LARGE DEVIATION PRINCIPLE FOR STOCHASTIC HEAT EQUATION WITH GENERAL ROUGH NOISE

RUINAN LI, RAN WANG*, AND BEIBEI ZHANG

Abstract We study Freidlin-Wentzell’s large deviation principle for one dimensional nonlinear stochastic heat equation driven by a Gaussian noise:

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} = \frac{\partial^2 u^\varepsilon(t,x)}{\partial x^2} + \sqrt{\varepsilon} \sigma(t,x,u^\varepsilon(t,x)) \dot{W}(t,x), \quad t > 0, \ x \in \mathbb{R},$$

where \( \dot{W} \) is white in time and fractional in space with Hurst parameter \( H \in (\frac{1}{4}, \frac{1}{2}) \). Recently, Hu and Wang (Ann. Inst. Henri Poincaré Probab. Stat. 58 (2022) 379-423) studied the well-posedness of this equation without the technical condition of \( \sigma(0) = 0 \) which was previously assumed in Hu et al. (Ann. Probab. 45 (2017) 4561-4616). We adopt a new sufficient condition proposed by Matoussi et al. (Appl. Math. Optim. 83 (2021) 849-879) for the weak convergence criterion of the large deviation principle.

Keywords Stochastic heat equation; fractional Brownian motion; large deviation principle; weak convergence approach.

Mathematics Subject Classification (2000) Primary 60F10; secondary 60H15.

CONTENTS

1. Introduction 1
2. Preliminaries and main result 3
2.1. Covariance structure and stochastic integration 3
2.2. Stochastic heat equation 5
2.3. A general criterion for the large deviation principle 6
2.4. Main result 7
3. Skeleton equation 7
4. Verification of Condition 2.8 (a) 24
5. Verification of Condition 2.8 (b) 31
6. Appendix 45
References 47

1. Introduction

In this paper, we consider the following nonlinear stochastic heat equation (SHE in short) driven by a Gaussian noise which is white in time and fractional in space:

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} = \frac{\partial^2 u^\varepsilon(t,x)}{\partial x^2} + \sqrt{\varepsilon} \sigma(t,x,u^\varepsilon(t,x)) \dot{W}(t,x), \quad t > 0, \ x \in \mathbb{R}, \quad (1.1)$$

*Corresponding author. E-mail: rwang@whu.edu.cn.
where $\epsilon > 0$, $u^\epsilon(0, x) = u_0(x)$ and $W(t, x) = \frac{\partial^2 W}{\partial t \partial x}$ with that $W(t, x)$ being a centered Gaussian field with covariance given by

$$E[W(t, x)W(s, y)] = \frac{1}{2} (s \wedge t) \left(|x|^{2H} + |y|^{2H} - |x - y|^{2H}\right),$$

for some $\frac{1}{4} < H < \frac{1}{2}$. The covariance of the noise $\hat{W}$ is given by

$$E[\hat{W}(t, x)\hat{W}(s, y)] = \delta_0(t - s)\Lambda(x - y),$$

where $\Lambda$ is a distribution, whose Fourier transform is the measure $\mu(d\xi) = c_{1,H}|\xi|^{1-2H}d\xi$ with

$$c_{1,H} = \frac{1}{2\pi} \Gamma(2H + 1) \sin(\pi H).$$

$\Lambda$ can be formally written as $\Lambda(x - y) = H(2H - 1) \times |x - y|^{2H-2}$. In addition, the measure $\mu$ satisfies the integrability condition $\int_\mathbb{R} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty$. However, the corresponding covariance $\Lambda$ is not locally integrable and fails to be non-negative when $H \in (\frac{1}{4}, \frac{1}{2})$. It does not satisfy the classical Dallang’s condition in [10] (there $\Lambda$ is given by a non-negative locally integrable function). See [1, 17] for more details. For this reason, the approaches used in references [8, 10, 11, 29, 30] cannot handle such rough covariance.

Recently, many authors studied the existence and uniqueness for the solutions of stochastic partial differential equations (SPDEs) driven by Gaussian noises rough in space, see, e.g., [17–19, 23, 34]. We refer to [16] and [33] for surveys. When the diffusion coefficient is an affine function $\sigma(x) = ax + b$ and the initial value function is bounded and $H$-Hölder continuous, Balan et al. [1, 2] proved the existence and uniqueness of a mild solution to SHE (1.1) by using technique of Fourier analysis, and established the Hölder continuity of the solution. For the general nonlinear coefficient $\sigma$, which has a Lipschitz derivative and satisfies $\sigma(0) = 0$, the existence and uniqueness of the solution were proved by Hu et al. [17]. Under this condition, the large deviations, the moderate deviations and the transportation inequalities were studied in [20], [22], [9], respectively.

In [19], Hu and Wang removed the technical and unusual condition of $\sigma(0) = 0$ and they proved the well-posedness of the solution to Eq. (1.1) under Condition (H) in Subsection 2.3 below. Without the condition of $\sigma(0) = 0$, the solution is no longer in the space $Z^p_T$ (see [17] or (2.12) in Subsection 2.3 of this paper with $\lambda(x) \equiv 1$). Thus, to relax the restriction, Hu and Wang [19] introduced a decay weight (as the spatial variable $x$ goes to infinity) to enlarge the solution space $Z^p_T$ to a weighted space $Z^p_{\Lambda,T}$ for some suitable power decay function $\lambda(x)$, see Subsection 2.3 below for details.

The aim of this paper is to establish a large deviation principle (LDP) for the solution $u^\epsilon$ of (1.1) as $\epsilon \to 0$. An important approach of investigating the LDP is the well-known weak convergence method (see, e.g., [3–7, 13]), which is mainly based on the variational representation formula for measurable functionals of Brownian motion. For some relevant LDP results by using the weak convergence method, we refer to [24, 32, 38] and references therein. In particular, Liu et al. [26] and Xiong and Zhai [37] proved the LDP for a large class of SPDEs with locally monotonic coefficients driven by Brownian motions or Lévy noises, respectively. However, the frameworks of [26] and [37] cannot be applied to SHE (1.1) because of the spatial rough noise with $H \in (\frac{1}{4}, \frac{1}{2})$.

Comparing with the large deviation principle for SHE (1.1) under the assumption of $\sigma(0) = 0$ in Hu et al. [20], our result and its proof are in the weighted space $Z^p_{\Lambda,T}$. Notice that the introduction of the weight brings many difficulties. See Hu and Wang [19] for the excellent analysis. In this paper, we adopt a new sufficient condition for the LDP (see Condition 2.8 below) which is proposed by Matoussi, Sabbagh and Zhang [28]. This method has been successfully applied to the study of LDP for SPDEs, see, e.g., [12, 14, 15, 25, 35, 36].
This paper is organized as follows. In Section 2, we first present the framework introduced in [19] and recall a general criterion for the LDP based on the weak convergence criterion in [28], then we formulate the main result of the present paper. In Section 3, the associated skeleton equation is studied. Sections 4 and 5 are devoted to verifying the two conditions for the weak convergence criterion. Finally, we give some useful lemmas in Appendix.

We adopt the following notations throughout this paper. We always use $C_\alpha$ to denote a constant dependent on the parameter $\alpha$, which may change from line to line. $A \leq B$ ($A \geq B$, resp.) means that $A \leq CB$ ($A \geq CB$, resp.) for some positive universal constant $C$, and $A \approx B$ if and only if $A \leq B$ and $A \geq B$.

2. Preliminaries and main result

In this section, we first give some preliminaries of SHE (1.1), then we state the weak convergence criterion for the LDP in [28] and give the main result of this paper.

2.1. Covariance structure and stochastic integration. Recall some notations from [17] and [19]. Denote by $\mathcal{D} = \mathcal{D}(\mathbb{R})$ the space of smooth functions on $\mathbb{R}$ with compact support, and by $\mathcal{D}'$ the dual of $\mathcal{D}$ with respect to the $L^2(\mathbb{R}, dx)$. The Fourier transform of a function $f \in \mathcal{D}$ is defined as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-i\xi x}f(x)dx,$$

and the inverse Fourier transform is then given by $\mathcal{F}^{-1}g(x) = \frac{1}{2\pi} \mathcal{F}g(-x)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. The noise $\hat{W}$ is a zero-mean Gaussian family $\{W(\phi), \phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})\}$ with covariance structure given by

$$\mathbb{E}[W(\phi)W(\psi)] = c_{1,H} \int_{\mathbb{R}_+} \mathcal{F}\phi(s, \xi)\mathcal{F}\psi(s, \xi) \cdot |\xi|^{1-2H}d\xi ds,$$  \hspace{1cm} (2.1)

where $H \in (\frac{1}{4}, \frac{1}{2})$, $c_{1,H}$ is given in (1.3), and $\mathcal{F}\phi(s, \xi)$ is the Fourier transform with respect to the spatial variable $x$ of the function $\phi(s, x)$. Let $\mathcal{F}_t$ be the filtration generated by $W$. That is

$$\mathcal{F}_t = \sigma\{W(\phi(x)1_{[0,r]}(s)) : r \in [0, t], \phi \in \mathcal{D}(\mathbb{R})\}.$$  

Then (2.1) defines a Hilbert scalar product on $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$. Denote $\mathfrak{H}$ the Hilbert space obtained by completing $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ with respect to this scalar product.

**Proposition 2.1.** [19, Proposition 2.1], [31, Theorem 3.1] The function space $\mathfrak{H}$ is a Hilbert space equipped with the scalar product

$$\langle \phi, \psi \rangle_{\mathfrak{H}} = c_{2,H} \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}} \mathcal{F}\phi(t, \xi)\mathcal{F}\psi(t, \xi) \cdot |\xi|^{1-2H}d\xi \right) dt$$

$$= c_{2,H} \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}^2} \left[ \phi(t, x+y) - \phi(t, x) \right] \cdot \left[ \psi(t, x+y) - \psi(t, x) \right] \cdot |y|^{2H-2}dxdy \right) dt,$$  \hspace{1cm} (2.2)

where

$$c_{2,H} = \left( \frac{1}{2} - H \right)^\frac{1}{2} H^\frac{1}{2} \left[ \Gamma \left( H + \frac{1}{2} \right) \right]^{-1} \left( \int_0^\infty \left[ (1 + t)^{H-\frac{1}{2}} - t^{H-\frac{1}{2}} \right]^2 dt + \frac{1}{2H} \right)^\frac{1}{2}. \hspace{1cm} (2.3)$$

The space $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ is dense in $\mathfrak{H}$. 
Let $\mathcal{H}$ be the Hilbert space obtained by completing $\mathcal{D}(\mathbb{R})$ with respect to the following scalar product:

$$
\langle \phi, \psi \rangle_{\mathcal{H}} = c_{1,H} \int_{\mathbb{R}} \mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)} \cdot |\xi|^{2H-2d} \, d\xi 
$$

$$
= c_{2,H} \int_{\mathbb{R}} [\phi(x,y) - \phi(x)] \cdot [\psi(x,y) - \psi(x)] \cdot |y|^{2H-2} \, dx \, dy, \quad \forall \phi, \psi \in \mathcal{D}(\mathbb{R}).
$$

(2.4)

**Definition 2.2.** An elementary process $g$ is a process given by

$$
g(t, x) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \mathbf{1}_{(a_i, b_i]}(t) \mathbf{1}_{(b_j, l_j]}(x),
$$

where $n$ and $m$ are finite positive integers, $0 \leq a_1 < b_1 < \cdots < a_n < b_n < \infty$, $h_j < l_j$ and $X_{i,j}$ are $\mathcal{F}_{a_i}$-measurable random variables for $i = 1, \ldots, n$, $j = 1, \ldots, m$. The stochastic integral of an elementary process with respect to $W$ is defined as

$$
\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(t,x)W(dt, dx) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} W(\mathbf{1}_{(a_i, b_i]} \otimes \mathbf{1}_{(h_j, l_j]})
$$

$$
= \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \left[ W(b_i, t_j) - W(a_i, t_j) - W(b_i, h_j) + W(a_i, h_j) \right].
$$

(2.5)

Let $\Lambda_H$ be the space of predictable processes $g$ defined on $\mathbb{R}_+ \times \mathbb{R}$ such that almost surely $g \in \mathcal{F}$ and $\mathbb{E}[\|g\|_{\mathcal{H}}^2] < \infty$. According to Proposition 2.3 of [19], we know that the space of the elementary processes defined in Definition 2.2 is dense in $\Lambda_H$. Hence, for any $g \in \Lambda_H$, the stochastic integral $\int_{\mathbb{R}_+ \times \mathbb{R}} g(t,x)W(dt, dx)$ can be defined as the $L^2(\Omega)$ limit of stochastic integrals of the elementary processes approximating $g(t, x)$ in $\Lambda_H$, and

$$
\mathbb{E} \left[ \left( \int_{\mathbb{R}_+ \times \mathbb{R}} g(t,x)W(dt, dx) \right)^2 \right] = \mathbb{E} [\|g\|_{\mathcal{H}}^2].
$$

(2.6)

Let $\{e_k; k \geq 1\}$ be an orthonormal basis of the Hilbert space $\mathcal{H}$. The Gaussian field $W$ in (1.2) admits a representation:

$$
W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k,
$$

where $\{\beta_k; k \geq 1\}$ is a sequence of independent standard Brownian motions. The stochastic integral against $W$ can be expressed as

$$
\int_{0}^{T} \int_{\mathbb{R}} g(s,x)W(ds, dx) = \sum_{k=1}^{\infty} \int_{0}^{T} \langle g(s, \cdot), e_k \rangle_{\mathcal{H}} d\beta_k(s).
$$

Let $(B, \| \cdot \|_B)$ be a Banach space with the norm $\| \cdot \|_B$. Let $H \in (\frac{1}{4}, \frac{1}{2})$ be a fixed number. For any function $f : \mathbb{R} \to B$, denote

$$
\mathcal{N}^{B}_{\frac{1}{2}-H} f(x) := \left( \int_{\mathbb{R}} \|f(x+h) - f(x)\|^2_B \cdot |h|^{2H-2} \, dh \right)^{\frac{1}{2}},
$$

(2.7)

if the above quantity is finite. When $B = \mathbb{R}$, we abbreviate the notation $\mathcal{N}_{\frac{1}{2}-H}^{\mathbb{R}} f$ as $\mathcal{N}_{\frac{1}{2}-H} f$. As in [17], when $B = L^p(\Omega)$, we denote $\mathcal{N}_{\frac{1}{2}-H}^{B}$ by $\mathcal{N}_{\frac{1}{2}-H, p}$:

$$
\mathcal{N}_{\frac{1}{2}-H, p} f(x) := \left( \int_{\mathbb{R}} \|f(x+h) - f(x)\|^2_{L^p(\Omega)} \cdot |h|^{2H-2} \, dh \right)^{\frac{1}{2}}.
$$

(2.8)
The following Burkholder-Davis-Gundy’s inequality is well-known (see, e.g., [17]).

**Proposition 2.3.** [17, Proposition 3.2] Let $W$ be the Gaussian noise with the covariance (2.1), and let $f \in \Lambda_H$ be a predictable random field. Then, we have for any $p \geq 2$,

$$
\left\| \int_0^t \int_\mathbb{R} f(s,y)W(ds,dy) \right\|_{L^p(\Omega)} \leq \sqrt{4pc_H} \left( \int_0^t \int_\mathbb{R} \left[ \mathcal{N}_{1/2-H,p} f(s,y) \right]^2 dy ds \right)^{\frac{1}{2}},
$$

(2.9)

where $c_H$ is a constant depending only on $H$ and $\mathcal{N}_{1/2-H,p} f(s,y)$ denotes the application of $\mathcal{N}_{1/2-H,p}$ to the spatial variable $y$.

### 2.2. Stochastic heat equation.

Let $\mathcal{C}([0,T] \times \mathbb{R})$ be the space of all continuous real-valued functions on $[0,T] \times \mathbb{R}$, equipped with the metric:

$$
d_{\mathcal{C}}(u, v) := \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0 \leq t \leq T, |x| \leq n} \left( |u(t,x) - v(t,x)| \wedge 1 \right).
$$

(2.10)

Define the weighted function

$$
\lambda(x) := c_H (1 + |x|^2)^H -1,
$$

(2.11)

where $c_H$ is a constant such that $\int_\mathbb{R} \lambda(x) dx = 1$. For any $p \geq 2$ and $\frac{1}{4} < H < \frac{1}{2}$, we introduce a norm $\| \cdot \|_{Z^p_{\lambda,T}}$ for a random field $v = \{v(t,x)\}_{(t,x) \in [0,T] \times \mathbb{R}}$ as follows:

$$
\|v\|_{Z^p_{\lambda,T}} := \sup_{t \in [0,T]} \left\| v(t, \cdot) \right\|_{L^p_\lambda(\Omega \times \mathbb{R})} + \sup_{t \in [0,T]} \mathcal{N}_{1/2-H,p}^* v(t),
$$

(2.12)

where

$$
\|v(t, \cdot)\|_{L^p_\lambda(\Omega \times \mathbb{R})} := \left( \int_\mathbb{R} \mathbb{E} \left[ |v(t,x)|^p \right] \lambda(x) dx \right)^{\frac{1}{p}}
$$

(2.13)

and

$$
\mathcal{N}_{1/2-H,p}^* v(t) := \left( \int_\mathbb{R} \left\| v(t, \cdot) - v(t, \cdot + h) \right\|_{L^p_\lambda(\Omega \times \mathbb{R})}^2 \cdot |h|^{2H-2} dh \right)^{\frac{1}{2}}.
$$

(2.14)

Denote $Z^p_{\lambda,T}$ the space of all random fields $v = \{v(t,x)\}_{(t,x) \in [0,T] \times \mathbb{R}}$ such that $\|v\|_{Z^p_{\lambda,T}}$ is finite.

For the well-posedness of the solution and the large deviation principle, we assume the following conditions.

**(H)** Assume that $\sigma(t,x,u) \in \mathcal{C}^{0,1,1}([0,T] \times \mathbb{R}^2)$ (the space of all continuous functions $\sigma$, with continuous partial derivatives $\sigma'_x, \sigma'_u$ and $\sigma''_{xu}$), and there exists a constant $C > 0$ such that

$$
\sup_{t \in [0,T], x \in \mathbb{R}} |\sigma(t,x,u)| \leq C(1 + |u|), \quad \forall u \in \mathbb{R};
$$

(2.15)

$$
\sup_{t \in [0,T], x \in \mathbb{R}} |\sigma(t,x,u) - \sigma(t,x,v)| \leq C|u - v|, \quad \forall u, v \in \mathbb{R};
$$

(2.16)

$$
\sup_{t \in [0,T], x \in \mathbb{R}, u \in \mathbb{R}} |\sigma'_u(t,x,u)| \leq C;
$$

(2.17)

$$
\sup_{t \in [0,T], x \in \mathbb{R}} |\sigma'_x(t,x,0)| \leq C;
$$

(2.18)

$$
\sup_{t \in [0,T], x \in \mathbb{R}, u \in \mathbb{R}} |\sigma''_{xu}(t,x,u)| \leq C.
$$

(2.19)
Moreover, there exist some constants \( p > \frac{6}{4H-1} \) and \( C > 0 \) such that
\[
\sup_{t \in [0, T], x \in \mathbb{R}} \lambda^{-\frac{1}{2}}(t)(x) \sigma_u'(t, x, u_1) - \sigma_u'(t, x, u_2) \leq C|u_1 - u_2|, \quad \forall u_1, u_2 \in \mathbb{R},
\]  
(2.20)
where \( \lambda(x) \) is the weighted function defined by (2.11).

Let \( p_t(x) := \frac{1}{4 \pi t} \exp \left(-\frac{x^2}{4t}\right) \) be the heat kernel associated with the Laplacian operator \( \Delta \). Recall the following definition of the solution to SHE (1.1) from [19].

**Definition 2.4.** [19, Definition 1.4] Given the initial value \( u_0(x) = 1 \), a real-valued adapted stochastic process \( u^\varepsilon \) is called a strong (mild) solution to (1.1), if for all \( t \geq 0 \) and \( x \in \mathbb{R} \),
\[
u^\varepsilon(t, x) = 1 + \sqrt{\varepsilon} \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \sigma(s, y, u^\varepsilon(s, y)) W(ds, dy), \quad \text{a.s.}
\]  
(2.21)

The following theorem follows from [19].

**Theorem 2.5.** [19, Theorem 1.6] Assume that the initial value \( u_0(x) = 1 \) and \( \sigma \) satisfies the hypothesis (H). Then (1.1) admits a unique strong solution in \( C([0, T] \times \mathbb{R}) \) almost surely.

2.3. **A general criterion for the large deviation principle.** Let \( \{\nu^\varepsilon\}_{\varepsilon > 0} \) be a family of random variables defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and taking values in a Polish space \( E \).

**Definition 2.6.** A function \( I : E \to [0, \infty] \) is called a rate function on \( E \), if for each \( M < \infty \) the level set \( \{y \in E : I(y) \leq M\} \) is a compact subset of \( E \).

**Definition 2.7.** Let \( I \) be a rate function on \( E \). The sequence \( \{\nu^\varepsilon\}_{\varepsilon > 0} \) is said to satisfy a large deviation principle on \( E \) with the rate function \( I \), if the following two conditions hold:
(a). for each closed subset \( F \) of \( E \),
\[
\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(\nu^\varepsilon \in F) \leq \inf_{y \in F} I(y);
\]
(b). for each open subset \( G \) of \( E \),
\[
\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(\nu^\varepsilon \in G) \geq -\inf_{y \in G} I(y).
\]

Set \( V = C([0, T]; \mathcal{H}) \subset C([0, T] \times \mathbb{R}) \), where \( C([0, T]; \mathcal{H}) \) is the space of all continuous \( \mathcal{H} \)-valued functions on \([0, T]\), and let \( \mathcal{U} \) be a Polish space. Let \( \{\Gamma^\varepsilon\}_{\varepsilon > 0} \) be a family of measurable maps from \( V \) to \( \mathcal{U} \). We recall a criterion for the LDP of the family \( Z^\varepsilon = \Gamma^\varepsilon(W) \) as \( \varepsilon \to 0 \), where \( W \) is the Gaussian process identified as an \( \mathcal{H} \)-cylindrical Brownian motion. Define the following space of stochastic processes:
\[
\mathcal{L}_2 := \left\{ \phi : \Omega \times [0, T] \to \mathcal{H} \text{ is predictable and } \int_0^T \|\phi(s)\|^2_{\mathcal{H}} ds < \infty, \quad \mathbb{P}\text{-a.s.} \right\},
\]  
(2.22)
and denote \( L^2([0, T]; \mathcal{H}) \) the space of square integrable \( \mathcal{H} \)-valued functions on \([0, T]\). For each \( N > 1 \), define
\[
S^N = \left\{ g \in L^2([0, T]; \mathcal{H}) : L_T(g) \leq N \right\},
\]  
(2.23)
where
\[
L_T(g) := \frac{1}{2} \int_0^T \|g(s)\|^2_{\mathcal{H}} ds,
\]  
(2.24)
and \( S^N \) is equipped with the topology of the weak convergence in \( L^2([0, T]; \mathcal{H}) \). Set \( \mathcal{S} = \bigcup_{N \geq 1} S^N \), and
\[
\mathcal{U}^N = \left\{ g \in \mathcal{L}_2 : g(\omega) \in S^N, \quad \mathbb{P}\text{-a.s. } \omega \right\}.
\]

**Condition 2.8.** There exists a measurable mapping \( \Gamma^0 : V \to \mathcal{U} \) such that the following hold.
(a) For every $N < +\infty$, let $g_n, \, g \in \mathcal{S}^N$ be such that $g_n \to g$ weakly as $n \to \infty$. Then $\Gamma^0 \left( \int_0 g_n(s)ds \right)$ converges to $\Gamma^0 \left( \int_0 g(s)ds \right)$ in the space $\mathcal{U}$.

(b) For every $N < +\infty$, $\{g^\varepsilon, \|\cdot\| \geq 0 \} \subset \mathcal{U}^N$ and $\delta > 0$,
\[
\lim_{\varepsilon \to 0} \mathbb{P} (\rho(Y^\varepsilon, Z^\varepsilon) > \delta) = 0,
\]
where $Y^\varepsilon = \Gamma^\varepsilon \left( W + \frac{1}{\sqrt{\varepsilon}} \int_0 g^\varepsilon(s)ds \right)$, $Z^\varepsilon = \Gamma^0 \left( \int_0 g^\varepsilon(s)ds \right)$ and $\rho(\cdot, \cdot)$ stands for the metric in the space $\mathcal{U}$.

Let $I : \mathcal{U} \to [0, \infty]$ be defined by
\[
I(\phi) := \inf_{g \in \mathcal{S}; \phi = \Gamma^0 \left( \int_0 g(s)ds \right)} \left\{ L_T(g) \right\}, \, \phi \in \mathcal{U},
\]
with the convention $\inf \emptyset = \infty$.

