Thermodynamic Behavior of Fuzzy Membranes in PP-Wave Matrix Model

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Abstract

We discuss a two-body interaction of membrane fuzzy spheres in a pp-wave matrix model at finite temperature by considering a fuzzy sphere rotates with a constant radius $r$ around the other one sitting at the origin in the SO(6) symmetric space. This system of two fuzzy spheres is supersymmetric at zero temperature and there is no interaction between them. Once the system is coupled to the heat bath, supersymmetries are completely broken and non-trivial interaction appears. We numerically show that the potential between fuzzy spheres is attractive and so the rotating fuzzy sphere tends to fall into the origin. The analytic formula of the free energy is also evaluated in the large $N$ limit. It is well approximated by a polylog-function.

Keywords: pp-wave matrix model, fuzzy sphere, giant graviton, thermodynamics
1 Introduction and Summary

The basic degrees of freedom of string theory and M-theory are fully encoded in matrix models [1–3]. The matrix models are believed to give non-perturbative formulations of string theory and M-theory. In particular, the BFSS matrix model is supersymmetric matrix quantum mechanics. It is believed to describe a discrete light-cone quantization of M-theory. It also describes the low-energy dynamics of \( N \) D0-branes of type IIA superstring theory [4]. In addition it is a matrix regularization of the light-cone action for the supermembrane in eleven dimensions [5].

Matrix model thermodynamics is also interesting in relation to black hole physics. One motivation for understanding the behavior of the BFSS matrix model at finite temperature comes from the conjecture that their finite temperature states are related to black hole states of type IIA supergravity [6, 7]. This idea was studied in a series of papers by Kabat, Lifschytz and Lowe [8]. The BFSS matrix model at finite temperature is also investigated in [9]. Some features are discussed in [10] (For a review of brane thermodynamics, see [11]).

By the way, a matrix model on a pp-wave background was proposed by Berenstein-Maldacena-Nastase (BMN) [12], and it has been intensively studied. The background of this matrix model is given by the maximally supersymmetric pp-wave background [13]:

\[
\begin{align*}
    ds^2 &= -2dx^+dx^- - \left( \sum_{i=1}^{3} \left( \frac{\mu}{3} \right)^2 (x^i)^2 + \sum_{a=4}^{6} \left( \frac{\mu}{6} \right)^2 (x^a)^2 \right) (dx^+)^2 + \sum_{I=1}^{9} (dx^I)^2, \\
    F_{+123} &= \mu.
\end{align*}
\]

The action of the matrix model on this background \( S_{\text{pp}} \) consists of two parts as follows:* \( S_{\text{pop}} = S_{\text{flat}} + S_\mu \),

\[
\begin{align*}
    S_{\text{flat}} &= \int dt \, \text{Tr} \left[ \frac{1}{2R} D_t X^I D_t X^J + \frac{R}{4} ([X^I, X^J])^2 + i \Theta^I D_t \Theta - R \Theta^I \gamma^I [\Theta, X^I] \right], \\
    S_\mu &= \int dt \, \text{Tr} \left[ - \frac{1}{2R} \left( \frac{\mu}{3} \right)^2 (X^i)^2 - \frac{1}{2R} \left( \frac{\mu}{6} \right)^2 (X^a)^2 - \frac{i}{2} \epsilon^{ijk} X^i X^j X^k - i \frac{\mu}{3} \Theta^I \gamma^I \Theta \right],
\end{align*}
\]

where the indices of the transverse nine-dimensional space are \( I, J = 1, \ldots, 9 \) and \( R \) is the radius of the circle compactified along \( x^- \). All degrees of freedom are \( N \times N \) Hermitian matrices and the covariant derivative \( D_t \) with the gauge field \( A \) is defined by \( D_t = \partial_t - i[A, \ _] \).

This matrix model is closely related to the supermembrane theory on the pp-wave background via the matrix regularization [5] (For works in this direction see [14, 15]). This matrix model

