CONFLICT, PRIVATE AND COMMUNAL PROPERTY

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Abstract. This paper develops a model where agents can create private property rights on a resource by making appropriative activities. We show that the value of the resource has a non-monotonic effect on the emergence of private property. When the resource is sufficiently valuable, agents have an incentive to leave a sharing agreement and private property can appear. However if the value of the resource increases beyond a given threshold, deviations from the sharing agreement leads to a very costly confrontation. In this case, private property is not sustainable. Our analysis also finds that population size has an important effect on the size of the parameter set in which private property is sustainable.

1. Introduction. Can property rights emerge as the result of a game? Sanchez Pages [15] shows that conflict resulting in the establishment of private property rights can be ex-ante Pareto superior to free access to a resource if the number of agents is large enough. In this paper, instead, we analyze the possibility of establishment of private property as the result of a game between agents. We focus on the relationship between the value of a resource and conflict among agents, and the creation of private property rights, and analyze its sensitivity to changes in the population size of the economy.

In our modelling framework we follow Grossman’s [9] model on the appearance of property rights.1 As in Sanchez Pages [16], “free access” is defined as an agreement (i.e. coalition) between all agents to share collectively a valuable resource. In this formulation private property appears as a rational deviation from this agreement.2

Our main results are as follows. We show that the value of the resource has a non monotonic effect on the emergence of private property. Specifically, when the resource is sufficiently valuable, agents have an incentive to leave the free access agreement. However, if the value of the resource increases sufficiently, deviations from the free access agreement lead to a very costly conflict so, in order to avoid it, agents prefer to stick to the agreement. Therefore, we show that private property of the resource is only sustainable for intermediate values of the resource.

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1Grossman [9] presents a general equilibrium model in which by making an effort people can obtain private property from a common pool of a valuable resource. The resource appropriated is used, together with labor, for production activities.

2Our concept of stability is based on Bloch [3]. For a discussion of stability see also Bolgomon-
aia and Jackson [2], Chwe [5] and Ray and Vohra [14].
We show that the set of parameter values for which the “free access” agreement is unstable (i.e. private property emerges as equilibrium) can increase or decrease with the number of agents in the economy. Therefore, increases in the number of agents are also shown to have a non-monotonic effect.

Our paper is closely related to Umbeck [19] who presents a theoretical investigation of how the initial distribution of property rights can arise starting from a situation of free access. In Umbeck [19] each agent can use time in violence to appropriate land or to collect gold. The marginal rate of substitution between land and labor (in the production of gold) is a measure of how much labor an agent is willing to allocate to maintain the exclusivity of a marginal unit of land. The equilibrium allocation of land is characterized by the equal willingness to fight among the set of agents (and no conflict). In a symmetric model, the equal willingness to allocate labor to conflict implies an equal distribution of land.

Our research is also related to the literature of conflict with coalition formation summarized in Bloch [4], and in the corresponding sections in Garfinkel and Skaperdas [8]. Furthermore, our research is complementary to the literature on the emergence of property rights (see, for instance, Alchian and Demsetz [1], Demsetz [7], de Meza and Gould [6], Grossman [9], and Skaperdas [17]).

The rest of the paper is organized as follows. Section 2 presents the basic model. Section 3 analyzes the formation of property rights in a three agent economy. Section 4 considers coalition formation with endogenous responses. Section 5 presents some results when the process of coalition formation is endogenous. Section 6 analyzes the change in the equilibrium coalition structure when the number of agents changes. Finally, Section 7 presents some conclusions. An Appendix presents most of the relevant proofs.

2. The model. Assume that there is a valuable resource (i.e. a pool of “land”) of size 1 in a n agent economy. Let \( N = \{1, 2, \ldots, n\} \) denote the set of players. Players are identical. Each agent \( i \) has a stock of time of size 1 that can be used either in production or in appropriative activities. Agents can participate individually or collectively in these activities.

A coalition structure \( \pi = [\{A_1, A_2, \ldots, A_k\}] \) is a partition of the set \( N \). In other words, in a coalition structure each \( A_m \subseteq N \), \( A_m \cap A_q = \emptyset \) and the union of all these coalitions \( \bigcup_{m=1}^k A_m \) is equal to the set \( N \).

Since all players are assumed to be identical, payoffs for each player are dependent on the group size rather than on the specific players that are in the group (i.e. the game is symmetric in the sense of Bloch [3]. Let \( a_m = |A_m| \) be the cardinality of the coalition \( m \), that also denotes the size of that alliance. Therefore we can characterize a coalition structure by \( \pi = \{\{a_1, a_2, \ldots, a_k\}\} \) which only depends on the number of members that are in each alliance\(^3\). From now on, \( i \) denotes the player and \( m \) the alliance \( a_m \) to which she belongs.

Agents that participate individually have a group size of \( a_m = 1 \), and agents that participate as members of a group have a group size \( a_m > 1 \). Each agent \( i \) that belongs to the alliance \( a_m \) divides her available time in production \( (l_{im}) \) and appropriative activities \( (e_{im}) \). For each agent, assume that \( 1 = l_{im} + e_{im} \) is satisfied.