Recall the following result from Matoussi et al. [28].

**Theorem 2.9.** ([28, Theorem 3.2]) For any $\varepsilon > 0$, let $X^\varepsilon = \Gamma^\varepsilon(W)$ and suppose that Condition 2.8 holds. Then, the family $\{X^\varepsilon\}_{\varepsilon > 0}$ satisfies an LDP in the space $\mathcal{U}$ with the rate function $I$ defined by (2.25).

### 2.4 Main result.

By Theorem 2.9, Eq. (1.1) admits a unique strong solution $u^\varepsilon \in \mathcal{C}([0, T] \times \mathbb{R})$. To state our main result, we need to introduce a map $\Gamma^0$. Given $g \in \mathcal{S}$, for any $t \in [0, T], x \in \mathbb{R}$, consider the following deterministic integral equation (the skeleton equation):

\[
u^\delta(t, x) = 1 + \int_0^t \langle p_{t-s}(x-\cdot) \sigma(s, \cdot, u^\delta(s, \cdot)), g(s, \cdot) \rangle_H ds.
\]

By Proposition 3.1 below, Eq. (2.26) admits a unique solution $u^\delta \in \mathcal{C}([0, T] \times \mathbb{R})$. For any $g \in \mathcal{S}$, define
\[
\Gamma^0 \left( \int_0 g(s)ds \right) := u^\delta(\cdot).
\]

The following is the main result of this paper.

**Theorem 2.10.** Assume that the initial value $u_0(x) = 1$ and the hypothesis (H) holds. Then, the family $\{u^\varepsilon\}_{\varepsilon > 0}$ in Eq. (1.1) satisfies an LDP in the space $\mathcal{C}([0, T] \times \mathbb{R})$ with the rate function $I$ given by
\[
I(\phi) := \inf_{g \in \mathcal{S}; \phi = \Gamma^0 \left( \int_0 g(s)ds \right)} \left\{ \frac{1}{2} \int_0^T \| g(s) \|^2_H ds \right\}.
\]

**Proof.** According to Theorem 2.9, it suffices to show that the conditions (a) and (b) in Condition 2.8 are satisfied. Condition (a) will be proved in Proposition 4.1, and Condition (b) will be verified in Proposition 5.1. The proof is complete. \( \Box \)

### 3. Skeleton equation

In this section, we study the skeleton equation (2.26). For $p \geq 2$, $H \in (\frac{1}{2}, 1]$, recall the space $Z^{p}_{\lambda, T}$ with the norm (2.12). The space of all non-random functions in $Z^{p}_{\lambda, T}$ is denoted by $Z^{\infty}_{\lambda, T}$, with the following norm:
\[
\| v \|_{Z^{p}_{\lambda, T}} := \sup_{t \in [0, T]} \| v(t, \cdot) \|_{L^p_{\lambda}(\mathbb{R})} + \sup_{t \in [0, T]} \mathcal{N}_{\frac{\lambda}{2}-H, p} v(t),
\]
where
\[
\| v(t, \cdot) \|_{L^p_{\lambda}(\mathbb{R})} := \left( \int_{\mathbb{R}} |v(t, x)|^p \lambda(x)dx \right)^{\frac{1}{p}}
\]
and
\[
\mathcal{N}^\varepsilon_{\frac{1}{2} - H, \rho} v(t) := \left( \int_{\mathbb{R}} \|v(t, \cdot) - v(t, \cdot + h)\|_{L^\infty_x(\mathbb{R})}^2 |h|^{2H - 2} \, dh \right)^{\frac{1}{2}}.
\] (3.2)

We have the following well-posedness result for the skeleton equation (2.26).

**Proposition 3.1.** Assume that \( \sigma \) satisfies the hypothesis (H). Then Eq. (2.26) admits a unique solution in \( C([0, T] \times \mathbb{R}) \).

Due to the complexity of the space \( \mathcal{H} \), it is difficult to prove Proposition 3.1 by using Picard’s iteration directly. We use the approximation method by introducing a new Hilbert space \( \mathcal{H}_\varepsilon \) as follows.

For every fixed \( \varepsilon > 0 \), let
\[
f_\varepsilon(x) := \mathcal{F}^{-1} \left( e^{-\varepsilon |\xi|^2 |\xi|^{1-2H}} \right) (x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} e^{-\varepsilon |\xi|^2 |\xi|^{1-2H}} d\xi.
\] (3.3)

For any \( \phi, \psi \in \mathcal{D}(\mathbb{R}) \), define
\[
\langle \phi, \psi \rangle_{\mathcal{H}_\varepsilon} := c_{1,H} \int_{\mathbb{R}} \mathcal{F} \phi(\xi) \overline{\mathcal{F} \psi(\xi)} e^{-\varepsilon |\xi|^2 |\xi|^{1-2H}} d\xi
\]
\[
= c_{1,H} \int_{\mathbb{R}^2} \phi(x) \psi(y) f_\varepsilon(x - y) \, dx \, dy,
\] (3.4)

where \( c_{1,H} \) is given by (1.3). Let \( \mathcal{H}_\varepsilon \) be the Hilbert space obtained by completing \( \mathcal{D}(\mathbb{R}) \) with respect to the scalar product given by (3.4). Notice that for any \( 0 < \varepsilon_1 < \varepsilon_2 \), we have that for any \( \phi \in \mathcal{H}_{\varepsilon_1} \),
\[
\|\phi\|_{\mathcal{H}_{\varepsilon_1}} \geq \|\phi\|_{\mathcal{H}_{\varepsilon_2}},
\] (3.5)

and for any \( \phi, \psi \in \mathcal{H} \), by the dominated convergence theorem,
\[
\lim_{\varepsilon \to 0} \langle \phi, \psi \rangle_{\mathcal{H}_\varepsilon} = \langle \phi, \psi \rangle_{\mathcal{H}}.
\]

For any \( g \in L^2([0, T]; \mathcal{H}) \), let
\[
u^\varepsilon g(t, x) = 1 + \int_0^t \langle p_{t-s}(x - \cdot) \sigma(s, \cdot, u^\varepsilon g(s, \cdot)), g(s, \cdot) \rangle_{\mathcal{H}_\varepsilon} \, ds.
\] (3.6)

Since \( |\xi|^{1-2H} e^{-\varepsilon |\xi|^2} \) is in \( L^1(\mathbb{R}) \), \( |f_\varepsilon| \) is bounded. Due to the regularity in space, the existence and uniqueness of the solution \( u^\varepsilon g \) to Eq. (3.6) is well-known, see, e.g., [27, Section 4].

The following lemma asserts that the approximate solution \( u^\varepsilon g \) is uniformly bounded in the space \( Z^p_{\lambda, t} \) with respect to \( \varepsilon > 0 \).

**Lemma 3.1.** Let \( H \in \left( \frac{1}{4}, \frac{1}{2} \right) \) and \( g \in L^2([0, T]; \mathcal{H}) \). Assume that \( \sigma \) satisfies the hypothesis (H). Then the approximate solution \( u^\varepsilon g \) satisfies that for any \( p \geq 2 \),
\[
\sup_{\varepsilon > 0} \|u^\varepsilon g\|_{Z^p_{\lambda, t}} := \sup_{\varepsilon > 0} \sup_{t \in [0, T]} \|u^\varepsilon g(t, \cdot)\|_{L^p_x(\mathbb{R})} + \sup_{\varepsilon > 0} \sup_{t \in [0, T]} \mathcal{N}^\varepsilon_{\frac{1}{2} - H, \rho} u^\varepsilon g(t) < \infty.
\] (3.7)

**Proof.** We use the similar argument as in [19] replacing the stochastic integral by the deterministic integral. In Steps 1 and 2, we will use Picard’s iteration to show that for each fixed \( \varepsilon > 0 \), \( u^\varepsilon g \in Z^p_{\lambda, t} \), and prove that \( u^\varepsilon g \) is uniformly bounded in \( Z^p_{\lambda, t} \) with respect to \( \varepsilon > 0 \) in Step 3. To this end, we define the Picard iteration sequence as follows: let
\[
u^\varepsilon g^0(t, x) = 1,
\]
and recursively for \( n = 0, 1, 2, \cdots \)
\[
u^\varepsilon g^{n+1}(t, x) = 1 + \int_0^t \langle p_{t-s}(x - \cdot) \sigma(s, \cdot, u^\varepsilon g^n(s, \cdot)), g(s, \cdot) \rangle_{\mathcal{H}_\varepsilon} \, ds.
\] (3.8)
Assume that there exists some constant $M > 0$ such that $\int_0^T \|g(s, \cdot)\|_{H^\varepsilon}^2 ds \leq M$. By (3.5), we know that
\[
\int_0^T \|g(s, \cdot)\|_{H^\varepsilon}^2 ds \leq M, \text{ for any } \varepsilon > 0. \tag{3.9}
\]

**Step 1.** We will bound $\|u^{g,n}_\varepsilon(t, \cdot)\|_{L^p_x(\mathbb{R})}$ uniformly in $n$ and show that $u^{\varepsilon}_g(t, \cdot)$ is in $L^p_{\lambda}(\mathbb{R})$ in this step. By Cauchy-Schwarz’s inequality, (2.16), (3.9), the boundedness of $f_\varepsilon$ and Jensen’s inequality respect to $p_{t-s}(x-y)$, we have that for any $t \in [0, T]$ and $x \in \mathbb{R}$,
\[
|u^{g,n+1}_\varepsilon(t, x) - u^{g,n}_\varepsilon(t, x)|^2 \\
= \left| \int_0^t \left< p_{t-s}(x - \cdot) \left[ \sigma(s, \cdot, u^{g,n}_\varepsilon(s, \cdot)) - \sigma(s, \cdot, u^{g,n-1}_\varepsilon(s, \cdot)) \right], g(s, \cdot) \right>_{H^\varepsilon} ds \right|^2 \\
\leq \int_0^t \|g(s, \cdot)\|^2_{H^\varepsilon} ds \cdot \int_0^t \left\| p_{t-s}(x - \cdot) \left[ \sigma(s, \cdot, u^{g,n}_\varepsilon(s, \cdot)) - \sigma(s, \cdot, u^{g,n-1}_\varepsilon(s, \cdot)) \right] \right\|^2_{H^\varepsilon} ds \\
\leq c_{1,H} M \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} p_{t-s}(x - y) \left[ \sigma(s, y, u^{g,n}_\varepsilon(s, y)) - \sigma(s, y, u^{g,n-1}_\varepsilon(s, y)) \right] f_\varepsilon(y-z) dydz ds \\
\leq c_{1,H} M \int_{\mathbb{R}} \cdot \int_0^t \int_{\mathbb{R}} p_{t-s}(x - y) \left[ \sigma(s, y, u^{g,n}_\varepsilon(s, y)) - \sigma(s, y, u^{g,n-1}_\varepsilon(s, y)) \right]^2 dy ds.
\]

Integrating with respect to the spatial variable with the weight $\lambda(x)$ and invoking (2.16), Jensen’s inequality respect to $p_{t-s}(x-y)dyds$ and an application of Lemma 6.1 yield that for any $p \geq 2$, $t \in [0, T]$,
\[
\|u^{g,n+1}_\varepsilon(t, \cdot) - u^{g,n}_\varepsilon(t, \cdot)\|_{L^p_x(\mathbb{R})}^p \\
= \int_{\mathbb{R}} \left| u^{g,n+1}_\varepsilon(t, x) - u^{g,n}_\varepsilon(t, x) \right|^p \lambda(x) dx \\
\leq C_{\varepsilon,M} \int_{\mathbb{R}} \left| \int_0^t \int_{\mathbb{R}} p_{t-s}(x - y) \left| u^{g,n}_\varepsilon(s, y) - u^{g,n-1}_\varepsilon(s, y) \right|^2 dy ds \right|^2 \lambda(x) dx \\
\leq C_{\varepsilon,M,p,T} \int_{\mathbb{R}} \left[ \int_0^t \int_{\mathbb{R}} p_{t-s}(x - y) \left| u^{g,n}_\varepsilon(s, y) - u^{g,n-1}_\varepsilon(s, y) \right|^p dy ds \right] \lambda(x) dx \\
\leq C_{\varepsilon,M,p,T} \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} \frac{1}{\lambda(y)} p_{t-s}(x-y) \lambda(x) dy ds \left| u^{g,n}_\varepsilon(s, y) - u^{g,n-1}_\varepsilon(s, y) \right|^p \lambda(y) dy ds \\
\leq C_{\varepsilon,M,p,T} \int_{\mathbb{R}} \int_0^t \|u^{g,n}_\varepsilon(s, \cdot) - u^{g,n-1}_\varepsilon(s, \cdot)\|_{L^p_x(\mathbb{R})}^p ds \\
\leq C_{\varepsilon,M,p,T} \frac{Tn}{n!} \sup_{s \in [0,T]} \|u^{g,1}_\varepsilon(s, \cdot) - u^{g,0}_\varepsilon(s, \cdot)\|_{L^p_x(\mathbb{R})}^p.
\]
Thus, (3.10) implies that
\[
\sup_{n \geq 1} \sup_{t \in [0, T]} \|u^{g,n}_\varepsilon(t, \cdot)\|_{L^p_x(\mathbb{R})} < \infty, \quad \text{for each } \varepsilon > 0,
\]
and $\{u^{g,n}_\varepsilon(t)\}_{n \geq 1}$ is a Cauchy sequence in $L^p_{\lambda}(\mathbb{R})$. Hence, there exists $u^{\varepsilon}_g(t) \in L^p_{\lambda}(\mathbb{R})$ such that $u^{g,n}_\varepsilon(t) \rightarrow u^{\varepsilon}_g(t)$ as $n \rightarrow \infty$.

**Step 2.** This step is devoted to estimating $N^*_{{\frac{1}{2}}-H,p} u^{\varepsilon}_g(t)$. For the simplicity of writing, denote
\[
D_t(x, h) := p_t(x + h) - p_t(x) \tag{3.11}
\]
and

\[
\square_t(x, y, h) := p_t(x + y + h) - p_t(x + y) - p_t(x + h) + p_t(x). \tag{3.12}
\]

Using the Cauchy-Schwarz inequality, (3.5) and (3.9), we have

\[
|u^\varphi_t(t, x) - u^\varphi_t(t, x + h)|^p \\
\leq \left( \int_0^1 \| D_{t-s}(x - \cdot, h) \sigma(s, \cdot, u^\varphi_t(s, \cdot)) \|_{\mathcal{H}}^2 ds \right)^{\frac{p}{2}} \\
\leq \left( \int_0^1 \| D_{t-s}(x - \cdot, h) \sigma(s, \cdot, u^\varphi_t(s, \cdot)) \|_{\mathcal{H}}^2 ds \right)^{\frac{p}{2}} \\
\simeq \left( \int_0^1 \int_{\mathbb{R}^2} \left| D_{t-s}(x - y - z, h) \sigma(s, y + z, u^\varphi_t(s, y + z)) \\
- D_{t-s}(x - z, h) \sigma(s, z, u^\varphi_t(s, z)) \right|^2 \cdot |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}} \\
\leq \mathcal{I}_1(t, x, h) + \mathcal{I}_2(t, x, h) + \mathcal{I}_3(t, x, h),
\]

where

\[
\mathcal{I}_1(t, x, h) := \left( \int_0^1 \int_{\mathbb{R}^2} \left| D_{t-s}(x - y - z, h) \right|^2 \\
\cdot \left| \sigma(s, y + z, u^\varphi_t(s, y + z)) - \sigma(s, z, u^\varphi_t(s, y + z)) \right|^2 \cdot |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}};
\]

\[
\mathcal{I}_2(t, x, h) := \left( \int_0^1 \int_{\mathbb{R}^2} \left| D_{t-s}(x - y - z, h) \right|^2 \\
\cdot \left| \sigma(s, z, u^\varphi_t(s, y + z)) - \sigma(s, z, u^\varphi_t(s, z)) \right|^2 \cdot |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}};
\]

\[
\mathcal{I}_3(t, x, h) := \left( \int_0^1 \int_{\mathbb{R}^2} \left| \square_{t-s}(x - z, y, h) \right|^2 \cdot \left| \sigma(s, z, u^\varphi_t(s, z)) \right|^2 \cdot |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}}.
\]

Therefore, by (3.2), we have

\[
\left[ \mathcal{N}^{\varphi}_{H, p, u^\varphi_t(t)} \right]^2 = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left| u^\varphi_t(t, x) - u^\varphi_t(t, x + h) \right|^p \lambda(x) dx \right)^{\frac{p}{2}} \cdot |h|^{2H-2} dh \\
\leq \sum_{j=1}^3 \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathcal{I}_j(t, x, h) \lambda(x) dx \right)^{\frac{p}{2}} \cdot |h|^{2H-2} dh. \tag{3.13}
\]

If \(|y| > 1\), then we have by (2.15)

\[
\left| \sigma(s, y + z, u^\varphi_t(s, y + z)) - \sigma(s, z, u^\varphi_t(s, y + z)) \right|^2 \\
\leq \left| \sigma(s, y + z, u^\varphi_t(s, y + z)) \right|^2 + \left| \sigma(s, z, u^\varphi_t(s, y + z)) \right|^2 \\
\leq 1 + \left| u^\varphi_t(s, y + z) \right|^2. \tag{3.14}
\]
If $|y| \leq 1$, then by (2.18) and (2.19) there exists some $\zeta \in (0,1)$ such that
\[
|\sigma(s, y + z, u^\varphi(s, y + z)) - \sigma(s, z, u^\varphi(s, y + z))|^2 
\leq |\sigma(s, y + z, 0) - \sigma(s, z, 0)|^2 + \left| \int_0^t \left[ \sigma'(s, y + z, \xi) - \sigma'(s, z, \xi) \right] d\xi \right|^2
\]
\[
\leq |\sigma'(s, y + \zeta z, 0)|^2 \cdot |y|^2 + |u^\varphi(s, y + z)|^2 \cdot |y|^2
\leq \left(1 + |u^\varphi(s, y + z)|^2\right) \cdot |y|^2.
\] (3.15)

For the first term of (3.13), by (3.14), (3.15) and a change of variable, we have
\[
\mathcal{I}_1(t,x,h) \leq \left( \int_0^t \int_{\mathbb{R}} \int_{|y| > 1} |D_{t-s}(x-y-z, h)|^2 \cdot \left(1 + |u^\varphi(s, y + z)|^2\right) \cdot |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}}
\]
\[
+ \left( \int_0^t \int_{\mathbb{R}} \int_{|y| \leq 1} |D_{t-s}(x-y-z, h)|^2 \cdot \left(1 + |u^\varphi(s, y + z)|^2\right) \cdot |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}}
\]
\[
= \left( \int_0^t \int_{\mathbb{R}} \int_{|y| > 1} |D_{t-s}(x-z, h)|^2 \cdot \left(1 + |u^\varphi(s, z)|^2\right) \cdot |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}}
\]
\[
+ \left( \int_0^t \int_{\mathbb{R}} \int_{|y| \leq 1} |D_{t-s}(x-z, h)|^2 \cdot \left(1 + |u^\varphi(s, z)|^2\right) \cdot |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}}
\]
\[
\leq \left( \int_0^t \int_{\mathbb{R}} \int_{|y| > 1} |p_{t-s}(z-h) - p_{t-s}(z)|^2 \cdot \left(1 + |u^\varphi(s, x + z)|^2\right) dz ds \right)^{\frac{p}{2}}.
\] (3.16)

By (3.16), a change of variable, Minkowski’s inequality, Lemma 6.5 and Jensen’s inequality with respect to $(t-s)^{1-H} |D_{t-s}(z, h)|^2 |h|^{2H-2} dz dh$, we have
\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_0^t \int_{\mathbb{R}} |D_{t-s}(z, h)|^2 \left(1 + |u^\varphi(s, x)|^2\right) dz ds \right)^{\frac{p}{2}} \lambda(x-z) dx \right)^{\frac{p}{2}} \cdot |h|^{2H-2} dh
\]
\[
\leq \int_{\mathbb{R}} \left( \int_0^t \int_{\mathbb{R}^2} |D_{t-s}(z, h)|^2 \cdot |h|^{2H-2} \left(\int_{\mathbb{R}} \left(1 + |u^\varphi(s, x)|^2\right)^p \lambda(x-z) dx \right)^{\frac{2}{p}} dz dh ds \right)^{\frac{p}{2}}
\]
\[
\leq \int_0^t (t-s)^{H-1} \cdot \left( \int_{\mathbb{R}^3} (t-s)^{1-H} \cdot |D_{t-s}(z, h)|^2 \cdot |h|^{2H-2} \cdot \left(1 + |u^\varphi(s, x)|^2\right)^p \lambda(x-z) dx dz dh \right)^{\frac{2}{p}} ds
\]
\[
\leq \int_0^t (t-s)^{H-1} \cdot \left(1 + \|u^\varphi(s, \cdot)|^2 \right)_{L^p_L(\mathbb{R})}^2 ds.
\] (3.17)

For the second term of (3.13), (2.16) and a change of variable imply that
\[
\mathcal{I}_2(t,x,h) \leq \left( \int_0^t \int_{\mathbb{R}^2} |D_{t-s}(x-y-z, h)|^2 \cdot |u^\varphi(s, y+z) - u^\varphi(s, z)|^2 \cdot |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}}
\]
\[
= \left( \int_0^t \int_{\mathbb{R}^2} |D_{t-s}(x-z, h)|^2 \cdot |u^\varphi(s, z) - u^\varphi(s, z-y)|^2 \cdot |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}}
\]
\[
= \left( \int_0^t \int_{\mathbb{R}^2} |D_{t-s}(x-z, h)|^2 \cdot |u^\varphi(s, y+z) - u^\varphi(s, z)|^2 \cdot |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}}.
\] (3.18)
By a change of variable, Minkowski’s inequality, Jensen’s inequality and Lemma 6.4, we have

\[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathcal{I}_2(t, x, h) \lambda(x) dx \right)^{\frac{2}{p}} \cdot |h|^{2H-2} dh \]

\[ \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} |D_{t-s}(z, h)|^2 \cdot |u^0_{s}(s, x + y) - u^0_{s}(s, x)|^2 \cdot |y|^{2H-2} dz dy ds \right)^{\frac{2}{p}} \cdot |h|^{2H-2} dh \]

\[ \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^3} |D_{t-s}(z, h)|^2 \cdot |y|^{2H-2} \cdot |h|^{2H-2} \right)^{\frac{2}{p}} \cdot |h|^{2H-2} dh \]

\[ \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left[ \mathcal{N}_{\frac{p}{2}-H} u^0_{s}(s) \right]^2 ds. \right) \]

For the last term of (3.13), by (2.15) and a change of variable, we have

\[ \mathcal{I}_3(t, x, h) \leq \left( \int_{0}^{t} \int_{\mathbb{R}^2} |\nabla_{t-s}(x - z, y, h)|^2 \cdot \left[ 1 + |u^0_{s}(s, z)|^2 \right] \cdot |y|^{2H-2} dxdz dh \right)^{\frac{2}{p}} \]

\[ = \left( \int_{0}^{t} \int_{\mathbb{R}^2} |\nabla_{t-s}(z, y, h)|^2 \cdot \left[ 1 + |u^0_{s}(s, x - z)|^2 \right] \cdot |y|^{2H-2} dxdz dh \right)^{\frac{2}{p}} \]

\[ \leq \left( \int_{0}^{t} \int_{\mathbb{R}^2} |\nabla_{t-s}(-z, y, h)|^2 \cdot |u^0_{s}(s, x + z) - u^0_{s}(s, x)|^2 \cdot |y|^{2H-2} dxdz dh \right)^{\frac{2}{p}} \]

\[ + \left( \int_{0}^{t} \int_{\mathbb{R}^2} |\nabla_{t-s}(-z, y, h)|^2 \cdot \left[ 1 + |u^0_{s}(s, x)|^2 \right] \cdot |y|^{2H-2} dxdz dh \right)^{\frac{2}{p}} \]

\[ =: \mathcal{I}_{31}(t, x, h) + \mathcal{I}_{32}(t, x, h). \]

By a change of variable, Minkowski’s inequality and Lemma 6.4, we have

\[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathcal{I}_{31}(t, x, h) \lambda(x) dx \right)^{\frac{2}{p}} \cdot |h|^{2H-2} dh \]

\[ \leq \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla_{t-s}(z, y, h)|^2 \cdot \left( \int_{\mathbb{R}} |u^0_{s}(s, x + z) - u^0_{s}(s, x)|^2 \cdot |\lambda(x)| dx \right)^{\frac{2}{p}} \cdot |h|^{2H-2} \cdot |y|^{2H-2} dxdz dy ds \]
\[
\lesssim \int_0^t (t-s)^{H-1} \cdot \left[ \mathcal{N}_{\frac{3}{2}-H, p}^* u_\varepsilon^2(s) \right] \frac{ds}{ds}.
\]

By a change of variable, Minkowski's inequality and Lemma 6.2, we have
\[
\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \mathcal{I}_{32}(t, x, h) \lambda(x) dx \right| \frac{\varepsilon}{p} \cdot |h|^{2H-2} dh
\lesssim \int_0^t \int_{\mathbb{R}^3} \Box_{t-s}(z, y, h) \cdot \left( 1 + \int_{\mathbb{R}} |u_\varepsilon^p(s, x, y)|^p \lambda(x) dx \right)^{\frac{2}{p}} \cdot |h|^{2H-2} \cdot |y|^{2H-2} dh dz dy ds.
\]

Thus, by (3.13), (3.17), (3.19), (3.20) and (3.21), we have
\[
\left[ \mathcal{N}_{\frac{3}{2}-H, p}^* u_\varepsilon^2(t) \right] \leq 1 + \int_0^t \left( (t-s)^{\frac{2H-2}{2}} + (t-s)^{H-1} \right) \cdot \|u_\varepsilon^p(s, \cdot, \cdot)^2 \|_{L^2(\mathbb{R})} \frac{ds}{ds}.
\]

Since \( u_\varepsilon^p(t) \in L^2_\lambda(\mathbb{R}) \), by the fractional Gronwall lemma ([21, Lemma 1]), we have
\[
\sup_{\varepsilon \in [0, T]} \mathcal{N}_{\frac{3}{2}-H, p}^* u_\varepsilon^2(t) \leq \infty, \quad \text{for each } \varepsilon > 0.
\]

Combining Steps 1 and 2, we can conclude that for each \( \varepsilon > 0 \), \( u_\varepsilon^p \in Z_{\lambda, T}^p \).

