Abstract

Let \( r \geq 2 \) be a fixed integer. For infinitely many \( n \), let \( \mathbf{k} = (k_1, \ldots, k_n) \) be a vector of nonnegative integers such that their sum \( M \) is divisible by \( r \). We present an asymptotic enumeration formula for simple \( r \)-uniform hypergraphs with degree sequence \( \mathbf{k} \). (Here “simple” means no loops and no repeated edges.) Our formula holds whenever the maximum degree \( k_{\text{max}} \) satisfies \( k_{\text{max}}^{2+\eta} = o(M) \), where \( \eta = 1 \) if \( r \in \{3, 4, 5, 6\} \) and \( \eta = 0 \) otherwise.

1 Introduction

Hypergraphs are combinatorial structures which can model very general relational systems, including some real-world networks \[3, 5, 7\]. Formally, a hypergraph or a set system is defined as a pair \((V, E)\), where \( V \) is a finite set and \( E \) is a multiset of multisubsets of \( V \). (We refer to elements of \( E \) as edges.) Note that under this definition, a hypergraph may contain repeated edges and an edge may contain repeated vertices.

Any 2-element multisubset of an edge \( e \in E \) is called a link in \( e \). If a vertex \( v \) has multiplicity at least 2 in the edge \( e \), we say that \( v \) is a loop in \( e \). (So every loop in \( e \) is also a

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link in e.) The multiplicity of a link \( \{x,y\} \) is the number of edges in \( E \) which contain \( \{x,y\} \) (counting multiplicities).

We will say that a hypergraph is weakly simple if it has no loops and no repeated edges. Here it is possible that distinct edges may have more than one vertex in common. A hypergraph is said to be strongly simple if it has no loops and each pair of distinct edges intersect in at most one vertex. Both of these definitions have appeared in the literature as definitions for “simple hypergraphs”. For example, weakly simple hypergraphs were studied in [1, 2], while strongly simple hypergraphs were considered in [4, 8].

In this paper we consider weakly simple hypergraphs, which we refer to from now on as simple.

Let \( r \) be a fixed positive integer. We say that the hypergraph \( (V,E) \) is \( r \)-uniform if each edge \( e \in E \) contains exactly \( r \) vertices (counting multiplicities). Uniform hypergraphs are a particular focus of study, not least because a 2-uniform hypergraph is precisely a graph. We seek an asymptotic enumeration formula for the number of \( r \)-uniform simple hypergraphs with a given degree sequence, when the maximum degree is not too large (the sparse range).

To state our result precisely, we need some definitions. Let \( k_{i,n} \) be a nonnegative integer for all pairs \((i, n)\) of integers which satisfy \( 1 \leq i \leq n \). Then for each \( n \geq 1 \), let \( \mathbf{k} = k(n) = (k_{1,n}, \ldots, k_{n,n}) \). We usually write \( k_i \) instead of \( k_{i,n} \). Define \( M = \sum_{i=1}^{n} k_i \). We assume that \( M \) is divisible by \( r \) for an infinite number of values of \( n \), and tacitly restrict ourselves to such \( n \).

We write \( (a)_m \) to denote the falling factorial \( a(a - 1) \cdots (a - m + 1) \), for integers \( a \) and \( m \). For each positive integer \( t \), let \( M_t = \sum_{i=1}^{n} (k_i)_t \). Notice that \( M_1 = M \) and that \( M_t \leq k_{\max} M_{t-1} \) for \( t \geq 2 \).

Let \( \mathcal{H}_r(\mathbf{k}) \) be the set of (weakly) simple \( r \)-uniform hypergraphs on the vertex set \( \{1, 2, \ldots, n\} \) with degrees given by \( \mathbf{k} = (k_1, \ldots, k_n) \). Define

\[
\eta = \eta(r) = \begin{cases} 
1 & \text{if } r \in \{3, 4, 5, 6\}, \\
0 & \text{if } r \geq 7.
\end{cases}
\]

Our main theorem is the following.

**Theorem 1.1.** Suppose that \( n \to \infty \), \( M \to \infty \) and that \( k_{\max} \) satisfies \( k_{\max} \geq 2 \) and \( k_{\max}^{2+\eta} = o(M) \). Then

\[
|\mathcal{H}_r(\mathbf{k})| = \frac{M!}{(M/r)! (r!)^{M/r} \prod_{i=1}^{n} k_i!} \exp \left( -\frac{(r-1)M_2}{2M} + O(k_{\max}^{2+\eta}/M) \right).
\]
It may be that the $O(k^3_{\text{max}}/M)$ error term can be reduced to $O(k^2_{\text{max}}/M)$ for some values of $r \in \{3, 4, 5, 6\}$, possibly for $r \geq 4$.

