Controlling entanglement in the interferometry of driven coupled flux qubits

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Abstract. We study the manipulation of quantum entanglement by periodic external fields in the Landau-Zener-Stuckelberg interferometry of coupled flux qubits. As an entanglement measure we compute numerically the concurrence of two coupled flux qubits, both driven by a dc+ac magnetic flux. We show that it is possible to create or destroy entanglement in a controlled way by tuning the system at or near multiphoton resonances.

1. Introduction
Among the rich variety of qubits that have been explored as candidates for quantum computation, solid state superconducting devices based on Josephson junctions circuits are promising due to their microfabrication techniques and downscalability. [1, 2, 3, 4]

To implement a quantum algorithm, one must be able to entangle qubits by means of an interaction term in the Hamiltonian describing a two qubit system. For superconducting flux qubits [1, 5], the natural interaction is between the magnetic fluxes, providing a coupling through their mutual inductance.[1, 5, 6]

Although coupled superconducting qubits have been experimentally realized [6, 7, 8], the generation and control of entanglement can be quite complicated and demanding, even requiring sequences of single- and two-qubit operations. Along this line, pulse sequences have been implemented for several superconducting qubits with fixed interaction energies [9, 10]. However, entangling operations can be much more efficient if the interaction can be varied and, ideally, turned off during parts of the manipulation. Some tunable coupling schemes have been proposed, [11, 12, 13, 14] but these approaches failed in controlling the coupling entirely.

Alternatively, engineering selection rules of transitions among different energy levels is a possible strategy for coupling and decoupling superconducting qubits. [15] Following this idea, a method to create artificial selection rules—by suppressing and/or exciting specific transitions—has been developed for a pair of superconducting flux qubits [11], simultaneously driven by a single resonant frequency with different amplitudes and phases.[16] Also, entangling gates via proper microwave driving have been considered. [17]

Our main goal is to study the possibility of manipulating entanglement between two flux qubits, by external driving fields of variable amplitude and fixed frequency non resonant with any specific transition. The sensitivity of the energy levels of a flux qubit driven by an external (dc+ac) magnetic field, has been extensively studied in recent years.[18, 19, 20, 21, 22] The experimental implementation of Landau-Zener-Stückelberg interferometry [5] has become a tool to analyze quantum coherence under strong driving and to access the multilevel structure of flux qubits, which exhibits several avoided crossings as a function of the magnetic flux.[22] In most
of these cases the Floquet formalism has been employed to solve the system dynamics in terms of quasienergies and Floquet states. \cite{23, 24, 25}

In Ref.\cite{26} it has been recently shown how to generate entangled Floquet states in weakly interacting two level systems by controlling the amplitude of the external driving field. However Floquet states are not accessible experimentally. Here we extend the study for arbitrary coupling beyond the weak interaction case, and for general pure states that can be prepared as initial states, like the ground state of the two qubit system or states of the computational basis.

The system of work consists in two coupled flux qubits driven by (the same) microwave field. As usual, each qubit is represented by a two level system \cite{27, 28, 29} and we focus here on the case of negligible dissipation and/or interaction with the environment. Therefore, our results are valid for time scales smaller than the decoherence time of the qubits. As a measure of entanglement we choose the concurrence \cite{30} which increases monotonically from 0 for non-entangled states to 1, for maximally entangled states. Experimentally there are some evidences of entanglement measurements in solid state physics, see for example \cite{31}, \cite{32}.

### 2. Concurrence for the two coupled flux qubits model

The dynamics of two inductively coupled flux qubits can be described by the Hamiltonian \cite{33, 34}

\[
\hat{H}_0 = -\frac{1}{2} \sum_{i=1}^{2} \left( \epsilon_i \sigma_z^{(i)} + \Delta_i \sigma_x^{(i)} \right) - \frac{J_z}{2} \sigma_z^{(1)} \otimes \sigma_z^{(2)},
\]

where \(\epsilon_i\) is the detuning energy (which is proportional to the difference between the magnetic flux through the qubit and half the quantum of flux), \(\Delta_i\) is the tunnel splitting energy and \(\sigma_z^{(i)}, \sigma_x^{(i)}\) the Pauli matrices, with \(i = 1, 2\) the index of each qubit, with \(J_z\) the qubit-qubit coupling constant. The coupling corresponds to an inductive coupling \cite{6, 13}, since \(J_z = \pm MI_1 I_2\) can be written in terms of the mutual inductance \(M\) and the qubits currents \(I_{1,2}\). For \(J_z < 0\) \(J_z > 0\) the coupling is antiferromagnetic (ferromagnetic).