**Step 3.** In this step, we will prove that \( u_\varepsilon^p \) is uniformly bounded in \( (Z_{\lambda, T}^p, \| \cdot \|_{Z_{\lambda, T}^p}) \) with respect to \( \varepsilon \). First, we will give an estimate for the norm of \( u_\varepsilon^p \in L^2_\lambda(\mathbb{R}) \). By the Cauchy-Schwarz inequality, (2.4), (3.4) and (3.9), we have

\[
|u_\varepsilon^p(t, x, y)| \leq 1 + \left( \int_0^t \|p_{t-s}(x-\cdot, y-\cdot)\sigma(s, \cdot, u_\varepsilon^p(s, \cdot))\|_{L^2_\lambda(\mathbb{R})} ds \right)^\frac{1}{2}
\leq 1 + \left( \int_0^t \|p_{t-s}(x-\cdot, y-\cdot)\sigma(s, \cdot, u_\varepsilon^p(s, \cdot))\|_{L^2_\lambda(\mathbb{R})} ds \right)^\frac{1}{2}
\approx 1 + \left( \int_0^t \int_{\mathbb{R}^2} \left| p_{t-s}(x-\cdot, y-\cdot) \sigma(s, y, u_\varepsilon^p(s, y+h)) - p_{t-s}(x-\cdot, y-\cdot) \sigma(s, y, u_\varepsilon^p(s, y+h)) \right| ds \right)^\frac{1}{2}
\leq 1 + A_1(t, x) + A_2(t, x) + A_3(t, x),
\]

where
\[
A_1(t, x) := \left( \int_0^t \int_{\mathbb{R}^2} p_{t-s}(x-\cdot, y-\cdot) \cdot \left| \sigma(s, y, u_\varepsilon^p(s, y+h)) - \sigma(s, y, u_\varepsilon^p(s, y+h)) \right|^2 \cdot |h|^{2H-2} dh dy ds \right)^\frac{1}{2};
\]

\[
A_2(t, x) := \left( \int_0^t \int_{\mathbb{R}^2} p_{t-s}(x-\cdot, y-\cdot) \cdot \left| \sigma(s, y, u_\varepsilon^p(s, y+h)) - \sigma(s, y, u_\varepsilon^p(s, y+h)) \right|^2 \cdot |h|^{2H-2} dh dy ds \right)^\frac{1}{2};
\]

\[
A_3(t, x) := \left( \int_0^t \int_{\mathbb{R}^2} p_{t-s}(x-\cdot, y-\cdot) \cdot \left| \sigma(s, y, u_\varepsilon^p(s, y+h)) - \sigma(s, y, u_\varepsilon^p(s, y+h)) \right|^2 \cdot |h|^{2H-2} dh dy ds \right)^\frac{1}{2};
\]
\[ \mathcal{A}_3(t, x) := \left( \int_{0}^{t} \int_{\mathbb{R}^2} |D_{t-s}(x - y, h)|^2 \cdot |\sigma(s, y, u_\varepsilon^g(s, y))|^2 \cdot |h|^{2H-2} dhdyds \right)^{\frac{2}{p}}. \]

By a change of variable, (3.14) and (3.15), we have
\[ \mathcal{A}_1(t, x) \lesssim \left( \int_{0}^{t} \int_{\mathbb{R}} p_{t-s}^2(x - y) \cdot \left( 1 + |u_\varepsilon^g(s, y)|^2 \right) dy ds \right)^{\frac{2}{p}}. \]

By a change of variable, (2.15), a change of variable, Minkowski’s inequality, Lemma 6.1 and Jensen’s inequality, we have
\[ \left( \int_{\mathbb{R}} \mathcal{A}_2(t, x) \lambda(x) dx \right)^{\frac{2}{p}} \lesssim \left( \int_{\mathbb{R}} \left( \int_{0}^{t} \int_{\mathbb{R}} p_{t-s}^2(x - y) \cdot |u_\varepsilon^g(s, x + h) - u_\varepsilon^g(s, y)|^2 \cdot |h|^{2H-2} dhdyds \right)^{\frac{2}{p}} \lambda(x) dx \right)^{\frac{2}{p}}. \]

By (2.16), a change of variable, Minkowski’s inequality, Jensen’s inequality and Lemma 6.1, we have
\[ \left( \int_{\mathbb{R}} \mathcal{A}_3(t, x) \lambda(x) dx \right)^{\frac{2}{p}} \lesssim \left( \int_{\mathbb{R}} \left( \int_{0}^{t} \int_{\mathbb{R}} |D_{t-s}(x - y, h)|^2 \cdot \left( 1 + \int_{\mathbb{R}} |u_\varepsilon^g(s, x)|^p \lambda(x - y) dx \right) \cdot |h|^{2H-2} dhdyds \right)^{\frac{2}{p}} \lambda(x) dx \right)^{\frac{2}{p}}. \]
\[ \leq \int_0^t (t-s)^{H-1} \cdot \left( 1 + \| u^\theta_\varepsilon (s, \cdot) \|_{L^p(X)}^2 \right) ds. \]

Thus, (3.23), (3.24), (3.25) and (3.26) yield that
\[
\| u^\theta_\varepsilon (t, \cdot) \|^2_{L^p(X)} = \left( \int \| u^\theta_\varepsilon (t, x) \|^p \lambda(x) dx \right) ^{2/p} \\
\leq 1 + \int_0^t \left( (t-s)^{H-1} + (t-s)^{-\frac{1}{2}} \right) \cdot \| u^\theta_\varepsilon (s, \cdot) \|^2_{L^p(X)} ds \\
+ \int_0^t (t-s)^{-1/2} \left[ \mathcal{N}_{\frac{1}{2} - H, p} u^\theta_\varepsilon (s) \right]^2 ds.
\]

Let
\[
\Psi_\varepsilon (t) := \| u^\theta_\varepsilon (t, \cdot) \|^2_{L^p(X)} + \left[ \mathcal{N}_{\frac{1}{2} - H, p} u^\theta_\varepsilon (t) \right]^2.
\]

Putting (3.22) and (3.27) together, there exists a constant \( C_{T,p,H} \) such that
\[
\Psi_\varepsilon (t) \leq C_{T,p,H} \left( 1 + \int_0^t (t-s)^{2H-\frac{3}{2}} \Psi_\varepsilon (s) ds \right).
\]

By the fractional Gronwall lemma ([21, Lemma 1]), we have
\[
\sup_{\varepsilon > 0} \sup_{t \in [0,T]} \Psi_\varepsilon (t) < \infty,
\]
which implies that
\[
\sup_{\varepsilon > 0} \| u^\theta_\varepsilon \|_{L^p_{\gamma,T}} < \infty.
\]

The proof is complete. \( \square \)

**Lemma 3.2.** Let \( u^\theta_\varepsilon \) be the approximate mild solution defined in (3.6).

(i). If \( p > \frac{6}{4H-1} \), then there exists a constant \( C_{T,p,H} > 0 \) such that
\[
\sup_{t \in [0,T], x \in \mathbb{R}} \lambda^{\frac{1}{p}}(x) \cdot \mathcal{N}_{\frac{1}{2} - H} u^\theta_\varepsilon (t, x) \leq C_{T,p,H} \left( 1 + \| u^\theta_\varepsilon \|_{Z^p_{\gamma,T}} \right). \tag{3.28}
\]

(ii). If \( p > \frac{3}{2H} \) and \( 0 < \gamma < H - \frac{3}{2} - \frac{2}{p} \), then there exists a constant \( C_{T,p,H,\gamma} > 0 \) such that
\[
\sup_{t, t+h \in [0,T], x \in \mathbb{R}} \lambda^{\frac{1}{p}}(x) \cdot |u^\theta_\varepsilon (t+h, x) - u^\theta_\varepsilon (t, x)| \leq C_{T,p,H,\gamma} |h|^{\gamma} \cdot \left( 1 + \| u^\theta_\varepsilon \|_{Z^p_{\gamma,T}} \right). \tag{3.29}
\]

(iii). If \( p > \frac{3}{2H} \) and \( 0 < \gamma < H - \frac{3}{2} \), then there exists a constant \( C_{T,p,H,\gamma} > 0 \) such that
\[
\sup_{t \in [0,T], x, y \in \mathbb{R}} \frac{|u^\theta_\varepsilon (t, x) - u^\theta_\varepsilon (t, y)|}{\lambda^{\frac{1}{p}}(x) + \lambda^{\frac{1}{p}}(y)} \leq C_{T,p,H,\gamma} |x-y|^{\gamma} \cdot \left( 1 + \| u^\theta_\varepsilon \|_{Z^p_{\gamma,T}} \right). \tag{3.30}
\]

**Proof.** (i). By (3.6), we have
\[
u^\theta_\varepsilon (t, x+h) - u^\theta_\varepsilon (t, x) \\
= \int_0^t \langle p_{t-s}(x+h-\cdot) \sigma(s, \cdot, u^\theta_\varepsilon (s, \cdot)), g(s, \cdot) \rangle_{\mathcal{H}_\varepsilon} ds \\
- \int_0^t \langle p_{t-s}(x-\cdot) \sigma(s, \cdot, u^\theta_\varepsilon (s, \cdot)), g(s, \cdot) \rangle_{\mathcal{H}_\varepsilon} ds \\
=: \Phi(t, x+h) - \Phi(t, x).
\]
To get our desired results, we need to deal with \( \Phi(t, x) \). Applying (3.4) and the Fubini theorem, for any \( \alpha \in (0, 1) \), we have

\[
\Phi(t, x) = \int_0^t \langle p_{t-s}(x - \cdot) \sigma(s, \cdot, u^q_2(s, \cdot)), g(s, \cdot) \rangle_{\mathcal{H}_c} ds
\]

\[
\approx \int_0^t \int_{\mathbb{R}^2} p_{t-s}(x - y) \sigma(s, y, u^q_2(s, y)) g(s, \bar{y}) f_\varepsilon(y - \bar{y}) dy d\bar{y} ds
\]

\[
= \frac{\sin(\pi \alpha)}{\pi} \int_0^t \int_{\mathbb{R}^2} p_{t-s}(x - y) \left[ \int_s^t (t - r)^{\alpha - 1} (r - s)^{-\alpha} dr \right] \cdot \sigma(s, y, u^q_2(s, y)) g(s, \bar{y}) f_\varepsilon(y - \bar{y}) dy d\bar{y} ds
\]

\[
= \frac{\sin(\pi \alpha)}{\pi} \int_0^t \int_{\mathbb{R}^2} \left[ \int_s^t (t - r)^{\alpha - 1} (r - s)^{-\alpha} dr \right] \cdot \sigma(s, y, u^q_2(s, y)) g(s, \bar{y}) f_\varepsilon(y - \bar{y}) dy d\bar{y} ds
\]

\[
\approx \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} \cdot p_{t-r}(x - z) \left( \int_0^r (r - s)^{-\alpha} \cdot \int_{\mathbb{R}^2} p_{t-r}(z - y) \sigma(s, y, u^q_2(s, y)) g(s, \bar{y}) f_\varepsilon(y - \bar{y}) dy d\bar{y} ds \right) dz dr
\]

(3.31)

\[
\approx \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} \cdot p_{t-r}(x - z) \left( \int_0^r (r - s)^{-\alpha} \langle p_{t-r}(z - \cdot) \sigma(s, \cdot, u^q_2(s, \cdot)), g(s, \cdot) \rangle_{\mathcal{H}_c} ds \right) dz dr
\]

\[
= \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} \cdot p_{t-r}(x - z) J_\alpha(r, z) dz dr,
\]

where

\[
J_\alpha(r, z) := \int_0^r (r - s)^{-\alpha} \langle p_{t-r}(z - \cdot) \sigma(s, \cdot, u^q_2(s, \cdot)), g(s, \cdot) \rangle_{\mathcal{H}_c} ds.
\]

Set

\[
\Delta_h J_\alpha(t, x) := J_\alpha(t, x + h) - J_\alpha(t, x).
\]

Applying a change of variable, we have

\[
\Phi(t, x + h) - \Phi(t, x)
\]

\[
\approx \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} \cdot p_{t-r}(x + h - z) \cdot J_\alpha(r, z) dz dr - \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} \cdot p_{t-r}(x - z) \cdot J_\alpha(r, z) dz dr
\]

\[
= \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} \cdot p_{t-r}(x - z) \cdot J_\alpha(r, z + h) dz dr - \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} \cdot p_{t-r}(x - z) \cdot J_\alpha(r, z) dz dr
\]

\[
= \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} \cdot p_{t-r}(x - z) \cdot \Delta_h J_\alpha(r, z) dz dr.
\]

Invoking Minkowski’s inequality and Hölder’s inequality with \( \frac{1}{p} + \frac{1}{q} = 1 \), we have

\[
\int_{\mathbb{R}} |\Phi(t, x + h) - \Phi(t, x)|^2 \cdot |h|^{2H - 2} dh
\]

\[
\approx \int_{\mathbb{R}} \left| \int_0^t (t - r)^{\alpha - 1} \cdot p_{t-r}(x - z) \Delta_h J_\alpha(r, z) dz dr \right|^2 \cdot |h|^{2H - 2} dh
\]
Applying the Cauchy-Schwarz inequality, (2.4), (3.4) and (3.9), we have
\[
\leq \left( \int_0^t \int_\mathbb{R} (t-r)^{\alpha-1} \cdot p_{t-r}(x-z) \left[ \int_\mathbb{R} |\Delta_h J_\alpha(r,z)|^2 \cdot |h|^{2H-2} dh \right]^{\frac{q}{2}} dzdr \right)^{\frac{2}{q}} 
\]
\[
\leq \left( \int_0^t \int_\mathbb{R} (t-r)^{q(\alpha-1)} \cdot p_{t-r}^q(x-z) \lambda^\frac{1}{q}(z) dzdr \right)^{\frac{2}{q}} 
\cdot \left( \int_0^T \int_\mathbb{R} |\Delta_h J_\alpha(r,z)|^2 \cdot |h|^{2H-2} dh \right)^{\frac{p}{2}} \lambda(z) dz dr 
\]
\[
\leq \lambda^\frac{2}{p}(x) \left( \int_0^t (t-r)^{q(\alpha-\frac{3}{2}+\frac{1}{2p})} dr \right)^{\frac{2}{q}} 
\cdot \left( \int_0^T \int_\mathbb{R} |\Delta_h J_\alpha(r,z)|^2 \cdot |h|^{2H-2} dh \right)^{\frac{p}{2}} \lambda(z) dz dr 
\]

(3.32)

where in the last inequality above we used \( p_{t-r}(x-z) \simeq (t-r)^{\frac{1-q}{2q}} p_{t-r}^q(x-z) \) and Lemma 6.1. If we take \( q(\alpha-\frac{3}{2}+\frac{1}{2p}) > -1 \), i.e., \( \alpha > \frac{3}{2} \frac{1}{p} \), then
\[
\sup \limits_{t \in [0,T], x \in \mathbb{R}} \lambda^\frac{p}{q}(x) \left( \int_\mathbb{R} |\Phi(t, x+h) - \Phi(t,x)|^2 \cdot |h|^{2H-2} dh \right)^{\frac{1}{2}} 
\leq \left( \int_0^T \int_\mathbb{R} |\Delta_h J_\alpha(r,z)|^2 \cdot |h|^{2H-2} dh \right)^{\frac{p}{2}} \lambda(z) dz dr 
\]

(3.33)

Thus, to prove part (i) we only need to prove that there exists some constant \( C \), independent of \( r \in [0,T] \), such that
\[
\mathcal{J} := \int_\mathbb{R} \int_\mathbb{R} |\Delta_h J_\alpha(r,z)|^2 \cdot |h|^{2H-2} dh \lambda(z) dz \leq C \left( 1 + \|u^\theta\|_{L^p_{\alpha,T}} \right).
\]

(3.34)

The rest of this part is to show that the inequality (3.34) holds. Recall \( D_t(x,h) \) defined in (3.11). Applying the Cauchy-Schwarz inequality, (2.4), (3.4) and (3.9), we have
\[
\Delta_h J_\alpha(r,z) \leq \int_0^r (r-s)^{-\alpha} \cdot \|D_{r-s}(z-\cdot, h) \sigma(s, \cdot, u^\theta(s, \cdot))\|_{H^\alpha} \cdot \|g(s, \cdot)\|_{H^\alpha} ds 
\]
\[
\leq \left( \int_0^r (r-s)^{-2\alpha} \cdot \|D_{r-s}(z-\cdot, h) \sigma(s, \cdot, u^\theta(s, \cdot))\|^2_{H^\alpha} ds \right)^{\frac{1}{2}} \cdot \left( \int_0^r \|g(s, \cdot)\|_{H^\alpha} ds \right)^{\frac{1}{2}} 
\leq \left( \int_0^r (r-s)^{-2\alpha} \cdot \|D_{r-s}(z-\cdot, h) \sigma(s, \cdot, u^\theta(s, \cdot))\|^2_{H^\alpha} ds \right)^{\frac{1}{2}} 
\leq \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \cdot |D_{r-s}(z-y-l, h) \sigma(s, y+l, u^\theta(s, y+l)) 
\right. 
\left. -D_{r-s}(z-y, h) \sigma(s, y, u^\theta(s, y))| |l|^{2H-2} dldy ds \right)^{\frac{1}{2}}. 
\]

Set
\[
\mathcal{J}_1(r, z, h) := \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \cdot |D_{r-s}(z-y-l, h)|^2 
\right. 
\left. -D_{r-s}(z-y, h) \sigma(s, y, u^\theta(s, y))| |l|^{2H-2} dldy ds \right)^{\frac{1}{2}}.
\]
By (3.35), (3.36), (3.37), (3.38) and by using the same method as that in the proof of (4.27) in [18] R. Li, R. Wang, and B. Zhang

By the same technique as that in Step 2 in the proof of Lemma 3.1, we have

Using the definition of \( \Phi \)

Once we have (3.33) with (3.39), we have (3.28).

\( J_2(r, z, h) := \left( \int_0^r \int_{\mathbb{R}^2} (r - s)^{-2\alpha} \cdot |D_{r-s}(z - y - l, h)|^2 \right) \phi \).

\( J_3(r, z, h) := \left( \int_0^r \int_{\mathbb{R}^2} (r - s)^{-2\alpha} \cdot |\square_{r-s}(z - y - l, h)|^2 \cdot |\sigma(s, y, u^\theta_q(s, y))|^2 \cdot |l|^{2H-2} dldyds \right) \phi \).

By Minkowski’s inequality, we have

\[
\mathcal{J} \lesssim \sum_{i=1}^3 \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} J_i(r, z, h) \lambda(z)dz \right)^\frac{2}{p} |h|^{2H-2} dh \right]^\frac{p}{2}.
\]

By the same technique as that in Step 2 in the proof of Lemma 3.1, we have

\[
\int_{\mathbb{R}} \left[ \int_{\mathbb{R}} J_1(r, z, h) \lambda(z)dz \right]^\frac{2}{p} |h|^{2H-2} dh \lesssim \int_0^r (t - s)^{-2\alpha + H - 1} \left( 1 + \|u^\theta_q(s, \cdot)\|_{L^p_L(\mathbb{R})}^2 \right) ds;
\]

\[
\int_{\mathbb{R}} \left[ \int_{\mathbb{R}} J_2(r, z, h) \lambda(z)dz \right]^\frac{2}{p} |h|^{2H-2} dh \lesssim \int_0^r (t - s)^{-2\alpha + H - 1} \left[ N^*_H u^\theta_q(s) \right]^2 ds;
\]

\[
\int_{\mathbb{R}} \left[ \int_{\mathbb{R}} J_3(r, z, h) \lambda(z)dz \right]^\frac{2}{p} |h|^{2H-2} dh \lesssim \int_0^r (t - s)^{-2\alpha + 2H - \frac{4}{q}} \left( 1 + \|u^\theta_q(s, \cdot)\|_{L^p_L(\mathbb{R})}^2 \right) ds
\]

+ \int_0^r (t - s)^{-2\alpha + H - 1} \left[ N^*_H u^\theta_q(s) \right]^2 ds.

By (3.35), (3.36), (3.37), (3.38) and by using the same method as that in the proof of (4.27) in [19], we have

\[
\mathcal{J} \lesssim C \left( \int_0^r (r - s)^{-2\alpha + 2H - \frac{4}{q}} + (r - s)^{-2\alpha + H - 1} ds \right)^\frac{p}{2} \left( 1 + \|u^\theta_q\|_{Z^p_{K,T}}^p \right).
\]

Once we have \( \alpha < H - \frac{1}{4} \), we obtain that (3.34) follows from (3.39). This condition on \( \alpha \) is combined with \( \alpha > \frac{3}{2p} \) to become \( \frac{3}{2p} < \alpha < H - \frac{1}{4} \), which requires that \( p > \frac{6}{(4H - 1)} \). Combining (3.33) with (3.39), we have (3.28).

