Performance of quantum heat engines under the influence of long-range interactions

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We examine a quantum heat engine with an interacting many-body working medium consisting of the long-range Kitaev chain to explore the role of long-range interactions in the performance of quantum engine. By analytically studying two type of thermodynamic cycles, namely Otto cycle and Stirling cycle, we demonstrate that the work output and efficiency of long-range interacting heat engine can be boosted by the long-range interactions, in comparison to the short-range counterpart. We further show that in the Otto cycle, there exist an optimal condition for which the largest enhancement of work output and efficiency can be achieved simultaneously by the long-range interactions. But, for the Stirling cycle, the condition that gives the largest enhancement in work output does not lead to the largest enhancement in efficiency. We also investigate how the parameter regimes under which the engine performance is enhanced by the long-range interactions is evolved with decrease in the range of interaction.

I. INTRODUCTION

Since the seminal work of Scovil and Schulz-Dubois [1], the concept of quantum heat engines has attracted lots of attention [2–11]. In particular, in the last few decades, spurred by the experimental and theoretical progresses in the field of quantum thermodynamics (see e. g. Refs. [12–14] and references therein), a tremendous amount of effort has been devoted to the studies of quantum heat engines, both experimentally [15–22] and theoretically [14, 23–27]. As the quantum versions of classical engines, quantum heat engines exploit quantum systems as working medium to extract work in quantum thermodynamic cycles [5, 9]. A wide range of quantum systems have been used to devise quantum heat engines, including harmonic oscillators [28, 29], spin systems [30–35], photons [4, 36], optomechanical systems [37], Dirac particles [38, 39], and quantum dots [40–42]. A few works have also considered the quantum engines with the heat reservoirs are replaced by quantum systems [24, 43, 44].

Numerous studies in quantum heat engines are focused on the possible enhancement in work output and efficiency via the utilization of quantum properties in working medium [4, 35, 45–47] or heat reservoirs [17, 24, 44]. Other efforts towards understanding of the fundamental differences between classical and quantum heat engines [9, 48, 49], the finite time cycles [50–55], and the applications of shortcuts to adiabaticity [56–63]. The effects of the quantum statistics on the performance of quantum heat engines have also been investigated [64, 65]. Very recently, the results in Ref. [66] further demonstrated that the wave function symmetry has nontrivial impacts on the performance of quantum heat engines.

Most of the aforementioned works are limited to the single-particle working medium. However, with the aim to scale-up quantum heat engine and related thermodynamic devices, it is necessary for us to consider the many-body quantum heat engines. Several recent studies have been reported that the engine performance can be enhanced by various quantum many body effects, such as quantum phase transitions [67–71] and many-body localization [72]. Importantly, the significant role played by interparticle interactions in quantum many-body heat engines has been revealed in Refs. [73–75].

On the other hand, the recent experimental realization of quantum many-body systems with tunable long-range interactions [76, 77] has triggered a surge of interest in the properties of quantum systems with long-range interactions [78–87]. In these systems, the interaction strength between two particles with distance $r$ usually decaying as $1/r^\alpha$ with $\alpha \geq 0$. Due to the long-range interactions, long-range interacting systems can host novel features that are not observed in their short-range counterparts, such as the anomalous dynamical phase [88], quantum time crystals [89], and the breakdown of locality (see, e. g. Refs. [83, 90, 91] and references therein). Give the fact that the long-range interactions are very relevant in experimental platforms that include Rydberg atoms [92], trapped ions [76, 77, 93, 94], polar molecules [95], and magnetic atoms [96]. It is thus interesting to see whether the long-range interactions can be used to enhance the engine performance.

In this work, we explore the enhancement effect of long-range interactions on the performance of a quantum heat engine, which employ the long-range Kitaev chain as its working medium. The long-range Kitaev chain can be considered as an extension of the well-known Kitaev chain [97] with a long-range superconducting pairing term, and has been used as a prototypical model in the studies of long-range interacting systems [98, 99]. By exploiting the integrability of the long-range Kitaev chain, we are able to obtain the explicit expressions for the work output and efficiency of the quantum engine. We will consider two type of thermodynamic cycles, namely, Otto cycle and Stirling cycle, respectively. For both type of cycles, we find that the long-range interactions can display a notable improvement in engine performance in comparison to the case of short-range interactions. We further...
demonstrate how the enhancement regions in the cycle parameter space evolve with the range of interactions. Here, we stress that we only consider the quasisatistc cycles, and leaving the cycles operate in finite time as an interesting topic of future study. Moreover, we focus on the engine performance in terms of work output and efficiency.

The remainder of this article is organized as follows. In Sec. II, we introduce the long-range interacting Kitaev chain and review briefly some of its basic features, we also specifies the exact expressions for the thermodynamic quantities of the long-range Kitaev chain in this section. In Sec. III, we describe two type of thermodynamic cycles (i. e. Otto cycle and Stirling cycle) in which our quantum heat engine operates, we extract the analytical expressions for work output and efficiency of these cycles, respectively. Then, we present our numerical results and discussions in Sec. IV. We finally conclude our study in Sec. V.

