THE YANG-BAXTER EQUATION AND THOMPSON’S GROUP $F$

FABIENNE CHOURAQI

Abstract. We define non-degenerate involutive partial solutions as an extension of non-degenerate involutive set-theoretical solutions of the quantum Yang-Baxter equation (QYBE). The induced operator is not a classical solution of the QYBE, but either a braiding operator as in conformal field theory. We define the structure inverse monoid of a non-degenerate involutive partial solution and prove that if the partial solution is square-free, then it embeds into the restricted product of a commutative inverse monoid and an inverse symmetric monoid. Furthermore, we show that there is a connection between partial solutions and the Thompson’s group $F$. This raises the question of whether there are further connections between partial solutions and Thompson’s groups in general.

Introduction

The quantum Yang-Baxter equation is an equation in mathematical physics and it lies in the foundation of the theory of quantum groups. One of the fundamental problems is to find all the solutions of this equation. In [31], Drinfeld suggested the study of a particular class of solutions, derived from the so-called set-theoretic solutions. A set-theoretic solution of the Yang-Baxter equation is a pair $(X, r)$, where $X$ is a set and

$$r : X \times X \rightarrow X \times X, \quad r(x, y) = (\sigma_x(y), \gamma_y(x))$$

is a bijective map satisfying $r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$, where $r^{12} = r \times Id_X$ and $r^{23} = Id_X \times r$. A set-theoretic solution $(X, r)$ is said to be non-degenerate if, for every $x \in X$, the maps $\sigma_x, \gamma_x$ are bijections of $X$ and it is said to be involutive if $r^2 = Id_{X \times X}$. Non-degenerate and involutive set-theoretic solutions give rise to solutions of the quantum Yang Baxter.

Indeed, by defining $V$ to be the real vector space spanned by $X$, and $R : V \otimes V \rightarrow V \otimes V$ to be a linear operator induced by $\tau \circ r$, where $\tau$ is the flip map $\tau(x, y) = (y, x)$, $R$ is a linear operator satisfying the equality $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$ in $V \otimes V \otimes V$, that is $R$ is a solution of the quantum Yang-Baxter equation. Non-degenerate and involutive set-theoretic solutions of the quantum Yang-Baxter equation are intensively investigated and they give rise to several algebraic structures associated to them. One of these is the structure group of a solution, $G(X, r)$, which is defined by $G(X, r) = Gp(X \mid x_i x_j = x_k x_l ; r(x_i, x_j) = (x_k, x_l))$ in [32]. The authors prove that, $G(X, r)$, the structure group of a non-degenerate, involutive set-theoretic solution $(X, S)$ embeds into the semidirect product $Z^X \rtimes \text{Sym}_X$, where $\text{Sym}_X$ denotes the symmetric group of $X$ and $Z^X$ is the free abelian group generated by $X$. Moreover they prove that if $X$ is finite, then $G(X, r)$ is a solvable group [32].

1
Another algebraic structure associated to a set-theoretic solution is a monoid of left \( I \)-type, or of right \( I \)-type, defined in [35]. T. Gateva-Ivanova and M. Van den Bergh prove that there is a correspondence between monoids of left \( I \)-type and non-degenerate, involutive set-theoretic solutions. Indeed, they show that a monoid \( M \) is of left \( I \)-type if and only if there exists a non-degenerate, involutive set-theoretic solution \((X, r)\), with \( X \) finite, such that \( M \simeq \text{Mon}(X \mid x_i x_j = x_k x_l ; r(x_i, x_j) = (x_k, x_l))\). Furthermore, they prove that in this case, the structure group of \((X, r)\), \( G(X, r) \), is the group of fractions of \( M \) and it is a Bieberbach group [35 Theorem 1.6]. In [42], E. Jespers and J. Okninski prove that a monoid \( M \) is of left \( I \)-type if and only if \( M \) is of right \( I \)-type. In [14], the authors initiate the study of IYB-groups, a special class of finite solvable groups. For each non-degenerate, involutive set-theoretic solution \((X, r)\), with \( X \) finite, there is a IYB-group, the group generated by the set \( \{\sigma_x \mid x \in X\} \). Furthermore, they raise the question whether every finite solvable group is IYB.

T. Gateva-Ivanova conjectured that every square-free set-theoretic solution is decomposable [34]. In [51], W. Rump proves the conjecture is true for square-free finite solutions and that an extension to infinite solutions is false. He defines the structure of cycle sets, in correspondence with non-degenerate, involutive set-theoretic solutions, and uses it to prove the conjecture. Cycle sets have been studied also in the context of another conjecture of Gateva-Ivanova in [59] and [13, 9].

In [18], we show there is a one-to-one correspondence between non-degenerate, involutive set-theoretic solutions and a particular class of groups, the so-called Garside groups, with a particular presentation. Garside groups have been defined by P. Dehornoy and L. Paris as a generalization in some sense of the braid groups, and the finite-type Artin groups [25]. In [19, 20], with E. Godelle, we deepen our understanding of the connection between these structures (see also [21, 22], [38], [29]).

In [53], W. Rump introduced braces as a generalization of radical rings related with solutions of the Yang-Baxter equation. He proves there is some correspondence between non-degenerate, involutive set-theoretic solutions and left braces. In his subsequent papers [54, 52], he deepened the study of this new structure. In [16], the authors give an equivalent definition of brace, and they prove some properties. In particular, they use braces to solve a problem arised by T. Gateva-Ivanova and P. Cameron in [40]. Braces are intensively studied and the following list of references on the topic is certainly not exhaustive [1, 2, 3, 17, 39, 55].

Roughly, a brace is a triple \((B, +, \cdot)\), where \((B, +)\) is an abelian group, \((B, \cdot)\) is a group, and there is a left-distributivity-like axiom that relates between the two operations in \( B \). For a left brace, this is the following relation: \( a \cdot (b + c) = a \cdot b + a \cdot c - a \), for every \( a, b, c \in B \). Several extensions of the structure of left brace have been defined. L. Guarnieri and L. Vendramin define a skew left brace to be a triple \((B, +, \cdot)\), with both \((B, +)\) and \((B, \cdot)\) groups, and a left-distributivity-like axiom, and they prove that there is some correspondence between skew left braces and non-degenerate set-theoretic solutions that are not necessarily involutive [41]. We refer to [56], [10] and others for more results on skew braces.
In [11], the authors define a left semi-brace to be a triple \((B, +, \cdot)\), with \((B, +)\) a left-cancellative semigroup and \((B, \cdot)\) a group, and a left-distributivity-like axiom, and they prove that there is some correspondence between left semi-braces and left non-degenerate (non-involutive) set-theoretic solutions. In [12], the authors define a left inverse semi-brace to be a triple \((B, +, \cdot)\), with \((B, +)\) a semigroup, \((B, \cdot)\) an inverse semigroup, a left-distributivity-like axiom, and they prove that there is some correspondence between left inverse semi-braces and (degenerate and non-involutive) set-theoretic solutions.

In this paper, we define a partial left brace, to be a triple \((B, \oplus, \cdot)\), where \((B, \oplus)\) is a commutative partial monoid in the sense of \([5]\), \((B, \cdot)\) is an inverse monoid, and the axiom is left-distributivity (whenever defined). This structure is very reminiscent of a left inverse semi-brace, but its motivation is completely different, as it does not correspond to a set-theoretical solution, but to an extension of a set-theoretic solution. Indeed, we consider a pair \((X, r)\), with \(X\) a set and \(r : D \to X \times X, r(x, y) = (\sigma_x(y), \gamma_x(y))\), \(D \subseteq X \times X\), a partial bijection, where \(\sigma_x : D_x \to R_{\sigma_x}, \gamma_y : D_{\gamma_y} \to R_{\gamma_y}\) are maps, and \(D_{\sigma_x}, R_{\sigma_x}, D_{\gamma_y}, R_{\gamma_y} \subseteq X\). We define a partial set-theoretic solution to be such a pair \((X, r)\) that satisfies an extension of the definition of braided, and show that a partial brace is the natural corresponding structure (Theorem 2).

The linear operator induced by a partial set-theoretic solution, \(R\), is defined as \(R : W \to V \otimes V\), where \(W\) is the real vector space spanned by \(X\), \(W\) is a subspace of \(V \otimes V\) and \(R\) satisfies the quantum Yang-Baxter equation in some subspace of \(V \otimes V \otimes V\). Such a kind of operators occur in the context of rational conformal field theory, and string theory. Indeed, if \(V\) is a vector space with a spanning vector for each coupling, the constraints on the allowed interactions or couplings take the form of relations satisfied by two linear operators, \(B\) and \(F\), called the braiding and the fusion operators respectively. Both operators are defined from subspaces of \(V \otimes V\) to \(V \otimes V\), and \(B\) satisfies the Yang-Baxter equation in a subspace of \(V \otimes V \otimes V\). We refer to [10] for more details.

We extend the definitions of non-degenerate and involutive to partial set-theoretic solutions. We define the structure group of a non-degenerate, involutive partial set theoretic solution to be \(G(X, r) = Gp(X \mid xy = \sigma_x(y)\gamma_y(x) ; (x, y) \in D)\). The structure inverse monoid of \((X, r)\) is \(IM(X, r) = Inv(X \mid xy = \sigma_x(y)\gamma_y(x) ; (x, y) \in D)\). We prove the following:

**Theorem 1.** Let \((X, r)\) be a square-free, non-degenerate, involutive partial set-theoretic solution, with structure inverse monoid \(IM(X, r)\). Let \(I_X\) denote the symmetric inverse monoid, that is the set of partial bijections of \(X\) (with respect to composition whenever it is defined). Let \(A\) denote a commutative inverse monoid generated by a set in bijection with \(X\). Then the restricted product \(A \rtimes I_X\) exists and is an inverse semigroup. Moreover, \(IM(X, r)\) embeds in \(A \rtimes I_X\).

We present the relation between partial set-theoretic solutions and partial braces.

**Theorem 2.** Let \((X, r)\) be a non-degenerate involutive partial solution with structure inverse monoid \(IM(X, r)\). Then there exists a partial left brace \((B, \oplus, \cdot)\) and a congruence \(\rho\) on \(IM(X, r)\), such that \(IM(X, r)/\rho\) is isomorphic \((B, \cdot)\).
In 1965, R. Thompson defined three groups, $F$, $T$ and $V$, that are nowadays called the Thompson groups. They were used to construct finitely-presented groups with unsolvable word problem \cite{58}. Thompson proved that $T$ and $V$ are finitely-presented, infinite simple groups. The group $F$ is $FP_\infty$ and it is the first example of a torsion-free infinite dimensional $FP_\infty$ group \cite{7}. The present work arose from an observation of an infinite presentation of the Thompson group $F$, $F \cong \text{Gp}(x_0, x_1, \ldots \mid x_n x_k = x_k x_{n+1}, k < n)$ and we found a surprising connection between this and partial solutions. This raises the question of whether there are further connections between partial solutions and Thompson’s groups in general, and in Section 6, we present some points in this direction.

**Theorem 3.** There exists a square-free, non-degenerate, involutive partial set-theoretic solution, $(X, r)$ such that its structure group $G(X, r)$ is isomorphic to the Thompson group $F$. Furthermore, $(X, r)$ is irretractable, decomposable and it induces a non-degenerate and square-free cycle set $(X, *)$.

The paper is organized as follows. In Section 1, we give some preliminaries on set-theoretic solutions of the Yang-Baxter equation, cycle sets and braces. In Section 2, we give some preliminaries on inverse semigroups and monoids. In Section 3, we give some preliminaries on the Thompson group $F$, and some of its properties. In Section 4, we define partial set-theoretic solutions, we extend the usual definitions of non-degenerate, braided and involutive to this context. We prove some properties of the square-free partial set-theoretic solutions and Theorem 1. In Section 5, we define partial braces and introduce the method of right reversing. This process was developed in the context of Garside monoids and groups and it is an important tool in the proof of Theorem 2. In Section 6, we prove Theorem 3, and at last we make an attempt to compare between the properties of set-theoretic solutions and those of partial set-theoretic solutions. Section 7 is an appendix.

**Acknowledgment.** I am very grateful to Mark Lawson for his great help in learning the domain of inverse semigroups, via his book and via the numerous questions I asked him. I am also grateful to Ben Steinberg for suggesting me the study of the restricted product of a commutative inverse monoid and the symmetric inverse monoid.

1. **Preliminaries on set-theoretic solutions of the quantum Yang-Baxter equation (QYBE)**

1.1. **Definition and properties of set-theoretic solutions of the QYBE.** We refer to \cite{32, 35, 37, 39, 40, 42, 43, 44}.

Let $X$ be a non-empty set. Let $r : X \times X \to X \times X$ be a map and write $r(x, y) = (\sigma_x(y), \gamma_y(x))$, where $\sigma_x, \gamma_x : X \to X$ are functions for all $x, y \in X$. The pair $(X, r)$ is **braided** if $r^{12} r^{23} r^{12} = r^{23} r^{12} r^{23}$, where the map $r^{i+1}$ means $r$ acting on the $i$-th and $(i + 1)$-th components of $X^3$. In this case, we call $(X, r)$ a **set-theoretic solution of the quantum Yang-Baxter equation**, and whenever $X$ is finite, we call $(X, r)$ a **finite set-theoretic solution of the quantum Yang-Baxter equation**. The pair $(X, r)$ is **non-degenerate** if for every $x \in X$, $\sigma_x$ and $\gamma_x$ are bijective and it is **involutive** if $r \circ r = \text{Id}_X$. If $(X, r)$
is a non-degenerate involutive set-theoretic solution, then $r(x, y)$ can be described as $r(x, y) = (\sigma_x(y), \gamma_y(x)) = (\sigma_x(y), \sigma_{\sigma_x(y)}^{-1}(x))$. A set-theoretic solution $(X, r)$ is square-free, if for every $x \in X$, $r(x, x) = (x, x)$. A set-theoretic solution $(X, r)$ is trivial if $\sigma_x = \gamma_x = \text{Id}_X$, for every $x \in X$.

**Lemma 1.1.** $^{[32]}$  
(i) $(X, r)$ is involutive if and only if $\sigma_{\sigma_x(y)}\gamma_y(x) = x$ and $\gamma_{\gamma_x(x)}\sigma_y(y) = y$, $x, y \in X$.  
(ii) $(X, r)$ is braided if and only if, for every $x, y, z \in X$, the following holds:

\[
\begin{align*}
\sigma_x\sigma_y &= \sigma_{\sigma_x(y)}\sigma_{\gamma_y(x)} \\
\gamma_y\gamma_x &= \gamma_{\gamma_y(x)}\gamma_{\sigma_x(y)} \\
\gamma_{\sigma_{\gamma_x(x)}(y)}\sigma_{\gamma_y(x)}(x) &= \sigma_{\gamma_{\sigma_{\gamma_x(x)}(y)}(x)}(\gamma_y(x))
\end{align*}
\]

**Definition 1.2.** Let $(X, r)$ be a non-degenerate involutive set-theoretic solution of the QYBE.

(i) A set $Y \subset X$ is invariant if $r(Y \times Y) \subset Y \times Y$.  
(ii) An invariant subset $Y \subset X$ is non-degenerate if $(Y, r_{|Y\times Y})$ is non-degenerate involutive set-theoretic solution of the QYBE.  
(iii) $(X, r)$ is decomposable if it is a union of two non-empty disjoint non-degenerate invariant subsets. Otherwise, it is called indecomposable.

A very simple class of non-degenerate involutive set-theoretic solutions of the QYBE is the class of permutation solutions. These solutions have the form $r(x, y) = (\sigma(y), \sigma^{-1}(x))$, where the bijections $\sigma_x : X \to X$ are all equal and equal to $\sigma$, the bijections $\gamma_x : X \to X$ are all equal and equal to $\sigma^{-1}$. If $\sigma$ is a cyclic permutation, $(X, r)$ is a cyclic permutation solution. A permutation solution is indecomposable if and only if it is cyclic $^{[32]}$ p.184.

**Definition 1.3.** Let $(X, r)$ be a set-theoretic solution of the QYBE. The structure group of $(X, r)$ is defined by $G(X, r) = \text{Gp}(X \mid xy = \sigma_x(y)\gamma_y(x) \ ; \ x, y \in X)$.

The structure group of the trivial solution is $\mathbb{Z}^X$. Two set-theoretic solutions $(X, r)$ and $(X', r')$ are isomorphic if there is a bijection $\alpha : X \to X'$ such that $(\alpha \times \alpha) \circ r = r' \circ (\alpha \times \alpha)$ $^{[32]}$. If $(X, r)$ and $(X', r')$ are isomorphic, then $G(X, r) \simeq G(X', r')$, with $G(X, r)$ and $G(X', r')$ their respective structure groups. An important characterisation of non-degenerate involutive set-theoretic solutions of the QYBE is presented in the following proposition.

