ON CYCLIC QUADRILATERALS
IN EUCLIDEAN AND HYPERBOLIC GEOMETRIES

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Abstract. Four points ordered in the positive order on the unit circle determine the
vertices of a quadrilateral, which is considered either as a euclidean or as a hyperbolic
quadrilateral depending on whether the lines connecting the vertices are euclidean or
hyperbolic lines. In the case of hyperbolic lines, this type of quadrilaterals are called ideal
quadrilaterals. Our main result gives a euclidean counterpart of an earlier result on the
hyperbolic distances between the opposite sides of ideal quadrilaterals. The proof is based
on computations involving hyperbolic geometry. We also found a new formula for the
hyperbolic midpoint of a hyperbolic geodesic segment in the unit disk. As an application
of some geometric properties, we provided a euclidean construction of the symmetrization
of random four points on the unit circle with respect to a diameter which preserves the
absolute cross ratio of quadruples.

1. Introduction

In complex analysis, quadruples of points have a very special role: the absolute cross
ratio of four points \(a, b, c, d\) in the complex plane \(\mathbb{C}\),

\[
|a, b, c, d| = \frac{|a - c||b - d|}{|a - b||c - d|}
\]

(1.1)
is preserved under Möbius transformations. This fact is of fundamental importance to
geometry and, in particular, to the hyperbolic geometry of the unit disk \(\mathbb{H}^2\) and of the
upper half plane \(\mathbb{H}^2\). Indeed, the hyperbolic metric of both domains can be defined in
terms of the absolute cross ratio.

Our goal is to study ordered quadruples \((a, b, c, d)\) of points on the unit circle, points
being listed in the order they occur when we traverse the unit circle in the positive direction.
These points are vertices of a quadrilateral. We analyse the points of intersection of the
lines through opposite sides of the quadrilateral and interpret our observations in terms of
the euclidean or the hyperbolic geometry.

In the course of this work we discover a number of formulas which are very useful for
our work and which, as far as we know, are new — at least we have not found them in
literature. We make frequent use of the well-known formula (2.3) which gives an expression
for the point of intersection \(LIS[a, b, c, d]\) of the two lines \(L[a, b]\) and \(L[c, d]\), through \(a, b\)
and \(c, d\), respectively.

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Quadrilaterals in \( \mathbb{B}^2 \) whose vertices are on the unit circle \( \partial \mathbb{B}^2 \) and whose boundary consists of four circular arcs, orthogonal to the unit circle, are called ideal hyperbolic quadrilaterals. An ideal hyperbolic quadrilateral can be divided into four Lambert quadrilaterals. In [VW1, W], the authors studied the bounds of the two adjacent sides of a hyperbolic Lambert quadrilateral in the unit disk. In particular, the following sharp inequalities hold for the product and the sum of the hyperbolic distances between the opposite sides of an ideal hyperbolic quadrilateral [VW1].

Let \( J^*[a, b] \) denote the hyperbolic line through two points \( a, b \in \partial \mathbb{B}^2 \). For \( A, B \subset \mathbb{B}^2 \), let

\[
d_{\rho}(A, B) = \inf_{x \in A, y \in B} \rho_{\mathbb{B}^2}(x, y),
\]

where \( \rho_{\mathbb{B}^2} \) is the hyperbolic metric in the unit disk defined as (2.11).

**Theorem 1.2.** [VW1, Corollary 1.4] Let \( Q(a, b, c, d) \) be an ideal hyperbolic quadrilateral in \( \mathbb{B}^2 \). Let \( d_1 = d_{\rho}(J^*[a, d], J^*[b, c]) \) and \( d_2 = d_{\rho}(J^*[a, b], J^*[c, d]) \) (see Figure 2). Then

\[
d_1d_2 \leq (2 \log(\sqrt{2} + 1))^2
\]

and

\[
d_1 + d_2 \geq 4 \log(\sqrt{2} + 1).
\]

In both cases equalities hold if and only if \( |a, b, c, d| = 2 \).