\(^3\)Note that with this notation there could be more than one partition of \( N \) that gives the same coalition structure. For example, suppose there are three players, \( N = \{1, 2, 3\} \), two different partition of this set are \( \{\{1, 2\}, \{3\}\} \) and \( \{\{1, 3\}, \{2\}\} \), but both of them have the same coalition structure \( \pi = \{\{2, 1\}\} \).
Then in a coalition \( a_m \) production time is the sum of individual production time for each member \( (L_m = \sum_{i \in A_m} l_{im}) \). Analogously, appropriative effort of a given coalition is the sum of individual effort within the coalition \( (E_m = \sum_{i \in A_m} e_{im}) \).

We assume the particular following functional form for appropriated land for a given coalition \( m \),

\[
 r_m = \begin{cases} 
 \frac{\sum_{i \in A_m} e_{im}}{(\sum_{m=1}^k \sum_{i \in A_m} e_{im})} & \text{if } \sum_{m=1}^k \sum_{i \in A_m} e_{im} > 0 \\
 0 & \text{otherwise} 
\end{cases}
\]

in which the amount appropriated \( r_m \) depends on the relative coalition effort on appropriative activities. This functional form is a trivial extension of the Grossman's [9] form to an economy with coalitions.\(^4\)

We consider a sequential game of two stages. In the first stage, coalitions are formed. In the second stage, agents decide how much time to spend in appropriative \( e_{im} \) and productive \( l_{im} \) activities, given the coalition structure. The benefit that an alliance gets from productive activities is shared between the alliance members according to a proportional sharing rule \( \frac{l_{im}}{L_m} \).\(^5\) We assume that all agents in a coalition can freely use the common land and they get consumption in function of the labor they supply. Thus the individual utility that agent \( i \in A_m \) obtains is

\[
 U_{im} = \frac{l_{im}}{L_m} r_m L_m^{1-\alpha} 
\]

Here the Cobb-Douglas parameter, \( \alpha \), measures the importance of land relative to time spent on productive activities. As the importance of land increases (relative to productive labor), agents spend relatively more time on appropriation than on productive activities.

Given a particular coalition structure \( \pi \), players maximize their individual utility subject to the time constraint. Therefore each agent solves

\[
 \max_{e_{im}, l_{im} | \pi} \frac{l_{im}}{L_m} \left( \frac{E_m}{\sum_{m=1}^k E_m} \right)^\alpha L_m^{1-\alpha} \quad \text{s.t.} \quad e_{im} + l_{im} = 1 \tag{2}
\]

The marginal rate of substitution obtained from the above maximization problem is\(^6\)

\[
 \frac{L_q - \alpha l_{iq}}{\alpha l_{iq} L_q^{1-r_q} r_q \left( \sum_{m} E_m \right)} = 1 \tag{3}
\]

From the solution of the maximization problem (2) we obtain the following lemma.

**Lemma 2.1.** In equilibrium, players that belong to the same alliance make the same appropriative effort, \( e_{im} \), and supply the same productive labor \( l_{im} \).

**Proof.** See the appendix.

From now on we omit the subindex \( i \) (therefore \( e_{im} = e_m \) and \( l_{im} = l_m \)) since these variables only depend on the alliance the agent belongs to. Lemma 2.1 also implies that \( L_m = a_m l_m \) and \( E_m = a_m e_m \). Then if two different alliances have the same group size, the level of efforts will be the same.

\(^4\)This is a simple form of what the literature knows as a Contest Success Function, and a particular simple form of the one analyzed in Skaperdas [18].

\(^5\)This is the typical assumption when agents exploit a common property resource. See, for instance, Miceli and Lueck [11].

\(^6\)See proof of lemma 2.1 in the appendix.
We can also show that, given a coalition structure, a Nash equilibrium in efforts exists.

**Lemma 2.2.** Given the coalition structure $\pi$, for $0 < \alpha < 1$, there is a Nash equilibrium $e^* = (e^*_1, e^*_2, \ldots, e^*_k)$ and $l^* = (l^*_1, l^*_2, \ldots, l^*_k)$ corresponding to the second stage of the game.

**Proof.** See the appendix.

3. A three agent economy. We fully work the process of coalition formation in a three agent economy in order to illustrate one of the main insights of the model.

### 3.1. Efforts and utilities for each coalition structure

The possible coalition structures are $\{\{1, 1, 1\}, \{\{1, 2\}\},$ and $\{3\}$. The first (degenerate coalition structure) occurs when all the three agents make individual appropriation efforts. The second coalition structure occurs when an agent makes individual appropriation efforts and the other two make collective appropriation efforts. The third coalition structure is the grand coalition that implies a free access agreement between the three agents. In the first and second cases private property arises and the valuable resource is divided in parts from which non coalition members can be excluded.

