AN OSCILLATORY FERMAT-TORRICELLI TREE IN \( \mathbb{R}^2 \)

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Abstract. We obtain an important generalization of the mechanical solution given by S. Gueron and R. Tessler w.r. to the weighted Fermat-Torricelli problem which derives a new structure of solutions which may be called oscillatory Fermat-Torricelli trees. The weighted Fermat-Torricelli problem in \( \mathbb{R}^2 \) states that: Given three points in \( \mathbb{R}^2 \) and a positive real number (weight) which correspond to each point, find the point (weighted Fermat-Torricelli point) such that the sum of the weighted distances to these three points is minimized. By applying the mechanical device of Pick and Polya the oscillatory tree solution is a new solution w.r to the weighted Fermat-Torricelli problem for a given isosceles triangle with corresponding two equal weights at the vertices of the base segment. It is worth mentioning that at after time \( t \) the oscillatory knot of the mechanical system passes from the weighted Fermat-Torricelli point with non zero velocity. Furthermore, we give a numerical example to verify the structure of an oscillatory Fermat-Torricelli tree for a given isosceles triangle with equal weights.

1. Introduction

We start by stating the weighted Fermat-Torricelli problem in \( \mathbb{R}^2 \):

**Problem 1.** Given three points \( A_1 = (x_1, y_1) \), \( A_2 = (x_2, y_2) \), \( A_3 = (x_3, y_3) \), find a point \( O \) which minimizes the objective function

\[
f(x, y) = \sum_{i=1}^{3} w_i \sqrt{(x-x_j)^2 + (y-y_j)^2} \quad (1.1)
\]

where \( w_i \) is a positive real number (weight) which corresponds to \( A_i \).

The solution of the weighted Fermat-Torricelli problem (Problem 1) is called the weighted Fermat-Torricelli tree, which consists of the

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union of the three edges (branches) $A_1O$, $A_2O$, $A_3O$ which meet at the weighted Fermat-Torricelli point $O$.

By replacing $w_1 = w_2 = w_3$ in (1.1), we obtain the (unweighted) Fermat-Torricelli tree. The (unweighted) Fermat-Torricelli problem was first stated by Pierre de Fermat (1643) and first solved by E. Torricelli.

The existence and uniqueness of the weighted Fermat-Torricelli tree and a complete characterization of the "floating case" and "absorbed case" has been established by Y. S Kupitz and H. Martini (see [7], theorem 1.1, reformulation 1.2 page 58, theorem 8.5 page 76, 77). A particular case of this result for three non-collinear points in $\mathbb{R}^2$, is given by the following theorem:

**Theorem 1.** [1], [7] Let there be given three non-collinear points $A_1, A_2, A_3 \in \mathbb{R}^2$ with corresponding positive weights $w_1, w_2, w_3$.

(a) The weighted Fermat-Torricelli point $O$ exists and is unique.

(b) If for each point $A_i \in \{A_1, A_2, A_3\}$

$$\left\| \sum_{j=1, j \neq i}^{3} w_j \vec{u}(A_i, A_j) \right\| > w_i, \quad (1.2)$$

for $i, j = 1, 2, 3$ holds, then

(b1) the weighted Fermat-Torricelli point $O$ (weighted floating equilibrium point) does not belong to $\{A_1, A_2, A_3\}$ and

(b2)

$$\sum_{i=1}^{3} w_i \vec{u}(O, A_i) = \vec{0}, \quad (1.3)$$

where $\vec{u}(A_k, A_l)$ is the unit vector from $A_k$ to $A_l$, for $k, l \in \{0, 1, 2, 3\}$ (Weighted Floating Case).

(c) If there is a point $A_i \in \{A_1, A_2, A_3\}$ satisfying

$$\left\| \sum_{j=1, j \neq i}^{3} w_j \vec{u}(A_i, A_j) \right\| \leq w_i, \quad (1.4)$$

then the weighted Fermat-Torricelli point $O$ (weighted absorbed point) coincides with the point $A_i$ (Weighted Absorbed Case).

