Analysis motion of borehole-expander - device for densifying subsurface walls of boreholes

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Abstract. The paper introduces an algorithm for analyzing the following solids of rotation and their motion: a bearing roller and a borehole-expander - a device for densifying subsurface walls of boreholes. These devices are moving in the environment with rheological properties typical for the Kelvin model, with nonholonomic constraints that impose restrictions on the velocity of the disk points. The research presents both mathematical models of the standard viscoelastic hereditary-deformed cylinder and a weakly singular model proposed by A.R. Rzhanitsin, representing a nonautonomous system of differential equations in the form of Routh. The fact that this system is nonautonomous complicates its sustainability analysis. The presentation of a rheological force in a differential form does not lead to a system of equations in the form of Chaplygin. The solution was found by numerical integration. The solution is multivariant as one initial value of generalized coordinates and generalized accelerations corresponds to a series of initial generalized speeds. It is shown that the disk is not able to fall in the course of motion because its point of contact being in a radial vibration is still within the tension region of the cylinder wall.

1. Introduction
At present, problems of solids of rotation (with rheological properties) and their motion are systematically analyzed in works of various authors [4-6]. Some papers provide analysis of solids of rotation motion together with research of stability of motion [7-12]. Though rheological models have a large research history there are still many unresolved questions.

2. Formulation of the problem
This paper goes into the problem of rolling motion without sliding and a circular disk rotation along the inside surface of a vertical cylinder with rheological properties. The disk can be a model that simulates both a roller motion in a rolling bearing and a borehole-expander model. The problem is obviously new, as the authors are not aware of a similar research. Let us consider the disk motion to a stationary coordinate system $O\xi\eta\zeta$ where $O\zeta$ axis is directed up and parallel to the element of the cylinder (see Figure 1).
The second coordinate system $Oe_1e_2e_3$ rotates around $O\zeta$ axis. $Cxyz$ coordinate system (not shown in the figure) is rigidly bound to the disk, its $C\zeta$ axis being perpendicular to the disk plane. The generalized coordinates here are three Eulerian angles and three coordinates $\gamma, \zeta, \rho$ determining the position of the point of contact of the cylinder and the disk. The $\rho$-coordinate monitors the deformation of the cylinder wall. As a result of power interaction of the disk and the cylinder, a rheological reaction force of $P$ is generated. This force can be presented in both integral and differential forms.

A nonholonomic constraint equation is derived in vector form:

$$\frac{d}{dt}(Re_1 + \zeta e_3 + \rho e_3 + DC) + \omega \times CD = 0,$$

where $\omega = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$. Instantaneous angular velocity values $\omega_1, \omega_2, \omega_3$ can be calculated while equating projection of the vector of disk angular velocity $\omega_1, \omega_2, \omega_3$ on fixed axes $\xi, \eta, \zeta$ to projections of vectors $\omega_1 e_1, \omega_2 e_2, \omega_3 e_3$ on the same axes.

To find vector $\overrightarrow{DC}$, let us calculated the product of two matrices displaying the rotation of $x, y, z$ axes which are bound to the disk by $\psi$ and $\theta$ angles. The vector’s action line $\overrightarrow{DC}$ matches the $y$ axis.

| Table 1. Table of direction cosines |
|------------------------------------|
| Angle cosines  | $x$          | $y$          | $z$         |
| $\xi$          | $\cos \psi$ | $-\sin \psi \cos \theta$ | $\sin \theta \sin \psi$ |
| $\eta$         | $\sin \psi$ | $\cos \psi \cos \theta$ | $-\sin \theta \cos \psi$ |
| $\zeta$        | $\theta$    | $\sin \theta$ | $\cos \theta$ |

Vector $\overrightarrow{DC}$ projection on the axis of the stationary system is:

$$\overrightarrow{DC} = r(-\sin \psi \cos \theta \xi^0 + \cos \psi \cos \theta \eta^0 + \sin \theta \zeta^0),$$

where $\xi^0, \eta^0, \zeta^0$ - the unit vectors of corresponding axes.
Vector $\overrightarrow{DC}$ projection on axes $e_1, e_2, e_3$ is:

$DC_{e_1} = r(\cos \theta \cos \psi \sin \gamma - \cos \theta \sin \psi \cos \gamma) ;$

