CONVERGENCE RESULTS FOR A COMMON SOLUTION OF A FINITE FAMILY OF EQUILIBRIUM PROBLEMS AND QUASI-BREGMAN NONEXPANSIVE MAPPINGS IN BANACH SPACE

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Abstract. In this paper, we introduce an iterative process for finding common fixed point of finite family of quasi-Bregman nonexpansive mappings which is a unique solution of some equilibrium problem.

1. INTRODUCTION

Let $E$ be a real reflexive Banach space, $C$ a nonempty subset of $E$. Let $T : C \rightarrow C$ be a map, a point $x \in C$ is called a fixed point of $T$ if $Tx = x$, and the set of all fixed points of $T$ is denoted by $F(T)$. The mapping $T$ is called $L$-Lipschitzian or simply Lipschitz if there exists $L > 0$, such that $\|Tx - Ty\| \leq L\|x - y\|$, $\forall x, y \in C$ and if $L = 1$, then the map $T$ is called nonexpansive.

Let $g : C \times C \rightarrow \mathbb{R}$ be a bifunction. The equilibrium problem with respect to $g$ is to find

$$z \in C \text{ such that } g(z, y) \geq 0, \forall y \in C.$$  
The set of solution of equilibrium problem is denoted by $EP(g)$. Thus

$$EP(g) := \{z \in C : g(z, y) \geq 0, \forall y \in C\}.$$ 

Numerous problems in Physics, Optimization and Economics reduce to finding a solution of the equilibrium problem. Some methods have been proposed to solve equilibrium problem in Hilbert spaces; see for example Blum and Oettli [5], Combettes and Hirstoaga [12]. Recently, Tada and Takahashi [29, 30] and Takahashi and Takahashi [31] obtain weak and strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and set of fixed points of nonexpansive mapping in Hilbert space. In particular, Tada and Takahashi [30] establish a strong convergence theorem for finding a common element of the two sets by using the hybrid method introduced in Nakajo and Takahashi [18]. They also proved such a strong convergence theorem in a uniformly convex and uniformly smooth Banach space.

In 1967, Bregman [7] discovered an elegant and effective technique for using so-called Bregman distance function $D_f$ see, [14] in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman’s technique has been applied in various ways in order to design and analyze iterative algorithms for solving feasibility and optimization.
problems.
Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \text{dom} f \times \text{int dom} f \to [0, +\infty)$ defined as follows:
\begin{equation}
D_f(y, x) := f(y) - f(x) - \langle \nabla f(x) , y - x \rangle
\end{equation}
is called the Bregman distance with respect to $f$ (see [10]). It is obvious from the definition of $D_f$ that
\begin{equation}
D_f(z, x) = D_f(z, y) + D_f(y, x) + \langle \nabla f(y) - \nabla f(x), z - y \rangle.
\end{equation}
We observed from (1.2), that for any $y_1, y_2, \cdots, y_N \in E$, the following holds
\begin{equation}
D_f(y_1, y_N) = \sum_{k=2}^{N} D_f(y_{k-1}, y_k) + \sum_{k=3}^{N} \langle \nabla f(y_{k-1}) - \nabla f(y_k), y_{k-1} - y_k \rangle.
\end{equation}
Recall that the Bregman projection [7] of $x \in \text{int dom} f$ onto the nonempty closed and convex set $C \subset \text{dom} f$ is the necessarily unique vector $P_C^f(x) \in C$ satisfying
\begin{equation}
D_f(P_C^f(x), x) = \inf \{D_f(y, x) : y \in C\}.
\end{equation}
A mapping $T$ is said to be Bregman firmly nonexpansive [20], if for all $x, y \in C$,\[
\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle
\]
or equivalently,
\begin{equation}
D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).
\end{equation}
A point $p \in C$ is said to be asymptotic fixed point of a map $T$, if for any sequence $\{x_n\}$ in $C$ which converges weakly to $p$, and $\lim_{n \to \infty} ||x_n - T x_n|| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of $T$. Let $f : E \to \mathbb{R}$, a mapping $T : C \to C$ is said to be Bregman relatively nonexpansive [15] if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$ and $D_f(p, T(x)) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$, $T$ is said to be quasi-Bregman relatively nonexpansive if $F(T) \neq \emptyset$, and $D_f(p, T(x)) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$.

Recently, by using the Bregman projection, in 2011 Reich and Sabach [20] proposed algorithms for finding common fixed points of finitely many Bregman firmly nonexpansive operators in a reflexive Banach space.

\begin{equation}
\begin{cases}
x_0 \in E \\
Q_0^i = E, i = 1, 2, \cdots, N \\
u_n \in C \text{ such that} \\
y_n^i = T_i(x_n + e_n^i) \\
Q_{n+1}^i = \{z \in Q_n^i : \langle \nabla f(x_n + e_n^i) - \nabla f(y_n^i), z - y_n^i \rangle \leq 0\}, \\
C_n = \bigcap_{i=1}^{N} C_n^i, \\
x_{n+1} = P_{C_{n+1}}^{f} x_0, n \geq 0.
\end{cases}
\end{equation}

Under some suitable conditions, they proved that the sequence generated by (1.4) converges strongly to $\bigcap_{i=1}^{N} F(T_i)$ and applied the result for the solution of convex feasibility and equilibrium problems.

In 2011, Chen et al. [11], introduced the concept of weak Bregman relatively nonexpansive mappings in a reflexive Banach space and gave an example to illustrate the existence of a weak Bregman relatively nonexpansive mapping and the difference between a weak Bregman relatively nonexpansive mapping and a Bregman
Let $E$ be a real Banach space. We consider the problem of finding a fixed point of a family of quasi-Bregman nonexpansive mappings $T_n : E \to E$ with $n \in \mathbb{N}$, where $E$ is a Banach space. The family $\{T_n\}$ is said to be strongly nonexpansive if
\[
\|T_n(x) - T_n(y)\| \leq \|x - y\|, \quad \forall x, y \in E.
\]
Recent results on this topic include those by Alghamdi et al. [1] and Pang et al. [19].

