What local supersymmetry can do for quantum cosmology

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1 Introduction

In all approaches to quantum gravity, one makes vital use of the classical theory, with the knowledge and intuition which this carries, in conjunction with the quantum formulation. When, in addition, the theory possesses local supersymmetry, this generally has profound consequences for the nature of classical solutions, as well as for the quantum theory.

We are used to the procedure in which a bosonic field, such as a massless scalar field $\phi$ in flat 4-dimensional space-time, obeying the wave equation

$$\Box \phi = 0$$

(1.1)
is paired with a fermionic field taken to be (say) an unprimed spinor field $\phi^A$, using 2-component language Penrose and Rindler 1984,1986.

The corresponding Weyl equation,

$$\nabla_{AA'}\phi^A = 0,$$

(1.2)
is a system of two coupled first-order equations, which further imply that each component of $\phi^A$ obeys the massless wave equation (1.2). These bosonic and fermionic fields may be combined (with an auxiliary field) into a multiplet under (rigid) supersymmetry [Wess and Bagger 1992]. The classical fermionic field equation (1.2) may be viewed as a 'square root' of the original second-order bosonic equation (1.1).

There is a further relation here, which will be examined in the following sections. This is with the possibility of curvature being self-dual in four-dimensional Riemannian geometry [Atiyah et al. 1978b]. Only in four dimensions, and only for signature $+4$ (Riemannian), $0$ (ultra-hyperbolic) and $-4$ (equivalent to Riemannian) is this property defined [Mason and Woodhouse 1996]. It applies both to the curvature or field strength $F^{(a)}_{\mu\nu}$ of Yang-Mills or Maxwell theory on a (possibly curved) Riemannian background geometry [Penrose and Rindler 1984, 1986], and to the conformally invariant Weyl curvature tensor $W_{\alpha\beta\gamma\delta}$ of the geometry, which contributes 10 of the 20 algebraic degrees of freedom contained in the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$. The other 10 degrees of freedom reside in the Ricci tensor $R_{\alpha\gamma} = g^{\beta\delta} R_{\alpha\beta\gamma\delta}$, where $g^{\beta\delta}$ describes the inverse metric. In Einstein’s theory, the Ricci tensor corresponds to the matter source for the curvature. The Weyl tensor $W_{\alpha\beta\gamma\delta}$ may be thought of as describing the ‘vacuum’ part of the gravitational field in General Relativity.

In both the Yang-Mills and the Weyl-tensor cases, one can describe the curvature simply in two-component spinor language [Penrose and Rindler 1984, 1986]. The Yang-Mills field-strength ten-
\( F^{(a)}_{\mu \nu} = F_{[\mu \nu]} \) corresponds to the spinor field

\[
F^{(a)}_{AA'B'B'} = \phi^{(a)}_{AB} \varepsilon_{AB'} + \bar{\phi}^{(a)}_{A'B'} \varepsilon_{AB},
\]

where \( \phi^{(a)}_{AB} = \phi^{(a)}_{(AB)} \) and \( \bar{\phi}^{(a)}_{A'B'} = \bar{\phi}^{(a)}_{(A'B')} \) are symmetric. Here, \( \varepsilon_{AB} \) is the unprimed alternating spinor, and \( \varepsilon_{A'B'} \) its primed counterpart. In the present Riemannian case, the fields \( \phi^{(a)}_{AB} \) and \( \bar{\phi}^{(a)}_{A'B'} \) are independent complex quantities. If the field strength \( F^{(a)}_{\mu \nu} \) is real, arising (say) from real Yang-Mills potentials \( v^{(a)}_{\mu} \), then each of \( \phi^{(a)}_{AB} \) and \( \bar{\phi}^{(a)}_{A'B'} \) is subject to a condition which halves the number of real components of each. [But in the Lorentzian case, for real \( F^{(a)}_{\mu \nu} \), \( \bar{\phi}^{(a)}_{A'B'} \) will be replaced by \( \bar{\phi}^{(a)}_{A'B'} \), the complex conjugate of \( \phi^{(a)}_{AB} \).]

In the Riemannian case, the Yang-Mills field strength is said to be self-dual if

\[
\phi^{(a)}_{AB} = 0.
\]

Similarly, an anti-self-dual field has \( \bar{\phi}^{(a)}_{A'B'} = 0 \). In the case (say) of a Maxwell field in flat Euclidean 4-space, the (anti-)self-dual condition can be written in terms of the electric and magnetic fields as \( E = \pm B \). Generally, if a Yang-Mills field is anti-self-dual, then the Yang-Mills field equations reduce to the set:

\[
D_{A'A'}^{B} \phi^{(a)}_{AB} = 0,
\]

where \( D_{AA'}^{B} \) is the covariant derivative [Penrose and Rindler 1984, 1986]. This set of equations is, of course, a generalisation of the Weyl (massless Dirac) equation (1.2).

Regular real solutions of the anti-self-dual Yang-Mills equations (1.5) on the four-sphere \( S^4 \) are known as instantons [Eguchi et al. 1980]. Since (1.5) is conformally invariant [Atiyah et al. 1978], such a solution corresponds to a ‘localised’ region of Yang-Mills curvature.
in flat Euclidean $\mathbb{E}^4$, with suitable asymptotic behaviour at large four-dimensional radius. For the simplest non-trivial gauge group $\text{SU}(2)$, Atiyah et al. (1978a) have, remarkably, given a construction of the general Yang-Mills instanton. Yang-Mills instantons can also be described in terms of twistor theory [Ward and Wells 1990].

In section 2, motivated by the Hartle-Hawking proposal in quantum cosmology [Hawking 1982; Hartle and Hawking 1983], we shall be led to consider Riemannian Einstein gravity, including possibly a negative cosmological constant $\Lambda$, in the case that the Weyl tensor is (anti-)self-dual [Capovilla et al. 1990]. The Einstein field equations

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

are the conditions for an Einstein space. The (anti-)self-duality condition then gives a further set of equations, closely related to the (anti-)self-dual Yang-Mills equations (1.5) in the SU(2) case. But in the case of quantum cosmology, the boundary conditions are usually specified on a compact connected three-surface, such as a three-sphere $S^3$, in contrast to the Yang-Mills instanton case, where they are specified at infinity. This gravitational version is therefore more complicated. Note that a treatment of the related boundary-value problem for Hermitian Yang-Mills equations over complex manifolds has been given by Donaldson (1992). In the case of 2-complex-dimensional manifolds with Kähler metric [Eguchi et al. 1980], this leads to anti-self-dual Yang-Mills connections.

The corresponding purely gravitational solutions, in the case where the boundary is at infinity (with suitable fall-off) or where the manifold is compact without boundary, are known as gravitational instantons [Hawking 1977; Eguchi et al. 1980]. The (anti-)self-dual
condition on the Weyl tensor $W_{\alpha\beta\gamma\delta}$ (in the Riemannian case) may again be described in spinor terms [Penrose and Rindler 1984, 1986]:

- $W_{\alpha\beta\gamma\delta}$ corresponds to

$$W_{AA'BB'CC'DD'} = \Psi_{ABCD} \varepsilon_{A'B'C'D'} + \tilde{\Psi}_{A'B'C'D'} \varepsilon_{AB} \varepsilon_{CD};$$

where the Weyl spinors $\Psi_{ABCD} = \Psi_{(ABCD)}$ and $\tilde{\Psi}_{A'B'C'D'} = \tilde{\Psi}_{(A'B'C'D')}$ are again totally symmetric. The Weyl tensor is self-dual if

$$\Psi_{ABCD} = 0,$$  \hspace{1cm} (1.8)

and anti-self-dual if

$$\tilde{\Psi}_{A'B'C'D'} = 0.$$

Thus, in the anti-self-dual case (say) arising in quantum cosmology, the Ricci tensor is restricted by Eq. (1.6) and the Weyl tensor by Eq. (1.9). The Bianchi identities [Penrose and Rindler 1984, 1986] then imply further that the remaining Weyl spinor $\Psi_{ABCD}$ obeys

$$\nabla^{AA'} \Psi_{ABCD} = 0.$$  \hspace{1cm} (1.10)

These equations are again a generalisation of the Weyl equation (1.2).

Thus, at least at the formal level, there are clear resemblances concerning supersymmetry and (anti-)self-dual classical Yang-Mills or Einstein theory. More detail will be given in sections 4–6.

Turning now to the quantum theory, one has an apparent choice in quantum cosmology between the Feynman path-integral approach [Hartle and Hawking 1983] and the differential approach given by Dirac’s theory of the quantisation of constrained Hamiltonian systems [Dirac, 1950, 1958a, 1958b, 1959, 1965]. Loosely speaking, the latter may be thought of as a description of quantum theories with
local invariance properties, such as gauge invariance and/or invariance under local coordinate transformations, although in fact it is more general than that. Historically, a large amount of work on quantum cosmology was carried out by relativists following the pioneering work of DeWitt (1967) and Wheeler (1968) based on the Dirac approach. The eponymous (Wheeler-DeWitt) equation is central to the resulting quantum treatment of spatially-homogeneous cosmologies, possibly containing bosonic matter, in which the classical dynamics involves a (typically small) number of functions of a time-coordinate $t$ only, and the resulting quantum field theory reduces to a quantum-mechanical theory, with a finite number of coordinate variables \cite{Ryan and Shepley 1975}. However, it has not been possible to make sense of the second-order functional Wheeler-DeWitt equation in the non-supersymmetric case of Einstein gravity plus possible bosonic fields, when the gravitational and any other bosonic fields are allowed to have generic spatial dependence.

