Consequences of a possible adiabatic transition between $\nu = 1/3$ and $\nu = 1$ quantum Hall states in a narrow wire.

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We consider the possibility of creating an adiabatic transition through a narrow neck, or point contact, between two different quantized Hall states that have the same number of edge modes, such as $\nu = 1$ and $\nu = 1/3$. We apply both the composite fermion and the Luttinger liquid formalism to analyze the transition. We suggest that using such adiabatic junctions one could build a DC step-up transformer, where the output voltage is higher than the input. Difficulties standing in the way of an experimental implementation of the adiabatic junction are addressed.

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It has long been understood that quantized Hall states with different Hall conductances cannot be connected adiabatically in the interior of a macroscopic two-dimensional electron system. For a pure system, where the quantized Hall states have energy gaps, the boundary between two quantized states must be characterized by a vanishing energy gap, with associated low energy excitations. In a disordered system there are generally localized low energy excitations in the interior of a quantized Hall region, which then become extended at the boundary between two quantized regions. The possible transitions between different quantized Hall states have been elucidated (in the case of a fully spin-polarized system) by the introduction of a "global phase diagram" based on a unitary transformation which introduces a Chern-Simons gauge field and which, at the mean field level, maps fractional quantized Hall states onto integer ones. [1,2]

In this Letter, we suggest that in a narrow quantum wire there can be an adiabatic transition between two different quantized Hall states, under certain conditions. The most important example, to which we restrict ourselves here, is the case of a transition between states with $\nu = 1$ and $\nu = 1/3$. It should be noted that for both these states, there is a single edge mode at a sharp sample boundary, [3] so one can have a single pair of oppositely moving modes running continuously through the transition region. We shall discuss the transition between the two states in a narrow wire using a fermion-Chern-Simons mean-field description [3,4], in which the effective magnetic field changes sign in the transition region, and using a bosonized Luttinger liquid formalism, in which the interaction coefficient $g$ is allowed to vary continuously within the transition region. We also show that the existence of an adiabatic junction between the two quantized Hall regions would allow construction of a DC step-up transformer, where the output voltage is larger than the input voltage supplied by the power source.

Consider the geometry illustrated in Fig.1, where there is a narrow wire (or "point contact") connecting two macroscopic quantized Hall regions, with different electron densities corresponding to $\nu = 1$ and $\nu = 1/3$ respectively. We assume that each of the edges is sufficiently long that local thermal equilibrium is established on the edge at a voltage labelled $V_j$, where $j = 1$ and $j = 2$ denote, respectively, the incoming and outgoing channels on the $\nu = 1$ side of the junction, and $j = 3$ and $j = 4$ denote the incoming and outgoing channels on the $\nu = 1/3$ side. We also assume that the external current contacts are "ideal" so $V_1$ and $V_3$ are equal to the voltages in the leads. [3]

![FIG. 1. Junction connecting quantum Hall states with different filling factors $\nu = 1$ and $\nu = 1/3$. Quantum point contact is produced by a narrow neck with the width of the order of the magnetic length. Arrows show the direction of the edge states.](image)

If the voltages of the external leads are equal to each other, then the system will be in global thermal equilibrium, with all $V_j$ being equal. More generally if $V_1 - V_3$ is sufficiently small, the voltages $V_2$ and $V_4$ will be linear functions of $V_1$ and $V_3$, and we may write

$$V_2 = \alpha V_1 + (1 - \alpha)V_3,$$

$$V_4 = \beta V_1 + (1 - \beta)V_3,$$

where $\alpha$ and $\beta$ depend on the characteristics of the connecting junction, including, in general, the temperature $T$.

The current on edge $j$ is given by $I_j = \nu_j V_j (e^2/h)$, and the energy flux along the edge is $I_j V_j / 2$. Thus current conservation through the junction requires that

$$\beta = 3(1 - \alpha),$$

where $\alpha$ and $\beta$ are related by

$\alpha = \frac{V_2 - V_4}{V_1 - V_3}$ and $\beta = \frac{V_3 - V_2}{V_1 - V_3}$.

This relationship indicates that $\alpha$ and $\beta$ are inversely proportional to $V_1 - V_3$.

If $\nu = 1$ and $\nu = 1/3$ were to be connected together, the total number of edge states would have to be conserved, and this would require $\alpha + \beta = 1$. However, in practice, it is difficult to achieve this condition exactly, especially when $\alpha$ and $\beta$ are small. This is because the voltages $V_1$ and $V_3$ are not fixed, but are determined by the external contacts. In general, the voltages $V_1$ and $V_3$ will be approximately equal, and the currents $I_1$ and $I_3$ will be equal to $\nu_1 V_1 (e^2/h)$ and $\nu_3 V_3 (e^2/h)$, respectively. Therefore, the ratio $\alpha / \beta$ will be approximately equal to $\nu_1 / \nu_3$. This ratio will be close to 1 when $\nu_1$ and $\nu_3$ are close to each other in value, but will increase as $\nu_1$ and $\nu_3$ become further apart.