In 2014, Alghamdi et al. [1] proved a strong convergence theorem for the common fixed point of finite family of quasi-Bregman nonexpansive mappings. Pang et al. [19] proved weak convergence theorems for Bregman relatively nonexpansive mappings. While, Zegeye and Shahzad in [34] and [35] proved a strong convergence theorem for the common fixed point of finite family of right Bregman strongly nonexpansive mappings and Bregman weak relatively nonexpansive mappings.

Recently in 2014, Alghamdi et al. [1] proved a strong convergence theorem for the common fixed point of finite family of quasi-Bregman nonexpansive mappings. They proved that the sequence $\{x_n\}$ which is generated by the algorithm (1.5) converges strongly to $T^\#_\Omega x$, where $\Omega := F(T) \cap EP(g)$.

Motivated and inspired by the above works, in this paper, we prove a new strong convergence theorem for finite family of quasi-Bregman nonexpansive mappings and system of equilibrium problem in a real Banach space.

2. Preliminaries

Let $E$ be a real reflexive Banach space with the norm $\|\cdot\|$ and $E^*$ the dual space of $E$. Throughout this paper, we shall assume $f : E \to (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function. We denote by $\text{dom} f := \{x \in E : f(x) < +\infty\}$ as the domain of $f$.

Let $x \in \text{int}\text{ dom} f$, the subdifferential of $f$ at $x$ is the convex set defined by
\[
\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \leq f(y), \forall y \in E\}
\]
where the Fenchel conjugate of $f$ is the function $f^* : E^* \to (-\infty, +\infty]$ defined by
\[
f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x) : x \in E\}.
\]
We know that the Young-Fenchel inequality holds:
\[
\langle x^*, x \rangle \leq f(x) + f^*(x^*), \quad \forall x \in E, \ x^* \in E^*.
\]
A function $f$ on $E$ is coercive [13] if the sublevel set of $f$ is bounded; equivalently,
\[
\lim_{\|x\| \to +\infty} f(x) = +\infty.
\]
A function $f$ on $E$ is said to be strongly coercive [33] if
\[
\lim_{\|x\| \to +\infty \|x\|} f(x) = +\infty.
\]

For any $x \in \text{int dom} f$ and $y \in E$, the right-hand derivative of $f$ at $x$ in the direction $y$ is defined by
\[
f^0(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}.
\]
The function \( f \) is said to be Gâteaux differentiable at \( x \) if \( \lim_{t\to0} \frac{f(x+ty)-f(x)}{t} \) exists for any \( y \). In this case, \( f^\circ(x,y) \) coincides with \( \nabla f(x) \), the value of the gradient \( \nabla f \) of \( f \) at \( x \). The function \( f \) is said to be Gâteaux differentiable if it is Gâteaux differentiable for any \( x \in \text{int dom} f \). The function \( f \) is said to be Fréchet differentiable at \( x \) if this limit is attained uniformly in \( ||y|| = 1 \). Finally, \( f \) is said to be uniformly Fréchet differentiable on a subset \( C \) of \( E \) if the limit is attained uniformly for \( x \in C \) and \( ||y|| = 1 \). It is known that if \( f \) is Gâteaux differentiable (resp. Fréchet differentiable) on int dom\( f \), then \( f \) is continuous and its Gâteaux derivative \( \nabla f \) is norm-to-weak* continuous (resp. continuous) on int dom\( f \) (see also [24] [6]).

We will need the following results.

**Lemma 2.1.** [21] If \( f : E \to \mathbb{R} \) is uniformly Fréchet differentiable and bounded on bounded subsets of \( E \), then \( \nabla f \) is uniformly continuous on bounded subsets of \( E \) from the strong topology of \( E \) to the strong topology of \( E^* \).

**Definition 2.2.** [3] The function \( f \) is said to be:

(i) essentially smooth, if \( \partial f \) is both locally bounded and single-valued on its domain.

(ii) essentially strictly convex, if \( (\partial f)^{-1} \) is locally bounded on its domain and \( f \) is strictly convex on every convex subset of \( \text{dom}\partial f \).

(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

**Remark 2.3.** Let \( E \) be a reflexive Banach space. Then we have

(i) \( f \) is essentially smooth if and only if \( f^* \) is essentially strictly convex (see [3], Theorem 5.4).

(ii) \( (\partial f)^{-1} = \partial f^* \) (see [6])

(iii) \( f \) is Legendre if and only if \( f^* \) is Legendre, (see [3], Corollary 5.5).

(iv) If \( f \) is Legendre, then \( \nabla f \) is a bijection satisfying

\[
\nabla f = (\nabla f^*)^{-1}, \quad \text{ran} \nabla f = \text{dom} \nabla f^* = \text{int dom} f^* \quad \text{and} \quad \text{ran} \nabla f^* = \text{dom} f = \text{int dom} f, \quad (\text{see [3], Theorem 5.10}).
\]

The following result was prove in [24], (see also [25]).

**Lemma 2.4.** Let \( E \) be a Banach space, \( r > 0 \) be a constant, \( \rho_r \) be the gauge of uniform convexity of \( g \) and \( g : E \to \mathbb{R} \) be a convex function which is uniformly convex on bounded subsets of \( E \). Then

(i) For any \( x, y \in B_r \) and \( \alpha \in (0,1) \),

\[
 g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha)\rho_r(||x-y||).
\]

(ii) For any \( x, y \in B_r \),

\[
 \rho_r(||x-y||) \leq D_g(x,y)
\]

(iii) If, in addition, \( g \) is bounded on bounded subsets and uniformly convex on bounded subsets of \( E \) then, for any \( x \in E, y^*, z^* \in B_{r^*} \) and \( \alpha \in (0,1) \),

\[
 V_g(x, \alpha y^* + (1-\alpha)z^*) \leq \alpha V_g(x, y^*) + (1-\alpha) V_g(x, z^*) - \alpha(1-\alpha)\rho_r(||y^*-x^*||).
\]

**Lemma 2.5.** [22] Let \( E \) be a Banach space, let \( r > 0 \) be a constant and let \( f : E \to \mathbb{R} \) be a continuous and convex function which is uniformly convex on bounded subsets of \( E \). Then

\[
f \left( \sum_{k=0}^{\infty} \alpha_k x_k \right) \leq \sum_{k=0}^{\infty} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(||x_i - x_j||)
\]
Lemma 2.8. Let $f$ that the identity mapping in Hilbert spaces. In the rest of this paper, we always assume

Theorem 2.9. Concerning the Bregman projection, the following are well known.