The path-integral approach of relevance here is that of Hartle and Hawking (1983). There is, formally speaking, a ‘preferred quantum state’ for the quantum theory of (say) a spatially-compact cosmology, where typically the coordinate variables, which become the arguments of the wave functional, are taken to be the Riemannian three-metric $h_{ij}$ of the compact three-manifold, together with (say) any other bosonic fields on the three-manifold, denoted schematically by $\phi_0$. One then considers all possible Riemannian metrics $g_{\mu \nu}$ and all other fields $\phi$ on all possible four-manifolds $\mathcal{M}$, such that the original three-manifold is the boundary $\partial \mathcal{M}$ of $\mathcal{M}$, and such that the ‘interior’ $\mathcal{M}$ together with its boundary $\partial \mathcal{M}$, namely $\mathcal{M} \cup \partial \mathcal{M}$ or $\bar{\mathcal{M}}$, is a compact manifold-with-boundary. The three-metric and
other fields inherited from \((g_{\mu\nu}, \phi)\) on the boundary must agree with the originally prescribed \((h_{ij}, \phi_0)\). For visualisation, the simplest example is the compact manifold \(S^3\) (the three-sphere), with interior the four-ball \(B^4\). The Hartle-Hawking state \(\Psi_{HH}\), also known as the ‘no-boundary state’, is then (formally) described by

\[
\Psi_{HH}(h_{ij}, \phi_0) = \int \mathcal{D}g_{\mu\nu} \mathcal{D}\phi \exp\left[ -I(g_{\mu\nu}, \phi)/\hbar \right]. \tag{1.11}
\]

Here the functional integral is over all suitable infilling fields \((g_{\mu\nu}, \phi)\), and \(I\) is the corresponding Euclidean action [Hartle and Hawking 1983; D’Eath 1996]. Since the integrand is an analytic (holomorphic) function of its arguments, this path integral may be regarded as a giant contour integral, with the set of suitable infilling fields deformed into the complex. The question of finding a suitable contour for which the integral is meaningful (convergent) is a major problem in this approach to quantum cosmology; in the above Riemannian case, the Euclidean action \(I\) is unbounded below [Gibbons et al. 1978], so that the integrand in Eq. (1.11) can become arbitrarily large and positive.

The Feynman-path-integral and Dirac-quantisation approaches are dual integral and differential attempts to describe the same quantum theory, here ‘quantum gravity’. As shown in the book of Feynman and Hibbs (1965), for non-relativistic quantum mechanics, the path integral gives a wave function or quantum amplitude for a particle to go from initial position and time \((x_a, t_a)\) to final \((x_b, t_b)\), with \(t_b > t_a\), which obeys the Schrödinger equation and also satisfies the boundary conditions as \(t_b \to t_a\). Similarly for the converse. Indeed Feynman’s Princeton Ph.D. thesis evolved from his continuing thought about the paper by Dirac (1933), which in effect derived the path integral for propagation in a short time-
interval $\Delta t$. Similar dual relations even hold, somewhat schemati-
cally, for the path-integral and differential versions of quantum grav-
ity [Hartle and Hawking 1983].

Given the above difficulties, which of these two approaches (if any) should we use and, perhaps, trust? The bad convergence prop-
erties of the gravitational path integral seem, at present, very diffi-
cult to overcome. Similarly for the question of defining the second-
order Wheeler-DeWitt operator in the non-supersymmetric case. But the Hartle-Hawking path integral provides a powerful concep-
tual, indeed visual, schema. And when local supersymmetry is in-
cluded, the Wheeler-DeWitt equation is replaced by its fermionic
‘square root’ [Teitelboim 1977], the quantum supersymmetry con-
straints [D’Eath 1984, 1986], which allow more sense to be made
of the quantum theory. One should expect to use both approaches
together, so far as is possible. Richard Feynman himself certainly
attended to the Dirac constrained-quantisation approach, notably in
his last substantial paper [Feynman 1981], on Yang-Mills theory in
2+1 dimensions, on which he worked for three years. Indeed, when
Feynman gave the first of the annual Dirac lectures in Cambridge,
in June 1986, he remarked “How could I refuse the invitation? –
Dirac was my hero”. Pragmatically, anyone who has to teach a
first undergraduate course in quantum mechanics will usually base
it on the Schrödinger equation (the differential approach). But, of
course, there is nothing to stop them from inducting the students
via the path-integral approach. Maybe this has been tried, at least
at CalTech!
2 No-Boundary State

To expand on the description (1.11) of the Hartle-Hawking state, we will need later to be able to include fermions with the gravitational and other bosonic fields, in the Riemannian context. This requires the introduction of an orthonormal tetrad $e^a_\mu$ of one-forms; here $a = 0, 1, 2, 3$ labels the four orthonormal co-vectors $e^a_\mu$, while $\mu$ is the ‘space-time’ or ‘world’ index. One has, by orthogonality and completeness:

$$g^{\mu\nu} e^a_\mu e^b_\nu = \delta^{ab}, \quad \delta_{ab} e^a_\mu e^b_\nu = g_{\mu\nu}. \quad (2.1)$$

Thus $e^a_\mu$ is a ‘square root’ of the metric $g_{\mu\nu}$, non-unique up to local SO(4) rotations acting on the internal index $a$.

In this case, instead of specifying the intrinsic three-metric $h_{ij}$ on the boundary ($i, j, ... = 1, 2, 3$), one would specify the four spatial one-forms $e^a_i$, with

$$h_{ij} = \delta_{ab} e^a_i e^b_j. \quad (2.2)$$

This seems a redundant description, since only three co-vectors $e^a_i$ are needed to take the ‘square root’ of $h_{ij}$ in Eq.(2.2). On and near the boundary surface, it is valid to work in the ‘time gauge’ [Nelson and Teitelboim 1978], using a triad $e^a_i$ ($a = 1, 2, 3$) to obey Eq.(2.2). Equivalently, one imposes $e^0_i = 0$ as a gauge condition. Instead of integrating over all Riemannian four-metrics $g_{\mu\nu}$ in Eq.(1.11), one integrates over all $e^a_\mu$, each of which corresponds to a Riemannian metric by Eq.(2.1).

The gravitational part of the Euclidean action $I$ in Eq.(1.11) is [Gibbons and Hawking 1977]

$$I_{grav} = -\frac{1}{2\kappa^2} \int_M d^4 x \, g^{\frac{1}{2}} (R - 2\Lambda) - \frac{1}{\kappa^2} \int_{\partial M} d^3 x \, h^{\frac{1}{2}} tr K. \quad (2.3)$$
Here $\kappa^2 = 8\pi$, $g = \text{det}(g_{\mu\nu}) = [\text{det}(e^a_\mu)]^2$, $R$ is the four-dimensional Ricci scalar, $\Lambda$ is the cosmological constant and $\text{tr} K = h^{ij} K_{ij}$, where $K_{ij}$ is the second fundamental form (or extrinsic curvature) of the boundary \cite{Misner1973,D'8996}.

In the path integral (1.11), one expects to sum over all four-manifolds $\mathcal{M}$, of different topology, which have the prescribed three-manifold as boundary $\partial \mathcal{M}$; for each topologically different $\mathcal{M}$, one then integrates over metrics $g_{\mu\nu}$ or tetrads $e^a_\mu$. For each choice of $\mathcal{M}$, one can ask whether there are any solutions of the classical field equations, for the given boundary data $h_{ij}$ or $e^a_i$, other bosonic data $\phi_0$ and possible fermionic data. In the simplest case of gravity without matter (for definiteness), there may be zero, one, two, ... real Riemannian solutions of Eq.(1.6) for a given topology $\mathcal{M}$. Of course, for the same $\mathcal{M}$ and real boundary data $h_{ij}$, there may be a larger number of complex classical solutions $g_{\mu\nu}$.

Suppose, again for definiteness, that we again have gravity without matter, and that there is a unique classical solution $g_{\mu\nu}$ (up to coordinate transformation) which is Riemannian (and hence real) on a particular four-manifold $\mathcal{M}_0$, corresponding to the boundary data $h_{ij}(x)$. Further, suppose that there are no other classical solutions on any other manifold $\mathcal{M}$. Then, were the path integral (1.11) to be meaningful, one would expect to have a semi-classical expansion of the Hartle-Hawking state, of the form

$$\Psi_{\text{HH}}[h_{ij}(x)] \sim (A_0 + \hbar A_1 + \hbar^2 A_2 + \ldots) \exp(-I_{\text{class}}/\hbar). \quad (2.4)$$

Here the wave function $\Psi_{\text{HH}}$, the ‘one-loop factor’ $A_0$, ‘two-loop factor’ $A_1$, ... and the Euclidean action $I_{\text{class}}$ of the classical solution [as in Eq.(2.3)] are all functionals of $h_{ij}(x)$. Technically, one might expect that such an expansion, if it existed, would only be
an asymptotic expansion valid in the limit as $\hbar \to 0_+$. Even in non-relativistic quantum mechanics, semi-classical expansions are typically only asymptotic but not convergent [Itzykson and Zuber 1980]. Note also that, if well-posed fermionic boundary data are included, and there is a unique corresponding coupled bosonic-fermionic classical solution, then one expects again a semi-classical wave function $\Psi_{HH}$ of the boundary data, of the form (2.4), except that each of $I_{class}, A_0, A_1, A_2,...$ will be a functional of the complete bosonic and fermionic boundary data.

