Dynamic behaviour of Bose–Einstein condensates in optical lattices with two- and three-body interactions

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Abstract

We study the dynamic behaviour of Bose–Einstein condensates with two- and three-atom interactions in optical lattices with analytical and numerical methods. It is found that the steady-state relative population displays tuning-fork bifurcation when the system parameters are changed to certain critical values. In particular, the existence of the three-body interaction not only transforms the bifurcation point of the system but also greatly affects the macroscopic quantum self-trapping behaviours associated with the critically stable steady-state solution. In addition, we investigated the influence of the initial conditions, three-body interaction, and the energy bias on the macroscopic quantum self-trapping. Finally, by applying the periodic modulation on the energy bias, we observed that the relative population oscillation exhibits a process from order to chaos, via a series of period-doubling bifurcations.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In recent years, Bose–Einstein condensates (BECs) in optical lattices have attracted enormous attention both experimentally and theoretically [1, 2]. This is mainly because the lattice parameters and interaction strength can be manipulated using a modern experimental technique. Researchers have discovered many novel phenomena, such as nonlinear Landau–Zener tunneling, energetic and dynamical instability and the strongly inhibited transport of a one-dimensional BEC in optical lattices [3–10]. More attractive phenomena, namely self-trapping, were recently observed experimentally in this system [11]. In such an experiment, a BEC cloud with repulsive interaction initially loaded in optical lattices was self-trapped. Many theoretical analyses have also been presented about self-trapping [12–15]. It is well known that macroscopic quantum self-trapping (MQST) means self-maintained population imbalance with a non-zero average value of the fractional population imbalance, which was discussed in detail in [16, 17]. Marino et al considered that the damping decays all different oscillations to the zero-phase mode [18]. Macroscopic quantum fluctuations have also been discussed taking advantage of second-quantization approaches [19]. However, chaotic behaviour emerges when the trapping potential is time dependent and the damping and finite-temperature effect cannot be neglected. Abdullaev and Kraenkel studied the nonlinear resonances and chaotic oscillation of the fractional imbalance between two coupled BECs in a double-well trap with a time-dependent tunneling amplitude for different damping [20]. When the asymmetry of the trap potential is time dependent and its amplitude is so small that it can be taken as a perturbation, Lee et al studied the chaotic and frequency-locked atomic population oscillation between two coupled BECs with a weak damping, and discovered that the system comes to a stationary frequency-locked atomic population oscillation from transient chaos [21].
It is important to note that theoretical studies of dynamic behaviours are mainly focused on the effect of two-body interactions. It is clear that, at low temperature and density where interatomic distance is much greater than the distance scale of atom—atom interactions, two-body s-wave scattering should be important, and three-body interactions can be neglected. On the other hand, in the past few years a great deal of work has also contributed to understanding the importance of three-body Efimov physics [22–24]. Today, it is widely accepted that, even at a very dilute limit, dominated by two-body interactions, few-body interactions are also important. More recently, ground-breaking studies have demonstrated this [25] and verified it experimentally [26].

Therefore, the purpose of this paper is to investigate the steady-state solution of a BEC in a one-dimensional periodic optical lattice when both two-body and three-body interactions are taken into account. Using the mean-field approximation and linear stability theorem, it is found that the tuning-fork bifurcation of steady-state relative population appears at the critical values. The existence of the three-body interaction not only transforms the bifurcation point of the system but also affects greatly its self-trapping behaviours associated with the critically stable steady-state solutions. We also study the effects of initial conditions, three-body interaction and the energy bias on MQST. We discuss the chaos behaviours of the system by applying the periodic modulation on the energy bias. The result shows that the relative population oscillation can undergo a process from order to chaos, via a series of period-doubling bifurcations.

This paper is organized as follows. In section 2, we introduce the mean-field description of a BEC in optical lattices with two- and three-atom interactions. In section 3, with the linear stability theorem, we analyse the stability of steady-state solutions and plot relevant phase trajectories. Then the influence of three-body interactions on MQST of the system is demonstrated in section 4. In section 5, by applying the periodic modulation in the energy bias, we discuss the chaotic behaviours of the system using the numerical simulation method. In the last section, a summary and conclusion of our work are presented.

