On the linearized system of equations for the condensate-normal fluid interaction near the critical temperature.

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Abstract: The Cauchy problem for the linearization of a system of equations arising in the kinetic theory of a condensed gas of bosons near the critical temperature around one of its equilibria is solved for radially symmetric initial data. It is proved that the linearized system has global classical solutions that satisfy the natural conservation laws for a large set of initial data. Some regularity properties of the solutions and their long time asymptotic behavior are described.

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1 Introduction

Correlations between the superfluid component and the normal fluid part in a uniform condensed Bose gas, at temperature below but close to the condensation temperature, and for a small number density of condensed atoms, may be described by the equation

\[ \frac{\partial n}{\partial t}(t,p) = n_c(t)I_3(n(t))(p) \quad t > 0, \ p \in \mathbb{R}^3, \]  

(1.1)

\[ I_3(n)(p) = \int_{(\mathbb{R}^3)^2} \left[ R(p,p_1,p_2) - R(p_1,p,p_2) - R(p_2,p_1,p) \right] dp_1 dp_2. \]  

(1.2)

\[ R(p,p_1,p_2) = \left[ \delta(|p|^2 - |p_1|^2 - |p_2|^2)\delta(p - p_1 - p_2) \right] \times \left[ n_1 n_2 (1 + n) - (1 + n_1)(1 + n_2)n \right], \]  

(1.3)

where \( n(t,p) \) represents the density of particles in the normal gas that at time \( t > 0 \) have momentum \( p \) and \( n_c(t) \) is the density of the condensate at time \( t \), that satisfies

\[ \frac{dn_c}{dt}(t) = -n_c(t) \int_{\mathbb{R}^3} I_3(n(t))(p) dp \quad t > 0. \]  

(1.4)
Equation (1.1) was first derived in [7] and [15] and their treatment was afterwards extended to a trapped Bose gas. By including Hartree–Fock corrections to the energy of the excitations the so called ZNG system was obtained (cf. [13]). On the interest of system (1.1), (1.4) for the description of condensed Bose gases see also [20, 18, 14]. Other theoretical models do exist to describe Bose gases in presence of a condensate (cf. [17]) but ZNG system, and (1.1), (1.4) in particular, are very appealing by their simplicity and are well suited for analytical PDE methods.

The two following functions of time,

\[ n(t) = \int_{\mathbb{R}^3} n(t, p) \, dp, \quad E(t) = \int_{\mathbb{R}^3} n(t, p) |p|^2 \, dp \] (1.5)

give respectively the total number of particles and the total energy of the normal fluid part in the gas, with density function \( n(t, p) \). The total number of particles at time \( t \) in the system condensate-normal fluid is \( n_c(t) + n(t) \) and its total energy is \( E(t) \). It formally follows from (1.1), (1.4) that these two quantities are constant in time:

\[ E(t) = E(0) \quad \text{and} \quad n_c(t) + n(t) = n_c(0) + n(0) \quad \text{for all} \quad t > 0. \]

This corresponds to the conservation of the total mass and energy property that is satisfied by the particle system in the physical description (cf. for example [7]). It is also well known that equation (1.1) has a family of non trivial equilibria,

\[ n_0(p) = \left( e^{\beta |p|^2} - 1 \right)^{-1} \] (1.6)

where the mass of the particles is taken to be \( m = 1/2 \) and \( \beta \) is a positive constant related to the temperature of the gas whose particle’s density is at the equilibrium \( n_0 \). It is easily checked that \( R(p, p_k, p_\ell) \equiv 0 \) in (1.3) for \( n = n_0 \).

Our purpose is to prove the existence of classical solutions to the Cauchy problem for the “radially symmetric linearization” of (1.1)–(1.4) around an equilibrium \( n_0 \), and describe some of their properties. Such linearization is deduced through the change of variables (cf. [8, 10])

\[ n(t, p) = n_0(p) + n_0(p)(1 + n_0(p)) \Omega(t, |p|) = n_0(p) + \frac{\Omega(t, |p|)}{4 \sinh^2 \left( \frac{\beta |p|^2}{2} \right)} \] (1.7)

\[ x = \frac{\sqrt{\beta} |p|}{\sqrt{2}} = \frac{\sqrt{\beta} k}{\sqrt{2}}, \quad u(t, x) = \frac{\Omega(t, |p|)}{k^2}, \] (1.8)

and keeping only linear terms with respect to \( u \) in (1.1). Since equation (1.1) is linear with respect to \( n_c \) its linearization (1.10) follows by just keeping the terms of \( I_3(n_0(p) + n_0(p)(1 + n_0(p))|p|^2u(t, |p|)) \) that are linear with respect to \( u \). It finally reads, for dimensionless variables in units which minimize the number of prefactors

\[ \frac{\partial u}{\partial t}(t, x) = p_c(t)\mathcal{L}(u(t)) \] (1.9)

\[ \mathcal{L}(u(t)) = \int_0^\infty (u(t, y) - u(t, x)) M(x, y) \, dy \] (1.10)

\[ \frac{dp_c}{dt}(t) = -p_c(t) \int_0^\infty \int_0^\infty W(x, y)(u(t, y) - u(t, x)) y^4 x^2 \, dy \, dx \] (1.11)
for where, for all $x > 0$, $y > 0$, $x \neq y$,

\[
M(x, y) = \left( \frac{1}{\sinh |x^2 - y^2|} - \frac{1}{\sinh(x^2 + y^2)} \right) \frac{y^2 \sinh x^2}{x^2 \sinh y^2},
\] (1.12)

\[
W(x, y) = \frac{M(x, y)}{(\sinh x^2)^2} \frac{x^2}{y^2} \left( \frac{1}{\sinh |x^2 - y^2|} - \frac{1}{\sinh(x^2 + y^2)} \right) \frac{1}{xy \sinh x^2 \sinh y^2}. \tag{1.13}
\]

\[
\Omega(0) = \frac{1}{k L_I 3} \int_{-\infty}^{\infty} \left( \mathcal{U}(k, k') \Omega(t, k') - \mathcal{V}(k, k') \Omega(t, k) \right) k'^2 dk',
\] (1.16)

\[
\mathcal{U}(k, k') = \frac{16 n_c a^2}{k k'} \left[ \theta(k - k') \times n_0(\omega(k)) [1 + n_0(\omega(k'))][1 + n_0(\omega(k) - \omega(k'))] + (k \leftrightarrow k') \right]
\]

\[
- n_0(\omega(k) + \omega(k')) [1 + n_0(\omega(k))][1 + n_0(\omega(k'))],
\] (1.17)

\[
\mathcal{V}(k, k') = \frac{16 n_c a^2}{k k'} \left[ \theta(k - k') \times n_0(\omega(k)) [1 + n_0(\omega(k'))][1 + n_0(\omega(k) - \omega(k'))] + (k \leftrightarrow k') \right]
\] (1.18)

where $a$ is the s-wave scattering length, $k = |p|$ and $k' = |p'|$. The functions $\mathcal{U}(k, k')$ and $\mathcal{V}(k, k')$ have a non integrable singularity along the diagonal $k = k'$. However, these singularities cancel each other when the two terms are combined as in (1.16) as far as it is assumed that, for all $t > 0$, $\Omega(t) \in C^\alpha(0, \infty)$ for some $\alpha > 0$. But the integrand $(\mathcal{U}(k, k') \Omega(t, k') - \mathcal{V}(k, k') \Omega(t, k))$ can not be split as for example in the linearization of Boltzmann equations for classical particles. However an explicit calculation shows that, for all $k > 0$,

\[
L_{I_3}(\omega)(k) = \int_0^\infty \left( \frac{\mathcal{U}(k, k') k'^2}{k^2} \Omega(t, k') - \frac{\mathcal{V}(k, k') k^2}{k^2} \right) k'^2 dk' = 0
\] (1.19)

from where we deduce, for all $k > 0$,

\[
\int_0^\infty \left( \frac{\mathcal{U}(k, k') k'^2}{k^2} \Omega(t, k') - \frac{\mathcal{V}(k, k') k^2}{k^2} \right) k'^2 dk' = \frac{\Omega(t, k)}{k} L_{I_3}(\omega)(k) = 0.
\]
We may then write,
\[
L_{I_3}(\Omega(t)) = \int_0^\infty \left( \mathcal{W}(k,k')\Omega(t,k') - \mathcal{V}(k,k')\Omega(t,k) \right) k'^2 dk' \\
= \int_0^\infty \mathcal{W}(k,k') \left( \frac{\Omega(t,k')}{k'^2} - \frac{\Omega(t,k)}{k^2} \right) k'^4 dk'
\]
Since equation (1.4) is linear with respect to \( n_c \) its linearization (1.11) follows by just keeping the terms of \( I_3(n_0(p) + n_0(p)(1 + n_0(p))|p|^2 u(t,|p|)) \) that are linear with respect to \( u \). The linearized system reads then,
\[
n_0(1 + n_0)\frac{\partial \Omega(t,k)}{\partial t} = p_c(t) \int_0^\infty \mathcal{W}(k,k') \left( \frac{\Omega(t,k')}{k'^2} - \frac{\Omega(t,k)}{k^2} \right) k'^4 dk' \\
p_c'(t) = -p_c(t) \int_0^\infty \int_0^\infty \mathcal{W}(k,k') \left( \frac{\Omega(t,k')}{k'^2} - \frac{\Omega(t,k)}{k^2} \right) k'^4 k^2 dk' dk,
\]
or, in terms of \( \hat{\Omega}(t,k) = \Omega(t,k)/k^2 \),
\[
\frac{\partial \hat{\Omega}(t,k)}{\partial t} = p_c(t) \int_0^\infty \frac{\mathcal{W}(k,k')}{n_0(k)(1 + n_0(k))k^2} \left( \hat{\Omega}(t,k') - \hat{\Omega}(t,k) \right) k'^4 dk' \\
p_c'(t) = -p_c(t) \int_0^\infty \int_0^\infty \mathcal{W}(k,k') \left( \hat{\Omega}(t,k') - \hat{\Omega}(t,k) \right) k'^4 k^2 dk' dk.
\]
Since \( (k^2 n_0(k)(1 + n_0(k)))^{-1} = 4k^{-2} \sinh^2 \left( \frac{\beta k^2}{2} \right) \),
\[
\frac{\partial \hat{\Omega}(t,k)}{\partial t} = 4p_c(t) \int_0^\infty \left[ \mathcal{W}(k,k') \sinh^2 \left( \frac{\beta k^2}{2} \right) k'^4 \right] \left( \hat{\Omega}(t,k') - \hat{\Omega}(t,k) \right) dk'
\]
Use of the change of variables (1.7-1.8) in (1.20) yields system (1.9), (1.11) for \( (u, p_c) \), after scaling the time variable to get rid of some positive numerical constants.

### 1.2 A nonlinear approximation.

Another approximation of the system (1.1), (1.4) is possible where, in the equation (1.4), the function \( n \) is replaced by \( n_0 + n_0(1 + n_0)x^2 u \) in the nonlinear collision term \( I_3 \) given by (1.2) to obtain the system
\[
\frac{\partial v}{\partial t}(t,x) = \hat{p}_c(t) \int_0^\infty (v(t,y) - v(t,x)) M(x,y) dy \\
\frac{\partial \hat{p}_c}{\partial t}(t) = -\hat{p}_c(t) \int_0^\infty I_3(n_0 + n_0(1 + n_0)x^2 v(t,x)) x^2 dx
\]
instead of (1.9), (1.11). In that way the non linearity of \( I_3 \) in the equation for \( \hat{p}_c \) is kept. But the conservation in time of \( \hat{p}_c(t) + N(t) \) does not hold, and so an important global property of the original system (1.1) - (1.4) is lost. As a consequence the time existence of the solutions to system (1.11), (1.23) cannot be proved to be \((0, \infty)\). Then, system (1.22), (1.23) is not too satisfactory to describe global properties of the particle's system. But it may be a better approximation of the local properties of the solutions to the nonlinear system of equations (1.1), (1.4). In order to avoid any confusion, system (1.22), (1.23) is considered in the Appendix.
1.3 Further Motivation

It is known that for all non negative measure \( n_{in} \) with a finite first moment, and for every constant \( \rho > 0 \), system (1.1)-(1.4) has a weak solution \((n(t), n_c(t))\) with initial data \((n_{in}, \rho)\) that satisfies the conservation of mass and energy (cf. [5]). For all \( t > 0 \), \( n(t) \) is a non negative measure that does not charge the origin, with finite first moment, and \( n_c(t) > 0 \). However, one basic aspect of the non equilibrium behavior of the system condensate–normal fluid is the growth of the condensate after its formation (cf. [13, 3, 17] and references therein). In the kinetic formulation (1.1)-(1.4), this behavior is driven by the integral of \( I_3(n) \) in the right hand side of equation (1.4). As shown in [19], the behaviour of that term crucially depends on the behavior of \( n(t,p) \) as \( |p| \to 0 \) (this was discussed also in [5, 16, 20]). If for example the measure \( n(t) \) is a radially symmetric, bounded function near the origin then, from a simple use of Fubini’s Theorem,

\[
\int_{\mathbb{R}^3} I_3(n(t))(p)dp = C \int_0^\infty x^3 n(t,x)dx
\]

for some constant \( C > 0 \) independent of \( n \), and this would give a monotone decreasing behavior of \( n_c(t) \). On the contrary, as it is shown in [19], if the measure \( n(t) \) is a function such that \( n(t,p) \sim a(t)|p|^{-2} \) for some \( a(t) > 0 \), and satisfies some Hölder regularity property with respect to \( p \) in a neighborhood of the origin, then for some other constant \( C_1 > 0 \) independent of \( n \),

\[
\int_{\mathbb{R}^3} I_3(n(t))(p)dp = -C_1 a^2(t) + C \int_0^\infty x^3 n(t,p)dx
\]

On the other hand, it was proved in [5] that if the measure \( |p|^2 n(t,p) \) has no atomic part and has an algebraic behavior as \( |p| \to 0 \) then it satisfies (1.24). Both results in [19] and [5] assume some regularity of the solution \( n \) with respect to \( p \), although no regular solutions to (1.1) are known yet. The existence of regular classical solutions to (1.1)-(1.4) satisfying (1.24) is one of the motivations of our present work.

Since (1.24) is the behavior of the equilibrium \( n_0 \) (with \( a(t) \equiv \beta \)), it is natural to first consider the existence of such regular solutions for the linearization of (1.1) around \( n_0 \). Because of the singular behavior (1.24) of \( n_0 \) near the origin, the linear operator \( L \) in (1.10) has regularizing effects. Similar regularizing effects may be expected also in the non linear equation (1.1).

1.4 Basic arguments and Main results.

The function \( p_c(t) \) in the right hand side of (1.9) may be absorbed by the change of variables,

\[
\tau = \int_0^t p_c(s)ds, \quad f(\tau,x) = u(t,x)
\]

to obtain

\[
\frac{\partial f(\tau,x)}{\partial \tau} = L(f(\tau))(x).
\]
This equation may be written,
\begin{align}
\frac{\partial f}{\partial \tau}(\tau, x) &= L(f(\tau))(x) + F(f(\tau))(x) \\
&= \frac{1}{x^2 - y^2} - \frac{1}{x^2 + y^2} \cdot \frac{y}{x} dy \tag{1.28}
\end{align}

where, from (1.10), (1.29) and (1.31), the operator \( F \) may be written,

\[ F(f)(x) = -f(x) \int_0^\infty T(x, y) dy + \int_0^\infty T(x, y) f(y) dy \tag{1.31} \]

\[ T(x, y) = \frac{y^3 \sinh x^2}{x^3 \sinh y^2} \left( \frac{1}{\sinh|x^2 - y^2|} - \frac{1}{\sinh(x^2 + y^2)} \right) - \frac{y}{x} \left( \frac{1}{|x^2 - y^2|} - \frac{1}{x^2 + y^2} \right). \tag{1.32} \]

The equation (1.28) is solved as a perturbation of
\[ \frac{\partial f}{\partial \tau}(\tau, x) = L(f(\tau))(x) \tag{1.33} \]

with forcing term in (1.31). To this end several results about equation (1.33) obtained in [8] are used, in particular the regularizing effects of the operator \( L \). If \( \Lambda \) denotes the fundamental solution of (1.33), for all initial data \( f_0 \in L^1 \) there exists a weak solution of (1.33),
\[ S(t)f_0(x) = \int_0^\infty \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) f_0(y) \frac{dy}{y}, \forall t > 0, \forall x > 0, \tag{1.34} \]

such that \( S(\cdot)f_0 \in C([0, \infty); L^1(0, \infty)) \), \( S(t)f_0 \in C([0, \infty)) \) for all \( t > 0 \) and, for all \( t > 0 \) and any compact interval \( [x_0, x_1] \subset (0, \infty) \) there exists \( \alpha = \alpha(t, x_0) > 0 \) such that \( S(t)f_0 \in C^\alpha([x_0, x_1]) \), and (1.33) is satisfied pointwise for almost every \( t > 0, x > 0 \).

Once the Cauchy problem for equation (1.28) is solved using the semigroup \( S(t) \), the change of time variable in (1.26) is inverted to obtain the function \( u(t) \), and deduce \( p_c(t) \) using the conservation of mass of system and equation (1.11). Our first result is then as follows.

**Theorem 1.1.** Suppose that \( u_0 \in L^1(0, \infty) \) satisfies
\[ |||u_0|||_\theta \equiv \sup_{0 < x < 1} x^\theta |u_0(x)| + \sup_{x > 1} |u_0(x)| < \infty \tag{1.35} \]

for some \( \theta \in (0, 1) \). Then, there exists a pair \((u, p_c)\),
\[ u \in C([0, \infty); L^1(0, \infty)) \cap L^\infty((\delta, \infty); L^\infty(0, \infty)), \forall \delta > 0, \tag{1.36} \]
\[ p_c \in C([0, \infty)) \quad \tag{1.37} \]
such that for each \( t > 0 \), \( u(t) \) is locally Lipschitz on \((0, \infty)\) and, for all almost every \( t > 0 \) and \( x > 0 \),

\[
\frac{\partial u(t,x)}{\partial t} = p_c(t) \mathcal{L}(u(t))(x), \quad \text{(1.38)}
\]

\[
\frac{dp_c(t)}{dt} = -p_c(t) \int_0^\infty \mathcal{L}(u(t))(x)n_0(x^2)(1 + n_0(x^2))x^4dx. \quad \text{(1.39)}
\]

Moreover:

\[
\frac{\partial u}{\partial t}, \mathcal{L}(u) \in L^\infty_{\text{loc}}((0, \infty) \times (0, \infty)) \cap L^1((0, T); L^1(0, \infty)), \forall T > 0, \quad \text{(1.40)}
\]

there exists a function \( H \in L^\infty((\delta, 0) \times (0, \delta)) \) for all \( \delta > 0 \), defined in \((5.12)\), such that

\[
\left| \frac{\partial u(t,x)}{\partial t} \right| + |\mathcal{L}(u)(t,x)| \leq C \left( \sup_{0 < y < 1} y^\theta |f_0(y)| + ||u_0||_{L^\infty(1, \infty)} + ||u_0||_1 \right) H(t,x) \quad \text{(1.41)}
\]

and for all \( \beta \in (0, 1) \) and \( \delta(0, 1) \) there exists \( \lambda_{\beta, \delta}(t,x), \) defined in \((5.15)\), such that,

\[
\left| \frac{\partial u(t,x)}{\partial x} \right| \leq \lambda_{\beta, \delta}(t,x)||f_0||_{s,1} + \sup_{0 \leq s \leq t} (||f(s)||_{s,0} + ||f(s)||_{1,1}) \int_0^t \lambda_{\beta, \delta}(s,x)ds. \quad \text{(1.42)}
\]

\( \forall T > 0, \exists C_T > 0; ||u(t)||_1 \leq C_T ||u_0||_1, \forall t \in (0, T). \quad \text{(1.43)}
\]

\( \forall \delta > 0, ||u(t)||_\infty \leq C(\theta) \left( ||u_0||_{L^\infty(1, \infty)} + \delta^{-\theta} \sup_{0 < y < 1} y^\theta |u_0(y)| + ||u_0||_1 \right), \forall t \geq \delta. \quad \text{(1.44)}
\]

For all \( \varphi \in C^1_0(0, \infty) \), the map \( t \mapsto \int_0^\infty \varphi(x)u(t,x)dx \) belongs to \( W^{1,1}_{\text{loc}}(0, \infty) \) and for almost every \( t > 0 \),

\[
\frac{d}{dt} \int_0^\infty \varphi(x)u(t,x)dx = \int_0^\infty \mathcal{L}(u(t))(x)\varphi(x)dx + \int_0^\infty F(u(t,x))\varphi(x)dx. \quad \text{(1.45)}
\]

In view of \((1.17), (1.8)\) and \((1.20)\), if \( u \) is a solution of \((1.28)\) given by Theorem \((1.1)\), the pair of functions

\[
u(t,p) = n_0(p) + n_0(p)(1 + n_0(p))|p|^2u(t,|p|) \quad \text{(1.46)}
\]

\[
p_c(t) = \exp \left( \int_0^t \int_0^\infty \int_0^\infty \frac{M(x,y)}{(\sinh x^2)^2} \{ u(t,y) - u(t,x) \} \frac{x^2}{y^4} dy dx ds \right) \quad \text{(1.47)}
\]

\[= \exp \left( \int_0^t \int_0^\infty \mathcal{L}(u(t))(x) \frac{x^2}{(\sinh x^2)^2} dx ds \right) \quad \text{(1.48)}
\]

may be seen as an approximated solution of \((1.11), (1.24)\), as far as \( n_0(p)(1 + n_0(p))|p|^2u(t,|p|) \) remains small compared to \( n_0 \). In view of \((1.3)\) it is natural to look at the quantities

\[
N(t) = \int_0^\infty n_0(x)(1 + n_0(x))u(t,x)x^4dx \quad \text{(1.49)}
\]

\[
E(t) = \int_0^\infty n_0(x)(1 + n_0(x))u(t,x)x^6dx. \quad \text{(1.50)}
\]
They represent respectively the variation of the total number of particles and of energy caused by the initial perturbation \( n_0(p)(1 + n_0(p))|p|^2u(0) \) of the equilibrium \( n_0 \). Let us also define,

\[
N_0 = \int_0^\infty n_0(x)(1 + n_0(x))x^4dx 
\]

(1.51)

\[
E_0 = \int_0^\infty n_0(x)(1 + n_0(x))x^6dx. 
\]

(1.52)

The two following properties, hold then true,

**Corollary 1.2.** Let \( u_0 \) and \( u \) be as in Theorem 1.1 and \( n \) defined by (1.46). Then,

\[
E(t) = E(0), \forall t > 0, 
\]

(1.53)

\[
p_c(t) + N(t) = p_c(0) + N(0), \forall t > 0. 
\]

(1.54)

**Corollary 1.3.** Let \( u_0 \) and \( u \) be as in Theorem 1.1. Then,

\[
\lim_{t \to \infty} \int_0^\infty |u(t, x) - C_*^2n_0(x)(1 + n_0(x))x^6dx = 0, 
\]

(1.55)

\[
\lim_{t \to \infty} \int_0^\infty |u(t, x) - C_* |n_0(x)(1 + n_0(x))x^4dx = 0, 
\]

(1.56)

where \( C_* = \frac{E(0)}{E_0} \).

(1.57)

It follows from Corollary 1.3 that the mass and the energy variations due to the perturbation \( n_0(1+n_0)|p|^2u(t) \) tend to the mass and energy of \( C_*n_0(1+n_0)|p|^2 \), and this however small the perturbation is at infinity, even if, for example, \( u_0 \) is compactly supported. This kind of flux of energy to infinity could be expected, since it is well known to happen in the nonlinear homogeneous version of wave turbulence type of the system (1.1), (1.4) and is called direct energy cascade ([6, 22] and [11, 21]).

**Corollary 1.4.** Let \( u_0 \) and \( u \) be as in Theorem 1.1. Then, the function \( p_c \in C[0, \infty) \) is bounded on \([0, \infty)\) and

\[
\lim_{t \to \infty} p_c(t) = p_c(0) \exp (-M_\infty) 
\]

(1.58)

\[
M_\infty = C_*\int_0^\infty n_0(1 + n_0)x^4dx - \int_0^\infty n_0(1 + n_0)u_0(x)x^4dx 
\]

(1.59)

1.5 Some Remarks.

Several remarks follow from the previous results.

1.5.1 On the formal approximation

The approximation of (1.1), (1.4) by (1.9), (1.11) may be expected to be reasonable only as long as the perturbation remains small with respect to \( n_0 \),

\[
n_0(1 + n_0)|\Omega(t)| << n_0, 
\]

(1.60)
and this requires $x^2|f(t,x)|$ small for $x \to \infty$. However, although it could be proved that (1.60) holds for small values of time if it holds at $t = 0$, it follows from (1.55) that it cannot be true for all $t > 0$. Notice indeed that, for all $R \geq R_0 > 0$,

$$
\left| \int_R^\infty x^6 n_0(1 + n_0)|u(t,x)|dx - \int_R^\infty x^6 n_0(1 + n_0)|C_*| \right| \\
\leq \int_R^\infty x^6 n_0(1 + n_0)|u(t,x)|dx - C_*| \right| \\
\leq \left( \int_R^\infty x^6 n_0(1 + n_0)|u(t,x)| - C_*|^2 dx \right)^{1/2} \left( \int_R^\infty x^6 n_0(1 + n_0)dx \right)^{1/2} \\
(1.61)
$$

and the right hand side tends to zero as $t \to \infty$. If, on the other hand, we had $x^2|u(t,x)| \leq C$, for some $C > 0$, $R_0 > 0$ and $t_0 > 0$ for all $x > R_0$ and all $t > t_0$,

$$
\int_R^\infty x^6 n_0(1 + n_0)|u(t,x)|dx \leq \frac{C}{R^2} \int_R^\infty x^6 n_0(1 + n_0)dx, \ \forall R > R_0, t > t_0
$$

and then, for $R > C/|C_*|$ and all $t > t_0$

$$
\left| \int_R^\infty x^6 n_0(1 + n_0)|C_*|dx - \int_R^\infty x^6 n_0(1 + n_0)|u(t,x)| \right| \\
\geq \left| C_* - \frac{C}{R^2} \right| \int_R^\infty x^6 n_0(1 + n_0)dx > 0, \ \forall t > t_0.
$$

and this would contradict (1.61). System (1.4), (1.11) may then be considered “close to” (1.1), (1.4) only for small values of $t$. Of course, $u$ could be such that, for some $C(t)$ that tends to $\infty$ with $t$, $x^2|u(t,x)| \leq C(t)$ for all $x > 0$.