We start computing utilities corresponding to the grand coalition $\{3\}$. In this case, appropriation efforts are zero for each agent and the resource is shared and exploited among the three agents. Then, given the parameter values, each agent $i$ has a payoff of

$$U_i = U_G = (1/3)^\alpha.$$  (4)

For the coalition structure $\{\{1, 1, 1\}\}$, from Grossman’s [9], all players receive the same payoff

$$U_i = U_d = \left(\frac{1}{3}\right)^\alpha \left(\frac{3(1 - \alpha)}{3 - \alpha}\right)^{1-\alpha}$$  (5)

Consider now the coalition structure $\{\{1, 2\}\}$. The agent remaining a singleton, $s$, decides the appropriative effort by maximizing consumption. The corresponding reaction function solves equation.

$$\frac{dU_s}{de_s} \alpha \left(\frac{l_s}{r_s}\right)^{1-\alpha} \left(\frac{\sum_{j=1}^{2} e_j}{(\sum_{j=1}^{2} e_j)^2}\right) - \left(1 - \alpha\right)(r_s l_s)^{\alpha} = 0$$  (6)

For an agent remaining in a coalition of two (i.e. for $i \in \{2\}$), the first order condition is:

$$\frac{dU_2}{de_2} = \frac{l_2}{L_2} \left[\alpha \left(\frac{L_2}{r_2}\right)^{1-\alpha} \frac{e_s}{(\sum_{j=1}^{2} e_j)^2} - \left(1 - \alpha\right)\left(\frac{L_2}{r_2}\right)^{-\alpha}\right] - \frac{L_2}{L_2} \frac{l_2}{r_2} \frac{1-\alpha}{L_2} = 0$$  (7)

Table 1 presents solutions of the equation system (6) and (7) for explicit values of $\alpha$. Finally, table 2 presents the explicit values of $\alpha$ (first column) and corresponding utility levels for the two different coalitions in the $\{\{2, 1\}\}$ coalition structure (second and third columns), the grand coalition (fourth column), and the degenerated coalition structure (fifth column). We have the following remarks about table 2.

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7The solution from the system of equations for general values of $\alpha$ can be obtained by using Lemma 5.1 presented in section 6.
Table 1. Appropriate Effort Values

| $\alpha$ | $e_2$ | $e_s$ |
|----------|-------|-------|
| 0.1      | 0.026 | 0.052 |
| 0.3      | 0.083 | 0.173 |
| 0.348    | 0.098 | 0.206 |
| 0.4      | 0.114 | 0.244 |
| 0.5      | 0.148 | 0.323 |
| 0.549    | 0.166 | 0.367 |
| 0.6      | 0.184 | 0.414 |
| 0.7      | 0.224 | 0.520 |
| 0.9      | 0.314 | 0.799 |

Table 2. Comparison of Utilities for Different Coalition Structures

| $\alpha$ | $U_2$ | $U_s$ | $U_G$ | $U_d$ |
|----------|-------|-------|-------|-------|
| 0.1      | 0.850 | 0.890 | 0.896 | 0.840 |
| 0.3      | 0.617 | 0.716 | 0.719 | 0.603 |
| 0.348    | 0.572 | 0.682 | 0.682 | 0.559 |
| 0.4      | 0.527 | 0.649 | 0.644 | 0.517 |
| 0.5      | 0.451 | 0.594 | 0.577 | 0.447 |
| 0.549    | 0.418 | 0.572 | 0.547 | 0.418 |
| 0.6      | 0.387 | 0.551 | 0.517 | 0.392 |
| 0.7      | 0.332 | 0.519 | 0.464 | 0.350 |
| 0.9      | 0.247 | 0.505 | 0.372 | 0.306 |

**Remark 1.** A particularly interesting value $\alpha = 0.348$ is obtained as the solution of the system of three equations (6), (7) and $U_G = U_s$ in $\alpha$. It is easy to show (see table 2) that for numerical values $\alpha < 0.348$, $U_G$ is larger than $U_s$. However for $\alpha > 0.348$ this inequality is reversed. The immediate consequence is that for low values of $\alpha$, it does not pay for agent $s$ to deviate from the grand coalition.

**Remark 2.** Another interesting value is $\alpha = 0.549$ which is obtained as the solution of the system of three equations (6), (7) and $U_2 = U_d$ in $\alpha$. It is easy to show (see table 2) that for $\alpha$ less than 0.549, $U_2 > U_d$, and that for $\alpha$ greater than 0.549, $U_2 < U_d$. Therefore, for large enough $\alpha$, doing private appropriation efforts is better than doing efforts in a coalition of two, when an agent deviates from the free access agreement.

**Remark 3.** A trivial observation is that, for each agent, the grand coalition structure is always better than the degenerate coalition structure (i.e. $U_G > U_d$) for every value of $\alpha$.

3.2. The sequential equilibrium. We find the equilibrium coalition structures as the result of a game of sequential coalition formation. Following Bloch [3], in a symmetric game, a perfect equilibrium coalition structure can be reached as the outcome of a finite game of choice of coalition sizes. In the Bloch’s game, an exogenous protocol sets an order in which agents propose coalition sizes. The

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8Following a bargaining protocol as proposed by Bloch [3] and Ray and Vohra [13].
initiator proposes a coalition size. All the prospective members of the coalition respond in turn to the offer. If all the agents accept the offer, the cooperative agreement takes effect and they leave the game. If one of the agents rejects the offer, the proposed coalition is not formed and the agent that rejected the offer becomes the initiator in the next round.

**Remark 4.** Assume that there are three agents in the economy. Private property is sustainable as a perfect equilibrium coalition structure for intermediate values of \( \alpha \) (i.e. \( 0.348 < \alpha < 0.549 \)). For the rest of the values of \( \alpha \) the grand coalition is the only perfect equilibrium coalition structure.

**Proof.** For \( \alpha \) small (i.e. \( \alpha < 0.348 \)) no individual agent has incentives to deviate from the grand coalition as \( U_2 < U_d < U_s < U_G \), so the grand coalition is the only stable coalition structure.