II. THE LONG-RANGE KITAEV CHAIN

We consider one-dimensional Kitaev model with long-range pairing interactions, its Hamiltonian reads [98–100]

\[ H = -J \sum_{j=1}^{L} (c_{j}^{\dagger}c_{j+1} + c_{j+1}^{\dagger}c_{j}) - \mu \sum_{j=1}^{L} \left( c_{j}^{\dagger}c_{j} - \frac{1}{2} \right) + \frac{\Delta}{2} \sum_{j=1}^{L} \sum_{\ell=1}^{L-1} \frac{1}{d_{\ell}^{2}} \left( c_{j}c_{j+\ell} + c_{j+\ell}^{\dagger}c_{j}^{\dagger} \right), \]  

(1)

Here, \( c_{j}^{\dagger}(c_{j}) \) are the fermionic creation (annihilation) operators on the \( j \)th site, \( L \) denotes the size of the model, and \( \mu \) is the on-site chemical potential. \( J \) represents the hopping strength of the fermion between nearest neighbor sites, while \( \Delta \) is the strength of the fermion pairing interactions. The power-law decaying pairing term is characterized by exponent \( \alpha \). \( d_{\ell} \) = \( \min(\ell, L - \ell) \) is the effective distance between the \( i \)th and \( (i + \ell) \)th sites. In our study, we consider a close chain with and antiperiodic boundary condition \( c_{j} = -c_{j+L} \) and assume the total number of sites \( L \) is even. Throughout this work, we set \( h = 1 \).

In the short range limit \( \alpha \rightarrow \infty \), the Hamiltonian in Eq. (1) reduces to the well-known short-range Kitaev chain [97], which can be mapped to the quantum transverse field Ising model via the Jordan-Wigner transformation [101]. In this case, by analytically diagonalizing the Hamiltonian, one can find that the model undergoes the topological phase transitions at the critical points \( \mu/J = \pm 1 \). For finite \( \alpha \), the above mentioned mapping does not exist anymore. However, the quadratic form of the Hamiltonian (1) implies that it can still be exactly diagonalized for any finite \( \alpha \).

To this end, we first recast the Hamiltonian (1) in the momentum space via the Fourier transform \( c_{j} = \frac{1}{L} \sum_{k} c_{k} e^{ikj} \), where due to an antiperiodic boundary condition, the lattice momentum \( k = \pm \pi (2n - 1)/L \) with \( n = 1, 2, \ldots, L/2 \). Then the Hamiltonian takes the following block diagonal form [99, 100, 102]

\[ H = \sum_{k} \mathcal{C}_{k}^{\dagger} \mathcal{H}_{k} \mathcal{C}_{k}, \]  

(2)

where \( \mathcal{C}_{k} = (c_{k}^{\dagger}, c_{-k}) \) and

\[ \mathcal{H}_{k} = \begin{pmatrix} -(J \cos k + \mu) & i \Delta f_{\alpha}(k)/2 \\ -i \Delta f_{\alpha}(k)/2 & J \cos k + \mu \end{pmatrix}, \]  

(3)

with the function \( f_{\alpha}(k) = \sum_{\ell=1}^{L} \sin(\ell k)/d_{\ell}^{2} \).

The Hamiltonian (2) can be diagonalized through the Bogoliubov transformation

\[ \begin{pmatrix} c_{k} \\ c_{-k}^{\dagger} \end{pmatrix} = \begin{pmatrix} \cos \theta_{k} & i \sin \theta_{k} \\ i \sin \theta_{k} & \cos \theta_{k} \end{pmatrix} \begin{pmatrix} d_{k}^{\dagger} \\ d_{-k} \end{pmatrix}. \]  

(4)

Here, the Bogoliubov angle \( \theta_{k} \) is defined as

\[ \tan(2\theta_{k}) = -\frac{\Delta f_{\alpha}(k)/2}{J \cos k + \mu}. \]  

(5)

The Hamiltonian (1) is finally diagonalized as

\[ H = \sum_{k} \varepsilon_{k} \left( d_{k}^{\dagger}d_{k} - \frac{1}{2} \right), \]  

(6)

where \( \varepsilon_{k} \) is the quasiparticle energy and given by

\[ \varepsilon_{k} = \sqrt{(J \cos k + \mu)^{2} + [\Delta f_{\alpha}(k)/2]^{2}}. \]  

(7)

For the short-range limit \( \alpha \rightarrow \infty \), we have \( f_{\alpha}(k) = 2 \sin k \) [99, 100]. Then, the quasiparticle energy \( \varepsilon_{k} \) takes

![FIG. 1. Energy spectrum as a function of \( \mu \) of the long-range Kitaev chain (1) with antiperiodic boundary condition for \( \alpha \) = 0.4, (b) \( \alpha \) = 1.7, and (c) \( \alpha \) = 4. Here the system size is \( L = 200 \), \( J = \Delta = 1 \) and \( h = 1 \). (d) Schematic phase diagram of the long-range Kitaev chain in the \( \mu - \alpha \) plan. Different phases are indicated by different colors with corresponding winding number \( w \).](image)
FIG. 2. Schematic diagram of the quantum Otto cycle studied in this work. The thermodynamic cycle operating between a hot bath at temperature $\beta^{-1}_A$ and a cold bath at temperature $\beta^{-1}_B$. It consists of two adiabatic ($A \rightarrow B$ and $C \rightarrow D$) and two isochoric ($B \rightarrow C$ and $D \rightarrow A$) strokes. $\mu_i$ and $\mu_f$ are the chemical potential in the two isochoric strokes. $\rho_c$ and $\rho_d$ are the state of the working medium at each stage of the cycle. At stages $A$ and $C$ the working medium is in the thermal equilibrium with the hot and cold baths, respectively.

the usual short-range form [97]. However, in the thermodynamic limit $L \rightarrow \infty$, the function $f_\alpha^w(x)$ becomes $f_\alpha^w(x) = (-i/2) [\text{Li}_0(e^{ik}) - \text{Li}_0(e^{-ik})]$, with $\text{Li}_0(x) = \sum_{n=1}^{\infty} (x^n/n^n)$ being the $n$th order polylogarithm function of $x$ [100, 102, 103]. It is known that for $\alpha > 1$, $f_\alpha^w(x)$ vanishes at $k = 0$ and $k = \pi$, while when $\alpha < 1$, $f_\alpha^w(x) = 0$ occurs only at $k = \pi$ [99, 100]. Thus, for finite $\alpha$, the spectrum is gapless at $\mu/J = \pm 1$ when $\alpha > 1$, whereas once $\alpha < 1$, we have $\varepsilon_k = 0$ only at $\mu/J = 1$. For brevity but without loss of generality, we set $J = \Delta = 1$ in the following of our study.

