WAP SYSTEMS AND LABELED SUBSHIFTS

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Abstract. The main object of this work is to present a powerful method of construction of subshifts which we use chiefly to construct WAP systems with various properties. Among many other applications of this so called labeled subshifts, we obtain examples of null as well as non-null WAP subshifts, WAP subshifts of arbitrary countable (Birkhoff) height, and completely scrambled WAP systems of arbitrary countable height. We also construct LE but not HAE subshifts, and recurrent non-tame subshifts.

Contents

Introduction 2
1. WAP Systems 5
2. Coalescence, LE, HAE and CT-WAP systems 15
3. Discrete suspensions and spin constructions 24
4. The space of labels 30
4.1. Finitary and simple labels 47
5. Labeled subshifts 55
5.1. Expanding functions 55
5.2. Labeled integers 58
5.3. Subshifts 63
6. Dynamical properties of $X(M)$ 78
6.1. Translation finite subsets of $\mathbb{Z}$ 78
6.2. Non-null and non-tame labels 81
6.3. Rare systems of expanding functions 87
6.4. Gamow transformations 91
6.5. Ordinal constructions 94
7. Scrambled sets 102
References 107
Index 109

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The notion of weakly almost periodic (WAP) functions on a LCA group $G$ was introduced by Eberlein [10], generalizing Bohr’s notion of almost periodic (AP) functions. As the theory of AP functions was eventually reduced to the study of the largest topological group compactification of $G$, so the theory of WAP functions can be reduced to the study of the largest semi-topological semigroup compactification of $G$. Following Eberlein’s work there evolved a general theory of WAP functions on a general topological group $G$, or even more generally, on various type of semigroups. From the very beginning it was realized that a dual approach, via topological dynamics, is a very fruitful tool as well as an end in itself. Thus in the more recent literature on the subject one is usually concerned with WAP dynamical systems $(X,G)$. These are defined as continuous actions of the group $G$ on a compact Hausdorff space $X$ such that, for every $f \in C(X)$, the weak closure of the orbit $\{f \circ g : g \in G\}$ is weakly compact. The turning point toward this view point is the paper of Ellis and Nerurkar [11], which used the famous double limit criterion of Grothendiek to reformulate the definition of WAP dynamical systems as those $(X, G)$ whose enveloping semigroup $E(X, G)$ consists of continuous maps (and is thus a semi-topological semigroup).

In the last two decades the theory of WAP dynamical systems was put into the broader context of hereditarily almost equicontinuous (HAE) and tame dynamical systems. The starting point for this direction was the proof, in the work [5] of Akin Auslander and Berg, that WAP systems are HAE. For later development along these lines see e.g. [17].

Most of the extensive literature on the subject of WAP functions and WAP dynamical systems has a very abstract flavor. The research in these works is mostly concerned with related questions in harmonic analysis, Banach space theory, and the topology and the algebraic structure of the universal WAP semigroup compactification. Very few papers deal with presentations and constructions of concrete WAP dynamical systems. As a few exceptions let us point out the works of Katzenelson-Weiss [28], Akin-Auslander-Berg [4], Downarowicz [9] and Glasner-Weiss [20, Example 1, page 349]. Even in these few works the attention is usually directed toward examples of recurrent WAP topologically transitive systems. These are the (usually metric) WAP dynamical systems $(X, G)$ which admit a recurrent transitive point.

A point $x$ in a metric dynamical system $(X, G)$ is a point of equicontinuity if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(gx, gx') < \epsilon$ for
every $x' \in B_\delta(x)$ and every $g \in G$. The system is called *almost equicontinuous* (AE) if it has a dense (necessarily $G_\delta$) subset of equicontinuity points. It is *hereditarily almost equicontinuous* (HAE) if every subsystem (i.e. non-empty closed invariant subset) is AE.

As our work deals almost exclusively with cascades (i.e. $\mathbb{Z}$-dynamical systems), in the sequel we will consider dynamical systems of the form $(X,T)$ where $T : X \to X$ is a surjective map, usually a homeomorphism. A large and important class of cascades is the class of symbolic systems or subshifts. We will deal only with subsystems of the *Bernoulli* dynamical system $(\{0,1\}^\mathbb{Z}, S)$, where $S$ is the shift transformation defined by

\[(0.1) \quad (Sx)_n = x_{n+1} \quad (x \in \{0,1\}^\mathbb{Z}, \ n \in \mathbb{Z}).\]

We will call such dynamical systems *subshifts*. It was first observed in [17] that a subshift is HAE iff it is countable (see Proposition 2.2 below). In particular it follows that WAP subshifts are countable. Since a dynamical system which admits a recurrent non-periodic point is necessarily uncountable, it follows that in a WAP subshift the only recurrent points are the periodic points. These considerations immediately raise the question which countable subshifts are WAP, and how rich is this class? This question was the starting point of our investigation.

As we proceeded with our study of that problem we were able to construct several simple examples of both WAP and non WAP topologically transitive countable subshifts, but particular constructions of WAP subshifts turned out to be quite complicated. After many trials we finally discovered the beautiful world of *labels*, which together with a system of *expanding functions* provide a powerful tool for constructing subshifts with various prescribed properties.

As the reader will see, once a system of expansion is fixed, there is a canonical map from the space of labels $\mathcal{L}A\mathcal{B}$ into $\{0,1\}^\mathbb{Z}, \ M \mapsto x[M]$, which assigns to a label $M$ a subshift $X(M)$, the orbit closure of $x[M]$ under $S$. The Polish space $\mathcal{L}A\mathcal{B}$ is naturally equipped with an action of a discrete semigroup $FIN(\mathbb{N})$, and we denote the compact orbit closure of a label $M$ under this action by $\Theta(M)$. The latter always contains the empty label $\emptyset$, for which $x[\emptyset] = 0$ (which we always denote by $e$). Thus $e \in X(M)$ for every label $M$.

The key lemma which connects the two actions (the $FIN(\mathbb{N})$ action on labels and the shift $S$ on subshifts) is lemma 5.14 which ensures that for certain sequences $\{t_i\}$ of expanding times with length $\mathbb{N}$-vectors $r(t_i) \in FIN(\mathbb{N})$, and a sequence of labels $\{M^i\}$, the sequences $\{S^{t_i}(x[M^i])\}$ and $\{x[M^i - r(t_i)]\}$ are asymptotic in $\{0,1\}^\mathbb{Z}$.
We show that for a $FIN(\mathbb{N})$-recurrent label the corresponding $x[\mathcal{M}]$ is an $S$-recurrent point. At the other extreme we have the labels of finite type. For such a label $\mathcal{M}$, $e$ is the only recurrent point in $X(\mathcal{M})$. These labels are particularly amenable to our analysis, which leads to a complete picture of the resulting subshift. In fact for a label $\mathcal{M}$ of finite type

$$X(\mathcal{M}) = \{S^k x[\mathcal{N}] : k \in \mathbb{Z}, \mathcal{N} \in \Theta(\mathcal{M})\} = \bigcup_{k \in \mathbb{Z}} S^k x[\Theta(X(\mathcal{M}))].$$

Two useful subcollections of the collection of finite type labels are the classes of the finitary labels and of the simple labels. For each label $\mathcal{M}$ in either one of these special classes, the corresponding subshift $X(\mathcal{M})$ is a countable WAP system whose enveloping semigroup structure is encoded in the structure of the label $\mathcal{M}$. This fact enables us to produce WAP subshifts with various dynamical properties by tinkering their labels.

The recurrent labels are far less transparent and for these labels the image $x[\Theta(X(\mathcal{M}))]$, which in this case is a Cantor set, forms only a meagre subset of the subshift $X(\mathcal{M})$. Nonetheless it seems that this image forms a kind of nucleolus which encapsulates the dynamical properties of $X(\mathcal{M})$.

The table of contents will now give the reader a rough notion of the structure of our work. In the first section we deal with abstract WAP systems, their enveloping semigroups, and, for an arbitrary separable metric system the hierarchies $z_{NW}$ and $z_{LIM}$ of non-wandering and $\alpha \cup \omega$ limiting procedures, which lead by transfinite induction to the Birkhoff center of the dynamical system. We call the ordinal at which the limiting $\alpha \cup \omega$ transfinite sequence stabilizes, the height of the system. We also consider various simple examples of some WAP and non-WAP systems. In the second section we study HAE systems and show, among other considerations, that topologically transitive WAP systems are coalescent and that a general WAP system is $E$-coalescent. The third section describes some general constructions like the discrete suspension, and the spin construction.

The space of labels is introduced and studied in section 4. The expanding function systems and the associated subshifts are introduced and studied in section 5. Finally, in sections 6 and 7 these tools are applied to obtain many interesting and subtle constructions of subshifts. Let us mention just a few. On the finite type side we obtain examples of null as well as non-null WAP subshifts, Example 6.21 (answering a question of Downarowicz); WAP subshifts of arbitrary (countable) height, Theorem 6.39; topologically transitive subshifts which are LE
but not HAE, Example 5.24 and Remark 6.22 (these seem to be the first such examples); and completely scrambled WAP systems (although not subshifts) of arbitrary countable height, Example 7.13 (answering a question which is left open in Huang and Ye’s work [27]). On the recurrent side we construct various examples of non-tame subshifts. Of course many questions are left open, especially when labels which are not of finite type are considered, and we present some of these throughout the work at the relevant places. We note however that many of the questions regarding recurrent labels can be answered when further restrictions are posed on the basic system of expanding functions. E.g. with this special kind of systems we show, in Theorem 6.30, that the collection of all expanding times is a set of upper Banach density zero which, in turn, implies that for every label $M$ the corresponding subshift $(X(M), S)$ is uniquely ergodic and has zero topological entropy.

1. WAP Systems

A compact dynamical system $(X, T)$ is a homeomorphism $T$ on a compact space $X$. We follow some of the notation of [1] concerning relations on a space. In particular, we will use the orbit relation

$$O_T = \{ (x, T^n(x)) : x \in X, n \in \mathbb{Z} \}$$

and the associated limit relation: $R_T = \omega_T \cup \alpha_T$, where

$$\omega_T = \{ (x, x') : x \in X, x' = \lim_{i \to \infty} T^{n_i}x \text{ with } n_i \nearrow \infty \},$$

and

$$\alpha_T = \{ (x, x') : x \in X, x' = \lim_{i \to \infty} T^{-n_i}x \text{ with } n_i \nearrow \infty \}.$$

For $A \subset X$ define

$$R_T^*(A) = \{ x \in X : R_T(x) \subset A \}.$$  

$R_T$ is a pointwise closed relation (each $R_T(x)$ is closed) but not usually a closed relation (i.e. $R_T$ is usually not closed in $X \times X$).

The dynamical system $(X, T)$ is called topologically transitive if for every two non-empty open sets $U, V$ in $X$ there is an $n \in \mathbb{Z}$ with $T^{-n}U \cap V \neq \emptyset$. When $X$ is metrizable this is equivalent to the requirement that $X_{tr}$, the set of points with dense orbit, is a dense $G_\delta$ subset of $X$. The points of $X_{tr}$ are the transitive points of $X$. The system $(X, T)$ is called weakly mixing when the product system $(X \times X, T \times T)$ is topologically transitive. We also recall the definitions of $\epsilon$-chains and
chain transitivity. Given $\epsilon > 0$ an $\epsilon$-chain from $x$ to $y$ is a finite sequence $x = x_0, x_1, \ldots, x_n = y$ such that $n > 0$ and $d(T(x_i), x_{i+1}) < \epsilon$ for $i = 0, \ldots, n-1$. The system $(X, T)$ is chain transitive if for any nonempty open sets $U, V$ and any $\epsilon > 0$ there is an $\epsilon$-chain going from a point in $U$ to a point in $V$. An asymptotic chain is an infinite sequence $x^i = x_0, x_1, \ldots$ such that $\lim_{i \to \infty} d(T(x_i), x_{i+1}) = 0$. It is a dense asymptotic chain if for every $N \in \mathbb{N} \{x^i : i \geq N\}$ is dense in $X$. If $(X, T)$ is chain transitive and $x \in X$ then there exists a dense asymptotic chain $\{x^i\}$ with $x = x_0$.

More generally given a monoid ( = a semigroup with an identity element) $\Gamma$, a $\Gamma$-dynamical system is a pair $(X, \Gamma)$ where $X$ is a compact Hausdorff space and $\Gamma$ acts on $X$ via a homomorphism of $\Gamma$ into the semigroup $C(X, X)$ of continuous maps from $X$ to itself, mapping $id_\Gamma$ to $id_X$.

The enveloping semigroup $E = E(X, \Gamma)$ of the dynamical system $(X, \Gamma)$ is defined as the closure in $X^X$ (with its compact, usually non-metrizable, pointwise convergence topology) of the image of $\Gamma$ in $C(X, X)$ considered as a subset of $X^X$.

It follows directly from the definitions that, under composition of maps, $E$ forms a compact semigroup in which the operations

$$p \mapsto pq \quad \text{and} \quad p \mapsto \gamma p$$

for $p, q \in E$, $\gamma \in \Gamma$, are continuous. Notice that this makes $\Gamma$ act on $E$ by left multiplication, so that $(E, \Gamma)$ is a $\Gamma$-system (though usually non-metrizable). It is easy to see that the subset $A(X, \Gamma) \subset E(X, \Gamma)$, consisting of the non-isolated points in $E(X, \Gamma)$, forms a closed left ideal, called the adherence semigroup of $(X, \Gamma)$.

The elements of $E$ may behave very badly as maps of $X$ into itself; usually they are not even Borel measurable. However our main interest in $E$ lies in its algebraic structure and its dynamical significance. A key lemma in the study of this algebraic structure is the following:

**Lemma 1.1** (Ellis-Numakura). Let $L$ be a compact Hausdorff semigroup in which all maps $p \mapsto pq$ are continuous. Then $L$ contains an idempotent; i.e., an element $v$ with $v^2 = v$.

Given two $\Gamma$ dynamical systems, say $(X, \Gamma)$ and $(Y, \Gamma)$, a continuous surjective map $\pi : X \to Y$ is a homomorphism or an action map if it intertwines the $\Gamma$ actions, i.e. $\gamma \pi(x) = \pi(\gamma x)$ for every $x \in X$ and $\gamma \in \Gamma$. An action map $\pi : X \to Y$ induces a surjective semigroup homomorphism (and an action map) $\pi_* : E(X, \Gamma) \to E(Y, \Gamma)$.

For more details see e.g. [13, Chapter 1, Section 4] and [6].
In our cascade case, from a dynamical system \((X, T)\), we let \((E(X, T), A(X, T))\) denote the enveloping semigroup and the ideal which is the adherence semigroup (= the limit points of \(\{T^n\}\) as \(n \to \pm \infty\)). Let \(T_\ast \) on \(E(X, T)\) be the homeomorphism given by \(T_\ast(p) = Tp = pT\). Thus, \(y \in R_T(x)\) iff \(y = px\) for some \(p \in A(X, T)\) and \(A(X, T) = R_T(id_X)\).

A point \(x \in X\) is recurrent when \(x \in R_T(x)\) and so when there exists an idempotent \(u \in A(X, T)\) such that \(ux = x\). On the other hand, \(x\) is non-recurrent iff the orbit \(O_T(x)\) is disjoint from the limit set \(R_T(x)\).

The closure of the set of recurrent points is the Birkhoff Center. We will denote by \(\text{Cent}_T\) the Birkhoff center for \((X, T)\).

A point \(x^*\) is a transitive point when the orbit \(O_T(x^*)\) is dense in \(X\). In that case, the evaluation map \(ev_{x^*}\) defined by \(p \to px^*\) is a surjective action map from \((E(X, T), T_\ast)\) to \((X, T)\).

**Lemma 1.2.** If \(p \in E(X, T)\) is continuous at \(x \in X\), then for any \(q \in E(X, T)\) and \(y \in X\) if \(qy = x\) then \(pqy = qpy\).

**Proof:** If \(T^{n_i} \to q\) pointwise then \(T^{n_i}y \to qy = x\) and so by continuity of \(p\) at \(x\), \(pT^{n_i}y \to pqy\). But \(pT^{n_i}y = T^{n_i}py \to qpy\).

\(\Box\)

**Proposition 1.3.** If \((X, T)\) is a compact system with a transitive point \(x^*\) and \(p \in E(X, T)\) then the following are equivalent.

(i) \(p\) is continuous on \(X\).

(ii) For all \(q \in E(X, T)\) \(pq = qp\) on \(X\).

(iii) For all \(q \in E(X, T)\) \(pqx^* = qpx^*\).

**Proof:** (i) \(\Rightarrow\) (ii): This follows from Lemma 1.2.

(ii) \(\Rightarrow\) (iii): Obvious.

(iii) \(\Rightarrow\) (i): Suppose \(x_i \to x\). To show that then \(px_i \to px\), it suffices to show that every convergent subnet has limit \(px\). So we can assume that \(\lim px_i\) exists. There are \(r_i \in E(X, T)\) with \(x_i = r_ix^*\). Let \(r_i \to r\) be a convergent subnet. Then, necessarily \(rx^* = x\) and \(\lim px_i = \lim px_i = \lim pr_ix^* = \lim r_ix^* = rpx^* = prx^* = px\).

\(\Box\)

\((X, T)\) is called WAP when the elements of \(E(X, T)\) are all continuous functions on \(X\).

**Corollary 1.4.** If \((X, T)\) is a compact system with a transitive point \(x^*\) then the following are equivalent.

(i) \((X, T)\) is WAP

(ii) \(E(X, T)\) is abelian.
(iii) For every \( p, q \in E(X, T) \) \( pqx^* = qp^x \).

When these conditions hold \( ev_{x^*} : (E(X, T), T_\ast) \rightarrow (X, T) \) is an isomorphism and there is a unique minimal subset of \( X \).

Proof: The equivalence of (i), (ii) and (iii) follows from Proposition 1.3.

If \((X, T)\) is WAP and \( px^* = qx^* \) then \( p = q \) on \( O(x^*) \) which is dense and so \( p = q \) by continuity. That is, \( ev_{x^*} \) is injective and so is an isomorphism.

If \( x_i \in M_i \) for minimal sets \( M_1, M_2 \) there exist \( p_1, p_2 \in E(X, T) \) s.t. \( p_i(x^*) = x_i \) and so \( p_2(x_1) = p_2(p_1(x^*)) \in M_1 \) while \( p_1(x_2) = p_1(p_2(x^*)) \in M_2 \). Since \( E(X, T) \) is abelian, \( M_1 \cap M_2 \neq \emptyset \) and so \( M_1 = M_2 \).

\( \Box \)

Proposition 1.5. If \( ev_{x^*} : E(X, T) \rightarrow X \) is an homeomorphism (e.g. if \((X, T)\) is WAP with transitive point \( x^* \)) and \( \{ T^m(x^*) \} \) is a net converging to a point \( x \in X \) then \( \{ T^m(z) \} \) is a net converging in \( X \) for every \( z \in X \). In fact, \( \{ T^m \} \) converges pointwise to the unique \( p \in E(X, T) \) such that \( p(x^*) = x \).

In general, if \( p \in E(X, T) \) and \( \{ q_i \} \) is a net in \( E(X, T) \) such that \( \{ q_i(x^*) \} \rightarrow p(x^*) \) in \( X \) then \( \{ q_i \} \rightarrow p \) in \( E(X, T) \).

Proof: Obvious by inverting the homeomorphism \( ev_{x^*} \).

\( \Box \)

Example 1.6. The surjection \( ev_{x^*} \) can be a homeomorphism in non-WAP cases.

Let \( X_0 \) be a compact, connected metric space and \( T_0 = id_{X_0} \), the identity map. So \( E(X_0, T_0) = \{ id_{X_0} \} \). Let \( \{ x_i : i \in \mathbb{Z} \} \) be a sequence of distinct points in \( X_0 \) such that \( \lim_{|i| \rightarrow \infty} d(x_i, x_{i+1}) = 0 \) and so that the positive and negative tails are dense in \( X_0 \). Identify \( X_0 \) with \( X_0 \times \{ 0 \} \subset X \times [0, 1] \) and let \( \hat{x}_i = (x_i, (|i| + 1)^{-1}) \). Let \( x^* = \hat{x}_0 = (x_0, 1) \).

Let \( X = X_0 \cup \{ \hat{x}_i : i \in \mathbb{Z} \} \). Define \( T \) by \( \hat{x}_i \mapsto \hat{x}_{i+1} \). Now assume that \( y \in X_0 \) and \( x_{n_k} \) converges to \( y \) (and so with \( |n_k| \rightarrow \infty \)). Then \( T^{n_k}(\hat{x}_N) = T^{n_k+N}(x^*) \) converges to \( T^N(y) = y \). Furthermore, for any \( z \in X_0 \), \( T^{n_k}(z) = z \) converges to \( z \). Thus, \( T^{n_k} \) converges pointwise to the function which is the identity on \( X_0 \) and which is constantly \( y \) on the orbit of \( x^* \). In particular, \( ev_{x^*} : (E(X, T), T_\ast) \rightarrow (X, T) \) is an isomorphism. On the other hand, \( E(X, T) \) is not abelian and none of the elements of \( E(X, T) \) are continuous except for the iterates \( T^n \).
Example 1.7. Abelian enveloping semigroup does not imply WAP in general. (a) Let the circle be $\mathbb{R}/\mathbb{Z}$. Let $X = \mathbb{R}/\mathbb{Z} \times \mathbb{Z}^*$ where $\mathbb{Z}^*$ is the one-point compactification of $\mathbb{Z}$. Define $T$ to be the identity on $\mathbb{R}/\mathbb{Z} \times \{\infty\}$ and by $(t, n) \mapsto (t + 3^{-(|n|+1)}, n)$. On each circle the map is just a rotation and so is WAP. Hence the enveloping semigroup is abelian. Consider the sequence $\{T^{\sum_{i=0}^{k} i}\}$. On $\mathbb{R}/\mathbb{Z} \times \{\infty\}$ this is eventually constant at the rotation $t \mapsto t + \sum_{i=0}^{n} 3^{-(|n|-i+1)}$. As $|n| \to \infty$ this approaches the rotation $t \mapsto t + \frac{1}{2}$. But on the circle $\mathbb{R}/\mathbb{Z} \times \{\infty\}$ the identity is the only element of the enveloping semigroup.

(b) A countable example is the spin $(\mathbb{Z}, T)$ of the identity on $\mathbb{Z}^*$. $Z$ is a subset of $\mathbb{Z}^* \times \mathbb{Z}^*$

$$Z = (\mathbb{Z}^* \times \{\infty\}) \cup \bigcup_{n \in \mathbb{Z}} \{[-|n|, +|n|] \times \{n\}\},$$

(1.2)

$$f(x) = \begin{cases} 
(\infty, \infty) & \text{for } x = (\infty, \infty) \\
(t + 1, \infty) & \text{for } x = (t, \infty) \\
(t + 1, 0) & \text{for } x = (t, n) \text{ with } t < |n|, \\
(-|n|, n) & \text{for } x = (|n|, n). 
\end{cases}$$

The following describes the equivalences to what might be called local WAP.

Proposition 1.8. For a system $(X, T)$ the following are equivalent.

(i) Multiplication for the enveloping semigroup is continuous in each variable.

(ii) Every element of the enveloping semigroup has a continuous restriction on the orbit closure of each element of $X$.

(iii) The enveloping semigroup is abelian.

(iv) Each orbit closure in $X$ is a WAP system.

Proof: Since each orbit closure is invariant for the enveloping semigroup and since the topology of the latter is pointwise convergence, each of these conditions holds $(X, T)$ iff it holds for the restriction to each orbit closure. This restricts to the topologically transitive case for which (iii) implies (ii) by Proposition 1.3. Because of pointwise convergence, (ii) implies (i) is obvious. If $p, q$ are in the enveloping semigroup and $T^{a_j}$ is a net converging to $q$ then $pT^{a_j} = T^{a_j}q$ and two-sided continuity at $p$ imply $pq = qp$. 

□
Let \((X,T)\) be an arbitrary separable metric system. Define \(z_{CAN}(X)\) to be the complement of the set of isolated points in \(X\). Let \(z_{NW}(X)\) be the complement of the union of all wandering open sets. Note that if a point is isolated and non-periodic then it is wandering. Thus, if there are no isolated periodic points, then \(z_{NW}(X) \subset z_{CAN}(X)\).

Let \(z_{LIM}(X) = R_T(X)\). If \(U\) meets \(R_T(X)\) then there exist \(x,y \in X\) such that \(y \in U \cap R_T(x)\) and so for infinitely many \(n \in \mathbb{Z}\), \(T^n(x) \in U\). It follows that \(z_{LIM}(X) \subset z_{NW}(X)\).

For each of these operators we define the descending transfinite sequence of closed sets by

\[
(1.3) \quad z_0(X) = X, \quad z_{\alpha+1}(X) = z(z_{\alpha}(X)), \quad z_\beta = \bigcap_{\alpha < \beta} z_{\alpha}(X),
\]

for \(\beta\) a limit ordinal. We say that the sequence stabilizes at \(\beta\) when \(z_\beta(X) = z_{\beta+1}(X)\) in which case it is constant from then on. The first \(\beta\) at which stabilization occurs for the \(CAN/NW/LIM\) sequence is called the \(CAN/NW/LIM\) level. Since \(X\) is a separable metric space, each level is a countable ordinal (because \(\{X \setminus z_{\alpha} : \alpha \leq \beta\}\) is an increasing open cover of the Lindelöf space \(X \setminus z_\beta\)).

We let \(z_{\infty}(X) = z_\beta(X)\) when the sequence stabilizes at \(\beta\). Clearly \(z_{LIM,\alpha}(X) \subset z_{NW,\alpha}(X)\) for all \(\alpha\). Recall that when \((X,T)\) is non-wandering, i.e. \(z_{NW}(X) = X\) then the recurrent points are dense. Since all recurrent points are contained in \(z_{LIM,\infty}(X)\) it follows that \(z_{LIM,\infty}(X) = z_{NW,\infty}(X)\) is the closure of the set of recurrent points, i.e. the Birkhoff Center.

If \(X\) is Polish then a nonempty \(G_\delta\) subset without isolated points contains a Cantor Set. Hence, if \(X\) is Polish, \(z_{CAN,\infty} = \{x : \text{every neighborhood of } x \text{ is uncountable}\}\). In particular, if \(X\) is Polish and countable then \(z_{CAN,\infty} = \emptyset\) and the isolated points are dense in \(X\). Since the intersection of the decreasing family of nonempty closed sets has a nonempty intersection, \(z_{CAN}(X)\) is not a limit ordinal if \(X\) is compact and countable.

**Definition 1.9.** We will call the ordinal \(\beta_{LIM}(X)\) at which the \(z_{LIM}\) sequence stabilizes, the **height** of \((X,T)\).

Call \((X,T)\) **semi-trivial** (hereafter ST) if \(R_T = X \times \{e\}\) for a point, a fixed point, \(e \in X\). That is, for every \(x \in X\), \(R_T(x) = \{e\}\). Call \((X,T)\) **center periodic** (hereafter CP) if the only recurrent points are periodic. Call \((X,T)\) **center trivial** (hereafter CT) if there is a unique recurrent point \(e\), necessarily a fixed point, and so the Birkhoff center
is \{e\}. Clearly, ST implies CT and CT implies CP. A nontrivial system is ST if it is a CT system of height 1. For a CT system we will denote by \(u\) the retraction to the fixed point \(e\).

In a CP system every point is isolated in its orbit closure. If \((X, T)\) is CP and non-wandering then the recurrent points and so the periodic points form a dense \(G_\delta\) and so for some finite positive \(n\) \(\{x : T^n(x) = x\}\) has nonempty interior. In fact the union of such interiors is dense in \(X\). If there are only countably many periodic points then this open dense set is countable and Polish and so the isolated points are dense in \(X\). If \(X\) is non-wandering then every isolated point must be periodic.

The identity on any compact space defines a CP system and the finite product of CP’s is CP (not the infinite product since the product of periodic orbits can contain an adding machine). Any subsystem and factor of CP is CP (since any recurrent point in the factor lifts to some recurrent point in the top). Inverse limit does not work. Again, an adding machine is the inverse limit of periodic orbits.

Remark 1.10. A nontrivial CP system \((X, T)\) can never be weak mixing, i.e. \((X \times X, T \times T)\) is never topologically transitive. If \(x^*\) is a transitive point for a CP system \((X, T)\) then it is isolated in \(X\). If \(x^*\) is an isolated, transitive point for a nontrivial system \((X, T)\) then \(T(x^*) \neq x^*\) and so \(U = \{x^*\}\) and \(V = \{T(x^*)\}\) are nonempty open subsets of \(X\), but \(N(U \times U, U \times V) = \emptyset\).

The CT condition is closed under arbitrary products and subsystems. In particular, the enveloping semigroup of a CT system is CT. The retraction \(u\) to the fixed point \(e \in X\) is the unique fixed point in \(E(X, T)\). Also, it is the unique idempotent in \(A(X, T)\). If \(\pi : (X, T) \rightarrow (Y, S)\) is an action map and \(X\) is CT then \(Y\) is. In general, \((X, T)\) is CT iff \((Y, S)\) is CT and \(\pi^{-1}(e)\) is a CT subsystem of \(X\). If \(Y\) is a metrizable CT then since it is chain transitive we can attach a single orbit of isolated points and obtain a metrizable CT which is topologically transitive (see, e. g. [1, Chapter 4, Exercise 29]. For an illustration, see Example 1.6 above). Mapping \((X, T)\) to the factor system on \(X/Cent_T\) defines a functor from compact systems to CT systems. An action map \(X \rightarrow Y\) with \(Y\) CT factors through the projection from \(X\) to \(X/Cent_T\) and so the functor is adjoint to the inclusion functor.

If \((X, T)\) is a countable CT system, then \(e\) is not isolated in any invariant closed subset, of \(X\) except \(\{e\}\) itself. Thus, if the Cantor sequence stabilizes at \(\beta + 1\) then \(z_{CAN, \beta} = \{e\}\) and conversely. In that
case, for any $\alpha \leq \beta$ $z_{\text{LIM},\alpha} \subset z_{\text{NW},\alpha} \subset z_{\text{CAN},\alpha}$. That is, up to $\alpha = \beta$ the isolated points are all non-wandering.

A CT system $(X, T)$ has height 0 iff $X = \{e\}$, i.e. the system is trivial, and has height 1 iff it is ST.

**Proposition 1.11.** (a) If $(X, T)$ is an ST system then it is WAP.
(b) If $(X, T)$ is a CT system with height at most 2 then $E(X, T)$ is abelian. If, in addition, $(X, T)$ is topologically transitive then it is WAP.

**Proof:** (a) $E(X, T) = \{T^n : n \in \mathbb{Z}\} \cup \{u\}$ if $(X, T)$ is ST.
(b) If $p, q \in A(X, T)$ then $pq = u = qp$ where $u$ is the retraction onto $e$. Hence, the semigroup is abelian. So if $(X, T)$ is topologically transitive, then it is WAP by Proposition 1.3.

Example 1.12. In his work [33] Shapovalov shows that within the class of countable subshifts one can find, for any countable ordinal $\alpha$, a subshift $X_\alpha \subset \{0, 1\}^\mathbb{Z}$ whose Birkhoff degree, i.e. its NW level, is $\alpha + 1$. Now it is easy to verify that all of these subshifts $X_\alpha$ constructed by Shapovalov are in fact CT and semi-trivial and therefore also WAP. One can make them topologically transitive by attaching a single orbit. Thus we conclude that the class of WAP, topologically transitive subshifts is rich enough to present every countable Birkhoff degree. Note however that being semi-trivial Shapovalov’s original examples all are of height 2 and they become of height 3 when an orbit is attached to make them topologically transitive. As we will show later (Theorem 6.39) the class of WAP, topologically transitive subshifts is also rich enough to present every countable height.

Let $S$ denote the shift homeomorphism on $\{0, 1\}^\mathbb{Z}$.

Example 1.13. Various non-WAP examples.
(a) Let $e = 0, x(0) = 0^\infty 10^\infty$. For $k = 1, 2, \ldots$ let $b_j^k = 1$ for $j = 10^{nk}, n \in \mathbb{Z}$ and $= 0$ otherwise. Let $(X, S)$ be the generated subshift. Let $X(0)$ be the ST subshift generated by $x(0)$, a copy of $(\mathbb{Z}^*, t)$. $R_S(X) = X(0)$ and so $(X, S)$ has height 2. $b^k \to x(0)$ as $k \to \infty$. The sequence $S^{10^d} \to p$ in $A(X, T)$ with $p(b^k) = x(0)$ for all $k$ and $p(x(0)) = e$. So $p$ is not continuous at $x(0)$, despite the fact that all of the points of $X \setminus X(0)$ are isolated. That is, the assumption of topological transitivity in Proposition 1.11 (b) is necessary.

(b) A topologically transitive system of height 1 with minimal set not a fixed point need not be WAP. Let $c$ be given by $c_i = 1$ for
\[ i = 2n, -1 - 2n \text{ for } n \in \mathbb{N} \text{ and } = 0 \text{ otherwise, i.e. } c = (01)^{\infty}(10)^{\infty}. \]

The orbit closure of \( c \) consists of \( O(c) \) together with the periodic orbit \( \{10, 01\} \). \( s^{-2k}(c) \rightarrow 01, S^{2k}(c) \rightarrow 10 \). \( S^{-2k} \rightarrow p \) and \( s^{2k} \rightarrow q \) both \( p, q \) are identity on the periodic orbit. Hence, \( pq \neq qp \) on \( c \).

(c) For a countable topologically transitive, height 3 CT subshift which is not WAP Let \( d_i = 1 \), for \( i = 0, \pm 2^k, (k = 0, 1, \ldots) \) and = 0 otherwise, \( X_2 \) be the orbit closure of \( d \) which is the orbit of \( d \) together with \( X_1 \).

Let \( \text{Block}_k \) denote the \( k^{th} \) block, of \( d \) defined to be the word of length \( 2^{k+1} + 1 \) which agrees with \( d_{[-2k, 2k]} \). Three successive 1’s uniquely determine where in the block a subblock is. It follows, that you can build \( x^* \) on the positive side as follows

\[(1.4) \quad \text{Block}_1 \quad N_1 0's \quad \text{Block}_2 \quad N_2 0's \quad \text{Block}_3 \quad \ldots \]

with \( \text{Block}_1 = 111 \) centered at position 0. Then reflect to define \( x^* \). Provided the sequence \( N_k \) increases much faster than \( 2^{k+1} + 1 \) then it can be chosen arbitrarily and the orbit closure \( X \) of \( x^* \) will consist of the dense orbit of \( x^* \) together with the limit point set which is equal to \( X_2 \). It clearly contains \( X_2 \) and I believe the three 1’s argument yields equality. Let \( m_k \) be the location of the center of \( \text{Block}_k \) in \( \hat{x} \). Since the \( N_k \)’s are pretty much arbitrary we can arrange that \( m_k \) is a power of 2 for \( k \) even and 1 plus a power of 2 for \( k \) odd. Then \( s^{m_k}(x^*) \) converges to \( d \), but \( s^{m_{2k}}(d) \) converges to \( a \) and \( f^{m_{2k+1}}(d) \) converges to \( T(a) \). Thus, \( s^{m_k}(x^*) \) is not a convergent sequence. From Proposition 1.5 it follows that \( (X, s) \) is not WAP.

(d) For \((Y, S)\) any compact metric system, let \((X, T)\) be the one point compactification of \((Y \times Z, S \times t)\) with \( t \) the translation on \( Z \). This is an ST system. If \( S = \text{id}_Y \) it is easy to build a countable sequence of periodic orbits with limit set \((X, T)\). The expanded system is CP with an uncountable center although there are only countably many periodic orbits.

(e) Let \( e = 0, X(0) = \overline{O(0^\infty 10^\infty)} \). Call \( x \) selective if for any \( n \) the word \( 10^n 1 \) occurs at most once in \( x \). Let \( X \) be the set of all selective \( x \). Clearly, if \( x \) is selective then \( R_S(x) \subset X(0) \). Note that if \( A \subset Z \) is such that all the nonzero differences \( a_i - a_j \) are distinct then extending any element of \( \{0, 1\}^A \) by \( x_i = 0 \) for \( i \not\in A \), we obtain a selective element. \((X, S)\) is an uncountable CT subshift with height 2.

(f) Let \((Y, S)\) be any CP subshift. Let \( \{w_i\} \) count the finite words in \( Y \). Then \( Y \cup \bigcup \{w_i\} \) is a CP subshift with dense periodic points.
Any CT metric WAP \((X, T)\) is chain transitive and so there exists \((X, T)\) topologically transitive consisting of \(X \subset X^*\) and the dense orbit of isolated points \(O(x^*)\). That is, \((X, T)\) topologically transitive with an isolated transitive point \(x^*\) and is such that \((X_1, T_1)\) is the subsystem with \(X_1 = R_T(x^*)\).

**Example 1.14.** It may happen that we cannot choose the extension so that \((X, T)\) is WAP.

Let \((X, T)\) be a CT WAP which is not semi-trivial, but with a fixed point \(e\). That is, there exists \(p \in A(X, T)\) with \(p(e) \neq u\) and so \(p(X) \setminus \{e\}\) is nonempty. Let \(X = X_1 \vee X_2\), two copies of \(X\) with the fixed points identified. For any map \(g\) on \(X\) which fixes \(e\), let \(\bar{g}\) on \(\bar{X}\) be copies of \(g\) on each term. The system \((\bar{X}, \bar{T})\) is clearly WAP and \(p \mapsto \bar{p}\) is an isomorphism from \(E(X, T)\) onto \(E(\bar{X}, \bar{T})\). Notice the \(\bar{p}(\bar{X}) \setminus X_i\) is nonempty for \(i = 1, 2\).

Now let \((\hat{X}, \hat{T})\) contain \((\bar{X}, \bar{T})\) and with \(\hat{X} \setminus \bar{X}\) consisting of a single dense orbit. Thus, if \(x^* \in \hat{X} \setminus \bar{X}\) then the orbit of \(x^*\) is dense. Let \(q\) be an element of the enveloping semigroup of \(E(\hat{X}, \hat{T})\) with \(q(x^*) \in X_1\). Then \(q\) maps the whole orbit of \(x^*\) into \(X_1\) and if \(q\) is continuous then \(q(\hat{X}) \subset X_1\). Thus, every continuous element of the enveloping semigroup \(E(\hat{X}, \hat{T})\) maps all of \(\hat{X}\) either into \(X_1\) or into \(X_2\). Every element of the enveloping semigroup of \((\bar{X}, \bar{T})\) extends to some element of the enveloping semigroup of \((\hat{X}, \hat{T}^*)\). Thus, if \(\hat{p}\) extends \(\bar{p}\) it cannot be continuous and so \((\hat{X}, \hat{T})\) is not WAP.

We will say that \(A(X, T)\) distinguishes points when \(p(x_1) = p(x_2)\) for all \(p \in A(X, T)\) implies \(x_1 = x_2\). It suffices that some \(p \in A(X, T)\) be injective. If \(X\) has any non-trivial, but semi-trivial subspace then \(A(X, T)\) does not distinguish points.

Let \(T_*\) be composition with \(T\) on \(E(X, T)\). Clearly \(id_X\) is a transitive point for \(T_*\). If \((X, T)\) is not weakly rigid, i.e. \(id_X\) is not a recurrent point for \(T_*\) then \(id_X\) is an isolated transitive point for \(T_*\) and \(A(X, T) = R_{T_*}(id_X)\) is a proper subset of \(E(X, T)\).

Assume that \(x^*\) is a transitive point for \((X, T)\). Then \(ev_{x^*} : (E(X, T), T_*) \rightarrow (X, T)\) is a factor map sending \(A(X, T)\) to \(X_1 = R_T(x^*)\). If \((X, T)\) is WAP then the map is an isomorphism by Proposition 1.3.

Now assume that the subspace \((X_1, T_1)\) is WAP. As is true for any subsystem the restriction map \(\rho : A(X, f) \rightarrow A(X_1, f)\) is surjective.
**Proposition 1.15.** Assume that \((X, T)\) is topologically transitive with an isolated transitive point \(x^*\) such that the subsystem \((X_1, T_1)\) with \(X_1 = R_T(x^*)\) is WAP. The map \(\rho\) is injective, and so is an isomorphism, iff \((X, T)\) is WAP and, in addition, \(A(X_1, T_1)\) distinguishes points of \(X_1\).

**Proof:** Since \(X_1\) is WAP, \(A(X_1, T_1)\) is abelian. If \(\rho\) is injective then \(A(X, T)\) is abelian and so \((X, T)\) is WAP by Proposition 1.3.

Now assume \((X, T)\) is WAP. We show that \(\rho\) is injective iff \(A(X_1, T_1)\) distinguishes the points of \(X_1\).

Let \(p_1, p_2 \in A(X, T)\). Since \(A(X, T)\) is abelian, \(p_1(q(x^*)) = p_2(q(x^*))\) for all \(q \in A(X, T)\) iff \(q(p_1(x^*)) = q(p_2(x^*))\) for all \(q \in A(X, T)\). The first says \(\rho(p_1) = \rho(p_2)\) and the second says \(p_1(x^*), p_2(x^*) \in X_1\) are not distinguished by \(A(X_1, T_1)\). \(\rho\) is injective says that the first implies \(p_1 = p_2\) while \(A(X_1, T)\) distinguishes points says that the second implies \(p_1(x^*) = p_2(x^*)\) and so by continuity \(p_1 = p_2\). This proves the equivalence.

\(\square\)

**Corollary 1.16.** Assume \((X_1, T_1)\) is WAP which is not weakly rigid and is such that \(A(X_1, T_1)\) distinguishes points. If there exists \((X, T)\) topologically transitive with an isolated transitive point \(x^*\) such that \((X_1, T_1)\) is the subsystem with \(X_1 = R_T(x^*)\) then \((X, X_1, T)\) is isomorphic to \((E(X_1, T_1), A(X_1, T_1), (T_1)_s)\).

**Proof:** Since \(X_1\) is not weakly rigid, \(\rho\) is an isomorphism from \((E(X, T), A(X, T), T_s)\) onto \((E(X_1, T_1), A(X_1, T_1), (T_1)_s)\). On the other hand, \(ev_{x^*}\) is an isomorphism from \((E(X, T), A(X, T), T_s)\) onto \((X, X_1, T)\).

\(\square\)

More generally, if we define \(E = \{(x_1, x_2) \in X_1 \times X_1 : q(x_1) = q(x_2)\}\) for all \(q \in A(X_1, T)\) then \(E\) is an ICER and the factor \((X/E, X_1/E, T)\) is isomorphic to \((E(X_1, T_1), A(X_1, T_1), (T_1)_s)\), because \(\rho(p_1) = \rho(p_2)\) iff \((p_1(x^*), p_2(x^*)) \in E\).

2. Coalescence, LE, HAE and CT-WAP systems

Given a metric dynamical system \((X, T)\), a point \(x \in X\) is an *equicontinuity point* if for every \(\epsilon > 0\) there is \(\delta > 0\) such that \(d(x', x) < \delta\) implies \(d(T^n x', T^n x) < \epsilon\) for every \(n \in \mathbb{Z}\). The system \((X, T)\) is
called *equicontinuous* if every point in $X$ is an equicontinuity point (and then it is already *uniformly equicontinuous* meaning that the $\delta$ in the above definition does not depend on $x$). It is called *almost equicontinuous*, hereafter AE, when there is a dense set of points in $X$ at which $\{T^n : n \in \mathbb{Z}\}$ is equicontinuous. Following [16] we will call $(X,T)$ *hereditarily almost equicontinuous*, hereafter HAE, when every subsystem (i.e. closed invariant subset) is again an AE system. As was shown in [5] every WAP is HAE (see also [13, Chapter 1, Sections 8 and 9]). An isolated point is an equicontinuity point and so if the isolated points are dense then the system is AE. Any countable Polish space has isolated points and so applying this to any nonempty open subset we see that the isolated points are dense. Hence, if $(X,T)$ is countable then every subsystem is AE i.e. it is HAE.

A compact, metric system $(X,T)$ is *expansive* if there exists $\epsilon > 0$ such that for every $x_1 \neq x_2$ there exists $n \in \mathbb{Z}$ such that $d(T^n(x_1), T^n(x_2)) > \epsilon$. Any subshift is expansive. The following is obvious.

**Lemma 2.1.** If $(X,T)$ is expansive then $x \in X$ is an equicontinuity point iff it is isolated.

□

From this follows the result from [17] that a subshift is HAE iff it is countable.

**Proposition 2.2.** An expansive, compact, metric dynamical system $(X,T)$ is HAE iff $X$ is countable. In particular, a subshift is HAE iff it is countable.

**Proof:** It was observed above that a countable system is HAE.

Now assume that $X$ is uncountable and so contains a Cantor set $C$. If $X_1$ is the closure of $\bigcup_n \{T^n(C)\}$, then the subsystem $(X_1,T)$ is expansive and contains no isolated points. So by Lemma 2.1 it has no equicontinuity points. Thus, $(X,T)$ is not HAE.

□

Thus:

**Proposition 2.3.** A WAP subshift is countable.

Following [20] we call $(X,T)$ *locally equicontinuous* (hereafter LE) if each point $x$ is an equicontinuity point in its orbit closure or, equivalently, if each orbit closure is an almost equicontinuous subsystem.
The equivalence follows from the Auslander-Yorke Dichotomy Theorem, [8], which says that in a topologically transitive system the set of equicontinuity points either coincides with the set of transitive points or else it is empty.

