Dispersion of inertial particles in cellular flows in the small-Stokes, large-Péclet regime

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We investigate the transport of inertial particles by cellular flows when advection dominates over inertia and diffusion, that is, for Stokes and Péclet numbers satisfying $\text{St} \ll 1$ and $\text{Pe} \gg 1$. Starting from the Maxwey–Riley model, we consider the distinguished scaling $\text{St Pe} = O(1)$ and derive an effective Brownian dynamics approximating the full Langevin dynamics. We then apply homogenisation and matched-asymptotics techniques to obtain an explicit expression for the effective diffusivity $\overline{D}$ characterising long-time dispersion. This expression quantifies how $\overline{D}$, proportional to $\text{Pe}^{-1/2}$ when inertia is neglected, increases for particles heavier than the fluid and decreases for lighter particles. In particular, when $\text{St} \gg \text{Pe}^{-1}$, we find that $\overline{D}$ is proportional to $\text{St}^{1/2}/(\log(\text{St Pe}))^{1/2}$ for heavy particles and exponentially small in $\text{St Pe}$ for light particles. We verify our asymptotic predictions against numerical simulations of the particle dynamics.

Key words: inertial particles, dispersion, homogenisation, cellular flow

1. Introduction

Over long time scales, the dispersion of particles and passive scalars in periodic incompressible flows is asymptotically a pure diffusive process, with an effective diffusivity tensor which can be computed, e.g. using the method of homogenisation (Majda & Kramer 1999; Pavliotis & Stuart 2005). The flows that have attracted most attention in this context are shear flows and cellular flows because of the remarkably different dependence of their effective diffusivities on the Péclet number and the availability of closed-form results. The cellular flow – on which this paper focuses – is the two-dimensional flow with the stream function

$$\psi(x, y) = U a \sin(x/a) \sin(y/a),$$

(1.1)

where $U$ is the maximum flow speed and $2\pi a$ is the cell period. This flow consists of a doubly periodic array of cells containing four vortices, with the fluid rotating alternatively clockwise and anti-clockwise in each quarter-cell. A classic result, due to Childress (1979), Shraiman (1987), Rosenbluth \textit{et al.} (1987) and Soward (1987), gives the asymptotic form of the (isotropic) effective diffusivity for non-inertial particles when advection dominates over diffusion. Using $a$ and $U/a$ as reference length and time, it reads

$$\overline{D} \sim 2\nu \text{Pe}^{-1/2} \quad \text{for } \text{Pe} \gg 1,$$

(1.2)

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where $\text{Pe} = U_a/D$ is the Péclet number and $D$ is the diffusion coefficient. The constant $\nu \approx 0.5327 \ldots$ was determined by Soward (1987) using Wiener–Hopf techniques. (See also the mathematical literature, e.g. Heinze (2003); Novikov et al. (2005) for rigorous bounds.)

In this paper, we investigate how the dispersion is affected by small-but-finite particle inertia. Numerical simulations (e.g. Pavliotis et al. 2006, 2009) show a strong enhancement of the effective diffusivity compared to the non-inertial case when the particles are denser than fluid. This is because inertia expels heavy particles away from high-vorticity regions, towards the cellular flow’s separatrices, enhancing transport between cells and hence large-scale dispersion. Conversely, particles less dense than the fluid tend to accumulate in the (high-vorticity) cell centres, leading to a smaller effective diffusivity. Our aim is to quantify this by generalising (1.2) to inertial particles.

The dynamics of non-inertial particles is Brownian, governed by the nondimensional equation

$$\text{d}X = u \text{d}t + (2/\text{Pe})^{1/2} \text{d}W,$$

where $X$ is the particle position, $W$ a two-dimensional Wiener process, and $u = (\partial_y \psi, -\partial_x \psi)$ is the fluid velocity. We model the dynamics of inertial particles by the Langevin equation based on the Maxey–Riley (1983) model (see Maxey 1987),

$$\text{d}X = \mathbf{V} \text{d}t,$$

$$\text{St} \text{d} \mathbf{V} = (\mathbf{u} (X, t) - \mathbf{V}) \text{d}t + \text{St} \beta D_t \mathbf{u} \text{d}t + (2/\text{Pe})^{1/2} \text{d}W,$$

where $\mathbf{V}$ is the particle velocity, $D_t = \partial_t + \mathbf{u} \cdot \nabla$ is the material derivative along the fluid velocity $\mathbf{u}$, $\text{St} = \tau U/a$ is the Stokes number, with $\tau = r_p^2/(9 \beta \nu)$ the Stokes timescale, $\beta = 3 \rho_t/(2 \rho_p + \rho_t) \in [0, 3]$ is the density parameter with $\rho_t$ and $\rho_p$ the mass density of the fluid and particle, $r_p$ is the particle radius, and $\nu$ is the kinematic viscosity of the fluid. The momentum equation (1.4b) includes Stokes drag (first term on the right-hand-side) and Auton’s added-mass force (second term, see Auton et al. (1988)) but neglects the
Boussinesq–Basset force and Faxen correction (see Maxey 1987, for a justification) as well as gravity.

Typical examples of single trajectories, obtained by solving (1.4) for light ($\beta > 1$) and heavy ($\beta < 1$) particles and (1.3) for non-inertial particles, are shown in the top row of Figure 1. The early-time dispersion is illustrated by the bottom row which shows the location at a fixed time $t$ of an ensemble of particles initially concentrated within a single quarter-cell. The figure demonstrates the expected enhancement of dispersion for heavy particles and inhibition for light particles.

The Brownian dynamics (1.3) is recovered from the Langevin dynamics (1.4) when inertial effects are negligible compared to diffusion, that is, for $St \ll Pe^{-1} \ll 1$. We consider the more general, distinguished regime $St \ll 1$, $Pe \gg 1$ with $St Pe = O(1)$, where both inertial and diffusive effects enter the problem at the same order. Our main result is an asymptotic formula for the effective diffusivity in this regime:

$$D \sim \frac{2\nu Pe^{-1/2}}{Z(\alpha)}$$

for $St, Pe^{-1} \ll 1$, $St Pe = O(1)$. (1.5)

Here,

$$\alpha = St Pe (1 - \beta)$$

and $Z(\alpha)$ is a function given explicitly in terms of elliptic integrals in (4.7) below and shown in Figure 2. Comparing (1.5) with its non-inertial counterpart (1.2) shows that $Z(\alpha)$ completely captures the effect of inertia. Since it is monotonically decreasing and satisfies $Z(0) = 1$, inertia is confirmed to decrease $D$ for light particles ($\alpha < 0$) and to increase $D$ for heavy particles ($\alpha > 0$).

