Three Dimensional Quantum Geometry and Deformed Poincaré Symmetry

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Abstract

We study a three dimensional non-commutative space emerging in the context of three dimensional Euclidean quantum gravity. Our starting point is the assumption that the isometry group is deformed to the Drinfeld double $D(SU(2))$. We generalize to the deformed case the construction of $E^3$ as the quotient of its isometry group $ISU(2)$ by $SU(2)$. We show that the algebra of functions on $E^3$ becomes the non-commutative algebra of $SU(2)$ distributions, $C(SU(2))^*$, endowed with the convolution product. This construction gives the action of $ISU(2)$ on the algebra and allows the determination of plane waves and coordinate functions. In particular, we show that: (i) plane waves have bounded momenta; (ii) to a given momentum are associated several $SU(2)$ elements leading to an effective description of an element in $C(SU(2))^*$ in terms of several physical scalar fields on $E^3$; (iii) their product leads to a deformed addition rule of momenta consistent with the bound on the spectrum. We generalize to the non-commutative setting the local action for a scalar field. Finally, we obtain, using harmonic analysis, another useful description of the algebra as the direct sum of the algebra of matrices. The algebra of matrices inherits the action of $ISU(2)$: rotations leave the order of the matrices invariant whereas translations change the order in a way we explicitly determine.

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1 Introduction

It is commonly believed that space-time cannot be described in terms of usual differential geometry at arbitrary small scales. A complete and consistent theory of quantum gravity that would give a precise description of space-time at the Planck scale is however still missing. Many models have been proposed in the literature where space-time appears non-commutative, fuzzy or discrete [1, 2]. In string theory for example [3], the low energy effective theory on D-branes with an external B-field is described in terms of quantum field theory (QFT) on the non-commutative Moyal space. In loop quantum gravity [2], space-time is argued to be discrete because geometrical operators such as the area and the volume operators have discrete spectra on the spin-networks states space. These two approaches are very different at the technical as well as the conceptual levels but raise the same fundamental question of the Poincaré invariance at the Planck scale. Is it broken, hidden or deformed? Whatever the answer is, one can consider the construction of a QFT on such spaces keeping in mind that invariance under the Poincaré group is crucial for standard QFT. In this respect, a twisted version of the Poincaré group was conjectured to replace the Poincaré group for the non-commutative field theory on the Moyal plane [4]. The consequences of the invariance under the twisted Poincaré group were studied [5] where it was shown that the twisting is not observable in the S-matrix.

The same issue of Poincaré invariance has also been discussed in the loop quantum gravity (LQG) and spin-foam (SF) frameworks. LQG and SF models offer respectively a canonical and covariant (or path integral) quantization of gravity. In particular, it has been shown, in the four dimensional case, that states of quantum geometries which are unphysical are eigenstates of area operators with discrete eigenvalues. This result has motivated the idea that Poincaré invariance must be broken at the Planck scale and a phenomenology of Poincaré invariant broken physics has emerged in the literature (see [6] for instance). However, it is worth mentioning that there is, up to our knowledge, no clear relationship between discrete quantum geometry emerging in the context of LQG and an eventual Poincaré symmetry breaking.

In the three dimensional case, the situation is somehow simpler [7, 8]. First, it was argued that a self-gravitating scalar Euclidean QFT (with no cosmological constant) is effectively described by a non-commutative QFT with no-gravity [7]. Newton’s constant $G$ or equivalently the Planck length $\ell_P = G\hbar$ encodes the non-commutativity of the spacetime: the algebra of functions on the Euclidean spacetime is non-commutative and its product, denoted $\star$, is a deformation of the standard point-wise product with deformation parameter $\ell_P$. Therefore, one explicitly sees, in such models, how quantum gravity effects could be encoded in a non-commutative space-time. The Drinfeld double $D(SU(2))$ of the classical Lie group $SU(2)$ plays the role of the isometry algebra of the non-commutative space and is a deformation of the group algebra of the Euclidean group $ISU(2)$ [9] which is the isometry group of the classical Euclidean space. Therefore, QFT is no longer covariant under the
usual Euclidean group but is covariant under a deformed version of it. More precisely
the action of any symmetry element $\xi$ (in the quantum double) on a field (viewed as
a representation of the quantum double) is not modified, but its action on a product
of two fields $\phi_1$ and $\phi_2$ is modified compared to the classical case and is such that

$$\xi \triangleright (\phi_1 \star \phi_2) = \sum_{(\xi)} (\xi_1 \triangleright \phi_1) \star (\xi_2 \triangleright \phi_2),$$

where $\Delta(\xi) = \sum_{(\xi)} \xi_{(1)} \otimes \xi_{(2)}$ is the $D(SU(2))$ co-product. This co-product is not
co-commutative. This was the starting point for the construction of a scalar self-
gravitating quantum field theory [8].

In fact, as usual in non-commutative geometry, the non-commutative space is
indirectly defined through its non-commutative algebra of functions. In the case
we are studying, the momentum space is no longer a vector space, it is a curved
manifold closely related to the Lie group $SU(2)$. More precisely, to each $SU(2)$
element $u$ one associates a three dimensional momentum $\vec{P}(u)$; the sum $\vec{P}_{tot}$ of two
momenta $\vec{P}(u)$ and $\vec{P}(v)$ is obtained from the $SU(2)$ group structure as follows:

$$\vec{P}_{tot} = \vec{P}(uv).$$

The momentum space is then curved and has the topology of the three dimensional
sphere. The compactness of momentum space has the important effect of eliminating
the potential ultra-violet divergences in QFT. The price to be paid is a deforma-
tion of the addition rule of momenta which is no more commutative. In a more
formal language, the co-product of the isometry algebra is deformed compared to
the standard case and becomes non co-commutative.

The aim of this article is to study the non-commutative geometry associated to
the quantum double. We identify the relevant non-commutative algebra replacing
the algebra of functions on the manifold $E^3$ and we exhibit its realizations in Eu-
clidean space with a star product. Moreover, we show that the non-commutative
space has a discrete structure: the coordinate functions satisfy the $su(2)$ lie alge-
bra and have a discrete spectrum. The discreteness is physically expected from the
compactness of the momentum space.

Our starting point is the deformed isometry group of the manifold, the Euclidean
group $ISU(2)$ is replaced by its deformed version the Drinfeld double $D(SU(2))$.
The latter contains $ISU(2)$ but has a twisted co-algebra structure. We then general-
ize the construction of $E^3$ as the quotient of its isometry group $ISU(2)$ by $SU(2)$ to
the deformed case. We show that the resulting algebra is the non-commutative
algebra of $SU(2)$ distributions endowed with the convolution product, denoted
$C(SU(2))^*$. This construction gives the action of $ISU(2)$ on the algebra and con-
sequently allows the determination of plane waves and coordinate functions. Then
we show that surprisingly $C(SU(2))^*$ cannot be represented as a deformation of the
commutative algebra of functions on $E^3$. This is due to the fact that the previously
mentionned map $\vec{P}(u)$ cannot be injective or in more formal terms $E^3$ and $SU(2)$ are
not homeomorphic. However, we exhibit an (family of) injective mapping(s) from
to the direct sum of three sub-spaces of $C(\mathbb{E}^3)$, each of them having a good physical interpretation. Furthermore, This mapping is compatible with the action of the Euclidean group and gives rise to a star product which can be used to construct a local action for a scalar field. We then consider another useful description of the algebra obtained using harmonic analysis as the direct sum of the algebra of matrices. This makes clear the discreteness of the non-commutative space. The algebra of matrices inherits the action of the $ISU(2)$ elements: rotations leave the order of the matrices invariant whereas translations change the order in a way we determine.

The plan of the paper is as follows. In Section 2, we recall the definition of the quantum double $\mathcal{D}(SU(2))$. We exhibit a family of morphisms between the group algebra $\mathbb{C}[ISU(2)]$ and $\mathcal{D}(SU(2))$ which makes clear that $\mathcal{D}(SU(2))$ is a deformation of the group algebra of the Euclidean group $ISU(2)$. We show that there is an ambiguity in the definition of the momenta in $\mathcal{D}(SU(2))$. A very similar ambiguity is also present in full Loop Quantum Gravity. In Section 3, we construct the non-commutative algebra using the quantum double. The basic idea mimics the classical situation: the classical space $\mathbb{E}^3$ can be defined as the homogeneous space $ISU(2)/SU(2)$. In the deformed case, we find that what replaces the algebra of functions on $\mathbb{E}^3$ is the non-commutative algebra $C(SU(2))^*$ of $SU(2)$ distributions endowed with the convolution product. In Section 4, we exhibit a morphism between $C(SU(2))^*$ and the space $C_{\ell_p}(\mathbb{E}^3)$ which, viewed as a vector space, is the direct sum of three-subspaces of $C(\mathbb{E}^3)$: two of them being the set of functions on $\mathbb{E}^3$ with a spectrum strictly bounded by $\ell_p^{-1}$ and the last one the set of functions on $\mathbb{E}^3$ whose spectrum belongs to the sphere of radius $\ell_p^{-1}$. The algebra structure of $C_{\ell_p}(\mathbb{E}^3)$ is obviously non-commutative and leads to a $\star$-product. We finally write an action for a scalar field in this non-commutative space, we postpone its detailed study for the future. In Section 5, we show that $C(SU(2))^*$ admits in fact a discrete structure. More precisely, we introduce a Fourier map on $C(SU(2))^*$ whose image is the space of complex matrices $\bigoplus_{n \in \mathbb{N}} \text{Mat}_{n \times n}(\mathbb{C})$. This makes explicit the fuzzy space formulation of the non-commutative space which appears as a collection of concentric fuzzy spheres of different radii. Then, we determine the action of the isometry algebra $\mathcal{D}(SU(2))$ on the fuzzy space: rotations leave each fuzzy sphere invariant whereas translations define finite difference operators which send points on a fuzzy sphere to points on neighboring spheres. In Section 6, we study the correspondence between the continuous formulation of $C(SU(2))^*$ and its discrete formulation in terms of matrices. This relation gives some insight on the fuzzy geometry and its classical limit. We conclude with some remarks on the construction of a QFT on the non-commutative space. This QFT is interesting for many reasons, the most important being the possibility of eliminating UV divergences. In the Appendix we use Schwinger’s oscillator representation of $SU(2)$ to determine the action of translations in terms of a couple of creation and annihilation operators.
The Quantum double $\mathcal{D}(SU(2))$: a deformation of the Euclidean group $ISU(2)$

The quantum double $\mathcal{D}(SU(2))$ is a deformation of the group algebra of the Lie group $ISU(2)$ [9]. This Hopf $*$-algebra was extensively studied in the context of combinatorial quantization of Chern-Simons theory [10] with the Euclidean group $ISU(2)$; such a theory is equivalent [11] (up to some discrepancies [12]) to three dimensional Euclidean gravity without a cosmological constant. In that context, the Newton constant $G$ or equivalently the Planck length $\ell_P = \hbar G$ plays the role of the deformation parameter.

2.1 Definition of the quantum double $\mathcal{D}(SU(2))$

The quantum double $\mathcal{D}(SU(2))$ is defined as a vector space by the tensor product:

$$\mathcal{D}(SU(2)) = C(SU(2)) \otimes \mathbb{C}[SU(2)],$$

where $C(SU(2))$ is a suitable set of complex functions on $SU(2)$ and $\mathbb{C}[SU(2)]$ is the group algebra of $SU(2)$. When $u \in SU(2)$, and for elements of $\mathcal{D}(SU(2))$ of the form, $(f \otimes u)$, the Hopf $*$-algebra structure is given by

Product : $(f_1 \otimes u_1) \cdot (f_2 \otimes u_2) = (f_1 \cdot \text{ad}_{u_1} f_2 \otimes u_1 u_2)$

Co-product : $\Delta_{\mathcal{D}(SU(2))}(f \otimes u) = \sum_{(f)} (f_1 \otimes u) \otimes (f_2 \otimes u)$

Unit : $(1 \otimes e)$

Co-unit : $\varepsilon(f \otimes u) = f(e)$

Antipode : $S(f \otimes u) = (\iota \text{ad}_{u^{-1}} f \otimes u^{-1})$

Complex conjugate : $(f \otimes u)^* = (\overline{\text{ad}_{u^{-1}} f} \otimes u^{-1})$,

where the adjoint map $\text{ad}$ and the inverse map $\iota$ are respectively defined by

$$(\text{ad}_u f)(v) = f(u^{-1}vu), \quad (\iota f)(u) = f(u^{-1}),$$

and we used the Sweedler notation. The explicit expression for the $C(SU(2))$ co-product is

$$\Delta_{C(SU(2))}(f)(u \otimes v) = \sum_{(f)} f_1(u) f_2(v) = f(uv).$$

These relations are then extended by linearity, morphism (for the product, co-product and co-unit) and anti-morphism (for the antipode and complex conjugate operation) when $u$ is a general element of $\mathbb{C}[SU(2)]$. Notice that $\mathbb{C}[SU(2)]$ and $C(SU(2))$ are two sub-Hopf $*$-algebras.