We will analyse this further and our discoveries here consist of finding counterparts of these formulas for quadrilaterals obtained from ideal quadrilaterals by replacing the sides, circular arcs, with the euclidean segments with the same endpoints.
Theorem 1.3. Let $w_1 = LIS[a, b, c, d]$, $w_2 = LIS[a, c, b, d]$, $w_3 = LIS[a, d, b, c]$. Let $t_1 = L[w_3, w_2] \cap L[a, b]$, $t_2 = L[w_1, w_2] \cap L[b, c]$, $t_3 = L[w_3, w_2] \cap L[c, d]$, $t_4 = L[w_1, w_2] \cap L[a, d]$ (see Figure 7). Let $d_1 = d_{\rho}(J^*[a, d], J^*[b, c])$, $d_2 = d_{\rho}(J^*[a, b], J^*[c, d])$ (see Figure 3). Then
\[
\frac{\text{sh} \frac{1}{2} \rho_{B^2}(w_2, t_2) + \text{sh} \frac{1}{2} \rho_{B^2}(w_2, t_4)}{\text{sh} \frac{1}{2} \rho_{B^2}(t_2, t_4)} = \frac{d_1}{2},
\]
\[
\frac{\text{sh} \frac{1}{2} \rho_{B^2}(w_2, t_1) + \text{sh} \frac{1}{2} \rho_{B^2}(w_2, t_3)}{\text{sh} \frac{1}{2} \rho_{B^2}(t_1, t_3)} = \frac{d_2}{2},
\]
and
\[
\frac{d_1}{2} + \frac{d_2}{2} = 1.
\]

The bisection problem in the classical hyperbolic geometry has been studied in [G, Construction 3.1] and [KV, 2.9]. Recently, Vuorinen and Wang [VW2] provided several geometric constructions based on euclidean compass and ruler to find the hyperbolic midpoint of a hyperbolic geodesic segment in the unit disk or in the upper half plane. In this paper, we give an explicit formula for the hyperbolic midpoint of a hyperbolic geodesic segment in the unit disk.

Theorem 1.4. For given $x, y \in \mathbb{B}^2$, the hyperbolic midpoint $z \in \mathbb{B}^2$ with $\rho_{B^2}(x, z) = \rho_{B^2}(y, z) = \rho_{B^2}(x, y)/2$ is given by
\[
(1.5) \quad z = \frac{y(1 - |x|^2) + x(1 - |y|^2)}{1 - |x|^2|y|^2 + A[x, y]\sqrt{(1 - |x|^2)(1 - |y|^2)}}
\]
where $A[x, y]$ is the Ahlfors bracket defined as (2.12).

2. Preliminary notation

We will give here some formulas about the geometry of lines and triangles on which our later work is based. In particular, Euler’s formula for the orthocenter of a triangle is very useful.

2.1. Geometry and complex numbers. The extended complex plane $\mathbb{C} = \mathbb{C} \cup \{\infty\}$ is identified with the Riemann sphere via the stereographic projection. The stereographic projection then can be used to define the chordal distance $q(x, y)$ between points $x, y \in \mathbb{C}$ [B]. If $f : (G_1, d_1) \rightarrow (G_2, d_2)$ is a homeomorphism between metric spaces, then the Lipschitz constant of $f$ is defined by
\[
(2.2) \quad \text{Lip}(f) = \sup \left\{ \frac{d_2(f(x), f(y))}{d_1(x, y)} : x, y \in G_1, x \neq y \right\}.
\]

Let $L[a, b]$ stand for the line through $a$ and $b \neq a$. For distinct points $a, b, c, d \in \mathbb{C}$ such that the lines $L[a, b]$ and $L[c, d]$ have a unique point $w$ of intersection, let
\[
w = LIS[a, b, c, d] = L[a, b] \cap L[c, d].
\]
This point is given by
\begin{equation}
(2.3) \quad w = LIS[a, b, c, d] = \frac{u}{v},
\end{equation}
where
\begin{equation}
\begin{aligned}
u &= (\overline{a - b})(c - d) - (a - b)(\overline{c - d}), \\
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
u &= (\overline{a - b})(c - d) - (a - b)(\overline{c - d}).
\end{aligned}
\end{equation}