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*Hereafter we will rescale the gauge field and parameters as \( A \to RA, \ t \to t/R, \ \mu \to R\mu \).
has a supersymmetric fuzzy sphere solution (which is called “giant graviton”) due to the Myers effects [16], because the constant 4-form flux is equipped with. This fuzzy sphere solution is given by

$$X^i_{\text{sphere}} = \frac{\mu}{3} J^i,$$

where $J^i$ satisfies the $SU(2)$ algebra $[J^i, J^j] = i \epsilon^{ijk} J^k$. The classical energy of this solution is zero and hence the fuzzy sphere can appear in classical vacua without loss of energy. Namely, the classical vacua of the pp-wave matrix model are enriched with fuzzy spheres, and it may be interesting to look deeper into the dynamics of fuzzy sphere solution. Then the fuzzy sphere solution $X^i_{\text{sphere}}$ preserves the full 16 dynamical supersymmetries of the pp-wave and hence is 1/2-BPS object. We note that actually there is another fuzzy sphere solution of the form $\mu^6 J^i$ but such solution does not have quantum stability and is thus non-BPS object [17] (For other classical solutions, see [18–21]). In [22] the interaction potential of membrane fuzzy spheres was shown to be the $1/r^7$-type.

In this letter we discuss a thermal interaction between fuzzy membrane solutions in the pp-wave matrix model. The two-body interaction of fuzzy spheres at zero temperature was studied in our previous work [23] (containing the extension of the work [24]) by considering the system that a fuzzy sphere rotates with a constant radius $r$ around the other one sitting at the origin in the transverse six-dimensional space (Fig. 1). In the case of zero temperature the interaction potential is zero basically because of the existence of the remaining supersymmetries. In the finite temperature case, however, non-trivial interaction should appear since the supersymmetries are completely broken due to the thermal effect. Here, we will be interested in the evaluation of the thermal potential at finite temperature by heating the system depicted in Fig. 1. Firstly we consider the exact one-loop free energy by using the $\mu \to \infty$ limit. In this computation we use the spectrum around the two-body system which has been obtained in [23]. By numerically plotting the free energy, we can see that the rotating fuzzy sphere tends to approach to the static fuzzy sphere. The attractive potential becomes stronger and stronger as temperature grows and the distance decreases. Furthermore, as the size of the matrix becomes larger and larger, the potential becomes deeper and deeper. In addition we evaluate the analytical expression of the free energy by taking the large $N$ limit. Then the discrete sum for the index of spin may be replaced by the integral for it. When we utilize the long-distance expansion, we find that the leading part of the free energy can be represented by a polylog function. The approximation of the free energy qualitatively well reproduces the behavior of
This paper is organized as follows: In section 2 we introduce the formula for the exact free energy of a pp-wave matrix model at finite temperature. In section 3 the free energy for the interaction between two fuzzy spheres is evaluated both in numerical and analytical methods.

2 One-Loop Exact Free Energy at Finite Temperature

We will briefly review the calculus of the one-loop free energy in the finite temperature case. By using the background field method and the $\mu \to \infty$ limit, it can be exactly computed.

We first decompose the variables as

$$X^I = B^I + Y^I, \quad \Theta = 0 + \Psi,$$

where $B^I$ are the classical background fields while $Y^I$ and $\Psi$ are the quantum fluctuations around them. In some cases we need to consider the background of the gauge field (gauge field moduli) due to the finite temperature effect. In the case we consider here, however, we do not need to take account of it, as we will explain later.

In order to perform the path integration, we take the background field gauge which is usually chosen in the matrix model calculation as

$$D^\mu_{bg} A^\mu_{qu} \equiv D_t A + i [B^I, X^I] = 0.$$  \hspace{1cm} (2.2)

Then the corresponding gauge-fixing $S_{GF}$ and Faddeev-Popov ghost $S_{FP}$ terms are given by

$$S_{GF} + S_{FP} = \int dt \text{Tr} \left( -\frac{1}{2} (D^\mu_{bg} A^\mu_{qu})^2 - \bar{C} \partial_t D_t C + [B^I, C][X^I, C] \right).$$  \hspace{1cm} (2.3)
Now by inserting the decomposition of the matrix fields into the matrix model action, we get the gauge fixed plane-wave action $S (\equiv S_{pp} + S_{GF} + S_{FP})$ expanded around the background. The resulting acting is read as $S = S_0 + S_2 + S_3 + S_4$, where $S_n$ represents the action of order $n$ with respect to the quantum fluctuations and, for each $n$, its expression is

$$S_0 = \int dt \operatorname{Tr} \left[ \frac{1}{2} \left( \dot{B}^I \right)^2 - \frac{1}{2} \left( \frac{\mu}{3} \right)^2 (B^I)^2 - \frac{1}{2} \left( \frac{\mu}{6} \right)^2 (B^b)^2 + \frac{1}{4} ([B^I, B^J])^2 - i \frac{\mu}{3} \epsilon^{ijk} B^i B^j B^k \right],$$