2.2 Drinfeld double $\mathcal{D}(SU(2))$ as a deformation of $\mathbb{C}[ISU(2)]$

In order to see how $\mathcal{D}(SU(2))$ can be viewed as a deformation of $\mathbb{C}[ISU(2)]$, we start by recalling the structure of the group algebra $\mathbb{C}[ISU(2)]$. 
**Group algebra \( \mathbb{C}[ISU(2)] \)**

As previously for the quantum double, \( \mathbb{C}[ISU(2)] \) is defined as a vector space by the tensor product:

\[
\mathbb{C}[ISU(2)] = \mathbb{C}[\mathbb{R}^3 \rtimes SU(2)] = \mathbb{C}[\mathbb{R}^3] \otimes \mathbb{C}[SU(2)],
\]

where \( \mathbb{R}^3 \) is the group of standard Euclidean translations whose elements will be denoted \( T_x \) for \( x \in \mathbb{R}^3 \). We define the group algebra \( \mathbb{C}[\mathbb{R}^3] \) as the set of elements \( \int d^3x \hat{f}(x) T_x \) with \( \hat{f} \) a function (or a distribution) of compact support. \( \mathbb{C}[\mathbb{R}^3] \) can be canonically identified with a sub-algebra \( C(\mathbb{R}^3) \) of the algebra of functions: to any element \( \int d^3x \hat{f}(x) T_x \), one associates the function \( f(p) = \int d^3x \hat{f}(x) e^{-i\vec{p} \cdot \vec{x}} \) whose Fourier transform is of compact support. In particular, to the translation \( T_x \) is associated the function \( T_x(p) = e^{-i\vec{p} \cdot \vec{x}} \). Finally, \( \mathbb{C}[ISU(2)] \) can be identified with the vector space:

\[
\mathbb{C}[ISU(2)] \simeq C(\mathbb{R}^3) \otimes \mathbb{C}[SU(2)].
\]

The group algebra \( \mathbb{C}[ISU(2)] \) is in fact a Hopf *-algebra. When \( u \in SU(2) \) the Hopf *-algebra structure is given for elements of the form \( (f \otimes u) \in C(\mathbb{R}^3) \otimes \mathbb{C}[SU(2)] \), by

- **Product**: \( (f_1 \otimes u_1) \cdot (f_2 \otimes u_2) = (f_1 \cdot R_{u_1} f_2 \otimes u_1 u_2) \)
- **Coproduct**: \( \Delta_{\mathbb{C}[ISU(2)]}(f \otimes u) = \sum_{(f)} (f_{(1)} \otimes u) \otimes (f_{(2)} \otimes u) \)
- **Unit**: \( (1 \otimes e) \)
- **Counit**: \( \varepsilon(f \otimes u) = f(e) \)
- **Antipode**: \( S(f \otimes u) = (i R_{u^{-1}} f \otimes u^{-1}) \)
- **Complex conjugate**: \( (f \otimes u)^* = (R_{u^{-1}} \hat{f} \otimes u^{-1}) \),

where the inverse map \( i \) is defined as for \( D(SU(2)) \) and \( R_u \) is given by \( (R_u f)(p) \equiv f(R(u^{-1})p) \), \( R(u) \) being the vectorial representation of \( SU(2) \). The co-product on \( C(\mathbb{R}^3) \) explicitly reads

\[
\Delta_{C(\mathbb{R}^3)}(f)(p \otimes q) = \sum_{(f)} f_{(1)}(p) f_{(2)}(q) = f(p + q).
\]

Contrary to the \( D(SU(2)) \) case, this co-product is co-commutative. These relations are extended by linearity, morphism or anti-morphism to any element of \( \mathbb{C}[SU(2)] \).

Notice that the set of elements \( \{ T_x \otimes u \} \subset \mathbb{C}[ISU(2)] \) where \( T_x(p) = e^{-i\vec{p} \cdot \vec{x}} \) equipped with the product inherited from the Hopf algebra is isomorphic to the Euclidean group.

**Algebra morphisms between \( \mathbb{C}[ISU(2)] \) and \( D(SU(2)) \)**

The algebras \( \mathbb{C}[ISU(2)] \) and \( D(SU(2)) \) are in fact closely related. To clarify this point, we are going to construct an algebra morphism \( \varphi \) between these two algebras. We require this morphism to be trivial on the \( \mathbb{C}[SU(2)] \) part, namely
\( \varphi(f \otimes u) \equiv \varphi(f) \otimes u \); therefore \( \varphi \) is viewed as a mapping from \( C(\mathbb{R}^3) \) to \( C(SU(2)) \).

From the algebra structures of \( \mathbb{C}[I SU(2)] \) and \( D(SU(2)) \), it is immediate to see that \( \varphi \) must satisfy the following property:

\[
\varphi(R_u f) = \text{ad}_u \varphi(f),
\]

for any \( SU(2) \) element \( u \).

In order to find the solutions \( \varphi \) to this equation, it will be useful to introduce explicit parameterizations of \( SU(2) \). Actually, two different parameterizations will be useful. In the first one, we identify \( SU(2) \) with the three-sphere \( S^3 = \{(\vec{y}, y_4) \in \mathbb{R}^4 \mid \vec{y}^2 + y_4^2 = 1\} \) and then the fundamental representation of any \( u \in SU(2) \) is given by

\[
u(\vec{y}, y_4) = y_4 - i \vec{y} \cdot \vec{\sigma},
\]

with \( \vec{\sigma} \) being the Pauli matrices. In the second one, any group element \( u \), but the identity \( u = e = 1 \) and its antipode \( u = e_A = -1 \), is characterized by a rotation angle (its conjugacy class) \( \theta \in [0, 2\pi] \) together with a unit vector \( \vec{n} \in S^2 \) as follows:

\[
u(\theta, \vec{n}) = \exp(-i \theta n^a J_a),
\]

where \( J_a \) are the generators of the Lie algebra \( su(2) \); they satisfy the relation \([J_a, J_b] = i \epsilon_{abc} J_c\). Note that the \((\theta, \vec{n})\)-parametrization is not well-defined at \( e \) and \( e_A \) which are respectively parameterized by \( (\theta = 0, \vec{n}) \) and \( (\theta = 2\pi, \vec{n}) \) for any unit vector \( \vec{n} \). We are now ready to give explicitly the solutions to eq. (11) for \( \varphi \): the map \( \varphi \) is completely characterized by an arbitrary function \( \Pi \) defined on \([-1, 1]\) according to:

\[
\varphi(f)(u(\vec{y}, y_4)) = f(\ell_P^{-1} \Pi(y_4) \vec{y}).
\]

The Planck length \( \ell_P \) has been introduced for dimensional purposes only and from now on we assume \( \Pi(1) = 1 \). Using the other parametrization, \( \varphi \) reads

\[
\varphi(f)(u(\theta, \vec{n})) = f(\ell_P^{-1} \Xi(\theta) \vec{n})
\]

with \( \Xi(\theta) = \sin(\theta/2) \Pi(\cos(\theta/2)) \).

To understand some fundamental properties of \( \varphi \), it is particularly interesting to consider the images of the momentum coordinates functions \( p_a \in C(\mathbb{R}^3) \) given by the \( SU(2) \) functions \( P_a \):

\[
P_a(u) \equiv \varphi(p_a)(u) = \ell_P^{-1} \Pi(y_4) y_a = \ell_P^{-1} \Xi(\theta) n_a.
\]

Note that the norm of the momentum vector \( |\vec{P}| \) is function of the conjugacy class \((y_4, \theta)\) of \( SU(2) \) and it vanishes on the identity \( e \) and its antipode \( e_A \). Mathematically, there is of course no way to distinguish between two different choices of \( \Pi \) (or \( \Xi \)). Therefore, there exists an ambiguity which is very similar to those ambiguities that exist in full Loop Quantum Gravity [13]. However, the choice of a morphism is physically rather important because it determines what will be the momenta in our theory. Indeed, as we will see in the sequel, \( \vec{P} \) generate translations in the non-commutative space whose isometry algebra is \( D(SU(2)) \).
Another important property of ϕ is that it is not invertible. In fact there is no continuous and monotonous function Π(y4) which vanishes on y4 = ±1. This is a signature of the fact that one cannot cover S3 ≃ SU(2) with a single coordinate patch. This point will have important physical consequences in the sequel. We can however assume that |⃗P| is invertible in a vicinity V of the identity e ∈ SU(2); then the restriction of ϕ on V defines a bijection from V to its image W ⊂ R3. As a result, given a vector ⃗p ∈ W, we reconstruct an element u(⃗y, y4) of SU(2) with:

\[ ⃗y = \ell_P \rho(p) \vec{p}, \quad y_4 = \sqrt{1 - \ell_P^2 p^2 \rho^2(p)}, \]

where \( \rho \) is function of \( p = |⃗p| \) related to Π by the relation:

\[ \rho(p) \Pi(\sqrt{1 - \ell_P^2 p^2 \rho^2(p)}) = 1. \]

The co-algebra structures: addition rule of momenta

Contrary to the algebra structure, the co-algebra structures of two Hopf *-algebras are obviously different. The co-product of the momenta coordinates functions \((P_a \otimes e) \in D(SU(2))\) can be expanded in powers of \( \ell_P \) as follows:

\[ \Delta_{C(SU(2))}(P_a) = P_a \otimes 1 + 1 \otimes P_a + \ell_P \epsilon_{abc} P_b \otimes P_c + O(\ell_P^2), \]

where \( \epsilon_{abc} \) is the totally antisymmetric tensor with \( \epsilon_{123} = +1 \). Comparing with the standard group-like \( C(\mathbb{R}^3) \) co-product,

\[ \Delta_{C(\mathbb{R}^3)}(p_a) = p_a \otimes 1 + 1 \otimes p_a, \]

it becomes clear that the co-algebra structure \([13]\) of \( D(SU(2)) \) is a deformation (in the sense of Drinfeld) of the co-algebra structure of \( C[ISU(2)] \) with a deformation parameter given by \( \ell_P \). According to Drinfeld the two co-products are related by a twist operator which can be explicitly found order by order in \( \ell_P \).

The co-product is of course related to the addition rule of momenta. Given two momenta \( \vec{p} \) and \( \vec{q} \) in \( W \) their sum is given by

\[ \Delta_{C(SU(2))}(\vec{P})(P^{-1}(\vec{p}) \otimes P^{-1}(\vec{q})) = \vec{P}(P^{-1}(\vec{p}) P^{-1}(\vec{q})). \]

It is important to note that this formula makes sense when \( P^{-1}(\vec{p}) P^{-1}(\vec{q}) \in V \). Using the parametrization \([12]\) and the inversion formula \([17]\), one can write the previous sum more explicitly as follows:

\[ \Pi\left( \sqrt{1 - (\ell_P \rho(p) p)^2} \sqrt{1 - (\ell_P \rho(q) q)^2} - (\ell_P)^2 \rho(p) \rho(q) \vec{p} \cdot \vec{q} \right) \times \]

\[ \times \left( \rho(q) \vec{q} \sqrt{1 - (\ell_P \rho(p) p)^2} + \rho(p) \vec{p} \sqrt{1 - (\ell_P \rho(q) q)^2} + \ell_P \rho(p) \rho(q) \vec{p} \wedge \vec{q} \right). \]

Let us illustrate the construction with explicit examples where the co-product and then the addition rule of momenta have quite simple expressions. The first
example introduced in the literature consists in taking $\Pi = 1$. It is clear that this choice makes the function $\varphi$ not invertible in the whole $SU(2)$ but we restrict it to a vicinity of the identity $V$ where it is invertible. The momenta coordinates functions are given by $\vec{P}(u) = \ell P^{-1} \vec{y}$ (resp. $\vec{P}(u) = \ell P^{-1} \sin(\theta/2) \vec{n}$). This choice leads to the following closed formula for the co-product of $P_a$:

$$\Delta_{C(SU(2))}(P_a) = P_a \otimes P_a + P_4 \otimes P_a + \ell P \epsilon_{abc} P_b \otimes P_c,$$

where $P_4$ is worth written in the $(\theta, \vec{n})$ parametrization: $P_4 = \ell P^{-1} \cos(\theta/2)$. The addition rule of momenta follows immediately

$$\vec{q} \sqrt{1 - (\ell P p)^2} + \vec{p} \sqrt{1 - (\ell P q)^2} + \ell P \vec{p} \wedge \vec{q},$$

when $P^{-1}(\vec{p}), P^{-1}(\vec{q})$ and $P^{-1}(\vec{p}) P^{-1}(\vec{q})$ belong to $V$.

Another possibility is to first consider $SO(3)$ instead of $SU(2)$ and then to choose $\Pi(y_4) = \text{sgn}(y_4) = y_4/|y_4|$. This choice has been made in the literature\cite{19} but leads to some difficulties related to the fact that $\varphi$ becomes not only discontinuous at $y_4 = 0$ but above all not defined at these points.

3 Convolution algebra $C(SU(2))^*$ from $\mathcal{D}(SU(2))$

In the previous sections, we introduced all the ingredients to construct the non-commutative space with isometry algebra given by the quantum double $\mathcal{D}(SU(2))$. This non-commutative space is characterized as usual by its algebra of functions interpreted as functions on the non-commutative space. The basic idea is to define the non-commutative algebra of functions as the space of $\mathcal{D}(SU(2))$-linear forms which are invariant under the action of $\mathbb{C}[SU(2)]$.

To illustrate this idea, let us recall how this works in the undeformed case: the Euclidean manifold $\mathbb{E}^3$ admits $ISU(2)$ as its (transitive) isometry group and can be identified with the coset $ISU(2)/SU(2)$. As a result, the space of complex functions on $\mathbb{E}^3$, denoted $C(\mathbb{E}^3)$, is the space of $SU(2)$-invariant functions on $ISU(2)$. Let us recall that $C(\mathbb{E}^3)$ is a commutative algebra endowed with the point-wise product. The non-commutative algebra we want to construct is a deformation of $C(\mathbb{E}^3)$.

This section is decomposed as follows. First, we recall the classical construction of $C(\mathbb{E}^3)$ from the convolution algebra $C(\mathbb{R}^3)^*$ of distributions on $\mathbb{R}^3$ before going to the deformed case. Then, we show that deforming $C(\mathbb{R}^3)^*$ and using the Hopf algebra duality principle we get the convolution algebra $C(SU(2))^*$ of distributions on $SU(2)$.