The corresponding asymptotic enumeration formula for strongly simple hypergraphs will be presented in a future paper.

As a corollary, we immediately obtain the corresponding formula for regular hypergraphs. Let $H_r(k, n)$ denote the set of all $k$-regular $r$-uniform hypergraphs on the vertex set $\{1, \ldots, n\}$, where $k \geq 2$ is an integer, which may be a function of $n$.

**Corollary 1.2.** Suppose that $n \to \infty$ and that $k$ satisfies $k \geq 2$ and $k^{1+\eta} = o(n)$. Then

$$|H_r(k, n)| = \frac{(kn)!}{(kn/r)! (r!)^{kn/r} (k!)^n} \exp \left( -\frac{1}{2} (k - 1)(r - 1) + O(k^{1+\eta}/n) \right).$$

### 1.1 History

In the case of graphs, the best asymptotic formula in the sparse range is given by McKay and Wormald [13]. See that paper for further history of the problem. Note that their formula has a similar form to ours, but with many more terms in the exponential factor. This is due to the fact that it is harder to avoid creating a repeated edge with a switching when $r = 2$. (These extra terms will also appear in the asymptotic enumeration of strongly simple hypergraphs which, as mentioned above, will be presented in a future paper.)

The dense range for $r = 2$ was treated in [11, 12], but there is a gap between these two ranges in which nothing is known.

An early result in the asymptotic enumeration of hypergraphs was given by Cooper et al. [1], who considered simple $k$-regular hypergraphs when $k = O(1)$. Very recently Dudek et al. [2] proved an asymptotic formula for (weakly) simple $k$-regular hypergraphs graphs with $k = o(n^{1/2})$. A restatement of their result in our notation is the following:

**Theorem 1.3.** ([2, Theorem 1]) For each integer $r \geq 3$, define

$$\kappa = \kappa(r) = \begin{cases} 1 & \text{if } r \geq 4, \\ \frac{1}{2} & \text{if } r = 3. \end{cases}$$

Let $H(r, k)$ denote the set of all (weakly) simple $k$-regular $r$-uniform hypergraphs on the vertex set $\{1, \ldots, n\}$. For every $r \geq 3$, if $k = o(n^{1/2})$ then

$$|H(r, k)| = \frac{(kn)!}{(kn/r)! (r!)^{kn/r} (k!)^n} \exp \left( -\frac{1}{2} (k - 1)(r - 1)(1 + O(\delta(n))) \right),$$

where $\delta(n) = (kn)^{-1/2} + k/n$. 

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Note that the factor outside the exponential part matches ours (see Corollary 1.2), and that the exponential part of their formula can be rewritten as
\[
\exp \left( -\frac{1}{2} (k - 1)(r - 1) + O(k\delta(n)) \right)
\]
with relative error
\[
O(k\delta(n)) = O\left( \sqrt{k/n} + k^2/n \right).
\]
This relative error is only \( o(1) \) when \( k^2 = o(n) \). The relative error from Corollary 1.2 is certainly \( o(1) \) when \( k^2 = o(n) \), but for \( r \geq 7 \) is also \( o(1) \) when the less restrictive condition \( k = o(n) \) holds.

For an asymptotic formula for the number of dense (weakly) simple \( r \)-uniform hypergraphs with a given degree sequence, see [9].

1.2 The model, some early results and a plan of the proof

We work in a generalisation of the configuration model. Let \( B_1, B_2, \ldots, B_n \) be disjoint sets, which we call cells, and define \( B = \bigcup_{i=0}^n B_i \). Elements of \( B \) are called points. Assume that cell \( B_i \) contains exactly \( k_i \) points, for \( i = 1, \ldots, n \). We assume that there is a fixed ordering on the \( M \) points of \( B \).