2. Mean-field description of a BEC in optical lattices with two- and three-atom interactions

We focus our attention on a BEC with both two- and three-body interactions that is subjected to one-dimensional (1D) optical potentials where the motion in the perpendicular direction is confined. In the mean-field approximation, the dynamics of the BEC can be modelled by the 1D Gross–Pitaevskii (GP) equation in the comoving frame of the lattice [3, 6, 27, 28]:

\[
\begin{align*}
\hbar \frac{\partial \Phi}{\partial t} &= -\frac{1}{2m} \left( \hat{h} \frac{\partial}{\partial t} - i m a t \right)^2 \Phi + \nu_0 \cos(2K_I x) \Phi \\
+ & \frac{2\hbar^2 a_s + \nu_0^2}{\hbar^2 m} |\Phi|^2 \Phi + \frac{g_2}{3\nu_0^2 a_s^3} |\Phi|^4 \Phi, \\
\end{align*}
\]

where \( \Phi \) is the wavefunction of the condensate, \( m \) is the mass of atoms, \( a_s \) is the two-body s-wave scattering length, \( \nu_0 \) is the strength of the periodic potential, and \( K_I \) is the wave number of the laser light which is used to generate an optical lattice. \( ma_t \) stands for either the inertial force in the comoving frame of an accelerating lattice or the gravity force, \( a_t = \sqrt{\hbar/(m \omega t)} \), where \( \omega t \) is the radial frequencies of the anisotropic harmonic trap. \( g_2 \) is an effective three-body contact interaction related to the GP equation. As discussed in [29, 30], \( g_2 \) gives a mean-field contribution \( g_2 |\Phi|^4/3 \) to the energy density in a three-body system. This results in the correction of the GP equation with a term \( g_2 |\Phi|^4/3 \).

So the three-body recombination expressed by the imaginary part of \( g_2 \) is negligible. In equation (1), all the variables can be rescaled to be dimensionless by the following system’s basic parameter \( x \sim 2K_I x, \Phi \sim \frac{\Phi}{\sqrt{\hbar m}}, t \sim \frac{\hbar}{m} K_I t \). Then we obtain the normalized 1D GP equation in optical lattices with cubic and quintic nonlinearities:

\[
\begin{align*}
\hbar \frac{\partial \Phi}{\partial t} &= -\frac{1}{2} \left( \frac{\partial}{\partial t} - i a t \right)^2 \Phi + \nu \cos(x) \Phi \\
+ c|\Phi|^2 \Phi + \lambda |\Phi|^4 \Phi, \\
\end{align*}
\]

where \( \nu = \frac{m a_t}{\hbar}, \lambda = \frac{m a_t}{\hbar K_I} a_t, c = \frac{N a_t}{\hbar K_I} \) is the effective two-body interaction, \( N \) is the total number of atoms. The three-body interactions \( \lambda = \frac{g_2 N^2}{\hbar K_I} \) have a critical dependence on the radial frequencies of the anisotropic harmonic trap and \( g_2 \). Recent theoretical studies [29, 30] estimated that, for \(^{87}\text{Rb}\) atoms, \( g_2 \approx \hbar \times 10^{-26} - \hbar \times 10^{-27} \text{cm}^6 \text{s}^{-1} \), taking \( N \approx 10^4 \)–\( 10^6 \), \( a_t \approx 10^{-6} \text{m} \), \( K_I = \frac{2\pi}{\lambda \text{mm}} \), the three-body interaction is expected to be positive with a value of \( 0 < \lambda < 1 \).