$$S_2 = \int dt \operatorname{Tr} \left[ \frac{1}{2} (\dot{Y}^I)^2 - 2i \dot{B}^I [A, Y^I] + \frac{1}{2} ([B^I, Y^J])^2 + [B^I, B^J] [Y^I, Y^J] - i \mu \epsilon^{ijk} B^i Y^j Y^k - \frac{1}{2} \left( \frac{\mu}{3} \right)^2 (Y^I)^2 - \frac{1}{2} \left( \frac{\mu}{6} \right)^2 (Y^a)^2 + \iota \varepsilon^{I\gamma^I} [\Psi, B^I] - i \frac{\mu}{4} \varepsilon^{I\gamma^I\delta^I} \Psi \right] - \frac{1}{2} \dot{A}^2 - \frac{1}{2} ([B^I, A])^2 + \dot{C} \dot{C} + [B^I, \bar{C}] [B^I, C],$$

$$S_3 = \int dt \operatorname{Tr} \left[ - \iota \dot{Y}^I [A, Y^I] - [A, B^I] [A, Y^I] + [B^I, Y^J] [Y^I, Y^J] + \Psi^I [A, \Psi] - \Psi^I \gamma^I [\Psi, Y^I] - i \frac{\mu}{3} \epsilon^{ijk} Y^i Y^j Y^k - i \dot{C} [A, C] + [B^I, \bar{C}] [Y^I, C] \right],$$

$$S_4 = \int dt \operatorname{Tr} \left[ - \frac{1}{2} ([A, Y^I])^2 + \frac{1}{4} ([Y^I, Y^J])^2 \right]. \tag{2.4}$$

For the justification of one-loop computation or the semi-classical analysis, it should be made clear that $S_3$ and $S_4$ of Eq. (2.4) can be regarded as perturbations. For this purpose, following [14], we rescale the fluctuations and parameters as

$$A \to \mu^{-1/2} A, \quad Y^I \to \mu^{-1/2} Y^I, \quad C \to \mu^{-1/2} C, \quad \bar{C} \to \mu^{-1/2} \bar{C}, \quad t \to \mu^{-1} t. \tag{2.5}$$

Under this rescaling, the action $S$ in the fuzzy sphere background becomes

$$S = S_2 + \mu^{-3/2} S_3 + \mu^{-3} S_4, \tag{2.6}$$

where $S_2, S_3$ and $S_4$ do not have $\mu$ dependence. Now it is obvious that, in the large $\mu$ limit, $S_3$ and $S_4$ can be treated as perturbations and the one-loop computation gives the sensible result. Note that the analysis in the $S_2$ part is exact in the $\mu \to \infty$ limit. We can calculate the exact spectra around an $N$-dimensional irreducible fuzzy sphere in the $\mu \to \infty$ limit, by following the method in the work [14] (For the detail calculation, see [14, 23]).

Now we are interested in the finite temperature case where the system couples to the thermal bath. In order to consider the thermal system with temperature $T$, let us move to the Euclidean formulation via the Wick rotation $t \to it$, and compactify the Euclidean time direction with a periodicity $\beta \equiv 1/T$. Note that $T$ is a dimensionless parameter now because of the scaling of
time variable \( t \rightarrow R^{-1}t \). This compactification leads us to encounter the summation instead of the momentum integral. We can easily compute this summation by using the formulae

\[
\sum_{n=\text{integer}} \ln \left( n^2 \pi^2 + M'^2 \right) = 2 \ln \sinh M' \quad \text{(for bosons)},
\]

\[
\sum_{n=\text{half integer}} \ln \left( n^2 \pi^2 + M'^2 \right) = 2 \ln \cosh M' \quad \text{(for fermions)},
\]

and the fact that the fuzzy sphere configuration under our consideration is supersymmetric. Here we should note that the ghost fields obey the periodic condition rather than anti-periodic condition. Hence its contribution gives the \( \sinh \) but the integration of the ghost field gives the inverse sign in comparison to bosons. This fact ensures the cancellation among unphysical and ghost degrees of freedom. Thus the free energy \( F = -T \ln Z \) is represented by

\[
F = T \sum_{i \in Y,A} N_i \ln \left( 1 - e^{-\beta M_i} \right) - \frac{1}{2} T N_\Psi \ln \left( 1 + e^{-\beta M_\Psi} \right) - 2 T N_C \ln \left( 1 - e^{-\beta M_C} \right). \tag{2.9}
\]

The symbols \( Y \) and \( A \) denote the bosonic fluctuations of \( X \) and \( A \), respectively. The \( \Psi \) and \( C \) denote the fermionic fluctuations of \( \Theta \) and \( C \). The free energy usually contains the zero temperature part, but this part does not appear in the present case since the fuzzy sphere background is supersymmetric at zero temperature.