**Remark 2.4.** From the latter condition, it follows that an HAE system is LE. Any CP system is LE since each point is isolated in its orbit closure. From Proposition 2.2 it follows that any uncountable CP subshift is LE but not HAE.

A system $(X, T)$ is coalescent when any surjective action map $\pi$ on $(X, T)$ is an isomorphism.

**Proposition 2.5.** A topologically transitive system which is WAP is coalescent.

**Proof:** There exists $p$ in the enveloping semigroup with $p(x^*) = \pi(x^*)$. Because $p$ is continuous it is an action map and so since $p$ and $\pi$ agree on the dense orbit of $x^*$, $p = \pi$. Since $p$ is surjective, $p(x^*)$ is a transitive point and so there exists $q$ such that $qp(x^*) = x^*$ and so $qp = id$. Hence, $p$ is injective with inverse $q$.

**Example 2.6.** In general a WAP system need not be coalescent.

If $(X, T)$ is WAP then the countable product $(X^\mathbb{N}, T^\mathbb{N})$ is WAP and the shift map is a surjective action map which is not injective. If $X$ is CT with fixed point $e$ then the infinite wedge which is $\{x \in X^\mathbb{N} : x_i \neq e \text{ for at most one } i\}$ is a closed invariant set which is shift invariant as well. This is also WAP and not coalescent. In addition, it is countable if $X$ is.

**Lemma 2.7.** If a dynamical system $(X, T)$ contains an increasing net of topologically transitive subsystems $\{X^i\}$ with $\bigcup_i \{X^i\}$ dense in $X$, then $X$ is also topologically transitive.

**Proof:** Let $U, V \subset X$ be two nonempty open subsets. For some $i$

$$U \cap X^i \neq \emptyset \quad \text{and} \quad V \cap X^i \neq \emptyset.$$ 

As $X^i$ is topologically transitive, there exists $k \in \mathbb{Z}$ with $T^k(U \cap X^i) \cap (V \cap X^i) \neq \emptyset$ and, a fortiori, also $T^k(U) \cap V \neq \emptyset$. 
Proposition 2.8. Every dynamical system is a union of maximal topologically transitive subsystems.

Proof: Let \((X, T)\) be a dynamical system and consider the family \(\mathcal{T}\) of topologically transitive subsystems of \(X\). Using Lemma 2.7 it is easy to check that this family is inductive. Hence, by Zorns lemma, every topologically transitive subsystem of \(X\) is contained in a maximal element of \(\mathcal{T}\). In particular, for \(x \in X\), the orbit closure of \(x\) is contained in a maximal element of \(\mathcal{T}\).

We then obtain the following results on \(E\)-Coalescence (i.e. the property that every continuous surjective element of \(E(X, T)\) is injective).

Recall that a dynamical system \((X, T)\) is called (i) weakly rigid if there is a net \(\{T_{n_i}\}\) with \(|n_i| \to \infty\) and \(\{T_{n_i}(x)\} \to x\) for every \(x \in X\), (ii) rigid if the net can be chosen to be a sequence, and (iii) uniformly rigid if the convergence can be taken to be uniform (see [15]). Recall that if \(X\) is a metrizable, topologically transitive AE system, and a fortiori a metrizable, topologically transitive WAP, then it is uniformly rigid (see [17], [18] and [4]).

Theorem 2.9. Let \((X, T)\) be an AE compact, metrizable system. Assume that \(p \in A(X, T)\) is continuous and surjective.

(i) If \((X, T)\) is topologically transitive then \(p\) is injective and so is an isomorphism. If \(T_{n_j}\) is a net converging pointwise to \(p\) then it converges uniformly to \(p\) and \(T^{-n_j}\) converges uniformly to \(p^{-1}\). Thus, \(p^{-1}, \text{id}_X \in E(X, T) = A(X, T)\).

(ii) If \((X, T)\) is HAE then \(p\) is an isomorphism and if \(T_{n_j}\) is a net converging pointwise to \(p\) then \(T^{-n_j}\) converges pointwise to \(p^{-1}\). Thus, \(p^{-1}, \text{id}_X \in E(X, T) = A(X, T)\) and the system is weakly rigid.

Proof: (i) Let \(x^*\) be a transitive point for \(X\). Since \(p\) is surjective and continuous, \(px^*\) is a transitive point and so there exists a sequence \(T_{n_i}\) with \(T_{n_i}px^*\) converging to \(x^*\). Let \(\epsilon > 0\). Since \(x^*\) is an equicontinuity point eventually \(T_{n_i}pT^kx^* = T^kT_{n_i}px^*\) is within \(\epsilon\) of \(T^kx^*\) for all \(k\). Since the orbit of \(x^*\) is dense it follows that \(T_{n_i}p\) converges uniformly to \(\text{id}_X\). Hence, \(p\) is injective and so is an isomorphism by compactness. If \(q\) is any limit point of \(T_{n_i}\) in \(E(X, T)\) then \(pq\) which is the limit of \(pT_{n_i} = T_{n_i}p\) is the identity and so \(q = p^{-1}\). Thus, \(p^{-1} \in E(X, T)\). Hence, \(\text{id}_X = p^{-1}p \in A(X, T)\) and so \((X, T)\)
is weakly rigid and \(E(X,T) = A(X,T)\). Since \(pT^m_i = T^m_ip\) converges uniformly to \(pp^{-1}\), uniform continuity of \(p^{-1}\) implies \(T^m_i\) converges uniformly to \(p^{-1}\). Hence, \(\lim_{n,j \to \infty} T^{m_i-n_j} = id_X\) and so the system is uniformly rigid. Finally, if a net \(T^m_i\) converges to \(p\) pointwise then \(p^{-1}T^m_ip^*\) is eventually close to \(p^*\) and so as above \(p^{-1}T^m_i\) converges to \(id_X\) uniformly and so \(T^m_i\) converges to \(p\) uniformly and \(T^{-m_i}\) converges uniformly to \(p^{-1}\).

(ii) Each point is contained in a maximal topologically transitive subset of \(X\), which is necessarily closed.

Now if for some points \(x_1, x_2 \in X\) we have \(z = px_1 = px_2\), then let \(X_i\) be a maximal topologically transitive subset of \(X\) which contains \(x_i\). Since they are closed, \(z \in X_1 \cap X_2\). By (i) there exist \(q_i \in A(X,T)\) such that on \(X_i\) \(q_ip\) is the identity. In particular, \(q_i z = x_i\). Hence, \(q_i p x_1 = x_2\) and so \(x_2\) as well as \(x_1\) is in \(X_1\). Since \(q_i p\) is the identity on \(X_1\) it follows that \(x_1 = x_2\). Thus, \(p\) is an isomorphism. Now let \(T^m_i\) be a net converging to \(p\) pointwise. By part (i) \(T^{-m_i}\) converges to \(p^{-1}\) uniformly on each orbit closure and so pointwise on \(X\). Hence, \(p^{-1}\) and \(id_X = p^{-1}p\) are in \(A(X,T)\). Hence, \(E(X,T) = A(X,T)\) and \((X,T)\) is weakly rigid.

\[\square\]

**Corollary 2.10.** Every WAP dynamical system is \(E\)-coalescent.

**Proof:** If \((X,T)\) is a WAP dynamical system then (i) it is is HAE, and (ii) every \(p \in E(X,T)\) is continuous. Now apply Theorem 2.9.

\[\square\]

**Corollary 2.11.** Let \((X,T)\) be a WAP system and \(p \in A(X,T)\).

1. The restriction of \(p\) to the subsystem \(Z = \bigcap_{n \in \mathbb{N}} p^nX\) is an automorphism of \(Z\). In particular the system \((Z,T)\) is weakly rigid.
2. If \((X,T)\) is WAP and CP, then every point of \(Z\) is \(T\)-periodic (i.e. has a finite \(T\)-orbit). If moreover \((X,T)\) is topologically transitive then \(Z\) consists of a single periodic \(T\)-orbit.

**Proof:** Assume first that \(X\) is metrizable. Hence, \((X,T)\) is HAE because it is WAP.

Clearly \(Z\) is a (nonempty) subsystem and \(pZ = Z\). Thus, by Theorem 2.9 \(p|Z\) is an automorphism of \(Z\). Since \(p \in A(X,T)\) it follows that \(Z\) is weakly rigid. The last assertion is now easily deduced.

Since the group \(Z\) is countable, \((X,T)\) is an inverse limit of a net \(\{(X^i,T^i)\}\) metrizable systems which are WAP since the latter property is preserved by factors. The set \(Z\) projects onto the corresponding set
Thus, the restriction of $p$ to $Z$ is the inverse limit of isomorphisms and so is an isomorphism.

Questions 2.12.

(1) Is there a metric WAP system with every point non-wandering (and so it is its own Birkhoff center) but which is not rigid?

(2) If a homeomorphism for $X$ is in $A(X, T)$ then is its inverse also in $A(X, T)$ (and so it is weakly rigid)? For WAP or even for HAE the answer is yes, as above.

The closure of orbits of idempotents, in various universal semigroups, is the center in the semigroup.

Theorem 2.13. Let $A$ be an Ellis semigroup (sensu Akin, Auslander and Glasner [6]).

(a) Assume there is a unique idempotent $u$ such that $pu = u$ for all $p \in A$. Then $up = u$ for all $p \in A$. In particular, if $up = p$ (i.e. $p$ is recurrent) then $p = u$.

(b) Assume there is a unique idempotent $u$ such that $pu = u$ for all minimal $p \in A$. Then $u$ is minimal and is the unique minimal element of $A$.

Proof: (a) Let $q = up$. Then $qq = qup = up = q$ and so $q$ is an idempotent. As $u$ is the only idempotent, $q = u$.

(b) Let $q = up$ with $p$ minimal and so $q$ is minimal. Then $qq = qup = up = q$ and so $q$ is a minimal idempotent. If $r$ is any minimal element then $rq = rup = up = q$. As $u$ is the only such idempotent, $q = u$. Now $up = u$ implies that the ideal $Au \subset \{p\} \cup Ap$ with equality because $p$ is minimal. So there exists $r \in A$ such that $p = ru$ and so $p = ru = ruu = pu = u$.

Remark 2.14. We are using as our definition that $p$ be a minimal element that $\{p\} \cup Ap$ be a minimal ideal, or equivalently, that $p \in Ap$ and the latter is a minimal ideal. Merely using $Ap$ being a minimal ideal does not work. Let $e \in A$ with $A \setminus \{e\} \neq \emptyset$ and let $pq = e$ for all $p, q \in A$. Then $Ap = \{e\}$ for all $p$ but $e$ is the only minimal element.

Lemma 2.15. Assume $(X, T)$ is a CP system. If there is an infinite sequence $\{x_i : i \in \mathbb{N}\}$ in $X$ such that $x_i \in R_T(x_{i+1})$ for all $i$, then $x_1$ is a periodic point and all $x_i$'s are in the orbit of $x_1$. 
Proof: First, assume that $X$ is metrizable.

If $x_i$ is periodic then all $x_j$'s with $j < i$ are in the orbit of $x_i$. Hence, if infinitely many of the $x_i$'s are periodic then they all are and all lie in the same periodic orbit. If $x_{i+1} \in R_T(x_{i+1})$ then it is positively or negatively recurrent. Since $X$ is CP the only recurrent points are periodic. Hence, if $x_{i+1}$ is not periodic then $x_{i+1}$ is not in the orbit closure of $x_i$.

We show that the alternative to finitely many periodic points cannot happen. As usual we can use inverse limits to reduce to the metric case. If instead there are only finitely many periodic points in the sequence then by omitting finitely many initial terms and re-numbering we can assume that none are periodic. Let $A_i$ be the orbit closure of $x_i$. Since $x_i \in A_{i+1}$ we have $A_i \subset A_{i+1}$. Since $x_{i+1} \in A_{i+1} \setminus A_i$, each inclusion is strict. Each $A_i$ is topologically transitive. Therefore the closure of the union $A = \bigcup A_i$ is topologically transitive. Let $z$ be a transitive point for $A$. It is isolated in $A = \bigcup A_i$ and so must lie in some $A_j$, but since the $\{A_i\}$ sequence is a strictly increasing sequence of closed invariant sets, it cannot be in any of them.

For the general case, we can assume that some $x_i$ is not periodic. There is a metrizable factor for which the image of $x_i$ is not periodic and this contradicts the metric result above. 

Proposition 2.16. Assume that $(X,T)$ is a CP-WAP. If $\{p_i : i \in \mathbb{N}\}$ is a sequence in $A(X,T)$ then $\bigcap_{n=1}^{\infty} p_np_{n-1} \cdots p_1 X$ is a closed subset consisting of periodic points.

Proof: For $i = 1,2,\ldots$ and $n \geq i$ let $X_{i,n} = p_ip_{i+1} \cdots p_n X$. $X_i = \bigcap_{n=i}^{\infty} X_{i,n}$. Since each $p_i$ is a continuous, each $X_{i,n}$ is a closed, invariant subspace and the sequence is decreasing in $n$ and $p_i(X_{i+1,n}) = X_{i,n}$ when $n \geq i + 1$. Hence, continuity and compactness imply that $p_i(X_{i+1}) = X_i$. Since the semigroup is abelian, $X_{i,n} = p_np_{n-1} \cdots p_1 X$. Let $x_1 \in X_1$. By induction we can build a sequence $x_i \in X_i$ such that $p_i(x_{i+1}) = x_i$ and so $x_i \in R_T(x_{i+1})$. From Lemma 2.15 it follows that $x_1$ is periodic.

Corollary 2.17. Assume that $(X,T)$ is a CP-WAP. If $\{x_i : i \in \mathbb{N}\}$ is a sequence in $X$ such that $x_{i+1} \in R_T(x_i)$ for all $i$, then $\bigcap O(x_i)$ (the orbit closures of the $x_i$'s) is a closed subset consisting of periodic points.
Proof: The sequence $\overline{O(x)}$ is decreasing and so we can restrict to the case $X = \overline{O(x)}$. As this is a transitive subspace and the system is WAP the elements of the semigroup are continuous on it. By assumption, there exists $p_i \in A(X, T)$ such that $p_i(x_i) = x_{i+1}$. In the notation of the proof of Proposition 2.16, $x_{n+1} \in X_{1,n}$ for all $n \geq 1$. Hence, $\bigcup_n \overline{O(x_n)} \subset \bigcup_n X_{1,n}$ and the latter consists of periodic points by Proposition 2.16.

\[ \square \]

Remark 2.18. Let $X$ be a countable CT-WAP subshift with $e = \{0\}$ as the unique minimal subset of $X$. If and $p_1, p_2, \ldots$ an infinite sequence of elements of $A(X, T)$ then there exists an $n$ with $p_1p_2 \cdots p_n(X) = \{0\}$. In fact, if $X = X_0 \supset X_1 \supset X_2 \supset \cdots$ is any descending sequence of closed invariant sets with $\bigcap_{n=0}^{\infty} X_n = \{0\}$, then there is a finite $n \in \mathbb{N}$ with $X_n = \{0\}$. To see this observe that from our assumptions it follows that for large enough $n$, we have $X_n \subset U_0 = \{x \in X : x_0 = 0\}$. Since $X_n$ is invariant, it follows that $X_n = \{0\}$. Now set $X_n = p_1p_2 \cdots p_n(X)$ and apply Proposition 2.16.

Now we restrict to the case where $(X, T)$ is a metrizable CT-WAP with a single minimal set consisting of a fixed point $e$.

For $\epsilon > 0$ let $A_{\epsilon}$ denote the maximal invariant subset of $X$ which is contained in the closed neighborhood $\overline{V_{\epsilon}(e)}$, i.e. $A_{\epsilon} = \bigcap_{i \in \mathbb{Z}} T^i(\overline{V_{\epsilon}(e)})$. [If $V_{\epsilon}(e)$ is clopen, $A_{\epsilon}$ is an isolated invariant set.] In the notation of the proof of Proposition 2.16 each $X_i = \{e\}$ and so for each $i$, there exists $n \geq i$ such that $X_{i,n} \subset V_{\epsilon}(e)$ and so $X_{i,n} \subset A_{\epsilon}$. Since $A_{\epsilon}$ is closed and invariant, $X_{j,m} \subset A_{\epsilon}$ for all $j \leq i, m \geq n$.

Proposition 2.19. Let $A$ be any invariant set which contains $A_{\epsilon}$. If $x \notin A$ then there exists $q \in E(X, T)$ such that $qx \notin A \cup V_{\epsilon}(e)$ but for all $p \in A(X, T), pqx \in A$, i.e. $R_T(qx) = q(R_T(x)) \subset A$.

Proof: Clearly,

\[
A \subset \bigcap_{\ell \in \mathbb{Z}} T^\ell(A \cup V_{\epsilon}(e)) \subset A \cup A_{\epsilon} = A
\]

So for some $i \in \mathbb{Z}$, $T^i(x) \notin A \cup V_{\epsilon}(e)$. If $R_T(T^i(x)) \subset A \cup V_{\epsilon}(e)$ then let $q = T^i$. Otherwise, there exists $q_1$ such that $q_1(T^i(x)) \notin A \cup V_{\epsilon}(e)$. Continue inductively defining $q_n$ such that $q_n \cdots q_1(T^i(x)) \notin A \cup V_{\epsilon}(e)$ whenever $R_f(q_n \cdots q_1(T^i(x)))$ is not contained in $A \cup V_{\epsilon}(e)$. This process must terminate because for any infinite sequence $\{q_1, \ldots, q_n, \ldots\}$
in \( A(X, T) \) the sequence \( \{q_1(T^i(x)), \ldots, q_n(T^i(x)), \ldots \} \) converges to \( e \) and so eventually is contained in \( V_\epsilon(e) \).

If \( q_n \cdots q_1(T^i(x)) \not\in A \cup V_\epsilon(e) \) but \( R_T(q_n \cdots q_1(T^i(x)) \subset A \cup V_\epsilon(e) \), then let \( q = q_n \cdots q_1T^i \). Since \( R_T(qx) \subset A \cup V_\epsilon(e) \) and \( R_T(qx) \) is an invariant set, equation 2.1 implies \( R_T(qx) \subset A \).

\( \Box \)

**Remark:** Notice that \( A \) need not be closed.

Recall that for the relation \( R_T \) on \( X \) and \( A \subset X \) we let \( R_T^n(A) = \{x : R_T(x) \subset A\} \). Let \( R_T^n(A) = R_T^*(R_T^{(n-1)}(A)) = \{x : R_T^n(x) \subset A\} \).

Observe that \( R_T^*(A) = \bigcap \{p^{-1}(A) : p \in A(X, f)\} \). Hence, in the case with all members of \( A(X, f) \) continuous, \( R_T^*(A) \) is closed when \( A \) is. Proposition 2.19 says exactly

**Proposition 2.20.** Let \( A \) be any invariant set which contains \( A_\epsilon \). If \( x \not\in A \) then there exists \( q \in E(X, T) \) such that \( qx \in R_T^*(A) \setminus (A \cup V_\epsilon(e)) \).

\( \Box \)

If \( A \) is not closed then \( R_T^*(A) \) need not be a closed invariant set, but it is always invariant and satisfies the following weakening of closure.

**Lemma 2.21.** Let \( A \subset X \).

\[ (2.2) \quad x \in R_T^*(A) \Rightarrow \overline{O(x)} \subset R_T^*(A) \]

**Proof:** Since \( R_T(T^i(x)) = R_T(x) \) for all \( i \in \mathbb{Z} \) it follows that \( R_T^*(A) \) is invariant. Since \( R_T \circ R_T \subset R_T \) it follows that \( x \in R_T^*(A) \) implies \( R_T(x) \subset R_T^*(A) \).

\( \Box \)

Inductively define \( A_0 = A_\epsilon \) and for an ordinal \( \alpha \), \( A_{\alpha+1} = R_T^*(A_\alpha) \) and \( A_\alpha = \bigcup_{\beta<\alpha} A_\beta \) for \( \alpha \) a limit ordinal. At every stage, \( x \in A_\alpha \Rightarrow \overline{O(x)} \subset A_\alpha \) although it is not clear that \( A_\alpha \) is closed when \( \alpha \) is infinite.

**Proposition 2.22.** For any ordinal \( \alpha \), if \( x \not\in A_\alpha \) then there exists \( q \in E(X, f) \) such that

\[ (2.3) \quad qx \not\in A_\alpha \cup V_\epsilon(e) \quad \text{and} \quad R_T(qx) \subset A_\alpha. \]

For any such \( q \), \( R_T(qx) \) meets \( A_{\beta+1} \setminus (A_\beta \cup V_\epsilon(e)) \) for every \( \beta < \alpha \).

**Proof:** We repeatedly apply Proposition 2.19. First we obtain \( q \in E(X, T) \) which satisfies (2.3).
For any $\beta < \alpha$ there exists $q_1 \in E(X,T)$ such that $q_1qx \in A_{\beta+1} \setminus (A_\beta \cup V(e))$. Since $\beta + 1 \leq \alpha$, $qx \not\in A_{\beta+1}$ and so $T^i(qx) \not\in A_{\beta+1}$ for any $i \in \mathbb{Z}$. Hence, $q_1 \in A(X,T)$ and so $q_1(qx) \in R_T(qx)$.

**Corollary 2.23.** If for any ordinal $\alpha$, $A_\alpha$ is a proper subset of $X$ then $A_{\alpha+1} \setminus A_\alpha$ is nonempty. So the transfinite sequence $\{A_\alpha\}$ is strictly increasing until the first ordinal $\alpha^*$ such that $A_{\alpha^*} = X$. If $(X,T)$ is topologically transitive with transitive point $x^*$ then $\alpha^*$ is the first ordinal such that $x^* \in A_{\alpha^*}$ and $\alpha^*$ is a non-limit ordinal.

**Proof:** That the sequence is strictly increasing until $A_\alpha = X$ is clear from Proposition 2.22. It is also clear that if $x^*$ is a transitive point then $x^* \in A_\alpha$ implies $X = \overline{O(x^*)} \subset A_\alpha$ by (2.2) and so $A_\alpha = X$ iff $x^* \in A_\alpha$. If $x^* \in \bigcup_{\beta < \alpha} A_\beta$ then $x^* \in A_\beta$ for some $\beta < \alpha$ and so $\alpha^*$ cannot then be a limit ordinal.

For the case when $(X,T)$ is a metrizable CT-WAP with the fixed point $e$ isolated (as a closed invariant subset), e.g. a CT-WAP subshift, we choose $\epsilon > 0$ so that $A_\epsilon = A_0 = \{e\}$. We then call the ordinal at which the $A_\alpha$ sequence stabilizes the *height* of $(X,T)$. Because $X$ is then countable it follows that the ordinal $\alpha^*$ is countable.

### 3. Discrete suspensions and spin constructions

For any $(X,T)$ and positive integer $N$ we define on $X \times [0,N-1]$ the homeomorphism $\tilde{T}$ by

\[
\tilde{T}(x,i) = \begin{cases} 
(x, i+1) & \text{for } i < N-1, \\
(T(x), 0) & \text{for } i = N-1. 
\end{cases}
\]

so that $\tilde{T}^N = T \times 1_{[0,N-1]}$. $(X \times [0,N-1], \tilde{T})$ is called the discrete suspension of height $N$. It is countable if $X$ is, it is CP if $(X,T)$ is CT. It is WAP if $(X,T)$ is. Apply the following

**Lemma 3.1.** A point $x \in X$ is an equicontinuity point for $T$ on $X$ iff it is an equicontinuity point for $T^N$ on $X$. Hence, $(X,T)$ is AE or HAE iff $(X,T^N)$ is. In general, $(X,T)$ is WAP iff $(X,T^N)$ is.
Proof: Since \( \{T_i : i \in \mathbb{Z} \} \subset \{T^i : i \in \mathbb{Z} \} \), an equicontinuity point for \( T \) is one for \( T^N \), \( E(X,T^N) \subset E(X,T) \) and so each of the conditions for \( T \) implies the corresponding condition for \( T^N \). In fact,

\[
\{T_i : i \in \mathbb{Z} \} = \{T_i \circ T^k : i \in \mathbb{Z}, k = 0, \ldots, N - 1 \} = \bigcup_{k=0}^{N-1} T^k \circ \{T_i : i \in \mathbb{Z} \}.
\]

(3.2)

It follows that if \( x \) is an equicontinuity point for \( T^N \) then it is for \( T \). Also, we obtain \( E(X,T) = \bigcup_{k=0}^{N-1} T^k \circ E(X,T^N) \) and so the elements of \( E(X,T) \) are continuous when those of \( E(X,T^N) \) are.

\( \Box \)

Theorem 3.2. If \((X,T)\) is a CP WAP system with a unique minimal set, a periodic orbit of period \( N \), then \((X,T)\) is isomorphic to the discrete suspension of height \( N \) of a CT WAP.

Proof: Let \( \{x_0, \ldots, x_{N-1}\} \) be the periodic orbit in \( X \). By Lemma spinlem1 \((X,T^N)\) is a WAP with \( N \) minimal fixed points \( x_0, \ldots, x_{N-1} \). For any \( x \in X \) the restriction of \( T^N \) to the \( T^N \) orbit closure of \( x \) is a point transitive WAP and so with a unique minimal set, necessarily one of the \( x_i \). Let \( X_i = \{x \in X : x_i \in \mathcal{O}_{T^N}(x) \} \). Clearly, \( T(X_i) = X_{i+1} \) (addition mod \( N \)) and the \( X_i \)'s are pairwise disjoint. Each is \( T^N \) invariant. Let \( u \) be a minimal element of \( E(X,T^N) \). If \( x \in X_i \) then \( u(x) \) is a minimal element of the \( T^N \) orbit closure of \( x \) and so \( u(x) = x_i \). That is, \( u \) retracts \( X_i \) to \( x_i \). Because \((X,T^N)\) is WAP, \( u \) is continuous and so each \( X_i = u^{-1}(x_i) \) is closed.

Define \( H : X_0 \times \{0, \ldots, N-1\} \to X \) by \( H(x,i) = T^i(x) \). This is continuous and surjective with inverse, \( x \mapsto (T^{-i}x,i) \) for \( x \in X_i \) and so \( H \) is bijective. Furthermore, \( H(x,i+1) = T(H(x,i)) \) for \( i < N - 1 \) and \( H(T^N(x),0) = T^N(x) = T(T^{N-1}(x)) = T(H(x,N-1)) \). Thus, \( H \) is an isomorphism from the discrete suspension of height \( N \) of \((X_0,T^N)\) onto \((X,T)\).

\( \Box \)

Call \((X,T)\) a minimal trivial (hereafter minCT) when there is a fixed point which is the unique minimal subset, i.e. the mincenter is a single point. A CT system is a minCT system.

Lemma 3.3. Let \((X,T)\) be a nontrivial, metric minCT system with fixed point \( e \) and let \( \epsilon > 0 \). There exists an \( \epsilon \)-dense sequence of distinct points \( \{e = x_0, \ldots, x_{N-1}\} \) in \( X \) such that with \( x_N = e \), \( \{x_0, \ldots, x_N\} \) is an \( \epsilon \) chain for \((X,T)\), i.e. \( d(T(x_i),x_{i+1}) < \epsilon \) for \( i = 0, \ldots, N - 1 \).
Proof: Since \( X \) is separable we can choose a finite or infinite sequence \( \{a_1, a_2, \ldots \} \) of points of \( X \setminus \{e\} \) with pairwise distinct orbits and such that the union of the orbits is dense in \( X \setminus \{e\} \). Since this set is nonempty the sequence contains at least one point. Since \( e \) is the only minimal point, \( e \in \alpha_T(x) \cap \omega_T(x) \) for every \( x \in X \). Now truncate so that the union of the orbits of the finite sequence \( \{a_1, \ldots, a_K\} \) is \( \epsilon/2 \) dense in \( X \setminus \{e\} \). For each \( a_i \) we can choose a finite piece of the orbit \( \{y_0, i, \ldots, y_{K+1}, i\} \) which begins and ends \( \epsilon/2 \) close to \( e \) and which is \( \epsilon/2 \) dense in the orbit of \( a_i \). We concatenate to obtain the sequence \( \{x_1, \ldots, x_{N-1}\} \). Then let \( x_0 = e \).

On the one-point compactification \( Z^* \) with \( e \) the point at infinity and \( T(t) = t + 1 \), define the ultra-metric \( d \) by

\[
    (3.3) \quad d(i, j) = \begin{cases} 
    0 & \text{if } i = j, \\
    \max(1/(|i| + 1), 1/(|j| + 1)) & \text{if } i \neq j,
    \end{cases}
\]

where \( 1/(|i| + 1) = 0 \) if \( i = e \). If \( (X, T) = (Z^*, T) \) then with \( K > 1/\epsilon \) and \( N = 2K + 2 \) we can use the sequence \( \{e, -K, -K + 1, \ldots, K\} \).

When \( (X, T) \) is a nontrivial metric minCT system we define a preparation for \( (X, T) \) to be a choice for each \( i = 0, 1, \ldots \) of a sequence \( \{e = x_0^i, \ldots, x_{N_i}^i\} \) which is an \( 2^{-i} \) dense sequence of distinct elements of \( X \) so that \( \{e = x_0^i, \ldots, x_{N_i}^i, e\} \) is an \( 2^{-i} \) chain. For \( i = 0 \) we let \( N_0 = 1 \) so that with \( i = 0 \) the sequence is \( \{e\} \).

An ultrametric minCT system is a minCT system \( (X, T) \) with \( d \leq 1 \) an ultra-metric on \( X \) (and so \( X \) is zero-dimensional).

Let \( (X_1, T_1), (X_2, T_2) \) be nontrivial ultrametric minCT systems with fixed points \( e_1, e_2 \). Assume that \( (X_2, T_2) \) is given a preparation.

Let \( 1 \leq \epsilon > 0 \). We will define the \( \epsilon \) spin of \( (X_2, T_2) \) into \( (X_1, T_1) \) to be the ultrametric system \( (X, T) \) where \( X \) is the closed subset of \( X_1 \times X_2 \) described below. On \( X_1 \times X_2 \) we will use the ultrametric \( \max(\pi_1^*d_1, \epsilon\pi_2^*d_2) \) so that \( \pi_1 \) has Lipschitz constant 1 and for any \( \delta > 0 \)

\[
    (3.4) \quad V_\delta(e_1, e_2) \subset \pi_1^{-1}(V_\delta(e_1))
\]

with equality if \( \delta \geq \epsilon \).

In \( X_1 \) we define the sequence of pairwise disjoint clopen sets: \( A_0 = X_1 \setminus V_\epsilon(e_1) \) and for \( i = 1, 2, \ldots, A_i = V_{\epsilon/2^{i+1}}(e_1) \setminus V_{\epsilon/2^i}(e_1) \). So \( X_1 = \)
\{e_1\} \cup \bigcup_{i=0}^{\infty} A_i.

(3.5)

\begin{align*}
X &= (\{e_1\} \times X_2) \cup \left( \bigcup_{i=0}^{\infty} A_i \times \{x_i^0, \ldots, x_i^{N_i-1}\}\right), \\
T(x, y) &= \begin{cases} (e_1, T_2(y)) & \text{when } x = e_1, \\
(x, x_k^{i+1}) & \text{when } x \in A_i, \ y = x_k^i \text{ with } k < N_i - 1, \\
(T_1(x), e_2) & \text{when } x \in A_i, \ y = x_N^{i-1}. 
\end{cases}
\end{align*}

It is obvious that \(X\) is a closed subset of \(X_1 \times X_2\) and easy to check that \(T\) is invertible with \(T^{-1}(x, e) = (T_1^{-1}(x), x_N^{i-1})\) when \(T^{-1}(x) \in A_i\).

Continuity of \(T\) is clear on each \(A_i \times \{x_i^0, \ldots, x_i^{N_i-1}\}\). Continuity at the points of \(\{e_1\} \times X_2\) follows from the following estimate.

Lemma 3.4. (a) For \(i \geq 1\), let \(\delta\) with \(2^{-i} > \delta > 0\) be a \(2^{-i}\) modulus of uniform continuity for \(T_2\) on \(X_2\). For \((x, y) \in X, \tilde{y} \in X_2\)

(3.6)

\[ x \in V_{2^{-i+1}}(e_1) \cap T_i^{-1}(V_{2^{-i+1}}(e_1)), y \in V_\delta(\tilde{y}) \implies T(x, y) \in V_{2^{-i+1}}(e_1, T_2(\tilde{y})). \]

(b) Let \(\delta \geq \epsilon\). If \((x, y) \in X\) with \(x \in V_\delta(e_1) \cap T_i^{-1}(V_\delta(e_1))\) then \((x, y) \in V_\delta(e_1, e_2) \cap T_i^{-1}(V_\delta(e_1, e_2))\).

Proof: (a) If \(x \in V_{2^{-i+1}}(e_1)\) then the fiber in \(X\) over \(x\) is a \(2^{-i}\) chain for \(T_2\) and so if \(y \in V_\delta(\tilde{y})\) then the second coordinate of \(T(x, y)\) is within \(2^{-i+1} = 2^{-i} + 2^{-i}\) of \(T(\tilde{y})\).

(b) follows from (3.4) applied to \(x\) and to \(T_1(x)\).

\[ \square \]

So we obtain the ultrametric system \((X, T)\).

From (3.4) it follows that the restriction

(3.7)

\[ \pi_1 : X \setminus V_\epsilon(e_1, e_2) = (\pi_1)^{-1}(X_1 \setminus V_\epsilon(e_1)) \to X_1 \setminus V_\epsilon(e_1) \text{ is bijective,} \]

and on it \(\pi_1 \circ T = T_1 \circ \pi_1\),

\[ x \in X_1 \setminus V_\epsilon(e_1) \implies (\pi_1)^{-1}(x) = \{(x, e_2)\}, \ T(x, e_2) = (T_1(x), e_2). \]

On the rest of the space the map \(\pi_1 : X \to X_1\) does not define an action map, but we obviously have for \(x \in X_1 \setminus \{e_1\}\):

(3.8)

\[ \pi_1^{-1}\left(\{T_i^0(x) : i = 0, 1, \ldots\}\right) = \{T_i^0(x, e_2) : i = 0, 1, \ldots\}, \]

\[ \pi_1^{-1}\left(\{T_i^{-i}(x) : i = 1, 2, \ldots\}\right) = \{T_i^{-i}(x, e_2) : i = 1, 2, \ldots\}. \]
Proposition 3.5. If \( x \in X_1 \setminus \{e_1\} \), then
\[
\pi^{-1}_1(\omega_{T_1}(x)) = \omega_T(x, e_2), \quad \pi^{-1}_1(\alpha_{T_1}(x)) = \alpha_T(x, e_2).
\]

If \( A \) is a \( T_1 \) invariant subset of \( X_1 \) then \( \pi^{-1}_1(A) \) is a \( T \) invariant subset of \( X \).

If \( B \) is a \( T \) invariant subset of \( X \) then \( \pi_1(B) \) is a \( T \) invariant subset of \( X_1 \).

**Proof:** The equations (3.8) clearly imply that \( \pi_1 \) maps the limit point set \( \omega_T(x, e_2) \) onto \( \omega_{T_1}(x) \). Then they imply that if \( z \in X_1 \setminus \{e_1\} \) then all the points of \( \pi^{-1}_1(z) \) all lie in the same orbit. Finally, for \( z \in V_{2^{-i}}(e_1) \setminus \{e_1\} \) the set \( \pi_2(\pi^{-1}_1(z)) \) is \( e^{2^{-i}} \) dense in \( X_2 \). This proves the result for \( \omega_T(x) \) and the result for \( \alpha_T(x) \) is similar.

The invariant set results are obvious from (3.8).

\( \Box \)

**Corollary 3.6.** \((X, T)\) is a minCT system.

If \((X_1, T_1)\) and \((X_2, T_2)\) are CT systems then \((X, T)\) is a CT system. Furthermore, if the Birkhoff center sequences for \((X_1, T_1)\) and \((X_2, T_2)\) stabilize at ordinals \( \omega_1 \) and \( \omega_2 \) respectively, then the Birkhoff center sequence for \((X, T)\) stabilizes at \( \omega_1 + \omega_2 \).

**Proof:** If \( M \) is a minimal subset of \( X \) then by Proposition 3.5 \( \pi_1 \) is a minimal subset of \( X_1 \) and so \( M \subset \{e_1\} \times X_2 \) where \( T \) is isomorphic to \( T_2 \) and so the \( M = \{(e_1, e_2)\} \).

Now assume \((X_1, T_1)\) and \((X_2, T_2)\) are CT systems. If \((x, y)\) is a recurrent point for \( T \) then \( x_1 \) is a recurrent point for \( X_1 \) by (3.9). Hence, \( x = e_1 \) and \( y \) is a recurrent point for \( T_2 \). So \( y = e_2 \). Hence, \((X, T)\) is CT. If \( A \) is a closed \( T_1 \) invariant subset of \( X_1 \) then the limit point set \( R_T(\pi^{-1}_1(A)) \) is the limit point set \( \pi^{-1}_1(R_{T_1}(A)) \). So exactly at \( \omega_1 \) the Birkhoff center sequence for \( X \) arrives at \( \{e_1\} \times X_2 \). It then stabilizes at \( (e_1, e_2) \) after \( \omega_2 \) more steps.

\( \Box \)

Notice that by replacing the metric \( d \) on \( X \) by the equivalent metric \( \min(1, \max_{n=0}^\infty 2^{-n}(T^n)^* d) \) we can assume that the metric is bounded by 1 and that \( T \) has Lipschitz constant at most 2. The new metric is an ultrametric if \( d \) was.

Let \( \{ (X_n, T_n) : n = 1, 2, \ldots \} \) be a sequence of ultrametric minCT systems with ultrametric \( d_n \leq 1 \) on \( X_n \) and with each \( T_n \) having Lipschitz constant at most 2. Assume that for \( n > 1 \) each \((X_n, T_n)\) is given a preparation. Let \((Z_1, U_1) = (X_1, T_1)\), let \((Z_2, U_2)\) be the \( 2^{-1} \) spin of \((X_2, T_2)\) into \((Z_1, U_1)\) with \( \xi_2 : Z_2 \rightarrow Z_1 \) be the first coordinate
projection. Thus, \( Z_2 \subset X_1 \times X_2 \). Inductively, let \((Z_{n+1}, U_{n+1})\) be the \( 2^{-n} \)-spin of \((X_{n+1}, T_{n+1})\) into \((Z_n, U_n)\) which we can regard as a subset of the product \( \Pi_n = X_1 \times \cdots \times X_n \times X_{n+1} \) equipped with the ultrametric \( \max(\pi_1^*d_1, 2^{-1}\pi_2^*d_2, \ldots, 2^{-n-1}\pi_{n+1}^*d_{n+1}) \). Let \( \xi_{n+1} : Z_{n+1} \to Z_n \) be the restriction of the coordinate projection from \( \Pi_{n+1} \to \Pi_n \) which has Lipschitz constant 1. Note again that the \( \xi_n \)'s are not action maps, but by (3.7) the restriction \( \xi_n : (\xi_n)^{-1}(Z_n \setminus V_{2^n}(e_1, \ldots, e_n)) \to Z_n \setminus V_{2^n}(e_1, \ldots, e_n) \) is injective and on it \( \xi_n \circ U_{n+1} = U_n \circ \xi_n \).

Let \( Z_\infty \) denote the inverse limit, regarded as a closed subset of \( \Pi_\infty = \Pi_{i=1}^\infty X_i \) equipped with the ultrametric \( \max\{2^{-i+1}\pi_i^*d_i : i = 1, 2, \ldots\} \) which yields the product topology. The space \( Z_\infty \) consists of the points \( z \) such that \( \xi_n(z) = (z_1, \ldots, z_n) \in Z_n \) for \( n = 1, 2, \ldots \). We let \( e \in Z_\infty \) denote the point \((e_1, e_2, \ldots)\).

Assume that \( z \in Z_\infty \) with \( z \neq e \) and let \( n \) be the smallest value such that \( x = \xi_n(z) \neq \xi_n(e) = (e_1, \ldots, e_n) \). Let \( \delta = \frac{1}{2}d(x, (e_1, \ldots, e_n)) \). Let \( i > n \) be such that \( 2^{-i} \leq \delta \). Since the projections have Lipschitz constant 1, \( (\xi_{n+k-1} \circ \cdots \circ \xi_n)^{-1}(V_\delta(x)) \) is disjoint from \( V_\delta(e_1, \ldots, e_{n+k}) \) for any positive integer \( k \). Once \( n + k \geq i \) it follows that \( (\xi_{n+k-1} \circ \cdots \circ \xi_n)^{-1}(V_\delta(x)) \) is disjoint from \( V_{2^{-n-k}}(e_1, \ldots, e_{n+k}) \). By (3.7) if \( \tilde{x} \in Z_{n+k} \setminus V_{2^{-n-k}}(e_1, \ldots, e_{n+k}) \) and \( \tilde{z} = (\tilde{x}, e_{n+k+1}, e_{n+k+2}, \ldots) \in Z_\infty \) then \( \xi_{n+k}^{-1}(\tilde{x}) = \{\tilde{z}\} \) and by (3.7) \( U(\tilde{z}) \) is unambiguously defined by \( U(\tilde{z}) = (U_{n+k}(\tilde{x}), e_{n+k+1}, e_{n+k+2}, \ldots) \). Since each \( U_{n+k} \) has Lipschitz constant at most 2, it follows that on each \( Z_\infty \setminus \xi_{n+k}^{-1}(V_{2^{-n-k}}(e_1, \ldots, e_{n+k})) \) \( U \) has Lipschitz constant at most 2. Finally,

\[
d(U(\tilde{z}), e) = d(U_{n+k}(\tilde{x}), (e_1, \ldots, e_{n+k})) \leq 2d(\tilde{x}, (e_1, \ldots, e_{n+k})) = 2d(\tilde{z}, e)
\]

shows that \( U \) has Lipschitz constant at most 2 on all of \( Z_\infty \).

Finally, with essentially the same proof as that of Corollary 3.6 we have

**Corollary 3.7.** \((Z_\infty, U)\) is a minCT system.

If each \((X_n, T_n)\) is a CT system then \((Z_\infty, U)\) is a CT system. Furthermore, if the Birkhoff center sequences for \((X_n, T_n)\) stabilize at the ordinals \( \omega_n \), then the Birkhoff center sequence for \((Z_\infty, U)\) stabilizes at \( \operatorname{Lim}_{n \to \infty} \omega_1 + \omega_2 + \cdots + \omega_n \).

\[\square\]
4. The space of labels

Let \( \mathbb{Z}, \mathbb{Z}_+, \mathbb{N} \) denote the sets of integers, of non-negative integers and of positive integers, respectively. On the vector space \( \mathbb{R}^N \) we will use the lattice structure, with \( x \geq y, x \leq y, x \lor y, x \land y \), the pointwise relations and the pointwise operations of maximum and minimum for vectors \( x, y \in \mathbb{R}^N \). As usual \( x > y \) means \( x \geq y \) and \( x \neq y \) so that the inequality is strict for at least one coordinate. The support of a vector \( x \in \mathbb{R}^N \), denoted \( \text{supp } x \), is \( \{ \ell : x_\ell \neq 0 \} \).

We will call \( m \in \mathbb{Z}^N \) an \( N \)-vector when it is non-negative and has finite support, that is, when \( m \geq 0 \) for all \( \ell \in \mathbb{N} \) and \( \text{supp } m = \{ \ell : m_\ell > 0 \} \) is finite. We call \( \# \text{supp } m \) the size of \( m \) and call \( |m| = \sum_\ell m_\ell \) the norm of \( m \).

If \( S \subset \mathbb{N} \) we let \( \chi(S) \) be the characteristic function of \( S \) with \( \chi(\{ \ell \}) = \chi(\ell) \). Thus, \( \chi(S) = \sum_{\ell \in S} \chi(\ell) \) is an \( N \)-vector when \( S \) is a finite set.

We denote by \( FIN(\mathbb{N}) \) the discrete abelian monoid of all \( \mathbb{N} \)-vectors with identity \( 0 \). It is also a lattice via the pointwise order relations described above.

For an \( N \)-vector \( m \) and a positive integer \( \ell^* \) we define \( m \land [1, \ell^*] \) to be the \( N \)-vector with

\[
(m \land [1, \ell^*])_\ell = \begin{cases} 
m_\ell & \text{for } \ell \leq \ell^*, \\
0 & \text{for } \ell > \ell^*. 
\end{cases}
\]

**Definition 4.1.** A set \( M \) of \( N \)-vectors is called a label of \( N \)-vectors when it satisfies the following two conditions.

(i) [Bound Condition] There exists \( \mu \in \mathbb{Z}_+^N \) such that \( 0 \leq m \leq \mu \) for all \( m \in M \).

(ii) [Heredity Condition] \( 0 \leq m_1 \leq m \) and \( m \in M \) imply \( m_1 \in M \).

**Definition 4.2.** If \( S \subset \mathbb{Z}_+^N \), we will say that \( S \) satisfies the Bound Condition when there exists \( \mu \in \mathbb{Z}_+^N \) such that \( 0 \leq \nu \leq \mu \) for all \( \nu \in S \). In that case, we define \( \langle S \rangle = \{ m \in FIN(\mathbb{N}) : m \leq \nu \text{ for some } \nu \in S \} \). We call \( \langle S \rangle \) the label generated by \( S \). We will write \( \langle \nu \rangle \) for \( \langle S \rangle \) when \( S = \{ \nu \} \).