To derive (1.5), we first reduce the Langevin dynamics (1.4) to an effective Brownian dynamics which captures weak inertia (§2). We then apply the method of homogenisation, formulate its cell problem in §3, and solve it asymptotically in §4, recasting the results of Childress (1979), Shraiman (1987), Rosenbluth et al. (1987) and Soward (1987) in the language of homogenisation along the way. We discuss the results, derive simplified versions of (1.5) valid when inertia dominates diffusion ($St Pe \gg 1$), and conclude in §5.
2. Effective Brownian dynamics

In this section, we derive an effective Brownian dynamics capturing inertial effects in
the distinguished regime \( \mathrm{St} \), \( \mathrm{Pe}^{-1} \ll 1 \) with \( \mathrm{St} \mathrm{Pe} = O(1) \) and \( \beta = O(1) \). The derivation
is similar to that carried out in Pavliotis et al. (2009) in the case \( \beta = 1 \). It is convenient
to introduce a small parameter \( \epsilon \ll 1 \) such that \( \mathrm{Pe}^{-1} = \epsilon^2 \), let \( \mathrm{St} = \gamma \epsilon^2 \) with \( \gamma = O(1) \),
and define the rescaled relative velocity \( \mathbf{P} = \gamma \epsilon (\mathbf{V} - \mathbf{u}) \) to rewrite (1.4) as

\[
\begin{align*}
\mathrm{dX} &= (\mathbf{u} + \mathbf{P}/(\gamma \epsilon)) \, \mathrm{d}t, \\
\mathrm{dP} &= -\left(\mathbf{P}/(\gamma \epsilon^2) + \gamma \epsilon (1 - \beta) \partial_t \mathbf{u} - (\mathbf{P} \cdot \nabla) \mathbf{u}\right) \, \mathrm{d}t + \sqrt{2} \, \mathrm{dW}.
\end{align*}
\]

We reduce the dynamics of (2.1) by considering the corresponding backward Kolmogorov
equation, namely

\[
\partial_t g - \mathbf{u} \cdot \nabla x g = \left(\frac{1}{\gamma \epsilon} \mathbf{p} \cdot \nabla x - \frac{1}{\gamma \epsilon^2} \mathbf{p} + \gamma \epsilon (1 - \beta) \partial_t \mathbf{u} - (\mathbf{p} \cdot \nabla) \mathbf{u}\right) \cdot \nabla p + \nabla^2 p \right) g 
\]

for an expectation \( g(x,p,t) \). We now introduce the multiple time scales \( t_n = \epsilon^n t \) with
\( n = 0,1,\cdots \) and the expansion

\[
g(p,x,t) = g_0(x,t_0,t_1,t_2,\cdots) + \epsilon g_1(p,x,t_0,t_1,t_2,\cdots) + \cdots. \tag{2.3}
\]

Substituting (2.3) into (2.2) yields, up to order \( \epsilon^2 \),

\[
\begin{align*}
\mathbf{p} &\cdot (\nabla x g_0 - \nabla p g_1) = 0, \tag{2.4a} \\
\mathbf{p} &\cdot (\nabla x g_1 - \nabla p g_2) = (\partial_{t_0} - \mathbf{u} \cdot \nabla x) g_0, \tag{2.4b} \\
\mathbf{p} &\cdot (\nabla x g_2 - \nabla p g_3) = (\partial_{t_0} - \mathbf{u} \cdot \nabla x) g_1 + \partial_{t_1} g_0 - \nabla^2 p g_1 - (\mathbf{p} \cdot \nabla x) \mathbf{u} \cdot \nabla p g_1, \tag{2.4c} \\
\mathbf{p} &\cdot (\nabla x g_3 - \nabla p g_4) = (\partial_{t_0} - \mathbf{u} \cdot \nabla x) g_2 + \partial_{t_1} g_1 - \nabla^2 p g_2 - (\mathbf{p} \cdot \nabla x) \mathbf{u} \cdot \nabla p g_2 + \partial_{t_2} g_0 + \gamma (1 - \beta) \partial_t \mathbf{u} \cdot \nabla x g_0. \tag{2.4d}
\end{align*}
\]

Using that \( g_0 \) is independent of \( \mathbf{p} \), we can solve (2.4) successively by equating powers of
\( \mathbf{p} \). This yields

\[
g_n = \left(\frac{\mathbf{p} \cdot \nabla x}{n!}\right)^n g_0, \quad \text{for } n = 1, 2, 3, \tag{2.5a}
\]

and

\[
\begin{align*}
\partial_{t_0} g_0 &= \mathbf{u} \cdot \nabla x g_0, \tag{2.5b} \\
\partial_{t_1} g_0 &= 0, \tag{2.5c} \\
\partial_{t_2} g_0 &= -\gamma (1 - \beta) \partial_t \mathbf{u} \cdot \nabla x g_0 + \nabla^2 x g_0. \tag{2.5d}
\end{align*}
\]

From (2.5b)-(2.5d), it is clear that advection by \( \mathbf{u} \) is the dominant process while inertia
and diffusion arise at the same, lower order. We reconstitute the dynamics of \( g_0 \) capturing
both all three effects by adding (2.5b)-(2.5d), setting \( \partial_{t_0} + \epsilon \partial_{t_1} + \epsilon^2 \partial_{t_2} = \partial_t \) and using
\( \epsilon^2 = \mathrm{Pe}^{-1} \) and \( \gamma \epsilon^2 = \mathrm{St} \) to obtain the backward Kolmogorov equation

\[
\partial_t g_0 = (\mathbf{u} - \mathrm{St}(1 - \beta) \partial_t \mathbf{u}) \cdot \nabla x g_0 + \frac{1}{\mathrm{Pe}} \nabla^2 g_0 \tag{2.6}
\]

for \( g_0 \). This corresponds to the effective Brownian (or overdamped Langevin) dynamics

\[
\mathrm{dX} = (\mathbf{u} - \mathrm{St}(1 - \beta) \partial_t \mathbf{u}) \, \mathrm{d}t + (2/\mathrm{Pe})^{1/2} \, \mathrm{dW}, \tag{2.7}
\]

which includes inertial correction through the term \( \mathrm{St}(1 - \beta) \partial_t \mathbf{u} \). In the absence of
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diffusion (Pe → ∞), (2.7) recovers the so-called first-order Eulerian approximation (e.g., J.Ferry & Balachandar 2001) or equivalently the slow manifold dynamics discussed by Rubin et al. (1995) and Haller & Sapsis (2008).