Denote by \( \Lambda_r(k) \) the set of all unordered partitions \( Q = \{U_1, \ldots, U_{M/r}\} \) of \( B \) into \( M/r \) parts, where each part has exactly \( r \) points. Then
\[
|\Lambda_r(k)| = \frac{M!}{(M/r)! (r!)^{M/r}}.
\]

Each partition \( Q \in \Lambda_r(k) \) defines a hypergraph \( G(Q) \) on the vertex set \( \{1, \ldots, n\} \) in a natural way: vertex \( i \) corresponds to the cell \( B_i \), and each part \( U \in Q \) gives rise to an edge \( e_U \) such that the multiplicity of vertex \( i \) in \( e_U \) equals \( |U \cap B_i| \), for \( i = 1, \ldots, n \). Then \( G(Q) \) is an \( r \)-uniform hypergraph with degree sequence \( k \). The partition \( Q = \Lambda_r(k) \) is called simple if \( G(Q) \) is simple.

Similarly, any two points in a part \( U \) is called a link of \( U \). The edge \( e_U \) has a loop at \( i \) if and only if \( |U \cap B_i| \geq 2 \). In this case, each pair of distinct points in \( U \cap B_i \) is called a loop in \( U \).

We reserve the letters \( e, f \) for edges in a hypergraph, and use \( U, W \) for parts in a partition \( Q \) (that is, in the configuration model).

Now we will consider random partitions. Each hypergraph in \( \mathcal{H}_r(k) \) corresponds to
partitions $Q \in \Lambda_r(k)$. Hence, when $Q \in \Lambda_r(k)$ is chosen uniformly at random, conditioned on $G(Q)$ being simple, the probability distribution of $G(Q)$ is uniform over $\Lambda_r(k)$. Let $P_r(k)$ denote the probability that a partition $Q \in \Lambda_r(k)$ chosen uniformly at random is simple. Then

$$|\mathcal{H}_r(k)| = \frac{M!}{(M/r)! (r!)^{M/r} \prod_{i=1}^n k_i!} P_r(k).$$

Hence it suffices to show that $P_r(k)$ equals the exponential factor in the statement of Theorem 1.1. As a first step, we identify several events which have probability $O(k^2 \max / M)$ in the uniform probability space over $\Lambda_r(k)$.

The following easy fact will be used repeatedly. If $c$ is a fixed positive integer and $U_1, \ldots, U_c$ are fixed, disjoint $r$-subsets of the set of points $B$, then the probability that a uniformly random $Q \in \Lambda_r(k)$ contains the parts $\{U_1, \ldots, U_c\}$ is

$$(1 + o(1)) \frac{((r - 1)!)^c}{M^{c(r-1)}}.$$ 

The multiplicity of a link in $Q$ is defined to be multiplicity of the corresponding link (pair of vertices) in $G(Q)$. (Recall that this equals the number of edges containing the link, counting multiplicities.) Denote the number of edges in a hypergraph $G$ by $h(G)$ (counting multiplicities) and for a given edge $e$ in $G$, let $\mu_G(e)$ denote the multiplicity of the edge $e$ in $G$. Let

$$N = \max\{\lceil \log M \rceil, \left\lceil 9(r - 1)M_2/M \right\rceil\}.$$ 

Now define $\Lambda^+_r(k)$ to be the set of partitions $Q \in \Lambda_r(k)$ which satisfy the following properties:

(i) For each part $U \in Q$ we have $|U \cap B_i| \leq 2$ for $i = 1, \ldots, n$.

(ii) For each part $U \in Q$ there is at most one $i \in \{1, \ldots, n\}$ with $|U \cap B_i| = 2$.

(iii) There are at most $N$ parts which contain loops.

(iv) If $r \geq 7$ then there is no 3-tuple of distinct parts $(U_1, U_2, U_3)$ such that the corresponding 3-tuple of edges $(e_1, e_2, e_3)$ of $G(Q)$ has the following properties:

- $e_1$ has a loop at some vertex $x$ which belongs to $e_2 \setminus e_3$, and
- there are $r - 1$ vertices in $e_2 \cap e_3$, counting multiplicities (so $x$ is the unique vertex in $e_2 \setminus e_3$).
(v) If $r \geq 7$ then there is no 4-tuple of distinct parts $(U_1, U_2, U_3, U_4)$ such that the corresponding 4-tuple of edges $(e_1, e_2, e_3, e_4)$ of $G(Q)$ has the following properties:

- $e_1$ has a loop at some vertex $x$ such that $x \in e_2 \cap e_3$ and $x \not\in e_4$, and
- there are $r - 2$ vertices in $e_1 \cap e_4$, counting multiplicities (so that the only vertex in $e_1 \setminus e_4$ is the loop vertex $x$).