Consider now intermediate values of \( \alpha \) (i.e \( 0.348 < \alpha < 0.549 \)). In the three agent economy the Bloch’s protocol could choose randomly any player. As \( U_s > U_G \), and \( U_2 > U_d \) (so, upon a deviation, the other two players would stick together in a complementary coalition), that player would rationally offer to form a coalition of size one that is accepted and the coalition formed. In a second stage, one of the remaining players offers to form a coalition of size two that is accepted by the other agent (as \( U_2 > U_d \) for the corresponding values of \( \alpha \)) and the coalition is formed. Then the coalition structure \( \{2,1\} \) arises.

Assume now that we are in the region \( \alpha > 0.549 \). In the first stage a player would not offer to form a coalition size of one as she knows that \( U_2 < U_d \) and, upon a deviation from the grand coalition, the rest of players would become singletons (and \( U_G > U_d \)). Therefore the grand coalition is an equilibrium structure for those values of \( \alpha \).

As \( \alpha \) is an index of the value of the resource (the share of the resource in production), the conclusion of this section is that private property is only sustainable as a perfect equilibrium coalition structure for intermediate values of the resource. Unless the resource is sufficiently valuable, agents do not have incentives to deviate from the free access agreement. However, if the value of the resource increases enough, deviations from the free access agreement are too costly in terms of conflict.

4. **Coalition formation with exogenous responses.** As we saw in subsection 3.2, testing stability is difficult as we would need to specify the responses of the other members of a coalition once a member or group of members deviated.

Hart and Kurz [10] present two models of stability\(^9\) that assume two types of responses of members of a coalition once a member deviates. Each model corresponds to a coalitional game and, in each one, stability is based on the strong equilibria concept. The first one is called the \( \gamma \) game and corresponds to the case in which each agent chooses the coalition to which she wants to belong, and a coalition forms if all its members have chosen to form it. The players not belonging to these unanimous consent coalitions become singletons. This means that, if a player leaves a given coalition, the rest of the players become singletons (the coalition breaks). As Hart and Kurz claim, this game is supported by the view of coalitions as the result of an unanimous agreement among all its members to act together. Then, if one of the players leave, the agreement breaks down.

\(^9\)Also analyzed in Bloch [4], for the particular case of contests by coalitions.
The second is called the $\delta$ game and corresponds to the case in which each player chooses the largest set of players she is willing to be associated with in the same coalition. Coalitions are formed among all the players that choose to be in the same coalition. In Hart and Kurz’s words “a coalition corresponds to an equivalence class, with respect to equality of strategies”. This means that if a player leaves a given coalition, the rest of the members form one new coalition. As Hart and Kurz claim, this model is justified specially in large games in which the fact that a player leaves a coalition has no influence in the others agreement to act together.

We characterize stable coalition structures for each of the games proposed by Hart and Kurz.

To analyze the appropriative efforts in the economy we have to compute the reaction functions of all the agents (including the deviating agent and also the agents remaining in the coalition).

**Proposition 1.** Assume that we are in the $\delta$ model (i.e. upon a deviation by one agent from the grand coalition the rest of agents remain in a complementary coalition). There is a finite $\bar{n}$ such that for any $n \geq \bar{n}$ the grand coalition is not stable for any $0 < \alpha < 1$.

**Proof.** See appendix.

This proposition establishes that if we are in the $\delta$ model defined by Hart and Kurz [10] private property would be sustainable for a large enough number of agents.

**Proposition 2.** Assume that we are in the $\gamma$ model (i.e. upon a deviation by one agent from the grand coalition the rest of agents become singletons). The only stable coalition is the grand coalition.

**Proof.** This results from the simple comparison between the individual utility in case all agents form the grand coalition $U_i = (\frac{1}{n})^\alpha$ and the individual utility in the case all agents make individual appropriative efforts $U_i = (\frac{1}{n})^\alpha (\frac{n(1-\alpha)}{n-\alpha})^{1-\alpha}$.

Therefore, if we are in the $\gamma$ model, private property would not be sustainable.

5. Endogenous coalition formation. In the general case with an arbitrary number of agents, closed form solutions for the strategies and utilities associated to each coalition structure are impossible to obtain. This is only possible for particular coalition structures. One of them is a coalition structure that divides the set of agents into two. The other is for symmetric coalition structures.

A closed form solution can be obtained when the alliance structure is $\pi = \{c, s\}$ where $s$ is an integer, $1 \leq s \leq \frac{n}{2}$, and $c + s = n$.

**Lemma 5.1.** The optimal effort level for the problem of $s$ players in the coalition structure $\pi = \{c, s\}$ is given by

$$e_s = \frac{2}{3} \sqrt{f(6 + f)} \cos \left(\frac{\theta}{3} + \frac{4\pi}{3}\right) - \frac{f}{3}$$

where $f(n, s, \alpha) = \frac{\cot \theta(n, s, \alpha)}{(n-\alpha)(s-\alpha)}$ and $\theta(n, s, \alpha) = \cos^{-1}\left(-\frac{1}{2} \frac{f(18f+27s+2f^2)}{\sqrt{(f(6+f))^3}} \right)$.

