FRAC TAL INTERPOLATION FUNCTION ON PRODUCTS OF THE SIERPİŃSKI GASKETS

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Abstract. In this paper, we aim to construct fractal interpolation function (FIF) on the product of two Sierpiński gaskets. Further, we collect some results regarding smoothness of the constructed FIF. We prove, in particular, that the FIF are Hölder functions under specific conditions. In the final section, we obtain some bounds on the fractal dimension of FIF.

1. Introduction

Barnsley [3] introduced the concept of Fractal Interpolation Functions (FIFs) on the interval using the theory of Iterated Function System (IFS). Since then, FIFs have proven to be an invaluable tool for interpolating experimental data by a non-smooth curve, with numerous applications in engineering [4], biological sciences [6], planetary science [7] and arts [8]. FIFs of various types, such as Hidden-variable FIFs, Coalescence Hidden-variable FIFs, Hermite FIFs, Spline FIFs, and Super FIFs, have been built in [9, 10, 13–16] and properties such as smoothness, approximation property, regularity, multiresolution analysis, reproducing kernel, node insertion, fractional calculus, dimension property have been studied in [5, 17, 18, 20–26, 28, 40]. The development of Fractal Interpolation Surfaces or multivariable FIFs formed by employing higher dimensional or recurrent IFSs has been addressed in [27, 29–31].

Celik et al. [32] defined FIFs on the Sierpinski Gasket (SG), a well-known fractal domain. Further, these FIFs were studied by Ruan [33] on the basis of p.c.f. self-similar sets. In [33], Ruan extended his work on p.c.f. self-similar sets which were introduced by Kigami by defining the FIFs on these sets. On SG, Rí and Ruan [35] explored some properties of uniform fractal interpolation functions, a particular class of FIFs. First, they investigated the min-max property of uniform FIFs. Then, they determined a necessary and sufficient condition for uniform FIFs to have finite energy. Uniform FIFs’s normal derivatives and laplacian were also discussed. Sahu and Priyadarshi [36] computed some bounds for box dimensions of harmonic functions on the SG. They also constructed a fractal interpolation function on the SG. Recently, Agrawal and Som [1] estimated box dimension of FIF on the SG. Further, they [2] used FIFs to approximate functions on SG. In [19], Navascués et al. constructed vector-valued FIFs on SG and studied several approximation theoretic results. Strichartz [38, 39] started analysis on the products of SGs and developed a new concept of

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2. Fractal Interpolation Function on $SG' \times SG''$

Let $\{X; W_i, i = 1, 2, \ldots, k\}$ be an Iterated Function System (IFS), where $(X, d)$ is a complete metric space and $W_i : X \to X$ are contractive mappings with contraction ratio $\alpha_i$ respectively. The IFS generates the mapping $W$ from $\mathcal{K}(X)$ into $\mathcal{K}(X)$ given by

$$W(K) = \bigcup_{i=1}^{k} W_i(K),$$

where $\mathcal{K}(X)$ is the collection of all non-empty compact subsets of $X$. The Hutchinson-Barnsley map $W$ defined above is then a contraction mapping, with respect to the Hausdorff metric $h_d$, the contraction ratio $\alpha$ of $W$ is equal to $\max\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. Then, by the Banach contraction principle, there exists a unique nonempty compact subset $K_*$ such that $K_* = \bigcup_{i=1}^{k} W_i(K_*).$ The set $K_*$ is termed as the attractor of the IFS. For further details, the reader is referred to [3].

Let $U_0 = \{p_1, p_2, p_3\}$ be the vertices of an equilateral triangle on the plane $\mathbb{R}^2$ and $L_i(t) = \frac{1}{2}(t + p_i), \ i = 1, 2, 3$ be three contractions of the plane which constitute an iterated functions system. For $n \in \mathbb{N}$, we denote the collection of all words with length $n$ by $\{1, 2, 3\}^n$, i.e. if $\omega \in \{1, 2, 3\}^n$ then $\omega = \omega_1, \omega_2, \ldots, \omega_n$ where $\omega_i \in \{1, 2, 3\}$. We define, for $\omega \in \{1, 2, 3\}^N$, $L_\omega = L_{\omega_1} \circ L_{\omega_2} \circ \cdots \circ L_{\omega_N}$. $SG'$ is the attractor of the system:

$$SG' = L_1(SG') \cup L_2(SG') \cup L_3(SG').$$

Similarly, consider $W_0 = \{q_1, q_2, q_3\}$ be the vertices of another equilateral triangle on another $\mathbb{R}^2$ plane and $K_i(t) = \frac{1}{2}(t + q_i), \ i = 1, 2, 3$ be three contractions of the second plane constituting another iterated functions system. Then, we have another attractor

$$SG'' = K_1(SG'') \cup K_2(SG'') \cup K_3(SG'').$$

Hence, we have

$$SG' \times SG'' = \bigcup_{i=1, j=1}^{3} L_i(SG') \times K_j(SG'').$$

Iteratively, we get

$$SG' \times SG'' = \bigcup_{\omega, \eta \in L^n} L_\omega(SG') \times K_\eta(SG'').$$
Define $V_N$ by $V_N = \{(L_\omega(p_i), K_\eta(q_j)) : \omega, \eta \in I^N, \ i, j \in I\}$. The union of images of $V_0 := U_0 \times W_0$ under these iterations $(L_\omega(.), K_\eta(.))$ with $|w| = N$ constitutes the set of N-th stage vertices $V_N$ of $SG' \times SG''$. Further, define $V_* = \bigcup_{N=1}^{\infty} V_N$.