According to (2.7), inertial particles behave as though advected by the effective flow

\[ \mathbf{u}_e = \mathbf{u} - \text{St}(1 - \beta)D_t \mathbf{u}. \]  

(2.8)

For incompressible fluids, \( \mathbf{u} \) is divergence free but \( \mathbf{u}_e \) rarely is. It can be shown that

\[ \nabla \cdot \mathbf{u}_e = -\text{St}(1 - \beta) \left( \| \mathbf{S} \|^2 - \| \Omega \|^2 \right), \]  

(2.9)

where \( \mathbf{S} \) and \( \Omega \) are the local strain-rate and rotation-rate tensors of \( \mathbf{u} \). Thus particles denser than the fluid (\( \beta < 1 \)) tend to accumulate (\( \nabla \cdot \mathbf{u}_e < 0 \)) in low-vorticity and high-strain regions, while less-dense particles (\( \beta > 1 \)) tend to accumulate in high-vorticity and low-strain regions. A detailed analysis of this clustering effect without diffusion is given by Sapsis & Haller (2010). Note that, at the order of accuracy of (2.7), the dynamics of isodense particles (\( \beta = 1 \)) is identical to that of non-inertial particles; capturing inertial correction would require a different scaling relation between St and Pe.

3. Periodic homogenisation

In this section, we apply the methodology of homogenisation to the Brownian dynamics (2.7) to obtain an expression for the corresponding effective diffusivity. The derivation is standard (e.g., Pavliotis & Stuart 2005) and recorded here for completeness and to set up notation.

We rewrite the associated backward Kolmogorov equation (2.6) as

\[ \partial_t g = \mathbf{u}_e \cdot \nabla g + \frac{1}{\text{Pe}} \nabla^2 g, \]  

(3.1)

where \( \mathbf{u}_e = \mathbf{u} - \text{St}(1 - \beta)\mathbf{u} \cdot \nabla \mathbf{u} \) is the effective velocity field (2.8) which is steady, periodic and divergent. We introduce the small parameter \( \delta \ll 1 \) along with the variables \( X = \delta x \), \( t_1 = \delta t \) and \( t_2 = \delta^2 t \). We seek for a solution of (3.1) in the form

\[ g(x,t) = g_0(X, t_1, t_2) + \delta g_1(x, X, t_1, t_2) + \delta^2 g_2(x, X, t_1, t_2) \cdots. \]  

(3.2)

where the \( g_i, i = 1, 2, \cdots \), are periodic functions of \( x \). We introduce the expansion (3.2) into (3.1) and collect terms at each order in \( \delta \). At orders \( \delta \) and \( \delta^2 \), we find

\[ \partial_{t_1} g_0 = L_1 g_0 + L_0 g_1, \]  

(3.3a)

\[ \partial_{t_2} g_0 = L_2 g_2 + L_1 g_1 + L_0 g_0, \]  

(3.3b)

with

\[ L_0 = \mathbf{u}_e \cdot \nabla_x + \frac{1}{\text{Pe}} \nabla^2_x, \quad L_1 = \mathbf{u}_e \cdot \nabla X + \frac{2}{\text{Pe}} \nabla X \cdot \nabla_x \]  

and \( L_2 = \frac{1}{\text{Pe}} \nabla^2_X \).  

(3.3c)

Let \( \phi^\dagger(x) \) be the invariant measure associated with \( L_0 \), that is, the periodic solution of

\[ L_0^\dagger \phi^\dagger = -\mathbf{u}_e \cdot \nabla_x \phi^\dagger + \frac{1}{\text{Pe}} \nabla^2_x \phi^\dagger = 0 \]  

(3.4)

normalised such that \( \langle \phi^\dagger \rangle = 1 \), with \( \langle \cdot \rangle \) denoting spatial average over the periodic cell. Multiplying (3.3a) by \( \phi^\dagger \) and averaging yields

\[ \partial_{t_1} g_0 = c \cdot \nabla_X g_0, \]  

(3.5)
where
\[ c = \langle \phi^\dagger u_e \rangle \]  \hspace{1cm} (3.6)
is the effective drift velocity vector.

Using (3.5) and (3.3a), \( g_1 \) is found to satisfy
\[ \mathcal{L}_0 g_1 + (u_e - c) \cdot \nabla x g_0 = 0. \]  \hspace{1cm} (3.7)
The solution takes the form \( g_1 = \phi \cdot \nabla x g_0 \), where \( \phi \) obeys the cell problem
\[ \mathcal{L}_0 \phi + u_e - c = 0 \]  \hspace{1cm} (3.8)
with periodic boundary conditions. Substituting this solution into (3.3b), multiplying by \( \phi^\dagger \) and averaging yields the effective equation
\[ \partial_t g_0 + c \cdot \nabla g_0 + \nabla \cdot (D_e \cdot \nabla x) g_0 = 0, \]  \hspace{1cm} (3.9)
where
\[ D_e = \left\langle \phi^\dagger \left( \frac{1}{\mathrm{Pe}} (\mathbb{I} + 2 \nabla x \otimes \phi) + (u_e - c) \otimes \phi \right) \right\rangle, \]  \hspace{1cm} (3.10)
is the effective diffusivity tensor with \( \mathbb{I} \) denoting the identity tensor and \( \otimes \) the tensorial product. Finally, we gather (3.5) and (3.9), set \( \delta \partial_{t_1} + \delta^2 \partial_{t_2} = \partial_t \) and \( \delta \nabla x = \nabla \) to obtain the effective backward Kolmogorov equation
\[ \partial_t g_0 = c \cdot \nabla g_0 + \nabla \cdot (D_e \cdot \nabla) g_0, \]  \hspace{1cm} (3.11)
This corresponds to a purely diffusive process, characterised by the effective drift velocity \( c \) and diffusivity tensor \( D_e \), which approximates the dynamics (2.7) over long time scales.

