Landau-Vavilov distribution (see appendix A.3 for a proof). Thus, existing numerical routines such as the ROOT [30] VavilovAccurate function cannot be used. For typical LArTPC detector configurations, the non-Landau region corresponds to the particle Bragg peak (as shown by the values of \(\zeta \cdot t\) in figure 5), so these considerations are also important for modeling the distribution of particle energy losses for particle identification.

### Table 1

| Detector       | Wire Pitch [mm] | Drift Time [ms] | Diff. Const. \(D_T\) [cm\(^2\)/s] | MPV \(dE/dx\), No Diffusion [MeV/cm] | MPV \(dE/dx\) at Cathode (Full Diff.) [MeV/cm] |
|----------------|----------------|-----------------|----------------------------------|--------------------------------------|-----------------------------------------------|
| MicroBooNE [4] | 3.00           | 2.33            | 5.85                             | 1.69                                 | 1.79                                          |
| ArgoNeuT [3]   | 4.00           | 0.295           | 12.0 (9.30)                      | 1.72 (1.72)                          | 1.76 (1.75)                                  |
| ICARUS [5]     | 3.00           | 0.960           | 12.0 (9.30)                      | 1.69 (1.69)                          | 1.78 (1.77)                                  |
| SBND [5]       | 3.00           | 1.28            | 12.0 (9.30)                      | 1.69 (1.69)                          | 1.79 (1.78)                                  |
| DUNE-FD (SP) [7]| 4.71           | 2.2             | 12.0 (9.30)                      | 1.74 (1.74)                          | 1.82 (1.81)                                  |

#### 4.2 Drift Direction Response Equalization

The effect of diffusion can bias measurements attempting to equalize the response of the detector as a function of drift time (removing effects such as LAr impurities). This is typically done by defining an observable of the \(dQ/dx\) distribution from cosmic muons (such as the median [29]) and applying a drift-dependent correction factor to make the observable flat across the detector. However, since the underlying distribution of energy loss is also drift-dependent in the presence of diffusion, these procedures may be dividing-out a physical effect rather than just the detector response. The effect

---

\(^2\)This is using the 0.3 cm wire pitch, 2.56m drift length, 0.1098 cm/\(\mu\)s drift velocity, and 5.85 cm\(^2\)/s transverse diffusion constant reported in [17]. This results in a transverse smearing width of 0.165 cm and a thickness of 0.771 cm. Both values of \(dE/dx\) neglect the density effect.
of smearing induced by transverse diffusion is to push the value of the $dQ/dx$ MPV up along the drift length, as opposed to LAr impurities which push it down.

There are two strategies one can take to minimize the effect of diffusion on these measurements. First, one can try to define a quantity of the $dQ/dx$ distribution that tracks to the mean energy loss more than the MPV (such as a truncated mean). The mean value of energy loss is not affected by diffusion. Second, one can “coarse-grain” the detector by summing hits across many consecutive channels along the track into each value of $dQ/dx$. By combining enough channels, the length of the effective channel spacing would dominate over the length of diffusion and thus the drift-dependent effect of diffusion could be made negligible. This process is demonstrated diagrammatically in figure 6.

There are two strategies one can take to minimize the effect of diffusion on these measurements. First, one can try to define a quantity of the $dQ/dx$ distribution that tracks to the mean energy loss more than the MPV (such as a truncated mean). The mean value of energy loss is not affected by diffusion. Second, one can “coarse-grain” the detector by summing hits across many consecutive channels along the track into each value of $dQ/dx$. By combining enough channels, the length of the effective channel spacing would dominate over the length of diffusion and thus the drift-dependent effect of diffusion could be made negligible. This process is demonstrated diagrammatically in figure 6.

Figure 6. Measuring the charge wire-by-wire in a LArTPC (top) means that the thickness of the wire is significantly impacted by diffusion. Summing the charge across a large number of consecutive channels (bottom) diminishes this effect, which can be a useful technique for calibrations.