**Proposition 1.4.** $^{[32]}$ p.176-180] Let $(X, r)$ be a non-degenerate involutive set-theoretic solution of the QYBE, defined by $r(x, y) = (\sigma_x(y), \gamma_y(x))$, $x, y \in X$, with structure group $G(X, r)$. Let $\mathbb{Z}^X$ denote the free abelian group with basis $\{t_x \mid x \in X\}$, and $\text{Sym}_X$ denote the symmetric group of $X$. Then

(i) The map $\varphi : G(X, r) \to \text{Sym}_X$, defined by $x \mapsto \sigma_x$, is a homomorphism of groups.  
(ii) The group $\text{Sym}_X$ acts on $\mathbb{Z}^X$.  
(iii) The group $G(X, r)$ acts on $\mathbb{Z}^X$: if $g \in G$, then $g \cdot t_x = t_{\alpha(x)}$, with $\alpha = \varphi(g)$.  

The map \( \pi : G(X, r) \rightarrow \mathbb{Z}^X \) is a bijective 1-cocycle, where \( \pi(x) = t_x \), for \( x \in X \), and 
\[ \pi(gh) = \pi(g) + g \cdot \pi(h), \] 
for \( g, h \in G(X, r) \).

Definition 1.7. A \( \pi \) is a natural induced solution.

There is a monomorphism of groups \( \psi : G(X, r) \rightarrow \mathbb{Z}^X \times \text{Sym}_X : \psi(x) = (t_x, \sigma_x) \), where 
\[ \psi(g) = (\pi(g), \varphi(g)). \]

The group \( G(X, r) \) is isomorphic to a subgroup of \( \mathbb{Z}^X \times \text{Sym}_X \) of the form 
\[ \{ (a, \phi(a)) \mid a \in \mathbb{Z}^X \}, \] 
where \( \phi : \mathbb{Z}^X \rightarrow \text{Sym}_X \), is defined by \( \phi(a) = \varphi(g) \), whenever 
\[ \pi(g) = a. \]

The subgroup of \( \text{Sym}_X \) generated by \( \{ \sigma_x \mid x \in X \} \) is denoted by \( G(X, r) \) and is called a 
IYB group \[ 13, 40]. \]

**Definition 1.5.** The retract relation \( \sim \) on the set \( X \) is defined by \( x \sim y \) if \( \sigma_x = \sigma_y \).

There is a natural induced solution \( \text{Ret}(X, r) = (X/ \sim, r) \), called the the retraction of 
\( (X, r) \), defined by \( r'([x], [y]) = ([\sigma_x(y)], [\gamma_y(x)]) \). A non-degenerate involutive set-theoretic solution \( (X, r) \) is called a multipermutation solution of level \( m \) if \( m \) is the smallest natural number such that the solution \( \text{Ret}^m(X, r) \sim \text{Ret}(\text{Ret}^{m-1}(X, r)) \), for \( k > 1 \). If such an \( m \) exists, \( (X, r) \) is also called retractable, otherwise it is called irretractable.

**Example 1.6.** Let \( X = \{ x_1, x_2, x_3, x_4 \} \), and \( r : X \times X \rightarrow X \times X \) be defined by \( r(x_i, x_j) = (x_{g(i)}, x_{f(j)}) \) where \( g_i \) and \( f_j \) are permutations on \( \{1, 2, 3, 4\} \) as follows: \( g_1 = (2, 3), g_2 = (1, 4), g_3 = (1, 2, 4, 3), g_4 = (1, 3, 4, 2); f_1 = (2, 4), f_2 = (1, 3), f_3 = (1, 4, 3, 2), f_4 = (1, 2, 3, 4). \) Then \( (X, r) \) is an indecomposable, and irretractable solution, with structure group \( G = \text{Gp}(X \mid x_1 x_2 = x_2^2; x_1 x_3 = x_2 x_4; x_2 x_1 = x_3^2; x_2 x_3 = x_3 x_1; x_1 x_4 = x_4 x_2; x_3 x_2 = x_4 x_1) \).

Structure monoids of non-degenerate, involutive set-theoretic solutions of the QYBE are Garside monoids, that satisfy interesting properties \[ 18, 19, 20, 38]. \]

1.2. The Yang-Baxter equation, Cycle sets, and Braces. Gateva-Ivanova conjectured that square-free, non-degenerate involutive set-theoretic solutions of the QYBE are decomposable \[ 44 \]. In \[ 51 \], Rump proved that Gateva-Ivanova’s conjecture is true for finite square-free solutions and not necessarily true whenever the finiteness assumption is removed. His proof is based on cycle sets, a new structure introduced in \[ 51 \].

**Definition 1.7.** A cycle set is a a non-empty set \( X \) with a binary operation \( \star \) that satisfies:

(i) The map \( \tau(x) \), defined by \( \tau(x)(y) = x \star y \) is invertible, for every \( x \in X \).

(ii) \( (x \star y) \star (x \star z) = (y \star x) \star (y \star z) \), for every \( x, y, z \in X \).

A cycle set is non-degenerate if the map \( x \mapsto x \star x \) is bijective, for all \( x \in X \).

A cycle set is square-free if \( x \star x = x \), for all \( x \in X \).

**Theorem 1.8.** \[ 51 \] Prop. 1] There is a bijective correspondence between non-degenerate cycle sets and non-degenerate involutive set-theoretic solutions of the QYBE.

Given \( X \) a non-degenerate cycle set. The pair \( (X, r) \) is a non-degenerate involutive set-theoretic solution of the YBE, with \( r(x, y) = ((\tau(x))^{-1}(y), (\tau(x))^{-1}(y) \star x) \). Given
a non-degenerate involutive set-theoretic solution of the YBE, \((X, r)\), with \(r(x, y) = (\sigma_y(x), \tau_y(x))\), \(X\) is a non-degenerate cycle set with \(\tau(x)(y) = x \star y = \sigma_x^{-1}(y)\).

In [52], Rump introduced braces as a generalization of radical rings related with non-degenerate involutive set-theoretic solutions of the QYBE. In subsequent papers, he developed the theory of this new algebraic structure. In [16], the authors give another equivalent definition of a brace and study its structure. We follow the terminology from [16].

**Definition 1.9.** A left brace is a set \(G\) with two operations, \(+\) and \(\cdot\), such that \((G+)\) is an abelian group, \((G, \cdot)\) is a group and for every \(a, b, c \in G\):

\[
(1.1) \quad a \cdot (b + c) = a \cdot b + a \cdot c - a
\]

The groups \((G, +)\) and \((G, \cdot)\) are called the additive group and the multiplicative group of the brace, respectively.

A right brace is defined similarly, by replacing Equation (1.1) by

\[
(1.2) \quad (a + b) \cdot c + c = a \cdot c + b \cdot c
\]

A two-sided brace is a left and right brace, that is both Equations (1.1) and (1.2) are satisfied. From the definition of a left brace \(G\), it follows that the multiplicative identity of the multiplicative group of \(G\) is equal to the neutral element of the additive group of \(G\). Additionally, for every \(a, b, c \in G\), \(a \cdot (b - c) = a \cdot b - a \cdot c + a\).

2. Preliminaries on Inverse semigroups and monoids

2.1. **Definition and properties of Inverse semigroups and monoids.** We refer to [15], [23], [60]. A regular semigroup is a semigroup \(S\) such that for every element \(s \in S\) there exists at least one element \(s^* \in S\) such that \(sss^* = s\) and \(s^*ss^* = s^*\). The element \(s^*\) is called the inverse of \(s\). An inverse semigroup is a semigroup \(S\) such that for every element \(s \in S\) there exists a unique element \(s^* \in S\) such that \(sss^* = s\) and \(s^*ss^* = s^*\), that is \(S\) is a regular semigroup such that every element in \(S\) has a unique inverse. An inverse monoid \(S\) is an inverse semigroup with \(1\), and if additionally, for every \(s \in S\), \(s^*s = ss^* = 1\), then \(S\) is a group. Another equivalent definition of an inverse semigroup is a regular semigroup in which all the idempotents commute, that is the set \(E(S)\) of idempotents of an inverse semigroup \(S\) is a commutative subsemigroup; it is ordered by \(e \leq f\) if and only if \(ef = e = fe\). The order on \(E(S)\) extends to \(S\) as the so-called natural partial order by putting \(s \leq t\) if \(s = et\) for some idempotent \(e\) (or equivalently \(s = tf\) for some idempotent \(f\)). This is equivalent to \(s = ts^*s\) or \(s = ss^*t\). For an idempotent \(e\), the set \(G_e = \{ s \in S \mid ss^* = s^*s = e \}\) is a group (called the maximal subgroup of \(S\) at \(e\)). Idempotents \(e\) and \(f\) are said to be \(\mathcal{D}\)-equivalent, written \(e \mathcal{D} f\), if there exists \(s \in S\) so that \(e = s^*s\) and \(f = ss^*\). If \(S\) is an inverse semigroup and \(\varphi : S \rightarrow T\) is a homomorphism of semigroups, then the homomorphic image of \(S\) is an inverse semigroup and the property \(\varphi(s^*) = (\varphi(s))^*\) holds for every \(s \in S\).

An important class of inverse monoids is the class of commutative inverse monoids (or semigroups). Each element in a commutative inverse monoid \(A\) generated by a set \(X\) is in one-to-one correspondence with a partial function with finite support \(f : X \rightarrow \mathbb{Z}\),
A partial bijection of a set $X$ is a bijection $f$ between two (non-necessarily proper) subsets of $X$, the domain and the range of $f$ are denoted by $\mathcal{D}_f$, and $\mathcal{R}_f$ respectively. If $f \neq Id$, where $Id$ denotes the identity function on $X$, the domain of $f$ is allowed to be the empty set. The set of all partial bijections of a set $X$, equipped with the operation of composition of functions $\circ$, is an inverse monoid with zero, the zero element being the vacuous map $\emptyset \to \emptyset$. It is called the symmetric inverse monoid and it is denoted by $I_X$.

If $f, g \in I_X$, then $f \circ g$ is the composition of partial maps in the largest domain where it makes sense, that is $\mathcal{D}_{fog} = g^{-1}(\mathcal{D}_f \cap \mathcal{R}_g)$, and $\mathcal{R}_{fog} = f(\mathcal{D}_f \cap \mathcal{R}_g)$. The idempotents of $I_X$ are precisely the partial identities on $X$ [45, p.6]. From the Wagner-Preston Theorem, the analogue of Cayley’s Theorem from group theory, any inverse semigroup $S$ can be embedded into some symmetric inverse semigroup, that is $S$ is isomorphic to the subsemigroup of a symmetric inverse semigroup [50, 60].

**Definition 2.1.** Let $M$ be an inverse monoid. Let $X$ be a set. Let $S$ be an inverse semigroup. We say that $M$ acts on $X$ by partial permutations if there exists a homomorphism of monoids $M \to I_X$.

**Definition 2.2.** [45] Let $S$ be a semigroup. The monoid $M$ is said to act on $S$ (on the left) (by endomorphisms) if there exists a map $M \times S \to S$, denoted by $(a, s) \mapsto a \bullet s$ satisfying the following conditions:

(i) for any $a, b \in M$, $s \in S$, $(ab) \bullet s = a \bullet (b \bullet s)$.

(ii) for any $a \in M$, $s, s' \in S$, $a \bullet (ss') = (a \bullet s)(a \bullet s')$.

(iii) for every $s \in S$, $1 \bullet s = s$.

(iv) for any $a \in M$, $a \bullet 0 = 0$.

This definition is equivalent to the existence of a homomorphism of monoids from $M$ to the monoid of endomorphisms of $S$. This homomorphism maps inverses to inverses and idempotents to idempotents. Let $M$ act on $S$ by endomorphisms. The (classical) semidirect product of $S$ by $M$ is the set $S \times M$ equipped with the product $(s, a)(s', b) = (s(a \bullet s'), ab)$. If $M$ and $S$ are inverse semigroups, their semidirect product is not necessarily an inverse semigroup. This leads to the definition of the $\lambda$-semidirect product and the restricted product of inverse semigroups or monoids.

**Definition 2.3.** [45] p.147] Let $M, S$ be inverse semigroups. Assume $M$ acts on $S$ by endomorphisms. Let $S \times^\lambda M$ be the following set with the binary operation defined below:

$$S \times^\lambda M = \{(s, m) \in S \times M \mid r(m) \bullet s = s\}$$

$$(s, m)(s', m') = ((r(mm') \bullet s)(m \bullet s'), mm')$$
2.2. The free inverse monoid, Inverse monoid presentations. We refer the reader to [15] Chapter 6 for more details.

Definition 2.4. [15] p.155 Let $M$, $S$ be inverse semigroups. Let $E(M)$ denote the set of idempotents of $M$ (ordered by $e \leq f$ if and only if $ef = e = fe$). Assume the following assumptions:

(i) $M$ acts on $S$ by endomorphisms.
(ii) There exists a surjective homomorphism $\epsilon : S \to E(M)$.
(iii) For each $s \in S$, there exists $\epsilon(s) \in E(M)$ such that

$$\epsilon(s) \leq e \iff e \cdot s = s, \forall e \in E(M)$$

Let $S \bowtie M$ be the following set with the binary operation defined below:

$$S \bowtie M = \{(s, m) \in S \times M \mid r(m) = \epsilon(s)\}$$

$$(s, m)(s', m') = (s(m \cdot s'), mm')$$

$S \bowtie M$ is an inverse semigroup. Furthermore, it is an inverse subsemigroup of $S \rtimes^\lambda M$.

2.2. The free inverse monoid, Inverse monoid presentations. We refer the reader to [15] Chapter 6 for more details.

Definition 2.5. [15] p.171 Let $X$ be a non-empty set. An inverse (semigroup) monoid $\text{FIM}(X)$, equipped with a function $i : X \to \text{FIM}(X)$, is said to be a free inverse (semigroup) monoid on $X$ if for every inverse (semigroup) monoid $M$ and function $\kappa : X \to M$ there exists a unique homomorphism $\theta : \text{FIM}(X) \to M$ such that $\theta \circ i = \kappa$.

Free inverse (semigroup) monoids exist, and $\text{FIM}(X)$ is constructed as follows. Let $X^* = \{x^* \mid x \in X\}$, a set in bijection with $X$ and disjoint from $X$. Let $FM$ be the free monoid generated by $X \cup X^*$, and define the following unary operation: first, if $y = x \in X$, then $y^* = x^*$ and if $y = x^* \in X^*$, $y^* = x$; next, if $y_1...y_k \in FM$, then $(y_1...y_k)^* = y_k^*...y_1^*$. This unary operation turns $FM$ into a free monoid with involution.

The free inverse monoid $\text{FIM}(X)$ is then obtained by factoring $FM$ by $\rho'$, the congruence generated by the following binary relation $\rho = \{(u, uu^*u), (uu^*uv^*, vv^*uu^*) \mid u, v \in FM\}$. If $u$ and $v$ belong to the same $\rho'$-class, then we shall say that $u$ and $v$ are equivalent or $u$ and $v$ represent the same element in $\text{FIM}(X)$.

The word problem is solvable in free inverse monoids. There exists an approach to the solution of the word problem in free inverse monoids which is very similar to the one in free groups, that is based on the existence of a unique normal (reduced) form. The existence of free inverse monoids gives the possibility to define the notion of an inverse monoid presentation.

Definition 2.6. An inverse monoid presentation is a pair $(X, \rho)$, where $X$ is a set and $\rho$ is a relation on $\text{FIM}(X)$. The inverse monoid $\text{FIM}(X)/\rho'$, where $\rho'$ is the congruence generated by $\rho$, is said to be presented by the generators $X$ and the relations $\rho$ and is denoted by $IM = \text{Inv}(X \mid \rho)$. If both $X$ and $\rho$ are finite, we say that $IM$ is finitely generated.
3. Preliminaries on Thomson group $F$

In 1965, R. Thompson defined three groups, $F$, $T$ and $V$, that are nowadays called the Thompson groups. They were used to construct finitely-presented groups with unsolvable word problem \cite{58}. Thompson proved that $T$ and $V$ are finitely-presented, infinite simple groups. The group $F$ is $FP_\infty$ and it is the first example of a torsion-free infinite dimensional $FP_\infty$ group \cite{7}. In this paper, we consider the Thompson group $F$. There are several descriptions of the Thompson group $F$, and we present two of them: as the group of dyadic rearrangements (or piecewise-linear homeomorphisms with additional properties), and as the group of tree diagrams with a certain product. We refer to \cite{58}, \cite{8}, \cite{6}, \cite{57}, \cite{4} and many others for more details.