For the long-range Kiteav chain with antiperiodic boundary condition, the energy spectrum with respect to the different chemical potential $\mu$ for several values of $\alpha$ are plotted in panels (a)-(c) of Fig. 1. Clearly, regardless of $\alpha$, the energy gap always close to zero at $\mu = 1$. In contrast, the gap at $\mu = -1$ is increased with decreasing $\alpha$. In the thermodynamic limit $L \rightarrow \infty$, one can expect that the energy gap closes at $\mu = \pm 1$ for $\alpha > 1$, while it closes only at $\mu = 1$ when $\alpha < 1$. The close of the energy gap in the spectrum corresponds to the transitions between different topological phases, which are characterized by different winding numbers. The winding number defines as $w = (1/2\pi) \sum_n (\partial_k \theta_k) dk$ with $\theta_k$ is the Bogoliubov angle given by Eq. (5). Fig. 1(d) shows the schematic phase diagram of the long-range Kiteav chain in the $\mu - \alpha$ plan. The topological phases with different winding numbers are discriminated by different color regions. Depending upon the values of $\mu$ and $\alpha$, distinct topological phases with $w = \pm 1/2, 0, 1$ are displayed in the model. It is worth mentioning that the effects of the topological phase transition on the work and efficiency of the quantum heat engine has been explored in Ref. [69, 104]. Here, in our study we mainly focus on the influences of the interaction range on the performance of the quantum heat engine.

Expressions of the thermodynamic quantities

Considering the model is in an equilibrium state at temperature $\beta^{-1}$, here and henceforth we set $k_B = 1$. The state of the model is, therefore, the Gibbs state $\rho = e^{-\beta H}/Z$ with $Z = \text{tr}(e^{-\beta H})$ is the partition function. From the diagonal form of the Hamiltonian in Eq. (6), the partition function can be written as

$$Z = 2^L \prod_{k>0} \cosh^2 \left( \frac{\beta \varepsilon_k}{2} \right).$$

(8)

Then the internal energy $U$ and the free-energy $F$ for the long-range Kiteav chain are, respectively, given by

$$U = \text{tr}(\rho H) = -\frac{\partial \ln Z}{\partial \beta} = -\sum_{k>0} \varepsilon_k \tanh \left( \frac{\beta \varepsilon_k}{2} \right),$$

(9)

$$F = -\frac{\ln Z}{\beta} = -\frac{1}{\beta} \left[ L \ln 2 + \sum_{k>0} 2 \ln \cosh \left( \frac{\beta \varepsilon_k}{2} \right) \right].$$

(10)

With the density of state $\rho$, one can find the entropy $S = -\text{Tr}[\rho \ln \rho]$ has the form

$$S = L \ln 2 + \sum_{k>0} \left[ 2 \ln \cosh \left( \frac{\beta \varepsilon_k}{2} \right) - \beta \varepsilon_k \tanh \left( \frac{\beta \varepsilon_k}{2} \right) \right].$$

(11)

Armed with these results, we turn to investigating the effects of the interacting range on the performance of quantum heat engine.
\[ \alpha = \Delta = 1 \]

in this process. At the end of the stroke, the working ability of each eigenstate of the working medium remains

\[ \text{Otto cycle consists of two isochoric and two adiabatic} \]

\[ \text{cycles, namely, Otto cycle and Stirling cycle.} \]

### III. THERMODYNAMIC CYCLES

In our study, we analyze the performance of a quantum heat engine with the long-range Kitaev chain in Eq. (1) as its working medium. We focus on two type of thermodynamic cycles, namely, Otto cycle and Stirling cycle.

#### A. Quantum Otto cycle

As a widely used cycle in both theoretical and experimental studies of quantum heat engines, the quantum Otto cycle consists of two isochoric and two adiabatic strokes [8, 9], see Fig. 2. In the isochoric branches, the working medium with fixed Hamiltonian is coupled to the hot and cold baths and thermalized with the baths. For the adiabatic strokes of the cycle, the working medium is detached from the hot and cold baths, and undergoes an unitary evolution by changing the the control parameter in the Hamiltonian in an infinitely slow way, so that the quantum adiabatic theorem holds. The details of the four strokes are described as follows.

(i) Adiabatic stroke \( A \to B \) (expansion process): The working medium is initially in the thermal state \( \rho_A = \sum_n P_n(\beta_h) |E_n^A \rangle \langle E_n^A| \) at temperature \( \beta_h^{-1} \) and Hamiltonian \( H_i \), which gives by Eq. (1) with \( \mu = \mu_i \). Here, \( |E_n^A \rangle \) is the nth eigenstate of the Hamiltonian \( H_i \) with corresponding eigenvalue \( E_n^A \), and \( P_n(\beta_h) = e^{-\beta_h E_n^A} / Z_h \) with \( Z_h = \sum_n e^{-\beta_h E_n^A} \). Then we decouple the working medium from the thermal bath and slowly vary the chemical potential from \( \mu_i \) to \( \mu_f \) (\( \mu_f \leq \mu_i \)), such that the quantum adiabatic theorem holds in this process. Hence, during the process the occupation probability of each eigenstate of the working medium remains the same, no heat is transferred. However, the change in the chemical potential \( \mu \) results in the work is done in this process. At the end of the stroke, the working medium reaches the state \( \rho_B = \sum_n P_n(\beta_h) |E_n^B \rangle \langle E_n^B| \), where \( \{ |E_n^B \rangle \} \) are the eigenstates of \( H_f \), which is the Hamiltonian (1) with \( \mu = \mu_f \).