Theorem 2.6. Let $E$ be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of $E$. Then the following assertions are equivalent:

1. $f$ is strongly coercive and uniformly convex on bounded subsets of $E$;
2. $\text{dom} f^* = E^*$, $f^*$ is bounded on bounded subsets and uniformly smooth on bounded subsets of $E^*$;
3. $\text{dom} f^* = E^*$, $f^*$ is Frechet differentiable and $\nabla f$ is uniformly norm-to-norm continuous on bounded subsets of $E^*$.

Theorem 2.7. Let $E$ be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:

1. $f$ is bounded on bounded subsets and uniformly smooth on bounded subsets of $E$;
2. $f^*$ is Frechet differentiable and $f^*$ is uniformly norm-to-norm continuous on bounded subsets of $E^*$;
3. $\text{dom} f^* = E^*$, $f^*$ is strongly coercive and uniformly convex on bounded subsets of $E^*$.

The following result was first proved in [8] (see also [16]).

Lemma 2.8. Let $E$ be a reflexive Banach space, let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function and let $V$ be the function defined by

$$V(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \quad x \in E, \ x^* \in E^*$$

Then the following assertions hold:

1. $D_f(x, \nabla f(x^*)) = V(x, x^*)$ for all $x \in E$ and $x^* \in E^*$.
2. $V(x, x^*) + (\langle \nabla f^*(x^*) - x, y^* \rangle) \leq V(x, x^* + y^*)$ for all $x \in E$ and $x^*, y^* \in E^*$.

Examples of Legendre functions were given in [3] [4]. One important and interesting Legendre function is $\frac{1}{p}||\cdot||^p(1 < p < \infty)$ when $E$ is a smooth and strictly convex Banach space. In this case the gradient $\nabla f$ of $f$ is coincident with the generalized duality mapping of $E$, i.e., $\nabla f = J_p(1 < p < \infty)$. In particular, $\nabla f = I$ the identity mapping in Hilbert spaces. In the rest of this paper, we always assume that $f : E \to (-\infty, +\infty]$ is Legendre.

Concerning the Bregman projection, the following are well known.

Lemma 2.9. [8] Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then

(a) $z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \forall y \in C$.
(b) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \forall x \in E, \ y \in C$.

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The modulus of total convexity of $f$ at $x \in \text{int} \text{ dom} f$ is the function $v_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$ defined by

$$v_f(x, t) := \inf \{D_f(y, x) : y \in \text{dom} f, ||y - x|| = t\}.$$
The function $f$ is called totally convex at $x$ if $v_f(x,t) > 0$ whenever $t > 0$. The function $f$ is called totally convex if it is totally convex at any point $x \in \operatorname{int} \operatorname{dom} f$ and is said to be totally convex on bounded sets if $v_f(B,t) > 0$ for any nonempty bounded subset $B$ of $E$ and $t > 0$, where the modulus of total convexity of the function $f$ on the set $B$ is the function $v_f : \operatorname{int} \operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$ defined by

$$v_f(B,t) := \inf \{v_f(x,t) : x \in B \cap \operatorname{dom} f \}.$$

**Lemma 2.10.** [28] If $x \in \operatorname{dom} f$, then the following statements are equivalent:

(i) The function $f$ is totally convex at $x$;

(ii) For any sequence $\{y_n\} \subset \operatorname{dom} f$,

$$\lim_{n \to +\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n \to +\infty} ||y_n - x|| = 0.$$

Recall that the function $f$ called sequentially consistent [8] if for any two sequence $\{x_n\}$ and $\{y_n\}$ in $E$ such that the first one is bounded

$$\lim_{n \to +\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n \to +\infty} ||y_n - x_n|| = 0.$$

**Lemma 2.11.** [9] The function $f$ is totally convex on bounded sets if and only if the function $f$ is sequentially consistent.

**Lemma 2.12.** [27] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.

**Lemma 2.13.** [27] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, $x_0 \in E$ and let $C$ be a nonempty, closed and convex subset of $E$. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to $C$. If $D_f(x_n, x_0) \leq D_f(P_C^f(x_0), x_0)$ for any $n \in \mathbb{R}$, then $\{x_n\}$ converges strongly to $P_C^f(x_0)$.

**Lemma 2.14.** [24] Let $E$ be a real reflexive Banach space, $f : E \to (-\infty, +\infty]$ be a proper lower semi-continuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semi-continuous and convex function. Thus, for all $z \in E$, we have

$$D_f(z, \nabla f^*(\sum_{i=1}^{N} t_i \nabla f(x_i))) \leq \sum_{i=1}^{N} t_i D_f(z, x_i) \tag{2.1}$$

In order to solve the equilibrium problem, let us assume that a bifunction $g : C \times C \to \mathbb{R}$ satisfies the following condition [17]

(A1) $g(x, x) = 0$, $\forall x \in C$.

(A2) $g$ is monotone, i.e., $g(x, y) + g(y, x) \leq 0$, $\forall x, y \in C$.

(A3) $\limsup_{t \to 0} g(x + t(z - x), y) \leq g(x, y)$ $\forall x, z, y \in C$.

(A4) The function $y \mapsto g(x, y)$ is convex and lower semi-continuous.

The resolvent of a bifunction $g$ [12] is the operator $\operatorname{Res}_g^f : E \to 2^C$ defined by

$$\operatorname{Res}_g^f(x) = \{z \in C : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \forall y \in C\}. \tag{2.2}$$

From (Lemma 1, in [23]), if $f : (-\infty, +\infty]$ is a strongly coercive and Gâteaux differentiable function, and $g$ satisfies conditions (A1)-(A4), then $\operatorname{dom}(\operatorname{Res}_g^f) = E$.

The following lemma gives some characterization of the resolvent $\operatorname{Res}_g^f$. 

