Coulomb Blockade-Effects on Quantization of Charge and Persistent Current in a Luttinger-Liquid Ring

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We show that in a ring-shaped Luttinger Liquid (LL) in contact with an electron reservoir — the chemical potential in the ring being controlled by a gate voltage \( V_g \) — both the average ring charge and the persistent current in the ring are step-like functions of \( V_g \) at low temperatures. The step positions are determined by the LL parameter \( \alpha = \frac{v_F}{s} \), which therefore can be directly measured. We study electrons both with and without spin, taking into account long-range interactions in the ring.

I. INTRODUCTION

Quantum transport of charge in mesoscopic systems has been intensively studied — both theoretically and experimentally — during the last decade. Two kinds of oscillatory phenomena have frequently been in the focus of interest:

i. Coulomb blockade of tunneling through small capacitance metallic grains and the oscillatory behavior of the conductance upon change of gate voltage (for a review see Ref. 1)

ii. Aharonov-Bohm oscillations of thermodynamic (persistent current) and transport (conductance) properties of mesoscopic rings in weak magnetic fields (see, e.g., the review articles Refs. 2,3).

Recently, both these effects were observed in semiconductor nanostructures. Unlike in the case of metals, semiconductor heterostructures enable one to study the properties of electron systems of reduced (2D and 1D) dimensionality. In nanostructures created by laterally constraining the motion of the 2D electron gas (2DEG) formed in an AlGaAs/GaAs heterostructure, both the characteristics of the system under measurement (electron density, geometric form and size) and the nature of the leads connected to it can be controlled. This is achieved by varying (i) the voltage, \( V_g \), on a gate that couples capacitively to the net charge in the device, and (ii) by adjusting voltages on additional (split gate) electrodes that are needed for creating the lateral confinement. Hence, one is able to vary the parameters of the electronic device at hand and its interaction with electron reservoirs over a wide range.

In systems of reduced dimensionality electron-electron correlations play a significant role and the naive picture of noninteracting (or weakly interacting) electrons moving in an external electrostatic potential does not apply.

In particular, under conditions of strong transverse confinement, when the charge motion in a nominally 2D quantum channel becomes essentially one-dimensional, one can expect to be able to realize the Luttinger liquid (LL) regime. Experimental studies of transport properties of laterally confined 2DEGs do not provide unambiguous proof of LL-like behavior of charges in quantum wires. Therefore it seems interesting and important to examine thermodynamic properties of quantum dots and wires and to suggest experiments which could reveal a non-Fermi-liquid character of the electron dynamics in systems of reduced dimensionality.

FIG. 1. Sketch of the ring-shaped conductor discussed in the text. It encloses a magnetic flux \( \Phi \) and is created in the 2D electron gas of a gated semiconductor heterostructure. The ring is in weak contact with an electron reservoir, which implies that the number of electrons in the ring is not fixed at finite temperatures. The difference in electrostatic potential, \( V_g \), between the conducting ring and the rest of the 2D electron gas (the reservoir) is produced by a charged gate electrode. Because of the circular symmetry no electric field appears along the circumference of the ring.

As a possible test of non-Fermi-liquid behavior of 1D electrons we propose to study persistent current oscillations and peculiarities of charge quantization in a quantum ring controlled by a gate voltage. The ring is assumed to be in contact with a reservoir of 2D electrons as illustrated in Fig. 1. Therefore the number of ring particles is not fixed at finite temperatures and interactions do affect transport properties even in an impurity-free ring. The voltage shift, \( V_g \), between the conducting ring and
the rest of the 2D electron gas is produced by a charged gate electrode. In contrast to the situation discussed in Ref. [10], where the connection between ring and electron reservoir is asymmetric, the exchange of electrons between ring and reservoir in our case takes place along the entire ring circumference. Hence, it is symmetric and does not lead to any electric field along the circumference of the ring.

To calculate persistent currents and Coulomb blockade effects in a ring of strongly correlated 1D electrons we use below the Luttinger liquid model[12] both for spinless and spin-1/2 fermions.

We first show that the persistent current for electrons without spin oscillates as a function of gate voltage, \( V_g \), the ring periodically exhibiting a dia- or paramagnetic response to the magnetic flux. The properties of these oscillations are sensitive to the value of the correlation parameter \( \alpha = v_F/s \) of the LL (\( v_F \) is the Fermi velocity, \( s \) is the plasmon velocity). At low temperatures, when the amplitude of the persistent current attains its maximum value \( I_0 \) (\( I_0 = cv_F/L \) for an impurity-free ring, \( L \) is the ring circumference) the oscillations for \( \alpha < 1 \) (repulsive interaction) have the form of periodically repeated current bumps and dips. The width, i.e. the range in gate voltage of the bumps depends on the parameter \( \alpha \) and the dimensionless flux \( \Phi/\Phi_0 \) (\( \Phi_0 = \hbar c/e \) is the flux quantum). In particular, at small flux values, (\( \Phi \rightarrow 0 \)), the plateau width for paramagnetic response is maximal for strong repulsion (\( \alpha \ll 1 \)) and reduces to narrow spikes in the limit of noninteracting particles (\( \alpha \rightarrow 1 \)). For an attractive interaction (\( \alpha > 1 \)) the low-\( T \) oscillations of persistent current with gate voltage disappear.

The above picture of current oscillations is correlated with the behavior of the average ring charge, \( Q \), when the gate voltage is varied. For a repulsive interaction (\( \alpha < 1 \)) and low temperatures, the total charge in the ring is quantized in units of the elementary charge \( e \). The function \( Q = Q(V_g) \) exhibits a typical behavior (Coulomb "staircase"), which is a distinct signature of the Coulomb blockade. The step positions and their widths (for even or odd number of particles in the ring) depend on the correlation parameter \( \alpha \). For a stiff system (\( \alpha \ll 1 \)) all steps are of equal width. The width of the steps corresponding to an even number of spinless electrons is decreased when \( \alpha \) is increased and collapses to a point when \( \alpha \rightarrow 1 \). For an attractive interaction between spinless particles the mean ring charge is quantized in units of \( 2e \) and the oscillations as a function of gate voltage vanish.