### 3 Free Energy of Fuzzy Membrane Interaction

We are interested in the configuration of classical solution in which a fuzzy sphere rotates with a constant radius \( r \) around the other fuzzy sphere (see Fig.1). This system is described by

\[
B^I = \begin{pmatrix} B^I_{(1)} & 0 \\ 0 & B^I_{(2)} \end{pmatrix}, \quad B^i_{(s)} = \frac{\mu}{3} J^i_{(s)} \quad (i = 1, 2, 3; \ s = 1, 2),
\]

\[
B^4_{(1)} = r \cos \left( \frac{\mu}{6} t \right) \mathbf{1}_{N_1 \times N_1}, \quad B^5_{(1)} = r \sin \left( \frac{\mu}{6} t \right) \mathbf{1}_{N_1 \times N_1}, \quad \text{otherwise} = 0. \tag{3.1}
\]

The \( B^I_{(s)} \) are \( N_s \times N_s \) matrices. We take \( B^I \) as \( N \times N \) matrices and then \( N = N_1 + N_2 \). Let us concentrate on the \( N_1 \neq N_2 \) case so that we do not have to take account of the gauge field moduli considered in [26–28]. If two fuzzy spheres of the same size are considered then we may have the gauge field moduli. This gauge field moduli plays an important role in the study of the Hagedorn transition in the transverse M5-brane vacua [29].

This system (3.1) was proposed in our previous work [23] and it was shown to be supersymmetric. That is, the interaction potential between them vanishes (one-loop flatness) [23] as well
are responsible for the interaction. Being at a distance in the transverse space and will consider the off-diagonal fluctuations which all the fuzzy spheres are considered to be located at the origin in the transverse six-dimensional space.

† Stability of fuzzy spheres is discussed in the IIB matrix model with Chern-Simons term [30] and mass term [31].

In the computation of the partition function for off-diagonal fluctuations, we should diagonalize the partition function for the rotation part, and we should consider the off-diagonal fluctuations which are responsible for the interaction.

| Fields                  | (Mass)$^2$ | Degeneracy | Spin                                                                 |
|-------------------------|------------|------------|----------------------------------------------------------------------|
| $SO(3)$ bosons          |            |            |                                                                      |
| $\alpha_{jm}$           | $r^2 + \frac{1}{32}(j + 1)^2$ | $2j + 1$   | $\frac{1}{2}|N_1 - N_2| \leq j \leq \frac{1}{2}(N_1 + N_2) - 2$ |
| $\beta_{jm}$            | $r^2 + \frac{1}{32}j^2$ | $2j + 1$   | $\frac{1}{2}|N_1 - N_2| + 1 \leq j \leq \frac{1}{2}(N_1 + N_2)$ |
| $\omega_{jm}$ (gauge)   | $r^2 + \frac{1}{32}(j + 1)^2$ | $2j + 1$   | $\frac{1}{2}|N_1 - N_2| \leq j \leq \frac{1}{2}(N_1 + N_2) - 1$ |
| \(SO(4)\) bosons       |            |            |                                                                      |
| $\phi'_{jm}$ (\(a' = 6, 7, 8, 9\)) | $r^2 + \frac{1}{32}(j + \frac{1}{2})^2$ | $4(2j + 1)$ | $\frac{1}{2}|N_1 - N_2| \leq j \leq \frac{1}{2}(N_1 + N_2) - 1$ |

Rotational Part

| (r1)$_{jm}$ (gauge)     | $r^2 + \frac{1}{32}j(j + 1)$ | $2j + 1$   | $\frac{1}{2}|N_1 - N_2| \leq j \leq \frac{1}{2}(N_1 + N_2) - 1$ |
| (r2)$_{jm}$             | $r^2 + \frac{1}{32}j^2$ | $2j + 1$   | $\frac{1}{2}|N_1 - N_2| \leq j \leq \frac{1}{2}(N_1 + N_2) - 1$ |
| (r3)$_{jm}$             | $r^2 + \frac{1}{32}(j + 1)^2$ | $2j + 1$   | $\frac{1}{2}|N_1 - N_2| \leq j \leq \frac{1}{2}(N_1 + N_2) - 1$ |

