Dielectric function with exact exchange contribution in the
electron liquid. II. Analytical expression

Zhixin Qian

Department of Physics, Peking University, Beijing 100871, China

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Abstract

The first-order, in terms of electron-interaction in the perturbation theory, of the proper linear response function $\Pi(k, \omega)$ gives rise to the exchange-contribution to the dielectric function $\epsilon(k, \omega)$ in the electron liquid. Its imaginary part, $Im\Pi_1(k, \omega)$, is calculated exactly. An analytical expression for $Im\Pi_1(k, \omega)$ is derived which after refinement has a quite simple form.

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I. INTRODUCTION WITH CONCLUDING REMARKS

Electronic excitations are one of major subjects in solid state physics [1]; the dielectric function $\epsilon(k, \omega)$ of the homogeneous electron liquid [2–4] has been playing a central role in the description of these excitations. In the preceding paper [5], referred to as I hereafter, the static dielectric function $\epsilon(k, 0)$ with exchange contribution was studied. A very simple expression for $\Pi_1(k, 0)$ the first order, in terms of electron-interaction in the perturbation theory, of the static proper linear response function $\Pi(k, 0)$ in the electron liquid, was derived. In this paper we set as our task to make like development for $\Pi_1(k, \omega)$, its dynamical counterpart. An analytical expression is obtained for $\text{Im}\Pi_1(k, \omega)$, the imaginary part of $\Pi_1(k, \omega)$.

The conceptual importance of $\epsilon(k, \omega)$ [and $\Pi(k, \omega)$] and previous progress made in the study of them have been briefly introduced in I, with emphasis on their static aspect. In general previous works in both of experimental and theoretical respects are enormous. We here limit ourselves to mentioning several of them which bear most close theoretical relation to the present paper [6–33]. Particularly noteworthy is the work by Holas et al in Ref. [23] in which an analytical expression for $\text{Im}\Pi_1(k, \omega)$ had been reported. Equation (2.18) in Ref. [23] deserves fully appreciation, for it is the first analytical expression obtained for $\text{Im}\Pi_1(k, \omega)$ in terms of one-fold integral. Our expression, given as Eq. (62) in Sec. IV, agrees numerically with Eq. (2.18) of Ref. [23]. The correctness of both of them thus should be beyond doubt. It can be hardly denied that the method invented to obtain Eq. (2.18) in Ref. [23] is ingenious. Our expression also in terms of one-fold integral has in contrast the character of simplicity. It is also the belief of the present author that this expression has been obtained in optimal way and the derivation is more or less straightforward. Overall, the exchange contribution included in the dielectric function makes a significant improvement over the random-phase approximation (RPA), as had already been shown in Ref. [23] in several important respects. This will get full confirmation in this series of papers. We must further mention that the singular behavior of $\Pi_1(k, \omega)$ near the characteristic frequencies $\omega_s = (h/2m)|\pm k_F k + k^2/2|$, which had been elucidated in Ref. [23] and apparently had made some negative impression of the many-body perturbation theory on those authors [24], is also confirmed. Indeed explicit expressions of both of the discontinuity jump of $\text{Im}\Pi_1(k, \omega)$ at $\omega = \omega_s$ and the corresponding logarithmic divergence there of its real counterpart are
obtained in this paper, which are presented in Sec. V.

In a series of papers, Brosens et al. [25] investigated the local field correction to the RPA. They calculated the property

\[ G(k, \omega) = -v^{-1}(k) \Pi_1(k, \omega)/\Pi_0^2(k, \omega) \]  

as an approximation to the local field factor [8]. This property is surely not the local field factor including the exact exchange contribution, a fact evidently appreciated by those authors. The latter is instead [according to Eq. (2) in I]

\[ G(k, \omega) = v(k)^{-1} \left[ \frac{1}{\Pi_0(k, \omega) + \Pi_1(k, \omega)} - \frac{1}{\Pi_0(k, \omega)} \right]. \]  