At high temperatures, i.e. when \( T > T_{c}^{*}(\alpha) \rightarrow T_{c}^{*} \), being the crossover temperature for current \( (j) \) or charge \( (c) \) oscillations — the oscillations of both persistent current and average charge in the ring become sinusoidal with an exponentially small amplitude, \( \sim \exp(-T/T_0^*) \). For current oscillations the crossover temperature is increased if the strength of the (repulsive) interaction is increased. In the limiting case of a stiff Wigner crystal (\( \alpha \ll 1 \)) it has a maximum value \( T^* = \Delta/\pi^2 \) (for spinless electrons; \( \Delta = 2\pi\hbar v_F/L \) is the energy level spacing at the Fermi level of non-interacting electrons at the relevant density) which coincides with the crossover temperature for an isolated LL-ring[3]. Note that now and below \( v_F \) always refer to the Fermi velocity of spinless electrons. The actual Fermi velocity for electrons with spin and the same density is therefore \( v_F/2 \). For the charge oscillations (as function of \( V_g \)) in the limit considered (\( \alpha \rightarrow 0 \)) the crossover temperature goes to infinity as \( T_0^* \sim 1/\alpha^2 \). This means that at any finite temperature a ring of strongly repulsive electrons exhibits sharp charge quantization. In the high-\( T \) region with strong attraction, \( \alpha^2 \gg 3 \), we predict also that the period of the dominant contribution to the current oscillations is reduced by a factor of two.

The above results correspond to the case of an interaction which is short-ranged. We show further that long-range Coulomb forces can be incorporated into our model if we replace the correlation parameter \( \alpha \) of the LL by the effective "coupling" \( \alpha_{eff}(L) = \alpha/[(1 + (2\alpha^2/\pi)(e^2/\hbar v_F)\ln(L/\lambda)]^{1/2} \), where \( \lambda \) is the width of 1D channel and \( L \) is the ring circumference (\( \lambda \ll L \)).

The magnetic field normally used in Aharonov-Bohm experiments is weak and the ring electrons are therefore unpolarized even at low temperatures. To describe a real situation involving interacting fermions with spin one needs to understand how the electron spin affects the oscillation phenomena of interest here. We take the spin degrees of freedom into consideration by using the two-component LL model (spin-“up” and spin-“down” electrons of equal density). Our calculation scheme can be easily adopted to this model. For the purpose of comparing with the case of spinless electrons it is useful to distinguish between the cases with different parity of the number of spin-“up” \( (N_{up}^0) \) and spin-“down” \( (N_{down}^0) \) electrons (referring to the situation at zero temperature when the particle number is fixed). For a repulsive interaction there are three different cases: (i) (odd,even) \( N_{up}^0 = N_{down}^0 = 2n + 1 \); (ii) (even,even) \( N_{up}^0 = N_{down}^0 = 2n \); (iii) (even,odd) or (odd,even) for \( N_{up}^0 = 2n, N_{down}^0 = 2n + 1 \) or vice versa.

We recall once more that we are dealing with a ring, which is weakly connected to an electron reservoir of fixed chemical potential. Therefore, for non-interacting electrons or for an attractive electron-electron interaction between electrons carrying spin, the ring at arbitrary flux will exchange pairs of particles with the reservoir as the gate voltage is varied. For special values of the flux, when the energy levels are fourfold degenerate \( (\Phi/\Phi_0 = \text{integer or half-integer}) \), this exchange will involve four electrons. Since for noninteracting electrons with spin an energy level is at least doubly occupied, the ground state \( (T = 0, V_g = 0, \Phi = 0) \) in the ring must have an odd number of both spin-”up” and spin-”down” electrons. We refer to this as the \( \text{^*} \) (odd,odd)-case, where obviously \( N_{tot} = N_{up}^0 + N_{down}^0 = 4n + 2 \). The formulas for the (even,even)- case \( (N_{tot} = 4n) \) can be derived from...
those valid for the (odd,odd)-case by a simple shift of the flux by half a flux quantum (it is the parity rule for electrons with spin). The (odd,even)-case describes an odd number of spin-1/2 fermions in a ring ($N_{\text{tot}} = 4n + 1$ or $4n + 3$). Such a ground state can be justified only for a repulsive interaction. Moreover, for weakly interacting particles ($1 - \alpha \ll 1$) it is destroyed due to the particle exchange with electron reservoir already at very low temperatures $T_d \sim (1 - \alpha) \Delta$.

For the (odd,odd)-case the persistent current as a function of flux behaves very much like the current of spinless electrons. It oscillates with the fundamental period $\Phi_0$ and amplitude $I_0$. At low temperatures the only difference is the change in oscillation period when the gate voltage is varied. In the limit of strong repulsion the period of oscillations is twice of that for spinless electrons. The inclusion of spin changes also the high-$T$ behavior of persistent current. The most striking effect is the decrease by a factor of three (compared to the spinless case) of the crossover temperature in the limit of strong repulsion.

A ring with an odd number of spin-1/2 fermions ($N_{\text{tot}} = 4n + 1$ or $4n + 3$) at fixed chemical potential can be conceived only at low temperatures. In such a situation we find that the persistent current oscillates with half the period ($\Phi_0/2$) and half the amplitude ($I_0/2$) compared with the case of spinless electrons. Notice that for non-interacting electrons the effect of period- and amplitude “halving” was predicted in Ref. 27. We generalize this prediction to the more realistic case of correlated electrons at finite (though small) temperatures.

The Coulomb staircase for strongly interacting spin-1/2 fermions ($\alpha \to 0$) at low temperatures looks like the one for spinless particles. The charge in a ring is quantized in units of the elementary charge, $e$, and all steps are of equal width. The spin degrees of freedom influence the Coulomb blockade effects at moderate and weak interaction strengths. The width of steps corresponding to the number $N_{\text{tot}} = 4n, 4n + 1, 4n + 3$ of electrons in a ring is decreased when the interaction strength is decreased and is contracted to a point in the limit $\alpha \to 1$. For non-interacting spin-1/2 electrons the total charge in a ring at small flux values is quantized in units of $4e$ (as it should be from the qualitative considerations).

II. THE MODEL

In order to study Coulomb blockade effects in a 1D ring of strongly correlated spinless electrons, we will use the Luttinger liquid (LL) model. For our purposes it is convenient to represent the Lagrangian of the model in the form:

$$\mathcal{L} = \frac{m_0}{8\pi^2\overline{\rho}} \left\{ \dot{\varphi}^2 - s^2(\varphi')^2 \right\} + \frac{\hbar}{L} \varphi \left( \frac{\Phi}{\Phi_0} + \frac{N - 1}{2} \right) + \frac{\mu_0}{2\pi} \varphi'. \tag{1}$$

Here $m_0$ is the electron (band) mass, $\overline{\rho} = N_0/L$ is the mean density of “vacuum” electrons in the ring. We assume that the LL-ring is in weak contact with a reservoir of 2D electrons. Then for repulsive interaction, $N_0$ is fixed by the chemical potential $\mu$ of the reservoir which is implicit in the formulation Eq. (1); $s$ is the plasmon velocity, $\Phi$ is the flux of magnetic field ($\Phi_0 = \hbar c/e$ is the flux quantum), $N$ is the total number of particles in the ring, $\mu_0 = eV_0$ ($V_0$ being the gate voltage). The Lagrangian (1) describes the long-wavelength dynamics of the electron displacement field $u(x,t)$ ($\varphi = 2\pi\overline{\rho}u(x,t)$). The total number, $N$, of electrons in the ring can be represented as a sum of “vacuum electron” number, $N_0$, and the number, $m$, of extra electrons which appear due to the particle exchange between the reservoir and the ring, $N = N_0 + m$. The integers $m = 0, \pm 1, \pm 2, \ldots$ can be connected with a winding number of topological excitations of the $\varphi$-field in a ring geometry. It enters into the boundary conditions imposed on the dynamical field $\varphi(x, \tau)$ in $(x, \tau)$ space ($\tau$ is an imaginary time)

$$\varphi(\tau + \beta, x) - \varphi(\tau, x) = 2m \pi \varphi(\tau, x + L) - \varphi(\tau, x) = 2m \pi \tag{2}.$$

