SPECTRA OF STRONG MAGNETOHYDRODYNAMIC TURBULENCE FROM HIGH-RESOLUTION SIMULATIONS

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ABSTRACT

Magnetohydrodynamic (MHD) turbulence is present in a variety of solar and astrophysical environments. Solar wind fluctuations with frequencies lower than 0.1 Hz are believed to be mostly governed by Alfvénic turbulence with particle transport depending on the power spectrum and the anisotropy of such turbulence. Recently, conflicting spectral slopes for the inertial range of MHD turbulence have been reported by different groups. Spectral shapes from earlier simulations showed that MHD turbulence is less scale-local compared with hydrodynamic turbulence. This is why higher-resolution simulations, and careful and rigorous numerical analysis is especially needed for the MHD case. In this Letter, we present two groups of simulations with resolution up to 4096^3, which are numerically well-resolved and have been analyzed with an exact and well-tested method of scaling study. Our results from both simulation groups indicate that the asymptotic power spectral slope for all energy-related quantities, such as total energy and residual energy, is around −1.7, close to Kolmogorov’s −5/3. This suggests that residual energy is a constant fraction of the total energy and that in the asymptotic regime of Alfvénic turbulence magnetic and kinetic spectra have the same scaling. The −1.5 slope for energy and the −2 slope for residual energy, which have been suggested earlier, are incompatible with our numerics.

Key words: magnetohydrodynamics (MHD) – turbulence

1. INTRODUCTION

Most astrophysical, stellar, and space plasmas are well-ionized and well-conductive. On large scales they are often described as an ideal magnetohydrodynamic (MHD) fluid—a perfectly conducting, inviscid fluid described by MHD equations. An initially unmagnetized well-conductive turbulent fluid generates its own magnetic field which becomes dynamically important on almost all relevant scales. The presence of a large-scale field, however, is qualitatively different from the presence of large-scale flows in hydrodynamics that can be excluded by the choice of reference frame. The inertial range of MHD turbulence, therefore, has to be dominated by the large-scale mean magnetic field, which is known as a strong field limit. Initial investigations of the strong field limit (Iroshnikov 1964; Kraichnan 1965) prematurely concluded that inertial-range MHD turbulence has to be weak turbulence, which happened not to be the case. The success of analytic weak turbulence theories (Ng & Bhattacharjee 1997; Galtier et al. 2000) demonstrated that MHD turbulence has a tendency to become stronger, not weaker, during the cascade. Similar arguments led Goldreich & Sridhar (1995; hereafter GS95) to conclude that the inertial range of MHD turbulence has to be the so-called strong critically balanced anisotropic cascade, which was tentatively confirmed in many earlier simulations of MHD turbulence, e.g., Cho & Vishniac (2000) and Maron & Goldreich (2001).

The properties of strong field anisotropic cascade can be rigorously argued to be governed by the Alfvénic part of MHD perturbations. This regime has been dubbed Alfvénic turbulence. The Alfvénic component evolves according to the so-called reduced MHD (RMHD; Kadomtsev & Pogutse 1974; Strauss 1976), which has been known in plasma physics for a long time and can be justified based on plasma drift approximation alone, without resorting to collisions (Schekochihin et al. 2009). The full compressible MHD also has fast mode cascade (Cho et al. 2003) which will not be considered here. Reduced MHD has an inherent symmetry, similar to hydrodynamic symmetry, which allows us to argue that the power-law scaling of turbulent spectra is indeed possible in the strong mean field and strong anisotropy limit, e.g., in the inertial range (Beresnyak 2012a).

While previous numerical work confirmed scale-dependent anisotropy of strong MHD turbulence, the precise value of the spectral slope was a matter of debate. As earlier simulations (Maron & Goldreich 2001; Müller & Grappin 2005) reported slopes shallower than the GS95’s standard −5/3, some adjustments have been proposed to accommodate this difference (Galtier et al. 2005; Boldyrev 2005; Gogoberidze 2007). A model with the so-called “dynamic alignment” (Boldyrev 2005, 2006) has been especially popular. Earlier simulations indicated that MHD turbulence is fairly scale-nonlocal (Beresnyak & Lazarian 2009). Also we observed that the scaling of the “alignment” tends to flatten out, and the energy spectrum restores its −5/3 scaling at sufficiently high Reynolds numbers (Re; Beresnyak 2011, 2012a). Another theoretical challenge was a different scaling of kinetic and magnetic energies in the strong field limit, with the magnetic energy typically being higher. The so-called residual energy, $E_R = E_k$, scaled approximately as $k^{-2}$ (Müller & Grappin 2005), which was problematic to explain in the framework of local cascade.