They apparently had never elucidated however, for the benefit of readers, that their approximation, obtained by them from the dynamic-exchange decoupling in the equation of motion for the Wigner distribution function, could be also obtained as an (sub-exchange in the sense explained above) approximation in the perturbation theory. (See also the comments made in Ref. [23] on the earlier ones of the series papers by Brosens et al.) They did point out definitely that several forms obtained before and after them [21, 26] were very close to or virtually identical to theirs. The relation between the theory of Rajagopal [21] and that by Tripathy and Mandal [26] was also pointed out in Ref. [26]. Tripathy and Mandal further elucidated the relation between their theory and that proposed in Ref. [19]. A critical analysis of the relation of the latter (in the static case) to the first order theory was given earlier in Ref. [20]. Finally we wish to mention that Richardson and Ashcroft [31] also had obtained an analytical expression for \( \Pi_1(k, \omega) \) but with \( \omega \) to be imaginary. Investigations beyond the first order had also been attempted in general, in Refs. [24, 30, 31] for instance, but mainly in limiting cases, in Refs. [13, 15, 18, 23, 29, 32] again for instance.

Expression (62) together with (64) for \( \text{Im} \Pi_1(k, \omega) \) is the main result of this paper. [We remind the reader that \( \text{Im} \Pi(k, \omega) \) determines fully \( \Pi(k, \omega) \), for its real conjugation can be determined from it via the dispersion relation.] The aim of this series of papers is to achieve a (relatively speaking) complete and final understanding of the role of the exchange contribution in the dielectric function, taking advantage of the explicit form of expression (62) and that for \( \Pi_1(k, 0) \) (Eq. (3) in I [5, 34]). As an example, we mention that it has been traditionally believed that \( \text{Im} \Pi_1(k, \omega) \) has the limiting form of \( \sim \omega \) for small \( \omega \) [18, 32, 33]. In fact, it was claimed by Mahan [3] and has been commonly accepted that this must be true.
also for $Im \epsilon(k, \omega)$, the imaginary part of $\epsilon(k, \omega)$, in general. We find that this is not the case and $Im \Pi_1(k, \omega)$ actually has the limiting form of $\sim \omega \ln \omega$, (details of which will be presented in a subsequent paper.) The deep subtlety of many-body effects often reveals itself against our intuitive understanding, and does so most definitely and convincingly in the perturbation theory indeed. We end the introduction by further remarking that calculations in the many-body perturbation theory are conventionally known to be notoriously complicated. In this sense, our expression appears quite simple. The derivation to obtain it has also been carried out in a quite manageable manner. Perhaps this is an enlightening revelation about the many-body perturbation theory.

We give our derivation in Sec. III, after presenting the starting formalism in Sec. II.

II. STARTING FORMALISM

The Feynman-diagrammatically obtained expression for $\Pi_1(k, \omega)$ has been shown as Eq. (4) in I. It is, as is well known, the sum of two contributions:

$$\Pi_1(k, \omega) = \Pi^SE_1(k, \omega) + \Pi^Ex_1(k, \omega);$$

(3)

$\Pi^SE_1(k, \omega)$ and $\Pi^Ex_1(k, \omega)$ arise, respectively, from the self-energy diagrams and the exchange diagram. We put down below the explicit expressions for them:

$$\Pi^SE_1(k, \omega) = \frac{2}{\hbar^2} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\mathbf{p}'}{(2\pi)^3} v(\mathbf{p} - \mathbf{p}') \frac{(n_p - n_{p+k})(n_{p'} - n_{p'+k})}{\omega + \omega_p - \omega_{p+k} + i0^+},$$

(4)

$$\Pi^Ex_1(k, \omega) = -\frac{2}{\hbar^2} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\mathbf{p}'}{(2\pi)^3} v(\mathbf{p} - \mathbf{p}') \frac{(n_p - n_{p+k})(n_{p'} - n_{p'+k})}{\omega + \omega_p - \omega_{p+k} + i0^+}.$$  

(5)

(See also Refs. 9, 23, 35.) The notations in this paper all follow I, and here we have explicitly written $\hbar$. With some manipulation, $\Pi^SE_1(k, \omega)$ can be cast in the following form:

$$\Pi^SE_1(k, \omega) = \frac{2m^2}{(2\pi)^3 \hbar^2} \int d\mathbf{p} \int d\mathbf{p}' n_{p-k/2} n_{p'-k/2} \frac{1}{(m\omega - \hbar \mathbf{p} \cdot \mathbf{k} + i0^+)^2} + \frac{1}{(m\omega + \hbar \mathbf{p} \cdot \mathbf{k} + i0^+)^2},$$