(vi) If $r \geq 7$ then there is no 4-tuple of distinct parts $(U_1, U_2, U_3, U_4)$ such that the corresponding 4-tuple of edges $(e_1, e_2, e_3, e_4)$ of $G(Q)$ have the following properties:

- there are $r - 1$ vertices in $e_1 \cap e_2$, counting multiplicities,
- the unique vertex in $e_1 \setminus e_2$ also belongs to $e_3$, and
- the unique vertex in $e_2 \setminus e_1$ also belongs to $e_4$.

The first three properties above are fairly natural, while the last three properties relate to specific structures which give rise to higher error bounds (as we will see in the proof of Lemma 2.2).

**Lemma 1.4.** Under the assumptions of Theorem 1.1, we have

$$\frac{|\Lambda^+_r(k)|}{|\Lambda_r(k)|} = 1 + O\left(\frac{k^2}{k_{\max}}\right).$$

**Proof.** Consider $Q \in \Lambda_r(k)$ chosen uniformly at random.

(i) The expected number of parts in $Q$ which contain three or more points from the same cell is

$$O\left(\frac{M_3 M^{r-3}}{M^{r-1}}\right) = O\left(\frac{k^2}{k_{\max}}\right),$$

using (1.2).

(ii) Similarly, the expected number of parts in $Q$ which contain two loops (where each loop is from a distinct cell) is

$$O\left(\frac{M_2^2 M^{r-4}}{M^{r-1}}\right) = O\left(\frac{k^2}{k_{\max}}\right).$$

(iii) Let $\ell = N + 1$. We bound the expected number of sets $\{U_1, \ldots, U_\ell\}$ of $\ell$ parts which each contain a loop. Given $(U_1, \ldots, U_{\ell-1})$, there are at most $M_2 M^{r-2} / (2(r-2)!)$ choices for $U_\ell$. Hence there are

$$O\left(\frac{1}{\ell!} \left(\frac{M_2 M^{r-2}}{2(r-2)!}\right)^\ell\right)$$
possible sets \{U_1, \ldots, U_\ell\} of parts which each contain a loop. Applying (1.2) and by definition of \(N\), the expected number of sets of \(\ell = N + 1\) parts which each contain a loop is

\[
O \left( \frac{1}{\ell!} \left( \frac{(r-1)M_2}{2M} \right)^\ell \right) = O \left( \left( \frac{e(r-1)M_2}{2\ell M} \right)^\ell \right) = O \left( \frac{(e/18)^{\log M}}{\ell M} \right) = o(1/M).
\]

(iv) Suppose that \(r \geq 7\). There are at most \(O(M^3 M^{r-1} M^{r-1})\) ways to choose the points of \((U_1, U_2, U_3)\) satisfying the given conditions. For each such 3-tuple of parts, the probability that \(Q\) contains these three parts is \(O(M^{-3(r-1)})\), by (1.2). Hence the expected number of such \((U_1, U_2, U_3)\) is

\[
O \left( \frac{M^r M^{r-1}}{M^{2r-2}} \right) = O \left( \frac{k_{\text{max}}^{r+1}}{M^{r-3}} \right) = O \left( \frac{k_{\text{max}}^2}{M} \right)
\]

since \(r \geq 7\). (Indeed, this bound holds for \(r \geq 5\) using only the assumption that \(k_{\text{max}} = o(M)\).)

(v) Arguing as above, using (1.2), the expected number of 4-tuples of parts \((U_1, U_2, U_3, U_4)\) with the given property is

\[
O \left( \frac{M_4 M_2^{r-2}}{M^{2r-4}} \right) = O \left( \frac{k_{\text{max}}^{r+1}}{M^{r-3}} \right) = O \left( \frac{k_{\text{max}}^2}{M} \right)
\]

when \(r \geq 7\).

(vi) Using (1.2) and arguing as above, when \(r \geq 7\) the expected number of 4-tuples of this form is

\[
O \left( \frac{M_2^{r+1}}{M^{2r-2}} \right) = O \left( \frac{k_{\text{max}}^{r+1}}{M^{r-3}} \right) = O \left( \frac{k_{\text{max}}^2}{M} \right).
\]

\[\square\]

In Section 2 we will calculate \(|\Lambda_+^+(k)|\) by analysing switchings which remove one loop from a given partition.