To carefully investigate these issues, we (1) performed simulations of RMHD turbulence with resolution up to 4096^3, (2) used well-resolved, numerically precise data, and (3) used a rigorous quantitative argument known as a scaling study to provide a “yes/no” test of any hypothesis of universal scaling.

2. SCALING STUDY

Kolmogorov (1941) suggested that if strong turbulence is universal and its scaling is only determined by the dissipation rate and viscosity, the dissipative range would have certain spatial, velocity, and timescales, known as Kolmogorov scales. This has been tested with a number of experimental and/or numerical data being expressed in units of these scales and presented on the same plot; see, e.g., Sreenivasan (1995) and...
et al. (2003), using a simulation group up to 40963 resolution, and best measured quantities in numerics. In particular, Kaneda Kolmogorov amplitude of the spectra is one of the most robust driving scale and the least affected by driving, but also has scale, therefore, is not only the most separated from the eddies in the highest-resolution simulations. The dissipation velocity should scale as Re

Technically, the scaling study investigates the scaling of all data on the same curve, validating Kolmogorov’s conjecture. The Astrophysical Journal Letters

structure function.

to volume and time-averaging goes as

e.g.,

is the energy dissipation rate. Checking the hypothesis that the

perpendicular box size of 2

are explicitly. If one wants to multiply the subjectively perceived flatness of the spectra to determine precision of the method, our work we aim to differentiate between the transitional scalings that looks subjectively flat.

Given a characteristic eddy scale \( l \), the number of eddies in a datacube goes as \( l^{-3} \), while the number of correlation timescales for strong turbulence goes as \( l^{-2/3} \), so the statistical error due to volume and time-averaging goes as \( l^{-1/6} \), which is about a factor of \( 10^{-4} \) between outer scale eddies and viscously damped eddies in the highest-resolution simulations. The dissipation scale, therefore, is not only the most separated from the driving scale and the least affected by driving, but also has the smallest statistical error. In combination with the very low numerical error of the pseudospectral method (see below), the Kolmogorov amplitude of the spectra is one of the most robust and best measured quantities in numerics. In particular, Kaneda et al. (2003), using a simulation group up to 40963 resolution, has been able to estimate the power slope of hydrodynamic turbulence within very small error and differentiate between slopes of \( -5/3 ≈ -1.667 \) and intermittency-corrected \(-1.7 \) in our work we aim to differentiate between the \(-3/2 \) and \(-5/3 \) slopes, which are different by \( \approx 0.167 \), much higher than the precision of the method, \( \approx 0.02 \). This is much better than using the subjectively perceived flatness of the spectra to determine the asymptotic scaling, as it could easily fail, e.g., due to the transitional scalings that looks subjectively flat.

The Kolmogorov scale can be expressed as

\[
\eta = \left( \frac{\nu^3}{\epsilon} \right)^{1/3(\sigma + i + 1)},
\]

where \( n \) is the order of dissipation, \( i \) is the spectral index, e.g., \(-5/3 \), \( \nu \) is the viscosity or magnetic diffusivity, and \( \epsilon \) is the energy dissipation rate. Checking the hypothesis that the Kolmogorov scale and the Kolmogorov velocity scale properly with the Re requires plotting the spectrum in Kolmogorov units, i.e., making the x- and y-axes dimensionless. The x-axis is expressed in \( k \eta \), where \( \eta \) is not necessarily the classic Kolmogorov scale, corresponding to \(-5/3 \) slope, but defined by the above formula, i.e., different for each spectral slope. The y-axis is usually expressed in units of \( E(k) k^{5/3+\sigma} L^{\sigma} e^{-2/3} \), where \( \sigma \) is the correction to the \(-5/3 \) slope and \( L \) is an outer scale, which is normally kept constant in a scaling study. This is, in fact, the only dimensionless expression for the spectrum that does not contain \( \eta \) explicitly. If one wants to multiply the above expression by some power of \((L/\eta) \) or Re, this would introduce explicit \( \eta \) dependence and would violate the so-called zeroth law of turbulence that claims that large-scale properties are largely independent of viscosity. We also note that the scaling study works perfectly well not only with spectra, but with other measures. In particular, we reproduced the results reported below by using the second order perpendicular structure function.