(ii) Isochoric stroke \( B \to C \) (cooling process): In this stroke, the chemical potential is fixed at \( \mu = \mu_f \) and the working medium is brought in contact with the cold bath at temperature \( \beta^{-1} \) until it attains the thermal equilibrium with the cold bath. The state of the working medium at the end of this process is, therefore, given by \( \rho_C = \sum_n P_n(\beta_c) |E_n^C \rangle \langle E_n^C| \) with \( P_n(\beta_c) = e^{-\beta_c E_n^C} / Z_c \) and \( Z_c = \sum_n e^{-\beta_c E_n^C} \). Because the chemical potential is fixed, there is no work has been done in this process. Whereas the heat \( Q_C^h \) is ejected from the working medium to the cold bath.

(iii) Adiabatic stroke \( C \to D \) (compression process): The working medium is detached from the cold bath and the chemical potential is adiabatically driven back from \( \mu_f \) to \( \mu_i \). The occupation probability of each energy levels keeps at the values \( P_n(\beta_c) \). The work is done in this process but no heat transfer. At the end of this stroke, the state of the working medium reads \( \rho_D = \sum_n P_n(\beta_i) |E_n^D \rangle \langle E_n^D| \).

(iv) Isochoric stroke \( D \to A \) (heating process): This is an isochoric heating process, in which the working medium, with chemical potential \( \mu = \mu_i \) and Hamiltonian \( H = H_i \), is attached with the hot bath at temperature \( \beta^{-1} \), relaxing to the thermal state \( \rho_A \). In this process, the fixed chemical potential means that the energy levels are unchanged, no work is done. But the change in the occupation probability of each eigenstate leads to the working medium absorbs heat \( Q_A^h \) from the hot bath.

As no work is done in the two isochoric strokes of the cycle, the heat transfer between the working medium and the heat bath in the isochoric heating and cooling strokes are equal to the vary of the internal energy during the processes. Therefore, the heat \( Q_A^h \) injected into the working medium in the heating stroke and heat \( Q_C^h \) ejected in the cooling stroke are calculated as \( Q_A^h = \text{Tr}[H_i(\rho_A - \rho_D)] \) and \( Q_C^h = \text{Tr}[H_f(\rho_C - \rho_B)] \) [8, 9, 30]. By using Eq. (9), one can explicitly write \( Q_A^h \) and \( Q_C^h \) as follows.

\[ Q_A^h = \sum_{k > 0} \varepsilon_k^f \left[ \tanh \left( \frac{\beta \varepsilon_k^f}{2} \right) - \tanh \left( \frac{\beta_h \varepsilon_k^f}{2} \right) \right], \]  
\[ Q_C^h = \sum_{k > 0} \varepsilon_k^f \left[ \tanh \left( \frac{\beta_h \varepsilon_k^f}{2} \right) - \tanh \left( \frac{\beta_c \varepsilon_k^f}{2} \right) \right], \]

where \( \varepsilon_k^f \) and \( \varepsilon_k^i \) are given by Eq. (7) with \( \mu \) is replaced by \( \mu_i \) and \( \mu_f \), respectively.

The work is done only in the quantum adiabatic strokes of cycle. From the first law of thermodynamics, the amount of net work performed by the Otto cycle gives as

\[ W_O = Q_A^h + Q_C^h = \sum_{k > 0} (\varepsilon_k^f - \varepsilon_k^i) \left[ \tanh \left( \frac{\beta \varepsilon_k^f}{2} \right) - \tanh \left( \frac{\beta_h \varepsilon_k^f}{2} \right) \right]. \]
Note that in order to make sure the cycle works as an engine, we must have \( W_0 > 0, Q^n_O > -Q^n_O > 0 \). Then, the efficiency of the engine gives by

\[
\eta_O = \frac{W_0}{Q^n_O}. \tag{15}
\]

It is known that the efficiency of quantum Otto cycle is bounded by Carnot efficiency \( \eta_c = 1 - \beta_h/\beta_c \) \cite{9}.

**B. Quantum Stirling cycle**

In this subsection we consider the quantum Stirling cycle, which has been used to explore the quantum criticality impacts on the performance of quantum heat engines \cite{68,69}. The Stirling cycle is composed of two isothermal and two isochemical (i.e., fixed value of chemical potential) processes. As shown in Fig. 3, the details of the cycle are given as follows.

(i) **Isothermal process** \((A \rightarrow B)\): In this process, the working medium is kept in contact with the hot bath at temperature \( \beta^{-1}_c \). The chemical potential is changed from \( \mu_i \) to \( \mu_f \). During this process, the amount of heat, denoted by \( Q_I \), absorbed by the working medium from the hot bath is \( Q_I = (S_f - S_i)/\beta_h \), where \( S_i \) (\( S_f \)) is the entropy of the working medium with \( \mu = \mu_i \) (\( \mu = \mu_f \)) at temperature \( \beta^{-1}_h \). From the expression of entropy in Eq. (11), \( Q_I \) can be calculated as

\[
Q_I = \sum_{k>0} \left\{ \frac{2}{\beta_h} \left[ \ln \cosh \left( \frac{\beta_h \epsilon_k^f}{2} \right) - \ln \cosh \left( \frac{\beta_h \epsilon_k^i}{2} \right) \right] - \epsilon_k^f \tanh \left( \frac{\beta_h \epsilon_k^f}{2} \right) - \epsilon_k^i \tanh \left( \frac{\beta_h \epsilon_k^i}{2} \right) \right\}. \tag{16}
\]

Here, \( \epsilon_k^i (\epsilon_k^f) \) is defined by Eq. (7) with \( \mu = \mu_i \) (\( \mu = \mu_f \)).