## 2 The switchings

For a given nonnegative integer \(\ell\), let \(C_\ell\) be the set of partitions \(Q \in \Lambda_+^+(k)\) with exactly \(\ell\) loops. It follows from Lemma 1.4 that

\[
\frac{1}{P_r(k)} = \left(1 + O(k_{\text{max}}^2/M)\right) \sum_{\ell=0}^N \frac{|C_\ell|}{|C_0|}.
\]

We estimate this sum using a switching designed to remove loops.
An ℓ-switching in a partition \( Q \) is specified by a 4-tuple \((x_1, x_2, y_1, y_2)\) of points where \( x_1 \) belongs to the part \( U \), and \( y_j \) belongs to the part \( W_j \) for \( j = 1, 2 \), such that:

- \( U, W_1 \) and \( W_2 \) are distinct parts of \( Q \), and
- \( U \) contains a loop \( \{x_1, x_2\} \).

The ℓ-switching maps \( Q \) to the partition \( Q' \) defined by

\[
Q' = (Q - \{U, W_1, W_2\}) \cup \{\hat{U}, \hat{W}_1, \hat{W}_2\}
\]

(2.2)

where

\[
\hat{U} = (U - \{x_1, x_2\}) \cup \{y_1, y_2\}, \quad \hat{W}_1 = (W_1 - \{y_1\}) \cup \{x_1\}, \quad \hat{W}_2 = (W_2 - \{y_2\}) \cup \{x_2\}.
\]

This operation is illustrated in Figure 1. It is the same operation used by Dudek et al. [2], but we will use it under slightly stricter conditions than they do (see below).

Let \( e \) be the edge of \( G(Q) \) corresponding to \( U \), and let \( f_j \) be the edge of \( G(Q) \) corresponding to \( W_j \) for \( j = 1, 2 \). Similarly, let \( \hat{e} \) be the edge of \( G(Q') \) corresponding to \( \hat{U} \), and let \( \hat{f}_j \) be the edge of \( G(Q') \) corresponding to \( \hat{W}_j \) for \( j = 1, 2 \). Suppose that \( Q \in \mathcal{C}_\ell \). Then \( Q' \in \mathcal{C}_{\ell-1} \) if the following conditions hold:

(I) there are \( 3\ell - 1 \) distinct vertices in \( e \cup f_1 \cup f_2 \),

(II) the new edges \( \hat{e}, \hat{f}_1, \hat{f}_2 \) do not equal any edge of \( G(Q') \setminus \{\hat{e}, \hat{f}_1, \hat{f}_2\} \).
If one of these conditions fails then the 4-tuple \((x_1, x_2, y_1, y_2)\) is said to be illegal.

Note that we insist that the only repeated vertex in \(e, f_1, f_2\) is the known loop within \(e\). This is where our switching operation differs from Dudek et al. \([2]\), who allow \(|f_1 \cap f_2|\) to contain up to \(r - 2\) vertices.

A reverse \(\ell\)-switching in a given partition \(Q'\) is the reverse of an \(\ell\)-switching. It is described by a 4-tuple \((x_1, x_2, y_1, y_2)\) of points, where \(\hat{W}_j\) is the part of \(Q'\) containing \(x_j\), for \(j = 1, 2\), and \(y_1, y_2\) are distinct points in the part \(\hat{U}\) of \(Q'\), such that

- \(\hat{U}, \hat{W}_1\) and \(\hat{W}_2\) are distinct parts of \(Q'\),
- \(x_1\) and \(x_2\) belong to the same cell.

This reverse \(\ell\)-switching acting on \(Q'\) produces the partition \(Q\) defined by (2.2), as depicted in Figure [1] by following the arrow in reverse. The conditions for a legal reverse \(\ell\)-switching are:

(I') there are \(3r - 1\) distinct vertices in \(\hat{e} \cup \hat{f}_1 \cup \hat{f}_2\),

(II') the new edges \(e, f_1\) and \(f_2\) are not equal to any edge of \(G(Q) \setminus \{e, f_1, f_2\}\).

The following summation lemma proved in \([6]\) will be used in our analysis, and for completeness we state it here.