To study the dynamics in the presence of driving fields, we replace as usual \(\epsilon_i \rightarrow \epsilon_i(t) = \epsilon_i + f(t)\) \cite{26, 29, 33, 34} where \(f(t) = A \cos(\omega t)\) is the microwave field (magnetic flux) of amplitude \(A\) and frequency \(\omega\) applied to each qubit.

According to the Floquet theorem \cite{23, 27, 28}, the solution of the Schrödinger equation can be spanned in the Floquet basis \(\{|u_{\alpha}(t)\}\) as \(\Psi(t) = \sum_{\alpha} a_{\alpha}(t_0) e^{-i\gamma_{\alpha} t/\hbar} |u_{\alpha}(t)\rangle\), with \(\gamma_{\alpha}\) the quasienergies and \(\alpha\) the index labeling the eigenstates of the time independent problem. For an initial condition \(|\Psi(t_0)\rangle\) at time \(t_0\), we define the coefficients \(a_{\alpha}(t_0) = \langle u_{\alpha}(t_0)|\Psi(t_0)\rangle\). The time-evolution for a Floquet state is given by \((H(t) - i\hbar \frac{\partial}{\partial t}) |u_{\alpha}(t)\rangle = \gamma_{\alpha} |u_{\alpha}(t)\rangle\), and the satisfy \(|u_{\alpha}(t + T)\rangle = |u_{\alpha}(t)\rangle\). Therefore after expanding the time periodic Floquet states in the Fourier basis, \(|u_{\alpha}(t)\rangle = \sum_{k} e^{ik\omega t} |u_{\alpha}(k)\rangle\) the time-dependent problem is reduced to a time-independent eigenvalue problem.

An entanglement measure quantifies the degree of quantum correlations present in a given quantum state. In the case of pure states \(|\Psi(t)\rangle\) a useful quantity is the concurrence \cite{30},

\[
C(t, t_0) = |\langle \Psi(t)|^\dagger \sigma_y^{(1)} \otimes \sigma_y^{(2)} |\Psi(t)\rangle|,
\]

that goes from 0 for non-entangled states, to 1 for maximally entangled states. Notice that Eq.(2) depends implicitly on the initial time \(t_0\) through \(|\Psi(t)\rangle\).

Using the extended Floquet basis in Fourier space \(|\{u_{\alpha}(k)\}\rangle\) with \(k \in \mathbb{Z}\), Eq.(2) can be written as

\[
C(t, t_0) = \left| \sum_{\alpha \beta k k' q q'} \tilde{C}_{\alpha\beta}(k, k') f_{\alpha\beta}(q, q') e^{-i\epsilon_{\alpha\beta}(k, k') (t - t_0)} \right|,
\]
where \( \tilde{C}_{\alpha\beta}(k, k') = \langle u_\alpha(k)|\sigma_y^{(1)} \otimes \sigma_y^{(2)}|u_\beta(k') \rangle \) and \( f_{\alpha\beta}(q, q') = a_\alpha(q)a_\beta(q') \), with \( a_\alpha(q) = \langle u_\alpha(q)|\Psi(t_0) \rangle \). Under general conditions, the initial time should be averaged out, thus we compute the time-averaged of Eq. (3) over \( t_0 \),

\[
\overline{C}(t) = \frac{1}{T} \int_0^T dt_0 C(t, t_0),
\]

which still presents an oscillating behaviour with time \( t \), typical for time dependent driven systems. However and unlike the occupation probability, the concurrence is not a periodic function of the driving period, as can be easily checked from its definition, Eq. (3). In order to obtain a representative value of concurrence, we eliminate the dependence on time \( t \) by performing an additional time average, obtaining

\[
\overline{C} = \lim_{t' \to \infty} \frac{1}{t'} \int_0^{t'} dt \overline{C}(t).
\]

3. Results

After computing numerically the Floquet states and quasienergies, we calculate the concurrence using Eq. (3) and the respective averages over \( t_0 \) and \( t \), given in Eqs. (4) and (5). Along this work, we fix \( \epsilon_i = \epsilon_0 \), \( \forall i = 1, 2 \), and choose without loss of generality, \( \Delta_1/\omega = 0.1 \) and \( \Delta_2/\omega = 0.15 \). We take \( \hbar = 1 \) and energy scales are normalized by \( \omega \).