Fermion 1

| (F1)$_{jm}$             | $r^2 + \frac{1}{32}(j + 1)^2$ | $2(2j + 1)$ | $\frac{1}{2}|N_1 - N_2| - \frac{3}{2} \leq j \leq \frac{1}{2}(N_1 + N_2) - \frac{3}{2}$ |
| (F2)$_{jm}$             | $r^2 + \frac{1}{32}(j + \frac{1}{2})^2$ | $2(2j + 1)$ | $\frac{1}{2}|N_1 - N_2| - \frac{1}{2} \leq j \leq \frac{1}{2}(N_1 + N_2) - \frac{1}{2}$ |

Fermion 2

| (F'1)$_{jm}$            | $r^2 + \frac{1}{32}(j + \frac{1}{2})^2$ | $2(2j + 1)$ | $\frac{1}{2}|N_1 - N_2| + \frac{1}{2} \leq j \leq \frac{1}{2}(N_1 + N_2) - \frac{1}{2}$ |
| (F'2)$_{jm}$            | $r^2 + \frac{1}{32}j^2$ | $2(2j + 1)$ | $\frac{1}{2}|N_1 - N_2| + \frac{3}{2} \leq j \leq \frac{1}{2}(N_1 + N_2) - \frac{3}{2}$ |

Ghost

| $c_{jm}$ (\(\bar{c}_{jm}\)) | $r^2 + \frac{1}{32}j(j + 1)$ | $2j + 1$   | $\frac{1}{2}|N_1 - N_2| \leq j \leq \frac{1}{2}(N_1 + N_2) - 1$ |

Tab. 1: The spectrum of the interaction part between two fuzzy spheres.

as the quantum fluctuations around each of the fuzzy spheres [14,17,23]. This result comes from the supersymmetries of the theory. However, the supersymmetries are completely broken down and so non-trivial potential should appear. In our previous work [25] we studied the stability of the fuzzy sphere \(^\dagger\) in the case of finite temperature. In particular, we compared the free energies between the fuzzy sphere vacuum \((X^i = \frac{4}{3}J^i (i = 1, 2, 3))\) and the trivial one \((X^i = 0)\), where all the fuzzy spheres are considered to be located at the origin in the transverse six-dimensional space. Then we are now interested in the interaction potential between the two fuzzy spheres being at a distance in the transverse space and will consider the off-diagonal fluctuations which are responsible for the interaction.

\(^\dagger\) Stability of fuzzy spheres is discussed in the IIB matrix model with Chern-Simons term [30] and mass term [31].
onalize the quadratic action. After the diagonalization, we see the spectrum. The resulting spectrum is summarized in Tab. 1. By using the spectrum the free energy is expressed as

\[
\frac{1}{2} \beta F = \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-2} (2j+1) \ln(1 - e^{-\beta \sqrt{r^2 + \frac{1}{32}(j+\frac{1}{2})^2}}) + \sum_{j=\frac{1}{2}|N_1-N_2|+1}^{\frac{1}{2}(N_1+N_2)} (2j+1) \ln(1 - e^{-\beta \sqrt{r^2 + \frac{1}{32}j^2}})
\]

\[
+ \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} 4(2j+1) \ln(1 - e^{-\beta \sqrt{r^2 + \frac{1}{32}(j+\frac{1}{2})^2}})
\]

\[
+ \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} (2j+1) \left[ \ln(1 - e^{-\beta \sqrt{r^2 + \frac{1}{32}j^2}}) + \ln(1 - e^{-\beta \sqrt{r^2 + \frac{1}{32}(j+1)^2}}) \right]
\]

\[
- \sum_{j=\frac{1}{2}|N_1-N_2|-\frac{3}{2}}^{\frac{1}{2}(N_1+N_2)-\frac{3}{2}} \ln(1 + e^{-\beta \sqrt{r^2 + \frac{1}{32}(j+\frac{1}{2})^2}}) + \ln(1 + e^{-\beta \sqrt{r^2 + \frac{1}{32}(j+1)^2}})
\]

\[
- \sum_{j=\frac{1}{2}|N_1-N_2|+\frac{1}{2}}^{\frac{1}{2}(N_1+N_2)-\frac{1}{2}} 2(2j+1) \left[ \ln(1 + e^{-\beta \sqrt{r^2 + \frac{1}{32}(j+\frac{1}{2})^2}}) + \ln(1 + e^{-\beta \sqrt{r^2 + \frac{1}{32}j^2}}) \right].
\]

Here the unphysical degrees of freedom are canceled out. That is, the above expression has no degrees of freedom for fuzzy sphere rotation, gauge field and ghosts.