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Lemma 2.15. 

Let $E$ be a real reflexive Banach space and $C$ be a nonempty closed convex subset of $E$. Let $f : E \to (-\infty, +\infty]$ be a Legendre function. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies the conditions (A1)-(A4). Then, the followings hold:

(i) $\text{Res}_g f$ is single-valued;
(ii) $\text{Res}_g f$ is a Bregman firmly nonexpansive operator;
(iii) $F(\text{Res}_g f) = \text{EP}(g)$;
(iv) $\text{EP}(g)$ is closed and convex subset of $C$;
(v) for all $x \in E$ and for all $q \in F(\text{Res}_g f)$, we have

\begin{equation}
D_f(q, \text{Res}_g f(x)) + D_f(\text{Res}_g f(x), x) \leq D_f(q, x).
\end{equation}

Lemma 2.16. 

Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n, \quad n \geq n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\}$ is a real sequence satisfying the following conditions:

$$\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \text{as} \quad \limsup_{n \to \infty} \delta_n \leq 0.$$

Then, $\lim_{n \to \infty} a_n = 0.$

Lemma 2.17. 

Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$.

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$  

In fact, $m_k = \max\{j : a_j < a_{j+1}\}$.

3. MAIN RESULTS

We now prove the following theorem.

Theorem 3.1. 

Let $C$ be a nonempty, closed and convex subset of a real reflexive Banach space $E$ and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of $E$. For each $j = 1, 2, \ldots, m$, let $g_j$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\{T^n_{i=1}\}$ be a finite family of quasi-Bregman nonexpansive self mapping of $C$ such that $F := \cap_{i=1}^{N}F(T_i) \neq \emptyset$, where $F = F(T_N T_{N-1} T_{N-2} \cdots T_2 T_1) = F(T_N T_{N-1} T_{N-2} \cdots T_2 T) = \cdots = F(T_N T_{N-1} T_{N-2} \cdots T_2 T_{i}) \neq \emptyset$ and $\Omega := \left( \cap_{j=1}^{m} \text{EP}(g_j) \right) \cap F \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by $x_1 = x \in C, C_1 = C$ and

\begin{equation}
x_{n+1} = P_C(\nabla f^*((1 - \alpha_n)\nabla f(u_{n,j}) + (1 - \beta_n)\nabla f(T_n[y_n])))
\end{equation}

where $T_n = T_{n(\text{mod } N)}$ and $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1), \{\beta_n\}_{n=1}^{\infty} \subset [c, d] \subset (0, 1)$ satisfying

$$\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$  

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $P^{f}_{\Omega}(x)$, where $P^{f}_{\Omega}$ is the Bregman projection of $C$ onto $\Omega$.  


Proof. Let $p = P^f_{\Omega} \in \Omega$ from Lemma 2.15, we obtain

$$D_f(p, u_{j,n}) = D_f(p, Res^f_{\Omega}, x_n) \leq D_f(p, x_n)$$

Now from (3.1), we obtain

$$D_f(p, y_n) \leq D_f(p, \nabla f^*(1 - \alpha_n)\nabla f(u_{j,n}))$$

$$= D_f(p, \nabla f^*(\alpha_n\nabla f(0) + (1 - \alpha_n)\nabla f(u_{j,n})))$$

$$\leq \alpha_n D_f(p, 0) + (1 - \alpha_n) D_f(p, u_{j,n})$$

(3.2)

$$\leq \alpha_n D_f(p, 0) + (1 - \alpha_n) D_f(p, x_n)$$

Also from (3.1), (2.1) and (3.2), we have

$$\int D_f(p, x_{n+1}) \leq D_f(p, \nabla f^*((1 - \beta_n)\nabla f(y_n) + \beta_n\nabla f(T_{[n]}y_n)))$$

$$\leq (1 - \beta_n) D_f(p, y_n) + \beta_n D_f(p, T_{[n]}y_n)$$

$$\leq (1 - \beta_n) D_f(p, y_n) + \beta_n D_f(p, y_n)$$

$$= D_f(p, y_n)$$

(3.3)

$$\leq \alpha_n D_f(p, 0) + (1 - \alpha_n) D_f(p, x_n)$$

$$\leq \max \{D_f(p, 0), D_f(p, x_n)\}$$

Thus, by induction we obtain

$$D_f(p, x_{n+1}) \leq \max \{D_f(p, 0), D_f(p, x_n)\}, \forall n \geq 0$$

which implies that $\{x_n\}$ is bounded and hence $\{y_n\}, \{T_{[n]}y_n\}, \{T_{[n]}x_n\}$ and $\{u_{j,n}\}$ are all bounded for each $j = 1, 2, \ldots, m$. Now from (3.1) let $z_n := \nabla f^*((1 - \alpha_n)\nabla f(u_{j,n}))$. Furthermore since $\alpha_n \to 0$ as $n \to \infty$, we obtain

(3.4) \[ ||\nabla f(z_n) - \nabla f(u_{j,n})|| = \alpha_n ||(-\nabla f(u_{j,n}))|| \to 0 \text{ as } n \to \infty. \]

Since $f$ is strongly coercive and uniformly convex on bounded subsets of $E$, $f^*$ is uniformly Fréchet differentiable on bounded sets. Moreover, $f^*$ is bounded on bounded sets, from (3.4), we obtain

(3.5) \[ \lim_{n \to \infty} ||z_n - u_{j,n}|| = 0. \]