Let $\{(y, z_y) : y \in V_N\} \subset SG' \times SG'' \times \mathbb{R}$ be a data set such that $z_y$ is zero on the boundary of $SG' \times SG''$. We construct an IFS whose attractor is the graph of a function $f : SG' \times SG'' \to \mathbb{R}$ such that

$$f(y) = z_y \text{ for all } y \in V_N.$$ 

Let $X = SG' \times SG'' \times \mathbb{R}$ and define maps $W_{\omega\eta} : X \to X$ by

$$W_{\omega\eta}(t, s, x) = \left( L_\omega(t), K_\eta(s), F_{\omega\eta}(t, s, x) \right), \ \omega, \eta \in I^N,$$

where $F_{\omega\eta} : X \to \mathbb{R}$ is given by

$$F_{\omega\eta}(t, s, x) = \alpha_{\omega\eta}(t, s)x + h_{\omega\eta}(t, s), \ \omega, \eta \in I^N,$$

where $h_{\omega\eta} : SG' \times SG'' \to \mathbb{R}$ is a function such that

- For $\omega = \omega_1\omega_2 \ldots \omega_{N-1}i, \tau = \omega_1\omega_2 \ldots \omega_{N-1}j, h_{\omega\eta}(p_j, s) = h_{\omega\eta}(p_i, s)$ for all $s \in SG''$.
- For $\eta = \eta_1\eta_2 \ldots \eta_{N-1}i, \xi = \eta_1\eta_2 \ldots \eta_{N-1}j, h_{\omega\eta}(t, q_i) = h_{\omega\eta}(t, q_j)$ for all $t \in SG'$.
- For $\omega = \omega_1\omega_2 \ldots \omega_{N-1}i, \tau = \omega_1\omega_2 \ldots \omega_{N-1}j$, and $\eta = \eta_1\eta_2 \ldots \eta_{N-1}k, \xi = \eta_1\eta_2 \ldots \eta_{N-1}l$, $h_{\omega\eta}(p_j, q_i) = h_{\omega\eta}(p_i, q_k)$, for all $i, j, k, l \in I$.

and for each $\omega, \eta \in I^N$, $\alpha_{\omega\eta} : SG' \times SG'' \to \mathbb{R}$ is a continuous function with $\|\alpha_{\omega\eta}\|_\infty < 1$. Then $\{X; W_{\omega\eta}, \omega, \eta \in I^N\}$ is an IFS and admits an invariant set.

**Theorem 2.1.** The IFS $\{X; W_{\omega\eta}, \omega, \eta \in I^N\}$ defined above has a unique attractor $Gr(f)$. The set $Gr(f)$ is the graph of a continuous function $f : SG' \times SG'' \to \mathbb{R}$ such that $f(y) = z_y$ for all $y \in V_N$. The function $f$ is known as the Fractal Interpolation Function on the product $SG' \times SG''$.

**Proof.** We know that $\mathcal{C}(SG' \times SG'')$ the space of all continuous functions forms a Banach space with respect to supnorm. Let

$$\mathcal{C}_*(SG' \times SG'') = \left\{ g \in \mathcal{C}(SG' \times SG'') : g(y) = z_y, \ y \in V_N, g(p_i, s) = g(t, q_i) = 0, \right\}$$

$$\forall \ t \in SG', s \in SG'', i \in I.$$ 

Define a Read-Bajracterevic map $T : \mathcal{C}_*(SG' \times SG'') \to \mathcal{C}_*(SG' \times SG'')$ by

$$(Tg)(t, s) = F_{\omega\eta}(L_\omega^{-1}(t), K_\eta^{-1}(s), g(L_\omega^{-1}(t), K_\eta^{-1}(s))), \ \forall \ (t, s) \in L_\omega(SG') \times K_\eta(SG''),$$

for every $\omega, \eta \in I^N$.

We first show that $T$ is well-defined. For this, we need to check the continuity of $Tg$ at the following point :

- $(t, s) \in \left(L_\omega(SG') \cap L_\tau(SG')\right) \times K_\eta(SG'')$
- $(t, s) \in \left(L_\omega(SG') \times (L_\eta(SG'') \cap L_\xi(SG''))\right)$
- $(t, s) \in \left((L_\omega(SG') \cap L_\tau(SG')) \times (L_\eta(SG'') \cap L_\xi(SG''))\right)$. 