A label is of finite type if it also satisfies the following

(iii) [Finite Chain Condition] There does not exist an infinite increasing sequence in \( M \), or equivalently, any infinite nondecreasing sequence in \( M \) is eventually constant.
Clearly, a finite set of \( \mathbb{N} \)-vectors is a label if it satisfies the Heredity Condition in which case it is of finite type.

We call a \( \mathcal{M} \) size bounded when

(iii') [Size Bound Condition] There exists \( n \in \mathbb{N} \) such that \( \text{size}(\mathbf{m}) \leq n \) for all \( \mathbf{m} \in \mathcal{M} \).

**Lemma 4.3.** If a label \( \mathcal{M} \) is size bounded then it is of finite type.

**Proof:** If \( \mathbf{m}_1 < \mathbf{m}_2 < \cdots \) is an infinite sequence of \( \mathbb{N} \)-vectors then at each step either some already positive entry increases or the size increases. Since the entries in \( \mathcal{M} \) are bounded by the vector \( \mu \) and the size is assumed bounded the sequence must eventually leave \( \mathcal{M} \). Hence, the Finite Chain Condition holds.

\( \square \)

For example, \( \emptyset \) and 0 = \( \{0\} \) are labels. \( \mathcal{M} \neq \emptyset \) iff 0 \( \in \mathcal{M} \). We call \( \mathcal{M} \) a positive label when it is neither empty nor 0.

Define the roof of a label \( \mathcal{M} \) by \( \rho(\mathcal{M}) = \max_{\mathbf{m} \in \mathcal{M}} \{\mathbf{m}\} \). That is, the roof is the minimum of the functions \( \mu \in \mathbb{Z}^\mathbb{N}_+ \) which bound the elements of \( \mathcal{M} \). Clearly, \( \rho(0) = 0 \) and by convention we let \( \rho(\emptyset) = 0 \) as well.

We let \( \text{Supp} \mathcal{M} = \{\text{supp} \mathbf{m} : \mathbf{m} \in \mathcal{M}\} \).

For a label \( \mathcal{M} \) and \( \ell^* \in \mathbb{N} \) we define

\[
(4.2) \quad \mathcal{M} \wedge [1, \ell^*] = \begin{cases} 
\emptyset & \text{when } \ell^* = 0, \\
\{ \mathbf{m} \wedge [1, \ell^*] : \mathbf{m} \in \mathcal{M} \} & \text{when } \ell^* > 0.
\end{cases}
\]

For a label \( \mathcal{M} \) and an \( \mathbb{N} \)-vector \( \mathbf{r} \) we define

\[
(4.3) \quad \mathcal{M} - \mathbf{r} = \{ \mathbf{w} : \mathbf{w} \geq 0, \mathbf{w} + \mathbf{r} \in \mathcal{M} \}.
\]

Thus, \( \mathcal{M} - \mathbf{r} \) is the set of all non-negative vectors of the form \( \mathbf{m} - \mathbf{r} \) for \( \mathbf{m} \in \mathcal{M} \). Clearly, \( (\mathcal{M} - \mathbf{r}) - \mathbf{s} = \mathcal{M} - (\mathbf{r} + \mathbf{s}) = (\mathcal{M} - \mathbf{s}) - \mathbf{r} \) (and so we can omit the parentheses) since each is the set of \( \mathbb{N} \)-vectors \( \mathbf{w} \) such that \( \mathbf{w} + \mathbf{r} + \mathbf{s} \in \mathcal{M} \).

Let \( \text{max} \mathcal{M} \) be the set of \( \mathbb{N} \)-vectors which are maximal in \( \mathcal{M} \). That is, \( \mathbf{n} \in \text{max} \mathcal{M} \) if

\[
(4.4) \quad \mathbf{m} \geq \mathbf{n} \text{ and } \mathbf{m} \in \mathcal{M} \iff \mathbf{m} = \mathbf{n}.
\]

**Proposition 4.4.** Let \( \mathcal{M} \) be a label and let \( \mathbf{r} > 0 \) be an \( \mathbb{N} \)-vector.

(a) If \( \mathcal{M} \) is of finite type then for every \( \mathbf{m} \in \mathcal{M} \) there exists \( \mathbf{n} \in \text{max} \mathcal{M} \) such that \( \mathbf{m} \leq \mathbf{n} \). Hence, \( \mathcal{M} = \langle \text{max} \mathcal{M} \rangle \). Thus, a label \( \mathcal{M} \) of finite type is determined by \( \text{max} \mathcal{M} \).

(b) If \( \mathcal{M}_1 \subset \mathcal{M} \) and \( \mathcal{M}_1 \) satisfies the Heredity Condition then \( \mathcal{M}_1 \) is a label and is of finite type if \( \mathcal{M} \) is.
32 ETHAN AKIN AND ELI GLASNER

(c) If $\ell^* \in \mathbb{N}$ then $M \land [1, \ell^*]$ is a finite label.

(d) $M - r$ is a label. If $M$ is nonempty then $M - r$ is a proper subset of $M$ with $\max M \subset M \setminus (M - r)$.

(e) $M - r \neq \emptyset$ iff $r \in M$.

(f) If $\Phi$ is any set of labels then $\bigcap \Phi$ is a label which is of finite type if one of the labels in $\Phi$ is. If $\Phi$ is finite then $\bigcup \Phi$ is a label which is of finite type if all the elements of $\Phi$ are.

Proof: (a): If $m$ is not maximal then there exists $m_1 \in M$ with $m_1 > m$. Continue if $m_1$ is not maximal. This sequence can continue only finitely many steps by the Finite Chain Condition. It terminates at a maximal vector $n$.

(b): Conditions (i) and (iii) for $M_1$ are clearly inherited from $M_1$.

(c): For any $\mu \in \mathbb{Z}^N$ there are only finitely many $m \in \mathbb{Z}^N_+$ such that $m \leq \mu$ and $\operatorname{supp} m \subset [1, \ell^*]$. The Heredity Condition is obvious and so $M \land [0, \ell^*]$ is a finite label.

(d) $M - r$ satisfies the Heredity Condition and so is a label. If $m \in M - r$ then $m + r$ is an element of $M$ with $m + r > m$ since $r > 0$. Hence, $m$ is not a maximal element of $M$.

Now assume $M$ is nonempty. Then $0 = 0r \in M$. If $r_\ell > 0$ then for $n \in \mathbb{N}$ such that $n > \rho(M)_\ell/r_\ell$, $nr \notin M$. So there is a maximum $n \geq 0$ such that $nr \in M$. It follows that $nr \in M \setminus (M - r)$.

(e) If $r \in M$ then $0 \in M - r$. If $m \in M - r$ then $m + r \in M$ and so $m + r \geq r$ implies $r \in M$.

(f) The Hereditary Condition is preserved by both intersection and union. Hence, the intersection result follows from (b).

$\rho(\bigcup \Phi)_\ell = \max \{\rho(M)_\ell : M \in \Phi\} < \infty$ when $\Phi$ is finite. So the Bound Condition for $\bigcup \Phi$ is satisfied when $\Phi$ is finite.

If $\Phi$ is finite collection of labels of finite type and $\{m^i\}$ is a strictly increasing sequence of $N$-vectors then it can remain in each member of $\Phi$ for only finitely many terms. As $\Phi$ is finite, the sequence must eventually leave $\bigcup \Phi$. Hence, the union satisfies the Finite Chain Condition.

Example 4.5. A label $M$ which is generated by $\max M$ need not be of finite type. If $M = \{0, \Sigma_{\ell \in A} \chi(\ell), \Sigma_{\ell \in A} \chi(\ell) + k \chi(\max A)\}$ with $A$ varying over all finite subsets of $\mathbb{N}$, and $1 \leq k \leq \max A$ then $M$ is not of finite type although every $m \in M$ is bounded by an element of $\max M$.
In terms of supports we do have an equivalence version of Proposition 4.4 (a). We will say that $\text{Supp } M$ $f$-contains a set $L \subset \mathbb{N}$ when every finite subset of $L$ is a member of $\text{Supp } M$. That is, $\mathcal{P}_f L \subset \text{Supp } M$ where $\mathcal{P}_f L$ is the set of finite subsets of $L$. Equivalently, $M \supset \langle \chi(L) \rangle$.

**Proposition 4.6.** (a) A label $M$ is not of finite type iff there exists an infinite subset $L$ of $\mathbb{N}$ such that $\chi(A) = \sum_{\ell \in A} \chi(\ell) \in M$ for any finite subset $A$ of $L$. Equivalently, $\text{Supp } M$ $f$-contains $L$.

(b) A label $M$ is of finite type iff every $A \in \text{Supp } M$ is contained in some $B \in \text{Supp } M$ which is maximal with respect to inclusion in $\text{Supp } M$.

**Proof:** (a) If such an $L = \{\ell_1, \ell_2, \ldots \}$ exists then $n^k = \sum_{i=1}^k \chi(\ell_i)$ is a strictly increasing infinite sequence in $M$ and so $M$ is not of finite type. Note, conversely, that if $n^k \in M$ for all $k$ then the characteristic function of every finite subset of $L$ is in $M$.

If $M$ is not of finite type then there is a strictly increasing sequence $\{m^i\}$. By the Bound Condition $\bigcup \{\text{supp } m^i\}$ is infinite. Hence, we can inductively choose $\ell_1, \ldots, \ell_k$ contained in the support of some $m^k$ with $i_k > i_{k-1}$. Hence, $\chi(\{\ell_1, \ldots, \ell_k\}) \in M$ for every $k$ and so $\chi(F) \in M$ for every finite subset of $\{\ell_1, \ell_2, \ldots \}$.

In general, a finite set $F \subset \mathbb{N}$ is in $\text{Supp } M$ iff $\chi(F) \in M$.

(b) If $\text{Supp } M$ $f$-contains an infinite set $L$ then none of the finite subsets of $L$ are contained in maximal elements of $\text{Supp } M$. Conversely, if $A \in \text{Supp } M$ is not contained in a maximal element then there is an increasing sequence in $\{A_0, A_1, \ldots \}$ in $\text{Supp } M$ with $A = A_0$. Then $L = \bigcup_i \{A_i\}$ is an infinite set and $\text{Supp } M$ $f$-contains $L$.

□

A set $\Phi$ (or a sequence $\{M^i\}$) of labels is said to be **bounded** when there exists a label $N$ such that $M \subset N$ for all $M \in \Phi$ (resp. for all $M = M^i$ for some $i$). In that case we will call $N$ a **bound** for the set or sequence or say that the set or sequence is bounded by $N$.

By Proposition 4.4 (b) this is equivalent to the condition that $\bigcup \Phi$ (resp. that $\bigcup \{M^i\}$) satisfies the Bound Condition and so is a label. That is, when for every $\ell \in \mathbb{N}$ the set $\{\rho(M)_{\ell} : M \in \Phi\}$ is bounded, (resp. the set $\{\rho(M^i)_{\ell} \}$ is bounded,) and so has a finite maximum value which is then $\rho(\bigcup \Phi)_{\ell}$ (resp. is then $\rho(\bigcup \{M^i\})_{\ell}$).

For any label $N$ the set $[[N]]$ of all labels which are contained in $N$ is a bounded set of labels. Thus a set or sequence is bounded iff it is contained in $[[N]]$ for some label $N$.

If $N$ is of finite type then all the members of $[[N]]$ are labels of finite type.
We denote by $\mathcal{LAB}$ the space of labels. On $\mathcal{LAB}$ we define an ultrametric by

$$d(M_1, M_2) = \inf \{ 2^{-\ell} : \ell \in \mathbb{Z}_+ \text{ and } M_1 \wedge [1, \ell] = M_2 \wedge [1, \ell] \}.$$ (4.5)

Since every $\mathbb{N}$-vector has finite support it is clear that $d(M_1, M_2) = 0$ iff $M_1 = M_2$. Furthermore, for any finite label $N$ the set $\{ M : N \subset M \}$ is clopen because there exists an $\ell$ such that $\bigcup \text{Supp } N \subset [1, \ell]$. Also, for any $\ell$, $\{ M : M \wedge [1, \ell] \subset N \}$ is clopen.

Let $M_i$ be a sequence of labels. Define the sets of $\mathbb{N}$-vectors

$$\text{LIMSUP } \{ M_i \} = \bigcap \bigcup_{j \geq i} \{ M_j \};$$ (4.6)
$$\text{LIMINF } \{ M_i \} = \bigcup \bigcap_{j \geq i} \{ M_j \}.$$

Clearly, $m \in \text{LIMSUP}$ iff frequently $m \in M_i$ and $m \in \text{LIMINF}$ iff eventually $m \in M_i$ and so \text{LIMINF} $\subset$ \text{LIMSUP}. As usual, if we go to a subsequence $\{ M'_i \}$ with \text{LIMSUP}' and \text{LIMINF}' then \text{LIMINF} $\subset$ \text{LIMINF}' $\subset$ \text{LIMSUP}. \text{LIMSUP} and \text{LIMINF} always satisfy the Heredity Condition. So they are labels if $\bigcup_i \{ M_i \}$ satisfies the Bound Condition. That is, if the sequence is bounded. Conversely, if \text{LIMSUP} is a label with roof $\rho$ then for every $\ell$ there exists $i$ such that $(\rho(\ell) + 1) \chi(\ell) \not\in \bigcup_{j \geq i} \{ M_j \}$ and so $\{ \rho(M_i)(\ell) \}$ is bounded. Thus, \text{LIMSUP} is a label only if $\bigcup_i \{ M_i \}$ satisfies the Bound Condition.

If the sequence $\{ M_i \}$ is bounded with $M_i \subset M$ for all $i$ then both \text{LIMINF} $\subset$ \text{LIMSUP} $\subset M$ and both \text{LIMSUP} and \text{LIMINF} are labels of finite type if $M$ is of finite type.

**Proposition 4.7.** Let $\{ M^i \}$ be a sequence of labels.

(a) The following are equivalent.

1. The sequence $\{ M^i \}$ is a Cauchy sequence.
2. For every $\ell \in \mathbb{N}$, the sequence $\{ M^i \wedge [1, \ell] \}$ of finite labels is eventually constant.
3. \text{LIMSUP} = \text{LIMINF} and the common value satisfies the Bound Condition.
4. The sequence $\{ M^i \}$ is bounded and \text{LIMSUP} = \text{LIMINF}. The common value \text{LIMSUP} = \text{LIMINF} is then the limit, and is then denoted $\text{LIM } \{ M^i \}$.

(b) A convergent sequence $\{ M^i \}$ is bounded by a label of finite type iff each $M^i$ and the limit are of finite type.
(c) If $\{M^i\}$ is a bounded sequence and $M^i \subset M^{i+1}$ then $\text{LIM}\{M^i\} = \bigcup \{M^i\}$. If $M^i \supset M^{i+1}$ then $\{M^i\}$ is bounded and $\text{LIM}\{M^i\} = \bigcap \{M^i\}$.

**Proof:** (a) (1) $\Longleftrightarrow$ (2): Obvious from the definition of the ultrametric.

(2) $\Rightarrow$ (4): Because $M^i \land [0, \ell]$ is eventually constant it is clear that $\mu_\ell = \max_i \{\rho(M^i)_\ell\} < \infty$ and so $m \leq \mu$ for all $m \in \bigcup \{M^i\}$. Thus the sequence is bounded. Furthermore, for every $N$-vector $m$ there exists $\ell$ such that $\text{supp} m \subset [1, \ell]$ and so either eventually $m \in M^i$ or eventually $m \not\in M^i$. This means $\text{LIMSUP} = \text{LIMINF}$.

(4) $\Rightarrow$ (3): If $M^i \subset M$ for all $i$ then $\text{LIMINF} \subset \text{LIMSUP} \subset M$ and so each is a label.

(3) $\Rightarrow$ (2): Assume that $M = \text{LIMSUP} = \text{LIMINF}$ and the $\mu \in \mathbb{Z}_+^N$ bounds all the elements in this set. Let $N_\ell$ be the finite label consisting of $\mathbb{N}$-vectors $n$ such that $n \leq \mu + 1$ and $\text{supp} n \subset [1, \ell]$. Because $\text{LIMSUP} = \text{LIMINF}$ there exists $I$ such that for $n \in N_\ell$ either $n \in M^i$ for all $i \geq I$ or $n \not\in M^i$ for all $i \geq I$. Now let $m$ be an $\mathbb{N}$-vector with $\text{supp} m \subset [1, \ell]$ but $m \not\in N_\ell$. Then for some $k \in [1, \ell]$ $m_k > \mu_k + 1$. Hence, $m \land (\mu + 1) \in N_\ell \setminus \text{LIMSUP}$. It follows that for all $i \geq I$, $m \land (\mu + 1) \not\in M^i$ and so $m \not\in M^i$. Thus, for $i \geq I$ the sequence $\{M^i \land [1, \ell]\}$ is constant and equals $M \land [1, \ell]$. Thus, (3) implies (2) and it shows that $M$ is the limit of $\{M^i\}$.

(b): If the sequence has a bound $N$ of finite type then $\text{LIMINF} \subset \text{LIMSUP} \subset N$ implies that each $M^i$ and the limit are of finite type. Now suppose that each $M^i$ is of finite type but there is a strictly increasing sequence $\{i_j\}$ in the union. It must leave each $M^i$ and so if $m^j \in M^{i_j}$ then $\{i_j\} \to \infty$ with $j$. Since $m^k \in M^{i_j}$ if $k \leq j$, it follows that each $m^k$ is frequently in a member of the sequence $\{M^i\}$. So by convergence the sequence $\{m^j\}$ is contained in the limit and so the limit is not of finite type.

(c): For an increasing sequence the $\text{LIMSUP} = \text{LIMINF}$ is the union and for a decreasing sequence $\text{LIMSUP} = \text{LIMINF}$ is the intersection.

$\Box$

**Corollary 4.8.** $\mathcal{LAB}$ is a complete, separable, zero-dimensional metric space with $\emptyset$ the only isolated point.

**Proof:** $\mathcal{LAB}$ is complete by Proposition 4.7. It is separable because the countable set of finite labels is dense. It is zero-dimensional because it has an ultrametric. Any nonempty label contains $0$ and so $\emptyset$ is isolated. Clearly, no other label is isolated.
Example 4.9. Limit examples.
(a) If $M^i = \{ k\chi(1) : 0 \leq k \leq i \}$ then $d(M^i, M^j) = \frac{1}{2}$ whenever $i \neq j$ and the sequence is not Cauchy. On the other hand, $\text{LIMSUP} = \text{LIMINF} = \{ k\chi(1) : 0 \leq k < \infty \}$ which is not a label since the Bound Condition fails.
(b) Any label $M$ is the limit of the sequence $\{ M \wedge [1, i] \}$ of finite labels and so a convergent sequence of labels of finite type need not have limit of finite type.

Proposition 4.10. Let $\{ M^i \}$ be a bounded sequence of labels.
- $m \in \text{LIMSUP}\{ M^i \}$ iff there is a subsequence $\{ M^{i'} \}$ which is convergent with $m \in LIM \{ M^{i'} \}$.
- $m \notin \text{LIMINF}\{ M^i \}$ iff there is a subsequence $\{ M^{i'} \}$ which is convergent with $m \notin LIM \{ M^{i'} \}$.

Proof: The $\text{LIMSUP}$ of a subsequence is contained in the $\text{LIMSUP}$ of the original sequence and the $\text{LIMINF}$ of a subsequence contains the $\text{LIMINF}$ of the original sequence Thus, sufficiency is clear in each case.

Let $\{ m_1, m_2, \ldots \}$ be a numbering of the countable set of $\mathbb{N}$-vectors with $m_1 = m$. Since $m \in M^i$ frequently, we can choose $\text{SEQ}_1$ an infinite subset of $\mathbb{N}$ so that $m_1 \in M^i$ for all $i \in \text{SEQ}_1$. If eventually $m_2 \in M^i$ for $i \in \text{SEQ}_1$ let these values of $i$ define $\text{SEQ}_2 \subset \text{SEQ}_1$. Otherwise, let $\text{SEQ}_2$ be the $i \in \text{SEQ}_1$ such that $m_2 \notin M^i$. Inductively we define a decreasing sequence $\text{SEQ}_n$ of infinite subsets of $\mathbb{N}$ such that $p \leq n$ implies either $m_p \in M^i$ for all $i \in \text{SEQ}_n$ or for no $i \in \text{SEQ}_n$. Diagonalizing, we obtain a convergent subsequence whose limit contains $m$. That is, if $i_n$ be the $n^{th}$ element of $\text{SEQ}_n$, then $\{ M^{i_n} \}$ is convergent and the limit contains $m$.

Alternatively, if $m \notin \text{LIMINF}$ we begin by choosing $\text{SEQ}_1$ so that $m_1 \notin M^i$ for all $i \in \text{SEQ}_1$ and continue as before.

For any compact subset of $\Phi \subset \text{LAB}$ we will denote by $[[\Phi]]$ the compact set $\text{Inc}(\Phi) = \{ N : N \subset M \text{ for some } M \in \Phi \}$. We write $[[\{ M \}]]$ for $[[\{ M \}]]$. 
Theorem 4.11. (a) A set of labels is compact iff it is closed and bounded. In particular, for any label $\mathcal{M}$, $[[\mathcal{M}]]$ is compact.

(b) The set $\text{Inc} = \{(\mathcal{M}, \mathcal{N}) : \mathcal{N} \subset \mathcal{M}\}$ is a closed subset of $\mathcal{LAB} \times \mathcal{LAB}$ and for any compact subset $\Phi$ of $\mathcal{LAB}$, the set $\text{Inc}(\Phi) = \bigcup_{\mathcal{M} \in \Phi} [[\mathcal{M}]]$ is compact.

Proof: (a) A sequence $\{\mathcal{M}^i\}$ is bounded when it is contained in $[[\mathcal{M}]]$ for some label $\mathcal{M}$ in which case $\text{LIMSUP} \in [[\mathcal{M}]]$. By Proposition 4.10 there is a convergent subsequence and the limit is in $[[\mathcal{M}]]$. Hence, $[[\mathcal{M}]]$ is compact as is any other closed subset of $[[\mathcal{M}]]$. That is, a closed bounded set is compact.

If $\Phi$ is compact then it is closed and for $\ell \in \mathbb{N}$ the collection of clopen sets $\{\mathcal{M} \land [1, \ell] \subset \mathcal{N}\}$ covers $\Phi$ as $\mathcal{N}$ varies over all finite labels. Hence, there is a finite list of finite labels $\{\mathcal{N}^i : i = 1, \ldots, N\}$ such that $\mathcal{M} \land [1, \ell] \subset \bigcup_{i=1}^N \mathcal{N}^i$ for all $\mathcal{M} \in \Phi$. It follows that $\{\mathcal{M}^i : \mathcal{M} \in \mathcal{M}\}$ is bounded for each $\ell$. Thus, $\bigcup \Phi$ satisfies the Bound Condition and so is a label. Hence, $\Phi$ is bounded.

(b) For every $\ell \in \mathbb{N}$ the set $\{(\mathcal{M}, \mathcal{N}) : \mathcal{N} \land [1, \ell] \subset \mathcal{M} \land [1, \ell]\}$ is clopen. Intersect over $\ell$ to get the closed set $\text{Inc}$. If $\Phi$ is bounded by $\mathcal{M}_i$, i.e. $\Phi \subset [[\mathcal{M}_i]]$ then $\text{Inc}(\Phi) \subset [[\mathcal{M}]]$ and so this set is bounded. If a sequence $\{\mathcal{N}^i\}$ in $\text{Inc}(\Phi)$ converges to $\mathcal{N}$ then there exists $\mathcal{M}^i \in \Phi$ such that $\mathcal{N}^i \subset \mathcal{M}^i$. By going to a subsequence we can assume that $\{\mathcal{M}^i\}$ converges to some $\mathcal{M}$ in the compact set $[[\mathcal{M}_i]]$. Since the convergent sequence $\{(\mathcal{M}^i, \mathcal{N}^i)\}$ is in the closed set $\text{Inc}$ the limit point $(\mathcal{M}, \mathcal{N}) \in \text{Inc}$. Because $\Phi$ is closed, $\mathcal{M} \in \Phi$ and so $\mathcal{N} \in \text{Inc}(\Phi)$.

Given an $\mathbb{N}$-vector $\mathbf{r}$ we define the map $P_\mathbf{r}$ on $\mathcal{LAB}$, by $P_\mathbf{r}(\mathcal{M}) = \mathcal{M} - \mathbf{r}$.

Proposition 4.12. The function $P_\mathbf{r}$ is continuous. In particular, if $\{\mathcal{M}^i\}$ is a convergent sequence of labels then $\{\mathcal{M}^i - \mathbf{r}\}$ is convergent with $\text{LIM}\{\mathcal{M}^i - \mathbf{r}\} = \text{LIM}\{\mathcal{M}^i\} - \mathbf{r}$.

Proof: Let $\ell_\mathbf{r}$ be the minimum value such that $\text{supp} \mathbf{r} \subset [1, \ell_\mathbf{r}]$. We show that

\begin{equation}
\ell \geq \ell_\mathbf{r} \implies \mathcal{M} \land [1, \ell] - \mathbf{r} = (\mathcal{M} - \mathbf{r}) \land [1, \ell].
\end{equation}

If $\mathbf{w} \in \mathcal{M} - \mathbf{r}$ then $\mathbf{w} \land [1, \ell] \in \mathcal{M} - \mathbf{r}$ and so $\mathbf{w} \land [1, \ell] + \mathbf{r} \in \mathcal{M} \land [1, \ell]$ since $\text{supp} \mathbf{r} \subset [1, \ell]$. On the other hand, if $\mathbf{w} \in \mathcal{M} \land [1, \ell] - \mathbf{r}$ then $\mathbf{w} + \mathbf{r} \in \mathcal{M}$ and $\text{supp} \mathbf{w} \subset [1, \ell]$. Hence, $\mathbf{w} \in (\mathcal{M} - \mathbf{r}) \land [1, \ell]$.

From (4.7) it follows that $d(\mathcal{M}_1, \mathcal{M}_2) < 2^{-\ell}$ implies $d(P_\mathbf{r}\mathcal{M}_1, P_\mathbf{r}\mathcal{M}_2) < 2^{-\ell}$. This implies that $P_\mathbf{r}$ is Lipschitz with Lipschitz constant at most $2^{\ell_\mathbf{r}}$. 

Corollary 4.13. The map $P : FIN(N) \times \mathcal{LAB} \to \mathcal{LAB}$ given by $(r, M) = M - r = P_r(M)$ is a continuous monoid action of $FIN(N)$ on $\mathcal{LAB}$.

The action is faithful i.e. if $P_r(M) = P_s(M)$ for all $M$ then $r = s$.

Proof: It is an action since $(M - r_1) - r_2 = M - (r_1 + r_2)$ for $N$-vectors $r_1, r_2$ and $M - 0 = M$. It is a continuous action by Proposition 4.12.

For a label $r$ let $M$ be the finite label $\langle r \rangle$. Since $\{r\} = \text{max} M$, $P_r(M) = 0$. If $P_s(M) = 0$ then $s \in M$ and so $s \leq r$. Clearly, $r - s \in P_s(M)$ and so $r - s = 0$.

If $\Phi$ is any compact subset of $\mathcal{LAB}$ then $[[\Phi]]$ is a compact, invariant set containing $\Phi$. In particular, this is true with $\Phi = \{M\}$ and $[[\Phi]] = [[M]]$ for any label $M$.

Notice that $FIN(N)$ is the free abelian monoid generated by $\{\chi(\ell) : \ell \in N\}$ and it is a submonoid of the free abelian group consisting of the members of $Z^N$ with finite support. In particular, it is a cancellation semigroup: $r_1 + s = r_2 + s$ implies $r_1 = r_2$. In particular, $r + r = r$ only when $r = 0$. Also, $r + s = 0$ if $r = s = 0$.

Giving $FIN(N)$ the discrete topology, we obtain on the Stone-Čech compactification $\beta FIN(N)$ the structure of an Ellis semigroup with product which extends the addition on $FIN(N)$ and is such that $Q \mapsto QR$ is continuous for any $R \in \beta FIN(N)$. Let $\beta^* FIN(N) = \beta FIN(N) \setminus \{0\}$ and $\beta^{**} FIN(N) = \beta FIN(N) \setminus FIN(N)$. Notice that since $FIN(N)$ is discrete, it is the set of isolated points in $\beta FIN(N)$. Since the elements of $FIN(N)$ commute with all elements of $\beta FIN(N)$, the submonoid $FIN(N)$ acts continuously on $\beta FIN(N)$.

Although $\mathcal{LAB}$ is not compact, the action of $FIN(N)$ extends to an action of $\beta FIN(N)$ on $\mathcal{LAB}$.

Theorem 4.14. (a) For any label $M$ the map $r \mapsto P_r(M)$ extends to a continuous map from $\beta FIN(N)$ to $[[M]]$ and this defines an Ellis action $\beta P : \beta FIN(N) \times \mathcal{LAB} \to \mathcal{LAB}$.

(b) If $N \subset M$ then $Q(N) \subset Q(M)$ for all $Q \in \beta FIN(N)$.

(c) The sets $\beta^* FIN(N)$ and $\beta^{**} FIN(N)$ are closed, invariant subsets of $\beta FIN(N)$ and so are ideals in the Ellis semigroup.

(d) Every nonempty, closed sub-semigroup of $\beta FIN(N)$ contains an idempotent and all the idempotents of $\beta^* FIN(N)$ lie in $\beta^{**} FIN(N)$. 
Proof: (a) The extension to $βFIN(\mathbb{N})$ of the map to the compact set $[\mathbb{M}]$ is a standard property of the Stone-Čech compactification. It thus defines a function $βFIN(\mathbb{N}) \times \mathcal{LAB} \to \mathcal{LAB}$. As usual the equation $(QR)(\mathcal{M}) = Q(R(\mathcal{M}))$ for $Q, R \in βFIN(\mathbb{N})$ holds when $Q, R \in FIN(\mathbb{N})$. With $Q$ fixed in $FIN(\mathbb{N})$, continuity of $Q$ implies that it holds for all $R \in βFIN(\mathbb{N})$. Then with $R \in βFIN(\mathbb{N})$ fixed it then extends to all $Q \in βFIN(\mathbb{N})$.

(b) Each $P_r$ preserves inclusions. Then with $N \subset M$ fixed, it follows that all $Q \in βFIN(\mathbb{N})$ preserve inclusions because $Inc$ is closed by Theorem 4.11 (b).

(c) Suppose that $rQ = Qr = s$ for some $r, s \in FIN(\mathbb{N})$. Assume that $\{r^i\}$ is a net in $FIN(\mathbb{N})$ converging to $Q$. Since $s$ is isolated and $\{r^i + r\}$ converges to $Qr$ it follows that eventually $r^i + r = s$. By cancelation, eventually $\{r^i\}$ is constant and so the limit $Q$ is in $FIN(\mathbb{N})$. Contrapositively, $Q \in β^{**}FIN(\mathbb{N})$ implies $P_rQ \in β^{**}FIN(\mathbb{N})$. Hence, $β^{**}FIN(\mathbb{N})$ is invariant. As the complement of the set of isolated points, it is closed. If $rQ = 0$ then $Q = s$ for some $s \in FIN(\mathbb{N})$ and $r + s = 0$. So $r = s = 0$. Hence, $β^{*}FIN(\mathbb{N})$ is invariant and since $0$ is isolated, it is closed. A closed, $FIN(\mathbb{N})$ invariant subset of $βFIN(\mathbb{N})$ is an ideal.

(d) The existence of idempotents is the Ellis-Namakura Lemma. We saw above that $0$ is the only idempotent in $FIN(\mathbb{N})$ and so there are no idempotents in $β^{*}FIN(\mathbb{N}) \setminus β^{**}FIN(\mathbb{N})$.

Define $β^{0}FIN(\mathbb{N}) = \{Q \in βFIN(\mathbb{N}) : Q(\mathcal{M}) = \emptyset$ for all $\mathcal{M} \in \mathcal{LAB}\}$. Let $γFIN(\mathbb{N})$ be the quotient space of $βFIN(\mathbb{N})$ obtained by collapsing $β^{0}FIN(\mathbb{N})$ to a point which we will denote $U$.

**Proposition 4.15.** $β^{0}FIN(\mathbb{N})$ is a closed, two-sided ideal in the semigroup $βFIN(\mathbb{N})$. The projection map $βFIN(\mathbb{N}) \to γFIN(\mathbb{N})$ induces an Ellis semigroup structure so that the projection becomes a continuous, surjective homomorphism. The images $γ^{*}FIN(\mathbb{N})$ and $γ^{**}FIN(\mathbb{N})$ of $β^{*}FIN(\mathbb{N})$ and $β^{**}FIN(\mathbb{N})$ are closed ideals in $γFIN(\mathbb{N})$. The action of $βFIN(\mathbb{N})$ on $\mathcal{LAB}$ factors to define an Ellis action of $γFIN(\mathbb{N})$ on $\mathcal{LAB}$. If $r > 0$ then any idempotent in the closed ideal $(βFIN(\mathbb{N}))P_r$ maps to $U$ in $γFIN(\mathbb{N})$.

Proof: By definition of an Ellis action $Q \mapsto Q(\mathcal{M})$ is continuous and so $Q(\mathcal{M}) = \emptyset$ is a closed condition. If $Q \in β^{0}FIN(\mathbb{N})$ and $Q_1 \in βFIN(\mathbb{N})$ then $Q_1(\emptyset) = \emptyset$ implies $Q_1Q \in β^{0}FIN(\mathbb{N})$ and $Q_1(\mathcal{M}) \in \mathcal{LAB}$ implies $QQ_1 \in β^{0}FIN(\mathbb{N})$. So multiplication is well-defined on $γFIN(\mathbb{N})$ so that the projection is a homomorphism and $Q_1 \mapsto Q_1Q_2$
and \( Q_1 \mapsto Q_1(M) \) are continuous by definition of the quotient topology. Finally, if \( Q = Q_1P_r \) is an idempotent in \( \beta FIN(N) \) then since \( P_r \) commutes with \( Q_1 \), we have \( Q = Q^n = Q^n_1P_{nr} \) for all positive integers \( n \). For any \( M \), \( P_{nr}(M) = \emptyset \) for \( n \) sufficiently large. Hence, \( Q \in \beta FIN(N) \) and so maps to \( U \).

Let \( \Theta(M) \) be the closure in the space of labels of the set \( \{ M - r : r \text{ an } N \text{-vector} \} \). That is, \( \Theta(M) \) is the orbit closure of \( M \) with respect to the \( FIN(N) \) action or, equivalently, \( \Theta(M) = \beta FIN(N)(M) \). Since \([M]\) is closed and invariant, \( \Theta(M) \subset [M] \). Even in the finite case, it can happen that the inclusion is proper.

**Example 4.16.** Set \( M = \langle \chi(1) + \chi(2) + \chi(3), 2\chi(3) + \chi(4) + \chi(5) \rangle \) and let \( N = \langle \chi(1) + \chi(2) + \chi(3), \chi(3) + \chi(4) + \chi(5) \rangle \). It is easy to check that \( N \in [M] \backslash \Theta(M) \).

**Lemma 4.17.** For any label \( M \), \( \emptyset \in \Theta(M) \). If \( M \) is nonempty then \( 0 \in \Theta(M) \).

**Proof:** If \( r \not\in M \) then \( M - r = \emptyset \) and so \( \emptyset \in \Theta(M) \). If \( r \in \text{max } M \) then \( M - r = 0 \). In general, let \( r^i \) be maximal in the finite label \( M \land [1, i] \). Clearly \( 0 \in LIMINF\{M - r^i\} \). On the other hand if \( w \in LIMSUP\{M - r^i\} \) then for some \( j \) we have \( \text{supp } w \subset [1, j] \) and, as frequently \( w + r^i \in M \), for \( i \geq j \), \( w + r^i \geq 0 \) implies \( w = 0 \).

**Remark:** Notice that \( M - r = 0 \) iff \( r \in \text{max } M \) and so if \( \text{max } M = \emptyset \) then \( M - r \neq 0 \) for any \( N \)-vector \( r \).

Let \( \Theta'(M) \) be the closure in the space of labels of the set \( \{ M - r : r > 0 \text{ an } N \text{-vector} \} \). Thus, \( \Theta(M) = \Theta'(M) \cup \{M\} \) and \( \Theta'(M) = \beta^* FIN(N)(M) \).

If \( m \in \text{max } M \) then \( \{N : m \in N\} \) is a clopen subset of \( LAB \) which is disjoint from \( \Theta'(M) \). In particular, if \( M \) is of finite type and nonempty then \( M \not\in \Theta'(M) \).

**Definition 4.18.** In general, call \( M \) a **recurrent label** if \( M \in \Theta'(M) \). So \( M \) is recurrent if there exists a sequence \( \{r^i > 0\} \) such that \( M = LIM\{M - r^i\} \) and so for all \( m \in M \) eventually \( m + r^i \in M \).

Clearly, if \( M \) is a recurrent label then \( \text{max } M = \emptyset \). Notice that

\[
(4.8) \quad M = LIM\{M - r^i\} \implies LIMSUP\{\text{supp } r^i\} = \emptyset.
\]
This is because \( \ell \in \text{supp } r^i \) implies \( \rho(M)(\ell) \chi(\ell) \in M \setminus (M - r^i) \). So it cannot happen that \( \ell \in \text{supp } r^i \) infinitely often.

Define

\[
(4.9) \quad \text{ISO}(M) = \{ Q \in \beta^*\text{FIN}(\mathbb{N}) : Q(M) = M \}
\]

Clearly, \( 0 \in \text{ISO}(M) \) and \( M \) is recurrent iff there exists \( Q \in \beta^*\text{FIN}(\mathbb{N}) \) such that \( Q(M) = M \) and so iff \( \text{ISO}(M) \cap \beta^*\text{FIN}(\mathbb{N}) \neq \emptyset \).

**Definition 4.19.** Call \( M \) a strongly recurrent label if \( M \) is infinite and for every \( m \in M \), there is a finite set \( F(m) \) such that \( M - m \supset \{ w \in M : \text{supp } w \cap F(m) = \emptyset \} \).

Call a label \( N \) a strongly recurrent set for a label \( M \) if \( N \) is infinite, \( N \subset M \) and for every \( m \in M \), there is a finite set \( F(m) \) such that \( M - m \supset \{ w \in N : \text{supp } w \cap F(m) = \emptyset \} \) and if \( m \in N \), \( N - m \supset \{ w \in N : \text{supp } w \cap F(m) = \emptyset \} \). Thus, the label \( N \) is strongly recurrent and \( M \) is strongly recurrent iff \( M \) itself is a strongly recurrent set for \( M \).

**Proposition 4.20.** (a) If \( \{m^i\} \) is a strictly increasing infinite sequence in \( M \) then \( \rho \) defined by \( \rho_i = \max_j \{ m^i(\ell) \} \) is an element of \( \mathbb{Z}_+^\mathbb{N} \) with \( \rho \leq \rho(M) \) and \( N = \langle \rho \rangle = \bigcup_i \{ \rho^i \} \) is a strongly recurrent label with \( N \subset M \) and \( m^i \in N \) for all \( i \).

(b) For any label \( M \) \( \text{ISO}(M) \) is a closed submonoid of \( \beta^*\text{FIN}(\mathbb{N}) \) such that for \( Q_1, Q_2 \in \beta^*\text{FIN}(\mathbb{N}) \) the product \( Q_1 Q_2 \) is in \( \text{ISO}(M) \) iff both \( Q_1 \) and \( Q_2 \) are in \( \text{ISO}(M) \).

(c) A label \( N \) is recurrent iff there exists an idempotent \( Q \in \beta^*\text{FIN}(\mathbb{N}) \) such that \( Q(N) = N \). If \( M \) is any label and \( Q \) is an idempotent in \( \beta^*\text{FIN}(\mathbb{N}) \) then \( Q(M) \) is a recurrent element of \( \Theta(M) \). In particular, if \( M \) is of finite type then \( Q(M) = \emptyset \).

(d) If \( N \) is any nonempty recurrent label with \( N \subset M \), then there is a recurrent label \( M_\infty \in \Theta(M) \) such that \( N \subset M_\infty \).

(e) If \( M \) is a recurrent label then \( M - r \) is a recurrent label for any \( \mathbb{N} \)-vector \( r \).

(f) If \( N \) is a strongly recurrent set for \( M \) then for every net \( \{r^i\} \) of elements of \( N \) such that \( \text{LIMSUP}_i \{ \text{supp } r^i \} = \emptyset \), \( M = \text{LIM}_i \{ M - r^i \} \) and \( N = \text{LIM}_i \{ N - r^i \} \). In particular, \( M \) is recurrent if it has a strongly recurrent set.

(g) An infinite label \( N \) is strongly recurrent iff for every sequence \( \{r^i\} \) of elements of \( N \) such that \( \text{LIMSUP}_i \{ \text{supp } r^i \} = \emptyset \), \( N = \text{LIM}_i \{ N - r^i \} \).

(h) If label \( M \) is recurrent then there exists an infinite set \( L \subset \mathbb{N} \) such that \( \langle \chi(L) \rangle = \{ \emptyset \} \cup \{ \chi(F) : F \in \mathcal{P}_f(L) \} \) is a strongly recurrent set for \( M \).
(i) For any nonempty label $M$ the following conditions are equivalent.

1. If $m_1, m_2 \in M$ with disjoint supports then $m_1 + m_2 \in M$.
2. $M$ is a sublattice of $FIN(N)$, i.e. if $m_1, m_2 \in M$ then $m_1 \lor m_2 \in M$.
3. $M = \langle \rho \rangle$ for some $\rho \in \mathbb{Z}_+^N$.
4. $M = \langle \rho(M) \rangle$.

When these conditions hold, $\text{Supp } M$ f-contains $\text{supp } \rho(M)$. If, in addition, $M$ is an infinite label, then it is strongly recurrent.

**Proof:**

(a) Let $r^i = m^{i+1} - m^i$. If $m \leq m^j$ and $i \geq j$ then $m + r^i \in N$. That is, $N = \text{LIM}\{N - r^i\}$. Thus, $N$ is recurrent. It is strongly recurrent by (g), proved below.

(b) It is clear that $ISO(M)$ is a closed subsemigroup. For $Q, P \in \beta FIN(N)$ if $Q(M) \neq M$ then $N = Q(M)$ is a proper subset of $M$ and so $PQ(M) = P(N) \in [[N]]$ and so it, too, is a proper subset of $M$. Also, $P(M) \subset M$ and so $QP(M) \subset Q(M) = N$, also a proper subset of $M$. Thus, $\{Q \in \beta FIN(N) : Q(M) \neq M\}$ is a two-sided ideal (though it is not closed when $M$ is recurrent). It follows that $Q_1Q_2(M) = M$ implies $Q_1(M) = M$ and $Q_2(M) = M$. The converse is true because $ISO(M)$ is a semigroup.

(c) $ISO(M) \cap \beta^* FIN(N)$ is a closed subsemigroup of $\beta^* FIN(N)$ which is nonempty iff $N$ is recurrent. In that case, the subsemigroup contains an idempotent which must lie in $\beta^{**} FIN(N)$. If $Q \in \beta^* FIN(N)$ is an idempotent then $Q(Q(M)) = Q(M)$ is recurrent and lies in $\Theta(M) = \beta^* FIN(N)(M)$. If $M$ is of finite type and $N \in [[M]]$ is nonempty then $N$ is of finite type and so $\text{max } N$ is nonempty. Thus, $\emptyset$ is the only recurrent label in $[[M]]$.

(d) Since $N$ is recurrent, there exists $Q$ an idempotent in $\beta^{**} FIN(N)$ such that $Q(N) = N$. Since $N \subset M$, $N = Q(N) \subset Q(M)$ which is recurrent by (c).

(e) If $Q(M) = M$, then $Q(P_r(M)) = P_r(Q(M)) = P_r(M)$ and so $P_r(M)$ is recurrent.

(f) For any $m \in M$, $m \in M - r^i$ as soon as $F(m) \cap \text{supp } r^i = \emptyset$ which happens eventually.

(g) If $M$ is infinite but not strongly recurrent then there exists $m \in M$ and for every $F$ finite subset of $N$ there exists $n \in M$ with $\text{supp } n \cap F = \emptyset$ but with $m + n \notin M$. Note that this implies $n > 0$.

Let $F_1 = \text{supp } m$ and choose a positive $r_1 \in M$ with support disjoint from $F_1$ and is such that $m + r^i \notin M$. Let $F_2 = F_1 \cup \text{supp } r_1$. Inductively, we build an increasing sequence of finite sets $\{F^i\}$ and positive elements $r^i \in M$ such that $\text{supp } r^i \subset F^{i+1} \setminus F^i$ and $m + r^i \notin M$. 


Since the supports are disjoint, \( LIM SUP \{ supp \ r^i \} = \emptyset \). Because \( m \not\in LIM \{ M - r^i \} \), the limit is not \( M \).