### 1.5.2 The behavior of the perturbation as $|p| \to 0$.

For all $t > 0$, the perturbation $n_0(1 + n_0)\Omega(t,p)$ of $n_0$ satisfies (1.24), for any $f_0$ as in the hypothesis of Theorem 1.1, where $a(t)$ is given in Proposition 4.15. The behavior $|p|^{-2}$ at the origin (that of the equilibria of (1.1), (1.4)) is then instantaneously fixed, whatever the behavior at the origin of $f_0$ may be, as far as the hypothesis of Theorem 1.1 are satisfied.

### 1.5.3 The function $p_c(t)$.

In view of Corollary 1.4 if the initial data $u_0$ is such that,

$$
\frac{E(0)}{E_0} N_0 < N(0), \quad (1.62)
$$

or equivalently,

$$
\mathcal{M}_\infty = C_* \int_0^\infty n_0(1 + n_0)x^4 dx - \int_0^\infty n_0(1 + n_0)u_0(x)x^4 dx < 0,
$$

then $\lim_{t \to \infty} p_c(t) > p_c(0)$, and conversely.
Condition (1.62) and its converse are both compatibles with \( n_0(1 + n_0)x^2u_0 \) being a small perturbation of \( n_0 \). For example
\[
n_0(1 + n_0)x^2u_0(x) = \left( \frac{1.02}{1 + x^2} - 1 \right) n_0,
\]
\[
N(0) \approx -0.344949, \quad E(0) \approx -0.523546
\]
\[
\frac{N(0)}{E(0)} \approx 0.658872 < \frac{N_0}{E_0} \approx 0.778949 \iff \frac{E(0)}{E_0} N_0 < N(0)
\]
and
\[
n_0(1 + n_0)x^2u_0(x) = 0.2\text{Arctg} \left( \frac{x}{10} \right) n_0,
\]
\[
N(0) \approx 0.0163705, \quad E(0) \approx 0.0238295
\]
\[
\frac{E(0)}{N(0)} \approx 1.45564 > \frac{E_0}{N_0} \approx 1.28378 \iff \frac{N(0)}{N_0} > N(0).
\]

1.6 Very low temperature and large \( n_c \).

Linearization of system (1.1), (1.4) for large number density of condensed atoms and very
low temperature may be performed following similar arguments as above (cf. [4] and [9]).
No regularizing effects have been observed and the existence of a first positive eigenvalue
and spectral gap for a suitable integrable operator ([4] and [12]) provide a convergence
rate to the equilibrium for a large set of initial data (cf. [9], Theorem 2.2) A necessary and
sufficient condition on \( p_c(0) \) to have a global solution.

2 The operator \( F \).

Equation (1.28) may be treated as a perturbation of (1.33) only whenever the term \( F(f) \)
in (1.31) is bounded in spaces where the properties of the solutions of (1.33) may be used.
The purpose of this Section is to establish that this is the case.

**Proposition 2.1.** (i) For all \( g \in L^\infty(0, \infty) \), \( F(g) \in L^\infty(0, \infty) \) and
\[
\| F(g) \|_\infty \leq 2M\| g \|_\infty.
\]
(ii) For all \( g \in L^1(0, \infty) \),
\[
\| F(g) \|_1 \leq C_F \| g \|_1 \quad C_F = (M + \tilde{M}),
\]
where
\[
M = \sup_{x > 0} \int_0^\infty |T(x, x')|dx', \quad \tilde{M} = \sup_{x' > 0} \int_0^\infty |T(x, x')|dx.
\]
(iii) For all \( \theta \in [0, 1) \) there exists a positive constant \( C(\theta) \) depending on \( \theta \), such that, if \( g \in L^\infty_{loc}(0, \infty) \) satisfies \( \| g \|_\theta < \infty \) then \( \| F(g) \|_\theta \leq C(\theta)\| g \|_\theta \).
(iv) For all \( g \in L^1(0, \infty) \cap L^\infty_{loc}(0, \infty) \), \( F(g) \in L^1(0, \infty) \cap L^\infty_{loc}(0, \infty) \).
(v) For all \( R > 0 \) there exists a constant \( C = C(R) > 0 \) such that:
\[
\| F(g) \|_\infty \leq C \left( \| g \|_{L^1(0, R)} + \| g \|_{L^\infty(R, \infty)} \right), \quad \forall g \in L^1(0, \infty) \cap L^\infty(R, \infty).
\]
Proposition 2.1 follows from estimates of the kernel $T$ defined in (1.32), that we split as follows,

$$T(x, x') = T_1(x, x') + T_2(x, x')$$

$$T_1(x, x') = \frac{x'}{x} \left( \frac{1}{\sinh |x^2 - x'^2|} - \frac{1}{|x^2 - x'^2|} - \frac{1}{\sinh(x^2 + x'^2)^2} + \frac{1}{|x^2 - x'^2|} \right)$$

$$T_2(x, x') = \frac{x'^3}{x^3} \left( \frac{\sinh x^2}{\sinh x'^2} - \frac{x^2}{x'^2} \left( \frac{1}{\sinh |x^2 - x'^2|} - \frac{1}{\sinh(x^2 + x'^2)^2} \right) \right).$$

The kernels $T_1$ and $T_2$ are estimated in the two next Propositions.

**Proposition 2.2.**

$$\forall R > 0, \exists C_R > 0: |T_1(x, x')| \leq C_R xx', \forall x' \in (0, R), \forall x \in (0, R) \tag{2.1}$$

$$|T_1(x, x')| \leq \frac{C|x|}{x} \left( \min(x^2, x'^2) + O(x^2 + x'^2)^3 \right), 0 \leq x \leq 1/2, 0 \leq x' \leq 1/2. \tag{2.2}$$

$$|T_1(x, x')| \leq C, \text{ if } x + x' > 1, |x' - x| \leq 1/8, \tag{2.3}$$

$$\forall x \in (0, 1), x' > \min(2, 3x/2), \ |T_1(x, x')| \leq \frac{C|x|^2}{x^2} \tag{2.4}$$

If $x + x' > 1, 1/8 < |x - x'| < x/2$

$$|T_1(x, x')| \leq \frac{C|x|}{x} \left( \frac{2 \min(x, x')^2}{(x^2 - x'^2)(x^2 + x'^2)} + \frac{1}{\sinh |x^2 - x'^2| - \sinh(x^2 + x'^2)} \right), \tag{2.5}$$

$$\forall |x - x'| > x/2,$$

$$|T_1(x, x')| \leq \frac{C|x|}{x} \left( \frac{e^{-C_1 \max(x, x')^2}}{(1 - e^{-\beta(x^2 + x'^2)})(1 - e^{-C_1 \max(x', x')^2}) + \min(x', x')^2} \max(x, x')^4 \right) \tag{2.6}$$

$$|T_1(x, x')| \leq \frac{C|x|}{x} \left( \frac{1}{\sinh \frac{3x^2}{4} + \min(x', x')^2} \max(x', x)^4 \right) \text{ if } 0 < x' < \frac{x}{2} \tag{2.7}$$

**Proof.**

0.- Proof of (2.1). When $x' \in (0, R)$ and $x \in (0, R)$. We may use the series expansion of the function $1/\sinh x$ to obtain,

$$h(z) = \frac{1}{z} - \frac{1}{\sinh z} = \sum_{n=1}^{\infty} \frac{2(2^{2n-1} - 1)B_n}{(2n)!} z^{2n-1}$$

$$B_0 = 1, B_n = \sum_{\ell=1}^{n-1} \left( \begin{array}{c} n \\ \ell \end{array} \right) \frac{B_{\ell}}{n + 1 - \ell}, n \geq 1, \text{ (Bernoulli’s numbers)},$$

$$h'(z) = \sum_{n=1}^{\infty} \frac{2(2^{2n-1} - 1)B_n}{(2n)!} z^{2n-1}.$$
and

$$|T_1(x, x^{'})| \leq 2xx' \sup_{\xi \in (0, 2R^2)} |h'(\xi)|, \ \forall x' \in (0, R), x \in (0, R).$$

1.- Proof of (2.2). Consider first the set where $x' < 1/2$ and $x < 1/2$, and use the Taylor’s expansion of $1/\sinh z$ around $z = 0$.

$$\frac{1}{\sinh|x^2-x^2|} = \frac{2}{\beta|x^2-x^2|} - \frac{\beta|x^2-x^2|}{12} + O(|x^2-x^2|^3), \ |x - x'| \to 0$$

$$\frac{1}{\sinh(x^2+x^2)} = \frac{2}{\beta(x^2+x^2)} - \frac{\beta(x^2+x^2)}{12} + O((x^2+x^2)^3), \ |x + x'| \to 0.$$

Then

$$\left( \frac{1}{\sinh|x^2-x^2|} - \frac{1}{\sinh(x^2+x^2)} \right) = \frac{\min(x^2, x^2)}{3} + O(x^2 + x^2)^3,$$

and that proves (2.2).

2.- Proof of (2.1). When $x \in (0, 1)$ and $x' > x$, in the identity,

$$\frac{1}{\sinh|x^2-x^2|} - \frac{1}{\sinh(x^2+x^2)} = \frac{\sin(x^2+x^2) - \sinh(x^2-x^2)}{\sin(x^2+x^2) \sinh(x^2-x^2)},$$

we use the mean value Theorem to obtain

$$\sin(x^2+x^2) - \sinh(x^2-x^2) = -2x^2 \cosh \xi$$

$$\xi \in \xi(x^2-x^2, x^2+x^2) \in (x^2-x^2, x^2+x^2) \subset (x^2-1, x^2+1)$$

and then,

$$\cosh \xi \leq C \cosh(x^2).$$

We deduce,

$$\left| \frac{1}{\sinh|x^2-x^2|} - \frac{1}{\sinh(x^2+x^2)} \right| \leq C \frac{x^2 \cosh x^2}{\sin(x^2+x^2) \sinh(x^2-x^2)} \leq C \frac{x^2}{\sinh x^2}.$$

On the other hand, since $x' > \min(2, 3x/2)$, $x^2-x^2 > Cx^2$ and,

$$\frac{1}{|x^2-x^2|} - \frac{1}{|x^2+x^2|} = \frac{2x^2}{(x^2-x^2)(x^2+x^2)} \leq C \frac{x^2}{x^4},$$

it follows that, $|T_1(x, x^{'})| \leq C \frac{x^2}{x^4} \leq C \frac{x^2}{x^4}$ and that proves (2.1).

3.- Proof of (2.3) Suppose now that $x + x' > 1$, and $|x - x'| \leq 1/8$. We may still use the Taylor’s expansion of $1/\sinh(|x^2-x^2|)$ around $|x^2-x^2| = 0$,

$$\frac{1}{\sinh|x^2-x^2|} - \frac{1}{|x^2-x^2|} = \frac{|x^2-x^2|}{6} + O(|x^2-x^2|^3), \ |x - x'| \to 0.$$
Then, if \( x + x' > 1 \), \( |x' - x| \leq 1/8 \),
\[
|T_1(x, x')| \leq \frac{C x'}{x} \left( \left| \frac{x^2 - x'^2}{x^2 + x'^2} \right| - \frac{1}{\sinh \left| \frac{x^2 - x'^2}{x^2 + x'^2} \right|} + \frac{1}{|x^2 + x'^2|} \right)
\leq \frac{C x'}{x} \left( 1 + \frac{1}{(x^2 + x'^2)} \right) \leq C x'
\]
and that proves (2.8).

4.- Proof of (2.5). Consider now, the cases where \( x + x' > 1 \), \( |x' - x| > 1/8 \) and \( |x - x'| < x/2 \). Since,
\[
\frac{1}{|x^2 - x'^2|} - \frac{1}{|x^2 + x'^2|} = \frac{2 \min(x, x')^2}{(x^2 - x'^2)(x^2 + x'^2)} \leq \frac{16 \min(x, x')^2}{(x' + x)(x^2 + x'^2)}
\]
If \( x + x' > 1 \), \( |x' - x| > 1/8 \) and \( |x - x'| < x/2 \),
\[
|T_1(x, x')| \leq \frac{C x'}{k} \left( \frac{2 \min(x, x')^2}{(x^2 - x'^2)(x^2 + x'^2)} + \frac{1}{\sinh \left| \frac{x^2 - x'^2}{x^2 + x'^2} \right|} - \frac{1}{\sinh(x^2 + x'^2)} \right)
\leq \frac{C x'}{x} \left( \frac{2 \min(x, x')^2}{(x' + x)(x^2 + x'^2)} + \frac{1}{\sinh \left( \frac{|x+x'|}{8} \right)} \right)
\leq \frac{C x'}{x} \left( \frac{2 \min(x, x')^2}{(x' + x)(x^2 + x'^2)} + \frac{1}{\sinh \left( \frac{|x+x'|}{8} \right)} \right)
\]
The estimate (2.8) is nothing but (2.5).

5.- Proof of (2.6). On the other hand, if \( x > 1 \) and \( |x' - x| > x/2 \) we may still use hat,
\[
\frac{1}{|x^2 - x'^2|} - \frac{1}{|x^2 + x'^2|} = \frac{2 \min(x, x')^2}{(x'^2 - x^2)(x^2 + x'^2)} \leq \begin{cases} \frac{C x^2}{x^2}, & \forall x' > 3x/2 > 0 \\ \frac{C x^2}{x^2}, & \forall x' < x/2. \end{cases}
\]
If \( y > x \),
\[
\left( \frac{1}{\sinh |y^2 - x^2|} - \frac{1}{\sinh |y^2 + x^2|} \right) = \frac{e^{x^2+y^2} \left( 1 - e^{-2(x^2+y^2)} - e^{-2x^2} + e^{-2y^2} \right)}{e^{x^2+y^2} \left( 1 - e^{-2(x^2+y^2)} \right) \left( e^{y^2-x^2} - e^{-y^2+x^2} \right)}
= \frac{1 - e^{-2(x^2+y^2)} - e^{-2x^2} + e^{-2y^2}}{\left( 1 - e^{-2(x^2+y^2)} \right) \left( 1 - e^{-2(y^2-x^2)} \right)} \leq e^{-2(y^2-x^2)}
\]
Then, if \( x' > 3x/2 \),
\[
\left( \frac{1}{\sinh |x'^2 - x^2|} - \frac{1}{\sinh |x'^2 + x^2|} \right) = \frac{e^{-(x'^2-x^2)}}{\left( 1 - e^{-2(x^2+x'^2)} \right) \left( 1 - e^{-2(x'^2-x^2)} \right)} \leq \frac{e^{-(x'^2-x^2)}}{\left( 1 - e^{-2(x^2+x'^2)} \right) \left( 1 - e^{-2(x'^2-x^2)} \right)} \leq \frac{1}{\left( e^{x^2} \right) \left( 1 - e^{-2(x^2+x'^2)} \right)}
\]
If \( x > 2x' \),
\[
\left( \frac{1}{\sinh |x'^2 - x^2|} - \frac{1}{\sinh |x'^2 + x^2|} \right) \leq \frac{e^{-(x^2-x'^2)}}{\left( 1 - e^{-2(x^2+x'^2)} \right) \left( 1 - e^{-2(x^2-x'^2)} \right)} \leq \frac{1}{\left( e^{x^2} \right) \left( 1 - e^{-2(x^2+x'^2)} \right)}
\]
and that proves (2.6). But also, in the case \( x > 2x' \), \( x^2 - x'^2 > 3x^2/4 \) and then,

\[
\frac{1}{\sinh x^2 - x'^2} \leq \frac{1}{\sinh \frac{3x^2}{4}}; \quad \frac{1}{\sinh(x^2 + x'^2)} \leq \frac{1}{\sinh x^2}
\]

and that proves (2.7).

\[\square\]

**Proposition 2.3.**

\[
\forall R > 0, \exists C_R > 0; \quad |T_2(x, x')| \leq C_R, \forall x \in (0, R), \forall x' \in (0, R) \tag{2.10}
\]

\[
\forall \delta > 0, \exists C_\delta > 0; \quad \forall x \in (0, 1), \forall x' > x + \delta : \quad |T_2(x, x')| \leq C_\delta \frac{x x'}{\sinh x'^2} \tag{2.11}
\]

\[
\forall x + x' > 1, |x - x'| < 1/8 : \quad |T_2(x, x')| \leq C \tag{2.12}
\]

\[
\forall x + x' > 1, \frac{1}{8} < |x - x'| < x/2 : \quad |T_2(x, x')| \leq C \frac{1}{\sinh x^2 - x'^2} - \frac{1}{\sinh(x^2 + x'^2)} \tag{2.13}
\]

If \( x + x' > 1 \) and \( |x - x'| > x/2 \),

\[
|T_2(x, x')| \leq \frac{C}{\sinh \frac{3x'^2}{4}}, x' < x/2 \tag{2.14}
\]

\[
|T_2(x, x')| \leq C e^{-(x^2 - x'^2)} \left(e^{-(x^2 - x'^2)} + \frac{x^2}{x'^2}\right) \frac{x'^3}{x^3}, x' \geq 3x/2 \tag{2.15}
\]

**Proof.**

1.- Proof of (2.10). Let us write first,

\[
\left|\frac{\sinh x^2}{\sinh x'^2} - \frac{x^2}{x'^2}\right| = \left|\frac{x^2 \sinh x - x^2 \sinh x'^2}{x'^2 \sinh x'^2}\right| = \left|\frac{x^2 \sinh x}{x'^2 \sinh x'^2} - \frac{x^2 \sinh x}{x'^2 \sinh x'^2}\right|
\]

and

\[
x'^2 \sinh x^2 - x^2 \sinh x'^2 = x'^2(\sinh x^2 - \sinh x'^2) + (x^2 - x'^2) \sinh x'^2.
\]

Using Taylor’s expansion we have, for some \( \xi(x^2, x'^2) \) between \( x^2 \) and \( x'^2 \),

\[
x'^2(\sinh x^2 - \sinh x'^2) = x'^2\left((x^2 - x'^2) \cosh x'^2 + \frac{1}{2}(x^2 - x'^2)^2 + \frac{1}{6}(x^2 - x'^2)^3 \cosh \xi(x^2, x'^2)\right)
\]

\[\square\]
and,

\[ x^2 (\sinh x^2 - \sinh x'^2) + (x'^2 - x^2) \sinh x^2 = \]
\[ = x^2 \left( (x^2 - x'^2) \cosh x^2 + \frac{1}{2} (x^2 - x'^2)^2 + \frac{1}{6} (x^2 - x'^2)^3 \cosh \xi(x^2, x'^2) \right) + \]
\[ + (x'^2 - x^2) \sinh x^2 \]
\[ = (x^2 - x'^2) (x'^2 \cosh x^2 - \sinh x^2) + \frac{x'^2}{2} (x^2 - x'^2)^2 + \]
\[ + \frac{x'^2}{6} (x^2 - x'^2)^3 \cosh \xi(x^2, x'^2). \]

We deduce that for all \( x \in (0, R) \) and \( x' \in (0, R) \),

\[ |x^2 (\sinh x^2 - \sinh x'^2) + (x'^2 - x^2) \sinh x^2| \leq C_R |x^2 - x'^2| \times \]
\[ \times |x^6 + x^2 |x^2 - x'^2| + |x^2 - x'^2|^2 |x^2| \leq C'_R |x^2 - x'^2| (x^6 + x^2 x'^2). \]

Similarly, using the change \( x \leftrightarrow x' \),

\[ |x'^2 (\sinh x'^2 - \sinh x^2) + (x^2 - x'^2) \sinh x'^2| \leq C_R |x^2 - x'^2| (x^6 + x^2 x'^2). \]

It follows that, for all \( x \in (0, R) \), \( x' \in (0, R) \),

\[ |x^2 (\sinh x^2 - \sinh x'^2) + (x'^2 - x^2) \sinh x^2| \leq C_R |x^2 - x'^2| \times \]
\[ \times \min \{ x'^2 (x^2 + x'^4), x^2 (x^4 + x'^2) \} \leq C_R |x^2 - x'^2| x^2 x'^2 \]

and

\[ \left| \frac{\sinh x^2}{\sinh x'^2} - \frac{x^2}{x'^2} \right| \leq C_R \frac{|x^2 - x'^2| x^2 x'^2}{x^4}. \]

On the other hand, since \( \sinh \) is locally Lipschitz,

\[ \left| \frac{1}{\sinh |x^2 - x'^2|} - \frac{1}{\sinh |x^2 + x'^2|} \right| \leq C_R \frac{\min(x^2, x'^2)}{|x^2 - x'^2|(x^2 + x'^2)} \]

for some constant \( C_R > 0 \). We deduce,

\[ \left| \frac{\sinh x^2}{\sinh x'^2} - \frac{x^2}{x'^2} \right| \left| \frac{1}{\sinh |x^2 - x'^2|} - \frac{1}{\sinh |x^2 + x'^2|} \right| \leq C_R \frac{x^2 x'^2 \min(x^2, x'^2)}{x^4 (x^2 + x'^2)} \]
\[ \leq C_R \frac{x^2 \min(x^2, x'^2)}{x^2 (x^2 + x'^2)} \]

and

\[ |T_2(x, x')| \leq C_R \frac{x' \min(x^2, x'^2)}{x (x^2 + x'^2)} \leq C_R, \ \forall x \in (0, R), \ x' \in (0, R). \]

This proves \textbf{[14.1]}. 

15
2.- Proof of (2.11). Suppose now that \( x \in (0, 1) \) and \( x' > x + \delta \) for any \( \delta > 0 \). Then, as we have seen in the proof of Proposition 2.2,
\[
\left| \frac{1}{\sinh |x^2 - x'^2|} - \frac{1}{\sinh(x^2 + x'^2)} \right| \leq \frac{x^2 \cosh x'^2}{\sinh(x^2 + x'^2) \sinh(x^2 - x'^2)} \leq C_8 \frac{x^2}{\sinh x'^2}.
\]
We have also, since \( \sinh x^2 \leq 1 \),
\[
\left| \frac{\sinh x^2}{\sinh x'^2} - \frac{x^2}{x'^2} \right| = \left| \frac{x^2 \sinh x^2 - x^2 \sinh x'^2}{x^2 \sinh x'^2} \right| \leq C \frac{x^2(x^2 + \sinh x'^2)}{x^2 \sinh x'^2} = C \frac{x^2}{\sinh x'^2} \left( \frac{1}{x'^2} + 1 \right) \leq C \frac{x^2}{x'^2}.
\]
Then,
\[
|T_2(x, x')| \leq C_8 \frac{x^2}{\sinh x'^2} \frac{x^2}{x'^3} = C_8 \frac{x}{x'^2}
\]
and this proves (2.11).

3.- Proof of (2.12) We consider now the case where \( x + x' > 1 \) and \( |x - x'| < 1/8 \). Suppose again that \( 0 < x^2 < x \). We may still use the Taylor’s expansion around \( x^2 = x'^2 \), and write,
\[
|x^2 \sinh x^2 - x^2 \sinh x'^2| \leq x^2 |\sinh x^2 - \sinh x'^2| + |x^2 - x'^2| |\sinh x^2|
\leq x^2 |x^2 - x'^2| \cosh \xi(x^2, x'^2) + |x^2 - x'^2| |\sinh x^2|
\]
and
\[
\left| \frac{\sinh x^2}{\sinh x'^2} - \frac{x^2}{x'^2} \right| \leq x^2 |x^2 - x'^2| \cosh \xi(x^2, x'^2) + |x^2 - x'^2| \sinh x'^2
\]
\[
= |x^2 - x'^2| \left( \frac{\cosh \xi(x^2, x'^2)}{\sinh x'^2} + 1 \right).
\]
If \( |x - x'| < 1/8 \) and \( x + x' > 1 \) then, for some \( x_0 > 0 \), \( x' \geq x_0 \) and \( x \geq x_0 \). Then, there exists \( C > 0 \) such that
\[
\frac{\cosh \xi(x^2, x'^2)}{\sinh x'^2} \leq C, \ \forall x, \ \forall x; x + x' > 1, |x - x'| < 1/8
\]
and,
\[
\left| \frac{\sinh x^2}{\sinh x'^2} - \frac{x^2}{x'^2} \right| \leq C |x^2 - x'^2| \ \forall x, \ \forall x'; x + x' > 1, |x - x'| < 1/8.
\]
On the other hand, if \( |x + x'| > 1 \) and \( |x - x'| < 1/8 \),
\[
\left| \frac{1}{\sinh |x^2 - x'^2|} - \frac{1}{\sinh(x^2 + x'^2)} \right| \leq \frac{C}{|x^2 - x'^2|} + \frac{1}{\sinh(x^2 + x'^2)}
\]
for some positive \( C \). It follows,
\[
\left| \frac{\sinh x^2}{\sinh x'^2} - \frac{x^2}{x'^2} \right| \left| \frac{1}{\sinh |x^2 - x'^2|} - \frac{1}{\sinh(x^2 + x'^2)} \right| \leq C; \ \forall x', \ \forall x; |k + k'| > 1, |k - k'| < 1/8.
and,
\[|T_2(x,x')| \leq C \frac{x'^3}{x^3} \leq C \frac{(x + 1/8)^3}{x^3} \leq C', \quad x + x' > 1, |x - x'| < 1/8.\]

and this shows (2.12).

4.- Proof of (2.13) Suppose now that \(x + x' > 1, |x - x'| > 1/8\) and \(|x - x'| < x/2\). Then
\[
\left| \frac{\sinh x^2}{\sinh x'^2} - \frac{x^2}{x'^2} \right| \leq C
\]

and,
\[
\left| \frac{\sinh x^2}{\sinh x'^2} - \frac{x^2}{x'^2} \right| \leq C \left| \frac{1}{\sinh |x^2 - x'^2|} - \frac{1}{\sinh |x^2 + x'^2|} \right| \leq C \left| \frac{1}{\sinh |x^2 - x'^2|} - \frac{1}{\sinh |x^2 + x'^2|} \right|
\]

and this shows (2.13).