In the neighbourhood of the Brillouin zone edge \( k = 1/2 \), the wavefunction can be approximated by [3]

\[
\Phi(x, t) = a(t) e^{ix} + b(t) e^{-ix},
\]

where \( a(t), b(t) \) are the probability amplitudes of atoms in each of the two wells, respectively, and \(|a|^2 + |b|^2 = 1\). By inserting such wavefunctions into equation (2) and performing some spatial integrals, we obtain the dynamical equations with two- and three-body interactions:

\[
\begin{align*}
\frac{\hbar}{2} \frac{\partial a}{\partial t} &= \lambda (1 + 2 |a|^2 |b|^2) a + \frac{\gamma}{2} a, \\
\frac{\hbar}{2} \frac{\partial b}{\partial t} &= -\lambda (1 + 2 |a|^2 |b|^2) b + \frac{\gamma}{2} b, \\
\end{align*}
\]

Here, the level bias \( \gamma(t) = \alpha t \), \( \alpha \) is the sweeping rate, \( c \) and \( \lambda \) represent the nonlinear parameters, and \( \nu \) is the coupling constant between the two condensates. We introduce the relative population variance

\[
\begin{align*}
s &= |b|^2 - |a|^2, \\
\text{with the parameters} \quad a &= |a| e^{i \theta_a}, \quad b = |b| e^{i \theta_b}, \\
\theta &= \theta_b - \theta_a.
\end{align*}
\]
Combining equations (4)–(7), one yields the equations of the relative population and relative phase:

\[ \dot{s} = -\nu \sqrt{1 - s^2} \sin \theta, \]

\[ \dot{\theta} = \gamma + (c + 2\lambda)s + \frac{\nu s}{\sqrt{1 - s^2}} \cos \theta, \tag{9} \]

where \( \dot{s} \) and \( \dot{\theta} \) denote the time derivative of the relative population and the relative phase. If we regard \( s \) and \( \theta \) as the canonically conjugate variables, equations (8) and (9) become a pair of Hamilton canonical equations with the conserved effective Hamiltonian:

\[ H = \gamma s + \frac{1}{2}(c + 2\lambda)s^2 + \nu \sqrt{1 - s^2} \cos \theta. \tag{10} \]

In the following section, we will discuss the stability of the steady state with the symmetric condition \( \gamma = 0 \) by the linear stability theorem and the Lyapunov direct method. We will investigate the stability of a nonlinear system: the linear stability analysis on the phase space of the classical Hamiltonian.

### 3. Stability analysis of the steady-state solutions

In section 2, we present the dynamical equations with three-body interaction and discuss the stability of the steady state with the symmetric condition. Generally, there are two ways to study the stability of a nonlinear system: the linear stability theorem and the Lyapunov direct method. We will investigate the stability of the system with the first method.

The steady-state solution of this system is obtained by setting equations (8) and (9) to zero. The forms of the steady-state solutions are very complicated when the level bias \( \gamma \neq 0 \). For simplicity, we set \( \gamma = 0 \), leading to

\[ \dot{s} = f_1(s, \theta) = -\nu \sqrt{1 - s^2} \sin \theta, \tag{11} \]

\[ \dot{\theta} = f_2(s, \theta) = (c + 2\lambda)s + \frac{\nu s}{\sqrt{1 - s^2}} \cos \theta, \tag{12} \]

and the conserved energy

\[ H = \frac{1}{2}(c + 2\lambda)s^2 + \nu \sqrt{1 - s^2} \cos \theta. \tag{13} \]

Taking \( \dot{s} = 0, \dot{\theta} = 0 \), we get the steady-state solutions:

\[ \theta_1 = 0, \quad s_1 = 0, \tag{14} \]

\[ \theta_2 = \pi, \quad s_2 = 0, \tag{15} \]

\[ \theta_{3,4} = \pi, \quad s_{3,4} = \pm \sqrt{1 - \left(\frac{\nu}{c + 2\lambda}\right)^2}. \tag{16} \]