From the construction of $SG'$ and $SG''$, for any $t \in L_\omega(SG') \cap L_\tau(SG')$, one may get the following relation: $\omega = \omega_1 \omega_2 \ldots \omega_{N-1} i$, $\tau = \omega_1 \omega_2 \ldots \omega_{N-1} j$ and $t = L_\omega(p_j) = L_\tau(p_i)$ for some $i, j \in I$. Now, since $g(p_k, s) = 0, \forall s \in SG'', k \in I$, we have

$$(Tg)(t, s) = F_{\omega \eta}(L_\omega^{-1}(t), K_\omega^{-1}(s), g(L_\omega^{-1}(t), K_\omega^{-1}(s)))$$

$$= F_{\omega \eta}(p_j, K_\eta^{-1}(s), g(p_j, K_\eta^{-1}(s)))$$

$$= \alpha_{\omega \eta}(p_j, K_\eta^{-1}(s))g(p_j, K_\eta^{-1}(s)) + h_{\omega \eta}(p_j, K_\eta^{-1}(s))$$

$$= h_{\omega \eta}(p_j, K_\eta^{-1}(s)),$$

and

$$(Tg)(t, s) = F_{\tau \eta}(L_\tau^{-1}(t), K_\eta^{-1}(s), g(L_\tau^{-1}(t), K_\eta^{-1}(s)))$$

$$= F_{\tau \eta}(p_i, K_\eta^{-1}(s), g(p_i, K_\eta^{-1}(s)))$$

$$= \alpha_{\tau \eta}(p_i, K_\eta^{-1}(s))g(p_i, K_\eta^{-1}(s)) + h_{\tau \eta}(p_i, K_\eta^{-1}(s))$$

$$= h_{\tau \eta}(p_i, K_\eta^{-1}(s)).$$

With the help of conditions on $h_{\omega \eta}$, it follows that $Tg$ is continuous at $(t, s) \in (L_\omega(SG') \cap L_\tau(SG')) \times K_\eta(SG'')$. Other cases can be dealt similarly. Further, since $F_{\omega \eta}$ is continuous for each $\omega, \eta \in I^N$, $Tg$ is continuous, and $Tg \in C_*(SG' \times SG'')$. Hence, $T$ is well-defined. Now, let $g, h \in C_*(SG' \times SG'')$. Then we have

$$||(Tg)(t, s) - (Th)(t, s)|| = \left| F_{\omega \eta}(L_\omega^{-1}(t), K_\omega^{-1}(s), g(L_\omega^{-1}(t), K_\omega^{-1}(s))) - F_{\omega \eta}(L_\omega^{-1}(t), K_\omega^{-1}(s), h(L_\omega^{-1}(t), K_\omega^{-1}(s))) \right|$$

$$\leq ||\alpha_{\omega \eta}(t, s)||\left| g(L_\omega^{-1}(t), K_\omega^{-1}(s)) - h(L_\omega^{-1}(t), K_\omega^{-1}(s)) \right|$$

$$\leq ||\alpha_{\omega \eta}||_\infty ||g - h||_\infty.$$

This in turn yields

$$\|Tg - Th\|_\infty \leq ||\alpha_{\omega \eta}||_\infty ||g - h||_\infty.$$

Since $||\alpha_{\omega \eta}||_\infty < 1$, $T$ is a contraction. Using the Banach fixed point theorem, $T$ has a unique fixed point $f \in C_*(SG' \times SG'')$. Since $Tf = f$,

$$f(t, s) = F_{\omega \eta}(L_\omega^{-1}(t), K_\eta^{-1}(s), f(L_\omega^{-1}(t), K_\eta^{-1}(s))), \forall (t, s) \in L_\omega(SG') \times K_\eta(SG''), \quad (1)$$

for every $\omega, \eta \in I^N$. By using the proof of Theorem 3.8 in [12] (motivated by Barnsley [3]) we can show that attractor associated with mentioned IFS is the graph of the above function. Now, we show that it is the graph of FIF $f$. With the help of functional equation (1) and $SG' \times SG'' = \cup_{\omega, \eta \in I^N} L_\omega(SG') \times K_\eta(SG'')$, we have

$$\cup_{\omega, \eta \in I^N} W_{\omega \eta}(Gr(f)) = \cup_{\omega, \eta \in I^N} \left\{(L_\omega(t), K_\eta(s), f(L_\omega(x), K_\eta(s))) : x \in I \right\}$$

$$= \cup_{\omega, \eta \in I^N} \left\{(t, s, f(t, s)) : (t, s) \in L_\omega(SG') \times K_\eta(SG'') \right\}$$

$$= Gr(f),$$

completing the proof. \qed
Remark 2.2. The assumption \( g(p_i, s) = g(t, q_i) = 0, \forall \ t \in SG', \ s \in SG'', \ i \in I \) can be weakened by taking constant and same scaling functions, that is, if \( \alpha_{\omega_i} = \alpha \) then \( g(p_i, s) = g(p_j, s) \) and \( g(t, q_i) = g(t, q_j) \) for all \( \forall \ t \in SG', \ s \in SG'', \ i, j \in I \) will be sufficient to show that the mapping \( T \) is well-defined. It may hint us that we can have many different constructions of FIF on \( SG' \times SG'' \) as we have for fractal surfaces \([27,30,34]\).

3. Smoothness of Fractal Interpolation Functions on \( SG \times SG \)

In this section, we aim to discuss the smoothness property of the FIF \( f: SG' \times SG'' \rightarrow \mathbb{R} \) obtained through the IFS:

\[
L_\omega(t) = \frac{1}{2N} t + \sum_{k=1}^{N} \frac{1}{2^k} p_{\omega_k}; K_\eta(s) = \frac{1}{2N} s + \sum_{k=1}^{N} \frac{1}{2^k} q_{\eta_k}; \ F_{\omega \eta}(t, s) = \alpha_{\omega \eta}(t, s)x + h_{\omega \eta}(t, s), \ \omega \in I^N.
\]

Targeting to give a simplified presentation, one may introduce the notation \( a = \frac{1}{2N}, \ a_\omega = \sum_{k=1}^{N} \frac{1}{2^k} p_{\omega_k} \) and \( b_\omega = \sum_{k=1}^{N} \frac{1}{2^k} q_{\eta_k} \). Then we have \( L_\omega(t) = at + a_\omega, \ K_\eta(s) = as + b_\omega \) for \( \omega \in I^N \). In this section, for computational convenience, we consider \( \|(t, s) - (t', s')\|_2 \leq 1 \) for all \( (t, s), (t', s') \in SG' \times SG'' \).