3.1. $F$ as the group of dyadic rearrangements, and tree diagrams. Any subdivision of the interval $[0,1]$ obtained by repeatedly cutting intervals in half is called a dyadic subdivision. The intervals of a dyadic subdivision are all of the form $\left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right]$, $k, n \in \mathbb{N}$. The set of all dyadic rearrangements forms a group under composition, the Thompson group $F$.

**Example 3.1.** To illustrate, take the interval $[0,1]$ and cut in half, like this:

![Figure 3.1. A standard dyadic interval](image)

We then cut each of the resulting intervals in half:

![Figure 3.1. A standard dyadic interval](image)

and then cut some of the new intervals in half to get a certain subdivision of $[0,1]$:

![Figure 3.1. A standard dyadic interval](image)

Given a pair of dyadic subdivisions, $\mathcal{D}$, $\mathcal{R}$, with the same number of cuts, a *dyadic rearrangement of* $[0,1]$ is a piecewise-linear homomorphism $f : [0,1] \to [0,1]$ that sends each interval of $\mathcal{D}$ linearly onto the corresponding interval of $\mathcal{R}$.
To each standard dyadic interval there corresponds a binary tree. The binary tree corresponding to the dyadic interval from Figure 3.1 is described in Figure 3.1.

Any element of the group $F$ can be described by a pair of finite binary trees, called a tree diagram. The two trees are aligned vertically so that corresponding leaves match up. The domain tree appears on the top and the range tree appears on the bottom. It is illustrated in the following figure.
Figure 3.4. The tree diagrams corresponding to $x_0$ and $x_1$ (as described in Fig. 3.1) at left and the tree diagram of $x_2 = x_0x_1^{-1}$ at right.

We do not get into details on the existence of a unique reduced tree diagram for any element of $F$, on the product of tree diagrams and the determination of the presentation of $F$. We refer to the literature on the topic, and in particular to [4] for a very detailed introduction to the topic.

3.2. Presentations of the group $F$ and some of its properties. In the following theorem, we describe two finite presentations of $F$.

**Theorem 3.2.** (i) The elements $x_0$ and $x_1$ generate Thompson’s group $F$ with presentation of the form $\langle x_0, x_1 \mid x_2x_1 = x_1x_3, x_3x_1 = x_1x_4, x_2 = x_0x_1x_0^{-1} \rangle$, where $x_3 = x_0^2x_1x_0^{-2}$, $x_4 = x_3^3x_1x_0^{-3}$ and more generally $x_n = x_0x_{n-1}x_0^{-1} = x_0^{n-1}x_1x_0^{-(n-1)}$, for every $n \geq 2$.

(ii) The elements $x_0$ and $x_1$ generate Thompson’s group $F$ with presentation of the form $\langle x_0, x_1 \mid x_2x_0 = x_0x_3, x_3x_0 = x_0x_4 \rangle$, where $x_2 = x_0^{-1}x_1x_0$ and more generally $x_{n+1} = x_n^{-1}x_0x_{n-1}$, for every $n \geq 2$.

Surprisingly, although it is usually far more convenient to work with a finite presentation, in this paper we work with the infinite presentation of $F$ presented in the following theorem.

**Theorem 3.3.** (i) The elements $\{x_0, x_1, x_2, \ldots\}$ generate Thompson’s group $F$ and $F$ has an infinite presentation of the form $\langle x_0, x_1, x_2, \ldots \mid x_nx_k = x_kx_{n+1}, \ k < n \rangle$.

(ii) Every element of $F$ can be expressed uniquely in the form $x_0^{a_0}x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}x_{n+1} \cdots x_0^{b_0}$, where $a_0, ..., a_n, b_0, ..., b_n \in \mathbb{N}$, exactly one of $a_n, b_n$ is non-zero and if both $a_i$ and $b_i$ are not equal 0, then either $a_{i+1} \neq 0$ or $b_{i+1} \neq 0$ for all $i$.

(iii) Every proper quotient of $F$ is abelian.

There are several questions about Thompson’s group $F$ that are still open. It is still not known whether $F$ is an automatic group. It is known that $F$ is not elementary amenable.
and that it does not contain the free group of rank 2, but it is still unknown whether $F$ is an amenable group.

4. Definition of partial set-theoretic solutions and proof of Theorem 1

4.1. Definition of partial set-theoretic solutions and their properties.

Definition 4.1. Let $X$ be a non-empty set. Let $r : D \to R$ be a map, where $D, R \subseteq X \times X$. We write $r(x, y) = (\sigma_x(y), \gamma_y(x))$, where $\sigma_x : D_{\sigma_x} \to R_{\sigma_x}$ and $\gamma_y : D_{\gamma_y} \to R_{\gamma_y}$, with $D_{\sigma_x}, R_{\sigma_x}, D_{\gamma_y}, R_{\gamma_y} \subseteq X$. With this notation, $(x, y) \in D$ if and only if $y \in D_{\sigma_x}$ and $x \in D_{\gamma_y}$. Let $r^{i,i+1} : X^{i-1} \times D \times X^{k-i-1} \to X^{i-1} \times R \times X^{k-i-1}$ be the maps defined by $r^{i,i+1} = I_dX^{-1} \circ r \circ I_dX^{k-i-1}$.

(i) The pair $(X, r)$ is non-degenerate, if for every $x, y \in X$, $\sigma_x : D_{\sigma_x} \to R_{\sigma_x}$ and $\gamma_y : D_{\gamma_y} \to R_{\gamma_y}$ are bijections of $X$.

(ii) The pair $(X, r)$ is involutive if for all pairs $(x, y) \in X^2$, $x \in D_{\gamma_y}$ if and only if $y \in D_{\sigma_x}$, and additionally if $r(x, y)$ is defined, then $r^2(x, y)$ is also defined and satisfies $r \circ r = I_dX$, that is $r^2(x, y) = (x, y)$.

(iii) The pair $(X, r)$ is braided if for every $x, y, z \in X$, $r^{12}_{r,23}r^{12}_{r,23}(x, y, z) = r^{23}_{r,12}r^{23}_{r,12}(x, y, z)$, whenever $r^{12}_{r,23}r^{12}_{r,23}(x, y, z)$ and $r^{23}_{r,12}r^{23}_{r,12}(x, y, z)$ are defined.

(iv) The pair $(X, r)$ is square-free, if for every $x \in X$, $(x, x) \in D$ and $r(x, x) = (x, x)$.

If $(X, r)$ is braided, we call $(X, r)$ a partial set-theoretic solution. If $(X, r)$ is a non-degenerate, involutive partial set-theoretic solution, we call it a partial solution.

Example 4.2. Let $X = \{x_0, x_1, x_2\}$. Let $r : D \to R$, be defined by $r(x, y) = (\sigma_x(y), \gamma_y(x))$, with $D = R = \{(x_0, x_2), (x_1, x_2), (x_2, x_0), (x_2, x_1), (x_0, x_0), (x_1, x_1), (x_2, x_2)\}$. The functions $\sigma_x$ and $\gamma_x$ are described like permutations, with a specification of their domain of definition. Let $\sigma_0 = (0)(2)$, $D_{\sigma_0} = D_{\gamma_0} = \{0, 2\}$; $\sigma_1 = (1)(2)$, $D_{\sigma_1} = D_{\gamma_1} = \{1, 2\}$; $\sigma_2 = (0, 1)(2)$, $D_{\sigma_2} = D_{\gamma_2} = \{0, 1, 2\} = X$. The functions $\sigma_0, \gamma_0, \sigma_1, \sigma_2$ are bijections of $X$ and $\sigma_2, \sigma_2$ are bijections of $X$. A technical computation shows that $(X, r)$ is a square-free partial solution, with $r(x_0, x_2) = (x_2, x_1)$, $r(x_2, x_1) = (x_0, x_2)$, $r(x_1, x_2) = (x_2, x_0)$, and $r(x_2, x_0) = (x_1, x_2)$.

Lemma 4.1 can be directly extended to partial solutions in the following way:

Lemma 4.3. Let $(x, y), (y, z) \in D$, that is $x \in D_{\gamma_y}$, $y \in D_{\sigma_x} \cap D_{\gamma_z}$, $z \in D_{\sigma_y}$.

(i) $(X, r)$ is involutive if and only if $\gamma_y(x) \in D_{\sigma_x} \cap D_{\gamma_z}$, $\sigma_x(y) \in D_{\gamma_y}$ and additionally

\[ \sigma_{\sigma_x(y)} \gamma_y(x) = x \quad (4.1) \]
\[ \gamma_{\gamma_y(x)} \sigma_x(y) = y \quad (4.2) \]

(ii) If $(X, r)$ is involutive, then $(x, y) \in D$ if and only if $y \in D_{\sigma_x}$ and $x \in D_{\sigma^{-1}_{\sigma_x}(y)}$, and in this case $\sigma_x(y) = \gamma_{\gamma_y(x)}$ and $\gamma_y(x) = \sigma^{-1}_{\sigma_x}(y)$.

(iii) $(X, r)$ is braided if and only if whenever $x \in D_{\gamma_z}$, $z \in D_{\sigma_x}$, $y \in D_{\sigma_{\sigma_y(z)}(x)} \gamma_z$, the following equations hold:
\begin{align*}
(4.3) & \quad \sigma_x \sigma_y = \sigma_{\sigma_x(y)} \sigma_y(x) \\
(4.4) & \quad \gamma_x \gamma_y = \gamma_{\gamma_x(y)} \gamma_y(x) \\
(4.5) & \quad \gamma \sigma_{\gamma_y(x)}(\gamma_x(y)) = \sigma_{\gamma_y(x)}(\gamma_x(y))
\end{align*}

**Definition 4.4.** Let \((X, r)\) be a partial set-theoretic solution. The **structure group** of \((X, r)\) is \(G(X, r) = \text{Gp}(X \mid x \in X, \langle x, r \rangle = (\sigma_x(y), \gamma_y(x)) \mid (x, y) \in \mathcal{D})\). The **structure inverse monoid** of \((X, r)\) is \(\text{IM}(X, r) = \text{Inv}(X \mid x \in X, \langle x, r \rangle = (\sigma_x(y), \gamma_y(x)) \mid (x, y) \in \mathcal{D})\).

A partial solution \((X, r)\) is **trivial** if for every \(x \in X\), \(\sigma_x = \text{Id}_{\mathcal{D}_{\sigma_x}}, \gamma_x = \text{Id}_{\mathcal{D}_{\gamma_x}}\). So, for all pairs \((x, y) \in \mathcal{D}, r(x, y) = (y, x)\), that is the structure group of a trivial partial solution is a partially commutative group (or a right-angled Artin group) with generating set \(X\), and defining relations that depend on \(\mathcal{D}_{\sigma_x}\) and \(\mathcal{D}_{\gamma_x}\).

**Example 4.5.** Let \(X = \{x_0, x_1, x_2\}\). Let \(r: \mathcal{D} \to \mathcal{R}\), be defined by \(r(x, y) = (\sigma_x(y), \gamma_y(x))\), with \(\mathcal{D} = \mathcal{R} = \{(x_0, x_2), (x_1, x_2), (x_2, x_0), (x_0, x_1), (x_1, x_2)\}\). Let \(\sigma_0 = 0, \mathcal{D}_{\sigma_0} = \mathcal{D}_{\gamma_0} = \{0, 2\}; \sigma_1 = (1, 2), \mathcal{D}_{\sigma_1} = \mathcal{D}_{\gamma_1} = \{1, 2\}; \sigma_2 = (0, 1)(2), \mathcal{D}_{\sigma_2} = \mathcal{D}_{\gamma_2} = \{0, 1, 2\}\). So, \(r(x_0, x_2) = (x_2, x_0), r(x_2, x_0) = (x_0, x_2), r(x_1, x_2) = (x_2, x_1), r(x_2, x_1) = (x_1, x_2)\), and \(r(x_i, x_i) = (x_i, x_i)\), for \(0 \leq i \leq 2\), that is \((X, r)\) is a trivial partial solution, with structure group the partially commutative group \(\text{Gp}(x_0, x_1, x_2 \mid x_0x_2 = x_2x_0, x_1x_2 = x_2x_1)\).

Two partial set-theoretic solutions \((X, r)\) and \((X', r')\) are **isomorphic** if there is a partial bijection \(\alpha: X \to X'\) such that if \((x, y) \in \mathcal{D}\), then \((\alpha(x), \alpha(y)) \in \mathcal{D}'\) and \(r(x, y) \in \mathcal{D}_{\alpha}(\mathcal{D}_{\alpha})\), and additionally \((\alpha \times \alpha) \circ r = r' \circ (\alpha \times \alpha)\). If \((X, r)\) and \((X', r')\) are isomorphic, then \(G(X, r) \simeq G(X', r')\), with \(G(X, r)\) and \(G(X', r')\) their respective structure groups.

**Definition 4.6.** Let \((X, r)\) be a non-degenerate involutive partial set-theoretic solution.

1. A set \(Y \subseteq X\) is **invariant** if \(r(Y \times Y) \subseteq Y \times Y\), whenever \(r(Y \times Y)\) is defined.
2. An invariant subset \(Y \subseteq X\) is **non-degenerate** if \((Y, r)\) is a non-degenerate involutive partial set-theoretic solution of the YBE.
3. \((X, r)\) is **decomposable** if it is a union of two non-empty disjoint non-degenerate invariant subsets. Otherwise, it is called **indecomposable**.

The **retract** relation \(\sim\) on the set \(X\) is defined by \(x \sim y\) if \(\mathcal{D}_{\sigma_x} = \mathcal{D}_{\sigma_y}\) and \(\sigma_x = \sigma_y\). There is a natural induced solution \(\text{Ret}(X, r) = (X/ \sim, r), \text{called the the retraction of } (X, r)\), defined by \(r'[\{x\}, [y]] = ([\sigma_x(y)], [\gamma_y(x)])\). A non-degenerate involutive partial set-theoretic solution \((X, r)\) is called a **multipermutation partial solution of level** \(m\) if \(m\) is the smallest natural number such that the solution \(\text{Ret}^m(X, r) = 1\), where \(\text{Ret}^k(X, r) = \text{Ret}(\text{Ret}^{k-1}(X, r))\), for \(k > 1\). If such an \(m\) exists, \((X, r)\) is also called **retractable**, otherwise it is called **irretractable**.

**4.2. Characterization of square-free partial solutions and proof of Theorem**

1. An important characterisation of non-degenerate involutive set-theoretic solutions is
presented in Proposition \textit{[14]} (\textit{[32]} p.176-180). We extend some of the properties of the classical set-theoretic solutions to the partial ones.

From now on, we use the following notation. Let \((X, r)\) be a non-degenerate involutive partial set-theoretic solution, defined by \(r(x, y) = (\sigma_x(y), \gamma_y(x))\), \((x, y) \in D\), with structure inverse monoid \(\text{IM}(X, r)\). We assume that for every \(x \in X\), \(x \in R_{\sigma_x}\). This property always holds for square-free solutions. Let \(I_X\) denote the symmetric inverse monoid of \(X\).

Let \(A\) denote the commutative inverse monoid, that is \(A\) is the set of partial functions \(f : X \to \mathbb{Z}\) with finite support, with the following operation: for any two elements \(f, f' \in A\), the domain of the sum \(f + f'\) is \(D_f \cap D_{f'}\) and the operation in \(A\) is defined pointwise, that is \((f + f')(x) = f(x) + f'(x)\), for every \(x \in D_f \cap D_{f'}\).