(6)
and $\Pi_{1}^{Ex}(k, \omega)$:

$$
\Pi_{1}^{Ex}(k, \omega) = -\frac{2m^2}{(2\pi)^6 \hbar^2} \int dp \int dp' n_{p-k/2} n_{p'-k/2} \left[ v(p - p') \left( \frac{1}{(m\omega - \hbar p \cdot k + i0^+)(m\omega - \hbar p' \cdot k + i0^+)} \right. \\
+ \frac{1}{(m\omega + \hbar p \cdot k + i0^+)(m\omega + \hbar p' \cdot k + i0^+)} \right] \\
- v(p + p') \left( \frac{1}{(m\omega - \hbar p \cdot k + i0^+)(m\omega + \hbar p' \cdot k + i0^+)} \right. \\
+ \frac{1}{(m\omega + \hbar p \cdot k + i0^+)(m\omega - \hbar p' \cdot k + i0^+)} \right] \right]. 
$$

(7)

The imaginary parts of them can be obtained, respectively, as

$$
Im\Pi_{1}^{SE}(k, \omega) = \frac{m}{(2\pi)^6 \hbar^2} \frac{\partial}{\partial \omega} \int dp \int dp' n_{p-k/2} n_{p'-k/2} \left[ v(p - p') - v(p + p') \right] \left[ \delta(m\omega - \hbar p \cdot k) + \delta(m\omega + \hbar p \cdot k) \right], 
$$

(8)

and

$$
Im\Pi_{1}^{Ex}(k, \omega) = \frac{2m^2}{(2\pi)^6 \hbar^2} \int dp \int dp' n_{p-k/2} n_{p'-k/2} \left[ v(p - p') \left( \frac{1}{m\omega - \hbar p \cdot k} \delta(m\omega - \hbar p \cdot k) + \frac{1}{m\omega + \hbar p' \cdot k} \delta(m\omega + \hbar p \cdot k) \right) \\
- v(p + p') \left( \frac{1}{m\omega - \hbar p' \cdot k} \delta(m\omega + \hbar p \cdot k) + \frac{1}{m\omega + \hbar p' \cdot k} \delta(m\omega - \hbar p \cdot k) \right) \right]. 
$$

(9)

These forms serve our purpose best.

III. DERIVATION

A. $Im\Pi_{1}^{SE}(k, \omega)$

The property $\Pi_1(k, \omega)$ depends only on the magnitude of $k$ in a uniform system, so it may be written as $\Pi_1(k, \omega)$. We first define a dimensionless quantity: $\Omega = m\omega/\hbar k_F^2$. From now on throughout the paper we put $k$ in units of $k_F$, i.e., $k$ will always be dimensionless.

The computation for $Im\Pi_{1}^{SE}(k, \omega)$ can be made very simple. The integral over the variable $p'$ in Eq. (8) can be carried out first, which leads to

$$
Im\Pi_{1}^{SE}(k, \omega) = \frac{m^2 e^2}{2\pi^2 \hbar^4} \frac{\partial}{\partial \Omega} \int_{-a}^{b} dz \int_{0}^{\lambda} dx \left[ \delta(\Omega - k \Delta) + \delta(\Omega + k \Delta) \right] \\
\left[ F(\sqrt{z^2 + x - k \Delta + k^2/4}) - F(\sqrt{z^2 + x + k \Delta + k^2/4}) \right],
$$

(10)
where
\[ F(q) = \frac{1}{4\pi} \int dp \frac{n_p}{|p - q|^2}. \] (11)

Explicitly,
\[ F(q) = \frac{1}{2} + \frac{1 - q^2}{4q} \ln \left| \frac{1 + q}{1 - q} \right|. \] (12)

We mention once again that the notations here follow I. The integration over \( z \) in Eq. (10) is trivial. After performing it, one gets
\[
\text{Im} \Pi^{SE}_1(k, \omega) = m^2 e^2 \frac{1}{2\pi^2 \hbar^2 k^2} \left[ \theta\{(b - \Omega/k)(a + \Omega/k)\} H^{SE}(k, \Omega/k) \\
- \theta\{(b + \Omega/k)(a - \Omega/k)\} H^{SE}(k, -\Omega/k) \right],
\] (13)

with
\[
H^{SE}(k, z) = \frac{\partial}{\partial z} \int_0^\lambda dx \left[ F(\sqrt{x} - \lambda + 1) - F(\sqrt{x} - \lambda + 1 + 2kz) \right].
\] (14)