Here $\beta = 1/T$ is the inverse temperature, $n = 0, \pm 1, \pm 2, \ldots$ is the winding number for the imaginary time “evolution”. The twisted boundary conditions (2) are analogous to the ones derived in Ref. 12. Notice that in Haldane’s description (1) of the LL, the homotopy indices $(n,m)$ represent topological current and charge excitations. At nonzero flux $\Phi$, the parity rule (see Ref. 1) that connects the two sectors of the model (charge and current) can be explicitly incorporated into the Lagrangian (1) by adding to the external flux, $\Phi$, a fictitious (statistical) flux $\Phi_{\text{st}} = 0$ ($\Phi_0/2$) for an odd (even) total number of electrons in the ring. This statistical flux is represented by the second term in the parenthesis in Eq. (1).

For the following analysis, it is convenient to represent the thermodynamic potential, $\Omega$, (grand canonical ensemble) as a path integral over the fluctuations of the dynamical field $\varphi(x, \tau)$

$$\Omega = -T \ln \left\{ \sum_{m,n = -\infty}^{\infty} \int D\varphi \exp\{-S_{\text{E}}(\varphi,m,n) / \hbar\} \right\}, \tag{3}$$

where $S_{\text{E}}$ is the Euclidean (imaginary time) action for the Lagrangian (1) and we explicitly include in the definition, a sum over the homotopy indices $(m,n)$, which represent the contributions of topological excitations. Plasmon modes.

Topological trajectories (zero modes) satisfying the twisted boundary conditions (2) take the form

$$\varphi_{m,n}(x, \tau) = 2\pi \frac{\tau}{\beta} n + 2\pi \frac{x}{L} m. \tag{4}$$

A more general solution can be obtained by adding to a zero mode of fixed $m$ and $n$ an additional part which then
must have periodic boundary conditions. This physically corresponds to having an intermediate state that contains plasmons. Because of the periodic boundary conditions these extra parts only depend on temperature. Since we are interested in derivatives of the thermodynamic potential with respect to flux (to get the persistent current) and gate voltage (to get the average charge) we will consider only the zero mode solutions in what follows.

So, since only zero modes are important, our problem reduces to the evaluation of a double sum in Eq. (1). Since the current and charge sectors of the model are connected only by a single (parity) term in Eq. (1), this summation can be done exactly. The result is

$$\Omega - \Omega_{\mu} = -T \ln \left\{ \left( \frac{4\pi}{\Delta} \right)^{1/2} \exp(\alpha^2 \frac{\mu^2 \beta}{\Delta}) \times \left[ \theta_3(\Theta, q) \theta_3(\mu, q^\alpha) + \theta_4(\Theta, q) \theta_4(\mu, q^\alpha) \right] \right\}. \quad (5)$$

Here \( \theta_3, \theta_4(n, q) \) are the Jacobi theta-functions (see e.g. Ref. [5] and

\[ \Theta \equiv \frac{\Phi}{\Phi_0} + \frac{N_0 - 1}{2}, \quad \mu \equiv \alpha^2 \frac{\mu_g}{\Delta}, \quad q \equiv \exp(-\pi^2 T / \Delta). \quad (6) \]

The dimensionless parameter \( \alpha = v_F / s \), where \( v_F = \pi \hbar N_0 / n_0 L \), characterizes the correlation properties of the LL.

### III. Persistent Current

The persistent current in a 1D LL ring and its dependence on enclosed magnetic flux and gate voltage can be obtained from the thermodynamic potential (5). One finds that

$$I(\Phi, V_g, T) = \frac{eT}{\hbar} \frac{1}{2\pi} \times \frac{d}{d\Theta} \ln \left\{ \theta_3(\Theta, q) \theta_3(\mu, q^\alpha) + \theta_4(\Theta, q) \theta_4(\mu, q^\alpha) \right\}. \quad (7)$$

At first we consider the case of free spinless electrons \( (\alpha = 1) \). In this case Eq. (7) can be shown to coincide with a well-known expression for the persistent current of non-interacting electrons at fixed chemical potential that was derived in a standard approach using the Fermi-Dirac distribution function. In order to show this equivalence we take advantage of the fact that for \( \alpha = 1 \) the sum of \( \theta \)-functions in Eq. (5) can be written as a product of \( \theta \)-functions with the help of addition formulae for \( \theta \)-functions. The derivative with respect to flux can then be performed to give the result

$$I_{\text{free}}(\Phi, V_g, T) = \frac{eT}{\hbar} \frac{1}{4\pi} \times \left\{ \frac{\theta'_3(\pm)}{\theta_3(\pm)} + \frac{\theta'_4(\pm)}{\theta_4(\pm)} \right\}. \quad (8)$$

valid in the limit \( \alpha = 1 \). In Eq. (8) \( \theta'(\nu, q) \) is the derivative of the theta-function with respect to its argument \( \nu \), and we have used the reduced notation

$$\theta_j(\pm) \equiv \theta_j \left( \frac{\Theta \pm \left( \frac{\mu}{2} \right)}{2}, q \right). \quad (9)$$

By making use of the formulae for logarithmic derivatives of the \( \theta \)-functions one finds

$$I_{\text{free}} = \frac{eT}{\hbar} T \sum_{n=1}^{\infty} ( -1)^n \frac{\sin(2\pi n \phi / \Phi)}{\sinh(2\pi n \Delta / \Delta)}. \quad (10)$$

This expression coincides with the well-known expression for the persistent current of free spinless electrons at fixed chemical potential \( \mu \) \( (\mu = \epsilon_F + \epsilon_{V_g}, \epsilon_F \) is the Fermi energy). Notice that the crossover temperature \( T^* = \Delta / 2 \pi^2 \hbar v_F / n_0 L \) in Eq. (10) is reduced by a factor 2 in comparison with the one for an isolated ring (fixed number of particles). In what follows we will show that for interacting fermions \( (\alpha \neq 1) \) the crossover temperature — for a ring at fixed chemical potential — smoothly interpolates between the result for a stiff Wigner crystal (when fluctuations of the electron number are strongly suppressed) and the result for a Fermi gas of spinless non-interacting electrons.