### 4.3 Future Outlook

Since this is a new effect relative to prior calibrations of LArTPC experiments, one may also want to verify in the data that there is a change in $dQ/dx$ as predicted by this effect. This may be challenging, as a number of different detector effects conspire to change the $dQ/dx$ by drift time, all in their own way. One possibility could be to leverage the fact that this effect only changes the MPV of particle charge depositions, as opposed to other effects which should change the mean and the MPV equally: measure the ratio of a truncated mean to the MPV of $dQ/dx$ by drift time. Another would be to compare the ratio of a coarse-grained measurement of $dQ/dx$ (where diffusion should be negligible) to a channel-by-channel measurement of $dQ/dx$. In any such measurement, an experiment where other drift-dependent factors are small (small field distortions and few LAr impurities) is ideal. Existing LArTPC experiments should be able to verify in their Monte-Carlo simulations that this effect occurs. The change to the MPV is an emergent effect of the Landau-Vavilov nature of particle
energy loss combined with diffusion. Any LArTPC detector simulation that includes these effects should output a $dE/dx$ distribution consistent with the results of this work. (This effect has already been demonstrated in a standalone LArTPC-like simulation using GEANT4 in [31].)

The smearing which warps the thickness is induced by transverse diffusion, which has never been measured directly in LAr. Existing estimates of the transverse diffusion constant are based on longitudinal diffusion measurements [17–19]. These measurements disagree with each other, which makes this extraction challenging. Until it is measured directly, the value of transverse diffusion represents a significant uncertainty for energy scale calibrations in a LArTPC.

5 Summary

In this work, we have derived the distribution of particle energy loss seen by a channel with a position dependent weight function to the particle ionization $w(x)$. In general, this distribution is not equivalent to the Landau-Vavilov distribution, but in the thin-film limit it does converge to a Landau distribution. In the Landau limit, the energy loss MPV can be computed using the thickness $t = \rho e^{-\int w(x)\log[w(x)]dx/\rho}$, where $\rho = \int w(x)dx$. In a LArTPC, $w$ is given by the convolution of the step-function wire slice and the Gaussian effect of transverse diffusion. In this case $t \neq \rho$. Prior calibrations in LArTPCs have used the track pitch $\rho$ as the input to the Landau MPV computation, which is incorrect. In addition, since the effect of transverse diffusion is drift dependent, this effect can bias attempts to equalize the detector response of a LArTPC in the drift direction. Coarse-graining the detector is likely an effective method to mitigate this bias. The value of transverse diffusion has never been quantitatively measured in LAr, and it represents a significant systematic uncertainty on any energy scale calibration.

Acknowledgments

Thank you to Mike Mooney, Joseph Zennamo, and Ed Blucher for valuable conversations on the content of the paper. And thank you to Mike Mooney for a helpful discussion on the current state of diffusion measurements. This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE-1746045 and the National Science Foundation under Grant No. PHY-1913983.

References

[1] C. Rubbia, The Liquid Argon Time Projection Chamber: A New Concept for Neutrino Detectors, .
[2] ICARUS collaboration, Design, construction and tests of the ICARUS T600 detector, Nucl. Instrum. Meth. A 527 (2004) 329.
[3] ArgoNeuT collaboration, The ArgoNeuT Detector in the NuMI Low-Energy beam line at Fermilab, JINST 7 (2012) P10019 [1205.6747].
[4] MicroBooNE collaboration, Design and Construction of the MicroBooNE Detector, JINST 12 (2017) P02017 [1612.05824].
[5] SBN collaboration, A Proposal for a Three Detector Short-Baseline Neutrino Oscillation Program in the Fermilab Booster Neutrino Beam, 1503.01520.
[6] DUNE collaboration, *Design, construction and operation of the ProtoDUNE-SP Liquid Argon TPC*, *JINST* 17 (2022) P01005 [2108.01902].

[7] DUNE collaboration, *Deep Underground Neutrino Experiment (DUNE), Far Detector Technical Design Report, Volume IV: Far Detector Single-phase Technology*, *JINST* 15 (2020) T08010 [2002.03010].

[8] J. Asaadi et al., *First Demonstration of a Pixelated Charge Readout for Single-Phase Liquid Argon Time Projection Chambers*, *Instruments* 4 (2020) 9 [1801.08884].