We show that for suitable choices of the parameters, the fractal function \( f \) preserves the Hölder continuity of FIF \( h \).

For \( t \in SG' \) and \( \omega_j \in I^N \), let

\[
L_{\omega_1 \omega_2 \ldots \omega_m}(t) = L_{\omega_1} \circ L_{\omega_2} \circ \cdots \circ L_{\omega_m}(t), \ L_{\omega_1 \omega_2 \ldots \omega_m}(SG') = L_{\omega_1} \circ L_{\omega_2} \circ \cdots \circ L_{\omega_m}(SG').
\]

Define a shift operator \( \sigma \) by

\[
\sigma(\omega_1 \omega_2 \ldots \omega_m) = (\omega_2 \omega_3 \ldots \omega_m).
\]

Let \( \sigma^k \) denote the \( k \)-fold autocomposition of \( \sigma \) such that for \( 1 \leq k \leq m-1 \), \( L_{\sigma^k(\omega_1 \omega_2 \ldots \omega_m)}(t) = L_{\omega_1 \omega_2 \ldots \omega_m}(t) \) otherwise \( L_{\sigma^k(\omega_1 \omega_2 \ldots \omega_m)}(t) = t \). With the help of the successive iteration and induction, we may prove the following lemma; see also \([37]\).

Lemma 3.1. Let \( f \) be a fractal function corresponding to the aforementioned IFS. For any \( (t, s) \in SG' \times SG'' \) and \( \omega_j \in I^N \),

\[
L_{\omega_1 \omega_2 \ldots \omega_m}(t) = a^m t + \sum_{k=1}^{m} a^{k-1} a_{\omega_k} \text{ and } K_{\eta_1 \eta_2 \ldots \eta_m}(s) = a^m s + \sum_{k=1}^{m} a^{k-1} b_{\eta_k}
\]

and

\[
f(L_{\omega_1 \omega_2 \ldots \omega_m}(t), K_{\eta_1 \eta_2 \ldots \eta_m}(s))
\]

\[
= \left( \prod_{k=1}^{m} \alpha_{\omega_k \eta_k}(L_{\sigma^k(\omega_1 \omega_2 \ldots \omega_m)}(t), K_{\sigma^k(\eta_1 \eta_2 \ldots \eta_m)}(s)) \right) f(t, s)
\]

\[
+ \sum_{r=1}^{m} \left( \prod_{k=1}^{r-1} \alpha_{\omega_k \eta_k}(L_{\sigma^k(\omega_1 \omega_2 \ldots \omega_m)}(t), K_{\sigma^k(\eta_1 \eta_2 \ldots \eta_m)}(s)) \right) h_{\omega_r \eta_r}(L_{\sigma^{r-1}(\omega_1 \omega_2 \ldots \omega_m)}(t), K_{\sigma^{r-1}(\eta_1 \eta_2 \ldots \eta_m)}(s)),
\]
where
\[
L_{\sigma}^{k}(\omega_{1}\omega_{2}\ldots\omega_{m})(t) = a^{m-k}t + \sum_{l=1}^{m-k} a^{l-1} a\omega_{k+l} \quad \text{and} \quad K_{\sigma}^{k}(\eta_{1}\eta_{2}\ldots\eta_{m})(s) = a^{m-k}s + \sum_{l=1}^{m-k} a^{l-1} b\omega_{k+l}.
\]

Let us now state the next result borrowed from [37].

**Lemma 3.2.** Let \( r_i, q_i, \ i = 1, 2, \ldots, m, \) be given real numbers. Then
\[
\prod_{i=1}^{m} r_i - \prod_{i=1}^{m} q_i = \sum_{i=1}^{m} \left( \prod_{k=1}^{m-1} c_k^{(i)} \right) (r_i - q_i),
\]
where, for all \( k = 1, 2, \ldots, m - 1, \) each \( c_k^{(i)}, i = 1, 2, \ldots, m, \) is a real number from the set
\[
\{ r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_m, q_1, q_2, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m \}.
\]

**Theorem 3.3.** Let the parameter maps \( h_{\omega}, \alpha_{\eta} \ (\omega, \eta \in I^N) \) be Hölder continuous functions with constants \( K_h, K_\alpha \) and Hölder exponents \( s_h, s_\alpha \) respectively. Define
\[
\|\alpha\|_{\infty} = \max\{\|\alpha_{\omega}\|_{\infty} : \omega, \eta \in I^N\}, \quad \delta = \frac{\|\alpha\|_{\infty}}{a} \quad \text{and} \quad s_\alpha = \min\{s_{\alpha_{\omega}} : \omega, \eta \in I^N\}.\]
Then the following hold:

1. If \( \|\alpha\|_{\infty} < \frac{1}{2^N} \), then \( f \) is a Hölder continuous function with exponent \( s \), where \( s = \min\{s_h, s_\alpha\} \).
2. If \( \|\alpha\|_{\infty} = \frac{1}{2^N} \), then \( f \) is a Hölder continuous function with exponent \( s - \mu \) for some \( 0 < \mu < 1 \).
3. If \( \|\alpha\|_{\infty} > \frac{1}{2^N} \), then \( f \) is a Hölder continuous function with exponent \( \lambda \), where \( 0 < \lambda \leq s - 1 + \frac{\ln \|\alpha\|_{\infty}}{\ln a} < 1 \).

