Cusa-Huygens, Wilker and Huygens Type Inequalities for Generalized Hyperbolic Functions

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Abstract

In this paper, we establish Cusa-Huygens, Wilker and Huygens type inequalities for certain generalizations of the hyperbolic functions. From the established results, we recover some previous results as particular cases.

1 Introduction

The inequality

\[
\frac{\sin z}{z} < \frac{\cos z + 2}{3}, \quad 0 < z < \frac{\pi}{2},
\]

(1)

is known in the literature as Cusa-Huygens inequality. Its hyperbolic counterpart, which is given as

\[
\frac{\sinh z}{z} < \frac{\cosh z + 2}{3}, \quad x > 0,
\]

(2)

was established by Neuman and Sandor \[14\]. The inequality

\[
\left(\frac{\sin z}{z}\right)^2 + \frac{\tan z}{z} > 2, \quad 0 < z < \frac{\pi}{2},
\]

(3)

which is known as Wilker inequality was first proposed in the classic work \[21\] p.55 and subsequently attracted the attention of other researchers. In \[23\], Wu
and Srivastava proved the Wilker-type inequality
\[
\left( \frac{z}{\sin z} \right)^2 + \frac{z}{\tan z} > 2, \quad 0 < z < \frac{\pi}{2}.
\] (4)

The hyperbolic counterpart of (3) was established by Zu [24] as
\[
\left( \frac{\sinh z}{z} \right)^2 + \frac{z}{\tanh z} > 2, \quad z \in \mathbb{R} \setminus \{0\}.
\] (5)

Also, the hyperbolic counterpart of (4) was established by Wu and Debnath [22] as
\[
\left( \frac{z}{\sinh z} \right)^2 + \frac{z}{\tanh z} > 2, \quad z \in \mathbb{R} \setminus \{0\}.
\] (6)

Another inequality of interest is the Huygens inequality which is given as [17]
\[
2\frac{\sin z}{z} + \frac{\tan z}{z} > 3, \quad 0 < z < \frac{\pi}{2},
\] (7)

and its hyperbolic counterpart given as [14]
\[
2\frac{\sinh z}{z} + \frac{\tanh z}{z} > 3, \quad z \in \mathbb{R} \setminus \{0\}.
\] (8)

Due to their usefulness, these elegant inequalities have been studied extensively and in diverse ways by several researchers. See for example [2], [3], [4], [5], [6], [7], [8], [12], [13], [15], [16], [18], [19], [20], [22], [24], [25], [26] and the related references therein.

Also, in a recent work, the Huygens-type inequality
\[
\frac{2}{\cosh z} + \cosh z > \frac{\sinh z}{z} + 2\frac{\tanh z}{z} > \frac{1}{\cosh z} + 2, \quad z \in \mathbb{R} \setminus \{0\},
\] (9)

was established among other things by Bagul and Chesneau [1].

Motivated by the results [2], [3], [4], [8] and [9], the objective of this paper is to establish analogous inequalities concerning certain generalizations of the hyperbolic functions. The established results serve as generalizations of the previous results.
2 Preliminary Definitions

In a bid to generalize a previous work [9], the authors of [10] gave the following generalizations of the hyperbolic functions.

**Definition 2.1.** The generalized hyperbolic cosine, hyperbolic sine and hyperbolic tangent functions are respectively defined as [10]

\[
\cosh_a(z) = \frac{a^z + a^{-z}}{2}, \quad (10)
\]

\[
\sinh_a(z) = \frac{a^z - a^{-z}}{2}, \quad (11)
\]

\[
\tanh_a(z) = \frac{\sinh_a(z)}{\cosh_a(z)} = \frac{a^z - a^{-z}}{a^z + a^{-z}} = 1 - \frac{2}{1 + a^{2z}}, \quad (12)
\]

where \( a > 1 \) and \( z \in \mathbb{R} \).