$DC_{e_2} = r(\cos \theta \sin \psi \sin \gamma + \cos \theta \cos \psi \cos \gamma) ;$

$DC_{e_3} = r \sin \theta .$

Let us project vector equality (1) on axis $e_1, e_2, e_3$ and put down equations of constraints:

$\dot{\rho} + r \cdot \cos(\gamma - \psi) \cdot \dot{\phi} = 0,$

$(R + \rho) \cdot \dot{\psi} - r \cdot \sin(\gamma - \psi) \cdot \dot{\phi} = 0,$

$\dot{\zeta} - 2 \cdot r \cdot \cos \theta \cdot \dot{\theta} = 0.$

The first two equations are non-integral, and the third equation is holonomic. To find the generalized rheological force corresponding coordinate $\rho$, let us make an equation of possible work on possible motion $\delta \rho$:

$\delta A(\overline{Q}_\rho) = -P \cdot \Delta \rho,$ where $Q_\rho = -P .

Equations of the disk motion with account of equations (2) are presented in the form of Routh:

$$d \frac{dT}{d\rho} - \frac{dT}{\rho} Q_\rho - \lambda_1 = 0 ,$$

$$d \frac{dT}{d\psi} - \frac{dT}{\psi} Q_\psi = 0 ,$$

$$d \frac{dT}{d\theta} - \frac{dT}{\theta} Q_\theta - 2 \lambda_r r \cos \theta = 0 ,$$

$$d \frac{dT}{d\phi} - \frac{dT}{\phi} Q_\phi = \lambda_r r \cos(\gamma - \psi) - \lambda_2 r \sin(\gamma - \psi) ,$$

$$d \frac{dT}{d\psi} - \frac{dT}{\psi} Q_\psi = \lambda_2 (R + \rho) ,$$

$$d \frac{dT}{d\zeta} - \frac{dT}{\zeta} Q_\zeta = \lambda_3 .$$

Here $T = \frac{1}{2} m \cdot (V_1^2 + V_2^2 + V_3^2) + \frac{1}{2} m \cdot (J_x \omega_x^2 + J_y \omega_y^2 + J_z \omega_z^2) - \text{kinetic energy of the disk}$,

$Q_{q_i} = -\frac{\partial \Pi}{\partial q_i} ,

V_1 = \dot{\rho} - r \cdot (\sin \theta \sin(\gamma - \psi) \dot{\theta} + \cos \theta \cos(\gamma - \psi) \dot{\psi}) .

$
\[ V_z = (R + \rho) \dot{\gamma} - r \cdot (\cos \theta (\gamma - \psi) \sin \theta \dot{\theta} r + \sin (\gamma - \psi) \cos \theta \dot{\psi}) , \]
\[ V_\lambda = \dot{\zeta} - r \cdot \cos \theta \dot{\theta} , \quad \text{and} \quad \omega_x , \omega_y , \omega_z \] are set by kinematic equations of Euler:
\[ \omega_x = \dot{\theta} \cos \phi + \dot{\psi} \sin \theta \sin \phi , \]
\[ \omega_y = -\dot{\theta} \sin \phi + \dot{\psi} \sin \theta \cos \phi , \]
\[ \omega_z = \dot{\phi} + \dot{\psi} \cos \theta , \]

where \( m \) is the mass of the disk; \( r, R \) are radii of the disk and the cylinder; \( Q_q = -\frac{\partial \Pi}{\partial q} - Q \)-th generalized force; \( \Pi = -mg \cdot (\zeta - r \sin \theta) \) - potential energy of the disk; \( \lambda_1, \lambda_2, \lambda_3 \) are joining factors; \( P \) is rheological force of the cylinder walls reaction, which is normally directed to the element of the cylinder and is presented in an integral form [1]

\[ P(t) = c \left[ \rho(t) - \frac{\dot{z} \Re(t-\tau)}{0} \mathrm{d} \tau \right] , \tag{4} \]

where \( \Re(t-\tau) = \frac{c-c_1}{nc} e^{-\frac{1}{n}(t-\tau)} \) is the relaxation core for a standard hereditarily deformable body [2], \( \Re(t-\tau) = (t-\tau)^{-0.05} e^{\frac{(t-\tau)}{n}} \) if for weakly singular models of hereditary bodies, \( c, c_1 \) are instantaneous and long-term modulus of elasticity, respectively, and \( \Re \) is the relaxation time.