Let \( C[a, b, c] \) be the circle through distinct non-collinear points \( a, b, c \). The formula \((2.3)\) gives easily the formula for the center \( m(a, b, c) \) of \( C[a, b, c] \). For instance, we can find two points on the bisecting normal to the side \([a, b]\) and another two points on the bisecting normal to the side \([a, c]\) and then apply \((2.3)\) to get \( m(a, b, c) \). In this way we see that the center \( m(a, b, c) \) of \( C[a, b, c] \) is
\begin{equation}
(2.5) \quad m(a, b, c) = \frac{|a|^2(b - c) + |b|^2(c - a) + |c|^2(a - b)}{a(c - d) + b(\overline{a - c}) + c(\overline{b - d})}.
\end{equation}

Euler’s formula gives the orthocenter \( o(a, b, c) \) of a triangle with vertices \( a, b, c \) as
\begin{equation}
(2.6) \quad o(a, b, c) = \frac{a(b + c - d) + b(c + a - d) + c(a + b - d)}{a(b - d) + b(c - a) + c(a - b)}.
\end{equation}

2.7. Möbius transformations. A Möbius transformation is a mapping of the form
\[ z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d, z \in \mathbb{C}, ad - bc \neq 0. \]
The special Möbius transformation
\begin{equation}
(2.8) \quad T_a(z) = \frac{z - a}{1 - \overline{a}z}, \quad a \in \mathbb{B}^2 \setminus \{0\},
\end{equation}
maps the unit disk \( \mathbb{B}^2 \) onto itself with \( T_a(a) = 0 \). Its Lipschitz constant \((2.2)\) as a mapping \( T_a : \mathbb{B}^2 \rightarrow \mathbb{B}^2 \) with respect to the euclidean metric is \([B, p.43]\)
\begin{equation}
(2.9) \quad \text{Lip}(T_a) = \frac{1 + |a|}{1 - |a|}.
\end{equation}

We sometimes use the notation \( x^* = x/|x|^2 = 1/\overline{x} \) for \( x \in \mathbb{C} \setminus \{0\} \).

2.10. Hyperbolic geometry. We recall some basic formulas and notation for hyperbolic geometry from \([B]\). The hyperbolic metric \( \rho_{\mathbb{B}^2} \) is defined by
\begin{equation}
(2.11) \quad \text{sh} \frac{\rho_{\mathbb{B}^2}(x, y)}{2} = \frac{|x - y|}{\sqrt{(1 - |x|^2)(1 - |y|^2)}},
\end{equation}
Since
\[ |x, x^*, y, y^*| = \frac{|x - y^*||y - x^*|}{|x - x^*||y - y^*|} = 1 + \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}, \]
another form of the formula for the hyperbolic metric follows

\[
\frac{\text{ch}_{\rho_{B^2}}(x, y)}{2} = \sqrt{|x, x^*, y^*, y|}.
\]

We also use the Ahlfors bracket notation \( A[x, y] \) \cite{AVV} 7.37, for \( x, y \in \mathbb{B}^2 \)

\[(2.12) \quad A[x, y]^2 = (1 - |x|^2)(1 - |y|^2) + |x - y|^2. \]

The circle which is orthogonal to the unit circle and contains two distinct points \( x, y \in \mathbb{C} \) is denoted by \( C[x, y] \). If \( x, y \in \mathbb{B}^2 \) are distinct points, then \( C[x, y] \cap \partial \mathbb{B}^2 = \{x^*, y^*\} \) where the points are labelled in such a way that \( x^*, x, y, y^* \) occur in this order on \( C[x, y] \). We denote by \( J[x, y] \) the hyperbolic geodesic segment joining two distinct points \( x, y \in \mathbb{B}^2 \).

3. Geometric Observations

Let \( a, b, c, d \in \partial \mathbb{B}^2 \) be points listed in the order they occur when one traverses the unit circle in the positive direction.

**Theorem 3.1.** (1) Let

\[
w_1 = LIS[a, b, c, d], w_2 = LIS[a, c, b, d], w_3 = LIS[a, d, b, c].
\]

Then the point \( w_2 \) is the orthocenter of the triangle with vertices at the points \( 0, w_1, w_3 \).

(2) The point of intersection of the hyperbolic lines \( J^*[a, c] \) and \( J^*[b, d] \) is given by

\[
Y = \frac{(ac - bd) \pm \sqrt{(a - b)(b - c)(c - d)(d - a)}}{a - b + c - d},
\]

where the sign "+" or "-" in front of the square root is chosen such that \( |Y| < 1 \).

