ON THE EXTENDED T-SYSTEM OF TYPE C

JIAN-RONG LI

Abstract. We continue the study of extended T-systems of quantum affine algebras. We find a sub-system of the extended T-system of the quantum affine algebra $U_q\hat{g}$ of type $C_3$. The sub-system consisting of four systems which are denoted by I, II, III, and IV. Each of the systems I, II, III, IV is closed. The systems I-IV can be used to compute minimal affinizations with weights of the form $\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3$, where at least one of $\lambda_1$, $\lambda_2$, $\lambda_3$ are zero. Using the systems I-IV, we compute the characters of the restrictions of the minimal affinizations in the systems to $U_qg$ and obtain some conjectural decomposition formulas for the restrictions of some minimal affinizations.

1. Introduction

The T-systems are some families of relations in the Grothendieck ring of the category of the finite-dimensional modules of quantum affine algebras (or Yangians), see [K83], [K84], [K87], [KR90], [KNS94], [Nak03], [Her06]. The T-systems are widely applied to representation theory, combinatorics and integrable systems, see the recent survey [KNS11].

The modules in the usual T-systems are the Kirillov-Reshetikhin modules. The Kirillov-Reshetikhin modules are simplest examples of irreducible finite-dimensional modules over quantum affine algebras. Recently, the usual T-systems have been generalized to the so called extended T-systems, see [MY12b], [LM12]. The extended T-systems contains minimal affinizations of quantum affine algebras. The family of minimal affinizations is an important family of irreducible modules which contains the Kirillov-Reshetikhin modules, see [C95], [CP95b], [CP96a], [CP96b]. The minimal affinizations are also interesting from the physical point of view, see Remark 4.2 of [FR92] and [C95].

Minimal affinizations are studied intensively in recently years, see for example, [CMY12], [CG11], [Her07], [LM12], [Mou10], [MF11], [MY12a], [MY12b], [MY12c], [Nao12]. The finite dimensional representations of $U_q\hat{g}$ and cluster algebras are closely related, see [IKKN13a], [IKKN13b], [HL10], [HL13], [Nak11]. The relations in the extended T-systems are special relations in the cluster algebras. A cluster algebra algorithm for computing $q$-characters of Kirillov-Reshetikhin modules for any untwisted quantum affine algebra is given in [HL13]. Their method uses the T-systems for Kirillov-Reshetikhin modules. We expect that the extended T-systems can be used to find new algorithms to compute $q$-characters of more general modules including minimal affinizations.

The extended T-systems of type $A$, $B$ has been found in [MY12b] and the extended T-system of type $G_2$ has been found in [LM12]. In [MY12b], it was conjectured that the extended T-systems exist in all types.
In this paper, we continue the study of extended $T$-systems of quantum affine algebras. Let $\text{Rep}(U_q\mathfrak{g})$ denote the Grothendieck ring of the category of finite-dimensional representations of $U_q\mathfrak{g}$. The irreducible finite-dimensional modules of quantum affine algebras are parameterized by the highest $l$-weights or Drinfeld polynomials.

Our method is similar as the method used in [MY12b], [LM12]. Let $a \in \mathbb{C}^*$ and $T$ an irreducible $U_q\mathfrak{g}$-module such that the zeros of the Drinfeld polynomials of $T$ belong to $aq^Z$. Following [MY12b], we define the left, right, and bottom modules, denoted by $\mathcal{L}$, $\mathcal{R}$, $\mathcal{B}$ respectively, as the modules whose Drinfeld polynomials has zeros obtained by dropping the rightmost, leftmost, and both left- and rightmost zeros of the union of zeros of the Drinfeld polynomials of the top module $T$.

The $q$-character theory and the FM algorithm are important tools to study the representation theory of quantum affine algebras, see [NT98], [FR98], [FM01]. The main tools in this paper are the $q$-character theory and the FM algorithm.

If the $q$-character of a module contains only one dominant monomial, then it is called a special module. Let $\mathfrak{g}$ be the simple Lie algebra of type $C_3$. The FM algorithm applies to special modules. Although in general the minimal affinizations of type $C_3$ are not special, there are some families of minimal affinizations of type $C$ which are special.

For $k, \ell, m \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$, let

$$T_{k, \ell, m}^{(s)} = \left( \prod_{i=0}^{k-1} 3s+4i \right) \left( \prod_{i=0}^{\ell-1} 2s+4k+2i+1 \right) \left( \prod_{i=0}^{m-1} 1s+4k+2\ell+2i+2 \right),$$

$$\tilde{T}_{k, \ell, m}^{(s)} = \left( \prod_{i=0}^{k-1} 1s+2i \right) \left( \prod_{i=0}^{\ell-1} 2s+2k+2i+1 \right) \left( \prod_{i=0}^{m-1} 3s+2k+2\ell+4i+4 \right).$$

We use $T$ to denote the highest $l$-weight module with a highest $l$-weight $T$. Then $T_{k, \ell, m}^{(s)}$ and $\tilde{T}_{k, \ell, m}^{(s)}$ are all minimal affinizations of type $C_3$.

We take minimal affinizations $T_{k, \ell, 0}^{(s)}$, $\tilde{T}_{k, \ell, 0}^{(s)}$ as top modules $T$ respectively. It turns out that the left, right, bottom modules of $T$ satisfy a relation $[\mathcal{L}][\mathcal{R}] = [T][\mathcal{B}] + [S]$. Here $[\cdot]$ denotes the equivalence class of a $U_q\mathfrak{g}$-module in $\text{Rep}(U_q\mathfrak{g})$ and $S$ is a tensor product of some irreducible modules. The factors of $S$ are called sources. The factors of the sources can be some modules which are not minimal affinizations. In order to obtain a closed system, we take the sources as top modules and compute new left, right, bottom modules, and sources. We continue this procedure until all modules in the sources are the modules obtained before. Then we obtain the desired closed systems $I$ and $III$.

The system $I$ contains minimal affinizations $T_{k, \ell, 0}^{(s)}$, $\tilde{T}_{k, 0, m}^{(s)}$, and $\tilde{T}_{k, \ell, 0}^{(s)}$ for all $k, \ell, m \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$. The system $III$ contains minimal affinizations $\tilde{T}_{k, \ell, 0}^{(s)}$ for all $k, \ell \in \mathbb{Z}$. We denote by II, IV the dual systems of I, III respectively. The system $II$ contains minimal affinizations which can be obtained from $\tilde{T}_{0, k, \ell}^{(s)}$, $k, \ell, m \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$, by shifting the upper-subscripts.

The system $IV$ contains minimal affinizations which can be obtained from $T_{0, k, \ell}^{(s)}$, $k, \ell \in \mathbb{Z}$, $s \in \mathbb{Z}$, by shifting the upper-subscripts. Therefore we find a sub-system of the extended
T-system of the quantum affine algebra $U_q \hat{\mathfrak{g}}$ of type $C_3$ which is the union of the systems I, II, III, IV and the sub-system can be used to compute minimal affinizations with weights of the form $\lambda_1 \omega_1 + \lambda_2 \omega_2 + \lambda_3 \omega_3$, where at least one of $\lambda_1$, $\lambda_2$, $\lambda_3$ are zero, see Section 3. We find that the modules in the systems I, III are special. We show that these modules are special. We show that every relation $[L][R] = [T][B] + [S]$ in the system holds by comparing the dominant monomials in both sides of $[L][R] = [T][B] + [S]$. Moreover, we show that the modules $T \otimes B$ and $S$ in the systems I-IV are irreducible.

The extended T-system is a powerful tool of studying the finite dimensional representations of $U_q \hat{\mathfrak{g}}$. Using the systems I-IV, we compute the characters of the restrictions of some minimal affinizations, see Section 8.

The paper is organized as follows. In Section 2 we give some background material. In Section 3 we describe the system I, II, III, and IV. In Section 4 we prove that the modules in the systems I and III are special. In Section 5 we prove Theorem 3.3 and Theorem 3.12. In Section 6 we prove that the module $T \otimes B$ is irreducible for each relation in the systems I, II, III, and IV. In Section 7 we prove Proposition 3.3. In Section 8 we give conjectural character formulas for $\text{Res}(T_{k,\ell,0}^{(s)}), \text{Res}(T_{k,0,m}^{(s)}), \text{Res}(T_{k,0,n}^{(s)}), s \in \mathbb{Z}, k, \ell, m \in \mathbb{Z}_{\geq 0}$.}

2. Background

2.1. Quantum affine algebra. Let $\mathfrak{g}$ be a complex simple Lie algebra of type $C_3$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let $I = \{1, 2, 3\}$. We choose simple roots $\alpha_1, \alpha_2, \alpha_3$ and scalar product $\langle \cdot, \cdot \rangle$ such that

\[
(\alpha_1, \alpha_1) = 2, \ (\alpha_2, \alpha_2) = 2, \ (\alpha_3, \alpha_3) = 4, \ (\alpha_1, \alpha_2) = -1, \ (\alpha_1, \alpha_3) = 0, \ (\alpha_2, \alpha_3) = -2.
\]

Let $\{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$ and $\{\omega_1, \omega_2, \omega_3\}$ be the sets of simple coroots and fundamental weights respectively. Let $C = (C_{ij})_{i,j \in I}$ denote the Cartan matrix, where $C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$. Let $r_1 = 1, r_2 = 1, r_3 = 2, D = \text{diag}(r_1, r_2, r_3)$ and $B = DC$. Then

\[
C = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -2 \\
0 & -1 & 2
\end{pmatrix}, \quad B = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -2 \\
0 & -2 & 4
\end{pmatrix}.
\]

Let $Q$ (resp. $Q^+$) and $P$ (resp. $P^+$) denote the $\mathbb{Z}$-span (resp. $\mathbb{Z}_{\geq 0}$-span) of the simple roots and fundamental weights respectively. There is a partial order $\leq$ on $P$ such that $\lambda \leq \lambda'$ if and only if $\lambda' - \lambda \in Q^+$. 

The quantum affine algebra $U_q\hat{g}$ is a $\mathbb{C}(q)$-algebra generated by $x_i^\pm (i \in I, n \in \mathbb{Z})$, $k_i^\pm$ ($i \in I$), $h_i$ ($i \in I, n \in \mathbb{Z}\setminus \{0\}$) and central elements $c^{\pm 1/2}$, subject to certain relations, see [Dri88]. The algebra $U_q\hat{g}$ is a Hopf algebra.

Let $U_q\hat{g}$ be the quantized enveloping algebra of $g$. The subalgebra of $U_q\hat{g}$ generated by $(k_i^\pm)_{i \in I}, (x_i^\pm)_{i \in I}$ is a Hopf subalgebra of $U_q\hat{g}$ and is isomorphic as a Hopf algebra to $U_qg$. Therefore $U_q\hat{g}$-modules restrict to $U_qg$-modules. We denote by $\text{Res}(V)$ the restriction of a $U_q\hat{g}$-module $V$ to $U_qg$.

### 2.2. Finite-dimensional representations of $U_q\hat{g}$ and $q$-characters

A $U_q\hat{g}$-module is called of type 1 if $c^{\pm 1/2}$ acts as the identity on $V$ and

$$V = \bigoplus_{\lambda \in P} V_\lambda, \quad V_\lambda = \{v \in V : k_i v = q^{(\alpha_i, \lambda)} v\}. \quad (2.1)$$

We will only consider finite-dimensional type 1 $U_q\hat{g}$-modules in this paper. The decomposition (2.1) of a finite-dimensional $U_q\hat{g}$-module $V$ into its $U_q\hat{g}$-weight spaces can be refined as follows, see [FR98]:

$$V = \bigoplus_{\gamma} V_\gamma, \quad \gamma = (\gamma_{i,\pm r})_{i \in I, r \in \mathbb{Z}_{\geq 0}}, \quad \gamma_{i,\pm r} \in \mathbb{C}, \quad (2.2)$$

where

$$V_\gamma = \{v \in V : \exists k \in \mathbb{N}, \forall i \in I, m \geq 0, (\phi_i^\pm - \gamma_i^\pm m)^k v = 0\}.$$ 

If $\dim(V_\gamma) > 0$, then $\gamma$ is called an l-weight of $V$.

Let $P$ denote the free Abelian multiplicative group of monomials in infinitely many formal variables $(Y_{i,a})_{i \in I, a \in \mathbb{C}^\times}$. In [FR98], it is shown that there is a bijection $\gamma$ from from $P$ to the set of l-weights of finite-dimensional $U_q\hat{g}$-modules. Therefore the monomials in $P$ are also called l-weights. For simplicity, we denote $V_m = V_{\gamma(m)}$.

Let $\mathbb{Z}P = \mathbb{Z}[Y_{i,a}]_{i \in I, a \in \mathbb{C}^\times}$. For $\chi \in \mathbb{Z}P$, we write $m \in P$ if the coefficient of $m$ in $\chi$ is non-zero.

The $q$-character of a $U_q\hat{g}$-module $V$ is given by

$$\chi_q(V) = \sum_{m \in P} \dim(V_m) m \in \mathbb{Z}P,$$

where $V_m = V_{\gamma(m)}$.

Let $\text{Rep}(U_q\hat{g})$ be the Grothendieck ring of finite-dimensional representations of $U_q\hat{g}$ and $[V] \in \text{Rep}(U_q\hat{g})$ the class of a finite-dimensional $U_q\hat{g}$-module $V$. The $q$-character map

$$\chi_q : \text{Rep}(U_q\hat{g}) \to \mathbb{Z}P, \quad V \mapsto \chi_q(V),$$

is an injective ring homomorphism, see [FR98].

Let $\mathcal{M}(V)$ denote the set of all monomials in $\chi_q(V)$ for any finite-dimensional $U_q\hat{g}$-module $V$. For each $j \in I$, a monomial $m = \prod_{a \in \mathbb{C}^\times} Y_{i,a}^{u_{i,a}}$, where $u_{i,a}$ are some integers, is said to be $j$-dominant (resp. $j$-anti-dominant) if and only if $u_{j,a} \geq 0$ (resp. $u_{j,a} \leq 0$).
for all $a \in \mathbb{C}^x$. A monomial is called dominant (resp. anti-dominant) if and only if it is $j$-dominant (resp. $j$-anti-dominant) for all $j \in I$. Let $\mathcal{P}^+ \subset \mathcal{P}$ denote the set of all dominant monomials.

Let $V$ be a $U_q\hat{g}$-module and $m \in \mathcal{M}(V)$ a monomial. Let $v \in V_m$ be a non-zero vector. If

$$x^+_{i,r} \cdot v = 0, \quad \phi^+_{i,\pm t} \cdot v = \gamma(m)^\pm_{i,\pm t} v, \quad \forall i \in I, r \in \mathbb{Z}, t \in \mathbb{Z}_{\geq 0},$$

then $v$ is called a highest $l$-weight vector with highest $l$-weight $\gamma(m)$. The module $V$ is called a highest $l$-weight representation if $V = U_q\hat{g} \cdot v$ for some highest $l$-weight vector $v \in V$.

In [CP94], [CP95a], it is shown that there is a bijection between the set of isomorphism classes of finite-dimensional irreducible highest $l$-weight $U_q\hat{g}$-modules of type 1. Let $L(m_\pm)$ denote the irreducible highest $l$-weight $U_q\hat{g}$-module corresponding to $m_\pm \in \mathcal{P}^+$. We use $\chi_q(m_\pm)$ to denote $\chi_q(L(m_\pm))$ for $m_\pm \in \mathcal{P}^+$.

We have the following well-known lemma.

**Lemma 2.1.** Let $m_1, m_2$ be two monomials. Then $L(m_1m_2)$ is a sub-quotient of $L(m_1) \otimes L(m_2)$. In particular, $\mathcal{M}(L(m_1m_2)) \subseteq \mathcal{M}(L(m_1)) \cdot \mathcal{M}(L(m_2))$. □

For $b \in \mathbb{C}^x$, we define the shift of spectral parameter map $\tau_b : \mathbb{Z}P \to \mathbb{Z}P$ to be a ring homomorphism sending $Y_{i,a}^{\pm 1}$ to $Y_{i,ab}^{\pm 1}$. Let $m_1, m_2 \in \mathcal{P}^+$. If $\tau_b(m_1) = m_2$, then

$$\tau_b \chi_q(m_1) = \chi_q(m_2). \quad (2.3)$$

A finite-dimensional $U_q\hat{g}$-module $V$ is said to be special if and only if $\mathcal{M}(V)$ contains exactly one dominant monomial. It is called anti-special if and only if $\mathcal{M}(V)$ contains exactly one anti-dominant monomial. It is called thin if and only if every $l$-weight space of $V$ has dimension less or equal to 1. Clearly, if a module is special or anti-special, then it is irreducible.

Define $A_{i,a} \in \mathcal{P}$, $i \in I, a \in \mathbb{C}^x$, by

$$A_{1,a} = Y_{1,aq} Y_{1,aq^{-1}} Y_{a}^{-1},$$
$$A_{2,a} = Y_{2,aq} Y_{2,aq^{-1}} Y_{1,a}^{-1} Y_{3,a}^{-1},$$
$$A_{3,a} = Y_{3,aq} Y_{3,aq^{-1}} Y_{2,aq} Y_{2,aq^{-1}}^{-1}.$$

Let $Q$ be the subgroup of $\mathcal{P}$ generated by $A_{i,a}, i \in I, a \in \mathbb{C}^x$. Let $Q^\pm$ be the monoids generated by $A_{i,a}^{\pm 1}, i \in I, a \in \mathbb{C}^x$. There is a partial order $\leq$ on $\mathcal{P}$ such that

$$m \leq m' \text{ if and only if } m' m^{-1} \in Q^+. \quad (2.4)$$

For all $m_+ \in \mathcal{P}^+, \mathcal{M}(L(m_+)) \subset m_+ Q^-$, see [EM01].