According to the linear stability theorem, we look for the perturbed solutions which are near the steady-state solutions, \( s(t) = s_i(t) + \varepsilon_1(t), \theta(t) = \theta_i(t) + \varepsilon_2(t) \), where \( s_i(t), \theta_i(t) \) for \( i = 1, 2, 3, 4 \) signify the steady-state solutions, \( |\varepsilon_1(t)| \ll |s_i(t)| \) and \( |\varepsilon_2(t)| \ll |\theta_i(t)| \) which are related to the first-order perturbed solutions. Inserting the above expression into equations (11) and (12), we obtain the linear equations near the steady states of the nonlinear equations:

\[ \dot{\varepsilon}_1 = \left( \frac{\partial f_1}{\partial s} \right)_1 \varepsilon_1 + \left( \frac{\partial f_1}{\partial \theta} \right)_1 \varepsilon_2, \]

namely

\[ \dot{\varepsilon}_1 = a_{11}\varepsilon_1 + a_{12}\varepsilon_2, \tag{17} \]

\[ a_{11} = \left( \frac{\partial f_1}{\partial s} \right)_1, \quad a_{12} = \left( \frac{\partial f_1}{\partial \theta} \right)_1. \]

Now, we make use of the above expression to investigate the stability of the steady states of equations (14)–(16). In the case of \( \theta_0 = 0 \) and \( s_0 = 0 \), corresponding to the characteristic equations, we calculate the matrix elements \( \{a_1, a_2, a_3, a_4\} \), then get the two eigenvalues \( \lambda_1 = -\nu(c + 2\lambda + \nu), \lambda_2 = -\nu(c + 2\lambda + \nu) \). When the eigenvalues are pure imaginary numbers, the stability of the steady-state solutions \( \theta_1, s_1 \) corresponds to a critical case \( \|33\| \), and dynamical bifurcations between the unstable and stable steady states appear with the changing three-body interaction parameters.

When the two eigenvalues are real numbers, \( \varepsilon_1 \) and \( \varepsilon_2 \) tend to infinity with the increase of time, and the steady-state solutions \( (\theta_2, s_2) \) are unstable. For \( \theta_2 = \pi \) and \( s_2 = 0 \), the corresponding matrix eigenvalues become

\[ \lambda_1 = \sqrt{-\nu(c + 2\lambda - \nu)}, \lambda_2 = -\sqrt{-\nu(c + 2\lambda + \nu)}. \]

It is similar when the two eigenvalues are both pure imaginary numbers.

The stability of the steady-state solutions \( (\theta_3, s_3) \) of the nonlinear equations is presented. When the two eigenvalues are positive or negative real numbers, respectively, \( \varepsilon_1, \varepsilon_2 \) tend to infinity with increasing time to infinity, and the steady-state solutions \( (\theta_2, s_2) \) lose their stability. For \( \theta_{3,4} = \pi, s_{3,4} = \pm \sqrt{1 - \left(\frac{\nu}{c + 2\lambda}\right)^2} \), the eigenvalues are \( \lambda_1 = \sqrt{-\nu^2 - (c + 2\lambda)^2}, \lambda_2 = -\sqrt{-\nu^2 - (c + 2\lambda)^2} \).

In equation (16), the population \( s_{3,4} \) are both real quantities which implies \( c + 2\lambda > \nu^2 \). Therefore, the two eigenvalues are pure imaginary numbers. The stability of the steady-state solutions \( (\theta_{3,4}, s_{3,4}) \) of the nonlinear equations is regarded as critical, and the dynamical bifurcations emerge at the bifurcation point \( (c + 2\lambda) = \nu, s = 0 \). Obviously, the existence of three-body interaction can change the bifurcation point of the system. It plays an important role for the stability analysis of the system, as shown in figure 1. For \( \frac{\nu}{c + 2\lambda} > 1 \), the system is in the critical stable steady state \( (\theta_3, s_3) \), and for \( \frac{\nu}{c + 2\lambda} < 1 \), \( (\theta_2, s_2) \) is unstable and the two steady-state solutions \( (\theta_{3,4}, s_{3,4}) \) are critically stable. This is a typical tuning-fork bifurcation, and the bifurcation point is \( \frac{\nu}{c + 2\lambda} = 1 \). However, the dynamical transition happens at the moment \( (c/\nu = 2) \) in [14]. The above process can be well understood from the analysis on the phase space of the classical Hamiltonian system. In figure 2, we plot the trajectories in phase space and classical energy profiles for the different parameters. In the case of \( \nu = 0.2, c = 0.1 \), and \( 0 \leq \lambda < 0.05 \) in the phase space, there are two stable points \( P_1, P_2 \) at \( s = 0, \theta = \pi \) and \( s = 0, \theta = 0 \), respectively (see figure 2(a)).