[9] MicroBooNE collaboration, *Measurement of space charge effects in the MicroBooNE LArTPC using cosmic muons*, *JINST* 15 (2020) P12037 [2008.09765].

[10] P.V. Vavilov, *Ionization losses of high-energy heavy particles*, Sov. Phys. JETP 5 (1957) 749.

[11] L. Landau, *On the energy loss of fast particles by ionization*, J. Phys. (USSR) 8 (1944) 201.

[12] H. Bethe, *Zur Theorie des Durchgangs schneller Korpuskularstrahlen durch Materie*, *Annalen der Physik* 397 (1930) 325.

[13] A. Einstein, *Über die von der molekularkinetischen theorie der wärme geforderte bewegung von in ruhenden flüssigkeiten suspendierten teilchen*, *Annalen der Physik* 322 (1905) 549.

[14] M. von Smoluchowski, *Zur kinetischen theorie der brownschen molekularbewegung und der suspensionen*, *Annalen der Physik* 326 (1906) 756.

[15] G.H. Wannier, *Motion of gaseous ions in strong electric fields*, *The Bell System Technical Journal* 32 (1953) 170.

[16] R.E. Robson, *A thermodynamic treatment of anisotropic diffusion in an electric field*, *Australian Journal of Physics* 25 (1972) 685.

[17] MicroBooNE collaboration, *Measurement of the longitudinal diffusion of ionization electrons in the MicroBooNE detector*, *JINST* 16 (2021) P09025 [2104.06551].

[18] Y. Li et al., *Measurement of Longitudinal Electron Diffusion in Liquid Argon*, *Nucl. Instrum. Meth. A* 816 (2016) 160 [1508.07059].

[19] ICARUS collaboration, *Performance of a 3-ton liquid argon time projection chamber*, *Nucl. Instrum. Meth. A* 345 (1994) 230.

[20] V.M. Atrazhev and I.V. Timoshkin, *Transport of electrons in atomic liquids in high electric fields*, *IEEE Transactions on Dielectrics and Electrical Insulation* 5 (1998) 450.

[21] P.A.Z. et. al. (Particle Data Group), *Review of Particle Physics*, *Progress of Theoretical and Experimental Physics* 2020 (2020).

[22] K.S. Kolbig and B. Schorr, *Asymptotic expansions for the landau density and distribution functions*, *Comput. Phys. Commun.* 32 (1984) 121.

[23] M. Miyajima, T. Takahashi, S. Konno, T. Hamada, S. Kubota, H. Shibamura et al., *Average energy expended per ion pair in liquid argon*, *Phys. Rev. A* 9 (1974) 1438.

[24] ArgoNeuT collaboration, *A Study of Electron Recombination Using Highly Ionizing Particles in the ArgoNeuT Liquid Argon TPC*, *JINST* 8 (2013) P08005 [1306.1712].

[25] ICARUS collaboration, *Study of electron recombination in liquid argon with the ICARUS TPC*, *Nucl. Instrum. Meth. A* 523 (2004) 275.
A The Distribution of Energy Loss Seen by a Channel with a Position-Dependent Sensitivity to Particle Energy

A.1 Derivation

Typically, the Landau and Landau-Vavilov distributions are derived by means of a Laplace transform leveraging the continuity property of the energy loss distribution [10]. Here we take an alternative approach, applying the convolution property (equation 2.1) by means of a Fourier transform. We also keep track of the channel ionization weight function as a function of the charged particle position \( w(x) \). We end at the same place, except for perturbations coming from the weight function. To start, we discretize \( w(x) \) into weights \( w_i \) over infinitesimal steps \( dx_i \), and build up the probability distribution over the full weight function by performing a product of convolutions:

\[
p_w(E) = \int dT_1 \int dT_2 \cdots \int dT_n p_{dx_0}(E-T_0-w_0) w_0 \times p_{dx_1}(T_0-w_1) w_1 \times \cdots \times p_{dx_n}(T_n-w_n) w_n. \tag{A.1}
\]

By use of a Fourier transform \( \mathcal{F} \), we can turn this into a regular product:

\[
p_w(E) = \mathcal{F}^{-1} \prod_{\tau \rightarrow E} \mathcal{F} \int p_{dx_i}(T/w_i) w_i. \]