By replacing $w_1 = w_2 = w_3$ in Theorem 1, we get:

**Corollary 1.** If $w_1 = w_2 = w_3$ and all three angles of the triangle $\triangle A_1A_2A_3$ are less than $120^\circ$, then $O$ is the isogonal point (interior point) of $\triangle A_1A_2A_3$ whose sight angle to every side of $\triangle A_1A_2A_3$ is $120^\circ$. 
Corollary 2. If \( w_1 = w_2 = w_3 \) and one of the angles of the triangle \( \triangle A_1A_2A_3 \) is equal or greater than \( 120^\circ \), then \( O \) is the vertex of the obtuse angle of \( \triangle A_1A_2A_3 \).

For an excellent historical exposition regarding the solution of the weighted Fermat-Torricelli problem we mention the works of [1], [7], [3] and [4] and for further generalizations classical works are given in [5] and [6].

In 2002, S Gueron and R. Tessler invented a mechanical solution in the sense of Polya and Varignon by applying the following construction ([3]):

Suppose that \( \{A_1, A_2, A_3\} \) lie on a horizontal table, and that holes are drilled at the vertices, where smooth pulleys are attached. The three massless strings referring to \( A_1, A_2 \) and \( A_3 \) that emanate from the knot are passed through the pulleys, and three masses \( w_1, w_2, w_3 \), are suspended from the ends of these strings.

Assume that the system is released and reaches its mechanical equilibrium, and that the knot stop at the interior point \( O \). Then, by applying the minimum energy principle at equilibrium, they obtain the weighted Fermat-Torricelli tree solution. Thus, the mechanical equilibrium of the system gives the condition for vectorial balance:

\[
w_1\vec{u}(O, A_1) + w_2\vec{u}(O, A_2) + w_3\vec{u}(O, A_3) = \vec{0}. \tag{1.5}
\]

We note that the mechanical system reaches its mechanical equilibrium, by taking into account friction.

In this paper, we generalize the mechanical solution of S. Gueron and R. Tessler for the case of an isosceles triangle \( \triangle A_1A_2A_3 \) where \( w_1 = 1 \) and \( w_2 = w_3 \) by introducing the oscillatory Fermat-Torricelli tree solution of the corresponding mechanical system by assuming it is frictionless.

We note that after time \( t \) the oscillatory knot of the mechanical system passes from the weighted Fermat-Torricelli point with non zero velocity, by releasing the mechanical system from the vertex \( A_1 \) with zero velocity (Theorem 2, Corollary 3).

Furthermore, we give a numerical example to verify the structure of an oscillatory Fermat-Torricelli tree for a given isosceles triangle with equal weights (Example 1).
2. A GENERALIZATION OF THE MECHANICAL SOLUTION OF S. GUERON AND R. TESSLER

We shall use the same mechanical system of S. Gueron and R. Tessler in the spirit of Pick and Polya, in order to solve the following mechanical problem:

**Problem 2.** A board is drilled with three holes corresponding to three given points \(A_1, A_2, A_3\) which form an isosceles triangle \(\triangle A_1A_2A_3\) where \(A_1A_2 = A_1A_3 = a\) and three strings are tied together in a knot with mass \(m_0\) at one knot and the loose ends are passed through the three holes attached to the physical weights \(w_1 = 1\) from \(A_1\), \(w_2\) from \(A_2\) and \(w_3\) from \(A_3\), where \(w_2 = w_3\). If we release \(m_0\) from \(A_1\) with zero velocity find the motion of the knot \(O\).

**Definition 1.** We call the motion of the knot with mass \(m_0\) w.r. to the mechanical system of Problem 2 an oscillatory Fermat-Torricelli tree.

We shall verify the oscillation of the knot with mass \(m_0\) numerically in example 1.