**Proof.** (1) Since all the points \( a, b, c, d \) are unimodular, i.e., \( a\overline{a} = b\overline{b} = c\overline{c} = d\overline{d} = 1 \), (2.3) is simplified to

\[
(3.2) \quad w_1 = L[a, b] \cap L[c, d] = \frac{ab(c + d) - cd(a + b)}{ab - cd}.
\]

This follows easily if we multiply \( u \) and \( v \) in (2.4) by the product \( abcd \) and use unimodularity.

From (3.2) there follows that

\[
(3.3) \quad w_2 = L[a, c] \cap L[b, d] = \frac{ac(b + d) - bd(a + c)}{ac - bd}
\]

and

\[
(3.4) \quad w_3 = L[a, d] \cap L[b, c] = \frac{ad(b + c) - bc(a + d)}{ad - bc}.
\]

These formulas together with (2.6) yield

\[
o(0, w_1, w_3) = \frac{w_1w_3 + w_1\overline{w}_3}{w_1w_3 - w_1\overline{w}_3}(w_3 - w_1) = w_2.
\]
Therefore, the point $w_2$ is the orthocenter of the triangle with vertices at the points 0, $w_1$, $w_3$, see Figure 3.

Figure 3.

(2) Let $z = C[a, c] \cap C[b, d]$ and $m(a, c)$ be the center of $C[a, c]$. Then

$$\left| \frac{a + c}{2} \right| |m(a, c)| = 1$$

and hence

(3.5) $$m(a, c) = \frac{2(a - c)}{a^c - ac} = \frac{2ac}{a + c}.$$ 

By (2.5), we have

\[ m(a, z, c) = \frac{ac(1 - |z|^2)}{(a + c) - z - ac\overline{z}}. \]

Since $m(a, c) = m(a, z, c)$, we obtain

\[ \frac{ac\overline{z} + z}{a + c} = \frac{1 + |z|^2}{2}. \]

A similar argument yields

\[ \frac{bd\overline{z} + z}{b + d} = \frac{1 + |z|^2}{2}. \]

Therefore,

\[ \overline{z} = \frac{a - b + c - d}{ac(b + d) - bd(a + c)}z \]

and the point $z$ satisfies the following quadratic equation

(3.6) $$ (a - b + c - d)z^2 - 2(ac - bd)z + (ac(b + d) - bd(a + c)) = 0. $$
Solving this equation, we have the result and the proof is complete. \(\square\)

Remark 3.7. Denote
\[
(3.8) \quad w = J^*[a,c] \cap J^*[b,d].
\]
Let \(u_1 = J^*[a,c] \cap J[0,w_2]\) and \(u_2 = J^*[b,d] \cap J[0,w_2].\) By \[VW1, Proposition 3.1\], \(u_1\) is the hyperbolic midpoint of \(J[0,w_2]\) and so is \(u_2.\) Hence \(u_1 = u_2 = w,\) i.e., the point \(w,\) intersection of the two hyperbolic lines \(J^*[a,c]\) and \(J^*[b,d],\) is the hyperbolic midpoint of the hyperbolic geodesic segment \(J[0,w_2],\) where \(w_2\) is the point of intersection of the two euclidean lines \(L[a,c]\) and \(L[b,d].\)

**Theorem 3.9.** Let \(w_2\) be as \((3.3)\). The centers of the four circles through the point \(w_2\)

\[C[a,b,w_2], C'[b,c,w_2], C'[c,d,w_2], C[a,d,w_2]\]

form the vertices of a rhomboid. Moreover, the euclidean center of the rhomboid is the euclidean midpoint of the segment \([0,w_2].\)

**Proof.** Let

\[p_1 = m[a,b,w_2], p_2 = m[b,c,w_2], p_3 = m[c,d,w_2], p_4 = m[a,d,w_2].\]

Clearly, the lines \(L[p_1,p_2]\) and \(L[p_3,p_4]\) are the bisecting normals to the segments \([b,w_2]\) and \([d,w_2],\) respectively, and hence \(L[p_1,p_2]\) is parallel to \(L[p_3,p_4].\) A similar argument yields that \(L[p_1,p_4]\) is parallel to \(L[p_2,p_3].\) Therefore, \(p_1, p_2, p_3, p_4\) form the vertices of a rhomboid.