(h) Since \( M \) is recurrent there exists a sequence \( \{ r^i > 0 \} \) be such that \( M = LIM \{ M - r^i \} \). It follows from (4.8) that \( \bigcup_i supp \ r^i \) is infinite. Choose \( \ell_1 \in supp \ r^i \) and let \( N_1 = \{ \chi(\ell_1) \} \cup M \cap [1, 1] \). There exists \( r^{i_2} \) with \( r^{i_2} > 0 \) and \( \ell_2 \) not in the support of a member of \( N_1 \) and is such that \( m + r^{i_2} \in M \) for all \( m \in N_1 \). Let \( N_2 = \{ m + \chi(\ell_2) : m \in N_1 \} \cup M \cap [1, 2] \). Inductively, we can choose \( \ell_{k+1} \) such that \( m + \chi(\ell_{k+1}) \in M \) for all \( m \in N_k \) and with \( \ell_{k+1} \) not in the support of any member of \( N_k \). Let \( N_{k+1} = \{ m + \chi(\ell_{k+1}) : m \in N_k \} \cup M \cap [1, k+1] \). By the inductive construction if \( m \in N_{k-1} \) then \( m + \Sigma_{i=k}^{\ell} \chi(\ell_i) \in M \) for all \( j \geq k \). With \( m = 0 \) this says that \( \Sigma_{i=1}^{\ell} \chi(\ell_i) \in M \) for all \( j \) and so \( Supp \ M \) f-contains \( L = \{ \ell_k \} \) and \( \langle \chi(L) \rangle \) is a strongly recurrent set for \( M \).

\[ \begin{align*}
(1) \implies (4) & \implies (3) \implies (2) : \text{Obvious.} \\
(2) \implies (1) : \text{If } m_1 \text{ and } m_2 \text{ have disjoint supports then } m_1 \lor m_2 = m_1 + m_2. \\
(1) \implies (4) : \text{For any } M, \rho(M) \chi(\ell) \in M. \text{ So if } m \leq \rho(M) \text{ then } (1) \text{ (and induction) implies that } m \leq \Sigma \{ \rho(M) \chi(\ell) : \ell \in supp \ m \} \text{ is in } M. \\
\end{align*} \]

If \( L = supp \rho(M) \) then \( \chi(L) \leq \rho(M) \) and so \( \langle \chi(L) \rangle \subset \langle \rho(M) \rangle \). This implies that \( Supp \langle \rho(M) \rangle \) f-contains \( L \).

Now assume that \( M \) is infinite. For any \( m \in M \) letting \( F(m) = supp \ m \) and apply (i) to get the strong recurrence condition.

\[ \square \]

**Remark 4.21.** Let \( FIN^\ell \ = \{ m \in FIN(N) : supp \ m \cap [1, \ell] = \emptyset \} \). It is clear from Proposition 4.20 (f) that , when \( N \) is a strongly recurrent set for \( M \), \( \bigcap_\ell \ \bar{N} \cap FIN^\ell \subset ISO(M) \cap ISO(N) \), where the closure is taken in \( \beta FIN(N) \). One can prove, by using (4.8), that the intersection is equal to \( ISO(N) \).

**Corollary 4.22.** (a) A label \( M \) is of finite type iff \( \emptyset \) is the only recurrent label contained in \( M \), iff \( \emptyset \) is the only recurrent label in \( \Theta(M) \).

(b) For a label \( M \), \( \max M = \emptyset \) iff \( M \) is the union of the recurrent labels contained in \( M \). Thus:

\[ \{ M : \text{finite type} \} \subset \{ M : \max M \neq \emptyset \} = \{ M : \text{not a union of recurrent labels} \} \subset \{ M : \text{not recurrent} \}. \]

**Proof:** (a): If \( N \) is a nonempty recurrent label then it is not of finite type and so if \( N \subset M \) then \( M \) is not of finite type. If \( M \) is not of finite
type then it contains a strictly increasing infinite sequence and so by Proposition 4.20 (a) it contains a recurrent label. For the second claim apply Proposition 4.20.(d).

(b) If \( m \in \max M \) and \( m \in N \subset M \) then \( m \in \max N \) and so \( N \) is not recurrent. That is, \( \max M \) is disjoint from all nonempty recurrent labels contained in \( M \).

If \( \max M = \emptyset \) and \( m \in M \) then inductively we can define a strictly increasing sequence \( \{m^i\} \) in \( M \) with \( m = m^1 \). By Proposition 4.20 there is a recurrent label \( N \subset M \) with \( m \in N \).

\[ \square \]

**Corollary 4.23.** If \( M \) is a nonempty recurrent label then \( \Theta(M) \) is a Cantor set. If \( M \) is a label not of finite type, then \( \Theta(M) \) is uncountable.

**Proof:** If \( M \) is a nonempty recurrent label then for some sequence \( \{r^i\} \) of positive elements of \( M \), \( \{M - r^i\} \) converges to \( M \). Since each of the \( M - r^i \) is a proper subset of \( M \) and each lies in \( \Theta(M) \) it follows that \( M \) is not an isolated point of \( \Theta(M) \). For each \( r \in M \), the label \( M - r \) is nonempty and it is recurrent by Proposition 4.20(e). Hence, \( M - r \) is not isolated in \( \Theta(M - r) \subset \Theta(M) \). It follows that \( \{M - r : r \in M\} \) is a dense subset of \( \Theta(M) \) no point of which is isolated and so no point of \( \Theta(M) \) is isolated. On the other hand, \( \Theta(M) \) is a nonempty, compact, ultra-metric space and so it is a Cantor set.

If \( M \) is not of finite type then there exists a nonempty recurrent \( N \in \Theta(M) \) by Corollary 4.22(a). Hence, \( \Theta(M) \) contains the Cantor set \( \Theta(N) \).

\[ \square \]

**Corollary 4.24.** For any label \( M \), \( M \) is the only \( \text{FIN}(\mathbb{N}) \)-transitive point in \( \Theta(M) \).

**Proof:** Suppose \( N \in \Theta(M) \) is a transitive point. Then there are \( P, Q \in \beta \text{FIN}(\mathbb{N}) \) with \( P(M) = N \) and \( Q(N) = M \). Thus \( QP \in \text{ISO}(M) \) and it follows, by Proposition 4.20(b), that \( N = M \).

\[ \square \]

**Remark 4.25.** Note the sharp contrast with the case where the acting semigroup is a group. In fact, for a dynamical system \((X,G)\), where \( X \) is a compact metric space and \( G \) is a group, the existence of one dense orbit implies that the set \( X_{tr} \) of transitive points forms a dense \( G_{\delta} \) subset of \( X \).
Example 4.26. It can happen that $\Theta(M)$ is uncountable even with $M$ of finite type.

Define a bijection $w \mapsto \ell_w$, from the set of finite words on the alphabet $\{0, 1\}$ onto $\mathbb{N}$. For $x \in \{0, 1\}^N$ let $w_k(x)$ be the initial word of length $k$ in $x$, that is, $w_k(x) = x_1x_2 \ldots x_k$, for $k \in \mathbb{N}$. Let

\[(4.10)\]

\[M_x = \{ \ell_{w_k(x)} : k \in \mathbb{N} \} \]
\[M_x = \{ 0 \} \cup \{ \chi(\ell) : \ell \in M_x \}, \]
\[M_x^{(2)} = M_x \oplus M_x = \{ \chi(\ell^1) + \chi(\ell^2) : \ell^1, \ell^2 \in M_x \} \cup M_x, \]
\[M = \bigcup \{ M_x^{(2)} : x \in \{0, 1\}^N \}. \]

Since $\rho(M) \leq 2$ and the size of the elements of $M$ are bounded by 2, it follows from Lemma 4.3 that $M$ is a label of finite type. Notice that if $x \neq y$ in $\{0, 1\}^N$ then $M_x \cap M_y$ is finite. It is easy to see that for each $x \in \{0, 1\}^N$, $LIM \{ M - \chi(\ell_{w_i(x)}) \} = M_x$ and it follows that $\Theta(M)$ is uncountable.

If $\Phi$ is a compact, invariant subset of $LAB$ then the action of $FIN(\mathbb{N})$ restricts to an action on $\Phi$. Since $\Phi^\Phi$ is an Ellis semigroup with an Ellis action on $\Phi$, the closure in $\Phi^\Phi$ of $\{ P_r : r \in FIN(\mathbb{N}) \}$, denoted $E(\Phi)$, is an Ellis semigroup with an Ellis action on $\Phi$. We call $E(\Phi)$ the *enveloping semigroup* of $\Phi$. A map $Q$ on $\Phi$ is an element of $E(\Phi)$ iff for every finite sequence $\{ M^i \}$ in $\Phi$ and any $\ell \in \mathbb{N}$ there exists $r \in FIN(\mathbb{N})$ such that $P_r(M^i) \cap [1, \ell] = Q(M^i) \cap [1, \ell]$ for all $i$. It follows that for $N \in \Phi, \ N_1 = Q(N)$ for some some $Q \in E(\Phi)$ iff $N_1$ is the limit of some sequence $\{ N - r^i \}$. Notice that this does not necessarily imply that $Q$ is a pointwise limit of some sequence $\{ P_r \}$.

The adherence semigroup $A(\Phi)$ is the closure in $\Phi^\Phi$ of $\{ P_r : r > 0 \}$. It follows that

\[(4.11)\]

$E(\Phi) = A(\Phi) \cup \{ P_0 = id_\Phi \}, \quad \Theta(M) = E(\Phi)M, \quad \Theta'(M) = A(\Phi)M,$

for $M \in \Phi$.

If $\Phi_1 \subset \Phi$ is also closed and invariant then the restriction map defines a continuous, surjective homomorphism from $E(\Phi)$ to $E(\Phi_1)$.

The homomorphism from $FIN(\mathbb{N})$ into $E(\Phi)$ extends to an Ellis semigroup homomorphism $\beta FIN(\mathbb{N}) \to E(\Phi)$ with $\beta^* FIN(\mathbb{N})$ mapping onto $A(\Phi)$.

The sequences $\{ M^i \}$ that we will find most useful will be given by $M^i = M - r^i = P_{r^i}(M)$ for a sequence $\{ r^i \}$ of $\mathbb{N}$-vectors. Such a
sequence is bounded by $M$ and $m \in \text{LIMSUP}$ iff frequently $m + r^i \in M$ and $m \in \text{LIMINF}$ iff eventually $m + r^i \in M$.

If $r^i > 0$ for all $i$ then $\max M$ is disjoint from $\text{LIMSUP}$.

If $r^i \in M$ for all $i$ then $0 \in \text{LIMINF}$ and so $\text{LIMINF} \neq \emptyset$.

**Lemma 4.27.** Assume that $\{M - r^i\}$ is convergent.

(a) Either eventually $r^i = 0$ and so $\text{LIM} \{M - r^i\} = M = M - 0$ or eventually $r^i > 0$ and max $M \cap \text{LIM} \{M - r^i\} = \emptyset$.

(b) Either eventually $r^i \in M$ and $0 \in \text{LIM} \{M - r^i\}$ or eventually $r^i \not\in M$ and $\text{LIM} \{M - r^i\} = \emptyset$.

(c) If $M - m_1 = M - m_2$ then $\text{LIM} \{M - r^i\} - m_1 = \text{LIM} \{M - r^i\} - m_2$.

(d) If there exists $r$ such that for infinitely many $i$, $M - r^i = M - r$ then the limit is $M - r$.

**Proof:** (a) Since $r > 0$ implies $\max M \cap (M - r) = \emptyset$, if $r^i = 0$ infinitely often then convergence implies that the limit is $M$ and so an element of $\max M$ is eventually in $M - r^i$ which can only happen when $r^i$ is eventually 0.

(b) Since $0 \in M - r$ iff $r \in M$ we see that if $r^i \in M$ infinitely often then convergence implies that eventually $0 \in M - r^i$ and so eventually $r^i \in M$.

(c) Since $P_{m_1}$ and $P_{m_2}$ are continuous,

$$\text{LIM} \{M - r^i\} - m_1 = \text{LIM} \{M - m_1 - r^i\} = \text{LIM} \{M - m_2 - r^i\} = \text{LIM} \{M - r^i\} - m_2.$$  

(4.12)

(d) By assumption there is a subsequence $r'^i$ such that $M - r'^i$ is constant at $M - r$ and so converges to $M - r$. By the assumption of convergence the limit of the original sequence is $M - r$.

\[ \square \]

**Remark:** (c) has the following interpretation: If $M - m_1 = M - m_2$ then as elements of $\mathcal{E} (\Theta (M))$, $P_{m_1} = P_{m_2}$ because the set of $N \in \Theta (M)$ on which they agree is closed and invariant and includes $M$.

**Corollary 4.28.** Assume that $M$ is a label and $\{r^i\}$ is a sequence of $\mathbb{N}$-vectors such that $\{M - r^i\}$ is convergent. If $\bigcup_i \text{supp } r^i$ is finite then the sequence $\{M - r^i\}$ is eventually constant. That is, there exists $I \in \mathbb{N}$ such that for $i, j \geq I$:

$$M - r^i = M - r^j,$$

(4.13)

and this common value is the limit.
Proof: If $\bigcup_i supp r^i$ is finite then by the Bound Condition the set of vectors $\{r^i\}$ is finite. There exists $I$ such that for $i \geq I$ each $r^i$ occurs infinitely often in the sequence. It follows from Lemma 4.27 (d) the limit is $M - r^i$ for all $i \geq I$. As the limit is unique, $M - r^i = M - r^j$ for all $i, j \geq I$.

Lemma 4.29. Let $M$ be a label. Assume that $\{r^i\}$ and $\{s^j\}$ are sequences of $\mathbb{N}$-vectors such that $\{M - r^i\}$ and $\{M - s^j\}$ are convergent with $LIM \{M - r^i\} = M - r$ for some $\mathbb{N}$-vector $r$ and $LIM \{M - s^j\}$ denoted $LIM_s$. If either $\{M - r^i\}$ is eventually constant at $r$ or $LIM_s = M - s$ for some $\mathbb{N}$-vector $s$, then

$$LIM_{i \to \infty} LIM_{j \to \infty} \{M - r^i - s^j\} = LIM_s - r$$

$$= LIM_{j \to \infty} LIM_{i \to \infty} \{M - r^i - s^j\}.$$  

(4.14)

Proof: By Proposition 4.12 applied twice $LIM_{i \to \infty} \{M - r^i - s^j\} = M - r - s^j$ and this sequence converges to $LIM_s - r$. If $LIM_s = M - s$ so that $LIM_s - r = M - s - r$, we similarly, $LIM_i LIM_j = M - r - s$. On the other hand, if eventually $M - r^i = M - r$ we can omit terms and assume this is true for all $i$. Then by 4.12 $LIM_{j \to \infty} \{M - r^i - s^j\} = LIM_s - r^i = LIM_s - r$. That is the sequence $\{LIM_i - r^i\}$ is eventually constant at $LIM_s - r$. Hence, in both cases $LIM_i LIM_j = LIM_s - r = LIM_j LIM_i$.

4.1. Finitary and simple labels.

Call a label $M$ finitary when it satisfies

(iv) [Finitary condition] Whenever $\{S_i\}$ is a sequence of finite subsets of $\mathbb{N}$ with $\bigcup_i S_i$ infinite, there are only finitely many subsets $S$ of $\mathbb{N}$ such that eventually $S \cup S_i \in Supp \ M$.

Clearly, if $M_1 \subset M$ is a label then $M_1$ is finitary if $M$ is.

For a label $M$ and $N$ a nonempty set of $\mathbb{N}$-vectors we define

$$M - N = \{ m : m + r \in M \text{ for all } r \in N \} = \bigcap_{r \in N} M - r.$$  

(4.15)

Proposition 4.30. Let $M$ be a label.

(a) The following conditions are equivalent.

(i) $M$ is finitary.
(ii) If \( \{ r^i \} \) is a sequence of \( \mathbb{N} \)-vectors with \( \bigcup_i \text{supp } r^i \) infinite then \( \text{LIMINF } \{ M - r^i \} \) is finite.

(iii) If \( r^i \) is a sequence of \( \mathbb{N} \)-vectors with \( \bigcup_i \text{supp } r^i \) infinite and \( \{ M - r^i \} \) convergent then then \( \text{LIM } \{ M - r^i \} \) is finite.

(iv) If \( N \) is infinite then \( M - N \) is finite, and there is no strictly increasing sequence of members of \( \{ M - N : N \text{ infinite } \} \).

(b) If \( M \) is finitary then it is of finite type.

(c) If \( M \) is finitary, then the following conditions on a finite subset \( F \) of \( M \) are equivalent.

(i) There exists \( r^i \) a sequence of finite vectors with \( \bigcup_i \text{supp } r^i \) infinite such that \( \text{LIM } \{ M - r^i \} = F \).

(ii) There exists \( r^i \) a sequence of distinct \( \mathbb{N} \)-vectors such that \( \text{LIM } \{ M - r^i \} = F \).

(iii) There is an infinite set \( N \) such that \( F = M - N_1 \) for every infinite subset \( N_1 \) of \( N \).

Proof: If \( N \subseteq M \) then by the Bound Condition \( N \) is infinite iff \( \bigcup \text{Supp } N \) is infinite. Hence, if \( \{ r^i \} \) is a sequence with \( \bigcup_i \{ \text{supp } r^i \} \) infinite then we can choose a subsequence of distinct vectors.

(a) (i) \(\Rightarrow\) (ii): If \( m \in \text{LIMINF } \) then eventually \( \text{supp } m \cup \text{supp } r^i \in \text{Supp } \ M \). Because \( M \) is finitary there are only finitely many such sets \( \text{supp } m \) and by the Bound Condition there are only finitely many \( m \in M \) with such supports.

(ii) \(\Leftrightarrow\) (iii): If \( \{ r^i \} \) is a sequence with \( \bigcup_i \{ \text{supp } r^i \} \) infinite then we can choose a subsequence of distinct vectors and then go to a further subsequence \( \{ r'' \} \) which is convergent. Assuming (iii) \( \text{LIM } \{ M - r'' \} = \text{LIMINF } \{ M - r'' \} \supset \text{LIMINF } \{ M - r' \} \) is finite. This shows that (iii) implies (ii). The converse is obvious.

(ii) \(\Rightarrow\) (iv): Suppose that \( \{ F^i \} \) is a nondecreasing sequence of subsets of \( M \) with each \( M - F^i \) finite. Inductively, choose \( r^i \in M - F^i \) distinct from the \( r^j \)'s with \( j \neq i \). Since the sequence \( \{ F^i \} \) is a nondecreasing, \( F^i \subset M - r^i \) for \( j \leq i \). Hence, \( \bigcup_i \text{F}^i \subset \text{LIMINF } \{ M - r^i \} \) and so it is finite by (ii). Thus, the sequence \( \{ F^i \} \) is eventually constant.

(iv) \(\Rightarrow\) (iii): By going to a subsequence we can assume that \( \{ r^i \} \) is a sequence of distinct elements. Then \( \bigcap_{j \geq i} \{ M - r^j \} \) is a nondecreasing sequence in \( \{ M - N : N \text{ infinite } \} \). Hence, its union, which is the limit, is finite by (iv).

(b) Assume that \( \{ m^i \} \) is a strictly increasing sequence in \( M \) and that \( r^i = m^{i+1} - m^i \). Since \( M \wedge [1, \ell] \) is finite and so is of finite type, it follows that \( \bigcup_i \{ \text{supp } m^i \} \) is infinite and hence so is \( \bigcup_i \{ \text{supp } r^i \} \). Since each \( m^i \in \text{LIMINF } \{ M - r^i \} \) it follows that \( M \) is not finitary.
(c) (i) $\iff$ (ii): This is obvious from our initial remarks.
(ii) $\Rightarrow$ (iii): Assume $r^i$ a sequence of $\mathbb{N}$-vectors with $LIM\{M-r^i\} = F$. By discarding finitely many terms $r^i$ we can assume that the finite set $F$ equals $M-N$ with $N = \{r^i\}$. If $N_1$ is any infinite subset of $N$ then $F = M - N \subset M - N_1$. On the other hand, if $m \in M - N_1$ then $m \in M - r^i$ for infinitely many $i$ and so by convergence $m \in LIM = F$.
(iii) If $N = \{r^1, r^2, \ldots\}$ is the infinite set given by (iii) then $\{r^i\}$ is a sequence of distinct elements with $F = LIM\{M-r^i\}$.
\begin{flushright}
\Box\end{flushright}

We will call $F$ an external limit set (or an external label) for a finitary label $M$ when it is a limit of a sequence $\{M-r^i\}$ for some sequence of distinct vectors $\{r^i\}$ in $M$ (and so $0 \in F$).

The value of the assumption that a label is finitary will come from the following result.

Lemma 4.31. Assume that $\{r^i\}$ and $\{s^j\}$ are sequences of $\mathbb{N}$-vectors such that $\{M-r^i\}$ and $\{M-s^j\}$ are convergent and with $LIM\{M-r^i\}$ and $LIM\{M-s^j\}$ are both finite. If $\bigcup_j supp\ s^j$ is infinite then for sufficiently large $i$, $(LIM\{M-s^j\}) - r^i = \emptyset$.

\textbf{Proof:} Suppose for some $w \in LIM_s = LIM\{M-s^j\}$ that $w - r^k \geq 0$ for $k$ in some infinite subset $SEQ$ of $\mathbb{N}$. For $k \in SEQ$, $0 \leq r^k \leq w$ and so each such $r^k \in LIM_s$. Since $LIM_s$ is finite there must be a $w_1 \in LIM_s$ such that $r^k = w_1$ for $k \in SEQ$ an infinite subset of $SEQ$. There exists $J$ such that $j \geq J$ implies $w_1 + s^j \in M$. Thus, for each $j \geq J s^j + r^k \in M$ for all $k \in SEQ$. By convergence of $\{M-r^i\}$, this means $s^j \in LIM_r = LIM\{M-r^i\}$ for all $j \geq J$. Since $LIM_r$ is also finite, this contradicts the assumption that $\bigcup_j supp\ s^j$ is infinite. Finally, as there are only finitely many $w \in LIM_s$ and for each such $w-r^i$ is eventually not nonnegative, it follows that eventually $LIM_s - r^i$ is empty.
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\Box\end{flushright}

This immediately yields
\begin{corollary}
Assume that $\{r^i\}$ and $\{s^j\}$ are sequences of $\mathbb{N}$-vectors such that for a label $M$ both $\{M-r^i\}$ and $\{M-s^j\}$ are convergent and both $LIM\{M-r^i\}$ and $LIM\{M-s^j\}$ are finite. If $\bigcup_i supp\ s^j$ and $\bigcup_i supp\ r^i$ are both infinite then
\begin{equation}
LIM_{i \to \infty}LIM_{j \to \infty}\{M-r^i-s^j\} = LIM_{j \to \infty}LIM_{i \to \infty}\{M-r^i-s^j\} = \emptyset.
\end{equation}
\begin{flushright}
\Box\end{flushright}

Call a label $M$ *simple* when it satisfies

(v) [Convergence Condition] If $\{r^i\}$ is a sequence of vectors in $M$ such that the sequence $\{M - r^i\}$ is convergent then $\text{LIM} \{M - r^i\} = M - r$ for some $\mathbb{N}$-vector $r$.

It follows from Corollary 4.28 that any finite label is simple as well as finitary.

**Proposition 4.33.** Let $M$ be a label.

(a) If $M$ is simple then $M - r$ is simple for all $\mathbb{N}$-vectors $r$.

(b) If $M$ is simple then it is of finite type.

(c) $M$ is simple iff $\Theta(M) = \{ M - r : r \in \text{FIN}(\mathbb{N}) \}$.

(d) If $M$ is simple then $\mathcal{E}(\Theta(M)) = \{ P_r : r \in \text{FIN}(\mathbb{N}) \}$ or, more precisely, the restrictions of these maps to $\Theta(M)$. Thus, $\mathcal{E}(\Theta(M))$ is an abelian semigroup whose members act continuously on $\Theta(M)$.

**Proof:** (a): If $\{M - r - s^j\}$ is a convergent sequence, then we can choose a convergent subsequence $\{M - r - s^{j'}\}$ with limit $\text{LIM} M - s^{j'}$. Because $M$ is simple, $\text{LIM} M - s^{j'} = M - s$ for some $s$. By Lemma 4.29 the subsequence $\{M - r - s^{j'}\}$ converges to $M - r - s$ and so this is the limit of the full convergent sequence $\{M - r - s^j\}$.

(c): Since $\Theta(M)$ consists exactly of the limits of convergent sequences $\{M - r^i\}$, the equivalence is clear.

(b) If $M$ is not of finite type then by Corollary 4.22 there exists $N \in \Theta(M)$ which is recurrent. Since $M$ is recurrent, $\text{max} N = \emptyset$. By Lemma 4.17 and the Remark thereafter $0 \in \Theta(N)$ but $0 \neq N - r$ for any $\mathbb{N}$-vector $r$. Hence, $N$ is not simple. Since $N \in \Theta(M)$, (a) and (c) imply that $M$ is not simple.

(d) We observe first that if $\{M^i\}$ is a net converging to some $N$, then there is a sequence of elements $\{M^{i'}\}$ which converges to $N$, although not necessarily a subnet. Now let $Q \in \mathcal{E}(\Theta(M))$ with $M$ of finite type. There is a net $\{P_{r_i}\}$ which converges pointwise to $Q$. For any $\mathbb{N}$-vector $r$ we can choose a sequence $\{i'\}$ so that $\{P_{r_{i'}}(M)\}$ converges to $Q(M)$ and $\{P_{r_{i'}}(M - r)\}$ converges to $Q(M - r)$. Because $M$ is simple, there exists $s$ such that $M - s = \text{LIM} \{M - r^{i'}\} = Q(M)$. Then Lemma 4.29 implies that $M - r - s = \text{LIM} \{M - r - r^{i'}\} = Q(M - r)$. That is, $Q = P_s$ on $\Theta(M)$.

From Corollary 4.13 it follows that $\mathcal{E}(\Theta(M))$ is abelian and acts continuously on $\Theta(M)$. Notice that $r \mapsto P_r$ is a homomorphism from the discrete monoid $\text{FIN}(\mathbb{N})$ onto the compact monoid $\mathcal{E}(\Theta(M))$. 

\qed
If the sequence \( \{ M - r^i \} \) is eventually constant then, of course, the limit is of the form \( M - r \). However, it can happen that \( \{ r^i \} \) is a sequence of distinct vectors in \( M \setminus \max M \) such that \( 0 = \text{LIM} \{ M - r^i \} \). In that case, the sequence is not eventually constant but the limit is \( M - r \) for \( r \in \max M \). Notice that because \( M \) is of finite type by (b), \( \max M \neq \emptyset \).

For labels \( M_1 \) and \( M_2 \) define \( M_1 \oplus M_2 = \{ m_1 + m_2 : m_1 \in M_1, m_2 \in M_2 \} \). Clearly, this is \( \emptyset \) if either is \( \emptyset \). If neither term is empty then (4.17) \( \rho(M_1 \oplus M_2) = \rho(M_1) + \rho(M_2) \).

and so \( M_1 \oplus M_2 \) is a label. Observe that for any \( \ell \in \mathbb{N} \), we have \( (M_1 \oplus M_2) \wedge [1, \ell] = (M_1 \wedge [1, \ell]) \oplus (M_2 \wedge [1, \ell]) \). It follows that the map \( \oplus : \text{LAB} \times \text{LAB} \to \text{LAB} \) is continuous.

Lemma 4.34. (a) If \( M \) is a recurrent label, then \( M_1 \oplus M \) is a recurrent label for any label \( M_1 \). If \( M \) is a strongly recurrent label, then \( M_1 \oplus M \) is a strongly recurrent label for any finite label \( M_1 \).

(b) If \( M \) is any label and \( \ell \in \mathbb{N} \) then there exists a recurrent label \( M_1 \) with \( M \subset M_1 \) and \( M \wedge [1, \ell] = M_1 \wedge [1, \ell] \). There exists a strongly recurrent label \( M_2 \) with \( M \wedge [1, \ell] = M_2 \wedge [1, \ell] \).

Proof: (a) Clearly \( M_1 \oplus (M - r) \subset (M_1 \oplus M) - r \) for any \( r \in M \). Hence, if \( \{ M - r^i \} \to M \) then \( \{ (M_1 \oplus M) - r^i \} \to M_1 \oplus M \) and so \( M_1 \oplus M \) is recurrent.

If \( m_1 \in M_1 \) and \( m \in M \) we let \( F(m_1 + m) = \bigcup \text{Supp} M_1 \cup F(m) \) (see Definition 4.19). If \( r_1 + r \) has support disjoint from this set then \( r_1 = 0 \) and \( r + m \in M \). Hence, \( r_1 + r + m \in M_1 \oplus M \).

(b) Build a recurrent label \( N_1 \) with \( N_1 \wedge [1, \ell] = 0 \). Then \( M_1 = M \oplus N_1 \) is a recurrent label with \( M \wedge [1, \ell] = M_1 \wedge [1, \ell] \). If you start with a strongly recurrent label \( N_2 \) with \( N_2 \wedge [1, \ell] = 0 \) then \( M_2 = (M \wedge [1, \ell]) \oplus N_2 \) is a strongly recurrent label.

Example 4.35. If \( L \) is an infinite set disjoint from \( \bigcup \text{Supp} M \). Let \( M_1 = \{ \emptyset \} \cup \{ \chi(\ell) : \ell \in L \} \). If \( M \) is strongly recurrent then \( M_1 \oplus M \) is recurrent by (a) but it is not strongly recurrent.

\[ \square \]

Proposition 4.36. The set \( \text{RECUR} \) of recurrent labels is a dense, \( G_\delta \) subset of \( \text{LAB} \).
Proof: For any $\ell \in \mathbb{N}$, the set of pairs $\{ (M, N) : M \land [1, \ell] \subset N \}$ is a clopen subset of $\mathcal{LAB} \times \mathcal{LAB}$. Since the map $P_r$ is continuous, the equation

$$RECUR = \bigcap_{\ell \in \mathbb{N}} \bigcup_{r \in \text{FIN}} \{ M : M \land [1, \ell] \subset M - r \}$$

implies that $RECUR$ is a $G_\delta$ subset of $\mathcal{LAB}$. It is dense by Lemma 4.34 (b).

Thus, the labels of finite type, upon which we focus most of our attention, comprise a subset of first category. Since the finite labels are dense, the labels of finite type are dense. Thus, the set of recurrent labels has empty interior.

We will need a combinatorial lemma. Let $M$ be a collection of finite subsets of $\mathbb{N}$. Call it hereditary if $A \subset B$ and $B \in M$ implies $A \in M$. We say that $M$ f-contains $L \subset \mathbb{N}$ if every finite subset of $L$ is in $M$, i.e. $\mathcal{P}_f L \subset M$. For two hereditary collections $M_1, M_2$ define $M_1 \oplus M_2 = \{ A \cup B : A \in M_1, B \subset M_2 \}$.

Lemma 4.37. Let $M_1$ and $M_2$ be hereditary collections of finite subsets of $\mathbb{N}$. If $M_1 \oplus M_2$ f-contains an infinite set then either $M_1$ or $M_2$ f-contains an infinite set.

Proof: Let $L = \{ \ell_1, \ell_2, \ldots \}$ be a counting for an infinite set f-contained in $M_1 \oplus M_2$. Define the directed binary tree with vertices at level $n = 0, 1, \ldots$ consisting of the $2^n$ ordered partitions $(A, B)$ of $\{ \ell_1, \ldots, \ell_n \}$. Connect $(A, B)$ to the $n + 1$ level vertices $(A \cup \{ \ell_{n+1} \}, B)$ and $(A, B \cup \{ \ell_{n+1} \})$. The set of paths to infinity form a Cantor set. Call a path good at level $n$ if for the partition $(A_n, B_n)$ at level $n$, $A_n \in M_1$ and $B_n \in M_2$. Since $M_1 \oplus M_2$ f-contains $L$, the set $G_n$ of paths good at level $n$ is nonempty. Each $G_n$ is closed and $G_{n+1} \subset G_n$. So the intersection contains a path $\{(A^i, B^i) : i = 0, 1, \ldots \}$. Let $A^\infty = \bigcup A^i, B^\infty = \bigcup B^i$. Clearly, $\{A^i\}$ is a nondecreasing sequence of finite sets in $M_1$ with union $A^\infty$ and so $A^\infty$ is contained in $M_1$. Similarly, $B^\infty$ is contained in $M_2$. Since $A^\infty \cup B^\infty = L$, at least one of them is infinite.

For a label $M$, if $m \in M$ then $0 \leq \chi(\text{supp } m) \leq m$ and so $\chi(\text{supp } m) \in M$. Hence, $\text{Supp } M = \{ A \subset \mathbb{N} : \chi(A) \in M \}$. Thus, $\text{Supp } M$ is a hereditary collection of finite subsets of $\mathbb{N}$ with
Supp $M = \emptyset$ iff $M = \emptyset$. Thus, Proposition 4.6 says that $M$ is not of finite type iff $\text{Supp} \ M$ f-contains some infinite subset.

**Theorem 4.38.** Let $N$ be a nonempty label.

(a) Assume $M$ is a nonempty label. The nonempty label $N \oplus M$ is of finite type iff $N$ and $M$ are labels of finite type. If $N \cap M = \emptyset$ and both $N$ and $M$ are simple, then $N \oplus M$ is simple. If $N$ is finite and $M$ is finitary, then $N \oplus M$ is finitary. If $\{M_a\}$ is a finite or infinite sequence of labels of finite type such that $M_a \cap M_b \subset N$ when $a \neq b$, then $M = \bigcup_a M_a$ is a label which is of finite type if all the $M_a$'s and $N$ are of finite type. If $N$ is finite and the $M_a$'s are all finitary then $\bigcup_a M_a$ is finitary. If $N = 0$ and the $M_a$'s are all simple then $\bigcup_a M_a$ is simple.

**Proof:** Notice first that for labels $M_1, M_2$ 

\[
\bigcup \text{Supp}(M_1 \cap M_2) = (\bigcup \text{Supp} \ M_1) \cap (\bigcup \text{Supp} \ M_2).
\]

For if $\ell \in (\bigcup \text{Supp} \ M_1) \cap (\bigcup \text{Supp} \ M_2)$ then $\chi(\ell) \in M_1 \cap M_2$.

With $M$ and $N$ labels with $N$ finite, we let $N^+$ be the finite label consisting of the $N$-vectors $m$ with support contained in $\bigcup \text{Supp} \ N$ and with $m \leq \rho(N) + \rho(M)$. Let $M \oplus N = \{m \in M \mid \text{supp } m \cap \bigcup \text{Supp} \ N = \emptyset\}$. Any finite vector $m \in N \oplus M$ can be written uniquely as $m_N + m_{M \cap N}$ where $m_N \in N^+$ and $m_{M \cap N} \in M \oplus N$.

(a) Assume that $M$ and $N$ are nonempty labels of finite type. If $N \oplus M$ is not of finite type then $\text{Supp} (N \oplus M)$ f-contains an infinite set by Proposition 4.6. Since $\text{Supp} (N \oplus M) = (\text{Supp} \ N) \oplus (\text{Supp} \ M)$, by Lemma 4.37, either $\text{Supp} \ N$ or $\text{Supp} \ M$ f-contains an infinite set. Thus, $N$ and $M$ of finite type implies $N \oplus M$ is of finite type. The converse is obvious since $N \cup M \subset N \oplus M$.

Now define the disjoint sets $L_0 = (\bigcup \text{Supp} \ N) \setminus (\bigcup \text{Supp} \ M)$, $L_1 = (\bigcup \text{Supp} \ M) \setminus (\bigcup \text{Supp} \ N)$ and $L_{01} = (\bigcup \text{Supp} \ N) \cap (\bigcup \text{Supp} \ M)$. If $\text{supp } m \subset (\bigcup \text{Supp} \ N) \cup (\bigcup \text{Supp} \ M)$ then we can decompose $m = m_0 + m_1 + m_{01}$ with $m_i = m \land L_i$ for $i = 0, 1$ or $01$.

Assume that $N$ and $M$ are simple and $N \cap M = \emptyset$ and so $L_{01} = \emptyset$. We have $w \in (N \oplus M) - r$ for $r \in N \oplus M$, iff $w_0 \in N - r_0$ and $w_1 \in M - r_1$. Thus, if $\{(N \oplus M) - r^i\}$ converges with a nonempty limit then the limit is $(LIM \ (N - r_0^i)) \oplus (LIM \ (N - r_1^i))$. Because $N$ and $M$ are simple this is $(N - n) \oplus (M - m) = (N \oplus M) - (n + m)$. Hence, $N \oplus M$ is simple. The empty limit is $(N \oplus M) - r$ with $r \notin (N \oplus M)$.

Now assume $M$ is finitary and $N$ is finite so that $L_0 \cup L_{01}$ is finite. Let $\{r^i\}$ be a sequence in $N \oplus M$ with $\bigcup_i \{\text{supp } r^i\}$ infinite. Then $\bigcup_i \{\text{supp } r^i\}$ is infinite since $\bigcup \text{Supp} \ N = L_0 \cup L_{01}$ is finite. If $m_0 +
\( m_1 + m_0 \downarrow + r^i \in N \oplus M \) then \( m_1 + r^i \in (N \oplus M) \land L_1 \subset M \). Because \( M \) is finitary, there are only finitely many \( m_1 \) such that eventually \( m_1 + r^i \in M \). Since there are only finitely many \( m_0 \)'s and \( m_0 \)'s it follows that \( \text{LIMINF}\{N \oplus M - r^i\} \) is finite. Since this contains \( \text{LIMINF}\{N \oplus M - r^i\} \) it follows that \( N \oplus M \) is finitary.

(b) Let \( a \neq b \). If \( \rho(M_a) \ell > \rho(M_b) \ell \) then \( \rho(M_a) \ell \chi(\ell) \in M_a \cap M_b \subset N \). Thus, for each \( \ell \) there is at most one index \( a \) with \( \rho(M_a) \ell > \rho(N) \ell \). Hence, \( \rho(M) \ell = \max_a \rho(M_a) \ell < \infty \) for all \( \ell \) and the Bound Condition is satisfied. If \( m_a \in M_a \), \( m_b \in M_b \) with \( m_a \geq m_b \) then \( m_b \in M_a \cap M_b \subset N \). Thus, an increasing sequence in \( M \) which is not contained in \( N \) is contained in some \( M_a \). Thus, the Finite Chain Condition for \( M \) follows from the condition for each \( M_a \).

Let \( L_{M \in N} = (\bigcup \text{Supp } M) \setminus (\bigcup \text{Supp } N) = \bigcup_a \{L_a\} \) and \( L_N = \bigcup \text{Supp } N \). For \( m \in M \) let \( m_N = m \land L_N \) and \( m_{M \in N} = m \land L_{M \in N} \). If \( m_{M \in N} > 0 \) then \( m \in M_a \) for a unique \( a \).

Assume that \( N = 0 \) and the \( M_a \)'s are simple. We have \( M = M - 0, 0 = M - r \) with \( r \in \max M \) and \( \emptyset = M - r \) with \( r \not\in M \). Now note that \( M_a \ni r > 0 \) implies \( M_b - r = \emptyset \) if \( b \neq a \) and so \( M - r = M_a - r \). Thus, if \( \{M - r^i\} \) is convergent with limit not equal to \( M, 0 \) or \( 0 \), there is then a unique \( a \) such that eventually \( r^i \in M_a \) and \( \{M_a - r^i\} \) is convergent and with the same limit. Since \( M_a \) is simple, this limit is \( M_a - r \) for some \( r \in M_a \) and \( r > 0 \). It follows that \( M - r = M_a - r \) is the limit. Hence, \( M \) is simple.

Now assume that \( N \) is finite and that the \( M_a \)'s are finitary. Let \( \{r^i \in M\} \) with \( \bigcup \{\text{supp } r^i\} \) infinite. Again \( \bigcup \{\text{supp } r^i_{M \in N}\} \) is infinite. If \( r^i_{M \in N} > 0 \) then it is contained in a unique \( M_{a(i)} \) and so if \( m + r^i_{M \in N} \in M \) then \( m + r^i_{M \in N} \in M_{a(i)} \) and so \( m \in M_{a(i)} \). Case (i) - There exists \( a \) such that eventually \( r^i_{M \in N} > 0 \) implies \( a(i) = a \). In that case, if \( m + r^i_{M \in N} \in M \) eventually then \( m + r^i_{M \in N} \in M_a \) eventually and so \( \text{LIMINF}\{M - r^i_{M \in N}\} = \text{LIMINF}\{M_a - r^i_{M \in N}\} \) and the latter is finite because \( M_a \) is finitary. Case (ii) - For every \( I \) there exist \( i_1, i_2 \geq I \) with \( r^i_{M \in N}, r^i_{M \in N} > 0 \) and \( a(i_1) \neq a(i_2) \). In that case, if \( m + r^i_{M \in N} \in M \) eventually then \( m \in M_{a(i_1)} \cap M_{a(i_2)} \) with \( a(i_1) \neq a(i_2) \) and so \( m \in N \). Hence, \( \text{LIMINF}\{M - r^i_{M \in N}\} \subset N \) which is finite.

Thus, in either case \( \text{LIMINF}\{M - r^i_{M \in N}\} \) is finite. As before this contains \( \text{LIMINF}\{M - r^i\} \) and so \( M \) is finitary.

\[ \square \]

**Remark 4.39.** (a) Since any label \( M \) is the union of the finite labels \( M \cap [1, \ell] \) it follows that some condition like that in (b) above is needed to get the finite type or finitary conditions.
(b) If $N \oplus M$ is finitary and $M$ is infinite then $N$ must be finite since if $\{r^i\}$ is an infinite sequence of distinct positive vectors in $M$ then $N \subset LIMINF \{(N \oplus M) - r^i\}$.

Example 4.40. (a) If $M$ is defined by $M = \langle \{\chi(1) + \chi(\ell) : \ell > 1\} \cup \{\chi(\ell) + \chi(\ell + 1) : \ell > 1\} \rangle$, then $M$ is size bounded and finitary, but not simple.

$$M - \chi(1) = \{0\} \cup \{\chi(\ell) : \ell > 1\};$$

(4.20)  \hspace{1cm}  M - \chi(\ell) = \{0\} \cup \{\chi(1), \chi(\ell - 1), \chi(\ell + 1)\}, \quad \text{for } \ell > 1,

$$\mathcal{F} = \{0, \chi(1)\} = LIM_{\ell \to \infty}\{M - \chi(\ell)\}.$$  

Notice that $\mathcal{F} \neq M - r$ for any $r \in M$.

(b) If $M$ be defined by $M = \langle \{\chi(2a - 1) + \chi(2b) : a, b \geq 1\} \rangle$, then $M$ is size bounded and is simple but is not finitary. In general, if $M = M_1 \oplus M_2$ with $M_1, M_2$ infinite simple labels with $M_1 \cap M_2 = 0$ then $M$ is simple but not finitary.

Summarizing we have the following inclusions:

\[
\begin{array}{c}
\text{simple labels} \\
\text{finite labels} \\
\text{finite type labels} \\
\text{finitary labels}
\end{array}
\]

5. Labeled subshifts

5.1. Expanding functions.

Call $h$ a function on $\mathbb{Z}$ when $h : \mathbb{Z} \to \mathbb{Z}$. A function $h$ on $\mathbb{Z}$ is odd when $h(-j) = -h(j)$ for all $j \in \mathbb{Z}$ and so, in particular, $h(0) = 0$. We call $h$ increasing when $h(j + 1) > h(j)$ for all $j \in \mathbb{Z}$. When $h$ is odd, then it is increasing when its restriction to $\mathbb{Z}_+$ is increasing.
Then, of course, \( h \) is positive on \( \mathbb{N} \). For an odd increasing function \( |h(j)| = h(|j|) \) for all \( j \in \mathbb{Z} \).

On \( \mathbb{Z} \) let \( \epsilon \) be the signum function so that

\[
\epsilon(N) = \begin{cases} 
+1 & N > 0, \\
0 & N = 0 \\
-1 & N < 0.
\end{cases}
\]

For integers \( a, b \) we will denote by \( [a \pm b] \) the interval \([a-|b|, a+|b|]\) in \( \mathbb{Z} \). When \( a = 0 \) we will write \( [\pm b] \) for the interval \([-|b|, +|b|]\).

Let \( b \geq 9 \) be an integer. Call a function \( k : \mathbb{Z} \to \mathbb{Z} \) a \( b \)-expanding function when it satisfies

1. \( k(-n) = -k(n) \) for all \( n \in \mathbb{Z} \).
2. \( k(n+1) > b \cdot \sum_{i=0}^{n} k(i) \) and, in particular, \( k(n) > 0 \), for all \( n \geq 0 \).

Thus, \( k \) is an odd, increasing function and so \( k(|n|) = |k(n)| \) for all \( n \in \mathbb{Z} \).

Define \( k_0(n) = \epsilon(n)(1 + b)^{|n|} - 1 \).

Define for \( k \) the function \( k^+ \) by

\[
k^+(n) = \epsilon(n)k(|n|+1).
\]

When \( k \) is \( b \)-expanding, \( k^+(n) > bk(n) \) for all \( n > 0 \).

Let \( L \) be a thick set of positive integers (i.e. a set containing arbitrarily large intervals) and \( \ell > 0 \). We say that a \( b \)-expanding function \( k \) is associated with \( L \) and \( \ell \) when

\[
\bigcup_{n=1}^{\infty} [ k(n) \pm b \cdot (\ell + \sum_{j=1}^{n-1} k(j)) ] \subset L.
\]

In particular, \( k(1) \geq 1 + b\ell \).

It is clear that for any thick set \( L \) of positive integers and positive integer \( \ell \) we can inductively define a \( b \)-expanding function \( k \) associated with \( L \) and \( \ell \).