(ii) **Isochemical potential process** \((B \rightarrow C)\): The chemical potential is fixed at \( \mu = \mu_f \). The temperature of the working medium is decreased from \( \beta^{-1}_h \) to \( \beta^{-1}_c \).

Because the chemical potential is fixed, the energy levels of the working medium remain invariant. Hence, no work is done during this process, but heat \( Q_{II} = U_f(\beta_c) - U_f(\beta_h) < 0 \) is released to the reservoir. Here, \( U_f(\beta_c) [U_f(\beta_h)] \) is the internal energy of the working medium at temperature \( \beta^{-1}_c (\beta^{-1}_h) \) with \( \mu = \mu_f \). By employing Eq. (9), \( Q_{II} \) can be expressed as

\[
Q_{II} = \sum_{k>0} \epsilon_k^f \left[ \frac{\tanh \left( \frac{\beta_h \epsilon_k^f}{2} \right) - \tanh \left( \frac{\beta_c \epsilon_k^f}{2} \right)}{\beta_c} \right]. \tag{17}
\]

(iii) **Isothermal process** \((C \rightarrow D)\): The working medium undergoes another isothermal process which restores the chemical potential to the value \( \mu_i \) from \( \mu_f \), while keeping the working medium in contact with the cold bath at temperature \( \beta^{-1}_c \). In this process, the working medium ejects heat \( Q_{III} = (S_i - S_f)/\beta_c < 0 \), which can be explicitly written as

\[
Q_{III} = \sum_{k>0} \left\{ \frac{2}{\beta_c} \left[ \ln \cosh \left( \frac{\beta_c \epsilon_k^i}{2} \right) - \ln \cosh \left( \frac{\beta_c \epsilon_k^f}{2} \right) \right] - \epsilon_k^i \tanh \left( \frac{\beta_c \epsilon_k^i}{2} \right) - \epsilon_k^f \tanh \left( \frac{\beta_c \epsilon_k^f}{2} \right) \right\}. \tag{18}
\]

(iv) **Isochemical potential process** \((D \rightarrow A)\): As the reverse of the process (II), the last process of the cycle is operated at fixed chemical potential \( \mu_i \), and the temperature of the working medium is increased from \( \beta^{-1}_c \) to \( \beta^{-1}_h \). During this process, the heat absorbed by the working medium is given by \( Q_{IV} = U_i(\beta_h) - U_i(\beta_c) \), where \( U_i(\beta_h) [U_i(\beta_c)] \) denotes the internal energy of the working medium at temperature \( \beta^{-1}_h (\beta^{-1}_c) \) with \( \mu = \mu_i \). Using Eq. (9), \( Q_{IV} \) takes the form

\[
Q_{IV} = \sum_{k>0} \epsilon_k^i \left[ \tanh \left( \frac{\beta_h \epsilon_k^i}{2} \right) - \tanh \left( \frac{\beta_h \epsilon_k^i}{2} \right) \right]. \tag{19}
\]

According to the first law of thermodynamics, the net
work extracted by the quantum Stirling cycle is
\[ W_S = Q_I + Q_{II} + Q_{III} + Q_{IV} \]
\[ = \sum_{k>0} \left\{ \frac{2}{\beta_h} \left[ \ln \cosh \left( \frac{\beta_h \varepsilon_k^I}{2} \right) - \ln \cosh \left( \frac{\beta_h \varepsilon_k^I}{2} \right) \right] \right. \]
\[ + \frac{2}{\beta_c} \left[ \ln \cosh \left( \frac{\beta_c \varepsilon_k^I}{2} \right) - \ln \cosh \left( \frac{\beta_c \varepsilon_k^I}{2} \right) \right] \} \right) \). \]

For each cycle, the amount of heat \( Q_S^h \) absorbed by the working medium gives as
\[ Q_S^h = Q_I + Q_{IV} \]
\[ = \sum_{k>0} \left\{ \frac{2}{\beta_h} \left[ \ln \cosh \left( \frac{\beta_h \varepsilon_k^I}{2} \right) - \ln \cosh \left( \frac{\beta_h \varepsilon_k^I}{2} \right) \right] \right. \]
\[ + \left[ \varepsilon_k^I \tanh \left( \frac{\beta_c \varepsilon_k^I}{2} \right) - \varepsilon_k^I \tanh \left( \frac{\beta_c \varepsilon_k^I}{2} \right) \right] \} \). \]

Therefore, the efficiency of the heat engine reads
\[ \eta_S = \frac{W_S}{Q_S^h}. \]

Here, we stress that in order to extract work from engine we should have \( W_S > 0 \), as in the Otto cycle. Moreover, the efficiency \( \eta_S \) is also limited by the Carnot efficiency \( \eta_c \). [68]

Based on the above results, in the next section, we will investigate the effects of the long-range interactions on the performance of a quantum heat engine, which drives through the quantum Otto and Stirling cycles, respectively.

**IV. NUMERICAL RESULTS AND DISCUSSION**

In our numerical calculation, the chemical potential varies from the initial value \( \mu_i = 2 \) to the final value \( \mu_f \) with \( 0 \leq \mu_f \leq 2 \). To unveil the influences of the long-range interactions on the performance of quantum engine, for both Otto and Stirling cycles, we consider the following ratios
\[ \mathcal{R}_W^\kappa = \frac{W_S}{W_S^\kappa}, \quad \mathcal{R}_\eta^\kappa = \frac{\eta_c}{\eta_c^\kappa}, \]

where \( \kappa = O \) (S) corresponds to the Otto (Stirling) cycle, \( W_S (\eta_c) \) is the work output (efficiency) with finite \( \alpha \), while \( W_S^\infty (\eta_c^\infty) \) represents its short-range (with \( \alpha = \infty \)) counterpart.