Computing \( c \) and \( D_e \), requires solving both (3.4) for \( \phi^\dagger \) and (3.8) for \( \phi \). In the non-inertial case, \( \nabla \cdot u_e = 0 \) and the invariant measure \( \phi^\dagger \) is simply a constant. For inertial particles, \( \phi^\dagger \) is non-trivial and reflects the clustering caused by the divergence of \( u_e \).

We remark that we obtained (3.6) and (3.10) by first deriving the effective Brownian dynamics (2.7) then applying homogenisation. Alternatively, we could have first applied homogenisation to the Langevin dynamics (1.4), then exploited the asymptotic parameters to simplify the cell problems. Pavliotis & Stuart (2005) and Martins Afonso et al. (2012) performed the first step but only considered the regime \( \mathrm{St} = O(\mathrm{Pe}^{-1}) \ll 1 \) rather than the more general regime \( \mathrm{St} = O(\mathrm{Pe}^{-1}) \ll 1 \) of this paper. We note that the cell problems for the Langevin dynamics are difficult to solve as they are defined within domains including unbounded momentum spaces. The sampling of trajectories is also delicate for the Langevin dynamics when \( \mathrm{St} \ll 1 \) since it requires exceedingly small time steps, and we make use of the effective Brownian dynamics for this purpose also below.

In the next section, we solve the cell problems (3.4) and (3.8) asymptotically in the distinguished regime \( \mathrm{St} = O(\mathrm{Pe}^{-1}) \ll 1 \) and compute the leading order expression of the effective diffusivity (3.10) for the cellular flow.

4. Effective diffusivity

In non-dimensional variables, the effective velocity is
\[ u_e = u - \alpha^2 u \cdot \nabla u \]  \hspace{1cm} with \( u = (\partial_y \psi, -\partial_x \psi) \) and \( \psi(x, y) = \sin x \sin y \),
\[ \hspace{1cm} (4.1) \]
where \( \alpha = (1 - \beta) \) and \( \gamma = (1 - \beta) \mathrm{St} \mathrm{Pe} = O(1) \). The inertial contribution turns out to have a gradient structure: we can write (4.1) as
\[ u_e = u + \alpha^2 \nabla \Phi \]  \hspace{1cm} with \( \Phi(x, y) = -(\cos(2x) + \cos(2y)) / 4 \).
\[ \hspace{1cm} (4.2) \]
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Figure 3. Potential $\Phi$ defined in (4.2) and invariant measure $\phi^\dagger$ for $\alpha = -10$, 0 and 100 computed from (4.6) (right panels). Both fields are $\pi$-periodic and shown here in a quarter-cell. The thin dark lines represent the streamlines $|\psi| = 0, 0.25$ and 0.75.

Figure 3 displays the $\pi$-periodic potential $\Phi$. In effect, inertia adds a weak potential flow which, depending on the sign of $\alpha$, attracts particles to either the minima or the maxima of $\Phi$ located at the corners or centres of the quarter-cells.

To compute the effective diffusivity (3.10), we first derive the leading-order approximation to $\phi^\dagger$, then to $\phi$, and finally combine the resulting expressions.

4.1. Asymptotic computation of $\phi^\dagger$

We rewrite Eq. (3.4) for $\phi^\dagger$ as

$$\epsilon^2 \nabla \phi^\dagger - u \cdot \nabla \phi^\dagger + \alpha \epsilon^2 \nabla \cdot (\phi^\dagger u \cdot \nabla u) = 0$$

and expand $\phi^\dagger$ in powers of $\epsilon^2$ according to $\phi^\dagger = \phi^\dagger_0 + \epsilon^2 \phi^\dagger_1 + \cdots$ to obtain

$$u \cdot \nabla \phi^\dagger_0 = 0,$$  \hspace{1cm} (4.4a)

$$u \cdot \nabla \phi^\dagger_1 = \nabla^2 \phi^\dagger_0 + \alpha \nabla \cdot (\phi^\dagger_0 u \cdot \nabla u).$$  \hspace{1cm} (4.4b)

Eq. (4.4a) implies that $\phi^\dagger_0$ must be constant along streamlines: $\phi^\dagger_0 = \phi^\dagger_0 (\psi)$. Eq. (4.4b) can be solved for $\phi^\dagger_1$ provided that a solvability condition, obtained by integrating (4.4b) along streamlines, is satisfied. We show in Appendix A that this solvability condition is

$$\frac{d}{d\psi} \left( a(\psi) \frac{d\phi^\dagger_0}{d\psi} + \alpha b(\psi) \phi^\dagger_0 \right) = 0,$$  \hspace{1cm} (4.5a)

where

$$a(\psi) = E'(\psi) - \psi^2 K'(\psi) \quad \text{and} \quad b(\psi) = \psi (K'(\psi) - E'(\psi)).$$  \hspace{1cm} (4.5b)

are defined in terms of (complementary) complete elliptic integrals $K'(\psi) = K(\sqrt{1 - \psi^2})$ and $E'(\psi) = E(\sqrt{1 - \psi^2})$ (see DLMF 2019). Taking into account that $a(\psi)$ vanishes at the quarter-cell centres $\psi = \pm 1$, specifically $a(\psi) \sim \pi (1 \mp \psi)/2$ as $\psi \to \pm 1$, we find that the (bounded) normalised solution to (4.5) is

$$\phi^\dagger_0 (\psi) = \frac{1}{Z(\alpha)} \exp \left( -\alpha \int_{0}^{\psi} d\psi' \frac{b(\psi')}{a(\psi')} \right),$$  \hspace{1cm} (4.6)
where
\[
Z(\alpha) = \frac{4}{\pi^2} \int_0^1 d\psi K'(\psi) \exp \left( -\alpha \int_0^\psi d\psi' \frac{\psi' (K'(\psi') - E'(\psi'))}{E'(\psi') - \psi'^2 K'(\psi')} \right). \tag{4.7}
\]

Figure 3 shows the function $\phi^\dagger_0$ for three values of $\alpha$ in one quarter-cell. This is of interest because $\phi^\dagger$ describes locally the spatial structure of the particle density which is given by $\phi^\dagger(x)$ times a large-scale diffusive envelope. The function is even and monotonous with $|\psi|$, decreasing from the centre of quarter-cells for light particles ($\alpha < 0$) and increasing from the centre for heavy particles ($\alpha > 0$) reflecting the expected clustering induced by inertia.