(ii) By Eq. (3.6), we have

\[
u^\theta_q(t + h, x) - u^\theta_q(t, x) = \Phi(\tau + h, x) - \Phi(\tau, x).
\]

Using the definition of \( \Phi(t, x) \), we have

\[
\Phi(\tau + h, x) - \Phi(\tau, x)
\]

\[
\approx \int_0^t \int_{\mathbb{R}} (t + h - r)^{\alpha - 1} p_{t+h-r}(x - z) J_\alpha(r, z) dz dr
\]

\[
- \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} p_{t-r}(x - z) J_\alpha(r, z) dz dr
\]

\[
+ \int_t^{t+h} \int_{\mathbb{R}} (t + h - r)^{\alpha - 1} p_{t+h-r}(x - z) J_\alpha(r, z) dz dr
\]

(3.41)
\begin{align*}
&= \int_0^t \int_{\mathbb{R}} (t + h - r)^{\alpha - 1} (p_{t+h-r}(x-z) - p_{t-r}(x-z)) J_\alpha(r,z) dz \, dr \\
&\quad + \int_0^t \int_{\mathbb{R}} ((t + h - r)^{\alpha - 1} - (t - r)^{\alpha - 1}) p_{t-r}(x-z) J_\alpha(r,z) dz \, dr \\
&\quad + \int_t^{t+h} \int_{\mathbb{R}} (t + h - r)^{\alpha - 1} p_{t+h-r}(x-z) J_\alpha(r,z) dz \, dr \\
&=: K_1(t, h, x) + K_2(t, h, x) + K_3(t, h, x).
\end{align*}

In the following, we give estimates for $K_i(t, h, x)$, $i = 1, 2, 3$, respectively. For $K_1(t, h, x)$, by Hölder’s inequality, we have

$$
K_1(t, h, x) = \int_0^t \int_{\mathbb{R}} (t + h - r)^{\alpha - 1} (p_{t+h-r}(x-z) - p_{t-r}(x-z)) \lambda^{-\frac{1}{p}}(z) J_\alpha(r,z) \lambda^{\frac{1}{p}}(z) dz \, dr
\leq \left( \int_0^t \int_{\mathbb{R}} (t + h - r)^{q(\alpha - 1)} |p_{t+h-r}(x-z) - p_{t-r}(x-z)|^q \lambda^{-\frac{q}{p}}(z) dz \, dr \right)^{\frac{1}{q}}
\cdot \left( \int_0^t \int_{\mathbb{R}} |J_\alpha(r,z)|^p \lambda(z) dz \, dr \right)^{\frac{1}{p}}
=: |K_{11}(t, h, x)|^{\frac{1}{p}} \cdot \left( \int_0^t \|J_\alpha(r, \cdot)\|_{L^p_\lambda(\mathbb{R})}^p \, dr \right)^{\frac{1}{p}}.
$$

By (6.9) and Lemma 6.1, for any $\gamma \in (0, 1)$, we have

$$
|K_{11}(t, h, x)| \lesssim |h|^{|\gamma| \int_0^t \int_{\mathbb{R}} (t - r)^{q(\alpha - 1 - \gamma)} z \, p_{2(t+h-r)}(x-z) \lambda^{-\frac{2}{p}}(z) dz \, dr
\cdot \left( \int_0^t \int_{\mathbb{R}} (r - t)^{q(\alpha - 1 - \gamma)} z \, p_{2(t+h-r)}(x-z) \lambda^{-\frac{2}{p}}(z) dz \, dr \right)^{\frac{1}{q}}
\cdot \left( \int_0^t \int_{\mathbb{R}} |J_\alpha(r,z)|^p \lambda(z) dz \, dr \right)^{\frac{1}{p}}
\lesssim |h|^{|\gamma| \int_0^t \int_{\mathbb{R}} (r - t)^{q(\alpha - 1 - \gamma)} z \, p_{2(t+h-r)}(x-z) \lambda^{-\frac{2}{p}}(z) dz \, dr
\cdot \left( \int_0^t \int_{\mathbb{R}} (r - t)^{q(\alpha - 1 - \gamma)} z \, p_{2(t+h-r)}(x-z) \lambda^{-\frac{2}{p}}(z) dz \, dr \right)^{\frac{1}{q}}
\cdot \left( \int_0^t \int_{\mathbb{R}} |J_\alpha(r,z)|^p \lambda(z) dz \, dr \right)^{\frac{1}{p}}.
$$

Hence, for any $0 < \gamma < \frac{H}{2} - \frac{3}{2p} < 1$, we have

$$
|K_{11}(t, h, x)|^{\frac{1}{p}} \lesssim |h|^{|\gamma| \int_0^t \int_{\mathbb{R}} (r - t)^{q(\alpha - 1 - \gamma)} z \, p_{2(t+h-r)}(x-z) \lambda^{-\frac{2}{p}}(z) dz \, dr
\cdot \left( \int_0^t \int_{\mathbb{R}} (r - t)^{q(\alpha - 1 - \gamma)} z \, p_{2(t+h-r)}(x-z) \lambda^{-\frac{2}{p}}(z) dz \, dr \right)^{\frac{1}{q}}
\cdot \left( \int_0^t \int_{\mathbb{R}} |J_\alpha(r,z)|^p \lambda(z) dz \, dr \right)^{\frac{1}{p}}.
$$

Applying the Cauchy-Schwarz inequality, (2.4), (3.4) and (3.9), we have

$$
\|J_\alpha(r, \cdot)\|_{L^p_\lambda(\mathbb{R})}^p
\leq \int_0^r \int_{\mathbb{R}} (r - s)^{-\alpha} \cdot \langle p_{r-s}(z - \cdot) \sigma(s, \cdot, u_\epsilon^g(s, \cdot)), g(s, \cdot) \rangle_{H^s} ds \lambda(z) dz
\leq \int_0^r \int_{\mathbb{R}} (r - s)^{-2\alpha} \cdot \|p_{r-s}(z - \cdot) \sigma(s, \cdot, u_\epsilon^g(s, \cdot))\|_{H^s}^2 ds \lambda(z) dz
\leq \int_0^r \int_{\mathbb{R}} \int_{\mathbb{R}^2} (r - s)^{-2\alpha} \cdot \left( p_{r-s}(z - y - l) \sigma(s, y + l, u_\epsilon^g(s, y + l)) - p_{r-s}(z - y) \sigma(s, y, u_\epsilon^g(s, y)) \right)^2 ||l|^{2H-2} dydl ds \lambda(z) dz
\leq \int_\mathbb{R} [B_1(r, z) + B_2(r, z) + B_3(r, z)] \lambda(z) dz,
$$

where $B_1, B_2, B_3$ are some integrable functions.
where
\[
B_1(r, z) := \left( \int_0^r \int_{\mathbb{R}^2} (r - s)^{-2\alpha} \cdot p_{r-s}^2(z - y - l) \cdot \left| \sigma(s, y + l, u_x^2(s, y + l) - \sigma(s, y, u_x^2(s, y)) \right|^2 \cdot |l|^{2H-2} dy \, dl \right)^{\frac{1}{2}}; \\
B_2(r, z) := \left( \int_0^r \int_{\mathbb{R}^2} (r - s)^{-2\alpha} \cdot p_{r-s}^2(z - y - l) \cdot \left| \sigma(s, y, u_x^2(s, y)) \right|^2 \cdot |l|^{2H-2} dy \, dl \right)^{\frac{1}{2}}; \\
B_3(r, z) := \left( \int_0^r \int_{\mathbb{R}^2} (r - s)^{-2\alpha} \cdot |D_{r-s}(z - y, l)|^2 \cdot \left| \sigma(s, y, u_x^2(s, y)) \right|^2 \cdot |l|^{2H-2} dy \, dl \right)^{\frac{1}{2}}.
\]

By the same technique as that in Step 3 in the proof of Lemma 3.1, we have
\[
\begin{align*}
\int_{\mathbb{R}} B_1(r, z) \lambda(z) \, dz &\lesssim \left( \int_0^r (r - s)^{-2\alpha - \frac{1}{2}} \cdot \left( 1 + \|u_x^2(s, \cdot)\|^2_{L^p_{x}(\mathbb{R})} \right) \, ds \right)^{\frac{1}{2}}; \\
\int_{\mathbb{R}} B_2(r, z) \lambda(z) \, dz &\lesssim \left( \int_0^r (r - s)^{-2\alpha - \frac{1}{2}} \cdot \left( N_{\frac{1}{2}}^{\frac{1}{2}} u_x^2(s) \right)^2 \, ds \right)^{\frac{1}{2}}; \\
\int_{\mathbb{R}} B_3(r, z) \lambda(z) \, dz &\lesssim \left( \int_0^r (r - s)^{-2\alpha + H - 1} \cdot \left( 1 + \|u_x^2(s, \cdot)\|^2_{L^p_{x}(\mathbb{R})} \right) \, ds \right)^{\frac{1}{2}}. 
\end{align*}
\] (3.45) (3.46) (3.47)

Putting (3.44), (3.45), (3.46) and (3.47) together, we have
\[
\|J_\alpha(r, \cdot)\|_{L^p_x(\mathbb{R})} \lesssim \left( \int_0^r (r - s)^{-2\alpha - \frac{1}{2}} + (r - s)^{-2\alpha + H - 1} \, ds \right)^{\frac{1}{2}} \left( 1 + \|u_x^2\|^p_{Z_{\lambda,T}^p} \right). 
\] (3.48)

If \( \alpha < \frac{H}{2} \), then
\[
\|J_\alpha(r, \cdot)\|_{L^p_x(\mathbb{R})} \lesssim 1 + \|u_x^2\|^p_{Z_{\lambda,T}^p}. 
\] (3.49)

If \( 0 < \gamma < \frac{H}{2} - \frac{3}{2p} \), then by putting (3.42), (3.43) and (3.49) together, we have
\[
\sup_{t, t + h \in [0, T], x \in \mathbb{R}} \lambda(x) \cdot |\mathcal{K}_1(t, h, x)| \leq C_{T, p, H, \gamma} \cdot |h|^\gamma \cdot \left( 1 + \|u_x^2\|^p_{Z_{\lambda,T}^p} \right). 
\] (3.50)

For \( \mathcal{K}_2(t, h, x) \), applying Hölder’s inequality and Lemma 6.1, we have
\[
\begin{align*}
\mathcal{K}_2(t, h, x) &\lesssim \left( \int_0^t \int_{\mathbb{R}} |t + h - r|^{\alpha - 1} - (t - r)^{\alpha - 1} |q| \cdot p_{t-r}^{\alpha - 1}(x - z) \lambda^{\frac{2q}{p}}(z) \, dz \, dr \right)^{\frac{1}{q}} \cdot \left( \int_0^t \|J_\alpha(r, \cdot)\|^p_{L^p_x(\mathbb{R})} \, dr \right)^{\frac{1}{p}} \\
&\lesssim \left( \int_0^t |t + h - r|^{\alpha - 1} - (t - r)^{\alpha - 1} |q| (t - r)^{\frac{1 - q}{2}} \lambda^{\frac{2q}{p}}(x) \, dr \right)^{\frac{1}{q}} \\
&\quad \cdot \left( \int_0^t \|J_\alpha(r, \cdot)\|^p_{L^p_x(\mathbb{R})} \, dr \right)^{\frac{1}{p}}. 
\end{align*}
\] (3.51)
By (6.8) and (3.49), for any $\gamma \in (0, 1)$, we have
\[
\sup_{t, t + h \in [0, T], x \in \mathbb{R}} \lambda^\frac{1}{p}(x) \cdot |\mathcal{K}_2(t, h, x)|
\leq |h|^\gamma \cdot \sup_{t \in [0, T]} \left( \int_0^t (t - r)^{(\alpha - 1 - \gamma)q}(t - r)^{-\frac{q}{2}} dr \right)^{\frac{1}{q}} \cdot \left( 1 + \|u^g\|_{Z^p_{\lambda, T}} \right).
\]
If $0 < \gamma < \frac{H}{2} - \frac{3}{2p}$, then there exists a positive constant $C_{T,p,H,\gamma}$ such that
\[
\sup_{t, t + h \in [0, T], x \in \mathbb{R}} \lambda^\frac{1}{p}(x) \cdot |\mathcal{K}_2(t, h, x)| \leq C_{T,p,H,\gamma} |h|^\gamma \cdot \left( 1 + \|u^g\|_{Z^p_{\lambda, T}} \right). \tag{3.52}
\]
For $\mathcal{K}_3(t, h, x)$, by Hölder’s inequality and Lemma 6.1, we have
\[
\mathcal{K}_3(t, h, x) \leq \left( \int_t^{t+h} (t + h - r)^{q(\alpha - 1)}(t + h - r)^{-\frac{q}{2}} \lambda^{-\frac{q}{p}}(x)dr \right)^{\frac{1}{q}} \cdot \left( \int_0^T \|J_{\alpha}(r, \cdot\|_{L^p_\lambda(\mathbb{R})})dr \right)^{\frac{1}{q}} \cdot \left( \int_0^T \|J_{\alpha}(r, \cdot\|_{L^p_\lambda(\mathbb{R})})dr \right)^{\frac{1}{q}}.
\]
If $0 < \gamma < \frac{H}{2} - \frac{3}{2p}$, then
\[
\sup_{t, t + h \in [0, T], x \in \mathbb{R}} \lambda^\frac{1}{p}(x) \cdot |\mathcal{K}_3(t, h, x)| \leq C_{T,p,H,\gamma} |h|^\gamma \cdot \left( 1 + \|u^g\|_{Z^p_{\lambda, T}} \right). \tag{3.54}
\]
If $p > \frac{3}{H}$ and $0 < \gamma < \frac{H}{2} - \frac{3}{2p}$, then by putting (3.41), (3.50), (3.52) and (3.54) together, we have
\[
\sup_{t, t + h \in [0, T], x \in \mathbb{R}} \lambda^\frac{1}{p}(x) \cdot |\Phi(t + h, x) - \Phi(t, x)| \leq C_{T,p,H,\gamma} |h|^\gamma \cdot \left( 1 + \|u^g\|_{Z^p_{\lambda, T}} \right). \tag{3.55}
\]
(iii). Notice that
\[
\Phi(t, x) - \Phi(t, y) = \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} \cdot (p_{t-r}(x - z) - p_{t-r}(y - z)) J_{\alpha}(r, z)dzdr.
\]
For any $\gamma \in (0, 1)$, by using the same method as that in the proof of (4.35) in [19] and (3.49), we have
\[
|\Phi(t, x) - \Phi(t, y)| \leq |x - y|^\gamma \cdot \left| \lambda^{-\frac{1}{p}}(x) + \lambda^{-\frac{1}{p}}(y) \right| \cdot \left( \int_0^t (t - r)^{q(\alpha - \frac{2q}{p}) + \frac{q}{2}} dr \right)^{\frac{1}{q}} \cdot \left( 1 + \|u^g\|_{Z^p_{\lambda, T}} \right). \tag{3.56}
\]
If $p > \frac{3}{H}$ and $0 < \gamma < H - \frac{3}{p}$, then we get the desired result (3.30). The proof is complete. \(\square\)
Proof of Proposition 3.1. (Existence). By Lemma 3.1 and Lemma 3.2 (ii) and (iii), we know that the family \{u^g_\varepsilon\}_{\varepsilon > 0} is relative compact on the space \((C([0,T] \times \mathbb{R}), d_C)\) by the Arzelà-Ascoli theorem. Thus, there is a subsequence \(\varepsilon_n \downarrow 0\) such that \(u^g_{\varepsilon_n}\) converges to a function \(u^g\) in \((C([0,T] \times \mathbb{R}), d_C)\). On the other hand, since \(\int_0^T \|g(s)\|_H^2 ds \leq M\), by using the Cauchy-Schwarz inequality, (2.16) and the dominated convergence theorem, we have as \(\varepsilon_n \downarrow 0\),

\[
\begin{align*}
u^g_{\varepsilon_n}(t,x) &= 1 + \int_0^t \langle p_{t-s}(x-\cdot) \sigma(s,\cdot, u^g_{\varepsilon_n}(s,\cdot)) - \sigma(s,\cdot, u^g(s,\cdot)), g(s,\cdot) \rangle_{H_2} ds \\
&\quad + \int_0^t \langle p_{t-s}(x-\cdot) \sigma(s,\cdot, u^g(s,\cdot)), g(s,\cdot) \rangle_{H_2} ds \\
\to &1 + \int_0^t \langle p_{t-s}(x-\cdot) \sigma(s,\cdot, u^g(s,\cdot)), g(s,\cdot) \rangle_{H_2} ds.
\end{align*}
\]

By the uniqueness of the limit of \(\{u^g_{\varepsilon_n}\}_{n \geq 1}\), we know that \(u^g\) satisfies Eq. (2.26).

(Uniqueness). Let \(u^g\) and \(v^g\) be two solutions of (2.26). Note that by using the same technique as that in the proof of Lemma 3.2 (i), we have that for any \(p > \frac{6}{2H-1}\),

\[
\begin{align*}
sup_{t \in [0,T], x \in \mathbb{R}} \lambda^\frac{1}{2}(x) \cdot \mathcal{N}_{\frac{1}{2}-H} u^g(t,x) &< \infty; \quad (3.57) \\
\sup_{t \in [0,T], x \in \mathbb{R}} \lambda^\frac{1}{2}(x) \cdot \mathcal{N}_{\frac{1}{2}-H} v^g(t,x) &< \infty. \quad (3.58)
\end{align*}
\]

Denote

\[
S_1(t) = \sup_{x \in \mathbb{R}} |u^g(t,x) - v^g(t,x)|^2
\]

and

\[
S_2(t) = \sup_{x \in \mathbb{R}} \left[ \int_{\mathbb{R}} |u^g(t,x) - v^g(t,x) - u^g(t,x+h) + v^g(t,x+h)|^2 \cdot |h|^{2H-2} dh \right].
\]

Recall \(D_t(x,h)\) defined in (3.11) and denote

\[
\Delta(t,x,y) := \sigma(t,x,u^g(t,y)) - \sigma(t,x,v^g(t,y)).
\]

Since \(\int_0^T \|g(s,\cdot)\|_H^2 ds \leq M\), by using the Cauchy-Schwarz inequality, (2.4) and a change of variable, we have

\[
\begin{align*}
|u^g(t,x) - v^g(t,x)|^2 \\
&\leq \left| \int_0^t \langle p_{t-s}(x-\cdot) \Delta(s,\cdot, \cdot), g(s,\cdot) \rangle_{H_2} ds \right|^2 \\
&\leq \int_0^t \|g(s,\cdot)\|_H^2 ds \cdot \int_0^t \|p_{t-s}(x-\cdot) \Delta(s,\cdot, \cdot)\|^2_{H_2} ds \\
&\leq \int_0^t \int_{\mathbb{R}^2} \left[ p_{t-s}(x-y-h) \Delta(s, y+h, y+h) - p_{t-s}(x-y) \Delta(s, y, y) \right]^2 \cdot |h|^{2H-2} dh dy ds \\
&\leq \int_0^t \int_{\mathbb{R}^2} |D_{t-s}(x-y,h)|^2 \Delta^2(s, y, y) \cdot |h|^{2H-2} dh dy ds + \int_0^t \int_{\mathbb{R}^2} p_{t-s}^2(x-y-h) \left( \Delta(s, y+h, y) - \Delta(s, y, y) \right)^2 \cdot |h|^{2H-2} dh dy ds \\
&\quad + \int_0^t \int_{\mathbb{R}^2} p_{t-s}^2(x-y) \left( \Delta(s, y+h) - \Delta(s, y, y) \right)^2 \cdot |h|^{2H-2} dh dy ds \\
=: &V_1(t,x) + V_2(t,x) + V_3(t,x).
\end{align*}
\]
Lemma 6.2 and (2.16) yield that
\[
V_1(t, x) \leq \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |D_{t-s}(x - y, h)|^2 \cdot |u^q(s, y) - v^q(s, y)|^2 \cdot |h|^{2H-2} dh dy ds
\]
\[
\leq \int_0^t (t-s)^{H-1} \cdot \sup_{y \in \mathbb{R}} \left[ |u^q(s, y) - v^q(s, y)|^2 \right] ds
\]
\[
= \int_0^t (t-s)^{H-1} \cdot S_1(s) ds.
\]
If \( h > 1 \), then we have by (2.19)
\[
(\triangle(s, y + h, y) - \triangle(s, y, y))^2
\]
\[
= \left| \int_{u^q} (\sigma'_\xi(s, y + h, \xi) - \sigma'_\xi(s, y, \xi)) d\xi \right|^2
\]
\[
\leq |u^q(s, y) - v^q(s, y)|^2.
\]
If \( h \leq 1 \), then we have by (2.19)
\[
(\triangle(s, y + h, y) - \triangle(s, y, y))^2
\]
\[
= \left| \int_{u^q} (\sigma'_\xi(s, y + h, \xi) - \sigma'_\xi(s, y, \xi)) d\xi \right|^2
\]
\[
\leq |u^q(s, y) - v^q(s, y)|^2 \cdot |h|^2.
\]
Thus, by (3.61) and (3.62), we have
\[
V_2(t, x) = \int_0^t \int_{\mathbb{R}} \int_{h > 1} p_{t-s}(x - y - h) \cdot |u^q(s, y) - v^q(s, y)|^2 \cdot |h|^{2H-2} dh dy ds
\]
\[
+ \int_0^t \int_{\mathbb{R}} \int_{h \leq 1} p_{t-s}(x - y - h) \cdot |u^q(s, y) - v^q(s, y)|^2 \cdot |h|^{2H} dh dy ds
\]
\[
\leq \int_0^t \left( \int_{\mathbb{R}} p_{t-s}(x - y) dy \right) \cdot \sup_{y \in \mathbb{R}} |u^q(s, y) - v^q(s, y)|^2 ds
\]
\[
= \int_0^t (t-s)^{-\frac{1}{2}} \cdot S_1(s) ds.
\]
Notice that
\[
|\triangle(s, y, y + h) - \triangle(s, y, y)|^2
\]
\[
= \left| \int_0^1 [u^q(s, y + h) - v^q(s, y + h)] \sigma'_\xi(s, y, \theta u^q(s, y + h) + (1 - \theta)v^q(s, y + h)) d\theta
\]
\[
- \int_0^1 [u^q(s, y) - v^q(s, y)] \sigma'_\xi(s, y, \theta u^q(s, y) + (1 - \theta)v^q(s, y)) d\theta \right|^2.
\]
By (2.17) and (2.20), we have
\[
|\triangle(s, y, y + h) - \triangle(s, y, y)|^2
\]
\[
\leq |u^q(s, y + h) - v^q(s, y + h) - u^q(s, y) + v^q(s, y)|^2
\]
\[
+ \lambda^2(y) |u^q(s, y) - v^q(s, y)|^2 \cdot \left[ |u^q(s, y + h) - u^q(s, y)|^2 + |v^q(s, y + h) - v^q(s, y)|^2 \right].
\]
This, together with (3.57) and (3.58), gives the following estimate:

\[ V_3(t, x) \lesssim \int_0^t (t - s)^{-\frac{1}{2}} \cdot [S_1(s) + S_2(s)]ds. \]  

(3.65)

By (3.59), (3.60), (3.63) and (3.65), we have

\[ S_1(t) \leq \int_0^t (t - s)^{H-1} \cdot [S_1(s) + S_2(s)]ds. \]  

(3.66)

The similar procedure can be applied to estimate the term \( S_2(t) \) to obtain that

\[ S_2(t) \leq \int_0^t (t - s)^{2H-\frac{3}{2}} \cdot [S_1(s) + S_2(s)]ds. \]  

(3.67)

Therefore, by (3.66) and (3.67), we have

\[ S_1(t) + S_2(t) \leq \int_0^t \left((t - s)^{H-1} + (t - s)^{2H-\frac{3}{2}}\right) \cdot [S_1(s) + S_2(s)]ds. \]

By the fractional Gronwall lemma ([21, Lemma 1]), we have \( S_1(t) + S_2(t) = 0 \) for all \( t \in [0, T] \). In particular, we have for all \( t \in [0, T], x \in \mathbb{R} \)

\[ |u^g(t, x) - v^g(t, x)|^2 = 0. \]

The proof is complete.

\[ \square \]

4. Verification of Condition 2.8 (a)

We now verify Condition 2.8 (a). Recall that \( \Gamma^0 \left( \int_0^T g(s)ds \right) = u^g \) for \( g \in \mathcal{S} \), where \( u^g \) is the solution of Eq. (2.26).

**Proposition 4.1.** Assume that \( \sigma \) satisfies the hypothesis (H). For any \( N > 1 \), let \( g_n, g \in S^N \) be such that \( g_n \to g \) weakly as \( n \to \infty \). Let \( u^{g_n} \) denote the solution to Eq. (2.26) replacing \( g \) by \( g_n \). Then, as \( n \to \infty \),

\[ u^{g_n} \to u^g \quad \text{in} \quad C([0, T] \times \mathbb{R}). \]

**Proof.** The proof is divided into three steps. In Step 1, we give some estimates for preparation.

In Step 2, we prove that there exists a subsequence of \( \{u^{g_n}\}_{n \geq 1} \) (still denoted by \( \{u^{g_n}\}_{n \geq 1} \) and \( u \in C([0, T] \times \mathbb{R}) \)) such that \( u^{g_n} \to u \) as \( n \to \infty \) in \( C([0, T] \times \mathbb{R}) \). In Step 3, we show that \( u = u^g \).