The \( H^{SE}(k, z) \) in the preceding equation can be readily refined into
\[
H^{SE}(k, z) = (k - 2z)F(\sqrt{-\lambda + 1}) + (k + 2z)F(\sqrt{-\lambda + 1 + 2kz}) - 2kF(\sqrt{1 + 2kz}).
\] (15)

Explicitly,
\[
H^{SE}(k, z) = \frac{1}{2} \left[ \frac{2k^2 z}{\sqrt{C_0}} Y(z) - \lambda W_1(z) - \tilde{\lambda} W_2(z) \right].
\] (16)

In Eq. (16) we have introduced (newly) the symbol \( \tilde{\lambda} = (b + z)(a - z) \).

B. \( \text{Im} \Pi^{Ex}_1(k, \omega) \)

Our labor lies mainly in the evaluation of \( \text{Im} \Pi^{Ex}_1(k, \omega) \) expressed in (9). Following paper I, we first carry out the integrals over the azimuthal angular variables of \( p \) and \( p' \). After that, we obtain
\[
\text{Im} \Pi^{Ex}_1(k, \omega) = m^2 e^2 \frac{1}{4\pi^2 \hbar^4} \int_{-\alpha}^\beta dz' \int_{-\alpha}^\beta dz \left[ \left\{ \frac{1}{\Omega - kz'} \delta(\Omega - kz) + \frac{1}{\Omega + kz'} \delta(\Omega + kz) \right\} L(\beta^2) \\
- \left\{ \frac{1}{\Omega - kz'} \delta(\Omega + kz) + \frac{1}{\Omega + kz'} \delta(\Omega - kz) \right\} L(\alpha^2) \right].
\] (17)
We then, taking advantage of the presence of the $\delta$—function, reduce the two-fold integral to one-fold. The $Im\Pi^{Ex}_1(k, \omega)$ becomes thus

$$Im\Pi^{Ex}_1(k, \omega) = -\frac{m^2 e^2}{4\pi^2 \hbar^4 k^2} \left[ \theta \{(b - \Omega/k)(a + \Omega/k)\} H^{Ex}_1(k, \Omega/k) - \theta \{(b + \Omega/k)(a - \Omega/k)\} H^{Ex}_1(k, -\Omega/k) \right],$$

with the function $H^{Ex}_1(k, z)$ defined as

$$H^{Ex}_1(k, z) = \int_{-a}^{b} dz' \left[ \frac{1}{\alpha} L(\alpha^2) - \frac{1}{\beta} L(\beta^2) \right].$$

The function $L$ has been given in Eq. (9) in I and in Ref. [36]. There are several components in it, and we separate them in the evaluation of the integral in Eq. (19). Accordingly we write $H^{Ex}_1(k, z)$ in the following manner:

$$H^{Ex}_1(k, z) = H^{Ex}_0(k, z) + H^{Ex}_1(k, z) + H^{Ex}_{23}(k, z),$$

with

$$H^{Ex}_0(k, z) = \int_{-a}^{b} dz' \left[ \lambda W(z) + (ab - z^2)W_2(z) - k \right],$$

$$H^{Ex}_1(k, z) = \int_{-a}^{b} dz' \left[ \frac{1}{\alpha} \sqrt{R(z, z')} - \frac{1}{\beta} |\beta| \right],$$

and

$$H^{Ex}_{23}(k, z) = \int_{-a}^{b} dz' \left[ \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \lambda \ln |4\lambda| - 2\lambda' \left( \frac{1}{\alpha} \ln |\alpha| - \frac{1}{\beta} \ln |\beta| \right) \right. \right.$$

$$\left. - \frac{1}{\alpha} \left( \lambda \ln |\alpha^2 + \lambda' - \lambda - 2\sqrt{R(z, z')}| - \lambda' \ln |\alpha^2 - \lambda' + \lambda + 2\sqrt{R(z, z')}| \right) \right] + \frac{1}{\beta} \left( \lambda \ln |\beta(k - 2z) + 2|\beta|| - \lambda' \ln |\beta(k - 2z') + 2|\beta|| \right).$$

The two terms of $J_2$ and $J_3$ in Eq. (11) of I were combined, for the simplicity of the computation, into one term [denoted as $J_{23}$ in Eq. (43) there]. The $H^{Ex}_{23}(k, z)$ here follows suit.

