Dynamical Properties of Potts Model with Invisible States

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Abstract. We study dynamic behavior of Potts model with invisible states near the first-order phase transition temperature. This model is regarded as a standard model to analyse nature of phase transition. We can control the energy barrier between the ordered state and disordered state without changing the symmetry which breaks at the transition point. We focus on melting process starting from the perfect ordered state. We calculate time-dependency of the order parameter, density of invisible state, and internal energy. All of them show two-step relaxation behavior. We also analyze the relationship between the characteristic melting time and characteristic scale of the energy barrier by changing the number of invisible states. We find that characteristic melting time increases as the energy barrier enlarges in this model. This model is regarded as a fundamental model to analyze dynamic behavior near the first-order phase transition point.

1. Introduction

Frustration causes many interesting static and dynamic behavior which are not observed in unfrustrated systems because of peculiar density of states [1–10]. In two-dimensional frustrated systems, there have been found many nontrivial phase transitions such as order by disorder [11, 12], reentrant phase transition [13–17], topological phase transition [18], and novel type of first-order phase transition. Recently, strange first-order phase transitions have been found in two-dimensional frustrated continuous spin systems [19–22].

In [19], the authors studied equilibrium properties of the classical Heisenberg model on triangular lattice with nearest neighbor ferromagnetic interaction $J_1$ and third-nearest neighbor antiferromagnetic interaction $J_3$. They found that a first-order phase transition with threefold symmetry breaking occurs at finite temperature. This looks a strange phase transition, since phase transition with threefold symmetry breaking is often second-order phase transition on two-dimensional lattice e.g. the three-state ferromagnetic Potts model [23]. After this study, similar nature of first-order phase transition have been found by a number of researchers [20–22]. Stoudenmire et al. found a first-order phase transition with breaking of threefold symmetry in $J_1 - J_3$ model with biquadratic interaction on triangular lattice. Okumura et al. also found
similar first-order phase transition in frustrated $J_1 - J_2$ model on hexagonal lattice. Such a first-order phase transition sometimes takes place in two-dimensional frustrated systems with a number of competed short-range interactions. It is an open problem why such a first-order phase transition appears in some two-dimensional frustrated systems.

To consider this problem, we constructed a model which exhibits a first-order phase transition with threefold symmetry breaking by introducing a new kind of degree of freedom into the standard ferromagnetic three-state Potts model [24, 25]. We introduced the invisible states which do not contribute to the internal energy. The ground state of the Potts model with invisible state and its degeneracy are the same as that of the standard Potts model. We found that the first-order phase transition with threefold symmetry breaking occurs by just adding the invisible states into the standard three-state ferromagnetic Potts model. The standard Potts model has been regarded as a fundamental model to consider the properties of phase transition in statistical physics [23, 26]. It is believed that the Potts model with invisible states is a potential model which can clarify inherent nature of first-order phase transition with breaking of threefold symmetry on two-dimensional lattice.

Our purpose of the present study is to clarify dynamic properties of Potts model with invisible states. It is an interesting topic in statistical physics to study dynamical nature of the systems which exhibit a first-order phase transition, since some similar points between these systems and glassy systems have been pointed out [27, 28]. It is expected that some dynamic nature in glassy systems can be explained in terms of dynamic behavior of the systems in which a first-order phase transition occurs. Potts model with invisible states can be considered a potential model to study dynamical nature of first-order phase transition. This is because we can control the energy barrier between the ordered state and disordered state without changing the symmetry which breaks at the transition temperature by modulating the number of invisible states. In this paper, we focus on melting process, then, we study dynamics of the Potts model with invisible states at fixed temperatures which are above the transition temperature.

2. Model

We consider the ferromagnetic Potts model with invisible states on square lattice. The Hamiltonian of this model is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta_{\sigma_i \sigma_j} \sum_{\alpha=1}^{q} \delta_{\sigma_i \alpha}, \quad (J > 0), \quad \sigma_i = 1, \ldots, q, q + 1, \ldots, q + r, \quad (1)$$