**Step 1.** Recall

\[ u^{g_n}(t, x) = 1 + \int_0^t \langle p_{t-s}(x - \cdot)\sigma(s, \cdot, u^{g_n}(s, \cdot)), g_n(s, \cdot) \rangle_{\mathcal{H}} ds. \]  

(4.1)

Since the norm \( \left\{ \int_0^T \|g_n(s)\|^2_{\mathcal{H}} ds \right\}_{n \geq 1} \) is bounded by a constant \( M \), invoking similar arguments as that in the proof of Lemma 3.1, we have

\[ \sup_{n \geq 1} \|u^{g_n}\|_{L^2_{t,x}} < \infty. \]  

(4.2)

Moreover, we claim that

\[ \sup_{n \geq 1} \sup_{t \in [0, T]} \|u^{g_n}(t, \cdot)\|_{\infty} < \infty, \]  

(4.3)

where \( \| \cdot \|_{\infty} \) is the uniform norm with respect to the spatial variable \( x \). Next, we prove (4.3). Define \( u^{g_n}_{m+1}(t, x) = 1 \) and

\[ u^{g_n}_{m+1}(t, x) = 1 + \int_0^t \langle p_{t-s}(x - \cdot)\sigma(s, \cdot, u^{g_n}_{m}(s, \cdot)), g_n(s, \cdot) \rangle_{\mathcal{H}} ds. \]  

(4.4)
By the Cauchy-Schwarz inequality and (4.4), we have

\[
|u_{m+1}^n(t, x)|^2 \lesssim 1 + \int_0^T \|g_n(s, \cdot)\|_H^2 \, ds \cdot \int_0^t \|p_{t-s}(x - \cdot)\|_H^2 \, ds.
\]  

(4.5)

Moreover, by (2.4), we have

\[
\|p_{t-s}(x - \cdot)\|_H^2 \lesssim \int_{\mathbb{R}^2} |p_{t-s}(x - y - z)|^2 \cdot |\sigma(s, y + z, u_{m}^n(s, y + z))| \cdot |z|^{2H-2} \, dy \, dz
\]

\[
+ \int_{\mathbb{R}^2} p_{t-s}(x - y) \cdot |\sigma(s, y + z, u_{m}^n(s, y + z))| \cdot |z|^{2H-2} \, dy \, dz
\]

\[
+ \int_{\mathbb{R}^2} p_{t-s}(x - y) \cdot \|\sigma(s, y, u_{m}^n(s, y + z)) - \sigma(s, y, u_{m}^n(s, y))\|^2 \cdot |z|^{2H-2} \, dy \, dz
\]

\[
=: I_1^m(t, s, x) + I_2^m(t, s, x) + I_3^m(t, s, x).
\]

For \(I_1^m(t, s, x)\), by (2.15) and Lemma 6.2, we have

\[
I_1^m(t, s, x) \lesssim \int_{\mathbb{R}^2} \left[ |D_{t-s}(x - y, z)|^2 \cdot |z|^{2H-2} \, dy \, dz \right] \cdot \left( 1 + \|u_{m}^n(s, \cdot)\|_\infty^2 \right)
\]

\[
\lesssim (t - s)^{\frac{1}{2} - H} \cdot \left( 1 + \|u_{m}^n(s, \cdot)\|_\infty^2 \right).
\]

(4.7)

For \(I_2^m(t, s, x)\), by (2.15), (2.18) and (2.19), applying the same calculation as that in (3.14) and (3.15), we have

\[
I_2^m(t, s, x) \lesssim \int_{\mathbb{R}^2} \int_{|y| > \epsilon} p_{t-s}(x - y) \cdot |z|^{2H-2} \, dy \, dz \cdot \left( 1 + \|u_{m}^n(s, \cdot)\|_\infty^2 \right)
\]

\[
+ \int_{\mathbb{R}^2} \int_{|y| \leq \epsilon} p_{t-s}(x - y) \cdot |z|^{2H} \, dy \, dz 
\]

\[
\lesssim (t - s)^{-\frac{1}{2} - H} \cdot \left( 1 + \|u_{m}^n(s, \cdot)\|_\infty^2 \right).
\]

(4.8)

For \(I_3^m(t, s, x)\), by (2.16), we have

\[
I_3^m(t, s, x) \lesssim \int_{\mathbb{R}^2} p_{t-s}(x - y) \cdot |u_{m}^n(s, y + z) - u_{m}^n(s, y)|^2 \cdot |z|^{2H-2} \, dy \, dz
\]

\[
\lesssim \int_{\mathbb{R}^2} p_{t-s}(x - y) \cdot \left| \mathcal{N}_{\frac{1}{2} - H} u_{m}^n(s, y) \right|^2 \, dy
\]

\[
\lesssim (t - s)^{-\frac{1}{2} - H} \cdot \left| \mathcal{N}_{\frac{1}{2} - H} u_{m}^n(s, \cdot) \right|_\infty^2.
\]

(4.9)

Putting (4.5), (4.6), (4.7), (4.8) and (4.9) together, we have

\[
\|u_{m}^n(t, \cdot)\|_\infty^2 \lesssim 1 + \int_0^t \left( (t - s)^{H-1} + (t - s)^{-\frac{1}{2} - H} \right) \cdot \|u_{m}^n(s, \cdot)\|_\infty^2 \, ds \]

\[
+ \int_0^t (t - s)^{-\frac{1}{2} - H} \cdot \left| \mathcal{N}_{\frac{1}{2} - H} u_{m}^n(s, \cdot) \right|_\infty^2 \, ds.
\]

(4.10)

Next, we bound \(\mathcal{N}_{\frac{1}{2} - H} u_{m}^n(t, x)\). Set

\[
\Phi_{m}^n(t, x) := \int_0^t \langle p_{t-s}(x - \cdot)\sigma(s, u_{m}^n(s, \cdot), g_n(s, \cdot) \rangle_H \, ds.
\]
Similarly to (3.59), we have
\[
\left| N_{\frac{1}{2}-H} u_{m+1}^g(t, x) \right|^2 = \left| N_{\frac{1}{2}-H} \Phi_{m}^g(t, x) \right|^2
= \int_{\mathbb{R}} \left| \Phi_{m}^g(t, x + z) - \Phi_{m}^g(t, x) \right|^2 |z|^{2H-2} dz
= \int_{\mathbb{R}} \left[ \int_{0}^{t} \langle D_{t-s}(x - \cdot, z) \sigma(s, \cdot, u_{m}^{g}(s, \cdot)), g_{m}(s, \cdot) \rangle_{\mathcal{H}} ds \right]^2 |z|^{2H-2} dz
\leq \int_{\mathbb{R}} \| f(s, \cdot) \|_{\mathcal{H}}^2 ds \cdot \int_{\mathbb{R}} \int_{0}^{t} \| D_{t-s}(x - \cdot, z) \sigma(s, \cdot, u_{m}^{g}(s, \cdot)) \|_{\mathcal{H}}^2 |z|^{2H-2} d\sigma dz.
\]
By (4.12), we have
\[
\| D_{t-s}(x - \cdot, z) \sigma(s, \cdot, u_{m}^{g}(s, \cdot)) \|_{\mathcal{H}}^2
\leq \int_{\mathbb{R}^2} \left| D_{t-s}(x - h - l, z) \sigma(s, h + l, u_{m}^{g}(s, h + l)) \right|^2 \cdot |l|^{2H-2} dh dl
+ \int_{\mathbb{R}^2} | D_{t-s}(x - h, z) |^2 \cdot | \sigma(s, h + l, u_{m}^{g}(s, h + l)) - \sigma(s, h, u_{m}^{g}(s, h)) |^2 \cdot |l|^{2H-2} dh dl
+ \int_{\mathbb{R}^2} | D_{t-s}(x - h, z) |^2 \cdot | \sigma(s, h, u_{m}^{g}(s, h + l)) - \sigma(s, h, u_{m}^{g}(s, h)) |^2 \cdot |l|^{2H-2} dh dl
= J_{1}^m(t, s, x, z) + J_{2}^m(t, s, x, z) + J_{3}^m(t, s, x, z).
\]
For $J_{1}^m(t, s, x, z)$, by (2.15), a change of variable and Lemma 6.2, we have
\[
\int_{\mathbb{R}} J_{1}^m(t, s, x, z) \cdot |z|^{2H-2} dz
\leq \int_{\mathbb{R}^3} \left| D_{t-s}(h, l, z) \right|^2 \cdot |l|^{2H-2} \cdot |z|^{2H-2} dldz dh \cdot \left( 1 + \| u_{m}^{g}(s, \cdot) \|_{\mathcal{H}}^2 \right)
\leq (t - s)^{H - \frac{3}{2}} \cdot \left( 1 + \| u_{m}^{g}(s, \cdot) \|_{\mathcal{H}}^2 \right).
\]
For $J_{2}^m(t, s, x, z)$, by (2.15), (2.18), (2.19) and Lemma 6.2, we have
\[
\int_{\mathbb{R}} J_{2}^m(t, s, x, z) \cdot |z|^{2H-2} dz
\leq \int_{\mathbb{R}} \int_{|l| > 1} | D_{t-s}(h, z) |^2 \cdot |l|^{2H-2} \cdot |z|^{2H-2} dldz dh \cdot \left( 1 + \| u_{m}^{g}(s, \cdot) \|_{\mathcal{H}}^2 \right)
+ \int_{\mathbb{R}^2} \int_{|l| \leq 1} | D_{t-s}(h, z) |^2 \cdot |l|^{2H-2} \cdot |z|^{2H-2} dldz dh \cdot \left( 1 + \| u_{m}^{g}(s, \cdot) \|_{\mathcal{H}}^2 \right)
\leq (t - s)^{H - 1} \cdot \left( 1 + \| u_{m}^{g}(s, \cdot) \|_{\mathcal{H}}^2 \right).
For $J^n_{3}(t,s,x,z)$, by (2.16) and Lemma 6.2, we have
\[
\int_{\mathbb{R}} J^n_{3}(t,s,x,z) \cdot |z|^{2H-2} \, dz \\
\leq \int_{\mathbb{R}^3} |D_t (h,z)|^2 \cdot |u^n_m (s,h) - u^n_m (s,l)|^2 \cdot |l|^{2H-2} \cdot |z|^{2H-2} \, dh \, dldz \\
\leq \int_{\mathbb{R}^2} |D_t (h,z)|^2 \cdot |z|^{2H-2} \, dh \cdot \|N_{\frac{1}{2}-H} u^n_m (s,\cdot)\|_\infty^2 \\
\leq (t-s)^{H-1} \cdot \|N_{\frac{1}{2}-H} u^n_m (s,\cdot)\|_\infty^2.
\] (4.15)

According to (4.11)-(4.15), we have
\[
\|N_{\frac{1}{2}-H} u^n_{m+1} (t,\cdot)\|_\infty^2 \leq 1 + \int_0^t (t-s)^{H-1} \cdot \|N_{\frac{1}{2}-H} u^n_m (s,\cdot)\|_\infty^2 \, ds \\
+ \int_0^t (t-s)^{H-1} + (t-s)^{2H-\frac{3}{2}} \cdot \|u^n_m (s,\cdot)\|_\infty^2 \, ds.
\] (4.16)

Thus, by (4.10), (4.16) and the fractional Gronwall lemma ([21, Lemma 1]), we obtain (4.3).

**Step 2.** In this step, we prove that the family $\{u^n_n\}_{n \geq 1}$ is relatively compact, which implies that there exists a subsequence of $\{u^n_n\}_{n \geq 1}$ (still denoted by $\{u^n_n\}_{n \geq 1}$) and $u \in C([0,T] \times \mathbb{R})$ such that $u^n_n \to u$ as $n \to \infty$ in $C([0,T] \times \mathbb{R})$. Set
\[
\Phi^n_n (t,x) := \int_0^t \langle p_t-s (x-y) \sigma(s,\cdot, u^n_n (s,\cdot)), g_n (s,\cdot) \rangle_{\mathcal{H}} \, ds.
\]

By using the same technique as that in (3.31), for any $\alpha \in (0,1)$, we have
\[
\Phi^n_n (t,x) \simeq \int_0^t \int_{\mathbb{R}^2} \left[ p_t-s (x-y-l) \sigma(s, y+l, u^n_n (s,y+l)) - p_t-s (x-y) \sigma(s, y, u^n_n (s,y)) \right] \\
\cdot \|g_n (s,y+l) - g_n (s,y)\| l^{2H-2} \, dldy \, ds \\
\simeq \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} \cdot p_{t-r} (x-z) \cdot \int_0^r \langle r-s \rangle^{-\alpha} p_{r-s} (z-\cdot) \sigma(s, z, u^n_n (s,\cdot)) \cdot g_n (s,\cdot) \rangle_{\mathcal{H}} ds \, dz \, dr \\
= \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} \cdot p_{r-t} (x-z) \cdot I_\alpha (r,z) \, dz \, dr,
\]

where
\[
I_\alpha (r,z) := \int_0^r \langle r-s \rangle^{-\alpha} p_{r-s} (z-\cdot) \sigma(s, z, u^n_n (s,\cdot)) \cdot g_n (s,\cdot) \rangle_{\mathcal{H}} ds.
\]

If $\alpha < \frac{H}{2}$, then by using the same technique as that in the proof of (3.49), we have
\[
\|I_\alpha (r,\cdot)\|^p_{L_\infty^p (\mathbb{R})} \\
= \int_{\mathbb{R}} \int_0^r \langle r-s \rangle^{-\alpha} p_{r-s} (z-\cdot) \sigma(s, z, u^n_n (s,\cdot)) \cdot g_n (s,\cdot) \rangle_{\mathcal{H}} ds \, dz \|_{\mathcal{H}}^p \lambda(z) \, dz \\
\leq \int_{\mathbb{R}} \int_0^r (r-s)^{-2\alpha} \cdot p_{r-s} (z-\cdot) \sigma(s, z, u^n_n (s,\cdot)) \|_{\mathcal{H}}^2 ds \, dz \lambda(z) \, dz \\
\leq 1 + \|u^n_n\|_{Z_{p,\mathcal{H}}^p}^p.
\] (4.17)
Fix $\gamma \in (0, 1)$. By using the same method as that in the proofs of (3.55) and (3.56), we have
\[
\sup_{t, t+h \in [0,T], x \in \mathbb{R}} \lambda^\frac{1}{p}(x) \cdot |\Phi^{g_n}(t + h, x) - \Phi^{g_n}(t, x)|
\]
\[
\leq |h|^{\gamma} \cdot \left( \int_0^t (t - r)^{q(\alpha - \gamma)} \frac{1}{2} dr \right)^{\frac{1}{q}} \cdot \left( 1 + \|u^{g_n}\|_{L^p_{\lambda,T}} \right),
\]
and
\[
|\Phi^{g_n}(t, x + h) - \Phi^{g_n}(t, x)|
\]
\[
\leq |h|^{\gamma} \cdot \left( \lambda^\frac{1}{p}(x + h) + \lambda^\frac{1}{p}(x) \right) \cdot \left( \int_0^{t} (t - r)^{(\alpha q - \frac{3q}{2} + \frac{1}{2}) - \frac{2q}{p}} dr \right)^{\frac{1}{q}} \cdot \left( 1 + \|u^{g_n}\|_{L^p_{\lambda,T}} \right).
\]
If $p > \frac{3}{2}$ and $0 < \gamma < \frac{H}{2} - \frac{3}{2p}$, then there exists a positive constant $C_{T, p, H, \gamma}$ such that
\[
\sup_{t, t+h \in [0,T], x \in \mathbb{R}} \lambda^\frac{1}{p}(x) \cdot |u^{g_n}(t + h, x) - u^{g_n}(t, x)| \leq C_{T, p, H, \gamma} \cdot |h|^{\gamma} \cdot \left( 1 + \|u^{g_n}\|_{L^p_{\lambda,T}} \right).
\]
If $p > \frac{3}{2}$ and $0 < \gamma < H - \frac{3}{2p}$, then there exists a positive constant $C_{T, p, H, \gamma}$ such that
\[
\sup_{t \in [0,T], x \in \mathbb{R}} \lambda^\frac{1}{p}(x) \cdot |u^{g_n}(t, x + h) - u^{g_n}(t, x)| \leq C_{T, p, H, \gamma} \cdot |h|^{\gamma} \cdot \left( 1 + \|u^{g_n}\|_{L^p_{\lambda,T}} \right). \tag{4.18}
\]
Therefore, $\{u^{g_n}\}_{n \geq 1}$ is relatively compact on the space $C([0, T] \times \mathbb{R})$ by the Arzelà-Ascoli theorem. Then, $u^{g_n} \rightarrow u$ as $n \rightarrow \infty$ in $C([0, T] \times \mathbb{R})$. By using Fatou’s lemma and taking into account (4.2) and (4.3), we have
\[
\|u\|_{L^p_{\lambda,T}} < \infty \quad \text{and} \quad \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{\infty} < \infty. \tag{4.19}
\]

**Step 3.** In this step, we prove that $u = u^g$. Denote
\[
\Delta_n(t, x, y) := \sigma(t, x, u^{g_n}(t, y)) - \sigma(t, x, u(t, y)).
\]
Since $g_n$ converges weakly to $g$, we have
\[
\lim_{n \rightarrow \infty} \int_0^t \langle p_{t-s}(x - \cdot) \sigma(s, \cdot, u^{g_n}(s, \cdot)), g_n(s, \cdot) \rangle_{\mathcal{H}} ds
\]
\[
= \lim_{n \rightarrow \infty} \int_0^t \langle p_{t-s}(x - \cdot) \Delta_n(s, \cdot, \cdot), g_n(s, \cdot) \rangle_{\mathcal{H}} ds
\]
\[
+ \lim_{n \rightarrow \infty} \int_0^t \langle p_{t-s}(x - \cdot) \sigma(s, \cdot, u(s, \cdot)), g_n(s, \cdot) \rangle_{\mathcal{H}} ds
\]
\[
= \lim_{n \rightarrow \infty} \int_0^t \langle p_{t-s}(x - \cdot) \Delta_n(s, \cdot, \cdot), g_n(s, \cdot) \rangle_{\mathcal{H}} ds
\]
\[
+ \int_0^t \langle p_{t-s}(x - \cdot) \sigma(s, \cdot, u(s, \cdot)), g(s, \cdot) \rangle_{\mathcal{H}} ds.
\]
By the Cauchy-Schwarz inequality, we have
\[
\left| \int_0^t \langle p_{t-s}(x - \cdot) \Delta_n(s, \cdot, \cdot), g_n(s, \cdot) \rangle_{\mathcal{H}} ds \right| \leq \left( \int_0^t \| p_{t-s}(x - \cdot) \Delta_n(s, \cdot, \cdot) \|^2_{\mathcal{H}} ds \right)^{\frac{1}{2}}. \tag{4.21}
\]
Furthermore, by (2.4) and a change of variable, we have
\[
\| p_{t-s}(x - \cdot) \Delta_n(s, \cdot, \cdot) \|^2_{\mathcal{H}}
\]
The dominated convergence theorem yields that

\[ \lim_{n \to \infty} D_{t-s}(x-y, h) \cdot |\Delta_n(s, y + h, y + h) - \Delta_n(s, y, y + h)| = 0, \]

and

\[ \lim_{n \to \infty} \int_{\mathbb{R}^2} |p_{t-s}(x-y) \cdot \Delta_n(s, y + h, y + h) - p_{t-s}(x-y) \cdot \Delta_n(s, y, y + h)| \cdot |h|^{2H-2} \, dhdy = 0. \]
Since
\[ \int_{\mathbb{R}^2} p^2_{t-s}(x-y) \cdot (1_{|h| \leq 1}|h|^{2H} + 1_{|h| > 1}|h|^{2H-2}) \, dh \, dy < +\infty, \]
by the dominated convergence theorem, we have
\[ \lim_{n \to \infty} D^n_2(t, s, x) = 0. \]
By (4.27), we have
\[ D^n_2(t, s, x) \leq \int_{\mathbb{R}^2} p^2_{t-s}(x-y) \cdot (1_{|h| \leq 1}|h|^{2H} + 1_{|h| > 1}|h|^{2H-2}) \, dh \, dy \]
\[ \leq \int_{\mathbb{R}} p^2_{t-s}(x-y) \, dy \]
\[ \leq (t-s)^{-\frac{1}{p}}. \]
The dominated convergence theorem yields that
\[ \lim_{n \to \infty} \int_0^t D^n_2(t, s, x) \, ds = 0. \]
By (2.16), we have
\[ \lim_{n \to \infty} |\triangle_n(s, y, y + h) - \triangle_n(s, y, y)| = 0. \]
By using the same technique as that in the proof of Lemma 3.2 (i), for any \( p > \frac{6}{2H-1} \), we have
\[ \sup_{t \in [0, T], x \in \mathbb{R}} \lambda^{-\frac{2}{p}}(x) \cdot \mathcal{N}_{\frac{2}{2} - H} u^{g^n}(t, x) < \infty. \]
Applying Fatou’s lemma, we have
\[ \sup_{t \in [0, T], x \in \mathbb{R}} \lambda^{-\frac{2}{p}}(x) \cdot \mathcal{N}_{\frac{2}{2} - H} u(t, x) < \infty. \]
By (2.16), we have
\[ |\triangle_n(s, y, y + h) - \triangle_n(s, y, y)|^2 \]
\[ \leq |u^{g^n}(s, y + h) - u^{g^n}(s, y)|^2 + |u(s, y + h) - u(s, y)|^2. \]
(4.29)
Since
\[ \int_{\mathbb{R}^2} p^2_{t-s}(x-y) \cdot |u^{g^n}(s, y + h) - u^{g^n}(s, y)|^2 \cdot |h|^{2H-2} \, dh \, dy \]
\[ + \int_{\mathbb{R}^2} p^2_{t-s}(x-y) \cdot |u(s, y + h) - u(s, y)|^2 \cdot |h|^{2H-2} \, dh \, dy \]
\[ \leq \sup_{y \in \mathbb{R}} \left[ \lambda^{-\frac{2}{p}}(x) \mathcal{N}_{\frac{2}{2} - H} u^{g^n}(s, y) \right]^2 \cdot \int_{\mathbb{R}} p^2_{t-s}(x-y) \lambda^{-\frac{2}{p}}(y) \, dy \]
\[ + \sup_{y \in \mathbb{R}} \left[ \lambda^{-\frac{2}{p}}(x) \mathcal{N}_{\frac{2}{2} - H} u(s, y) \right]^2 \cdot \int_{\mathbb{R}} p^2_{t-s}(x-y) \lambda^{-\frac{2}{p}}(y) \, dy \]
\[ \leq (t-s)^{-\frac{1}{p}} \cdot \lambda^{-\frac{2}{p}}(x), \]
the dominated convergence theorem implies that
\[ \lim_{n \to \infty} D^n_3(t, s, x) = 0. \]
Furthermore, by (4.30), we have
\[ D^n_3(t, s, x) \leq (t-s)^{-\frac{1}{p}} \cdot \lambda^{-\frac{2}{p}}(x). \]
By the dominated convergence theorem, we have
\[
\lim_{n \to \infty} \int_0^t D_n^2(t, s, x) ds = 0. \tag{4.31}
\]
By (4.21), (4.22), (4.25), (4.28) and (4.31), we have
\[
\lim_{n \to \infty} \int_0^t \langle p_{t-s}(x - \cdot) \Delta_n(s, \cdot, \cdot), g_n(s, \cdot) \rangle_{\mathcal{H}} ds = 0. \tag{4.32}
\]
Hence, by (4.20), we have
\[
u(t, x) = \lim_{n \to \infty} u^{g_n}(t, x)
= 1 + \lim_{n \to \infty} \int_0^t \langle p_{t-s}(x - \cdot) \sigma(s, \cdot, u^{g_n}(s, \cdot)), g_n(s, \cdot) \rangle_{\mathcal{H}} ds
= 1 + \int_0^t \langle p_{t-s}(x - \cdot) \sigma(s, \cdot, u(s, \cdot)), g(s, \cdot) \rangle_{\mathcal{H}} ds,
\]
which implies \( u = u^g \). This completes the proof. \( \square \)

5. Verification of Condition 2.8 (b)

For any \( \varepsilon > 0 \), define the solution functional \( \Gamma^\varepsilon : L^2([0, T]; \mathcal{H}) \to C([0, T] \times \mathbb{R}) \) by
\[
\Gamma^\varepsilon (W(\cdot)) := u^\varepsilon, \tag{5.1}
\]
where \( u^\varepsilon \) stands for the solution of Eq. (1.1).

Let \( \{g^\varepsilon\}_{\varepsilon > 0} \subset \mathcal{U}^N \) be a given family of stochastic processes. By the Girsanov theorem, it is easily to see that \( \tilde{u}^\varepsilon := \Gamma^\varepsilon \left( W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^T g^\varepsilon(s) ds \right) \) is the unique solution of the following equation:
\[
\frac{\partial \tilde{u}^\varepsilon(t, x)}{\partial t} = \frac{\partial^2 \tilde{u}^\varepsilon(t, x)}{\partial x^2} + \sqrt{\varepsilon} \sigma(t, x, \tilde{u}^\varepsilon(t, x)) W(t, x) + \langle \sigma(t, \cdot, \tilde{u}^\varepsilon(t, \cdot)), g^\varepsilon(t, \cdot) \rangle_{\mathcal{H}}, \quad t > 0, \quad x \in \mathbb{R},
\]
with the initial value \( \tilde{u}^\varepsilon(0, x) = 1 \). Moreover, \( \tilde{u}^\varepsilon := \Gamma^0 \left( \int_0^T g^\varepsilon(s) ds \right) \), with \( \Gamma^0 \) defined by (2.27), solves the following equation:
\[
\frac{\partial \tilde{u}^0(t, x)}{\partial t} = \frac{\partial^2 \tilde{u}^0(t, x)}{\partial x^2} + \langle \sigma(t, \cdot, \tilde{u}^\varepsilon(t, \cdot)), g^\varepsilon(t, \cdot) \rangle_{\mathcal{H}}, \quad t > 0, \quad x \in \mathbb{R},
\]
with the initial value \( \tilde{u}^\varepsilon(0, x) = 1 \). Equivalently, \( \tilde{u}^\varepsilon \) and \( \tilde{u}^\varepsilon \) satisfy the following two integral equations, respectively:
\[
\tilde{u}^\varepsilon(t, x) = 1 + \sqrt{\varepsilon} \int_0^t \int_\mathbb{R} p_{t-s}(x - y) \sigma(s, y, \tilde{u}^\varepsilon(s, y)) W(ds, dy)
+ \int_0^t \langle p_{t-s}(x - \cdot) \sigma(s, \cdot, \tilde{u}^\varepsilon(s, \cdot)), g^\varepsilon(s, \cdot) \rangle_{\mathcal{H}} ds, \tag{5.2}
\]
and
\[
\tilde{u}^\varepsilon(t, x) = 1 + \int_0^t \langle p_{t-s}(x - \cdot) \sigma(s, \cdot, \tilde{u}^\varepsilon(s, \cdot)), g^\varepsilon(s, \cdot) \rangle_{\mathcal{H}} ds. \tag{5.3}
\]

**Proposition 5.1.** Assume that \( \sigma \) satisfies the hypothesis (H). Then, for every \( N < +\infty \), \( \{g^\varepsilon\}_{\varepsilon > 0} \subset \mathcal{U}^N \) and \( \delta > 0 \),
\[
\lim_{\varepsilon \to 0} \mathbb{P} \left( d_C(\tilde{u}^\varepsilon, \bar{u}^\varepsilon) > \delta \right) = 0.
\]
Before proving Proposition 5.1, we give the following lemmas in preparation.