\textbf{Lemma 4.7.} The symmetric inverse monoid \(I_X\) acts (totally) on \(A\) by endomorphisms:
\[
\tau \circ f = f \circ \tau^{-1}
\]
where \(\tau \in I_X\) is a partial bijection of \(X\) and \(f \in A\), \(f : X \to \mathbb{Z}\) is a partial function with finite support. It is described by the following diagram,

\[
\begin{array}{ccc}
X & \xrightarrow{\tau^{-1}} & X \\
\downarrow f & & \downarrow f \\
\mathbb{Z} & & \mathbb{Z}
\end{array}
\]

Proof. Let \(\varphi\) be the map from \(I_X\) to the monoid of endomorphisms of \(A\), defined by \(\varphi(\tau)(f) = f \circ \tau^{-1}\), \(f \in A\), \(\tau \in I_X\). First, we show that \(\varphi(\tau)\) is indeed an endomorphism of \(A\). Let \(f, f' \in A\). Then \(\varphi(\tau)(f + f') = (f + f') \circ \tau^{-1} = f \circ \tau^{-1} + f' \circ \tau^{-1} = \varphi(\tau)(f) + \varphi(\tau)(f')\). We show that \(\varphi\) is a homomorphism of monoids. Let \(\tau, \nu \in I_X\), then \(\varphi(\tau \circ \nu)(f) = f \circ (\tau \circ \nu)^{-1} = (f \circ \nu^{-1}) \circ \tau^{-1} = \varphi(\tau)(f \circ \nu^{-1}) = \varphi(\tau)\varphi(\nu)(f)\).

Additionally, \(\text{Id}_X\) acts trivially on every \(f \in A\), that is \(f \circ \text{Id}_X^{-1} = f\), and for every \(\tau \in I_X\), \(\tau \circ 0_X = 0_X\), where \(0_X \in A\) is the zero function. So, from Definition \textit{[22]} \(I_X\) acts on \(A\) by endomorphisms.

Note that \(\varphi(\tau)\) is not necessarily an automorphism of \(A\), since, for any two elements \(f, f' \in A\), the equality of the maps \((\varphi(\tau))(f) = (\varphi(\tau))(f')\) does not necessarily imply \(f = f'\). Indeed, if \(f \circ \tau^{-1} = f' \circ \tau^{-1}\), then by precomposing each function with \(\tau\), we have \(f \circ \text{Id}_{D_{\tau}} = f' \circ \text{Id}_{D_{\tau}},\) that is these two partial functions have the same domain \(D_\tau \cap D_f = D_\tau \cap D_{f'}\) and \(f(x) = f'(x)\), for every \(x \in D_\tau \cap D_f\). But this does not imply necessarily that \(D_f = D_{f'}\), nor \(f = f'\).

We show the structure inverse monoid \(\text{IM}(X, r)\) acts on \(X\) by partial permutations and that this action extends to an action of \(\text{IM}(X, r)\) on \(A\).

\textbf{Lemma 4.8.} \(i\) The map \(\alpha : \text{IM}(X, r) \to I_X\), defined by \(\alpha(x) = \sigma_x, x \in X\), is a homomorphism of monoids.
(ii) There is an action of \( \text{IM}(X, r) \) on \( X \) by partial permutations, defined by:

\[
\begin{align*}
  x_i \cdot x_j &= x_{\sigma_i(j)} \\
  g \cdot x_j &= x_{\sigma_g(j)}
\end{align*}
\]

where \( x_i, g \in \text{IM}(X, r), \ x_j \in X, \ \sigma_g = \alpha(g) \).

(iii) There is an action of \( \text{IM}(X, r) \) on itself by endomorphisms, defined by:

\[
\begin{align*}
  g \cdot x_j &= x_{\sigma_g(j)} \\
  g \cdot h &= g \cdot x_{j_1} \ldots x_{j_k} = x_{\sigma_g(j_1)} \ldots x_{\sigma_g(j_k)}
\end{align*}
\]

where \( g, h \in \text{IM}(X, r) \).

(iv) There is an action of \( \text{IM}(X, r) \) on \( A \) by endomorphisms, defined by:

\[
\begin{align*}
  g \cdot f &= \alpha(g) \cdot f = f \circ \sigma_g^{-1}
\end{align*}
\]

where \( g \in \text{IM}(X, r), \ f \in A \).

Proof. (i), (ii), (iii) From the definition of the structure inverse monoid, the defining relations in \( \text{IM}(X, r) \), have the form \( xy = \sigma_x(y) \gamma_x(x), \ (x, y) \in D \). So, from Equation ??, \( \alpha : \text{IM}(X, r) \to I_X \) is a homomorphism of monoids. That is, \( G(X, r) \) acts on \( X \) by partial permutations, and this action extends to an action by endomorphisms on itself.

(iv) Let \( g \in \text{IM}(X, r) \). Then \( g \) has a homomorphic image \( \alpha(g) = \sigma_g \in I_X \), and there is an action of \( I_X \) on \( A \) by endomorphisms, from Lemma 4.7. So, \( \text{IM}(X, r) \) acts on \( A \) by endomorphisms. \( \square \)

Note that, since \( \alpha \) is a homomorphism of monoids, for every \( x \in X \), \( \alpha(x_i^*) = \sigma_i^{-1} \), and \( \alpha(x_i x_i^*) = \sigma_i \sigma_i^{-1} = \text{Id}_{\mathcal{R}_{\sigma_i}}, \ \alpha(x_i^* x_i) = \sigma_i^{-1} \sigma_i = \text{Id}_{\mathcal{D}_{\sigma_i}} \).

Definition 4.9. For every \( x \in X \), we define a partial function with finite support, \( \delta_x \), such that \( \mathcal{D}_{\delta_x} = \mathcal{R}_{\sigma_x} \subseteq X \) and \( \delta_x : \mathcal{D}_{\delta_x} \to \mathbb{Z} \) is defined by:

\[
\delta_x(y) = \begin{cases} 
1 & y = x \\
0 & y \in \mathcal{R}_{\sigma_x}, \ y \neq x 
\end{cases}
\]

Furthermore, \( \delta_x(y) \) is not defined for \( y \in X \setminus \mathcal{R}_{\sigma_x} \).

In what follows, we give a characterization of square-free partial solutions, in analogy with the characterization of classical solutions given in Proposition 1.4.

Theorem 4.10. Let \( (X, r) \) be a square-free, non-degenerate involutive partial set-theoretic solution, defined by \( r(x, y) = (\sigma_x(y), \gamma_y(x)), \ (x, y) \in D \), with structure inverse monoid \( \text{IM}(X, r) \). Let \( \pi \) be the map defined by:

\[
\begin{align*}
  \pi : \text{IM}(X, r) &\to A \\
  \pi(x_i) &= \delta_i; \ \pi(x_i^*) = -\delta_i \circ \sigma_i \\
  \pi(g h) &= \pi(g) + g \cdot \pi(h); \pi(g^*) = -\pi(g) \circ \sigma_g
\end{align*}
\]

Then \( \pi \) is an injective map.
For convenience, we divide the proof of Theorem 4.10 into several lemmas. First, we show that \( \pi \) is well-defined.

**Lemma 4.11.** Let \( g, h \in \text{IM}(X,r) \). Then

(i) \( \pi(gg^* g) = \pi(g) \) in \( A \).
(ii) \( \pi(gg^* hh^*) = \pi(hh^* gg^*) \) in \( A \).

**Proof.** (i), (ii) Let \( x_i \in X \). We show that \( \pi(x_i x_i^* x_i) = \pi(x_i) = \delta_i \). From the definition of \( \pi \), \( \pi(x_i x_i^* x_i) = \pi(x_i^* + x_i^*) \pi(x_i) = -\delta_i \circ \sigma_i + \delta_i \circ \sigma_i = 0_{D_{\sigma_i}} \), and \( \pi(x_i x_i^* x_i) = \pi(x_i) + x_i \pi(x_i^* x_i) = \delta_i + 0_{D_{\sigma_i}} \circ \sigma_i^{-1} = \delta_i + 0_{R_{\sigma_i}} = \delta_i \). In the same way, we show \( \pi(x_i x_i^* x_j) = \pi(x_j x_j^* x_i) = 0_{D_{\sigma_i} \cap D_{\sigma_j}} \), and for any \( g, h \in \text{IM}(X,r) \), \( \pi(gg^* g) = \pi(g) \), and \( \pi(gg^* hh^*) = \pi(hh^* gg^*) \). \( \square \)

Let \( (x_i, x_j) \in D, i \neq j \). So, \( x_i x_j = x_{\sigma_i(j)} x_{\gamma_j(i)} \) is a defining relation in \( \text{IM}(X,r) \). We show that \( \pi(x_i x_j) = \pi(x_{\sigma_i(j)} x_{\gamma_j(i)}) \) in \( A \). From the definition of \( \pi \):

\[
\begin{align*}
\pi(x_i x_j) &= \delta_i + \delta_j \circ \sigma_i^{-1} \\
\pi(x_{\sigma_i(j)} x_{\gamma_j(i)}) &= \delta_{\sigma_i(j)} + \delta_{\gamma_j(i)} \circ \sigma_{\sigma_i(j)}^{-1}
\end{align*}
\]

**Lemma 4.12.** Let denote \( f' = \delta_{\gamma_j(i)} \circ \sigma_{\sigma_i(j)}^{-1} \) and \( f = \delta_{\sigma_i(j)} + f' \). Then \( D_{\delta_i + \delta_j \circ \sigma_i^{-1}} = D_f \).

**Proof.** We compute both \( D_{\delta_i + \delta_j \circ \sigma_i^{-1}} \) and \( D_f \):

\[
\begin{align*}
D_{\delta_i + \delta_j \circ \sigma_i^{-1}} &= D_{\delta_i} \cap D_{\delta_j \circ \sigma_i^{-1}} = R_{\delta_i} \cap \sigma_i(D_{\delta_i} \cap D_{\delta_j}) = R_{\delta_i} \cap \sigma_i(D_{\delta_j}) = R_{\delta_i} \cap \sigma_i(R_{\delta_j}) = R_{\sigma_i \sigma_j} \\
D_f &= D_{\delta_{\sigma_i(j)}} \cap D_f' = D_{\delta_{\sigma_i(j)}} \cap \sigma_{\sigma_i(j)}(D_{\sigma_{\sigma_i(j)}} \cap D_{\delta_{\gamma_j(i)}}) = R_{\sigma_{\sigma_i(j)}} \cap \sigma_{\sigma_i(j)}(R_{\delta_{\gamma_j(i)}}) = R_{\sigma_{\sigma_i(j)}} \sigma_{\gamma_j(i)}
\end{align*}
\]

From Lemma 4.3 (iii), \( \sigma_{\sigma_j} = \sigma_{\sigma_i(j)} \sigma_{\gamma_j(i)} \), so \( D_{\delta_i + \delta_j \circ \sigma_i^{-1}} = D_f \). \( \square \)

**Lemma 4.13.** The partial maps \( \delta_i + \delta_j \circ \sigma_i^{-1} \) and \( f \) are equal.

**Proof.** For all the values in \( D_f \), except maybe for \( i, \sigma_i(j) \) and \( \sigma_{\sigma_i(j)} \gamma_j(i) \) (assuming they belong to \( D_f \)), both functions are zero. As, from Equation 4.11, \( \sigma_{\sigma_i(j)} \gamma_j(i) = i \), we need to check the value of the partial maps \( \delta_i + \delta_j \circ \sigma_i^{-1} \) and \( f \) at two values: \( i \) and \( \sigma_i(j) \).

For \( i \in D_{\delta_i + \delta_j \circ \sigma_i^{-1}} \). As the partial solution is square-free, \( \sigma_i^{-1}(i) = i, \sigma_i(j) \neq i, \neq j \):

\[
\begin{align*}
\begin{cases}
(\delta_i + \delta_j \circ \sigma_i^{-1}(i)) &= \delta_i(i) + \delta_j \circ \sigma_i^{-1}(i) = 1 + 0 = 1 \\
(f(i)) &= (\delta_{\sigma_i(j)} + \delta_{\gamma_j(i)} \circ \sigma_{\sigma_i(j)}^{-1})(i) = \delta_{\sigma_i(j)}(i) + \delta_{\gamma_j(i)} \circ \sigma_{\sigma_i(j)}^{-1}(i) = 0 + 0 = 1
\end{cases}
\end{align*}
\]

The last step comes from Lemma 4.3 (ii). Indeed, \( \sigma_{\sigma_i(j)}^{-1}(i) = \gamma_j(i) \), so \( f(i) = \delta_{\gamma_j(i)}(\gamma_j(i)) = 1 \), that is \( (\delta_i + \delta_j \circ \sigma_i^{-1})(i) = f(i) = 1 \).

For \( \sigma_i(j) \in D_{\delta_i + \delta_j \circ \sigma_i^{-1}} \). As the partial solution is square-free, \( \sigma_{\sigma_i(j)}^{-1}(\sigma_i(j)) = \sigma_i(j) \), for \( i \neq j, \sigma_i(j) \neq j \):

\[
\begin{align*}
\begin{cases}
(\delta_i + \delta_j \circ \sigma_i^{-1}(\sigma_i(j))) &= \delta_i(\sigma_i(j)) + \delta_j \circ \sigma_i^{-1}(\sigma_i(j)) = 0 + \delta_j(j) = 0 + 1 = 1 \\
(f(\sigma_i(j))) &= (\delta_{\sigma_i(j)} + \delta_{\gamma_j(i)} \circ \sigma_{\sigma_i(j)}^{-1})(\sigma_i(j)) = \delta_{\sigma_i(j)}(\sigma_i(j)) + \delta_{\gamma_j(i)}(\sigma_i(j)) = 1 + 0 = 1
\end{cases}
\end{align*}
\]

So, \( f(\sigma_i(j)) = (\delta_i + \delta_j \circ \sigma_i^{-1})(\sigma_i(j)) = 1 \), and the partial maps \( \delta_i + \delta_j \circ \sigma_i^{-1} \) and \( f \) are equal, that is \( \pi(x_i x_j) = \pi(x_{\sigma_i(j)} x_{\gamma_j(i)}) \) in \( A \). \( \square \)
Lemma 4.14. \( \pi \) is an injective map.

Proof. We define a map \( \omega : A \to \text{IM}(X, r) \) in the following way:

\[
\omega(\delta_i) = x_i \quad \omega(-\delta_i \circ \sigma_i) = x_i^* \quad \omega((-\delta_i \circ \sigma_i) | D) = x_i^* \quad D \subseteq D_{\delta_i} \\
\omega(f + f') = \omega(f) \cdot \omega((\omega(f))^* \cdot f'), \ f, f' \in A
\]

We show that \( \omega \circ \pi = \text{Id}_{\text{IM}(X, r)} \). Let \( g = x_{j_1}^{e_1}x_{j_2}^{e_2}...x_{j_k}^{e_k} \in \text{IM}(X, r) \), where for every \( 1 \leq i \leq k, x_{j_i} \in X, \epsilon_i = \pm 1 \) (we replace here the * by -1). Let \( g_i = x_{j_i}^{e_i}x_{j_2}^{e_2}...x_{j_k}^{e_k} \), \( 1 \leq i \leq k, \) and \( g_0 = 1 \). So, \( \sigma_{g_i-1} = \sigma_{j_1}^{e_1}...\sigma_{j_i}^{e_i-1} \) and \( \sigma_{g_0} = \text{Id}_X \). Let \( \mu_i = \epsilon_i - 1 (\equiv \text{mod} 3) \).

We show by induction on \( k \) that \( \omega \circ \pi(g) = g \), where \( \pi(g) = \sum_{i=1}^{i=k} \epsilon_i (\delta_{j_i} \circ \sigma_{g_i}^{e_i}) \circ \sigma_{g_{i-1}}^{-1} \).

For \( k = 1 \), it holds from the definition of \( \pi \) and \( \omega \). Assume \( \omega \circ \pi(g') = g' \), for every \( g' \in \text{IM}(X, r) \) of the form \( x_{j_1}^{e_1}x_{j_2}^{e_2}...x_{j_k}^{e_k} \), \( 1 \leq i \leq k, \) and \( g = g'x_{j_k}^{e_k} \). Let \( \delta = \pi(x_{j_k}^{e_k}) \). So, \( \omega \circ \pi(g) = \omega(\pi(g') + g' \cdot \delta) = \omega(\pi(g')) \omega((\omega(\pi(g'))^* \cdot g' \cdot \delta) \). From the induction assumption, we have \( \omega \circ \pi(g) = g \omega(g'^* \cdot g' \cdot \delta) = g' \omega(\delta \circ \sigma_{g'}^{-1} \circ \sigma_{g'}) = g' \omega(\delta \circ \text{Id}_{D_{g'}}) = g'x_{j_k}^{e_k} = g \), since \( \delta \circ \text{Id}_{D_{g'}} \) is equal \( \delta | D \), with \( D = D_{x_{j_k}^{e_k}} \cap D_{\delta} \).

Note that the map \( \pi \) is injective, but certainly not surjective. Indeed, the function \( \omega \), as defined, is not injective. Finally, we prove Theorem 4.10.

Proof of Theorem 4.10. From Lemmas 4.11, 4.12, 4.13, \( \pi \) is well-defined and from Lemma 4.14, \( \pi \) is injective.