These generalized functions satisfy the following identities.

\[
\cosh_a(z) + \sinh_a(z) = a^z, \quad (13)
\]

\[
\cosh_a(z) - \sinh_a(z) = a^{-z}, \quad (14)
\]

\[
(\cosh_a(z))' = (\ln a) \sinh_a(z), \quad (15)
\]

\[
(\sinh_a(z))' = (\ln a) \cosh_a(z), \quad (16)
\]

\[
(\tanh_a(z))' = \frac{\ln a}{\cosh_a^2(z)}, \quad (17)
\]

\[
(\cosh_a(z))'' + (\sinh_a(z))'' = (\ln a)^2 a^z, \quad (18)
\]

\[
(\cosh_a(z))'' - (\sinh_a(z))'' = (\ln a)^2 a^{-z}, \quad (19)
\]

\[
\cosh_a^2(z) + \sinh_a^2(z) = \cosh_a(2z), \quad (20)
\]

\[
\cosh_a^2(z) - \sinh_a^2(z) = 1, \quad (21)
\]

\[
2 \sinh_a(z) \cosh_a(z) = \sinh_a(2z), \quad (22)
\]

\[
\cosh_a^2(z) = \frac{\cosh_a(2z) + 1}{2}, \quad (23)
\]
The generalized hyperbolic secant, hyperbolic cosecant and hyperbolic cotangent functions are respectively defined as
\[
\text{sech}_a(z) = \frac{1}{\cosh_a(z)}, \quad \text{cosech}_a(z) = \frac{1}{\sinh_a(z)}, \quad \text{coth}_a(z) = \frac{1}{\tanh_a(z)}.
\] (25)

As pointed out in [10], several other identities can be derived from (10), (11) and (12). When \( a = e \), where \( e = 2.71828... \) is the Euler’s number, then the above definitions and identities reduce to their ordinary counterparts.

3 Results and Discussion

Lemma 3.1. The inequality
\[
\cosh_a(z) < \left( \frac{\sinh_a(z)}{z} \right)^3,
\] (26)
holds for \( z \in \mathbb{R} \setminus \{0\} \).

Proof. Inequality (26) has been proved in [11] for \( z > 0 \). Now let \( z < 0 \) so that \( -z > 0 \). Then
\[
\cosh_a(z) = \cosh_a(-z) < \left( \frac{\sinh_a(-z)}{-z} \right)^3 = \left( \frac{\sinh_a(z)}{z} \right)^3,
\]
which completes the proof. \( \square \)

Since the function \( \frac{\sinh_a(z)}{z} \) is increasing for \( z > 0 \) and decreasing for \( z < 0 \), then Lemma 3.1 implies the following generalized result.

Lemma 3.2. The inequality
\[
\cosh_a(z) < \left( \frac{\sinh_a(z)}{z} \right)^v,
\] (27)
holds for \( z \in \mathbb{R} \setminus \{0\} \) and \( v \geq 3 \).
Lemma 3.3 (\cite{11}). For \( z \in \mathbb{R} \setminus \{0\} \), the inequality
\[
\frac{\ln a}{\cosh_a(z)} < \frac{\sinh_a(z)}{z} < \frac{(\ln a) \cosh_a(z)}{z},
\]
holds.

Lemma 3.4 (Young’s Inequality). Let \( x, y \geq 0, r, s \in (0, 1) \) such that \( r + s = 1 \). Then,
\[
x^r y^s \leq rx + sy.
\]

Theorem 3.5. The inequality
\[
\frac{\sinh_a(z)}{z} < \frac{2 \ln a + (\ln a) \cosh_a(z)}{3},
\]
holds for \( z \in \mathbb{R} \setminus \{0\} \).

Proof. Since the functions in each term of the inequality are even, it suffices to prove the case for \( z > 0 \). Let \( z > 0 \) and \( h \) be defined as
\[
h(z) = \frac{2z + z \cosh_a(z)}{\sinh_a(z)}.
\]
Then
\[
h'(z) = \frac{1}{\sinh^2_a(z)} [2 \sinh_a(z) + \cosh_a(z) \sinh_a(z) - 2z(\ln a) \cosh_a(z) - (\ln a)z]
\[
= \frac{1}{\sinh^2_a(z)} \phi(z)
\]
and then
\[
\phi'(z) = (\ln a) [\cosh_a(2z) - 2(\ln a)z \sinh_a(z) - 1]
\[
= 2(\ln a) \sinh_a(z) [\sinh_a(z) - (\ln a)z] > 0,
\]
since \( \frac{\sinh_a(z)}{z} > \ln a \) for \( z \in (0, \infty) \). Hence \( \phi(z) \) is increasing and consequently, \( \phi(z) > \phi(0) = 0 \). Thus \( h(z) \) is increasing. Hence
\[
h(z) > \lim_{z \to 0} h(z) = \frac{3}{\ln a},
\]
which gives \( (30) \).