**Proof.** For any \((t, s), (t', s') \in SG' \times SG''\), it is possible to find \( m \geq 0 \) such that \( t \in L_{\omega_{1}\omega_{2}\ldots\omega_{m}}(SG') \), \( s \in K_{\eta_{1}\eta_{2}\ldots\eta_{m}}(SG'') \) and
\[
a^{m+1} \leq \|(t, s) - (t', s')\|_{2} \leq a^{m}.
\]
If \( m = 0 \), we set \( L_{\omega_{1}\omega_{2}\ldots\omega_{m}}(SG') = SG' \). Let \( t, t' \in L_{\omega_{1}\omega_{2}\ldots\omega_{m}}(SG') \). Since \( t \in L_{\omega_{1}\omega_{2}\ldots\omega_{m}}(SG') \), \( s \in K_{\eta_{1}\eta_{2}\ldots\eta_{m}}(SG'') \) there exists elements \( \bar{t} \in SG', \bar{s} \in SG'' \) such that the following conditions hold due to Lemma 3.1,
\[
t = L_{\omega_{1}\omega_{2}\ldots\omega_{m}}(\bar{t}) = a^{m} \bar{t} + \sum_{h=1}^{m} a^{h-1} a_{\omega_{h}}, \quad s = K_{\eta_{1}\eta_{2}\ldots\eta_{m}}(\bar{s}) = a^{m} \bar{s} + \sum_{h=1}^{m} a^{h-1} b_{\eta_{h}} \quad (2)
\]
and
\[
f(t, s) = f(L_{\omega_{1}\omega_{2}\ldots\omega_{m}}(\bar{t}), K_{\eta_{1}\eta_{2}\ldots\eta_{m}}(\bar{s}))
\]
\[
= \left( \prod_{k=1}^{m} \alpha_{\omega_{k}\eta_{h}}(L_{\sigma}^{k}(\omega_{1}\omega_{2}\ldots\omega_{m})(t), K_{\sigma}^{k}(\eta_{1}\eta_{2}\ldots\eta_{m})(s)) \right) f(\bar{t}, \bar{s})
\]
\[
+ \sum_{r=1}^{m} \left( \prod_{k=1}^{r-1} \alpha_{\omega_{k}\eta_{h}}(L_{\sigma}^{k}(\omega_{1}\omega_{2}\ldots\omega_{m})(\bar{t}), K_{\sigma}^{k}(\eta_{1}\eta_{2}\ldots\eta_{m})(\bar{s})) \right) h_{\omega_{r}\eta_{r}}(L_{\sigma^{r-1}}(\omega_{1}\omega_{2}\ldots\omega_{m})(\bar{t}), K_{\sigma^{r-1}}(\eta_{1}\eta_{2}\ldots\eta_{m})(\bar{s})).
\]
From (2), we may write \( \tilde{t} \) and \( \tilde{s} \) as follows:

\[
\tilde{t} = a^{-m} \left[ t - \sum_{l=1}^{m} a^{l-1} a_{\omega_l} \right] \quad \text{and} \quad \tilde{s} = a^{-m} \left[ s - \sum_{l=1}^{m} a^{l-1} b_{\eta_l} \right].
\]

In view of Lemma 3.1 and the above equation, we get

\[
L^{\alpha_k}(\omega_1 \omega_2 \ldots \omega_m) (\tilde{t}) = a^{-k} \left[ t - \sum_{l=1}^{m} a^{l-1} a_{\omega_l} \right] + \sum_{l=1}^{m-k} a^{l-1} a_{\omega_{k+l}}
\]

and

\[
K^{\alpha_k}(\eta_1 \eta_2 \ldots \eta_m) (\tilde{s}) = a^{-k} \left[ s - \sum_{l=1}^{m} a^{l-1} b_{\eta_l} \right] + \sum_{l=1}^{m-k} a^{l-1} b_{\eta_{k+l}}.
\]

Similarly, since \((t', s') \in L^{\alpha_k}(\omega_1 \omega_2 \ldots \omega_m) (SG') \times K^{\alpha_k}(\eta_1 \eta_2 \ldots \eta_m) (SG'')\), there exists \((\tilde{t}', \tilde{s}') \in SG' \times SG''\) such that \(t', s'\) and \(f(t', s')\) have expressions similar to the above. Consequently,