In the analysis of weakly singular models, the relaxation core was approximated by Newton binomial series with the first two parts restrained. As a result, the rheological force of the cylinder walls reaction can be presented as:

\[ P(t) = c \left[ \rho(t) - \frac{1}{150} e^{-\frac{1}{n}(t-0.05 f_1 - t^{-0.05 f_2})} \right] , \tag{5} \]

where \( f_1 = \int_0^t e^{\frac{1}{n} \rho(\tau)d\tau} , f_2 = \int_0^t \tau e^{\frac{1}{n} \rho(\tau)d\tau} \).

As Formula (5) shows, the relaxation deformation rate \( P(t) \) at the initial time moment is equal to infinity, which is consistent with the results of other authors [3].

Excluding from the system of equations (3) and (4) joining factors and force \( P \), it is possible to obtain equations of the disk motion that make the complete system of motion equations:

\[ \dot{\rho} + r \cdot \cos (\gamma - \psi) \cdot \dot{\phi} = 0 , \]
\[ (R + \rho) \cdot \dot{\gamma} - r \cdot \sin (\gamma - \psi) \cdot \dot{\phi} = 0 , \]
\[ \dot{\zeta} - 2 \cdot r \cdot \cos \theta \cdot \dot{\theta} = 0 . \]
\[ L_\phi + 2\lambda_r r \cos \theta = 0 , \]
\[ L_\varphi = \lambda_1 r \cos(\gamma - \psi) - \lambda_2 r \sin(\gamma - \psi) , \]

\[ L_\psi = 0 , \]

\[ -\dot{f}_1 + e^{\frac{1}{2}} \cdot \rho = 0 , \]

\[ -\dot{f}_2 + e^{\frac{1}{2}} \cdot \rho = 0 . \]  \hspace{1cm} (6)

where \( \lambda_1 = L_\varphi - c(\rho - \frac{1}{150} e^{-2t}) ; \lambda_2 = \frac{L_\psi}{R + \rho} ; \lambda_3 = L_n ; L_n = \frac{d}{dt} \frac{\partial T}{\partial q_i} - \frac{\partial T}{\partial q_i} = Q(q_i = \psi , \theta , \gamma , \rho , \zeta) . \)

The system of equations (6) was numerically integrated under the following operating and initial conditions:

\[ n = 10 s , \ c = 0.1 \frac{N}{cm} , \ c_1 = 0.75c \frac{N}{cm} , \ \ r = 0.2 m , \ R = 0.4 m , \ m = 0.3 kg , \ g = 9.8 \frac{m}{s^2} , \]

\[ \theta(0) = \frac{\pi}{6} \ rad , \ \psi(0) = \pi \ rad , \ \gamma(0) = \frac{r \cdot \phi(0)}{R} \ rad , \ \zeta(0) = 0.1 cm , \ \rho(0) = 0.4 cm . \]

The picture of the rheological process is presented in graphs and is shown in Figure 2.

**Figure 2.** Graphs of variance: a – self-rotation angle; b – angle of rotation of coordinate system
\(e_1, e_2, e_3,\)

c – coordinates that define the deformation of the cylinder wall; d – coordinates that define the height of the point of contact of the disk; e – rheological force of the cylinder walls reaction.

The graphical dependencies of rheological parameters in the analysis of weakly singular models are shown in Figure 3.

![Graphs of variance of weakly singular models parameters](image)

**Figure 3.** Graphs of variance of weakly singular models parameters

### 3. Conclusion

It follows from the graphs analysis that, by making radial vibrations, the disk moves inside the cylinder at the sinusoid, with the point of the disk contact is within the tension region of the cylinder wall. It all proves that there is no phase of the disk break-away. The motion of the disk can be described as a set of three even rotations at rotation angles: the angle of self-rotation, the angle of precession, and the angle of the coordinate system rotation \(e_1, e_2, e_3\) to which high frequency vibrations are applied. A characteristic feature of weakly singular models is the infinite value of the examined parameter in the initial time moment, because at this point the relaxation core goes into infinity [3].

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