The evaluation for $H^{Ex}_0(k, z)$ and $H^{Ex}_1(k, z)$ is a routine job. It is quite straightforward to get the following result:

$$H^{Ex}_0(k, z) = (2 \ln 2 + 1)[\lambda W_1(z) + (ab - z^2)W_2(z) - k] - k,$$

(24)
and

\[ H_1^{Ex}(k, z) = k[2 - zW_2(z) - z(2z + k)C_0^{-1/2}Y(z)]. \]  (25)

We next attack \( H_{23}^{Ex}(k, z) \). With a little algebra, we rewrite Eq. (23) in the following form:

\[ H_{23}^{Ex}(k, z) = -[W_1(z) + W_2(z)]\lambda \ln |4\lambda| - 2\zeta_1(z) - \zeta_1(-z) + \zeta_2(z) - \zeta_3(z), \]  (26)

with

\[ \zeta_1(z) = \int_{-a}^{b} dz' \frac{1}{\alpha} \ln |\alpha|, \]  (27)

\[ \zeta_2(z) = \int_{-a}^{b} dz' \frac{1}{\beta} \left[ \lambda \ln |\beta(k - 2z) + 2|\beta|| - \lambda' \ln |k - 2z' + 2\beta/|\beta|| \right], \]  (28)

and

\[ \zeta_3(z) = \int_{-a}^{b} dz' \frac{1}{\alpha} \left[ \lambda \ln |\alpha^2 + \lambda' - \lambda - 2\sqrt{R(z, z')}| - \lambda' \ln |\alpha^2 + \lambda + 2\sqrt{R(z, z')}| \right]. \]  (29)

The reader should not confuse the functions \( \zeta_n(z) \) here with the Riemann's function \( \zeta(n) \) that appeared in I. The evaluation for \( \zeta_1(z) \) and \( \zeta_2(z) \) is a little tedious but clearly straightforward. We thus present only the results:

\[ \zeta_1(z) = \frac{1}{2} [(2z + k)(\ln |\tilde{\lambda}| - 3) + \{\tilde{\lambda}(3 - \ln |\tilde{\lambda}|) - 2\}W_2(z)], \]  (30)

and

\[ \zeta_2(z) = \frac{1}{2} [\lambda(\ln \lambda - 2 \ln 2 - 3) + 2W_1(z) + \frac{1}{2}(2z - k)(\ln \lambda - 6 \ln 2 + 3) - \lambda v_1(z)], \]  (31)

where

\[ v_1(z) = \int_{-a}^{z} dz' \frac{1}{z' - b} \ln \beta + \int_{z}^{b} dz' \frac{1}{z' + a} \ln \beta. \]  (32)

We now turn to \( \zeta_3(z) \). We first rewrite Eq. (29) as

\[ \zeta_3(z) = \lambda \zeta_{3a}(z) - \zeta_{3b}(z), \]  (33)

with

\[ \zeta_{3a}(z) = \int_{-a}^{b} dz' \frac{1}{\alpha} \ln |\alpha^2 + \lambda' - \lambda - 2\sqrt{R(z, z')}|, \]  (34)
\[ \zeta_{3b}(z) = \int_{-a}^{b} \frac{dz' \lambda'}{\alpha} \ln |\alpha^2 - \lambda' + \lambda + 2\sqrt{R(z, z')}|. \] (35)

The evaluation of \( \zeta_{3a}(z) \) is also a routine job. It can be effected with partial integration. One gets in this manner

\[ \zeta_{3a}(z) = -W_2(z) \ln |2\lambda| - \int_{-a}^{b} dz' D_1(z, z') \ln |\alpha|, \] (36)

where

\[ D_1(z, z') = \frac{\partial}{\partial z'} \ln |\alpha^2 + \lambda' - \lambda - 2\sqrt{R(z, z')}|. \] (37)

Explicitly,

\[ D_1(z, z') = \frac{1}{\alpha} \left[ 1 - \frac{kz}{\sqrt{R(z, z')}} \right]. \] (38)

The equation (36) can now be readily refined into

\[ \zeta_{3a}(z) = \frac{1}{2} [\ln |\tilde{\lambda}| - 2 \ln |2\lambda|] W_2(z) + kz v_2(z), \] (39)

with

\[ v_2(z) = \int_{-a}^{b} dz' \frac{1}{\alpha \sqrt{R(z, z')}} \ln |\alpha|. \] (40)