We denote by \(O\) the corresponding weighted Fermat-Torricelli point of the isosceles triangle \(\triangle A_1A_2A_3\) where \(w_2 = w_3\) and \(w_1 = 1\). The point \(O\) belongs to the height \(A_1A_4\) w.r. to the base \(A_2A_3\). By applying theorem 1 for \(w_2 = w_3\) and \(w_1 = 1\) we get:

**Lemma 1.** If \(\angle A_4A_1A_3 < \frac{\arccos\left(\frac{1}{2w_3^2} - 1\right)}{2}\), then the weighted Fermat-Torricelli point \(O\) of \(\triangle A_1A_2A_3\), belongs to the height \(A_1A_4\) w.r. to the base \(A_2A_3\) and

\[
\angle A_4OA_3 = \frac{\arccos\left(\frac{1}{2w_3^2} - 1\right)}{2}. \tag{2.1}
\]

We assume that \(\angle A_4A_1A_3 < \frac{\arccos\left(\frac{1}{2w_3^2} - 1\right)}{2}\), such that Lemma 1 holds.

Suppose that we release mass \(m_0\) from the vertex \(A_1\) with zero velocity \(\dot{x}(0) = 0\). After time \(t\), \(m_0\) reaches at the point \(S\) which lies on \(A_1O\), because \(F_2 = F_3 = w_2\). Thus, the knot will move along the line defined by \(A_1O\) via the force \(\vec{F}_{23} - \vec{F}_1\), such that \(\vec{F}_{23} = \vec{F}_2 + \vec{F}_3\) and \(\Delta F = F_{23} - F_1 = 2\cos \angle OSA_3 - 1\) (see fig. 1).

We set \(\angle OSA_3 := \phi(t)\), \(x[t] := A_1S\) and \(\angle O\triangle A_1A_3 := \phi(0)\).

**Theorem 2.** The solution of the mechanical system of Problem 2 is an oscillatory Fermat-Torricelli tree which is described by the motion of the knot with mass \(m_0\) along the line defined by the \(A_1O\) having non zero velocity after time \(t_0\) at the weighted Fermat-Torricelli point \(O\) of \(\triangle A_1A_2A_3\) which is given by:
Figure 1. Motion of the oscillatory Fermat-Torricelli tree $A_1S, A_2S, A_3S$ along $A_1O$ for the boundary isosceles $\triangle A_1A_2A_3$

\[ \dot{x}(t_0) = \sqrt{\frac{2}{m_0}(2aw_2 - x(t_0) - \frac{2aw_2 \sin \phi(0)}{\sin(\angle A_4O_A_3)}.} \]  \quad (2.2)

**Proof.** By applying the sine law in $\triangle SOA_2$, we obtain:

\[
\frac{OS}{\sin(\frac{\angle A_4OA_3}{2}) - \phi(t)} = \frac{OA_3}{\sin \phi(t)} \tag{2.3}
\]

where

\[
OS = a(\cos \phi(0) - \frac{\sin \phi(0)}{\cot(\frac{\angle A_4OA_3}{2})}) - x \tag{2.4}
\]

By replacing (2.4) in (2.3), we get:

\[
x(t) = a \cos \phi(0) - a \sin \phi(0) \cot \phi(t). \tag{2.5}
\]

At time $t$ the force along the path $A_1O$ is given by:

\[
m_0 \ddot{x} = 2w_2 \cos \phi(t) - 1. \tag{2.6}
\]

By differentiating (2.5), we have:

\[
dx = \frac{a \sin \phi(0)}{\sin^2 \phi(t)} d\phi \tag{2.7}
\]
It is well known that:

$$\ddot{x} = \frac{d\dot{x}}{dx}.$$ 

Thus, by integrating both parts of (2.6) from $A_1$ to $S$, w.r. to $x$ and taking into account (2.5), we obtain:

$$m_0 \left( \frac{\dot{x}(t)^2 - \dot{x}(0)^2}{2} \right) = \int_{\phi(0)}^\phi 2aw_2 \sin(0) \frac{\cos \phi}{\sin^2 \phi(0)} \, d\phi - \int_0^x dx \tag{2.8}$$

or

$$m_0 \left( \frac{\dot{x}(t)^2}{2} \right) = 2aw_2 - 2aw_2 \frac{\sin \phi(0)}{\sin \phi} - \int_0^x dx \tag{2.9}$$