5.- Proof of (2.14) and (2.15). Suppose now \(x + x' > 1, |x - x'| > 1/8\). As we have seen in the proof of Proposition 2.2, if \(x' > x\),
\[
\left| \frac{1}{\sinh |x^2 - x'^2|} - \frac{1}{\sinh |x^2 + x'^2|} \right| \leq \frac{e^{-(x^2-x'^2)}}{(1 - e^{-2(x^2+x'^2)}) (1 - e^{-2(x'^2-x^2)})} \leq C e^{-(x^2-x'^2)}
\]

and
\[
\left| \frac{\sinh x^2}{\sinh x'^2} - \frac{x^2}{x'^2} \right| \leq C e^{-(x^2-x'^2)} + \frac{x^2}{x'^2}.
\]

Then,
\[
\left| \frac{\sinh x^2}{\sinh x'^2} - \frac{x^2}{x'^2} \right| \left| \frac{1}{\sinh |x^2 - x'^2|} - \frac{1}{\sinh |x^2 + x'^2|} \right| \leq C e^{-(x^2-x'^2)} \left( e^{-(x^2-x'^2)} + \frac{x^2}{x'^2} \right).
\]

If on the other hand, \(x' < x/2, x^2 - x'^2 > 3x^2/4\), and then
\[
\left| \frac{1}{\sinh |x^2 - x'^2|} - \frac{1}{\sinh |x^2 + x'^2|} \right| \leq \frac{1}{\sinh \frac{x^2}{4}} + \frac{1}{\sinh \frac{x^2}{2}} \leq \frac{2}{\sinh \frac{x^2}{2}}
\]

We deduce,
\[
|T_2(x,x')| \leq \frac{C}{\sinh \frac{3x^2}{4}} \frac{x'^3}{x^3} \leq \frac{C}{\sinh \frac{3x^2}{4}}, \quad x' < x/2
\]
\[
|T_2(x,x')| \leq C e^{-(x^2-x'^2)} \left( e^{-(x^2-x'^2)} + \frac{x^2}{x'^2} \right) \frac{x'^3}{x^3}, \quad x' \geq 3x/2
\]

and this proves (2.14) and (2.15). \(\square\)

Two Corollaries follow from Proposition (2.2) and Proposition (2.3). First,

**Corollary 2.4.**
\[
M = \sup_{x>0} m(x) = \sup_{x>0} \int_0^\infty |T(x,x')|dx' < \infty
\]

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Proof. By definition, for all \( x > 0 \),
\[
m(x) = m_1(x) + m_2(x): \ |m_1(x)| \leq \int_0^\infty |T_1(x, x')| dx' \quad |m_2(x)| \leq \int_0^\infty |T_2(x, x')| dx'.
\]
Suppose first that \( x \in (0,1) \). Then, by (2.10) and (2.11),
\[
m_1(x) \leq \int_0^2 |T_1(x, x')| dx' + \int_2^\infty |T_1(x, x')| dx' \leq Cx \int_0^1 x' dx' + x \int_1^\infty \frac{dx'}{x'^3} = Cx, \forall x \in (0,1).
\]
(2.16)

With a similar argument, using instead (2.10) and (2.11),
\[
m_2(x) \leq C + Cx \int_2^\infty \frac{x' dx'}{\sinh x'^2} \leq C, \forall x \in (0,1).
\]
(2.17)

Suppose now that \( x > 1 \) and write, for \( \ell = 1,2 \),
\[
m_{\ell}(x) = \int_0^{x/2} dx' + \int_{x/2}^{x-1/8} dx' + \int_{x-1/8}^{x+1/8} dx' + \int_{x+1/8}^{3x/2} dx' + \int_{3x/2}^\infty dx'
\]
(2.18)

When \( x' \in (0,x/2) \) we may use (2.7) to obtain,
\[
\int_0^{x/2} |T_1(x, x')| \leq \frac{C}{x} \int_0^{x/2} x' \left( \frac{1}{\sinh \frac{3x^2}{4}} + \frac{x'^2}{x^4} \right) \leq C \left( \frac{x}{\sinh \frac{3x^2}{4}} + \frac{1}{x} \right) \leq C, \forall x > 1.
\]
(2.19)

If we use (2.14) in the same region we obtain,
\[
|T_2(x, x')| \leq \frac{C}{\sinh \frac{3x^2}{4}}
\]

and then,
\[
\int_0^{x/2} |T_2(x, x')| \leq \frac{Cx}{\sinh \frac{3x^2}{4}}, \forall x > 1.
\]
(2.20)

When \( x' \in (x/2, x-1/8) \), by (2.14),
\[
|T_1(x, x')| \leq \frac{C x'}{x} \left( \frac{2 \min(x, x')^2}{(x' + x)(x'^2 + x^2)} + \frac{1}{\sinh \frac{|x + x'|}{8}} \right) \leq C \frac{x'}{x^2}
\]
(2.21)

and then,
\[
\int_{x/2}^{x-1/8} |T_1(x, x')| dx' \leq C, \forall x > 1.
\]
(2.22)

We use now (2.13) in the same region,
\[
|T_2(x, x')| \leq C \left( \frac{1}{\sinh \frac{|x^2 - x'^2|}{2}} - \frac{1}{\sinh(x^2 + x'^2)} \right) \leq C \left( \frac{1}{\sinh \frac{x}{8}} + \frac{1}{\sinh x^2} \right),
\]
from where:
\[
\int_{x/2}^{x-1/8} |T_2(x, x')| dx' \leq C \frac{x}{\sinh \frac{x}{8}}, \forall x > 1.
\]
(2.23)
Suppose now that \( x' \in (x - 1/8, x + 1/8) \), by (2.3) and (2.12) \( T_1(x, x') \) and \( T_2(x, x') \) are both bounded on \([x - 1/8, x + 1/8]\) and then
\[
\int_{x - 1/8}^{x + 1/8} (|T_1(x, x')| + |T_2(x, x')|) \, dx' \leq C, \; \forall x > 1. \tag{2.24}
\]
In the region \( x' \in (x + 1/8, 3x/2) \), we may use again (2.9) to obtain, as in (2.21),
\[
|T_1(x, x')| \leq C \left( \frac{2 \min(s, x')^2}{(x' + x)(x^2 + x'^2)} + \frac{1}{\sinh \frac{|x + x'|}{8}} \right) \leq \frac{C x'}{x^2},
\]
from where
\[
\int_{x - 1/8}^{3x/2} |T_1(x, x')| \, dx' \leq C, \; \forall x > 1. \tag{2.25}
\]
By (2.13) we have in the same region,
\[
|T_2(x, x')| \leq C \left| \frac{1}{\sinh (|x' - x|)} - \frac{1}{\sinh (x^2 + x'^2)} \right| \leq C \left( \frac{1}{\sinh \frac{x}{8}} + \frac{1}{\sinh x^2} \right), \tag{2.26}
\]
and then
\[
\int_{x - 1/8}^{3x/2} |T_2(x, x')| \, dx' \leq C \frac{x}{\sinh \frac{x}{8}}, \; \forall x > 1. \tag{2.27}
\]
If \( x' > 3x/2 \), by (2.6),
\[
|T_1(x, x')| \leq \frac{C x'}{x} \left( e^{-C_1 x'^2} + \frac{x^2}{x'^3} \right)
\]
and we may write,
\[
e^{-C_1 x'^2} \leq e^{-C_1 x'^2} e^{-C_1 x'^2} \text{ and } \frac{x^2}{x'^3} \leq \frac{1}{x'^2}
\]
from where,
\[
\int_{x - 1/8}^{3x/2} |T_1(x, x')| \, dx' \leq C e^{-C_1 x'^2} + \int_{x - 1/8}^{3x/2} \frac{dx'}{x'^2} \leq C \left( e^{-C_1 x'^2} + \frac{1}{x} \right), \; \forall x > 1. \tag{2.28}
\]
By (2.14),
\[
|T_2(x, x')| \leq \begin{cases} 
Ce^{-x^2 - x'^2} \left( e^{-x'^2 - x^2} + \frac{x^2}{x'^2} \right) \frac{x'^3}{x^3}, & \forall x > \frac{3x}{2} \\
Ce^{-x'^2} \left( e^{-x'^2} + 1 \right) \frac{x'^3}{x^3}, & \forall x > \frac{3x}{2},
\end{cases}
\]
\[
\int_{x - 1/8}^{3x/2} |T_2(x, x')| \, dx' \leq C e^{- \frac{5x^2}{9}}, \; \forall x > 1.
\]
Similar arguments show the second Corollary,

**Corollary 2.5.**
\[
\tilde{M} = \sup_{x' > 0} \mu(x') = \sup_{x' > 0} \int_{0}^{\infty} |T(x, x')| \, dx < \infty
\]
Proof. As before,

\[ \mu(x') \leq \mu_1(x') + \mu_2(x') \]
\[ \mu_\ell(x') = \int_0^\infty |T_\ell(x, x')|dx, \quad \ell = 1, 2. \]

Suppose first that \( x' \in (0, 2) \). We have then,

\[ \mu_1(x') \leq \int_0^5 |T_1(x, x')|dx + \int_5^\infty |T_1(x, x')|dx \tag{2.29} \]

We use (2.1) in the first integral of the right hand side of (2.29). Since \( x > 5 > 2x' \) in the second integral of the right hand side of (2.29), we may use (2.7) and deduce that \( \sup_{x' \in (0, 2)} \mu_1(x') < \infty \). Similarly,

\[ \mu_2(x') \leq \int_0^5 |T_2(x, x')|dx + \int_5^\infty |T_2(x, x')|dx \]

where (2.10) and (2.14) yield \( \sup_{x' \in (0, 2)} \mu_2(x') < \infty \).

Suppose now that \( x' \in (2, 15/2) \). We write then,

\[ \int_5^\infty |T_1(x, x')|dx = \int_5^{2x'} |T_1(x, x')|dx + \int_5^\infty |T_1(x, x')|dx \tag{2.30} \]

The first integral in the right hand side of (2.30) is estimated using (2.1),

\[ \int_5^{2x'} |T_1(x, x')|dx \leq \int_5^{15} |T_1(x, x')|dx \leq C. \]

The second integral in the right hand side of (2.30) may be estimated using (2.7) to obtain,

\[ \int_5^{2x'} |T_1(x, x')|dx \leq Cx' \int_5^{\infty} \frac{1}{x} \left( \frac{1}{\sinh \frac{3x^2}{4}} + \frac{x'^2}{x^3} \right) dx \leq C \int_0^\infty e^{-\frac{3x^2}{8}} dx + Cx'^{-1}. \]

and it follows that \( \sup_{x' \in (2, 15/2)} \mu_1(x') < \infty \). A similar argument using (2.10) and (2.14) gives \( \sup_{x' \in (2, 15/2)} \mu_2(x') < \infty \).

Suppose now that \( x' > 15/2 \). Then,

\[ \int_5^{\infty} |T_1(x, x')|dx = \int_5^{2x'} |T_1(x, x')|dx + \int_5^{2x'} |T_1(x, x')|dx + \int_5^{\infty} |T_1(x, x')|dx \tag{2.31} \]

In the first and third integrals of the righthand side of (2.31) we use (2.6),

\[ \int_5^{2x'} |T_1(x, x')|dx \leq Cx' \int_5^{2x'} \frac{1}{x} \left( e^{-C_1x'^2} + \frac{x'^2}{x^4} \right) dx < C \]
\[ \int_2^{\infty} |T_1(x, x')|dx \leq Cx' \int_2^{\infty} \frac{1}{x} \left( e^{-C_1x'^2} + \frac{x'^2}{x^4} \right) dx < C \]
Since $|x - x'| < x/2$ in the second integral at the right hand side of (2.31) we write,

$$\int_{\frac{x}{2}}^{2x'} |T_1(x, x')|dx = \int_{|x' - x| < \frac{x}{2}} |T_1(x, x')|dx + \int_{\frac{x}{2} < |x' - x| < x/2} |T_1(x, x')|dx$$

(2.32)

Using (2.3),

$$\int_{|x' - x| < \frac{x}{2}} |T_1(x, x')|dx \leq C x'$$

We use now (2.5) in the second integral at the right hand side of (2.31) we write,

$$\int_{\frac{x}{2} < |x' - x| < x/2} \frac{1}{x} \left( \frac{2 \min(x', x)^2}{(x^2 - x'^2)(x^2 + x'^2)} + \frac{1}{\sinh |x^2 - x'^2|} - \frac{1}{\sinh(x^2 + x'^2)} \right) dx \leq C x'$$

(2.33)

from where sup$_{x' > 15/2} \mu_1(x') < \infty$. A similar argument shows sup$_{x' > 15/2} \mu_2(x') < \infty$, using (2.13), (2.12) instead of (2.3) and (2.15), (2.14) instead of (2.3).

Proof of Proposition 2.1. The proof of (i) and (ii) are now straightforward:

$$|F(g(x))| \leq |g(x)| \int_0^\infty |T(x, x')|dx' + \int_0^\infty |g(x')| |T(x, x')|dx'$$

(2.33)

$$||F(g)||_1 \leq \int_0^\infty |g(x)| \int_0^\infty |T(x, x')|dx'dx + \int_0^\infty \int_0^\infty |g(x')| |T(x, x')|dx'dx dx \leq \tilde{M} ||g||_1 + M ||g||_1.$$

Proof of (iii). Consider again the right hand side of (2.33) and notice that, by (2.4) and (2.10), for $x \in (0, 1)$,

$$\int_0^1 |g(x')||T(x, x')|dx' \leq \int_0^1 |g(x')| \left( |T_1(x, x')| + |T_2(x, x')| \right) dx' \leq C \int_0^1 |g(x')| \left( 1 + xx' \right) dx' \leq C ||f(s)||_\theta.$$

by Corollary 2.4

$$\int_1^\infty |g(x')||T(x, x')|dx' \leq ||g||_{L^\infty(1, \infty)} \int_0^\infty |T(x, x')|dx' \leq M ||g||_{L^\infty(1, \infty)}$$

and then,

$$\sup_{0 \leq x' \leq 1} x'^\theta |F(g)(x')| \leq M \sup_{0 \leq x' \leq 1} x'^\theta |g(x')| + C ||g||_\theta.$$
By (2.7) and (2.14), for \( x > 2 \),
\[
\int_0^1 |g(x')| \left( |T_1(x, x')| + |T_2(x, x')| \right) \, dx' \leq \sup_{0 \leq x' \leq 1} x'^\theta |g(x')| \times \\
\times \int_0^1 x'^{-\theta} \left( |T_1(x, x')| + |T_2(x, x')| \right) \, dx' \leq C \sup_{0 \leq x' \leq 1} x'^\theta |g(x')| \\
\int_0^1 x'^{1-\theta} \left( 1 + x'^2 \right) \, dx' = C \sup_{0 \leq x' \leq 1} x'^\theta |g(x')|
\]
and by, Corollary 2.4
\[
\int_1^\infty |g(x')| |T(x, x')| \, dx' \leq \|g\|_{L^\infty(1, \infty)} \int_0^\infty |T(x, x')| \, dx' \leq M \|g\|_{L^\infty(1, \infty)}.
\]
and \(||F(g)||_\theta \leq C||g||_\theta\). Proof of (iv). By (ii) only \( F(g) \in L^\infty_{loc}(0, \infty) \) remains to be proved. For \( K = [a, b] \subset (0, \infty) \), \( [a, b] \in (\alpha, \beta) \) and all \( x \in [a, b] \),
\[
\int_0^\infty T(x, x') g(x') \, dx' = \int_{g \in (\alpha, \beta)} T(x, x') g(x') \, dx' + \int_{g \in (\alpha, \beta)c} T(x, x') g(x') \, dx' = I_1 + I_2.
\]
where,
\[
|I_1(x)| = \int_\alpha^\beta T(x, x') g(x') \, dx' \leq \|g\|_{L^\infty(\alpha, \beta)} \int_\alpha^\beta |T(x, x')| \, dx' \leq m(x) \|g\|_{L^\infty(\alpha, \beta)} \leq M \|g\|_{L^\infty(\alpha, \beta)}.
\]
and
\[
|I_2(x)| \leq \int_0^\alpha |T(x, x')| |g(x')| \, dx' + \int_\beta^\infty |T(x, x')| |g(x')| \, dx'.
\]
By (2.1) and (2.10),
\[
\int_0^\alpha |T(x, x')| |g(x')| \, dx' \leq \sup\{|T(x, y)|; x \in [a, b], y < \alpha\} \int_0^\infty |g(x')| \, dx' \leq C_{\beta}(1 + b\alpha) \int_0^\infty |g(x')| \, dx' \leq C_{\beta}(1 + b\alpha) \int_0^\infty |g(x')| \, dx'.
\]
A simple inspection of the expression of \( T(x, x') \) given by (2.32) shows that \( C(K) = \sup\{|T(x, y)|; x \in [a, b], y > \beta\} < \infty \). Then,
\[
\int_\beta^\infty |T(x, x')| |g(x')| \, dx' \leq C(K) \int_0^\infty |g(x')| \, dx'.
\]
and,
\[
\sup_{x \in K} |F(g)(x)| \leq \|g\|_{L^\infty((\alpha, \beta))} M + (C_{\beta} + C(K))(1 + b\alpha)|g|_1.
\]
Proof of (v). For all \( x > 0 \),
\[
\left| \int_0^\infty T(x, x') g(x') \, dx' \right| \leq \int_0^R T(x, x') |g(x')| \, dx' + \int_R^\infty T(x, x') |g(x')| \, dx' \leq \int_0^R T(x, x') |g(x')| \, dx' + M \|g\|_{L^\infty(R, \infty)}.
\]
If \( x \leq 2R \), by (2.1) and (2.10)
\[
\int_0^R T(x,x')|g(x')|dx' \leq C_{2R} \int_0^R x'|g(x')|dx' \leq 2C_{2R}R^2 ||g||_1.
\]
For \( x > 2R \), we use the original expression of \( T(x,x') \) in (1.32):
\[
0 \leq \frac{x^3 \sinh x^2}{x^3 \sinh x^2} \left( \frac{1}{\sinh |x^2 - x'^2|} - \frac{1}{\sinh(x^2 + x'^2)} \right) \leq C \frac{e^{x^2}}{x^3} \left( e^{-x^2 + x'^2} \right) \leq \frac{C_R}{x^3}
\]
\[
0 \leq \frac{x'}{x} \left( \frac{1}{|x^2 - x'^2|} - \frac{1}{x^2 + x'^2} \right) \leq \frac{R}{x^3}
\]
then,
\[
|T(x,x')| \leq \frac{C_R}{x^3}, \quad \forall x > 2R, \quad x' \in (0, R).
\]
and (iv) follow since,
\[
\int_0^R T(x,x')|g(x')|dx' \leq \frac{C_R}{x^3} ||g||_1, \quad \forall x > 2R.
\]

### 3 Existence of global solution \( f \).

Using the properties of the operator \( L \), Proposition 2.1 and a fixed point argument, classical solutions \( f \in C([0, \infty); L^1(0, \infty)) \) of the Cauchy problem for (1.28) with initial data \( f_0 \in L^1(0, \infty) \) are obtained. If moreover \( f_0 \in L^1(0, \infty) \cap L^\infty(0, \infty) \) then \( f \in C([0, \infty); L^1(0, \infty)) \cap L^\infty(0, \infty; L^\infty) \). However it is interesting to consider initial data slightly more general than in \( f_0 \in L^1(0, \infty) \cap L^\infty(0, \infty) \) but whose solutions are more regular than just integrable with respect to \( x \) in \((0, \infty)\).

**Theorem 3.1.** Suppose that \( f_0 \in L^1(0, \infty) \) satisfies
\[
|||f_0|||_\theta \equiv \sup_{0 < x < 1} x^\theta |f_0(x)| + \sup_{x > 1} |f_0(x)| < \infty \quad (3.1)
\]
for some \( \theta \in (0, 1) \). Then, there exists a function
\[
f \in C([0, \infty); L^1(0, \infty)) \cap L^\infty_{loc}((0, \infty); L^\infty(0, \infty)) \quad (3.2)
\]
satisfying,
\[
f(t) = S(t)f_0 + \int_0^t S(t-s)F(f(s))ds. \quad (3.3)
\]
For all \( t > 0 \), \( f(t) \in C(0, \infty) \) and for all \( T > 0 \) and \( t \in (0, T) \),
\[
||f(t)||_1 \leq C_T ||f_0||_1 \quad (3.4)
\]
\[
||f(t)||_1 + |||f(t)|||_\theta \leq C(T)(||f_0||_1 + |||f_0|||_\theta), \quad \forall t \in (0, T) \quad (3.5)
\]
\[
||f(t)||_\infty \leq C(T, \theta) \left( ||f_0||_{L^\infty(1, \infty)} + t^{-\theta} \sup_{0 < y < 1} y^\theta |f_0(y)| + ||f_0||_1 \right). \quad (3.6)
\]
Proof. Given $f_0$ fixed and satisfying the hypothesis, consider the operator

$$\mathcal{L}(f)(t, x) = S(t)f_0(x) + \int_0^t S(t-s)(F(f(s)))ds$$
on the space

$$Z_T = C((0,T); L^1(0,\infty)) \cap C((0,T); L^\infty(0,\infty)),$$

$$\|f\|_{Z_T} = \sup_{s \in (0, T)} \left( \|f(s)\|_1 + \|f(s)\|_{L^\infty(1,\infty)} + s^\theta \sup_{0 < x < 1} |f(s, x)| \right)$$

By (ii) in Proposition (2.1),

$$\|\mathcal{L}(f(t))\|_1 \leq C_s \|f_0\|_1 + C_s C_F \int_0^t \|f(s)\|_1 ds \quad (3.7)$$

On the other hand,

$$|\mathcal{L}(f(t))(x)| \leq |S(t)f_0(x)| + \int_0^t |S(t-s)F(f(s))(x)|ds.$$

By (iv) in Proposition (2.1) and Proposition 6.2 in the Appendix, for all $t > 0$, $s \in (0, t)$ and $x \in (0, 2)$,

$$|S(t-s)F(f(s))(x)| \leq C\|F(s)\|_{L^\infty(1,\infty)} + C(t-s)^{-\theta} \sup_{0 < x < 1} x^\theta |F(f(s))(x)|$$

$$\leq C(1 + (t-s)^{-\theta})\|F(f)\|_{L^\infty} \leq C(1 + (t-s)^{-\theta})(\|f(t)\|_{L^\infty(1,\infty)} + \|f(t)\|_1)$$

Since we also have, for $x \in (0, 2)$,

$$|S(t)f_0(x)| \leq C \left( \|f_0\|_{L^\infty(1,\infty)} + t^{-\theta} \sup_{0 < x < 1} x^\theta |f_0(x)| \right) \quad (3.8)$$

we deduce, for $t > 0$, $s \in (0, t)$ and $x \in (0, 2)$,

$$|\mathcal{L}(f(t))(x)| \leq C \left( \|f_0\|_{L^\infty(1,\infty)} + t^{-\theta} \sup_{0 < x < 1} x^\theta |f_0(x)| \right) +$$

$$+ C \int_0^t (1 + (t-s)^{-\theta}) \left( \|f(s)\|_1 + \|f(s)\|_{L^\infty(1,\infty)} \right) ds$$

and then,

$$\sup_{0 < x < 2} t^\theta |\mathcal{L}(f(t))(x)| \leq C \left( t^\theta \|f_0\|_{L^\infty(1,\infty)} + \sup_{0 < x < 1} x^\theta |f_0(x)| \right) +$$

$$+ C t^\theta \int_0^t (1 + (t-s)^{-\theta}) \left( \|f(s)\|_1 + \|f(s)\|_{L^\infty(1,\infty)} \right) ds. \quad (3.9)$$

If $x > 2$,

$$|S(t-s)F(f(s))(x)| \leq C\|F(s)\|_{L^\infty(1,\infty)} + C\|F(f(s))\|_1$$

$$\leq C \left( \|f(s)\|_1 + \|f(s)\|_{L^\infty(1,\infty)} \right)$$

$$|S(t)f_0(x)| \leq C \left( \|f_0\|_{L^\infty(1,\infty)} + \|f_0\|_1 \right),$$

$$24$$
\[ \sup_{x>2} |L(f(t))(x)| \leq C \left( \|f_0\|_{L^{\infty}(1,\infty)} + \|f_0\|_1 \right) + C \int_0^t \left( \|f(s)\|_1 + \|f(s)\|_{L^{\infty}(1,\infty)} \right) ds. \quad (3.10) \]

Adding (3.17), (3.14) and (3.16),
\[
\begin{align*}
\|L(f(t))\|_1 + t^\theta &\sup_{0<x<1} x^\theta \left| L(f(t))(x) \right| + \left| \sup_{x>1} L(f(t))(x) \right| \\
&\leq \|L(f(t))\|_1 + t^\theta \left| \sup_{0<x<1} L(f(t))(x) \right| + \left| \sup_{x>1} L(f(t))(x) \right| \\
&\leq C \left( \|f_0\|_{L^{\infty}(R,\infty)} + \|f_0\|_1 \right) + C \left( t^\theta \|f_0\|_{L^{\infty}(1,\infty)} + \sup_{0<x<1} x^\theta |f_0(x)| \right) + \\
&+ C t^\theta \int_0^t \left( 1 + (t-s)^{-\theta} \right) \left( \|f(s)\|_1 + \|f(s)\|_{L^{\infty}(1,\infty)} \right) ds + \\
&+ C \int_0^t \left( \|f(s)\|_1 + \|f(s)\|_{L^{\infty}(1,\infty)} \right) ds \\
&\leq C (1 + t^\theta) \|f_0\|_{L^{\infty}(1,\infty)} + \|f_0\|_1 + C \sup_{0<x<1} x^\theta |f_0(x)| + \\
&+ C t^\theta \sup_{0<s<t} \left( \|f(s)\|_1 + \|f(s)\|_{L^{\infty}(R,\infty)} \right) (t + t^{1-\theta}) + T \sup_{0<s<1} s^\theta |f(t, x)| \\
&\leq C_1 \left( 1 + T^\theta \right) \|f_0\|_{L^{\infty}(1,\infty)} + \|f_0\|_1 + \sup_{0<x<1} x^\theta |f_0(x)| + C_2 T \|f\|_{Z_T}. \quad (3.11) \end{align*} \]

If we denote
\[ \gamma_0 = C_1 (2 \|f_0\|_{L^{\infty}(1,\infty)} + \|f_0\|_1 + \sup_{0<x<1} x^\theta |f_0(x)|) \quad (3.12) \]
we have then proved,
\[ \|L(f)\|_{Z_T} \leq \gamma_0 + C_2 T \|f\|_{Z_T}, \ \forall T \in (0, 1). \quad (3.13) \]

Let \( \rho > 0 \) and \( T > 0 \) be such that:
\[ T < \min \left( 1, \frac{1}{2 C_2} \right) \quad (3.14) \]
\[ \rho > 2 \gamma_0. \quad (3.15) \]

Then, for all \( f \in Z_T \) such that \( \|f\|_{Z_T} \leq \rho \),
\[ \|L(f)\|_{Z_T} \leq \gamma_0 + C_2 T \rho \leq \rho \quad (3.16) \]
and then,
\[ L : B_{Z_T}(0, \rho) \rightarrow B_{Z_T}(0, \rho), \quad (3.17) \]
\[ B_{Z_T}(0, \rho) = \{ f \in Z_T; \|f\|_{Z_T} \leq \rho \}. \quad (3.18) \]
On the other hand,

$$\mathcal{L}(f(t)) - \mathcal{L}(g(t)) = \int_0^t S(t-s)F(f(s) - g(s))ds$$

and arguing as before,

$$||\mathcal{L}(f) - \mathcal{L}(g)||_{Z_T} \leq CT||f - g||_{Z_T}$$

The map $\mathcal{L}$ is then a contraction form $B_{Z_T}(0, \rho)$ into itself if $T$ is small enough, and has a fixed point $u \in B_{Z_T}(0, \rho)$ that satisfies

$$f(t) = S(t)f_0 + \int_0^t S(t-s)F(f(s))ds$$

in $Z_T$. Property (3.4) follows from and Gronwall’s Lemma on $(0, T)$ and by (3.9) and (3.10),

$$||f(t)||_1 + ||f(t)||_\theta \leq C(||f_0||_1 + ||f_0||_\theta) +$$

$$+ C \left(1 + t^\theta\right) \int_0^t \left(1 + (t-s)^{-\theta}\right) \left(||f(s)||_{L^\infty(1, \infty)} + ||f(s)||_1\right), \forall t \in (0, T).$$

Then, there exists a constant $C = C(T) > 0$ such that, (3.5) holds true.