\[
\begin{align*}
& \left| f(t, s) - f(t', s') \right| \\
& \leq \left\| \left( \prod_{k=1}^{m} \alpha_{\omega_k \eta_k} \left( L^{\alpha_k}(\omega_1 \omega_2 \ldots \omega_m) (\tilde{t}), K^{\alpha_k}(\eta_1 \eta_2 \ldots \eta_m) (\tilde{s}) \right) \right) f(\tilde{t}, \tilde{s}) \right\| \\
& - \left. \left( \prod_{k=1}^{m} \alpha_{\omega_k \eta_k} \left( L^{\alpha_k}(\omega_1 \omega_2 \ldots \omega_m) (\tilde{t}), K^{\alpha_k}(\eta_1 \eta_2 \ldots \eta_m) (\tilde{s}) \right) \right) \right| f(\tilde{t}, \tilde{s}) \right\|_2 \\
& + \sum_{r=1}^{m} \left| \left( \prod_{k=1}^{r-1} \alpha_{\omega_k \eta_k} \left( L^{\alpha_k}(\omega_1 \omega_2 \ldots \omega_m) (\tilde{t}), K^{\alpha_k}(\eta_1 \eta_2 \ldots \eta_m) (\tilde{s}) \right) \right) h_{\omega_r \eta_r} \left( L^{\alpha_{r-1}}(\omega_1 \omega_2 \ldots \omega_m) (\tilde{t}), K^{\alpha_{r-1}}(\eta_1 \eta_2 \ldots \eta_m) (\tilde{s}) \right) \right| \\
& - \left( \prod_{k=1}^{r-1} \alpha_{\omega_k \eta_k} \left( L^{\alpha_k}(\omega_1 \omega_2 \ldots \omega_m) (\tilde{t}), K^{\alpha_k}(\eta_1 \eta_2 \ldots \eta_m) (\tilde{s}) \right) \right) h_{\omega_r \eta_r} \left( L^{\alpha_{r-1}}(\omega_1 \omega_2 \ldots \omega_m) (\tilde{t}), K^{\alpha_{r-1}}(\eta_1 \eta_2 \ldots \eta_m) (\tilde{s}) \right) \right|_2 \\
& + \sum_{r=1}^{m} \left| \left( \prod_{k=1}^{r-1} \alpha_{\omega_k \eta_k} \left( L^{\alpha_k}(\omega_1 \omega_2 \ldots \omega_m) (\tilde{t}), K^{\alpha_k}(\eta_1 \eta_2 \ldots \eta_m) (\tilde{s}) \right) \right) \right| h_{\omega_r \eta_r} \left( L^{\alpha_{r-1}}(\omega_1 \omega_2 \ldots \omega_m) (\tilde{t}), K^{\alpha_{r-1}}(\eta_1 \eta_2 \ldots \eta_m) (\tilde{s}) \right) \right|_2 \\
& - h_{\omega_r \eta_r} \left( L^{\alpha_{r-1}}(\omega_1 \omega_2 \ldots \omega_m) (\tilde{t}), K^{\alpha_{r-1}}(\eta_1 \eta_2 \ldots \eta_m) (\tilde{s}) \right) \right|_2 \\
& + \sum_{r=1}^{m} \left| \left( \prod_{k=1}^{r-1} \alpha_{\omega_k \eta_k} \left( L^{\alpha_k}(\omega_1 \omega_2 \ldots \omega_m) (\tilde{t}), K^{\alpha_k}(\eta_1 \eta_2 \ldots \eta_m) (\tilde{s}) \right) \right) \right| \\
& - \left( \prod_{k=1}^{r-1} \alpha_{\omega_k \eta_k} \left( L^{\alpha_k}(\omega_1 \omega_2 \ldots \omega_m) (\tilde{t}), K^{\alpha_k}(\eta_1 \eta_2 \ldots \eta_m) (\tilde{s}) \right) \right) \right|_2 \\
& + 2 \left( \| \alpha \|_\infty \right)^m \| f \|_\infty,
\end{align*}
\]
Set $s = \min \{s_h, s_a\}$. Using $\|L_{\sigma^r(\omega_1, \omega_2, \ldots, \omega_m)}(t) - L_{\sigma^r(\omega_1, \omega_2, \ldots, \omega_m)}(\bar{t})\|_2 = a^{-r} \|t - t'\|_2$ and Hölder continuity of $h_{\omega, \eta}$, we have

$$
\left\| h_{\omega, \eta} \left( L_{\sigma^{r-1}(\omega_1, \omega_2, \ldots, \omega_m)}(t), K_{\sigma^{r-1}(\eta_1, \eta_2, \ldots, \eta_m)}(\bar{s}) \right) - h_{\omega, \eta} \left( L_{\sigma^{r-1}(\omega_1, \omega_2, \ldots, \omega_m)}(\bar{t}), K_{\sigma^{r-1}(\eta_1, \eta_2, \ldots, \eta_m)}(\bar{s}') \right) \right\|_2 \\
\leq K_h \left\| \left( L_{\sigma^{r-1}(\omega_1, \omega_2, \ldots, \omega_m)}(t), K_{\sigma^{r-1}(\eta_1, \eta_2, \ldots, \eta_m)}(\bar{s}) \right) - \left( L_{\sigma^{r-1}(\omega_1, \omega_2, \ldots, \omega_m)}(\bar{t}), K_{\sigma^{r-1}(\eta_1, \eta_2, \ldots, \eta_m)}(\bar{s}') \right) \right\|_2^{s_h} \\
= K_h \left[ a^{-(r-1)} \| (t, s) - (t', s') \|_2 \right]^{s_h} \\
\leq K_h a^{-(r-1)} \| (t, s) - (t', s') \|_2.
$$