\[ \square \]
Remark 3.6. When \( a = e \), then inequality (30) reduces to the hyperbolic Cusa-Huygens inequality (2).

**Theorem 3.7.** The inequalities

\[
\left( \frac{\sinh_a(z)}{z} \right)^2 + \frac{\tanh_a(z)}{z} > 2,
\]

\[
\left( \frac{z}{\sinh_a(z)} \right)^2 + \frac{z}{\tanh_a(z)} > 1 + \ln a \left( \frac{\ln a}{a} \right)^2, \quad 1 < a \leq e,
\]

hold for \( z \in \mathbb{R} \setminus \{0\} \).

**Proof.** Let \( z \in \mathbb{R} \setminus \{0\} \). Then by the AM-GM inequality and Lemma 3.1, we obtain

\[
\left( \frac{\sinh_a(z)}{z} \right)^2 + \frac{\tanh_a(z)}{z} \geq 2 \sqrt{\left( \frac{\sinh_a(z)}{z} \right)^2 \frac{\tanh_a(z)}{z}}
\]

\[
= 2 \sqrt{\left( \frac{\sinh_a(z)}{z} \right)^3 \frac{1}{\cosh_a(z)}} > 2,
\]

which gives (31). To prove (32), it suffices to prove the case for \( z > 0 \). Let \( z > 0 \) and define \( \psi(z) \) by

\[
\psi(z) = \left( \frac{z}{\sinh_a(z)} \right)^2 + \frac{z}{\tanh_a(z)},
\]

where \( 1 < a \leq e \). Then by differentiating, applying the AM-GM inequality and Lemma 3.1 we obtain

\[
\psi'(z) = \frac{1}{\sinh_a(z)} \left[ \sinh_a^2(z) \cosh_a(z) + (2 - \ln a) z \sinh_a(z) - 2(\ln a) z^2 \cosh_a(z) \right]
\]

\[
\geq \frac{1}{\sinh_a(z)} \left[ 2 \sqrt{\sinh_a^2(z) \cosh_a(z) \cdot (2 - \ln a) z \sinh_a(z) - 2(\ln a) z^2 \cosh_a(z)} \right]
\]

\[
= \frac{2z^2}{\sinh_a(z)} \sqrt{\cosh_a(z)} \left[ \sqrt{2 - \ln a} \left( \frac{\sinh_a(z)}{z} \right)^3 - (\ln a) \sqrt{\cosh_a(z)} \right]
\]

\[> 0.\]
Thus $\psi(z)$ is increasing. Hence

$$\psi(z) > \lim_{z \to 0} \psi(z) = \frac{1 + \ln a}{(\ln a)^2},$$

which gives (32).

**Remark 3.8.** When $a = e$, then the inequalities (31) and (31) reduce to the hyperbolic Wilker-type inequalities (5) and (6) respectively.

**Theorem 3.9.** The inequality

$$\left(\frac{\sinh_a(z)}{z}\right)^2 + \frac{\tanh_a(z)}{z} > \left(\frac{z}{\sinh_a(z)}\right)^2 + \frac{z}{\tanh_a(z)},$$

holds for $z \in \mathbb{R} \setminus \{0\}$.

**Proof.** Using the fact that $(A^2 + B)/(1/A^2 + 1/B) = A^2B$, together with Lemma 3.1, we obtain

$$\left(\frac{\sinh_a(z)}{z}\right)^2 + \frac{\tanh_a(z)}{z} = \left(\frac{z}{\sinh_a(z)}\right)^2 + \frac{z}{\tanh_a(z)} = \left(\frac{\sinh_a(z)}{z}\right)^3 \frac{1}{\cosh_a(z)} > 1,$$

which concludes the proof.