3 The classical Riemannian boundary-value problem

3.1 The general boundary problem

As follows from section 2, it is important to understand the nature of the Riemannian boundary-value problem for Einstein gravity, possibly including a $\Lambda$-term, matter fields and local supersymmetry. Only very partial results are available in the generic case for which the boundary data has no symmetries. Reula (1987) proved an existence theorem for the vacuum Riemannian Einstein equations ($\Lambda = 0$) on a slab-like region, where suitable data on two parallel planes enclosing a slab of Euclidean $\mathbb{E}^4$ are slightly perturbed. For weak perturbations of a suitable known compact manifold-with-boundary, the case $\Lambda \leq 0$ was studied by Schlenker (1998). To fix one’s intuition, consider the case in which the unperturbed boundary is a metric three-sphere $S^3$, bounding part of flat $\mathbb{E}^4$ (if $\Lambda = 0$) or of a hyperbolic space $\mathbb{H}^4$ (if $\Lambda < 0$). Then any sufficiently weak perturbation of the boundary metric $h_{ij}$ yields a corresponding (per-
turbed but non-linear) interior solution \( g_{\mu\nu} \) for the 4-metric, obeying \( R_{\mu\nu} = \Lambda g_{\mu\nu} \).

Boundary-value problems ‘at infinity’ have also been studied, for \( \Lambda < 0 \), when a 4-dimensional Riemannian geometry can be given a conformal infinity [Graham and Lee 1991]. The canonical example is hyperbolic space \( \mathbb{H}^4 \), with its metric of constant curvature, here normalised such that \( \Lambda = -\frac{1}{12} \). Let \( ds_b^2 \) denote the flat Euclidean metric on \( \mathbb{R}^4 \), expressed in terms of Cartesian coordinates \( x^\mu (\mu = 0,1,2,3) \); then define the conformal function \( \rho = \frac{1}{2}(1 - |x|^2) \) on the unit ball \( B^4 \subset \mathbb{R}^4 \). The conformal metric \( ds^2 = \rho^{-2}ds_b^2 \) is the hyperbolic metric on the open set \( B^4 \), and the ‘conformal metric at infinity (\( |x| = 1 \))’ can be taken to be the standard metric \( H_{ij} \) on \( S^3 \). Graham and Lee (1991) have shown that this ‘conformal Einstein’ problem is also well-behaved for small perturbations of the unit-sphere metric \( H_{ij} \) on the conformal boundary \( S^3 \). As in the previous paragraph, for 3-metrics \( h_{ij} \) sufficiently close to \( H_{ij} \), there is a corresponding conformal metric on the interior \( B^4 \), close to the unperturbed hyperbolic metric.

The case with conformal infinity, imposing the Einstein condition \( R_{\mu\nu} = \Lambda g_{\mu\nu} \) with \( \Lambda < 0 \), has also been studied subject to the additional requirement of (say) self-duality of the Weyl tensor, \( \Psi_{ABCD} = 0 \) [Eq.(1.9)]. LeBrun (1982) has shown that, when the conformal infinity \( \partial M \) is a real-analytic 3-manifold with conformal metric \( h_{ij} \), then, in a neighbourhood of \( \partial M \), there is a conformal 4-metric \( g_{\mu\nu} \) on a real-analytic 4-manifold \( M \), satisfying the Einstein equations with \( \Lambda < 0 \) and self-dual Weyl curvature. The real-analytic condition on the conformal boundary \( \partial M \) is essential, since the Einstein-space condition \( R_{\mu\nu} = \Lambda g_{\mu\nu} \) together
with self-duality imply that the 4-manifold must be real-analytic \cite{Atiyah}. Further, LeBrun (1991) has shown that there is an infinite-dimensional space of conformal metrics \( h_{ij} \) on \( S^3 \) which bound complete Einstein metrics on the 4-ball, with (anti-)self-dual Weyl curvature; that is, \( S^3 \) is again conformal infinity, but now the result is not just local, in a neighbourhood of the \( S^3 \) boundary, but extends smoothly across the interior, the 4-ball.

Finally, note that LeBrun (1982) also proved a local result in which the conformal infinity \( \partial M \) is taken to be a suitable complex 3-manifold with given holomorphic metric \cite{Wells}, and a complex 4-manifold \( M \) in a neighbourhood of \( \partial M \) is then guaranteed to exist, with holomorphic metric obeying \( R_{\mu\nu} = \Lambda g_{\mu\nu} \) and \( \Lambda \neq 0 \) (possibly complex), together with self-duality of the Weyl tensor. This is of potential interest in quantum gravity, since, as in section 1, the Hartle-Hawking path integral is a contour integral, and there may be stationary points (classical solutions) with holomorphic 4-metrics; further, one would expect to be able to continue the boundary data, such as \( h_{ij} \), into the complex (i.e., holomorphically).

### 3.2 Example - Biaxial Riemannian Bianchi-IX Models

As an example, consider the family of Riemannian 4-metrics with isometry group \( SU(2) \times U(1) \), given (locally in the coordinate \( r \)) by:

\[
 ds^2 = dr^2 + a^2(r)(\sigma_1^2 + \sigma_2^2) + b^2(r)\sigma_3^2. \tag{3.1}
\]

Here, \( a(r) \) and \( b(r) \) are two functions of the ‘radial’ coordinate \( r \), and \( \{\sigma_1, \sigma_2, \sigma_3\} \) denotes the basis of left-invariant 1-forms (co-vector fields) on the three-sphere \( S^3 \), regarded as the group \( SU(2) \), with the conventions of \cite{Eguchi}. The more general triaxial
Bianchi-IX metric \cite{Kramer1979} – see below – would have three different functions multiplying $\sigma_1^2$, $\sigma_2^2$ and $\sigma_3^2$ in Eq. (3.1).

In the biaxial case, the boundary data at a value $r = r_0$ are taken to be the intrinsic 3-metric

$$ds^2 = a^2(r_0)(\sigma_1^2 + \sigma_2^2) + b^2(r_0)\sigma_3^2,$$

(3.2)
determined by the positive numbers $a^2(r_0)$ and $b^2(r_0)$. This gives a ‘squashed 3-sphere’ or Berger sphere. Thus one wishes to find a regular solution of the Einstein-$\Lambda$ field equations (1.6) in the interior $\mathcal{M}$ of the boundary $\partial\mathcal{M} \cong S^3$ (denoting ‘$\partial\mathcal{M}$ is diffeomorphic to $S^3$’), subject to the boundary data (3.2). There are two possible ways in which such a 4-geometry could close in a regular way as $r$ is decreased from $r_0$, to give a compact manifold-with-boundary $\mathcal{M} \cup \partial\mathcal{M} = \bar{\mathcal{M}}$. Either $\mathcal{M}$ is (diffeomorphically) a 4-ball $B^4$, with standard polar-coordinate behaviour

$$a(r) \sim r, \quad b(r) \sim r \quad \text{as} \quad r \to 0$$

(3.3)
near the ‘centre’ $r = 0$ of the 4-ball. Or $\mathcal{M}$ has a more complicated topology, still with boundary $\partial\mathcal{M} \cong S^3$, such that

$$a(r) \to c(\text{constant} > 0), \quad b(r) \sim r \quad \text{as} \quad r \to 0.$$  

(3.4)

Here the 4-metric degenerates to the metric of a round 2-sphere $S^2$, as $r \to 0$. The first case is described as NUT behaviour as $r \to 0$, and the second as BOLT behaviour \cite{Gibbons1979, Eguchi1980}. In both cases, the apparent singularity at $r = 0$ is a removable coordinate singularity.

The general Riemannian solution of the Einstein field equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$ for biaxial Bianchi-IX metrics can be written in the
ds^2 = \frac{(\rho^2 - L^2)}{4\Delta} d\rho^2 + (\rho^2 - L^2)(\sigma_1^2 + \sigma_2^2) + \frac{4L^2\Delta}{(\rho^2 - L^2)}\sigma_3^2, \quad (3.5)

where

\Delta = \rho^2 - 2M\rho + L^2 + \frac{1}{4}\Lambda(L^4 + 2L^2\rho^2 - \frac{1}{3}\rho^4). \quad (3.6)

This 2-parameter family of metrics, labelled (for given \(\Lambda\)) by the constants \(L, M\), is known as the Taub-NUT-(anti)de Sitter family.

It was found by Jensen et al. (1991) that a 4-geometry in this family has NUT behaviour (near \(\rho^2 = L^2\)) precisely when one of the relations

\[ M = \pm L\left(1 + \frac{4}{3}\Lambda L^2\right) \quad (3.7) \]

holds. Further, these are the conditions for (anti-)self-duality of the Weyl tensor. In the classical NUT boundary-value problem, positive values of \(A = a^2(r_0)\), \(B = b^2(r_0)\) are specified on the boundary \(\partial\mathcal{M}\), and the geometry must fill in smoothly on a 4-ball interior, subject to the Einstein equations \(R_{\mu\nu} = \Lambda g_{\mu\nu}\). As remarked by Jensen et al. (1991), NUT regularity corresponds to one further requirement, given by a cubic equation, beyond Eq. (3.7). This is investigated further in Akbar and D’Eath (2002a); see also Chamblin et al. (1999). Taking (say) the anti-self-dual case in Eq. (3.7), assuming also \(\Lambda < 0\) for the sake of argument, and given positive boundary values \((A, B)\), the cubic leads to three regular NUT solutions (counting multiplicity). Depending on the values \((A, B)\), from zero to three of these are real Riemannian solutions of the type (3.5). The remaining NUT solutions are inevitably complex (holomorphic) geometries. In the physically interesting limit, where both ‘cosmological’ (radii)^2 \(A\) and \(B\)
are large and comparable, all three solutions are real. In the Hartle-Hawking path integral (1.11) and its semi-classical expansion (2.4), with Euclidean action $I_{\text{grav}}(\cdot)$, this would give an estimate, say for the isotropic case $A = B$:

$$I_{\text{class}} \sim -\frac{\pi}{12|\Lambda|} A^2 \text{ as } A \to \infty,$$

with

$$\Psi_{HH} \sim (\text{slowly varying prefactor}) \times \exp(-I_{\text{class}}/\hbar).$$

(3.8)

In such an Einstein-negative-$\Lambda$ model, without further matter, the relative probability of finding a universe with a given $A$ would increase enormously with $A$.