Now, let $K_\alpha = \max\{K_{\alpha, \eta} : \omega, \eta \in \mathbb{N}^L\}$. Since $\alpha_{\omega, \eta}$ is Hölder continuous with exponent $s_{\alpha, \omega, \eta}$ and Hölder constant $K_{\alpha, \omega, \eta}$, in view of Lemma 3.2, for $r \geq 2$, we have

$$
\left| \prod_{k=1}^{r-1} a_{\omega_k, \eta_k} \left( L_{\sigma^k(\omega_1, \omega_2, \ldots, \omega_m)}(t), K_{\sigma^k(\eta_1, \eta_2, \ldots, \eta_m)}(\bar{s}) \right) - \prod_{k=1}^{r-1} a_{\omega_k, \eta_k} \left( L_{\sigma^k(\omega_1, \omega_2, \ldots, \omega_m)}(\bar{t}), K_{\sigma^k(\eta_1, \eta_2, \ldots, \eta_m)}(\bar{s}') \right) \right| \\
\leq \sum_{k=1}^{r-1} (\|\alpha\|_\infty)^{r-2} K_\alpha \left| \left( L_{\sigma^k(\omega_1, \omega_2, \ldots, \omega_m)}(t), K_{\sigma^k(\eta_1, \eta_2, \ldots, \eta_m)}(\bar{s}) \right) - \left( L_{\sigma^k(\omega_1, \omega_2, \ldots, \omega_m)}(\bar{t}), K_{\sigma^k(\eta_1, \eta_2, \ldots, \eta_m)}(\bar{s}') \right) \right|^{s_\alpha} \\
\leq \sum_{k=1}^{r-1} (\|\alpha\|_\infty)^{r-2} K_\alpha \left[ a^{-k} \| (t, s) - (t', s') \|_2 \right]^{s_\alpha} \\
\leq K_\alpha (\|\alpha\|_\infty)^{r-2} \| (t, s) - (t', s') \|_2 s_\alpha a^{-r} \\
= \frac{K_\alpha}{a(1-a)} \| (t, s) - (t', s') \|_2^s.
$$

Using the above estimates, we get

$$
|f(t, s) - f(t', s')| \leq \sum_{r=1}^m (\|\alpha\|_\infty)^{r-1} K_h a^{-(r-1)} \| (t, s) - (t', s') \|_2^s \\
+ \sum_{r=1}^m K_\alpha \|h\|_\infty \delta^{r-2} \| (t, s) - (t', s') \|_2^s + 2 (\|\alpha\|_\infty)^m \|f\|_\infty \\
= \sum_{r=1}^m (\|\alpha\|_\infty)^{r-1} K_h a^{-(r-1)} \| (t, s) - (t', s') \|_2^s \\
+ \sum_{r=2}^m K_\alpha \|h\|_\infty \delta^{r-2} \| (t, s) - (t', s') \|_2^s + 2 \|f\|_\infty \delta^m a^m.
$$
Since $a^{m+1} \leq \|(t, s) - (t', s')\|_2$, we have
\[
\|f(t, s) - f(t', s')\|_2 \leq \sum_{r=1}^{m} \delta^{(r-1)}K_h\|(t, s) - (t', s')\|_2 + \sum_{r=2}^{m} \frac{K_h}{a(1-a)}\|t, s\|_\infty \delta^{r-2} + \frac{2\|f\|_\infty \delta^m}{a}\|(t, s) - (t', s')\|_2
\]
\[
\leq 4K\|(t, s) - (t', s')\|_2 \sum_{r=1}^{m+1} \delta^{r-1},
\]
for a fixed suitable constant $K$.

Case 1: If $\delta < 1$, then we have
\[
\sum_{r=1}^{m+1} \delta^{r-1} \leq \frac{1}{1-\delta}
\]
and hence we obtain
\[
|f(t, s) - f(t', s')| \leq \frac{4K}{1-\delta}\|(t, s) - (t', s')\|_2^s.
\]
That is, $f$ is Hölder continuous with exponent $s$.

Case 2: If $\delta = 1$, then
\[
\|f(t, s) - f(t', s')\|_2 \leq 4K(m+1)\|(t, s) - (t', s')\|_2^s.
\]
Since $\|(t, s) - (t', s')\|_2 \leq a^m < 1$, we get $m \leq \ln\|(t, s) - (t', s')\|_2 / \ln(a)$. With the help of a known inequality:
\[
0 < -x^\mu \ln x \leq \frac{1}{\mu e} \quad \text{for } 0 < x \leq 1 \text{ and } 0 < \mu < 1,
\]
we select a suitable $0 < \mu < 1$ such that
\[
(m+1)\|(t, s) - (t', s')\|_2^s \leq \left(1 + \frac{\ln\|(t, s) - (t', s')\|_2}{\ln a}\right)\|(t, s) - (t', s')\|_2^s
\]
\[
= \|(t, s) - (t', s')\|_2^s + \frac{-\|(t, s) - (t', s')\|_2^\mu \ln\|(t, s) - (t', s')\|_2}{\ln a}\|(t, s) - (t', s')\|_2^{s-\mu}
\]
\[
\leq \|(t, s) - (t', s')\|_2^s + \frac{1}{\mu e \ln a}\|(t, s) - (t', s')\|_2^{2-\mu}
\]
\[
\leq \left(1 + \frac{1}{\mu e \ln a}\right)\|(t, s) - (t', s')\|_2^{s-\mu}.
\]
Hence, we have
\[
|f(t, s) - f(t', s')| \leq 4K\left(1 + \frac{1}{\mu e \ln a}\right)\|(t, s) - (t', s')\|_2^{2-\mu}.
\]
Case 3: If $\delta > 1$, then

$$|f(t, s) - f(t', s')| \leq 4K \frac{\delta^{m+1}}{\delta - 1} \|\|(t, s) - (t', s')\|_2^s.$$  