Let $m$ be a monomial. If for all $a \in \mathbb{C}^x$ and $i \in I$, we have the property: if the power of $Y_{i,a}$ in $m$ is non-zero and the power of $Y_{j,ak}$ in $m$ is zero for all $j \in I, k \in \mathbb{Z}_{>0}$, then the power of $Y_{i,a}$ in $m$ is negative, then the monomial $m$ is called right negative. For $i \in I, a \in \mathbb{C}^x, A_{i,a}^{-1}$ is right-negative. A product of right-negative monomials is right-negative. If $m$ is right-negative and $m' \leq m$, then $m'$ is right-negative.
2.3. Minimal affinizations of $U_q\mathfrak{g}$-modules. The minimal affinizations of type $C_3$ are classified in \cite{CP95b}. Let $\mathfrak{g}$ be the simple Lie algebra of type $C_3$ and $V(\lambda)$ the $U_q(\mathfrak{g})$ module of weight $\lambda$, where $k, \ell, m \in \mathbb{Z}_{\geq 0}$ and $\lambda = m\omega_1 + \ell \omega_2 + k \omega_3$. A simple $U_q\mathfrak{g}$-module $L(m_+)$ is a minimal affinization of $V(\lambda)$ if and only if $m_+$ is one of the following monomials

$$T_{k,\ell,m}^{(s)} = \left( \prod_{i=0}^{k-1} Y_{3,aq^{2i+1}} \right) \left( \prod_{i=0}^{\ell-1} Y_{2,aq^{4i+2}+2i+1} \right) \left( \prod_{i=0}^{m-1} Y_{1,aq^{4k+2i+2}} \right),$$

$$\tilde{T}_{k,\ell,m}^{(s)} = \left( \prod_{i=0}^{k-1} Y_{1,aq^{2i+2}} \right) \left( \prod_{i=0}^{\ell-1} Y_{2,aq^{2k+2i+1}} \right) \left( \prod_{i=0}^{m-1} Y_{3,aq^{2k+4i+4}} \right),$$

for some $a \in \mathbb{C}^\times$, see \cite{CP95b}. In particular, when $k = 0$ or $\ell = 0$ or $m = 0$, the minimal affinization $L(m_+)$ is a Kirillov-Reshetikhin module.

Let $L(m_+)$ be a Kirillov-Reshetikhin module. It is shown in \cite{Her06} that any non-highest monomial in $\mathcal{M}(L(m_+))$ is right-negative and hence $L(m_+)$ is special.

2.4. $q$-characters of $U_q\mathfrak{sl}_2$-modules and the FM algorithm. The $q$-characters of $U_q\mathfrak{sl}_2$-modules are well-understood, see \cite{CP91, FR98}. We recall the results here.

Let $W_k^{(a)}$ be the irreducible representation $U_q\mathfrak{sl}_2$ with highest weight monomial

$$X_k^{(a)} = \prod_{i=0}^{k-1} Y_{aq^{k-2i+1}},$$

where $Y_a = Y_{1,a}$. Then the $q$-character of $W_k^{(a)}$ is given by

$$\chi_q(W_k^{(a)}) = X_k^{(a)} \prod_{j=0}^{k-1} A_{aq^{k-2j}}^{-1},$$

where $A_a = Y_{aq^{-1}}Y_{aq}$.

For $a \in \mathbb{C}^\times$, $k \in \mathbb{Z}_{\geq 1}$, the set $\Sigma_k^{(a)} = \{aq^{k-2i-1}\}_{i=0,...,k-1}$ is called a string. Two strings $\Sigma_k^{(a)}$ and $\Sigma_k^{(a')} \subset \Sigma_{k'}^{(a')}$ are said to be in general position if the union $\Sigma_k^{(a)} \cup \Sigma_k^{(a')}$ is not a string or $\Sigma_k^{(a)} \subset \Sigma_{k'}^{(a')}$ or $\Sigma_k^{(a')} \subset \Sigma_{k}^{(a)}$.

Let $L(m_+)$ denote the irreducible $U_q\mathfrak{sl}_2$-module with highest weight monomial $m_+$. Let $m_+ \neq 1$, $m_+ \in \mathbb{Z}[Y_a]_{a \in \mathbb{C}^\times}$, be a dominant monomial. Then $m_+$ can be uniquely written in the form

$$m_+ = \prod_{i=1}^{s} \left( \prod_{b \in \Sigma_{k_i}^{(a_i)}} Y_b \right),$$
up to a permutation, where \( s \) is an integer, \( \Sigma_{k_i}^{(a_i)} \), \( i = 1, \ldots, s \), are strings which are pairwise in general position and

\[
L(m_+) = \bigotimes_{i=1}^{s} W_{k_i}^{(a_i)} , \quad \chi_q(m_+) = \prod_{i=1}^{s} \chi_q(W_{k_i}^{(a_i)}).
\]

For \( j \in I \), let

\[
\beta_j : \mathbb{Z}[Y_{i,a}^\pm]_{i \in I, a \in \mathbb{C}^\times} \to \mathbb{Z}[Y_{a}^\pm]_{a \in \mathbb{C}^\times}
\]

be the ring homomorphism which sends, for all \( a \in \mathbb{C}^\times \), \( Y_{k,a} \mapsto 1 \) for \( k \neq j \) and \( Y_{j,a} \mapsto Y_{a} \).

Let \( V \) be a \( U_q\hat{g} \)-module. Then \( \beta_i(\chi_q(V)) \), \( i = 1, 2 \), is the \( q \)-character of \( V \) considered as a \( U_q\hat{g} \)-module. In Section 5 of \([FM01]\), a powerful algorithm is proposed to compute the \( q \)-characters of \( U_q\hat{g} \)-modules. The algorithm is called the FM algorithm. The FM algorithm recursively computes the minimal possible \( q \)-character which contains \( m_++ \) and is consistent when restricted to \( U_q\hat{sl}_2 \), \( i \in I \). If a module \( L(m_+) \) is special, then the FM algorithm produces the correct \( q \)-character \( \chi_q(m_+) \), see \([FM01]\).

2.5. Truncated \( q \)-characters. In this paper, we need to use the concept truncated \( q \)-characters, see \([HL10]\), \([MY12b]\). Given a set of monomials \( \mathcal{R} \subset \mathcal{P} \), let \( \mathbb{Z}\mathcal{R} \subset \mathbb{Z}\mathcal{P} \) denote the \( \mathbb{Z} \)-module of formal linear combinations of elements of \( \mathcal{R} \) with integer coefficients. Define

\[
\text{trunc}_\mathcal{R} : \mathcal{P} \to \mathcal{R}; \quad m \mapsto \begin{cases} m & \text{if } m \in \mathcal{R}, \\ 0 & \text{if } m \notin \mathcal{R}, \end{cases}
\]

and extend \( \text{trunc}_\mathcal{R} \) as a \( \mathbb{Z} \)-module map \( \mathbb{Z}\mathcal{P} \to \mathbb{Z}\mathcal{R} \).

Given a subset \( U \subset I \times \mathbb{C}^\times \), let \( \mathcal{Q}_U \) be the subgroups of \( \mathcal{Q} \) generated by \( A_{i,a} \) with \( (i,a) \in U \). Let \( \mathcal{Q}_U^\pm \) be the monoid generated by \( A_{i,a}^{\pm 1} \) with \( (i,a) \in U \). The polynomial \( \text{trunc}_{\mathcal{M}_+}^{\mathcal{Q}_U^-} \chi_q(m_+) \) is called the \( q \)-character of \( L(m_+) \) truncated to \( U \).

The following theorem can be used to compute some truncated \( q \)-characters.

**Theorem 2.2** (Theorem 2.1, \([MY12b]\)). Let \( U \subset I \times \mathbb{C}^\times \) and \( m_+ \in \mathcal{P}^+ \). Suppose that \( \mathcal{M} \subset \mathcal{P} \) is a finite set of distinct monomials such that

1. \( \mathcal{M} \subset m_+ \mathcal{Q}_U^- \),
2. \( \mathcal{P}^+ \cap \mathcal{M} = \{m_+\} \),
3. for all \( m \in \mathcal{M} \) and all \( (i,a) \in U \), if \( mA_{i,a}^{-1} \notin \mathcal{M} \), then \( mA_{i,a}^{-1}A_{j,b} \notin \mathcal{M} \) unless \( (j,b) = (i,a) \),
4. for all \( m \in \mathcal{M} \) and all \( i \in I \), there exists a unique \( i \)-dominant monomial \( M \in \mathcal{M} \) such that

\[
\text{trunc}_{\beta_i(M\mathcal{Q}_U^-)} \chi_q(\beta_i(M)) = \sum_{m' \in m\mathcal{Q}_U^{-} \cap \mathcal{M}} \beta_i(m').
\]
Then
\[ \text{trunc}_{r+ \mathcal{Q}^\ominus_q} \chi_q(m_+) = \sum_{m \in M} m. \]

Here \( \chi_q(\beta_i(M)) \) is the \( q \)-character of the irreducible \( U_q(\mathfrak{sl}_2) \)-module with highest weight monomial \( \beta_i(M) \) and \( \text{trunc}_{\beta_i(M) \mathcal{Q}^\ominus_q} \) is the polynomial obtained from \( \chi_q(\beta_i(M)) \) by keeping only the monomials of \( \chi_q(\beta_i(M)) \) in the set \( \beta_i(M \mathcal{Q}^\ominus_q) \).

3. Extended T-system of type \( C_3 \)

In this section, \( \mathfrak{g} \) is the simple Lie algebra of type \( C_3 \). We will describe the extended T-system which contains all minimal affinizations of type \( C_3 \) whose weights are of the form \( \lambda_1 \omega_1 + \lambda_2 \omega_2 + \lambda_3 \omega_3 \), where at least one of \( \lambda_1, \lambda_2, \lambda_3 \) are 0. The extended T-system consisting of four sub-systems I, II, III, IV. Each of the systems I, II, III, IV is closed.

3.1. Fundamental \( q \)-characters of type \( C_3 \). First we use the FM algorithm to compute the \( q \)-characters of the fundamental modules type \( C_3 \).

**Lemma 3.1.** The \( q \)-characters of the fundamental modules type \( C_3 \) are
\[
\chi_q(1_0) = 1_0 + 1_2^{-1}2_1 + 2_3^{-1}3_2 + 2_53_6^{-1} + 1_62_7^{-1} + 1_8^{-1},
\]
\[
\chi_q(2_0) = 2_0 + 1_22_1^{-1}3_1 + 1_3^{-1}3_1 + 1_12_43_5^{-1} + 1_3^{-1}2_43_5^{-1} + 1_15_26^{-1} + 1_31_52_62^{-1} + 1_52_62^{-1} + 1_62_7^{-1} + 1_72_43_3^{-1}
\]
\[
\chi_q(3_0) = 3_0 + 1_23_3^{-1} + 1_42_5^{-1} + 1_21_43_5^{-1} + 1_62_7^{-1} + 1_21_43_5^{-1} + 1_62_7^{-1} + 1_41_62_53_6^{-1} + 1_41_62_53_6^{-1} + 1_21_43_5^{-1} + 1_62_7^{-1} + 1_41_62_53_6^{-1} + 1_41_62_53_6^{-1}.
\]

3.2. The system I. For \( s \in \mathbb{Z}, k, \ell, m \in \mathbb{Z}_{\geq 0} \), we define the following monomials.
\[
\mathcal{T}^{(s)}_{k,\ell,m} = (\prod_{i=0}^{k-1}3_{s+4i}) (\prod_{i=0}^{\ell-1}2_{s+4k+2i+1}) (\prod_{i=0}^{m-1}1_{s+4k+2\ell+2i+2}),
\]
\[
\tilde{\mathcal{T}}^{(s)}_{k,\ell,m} = (\prod_{i=0}^{k-1}1_{s+2i}) (\prod_{i=0}^{\ell-1}2_{s+2k+2i+1}) (\prod_{i=0}^{m-1}3_{s+2k+2\ell+4i+4}),
\]
\[
\mathcal{S}^{(s)}_{k,\ell} = (\prod_{i=0}^{k-1}2_{s+2i}) (\prod_{i=0}^{\ell-1}2_{s+2k+2i+4}).
\]
For $T_k \in -s \geq \tilde{\mathcal{T}}(s,\ell) = \begin{bmatrix} k, \ell \end{bmatrix}$, we have the following relations in $\text{Rep}(s)_0:\ k, \ell, m$

\[ R_{k,\ell,m}^{(s)} = \left( \prod_{i=0}^{k-1} 2s+2i \right) \left( \prod_{i=0}^{\ell-1} 1s+2k+2i+1 \right) \left( \prod_{i=0}^{m-1} 3s+2k+4i+3 \right). \]

Let $T$ be a monomial and $\mathcal{T}$ the highest l-weight module of $U_q\mathfrak{g}$ with highest l-weight $T$. For $k, \ell \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}$, we have the following trivial relations:

\[ \text{\begin{align*}
\mathcal{T}_{0,0,0}^{(s)} &= \mathcal{T}_{0,0,0}^{(s-2)}, \\
\mathcal{T}_{0,k,0}^{(s)} &= \mathcal{T}_{0,k,0}^{(s-1)}, \\
\mathcal{T}_{0,0,k}^{(s)} &= \mathcal{T}_{0,0,k}^{(s+1)}, \\
\mathcal{S}_{0,k,0}^{(s)} &= \mathcal{S}_{0,k,0}^{(s-1)}, \\
\mathcal{R}_{0,k,0}^{(s)} &= \mathcal{R}_{0,k,0}^{(s+1)}.
\end{align*}} \]

(3.1)

Theorem 3.2. The modules $\mathcal{T}_{k,\ell,0}^{(s)}, \mathcal{T}_{k,0,m}^{(s)}, \tilde{\mathcal{T}}_{0,\ell,r}^{(s)}, \tilde{\mathcal{T}}_{k,0,m}^{(s)}, \mathcal{S}_{k,\ell}^{(s)}, \mathcal{R}_{k,2\ell,\ell}^{(s)}, \mathcal{R}_{k,2\ell+1,\ell}^{(s)}, \mathcal{R}_{k,2\ell+2,\ell}^{(s)}$ are special. In particular, we can use the FM algorithm to compute these modules.

We will prove Theorem 3.2 in Section 4.

For $k \in \mathbb{Z}_{\geq 1},$ we have the following relations in $\text{Rep}(U_q\mathfrak{g})$ in the usual T-system, see [KR90], [Her06],

\[ \mathcal{T}_{k-1,0,0}^{(s)}\mathcal{T}_{k-1,0,0}^{(s+4)} = \mathcal{T}_{k,0,0}^{(s)}\mathcal{T}_{k-2,0,0}^{(s+4)} + \mathcal{T}_{0,2k-2,0}^{(s+1)}, \]

(3.2)

\[ \mathcal{T}_{0,k-1,0}^{(s)}\mathcal{T}_{0,k-1,0}^{(s+2)} = \mathcal{T}_{0,k,0}^{(s)}\mathcal{T}_{0,k-2,0}^{(s+2)} + \mathcal{T}_{0,k-1,0}^{(s)}\mathcal{T}_{0,\ell,0}^{(s+1)}\mathcal{T}_{0,0,\frac{1}{2}}^{(s+3)} + \mathcal{T}_{0,0,\frac{1}{2}}^{(s+3)}, \]

(3.3)

\[ \mathcal{T}_{k,0,0}^{(s)}\mathcal{T}_{k-1,0,0}^{(s+2)} = \mathcal{T}_{k,0,0}^{(s)}\mathcal{T}_{k-2,0,0}^{(s+2)} + \mathcal{T}_{k,0,0}^{(s)}\mathcal{T}_{k,0,0}^{(s+1)}\mathcal{T}_{0,0,\frac{1}{2}}^{(s+1)} + \mathcal{T}_{k,0,0}^{(s+1)}\mathcal{T}_{0,0,\frac{1}{2}}^{(s+1)} \]

(3.4)

Let $\sigma : \mathbb{Z}_{\geq 0} \to \{0, 1\}$ be the function such that $\sigma(k) = 0$ if $k$ is even and $\sigma(k) = 1$ if $k$ is odd.

Our first main result is the following theorem.

Theorem 3.3. For $k, \ell, m \in \mathbb{Z}_{\geq 1}, r \in \{1, 2\}$, we have the following relations in $\text{Rep}(U_q\mathfrak{g})$.

\[ \mathcal{T}_{k,\ell-1,0}^{(s)}\mathcal{T}_{k-1,\ell,0}^{(s+4)} = \mathcal{T}_{k,\ell,0}^{(s)}\mathcal{T}_{k-1,\ell-1,0}^{(s+4)} + \mathcal{R}_{k,\ell-1,\ell,0}^{(s+1)}\mathcal{T}_{k,\ell,0}^{(s+1)} + \mathcal{S}_{k,\ell,0}^{(s+1)}\mathcal{T}_{k,\ell-1,0}^{(s+1)} + \mathcal{S}_{k,\ell,0}^{(s+1)}\mathcal{T}_{k,\ell-1,0}^{(s+1)}, \]

(3.5)

\[ \mathcal{T}_{k,0,m-1}^{(s)}\mathcal{T}_{k-1,0,m}^{(s+4)} = \mathcal{T}_{k,0,m}^{(s)}\mathcal{T}_{k-1,0,m-1}^{(s+4)} + \mathcal{S}_{k,\ell,0}^{(s+1)}\mathcal{T}_{k,0,m}^{(s+1)} + \mathcal{S}_{k,\ell,0}^{(s+1)}\mathcal{T}_{k,0,m}^{(s+1)} + \mathcal{S}_{k,\ell,0}^{(s+1)}\mathcal{T}_{k,0,m}^{(s+1)}, \]

(3.6)

\[ \mathcal{S}_{k,\ell-1}^{(s)}\mathcal{S}_{k-1,\ell}^{(s+2)} = \mathcal{S}_{k,\ell}^{(s)}\mathcal{S}_{k-1,\ell-1}^{(s+2)} + \mathcal{T}_{k,0,\ell}^{(s+1)}\mathcal{T}_{k,0,\ell}^{(s+1)}\mathcal{T}_{k,0,\ell}^{(s+1)}\mathcal{T}_{k,0,\ell}^{(s+1)} + \mathcal{T}_{k,0,\ell}^{(s+1)}\mathcal{T}_{k,0,\ell}^{(s+1)}\mathcal{T}_{k,0,\ell}^{(s+1)}\mathcal{T}_{k,0,\ell}^{(s+1)}, \]

(3.7)

\[ \mathcal{T}_{0,\ell,r-1}^{(s)}\mathcal{T}_{0,\ell-1,r}^{(s+2)} = \mathcal{T}_{0,\ell,r-1}^{(s+1)}\mathcal{T}_{0,\ell-1,r}^{(s+1)} + \mathcal{T}_{0,\ell,r-1}^{(s+1)}\mathcal{T}_{0,\ell-1,r}^{(s+1)} + \mathcal{T}_{0,\ell,r-1}^{(s+1)}\mathcal{T}_{0,\ell-1,r}^{(s+1)} + \mathcal{T}_{0,\ell,r-1}^{(s+1)}\mathcal{T}_{0,\ell-1,r}^{(s+1)}, \]

(3.8)
$$[\mathcal{R}^{(s)}_{k,2\ell,\ell-1}] [\mathcal{R}^{(s+2)}_{k-1,2\ell,\ell}] = [\mathcal{R}^{(s)}_{k,2\ell,\ell}] [\mathcal{R}^{(s+2)}_{k-1,2\ell,\ell-1}] + [\mathcal{T}^{(s-1)}_{0,0,0,0}] [\mathcal{T}^{(s+2\sigma(k+1)+1)}_{\frac{1}{2^{k}},\ell,\ell-1}] [\mathcal{T}^{(s+2\sigma(k+1)+1)}_{\frac{1}{2^{k}},2\ell-1,0}],$$

(3.10)

$$[\mathcal{R}^{(s)}_{k,2\ell,\ell}] [\mathcal{R}^{(s+2)}_{k-1,2\ell+\ell}] = [\mathcal{R}^{(s)}_{k,2\ell+1,\ell}] [\mathcal{R}^{(s+2)}_{k-1,2\ell,\ell}] + [\mathcal{T}^{(s+1)}_{k-1,0,\ell}] [\mathcal{T}^{(s+2\sigma(k+1)+1)}_{\frac{1}{2^{k+1}},\ell+\ell,0}] [\mathcal{T}^{(s+2\sigma(k+1)+1)}_{\frac{1}{2^{k}},2\ell,0}],$$

(3.11)

$$[\mathcal{R}^{(s)}_{k,2\ell,\ell}] [\mathcal{R}^{(s+2)}_{k-1,2\ell+2,\ell}] = [\mathcal{R}^{(s)}_{k,2\ell+2,\ell}] [\mathcal{R}^{(s+2)}_{k-1,2\ell+1,\ell}] + [\mathcal{T}^{(s+1)}_{k-1,0,\ell}] [\mathcal{T}^{(s+2\sigma(k+1)+1)}_{\frac{1}{2^{k+1}},\ell+\ell,0}] [\mathcal{T}^{(s+2\sigma(k+1)+1)}_{\frac{1}{2^{k}},2\ell+1,0}],$$

(3.12)

$$[\mathcal{R}^{(s)}_{0,2\ell+i,\ell}] = [\mathcal{T}^{(s-2)}_{0,0,2\ell+i}] [\mathcal{T}^{(s+2)}_{\ell,0,0}], \quad i = 0, 1, 2. \tag{3.13}$$

We will prove Theorem 3.3 in Section 5.