Let

\[p_5 = LIS[p_1,p_3,p_2,p_4].\]

By \((2.5),\) we have

\[p_1 = m(a,b,w_2) = \frac{ab(c-d)}{ac-bd} \quad \text{and} \quad p_3 = m(c,d,w_2) = \frac{cd(a-b)}{ac-bd}.\]

Hence

\[p_5 = \frac{1}{2}(p_1 + p_3) = \frac{1}{2}w_2.\]

Namely, \(p_5\) is the euclidean midpoint of the segment \([0,w_2],\) see Figure 4. \(\square\)

Remark 3.10. Let \(w_1, w_3\) be as \((3.3)\) and \((3.4),\) respectively. Denote

\[\{v_1, w_2\} = C[a,b,w_2] \cap C[c,d,w_2],\]

\[\{v_2, w_2\} = C[b,c,w_2] \cap C[a,d,w_2].\]

Since \(p_5\) is on \(L[p_1,p_3]\) and \(L[p_2,p_4]\) which are the bisecting normals to the segments \([v_1,w_2]\) and \([v_2,w_2],\) respectively, by Theorem 3.9

\[|p_5 - v_1| = |p_5 - v_2| = |p_5 - w_2| = |p_5 - 0|= \frac{1}{2}|w_2|.
\]

Hence \(0, v_1, w_2, v_2\) are on the circle centered at \(p_5\) with diameter \([0,w_2].\)
By intersecting-secants theorem, \(w_3, w_2, v_2\) are collinear and so are \(w_1, w_2, v_1\), see Figure 3 and Figure 4. Since \(0, v_2\) is perpendicular to \([w_2, v_2]\), together with Theorem 3.1(1), we have that \(0, v_2, w_1\) are collinear. A similar argument yields that \(0, v_1, w_3\) are collinear.

**Theorem 3.11.** Let \(w_1, w_3, w\) be as (3.2), (3.4) and (3.8), respectively. Then the circle orthogonal to the unit circle goes through \(w\) if and only if the center of the circle is on the line \(L[w_1, w_3]\).

**Proof.** Let \(c_1 = m(a, c)\) and \(c_2 = m(b, d)\). Apparently, the circle orthogonal to the unit circle goes through \(w\) if and only if the center of the circle is on the bisecting normal \(L[c_1, c_2]\) to the segment \([w, 1/w]\). It suffices to show that the four points \(c_2, w_1, c_1, w_3\) are collinear, see Figure 5.

By (3.2) and (3.5),

\[
\frac{c_2 - c_1}{w_1 - c_1} = \frac{2(ab - cd)}{(a - c)(b + d)}. 
\]

Since \(a\overline{a} = b\overline{b} = c\overline{c} = d\overline{d} = 1\), we have

\[
\left( \frac{ab - cd}{(a - c)(b + d)} \right) = \frac{abcd(\overline{a}\overline{b} - \overline{c}\overline{d})}{abcd(\overline{a} - \overline{c})(\overline{b} + \overline{d})} = \frac{(ab - cd)}{(a - c)(b + d)}. 
\]
Therefore, $\frac{c_2 - c_1}{w_1 - c_1} \in \mathbb{R}$ and hence $w_1 \in L[c_1, c_2]$. A similar argument yields $w_3 \in L[c_1, c_2]$. Hence, the four points $c_2, w_1, c_1, w_3$ are collinear.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Figure 5.}
\end{figure}

Remark 3.12. Let $C_o$ be the circle orthogonal to the unit circle centered at $w_1$ and denote its points of intersection with the unit circle by

\begin{equation}
\{p, q\} = C_o \cap \partial \mathbb{B}^2.
\end{equation}

By Remark 3.10, $w_2 \in [w_3, v_2]$ and $L[w_3, v_2]$ is perpendicular to $L[0, w_1]$. By Theorem 3.11, $w \in J^*[p, q]$. By Remark 3.7 and \[VW1\] Proposition 3.1, $w_2 \in [p, q]$. Since $L[p, q]$ is also perpendicular to $L[0, w_1]$, we see that the five points $w_3, p, w_2, v_2, q$ are collinear, see Figure 5.