We show that \( A \bowtie I_X \), the restricted product of \( A \) and \( I_X \), can be defined, and is an inverse monoid. Furthermore, we prove that, if \( (X, r) \) is square-free, \( \text{IM}(X, r) \) embeds in the inverse monoid \( A \bowtie I_X \).

Lemma 4.15. Let \( E(I_X) \) denote the set of idempotents of \( I_X \). Let \( \epsilon \) be defined by:

\[
\epsilon : A \to E(I_X) \quad \epsilon(f) = \text{Id}_{D_f}
\]

Then \( \epsilon \) is a surjective homomorphism of monoids and it satisfies the following condition: for each \( f \in A \), there exists \( \epsilon(f) \in E(I_X) \) such that \( \epsilon(f) \leq \text{Id}_X \iff \text{Id}_X \circ f = f \), for all \( \chi \subseteq X \).

Proof. Clearly, the map \( \epsilon \) is surjective, since the idempotents of \( I_X \) are precisely the partial identities of \( X \), and for each subset \( \chi \) of \( X \), there is \( f \in A \) such that \( \chi = D_f \).

We show that \( \epsilon \) is a homomorphism of monoids. Let \( f, f' \in A \). So,

\[
\epsilon(f + f') = \text{Id}_{D_{f+f'}} = \text{Id}_{D_f \cap D_{f'}} = \text{Id}_{D_f} \circ \text{Id}_{D_{f'}} = \epsilon(f) \circ \epsilon(f')
\]

Let \( \chi \subseteq X \). We show that \( \epsilon(f) \leq \text{Id}_X \iff \text{Id}_X \circ f = f \). From the definition of the action of \( I_X \) on \( A \), \( \text{Id}_X \circ f = f \circ \text{Id}_X^{-1} \), so \( \text{Id}_X \circ f = f \iff f \circ \text{Id}_X = f \iff f \circ \text{Id}_X = f \),
which occurs if and only if $\mathcal{D}_f \subseteq \chi$. On the other hand, $\epsilon(f) \leq \text{Id}_\chi \iff \text{Id}_{\mathcal{D}_f} \leq \text{Id}_\chi \iff \text{Id}_{\mathcal{D}_f} = \text{Id}_{\mathcal{D}_f} \circ \text{Id}_\chi = \text{Id}_\chi \circ \text{Id}_{\mathcal{D}_f}$, and this occurs if and only if $\mathcal{D}_f \subseteq \chi$.

**Theorem 4.16.** Let $\mathcal{I}$ be the following set and with the following operation

$$\mathcal{I} = \{(f, \tau) \in A \times I_X \mid \mathcal{R}_\tau = \mathcal{D}_f\}$$

$$(f, \tau)(f', \nu) = (f + (\tau \bullet f'), \tau \nu)$$

Then $\mathcal{I}$ is the restricted product $A \rtimes I_X$. Furthermore, $\mathcal{I}$ is an inverse monoid.

**Proof.** From Lemma 4.14, $I_X$ acts on $A$ by endomorphisms and from Lemma 4.15, this action satisfies additional conditions that ensure the existence of the restricted product $A \rtimes I_X$, so $A \rtimes I_X$ exists and is by definition $A \rtimes I_X = \{(f, \tau) \in A \times I_X \mid \tau(\tau) = \epsilon(f)\}$ (see Defn. 2.14). Since, $r(\tau) = \tau^* = \text{Id}_\mathcal{R}$ and $\epsilon(f) = \text{Id}_{\mathcal{D}_f}$, the condition $r(\tau) = \epsilon(f)$ is equivalent to $\mathcal{R}_\tau = \mathcal{D}_f$. That is, $\mathcal{I} = \{(f, \tau) \in A \times I_X \mid \mathcal{R}_\tau = \mathcal{D}_f\} = A \rtimes I_X$, an inverse monoid.

**Theorem 4.17.** Let $(X, r)$ be a square-free, non-degenerate involutive partial set-theoretic solution, defined by $r(x, y) = (\sigma_x(y), \gamma_y(x))$, $(x, y) \in \mathcal{D}$, with structure inverse monoid $\text{IM}(X, r)$. Then the map

$$\psi : \text{IM}(X, r) \to A \rtimes I_X$$

$$\psi(x) = (\delta_x, \sigma_x)$$

$$\psi(g) = (\pi(g), \sigma_g)$$

is an injective homomorphism of monoids. Furthermore, $\text{Im}(\psi)$, the image of $\text{IM}(X, r)$ in $A \rtimes I_X$, is an inverse monoid.

**Proof.** Let $g \in \text{IM}(X, r)$, with $\psi(g) = (\pi(g), \sigma_g)$. First, we show that $(\pi(g), \sigma_g)$ belongs indeed to $A \rtimes I_X$, that is $\mathcal{D}_{\pi(g)} = \mathcal{R}_{\sigma_g}$ is satisfied. Let $g = x_{j_1}^{\epsilon_1}x_{j_2}^{\epsilon_2}...x_{j_k}^{\epsilon_k} \in \text{IM}(X, r)$, where for every $1 \leq i \leq k$, $x_{j_i} \in X$, $\epsilon_i = \pm 1$ (here the $*$ is $-1$). The proof is by induction on $k$.

For $k = 1$, if $g = x_{j_1}$, then $\psi(x_{j_1}) = (\delta_{j_1}, \sigma_{j_1})$ and $\mathcal{D}_{\delta_{j_1}} = \mathcal{R}_{\sigma_{j_1}}$, from the definition of $\delta_{j_1}$. If $g = x_{j_1}^{-1}$, then $\psi(x_{j_1}^{-1}) = (-\delta_{j_1} \circ \sigma_{j_1}, \sigma_{j_1}^{-1})$ and we have $\mathcal{R}_{\sigma_{j_1}^{-1}} = \mathcal{D}_{\sigma_{j_1}}$ and $\mathcal{D}_{\delta_{j_1} \circ \sigma_{j_1}} = \mathcal{D}_{\sigma_{j_1}}$ also, since $\mathcal{R}_{\sigma_{j_1}} = \mathcal{D}_{\delta_{j_1}}$. For $k > 1$, assume $\mathcal{D}_{\pi(g')} = \mathcal{R}_{\sigma_{g'}}$, where $g' = x_{j_1}^{\epsilon_1}x_{j_2}^{\epsilon_2}...x_{j_{k-1}}^{\epsilon_{k-1}}$. $g = g' x_{j_k}^{\epsilon_k}$ and $\delta = \pi(x_{j_k}^{\epsilon_k})$. So, $\pi(g) = \pi(g') + g' \bullet \delta$ and from the induction assumption,

$$\mathcal{D}_{\pi(g)} = \mathcal{D}_{\pi(g')} \cap \mathcal{D}_{g' \bullet \delta} = \mathcal{R}_{\sigma_{g'}} \cap \mathcal{D}_{\delta \circ \sigma_{g'}^{-1}}.$$

If $\epsilon_k = 1$, then $\delta = \delta_k$, and:

$$\mathcal{D}_{\delta \circ \sigma_{g'}^{-1}} = \sigma_{g'}(\mathcal{D}_{\sigma_{g'}} \cap \mathcal{D}_{\delta_k}) = \sigma_{g'}(\mathcal{D}_{\sigma_{g'}} \cap \mathcal{R}_{\sigma_{j_k}})$$

$$\Rightarrow \mathcal{D}_{\pi(g)} = \mathcal{R}_{\sigma_{g'}} \cap \sigma_{g'}(\mathcal{D}_{\sigma_{g'}} \cap \mathcal{R}_{\sigma_{j_k}}) = \mathcal{R}_{\sigma_{g'} \sigma_{j_k}} = \mathcal{R}_{\sigma_g}$$

If $\epsilon_k = -1$, then $\delta = -\delta_{j_k} \circ \sigma_{j_k}$, and:
that maps a subset $\text{IM}(\phi)$. As the homomorphic image of an inverse monoid is an inverse monoid ([45, p.30]), $g$ is a partial monoid that is defined similarly: for every $a \in A$ and in particular to [27] for a wider understanding of this topic.

Next, we show that $\psi$ is an injective homomorphism of monoids. Let $g, h \in \text{IM}(X, r)$. Then $\psi(gh) = (\pi(gh), \sigma_{gh}) = (\pi(g) + g \bullet \psi(h), \sigma_g \circ \sigma_h) = \psi(g) \psi(h)$, from the definition of the product in $A \rtimes 1_X$. The injectivity of $\psi$ results from the injectivity of $\pi$ (see Theorem [4,10]). As the homomorphic image of an inverse monoid is an inverse monoid ([45, p.30]), $\text{Im}(\psi)$ is an inverse monoid.

Note that from the definition of the inverse in an inverse semigroup, $\psi(g)^* \psi(g) \psi(g)^* = \psi(g)$, so $\psi(g)^* = (-\pi(g) \circ \sigma_g, \sigma_g^{-1})$.

5. Definition of partial braces and their properties

In [5], the authors define a partial semigroup to be a set $S$ together with an operation $\oplus$ that maps a subset $D \subset S \times S$ into $S$ and satisfies the associative law $(g \oplus h) \oplus k = g \oplus (h \oplus k)$, in the sense that if either side is defined then so is the other and they are equal. We define a a partial monoid to be a partial semigroup, with an identity 1 such that $g \oplus 1, 1 \oplus g$ are always defined, equal and equal to $g$. We say that a partial monoid is commutative, if $g \oplus h = h \oplus g$, whenever both are defined.

Definition 5.1. A partial left brace is a set $B$ with two operations, $\oplus$ and $\cdot$, such that $(B, \oplus)$ is a commutative partial monoid, $(B, \cdot)$ is an inverse monoid and for every $a, b, c \in B$, such that $b \oplus c$ and $a \cdot b \oplus a \cdot c$ are defined, the following holds:

\[(5.1) \quad a \cdot (b \oplus c) = a \cdot b \oplus a \cdot c\]

$(B, \oplus)$ is called the partial monoid of the brace and $(B, \cdot)$ is called the multiplicative inverse monoid of the brace.

A partial right brace is defined similarly: for every $a, b, c \in B$, $(**): (a \oplus b) \cdot c = a \cdot c \oplus b \cdot c$.

In what follows, we show that a non-degenerate involutive partial set-theoretic solution, with structure inverse monoid $\text{IM}(X, r)$, induces a partial left brace $(\text{IM}(X, r), \oplus, \cdot)$. We define a new partial operation $\oplus$ in $\text{IM}(X, r)$, and in order to make the definition of $\oplus$ easier we use the method of right reversing. P. Dehornoy introduced this tool in the context of braids and Garside groups, and we refer the reader to [30], [26], and in particular to [27] for a wider understanding of this topic.

5.1. A very brief presentation of the method of right-reversing. Roughly, reversing can be used as a tool for constructing van Kampen diagrams in the context of presented semigroups or monoids. Let $M = \text{Mon}(X \mid R)$, with $R$ a family of pairs of nonempty words in the alphabet $X$, called relations. As is well-known, two strings $w$ and $w'$ are equal in $M$ if and only if there exists an $R$-derivation from $w$ to $w'$, defined to be a finite sequence of words $(w_0, ..., w_p)$ such that $w_0$ is $w$, $w_p$ is $w'$, and, for each $i$, $w_{i+1}$
is obtained from \( w_i \) by substituting some subword that occurs in a relation of \( R \) with the other element of that relation. A van Kampen diagram for a pair of words \((w, w')\) is a planar oriented graph with a unique source vertex and a unique sink vertex and edges labeled by letters of \( X \) so that the labels of each face correspond to a relation of \( R \) and the labels of the bounding paths form the words \( w \) and \( w' \), respectively. The strings \( w \) and \( w' \) are equal in \( M \) if and only if there exists a van Kampen diagram for \((w, w')\). It is convenient to standardize van Kampen diagrams, so that they only contain vertical and horizontal edges, plus dotted arcs connecting vertices that are to be identified. Such standardized diagrams are called reversing diagrams. Here we consider only right-reversing diagrams. We illustrate in the following example the construction of a right-reversing diagram.

**Example 5.2.** We consider the solution from Example 1.6 with structure monoid \( M = \text{Mon}(X \mid x_1 x_2 = x_3^2; x_1 x_3 = x_2 x_4; x_2 x_1 = x_3^3; x_2 x_3 = x_3 x_1; x_1 x_4 = x_4 x_2; x_3 x_2 = x_4 x_1) \).

We illustrate with the following figure how to construct a reversing diagram, that represents a van kampen diagram for a pair of words \((w, w')\), such that \( x_1 x_2 \) is the prefix of \( w \) and \( x_2 x_1 \) is the prefix of \( w' \). We begin with the left-most figure, with source the star, and two paths labelled \( x_1 x_2 \) and \( x_2 x_1 \) respectively. We complete the first left square with the defining relation \( x_1 x_3 = x_2 x_4 \). At the next step, we complete simultaneously two squares, using \( x_2 x_3 = x_3 x_1 \) and \( x_1 x_4 = x_4 x_2 \). At the last step, we close the diagram using \( x_1 x_3 = x_2 x_4 \). In the down right diagram, the labels of all the directed paths from the upper left star to the other star represent the same element in \( M \).

![Figure 5.1. Right reversing to build a van kampen diagram for the pair \((x_1 x_2 x_3^2, x_2 x_1 x_4^2)\).](image)

The process of reversing is not successful for every monoid, and for every pair of elements. Indeed, if a monoid is a Garside monoid, as in the case of the monoid \( M \) in Example 5.2 then right and left reversing are successful for every pair of elements in the monoid [26, 27]. But in an arbitrary monoid, it is not necessarily true anymore, and it may occur that the process never terminates, and even if it terminates there may be some obstructions. Assume we have a subdiagram with horizontal edge labelled \( x \) and vertical
edge labelled $y$. If there is no relation $x... = y...$ in $R$, then the subdiagram cannot be completed, and so the diagram neither. On the opposite, if there are more than one relation $x... = y...$ in $R$, then there may be several different ways to close the diagram. We illustrate with the following example an obstruction of the first kind in the use of right reversing that occurs for the structure inverse monoid of a partial solution.

**Example 5.3.** Let $\text{IM}(X, r) = \text{Inv}(x_0, x_1, x_2 \mid x_0x_2 = x_2x_1; x_1x_2 = x_2x_0)$ be the structure inverse monoid of the partial solution described in Example 4.2. As there is no defining relation $x_0... = x_1...$, the mostright diagram in Figure 5.2 cannot be completed.

### 5.2. Partial braces and partial solutions

Given $(X, r)$ a non-degenerate involutive partial solution with structure inverse monoid $\text{IM}(X, r)$, the set of defining relations can be described in several forms that are useful for computations.

**Lemma 5.4.** Let $(X, r)$ be a non-degenerate involutive partial solution with structure inverse monoid $\text{IM}(X, r)$. Let $R$ denote the set of defining relations of $\text{IM}(X, r)$. Then

\[
R = \{ x_ix_{\sigma_i^{-1}(j)} = x_jx_{\sigma_j^{-1}(i)} \mid (x_i, x_{\sigma_i^{-1}(j)}) \in \mathcal{D} \} \tag{5.2}
\]

\[
R = \{ x_{\alpha_k^{-1}(l)}x_k = x_{\gamma_k^{-1}(l)}x_l \mid (x_{\alpha_k^{-1}(l)}, x_k) \in \mathcal{D} \} \tag{5.3}
\]

**Proof.** If $(x_i, x_{\sigma_i^{-1}(j)}) \in \mathcal{D}$, then $r(x_i, x_{\sigma_i^{-1}(j)}) = (x_{\sigma_i(j)}, x_{\sigma_j^{-1}(i)}(i)) = (x_j, x_{\sigma_j^{-1}(i)}(i))$. From Lemma 4.3 by replacing $x$ by $i$ and $y$ by $\sigma_i^{-1}(j)$ in (4.4), we have $\gamma_i^{-1}(j)(t) = \sigma_i^{-1}(i)$, that is $r(x_i, x_{\sigma_i^{-1}(j)}) = (x_j, x_{\sigma_j^{-1}(i)})$ and $R$ can be described by Equation 5.2. If $(x_{\alpha_k^{-1}(l)}, x_k) \in \mathcal{D}$, then $r(x_{\alpha_k^{-1}(l)}, x_k) = (x_{\sigma_k^{-1}(l)}(k), x_{\gamma_k^{-1}(l)}) = (x_{\sigma_k^{-1}(l)}(k), x_l)$. From Lemma 4.3 by replacing $x$ by $\gamma_k^{-1}(l)$ and $y$ by $k$ in (4.2), we have $\sigma_{\gamma_k^{-1}(l)}(k) = \gamma_l^{-1}(k)$, that is $r(x_{\gamma_k^{-1}(l)}, x_k) = (x_{\gamma_l^{-1}(k)}, x_l)$ and the set $R$ can be described by Equation 5.3 \hfill \square

Note that the set $R$ as defined from $(X, r)$ is unique and Equations 5.2, 5.3 are just different ways to describe it.