Now we proceed to the case of interacting fermions. We will use the general result (8) to find analytic expressions for the low- and high temperature limits and rely on numerical evaluations of (8) in the intermediate temperature region. At low temperatures, oscillations of the persistent current as a function of gate voltage (for \( \Phi \rightarrow 0 \)) are shown in Fig. 2(a). The traces shown refer to a ring with a fixed number, \( N_0 \), of “vacuum” fermions. It is easy to see from the general expression Eq. (8) that the result for an “even” ground state can be obtained from those corresponding to \( N_0 \) being an odd integer by a shift of the gate voltage, \( V_g \rightarrow V_g + \Delta / 2 e \alpha^2 \). Therefore in what follows we will analyze only the case when the number of “vacuum” electrons is odd.

The oscillations of the persistent current when the gate voltage is changed in the presence of a small magnetic flux amount to a series of transitions between a dia- and a paramagnetic response. At low temperatures these transitions have the form of periodic rectangular bumps as shown in Fig. 2(a). The width in gate voltage of the bumps and their positions depend on the correlation parameter \( \alpha \) of the LL and the dimensionless flux \( \Phi / \Phi_0 \). The ratio of the width of paramagnetic response to the width of diamagnetic response can be simply expressed as

$$\frac{\Delta V^{(p)}_{\text{g}}}{\Delta V^{(d)}_{\text{g}}} = \frac{1}{\alpha^2} - 1 + \frac{4 \Phi}{\alpha^4}. \quad (11)$$

For a stiff Wigner crystal-ring \( (\alpha \ll 1) \) the width of the bumps and the dips are equal since the ground state of a strongly repulsive LL-ring does not depend on the
parity of the particle number. With less stiffness (larger $\alpha$), the width of the current bumps (corresponding to a paramagnetic response) decreases and for $\alpha \to 1$ they reduce to narrow spikes located at $V'_g(p) = (2k + 1)\Delta/2$, where $k$ is an integer. For an attractive electron-electron interaction ($\alpha > 1$), the oscillatory dependence on gate voltage vanish. The most plausible values of the correlation parameter should be $\alpha \lesssim 1$ (we have in mind possible experiments in AlGaAs heterostructures). In this regime the shape of the current oscillations is sensitive to $\alpha$ and the mere detection of such oscillations could be a strong argument in favour of a non-Fermi liquid like behavior of quantum rings.

According to Eq. (13), the main contribution to the current is given by the first term if $\alpha^2 < 3$. The current then oscillates as a function of flux with the fundamental period $\Phi_0 = hc/e$. At the same time it is parity sensitive and has an oscillatory dependence on the gate voltage. From Eq. (13) one finds the crossover temperature to be a function of the interaction parameter $\alpha$:

$$T^*_j(\alpha) = \frac{1}{\pi^2} \frac{\Delta}{n + 1 - \alpha^2}$$  \hspace{1cm} (13)

This temperature attains its maximum ($T^* = \Delta/\pi^2$) for a stiff ($\alpha \ll 1$) Wigner crystal-ring, when the fluctuations of particle number in the ring are suppressed and the system can be considered as isolated.

If $\alpha^2 > 3$, on the other hand, the second harmonic in Eq. (13) gives the main contribution to the persistent current. There is no dependence on gate voltage and no parity effects in this term. The current is periodic in flux with a halved period ($\Phi_0/2$) and is diamagnetic. The crossover temperature saturates at $T^*_j(\alpha = \sqrt{3}) = \Delta/4\pi^2$, which is the lowest crossover temperature for spinless particles. The current oscillations at $\alpha > \sqrt{3}$ resemble the Aharonov-Bohm oscillations in a superconducting ring (period halving, absence of parity effect). This analogy appears also when studying the effects of Coulomb blockade in quantum rings.

### IV. CHARGE OSCILLATIONS AND COULOMB BLOCKADE

The average number of particles in a ring connected to a reservoir of electrons with a chemical potential $\mu$ can be found by making use of the general thermodynamic relation

$$\overline{N} = N_0(\mu) - \frac{1}{e} \frac{\partial Q}{\partial V_g}.$$  \hspace{1cm} (14)

Since we have an exact expression (3) for the thermodynamic potential $\Omega$ of a quantum ring, it is straightforward to find out how $\Delta N = \overline{N} - N_0(\mu)$ depends on temperature, magnetic field (flux) and gate voltage.

At low temperatures the gate voltage-dependence of charge $Q = e\Delta N = e\int(V_g)$ takes the form of a Coulomb staircase\textsuperscript{24} with steps of even and odd heights (in units of $e$) and different widths (Fig. 3a,b). Using Eqs. (3) and (14) one can derive an analytic expression for $\Delta N(V_g)$ in the limit of zero magnetic flux one finds

$$\Delta N(\overline{\mu}) = 2\overline{\mu} + \times \hspace{1cm} (15)$$

$$2 \sum_{n=1}^{\infty} \frac{(-1)^n \sin(2n\pi \overline{\mu})}{n} + 2k_{1/2} \leq \overline{\mu} < k + 2\overline{\mu}_c,$$  \hspace{1cm} (16)

where $k$ is an integer and

$$\overline{\mu} = \alpha^2 \frac{V_g}{\Delta}, \overline{\mu}_c = \left\{ \begin{array}{ll}
\frac{1}{2}(1 + \alpha^2) & , \alpha \leq 1 \\
\frac{1}{2} & , \alpha > 1.
\end{array} \right.$$
According to Eq. (13) the electric charge in a quantum ring is quantized in units of electron charge e if the interaction is repulsive (α > 1; see Fig. 3). The step widths corresponding to even (∆V_{g(e)}) and odd (∆V_{g(o)}) values of ∆N depend on the correlation parameter α and the dimensionless flux Φ/Φ_0. The ratio of the widths is described by the simple equation

\[
\frac{\Delta V_{g(e)}}{\Delta V_{g(o)}} = \frac{1 + 4\Phi/\Phi_0}{2}.
\]  

(17)

Recall that when deriving expression (14) for the average number of ring particles, the number of "vacuum" electrons was assumed to be odd. Therefore the odd(even)-step corresponds to an even(odd) total number of electrons in a ring.

\[\frac{\Delta N(V_g, T)}{\Delta} = \frac{2\alpha^2}{\Delta} \left\{ eV_g - 4\pi T e^{-4\pi^2 \alpha T/\Delta} \sin(4\pi \alpha^2 eV_g/\Delta) + 4\pi T(-1)^N_0 \cos(2\pi \Phi/\Phi_0)e^{-\pi^2 (1+\alpha^2) T/\Delta} \sin(2\pi\alpha^2 eV_g/\Delta) \right\}. \]  

(18)

It follows immediately that when α^2 < 1/3, the periodic modulation mainly comes from the second term of Eq. (18). It depends neither on magnetic flux nor on the parity of "vacuum" electrons. The crossover temperature for charge oscillations is equal to

\[T_c^* = \frac{\Delta}{4\pi^2 \alpha^*}, \quad \alpha^2 < 1/3. \]  

(19)

In the limit of strong repulsive interactions (α → 0), \(T_c^* \to \infty\), which implies that a sharp charge quantization is restored at any finite temperature.