We will consider the free energy of the fuzzy membrane interaction in both numerical and analytical methods below. To begin with, we numerically study the behavior of the free energy. Then we evaluate the analytical expression of the free energy by using the large $N$ limit.

### 3.1 Numerical Study of the Free Energy

The expression of the free energy for the fuzzy membrane interaction is quite complicated and so it is difficult to study analytically the behavior of the free energy without any approximation. We will analyze the analytic behavior of the free energy in the next subsection by using the large $N$ limit and the long-range expansion. Here let us however investigate numerically the behavior of the free energy. In Fig. 2 we give the numerical plot of the free energy for some cases. From the results we can see that the interaction between two fuzzy spheres is attractive.

### 3.2 Analytic Behavior of the Free Energy in Large $N$ Limit

In the previous subsection we presented the numerical study of the thermodynamic behavior of the free energy for the interaction between two fuzzy spheres. From now on we will evaluate
Fig. 2: The numerical plots of the free energy $F$ for the two-body interaction. The $F$ is plotted in the vertical axis with respect to the distance $r$ and the temperature $T$. The left one is for the $N_1 = 3$ and $N_2 = 2$, and the right is for $N_1 = 20$ and $N_2 = 10$.

the analytical expression by considering the large $N$ limit. In such situations we can replace the discrete summation for the spin $j$ by an integral with a continuum variable and so it is possible to evaluate an analytical expression for the free energy. For example, let us focus upon a part of the $SO(3)$ bosons, $\alpha_{jm}$. We consider the large $N$ limit:

$$N_1, N_2 \gg 1 \text{ with } N_1 - N_2 = c,$$

where $c$ is a non-vanishing constant. Then the summation for $\alpha_{jm}$ may be replaced as follows:

$$\sum_{j = \frac{1}{2}|N_1 - N_2|}^{\frac{1}{2}(N_1 + N_2) - 2} (2j + 1) \ln \left(1 - e^{-\beta \sqrt{r^2 + \frac{1}{3} (j+1)^2}}\right) \rightarrow \int_{\frac{1}{2}|c|}^{N} dx \left(2x + 1\right) \ln \left(1 - e^{-\beta \sqrt{r^2 + \frac{1}{3} (x+1)^2}}\right).$$

In addition, in order to evaluate the above integral analytically, we utilize the long-distance expansion (large $r$ limit) and expand the square root as

$$\sqrt{r^2 + \frac{1}{3^2} (x+1)^2} \cong r + \frac{1}{18r} (x + 1)^2 + O\left(\frac{1}{r^3}\right).$$

By using the Taylor expansion,

$$\ln(1 - y) = - \left( y + \frac{1}{2} y^2 + \frac{1}{3} y^3 + \cdots \right),$$

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the free energy for \( \alpha_{jm} \) can be evaluated as

\[
F_{\alpha} = 36T^2r \sum_{n=1}^{\infty} \frac{1}{n^2} \left\{ e^{-\frac{1}{18} \left( \frac{n\beta}{r} \right) (18r^2+1+2N+N^2)} - e^{-\frac{1}{72} \left( \frac{n\beta}{r} \right) (72r^2+4+4|c|+|c|^2)} \right\}
+ 3\sqrt{2\pi r} T^{3/2} \sum_{n=1}^{\infty} \left\{ \text{erf} \left( \frac{N + 1}{6} \sqrt{\frac{2n\beta}{r}} \right) - \text{erf} \left( \frac{|c| + 2}{12} \sqrt{\frac{2n\beta}{r}} \right) \right\} \frac{e^{-n\beta r}}{n^{3/2}}
\approx -36T^2r \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n\beta r} + 3\sqrt{2\pi r} T^{3/2} \sum_{n=1}^{\infty} \left\{ 1 - \text{erf} \left( \frac{|c| + 2}{12} \sqrt{\frac{2n\beta}{r}} \right) \right\} \frac{e^{-n\beta r}}{n^{3/2}},
\] (3.7)

where we have used that \( N \) and \( r \) is sufficiently large so that \( \text{erf}(x) \approx 1 \ (x \gg 1) \) as shown in Fig. 3, but we have supposed that \( r \gg N \). When we consider the low temperature region \( T \ll 1/r \), the second summation in the expression (3.7) can be ignored. Then, using a polylog function,

\[
\text{polylog}(a, z) \equiv \sum_{n=1}^{\infty} \frac{z^n}{n^a}.
\]

we can write down the analytical expression of the free energy as follows:

\[
F \approx -36T^2r \cdot \text{polylog}(2, e^{-\beta r}).
\] (3.8)