Applying the scale property of a Fourier Transform:

\[
p_w(E) = \mathcal{F}^{-1} \prod_{\tau \rightarrow E} \mathcal{F} \int p_{dx_i}(T). \]

By taking the exponential of the log of the RHS, we can manipulate it into a sum:

\[
p_w(E) = \mathcal{F}^{-1} \exp \left[ \sum_i \log \mathcal{F} \int p_{dx_i}(T) \right].
\]
Using the small-\( \ell \) formula (2.2) for \( p_{\text{d}x}(T) \), we can simplify its Fourier transform:

\[
p_w(E) = \frac{F^{-1}}{\tau \to E} \exp \left[ \sum_{i} \log(1 - \sigma \rho dx_i + \rho dx_i \mathcal{F}_{T \to w_i \tau} \frac{d\sigma}{dT} \right].
\]

By applying \( \log(1 + \epsilon) \approx \epsilon \), we obtain:

\[
p_w(E) = \frac{F^{-1}}{\tau \to E} \exp \left[ \sum_{i} -\sigma \rho dx_i + \rho dx_i \mathcal{F}_{T \to w_i \tau} \frac{d\sigma}{dT} \right].
\]

Which can be neatly turned into an integral:

\[
p_w(E) = \frac{F^{-1}}{\tau \to E} \exp \left[ \int dx(-\rho \sigma + \rho \mathcal{F}_{T \to w(\tau)} \frac{d\sigma}{dT} \right].
\]

Now we apply the definitions of \( \sigma \) and \( \mathcal{F} \) to evaluate the integral. Noting \( d\sigma/dT \neq 0 \) only for \( 0 < T < T_{\text{max}} \), we obtain

\[
p_w(E) = \frac{F^{-1}}{\tau \to E} \exp \left[ \int dx \frac{2\pi r^2 m_e}{\beta^2} \int dx \int_0^{T_{\text{max}}} dT \frac{d\sigma}{dT} \left( e^{-2\pi i \tau w(x)T} - 1 \right) \right].
\]

where we have used the formula for the bare cross section (equation 2.3) in place of \( d\sigma/dT \).

The integrand diverges as \( 1/T \) as \( T \to 0 \). This divergence appears because we have used the cross section of scattering on bare electrons instead of atomic electrons. At low energy transfer the cross section is modified by atomic effects, which in particular cut off the cross section near the excitation energy (above \( T = 0 \)) to remove the divergence. We can remove the divergent behavior of the integrand by adding and subtracting the mean energy loss \( \bar{E} = \int dx \int dT \rho \frac{d\sigma}{dT} T w(x) \). That subtracting the mean energy loss makes the integrand converge indicates that the shape of the distribution is not sensitive to atomic effects; these only act to change the mean. So once the impact of atomic effects on the mean energy loss is accounted for, it is safe to apply the bare cross section to find the shape of the distribution. Thus, for the purpose of this derivation, we take the mean energy loss as an external input (from Bethe-Bloch theory [12]) and find how the mean relates to the shape of the distribution. (There is also nothing new here coming from the weight function; this is precisely the same approximation that is made by Vavilov [10]). Applying this substitution, we get

\[
p_w(E) = \frac{F^{-1}}{\tau \to E} \exp \left[ -2\pi i \bar{E} \tau + \frac{2\pi r^2 m_e}{\beta^2} \int dx \int_0^{T_{\text{max}}} dT \frac{1 - \beta^2 T / T_{\text{max}}}{T^2} \left( e^{-2\pi i \tau w(x)T} + 2\pi i \tau w(x)T - 1 \right) \right].
\]

This integral converges to

\[
p_w(E) = \frac{F^{-1}}{\tau \to E} \exp \left[ -2\pi i \bar{E} \tau + \frac{2\pi r^2 m_e}{\beta^2 T_{\text{max}}} \int dx \left( 1 - e^{-2\pi i T_{\text{max}} w(x) \tau} - 2\pi i \tau w(x) T_{\text{max}} (1 + \beta^2) + (\beta^2 + 2i \pi T_{\text{max}} w(x) \tau) (-Ei(-2\pi i T_{\text{max}} w(x) \tau) + \log(2 \pi T_{\text{max}} w(x) \tau) + \gamma_{\text{EM}}) \right) \right].
\]
where $\gamma_{\text{EM}}$ is the Euler constant and $\text{Ei}$ is the exponential integral function, $\text{Ei}(x) = -\int_{-\infty}^{x} dt \, e^{-t}/t$.