Finally, when α^2 > 1/3 the charge oscillations are determined by the third term in Eq. (18) rather than the second. The oscillation period with respect to \(V_g\) is then doubled and the crossover temperature decreases with the increase of α

\[T_c^* = \frac{\Delta}{\pi^2 (1 + \alpha^2)}, \quad \alpha^2 \geq 1/3. \]  

(20)

V. INFLUENCE OF LONG-RANGE COULOMB INTERACTIONS ON THE THERMODYNAMICS OF QUANTUM RINGS

So far it has been assumed that the interactions between electrons in the quantum ring is short ranged. Now we consider long-range Coulomb forces and demonstrate that this interaction can be included in our scheme by replacing the "bare" coupling constant \(\alpha\) by an effective size-dependent "interaction constant" \(\alpha_{eff}(L)\),

\[\alpha_{eff}^2(L) = \frac{\alpha^2}{1 + \alpha^2 \frac{\pi^2}{\lambda \ln(\frac{L}{\lambda})}}. \]  

(21)

Here \(\lambda\) is an ultraviolet cutoff (\(\lambda \ll L\)) which in our case has the physical meaning of the width of the quantum wire.

Long-range Coulomb interactions can be incorporated in our model by adding to the local Lagrangian (11) a nonlocal Coulomb term

\[\mathcal{L}_c = -\frac{e^2}{8\pi} \int_0^L dy \frac{\varphi'(y)}{\sqrt{(x-y)^2 + \lambda^2}}. \]  

(22)
This term (see e.g. Refs. [21,22] describes the electrostatic interaction of charge densities \( \rho(x) = \varphi'(x)/2\pi \) localized in a long 1D quantum wire of width \( \lambda \).

In the presence of unscreened Coulomb forces, the equation of motion for the dynamical field \( \varphi(x, \tau) \) in imaginary time takes the form

\[
\ddot{\varphi} + s^2\varphi'' + \frac{1}{2}\frac{s_2}{s_1^2} \frac{\partial}{\partial x} \left( \int_0^L \frac{\partial \varphi(y, \tau)}{\sqrt{(x-y)^2 + \lambda^2}} \right) = 0,
\]

(23)

where \( s_2 = 2e^2\beta/\hbar m_0 (\beta = m_0/\hbar) \). Since we are interested only in topological excitations, we will seek solutions to Eq. (23) of the form

\[
\varphi(x, \tau) = 2\pi \frac{\tau}{\beta} n + \psi(x), \quad n = 0, \pm 1, ...
\]

(24)

where the spatial derivative of \( \psi \) (\( \psi(x) = \partial_x \psi(x)/2\pi \)) obeys the integral equation

\[
\rho(x) = \gamma \int_0^L \frac{dy \rho(y)}{\sqrt{(x-y)^2 + \lambda^2}} = C(\gamma).
\]

(25)

Here \( \gamma = (4\alpha^2/\pi)(e^2/\hbar v_F) \); \( C(\gamma) \) is an integration constant, which will be determined in what follows by the requirement that the desired solution \( \psi(x) \) has to satisfy the twisted boundary conditions (4). One readily verifies that

\[
\rho(\gamma) \simeq \frac{C(\gamma)}{1 + 2\gamma \ln \left( \frac{\alpha}{\lambda} \right)}
\]

(26)

is an approximate solution of the integral equation (25).

It should be noted that when modelling the effects of long-range Coulomb interactions in a quantum ring, we substitute the ring geometry by a line when calculating the Coulomb energy, Eq. (24). Such a substitution is justified only for a long enough ring \( L \gg \lambda \), when one can neglect numerical factors in the argument of the big logarithm in Eq. (26). In this case the precise values of the integration limits (\( L \rightarrow \alpha L, \alpha \sim 1 \)) in the Coulomb integrals Eqs. (23) and (25) do not affect the final expression evaluated in the logarithmic approximation. It is evident also from a physical point of view that in a perfect ring, a homogenous distribution of charges in the long wavelength limit can be inferred from symmetry considerations.

By determining the integration constant \( C(\gamma) \) from the requirement that the desired solution \( \psi(x) \) satisfies the twisted boundary conditions (4), one concludes that long-range Coulomb forces do not deform the zero modes which in the ideal (impurity free) ring remain of the the form of Eq. (4). However, the appearance of a Coulomb energy in the Lagrangian of the model changes the Euclidean action for these trajectories. This effect can be taken into account by replacing the “bare” constant \( \alpha \) by the effective size-dependent coupling constant \( \alpha_{\text{eff}}(L) \) of Eq. (21).

Equation (21) for \( \alpha_{\text{eff}}(L) \) requires a comment. Let us define the correlation parameter, \( \alpha \), of the LL as the value of \( \alpha_{\text{eff}} \) at \( L = \lambda (\alpha \equiv \alpha_{\text{eff}}(\lambda)) \). Then Eq. (21) takes the typical form of a running interaction constant in the renormalization group sense (see e.g. Ref. [23]). As \( L \) goes to infinity, \( \alpha_{\text{eff}} \rightarrow 0 \) irrespective of the value of the “bare” coupling constant \( \alpha \). Hence, in the presence of long range Coulomb interactions, a LL-system of long enough length behaves like a stiff (\( \alpha_{\text{eff}} \ll 1 \)) Wigner crystal (see the discussion in Ref. [23]). In particular, the crossover temperature for current oscillations,

\[
T_j^*(\alpha) = \frac{\Delta}{\pi^2} \frac{1 + \alpha^2(1 + \frac{e^2}{\hbar v_F} \ln \left( \frac{\lambda}{\lambda} \right))}{1 + \alpha^2(1 + \frac{e^2}{\hbar v_F} \ln \left( \frac{\lambda}{\lambda} \right))}
\]

(27)

ceases to depend on \( \alpha \) when \( (e^2/\hbar v_F) \ln (L/\lambda) \gg 1 \) and coincides with the crossover temperature of isolated LL-rings [24].

In experiments involving quantum wires and dots in the 2D electron gas of GaAs heterostructures, long-range Coulomb interactions can be partially screened by a large metallic electrode (gate) which controls the density of charge carriers. If the distance, \( D \) (\( D \gg \lambda \)), between the quantum ring and the gate is smaller than the characteristic size of the ring, Coulomb forces are screened on distances of the order of \( D \) and one has to replace \( L \) by \( D \) in Eq. (21). In the experiments considered, the distance \( D \) can be regarded as a controllable parameter. Thus one has an interesting possibility to study oscillation effects in quantum rings of different stiffness \( \alpha_{\text{eff}}(D) < 1 \).