![Figure 1: Eigenenergies in the absence of driving, \( A/\omega = 0 \) (red lines) and quasienergies for \( A/\omega = 3.8 \) (black lines) as a function of \( \epsilon_0/\omega \), for the coupling strength \( J^z/\omega = -3 \). Parameters are \( \Delta_1/\omega = 0.1 \) and \( \Delta_2/\omega = 0.15 \).](image)

Fig.1 shows as a function of the detuning \( \epsilon_0/\omega \) and for the coupling strength \( J^z/\omega = -3 \), the eigenenergies \( E_i, i = 0,...,3 \) for the time independent Hamiltonian (red lines), and in black the quasienergies for the driven Hamiltonian for \( A/\omega = 3.8 \). We work with the (antiferromagnetic) coupling \( J^z < 0 \), which reduces the energy of states \( |01 \rangle, |10 \rangle \) while it increases the energy of states \( |00 \rangle, |11 \rangle \), with \( \mathcal{E} = \{|00 \rangle, |01 \rangle, |10 \rangle, |11 \rangle \} \) the computational basis in the product space of the two qubits. The quasienergies display avoided crossings (quasidegeneracies), where the Floquet states are strongly mixed. These quasidegeneracies will play a central role in the structure of the concurrence as we will discuss in the following. In the limit of \( \Delta_i/\omega \to 0 \), the quasidegeneracies transform in exact crossings of quasienergies, giving the resonance condition \( \gamma_\alpha - \gamma_\beta = n\omega, n \in \mathbb{Z} \). [27] The quasienergies of the two qubit system computed analytically for \( \Delta_i/\omega \to 0 '[27] \) are \( \gamma_\alpha \sim \pm \epsilon_0 + J^z/2 + m\omega \) and the (quasi) degenerate pair \( -J^z/2 + m\omega \) with \( m \in \mathbb{Z} \). The resonance conditions are thus satisfied respectively for \( 2\epsilon_0 \sim n\omega \) and \( \epsilon_0 \pm J^z \sim n\omega \), and correspond to multiphoton processes where the population probability is modulated by the
zeros of the Bessel functions of order \( n \), \( J_n(A/\omega) = 0. \) Notice that while the first condition gives half integer and integer values of \( \epsilon_0/\omega \), the second one depends on \( J^z/\omega \). For integer values of \( J^z/\omega \) the quasidegeneracies are located at integer values of \( \epsilon_0/\omega \), as can be clearly seen from Fig. 1.

In Fig. 2 we show \( \overline{C} \) as a function of \( \epsilon_0/\omega \). The initial state is chosen as the ground state for the correspondent \( \epsilon_0/\omega \). In the absence of driving, \( A/\omega = 0 \) (black line), the ground state is entangled for detuning energies satisfying \( |\epsilon_0/\omega| \lesssim |J^z/\omega| = 3 \), where the concurrence takes values close to 1. In particular, for \( \epsilon_0 = 0 \), the ground state is the Bell’s state \( (|01\rangle + |10\rangle)/\sqrt{2} \), which is known to be a maximally entangled state. On the other hand, for values of detuning energy \( |\epsilon_0/\omega| > |J^z/\omega| = 3 \) the ground state is almost disentangled, i.e. for large values of \( \epsilon_0 \) the ground state is asymptotically a separable state of the computational basis, corresponding to \( |00\rangle \) for \( \epsilon_0 \gg 0 \) and \( |11\rangle \) for \( \epsilon_0 \ll 0 \), see Fig. 1.

When the driving is turned on \( (A/\omega = 3.8) \), a pattern of resonances is clearly visible in \( \overline{C} \), where entanglement is either created or destroyed. For \( |\epsilon_0/\omega| > |J^z/\omega| = 3 \) the initial condition corresponds to a separable state, we see that it is possible to generate entanglement in a controlled way around a given resonance. Otherwise entanglement is reduced.

The positions of the resonances in \( \overline{C} \) are determined from the already mentioned conditions: \( 2\epsilon_0/\omega \sim n \) and \( \epsilon_0/\omega + J^z/\omega \sim n \), with \( n \in \mathbb{Z} \). Therefore is around a (quasi) degeneracy where the Floquet states can be strongly mixed given rise to significant deviations in the behaviour of the concurrence compared to the undriven case.