**Remark 5.5.** From these presentations of $R$ and the properties of a partial solution ($\sigma_i, \gamma_i$, $1 \leq i \leq n$, are partial bijections), for every pair $x_i, x_j \in X$, there is at most one defining relation of the form $x_it = x_jz$, with $t, z \in X$, and for every pair $x_k, x_l \in X$, there is at most one defining relation of the form $t'x_k = z'x_l$, with $t', z' \in X$.

---

**Figure 5.2.** Right reversing in $\text{IM}(X, r)$, $(X, r)$ a partial solution: the mostright cube could not be completed.
Let $\mathcal{R}^*$ denote the following set of relations, described in two ways (as in Lemma 5.4):

\begin{align}
(5.4) & \quad \mathcal{R}^* = \{ x^*_{\sigma_i^{-1}(j)} x^*_i = x^*_{\sigma_j^{-1}(i)} x^*_j \mid (x_i, x_{\sigma_i^{-1}(j)}) \in \mathcal{D} \} \\
(5.5) & \quad \mathcal{R}^* = \{ x^*_k x^*_j = x^*_j x^*_k \mid (x^*_{\gamma_k^{-1}(l)}, x_k) \in \mathcal{D} \}
\end{align}

Let $\mathcal{R}_\rho$ denote the following set of relations, described in two ways (as in Lemma 5.4):

\begin{align}
R_\rho &= R \cup R^* \cup \{ x^*_i x_j = x x_{\sigma_i^{-1}(j)} x^*_{\sigma_j^{-1}(i)} \mid (x_i, x_{\sigma_i^{-1}(j)}) = x_j x_{\sigma_j^{-1}(i)} \in \mathcal{R} \} \\
R_\rho &= R \cup R^* \cup \{ x^*_k x_j^{\ast -1}(k) = x^*_j x^*_k \mid (x^*_{\gamma_k^{-1}(l)}, x_k) = x^*_j x^*_k \in \mathcal{D} \}
\end{align}

Let $\rho$ be the congruence generated by $\mathcal{R}_\rho$ on $\text{IM}(X, r)$ and $\text{IM}(X, r)/\rho$ the corresponding quotient monoid. We can reformulate now Theorem 2 in a more precise way:

**Theorem 5.6.** Let $(X, r)$ be a non-degenerate involutive partial solution with structure inverse monoid $\text{IM}(X, r)$. Let $\rho$ be the congruence generated by $\mathcal{R}_\rho$ on $(X, r)$. Then there exists a partial left brace $(B, \oplus, \cdot)$ such that $\text{IM}(X, r)/\rho$ is isomorphic $(B, \cdot)$.

**Proof.** Let $\text{IM}_\rho$ denote the quotient monoid $\text{IM}(X, r)/\rho$. We define in $\text{IM}_\rho$ a partial operation, $\oplus$, and show that $(\text{IM}_\rho, \oplus, \cdot)$ is a partial brace. Clearly, $(\text{IM}_\rho, \cdot)$ is an inverse monoid (and we omit the $\cdot$). It remains to show that $(\text{IM}_\rho, \oplus)$ is a partial commutative monoid. The proof contains two parts: the iterative definition of $\oplus$ in $\text{IM}_\rho$ and the proof $\oplus$ is well-defined and satisfies the relevant properties.

First, we define $\oplus$ for pairs of elements of $X$. Let $x_i, x_j \in X$ such that $(x_i, x_{\sigma_i^{-1}(j)}) \in \mathcal{D}$. Then $x_i \oplus x_j$ and $x_j \oplus x_i$ exist and are defined by:

\begin{align}
(5.6) & \quad x_i \oplus x_j = x_i x_{\sigma_i^{-1}(j)} \quad x_j \oplus x_i = x_j x_{\sigma_j^{-1}(i)}
\end{align}

Since $x_i x_{\sigma_i^{-1}(j)} = x_j x_{\sigma_j^{-1}(i)}$ in $\text{IM}(X, r)$, from Lemma 5.4 $x_i \oplus x_j = x_j \oplus x_i$ in $\text{IM}_\rho$.

Second, we define $\oplus$ for pairs of elements of $X^*$, where $X^* = \{ x^* \mid x \in X \}$. Let $x_k, x_l \in X$, such that $(x_{\gamma_k^{-1}(l)}, x_k) \in \mathcal{D}$. Then $x^*_k \oplus x^*_l$ and $x^*_l \oplus x^*_k$ exist and are defined by:

\begin{align}
(5.7) & \quad x^*_k \oplus x^*_l = x^*_k x^*_{\gamma_k^{-1}(l)} \quad x^*_l \oplus x^*_k = x^*_l x^*_{\gamma_k^{-1}(k)}
\end{align}

From Lemma 5.4, $x_{\gamma_k^{-1}(l)} x_k = x_{\gamma_k^{-1}(k)} x_l$, so $x^*_k x^*_{\gamma_k^{-1}(l)} = x^*_l x^*_{\gamma_k^{-1}(k)}$ also in $\text{IM}(X, r)$, that is $x^*_k \oplus x^*_l = x^*_l \oplus x^*_k$ in $\text{IM}_\rho$.

Third, we define $\oplus$ for pairs of elements of $X \cup X^*$. Let $x_i, x_j \in X$. If $x_i \oplus x_j$ exists and $x_i \oplus x_j = x_i t = x_j z$, with $t, z \in X$, then $t \oplus x^*_i$ and $z \oplus x^*_j$ exist and, for $x^*_i, x^*_j, t^*, z^* \in X^*$:

\begin{align}
(5.8) & \quad t \oplus x^*_i = t z^* \quad x^*_i \oplus t = x^*_i x_j \\
(5.9) & \quad z \oplus x^*_j = z t^* \quad x^*_j \oplus z = x^*_j x_i
\end{align}

From the definition of $\mathcal{R}_\rho$, $t z^* = x^*_i x_j$ and $z t^* = x^*_j x_i$ in $\text{IM}_\rho$, that is $z \oplus x^*_j = x^*_j z^* \oplus z$ and $t \oplus x^*_i = x^*_i t^* \oplus t$ in $\text{IM}_\rho$. Note that whenever defined, $\oplus$ is well-defined for pairs of elements in $X \cup X^*$ (see Remark 5.5). In Figure 5.3 we illustrate diagrammatically the motivation for the equations (5.6) - (5.9).
Next, we define $\oplus$ for pairs of elements in $\text{IM}_\rho$. For every $g \in \text{IM}_\rho$, we define

\begin{align}
(5.10) & \quad g \oplus 1 = g1 = 1g \\
(5.11) & \quad g \oplus g = g1 = 1g
\end{align}

For any two elements $g, h \in \text{IM}_\rho$, we use an extended version of a reversing diagram in which we allow arrows labelled with elements from $X^\ast$, to define iteratively $g \oplus h$. 

We begin at the left-most upper square and whenever possible we complete each square according to the unique definition of $\oplus$ for pairs of elements in $X \cup X^\ast$. If each square can be closed, then $g \oplus h$ is defined and can be read from the directed path from the upper star to the lower star, going right and then down, $h \oplus g$ is also defined and it can be read from the directed path from the upper star to the lower star, going down and then right. Clearly, $g \oplus h$ and $h \oplus g$ are equal in $\text{IM}_\rho$. If the process of reversing to define $g \oplus h$ is not terminating, or if at any step, any square could not be completed, then $g \oplus h$ is not defined. If $g \oplus h$ exists and $g \oplus h = gu = hv$, then $g^\ast \oplus u$ and $h^\ast \oplus v$ exist and using
the extended version of a reversing diagram in which the inversion of the direction of an arrow labelled $x$ gives an arrow labelled $x^*$ (as in Figure 5.4) we have:

\[(5.12)\] $g \oplus h = gu = hv$

\[(5.13)\] $g^* \oplus u = g^*h = uv^*$

\[(5.14)\] $h^* \oplus v = h^*g = vu^*$

We show that $\oplus$ is well-defined. As $\text{IM}_\rho$ is an inverse monoid with respect to the product, for every $g \in \text{IM}_\rho$, $g$ and $gg^*g$ are equivalent, so we need to show that whenever $g \oplus h$ and $gg^*g \oplus h$ exist, then $g \oplus h$ and $gg^*g \oplus h$ are also equivalent (see Figure 5.5).

\[\text{Figure 5.5. Right reversing to compute } gg^*g \oplus h \text{ at left and } g \oplus h \text{ at right.}\]

Note that if $g \oplus h$ exists and $g \oplus h = gu = hv$, then $g^* \oplus u$ exists and so $gg^*g \oplus h$ exists also. From Equations \[(5.12)\] \[(5.13)\] we have $gg^*g \oplus h = gg^*gu = hvv^*v$. Since $hv^*v$ and $hv$ are equivalent, $gg^*g \oplus h = gu = hv = g \oplus h$ in $\text{IM}_\rho$. In the same way, we can show that for any $g, h, w, z \in \text{IM}_\rho$, $gzw^*ww^* \oplus h = gw^*zz^* \oplus h$.

Let $g, g', h \in \text{IM}_\rho$ and assume $g$ and $g'$ are equal in $\text{IM}_\rho$. We show that $g' \oplus h = g \oplus h$ in $\text{IM}_\rho$. From the assumption, $g'$ can be obtained from $g$ by successive applications of relations from $\mathcal{R}_\rho$, so it is enough to show that after each single application there is equality. That is, we can assume that $g'$ is obtained from $g$ after the application of a single relation ($l = r$) from $\mathcal{R}_\rho$: $g = ab$, $g' = ab$, $a, b, l, r \in \text{IM}_\rho$. Assume $h = x^*_l w$, with $x^*_l \in X \cup X^*$, $w \in \text{IM}_\rho$, as in Figure 5.6. Then, if we show that $(l' = r')$ belongs to $\mathcal{R}_\rho$ and $x_{r'} = x_r$, then $x_{s'} = x_s$ and $b'_2$ is $b'_1$, that is $g' \oplus x^*_l = g \oplus x^*_l$ and $a' \star b'_1 = a' \star b'_1$ in $\text{IM}_\rho$.

\[\text{Figure 5.6. The iterative computation of } g \oplus h \text{ at left and } g' \oplus h \text{ at right, with } g = ab, g' = ab, h = x^*_l w.\]

Using the same argument in the completion of the diagram row by row in the same way, we then obtain $g' \oplus h = g \oplus h$ in $\text{IM}_\rho$. We prove that $(l' = r')$ belongs to $\mathcal{R}_\rho$ and
that $x_{v'} = x_v$ in a case by case proof which relies entirely on the properties of the partial solution from Lemma 4.3. The proof is in Appendix 2. So, $\oplus$ is well-defined in $\text{IM}_\rho$.

We show now that for every $g, h, k \in \text{IM}_\rho$, $(g \oplus h) \oplus k = g \oplus (h \oplus k)$, whenever they are defined. Assume $g \oplus h = gu = hv$, $h \oplus k = hw = kz$, $v \oplus w = v'k' = wv'$. So, we can compute $(g \oplus h) \oplus k$ and $g \oplus (h \oplus k)$ as described in Figure 5.7. As one can see, along the thick path from the upper left star to the down right star, we read in both diagrams a common element, so $(g \oplus h) \oplus k = g \oplus (h \oplus k)$ in $\text{IM}_\rho$ (whenever defined). So, $(\text{IM}_\rho, \oplus)$ is a commutative partial monoid, with identity element equal to 1, the identity of $(\text{IM}_\rho, \cdot)$.

**Figure 5.7.** Computation of $(g \oplus h) \oplus k$ at left and $g \oplus (h \oplus k)$ at right

It remains to show that Equation 5.1 holds, that is $a(g \oplus h) = ag \oplus ah$, for every $a, g, h \in \text{IM}_\rho$, such that this expression is defined. From $g \oplus h = gu = hv$, multiplying at left by $a$, we have $a(g \oplus h) = agu = ahv$, and using the reversing diagram for $ag \oplus ah$, and Equations 5.10, 5.11 we obtain $ag \oplus ah = agu = ahv$, that is $a(g \oplus h) = ag \oplus ah$. So, $(\text{IM}_\rho, \cdot, \oplus)$ is a partial left brace.

**Remark 5.7.** Note that in the proof of Theorem 5.6, the operation $\oplus$ defined satisfies a monoidal version of the left-distributivity-like axiom in a left brace as described in Equation 1.1, that is $a \oplus a \cdot (g \oplus h) = a \cdot g \oplus a \cdot h$, for every $a, g, h \in \text{IM}_\rho$ (see Figure 5.8).

**Figure 5.8.** Computation of $g \oplus h$, $ag \oplus ah$, and $a \oplus a(g \oplus h)$ from left to right

Note that the same process could be done with left-reversing instead of right-reversing, and in this case $(\text{IM}_\rho, \cdot, \oplus)$ would have been a partial right brace. Although the structure of $(\text{IM}_\rho, \oplus)$ is reminiscent of that of a category, it is not one. Indeed, in a category, a strong associativity axiom is satisfied: if $a \oplus b$ and $b \oplus c$ are defined, then $a \oplus c$ is also defined, which is not satisfied in $(\text{IM}_\rho, \oplus)$. 
Given a partial left brace $\mathcal{B}$, it is not clear how to define a non-degenerate involutive partial solution associated to $\mathcal{B}$. If we try, like in the case of the classical brace, to define a homomorphism $\lambda : (\mathcal{B}, \cdot) \to \text{Aut}(\mathcal{B}, \oplus)$ such that $r(x, y) = (\lambda_x(y), \lambda_{x}(y^{-1})(x))$ is a non-degenerate involutive partial set-theoretic solution, we encounter many difficulties. Indeed, as $(\mathcal{B}, \cdot)$ is an inverse monoid and not a group, there is no left or right cancellation rules, and the maps $\lambda_x$ are not necessarily injective.

6. **The Thompson group $F$ as the structure group of a partial set-theoretical solution of the QYBE**

6.1. **Proof of Theorem 3.** We consider the infinite presentation of the group $F$ from Theorem 3.3 and show that $F$ is the structure group of a square-free, non-degenerate, involutive partial set-theoretic solution. We recall that $F = \text{Gp}\langle x_0, x_1, x_2, \ldots \mid x_n x_k = x_k x_{n+1}, 0 \leq k < n \rangle$. Let $X = \{x_0, x_1, x_2, \ldots\}$. For brevity of notation, we often denote these elements by $\{0, 1, 2, \ldots\}$. Let $\sigma_n : X \to X$ and $\gamma_n : X \to X$ be the following partial functions.

\[
\begin{align*}
\sigma_n(k) &= \begin{cases} 
  k & k \leq n \\
  \text{not defined} & k = n + 1 \\
  k - 1 & k \geq n + 2 
\end{cases} \\
\gamma_n(k) &= \begin{cases} 
  \text{not defined} & k \leq n - 2 \\
  k & k = n - 1 \\
  n & k = n \\
  k + 1 & k \geq n + 1 
\end{cases}
\end{align*}
\]

Let define the following subsets of $X$: $D_{\sigma_n} = X \setminus \{x_{n+1}\}$, $R_{\sigma_n} = X$, $D_{\gamma_n} = X \setminus \{x_{n-1}\}$, and $R_{\gamma_n} = X \setminus \{x_{n-1}, x_{n+1}\}$. So, $\sigma_n : D_{\sigma_n} \to R_{\sigma_n}$, and $\gamma_n : D_{\gamma_n} \to R_{\gamma_n}$ are functions.

Let define the following partial function:

\[
r : X \times X \to X \times X \\
r(x_i, x_j) = (x_{\sigma(i)}, x_{\gamma(j)})
\]

We denote by $D \subset X \times X$ and $R \subset X \times X$ the domain and range of $r$ respectively. We show that the pair $(X, r)$ just defined is a square-free, non-degenerate, involutive partial set-theoretical solution.

**Lemma 6.1.** Let $r : D \to R$ as defined in Eq. 6.2. Then $(X, r)$ satisfies the following properties:

(i) $D = R = X^2 \setminus \{(x_n, x_{n+1}) \mid n \geq 0\}$.

(ii) $r : D \to R$ is bijective.