On the other hand, in view of (3) in Theorem 2.6, we know that $\text{dom} f^* = E^*$ and $f^*$ is strongly coercive and uniformly convex on bounded subsets. Let $s = \sup\{||\nabla f(y_n)||, ||\nabla f(T_{[n]}y_n)||\}$ and $\rho^*_s : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function $f^*$. Now from (3.1), Lemma 2.4 and 2.8, we obtain

$$D_f(p, y_n) \leq D_f(p, z_n) = V(p, \nabla f(z_n))$$

$$\leq V(p, \nabla f(z_n) + \alpha_n \nabla f(p) + \alpha_n (\nabla f(p), z_n - p)$$

$$= D_f(p, \nabla f^*((1 - \alpha_n)\nabla f(u_{j,n}) + \alpha_n \nabla f(p)))$$

$$+ \alpha_n (\nabla f(p), z_n - p)$$

$$\leq \alpha_n D_f(p, p) + (1 - \alpha_n) D_f(p, u_{j,n})$$

$$+ \alpha_n (\nabla f(p), z_n - p)$$

(3.6)

$$\leq (1 - \alpha_n) D_f(p, x_n) + \alpha_n (\nabla f(p), z_n - p)$$
and

$$D_f(p, x_{n+1}) \leq D_f(p, \nabla f^*(1 - \beta_n)\nabla f(y_n) + \beta_n\nabla f(T_n y_n))$$

$$= V(p, (1 - \beta_n)\nabla f(y_n) + \beta_n\nabla f(T_n y_n))$$

$$= f(p) - (p, (1 - \beta_n)\nabla f(y_n) + \beta_n\nabla f(T_n y_n))$$

$$+ f^*(1 - \beta_n)\nabla f(y_n) + \beta_n\nabla f(T_n y_n))$$

$$\leq (1 - \beta_n)f(p) + \beta_n f(p) - (1 - \beta_n)(p, \nabla f(y_n)) - \beta_n(p, \nabla f(T_n y_n))$$

$$+ (1 - \beta_n)f^*(\nabla f(T_n y_n)) + \beta_n f^*(\nabla f(T_n y_n))$$

$$- \beta_n(1 - \beta_n)\rho_s^*(||\nabla f(y_n) - \nabla f(T_n y_n)||)$$

$$= (1 - \beta_n)\nabla f(y_n) + \beta_n\nabla f(T_n y_n))$$

$$- \beta_n(1 - \beta_n)\rho_s^*(||\nabla f(y_n) - \nabla f(T_n y_n)||)$$

$$\leq (1 - \beta_n)D_f(p, y_n) + \beta_nD_f(p, T_n y_n)$$

$$- \beta_n(1 - \beta_n)\rho_s^*(||\nabla f(y_n) - \nabla f(T_n y_n)||)$$

$$= D_f(p, y_n) - \beta_n(1 - \beta_n)\rho_s^*(||\nabla f(y_n) - \nabla f(T_n y_n)||)$$

$$\leq (1 - \alpha_n)D_f(p, x_n) + \alpha_n(-\nabla f(p), z_n - p)$$

$$- \beta_n(1 - \beta_n)\rho_s^*(||\nabla f(y_n) - \nabla f(T_n y_n)||)$$

$$(3.7)$$

$$\leq (1 - \alpha_n)D_f(p, x_n) + \alpha_n(-\nabla f(p), z_n - p)$$

$$(3.8)$$

Now, we consider two cases:

**Case 1.** Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(p, x_n)\}$ is non increasing. In this situation $\{D_f(p, x_n)\}$ is convergent. Then from $$(3.7)$$ we obtain

$$\beta_n(1 - \beta_n)\rho_s^*(||\nabla f(y_n) - \nabla f(T_n y_n)||) \to 0 \text{ as } n \to \infty,$$

which implies, by the property of $\rho_s$ and since $\beta_n \in [c, d] \subset (0, 1)$, we obtain

$$\lim_{n \to \infty} ||\nabla f(y_n) - \nabla f(T_n y_n)|| = 0$$

Since $f$ is strongly coercive and uniformly convex on bounded subsets of $E$, $f^*$ is uniformly Fréchet differentiable on bounded sets. Moreover, $f^*$ is bounded on bounded sets, from $$(3.10)$$, we obtain

$$\lim_{n \to \infty} ||y_n - T_n y_n|| = 0.$$  

Now from $$(1.2)$$, we obtain

$$D_f(y_n, T_n y_n) = D_f(p, T_n y_n) - D_f(p, y_n)$$

$$+ \langle \nabla f(T_n y_n) - \nabla f(y_n), p - y_n \rangle$$

$$\leq D_f(p, y_n) - D_f(p, y_n)$$

$$+ \langle \nabla f(T_n y_n) - \nabla f(y_n), p - y_n \rangle$$

therefore

$$(3.12) D_f(y_n, T_n y_n) \leq ||\nabla f(y_n) - \nabla f(T_n y_n)|| ||p - y_n|| \to 0 \text{ as } n \to \infty.$$
Also, from (2.15), we have
\[ D_f(x_n, u_{j,n}) = D_f(p, \text{Res}_j f, x_n) \leq D_f(p, \text{Res}_j f, x_n) - D_f(p, x_n) \leq D_f(p, x_n) - D_f(p, x_n) \to 0 \text{ as } n \to \infty. \]
(3.13)

Then, we have from Lemma 2.10 that
\[
\lim_{n \to \infty} ||x_n - u_{j,n}|| = 0
\]
(3.14)

Also, from (b) of Lemma 2.9, we have
\[
D_f(y_n, PC z_n) = D_f(y_n, z_n) = D_f(u_{j,n}) \leq \alpha_n D_f(0) + (1 - \alpha_n) D_f(u_{j,n}) \to 0 \text{ as } n \to \infty.
\]
(3.15)

Then, we have from Lemma 2.10 that
\[
\lim_{n \to \infty} ||y_n - z_n|| = 0
\]
(3.16)

From (3.15) and (3.14), we obtain
\[
\lim_{n \to \infty} ||x_n - z_n|| = 0
\]
(3.17)

From (3.16) and (3.17), we obtain
\[
\lim_{n \to \infty} ||x_n - y_n|| = 0
\]
(3.18)

Since \( f \) is strongly coercive and uniformly convex on bounded subsets of \( E \), \( f^* \) is uniformly Fréchet differentiable on bounded sets. Moreover, \( f^* \) is bounded on bounded sets, from (3.18), we obtain
\[
\lim_{n \to \infty} ||\nabla f(x_n) - \nabla f(z_n)|| = 0.
\]
(3.19)

Also from (3.11) and (3.18)
\[
\lim_{n \to \infty} ||x_n - T[n] y_n|| = 0.
\]
(3.20)