If instead $\Lambda = 0$, the solution (3.5,6) reduces to the Euclidean Taub-NUT solution [Hawking 1977]. For $\Lambda > 0$, one may visualise the isotropic case $A = B$, with a metric 4-sphere $S^4$ as Riemannian solution, the radius being determined in terms of $\Lambda$. When the (radii)$^2$ $A$ and $B$ become too large relative to $\Lambda^{-1}$, there will be no real Riemannian solution [Jensen et al. 1991], but only complex (holomorphic) geometries.

The alternative regular BOLT solutions are studied in [Akbar and D’Eath 2002b], particularly in the case $\Lambda < 0$, for given positive boundary data $(A, B)$. These solutions do not have an (anti-)self-dual Weyl tensor. Further, their topology is more complicated than that of the 4-ball $B^4$ – the simplest way of filling in an $S^3$. This difference can be seen, for example, by computing the topological invariants $\chi$, the Euler invariant, and $\tau$, the Pontryagin number, each of which is given by a volume integral quadratic in the Riemann tensor, together with a suitable surface integral [Eguchi et al. 1980]. For the 4-ball, one has $\chi = 1, \tau = 0$; for a
BOLT solution, \( \chi = 2, \tau = -1 \). The problem of finding BOLT solutions depends on studying a seventh-degree polynomial! The number of real regular BOLT solutions, for given positive boundary data \((A, B)\), must be twice a strictly positive odd number; other solutions are necessarily complex. When the boundary is not too anisotropic, i.e., when \( A \) and \( B \) are sufficiently close to one another, there are exactly two regular BOLT solutions.

Of course, one could in principle study the corresponding much more elaborate triaxial boundary-value problem. This has at least been done for the case of a conformal boundary at infinity, as in section \( 3.1 \), with conformal 3-metric of triaxial Bianchi-IX type [Hitchin 1993]. The solutions involve Painlevé's sixth equation [Mason and Woodhouse 1996]; see also [Tod 1994].

4 Self-duality

4.1 Hamiltonian approach; Ashtekar variables

Consider now a Hamiltonian treatment of Einstein gravity with a \( \Lambda \)-term, modified for use in the Riemannian or 'imaginary-time' case. Since we shall later include fermions, a tetrad (or triad) description of the geometry must be used, as in Eqs. (2.12), except that we shall use spinor-valued one-forms \( e^{AA'}{}_{\mu} \) instead of the tetrad \( e^a{}_{\mu} \). Here

\[
e^{AA'}{}_{\mu} = \sigma_a^{AA'} e^a{}_{\mu}, \quad (4.1)
\]

where \( \sigma_a^{AA'} \) are appropriate Infeld-van der Waerden translation symbols [Penrose and Rindler 1984,1986]. The spatial 3-metric \( h_{ij} \) is given by

\[
h_{ij} = -e_{AA'} e^{AA'}{}^j, \quad (4.2)
\]
where the spinor-valued spatial one-forms $e^A A' i$ are regarded as the coordinate variables in a ‘traditional’ Hamiltonian treatment. The Lorentzian normal vector $n^\mu$ has spinor version $n^{AA'}$, which is determined once the $e^{AA'} i$ are known [D’Eath 1996]. In our Riemannian context, the corresponding Euclidean normal vector $e n^\mu$ corresponds to

$$e n^{AA'} = -i n^{AA'}.$$  \hfill (4.3)

In the ‘time gauge’ of section 21 one has only a triad $e^a i (a = 1, 2, 3)$, and the remaining one-form $e^0 i$ is constrained by $e^0 i = 0$; equivalently $e n^\mu = \delta^\mu_0$. The local invariance group of the theory becomes SO(3) in the triad version.

For Riemannian 4-geometries, the torsion-free connection is given by the connection 1-forms $\omega^{ab} \mu = \omega^{(ab)} \mu$ [D’Eath 1996]. In spinor language, these correspond to

$$\omega^{AA'BB'} \mu = \omega^{AB} \mu \varepsilon^{A'B'} + \tilde{\omega}^{A'B'} \mu \varepsilon^{AB},$$  \hfill (4.4)

where $\omega^{AB} \mu = \omega^{(AB)} \mu$ is a set of 1-forms taking values in the Lie algebra $\text{su}(2)$ of the group SU(2); similarly for the independent quantity $\tilde{\omega}^{A'B'} \mu = \tilde{\omega}^{(A'B')} \mu$, with a different copy of SU(2). Then the curvature is described by the 2-forms $R^{AB} \mu \nu = R^{(AB)} [\mu \nu]$, defined by

$$R^{AB} \mu \nu = 2(\partial_\mu \omega^{AB} \nu + \omega^A C[\mu \omega^{CB} \nu]),$$  \hfill (4.5)

and a corresponding $\tilde{R}^{A'B'} \mu \nu$. In the language of forms [Eguchi et al. 1980], the spinor-valued 2-form $R^{AB}$ is defined equivalently as

$$R^{AB} = d\omega^{AB} + \omega^C A \wedge \omega_C B$$  \hfill (4.6)

and corresponds to the anti-self-dual part of the Riemann tensor. Similarly, $\tilde{R}^{A'B'}$ corresponds to the self-dual part.
In the Hamiltonian formulation of Ashtekar (1986, 1987, 1988, 1991), one defines the spatial spinor-valued 1-forms $\sigma^{AB}_i = \sigma^{(AB)}_i$ as

$$\sigma^{AB}_i = \sqrt{2}ie^{A}_{Ai}n^{BA_i}.$$  \hspace{1cm} (4.7)

These can equally be described, in the time gauge, in terms of the spatial triad $e^a_i$, the translation symbols $\sigma^a_{AB}'$ and the unit matrix $\delta^{AA'}$. Then, with $\sigma^{AB}_i = h^{ij}\sigma^{AB}_j$, one defines the density

$$\tilde{\sigma}^{AB}_i = h^iJ\sigma^{AB}_i,$$  \hspace{1cm} (4.8)

where $h = det(h_{ij})$. The Ashtekar canonical variables are then $\tilde{\sigma}^{AB}_i$ and $\omega_{ABi}$, the spatial part of the unprimed connection 1-forms with spinor indices lowered. It can be verified that these are canonically conjugate. Of course, since, they contain only unprimed spinor indices, they are very well adapted for a description of (anti-)self-duality. For this purpose, we shall also need the spinor-valued 2-form

$$\Sigma_{AB} = e^{A'}_{Ai} \wedge e_{BA'},$$  \hspace{1cm} (4.9)

obeying $\Sigma_{AB} = \Sigma_{(AB)}$.

In the Hamiltonian approach, the action can be decomposed in terms of the spatial ‘coordinate variables’ $\omega_{ABi} = \omega^{(AB)i}$ and ‘momentum variables’ $\tilde{\sigma}^{AB}_i = \tilde{\sigma}^{(AB)i}$, together with the Lagrange multipliers $N$ (lapse), $N^i$ (shift) and $\omega_{AB0}$ [specifying local SU(2) transformations]. The (Lorentzian) action $S$ has the form

$$S = \int Tr[\tilde{\sigma}^i\dot{\omega}_i + (N\tilde{\sigma}^i\tilde{\sigma}^j(R_{ij} - \frac{1}{3}\Lambda\Sigma_{ij}) + N^i\tilde{\sigma}^jR_{ij} + \omega_0D_i\tilde{\sigma}^i)].$$  \hspace{1cm} (4.10)

Here, all spinor indices have been suppressed, but spatial indices are left explicit. The conventions $(MN)_A^C = M^B_A N^C_B$ and $Tr(M^B_A) =$
$M_A^A$ are being used. The spatial curvature 2-forms $R_{ij}^{AB} = R^{(AB)}_{[ij]}$ are constructed as in Eq. (4.5) from the spatial connection 1-forms $\omega^{AB}_i$, and $D_i$ denotes the spatial covariant derivative. From variation of the Lagrange multipliers, one finds as usual that each of their coefficients vanish, giving the constraint equations which restrict the form of the allowed data ($\omega_{ABi}, \tilde{\sigma}^{ABi}$) for classical solutions (whether anti-self-dual or more general). Further, the spatial 2-forms $\Sigma^{(AB)}_{[ij]}$ of Eq. (4.9) are related to the variables $\tilde{\sigma}^{ABi}$ by

$$\tilde{\sigma}^{ABi} = \varepsilon^{ijk} \Sigma^{AB}_{jk}.$$  \hfill (4.11)

The generalisation of Ashtekar’s approach to supergravity was given by Jacobson (1988), and will be used in section 6.