Theorem 3.14. Let $p, q$ be as \eqref{3.13}. The hyperbolic line $J^*[p, q]$ is the angular bisector of the angle $\angle(J^*[a, c], w, J^*[b, d])$.

Proof. Let $C_o$ be as in Remark 3.12. Tangent-secant theorem yields

$$|w_1|^2 - 1 = |w_1 - q|^2 = |w_1 - a||w_1 - b| = |w_1 - c||w_1 - d|.$$
Then $a$ and $b$ are a pair of inverse points with respect to $C_p$, and so are $c$ and $d$. Hence the inversion in $C_o$ maps $J^*[a, c]$ onto $J^*[b, d]$ and $J^*[p, q]$ onto itself, see Figure 5. Therefore, $J^*[p, q]$ is the angular bisector of the angle $\angle(J^*[a, c], w, J^*[b, d])$ since Möbius transformations preserve angles.

\[ \square \]

**Remark 3.15.** Since $|w_1 - w| = |w_1 - q|$, the proof of Theorem 3.14 yields that $C[a, b, w]$ is tangent to $C[c, d, w]$ at the point $w$.

### 4. Proofs of Main Results

#### 4.1. Proof of Theorem 1.3

By [VW1, (4.1) in the proof of Corollary 1.4],

$$
\frac{1}{|a, b, c, d|} = 2 \arh r,
$$

Therefore,

$$
\frac{1}{|d, a, b, c|} = 2 \arh r',
$$

where $r^2 + r'^2 = 1$. Clearly,

$$
2 \arh^2 d_1 + 2 \arh^2 d_2 = 1.
$$

By [2.3], we have

$$
t_2 = L[w_1, w_2] \cap L[b, c] = \frac{2bc(ad - bc) + (b + c)(bc(a + d) - ad(b + c))}{bc((a + d) - (b + c)) + (bc(a + d) - ad(b + c))}.
$$

Hence

$$
\frac{|w_2 - t_2|^2}{1 - |t_2|^2} = \frac{(w_2 - t_2)(\overline{w_2} - \overline{t_2})}{1 - t_2\bar{t}_2} = \frac{(a - b)(c - d)((a + d) - (b + c))(bc(a + d) - ad(b + c))}{(a - c)(b - d)(ac - bd)^2}.
$$

In a similar way, we have

$$
t_4 = L[w_1, w_2] \cap L[a, d] = \frac{2ad(ad - bc) + (a + d)(bc(a + d) - ad(b + c))}{ad((a + d) - (b + c)) + (bc(a + d) - ad(b + c))}
$$

and hence

$$
\frac{|w_2 - t_4|^2}{1 - |t_4|^2} = \frac{|w_2 - t_2|^2}{1 - |t_2|^2}.
$$

By [2.11], we have

$$
\frac{1}{2} \rho_{\mathbb{B}}(w_2, t_2) = \frac{1}{2} \rho_{\mathbb{B}}(w_2, t_4) = \frac{(a + d) - (b + c))(bc(a + d) - ad(b + c))}{(a - c)(a - d)(b - d)(b - c)}.
$$

Since

$$
\frac{1}{2} \rho_{\mathbb{B}}(t_2, t_4) = \frac{4((a + d) - (b + c))(bc(a + d) - ad(b + c))}{(a - d)^2(b - c)^2},
$$

we obtain

$$
\frac{\frac{1}{2} \rho_{\mathbb{B}}(w_2, t_2) + \frac{1}{2} \rho_{\mathbb{B}}(w_2, t_4)}{\frac{1}{2} \rho_{\mathbb{B}}(t_2, t_4)} = \sqrt{\frac{1}{|a, b, c, d|}} = \frac{d_3}{2}.
$$
A similar argument yields
\[
\frac{\operatorname{sh} \frac{1}{2} \rho_{B^2}(w_2, t_1) + \operatorname{sh} \frac{1}{2} \rho_{B^2}(w_2, t_3)}{\operatorname{sh} \frac{1}{2} \rho_{B^2}(t_1, t_3)} = \sqrt{\frac{1}{|d, a, b, c|}} = \text{th} \frac{d_2}{2}.
\]
This completes the proof. □