On the other hand, since $f_0 \in L^1(0, \infty)$ and $||f_0||_\theta < \infty$, by Proposition 6.2 in the Appendix, $S(t)f_0 \in L^\infty(0, \infty)$. Moreover, by Proposition 6.2 and Proposition 2.1, for $t \in (0, T)$ and $x \in (0, 2)$,

$$|S(t-s)F(f(s))(x)| \leq C(t-s)^{-\theta} \sup_{0 < y < 1} x^\theta |F(f(s))(x)| + ||F(f(s)||_{L^\infty(1, \infty)}$$

$$\leq C(1 + (t-s)^{-\theta}) ||F(f(s))||_{\infty}$$

$$\leq C(1 + (t-s)^{-\theta})(||f(s)||_1 + ||f(s)||_{L^\infty(1, \infty)})$$

$$\leq C(1 + (t-s)^{-\theta}) \rho$$

(3.22)

It immediately follows that for all $t \in (0, T)$ and $x \in (0, 2),$

$$|f(t, x)| \leq C||f_0||_{L^\infty(1, \infty)} + Ct^{-\theta} \sup_{0 < y < 1} y^\theta |f_0(y)| + C\rho \int_0^t (1 + (t-s)^{-\theta})ds.$$

and then, $f(t) \in L^\infty(0, \infty)$ for all $t \in (0, T)$. We wish to extend now this function $f$ for all $t > 0$. We notice to this end that, for all $x > 1,$

$$|f(t, x)| \leq C||f_0||_{L^\infty(1, \infty)} + Ct^{-\theta} \sup_{0 < y < 1} y^\theta |f_0(y)| + \int_0^t |S(t-s)F(f(s))(x)|ds$$

$$\leq C||f_0||_{L^\infty(1, \infty)} + Ct^{-\theta} \sup_{0 < y < 1} y^\theta |f_0(y)| +$$

$$+ C \int_0^t (1 + (t-s)^{-\theta})(||f(s)||_1 + ||f(s)||_{L^\infty(1, \infty)})$$

(3.23)
Since by Proposition 2.1,

$$||f(t)||_1 \leq C||f_0||_1 + C \int_0^t ||f(s)||_1 ds,$$  \hspace{1cm} (3.23)

we obtain,

$$\sup_{x>1} |f(t,x)| + ||f(t)||_1 \leq C \left( t^{-\theta} \sup_{x \in (0,1)} x^\theta |f_0(x)| + ||f_0||_1 \right) +$$

$$+ C \int_0^t (1 + (t-s)^{-\theta})(||f(s)||_1 + ||f(s)||_{L^\infty(1,\infty)}) ds.$$

It follows by Gronwall’s Lemma, that for some constant $C$ depending on $T$ and $\theta$,

$$\sup_{x>1} |f(t,x)| + ||f(t)||_1 \leq C(T, \theta) \left( ||f_0||_{L^\infty(1,\infty)} + t^{-\theta} \sup_{0<s<1} x^\theta |f_0(x)| + ||f_0||_1 \right).$$  \hspace{1cm} (3.24)

On the other hand, for $x \in (0, 2)$, using (3.22)

$$|f(t,x)| \leq C||f_0||_{L^\infty(1,\infty)} + Ct^{-\theta} \sup_{0<y<1} y^\theta |f_0(y)| +$$

$$+ C \int_0^t (1 + (t-s)^{-\theta})(||f(s)||_1 + ||f(s)||_{L^\infty(1,\infty)}) ds$$

and by (3.24), for all $x \in (0, 2)$,

$$|f(t,x)| \leq C||f_0||_{L^\infty(1,\infty)} + Ct^{-\theta} \sup_{0<y<1} y^\theta |f_0(y)| +$$

$$+ C(T, \theta) \int_0^t (1 + (t-s)^{-\theta}) \left( ||f_0||_{L^\infty(1,\infty)} + s^{-\theta} \sup_{0<s<1} x^\theta |f_0(x)| + ||f_0||_1 \right) ds$$

$$\leq C(T, \theta) \left( ||f_0||_{L^\infty(1,\infty)} + t^{-\theta} \sup_{0<y<1} y^\theta |f_0(y)| + ||f_0||_1 \right).$$  \hspace{1cm} (3.25)

We obtain from (3.23), (3.21) and (3.24) for all $t \in (0, T)$,

$$||f(t)||_1 + ||f(t)||_\infty \leq C(T, \theta) \left( ||f_0||_{L^\infty(1,\infty)} + t^{-\theta} \sup_{0<y<1} y^\theta |f_0(y)| + ||f_0||_1 \right)$$  \hspace{1cm} (3.26)

By a classical argument it follows that the function $f$ may be extended to a function, still denoted $f$, for all $t > 0$ such that $f \in Z_t$ for all $t > 0$ and satisfies (3.19) for all $t > 0$.

The same arguments used to prove the estimates (3.4), (3.5) and (3.6) on the interval of time given by (3.14) may now be applied to obtain (3.4), (3.5), (3.6) on all finite interval $(0, T)$ for all $T > 0$.

Since $f_0 \in L^1(0, \infty)$ and $||f_0||_\theta < \infty$ it follows from Proposition 6.1 that $S(t)f_0 \in C(0, \infty)$ for every $t > 0$. Since $f(s) \in L^\infty(0, \infty)$ for all $s > 0$ it follows by Proposition 2.1 that $F'(f(s)) \in L^\infty(0, \infty)$ too and therefore, again by 6.1 $S(t-s)F'(f(s)) \in C(0, \infty)$ for all $t > 0$ and $s \in (0, t)$. This shows that $f(t) \in C(0, \infty)$ for all $t > 0$.  \hspace{1cm} $\Box$
Theorem 3.2. Suppose that \( f_0 \in L^1(0, \infty) \) satisfies (3.1) and \( f \) is the function given by Theorem 3.2. Then,

\[
\frac{\partial f}{\partial t}, \mathcal{L}(f) \in L^\infty_{\text{loc}}((0, \infty) \times (0, \infty)) \cap L^1((0, T) \times (0, \infty)), \forall T > 0,
\]
and, for all almost every \( t > 0, x > 0, \)

\[
\frac{\partial f(t, x)}{\partial t} = \mathcal{L}(f(t))(x).
\]

There exists a function \( \tilde{H} \in L^\infty((\delta, \infty) \times (\bar{\delta}, \infty)) \) for all \( \delta > 0 \) such that

\[
\left| \frac{\partial f(t, x)}{\partial t} \right| + |\mathcal{L}(f)(t, x)| \leq C \left( \sup_{0 < y < 1} y^\theta |f_0(y)| + ||f_0||_{L^\infty(1, \infty)} + ||f_0||_1 \right) \tilde{H}(t, x)
\]

(\( \tilde{H} \) is defined in (3.40) below), and

\[
\left| \frac{\partial f}{\partial x}(t, x) \right| \leq \tilde{\lambda}_{\beta, \delta}(t, x)||f_0||_{\theta, 1} + \sup_{0 \leq s \leq t} \left( ||f(s)||_\theta + ||f(s)||_1 \right) \int_0^t \tilde{\lambda}_{\beta, \delta}(s, x)ds,
\]

where: \( \forall \delta \in (0, 1), \forall \beta \in (0, 1), \quad \tilde{\lambda}_{\beta, \delta}(t, x) = \begin{cases} x^{-1+\delta} t^{-\theta-\delta}, & 0 < x < t < 1 \\ t^2, & t > 1, x \in (0, t) \\ x^{-1-\beta} t^\beta, & x > t. \end{cases} \)

For all \( \varphi \in C^1_0([0, \infty)), \) the map \( t \mapsto \int_0^\infty \varphi(x)f(t, x)dx \) belongs to \( W^{1,1}_{\text{loc}}(0, \infty) \) and for almost every \( t > 0, \)

\[
\frac{d}{dt} \int_0^\infty \varphi(x)f(t, x)dx = \int_0^\infty \mathcal{L}(f(t))(x)\varphi(x)dx
\]

Proof. We begin proving (3.27), (3.28), and (3.29). Since \( f_0 \in L^1(0, \infty) \) and \( ||f_0||_{\theta} < \infty, \) by Proposition 6.2 in the Appendix \( S(t)u_0 \in L^\infty(0, \infty) \) and Theorem 1.2 in [8],

\[
\frac{\partial}{\partial t}((S(t)f_0) \in L^\infty(0, \infty); \quad L(S(t)f_0)(x) \in L^\infty(0, \infty), \forall t > 0
\]

\[
\frac{\partial}{\partial t}((S(t)f_0)(x)) = L(S(t)f_0)(x), \text{ for a. e. } t > 0, x > 0.
\]

Since for all \( s > 0, F(f(s)) \in L^\infty(0, \infty) \cap L^1(0, \infty), \) for almost every \( x > 0, t \in (0, T) \) and \( s \in (0, t), \) by the same argument,

\[
\frac{\partial}{\partial t}[(S(t-s)F(f(s)))(x)] = L(S(t-s)F(f(s)))(x),
\]

where both terms belong to \( L^\infty(0, \infty). \) Let us define

\[
\xi(t, x) = \frac{t^3}{\max(t^4, x^4)}
\]

\[
\zeta_\theta(t, x) = \frac{\min(t, t^2-\theta)}{x^3} \mathbb{1}_{t < 2x/3} + \frac{x \mathbb{1}_{2x < t}}{\max(t^2+\theta, t)} + \frac{\mathbb{1}_{2x/3 < t < 2x}}{\max(x^2, x^4+\theta)}.
\]
By (6.19), and (iv) of Proposition (2.1), for all \( t \in (0, T) \), \( s \in (0, t) \) and \( x > 0 \),

\[
\left| \frac{\partial}{\partial t} (S(t-s)F(f(s)))(x) \right| \leq C \|F(f(s))\|_\infty (1 + \xi(t-s, x)) + \zeta_0(t-s, x) \|F(f(s))\|_{1,0} \\
\leq C \left( \|f(s)\|_{L^\infty(R, \infty)} + \|f(s)\|_1 \right) (1 + \xi(t-s, x)) + \\
+ \zeta_0(t-s, x) \left( \|f(s)\|_1 + \|f(s)\|_{L^\infty(R, \infty)} \right) \\
\leq C(1 + \rho) \left( 1 + \xi(t-s, x) + \zeta_0(t-s, x) \right). \tag{3.36}
\]

The right hand side of (3.36) may now be estimated for all \( s \in (0, t) \) using:

\[
\zeta_0(t-s, x) + \xi(t-s, x) \leq \begin{cases} 
\frac{C}{x} + \frac{1_{0<s<t-x}}{x} + \frac{1_{t-x<s<x}t^3}{x^4}, & \text{when } x \in (0, t) \\
\frac{C}{x} + \frac{1}{x^4}, & \text{when } x > t
\end{cases}
\]

We deduce, for all \( t > 0, x > 0 \),

\[
\frac{\partial}{\partial t} \left( \int_0^t S(t-s)F(f(s))(x)ds \right) = F(f(t))(x) + \int_0^t \frac{\partial}{\partial t} (S(t-s)F(f(s)))(x) ds. \tag{3.37}
\]

and,

\[
\left| \frac{\partial}{\partial t} (S(t-s)F(f(s)))(x) \right| \leq C \|F(f(s))\|_\infty (1 + \xi(t-s, x)) + \zeta(t-s, x) \|F(f(s))\|_1 \\
\leq C \left( \|f(s)\|_{L^\infty(R, \infty)} + \|f(s)\|_1 \right) (1 + \xi(t-s, x) + \zeta_0(t-s, x)). \tag{3.38}
\]

Then,

\[
\left| \frac{\partial f(t,x)}{\partial t} \right| \leq \left| \frac{\partial}{\partial t} (S(t)u_0)(x) \right| + \|F(f(t))(x)\| + \left| \int_0^t \frac{\partial}{\partial t} (S(t-s)F(f(s)))(x) ds \right| \\
\leq C \|S(t)u_0\|_\infty (1 + \xi(t,x)) + \zeta_0(t,x) \|S(t)f_0\|_{1,0} + \|F(f(t))\|_\infty + \\
+ C \int_0^t \left( \|f(s)\|_{L^\infty(1, \infty)} + \|f(s)\|_{1,0} \right) (1 + \xi(t-s, x) + \zeta_0(t-s, x)) ds
\]

Then, for all \( T > 0 \) and \( t \in (0, T) \), there exists \( C = C(T, \theta) \) such that,

\[
\left| \frac{\partial f(t,x)}{\partial t} \right| \leq C \left( \|f_0\|_{L^\infty(1, \infty)} + t^{-\theta} \sup_{0<y<1} y^\theta |f_0(y)| + \|f_0\|_1 \right) (1 + \xi(t,x) + \zeta_0(t,x)) + \\
+ C \int_0^t \left( \|f_0\|_{L^\infty(1, \infty)} + s^{-\theta} \sup_{0<y<1} y^\theta |f_0(y)| + \|f_0\|_1 \right) (1 + \xi(t-s, x) + \zeta_0(t-s, x)) ds
\]

Using Proposition (6.8) in the Appendix it follows,

\[
\int_0^t \left( \|f_0\|_{L^\infty(1, \infty)} + s^{-\theta} \sup_{0<y<1} y^\theta |f_0(y)| + \|f_0\|_1 \right) (1 + \xi(t-s, x) + \zeta_0(t-s, x)) ds \\
\leq C(\|f_0\|_{L^\infty(1, \infty)} + \|f_0\|_1) (\Xi_1(t,x) + \omega_{1,0}(t,x)) + \sup_{0<y<1} y^\theta |f_0(y)| (\Xi_2(t,x) + \omega_2(t,x)).
\]

We deduce,
\[
\frac{|\partial f(t, x)|}{\partial t} \leq C \left( ||f_0||_{L^\infty_1} + t^{-\theta} \sup_{0 < y < 1} y^\theta |f_0(y)| + ||f_0||_1 \right) (1 + \xi(t, x) + \zeta(t, x)) + \\
+ C(||f_0||_{L^\infty_1} + ||f_0||_1) (\Xi_1(t, x) + \omega_1(t, x)) + \sup_{0 < y < 1} y^\theta |f_0(y)| (\Xi_2(t, x) + \omega_2(x, t))
\]
\[
\leq C \left( ||f_0||_{L^\infty_1} + t^{-\theta} \sup_{0 < y < 1} y^\theta |f_0(y)| + ||f_0||_1 \right) \tilde{H}(t, x) \tag{3.39}
\]

with
\[
\tilde{H}(t, x) = (1 + \xi(t, x) + \zeta(t, x)) + \Xi_1(t, x) + \omega_1(t, x) + \Xi_2(t, x) + \omega_2(x, t). \tag{3.40}
\]

Estimates (3.27) and (3.28) for \( \partial f \) follow using again Proposition 6.8. Moreover, by (3.37), (3.38) and (3.39),
\[
\frac{\partial}{\partial t} \left( \int_0^t S(t-s)F(f(s))ds \right) = F(f(t)) + \int_0^t \frac{\partial}{\partial t} (S(t-s)F(f(s)))\, ds,
\]
for almost every \( t > 0 \) and \( x > 0 \),
\[
\frac{\partial f(t, x)}{\partial t} = \frac{\partial}{\partial t} ((S(t)f_0)(x)) + \frac{\partial}{\partial t} \left( \int_0^t S(t-s)F(f(s))(x)\, ds \right)
\]
\[
= L(S(t)f_0)(x) + F(f(t))(x) + \int_0^t L(S(t-s)F(f(s)))(x)\, ds
\]
\[
= L \left( S(t)f_0 + \int_0^t S(t-s)F(f(s))\, ds \right)(x) + F(f(t))(x)
\]
\[
= L(f(t))(x) + F(f(t))(x).
\]

By Proposition 2.1 this shows \( L(f) \in L^\infty_{loc}((0, \infty); L^\infty(\delta, \infty) \cap L^1(0, \infty)) \) for all \( \delta > 0 \). This ends the proof of (3.27) and (3.28), and proves (3.29).

To prove (3.30) notice that by Corollary 6.4 \( \partial x(S(t)f_0) \) exists for almost every \( t > 0 \) and \( x > 0 \) and satisfies (6.17) and by Proposition 2.1 and Corollary 6.4 \( \partial x(S(t-s)F(f(s))) \) exists for almost every \( t > 0 \) and \( s \in (0, t) \) and \( x > 0 \) and satisfies:
\[
\frac{|\partial S(t-s)f(s)(x)|}{\partial x} \leq h(t-s)g(x)||F(f(s))||_1, \theta
\]
\[
\leq C h(t-s)g(x) \left( ||f(s)||_1 + ||f(s)||_\theta \right)
\]
\[
\leq C h(t-s)g(x) \sup_{0 \leq s \leq t} \left( ||f(s)||_1 + ||f(s)||_\theta \right).
\]

Since \( h(t-s) \in L^1(0, t) \), (3.30) easily follows.

On the other hand, if we multiply both sides of (3.19) by \( \varphi \in C^1_0([0, \infty)) \) and integrate,
\[
\int_0^\infty f(t, x)\varphi(x)\, dx = \int_0^\infty \varphi(x)S(t)f_0(x)\, dx + \int_0^\infty \varphi(x) \int_0^t S(t-s)F(f(s))(x)\, ds dx.
\]
In order to derivate this expression with respect to $t$, we use again (3.33) and (3.38) to obtain,

\[
\frac{d}{dt} \int_0^\infty f(t,x)\varphi(x)dx = \int_0^\infty \varphi(x)L(S(t)f_0)(x)dx + \int_0^\infty \varphi(x)F(f(t,x))dx + \int_0^\infty \varphi(x) \int_0^t L(S(t-s)F(f(s))(x))ds dx,
\]

and for all $t \in (0,T)$,

\[
\left| \frac{d}{dt} \int_0^\infty f(t,x)\varphi(x)dx \right| \leq C||\varphi||_1||f_0||_\infty + C(1+T)||\varphi||_1||f||_\infty.
\]

which shows

\[
\frac{d}{dt} \int_0^\infty f(t,x)\varphi(x)dx \in L^\infty_{\text{loc}}([0,\infty)).
\]

Identity (3.32) follows now, since

\[
\frac{d}{dt} \int_0^\infty f(t,x)\varphi(x)dx = \int_0^\infty \varphi(x)L((S(t)f_0)(x) + \int_0^t S(t-s)F(f(s))(x))ds dx + \int_0^\infty \varphi(x)F(f(t,x))dx.
\]

\[
\square
\]

4 Further properties of the solution $f$.

We describe in this Section some further properties of the solutions $f$ given by Theorem 3.1. We first consider what are the variations of mass and energy induced by the initial perturbation $n_0(1+n_0)x^2f(\tau)$ of the equilibrium $n_0$ introduced in (1.7), (1.8). Then we prove that for all $\delta > 0$, $f \in L^\infty((\delta,\infty) \times (0,\infty))$ and that for every $t > 0$ the function $f(t)$ has a limit as $x \to 0$.

4.1 Mass and Energy.

It will be sometimes denoted in what follows

\[
N_0(x) = \nu_0(x^2)(1+\nu_0(x^2))x^6; \quad d\mu(x) = N_0(x)dx, \quad \int_0^\infty \int_0^\infty W(x,x')|f(\tau,x') - f(\tau,x)|^2x'^4x^4dx'dx = D(f(\tau)).
\]

A first basic property is the following,

\textbf{Proposition 4.1.} Let $f_0$ and $f$ be as in Theorem 3.1. Then for all $p > 1$,

\[
\frac{d}{dt} \int_0^\infty |f(t,x)|^p d\mu(x) \leq 0, \forall t > 0.
\]

\[
||f(t_2)||_{L^\infty(d\mu)} \leq ||f(t_1)||_{L^\infty(d\mu)}, \forall t_2 > t_1 \geq 0.
\]
Proof. Since \( f \) satisfies (3.27), (3.29), and \( x^6n_0(1+n_0)|f|^{p-2}f \in L^1(0,\infty) \), multiplication of both sides of (3.29) and integration over \((0,\infty)\) gives, using (1.27) and the symmetry of \( W(x,x') \) as follows,

\[
\frac{d}{dt} \int_0^\infty \nu_0(x^2)\left(1+\nu_0(x^2)\right)|f(t,x)|^p x^6 dx =
\]

\[
= \int_0^\infty \int_0^\infty W(x,x')(f(\tau,x')-f(\tau,x))|f|^{p-2} f x^4 x' dx' dx
\]

\[
= -\frac{1}{2} \int_0^\infty \int_0^\infty W(x,x')(f(\tau,x')-f(\tau,x)) \left( |f'|^{p-2} f' - |f|^{p-2} f \right) x^4 x' dx' dx \leq 0. \quad (4.3)
\]

Since \( d\mu \) is a non negative finite measure on \((0,\infty)\),

\[
||f(t_2)||_{L^\infty(d\mu)} = \lim_{p \to \infty} ||f(t_2)||_{L^p(d\mu)} \leq \lim_{p \to \infty} ||f(t_1)||_{L^p(d\mu)} = ||f(t_1)||_{L^\infty(d\mu)}.
\]

\[\square\]

Lemma 4.2. For all \( \theta \in [0,3) \),

\[
(i) \quad ||f||_{L^\infty(d\mu)} \leq \max \left(||f||_{L^\infty(1,\infty)}, \sup_{x(0,1)} x^\theta |f(x)|\right) \quad (4.4)
\]

\[
(ii) \quad ||x^6 n_0(1+n_0)f||_{L^\infty} \leq ||f||_{L^\infty(d\mu)}, \forall f \in L^\infty(d\mu) \cap L^\infty(0,\infty) \quad (4.5)
\]

Proof. (i) For all \( C > 0 \) denote,

\[A_C = \{ x \in (0,\infty); |f(x)| > C \}.\]

Then, if \( C > ||f||_{L^\infty(1,\infty)} \), the Lebesgue measure of \( A_C \cap (1,\infty) \) is zero and

\[
d\mu(A_C) = \int_{A_C \cap (0,1)} n_0(1+n_0)x^6 dx + \int_{A_C \cap (1,\infty)} n_0(1+n_0)x^6 dx
\]

\[
= \int_{A_C \cap (0,1)} n_0(1+n_0)x^6 dx
\]

On the other hand,

\[
\int_{A_C \cap (0,1)} n_0(1+n_0)x^6 dx \leq \sup_{x>0} (n_0(1+n_0)x^4) \int_{A_C \cap (0,1)} x^2 dx
\]

\[
\leq \sup_{x>0} (n_0(1+n_0)x^4) \frac{1}{C} \int_{A_C \cap (0,1)} |f(x)|x^2 dx
\]

Then, since \( x^\theta f \in L^1(0,1) \), if \( C > \sup_{x(0,1)} x^\theta |f(x)| \),

\[
\int_{A_C \cap (0,1)} x^\theta |f(x)| dx = 0.
\]
It follows that for all $C > \max(||f||_{L^\infty(1,\infty)}, \sup_{x \in (0,1)} x^6 |f(x)|)$,

$$|f(x)| \leq C,$$

except in a set of zero measure, and this proves (4.4).

(ii) Let us denote now,

$$B_C = \{ x > 0; x^6 n_0(1 + n_0)|f(x)| \geq C \}$$

and suppose that $C > ||f||_{L^\infty(d\mu)}$. Then

$$m(B_C) = \int_{B_C} dx \leq \frac{1}{C} \int_{B_C} x^6 n_0(1 + n_0)|f(x)|dx \leq \frac{||f||_\infty}{C} \int_{B_C} x^6 n_0(1 + n_0)dx = \frac{||f||_\infty}{C} d\mu(B_C) = 0.$$

The following easily follows now,

**Corollary 4.3.** Let $f_0$ and $f$ be as in Theorem [3.1]. Then,

$$\frac{d}{dt} \int_0^\infty n_0(x)(1 + n_0(x)f(t,x)x^6 dx = 0, \forall t > 0.$$

**Proof.** Since $n_0(1 + n_0)x^6 \in C^1_0([0,\infty))$, identity [3.32] with $\varphi = n_0(1 + n_0)x^6$ gives, by the definition of $\mathcal{L}$ (cf. [1.27]),

$$\frac{d}{dt} \int_0^\infty n_0(1 + n_0)x^4 f(\tau, x)x^2 dx = \int_0^\infty n_0(1 + n_0)x^4 \mathcal{L}(f(\tau))(x)dx = \int_0^\infty \int_0^\infty W(x, x')(f(\tau, x') - f(\tau, x))x^4 dx'dx^4.$$

The result follows from the symmetry of the kernel $W(x, y)$.