4.2 Non-zero $\Lambda$: the anti-self-dual case and the Chern-Simons functional

It was shown in [Capovilla et al. 1990, 1991; Samuel 1988] that the anti-self-dual Einstein field equations (1.6,9), with a non-zero cosmological constant $\Lambda$, can be re-expressed in terms of the (4-dimensional) 2-form $\Sigma^{AB} = \Sigma^{(AB)}$. Note first that, for any set of orthonormal 1-forms $e^{AA'}_\mu$, the 2-forms $\Sigma^{AB}$ defined in Eq.(4.9) automatically obey

$$\Sigma^{(AB} \wedge \Sigma^{CD)} = 0.$$  \hfill (4.12)

Equivalently, for a real $SO(3)$ triad $e^a_\mu$, where $a = 1, 2, 3$ here, the conditions (4.12) read

$$\Sigma^a \wedge \Sigma^b - \frac{1}{3} \delta^{ab} \Sigma^c \wedge \Sigma_c = 0.$$  \hfill (4.13)

In the case of anti-self-dual Weyl curvature ($\Psi_{ABCD} = 0$), the Ein-
stein field equations reduce to
\[ R^{AB} = \frac{1}{3} \Lambda \Sigma^{AB}. \] (4.14)

Conversely, these authors show that, given any \( su(2) \)-valued un-primed connection 1-form \( \omega^{AB} = \omega^{(AB)} \), with corresponding curvature 2-forms \( R^{AB} = R^{(AB)} \) defined by Eq.(4.16), it is sufficient (locally) that the \( R^{AB} \) further obey the algebraic conditions
\[ R^{(AB)} \wedge R^{CD)} = 0, \] (4.15)
or the equivalent of Eq.(4.13), with \( R^a \) replacing \( \Sigma^a \), in the real SO(3) triad case. Then, defining \( \Sigma^{AB} \) by the inverse of Eq.(4.15):
\[ \Sigma^{AB} = 3 \Lambda^{-1} R^{AB}, \] (4.16)
it is shown that (locally) this defines, \( \text{via} \) Eq.(4.9), a metric which obeys the Einstein-\( \Lambda \) equations, with anti-self-dual Weyl curvature.

The Hamiltonian approach with \( \Lambda \neq 0 \), taking canonical variables \( (\omega_{ABi}, \tilde{\sigma}^{ABi}) \) with action \( S \) given by Eq.(5.4), has been further discussed by [Koshti and Dadhich 1990]. A necessary and sufficient condition for an initial-data set to correspond locally to a solution of the Einstein equations with anti-self-dual Weyl curvature is that
\[ \tilde{\sigma}^{ABi} = -\frac{3}{\Lambda} \tilde{B}^{ABi}. \] (4.17)
Here, \( \tilde{B}^{ABi} \) is a densitised version of the magnetic part of the Weyl tensor [Misner et al. 1973], defined by
\[ \tilde{B}^{ABi} = \frac{1}{2} \varepsilon^{ijk} R^{AB}{}_{jk}, \] (4.18)
where \( \varepsilon^{ijk} \) is the alternating symbol in 3 dimensions, and \( R^{AB}{}_{jk} = R^{(AB)}{}_{[jk]} \) gives, as usual, the spatial part of the curvature 2-form, following Eq.(4.5).
The evolution equations are most easily described in terms of the equivalent variables $(\omega_{ai}, \tilde{\sigma}^{ai})$, where $a = 1, 2, 3$ is a local SO(3) index. Recall that $\omega_{ai}$ and $\tilde{\sigma}^{ai}$ are defined by $\omega_{ai} = \Sigma^A_a \omega_{ABi}$, $\tilde{\sigma}^{ai} = \Sigma^a_{AB} \omega^{ABi}$, where $\Sigma^A_a$ and $\Sigma^a_{AB}$ are the triad Infeld-van der Waerden translation symbols. Then, in the anti-self-dual Riemannian case, the normal derivative of $\omega_{ai}$ is given by

$$\dot{\omega}_{ai} = \left(\frac{3}{4\Lambda}\right) \frac{N}{\varepsilon_{ijk} \varepsilon_{abc}} \tilde{B}^{bi} \tilde{B}^{ck}. \quad (4.19)$$

Here, in the language of Eq.(5.4), only the ‘lapse’ Lagrange multiplier $N$ is taken non-zero. Hence, if $\omega_{ai}$ is specified on a hypersurface, then, assuming anti-self-duality, the conjugate variable $\tilde{\sigma}^{ai}$ is determined by Eq.(4.17). The evolution of $\omega_{ai}$ away from the hypersurface is then determined by solving the set of partial differential equations

$$\dot{\omega}_{ai} = \left(\frac{3}{4\Lambda}\right) \frac{N}{\varepsilon_{ijk} \varepsilon_{abc}} \tilde{B}^{bi} \tilde{B}^{ck}, \quad (4.19)$$

which involves no more than first derivatives of $\omega_{ai}$, in the form $\dot{\omega}_{ai}$ and $\varepsilon^{ijk} \partial \omega_{aj} / \partial x^k$, the latter quadratically. Away from the bounding hypersurface, the conjugate variables $\tilde{\sigma}^{ai}$ continue to be given in terms of $\omega_{ai}$ by Eq.(4.17).

As usual for Hamiltonian systems with first-order evolution for the ‘coordinates’ alone (here $\omega_{ai}$), the classical action $I[\omega_{ai}]$, regarded as a functional of the boundary data $\omega_{ai}$, is the principal generating function [Arnold 1980; Goldstein 1980; Landau and Lifshitz 1976], with (in spinor language):

$$\frac{\delta I}{\delta \omega_{ABi}} = \tilde{\sigma}^{ABi}, \quad (4.20)$$

together with the correct evolution equations. Up to an additive constant, the classical action $I[\omega_{ai}]$ is precisely the Chern-Simons action

$$I_{CS}[\omega_{ai}] = -\frac{3}{2\Lambda} \int \varepsilon^{ijk} [\omega^{AB} \varepsilon_{ij} (\partial_j \omega_{ABk}) + \frac{2}{3} \omega^{AB} \omega^{ij}_{B} \omega^{ij}_{CAk}], \quad (4.21)$$
as studied in this general context by \[\text{Ashtekar} \text{ et al.} \ 1989; \ \text{Kodama} \ 1990\] and others. Here, for comparison, we again assume that the bounding hypersurface $\partial M$ is diffeomorphic to $S^3$. Note further that the value of $I_{CS}$ for a particular classical solution does not change as one evolves the boundary data $\omega_{ai}$ in (say) the normal direction, because of the Hamiltonian (normal) constraint $\tilde{\sigma}^i \tilde{\sigma}^j R_{ij} = 0$, arising from Eq. (2.4). In the case of Bianchi-IX symmetry, for Einstein-$\Lambda$ gravity, the corresponding Chern-Simons quantum states

$$\Psi_{CS}[\omega_{ai}] = \exp(\pm I_{CS}[\omega_{ai}]/\hbar) \quad (4.22)$$

in quantum cosmology have been further studied by \[\text{Louko} \ 1995; \ \text{Graham and Paternoga} \ 1996; \ \text{Cheng and D’Eath} \ 1997\]. For $N = 1$ (simple) supergravity, including a non-zero positive cosmological constant $\Lambda$ \[\text{Jacobson} \ 1988\], this state has been studied in the case of $k = +1$ cosmology (round $S^3$) by \[\text{Sano and Shiraishi} \ 1993\]; see also \[\text{Sano} \ 1992\]. In all of these mini-superspace treatments, it is clear that the Chern-Simons state(s) are at least WKB or semi-classical approximations to exact quantum states; similarly in the full theory \[\text{Ashtekar} \text{ et al.} \ 1989\]. An excellent review of Yang-Mills theory in Hamiltonian form, the Yang-Mills Chern-Simons action and its rôle in topology and the quantum theory, is given by Jackiw (1984).

There has been some discussion as to whether the Chern-Simons state $\Psi_{CS}$ with the minus sign in Eq. (4.22) is also the Hartle-Hawking state \[\text{Louko} \ 1995; \ \text{Cheng and D’Eath} \ 1997\], at least within the context of Bianchi-IX symmetry. I doubt whether the last word has been said on this subject, despite the definite tone of the latter paper. The argument involves the stability of the $\text{SO}(4)$-spherically symmetric solution of the evolution equation (4.19) for $\omega_{ai}$; for definiteness, assume here that $\Lambda < 0$, giving a hyperboloid $H^4$ as the
maximally symmetric solution. The corresponding anti-self-dual evolution has the form \( \omega_{ai} = A(t)(\sigma_a)_i \), where \( \sigma_a \) (\( a = 1, 2, 3 \)) denotes the orthonormal basis of left-invariant 1-forms on \( S^3 \), as used in section 3.2 and [Cheng and D’Eath 1997]

\[
A(t) = \frac{1}{2} [1 + \cosh(4mt)]; \quad \Lambda = -12m^2. \quad (4.23)
\]

Here \( t \) is a ‘Euclidean time coordinate’, with its ‘origin’ chosen to be at \( t = 0 \). Correspondingly, the form of the resulting \( \tilde{\sigma}^{ai} \) shows that the intrinsic radius \( a(t) \) of the \( S^3 \) at ‘time’ \( t \) is given by

\[
a^2(t) = \frac{1}{8m^2} \sinh^2(4mt), \quad (4.24)
\]
as it should be for a 4-space of constant negative curvature. Small gravitational perturbations of \( \mathbb{H}^4 \), whether of Bianchi-IX type or generic inhomogeneous distortions, give a linearised unprimed Weyl spinor \( \Psi_{ABCD} \), obeying the linearised Bianchi identity (1.10) in the background \( \mathbb{H}^4 \). The appropriate spherical harmonics \( X_{ABCD} \) and \( \tilde{Y}_{ABCD} \) on \( S^3 \) are, respectively, of positive and of negative frequency with respect to the spatial first-order (Dirac-like) projection of \( \nabla_{AA'} \).

