Towards a String Dual of SYK

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SYK Model

- Model of $N$ interacting Majorana fermions with Gaussian random couplings - [Sachdev, Ye; Kitaev; Maldacena, Stanford; ...]

$$S_{\text{SYK}} = \int_0^{\beta=2\pi} d\tau \left( \sum_i \psi_i \partial_\tau \psi_i - H_{\text{SYK}} \right),$$

$$H_{\text{SYK}} = \frac{i^p}{2} \sum_{i_1...i_p} J_{i_1...i_p} \psi_{i_1} \psi_{i_2} \ldots \psi_{i_p}, \quad \langle J_{i_1...i_p}^2 \rangle = \mathcal{J}^2 \frac{p!}{2p^2 N^{p-1}}.$$

- Admits a bilocal effective action -

$$\frac{S}{N} = \log \text{Pf} \left( \partial_\tau - \Sigma \right) - \int d\tau_1 d\tau_2 \left[ G\Sigma - \frac{\mathcal{J}^2}{2p^2} G^p \right].$$

- Has an effective Lorentzian Liouville description in the double scaling limit - $p \to \infty$, $N \to \infty$ with $\frac{2p^2}{N} = \lambda$ fixed.

- Is a useful toy model for quantum gravity - would like to embed within string theory.
$(p, q)$ Minimal String Theory

- Minimal CFT + Liouville gravity, $c_{MM} + c_L = 26$, with $c_{MM} = 1 - 6 \left( b - b^{-1} \right)^2$, $b^2 = p/q$.

- Operator Spectrum
  - Tachyons, $\hat{T}_{r,s} = c\bar{c}O_{r,s}e^{2\alpha_{r,s}\phi}$.
  - Ground ring - $\hat{O}_{r,s} = \mathcal{L}_{r,s} \cdot O_{r,s}e^{2\beta_{r,s}\phi}$ generated by $\hat{O}_{1,2} = \hat{x}$, $\hat{O}_{2,1} = \hat{y}$.

- Branes [Seiberg, Shih '04]
  - FZZT$_{\sigma}$ branes with boundary cosmological constant, $x = \mu_B = \cosh(\pi b\sigma)$, $y = \partial_{\mu_B} Z$.
  - ZZ$_{r,s}$ branes.
  - Live on semiclassical target space - $T_q(x) = T_p(y)$.

- Two-matrix model dual [Daul, Kazakov, Kostov '93]

$$Z_{p,q} = \int dA dB \ e^{-\text{Tr}(V_p(A) + V_q(B) - AB)}, \quad V(A) = \sum_{n=1}^{p} g_n A^n, \ W(B) = \sum_{k=1}^{q} t_k B^k.$$ 

- $(p, q)$ minimal string - tune to $(p, q)$ multicritical point in double scaling limit.
Setup: Brane Construction

- FZZT brane insertion at location $x$,

$$\psi(x) = \det(x - B) = \exp(\text{Tr} \log(x - B)).$$

- Insertion of $Q$ FZZT branes,

$$\psi(X_Q) = \det(X_Q \otimes 1_N - 1_Q \otimes B) = \int d\psi^\dagger d\psi e^{\psi^\dagger (X_Q \otimes 1_N - 1_Q \otimes B) \psi}.$$ 

- $\psi_{ia}$ ZZ-FZZT open string with $i = 1, \ldots, N$, $a = 1, \ldots, Q$.

- Consider $(p, 1)$ MST with $Q$ FZZT branes

$$\int dA dB e^{-\text{Tr}(V_p(A) - AB)} \int d\psi^\dagger d\psi e^{\psi^\dagger (X_Q \otimes 1_N - 1_Q \otimes B) \psi}$$

- Colour-flavour map: $\text{Tr}_N (\psi \psi^\dagger)^k = \text{tr}_Q (\psi^\dagger \psi)^k$.

- FZZT worldvolume description in double scaling limit equivalent to Kontsevich matrix integral $A_i (X_Q)$.

[Refs: Maldacena, Moore, Seiberg, Shih '04; Hashimoto, Huang, Klemm, Shih '05; ...]
Mapping to Non-Commutative Torus

- Matrix SYK model: \( \int d\psi d\psi^\dagger e^{i \sum \psi^\dagger X_Q \psi - J^2 \text{tr}_Q G^p} \), \( G_{ab} = \frac{1}{N} \sum_{i=1}^{N} \psi_{ia}^\dagger \psi_{ib} \).

- Map \( G_{Q \times Q} \) to bilocal functions,
  - Introduce clock and shift matrices.

\[
UV = VU \xi, \quad U^Q = V^Q = 1, \quad \xi = e^{i \hbar}, \quad \hbar = 2\pi / Q,
\]

\[
U = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & \xi & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \xi^{Q-1}
\end{bmatrix}, \quad V = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

- \( G = \sum_{n,m=1}^{Q} G_{nm} U^n V^m \rightarrow G(u, v) = \sum_{n,m=1}^{Q} G_{nm} e^{inu} e^{imv} \).
- \( G \cdot G \rightarrow G \ast G = e^{i\hbar(\partial_u \partial_v - \partial_v \partial_u)} G(u, v) G(\tilde{u}, \tilde{v}) \big|_{u=\tilde{u}, v=\tilde{v}}; \quad \text{tr}_Q \rightarrow \int \frac{du \, dv}{2\pi \tilde{\hbar}}. \)
- \( \psi_a \rightarrow \sum_u \psi(u) |u\rangle, \quad U|u\rangle = e^{iu} |u\rangle, \quad V|u\rangle = |u - \hbar\rangle. \)

- We choose \( iX_Q = V^{-1/2} - V^{1/2} \) such that,
  \[
iX_Q\psi(u) = \psi\left(u + \frac{\hbar}{2}\right) - \psi\left(u - \frac{\hbar}{2}\right) \equiv \hbar \hat{\partial}_u \psi(u).
\]
Conclusions and Comments

- \[ S = \int \frac{dudv}{2\pi \hbar} \left[ \sum_{i=1}^{N} \psi_i^\dagger \left( \hat{\partial}_u - \Sigma \right) \psi_i + N \left( \Sigma \star G - \frac{J^2}{2p^2} G^\star p \right) \right], \]
  \[ G(u, v) = \frac{1}{N} \sum_i \psi_i^\dagger(u) \psi_i(v) \text{ with } G(u, v)^\dagger = G(v, u). \]

- \( X_Q \) - singularities of spectral curve of \((p, Q)\) minimal string.

- Phase structure -

- Double-scaled SYK limit \( \rightarrow S[g_+, g_-] \), with \( c_\pm = 13 \pm i\gamma \) satisfying \( c_+ + c_- = 26 \).