**Proof.** See the appendix.

A symmetric coalition structure is $\pi = \{a, a, \ldots, a\}$ where $a$ is repeated $k$ times and $a = \frac{n}{k}$ with $\frac{n}{k}$ an integer. For symmetric coalition structures it is also possible to compute closed form solutions for arbitrary number of agents.
Lemma 5.2. In a symmetric coalition structure every player obtains a payoff given by
\[
U_{i \in A_m} = \frac{1}{a} \left( \frac{1}{k} \right)^{\alpha} \left( \frac{ak(a - \alpha)}{ak - \alpha} \right)^{1-\alpha}.
\]

Proof. See the appendix. □

A symmetric coalition structure cannot be an equilibrium of the game of sequential coalition formation.

Proposition 3. A symmetric coalition structure is strictly dominated by the grand coalition.

Proof. This result comes from the observation that \( ak = n \). Therefore
\[
U_{i \in A_m} = \frac{1}{a} \left( \frac{1}{k} \right)^{\alpha} \left( \frac{ak(a - \alpha)}{ak - \alpha} \right)^{1-\alpha}
\]
can be written as
\[
U_{i \in A_m} = \left( \frac{1}{n} \right)^{\alpha} \left( \frac{n(a - \alpha)}{a(n - \alpha)} \right)^{1-\alpha}
\]
and \( \left( \frac{n(a - \alpha)}{a(n - \alpha)} \right) < 1 \). Therefore \( U_{i \in A_m} < \left( \frac{1}{n} \right)^{\alpha} \). □

6. The role of changing the number of players. The closed form computation of coalitional equilibria with arbitrary number of agents is impossible as the number of coalitions to be considered is also arbitrary. We analyze the role of changing the number of agents by computing equilibrium coalition structures for economies with different number of agents. We only detail the computation of equilibria for the four and five agents cases. The six, seven and eight agent cases are solved similarly.

6.1. Four and five agent example. In the four agent example the possible coalition structures are \( \{\{1, 1, 1, 1\}\}, \{\{1, 1, 2\}\}, \{\{2, 2\}\}, \{\{1, 3\}\} \) and \( \{\{4\}\} \). Without loss of generality we will assume that a Bloch’s protocol sets and order \( (a, b, c, d) \) among agents and, to simplify our exposition, we denote the possible coalition structures consequently (this is, in fact, the notation used by Sanchez Pages [16]). Therefore we will refer, from now on, to the coalition structures \( \{\{a, b, c, d\}\}, \{\{a, b, d\}\}, \{\{ab, cd\}\}, \{\{a, b, cd\}\} \) and \( \{\{ab, cd\}\} \). We omit the coalition structures that are associated with the same payoff. In the first stage of the game players compare five possible outcomes. Table 3 shows the utility levels for each coalition structure at different levels of \( \alpha \).
Remark 5. Consider a four agent economy. Private property is sustainable as a perfect equilibrium solution for intermediate values of \( \alpha \) (i.e \( 0.074 \leq \alpha \leq 0.493 \)). For the rest of the values of \( \alpha \) the grand coalition is the only perfect equilibrium coalition structure.

Proof. First notice that player \( a \) would never propose to form a coalition of size two or three. Thus, in the following analysis we do not consider these strategies.

If \( \alpha \leq 0.073 \), player \( a \) proposes to form the grand coalition and all players receive the same payoff. If she deviates and chooses to form a singleton, then player \( b \) forms a singleton also. Player \( c \) offers a two-member coalition to player \( d \) which is accepted because the payoffs for the two-member coalition in the coalition structure \( \{a, b, cd\} \) are bigger than those corresponding to the degenerated coalition structure. However, in the coalition structure in \( \{a, b, cd\} \), player \( a \) receives \( U_{b}^{\{a, b, cd\}} \) which is lower than the payoff corresponding to the grand coalition. Therefore the grand coalition is stable.

If \( 0.074 \leq \alpha \leq 0.492 \), player \( a \) chooses to form a singleton. Player \( b \) proposes a three-member coalition to players \( c \) and \( d \), which is accepted. Notice that Player \( b \) never proposes a two-member coalition because the resulting payoff is lower than the corresponding to the three-member coalition. If player \( b \) deviates to a one-member coalition, then player \( c \) and \( d \) become singletons, and all players would receive the degenerated coalition structure payoff, which is lower than \( U_{b}^{\{a, b, cd\}} \). Therefore player \( b \) would not deviate (and symmetrically, neither \( c \) nor \( d \) and, as player \( a \) receives her best payoff, she would not deviate either.

If \( 0.493 \leq \alpha \), player \( a \) proposes to form the grand coalition which is accepted by all players. Notice that, for those values of \( \alpha \), the degenerated structure dominates coalitions with three and two members if a singleton is formed. Therefore, if player \( a \) deviates and chooses to form a singleton, then players \( b \) to \( d \) would do the same, and all of them would receive the payoff corresponding to the degenerated coalition structure. Hence player \( a \) does not deviate in the first place.

The five agent case is analyzed similarly. Table 4 shows the utility levels for each coalition structure for different levels of \( \alpha \). There are only four equilibrium coalition structures.

Remark 6. Consider a five agent economy. Private property is sustainable as a perfect equilibrium solution for intermediate values of \( \alpha \) (i.e \( 0.081 \leq \alpha \leq 0.469 \)). If \( \alpha \in (0.469, 0.566) \), the perfect equilibrium coalition structure is \( \pi = \{ab, cde\} \). Finally, for the rest of values of \( \alpha \) the grand coalition is the perfect equilibrium coalition structure.