For the other bosonic sector we can evaluate the free energy in the similar way and the same result as (3.8). Then the fermionic sector and the ghost sector the contributions are evaluated as

\[
F \approx -36T^2r \cdot \text{polylog}(2, -e^{-\beta r}).
\] (3.9)

Thus the total free energy is given by

\[
F \sim -36 \times 8T^2r \left[ \text{polylog}(2, e^{-\beta r}) + \text{polylog}(2, -e^{-\beta r}) \right]
= -144T^2r \cdot \text{polylog}(2, e^{-2\beta r}),
\] (3.10)
i) the approximated free energy in large $N$  

ii) the exact free energy for $N_1 = 50$ and $N_2 = 30$

Fig. 4: A comparison of the approximated free energy in large $N$ and the exact free energy. In i) the approximated free energy is numerically plotted. In ii) the exact free energy in the same region as i) is shown in order to see the validity of the approximation.

in the region:

$$T \ll \frac{1}{r}, \quad N \ll r, \quad r, N : \text{sufficiently large}. \quad (3.11)$$

Here we have used the formula:

$$\text{polylog}(2, z) + \text{polylog}(2, -z) = \frac{1}{2} \text{polylog}(2, z^2).$$

In particular, $\text{polylog}(2, 1) = \pi^2/6$ and $\text{polylog}(2, 0) = 0$. The numerical plot of (3.10) is shown in Fig. 4. We see that the behavior of (3.10) is quantitatively similar to that of the exact free energy with large $N$ values. In the low temperature region the large $N$ free energy for the membrane fuzzy sphere interaction is well described by a polylog function.

Note that we cannot naively evaluate the high-temperature behavior from (3.7) since the $n \approx \infty$ contributes and so we cannot take the error function as erf(0)=0.

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References

[1] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A conjecture,” Phys. Rev. D 55 (1997) 5112 [arXiv:hep-th/9610043].

[2] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, “A large-N reduced model as superstring,” Nucl. Phys. B 498 (1997) 467 [arXiv:hep-th/9612115].

[3] R. Dijkgraaf, E. Verlinde and H. Verlinde, “Matrix string theory,” Nucl. Phys. B 500 (1997) 43 [arXiv:hep-th/9703030].

[4] E. Witten, “String theory dynamics in various dimensions,” Nucl. Phys. B 443 (1995) [arXiv:hep-th/9503124].

[5] B. de Wit, J. Hoppe and H. Nicolai, “On the quantum mechanics of supermembranes,” Nucl. Phys. B 305 (1988) 545.

[6] I. R. Klebanov and L. Susskind, “Schwarzschild black holes in various dimensions from matrix theory,” Phys. Lett. B 416 (1998) 62 [arXiv:hep-th/9709108].

[7] T. Banks, W. Fischler, I. R. Klebanov and L. Susskind, “Schwarzschild black holes in matrix theory. II,” JHEP 9801 (1998) 008 [arXiv:hep-th/9711005].

[8] D. Kabat, G. Lifschytz and D. A. Lowe, “Black hole thermodynamics from calculations in strongly coupled gauge theory,” Phys. Rev. Lett. 86 (2001) 1426 [Int. J. Mod. Phys. A 16 (2001) 856] [arXiv:hep-th/0007051].

[9] J. Ambjorn, Y. M. Makeenko and G. W. Semenoff, “Thermodynamics of D0-branes in matrix theory,” Phys. Lett. B 445 (1999) 307 [arXiv:hep-th/9810170]. Y. Makeenko, “Formulation of matrix theory at finite temperature,” Fortsch. Phys. 48 (2000) 171 [arXiv:hep-th/9903030].

[10] M. Li, “Ten dimensional black hole and the D0-brane threshold bound state,” Phys. Rev. D 60 (1999) 066002 [arXiv:hep-th/9901158].

S. Bal and B. Sathiapalan, “High temperature limit of the N = 2 matrix model,” Mod. Phys. Lett. A 14 (1999) 2753 [arXiv:hep-th/9902087]; “High temperature limit of the N = 2 IIA matrix model,” Nucl. Phys. Proc. Suppl. 94 (2001) 693 [arXiv:hep-lat/0011039].
[11] E. J. Martinec, “Black holes and the phases of brane thermodynamics,” arXiv:hep-th/9909049.