The following property will show the boundedness of the variation of the mass.

**Proposition 4.4.** Let $f_0$ and $f$ be as in Theorem [3.1]. Then for all $t > 0$ and all $p > 3$,

$$\int_0^\infty n_0(1 + n_0)x^4 |f(t, x)|dx \leq C_p \max \left( ||f_0||_{L^\infty((0,1)), \sup_{x \in (0,1)} x^6 |f_0(x)|}, \frac{||f_0||_{L^1(d\mu)}}{||f_0||_{L^1(d\mu)}} \right) \cdot \frac{||f_0||_{L^1(d\mu)}}{p}.$$

**Proof.** By Holder’s inequality,

$$\int_0^\infty n_0(1 + n_0)x^4 |f(t, x)|dx \leq \left( \int_0^\infty n_0(1 + n_0)x^6 |f(t, x)|^p dx \right)^\frac{1}{p} \times$$

$$\times \left( \int_0^\infty n_0(1 + n_0)x^{\left(4 - \frac{6}{p-1}\right)} dx \right)^\frac{p-1}{p}$$

$$= ||f(t)||_{L^p(d\mu)} \left( \int_0^\infty n_0(1 + n_0)x^{\left(\frac{6p}{p-1} - \frac{6}{p-1}\right)} dx \right)^\frac{p-1}{p}.$$
If $p > 3$,
\[
\frac{4p}{p-1} - \frac{6}{p-1} > 3, \quad \text{and then,}
\int_0^\infty n_0(1 + n_0)x^{\frac{4p}{p-1} - \frac{6}{p-1}} \, dx = C_p^{\frac{p}{p-1}} < \infty,
\]
\[
\lim_{p \to 3} C_p = \infty.
\]
It follows by Proposition 4.1 for all $t > 0$,
\[
\int_0^\infty n_0(1 + n_0)x^4|f(t, x)| \, dx \leq C_p\|f(t)\|_{L^p(\mu)} \leq C_p\|f_0\|_{L^p(\mu)}
\]
Since,
\[
\|f_0\|_{L^p(\mu)} \leq \|f_0\|_{L^\infty(\mu)}^{\frac{p-1}{p}} \|f_0\|_{L^1(\mu)}^{1/p}
\]
it follows from Lemma 4.2 that
\[
\|f_0\|_{L^p(\mu)} \leq \max \left( \|f_0\|_{L^\infty(1, \infty)}, \sup_{x \in (0, 1)} x^\theta |f_0(x)| \right)^{\frac{p-1}{p}} \|f_0\|_{L^1(\mu)}^{1/p}
\]
and the Proposition follows.

It immediately follows from Proposition 4.4.

Corollary 4.5. Let $f_0$ and $f$ be as in Theorem 3.1. Then, for $N(\tau)$ defined in (3.51), and all $p > 3$,
\[
N(\tau) \leq C_p \max \left( \|f_0\|_{L^\infty(1, \infty)}, \sup_{x \in (0, 1)} x^\theta |f_0(x)| \right)^{\frac{p-1}{p}} \|f_0\|_{L^1(\mu)}^{1/p}, \quad \forall \tau > 0,
\]
where $C_p$ is given in Proposition 4.4.

The following property also follows from similar arguments.

Proposition 4.6. Suppose $f_0$ and $f$ are as in Theorem 3.1. $C > 0$ is a constant and $f_0 \leq C$. Then $f(t) \leq C$ for all $t > 0$.

Proof. Since $L(C) \equiv 0$ it follows
\[
\frac{\partial(f(t, x) - C)}{\partial t} = L \left( f(t) - C \right)(x).
\]
If we multiply the equation by $n_0(1 + n_0)x^6f_C(t)^+\), with $f_C(t) = (f(t) - C)$
\[
\frac{d}{dt} \int_0^\infty n_0(1 + n_0)|f_C^+(t, x)|^2 x^6 \, dx = \int_0^\infty \int_0^\infty W(x, y)(f_C(t, y) - f_C(t, x))f_C^+(t, x)x^4y^4 \, dy \, dx
\]
\[
= -\int_0^\infty \int_0^\infty W(x, y)(f_C(t, y) - f_C(t, x))f_C^+(t, y)x^4y^4 \, dy \, dx
\]
\[
= -\frac{1}{2} \int_0^\infty \int_0^\infty W(x, y)(f_C(t, y) - f_C(t, x))(f_C^+(t, y) - f_C^+(t, x))x^4y^4 \, dy \, dx
\]
If \( f_C(t, y) > f_c(t, x) \) and \( f_C^+(t, y) \leq f_C^+(t, x) \) then \( f_C(t, y) \leq 0 \) because if \( f_C(t, y) > 0 \) we would have \( f_C^+(t, y) = f_c(t, y) > f(t, x) = f_C^+(t, x) \), a contradiction. But this implies \( f_C^+(t, x) = f_C^+(t, y) = 0 \). On the other hand, if \( f_C(t, y) < f_c(t, x) \) and \( f_C^+(t, y) \geq f_C^+(t, x) \) then, \( f_C(t, x) < 0 \) because if \( f_C(t, x) > 0 \) then \( f_C^+(t, x) = f_C(t, x) > 0 \) then, as before we would have, \( f_C^+(t, y) \geq f_C^+(t, x) > 0 \) and so \( f_C(t, y) = f_C^+(t, y) \) but this would give \( C f^+(t, y) = f_C(t, y) < f_C(t, x) = f_C^+(t, x) \), which is a contradiction. So in that case again \( f_C^+(t, x) = f_C^+(t, y) = 0 \). We deduce,

\[
\frac{d}{dt} \int_0^\infty n_0(1 + n_0) |f_C^+(t, x)|^2 x^\delta dx \leq 0
\]

\[
\int_0^\infty n_0(1 + n_0) |f_C^+(t, x)|^2 x^\delta dx = 0,
\]

from where \( f_C^+(t)(x) = (f(t, x) - C)^+ = 0 \) a.e. and the Proposition follows. \( \square \)

The following Corollary immediately follows,

**Corollary 4.7.** Suppose that \( f \) and \( g \) are two solutions of \([3.24]\) given by Theorem 3.1 with initial data \( f_0 \) and \( g_0 \) satisfying the hypothesis of Theorem 3.1 and such that \( f_0 \leq g_0 \). Then, \( f(t) \leq g(t) \) for all \( t > 0 \).

And let us also deduce,

**Corollary 4.8.** Suppose that \( f_0 \) and \( f \) are as in Theorem 3.1. Then, for all \( \delta > 0 \) and all \( t \geq \delta \),

\[
||f(t)||_{\infty} \leq C(\theta) \left( ||f_0||_{L^\infty(1, \infty)} + \delta^{-\theta} \sup_{0 \leq y \leq 1} y^\theta |f_0(y)| + ||f_0||_1 \right).
\]

**Proof.** By Theorem 3.1 for all \( \delta > 0 \), \( f(\delta) \in L^\infty(0, \infty) \cap L^1(0, \infty) \) and,

\[
||f(\delta)||_{\infty} \leq C(\theta) \left( ||f_0||_{L^\infty(1, \infty)} + \delta^{-\theta} \sup_{0 \leq y \leq 1} y^\theta |f_0(y)| + ||f_0||_1 \right).
\]

Since the constant \( ||f(\delta)||_{\infty} \) is a solution of \([3.29]\), the result follows by Corollary 4.7. \( \square \)

Our next result concerns the long time behavior of the solution \( f \). The \( L^2(d\mu) \) norm and \( D(f(\tau)) \) play here the usual roles of entropy and entropy’s dissipation through the identity (cf. \([4.33]\))

\[
\frac{d}{dt} \int_0^\infty |f(t, x)|^2 d\mu(x) = -D(f(\tau)) \tag{4.6}
\]

Consider the convex, proper lower semi continuous function

\[
j(u, v) = \begin{cases} 
|u - v|, & \text{if } u > 0, v > 0, \\
0, & \text{if } u \leq 0, v \leq 0 \\
\infty, & \text{elsewhere}
\end{cases}
\]

and define the following regularized kernel for all \( n \in \mathbb{N} \setminus \{0\} \),

\[
W_n(x, x') = \left( \frac{\sinh(x^2 + y^2) - \sinh |x^2 - y^2|}{\sinh(x^2 + y^2) \sinh |x^2 - y^2| + \frac{1}{n}} \right) \frac{1}{xy \sinh(x^2) \sinh(y^2)}
\]

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so that, \(d\sigma_n(x, x') = W_n(x, x')x^4x'dx'dx\) is now a bounded measure on \(\mathbb{R}^2\). Consider then for any pair of functions \(f, g\) defined on \((0, \infty)\) the function \(U\) defined on \((0, \infty)^2\) as,

\[
U(x, y) = (f(x), g(y)) \in \mathbb{R}^2
\]

and denote,

\[
J_n(U) = \begin{cases} 
\int_0^1 \int_0^1 j(f(x), g(x'))d\sigma_n(x'x), & \text{if } j(f, g') \in L^1(d\sigma_n) \\
\infty, & \text{elsewhere.}
\end{cases}
\]

It follows that \(J_n\) is convex and l.s.c. on \(L^2(d\mu)\).

**Proposition 4.9.** Let \(f_0\) and \(f\) be as in Theorem [3.1]. Then, for all \(\varphi \in L^2(d\mu)\),

\[
\lim_{\tau \to \infty} \int_0^\infty f(\tau, x)\varphi(x)d\mu(x) = \frac{E(0)}{\int_0^\infty d\mu(x)} \int_0^\infty \varphi(x)d\mu(x)
\]

**Remark 4.10.** Notice that, since \(d\mu\) is a non negative bounded measure on \((0, \infty)\), \(L^2(d\mu) \subset L^1(d\mu)\). Corollary 4.9 shows the weak convergence,

\[
n_0(1 + n_0)x^6 f(\tau) \to \tau \to \infty \frac{E(0)}{\int_0^\infty d\mu(x)}, \text{ weakly in } L^2(0, \infty).
\]

**Proof.** Consider any sequence \(\{\tau_k\}_{k \in \mathbb{N}}\) where \(\tau_k \to \infty\) as \(k \to \infty\) and define

\[
f_k(\tau) = f(\tau + \tau_k).
\]

By (4.8), for all \(T > 0\),

\[
\frac{1}{2} \int_0^T D(f(s))dt = -\int_0^\infty |f(T, x)|^2d\mu(x) + \int_0^\infty |f(0, x)|d\mu(x) \leq \int_0^\infty |f(x)|^2d\mu(x)
\]

from where we deduce \(D(f(s)) \in L^1(0, \infty)\) and

\[
\int_0^T D(f_k(t))dt = \int_{t_k}^{t_k+T} |Df(t)|dt \to 0, \text{ as } k \to \infty.
\]

Then, for all \(T > 0\),

\[
\lim_{k \to \infty} \int_0^T D_n(f_k(t))dt = 0, \forall n.
\]

and since \(D_n(f_k(t)) \geq 0\) for all \(t > 0\),

\[
\lim_{k \to \infty} D_n(f_k(t)) = 0, \text{ for a. e. } t \in (0, T).
\]

On the other hand, by (4.8) again, the sequence \(\{f_k\}_{k \in \mathbb{N}}\) is bounded in \(L^\infty((0, \infty); L^2(d\mu))\) and there exists a subsequence still denoted \(\{t_k\}_{k \in \mathbb{N}}, \text{ such that } t_k \to \infty\) as \(k \to \infty\) and \(g \in L^\infty((0, \infty); L^2(d\mu))\) satisfying

\[
f_k \rightharpoonup g, \text{ in the weak}^* \text{ topology of } L^\infty((0, \infty); L^2(d\mu)).
\]

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It then follows by the lower semicontinuity of $D_n$, 

$$\int_0^T D_n(g(s))ds \leq \liminf_{k \to \infty} \int_0^T D_n(f_k(s))ds = 0.$$ 

Then, 

$$D_n(g(s)) = \int_0^\infty \int_0^\infty W_n(x, x')|g(s, x') - g(s, x)|^2 x'^4 dx' dx = 0$$

and, for almost every $s \in (0, T)$ the measure $|g(s, x') - g(s, x)|^2 x'^4 dx' dx$ is concentrated on the diagonal $\{(x, x') \in (0, \infty)^2; x' = x\}$. Since $g(s) \in L^2(\mu)$ for almost every $s \in (0, T)$, it follows that $g = C_*$ for some constant $C_* \in \mathbb{R}$. Since $\mathbb{1}_{(0,1)} \in L^1(0, \infty; L^2(\mu))$, it follows that:

$$C \int_0^\infty d\mu(x) = \lim_{k \to \infty} \int_0^1 \int_0^\infty f_k(t, x)d\mu(x)dt = \lim_{k \to \infty} \int_0^1 \int_0^\infty f_k(t, x)n_0(x)(1 + n_0(x)x^6 dxdt$$

and then,

$$C_* = \frac{E(0)}{\int_0^\infty d\mu(x)}.$$

For all $\varphi \in C^1_c(0, \infty)$, the function:

$$t \to \int_0^\infty f(t, x)\varphi(x)n_0(x)(1 + n_0(x))x^6 dx \in C(0, \infty; \mathbb{R}) \cap L^\infty(0, \infty).$$

and,

$$\left| \frac{d}{dt} \int_0^\infty f(t, x)\varphi(x)n_0(x)(1 + n_0(x))x^6 dx \right| \leq \int_0^\infty \int_0^\infty W(x, y)(f(t, y) - f(t, x))y^4 dy\varphi(x)dx$$

$$\leq ||L(f)(t)||_1 ||\varphi||_1 \in L^\infty_{loc}(0, \infty).$$

There exists then a sequence of $\{t_j\}_{j \in \mathbb{N}}$, and $C(\varphi)$ such that,

$$\lim_{j \to \infty} \int_0^\infty f(t_j)\varphi(x)d\mu(x) = C(\varphi).$$

Since $\mathbb{1}_{(0,1)}(t)\varphi(x) \in L^1(0, \infty; L^2(\mu))$, by (4.10)

$$\lim_{j \to \infty} \int_0^1 \int_0^\infty f(t_j)\varphi(x)d\mu(x)dt = C_* \int_0^1 \int_0^\infty \varphi(x)d\mu(x)dt = C_* \int_0^\infty \varphi(x)d\mu$$

and

$$C(\varphi) = C_* \int_0^\infty \varphi(x)d\mu.$$
The following auxiliary result is used in the proof of the next Proposition.

**Lemma 4.11.** For all \( \epsilon > 0 \), there exists a constant \( C_{\epsilon} > 0 \) such that

\[
\forall (x, y) \in \left( \frac{\epsilon}{1}, \frac{1}{\epsilon} \right) \times \left( \frac{\epsilon}{1}, \frac{1}{\epsilon} \right),
\]

\[
W(x, y) \geq \frac{C_{\epsilon}}{\sinh(x^2 + y^2) \sinh|y^2 - x^2|} \cdot \frac{1}{xyy_2(x^2) (\sinh y^2)}
\]

**Proof.** There certainly exists \( C_{1, \epsilon} > 0 \) such that

\[
x^2 + y^2 - |x^2 - y^2| \geq C_{\epsilon}', \ \forall (x, y) \in \left( \epsilon, \frac{1}{\epsilon} \right) \times \left( \epsilon, \frac{1}{\epsilon} \right)
\]

Then, by continuity,

\[
\sinh(x^2 + y^2) - \sinh|y^2 - x^2| \geq C_{\epsilon}, \ \forall (x, y) \in \left( \epsilon, \frac{1}{\epsilon} \right) \times \left( \epsilon, \frac{1}{\epsilon} \right)
\]

and the result follows. \( \square \)

**Proposition 4.12.** SeEC2f3 Let \( f_0 \) and \( f \) be as in Theorem 4.1. Then,

\[
\lim_{\tau \to \infty} \int_{0}^{\infty} |f(t, x) - C_{\epsilon}^2| d\mu(x) = 0
\]

**Proof.** It follows from Lemma 4.11 for all \( \epsilon > 0 \), for all \( x \in (\epsilon/2, 2/\epsilon) \) and \( y \in (\epsilon/2, 2/\epsilon) \)

\[
x^6 y^4 W_n(x, y) \geq \frac{C_{\epsilon}}{\sinh(x^2 + y^2) \sinh|y^2 - x^2| + 1/n} \cdot \frac{x^3 y^3}{\sinh x^2 (\sinh y^2)}
\]

\[
= \frac{x^6}{\sinh x^2} \cdot \left\{ \left( \frac{C_{\epsilon}}{\sinh(x^2 + y^2) \sinh|y^2 - x^2| + 1/n} \right) \cdot \frac{y^3 (\sinh x^2)}{\sinh y^2} \right\}
\]

\[
\geq \frac{x^6}{\sinh x^2} \cdot \left\{ \left( \frac{C_{\epsilon}}{\sinh(x^2 + y^2)^2 + 1/n} \right) \cdot \frac{y^3 (\sinh x^2)}{\sinh y^2} \right\}
\]

\[
\geq \frac{x^6}{\sinh x^2} \cdot \left\{ \left( \frac{C_{\epsilon}}{\sinh(x^2 - y^2)^2 + 1/n} \right) \cdot \frac{y^3 (\sinh x^2)}{\sinh y^2} \right\} = \gamma_{\epsilon, n} \frac{x^6}{\sinh x^2} \sinh
\]

Let then be \( \theta \in C_0(0, \infty), \theta_{\epsilon}(x) = 1 \) for \( x \in (\epsilon, 1/\epsilon) \) and \( \theta_{\epsilon}(x) = 0 \) if \( x \in (0, \epsilon/2) \) or \( x > 2/\epsilon \). Consider \( \{f_k\}_{k \in \mathbb{N}} \) the sequence constructed in 4.7.

\[
\gamma_{\epsilon, n} \int_{0}^{T} \int_{0}^{\infty} \theta_{\epsilon}(x) n_0(1 + n_0) t^6 |f_k(t, x) - C_{\epsilon}| dx \leq
\]

\[
\leq \int_{0}^{T} \int_{0}^{\infty} \int_{0}^{\infty} \theta_{\epsilon}(x) \theta_{\epsilon}(y) W_n(x, y) \left( |f_k(t, x) - C_{\epsilon}|^2 + |f_k(t, y) - C_{\epsilon}|^2 \right) x^4 y^4 dx dy dt
\]

\[
\leq \int_{0}^{T} D(f_k(t)) dt + 2 \int_{0}^{\infty} \int_{0}^{\infty} \theta_{\epsilon}(x) \theta_{\epsilon}(y) W_n(x, y) 	imes
\]

\[
\times \left( |f_k(t, x) f_k(t, y) - C_{\epsilon}^2| + |f_k(t, x) f_k(t, y) - C_{\epsilon} f_k(t, x) - C_{\epsilon} f_k(t, y)| \right) x^4 y^4 dx dy dt
\]  

(4.11)

In order to prove that the last term in the right hand side tends to zero the following Lemma is needed,
Lemma 4.13. For all $\varphi \in C_c(0, \infty)$ and all $T > 0$,
\[
\lim_{k \to \infty} \int_0^T \left| \int_0^\infty f_k(t, x) \varphi(x) d\mu(x) - C_t \int_0^\infty \varphi(x) d\mu(x) \right| dt = 0
\]
Proof. For all $\varphi \in C_0(0, \infty)$,
\[
\int_0^\infty \int_0^\infty U(x, x')(f' - f) \varphi x^4 x'^4 dx' dx = - \int_0^\infty \int_0^\infty U(x, x')(f' - f) \varphi x^4 x'^4 dx' dx
\]
\[
= \frac{1}{2} \int_0^\infty \int_0^\infty U(x, x')(f' - f)(\varphi - \varphi') x^4 x'^4 dx' dx
\]
\[
= \frac{1}{2} \int_0^\infty \int_0^\infty U(x, x') f'(\varphi - \varphi') x^4 x'^4 dx' dx - \frac{1}{2} \int_0^\infty \int_0^\infty U(x, x') f(\varphi - \varphi') x^4 x'^4 dx' dx
\]
\[
= \int_0^\infty \int_0^\infty U(x, x') f'(\varphi - \varphi') x^4 x'^4 dx' dx
\]
\[
I_k(t) = \left| \frac{d}{dt} \int_0^\infty n_0(1 + n_0) x^4 f_k(t, x) \varphi(x) dx \right| \leq
\]
\[
\leq \int_0^\infty \int_0^\infty U(x, x') |\varphi(x') - \varphi(x)| |f_k(t, x)| x^4 x'^4 dx' dx
\]
If $\rho > 0$ and $R > 0$ are such that supp$\varphi \subset (\rho, R)$, then $|\varphi(y) - \varphi(x)| = 0$ for $x \in (0, \rho)$ and $y \in (0, \rho)$. If on the other hand, $x \geq \rho$ or $y \geq \rho$,
\[
\sinh |x^2 - y^2| = \sinh ((x + y)|x - y|) \geq \sinh(2\rho|x - y|)
\]
\[
\frac{|\varphi(y) - \varphi(x)|}{\sinh |x^2 - y^2|} \leq C_{\varphi} \frac{|\varphi(y) - \varphi(x)|}{\sinh(2\rho|x - y|)}
\]
If $x > R$ and $y > R$, then $|\varphi(y) - \varphi(x)| = 0$ again and therefore
\[
I_k(t) \leq
\]
\[
\leq \int_0^R \int_0^\infty \left( \frac{\sinh(x^2 + y^2) - \sinh |x^2 - y^2|}{\sinh(x^2 + y^2)} \right) \frac{|\varphi(y) - \varphi(x)| |f_k(t, x)| x^3 x'^3 dx' dx}{\sinh |x^2 - y^2| (\sinh x^2)(\sinh y^2)} +
\]
\[
+ \int_0^\infty \int_0^R \left( \frac{\sinh(x^2 + y^2) - \sinh |x^2 - y^2|}{\sinh(x^2 + y^2)} \right) \frac{|\varphi(y) - \varphi(x)| |f_k(t, x)| x^3 x'^3 dx' dx}{\sinh |x^2 - y^2| (\sinh x^2)(\sinh y^2)}
\]
\[
= I_{k,1}(t) + I_{k,2}(t)
\]
The term $I_{k,1}$ is easily estimated as follows,
\[
I_{k,1}(t) \leq
\]
\[
= \int_0^R \int_0^\infty \left( \frac{\sinh(x^2 + y^2) - \sinh |x^2 - y^2|}{\sinh(x^2 + y^2)} \right) \frac{|\varphi(y) - \varphi(x)| |f_k(t, x)| x^3 x'^3 dx' dx}{\sinh |x^2 - y^2| (\sinh x^2)(\sinh y^2)}
\]
\[
\leq C_{\varphi} \int_0^R \int_0^\infty \frac{|f_k(t, x)| x^3 dx}{\sinh x^2} \frac{y^3 dy}{\sinh y^2}
\]
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Using Hölder’s inequality and Proposition 4.1

\[ I_{k,1}(t) \leq C_\varphi \sqrt{R} \left( \int_0^R |f(t,x)|^2 x^6 dx \right)^{1/2} \int_0^\infty \frac{y^3 dy}{\sinh y^2} \]
\[ \leq C_\varphi' \left( \int_0^\infty |f_0(x)|^2 x^6 dx \right)^{1/2} \cdot \]

The integral \( I_{k,2} \) may be split again as follows,

\[ I_{k,2} = I_{k,2,1} + I_{k,2,2}, \]

where,

\[ I_{k,2,1} \leq \int_0^{2R} \int_0^R \frac{\varphi(y) - \varphi(x)}{\sinh |x^2 - y^2|} \frac{|f_k(t,x)| x^3 x^3 dx'dx}{(\sinh x^2)(\sinh y^2)} \]

and \( I_{k,2,1} \) is then estimated as \( I_{k,1} \). The estimate of the term \( I_{k,2,2} \) uses that when \( y \leq R < x/2 \) then \( \sinh |x^2 - y^2| \geq \sinh(3x^2/4) \) as follows

\[ I_{k,2,1} \leq \int_2^\infty \int_0^R \frac{\varphi(y) - \varphi(x)}{\sinh |x^2 - y^2|} \frac{|f_k(t,x)| x^3 x^3 dx'dx}{(\sinh x^2)(\sinh y^2)} \]
\[ \leq C_\varphi \int_2^\infty \int_0^R \frac{|f_k(t,x)| x^3 x^3 dx'dx}{\sinh(3x^2/4)(\sinh x^2)(\sinh y^2)} \]

and using Holder’s inequality again,

\[ I_{k,2,1} \leq C_\varphi \left( \int_2^\infty |f_k(t,x)|^2 x^6 dx \right)^{1/2} \left( \int_2^\infty \frac{dx}{(\sinh(3x^2/4))^2} \right)^{1/2} \int_0^R \frac{y^3 dy}{\sinh y^2} \]

We deduce, that

\[ \int_0^T \left. \frac{d}{dt} \int_0^\infty f_k(t,x) \varphi(x) d\mu(x) \right| dt \leq C_\varphi T \left( \int_0^\infty |f_0(x)|^2 x^6 dx \right)^{1/2}. \]

We have then, for all \( T > 0 \),

\[ \int_0^\infty f_k(t,x) \varphi(x) d\mu(x) \in W^{1,1}(0,T), \]

and by compactness of the injection \( W^{1,1}(0,T) \subset L^1(0,T) \), there exists a sequence \( t_j \xrightarrow{j \to \infty} \infty \)

and \( h \in L^1(0,T) \) satisfying

\[ \lim_{k \to \infty} \int_0^T \left| \int_0^\infty f_k(t,x) \varphi(x) d\mu(x) - h(t) \right| dt = 0. \]

But, since Corollary 4.9

\[ \lim_{k \to \infty} \int_0^T \int_0^\infty f_k(t,x) \varphi(x) d\mu(x) dt = C_\varphi T \int_0^\infty \varphi(x) d\mu(x) \]
we deduce that,

\[
\int_0^T h(t) dt = C_s T \int_0^\infty \varphi(x) d\mu(x), \ \forall T > 0
\]

and by the fundamental Theorem of Calculus,

\[
C_s T \int_0^\infty \varphi(x) d\mu(x) = \frac{d}{dT} \int_0^T h(t) dt = h(T).
\]