**Lemma 5.1.** If \( p > \frac{6}{4H-1} \), then there exists a constant \( C_{T,p,H} \) such that

(i). \[
\sup_{\varepsilon > 0} \| \tilde{u}^\varepsilon \|_{z^{p}_{\lambda,T}} < \infty, \quad \sup_{\varepsilon > 0} \| \tilde{u}^\varepsilon \|_{L^{p}_{\lambda,T}} < \infty; \tag{5.4}
\]

(ii). \[
\left\| \sup_{t \in [0,T], x \in \mathbb{R}} \lambda^\varepsilon(x) N_{\frac{1}{2}-H} \tilde{u}^\varepsilon(t, x) \right\|_{L^p(\Omega)} \lesssim C_{T,p,H} \left( 1 + \| \tilde{u}^\varepsilon \|_{z^{p}_{\lambda,T}} \right); \tag{5.5}
\]

(iii). \[
\left\| \sup_{t \in [0,T], x \in \mathbb{R}} \lambda^\varepsilon(x) N_{\frac{1}{2}-H} \tilde{u}^\varepsilon(t) \right\|_{L^p(\Omega)} \lesssim C_{T,p,H} \left( 1 + \| \tilde{u}^\varepsilon \|_{z^{p}_{\lambda,T}} \right). \tag{5.6}
\]

**Proof.** We give the details of the proofs for \( \tilde{u}^\varepsilon \), while the proofs for \( u^\varepsilon \) are similar but simpler which are omitted here.

Define \[
\Phi_1^\varepsilon(t, x) := \int_0^t \int_{\mathbb{R}} p_{t-s}(x - y)\sigma(s, y, \tilde{u}^\varepsilon(s, y)) W(ds, dy) \tag{5.7}
\]

and

\[
\Phi_2^\varepsilon(t, x) := \int_0^t \langle p_{t-s}(x - \cdot)\sigma(s, \cdot, \tilde{u}^\varepsilon(s, \cdot)), g^\varepsilon(s, \cdot) \rangle_{H} ds. \tag{5.8}
\]

Thus, we can rewrite (5.2) as follows:

\[
\tilde{u}^\varepsilon(t, x) = 1 + \sqrt{\varepsilon} \Phi_1^\varepsilon(t, x) + \Phi_2^\varepsilon(t, x). \tag{5.9}
\]

Recall that

\[
\| \tilde{u}^\varepsilon \|_{z^{p}_{\lambda,T}} = \sup_{t \in [0,T]} \| \tilde{u}^\varepsilon(t, \cdot) \|_{L^p(\Omega \times \mathbb{R})} + \sup_{t \in [0,T]} N_{\frac{1}{2}-H} \tilde{u}^\varepsilon(t).
\]

We divide our proofs into the following four steps.

**Step 1.** In this step, we estimate \( \| \Phi_1^\varepsilon(t, \cdot) \|_{L^p(\Omega \times \mathbb{R})} \). By the Burkholder-Davis-Gundy inequality, we have

\[
\mathbb{E} \left[ \| \Phi_1^\varepsilon(t, x) \|^p \right] \lesssim \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} |p_{t-s}(x - y - h)\sigma(s, y + h, \tilde{u}^\varepsilon(s, y + h)) - p_{t-s}(x - y)\sigma(s, y, \tilde{u}^\varepsilon(s, y))|^2 \right) \lesssim \mathcal{L}_1(t, x) + \mathcal{L}_2(t, x) + \mathcal{L}_3(t, x), \tag{5.10}
\]

where

\[
\mathcal{L}_1(t, x) := \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} p_{t-s}(x - y - h)\sigma(s, y + h, u_0^\varepsilon(s, y + h)) - \sigma(s, y, u_0^\varepsilon(s, y + h)) \right) \lesssim \mathcal{L}_1(t, x),
\]

\[
\mathcal{L}_2(t, x) := \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} p_{t-s}(x - y - h)\sigma(s, y, u_0^\varepsilon(s, y + h)) - \sigma(s, y, u_0^\varepsilon(s, y)) \right) \lesssim \mathcal{L}_2(t, x),
\]

and

\[
\mathcal{L}_3(t, x) := \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} p_{t-s}(x - y - h)\sigma(s, y, u_0^\varepsilon(s, y + h)) - \sigma(s, y, u_0^\varepsilon(s, y)) \right) \lesssim \mathcal{L}_3(t, x).
\]
\[ \mathcal{L}_3(t, x) := \mathbb{E} \left( \int_{0}^{t} \int_{\mathbb{R}^2} |D_{t-s}(x - y, h)|^2 \cdot |\sigma(s, y, u^\varepsilon(s, y))|^2 \cdot |h|^{2H-2}dhdyds \right)^{\frac{2}{p}}. \]

By using the same arguments as that in Step 3 of the proof of Lemma 3.1, we have

\[
\frac{1}{2} \int_{\mathbb{R}} \mathcal{L}_2(t, x) \lambda(x)dx \leq \int_{0}^{t} \int_{\mathbb{R}^2} p_{t-s}(y) \left( \int_{\mathbb{R}} \|u^\varepsilon(s, x + y + h) - \bar{u}^\varepsilon(s, x + y)\|_{L^p(\Omega)}^p \lambda(x)dx \right)^{\frac{2}{p}} dyds \leq \int_{0}^{t} (t - s)^{-\frac{1}{2}} \cdot \left( \int_{\mathbb{R}^2} p_{t-s}(y) \|u^\varepsilon(s, x + h) - \bar{u}^\varepsilon(s, x)\|_{L^p(\Omega)}^p \lambda(x - y)dxdy \right)^{\frac{2}{p}} ds \leq \int_{0}^{t} (t - s)^{-\frac{1}{2}} \cdot \left( 1 + \|u^\varepsilon(s, \cdot)\|_{L^2_x(\Omega \times \mathbb{R})}^2 \right) ds,
\]

and

\[
\frac{1}{2} \int_{\mathbb{R}} \mathcal{L}_3(t, x) \lambda(x)dx \leq \int_{0}^{t} \int_{\mathbb{R}^2} |D_{t-s}(y, h)|^2 \cdot \left( \int_{\mathbb{R}} \|u^\varepsilon(s, x + y)\|_{L^p(\Omega)}^p \lambda(x)dx \right)^{\frac{2}{p}} dyds \leq \int_{0}^{t} (t - s)^{H-1} \cdot \left( \int_{\mathbb{R}^2} (t - s)^{1-H} \cdot |D_{t-s}(y, h)|^2 \cdot \left( 1 + \|u^\varepsilon(s, x)\|_{L^p(\Omega)}^p \lambda(x - y) \right) \right)^{\frac{2}{p}} ds \leq \int_{0}^{t} (t - s)^{H-1} \cdot \left( 1 + \|u^\varepsilon(s, \cdot)\|_{L^2_x(\Omega \times \mathbb{R})}^2 \right) ds.
\]

Therefore, by (5.10), (5.11), (5.12) and (5.13), we have

\[
\|\Phi_1(t, \cdot)\|_{L^2_x(\Omega \times \mathbb{R})}^2 = \left( \int_{\mathbb{R}} \mathbb{E} \left[ \left| \Phi_1(t, x) \right|^p \right] \lambda(x)dx \right)^{\frac{2}{p}} \leq 1 + \int_{0}^{t} \left( (t - s)^{-\frac{1}{2}} + (t - s)^{H-1} \right) \cdot \|u^\varepsilon(s, \cdot)\|_{L^2_x(\Omega \times \mathbb{R})}^2 ds
\]

(5.14)
+ \int_0^t (t-s)^{-\frac{H}{2}} \left[ \mathcal{N}^\varepsilon_{\frac{\alpha}{2}-H,p} \tilde{u}^\varepsilon(s) \right]^2 ds.

**Step 2.** In this step, we deal with $\mathcal{N}^\varepsilon_{\frac{\alpha}{2}-H,p} \Phi_1^\varepsilon(t)$. By the Burkholder-Davis-Gundy inequality and the hypothesis (H), the similar calculation as that in Step 2 of the proof of Lemma 3.1 implies that

$$
\mathbb{E} \left[ |\Phi_1^\varepsilon(t,x) - \Phi_1^\varepsilon(t,x+h)|^p \right] \\
\lesssim \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} |D_{t-s}(x-y,z,h)\sigma(s,y+z,\tilde{u}^\varepsilon(s,y+z)) \\
- D_{t-s}(x-z,h)\sigma(s,z,\tilde{u}^\varepsilon(s,z))|^2 \cdot |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}} \\
\lesssim \mathcal{M}_1(t,x,h) + \mathcal{M}_2(t,x,h) + \mathcal{M}_3(t,x,h),
$$

where

$$
\mathcal{M}_1(t,x,h) := \mathbb{E} \left( \int_0^t \int_{\mathbb{R}} |D_{t-s}(x-z,h)|^2 \cdot (1 + |\tilde{u}^\varepsilon(s,z)|^2) dz ds \right)^{\frac{p}{2}};
$$

$$
\mathcal{M}_2(t,x,h) := \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} |D_{t-s}(x-y,z,h)|^2 \cdot |\tilde{u}^\varepsilon(s,y+z) - \tilde{u}^\varepsilon(s,z)|^2 \cdot |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}};
$$

$$
\mathcal{M}_3(t,x,h) := \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} \left| D_{t-s}(x-z,y,h) \right|^2 \cdot (1 + |\tilde{u}^\varepsilon(s,z)|^2) \cdot |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}}.
$$

By (2.13) and (3.2), we have

$$
\left[ \mathcal{N}^\varepsilon_{\frac{\alpha}{2}-H,p} \Phi_1^\varepsilon(t) \right]^2 \lesssim \sum_{i=1}^3 \mathbb{E} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathcal{M}_i(t,x,h) \lambda(x) dx \right)^{\frac{2}{p}} \cdot |h|^{2H-2} dh \right).
$$

Applying a change of variable, Minkowski’s inequality, Jensen’s inequality and Lemma 6.5, we have

$$
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathcal{M}_1(t,x,h) \lambda(x) dx \right)^{\frac{2}{p}} |h|^{2H-2} dh \lesssim \int_0^t \left( t-s \right)^{H-1} \left( 1 + |\tilde{u}^\varepsilon(s,)|^2 \right)^{\frac{2}{L^p(\Omega \times \mathbb{R})}} ds;
$$

$$
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathcal{M}_2(t,x,h) \lambda(x) dx \right)^{\frac{2}{p}} |h|^{2H-2} dh \lesssim \int_0^t \left( t-s \right)^{H-1} \left[ \mathcal{N}^\varepsilon_{\frac{\alpha}{2}-H,p} \tilde{u}^\varepsilon(s) \right]^2 ds;
$$

$$
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathcal{M}_3(t,x,h) \lambda(x) dx \right)^{\frac{2}{p}} \cdot |h|^{2H-2} dh \lesssim \int_0^t \left( t-s \right)^{2H-\frac{3}{2}} \left( t-s \right)^{\frac{3}{2}-2H} \left( t-s \right)^{2H-\frac{3}{2}} \cdot |y|^{2H-2} \cdot |h|^{2H-2} \\
\cdot \left( 1 + |\tilde{u}^\varepsilon(s,z)|^2 \right)^{\frac{2}{L^p(\Omega \times \mathbb{R})}} ds
$$

$$
\lesssim \int_0^t \left( t-s \right)^{2H-\frac{3}{2}} \cdot \left( 1 + |\tilde{u}^\varepsilon(s,)|^2 \right)^{\frac{2}{L^p(\Omega \times \mathbb{R})}} ds.
$$
Therefore, by (5.15), (5.16), (5.17) and (5.53), we have
\[
\left[ \mathcal{N}_{\frac{1}{2}-H,p} \Phi_1(\varepsilon) \right]^2 \lesssim 1 + \int_0^t \left( (t-s)^{H-1} + (t-s)^{2-H-\frac{3}{2}} \right) \cdot \| \tilde{u}^{\varepsilon}(s,\cdot) \|^2_{L_*^p(\Omega\times\mathbb{R})} ds \\
+ \int_0^t (t-s)^{H-1} \cdot \left[ \mathcal{N}_{\frac{1}{2}-H,p} \tilde{u}^{\varepsilon}(s) \right]^2 ds.
\] (5.19)

**Step 3.** By (5.14) and (5.19), we obtain that
\[
\| \Phi_2(t,\cdot) \|^2_{L_*^p(\Omega\times\mathbb{R})} + \left[ \mathcal{N}_{\frac{1}{2}-H,p} \Phi_2(\varepsilon) \right]^2 \\
\lesssim 1 + \int_0^t \left( (t-s)^{2H-\frac{3}{2}} + (t-s)^{H-1} + (t-s)^{-\frac{1}{2}} \right) \cdot \| \tilde{u}^{\varepsilon}(s,\cdot) \|^2_{L_*^p(\Omega\times\mathbb{R})} ds \\
+ \int_0^t \left( (t-s)^{H-1} + (t-s)^{-\frac{1}{2}} \right) \cdot \left[ \mathcal{N}_{\frac{1}{2}-H,p} \tilde{u}^{\varepsilon}(s) \right]^2 ds.
\] (5.20)

By using the same method as that in the proof of Lemma 3.1, we obtain that
\[
\| \Phi_2(t,\cdot) \|^2_{L_*^p(\Omega\times\mathbb{R})} + \left[ \mathcal{N}_{\frac{1}{2}-H,p} \Phi_2(\varepsilon) \right]^2 \\
\lesssim 1 + \int_0^t \left( (t-s)^{2H-\frac{3}{2}} + (t-s)^{H-1} + (t-s)^{-\frac{1}{2}} \right) \cdot \| \tilde{u}^{\varepsilon}(s,\cdot) \|^2_{L_*^p(\Omega\times\mathbb{R})} ds \\
+ \int_0^t \left( (t-s)^{H-1} + (t-s)^{-\frac{1}{2}} \right) \cdot \left[ \mathcal{N}_{\frac{1}{2}-H,p} \tilde{u}^{\varepsilon}(s) \right]^2 ds.
\] (5.21)

Set
\[
\tilde{\Psi}_\varepsilon(s) := \| \tilde{u}^{\varepsilon}(s,\cdot) \|^2_{L_*^p(\Omega\times\mathbb{R})} + \left[ \mathcal{N}_{\frac{1}{2}-H,p} \tilde{u}^{\varepsilon}(s) \right]^2.
\]

By (5.9), (5.20) and (5.21), there exists a constant $C_{T,p,H} > 0$ such that
\[
\tilde{\Psi}_\varepsilon(t) \leq C_{T,p,H} \left( 1 + \int_0^t (t-s)^{2H-\frac{3}{2}} \cdot \tilde{\Psi}_\varepsilon(s) ds \right).
\]

Hence, by the fractional Gronwall lemma ([21, Lemma 1]), we have
\[
\sup_{\varepsilon > 0} \sup_{t \in [0,T]} \tilde{\Psi}_\varepsilon(t) < \infty.
\]

Therefore, we have \( \sup_{\varepsilon > 0} \| \tilde{u}^{\varepsilon} \|^p_{L_*^p(\Omega\times\mathbb{R})} < \infty. \)

**Step 4.** Since
\[
\mathcal{N}_{\frac{1}{2}-H} \tilde{u}^{\varepsilon}(t,x) \leq \sqrt{\mathcal{N}_{\frac{1}{2}-H} \Phi_1(t,x)} + \mathcal{N}_{\frac{1}{2}-H} \Phi_2(t,x),
\]
taking into account that by Lemma 4.7 (i) in [19], if \( p > \frac{6}{4H-1} \), then there exists a constant $C_{T,p,H}$ such that
\[
\sup_{t \in [0,T], x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} \Phi_1(t,x) \leq C_{T,p,H} \left( 1 + \| \tilde{u}^{\varepsilon} \|^p_{L_*^p(\Omega\times\mathbb{R})} \right),
\] (5.22)
we only need to deal with \( \mathcal{N}_{\frac{1}{2}-H} \Phi_2(t,x) \). By using the similar procedure as that in (3.31), for any \( \alpha \in (0,1) \), we have
\[
\Phi_2(t,x) \simeq \int_0^t \int_{\mathbb{R}^2} \left[ p_{t-s}(x-y-l)\sigma(s,y+l,\tilde{u}^{\varepsilon}(s,y+l)) - p_{t-s}(x-y)\sigma(s,y,\tilde{u}^{\varepsilon}(s,y)) \right] \\
\cdot [g^\varepsilon(s,y+l) - g^\varepsilon(s,y)] \cdot |l|^{2H-2} dl dy ds
\]
where

\[ \tilde{I}_\alpha(r, z) := \int_0^r \langle (r-s)^{-\alpha} p_{r-s}(z-\cdot) \sigma(\cdot, \cdot, \tilde{\mu}^\varepsilon(\cdot, \cdot)), g^\varepsilon(\cdot, \cdot) \rangle \mathcal{H} ds. \]

Set

\[ \Delta_h \tilde{I}_\alpha(r, z) := \tilde{I}_\alpha(r, z + h) - \tilde{I}_\alpha(r, z). \]

Then by a change of variable, we have

\[ \Phi^\varepsilon_2(t, x + h) - \Phi^\varepsilon_2(t, x) \approx \int_0^t \int_\mathbb{R} (t-r)^{\alpha-1} \cdot p_{t-r}(x-z) \cdot \Delta_h \tilde{I}_\alpha(r, z) dz dr. \]

Using the same technique as that in the proof of (3.32), we have

\[ \int_\mathbb{R} |\Phi^\varepsilon_2(t, x + h) - \Phi^\varepsilon_2(t, x)|^2 \cdot |h|^{2H-2} dh \]

\[ \lesssim \lambda^\frac{2}{p} \left( \int_0^t (t-r)^{q(\alpha-\frac{1}{4}+\frac{1}{2p})} dr \right)^{\frac{2}{p}} \cdot \left( \int_0^T \int_\mathbb{R} \int_\mathbb{R} \left| \Delta_h \tilde{I}_\alpha(r, z) \right|^2 \cdot |h|^{2H-2} dh \right)^{\frac{q}{2}} \lambda(z) dz dr. \]

Consequently, if \( \alpha > \frac{3}{2p} \), then

\[ \sup_{t \in [0, T], x \in \mathbb{R}} \lambda^\frac{1}{p} (x) \left( \int_\mathbb{R} \left| \Phi^\varepsilon_2(t, x + h) - \Phi^\varepsilon_2(t, x) \right|^2 \cdot |h|^{2H-2} dh \right)^{\frac{1}{2}} \]

\[ \lesssim \left( \int_0^T \int_\mathbb{R} \left( \int_\mathbb{R} \left| \Delta_h \tilde{I}_\alpha(r, z) \right|^2 \cdot |h|^{2H-2} dh \right)^{\frac{q}{2}} \lambda(z) dz dr \right)^{\frac{1}{p}}. \]

By using the same method as that in the proof of Lemma 3.2, there exists some constant \( C \), independent of \( r \in [0, T] \), such that

\[ \mathbb{E} \left[ \int_\mathbb{R} \left( \int_\mathbb{R} \left| \Delta_h \tilde{I}_\alpha(r, z) \right|^2 \cdot |h|^{2H-2} dh \right)^{\frac{q}{2}} \lambda(z) dz \right] \leq C \left( 1 + \| \tilde{\mu}^\varepsilon \|^p_{L^\infty(\mathbb{R})} \right). \]  

(5.23)

Thus, if \( p > \frac{6}{H-1} \), then there exists some positive constant \( C_{T,p,H} \) such that

\[ \left\| \sup_{t \in [0, T], x \in \mathbb{R}} \lambda^\frac{1}{p} (x) N^{\frac{1}{2}-H} \Phi^\varepsilon_2(t, x) \right\|_{L^p(\Omega)} \leq C_{T,p,H} \left( 1 + \| \tilde{\mu}^\varepsilon \|^p_{L^\infty(\mathbb{R})} \right). \]  

(5.24)

Therefore, the inequality (5.5) holds. The proof is complete. \( \Box \)

**Lemma 5.2.** (i) If \( p > \frac{3}{H} \) and \( 0 < \gamma < \frac{H}{2} - \frac{3}{2p} \), then there exists a positive constant \( C_{T,p,H,\gamma} \) such that

\[ \left\| \sup_{t, t+h \in [0, T], x \in \mathbb{R}} \lambda^\frac{1}{p} (x) \left[ \tilde{\mu}^\varepsilon(t+h, x) - \tilde{\mu}^\varepsilon(t+h, x) - \tilde{\mu}^\varepsilon(t, x) + \tilde{\mu}^\varepsilon(t, x) \right] \right\|_{L^p(\Omega)} \]

\[ \leq C_{T,p,H,\gamma} |h|^\gamma \cdot \| \tilde{\mu}^\varepsilon - \tilde{\mu}^\varepsilon \|^p_{L^\infty(\Omega)}. \]
(ii) If $p > \frac{3}{4}$ and $0 < \gamma < H - \frac{3}{2p}$, then there exists a positive constant $C_{T,p,H,\gamma}$ such that

$$
\sup_{t \in [0,T], x,y \in \mathbb{R}} \| \bar{u}^\varepsilon(t, x) - \bar{u}^\varepsilon(t, x) - \bar{u}^\varepsilon(t, y) + \bar{u}^\varepsilon(t, y) \|_{L^p(\Omega)} \\
\leq C_{T,p,H,\gamma} |x - y|^{\gamma} \cdot \| \bar{u}^\varepsilon - \bar{u}^{\varepsilon} \|_{Z_{x,T}^p}.
$$

Proof. (i). Denote $\Delta(t, x, y) = \sigma(t, x, \bar{u}^\varepsilon(t, y)) - \sigma(t, x, \bar{u}^\varepsilon(t, y))$. Set

$$
\Phi_{3}^\varepsilon(t, x) := \int_0^t \langle p_{t-s}(x - \cdot) \Delta(s, \cdot, \cdot), g^\varepsilon(s, \cdot) \rangle_{\mathcal{H}} ds. \tag{5.25}
$$

By (5.2), (5.3), (5.7) and (5.25), we have

$$
\bar{u}^\varepsilon(t + h, x) - \bar{u}^\varepsilon(t, x) - \bar{u}^\varepsilon(t, x) + \bar{u}^\varepsilon(t, x) = \sqrt{\varepsilon}(\Phi_1^\varepsilon(t + h, x) - \Phi_1^\varepsilon(t, x)) + (\Phi_2^\varepsilon(t + h, x) - \Phi_2^\varepsilon(t, x)). \tag{5.26}
$$

If $p > \frac{3}{4}$ and $0 < \gamma < H - \frac{3}{2p}$, then by Lemma 4.2 (iii) in [19], we have

$$
\sup_{t, t+h \in [0,T], x \in \mathbb{R}} \lambda^\frac{1}{\frac{3}{2p}}(x) \left[ \Phi_1^\varepsilon(t + h, x) - \Phi_1^\varepsilon(t, x) \right] \right\|_{L^p(\Omega)} \\
\leq C_{T,p,H,\gamma} |h|^{\gamma} \cdot \| \bar{u}^\varepsilon - \bar{u}^{\varepsilon} \|_{Z_{x,T}^p},
$$

hence we only need to show that (5.27) also holds for $\Phi_3^\varepsilon(t, x)$. By using the same technique as that in the proof of (3.31), for any $\alpha \in (0, 1)$, we have

$$
\Phi_3^\varepsilon(t, x) \simeq \int_0^t J_{\alpha}^\varepsilon(r, z) dz dr,
$$

where

$$
J_{\alpha}^\varepsilon(r, z) := \int_0^r \langle (r - s)^{-\alpha} p_{t-r-s}(z - \cdot) \Delta(s, \cdot, \cdot), g^\varepsilon(s, \cdot) \rangle_{\mathcal{H}} ds.
$$

Similarly to (3.41), we have

$$
\Phi_3^\varepsilon(t + h, x) - \Phi_3^\varepsilon(t, x) \\
= \int_0^t \int_\mathbb{R} (t + h - r)^{-\alpha-1} \cdot (p_{t+h-r}(x - z) - p_{t-r}(x - z)) \cdot J_{\alpha}^\varepsilon(r, z) dz dr \\
+ \int_0^t \int_\mathbb{R} ((t + h - r)^{-\alpha-1} - (t - r)^{-\alpha-1}) \cdot p_{t-r}(x - z) \cdot J_{\alpha}^\varepsilon(r, z) dz dr \\
+ \int_t^{t+h} \int_\mathbb{R} (t + h - r)^{-\alpha-1} \cdot p_{t+h-r}(x - z) \cdot J_{\alpha}^\varepsilon(r, z) dz dr \\
=: \mathcal{K}_1^\varepsilon(t, h, x) + \mathcal{K}_2^\varepsilon(t, h, x) + \mathcal{K}_3^\varepsilon(t, h, x).
$$

If $0 < \gamma < H - \frac{3}{2p}$, then by using the same technique as that in the proofs of (3.43), (3.51) and (3.53), we have

$$
\sup_{t, t+h \in [0,T], x \in \mathbb{R}} \lambda^\frac{1}{\frac{3}{2p}}(x) |\mathcal{K}_i^\varepsilon(t, h, x)| \leq C_{T,p,H,\gamma} |h|^{\gamma} \cdot \left( \int_0^t \| J_{\alpha}^\varepsilon(r, \cdot) \|_{L^p(\mathbb{R})}^p \right)^\frac{1}{p}, \ i = 1, 2, 3. \tag{5.29}
$$
We claim that
\[
\mathbb{E}\left[\left\| J_\varepsilon^x(r, \cdot) \right\|_{L^p_{\gamma}(\mathbb{R})}^p \right] \leq \left\| \overline{u}^\varepsilon - \tilde{u}^\varepsilon \right\|_{L^p_{\lambda, T}, T}^p, \quad \forall r \in [0, T].
\] (5.30)

If \(0 < \gamma < \frac{H}{2} - \frac{3}{2p}\), then by (5.28), (5.29) and (5.30), we have
\[
\left\| \left. \sup_{t, t+h \in [0, T], x \in \mathbb{R}} \frac{1}{h} \Phi_3^x(t+h, x) - \Phi_3^x(t, x) \right\|_{L^p(\Omega)} \leq C_{T, p, H, \gamma} |h|^{\gamma} \cdot \left\| \overline{u}^\varepsilon - \tilde{u}^\varepsilon \right\|_{L^p_{\lambda, T}.}
\] (5.31)
Puting (5.26), (5.27) and (5.31) together, we obtain (i).