### 4.2. Asymptotic computation of $\varphi$

It is clear from the symmetries of the cellular flow that $c = 0$ and that the effective diffusivity is isotropic. Therefore, we focus on a single component of $\varphi$, say $\varphi = \varphi \cdot e_x$, which satisfies
\[
\epsilon^2 \nabla^2 \varphi + (u - \alpha \epsilon^2 u \cdot \nabla u) \cdot \nabla \varphi + u - \alpha \epsilon^2 u \cdot \nabla u = 0, \tag{4.8}
\]

where $u = u \cdot e_x$. Making use of the symmetries $(x, y, \varphi) \mapsto (\pi - y, \pi - x, -\varphi)$, $(x, y, \varphi) \mapsto (x, -y, \varphi)$ and $(x, y, \varphi) \mapsto (-x, y, -\varphi)$ we can focus on the quarter-cell $[0, \pi]^2$ and look for a solution satisfying the boundary conditions
\[
\varphi(0, y) = \varphi(\pi, y) = 0 \quad \text{and} \quad \partial_y \varphi(x, 0) = \partial_y \varphi(x, \pi) = 0. \tag{4.9}
\]

The general solution of the problem is identical to the solution of this simpler problem up to an irrelevant constant.

We now introduce $\Theta = 2(\varphi + x)/\pi - 1$ satisfying
\[
\epsilon^2 \nabla^2 \Theta + (u - \alpha \epsilon^2 (u \cdot \nabla) u) \cdot \nabla \Theta = 0, \tag{4.10}
\]

with the boundary conditions
\[
\Theta(0, y) = -1, \quad \Theta(\pi, y) = 1, \quad \partial_y \Theta(x, 0) = 0 \quad \text{and} \quad \partial_y \Theta(x, \pi) = 0. \tag{4.11}
\]

We obtain an approximation for $\Theta$ for $\epsilon \ll 1$ using matched asymptotics.

#### 4.2.1. Interior solution

We first consider the solution in the quarter-cell interior. Introducing the expansion $\Theta = \Theta_0 + \epsilon^2 \Theta_1 + \cdots$, we obtain $\Theta_0 = \Theta_0(\psi)$ at leading order and, at the next order,
\[
u \cdot \nabla \Theta_1 = -\nabla^2 \Theta_0 + \alpha (u \cdot \nabla u) \cdot \nabla \Theta_0. \tag{4.12}
\]

Integrating along a streamline yields the solvability condition
\[
\frac{d}{d\psi} \left( a(\psi) \frac{d\Theta_0}{d\psi} \right) - \alpha b(\psi) \frac{d\Theta_0}{d\psi} = 0, \tag{4.13}
\]

with $a(\psi)$ and $b(\psi)$ given in (4.5b). The only bounded solution of (4.13) is a constant. This constant interior solution must be matched with a boundary layer solution around the separatrix $\psi = 0$ so as to satisfy the boundary conditions (4.11).

#### 4.2.2. Boundary-layer solution

Following Childress (1979), we introduce the variables
\[
\zeta = \epsilon^{-1} \psi \quad \text{and} \quad \sigma = -\int_0^\psi |u| dl, \tag{4.14}
\]
where \( l \) is the arclength along a streamline. At the separatrix, \( 0 < \sigma < 8 \) parameterises the boundary of the quarter-cell clockwise, starting from 0 at (0, 0) and taking values 2, 4 and 6 at successive corners. Introducing the expansion \( \Theta = \Theta_0(\sigma, \zeta) + O(\epsilon) \), we obtain at leading order, away from the corners,

\[
\partial^2_{\zeta} \Theta_0 - \partial_\sigma \Theta_0 = 0,
\]  

(4.15a)

with the boundary conditions

\[
\begin{align*}
\Theta_0(\sigma, 0) &= -1 \quad \text{for } \sigma \in [0, 2], \\
\Theta_0(\sigma, 0) &= 1 \quad \text{for } \sigma \in [4, 6], \\
\partial_\zeta \Theta_0(\sigma, 0) &= 0 \quad \text{for } \sigma \in [2, 4] \cup [6, 8].
\end{align*}
\]  

(4.15b, c)

Eqs. (4.15) make up the so-called Childress problem, solved in closed form by Soward (1987). Using the symmetry of the boundary conditions (4.15b), it can be shown that \( \Theta_0 \) can only match the constant interior solution if this vanishes. Thus we conclude that, to leading order, \( \Theta \) is non-zero only within the boundary layer. As a result, as we now show, we can compute the effective diffusivity by combining Eq. (4.6) for \( \phi^\dagger \) with the solution of the Childress problem (4.15).

### 4.3. Computation of the effective diffusivity \( D_e \)

Using the symmetry of the problem, (3.8) and integration by parts, the effective diffusivity (3.10) can be recast into the isotropic form

\[
D_e = D \mathbb{I} \quad \text{with} \quad D = \epsilon^2 \left\langle \phi^\dagger \left( 2 \partial_\sigma \varphi + |\nabla \varphi|^2 + 1 \right) \right\rangle.
\]  

(4.16)

Since the four quarter-cells are equivalent, we can focus on \([0, \pi]^2\) where \( \varphi = \pi(1+\Theta)/2-x \) to obtain

\[
D = \frac{\pi^2 \epsilon^2}{4} \left\langle \phi^\dagger |\nabla \Theta|^2 \right\rangle_{[0, \pi]^2}
\]  

(4.17)

where \( \left\langle \cdot \right\rangle_{[0, \pi]^2} \) denotes the spatial average over the quarter-cell \([0, \pi]^2\). In the limit \( \epsilon \ll 1 \), \( \Theta \) becomes a boundary layer term localised in the \( O(\epsilon) \) boundary layer around the separatrix \( \psi = 0 \). Within this boundary layer, \( \phi^\dagger \sim Z(\alpha)^{-1} \) as (4.6) shows. As a result, to leading order in \( \epsilon \), the effective diffusivity reduces to

\[
D_{\phi^\dagger} \sim \frac{\epsilon}{4Z(\alpha)} \int_0^{\pi} d\sigma \int_0^\infty d\zeta \left( \partial_\zeta \Theta_0 \right)^2
\]  

(4.18)

using that \( d\sigma d\zeta = \epsilon^{-1} |\nabla \psi|^{-2} d\sigma d\zeta \). In Appendix B, we relate the integral in (4.18) to Soward’s constant \( \nu \) appearing in the asymptotic expression of the non-inertial effective diffusivity (1.2) to rewrite the effective diffusivity in the fully explicit form (1.5). Note that it is possible to bypass to the computation in Appendix B by observing that the integral in (4.18) is independent of \( \alpha \) so that it can be determined by setting \( \alpha = 0 \) in (4.18) and matching to the familiar non-inertial result (1.2), noting that \( Z(0) = 1 \).