Now from (1.2) and (3.2), we obtain
\[
D_f(x_n, y_n) = D_f(p, y_n) - D_f(p, x_n) + \langle \nabla f(x_n) - \nabla f(y_n), p - x_n \rangle \leq \alpha_n D_f(p, 0) + (1 - \alpha_n) D_f(p, x_n) + \langle \nabla f(x_n) - \nabla f(y_n), p - x_n \rangle
\]
\[
= \alpha_n (D_f(p, 0) - D_f(p, x_n)) + \langle \nabla f(x_n) - \nabla f(y_n), p - x_n \rangle
\]
therefore, from (3.19), we obtain
\[
D_f(x_n, y_n) \leq \alpha_n (D_f(p, 0) - D_f(p, x_n)) + ||\nabla f(x_n) - \nabla f(y_n)|| ||p - x_n|| \to 0 \text{ as } n \to \infty.
\]
(3.21)
and also
\[
D_f(x_n, T_{[n]}y_n) = D_f(p, T_{[n]}y_n) - D_f(p, x_n) \\
+ \langle \nabla f(x_n) - \nabla f(T_{[n]}y_n), p - x_n \rangle
\]
\[
\leq D_f(p, y_n) - D_f(p, x_n) \\
+ \langle \nabla f(x_n) - \nabla f(T_{[n]}y_n), p - x_n \rangle
\]
\[
\leq \alpha_n D_f(p, 0) + (1 - \alpha_n) - D_f(p, x_n) \\
+ \langle \nabla f(x_n) - \nabla f(T_{[n]}y_n), p - x_n \rangle
\]
\[
= \alpha_n(D_f(p, 0) - D_f(p, x_n)) + \langle \nabla f(x_n) - \nabla f(T_{[n]}y_n), p - x_n \rangle
\]
thus
\[
D_f(x_n, T_{[n]}y_n) \leq \alpha_n|D_f(p, 0) - D_f(p, x_n)| \\
+ ||\nabla f(T_{[n]}y_n) - \nabla f(x_n)|| ||p - x_n|| \to 0 \text{ as } n \to \infty.
\]
(3.22)

Also, from (3.21)
(3.23)
\[
D_f(T_{[n]}x_n, T_{[n]}y_n) \leq D_f(x_n, y_n) \to 0 \text{ as } n \to \infty.
\]
Then, we have from Lemma 2.10 that
(3.24)
\[
\lim_{n \to \infty} ||T_{[n]}x_n - T_{[n]}y_n|| = 0.
\]
Then from (3.21) and (3.11), we have
(3.25)
\[
||\nabla f(x_{n+1}) - \nabla f(y_{n})|| = \beta_n ||\nabla f(T_{[n]}y_n) - \nabla f(y_{n})|| \to 0 \text{ as } n \to \infty.
\]
which implies
(3.26)
\[
||x_{n+1} - y_{n}|| \to 0 \text{ as } n \to \infty.
\]
and
\[
||x_{n} - T_{[n]}x_{n}|| \leq ||x_{n} - y_{n}|| + ||y_{n} - T_{[n]}y_{n}|| + ||T_{[n]}y_{n} - T_{[n]}x_{n}||
\]
from (3.11), (3.18) and (3.24), we obtain
(3.27)
\[
\lim_{n \to \infty} ||x_{n} - T_{[n]}x_{n}|| = 0.
\]
which implies that
(3.28)
\[
\lim_{n \to \infty} ||\nabla f(x_{n}) - \nabla f(T_{[n]}x_{n})|| = 0.
\]
Also from (3.18) and (3.20), we obtain
(3.29)
\[
||x_{n+1} - x_{n}|| \leq ||x_{n+1} - y_{n}|| + ||y_{n} - x_{n}|| \to 0 \text{ as } n \to \infty.
\]
But
\[
||x_{n+N} - x_{n}|| \leq ||x_{n+N} - x_{n+N-1}|| + ||x_{n+N-1} - x_{n+N-2}|| + \cdots + ||x_{n+1} - x_{n}|| \to 0
\]
as \(n \to \infty\). Hence
(3.30)
\[
\lim_{n \to \infty} ||x_{n+N} - x_{n}|| = 0.
\]
From the uniformly continuous of \(\nabla f\), we have from (3.29) that
(3.31)
\[
\lim_{n \to \infty} ||\nabla f(x_{n+1}) - \nabla f(x_{n})|| = 0.
\]
From (1.2), (3.3) and (3.31), we obtain
\[ \begin{align*}
D_f(x_n, x_{n+1}) &= D_f(p, x_{n+1}) - D_f(p, x_n) \\
&= \langle \nabla f(x_n) - \nabla f(x_{n+1}), p - x_n \rangle \\
&\leq \alpha_n D_f(p, 0) + (1 - \alpha_n) D_f(p, x_n) - D_f(p, x_n) \\
&= \langle \nabla f(x_n) - \nabla f(x_{n+1}), p - x_n \rangle
\end{align*} \]
which implies
\[ D_f(x_n, x_{n+1}) \leq \alpha_n|D_f(p, 0) - D_f(p, x_n)| + ||\nabla f(x_{n+1}) - \nabla f(x_n)||||p - x_n|| \to 0 \text{ as } n \to \infty. \] (3.32)

Also from (1.2) and (3.27), we obtain
\[ \begin{align*}
D_f(T_{[n]}x_n, x_{n+1}) &\leq D_f(x_n, x_{n+1}) \to 0 \text{ as } n \to \infty.
\end{align*} \]
which implies
\[ \lim_{n \to \infty} ||T_{[n]}x_n - T_{[n]}x_{n+1}|| = 0. \] (3.33)

and from the uniform continuous of \( \nabla f \), we obtain
\[ \lim_{n \to \infty} ||\nabla f(T_{[n]}x_n) - \nabla f(T_{[n]}x_{n+1})|| = 0. \] (3.35)