Next, we expand the inverse Fourier transform $\mathcal{F}^{-1}$:

$$p_w(E) = \int_{-\infty}^{\infty} d\tau \exp\left[2\pi i E\tau - 2\pi i \bar{E}\tau + \frac{2\pi \nu^2 m_e}{\beta^2 T_{\text{max}}} \int dx \left(1 - e^{-2i\pi T_{\text{max}} w(x)\tau} - 2i\pi T_{\text{max}} w(x)(1 + \beta^2) + \beta^2 + 2i\pi T_{\text{max}} w(x)\tau)(-\text{Ei}[-2i\pi T_{\text{max}} w(x)\tau] + \log[2i\pi T_{\text{max}} w(x)\tau] + \gamma_{\text{EM}})\right]\right].$$

To simplify these integrals, we can leverage the $\zeta$ quantize defined earlier (equation 2.4) and $z \equiv 2\pi \zeta T_{\text{max}}$:

$$p_w(E) = \frac{1}{2\pi i \zeta T_{\text{max}}} \int_{-\infty}^{\infty} dz \exp\left[\frac{iz}{\zeta T_{\text{max}}}(E - \bar{E}) + \int dx \zeta(1 - e^{-iw(x)z/\zeta}) - izw(x)(1 + \beta^2) + (\zeta \beta^2 + iw(x)z)(-\text{Ei}[-iw(x)z/\zeta] + \log[iw(x)z/\zeta] + \gamma_{\text{EM}})\right].$$

This equation can be recognized as the Landau distribution for a parameter $\lambda$. In section A.2 we find that only for specific channel sensitivities is the distribution equivalent to a Landau-Vavilov in the general case.

### A.2 The Landau Limit

To restrict to the Landau case, we take the thin-film approximation. In the usual derivation, one takes the limit that $\zeta \cdot \ell \ll 1$, where $\ell$ is the width of the step function. In our case, since we don’t have a single such width, we have to be more careful about this approximation. In this case we can make a requirement on $w$ – that the range of values where $w$ is not $\ll 1$, $r$, has the property that $\zeta \cdot r \ll 1$. Then, inside the integrand of $\int dx$, we can take the limit that $\zeta$ is small. In this limit, $p_w(E)$ converges to

$$p_w(E) = \frac{1}{2\pi i \zeta T_{\text{max}} \nu} \int_{-\infty}^{\infty} dz' \exp\left[z'(\lambda + \log|z'|)\right],$$

where $z' = iz/\zeta T_{\text{max}}$, $\nu \equiv \int w(x)dx$, and $\lambda = \frac{E - \bar{E}}{\nu T_{\text{max}}} - \log\zeta \nu + \gamma_{\text{EM}} - 1 - \beta^2 + \int dx \frac{w(x)\log[w(x)]}{\nu}$. This equation can be recognized as the Landau distribution for a parameter $\lambda$. 

---

This integral definition gives the general result of the probability distribution of energy loss observed by some channel with a position-dependent weight function $w(x)$. In the nominal case, we would replace $\int dx \rightarrow \nu$ for some channel pitch $\nu$ and would obtain the Landau-Vavilov distribution. From here, we will consider for which channel sensitivities the distribution is equal to the Landau distribution (in the thin film case) or the Landau-Vavilov distribution (in the general case). In section A.2 it is shown that for all channel sensitivities that satisfy the thin film approximation, the distribution is a Landau. Finally, in section A.3 we find that only for specific channel sensitivities is the distribution equivalent to a Landau-Vavilov in the general case.
A.3 General / Landau-Vavilov Case