3.1 Construction in the classical case

Here we recall the construction in the undeformed case where $\mathcal{D}(SU(2))$ is replaced by $\mathbb{C}[ISU(2)]$. We introduce the dual of the coset $\mathbb{C}[ISU(2)]/\mathbb{C}[SU(2)] = \mathbb{C}[\mathbb{R}^3]$ which is defined as $SU(2)$-invariant sesquilinear forms on $\mathbb{C}[ISU(2)]$ with compact
Thus, one recovers the usual action of the Euclidean group. Consider given by

\[ T \in C(\mathbb{R}^3) \sim \Delta \] to its evaluation on a pure translational element. Furthermore, its co-product is follows:

\[ \langle f, T\psi \rangle = \langle f \otimes e, T \rangle \psi \]

Fourier transform are of compact support. Indeed, the Fourier map is defined as

\[ T \psi \equiv \langle f \otimes u, \psi \rangle \]

composition product on the space of distributions:

\[ \langle f, T\psi \rangle = \langle f \otimes e, T \rangle \psi \]

support:

\[ F \in C[ISU(2)]^* \quad | \quad \forall (f \otimes u) \in C[ISU(2)], \forall v \in SU(2), \quad \langle (f \otimes u), (1 \otimes v), F \rangle \equiv \langle (f \otimes uv), F \rangle = \langle (f \otimes u), F \rangle \]  

with \( \langle c_1 a, c_2 F \rangle = c_1 c_2 \langle a, F \rangle \) for any \( c_1, c_2 \in \mathbb{C} \). This set can be identified with the space of distributions on \( \mathbb{R}^3 \) of compact support as follows:

\[ C(\mathbb{R}^3)^* = \{ \psi \mid \langle f, \psi \rangle = \langle (f \otimes e), F \rangle, \quad \forall f \in C(\mathbb{R}^3) \} . \]  

Furthermore, \( C(\mathbb{R}^3)^* \) is equipped with an algebra structure obtained from the Hopf duality principle. This principle allows to define a Hopf algebra structure to the dual \( \mathcal{H}^* \) of a given Hopf algebra \( \mathcal{H} \). In particular, \( \mathcal{H}^* \) inherits an algebra structure from the co-algebra structure of \( \mathcal{H} \). In our particular case, the product \( \psi_1 \circ \psi_2 \) of two distributions \( \psi_1 \) and \( \psi_2 \) of \( C(\mathbb{R}^3)^* \) is given by

\[ \langle f, \psi_1 \circ \psi_2 \rangle = \langle \Delta_C(\mathbb{R}^3)(f, e), F_1 \otimes F_2 \rangle = \langle \Delta_C(\mathbb{R}^3)(f), \psi_1 \otimes \psi_2 \rangle , \]  

where \( \Delta_C(\mathbb{R}^3) \) is the usual co-product on \( C(\mathbb{R}^3) \) defined in eq.(10). In fact, this product is the convolution product on \( C(\mathbb{R}^3)^* \) since it can be written as the standard definition of the convolution product on the space of distributions:

\[ \langle f, \psi_1 \circ \psi_2 \rangle = \langle \langle f, T[\psi_2] \rangle, \psi_1 \rangle , \]  

where \( T[\psi] \) is the \( C(\mathbb{R}^3) \)-valued distribution defined by \( \langle f, T[\psi](p) \rangle = \langle T_p f, \psi \rangle \) and \( T_p \) is the translation operator: \( (T_p f)(q) = f(p + q) \).

To complete the \( C[ISU(2)] \) case study, we introduce the Fourier transform on \( C(\mathbb{R}^3)^* \) and we get a subspace of functions on the Euclidean space \( \mathbb{E}^3 \), \( C(\mathbb{E}^3) \) whose Fourier transform are of compact support. Indeed, the Fourier map is defined as follows:

\[ \mathcal{F} : C(\mathbb{R}^3)^* \rightarrow C(\mathbb{E}^3), \quad \psi \mapsto \mathcal{F}[\psi] = \langle T_x, \psi \rangle , \]

with \( T_x(p) = e^{-i \vec{p} \cdot \vec{x}} \). We see that the Fourier transform of a distribution \( \psi \) reduces to its evaluation on a pure translational element. Furthermore, its co-product is simply \( \Delta_C(\mathbb{R}^3)(T_x) = T_x \otimes T_x \) and then the product of two elements \( \Psi_1 \) and \( \Psi_2 \) is given by

\[ \langle \Psi_1 \circ \Psi_2 \rangle(x) \equiv \langle T_x, \mathcal{F}^{-1}(\Psi_1) \circ \mathcal{F}^{-1}(\Psi_2) \rangle = \Psi_1(x) \Psi_2(x) . \]

An important consequence of this construction is the action of the symmetry group. The action of an element \( (T_y, u) \) of \( ISU(2) \) on \( C(\mathbb{E}^3) \) is induced by its action on \( C[ISU(2)]^* \) as follows:

\[ ((T_y \otimes u) \triangleright \Psi)(x) \equiv \langle (T_y \otimes u) \triangleright \Psi \rangle(x) = \Psi(R(u^{-1})(\vec{x} - \vec{y})). \]

Thus, one recovers the usual action of the Euclidean group.
3.2 Construction in the deformed case

Now, we go back to the deformed case of \( \mathcal{D}(SU(2)) \): the construction follows the same steps. We consider the dual of the coset \( \mathcal{D}(SU(2))/\mathbb{C}[SU(2)] \) which replaces its undeformed counterpart, the dual of \( \mathbb{C}[ISU(2)]/\mathbb{C}[SU(2)] \):

\[
\left\{ F \in \mathcal{D}(SU(2))^* \mid \forall (f \otimes u) \in \mathcal{D}(SU(2)) , \ \forall v \in SU(2) , \ \langle (f \otimes u) \cdot (1 \otimes v) , F \rangle = \langle (f \otimes u) , F \rangle \right\}.
\]  

This space can be identified with the space of distributions on \( SU(2) \):

\[
C(SU(2))^* = \{ \phi \mid \langle f , \phi \rangle = \langle (f \otimes e) , F \rangle , \ \forall f \in C(SU(2)) \},
\]

and the algebra structure of \( C(SU(2))^* \) is obtained, as in the \( \mathbb{C}[ISU(2)] \) case, from the Hopf duality procedure. The product of two elements \( \phi_1 , \phi_2 \) of \( C(SU(2))^* \) is given by

\[
\langle f , \phi_1 \ast \phi_2 \rangle \equiv \langle \Delta_{\mathcal{D}(SU(2))}(f \otimes e) , F_1 \otimes F_2 \rangle = \langle \Delta_{C(SU(2))}(f) , \phi_1 \otimes \phi_2 \rangle ,
\]

where \( \Delta_{C(SU(2))} \) is the co-product defined in eq. (33). One recognizes that this product is the usual convolution product on \( C(SU(2))^* \) by recasting the above equation as follows:

\[
\langle f , \phi_1 \ast \phi_2 \rangle = \langle \langle f , L[\phi_2] \rangle , \phi_1 \rangle ,
\]

where \( L[\phi] \) is a \( C(SU(2)) \)-valued distribution defined by \( \langle f , L[\phi](u) \rangle = \langle L_u f , \phi \rangle \) and the \( L_u \) is the left deformed translation: \( (L_u f)(v) = f(uv) \).

The symmetry action of \( \mathcal{D}(SU(2)) \) on \( C(SU(2))^* \) is induced by its action on \( \mathcal{D}(SU(2))^* \). The action of an element \( (g \otimes u) \) of \( \mathcal{D}(SU(2)) \) on \( \phi \in C(SU(2))^* \) is defined by

\[
\langle f , (g \otimes u) \triangleright \phi \rangle \equiv \langle (f \otimes e) , (g \otimes u) \triangleright F \rangle \equiv \langle (g \otimes u)^* \cdot (f \otimes e) , F \rangle
\]

and in particular the action of \( C(SU(2)) \subset \mathcal{D}(SU(2)) \) is just the multiplication by the \( C(SU(2)) \) element: \( (g \otimes e) \triangleright \phi = g \phi \). The \( \mathcal{D}(SU(2)) \) action on the product \( \phi_1 \ast \phi_2 \) is defined using the co-algebra structure by

\[
(g \otimes u) \triangleright (\phi_1 \ast \phi_2) \equiv \sum_{(g)} \langle (g(1) \otimes u) \triangleright \phi_1 \rangle \ast \langle (g(2) \otimes u) \triangleright \phi_2 \rangle ,
\]

where we used the Sweedler notation for the co-product of \( C(SU(2)) \).

To finish, let us note that the algebra \( C(SU(2))^* \) possesses an antimorphic involution \( \phi \mapsto \phi^b \) which plays the role of the complex-conjugation. It is explicitly defined by

\[
\langle f , \phi^b \rangle \equiv \langle S(f \otimes e)^* , F \rangle ,
\]

using previous notations.

In order to get an intuition about the previous construction let us consider some examples.
Of particular interest is the subalgebra of delta distributions. The delta distributions \( \delta_u \in C(SU(2))^* \) are defined for any \( u \in SU(2) \) by
\[
\langle f, \delta_u \rangle \equiv f(u),
\]
and these delta distributions are in fact the eigenfunctions of the momentum elements \((P_a \otimes e)\) of \( D(SU(2)) \), namely \((P_a \otimes e) \triangleright \delta_u = P_a(u) \delta_u \). As a result, \( \delta_u \) can be interpreted as a pure momentum state of momentum \( P_a(u) \). The product between two such distributions reads
\[
\delta_{u_1} \star \delta_{u_2} = \delta_{u_1 u_2},
\]
giving the composition law of momenta recalled in the Introduction.

We can now introduce the coordinate distributions \( \chi^a \). They are defined as the elements of the algebra on which the infinitesimal translation, \(((1 - ix^b P_b) \otimes e)\), in the direction \( x^b \) acts as
\[
((1 - ix^a P_a) \otimes e) \triangleright \chi^b = (1 - ix^a P_a) \chi^b = \chi^b - x^b \delta_e.
\]
This equality can be recasted using a left-invariant vector field \( \xi_a \) (whose normalization is implicitly given in the eq. (43) below) as follows:
\[
P_a(u) \left( \chi^b + 2i \ell_P \xi^b \delta_e \right) = 0.
\]
The solution for \( \chi^a \) which is odd under parity, \( u \rightarrow u^{-1} \), is given by
\[
\chi^a = -2i \ell_P \xi^a \delta_e.
\]
When acting on a test function \( f \), we have
\[
\langle f, \chi^a \rangle = -2i \ell_P \xi^a \overline{f}|_e = -2i \ell_P \frac{d}{d\theta} \overline{f(e^{-i\theta J_a})}|_{\theta=0}.
\]
Notice that the sub-algebra generated by the \( \chi^a \) is given by the distributions whose support reduces to the identity element. According to a theorem by L. Schwartz [14] the latter sub-algebra is isomorphic to the universal enveloping algebra of \( su(2) \), \( U(su(2)) \), the image of \( \chi^a \) being the generators of \( su(2) \) and then satisfy
\[
[\chi^a, \chi^b] \star \equiv \chi^a \star \chi^b - \chi^b \star \chi^a = 2i \ell_P \epsilon_{abc} \chi^c.
\]
This relation is a consequence of the relation giving the product of two coordinates:
\[
\langle f, \chi^a \star \chi^b \rangle = -4 \ell_P^2 \xi^a \xi^b \overline{f}|_e = -4 \ell_P^2 \frac{\partial^2}{\partial \theta \partial \theta'} \overline{f(e^{-i\theta J_a} e^{-i\theta' J_b})}|_{\theta=\theta'=0}
\]
and the standard commutation relation of \( su(2) \).

Another interesting subalgebra of \( C(SU(2))^* \) is the algebra of functions \( \phi \) on \( SU(2) \) viewed as distributions in the standard way:
\[
\langle f, \phi \rangle = \int d\mu(u) \overline{f(u)} \phi(u),
\]
where \( d\mu(u) \) is the Haar measure on \( SU(2) \) with the normalization \( \int d\mu(u) = 1 \). The product between two such elements is the convolution product on \( C(SU(2)) \):

\[
(\phi_1 \ast \phi_2)(u) = \int d\mu(v) \, \phi_1(v) \, \phi_2(uv^{-1}).
\] (47)

We end this paragraph by noting that when restricted to the space of functions \( C(SU(2)) \), the non-commutative algebra we constructed is equipped with a Hermitian scalar product defined from the \( SU(2) \) Haar measure. Given two functions \( \phi_1 \) and \( \phi_2 \), their scalar product is given by the sesquilinear form \( \langle \phi_1, \phi_2 \rangle \) where \( \phi_2 \) is viewed as a distribution. The action (36) is unitary with respect to this Hermitian form, namely, for any element \((g \otimes u)\) in \( D(SU(2)) \), it satisfies

\[
\langle \phi_1, (g \otimes u) \triangleright \phi_2 \rangle = \overline{\langle \phi_2, (g \otimes u)^* \triangleright \phi_1 \rangle}.
\] (48)

4 The non-commutative algebra \( C_{\ell_P}(\mathbb{R}^3) \)

This section aims at linking, having physical applications in mind, the sets \( C(SU(2))^* \) and \( C(\mathbb{R}^3)^* \) viewed, first, as vector spaces. In other words, we want to interpret any distribution \( \phi \in C(SU(2))^* \) in terms of functions on the classical Euclidean space \( \mathbb{R}^3 \). To do so, it is in fact sufficient to make a link between \( C(SU(2))^* \) and \( C(\mathbb{R}^3)^* \) and then to make use of the classical Fourier map \( \mathfrak{F} \) to go from \( C(\mathbb{R}^3)^* \) to \( C(\mathbb{E}^3)^* \). Then, we extend these linear maps between \( C(SU(2))^* \) and \( C(\mathbb{R}^3)^* \) to morphisms in order to endow the two later spaces with non-commutative products. The obtained algebras will be respectively denoted by \( C_{\ell_P}(\mathbb{R}^3)^* \) and \( C_{\ell_P}(\mathbb{E}^3)^* \): they are deformations of \( C(\mathbb{R}^3)^* \) and \( C(\mathbb{E}^3)^* \) with deformation parameter \( \ell_P \).

4.1 Relation between \( C(SU(2))^* \) and \( C(\mathbb{R}^3)^* \)

Let us concentrate on the link between \( C(SU(2))^* \) and \( C(\mathbb{R}^3)^* \). As we have already emphasized in Section 2.2, there is no one-to-one mapping between \( SU(2) \) and \( (\text{subsets of}) \ \mathbb{R}^3 \) because they are not homeomorphic. As a result, it is impossible to find a one-to-one mapping between \( C(SU(2))^* \) and \( C(\mathbb{R}^3)^* \) and the construction of a map between these two spaces is rather involved.