**Lemma 2.1** \(([6, Corollary 4.5])\). Let \(N \geq 2\) be an integer and, for \(1 \leq i \leq N\), let real numbers \(A(i), B(i)\) be given such that \(A(i) \geq 0\) and \(1 - (i - 1)B(i) \geq 0\). Define \(A_1 = \min_{i=1}^{N} A(i), A_2 = \max_{i=1}^{N} A(i), C_1 = \min_{i=1}^{N} A(i)B(i)\) and \(C_2 = \max_{i=1}^{N} A(i)B(i)\). Suppose that there exists a real number \(\hat{c}\) with \(0 < \hat{c} < \frac{1}{3}\) such that \(\max\{A/N, |C|\} \leq \hat{c}\) for all \(A \in [A_1, A_2], C \in [C_1, C_2]\). Define \(n_0, \ldots, n_N\) by \(n_0 = 1\) and

\[
\frac{n_i}{n_{i-1}} = \frac{A(i)}{i} \left(1 - (i - 1)B(i)\right)
\]

for \(1 \leq i \leq N\), with the following interpretation: if \(A(i) = 0\) or \(1 - (i - 1)B(i) = 0\), then \(n_j = 0\) for \(i \leq j \leq N\). Then

\[
\Sigma_1 \leq \sum_{i=0}^{N} n_i \leq \Sigma_2,
\]

where

\[
\Sigma_1 = \exp\left(A_1 - \frac{1}{2}A_1C_2\right) - (2\hat{c})^N,
\]

\[
\Sigma_2 = \exp\left(A_2 - \frac{1}{2}A_2C_1 + \frac{1}{2}A_2C_1^2\right) + (2\hat{c})^N.
\] □
Lemma 2.2. Under the conditions of Theorem 1.1 we have
\[
\sum_{\ell=0}^{N} \frac{|C_{\ell}|}{|C_0|} = \exp \left( \frac{(r - 1)M_2}{2M} + O \left( \frac{k_{\text{max}}^{2+\eta}}{M} \right) \right)
\]
if $C_0$ is nonempty.

Proof. Let $\ell \in \{1, \ldots, N\}$ be such that $C_{\ell-1} \neq \emptyset$, and let $Q \in C_{\ell}$ be given. We count the number of 4-tuples $(x_1, x_2, y_1, y_2)$ which give a legal $\ell$-switching from $Q$. There are $2\ell$ ways to choose a point $x_1$ contained in a loop, and then at most $(M - r\ell)_2$ ways to choose $(y_1, y_2)$, avoiding points which lie in parts containing loops. Hence there are at most
\[
2\ell (M - r\ell)_2 = 2\ell M^2 \left( 1 + O(\ell/M) \right)
\]
4-tuples which give legal $\ell$-switchings in $Q$.

From this upper bound, we must subtract the number of illegal 4-tuples. If condition (I) fails then there is an unwanted vertex coincidence. There are at most $O(\ell k_{\text{max}} M)$ such 4-tuples $(x_1, x_2, y_1, y_2)$. If condition (II) fails then one of the edges $\widehat{e}, \widehat{f}_1, \widehat{f}_2$ is an edge of $G(Q)$. If $\widehat{e}$ is an edge of $G(Q)$ then it shares $r-2$ vertices with $e$, and intersects each of $f_1, f_2$ in one vertex each. Hence there are at most $O(\ell k_{\text{max}}^3)$ possibilities for 4-tuples $(x_1, x_2, y_1, y_2)$ which fail condition (II) because $\widehat{e} \in G(Q)$.

Next, if $\widehat{f}_1$ is an edge of $G(Q)$ then it shares $r-1$ vertices with $f_1$, and shares its remaining vertex, $x$ with $e$. Since $e$ has a loop at $x$, we see that $(U, W_1, \widehat{W}_1)$ is a 3-tuple of parts of the kind considered in Lemma 1.4 (iv). Hence when $r \geq 7$ it is not possible for $\widehat{f}_j$ to be an edge of $G(Q)$, for $j = 1, 2$. When $r \in \{3, 4, 5, 6\}$ we observe that there are $O(\ell k_{\text{max}}^2)$ choices for $(x_1, x_2, y_1, y_2)$ for which condition (II) fails because $\widehat{f}_j \in G(Q)$.