Next, it remains to prove (5.30). For any \(k \in \mathbb{N}\), define the stopping time
\[
T_k := \inf \left\{ r \in [0, T] : \sup_{0 \leq s \leq r, x \in \mathbb{R}} \lambda^\alpha_r(x) \mathcal{N}_{1/2 - H} \overline{u}^\varepsilon(s, x) \geq k \right\}
\] (5.32)

Since \(g^\varepsilon \in \mathcal{U}^N\), by the Cauchy-Schwarz inequality and (2.4), we have
\[
\mathbb{E}\left[ 1_{\{r < T_k\}} \left| J_\alpha^x(r, z) \right|^p \right] \leq \mathbb{E}\left[ 1_{\{r < T_k\}} \left( \int_0^r \left\langle (r - s)^{-\alpha} p_{r-s}(z - \cdot) \Delta(s, \cdot, \cdot), g^\varepsilon(s, \cdot) \right\rangle_{H^1} ds \right|^p \right]
\]
\[
\leq \mathbb{E}\left[ 1_{\{r < T_k\}} \left( \int_0^r (r - s)^{-2\alpha} \left| p_{r-s}(z - y - h) \Delta(s, y, h, y) \right|^2 \cdot |h|^{2H-2} dhdys \right)^{\frac{p}{2}} \right]
\]
\[
\leq \mathcal{N}_{\alpha}^\varepsilon(r, z) + \mathcal{N}_2^\varepsilon(r, z) + \mathcal{N}_3^\varepsilon(r, z),
\]
where
\[
\mathcal{N}_1^\varepsilon(r, z) := \mathbb{E}\left[ \int_0^r \int_{\mathbb{R}^2} 1_{\{r < T_k\}} (r - s)^{-2\alpha} \cdot p_{r-s}^2(z - y) \cdot |\Delta(s, y, y) - \Delta(s, y, y) + h)|^2 \cdot |h|^{2H-2} dhdys \right]^{\frac{p}{2}};
\]
\[
\mathcal{N}_2^\varepsilon(r, z) := \mathbb{E}\left[ \int_0^r \int_{\mathbb{R}^2} 1_{\{r < T_k\}} (r - s)^{-2\alpha} \cdot |D_{r-s}(z - y, -h)|^2 \cdot |\Delta(s, y, y)|^2 \cdot |h|^{2H-2} dhdys \right]^{\frac{p}{2}};
\]
\[
\mathcal{N}_3^\varepsilon(r, z) := \mathbb{E}\left[ \int_0^r \int_{\mathbb{R}^2} 1_{\{r < T_k\}} (r - s)^{-2\alpha} \cdot p_{r-s}^2(z - y) \cdot |\Delta(s, y, y) - \Delta(s, y, y) - \Delta(s, y, y) + h)|^2 \cdot |h|^{2H-2} dhdys \right]^{\frac{p}{2}}.
\]
By using the similar technique as that in (3.64), we have
\[
|\Delta(s, y, y) - \Delta(s, y, y)|^2 \leq |\tilde{u}^\varepsilon(s, y + h) - \tilde{u}^\varepsilon(s, y + h) - \tilde{u}^\varepsilon(s, y)|^2
\]
\[
+ \lambda^\alpha_r(y) |\tilde{u}^\varepsilon(s, y) - \tilde{u}^\varepsilon(s, y)|^2 \cdot \left[ |\tilde{u}^\varepsilon(s, y + h) - \tilde{u}^\varepsilon(s, y)|^2 + |\tilde{u}^\varepsilon(s, y + h) - \tilde{u}^\varepsilon(s, y)|^2 \right].
\] (5.34)
Therefore, by (5.34) and a change of variable, we have
\[
\mathcal{N}_1^\varepsilon(r, z) \lesssim \mathcal{N}_{11}^\varepsilon(r, z) + \mathcal{N}_{12}^\varepsilon(r, z) + \mathcal{N}_{13}^\varepsilon(r, z),
\] (5.35)
where
\[
\mathcal{N}_{11}^\varepsilon(r, z) := \mathbb{E}\left[ \int_0^r \int_{\mathbb{R}^2} 1_{\{r < T_k\}} (r - s)^{-2\alpha} \cdot p_{r-s}^2(y) \cdot |\tilde{u}^\varepsilon(s, y + h) - \tilde{u}^\varepsilon(s, y + h)|^2 \right].
\]
\[
- \bar{u}^\varepsilon(s, z + y) + \bar{u}^\varepsilon(s, z + y)^2 \cdot |h|^{2H-2} dhyds \]

\[
\mathcal{N}_{12}^\varepsilon(r, z) := E \left[ \int_0^r \int_{\mathbb{R}} 1_{\{r < T_k\}}(r - s)^{-2\alpha} \cdot p_{r-s}^2(y) \cdot |\bar{u}^\varepsilon(s, z + y) - \bar{u}^\varepsilon(s, z + y)|^2 dy ds \right]^{\frac{p}{2}};
\]

\[
\mathcal{N}_{13}^\varepsilon(t, x) := E \left[ \int_0^t \int_{\mathbb{R}} 1_{\{r < T_k\}}(r - s)^{-2\alpha} \cdot p_{r-s}^2(y) \cdot \|1_{\{r < T_k\}}(\bar{u}^\varepsilon(s, z + y) - \bar{u}^\varepsilon(s, z + y))\|^2_{L^p(\Omega)} dy ds \right]^{\frac{p}{2}};
\]

By using the Minkowski inequality, a change of variable, Jensen’s inequality, Lemmas 6.1 and 6.5, we have

\[
\left[ \int_{\mathbb{R}} \mathcal{N}_{11}^\varepsilon(r, z) \lambda(z) dz \right]^{\frac{2}{p}} \leq \int_0^r \int_{\mathbb{R}}^2 (r - s)^{-2\alpha} \cdot p_{r-s}^2(y) \cdot \left( \int_{\mathbb{R}} \|1_{\{r < T_k\}}(\bar{u}^\varepsilon(s, z + y + h) - \bar{u}^\varepsilon(s, z + y + h)) \cdot \bar{u}^\varepsilon(s, z + y) \|_{L^p(\Omega)} \lambda(z) dz \right)^{\frac{2}{p}} \cdot |h|^{2H-2} dhyds
\]

\[
\leq \int_0^r \int_{\mathbb{R}} (t - s)^{-2\alpha - \frac{1}{2}} \cdot \left( \int_{\mathbb{R}} p_{t-s}^2(y) \lambda(z - y) dy \right) \|1_{\{r < T_k\}}(\bar{u}^\varepsilon(s, z + h) - \bar{u}^\varepsilon(s, z) + \bar{u}^\varepsilon(s, z))\|^p_{L^p(\Omega)} \frac{d}{dz} \cdot |h|^{2H-2} dhds
\]

\[
\leq \int_0^t (r - s)^{-2\alpha - \frac{1}{2}} \cdot \left[ \mathcal{N}_{12}^\varepsilon(\frac{r}{2-H}, p) \left(1_{\{r < T_k\}}(\bar{u}^\varepsilon(s) - \bar{u}^\varepsilon(s))\right) \right]^2 ds;
\]

\[
\left[ \int_{\mathbb{R}} \mathcal{N}_{12}^\varepsilon(r, z) \lambda(z) dz \right]^{\frac{2}{p}} \leq k^2 \int_0^r \int_{\mathbb{R}} (r - s)^{-2\alpha} \cdot p_{r-s}^2(y) \cdot \left( \int_{\mathbb{R}} \|1_{\{r < T_k\}}(\bar{u}^\varepsilon(s, z + y) - \bar{u}^\varepsilon(s, z + y))\|^p_{L^p(\Omega)} \lambda(z) dz \right)^{\frac{2}{p}} dy ds
\]

\[
\leq k^2 \int_0^t (r - s)^{-2\alpha - \frac{1}{2}} \cdot \left( \int_{\mathbb{R}}^2 p_{t-s}(y) \|1_{\{r < T_k\}}(\bar{u}^\varepsilon(s, z) - \bar{u}^\varepsilon(s, z))\|^p_{L^p(\Omega)} \lambda(z - y) dz dy \right)^{\frac{p}{2}} ds
\]
By using (2.16), a change of variable, Minkowski’s inequality, we have

\[
\leq k^2 \int_0^t (r-s)^{-2\alpha - \frac{1}{2}} \left( \int_\mathbb{R} \|1_{\{r<T_k\}}(\bar{u}^\varepsilon(s, z) - \bar{u}^\varepsilon(s, z))\|_{L^p(\Omega)}^p \lambda(z) dz \right)^{\frac{2}{p}} ds
\]

\[
= k^2 \int_0^t (t-s)^{-2\alpha - \frac{1}{2}} \cdot \left\|1_{\{r<T_k\}}(\bar{u}^\varepsilon(s, \cdot) - \bar{u}^\varepsilon(s, \cdot))\right\|_{L^p_t(\Omega \times \mathbb{R})}^2 ds;
\]

\[
\left[\int_\mathbb{R} N^\varepsilon_{I_3}(t, z) \lambda(z) dz \right]^{\frac{2}{p}} \leq k^2 \int_0^t (r-s)^{-2\alpha - \frac{1}{2}} \cdot \left\|1_{\{r<T_k\}}(\bar{u}^\varepsilon(s, \cdot) - \bar{u}^\varepsilon(s, \cdot))\right\|_{L^p_t(\Omega \times \mathbb{R})}^2 ds.
\]  
(5.38)

Thus, by (5.35), (5.36), (5.37) and (5.38), we have

\[
\left[\int_\mathbb{R} N^\varepsilon_{I_3}(r, z) \lambda(z) dz \right]^{\frac{2}{p}} \leq k^2 \int_0^t (r-s)^{-2\alpha - \frac{1}{2}} \cdot \left\|1_{\{r<T_k\}}(\bar{u}^\varepsilon(s, \cdot) - \bar{u}^\varepsilon(s, \cdot))\right\|_{L^p_t(\Omega \times \mathbb{R})}^2 ds
\]

\[
+ \int_0^t (r-s)^{-2\alpha - \frac{1}{2}} \cdot \left[ N^\varepsilon_{I_3-H} \left(1_{\{r<T_k\}}(\bar{u}^\varepsilon(s) - \bar{u}^\varepsilon(s))\right) \right]^{\frac{2}{p}} ds.
\]

(5.39)

By using (2.16), a change of variable, Minkowski’s inequality, we have

\[
N^\varepsilon_{I_3}(r, z)
\]

\[
\approx \left[ \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \cdot |D_\tau - s(y, h)|^2 \cdot \left\|1_{\{r<T_k\}}(\bar{u}^\varepsilon(s, z + y) - \bar{u}^\varepsilon(s, z + y))\right\|_{L^p(\Omega)}^2 \cdot |h|^{2H-2} dy ds \right]^{\frac{2}{p}}.
\]

Then, by using Minkowski’s inequality, Jensen’s inequality, a change of variable and Lemma 6.5, we have

\[
\left[\int_\mathbb{R} N^\varepsilon_{I_3}(r, z) \lambda(z) dz \right]^{\frac{2}{p}} \leq \left[ \int_\mathbb{R} \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \cdot |D_\tau - s(y, h)|^2 \cdot \left\|1_{\{r<T_k\}}(\bar{u}^\varepsilon(s, z + y) - \bar{u}^\varepsilon(s, z + y))\right\|_{L^p(\Omega)}^2 \cdot |h|^{2H-2} dy ds \right]^{\frac{2}{p}} \lambda(z) dz \right]^{\frac{2}{p}}
\]

\[
\lesssim \left[ \int_0^r (r-s)^{-2\alpha + H-1} \cdot \left\|1_{\{r<T_k\}}(\bar{u}^\varepsilon(s, \cdot) - \bar{u}^\varepsilon(s, \cdot))\right\|_{L^p_t(\Omega \times \mathbb{R})}^2 ds.
\]

(5.40)

By using the similar technique as that in (3.61) and (3.62), we have

\[
N^\varepsilon_{I_3}(r, z) \lesssim \mathbb{E} \left[ \int_0^r \int_{\mathbb{R}^2} 1_{\{r<T_k\}}(r-s)^{-2\alpha} \cdot p^\gamma_{r-s}(z-y) \cdot |\bar{u}^\varepsilon(s, y) - \bar{u}^\varepsilon(s, y)|^2 dy ds \right]^{\frac{2}{p}}.
\]

By the Minkowski inequality, a change of variable, Jensen’s inequality and Lemma 6.1, we have

\[
\left[\int_\mathbb{R} N^\varepsilon_{I_3}(r, z) \lambda(z) dz \right]^{\frac{2}{p}} \leq \left[ \int_\mathbb{R} \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \cdot p^\gamma_{r-s}(y) \cdot \left\|1_{\{r<T_k\}}(\bar{u}^\varepsilon(s, z + y) - \bar{u}^\varepsilon(s, z + y))\right\|_{L^p(\Omega)}^2 dy ds \right]^{\frac{2}{p}} \lambda(z) dz \right]^{\frac{2}{p}}
\]

\[
\lesssim \int_0^r (r-s)^{-2\alpha - \frac{1}{2}} \cdot \left[ \int_{\mathbb{R}^2} p^\gamma_{r-s}(y) \cdot \left\|1_{\{r<T_k\}}(\bar{u}^\varepsilon(s, z) - \bar{u}^\varepsilon(s, z))\right\|_{L^p(\Omega)}^p \lambda(z-y) dz dy \right]^{\frac{2}{p}} ds
\]

(5.41)
\[ \approx \int_0^r (r-s)^{-2\alpha - \frac{1}{2}} \| \mathbf{1}_{(r<T_k)}(\tilde{u}^\varepsilon(s) - \bar{u}^\varepsilon(s))\|_{L^p(\Omega \times \mathbb{R})}^2 ds. \]

Recall that
\[ \|J^\varepsilon_{\alpha}\|_{Z^p_{\lambda,T}} = \sup_{t \in [0,T]} \|J^\varepsilon_{\alpha}(t,\cdot)\|_{L^p(\Omega \times \mathbb{R})} + \sup_{t \in [0,T]} N^p_{\frac{1}{2}-H,p}J^\varepsilon_{\alpha}(t). \]

Therefore, by (5.33), (5.39), (5.40) and (5.41), we have
\[
\mathbb{E} \left[ \|\mathbf{1}_{(r<T_k)}J^\varepsilon_{\alpha}(r,\cdot)\|_{L^p(\mathbb{R})}^p \right] \\
\leq \|\bar{u}^\varepsilon - \tilde{u}^\varepsilon\|_{Z^p_{\lambda,T}}^p \cdot \left( \int_0^r (r-s)^{-2\alpha - \frac{1}{2}} + (r-s)^{-2\alpha + H-1} ds \right)^\frac{p}{2}. \tag{5.42}
\]

If \( \alpha < \frac{H}{2} \), then we have
\[
\mathbb{E} \left[ \|\mathbf{1}_{(r<T_k)}J^\varepsilon_{\alpha}(r,\cdot)\|_{L^p(\mathbb{R})}^p \right] \leq \|\bar{u}^\varepsilon - \tilde{u}^\varepsilon\|_{Z^p_{\lambda,T}}^p.
\]

By Lemma 5.1, we obtain that \( T_k \uparrow T \) as \( k \) tends to infinity, which implies that (5.30) holds for every \( r \in [0,T] \).

(ii). According to (5.2), (5.3), (5.7) and (5.25), we have
\[
\bar{u}^\varepsilon(t,x) - \tilde{u}^\varepsilon(t,x) - \bar{u}^\varepsilon(t,y) + \tilde{u}^\varepsilon(t,y) \\
= \sqrt{\varepsilon}\left( \Phi^\varepsilon_1(t,x) - \Phi^\varepsilon_1(t,y) \right) + \left( \Phi^\varepsilon_3(t,x) - \Phi^\varepsilon_3(t,y) \right). \tag{5.43}
\]

If \( p > \frac{3}{H} \) and \( 0 < \gamma < H - \frac{3}{p} \), then by Lemma 4.2 (iv) in [19], we have
\[
\left\| \Phi^\varepsilon_3(t,x) - \Phi^\varepsilon_3(t,y) \right\|_{L^p(\Omega)} \leq C_{T,p,H,\gamma} |x-y|^\gamma \cdot \|\bar{u}^\varepsilon - \tilde{u}^\varepsilon\|_{Z^p_{\lambda,T}}, \tag{5.44}
\]

hence we only need to show that (5.44) also holds for \( \Phi^\varepsilon_3(t,x) \). By Hölder’s inequality, we have
\[
\Phi^\varepsilon_3(t,x) - \Phi^\varepsilon_3(t,y) \\
\leq \left( \int_0^t \int_{\mathbb{R}} (t-r)^{q(\alpha-1)} \cdot |p_{t-r}(x-z) - p_{t-r}(y-z)|^{q\lambda^{-\frac{q}{p}}(z)} dz \cdot dr \right)^\frac{1}{q} \cdot \left( \int_0^t \|J^\varepsilon_{\alpha}(r,\cdot)\|_{L^p(\mathbb{R})}^p dr \right)^\frac{1}{p}. \]

If \( p > \frac{3}{H} \) and \( 0 < \gamma < H - \frac{3}{p} \), then by (5.30) and by using the same method as that in the proof of (4.35) in [19], we have
\[
\left\| \Phi^\varepsilon_3(t,x) - \Phi^\varepsilon_3(t,y) \right\|_{L^p(\Omega)} \leq C_{T,p,H,\gamma} |x-y|^\gamma \cdot \|\bar{u}^\varepsilon - \tilde{u}^\varepsilon\|_{Z^p_{\lambda,T}}. \tag{5.45}
\]

By (5.43), (5.44) and (5.45), we get (ii) when \( p > \frac{3}{H} \) and \( 0 < \gamma < H - \frac{3}{p} \). The proof is complete. \( \square \)

**Lemma 5.3.** For any \( \varepsilon > 0 \),
\[
\lim_{\varepsilon \to 0} \|\bar{u}^\varepsilon - \tilde{u}^\varepsilon\|_{Z^p_{\lambda,T}} = 0. \tag{5.46}
\]

**Proof.** Notice that
\[
\|\bar{u}^\varepsilon - \tilde{u}^\varepsilon\|_{Z^p_{\lambda,T}} \leq \sqrt{\varepsilon} \|\Phi^\varepsilon_1\|_{Z^p_{\lambda,T}} + \|\Phi^\varepsilon_3\|_{Z^p_{\lambda,T}}.
\]
According to Lemma 4.5 and Lemma 4.6 in [19], we have \( \| \Phi_1^\varepsilon \|_{Z_{\lambda,T}^p} < \infty \), which yields that

\[
\lim_{\varepsilon \to 0} \| \tilde{u}^\varepsilon - \bar{u}^\varepsilon \|_{Z_{\lambda,T}^p} \leq \lim_{\varepsilon \to 0} \| \Phi_3^\varepsilon \|_{Z_{\lambda,T}^p}.
\] (5.47)

We give the estimate for \( \Phi_3^\varepsilon \|_{Z_{\lambda,T}^p} \) as follows. Recall the definition in (2.12)

\[
\| \Phi_3^\varepsilon \|_{Z_{\lambda,T}^p} = \sup_{t \in [0,T]} \| \Phi_3^\varepsilon(t, \cdot) \|_{L^p_{\lambda}(\Omega \times \mathbb{R})} + \sup_{t \in [0,T]} \mathcal{N}^*_\frac{1}{2-H,p} \Phi_3^\varepsilon(t),
\]

and the stopping time \( T_k \) defined in (5.32). Since \( g^\varepsilon \in \mathcal{U}^N \), by the Cauchy-Schwarz inequality, we have

\[
\mathbb{E} \left[ \mathbf{1}_{\{t < T_k\}} \| \Phi_3^\varepsilon(t, x) \|^p \right] \\
= \mathbb{E} \left[ \mathbf{1}_{\{t < T_k\}} \left( \int_0^t |p_{t-s}(s-\cdot)\triangle(s, \cdot, \cdot, g^\varepsilon(s, \cdot))\|_{\mathcal{H}} ds \right)^p \right] \\
\leq \mathbb{E} \left[ \mathbf{1}_{\{t < T_k\}} \left( \int_0^t \int_{\mathbb{R}^2} |p_{t-s}(s-x-y)\triangle(s, y+h, y+h) - p_{t-s}(s-y)\triangle(s, y, y)|^2 ds dy \right)^p \right] \cdot |h|^{2H-2} \int dh dy ds \\
\leq \mathcal{O}_1^\varepsilon(t, x) + \mathcal{O}_2^\varepsilon(t, x) + \mathcal{O}_3^\varepsilon(t, x),
\]

where

\[
\mathcal{O}_1^\varepsilon(t, x) := \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^2} \mathbf{1}_{\{t < T_k\}} \left| p_{t-s}(s-x-y) \cdot \triangle(s, y, y) - \triangle(s, y, y + h) \right|^2 \cdot |h|^{2H-2} \int dh dy ds \right]^\frac{p}{2};
\]

\[
\mathcal{O}_2^\varepsilon(t, x) := \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^2} \mathbf{1}_{\{t < T_k\}} \left| D_{t-s}(s-x, -h) \right|^2 \cdot \left| \triangle(s, y + h, y) \right|^2 \cdot |h|^{2H-2} \int dh dy ds \right]^\frac{p}{2};
\]

\[
\mathcal{O}_3^\varepsilon(t, x) := \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^2} \mathbf{1}_{\{t < T_k\}} \left| p_{t-s}(s-x-y) \cdot \triangle(s, y + h, y) - \triangle(s, y, y) \right|^2 \cdot |h|^{2H-2} \int dh dy ds \right]^\frac{p}{2}.
\]