From the relations (3.10), (3.11), and (3.12), we see that if $\mathcal{R}_{k,\ell,0}$ is a module in the relations of Theorem 3.3 for some $k, \ell, s$, then $\ell \in \{0, 1, 2\}$. Therefore we need the relations for the modules $\mathcal{R}^{(s)}_{k,\ell,0} = \mathcal{T}^{(s-1)}_{0,\ell,0}$, $\ell \in \{0, 1, 2\}$, in order to obtain a closed system. These relations are (3.9).

Let us use $I$ to denote the system consisting of the relations in Theorem 3.3 and the relations (3.1), (3.2), (3.4). Then the system I is closed.

All relations except (3.13) in Theorem 3.3 are written in the form $[\mathcal{L}] [\mathcal{R}] = [\mathcal{T}] [\mathcal{B}] + [\mathcal{S}]$, where $\mathcal{L}, \mathcal{R}, \mathcal{T}, \mathcal{B}$ are irreducible modules which are called left, right, top and bottom modules and $\mathcal{S}$ is a tensor product of some irreducible modules. The factors of $\mathcal{S}$ are called sources, see [MY12], [LMT12]. Moreover, we have the following theorem.

**Theorem 3.4.** For each relation in Theorem 3.3 all summands on the right hand side, $\mathcal{T} \otimes \mathcal{B}$ and $\mathcal{S}$, are irreducible.

We will prove Theorem 3.4 in Section 6.

The system I can be used to compute the q-characters of the modules in the system. We have the following proposition.

**Proposition 3.5.** One can compute the q-characters of $\mathcal{T}^{(s)}_{k,\ell,0}$, $\mathcal{T}^{(s)}_{k,0,m}$, $\mathcal{T}^{(s)}_{0,\ell,r}$, $\mathcal{T}^{(s)}_{k,0,m}$, $\mathcal{S}^{(s)}_{k,\ell}$, $\mathcal{R}^{(s)}_{k,2\ell+j,\ell}$, $j = 0, 1, 2$, $s \in \mathbb{Z}$, $k, \ell, m \in \mathbb{Z}_{\geq 0}$, recursively, from $\chi_{q}(1_{0})$, $\chi_{q}(2_{0})$, and $\chi_{q}(3_{0})$, by using the relations in the system I.

We will prove Proposition 3.5 in Section 7.

3.3. The system II. We will describe the system II which is dual to the system I. Let $\tilde{T}^{(s)}_{k,\ell,m}$, $\tilde{T}^{(s)}_{k,\ell,m}$, $\tilde{S}^{(s)}_{k,\ell}$, $\tilde{R}^{(s)}_{k,\ell,m}$ be the monomials obtained from $T^{(s)}_{k,\ell,m}$, $\tilde{T}^{(s)}_{k,\ell,m}$, $S^{(s)}_{k,\ell}$, $R^{(s)}_{k,\ell,m}$ by replacing $i_{n}$ with $i_{-n}$, $i = 1, 2, 3$. Namely,

$$\tilde{T}^{(s)}_{k,\ell,m} = \left( \prod_{i=0}^{m-1} 1_{s-4k-2\ell-2i-2} \right) \left( \prod_{i=0}^{\ell-1} 2_{s-4k-2i-1} \right) \left( \prod_{i=0}^{k-1} 3_{s-4i} \right).$$
\[
\tilde{\mathcal{T}}_{k,\ell,m}^{(s)} = \left( \prod_{i=0}^{m-1} 3 - 2k - 2\ell - 4i - 4 \right) \left( \prod_{i=0}^{\ell-1} 2 - 2k - 2i - 1 \right) \left( \prod_{i=0}^{k-1} 1 - s - 2i \right),
\]

\[
\tilde{S}_{k,\ell}^{(s)} = \left( \prod_{i=0}^{\ell-1} 2 - 2k - 2i - 4 \right) \left( \prod_{i=0}^{k-1} 2 - s - 2i \right),
\]

\[
\tilde{R}_{k,\ell,m}^{(s)} = \left( \prod_{i=0}^{m-1} 3 - 2k - 2\ell - 4i - 3 \right) \left( \prod_{i=0}^{\ell-1} 1 - 2k - 2i - 1 \right) \left( \prod_{i=0}^{k-1} 2 - s - 2i \right).
\]

For \( k, \ell \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z} \), we have the following trivial relations

\[
\begin{align*}
\tilde{T}_{k,0,0}^{(s)} &= T_{0,0,k}^{(-s-2k)}, & \tilde{T}_{0,0,0}^{(s)} &= T_{0,0,k}^{(-s-2k)}, & \tilde{T}_{0,0,k}^{(s)} &= T_{k,0,0}^{(-s-4k)}, \\
\tilde{S}_{0,k}^{(s)} &= \tilde{S}_{k,0}^{(s)}, & \tilde{S}_{k,0,0}^{(s)} &= T_{0,k,0}^{(-s-2k-3)}, \\
\tilde{R}_{k,0,\ell}^{(s)} &= \tilde{R}_{0,k,\ell}^{(s)}, & \tilde{R}_{0,0,0}^{(s)} &= T_{0,k,0}^{(-s-2k-1)}, & \tilde{R}_{0,0,k}^{(s)} &= T_{k,0,0}^{(-s-4k+1)}.
\end{align*}
\]  

(3.14)

**Theorem 3.6.** The modules \( \tilde{T}_{k,0,0}^{(s)}, \tilde{T}_{0,0,m}^{(s)}, \tilde{T}_{k,0,\ell}^{(s)}, \tilde{T}_{k,0,m}^{(s)}, \tilde{S}_{k,\ell}^{(s)}, \tilde{S}_{k,0}^{(s)}, \tilde{S}_{k,\ell}^{(s)}, \tilde{R}_{k,0,\ell}^{(s)} \), \( k, \ell, m \in \mathbb{Z}_{\geq 0}, r \in \{1, 2\}, j \in \{0, 1, 2\} \), are anti-special.

**Proof.** This theorem can be proved using the dual arguments of the proof of Theorem 3.7.

**Lemma 3.7.** Let \( \iota : \mathbb{Z}\mathcal{P} \rightarrow \mathbb{Z}\mathcal{P} \) be a ring homomorphism such that \( Y_{i,aq^{s}} \mapsto Y_{i,aq^{s-1}}^{-1} \), \( i = 1, 2, 3 \), for all \( a \in \mathbb{C}^{\times}, s \in \mathbb{Z} \). Let \( m_{+} \) be one of the monomials

\[
T_{k,0,\ell}^{(s)}, T_{0,\ell,r}^{(s)}, T_{k,m}^{(s)}, S_{k,\ell}^{(s)}, R_{k,2\ell+j,\ell}^{(s)}, j = 0, 1, 2.
\]

Then \( \chi_{q}(m_{+}) = \iota(\chi_{q}(m_{+})) \).

**Proof.** The lemma is proved by using similar arguments of the proof of Lemma 7.3 of [LM12].

The modules \( \tilde{T}_{k,0,0}^{(s)}, \tilde{T}_{0,0,m}^{(s)}, \tilde{T}_{k,0,\ell}^{(s)}, \tilde{T}_{k,0,m}^{(s)}, \tilde{S}_{k,\ell}^{(s)}, \tilde{S}_{k,0}^{(s)}, \tilde{S}_{k,\ell}^{(s)}, \tilde{R}_{k,0,\ell}^{(s)} \) satisfy the same relations as in Theorem 3.3 but the roles of left and right modules are exchanged.

**Theorem 3.8.** For \( k, \ell, m \in \mathbb{Z}_{\geq 1}, r \in \{1, 2\} \), we have the following relations in \( \text{Rep}(U_{q}\mathfrak{g}) \).

\[
[\tilde{T}_{k-1,0,\ell,0}^{(s+1)}][\tilde{T}_{k,\ell,1}] = [\tilde{T}_{k,\ell,0}^{(s+4)}][\tilde{T}_{k-1,\ell,-1,0}] + [\tilde{R}_{2k-1,\ell,1}^{(s+1)}][\tilde{T}_{k-1,\ell,-1,0}][\tilde{T}_{k,\ell,1}],
\]

(3.15)

\[
[\tilde{T}_{k-1,0,m}][\tilde{T}_{k,0,m-1}^{(s+4)}] = [\tilde{T}_{k,0,m}^{(s+4)}][\tilde{T}_{k-1,0,m-1}] + [\tilde{S}_{2k-1,1,m-1}^{(s+1)}][\tilde{T}_{k-1,0,m-1}],
\]

(3.16)
Moreover, the modules corresponding to each summand on the right hand side of the above relations (3.14), (3.2)-(3.4). Then the system II is closed.

One can compute the propositions.

Proof. The theorem follows from the relations in Theorem 3.3, Theorem 3.4, and Lemma 3.7.

Let us use II to denote the system consisting of the relations in Theorem 3.8 and the relations (3.14), (3.2)-(3.4). Then the system II is closed.

Since \( \tilde{T}_{k,0}^{(s)} = \tilde{T}_{0,0}^{(-s-4k-2\ell)} \), the system II contains minimal affinizations \( \tilde{T}_{0,0}^{(-s-4k-2\ell)} \), \( k, \ell \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z} \). Using the shift defined in (2.3), we can obtain minimal affinizations \( \tilde{T}_{0,0}^{(s)} \) from \( \tilde{T}_{0,0}^{(-s-4k-2\ell)} \), \( k, \ell \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z} \).

The following proposition is similar to Proposition 3.5.

Proposition 3.9. One can compute the q-characters of
\[
\tilde{T}_{k,\ell,0}^{(s)}, \tilde{T}_{k,0,m}^{(s)}, \tilde{T}_{0,\ell,r}^{(s)}, \tilde{T}_{k,0,m}^{(s)}, \tilde{S}_{k,\ell}^{(s)}, \tilde{R}_{k,2\ell+r,\ell}^{(s)}, \quad s \in \mathbb{Z}, k, \ell, m \in \mathbb{Z}_{\geq 0}, r \in \{1, 2\},
\]
recursively, from \( \chi_q(1) \), \( \chi_q(2), \chi_q(3) \) by using the relations in the system II. □
3.4. The system III. For \( s \in \mathbb{Z}, k, \ell \in \mathbb{Z}_{\geq 0} \), we define the following monomials.

\[
U^{(s)}_{k, \ell} = \left( \prod_{i=0}^{k-1} 2s_{i+2i} \right) \left( \prod_{i=0}^{\ell-1} 3s_{i+2k+2i+1} \right),
\]

\[
V^{(s)}_{k, \ell} = \left( \prod_{i=0}^{k-1} 3s_{i+4i} \right) \left( \prod_{i=0}^{\ell-1} 3s_{i+4k+4i+2} \right),
\]

\[
P^{(s)}_{k, \ell} = \left( \prod_{i=0}^{k-1} 3s_{i+4i} \right) 2^{k+1} \left( \prod_{i=0}^{\ell-1} 3s_{i+4k+4i+6} \right),
\]

\[
O^{(s)}_{k, \ell} = \left( \prod_{i=0}^{k-1} 2s_{i+2i} \right) 1_{2k+1} 1_{2k+3} \left( \prod_{i=0}^{\ell-1} 2s_{i+2k+2i+6} \right).
\]

For \( k \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z} \), we have the following trivial relations

\[
U^{(s)}_{k,0} = T_{0,0,0}^{(s-1)}, \quad V^{(s)}_{k,0} = T_{k,0,0}^{(s)}, \quad O^{(s)}_{k,0} = T_{k,0,0}^{(s+2)}, \quad O^{(s)}_{0,0} = T_{0,0,0}^{(s+6)}.
\]

\[
\text{(3.24)}
\]

Theorem 3.10 ([Her07]). The modules \( U^{(s)}_{k,\ell}, V^{(s)}_{k,\ell}, P^{(s)}_{k,\ell}, O^{(s)}_{k,\ell}, s \in \mathbb{Z}, k, \ell \in \mathbb{Z}_{\geq 0} \), are special.

Theorem 3.11. The modules \( U^{(s)}_{k,\ell}, V^{(s)}_{k,\ell}, P^{(s)}_{k,\ell}, O^{(s)}_{k,\ell}, s \in \mathbb{Z}, k, \ell \in \mathbb{Z}_{\geq 0} \), are special.

We will prove Theorem 3.11 in Section 4.

Our second main result is the following theorem.

Theorem 3.12. For \( s \in \mathbb{Z}, k, \ell \in \mathbb{Z}_{\geq 1}, p \in \mathbb{Z}_{\geq 2}, \) we have the following relations in \( \text{Rep}(U_{k,\ell}) \).

\[
[\tilde{T}^{(s)}_{k,\ell-1,0}, \tilde{T}^{(s+2)}_{k-1,\ell,0}] = [\tilde{T}^{(s)}_{k,\ell-1,0}, \tilde{T}^{(s+2)}_{k-1,\ell,0}] + [\tilde{T}^{(s+2k+2)}_{\ell-1,0,0}] [U^{(s+1)}_{k-1,\ell}], \quad (3.25)
\]

\[
[U^{(s)}_{r,\ell}] = [T^{(s+2r+1)}_{0,\ell,0}], \quad r = 0, 1. \quad (3.26)
\]

\[
[U^{(s)}_{p,\ell-1}] [U^{(s+2)}_{p-1,\ell-1}] = [U^{(s)}_{p,\ell}] [U^{(s+2)}_{p-1,\ell-1}] + \begin{cases} [T^{(s+1)}_{p,\ell-1,0}] [T^{(s+2p+1)}_{p,\ell-1,0}] [V^{(s+2\sigma(p+1)+1)}_{\ell,0,0}], & \text{if } \ell \text{ is even}, \\ [T^{(s+1)}_{p,\ell-1,0}] [T^{(s+2\sigma(p+1)+1)}_{p,\ell-1,0}] [P^{(s+2\sigma(p+1)+1)}_{\ell,0}], & \text{if } \ell \text{ is odd}, \end{cases} \quad (3.27)
\]

\[
[V^{(s)}_{k,\ell-1}] [V^{(s+4)}_{k-1,\ell-1}] = [V^{(s)}_{k,\ell-1}] [V^{(s+4)}_{k-1,\ell-1}] + [O^{(s+1)}_{2(k-1),2(\ell-1)}], \quad (3.28)
\]

\[
[O^{(s+2)}_{k,\ell-1}] [O^{(s+2)}_{k-1,\ell-1}] = [O^{(s+2)}_{k,\ell}] [O^{(s+2)}_{k-1,\ell-1}] + [P^{(s+\sigma(k+1)+1)}_{\ell,0,0}] [T^{(s+1)}_{\ell-1,0,0}] [T^{(s+2k+7)}_{\ell-1,0,0}], \quad (3.29)
\]
We will prove Theorem 3.12 in Section 5.

Note that the relations for the modules $\mathcal{O}^{(s)}_{0,k} = \mathcal{T}^{(s+6)}_{2,k,0}$ are contained in (3.25). The modules $\mathcal{T}^{(s-1)}_{0,k,1}$, $\mathcal{T}^{(s)}_{k,1,0}$ in (3.24) can be computed by using the relations (3.5, $\ell = 1$), (3.9), (3.11, $\ell = 0$) in the system I. The modules $\mathcal{T}^{(s)}_{0,1,k}$ in (3.24) can be computed by using the relations (3.15, $\ell = 1$) and (3.21, $\ell = 0$) in the system II.

Let us use III to denote the system consisting of the relations in Theorem 3.12, the relations in (3.25), $\ell = 1$, (3.9), $\ell = 0$ in the system I, the relations (3.15, $\ell = 1$), (3.21, $\ell = 0$) in the system II, and the relations (3.24), (3.25)-(3.4). Then the system III is closed.

All relations except (3.26) in Theorem 3.12 are written in the form $[\mathcal{T}][\mathcal{R}] = [\mathcal{T}][\mathcal{B}] + [\mathcal{S}]$. We have the following theorem.

**Theorem 3.13.** For each relation in Theorem 3.12, all summands on the right hand side, $\mathcal{T} \otimes \mathcal{B}$ and $\mathcal{S}$, are irreducible.

We will prove Theorem 3.13 in Section 6.

The system III can be used to compute the $q$-characters of the modules in the system III. The following proposition can be proved by similar arguments as the proof of Proposition 3.5.