[12] D. Berenstein, J. M. Maldacena and H. Nastase, “Strings in flat space and pp waves from N = 4 super Yang Mills,” JHEP 0204 (2002) 013 arXiv:hep-th/0202021.

[13] J. Kowalski-Glikman, “Vacuum states in supersymmetric Kaluza-Klein theory,” Phys. Lett. B 134 (1984) 194.

[14] K. Dasgupta, M. M. Sheikh-Jabbari and M. Van Raamsdonk, “Matrix perturbation theory for M-theory on a PP-wave,” JHEP 0205 (2002) 056 arXiv:hep-th/0205185.

[15] K. Sugiyama and K. Yoshida, “Supermembrane on the pp-wave background,” Nucl. Phys. B 644 (2002) 113 arXiv:hep-th/0206070; “BPS conditions of supermembrane on the pp-wave,” Phys. Lett. B 546 (2002) 143 arXiv:hep-th/0206132; N. Nakayama, K. Sugiyama and K. Yoshida, “Ground state of the supermembrane on a pp-wave,” Phys. Rev. D 68 (2003) 026001, arXiv:hep-th/0209081.

[16] R. C. Myers, “Dielectric-branes,” JHEP 9912 (1999) 022 arXiv:hep-th/9910053.

[17] K. Sugiyama and K. Yoshida, “Giant graviton and quantum stability in matrix model on PP-wave background,” Phys. Rev. D 66 (2002) 085022 arXiv:hep-th/0207190.

[18] D. Bak, “Supersymmetric branes in PP wave background,” Phys. Rev. D67 (2003) 045017, hep-th/0204033.

[19] S. Hyun and H. Shin, “Branes from matrix theory in pp-wave background,” Phys. Lett. B543 (2002) 115–120, hep-th/0206090.

[20] J. H. Park, “Supersymmetric objects in the M-theory on a pp-wave,” JHEP 0210 (2002) 032 arXiv:hep-th/0208161.

[21] D. Bak, S. Kim and K. Lee, “All higher genus BPS membranes in the plane wave background,” arXiv:hep-th/0501202.

[22] H. Shin and K. Yoshida, “Membrane fuzzy sphere dynamics in plane-wave matrix model,” Nucl. Phys. B 709 (2005) 69 arXiv:hep-th/0409045.
[23] H. Shin and K. Yoshida, “One-loop flatness of membrane fuzzy sphere interaction in plane-wave matrix model,” Nucl. Phys. B 679 (2004) 99 [arXiv:hep-th/0309258].

[24] W. H. Huang, “Thermal instability of giant graviton in matrix model on pp-wave background,” [arXiv:hep-th/0310212].

[25] H. Shin and K. Yoshida, “Thermodynamics of fuzzy spheres in pp-wave matrix model,” Nucl. Phys. B 701 (2004) 380 [arXiv:hep-th/0401014].

[26] K. Furuuchi, E. Schreiber and G. W. Semenoff, “Five-brane thermodynamics from the matrix model,” [arXiv:hep-th/0310286].

[27] G. W. Semenoff, “Matrix model thermodynamics,” [arXiv:hep-th/0405107].

[28] S. Hadizadeh, B. Ramadanovic, G. W. Semenoff and D. Young, “Free energy and phase transition of the matrix model on a plane-wave,” Phys. Rev. D 71 (2005) 065016 [arXiv:hep-th/0409318].

[29] J. Maldacena, M. M. Sheikh-Jabbari and M. Van Raamsdonk, “Transverse fivebranes in matrix theory,” JHEP 0301 (2003) 038 [arXiv:hep-th/0211139].

[30] T. Imai, Y. Kitazawa, Y. Takayama and D. Tomino, “Quantum corrections on fuzzy sphere,” Nucl. Phys. B 665 (2003) 520 [arXiv:hep-th/0303120]; T. Azuma, S. Bal, K. Nagao and J. Nishimura, “Nonperturbative studies of fuzzy spheres in a matrix model with the Chern-Simons term,” JHEP 0405 (2004) 005 [arXiv:hep-th/0401038].

[31] T. Azuma, S. Bal and J. Nishimura, “Dynamical generation of gauge groups in the massive Yang-Mills-Chern-Simons matrix model,” [arXiv:hep-th/0504217].