**Lemma 5.1.**

(i) If \( k \) is \( b \)-expanding and \( C \) is a positive integer then \( C \cdot k \) is \( b \)-expanding. If \( h, k \) are \( b \)-expanding then \( h + k \) is \( b \)-expanding.

(ii) \( k_0 \) is \( b \)-expanding and if \( k \) is \( b \)-expanding then \( k(n) \geq k(1)k_0(n) \) for all \( n \geq 0 \).

(iii) Assume that \( h : \mathbb{Z} \to \mathbb{Z} \) is an increasing, odd function. If \( k \) is \( b \)-expanding then \( k \circ h \) is \( b \)-expanding, and so, if \( h, k \) are both \( b \)-expanding then \( k \circ h \) is \( b \)-expanding. If for a thick set \( L \) of positive integers and positive integer \( \ell \) the expanding function \( k \)
is associated with \( L \) and \( \ell \), then \( k \circ h \) is associated with \( L \) and \( \ell \).

(iv) If \( k \) is \( b \)-expanding, then \( k^+ \) is \( b \)-expanding.

(v) If \( k \) is \( b \)-expanding, then \( n \geq 0 \) implies

\[
k(n + 2) - k(n + 1) > (b - 1)(k(n + 1) - k(n))
\]

and so the sequence of successive differences \( \{ k(n + 1) - k(n) : n \in \mathbb{Z}_+ \} \) is a strictly increasing sequence of positive integers with \( k(n + 1) - k(n) \geq k(1)(b - 1)^n \).

(vi) \( \{ [k(n) ± 4\Sigma_{0 \leq i < |n|} k(i) : n \in \mathbb{Z}\} \) is a pairwise disjoint sequence of intervals in \( \mathbb{Z} \) (for \( n = 0 \) the empty sum is zero and the interval is \([0]\)).

**Proof:** (i) Obvious.

(ii) By summing the geometric series we see that for \( n \geq 0, k_0(n + 1) = 1 + b\Sigma_{i=0}^n k_0(i) \). So \( k_0 \) is \( b \)-expanding. If \( k \) is \( b \)-expanding then \( k(1) = k(1)k_0(1) \) and so by induction \( k(n + 1) > b\Sigma_{j=0}^n k(j) \geq k(1)b\Sigma_{j=0}^{n-1} k_0(j) = k(1)(k_0(n) - 1) > k(1)k_0(n) \).

(iii) Assume \( n \geq 0 \). Since \( h \) is a strictly increasing function,

\[
\{h(0), h(1), \ldots, h(n)\} \subset \{0, 1, \ldots, h(n + 1) - 1\}.
\]

Hence,

\[
k(h(n + 1)) > b\Sigma_{i=0}^{h(n+1)-1} k(i) \geq b\Sigma_{j=0}^{n} k(h(j)),
\]

and so

\[
[ k(h(n + 1)) \pm b(\ell + \Sigma_{j=0}^{n} k(h(j))) \subset \]

\[
[ k(h(n + 1)) \pm b(\ell + \Sigma_{i=0}^{h(n+1)-1} k(i)) ].
\]

Thus, \( k \circ h \) is \( b \)-expanding and is associated with \( L \) and \( \ell \) if \( k \) is.

(iv) Define \( h(n) = \epsilon(n)(|n| + 1) \). From (iii) it follows that \( k^+ = k \circ h \) is \( b \)-expanding.

(v) \( k(n + 2) > bk(n + 1) \) and so \( k(n + 2) - k(n + 1) > (b - 1)k(n + 1) \geq (b - 1)(k(n + 1) - k(n)) \). So the sequence of differences is increasing and, by induction, \( k(n + 1) - k(n) \geq (b - 1)^nk(1) \).

(vi) This follows from \( k(|n + 1|) - \frac{b-1}{2}\Sigma_{i=0}^{|n|} k(i) > \frac{b-1}{2} (k(|n|) + \Sigma_{i=0}^{|n|-1} k(i)) \), and \( b \geq 9 \). \( \square \)

We now fix a sequence \( \{ L^\ell : \ell \in \mathbb{N} \} \) of pairwise disjoint thick subsets of \( \mathbb{N} \) and for each \( \ell \in \mathbb{N} \) a \( b \)-expanding function \( k^\ell \) which is
associated with $L^\ell$ and $\ell$. In particular,
\begin{equation}
\begin{aligned}
k^{\ell}(1) & \geq 1 + b \cdot \ell, \\
k^{\ell}(n) & \geq (1 + b)^n \geq n + b \quad \text{for } n > 0.
\end{aligned}
\end{equation}

Finally, because each $k^{\ell}$ is increasing and different $k^{\ell}$'s are supported by disjoint thick sets, we have
\begin{equation}
n \neq 0 \quad \text{and} \quad k^{\ell}(n) = k^{\ell_1}(n_1) \implies \ell = \ell_1, \ n = n_1.
\end{equation}

5.2. Labeled integers.

**Definition 5.2.** (a) An expansion of length $r \geq 0$ for $t \in \mathbb{Z}$ is a sequence of pairs $(j_1, \ell_1), \ldots, (j_r, \ell_r) \in \mathbb{Z} \times \mathbb{N}$ such that $|j_i| \geq k^{\ell_i+1}(|j_{i+1}|)$ for $i = 1, \ldots, r-1$, $|j_i| \geq \ell_i$ for $i = 1, \ldots, r$ and with
\[ t = k^{\ell_1}(j_1) + k^{\ell_2}(j_2) + \cdots + k^{\ell_r}(j_r). \]
The length vector for the expansion is the $\mathbb{N}$-vector
\[ r = \chi(\ell_1) + \chi(\ell_2) + \cdots + \chi(\ell_r) \]
so that $|r| = r \in \mathbb{Z}$ has the empty expansion with length 0 and length vector 0.

(b) For $0 \leq \tilde{r} \leq r$ the $\tilde{r}$ truncation is the sequence $(j_1, \ell_1), \ldots, (j_{\tilde{r}}, \ell_{\tilde{r}})$ for $\tilde{t} = k^{\ell_1}(j_1) + k^{\ell_2}(j_2) + \cdots + k^{\ell_{\tilde{r}}}(j_{\tilde{r}})$, with length vector $\tilde{r} = \chi(\ell_1) + \chi(\ell_2) + \cdots + \chi(\ell_{\tilde{r}})$, so that $|\tilde{r}| = \tilde{r}$. The sequence $(j_{\tilde{r}+1}, \ell_{\tilde{r}+1}), \ldots, (j_r, \ell_r)$ is the expansion for $t - \tilde{t}$ with length vector $r - \tilde{r} \geq 0$ and so with length $r - \tilde{r}$. We call $\tilde{t}$ a truncation of $t$ and $t$ an extension of $\tilde{t}$.

When there is an expansion for $t$ we will say that $t$ is expanding or that $t$ is an expanding time.

**Remark 5.3.** Observe that for a sequence $\{t_n\}^\infty_{n=1}$ of expanding integers the corresponding sequence of lowest terms $|k^{\ell_r}(j_r)(n)|$ tends to $\infty$ iff the sequence $|j_r(n)|$ does. For clearly $|k^{\ell_r}(j_r)(n)| \geq |j_r(n)|$. On the other hand, if $|j_r(n)| \not \to \infty$, then passing to a subsequence if necessary, we can assume that $|j_r(n)| = j$ is constant. Now by the definition of expanding integers we also have $|j_r(n)| = j \geq \ell_r(n)$ and it follows that there are only finitely many $\ell_r(n)$’s. Thus the sequence $|k^{\ell_r}(j_r)(n)|$ is also bounded and in particular $|k^{\ell_r}(j_r)(n)| \not \to \infty$.

Using properties of $b$-expanding functions we obtain various useful estimates for the expansion and its truncations. The key idea is that the expanding functions grow so rapidly that the expansion of $t$ is dominated by its leading term $k^{\ell_1}(j_1)$.
Lemma 5.4. If \( t = k^{\ell_1}(j_1) + k^{\ell_2}(j_2) + \cdots + k^{\ell_r}(j_r) \) as above, then
\[
|t| \geq (b - 1) \cdot \Sigma_{i=2}^{r} |k^{\ell_i}(j_i)|,
\]
and also
\[
|t| \pm (b - 1) \cdot \Sigma_{p=0}^{j_1-1} k^{\ell_1}(p) \subset [k^{\ell_1}(|j_1|) \pm b \cdot \Sigma_{p=0}^{j_1-1} k^{\ell_1}(p)] \subset L^{\ell_1}.
\]
If \( \tilde{t} \) is its \( \tilde{r} \) truncation then
\[
|t| - b \cdot \Sigma_{p=0}^{j_1-1} k^{\ell_1}(p) < |t - \tilde{t}| < b \cdot \Sigma_{p=0}^{j_1-1} k^{\ell_1}(p).
\]

Proof: Clearly,
\[
|t| - k^{\ell_1}(j_1) | \leq \Sigma_{i=2}^{r} |k^{\ell_i}(j_i)|.
\]
Now we use that \( k^{\ell_1} \) is \( b \)-expanding and that \( k^{\ell_1}(p) \geq p + b \) for all positive integers \( p \). Then we observe that the sequence \{ \( |k^{\ell_i}(j_i)| : i = 2, \ldots, r \) \} consists of distinct positive integers all less than \( |j_1| \). So we obtain
\[
|k^{\ell_1}(j_1)| \geq b \cdot \Sigma_{p=0}^{j_1-1} k^{\ell_1}(p) \geq b \cdot \Sigma_{p=0}^{j_1-1} (p + b) \geq b \cdot \Sigma_{i=2}^{r} |k^{\ell_i}(j_i)|.
\]
Comparing (5.11) and (5.12) we see that
\[
|t - k^{\ell_1}(j_1)| \leq \frac{1}{b} |k^{\ell_1}(j_1)|.
\]
The estimates of (5.8) and (5.9) follow from these. Now we apply (5.12) to the expansion of \( t - \tilde{t} \) to get
\[
\frac{1}{b} |k^{\ell_{\tilde{r}+1}}(|j_{\tilde{r}+1}|)| \geq \Sigma_{\tilde{r}+1<i\leq r} |k^{\ell_i}(j_i)|.
\]
Thus, (5.10) follows from
\[
k^{\ell_{\tilde{r}+1}}(|j_{\tilde{r}+1}|) - \Sigma \leq t - \tilde{t} \leq k^{\ell_{\tilde{r}+1}}(|j_{\tilde{r}+1}|) + \Sigma,
\]
where \( \Sigma = \Sigma_{\tilde{r}+1<i\leq r} |k^{\ell_i}(j_i)| \).
\[
\Box
\]

Proposition 5.5. If \( t \in \mathbb{Z} \) is expanding with expansion sequence \((j_1, \ell_1), \ldots, (j_r, \ell_r)\) then the expansion is unique. In fact, \( t \) determines
the terms of the sequence $k_{\ell_1}(j_1), k_{\ell_2}(j_2), \ldots, k_{\ell_r}(j_r)$ uniquely. If $s \in \mathbb{Z}$ is also expanding and $s$ is not an extension of $t$ then

$$|t - s| > 3k_{\ell_r}(|j_r| - 1),$$

**Proof:** Suppose that $s$ admits an expansion

$$(j_1(s), \ell_1(s)), \ldots, (j_r(s), \ell_r(s))$$

which is different from the given one for $t$. Note first that if $r = r(s)$ and $k_{\ell_i}(j_i) = k_{\ell_i(s)}(j_i(s))$ for all $i$ then the expansions agree by (5.7). The order of the terms is determined so that the absolute values strictly decrease.

Let $\bar{r}$ be the smallest positive integer such that $k^{\ell_r}(j_r) \neq k^{\ell_r(s)}(j_r(s))$ so that $1 \leq \bar{r} \leq \min(r(s), r) + 1$. By convention, we use $j_{\bar{r}} = 0$ (or $j_{\bar{r}}(s) = 0$) if $\bar{r} = r + 1$ (resp. if $\bar{r} = r(s) + 1$). By (5.7) $(j_p(s), \ell_p(s)) = (j_p, \ell_p)$ for $1 \leq p < \bar{r}$.

Let $u$ be the common $\bar{r} - 1$ truncation of $t$ and $s$. If $\bar{r} = 1$, let $u = 0$.

Case 1 - Assume $r = \bar{r} - 1$. In that case, $s$ is an extension of $t$. Because the two expansions are different the length $r(s)$ is greater than $r$ and so $t \neq s$ by (5.10).

If $\bar{r} \leq r$ then $j_{\bar{r}} \neq 0$ and also $s$ is not an extension of $t$.

Case 2a - Assume $r \leq \bar{r}$, $j_{\bar{r}}(s) \neq 0$ and the indices of $\ell_{\bar{r}} \neq \ell_{\bar{r}}(s)$. In that case, $|t - u|$ and $|s - u|$ lie in different thick sets by (5.9) We have $|t - s| = |(t - u) - (s - u)| \geq |t - u| - |s - u|$. So by (5.9) again applied to the expansion of $t - u$

$$|t - s| > (b - 1)k^{\ell_{\bar{r}}}(|j_{\bar{r}}| - 1) \geq 3k_{\ell_{\bar{r}}}(|j_{\bar{r}}| - 1)$$

The last because if $\bar{r} < r$ then $k^{\ell_{\bar{r}}}(|j_{\bar{r}}| - 1) \geq |j_{\bar{r}}| - 1 + 1 > k^{\ell_r}(|j_r|)$.

Case 2b - Assume $\bar{r} \leq r$ and $\ell_{\bar{r}} = \ell_r(s)$ or $j_{\bar{r}}(s) = 0$. Let $k = k^{\ell_{\bar{r}}}$ and let $\hat{t}, \hat{s}$ be $\bar{r}$ truncations of $t$ and $s$. Then $\hat{t} = u + k(j_{\bar{r}}), \hat{s} = u + k(j_{\bar{r}}(s))$ and $j_{\bar{r}} \neq j_{\bar{r}}(s)$ by definition of $\bar{r}$. Let $I_t = [\hat{t} \pm 4\sum_{i=0}^{\ell_{\bar{r}}-1} k(i)]$ and $I_s = [\hat{s} \pm 4\sum_{i=0}^{\ell_{\bar{r}}(s)-1} k(i)]$. When $j_{\bar{r}}(s) = 0$ the latter sum is empty, $\hat{s} = s$ and $I_s = [s \pm 0]$.

By Lemma 5.1 (vi) these intervals are disjoint. By (5.11) applied to the expansions of $t - u$ and $s - u$, $s \in I_s$ and $[t \pm 3\sum_{i=0}^{\ell_{\bar{r}}(s)-1} k(i)] \subset I_t$. Hence, $|t - s| \geq 3k^{\ell_{\bar{r}}}(|j_{\bar{r}}| - 1)$ and this is not smaller than $3k^{\ell_r}(|j_r| - 1)$ as before.

In all of these cases, $t \neq s$ which implies that the expansion of $t$ is unique.

$\Box$
When we are looking at various \( t \in \mathbb{Z} \) we will label the corresponding expansion, as in the proof above, \( (j_1(t), \ell_1(t)), \ldots, (j_r(t), \ell_r(t)) \) and \( r(t) \) for the length vector.

**Corollary 5.6.** If \( t \) is expanding with expansion length \( r \) and \( s \) is expanding then \( s \) is an extension of \( t \) iff \( |s - t| \leq k^{\ell_r(t)}(|j_r(t)| - 1) \).

**Proof:** If \( t = \bar{s} \) then \( r = \bar{r}(s) \) and so by (5.10) applied to \( s \)

\[
|s - t| \leq k^{\ell_{r+1}(s)}(|j_{r+1}(s)|) \\
\leq k^{\ell_r(s)}(|j_r(s)| - 1) = k^{\ell_r(t)}(|j_r(t)| - 1).
\]

If \( s \) is not an extension of \( t \) then \( |s - t| > k^{\ell_r(t)}(|j_r(t)| - 1) \) by (5.16). \( \square \)

**Definition 5.7.** For a label \( M \) define \( A[M] \subset \mathbb{Z} \) to consist of those \( t \in \mathbb{Z} \) which are expanding and have length vector \( r(t) \in M \).

If \( M = \emptyset \) then \( A[M] = \emptyset \). Otherwise \( 0 \in A[M] \) since \( 0 \in M \).

**Proposition 5.8.** Assume \( t \in A[M] \) with expansion length \( r \). For any positive integer \( N \),

\[
2N < |j_r(t)| \implies [t \pm N] \cap A[M] = t + ([\pm N] \cap A[M - r(t)]),
\]

and the elements of \( [t \pm N] \cap A[M] \) are extensions of \( t \) in \( A[M] \).

**Proof:** If \( s \in [t \pm k^{\ell_r(t)}(|j_r(t)| - 1)] \cap A[M] \) then \( s \) is an extension of \( t \) with \( r(s) \in M \) and \( r(s) = r(t) + r(s - t) \) so \( s - t \in A[M - r(t)] \).

On the other hand, if \( u = k^{\ell_1(u)}(j_1(u)) + \cdots + k^{\ell_p(u)}(j_p(u)) \in A[M - r(t)] \) with length \( p \) then \( r(u) + r(t) \in M \).

By (5.8) applied to \( u \), \( |u| \leq N \) implies \( |k^{\ell_1(u)}(j_1(u))| \leq 2|u| \leq 2N \leq |j_r(t)| \). It follows that

\[
(j_1(t), \ell_1(t)), \ldots, (j_r(t), \ell_r(t)), (j_1(u), \ell_1(u)), \ldots, (j_p(u), \ell_p(u))
\]

is an expansion for \( t + u \) with length \( r + p \) and length vector \( r(t) + r(u) \).

Hence, \( t + u \in [t \pm N] \cap A[M] \).

\( \square \)

**Lemma 5.9.** If \( t \) is expanding and \( |t| < k^\ell(1) \) then \( r(t) = 0 \).

**Proof:** We may assume \( t \neq 0 \). If for some \( i = 2, \ldots, r \), \( \ell_i(t) = \ell \) then by (5.8) \( |t| \geq (b - 1)|k^\ell(t)(j_i(t))| \geq k^\ell(1) \). If \( \ell_i(t) = \ell \) and \( |j_i(t)| \geq 2 \) then \( |t| \geq \frac{b-1}{b}|k^\ell(t)(j_1(t))| \geq (b - 1)k^\ell(1) \). Finally, if \( |j_1(t)| = 1 \) (which
can only happen with $\ell = 1$) then $|t| = |k^{\ell}(j_1(t))| = k^{\ell}(1)$. Thus, $r(t)_{\ell} > 0$ implies $|t| \geq k^{\ell}(1)$.

\[ \square \]

**Proposition 5.10.** If $N$ is a positive integer, then $\ell \geq N$ implies $k^{\ell}(1) > N$. For any label $M$,

\[ [ \pm N ] \cap A[M] = [ \pm N ] \cap A[M \land [1,N]]. \]

**Proof:** If $N \leq \ell$ then $k^{\ell}(1) > N$ by (5.6). Hence, $|t| \leq N$ implies $r(t)_{\ell} = 0$ by Lemma 5.9. Thus, $r(t) \in M \land [1,N]$.

\[ \square \]

**Lemma 5.11.** (a) For any $\mathbb{N}$-vector $m$ there exist $t \geq 0$ such that $r(t) = m$.

(b) For labels $M_1, M_2$ if $m \in M_1 \setminus M_2$ and $t$ is expanding with $r(t) = m$ then $t \in A[M_1] \setminus A[M_2]$. In particular, $M_1 = M_2$ iff $A[M_1] = A[M_2]$.

(c) For any label $M$ and $\ell \in \mathbb{N}$ there exists a positive integer $N$ such that for all labels $M_1$

\[ [ \pm N ] \cap A[M] = [ \pm N ] \cap A[M_1] \implies M \land [1,\ell] = M_1 \land [1,\ell] \]

**Proof:** (a) If $m = 0$ then $t = 0$. Otherwise, write $m = \chi(\ell_1) + \cdots + \chi(\ell_r)$ where $r = |m|$. Choose $j_r = \ell_r, j_{r-1} = \max(k^{\ell_1}(j_r), \ell_{r-1}), \ldots, j_1 = \max(k^{\ell_1}(j_2), \ell_1)$. Then $(j_1, \ell_1), \ldots, (j_r, \ell_r)$ is an expansion for $t = k^{\ell_1}(j_1) + \cdots + k^{\ell_r}(j_r) \in \mathbb{N}$ with $r(t) = m$.

(b) If $m \in M_1 \setminus M_2$ then any $t$ with $r(t) = m$ then by definition $t$ is in $A[M_1] \setminus A[M_2]$. Hence, if $M_1 \neq M_2$, then $A[M_1] \neq A[M_2]$. The reverse implication is obvious.

(c) Let $N$ be the finite label of all $m$ with $\text{supp } m \subset [1,\ell]$ and $m \leq \rho(M) + 1$. For each $m \in N$ choose $t(t(m)) \in \mathbb{Z}_+$ with $r(t) = m$ and let $N = \max \{t(m) : m \in N\}$. Now assume $[ \pm N ] \cap A[M] = [ \pm N ] \cap A[M_1]$.

If $m \in M \land [0,\ell]$ then $m \in N$ and so $0 \leq t(m) \leq N$. Hence $t(m) \in [ \pm N ] \cap A[M] = [ \pm N ] \cap A[M_1]$. Because $t(m) \in A[M_1], m \in M_1$. That is, $M \land [1,\ell] \subset M_1 \land [1,\ell]$.

On the other hand assume there were $m \in M_1 \land [1,\ell]\setminus M \land [1,\ell]$. Let $m_1 = m \land (\rho(M) + 1)$ Then $m_1 \leq m$ and so $m_1 \in (M_1 \land [1,\ell]) \setminus N$. If $m_1 \neq m$ then for some $\ell$, $m_\ell > \rho(M) + 1$ and so $(m_1)_\ell = \rho(M)_\ell + 1$ for some $\ell$ which implies $m_1 \notin M$. Because $m_1 \in N, t(m_1) \in [ \pm N ] \cap A[M_1]$. But then $t(m_1) \in A[M]$ and so $m_1 \in M$ which contradicts our assumption.
5.3. Subshifts.

On \( \{0, 1\}^\mathbb{Z} \) define the ultrametric \( d \) by

\[
d(x, y) = \inf \{ 2^{-\ell-1} : \ell \in \mathbb{Z} \text{ and } x_t = y_t \text{ for all } |t| \leq \ell \}.
\]

We denote by \( S \) the shift homeomorphism on \( \{0, 1\}^\mathbb{Z} \). That is, \( S(x)_t = x_{t+1} \). Hence, for any \( k \in \mathbb{Z} \) \( S^k(x)_t = x_{t+k} \).

For a label \( \mathcal{M} \) define \( x[\mathcal{M}] \in \{0, 1\}^\mathbb{Z} \) by \( x[\mathcal{M}]_t \) is \( 1 \) iff \( t \in A[\mathcal{M}] \) that is, \( x[\mathcal{M}] \) is the characteristic function \( \chi(\mathcal{A}[\mathcal{M}]) \).

For example, \( x[0] = \bar{0} = (\ldots, 0, 0, 0, \ldots) \), a fixed point of \( S \) which we will denote \( e \). Note also that \( x[0] = (\ldots, 0, 0, 1, 0, 0, \ldots) \).

Let \( (X(\mathcal{M}), S) \) be the associated subshift, i.e. \( X(\mathcal{M}) \) is the \( S \) orbit closure of \( x[\mathcal{M}] \).

**Theorem 5.12.** The map \( x[] \) defined by \( \mathcal{M} \mapsto x[\mathcal{M}] \) is a homeomorphism from \( \mathcal{L} \mathcal{A} \mathcal{B} \) onto its image in \( \{0, 1\}^\mathbb{Z} \).

**Proof:** Let \( \mathcal{M}_1, \mathcal{M}_2 \) be labels and \( N \) be an arbitrary positive integer.

By Lemma 5.10 \( \mathcal{M}_1 \cap [1, N] = \mathcal{M}_2 \cap [1, N] \) iff \( A(\mathcal{M}_1 \cap [1, N]) = A(\mathcal{M}_2 \cap [1, N]) \). By Proposition 5.10 this implies \( [\pm N] \cap A(\mathcal{M}_1) = [\pm N] \cap A(\mathcal{M}_2) \) which is equivalent to \( x[\mathcal{M}_1]_t = x[\mathcal{M}_2]_t \) for all \( t \in [\pm N] \). Since \( d(0, 0) = 1 = d(x[0], x[0]) \), it follows that \( x[] \) has Lipschitz constant 1.

Continuity of the inverse map at \( x[\mathcal{M}] \) for a label \( \mathcal{M} \) follows from Lemma 5.11 (c).

\( \square \)

**Corollary 5.13.** Let \( \{\mathcal{M}^i\} \) be a sequence of labels:

(a) If \( \mathcal{M} \) is a nonempty label and \( k \in \mathbb{Z} \) then \( \{S^k(x[\mathcal{M}])\} \) converges to \( x[\mathcal{M}] \) in \( \{0, 1\}^\mathbb{Z} \) iff \( k = 0 \) and \( \{\mathcal{M}^i\} \) converges to \( \mathcal{M} \).

(b) If \( \{\mathcal{M}^i\} \) is bounded, then it is convergent iff \( \{x[\mathcal{M}^i]\} \) is convergent.

**Proof:** (a) When \( k = 0 \) the result follows from Theorem 5.12.

Now assume \( k \neq 0 \). The points \( x[\mathcal{M}] \) are symmetric about 0 and for every \( t \), \( \{S^k(x[\mathcal{M}])\}_t = x[\mathcal{M}]_{t+k} \rightarrow x[\mathcal{M}]_t \). Hence, \( \{x[\mathcal{M}^i]_{-t-k} = x[\mathcal{M}^i]_{t+k}\} \) tends to \( x[\mathcal{M}]_t \) and to \( x[\mathcal{M}]_{-t+2k} = x[\mathcal{M}]_{t+2k} \). This implies that \( x[\mathcal{M}] \) is a periodic point with period \( 2k \). Since \( \mathcal{M} \neq \emptyset \), \( x[\mathcal{M}] \neq 0 \). Since it has arbitrarily long runs of zeroes it is not periodic. The contradiction implies \( k = 0 \).
(b) When \( \{M^i\} \) converges to \( M \) then \( \{x[M^i]\} \) converges to \( x[M] \) by (a). When \( \{M^i\} \) is a sequence in some \([N]\) for some label \( N \) then compactness implies that convergence fails only when there are two subsequences with different limit points \( M_1 \) and \( M_2 \). Then (a) implies that \( \{x[M^i]\} \) has subsequences which converge to \( x[M_1] \) and \( x[M_2] \).

\[ \Box \]

**Lemma 5.14.** Let \( \{t^i\} \) be a sequence of expanding times with \( r^i \) the length of \( t^i \) and let \( \{M^i\} \) be a sequence of labels. If \( \{|j_{r^i}(t^i)|\} \to \infty \) then

\[
(5.23) \quad \lim_{i \to \infty} d(S^{t^i}(x[M^i]), x[M^i] - r(t^i)) = 0.
\]

That is, the sequences \( \{S^{t^i}(x[M^i])\} \) and \( \{x[M^i] - r(t^i)\} \) are asymptotic in \( \{0,1\}^Z \).

**Proof:** For any positive integer \( N \), eventually \( |j_{r^i}(t^i)| > 2N \) and then Proposition 5.8 implies \( [t^i \pm N] \cap A[M^i] = t^i + ([ \pm N] \cap A[M^i - r(t^i)]) \) This says that for all \( t \) with \( |t| \leq N \), \( x[M^i] - r(t^i) \) is asymptotic in \( \{0,1\}^Z \). Thus, eventually \( x[M^i - r(t^i)] = (S^{t^i}(x[M^i]))_t \) for all \( t \) with \( |t| \leq N \). So (5.23) follows from the definition (5.22) of the metric on \( \{0,1\}^Z \).

\[ \Box \]

**Remark 5.15.** For later semigroup applications we note that the proofs of Corollary 5.13 and Lemma 5.14 work just the same for nets instead of sequences.

**Lemma 5.16.** Assume \( M \) is a label of finite type, \( I \) is a directed set and \( \{t^i : i \in I\} \) is a net in \( A[M] \) with \( r^i \) the length of \( t^i \). If \( \{|t^i|\} \to \infty \) but \( \{|j_{r^i}(t^i)|\} \) is bounded then there exists \( \tilde{t} \in A[M] \) with length \( \tilde{r} > 0 \) (and so \( \tilde{t} \neq 0 \)) and a subnet \( \{t^{i'}\} \), defined by restricting to \( i' \in I' \) a cofinal subset of \( I \), such that \( t^{i'} - \tilde{t} = \bar{t} \) for all \( i' \), where \( \tilde{t}^{i'} = r^{i'} - \tilde{r} \) truncation of \( t^{i'} \) and, in addition, \( \{|j_{r^{i'}}(\bar{t}^{i'})| = j_{r^{i'}}(t^{i'})\} \) \( \to \infty \).

**Proof:** By definition \( |j_{r^i}(t^i)| \geq \ell_{r^i}(t^i) \). Hence, there are only finitely many \( \ell_{r^i} \) s associated with the bounded set \( \{|j_{r^i}(t^i)|\} \). It follows that there exists \( (j^{-1}, \ell^{-1}) \) such that \( (j_{r^i}(t^i), \ell_{r^i}(t^i)) = (j^{-1}, \ell^{-1}) \) for \( i \) in a cofinal subset \( SEQ_{-1} \) of \( I \). Let \( \bar{t}_{-1} = k^{-1} \) and let \( \tilde{t}_{-1} \) be the \( r_{-1} = r - 1 \) truncation of \( t^i \) for all \( i \) in \( SEQ_{-1} \). Since the truncation is proper, \( \bar{t}_{-1} \neq 0 \). Furthermore, \( r(t^i) = r(\bar{t}_{-1}) + \chi(\ell^{-1}) \) for all \( i \in SEQ_{-1} \).
If \( \{j_{i^{-1}}(\tilde{r}_{-1})\} \rightarrow \infty \) as \( i \rightarrow \infty \) in \( \text{SEQ}_{-1} \) then let \( I' = \text{SEQ}_{-1} \) use this as the required subnet. Otherwise, it is bounded on some cofinal subset and we can choose \((j^{-2}, \ell^{-2})\) such that \((j_{\ell^{-1}}(\tilde{r}_{-1}), \ell_{\ell^{-1}}(\tilde{r}_{-1})) = (j^{-2}, \ell^{-2})\) for \( i \) in a cofinal subset \( \text{SEQ}_{-2} \) of \( \text{SEQ}_{-1} \). Let \( \tilde{t}_{-2} = k^{\ell^{-2}}(j^{-2}) + k^{\ell^{-1}}(j^{-1}) \) and let \( \tilde{r}_{-2} = r^{-2} \) be the \( \tilde{r}_{-2} = r^{-2} \) truncation of \( t^i \) for all \( i \) in \( \text{SEQ}_{-2} \). Since the truncation is proper, \( \tilde{t}_{-2} \neq 0 \). Furthermore, \( r(t^i) = r(\tilde{r}_{-2}) + \chi(\ell^{-1}) + \chi(\ell^{-2}) \in M \) for all \( i \in \text{SEQ}_{-2} \).

Because \( M \) is of finite type, the Finite Chain Condition implies that this procedure must halt after finitely many steps. That is, for some \( k \geq 1 \), \( \{j_{i^{-k}}(\tilde{r}_{-k})\} \rightarrow \infty \) as \( i' \rightarrow \infty \) in \( \text{SEQ}_{-k} \). The subnet is obtained by restricting to \( I' = \text{SEQ}_{-k} \) and \( \tilde{t} = \tilde{t}_{-k} \) with length \( \tilde{r} = k \).

Since \( \tilde{t}' \) is the truncation of \( t' \) to \( \tilde{t}' \), we have \( |j_{\tilde{r}'(i')} = j_{\tilde{r}'(i')}| \rightarrow \infty \).

\( \square \)

**Theorem 5.17.** Let \( M \) be a label. Assume \( \{t^i\} \) is a sequence of expanding times with \( |t_i| \rightarrow \infty \) and \( \{u^i\} \) is a sequence in \( \mathbb{Z} \) with \( \{u^i - t^i\} \) bounded. Let \( r(t^i) \) be the length vector of the expansion for \( t^i \) with sum \( r_i \).

1. Assume \( |j_{r_i}(t^i)| \rightarrow \infty \).
   
   (a) \( \{S^u(x[M])\} \) is convergent iff \( \{M - r(t^i)\} \) is convergent in which case
   
   \[ \lim_{i \to \infty} S^t(x[M]) = x[LIM\{M - r(t^i)\}] . \]

   (b) \( \lim_{i \to \infty} S^u(x[M]) = e \) iff \( LIM\{M - r(t^i)\} = \emptyset \).

   (c) \( \{S^u(x[M])\} \) is convergent with \( \lim_{i \to \infty} S^u(x[M]) \neq e \) iff \( \{M - r(t^i)\} \) is convergent with a nonempty limit and there exists an integer \( k \) such that eventually \( u^i = t^i + k \). In that case,
   
   \[ \lim_{i \to \infty} S^u(x[M]) = S^k(x[LIM\{M - r(t^i)\}]) . \]

2. If \( \{v^i\} \) is a sequence in \( \mathbb{Z} \) such that \( |v^i| \rightarrow \infty \) and \( \lim_{i \to \infty} S^u(x[M]) = z \neq e \) then there exists \( k \in \mathbb{Z} \) and a sequence \( s^i \in A[M] \) such that eventually \( v_i = s_i + k \) and \( |s_i| \to \infty \).

3. If \( \{v^i\} \) is a sequence in \( \mathbb{Z} \) such that \( |v^i| \rightarrow \infty \) and \( \lim_{i \to \infty} S^u(x[M]) = e \) then \( \lim_{i \to \infty} S^u(x[M]) = e \) whenever \( M_1 \) is a label with \( M_1 \subset M \) and \( \{|u^i - v^i|\} \) is bounded.

4. Assume that \( M \) is of finite type and eventually \( t^i \in A[M] \). If \( \{S^u(x[M])\} \) converges to \( z \) then \( z \neq e \) and there exist \( \tilde{r}_i \leq r_i \) and \( \tilde{t} \) such that with \( t^i \) the \( \tilde{r}_i \) truncation of \( t^i \) we have \( |j_{\tilde{r}_i}(t^i)| = |j_{\tilde{r}_i}(t^i)| \rightarrow \infty \) and eventually \( t^i - \tilde{t} = \tilde{r} \). So if \( r(\tilde{r}) \) is the length
vector of \( \bar{v} \) then \( \{ \mathcal{M} - r(\bar{v}) \} \) is convergent and if \( \bar{r} < \bar{t} \neq 0 \). Thus,

\[
(5.26) \quad z = S^\bar{t}(x[LIM \{ \mathcal{M} - r(\bar{t}) \}]).
\]

Furthermore,

\[
(5.27) \quad \bar{t} = 0 \quad \iff \quad |j_{r_i}(t^i)| \to \infty.
\]

\[
\bar{t} \neq 0 \quad \iff \quad \{ j_{r_i}(t^i) \} \text{ is bounded.}
\]

**Proof:** (i): By Lemma 5.14 \( \{ S^t(x[M]) \} \) is asymptotic to \( \{ x[M - r(t^i)] \} \) and by Corollary 5.13 (b) the latter converges iff \( \{ \mathcal{M} - r(t^i) \} \) converges in which case the common limit is \( x[LIM \{ \mathcal{M} - r(t^i) \}] \).

If \( \{ \mathcal{M} - r(t^i) \} \) is not convergent. Then then by compactness there exist two convergent subsequences with limits \( LIM_1 \neq LIM_2 \) such that \( m \in LIM_1 \setminus LIM_2 \). Because \( LIM_1 \neq LIM_2 \) it follows that \( x[LIM_1] \neq x[LIM_2] \) by Theorem 5.12. Hence, \( \{ S^{t^i}(x[M]) \} \) has subsequences converging to different limits and so is not convergent.

It is clear that if \( \{ S^{t^i}(x[M]) \} \) is convergent and eventually \( u^i = t^i + k \) then \( \{ S^{u^i}(x[M]) \} \) converges to \( S^k(x[LIM]) \). If \( LIM = \emptyset \) or, equivalently, \( LIM sup = \emptyset \) then \( \{ u^i \} \) can be partitioned into finitely many subsequences on each of which \( u^i - t^i \) is constant and so each of these \( \{ S^{u^i}(x[M]) \} \) subsequences converges to \( e \).

Finally, if \( \{ S^{u^i}(x[M]) \} \) converges to \( z \neq e \) then the now familiar subsequence argument implies that \( \{ \mathcal{M} - r(t^i) \} \) is convergent and eventually \( u^i - t^i \) is constant.

(ii) Since \( z \neq 0 \) there exists \( k \in \mathbb{Z} \) such that \( z_{-k} = 1 \). Hence, \( S^{-k}(z) \) converges to \( S^k(z) \), \( x[M],^u \) is eventually 1 and so eventually \( t^i \in A[M] \). Since \( |v_i| \to \infty \), \( |t_i| \to \infty \).

(iii) To say that \( S^{u^i}(x[M]) = e \) says that for all positive integers \( N \), eventually \( [u^i + \pm N] \cap A[M] = \emptyset \). Since \( A[M_1] \subset A[M] \) when \( M_1 \subset M \), it follows that \( [u^i + \pm N] \cap A[M_1] = \emptyset \).

(iv) If \( \{ j_{r_i}(t^i) \} \to \infty \) then by (i), we can choose \( \bar{r}_i = r_i, \bar{t} = 0 \).

Otherwise, there is a bounded subsequence to which we apply Lemma 5.16. Because \( \bar{t} \neq 0 \) it follows that there is no subsequence on which \( |j_{r_i}(t^i)| \to \infty \). Hence, \( \{ |j_{r_i}(t^i)| \} \) is bounded. All of the convergent subsequences have the same limit and so \( \bar{t} \) is defined independent of the choice of subsequence by Corollary 5.13 (a). Because the length \( \bar{r} \) of the expansion of \( \bar{t} \) is uniquely determined by \( \bar{t} \) it follows that the \( \bar{r}_i = r_i - \bar{r} \) is the truncation used in every subsequence.

\( \square \)
Corollary 5.18. Assume $\mathcal{M}$ is a label of finite type, $I$ is a directed set, and the net $\{S^i x[\mathcal{M}] : i \in I\}$ converges to $x[N]$ with $N \neq \emptyset$. If eventually $t^i = 0$ then $N = \mathcal{M}$. Otherwise, eventually $t^i \in A[\mathcal{M}]$ with length $r_i$ and length vector $r(t^i) > 0$ and $\{ |j_{r_i}(t^i)| \} \to \infty$, $\{ \mathcal{M} - r(t^i) \}$ is convergent and $\text{LIM} \{ \mathcal{M} - r(t^i) \} = N$. In particular, $N \in \Theta'(\mathcal{M})$.

Proof: Since $0 \in A(N)$ it follows that eventually $x[N]t^i = 1$ and so eventually $t^i \in A[\mathcal{M}]$. If $\{t^i\}$ is bounded on some cofinal subset of $I$ then there is cofinal subset on which $t^i = k$ for some $k \in \mathbb{Z}$. By Corollary to cor10 (b), $k = 0$ and so the limit is $x[\mathcal{M}]$. Hence, $N = \mathcal{M}$.

When $\{|t^i|\} \to \infty$ we apply Lemma 5.16 and notice that $\bar{t}$ must equal 0 by Corollary 5.13 (b) again. Hence, the truncation is trivial and so $\{ |j_{r_i}(t^i)| \} \to \infty$ on every cofinal subset and so $\{ |j_{r_i}(t^i)| \} \to \infty$. In that case $N = \text{LIM} \{ \mathcal{M} - r(t^i) \}$ by Lemma 5.14. Since $r(t^i) > 0$ the limit $N$ is in $\Theta'(\mathcal{M})$.

Since these two results are different it follows that that either $\{|t^i|\}$ is bounded on $I$ and so eventually $t_i = 0$ or $\{|t^i|\} \to \infty$ and so $\{|j_{r_i}(t^i)|\} \to \infty$.

$\Box$

Definition 5.19. If $Y$ is an invariant subset of $X(\mathcal{M})$ we let $\Phi(Y) = \{ N \subset \mathcal{M} : x[N] \in Y \}$.

That is, $\Phi(Y)$ is the preimage of $Y$ by the map $x[\cdot]$. By Theorem 5.12 $N$ is uniquely determined by $x[N]$.

Proposition 5.20. (a) If $Y$ is a closed invariant subset of $X(\mathcal{M})$ then $\Phi(Y)$ is a closed, invariant subset of $\text{LAB}$.

(b) If $N \in \Theta(\mathcal{M})$, then $x[N] \in X(\mathcal{M})$. That is, $\Theta(\mathcal{M}) \subset \Phi(X(\mathcal{M}))$.

Proof: (a): Because the map $x[\cdot]$ is continuous, $\Phi(Y)$ is closed when $Y$ is.

If $x[N] \in Y$ and $r$ is a nonzero $\mathbb{N}$-vector then we an choose $\{t^i\}$ with length vector $r$ and with $|j_r(t^i)| \to \infty$. By Theorem 5.17(i)(a) $S^i x[N] \to x[N - r]$ and since $Y$ is closed, $N - r \in \Phi$.

(b): $\mathcal{M} \in \Phi(X(\mathcal{M}))$ and (a) implies that $\Phi(X(\mathcal{M}))$ is closed and invariant. Hence, it contains $\Theta(\mathcal{M})$, the smallest closed, invariant set containing $\mathcal{M}$.

$\Box$
Corollary 5.21. Let $M$ be a label of finite type with $(X(M), S)$ the associated subshift, so that $X(M)$ is the $S$ orbit closure of $x[M]$, and $\Theta(M)$ is the $FIN(N)$ orbit closure of $M$.

(a) $X(M) = \{S^k x[N] : k \in \mathbb{Z}, N \in \Theta(M)\}$. Thus, $\Theta(M) = \Phi(X(M))$.

(b) If $\Phi \subset \Theta(M)$ then $\Phi = \Phi(Y)$ for some closed, invariant subset $Y$ of $X(M)$ iff $\Phi$ is closed and invariant. In that case, $Y = \{S^k x[N] : k \in \mathbb{Z}, N \in \Phi\}$.

Proof: From Theorem 5.17 (iv) it follows that when $M$ is of finite type every element of the orbit closure of $x[M]$ is on the orbit of some $x[N]$ for $N$ a unique element of $\Theta(M)$. Hence, $\Phi(X(M)) = \Theta(M)$. If $Y$ is an invariant subset of $X(M)$ then $Y$ consists of the orbits of some of these $x[N]$. That is, $\Phi(Y) \subset \Theta(M)$ and $Y$ consists of the orbits of the points $x[N]$ for $N \in \Phi(Y)$. Conversely, if $\Phi \subset \Theta(M)$ then together the orbits of $\{x[N] : N \in \Phi\}$ form an invariant set with $\Phi(Y) = \Phi$. Proposition 5.20 implies that $\Phi(Y)$ is closed when $Y$ is. To complete the proof of (b) we must show that $Y$ is closed if $\Phi(Y)$ is closed and invariant.

A sequence in $Y$ is of the form $S^t x[N]$ with $N \in \Phi(Y)$. Shifting by a finite amount if necessary we can assume the limit point is $x[N]$ for some $N \in \Theta(M)$. If there is some subsequence of $\{t^i\}$ which is bounded then by going to a further subsequence we obtain a subsequence with $t'^i = k$ and $S^k x[N] \rightarrow x[N]$. Then Corollary 5.13(a) implies that $k = 0$ and $\{N'^i\} \rightarrow N$. Since $\Phi(Y)$ is closed, $N \in \Phi(Y)$ and so $x[N] \in Y$, by Lemma 5.11(b).

There remains the case with $\{|t_i|\} \rightarrow \infty$. Since $x[N]_0 = 1$ we can assume that $S^t x[N]_0 = 1$ for all $i$ and so $t_i \in A[N^i]$ for all $i$. We proceed as in the proof of Theorem 5.17.

First assume $\{|j_t(x[M^i])|\} \rightarrow \infty$. By Lemma 5.14 $\{S^t\langle x[M^i]\rangle\}$ and $\{x[M^i - r(t^i)]\}$ are asymptotic. So $LIM\{x[M^i - r(t^i)]\} = x[N]$. Hence, $\{M^i - r(t^i)\}$ converges to $N$. Since $\{M^i\}$ is a sequence in $\Phi(Y)$ and $\Phi(Y)$ is closed and invariant, $N \in \Phi(Y)$ and so $x[N] \in Y$.

If the sequence $\{|j_t(x[M^i])|\}$ had a bounded subsequence then Lemma 5.16 and Lemma 5.14 would imply that for some $\tilde{t} \neq 0$ there is a truncation $\tilde{t}'$ of some further subsequence so that $\{S^t\langle x[M^i - r(\tilde{t}')]\rangle\}$ converges to $x[N]$. Since $\tilde{t} \neq 0$, this contradicts Corollary 5.13 (a).

\[\square\]

Theorem 5.22. If $M$ is a recurrent label then $x[M]$ is a recurrent point in $(\{0, 1\}^\mathbb{Z}, S)$. 