VI. THERMODYNAMICS OF QUANTUM RINGS FOR SPIN-1/2 ELECTRONS

In the previous section we studied the thermodynamic properties of 1D mesoscopic rings of strongly interacting spinless electrons. Now we consider the influence of electron spin on the persistent current and charge quantization in a quantum ring. It is well known that in a perfect (impurity free) Luttinger liquid charge and spin degrees of freedom are separated and their dynamics can be described by independent quadratic Lagrangians (see e.g. Ref. [23]). Though the local dynamics of charge and spin excitations is independent, globally the two sectors of the LL model are connected by the “parity rules” which reflect the simple fact that the total spin and charge of the system are determined by the numbers of spin-“up” and spin-“down” electrons. Therefore the spin degrees of freedom affect persistent current- and charge oscillations through the parity term in the Lagrangian.

It is easy to generalize the model Lagrangian (1) to describe a Luttinger liquid of spin-1/2 electrons. Let \( N_{1,1}^{(0)} \) be the number of “up”- and “down”- spin electrons in the ground state (\( T=0, V_0=0 \)). We will assume that the
In what follows we will assume it to be local kernel of the electrostatic electron-electron interaction. The actual Fermi velocity here is therefore 

$$v = \frac{\Phi}{\Phi_0} \left( \frac{N_{\uparrow}^{(0)} + m_j - 1}{2} + \frac{eV_g}{2\pi S j} \right)$$

Here \( \nu_E \) — as above — refers to the Fermi velocity of spinless electrons of the same density. The actual Fermi velocity in the 2-component LL is of the form

$$\nu_E = \frac{v}{2\pi} \sum_{j=\uparrow,\downarrow} \left\{ \phi_j^2 - \frac{1}{4} v_F^2 (\phi_j')^2 + \frac{\hbar}{L} \phi_j \left( \frac{\Phi}{\Phi_0} \frac{N_{\uparrow}^{(0)} + m_j - 1}{2} + \frac{eV_g}{2\pi S j} \right) \right\}$$

The topological excitations which determine the current and charge oscillations in the 2-component LL are of the form

$$\varphi_j(x, \tau) = n_j \frac{2\pi}{\beta} + m_j \frac{2\pi x}{L},$$

where \((n_j, m_j)\) are winding numbers, which independently run over integer values. The Euclidean action for the trajectories takes the form

$$S_E(n_{\uparrow,\downarrow}, m_{\uparrow,\downarrow}) = \sum_{j=\uparrow,\downarrow} \left\{ 2\pi^2 \frac{n_j^2}{\beta \Delta} + i2\pi n_j (\Theta_j + \frac{m_j}{2}) \right\} + \frac{\hbar}{L} \Delta (1 + 4V_0/\pi \nu_E) m_j^2 + \beta \mu_0 m_j$$

where

$$\Delta = \frac{2\pi \hbar v_F}{L}, \quad \Theta_j = \frac{\Phi}{\Phi_0} + \frac{N_{\uparrow}^{(0)} - 1}{2}, \quad \mu_0 = eV_g$$

When calculating the thermodynamic potential \(\Omega(\Phi, V_g, T)\) of a 1D ring of strongly correlated electrons

$$\Omega - \Omega_p = -T \ln \left\{ \sum_{m_{\uparrow,\downarrow}, m_{\uparrow,\downarrow} = -\infty}^{\infty} \exp \left\{ -S_E(n_{\uparrow,\downarrow}, m_{\uparrow,\downarrow})/\hbar \right\} \right\}$$

it is convenient to replace the sum over \((m_{\uparrow,\downarrow}, n_{\uparrow,\downarrow})\) by a sum over linear combinations of winding numbers which describe the charge (c) and spin (s) channels:

$$m_c = m_\uparrow + m_\downarrow, \quad m_s = m_\uparrow - m_\downarrow.$$

The summation in Eq. (33) can be performed exactly. We omit the straightforward but cumbersome intermediate calculations and simply give the final expression for the oscillating part of \(\Omega\). The form of this expression depends significantly on the parity of the numbers \((N_{\uparrow}^{(0)}, N_{\downarrow}^{(0)})\) of spin-“up” and “down” vacuum electrons in the ring. Because of this we separately analyze the cases when the above numbers are of equal parity — (odd,odd) and (even,even) or they have opposite parities (even,odd) or vice versa. To compare the results for electrons with and without spin we express all quantities in terms of \(I_0\) determined for the spinless case:

$$I_0 = \frac{e\hbar N_{\uparrow}^{(0)}}{m_0 L^2}.$$
\begin{equation}
\Omega^{\text{osc}}(\alpha_c = 1) = -T \ln \{2 \exp(2\pi T) \times \left[ \theta_3(f, q^2)\theta_3(\overline{m}, q^2) + \theta_4(f, q^2)\theta_4(\overline{m}, q^2) \right] \}.
\end{equation}

By making use of the addition formulae for the Jacobi theta-functions (see e.g. Ref. [15]) we get the following expression for the persistent current of free spin-1/2 fermions at finite temperature (Cf. Eqs. (6) and (11))

\begin{equation}
I_{\text{free}}^-(\Phi, V_g, T) = \frac{eT}{\hbar} \frac{1}{\pi} \times \frac{d}{df} \ln \left\{ \theta_3(f, q^2)\theta_3(\overline{m}, q^2) + \theta_4(f, q^2)\theta_4(\overline{m}, q^2) \right\}
= 4 \frac{eT}{\hbar} \sum_{n=1}^{\infty} (-1)^n \frac{\sin(2\pi n \frac{\Phi}{\Phi_0}) \cos(4\pi n \frac{V_g}{\Delta})}{\sinh(4\pi^2 n \frac{\Delta}{\Delta})}.
\end{equation}

This result coincides with the one derived by Kulik in a standard approach using the Fermi-Dirac distribution function \( f_{FD}(p_n) \),

\begin{equation}
I(\Phi, \mu, T) = -e \sum_n \frac{\partial \epsilon_n}{\partial \Phi} f_{FD}(p_n)
= 4 \frac{eT}{\hbar} \sum_{k=1}^{\infty} \frac{\sin(2\pi k \frac{\Phi}{\Phi_0}) \cos(4\pi k \frac{k F}{\Delta})}{\sinh(4\pi^2 k \frac{\Delta}{\Delta})},
\end{equation}

where \( \epsilon \) is the one-particle spectrum. Since the persistent current is a property of the Fermi surface, it does not depend on the precise form of the spectrum far from the Fermi momentum. When expressed in terms of the Fermi velocity, it is the same for quadratic and linear energy dispersion.