3. NUMERICS

We performed two series of RMHD-driven simulations with a strong mean field \( B_0 \) in code units, rms fields \( \nu_{rms} ≈ B_{rms} ≈ 1 \), perpendicular box size of \( 2\pi \), and parallel box size of \( 2\pi B_0 \). The driving was correspondingly anisotropic with anisotropy \( B_0/\sqrt{v_{rms}} \), so that turbulence starts strong from the outer scale. Technically, \( B_0 \) is arbitrary; however, the RMHD limit is only applicable to very large \( B_0 \) (Beresnyak 2012a). Our previous simulations showed rapid decrease of the parallel correlation length right after the driving scale, which indicates efficiency of nonlinear interaction and the regime of strong turbulence. The correlation timescale for \( \nu \) and \( B \) was around \( \tau \approx 0.97 \), so the box contained roughly 6.5 parallel correlation lengths in the parallel direction and about 3–5 in the perpendicular direction. Each simulation was started from long-evolved low-resolution simulation, and was subsequently evolved for \( \Delta t = 13.5 \) in high resolution. Overall, our setup is very similar to our previous simulations (Beresnyak 2011, 2012a) with the exception for driving that was limited to lower \( k < 1.42 \) wavenumbers in these simulations. This Letter mostly deals with the issue of perpendicular spectrum, which has been hotly debated. The measurements of anisotropy, however, were mostly consistent; see Beresnyak (2012a) for the latest data. We will present parallel structure functions for the present simulations in a future publication. We used the last seven dynamical times for averaging. In our previous simulations, we found that averaging over \( \sim 7 \) correlation timescales gives a reasonably good statistic on the outer scale and excellent statistics on smaller scales (see the above estimates). The simulation parameters are listed in Table 1. Numerically, we used the \( k_{\text{max}} \eta \) > 1 resolution criterion, with \( \eta \) being the classic Kolmogorov scale, which has been shown to be sufficient in normal viscous simulations (e.g., Gotof et al. 2002) and was a better resolution than the one used in Perez et al. (2012).

For the hyperdiffusive series we used the same criterion. Additionally, we checked the numerical precision of the spectra by performing a resolution study on lower resolutions. In particular we saw spectral errors lower than \( 8 \times 10^{-3} \) up to \( k\eta = 0.5 \) when increasing the resolution from \( 576^3 \) to \( 960^3 \) and spectral errors lower than \( 3 \times 10^{-3} \) when we increased the parallel resolution in a 11523 simulation by a factor of two. We did not use any data above \( k\eta = 0.5 \) for fitting as the spectrum sharply declines after this point and contains negligible energy. We conclude that for our purposes, using \( k_{\text{max}} \eta \) = 1 is sufficient, and using cubic resolution, i.e., parallel resolution equal to perpendicular resolution, is also sufficient or even somewhat excessive. Note that increasing the resolution while keeping \( k_{\text{max}} \eta \) > 1 with \( \eta \) corresponding to the \(-5/3 \) slope is a conservative choice for all types of turbulence with slopes shallower than \(-5/3 \), including the \(-3/2 \) model. Table 1 also lists \( k_{\text{max}} \eta \) with \( \eta \) corresponding to the \(-3/2 \) model.