Proof: Let \( \{ r^i > 0 \} \) be a sequence in \( M \) with \( M = \text{LIM} \{ M - r^i \} \). Choose \( t^i \) with \( r(t^i) = r^i \) and with \( \{|j_{t^i}(M)|\} \to \infty \). By Lemma 5.14 \( \{ S^{t^i}(x[M]) \} \) is asymptotic to \( \{ x[M - r^i] \} \) which converges to \( x[M] \) by Corollary 5.13(b).

\[ \square \]

Corollary 5.23. If a label \( M \) is not of finite type then \( X(M) \) contains a non-periodic recurrent point.

If a label \( M \) is of finite type then \( e \) is the only recurrent point of \( X(M) \) and so \( (X(M), S) \) is a CT system. In that case, \( (X(M), S) \) is LE and topologically transitive but not weak mixing.

Proof: If \( M \) is not of finite type then Proposition 4.20 implies that there is a positive recurrent label \( N \) with \( N \in \Theta(M) \). Hence, \( x[N] \in X(M) \) by Proposition 5.20(b). Theorem 5.22 implies that \( x[N] \) is recurrent. Since \( \emptyset \in \Theta(N) \), \( x[N] \) is not periodic.

If \( M \) is of finite type then by Corollary 5.21 every point of \( X(M) \) is on the orbit of some \( x[M_1] \) with \( M_1 \subset M \). These are all labels of finite type and so none are recurrent except for \( M_1 = \emptyset \).

Since \( x[M] \) is always a transitive point for \( X(M) \), \( (X(M), S) \) is always topologically transitive. In the finite type case, it is CT and so is LE and not weak mixing (see Remarks 1.10 and 2.4).

Example 5.24. With \( M \) as in Example 4.26 the subshift \( X = X(M) \) is uncountable, CT and LE. In fact, each point of \( X \) is an isolated point in its orbit closure, and \( e \) is the unique recurrent point.

Let \( SYM = \{ x \in \{0,1\}^\mathbb{Z} : x_{-t} = x_t \quad \text{for all} \quad t \in \mathbb{Z} \} \).

Lemma 5.25. \( SYM \) is a closed subset of \( \{0,1\}^\mathbb{Z} \) which contains \( x[LAB] \). Each non-periodic \( S \) orbit in \( \{0,1\}^\mathbb{Z} \) meets \( SYM \) at most once.

Proof: Every \( A[M] \) is symmetric about 0 and so \( x[LAB] \) is contained in \( SYM \). \( SYM \) is clearly a closed set. Since \( LAB \) is complete and \( x[\cdot] \) is a homeomorphism onto its image, \( x[LAB] \) is a \( G_\delta \) subset although it is not closed.

If \( S^{k_1}x, S^{k_2}x \in SYM \) with \( k_2 \neq k_1 \) then for all \( t \in \mathbb{Z} \),

\[
(5.28) \quad x_{t+k_1} = x_{-t+k_1} = x_{(-t+k_1-k_2)+k_2} = x_{(t-k_1+k_2)+k_2}
\]

Letting \( s = t + k_1 \) we have that \( x_s = x_{s+2(k_2-k_1)} \) for all \( s \in \mathbb{Z} \). Since \( k_2 \neq k_1 \) it follows that \( x \) is periodic.
Remark 5.26. It follows that if $\mu$ is any non-atomic, shift-invariant probability measure on $\{0,1\}^Z$ then $\mu(SYM) = 0$. Observe first that the countable set $PER$ of periodic points in $\{0,1\}^Z$ has measure zero because $\mu$ is non-atomic. Since $\{S^k(SYM \setminus PER) : k \in \mathbb{Z}\}$ is a pairwise disjoint sequence of sets with identical measure the common value must be zero. Consequently, $\mu(x[LAB]) = 0$.

Proposition 5.27. Let $x^*$ be a non-periodic recurrent point of $\{0,1\}^Z$ and let $X$ be its orbit closure, so that $(X,S)$ is the closed subshift generated by $x^*$. If $K$ is a closed subset of $X$ such that every non-periodic orbit in $X$ meets $K$ at most once, then $K$ is nowhere dense. The set $X \setminus \bigcup_{k \in \mathbb{Z}} \{S^{-k}(K)\}$ is a dense $G_\delta$ subset of $X$.

In particular, $SYM \cap X$ is nowhere dense in $X$ and the set of points of $X$ whose orbits does not meet $SYM$ forms a dense $G_\delta$ subset of $X$.

Proof: Since the orbit of $x^*$ is dense in $X$, it meets any nonempty open subset of $X$. If the interior of $K$ contained more than one point then there would be two disjoint open sets $U_1,U_2$ contained in $K$ and so there would exist $k_1,k_2 \in \mathbb{Z}$ such that $S^{k_a}x^* \in U_a \subset K$ for $a = 1,2$. Since $U_1$ and $U_2$ are disjoint $k_2 \neq k_1$. This contradicts the assumption on $K$. Hence, if $K$ has nonempty interior then the interior consists of a single point which is on the orbit of $x^*$. This implies that $x^*$ is an isolated point and so cannot be recurrent unless it is periodic. Hence, the interior of $K$ is empty.

Since $K$ is nowhere dense, the $G_\delta$ set $X \setminus \bigcup_{k \in \mathbb{Z}} \{S^{-k}(K)\}$ is dense by the Baire Category Theorem. A point lies in this set exactly when its orbit does not meet $K$.

The result applies to $K = SYM \cap X$ by Lemma 5.25.

Corollary 5.28. For any label $\mathcal{M}$, the set $X(\mathcal{M}) \cap SYM$ is a compact subset of $X(\mathcal{M})$ which meets each orbit in at most one point. The set $X(\mathcal{M}) \cap SYM$ contains $x[\Theta(\mathcal{M})]$.

If $\mathcal{M}$ is of finite type then $x[\Theta(\mathcal{M})] = X(\mathcal{M}) \cap SYM$ meets each orbit in $X(\mathcal{M})$.

If $\mathcal{M}$ is not of finite type then $X(\mathcal{M}) \setminus \bigcup\{S^i(SYM)\}$ is non-empty and so $\{S^kx[\mathcal{N}] : k \in \mathbb{Z}, \mathcal{N} \in \Phi(X(\mathcal{M}))\}$ is a proper subset of $X(\mathcal{M})$. 

□
Proof: By Lemma 5.25 a non-periodic orbit meets \( SYM \) in at most one point. \( X(\mathcal{M}) \) is compact and \( SYM \) is closed and so the intersection is compact. \( x[\Theta(\mathcal{M})] \subset X(\mathcal{M}) \) and \( x[\mathcal{L}A\mathcal{B}] \subset SYM \).

If \( \mathcal{M} \) is of finite type then each orbit of \( X(\mathcal{M}) \) meets \( x[\Theta(\mathcal{M})] \). In particular, any point of \( X(\mathcal{M}) \cap SYM \) must lie in \( x[\Theta(\mathcal{M})] \).

If \( \mathcal{M} \) is not of finite type then by Corollary 5.23 there exists a non-periodic recurrent point \( x^* \in X(\mathcal{M}) \). If \( X^* \) is the orbit closure of \( x^* \) then \( X^* \subset X(\mathcal{M}) \) and by Proposition 5.27 \( X^* \setminus \bigcup \{S^i(SYM)\} \) is nonempty.

\( \square \)

Remark 5.29. Corollaries 5.21 (a) and 5.28 show that for any label \( \mathcal{M} \) of finite type the dynamical system \( (X(\mathcal{M}), S) \) admits \( x[\Theta(\mathcal{M})] \) as a closed cross-section; i.e. the orbit of any point \( x \in X(\mathcal{M}) \) meets \( x[\Theta(\mathcal{M})] \) exactly at one point. This is in accordance with the following general theorem (see [22, Section 1.2]).

Theorem 5.30. For a system \( (X, T) \), with \( X \) a completely metrizable separable space, there exists a Borel cross-section if and only if the only recurrent points are the periodic ones.

\( \square \)

Notice, too, that if \( \mu \) is an invariant probability measure on \( (X, T) \) such that the measure of the set of periodic points is zero, then any cross-section is a non-measurable set. The special case of translation by rationals on \( \mathbb{R}/\mathbb{Z} \) is used in the usual proof of the existence of a subset of \( \mathbb{R} \) which is not Lebesgue measurable.

These observations lead to the following:

Corollary 5.31. For a label \( \mathcal{M} \) of finite type the dynamical system \( (X(\mathcal{M}), S) \) is uniquely ergodic with \( \delta_e \), the point mass at \( e \), as the unique invariant probability measure. In particular \( (X(\mathcal{M}), S) \) has zero topological entropy.

Proof: The last assertion follows from the variational principle (see e.g. [13, Theorem 17.1]).

\( \square \)

On the other hand, when \( \mathcal{M} \) is recurrent, \( \Theta(\mathcal{M}) \) is a Cantor subset of \( X(\mathcal{M}) \) which meets each orbit at most once. This just says that the Cantor set \( \Theta(\mathcal{M}) \) is wandering in \( X(\mathcal{M}) \), i.e. \( S^i(\Theta(\mathcal{M})) \cap S^j(\Theta(\mathcal{M})) = \emptyset \) whenever \( i \neq j \) in \( \mathbb{Z} \). While this explicit construction may be of
interest, in fact any system \((X, T)\) admits wandering Cantor sets when \(X\) is perfect and the set of periodic points has empty interior, see [3] Theorem 1.4.

For a subset \(A\) of \(\mathbb{Z}\) the upper Banach density of \(A\) is

\[
\limsup_{#I \to \infty} \frac{#(I \cap A)}{#I}
\]

as \(I\) varies over intervals in \(\mathbb{Z}\). If \(x = \chi(A) \in \{0, 1\}^\mathbb{Z}\) so that \(x_i = 1\) iff \(i \in A\), then \(A\) has positive upper Banach density iff there exists an invariant measure \(\mu\) on the orbit closure of \(x\) such that \(\{z \in \{0, 1\}^\mathbb{Z} : z(0) = 1\}\) has positive measure and so \(\mu \neq \delta_e\), see [12, Lemma 3.17]. Thus, for example, if the orbit closure contains a periodic point other than \(e\) then \(A\) has positive upper Banach density.

**Corollary 5.32.** For a label \(M\), the set \(A[M]\) of expanding times \(t\) with \(r(t) \in M\) has upper Banach density zero iff the dynamical system \((X(M), S)\) is uniquely ergodic with \(\delta_e\), the point mass at \(e\), as the unique invariant probability measure. In particular, if \(M\) is of finite type, then \(A[M]\) has upper Banach density zero.

**Example 5.33.** For positive integers \(M, N\) let \(\mathcal{M}_{M,N} = \{ m \in FIN(N) : m \leq N \text{ and } \#supp m \leq M \}\). Each \(\mathcal{M}_{M,N}\) is size-bounded and so is of finite type. Hence, \(A[\mathcal{M}_{M,N}]\) has upper Banach density zero. \(\mathcal{M}_{*,N} = \bigcup_M \mathcal{M}_{M,N}\) is a recurrent label, with \(A[\mathcal{M}_{*,N}] = \bigcup_M A[\mathcal{M}_{M,N}]\) (even strongly recurrent see Proposition 6.15 (a) below). \(\bigcup_N \mathcal{M}_{M,N}\) is not a label since it does not satisfy the Bounded Condition. The union \(\bigcup_{M,N} A[\mathcal{M}_{M,N}]\) is the set of all expanding times.

**Questions 5.34.** Assume \(M\) is a label not of finite type.

1. Can it happen that \(x[M]\) is recurrent and \(M\) is not recurrent?
2. Can it happen that \(\Theta(M)\) is a proper subset of \(\Phi(X(M))\)? Can there be an \(x[N] \in X(M)\) with \(N\) not in \([M]\)?
3. Can there be any periodic points in \(X(M)\) other than \(e\)? Since \(\Theta(M)\) always contains \(\emptyset\), it follows from Proposition 5.20 that \(e \in X(M)\) for any label \(M\) and so no \(x[M]\) with \(M\) nonempty is periodic or even minimal.
4. Can \(A[M]\) have positive upper Banach density?
5. Is the set of all expanding times a set of upper Banach density zero? If so this would imply that each \(A[M]\) has upper Banach density zero and so that each \((X(M), S)\) is uniquely ergodic and has zero topological entropy.
These questions all arise because the process in the proof of Lemma 5.16 need not terminate and it is then not clear to us how to characterize the limits.

Remark 5.35. Although we don’t have, in general, an answer to questions 5.34 (4) and (5) above, we will show later (see Theorem 6.30 below) that for a particular choice of a system of thick sets and expanding functions \((k^\ell, L^\ell : \ell \in \mathbb{N})\) the set of all expanding times has upper Banach density zero.

Example 5.36. Recall that for \(L \subset \mathbb{N}\) we let \(\langle L \rangle = \{ \chi(A) : A \subset L \text{ with } A \text{ finite} \} \). For example, \(\langle \emptyset \rangle = \{0\}\). If \(\{A^i\}\) is a sequence of subsets of \(L\) then \(\text{Limsup}\{L \setminus A^i\} = L \setminus \text{liminf}\{A^i\}\) and \(\text{liminf}\{L \setminus A^i\} = L \setminus \text{limsup}\{A^i\}\). It then follows that \(\{(L) - \chi(A^i)\}\) converges iff \(\text{Limsup}\{A^i\} = \text{liminf}\{A^i\}\), which we then denote \(\text{Lim}\{A^i\}\) and then \(\text{Lim}\{(L) - \chi(A^i)\} = (L - \text{lim}\{A^i\})\). In particular, if \(\text{Lim}\{A^i\} = \emptyset\) then \(\text{Limsup}\{(L) - \chi(A^i)\} = (L)\). Thus, \((L)\) is a recurrent label whenever \(L\) is infinite. When \(L\) is finite then \(\langle L \rangle\) is a finite label and so is not recurrent. We see that \(\Theta(L) = \{\langle L_1 \rangle : L_1 \subset L\} \cup \{\emptyset\}\). These are all recurrent labels except for the countable collection of \(\langle L_1 \rangle\) with \(L_1\) finite.

\(\square\)

Theorem 5.37. Let \(M\) be a label of finite type. Let \(\{t^i\}\) is a sequence of expanding times with \(r_i\) the length of \(t^i\). If \(|j_i(t^i)| \to \infty\) and \(\{M - r(t^i)\}\) is eventually constant at \(M - r\), then \(\{S^{t^i}\}\) on \(X(M)\) converges pointwise to \(p_r \in E(X(M), S)\) such that \(p_r x[M] = x[M - r] = x[P_r(M)]\) and \(p_r\) is continuous on \(X(M)\).

Proof: Discarding the initial values we can assume \(M - r(t^i) = M - r\) for all \(i\).

Let \(p\) be any pointwise limit point of \(\{S^{t^i}\}\) in \(E(X(M), S)\), i.e. the limit of a convergent subnet. By Theorem 5.17 (i)\(\text{(a)}\) we have that \(p x[M] = x[M - r]\). It suffices to prove that \(p\) is continuous because it is then determined by its value on the transitive point \(x[M]\) and so the same \(p\) is the limit of every convergent subnet of \(\{S^{t^i}\}\). By compactness this implies that the sequence \(\{S^{t^i}\}\) converges to \(p\).

To prove continuity of \(p\) it suffices by Proposition 1.3 to show that \(p q x[M] = q p x[M]\) for any \(q \in E(X(M), S)\). We assume that \(q\) is a limit of a subnet of the sequence \(\{S^{t^i}\}\).
By Theorem 5.17 (iii), if \( \{S^ix[M]\} \rightarrow e \) then \( \{S^ix[M-r]\} \rightarrow e \) and so \( px[M] = e = qpx[M] \). We now consider the case when \( qx[M] = z \neq e \).

By Theorem 5.17 again we are reduced to the case when \( \{s^j\} \) is a sequence in \( A[M] \), \( |j_{s^j}| \rightarrow \infty \) and \( \{M-r(s^j)\} \) converges to \( \text{LIM}_s \). We show that the two double limits of \( \{S^{t+s}(x[M])\} \) exist and have the same value.

\[
\text{(5.29)}
\]

\[
\lim_j \lim_i \{S^{t+s}(x[M])\} = \lim_j \{S^i(x[M-r])\} = \lim_j x[M-r-r(s^j)]
\]

By Proposition 4.12 applied with \( M^j = M - r(s^j) \) \( \{M-r-r(s^j)\} \) converges to \( \text{LIM}_s - r \) and so this limit is \( x[\text{LIM}_s - r] \).

\[
\text{(5.30)}
\]

\[
\lim_i \lim_j \{S^{t+s}(x[M])\} = S^i(x[\text{LIM}_s] = \lim_i x[\text{LIM}_s - r(t^i)].
\]

By Lemma 4.27(c), \( \text{LIM}_s - r(t^i) = \text{LIM}_s - r \) for all \( i \). So this limit is also \( x[\text{LIM}_s - r] \).

So the common value of the double limits of \( \{S^{t+s}(x[M])\} \) is \( x[\text{LIM}_s - r] \). That is \( px[M] = qpx[M] = x[\text{LIM}_s - r] \).

\[\square\]

In particular, for any \( \ell \in \mathbb{N} \) there is a continuous element \( p_\ell \) of the semigroup with \( p_\ell x[M] = x[M - \chi(\ell)] \). For \( r \in M \), \( p_r = \Pi_\ell p_\ell^r \) is the unique continuous element of the semigroup such that \( p_r x[M] = x[M - r] = x[p_r M] \). The set of labels \( N \) in \( \Theta(M) \) such that \( p_r x[N] = x[p_r N] \) is closed and invariant since the elements of each set \( \{p_s \in E(X(M), S)\} \) and \( \{p_s\} \) commute with one another. Hence, the equation holds for all \( N \in \Theta(M) \). The relations on these elements of the semigroup are given by \( p_{r_1} = p_{r_2} \) iff \( M-r_1 = M-r_2 \). Compare the remark after Lemma 4.27.

Corollary 5.38. If \( M \) is a label of finite type, then \( \text{FIN}(\mathbb{N}) \times X(M) \rightarrow X(M) \) given by \( (r, x) \mapsto p_rx \) is a continuous monoid action and \( x[.] : \Theta(M) \rightarrow X(M) \) is an injective, continuous action map. It induces a homomorphism \( J_M : \mathcal{E}(\Theta(M)) \rightarrow E(X(M), S) \) which is a homomorphism onto its image. Except for the retraction to \( e \), every \( q \in E(X(M), S) \) can be expressed as \( q = S^k J_M(Q) = J_M(Q) \) with \( k \in \mathbb{Z} \) and \( Q \in \mathcal{E}(\Theta(M)) \) uniquely determined by \( q \). In particular, \( q \in J_M(\mathcal{E}(\Theta)) \) iff \( qx[M] \in x[\Theta(M)] \).

**Proof:** For \( r, s \in \text{FIN}(\mathbb{N}) \) we see that \( p_r \circ p_s = p_{r+s} \) on \( x[M] \). As these are continuous action maps of the \( S \) actions and \( x[M] \) is a
transitive point for \(X(M)\) it follows that equality holds on all of \(X(M)\). Similarly, \(p_0 = id_{X(M)}\). Thus, the map \((r, x) \mapsto p_r x\) is a continuous monoid action.

Because the continuous maps \(x[\cdot] \circ P_r = p_r \circ x[\cdot]\) on every \(P_r(M)\) and these points are dense in \(\Theta(M)\), it follows by continuity that the equation holds on all of \(\Theta(M)\). Hence, \(x[\cdot]\) is an action map.

If a net \(\{P_r\}\) converges pointwise to \(Q \in \mathcal{E}(\Theta(M))\), then for any limit point \(q\) of the net \(\{p_r\}\) in \(E(X(M), S)\), we have \(q(x[N]) = x[Q(N)]\) for all \(N \in \Theta(M)\). By Corollary 5.21 every point \(x\) of \(X(M)\) can be expressed uniquely as \(x = S^k(x[N])\) for some \(N \in \Theta(M)\). Hence, \(qx = S^k(qx[N]) = S^k x[Q(N)]\). That is, \(Q\) on \(x[\Theta(M)]\) has a unique extension \(q\) to an element of \(E(X(M), S)\). Since \(x[\cdot]\) is an action map, \(J_M\) is a homomorphism and since \(q\) is determined by its restriction to \(x[\Theta(M)]\), \(J_M\) is injective. If \(\{Q_i\}\) is a net converging to \(Q\) in \(\mathcal{E}(\Theta(M))\) then the net \(\{J_M(Q_i)\}\) converges to \(J_M(Q)\) pointwise on \(x[\Theta(M)]\). Since \(X(M) = \bigcup_k \{S^k(x[\Theta(M)])\}\) it follows that \(\{J_M(Q_i)\}\) converges to \(J_M(Q)\) pointwise on all of \(X(M)\). Hence, \(J_M\) is an injective, continuous map. Because \(\mathcal{E}(\Theta(M))\) is compact, \(J_M\) is a homeomorphism onto its image.

Now suppose that \(\{S^{t_i} : i \in I\}\) is a net converging to \(q_1 \in E(X(M), S)\). By Corollary 5.21 every point \(x\) of \(X(M)\) can be expressed uniquely as \(q_1 X(M) = S^k(x[N])\) for some \(N \in \Theta(M)\) and \(k \in \mathbb{Z}\). Hence, \(\{S^{t_i}(x[M]) : i \in I\}\) converges to \(x[N]\) with \(t_i = s_i - k\). By Corollary 5.18 we may have eventually \(t^i = 0\) in which case \(q_1 = S^k\). If not then eventually \(t^i \in A[M]\) with length \(r_i\) and length vector \(r(t^i) > 0\) and \(\{|r_i(t^i)|\} \to \infty\). \(M - r(t^i)\) convergent and \(\text{LIM} \{M - r(t^i)\} = N\). By Lemma 5.14 for any \(M_1 \in \Theta(M)\), \(\{S^{t_i}(x[M_1]) : i \in I\}\), which converges to \(S^{-k}q_1 x[M_1]\), is asymptotic to \(\{x[M_1] - r(t_i)\} : i \in I\) and so the latter is convergent to \(S^{-k}q_1 x[M_1] \in X(M)\) which is of the form \(S^{k_1} x[N_1]\) for some \(N_1 \in \Theta(M)\) and \(k_1 \in \mathbb{Z}\). By Corollary 5.13, \(k_1 = 0\) and \(\{M_1 - r(t^i) : i \in I\}\) converges to \(N_1\). That is, the net \(\{P_{r(t^i)} : i \in I\}\) converges pointwise in \(\mathcal{E}(\Theta(M))\) to \(Q \in \mathcal{E}(\Theta(M))\) with \(Q(M_1) = \text{LIM} / \{M_1 - r(t^i) : i \in I\}\). Thus, \(S^{-k}q_1\) is the image of \(Q\) via the injection \(J_M : \mathcal{E}(\Theta(M)) \to E(X(M), S)\). This shows that \(q_1 = S^k J_M(Q)\). Lemma 5.25 implies that \(k\) and hence \(Q\) are uniquely determined except when \(q_1\) is the retraction to \(e\) in which case \(Q = P_r\) with \(r \notin M\) but \(k\) can be anything in \(\mathbb{Z}\). Hence, if \(q_1 x[M] \in x[\Theta(M)]\) and \(q_1 x[M] \neq e\) then \(k = 0\) and so \(q_1 = J_M(Q)\).
If $M$ is a finitary label then $\Theta(M)$ consists of $\{M-r : r \in FIN(\mathbb{N})\}$ together with the finite external labels for $M$. As $M$ is of finite type, $X(M)$ consists of the orbits of the points of the compact set $x[\Theta(M)]$ and except for the fixed point $e$ each orbit intersects $x[\Theta(M)]$ in a single point.

**Theorem 5.39.** If $M$ is a finitary label, then $(X(M),S)$ is a countable, WAP subshift. For $\mathcal{F}$ an external limit set there is a unique element $q_{\mathcal{F}} \in E(X(M),S)$ with $q_{\mathcal{F}}x[M] = x[\mathcal{F}]$. There is a unique $Q_{\mathcal{F}} \in \mathcal{E}(\Theta(M))$ such that $Q_{\mathcal{F}}(M) = \mathcal{F}$ and $q_{\mathcal{F}} = J_M(Q_{\mathcal{F}})$. The semi-groups $\mathcal{E}(\Theta(M))$ and $E(X(M),S)$ are abelian and act continuously on the spaces $\Theta(M)$ and $X(M)$, respectively.

**Proof:** Since the external labels are finite, $\Theta(M)$ is countable. Since a finitary label is of finite type, every point of $X(M)$ lies on the orbit of an $x[N]$ with $N \in \Theta(M)$. It follows that $X(M)$ itself is countable.

To show that the shift is WAP it suffices by Corollary 1.4 to show that $pqx[M] = qpx[M]$ for any $p, q \in E(X(M),S)$.

By Theorem 5.17 (iii), if $S^i x[M] \to e$ then $S^i x[M_i] = e$ for all $M_i \subset M$. Hence $px[M] = e$ implies $pqx[M] = e = qpx[M]$ for all $q \in E(X(M),S)$.

We need to show that for sequences, $\{t^i\}, \{s^j\}$ in $\mathbb{N}$, if $lim_i S^{t^i}(x[M])$ and $lim_j S^{s^j}(x[M])$ exist then the two double limits of $\{S^{t^i+s^j}(x[M])\}$ exist and have the same value. By Theorem 5.17(iv) we are reduced to the case when $|j_{r_i}(t^i)|, |j_{r_j}(s^j)| \to \infty$ and $\{M-r(t^i)\}$ and $\{M-r(s^j)\}$ converge in which case $S^{t^i}(x[M]) \to x[lim \{M-r(t^i)\}] = x[lim\{M_i\}]$ and $S^{s^j}(x[M]) \to x[lim \{M-r(s^j)\}] = x[lim\{M_s\}]$.

If $\{M-r(t^i)\}$ is eventually constant at $M-r$ then by Lemma 4.29 implies that the two double limits of $\{M-r(t^i)-r(s^j)\}$ both exist and equal $lim\{M_s-r\}$. Applying the continuous map $x[\cdot]$ we see that the double limits of $\{S^{t^i+s^j}(x[M])\}$ both exist and equal $x[lim\{M_s-r\}]$.

By Corollary 4.28 when neither sequence is eventually constant both $\bigcup_i sup r(t^i)$ and $\bigcup_j sup r(s^j)$ are infinite. Since $M$ is finitary both $lim\{M_s-r\}$ and $lim\{M_t\}$ are finite by Proposition 4.30(a). Lemma 4.31 then implies that each double limit of $\{M-r(t^i)-r(s^j)\}$ is $\emptyset$ and so, applying $x[\cdot]$ again, each double limit of $\{S^{t^i+s^j}(x[M])\}$ is $e$.

Hence, $(X(M),S)$ is WAP and so each element of $E(X(M),S)$ is determined by its value on $x[M]$. Hence, there is a unique $q_{\mathcal{F}} \in E(X(M),S)$ such that $q_{\mathcal{F}}x[M] = x[\mathcal{F}]$. By Corollary 5.38 there is a unique $Q_{\mathcal{F}} \in \mathcal{E}(\Theta(M))$ such that $J_M(Q_{\mathcal{F}}) = q_{\mathcal{F}}$. Since $q_{\mathcal{F}}$ is continuous and $x[\cdot]$ is a homeomorphism, $Q_{\mathcal{F}}$ is continuous. 

$\square$
Remark 5.40. In (b) the additional relations are $q_{\mathcal{F}_1} p_{r_1} = q_{\mathcal{F}_2} p_{r_2}$ if $\mathcal{F}_1 - r_1 = \mathcal{F}_2 - r_2$. Also, $q_{\mathcal{F}_1} q_{\mathcal{F}_2} = u$, the retraction to $e$ for any pair of external limit sets $\mathcal{F}_1, \mathcal{F}_2$.

Corollary 5.41. If $\mathcal{F} = LIM \{M - r^i\}$ is an external label for a finitary label $M$, then $\{p_{r^i}\}$ converges to $q_{\mathcal{F}}$ in $E(X(M), S)$ and $\{P_{r^i}\}$ converges to $Q_{\mathcal{F}}$ in $E(\Theta(M))$.

Proof: $\{p_{r^i}(x[M] = x[M - r^i]\} converges to $x[\mathcal{F}] = q_{\mathcal{F}}(x[M])$. It follows from Proposition 1.5 that $\{p_{r^i}\} \rightarrow q_{\mathcal{F}}$ pointwise. Since $J_M$ is a homeomorphism onto its image, $\{P_{r^i}\} \rightarrow Q_{\mathcal{F}}$ pointwise.

Corollary 5.42. For a finitary label $M$, the abelian enveloping semigroup $E(\Theta(M)) = \{P_r : r \in \mathbb{N} \text{-vector}\} \cup \{Q_{\mathcal{F}} : \mathcal{F} \text{ an external label}\}$. The relations are given by $P_{r_1} = P_{r_2}$ iff $M - r_1 = M - r_2$ and $P_{r_1}Q_{\mathcal{F}} = P_{r_2}Q_{\mathcal{F}}$ iff $\mathcal{F} - r_1 = \mathcal{F} - r_2$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are external labels then $Q_{\mathcal{F}_1}Q_{\mathcal{F}_2}$ is the constant map to $\emptyset$.

Proof: By continuity there is a unique member of the enveloping semigroup which maps $M$ to $N \in \Theta(M)$. The relations follow from this uniqueness.

The simple case is easier:

Theorem 5.43. If $M$ is a simple label, then $(X(M), S)$ is a countable, WAP subshift. The semigroups $E(\Theta(M))$ and $E(X(M), S)$ are abelian and act continuously on the spaces $\Theta(M)$ and $X(M)$, respectively. Using the continuous, injective, homomorphism $J_M : E(\Theta(M)) \rightarrow E(X(M), S)$ every element of $E(X(M), S)$ can be expressed in the form $S^k p_r = S^k J_M(P_r)$ with $k \in \mathbb{Z}$ and $r \in FIN(\mathbb{N})$.

Proof: By Corollary 5.38 $J_M : E(\Theta(M)) \rightarrow E(X(M), S)$ is a continuous injective homomorphism and every element of $E(X(M), S)$ is of the form $S^k J_M(Q)$ for some $Q \in E(\Theta(M))$. Because $M$ is simple $E(\Theta(M)) = \{P_r : r \in FIN(\mathbb{N})\}$ by Proposition 4.33 (d). It follows that $E(X(M), S)$ is abelian and so $(X(M), S)$ is WAP and consists of the orbits of the points $x[M - r]$ for $r \in FIN(\mathbb{N})$. □
Example 5.44. (a) If \( M \) is neither finitary nor simple then \( X(M) \) need not be WAP even with \( M \) of finite type and \( X(M) \) countable. Let \( M \) be defined by \( \max \ M = \{ \chi(3) + \chi(2a + 1) + \chi(2b) : a \geq b \geq 1 \} \cup \{ \chi(1) + \chi(3) + \chi(2b) : b \geq 1 \} \).

\[
M - \chi(1) = \langle \{ \chi(3) + \chi(2b) : b \geq 1 \} \rangle,
\]
\[
M - \chi(3) = \langle \{ \chi(2a + 1) + \chi(2b) : a \geq b \geq 1 \} \cup \{ \chi(1) + \chi(2b) : b \geq 1 \} \rangle
\]
\[
M - \chi(2\ell + 1) = \langle \{ \chi(1) + \chi(2b) : \ell \geq b \geq 1 \} \rangle
\]
\[
M - \chi(2\ell) = \langle \{ \chi(1) + \chi(2a + 1) : a \geq \ell \geq 1 \} \cup \{ \chi(1) + \chi(3) \} \rangle.
\]

It follows that

\[
LIM_{a \to \infty} \{ M - \chi(2a + 1) \} = \langle \{ \chi(3) + \chi(2b) : b \geq 1 \} \rangle,
\]
\[
LIM_{b \to \infty} \{ M - \chi(2b) \} = \langle \{ \chi(1) + \chi(3) \} \rangle,
\]
\[
LIM_{b \to \infty} LIM_{a \to \infty} \{ M - \chi(2a + 1) - \chi(2b) \} = \{ \chi(3), 0 \},
\]
\[
LIM_{a \to \infty} LIM_{b \to \infty} \{ M - \chi(2a + 1) - \chi(2b) \} = \emptyset.
\]

So the enveloping semigroup \( E(X(M), S) \) is not abelian and \( (X(M), S) \) is not WAP. Notice that if \( N \subset M \) is infinite then \( M - N \) is finite. Thus, this condition does not suffice to yield \( M \) finitary.

In addition, notice that with \( r^i = \chi(2i + 1), r = \chi(1), s^j = \chi(2j) \), we have that \( LIM M - s^j \) is finite, and \( LIM M - r^i = M - r \), but the two double limits disagree. This shows that the conditions given in Lemma 4.29 are needed to get the resulting commutativity.

(b) Let \( M \) be defined by \( M = \{ \chi(3a) + \chi(3b + 1) + \chi(3(5^a7^b) + 2) : a, b \geq 1 \} \). It can be shown that the subshift \( (X(M), S) \) is WAP even though \( M \) is not finitary or simple. Notice that if \( \ell_1 \neq \ell_2 \) then

\[
(M \setminus \chi(3\ell_1)) \cap (M \setminus \chi(3\ell_2)) = \{ \chi(3b + 1) : b \geq 1 \}
\]
\[
(M \setminus \chi(3\ell_1 + 1)) \cap (M \setminus \chi(3\ell_2 + 1)) = \{ \chi(3a) : a \geq 1 \}.
\]

\( \Box \)

6. Dynamical properties of \( X(M) \)

6.1. Translation finite subsets of \( \mathbb{Z} \).

We recall the following combinatorial characterization of WAP subsets of \( \mathbb{Z} \) ([32]).
Theorem 6.1 (Ruppert). For a subset $A \subset \mathbb{Z}$ the following conditions are equivalent:

1. The subshift $\overline{O}(\chi(A)) \subset \{0, 1\}^\mathbb{Z}$ is WAP.
2. For every infinite subset $B \subset \mathbb{Z}$ either:
   (i) there exists $N \geq 1$ such that
   $$(6.1) \bigcap_{b \in B \cap [-N,N]} A - b \text{ is finite},$$
   or:
   (ii) there exists $N \geq 1$ and $n \in \mathbb{Z}$ such that
   $$A - n \supset B \cap (-N,N).$$

Definition 6.2 (Ruppert). We say that a subset $A \subset \mathbb{Z}$ is translation finite (TF hereafter) if for every infinite subset $B \subset \mathbb{Z}$ there exists an $N \geq 1$ such that

$$(6.3) \bigcap_{b \in B \cap [-N,N]} A - b = \{ n \in \mathbb{N} : A - n \supset B \cap [-N,N] \} \text{ is finite}.$$

Example 6.3. It is easy to check that the set $A = 2\mathbb{N} \cup -(2\mathbb{N}+1)$ (with $c = \chi(A) = (\ldots, 1, 0, 1, 0, 1, \hat{1}, 0, 1, 0, \ldots)$) does not satisfy Ruppert’s condition (and a fortiori is not translation finite), hence $\overline{O}(\chi(A))$ is not WAP. (See Example 1.13.(b).)

□

Proposition 6.4. Let $A$ be a subset of $\mathbb{Z}$. The following conditions are equivalent.

1. The subset $A$ is TF.
2. Every point in $R_S(\chi(A)) = (\omega_T \cup \alpha_T)(\chi(A))$ has finite support.
3. The subshift $\overline{O}(\chi(A))$ is CT of height at most 2.

Proof: (1) $\implies$ (2): Suppose first that $A$ is TF and suppose that for some sequence $\{n_i\}_{i=1}^\infty$, with $|n_i| \to \infty$, we have $S^{n_i}\chi(A) = x$ with $\text{supp } x$ an infinite set. Let $B = \text{supp } x$ and observe that for every $N \geq 1$, eventually,

$$(6.4) S^{n_i}\chi(A) \wedge [-N,N] = (x_{-N}, \ldots, x_N),$$

whence $A - n_i \supset B \cap [-N,N]$. But this contradicts our assumption that $A$ is TF.

(2) $\implies$ (1): Conversely, suppose $A$ is not TF. Then there exists an infinite $B \subset \mathbb{Z}$ such that for every $N \geq 1$ the intersection

$$(6.5) \{ n \in \mathbb{Z} : A - n \supset B \cap [-N,N] \} \text{ is infinite}.$$
We can construct a strictly increasing sequence \( \{n_i\}_{i=1}^{\infty} \) with \( A - n_i \supset B \land [-i, i] \) and so for any limit point \( x \in \{0, 1\}^Z \) of sequence \( \{S^{n_i} \chi(A) = \chi(A - n_i)\} \) the support \( \text{supp} \ x \supset B \) and so is infinite.

(2) \( \Rightarrow \) (3): is obvious.

(3) \( \Rightarrow \) (2): Suppose finally that that \( \overline{\Theta} (\chi(A)) \) is CT of height at most 2. Suppose to the contrary that \( x \in (\omega_T \cup \alpha_T) (\chi(A)) \) has infinite support, say \( \text{supp} \ x = B \). By compactness there exists a sequence \( \{n_i\}_{i=1}^{\infty} \subset B \) such that the sequence \( S^{n_i} x \) converges. Let \( y = \lim_{i \to \infty} S^{n_i} x \). Then \( y \in R_T(x) \) and \( y_0 = 1 \). Thus \( y \neq 0 \) and this contradicts our assumption that \( \overline{\Theta} (\chi(A)) \) is of height at most 2.

\( \square \)

We next address the question ‘when is \( A[M] \) TF?’. This turns out to be a rather restrictive condition, because \( \emptyset \) and 0 are the only labels \( N \) such that \( A[N] \) has finite support. For \( M = \emptyset \) or 0, \( R_S(A[M]) = \{e\} \) where \( e \) is the fixed point \( 0 = A[\emptyset] \). Thus, in these cases \( x[M] \) is TF.

**Proposition 6.5.** For a positive label \( M \) the following conditions are equivalent.

(i) \( \Theta(M) = \{M, 0, \emptyset\} \).

(ii) For all \( r > 0 \in M \), \( M - r = 0 \).

(iii) There exists \( L \) a nonempty subset of \( N \) such that \( M = \{\emptyset, 0\} \cup \{\chi(\ell) : \ell \in L\} \).

(iv) \( A[M] \) is TF.

(v) \( (X(M), S) \) has height 2.

When these conditions hold, \( M \) is finitary and simple.

**Proof:** (iii) \( \Rightarrow \) (ii) : Obvious.

(ii) \( \Rightarrow \) (i) : From (ii) it is clear that consists of \( P_r(M) = \emptyset \) for \( r \not\in M \) and \( P_r(M) = 0 \) for \( r > 0 \in M \) and finally, \( P_0(M) = M \). Hence, the only limit labels possible in \( \Theta(M) \) are \( \emptyset, 0 \) and \( M \). In passing, we see that \( A(\Theta(M)) \) contains one nontrivial element which maps \( M \) to 0 and maps 0 and \( \emptyset \) to \( \emptyset \).

(i) \( \Rightarrow \) (iii) : If \( r \in M \) with \( |r| \geq 2 \) then there exists \( \ell \in N \) such that \( r - \chi(\ell) > 0 \). Hence, \( M - \chi(\ell) \in \Theta(M) \) is neither 0 nor \( \emptyset \). Thus, every positive element of \( M \) is \( \chi(\ell) \) for some \( \ell \in N \).

(iii) \( \Rightarrow \) (v) : \( M \) is size-bounded and so is of finite type. By Corollary 5.21 the limit points of \( x[M] \) lie on the orbits of \( x[N] \) for some \( N \in \Theta(M) \). Since, \( M \) is not recurrent, the set \( R_S(x[M]) \) of limit points consists of the orbits of \( x[0] \) and \( x[\emptyset] \). These in turn map to the fixed point \( x[0] \) and so \( (X(M), S) \) has height 2.

(v) \( \Rightarrow \) (iv) : This follows from Proposition 6.4.
(iv) ⇒ (iii) : We prove the contrapositive, assuming, as above, that there exist \( r \in \mathcal{M} \) and \( \ell \in \mathbb{N} \) such that \( r - \chi(\ell) > 0 \). Choose an increasing sequence \( \{ t^i \} \) of expanding times with length \( r(t^i) = \chi(\ell) \) and with \( |j_r(t^i)| \to \infty \). By \( \{ S^{t^i}(x[M]) \} \) converges to \( x[M - \chi(\ell)] \) which does not have finite support since \( r - \chi(\ell) \in M - \chi(\ell) \). By Proposition 6.4 again \( A[M] \) is not TF.

Finally, it is clear that the labels described in (iii) are finitary and simple.

\( \square \)

6.2. Non-null and non-tame labels.

**Definition 6.6.** (a) For a subshift \( (X, S) \) a subset \( K \subset \mathbb{Z} \) is called an independent set if the restriction to \( X \) of the projection \( \pi_K : \{0, 1\}^\mathbb{Z} \to \{0, 1\}^K \) is surjective. The subshift is called null if there is a finite bound on the size of the independent sets for \( (X, S) \). It is called tame if there is no infinite independent set for \( (X, S) \).

(b) For a label \( \mathcal{M} \) a subset \( L \subset \mathcal{M} \) is called an independent set if for every subset \( L_1 \) of \( L \) there exists \( N \in \Theta(\mathcal{M}) \) such that \( L \cap N = L_1 \).

(c) A label \( \mathcal{M} \) is called non-null if for every \( n \in \mathbb{N} \) there is a finite independent subset \( F \subset \mathcal{M} \) with \( \#F \geq n \). It is non-tame if there is an infinite set \( L \subset \mathcal{M} \) such that every finite subset \( F \) of \( L \) is an independent set.

**Remark 6.7.** The concepts ‘null’ and ‘tame’ are defined for any dynamical system. The first is defined in terms of sequential topological entropy (see e.g. [24] and the review [23]) and the latter in terms of the dynamical Bourgain-Fremlin-Talagrand dichotomy for enveloping semigroups ([14]). The convenient criteria which we use here for subshifts to be non-null and non-tame, are due basically to Kerr and Li [30] (see [17, Theorem 6.1.(3)]).

**Lemma 6.8.** (a) If \( \mathcal{M} \) is a label and \( F \) is a finite subset of \( \mathcal{M} \) then for any \( N \in \Theta(\mathcal{M}) \) there exists an \( \mathbb{N} \)-vector \( r \) such that \( N \cap F = \mathcal{M} - r \cap F \). In particular, if \( F \) is a finite independent subset of \( \mathcal{M} \) then for every \( A \subset F \) there exists \( r \) such that \( F \cap \mathcal{M} - r = A \).

(b) If every finite subset of \( L \subset \mathcal{M} \) is an independent set then \( L \) is an independent set.
(c) If $L$ is an independent set for a label $M$, and if for every $m \in L$, $t(m) \in \mathbb{Z}$ is an expanding time such that $r(t(m)) = m$ then $\{ t(m) : m \in L \}$ is an independent set for the subshift $(X(M), S)$.

**Proof:** (a) If $[1, \ell]$ contains all the supports of elements of $F$ then $m \in F$ is in $M_1 \in \mathcal{LAB}$ iff it is in $M_1 \cap [1, \ell]$. Hence, for any $A \subset F$ the set $\{ M_1 : M_1 \cap F = A \}$ is clopen in $\mathcal{LAB}$. Since $\{ M - r \}$ is dense in $\Theta(M)$, the result follows.

(b) Let $L_1 \subset L$. Let $\{ F_i \}$ be an increasing sequence of finite subsets of $L$ with union $L$. Because $F_i$ is an independent set, part (a) implies there exists $r^i$ such that $F_i \cap M - r^i = L_1 \cap F_i \cap M - r^i$. It follows that if $m \in L_1$ then eventually $m \in M - r^i$. If $m \not\in L_1$ then eventually $m \not\in M - r^i$. By going to a subsequence, we can assume that $\{ M - r^i \}$ converges to some $N \in \Theta(M)$. Clearly, $L \cap N = L_1$.

(c) For any label $N \in \Theta(M)$, $t \in A[N]$ iff $t$ is expanding with $r(t) \in N$ and so $x[N]_t = 1$ iff $r(t) \in N$.

From this we obviously have

**Proposition 6.9.** (a) A label $M$ is non-null iff for every $n \in \mathbb{N}$ there is a finite subset $F \subset M$ with $\#F \geq n$ such that for every $A \subset F$ there exists $r$ such that $F \cap M - r = A$.

(b) A label $M$ is non-tame if there is an infinite set $L \subset M$ such that for any finite $A \subset F \subset L$ there exists $r$ such that $F \cap M - r = A$. In that case if $L_1$ is any subset of $L$ then there exists $N \in \Theta(M)$ such that $L \cap N = L_1$. In particular, $\Theta(M)$ is uncountable.

**Remark 6.10.** It follows that if $M$ is a non-tame label then $X(M)$ is uncountable and so $(X(M), S)$ cannot be WAP.

**Corollary 6.11.** Given a label $M$, if any label $N \in \Theta(M)$ is non-null (or non-tame) then the subshift $(X(M), S)$ is not null (resp. not tame).