**Proposition 3.14.** One can compute the $q$-characters of $U^{(s)}_{k,\ell}$, $V^{(s)}_{k,\ell}$, $P^{(s)}_{k,\ell}$, $\mathcal{O}^{(s)}_{k,\ell}$, $s \in \mathbb{Z}$, $k, \ell \in \mathbb{Z}_{\geq 0}$, recursively, from $\chi_{q}(10)$, $\chi_{q}(20)$, and $\chi_{q}(30)$ by using the relations in the system III. $\square$

3.5. The system IV. We will describe the system IV which is dual to the system III. Let $\bar{U}^{(s)}_{k,\ell}$, $\bar{V}^{(s)}_{k,\ell}$, $\bar{P}^{(s)}_{k,\ell}$, $\bar{O}^{(s)}_{k,\ell}$ be the monomials obtained from $U^{(s)}_{k,\ell}$, $V^{(s)}_{k,\ell}$, $P^{(s)}_{k,\ell}$, $\mathcal{O}^{(s)}_{k,\ell}$ by replacing $i_{a}$ with $\bar{i}_{a}$, $i = 1, 2, 3$. Namely,

$$
\bar{U}^{(s)}_{k,\ell} = \left( \prod_{i=0}^{\ell-1} 3^{s-2k-2i-1} \right) \left( \prod_{i=0}^{k-1} 2^{s-2i} \right),
$$

$$
\bar{V}^{(s)}_{k,\ell} = \left( \prod_{i=0}^{\ell-1} 3^{s-4k-4i-2} \right) \left( \prod_{i=0}^{k-1} 3^{s-4i} \right),
$$

$$
\bar{P}^{(s)}_{k,\ell} = \left( \prod_{i=0}^{\ell-1} 3^{s-4k-4i-6} \right) \left( 2^{4k-1} \prod_{i=0}^{k-1} 3^{s-4i} \right),
$$

$$
\bar{O}^{(s)}_{k,\ell,m} = \left( \prod_{i=0}^{\ell-1} 2^{s-2k-2i-6} \right) \left( 1-2k-1 \prod_{i=0}^{k-1} 2^{s-2i} \right).$$
For $k \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}$, we have the following trivial relations
\[
U^{(s)}_{k,0} = \mathcal{T}^{(s-1)}_{0,k,0}, \quad V^{(s)}_{k,0} = \mathcal{T}^{(s)}_{0,k,0}, \quad \mathcal{O}^{(s)}_{0,k} = \mathcal{T}^{(s+2)}_{k,0,0},
\]
\[
\mathcal{O}^{(s)}_{k,0} = \mathcal{T}^{(s-1)}_{k,0,0}, \quad \mathcal{O}^{(s)}_{0,k} = \mathcal{T}^{(s+6)}_{2,k,0},
\]
\[
\mathcal{O}^{(s)}_{k,0} = \mathcal{T}^{(s)}_{k,1,0}, \quad \mathcal{O}^{(s)}_{0,k} = \mathcal{T}^{(s)}_{0,1,k} = \mathcal{T}^{(s-4k-2)}_{1}.
\]

(3.31)

The following theorem is proved by using the dual arguments of the proof of Theorem 3.12.

**Theorem 3.15.** The modules $\tilde{T}^{(s)}_{k,\ell,0}, \tilde{U}^{(s)}_{k,\ell}, \tilde{V}^{(s)}_{k,\ell}, \tilde{P}^{(s)}_{k,\ell}, \tilde{O}^{(s)}_{k,\ell}, s \in \mathbb{Z}, k, \ell \in \mathbb{Z}_{\geq 0}$, are anti-special. □

**Lemma 3.16.** Let $\iota : \mathbb{Z} \mathcal{P} \rightarrow \mathbb{Z} \mathcal{P}$ be a homomorphism of rings such that $Y^{-i}_{a,\ell} \mapsto Y^{a}_{i,\ell}$, $i = 1, 2, 3$, for all $a \in \mathbb{C}^{\times}, s \in \mathbb{Z}$. Let $m_{\pm}$ be one of the monomials
\[
\tilde{T}^{(s)}_{k,\ell,0}, \tilde{U}^{(s)}_{k,\ell}, \tilde{V}^{(s)}_{k,\ell}, \tilde{P}^{(s)}_{k,\ell}, \tilde{O}^{(s)}_{k,\ell}, s \in \mathbb{Z}, k, \ell \in \mathbb{Z}_{\geq 0},
\]
Then $\chi_{\mathbb{Z}}(m_{\pm}) = \iota(\chi_{\mathbb{Z}}(m_{\pm}))$.

**Proof.** The lemma is proved by using similar arguments of the proof of Lemma 7.3 of [LMT]. □

The modules $\tilde{T}^{(s)}_{k,\ell,0}, \tilde{U}^{(s)}_{k,\ell}, \tilde{V}^{(s)}_{k,\ell}, \tilde{P}^{(s)}_{k,\ell}, \tilde{O}^{(s)}_{k,\ell}$ satisfy the same relations as in Theorem 3.12 but the roles of left and right modules are exchanged. More precisely, we have the following theorem.

**Theorem 3.17.** For $s \in \mathbb{Z}, k, \ell \in \mathbb{Z}_{\geq 1}, p \in \mathbb{Z}_{\geq 2}$, we have the following relations in $\text{Rep}(U_{q}\mathcal{B})$.

\[
[\tilde{T}^{(s+2)}_{k-1,\ell,0}] [\tilde{T}^{(s)}_{k,\ell-1,0}] = [\tilde{T}^{(s)}_{k,\ell,0}] [\tilde{T}^{(s+2)}_{k-1,\ell-1,0}] + [\tilde{T}^{(s+2k+2)}_{\ell-1,0}] [\tilde{U}^{(s+1)}_{k-1,\ell}],
\]

(3.32)

\[
[\tilde{U}^{(s)}_{r,\ell}] = [\tilde{T}^{(s-1)}_{0,r,\ell}] [\tilde{T}^{(s+2r+1)}_{0}], \quad r = 0, 1,
\]

(3.33)

\[
[\tilde{U}^{(s+2)}_{p-1,\ell,0}] [\tilde{U}^{(s)}_{p,\ell-1}] = [\tilde{U}^{(s)}_{p,\ell}] [\tilde{U}^{(s+2)}_{p-1,\ell-1}] + \left\{ [\tilde{T}^{(s+1)}_{p,\ell-1,0}] [\tilde{T}^{(s+2p+1)}_{\ell}]; \text{if } \ell \text{ is even,} \right.
\]

\[
[\tilde{T}^{(s+1)}_{p,\ell-1,0}] [\tilde{T}^{(s+2p+1)}_{\ell}]; \text{if } \ell \text{ is odd,}
\]

(3.34)

\[
[\tilde{V}^{(s+4)}_{k-1,\ell}] [\tilde{V}^{(s)}_{k,\ell-1}] = [\tilde{V}^{(s)}_{k,\ell}] [\tilde{V}^{(s+4)}_{k-1,\ell-1}] + [\tilde{O}^{(s+1)}_{2(k-1),2}] [\tilde{T}^{(s+4)}_{k-1,\ell}];
\]

(3.35)

\[
[\tilde{O}^{(s+2)}_{k-1,\ell}] [\tilde{O}^{(s)}_{k,\ell-1}] = [\tilde{O}^{(s)}_{k,\ell}] [\tilde{O}^{(s+2)}_{k-1,\ell-1}] + [\tilde{P}^{(s+2k+1)}_{\ell}]; [\tilde{T}^{(s+2)}_{k-1,\ell}] [\tilde{T}^{(s+2k+7)}_{\ell}];
\]

(3.36)
Moreover, the modules corresponding to each summand on the right hand side of the above relations are all irreducible.

Proof. The theorem follows from the relations in Theorem 3.12, Theorem 3.13, and Lemma 3.16.

Note that the relations for the modules $\tilde{O}^{(s)}_{0,k} = \tilde{T}^{(s+6)}_{2,k,0}$ are contained in (3.32). The modules $\tilde{T}^{(s-1)}_{0,k,2}$, $\tilde{T}^{(s)}_{0,k,1,0}$ in (3.31) can be computed by using the relations (3.15), $\ell = 1$, (3.19), (3.21), $\ell = 0$ in the system II. The modules $\tilde{T}^{(-s-4k-2)}_{k,1,0}$ in (3.31) can be computed by using the relations (3.5), $\ell = 1$, and (3.11), $\ell = 0$ in the system I.

Let us use IV to denote the system consisting of the relations in Theorem 3.17, the relations (3.15), $\ell = 1$, (3.19), (3.21), $\ell = 0$ in the system II, the relations (3.5), $\ell = 1$, (3.11), $\ell = 0$ in the system I, and the relations (3.24), (3.2), (3.4). Then the system IV is closed.

Since $\tilde{T}^{(s)}_{k,\ell,0} = \tilde{T}^{(-s-2k-2\ell)}_{0,\ell,k}$, the system II contains minimal affinizations $\tilde{T}^{(-s-2k-2\ell)}_{0,\ell,k}$, $k, \ell \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$. Using the shift defined in (2.3), we can obtain minimal affinizations $\tilde{T}^{(s)}_{0,\ell,k}$ from $\tilde{T}^{(-s-2k-2\ell)}_{0,\ell,k}$, $k, \ell \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$.

The following proposition is similar to Proposition 3.5.

Proposition 3.18. One can compute the $q$-characters of

$$
\tilde{T}^{(s)}_{k,\ell,0}, \tilde{U}^{(s)}_{k,\ell,0}, V^{(s)}_{k,\ell}, \tilde{P}^{(s)}_{k,\ell}, \tilde{O}^{(s)}_{k,\ell}, s \in \mathbb{Z}, k, \ell \in \mathbb{Z}_{\geq 0}, r \in \{1, 2\},
$$

recursively, from $\chi_q(1_0), \chi_q(2_0), \chi_q(3_0)$ by using the relations in the system IV. \qed

4. Special modules

In this section, we prove Theorem 3.2 and Theorem 3.11. Namely, we will prove that for $s \in \mathbb{Z}$, $k, \ell, m \in \mathbb{Z}_{\geq 0}$, $r \in \{0, 1, 2\}$, the modules

$$
\tilde{T}^{(s)}_{k,0,0}, \tilde{T}^{(s)}_{k,0,m}, \tilde{T}^{(s)}_{0,k,0}, \tilde{T}^{(s)}_{0,k,0}, S^{(s)}_{k,\ell}, R^{(s)}_{k,\ell,0}, R^{(s)}_{k,2\ell+1,0}, R^{(s)}_{k,2\ell+2,0}, U^{(s)}_{k,\ell}, V^{(s)}_{k,\ell}, \tilde{P}^{(s)}_{k,\ell}, \tilde{O}^{(s)}_{k,\ell},
$$

are special. Since the modules

$$
\tilde{T}^{(s)}_{0,0,0}, \tilde{T}^{(s)}_{0,0,0}, \tilde{T}^{(s)}_{0,k,0}, \tilde{T}^{(s)}_{0,k,0}, S^{(s)}_{k,0}, R^{(s)}_{k,0,0}, R^{(s)}_{0,1,0}, R^{(s)}_{0,2,0}, U^{(s)}_{0,0}, V^{(s)}_{0,0}, V^{(s)}_{0,0},
$$

are Kirillov-Reshetikhin modules, they are special. In the following, we will prove that the other modules in (4.1) are special.
4.1. The case of \( \mathcal{T}_{k,\ell,0}^{(s)} \). Let \( m_+ = T_{k,\ell,0}^{(s)} \). Without loss of generality, we may assume that \( s = 0 \). Then

\[
m_+ = (3_03_4 \cdots 3_{4k-4})(2_{4k+1}2_{4k+3} \cdots 2_{4k+2\ell-1}).
\]

Let

\[
U = I \times \{ a^s : s \in \mathbb{Z}, s \leq 4k + 2\ell - 1 \}.
\]

Since all monomials in \( \mathcal{M}(\chi_q(m_+) - \text{trunc}_{m_+Q_U} \chi_q(m_+)) \) are right-negative, it is sufficient to show that \( \text{trunc}_{m_+Q_U} \chi_q(m_+) \) is special.

Let \( \mathcal{M} \) be the finite set consisting of the following monomials

\[
m_0 = m_+, \ m_1 = m_0A_{3,4k-2}^{-1}, \ m_2 = m_1A_{3,4k-6}^{-1}, \ldots, \ m_k = m_{k-1}A_{3,2}^{-1}.
\]

It is easy to see that \( \mathcal{M} \) satisfies the conditions in Theorem 2.2. Therefore

\[
\text{trunc}_{m_+Q_U} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m
\]

and hence \( \text{trunc}_{m_+Q_U} \chi_q(m_+) \) is special.

4.2. The case of \( \mathcal{T}_{k,0,m}^{(s)} \). Let \( m_+ = T_{k,0,m}^{(s)} \). Without loss of generality, we may assume that \( s = 0 \). Then

\[
m_+ = (3_03_4 \cdots 3_{4k-4})(1_{4k+2}1_{4k+4} \cdots 1_{4k+2\ell}).
\]

**Case 1.** \( k = 1 \). In this case,

\[
m_+ = 3_0(1_{6}1_{4k+4} \cdots 1_{2\ell+4}).
\]

Let

\[
U = I \times \{ a^s : s \in \mathbb{Z}, s \leq 2\ell + 4 \}.
\]

Let \( \mathcal{M} \) be the finite set consisting of the following monomials

\[
m_0 = m_+, \ m_1 = m_0A_{3,2}^{-1}, \ m_2 = m_1A_{2,4}^{-1}, \ m_3 = m_2A_{2,2}^{-1}, \ m_4 = m_3A_{3,3}^{-1}.
\]

By Theorem 2.2

\[
\text{trunc}_{m_+Q_U} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m
\]

and hence \( \text{trunc}_{m_+Q_U} \chi_q(m_+) \) is special. Therefore \( \mathcal{T}_{1,0,m}^{(s)} \) is special.

**Case 2.** \( k > 1 \). We embed \( L(m_+) \) into two different tensor products. Since each factor in the tensor product is special, we can use the FM algorithm to compute the \( q \)-characters of the factors. We classify the dominant monomials in the first tensor product and prove that the only dominant monomial in the first tensor product which occurs in the second tensor product is \( m_+ \). Hence \( L(m_+) \) is special.

The first tensor product is

\[
L(m_1') \otimes L(m_2'), \text{ where } m_1' = 3_03_4 \cdots 3_{4k-8}, \ m_2' = 3_{4k-4}1_{4k+4} \cdots 1_{4k+2\ell}.
\]
In Case 1, we have shown that \( L(m'_1) \) is special. Therefore the FM algorithm works for \( L(m'_2) \). We will use the FM algorithm to compute \( \chi_q(L(m'_1)) \), \( \chi_q(L(m'_2)) \) and classify all dominant monomials in \( \chi_q(L(m'_1))\chi_q(L(m'_2)) \). Let \( m = m_1m_2 \) be a dominant monomial, where \( m_i \in \mathcal{M}(L(m'_i)), i = 1, 2 \).

Suppose that \( m_2 \neq m'_2 \). If \( m_2 \) is right-negative, then \( m \) is a right negative monomial and therefore \( m \) is not dominant. This is a contradiction. Hence \( m_2 \) is not right-negative. By Case 1, \( m_2 \) is one of the following monomials

\[
\begin{align*}
\bar{m}_1 &= m'_2 A_{3, 4k-2}^{-1} = 3^{-1} 2^{4k-3} 3^{-1} 2^{4k-1} 1^{4k+2} 2^{4k+3} 1 \cdots 4k+2 \ell, \\
\bar{m}_2 &= \bar{m}_1 A_{2, 4k}^{-1} = 2^{4k-3} 3^{-1} 2^{4k+1} 1^{4k+2} 2^{4k+3} 1 \cdots 4k+2 \ell, \\
\bar{m}_3 &= \bar{m}_2 A_{2, 4k-2}^{-1} = 2^{-1} 3^{-1} 2^{4k-1} 3^{4k-2} 1^{4k+2} 2^{4k+3} 1 \cdots 4k+2 \ell, \\
\bar{m}_4 &= \bar{m}_3 A_{4, 4k}^{-1} = 3^{-1} 2^{4k+1} 2^{4k-1} 3^{4k+2} 1^{4k+2} 2^{4k+3} 1 \cdots 4k+2 \ell.
\end{align*}
\]

By Lemma 3.1, \( 3^{-1} 4k \) cannot be canceled by any monomial in \( \chi_q(m'_1) \). Therefore \( m = m_1m_2 \) (\( m_1 \in \chi_q(m'_1) \)) is not dominant. This is a contradiction. Hence \( m_2 = m'_2 \). Similarly, \( m_2 \) cannot be \( m'_2 \), \( i = 2, 3, 4 \). This is a contradiction. Therefore \( m_2 = m'_2 \).

If \( m_1 \neq m'_1 \), then \( m_1 \) is right negative. Since \( m \) is dominant, each factor with a negative power in \( m_1 \) needs to be canceled by a factor in \( m'_2 \). By Lemma 3.1, the only factor in \( m'_2 \) which can be canceled is \( 3^{4k-4} \). We have \( \mathcal{M}(L(m'_1)) \subseteq \mathcal{M}(\chi_q(3_0^{34} \cdots 3_{4k-12}))\chi_q(L(3_{4k-8})) \). Only monomials in \( \chi_q(L(3_{4k-8})) \) can cancel \( 3^{4k-4} \). The only monomial in \( \chi_q(L(3_{4k-8})) \) which can cancel \( 3^{4k-4} \) is \( 3^{-1} 4k - 2^{4k-7} 4k-5 \). Therefore \( m_1 \) is in the set

\[
\mathcal{M}(\chi_q(3_0^{34} \cdots 3_{4k-12}))3^{-1} 4k - 2^{4k-7} 4k-5.
\]

If \( m_1 = (3_0^{34} \cdots 3_{4k-12})3^{-1} 4k - 2^{4k-7} 4k-5 \), then

\[
m = m_1m_2 = 3_0^{34} \cdots 3_{4k-12} 2^{4k-7} 4k-5 1^{4k+2} 2^{4k+1} \cdots 1 \quad \text{(4.2)}
\]

is dominant. Suppose that

\[
m_1 = (3_0^{34} \cdots 3_{4k-12})3^{-1} 4k - 2^{4k-7} 4k-5.
\]

Then \( m_1 = n_1 3^{-1} 4k - 2^{4k-7} 4k-5 \), where \( n_1 \) is a non-highest monomial in \( \chi_q(3_0^{34} \cdots 3_{4k-12}) \). Since \( n_1 \) is right negative, \( 2^{4k-7} \) or \( 2^{4k-5} \) should cancel a factor of \( n_1 \) with a negative power. It is easy to see that there exists either a factor \( 2^{4k-7} \) or \( 2^{4k-5} \) in a monomial in \( \chi_q(3_0^{34} \cdots 3_{4k-12})3^{-1} 4k - 2^{4k-7} 4k-5 \) by using the FM algorithm. Therefore we need a factor \( 2^{4k-9} \) or \( 2^{4k-7} \) in a monomial in \( \chi_q(3_0^{34} \cdots 3_{4k-12}) \). We have

\[
\chi_q(3_0^{34} \cdots 3_{4k-12})3^{-1} 4k - 2^{4k-7} 4k-5 \subseteq \chi_q(3_0^{34} \cdots 3_{4k-16})\chi_q(3_{4k-12})3^{-1} 4k - 2^{4k-7} 4k-5.
\]

The factors \( 2^{4k-9} \) and \( 2^{4k-7} \) can only come from the monomials in \( \chi_q(3_{4k-12}) \). By Lemma 3.1, any monomial in \( \chi_q(3_{4k-12}) \) does not have a factor \( 2^{4k-9} \). The only monomial in \( \chi_q(3_{4k-12}) \) which contains a factor \( 2^{4k-7} \) is \( 1^{4k-10} 1^{4k-6} 2^{4k-7} 3^{-1} 4k - 2^{4k-7} 4k-5 \). Therefore \( m_1 \) is in the set

\[
\mathcal{M}(\chi_q(3_0^{34} \cdots 3_{4k-16}))1^{4k-10} 1^{4k-6} 2^{4k-7} 3^{-1} 4k - 2^{4k-7} 4k-5 \quad \text{or} \quad \mathcal{M}(\chi_q(3_0^{34} \cdots 3_{4k-16}))1^{4k-10} 3^{-1} 4k - 2^{4k-7} 4k-5.
\]
Since \( m = m_1 m_2 \) is dominant, \( 1^{-1} 3^{-1} 6^{-1} 4k-6 \) should be canceled by some monomial in \( \chi_q(3_0 3_1 \cdots 3_{4k-16}) \). But by Lemma 3.1, \( 1^{-1} 3^{-1} 6^{-1} 4k-6 \) cannot be canceled by any monomial in \( \chi_q(3_0 3_1 \cdots 3_{4k-16}) \). This is a contradiction.