Obviously, for the stable points, \( P_1 \) and \( P_2 \), the atom distributions are equal in the two adjacent wells. It means that atoms oscillate between two adjacent wells. In the case of \( \nu = 0.2, c = 0.1 \) and \( 0.05 \leq \lambda < 0.15 \), two more fixed points emerge in the line \( \theta = \pi \) marked by \( P_3 \) and \( P_4 \). They are located in \( s = \pm \sqrt{1 - \left(\frac{\nu}{c + 2\lambda}\right)^2} \). Moreover, \( P_4 \) is an unstable state.
point, which lies in \( s = 0 \). As plotted in figure 2(b), for the stable points \( P_1 \) and \( P_3 \), the atom distributions are not in equilibrium between the two adjacent wells. This indicates that atoms are self-trapped in one well. We take it to be an oscillating-phase type because the relative population \( s \) and the relative phase \( \theta \) oscillate around the fixed points [14]. For \( \nu = 0.2, c = 0.1 \) and \( \lambda \geq 0.15 \), there emerge some new trajectories, i.e., the trajectories cross \( P_c \), figure 2(c). Only the fixed point \( P_2 \) is stable. So for these trajectories, \( s \) varies with time from a region of \([-1, 0]\) to \([0, 1]\). It appears that for \( s \neq 0 \), the atoms are self-trapped in one well. We regard it as running-phase-type MQST, as described in [34, 35] and observed in the experiment [36].

We conclude that the dynamic behaviour in the system changes constantly with the increase of \( \lambda \). We obtain a general criterion from figure 2 (second row) for the MQST trajectories, namely, \( H(s, \theta) < -\nu \) with an initial condition \( H(s = \pm 1) = (c + 2\lambda)/2 \). It is helpful to find the transition parameters of macroscopic quantum self-tapping. In the following section, we will illustrate MQST of the non-stationary states in detail by numerical simulations.

4. Macroscopic quantum self-trapping of a BEC with two- and three-atom interactions

In this section, we investigate MQST by the time evolution of the relative population of the system. MQST refers to the phase space trajectories whose relative population is not equal to zero. This can be well understood from the analysis of equations \((8)–(10)\), corresponding to the critically stable steady-state solutions discussed in section 2. Now, we focus on the dynamic behaviour dominated by equations \((8) \) and \((9)\) without the time-dependent system parameters. We study the effect of parameters of the system on MQST with the numerical method starting from equations \((8) \) and \((9)\).

For the initial conditions \( s(0) = 0 \) and \( \theta(0) = \pi/2 \), the time evolutions of the relative population, figures 3(a)–(d) show some very absorbing features. In figure 3(a), the oscillations are regular, and the average relative population \( \bar{s} \) is zero in the case of a symmetric well \( (\gamma = 0) \) with a special parameter, but the corresponding MQST does not appear.

Figure 1. The tuning-fork bifurcation from equations \((17) \) and \((18)\), where \( s_2, s_3, s_4 \) are the steady-state solutions, and the bifurcation point is \( s/c\lambda = 1 \).

Figure 2. Trajectories on the phase space of the system with the three-body interaction varying from \( \lambda = 0 \) to 0.25 (the first row). In the second row, we plot the energy profiles for the relative phase \( \theta = 0 \) (red dashed) and \( \theta = \pi \) (blue solid).
If we increase $\lambda$ from 0.45 to 0.95 in figure 3(b), MQST still does not appear, but the oscillating period reduces. Similarly, increasing $\nu$, we obtain the same result as shown in figure 3(c).