We now consider a positive number $\lambda$ with $0 < \lambda < 1$ such that

$$\delta^{m+1} \|\|(t, s) - (t', s')\|_2^s \leq \|\|(t, s) - (t', s')\|_2^\lambda.$$  

Further, one obtains

$$\lambda \leq s + \frac{(m + 1) \ln \delta}{\ln(\|\|(t, s) - (t', s')\|_2^s)}.$$  

Since $a^{m+1} \leq \|\|(t, s) - (t', s')\|_2^s$, we obtain

$$\frac{1}{\ln \|\|(t, s) - (t', s')\|_2^s} \leq \frac{1}{(m + 1) \ln(a)}.$$  

This in turn yields

$$\lambda \leq s + \frac{\ln \delta}{\ln a} = s - 1 + \frac{\ln \|\|\alpha\|\|_\infty}{\ln a} < 1.$$  

Therefore, we get

$$|f(t, s) - f(t', s')| \leq \frac{4K}{\delta - 1} \|\|(t, s) - (t', s')\|_2^\lambda.$$  

Thus, the proof of the theorem is completed. \qed

4. Dimension of FIF

In this section, we focus on estimating fractal dimension such as Hausdorff dimension and box dimension of the constructed FIFs. Here, as usual we denote by $\dim_H(A)$ and $\dim_B(A)$ the Hausdorff dimension and box dimension of a set $A$ respectively. For more details and definitions of these dimensions, we encourage the reader to see the book [11] of Falconer.

**Theorem 4.1.** Under the hypotheses of Theorem 3.3, if $\|\alpha\|_\infty < \frac{1}{2^n}$ then

$$2 \frac{\log 3}{\log 2} \leq \dim_H(Gr(f)) \leq \dim_B(Gr(f)) \leq 1 + 2 \frac{\log 3}{\log 2}.$$  

**Proof.** Let us first define the maximum range of the function $f$ over $X \subseteq SG' \times SG''$ as $R_f[X] = \sup_{(t,s),(t',s') \in X} |f(t, s) - f(t', s')|$. In view of Theorem 3.3, $f$ satisfies Hölder condition, we have

$$R_f[L_\omega(SG') \times K_\eta(SG'')] \leq \frac{c}{2^{ns}}$$  

for every $\omega, \eta \in \mathbb{I}^n$, where $c$ is suitable constant depending the Hölder constant of $f$ and $s = \min\{s_h, s_\alpha\}$. Suppose that $\delta = \frac{1}{2^n}$ for some $n \in \mathbb{N}$. If $N_\delta(Gr(f))$ denotes the number of $\delta-$cubes that intersect graph of $f$, then

$$N_\delta(Gr(f)) \leq 2.3^n \cdot 3^n + 2^n \sum_{\omega, \eta \in \mathbb{I}^n} R_f[L_\omega(SG') \times K_\eta(SG'')]$$.
We obtain \( N_\delta(Gr(f)) \leq 2.3^n.3^n + c2^n(1-s)3^n.3^n \). Upper box-dimension of \( Gr(f) \) can be estimated in the following way

\[
\lim_{\delta \to 0} \frac{\log N_\delta(Gr(f))}{-\log \delta} \leq \lim_{n \to \infty} \frac{\log(2.3^n.3^n + c2^n(1-s)3^n.3^n)}{\log 2^n}
\]

which produces \( \lim_{\delta \to 0} \frac{\log N_\delta(Gr(f))}{-\log \delta} \leq 1 - s + 2 \frac{\log 3}{\log 2} \).

For the lower bound, we begin by defining a mapping \( T : Gr(f) \to SG' \times SG'' \) by

\[
T((t, s, f(t, s))) = (t, s).
\]

Then

\[
\|T((t, s, f(t, s))) - T((t', s', f(t', s')))\|_2 = \|(t, s) - (t', s')\|_2
\]

\[
\leq \sqrt{\|(t, s) - (t', s')\|_2^2 + (f(t, s) - f(t', s'))^2}
\]

\[
= \|(t, s, f(t, s)) - (t', s', f(t', s'))\|_2,
\]

implies that \( T \) is Lipschitz. Now, for any \((t, s) \in SG' \times SG''\), we can choose \((t, s, f(t, s)) \in Gr(f)\) such that \((t, s) = T((t, s, f(t, s)))\) which implies the surjectiveness of \( T \). Recall (Cf. [11, Corollary 2.4(a)]) that if \( g : A \subseteq \mathbb{R}^n \to \mathbb{R}^m \) is a Lipschitz function, then \( \dim_H(g(A)) \leq \dim_H(A) \). Using [11, Corollary 7.4], it is immediate that \( \dim_H(SG' \times SG'') = 2 \frac{\log 3}{\log 2} \). In view of the previous two observations, we get

\[
\dim_H(Gr(f)) \geq \dim_H(T(Gr(f))) = \dim_H(SG' \times SG'') = 2 \frac{\log 3}{\log 2},
\]

proving the assertion. □

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