Notice that in Fig. 2, the resonances are at integer and half integer values of \( \epsilon_0/\omega \), since \( J^z/\omega = -3 \). Furthermore, when studying \( C \) as a function of \( A \), for a particular multiphoton resonance, we have found that the concurrence is modulated by the driving amplitude, where full (or partial) recovery of the initial entanglement is possible. [35]

Figure 2: Plots of \( \overline{C} \) versus \( \epsilon_0/\omega \) for \( A/\omega = 0 \) (black line) and \( A/\omega = 3.8 \) (red line). The initial condition corresponds to the ground state for the correspondent \( \epsilon_0/\omega \). \( J^z/\omega = -3 \) is the coupling strength and other qubits parameters are the same as in Fig. 1.

So far, we have studied the entanglement for a fixed value of the coupling strength. However in practical implementations with flux qubits the intensity of the inductive coupling can be controlled, and it is interesting to analyse whether a different static coupling would induce qualitative changes in the above description, taking into account that the spectrum of quasienergies is sensitive to this change (see Fig. 1). In the absence of the microwave field two well separated behaviours are observed, corresponding to positive and negative values of \( J^z \) respectively. [35] For \( J^z < 0 \) (antiferromagnetic coupling) the ground state is entangled for \( |\epsilon_0| < |J^z| \) as we already described. On the other hand, for the ferromagnetic coupling \( J^z > 0 \), which increases the energy of states \( |01\rangle, |10\rangle \) and decreases the energy of states \( |00\rangle, |11\rangle \), the ground state is entangled only for values \( \epsilon_0 \sim 0 \), being approximately the Bell’s state.
\((|00\rangle + |11\rangle)/\sqrt{2}\). When the microwave field is on, \(\overline{C}\) exhibits the structure of resonances where the entanglement is created or destroyed in a well controlled way with resonances located at half integer or integer values of \(\epsilon_0/\omega\) and others located at positions determined by the values of \(J^z/\omega\), as we already mentioned. For integer values of \(J^z/\omega\) the resonances are at integers \(\epsilon_0/\omega\), while for arbitrary real values of \(J^z/\omega\) they are respectively shifted to non integer values of \(\epsilon_0/\omega\). These features are clearly observed in Fig.3, where \(\overline{C}\) versus \(\epsilon_0/\omega\) is plotted for \(J^z/\omega > 0\) and \(A/\omega = 3.8\).

Figure 3: Plot of \(\overline{C}\) versus \(\epsilon_0/\omega\) for \(A/\omega = 3.8\) for ferromagnetic coupling strengths \(J^z > 0\). See text for details.

4. Conclusions
In this work we have shown that entanglement can be manipulated by external periodic driving fields. In particular we presented numerical results for the concurrence of a system composed by two coupled flux qubits driven by an external ac magnetic flux.

The main result is that when the system is tuned at or near a multiphoton resonance full control of entanglement is possible: (a) when the initial state is disentangled one can drive it towards a highly entangled state, and (b) when the initial state is entangled one can strongly reduce entanglement with the driving. This will show, for example, as well-defined lines of “entanglement resonance” in the \(\{\epsilon_0, J^z\}\) plane. One advantage of this case is that once the resonance is exactly tuned, the control of entanglement is almost complete.