Proof. If \( \alpha \leq 0.081 \), player \( a \) does not have incentives to deviate from the grand coalition. Player \( a \) never deviates and chooses a coalition of size two, three, or four,
because the grand coalition gives a higher payoff. If player $a$ deviates and chooses a coalition of size one, then player $b$ and $d$ would offer to form a coalition of size two which is accepted by players $c$ and $e$, respectively, because $U_b^{\{a, b, c, d, e\}} > U_b^{\{a, b, c, d\}}$ and $U_b^{\{a, b, c, d\}} > U_b^{\{a, b, c, d\}}$. Hence player $a$ obtains a payoff of $U_b^{\{a, b, c, d, e\}}$ which is worse than $U_b^{\{a, b, c, d\}}$. Player $b$ does not choose a coalition of size one because she knows that the other three players would choose to form singletons in that case. She never proposes a three or four member coalition because this gives a lower payoff than the grand coalition.

If $0.081 < \alpha \leq 0.469$, then player $a$ prefers to form a singleton, and player $b$ proposes a four member coalition which is accepted. If player $b$ deviates and proposes a one member coalition, then the other three players would choose to form singletons, and all players would obtain the degenerated payoff which is worse than $U_b^{\{a, b, c, d, e\}}$. If she proposes a two member coalition to player $c$, then the proposal is rejected because players $d$ and $e$ would choose to form singletons, as $U_d^{\{a, b, c, d, e\}} > U_d^{\{a, b, c, d, e\}}$, and player $b$ and $c$ would receive a lower payoff than $U_b^{\{a, b, c, d, e\}}$. She never proposes to form a three member coalition because, in that case, her payoff would be worse than the corresponding to the degenerated coalition structure. Player $a$ does not have incentives to deviate as $U_a^{\{a, b, c, d, e\}}$ is the best payoff for her.

If $0.469 < \alpha \leq 0.566$, player $a$ proposes a two member alliance to player $b$ which is accepted, and player $c$ proposes to form a three member coalition which is accepted by players $d$ and $e$. If player $c$ deviates and chooses to form a singleton then players $d$ and $e$ prefer to be singletons, and they obtain $U_e^{\{a, b, c, d\}} < U_e^{\{a, b, c, d\}}$. This player never proposes a two member coalition because it gives a lower payoff, given that $U_e^{\{a, b, c, d\}} < U_e^{\{a, b, c, d\}}$. Player $a$ never chooses to form a coalition of three, four or five members, because it gives lower payoffs in any structure that it is formed. If she deviates and chooses to form a singleton, then the remaining players become singletons too, and the degenerate structure is formed. That coalition structure gives worse payoffs than $\pi = \{\{a, b, c, d, e\}\}$. In the case that player $a$ chooses to form a singleton, player $b$ never chooses a two, three or four members coalition because, if it is accepted, the remaining players would form singletons and then they would obtain lower payoffs. The same argument applies for players $c$ and $d$.

If $0.566 < \alpha \leq 1$, the grand coalition is formed again. Player $a$ never deviates and proposes to form a coalition of three or four members because it would result in a lower payoff than the corresponding to the grand coalition. If she deviates and chooses to form a singleton, then the rest of the players would have incentives to form singletons too, and they would obtain the payoff corresponding to the degenerate coalition structure, which is lower than the payoff corresponding to the grand coalition. If she deviates and proposes to form a two member coalition, that offer would be rejected, because, if it were accepted, the remaining players would prefer to form singletons, since $U_c^{\{a, b, c, d, e\}} > U_c^{\{a, b, c, d, e\}} > U_c^{\{a, b, c, d, e\}}$. Players’ strategies are symmetric so the same arguments apply for the rest of the players.

The two remarks above show that private property is sustainable as a perfect equilibrium for intermediate values of $\alpha$. Another conclusion is that, if the number of agents increases, there are new equilibria that neither imply strictly private property nor free access land.

Table 5 shows the $\bar{\alpha}$ values such that for $\alpha \geq \bar{\alpha}$ the grand coalition is a perfect equilibrium structure for economies with different number of agents. We calculate
Table 5. Values of $\bar{\alpha}$ such that for $\alpha \geq \bar{\alpha}$, the Grand Coalition is a Perfect Equilibrium in a Sequential Coalition Formation Game

| n  | $\bar{\alpha}$ |
|----|----------------|
| 3  | 0.550          |
| 4  | 0.580          |
| 5  | 0.566          |
| 6  | 0.568          |
| 7  | 0.558          |
| 8  | 0.540          |

the perfect equilibrium for the $n = 3$ to $n = 8$ economies as described previously. In the table we can see that the value of $\bar{\alpha}$ is not always monotonic in the number of agents.

7. Final remarks. This paper analyzes the conditions under which private property arises in equilibrium employing the model of Grossman [9]. It is shown that for private property to arise as a coalitional equilibrium, the resource has to be valuable enough to provide incentives for agents to engage in private appropriation efforts on the resource. However, if the resource is too valuable, too many agents will be engaged in appropriation efforts and too much effort in conflict is wasted in equilibrium. This high cost in terms of conflict implies that it is not worthwhile for agents to attain private property rights on the resource.