Our starting point is the mapping \( \varphi \) which provides a mapping from \( SU(2) \) to \( \mathbb{R}^3 \). The map \( \varphi \) is characterized by a function \( \Pi \) that we will take to be one for simplicity. As we have already emphasized there is no way to find a bijection between \( C(\mathbb{R}^3) \) and any subset of \( C(SU(2)) \) for topological reasons; then this prevents \( \varphi \) from being an injective map from the whole \( SU(2) \) to \( \mathbb{R}^3 \). However, it becomes injective when restricted to some subsets of \( SU(2) \). In that case, \( \varphi \) allows to construct the following 3 bijections from subsets of \( SU(2) \) to subsets of \( \mathbb{R}^3 \):

\[
P_\pm : \quad U_\pm \quad \rightarrow \quad B_{\ell_P},
\]

\[
P_0 : \quad U_0 \simeq S^2 \quad \rightarrow \quad \partial B_{\ell_P} \simeq S^2,
\] (49)
where \( U_\pm, U_0 \) and \( U_0 \) are respectively the northern hemisphere, southern hemisphere and equator of \( S^3 \simeq SU(2) \) and \( \partial B_{\ell_p} \) is the open ball of \( \mathbb{R}^3 \), \( \partial B_{\ell_p} \) its boundary:

\[
U_\varepsilon = \{ u(\vec{y}, y_4) \in SU(2) \mid \text{sgn}(y_4) = \varepsilon \}, \\
B_{\ell_p} = \{ \vec{p} \in \mathbb{R}^3 \mid |\vec{p}| < \ell_p^{-1} \},
\]

with \( \text{sgn} \) being the sign function such that \( \text{sgn}(0) = 0 \). As a consequence, we have a natural decomposition of \( C(SU(2))^* \) into \( C_{U_+}(SU(2))^* \), \( C_{U_-}(SU(2))^* \) and \( C_{U_0}(SU(2))^* \) where \( C_V(SU(2))^* \) denotes the space of \( SU(2) \)-distributions with support on \( V \subset SU(2) \); since \( U_\pm \) are open subsets, \( C_{U_\pm}(SU(2))^* \) are in fact the spaces \( C(U_\pm)^* \) of distributions on \( U_\pm \). As a result, we have the following identification:

\[
C(SU(2))^* \simeq C(U_+)^* \oplus C(U_-)^* \oplus C_{U_0}(SU(2)), \quad \phi \simeq \phi_+ \oplus \phi_- \oplus \phi_0, \tag{51}
\]

where the components \( \phi_\pm \) are explicitly obtained from the characteristic functions \( I_\pm \) on \( U_\pm \) by the formulae \( \phi_\pm = I_\pm \phi_{\parallel} \). The remaining component \( \phi_0 \) is obtained by difference: \( \phi_0 = \phi - \phi_+ - \phi_- \). We can characterize more explicitly the space \( C_{U_0}(SU(2))^* \). Contrary to the other cases, this space does not reduce to the space \( C(U_0)^* \) of distributions on \( U_0 \) but decomposes into an infinite sum of them. More precisely, any \( \phi_0 \in C_{U_0}(SU(2))^* \) decomposes as follows:

\[
\phi_0 = \sum_s \phi_{0s} \delta^{(s)}(y_4) \tag{53}
\]

where the sum is finite, \( \delta^{(s)}(y_4) = d^s \delta(y_4)/(dy_4)^s \) and \( \phi_{0s} \in C(U_0)^* \) which is obtained from \( \phi_0 \) by the integral:

\[
\phi_{0s}(\vec{n}) = \frac{1}{s!} \int_{-1}^{1} dy_4 (-y_4)^s \phi_0(\vec{n}, y_4). \tag{54}
\]

For later convenience, we introduce the polarization vectors \( \xi_\varepsilon \) with \( \varepsilon \in \{+, -, 0\} \) defined by \( \phi \simeq \phi_\varepsilon \xi_\varepsilon \), i.e.

\[
\xi_+ = 1 \oplus 0 \oplus 0, \quad \xi_- = 0 \oplus 1 \oplus 0, \quad \xi_0 = 0 \oplus 0 \oplus 1. \tag{55}
\]

It is also convenient to introduce the dual polarization vector \( \xi_\varepsilon^t \) defined by the relation \( \xi_\varepsilon^t \xi_{\varepsilon'} = \delta_{\varepsilon \varepsilon'} \).

The idea is now to map the spaces \( C(U_\pm)^* \) and \( C_{U_0}(SU(2))^* \) to spaces of distributions on \( \mathbb{R}^3 \). This will make the link between the curved space of momenta and

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\(^1\) To explicitly define the characteristic functions \( I_\pm \) on \( U_\pm \), let us consider the limit of a test function \( I^\varepsilon \in C^\infty([-1, 1]) \) with \( \varepsilon < 1 \) defined by the fact that, \( I^\varepsilon(y_4) = 1 \) for \( y_4 > \varepsilon \) and \( d^s/dy_4^s I^\varepsilon(0) = 0 \) for \( s \geq 0 \). As a result, \( \phi_\pm \) are given by the relations:

\[
I_\pm \phi = \lim_{\varepsilon \to 0} I^\varepsilon(\pm y_4) \phi. \tag{52}
\]
the ordinary flat momentum space in 3 dimensions. More precisely, we look for the
following linear mappings:

$$
\begin{align*}
\mathbf{a}_\pm & : \quad C(U_\pm)^* \longrightarrow C_{B\ell_p}(\mathbb{R}^3)^*, \\
\mathbf{a}_0 & : \quad C_{U_0}(SU(2))^* \longrightarrow C_{\partial B\ell_p}(\mathbb{R}^3)^*.
\end{align*}
$$

(56)

For purposes of simplicity, let us define a mapping $\mathbf{a}$ as a multiplet of the above
mappings: $\mathbf{a} \equiv \mathbf{a}_+ \oplus \mathbf{a}_- \oplus \mathbf{a}_0$, and denote the image of $C(SU(2))^*$ by $\mathbf{a}$ as $C_{\ell_p}(\mathbb{R}^3)^*$:

$$
\begin{align*}
\mathbf{a} & : \quad C(SU(2))^* \longrightarrow C_{\ell_p}(\mathbb{R}^3)^* \equiv C_{B\ell_p}(\mathbb{R}^3)^* \oplus C_{\partial B\ell_p}(\mathbb{R}^3)^*.
\end{align*}
$$

(57)

Note that $C_{\ell_p}(\mathbb{R}^3)^*$ is interpreted as a deformation of the space $C(\mathbb{R}^3)^*$ of distributions on $\mathbb{R}^3$. We require, in addition, that the action of the Poincaré group
\( ISU(2) \subset D(SU(2)) \) induced by the mapping $\mathbf{a}$ on each component of $C_{\ell_p}(\mathbb{R}^3)^*$ is the standard covariant one (31). Therefore, we have the following conditions:

$$
(T_x \otimes u) \triangleright \mathbf{a}(\phi) \equiv \mathbf{a}(\varphi(T_x \otimes u) \triangleright \phi) \equiv \mathbf{a}(\varphi(T_x) \, \text{ad}_u(\phi)) = T_x \, R_u(\mathbf{a}(\phi)).
$$

(58)

Note that each components $C(U_\pm)^*$ and $C_{U_0}(SU(2))^*$ of $C(SU(2))^*$ are stable under
the action of $D(SU(2))$. As a result, the solutions of the above condition for $\mathbf{a}_\pm$ are

$$
\begin{align*}
\mathbf{a}_\pm(\phi) &= g_\pm \varphi_*(\phi),
\end{align*}
$$

(59)

where $g_\pm$ are functions of the norm $|\vec{p}|$ and $\varphi_*$ is the pull-back of $\varphi$ on the space of
distributions $C(SU(2))^*$ defined by

$$
\forall f \in C(\mathbb{R}^3), \quad \langle f , \varphi_*(\phi) \rangle = \langle \varphi(f) , \phi \rangle.
$$

(60)

For simplicity, we make the choice $g_\pm = 1$.

We can proceed in the same way to find the general solution for $\mathbf{a}_0$. Unfortunately,
with the same choice of $\varphi$, related to $\vec{P}(u)$, $\mathbf{a}_0$ admits a non-trivial kernel and then
is not a bijection: for example, $\mathbf{a}_0(\delta'(\theta - \pi)) = 0$ where we used the parametrization
$(\theta, \vec{n})$ for $SU(2)$. This is due to the fact that the derivative of $|\vec{P}|(\theta)$ vanishes for
$\theta = \pi$, i.e. on $U_0$. To construct a bijective mapping $\mathbf{a}_0$, it is necessary to make
a new choice $\vec{P}_0(u) = P_0(\theta) \vec{n}$ around $U_0$ which induces a new mapping $\varphi_0$. More
precisely, we consider an open set of $SU(2)$ containing $U_0$ and we require $P_0(\theta)$
strictly monotone with $P_0(\pi) = P(\pi) = \ell p^{-1}$. Then, the solution of the above
condition for $\mathbf{a}_0$ leads to

$$
\begin{align*}
\mathbf{a}_0(\phi) &= g_0 \varphi_0(\phi),
\end{align*}
$$

(61)

where $g_0$ is a constant; we will make the choice $g_0 = 1$ for simplicity. Note that the
choice of $\vec{P}_0$ is arbitrary and therefore there exist ambiguities in the definition of the
mapping $\mathbf{a}_0$. In fact, these ambiguities are similar to those we have already discussed
concerning the choice of the momentum coordinates in the deformed theory.

Using the usual Fourier transform $\mathcal{F}$ defined in eq. (29), one maps any multiplet of
distributions in $C_{\ell_p}(\mathbb{R}^3)^*$ into a multiplet of functions on $\mathbb{R}^3$. The image of $C_{\ell_p}(\mathbb{R}^3)^*$
is denoted $C_{\ell_p}(\mathbb{R}^3)$: it is interpreted as a deformation of the space $C(\mathbb{R}^3)$ of classical
functions on \( \mathbb{E}^3 \). It will be convenient to denote by \( \mathbf{m} \equiv \mathbf{F} \circ \mathbf{a} \) the mapping from \( C(SU(2))^* \) to \( C_{\ell_P}(\mathbb{E}^3) \) and \( \mathbf{m} \) defines a bijection between \( C(SU(2))^* \) and the space \( C_{\ell_P}(\mathbb{E}^3) \). The image of an element \( \phi \) in \( C(SU(2))^* \) by the mapping \( \mathbf{m} \) is explicitly given by

\[
\mathbf{m}(\phi)(x) = \langle e^{-i\vec{P} \cdot \vec{x}}, \phi_\perp \rangle + \langle e^{-i\vec{P} \cdot \vec{x}}, \phi_- \rangle + \langle e^{-i\vec{P}_0 \cdot \vec{x}}, \phi_0 \rangle \equiv \Phi(x) \tag{62}
\]

As a vector space, \( C_{\ell_P}(\mathbb{E}^3) \cong \hat{C}_{\ell_P}(\mathbb{R}^3)^* = \hat{C}_{B_{\ell_P}}(\mathbb{R}^3)^* \oplus \hat{C}_{\partial B_{\ell_P}}(\mathbb{R}^3)^* \) where \( \hat{C} \) is the Fourier image of the vector space \( C \). The space \( C_{\ell_P}(\mathbb{E}^3) \) inherits a non-commutative algebra structure we will describe in the next Section.

To illustrate the previous notions, let us consider some clarifying examples.

**Example 1: functions on \( SU(2) \)**

First, let us assume that \( \phi \) is a function, i.e. we restrict \( C(SU(2))^* \) to the subalgebra of functions \( C(SU(2)) \): in that case, \( \phi_\pm \) are both functions and \( \phi_0 = 0 \) in the decomposition of \( \phi \). The images of \( \phi_\pm \in C(SU(2)) \) by \( \mathbf{a}_\pm \) are completely determined by \( \vec{P}(u) \) and given by

\[
\mathbf{a}_\pm(\phi_\pm)(\vec{p}) = \int d\mu(u) \delta^3(\vec{p} - \vec{P}(u)) \phi_\pm(u) = \frac{v_{\ell_P}}{\sqrt{1 - (\ell_P |\vec{p}|)^2}} \phi_{\pm}(\ell_P \vec{p}, \pm \sqrt{1 - (\ell_P |\vec{p}|)^2}), \tag{63}
\]

where we have introduced the constant of volume dimension \( v_{\ell_P} = \ell_P^3/(2\pi^2) \).

Furthermore, the Fourier transform of \( \mathbf{a}_\pm(\phi_\pm) \) is a function of \( x \) which is related to the function \( \phi \in C(SU(2)) \) by the integral:

\[
\Phi_\pm(x) = \mathbf{m}_\pm(\phi_\pm)(x) = \int d\mu(u) \phi_\pm(u) e^{i\vec{P}(u) \cdot \vec{x}}. \tag{64}
\]

This transform is invertible and the inverse relation can be obtained performing the following classical Lebesgue integral on \( \mathbb{R}^3 \):

\[
\phi_\pm(u) = \sqrt{1 - (\ell_P P(u))^2} \int \frac{d^3 x}{(2\pi)^3 v_{\ell_P}} \Phi(x) e^{i\vec{P}(u) \cdot \vec{x}}. \tag{65}
\]

As we will see in the sequel, it is possible to inverse the relation \(\text{[64]}\) making use of a non-commutative \( \ast \)-product defined in Section 4.2.

**Example 2: elements of \( C_{U_0}(SU(2))^* \)**

A second example is the case where \( \phi \) has a support on \( y_4 = 0 \) that is \( \phi = \sum_s \delta^{(s)}(y_4) \phi_s \) where \( \phi_s \) are functions on \( U_0 \simeq S^2 \). A short calculation shows that the images of \( \phi \) by the mapping \( \mathbf{a}_0 \) is a distribution on \( \mathbb{R}^3 \) whose support is the two-sphere \( \partial B_{\ell_P} \):

\[
\mathbf{a}_0(\delta^{(s)}(y_4) \phi_s)(\vec{p}) = \frac{\phi_s(\vec{p}/p)}{4\pi p^2} \sin^2\left(\frac{\theta_0(p)}{2}\right) \frac{\theta_0'(p)}{\theta_0'(\ell_P^{-1})} \left(\frac{1}{\theta_0'(p)} d\right)^s \delta(p - \ell_P^{-1}), \tag{66}
\]

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where the function $\theta_0(p)$ is the inverse function of $P_0(\theta) : P_0(\theta_0(p)) = p$.

In the particular case where $P_0$ is linear, let us say $P_0(\theta) = \ell_0^{-1}(\theta - \pi) + \ell_{p^{-1}}$ for instance, then the mapping $a_0$ reduces to:

$$a_0(\delta^{(s)}(y_4) \phi_s)(p) = \frac{\phi_s(\vec{p}/p)}{4\pi \ell_0^2 \ell_0^p} \cos^2\left(\frac{\ell_0}{2} (p - \ell_{p^{-1}})\right) \delta^{(s)}(p - \ell_{p^{-1}})$$

Thus, the image of $\delta^{(s)}(y_4)$ is a sum of several $\delta^{(k)}(p - \ell_{p^{-1}})$ with $k \leq s$.