The integral on the right hand side of Eq. (35) can also be effected with partial integration. To this end, we employ the following identity:

\[ \frac{\lambda'}{\alpha} dz' = \frac{1}{2} d[2\tilde{\lambda} \ln |\alpha| + \lambda' + (2z + k) \alpha], \] (41)

with the symbol \( d \) here denoting the differential operating only on the variable \( z' \). We perform in this manner the partial integration and get for \( \zeta_{3b}(z) \):

\[ \zeta_{3b}(z) = \frac{1}{2} (2z + k)[4 \ln 2 + (b + z) \ln r_1 + (a - z) \ln r_2] + \tilde{\lambda}[-2W_2(z) \ln 2 + v_0(z)] \]

\[ - \frac{1}{2} \int_{-a}^{b} dz' [2\tilde{\lambda} \ln |\alpha| + \lambda' + (2z + k) \alpha] D_2(z, z'), \] (42)

where

\[ v_0(z) = \ln |b + z| \ln r_1 - \ln |a - z| \ln r_2, \] (43)
and
\[ D_2(z, z') = \frac{\partial}{\partial z'} \ln |\alpha^2 - \lambda' + \lambda + 2\sqrt{R(z, z')}|. \] (44)

We have introduced the following symbols:
\[ r_1 = \sqrt{R(z, b)}, \quad r_2 = \sqrt{R(z, -a)} \] (45)
in Eqs. (42) and (43). Explicitly
\[ D_2(z, z') = \frac{1}{\alpha} \left[ 1 - \frac{kz}{\sqrt{R(z, z')}} \right] - \frac{1}{2(b - z')} \left[ 1 - \frac{r_1}{\sqrt{R(z, z')}} \right] + \frac{1}{2(a + z')} \left[ 1 - \frac{r_2}{\sqrt{R(z, z')}} \right]. \] (46)

Making the use of Eq. (46) in Eq. (42), the expression for \( \zeta_{3b}(z) \) can be organized in the form:
\[ \zeta_{3b}(z) = \left[ (2z + k) \{ 4 \ln 2 + (b + z) \ln r_1 + (a - z) \ln r_2 \} + \tilde{\lambda} \{ -4W_2(z) \ln 2 + 2v_0(z) \ight. \\
+ W_2(z) \ln |\tilde{\lambda}| + 2kzv_2(z) + v_3(z) \} - \tilde{\zeta}_{3b}(z)]/2, \] (47)
where \( v_3(z) \) and \( \tilde{\zeta}_{3b}(z) \) are defined as
\[ v_3(z) = \int_{-a}^{b} dz' \ln |\alpha| \left[ \frac{1}{b - z'} \left( 1 - \frac{r_1}{\sqrt{R(z, z')}} \right) + \frac{1}{a + z'} \left( -1 + \frac{r_2}{\sqrt{R(z, z')}} \right) \right], \] (48)
and
\[ \tilde{\zeta}_{3b}(z) = \int_{-a}^{b} dz'[\lambda' + (2z + k)\alpha]D_2(z, z'), \] (49)
respectively. The integral on the right hand side of Eq. (49) [with \( D_2(z, z') \) given explicitly in Eq. (46)] is basic, although it looks somewhat tedious. With some algebra, we can put it in the following form:
\[ \tilde{\zeta}_{3b}(z) = -\tilde{\lambda}W_2(z) - (z/2)[3(k^2 - 1) + C_0V_0(z) + \lambda_0V_1(z)] - k\tilde{\lambda}V_{-1}(z, -z) \\\n- \frac{1}{2} (2z + k) \left[ -10 + 4C_0V_0(z) + (z + b) \left( \int_{-a}^{b} dz' \frac{1}{b - z'} + r_1V_{-1}(z, b) \right) \right. \\
- (z - a) \left( \int_{-a}^{b} dz' \frac{1}{a + z'} - r_2V_{-1}(z, -a) \right) \right], \] (50)
where
\[ V_n(z) = \int_{-a}^{b} dz' \frac{z'^n}{\sqrt{R(z, z')}} \] (51)
for \( n = 0, 1, \) and
\[
V_{-1}(z, x) = \int_{-a}^{b} \frac{dz'}{z' - x} \frac{1}{\sqrt{R(z, z')}} .
\]  
(52)
Explicitly,  
\[
V_0(z) = 2C_0^{-1/2}Y(z), \quad V_1(z) = [2z + k - z(kz + 1 - k^2/2)V_0(k, z)]C_0^{-1},
\]  
(53)
and
\[
V_{-1}(z, -z) = -W_2(z)/kz.
\]  
(54)