To understand the general case, we will examine the cumulants of the probability distribution. To do this, it is useful to go back to the definition in equation A.2, modified slightly so that we obtain the cumulant-generating function $K(\tau) = \log E[e^{-i\tau E}]$:

$$K(\tau) = -i\bar{E}\tau + \xi T_{\text{max}} \int dx \int_0^{T_{\text{max}}} dT \frac{1 - \beta^2 T / T_{\text{max}}}{T^2} \left( e^{-i\tau w(x)T} + i\tau w(x)T - 1 \right). \quad (A.5)$$

We expand the term in parentheses in a Taylor series:

$$K(\tau) = -i\bar{E}\tau + \xi T_{\text{max}} \sum_{n=2}^{\infty} \frac{(-i\tau)^n}{n!} T_{\text{max}}^{n-1} \left( \frac{1}{n-1} - \frac{\beta^2}{n} \right) \int dx \ w(x)^n.$$  

Which can be simplified to

$$K(\tau) = -i\bar{E}\tau + \xi T_{\text{max}} \sum_{n=2}^{\infty} \frac{(-i\tau)^n}{n!} T_{\text{max}}^{n-1} \left( \frac{1}{n-1} - \frac{\beta^2}{n} \right) \int dx \ w(x)^n.$$ 

From here, the nth cumulant $\kappa_n = i^n K^{(n)}(0)$ can be directly read off:

$$\kappa_1 = \bar{E},$$

$$\kappa_n = \xi T_{\text{max}} \sum_{n=2}^{\infty} \frac{(-i\tau)^n}{n!} T_{\text{max}}^{n-1} \left( \frac{1}{n-1} - \frac{\beta^2}{n} \right) \int dx \ w(x)^n. \quad (A.6)$$

The cumulants of the Landau-Vavilov distribution are given for $w(x) = \Theta(x) - \Theta(x + \rho)$ for some pitch $\rho$. Thus, the cumulants of the Landau-Vavilov distribution are

$$\kappa_1^{LV} = \bar{E},$$

$$\kappa_n^{LV} = \xi \rho T_{\text{max}} \sum_{n=2}^{\infty} \frac{(-i\tau)^n}{n!} T_{\text{max}}^{n-1} \left( \frac{1}{n-1} - \frac{\beta^2}{n} \right) \int dx \ w(x)^n. \quad (A.7)$$

A necessary but not sufficient condition for two probability distributions to be equivalent is that they have the same cumulants. Allowing for the distributions to be different by location and scale parameters, the n-th cumulant must be equal up to a multiplicative (scale) factor $c^n$. Thus, we need $\kappa_n = \kappa_n^{LV} c^n$ for a distribution (with cumulants $\kappa$) to be equivalent to the Landau-Vavilov distribution up to location and scale parameters. This puts a requirement on $w$ that $\int dx \ w(x)^n = \rho c^{n-1}$ for all integers $n \geq 1$ for some constant $c$ and the pitch $\rho$.

The cumulants being equivalent is not by itself a sufficient condition for the probability distributions to be the same. However, given this property on $w$ we can simplify the distribution further – starting from equation A.2:

$$p_w(E) = \mathcal{F}^{-1} \exp \left[ -2\pi i \bar{E} \tau + \rho \frac{2\pi \nu^2 m e}{\beta^2} \int dx \int_0^{T_{\text{max}}} dT \frac{1 - \beta^2 T / T_{\text{max}}}{T^2} \left( e^{-2\pi i \tau w(x)T} + 2\pi i \tau w(x)T - 1 \right) \right]$$

$$= \mathcal{F}^{-1} \exp \left[ -2\pi i \bar{E} \tau + \rho \frac{2\pi \nu^2 m e}{\beta^2} \frac{\rho}{c} \int_0^{T_{\text{max}}} dT \frac{1 - \beta^2 T / T_{\text{max}}}{T^2} \left( e^{-2\pi c i \tau T} + 2\pi c i \tau T - 1 \right) \right],$$