In conclusion, we have suggested that adiabatic transitions between different quantized Hall states can be achieved in narrow wires, allowing the construction of a DC step-up transformer. The difficulties standing in the way of experimental implementation of these junctions are addressed in the text.
while the requirement that the outgoing power be equal to or less than the power incident on the junction implies

$$1/2 \leq \alpha \leq 1$$

(4)

The two limiting situations, where there is no energy loss in the junction region, are $\alpha = 1, \beta = 0$, which corresponds to zero current transmission through the junction, and $\alpha = 1/2, \beta = 3/2$, which is what we mean by an "adiabatic junction." The more familiar case of a wide junction, where equilibration is established along a relatively long boundary separating bulk regions with $\nu = 1$ and $\nu = 1/3$, corresponds to parameters $\alpha = 2/3, \beta = 1$, which is not dissipationless.

If we set $V_3 = 0$, and supply a small voltage $V_1$ to the other current lead, then a voltmeter connected between the opposite edges of the $\nu = 1/3$ wire will measure the voltage $V_4 = \beta V_1$. Moreover, the two-terminal conductance $G$, defined as the ratio between the current $I$ in the leads and the input voltage $V_1$, is given by

$$G = \beta e^2 / 3h.$$  

(5)

If we can construct a junction with $\beta > 1$, then we can obtain a voltage $V_4$ which is larger than the input voltage, and we obtain $G > e^2 / 3h$. This last result violates the common belief that the two terminal conductance of a quantum Hall system is always less than the bottleneck conductance, as the two-contact resistance of ideal leads connected to a single $\nu = 1/3$ region would be $e^2 / 3h$. This also emphasizes an important point made by several authors that the question of conductance is subtle and should be formulated with a definite experimental arrangement in mind.

A more efficient voltage-transformer may be realized with the ring geometry illustrated in Fig.2. If a battery with voltage $V$ is connected to ideal current contacts at points 1 and 2, and a load with resistance $R$ is connected to points 3 and 4, then if the junctions between the regions of $\nu = 1$ and $\nu = 1/3$ are perfectly adiabatic ($\beta = 3/2$), the voltage across the load resistor will be equal to $3V/(1 + 12h/e^2 R)$. When $R = \infty$, this device draws no current from the battery, and the output voltage is $3V$. More generally, the output current is one-third of the input current. If $R \gg 12h/e^2$, the output voltage is close to $3V$, and the power lost in the transformer is small compared to the power delivered to the load.

To demonstrate the possibility of an adiabatic junction between $\nu = 1/3$ and $\nu = 1$ states, we first use the fermion-Chern-Simons approach.\[4\] In the mean field approximation the field $\nu = 1/3$ state is viewed as a completely filled Landau level for composite fermions. This also holds for the $\nu = 1$ state except that the effective magnetic field is opposite to the direction of the external magnetic field. Therefore, a narrow wire at either filling factor with sufficiently sharp boundaries can be described in the Landau gauge by a single energy band with two chiral edge channels. The two filling factors can be easily distinguished in a wire much wider than the magnetic length. In particular, the local electron density is three times greater in the $\nu = 1$ state. However, when the width of the wire is of the order of the magnetic length the distinction between the two states disappears. Then the density is not a good way to differentiate between the states. In fact, on the mean field level the two states look almost identical.

The transition between the two states can be carried out in the following way. On one side we have a wide $\nu = 1$ state with a single energy band in the Landau gauge. The wire is then narrowed gradually on the scale of the magnetic length. When the width of the wire is of the order of the magnetic length the energy spectrum is mainly determined by the confinement potential rather than magnetic field. Therefore reducing the effective magnetic field along the wire by reducing the density should not change radically the energy spectrum. Higher composite fermion energy bands corresponding to other fractions remain unfilled so that there is a single pair of edge channels. As the filling factor is reduced below 1/2 the effective magnetic field changes sign and is slowly brought to its $\nu = 1/3$ value. Then the wire is widened and represents a well-defined $\nu = 1/3$ state.

Although the composite fermion analysis can be extended to find the chemical potentials of edge channels \[3\], we take a different approach here. It has been argued by several authors \[4\,\,8\] that a quantum wire with filling factor $\nu = 1$ or $\nu = 1/3$ can be modelled by a Luttinger Hamiltonian of the form:

$$H = \frac{\hbar}{4\pi} \int_{-\infty}^{+\infty} \nu_F d\alpha \left[ \left( \frac{d\alpha L}{dx} \right)^2 + \left( \frac{d\alpha R}{dx} \right)^2 + \frac{g}{2} \left( \frac{d\alpha L}{dx} + \frac{d\alpha R}{dx} \right)^2 \right].$$

(6)

We define charge-density operators $\rho_j$ by $d\phi_j / dx = 2\pi \rho_j$, and we assume commutation relations
\[ \phi_j(x), \phi_j(x') = (-1)^j i\pi \text{sgn}(x-x') \delta_{jj'}, \]

where \( j = 1, 2 \) refers to the indices \( R \) and \( L \), respectively.