We note that the sum in each of the big curve bracket in Eq. (50) is well defined, though their respective components are not. The reader excuses us for the sake of a compact presentation. Indeed,
\[
\int_{-a}^{b} dz' \frac{1}{b - z'} + r_1V_{-1}(z, b) = 2 \ln |kz|/r_1,
\]  
(55)
and
\[
\int_{-a}^{b} dz' \frac{1}{a + z'} - r_2V_{-1}(z, -a) = 2 \ln |kz|/r_2.
\]  
(56)

In virtue of the foregoing results, the integral on the right hand side of Eq. (50) has now been fully carried out. The final result for \( \bar{\zeta}_{3b}(z) \) can after refinement be written as
\[
\bar{\zeta}_{3b}(z) = (2z + k)[6 - 2(3kz + 1)C_0^{-1/2}Y(z) + (z + b) \ln r_1 - (z - a) \ln r_2 - 2 \ln |kz|].
\]  
(57)

The substitution of Eq. (57) into Eq. (47) will give the result for \( \zeta_{3b}(z) \). Further substitution of thus obtained result for \( \zeta_{3b}(z) \) and the previously obtained one for \( \zeta_{3a}(z) \) in Eq. (39) into Eq. (33) then yields the final result for \( \zeta_3(z) \), which turns out to be
\[
\zeta_3(z) = [kz \ln |\bar{\lambda}| - \lambda \ln |2\lambda| + 2\bar{\lambda} \ln 2]W_2(z) + 2k^2z^2v_2(z) - (\bar{\lambda}/2)[2v_0(z) + v_3(z)]
- (2z + k)[2 \ln 2 - 3 + (3kz + 1)C_0^{-1/2}Y(z) + \ln |kz|].
\]  
(58)

One then substitutes \( \zeta_1(z) \) expressed in Eq. (30), \( \zeta_2(z) \) in Eq. (31), and \( \zeta_3(z) \) in the above equation into Eq. (26) to get
\[
H_{23}^{Ez}(k, z) = (-2z + 5k) \ln 2 + \mu_1(k, z) + \mu_1(k, -z) - 2(1 + \ln 2)[\lambda W_1(z) + \bar{\lambda} W_2(z)]
- [2kz \ln 2 + (kz - \bar{\lambda}) \ln |\bar{\lambda}|]W_2(z) - \lambda v_1(z) + (\bar{\lambda}/2)[2v_0(z) + v_3(z)]
+ (2z + k)[(3kz + 1)C_0^{-1/2}Y(z) + \ln |kz|] - 2k^2z^2v_2(z),
\]  
(59)
where
\[ \mu_1(k, z) = (2z - k) \ln \lambda + [2 - (1 + \ln 2)\lambda]W_1(z). \] (60)

One then advances further to add [according to Eq. (20)] \( H_0^{E_x}(k, z) \) of Eq. (24), \( H_1^{E_x}(k, z) \) of Eq. (25), and \( H_2^{E_x}(k, z) \) of Eq. (59) to get the result for \( H^{E_x}(k, z) \) which can be in the final form written as
\[ H^{E_x}(k, z) = (-2z + 3k) \ln 2 + \mu_1(k, z) + (\lambda - k)W_2(z) \ln |\lambda| \]
\[ -\lambda W_1(z) - \tilde{\lambda} W_2(z) - \lambda v_1(z) + (\tilde{\lambda}/2)[2v_0(z) + v_3(z)] \]
\[ + (2z + k)[\sqrt{C_0Y(z) + \ln |kz|} - 2k^2z^2v_2(z)]. \] (61)