The standard expression (40) for the persistent current in the case of noninteracting electrons with spin is transformed to our result (38) if we put the chemical potential equal to \( \mu = \epsilon_F(0) + eV_g(0) \) is the Fermi energy of the ground state with an odd number of spin-"up","down" electron pairs. From a physical point of view, it is evident that for noninteracting spin-1/2 electrons, the energy levels are doubly degenerate. Therefore the ground state \((T = 0, V_g = 0, \Phi = 0)\) of free electrons at fixed chemical potential has an odd number of electron pairs. We will prove this fact explicitly below when studying the effects of the Coulomb blockade.

Notice that at \( T = 0 \) and \( V_g = 0 \), Eq. (38) describes the persistent current for a system with a fixed number of particles. In the case considered (odd number of pairs), our formula exactly coincides with the one obtained in Ref. [14].

We conclude that the persistent current in a ring with an odd number of spin-"up", "down" pairs oscillates with the fundamental period \( \Phi_0 = \hbar c/e \). It is diamagnetic and at low temperatures has the same amplitude \( I_0 = eV_g/L \) as the current of spinless fermions (at the same particle density \( p \)). The influence of spin on the thermodynamic properties of quantum rings of free electrons is straightforward — the crossover temperature and oscillation period on gate voltage are diminished by a factor 2 (in comparison with the spinless case, Eq. (10)) which is a trivial consequence of the "halving" of the Fermi velocity for electrons with spin and same density.

What new effects do the electron-electron correlations lead to? At low temperatures \((T \to 0)\) the windings of the spatial sector are frozen (due to the suppression of particle fluctuations) and we get a simple expression for the persistent current of a LL-ring at \( T = 0 \) of the form

\begin{equation}
I(T = 0) = \frac{2}{\pi} I_0 \sum_{n=1}^{\infty} (-1)^n \frac{\sin(2\pi n \frac{\Phi}{\Phi_0})}{n} \cos(4\pi n \alpha c^2 \frac{eV_g}{\Delta}).
\end{equation}

Although the oscillations have the same sawtooth-like shape as for noninteracting particles, the period of the current oscillations when the gate voltage is varied is drastically different in the case of strongly correlated electrons. According to Eqs. (37) and (41) one gets for strong repulsion current oscillations with a period which is twice the oscillation period for a LL-ring of spinless fermions (see Fig. 2b). Electron-electron correlations significantly affect the Aharonov-Bohm oscillations (even for an impurity free ring) in the high temperature regime. In particular, the crossover temperature becomes a function of the correlation parameter \( \alpha \). Using a small-\( \alpha \) expansion of the \( \theta \)-functions, one can from Eq. (36) derive the high-\( T \) expansion for the persistent current of correlated electrons. As in the case of spinless fermions (see Eq. (12)) it is sufficient to keep only two harmonics in the expansion,

\begin{equation}
I_{- -} \simeq \frac{32 eT}{3 \hbar} \left\{ e^{-\pi^2(3+\alpha c^2)T/\Delta} \cos \left( 4\pi \alpha c^2 \frac{eV_g}{\Delta} \right) \times \sin \left( 2\pi \frac{\Phi}{\Phi_0} \right) + e^{-8\pi^2 T/\Delta} \sin \left( 4\pi \frac{\Phi}{\Phi_0} \right) \right\}. \tag{42}
\end{equation}

Note that the current oscillations are determined mainly by the first term in (12) if \( \alpha c^2 < 5 \). The current is diamagnetic and persists with fundamental period \( \Phi_0 \). The crossover temperature is

\begin{equation}
T^*(\alpha c^2) = \begin{cases}
\frac{\Delta}{8\pi^2}, & \alpha c^2 < 5 \\
\frac{\Delta}{8\pi^2}, & \alpha c^2 \geq 5
\end{cases} \tag{43}
\end{equation}

The transition to the regime where the oscillation period is reduced to \( \Phi_0/2 \) occurs only when the interaction is strong and attractive, i.e. when \( \alpha c^2 > 5 \). However, it should be noted that unlike in the spinless case, the two-component model (28) can not be directly applied for describing attractive interactions. It is known (see e.g. Ref. [24]) that in the last case backscattering processes become relevant and result in a gap for the spin excitation spectrum. The appearance of the gap rules out Luttinger liquid-like behavior of spin-1/2 attractive electrons. Therefore, in what follows we will analyze only the regime of repulsive interactions \( \alpha c < 1 \).
It is worth to note here that the inclusion of spin degrees of freedom changes the maximum crossover temperature, which is attained in the limit of strong repulsion $\alpha_c \ll 1$ (stiff Wigner crystal). As one can see from Eq. (13), for spin-1/2 electrons this temperature is three times smaller than for the Wigner crystal-ring of spinless fermions, Eq. (13).

Now we proceed to the analysis of charge quantization in a LL-ring of electrons with spin. For the case in question (an odd number of electron pairs: $N_\uparrow^{(0)} = N_\downarrow^{(0)} = 2n+1$) the analytic formulae for $\overline{N}(V_g)$ are too cumbersome for practical use and we will consider only numerical results. Some results for electrons with spin are shown in Fig. 4.

FIG. 4. Splitting of the Coulomb staircase at low temperatures and no flux due to repulsive interactions between spin-1/2 electrons. Traces have been vertically offset for clarity. In the absence of a magnetic flux the energy levels of the noninteracting electrons ($\alpha_c = 1$) are fourfold degenerate (with a factor of two from spin). Hence four electrons at a time are entering the ring each time the gate voltage $V_g$ is such that a new degenerate ring level is pulled below the chemical potential of the reservoir. As the strength of the repulsive electron-electron interactions is increased ($\alpha_c < 1$) the noninteracting energy levels become increasingly irrelevant and electron correlations dominate. As shown this implies that electrons are entering the ring one by one, and in the limit $\alpha_c \ll 1$ the steps in the Coulomb staircase become equidistant.

In the strongly repulsive limit ($\alpha_c \ll 1$) and at low temperatures the spin degree of freedom is insignificant and we have the same Coulomb staircase as for spinless electrons. The charge quantization for the two cases considered (electrons with and without spin) differs drastically in the opposite limit — weakly interacting particles. For spin-1/2 fermions, the steps pertaining to the numbers $N = 4n, 4n + 1, 4n + 3$ electrons on the ring are decreased with the decrease of interaction strength and disappear (the width of the steps is contracted to a point) at $\alpha_c \rightarrow 1$.

The above picture demonstrates the physical fact that at fixed chemical potential (grand canonical ensemble) the stable ground state of the Fermi gas of spin-1/2 fermions always contains an odd number of spin-“up” and spin-“down” electrons $N_\uparrow^{(0)} = N_\downarrow^{(0)} = 2n+1$. It leads to a diamagnetic response of a ring when applying an external magnetic field.