where $\langle i, j \rangle$ denotes the nearest neighbor pairs on square lattice. In this paper we take $J$ as an energy unit. If and only if $1 \leq \sigma_i = \sigma_j \leq q$, the interaction works. Obviously this model for $r = 0$ corresponds to the standard $q$-state ferromagnetic Potts model. We call the state in $1 \leq \sigma_i \leq q$ “colored state” and the state in $q + 1 \leq \sigma_i \leq q + r$ “invisible state”. Hereafter we call this model $(q,r)$-state Potts model. The invisible state does not affect the internal energy. It should be noted that the number of ground states of the $(q,r)$-state Potts model is the same as that of the standard ferromagnetic $q$-state Potts model. Then the $(q,r)$-state Potts model exhibits a phase transition with $q$-fold symmetry breaking. To change the number of invisible states $r$ corresponds to changing form of the density of states. The invisible states contribute the entropy as $\log r$. Then, it is expected that the order of phase transition can be changed by adding the invisible states. Actually, a first-order phase transition with $q$-fold symmetry breaking occurs for large enough the number of invisible states $r$ even for $q = 2, 3, \text{and} 4$ on two-dimensional lattice [24]. As the number of the invisible states $r$ increases, the latent heat increases and the transition temperature decreases [25].
3. Result

In this paper, we study melting dynamics of the \((q, r)\)-state Potts model on square lattice whose size is \(N = 128 \times 128\) at fixed temperatures which are above the transition temperature following [28]. We impose periodic boundary condition. We prepare independent 1024 samples for obtaining data. It should be noted that since the width of error bars obtained from 1024 samples is half the size of that obtained from 256 samples, it is enough to consider a statistic error by using 1024 samples. The initial state is set to be perfect ordered state such as \(\sigma_i(0) = 1\) for all \(i\). We adopt single-spin-flip Metropolis type of Monte Carlo method as the time-evolution rule.

To consider dynamic behavior of the \((q, r)\)-state Potts model, we define the time-dependent order parameter \(m(t)\), density of invisible states \(\rho_{\text{inv}}(t)\), and internal energy \(e(t)\) as follows:

\[
m(t) = \frac{(q + r) \sum_{i=1}^{N} \left[ \delta_{\sigma_i(t), 1} - \frac{1}{q + r} \right]}{N(q + r - 1)},
\]

\[
\rho_{\text{inv}}(t) = \frac{1}{N} \sum_{i=1}^{N} \sum_{\alpha=q+1}^{q+r} \delta_{\sigma_i(t), \alpha},
\]

\[
e(t) = -\frac{J}{N} \sum_{(i,j)} \delta_{\sigma_i(t), \sigma_j(t)} \sum_{\alpha=1}^{q} \delta_{\sigma_i(t), \alpha},
\]

where \(t\) denotes Monte Carlo step and \(\sigma_i(t)\) is the state at time \(t\). Figure 1 shows the time-evolution of the order parameter \(\langle m(t) \rangle\), density of the invisible states \(\langle \rho_{\text{inv}}(t) \rangle\), and the internal energy \(\langle e(t) \rangle\) of the \((3, 27)\)-state Potts model starting from the completely ordered state \(\sigma_i(0) = 1\) for all \(i\) for fixed different temperatures. From left to right, the temperatures are \(T = T_c + 0.5, T_c + 0.2, T_c + 0.1, T_c + 0.05, T_c + 0.02, T_c + 0.01, T_c + 0.005, T_c + 0.002, T_c + 0.001, T_c + 0.0005, T_c + 0.0002, \) and \(T_c + 0.0001\), where \(T_c = 0.58513\) [25].
energy \langle e(t) \rangle of the (3,27)-state Potts model starting from the completely ordered state \( \sigma_i(0) = 1 \) (for all \( i \)) for fixed different temperatures. Here \( \langle \cdot \rangle \) denotes an ensemble average. In Fig. 1, there are obvious two-step relaxations near the transition temperature. This nature comes from a first-order phase transition. Two-step relaxation often appears in systems where a first-order phase transition takes place and also in glassy systems [27–29]. Melting occurs after stabilization in the ordered state which corresponds to plateau region, in other words, first-step relaxation. As the temperature approaches to the transition point, the plateau region enlarges and as a result, the characteristic melting time becomes long.