**Example 3: delta distributions**

Another simple but important example is when $\phi$ is a delta distribution $\delta_u$, i.e. a plane wave. A trivial calculation yields for any $u \in U_\epsilon$ with $\epsilon = +, -, 0$

$$a(\delta_u)(\vec{p}) = \delta^3(\vec{p} - \vec{P}(u)) \xi_\epsilon.$$  

Note that $\vec{P}_0(u) = \vec{P}(u)$ for $u \in U_0$; for that reason, we write the same formula for each $\epsilon$.

It is now possible to compute the Fourier transform of these distributions to obtain the analogous of the classical expressions of the plane waves. A simple calculation leads to the following expressions:

$$w_u(\vec{x}) \equiv m(\delta_u)(\vec{x}) = e^{i \vec{P}(u) \cdot \vec{x}} \xi_\epsilon.$$  

Formally, the plane waves have the same expression as the classical ones.

**Example 4: coordinate distributions**

Of particular interest are the coordinate distributions. Their expressions in $C(\ell_p(\mathbb{R}^3)^\ast)$ or $C(\ell_p(\mathbb{E}^3))$ are also easy to obtain. Indeed, we have shown that, in the $C(SU(2))^\ast$ representation, coordinates are the left-invariant vector fields $\chi^a$ which admit only one non-trivial component in the decomposition (41), $\chi_+^a = \chi^a$, because they are distributions localized at the origin $e$. The images by $a$ and $m$ are therefore singlet as well respectively given by

$$a(\chi^a) = i \frac{d}{dp_a} \delta^3(\vec{p}) \xi_+,$$

$$m(\chi^a) = x_a \xi_+.$$  

These expressions agree with their classical counterpart.

### 4.2 The $\ast$-product

Now the idea is to require, in some sense we will precise in the sequel, the mapping $a$ to be algebra morphisms. The product between two elements $\psi_1$ and $\psi_2$ in $C(\ell_p(\mathbb{R}^3)^\ast)$ is induced from that of $C(SU(2))^\ast$ as follows

$$\psi_1 \ast \psi_2 = a(a^{-1}(\psi_1) \ast a^{-1}(\psi_2)).$$
This $\ast$-product is far from being equal to the classical convolution product $\psi_1 \circ \psi_2$ presented in eq. (28) for two reasons. First, $\psi_i$ are now triplets $\psi_{i+} \oplus \psi_{i-} \oplus \psi_{i0}$ contrary to classical functions. Second, even if $\psi_1$, $\psi_2$ and $\psi_1 \ast \psi_2$ admit only one non-trivial component for each, let us say $\psi_{1+}$, $\psi_{2+}$ and $(\psi_1 \ast \psi_2)_{+}$, the resulting $\ast$-product is no-longer the classical convolution product.

To be more precise, let us compute explicitly the product of two plane waves, viewed as elements of $C_{\ell P}(\mathbb{R}^3)^{\ast}$. A plane wave is the image by $a$ of $\delta_u$ and is thus characterized by the vector $\vec{P}(u)$ and the space $U_\varepsilon$ to which $u$ belongs. As a result, a plane wave is given by $\delta_\vec{P}^\varepsilon \xi_\pm$ with $| \vec{P} | < \ell P^{-1}$ or by $\delta_\vec{P}^\varepsilon \xi_0$ with $| \vec{P} | = \ell P^{-1}$. A short calculation leads to the following product of two plane waves:

$$\delta_\vec{P}^\varepsilon \xi_\varepsilon \ast \delta_\vec{q}^\varepsilon \xi_\zeta = \delta_\vec{k}^\eta \xi_\eta,$$

where

$$\vec{k} \equiv \zeta \sqrt{1 - (\ell P | \vec{q} |)^2} \vec{p} + \epsilon \sqrt{1 - (\ell P | \vec{p} |)^2} \vec{q} + \ell P \vec{p} \wedge \vec{q},$$

$$\eta \equiv \text{sgn} \left( \epsilon \zeta \sqrt{1 - (\ell P | \vec{p} |)^2} \sqrt{1 - (\ell P | \vec{q} |)^2} - \ell P \vec{p} \cdot \vec{q} \right).$$

In the dual point of view, the plane waves represent deformed momenta and their products define in fact an addition rule of momenta.

The first important difference with the classical theory is that a deformed momentum is now characterized by a couple $(\vec{P}, \varepsilon)$ where $\vec{P}$ is bounded by $\ell P^{-1}$ and $\varepsilon$ is a discrete internal variable. Second, the addition rule is deformed compared to its classical counterpart and the resulting momentum vector depends on the initial polarization vectors. Nevertheless, this addition rule agrees with the classical one at the classical limit where $\ell P$ goes to zero and the parameter $\varepsilon$ is fixed to the value 1.

Now, let us see how this $\ast$-product is expressed in the position representation. The Fourier map $\mathfrak{F}$ induces an algebraic structure on $C_{\ell P}(\mathbb{R}^3)$ and we keep the notation $\ast$ for the product between two elements of $C_{\ell P}(\mathbb{R}^3)$. The $\ast$ product between two plane waves, viewed as elements of $C_{\ell P}(\mathbb{R}^3)$ reads

$$e^{i \vec{p} \cdot \vec{x}} \xi_\varepsilon \ast e^{i \vec{q} \cdot \vec{x}} \xi_\zeta = e^{i \vec{k} \cdot \vec{x}} \xi_\eta,$$

where $\vec{k}$ and $\eta$ are given by the formulæ (73).

Now, let us compute the $\ast$ product of two elements $\psi_1, \psi_2 \in C_{\ell P}(\mathbb{R}^3)$ which are the images by $a$ of two functions on $SU(2)$, $\phi_1$ and $\phi_2$. In other words, $C(SU(2))^\ast$ is restricted to the sub-algebra of functions $C(SU(2))$, and we study the algebraic properties of the image of $C(SU(2))$ by $a$. First, we remark that $C(SU(2))$ is stable by the convolution product. Then, we recall that the only non-vanishing components of $\phi \in C(SU(2))$ are $\phi_+$ and $\phi_-$. Therefore, the image of $\phi$ consists only in a couple of functions given, in terms of delta distributions, by

$$\psi = \sum_{\varepsilon = \pm} \int_{B_{\ell P}} d^3 \vec{p} \, \psi \varepsilon(\vec{p}) \delta_\vec{P}^\varepsilon \xi_\varepsilon .$$

(75)
To compute the $\star$ product $\psi_1 \star \psi_2$, one uses the above decomposition in terms of delta distributions and making use of the formula (72) of the $\star$-product between delta distributions, one obtains

$$\psi_1 \star \psi_2 = \sum_{\epsilon, \zeta = \pm} \int_{B_{\ell P}^2} d^3 \vec{p} d^3 \vec{q} \psi_1 \epsilon(\vec{p}) \psi_2 \zeta(\vec{q}) \delta^3_{\vec{k}} \xi_\eta, \quad (76)$$

where $\vec{k}$ and $\eta$ are given by (73). At the classical limit, this $\star$-product reduces to the classical convolution product on $\mathbb{R}^3$.

The application $\mathcal{F}$ maps $\psi_+ \oplus \psi_- \oplus 0 \in C_{\ell P}(\mathbb{R}^3)$ to a pair of functions $\Phi_+ \oplus \Phi_- \oplus 0 \in C_{\ell P}(\mathbb{E}^3)$ where $\Phi_{\pm}$ are functions of the variable $x$:

$$\Phi_{\pm}(x) = \int_{U_{\pm}} d\mu(u) \phi(u) e^{i \ell P(u) \cdot \vec{x}}$$

$$= \int \frac{d^3 \vec{y}}{2 \sqrt{1 - |\vec{y}|^2}} \phi(\vec{y}, \pm \sqrt{1 - y^2}) e^{i \ell P^{-1} \cdot \vec{y}} \cdot \xi_\eta, \quad (77)$$

where we have identified $u$ with $(\vec{y}, y_4)$ where the integral is defined over the vector $\vec{y}$ such that $y < 1$. Thus, the image of $C(SU(2)) \subset C(SU(2))^*$ by $m = \mathcal{F} \circ a$ is the sub-space of $C_{\ell P}(\mathbb{E}^3)$ where the third component is null. The product between two such functions $\Phi_1$ and $\Phi_2$ is implicitly given by

$$\Phi_1 \star \Phi_2 = m( m^{-1}(\Phi_1) \circ m^{-1}(\Phi_2) ). \quad (78)$$

In fact, the product is a couple of functions. The explicit expression of each term of the couple is neither needed nor simple but can be given in terms of a non-local integral of the form:

$$(\Phi_1 \star \Phi_2)(x) = \sum_{\epsilon, \zeta = \pm} \int d^3 y \, d^3 z \, K_{\epsilon \zeta}(x; y, z) \Phi_{\epsilon}(x_1) \Phi_{\zeta}(x_2), \quad (79)$$

where the kernel $K_{\epsilon \zeta}$ is defined by the double integral

$$K_{\epsilon \zeta}(x; y, z) = \int d^3 \vec{p} d^3 \vec{q} e^{i (\vec{p} \cdot \vec{y} + \vec{q} \cdot \vec{z} + \vec{k} \cdot \vec{x})} \xi_\eta, \quad (80)$$

with $\vec{k}$ and $\eta$ are given by the sum rule (73). This formula simplifies drastically when $\Phi_i$ are chosen to be coordinates functions. In that case, the integral can be performed explicitly and one obtains

$$x^a \xi_+ \star x^b \xi_+ = (x^a x^b + i \ell_P \epsilon_{abc} x^c) \xi_+, \quad (81)$$

which illustrates the non-commutativity of the coordinates. As expected from (44), one sees immediately that

$$[x^a \xi_+, x^b \xi_+] \equiv x^a \xi_+ \star x^b \xi_+ - x^b \xi_+ \star x^a \xi_+ = 2i \epsilon_{abc} \ell_P x^c \xi_+, \quad (82)$$

which is the $su(2)$ algebra.
4.3 Invariant measure on $C_{\ell_p}(E^3)$ and scalar action

In order to construct a local action, it is necessary to define an invariant measure on $C(SU(2))^*$. By invariant, we mean invariant under translations and rotations. As in the undeformed case, there is no hope to define an invariant measure on the whole distributional algebra $C(SU(2))^*$: we will focus on the sub-algebra of functions $C(SU(2))$.

From the very construction, elements of $C(SU(2))$ are linear forms on a subset of $\mathcal{D}(SU(2))$ which is, itself, endowed with (a two dimensional space of) invariant measures in the sense of Hopf algebras. Thus $C(SU(2))$ inherits naturally invariant measures which depend on $\alpha, \beta \in \mathbb{C}$ as follows:

$$h : C(SU(2)) \rightarrow \mathbb{C},$$

$$\phi \mapsto h(\phi) = \alpha \phi(e) + \beta \phi(e_A).$$

It is immediate to check that $h$ is invariant under rotations and translations. It is also easy to see that $h$ does not extend to the whole distributional algebra $C(SU(2))^*$, the delta distribution at $e$ or $e_A$ for instance being non-normalizable with respect to this measure. Furthermore, $h$ allows to define a bilinear hermitian form on $C(SU(2))$ such that the scalar product between $\phi_1$ and $\phi_2$ is given by:

$$h(\phi_1^\flat \circ \phi_2) = \alpha \int d\mu(u) \overline{\phi_1(u)} \phi_2(u) + \beta \int d\mu(u) \overline{\phi_1(u)} \phi_2(u e_A)$$

where we have used the notation $\phi^\flat = \iota(\phi)$ introduced in eq. (83). In order to have a positive definite bilinear form, one has to assume that $\beta$ vanishes. We will make this assumption in the sequel, together with the choice $\alpha = 1$.

Having in mind the construction of a non-commutative QFT, it is useful to export $h$ in the $C_{\ell_p}(E^3)$ formulation of $C(SU(2))^*$ via the map $m$:

$$H(\Phi) \equiv h(m^{-1}(\Phi)) = \int \frac{d^3x}{(2\pi)^3 v_{\ell_p}} \xi_+^t \Phi(x),$$

where the integral in the r.h.s. is defined from the standard Lebesgue measure on $\mathbb{R}^3$. Let us remind that, as $\Phi = m(\phi)$ is the image of a function $\phi$, it admits only two components $\Phi_\pm$ and then the measure $h$ involves only one of these two components. Let us note however that the norm of $\Phi$ involves its two components, namely:

$$h(\phi^\flat \circ \phi) = \xi_+^t \int \frac{d^3x}{(2\pi)^3 v_{\ell_p}} (\Phi_+(x) \xi_+ \Phi_+(x) \xi_+ + \Phi_-(x) \xi_- \Phi_-(x) \xi_-).$$

Such a measure is necessary to define an action for a QFT in the non-commutative space $C_{\ell_p}(E^3)$. For instance, the free action for a scalar field $\Phi = m(\phi)$, where $\phi$ is a function on $SU(2)$, is given by the following integral:

$$\frac{2S[\phi]}{(2\pi)^3 v_{\ell_p}} = -h(\phi^\flat \circ (P^2 + m^2)\phi) = h((P\phi)^\flat \circ (P\phi)) - m^2 h(\phi^\flat \circ \phi)$$
where \( P \) is the momentum function [16]. This action is constructed by analogy with its classical undeformed counterpart: it is quadratic, of second order and local according to the \( \star \)-product in the position representation. It is immediate to notice that the components \( \Phi_+ \) and \( \Phi_- \) decouple and then, using (86), one shows that the action reads

\[
S[\Phi] = S_+[\Phi_+] + S_-[\Phi_-]
\]

where:

\[
S_{\pm}[\Phi_{\pm}] = \frac{1}{2} \xi_{\pm} \int d^3x \left( \partial^\alpha \Phi_{\pm} \xi_{\pm} \star \partial_\alpha \Phi_{\pm} \xi_{\pm} - m^2 \Phi_{\pm} \xi_{\pm} \star \Phi_{\pm} \xi_{\pm} \right) .
\]

(88)

We have assumed for simplicity that the fields \( \Phi_{\pm} \) are real, i.e. \( \overline{\Phi_{\pm}} = \Phi_{\pm} \) which is equivalent to the condition \( \phi = \phi^\dagger \).