The second term in the right hand side of (4.11) may be split as follows

\[
2 \int_0^\infty \int_0^\infty \theta_\varepsilon(x) \theta_\varepsilon(y) \mathcal{U}_n(x, y) \times
\]

\[
\times \left( f_k(t, x) f_k(t, y) + C_s^2 f_k(t, x) - C_s f_k(t, y) \right) x^4 y^4 dx dy dt = A_1 + A_2
\]

\[
A_1 = \int_0^T \int_0^\infty \theta_\varepsilon(x) \theta_\varepsilon(y) \mathcal{U}_n(x, y) \frac{f_k(t, x) (f_k(t, y) - C_s) x^4 y^4 dx dy dt}{xy (\sinh x^2) (\sinh y^2)}
\]

\[
A_2 = \int_0^T \int_0^\infty \theta_\varepsilon(x) \theta_\varepsilon(y) \mathcal{U}_n(x, y) \frac{C_s (C_s - f_k(t, y)) x^4 y^4 dx dy dt}{xy (\sinh x^2) (\sinh y^2)}
\]

The first term may be written,

\[
A_1 = \int_0^T \int_0^\infty \theta_\varepsilon(y) \frac{y^3 (f_k(t, y) - C_s)}{\sinh y^2} dy \times
\]

\[
\times \int_0^\infty \theta_\varepsilon(x) f_k(t, x) x^3 \left( \frac{\sinh(x^2 + y^2) - \sinh |x^2 - y^2|}{\sinh(x^2 + y^2)(\sinh |x^2 - y^2| + 1/n)} \right) dx dt
\]

Since,

\[
\int_0^\infty \theta_\varepsilon(x) \frac{|f_k(t, x)| x^3}{\sinh x^2} \left( \frac{\sinh(x^2 + y^2) - \sinh |x^2 - y^2|}{\sinh(x^2 + y^2)(\sinh |x^2 - y^2| + 1/n)} \right) dx \leq n \int_0^\infty \theta_\varepsilon(x) \frac{|f_k(t, x)| x^3}{\sinh x^2}
\]

\[
\leq \frac{n}{\varepsilon^2} \left( \int_0^\infty |f_k(t, x)|^2 d\mu \right)^{1/2} \leq \frac{n}{\varepsilon^2} \left( \int_0^\infty |f_0(x)|^2 d\mu \right)^{1/2}
\]

we deduce,

\[
|A_1| \leq \frac{n}{\varepsilon^2} \|f_0\|_{L^2(d\mu)} \int_0^T \left| \int_0^\infty \theta_\varepsilon(y) \frac{y^3 (f_k(t, y) - C_s) dy}{\sinh y^2} \right| dt
\]

\[
= \frac{n}{\varepsilon^2} \|f_0\|_{L^2(d\mu)} \int_0^T \left| \int_0^\infty (f_k(t, y) - C_s) \theta_\varepsilon(y) \frac{\sinh y^2}{y^3} dy \right| dt \to 0 \text{ as } k \to \infty.
\]

The same argument shows that \( A_2 \) tends to zero too as \( k \) goes to \( \infty \) and this shows that, for all \( \varepsilon > 0 \) fixed,

\[
\lim_{k \to \infty} \int_0^T \int_0^\infty \theta_\varepsilon(x) n_0 (1 + n_0) x^6 |f_k(t, x) - C_s|^2 dx = 0.
\]
On the other hand, for $\varepsilon > 0$,
\[
\int_0^T \int_0^T n_0(1 + n_0)x^6|f_k(t, x) - C_*|^2 dx = \int_0^T \int_0^T |f_k(t, x) - C_*|^2 d\mu(x)
\leq T||f_k(t) - C_*||_{L^2(\mu)} \int_0^\varepsilon d\mu(x) \leq T\left(C_* + ||f_0||_{L^2(\mu)}\right) \int_0^\varepsilon d\mu(x)
\tag{4.13}
\]
and similarly,
\[
\int_0^T \int_0^\varepsilon n_0(1 + n_0)x^6|f_k(t, x) - C_*|^2 dx \leq T\left(C_* + ||f_0||_{L^2(\mu)}\right) \int_{1/\varepsilon}^\infty d\mu(x)
\tag{4.14}
\]
We deduce from (4.12)-(4.14),
\[
\lim_{k \to \infty} \int_0^T \int_0^\varepsilon |f(t, x) - C_*|^2 d\mu(x) = 0.
\]
Since by Proposition 4.11
\[
\int_0^\infty |f(t, x) - C_*|^2 d\mu(x) \geq \int_0^\infty |f(T, x) - C_*|^2 d\mu(x), \forall t \in (0, T),
\]
and Corollary 4.12 follows since the right hand sides of (4.13) and (4.14) may be done arbitrarily small for $\varepsilon > 0$ small enough. 

**Corollary 4.14.** Let $f_0$ and $f$ be as in Theorem 3.1. Then,
\[
\lim_{\tau \to \infty} \int_0^\infty |f(\tau, x) - C_*|^4 n_0(x)(1 + n_0(x)) dx = 0
\]

**Proof.** Arguing as in Proposition 4.4 for all $\varepsilon > 0$ and $t > 1$,
\[
\int_0^\varepsilon n_0(1 + n_0)x^4|f(t, x) - C_*| dx \leq \left(\int_0^\varepsilon n_0(1 + n_0)x^6(|f(t, x)|^p + C^n_p) dx\right)^{\frac{1}{p}} \times
\int_0^\varepsilon n_0(1 + n_0)x \left(1 - \frac{2}{p}\right) \frac{p}{p-1} dx \leq (C + ||f(1)||_{L^p(\mu)} \left(\int_0^\varepsilon n_0(1 + n_0)x \left(\frac{4p - 6}{p-1}\right) dx \right)^{\frac{p-1}{p}}.
\]
For $p > 3$, $n_0(1 + n_0)x^{\left(\frac{4p - 6}{p-1}\right)} \in L^\infty(0, \infty)$ and then
\[
\lim_{\varepsilon \to 0} \int_0^\varepsilon n_0(1 + n_0)x^4|f(t, x) - C_*| dx = 0.
\]
On the other hand,
\[
\int_\varepsilon^\infty n_0(1 + n_0)x^4|f(t, x) - C_*| dx \leq \left(\int_\varepsilon^\infty x^6n_0(x)(1 + n_0(x))|f(t, x) - C_*|^2 dx\right)^{1/2} \times \left(\int_\varepsilon^\infty n_0(1 + n_0)x^2 dx\right)^{1/2} \leq C_\varepsilon \left(\int_0^\infty |f(t, x) - C_*|^2 d\mu(x)\right)^{1/2}
\]
and the Corollary 4.14 follows from Corollary 4.12. 

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4.2 The limit of $f(t, x)$ as $x \to 0$ for $t > 0$.

We show now the existence of the limit,

$$b(t) = \lim_{x \to 0} f(t, x) \ \forall t > 0,$$

for all $f_0$ and $f$ as in Theorem 3.1 and describe its time evolution. The following property, proven in Proposition 1.5 in [3], is used

$$\forall f_0 \in L^1(0, \infty), \forall t > 0, \lim_{x \to 0} S(t)f_0(x) = \ell(f_0; t) \in (-\infty, \infty)$$

for some constants $A_1, A_2$ and a function $b_1$ such that $b_1(t) = O(t^{-8})$ for $t > 1$. A slightly more precise information may be obtained and is shown here, since it is of further interest.

**Proposition 4.15.** Suppose that $f_0$ and $f$ are as in Theorem 3.1. Then for all $t > 0$, and $\delta > 0$ as small as desired,

$$f(t, x) = b(t) + ||f_0||_{L^\infty(1, \infty)} + ||f_0||_1 O\left(t^\delta x^{1-\delta}\right) +$$

$$+ \sup_{0 < x < 1} x^\theta |f_0(x)| O\left(t^{\delta-\theta} x^{1-\delta}\right), \ x \to 0,$$

where,

$$b(t) = \ell(f_0; t) + \int_0^t \ell(F(f(s)); t - s) ds$$

$$\ell(F(f(s)); t - s) = \frac{A_1}{(t - s)^3} \int_0^{t-s} F(f(s, y)) y^2 dy + \frac{A_2}{(t-s)^4} \int_0^{t-s} F(f(s, y)) y^3 dy +$$

$$+ \int_0^{t-s} F(f(s, y)) b_1\left(\frac{t - s}{y}\right) \frac{dy}{y}$$

and there exists a constant $C > 0$ such that

$$|b(t)| \leq C||f_0||_0 \left(t^{-\theta} + t\right), \ \forall t > 0.$$

**Proof.** By construction, for all $t > 0$ and $x > 0$,

$$f(t, x) = S(t)f_0(x) + \int_0^t S(t - s) F(f(s))(x) ds.$$
By (iv) in Proposition (2.1)
\[
(t-s)^{-2+\delta} \int_0^{t-s} |F(f(s,y))| dy \leq (t-s)^{-2+\delta} \int_0^{t-s} (||f(s)||_{1+ ||f(s)||_{L^{\infty}(1,\infty)}}) dy
\leq (t-s)^{-1+\delta} (||f(s)||_{1+ ||f(s)||_{L^{\infty}(1,\infty)}})
(t-s)^{-1+\delta} \int_{t-s}^{\infty} \frac{|F(f(s,y))| dy}{y^2}
= \frac{(t-s)^{-1+\delta}}{6} (||f(s)||_{1+ ||f(s)||_{L^{\infty}(1,\infty)}})
\]
and, by (3.24) for $T > 0$,
\[
\sup_{x>1} |f(t,x)| + ||f(t)||_{1} \leq C(T, \theta) \left( ||f_0||_{L^{\infty}(1,\infty)} + t^{-\theta} \sup_{0<\tau<1} x^\theta |f_0(\tau)| + ||f_0||_{1} \right).
\]
\[
S(t-s)F(f(s))(x) = \ell(F(f(s)); t-s) + (||f(s)||_{1+ ||f(s)||_{L^{\infty}(1,\infty)}})O(t-s)^{-1+\delta} x^{1-\delta}
= \ell(F(f(s)); t-s) + \left( ||f_0||_{L^{\infty}(1,\infty)} + s^{-\theta} \sup_{0<\tau<1} x^\theta |f_0(\tau)| + ||f_0||_{1} \right) \times O_T(t-s)^{-1+\delta} x^{1-\delta}
\]
and then,
\[
\int_0^t S(t-s)F(f(s))(x) = \int_0^t \ell(F(f(s)); t-s) ds + \left( ||f_0||_{L^{\infty}(1,\infty)} + ||f_0||_{1} \right) O \left( t^{\delta} x^{1-\delta} \right) + \sup_{0<\tau<1} x^\theta |f_0(\tau)| O \left( t^{\delta-\theta} x^{1-\delta} \right)
\]
This shows (4.18), (4.19), (4.20). On the other hand, by property (iv) in Proposition 2.1
there exists a constant $C$ that depends on $\theta$ such that,
\[
|||F(f(s))|||_\theta \leq C|||f(s)|||_\theta, \forall s > 0.
\]
Then, for $t \in (0,1)$ and $s \in (0, t)$,
\[
\int_0^t (t-s)^{-n} \int_0^{t-s} F(f(s,y)) y^{n-1} dy ds \leq C|||f_0|||_\theta \int_0^t (t-s)^{-n} \int_0^{t-s} y^{n-1-\theta} dy ds
= C|||f_0|||_\theta t^{-\theta}.
\]
and for $t > 1$,
\[
\int_0^t (t-s)^{-n} \int_0^{t-s} F(f(s,y)) y^{n-1} dy ds = J_1 + J_2 + J_3
J_1 = \int_0^{t-1} (t-s)^{-n} \int_0^{t} F(f(s,y)) y^{n-1} dy ds
J_2 = \int_0^{t-1} (t-s)^{-n} \int_0^{t-s} F(f(s,y)) y^{n-1} dy ds
J_3 = \int_{t-1}^t (t-s)^{-n} \int_0^{t-s} F(f(s,y)) y^{n-1} dy ds.
\]
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with,
\[ |J_1| \leq C\|f_0\|_{\theta} \int_0^{t-1} (t-s)^{-n} \int_0^1 y^{n-1-\theta} dy ds = C\|f_0\|_{\theta}, \]
\[ |J_2| \leq C\|f_0\|_{\theta} \int_0^{t-1} (t-s)^{-n} \int_1^{1-s} y^{n-1} dy ds \leq C\|f_0\|_{\theta} t \]
and
\[ |J_3| \leq C\|f_0\|_{\theta} \int_{t-1}^t (t-s)^{-n} \int_0^{1-s} y^{n-1-\theta} dy ds = C\|f_0\|_{\theta}, \]
It follows, for \( t > 1 \)
\[ \left| \int_0^{t-s} (t-s)^{-n} \int_0^{1-s} F(f(s,y)) y^{n-1} dy ds \right| \leq C\|f_0\|_{\theta} t, \]
and,
\[ |b(t) - \ell(f_0; t)| \leq C\|f_0\|_{\theta} \left( t^{-\theta} + t \right), \forall t > 0. \]
The same arguments show that \( |\ell(f_0; t)| \leq C\|f_0\|_{\theta}(1 + t^{-\theta}) \) for all \( t > 0 \) and (4.21) follows.

5 The functions \( u(t) \) and \( p_c(t) \).

We now return to the notation of the time variable as in sub Section 1.4. Then, given the function \( f(\tau, x) \) obtained in Theorem 3.1, \( t = t(\tau) \) and \( p_c(t) \) must be determined in order to define
\[ u(t, x) = f(\tau, x), \forall t > 0, \forall x > 0. \]
The functions \( t, \tau \) and \( p_c(t) \) are related by the change of time variable (1.26), i.e.
\[ \tau = \int_0^t p_c(s) ds. \]

Proposition 5.1. For all \( \tau > 0 \),
\[ \int_0^\tau \int_0^\infty |\mathcal{L}(f(\sigma))(x)| n_0(x^2)(1 + n_0(x^2)) x^4 dx d\sigma < \infty, \]
and, if
\[ m(\tau) = \int_0^\infty \mathcal{L}(f(\tau))(x)n_0(x^2)(1 + n_0(x^2))x^4 dx, \forall \tau > 0 \]
\[ |m(\tau)| < \infty, \forall \tau > 0. \]
Proof. The proposition is a direct and straightforward consequence of the integrability property (3.27) of $L(f)$.

Let us denote,

$$M(r) = \int_0^r m(\rho) d\rho, \quad \forall r > 0. \quad (5.5)$$

$$q_c(\tau) = q_c(0)e^{-M(\tau)}, \quad \forall \tau > 0. \quad (5.6)$$

**Proposition 5.2.** For all $t > 0$ there exists a unique $\tau > 0$ such that

$$t = \int_0^\tau \frac{d\sigma}{q_c(\sigma)}, \quad \forall \tau > 0. \quad (5.7)$$

**Proof.** By Proposition 5.1, $|M(\tau)| < \infty$ for all $\tau > 0$ and then $q_c(\tau) \in (0, \infty)$ for all $\tau > 0$ and the integral in the right hand side of (5.7) is well defined and convergent. Since $q_c(t) > 0$ this integral is a monotone increasing function of $\tau$. It only remains to check that its range is $[0, \infty)$.

By Corollary 4.14, for $\varepsilon > 0$ and $\tau_\varepsilon$ large enough,

$$\int_0^\infty f(\tau, x)n_0(1 + n_0)x^4 dx > C_* \int_0^\infty n_0(1 + n_0)x^4 dx - \varepsilon, \quad \forall \tau \geq \tau_\varepsilon.$$  

Since, on the other hand,

$$M(\tau) = \int_0^\tau \int_0^\infty \int_0^\infty W(x, y)(f(\sigma, y) - f(\sigma, x))y^4 x^2 dy dx d\sigma$$

$$= \int_0^\tau \frac{d}{d\sigma} \int_0^\infty n_0(1 + n_0)f(\sigma, x)x^4 dx d\sigma$$

$$= \int_0^\infty n_0(1 + n_0)f(\tau, x)x^4 dx - \int_0^\infty n_0(1 + n_0)f_0(x)x^4 dx. \quad (5.8)$$

It follows, for $\tau > \tau_\varepsilon$,

$$M(\tau) \geq C_* \int_0^\infty n_0(1 + n_0)x^4 dx - \varepsilon - \int_0^\infty n_0(1 + n_0)f_0(x)x^4 dx.$$

Therefore, the function $e^{M(\sigma)}$ is not integrable at infinity and

$$\lim_{\tau \to \infty} \int_0^\tau \frac{d\sigma}{q_c(\sigma)} = \infty$$

and, for all $t > 0$, there exists a unique $\tau > 0$ satisfying (5.7).  

**Proof of Theorem 1.1.** For all $t > 0$, let $\tau > 0$ be given by Proposition 5.2 and define,

$$u(t, x) = f(\tau, x), \quad \forall x > 0, \quad (5.9)$$

$$p_c(t) = q_c(\tau). \quad (5.10)$$

where $f$ is obtained by Theorem 3.1 with initial data $f_0 = u_0$. From the definition of $p_c$,

$$\frac{dt}{d\tau} = \frac{1}{q_c(\tau)} \implies \frac{d\tau}{dt} = q_c(\tau) = p_c(t)$$

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and (5.1) is satisfied. On the other hand,
\[
\frac{dp_c(t)}{dt} = \frac{dq_c(\tau)}{d\tau} \frac{d\tau}{dt} = -q_c(\tau)m(\tau) \frac{d\tau}{dt} = -p_c(t)m(\tau) \frac{d\tau}{dt}. \tag{5.11}
\]
By (5.3) and (5.9),
\[
m(\tau) = \int_0^\infty L(f(\tau))n_0(x^2)(1 + n_0(x^2))x^4 \, dx
\]
\[
= \int_0^\infty L(u(t))n_0(x^2)(1 + n_0(x^2))x^4 \, dx = \mu(t).
\]
It follows from (5.6)
\[
\frac{dp_c(t)}{dt} = -p_c(t)m(\tau) \frac{d\tau}{dt} = -p_c(t) \frac{d\mu(t)}{dt}
\]
\[
= -p_c(t) \frac{d}{dt} \int_0^\infty L(u(s))n_0(x^2)(1 + n_0(x^2))x^4 \, dx \, ds
\]
and then, \( p_c \in C^1(0, \infty) \) and satisfies (1.11). On the other hand, by (5.1) and Theorem 3.2
\[
\frac{\partial u}{\partial t}(t, x) = \frac{d\tau}{dt} \frac{\partial f}{\partial \tau}(\tau, x) = -p_c(t) \frac{\partial f}{\partial \tau}(\tau, x).
\]
Then, Theorem 1.1 follows from Theorem 3.1 and Theorem 3.2, where the functions \( H \) in (1.41) and \( \lambda_{\beta, \delta} \) in (1.42) are given by,
\[
\frac{\partial u}{\partial t}(t, x) = H(t, x, a.e. \ (0, \infty) \times (0, \infty)) \tag{5.12}
\]
\[
\lambda_{\beta, \delta}(t, x) = \lambda_{\beta, \delta}(\tau, x) \text{ a.e. } (0, \infty) \times (0, \infty), \tag{5.13}
\]
with \( \bar{H} \) given in (3.40), and \( \bar{\lambda}_{\beta, \delta} \) defined in (3.30).

**Proof of Corollary 1.2, Corollary 1.3 and Corollary 1.4.** It follows from (1.38), and Corollary 1.3,
\[
\frac{dE(t)}{dt} = p_c(t) \int_0^\infty n_0(x^2) \mathcal{L}(u(t)) \, dx
\]
\[
= p_c(t) \int_0^\infty n_0(x)(1 + n_0(x)) \mathcal{L}(f(\tau, x)) \, dx
\]
\[
= p_c(t) \frac{d}{d\tau} \int_0^\infty n_0(x^2) \mathcal{L}(f(\tau, x)) \, dx = 0
\]
and this yields property (1.53) in Corollary 1.2. Property (1.3) of Corollary 1.2 follows from (1.9) and (1.11), since by (1.40), integration of (1.9) on \((0, \infty)\) yields,
\[
\frac{d}{dt} \int_0^\infty n_0(x^2) u(t, x) x^4 \, dx = -\frac{dp_c(t)}{dt}.
\]
From Proposition 1.12 property (1.55) in Corollary 1.3 follows, and property (1.56) is deduced from Corollary 1.14.

Since \( f \in C([0, \infty); L^1(0, \infty)) \), by the identity (5.8) and (5.9), \( q_c \in C([0, \infty)) \). It follows that \( p_c \in C([0, \infty)) \) and it is bounded on any compact subset of \([0, \infty)\). Passing to the limit in (5.8) as \( t \to \infty \) and using Corollary 1.14 property (1.58) is obtained. Then, it also follows that \( p_c \) is bounded on \((0, \infty)\).
Proposition 5.3. Suppose that $u_0$ and $u$ are as in Theorem 1.1. Then for all $t > 0$, and $\delta > 0$ as small as desired,

$$u(t, x) = a(t) + (||f_0||_{L^\infty(1, \infty)} + ||f_0||_1) O\left(\tau^\delta x^{1-\delta}\right) +$$

$$+ \sup_{0 < x < 1} x^\theta |f_0(x)| O\left(\tau^{\delta-\theta}x^{1-\delta}\right), \ x \to 0,$$

where, $a(t) = b \left(\int_0^t p_c(s)ds\right)$

satisfies, for some constant $C > 0$,

$$|a(t)| \leq C ||u_0||_\theta \left(\left(\int_0^t p_c(s)ds\right)^{-\theta} + \int_0^t p_c(s)ds\right), \ \forall t > 0.$$  

Proof. By construction, $u(t, x) = f(\tau, x)$ where $\tau$ is given in terms of $t$ by (5.1) and therefore, (5.14), (5.15) follow from (4.18) and (5.16) follows from (4.21).

6 Appendix

6.1 Some further properties of $S(t)$.

We prove in this Appendix two properties of the solution $S(t)f_0$ of (1.33) with initial data $f_0$, that are not given in [8]. The first is just an elementary continuity result. We seek for the continuity of $S(t)f_0$ with respect to $x$ in order to deduce the same property for the solution $u$ of (1.9) and to be able later to speak of $\ell(u(t))$.

We first briefly recall ...

$$W(s) = -2\gamma_e - 2\psi\left(\frac{s}{2}\right) - \pi \cot\left(\frac{\pi s}{4}\right), \ s \in \mathbb{C}, \ \Re(s) \in (-2, 4)$$

where $\gamma_e$ is the Euler constant and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the Digamma function.

For any $\beta \in (0, 2)$ fixed, the function

$$B(s) = \exp\left(\int_{\Re(\rho)=\beta} \log(-W(\rho)) \left(\frac{1}{1 - e^{2\pi(\rho-s})} - \frac{1}{1 + e^{-2\pi \rho}}\right) d\rho\right)$$

is analytic in the domain $\{s \in \mathbb{C}; \Re(s) \in (\beta, \beta + 1)\}$ and satisfies

$$B(s) = -W(s-1)B(s-1), \ \forall s \in \mathbb{C}; \Re(s) \in (\beta, \beta + 1)$$

Proposition 6.1. If $f_0 \in L^1(0, \infty) \cap L^\infty_{loc}(0, \infty)$, then $S(t)f_0 \in L^\infty_{loc} \cap C((0, \infty))$ for every $t > 0$ and $S(t)f_0(x) \in C(0, \infty)$ for all $x > 0$.

Proof. By definition,

$$S(t)f_0(x) = S(t)f_0(x') = \int_0^\infty \left(\Lambda\left(\frac{t, x}{y}, \frac{x'}{y}\right) - \Lambda\left(\frac{t, x'}{y}, \frac{x}{y}\right)\right) f_0(y) \frac{dy}{y}$$
\[ |S(t)f_0(x) - S(t)f_0(x')| \leq \int_{y<\delta_1} \left( \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) - \Lambda \left( \frac{t}{y}, \frac{x'}{y} \right) \right) f_0(y) \frac{dy}{y} + \int_{y>R} \left( \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) - \Lambda \left( \frac{t}{y}, \frac{x'}{y} \right) \right) f_0(y) \frac{dy}{y} + \int_{\left|\frac{1-x}{y}\right|>\rho, y \in (\delta_1, R)} \left( \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) - \Lambda \left( \frac{t}{y}, \frac{x'}{y} \right) \right) f_0(y) \frac{dy}{y} + \int_{\left|\frac{1-x}{y}\right|<\rho, y \in (\delta_1, R)} \left( \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) - \Lambda \left( \frac{t}{y}, \frac{x'}{y} \right) \right) f_0(y) \frac{dy}{y} = \int_{y<\delta_1} \vert...dy + \int_{y>R} \vert...dy + J_1 + J_2\]

We choose \( \delta_1 \) small enough and \( R \) large enough, both depending on \( t \), to have:

\[ \int_{y<\delta_1} \vert...dy + \int_{y>R} \vert...dy \leq \varepsilon \]

using that \( f_0 \in L^1 \) and the asymptotics of \( \Lambda \) for large and small arguments. Consider for example, the integral for \( y < \delta \), with \( \delta < t \). Then \( t/y > 1 \) and

\[ \left| \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) + \Lambda \left( \frac{t}{y}, \frac{x'}{y} \right) \right| \leq C y^3 (\max(x, t)^{-3} + \max(x', t)^{-3}) \]

and, if we use that \( x' > x/2 \),

\[ \left| \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) - \Lambda \left( \frac{t}{y}, \frac{x'}{y} \right) \right| \int_{y<\delta_1} \frac{|f_0(y)|}{y} dy \leq C y^2 (\max(x, t)^{-3} + \max(x/2, t)^{-3}) \]

\[ \lim_{\delta \to 0} \int_{y<\delta_1} \left( \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) - \Lambda \left( \frac{t}{y}, \frac{x'}{y} \right) \right) f_0(y) \frac{dy}{y} = 0 \]

A similar argument yields the limit of the integral for \( y > R \) as \( R \to \infty \).