IV. ANALYTICAL RESULT

In virtue of Eq. (3), \( \text{Im} \Pi_1(k, \omega) \) can be obtained from Eq. (13) and Eq. (18) as
\[ \text{Im} \Pi_1(k, \omega) = \frac{m^2 e^2}{(2\pi)^2 \hbar^4} \frac{1}{k^2} \left[ \theta(1 - \nu_+^2)H(k, \Omega/k) - \theta(1 - \nu_-^2)H(k, -\Omega/k) \right], \] (62)
with \( \nu_+ = \Omega/k - k/2 \), \( \nu_- = -\Omega/k - k/2 \), and
\[ H(k, z) = 2H^{SE}(k, z) - H^{E_x}(k, z). \] (63)

The substitution from Eqs. (16) and (61) will give the result for \( H(k, z) \), which we then further refine into the following form:
\[ H(k, z) = -(2zC_0 + k)C_0^{-1/2}Y(z) + (2z - 3k) \ln 2 - (k + 2z) \ln |kz| \]
\[ + (kz - \bar{\lambda}) \ln |\bar{\lambda}|W_2(z) - \mu_1(k, z) - \mu_1(k, -z) - \mu_2(k, z) + \mu_2(-k, -z) \]
\[ + 2k^2z^2 \int_{-a}^{b} dz' \frac{1}{\alpha \sqrt{R(z, z')}} \ln |\alpha|, \] (64)
where
\[ \mu_2(k, z) = \bar{\lambda} \ln |z + b| \ln |(k + 1)z + b| - \lambda \int_{z}^{b} dz' \frac{1}{z' + a} \ln |\beta| \]
\[ + \frac{1}{2} \bar{\lambda} \int_{-a}^{b} dz' \frac{1}{b - z'} \left[ 1 - \frac{(k + 1)z + b}{\sqrt{R(z, z')}} \right] \ln |\alpha|. \] (65)
V. SINGULARITY OF $\Pi_1(k, \omega)$ AT $\omega = \omega_s$

One can immediately see that $\text{Im}\Pi_1(k, \omega)$ has the same nonvanishing region as $\text{Im}\Pi_0(k, \omega)$, the Lindhard function [2, 3, 7]. In other words, the region of the single particle-hole continuum remains unchanged with the inclusion of the exchange contribution. The long-wavelength plasmon which has zero linewidth in RPA up to wavevector $k_c$, at which the damping sets in, accordingly remains up to $k_c$ infinitely robust against exchange effect. This truth has been recognized before [13–15, 18, 23, 29, 30, 32]. Such a distinctly drawn conclusion, if understood in an appropriate manner, must also be appreciated as one of the merits of the perturbation theory.

While $\text{Im}\Pi_0(k, \omega)$ approaches to zero on the edge of the single particle-hole continuum, $\text{Im}\Pi_1(k, \omega)$ shows a discontinuity jump there. In other words, $\Pi_1(k, \omega)$ exhibits singular behavior at $\omega = \omega_s$ with $\omega_s = (\hbar k_F^2 / 2m) |\pm k + k^2/2|$. This singularity was noticed by Glick [12] before Holas et al. [23], and also by Awa et al. [28] after them, and Holas et al. had made the most elaborate investigation of it. In fact, all of the three groups of authors had adopted a similar approach in order to remove it. The jump discontinuity, defined as $\Delta_s(k) = \text{Im}\Pi_1(k, \omega_s^+) - \text{Im}\Pi_1(k, \omega_s^-)$, can be explicitly calculated by the use of Eq. (62). For $\omega_s = (\hbar k_F^2 / 2m)(k + k^2/2)$,

$$\Delta_s(k) = \frac{m^2 e^2}{2\pi^2 \hbar^4} \frac{b}{1 + k} \ln \left| \frac{2b}{k} \right|;$$

(66)

and, for $\omega_s = (\hbar k_F^2 / 2m)| - k + k^2/2|$,

$$\Delta_s(k) = -\frac{m^2 e^2}{2\pi^2 \hbar^4} \frac{a}{1 - k} \ln \left| \frac{2a}{k} \right|.$$  

(67)

The discontinuity in $\text{Im}\Pi_1(k, \omega)$ gives rise to a logarithmic divergence in $\text{Re}\Pi_1(k, \omega)$, which has the following form (to the accuracy of the leading logarithmic order):

$$\text{Re}\Pi_1(k, \omega) = \frac{1}{\pi} \Delta_s(k) \ln \left| 2m(\omega - \omega_s) / \hbar k_F^2 \right|$$

(68)

for $\omega \to \omega_s$.

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