Contrary to the Moyal case, the integral of the \( \star \)-product of two functions is different from the classical integral. To illustrate some differences between the deformed and the classical integrals of the product of two functions, let us consider the example where the functions are plane waves. In the case of the \( \star \)-product, one gets

\[
\int d^3x \left( \frac{2\pi}{\ell_P} \right)^3 \psi_\ell P(u) \psi_\ell P(v) \delta_\ell(uv) \xi_+, \quad (89)
\]

whereas for the point-wise product, one gets

\[
\int d^3x \left( \frac{2\pi}{\ell_P} \right)^3 \psi_\ell P(u) e^{i \vec{P}(u) \cdot \vec{x}} \psi_\ell P(v) e^{i \vec{P}(v) \cdot \vec{x}} = \frac{1}{\sqrt{1 - (\ell_P P(u))^2}} \delta_\ell(uv) .
\]

(90)

The difference between the non-commutative and standard commutative QFT is even deeper when one considers self-interactions. As an example, let us consider a cubic self-interaction which is defined such that the momenta addition rule is satisfied at each vertex. In the classical case, this requirement leads to the standard action whereas, in the non-commutative case, it leads to the following simple interacting term with coupling constant \( \lambda \):

\[
S^{(3)}[\phi] = (2\pi)^3 v_\ell P \lambda \frac{3!}{3!} h(\phi \circ \phi \circ \phi),
\]

(91)

in the momentum representation. The generalization to any polynomial interaction is immediate. It is interesting to write this integral in terms of the functions \( \Phi_\pm \) and, after some calculations, one gets

\[
S^{(3)}[\Phi] = \frac{\lambda}{3!} \xi_+ \int d^3x \left( \Phi \star \Phi \star \Phi \right) (x)
\]

(92)

In the momentum representation where \( \psi = \mathfrak{F}^{-1}[\Phi] \) this vertex reads

\[
S^{(3)}[\psi] = \frac{\lambda}{3!} \sum_{\epsilon, \zeta = \pm} \int d^3p d^3q \psi_\epsilon(p) \psi_\zeta(q) \psi_\eta(-k) ,
\]

(93)

where \( \overline{k} \) and \( \eta \) are related to \( \overline{\vec{p}}, \overline{\vec{q}} \) and \( \epsilon, \zeta \) by eq. (73). Contrary to the free action, this interaction term couples the two components of the field \( \Phi \); there are four
different vertices in the theory. Only the case where one out of the two components is non-trivial has been investigated so far.

We leave the precise study of this action for future investigations but we can give some preliminary interesting results. First, note that at each vertex the momentum conservation holds with the deformed addition rule. Next, let us consider the Feynman propagator of the free field theory which has been studied in [15]. In the momentum representation, it is simply given by

$$\Gamma(u) = \frac{1}{P^2(u) + m^2},$$

with $P(u) = \ell_P^{-1} \sin(\theta/2)$. An immediate analysis leads to the fact that the associated functions $G(x; x')$ depend only on the distance $|\vec{x} - \vec{x}'|$ between two positions $x$ and $x'$, and are given by the following equivalent expressions:

$$G(x; x') = \frac{1}{(2\pi)^3} \int_{U_+} d\mu(u) \frac{e^{i\vec{P}(u) \cdot (\vec{x} - \vec{x}')}}{P^2(u) + m^2} = \frac{1}{8r} \int_0^1 \frac{t dt}{\sqrt{1 - t^2}} \sin(\ell^{-1}_p r t),$$

with $r = |\vec{x} - \vec{x}'|$. It is interesting to compare this function with its classical counterpart:

$$G_{cl}(x; x') = \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}(\vec{x} - \vec{x}')}}{p^2 + m^2} = \frac{\exp(-mr)}{4\pi r}.$$  

Contrary to the classical case, the integral over the momentum $p$ is definite and the upper bound depends on the Planck length $\ell_P$. This important fact makes the Feynman propagator well-defined at the coincident point limit $r \to 0$:

$$G(x; x) = G(0; 0) = \frac{1}{4\pi \ell_P} \left[ 1 - \frac{m \ell_P}{\sqrt{2 + (m \ell_P)^2}} \right].$$

Physically, the non-commutativity or equivalently the boundness of the space of momenta regularizes the ultra-violet divergences of the classical propagator.

At large distances, the propagator $G(x; x')$ should coincide with $G_{cl}(x; x')$. To see this is indeed the case, let us note that $G(x; x')$ is in fact a function $F(r/\ell_P, m \ell_P)$ from the expression [95]. Then, it becomes clear that the large distance limit is defined by the condition $\ell_P \to 0$ with $m$ and $x$ fixed. As a result, we have

$$G(x; x') \sim G_{cl}(x; x').$$

Therefore, we recover the classical behavior of the propagator.

5 Fuzzy space formulation of $C(SU(2))^*$

This section is devoted to show that $C(SU(2))^*$ can be described as a fuzzy space. This is done, in a first part, by defining a notion of Fourier transform on the non-commutative algebra of distributions $C(SU(2))^*$. Their image by this Fourier transform is an algebra of matrices. In a second part, we concentrate on symmetry aspects and show how the action of $\mathcal{D}(SU(2))$ is expressed in the fuzzy formulation.
5.1 Fourier transform of $C(SU(2))^*$

Let us start by recalling that $C(SU(2))^*$ is the convolution algebra of distributions on $SU(2)$. The Fourier transform on $C(SU(2))^*$ can be induced by that on $C(SU(2))$.

Given a compact Lie group $G$, the Fourier transform on $C(G)$ is defined using harmonic analysis on the group: the Fourier transform of a given function in $C(G)$ is its decomposition on the unitary irreducible representations (UIR) of the group $G$. In the case of $SU(2)$, UIRs are characterized by a half-integer spin $j$ and their basis vectors are labelled by the magnetic number $m \in [-j, +j]$:

$$J^2|j, m\rangle = j(j+1)|j, m\rangle, \quad J_3|j, m\rangle = m|j, m\rangle.$$

(99)

It is useful to introduce the Wigner $D$-matrices $D_{nm}^{j}(u)$ as matrix elements of representations.

The $SU(2)$ Fourier transform is then defined as the following map:

$$\mathcal{F} : C(SU(2)) \longrightarrow \text{Mat}(\mathbb{C}) \equiv \bigoplus_{n \in \mathbb{N}} \text{Mat}_{n \times n}(\mathbb{C})$$

$$f \longmapsto \mathcal{F}[f] \equiv \bigoplus_{2j+1 \in \mathbb{N}} \int d\mu(u) D^{j}(u^{-1}) f(u).$$

(100)

The inverse map $\mathcal{F}^{-1} : \text{Mat}(\mathbb{C}) \rightarrow C(SU(2))$ is given by

$$\mathcal{F}^{-1}[M](u) \equiv \text{Tr} (D(u) M), \quad \forall M = \bigoplus_{n \in \mathbb{N}} M_{(n)} \in \text{Mat}(\mathbb{C}),$$

(101)

where $M_{(n)} \in \text{Mat}_{n \times n}(\mathbb{C})$ and the trace $\text{Tr}$ in $\text{Mat}(\mathbb{C})$ is defined as

$$\text{Tr} M \equiv \sum_{n \in \mathbb{N}} n \, \text{tr} M_{(n)}.$$

(102)

We have also introduced the notation $D$ for the $\text{Mat}(\mathbb{C})$-valued $SU(2)$ functions defined by $D(u) = \bigoplus_{j} D^{j}(u)$.

The Fourier transform $\mathcal{F}[\phi]$ of a distribution $\phi \in C(SU(2))^*$ is a linear map on $\text{Mat}(\mathbb{C})$ defined by the relations

$$\text{Tr} \left( M^\dagger \mathcal{F}[\phi] \right) \equiv \langle \mathcal{F}^{-1}[M], \phi \rangle, \quad \forall M \in \text{Mat}(\mathbb{C}).$$

(103)

Using the equation (101), we get a more explicit and simple expression of $\mathcal{F}[\phi]$:

$$\mathcal{F}[\phi] = \langle D^{t}, \phi \rangle.$$

(104)

When $\phi$ belongs to $C(SU(2))$, this formula coincides with (100).

Finally, we have the ingredients to write explicitly the Fourier transform of $C(SU(2))^*$: to any distribution $\phi$, we associate an element $\hat{\Phi}$ of $\text{Mat}(\mathbb{C})$ as

$$\hat{\Phi} \equiv \mathcal{F}[\phi] = \langle D^{t}, \phi \rangle.$$

(105)
This map is invertible and its inverse reads
\[ \phi = \text{Tr} (\Phi D) . \tag{106} \]

The Fourier map defines a morphism and a co-morphism. As a result, the space \( \text{Mat}(\mathbb{C}) \) inherits a Hopf-algebra structure. Of particular interest is the algebra structure of \( \text{Mat}(\mathbb{C}) \) inherited from the convolution product in \( C(SU(2))^* \) which is formally defined as follows:
\[ \hat{\Phi}_1 \star \hat{\Phi}_2 \equiv \langle D^t, \phi_1 \star \phi_2 \rangle = \langle \Delta_{C(SU(2))}(D^t), \phi_1 \otimes \phi_2 \rangle . \tag{107} \]

In fact, this product is the standard matrix product in \( \text{Mat}(\mathbb{C}) \):
\[ \hat{\Phi}_1 \star \hat{\Phi}_2 = \hat{\Phi}_1 \hat{\Phi}_2 . \tag{108} \]

This is a consequence of \( \Delta_{C(SU(2))}(D^t_{mm'}) = \sum_n D^i_{mn} \otimes D^i_{mn'} \). Therefore the algebra \( C(SU(2))^* \) is isomorphic to the algebra of matrices \( \text{Mat}(\mathbb{C}) \) and in the following we will omit the notation \( \star \) for the matrix product.

As a final remark about the Fourier transform, let us write the scalar product between two distributions \( \phi_1 \) and \( \phi_2 \) in terms of their Fourier modes:
\[ \langle \phi_1, \phi_2 \rangle = \text{Tr} (\hat{\Phi}_1^\dagger \hat{\Phi}_2) . \tag{109} \]

The obtained scalar product is the usual one defined in the fuzzy sphere when restricted to one single space (or fuzzy sphere) \( \text{Mat}_{n\times n}(\mathbb{C}) \).

### 5.2 \( \mathcal{D}(SU(2)) \) symmetry in the Fuzzy space

From the very beginning, we know that \( \mathcal{D}(SU(2)) \) plays a crucial role in the construction of \( C(SU(2))^* \): \( \mathcal{D}(SU(2)) \) can be viewed as the isometry algebra of the non-commutative space. We have already expressed the action of \( \mathcal{D}(SU(2)) \) on \( C(SU(2))^* \) in eq.(106). It is easy to export this action to the matrix algebra \( \text{Mat}(\mathbb{C}) \) as follows:
\[ (f \otimes u) \triangleright \hat{\Phi} \equiv \mathcal{F}[(f, u) \triangleright \phi] = \langle \text{ad}_{u^{-1}}(\tilde{f} D^t), \phi \rangle , \tag{110} \]
where \( (f \otimes u) \in \mathcal{D}(SU(2)) \), \( \phi \) is a distribution in \( C(SU(2))^* \) and \( \hat{\Phi} \) its Fourier transform. In this section, we want to express this action explicitly. For purposes of clarity, we will study separately actions of rotations in \( SU(2) \subset \mathcal{D}(SU(2)) \) and those of translation-like elements in \( C(SU(2)) \subset \mathcal{D}(SU(2)) \).

**Action of \( SU(2) \)**

We know that rotations act by conjugacy on \( C(SU(2))^* \) (see eq.(106)). As a consequence, the action of \( u \in SU(2) \) on a matrix family \( \hat{\Phi} \) simply reads
\[ (1 \otimes u) \triangleright \hat{\Phi} = D(u) \hat{\Phi} D(u^{-1}) , \tag{111} \]
As expected, rotations leaves the fuzzy spheres \( \text{Mat}_{n\times n}(\mathbb{C}) \) invariant. We can deduce the action of an infinitesimal rotation by \( \mathcal{J}_a \in \mathfrak{su}(2) \) on \( \hat{\Phi} \):
\[ (1 \otimes \mathcal{J}_a) \triangleright \hat{\Phi} = [D(\mathcal{J}_a), \hat{\Phi}] . \tag{112} \]
Action of $C(SU(2))$

Translation-like elements $(f \otimes e)$ act by multiplication of $f$ in the space $C(SU(2))^*$. This action induces the following one on the space of matrices:

$$
(f \otimes e) \triangleright \hat{\Phi} = \langle f D^t, \phi \rangle = \text{Tr}(F[\hat{f} D]^\dagger \hat{\Phi})
= \int d\mu(u) f(u) D(u^{-1}) \text{Tr}(D(u) \hat{\Phi}) .
$$

(113)

To have a more intuitive idea, let us look at infinitesimal translations generated by the momentum $P_a = (P_a, e)$ defined by $P_a(u) = \ell P_a \sin(\theta/2) = \ell P_a i \text{tr}^{12}(J a u)$.

(114)

The notation $\text{tr}^{12}$ holds for the trace in the fundamental $SU(2)$ representation. The calculation of $P_a \triangleright \hat{\Phi}$ mimics the previous one and we get after a straightforward calculation that

$$
(P_a \triangleright \hat{\Phi})^j_{st} = \frac{i}{\ell P_a} \sum_k (-1)^{m-n} \hat{\Phi}^j_{mn} D^{1/2}_{pq}(J_a) \left( \begin{array}{ccc} 1/2 & j & k \\ q & s & -m \end{array} \right) \left( \begin{array}{ccc} 1/2 & j & k \\ p & t & -n \end{array} \right)
= \frac{i D^{1/2}_{pq}(J_a)}{\ell P_a (2j+1)} \left( \sqrt{(j+1+2qs)(j+1+2tp)} \hat{\Phi}^{j+1/2}_{q+s+p+t} \right.
+ (-1)^{q-p} \sqrt{(j-2qs)(j-2pt)} \hat{\Phi}^{j-1/2}_{q+s+p+t} ,
$$

(115)

where we have used the notations of the book [10] for the Clebsch-Gordan coefficients. From the above formula, we see that infinitesimal translations move points of the sphere $\text{Mat}(2j+1 \times (2j+1))(\mathbb{C})$ into points of the nearest neighbor spheres. This is exactly what one would expect.

6 Relation between $\text{Mat}(\mathbb{C})$ and $C_{\ell P}(\mathbb{E}^3)$

This section is devoted to establish a general correspondence between the space of matrices $\text{Mat}(\mathbb{C})$ and the deformed algebra of functions $C_{\ell P}(\mathbb{E}^3)$. This correspondence is important in order to understand for instance the undeformed limit of the space of matrices.