**Theorem 3.10.** Let $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta = 1$. Then the inequality

$$\alpha \left(\frac{\sinh_a(z)}{z}\right)^2 + \beta \left(\frac{\tanh_a(z)}{z}\right) > (\ln a)^2(\alpha - \beta),$$

holds for $z \in \mathbb{R} \setminus \{0\}$. 

*Earthline J. Math. Sci. Vol. 5 No. 2 (2021), 277-289*
Proof. Let \( z \in \mathbb{R} \setminus \{0\} \). Then Youngs inequality (29) and Lemma 3.1 imply that

\[
\alpha \left( \frac{\sinh a(z)}{z} \right)^2 + \beta \left( \frac{\tanh a(z)}{z} \right) \geq \left( \frac{\sinh a(z)}{z} \right)^{2\alpha} \left( \frac{\tanh a(z)}{z} \right)^{\beta} \\
= \left( \frac{\sinh a(z)}{z} \right)^{2\alpha + \beta} \left( \frac{1}{\cosh a(z)} \right)^{\beta} \\
> \left( \frac{\sinh a(z)}{z} \right)^{2\alpha + \beta} \left( \frac{\sinh a(z)}{z} \right)^{-3\beta} \\
= \left( \frac{\sinh a(z)}{z} \right)^{2(\alpha - \beta)} \\
> (\ln a)^2(\alpha - \beta),
\]

which completes the proof. \( \square \)

Remark 3.11. If \( \alpha = \beta = \frac{1}{2} \), then (34) reduces to (31).

Theorem 3.12. The inequality

\[
2 \frac{\sinh a(z)}{z} + \frac{\tanh a(z)}{z} > 3 \ln a,
\]

holds for \( z \in \mathbb{R} \setminus \{0\} \).

Proof. It suffices to prove the case for \( z > 0 \). Let \( z > 0 \) and let \( h \) be defined as

\[
h(z) = 2 \frac{\sinh a(z)}{z} + \frac{\tanh a(z)}{z}.
\]

Then

\[
z^2 h'(z) = 2(\ln a) z \cosh a(z) - 2 \sinh a(z) + (\ln a) z \text{sech}^2 a(z) - \tanh a(z)
\]

\[
= \theta(z),
\]

and

\[
\theta'(z) = 2(\ln a)^2 z \sinh a(z) - 2(\ln a)^2 z \tanh a(z) \text{sech}^2 a(z)
\]

\[
= 2(\ln a)^2 z \sinh a(z) \left[ 1 - \frac{1}{\cosh^2 a(z)} \right] > 0,
\]

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which shows that $\theta(z)$ is increasing. Hence $\theta(z) > \theta(0) = 0$. Thus, $h(z)$ is increasing and consequently,

$$h(z) > \lim_{z \to 0} h(z) = 3 \ln a$$

which yields (35).

**Remark 3.13.** When $a = e$, then inequality (35) reduces to (8).

**Theorem 3.14.** The inequality

$$2 \frac{z}{\sinh_a(z)} + \frac{z}{\tanh_a(z)} > 3 \ln a,$$

holds for $z \in \mathbb{R} \setminus \{0\}$.

**Proof.** It suffices to prove the case for $z > 0$. Let $z > 0$ and $\delta$ be defined as

$$\delta(z) = 2 \frac{z}{\sinh_a(z)} + \frac{z}{\tanh_a(z)}.$$

Then

$$\delta'(z) = \frac{1}{\sinh_a^2(z)} [2 \sinh_a(z) - z \ln a + \cosh_a(z)(\sinh_a(z) - z \ln a)] > 0$$

since $\sinh_a(z) > z \ln a$. Hence $\delta(z)$ is increasing and consequently,

$$\delta(z) > \lim_{z \to 0} \delta(z) = \frac{3}{\ln a}$$

which yields (36).