The linearised \( t \)-evolution of such a harmonic is regular at \( t = 0 \) for positive frequency, but singular for negative frequency. As was seen in section 3.2, there is a one-parameter family of regular anti-self-dual Einstein metrics, containing the reference \( \mathbb{H}^4 \), of biaxial Bianchi-IX type. When linearised about \( \mathbb{H}^4 \), they give for \( \omega_{ai} \) or \( \Psi_{ABCD} \) a lowest-order (homogeneous) positive-frequency harmonic, multiplying a function of \( t \), with regular behaviour near the ‘origin’ \( t = 0 \) [Louko 1995]. But generic linearised data on a bounding \( S^3 \), in the \( \omega_{ai} \) description, will give perturbations of \( \omega_{ai} \) away from the background value \( A(t)(\sigma_a)_i \), diverging as \( t \to 0 \) within a linearised approximation.
Hence, the linear régime does not give enough information, and one must confront the full non-linear (but first-order) evolution equation (4.19) for $\omega_{ai}$. This partial differential equation is somewhat reminiscent of the heat-like equation for the Ricci flow on Riemannian manifolds, studied by Hamilton (1982,1986,1988), and it is possible that it might be susceptible to related techniques. This is under investigation; see also Mason and Woodhouse 1996. Of course, there are also descriptions of the general solution of the Riemannian anti-self-dual Einstein equations, for $\Lambda \neq 0$, in terms of twistor theory Ward 1980, Ward and Wells 1990, and in terms of $\mathcal{H}$-space LeBrun 1982.

At least, in the much simpler case of (abelian) Maxwell theory, when one takes the (anti-)self-dual part of the spatial connection (vector potential) $A_i$ to be the ‘coordinate variables’, the normalis-able Chern-Simons state $\Psi_{CS}$ is the ground state Ashtekar et al. 1992. This corresponds to the wormhole state in quantum cosmology Hawking 1988, D’Eath 1996. Similarly, one expects that the non-normalisable ‘state’ $\Psi_{CS}$, corresponding to the opposite sign in (4.22), gives the Maxwell version of the Hartle-Hawking ‘state’. Note here that, when the Maxwell field in this representation is split up into an infinite sum of harmonic oscillators, the description of each oscillator is that of the holomorphic representation Faddeev and Slavnov 1980; this recurs in supergravity.

In gravity, the ubiquitous Chern-Simons action $I_{CS}$ of Eq. 4.21 re-appears (naturally) as the generating function in the transformation from ‘traditional’ coordinates $e^{AA'}$, and conjugate momenta $p_{AA'}^i$ to Ashtekar variables $(\omega_{ABi}, \tilde{\sigma}_{ABi})$ Mielke 1990. The corresponding property for N=1 (simple) supergravity is described by
Macíás (1996).

One might then ask whether, for further generalisations containing Einstein gravity and other fields, corresponding (Euclidean) actions $I_{CS}$ can be found from descriptions of Ashtekar type. This requires (first) a suitably ‘form-al’ geometric treatment of the Lagrangian. Robinson (1994, 1995) has done this for, respectively, Einstein-Maxwell and Einstein-Yang-Mills theory, both with Λ-term; see also Gambini and Pullin (1993). For relations between anti-self-dual Yang-Mills theory and anti-self-dual gravity, see, for example, Bengtsson (1990). It would be extremely interesting if the generality could be increased to include, for example, N=1 supergravity with gauged supermatter, with gauge group SU(2), SU(3), ... [Wess and Bagger 1992; D’Eath 1996] — see the following sections 5 and 6.

5 Canonical quantum theory of N=1 supergravity: ‘traditional variables’

5.1 Dirac approach

Turning back to supergravity, consider the Dirac canonical treatment of simple N=1 supergravity, using the ‘traditional variables’ $(e^{AA′}_i, p_{AA′}_i, ψ^A_i, ˜ψ^{A′}_i)$ [D’Eath 1984, 1996], which are the natural generalisation of the ‘traditional’ variables $(e^{AA′}_i, p_{AA′}_i)$ for Einstein gravity, based on the spatial tetrad components $e^a_i$ and their conjugate momenta $p^a_i$ ($a = 0, 1, 2, 3$). In the supergravity version, the fermionic quantities $(ψ^A_i, ˜ψ^{A′}_i)$ are the spatial projections of the spin-3/2 potentials $(ψ^A_μ, ˜ψ^{A′}_μ)$. As classical quantities, they are odd Grassmann quantities, anti-commuting among themselves,
but commuting with bosonic quantities such as $e^{AA'}_i$ and $p_{AA'}^i$. The bosonic quantities, such as $e^{AA'}_i, p_{AA'}^i$, are not necessarily Hermitian complex (in Lorentzian signature, say), but are generally even elements of a Grassmann algebra; that is, they are (schematically) of the form (complex) + (complex)×(bilinear in $\psi, \psibar$) + analogous fourth-order terms + ... The bosonic fields $e^{AA'}_i(x), p_{AA'}^i(x)$ are canonical conjugates, and the canonical conjugate of $\psi^A_i(x)$, in the fermionic sense of Casalbuoni (1976), is

$$\pi^A_i = -\frac{1}{2} \epsilon^{ijk} \psibar^A_j e_{AA'k}. \quad (5.1)$$

In the quantum theory, one can, for example, consider wavefunctionals of the ‘traditional’ form:

$$\Psi = \Psi[e^{AA'}_i(x), \psi^A_i(x)], \quad (5.2)$$

living in a Grassmann algebra over the complex numbers $\mathbb{C}$. Following the Dirac approach, a wave-function $\Psi$, describing a physical state, must obey the quantum constraints, corresponding to the classical constraints appearing in a Hamiltonian treatment of a theory with local invariances, as seen in section 4.1 for Ashtekar variables. Taking the case of Lorentzian signature, for definiteness, the only relevant quantum constraints to be satisfied are the local Lorentz constraints

$$J^{AB} \Psi = 0, \quad \bar{J}^{A'B'} \Psi = 0, \quad (5.3)$$

together with the local supersymmetry constraints

$$S^A \Psi = 0, \quad \bar{S}^{A'} \Psi = 0, \quad (5.4)$$

The local Lorentz constraints (5.3) simply require that the wavefunctional $\Psi$ be constructed in a locally Lorentz-invariant way from
its arguments; that is, that all unprimed and all primed spinor indices be contracted together in pairs. Classically, the fermionic expression \( \tilde{S}_A' \) is given by

\[
\tilde{S}_A' = \varepsilon^{ijk} e_{AA'i}(3s D_j \psi^A_k) + \frac{1}{2} i \kappa^2 \psi^A i p_{AA'} i ,
\]

(5.5)

with \( \kappa^2 = 8\pi \), where \( 3s D_i() \) denotes a suitable torsion-free spatial covariant derivative [D’Eath 1984, 1996]. The classical \( S^A \) is given formally by Hermitian conjugation of (5.5).

In the quantum theory, the operator \( \bar{S}_A' \) contains only a first-order bosonic derivative:

\[
\bar{S}_A' = \varepsilon^{ijk} e_{AA'i}(3s D_j \psi^A_k) + \frac{1}{2} \frac{\hbar}{\kappa^2} \psi^A \delta e_{AA'} i .
\]

(5.6)

The resulting constraint, \( \bar{S}_A' \Psi = 0 \), then has a simple interpretation in terms of the transformation of \( \Psi \) under a local primed supersymmetry transformation, parametrized by \( \tilde{\varepsilon}^A'(x) \), acting on its arguments. One finds [D’Eath 1984, 1996] that, under the supersymmetry transformation with

\[
\delta e^{AA'} i = i \kappa \tilde{\varepsilon}^A' \psi^A i, \quad \delta \psi^A i = 0 ,
\]

(5.7)

the change \( \delta \Psi \) is given by

\[
\delta (\log \Psi) = \frac{-2i}{\hbar \kappa} \int d^3x \varepsilon^{ijk} e_{AA'i}(3s D_j \psi^A_k) \tilde{\varepsilon}^A' .
\]

(5.8)

The quantum version of \( S^A \) is more complicated in this representation, involving a mixed second-order functional derivative, schematically \( \delta^2 \Psi / \delta e \delta \psi \). However, one can move between the \( (e^{AA'} i, \psi^A i) \) representation and the \( (e^{AA'} i, \tilde{\psi}^A i) \) representation, by means of a suitable functional Fourier transform. In the latter representation, the operator \( S^A \) appears simple, being of first order, while the operator \( \bar{S}_A' \) appears more complicated. Finally, note that in N=1
supergravity, there is no need to study separately the quantum constraints \( \mathcal{H}^{AA'}\Psi = 0 \), corresponding to local coordinate invariance in 4 dimensions, and summarised classically in the Ashtekar representation by the vanishing of the quantities multiplying the Lagrange multipliers in Eq. (5.4). This is because the anti-commutator of the fermionic operators \( S^A \) and \( \bar{S}^{A'} \) gives \( \mathcal{H}^{AA'} \), in a suitable operator ordering, plus quantities multiplying \( J^{AB} \) or \( \bar{J}^{A'B'} \); hence, the annihilation of \( \Psi \) by \( S^A, S'^A, J^{AB} \) and \( \bar{J}^{A'B'} \) implies further that \( \mathcal{H}^{AA'}\Psi = 0 \).