$\text{– 17 –}$
which integrates to:

\[
p_w(E) = \mathcal{F}^{-1} \left[ \frac{2\pi r_m^2 me}{\beta^2 T_{\text{max}}} \frac{\rho}{c} \left( 1 - e^{-2i\pi T_{\text{max}}c\tau} - 2\pi i c T_{\text{max}}(1 + \beta^2) + (\beta^2 + 2i\pi T_{\text{max}}c\tau)(-\text{Ei}[-2i\pi T_{\text{max}}c\tau] + \log[2\pi T_{\text{max}}c\tau] + \gamma_{\text{EM}}) \right) \right].
\]

Then, defining \( \zeta' = \rho \frac{2\pi r_m^2 me}{\beta^2 T_{\text{max}}} \frac{\rho}{c} \) and \( z' = 2\pi i T_{\text{max}}c\tau \):

\[
p_w(E) = \frac{1}{2\pi i T_{\text{max}}c} \int_{-i\infty}^{i\infty} dz' \exp \left[ \frac{z'}{T_{\text{max}}c} (E - \bar{E}) + \zeta'(1 - e^{-z'}) - z' \zeta'(1 + \beta^2) + \zeta'(\beta^2 + z')(-\text{Ei}[-z'] + \log[z'] + \gamma_{\text{EM}}) \right],
\]

which is the Landau-Vavilov distribution with a scale parameter \( c \) (this can be verified against [10] equation 8, with somewhat different notation). Thus, the probability distribution of energy loss seen by a channel with a weight function \( w(x) \) is equal to the Landau-Vavilov distribution precisely when

\[
\int dx' w(x)^n = \rho c^{-n-1} \quad \text{(A.8)}
\]

for all integers \( n \geq 1 \) for some \( \rho, c \).

We can show further that this requirement means \( w(x) \) is equal to some number of non-overlapping step functions multiplied by a scale factor \( c \). First, define \( \rho' = c \rho \) and \( w'(x) = w(x)/c \). Then A.8 being satisfied means \( \int dx w'(x)^n = \rho' \) for all \( n \geq 1 \). We show this means \( w'(x) \) is equal to 1 or 0 for all \( x \). Take \( n \) large enough that for \( w' < 1 \), \( w'^n \approx 0 \) and for \( w' > 1 \), \( w'^n \approx \infty \). Then \( w'(x) \) can’t have some compact region where \( w' > 1 \), or else the integral would diverge. In this case, the integral breaks down to a sum of the compact regions where \( w' = 1 \):

\[
\lim_{n \to \infty} \int dx w'(x)^n = \sum_{\text{region}} r_i = \rho', \quad \text{where} \quad r_i \text{ is the length of region } i \text{ where } w' = 1.
\]

Then, consider the integral \( \int dx w'(x) \). This is equal to those same regions plus the integral of \( w'(x) \) outside those regions:

\[
\int dx w'(x) = \sum_{\text{region}} r_i + \sum_{w' \neq 1} \int w'(x). \quad \text{Since} \quad \sum_{\text{region}} r_i = \rho', \text{ we need} \quad \int w'(x) = 0.
\]

Since \( w' \) is positive for all \( x \), this means \( w' \) must be equal to 0 wherever \( w' 
eq 1 \). This means that \( w' \) should be given by the sum of some number of non-overlapping step functions at location \( x_i \) of length \( a_i \): \( w'(x) = \sum_i \Theta(x - x_i) - \Theta(x - x_i - a_i) \). Translating this back to \( w \), this means \( w \) must be given by the sum of those step functions multiplied by some constant \( c \) where \( 0 \leq c \leq 1 \).

Thus, when not considering the relativistic limit, the distribution of particle energy loss seen by a channel with a weight function \( w(x) \) is only equal to a Landau-Vavilov distribution (up to location and scale factors) when \( w(x) \) is given by the sum of step functions multiplied by some overall constant. In general, the probability distribution will be different and is given by equation A.3 (equation 2.6 in the main text). The cumulants of this distribution are given by equation A.6.

## B Implementation Details of the Monte Carlo

The theoretical results of this note have been augmented by a simple Monte-Carlo simulation of muon energy loss seen by a LArTPC-like channel weight function. The values of the parameters of
The computation (explained below) are provided in table 2. The Monte Carlo simulation consisted of sampling energy loss for muons in steps of $\Delta x$ over a total length $\ell_{MC}$, then summing up the energy loss from a weight function $w(x)$ computed at each $\Delta x$ step. The step-size $\Delta x$ was chosen so that the weight function did not appreciably vary across it. At each $\Delta x$ step, the distribution of energy loss was modeled by a Landau-Vavilov distribution with the parameters of the muon kinematics and LAr properties as input.