6.1 The general relation

For that purpose, it is necessary to decompose the matrix $\hat{\Phi}$, associated to a distribution $\phi$, into a multiplet $\hat{\Phi}_+ \oplus \hat{\Phi}_- \oplus \hat{\Phi}_0$ of matrices associated to the multiplet $\phi_+ \oplus \phi_- \oplus \phi_0$. Each component $\hat{\Phi}_\epsilon$ is the matrix representation of $\phi_\epsilon$ and can be obtained from the matrix $\hat{\Phi}$ as follows:

$$
\hat{\Phi}_\pm = \int d\mu(g) \text{Tr}(\hat{\Phi} D(g)) D^i(g) I_\pm(g) \quad \text{and} \quad \hat{\Phi} = \hat{\Phi}_+ + \hat{\Phi}_- + \hat{\Phi}_0
$$

(116)
where \( I_\pm (g) = \Theta (\pm (\pi - \theta)) \) is characteristic functions written in terms of the theta function: \( \Theta (x) = 1 \) if \( x > 0 \) otherwise 0.

Now, we can establish a link between each component \( \Phi_\varepsilon \) and the functions \( \Phi_\varepsilon (x) \). We start with the cases \( \varepsilon = \pm \). Using the relations (116) and (117), we obtain the mapping from \( \Phi \in \text{Mat}(\mathbb{C}) \) to \( \Phi_\pm \in \mathcal{C}_\ell (\mathbb{E}^3) \):

\[
\Phi_\pm (x) = \langle e^{iP_\varepsilon x} \chi_\pm , \phi_\pm \rangle = \text{Tr} \left( \mathcal{F}^\dagger [e^{iP_\varepsilon x} I_\pm] \mathcal{F}[\phi_\pm] \right) = \text{Tr} \left( K_\pm^\dagger (x) \tilde{\Phi}_\pm \right)
\]

where \( e^{iP_\varepsilon x} \) is the plane wave viewed as a function on \( SU(2) \). This mapping is by construction a morphism. In the last identity, we introduced the notation \( K_\pm \) for the \( \text{Mat}(\mathbb{C}) \)-valued function on \( \mathbb{E}^3 \) explicitly defined by \( K_\pm (x) \equiv \mathcal{F}[e^{iP_\varepsilon x} I_\pm] \). The functions \( K_\pm \) can be interpreted as the components of an element \( K \in \mathcal{C}_\ell (\mathbb{E}^3) \otimes \text{Mat}(\mathbb{C}) \) given by:

\[
K(x) = K_+ (x) \xi_+ + K_- (x) \xi_- = \langle w_x , D \rangle = \int d\mu (u) D(u) \overline{w_x (u)}.
\]

It remains to obtain the relation between \( \tilde{\Phi}_0 \) and \( \Phi_0 (x) \). To do so, we follow the same idea as previously. We start by writing the relation between \( \phi_0 \) and \( \Phi_0 (x) \) making use of the momentum \( \vec{P}_0 \) and then we expand \( \phi_0 \) into its Fourier modes \( \tilde{\Phi}_0 \) to obtain the relation:

\[
\Phi_0 (x) = \langle e^{iP_0 x} , \phi_0 \rangle = \text{Tr} (K_\nu_0^\dagger (x) \tilde{\Phi}_0)
\]

where the matrix valued function \( K_\nu_0 \) depends on a vicinity \( \nu_0 \subset SU(2) \) of \( U_0 \) where \( P_0 \) is well-defined according to the integral:

\[
K_\nu_0 (x) = \int d\mu (u) D(u) \exp (-iP_0 (u) \cdot x) I_\nu_0 (u)
\]

where \( I_\nu_0 \) is the characteristic function on \( \nu_0 \). It is clear that \( K_\nu_0 \) depends on the choice of \( \nu_0 \) but the relation (119) is independent of that choice.

The relations (117) and (119) are invertible and their inverse can be obtained using inverse Fourier transform. Thus, one uniquely associates to any element of \( \mathcal{C}_\ell (\mathbb{E}^3) \), a family of matrices in \( \text{Mat}(\mathbb{C}) \).

One can interpret the function \( \Phi_\varepsilon (x) \) as a kind of continuation to the whole Euclidean space of a discrete function \( \Phi_{\varepsilon, nm}^j \) which is a priori defined only on an infinite but denumerable set of points: there are obviously \( (2j + 1)^2 \) points on each fuzzy sphere of dimension \( 2j + 1 \). Given any \( x \in \mathbb{E}^3 \), each matrix element \( \tilde{\Phi}_{\varepsilon, nm}^j \) contributes to the definition of \( \Phi_\varepsilon (x) \) with a complex weight \( K_{\pm, nm}^j (x) \) or \( K_{\varepsilon, nm}^j (x) \).

We conclude this section with some properties of the elements \( K_\pm \) and \( K_\nu_0 \).

1. \( K_\pm \) are normalized elements in both \( \mathcal{C}_\ell (\mathbb{E}^3) \) and \( \text{Mat}(\mathbb{C}) \) in the sense that:

\[
\int \frac{d^3 x}{(2\pi)^3} \nu_\ell \mu \nu_\ell K_\pm (x) = I_\pm, \quad \text{Tr} (K(x)) = \xi_+,
\]

where \( I_\pm = \bigoplus_{n \in \mathbb{N}} (-1)^{n+1} \mathbb{I}_{(n)} \) with \( \mathbb{I}_{(n)} \) the \( n \times n \) identity matrix.
2. The previous properties can be extended to the matrix-valued function $K_{\nu_0}$. Assuming that $P_0$ does not vanish in $\nu_0$ and $\nu_0$ does not contain the identity, then we have:

$$\int \frac{d^3x}{(2\pi)^3} v_\ell P K_{\nu_0}(x) = 0, \quad \text{Tr} (K_{\nu_0}(x)) = 0.$$ (122)

3. The matrix elements of the functions $K$ form clearly a sub-algebra of $C_{\ell P}(E^3)$ and their product is given by

$$\left(K^j_{\ell m} \star K_{\ell m'}^{j'}\right)(x) = \frac{\delta_{jj'}}{2j+1} \delta_{m'n} K^j_{mn}(x).$$ (123)

These relations are consequences of the fact that the matrix elements of $SU(2)$ UIRs are orthonormal when viewed as functions on $SU(2)$.

6.2 The coordinates on the Fuzzy space

In this section we determine and study the coordinate functions $x_a \xi_+ \in C_{\ell P}(E^3)$ in the fuzzy space representation $\text{Mat}(\mathbb{C})$. To construct the coordinate functions, we use the plane waves $w_u(x)$ which can be viewed as a coordinate generating function.

Let us start by finding the matrix representation of plane waves. Plane waves have been defined previously as the image in $C_{\ell P}(E^3)$ of delta distributions $\delta_u$. As a result, the matrix representation of a plane wave reads:

$$\hat{w}_u = \langle D^t, \delta_u \rangle = D(u^{-1}) .$$ (124)

Using the parametrization $\{13\}$ of $SU(2)$ and doing the following expansion in the angle $\theta$ of each side of the above equation:

$$\hat{w}_u = I + 2i \theta n^a \left[ \ell^{-1} \hat{x}_a \right] - 2 \theta^2 n^a n^b \left[ \ell^{-2} \hat{x}_a \hat{x}_b \right] + O(\theta^3) ,$$

$$D(u^{-1}) = I + i \theta n^a \left[ D(J_a) \right] - \frac{\theta^2}{2} n^a n^b \left[ \frac{1}{2} D(J_a J_b + J_b J_a) \right] + O(\theta^3)$$ (125)

one finds immediately (at the first order in $\theta$) the matrix representations $\hat{x}_a$ of the coordinate functions $x_a \xi_+ \in C_{\ell P}(E^3)$ which are just the matrix representation of the $su(2)$ generators $J_a$:

$$\hat{x}_a = 2 \ell P D(J_a) .$$ (126)

As expected, the coordinates are non-commutative functions which satisfy the $su(2)$ Lie-algebra relation. This relation holds as it should in the $C_{\ell P}(E^3)$ representation and read the commutator $\{82\}$ between coordinate functions with $\star$-product. Furthermore, at the second order in $\theta$, one obtains more than the commutator between coordinates but the product of coordinates $\{81\}$.

We recover the well-known fact that coordinates functions on fuzzy spheres are obtained from representations of the $su(2)$ Lie algebra generators. Note that, in our
construction, this fact is not put by hand but is a consequence of some more fundamental principles. The geometrical consequences of this result are immediate: the spectrum of coordinates is discrete and can be obtained from $\mathfrak{su}(2)$ representations theory. In particular, the radius (squared) matrix $\hat{r}_r^2$ with $r_r^2 \xi_+ = x_a \xi_+ \star x^a \xi_+$ is proportional to the identity matrix on sphere $\text{Mat}_{n \times n}(\mathbb{C})$ and its values are given by the $\mathfrak{su}(2)$ Casimir evaluated in the representation $j = (n - 1)/2$, i.e. $\hat{r}_r = 2 \sqrt{j(j + 1)} \ell_P \mathbb{1}_{n \times n}$. Thus the algebra of matrices $\text{Mat}(\mathbb{C})$ can be thought as the dual of concentric fuzzy spheres or a fuzzy onion.

### 6.3 Examples

To have a more intuitive idea of the non-commutative space, let us now give some simple examples of functions and their equivalent formulations in $C(SU(2))^*$, $C_{\ell_P}(\mathbb{E}^3)$ and $\text{Mat}(\mathbb{C})$.

**Constant function**

Constant functions are the simplest examples which correspond in $C(SU(2))^*$ to the elements $\phi = c_+ \delta_\epsilon + c_- \delta_{\epsilon A}$ where $c_+, c_-$ are constant. We deduce immediately its $C_{\ell_P}(\mathbb{E}^3)$ and $\text{Mat}(\mathbb{C})$ representations given by

$$\hat{\Phi}^j = c_+ \mathbb{1}^j + (-1)^j c_- \mathbb{1}^j, \quad \Phi^\pm(x) = c^\pm.$$  \hspace{1cm} (127)

One-dimensional functions are also simple but less trivial examples. Their set is defined as the kernel of two out of the three derivative operators $\partial_a$. They form a commutative algebra which is not, nonetheless, the point-wise product. These functions and some applications have been studied in [17].

**Radial function**

A classical radial function depends only on the radius $r = \sqrt{x^a x_a}$ and is obviously invariant under rotations. In the deformed context, a radial function is also defined to be invariant under rotations. As a consequence, in the $C(SU(2))^*$ representation, a radial function corresponds to a distribution $\phi$ on the conjugacy classes of $SU(2)$. In the sequel, we consider only the cases where $\phi$ is a function for simplicity. One immediately obtains that the matrices $\hat{\Phi}^j_{\pm}$ are proportional to the identity matrix:

$$\hat{\Phi}^j_{\pm} = \lambda^j_{\pm} \mathbb{1}_{(2j + 1)}.$$  \hspace{1cm} (128)

In the $C_{\ell_P}(\mathbb{E}^3)$ representation, $\Phi^\pm$ can be expressed as a series:

$$\Phi^\pm(r) = 8\pi \sum_{2j + 1 \in \mathbb{N}} \ell_P r_j^2 \lambda^j_{\pm} R^j_{\pm}(r),$$  \hspace{1cm} (129)

with the discrete radius $r_j \equiv (2j + 1) \ell_P$ and the radial modes $R^j_{\pm}(r)$ are given by

$$R^j_{\pm}(r) \equiv \frac{1}{2(2\pi)^3} \frac{1}{(2j + 1) v_{\ell_P}} \text{tr}(K^j_{\pm}(x)).$$  \hspace{1cm} (130)
Let us underline some properties of the functions $R^j_{\pm}$:

1. A more explicit integral formula for the functions $R^j_{\pm}$ can be obtained as follows:

\[
R^j_{\pm}(r) = \frac{1}{2(2\pi)^3} \frac{1}{(2j + 1)v_{\ell_P}} \int d\mu(u) C^j(u) w_{u-1}(r) I_{\pm}(u) \tag{131}
\]

\[
= \frac{(\pm 1)^{2j}}{4\pi r r_j} \left[ \ell_P^{-1} \int_0^{\ell_P} \frac{d\theta}{2\pi} \sin\left(\frac{r_j}{2 \ell_P} \theta\right) \sin\left(\frac{r}{\ell_P} \sin(\theta/2)\right) \right],
\]

where $C^j(u) = \text{tr}(D^j(u))$ are the $SU(2)$ characters. In fact, when $j$ is an integer $R^j_{\pm}$ can be expressed in terms of the Bessel function $J_n$ as follows:

\[
R^j_{\pm}(r) = \frac{1}{4\pi r r_j} \left[ \frac{1}{2 \ell_P} J_{r_j/\ell_P}(r/\ell_P) \right]. \tag{132}
\]

This is not true when $j$ is a half-integer.

2. The functions $R^j_{\pm}$ are normalized when viewed as elements of $\text{Mat}(\mathbb{C})$ as well as when viewed as elements of $C_{\ell_P}(\mathbb{E}^3)$:

\[
4\pi \int dr^2 R^2_{\pm}(r) = \frac{(\pm 1)^{2j}}{2}, \quad 4\pi \sum_{2j+1 \in \mathbb{N}} (2 \ell_P) r_j^2 R^j_{\pm}(r) = \frac{1 \pm 1}{2}. \tag{133}
\]

This is a consequence of the eq. (121).