Remark 4.2. From the proof of Theorem 1.3, it is easy to see that
\[ \rho_{B^2}(w_2, t_1) = \rho_{B^2}(w_2, t_3) \quad \text{and} \quad \rho_{B^2}(w_2, t_2) = \rho_{B^2}(w_2, t_4). \]

4.3. Proof of Theorem 1.4. Let \( w = J^*[a, c] \cap J^*[b, d] \) and \( w_2 = L[a, c] \cap L[b, d] \), see Figure 3 or Figure 5. By (3.6), we have

\[
\begin{cases}
    w + \frac{1}{w} = \frac{2(ac - bd)}{a - b + c - d} \\
    w = \frac{ac(b + d) - bd(a + c)}{a - b + c - d},
\end{cases}
\]

and hence
\[ w_2 = \frac{2w}{1 + |w|^2}. \]

A simple calculation yields
\[ w = \frac{w_2}{1 + \sqrt{1 - |w|^2}}. \]

Figure 6. The hyperbolic midpoint \( z \) of \( J[x, y] \) is also the hyperbolic midpoint of \( J[0, u] \).

By Remark 3.7, the point \( w \) is the hyperbolic midpoint of \( J[0, w_2] \). By [VW1, Proposition 3.1], the hyperbolic midpoint \( z \) of \( J[x, y] \) is also the hyperbolic midpoint of \( J[0, u] \) (see Figure 6), where
\[
(4.4) \quad u = \text{LIS}[x, y^*, y, x^*] = \frac{y(1 - |x|^2) + x(1 - |y|^2)}{1 - |x|^2|y|^2},
\]
see [VW2, Lemma 4.6(2)]. Then
\[ z = \frac{u}{1 + \sqrt{1 - |u|^2}}. \]
These formulas together with (2.12) imply (1.5). □

**Remark 4.5.** Let \( 0, x, y \) be non-collinear points in the unit disk \( \mathbb{B}^2 \) and let \( z \) be the hyperbolic midpoint of the hyperbolic geodesic segment joining \( x \) and \( y \). As shown in [VW2, Fig.12-16], the five points
\[
k = LIS[x_*, x^*, y_*, y^*], \quad v = LIS[x, x_*, y, y_*], \quad s = LIS[x, y_*, y, x_*],
\]
\[
u = LIS[x, y^*, y, x_*], \quad t = LIS[x_*, y^*, y_*, x^*]
\]
are all on the line through the origin and the point \( z \).

![Figure 7. Five collinear points.](image)

As the proof of Theorem 1.4 shows, we have an explicit formula for \( z \) in terms of \( u \), but finding similar formulas for the other points seems to lead to tedious computations.

**Remark 4.6.** In this paper we treated problems in hyperbolic geometry analytically. However, as the referee pointed out to us, there is an approach to hyperbolic geometry that is fully analogous to the common vector space approach to Euclidean geometry. Indeed, Ungar [U1, U2] introduced the notion of a gyrovector space and developed its theory and applications in many papers and books, see [U1, U2, U3] for further references. The gyrovector formalism provides a framework which enables one to operate with hyperbolic geometric objects in the style of linear algebra. In particular, there is an elegant formula for the gyromidpoint \( z \) of two points \( x, y \in \mathbb{B}^2 \) (see [U1, (6.91)] or [U3]):
\[ z = \frac{1}{2} \odot (x \boxplus y), \]
where \( \frac{1}{2} \otimes x \) is defined as \( \frac{1}{2} \otimes x \oplus \frac{1}{2} \otimes x = x \), and \( x \boxplus y = x \oplus [(\ominus x \oplus y) \oplus x] \) with the Möbius addition \( x \oplus y = \frac{x+y}{1+xy} \) and \( \ominus x = -x \). Calculations yield that

\[
\frac{1}{2} \otimes x = \frac{x}{1 + \sqrt{1 - |x|^2}}
\]

and

\[
x \boxplus y = \frac{y(1 - |x|^2) + x(1 - |y|^2)}{1 - |x|^2 |y|^2}.
\]

Combining (4.7) and (4.8), we obtain

\[
\frac{1}{2} \otimes (x \boxplus y) = \frac{y(1 - |x|^2) + x(1 - |y|^2)}{1 - |x|^2 |y|^2 + A[x, y] \sqrt{(1 - |x|^2)(1 - |y|^2)}};
\]

which is coincident with the formula in Theorem 1.4. Hence we conclude that the gyromidpoint of \( x \) and \( y \) is the hyperbolic midpoint of these two points. According to the formulas (4.4) and (4.8), we see that \( u = x \boxplus y \) and our Figure 6 gives a Euclidean geometric construction for the gyroaddition \( x \boxplus y \) (also see [VW2]).