Therefore the only dominant monomials in \( \chi_q(L(m'_1)) \chi_q(L(m'_2)) \) are \( m_+ \) and \( m'' \).

The second tensor product is \( L(m'_1) \otimes L(m'_2) \), where

\[
m'_1 = 3_0 3_1 \cdots 3_{4k-4}, \quad m'' = 1_{4k+2} 2_{4k+4} \cdots 4_{2k+2}.
\]

The monomial \( m'' \) is

\[
n = m_+ A_{3,4k-2}^{-1}.
\]

Since \( A_{i,a}, i \in I, a \in \mathbb{C}_x \) are algebraically independent, the expression (4.3) of \( n \) of the form \( n = \prod_{(i,a) \in I} A_{i,a}^{-v_{i,a}} \), where the \( v_{i,a} \) are some integers, is unique. Suppose that the monomial \( n \) is in \( \chi_q(L(m'_1)) \chi_q(L(m'_2)) \). Then \( n = n_1 n_2 \) where \( n_i \in \mathcal{M}(L(m''_i)), i = 1, 2 \).

By the expression (4.3), we have \( n_2 = m'' \) and \( n_1 = m'' A_{3,4k-2}^{-1} \). By the FM algorithm, the monomial \( m'' A_{3,4k-2}^{-1} \) is not in \( \mathcal{M}(L(m'')) \). This contradicts the fact that \( n_1 \in \mathcal{M}(L(m'')) \).

Therefore \( n \) is not in \( \chi_q(L(m'_1)) \chi_q(L(m'_2)) \).

### 4.3. The case of \( \tilde{T}_{k,0,m}^{(s)} \)

Let \( m_+ = \tilde{T}_{k,0,m}^{(s)} \) with \( k, m \in \mathbb{Z}_{\geq 0} \). Without loss of generality, we may assume that \( s = 0 \). Then

\[
m_+ = (10_{12} \cdots 1_{2k-2})(3_{2k+4}3_{2k+8} \cdots 3_{2k+4m})
\]

**Case 1.** \( k = 1 \). In this case,

\[
m_+ = 10_{3(30_{10} \cdots 3_{4m+2})}.
\]

Let

\[
U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 4m + 2\}.
\]

Let \( \mathcal{M} \) be the finite set consisting of the following monomials

\[
m_0 = m_+, \quad m_1 = m_0 A_{1,1}^{-1}, \quad m_2 = m_1 A_{2,2}^{-1}.
\]

By Theorem 2.2

\[
\text{trunc}_{m_+} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m
\]

and hence \( \text{trunc}_{m_+} \chi_q(m_+) \) is special. Therefore \( \tilde{T}_{1,0,m}^{(s)} \) is special.

**Case 2.** \( k > 1 \). Let

\[
m'_1 = 10_{12} \cdots 1_{2k-4}, \quad m'_2 = 2_{2k-3} 3_{2k+43_{2k+8} \cdots 3_{2k+4m}},
\]

\[
m''_1 = 10_{12} \cdots 1_{2k-2}, \quad m''_2 = 3_{2k+43_{2k+8} \cdots 3_{2k+4m}}.
\]

Then \( \mathcal{M}(L(m_+)) \subset \mathcal{M}(\chi_q(m'_1) \chi_q(m'_2)) \cap \mathcal{M}(\chi_q(m''_1) \chi_q(m''_2)) \).

By using similar arguments as the case of \( \tilde{T}_{k,0,m}^{(s)} \), we show that the only possible dominant monomials in \( \chi_q(m'_1) \chi_q(m'_2) \) are \( m_+ \) and

\[
n_1 = 10_{12} \cdots 1_{2k-6} 2_{2k-3} 3_{2k+43_{2k+8} \cdots 3_{2k+4m}} = m_+ A_{1,2k-3}^{-1}.
\]
Moreover, \( n_1 \) is not in \( \chi_q(m'^{n}_0)\chi_q(m'^{n}_2) \). Therefore the only dominant monomial in \( \chi_q(m_+) \) is \( m_+ \).

4.4. The case of \( T^{(s)}_{0,\ell,r} \). Let \( m_+ = T^{(s)}_{0,\ell,r} \) with \( \ell \in \mathbb{Z}_{\geq 0}, r \in \{1, 2\} \). Without loss of generality, we may assume that \( s = 0 \). Then
\[
m_+ = (2^1 3^1 \cdots 2^{k-1})1^{2k+2}
\]
or
\[
m_+ = (2^2 3^2 \cdots 2^{k-1})1^{2k+2}2^{1k+4}.
\]

Case 1. \( m_+ = (2^1 3^1 \cdots 2^{k-1})1^{2k+2} \). Let
\[
U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 2k + 2 \}.
\]
Let \( \mathcal{M} \) be the finite set consisting of the following monomials
\[
m_0 = m_+, m_1 = m_0 A^{-1}_{2,2k}, m_2 = m_1 A^{-1}_{2,2k-2}, \cdots, m_k = m_{k-1} A^{-1}_{2,2}.
\]
By Theorem 2.2
\[
\text{trunc}_{m_+ \mathcal{Q}^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m
\]
and hence \( \text{trunc}_{m_+ \mathcal{Q}^-} \chi_q(m_+) \) is special. Therefore \( T^{(s)}_{0,\ell,r}, r = 1 \), is special.

Case 2. \( m_+ = (2^2 3^2 \cdots 2^{k-1})1^{2k+2} \). Let \( U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 2k + 4 \} \). It is sufficient to show that \( \text{trunc}_{m_+ \mathcal{Q}^-} \chi_q(m_+) \) is special.

Let \( \mathcal{M} \) be the finite set consisting of the following monomials
\[
m_0 = m_+, m_1 = m_0 A^{-1}_{2,2k}, m_2 = m_1 A^{-1}_{2,2k-2}, \cdots, m_k = m_{k-1} A^{-1}_{2,2},
\]
\[
n_1 = m_1 A^{-1}_{3,2k+2}, n_2 = n_1 A^{-1}_{2,2k-2}, n_3 = n_2 A^{-1}_{2,2k-4}, \cdots, n_k = n_{k-1} A^{-1}_{2,2}.
\]
It is clear that \( \mathcal{M} \) satisfies the conditions in Theorem 2.2. Therefore
\[
\text{trunc}_{m_+ \mathcal{Q}^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m
\]
and \( \text{trunc}_{m_+ \mathcal{Q}^-} \chi_q(m_+) \) is special.

4.5. The case of \( S^{(s)}_{k,\ell} \). Let \( m_+ = S^{(s)}_{k,\ell} \) with \( k, \ell \in \mathbb{Z}_{\geq 0} \). Without loss of generality, we may assume that \( s = 0 \). Then
\[
m_+ = (2^0 2^1 \cdots 2^{k-2})(2^{2k+4} 2^{2k+6} \cdots 2^{2k+2\ell+2}).
\]

Case 1. \( k = 1 \). In this case,
\[
m_+ = 2^0 (2_6 2_8 \cdots 2_{2\ell+4}).
\]

Let
\[
U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 2\ell + 4 \}.
\]
Let \( \mathcal{M} \) be the finite set consisting of the following monomials
\[
m_0 = m_+, m_1 = m_0 A^{-1}_{2,2}, m_2 = m_1 A^{-1}_{1,2}, m_3 = m_1 A^{-1}_{3,3}, m_4 = m_3 A^{-1}_{1,2}.
\]
By Theorem 2.2

\[ \text{trunc}_{m+Q_U} \chi_q(m+) = \sum_{m \in \mathcal{M}} m \]

and hence \( \text{trunc}_{m+Q_U} \chi_q(m+) \) is special. Therefore \( S^{(s)}_{1,\ell} \) is special.

**Case 2.** \( k > 1 \). Let

\[ m'_1 = 2022 \cdots 2k - 4, \quad m'_2 = 2k - 22k + 4 \cdot 2k + 6 \cdots 2k + 2\ell + 2, \quad m''_1 = 2022 \cdots 2k - 2, \quad m''_2 = 2k + 42k + 6 \cdots 2k + 2\ell + 2. \]

Then \( \mathcal{M}(L(m_+)) \subset \mathcal{M}(\chi_q(m'_1) \chi_q(m'_2)) \cap \mathcal{M}(\chi_q(m''_1) \chi_q(m''_2)). \)

By using similar arguments as the case of \( T^{(s)}_{k,0,m} \), we show that the only possible dominant monomials in \( \chi_q(m'_1) \chi_q(m'_2) \) are \( m_+ \) and

\[ n_1 = 2022 \cdots 2k - 612k - 532k - 32k + 42k + 6 \cdots 2k + 2\ell + 2 = m_+ A^{-1}_{2,2k - 3}, \]
\[ n_2 = 2022 \cdots 2k - 62k + 42k + 82k + 10 \cdots 2k + 2\ell + 2 = m_+ A^{-1}_{2,2k - 2} - A^{-1}_{3,2k - 1} A^{-1}_{2,2k - 1} \]
\[ n_3 = 2022 \cdots 2k - 1012k - 732k - 52k + 82k + 10 \cdots 2k + 2\ell + 2 = m_+ A^{-1}_{2,2k - 5} A^{-1}_{3,2k - 4} A^{-1}_{2,2k - 3}. \]

Moreover, \( n_1, n_2, n_3 \) are not in \( \chi_q(m'_1) \chi_q(m'_2) \). Therefore the only dominant monomial in \( \chi_q(m+) \) is \( m_+ \).

4.6. The case of \( R^{(s)}_{k,2\ell,\ell} \). Let \( m_+ = R^{(s)}_{k,2\ell,\ell} \) with \( k \in \mathbb{Z}_{\geq 1}, \ell \in \mathbb{Z}_{\geq 0} \). Without loss of generality, we may assume that \( s = 0 \). Then

\[ m_+ = (2022 \cdots 2k - 2)(12k + 12k + 3 \cdots 12k + 4\ell - 1)(32k + 32k + 7 \cdots 32k + 4\ell - 1). \]

Let

\[ U = I \times \{ aq^s : s \in \mathbb{Z}, s \leq 2k + 4\ell - 1 \}. \]

Let \( \mathcal{M} \) be the finite set consisting of the following monomials

\[ m_0 = m_+ , m_1 = m_0 A^{-1}_{2,2k - 1} , m_2 = m_1 A^{-1}_{2,2k - 3} , \ldots , m_k = m_{k-1} A^{-1}_{2,1}. \]

By Theorem 2.2

\[ \text{trunc}_{m+Q_U} \chi_q(m+) = \sum_{m \in \mathcal{M}} m \]

and hence \( \text{trunc}_{m+Q_U} \chi_q(m+) \) is special. Therefore \( R^{(s)}_{k,2\ell,\ell} \) is special.
4.7. The case of $R_{k,2\ell+1,\ell}^{(s)}$. Let $m_+ = R_0^{(s)}_{k,2\ell+1,\ell}$ with $k \in \mathbb{Z}_{\geq 1}, \ell \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (2_02_2 \cdots 2_{2k-2})(1_{2k+1}1_{2k+3} \cdots 1_{2k+4\ell-1}1_{2k+4\ell+1})(3_{2k+3}3_{2k+7} \cdots 3_{2k+4\ell+1})$$

Let

$$U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 2k + 4\ell - 1\}.$$ 

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = m_+, \ m_1 = m_0A_{2,2k-1}^{-1}, \ m_2 = m_1A_{2,2k-3}^{-1}, \ldots, \ m_k = m_{k-1}A_{2,1}^{-1}.$$ 

By Theorem 2.2

$$\text{trunc}_{m+Q_U^+} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and hence $\text{trunc}_{m+Q_U^+} \chi_q(m_+)$ is special. Therefore $R_{k,2\ell+1,\ell}^{(s)}$ is special.

4.8. The case of $R_{k,2\ell+2,\ell}^{(s)}$. Let $m_+ = R_0^{(s)}_{k,2\ell+2,\ell}$ with $k \in \mathbb{Z}_{\geq 1}, \ell \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (2_02_2 \cdots 2_{2k-2})(1_{2k+1}1_{2k+3} \cdots 1_{2k+4\ell-1}1_{2k+4\ell+1})(3_{2k+3}3_{2k+7} \cdots 3_{2k+4\ell+1}).$$

Let

$$U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 2k + 4\ell - 1\}.$$ 

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_+, \ m_1 = m_+A_{2,2k-1}^{-1}, \ m_2 = m_1A_{2,2k-3}^{-1}, \ldots, \ m_k = m_{k-1}A_{2,1}^{-1},$$

$$m_{11} = m_1A_{3,2k+4\ell-1}^{-1}, \ m_{21} = m_{11}A_{3,2k+4\ell-3}^{-1}, \ldots, \ m_{1\ell,1} = m_{\ell-1,1}A_{3,2k+5}^{-1},$$

$$m_{12} = m_2A_{3,2k+4\ell+1}^{-1}, \ m_{22} = m_{12}A_{3,2k+4\ell-3}^{-1}, \ldots, \ m_{1\ell,2} = m_{\ell-1,2}A_{3,2k+5}^{-1},$$

$$
\vdots
$$

$$m_{1k} = m_{1k}A_{3,2k+4\ell+1}^{-1}, \ m_{2k} = m_{1k}A_{3,2k+4\ell-3}^{-1}, \ldots, \ m_{1\ell,k} = m_{\ell-1,k}A_{3,2k+5}^{-1}.$$ 

By Theorem 2.2

$$\text{trunc}_{m+Q_U^+} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and hence $\text{trunc}_{m+Q_U^+} \chi_q(m_+)$ is special. Therefore $R_{k,2\ell+2,\ell}^{(s)}$ is special.

4.9. The cases of $R_{0,2\ell+i,\ell}^{(s)}, \ i = 0, 1, 2$. These cases can be proved by using similar arguments as the cases for $R_{k,2\ell+i,\ell}^{(s)}, \ i = 0, 1, 2, k \in \mathbb{Z}_{\geq 1}, \ell \in \mathbb{Z}_{\geq 0}$. 

4.10. **The case of** $\mathcal{U}_{k,\ell}^{(s)}$. Let $m_+ = U_{k,\ell}^{(s)}$ with $k, \ell \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (2_02_2 \cdots 2_{2k-2})(3_{2k+1}3_{2k+3} \cdots 3_{2k+2\ell-1}).$$

Let

$$U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 2k + 2\ell - 1\}.$$

Let $\mathcal{M}$ be the finite set consisting of the following monomials

- $m_+$, $m_1 = m_+ A_{2,2k-1}^{-1}$, $m_2 = m_1 A_{2,2k-3}^{-1}$, \ldots, $m_k = m_{k-1} A_{2,1}^{-1}$,
- $m_{11} = m_1 A_{1,2k}^{-1}$,
- $m_{12} = m_2 A_{1,2k}^{-1}$, $m_{22} = m_{12} A_{1,2k-2}^{-1}$,
  
- $m_{1k} = m_k A_{1,2k}^{-1}$, $m_{2k} = m_{1k} A_{1,2k-2}^{-1}$, \ldots, $m_{kk} = m_{k-1,k} A_{1,2}^{-1}$.

By Theorem 2.2

$$\text{trunc}_{m_+, q_U} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and hence $\text{trunc}_{m_+, q_U} \chi_q(m_+)$ is special. Therefore $\mathcal{U}_{k,\ell}^{(s)}$ is special.

4.11. **The case of** $\mathcal{V}_{k,\ell}^{(s)}$. Let $m_+ = V_{k,\ell}^{(s)}$ with $k, \ell \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (3_03_4 \cdots 3_{4k-4})(3_{4k+2}3_{4k+6} \cdots 3_{4k+4\ell-2}).$$

**Case 1.** $k = 1$. In this case,

$$m_+ = 3_0(3_03_4 \cdots 3_{4\ell+2}).$$

Let

$$U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 4\ell + 2\}.$$

Let $\mathcal{M}$ be the finite set consisting of the following monomials

- $m_0 = m_+$, $m_1 = m_0 A_{3,2}^{-1}$, $m_2 = m_1 A_{3,4}^{-1}$,
- $m_3 = m_2 A_{2,2}^{-1}$, $m_4 = m_3 A_{1,5}^{-1}$, $m_5 = m_4 A_{2,2}^{-1}$, $m_6 = m_5 A_{1,3}^{-1}$.

By Theorem 2.2

$$\text{trunc}_{m_+, q_U} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and hence $\text{trunc}_{m_+, q_U} \chi_q(m_+)$ is special. Therefore $\mathcal{V}_{k,\ell}^{(s)}$ is special.

**Case 2.** $k > 1$. Let

- $m_1' = 3_03_4 \cdots 3_{4k-8}$, $m_2' = 3_{4k-4}3_{4k+2}3_{4k+6} \cdots 3_{4k+4\ell-2}$,
- $m_1'' = 3_03_4 \cdots 3_{4k-4}$, $m_2'' = 3_{4k+2}3_{4k+6} \cdots 3_{4k+4\ell-2}$.
Then \( \mathcal{M}(L(m_+)) \subset \mathcal{M}(\chi_q(m'_1)\chi_q(m'_2)) \cap \mathcal{M}(\chi_q(m''_1)\chi_q(m''_2)) \).