### 5. Discussion and conclusion

The key result of the paper is the asymptotic expression (1.5) for the effective diffusivity \( D_{\phi^\dagger} \) in the distinguished regime \( St = O(\text{Pe}^{-1}) \ll 1 \). We test it against the direct numerical sampling of the dynamics of the particles. The Langevin equation (1.4) can be costly for \( St \ll 1 \) so we restrict its use to moderately small \( St \); for smaller \( St \), we instead integrate the effective Brownian dynamics (2.7). We use numerical schemes based on those of Pavliotis et al. (2009). The results are summarised in Figure 4. They demonstrate a very
Figure 4. Effective diffusivity $\overline{D}$ as a function of $St$ for $Pe = 1000$ (left) and as a function of $Pe$ for $St = 0.1$ (right). The values estimated by direct numerical sampling of the Langevin dynamics (1.4) for $\beta = 0$ ($*$), the effective Brownian dynamics (2.7) for $\beta = 0$ ($\nabla$) and $\beta = 2$ ($\triangle$), and the non-inertial dynamics (1.3) ($\circ$) are compared to the asymptotic prediction (1.5). The simplified asymptotic expressions (5.1) and (5.2) valid for $St Pe \gg 1$ are shown (dotted lines), as well as the refinement of (5.1) that includes the logarithmic correction in (C 6) (dot-dashed line).

good agreement between direct estimations and asymptotic predictions of $\overline{D}$. The near-coincidence of estimates obtained with the Langevin and effective Brownian dynamics also confirms the validity of the latter.

The impact of inertia is entirely captured by the function $Z(\alpha)$ defined by (4.7) and shown in Figure 2. We now analyse the behaviour of this function in detail. As noted in §1, $Z(\alpha)$ is decreasing with $\alpha$, with $Z(\alpha) < 1$ for $\alpha < 0$ and $Z(\alpha) > 1$ for $\alpha > 0$. Consequently, the effective diffusivity of particles denser (resp. less dense) than the fluid is always larger (resp. smaller) than that of non-inertial particles. This can be attributed to the divergence (2.9) of the effective velocity which, when averaged along streamlines, leads to an accumulation (resp. depletion) of particles in the separatrix region (cf. Figure 3), which controls the cell-to-cell and hence global transport.

It is interesting to consider the limiting behaviour of $Z(\alpha)$ and hence $\overline{D}$ as $\alpha = St Pe(1 - \beta) \to \pm\infty$, that is, when inertia dominates over diffusion. In the heavy-particle case $\alpha \to \infty$, the asymptotics of $Z(\alpha)$ derived in Appendix C gives

$$D \sim \nu\pi^{3/2} \left( \frac{St(1 - \beta)}{\log (Pe St(1 - \beta))} \right)^{1/2} \text{ for } Pe^{-1} \ll St \ll 1 \text{ and } \beta < 1. \quad (5.1)$$

This corresponds to an effective diffusivity that depends only weakly on the Péclet number or, equivalently on molecular diffusivity, and is instead controlled by inertia through the dependence on $St^{1/2}$. Note that the asymptotics (5.1) is rather poor because it ignores logarithmic corrections that are negligible only for exceedingly large $\alpha$. A more accurate formula can be obtained by using an improved approximation to $Z(\alpha)$ given in (C 6) and shown in Figure 2. The predictions of (5.1) and its improvement are compared against simulations results and the full asymptotic approximation (1.5) in Figure 4.
Dispersion of inertial particles in cellular flows

In the light-particle case $\alpha \to -\infty$, the asymptotics of $Z(\alpha)$ in Appendix C gives

$$D \sim \nu \pi \text{Pe}^{1/2} \text{St}(\beta - 1) \, e^{-\Upsilon \text{St} \text{Pe}(\beta - 1)} \quad \text{for} \quad \text{Pe}^{-1} \ll \text{St} \ll 1 \quad \text{and} \quad \beta > 1,$$

where $\Upsilon \approx 0.655$, which is also shown in Figure 4. The effective diffusivity decreases exponentially with $\text{Pe} \, \text{St}$ corresponding to a dramatic inhibition of dispersion caused by inertia. Physically, particles are trapped by inertia near the quarter-cell centres and only escape by crossing the separatrix for rare realisations of the noise. The rate of these rare escapes and hence $D$ could be estimated using small-noise large-deviation theory (e.g. Freidlin & Wentzell 2012). Note that the key parameter $\text{St} \, \text{Pe} = U^2 \tau / D$ (which is independent from the flow scale $a$) can be rewritten as the Arrhenius-like number

$$\text{St} \, \text{Pe} = m U^2 / (k_B T),$$

where $m = m_p + m_f / 2$ is the effective mass of the particles, $k_B$ is the Boltzmann constant and $T$ the temperature, on using the Einstein–Smoluchowski relation $D = \tau k_B T / m_p$. Thus (5.2) can be interpreted as a form of Arrhenius law, with the particle kinetic energy playing the role of activation energy.

We can assess the relative importance of inertia and diffusion by thermal noise for particles in a flow using (5.3). For instance, at room temperature, for particles with an effective density similar to that of water, we find that $\text{St} \, \text{Pe} \approx r_p^3 U^2 \times 10^{24}$ in SI units. Thus, inertia dominates diffusion ($\text{St} \, \text{Pe} \gtrsim 1$) for particles of radius $r_p = 10^{-6}$ m and $10^{-3}$ m when $U \gtrsim 10^{-3}$ m/s and $U \gtrsim 10^{-8}$ m/s, respectively. Since typical flow velocities are likely to exceed these small values, the simplified asymptotic expressions (5.1)–(5.2) will be valuable. We emphasise that, though subdominant, diffusion plays a crucial role in setting the effective diffusivity of cellular flows, as the dependence in $\text{Pe}$ of (5.1)–(5.2) indicates. This is because diffusion is indispensible for particles to move between cells.