Combining these contributions, we find that there are
\[
2\ell M^2 \left( 1 + O \left( \frac{\ell + k_{\text{max}}^{1+\eta}}{M} \right) \right)
\]
4-tuples $(x_1, x_2, y_1, y_2)$ which give a legal $\ell$-switching from $Q$. (That is, the lower bound matches the upper bound, within the accuracy afforded by the error term.) We remark that the $k_{\text{max}}^2/M$ term here can be replaced by $k_{\text{max}}/M$ when $r = 5, 6$, but for simplicity of presentation we leave the larger bound in place, since it will arise in later calculations.

Next, suppose that $Q' \in C_{\ell-1}$. We count the number of 4-tuples $(x_1, x_2, y_1, y_2)$ which give a legal reverse $\ell$-switching from $Q'$. There are at most $M_2$ ways to choose $(x_1, x_2)$, then there are at most $M$ ways to choose $y_1$ and at most $r - 1$ ways to choose $y_2 \in U_1 \setminus \{y_1\}$. Hence there are at most
\[
(r - 1) M_2 M
\]
legal 4-tuples for $Q'$.

From this we must subtract choices which lead to illegal 4-tuples. There are at most $O(k_{\text{max}}^2 M_2)$ 4-tuples which fail condition (I'). Next, suppose that condition (II') fails. Then one of the edges $e, f_1, f_2$ belongs to $G(Q')$. If $e \in G(Q')$ then $e$ shares $r - 2$ of its vertices with $\hat{e}$, and $e \setminus \hat{e}$ equals the loop at the vertex $x$, where $x$ also belongs to $\hat{f}_1$ and $\hat{f}_2$. It follows that $(U, \hat{W}_1, \hat{W}_2, \hat{U})$ is a 4-tuple of parts of the kind considered in Lemma 1.4 (v). Therefore, this case cannot arise if $r \geq 7$. If $r \in \{3, 4, 5, 6\}$ then we use the bound $O(k_{\text{max}}^2 M_2)$ for the number of illegal 4-tuples in this case. Hence there are $O(\eta k_{\text{max}}^2 M_2)$ illegal 4-tuples which fail condition (II') because $e \in G(Q')$.

Similarly, if $f_1 \in G(Q')$ then $f_1$ shares $r - 1$ vertices with $\hat{f}_1$, and shares its remaining vertex (the one vertex in $f_1 \setminus \hat{f}_1$) with $\hat{e}$. Furthermore, $\hat{f}_1 \setminus f_1 = \{x\}$ and $x \in \hat{f}_2$. Such a 4-tuple of edges $(f_1, \hat{f}_1, \hat{e}, \hat{f}_2)$ satisfies the properties considered in Lemma 1.4 (vi). Therefore this case cannot arise when $r \geq 7$. If $r \in \{3, 4, 5, 6\}$ then we bound the number of illegal 4-tuples in this case by $O(k_{\text{max}}^2 M_2)$. Hence there are at most $O(\eta k_{\text{max}}^2 M_2)$ illegal 4-tuples $(x_1, x_2, y_1, y_2)$ which fail condition (II') because $f_j \in G(Q')$ for some $j = 1, 2$.

Subtracting these bounds on the number of illegal 4-tuples for $Q'$, we find that there are

$$(r - 1)MM_2 \left(1 + O \left(\frac{k_{\text{max}}^{1+\eta}}{M}\right)\right)$$

4-tuples which correspond to a legal reverse $\ell$-switching in $Q'$.

Combining (2.3) and (2.4) gives

$$\frac{|C_\ell|}{|C_{\ell - 1}|} = \frac{(r - 1)M_2}{2\ell M} \left(1 + O \left(\frac{\ell + k_{\text{max}}^{1+\eta}}{M}\right)\right).$$

(2.5)

We complete the proof by summing over $\ell$ using Lemma 2.1. Let $\ell'$ be the first value of $\ell \leq N$ for which $C_\ell = \emptyset$, or $\ell' = N + 1$ if there is no such value. Define the constant $\beta_\ell$ for $\ell = 1, \ldots, \ell' - 1$ by

$$\frac{|C_\ell|}{|C_{\ell - 1}|} = \frac{(r - 1)M_2 - \beta_\ell \left(k_{\text{max}}^{2+\eta} + k_{\text{max}}(\ell - 1)\right)}{2\ell M}$$

(2.6)