In the \( \nu = 1 \) state the density operators \( \rho_j \) correspond to the real electron density at a given edge, and \( g = 0 \) for a sufficiently wide wire. In the \( \nu = 1/3 \) state, however, the physical density at a given edge is a linear combination of \( \rho_j \), and \( g = 8 \). By choosing fields \( \phi_j \) which are corresponding linear combinations of \( \phi_j \) one can get rid of the cross-term in the Hamiltonian and express it in terms of decoupled edge excitations, as we shall see explicitly below. Therefore, the presence of nonzero \( g \) does not necessarily imply interaction between the two edges but rather the effective mixing of modes. The general relation between \( g \) and the filling factor valid for the simplest fractions, with a single edge state, is:

\[ \nu = (1 + g)^{-1/2}, \]

where \( \nu^{-1} \) must be an odd integer. 

We take this idea further and postulate that the transition region between the two regions with \( g = g_1 \) for \( x < -L/2 \), \( g = g_2 \) for \( x > L/2 \) and \( g \) varying continuously from \( g_1 \) to \( g_2 \) for \(-L/2 < x < L/2 \). This is a natural consequence of the fact that if the translational invariance is not spontaneously broken in the wire and the conditions in the wire are changed adiabatically then there are two chiral boson modes running in opposite directions.

By using commutation relations (6) with the Hamiltonian we get the following equations of motion:

\[ \begin{align*}
\frac{d\phi_j}{dt} &= -v_F \left[ \left( 1 + \frac{g}{2} \right) \frac{d\phi_j}{dx} + g \frac{d\phi_j}{dx} \right] \\
\frac{d\phi_j}{dt} &= v_F \left[ \left( 1 + \frac{g}{2} \right) \frac{d\phi_j}{dx} + \frac{g}{2} \frac{d\phi_j}{dx} \right]
\end{align*} \]

where \( g \) and \( v_F \) are functions of \( x \). The solution of these equations depends on the particular form of \( g \). However, there are two limits when they can be solved exactly, independent of the way \( g \) varies in the transition region. 

The first limit is when the wavelength \( \lambda \) of the incoming pulse is smaller that the length \( L \) of the transition region. In this case the solution can be found by making a Bogoliubov transformation to chiral modes, which correspond to the density waves confined to a single edge:

\[ \begin{align*}
\tilde{\phi}_L &= \frac{1}{2} (1 + 1/\sqrt{1 + g}) \phi_L + \frac{1}{2} (1 - 1/\sqrt{1 + g}) \phi_R \\
\tilde{\phi}_R &= \frac{1}{2} (1 - 1/\sqrt{1 + g}) \phi_L + \frac{1}{2} (1 - 1/\sqrt{1 + g}) \phi_R
\end{align*} \]

Substituting this in the equations of motion and neglecting the variation in \( g \) on the length of the pulse we find:

\[ \frac{d\tilde{\phi}_j}{dt} = (-1)^j v_F \sqrt{1 + g} \frac{d\tilde{\phi}_j}{dx} \]

Solutions of these equations are chiral waves, which implies that a short density pulse goes through the transition region without any reflection. In this sense we call the transition adiabatic.

The other limit is when the wavelength \( \lambda \) of the incoming pulse is greater than the length \( L \) of the transition region. Then we can solve the problem separately in the two regions and then match the solutions assuming that the transition region is infinitely sharp. The chiral wave solutions are found for the transformed variables \( \tilde{\phi}_L \) and \( \tilde{\phi}_R \) with the values of \( g \) in Eq. (10) corresponding to the particular regions. The continuity of the \( \tilde{\phi}_j \) requires that these fields are equal on the two sides of the transition.