Figure 1 presents a convergence test for the \(-5/3 \) model and the convergence is reasonable, while the best convergence is reached at the \(-1.7 \) scaling. Figure 2 has a test for the \(-3/2 \) model and the convergence is poor in either low or high wavenumbers. The former is due to the scaling being actually quite shallower than \(-1.5 \) at low wavenumbers. Figure 3

| Run  | \( N^3 \) | Dissipation | \( \epsilon \) | \( k_{\text{max}} \eta \) | \( k_{\text{max}} \eta \# \) |
|------|------------|-------------|--------------|----------------|----------------|
| M1   | 10243      | \(-1.75 \times 10^{-4} k^2 \) | 0.06 | 1.05 | 2.00 |
| M2   | 20483      | \(-7 \times 10^{-5} \) | 0.06 | 1.06 | 2.17 |
| M3   | 40963      | \(-2.78 \times 10^{-5} \) | 0.06 | 1.06 | 2.34 |
| M1H  | 10243      | \(-1.6 \times 10^{-3} \) | 0.06 | 1.04 | 1.37 |
| M2H  | 20483      | \(-6 \times 10^{-4} \) | 0.06 | 1.04 | 1.42 |
| M3H  | 40963      | \(-6 \times 10^{-5} \) | 0.06 | 1.04 | 1.47 |

Three-dimensional RMHD Simulations
Figure 1. Checking the $-5/3$ hypothesis with the scaling study. Solid, dashed, and dash-dotted lines are spectra from the $4096^3$, $2048^3$, and $1024^3$ simulations, respectively. The upper plot shows normal diffusion M1–3 simulations and the lower plot shows hyperdiffusive M1–3H simulations. The convergence is reasonable around the dissipation scale. The scaling that achieves the best convergence is $\approx -1.70$. The Kolmogorov constant is around 3.5, which is compatible with our previous measurement (Beresnyak 2011).

Figure 2. Checking the $-3/2$ hypothesis with the resolution study. The convergence is poor. Convergence is required starting with dissipation scales, as the opposite would mean that either of the Kolmogorov scales depends differently on Re, as expected. Note that there is no convergence for the lower wavenumbers either, contrary to what was claimed in Perez et al. (2012). This was due to the scaling near driving scale being shallower than $-3/2$, around $-1.4$. Such scaling near the driving scale is always affected by driving, and the fine-tuning of driving is required to achieve the $-3/2$ slope.

shows a convergence study for the residual energy spectrum (magnetic energy minus kinetic energy). The best convergence is, again, near the $-1.7$ slope. Note that in all three cases the convergence or the lack of convergence is consistent across two simulation groups with different dissipation prescriptions, M1–3 and M1–3H. This is expected as different dissipation schemes only affect the shape of the dissipation range, while the nature of the convergence argument has nothing to do with a particular shape of the dissipation range.

4. DISCUSSION

Our results suggest that the residual energy scales as the total energy and is simply a constant fraction of the total energy. Our best estimate for this fraction is $0.15 \pm 0.03$. The discussion of the fraction of the residual energy and its scale-dependence dates back a couple of decades and has recently been connected to other dimensionless measures called alignment measures in simulations (Beresnyak & Lazarian 2009) and
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Figure 3. Residual energy convergence. Best convergence is for the $k^{-1.70}$ scaling for M1–3 and $k^{-1.69}$ scaling for M1–3H.

in solar wind measurements (Wicks et al. 2013). Technically, the puzzle of scale-dependence of the fraction of residual energy is as important as the question of scale-dependence of a particular measure introduced by Mason et al. (2006), which has been claimed (Mason et al. 2008; Perez et al. 2012) to be the exact measure that reduces nonlinear interaction and is responsible for the modification of the spectral slope. In our previous work, we found that although the “dynamic alignment” slope indeed somewhat correlates with the correction to the spectral slope, this relation is not exact. We are also not aware of any convincing theoretical argument why “dynamic alignment” should be preferred to other alignment measures; see Beresnyak & Lazarian (2009) for a discussion. The recent results (Beresnyak 2011, 2012b) that in higher Re simulations the alignment measures become constant and the slope approaches Kolmogorov slope significantly simplify the picture and make the discussion of the alignment influence on the nonlinear cascade largely irrelevant. Explaining the previously reported $-2$ scaling (Müller & Grappin 2005) for the residual energy is challenging from a theoretical standpoint. Indeed, if we assume a particular residual energy fraction on the outer scale and the $-2$ scaling, its value will depend on the separation from the outer scale, which will suggest nonlocal dynamics of this quantity. Our work, confirming that the residual energy is likely to be just a fraction of the total energy in the asymptotic regime, resolves this conceptual difficulty and makes theories suggesting different scalings for magnetic and kinetic energies obsolete.