By using the same technique as that in the proof of (5.42), we have

\[
\left\| \mathbf{1}_{\{t < T_k\}} \Phi_3^\varepsilon(t, \cdot) \right\|_{L^p_{\lambda}(\Omega \times \mathbb{R})}^2 \\
\leq \int_0^t \left( (t-s)^{H-1} + (t-s)^{-\frac{1}{2}} \right) \cdot \left\| \mathbf{1}_{\{t < T_k\}} (\tilde{u}^\varepsilon(s, \cdot) - \bar{u}^\varepsilon(s, \cdot)) \right\|_{L^p_{\lambda}(\Omega \times \mathbb{R})}^2 ds \\
+ \int_0^t (t-s)^{-\frac{1}{2}} \cdot \left[ \mathcal{N}^*_\frac{1}{2-H,p} \left( \mathbf{1}_{\{t < T_k\}} (\tilde{u}^\varepsilon(s) - \bar{u}^\varepsilon(s)) \right) \right]^2 ds.
\] (5.49)
Next, we deal with the term \( N_{t-H,p}^* \left( 1_{t<T_k} \Phi_5^\varepsilon(t) \right) \). Since \( g^\varepsilon \in \mathcal{U}^N \), by the Cauchy-Schwarz inequality and (2.4), we have

\[
\mathbb{E} \left[ 1_{t<T_k} \left| \Phi_5^\varepsilon(t, x) - \Phi_5^\varepsilon(t, x + h) \right|^p \right]
\leq \mathbb{E} \left[ \int_0^t 1_{t<T_k} \left| \left\langle D_{t-s}(x - s, h) \nabla \phi(s, \cdot), g^\varepsilon(s, \cdot) \right\rangle_{H^s} \right|^2 \right]^{\frac{q}{2}}
\leq \mathbb{E} \left[ \int_0^t 1_{t<T_k} \left| D_{t-s}(x - s, h) \nabla \phi(s, \cdot) \right|^2 \right]^{\frac{q}{2}}
\leq \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^2} 1_{t<T_k} \left| D_{t-s}(x - y - l, h) \nabla \phi(s, \cdot) \right|^2 \cdot |l|^{2H-2} dldyds \right]^{\frac{q}{2}}
\leq \mathcal{R}_1^\varepsilon(t, x, h) + \mathcal{R}_2^\varepsilon(t, x, h) + \mathcal{R}_3^\varepsilon(t, x, h)
\]

where

\[
\mathcal{R}_1^\varepsilon(t, x, h) := \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^2} 1_{t<T_k} \left| D_{t-s}(x - y - l, h) \right|^2 \cdot |\nabla \phi(s, \cdot) - \nabla \phi(s, y)|^2 \cdot |l|^{2H-2} dldyds \right]^{\frac{q}{2}}
\]

\[
\mathcal{R}_2^\varepsilon(t, x, h) := \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^2} 1_{t<T_k} \left| \nabla \phi(s, \cdot) - \nabla \phi(s, y) \right|^2 \cdot |l|^{2H-2} dldyds \right]^{\frac{q}{2}}
\]

\[
\mathcal{R}_3^\varepsilon(t, x, h) := \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^2} 1_{t<T_k} \left| D_{t-s}(x - y - l, h) \right|^2 \cdot |\nabla \phi(s, \cdot) - \nabla \phi(s, y)|^2 \cdot |l|^{2H-2} dldyds \right]^{\frac{q}{2}}
\]

By (5.34) and a change of variable, we have

\[
\mathcal{R}_1^\varepsilon(t, x, h) \leq \mathcal{R}_{11}^\varepsilon(t, x, h) + \mathcal{R}_{12}^\varepsilon(t, x, h) + \mathcal{R}_{13}^\varepsilon(t, x, h)
\]

where

\[
\mathcal{R}_{11}^\varepsilon(t, x, h) := \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^2} 1_{t<T_k} \left| D_{t-s}(x - y - l, h) \right|^2 \cdot |\tilde{u}^\varepsilon(s, y + l) - \tilde{u}^\varepsilon(s, y + l) - \tilde{u}^\varepsilon(s, y) + \tilde{u}^\varepsilon(s, y) - \tilde{u}^\varepsilon(s, y)|^2 \cdot |l|^{2H-2} dldyds \right]^{\frac{q}{2}}
\]

\[
\mathcal{R}_{12}^\varepsilon(t, x, h) := \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} 1_{t<T_k} \left| D_{t-s}(x - y, h) \right|^2 \cdot |\tilde{u}^\varepsilon(s, y) - \tilde{u}^\varepsilon(s, y)|^2 \cdot \left( \int_{\mathbb{R}} \tilde{\lambda}^\varepsilon(y) |\tilde{u}^\varepsilon(s, y + l) - \tilde{u}^\varepsilon(s, y)|^2 \cdot |l|^{2H-2} dl \right) dyds \right]^{\frac{q}{2}}
\]

\[
\leq k^p \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} 1_{t<T_k} \left| D_{t-s}(x - y, h) \right|^2 \cdot |\tilde{u}^\varepsilon(s, y) - \tilde{u}^\varepsilon(s, y)|^2 dyds \right]^{\frac{q}{2}}
\]
\[ R_{13}^\varepsilon(t, x, h) := \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} 1_{\{t < T_k\}} |D_{t-s}(x - y, h)|^2 \cdot |\tilde{u}^\varepsilon(s, y) - \tilde{v}^\varepsilon(s, y)|^2 \right. \]
\[ \cdot \left( \int_{\mathbb{R}} \lambda^2(y) |\tilde{u}^\varepsilon(s, y + l) - \tilde{v}^\varepsilon(s, y)|^2 \cdot |l|^{2H-2} \, dy \right) \, ds \right]^{\frac{2}{p}} \]
\[ \leq k_p \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} 1_{\{t < T_k\}} |D_{t-s}(x - y, h)|^2 \cdot |\tilde{u}^\varepsilon(s, y) - \tilde{v}^\varepsilon(s, y)|^2 \, dy \, ds \right]^{\frac{2}{p}}. \]

By using a change of variable, the Minkowski inequality, Jensen’s inequality and Lemma 6.5, we have
\[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} R_{11}^\varepsilon(t, x, h) \lambda(x) \, dx \right)^{\frac{2}{p}} \cdot |h|^{2H-2} \, dh \]
\[ \leq \int_0^t \int_{\mathbb{R}} (t - s)^{H-1} \left( \int_{\mathbb{R}^3} (t - s)^{1-H} |D_{t-s}(y, h)|^2 \cdot |h|^{2H-2} \right) \]
\[ \cdot \mathbb{E} \left[ 1_{\{t < T_k\}} |\tilde{u}^\varepsilon(s, x + l) - \tilde{v}^\varepsilon(s, x + l) - \tilde{v}^\varepsilon(s, x) + \tilde{u}^\varepsilon(s, x)|^p \right] \cdot \lambda(x - y) \, dxdydh \right]^{\frac{2}{p}} \cdot |l|^{2H-2} \, dd \]
\[ \leq \int_0^t (t - s)^{H-1} \left[ N_{\frac{2}{p}-H, p}^2 \left( 1_{\{t < T_k\}} (\tilde{u}^\varepsilon(s) - \tilde{v}^\varepsilon(s)) \right) \right] \, ds, \]

and
\[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} (R_{12}^\varepsilon(t, x, h) + R_{13}^\varepsilon(t, x, h)) \lambda(x) \, dx \right)^{\frac{2}{p}} \cdot |h|^{2H-2} \, dh \]
\[ \leq k_p \int_0^t (t - s)^{H-1} \left( \int_{\mathbb{R}^3} (t - s)^{1-H} |D_{t-s}(y, h)|^2 \cdot |h|^{2H-2} \right) \]
\[ \cdot \mathbb{E} \left[ 1_{\{t < T_k\}} |\tilde{u}^\varepsilon(s, x) - \tilde{v}^\varepsilon(s, x)|^p \right] \cdot \lambda(x - y) \, dxdydh \right]^{\frac{2}{p}} \right) \, ds \]
\[ \leq k_p \int_0^t (t - s)^{H-1} \left\| 1_{\{t < T_k\}} (\tilde{u}^\varepsilon(s, \cdot) - \tilde{v}^\varepsilon(s, \cdot)) \right\|_{L^p_{\lambda}(\Omega \times \mathbb{R})} \, ds. \]

By (2.16), we have
\[ R_{2}^\varepsilon(t, x, h) \leq \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^2} 1_{\{t < T_k\}} |\Box_{t-s}(x - y, -l, h)|^2 \cdot |\tilde{u}^\varepsilon(s, y) - \tilde{v}^\varepsilon(s, y)|^2 \cdot |l|^{2H-2} \, dldydh \right]^{\frac{2}{p}} \].

Hence, by a change of variable, Minkowski’s inequality, Jensen’s inequality and Lemma 6.5, we have
\[ \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} R_{2}^\varepsilon(t, x, h) \lambda(x) \, dx \right\|^{\frac{2}{p}} \cdot |h|^{2H-2} \, dh \]
\[ \leq \int_0^t (t - s)^{\frac{3}{2} - H} \cdot \left( t - s \right)^{\frac{3}{2} - 2H} \int_{\mathbb{R}^4} |\Box_{t-s}(y, l, h)|^2 \cdot |l|^{2H-2} \cdot |h|^{2H-2} \]
\[ \cdot \mathbb{E} \left[ 1_{\{t < T_k\}} |\tilde{u}^\varepsilon(s, x) - \tilde{v}^\varepsilon(s, x)|^p \right] \lambda(x - y) \, dxdydh \right]^{\frac{2}{p}} \right) \, ds \]
\[ \leq \left( t - s \right)^{\frac{3}{2} - H} \cdot \left( t - s \right)^{\frac{3}{2} - 2H} \int_{\mathbb{R}^4} |\Box_{t-s}(y, l, h)|^2 \cdot |l|^{2H-2} \cdot |h|^{2H-2} \]
\[ \cdot \mathbb{E} \left[ 1_{\{t < T_k\}} |\tilde{u}^\varepsilon(s, x) - \tilde{v}^\varepsilon(s, x)|^p \right] \lambda(x - y) \, dxdydh \right]^{\frac{2}{p}} \right) \, ds \]
\[ \leq \left( t - s \right)^{\frac{3}{2} - H} \cdot \left( t - s \right)^{\frac{3}{2} - 2H} \int_{\mathbb{R}^4} |\Box_{t-s}(y, l, h)|^2 \cdot |l|^{2H-2} \cdot |h|^{2H-2} \]
\[ \cdot \mathbb{E} \left[ 1_{\{t < T_k\}} |\tilde{u}^\varepsilon(s, x) - \tilde{v}^\varepsilon(s, x)|^p \right] \lambda(x - y) \, dxdydh \right]^{\frac{2}{p}} \right) \, ds \]
\[ \leq \left( t - s \right)^{\frac{3}{2} - H} \cdot \left( t - s \right)^{\frac{3}{2} - 2H} \int_{\mathbb{R}^4} |\Box_{t-s}(y, l, h)|^2 \cdot |l|^{2H-2} \cdot |h|^{2H-2} \]
\[ \cdot \mathbb{E} \left[ 1_{\{t < T_k\}} |\tilde{u}^\varepsilon(s, x) - \tilde{v}^\varepsilon(s, x)|^p \right] \lambda(x - y) \, dxdydh \right]^{\frac{2}{p}} \right) \, ds \]
Putting (5.50), (5.51), (5.52), (5.53) and (5.54) together, we have

\[ N \]  

According to Lemma 5.1, we obtain that

\[ T \]  

Theations of Lemma 6.7 are satisfied. Hence, the family of the laws of

Proof of Proposition 5.1. According to Lemma 5.2 (i) and (ii), it follows that the two conditions of Lemma 6.7 are satisfied. Hence, the family of the laws of \( \{ \tilde{u}^\varepsilon \} \) is tight. On the other hand, Lemma 5.3 shows that\( \lim \varepsilon \to 0 \) \( \| \tilde{u}^\varepsilon - \bar{u}^\varepsilon \|_{L_p^\varepsilon(\Omega \times \mathbb{R})} = 0. \)

By (5.47), (5.49), (5.55) and the fractional Gronwall lemma ([21, Lemma 1]), we have

\[ \lim_{\varepsilon \to 0} \| (1_{T_k, T}) (\tilde{u}^\varepsilon - \bar{u}^\varepsilon) \|_{L_p^\varepsilon(\Omega \times \mathbb{R})} = 0. \]

According to Lemma 5.1, we obtain that \( T_k \uparrow T \) as \( k \) tends to infinity, which implies that

\[ \lim_{\varepsilon \to 0} \| \tilde{u}^\varepsilon - \bar{u}^\varepsilon \|_{L_p^\varepsilon(\Omega \times \mathbb{R})} = 0. \]

The proof is complete.

We now give the proof of Proposition 5.1.

Proof of Proposition 5.1. According to Lemma 5.2 (i) and (ii), it follows that the two conditions of Lemma 6.7 are satisfied. Hence, the family of the laws of \( \{ \tilde{u}^\varepsilon \} \) is tight. On the other hand, Lemma 5.3 shows that

\[ \lim_{\varepsilon \to 0} \| \tilde{u}^\varepsilon - \bar{u}^\varepsilon \|_{L_p^\varepsilon(\Omega \times \mathbb{R})} = 0. \]

By the uniqueness of the limit, we know that as \( \varepsilon \to 0, \)

\[ d_C(\tilde{u}^\varepsilon, \bar{u}^\varepsilon) \to 0 \] in distribution.

This implies that as \( \varepsilon \to 0, \)

\[ d_C(\tilde{u}^\varepsilon, \bar{u}^\varepsilon) \to 0 \] in probability.

The proof is complete.

6. Appendix

In this section, we give some lemmas related to the heat kernel \( p_t(x) \). Recall that \( \lambda(x) = c_H(1 + |x|^2)^{H-1} \) and \( D_t(x, h), \mathbb{I}_t(x, y, h) \) are defined in (3.11) and (3.12), respectively.

Lemma 6.1. [19, Lemma 2.5] For any \( T > 0, \)

\[ \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}} \frac{1}{\lambda(x)} \int_{\mathbb{R}} p_t(x - y) \lambda(y) dy < \infty. \] (6.1)
Lemma 6.2. [19, Lemma 2.8] For any $H \in \left(\frac{1}{4}, \frac{1}{2}\right)$, there exists some constant $C_H$ such that
\[
\int_{\mathbb{R}^2} |D_t(x,h)|^2 \cdot |h|^{2H-2} dh dx = C_H t^{H-1}. \tag{6.2}
\]
and
\[
\int_{\mathbb{R}^3} |\square_t(x,y,h)|^2 \cdot |h|^{2H-2} \cdot |y|^{2H-2} dy dh dx = C_H t^{2H-\frac{3}{2}}. \tag{6.3}
\]

Lemma 6.3. [19, Lemma 2.10] For any $t > 0$, there exists some constant $C_H$ such that
\[
\int \mathcal{D}_t(x,h)^2 \cdot |h|^{2H-2} dh \leq C_H \left(t^{H-\frac{3}{2}} \wedge \frac{|x|^{2H-2}}{\sqrt{t}}\right), \tag{6.4}
\]
where $0 < H < \frac{1}{2}$. 

Lemma 6.4. [19, Lemma 2.11] For any $t > 0$, there exists some constant $C_H$ such that
\[
\int |D_t(x,h)|^2 \cdot |h|^{2H-2} dh \leq C_H \left(t^{H-2} \wedge \frac{|x|^{2H-2}}{t^{1-H}}\right). \tag{6.5}
\]

Lemma 6.5. [19, Lemma 2.12] For any $t > 0$, there exists some constant $C_{T,H}$ such that
\[
\int_{\mathbb{R}^2} |D_t(x,h)|^2 \cdot |h|^{2H-2} \lambda(z-x) dx dh \leq C_{T,H} t^{H-1} \lambda(z) \tag{6.6}
\]
and
\[
\int_{\mathbb{R}^3} |\square_t(x,y,h)|^2 \cdot |h|^{2H-2} \cdot |y|^{2H-2} \lambda(z-x) dy dh dx \leq C_{T,H} t^{2H-\frac{3}{2}} \lambda(z). \tag{6.7}
\]

Lemma 6.6. [19, (4.29)], [19, (4.32)] For some fixed $\gamma \in (0,1)$ and $\alpha \in (0,1)$, the following two inequalities hold:
\[
|\langle t+h \rangle^{\alpha-1} - t^{\alpha-1}| \leq |t|^{\alpha-1-\gamma} h^\gamma \tag{6.8}
\]
and
\[
|p_{t+h}(x) - p_t(x)| \leq C_{\gamma} h^{\gamma} t^{-\gamma} \left[p_{2\gamma(t+h)}(x) + p_{2\gamma}(x)\right]. \tag{6.9}
\]

Lemma 6.7. [19, Theorem 4.4] A sequence $\{\mathbb{P}_n\}_{n=1}^{\infty}$ of probability measures on $(\mathcal{C}([0,T] \times \mathbb{R}), \mathcal{B}(\mathcal{C}([0,T] \times \mathbb{R})))$ is tight if and only if the following conditions hold:
1. $\limsup_{\lambda \to \infty} \lambda^{\gamma} \mathbb{P}_n \{\omega \in \mathcal{C}([0,T] \times \mathbb{R}) : |\omega(0,0)| > \lambda\} = 0$;
2. for any $T > 0$, $R > 0$ and $\varepsilon > 0$
\[
\limsup_{\delta \to 0} \mathbb{P}_n \{\omega \in \mathcal{C}([0,T] \times \mathbb{R}) : m_{T,R}(\omega, \delta) > \varepsilon\} = 0.
\]
where
\[
m_{T,R}(\omega, \delta) := \max_{0 \leq t, s \leq T; 0 \leq |x|, |y| \leq R} \max_{0 \leq |t-s| + |x-y| \leq \delta} |\omega(t, x) - \omega(s, y)|
\]
is the modulus of continuity on $[0,T] \times [-R,R]$.

Acknowledgments: The research of R. Li is partially supported by Shanghai Sailing Program grant 21YF1415300 and NNSFC grant 12101392. The research of R. Wang is partially supported by NNSFC grants 11871382 and 12071361. The research of B. Zhang is partially supported by NNSFC grants 11971361 and 11731012.
References

[1] Balan, R., Jolis, M., Quer-Sardanyons, L.: SPDEs with affine multiplicative fractional noise in space with index $\frac{1}{4} < H < \frac{1}{2}$. Electron. J. Probab., 20(54), 1-36 (2015)
[2] Balan, R., Jolis, M., Quer-Sardanyons, L.: SPDEs with rough noise in space: Hölder continuity of the solution. Statist. Probab. Lett., 119, 310-316 (2016)
[3] Budhiraja, A., Chen, J., Dupuis, P.: Large deviations for stochastic partial differential equations driven by a Poisson random measure. Stoch. Process. Appl., 123(2), 523-560 (2013)
[4] Budhiraja, A., Dupuis, P.: A variational representation for positive functionals of infinite dimensional Brownian motion. Probab. Math. Statist., 20(1), 39-61 (2000)
[5] Budhiraja, A., Dupuis, P.: Analysis and approximation of rare events: representations and weak convergence methods, Springer, New York, 2019
[6] Budhiraja, A., Dupuis, P., Maroulas, V.: Large deviations for infinite dimensional stochastic dynamical systems. Ann. Probab., 36(4), 1390-1420 (2008)
[7] Budhiraja, A., Dupuis, P., Maroulas, V.: Variational representations for continuous time processes. Ann. Inst. Henri Poincaré Probab. Stat., 47(3), 725-747 (2011)
[8] Da Prato, G., Zabczyk, J.: Stochastic equations in infinite dimensions, Cambridge University Press, Cambridge, 1992
[9] Dai, Y., Li, R.: Transportation inequality for stochastic heat equation with rough dependence in space. Acta Math. Sin. (Engl. Ser.), in press
[10] Dalang, R. C.: Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.’s. Electron. J. Probab., 4(6), 1-29 (1999)
[11] Dalang, R. C., Quer-Sardanyons, L.: Stochastic integrals for spde’s: a comparison. Expo. Math., 29(1), 67-109 (2011)
[12] Dong, Z., Wu, J., Zhang, R., Zhang T.: Large deviation principles for first-order scalar conservation laws with stochastic forcing. Ann. Appl. Probab., 30(1), 324-367 (2020)
[13] Dupuis, P., Ellis, R.: A weak convergence approach to the theory of large deviations, Wiley, New York, 1997
[14] Hong, W., Hu S., Liu, W.: McKean-Vlasov SDEs and SPDEs with locally monotone coefficients. arXiv:2205.01043, (2022)
[15] Hong, W., Li, S., Liu, W.: Large deviation principle for McKean-Vlasov quasilinear stochastic evolution equation. Appl. Math. Optim., 84, S1119-S1147 (2021)
[16] Hu, Y.: Some recent progress on stochastic heat equations. Acta Math. Sci. Ser. B (Engl. Ed.), 39(3), 874-914 (2019)
[17] Hu, Y., Huang, J., Lé, K., Nualart, D., Tindel, S.: Stochastic heat equation with rough dependence in space. Ann. Probab., 45(6), 4561-616 (2017)
[18] Hu, Y., Huang, J., Lé, K., Nualart, D., Tindel, S.: Parabolic Anderson model with rough dependence in space. Computation and combinatorics in dynamics, stochastics and control, 477-498. Abel Symp., 13, Springer, Cham, 2018
[19] Hu, Y., Wang, X.: Stochastic heat equation with general rough noise. Ann. Inst. Henri Poincaré Probab. Stat., 58(1), 379-423 (2022)
[20] Hu, Y., Nualart, D., Zhang, T.: Large deviations for stochastic heat equation with rough dependence in space. Bernoulli, 24(1) 354-385 (2018)
[21] Li, M., Huang C., Hu Y.: Asymptotic separation for stochastic Volterra integral equation with doubly singular kernels. App. Math. Lett., 113, 106880 (2021)
[22] Liu, J.: Moderate deviations for stochastic heat equation with rough dependence in space. Acta Math. Sin. (Engl. Ser.), 35(9), 1491-1510 (2019)
[23] Liu, S., Hu, Y., Wang, X.: Nonlinear stochastic wave equation driven by rough noise. arXiv:2110.13800, (2021)
[24] Liu, W.: Large deviations for stochastic evolution equations with small multiplicative noise. *Appl. Math. Optim.*, 61(1), 27-56 (2010)

[25] Liu, W., Song, Y., Zhai, J., Zhang, T.: Large and moderate deviation principles for McKean-Vlasov with jumps. *Potential Anal.*, in press

[26] Liu, W., Tao, C., Zhu, J.: Large deviation principle for a class of SPDE with locally monotone coefficients. *Sci. China Math.*, 63(6), 1181-1202 (2020)

[27] Márquez-Carreras, D., Sarrà, M.: Large deviation principle for a stochastic heat equation with spatially correlated noise. *Electron. J. Probab.*, 8(12), 1-39 (2003)

[28] Matoussi, A., Sabbagh, W., Zhang, T.: Large deviation principles of obstacle problems for quasilinear stochastic PDEs. *Appl. Math. Optim.*, 83(2), 849-879 (2021)

[29] Peszat, S., Zabczyk, J.: Stochastic evolution equations with a spatially homogeneous Wiener process. *Stoch. Process. Appl.*, 72, 187-204 (1997)

[30] Peszat, S., Zabczyk, J.: Nonlinear stochastic wave and heat equations. *Probab. Theory Relat. Fields*, 116, 421-443 (2000)

[31] Pipiras, V., Taqqu, M. S.: Integration questions related to fractional Brownian motion. *Probab. Theory Relat. Fields*, 118(2), 251-91 (2000)

[32] Ren, J., Zhang, X.: Freidlin-Wentzell’s large deviations for stochastic evolution equations. *J. Funct. Anal.*, 254, 3148-3172 (2008)

[33] Song, J.: SPDEs with colored Gaussian noise: a survey. *Commun. Math. Stat.*, 6(4), 481-492 (2018)

[34] Song, J., Song X., Xu, F.: Fractional stochastic wave equation driven by a Gaussian noise rough in space. *Bernoulli*, 26(4), 2699-2726 (2020)

[35] Wang, R., Zhang, S., Zhai, J.: Large deviation principle for stochastic Burgers type equation with reflection. *Commun. Pure Appl Anal.*, 21(1), 213-238 (2022)

[36] Wu, W., Zhai, J.: Large deviations for stochastic porous media equation on general measure space. *J. Differential Equations*, 269, 10002-10036 (2020)

[37] Xiong, J., Zhai, J.: Large deviations for locally monotone stochastic partial differential equations driven by Lévy noise. *Bernoulli*, 24, 2842-2874 (2018)

[38] Xu, T., Zhang, T.: White noise driven SPDEs with reflection: existence, uniqueness and large deviation principles. *Stoch. Process. Appl.*, 119, 3453-3470 (2009)

Ruinan Li, School of Statistics and Information, Shanghai University of International Business and Economics, Shanghai, 201620, China

Email address: ruinanli@amss.ac.cn

Ran Wang, School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China.

Email address: rwang@whu.edu.cn

Beibei Zhang, School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China.

Email address: zhangbb@whu.edu.cn