Our results would imply that, as in Demsetz [7], increases in the value of land can lead to the emergence of private property. However, if land value increases too much then the emergence of private property is via too much conflict. The loss of resources can be large enough and could make private property not desirable for any of the agents. The implication is that the creation of private property, apart from private gains, may also require the existence of institutions that reduce the amount of conflict. One of such institutions can be a superior authority. Others can be family links between the agents that reduce conflict.

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Appendix.

Proof. Lemma 2.1. The Lagrangian of the problem for player $i$ in alliance $m$ is

$$L(e_{im}, l_{im}, \lambda) = \frac{l_{im}}{L_m} \left( \frac{E_m}{\sum_{m=1}^{k} E_m} \right)^{\alpha} L_m^{1-\alpha} - \lambda(1 - e_{im} - l_{im}),$$

the first order conditions of this problem are

$$\frac{\alpha l_{im}}{L_m} \frac{1}{\sum_{m=1}^{k} E_m} \left( \sum_{-m}^{k} r_m \right) r_m^{\alpha-1} L_m^{1-\alpha} = \lambda, \quad (10)$$

and

$$\frac{r_m^{\alpha} L_m^{1-\alpha}}{L_m} (L_m - \alpha l_{im}) = \lambda. \quad (11)$$
Without loss of generality suppose that agents $i$ and $j$ belong to the same alliance, $i, j \in a_m$. In equilibrium, $L_m$ and $r_m$ are the same for $i$ and $j$. Then from (11), it is easy to see that $l_{im} = l_{jm}$. This implies that $e_{im} = e_{jm}$, because of the time constraint.

**Proof. Lemma 2.2.** From (10) and (11), we obtain the marginal rate of substitution

$$\frac{L_q - \alpha l_{iq}}{\alpha l_{iq} L_q \frac{1 - r_q}{r_q} \sum_m E_m} = 1. \quad (12)$$

From (12) and lemma 2.1 we have

$$\frac{(a_m - \alpha)}{\alpha} e_m = \sum_{m=1}^{k} r_{-m}, \quad (13)$$

Using the time constraint $l_m = 1 - e_m$ and $\sum_{m=1}^{k} r_{-m} = 1 - r_m$

$$e_m = \frac{1}{1 + \frac{(a_m - \alpha)^2}{\alpha(1 - r_m)}}. \quad (14)$$

In the above expression, the right hand side is a function that depends on the appropriative effort level through $r_m(e)$. The Contest Success Function (CSF), i.e. $r_m(e)$, is an increasing convex function with respect to $e_m$.

Define $f(e) \equiv \frac{\alpha(1 - r_m(e))}{\alpha(1 - r_m(e)) + (a_m - \alpha)}$.

**Claim.** The function $f(e)$ is bounded and twice differentiable with respect to $e_m$.

**Proof.** The CSF is a twice differentiable function, then $f(e)$ is a twice differentiable function with respect to $e_m$. The derivatives are

$$f(e) e_m = \frac{-\alpha r_m (e_m (a_m - \alpha))}{(\alpha(1 - r_m) + (a_m - \alpha))^2} < 0,$$

and

$$f(e) e_m^2 = \frac{-\alpha^2 (a_m - \alpha)^2 r_m (e_m^2)}{(\alpha(1 - r_m) + (a_m - \alpha))^3} > 0.$$

Indeed, $f(e)$ is decreasing and concave. By definition, $a_m \geq 1$ and $r_m(e) \leq 1$, then $f(e) \in [0, 1]$ for any $\alpha \in (0, 1)$.

From above, $e = f(e)$, has a fixed point. Hence $l$ also has a fixed point.

**Proof. Lemma 5.1.** From equation (13), for a coalition structure $\pi\{c, s\}$ the fraction of land that each coalition obtains is

$$r = \frac{s - \alpha e_s}{\alpha} \frac{l_s}{l_s}, \quad (15)$$

and

$$r = \frac{c - \alpha e_c}{\alpha} \frac{l_c}{l_c}, \quad (16)$$

using that $r = \frac{c e_c}{c e_c + e_s}$ and $l_m = 1 - e_m$

$$e_c = \frac{\left(\frac{s - \alpha}{\alpha}\right)}{1 - \frac{c e_s}{\alpha e_s}}. \quad (17)$$
Notice that $0 \leq e_s \leq \frac{\alpha}{s}$. If $e_s > \frac{\alpha}{s}$ then $e_c = 0$ (By assumption $e_i \geq 0$), but this is not an equilibrium. Analogously $e_c < \frac{\alpha}{s}$. From equation (17) and $r_c + r_s = 1$, the following polynomial is obtained,

$$e_s^3 + f(n, s, \alpha)e_s^2 - 2f(n, s, \alpha)e_s + f(n, s, \alpha)h(s, \alpha) = 0,$$

where $f(n, s, \alpha) \equiv \frac{c\alpha}{(n-\alpha)(s-\alpha)}$ and $h(s, \alpha) \equiv \frac{\alpha}{s}$. By Descartes' rule, this polynomial has one negative real root and two or none positive real roots. Evaluating the polynomial at $e_s = \frac{s}{2\alpha}$, $e_s = \frac{s}{\alpha}$ and $e_s = 1$, it has three real roots. One of them in the interval $[\frac{s}{2\alpha}, \frac{s}{\alpha}]$, and the other in the interval $[\frac{s}{\alpha}, 1]$.