Also from (1.2) and (3.27), we obtain
\[ \begin{align*}
D_f(x_n, T_{[n]}x_n) &= D_f(p, T_{[n]}x_n) - D_f(p, x_n) \\
&= \langle \nabla f(x_n) - \nabla f(T_{[n]}x_n), p - x_n \rangle \\
&\leq D_f(p, x_n) - D_f(p, x_n) \\
&= \langle \nabla f(x_n) - \nabla f(T_{[n]}x_n), p - x_n \rangle
\end{align*} \]
\[ \leq ||\nabla f(x_{n+1}) - \nabla f(x_n)||||p - x_n|| \to 0 \text{ as } n \to \infty. \] (3.36)

From (3.27), (3.29) and (3.34), we obtain
\[ \begin{align*}
||x_n - T_{[n]}x_n|| &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - T_{[n]}x_{n+1}|| \\
&+ ||T_{[n]}x_{n+1} - T_{[n]}x_n|| \to 0 \text{ as } n \to \infty.
\end{align*} \] (3.37)

which from uniform continuous of \( \nabla f \) implies
\[ \lim_{n \to \infty} ||\nabla f(T_{[n]}x_n) - \nabla f(T_{[n]}x_{n+1})|| = 0. \] (3.38)

from (1.2) and (3.38), we obtain
\[ \begin{align*}
D_f(x_n, T_{[n]}x_n) &\leq D_f(p, T_{[n+1]}x_n) - D_f(p, x_n) \\
&= \langle \nabla f(x_n) - \nabla f(T_{[n]}x_n), p - x_n \rangle \\
&\leq D_f(p, x_n) - D_f(p, x_n) \\
&= \langle \nabla f(x_{n+1}) - \nabla f(x_n), ||p - x_n|| \rangle \to 0 \text{ as } n \to \infty.
\end{align*} \] (3.39)

From (1.2), (3.32), (3.38) and (3.39), we obtain
\[ \begin{align*}
D_f(x_n, T_{[n]}x_n) &= D_f(x_{n+1, n+1}) + D_f(x_n, T_{[n]}x_{n+1}) \\
&= \langle \nabla f(T_{[n]}x_n) - \nabla f(x_n), x_n - x_{n+1} \rangle \\
&\leq D_f(x_{n+1, n} + D_f(x_n, T_{[n]}x_{n+1}) \\
&+ ||\nabla f(T_{[n]}x_n) - \nabla f(x_n)||||x_n - x_{n+1}|| \to 0 \text{ as } n \to \infty.
\end{align*} \] (3.40)
Also \( (3.12) \), \( (3.32) \), and \( (3.30) \)

\[
D_f(x_n, T_{[n+1]}x_n) = D_f(x_n, x_{n+1}) + D_f(x_{n+1}, T_{[n+1]}x_n) \\
+ \langle \nabla f(x_{n+1}) - \nabla f(T_{[n+1]}x_{n+1}), x_{n+1} - x_n \rangle \\
= D_f(x_n, x_{n+1}) + D_f(x_{n+1}, T_{[n+1]}x_n) \\
+ ||\nabla f(T_{[n+1]}x_n) - \nabla f(x_{n+1})|| ||x_{n+1} - x_n|| \to 0 \text{ as } n \to \infty.
\]

(3.41)

Using the quasi-Bregman nonexpansivity of \( T_{(i)} \) for each \( i \), we obtain, we obtain the following finite table

\[
D_f(x_{n+N}, T_{(n+N)}x_{n+N-1}) \to 0 \text{ as } n \to \infty \\
D_f(T_{(n+N)}x_{n+N-1}, T_{(n+N)}T_{(n+N-1)}x_{n+N-2}) \to 0 \text{ as } n \to \infty \\
\vdots \\
D_f(T_{(n+N)} \cdots T_{(n+2)}x_{n+1}, T_{(n+N)} \cdots T_{(n+1)}x_n) \to 0 \text{ as } n \to \infty
\]

then, applying Lemma \( 2.10 \) on each line above, we obtain

\[
x_{n+N} - T_{(n+N)}x_{n+N-1} \to 0 \text{ as } n \to \infty \\
T_{(n+N)}x_{n+N-1} - T_{(n+N)}T_{(n+N-1)}x_{n+N-2} \to 0 \text{ as } n \to \infty \\
\vdots \\
T_{(n+N)} \cdots T_{(n+2)}x_{n+1} - T_{(n+N)} \cdots T_{(n+1)}x_n \to 0 \text{ as } n \to \infty
\]

and adding up this table, we obtain

\[
x_{n+N} - T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}x_n \to 0 \text{ as } n \to \infty.
\]

Using this and \( (3.30) \), we obtain

\[
\lim_{n \to \infty} ||x_n - T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}x_n|| = 0.
\]

Also from quasi-Bregman nonexpansive of \( T_{(i)} \), for each \( i \), we have

\[
(3.43) \quad D_f(T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}x_n, T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}y_n) \leq D_f(x_n, y_n) \to 0
\]

as \( n \to \infty \). Then, we have from Lemma \( 2.10 \) that

\[
(3.44) \quad T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}x_n - T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}y_n \to 0 \text{ as } n \to \infty.
\]

Since

\[
||y_n - T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}y_n|| \leq ||y_n - x_n|| \\
+ ||x_n - T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}x_n|| \\
+ ||T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}x_n - T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}y_n||
\]

then, from \( (3.18) \), \( (3.32) \) and \( (3.44) \), we obtain

\[
(3.45) \quad \lim_{n \to \infty} ||y_n - T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}y_n|| = 0.
\]