On the other hand, if we suppose \( y \in (\delta_1, R) \), \( \vert x - x' \vert < \delta \) then,

\[ \left| \frac{1-x}{y} < \rho \Rightarrow \right| \left| \frac{1-x'}{y} \leq 1 - \frac{x}{y} + \frac{\delta}{\delta_1} \right| \left| \frac{x}{y} - \frac{x'}{y} \leq \rho + \frac{\delta}{\delta_1} \right| \]

(6.4)

\[ \left| \frac{1-x}{y} > \rho \Rightarrow \right| \left| \frac{1-x'}{y} \geq 1 - \frac{x}{y} - \frac{\delta}{\delta_1} \right| \left| \frac{x}{y} - \frac{x'}{y} \geq \rho - \frac{\delta}{\delta_1} \right| \]

(6.5)

Then,

\[ J_{2,1} = \int_{\left|\frac{1-x}{y}\right|<\rho, y \in (\delta_1, R)} \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \left| f_0(y) \right| \frac{dy}{y} \leq C(f_0) \int_{\left|\frac{1-x}{y}\right|<\rho, y \in (\delta_1, R)} \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \frac{dy}{y} \]

Since \( t/y > t/R \) for all \( y \in (\delta_1, R) \). If \( t/R > 1/2 \), then the function,

\[ y \mapsto \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \]

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is continuous on 
\[ D = \{ y \in (\delta_1, R) \} \]
and by the Lebesgue’s convergence Theorem,
\[ \lim_{\rho \to 0} J_{2,1} = 0. \]

On the other hand, for \(|x - x'| < \delta\), by (6.4):

\[ J_{2,2} = \int_{\left|1 - \frac{x}{y}\right| < \rho, y \in (\delta_1, R)} \Lambda \left( \frac{t}{y}, \frac{x'}{y} \right) |f_0(y)| \frac{dy}{y} \leq C(f_0) \int_{\left|1 - \frac{x'}{y}\right| < \rho + \frac{\delta}{R}, y \in (\delta_1, R)} \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \frac{dy}{y} \]

and

\[ \lim_{\rho \to 0} J_{2,2} = 0. \]

If \(t/R < 1/2\) we must divide the domain \(D\) in two sub domains

\[ D_+ = \left\{ y \in (\delta_1, R); \frac{t}{y} > \frac{1}{2} \right\}, \quad D_- = \left\{ y \in (\delta_1, R); \frac{t}{R} < \frac{t}{y} < \frac{1}{2} \right\}. \]

With the previous argument,

\[ \lim_{\rho \to 0} \int_{\left|1 - \frac{x}{y}\right| < \rho, y \in D_+} \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \frac{dy}{y} = 0 \]

\[ \lim_{\rho \to 0} \int_{\left|1 - \frac{x}{y}\right| < \rho, y \in D_+} \Lambda \left( \frac{t}{y}, \frac{x'}{y} \right) \frac{dy}{y} = 0. \]

In the domain \(D_-\), with \(r = t/R > 0\) and \(\alpha = r/2\)

\[ \left| \frac{\log \left( \frac{x}{y} \right) \Lambda \left( \frac{t}{y}, \frac{x}{y} \right)}{1 - x/y} \right| \leq C \]

\[ \left| \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \right| \leq C' \frac{1 - x/y}{\log \left( \frac{x}{y} \right)} \leq C' |1 - x/y|^{-1} \]

and then,

\[ J_{2,1} \leq C(f_0) \int_{\left|1 - \frac{x}{y}\right| < \rho, y \in D_-} |1 - x/y|^{-1} dy \to 0, \text{ as } \rho \to 0 \]

and arguing as before,

\[ \lim_{\rho \to 0} \int_{\left|1 - \frac{x}{y}\right| < \rho, y \in D_-} \Lambda \left( \frac{t}{y}, \frac{x'}{y} \right) |f_0(y)| \frac{dy}{y} = 0. \]
By (6.5), if $|x - x'| < \delta$ and $|1 - x/y| > \rho$ then $|1 - x'/y| > \rho - \delta/\delta_1$ and, by...

$$\Lambda \in C((0, \infty) \times \tilde{D})$$

$$\tilde{D} = \left\{ z > 0; |z - 1| > \rho - \delta/\delta_1 \right\}$$

It follows that, for all $y \in (\delta_1, R)$, such that $|1 - x/y| > \rho$ the function

$$x' \mapsto \Lambda \left( \frac{t}{y}, \frac{x'}{y} \right)$$

is continuous at $x$. Then, for all $t > 0$ fixed and $y \in (\delta_1, R)$ such that $|1 - x/y| > \rho$:  

$$\lim_{x' \to x} \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) = \Lambda \left( \frac{t}{y}, \frac{x}{y} \right)$$

$$\left| \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) - \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \right| \leq 2 \sup \left\{ \Lambda(\tau, z); \tau \in \left( \frac{t}{R}\frac{t}{\delta_1}, z \in \tilde{D} \right) \right\}.$$  

We deduce from the Lebesgue's convergence Theorem,

$$\lim_{{x \to x'}} J_1 = 0$$

and this gives the continuity of with respect to $x$. On the other hand, fix $x > 0$ and $t > 0$ and suppose that $t_n \to t$.

$$\int_0^\infty \left| \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) - \Lambda \left( \frac{t_n}{y}, \frac{x}{y} \right) \right| f_0(y) \frac{dy}{y} \leq \int_{|1 - x| > \rho} \left| \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) - \Lambda \left( \frac{t_n}{y}, \frac{x}{y} \right) \right| f_0(y) \frac{dy}{y} +$$

$$+ \int_{|1 - x| \leq \rho} \left| \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) - \Lambda \left( \frac{t_n}{y}, \frac{x}{y} \right) \right| f_0(y) \frac{dy}{y} = I_1 + I_2.$$  

In the first integral, for $\rho \in (0, 1)$, we have

$$\frac{x}{1 + \rho} < y < \frac{x}{1 - \rho}.$$  

In that range of values of $y$, we have the estimates,

$$\left| \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \right| \leq C \left| 1 - \frac{x}{y} \right|^{-1 + 2\delta/y} \leq C \left| 1 - \frac{x}{y} \right|^{-1 + 2(1 - \rho)/y} \leq C \left| 1 - \frac{x}{y} \right|^{-1 + \frac{1}{\delta_1}}$$

$$\left| \Lambda \left( \frac{t_n}{y}, \frac{x}{y} \right) \right| \leq C \left| 1 - \frac{x}{y} \right|^{-1 + 2\delta/n} \leq C \left| 1 - \frac{x}{y} \right|^{-1 + 2(1 - \rho)/y} \leq C \left| 1 - \frac{x}{y} \right|^{-1 + \frac{1}{\delta_1}}$$

since we may assume that $t_n \geq t$ and $1 - \rho > 1/2$. We deduce,

$$I_1 \leq C \int_{|1 - x| < \rho} \left| 1 - \frac{x}{y} \right|^{-1 + \frac{1}{\delta_1}} f_0(y) \frac{dy}{y}$$

$$\leq C \|g\|_{L^\infty(x/2, 2x)} \int_{|1 - z| \leq \rho} (1 - z)^{-1 + \frac{1}{\delta_1}} \frac{dz}{z} \to 0, \text{ as } \rho \to 0.$$  

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This fixes \( \rho \). We then have,

\[
\lim_{n \to \infty} \Lambda \left( \frac{t_n}{y}, \frac{x}{y} \right) = \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) = 0, \quad \text{for } \left| 1 - \frac{x}{y} \right| > \rho
\]

\[
\Lambda \left( \frac{t_n}{y}, \frac{x}{y} \right) \frac{|f_0(y)|}{y} \leq C(t, x)|f_0(y)|
\]

and by the Lebesque’s convergence Theorem, \( J_2 \to 0 \) as \( n \to \infty \).

The next result is useful to consider initial data \( f_0 \) that are unbounded near the origin.

**Proposition 6.2.** Suppose that \( g \in L^1(0, \infty) \) is such that, for some \( \theta > 0 \),

\[
|||g|||_{\theta} = \sup_{0 < x < 1} x^\theta |g(x)| + \sup_{x > 1} |g(x)| < \infty
\]

Then, \( S(t)g \in L^\infty(0, \infty) \) and more precisely,

\[
|S(t)(g)(x)| \leq CG |||g|||_{L^\infty(2, \infty)} + Ct^{-\theta} \sup_{0 < y < 1} y^\theta |g(y)| \quad \forall x \in (0, 2)
\]

(6.6)

\[
|S(t)(g)(x)| \leq CG |||g|||_{L^\infty(1, \infty)} + C ||g||_1 \quad \forall x > 2
\]

(6.7)

**Proof.** By hypothesis, for all \( \delta > 0 \), \( g \in L^\infty_{\text{loc}}(\delta, \infty) \) and for all \( x > 0 \),

\[
\left| \int_2^\infty \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) g(y) \frac{dy}{y} \right| \leq CG |||g|||_{L^\infty(2, \infty)}
\]

On the other hand, for \( x > 2 \)

\[
\left| \int_0^1 \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) g(y) \frac{dy}{y} \right| \leq \sup_{y \in (0, 1)} \left( \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \frac{1}{y} \right) ||g||_1
\]

where \( x/y > 2 \) for \( y \in (0, 1) \). Several cases are now possible. If \( t > x \), then \( t > y \) and it follows from Proposition 3.1 and Proposition 3.2 of [S],

\[
\Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \frac{1}{y} \leq C t^{-3} y^2 \leq C
\]

The same argument yields, for \( t \in (2, x) \),

\[
\Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \frac{1}{y} \leq C x^{-3} y^2 \leq C
\]

and, for \( 0 < y < t < 2 \),

\[
\Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \frac{1}{y} \leq C \left( \frac{t}{y} \right)^{-3} \left( \frac{x}{t} \right)^{-1} \left( \frac{y}{t} \right) \leq C t^{-2} x^{-1} y^2 \leq C
\]

By Proposition 3.5 of [S], when \( t < 1 \), and \( y \in (t, 1) \),

\[
\Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \frac{1}{y} \leq C_\varepsilon \left( \frac{x}{y} \right)^{-3+\varepsilon} \left( \frac{t}{y} \right)^{9-\varepsilon} + C \left( \frac{x}{y} \right)^{-5} \left( \frac{t}{y} \right)^7
\]

\[
= C_\varepsilon t^{9-\varepsilon} x^{-3+\varepsilon} y^{-6} + C t^7 x^{-5} y^{-2} \leq C_\varepsilon t^{3-\varepsilon} x^{-3+\varepsilon} + C t^5 x^{-5} \leq C
\]

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and this ends the proof of (6.7).

When $x \in (0, 2)$ we first write,

$$\left| \int_0^1 \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) g(y) \frac{dy}{y} \right| \leq \sup_{0 < y < 1} y^\theta |g(y)| \int_0^1 \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \frac{dy}{y^{1+\theta}}.$$  

When $t > y$ Proposition 3.1 and Proposition 3.2 may be applied. It follows that, for $x < t$,

$$\left| \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \frac{1}{y^{1+\theta}} \right| \leq C t^{-3} y^{2-\theta} \leq C t^{-1-\theta}$$

and for $x > t$,

$$\left| \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \frac{1}{y^{1+\theta}} \right| \leq C x^{-3} y^{2-\theta} \leq C t^{-1-\theta}.$$

in both cases,

$$\int_0^1 \left| \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \frac{1}{y^{1+\theta}} \right| dy \leq C t^{-\theta}.$$

Similar arguments show the same estimate for $y \in (t, 1)$. Therefore,

$$\int_0^1 \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \frac{dy}{y^{1+\theta}} \leq C t^{-\theta} \ \forall x \in (0, 2)$$

and (6.6) follows.

\[ \square \]

**Lemma 6.3.** There exists a constant $\sigma_0^* \in (-2, -1)$ such that, for $\theta \in [0, 1)$, $\beta \in (0, 1)$

(i) there exists $C > 0$ such that for all $f_0 \in L^1(0, \infty)$ satisfying (1.35) and for all $t \in (0, 1)$

$$\left| \frac{\partial}{\partial x} S(t) f_0(x) \right| \leq C \left( t^{-\theta-1} + x^{-1-\sigma_0^*} t^{\sigma_0^*-\theta} \right) \sup_{0 \leq y \leq t} (y^\theta |f_0(y)|) + C x^{-1-\sigma_0^*} ||f_0||_1 + C t^2 ||f_0||_1, \ \forall x \in (0, t)$$

(6.8)

and

$$\left| \frac{\partial}{\partial x} S(t) f_0(x) \right| \leq C x^{-1-\beta} t^{\beta-1} ||f_0||_1, \ \forall x > t.$$  

(6.9)

(ii) There also exists a constant $C > 0$ such that for all $f_0 \in L^1(0, \infty)$ satisfying (1.35) for all $t > 1$,

$$\left| \frac{\partial}{\partial x} S(t) f_0(x) \right| \leq C t^{-4} \sup_{0 \leq y \leq t} (y^\theta |f_0(y)|) + C t^2 ||f_0||_1, \ \forall x \in (0, t),$$

(6.10)

and

$$\left| \frac{\partial}{\partial x} S(t) f_0(x) \right| \leq C t^{-1-\beta} x^{-1-\beta} ||f_0||_1, \ \forall x > t.$$  

(6.11)
Proof. Suppose first that \( x \in (0, t) \) and write

\[
\frac{\partial f}{\partial x}(t, x) = I_1 + I_2
\]

\[
I_1 = \int_0^t \frac{\partial \Lambda}{\partial z} \left( \frac{t}{y} \frac{x}{y} \right) f_0(y) \frac{dy}{y^2}, \quad I_2 = \int_t^\infty \frac{\partial \Lambda}{\partial z} \left( \frac{t}{y} \frac{x}{y} \right) f_0(y) \frac{dy}{y^2}.
\]

In the term \( I_1 \), \( y \in (0, t) \) and then, Proposition 3.4 in [8] may be applied to obtain,

\[
|I_1| \leq C \int_0^t \left( \frac{t}{y} \right)^{-4} \left( A + O \left( \left| \frac{x}{t} \right|^\delta \right) \right) f_0(y) \frac{dy}{y^2} \leq Ct^{-4} \int_0^t f_0(y) y^2 dy
\]

\[
\leq \begin{cases}
C \sup_{0 \leq y \leq t} (y^\theta f_0(y)) t^{-4} \int_0^t y^{2-\theta} dy = Ct^{-1-\theta} \sup_{0 \leq y \leq t} (y^\theta f_0(y)), & t < 1, \\
Ct^{-4} \int_0^t f_0(y) y^2 dy + Ct^{-4} \int_t^\infty f_0(y) y^2 dy & t > 1
\end{cases}
\]

\[
\leq \begin{cases}
C \sup_{0 \leq y \leq t} (y^\theta f_0(y)) t^{-4} \int_0^t y^{2-\theta} dy = Ct^{-1-\theta} \sup_{0 \leq y \leq t} (y^\theta f_0(y)), & t < 1, \\
Ct^{-4} \sup_{0 \leq y \leq t} (y^\theta f_0(y)) + Ct^{-2} \left( \frac{||f_0||_1}{t||f_0||_{L^\infty(1, \infty)}} \right) & t > 1
\end{cases}
\]  \[(6.12)\]

In the term \( I_2 \), \( t/y < 1 \) and then, by (3.32) in Proposition 3.5 of [8], there exists constant \( C \) such that,

\[
\left| \frac{\partial \Lambda}{\partial z} \left( \frac{t}{y} \frac{x}{y} \right) \right| \leq C \left( tx^{-1-\sigma_0^*} y^{\sigma_0^*} + tx^{-1-\sigma_1^*} y^{\sigma_1^*} + t^2 y^{-2} \right), \forall x \in (0, t).
\]

where \( \sigma_j^* \) are given real numbers such that \( \sigma_j^* \in (-2(2j+1), -2(2j+1)) \). Since \( y > t > x \) in the integration’s domain of \( I_2 \) and \( \sigma_1^* < \sigma_0^* < 0 \) it follows that

\[
tx^{-1-\sigma_1^*} y^{\sigma_1^*} \leq tx^{-1-\sigma_0^*} y^{\sigma_0^*}
\]

Then,

\[
|I_2| \leq C tx^{-1-\sigma_0^*} \int_t^\infty y^{\sigma_0^*-2} |f_0(y)| dy + Ct^2 \int_t^\infty |f_0(y)| dy
\]

where the first term in the right hand side may estimated as follows. If \( t \in (0, 1) \),

\[
\int_t^\infty y^{\sigma_0^*-2} |f_0(y)| dy \leq \int_t^1 y^{\sigma_0^*-2} |f_0(y)| dy + \int_1^\infty y^{\sigma_0^*-2} |f_0(y)| dy
\]

\[
\leq C \sup_{0 \leq y \leq 1} (y^\theta f_0(y)) t^{-1+\sigma_0^*-\theta} + ||f_0||_1
\]

and then

\[
\int_t^\infty y^{\sigma_0^*-2} |f_0(y)| dy \leq C t^{\sigma_0^*-\theta} x^{-1-\sigma_0^*} \sup_{0 \leq y \leq 1} (y^\theta f_0(y)) + tx^{-1-\sigma_0^*} ||f_0||_1.
\]

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Since \( x \in (0,t) \) and \( \sigma_0^* < 0 \), it follows that,

\[
\begin{align*}
& t e^{\sigma_0^* - \theta} x^{-1-\sigma_0^*} = t^{-\theta} x^{-1} \left( \frac{x}{t} \right)^{-\sigma_0^*} < t^{-\theta} x^{-1}, \\
& t x^{-1-\sigma_0^*} = t^{-\theta} x^{-1} \left( x^{-\sigma_0^*} t^{1+\theta} \right) \leq t^{-\theta} x^{-1} t^{-\sigma_0^*+1+\theta} \leq t^{-\theta} x^{-1}
\end{align*}
\]

and then,

\[
|I_2| \leq C t^{-\theta} x^{-1} \left( \sup_{0 \leq y \leq t} (y^\theta |f_0(y)|) + ||f_0||_1 \right) + C t^2 ||f_0||_1. \tag{6.13}
\]

If on the contrary \( t > 1 \),

\[
|I_2| \leq C t x^{-1-\sigma_0^*} \int_t^\infty y^{\sigma_0^*-2} |f_0(y)| dy + C t^2 \int_t^\infty |f_0(y)| dy \\
\leq t x^{-1-\sigma_0^*} \left( C t^{\sigma_0^*-1} ||f_0||_{L^\infty(1,\infty)} \right) + C t^2 ||f_0||_1. \tag{6.14}
\]

This shows the result for all \( t > 0 \) and \( x < t \). If \( x > t > y \), by Proposition 3.4, for \( \varepsilon > 0 \) as small as desired there exists \( C_\varepsilon > 0 \) such that,

\[
\left| \frac{\partial \Lambda}{\partial z} \left( \frac{t}{x}, \frac{y}{y} \right) \right| \leq C_\varepsilon (x^{-4} y^4 + y^4 x^{-1-\varepsilon}) \leq C_\varepsilon x^{-4} y^4, \forall x > t.
\]

Then,

\[
|I_1| \leq C x^{-4} \int_0^t y^2 f_0(y) dy \\
\leq \begin{cases} 
C \sup_{0 \leq y \leq t} (y^\theta |f_0(y)|) x^{-4} \int_0^t y^2 dy = C x^{-4} \int_0^t y^{3-\theta} dy \sup_{0 \leq y \leq t} (y^\theta |f_0(y)|), & t < 1, \\
C x^{-4} \int_0^1 f_0(y) y^2 dy + C x^{-4} \int_1^t f_0(y) y^2 dy & t > 1 \\
\end{cases} \tag{6.15}
\]

For \( y > t \) and \( x > t \), we must argue as in the previous cases, unfortunately there is an error in the estimate (3.32) of Proposition 3.5 in [8]. The error is corrected in Lemma 6.6 below and gives

\[
\left| \frac{\partial \Lambda}{\partial z} \left( \frac{t}{x}, \frac{y}{y} \right) \right| \leq C \left( \left( \frac{x}{y} \right)^{-\beta-1} + \left( \frac{y}{y} \right)^{-1-c'} \left( \frac{t}{y} \right)^{-\beta+c'} \right)
\]

Therefore

\[
|I_2| \leq C\beta x^{-\beta-1} \int_t^\infty y^{\beta-1} |f_0(y)| dy + C x^{-1+c'} t^{-\beta+c'} \int_t^\infty y^{-1+\beta} |f_0(y)| dy
\]

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and then, for $\beta \in (0, 1)$,

$$|I_2| \leq C \left( e^{1-x^1-x^1-\beta} + x^{1-c^1-t^c-1}\right) ||f_0||_1, \ \forall x > t > 0$$  \hspace{1cm} (6.16)

Since $x^{-4}e^{3-\theta} < x^{-1-\sigma t^1-s^2-\theta}$ when $x > t$, this ends the proof for all $t > 0$ and $x > t$. \hspace{1cm} \Box

The following straightforward Corollary simplifies somewhat Lemma’s 6.3 statement, at the price of a little loss in the estimates.

**Corollary 6.4.** There exists a constant $C > 0$ such that, for all $f_0 \in L^1$, $y^0f_0 \in L^\infty(0, 1)$,

$$\left| \frac{\partial}{\partial x} S(t)f_0(x) \right| \leq \begin{cases} Cx^{1+\delta-t^{-\theta-\delta}}||f_0||_1, & 0 < x < t < 1, \\ Ct^\delta||f_0||_1, & t > 1, \ 0 < x < t, \\ Ct^{\beta-1}x^{-\beta}||f_0||_1, & x > t. \end{cases}$$  \hspace{1cm} (6.17)

for all $\delta \in (0, 1)$, where $||f_0||_1 = \sup_{0 \leq y \leq t}(y^0f_0(y)) + ||f_0||_1$.

**Remark 6.5.** The right hand side of (6.17) may be denoted as $h(t) g(x) ||f_0||_1$ and the expressions of the functions $g$ and $h$ in the different domains of $t$ and $x$ are given in (6.17).

**Lemma 6.6.** For all $\beta \in (0, 2), \beta' \in (-1, 0)$ and $c'$ such that $-1 < \beta' < 0 < c' < 1 + \beta' < 1$ there exists a constant $C_0$ such that

$$\left| \frac{\partial \Lambda}{\partial x} (t, x) \right| \leq C_\beta \left( x^{-\beta-1} + x^{-1-c' t-\beta'+c'} \right) \ \forall t \in (0, 1), \ x > t.$$

**Proof.** The proof is based on the following representation formula of the function $\partial_x \Lambda(t, x)$:

$$\frac{\partial \Lambda}{\partial x}(t, x) = \left( \frac{x}{\partial_x} \right)^3 (J(t, x))$$

$$J(t, x) = -\frac{1}{4\pi^2} \int_{c-i\infty}^{c+i\infty} \int_{\Re(e(\sigma))=\beta} t^{-\sigma+s} B(s) \Gamma(\sigma-s) \frac{B(\sigma)}{s^2} x^{-s-1} ds d\sigma.$$  \hspace{1cm} (6.17)

for all $\beta \in (0, 2)$ and $c \in (0, \beta)$. As indicated in [8], when $x/t > x > 1$ it is suitable to deform first the $s$ contour integrals in $J$ towards larger values of $\Re(e(\sigma))$. Since by construction $c < \beta$ this first requires to cross the pole of $\Gamma(\sigma-s)$ at $s = \sigma$, from where, for $c' \in (\beta, 2)$,

$$J(t, x) = \frac{1}{2\pi i} \int_{\Re(e(\sigma))=\beta} \sigma^{-2} x^{-\sigma-1} ds + J_1(t, x)$$

$$J_1(t, x) = -\frac{x-1}{4\pi^2} \int_{\Re(e(\sigma))=\beta}^c \int_{c-i\infty}^{c+i\infty} \frac{t^{-\sigma}s^{-2} B(s) \Gamma(\sigma-s)}{B(\sigma)s^2} \frac{1}{t} (x/t)^{-s} ds d\sigma$$

Since $B(s)$ is analytic on $\Re(e(\sigma)) \in (0, 1)$ and $(B(s))^{-1}$ is analytic on $\Re(e(\sigma)) \in (-1, 2)$ it is possible to move the $\sigma$-integration contour towards lower values of $\Re(e(\sigma))$ and

$$J_1(t, x) = -\frac{x-1}{4\pi^2} \int_{\Re(e(\sigma))=\beta}^{c' \in \infty} \int_{c'-i\infty}^{c'+i\infty} \frac{t^{-\sigma}s^{-2} B(s) \Gamma(\sigma-s)}{B(\sigma)s^2} \frac{1}{t} (x/t)^{-s} ds d\sigma$$  \hspace{1cm} (6.18)
where $\beta'$ and $c'$ satisfy $-1 < \beta' < 0 < c' < 1 + \beta' < 1$. For $\beta \in (0, 2)$

$$\left| \int_{\mathbb{R}e(\sigma) = \beta} \sigma^{-2}x^{-\sigma-1}d\sigma \right| \leq \sqrt{2}x^{-1-\beta} \int_{\mathbb{R}} \frac{dv}{(\beta^2 + v^2)}.$$  

Since $x > t$, $c' > 0$ and $0 < t < 1$, $\beta' < 0$, $J_1(t, x) = \mathcal{O}\left(x^{-1-c'}t^{-\beta'+c'}\right)$ and the result follows.

We close this Appendix with the following Remark and some simple estimates.

**Remark 6.7.** If $f_0 \in L^\infty \cap L^1(0, \infty)$ it follows from the estimate (1.37) in \[8\]

$$\frac{\partial}{\partial t} \int_0^\infty f_0(z) \Lambda \left(\frac{t - \frac{x}{z}}{z}\right) \frac{dz}{z} \leq C t^{3-\theta} \frac{||x^\theta f_0||_{L^\infty(0, t)}}{\max(t^4, x^4)} + C ||f_0||_{\infty} 1_{2x/3 < t < 2x} + C\zeta_0(t, x)||f_0||_{1, \theta} \quad (6.19)$$

where the function $\zeta_0(t, x)$ is defined in \[8, 37\].