The ratios defined in Eq. (23) compare the net work output and efficiency of a long-range interacting heat engine with that of short-range interacting heat engine. If \( \mathcal{R}_W^\kappa = 1 \) and \( \mathcal{R}_\eta^\kappa = 1 \), the long-range interacting heat engine has same performance as its short-range counterpart. Conversely, having \( \mathcal{R}_W^\kappa > 1, \mathcal{R}_\eta^\kappa > 1 \) indicate that there is an enhancement in engine performance induced by the long-range interactions. Finally, \( \mathcal{R}_W^\kappa < 1, \mathcal{R}_\eta^\kappa < 1 \) mean that the quantum heat engine with short-range interacting working medium is more beneficial. In our following study, we will investigate the dependence of these ratios on the interacting range \( \alpha \) for the Otto and Stirling cycles, respectively.

We separately consider the cases in which the quantum engine operators at low- and high-temperature with the ratio of bath inverse temperature \( \beta_h/\beta_c \) is fixed. For the low-temperature case, we take the inverse temperature of the cold bath as \( \beta_c = 5 \), while for the high-temperature case we chose \( \beta_c = 0.05 \). Moreover, we restrict our study for \( \alpha > 1 \), where a transition between long- and short-ranges occurs.

Before we present our results for specific thermodynamic cycle, we first illustrate the notable differences in the behavior of quasiparticle energy \( \varepsilon_k \) [cf. Eq. (7)] arising from the long-range interactions. Fig. 4 depicts the quasiparticle energy as a function of momentum \( k \) and \( \mu_f/\mu_i \) for several \( \alpha \). By comparing with the short-range interactions case [panel (d)], one can see that the long-range interactions have strong impacts on the behavior of quasiparticle energy. As the work output and efficiency of a heat engine involve the sum of quasiparticle energy, the
differences in the quasiparticle energy from the interaction range allow us to expect that long-range interactions should affect the performance of the quantum engine.

A. Results for quantum Otto cycle

By using the results outline in Sec. III A, we are able to calculate the work output $W_O$ and efficiency $\eta_O$ of Otto cycle for different interacting ranges and cycle parameters. We first focus on the behavior of work output ratio $R^O_W$ for several ranges of interaction.

In Fig. 5, we plot $R^O_W$ as a function of $\mu_f/\mu_i$ for different values of $\alpha$ at $\beta_c = 5$ in panel (a) and $\beta_c = 0.05$ in panel (b). We first note that for some chemical potential regimes the value of $R^O_W$ can be greater than 1 and increases with decreasing $\alpha$. This means that the long-range interactions in Kitaev chain can enhance the performance of the Otto engine. Specifically, for the engine operators at low-temperature with $\beta_c = 5$, the work output will get a significant enhancement with increasing (decreasing) interaction range ($\alpha$) as long as $\mu_f/\mu_i > 0.5$, as shown in Fig. 5(a). On the contrary, the work output is enhanced by the long-range interactions only when $\mu_f/\mu_i < 0.5$ for the high-temperature case with $\beta_c = 0.05$ (see panel (b) in Fig. 5).

We further observe a remarkable decrease in $R^O_W$ around $\mu_f/\mu_i = 0.5$, which corresponds to the critical point of the system. In particular, for the low-temperature case, $R^O_W$ shows a significantly larger decrease in the case of long-range interactions. It is known that the work output takes its local minimum value at the critical point of topology phase transition [69]. The dramatic drop in $R^O_W$ in the neighborhood of the critical point implies that the long-range interactions have strong negative impacts on the engine performance around the critical point.

The efficiency ratio $R^O_\eta$ for different interaction ranges at low- and high-temperature cases are shown in Fig. 6. For the low-temperature case with $\beta_c = 5$, except the critical point, we see that the efficiency for long-range interactions is always superior to the efficiency for short-range interaction. Around the critical point, we see $R^O_\eta$ behaves in the same pattern as $R^O_W$, an obvious decrease in efficiency ratio is enhanced by the long-range interactions.

When the heat engine operators at high-temperature case with $\beta_c = 0.05$, we observe the enhancement in efficiency over short-range interacting engine at initial control parameter ratios for long-range interacting engine with $\alpha > 1$. However, in this case, the efficiency of long-range interacting engine with $\alpha = 1$ shows a universal decrease as compared to the short-range interacting engine. Again, near the critical point, we see a larger decrease of efficiency ratio in the case of long-range interactions. We stress that the efficiency is always less than Carnot efficiency, regardless of interaction range and/or bath temperatures.