By setting $t = t_0$ in (2.9), we obtain (2.2) □

**Corollary 3.** For $w_2 = w_3 = 1$, the velocity of the knot with mass $m_0$ which passes from the unweighted Fermat-Torricelli point is given by:

$$\dot{x}(t_0) = \sqrt{\frac{2}{m_0} \left( 2 - x(t_0) - \frac{4a \sin \phi(0)}{\sqrt{3}} \right)}. \tag{2.10}$$

**Proof.** By replacing $w_2 = w_3 = 1$, and $\phi = 60^\circ$, we derive (2.10). □

**Proposition 1.** The movement of the mechanical system is determined by the following differential equation:

$$m_0 \left( \frac{\dot{\phi}(t)^2 - \dot{\phi}(0)^2}{2} \right) = \int_{\phi(0)}^\phi 2aw_2 \sin(0) \frac{\cos \phi}{\sin^2 \phi(0)} \, d\phi - \int_0^x dx \tag{2.11}$$

with initial conditions $\phi(0) = \phi_0$ and $\dot{\phi} = 0$.

**Proof.** Differentiating twice (2.5) and by replacing in (2.6) we derive (2.11). □

**Proposition 2.** The work of the force of the mechanical system along the path $A_1O$ starting from $A_1$ with $\dot{x}(0) = 0$ is given by:

$$W = 2w_2(a - A_2O) - A_1O \neq 0. \tag{2.12}$$

**Proof.** We start with the work of the force $F_{23} - F_1$ along $A_1O$ :

$$W = \int_{A_1O} (F_{23} - F_1) \, dx. \tag{2.13}$$

By replacing (2.7) in (2.13), we get:

$$W = \int_{\phi(0)}^{\angle A_4OA_3} 2w_2 \cos \phi \frac{a \sin \phi(0)}{\sin^2 \phi} \, d\phi - A_1O, \tag{2.14}$$
Figure 2. Graph of $\phi(t)$ for $a = 5, \phi(0) = 40^\circ$

Figure 3. Graph of $x(t)$, for $a = 5, \phi(0) = 40^\circ$

or

$$W = 2w_2(a - a \frac{\sin \phi(0)}{\sin \angle A_4OA_3}) - 1$$

which yields (2.12).

For $w_2 = w_3 = 1$, we get:

**Corollary 4.** The work of the force of the mechanical system along the path $A_1O$ starting from $A_1$ with $\dot{x}(0) = 0$ is given by:

$$W = 2(a - A_2O) - A_1O \neq 0,$$

where $O$ is the unweighted Fermat-Torricelli point.

**Example 1.** Given an isosceles triangle $\triangle A_1A_2A_3$ where $a = 5, \phi(0) = 40^\circ, w_1 = w_2 = w_3 = 1$, we derive that $\angle A_4OA_3 = 60^\circ$.

Suppose that we release mass $m_0$ from the vertex $A_1$ with zero velocity $\dot{x}(0) = 0$. After time $t$, $m_0$ reaches at the point $S$ which lies on $A_1O$. 
By replacing $a, \phi(0)$ in (2.11), we obtain a numerical solution using Mathematica of $\phi(t)$ and $x(t)$ (see fig. 2, 3).

By replacing $a, \phi(0), m_0 = 1$ in (2.10), we derive a numerical solution using Mathematica of $\dot{x}(t)$ (see fig. 4).

We note that an approximation of the periodical function $x(t)$ may be of the form $a \sin(b(t + c)) + d:

\[ x[t] \approx 1.77363 + 1.77363 \sin(0.61133(−2.56947+t)) \]

By differentiating $x(t)$ w.r. to $t$, we get a good approximation of $\dot{x}(t)$:

\[ \dot{x}(t) \approx |1.08427 \cos(0.61133(−2.56947+t))| \]

The deviation of $x(t)$ is given by fig. 5 and the deviation of the corresponding velocity $\dot{x}(t)$ is given by fig. 6.
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Figure 6. Graph of $\dot{x}(t) - |1.08427 \cos(0.61133(-2.56947 + t))|$ for $a = 5, \phi(0) = 40^\circ, m_0 = 1$.

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