**Proposition 6.8.** For all $t > 0$ fixed,

$$\int_0^t \zeta_0(s, x)ds \leq C \min\left(\frac{t^2}{x^3}, \frac{t^{3-\theta}}{x^3}\right) 1_{t < 2x/3} + C \min(x^{-1}, x^{-\theta}) 1_{2x/3 < t} =: \omega_1(t, x) \quad (6.20)$$

$$\int_0^t \zeta_0(s, x)(t - s)^{-\theta} ds \leq C \min(t^{2-\theta}, t^{3-\theta}) \frac{1}{x^3} 1_{t < 2x/3} + C \min(x^{-1-\theta}, x^{-\theta}) 1_{2x/3 < t} =: \omega_2(t, x) \quad (6.21)$$

$$\int_0^t (1 + \xi(t - s, x)) ds = \left(t + \frac{\min(t^4, x^4)}{4x^4}\right) + \log\left(\frac{t}{x}\right) =: \Xi_1(t, x) \quad (6.22)$$

$$\int_0^t s^{-\theta} \xi(t - s, x) ds \leq C x^{-4} t^{-\theta} 1_{x \geq t} + C t^{-\theta} \left(1 + \log\left(\frac{t}{x}\right)\right) 1_{x \leq t} =: \Xi_2(t, x) \quad (6.23)$$

**Proof.**

$$\int_0^t \zeta_0(s, x)ds = \frac{1}{x^3} \int_0^t \min(s, s^{-2-\theta})ds \leq C \frac{1}{x^3} \min\left(t^2, t^{3-\theta}\right), \forall t < 2x/3$$

$$\int_0^t \zeta_0(s, x)ds = \frac{1}{x^3} \int_0^{2x/3} \min(s, s^{-2-\theta})ds + \frac{1}{\max(x^2, x^{1+\theta})} \int_{2x/3}^t ds$$

$$\leq C \frac{1}{x^3} \min\left(x^2, x^{3-\theta}\right) + \frac{C(t - 2x/3)}{\max(x^2, x^{1+\theta})}, \text{ if } 2x/3 < t < 2x.$$  

$$\int_0^t \zeta_0(s, x)ds = \frac{1}{x^3} \int_0^{2x/3} \min(s, s^{-2-\theta})ds + \frac{1}{\max(x^2, x^{1+\theta})} \int_{2x/3}^{2x} ds + x \int_{2x/3}^t \min(s^{-2-\theta}, s^{-3})ds \leq C \frac{1}{x^3} \min\left(x^2, x^{3-\theta}\right) + \frac{C}{\max(x, x^\theta)} +$$

$$+ C x \min\left((2x)^{-1-\theta} - t^{-1-\theta}, (2x)^{-2} - t^{-2}\right), \text{ if } 2x < t.$$
On the other hand, we easily obtain if $t < 2x/3$, 
\[ \int_0^t \zeta_0(s, x)(t - s)^{-\theta} ds = \frac{1}{x^\theta} \int_0^t \min(s, s^2)(t - s)^{-\theta} ds \leq \frac{C}{x^\theta} \min(t^{2-\theta}, t^{3-\theta}) \]
since:
\[ \int_0^t s(t - s)^{-\theta} ds = t^{2-\theta} \int_0^1 r(1 - r)^{-\theta} dr \]
\[ \int_0^t s^2(t - s)^{-\theta} ds = t^{3-\theta} \int_0^1 r^2(1 - r)^{-\theta} dr, \]
and, if $2x/3 < t < 2x$,
\[ \int_0^t \zeta_0(s, x)(t - s)^{-\theta} ds \leq \frac{C}{x^\theta} \min(x^{2-\theta}, x^{3-\theta}) + \frac{1}{\max(x^2, x)} \int_0^t (t - s)^{-\theta} ds \]
\[ \leq C \min(x^{-1-\theta}, x^{-\theta}) + \frac{Ct^{1-\theta}}{\max(x^2, x)}. \]

For $2x < t$,
\[ \int_0^t \zeta_0(s, x)(t - s)^{-\theta} ds \leq C \min(x^{-1-\theta}, x^{-\theta}) + \frac{Ct^{1-\theta}}{\max(x^2, x)} + x \int_{2x}^t \min(s^{-2}, s^{-3})(t - s)^{-\theta} ds \leq C \min(x^{-1-\theta}, x^{-\theta}) + \]
\[ + C \min(x^{-1-\theta}, x^{-\theta}) + \min \left( Ct^{-\theta}, \frac{Ct^{-\theta}}{x} \right) \]
since:
\[ \int_{2x}^t s^{-3}(t - s)^{-\theta} ds = t^{-2-\theta} \int_{2x/t}^1 \rho^{-3}(1 - \rho)^{-\theta} d\rho \leq Ct^{-2-\theta} \frac{t^2}{x^2} = \frac{Ct^{-\theta}}{x^2} \]
\[ \int_{2x}^t s^{-2}(t - s)^{-\theta} ds = t^{-1-\theta} \int_{2x/t}^1 \rho^{-2}(1 - \rho)^{-\theta} d\rho \leq Ct^{-1-\theta} \frac{t}{x} = \frac{Ct^{-\theta}}{x} \]

On the other hand,
\[ \int_0^t (1 + \xi(t - s, x)) ds = \left( t + \frac{\min(t^4, x^4)}{4x^4} \right) + \log \left( \frac{t}{x} \right) + \forall x > 0, \]
\[ \int_0^t s^{-\theta} \xi(t - s, x) ds = \frac{6x^{-4}t^{4-\theta}}{24 - 50\theta + 35\theta^2 - 10\theta^3 + \theta^4}, \forall x \geq t > 0 \]
\[ \int_0^t s^{-\theta} \xi(t - s, x) ds = t^{-\theta} \beta \left( 1 - \frac{x}{t}, 1 - \theta, 0 \right) + r(t, x), \forall t > x > 0 \]

\[ r(t, x) = \frac{x^{-t}(t(t - x))^{-\theta} (4 - \theta)(3 - \theta)(2 - \theta)(1 - \theta)}{(4 - \theta)(3 - \theta)(2 - \theta)(1 - \theta)} \times \]
\[ \times \left( -6t^{4+\theta} + 6t^4(t - x)^{\theta} + 6\theta t^{3+\theta} x + 3(1 - \theta)\theta t^{2+\theta} x^2 + (2 - \theta)(1 - \theta)\theta t^{1+\theta} x^3 + (3 - \theta)(2 - \theta)(1 - \theta)\theta t^{x} x^4 \right), \forall t > x > 0 \]
\[ = (t - x)^{-\theta} \left( C_{1, \theta} \left( \frac{t}{x} \right)^4 + C_{3, \theta} \left( \frac{t}{x} \right)^3 + C_{4, \theta} \left( \frac{t}{x} \right)^2 + C_{4, \theta} \left( \frac{t}{x} \right) + C_{5, \theta} \right) + \]
\[ + C_{2} x^{-4} t^{4-\theta} \]
\[
= t^{-\theta} \left(1 - \frac{x}{t}\right)^{-\theta} \left( C_{1,\theta} \left(\frac{t}{x}\right)^4 + C_{2,\theta} \left(\frac{t}{x}\right)^3 + C_{3,\theta} \left(\frac{t}{x}\right)^2 + C_{4,\theta} \left(\frac{t}{x}\right) + C_{5,\theta}\right) + \\
+ C_2 x^{-4} t^{4-\theta} = \frac{1}{4} (24 - 50\theta + 35\theta^2 - 10\theta^3 + \theta^4) t^{-\theta} + \\
+ \theta t^{-\theta} \left(24 - 50\theta + 35\theta^2 - 10\theta^3 + \theta^4\right) \left(\frac{x}{t}\right)^2 + t^{-\theta} \mathcal{O}\left(\frac{x}{t}\right)^2, \quad \frac{t}{x} \to \infty \\
= t^{-\theta} (24 - 50\theta + 35\theta^2 - 10\theta^3 + \theta^4) \left(\frac{1}{4} - \frac{\theta x}{3t}\right) + t^{-\theta} \mathcal{O}\left(\frac{x}{t}\right)^2, \quad \frac{t}{x} \to \infty.
\]

6.2 The system (1.22), (1.23).

We briefly present in this Section the results for the non linear system (1.22), (1.23),

\[
\begin{align*}
\frac{\partial v}{\partial t}(t,x) &= \tilde{p}_c(t) \int_0^\infty (v(t,y) - v(t,x)) M(x,y) dy \\
\frac{\partial \tilde{p}_c}{\partial t}(t) &= -\tilde{p}_c(t) \int_0^\infty I_3 \left(n_0 + n_0(1 + n_0)x^2 v(t,x)\right) x^2 dx.
\end{align*}
\]

The following change of time variable, similar to (1.26),

\[
\tau = \int_0^t \tilde{p}_c(s) ds, \quad f(\tau,x) = v(t,x)
\]

leads again to the equation (1.27). All the results of Section 4 are then available.

The argument goes now as in Section 5. First, define the auxiliary function,

\[
\tilde{m}(\tau) = \int_0^\infty I_3 \left(n_0 + n_0(1 + n_0)x^2 f(t,x)\right) x^2 dx.
\]

6.2.1 The function \( I_3(n_0 + n_0(1 + n_0)|p|^2 f(t,|p|)) \).

The first result is a simpler expression of the term \( I_3(n_0 + n_0(1 + n_0)|p|^2 f(t,x)) \) when \( f \) is the solution obtained in Section 4.

**Proposition 6.9.** There exists two numerical constants \( C_1 > 0 \) and \( C_2 > 0 \) such that, if \( u \) is the solution of (1.9)–(1.11) obtained in Theorem 1.1 for \( u_0 \) satisfying (1.35) and \( \rho > 0 \),

\[
\lim_{\delta \to 0} \int_{|p|>\delta} I_3 \left(n_0 + n_0(1 + n_0)x^2 f(t,x)\right) x^2 dx = -C_1 (1 + a(t))^2 + \\
+ C_2 \int_0^\infty x^3 \left(n_0 + n_0(1 + n_0)x^2 f(t,x)\right) dx, \quad \forall t > 0.
\]

**Proof of Proposition 6.9.** Following [19, 20] the argument is more clear and the calculations simpler in the energy variable \( \omega \). Define then the function \( F = F(\omega) \) as follows

\[
n_0 + n_0(1 + n_0)x^2 f(t,x) = F(t,\omega), \quad \omega = x^2.
\]

\[
= t^{-\theta} \left(1 - \frac{x}{t}\right)^{-\theta} \left( C_{1,\theta} \left(\frac{t}{x}\right)^4 + C_{2,\theta} \left(\frac{t}{x}\right)^3 + C_{3,\theta} \left(\frac{t}{x}\right)^2 + C_{4,\theta} \left(\frac{t}{x}\right) + C_{5,\theta}\right) + \\
+ C_2 x^{-4} t^{4-\theta} = \frac{1}{4} (24 - 50\theta + 35\theta^2 - 10\theta^3 + \theta^4) t^{-\theta} + \\
+ \theta t^{-\theta} \left(24 - 50\theta + 35\theta^2 - 10\theta^3 + \theta^4\right) \left(\frac{x}{t}\right)^2 + t^{-\theta} \mathcal{O}\left(\frac{x}{t}\right)^2, \quad \frac{t}{x} \to \infty
\]
After suitable time rescaling to absorb a positive constant,

\[
I_{3}(n_{0} + n_{0}(1 + n_{0})|p|^{2}u(t, |p|)) = \frac{2}{\sqrt{\omega}} \int_{0}^{\omega} \left( F(\omega - \omega') - F(\omega) \right) d\omega + \frac{4}{\sqrt{\omega}} \int_{\omega}^{\infty} \left( F(\omega') + F(\omega) \right) d\omega'. \tag{6.25}
\]

We arrive now to the point. The integral in the right hand side of (1.4) is obtained by integration over \((0, \infty)\) of (6.25) multiplied by \(\sqrt{\omega}\). The singular behavior of the function \(F(t, \omega)\) as \(\omega \to 0\) makes delicate the estimate of that integral. This was done in detail in [19] under some Hölder conditions on \(F\) that are not known to hold true for the function \(F\). Following the notations of [19], let us define

\[
A_{3}(F, G) = A_{3}^{1}(F, G) + A_{3}^{2}(F, g),
\]

\[
A_{3}^{1}(F, G) = 2 \int_{\delta}^{\infty} \int_{0}^{\omega} \left( F(\omega - \omega')G(\omega') - F(\omega)G(\omega - \omega') - F(\omega)G(\omega') - F(\omega) \right) d\omega' d\omega,
\]

\[
A_{3}^{2}(F, G) = 4 \int_{\delta}^{\infty} \int_{\omega}^{\infty} \left( F(\omega') + F(\omega)G(\omega') + F(\omega - \omega)G(\omega') - F(\omega)G(\omega' - \omega) \right) d\omega' d\omega.
\]

The right hand side of (6.25) is then strictly speaking,

\[
\lim_{\delta \to 0} \left( A_{3}(F(t), F(t)) + 4 \int_{\delta}^{\infty} F(\omega')(\omega' - \delta) d\omega - 2 \int_{\delta}^{\infty} \omega F(\omega) d\omega \right) \tag{6.26}
\]

If we define, as in [19], for some \(d > 0\) small fixed,

\[
g_{<}(t, \omega) = F(t, \omega) \mathbb{1}_{\omega < d}; \quad g_{>}(t, \omega) = F(t, \omega) \mathbb{1}_{\omega > d}
\]

\[
g_{<}(t, \omega) = h_{0}(t, \omega) \mathbb{1}_{\omega < d} + h(t, \omega), \quad h_{0}(t, \omega) = \frac{1 + a(t)}{\omega}.
\]

Then,

\[A_{3}(F, F) = A_{3}(g_{<}, g_{<}) + A_{3}(g_{<}, g_{>}) + A_{3}(g_{>, g_{<}}) + A_{3}(g_{>, g_{>}})\]

As in [19], by Lebesgue’s convergence,

\[
\lim_{\delta \to 0} A_{3}(g_{<}, g_{>}) = \lim_{\delta \to 0} A_{3}(g_{>, g_{<}}) = \lim_{\delta \to 0} A_{3}(g_{<}, g_{>) = 0},
\]

and we are then left with, \(A_{3}(g_{<}, g_{<})\). Define now the function \(h(t, \omega)\) such that,

\[h(t, \omega) = g_{<}(t, \omega) - h_{0}(t, \omega), \quad h_{0}(t, \omega) = \frac{1 + a(t)}{\omega} \mathbb{1}_{\omega < d}.
\]

Then,

\[h(t, \omega) = \left( F(t, \omega) - \frac{1 + a(t)}{\omega} \right) \mathbb{1}_{\omega < d} = \left( n_{0} + n_{0}(1 + n_{0})\omega u(t) - \frac{1 + a(t)}{\omega} \right) \mathbb{1}_{\omega < d}
\]

and

\[A_{3}(g_{<}, g_{<}) = A_{3}(h, h) + A_{3}(h_{0}, h) + A_{3}(h, h_{0}) + A_{3}(h_{0}, h_{0}).\]
The last term is explicit and gives \( A_\delta(h_0, h_0) = -\pi^2 a(t)^2 / 3 \). On the other hand, by \([1.18]\),

\[
 u(t, x) = a(t) + O(t^{-1-\theta} \omega^{1/2-\delta}), \quad \omega \to 0, \ t \to 0
\]

we deduce,

\[
 h(t, \omega) = n_0 + n_0(1 + n_0)\omega(a(t) + O(t^{-1-\theta} \sqrt{\omega})) - \frac{1 + a(t)}{\omega}, \quad \omega \to 0, \ t \to 0
\]

\[
 = O(t^{-1-\theta} \omega^{-1/2+\delta}), \quad \omega \to 0, \ t \to 0.
\]

The function \( h(t, \cdot) \) is then integrable for all \( t > 0 \) fixed and,

\[
 \lim_{\delta \to 0} A_\delta(h, h) = A_0(h, h) = 0.
\]

We slightly rearrange now the term \( A_\delta(h_0, h) + A_\delta(h, h_0) \) as

\[
 A_\delta(h_0, h) + A_\delta(h, h_0) = A_\delta^1(h_0, h) + A_\delta^2(h_0, h) + A_\delta^1(h, h_0) + A_\delta^2(h, h_0)
\]

\[
 = (A_\delta^1(h_0, h) + A_\delta^1(h, h_0)) + (A_\delta^2(h_0, h) + A_\delta^2(h, h_0)).
\]

with

\[
 A_\delta^1(h_0, h) + A_\delta^1(h, h_0) = 2 \int_\delta^\infty \int_0^\omega \left( (h(\omega' - \omega))h_0(\omega') + (h(\omega'))h_0(\omega - \omega') - (h(\omega - \omega'))h_0(\omega) \right) d\omega'd\omega
\]

and

\[
 A_\delta^2(h_0, h) + A_\delta^2(h, h_0) = 4 \int_\delta^\infty \int_\omega^\infty \left( (h(\omega') - h(\omega'))h_0(\omega) + (h(\omega') - h(\omega))h_0(\omega') + (h(\omega') - h(\omega))(\omega')h_0(\omega) \right) d\omega'd\omega.
\]

The argument still follows as in \([19]\), even if our function \( h \) satisfies slightly different conditions than \((A.13)\) and \((A.14)\). Indeed we claim that here also, the two functions under the integral signs in \((6.27)\) and \((6.28)\) are integrable on \((0, \infty)\). The only delicate region is where \( \omega \) and \( |\omega' - \omega| \) are arbitrarily small.

Consider for example the term \((h(t, \omega) - h(t, \omega'))h_0(t, \omega - \omega')\) for \( \omega' \in (0, \omega) \) in \((6.27)\). Rewrite first, with \( u(t) = u(t, \sqrt{\omega}) \) and \( u'(t) = u(t, \sqrt{\omega'}) \)

\[
 h(\omega) - h(\omega') = n_0 + n_0(1 + n_0)\omega u(t) - n_0' - n_0'(1 + n_0')\omega u'(t) - \frac{1 + a(t)}{\omega} + \frac{1 + a(t)}{\omega'} = \varphi(\omega) - \varphi(\omega') + \psi(\omega) - \psi(\omega')
\]

\[
 \varphi(\omega) = \left( n_0 - \frac{1}{\omega} \right)
\]

\[
 \psi(\omega) = \left( n_0(1 + n_0)\omega u(t) - \frac{a(t)}{\omega} \right)
\]

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It is now immediate that the function \( \varphi \) is globally Lipschitz on \([0, \infty)\). On the other hand, if the difference \( \psi(\omega) - \psi(\omega') \) is written as

\[
\psi(\omega) - \psi(\omega') = n_0(1 + n_0)\omega(u(t) - u'(t)) + (n_0(1 + n_0)\omega - n_0'(1 + n_0')\omega') u'(t) - \frac{a(t)}{\omega} + \frac{a(t)}{\omega'}
\]  

(6.32)

the two terms in the right hand side of (6.32) are estimated as follows. In the first one, for \( \omega \) and \( \omega' \) small

\[
|n_0(1 + n_0)\omega(u(t) - u'(t))| \leq C|u(t, \sqrt{\omega}) - u(t, \sqrt{\omega'})| / \omega
\]

By Lemma 6.3 for each \( t > 0 \) fixed, if \( x \) and \( x' \) are small enough,

\[
|f(t, x) - f(t, x')| \leq Ct^{-2}|x - x'|
\]

then, if \( \omega \) is small enough and \( \omega' \in (0, \omega) \),

\[
|u(t, \sqrt{\omega}) - u(t, \sqrt{\omega'})| \leq Ct^{-2}(\sqrt{\omega} - \sqrt{\omega'})
\]

and

\[
|n_0(1 + n_0)\omega(u(t) - u'(t))| \leq \frac{C(\sqrt{\omega} - \sqrt{\omega'})}{t^2\omega} \leq \frac{C\sqrt{\omega - \omega'}}{t^2\omega}
\]  

(6.33)

The second term in the right hand side of (6.32), is written,

\[
(n_0(1 + n_0)\omega - n_0'(1 + n_0')\omega') u'(t) - \frac{a(t)}{\omega} + \frac{a(t)}{\omega'} =
\]

\[
= \left( n_0(1 + n_0)\omega - n_0'(1 + n_0')\omega' \right) \left( a(t) + \mathcal{O} \left( \tau(t)^{-1-\theta \omega^{1/2-\delta}} \right) \right) - \frac{a(t)}{\omega} + \frac{a(t)}{\omega'}
\]

\[
= \left( n_0(1 + n_0)\omega - \frac{1}{\omega} - n_0'(1 + n_0')\omega' + \frac{1}{\omega'} \right) a(t) +
\]

\[
+ \left( n_0(1 + n_0)\omega - n_0'(1 + n_0')\omega' \right) \mathcal{O} \left( \tau(t)^{-1-\theta \omega^{1/2-\delta}} \right).
\]  

(6.34)

The function \( \omega \mapsto n_0(1 + n_0)\omega - \omega^{-1} \) is Lipschitz and then, the factor of \( a(t) \) in the right hand side of (6.34) yields,

\[
\left| n_0(1 + n_0)\omega - \frac{1}{\omega} - n_0'(1 + n_0')\omega' + \frac{1}{\omega'} \right| \leq C(\omega - \omega').
\]  

(6.35)

For the last term in the right hand side of (6.34), we notice that,

\[
\frac{d}{d\omega} n_0(1 + n_0)\omega = -\frac{1}{4} \left( -1 + \omega \coth(\omega/2) \right) (\text{csch}(\omega/2))^2 = -\frac{1}{\omega^2} + \mathcal{O}(1), \ \omega \to 0.
\]

from where, if we call \( g(\omega) \equiv n_0(1 + n_0)\omega \), for \( \omega \) and \( \omega' \) small,

\[
\left| n_0(1 + n_0)\omega - n_0'(1 + n_0')\omega' \right| \leq (\omega - \omega') \int_0^1 \left| \frac{dg}{d\omega} (r\omega + (1 - r)\omega') \right| dr
\]

\[
\leq C(\omega - \omega') \int_0^1 (r\omega + (1 - r)\omega')^{-2} dr \leq C|\omega - \omega'| / \omega\omega'.
\]  

(6.36)
It follows from (6.29), (6.32)–(6.36) that for $d$ small, $\omega \in (0, d)$ and $\omega' \in (0, \omega)$

$$|(h(t, \omega) - h(t, \omega'))h_0(t, \omega - \omega')| \leq \frac{C}{(\omega - \omega')} \left( \frac{\sqrt{\omega - \omega'}}{t^2\omega} + \frac{(\omega - \omega')}{\omega^{1/2} + \delta} \right)$$

$$\leq C \left( 1 + \frac{1}{t^2\omega(\omega - \omega')^{1/2}} + \frac{1}{\omega^{1/2} + \delta} \right).$$

Moreover, if $\omega' \in (0, \omega)$, were such that $\omega > d + \omega'$ or $\omega' > d$ it would follow that $(h(t, \omega) - h(t, \omega'))h_0(t, \omega - \omega') = 0$. Therefore,

$$\int_0^\infty \int_0^\omega |(h(t, \omega) - h(t, \omega'))h_0(t, \omega - \omega')|d\omega' d\omega =$$

$$= \int_0^{2d} \int_0^\omega |(h(t, \omega) - h(t, \omega'))h_0(t, \omega - \omega')|d\omega' d\omega$$

$$\leq C \int_0^{2d} \int_0^\omega \left( 1 + \frac{1}{t^2\omega(\omega - \omega')^{1/2}} + \frac{1}{\omega^{1/2} + \delta} \right) d\omega' d\omega \to 0, \text{ as } d \to 0.$$

Arguing in the same way for all the other terms in (6.27) and (6.28) it follows that,

$$\lim_{\delta \to 0} A_\delta(h_0, h) + A_\delta(h, h_0) = 0$$

and then

$$\lim_{\delta \to 0} A_\delta(F, F) = -\frac{\pi^2}{3} a(t)^2 + \int_0^\infty \frac{f(t,x)x^5}{(\sinh(x^2/2))^2}dx.$$

Proposition follows when the initial time rescaling is inverted. \hfill \Box

By Proposition 6.9, $\bar{m}$ is well defined and finite for all $\tau > 0$. Let us then define

$$\bar{M}(\tau) = \int_0^\tau \bar{m}(\sigma)d\sigma; \quad \bar{q}_c(\tau) = q_c(0)e^{-\bar{M}(\tau)}, \ \forall \tau > 0.$$

**Proposition 6.10.** There exists $T_\ast \in [0, \infty)$, such that for all $t \in (0, T_\ast)$ there exists a unique $\tau > 0$ such that

$$t = \int_0^\tau \frac{d\sigma}{\bar{q}_c(\sigma)}, \ \forall \tau > 0. \quad (6.37)$$

**Proof.** By Proposition 6.9 $|M(\tau)| < \infty$ for all $\tau > 0$ and then $q_c(\tau) \in (0, \infty)$ for all $\tau > 0$ and the integral in the right hand side of (6.37) is well defined and convergent. Since $q_c(t) > 0$ this integral is a monotone increasing function of $\tau$. The value of $T_\ast$ is then given by

$$T_\ast = \int_0^\infty \frac{d\sigma}{\bar{q}_c(\sigma)} \quad \Box$$

**Remark 6.11.** The function $\bar{M}$ can not be estimated as $M$ in Proposition 5.2, using the conservation of the total number of particles. By Proposition 6.7 and Proposition 5.12

$$\bar{m}(\sigma) = -C_1(1 + b(\sigma))^2 + C_2 \int_0^\infty (n_0 + n_0(1 + n_0)x^2f(\sigma,x))x^3dx. \quad (6.38)$$
By Corollary 4.12 and Corollary 4.14,

$$
\lim_{t \to \infty} \int_0^\infty (n_0 + n_0(1 + n_0)x^2f(\sigma, x))x^3dx = C_* \int_0^\infty (n_0 + n_0(1 + n_0)x^2x^3dx
$$

from where, for some constant $C > 0$,

$$
C_2 \int_0^\tau \int_0^\infty (n_0 + n_0(1 + n_0)f(\sigma, x))x^5dx \leq C\tau, \forall \tau > 0.
$$

But, the first term in the right hand side of (6.38) may only be estimated using (4.21),

$$
\int_0^\tau (1 + b(\sigma))^2d\sigma \leq \int_0^\tau \left(1 + C|||f_0|||_\theta \left(\sigma^{-\theta} + \sigma\right)^2\right) d\sigma \leq C\left(\tau + |||f_0|||_\theta \left(\tau^{1-2\theta} + \tau^3\right)\right).
$$

It then follows

$$
\tilde{m}(\tau) \geq C\left(-\left(\tau + |||f_0|||_\theta \left(\tau^{1-2\theta} + \tau^3\right)\right) + \tau\right), \quad (6.39)
$$

but this estimate is not sufficient to deduce that $T^* = \infty$.

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