Here, we study the effect of an asymmetric well ($\gamma \neq 0$) on MQST. When we enhance the level bias to $\gamma = 0.5$, the average of the relative population tends to $-0.41$ in figure 3(d). Correspondingly, the oscillating period of $s$ is longer and MQST appears. Note that parameters $c$, $\lambda$, and $\nu$ impact greatly on MQST (see figures 3(e)–(f)). In figure 3(e), when $\lambda$ is from 0.45 to 0.95, MQST is suppressed with a shorter oscillating period. Similarly, with increasing $\nu$, the average relative population changes to $-0.21$ and the oscillating period becomes shorter again, as shown in figure 3(f). Thus, the influence of parameters $c$, $\mu$, $\nu$ and $\gamma$ on the MQST of the system is very dramatic. In the case of $\gamma = 0$, fixing the other parameters and changing the initial conditions from $s(0) = 0$, $\theta(0) = \pi / 2$ of figure 3 to $s(0) = 0.8$, $\theta(0) = \pi / 2$ and $s(0) = 0.8$, $\theta(0) = \pi$, we observe that MQST always emerges with varying $s(0)$ and $\theta(0)$. The oscillating period is reduced compare to figures 3(a) and (d), but $\bar{s}$ increases to $-0.86$ or $-0.72$ as shown in figure 4.

According to the above analysis, we conjecture that, for the initial conditions $s(0) = 0$ and $\theta(0) = \pi / 2$, the parameters $c$, $\lambda$, and $\nu$ can impact on MQST for the asymmetric well ($\gamma \neq 0$). In addition, in the symmetric case, MQST does not appear and those parameters only affect the oscillating period of the system. The initial conditions can impact on MQST for any parameter set.

5. Numerical simulation of chaos by applying periodic modulation on the lever bias

As a whole, the elementary features of chaos are that the dynamic behaviour is unpredictable for a deterministic system.
Figure 5. Dynamical phase orbits of the dimensionless variables \( s, \frac{ds}{dt} \) from equations (19) and (20) with parameters \( \nu = 0.001, c = 0.1, \lambda = 0.45, \omega = 0.1, s(0) = 0, \theta(0) = \pi \), and (a) \( A_1 = 0.002 \), (b) \( A_1 = 0.009 \), (c) \( A_1 = 0.04 \), (d) \( A_1 = 0.12 \), (e) \( A_1 = 0.3 \), (f) \( A_1 = 1 \). Here, \( A_1 \) denotes the amplitude of the time-dependent relative energy.

It is very sensitive to the initial conditions and system parameters. So, according to these characteristics, we adjust the parameters to make the system go into or come out of chaos. In other words, we can control the regime where chaos appears. In this section, we discuss the chaotic behaviours of the system by the numerical method.

If we apply periodic modulation on the lever bias \( \gamma = A_0 + A_1 \sin(\omega t) \), the chaos appears in a special region, where \( A_0 \) and \( A_1 \) stand for initial phase and amplitude, respectively. Inserting this into equations (8) and (9), one derives the following dynamic equation:

\[
\dot{s} = -\nu \sqrt{1 - s^2} \sin \theta, \quad (19)
\]

\[
\dot{\theta} = A_0 + A_1 \sin(\omega t) + (c + 2\lambda)s + \frac{\nu s}{\sqrt{1 - s^2}} \cos \theta. \quad (20)
\]

Starting from equation (20), it is found that the dynamic behaviour of the system is periodic in some special parameters region, and it changes from order to chaos with the increase of \( A_1 \), as shown in figure 5. With initial conditions \( s(0) = 0 \) and \( \theta(0) = \pi \), the phase orbit is a period-1 cycle, and the corresponding oscillation is a Rabi oscillation for the set of parameters with an amplitude \( A_1 = 0.002 \). Figure 5(a). In this case, we set the oscillating period of the relative population \( T \). For \( A_1 = 0.009 \), the phase orbit becomes period-2 in figure 5(b). This means that the oscillating period of \( s \) reaches \( 2T \). Then the phase orbit increases from that of period-4 to period-8 with increasing \( A_1 \) as shown in figures 5(c) and (d). Figures 5(e) and (f) plotted for \( A_1 = 0.3 \) and \( A_1 = 1 \), where the phase orbit does not show a clear periodicity, which signifies the emergence of chaos.