Next we consider the dynamical susceptibility to consider the characteristic melting time systematically. The dynamical susceptibility of the time-dependent physical quantity \( A(t) \) is defined as

\[
\chi_A(t) = N \beta (\langle A(t)^2 \rangle - \langle A(t) \rangle^2),
\]

where \( \beta \) denotes the inverse temperature. We calculate the dynamical susceptibility of the order parameter \( \chi_m(t) \) and density of the invisible states \( \chi_{\rho_{inv}}(t) \). Figure 2 shows the dynamical susceptibility \( \chi_m(t) \) and \( \chi_{\rho_{inv}}(t) \) starting from the completely ordered state \( \sigma_i(0) = 1 \) (for all \( i \)) for fixed different temperatures of the (3,27)-state Potts model. In Fig. 2, the dynamical susceptibilities have a peak, and the peak position indicates characteristic melting time. Characteristic melting time defined from \( \chi_m(t) \) and \( \chi_{\rho_{inv}}(t) \) are almost same as shown in Fig. 2. Then we define \( \tau_{\text{max}} \) as the peak position of dynamical susceptibility of the order parameter \( \chi_m(t) \). As the temperature decreases, the peak height increases. This result means that characteristic length scale becomes large as the temperature approaches to the transition point, since the peak height relates to characteristic length scale. The characteristic melting time \( \tau_{\text{max}} \) also increases as the temperature decreases. The behavior of the dynamical susceptibilities is quantitatively similar with that of \( \chi_4 \) which is often used in analysis of glassy systems [28].

We also study temperature-dependency of \( \tau_{\text{max}} \) for the (3,25)-state Potts model and (3,27)-state Potts model to consider the effect of the number of invisible states. Figure 3 shows the
Figure 3. Peak position of the dynamical susceptibility of the order parameter $\tau_{\text{max}}$ as a function of $1/(T - T_c)$, where $T_c$ is the transition temperature. The red squares and blue circles indicate the case for the (3,25)-state Potts model and the (3,27)-state Potts model, respectively. Transition temperatures for the (3,25)-state Potts model and the (3,27)-state Potts model are 0.59630 and 0.58513, respectively [25].

characteristic melting time as a function of $1/(T - T_c)$, where $T_c$ denotes transition temperature. The transition temperatures for $(q,r)=$(3,25) and (3,27) are 0.59630 and 0.58513, respectively [25]. In high temperature region, behavior of $\tau_{\text{max}}$ is almost same both for (3,25)-state Potts model and for (3,27)-state Potts model. However, there is a difference between the behavior of $\tau_{\text{max}}$ for (3,25)-state Potts model and that of (3,27)-state Potts model near the transition temperature. The characteristic time $\tau_{\text{max}}$ for the (3,27)-state Potts model is larger than that for the (3,25)-state Potts model, since the energy barrier between the ordered state and paramagnetic state for the (3,27)-state Potts model is larger than that for the (3,25)-state Potts model [24,25]. From this result, we expect that $\tau_{\text{max}}$ enlarges as the number of invisible state $r$ increases. Thus, in the the Potts model with invisible states, we can change the time-scale and length-scale of first-order phase transition without changing the symmetry which breaks at the transition point. Since the time-scale and length-scale are very important physical properties for dynamic nature, the Potts model with invisible states can be regarded as a standard model to analyze dynamic behavior of systems which exhibits a first-order phase transition.

Before the conclusion, we should mention the size-dependency of melting dynamics. At high temperature, there is almost no size-dependency. On the other hand, size-dependence clearly appears in the second-step relaxation near the transition temperature whereas the first-step relaxation does not depend on system size. From this result, the first-step relaxation is just a local event, however, the second-step relaxation comes from cooperative phenomena. Note that it is definitely consistent that the characteristic melting time $\tau_{\text{max}}$ becomes large, as the number of invisible states $r$ increases for fixed lattice sizes. This size-dependency is very complicated and it will be reported elsewhere [30].

4. Conclusion and Future Perspective

We studied dynamic properties of the Potts model with invisible states at fixed temperatures which are above the transition temperature. In this paper, we focused on melting process which is an important nature of first-order phase transitions. First, we considered the dynamical properties of the order parameter, density of invisible states, and the internal energy. We found that there is a two-step relaxation which is typical behavior in systems which exhibit a first-order phase transition and also glassy systems. Next we calculated the dynamical susceptibility of the order parameter and density of invisible states. As the temperature approaches the transition temperature, the peak height of the dynamical susceptibility grows. Furthermore, we studied the relation between the peak position of the dynamical susceptibility of the order parameter
We found that $\tau_{\text{max}}$ depends on the energy barrier between the ordered state and paramagnetic state. We also found that in high temperature region, $\tau_{\text{max}}$ obviously depends on $r$ near the transition temperature.

In the Potts model with invisible states, we can control the energy barrier between the ordered state and paramagnetic state without changing the symmetry which breaks at the transition point. We should study the relation between the energy barrier and the time-scale and length-scale in this model more carefully.

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