Define,

$$Q \equiv -f^6 + f^3$$

and

$$R \equiv -f^{18} + 27h + 2f^2$$

Let $D \equiv Q^3 + R^2$ be the discriminant. If $D < 0$ all roots are real and unequal. In this case the discriminant is,

$$D = \left(\frac{c\alpha}{(n-\alpha)(s-\alpha)}\right)^2 \frac{s^3 \alpha}{3^3} \left[\frac{3^3 \alpha}{4} \frac{n-s(8n-9\alpha)}{(n-\alpha)^2}\right]$$

Claim. $D$ is always negative.

Proof. For any $\alpha$, $n \geq 3$ and $s \geq 1$, we only need to show that

$$\frac{27\alpha}{4} \frac{s^3 \alpha}{3^3} < \frac{(n-s)(8n-9\alpha)}{(n-\alpha)^2},$$

the left hand side (LHS) and the right hand side (RHS) are increasing on $\alpha$. Moreover the second derivative on the RHS is

$$\frac{\partial^2}{\partial \alpha^2} \frac{(n-s)(8n-9\alpha)}{(n-\alpha)^2} = \frac{6(n-s)(2n-3\alpha)}{(n-\alpha)^4} > 0.$$

The above expression implies that the RHS increases faster than the LHS. At $\alpha = 0$ the RHS is greater than the LHS. Therefore, the RHS is always greater than the LHS for any $\alpha \in (0, 1)$. 

In conclusion, there are three real roots,

$$e_{s1} = 2\sqrt{-Q\cos \left(\frac{\theta}{3}\right)} - \frac{f}{3},$$

$$e_{s2} = 2\sqrt{-Q\cos \left(\frac{\theta + 2\pi}{3}\right)} - \frac{f}{3},$$

and

$$e_{s3} = 2\sqrt{-Q\cos \left(\frac{\theta + 4\pi}{3}\right)} - \frac{f}{3},$$

Where $\theta \equiv \cos^{-1} \left(\frac{R}{\sqrt{-Q^3}}\right)$. In this case, $D < 0$ and $R \leq 0$, then $-1 < \frac{R}{\sqrt{-Q^3}} < 0$. Therefore $\theta \in \left(\frac{\pi}{2}, \pi\right)$, which implies that $\frac{\theta}{3} \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$, $\frac{\theta + 2\pi}{3} \in \left(\frac{5\pi}{6}, \pi\right)$, and $\frac{\theta + 4\pi}{3} \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right)$. From above $e_{s2}$ is the negative root, and $0 \leq e_{s3} \leq e_{s1}$. Therefore $e_{s3}$ is the root in the interval $[\frac{s}{2\alpha}, \frac{s}{\alpha}]$. 

\[\square\]
Proof. Lemma 5.2. From lemma 2.1, players that belong to the same coalition make the same appropriative effort and supply the same productive labor. Therefore, in a symmetric coalition structure every player makes the same effort and obtains the same fraction of land. Then $e_m = e$ and

$$r_m = \frac{1}{k}.$$  

(19)

Using equation (13), the appropriative effort is

$$e = \frac{\alpha(k - 1)}{ak - \alpha}.$$  

(20)

The individual labor supply is $l_m = l = 1 - e$ and the labor supply from one coalition of size $a$ is

$$l_a = \frac{ak(a - \alpha)}{ak - \alpha}. $$  

(21)

Using equations (19), (20) and (21) in the utility function, the equation (9) is obtained.

Proof. Proposition 1. We want to show that the utility that would obtain a singleton (when the rest of agents remain in a complementary coalition), $U_s$, is larger than the one corresponding to the grand coalition, $U_G$ when $n \geq 9$ for any $\alpha$ with $s = 1$ and $c = n - 1$.

Claim. A lower bound for the single deviator utility is $\left(\frac{1}{3}\right)^\alpha (1 - \alpha)^{1-\alpha}$. 

Proof. From lemma 5.1, $e_s \in \left[\frac{\alpha}{2}, \alpha\right)$, and $ce_s < \alpha$, given the coalition structure $\pi = \{\{s, c\}\}$. From equation (1) the fraction of land that the single deviator would obtain is increasing with $e_s$. Therefore $r_s \geq \frac{1}{3}$ and $l_s \geq 1 - \alpha$.

From lemma 5.1, $e_s$ is the solution from the maximization problem (2). Hence $U_s > \left(\frac{1}{3}\right)^\alpha (1 - \alpha)^{1-\alpha}$.

For this lower bound, $\bar{n} = 9$. To see this, notice that $\left(\frac{1}{3}\right)^\alpha (1 - \alpha)^{1-\alpha} > U_G$ and

$$(1 - \alpha)^{\frac{1-\alpha}{\alpha}} > \frac{3}{n}.$$  

The McLaurin series of $(1 - \alpha)^{\frac{1-\alpha}{\alpha}}$ is $\frac{1}{e} + \sum_{p=1}^{\infty} \frac{\beta(n)}{\alpha^{3p}} \frac{(1 - \alpha)^{\frac{1-\alpha}{\alpha}}}{\alpha^{3p}}$. Then $\frac{1}{e} > \frac{3}{n}$, if $n \geq 9$ at any $\alpha$. Hence $U_s > U_G$ at least for $n \geq 9$.

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