Following the argument from \( (3.43) \) to \( (3.45) \) by replacing \( y_n \) with \( z_n \) and using \( (3.17) \), we obtain

\[
(3.46) \quad \lim_{n \to \infty} ||z_n - T_{(n+N)}T_{(n+N-1)} \cdots T_{(n+1)}z_n|| = 0.
\]
Let \( \{x_n\} \) be a subsequence of \( \{x_n\} \). Since \( \{x_n\} \) is bounded and \( E \) is reflexive, without loss of generality, we may assume that \( x_n \to q \) for some \( q \in F \) and since \( x_n \to z_n \to 0 \) as \( n \to \infty \), then \( z_n \to q \). Since the pool of mappings of \( T_{[n]} \) is finite, passing to a further subsequence if necessary, we may further assume that, for some \( i \in \{1, 2, \cdots, N\} \), from (3.10), we get
\[
z_{n_i} - T_{(i+N)} \cdots T_{(i+1)} z_{n_i} \to 0 \text{ as } i \to \infty
\]
and also
\[
\limsup_{n \to \infty} \langle -\nabla f(p), z_n - p \rangle = \lim_{i \to \infty} \langle -\nabla f(p), z_{n_i} - p \rangle
\]
Noticing that \( u_{j,n} = \text{Res}_{f_j}^E(x_n) \) for each \( j = 1, 2, \cdots, m \), we obtain
\[
g_j(u_{j,n}, y) + \langle y - u_{j,n}, \nabla f(u_{j,n}) - \nabla f(x_n) \rangle \geq 0, \quad \forall y \in C
\]
Hence
\[
g_j(u_{j,n}, y) + \langle y - u_{j,n}, \nabla f(u_{j,n}) - \nabla f(x_n) \rangle \geq 0, \quad \forall y \in C.
\]
From the (A2), we note that for each \( j = 1, 2, \cdots, m \),
\[
\|y - u_{j,n}\| \frac{\|\nabla f(u_{j,n}) - \nabla f(x_n)\|}{\tau_n} \geq \langle y - u_{j,n}, \nabla f(u_{j,n}) - \nabla f(x_n) \rangle \geq -g_j(u_{j,n}, y) \geq g_j(y, u_{j,n}), \quad \forall y \in C.
\]
Taking the limit as \( i \to \infty \) in above inequality and from (A4) and \( u_{j,n_i} \to q \), we have \( g_j(y, q) \leq 0 \) for each \( j = 1, 2, \cdots, m \). For \( 0 < t < 1 \) and \( y \in C \), define \( y_t = ty + (1-t)q \). Noticing that \( y, q \in C \), we obtain \( y_t \in C \), which yield that \( g_j(y_t, q) \leq 0 \). It follows from (A1) that
\[
0 = g_j(y_t, y_t) \leq tg_j(y, y) + (1-t)g_j(y_t, q) \leq tg_j(y_t, y).
\]
That is for each \( j = 1, 2, \cdots, m \), we have \( g_j(y_t, y) \geq 0 \).
Let \( t \downarrow 0 \), from (A3), we obtain \( g_j(q, y) \geq 0 \) for any \( y \in C \), for each \( j = 1, 2, \cdots, m \). This implies that \( q \in \cap_{j=1}^m \text{EP}(g_j) \). Hence \( q \in \Omega \). It follows from the definition of the Bregman projection that
\[
\limsup_{n \to \infty} \langle -\nabla f(p), z_n - p \rangle = \lim_{i \to \infty} \langle -\nabla f(p), z_{n_i} - p \rangle \leq \langle -\nabla f(p), q - p \rangle \leq 0.
\]
It follows from Lemma 2.13 and (3.8) that \( D_f(p, x_n) \to 0 \) as \( n \to \infty \). Consequently, from Lemma 2.10, we obtain \( x_n \to p \) as \( n \to \infty \).

**Case 2.** Suppose \( D_f(p, x_n) \) is not monotone decreasing sequences, then set \( \Phi_n := D_f(p, x_n) \) and let \( \tau : \mathbb{N} \to \mathbb{N} \) be a mapping defined for all \( n \geq N_0 \) for some sufficiently large \( N_0 \) by
\[
\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Phi_k \leq \Phi_{k+1}\}.
\]
Then by Lemma 2.17, \( \tau(n) \) is a non-decreasing sequence such that \( \tau(n) \to \infty \) as \( n \to \infty \) and \( \Phi_{\tau(n)} \leq \Phi_{\tau(n)+1} \) for \( n \geq N_0 \). Then from (3.3) and the fact that \( \alpha_{\tau(n)} \to 0 \), we obtain that
\[
\rho^*_n\|\nabla f(y_{\tau(n)}) - \nabla f(T_{\tau(n)})y_{\tau(n)}\| \to 0 \text{ as } \tau(n) \to \infty.
\]
Following the same argument as in Case 1, we obtain
\[
y_{\tau(n)} - T_{(i+N)} \cdots T_{(i+1)} y_{\tau(n)} \to 0 \text{ as } \tau(n) \to \infty.
\]
and also we obtain
\[
\limsup_{\tau(n) \to \infty} \langle -\nabla f(p), y_{\tau(n)} - p \rangle \leq 0.
\]

Then from (3.48), we obtain that
\[
0 \leq D_f(p, x_{\tau(n)+1}) - D_f(p, x_{\tau(n)}) \\
\leq \alpha_{\tau(n)} \langle -\nabla f(p), y_{\tau(n)} - p \rangle - D_f(p, x_{\tau(n)}) \\
\tag{3.47}
\]
It follows from (3.47) and \( \Phi_n \leq \Phi_{\tau(n)+1}, \alpha_{\tau(n)} > 0 \) that
\[
D_f(p, x_{\tau(n)}) \leq \langle -\nabla f(p), y_{\tau(n)} - p \rangle \to 0
\]
as \( \tau(n) \to \infty \). Thus
\[
\lim_{\tau(n) \to \infty} \Phi_{\tau(n)} = \lim_{\tau(n) \to \infty} \Phi_{\tau(n)+1} = 0.
\]

Furthermore, for \( n \geq N_0 \), if \( n \neq \tau(n) \) (i.e., \( \tau(n) < n \)), because \( \Phi_j > \Phi_{j+1} \) for \( \tau(n) + 1 \leq j \leq n \), it then follows that for all \( n \geq N_0 \) we have
\[
0 \leq \Phi_n \leq \max \{ \Phi_{\tau(n)}, \Phi_{\tau(n)+1} \} = \Phi_{\tau(n)+1}.
\]
This implies that \( \lim_{n \to \infty} \Phi_n = 0 \), and hence \( D_f(p, x_{\tau(n)}) \to 0 \) as \( n \to \infty \). Consequently, from Lemma 2.4, we obtain \( x_n \to p \) as \( n \to \infty \). Therefore from the above two cases, we conclude that \( \{ x_n \} \) converges strongly to \( p \in \Omega \) and this complete the proof. \( \square \)

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