To gain further understanding of the long-range interactions impact on the performance of quantum engine, we investigate the behavior of the maximum work output and efficiency ratios, denoted by $R^O_{W,m}$ and $R^O_{\eta,m}$, as a function of $\alpha$ and $\beta_h/\beta_c$, respectively. In the left column of Fig. 7 we display $R^O_{W,m}$ and $R^O_{\eta,m}$ as a function of $\alpha$ at different values of $\beta_h/\beta_c$ with $\beta_c = 5$. On can immediately see striking similarity between $R^O_{W,m}$ and $R^O_{\eta,m}$. Regardless of the value of $\beta_h/\beta_c$, both of them converge towards unity in the short-range limit $\alpha \rightarrow \infty$, as expected. However, with increasing $\alpha$, the behaviors of $R^O_{W,m}$ and $R^O_{\eta,m}$ at $\beta_h/\beta_c = 0.4$ looks remarkably different from the monotonous decrement seen in other values of $\beta_h/\beta_c$. Specifically, both $R^O_{W,m}$ and $R^O_{\eta,m}$ firstly experience a growth, and then decrease to the unity as $\alpha$ increases for $\beta_h/\beta_c = 0.4$ case. Moreover, the observed $\beta_h/\beta_c$ dependence in the behaviors of $R^O_{W,m}$ and $R^O_{\eta,m}$ imply that the bath temperature has strong impacts on the enhancement ability of long-range interactions. In the right column of Fig. 7, we show $\beta_h/\beta_c$ dependence of $R^O_{W,m}$ and $R^O_{\eta,m}$, respectively, for several values of $\alpha$. We see again the remarkable resemblance between $R^O_{W,m}$ and $R^O_{\eta,m}$, except for the case with $\alpha = 1$. In fact, at $\alpha = 1$, in contrast to the monotonically increasing behavior of $R^O_{W,m}$, $R^O_{\eta,m}$ shows a growth followed by a decrement (right column of Fig. 7). The above features of Fig. 7 indicate that for Otto cycle operating at low-temperature, there exist an optimal condition for which the long-range interactions lead to the largest enhancement in the work output and efficiency at the same time. For the case considered here, it approximately given by $\alpha \approx 1.5$, $\beta_h/\beta_c \approx 0.4$.

For high-temperature case, the $\alpha$ and $\beta_h/\beta_c$ dependences of $R^O_{W,m}$ and $R^O_{\eta,m}$, respectively, are shown in Fig. 8. As seen in the left column of Fig. 8, as the interaction range is decreased, the behavior of $R^O_{W,m}$ is qualitatively similar to $R^O_{\eta,m}$. Both of them are decreased with increasing $\alpha$ and approach unity as $\alpha \rightarrow \infty$, irrespective of the value of $\beta_h/\beta_c$. On the other hand, for different values of $\alpha$, the observed $\beta_h/\beta_c$ dependence of $R^O_{W,m}$ is also very similar to $R^O_{\eta,m}$, as shown in the right column of Fig. 8. The features in Fig. 8 further imply that the above mentioned optimal condition can still be identified in the high-temperature case. For the case studied here, it given by $\alpha = 1$, $\beta_h/\beta_c = 0.4$, different from the low-temperature case. Note that at the optimal condition the enhancement of engine performance in the high-temperature case is significantly larger than the low-temperature case.

It is also of interest to study how the regions of enhancement in parameter space evolve as the range of interaction is changed from long-range to short-range limit. In order to do so we plot, as a function of $\beta_h/\beta_c$ and $\mu_f/\mu_i$, of the regions where the enhancement is absent (dark green) or present (yellow) for several values of $\alpha$ with $\beta_c = 5$ and $\beta_c = 0.05$, respectively. Here, in our study, the enhancement regions are the regions with $R^O_W > 1$ and $R^O_\eta > 1$. The results are shown in Fig. 9.
FIG. 9. The enhancement regions of Otto cycle in parameter space for different values of $\alpha$. Here, the enhancement regions are identified as the regions with $R_{OW}^O > 1$ and $R_{OW}^S > 1$. The yellow regions indicate the enhancement regions in parameter space under which the performance of heat engine is enhanced by long-range interactions. The upper panels are the results for the low-temperature case with $\beta_c = 5$, the bottom panels represent the high-temperature case with $\beta_c = 0.05$. The remain parameters are: $\hbar = k_B = 1$, $\mu_i = 2$, $J = \Delta = 1$, and $L = 2000$.

FIG. 10. The work output ratio $R_{SW}^O$ of Stirling cycle as a function of the ratio of final to initial chemical potentials $\mu_f/\mu_i$ for different values of $\alpha$ with $\beta_c = 5$ (a) and $\beta_c = 0.05$ (b). The efficiency ratio $R_{S\eta}^O$ of Stirling cycle as a function of $\mu_f/\mu_i$ for several $\alpha$ with $\beta_c = 5$ (c) and $\beta_c = 0.05$ (d). The remain parameters are: $h = k_B = 1$, $\mu_i = 2$, $J = \Delta = 1$, and $L = 2000$.

FIG. 11. Left column: The maximum work output ratio $R_{SW,m}^O$ (a) and the maximum efficiency ratio $R_{S\eta,m}^S$ (c) of the Stirling cycle as a function of $\alpha$ for different values of $\beta_h/\beta_c$. Right column: $R_{SW,m}^S$ (b) and $R_{S\eta,m}^S$ (d) as a function of $\beta_h/\beta_c$ for several $\alpha$. The remain parameters are: $h = k_B = 1$, $\beta_c = 5$, $J = \Delta = 1$, and $L = 2000$.

We see that for both low- and high-temperature cases, the enhancement regions firstly undergo an expansion in parameter space, and followed by a quickly shrink with increasing $\alpha$. One can expect that such regions will vanish as $\alpha \to \infty$. Here, it is worth mentioning that since $R_{OW}^O$ and $R_{OW}^S$ are decreased with increasing $\alpha$ and approach 1 as $\alpha \to \infty$. The expansion of the enhancement regions is, therefore, accompanied by the reduction in the enhancement ability of the long-range interactions. We finally note that the size of enhancement regions in the low-temperature case is always larger than the corresponding high-temperature case.
As we focus on the enhancement effect of the long-range interactions, we will only consider the low-temperature case. First, for the low-temperature case with $\beta_c = 5$, both the work output and efficiency of the Stirling cycle are enhanced by the long-range interactions at larger values of $\mu_f/\mu_i$, see panels (a) and (c) of Fig. 10. Moreover, different from the results in the Otto cycle, we see that the maximum values of $R^S_W$ and $R^S_\eta$ occur near the critical point of the topology phase transition, and increased with decreasing $\alpha$. Hence, for the Stirling cycle, the enhancement effect of long-range interactions can be boosted by the topology phase transition.