From the above analysis, we find that the oscillating period of the relative population varies from a period-1 limit cycle to period-2 to period-4 and then to period-8, and finally all limit cycles tend to infinity with increasing \( A \). It exhibits a process from order to chaos, through the period-doubling bifurcations [33]. That is to say, for a set of fixed parameters \( \nu, c, \lambda, A_0, A_1, s(0), \theta(0) \) and \( \omega \), the first-order derivative of relative population transforms from single period to multiple period and goes into chaos at last with the increase of vibration amplitude \( A_1 \). However in [15], chaos appears in a strong resonating region \( \omega \approx \nu \). In the chaotic sea, there are several stable islands and they appear alternately with the increase of the interaction parameter.

In order to show chaotic MQST, we present the plots of the time evolution of the relative population and corresponding plots of the power spectrum from equations (19) and (20) in figure 6. The size of the parameter in figure 5(a) is in accord with figures 6(a) and (c) where the system oscillates periodically with MSQT. The power spectrum inclines to a constant with the increase of frequency. In the course of frequency changing, there is only one maximal value of power. It means that the system oscillates periodically in accord with the result of figure 6(a). Making use of those parameters of figure 5(e), we plot figures 6(b) and (d). It shows that the power spectrum exhibits confusion, and the weight of each spectrum is comparable, and there does not exist a spectrum of
Figure 6. The time evolution of the relative population from equations (19) and (20) with the parameters $\nu = 0.001$, $A_0 = 0.4$, $c = 0.1$, $\lambda = 0.45$, $\omega = 0.1$, $s(0) = 0$, $\theta(0) = \pi$, and (a) $A_1 = 0.002$ and (b) $A_1 = 0.3$. Panels (c) and (d) are the corresponding power spectrum. The parameters in panel (c) are the same as in (a) and those in (d) are the same as in (b).

6. Summary and discussion

In this paper, we study the dynamics of a BEC with repulsive two- and three-body interactions immersed in a one-dimensional optical lattice. The stability of the steady-state solution is analysed with the linear stability theorem. The analytical results show that, satisfying certain relationships, the corresponding steady-state solution is stable. When these relationships are not satisfied, the corresponding steady-state solution is unstable. A typical tuning-fork bifurcation of steady-state relative population appears in a special parameter region. The existence of three-body interaction can change the bifurcation point of the system, which is shown in figure 1. It plays an important role for stability analysis of the system.

The critically stable steady-state solution indicates the existence of stationary MSQT. The stable behaviour of the system changes constantly with the increase of $\lambda$ and yields a general criterion for the self-trapping trajectories, $H < -\nu$. In addition, we also investigate the effects of the initial conditions, a set of parameters $c$, $\nu$, $\lambda$, $\gamma$, on MQST. It shows that $c$, $\nu$ and $\lambda$ could affect MQST when $s(0) = 0$ and $\theta_0 = \pi$ for $\gamma \neq 0$. Particularly, the initial values $s(0) = 0$, $\theta_0 = \pi$ or $s(0) = 0$, $\theta_0 = \pi/2$ can directly act on MQST. Finally, we discuss the chaotic behaviour by applying the modulation on the energy bias ($\gamma = A_0 + A_1 \sin \omega t$). In this case, the system comes into chaos through the period-doubling bifurcations with the increase of $\lambda$, and the time evolution of the relative population and power spectra indicate the existence of chaotic MQST. It suggests that one can adjust the lasing detuning and intensity to change the values of the parameters in experiments. These adjustable parameters supply the possibility for controlling the system instabilities, the MQST state and the chaotic behaviours [36].

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