**Proof:** If $N \in \Theta(M)$ then $X(N) \subset X(M)$ and so if $X(N)$ projects onto $\{0,1\}^L$ then $X(M)$ does.

There are some simple conditions which allow us to find non-tame labels.
Definition 6.12. A label $\mathcal{M}$ with roof $\rho(\mathcal{M})$ is called flat over a set $L \subset \mathbb{N}$ if for all $F \in \text{Supp}(\mathcal{M})$ with $F \subset L$, $\rho(\mathcal{M})|F \in \mathcal{M}$. Equivalently, if $\mathbf{m} \in \mathcal{M}$ with $\text{supp} \mathbf{m} \subset L$ then $\rho(\mathcal{M})|\text{supp} \mathbf{m}) \in \mathcal{M}$. The label is called flat when it is flat over $\mathbb{N}$.

Lemma 6.13. Let $L \subset \mathbb{N}$.

(a) If the label $\mathcal{M}$ is flat over $L$ and $\mathbf{r}$ is an $\mathbb{N}$-vector with $\text{supp} \mathbf{r} \subset L$ then $\mathcal{M} - \mathbf{r}$ is flat over $L$.

(b) If $\{\mathcal{M}_i\}$ is a collection of labels each flat over $L$ then $\bigcap \{\mathcal{M}_i\}$ is a label which is flat over $L$.

(c) If $\{\mathcal{M}_i\}$ is a bounded, increasing sequence of labels flat over $L$ then $\bigcup \{\mathcal{M}_i\}$ is a label flat over $L$.

(d) If $\{\mathcal{M}_i\}$ is a bounded sequence of labels flat over $L$ then $\text{Lim}\inf \{\mathcal{M}_i\}$ is a label flat over $L$.

(e) The set of labels which are flat over $L$ is a closed subset of $\text{Lab}$. $\mathcal{M}$.

(f) If $\mathcal{M}$ is a flat label then the elements of $\Theta(\mathcal{M})$ are all flat labels.

Proof: (a) If $\mathbf{m} \in \mathcal{M} - \mathbf{r}$ with $\text{supp} \mathbf{m} \subset L$ and $F = \text{supp}(\mathbf{m} + \mathbf{r})$ then $F \subset L$ and so $\rho(\mathcal{M})|F \in \mathcal{M}$. Hence, $\rho(\mathcal{M})|F - \mathbf{r} \in \mathcal{M} - \mathbf{r}$ and $\rho(\mathcal{M})|F - \mathbf{r} = (\rho(\mathcal{M}) - \mathbf{r})|F$. Clearly, $\rho(\mathcal{M}) - \mathbf{r} \geq \rho(\mathcal{M} - \mathbf{r})$.

(b) If $\mathcal{M}$ is the intersection then $\rho(\mathcal{M}) = \min \{\rho(\mathcal{M}_i)\}$. If $\mathbf{m} \in \mathcal{M}$ with $\text{supp} \mathbf{m} \subset L$ then $\rho(\mathcal{M})|\text{supp} \mathbf{m}) \in \mathcal{M}_i$ for all $i$ and so is in $\mathcal{M}$.

(c) If $\mathcal{M}$ is the union then $\rho(\mathcal{M}) = \max \{\rho(\mathcal{M}_i)\}$ and this is a non-decreasing sequence of functions. On any finite set $F$, eventually $\rho(\mathcal{M}) = \rho(\mathcal{M}_i)$. If $\mathbf{m} \in \mathcal{M}$ then eventually $\mathbf{m} \in \mathcal{M}_i$ and so eventually $\rho(\mathcal{M}_i)|\text{supp} \mathbf{m}) \in \mathcal{M}$ and eventually these equal $\rho(\mathcal{M})|\text{supp} \mathbf{m})$.

(d) Obvious from (b) and (c).

(e) Obvious from (d).

(f) Obvious from (a) with $L = \mathbb{N}$ and (e).

Recall that $\text{Supp} \mathcal{M}$ f-contains $L$ when $\mathcal{P}_f L \subset \text{Supp} \mathcal{M}$.

Lemma 6.14. Let $L \subset \mathbb{N}$. A label $\mathcal{M}$ is flat over $L$ and $\text{Supp} \mathcal{M}$ f-contains $L$ exactly when for any finite subset $F$ of $L$, $\rho(\mathcal{M})|F \in \mathcal{M}$. In that case, $\{ \chi(\ell) : \ell \in L \}$ is an independent set in $\mathcal{M}$.

Proof: The first sentence is clear from the definitions. If $F$ is a finite subset of $L$ and $A \subset F$, then $\rho(\mathcal{M})|F = \rho(\mathcal{M})|A + \rho(\mathcal{M})|(F \setminus A) \in \mathcal{M}$. So $\rho(\mathcal{M})|A \in \mathcal{M} - \rho(\mathcal{M})|(F \setminus A)$ and so $\{ \chi(\ell) : \ell \in A \} \subset \mathcal{M} - \rho(\mathcal{M})|(F \setminus A)$. But if $\ell \in F \setminus A$ then $\chi(\ell) \notin \mathcal{M} - \rho(\mathcal{M})|(F \setminus A)$. This means that $\{ \chi(\ell) : \ell \in L \}$ is an independent set.
Proposition 6.15. Let \( \mathcal{M} \) be a label with \( L = \text{supp} \, \rho(\mathcal{M}) \).

(a) If \( \mathcal{M} \) is flat and \( f \)-contains \( L \), then \( \mathcal{M} \) is a strongly recurrent label.
(b) If \( \mathcal{M} \) is a strongly recurrent label, then there exists an infinite set \( L_1 \subset L \) such that \( \mathcal{M} \) is flat over \( L_1 \) and \( \text{Supp} \, \mathcal{M} \) \( f \)-contains \( L_1 \).

Proof: (a) In this case, \( \mathcal{M} \) is a sublattice of \( \text{FIN}(\mathbb{N}) \) and so it is a strongly recurrent label by Proposition 4.20 (i).

(b) Assume that inductively that we have defined \( F_k = \{ \ell_1, \ldots, \ell_k \} \) of distinct points of \( \text{supp} \, \rho(\mathcal{M}) \) such that \( \rho(\mathcal{M})|F_k \in \mathcal{M} \). Because \( \mathcal{M} \) is strongly recurrent we can add, in \( \mathcal{M} \) any element with support outside of the finite set \( F(\rho(\mathcal{M})|F_k) \). So for sufficiently large \( \ell_{k+1} \) we have that \( \rho(\mathcal{M})|F_{k+1} = \rho(\mathcal{M})|F_k + r(\mathcal{M})\ell_{k+1}\chi(\ell_{k+1}) \in \mathcal{M} \) with \( F_{k+1} = F_k \cup \{ \ell_{k+1} \} \). Let \( L_1 = \bigcup_k \{ F_k \} \).

\[ \square \]

Corollary 6.16. Assume that \( \mathcal{M} \) is a label not of finite type. If \( \mathcal{M} \) is flat or strongly recurrent then it is non-tame.

Proof: If \( \mathcal{M} \) is strongly recurrent then by Proposition 6.15 there exists an infinite subset \( L_1 \) of \( \text{supp} \, \rho(\mathcal{M}) \) such that \( \mathcal{M} \) is flat over \( L_1 \) and \( \text{Supp} \, \mathcal{M} \) \( f \)-contains \( L_1 \). By Proposition 4.6 any label not of finite type \( f \)-contains some infinite set \( L_1 \) and if \( \mathcal{M} \) is flat then it is flat over \( L_1 \). By Lemma 6.14 \( \{ \chi(\ell) : \ell \in L_1 \} \) is an independent set in \( \mathcal{M} \).

\[ \square \]

If we define

\[ (6.6) \quad \mathcal{F}(\mathcal{M}, L) = \{ F : F \text{ is a finite subset of } L \text{ and } \rho(\mathcal{M})|F \in \mathcal{M} \}, \]

then \( \mathcal{M} \) is flat over \( L \) and \( \text{Supp} \, \mathcal{M} \) \( f \)-contains \( L \) exactly when \( \mathcal{F}(\mathcal{M}, L) = \mathcal{P}_fL \).

We conjecture that for every \( \mathcal{M} \) not of finite type there exists \( \mathcal{N} \in \Theta(\mathcal{M}) \) which is non-tame and so that \( (X(\mathcal{M}), S) \) is not tame. Beyond the above corollary the best we can do is the following.

Proposition 6.17. Let \( \mathcal{M} \) be a label not of finite type. If \( \rho(\mathcal{M}) \) is bounded then there exists \( \mathcal{N} \in \Theta(\mathcal{M}) \) which is non-tame.

Proof: If \( K \in \mathbb{N} \) and \( \rho(\mathcal{M}) \leq K \) then \( \rho(\mathcal{N}) \leq K \) for all \( \mathcal{N} \in [[\mathcal{M}]] \). By Proposition 4.20 (d) there is a positive recurrent label in \( \Theta(\mathcal{M}) \) and so we can assume that \( \mathcal{M} \) itself is recurrent. By Proposition 4.20 (h)
we can choose an infinite set $L \subseteq \text{supp } \rho(M)$ such that $\{ \chi(F) : F \in \mathcal{P}_f(L) \}$ is a strongly recurrent set for $M$. Consider $\mathcal{F}(M, L)$.

Case (i): If there exists $\{F^i\}$ a strictly increasing sequence of elements of $\mathcal{F}(M, L)$ with $L_1 = \bigcup \{F^i\}$ then $\mathcal{F}(M, L_1) = \mathcal{P}_f L_1$ and so $M$ itself is non-tame by Lemma 6.14.

Case (ii): If $F$ is a maximal element of $\mathcal{F}(M, L)$ then $\rho(M - \rho(M)|F) = 0$ for $\ell \in F$ and for $\ell \in L$ with

$$(6.7) \quad \rho(M)_{\ell} > 0 \quad \implies \quad \rho(M)_{\ell} > \rho(M - \rho(M)|F)_{\ell}$$

because for $\ell \in L \setminus F$, $\rho(M)_{\ell} \not\in M - \rho(M)|F$ by maximality of $F$. Let $M_1 = M - \rho(M)|F$ and $L_1 = L \setminus F(\rho(M)|F)$. We see that $M_1$ is a recurrent element of $\Theta(M)$ with $\{ \chi(F) : F \in \mathcal{P}_f(L_1) \}$ a strongly recurrent set for $M_1$. Furthermore, $\rho(M_1) \leq K - 1$ by $(6.7)$. In particular, this cannot happen if $K = 1$.

If Case (ii) occurs then we repeat the procedure with $M$ and $L$ replaced by $M_1$ and $L_1$. Eventually, we must terminate in a Case i situation and so at some $M_k \in \Theta(M)$ which is non-tame.

Remark 6.18. Notice that Case (i) did not require that $\rho(M)$ is bounded. Furthermore, once we have the set $L$ associated with the strongly recurrent subset of $M$ we can replace it by any infinite sub-

set. In particular, if there is any infinite subset of $L$ on which $\rho(M)$ is bounded then the above argument will apply. Thus, the obstruction to proving the conjecture in general arises when $\text{Lim} \rho(M)_{\ell} = \infty$ as $\ell \to \infty$ in $L$ and every element of $\mathcal{F}(M, L)$ is contained in a maximal element of $\mathcal{F}(M, L)$ and these conditions continue to hold as we replace $M$ and $L$ by $M - \rho(M)|F, L_1$ for $F$ any maximal element of $\mathcal{F}(M, L)$. Finally, we notice that if $\mathcal{F}(M, L)$ contains sets of arbitrarily large cardinality then $M$ is at least non-null by Lemma 6.14.

Corollary 6.19. Let $M$ be a label not of finite type.

1. There is a label $N \in [[M]]$ (i.e. $N \subset M$) which is not of finite type and is bounded, hence not tame.
2. There is a label $N \supset M$ which is flat, hence not tame.

Proof: (1). As $M$ is not of finite type there is a strictly increasing sequence $\{\mathbf{m}_i\}_{i=1}^{\infty}$ of elements of $M$. Let $N = \langle \{\text{supp } \mathbf{m}_i : i = 1, 2, \ldots \} \rangle$. Then clearly $N$ is not of finite type and $\rho(N) \equiv 1$. The non-tameness follows from 6.17.
(2). Let \( \mathcal{N} = \langle \{ \rho(\mathcal{M}) | (\text{supp } \mathcal{m}) : \mathcal{m} \in \mathcal{M} \} \rangle \). Clearly \( \mathcal{N} \supset \mathcal{M} \) and is flat, hence not tame by 6.16.

\[ \square \]

**Questions 6.20.**

1. Is it true that for every label \( \mathcal{M} \) not of finite type there exists \( \mathcal{N} \in \Theta(\mathcal{M}) \) which is non-tame (hence also so that \((X(\mathcal{M}), S)\) is not tame) ?
2. Is there a label \( \mathcal{M} \) not of finite type such that \( X(\mathcal{M}) \) is tame or even null ?
3. Is there a recurrent such label ?

A positive answer to the second question (in the null case) will yield an example of a null dynamical system with a recurrent transitive point which is not minimal. The question whether such a system exists is a long standing open question.

In a private conversation Tomasz Downarowicz asked us whether it is the case that every WAP system is null. Our next example shows that there are (a) non-null simple labels, hence topologically transitive WAP subshifts which are non-null; (b) non-tame labels of finite type, hence subshifts arising from finite type labels which are not tame.

**Example 6.21.** There are simple, finitary labels which are non-null. Accordingly, by Corollary 6.11, the corresponding subshifts are topologically transitive WAP subshifts which are non-null. Also, there are labels of finite type which are non-tame. Again the corresponding subshifts are topologically transitive and non-tame. Note that by Remark 6.10 these latter subshifts are not WAP.

(a) Partition \( \mathbb{N} \) into disjoints sets \( \{ A_n : n \in \mathbb{N} \} \) \( \cup \) \( \{ B_n : n \in \mathbb{N} \} \) with \( \#A_n = n \) and \( \#B_n = 2^n \) for all \( n \). Define a bijection \( A \mapsto \ell_A \) from the power set of \( A_n \) onto \( B_n \). Now define \( \mathcal{M}_n \) by \( \mathcal{M}_n = \langle \{ \chi(\ell_A) + \chi(i) : i \in A, A \subset A_n \} \rangle \). \( \mathcal{M}_n \) is a finite label and since \( \mathcal{M}_n - \chi(\ell_A) = \{ \chi(i) : i \in A \} \cup \{ \emptyset \} \) it follows that \( \chi(i) : i \in A_n \) is an independent set for \( \mathcal{M}_n \). Since \( \{ \mathcal{M}_n \} \) is a pairwise disjoint sequence of finite labels, \( \mathcal{M} = \bigcup_n \{ \mathcal{M}_n \} \) is a simple, finitary label which is clearly non-null.

Instead we can define \( \mathcal{N}_n \) by \( \mathcal{N}_n = \langle \chi(A_n) \rangle \). Again \( \mathcal{N} = \bigcup_n \{ \mathcal{N}_n \} \) is a simple, finitary label which is non-null by Lemma 6.14.

(b) Partition \( \mathbb{N} \) into two disjoint infinite sets \( L, B \) and define a bijection \( A \mapsto \ell_A \) from the set of finite subsets of \( L \) onto \( B \). Define \( \mathcal{M} = \{ \chi(\ell_A) + \chi(i) : i \in A, A \text{ a finite subset of } L \} \). Because it is size
bounded, the label \( M \) is of finite type. Just as in (a), \( \{ \chi(i) : i \in L \} \) is an independent set for \( M \).

\[ \square \]

**Remark 6.22.** By Proposition 6.9 the label \( M \) of example (b) has \( \Theta(M) \) uncountable. So this and Example 4.26 are labels \( M \) of finite type with \( \Theta(M) \) uncountable. It follows that \( (X(M), S) \) are subshifts which are LE but not HAE (see Remark 2.4).

\[ \square \]

**Questions 6.23.** According to [19], [30] and [17] a subshift \( (X, S) \) has positive entropy iff it admits an independent subset \( K \subset \mathbb{Z} \) which has positive density. Is there an expanding function system and a label \( M \) such that \( (X(M), S) \) has positive entropy? Can \( (X(M), S) \) be weakly mixing?

### 6.3. Rare systems of expanding functions.

Our aim in this subsection is to show that with some extra care one can construct a sequence of thick sets and associated expanding functions for which the set of all expanding times has upper Banach density zero.

Let \( A \) and \( F \) be subsets of \( \mathbb{Z} \) and \( f \in \mathbb{Z} \). Define \( -A = \{-t : t \in A\} \), \( A - f = \{ t - f : t \in A \} \), \( F_+ = F \cup \{0\} \) and \( t^{-F}A = \bigcap_{f \in F} A - f \).

If \( F \) is finite, then \( \Sigma F = \Sigma \{ f : f \in F \} \).

**Definition 6.24.** Let \( A \) be a nonempty subset of \( \mathbb{Z} \).

(i) \( A \) is symmetric if \( 0 \in A \) and \( A = -A \).

(ii) \( A \) is semiadditive if for every finite \( F \subset A \), the set \( t^{-F_+}A = A \cap \bigcap_{f \in F} A - f \) is infinite. Thus, \( s \in t^{-F}A \) iff \( s + f \in A \) for all \( f \in F_+ \).

(iii) \( IP(A) = \{ \Sigma F : F \subset A \text{ and } F \text{ is finite and nonempty} \} \). Letting \( F \) vary over singletons we see that \( A \subset IP(A) \).

Note that if \( F \) is symmetric then \( F_+ = F \). For more details on semiadditive sets see [2].

**Lemma 6.25.** Let \( A \) be a semiadditive subset of \( \mathbb{Z} \).

(a) If \( F \) is a finite subset of \( A \) then \( t^{-F_+}A \) is semiadditive. If \( A \) and \( F \) are symmetric then \( t^{-F}A \) is symmetric.
(b) If \( A \) is symmetric then \( IP(A) \) is symmetric.
(c) [See Hindman, [26]] There exists an infinite set \( B \) such that \( IP(B) \subset A \). If \( A \) is symmetric then \( B \) can be chosen to be symmetric.

**Proof:** (a) Let \( G \) be a finite nonempty subset of \( t^{-F}A \). By definition \( G + F = \{ g + f : g \in G, f \in F \} \) is a subset of \( t^{-F}A \). Hence, \( t^{-G}(t^{-F}A) \) is infinite.

If \( A \) and \( F \) are symmetric then \( 0 \in F \subset A \) implies \( 0 \in t^{-F}A \). If \( s \in t^{-F}A \) then \( s + f \in A \) for all \( f \in F \) and so, since \( F \) is symmetric \( s - f \in A \) for all \( f \in F \). Since \( A \) is symmetric, \( -s + f \in A \) for all \( f \in F \) and so \( -s \in t^{-F}A \). That is, \( t^{-F}A \) is symmetric. Since \( 0 \in F \), \( s \in t^{-F}A \) implies \( s \in A \).

(b) If \( F = \{0\} \) then \( \Sigma F = 0 \) and so \( 0 \in IP(A) \). If \( F \subset A \) then \( -F \subset -A = A \) and so \( -\Sigma F = \Sigma - F \in IP(A) \).

(c) We will first prove the symmetric case, which is what we will need. Let \( B_0 = \{0\} \). Inductively, we define \( \{B_n\} \) an increasing sequence of symmetric finite subsets of \( \mathbb{Z} \) such that \( IP(B_n) \subset A \). Since \( B_n \) is finite and symmetric, \( IP(B_n) \) is finite and symmetric. Since \( t^{-IP(B_n)}A \) is infinite we can choose \( t_{n+1} \in t^{-IP(B_n)}A \setminus IP(B_n) \). Let \( B_{n+1} = B_n \cup \{t_{n+1}, -t_{n+1}\} \). By definition of \( t^{-IP(B_n)}A \) it follows that \( IP(B_{n+1}) \subset A \).

Let \( B = \bigcup_n B_n \). This set is infinite since the sets \( B_n \)'s are increasing. Any finite subset \( F \) of \( B \) is contained in \( B_n \) for \( n \) sufficiently large. Hence, \( IP(B) = \bigcap_n IP(B_n) \subset A \). \( B \) is symmetric because the \( B_n \)'s are.

When symmetry is not involved, we let \( B_0 \) be any singleton in \( A \) and then choose \( t_{n+1} \in t^{-IP(B_n)}A \setminus IP(B_n) \) and let \( B_{n+1} = B_n \cup \{t_{n+1}\} \).

\( \square \)

Recall that for a dynamical system \( (X,T) \) and subsets \( U,V \) of \( X \), \( N_T(V,U) = \{t \in \mathbb{Z} : T^t(V) \cap U \neq \emptyset \} \). If \( V = \{x\} \) then we write \( N_T(x,U) = N_T(\{x\},U) = \{t \in \mathbb{Z} : T^t(x) \in U\} \).

**Proposition 6.26.** Let \( (X,T) \) be a dynamical system, \( x \in X \) and \( U \) an open set containing \( x \). If \( x \) is a recurrent point for \( (X,T) \) then \( N_T(x,U) \) is a semiadditive subset of \( \mathbb{Z} \). If \( (x,x) \) is a recurrent point for \( (X \times X,T \times T^{-1}) \) then \( N_{T \times T^{-1}}((x,x),U \times U) \) is a symmetric semiadditive subset of \( \mathbb{Z} \).

**Proof:** The point \( x \) is recurrent iff \( N_T(x,U) \) is infinite for every neighborhood \( U \) of \( x \). If \( F \) is a finite subset of \( N_T(x,U) \) then let \( t^{-F+}(U) = \bigcap \{T^{-f}(U) : f \in F_+\} \). By definition of \( N_T(x,U) \), \( x \) is a point of the open set \( t^{-F+}(U) \). Observe that \( t^{-F+}N_T(x,U) = N_T(x,t^{-F+}(U)) \) and so it is infinite. Hence, \( N_T(x,U) \) is semiadditive.
Since \( t \in N_{T \times T^{-1}}((x, x), U \times U) \) iff \( T^t(x) \in U \) and \( T^{-t}(x) \in U \), it follows that \( N_{T \times T^{-1}}((x, x), U \times U) \) is symmetric.

\[ \square \]

**Theorem 6.27.** There exists an infinite symmetric subset \( B \) of \( \mathbb{Z} \) such that \( IP(B) \) has upper Banach density zero.

**Proof:** By Lemma 6.25 (c) it suffices to construct a symmetric semi-additive subset of \( \mathbb{Z} \) which has upper Banach density zero.

Let \((X, T)\) be a topologically transitive, almost equicontinuous, non-minimal system and let \( x \) be a transitive point. By [20, Lemma 1.2 and Theorem 1.3] the system is uniformly rigid and \( x \) is not in the support of any invariant measure. Notice that if \( X = x \) and Theorem 1.3] the system is uniformly rigid and minimal system and let \( x \) and \( x \neq x \). Thus, if \( \{ U \times U \} \) is a symmetric, semiadditive subset of \( \mathbb{Z} \). Since \( N_{T \times T^{-1}}((x, x), U \times U) \) is a symmetric, semiadditive subset of \( \mathbb{Z} \). Since \( N_{T \times T^{-1}}((x, x), U \times U) \subset N_T(x, U) \) it has upper Banach density zero.

\[ \square \]

Now let \( B \) be an infinite, symmetric subset of \( \mathbb{Z} \) such that the symmetric set \( IP(B) \) has upper Banach density zero. Let \( B \cap \mathbb{N} = \{ u_1 < u_2 < \cdots \} \) be the infinite sequence of positive integers in \( B \).

Inductively, let \( h(0) = 0, h(1) = u_1 \) and \( h(i+1) = u_{m_i+1} = b\Sigma_{j=1}^i h(i) \) and let \( h(-i) = -h(i) \) for all positive integers \( i \). Thus, \( h \) is a \( b \)-expanding function and \( \{ h(i) = u_{m_i} \} \) is a subsequence of \( u_m \). Hence, the set \( h(\mathbb{Z}) \) is an infinite symmetric set with \( h(\mathbb{Z}) \subset B \). It follows that \( IP(h(\mathbb{Z})) \subset IP(B) \) has upper Banach density zero.

Now assume that \( \{ L_{\ell} : \ell \in \mathbb{N} \} \) is a partition of \( \mathbb{N} \) by thick sets and that \( \{ k^\ell \} \) is a sequence of \( b \)-expanding functions associated with the sequence of pairs \( \{ (L_{\ell}, \ell) \} \). Define \( \tilde{L}_\ell \) as follows: If \( 1 \in L_{\ell_1} \) then \( \{ 1, h(1) \} \subset \tilde{L}_{\ell_1} \) and if \( i + 1 \in L_{\ell_{i+1}} \) then \( \{ h(i) + 1, h(i + 1) \} \subset \tilde{L}_{\ell_{i+1}} \). Thus, if \( \{ a, b \} \subset L_{\ell} \) then \( \{ h(a), h(b) \} \subset \tilde{L}_{\ell} \). Note that the length of \( \{ h(a), h(b) \} \) is greater than the length of \( \{ a, b \} \). Thus, \( \{ \tilde{L}_{\ell} \} \) is a partition of \( \mathbb{N} \) by thick sets. Let \( \tilde{k}^\ell = h \circ k^\ell \). By Lemma 5.1 (iii) each \( \tilde{k}^\ell \) is a \( b \)-expanding function. By (5.7) \( (\ell, i) \mapsto \tilde{k}^\ell(i) \) defines an injective map from \( \mathbb{N} \times (\mathbb{Z} \setminus \{ 0 \}) \) to \( \mathbb{N} \). It follows that the set \( \tilde{U} = \{ \tilde{k}^\ell(i) : (\ell, i) \in \mathbb{N} \times (\mathbb{Z} \setminus \{ 0 \}) \} \) has upper Banach density zero.

\[ \square \]
\( \mathbb{N} \times \mathbb{Z} \) is a symmetric subset of subsequence of \( h(\mathbb{Z}) \) and so \( IP(\tilde{U}) \) has upper Banach density zero.

**Lemma 6.28.** If \( h \) is a \( b \)-expanding function then for any positive integer \( s \) and any finite sequence of not necessarily distinct positive integers \( \{a_1, \ldots, a_n\} \),

\[
h(s \sum_{i=1}^{n} a_i) \geq s \sum_{i=1}^{n} h(a_i).
\]

In particular, \( h(s) \geq sh(1) \geq s \).

**Proof:** By induction, it suffices to show that for positive integers \( a_1, a_2, h(a_1 + a_2) \geq h(a_1) + h(a_2) \). If \( a_1 \neq a_2 \) then \( h(a_1 + a_2) \geq b \sum_{i=1}^{a_1+a_2-1} h(i) \geq h(a_1) + h(a_2) \). On the other hand, \( h(2a_1) \geq b \sum_{i=1}^{2a_1-1} h(i) \geq bh(a_1) \geq 2h(a_1) \).

\( \square \)

**Proposition 6.29.** For each \( \ell \in \mathbb{N} \), \( \tilde{k}^{\ell} \) is a \( b \)-expanding function associated with the thick set \( \tilde{L}_\ell \) and the number \( \ell \).

**Proof:** Let \( n \in \mathbb{N} \). Since \( k^{\ell} \) is associated with \( L_\ell \) and \( \ell \) there exists an interval \( [a, b] \subseteq L_\ell \) such that \( a \leq a + \ell + \sum_{i=1}^{n-1} k^{\ell}(i) \leq k^{\ell}(n) \) and \( \ell + \sum_{i=1}^{n} k^{\ell}(i) \leq b \). From the construction above, \( [h(a), h(b)] \subseteq \tilde{L}_\ell \).

Because \( h \) is increasing, Lemma 6.28 implies

\[
h(k^{\ell}(n)) \geq h(a + \ell + \sum_{i=1}^{n-1} k^{\ell}(i)) \geq h(a) + \ell + \sum_{i=1}^{n-1} h(k^{\ell}(i))
\]

and so \( h(a) \leq h(a) + \ell + \sum_{i=1}^{n-1} \tilde{k}^{\ell}(i) \leq \tilde{k}^{\ell}(n) \).

Similarly,

\[
h(b) \geq h(\ell + \sum_{i=1}^{n} k^{\ell}(i)) \geq \ell + \sum_{i=1}^{n} h(k^{\ell}(i))
\]

and so \( \ell + \sum_{i=1}^{n} \tilde{k}^{\ell}(i) \leq h(b) \).

Thus, \( [\tilde{k}^{\ell}(n) \pm (\ell + \sum_{i=1}^{n-1} \tilde{k}^{\ell}(i))] \subseteq [h(a), h(b)] \subseteq \tilde{L}_\ell \). \( \square \)

Because the expanding times for the functions \( \{\tilde{k}^{\ell}\} \) form a subset of \( IP(\tilde{U}) \) they have upper Banach density zero. Thus, we have completed the proof of the following:

**Theorem 6.30.** There exists a sequence \( \{L_\ell : \ell \in \mathbb{N}\} \), of thick sets and a sequence \( \{k^{\ell} : \ell \in \mathbb{N}\} \) of \( b \)-expanding functions, such that

- For each \( \ell \) the function \( k^{\ell} \) is associated with the set \( L_\ell \) and the number \( \ell \).
- The thick sets \( \{L_\ell : \ell \in \mathbb{N}\} \) form a partition of \( \mathbb{N} \).
- The set of all expanding times has upper Banach upper density zero.
Thus, with this system of expanding functions, for every $M \in \mathcal{LAB}$ the corresponding subshift $(X(M), S)$ is (i) uniquely ergodic with unique invariant measure $\delta_e$ and (ii) has zero topological entropy.

$\square$

In fact, it follows that $\frac{\#I \cap x^{-1}(1)}{\#I}$ tends to zero as $\#I \to \infty$ uniformly as $I$ varies over intervals in $\mathbb{Z}$ and $x$ varies over the elements of $X(\mathcal{LAB})$, the smallest closed, shift invariant subset of $\{0,1\}^\mathbb{Z}$ which contains $x[\mathcal{LAB}]$. Thus, this entire subshift $(X(\mathcal{LAB}), S)$ satisfies properties (i) and (ii) described in the theorem above.

### 6.4. Gamow transformations.

For $L \subset \mathbb{N}$ we let $FIN(L) = \{ m \in FIN(\mathbb{N}) : \text{supp } m \subset L \}$ and $\mathcal{LAB}(L) = \{ M \in \mathcal{LAB} : \bigcup Supp M \subset L \}$. Clearly, $M \in \mathcal{LAB}(L)$ implies $[M] \subset \mathcal{LAB}(L)$. If $M \notin \mathcal{LAB}(L)$ then for some $\ell \in \mathbb{N}$ $M \cap [1, \ell] \notin \mathcal{LAB}(L)$ and so $d(M, M_1) < 2^{-\ell}$ implies $M_1 \notin \mathcal{LAB}(L)$. Thus, $\mathcal{LAB}(L)$ is a closed subset of $\mathcal{LAB}$. For example, $\mathcal{LAB}(\emptyset) = \{0, \emptyset\}$.

$FIN(L)$ is a submonoid of $FIN(\mathbb{N})$ and it acts on $\mathcal{LAB}(L)$. Furthermore, if $r \notin FIN(L)$ then $P_r(M) = \emptyset$ for all $M \in \mathcal{LAB}(L)$. Hence, we can restrict attention to this action and for $\Phi$ a closed, invariant subset of $\mathcal{LAB}(L)$, the enveloping semigroup $E(\Phi)$ is the closure of $FIN(L)$ in $\Phi^\mathcal{B}$.

Let $\tau : L_1 \to L_2$ be a bijection with $L_1, L_2 \subset \mathbb{N}$. In honor of the book *One, Two, Three...Infinity* we will refer to the following as the Gamow transformation induced by $\tau$. For an $\mathbb{N}$-vector $m$ with $\text{supp } m \subset L_2$ we let $\tau^* m = m \circ \tau$ so that $\text{supp } \tau^* m = \tau^{-1} \text{supp } m \subset L_1$. Thus, $\tau^* : FIN(L_2) \to FIN(L_1)$ is a monoid isomorphism which also preserves the lattice properties.

For $M \in \mathcal{LAB}(L_2)$ we let $\tau^* M = \{ \tau^* m : m \in M \}$ and for $\Phi \subset \mathcal{LAB}(L_2)$ we will let $\tau^* \Phi = \{ \tau^* N : N \in \Phi \}$. Thus, $\tau^*$ is a bijection from $\mathcal{LAB}(L_2)$ to $\mathcal{LAB}(L_1)$ with inverse $(\tau^{-1})^*$.

Given $\ell \in \mathbb{N}$, let $\ell' = \max \tau([1, \ell] \cap L_1)$. It follows from the definition (4.5) of the metric on $\mathcal{LAB}$ that $d(M_1, M_2) \leq 2^{-\ell'}$ implies $d(\tau^*(M_1), \tau^*(M_2)) \leq 2^{-\ell'}$. Thus, the $\tau^*$ is uniformly continuous on $\mathcal{LAB}(L_2)$ and so is a homeomorphism from $\mathcal{LAB}(L_2)$ onto $\mathcal{LAB}(L_1)$.

Clearly, $\tau^*$ preserves all label operations, e.g. $\tau^*(M + r) = \tau^* M - \tau^* r$ for $M \in \mathcal{LAB}(L_2)$ and $\text{supp } r \subset L_2$. Thus, $\tau^*$ is an action isomorphism relating the $FIN(L_2)$ action on $\mathcal{LAB}(L_2)$ to the $FIN(L_1)$ action on $\mathcal{LAB}(L_1)$. Hence, it induces an Ellis semigroup isomorphism from $E(\Phi)$
to $E(\tau^*(\Phi))$ where $\Phi$ is a compact, invariant subset of $\mathcal{LAB}(L_2)$. Also $\Theta(\tau^*M) = \tau^*\Theta(M)$ for $M \in \mathcal{LAB}(L_2)$.

Finally, $\tau^*M$ is of finite type, finitary, recurrent or strongly recurrent iff $M$ satisfies the corresponding property. In the finitary case, $\mathcal{F}$ is an external element for $M$ iff $\tau^*\mathcal{F}$ is an external element for $\tau^*M$.

On the other hand, the sets $A(\tau^*m)$ and $A(m)$ are only analogous. The thick set $L^\ell$ is different from $L^{\tau\ell}$ - unless $\tau\ell$ happens to equal $\ell$ - and so the associated $b$-expanding functions are completely different.

Now define $\mathcal{F} S^k x[M] = S^k x[\tau^*M]$. This is a well-defined map defined on the (not closed) subshift generated by $x[\mathcal{LAB}]$ and it commutes with the shift map $S$. It is not at all continuous. However, it has some very nice dynamical properties. If $M$ is of finite type Corollary 5.21 implies that the map $\mathcal{F}$ restricts to a bijection from $X(\mathcal{M})$ to $X(\tau^*\mathcal{M})$ which commutes with the shift. Furthermore, $Y$ is a closed, invariant subset of $X(\mathcal{M})$ iff $\mathcal{F}(Y)$ is a closed invariant subset of $\tau^*\mathcal{F}(Y)$. This again follows from Corollary 5.21 because $\Phi(\mathcal{F}(Y)) = \tau^*\Phi(Y)$. Recall that $r$ induces a continuous element $p_r \in E(X(\mathcal{M}), S)$ uniquely defined by $p_r x[M] = x[P_r M]$ and which satisfies $p_r x[N] = x[N - r] = x[P_r N]$ for every $N \in \Theta(\mathcal{M})$. When $\mathcal{M}$ is finitary, for each external element $\mathcal{F}$ there is a continuous element $q_{\mathcal{F}}$ of $E(X(\mathcal{M}), S)$ characterized by $q_{\mathcal{F}} x[M] = x[\mathcal{F}] = x[Q_{\mathcal{F}} M]$. We clearly have

$$
\mathcal{F} \circ p_r = p_{\tau^* r} \circ \mathcal{F}
$$

and when $\mathcal{M}$ is finitary $\mathcal{F} \circ q_{\mathcal{F}} = q_{\tau^* \mathcal{F}} \circ \mathcal{F}$. Thus, in the finitary case, $\mathcal{F}$ induces an algebraic isomorphism from $E(X(\mathcal{M}), S)$ to $E(X(\tau^*\mathcal{M}), S)$ which relates the actions on $X(\mathcal{M})$ and $X(\tau^*\mathcal{M})$.

To understand the failure of continuity, observe that $p_r$ is the limit of the sequence $S^i t_i$ on $X(\mathcal{M})$ where $r(i) = r$ and $|j_r(t_i)| \to \infty$. The associated sequence $S^i_{s_i}$ on $X(\tau^*\mathcal{M})$ has $r_{s_i} = \tau^* r$ and $|j_r(s'_i)| \to \infty$. Because the $b$-expanding functions are completely different, these are unrelated numerical sequences. For example, if $r = \chi(\ell)$ then $\tau^* r = \chi(\ell')$ with $\tau(\ell') = \ell$. The sequences $t_i = k^\ell(i)$ and $s_i = k^{\ell'}(i)$ don’t overlap at all unless $\ell = \ell'$.

Let $S_\infty$ denote the group of all permutations on $\mathbb{N}$. On $S_\infty$ we define an ultrametric by

$$
d(\tau_1, \tau_2) = \inf \{ 2^{-\ell} : \ell \in \mathbb{Z}_+ \text{ and } \tau_1[1, \ell] = \tau_2[1, \ell] \}.
$$

Clearly, for any $\gamma \in S_\infty$, $d(\gamma \circ \tau_1, \gamma \circ \tau_2) = d(\tau_1, \tau_2)$. If $\gamma([1, \ell]) \subset [1, \ell]$ then $\tau_1[1, \ell] = \tau_2[1, \ell]$ implies $\tau_1 \circ \gamma[1, \ell] = \tau_2 \circ \gamma[1, \ell]$ and $\gamma_1[1, \ell] = \gamma[1, \ell]$ then $\gamma_1^{-1}[1, \ell] = \gamma^{-1}[1, \ell]$. It follows that $S_\infty$ is a
topological group with left invariant ultrametric $d$. Furthermore, the equivalent metric $\bar{d}$ given by $\bar{d}(\tau_1, \tau_2) = \max(d(\tau_1, \tau_2), d(\tau_1^{-1}, \tau_2^{-1}))$ is complete. Finally, the set of permutations $S_{\text{fin}}$ consisting of permutations are the identity on the complement of a finite set, is a countable dense subgroup of $S_{\infty}$. Thus, $S_{\infty}$ is a Polish group, which is clearly perfect.

Furthermore, if $M$ and $M_1$ are labels with $M \land [1, \ell] = M_1 \land [1, \ell]$ and $\gamma_1|[1, \ell] = \gamma|[1, \ell]$ then $\gamma^* M \land [1, \ell] = \gamma_1^* M_1 \land [1, \ell]$. This implies that the action $S_{\infty} \times \mathcal{LAB} \to \mathcal{LAB}$ given by $(\tau, M) \mapsto (\tau^{-1})^* M$ is a continuous action. The empty label $\emptyset$ is an isolated fixed point for the action. Let $\mathcal{LAB}_+$ denote the perfect set of nonempty labels. We show that this action is topologically transitive on $\mathcal{LAB}_+$ by constructing explicitly a transitive point.

Example 6.31. Let $\Xi$ be the countable set of all pairs $(N_\xi, \ell_\xi)$ with $N_\xi$ a finite label such that $\bigcup \text{Supp} N_\xi \subset [1, \ell_\xi]$. Partition $N$ by disjoint intervals indexed by $\Xi$ such that $I_\xi$ has length $\ell_\xi$. Let $\tau_\xi : I_\xi \to [1, \ell_\xi]$ be the increasing linear bijection and let $M_\xi = \tau_\xi^* N_\xi$ so that $\bigcup \text{Supp} M_\xi \subset I_\xi$. Let $M_{\text{trans}} = \bigcup_\xi M_\xi$. By Theorem 4.38(b) $M_{\text{trans}}$ is finitary and simple and so is of finite type. On the other hand, given any nonempty label $M$ and any $\ell \in N$ there exists $\xi \in \Xi$ such that $(N_\xi, \ell_\xi) = (M \land [1, \ell], \ell)$. It follows that if $\gamma \in S_{\text{fin}}$ with $\gamma = \tau_\xi$ on $I_\xi$ then $(\gamma^{-1})^* M_{\text{trans}} \land [1, \ell] = M \land [1, \ell]$. Thus, $M_{\text{trans}}$ is a transitive point for the action of $S_{\infty}$ on $\mathcal{LAB}_+$.

Because $\mathcal{LAB}_+$ is a perfect Polish space, the set $\text{TRANS}$ of transitive points is a dense $G_\delta$ subset of $\mathcal{LAB}_+$. By Proposition 4.36 the set $\text{RECUR}$ of recurrent labels is a dense $G_\delta$ subset of $\mathcal{LAB}$. Hence, $\text{TRANS} \cap \text{RECUR}$ is a dense $G_\delta$ subset of $\mathcal{LAB}_+$. The transitive point $M_{\text{trans}}$ is of finite type and so is not recurrent. On the other hand, the set of flat labels is a proper, closed $S_{\infty}$ invariant subset which contains recurrent labels (see Proposition 6.15 (a)) which are thus not transitive with respect to the $S_{\infty}$ action.

For background regarding our next question we refer the reader to the works [29] and [21].

Question 6.32. Does there exist a label $M$ such that its $S_{\infty}$ orbit is residual, or are all the orbits meager? If such a residual orbit exists then it would be unique. It is well known that the adjoint action of $S_{\infty}$ on itself does have a dense $G_\delta$ orbit (see e.g. [21]).
6.5. Ordinal constructions.

For a label $M$ define $z_{LAB}(M) = M \setminus \max M = \{ m : m + r \in M \text{ for some } r > 0 \}$. If $M$ is of finite type and nonempty then $z_{LAB}(M)$ is a proper subset of $M$ and it is a label because it clearly satisfies the Heredity Condition. If $M$ is positive, then $0 \in z_{LAB}(M)$ and so $z_{LAB}(M) \neq \emptyset$. Hence, $z_{LAB}(M) = \emptyset$ iff $M = 0$ or $\emptyset$. In general, $z_{LAB}(M) = M$ iff $\max M = \emptyset$.

Define the descending transfinite sequence of labels by

$$
\begin{align*}
z_{LAB,0}(M) &= M, \\
z_{LAB,\alpha+1}(M) &= z_{LAB}(z_{LAB,\alpha}(M)), \\
z_{LAB,\beta}(M) &= \bigcap_{\alpha < \beta} \{z_{LAB,\alpha}(X)\} & \text{for } \beta \text{ a limit ordinal.}
\end{align*}
$$

The sequence stabilizes at $\beta$ when $z_{LAB,\beta}(M) = z_{LAB,\beta+1}(M)$ in which case $z_{LAB,\alpha}(M) = z_{LAB,\beta}(M)$ for all $\alpha \geq \beta$. So $\emptyset$ stabilizes at 0 and if $M$ is nonempty and of finite type then the sequence stabilizes at $\beta + 1$ where $\beta$ is the first ordinal for which $z_{LAB,\beta} = 0$.

If $\Phi$ is a closed, bounded, invariant set of labels define $z_{LAB}(\Phi)$ to be the closure of $\bigcup \{ P_r(\Phi) : r > 0 \}$, a closed, invariant subset of $\Phi$. Thus, $z_{LAB}(\Theta(M)) = \Theta(M)$.

Define the nonincreasing transfinite sequence of closed, bounded subsets of $LAB$ by

$$
\begin{align*}
z_{LAB,0}(\Phi) &= \Phi, \\
z_{LAB,\alpha+1}(\Phi) &= z_{LAB}(z_{LAB,\alpha}(\Phi)), \\
z_{LAB,\beta}(\Phi) &= \bigcap_{\alpha < \beta} \{z_{LAB,\alpha}(\Phi)\} & \text{for } \beta \text{ a limit ordinal.}
\end{align*}
$$

**Theorem 6.33.** If $M$ be a label then $z_{LAB,\alpha}(M) = \bigcup z_{LAB,\alpha}(\Theta(M))$ for every countable ordinal $\alpha$. That is, $m \in z_{LAB,\alpha}(M)$ iff there exists $N \in z_{LAB,\alpha}(\Theta(M))$ such that $m \in N$.

**Proof:** We use transfinite induction. Both procedures stabilize at a countable ordinal and so we need only consider countable ordinals.

Since $M = \bigcup \Theta(M)$ the result is true for $\alpha = 0$.

If $m \in z_{LAB,\alpha+1}(M) = z_{LAB}(z_{LAB,\alpha}(M))$, then there exists $r > 0$ such that $m + r \in z_{LAB,\alpha}(M)$. By induction hypothesis, there exists $N \in z_{LAB,\alpha}(\Phi)$ such that $m + r \in N$ and so $m \in N - r \in z_{LAB,\alpha+1}(\Phi)$.

Conversely, if $N \in z_{LAB,\alpha+1}(\Phi) = z_{LAB}(z_{LAB,\alpha}(\Phi))$ then there exists sequences $N^i \in z_{LAB,\alpha}(\Phi)$ and $r^i > 0$ such that $N = LIM\{N^i - r^i\}$. By
induction hypothesis $\bigcup \{N^i\} \subset z_{LAB,\alpha}(M)$. If $m \in N$ then eventually $m \in N^i - r^i$ and so $m \in z_{LAB,\alpha+1}(M)$.

Now let $\beta$ be a limit ordinal. If $m \in N \in z_{LAB,\beta}(\Phi)$ then by definition $N \in z_{LAB,\alpha}(\Phi)$ for all $\alpha < \beta$. So by induction hypothesis, $m \in z_{LAB,\alpha}(M)$ for all $\alpha < \beta$ and hence $m \in z_{LAB,\beta}(M)$.