We also note that, while (5.3) is restricted to diffusion by thermal noise, our results are also relevant to dispersion problems where diffusion models small-scale turbulent mixing, in which case the (turbulent) diffusivity is orders of magnitude larger than the molecular one and $\text{St} \, \text{Pe}$ is not necessarily large.

To conclude, we remark that the $\text{Pe}^{-1/2}$ scaling for the effective diffusivity of non-inertial particles in (1.2) is a universal feature of periodic flows with closed streamlines (Heinze 2003; Novikov et al. 2005). We expect that the conclusion that this is corrected to $\text{Pe}^{-1/2} / Z(\alpha)$ when inertia is taken into account, which we draw for cellular flows, also generalises. The form of $Z(\alpha)$ will be specific to each flow but its qualitative dependence on $\alpha$ and its asymptotic scalings for $\alpha \to \pm \infty$ are likely to be as in the cellular-flow case. Flows with open streamlines behave very differently. In shear flows, numerical results from Pavliotis et al. (2006) suggests that inertia only has a negligible effect on the (Taylor) effective diffusivity. It would be of interest to examine the impact of inertia on more complex flows such as the cat’s-eye flows of Childress & Soward (1989).

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Appendix A. Derivation of the solvability condition (4.5)

We introduce the time-like coordinate $s$ such that

$$\frac{d}{ds} = u \cdot \nabla \quad \text{i.e.,} \quad ds = \frac{dx}{\sin x \cos y} = -\frac{dy}{\cos x \sin y}. \quad (A1)$$
Integrating (4.4b) along streamlines gives
\[ \oint_{\psi} ds \nabla \cdot (\nabla \phi_0 + \alpha \phi_0 u \cdot \nabla u) = 0. \] (A 2)

Following Haynes & Vanneste (2014), we use the fact that the change of variables \((x, y) \mapsto (s, \psi)\) is area preserving and the divergence theorem to write
\[ \oint_{\psi} ds \nabla \cdot f = \frac{d}{d\psi} \left( \int \int dx dy \nabla \cdot f \right) = \frac{d}{d\psi} \left( \oint_{\psi} ds \nabla \psi \cdot f \right), \] (A 3)
for an arbitrary vector field \(f\). This reduces (A 2) to
\[ \frac{d}{d\psi} \left( a(\psi) \frac{d\phi_0}{d\psi} + \alpha b(\psi) \phi_0 \right) = 0, \] (A 4)
where
\[ a(\psi) = \frac{1}{8} \oint_{\psi} ds |\nabla \psi|^2 \quad \text{and} \quad b(\psi) = \frac{1}{8} \oint_{\psi} ds \nabla \psi \cdot (u \cdot \nabla u). \] (A 5)

The derivation of (4.5b) for \(a(\psi)\) can be found in Haynes & Vanneste (2014). To compute \(b(\psi)\), we use the symmetry of the streamline, (A 1) and \(\psi = \sin x \sin y\), to write
\[ b(\psi) = \psi \int_{\sin^{-1} \psi}^{\pi/2} \cos^2 x dx \sqrt{\sin^2 x - \psi^2}. \] (A 6)

The substitution \(t^2 = (1 - \sin^2 x)/(1 - \psi^2)\) then gives
\[ b(\psi) = \psi (1 - \psi^2) \int_0^1 \frac{t^2 dt}{\sqrt{1 - t^2} \sqrt{1 - (1 - \psi^2)t^2}}. \] (A 7)

Using formula (19.2.6) in DLMF (2019), this reduces to the expression in (4.5b).

**Appendix B. Computation of the integral in (4.18)**

We show that the integral
\[ K = \int_0^8 d\sigma \int_0^\infty d\zeta (\partial_\zeta \Theta_0)^2, \] (B 1)
where \(\Theta_0\) solves (4.15), is related to
\[ \int_0^\infty d\zeta \Theta_0 (2, \zeta) = -2\nu \quad \text{with} \quad \nu = \sqrt{\frac{2}{\pi}} \sum_{n=0}^\infty \frac{(-1)^n}{\sqrt{2n + 1}}, \] (B 2)
as calculated by Soward (1987). Integrating (B 1) by parts in \(\zeta\) gives
\[ K = -\int_0^8 d\sigma \Theta_0(\sigma, 0) \partial_\zeta \Theta_0(\sigma, 0) - \int_0^8 d\sigma \int_0^\infty d\zeta \Theta_0 \partial_\zeta^2 \Theta_0. \] (B 3)
The second term can be shown to vanish using (4.15) and periodicity in \(\sigma\). Using the boundary condition (4.15b) reduces the first term to
\[ K = \int_0^2 d\sigma \left( \partial_\zeta \Theta_0(\sigma, 0) - \partial_\zeta \Theta_0(\sigma + 4, 0) \right). \] (B 4)
Now, integrating (4.15) for $\zeta \in [0, \infty)$ gives $\int_0^\infty d\zeta \partial_\zeta \Theta_0 = -\partial_\zeta \Theta_0(\sigma, 0)$ which can be introduced in (B 4) to obtain

$$K = \int_0^\infty d\zeta \left( \Theta_0(0, \zeta) - \Theta_0(2, \zeta) + \Theta_0(6, \zeta) - \Theta_0(4, \zeta) \right).$$

Finally, using that $\int_0^\infty d\zeta \Theta_0(\sigma, \zeta)$ is constant for $\sigma \in [2, 4]$ and [4, 8] and the symmetry $\Theta_0(\sigma + 4, \zeta) = -\Theta_0(\sigma, \zeta)$, we obtain

$$K = -4 \int_0^\infty d\zeta \Theta_0(2, \sigma) = 8\nu.$$  

### Appendix C. Asymptotic calculation of $Z(\alpha)$ for $\alpha \to \pm \infty$

#### C.1. Limit $\alpha \to \infty$

The outer integral in (4.7) is dominated by a neighbourhood of the minimum of the integral multiplying $\alpha$. This minimum can be verified to be the left endpoint $\psi = 0$. Using asymptotic expansions for $K'(\psi)$ and $E'(\psi)$ as $\psi \to 0^+$, we find the approximation

$$Z(\alpha) \sim \frac{4}{\pi^2} \int_0^1 d\psi \left( -\log \frac{\psi}{4} \right) \exp \left( \frac{\alpha \psi^2}{4} \left( 1 + 2 \log \frac{\psi}{4} \right) \right).$$