5.2 The quantum amplitude

Consider, within \( N=1 \) simple supergravity, the ‘Euclidean’ quantum amplitude to go from given asymptotically flat initial data, specified by \( (e^{AA'}_{iI}(x), \tilde{\psi}^{A'}_{iI}(x)) \) on \( \mathbb{R}^3 \), to given final data \( (e^{AA'}_{iF}(x), \psi^{A'}_{iF}(x)) \), within a Euclidean time-separation \( \tau > 0 \), as measured at spatial infinity. Formally, this is given by the path integral

\[
K(e_F, \psi_F; e_I, \tilde{\psi}_I; \tau) = \int \exp(-I/\hbar) \mathcal{D}e \mathcal{D}\psi \mathcal{D}\tilde{\psi},
\]

where \( I \) denotes a version of the Euclidean action of supergravity, appropriate to the boundary data [D’Eath 1984, 1996], and Berezin integration is being used for the fermionic variables [Faddeev and Slavnov 1980]. Of course, this is very close to being a Hartle-Hawking integral, as in (1.11), except that part of the boundary has been pushed to spatial infinity, and that the fermionic data have been taken in different forms on the initial and final \( \mathbb{R}^3 \).

As in any theory with local (gauge-like) invariances, when treated by the Dirac approach, the quantum constraint operators at the initial and final surfaces annihilate the quantum amplitude \( K \) above.
In particular, on applying (say) the supersymmetry constraint $\bar{S}_{A^I} K = 0$ at the final surface, one obtains
\[ \varepsilon^{ijk} e_{AA' iF}(3sD_j \psi^A_{kF}) K + \frac{1}{2} \hbar k^2 \psi_{iF} A F \frac{\delta K}{\delta e_{AA' iF}} = 0. \] (5.10)

As in section 5.1, this describes how $K$ changes (in a simple way) when a local primed supersymmetry transformation (5.7) is applied to the final data $(e_{AA' iF}, \psi^A_{iF})$. One then considers the semi-classical expansion of this ‘Euclidean’ quantum amplitude, by analogy with Eq.(2.4). However, one should first note that, in general, there is no classical solution $(e^{AA' \mu}, \psi^A \mu, \tilde{\psi}^{A' \mu})$ of the supergravity field equations, agreeing with the initial and final data as specified above, and corresponding to a Euclidean time interval $\tau$ at spatial infinity. This was not appreciated in [D’Eath 1984], but was later corrected in [D’Eath 1996]. The difficulty resides in the classical $\bar{S}_{A'} = 0$ constraint at the final surface [Eq.(5.5)], and similarly $S^A = 0$ at the initial surface; it is precisely related to the primed supersymmetry behaviour of Eqs.(5.7,8) finally, and similarly for unprimed supersymmetry initially.

Suppose now that we start from a purely bosonic (Riemannian) solution $e^{AA' \mu}$ of the vacuum Einstein field equations, while $\psi^A \mu = 0$, $\tilde{\psi}^{A' \mu} = 0$. Then, for the corresponding bosonic boundary data $e^{AA' \mu}$ and $e^{AA' iF}$, we expect there to exist a semi-classical expansion of $K(e_F, 0; e_I, 0; \tau)$, as in Eq.(2.4). By studying (say) the quantum supersymmetry constraint $\bar{S}_{A^I} K = 0$ of Eq.(5.10) at the final surface, and allowing the variable $\psi^A_{iF}(x)$ at the final surface to become small and non-zero, while still obeying the classical $\bar{S}_{A^I} = 0$ constraint (5.5) at the final surface, one finds:
\[ A_0 = \text{const.}, \quad A_1 = A_2 = \ldots = 0, \] (5.11)
for the loop prefactors $A_0, A_1, A_2, \ldots$ in the expansion (2.4). Thus, in N=1 supergravity, for purely bosonic boundary data, the semi-classical expansion of the ‘Euclidean’ amplitude $K$ is exactly semi-classical; that is,

$$K \sim A_0 \exp(-I_{\text{class}}/\hbar),$$

(5.12)

where $A_0$ is a constant. The symbol $\sim$ for an asymptotic expansion, has been used in Eq.(5.12), rather than equality $=$, as there will sometimes be more than one inequivalent complex solution of the vacuum Einstein equations joining $e^{AA'iI}$ to $e^{AA'iF}$. In that case, there will be more than one classical action $I_{\text{class}}$, but only the leading contribution, corresponding to the most negative value of $\text{Re}[I_{\text{class}}]$, will appear in Eq.(5.12). Since there has been some disagreement in the past about the result described in this paragraph, it should be noted that no published paper, since the publication of the revised argument in D’Eath 1996, has given a substantive contrary argument.

Finally, one might ask whether more general amplitudes $K(e_F, \psi_F; e_I, \bar{\psi}_I; \tau)$ in N=1 supergravity share some of the simplicity of the purely bosonic amplitude above. Here, non-trivial fermionic data $\psi^A_F$ and $\bar{\psi}^{A'}_I$ should be chosen such that there is a classical solution joining the data in Euclidean time $\tau$, whence a semi-classical expansion of $K$ should exist, by analogy with Eq.(2.4). By considering the possible form of locally supersymmetric counterterms, formed from volume and surface integrals of various curvature invariants, at different loop orders, one is led to expect that the full amplitude $K$ might well be finite on-shell; certainly, the purely bosonic part of each invariant must be identically zero, by the property (5.11), and it would then be odd if some of its part-

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ners, which are quadratic, quartic, \ldots in fermions, managed not to be zero identically. A more detailed investigation of fermionic amplitudes, at one loop in N=1 supergravity with gauged supermatter, is given in [D'Eath 1999].

5.3 N=1 supergravity with gauged supermatter

Gauge theories of ‘ordinary matter’, with spins $0, \frac{1}{2}, 1$, can be combined with N=1 supergravity (say) in a very geometrical way, to give a theory with four types of local invariance:– local coordinate, local tetrad rotation, local N=1 supersymmetry, and (say) local SU($n$) invariance [Wess and Bagger 1992]. The resulting theory is uniquely defined once the coupling constant $g$ is specified (with $g^2 = 1/137.03 \ldots$), except for an analytic potential $P(a^I)$, where the $a^I$ are complex scalar fields, which live on complex projective space $\mathbb{C}P^{n-1}$ (for $n \geq 2$). In the simplest non-trivial case, with SU(2) gauge group, there is one complex scalar field $a$, with complex conjugate $\bar{a}$. There is a natural Kähler metric on $\mathbb{C}P^1$, or the Riemann sphere, parametrised by $a$ (provided one includes the point $a = \infty$ at the North pole, while $a = 0$ corresponds to the South pole). The Kähler potential is

$$K = \log(1 + a\bar{a}),$$

(5.13)

giving the Kähler metric

$$g_{11^*} = \frac{\partial^2 K}{\partial a \partial \bar{a}} = \frac{1}{(1 + a\bar{a})^2},$$

(5.14)

Equivalently, this metric reads as

$$ds^2 = \frac{da \, d\bar{a}}{(1 + a\bar{a})^2},$$

(5.15)
which is the metric on the unit round 2-sphere (really, \( \mathbb{C}P^1 \)). Not surprisingly, the isometry group for this geometry is just the original gauge group SU(2).

The other fields in the SU(2) theory may be summarised as follows. There is a spin-1/2 field \((\chi_A, \tilde{\chi}_A')\), which has no Yang-Mills index in this case, and which is the partner of \((a, \bar{a})\). The Yang-Mills potential(s) \(v^{(a)}_\mu\), with \((a) = 1, 2, 3\), have fermionic spin-1/2 partners \((\lambda^{(a)}_A, \tilde{\lambda}^{(a)}_{A'})\); thus there is a distinction between two different types of underlying spin-1/2 field - - the \(\chi\)'s and the \(\lambda\)'s. As usual, gravity is described by the tetrad \(e^{A'}\mu\), with spin-3/2 supersymmetry partner \((\psi^{A}_\mu, \tilde{\psi}^{A'}_{\mu})\). The relevant Lagrangian may be found in [Wess and Bagger 1992].

This model can be extended to the group SU(3), for example, by using the corresponding Kähler metric given in [Gibbons and Pope 1978]. A suitable basis of 8 generators of the Lie algebra \(su(3)\), as employed in the Lagrangian of [Wess and Bagger 1992], is given by the Gell-Mann matrices in [Itzykson and Zuber 1980].

Perhaps the most immediately striking feature of the resulting Lagrangian is the enormous negative cosmological constant \(\Lambda\), with

\[
\Lambda = -\frac{g^2}{8}
\]

in Planck units, for SU(2) and SU(3). However, at least in the case of zero potential \([P(a) = 0]\), when the theory is written out in Hamiltonian form, and the Dirac approach to canonical quantisation is taken [D’Eath 1996, 1999], then the N=1 local supersymmetry again implies that quantum amplitudes \(K\) to go between initial and final purely bosonic configurations in a Euclidean time-at-infinity \(\tau\)
are exactly semi-classical:

\[ K \sim \text{const.} \exp(-I_{\text{class}}/\hbar). \]  \hspace{1cm} (5.17)

Correspondingly, one might again expect some related simplification in amplitudes \( K \) for which there are non-trivial fermionic boundary data, in addition to gravity, Yang-Mills and scalars. Some evidence of this has been found in an investigation of one-loop corrections in the SU(2) theory, where the ‘background’ purely bosonic classical solution is taken to be a suitable hyperboloid \( \mathbb{H}^4 \), corresponding to the negative value of \( \Lambda \), and the ‘unperturbed’ initial and final boundaries are taken to be two round 3-spheres \( S^3 \) at different radii from the ‘centre’ of the \( \mathbb{H}^4 \) \cite{DEath1999}. Consistent weak-field fermionic boundary data are put on the spheres, and the one-loop corrections to the quantum amplitude are studied with the help of both (local) quantum supersymmetry constraint operators \( S^A \) and \( \bar{S}^{A'} \). The Dirac approach to the computation of loop terms in such a locally supersymmetric theory was found to be extremely streamlined by comparison with the corresponding path-integral calculation. Typically, fermionic one-loop examples of this type are often very simple, sometimes not even involving an infinite sum or integral. In non-trivial examples, the amplitudes appear to be exponentially convergent \cite{DEath1999}, and the structure suggests that this will continue at higher loop order.