At the $\Delta x$ value required by this simulation, the Vavilov-$\kappa$ parameter of the Landau-Vavilov distribution was too small to be computed by widely available libraries. Thus we implemented our own computation of the energy loss distribution. First, we computed the small-length distribution of energy loss (using equation 2.2) over a distance $\epsilon$. This distribution was computed on a grid of $N_{MC}$ points up to an energy cutoff $E_{\text{max}}$. To incorporate the effect of atomic effects on the cross section, we cut off the bare cross section at the mean ionization energy $I_0$ and added a term of constant energy loss to fix the value of mean energy loss. This is a valid procedure because, as discussed in appendix A, the shape of the Landau-Vavilov distribution is independent of atomic effects as long as the mean is correct. After computing the small-length energy loss distribution, we performed a series of discrete convolutions, doubling the length of the probability distribution at each iteration, until we obtained the desired $\Delta x$.

To validate this procedure, we also compared the energy loss distribution to the ROOT [30] Landau-Vavilov function `ROOT::Math::VavilovAccurate` at a length it was able to compute for the simulated energies (1 cm): see figure 7. Differences between the paper and the ROOT Landau-Vavilov distributions are on the order of a tenth of a percent, and are smaller in the region of the distribution near the peak.

![Figure 7](image)

**Figure 7.** Validation of the Landau-Vavilov distribution implemented for this paper compared to the ROOT VavilovAccurate implementation for a muon with 1 GeV of energy for a thickness of 1 cm.

In each run, the simulation was performed 1 million times to provide a large distribution of muon energy losses. Then, a Landau function was fit to the resulting distribution to obtain an MPV.
This fit was only done on the 20 bins on either side of the peak bin in order to avoid difficulties in the fit trying to directly model the tail. Statistical uncertainties are taken from the fit. Distributions for example model parameters are shown in figure 8. Note that at small energy and large thickness (where the thin-film approximation breaks down), the Landau no longer provides a very good fit to the distribution. This can be seen from the fact that the Landau over-estimates the size of the tail, especially for $E = 0.2 \, \text{GeV}$. The extraction of the MPV from the Landau fit may not be as accurate in this region, which is reflected by the larger uncertainty in the fit.

**Figure 8.** Distribution of energy loss values for various model parameters of the numerical computation. The fitted MPV of each run is shown.
| Quantity                        | Value (Units are MeV, cm, s, g) |
|--------------------------------|----------------------------------|
| **Fundamental Constants**      |                                  |
| Muon Mass ($M$)                | 105.6                            |
| Electron Mass ($m_e$)          | 0.5110                           |
| Classical Electron Radius ($r_e$) | 2.817940 $\cdot 10^{-13}$        |
| Avogadro Number                | 6.0221409 $\cdot 10^{23}$        |
| **Material Properties (LAr)**  |                                  |
| Mean Excitation Energy ($I_0$) | 188.0 $\cdot 10^{-6}$            |
| Mass Density                   | 1.396                            |
| Mass Number                    | 39.9623                          |
| **MC Simulation Properties**   |                                  |
| Small-$\ell$ distribution length ($\epsilon$) | $0.01 \cdot 2^{-20} \approx 9.537 \cdot 10^{-9}$ |
| Distribution step size ($\Delta x$) | 0.01                            |
| Energy cutoff ($E_{\text{max}}$) | 10                              |
| #Points in distribution ($N_{\text{MC}}$) | $50 \cdot 10^6$                |
| Simulated particle length ($\ell_{\text{MC}}$) | 10                             |
| #Simulations per data point    | $10^6$                           |
| Wire separation in weight function ($a$) | 0.3                             |

*Table 2.* Values of parameters in the numerical computation of muon energy loss measured by a LArTPC-like channel. Where applicable, the symbol of the quantity used in the text is also provided.