It is well-known, however, that solar wind spectra often feature different kinetic and magnetic scalings. Furthermore, the amount of residual energy is not stable from measurement to measurement and is different for the fast and the slow solar wind. Such deviations from idealized numerical experiments will be the subject of future careful investigation. While we believe that RMHD is a good description for low-frequency solar wind fluctuations, the mismatch between simulations and the solar wind could be due to the latter having significant deviations from homogeneity, anisotropy with respect to the sunward direction (Grappin & Velli 1996), the presence of large amounts of discontinuities (Borovsky 2010), or the fact that, unlike simulations, solar wind is not driven on any particular spatial scale.

Using scaling study is relatively new in MHD, and the first tentative confirmation of the $-5/3$ slope was published in Beresnyak (2011). The latter publication was heavily criticized in Perez et al. (2012) for being numerically unresolved in the parallel direction. This criticism was misguided, however, as high numerical accuracy is not required for the scaling study argument, involving confirmation of a particular scaling as long as $\eta$ for this particular model scales precisely with the grid scale. This is because the numerical error on the grid scale depends only on $k_{\text{max}}\eta$, and the distortion of the spectra will be exactly the same function of $k\eta$ for each simulation. Therefore as long as the hypothesis is correct and the scaling corresponds to this hypothesis, the numerical spectra will still collapse into the same curve. Logically, however, this argument does not work if one wants to reject a hypothesis of a different scaling because if the scaling is different, the $k_{\text{max}}\eta$ will be different for each simulation, and the numerical error that corresponds to it, or to the unresolved parallel direction, will be different. This being said, in Beresnyak (2011) $k_{\text{max}}\eta$ was close to unity for either model, owing to the high order of the dissipation term. So, in practicality Beresnyak (2011) also rejected the $-3/2$ scaling. The subsequent work (Beresnyak 2012a) actually contained fully resolved simulations, such as R4–5, which conclusively rejected the $-3/2$ scaling. In this Letter, we have opted to perform fully resolved, numerically accurate simulations in order to avoid delving into such complicated matters. However, in our opinion Beresnyak (2011, 2012a) conclusively and rigorously supported $-5/3$ scaling and rejected $-3/2$ scaling for high Re simulations. The other part of the criticism of Perez et al. (2012), that Beresnyak (2011) measured spectral slopes that were distorted due to hyperdiffusion, was the result of a misunderstanding of the scaling study argument. The scaling
study does not measure any particular slope at any point of the numerical spectrum. Instead, as we explained in Beresnyak (2012a) and this Letter, it measures how \( v_\eta \) and \( \eta \) scale with Re. Such scalings are expected to be universal for high Re and are insensitive to the type of dissipation that was used. A simple way to confirm this is to formulate the scaling convergence in terms of a different type of spectrum, e.g., one-dimensional (1D) spectrum. It is easy to show that if convergence is present for a three-dimensional (3D) spectrum, it will also be present for a 1D spectrum, despite a 1D spectrum having very different spectral distortions due to the bottleneck effect. This has been well-known for a long time in hydrodynamics, as both 1D and 3D spectra have been used for scaling studies, e.g., Gotoh et al. (2002).

The simulations presented in this Letter and also in Beresnyak (2011, 2012a) use the same equations, and similar box size prescription and large-scale driving prescription as the ones used in Perez et al. (2012). We are not aware of any significant differences in terms of raw spectra presented by their group and ours, except for the data anomaly for the highest-resolution spectrum on Figure 8 in Perez et al. (2012), which showed an unusually high level of noise; see Beresnyak (2013). What is the source of the radically different claims about the spectral slope between these works? First, theirs are lower-resolution data (up to 2048\(^3\), versus ours 4096\(^3\)). Second, the claim of convergence on Figure 8 in Perez et al. (2012) is at visible odds with the figure itself, i.e., the convergence is indeed absent for the \(-3/2\) slope. There was also a number of logically incorrect and misleading statements regarding the length of the inertial range, purportedly confirming the Boldyrev scaling, but instead being a logical loop argument (Beresnyak 2013).

To summarize, the highest-resolution MHD simulations to date, with Re up to 36,000, exhibit asymptotic spectral scaling of around \(-1.7\), slightly steeper than Kolmogorov. The residual energy and also kinetic and magnetic energies separately exhibit the same scaling.

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