On the other hand, for the high-temperature case with $\beta_c = 0.05$, a quite remarkable that as shown in panels (b) and (d) of Fig. 10, over the whole parameter region $R^S_W$ and $R^S_\eta$ are always less than 1. Therefore, the long-range interactions are useless to improve the performance of Stirling cycle if the cycle operating at high-temperature. As we focus on the enhancement effect of the long-range interactions, we will only consider the low-temperature case in the following studies of Stirling cycle.

Fig. 11 depicts the maximum work output ratio $R^S_{W,m}$ and the maximum efficiency ratio $R^S_{\eta,m}$ in the Stirling cycle. We see that as $\alpha$ increases, $R^S_{W,m}$ and $R^S_{\eta,m}$ converge towards 1 in a similar way, see panels (a) and (c) in Fig. 11. However, the dependence of $R^S_{W,m}$ on $\beta_h/\beta_c$ is prominent different to the maximum efficiency ratio $R^S_{\eta,m}$ case, as visible in panels (b) and (d) of Fig. 11. Specifically, $R^S_{W,m}$ increases monotonously as $\beta_h/\beta_c$ increases, while $R^S_{\eta,m}$ first increases with $\beta_h/\beta_c$, and finally decreases to 1 at larger values of $\beta_h/\beta_c$. As a consequence, the optimal condition extracted from the $\alpha$ and $\beta_h/\beta_c$ dependencies of $R^S_{W,m}$ in Figs. 11(a) and 11(b) is inconsistent with the optimal condition obtained from results of $R^S_\eta$ [see Figs. 11(c) and 11(b)]. This disagreement means that the largest enhancement in work output and efficiency cannot be achieved simultaneously by the long-range interactions in the Stirling cycle. This is in sharp contrast to the Otto cycle, where the long-range interactions can lead to the largest enhancement in work output and efficiency at the same time.

We finally investigate the evolution of enhancement regions in the parameter space of Stirling cycle with decreasing the range of interactions. As we have done in the Otto cycle, the enhancement regions are identified as the regions with $R^S_W > 1$ and $R^S_\eta > 1$. Fig. 12 plots the enhancement regions of Stirling cycle for several values of $\alpha$. We see that as $\alpha$ increases the enhancement regions firstly experience an expansion, and then shrink in the parameter space, similar to the Otto cycle case. As expect, the enhancement regions will disappear in the short-range limit $\alpha \to \infty$. Moreover, such as in the Otto cycle, due to the values of $R^S_W$ and $R^S_\eta$ decrease with increasing $\alpha$, the expansion of the enhancement regions will accompany with a larger reduction in the enhancement ability of the long-range interactions.

V. CONCLUSIONS

In conclusion, we have examined how the long-range interactions in quantum many-body systems affect the performance of quantum heat engines. To this end, we have introduced a many-body quantum heat engine with a long-range Kitaev chain as its working medium. The integrability nature of the long-range interacting Kitaev chain allows us to study the performance of heat engine through fully analytical results. We have demonstrated that the long-range interactions exhibit remarkable and nontrivial impacts on the performance of the quantum heat engine. We have considered a quantum heat engine operators with different thermodynamic cycles, one is the Otto cycle and the other is Stirling cycle.

For the Otto cycle operating at low temperature, we found that in all studied figures of merit of engine performance, including net work output and efficiency, the long-range interactions enhance the engine performance for certain cycle parameter ranges. This enhancement effect still persists when the engine working at high temperature. However, near the critical point of the topology phase transition, the work output and efficiency decrease dramatically with increasing the range of interaction, irrespective of the bath temperatures. In the case of Stirling cycle at low-temperature, the long-range interactions exhibit the same enhancement effect on the performance of engine. However, different from the Otto cycle, in this case the long-range interactions lead to the maximum enhancement on the performance of Stirling cycle.

\[ \text{FIG. 12. The regions of enhancement in the parameter space of Stirling cycle for different values of } \alpha. \text{ Here, the enhancement regions are defined as the regions with } R^S_W > 1 \text{ and } R^S_\eta > 1. \text{ Yellow and dark green shaded domains indicate the regions with enhancement and no enhancement. The parameters are: } h = k_B = 1, \mu_i = 2, \beta_c = 5, \beta_h/\beta_c = 0.2, J = \Delta = 1, \text{ and } L = 2000. \]
around the critical point of topology phase transition. We further found the universal negative effect of long-range interactions when Stirling cycle operating at high temperature.

For both cycles, we investigated the possibility of the long-range interactions that leads to the largest enhancement of work output and efficiency of the quantum heat engine at the same time. We found that in Otto cycle the largest enhancement in the performance of engine can be achieved simultaneously by the long-range interactions, while this is not true in the case of Stirling cycle. In addition, we also explored the depends of the enhancement regions in the cycle parameter space on the range of interaction and shown that with decrease in range of interaction the enhancement regions first expand and then quickly shrink. Such regions finally vanish as $\alpha \to \infty$.

Our results evidence that the long-range interactions in many-body systems has strong effects on the performance of quantum heat engine. Therefore, our work provides an additional insight into how to improve the engine performance in quantum many-body heat engines [58, 63, 70–75]. We expect our results are still hold true in other long-range interacting systems, such as long-range Ising model. It will be interesting to systematically study how the range of interaction affects the engine performance in different long-range interacting systems. An interesting extension of the present work is to analyze the effects of the long-range interactions on engine performance for finite time cycles. By considering the long-range interacting systems have ben realized in recent experiments [76, 77], we hope that our present results would be able to trigger more experimental efforts to investigate the effects of long-range interactions on quantum heat engines.

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