Introducing the new integration variable $x = -\alpha \psi^2(1 + 2 \log(\psi/4))/4$ yields, after cumbersome calculations (carried out using the symbolic-algebra software Mathematica),

$$Z(\alpha) \sim \frac{2}{\pi^2 \sqrt{\alpha}} \int_0^{\infty} \frac{(w(x) + 1) \sqrt{w(x)} e^{-x}}{w(x) - 1} dx,$$

where $c = \log 2 - 1/4 > 0$ and $w(x) = -W_{-1}(-e x/(4\alpha))$, with $W_{-1}$ the -1-branch of the Lambert-W function, namely the inverse function of $xe^x$ on $(-\infty, -1]$ (see DLMF 2019). Using the asymptotic expansion

$$W_{-1}(-x) = \log x + \log(-\log x) + \frac{\log(-\log x)}{\log x} + o\left( \frac{\log(\log x)}{\log x} \right) \text{ as } x \to 0^+,$$

we expand the integrand in (C 2) for $\alpha \to \infty$ then extend the integration interval to $[0, \infty)$ to find

$$Z(\alpha) \sim \frac{2\sqrt{\log \alpha}}{\pi^2 \sqrt{\alpha}} \left( 1 + \frac{3 + 2 \log 2 + \log \log \alpha}{2 \log \alpha} \right) I_0 + \frac{1}{2 \log \alpha} I_1,$$

where

$$I_0 = \int_0^\infty \frac{e^{-x} dx}{\sqrt{x}} = \sqrt{\pi} \quad \text{and} \quad I_1 = -\int_0^\infty \frac{e^{-x} \log x dx}{\sqrt{x}} = \sqrt{\pi} (\gamma_E + 2 \log 2)$$

with $\gamma_E = 0.577 \cdots$ the Euler–Mascheroni constant. Combining (C 4) with (C 5) finally yields

$$Z(\alpha) \sim \frac{2\sqrt{\log \alpha}}{\pi^{3/2} \sqrt{\alpha}} \left( 1 + \frac{3 + \gamma_E + \log(16 \log \alpha)}{2 \log \alpha} \right) \text{ as } \alpha \to \infty.$$ 

Note that the corrections up to $(\log \alpha)^{-1}$ are necessary for numerical applications as they decay very slowly with $\alpha$ (see Figure 2).

#### C.2. Limit $\alpha \to -\infty$

In this case, the outer integral in (4.7) is dominated by a neighbourhood of the maximum of the integral multiplying $\alpha$, located at the right endpoint $\psi = 1$. Expanding
the inner integral near $\psi = 1$ we find

$$Z(\alpha) \sim \frac{2}{\pi|\alpha|} e^{\frac{\alpha}{\sqrt{2}}}$$

as $\alpha \to -\infty$, where $\Upsilon = \int_0^1 \frac{b(\psi)}{a(\psi)} d\psi = 0.655 \cdots$ (C7)

using that $K'(1) = \pi/2$ and $b(\psi)/a(\psi) \to 1$ as $\psi \to 1$.

REFERENCES

AUTON, T. R., HUNT, J. C. R. & PRUD’HOMME, M. 1988 The force exerted on a body in inviscid unsteady non-uniform rotational flow. *Journal of Fluid Mechanics* 197, 241–257.

CHILDRESS, S. 1979 Alpha-effect in flux ropes and sheets. *Physics of the Earth and Planetary Interiors* 20 (2), 172 – 180.

CHILDRESS, S. & SOWARD, A. M. 1989 Scalar transport and alpha-effect for a family of cat’s-eye flows. *Journal of Fluid Mechanics* 205, 99–133.

DLMPF 2019 NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.24 of 2019-09-15, f. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

FREIDLIN, M. & WENTZELL, A. 2012 Random perturbations of dynamical systems, 3rd edn. Springer.

HALLER, G. & SAPSIS, T. 2008 Where do inertial particles go in fluid flows? *Physica D: Nonlinear Phenomena* 237 (5), 573 – 583.

HEINZE, S. 2003 Diffusion-advection in cellular flows with large Pécel number. *Journal of Fluid Mechanics* 745, 351–377.

MAXEY, M. R. 1987 The motion of small spherical particles in a cellular flow field. *Physics of Fluids* 30 (4), 1915–1928.

MAXEY, M. R. & RILEY, J. J. 1983 Equation of motion for a small rigid sphere in a nonuniform flow. *Physics of Fluids* 26 (4), 883–889.

NOVIKOV, A., PANAPICOLAOU, G. & RYZHIK, L. 2005 Boundary layers for cellular flows at high Péclet numbers. *Comm. Pure Appl. Math.* 687–922, 563–580.

PAVLIOTIS, G.A. & STUART, A.M. 2005 Periodic homogenization for inertial particles. *Physica D: Nonlinear Phenomena* 204 (3), 161 – 187.

Pavliotis, G.A., Stuart, A.M. & Band, L. 2006 Monte Carlo studies of effective diffusivities for inertial particles. In *Monte Carlo and Quasi-Monte Carlo Methods 2004* (ed. H. Niederreiter & D. Talay), pp. 431–441. Berlin, Heidelberg: Springer Berlin Heidelberg.

Pavliotis, G.A., Stuart, A.M. & Zygalakis, K.C. 2009 Calculating effective diffusivities in the limit of vanishing molecular diffusion. *Journal of Computational Physics* 228 (4), 1030 – 1055.

ROSENBLUTH, M. N., BERK, H. L., DOXAS, I. & HORTON, W. 1987 Effective diffusion in laminar convective flows. *The Physics of Fluids* 30 (9), 2636–2647.

RUBIN, J., JONES, C. K. R. T. & MAXEY, M. 1995 Settling and asymptotic motion of aerosol particles in a cellular flow field. *Journal of Nonlinear Science* 5 (4), 337–358.

SAPIS, T. & HALLER, G. 2010 Clustering criterion for inertial particles in two-dimensional time-periodic and three-dimensional steady flows. *Chaos: An Interdisciplinary Journal of Nonlinear Science* 20 (1), 017515.

SHRAMAN, B. I. 1987 Diffusive transport in a Rayleigh-Bénard convection cell. *Phys. Rev. A* 36, 261–267.
Soward, A. M. 1987 Fast dynamo action in a steady flow. *Journal of Fluid Mechanics* **180**, 267-295.