5. Normalisation of quadruples

Quadruples of points \( a, b, c, d \in \mathbb{C} \) in general positions are often more convenient to handle if they are normalised, brought by a preliminary Möbius transformation to a canonical position. The most commonly used reduction to a canonical position is to find a Möbius transformation \( h \) such that the points are mapped like this

\[
(a, b, c, d) \mapsto (h(a), h(b), h(c), h(d)) = (0, 1, p, \infty), \quad p > 1.
\]

Another possibility is to find a Möbius transformation \( g \) mapping the points to positions symmetric with respect to the origin (see e.g., [AVV, Lemma 7.24] for a formula for \( y \))

\[
(a, b, c, d) \mapsto (g(a), g(b), g(c), g(d)) = (-1, -y, y, 1), \quad |y| \leq 1.
\]

Both of these normalisations (5.1) and (5.2) preserve the cross ratio of the quadruples, but in general change euclidean or chordal distances. We can measure the change of distances utilising the Lipschitz constant (2.2). For each of the canonical forms we can indicate the "cost" of normalisation as the Lipschitz constant of the normalising map. The Lipschitz constant of every Möbius transformation is a finite number [B, Theorem 3.6.1].

Here we will study the normalisation in the special case when the points of the quadruple are on the unit circle and we will emphasise a symmetrization procedure from euclidean geometry. We have two canonical positions: the normalised quadruple is (a) symmetric with respect to a diameter of the unit disk or (b) symmetric with respect to the origin. It is clear that if (b) holds, then also (a) holds.

To see the existence of such kind of normalisation, we can use an appropriate hyperbolic rotation around the point of intersection of the hyperbolic lines which is a composition of two canonical mappings of the unit disk and a rotation around the origin as shown in Figure 8.
As an application of the results from Section 3, mainly from Theorem 3.11 and Theorem 3.14, we provide a euclidean construction of the symmetrization of random four points $a, b, c, d \in \partial \mathbb{B}^2$ with respect to a diameter, see Figure 9 and Figure 10.

(1) Find the points $w_1 = LIS[a, b, c, d]$, $w_3 = LIS[a, d, b, c]$, and $w = J^*[a, c] \cap J^*[b, d]$;
(2) Construct the lines $L[w_1, w_3]$ and $L[0, w]$;
(3) Find the centers $c_1$ and $c_2$ of the circles $C[a, c]$ and $C[b, d]$ on $L[w_1, w_3]$, respectively;
(4) Construct the lines $l_{ac}, l_{pq}, l_{bd}$ passing through the point $w$ such that they are perpendicular to $[c_1, w], [w_1, w], [c_2, w]$, respectively;
(5) Rotate the three lines $l_{ac}, l_{pq}, l_{bd}$ with respect to the point $w$ to the lines $l_{acs}, l_{pqs}, l_{bds}$, respectively, such that $l_{pqs}$ is perpendicular to $L[0, w]$;
(6) Find the points $c_{1s}$ and $c_{2s}$ on $L[w_1, w_3]$ by constructing the lines passing through $w$ and orthogonal to $l_{acs}$ and $l_{bds}$, respectively;
(7) Construct the circle centered at $c_1$ with radius $|c_1 - w|$ intersecting the unit circle at points $a_1$ and $c_1$.
(8) Construct the circle centered at $c_2$ with radius $|c_2 - w|$ intersecting the unit circle at points $b_2$ and $c_2$.

Then the points $a_2, b_2, c_2, d_2$ are the symmetric points of $a, b, c, d$. Specifically, the points $a_2$ and $d_2$ are symmetric with respect to the diameter through $w$, and so are $b_2$ and $c_2$.

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