There is a nice interpretation of the decomposition formula (129) as the Riemann sum of the classical integral:

\[
\Phi(r) = \int_0^\infty d\rho \Phi(\rho) \delta(r - \rho). \tag{134}
\]

For such an interpretation to be true, it is necessary to compute the classical limit of the radius modes $R^j_{\pm}$. First of all, we defined the classical limit by $\ell_P \to 0$ but $r$ and $r_j$ finite. We note that, in the integral formula of $R^j_{\pm}$, the two sine functions in the integrand oscillate very fast and the integral vanishes unless $r = r_j$. Since $R^j_{\pm}(r)$ is normalized to $(\pm 1)^{2j}$ according to the formula (133), the radial modes $R^j_{\pm}(r)$ tend to the delta distribution $\delta(r - r_j)$, up to a global factor, in the undeformed limit:

\[
R^j_{\pm}(r) \xrightarrow{\ell_P \to 0} \frac{(\pm 1)^{2j}}{8\pi r^2} \delta(r - r_j). \tag{135}
\]

This formula shows that in the undeformed limit, the value of radial function $\Phi_{\pm}$ at $r = r_j$ is given by the matrix element $\lambda^{(j)}_{\pm}$: $\Phi_{\pm}(r_j) = (\pm 1)^{2j} \lambda^{(j)}_{\pm}$.
Delta function

Other important examples are the delta distributions for the $\ast$-product. As in the classical case, the delta-distribution localized at the point $x$ is, in the $C(SU(2))^*$ formulation, the plane wave $w_x(u)$. Thus, as one could expect, the delta-distributions localized at $x$ are related to the delta distribution localized at the origin by a translation, namely $w_x = T_{-x} \ast w_0$. For that reason we concentrate first on the delta distributions localized at the origin which is in fact the constant function 1 in the $C(SU(2))^*$ formulation.

In the fuzzy space formulation, the delta distribution at the origin is given by the matrix $\hat{\delta}^0_0$ whose elements are

$$\hat{\delta}^j_0 = \delta_{j,0} . \quad (136)$$

As the result, the matrix is completely localized at the fuzzy origin.

In order to get its $C_{\ell P}(E^3)$ formulation, one has to decompose the constant function 1 into its three components $1_+ \oplus 1_- \oplus 1_0$. It is immediate to see that $1_\pm(g) = I_\pm(g)$ and $1_0 = 0$ where $I_\pm$ have been introduced in (116). Note that there is no zero component for the delta distribution. Furthermore, the functions $\delta_{0\pm} \in C_{\ell P}(E^3)$ are radial. One can easily show that they are equal and given by:

$$\delta_{0\pm}(r) = \frac{1}{\pi r} \int_0^\pi d\theta \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{r}{\ell_P} \sin\left(\frac{\theta}{2}\right)\right) = \frac{1}{2} R^0(r) \quad (137)$$

where $R^j_{\pm}$ have been introduced in the previous examples. The delta-distribution satisfies the following expected relation:

$$\xi_+^t \int \frac{d^3x}{(2\pi)^3 v_{\ell P}} (\Phi \ast \delta_0)(x) = \Phi_+(0) + \Phi_-(0), \quad \forall \Phi_\xi \in C_{\ell P}(E^3). \quad (138)$$

Since $w_y = T_{-y} w_0$ we find that the delta distributions localized at any point $y$ satisfy

$$\delta_{y\pm}(x) = (T_y \ast \delta_{0\pm})(x) = \delta_{0\pm}(x - y) . \quad (139)$$

The main difference with the classical case is that $\delta_{0\pm}$ is spread around the origin. It is nevertheless the most localized function at the origin: we cannot localized the origin of the fuzzy space with a better precision due to the non-commutativity of the coordinates. Besides, $\delta_{0\pm}$ tends to the classical delta distribution on $E^3$ in the undeformed limit because $R^0$ does as we have seen it previously.

When considering in (138) $\Phi(x) = x_a \xi_+$ we recover the fact that $\delta^*_{y\pm}$ is localized at the point $y$. Nevertheless, as for the function $\delta^*_0$, $\delta^*_y$ is spread. This fact can be illustrated computing the previous integral with $\Phi(x) = x_a \xi_+ \ast x_b \xi_+$:

$$\int \frac{d^3x}{(2\pi)^3 v_{\ell P}} x_a \xi_+ \ast x_b \xi_+ \ast \delta^*_y(x) = y_a \xi_+ \ast y_b \xi_+ \neq y_a y_b \xi_+ . \quad (140)$$

As a result, the integral is different from its undeformed analog.
6.4 Maximally localized state

As shown in the previous lines, $\delta^\star_0$ distributions are interpreted as the most localized functions at the origin in $C_{\ell_P}(E^3)$. In this section we determine the functions which are the most localized around a classical point.

To do so, we work in the fuzzy space representation. We define the most localized function at a given classical point $x \in E^3$ as the one which corresponds to the minimum of the operator $(\hat{x}_a - x_a)(\hat{x}^a - x^a)$, viewed as a family of matrices.

The minimum of the previous operator is realized on the state $|j\vec{e}_x\rangle$ where $\vec{e}_x$ is the unit vector in the direction of $\vec{x}$ and the state is such that: first, it is an eigenvector of $\vec{J} \cdot \vec{e}_x$ with eigenvalues $j$; second, it is an eigenvector of $J^2$ with eigenvalue $j(j+1)$; third, the eigenvalues are such that:

$$2j + \frac{1}{2} = \ell_P - \frac{1}{2} = \left\lfloor \frac{r}{\ell_P} - \frac{1}{2} \right\rfloor. \quad (141)$$

where $\left\lfloor x \right\rfloor$ is the Ceiling function of $x$, namely the smallest integer not less than $x$.

The minimum of the previous operator is thus given by

$$\min[(\vec{x} - x)^2] = \ell_P^2 \left\{ (\left\lfloor a \right\rfloor - a)^2 + 2a \right\}, \quad a = \frac{r}{\ell_P} - \frac{1}{2}. \quad (142)$$

Notice that as a function of $r$ this minimum has local minima for $a = \left\lfloor a \right\rfloor$, i.e. for integer radius $r = r_{j'}$ ($2j' + 1$ being an integer).

Therefore, the projector in $\text{Mat}(\mathbb{C})$ onto the state $|j\vec{\omega}\rangle$ is the function which is the most localized around a classical point $r_{j}\vec{\omega}$. Its associated $C(SU(2))^*$ element is obtained via Fourier transform and is explicitly given by:

$$\phi_{j,\vec{\omega}}(u(\theta, \vec{n})) = \langle j\vec{\omega} | D^j(u(\theta, \vec{n})) | j\vec{\omega} \rangle = \left( \cos \frac{\theta}{2} - i \vec{n} \cdot \vec{\omega} \sin \frac{\theta}{2} \right)^{2j}. \quad (143)$$

It is interesting to note that the classical limit of such a function, obtained by considering the limit $\ell_P \to 0$ with $r_j$ fixed is given by

$$\lim_{\ell_P \to 0} \phi_{j,\vec{\omega}}(u) = \exp \left( -i r_j \omega^a P_a(u) \right). \quad (144)$$

This result agrees with the fact that the limit of the $C_{\ell_P}(E^3)$ representation $\Phi_{r_j,\vec{\omega}}(x)$ of $\phi_{j,\vec{\omega}}/\ell_P$ is the usual delta distribution:

$$\lim_{\ell_P \to 0} \Phi_{r_j,\vec{\omega}}(x) = (2\pi)^3 \delta^3(\vec{x} - r_j \vec{\omega}). \quad (145)$$

7 Conclusion

In this paper, starting from the deformation of the Euclidean group we constructed a non-commutative space carrying the action of $ISU(2)$.

This non-commutative space has been described, as usual in non-commutative geometry, by its algebra of functions which has been shown to be the algebra $C(SU(2))^*$.
of \( SU(2) \) distributions endowed with the convolution product. We found two additional representations of this algebra. The first is given by \( C_{\ell P}(E^3) \). Its elements are given by three functions on \( E^3 \), two of them having a spectrum strictly bounded by \( \ell_p^{-1} \), the last one having its spectrum on the sphere of radius \( \ell_p^{-1} \). This representation makes a bridge between the non-commutative space and the standard classical manifold \( E^3 \). The important point is that the product of the algebra \( C_{\ell P}(E^3) \) is no longer the point-wise product but a \( \star \)-deformation of it with \( \ell_p \) as a deformation parameter. The second representation is obtained by means of matrices. To get this formulation we introduced a Fourier transform on the algebra \( C(SU(2))^* \) whose Fourier space is the algebra of complex matrices \( \text{Mat}(\mathbb{C}) \). This shows that the quantum space we constructed is in fact discrete or fuzzy, as expected from the boundedness of the momenta space.

Finally, we showed the correspondence between the continuous formulation in terms of \( C_{\ell P}(E^3) \) and the fuzzy formulation in terms of \( \text{Mat}(\mathbb{C}) \). This correspondence gives some insights regarding the geometry of the fuzzy space and its classical limit. Furthermore we illustrated our construction with examples. Some of these examples have been recently considered in [19].

This study is the starting point for a construction of a Quantum Field Theory on the non-commutative space. A local action for a scalar field has been proposed in Section 4. Similar actions have been partially investigated in [7, 8, 18] in the momentum representation. We showed that the appearance of a discrete degree of freedom is unavoidable in this context leading to a multiplet of scalar fields. It would be interesting to consider the formulation of such QFTs in the fuzzy space representation and its links to matrix models for instance.

Our construction is very general and opens the possibility to extend our result to the Lorentzian sector and to the non-vanishing cosmological constant cases, namely de Sitter and anti de Sitter space times. Indeed, in three dimensions, whatever the sign of the cosmological constant and the signature of the metric, quantum gravity is argued to have quantum doubles as quantum isometry algebras [20]. In the Lorentzian sector, the momentum space is no longer compact, therefore the fuzzy spacetime is expected not to be fully discrete and the ultraviolet divergences may not completely disappear. For de Sitter or anti de Sitter spacetime, we expect the quantum deformations of the classical isometry groups to play an important role. In these cases, the momentum space is no longer a curved manifold but a quantum group. It would be very interesting to look at the associated quantum geometries which would be interpreted as de Sitter or anti de Sitter quantum spaces.

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A Schwinger’s Oscillator representation of $SU(2)$ and the translations action

In this appendix, we recall Schwinger’s construction of the UIR of $\mathfrak{su}(2)$ from a couple of commuting harmonic oscillators defined by annihilation and creation operators $a_p$ and $a_p^\dagger$ with $p \in \{+, -\}$ (see e.g.

$$\mathcal{J} = \sum_{p,q} a_p^\dagger (\sigma)_{pq} a_q,$$

where $\sigma \equiv 2 D^{1/2}(\mathcal{J})$ are the Pauli matrices. Furthermore, one can recover the unitary irreducible representations of $\mathfrak{su}(2)$ from those of the harmonic oscillators. The Fock space generated by $a_\pm$ and $a_\pm^\dagger$ is spanned by given by the states $|n_+; n_-\rangle$ with

$$a_+|n_+; n_-\rangle = \sqrt{n_+}|n_+ - 1; n_-\rangle, \quad a_-|n_+; n_-\rangle = \sqrt{n_++1}|n_+ + 1; n_-\rangle,$$

$$a_-|n_+; n_-\rangle = \sqrt{n_-}|n_+; n_- - 1\rangle, \quad a_+|n_+; n_-\rangle = \sqrt{n_-+1}|n_+; n_- + 1\rangle,$$

where $(n_+, n_-)$ is a couple of non-negative integers. If one identifies the states $|n_+; n_-\rangle$ and $|j, m\rangle$ with the relations $2j = n_+ + n_-$ and $2m = n_+ - n_-$, then one recovers the action of $\mathfrak{su}(2)$. As a result, the representation (147) is not irreducible for $\mathfrak{su}(2)$: it is the direct sum of the whole set of finite dimensional representations of $\mathfrak{su}(2)$, each representation appearing only once. Finally, let us recall the link between occupation number $N = a_+a_+ + a_-a_-$ and the $\mathfrak{su}(2)$ spin $j$:

$$N|j, m\rangle = \left(a_+^\dagger a_+ + a_-^\dagger a_-\right) |j, m\rangle = 2j |j, m\rangle.\quad (148)$$

It is interesting to see how the infinitesimal translation can be written in a more compact form within Schwinger’s representation of the $SU(2)$ UIRs:

$$\mathcal{P}_a \triangleright \Phi = \frac{i}{\ell_p (N + 1)} \left( a_n \Phi a_m^\dagger - a_m^\dagger \Phi a_n \right),\quad (149)$$

where $a_p$ and $a_p^\dagger$ are a couple of annihilation and creation operators and $N = a_p^\dagger a_p$ is the occupation number. Let us see that this is indeed the case and let us verify that $\mathcal{P}_a$ satisfies the good properties of the infinitesimal translation operator:

1. The only non-vanishing matrix elements of $\delta_q \Phi \equiv iq^a \mathcal{P}_a \triangleright \Phi$ are $\langle j, s | \delta_q \Phi | j, t \rangle$. Each term involving the operators $a_p$ and $a_p^\dagger$ can be computed using (147):

$$\langle j, s | a_p \Phi a_p^\dagger | j, t \rangle = \sqrt{(j + 1 + qs)(j + 1 + pt)} \Phi^{j+1/2} a_{s+1/2}^\dagger a_{t+1/2},$$

$$\langle j, s | a_p^\dagger \Phi a_p | j, t \rangle = \sqrt{(j + qt)(j + ps)} \Phi^{j-1/2} a_{s-1/2} a_{t-1/2}.$$

If we use the fact that $(-1)^{m-n}D^{1/2}_{m-n}(\mathcal{J}_a) = -D^{1/2}_{nm}(\mathcal{J}_a)$, then we recover the expression (115) for $\delta_q \Phi$ from (149).
2. The set of hermitian matrices is stable under the action of translations $\delta_q$ as one can trivially see from the eq. (149).

3. The result of the translation of a coordinate variable $\hat{x}_a$, represented by the matrices $2 \ell_P D(J_a)$ in the fuzzy representation (see section 6.1 below), by a vector $q$ can be computed explicitly:

$$
\delta_q \hat{x}_a = -\frac{2 q^b}{N + 1} D^{1/2}(\frac{J_b}{2}) D^{1/2}(\frac{J_a}{2}) (a_q a_m a_n a_p - a_p a_m a_n a_q) .
$$

As expected, one can show that $\delta_q \hat{x}_a = q_a$. To do so, one has to use the standard commutation relations between creation and annihilation operators and the familiar property $D^{1/2}(J_a J_b) = 1/2 \epsilon_{abc} D^{1/2}(J_c) + 1/4 \delta_{ab}$.

It is also interesting to remark that $\delta_q N = 4 (q^a J_a) / (N + 1)$ is the quantum analog of $\delta_q \tilde{r} = (q^a \tilde{x}_a) / \tilde{r}$. Indeed, $N$ and the radius $\tilde{r}$ are both scalar on each subspace Mat$_{d_j \times d_j}(\mathbb{C})$ and simply related by $N + 1 = \sqrt{4 \tilde{r}^2 + 1}$.

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