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References
[1] Orlando T P, Mooij J E, Tian L, van der Wal C H, Levitov L S, Lloyd S and Mazo J J, 1999 Phys. Rev. B 60, 15398.
[2] Friedman J R, Patel V, Chen W, Tolpygo S K, and Lukens J E, 2000 Nature 406, 43.
[3] van der Wal C H, ter Haar A C J, F K Wilhelm, Schouten R N, Harmans C J P M, Orlando T P, Lloyd S and Mooij J E, 2000 Science 290, 773.
[4] Martinis J M, Nam S, Aumentado J, and Urbina C, 2002 Phys. Rev. Lett. 89, 117901.
[5] Oliver W D, Yu Y, Lee J C, Berggren K K, Levitov L S and Orlando T P, 2005 Science 310, 1653.
[6] Majer J B , Pauw F G, ter Haar A C J, Harmans C J P M and Mooij J E, 2005 Phys. Rev. Lett. 94, 090501.
[7] Izmalkov A, Grajcar M, Iliechev E, Wagner T, Meyer H G, Smirnov A Y, Amin M H S , van den Brink A M and Zagoskin A M, 2004 Phys. Rev. Lett. 93, 037003.
[8] Grajcar M, Izmalkov A, van der Ploeg S H W, Linzen S, Iliechev E, Wagner T, H Imer U, Meyer H G, Maassen van den Brink A, Uchaikin S and Zagoskin A M, 2005 Phys. Rev. B 72, 020503.
[9] Yamamoto T, Pashkin Y A, Astafiev O, Nakamura Y, and Tsai J S, 2003 Nature 425, 941.
[10] Strauch F W, Johnson P R, Dragt A J, Lobb C J, Anderson J R and Wellstood F C, 2003 Phys. Rev. Lett. 91, 167005.
[11] Mooij J E, Orlando T P, Levitov L, Tian L, van der Wal C H and Lloyd S, 1999 Science 285, 1036.
[12] van der Ploeg S H W, Izmalkov A, van den Brink A M, Hübner U, Grajcar M, Ilichev E, Meyer H G and Zagoskin A M, 2007 Phys. Rev. Lett. 98, 057004.
[13] Superconducting Quantum Circuits, Qubits and Computing, Wendin G and Shumeiko V, arXiv:cond-mat/0508729v1, 2005.
[14] Plourde B L T, Zhang J, Whaley K B, F K Wilhelm, Robertson T L, Hime T, Linzen S, Reichardt P A, Wu C E, and Clarke J, 2004 Phys. Rev. B 70, 140501.
[15] Harrabi K, Yoshihara F, Niskanen A O, Nakamura Y, and Tsai J S, 2009 Phys. Rev. B 79, 020507.
[16] de Groot P C, Lisenfeld J, Schouten R N, Ashhab S, Lupascu A, Harnans C J P M and Mooij J E, 2010 Nat Phys 6, 763.
[17] Poletto S, Gambetta J M, Merkel S T, Smolin J A, Chow J M, Córcoles A D, Keefe G A, Rothwell M B, Rozen J R, Abraham W D, Valenzuela S O, Shytov A V, Berggren K K, Levitov L S, and Orlando T P, 2006 Phys. Rev. Lett. 97, 150502.
[18] M S Rudner, Shytov A V, Levitov L S, Berns D M, Oliver W D, Valenzuela S O and Orlando T P, 2008 Phys. Rev. Lett. 101, 190502.
[19] Izmalkov A, van der Ploeg S H W, Shevchenko S N, Grajcar M, Iláichev E, Hübner U, Omelyanchouk A N, and Meyer H G, 2008 Phys. Rev. Lett. 101, 017003.
[20] Ferrón A , Domínguez D and Sánchez M J, 2010 Phys. Rev. B 82, 134522.
[21] Shirley J H, 1965 Phys. Rev. 138, B979.
[22] Ferrón A, Domínguez D and Sánchez M J, 2012 Phys. Rev. Lett. 109, 237005.
[23] Ferrón A , Domínguez D and Sánchez M J, 2016 Phys. Rev. B 93, 064521.
[24] Sauer S, Mintert F, Gneiting C and Buchleitner A, 2012 Journal of Physics B: Atomic, Molecular and Optical Physics 45, 154011.
[25] Son S K, Han S and Chu S I, 2009 Phys. Rev. A 79, 032301.
[26] Hausinger J and Grifoni M, 2010 Phys. Rev. A 81, 022117.
[27] Ashhab S, Johansson J R, Zagoskin A M and Nori F, 2007 Phys. Rev. A 75, 063414.
[28] Wootters W K, 1998 Phys. Rev. Lett. 80, 2245.
[29] Chow J M, DiCarlo L, Gambetta J M, Nunnenkamp A, Bishop L S, Frunzio L, Devoret M H, Girvin S M, and Schoelkopf R J, 2010 Phys. Rev. A 81, 062325.
[30] Shulman M D, Dial O E, Harvey S P, Bluhm H, Umansky V and Yacoby A, 2012 Science 336, 202.
[31] Macroscopic quantum resonance of coupled flux qubits: a quantum computation scheme, Akisato H, 2008.
[32] Satanin A M, Denisenko M V, Ashhab S and Nori F, 2012 Phys. Rev. B 85, 184524.
[33] Gramajo A L, Domínguez D and Sánchez M J, to be published.