Using (2.5) we see that $\beta_\ell$ is uniformly bounded (independently of $M$). For $1 \leq \ell < \ell'$, define

$$A(\ell) = \frac{(r - 1)M_2 - \beta_\ell k_{\text{max}}^{2+\eta}}{2M}, \quad C(\ell) = \frac{\beta_\ell k_{\text{max}}^{2+\eta}}{2M}.$$ 

If $\beta_\ell \leq 0$ then clearly $A(\ell) > 0$. Now suppose that $\beta_\ell > 0$. Then $C(\ell) > 0$. Since the left hand side of (2.6) is positive, it follows that $A(\ell) - (\ell - 1)C(\ell)$ is positive, which implies
that $A(\ell) > 0$. Therefore $A(\ell) > 0$ for $1 \leq \ell \leq \ell'$. Define $B(\ell) = C(\ell) / A(\ell)$ for $1 \leq \ell < \ell'$. Also define $A(\ell) = B(\ell) = 0$ for $\ell' \leq \ell \leq N$.

Define $A_1, A_2, C_1, C_2$ by taking the minimum and maximum of $A(\ell)$ and $C(\ell)$ over $1 \leq \ell \leq N$, as in Lemma 2.1. Let $A \in [A_1, A_2]$ and $C \in [C_1, C_2]$ and set $\hat{c} = \frac{1}{16}$. Since $A = (r - 1)M_2 / 2M + o(1)$ and $C = o(1)$, we have that $\max\{|A/N|, |C|\} \leq \hat{c}$ for $M$ sufficiently large, by the definition of $N$. Lemma 2.1 applies and says that

$$\sum_{\ell=0}^{N} \frac{|C_\ell|}{|C_0|} = \exp \left( \frac{(r - 1)M_2}{2M} + O \left( \frac{k_{\max}^2 + \eta}{M} \right) \right) + O((e/8)^N).$$

Now $(e/8)^N \leq (e/8)^{\log M} \leq M^{-1}$, and as the sum we are estimating is at least equal to 1, this additive error term is covered by the error term inside the exponential.

Theorem 1.1 now follows immediately, by combining (2.1) and Lemma 2.2.

References

[1] C. Cooper, A. Frieze, M. Molloy and B. Reed, Perfect matchings in random $r$-regular, $s$-uniform hypergraphs, Combinatorics, Probability and Computing 5 (1996), 1–14.

[2] A. Dudek, A. Frieze, A. Ruciński and M. Šileikis, Approximate counting of regular hypergraphs, Preprint, 2013. http://arxiv.org/abs/1303.0400

[3] E. Estrada, J.A. Rodríguez-Velázquez, Subgraph centrality and clustering in complex hyper-networks, Physica A: Statistical Mechanics and its Applications 364 (2006), 581–594.

[4] A. Frieze and P. Melsted, Randomly coloring simple hypergraphs, Information Processing Letters 111 (2011), 848–853.

[5] G. Ghoshal, V. Zlatić, G. Caldarelli and M.E.J. Newman, Random hypergraphs and their applications, Physical Review E 79 (2009), 066118.

[6] C. Greenhill, B.D. McKay and X. Wang, Asymptotic enumeration of sparse 0-1 matrices with irregular row and column sums, Journal of Combinatorial Theory (Series A) 113 (2006), 291–324.

[7] S. Klamt, U.-U. Haus and F. Theis, Hypergraphs and cellular networks, PLoS Comput. Biol. 5 (2009) e31000385.
[8] A.V. Kostochka and M. Kumbhat, Coloring uniform hypergraphs with few edges, *Random Structure and Algorithms* **35** (2009), 348–368.

[9] G. Kuperberg, S. Lovett and R. Peled, Probabilistic existence of regular combinatorial structures, Preprint, 2013, [http://arxiv.org/abs/1303.4295](http://arxiv.org/abs/1303.4295)

[10] B.D. McKay, Asymptotics for symmetric 0–1 matrices with prescribed row sums, *Ars Combinatoria* **19** (1985), 15–25.

[11] B. D. McKay, Subgraphs of dense random graphs with specified degrees, *Combin. Probab. Comput.*, **20** (2011) 413–433.

[12] B. D. McKay and N. C. Wormald, Asymptotic enumeration by degree sequence of graphs of high degree, *European J. Combin.*, **11** (1990) 565–580.

[13] B.D. McKay and N.C. Wormald, Asymptotic enumeration by degree sequence of graphs with degrees $o(n^{1/2})$, *Combinatorica* **11** (1991), 369–383.