(iii) $r(x_i, x_i) = (x_i, x_i)$, for all pairs $(x_i, x_i) \in X^2$.

(iv) For every $x \in X$, $\sigma_x : D_{\sigma_x} \to R_{\sigma_x}$ and $\gamma_y : D_{\gamma_y} \to R_{\gamma_y}$ are bijective.

(v) $r^2(x_i, x_j) = (x_i, x_j)$, for all pairs $(x_i, x_j) \in X^2$ such that $r^2(x_i, x_j)$ is defined.

(vi) $r^{12} r^{23} r^{12}(x_i, x_j, x_k) = x^{23} r^{12} r^{23}(x_i, x_j, x_k)$, for all triples $(x_i, x_j, x_k) \in X^3$ such that both $r^{12} r^{23} r^{12} x_i, x_j, x_k$ and $r^{23} r^{12} r^{23} x_i, x_j, x_k$ are defined.

That is, $(X, r)$ is a a non-degenerate, square-free, involutive partial set-theoretical solution.
This process can be iterated infinitely.

(iii) For and furthermore, \( \{ \sigma_i, \gamma_i \} \) are partial bijections of \( X \), for every \( i \geq 0 \).

(ii) Clearly, from their definition, \( \sigma_i \) and \( \gamma_i \) are partial bijections of \( X \), for every \( i \geq 0 \).

(v) \( r^2(x_i, x_j) = r(\begin{cases} (x_{j-1}, x_i) & i \leq j-2 \\ (x_j, x_i) & i = j-1 \\ (x_j, x_{i+1}) & i \geq j+1 \end{cases}) \) = \( (x_i, x_j) \) if \( i \leq j-2 \) and \( (x_i, x_j) \) if \( i = j-1 \) and \( (x_i, x_j) \) if \( i \geq j+1 \).

(vi) is a case by case proof and it appears in the appendix. \( \square \)

We denote by \( F \) the non-degenerate involutive partial set-theoretic solution defined above. We prove some properties of \( F \).

Theorem 6.2. Let \( r : D \to R \) as defined in Eq. (1). Then

(i) \( G(X, r) \), the structure group of \( F \), is isomorphic to the Thompson group \( F \).

(ii) \( \text{IM}(X, r) \), the structure inverse monoid of \( F \), embeds into the inverse monoid \( A \times \text{IA} \), where \( A \) is the commutative inverse monoid \( \{ f : D_f \to \mathbb{Z} \mid D_f \subseteq X \} \), with pointwise operation, and \( \text{IA} \) is the inverse symmetric monoid.

Proof. (i) From its definition, \( G(X, r) = \text{Gp}(x_0, x_1, x_2, \ldots \mid x_i x_j = x_{\sigma_i(j)} x_{\gamma_i(j)} \mid i, j \geq 0) \). That is, using the description of \( r \) in the proof of Lemma 6.1, the defining relations of \( G(X, r) \) have the following form:

\[
\begin{align*}
\sigma_i x_i & = x_{i+1} x_i, & i \leq j-2 \\
\gamma_i x_i & = x_{i+1} x_i, & i \geq j+2 \\
x_i x_j & = x_{j+1} x_i, & \text{all other cases} \\
x_0 & = 1
\end{align*}
\]

There are trivial relations of the form \( x_i x_i = x_i x_i \), for every \( i \geq 0 \), and there are no relations of the form \( x_i x_{i+1} = \ldots \). Rewriting the above relations in a unified form, we have \( G(X, r) = \text{Gp}(x_0, x_1, x_2, \ldots \mid x_i x_k = x_k x_{i+1} \mid 0 \leq k < n) \), the infinite presentation of the Thompson group \( F \) (from Theorem 3.3), that is \( G(X, r) \) is isomorphic to \( F \).

(ii) results directly from Theorem 4.1. \( \square \)

Lemma 6.3. Let \( F \) as defined above, with \( X = \{ x_0, x_1, \ldots \} \). Then

(i) \( F \) is irretractable.

(ii) \( F \) is decomposable.

Proof. (i) results directly from the definition of the functions \( \{ \sigma_x \mid x \in X \} \).

(ii) Let \( X = \{ x_0 \} \cup Y_1 \), where \( Y_1 = \{ x_1, x_2, \ldots \} \). We show that \( \{ x_0 \} \) and \( Y_1 = \{ x_1, x_2, \ldots \} \) are non-degenerate invariant subsets of \( X \). Clearly, \( r(x_0, x_0) = (x_0, x_0) \), and furthermore, \( r(Y_1, Y_1) \subseteq (Y_1, Y_1) \), since for \( 0 < k < n \), \( r(x_k, x_n) = (x_k, x_{n+1}) \).

For \( n \geq 1 \), the partial bijections of \( X \), written in the form of (infinite) permutations, \( \sigma_n = (0)(1)\ldots(n)\ldots(n+3, n+2) \) and \( \gamma_n = (0)(1)\ldots(n-2)(n)\ldots(n+1, n+2, \ldots) \) restricted to \( Y_1 \) are partial bijections of \( Y_1 \). That is, the restriction of \( r \) to \( Y_1 \) is a non-degenerate partial solution. This solution itself is decomposable. Indeed, \( Y_1 = \{ x_1 \} \cup Y_2 \), where \( Y_2 = \{ x_3, x_4, \ldots \} \), and \( \{ x_1 \} \) and \( Y_2 = \{ x_3, \ldots \} \) are non-degenerate invariant subsets of \( Y_1 \). This process can be iterated infinitely. \( \square \)
Remark 6.4. In the proof of Lemma 6.3, we have that \( \{x_0\} \) and \( Y_1 = \{x_1, x_2, \ldots\} \) are non-degenerate invariant subsets of \( X \). Let \( G_0 = Gp(x_0 | \{\}) \) and \( G_1 = G(Y_1, r | y^2) \), the structure group of \((Y_1, r | y^2)\). For a classical set-theoretic solution, if there is an action of \( G_0 \) on \( G_1 \) by conjugation, we have \( G(X, r) \sim G_1 \rtimes G_0 \), and iteratively \( G(X, r) \sim (\langle G_n \rtimes \langle x_{n-1} \rangle \rangle \rtimes \langle x_{n-2} \rangle \rangle \rtimes \cdots \rtimes \langle x_1 \rangle \rangle \rtimes \langle x_0 \rangle \). In the case of the partial solution \( \mathcal{F} \), as we have \( x_0 x_n x_0^{-1} = x_{n-1} \), only for every \( n \geq 2 \) (i.e. no relation \( x_0 x_1 x_0^{-1} = \ldots \)), the action of \( G_0 \) on \( G_1 \) by conjugation is not total. It is possible to consider the inverse monoids, \( M_0 = \text{Inv}(x_0 | -) \) and \( M_1 = \text{IM}(Y_1, r | y^2) \), and the action of \( M_0 \) on \( M_1 \) by partial permutations, but nevertheless it is not clear which kind of decomposition is obtained.

We now turn to the study of the existence of a cycle set corresponding to \( \mathcal{F} \).

Theorem 6.5. Let \( \mathcal{F} \) as defined above, with \( X = \{x_0, x_1, \ldots\} \). Then \( \mathcal{F} \) induces a non-degenerate and square-free cycle set \((X, \star)\).

6.2. Comparison between solutions, partial solutions and \( \mathcal{F} \). A question that arises naturally is which properties do the structure groups of set-theoretic solutions and of partial set-theoretic solutions share in common. In particular, as set-theoretic solutions and their structure groups have been intensively studied, it would be interesting to investigate which properties satisfied by the structure group of a set-theoretic solution is also satisfied by the structure group of a partial solution. The structure inverse monoid has not been defined for set-theoretic solutions, so it would be interesting to understand whether its definition would provide some interesting information on the set-theoretic solutions, and on the comparison between properties of set-theoretic solutions and partial solutions. As Thompson’s group \( F \) is the structure group of a square-free partial set-theoretic solution, we can examine some of its properties in comparison with those of structure groups of set-theoretic solutions. It is not clear if we can generalize these observations, since on one hand \( F \) is the structure group of \((X, r)\), with \((X, r)\) partial and also \( X \) infinite, and on the other hand structure groups of solutions form an infinite family of groups. Nevertheless, it is interesting to see the differences and we present some points below.

- The structure monoid of a solution \((X, r)\), with \( X \) finite, is Garside, and its group of fractions is a Garside group. Garside groups are torsion-free and biautomatic. Thompson’s group \( F \) is also torsion-free, and it is not known whether it is automatic.
- The structure group of a solution \((X, r)\), with \( X \) finite, is solvable. Thompson’s group \( F \) commutator subgroup \( F' \) is simple, so \( F'' = F' \), and \( F \) is not nilpotent, nor solvable. The centre of \( F \) is trivial. The structure group of a solution \((X, r)\), with \( X = n \), is a Bieberbach group, as it acts on the Euclidean \( n \)-dimensional space. As far as we know, there is no result of this kind for \( F \).
• The quotient group $F/F'$ is isomorphic to $\mathbb{Z}^2$, and so any proper quotient of $F$ is abelian [7, 4]. This is not necessarily the case for the structure group of a solution. Indeed, we describe the structure group $G(X, r)$ of a square-free solution $(X, r)$, with $X$ infinite, and with a quotient group not abelian. Let $G = \text{Gp}(x_0, x_1, x_2 \mid x_0x_1 = x_1x_0; x_2x_1 = x_0x_2; x_2x_0 = x_1x_2)$, the structure group of a square-free solution with set $\{x_0, x_1, x_2\}$, and $G \simeq \mathbb{Z}^2 \times \mathbb{Z}$. The subgroup $N = \langle x_0^2, x_1^2, x_2^2 \rangle$ is normal and free abelian of rank 3, and the quotient group of $G$ by $N$ is a finite group of order 8 [20]. We do not get into the details of the construction of this kind of quotient group and refer the reader to [20], and [29] for a generalisation. Let $G(X, r)$ and $N(X, r)$ be the direct product of an infinite number of copies of $G$ and $N$, respectively. So, $G(X, r)$ is the structure group of a square-free infinite solution, and quotienting it by $N(X, r)$, it has an infinite quotient group $W$ which is not abelian, since $W$ isomorphic to the direct product of an infinite number of copies of $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_2$.

To conclude, there are three Thompson’s groups $F, T, V$, with $F \subset T \subset V$. There have been also several generalisations of these groups. It would be interesting to know if there is some kind of solution such that these groups are derived from them.

7. Appendix

7.1. Appendix 1: Proofs of Lemma 6.1 (vi) and Theorem 6.5

Proof of Lemma 6.1 (vi). We show by a case by case proof that the conditions from Lemma 6.3(ii) are satisfied. For brevity of notation, instead of $x_m, x_n, x_k$, we write $m, n, k$

$$\sigma_m\sigma_n(k) = \begin{cases} k & k < n < m \\ k & k < m < n, n \neq m + 1 \\ k - 2 & m < n < k, k \neq n + 1, m + 2 \\ k - 1 & m < k < n, n \neq m + 1 \\ k - 1 & n < k < m, k \neq n + 1 \\ k - 2 & n < m < k, k > m + 2 \\ \text{not defined} & n < m < k, k = m + 1 \end{cases}$$

$$\sigma_{\sigma_m(n)}\sigma_{\sigma_n(m)}(k) = \begin{cases} \sigma_n\sigma_{m+1}(k) = k & k < n < m \\ \sigma_{n-1}\sigma_m(k) = k & k < m < n, n \neq m + 1 \\ \sigma_n\sigma_{m-1}(k) = k - 2 & m < n < k, k \neq n + 1, m + 2 \\ \sigma_{n-1}\sigma_m(k) = k - 1 & m < k < n, n \neq m + 1 \\ \sigma_n\sigma_{m+1}(k) = k - 1 & n < k < m, k \neq n + 1 \\ \sigma_n\sigma_{m-1}(k) = k - 2 & n < m < k, k > m + 2 \\ \sigma_m(k) = \text{not defined} & n < m < k, k = m + 1 \end{cases}$$
For every $\sigma$, partial bijection of $n$, from its definition, the map

$$\sigma_{\gamma_\sigma(n)}(\gamma_k(n)) = \begin{cases} 
\sigma_{m+1}\gamma_k(n) = n + 1 & k < n < m \\
\sigma_{m+1}\gamma_k(n) = n & k < m < n, n \neq m + 1 \\
\sigma_m\gamma_k(n) = n - 1 & m < n < k, k \neq n + 1, m + 2 \\
\sigma_m\gamma_k(n) = n & m < k < n, n \neq m + 1 \\
\sigma_{m+1}\gamma_k(n) = n & n < k < m, k \neq n + 1 \\
\sigma_m\gamma_k(n) = n & n < m < k, k > m + 2 \\
\sigma_m\gamma_k(n) = n & n < m < k, k = m + 1 \\
\gamma_k\sigma_m(n) = n + 1 & k < n < m \\
\gamma_k\sigma_m(n) = n & k < m < n, n \neq m + 1 \\
\gamma_{k-1}\sigma_m(n) = n - 1 & m < n < k, k \neq n + 1, m + 2 \\
\gamma_{k-1}\sigma_m(n) = n & m < k < n, n \neq m + 1 \\
\gamma_k\sigma_m(n) = n & n < k < m, k \neq n + 1 \\
\gamma_{k-1}\sigma_m(n) = n & n < m < k, k > m + 2 \\
\gamma_{k-1}\sigma_m(n) = n & n < m < k, k = m + 1 
\end{cases}$$

The third relation is proved in the same way. \(\square\)

**Proof of Theorem 6.5** For every $x_n, x_k \in X$, let

$$x_n \ast x_k = x_{\sigma_n^{-1}(k)}$$

with $\sigma_n^{-1} : X \to X \setminus \{x_{n+1}\}$ the map defined by:

$$\sigma_n^{-1}(k) = \begin{cases} 
k & k \leq n \\
k + 1 & k \geq n + 1
\end{cases}$$

Le $x_m, x_n, x_k \in X$. We show $(x_n \ast x_k) \ast (x_n \ast x_m) = (x_k \ast x_n) \ast (x_k \ast x_m)$. Instead of $x_m, x_n, x_k$, we write $m, n, k$. So, we need to show

$$\sigma_{\gamma_{\sigma_n^{-1}(k)}(m)} = \sigma_{\gamma_{\sigma_k^{-1}(n)}(m)}$$

From its definition, the map $\sigma_n^{-1} : X \to X \setminus \{x_{n+1}\}$ is bijective for every $x_n \in X$ (a partial bijection of $X$). So, $(X, \ast)$ is a cycle set. The cycle set $(X, \ast)$ is square-free, since
$x_n \star x_n = x_n$, for all $x_n \in X$, indeed $x_n \star x_n = x_{\sigma^{-1}_n(n)} = x_n$, and it is non-degenerate since the map $x_n \mapsto x_n \star x_n = x_n$ is bijective, for all $x_n \in X$.

7.2. Appendix 2: Case by case proof in Theorem 5.6 (for Theorem 2). Let $(\mathcal{I} = r)$ in $\mathcal{R}_\rho$. Let $x_k^c \in X \cup X^*$. We show that $(l' = r')$ belongs to $\mathcal{R}_\rho$ and $x_{r'} = x_r$.

![Figure 7.1](image1.png)

There are six cases to check, from the form of the defining relations in $\mathcal{R}_\rho$ and $x_k^c \in \{x_k, x_k^c\}$. For the proof, we assume all the diagrams can be completed, which means that the relations needed to close squares belong to $\mathcal{R}_\rho$ and that the partial bijections $\sigma_i, \gamma_i$ are computed in their domain of definition.

**Case 1:** $\{l = x_i \sigma^{-1}_i(j) ; r = x_j \sigma^{-1}_j(i) ; x_k\}$:

![Figure 7.2](image2.png)

From Lemma 5.3, the relations have the form $x_i x_{\sigma^{-1}_i(j)} = x_j x_{\sigma^{-1}_j(i)}$, so from Equation 4.3 $\sigma_i \sigma^{-1}_i(j) = \sigma_j \sigma^{-1}_j(i)$, for every $k$ in their domain of definition. This implies that $r = \sigma^{-1}_i(i) \sigma^{-1}_j(j)$ and $r' = \sigma^{-1}_j(i) \sigma^{-1}_i(j)$ are equal.