Since the loop behaviour of this SU(2) model appears reasonable, it would seem worthwhile to investigate this and other SU(\( n \)) models further, to understand better their physical consequences, and to try to predict effects which are observable at accelerator energies.
6 Canonical quantisation of N=1 supergravity: Ashtekar-Jacobson variables

Now consider the canonical variables introduced by Jacobson (1988) for N=1 supergravity, following Ashtekar’s approach, possibly including a positive cosmological constant written as $\Lambda = 12\mu^2$ [cf. Eq. (4.23) for negative $\Lambda$]. For consistency, the fermionic variables have been re-normalised as in D’Eath (1996). The bosonic variables are again taken to be the connection 1-forms $\omega_{ABi} = \omega^{(AB)i}$, together with the canonically-conjugate variables $\tilde{\sigma}^{ABi} = \tilde{\sigma}^{(AB)i}$. The fermionic variables are taken to be the unprimed spatial 1-forms $\psi_{Ai}$ and their conjugate momenta $\tilde{\pi}_{Ai}$. Once again, all variables only involve unprimed spinor indices, and so are well adapted to a treatment of (anti-)self-duality. Note that $\tilde{\pi}^{Ai}$ is given in terms of the ‘traditional’ variables of section 5 by

$$\tilde{\pi}^{Ai} = \frac{i}{\sqrt{2}} \varepsilon^{ijk} e^{AA'} j \psi_{A'A''}.$$

(6.1)

The classical supersymmetry constraints involve

$$S^A = \mathcal{D}_i \tilde{\pi}^{Ai} + 4i\mu(\tilde{\sigma}^k \psi_k)^A = 0,$$

(6.2)

where $\mathcal{D}_i$ is a spatial covariant derivative involving the connection $\omega_{ABi}$, and

$$S^+_A = (\tilde{\sigma}^j \tilde{\sigma}^k \mathcal{D}_j \psi_k)^A - 4i\mu(\tilde{\sigma}_k \tilde{\pi}^k)^A = 0.$$

(6.3)

Note here that $\tilde{\sigma}_k^{AB} = (1/\sqrt{2})\varepsilon_{kmn}\tilde{\sigma}^{ACm} \tilde{\sigma}^{Bn}$ [Jacobson 1988, D’Eath 1996, 1988]. Quantum-mechanically, for a
wave-functional $\Psi[\omega_{ABi}, \psi_{Ai}]$, the constraint $S^A \Psi = 0$ is a first-order functional differential equation, namely:

$$D_i \left( \frac{\delta \Psi}{\delta \psi_{Ai}} \right) - 4\mu \psi^B_k \left( \frac{\delta \Psi}{\delta \omega_{ABk}} \right) = 0.$$ (6.4)

This simply describes the invariance of the wave-functional $\Psi$ under a local unprimed supersymmetry transformation, parametrised by $\varepsilon^A(x)$, applied to its arguments $\omega_{ABi}(x), \psi_{Ai}(x)$. Note that the unprimed transformation properties of ‘traditional’ variables include

$$\delta e_{AA'i} = -i\bar{\psi}_{A'i} \varepsilon_A, \quad \delta \bar{\psi}_{A'i} = 2D_i \varepsilon_A, \quad \delta \bar{\psi}_{A'i} = 0.$$ (6.5)

One further deduces, following [Capovilla et al. 1991], the variation

$$\delta \omega_{ABi} = \mu \psi_{(Ai}\varepsilon_{B)i}.$$ (6.6)

However, the quantum constraint $S^A \Psi = 0$ is described by a complicated second-order functional differential equation. One can (say) transform from ‘coordinate’ variables $(\omega_{ABi}, \psi_{Ai})$ to the opposite ‘primed’ coordinates $(\tilde{\omega}_{A'B'i}, \tilde{\psi}_{A'i})$, via ‘traditional’ coordinates $(e_{AA'i}, \psi_{Ai})$ [Macías 1996], using functional Fourier transforms [D’Eath 1996], with Berezin integration over fermionic variables [Faddeev and Slavnov 1980]. In the ‘primed’ coordinates $(\tilde{\omega}_{A'B'i}, \tilde{\psi}_{A'i})$, the quantum constraint operator $S^A$ will appear complicated and second-order, while the operator $S^{1A}$ becomes simple and first-order.

In the unprimed representation $(\omega_{ABi}, \psi_{Ai})$, in the case $\Lambda = 12\mu^2 > 0$, one can again define the Chern-Simons action $I_{CS}$ for N=1 supergravity [Sano and Shiraishi 1993; Sano 1992] as:

$$I_{CS}[\omega_{ABi}, \psi_{Ai}] = \frac{3}{2\Lambda} \int W,$$ (6.7)
\[ W = \omega_{AB} \wedge d\omega^{AB} + \frac{2}{3} \omega_{AC} \wedge \omega^{CB} \wedge \omega^{AB} - \mu \psi^A \wedge D\psi_A. \] (6.8)

Here, we assume that the integration is over a compact (boundary) 3-surface. The notation \( D\psi_A \) denotes the covariant exterior derivative of \( \psi_{Ai} \), using the connection \( \omega_{ABi} \). Note that the functional \( I_{CS}[\omega_{ABi}, \psi_{Ai}] \) is invariant under unprimed local supersymmetry transformations (6.5), (6.6) applied to its arguments, with parameter \( \varepsilon^A(x) \).

Correspondingly, the Chern-Simons wave function,

\[ \Psi_{CS} = \exp(-I_{CS}[\omega_{ABi}, \psi_{Ai}] / \hbar), \] (6.9)

obeys the first quantum supersymmetry constraint

\[ S^A \Psi_{CS} = 0. \] (6.10)

But, by symmetry, when one transforms this wave function into the opposite primed \( (\tilde{\omega}_{AB'i}, \tilde{\psi}_{Ai}) \) representation, it will have the same form, and hence is also annihilated by the \( S^\dagger_A \) constraint operator. Hence, since \( \Psi_{CS} \) is automatically invariant under local tetrad rotations, this Chern-Simons wave function obeys all the quantum constraints, and so defines a physical ‘state’. Here inverted commas have been used, since it is not clear whether or not \( \Psi_{CS} \) is normalisable.

The classical action \( I_{CS}[\omega_{ABi}, \psi_{Ai}] \) is the generating function for anti-self-dual evolution of boundary data \( (\omega_{ABi}(x), \psi_{Ai}(x)) \) given on the compact spatial boundary, just as \( I_{CS}[\omega_{ABi}] \) generated the classical evolution for Einstein gravity with a non-zero \( \Lambda \) term in section 4.2. This certainly justifies further investigation.
Dirac’s approach to the quantisation of constrained Hamiltonian systems can be applied to boundary-value problems, whether the Hartle-Hawking path integral of quantum cosmology, or the transition amplitude to go from given initial to final asymptotically-flat data in Euclidean time $\tau$. When the crucial ingredient of local supersymmetry is added, the main quantum constraints to be satisfied by the wave-functional $\Psi$ become the supersymmetry constraints $S^A\Psi = 0$ and $\bar{S}^A\Psi = 0$, each of which is only of first order in bosonic derivatives. Using ‘traditional’ canonical variables, one finds that, both for $N=1$ simple supergravity and for $N=1$ supergravity with gauged $SU(n)$ supermatter (at least in the simplest case of zero potential), transition amplitudes to go from initial to final purely bosonic data are exactly semi-classical, without any loop corrections. This, in turn, strongly suggests that quantum amplitudes including fermionic boundary data are also finite in these theories.

Ashtekar’s different (essentially spinorial) choice of canonical variables, for Einstein gravity with cosmological constant $\Lambda$, allows a very efficient treatment of (anti-)self-dual Riemannian ‘space-times’. For $\Lambda \neq 0$, anti-self-dual evolution arises from the Chern-Simons generating functional $I_{CS}$ of section 4.2. Jacobson’s extension of Ashtekar canonical variables to include $N=1$ supergravity for $\Lambda \geq 0$ is again adapted to the study of anti-self-dual supergravity in the Riemannian case. For $\Lambda > 0$, anti-self-dual evolution in $N=1$ supergravity similarly arises from the Chern-Simons functional $I_{CS}$ of section 6. Further, when Ashtekar-Jacobson variables are used, the Chern-Simons wave function $\Psi_{CS} = \exp(-I_{CS})$, of $N=1$ su-
pergravity with $\Lambda > 0$, gives an exact solution of all the quantum constraints. (In contrast, it is not clear whether such a statement is meaningful in the non-supersymmetric case of section 4.2.) There remains the difficult question of the relation between $\Psi_{CS}$ and the Hartle-Hawking state.

Such inter-connections between (anti-)self-duality in Riemannian geometry and meaningful states in quantum cosmology may well provide an enduring Union (or, possibly, an Intersection) between Oxford and Cambridge and other such Centres of Gravity.

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