By using similar arguments as the case of \( T_{k,0,m}' \), we show that the only possible dominant monomials in \( \chi_q(m'_1)\chi_q(m'_2) \) are \( m_+ \) and

\[
n_1 = (3_03_4 \cdots 3_{4k-12})2_{4k-7}2_{4k-5}(3_{4k+2}3_{4k+6} \cdots 3_{4k+4\ell-2}) = m_+A_{3,4k-6}^{-1}.
\]

Moreover, \( n_1 \) is not in \( \chi_q(m''_1)\chi_q(m''_2) \). Therefore the only dominant monomial in \( \chi_q(m_+) \) is \( m_+ \).

4.12. The case of \( P_{k,t}' \). Let \( m_+ = P_{k,t}' \) with \( k, \ell \in \mathbb{Z}_{\geq 0} \). Without loss of generality, we may assume that \( s = 0 \). Then

\[
m_+ = (3_03_4 \cdots 3_{4k-4})2_{4k+1}(3_{4k+6}3_{4k+10} \cdots 3_{4k+4\ell+2}).
\]

**Case 1.** \( k = 0 \). In this case,

\[
m_+ = 2_1(3_03_10 \cdots 3_{4\ell+2}).
\]

Let

\[
U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 4\ell + 2\}.
\]

Let \( \mathcal{M} \) be the finite set consisting of the following monomials

\[
m_0 = m_+, \; m_1 = m_0A_{2,2}^{-1}, \; m_2 = m_1A_{1,3}^{-1}.
\]

It is clear that \( \mathcal{M} \) satisfies the conditions in Theorem 2.2. Therefore

\[
\text{trunc}_{m_+Q_U} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m
\]

and hence \( \text{trunc}_{m_+Q_U} \chi_q(m_+) \) is special. Therefore \( P_{0,t}' \) is special.

**Case 2.** \( k > 0 \). Let

\[
m'_1 = 3_03_4 \cdots 3_{4k-4}, \; m'_2 = 2_{4k+1}3_{4k+6}3_{4k+10} \cdots 3_{4k+4\ell+2};
\]

\[
m''_1 = 3_03_4 \cdots 3_{4k-4}2_{4k+1}, \; m''_2 = 3_{4k+6}3_{4k+10} \cdots 3_{4k+4\ell+2}.
\]

Then \( \mathcal{M}(L(m_+)) \subset \mathcal{M}(\chi_q(m'_1)\chi_q(m'_2)) \cap \mathcal{M}(\chi_q(m''_1)\chi_q(m''_2)) \). Note that we have shown that \( m''_2 = T_{k,1,0}' \) is special in the case of \( T_{k,t}' \). Therefore the FM algorithm applies to \( m''_2 \).

By using similar arguments as the case of \( T_{k,0,m}' \), we show that the only possible dominant monomials in \( \chi_q(m'_1)\chi_q(m'_2) \) are \( m_+ \) and

\[
n_1 = (3_03_4 \cdots 3_{4k-8})2_{4k-3}4_k(3_{4k+6}3_{4k+10} \cdots 3_{4k+4\ell+2}) = m_+A_{3,4k-2}^{-1}A_{2,4k}^{-1}.
\]

Moreover, \( n_1 \) is not in \( \chi_q(m'_1)\chi_q(m''_2) \). Therefore the only dominant monomial in \( \chi_q(m_+) \) is \( m_+ \).
4.13. The case of \( \mathcal{O}_{k,\ell}^{(s)} \). Let \( m_+ = \mathcal{O}_{k,\ell}^{(s)} \) with \( k, \ell \in \mathbb{Z}_{\geq 0} \). Without loss of generality, we may assume that \( s = 0 \). Then

\[
m_+ = (2_02_2\cdots2_{2k-2})1_{2k+1}1_{2k+3}(2_{2k+6}2_{2k+8}\cdots2_{2k+2\ell+4}).
\]

Case 1. \( k = 0 \). In this case,

\[
m_+ = 1_11_3(2_02_8\cdots2_{2\ell+4}) = T_{2,\ell,0}^{(1)}.
\]

By Theorem 3.10 \( L(m_+) = \mathcal{O}_{0,\ell}^{(s)} \) is special.

Case 2. \( k > 0 \). Let

\[
m'_1 = 2_02_2\cdots2_{2k-2}, \quad m'_2 = 1_{2k+1}1_{2k+3}2_{2k+6}2_{2k+8}\cdots2_{2k+2\ell+4},
\]

\[
m''_1 = 2_02_2\cdots2_{2k-2}1_{2k+1}3\quad m''_2 = 2_{2k+6}2_{2k+8}\cdots2_{2k+2\ell+4}.
\]

Then \( \mathcal{M}(L(m_+)) \subset \mathcal{M}(\chi_q(m'_1)\chi_q(m'_2)) \cap \mathcal{M}(\chi_q(m''_1)\chi_q(m''_2)) \). Note that we have shown that \( m''_2 = T_{0,k,2}^{(0)} \) is special in the case of \( T_{0,k,r}^{(s)} \), \( r \in \{1,2\} \). Therefore the FM algorithm applies to \( m''_2 \).

By using similar arguments as the case of \( T_{k,0,m}^{(s)} \), we show that the only possible dominant monomials in \( \chi_q(m'_1)\chi_q(m'_2) \) are \( m_+ \) and

\[
n_1 = (2_02_2\cdots2_{2k-4})3_{2k-1}1_{2k+3}(2_{2k+6}2_{2k+8}\cdots2_{2k+2\ell+4})
\]

\[
= m_+ A_{2,2k-1}^{-1}A_{1,2k}^{-1},
\]

\[
n_2 = (2_02_2\cdots2_{2k-4})1_{2k+1}1_{2k+3}(2_{2k+6}2_{2k+10}\cdots2_{2k+2\ell+4})
\]

\[
= n_1 A_{2,2k+1}^{-1}A_{2,2k+3}^{-1}A_{2,2k+1}^{-1}A_{3,2k+3}^{-1}A_{1,2k+4}^{-1}A_{2,2k+5}^{-1},
\]

\[
n_3 = (2_02_2\cdots2_{2k-6})1_{2k-3}1_{2k+1}(2_{2k+8}2_{2k+10}\cdots2_{2k+2\ell+4})
\]

\[
= n_2 A_{2,2k-3}^{-1}A_{3,2k-1}^{-1}A_{2,2k+1}^{-1}A_{1,2k+2}^{-1},
\]

\[
n_4 = (2_02_2\cdots2_{2k-8})1_{2k-5}1_{2k-3}(2_{2k+8}2_{2k+10}\cdots2_{2k+2\ell+4})
\]

\[
= n_3 A_{2,2k-5}^{-1}A_{3,2k-3}^{-1}A_{2,2k-1}^{-1}A_{1,2k}^{-1}.
\]

Moreover, \( n_1, n_2, n_3, n_4 \) are not in \( \chi_q(m'_1)\chi_q(m''_2) \). Therefore the only dominant monomial in \( \chi_q(m_+) \) is \( m_+ \).

5. Proof Theorem 3.3 and Theorem 3.12

In this section, we will prove Theorem 3.3 and Theorem 3.12. We use the FM algorithm to classify dominant monomials in \( \chi_q(\mathcal{L})\chi_q(\mathcal{R}), \chi_q(\mathcal{T})\chi_q(\mathcal{B}) \), and \( \chi_q(\mathcal{S}) \).

5.1. Classification of dominant monomials in \( \chi_q(\mathcal{L})\chi_q(\mathcal{R}) \) and \( \chi_q(\mathcal{T})\chi_q(\mathcal{B}) \).

Lemma 5.1. We have the following cases.

(1) Let

\[
M = T_{k,\ell-1,0}^{(s)} T_{k-1,\ell}^{(s+4)}, \quad k \geq 1, \ell \geq 1.
\]
Then dominant monomials in $\chi_q(T_{k,\ell-1,0}^{(s)})\chi_q(T_{k-1,\ell,0}^{(s+4)})$ are

$M_0 = M, \ M_1 = MA_{2,s+4k+2\ell-2}^{-1}, \ M_2 = M_1 A_{2,s+4k+2\ell-4}^{-1}, \ldots,$

$M_{\ell-1} = M_{\ell-2} A_{2,s+4k+2}^{-1}, \ M_\ell = M_{\ell-1} A_{3,s+4k-2}^{-1} A_{2,s+4k}^{-1},$

$M_{\ell+1} = M_{\ell+1} A_{3,s+4k-6}^{-1}, \ M_{\ell+2} = M_{\ell+1} A_{3,s+4k-10}^{-1}, \ldots, \ M_{k+\ell-1} = M_{k+\ell-2} A_{3,s+2}^{-1}.$

The dominant monomials in $\chi_q(T_{k,\ell,0}^{(s)})\chi_q(T_{k-1,\ell-1,0}^{(s+4)})$ are $M_0, \ldots, M_{k+\ell-2}.$

(2) Let $M = T_{k,0,m-1}^{(s)} T_{k-1,0,m}^{(s+4)}, k \geq 1, m \geq 1.$

Then dominant monomials in $\chi_q(T_{k,0,m-1}^{(s)})\chi_q(T_{k-1,0,m}^{(s+4)})$ are

$M_0 = M, \ M_1 = MA_{1,s+4k+2m-1}^{-1}, \ M_2 = M_1 A_{1,s+4k+2m-3}^{-1}, \ldots,$

$M_{m-1} = M_{m-2} A_{1,s+4k+2m+3}^{-1}, \ M_m = M_{m-1} A_{3,s+4k-2}^{-1} A_{2,s+4k}^{-1} A_{1,s+4k+1}^{-1},$

$M_{m+1} = M_{m+1} A_{3,s+4k-6}^{-1}, \ M_{m+2} = M_{m+1} A_{3,s+4k-10}^{-1}, \ldots, \ M_{k+m-1} = M_{k+m-2} A_{3,s+2}^{-1}.$

The dominant monomials in $\chi_q(T_{k,0,m}^{(s)})\chi_q(T_{k-1,0,m-1}^{(s+4)})$ are $M_0, \ldots, M_{k+m-2}.$

(3) Let $M = S_{k,\ell-1}^{(s)} S_{k-1,\ell}^{(s+2)}, k, \ell \geq 1.$

Then dominant monomials in $\chi_q(S_{k,\ell-1}^{(s)})\chi_q(S_{k-1,\ell}^{(s+2)})$ are

$M_0 = M, \ M_1 = MA_{2,s+2k+2\ell+1}^{-1}, \ M_2 = M_1 A_{2,s+2k+2\ell+3}^{-1}, \ldots,$

$M_{\ell-1} = M_{\ell-2} A_{2,s+2k+5}^{-1}, \ M_\ell = M_{\ell-1} A_{2,s+2k-1}^{-1} A_{3,s+2k+1}^{-1} A_{2,s+2k+3}^{-1},$

$M_{\ell+1} = M_{\ell+1} A_{2,s+2k-3}^{-1}, \ M_{\ell+2} = M_{\ell+1} A_{2,s+2k-5}^{-1}, \ldots, \ M_{k+\ell-1} = M_{k+\ell-2} A_{2,s+1}^{-1}.$

The dominant monomials in $\chi_q(S_{k,\ell}^{(s)})\chi_q(S_{k-1,\ell-1}^{(s+2)})$ are $M_0, \ldots, M_{k+\ell-2}.$

(4) Let $M = T_{0,\ell,r-1}^{(s)} T_{0,\ell-1,r}^{(s+2)}, \ell \geq 1, r \in \{1, 2\}.$

Then dominant monomials in $\chi_q(T_{0,\ell,r-1}^{(s)})\chi_q(T_{0,\ell-1,r}^{(s+2)})$ are

$M_0 = M, \ M_1 = MA_{1,s+2f+2r-2}^{-1}, \ M_2 = M_1 A_{1,s+2f+2r-4}^{-1}, \ldots,$

$M_{r-1} = M_{r-2} A_{1,s+2f+2}^{-1}, \ M_r = M_{r-1} A_{1,s+2f+2}^{-1} A_{1,s+2f+4}^{-1},$

$M_{r+1} = M_{r+1} A_{2,s+2f-2}^{-1}, \ M_{r+2} = M_{r+1} A_{2,s+2f-4}^{-1}, \ldots, \ M_{\ell+r-1} = M_{\ell+r-2} A_{2,s+2}^{-1}.$

The dominant monomials in $\chi_q(T_{0,\ell,r}^{(s)})\chi_q(T_{0,\ell-1,r-1}^{(s+2)})$ are $M_0, \ldots, M_{\ell+r-2}.$

(5) Let $M = \tilde{T}_{k,0,m-1}^{(s)} \tilde{T}_{k-1,0,m}^{(s+2)}, k, m \geq 1.$
Then dominant monomials in $\chi_q(\tilde{T}_{k,0,m-1}^{(s)})\chi_q(\tilde{T}_{k-1,0,m}^{(s+2)})$ are

$$M_0 = M, \quad M_1 = MA_{3,s+2k+4m-2}^{-1}, \quad M_2 = M_1A_{3,s+2k+4m-6}^{-1}, \quad \ldots, \quad M_{m-1} = M_{m-2}A_{3,s+2k+6}^{-1}, \quad M_m = M_{m-1}A_{3,s+2k-1}^{-1}A_{2,s+2k}^{-1}, \quad M_{m+1} = M_mA_{1,s+2k-3}^{-1}, \quad M_{m+2} = M_{m+1}A_{1,s+2k-5}^{-1}, \quad \ldots, \quad M_{k+m-1} = M_{k+m-2}A_{1,s+1}^{-1}.$$  

The dominant monomials in $\chi_q(\tilde{T}_{k,0,m}^{(s)})\chi_q(\tilde{T}_{k-1,0,m-1}^{(s+2)})$ are $M_0, \ldots, M_{\ell+m-2}$.

(6) Let

$$M = R_{k,2\ell,\ell-1}^{(s)}R_{k-1,2\ell,\ell}^{(s+2)}, \quad k \geq 1, \quad \ell \geq 1.$$  

Then dominant monomials in $\chi_q(R_{k,2\ell,\ell-1}^{(s)})\chi_q(R_{k-1,2\ell,\ell}^{(s+2)})$ are

$$M_0 = M, \quad M_1 = MA_{3,s+2k+4\ell-3}^{-1}, \quad M_2 = M_1A_{3,s+2k+4\ell-7}^{-1}, \quad \ldots, \quad M_{\ell-1} = M_{\ell-2}A_{3,s+2k+5}^{-1}, \quad M_\ell = M_{\ell-1}A_{3,s+2k-1}^{-1}A_{2,s+2k}^{-1}, \quad M_{\ell+1} = M_\ell A_{2,s+2k-3}^{-1}, \quad M_{\ell+2} = M_{\ell+1}A_{2,s+2k-5}^{-1}, \quad \ldots, \quad M_{k+\ell-1} = M_{k+\ell-2}A_{2,s+1}^{-1}.$$  

The dominant monomials in $\chi_q(R_{k,2\ell,\ell-1}^{(s)}\chi_q(R_{k-1,2\ell,\ell}^{(s+2)})$ are $M_0, \ldots, M_{k+\ell-2}$.

(7) Let

$$M = R_{k,2\ell,\ell}^{(s)}R_{k-1,2\ell+1,\ell}^{(s+2)}, \quad k \geq 1, \quad \ell \geq 1.$$  

Then dominant monomials in $\chi_q(R_{k,2\ell,\ell}^{(s)})\chi_q(R_{k-1,2\ell+1,\ell}^{(s+2)})$ are

$$M_0 = M, \quad M_1 = MA_{1,s+2k+4\ell}^{-1}, \quad M_2 = M_1A_{1,s+2k+4\ell-2}^{-1}, \quad \ldots, \quad M_{2\ell} = M_{2\ell-1}A_{1,s+2k+2}^{-1}, \quad M_{2\ell+1} = M_{2\ell}A_{2,s+2k-1}^{-1}A_{1,s+2k}, \quad M_{2\ell+2} = M_{2\ell+1}A_{2,s+2k-3}^{-1}, \quad M_{2\ell+3} = M_{2\ell+2}A_{2,s+2k-5}^{-1}, \quad \ldots, \quad M_{k+2\ell} = M_{k+2\ell-1}A_{2,s+1}^{-1}.$$  

The dominant monomials in $\chi_q(R_{k,2\ell,\ell}^{(s)}\chi_q(R_{k-1,2\ell+1,\ell}^{(s+2)})$ are $M_0, \ldots, M_{k+2\ell}$.

(8) Let

$$M = R_{k,2\ell+1,\ell}^{(s)}R_{k-1,2\ell+2,\ell}^{(s+2)}, \quad k \geq 1, \quad \ell \geq 1.$$  

Then dominant monomials in $\chi_q(R_{k,2\ell+1,\ell}^{(s)})\chi_q(R_{k-1,2\ell+2,\ell}^{(s+2)})$ are

$$M_0 = M, \quad M_1 = MA_{1,s+2k+4\ell+2}^{-1}, \quad M_2 = M_1A_{1,s+2k+4\ell}^{-1}, \quad \ldots, \quad M_{2\ell+1} = M_{2\ell}A_{1,s+2k+2}^{-1}, \quad M_{2\ell+2} = M_{2\ell+1}A_{2,s+2k-1}^{-1}A_{1,s+2k}, \quad M_{2\ell+3} = M_{2\ell+2}A_{2,s+2k-3}^{-1}, \quad M_{2\ell+4} = M_{2\ell+3}A_{2,s+2k-5}^{-1}, \quad \ldots, \quad M_{k+2\ell+1} = M_{k+2\ell}A_{2,s+1}^{-1}.$$  

The dominant monomials in $\chi_q(R_{k,2\ell+1,\ell}^{(s)}\chi_q(R_{k-1,2\ell+2,\ell}^{(s+2)})$ are $M_0, \ldots, M_{k+2\ell}$.