We show that $l' = x_p x_s$ and $r' = x_m x_l$ are equal and belong to $\mathcal{R}$. For that, we show that $s = \sigma^{-1}_p(m) = \sigma^{-1}_p(i) \sigma^{-1}_j(j)$ and $l = \sigma^{-1}_m(p) = \sigma^{-1}_m(i) \sigma^{-1}_k(i)$. This is true from Equation 4.3 since $(x_i x_{\sigma^{-1}_i(j)} = x_j x_{\sigma^{-1}_j(i)})$ and $(x_j x_{\sigma^{-1}_j(i)} = x_k x_{\sigma^{-1}_k(i)})$ belong to $\mathcal{R}$.

**Case 2:** $\{l = x_i \sigma^{-1}_i(j) x_{\sigma^{-1}_i(j)} ; r = x_i^* x_j ; x_k\}$:

From Equation 4.3 $r = \sigma^{-1}_j(i) \sigma^{-1}_j(j)$ and $r' = \sigma^{-1}_j(i)$ are equal. We show that $l' = x_p x_s^*$ and $r' = x_m x_l^*$ are equal and belong to $\mathcal{R}_\rho$, which is equivalent to $x_m x_p = x_m x_p$. 


From Equation 4.3, $x_1$ belongs to Case 3:

\[ x_1 = x_{\sigma_1^{-1}(j)} \sigma_1^{-1}(k) \]

This is true, from Equation 4.3, since $x_1x_s = x_nx_m$ in $\mathcal{R}$. For that, we show $p = \sigma^{-1}_m(l) = \sigma^{-1}_m(j)$ and $s = \sigma^{-1}_l(m) = \sigma^{-1}_l s^{-1}_n(i)$. This is true, from Equation 4.3, since $(x_1x_k = x_nx_m)$ and $(x_jx_r = x_nx_l)$ in $\mathcal{R}$.

Case 3: \( \{ l = x^*_{\sigma_1^{-1}(j)} x^*_i ; \ r = x^*_{\sigma_1^{-1}(i)} x^*_j ; \ x_k \} \):

\[ x_1 = x^*_{\sigma_1^{-1}(j)} x^*_i \]
\[ x_n = x^*_{\sigma_1^{-1}(i)} x^*_j \]
\[ x_k = x^*_{\sigma_1^{-1}(j)} \]

From Equation 4.3, \( r = \sigma^1_i \sigma^1_1(k) \) and \( r' = \sigma^1_j \sigma^1_1(j) \) are equal. We show that \( l' = x^*_{\sigma_1^{-1}(j)} \sigma^1_1(k) \) and \( l'' = x^*_{\sigma_1^{-1}(i)} \sigma^1_1(j) \) are equal and belong to $\mathcal{R}$, which is equivalent to $(x_1x_p = x_i x_m)$ in $\mathcal{R}$. We show that $p = \sigma^{-1}_q s^{-1}_1(j)$ and $m = \sigma^{-1}_q s^{-1}_1(i)$. Since $(x_1x_q = x_r x_s)$ and $(x_j x_n = x_r x_l)$ in $\mathcal{R}$, $p = \sigma^{-1}_q s^{-1}_1(j) = \sigma_q^{-1} s^{-1}_1(j)$ and $m = \sigma^{-1}_q s^{-1}_1(i) = \sigma^{-1}_q s^{-1}_1(i)$, from Equation 4.3.

Case 4: \( \{ l = x_i x_{\sigma_1^{-1}(j)} ; \ r = x_j x_{\sigma_1^{-1}(i)} ; \ x_k \} \):

\[ x_1 = x_i x_{\sigma_1^{-1}(j)} \]
\[ x_n = x_j x_{\sigma_1^{-1}(i)} \]
\[ x_k = x^* \]

From Equation 4.3, \( p = \sigma^1_i(l) \) and \( q = \sigma^1_k(k) \), $n = \sigma^{-1}_m(k)$, $l = \sigma_n \sigma^{-1}_m(i)$,

\[ s = \sigma_q \sigma^{-1}_i(j) \]
\[ r = \sigma^{-1}_p \sigma^{-1}_j(k) \]
\[ r' = \sigma^{-1}_i s^{-1}_m(k) \]
We show that \( t' = x_p x_s \) and \( t' = x_m x_l \) are equal and belong to \( \mathcal{R} \). For that, we show that \( s = \sigma_p^{-1}(m) = \sigma_m^{-1}(p) \) and \( l = \sigma_m^{-1}(p) = \sigma_m^{-1}(m) \). This is true from Equation 4.3 since \((x_k x_i = x_p x_q)\) and \((x_k x_j = x_m x_n)\) in \( \mathcal{R} \). So, \( x_p x_s = x_m x_l \) and this implies \( r = \sigma_s^{-1}(\sigma_p^{-1}(k)) \), \( r' = \sigma_l^{-1}(\sigma_m^{-1}(k)) \) are equal.

Case 5: \( \{ t = x_i x_j^* ; \ t = x_j^* x_i^* ; \ x_k^* \} \):

![Figure 7.6](image)

Case 5: \( q = \gamma_i(k) \), \( p = \gamma_q^{-1}(i) \), \( n = \gamma_q^{-1}(j) \), \( s = \gamma_q^{-1}(j) \), \( l = \gamma_{p'} \gamma_j^{-1}(i) \), \( m = \gamma_k^{-1} \gamma_j^{-1}(j) \), \( r = \gamma_j^{-1}(q) \), \( r' = \gamma_j^{-1}(n) \)

Since \( r = \gamma_j^{-1}(q) = \gamma_j^{-1} \gamma_i(k) \) and \( r' = \gamma_j^{-1} \gamma_i(n) = \gamma_j^{-1} \gamma_i^{-1}(j) \), we have \( r = r' \) from Equation 4.4

We show that \( t' = x_p x_s^* \) and \( t' = x_m x_l^* \) are equal and belong to \( \mathcal{R}_p \), which is equivalent to \((x_m x_p = x_l x_s)\) in \( \mathcal{R} \). For that, we show that \( l = \gamma_{p'}^{-1}(p) = \gamma_{p'}^{-1} \gamma_j^{-1}(i) \) and \( m = \gamma_p^{-1}(s) = \gamma_p^{-1} \gamma_q^{-1}(j) \). This is true, from Equation 4.3 since \((x_k x_i = x_p x_q)\) and \((x_m x_j = x_s x_q)\) in \( \mathcal{R} \).

Case 6: \( \{ t = x_i^* x_j^* ; \ t = x_j^* x_i^* ; \ x_k^* \} \):

![Figure 7.7](image)

Case 6: \( p = \gamma_k^{-1}(i) \), \( q = \gamma_i^{-1}(k) \), \( m = \gamma_k^{-1}(j) \), \( n = \gamma_j^{-1}(k) \), \( s = \gamma_q^{-1} \gamma_j^{-1}(j) \), \( l = \gamma_q^{-1} \gamma_j^{-1}(i) \)

From Lemma 5.3, the relations have the form \( x_j^{-1} x_i = x_j^{-1} x_i \), so from Equation 4.3 \( \gamma_i \gamma_j^{-1}(j) \( \gamma_i^{-1}(i) \), for every \( k \) in their domain of definition. This implies \( r = \gamma_i^{-1} \gamma_j^{-1}(i) \gamma_j^{-1}(k) \) and \( r' = \gamma_i^{-1} \gamma_j^{-1}(i) \gamma_j^{-1}(k) \) are equal. We show \( t' = x_p x_s^* \) and \( t' = x_m x_l^* \) are equal and belong to \( \mathcal{R}_p \), that is \((x_s x_p = x_l x_m)\) in \( \mathcal{R} \). For that, we show \( s = \gamma_p^{-1}(m) = \gamma_p^{-1} \gamma_k^{-1}(j) \) and \( l = \gamma_m^{-1}(p) = \gamma_m^{-1} \gamma_k^{-1}(i) \). This is true from Equation 4.3 since \((x_q x_i = x_p x_k)\) and \((x_m x_j = x_m x_k)\) in \( \mathcal{R} \).
References

[1] D. Bachiller, *Study of the algebraic structure of left braces and the Yang-Baxter equation*, Ph.D. thesis 2016, Universitat Autonoma de Barcelona.

[2] D. Bachiller, *Counterexample to a conjecture about braces*, J. Algebra 453 (2016), 160-176.

[3] D. Bachiller, *Extensions, matched products and simple braces*, J. Pure Appl. Algebra 222 (2018) 1670-1691.

[4] J. Belk, *Thompson’s group F*, Ph.D. Thesis (Cornell University), ArXiv 0708.3609.

[5] V. Bergelson, N. Hindman, A. Blass, *Partition Theorems for Spaces of Variable Words*, Proc. London Math. Soc. 68 (1994), n.3, 449-476.

[6] M.G. Brin, C.C. Squier, *Groups of piecewise linear homeomorphisms of the real line*, Invent. Math. 79 (1985), n. 3, 485-498.

[7] K.S. Brown, R. Geoghegan, *An infinite-dimesional torsion-free $FP_\infty$ group*, Invent. Math. 77 (1984), 367-381.

[8] J.W. Canon, W.J. Floyd, W.R. Parry, *Introductory notes to Richard Thompson’s groups*, L’enseignement Mathematique 42 (1996), 215-256.

[9] M. Castelli, F. Catino, G. Pinto, *A new family of set-theoretic solutions of the Yang-Baxter equation*, Comm. Algebra 46 (2018), 1622-1629.

[10] F. Catino, I. Colazzo, P. Stefanelli, *Skew left braces with non-trivial annihilator*, to appear J.Algebra Appl.

[11] F. Catino, I. Colazzo, P. Stefanelli, *Semi-braces and the Yang-Baxter equation*, J. Algebra 483 (2017), 163-187.

[12] F. Catino, M. Mazzotta, P. Stefanelli, *Inverse Semi-braces and the Yang-Baxter equation*, ArXiv 2007.05730.

[13] F. Catino, M.M. Miccoli, *Construction of quasi-linear left cycle sets*, J. Algebra Appl. 14, no.1, (2015).

[14] F. Cedó, E. Jespers, A. del Rio, *Involutive Yang-Baxter Groups*, Trans. Amer. Math. Soc. 362 (2010), 2541-2558.

[15] F. Cedó, E. Jespers, J. Okninski, *Retractability of set theoretic solutions of the Yang-Baxter equation*, Advances in Mathematics 224 (2010), 2472-2484.

[16] F. Cedó, E. Jespers, J. Okninski, *Braces and the Yang-Baxter equation*, Comm. Math. Phys. 327 (2014), 101-116.

[17] F. Cedó, T. Gateva-Ivanova, A. Smoktunowicz, *On the Yang-Baxter equation and left nilpotent left braces*, J. Pure App. Algebra 221 (2017), n.4, 751-756.

[18] F. Chouraqui, *Garside groups and the Yang-Baxter equation*, Comm. in Algebra 38 (2010) 4441-4460.

[19] F. Chouraqui and E. Godelle, *Folding of set theoretical solutions of the Yang-Baxter Equation, Algebra and Representation Theory 15* (2012) 1277-1290.

[20] F. Chouraqui and E. Godelle, *Finite quotients of I-type groups*, Adv. Math. 258 (2014), 46-68.

[21] F. Chouraqui, *An algorithmic construction of group automorphisms and the quantum Yang-Baxter equation*, Comm. in Algebra 46 (2018), n.11, 4710-4723.

[22] F. Chouraqui, *Left orders in Garside groups*, Int. J. of Alg. and Comp. (2016), vol.26, n.7, 1349-1359.

[23] A.H. Clifford, G.B. Preston, *The Algebraic Theory of Semigroups*, Math. Surveys of the American Math. Soc. 7, 1967.

[24] P. Dehornoy, *Gaussian groups are torsion free*, J. of Algebra 210 (1998) 291-297.

[25] P. Dehornoy and L. Paris, *Gaussian groups and Garside groups, two generalisations of Artin groups*, Proc. London Math Soc. 79 (1999) 569-604.

[26] P. Dehornoy, *Groupes de Garside*, Ann. Scient. Ec. Norm. Sup. 35 (2002), 267-306.

[27] P. Dehornoy, *The subword reversing method*, Int. J. of Alg. and Comp. 21 (2011), n.1, 71-118.
[28] P. Dehornoy, F. Digne and J. Michel, Garside families and Garside germs, J. Algebra 380 (2013), 109-145.
[29] P. Dehornoy, Coxeter-like groups for groups of set-theoretic solutions of the Yang-Baxter equation, Comptes Rendus Mathematiques 351 (2013) 419-424.
[30] P. Dehornoy, F. Digne, E. Godelle, D. Krammer, J. Michel, Foundations of Garside theory, EMS Tracts in Mathematics (2015), volume 24.
[31] V.G. Drinfeld, On some unsolved problems in quantum group theory, Lec. Notes Math. 1510 (1992) 1-8.
[32] P. Etingof, T. Schedler, A. Soloviev, Set-theoretical solutions to the Quantum Yang-Baxter equation, Duke Math. J. 100 (1999) 169-209.
[33] P. Etingof, R. Guralnick, A. Soloviev, Indecomposable set-theoretical solutions to the quantum Yang-Baxter equation on a set with a prime number of elements, J. Algebra 249 (2001), 709-719.
[34] T. Gateva-Ivanova, Regularity of skew-polynomials rings with binomial relations, Talk at the International Algebra Conference, Miskolc, Hungary, 1996.
[35] T. Gateva-Ivanova and M. Van den Bergh, Semigroups of I-type, J. Algebra 206 (1998) 97-112.
[36] T. Gateva-Ivanova, E. Jespers, J. Okninski, Quadratic algebras of skew type and the underlying semigroups, J. Algebra 270 (2003), no. 2, 635-659.
[37] T. Gateva-Ivanova, A combinatorial approach to the set-theoretic solutions of the Yang–Baxter equation, J. Math. Phys. 45 (2004), 3828-3858.
[38] T. Gateva-Ivanova, Garside Structures on Monoids with Quadratic Square-Free relations, Algebra and Representation Theory 14 (2011) 779-802.
[39] T. Gateva-Ivanova, Set-theoretical solutions of the Yang–Baxter equation, braces and symmetric groups, Adv. Math. 388 (2018), n.7, 649-701.
[40] T. Gateva-Ivanova and P.J. Cameron, Multipermutation solutions of the Yang–Baxter equation, Comm. in Math. Phys. 309(3) (2012) 583-621.
[41] L. Guarnieri, L. Vendramin, Skew braces and the Yang-Baxter equation, Math. Comp. 86 (2017), 2519-2534.
[42] E. Jespers, J. Okninski, Monoids and groups of I-type, Algebra Rep. Theory 8(2005), 709-729.
[43] E. Jespers, J. Okninski, Noetherian Semigroup Algebras, Algebra and applications, vol.7 (2007).
[44] E. Jespers, Groups and set-theoretical solutions of the Yang Baxter equation, Note Mat. 30 (2010) n. 1, 9-20.
[45] M.V. Lawson, Inverse semigroups: The Theory of Partial Symmetries, World Scientific, 1998.
[46] V. Lebed, L. Vendramin, Homology of left non-degenerate set-theoretic solutions to the Yang-Baxter equation, Advances Math. 304 (2017), 1219-1261.
[47] V. Lebed, L. Vendramin, On Structure Groups of Set-Theoretical Solutions to the Yang-Baxter Equation, Proc. Edinb. Math. Soc. 62 (2019), n. 3, 683-717.
[48] R. MacKenzie, R.J. Thompson, An elementary construction of unsolvable word problems in group theory, Word Problems: (W.W. Boone, F.B. Cannonito, and R.C. Lyndon eds.), Studies in Logic and the foundations of Mathematics, vol.71, North-Holland, Amsterdam, 1973, p. 457-478.
[49] G. Moore, N. Seiberg, Polynomial equations for rational conformal field theories, Physics Letters 212 (1988), n.4, 451-460.
[50] G.B. Preston, Inverse semi-groups, J. of the London Math. Soc. 29 (1954), n. 4, 396-403.
[51] W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation, Adv. Math. 193(2005), 40-55.
[52] W. Rump, Modules over braces, Algebra Dicrete Math. (2006), 127-137.
[53] W. Rump, Braces, radical rings, and the quantum Yang-Baxter equation, J. Algebra 307(2007), 153-170.
[54] W. Rump, Classification of cyclic braces, J. Pure Appl.Algebra 209(2007), 671-685.
[55] A. Smoktunowicz, On Engel groups, nilpotent groups, rings, braces and the Yang-Baxter equation, Trans. Amer. Math. Soc. 370 (2018), 6535-6564.
Fabienne Chouraqui
University of Haifa at Oranim, Israel.
E-mail: fabienne.chouraqui@gmail.com
fchoura@sci.haifa.ac.il