**Theorem 3.15.** The inequality

$$(m + 1) \frac{\sinh_a(z)}{z} + m \frac{\tanh_a(z)}{z} > m \frac{\sinh_a(z)}{z} + (m + 1) \frac{\tanh_a(z)}{z}, \quad (37)$$

holds for $z \in \mathbb{R} \setminus \{0\}$ and $m \in \mathbb{N} \cup \{0\}$.
Proof. \( z \in \mathbb{R} \setminus \{0\} \). Then
\[
\frac{\sinh_a(z)}{z} - \frac{\tanh_a(z)}{z} = \frac{\sinh_a(z)}{z} \left[1 - \frac{1}{\cosh_a(z)}\right] > 0,
\]
since \( \cosh_a(z) > 1 \) for \( z \in \mathbb{R} \). That is
\[
\frac{\sinh_a(z)}{z} > \frac{\tanh_a(z)}{z}.
\]
Adding \( m \left( \frac{\sinh_a(z)}{z} + \frac{\tanh_a(z)}{z} \right) \) to both sides of (38) completes the proof. \( \square \)

**Theorem 3.16.** The inequality
\[
\frac{2 \ln a}{\cosh_a(z)} + (\ln a) \cosh_a(z) > \frac{\sinh_a(z)}{z} + 2 \frac{\tanh_a(z)}{z} > \frac{\ln a}{\cosh_a(z)} + 2 \ln a,
\]
holds for \( z \in \mathbb{R} \setminus \{0\} \).

**Proof.** It suffices to prove the case for \( z > 0 \). Let \( z > 0 \) and \( f \) be defined as
\[
f(z) = 2(\ln a)z + (\ln a)z \cosh^2_a(z) - \sinh_a(z) \cosh_a(z) - 2 \sinh_a(z).
\]
Then
\[
f'(z) = 2(\ln a) + 2(\ln a)^2 z \cosh_a(z) \sinh_a(z) - (\ln a) \sinh^2_a(z) - 2(\ln a) \cosh_a(z),
\]
and then
\[
f''(z) = 2(\ln a)^3 z \sinh^2_a(z) + 2(\ln a)^2 \left[(\ln a)z \cosh^2_a(z) - \sinh_a(z)\right] > 0,
\]
since \( \sinh_a(z) < (\ln a)z \cosh_a(z) \) (see Lemma 3.3) and \( \cosh_a(z) < \cosh^2_a(z) \). Hence \( f''(z) \) is increasing and so, \( f'(z) > f'(0) = 0 \). Thus, \( f(z) \) is increasing and so \( f(z) > f(0) = 0 \). This yields the left-hand side of (39). Next, for \( z > 0 \), let \( g \) be defined as
\[
g(z) = \sinh_a(z) \cosh_a(z) + 2 \sinh_a(z) - (\ln a)z - 2(\ln a)z \cosh_a(z).
\]
Then
\[
g'(z) = (\ln a) \cosh^2_a(z) + (\ln a) \sinh^2_a(z) - \ln a - 2(\ln a)^2 z \sinh_a(z)
\]
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and then
\[ g''(z) = 2(\ln a)^2 \cosh_a(z) \left[ \sinh_a(z) - (\ln a)z \right] + 2(\ln a)^2 \sinh_a(z) \left[ \cosh_a(z) - 1 \right] > 0, \]

since \( \sinh_a(z) > (\ln a)z \) and \( \cosh_a(z) > 1 \). Hence \( g'(z) \) is increasing and so, \( g'(z) > g'(0) = 0 \). Thus, \( g(z) \) is increasing and so \( g(z) > g(0) = 0 \). This yields the right-hand side of (39) and that completes the proof. \( \square \)

**Remark 3.17.** When \( a = e \), then inequality (39) reduces to (9).

**Remark 3.18.** By (37) and (39), we have
\[ 2 \frac{\sinh_a(z)}{z} + \frac{\tanh_a(z)}{z} > \frac{\ln a}{\cosh_a(z)} + 2 \ln a. \]

This is however weaker than inequality (35).

**Acknowledgement**

The author is grateful to the anonymous referees for thorough reading of the manuscript.

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