(9) Let

$$M = \tilde{T}_{k,\ell-1}^{(s)}\tilde{T}_{k-1,\ell}^{(s+2)}, \quad k \geq 1, \quad \ell \geq 1.$$  

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Then dominant monomials in $\chi_q(\tilde{T}^{(s)}_{k,\ell,-1,0})\chi_q(\tilde{T}^{(s+2)}_{k,-1,\ell,0})$ are

$$M_0 = M, \quad M_1 = MA^{-1}_{3,s+4k+4\ell-4}, \quad M_2 = MA^{-1}_{3,s+4k+4\ell-8}, \quad \ldots,$$

$$M_{l-1} = M_{l-2}A^{-1}_{3,s+4k+4\ell-4}, \quad M_{l} = M_{l-1}A^{-1}_{3,s+4k+4\ell-8}, \quad \ldots,$$

$$M_{l+1} = M_{l}A^{-1}_{3,s+4k+4\ell-10}, \quad M_{l+2} = M_{l+1}A^{-1}_{3,s+4k+4\ell-14}, \quad \ldots,$$

$$M_{k} = M_{k-1}A^{-1}_{3,s+4k+4\ell-30}, \quad \ldots, \quad M_{k+\ell-1} = M_{k+\ell-2}A^{-1}_{3,s+4k+4\ell-30}.$$

The dominant monomials in $\chi_q(\tilde{T}^{(s)}_{k,\ell,0})\chi_q(\tilde{T}^{(s+2)}_{k,-1,\ell,0})$ correspond to $M_0, \ldots, M_{k+\ell-2}$.

Recall that $\sigma(x) = 0$ if $x$ is even and $\sigma(x) = 1$ if $x$ is odd.

(10) Let $M = U^{(s)}_{p,\ell-1}U^{(s+2)}_{p-1,\ell}$, $p \geq 2, \ell \geq 1$.

Then dominant monomials in $\chi_q(U^{(s)}_{p,\ell-1})\chi_q(U^{(s+2)}_{p-1,\ell})$ are

$$M_0 = M, \quad M_1 = MA^{-1}_{3,s+2p+2\ell-3}, \quad M_2 = MA^{-1}_{3,s+2p+2\ell-7}, \quad \ldots,$$

$$M_{\lfloor \frac{p-1}{2} \rfloor} = M_{\lfloor \frac{p-1}{2} \rfloor-1}A^{-1}_{3,s+2p+2\ell-4+2\ell+(\ell+1)+3}, \quad M_{\lfloor \frac{p-1}{2} \rfloor+1} = M_{\lfloor \frac{p-1}{2} \rfloor}A^{-1}_{3,s+2p+2\ell-4+2\ell+1},$$

$$M_{p+\lfloor \frac{p-1}{2} \rfloor} = M_{\lfloor \frac{p-1}{2} \rfloor+1}A^{-1}_{3,s+2p+2\ell-4+2\ell+5}, \quad \ldots,$$

The dominant monomials in $\chi_q(U^{(s)}_{p,\ell-1})\chi_q(U^{(s+2)}_{p-1,\ell-1})$ are $M_0, \ldots, M_{p+\lfloor \frac{p-1}{2} \rfloor-1}.$

(11) Let $M = V^{(s)}_{k,\ell-1}V^{(s+4)}_{k-1,\ell}$, $k \geq 1, \ell \geq 1$.

Then dominant monomials in $\chi_q(V^{(s)}_{k,\ell-1})\chi_q(V^{(s+4)}_{k-1,\ell})$ are

$$M_0 = M, \quad M_1 = MA^{-1}_{3,s+4k+4\ell-4}, \quad M_2 = MA^{-1}_{3,s+4k+4\ell-8}, \quad \ldots,$$

$$M_{\ell-1} = M_{\ell-2}A^{-1}_{3,s+4k+4\ell-10}, \quad M_{\ell} = M_{\ell-1}A^{-1}_{3,s+4k+4\ell-14}, \quad \ldots,$$

$$M_{\ell+1} = M_{\ell+2}A^{-1}_{3,s+4k+4\ell-18}, \quad \ldots, \quad M_{k} = M_{k-1}A^{-1}_{3,s+4k+4\ell-30}, \quad \ldots, \quad M_{k+\ell-1} = M_{k+\ell-2}A^{-1}_{3,s+4k+4\ell-30}.$$

The dominant monomials in $\chi_q(V^{(s)}_{k,\ell})\chi_q(V^{(s+4)}_{k-1,\ell})$ are $M_0, \ldots, M_{k+\ell-2}$.

(12) Let $M = P^{(s)}_{k,\ell-1}P^{(s+4)}_{k-1,\ell}$, $k \geq 1, \ell \geq 1$.

Then dominant monomials in $\chi_q(P^{(s)}_{k,\ell-1})\chi_q(P^{(s+4)}_{k-1,\ell})$ are

$$M_0 = M, \quad M_1 = MA^{-1}_{3,s+4k+4\ell-4}, \quad M_2 = MA^{-1}_{3,s+4k+4\ell-8}, \quad \ldots,$$

$$M_{\ell-1} = M_{\ell-2}A^{-1}_{3,s+4k+4\ell-10}, \quad M_{\ell} = M_{\ell-1}A^{-1}_{3,s+4k+4\ell-14}, \quad \ldots,$$

$$M_{\ell+1} = M_{\ell+2}A^{-1}_{3,s+4k+4\ell-18}, \quad \ldots, \quad M_{k} = M_{k-1}A^{-1}_{3,s+4k+4\ell-30}, \quad \ldots, \quad M_{k+\ell-1} = M_{k+\ell-2}A^{-1}_{3,s+4k+4\ell-30}.$$

The dominant monomials in $\chi_q(P^{(s)}_{k,\ell})\chi_q(P^{(s+4)}_{k-1,\ell})$ are $M_0, \ldots, M_{k+\ell-1}$. 
(13) Let

$$M = O_{k, \ell - 1}^{(s)} O_{k-1, \ell}^{(s+2)}, k \geq 1, \ell \geq 1.$$  

Then dominant monomials in $\chi_q(O_{k, \ell - 1}^{(s)}) \chi_q(O_{k-1, \ell}^{(s+2)})$ are

$$M_0 = M, \quad M_1 = MA_{2,s+2k+2\ell+2}^{-1}, \quad M_2 = M_1 A_{2,s+2k+2\ell+2}^{-1}, \ldots,$$

$$M_{\ell - 1} = M_{\ell - 2} A_{2,s+2k+7}^{-1}, \quad M_{\ell} = M_{\ell - 1} A_{1,s+2k+4}^{-1} A_{2,s+2k+5}^{-1},$$

$$M_{\ell + 1} = M_1 A_{1,s+2k+2}^{-1}, \quad M_{\ell + 2} = M_{\ell + 1} A_{2,s+2k+2}^{-1} A_{1,2k}^{-1},$$

$$M_{\ell + 3} = M_{\ell + 2} A_{2,s+2k-3}^{-1}, \quad M_{\ell + 4} = M_{\ell + 3} A_{2,s+2k-5}^{-1}, \ldots, \quad M_{k + \ell + 1} = M_{k + \ell} A_{2,s+1}^{-1}.$$

The dominant monomials in $\chi_q(O_{k, \ell}^{(s)}) \chi_q(O_{k-1, \ell - 1}^{(s+2)})$ are $M_0, \ldots, M_{k+\ell}.$

In each case, for each $i,$ the multiplicity of $M_i$ in the corresponding product of $q$-characters is 1.

Proof. We prove the case of $\chi_q(T_{k, \ell - 1, 0}^{(s)}) \chi_q(T_{k-1, \ell, 0}^{(s+2)}).$ The other cases are similar. Let $m'_1 = T_{k, \ell - 1, 0}^{(s)}$, $m'_2 = T_{k-1, \ell, 0}^{(s+2)}$. Without loss of generality, we assume that $s = 0$. Then

$$m'_1 = (3_0 3_4 \cdot \cdot \cdot 3_{4k-4})(2_{4k+1} 2_{4k+3} \cdot \cdot \cdot 2_{4k+2\ell-3}),$$

$$m'_2 = (3_3 3_6 \cdot \cdot \cdot 3_{4k-4})(2_{4k+1} 2_{4k+3} \cdot \cdot \cdot 2_{4k+2\ell-1}).$$

Let $m = m_1 m_2$ be a dominant monomial, where $m_i \in \chi_q(m'_i), i = 1, 2.$ Let

$$m_3 = 3_3 3_6 \cdot \cdot \cdot 3_{4k-4}, \quad m_4 = 2_{4k+1} 2_{4k+3} \cdot \cdot \cdot 2_{4k+2\ell-1}.$$ 

If $m_2 \in \chi_q(m_3) (\chi_q(m_4) - m_4),$ then $m = m_1 m_2$ is right negative and hence $m$ is not dominant. Therefore $m_2 \in \chi_q(m_3) m_4.$

Suppose that $m_2 \in M(L(m'_2)) \cap \mathcal{M}(\chi_q(m_3) - m_3) m_4$. By the FM algorithm for $L(m'_2),$ $m_2$ must have a factor $3_{4k}$. By Lemma 3.1, $m_1$ does not have the factor $3_{4k}$. Therefore $m = m_1 m_2$ is not dominant. This is a contradiction. Therefore $m_2 = m_3 m_4 = m'_2$.

If

$$m_1 \in \chi_q(3_0 3_4 \cdot \cdot \cdot 3_{4k-4} 2_{4k+1} 2_{4k+3} \cdot \cdot \cdot 2_{4k+2\ell-5}) \times$$

$$\times \chi_q(2_{4k+2\ell-3} - 2_{4k+2\ell-3} - 2_{4k+2\ell-3} 3_{4k+2\ell-4}),$$

then $m = m_1 m_2$ is right-negative and hence not dominant. Therefore $m_1$ is in one of the following polynomials

$$\chi_q(3_0 3_4 \cdot \cdot \cdot 3_{4k-4} 2_{4k+1} 2_{4k+3} \cdot \cdot \cdot 2_{4k+2\ell-5}) 2_{4k+2\ell-3}, \quad (5.1)$$

$$\chi_q(3_0 3_4 \cdot \cdot \cdot 3_{4k-4} 2_{4k+1} 2_{4k+3} \cdot \cdot \cdot 2_{4k+2\ell-5}) 2_{4k+2\ell-1} - 1 3_{4k+2\ell-2}. \quad (5.2)$$

If $m_1$ is in (5.1), then $m_1 = m'_1$. The dominant monomial we obtain is $M_0 = m'_1 m'_2$. If $m_1$ is the highest monomial in (5.2), then we obtain the dominant monomial $M_1 = m'_1 m'_2$. Suppose that $m_1$ is in

$$M(L(m'_1)) \cap \mathcal{M}(\chi_q(3_0 3_4 \cdot \cdot \cdot 3_{4k-4} 2_{4k+1} 2_{4k+3} \cdot \cdot \cdot 2_{4k+2\ell-7}) \times$$

$$\times (\chi_q(2_{4k+2\ell-5} - 2_{4k+2\ell-5} 3_{4k+2\ell-4})).$$
By the FM algorithm for $L(m'_1)$, $m_1$ is in one of the following polynomials

$$\chi_q(3_03_4\cdots 3_{4k-4}2_{4k+1}2_{4k+3}\cdots 2_{4k+2\ell-7})2_{4k+2\ell-3}^{-1}1_{4k+2\ell-4}3_{4k+2\ell-4}^{-1}2_{4k+2\ell-1}^{-1}1_{4k+2\ell-2}3_{4k+2\ell-2},$$

(5.3)

$$\chi_q(3_03_4\cdots 3_{4k-4}2_{4k+1}2_{4k+3}\cdots 2_{4k+2\ell-7})1_{4k+2\ell-3}^{-1}3_{4k+2\ell-2}^{-1}1_{4k+2\ell-2}3_{4k+2\ell-2}.$$  

(5.4)

If $m_1$ is in (5.4), then $m = m_1m_2$ is right-negative and hence $m$ is not dominant. This is a contradiction. Therefore $m_1$ is in (5.3). If $m_1$ is the highest monomial in (5.3), then we obtain the dominant monomial $M_1 = m_1m'_2$. Suppose that $m_1$ is not the highest monomial in (5.3). Then by the FM algorithm, $m_1$ is in the set

$$\chi_q(3_03_4\cdots 3_{4k-4}2_{4k+1}2_{4k+3}\cdots 2_{4k+2\ell-9}) \times
\times 2_{4k+2\ell-5}1_{4k+2\ell-6}3_{4k+2\ell-2}2_{4k+2\ell-3}^{-1}1_{4k+2\ell-4}3_{4k+2\ell-5}^{-1}2_{4k+2\ell-6}^{-1}1_{4k+2\ell-2}3_{4k+2\ell-2}.$$  

If $m_1$ is the highest monomial in the above set, then we obtain the dominant monomial $M_2 = m_1m'_2$. Continue this procedure, we obtain dominant monomials $M_1, \ldots, M_{\ell-1}$ and the remaining dominant monomials are of the form $m_1m'_2$, where $m_1$ is a non-highest monomial in

$$\mathcal{M}(L(m'_1)) \cap \mathcal{M}(\chi_q(3_03_4\cdots 3_{4k-4}) \times
\times 2_{4k+2\ell-5}1_{4k+2\ell-6}3_{4k+2\ell-2}2_{4k+2\ell-3}^{-1}1_{4k+2\ell-4}3_{4k+2\ell-5}^{-1}2_{4k+2\ell-6}^{-1}1_{4k+2\ell-2}3_{4k+2\ell-2}).$$

Suppose that $m_1$ is a non-highest monomial in the above set. Since the non-highest monomials in $\chi_q(3_03_4\cdots 3_{4k-4})$ are right-negative, we need cancelations of factors with negative powers of some monomial in $\chi_q(3_03_4\cdots 3_{4k-4})$ with

$$1_{4k+2\ell-1}3_{4k+2\ell-2}2_{4k+1}.$$  

The only cancelation can happen is to cancel $1_{4k+2}$ or $3_{4k+2}$ or $3_{4k+4}$ or $2_{4k+1}$. Since $1^2_{4k}$, $3^2_{4k}$, and $3^2_{4k-2}$ do not appear in

$$\mathcal{M}(L(m'_1)) \cap \mathcal{M}(\chi_q(3_03_4\cdots 3_{4k-4}) \times
\times 2_{4k+2\ell-5}1_{4k+2\ell-6}3_{4k+2\ell-2}2_{4k+2\ell-3}^{-1}1_{4k+2\ell-4}3_{4k+2\ell-5}^{-1}2_{4k+2\ell-6}^{-1}1_{4k+2\ell-2}3_{4k+2\ell-2}),$$

$1_{4k+2}$, $3_{4k+2}$, and $3_{4k+4}$ cannot be canceled. Therefore we need a cancelation with $2_{4k+1}$. The only monomials in $\chi_q(3_03_4\cdots 3_{4k-4})$ which can cancel $2_{4k+1}$ is in one of the following polynomials

$$\chi_q(3_03_4\cdots 3_{4k-8})1_{4k}2_{4k-3}2_{4k+1}^{-1},$$

$$\chi_q(3_03_4\cdots 3_{4k-8})1_{4k-2}4_{k-1}2_{4k-1}2_{4k+1}^{-1}3_{4k-2},$$

$$\chi_q(3_03_4\cdots 3_{4k-8})2_{4k+1}^{-1}2_{4k+3}3_{4k}.$$
Therefore $m_1$ is in one of the following sets

\[
\mathcal{M}(L(m'_1)) \cap \mathcal{M}(\chi_q(3_03_1\cdots3_k-8)1_{4k2_{4k-3}2_{4k+1}}\times \\
2_{4k+3}2_{4k+5} \cdots 2_{4k+2_{\ell-1}}1_{4k+2_{\ell-1}4k+4} \cdots 1_{4k+2_{\ell-1}4k+4+\cdots4k+2_{\ell-2}}),
\]

(5.5)

\[
\mathcal{M}(L(m'_1)) \cap \mathcal{M}(\chi_q(3_03_1\cdots3_k-8)1_{4k-2}1_{4k-1}2_{4k+3}3_{4k-2} \times \\
2_{4k+3}2_{4k+5} \cdots 2_{4k+2_{\ell-1}}1_{4k+2_{\ell-1}4k+4} \cdots 1_{4k+2_{\ell-1}4k+4+\cdots4k+2_{\ell-2}}),
\]

(5.6)

\[
\mathcal{M}(L(m'_1)) \cap \mathcal{M}(\chi_q(3_03_1\cdots3_k-8)2_{4k+1}2_{4k+3}3_{4k} \times \\
2_{4k+3}2_{4k+5} \cdots 2_{4k+2_{\ell-1}}1_{4k+2_{\ell-1}4k+4} \cdots 1_{4k+2_{\ell-1}4k+4+\cdots4k+2_{\ell-2}}).
\]

(5.7)

If $m_1$ is in (5.6), then we need to cancel $2_{4k-1}^{-1}$. But $2_{4k-1}^{-1}$ cannot be canceled by any monomials in $\chi_q(3_03_1\cdots3_k-8)$ or by $m'_2$. Hence $m_1$ is not in (5.6).

If $m_1$ is in (5.7), then we need to cancel $2_{4k+3}^{-1}$. But $2_{4k+3}^{-1}$ cannot be canceled by any monomials in $\chi_q(3_03_1\cdots3_k-8)$ or by $m'_2$. Therefore $m_1$ is not in (5.7). Hence $m_1$ is in (5.5).

If $m_1$ is the highest monomial in (5.5) with respect to \( \leq \) defined in (2.4), then we obtain the dominant monomial $M_{\ell} = m_1m'_2$. Suppose that $m_1$ a non-highest monomial in (5.5). By the FM algorithm, $m_1$ must in

\[
\mathcal{M}(L(m'_1)) \cap \mathcal{M}(\chi_q(3_03_1\cdots3_k-12)3_{4k-4}2_{4k-7}2_{4k-5}1_{4k-2_{4k+3}2_{4k+1}}\times \\
2_{4k+3}2_{4k+5} \cdots 2_{4k+2_{\ell-1}}1_{4k+2_{\ell-1}4k+4} \cdots 1_{4k+2_{\ell-1}4k+4+\cdots4k+2_{\ell-2}}).
\]

If $m_1$ is the highest monomial in the above set, then we obtain the dominant monomial $M_{\ell+1} = m_1m'_2$. Continue this procedure, we can show that the only remaining dominant monomials are $M_{\ell+2}, \ldots, M_{k+\ell-1}$.