Since the width of the other steps goes to zero as $\alpha_c \to 1$, one can estimate the destruction temperature of the corresponding “vacua” for $\alpha_c \lesssim 1$, $T_d \sim (1 - \alpha_c^2)\Delta/2$ ($\Delta/2$ is the level spacing for the Fermi gas of electrons with spin).

VIII. QUANTUM RING WITH AN EVEN NUMBER OF ELECTRON PAIRS

The thermodynamic potential, $\Omega_{++}$, for an even number of pairs of ring electron with opposite spin projections $N_\uparrow^{(0)} = N_\downarrow^{(0)} = 2n$ can be derived from Eq. (36) by a shift of flux, $f \to f + 1/2$. This is the parity rule for electrons with spin. In general, the thermodynamic potentials (or more correctly their oscillating parts) for the two cases in question — (odd,odd) and (even,even) — obey the following symmetry relations:

$$\Omega_{++}(f, \Phi) = \Omega_{++}^{osc}(f + 1/2, \Phi) = \Omega_{++}^{osc}(f, \Phi + 1/2).$$

(44)

By shifting arguments in the expression valid for an odd number of noninteracting electrons we find the persistent current for noninteracting electrons in a ring with an even number of pairs to be

$$I_{++}^{free}(\Phi, V_g, T) = \frac{4eT}{\hbar} \sum_{n=1}^{\infty} \frac{\sin(2\pi n \frac{\Phi}{\Phi_0}) \cos(4\pi n \frac{V_g}{\Delta})}{\sinh(4\pi^2 n \frac{\Delta}{\hbar} T)}.$$  

(45)

This expression transforms into the standard result Eq. (40) for $\mu = \epsilon_F^{(c)} + eV_g$, where $\epsilon_F^{(c)}$ is the Fermi energy of the ground state with an even number of electron pairs. As was discussed in section VIII, such a ground state ($N_\uparrow^{(0)} = N_\downarrow^{(0)} = 2n$) for noninteracting particles is destroyed due to particle exchange with the reservoir. Thus Eq. (45) makes sense only in the $T=0$, $V_g=0$ limit:

$$I_{++}^{free}(T = 0, V_g = 0) = \frac{2}{\pi} \frac{eI_0}{\hbar} \sum_{n=1}^{\infty} \frac{\sin(2\pi n \frac{\Phi}{\Phi_0})}{n},$$

(46)

when it describes the persistent current of a ring with fixed ($N = 4n$) number of particles. Eq. (46) coincides with the one obtained in Ref. [27]. The current is paramagnetic, $\Phi_0$-periodic and has the same amplitude, $I_0$ as the persistent current of spinless fermions. The temperature behavior of the persistent current for a LL-ring in contact with a reservoir is described by Eqs. (43) and (44), independent of the parity of the number of “vacuum” electrons ($N_\uparrow^{(0)}$, $N_\downarrow^{(0)}$).
IX. QUANTUM RING WITH AN ODD NUMBER OF PARTICLES

For a ring with an odd number of spin-1/2 fermions ($N^{(0)} = 4n + 1$ or $4n + 3$), the oscillating part of the thermodynamic potential $\Omega_{osc}$, takes the form

$$\Omega_{+/-} = -T \ln \left\{ \exp \left( 4\alpha_c \mu_g \frac{\Delta}{\Delta} \right) \times \left( \theta_3(2\frac{\Phi}{\Phi_0}, q^4) \theta_4(4\alpha_c \mu_g \frac{\Delta}{\Delta}, q^{4\alpha_c^2}) + \theta_4(2\frac{\Phi}{\Phi_0}, q^4) \theta_3(4\alpha_c \mu_g \frac{\Delta}{\Delta}, q^{4\alpha_c^2}) \right) \right\}$$

The principal difference between Eqs. (47) and (36) is concerned with the periodicity in flux and gate voltage. Using the properties of the Jacobi theta-functions, one can immediately see from Eq. (47) that in a ring with an odd number of particles, the persistent current is periodic in magnetic flux with period $\Phi_0/2$, half the fundamental period.

Period “halving” for the persistent current of an odd number of spin-1/2 noninteracting fermions was predicted (for $T = 0$) in Ref. [2]. From the general properties of Eq. (47) we can conclude that this effect survives in the presence of interactions (see also Ref. [30][31]) and even for finite temperatures. However, Eq. (47) pertains to the case of an odd number of “ground state” electrons in the ring. Therefore one has to be sure that vacuum states corresponding to an odd number of particles ($N = 4n + 1$ or $4n + 3$) are stable against particle number fluctuations since we study the situation with a fixed chemical potential. We have already seen that it is not the case for noninteracting spin-1/2 fermions. In the last case, the ground state corresponds to an odd number of particles ($N^{(0)} = 4n + 2$ number of electrons. Thus the $\alpha_c \to 1$ limit of Eq. (47) can be justified only for strictly zero temperature and $V_g = 0$ when it describes properties of a ring with fixed (odd) number of noninteracting electrons. In this limit we get from Eq. (47) a simple expression for the persistent current,

$$I_{free}^{+/-}(T = 0, V_g = 0) = \frac{1}{\pi} I_0 \sum_{n=1}^{\infty} \frac{\sin(4\pi n \frac{\Phi}{\Phi_0})}{n},$$

which coincides exactly with the one in Ref. [2]. The current is paramagnetic and periodic with half the period, $\Phi_0/2$, and half the amplitude, $\frac{1}{2}I_0$, in comparison with the case of spinless electrons.

For repulsive interactions, the “odd” ground state is stable only at temperatures $T < T_d \sim (1 - \alpha_c^2)\Delta$. Therefore it is reasonable to study current oscillations for the (even-odd)-case only in the low temperature regime. The high temperature behavior of the current is always described by Eq. (42).

At low temperatures the oscillations of current with varying gate voltage at $\Phi \to 0$ can be interpreted as oscillations between a dia- and a paramagnetic response. Qualitatively these oscillations are of the same form as for the spinless particles (see Fig. 4). For strong repulsion ($\alpha_c \ll 1$, stiff Wigner crystal) the bumps of paramagnetic response and dips of diamagnetic response have the same width in gate voltage. This property is closely related to the effects of charge quantazation. For a stiff Wigner crystal, the “even” and “odd” steps of the function $e\Delta N(V_g, T \to 0) = -(\partial \Omega_{+/-}/\partial V_g)$ have equal widths since in this strongly repulsive limit considered the energy does not depend on the parity of the electron number. With the increase of $\alpha_c$ the steps corresponding to an odd total number of electrons on a ring get narrow and paramagnetic bursts of current reduce to spikes. In the region $\alpha_c \gtrsim 1$ (even,odd), the ground state becomes absolutely unstable and to study the persistent current of noninteracting electrons at fixed chemical potential, one has to start from a different expression for $\Omega$, namely Eq. (38), “built” on the stable ground state.

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