Conversely, if $m \in z_{LAB,\beta}(M)$ and so in $z_{LAB,\alpha}(M)$ for all $\alpha < \beta$. Let $\{\alpha^i\}$ be an increasing sequence of ordinals converging to $\beta$. By induction hypothesis there exists $N^i \in z_{LAB,\alpha^i}(\Phi)$ such that $m \in N^i$. Since $\{\alpha^i\}$ is increasing $N^i \in z_{LAB,\alpha^i}(M)$ for all $j < i$. Let $\{N^i\}$ be a convergent subsequence with limit $N$. Since $m \in N^i$ for all $i$, $m \in N$. For every $\alpha < \beta$ there exists $\alpha^j > \alpha$. For $i \geq j$ the sequence $N^i \in z_{LAB,\alpha^i}(\Phi) \subset z_{LAB,\alpha}(\Phi)$ and so the limit of the subsequence $N$ is in the closed set $z_{LAB,\alpha}(\Phi)$. Since this is true for all $\alpha < \beta$, $N \in z_{LAB,\beta}(\Phi)$.

It follows that the sequences stabilize at the same countable ordinal. If $M = \emptyset$ then $\Theta(M) = \{\emptyset\}$ and the sequence stabilizes at 0. If $M$ is nonempty and of finite type then the sequence stabilizes at $\beta + 1$ where $\beta$ is the ordinal with $z_{LAB,\beta}(M) = 0$ and $z_{LAB,\beta}(\Theta(M)) = \{0, \emptyset\}$. In this nonempty finite type case, we call $\beta$ the height of $\Theta(M)$.

**Theorem 6.34.** Assume that $M$ is a label of finite type. For every countable ordinal $\alpha$ and every closed, invariant $Y \subset X(M)$,

$$\Phi(z_{LIM,\alpha}(Y)) = z_{LAB,\alpha}(\Phi(Y)).$$

**Proof:** The equation is clear for $\alpha = 0$.

Since $\Phi(Y)$ is the preimage of $Y$ with respect to the continuous map $x[\cdot]$ it follows that $\Phi(z_{LIM,Y})$ is a closed invariant set containing $P_rN$ whenever $x[N] \in Y$ and $r > 0$ since $x[P_rN] = p_rx[N]$. That is, $z_{LAB}(\Phi(Y)) \subset \Phi(z_{LIM,Y})$.

On the other hand, Corollary 5.21 (b) implies that $z_{LAB}(\Phi(Y)) = \Phi(Y)$ for a closed, invariant subspace $\tilde{Y}$ of $X(M)$. If $r > 0$ and $y \in \tilde{Y}$ then $y = S^k(x[N])$ with $k \in \mathbb{Z}$ and $N \in \Phi(Y)$. So $p_ry = S^k(x[P_rN])$. Since $P_rN \in z_{LAB}(\Phi(Y))$, it follows that $p_ry \in \tilde{Y}$. Hence, $z_{LIM,\alpha}(Y) \subset \tilde{Y}$ and so $\Phi(z_{LIM,\alpha}(Y)) \subset \Phi(\tilde{Y}) = z_{LAB}(\Phi(Y))$.

This proves equation (6.12) with $\alpha = 1$. Assuming the result for an ordinal $\alpha$ it follows for $\alpha + 1$. For a limit ordinal, $\beta$ we use the fact
that $\Phi$ commutes with intersection and so by the induction hypothesis
(6.13)
$$
\Phi(z_{\text{LIM},\beta}(Y)) = \Phi(\bigcap_{\alpha<\beta} z_{\text{LIM},\alpha}(Y)) = \bigcap_{\alpha<\beta} \Phi(z_{\text{LIM},\alpha}(Y)) = \bigcap_{\alpha<\beta} z_{\text{LAB},\alpha}(\Phi(Y)) = z_{\text{LAB},\beta}(\Phi(Y)).
$$

This completes the induction. 

We will say that $\Phi \subset \mathcal{LAB}$ is $\Theta$ invariant when it is nonempty and $M \in \Phi$ implies $\Theta(M) \subset \Phi$. If $\Phi$ is closed and invariant then it is $\Theta$ invariant. $\Theta$ invariance implies invariance but is usually a stronger condition since $\{P_t(M)\}$ is usually a proper subset of $\Theta(M)$.

Let $M$ be a label of finite type. For a $\Theta$ invariant $\Phi \subset \Theta(M)$, we define $z^*_M(\Phi) = \{ N \in \Theta(M) : \Theta'(N) \subset \Phi \}$. Equivalently, $N \in z^*_M(\Phi)$ iff $Q(N) \in \Phi$ for all $Q \in A(\Theta(M)) = E(\Theta(M)) \{ id_{\Theta(M)} \}$. For example, $z^*_M(\{\emptyset\}) = \{\emptyset, 0\} = [[0]]$.

Starting with a $\Theta$ invariant $\Phi \subset \Theta(M)$, define the nondecreasing transfinite sequence of $\Theta$ invariant subsets of $\Theta(M)$ by

$$
\begin{align*}
z^*_{M,0}(\Phi) &= \Phi, \\
z^*_{M,\alpha+1}(\Phi) &= z^*_{M,\alpha}(\Phi), \\
z^*_{M,\beta}(\Phi) &= \bigcup_{\alpha<\beta} \{ z^*_{M,\alpha}(\Phi) \} \text{ for } \beta \text{ a limit ordinal.}
\end{align*}
$$

Recall that $Y \subset X(M)$ is called orbit-closed when $x \in Y$ implies $\overline{O_S(x)} \subset Y$. For $M$ of finite type it is easy to adjust the proof of Corollary 5.21 to show that $Y \subset X(M)$ is orbit-closed iff $\Phi(Y)$ is $\Theta$ closed.

**Theorem 6.35.** Assume that $M$ is a label of finite type and that $Y$ is an orbit closed subset of $X(M)$. For every countable ordinal $\alpha$,

(6.15) $\Phi(R_{S,\alpha}^*(Y)) = z^*_{M,\alpha}(\Phi(Y))$.

**Proof:** This is obvious for $\alpha = 0$.

A point $x \in R^*_S(Y)$ iff $qx \in Y$ for all $q \in A(X(M), S)$. Since $Y$ is $S$-invariant, Corollary 5.38 implies that this is true iff $J_M(Q)x \in Y$ for all $Q \in A(\Theta(M))$. Hence, $x[N] \in R^*_S(Y)$ iff $x[Q(N)] \in Y$ for all $Q \in A(\Theta(M))$, i.e. iff $Q(N) \in \Phi(Y)$ for all $Q \in A(\Theta(M))$ and so iff $N \in z^*_M(\Phi(Y))$. This proves equation (6.15) for $\alpha = 1$ and so inductively for any $\alpha + 1$. 


Since \( \Phi \) is the preimage operator with respect to the map \( x[\cdot] \), it commutes with union. So the equation for a limit ordinal \( \beta \) follows because it is assumed, inductively, to hold for all \( \alpha < \beta \). The equations are analogous to those of (6.13) with intersection replaced by union.

\[ \square \]

The constructions of (6.10), (6.11) and (6.14) are label constructions and so they commute with Gamow transformations. We can use Gamow transformations to assure that a countable number of labels all occur with supports on disjoint sets. Let \( \tau_0 : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \) be a bijection and let \( L_i = \tau_0^{-1}(\mathbb{N} \times \{i\}) \) for \( i \in \mathbb{N} \). Define \( \tau_i : L_i \to \mathbb{N} \) to be the bijection \( \tau_i = \pi_1 \circ \tau_0 \) where \( \pi_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) is the first coordinate projection. Given a sequence \( \{M^i\} \) of labels, the label \( \{\tau^*i M^i\} \) is Gamow equivalent to \( M_i \) and \( \bigcup \text{Supp} M_i \subset L_i \).

For nonempty labels \( M_1, M_2 \) we have \( \bigcup \text{Supp} M_1 \cap \bigcup \text{Supp} M_2 = \emptyset \) iff \( M_1 \cap M_2 = 0 = \{0\} \). In that case we will say that \( M_1 \) and \( M_2 \) are disjoint. Recall that for labels \( M_1, M_2, M_1 \oplus M_2 = \{m_1 + m_2 : m_1 \in M_1, m_2 \in M_2\} \).

Proposition 6.36. If \( \{M_a\} \) is a finite or infinite pairwise disjoint collection of nonempty labels of finite type, then \( M = \bigcup \{M_a\} \) is a label of finite type which is finitary (or simple) if the \( M_a \)'s are all finitary (resp. are all simple). Moreover \( M \) has the following properties:

1. If \( r > 0 \) then \( M - r = \emptyset \) unless \( r \in M_a \) for some \( a \). If \( r \in M_a \), then \( M - r = M_a - r \). Furthermore,

\[
\Theta'(M) = \bigcup \{\Theta'(M_a)\},
\]

\[
\max M = \bigcup \{\max M_a\}.
\]

2. For \( r > 0 \in M_a \), the map sending \( P_r \) on \( \Theta(M_a) \) to \( P_r \) on \( \Theta(M) \) extends to a continuous injective homomorphism \( j_a : A(\Theta(M_a)) \to A(\Theta(M)) \). If \( a \neq b \) then:

\[
j_a(A(\Theta(M_a))) \cap j_b(A(\Theta(M_b))) \subset \{q : q(A(\Theta(M))) \subset \{0, \emptyset\}\}.
\]

Furthermore,

\[
A(\Theta(M)) \setminus \bigcup \{j_a(A(\Theta(M_a)))\}
\]

contains at most one point \( Q^* \) in which case \( Q^* M = 0 \) and \( Q^* = \emptyset \) on \( \Theta'(M) \).

3. For every \( \alpha \geq 1 \)

\[
z_{LAB,\alpha}(\Theta(M)) = \bigcup \{z_{LAB,\alpha}(\Theta(M_a))\}.
\]
4. If for all $a$, $\Phi_a$ is a $\Theta$ invariant subspace of $\Theta'(M_a)$ then $\bigcup \{\Phi_a\}$ is a $\Theta$ invariant subset of $\Theta'(M)$. If it is a proper subset then

$$
(6.19) \quad z^*_M(\bigcup \{\Phi_a\}) = \bigcup \{z^*_M(\Phi_a)\}.
$$

**Proof:** By Theorem 4.38 $M$ is of finite type and is finitary or simple if all of the $M_a$’s are.

If $\{r^i > 0\}$ is a sequence in $M$ and $m > 0$ is an $N$-vector in $M$ then $m \in M_a$ for some $a$. If $\{M - r^i\}$ converges with $m$ in the limit then eventually $r^i \in M_a$ in which case $M - r^i = M_a - r^i$ and $LIM\{M - r^i\} = LIM\{M_a - r^i\} \in \Theta'(M_a)$. Hence, $\Theta'(M) \subset \bigcup \{\Theta'(M_a)\}$. The reverse inclusion is obvious. That $max M$ is the union of the $max M_a$’s is obvious.

Now suppose that $\{P_r\}$ is a net with $r^i > 0$ in $M$ converging to $Q \in A(\Theta(M))$. Assume first that for some $N \in A(\Theta(M))$ that $m > 0 \in QN$. Let $M_b$ be the unique member to the sequence which contains $r$. Eventually $r^i \in M_b$ so that eventually $M_1 - r^i = (M_1 \cap M_b) - r^i$ for all $M_1 \in [[M]]$. In particular, $Q(M) = LIM\{M - r^i\} = LIM\{M_b - r^i\}$. Notice that $M_1 = \bigcup\{M_1 \cup M_a\}$ and by (6.16) $M_1 \in \Theta'(M)$ if $M_1 = M_1 \cap M_a \in \Theta(M_a)$ for some $a$. Clearly $Q(M_1) = \emptyset$ if $a \not= b$ and $Q(M_1) = LIM\{M_1 - r_b\}$ for $M_1 \in \Theta(M_a)$. Hence, $Q = j_b(Q)$ where $Q$ is the pointwise limit of $\{P_r\} \in A(\Theta(M_b))$. On the other hand if $\{P_r\} \in A(\Theta(M_b))$ converges to $Q$ with $r^i > 0$ in $M_b$ then $P_r$ in $A(\Theta(M))$ converges pointwise to $Q(N) = LIM\{N \cap M_b - r^i\}$ for all $N \in \Theta(M)$. This defines the injection $j_b : A(\Theta(M_b)) \rightarrow A(\Theta(M))$. Because $\{u\} \cup \{P_r : r > 0\}$ is dense in $A(\Theta(M_b))$ it follows that $j_b$ is continuous and is a homomorphism. It is obviously injective.

Now assume that $Q(M) = 0$. If for some cofinal set $\{r^i \in M_b\}$ then $Q = j_b(Q)$ with $Q = LIM\{P_{r^i}\}$ in $A(\Theta(M_b))$. Otherwise, eventually $r^i \notin M_b$ for every $b$ and so $Q(N) = \emptyset$ for all $N \in \Theta(M)$. This is $Q^*$ which exists iff the sequence $\{M_a\}$ is infinite. Notice that if some $M_b$ is infinite then $Q^* = j_b(Q)$ where $Q = LIM\{P_{s_i}\}$ with $\{s^i\}$ a sequence of distinct members of $max M_b$. However, if the sequence $\{M_a\}$ is infinite and each $M_a$ is finite then $Q^*$ is not in $\bigcup \{j_a(A(\Theta(M_a))))\}.

In general, let $\{M_b\}$ be a convergent sequence of nonempty labels with $M^i \subset M_{a(i)}$. If $a(i')$ has some common value $a$ for $i'$ in an infinite subset, then $\{M^{i'}\}$ is a sequence in $[[M_a]]$ and the limit is in $[[M_a]]$. Otherwise, the limit is $0$. It follows that if $\Phi_a$ is a closed subset of $\Theta'(M_a)$ for each $a$, and $0, \emptyset \in \bigcup \Phi_a$ then $\bigcup \Phi_a$ is a closed subset of $\Theta'(M)$. 


If \( \Phi \) is a closed subset of \( \mathcal{A}(\Theta(\mathcal{M})) \) then \( z_{\text{LAB}}(\Phi) \) is the closure of \( \mathcal{A}(\Theta(\mathcal{M})))\Phi \). Hence, (6.18) follows by transfinite induction starting with \( z_{\text{LAB}}(\Theta(\mathcal{M})) = \Theta'(\mathcal{M}) \).

Finally, the \( \Theta \) invariance of \( \bigcup \{ \Phi_a \} \) and (6.19) follow from (6.16) and (6.17).

\( \square \)

Since \((\mathcal{M}_1 \oplus \mathcal{M}_2) \land [1, \ell] = (\mathcal{M}_1 \land [1, \ell]) \oplus (\mathcal{M}_2 \land [1, \ell])\) it follows that the map \((\mathcal{M}_1, \mathcal{M}_2) \mapsto \mathcal{M}_1 \oplus \mathcal{M}_2\) is a continuous map from \( \mathcal{LAB} \times \mathcal{LAB} \to \mathcal{LAB} \) and so if \( \Phi_1, \Phi_2 \subset \mathcal{LAB} \) are compact then \( \Phi_1 \oplus \Phi_2 \) is compact and is therefore closed.

**Proposition 6.37.** If \( \mathcal{N} \) and \( \mathcal{M} \) are positive disjoint labels with \( \mathcal{N} \) finite and \( \mathcal{M} \) of finite type then \( \mathcal{N} \oplus \mathcal{M} \) is a label of finite type which is finitary (or simple) if \( \mathcal{M} \) is finitary (resp. simple). Moreover \( \mathcal{N} \oplus \mathcal{M} \) has the following properties:

1. If \( r \in \mathcal{N} \oplus \mathcal{M} \) then \( r = n + m \) with \( n \in \mathcal{N} \) and \( m \in \mathcal{M} \) uniquely determined by \( r \). In that case, \( \mathcal{N} \cap \mathcal{M} - r = (\mathcal{N} - n) \oplus (\mathcal{M} - m) \).

Furthermore,

\[
\Theta'(\mathcal{N} \oplus \mathcal{M}) = \Theta'(\mathcal{N}) \oplus \Theta(\mathcal{M}) \cup \Theta(\mathcal{N}) \oplus \Theta'(\mathcal{M}),
\]

\[
\max \mathcal{N} \oplus \mathcal{M} = (\max \mathcal{N}) \oplus (\max \mathcal{M}).
\]

\( \Theta(\mathcal{N}) \) is the finite set \( \{ \mathcal{N} - r \} \).

2. The map \( \oplus : \Theta(\mathcal{N}) \times \Theta(\mathcal{M}) \to \Theta(\mathcal{N} \oplus \mathcal{M}) \) is a homeomorphism inducing an isomorphism of compact semigroups \( j_{\oplus} : \mathcal{E}(\Theta(\mathcal{N})) \times \mathcal{E}(\Theta(\mathcal{M})) \to \mathcal{E}(\Theta(\mathcal{N} \oplus \mathcal{M})) \).

3. If \( \Psi \subset \mathcal{N} \) and \( \Phi \subset \mathcal{M} \) are closed and invariant then

\[
z_{\text{LAB}}(\Psi \oplus \Phi) = z_{\text{LAB}}(\Psi) \oplus \Phi \cup \Psi \oplus z_{\text{LAB}}(\Phi).
\]

And for every limit ordinal \( \alpha \)

\[
z_{\text{LAB},\alpha}(\Theta(\mathcal{N} \oplus \mathcal{M})) = \{ \mathcal{N} \} \oplus z_{\text{LAB},\alpha}(\Theta(\mathcal{M})).
\]

For \( \alpha = 0 \) or a limit ordinal and \( k \in \mathbb{N} \)

\[
z_{\text{LAB},\alpha+k}(\Theta(\mathcal{N} \oplus \mathcal{M})) = \bigcup_{r=0}^{k} z_{\text{LAB},\alpha+r}(\Theta(\mathcal{N})) \oplus z_{\text{LAB},\alpha+k-r}(\Theta(\mathcal{M})).
\]

4. For all \( k \in \mathbb{N} \)

\[
z_{\mathcal{N} \oplus \mathcal{M},k}(\mathcal{[[0]]}) = \bigcup_{r=0}^{k} z_{\mathcal{N},r}(\mathcal{[[0]]}) \oplus z_{\mathcal{M},k-r}(\mathcal{[[0]]}).
\]
For every limit ordinal $\alpha$ and $k \in \mathbb{Z}_+$

\[
z_{\mathcal{N} \oplus \mathcal{M}, \alpha + k}([[0]]) = \{\mathcal{N}\} \oplus z_{\mathcal{M}, \alpha}([[0]]) \cup \bigcup_{r=1}^{k} z_{\mathcal{N}, k-r}([[0]]) \oplus z_{\mathcal{M}, \alpha + r}([[0]])
\]

**Proof:** Again Theorem 4.38 says that $\mathcal{M}$ is of finite type and is finitary if $\mathcal{M}$ is.

The closed set $\Theta(N) \oplus \Theta(M) \cup \Theta(N) \oplus \Theta(M')$ contains $N \oplus M - r$ for all $r > 0$ and so contains $\Theta(N \oplus M)$. On the other hand, if either $n > 0$ or $m > 0$ then $r > 0$ and so $\Theta(N \oplus M)$ contains all $(N - n) \oplus (M - m)$ with either $n > 0$ or $m > 0$. Since $\oplus$ is continuous, it follows that $\Theta(N \oplus M)$ contains $\Theta(N) \oplus \Theta(M) \cup \Theta(N) \oplus \Theta(M')$.

That $\Theta(N) = \{N - r\}$ follows from Corollary 4.28.

Because of the disjoint support assumption, the continuous map $\oplus : [[N]] \times [[M]] \rightarrow [[N \oplus M]]$ is injective and so, by (6.19), it restricts to a homeomorphism from $\Theta(N) \times \Theta(M)$ onto $\Theta(N \oplus M)$. Notice that labels such as $N \cup M$ do not lie in $\Theta(N \oplus M)$. Furthermore, if \{N\} and \{M\} are nets in $[[N]]$ and $[[M]]$ then \{N \oplus M\} converges iff both \{N\} and \{M\} do, in which case the limit is $\text{LIM} \ \{N\} \oplus \text{LIM} \ \{M\}$. In particular, $P_r$ converges in $E(\Theta(N \oplus M))$ iff $(P_{n'}, P_{m'})$ converges in $E(\Theta(N)) \times E(\Theta(M))$. This implies that $j_{\mathfrak{I}}$ is continuous and surjective. Hence, it is a homeomorphism. It is a homomorphism by continuity since it is clearly a homomorphism on the dense submonoid \{(P_n, P_m)\}.

It therefore follows that for $\Phi_1 \subset \Theta(N)$ and $\Phi_2 \subset \Theta(M)$ closed invariant subspaces, $z_{\text{LAB}}(\Phi_1 \oplus \Phi_2) = z_{\text{LAB}}(\Psi \oplus \Phi \cup \Psi \cup z_{\text{LAB}}(\Phi))$. Then (6.23) follows by induction on $k$ since the operator $z_{\text{LAB}}$ commutes with union. For (6.22) we use induction on the limit ordinals together with 0 starting with $\alpha = 0$. Assume the result for all $\beta < \alpha$. From (6.23) we have

\[
\{\mathcal{N}\} \oplus z_{\text{LAB}, \beta + k}((\Theta(M))) \subset z_{\text{LAB}, \beta + k}((\Theta(N \oplus M)) \subset \{\mathcal{N}\} \oplus z_{\text{LAB}, \beta + k - r_0}((\Theta(M))
\]

where $z_{\text{LAB}, \beta}(\Theta(N)) = [[0]]$. Intersecting we obtain (6.22) for $\alpha = \beta + \omega$. Otherwise, $\alpha$ is an increasing limit of limit ordinals and the result follow from the induction hypothesis by intersecting.

For (6.24) and (6.25) we proceed by transfinite induction. First, starting from 0 or a limit ordinal $\alpha$ we use induction on $k \geq 1$. Observe that if $n > 0$ and $m \in z_{\text{LAB}, \alpha + 1}([[0]]) \setminus z_{\mathcal{M}, \alpha + r}([[0]])$ then $n + r \notin z_{\mathcal{N} \oplus \mathcal{M}, \alpha + r + 1}([[0]])$. This yields for $\beta$ a limit ordinal less than $\alpha$ and
$k \in \mathbb{N}$ that

\begin{equation}
\{N\} \oplus z_{M,\beta+k-r_0}^{*}([0]) \subset z_{N\oplus M,\beta+k}^{*}([0])
\end{equation}

where $z_{N,r_0}^{*}([0]) = \Theta(N)$. So by taking the unions we obtain the result for a limit ordinal $\alpha$ and $k = 0$.

\begin{definition}
For $M$ a nonzero finitary or simple label, the height of $\Theta(M)$ is $\alpha + 1$ where $\alpha$ is the ordinal with $z_{LAB,\alpha}(\Theta(M)) = [[0]]$. The height$^*$ is $\alpha + 1$ where $\alpha$ is the ordinal where $z_{M,\alpha}^{*}([\emptyset]) = \Theta'(M)$. Notice that $z_{M,\alpha}^{*}([\emptyset]) = z_{M,1+\alpha}^{*}([[0]])$ and so if $\alpha \geq \omega$ then $z_{M,\alpha}^{*}([\emptyset]) = z_{M,\alpha}^{*}([[0]])$.
\end{definition}

\begin{theorem}
For any countable ordinal $\alpha$ there exists a label $M$ which is both simple and finitary with height = height$^* = \alpha + 1$. Hence, $(X(M), S)$ is a topologically transitive WAP subshift with height = height$^* = \alpha + 1$.
\end{theorem}

\textbf{Proof:} First, let $N_n = \{k\chi(\ell_1) : 0 \leq k \leq n\}$. It is easy to see that $\Theta(N_n) = \{N_k : k \leq n\} \cup \{\emptyset\}$ has height and height$^*$ equal to $n + 1$. These are finite labels and so are both simple and finitary.

Now suppose that $\alpha$ is a countable limit ordinal, the limit of an increasing sequence $\beta^i$. By inductive hypothesis, we can choose for each $i$ a finitary and simple label $M^i$ so that $\Theta(M^i)$ has height $\beta^i + 1$, and height$^* \beta^i + 1$. By using a Gamow transformation we can assume that $\{M^i\}$ is a sequence with disjoint supports. By Proposition 6.36 $M = \bigcup \{M^i\}$ is finitary and simple and by (6.18) $\zeta_{LAB,\alpha}(\Theta(M)) = [[0]]$ and so $\Theta(M)$ has height $\alpha + 1$. By (6.19) it follows that $\zeta_{M,\alpha}^{*}([\emptyset]) = \Theta'(M)$ and so $\Theta(M)$ has height$^*$ equal to $\alpha + 1$.

Now for a countable limit ordinal $\alpha$ assume that $M$ is a finitary and simple label with height and height$^*$ equal to $\alpha + 1$. By using a Gamow transformation, we can assume that $\ell_1$ is not in the support of $M$. For $n \geq 1$, $N_n$ is a positive, finite label disjoint from $M$. By Proposition 6.36 $N_n \oplus M$ is finitary and simple and by (6.22) $z_{LAB,\alpha}(\Theta(N_n \oplus M)) = \Theta(N_n \oplus 0)$. Hence, $\Theta(N_n \oplus M)$ has height $\alpha + n + 1$. From (6.25) it follows that $\Theta(N_n \oplus M)$ has height$^* \alpha + n + 1$.

The results for $(X(M), S)$ follow from Theorem 6.34 and Theorem 6.35.

\vdash
7. Scrambled sets

Following Li and Yorke [31] a subset \( S \subset X \) is called scrambled for a dynamical system \((X,T)\) when every pair of distinct points of \( S \) is proximal but not asymptotic.

Recall that \( A(X,T) \) is the ideal of the enveloping semigroup \( E(X,T) \) consisting of the limit points of \( \{T^n\} \) as \(|n| \to \infty\). Let \( A_+(X,T) \) be the set of limit points of \( \{T^n\} \) as \( n \to \infty \), that is, we move only in the positive direction. Thus, \( \omega_T(x) = A_+(X,T)x \) for every \( x \in X \).

**Definition 7.1.** For a compact, metric dynamical system \((X,T)\) let \((x,y)\) be a pair in \( X \times X \).

(a) We call the pair \((x,y)\) proximal when it satisfies the following equivalent conditions:

- (i) \( \lim \inf_{n>0} d(T^n(x),T^n(y)) = 0 \).
- (ii) There exists a sequence \( n_i \to \infty \) such that \( \lim d(T^{n_i}(x),T^{n_i}(y)) = 0 \).
- (iii) There exists \( p \in A_+(X,T) \) such that \( px = py \).
- (iv) There exists \( u \) a minimal idempotent in \( A_+(X,T) \) such that \( ux = uy \).

We denote by \( PROX(X,T) \) (or just \( PROX \) when the system is clear) the set of all proximal pairs.

(b) We call the pair \((x,y)\) asymptotic when it satisfies the following equivalent conditions:

- (i) \( \lim \sup_{n>0} d(T^n(x),T^n(y)) = 0 \).
- (ii) \( \lim_{n>0} d(T^n(x),T^n(y)) = 0 \).
- (iii) For all \( p \in A_+(X,T) \) \( px = py \).

We denote by \( ASYMP(X,T) \) (or just \( ASYMP \) when the system is clear) the set of all asymptotic pairs.

(c) We call the pair \((x,y)\) a Li-Yorke pair when it is proximal but not asymptotic.

(d) The system \((X,T)\) is called proximal when all pairs are proximal, i.e. \( PROX = X \times X \). It is called completely scrambled when all non-diagonal pairs are Li-Yorke. That is, the system is proximal, but \( ASYMP = \Delta_X \).

Observe that the set \( \{p \in A_+(X,T) : px = py\} \) is a closed left ideal if it is nonempty and so it then contains minimal idempotents. This shows that (iii) \( \iff \) (iv) in (a). The remaining equivalences are obvious.
Remark 7.2. This notion of proximality actually refers to the action of the semigroup \( \{ T^n : n \in \mathbb{Z}_+ \} \). The usual definition of proximality would be: \( x \) and \( y \) are proximal if there exists a sequence \( n_i \in \mathbb{Z} \) with \( |n_i| \to \infty \) such that \( \lim d(T^{n_i}(x), T^{n_i}(y)) = 0 \).

Lemma 7.3. For \( x \in X \) and \( n \in \mathbb{N} \) the pair \( (x, T^n(x)) \) is proximal iff \( T^n(y) = y \) for some \( y \in \omega_T(x) \). The pair \( (x, T^n(x)) \) is asymptotic iff \( T^n(y) = y \) for every \( y \in \omega_T(x) \).

Proof: \( pT^n(x) = T^n(px) \) and so \( px = pT^n(x) \) iff \( px = T^n(px) \). The pair is proximal (or asymptotic) iff \( px = T^n(px) \) for some \( p \in A_+(X, T) \) (resp. for all \( p \in A_+(X, T) \)).

\( \square \)

We recall the following, see, e.g. [7] Proposition 2.2.

Proposition 7.4. A compact, metric dynamical system \( (X, T) \) is proximal iff there exists a fixed point \( e \in X \) which is the unique minimal subset of \( X \), i.e. \( (X, T) \) is a minCT system. Consequently, \( (X, T^{-1}) \) is proximal if \( (X, T) \) is.

Proof: If \( u \) is a minimal idempotent then \( ux \) is a minimal point for every \( x \in X \). So if \( e \) is the unique minimal point of \( X \), then \( ux = e \) for every \( x \in X \) and every minimal idempotent \( u \). Hence, every pair is proximal.

Assume now that \( (X, T) \) is proximal. For any \( x \in X \), the pair \( (x, T(x)) \) is proximal and so there exists \( p \in A_+(X, T) \) such that \( px = pT(x) = T(px) \) and so \( e = px \) is a fixed point. A pair of distinct fixed points is not proximal and so \( e \) is the unique fixed point. Hence, \( e \) is in the orbit closure of every point and so \( \{e\} \) is the only minimal set.

Since the minimal subsets for \( T \) and \( T^{-1} \) are the same, it follows that \( (X, T^{-1}) \) is proximal.

\( \square \)

Thus, we obtain the following obvious corollary. Compare Proposition 1.15

Corollary 7.5. A compact, metric dynamical system \( (X, T) \) is completely scrambled iff it is a minCT system and \( A_+(X, T) \) distinguishes points of \( X \).

\( \square \)

Completely scrambled systems were introduced by Huang and Ye [27] who provided a rich supply of examples, but all appear to be of height the first countable ordinal.
In contrast with proximal systems there exist completely scrambled systems \((X, T)\) whose inverse \((X, T^{-1})\) is not completely scrambled.

**Example 7.6.** Begin with \((Y, F)\) a completely scrambled system with fixed point \(e\). Let \((X, T)\) be the quotient space of the product system \((Y \times \{0, 1\}, F \times id_{\{0,1\}})\) obtained by identifying \((e, 0)\) with \((e, 1)\) to obtain the fixed point denoted \(e\) in \(X\). Let \(X_0\) and \(X_1\) be the images of \(Y \times \{0\}\) and \(Y \times \{1\}\) in \(X\). Since \((X, T)\) has a unique fixed point \(e\) we can construct a sequence \(\{x^n : n \in \mathbb{Z}_+\}\) so that

\[
\begin{align*}
&\bullet x^0 = e. \\
&\bullet \{d(x^n, T(x^n))\} \to 0 \text{ as } n \to \infty. \\
&\bullet \text{For every } N \in \mathbb{N} \text{ the set } \{x^i : i \geq N\} \text{ is dense in } X.
\end{align*}
\]

Now for \(n \in \mathbb{Z}\) let \(z^n \in X \times [0,1]\) be defined by

\[
\begin{align*}
z^n &= \begin{cases} 
(x^n, 1/(n+1)) & \text{for } n \geq 0, \\
(e, 1/(-n+1)) & \text{for } n < 0.
\end{cases}
\end{align*}
\]

Let 
\[
\hat{X} = X \times \{0\} \cup \{z^n : n \in \mathbb{Z}\}.
\]

Let \(\hat{T}(x, 0) = (T(x), 0)\) and \(\hat{T}(z^n) = z^{n+1}\).

The system \((\hat{X}, \hat{T})\) is topologically transitive with fixed point \((e, 0)\) the unique minimal set. Hence, the system is proximal. Since every orbit in \(X\) is confined to either \(X_0\) or \(X_1\) it follows that no point \(z^n\) is asymptotic to a point in \(X \times \{0\}\). By Lemma 7.3 no two distinct points on the \(z^n\) orbit are asymptotic. Hence, \((\hat{X}, \hat{T})\) is completely scrambled. However, the inverse \((\hat{X}, \hat{T}^{-1})\) is not since \(\{z^n\} \to (e, 0)\) as \(n \to -\infty\).

\[
\square
\]

By a result of Schwartzman (see Gottschalk and Hedlund [25, Theorem 10.36]) an expansive system admits non-diagonal asymptotic pairs. It follows that no subshift can be completely scrambled. However, we note that the subshifts which arise from labels of finite type are pretty close.

**Proposition 7.7.** If \(M_1, M_2\) are two different labels then the following are equivalent.

(i) The pair \((S^{n_1}x[M_1], S^{n_2}x[M_2])\) is asymptotic for \(S\) or \(S^{-1}\) for some \(n_1, n_2 \in \mathbb{Z}\).

(ii) The pair \((S^{n_1}x[M_1], S^{n_2}x[M_2])\) is asymptotic for both \(S\) and \(S^{-1}\) for all \(n_1, n_2 \in \mathbb{Z}\).

(iii) \(\{M_1, M_2\} = \{\emptyset, 0\}\).
Proof: Since \( R_S(x[0]) = e = x[\emptyset] \), it is clear that (iii) implies (ii) and (ii) implies (i) is obvious.

Now assume that \( \{M_1, M_2\} \neq \{\emptyset, 0\} \). By renumbering we can assume that there exists \( r > 0 \) such that \( r \in M_1 \setminus M_2 \). Let \( \{t^i\} \) be a sequence of expanding times with length vector \( r \) and such that \( |j_\nu(t^i)| \to \infty \). Then by Theorem 5.17 \( \text{Lim} \ S^{t^i}(x[M_1]) = x[M_1 - r] \) and this is not the fixed point \( e \) since \( r \in M_1 \). Hence, for any \( n_1 \in \mathbb{Z} \), \( \text{Lim} S^{t^i}(S^{n_1}(x[M_1])) = S^{n_1}(x[M_1 - r]) \neq e \). On the other hand, since, \( r \not\in M_2 \), \( \text{Lim} S^{t^i}(S^{n_2}(x[M_2])) = S^{n_2}(x[M_2 - r]) = e \). Thus, the pair \( (S^{n_1}x[M_1], S^{n_2}x[M_2]) \) is not asymptotic for \( S \) or for \( S^{-1} \). This prove the contrapositive of \((i) \Rightarrow (iii)\).

\[ \square \]

Corollary 7.8. For any positive label \( M \) the set \( \{ S^n(x[N]) : N \neq 0 \in \Theta(M), n \in \mathbb{Z} \} \) is a scrambled subset for \((X(M), S)\) and for \((X(M), S^{-1})\). If \( M \) is a label of finite type then this set is the complement of the orbit of \( x[0] \) in \( X(M) \).

Proof: That the set is scrambled is clear from Proposition 7.7. In the finite type case, Corollary 5.21 (a) implies that we are excluding only the orbit of \( x[0] \) from the set.

\[ \square \]

Definition 7.9. An inverse sequence in \( \mathcal{LAB} \) is a sequence \( \{M^i, r^i : i \in \mathbb{Z}_+\} \) with \( r^i > 0 \) in \( M^i \) and such that \( M^i = M^{i+1} - r^{i+1} \) for \( i > 0 \). For the associated inverse sequence \( p_{i+1} : (X(M^{i+1}), S) \to (X(M^i), S) \) we let \((X(\{M^i, r^i\}), S)\) denote the inverse limit of the system.

Theorem 7.10. Let \( \{M^i, r^i\} \) be an inverse sequence in \( \mathcal{LAB} \). The inverse limit system \((X(\{M^i, r^i\}), S)\) is a topologically transitive, compact metrizable system. If each \( M^i \) is of finite type then the limit system \((X(\{M^i, r^i\}), S)\) and its inverse \((X(\{M^i, r^i\}), S^{-1})\) are completely scrambled. If each \( M^i \) is either finitary or simple then the limit system is WAP.

Proof: Topologically transitive and WAP systems are closed under inverse limits. In this case, each map \( p_{i+1} \) is surjective as required because it maps the transitive point \( x[M^{i+1}] \) of \( X(M^{i+1}) \) onto the transitive point \( x[M^i] \) of \( X(M^i) \) since \( M^i = M^{i+1} - r^{i+1} \). In particular, the sequence \( x^* = \{x[M^i]\} \) is a transitive point for the inverse limit.
The point \( e \) associated with the sequence \( \{ x[0] \} \) is a fixed point in \( X(\{M^i, r_i\}) \). A minimal subset of the limit space projects to a minimal subset of \( X(M^i) \) for each \( i \). If \( M^i \) is of finite type then this minimal subset is \( \{ e \} \subset X(M^i) \). Thus, if all are of finite type, the fixed point is the only minimal subset of the limit and so \( (X(\{M^i, r_i\}), S) \) is proximal by Proposition 7.4.

Notice that if \( x \in X(M^i) \) is not equal to the fixed point \( e \) then \( x[0] \not\in P_{r_i^{-1}}(x) \). If \( x, y \) are distinct points of \( X(\{M^i, r_i\}) \) then for sufficiently large \( i \) they project to distinct points of \( X(M^i) \) with neither projecting to \( x[0] \) in \( X(M^i) \). In the finite type case it then follows from Corollary 7.8 that for sufficiently large \( i \), \( x \) and \( y \) project to a non-asymptotic pair. Consequently, the pair \( (x, y) \) is not asymptotic in \( X(\{M^i, r_i\}) \).

\[ \square \]

**Remark 7.11.** Since a transitive point for \( X(\{M^i, r_i\}) \) projects to a transitive point on each \( X(M^i) \) it follows that the transitive points for \( X(\{M^i, r_i\}) \) are all on the orbit of \( x^* \) described above and so \( x^* \) is isolated when the labels \( M^i \) are of finite type.

For the construction of our examples, we need the following. Recall that (6.20) implies that if \( M_1 \) is a finite label and \( M_2 \) is a label with supports disjoint from those of \( M_1 \), then

\[
\Theta'(M_1 \oplus M_2) = \Theta'(M_1) \oplus \Theta(M_2) \cup \Theta(M_1) \oplus \Theta'(M_2).
\]

**Lemma 7.12.** Let \( r \) be a positive finite vector with support disjoint from those in \( \text{Supp } M \) for some nonempty label \( M \). Then \( P_r(\Theta'(\langle r \rangle) \oplus \Theta(M)) = \{ \emptyset \} \) and on \( \{ \langle r \rangle \} \oplus \Theta(M) \) \( P_r \) is a bijection onto \( \Theta(M) \).

**Proof:** Since \( r \) is not an element of any label in \( \Theta'(\langle r \rangle) \oplus \Theta(M) \) it follows that all of these labels are mapped to \( \emptyset \).

By (6.20) every label of \( \Theta(\langle r \rangle \oplus M) \) is of the form \( N_1 \oplus N_2 \) with \( N_1 \in \Theta(\langle r \rangle) \) and \( N_2 \in \Theta(M) \). If \( N_1 \neq \langle r \rangle \) then \( P_r \) maps \( N_1 \oplus N_2 \) to \( \emptyset \). If \( N_1 = \langle r \rangle \) then \( P_r \) maps \( N_1 \oplus N_2 \) to \( N_2 \). Hence, for any \( N_2 \in \Theta(M) \) the unique label of the form \( \langle r \rangle \oplus N \) which is mapped to \( N_2 \) has \( N = N_2 \).

\[ \square \]

**Example 7.13.** Let \( \{ r_i \} \) a sequence of positive \( N \)-vectors all with disjoint supports and let \( M \) be a finitary label with the sets in \( \text{Supp } M \) disjoint from the supports of the sequence.

Let \( N^0 = \{ \emptyset \} \) and \( N^{i+1} = (r^{i+1}) \oplus N^i \) define an increasing sequence of finite labels. Define \( \{ M^i = N^i \oplus M, r_i \} \), an inverse sequence of finitary
labels. For each $i$ Lemma 7.12 implies that the preimage of $\emptyset$ by $P^i_{r+1}: \Theta(M^{i+1}) \to \Theta(M^i)$ is countable and the preimage of every other point is a singleton. It follows that the limit system $(X(\{M^i, r^i\}), S)$ and its inverse $(X(\{M^i, r^i\}), S^{-1})$ are completely scrambled, topologically transitive, countable WAPs.

Notice that $X(M)$ is a factor of $X(\{M^i, r^i\})$. Hence, if we choose $M$ with height greater than some countable ordinal $\alpha$ then $X(\{M^i, r^i\})$ has height greater than $\alpha$.

Following Huang and Ye we can take countable products of copies of these examples to get completely scrambled WAP systems on the Cantor set with arbitrarily large heights. However, these examples will not be topologically transitive.

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Index

Index of terms

action map (homomorphism) 6
adherence semigroup 5
almost periodic (AP) 2
asymptotic pair 102
Bernoulli shift, $S$ 3
Birkhoff center 7
coaescence 17
E-coalescence 18
center periodic (CP) 11
center trivial (CT) 11
completely scrambled 102
cross-section 71
expansive 15
external limit 49
enveloping semigroup 5
equicontinuous 15
equicontinuity point 15
almost equicontinuous (AE) 15
hereditarily almost equicontinuous (HAE) 15
locally equicontinuous (LE) 16
uniformly equicontinuous 15
expanding (or b-expanding)
functions 56
associated with $L$ and $\ell$ 56
expanding times 58
expansion of length $r$ 58
length vector 58
truncations 58
extensions 58
lowest term $k^r(j_r)$ 58
f-contains 33
Gamow transformations 91
height (for dynamical systems) 10
height* (for dynamical systems) 24
height (for labels) 95, 101
height* (for labels) 101
hereditary collection (of subsets of $\mathbb{N}$) 52
idempotent 6
independent set (for a dynamical system) 81
independent set (for a label) 81
labels 30
of finite type 30
size bounded 31
recurrent 40
strongly recurrent 41
strongly recurrent set 41
finitary 47
external limit set 49
external 49
simple 49
flat 83
non-null 81
non-tame 81
positive 31
Li-Yorke pair 102
minimal trivial (minCT) 25
null (dynamical system) 81
proximal (pair) 102
proximal (system) 102
recurrent point 7
rigid 18
weakly rigid 18, 14
uniformly rigid 18
roof (of a label) 30
scrambled 102
semiadditive (subset of $\mathbb{Z}$) 87
semi-trivial (ST) 11
size of $m$ 30
strongly recurrent set 41
subshift 3
symmetric (subset of $\mathbb{Z}$) 87
tame 81
thick set 56
topological transitivity 5
chain transitivity 5
transitive point 5
translation finite (TF) 79
weakly almost periodic (WAP) 2, 7
weak mixing 5, 11
Index of symbols

\[ [a \pm b], [\pm b] \]
\[ A_r \]
\[ A[M] \]
\[ A(\Phi) \]
\[ A(X, T) \]
\[ A_+(X, T) \]
\[ \text{ASYMP} \]
\[ \mathcal{E}(\Phi) \]
\[ E(X, T) \]
\[ F(m) \]
\[ FIN(\mathbb{N}) \]
\[ \beta FIN(\mathbb{N}) \]
\[ \beta^* FIN(\mathbb{N}) \]
\[ \beta^{**} FIN(\mathbb{N}) \]
\[ \gamma FIN(\mathbb{N}) \]
\[ \mathcal{F}(M, L) \]
\[ \Phi(Y) \]
\[ \|\Phi\| \]
\[ ISO(M) \]
\[ Inc(\Phi) \]
\[ IP(A) \]
\[ k^f(n) \]
\[ LAB \]
\[ LAB_+ \]
\[ LIMINF \]
\[ LIMSUP \]
\[ LIM \]
\[ M, N \text{ labels} \]
\[ [\![M]\!] \]
\[ max M \]
\[ M \oplus N \]
\[ \rho(M) \]
\[ m, r \text{ N-vectors} \]
\[ |m| \]
\[ OT \text{ (relations)} \]
\[ \rho_T \]
\[ \rho_T^* \]
\[ \alpha_T \]
\[ \omega_T \]
\[ \mathcal{O}_T(x) \text{ or } \mathcal{O}(x), \overline{\mathcal{O}_T}(x) \text{ (orbit and orbit closure of } x) \]
\[ p_x \]
\[ P_r \]
\[ \mathcal{P}_f(L) \]

PROX
RECUR
S (shift)
\[ supp m \]
\[ supp x \]
\[ Supp M \]
SYM
\[ \langle S \rangle \text{ (label generated by } S) \]
\[ t^{-F} \]
\[ \Theta(M) \]
\[ \Theta'(M) \]
\[ x[M] \]
\[ X(M) \]
\[ (X, T) \text{ (dynamical system)} \]
\[ (X, S) \text{ (subshift)} \]
\[ \chi(A) \]
set operators:
\[ z_{\text{CAN}} \]
\[ z_{\text{LAB}} \]
\[ z_{\text{LIM}} \]
\[ z_{\text{NW}} \]
\[ z_M \]
WAP SYSTEMS AND LABELED SUBSHIFTS

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