We formulate a scattering problem by forming an incoming wave with a current of unit amplitude from the filling factor \( \nu_1 \) side. Then the current in the reflected wave is given by the reflection coefficient \( r \) and the transmitted wave by the transmission coefficient \( t \). We find the current reflection and transmission coefficients:

\[ t = 2\nu_2/(\nu_1 + \nu_2) \]
\[ r = (\nu_1 - \nu_2)/(\nu_1 + \nu_2), \]

where \( \nu_1 \) and \( \nu_2 \) are related to \( g_1 \) and \( g_2 \) according to Eq. (6). It is easy to see that these coefficients satisfy the law of current conservation: \( r + t = 1 \). It also satisfies the law of energy conservation. In fact the coefficients can be obtained from these two conditions. For the particular values \( \nu_1 = 1 \), \( \nu_2 = 1/3 \), we find that the reflection coefficient is \( 1/2 \). The incoming wave originates from the filling factor \( 1/3 \) side \( (\nu_1 = 1/3, \nu_2 = 1) \), the reflection coefficient is \( -1/2 \). Minus implies that the reflected pulse has the opposite sign of density. The transmission coefficient is \( 3/2 \) in this case, which may appear to be a very unusual result. However, this is similar to a wave reflection in a classical string with an impedance discontinuity, the impedance being the inverse of the filling factor.

Knowing the reflection coefficients for the currents also allows us to find edge state chemical potentials on the two sides of the transition for DC transport. Let us send an infinite wavelength pulse from the \( \nu = 1 \) side with a current such that the voltage on that edge is \( V_1 \) and a pulse from the \( \nu = 1/3 \) with voltage \( V_2 \). Then the outgoing currents can be found from Eqs. (12, 13). The voltages on the outgoing channels are seen to obey Eqs. (12, 13), with \( \alpha = 1/2, \beta = 3/2 \).

Next, we consider the effects of disorder. An impurity, or an irregularity in the confining potential, at point \( x \) in the narrow-neck region can give rise to backscattering, through a term in the Hamiltonian of form

\[ H' = \gamma \exp[i\phi_L(x) + i\phi_R(x)] + \text{h.c.} \]

The phase of the coefficient \( \gamma \) will depend on the position \( x \), and its magnitude will depend sensitively on the width of the strip at that point. The amplitude will be very small if \( x \) is in a wide region, as there will then
be little overlap between the wavefunctions for states on opposite edges of the wire.

The resistance due to backscattering is proportional to $|\gamma|^2$, if $|\gamma|$ is small. According to the standard renormalization group analysis, however, for a wire of constant width, if $g > 0$, the value of $|\gamma|$ will increase with decreasing energy scale. Specifically, for voltages sufficiently small so that one is in the linear regime, the backscattering resistance of a wire should vary as $T^{-y}$, with $y = (1 + g)^{1/2} - 1$. (15)

For the present situation, where $g$ varies with $x$, if the temperature is sufficiently high that the thermal length scale $h v_F/k_B T$ is small compared to the size $L$ of the transition region, Eq. (15) still holds, with $g$ evaluated at the position of the impurity. The value of $y$ obtained in this way would be intermediate between the values $y = 0$ and $y = 2$ that correspond to uniform quantum Hall strips with $\nu = 1$ and $\nu = 1/3$ respectively. If the temperature is sufficiently low that the thermal length is large compared to $L$, however, then we find, from a normal-mode analysis, that the exponent $y$ becomes equal to 1, independent of the precise location $x$ of the scatterer.

In any case, we find that the adiabatic fixed point, where $\beta = 3/2$ and there is no backscattering, is unstable, according to a Luttinger-liquid analysis, so that any non-zero value of $(3/2 - \beta)$ will grow with decreasing temperature and voltage. Thus, to observe the effect of voltage amplification, one must fabricate a junction with a value of $(3/2 - \beta)$ as close as possible to zero, and then make the measurement at a temperature which is not too low.

There are several difficulties standing in the way of the experimental implementation of the DC transformer. First, the quantum point contacts must be approximately a magnetic length wide yet adiabatic. Second, the edges of the $\nu = 1$ and $\nu = 1/3$ states must be sufficiently sharp to support only a single edge channel.

In order to make the junction as close as possible to adiabatic, one would like to avoid any roughness in the confining potential, as well as impurities, which could lead to backscattering. One must also worry, however, about the possibility of an abrupt change in the electron density or its profile across the width of the wire that could occur due to a spontaneously-formed domain wall, if the electron system goes through a first-order phase transition in the neck region.

Although we do not find any symmetry change between the $\nu = 1/3$ and $\nu = 1$ states in a narrow wire, one cannot rule out the possibility of having several phases separated by first-order transitions. In fact, exact-diagonalization studies of systems with up to six electrons in a narrow wire suggest that there might be several distinct phases, separated by sharp transitions, between the densities which correspond to $\nu = 1$ and $\nu = 1/3$. (The calculated states have different density profiles across the wire, corresponding roughly to phases with one, two or three distinct rows of atoms.)

Even if there is a sharp transition in a long wire, however, it might be possible to obtain a smooth transition in a properly engineered point contact. Moreover, it is possible in principle to cancel the reflected amplitude from one density-discontinuity with a wave reflected by a second discontinuity or by an impurity placed at an appropriate position, using destructive interference. Such a complicated structure may be difficult to achieve by design, but might occur naturally in some fraction of samples due to random fluctuations during fabrication.

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