It is clear that the multiplicity of $M_i, i = 1, \ldots, k + \ell - 1$, in $\chi_q(3_03_1\cdots3_k-8)$ is 1. \( \square \)

### 5.2. Products of sources are special.

#### Lemma 5.2. Let $[S]$ be the last summand in one of the relations in Theorem 3.3. Then $S$ is special.

**Proof.** We prove the case of $S$ in the first relation in Theorem 3.3. The other cases are similar.

Let $S = \chi_q(R^{(s+1)}_{2k-1,\ell,|M_{\ell+1}})\chi_q(T^{(s+4k+4)}_{|M_{\ell+1}|,0,0}, \ell = 2m, m \in \mathbb{Z}_{\geq 0}$. Let

\[
n_1 = 2_{s+1}2_{s+3} \cdots 2_{s+4k-3}, \quad n'_1 = 1_{s+4k}1_{s+4k+2} \cdots 1_{s+4k+4m-2},
\]

\[
n''_1 = 3_{s+4k+2}3_{s+4k+6} \cdots 3_{s+4k+4m-2}, \quad n_2 = 3_{s+4k+4}3_{s+4k+8} \cdots 3_{s+4k+4m-4}.
\]
Then $R_{2k-1,1,\ell,\frac{r}{s}}^{(s+1)} = n_1 n'_1 n''_1 T_{\frac{r}{s},0,0}^{(s+4k+4)} = n_2$. Let $m = m_1 m_2$ be a dominant monomial, where
\[
m_1 \in \chi_q(R_{2k-1,1,\ell,\frac{r}{s}}^{(s+1)}), \quad m_2 \in \chi_q(T_{\frac{r}{s},0,0}^{(s+4k+4)}).
\]

If $m_2 \neq n_2$ or $m_1 \in \chi_q(n_1 n''_1) (\chi_q(n'_1) - n'_1)$ or $m_1 \in \chi_q(n_1 n'_1) (\chi_q(n''_1) - n''_1)$, then $m$ is right-negative which contradicts the fact that $m$ is dominant. Therefore $m_2 = n_2$, $m_1 \in \chi_q(n_1 n'_1 n''_1)$.

If $m_1$ is the highest monomial in $\chi_q(n_1 n'_1 n''_1)$, then we obtain a dominant monomial $n_1 n'_1 n''_1 n_2$. Suppose that $m_1$ is a non-highest monomial in $\chi_q(n_1 n'_1 n''_1)$. Then $m_1$ is in $(\chi_q(n_1) - n_1) n'_1 n''_1$. By the FM algorithm, $m_1$ is one of the following monomials
\[
m_1 = n_1 n'_1 n''_1 A_{s,+4k-2}^{-1}, \quad m_2 = \bar{m}_1 A_{s,+4k-4}^{-1}, \quad \ldots, \quad m_{2k-1} = \bar{m}_{2k-2} A_{s+2}^{-1}.
\]
Therefore $m = m_1 m_2 = m_1 n_2$ is one of the following monomials
\[
\bar{m}_1 n_2 = n_1 n'_1 n''_1 A_{s,+4k-2}^{-1}, \quad m_2 n_2 = \bar{m}_1 A_{s,+4k-4}^{-1} n_2, \quad \ldots, \quad m_{2k-1} n_2 = \bar{m}_{2k-2} A_{s+2}^{-1} n_2.
\]
Hence $m$ is not a dominant monomial. This is a contradiction. Therefore the only dominant monomial in $S$ is $R_{2k-1,1,\ell,\frac{r}{s}}^{(s+1)} T_{\frac{r}{s},0,0}^{(s+4k+4)}$. \hfill \Box

5.3. **Proof of Theorem 3.3** In Section 4, we have shown that $R_{0,2\ell+i,\ell}^{(s)}$, $i = 0, 1, 2$, and $U_{0,\ell}^{(s)}$ are special. Therefore by Lemma 5.1, 5.13, and 5.26, and (3.26) are true. By Lemmas 5.1 and 5.2, the dominant monomials in the $q$-characters of the left hand side and of the right hand side of every relation in Theorem 3.3 are the same. The theorem follows.

6. **Proof of Theorem 3.4 and Theorem 3.13**

By Lemma 5.2, $S$ is special and hence irreducible. Therefore we only have to show that each $T \otimes B$ in Theorem 3.4 and Theorem 3.13 is irreducible. It suffices to prove that for each non-highest dominant monomial $M$ in $T \otimes B$, we have $\mathcal{M}(L(M)) \not\subseteq \mathcal{M}(T \otimes B)$. The idea is similar as in Her06, MY12b, LM12.

**Lemma 6.1.** We consider the same cases as in Lemma 5.1. In each case $M_1$ are the dominant monomials described by that lemma.

1. For $k \geq 1$, $\ell \geq 1$, let
\[
n_1 = M_1 A_{s+4k+2\ell-2}^{-1}, \quad n_2 = M_2 A_{s+4k+2\ell-4}^{-1}, \quad \ldots,
\]
\[
n_{\ell-1} = M_{\ell-1} A_{s+4k+2\ell-2}^{-1}, \quad n_\ell = M_\ell A_{s+4k+2\ell-4}^{-1},
\]
\[
n_{\ell+1} = M_{\ell+1} A_{s+4k+2\ell-6}^{-1}, \quad n_{\ell+2} = M_{\ell+2} A_{s+4k+2\ell-10}^{-1}, \quad \ldots, \quad n_{k+\ell-2} = M_{k+\ell-2} A_{s+6}^{-1}.
\]

Then for $i = 1, \ldots, k + \ell - 2$, $n_i \in \chi_q(M_i)$ and $n_i \not\in \chi_q(T_{k,\ell,0}^{(s)}) \chi_q(T_{k-1,\ell-1,0}^{(s+4)}).$
(2) For \( k \geq 1, m \geq 1 \), let
\[
\begin{align*}
n_1 &= M_1 A_{1, s + 2k + 4\ell - 1}^{-1}, \quad n_2 = M_2 A_{1, s + 2k + 4\ell - 2}^{-1}; \\
n_{m-1} &= M_{m-1} A_{1, s + 2k + 4\ell - 3}^{-1}, \quad n_m = M_m A_{1, s + 2k - 1}^{-1} A_{2, s + 2k}^{-1} A_{1, s + 2k + 1}^{-1}; \\
n_{m+1} &= M_{m+1} A_{1, s + 2k + 4\ell - 5}^{-1}, \quad n_{m+2} = M_{m+2} A_{1, s + 2k + 4\ell - 6}; \ldots; \quad n_{k+m-2} = M_{k+m-2} A_{1, s + 2k + 6}^{-1} + 1.
\end{align*}
\]
Then for \( i = 1, \ldots, k + m - 2 \), \( n_i \in \chi_0(M_i) \) and \( n_i \not\in \chi_0(T_{k,0,m}^{(s)} \chi_0(T_{k-1,0,m-1}^{(s+1)}).
\]

(3) For \( k \geq 1, \ell \geq 1 \), let
\[
\begin{align*}
n_1 &= M_1 A_{2, s + 2k + 4\ell - 4}, \quad n_2 = M_2 A_{2, s + 2k + 4\ell - 5}; \\
n_{\ell-1} &= M_{\ell-1} A_{2, s + 2k + 4\ell - 6}, \quad n_{\ell} = M_{\ell} A_{2, s + 2k + 4\ell - 7} A_{2, s + 2k - 1}^{-1} A_{3, s + 2k + 1}^{-1} A_{2, s + 2k + 1}^{-1}; \\
n_{\ell+1} &= M_{\ell+1} A_{2, s + 2k + 4\ell - 8}, \quad n_{\ell+2} = M_{\ell+2} A_{2, s + 2k - 1}^{-1} A_{3, s + 2k - 2}^{-1} A_{2, s + 2k - 2}^{-1} \ldots; \quad n_{k+\ell-2} = M_{k+\ell-2} A_{2, s + 4}^{-1}.
\end{align*}
\]
Then for \( i = 1, \ldots, k + \ell - 2 \), \( n_i \in \chi_0(M_i) \) and \( n_i \not\in \chi_0(T_{0,\ell,\ell-1}^{(s)} \chi_0(T_{0,\ell,\ell-2}^{(s+1)}).
\]

(4) For \( \ell \geq 1, r \in \{1, 2\} \), let
\[
\begin{align*}
n_1 &= M_1 A_{3, s + 2k + 4m - 2}, \quad n_2 = M_2 A_{3, s + 2k + 4m - 3}; \\
n_{m-1} &= M_{m-1} A_{3, s + 2k + 4m - 4}, \quad n_m = M_m A_{3, s + 2k - 1}^{-1} A_{2, s + 2k - 1}^{-1} A_{3, s + 2k + 1}^{-1}; \\
n_{m+1} &= M_{m+1} A_{3, s + 2k - 2}^{-1}, \quad n_{m+2} = M_{m+2} A_{3, s + 2k - 3}; \ldots; \quad n_{k+m-2} = M_{k+m-2} A_{1, s + 2k}^{-1}.
\end{align*}
\]
Then for \( i = 1, \ldots, k + m - 2 \), \( n_i \in \chi_0(M_i) \) and \( n_i \not\in \chi_0(T_{k-1,\ell,\ell-1}^{(s)} \chi_0(T_{k-1,\ell,\ell-2}^{(s+1)}).
\]

(5) For \( k, m \geq 1, \ell \geq 1 \), let
\[
\begin{align*}
n_1 &= M_1 A_{4, s + 2k + 4\ell - 3}, \quad n_2 = M_2 A_{4, s + 2k + 4\ell - 4}; \\
n_{\ell-1} &= M_{\ell-1} A_{4, s + 2k + 4\ell - 5}, \quad n_{\ell} = M_{\ell} A_{4, s + 2k + 4\ell - 6} A_{4, s + 2k - 1}^{-1} A_{3, s + 2k + 1}^{-1} \ldots; \quad n_{k+\ell-2} = M_{k+\ell-2} A_{2, s + 3}^{-1}.
\end{align*}
\]
Then for \( i = 1, \ldots, k + \ell - 2 \), \( n_i \in \chi_0(M_i) \) and \( n_i \not\in \chi_0(T_{k,0,0}^{(s)} \chi_0(T_{k-1,0,0}^{(s+1)}).
\]

(6) For \( k \geq 1, \ell \geq 1 \), let
\[
\begin{align*}
n_1 &= M_1 A_{5, s + 2k + 4\ell - 3}, \quad n_2 = M_2 A_{5, s + 2k + 4\ell - 4}; \\
n_{2\ell-1} &= M_{2\ell-1} A_{5, s + 2k + 4\ell - 5}, \quad n_{2\ell} = M_{2\ell} A_{5, s + 2k + 4\ell - 6} A_{5, s + 2k - 1}^{-1} A_{4, s + 2k + 1}^{-1} \ldots; \quad n_{k+2\ell-2} = M_{k+2\ell-2} A_{2, s + 3}^{-1}.
\end{align*}
\]
Then for \( i = 1, \ldots, k + 2\ell - 2 \), \( n_i \in \chi_0(M_i) \) and \( n_i \not\in \chi_0(T_{k,2\ell,\ell-1}^{(s)} \chi_0(T_{k-1,2\ell,\ell-1}^{(s+1)}).
\]

(7) For \( k \geq 1, \ell \geq 1 \), let
\[
\begin{align*}
n_1 &= M_1 A_{6, s + 2k + 4\ell - 3}, \quad n_2 = M_2 A_{6, s + 2k + 4\ell - 4}; \\
n_{2\ell-2} &= M_{2\ell-2} A_{6, s + 2k + 4\ell - 5}, \quad n_{2\ell-1} = M_{2\ell-1} A_{6, s + 2k + 4\ell - 6} A_{6, s + 2k - 1}^{-1} A_{5, s + 2k + 1}^{-1} \ldots; \quad n_{k+2\ell-1} = M_{k+2\ell-1} A_{2, s + 3}^{-1}.
\end{align*}
\]
Then for \( i = 1, \ldots, k + 2\ell - 1 \), \( n_i \in \chi_0(M_i) \) and \( n_i \not\in \chi_0(R_{k,2\ell,\ell-1}^{(s)} \chi_0(R_{k-1,2\ell,\ell-1}^{(s+1)}).
\]
For $k \geq 1$, $\ell \geq 1$, let
\[
\begin{align*}
n_1 &= M_1A_{3, s+2k+4\ell+2}^{-1}, \quad n_2 = M_2A_{3, s+2k+4\ell}^{-1}, \quad \cdots, \\
n_{2\ell+1} &= M_{2\ell+1}A_{1, s+2k+2}, \quad n_{2\ell+2} = M_{2\ell+2}A_{3, s+2k-1}^{-1}A_{1, s+2k}, \\
n_{2\ell+3} &= M_{2\ell+3}A_{2, s+2k-3}^{-1}, \quad n_{2\ell+4} = M_{2\ell+4}A_{2, s+2k-5}, \quad \cdots, \quad n_{k+2\ell} = M_{k+2\ell}A_{2, s+3}^{-1}.
\end{align*}
\]
Then for $i = 1, \ldots, k + 2\ell$, $n_i \in \chi_q(M_i)$ and $n_i \not\in \chi_q(P_{k, 2\ell+2, \ell}^{(s)}) \chi_q(P_{k-1, 2\ell+1, \ell}^{(s+2)}).

For $p \geq 2$, $\ell \geq 1$, let
\[
\begin{align*}
n_1 &= M_1A_{3, s+2p+2\ell-3}^{-1}, \quad n_2 = M_2A_{3, s+2p+2\ell-7}, \quad \cdots, \\
n_{\ell-1} &= M_{\ell-1}A_{3, s+2p+2\ell-9}, \quad n_{\ell} = M_{\ell}A_{3, s+2p+2\ell-11}^{-1}A_{3, s+2p+1}^{-1}, \\
n_{\ell+1} &= M_{\ell+1}A_{3, s+2p+2\ell-13}^{-1}, \quad n_{\ell+2} = M_{\ell+2}A_{3, s+2p+2\ell-15}, \quad \cdots, \quad n_{p+\ell-1} = M_{p+\ell-1}A_{3, s+3}^{-1}.
\end{align*}
\]
Then for $i = 1, \ldots, p + \lfloor \frac{\ell-1}{2} \rfloor - 1$, $n_i \in \chi_q(M_i)$ and $n_i \not\in \chi_q(U_{p, \ell}^{(s)}) \chi_q(U_{p-1, \ell-1}^{(s+2)}).

For $k \geq 1$, $\ell \geq 1$, let
\[
\begin{align*}
n_1 &= M_1A_{3, s+4k+4\ell-4}, \quad n_2 = M_2A_{3, s+4k+4\ell-8}, \quad \cdots, \\
n_{\ell-1} &= M_{\ell-1}A_{3, s+4k+4\ell-6}, \quad n_{\ell} = M_{\ell}A_{3, s+4k+4\ell-10}^{-1}A_{3, s+4k+4\ell-8}, \\
n_{\ell+1} &= M_{\ell+1}A_{3, s+4k+4\ell-12}^{-1}, \quad n_{\ell+2} = M_{\ell+2}A_{3, s+4k+4\ell-14}, \quad \cdots, \quad n_{k+\ell-2} = M_{k+\ell-2}A_{3, s+6}^{-1}.
\end{align*}
\]
Then for $i = 1, \ldots, k + \ell - 2$, $n_i \in \chi_q(M_i)$ and $n_i \not\in \chi_q(V_{k, \ell}^{(s)}) \chi_q(V_{k-1, \ell-1}^{(s+4)})

For $k \geq 1$, $\ell \geq 1$, let
\[
\begin{align*}
n_1 &= M_1A_{3, s+4k+4\ell}, \quad n_2 = M_2A_{3, s+4k+4\ell-4}, \quad \cdots, \\
n_{\ell-1} &= M_{\ell-1}A_{3, s+4k+4\ell-8}, \quad n_{\ell} = M_{\ell}A_{3, s+4k+4\ell-12}^{-1}A_{3, s+4k+4\ell-4}, \\
n_{\ell+1} &= M_{\ell+1}A_{3, s+4k+4\ell-16}, \quad n_{\ell+2} = M_{\ell+2}A_{3, s+4k+4\ell-20}, \quad \cdots, \quad n_{k+\ell-1} = M_{k+\ell-1}A_{3, s+6}^{-1}.
\end{align*}
\]
Then for $i = 1, \ldots, k + \ell - 1$, $n_i \in \chi_q(M_i)$ and $n_i \not\in \chi_q(P_{k, \ell}^{(s)}) \chi_q(P_{k-1, \ell-1}^{(s+4)}).
For \( k \geq 1, \ell \geq 1 \), let
\[
\begin{align*}
n_1 &= M_1 A_{2,s+2k+2\ell+4}^{-1}, \quad n_2 = M_2 A_{2,s+2k+2\ell+2}^{-1}, \quad \cdots, \\
n_{\ell-1} &= M_{\ell-1} A_{2,s+2k+2\ell+7}^{-1}, \quad n_\ell = M_\ell A_{1,s+2k+4}^{-1} A_{2,s+2k+5}^{-1}, \\
n_{\ell+1} &= M_{\ell+1} A_{1,s+2k+2}^{-1}, \quad n_{\ell+2} = M_{\ell+2} A_{2,s+2k-1}^{-1} A_{1,2k}^{-1}, \\
n_{\ell+3} &= M_{\ell+3} A_{2,s+2k-3}^{-1}, \quad n_{\ell+4} = M_{\ell+4} A_{2,s+2k-5}^{-1}, \quad \cdots, \quad n_{k+\ell} = M_{k+\ell} A_{2,s+3}^{-1}.
\end{align*}
\]

Then for \( i = 1, \ldots, k+\ell \), \( n_i \in \chi_q(M_i) \) and \( n_i \not\in \chi_q(O_{k,\ell}^{(s)}) \chi_q(O_{k-1,\ell-1}^{(s+2)}) \).

**Proof.** We prove the case of \( \chi_q(T_{k,\ell,0}^{(s)}) \chi_q(T_{k-1,\ell-1,0}^{(s+4)}) \). The other cases are similar. By definition, we have
\[
\begin{align*}
T_{k,\ell,0}^{(s)} &= \left(3,3s+4 \cdots 3s+4k-4\right) (2s+4k+12s+4k+3 \cdots 2s+4k+2\ell-32s+4k+2\ell-1), \\
T_{k-1,\ell-1,0}^{(s+4)} &= \left(3,3s+4 \cdots 3s+4k-4\right) (2s+4k+12s+4k+3 \cdots 2s+4k+2\ell-3), \\
M_1 &= T_{k,\ell,0}^{(s)} T_{k-1,\ell-1,0}^{(s+4)} A_{2,s+4k+2\ell-2}^{-1} \\
&= T_{k,\ell,0}^{(s)} T_{k-1,\ell-1,0}^{(s+4)} 2^{-1}s+4k+2\ell-1 3^{-1}s+4k+2\ell-2.
\end{align*}
\]

By \( U_{q_2}(\hat{s}_2) \) argument, it is clear that \( n_1 = M_1 A_{2,s+4k+2\ell-2}^{-1} \) is in \( \chi_q(M_1) \).

If \( n_1 \) is in \( \chi_q(T_{k,\ell,0}^{(s)}) \chi_q(T_{k-1,\ell-1,0}^{(s+4)}) \), then \( T_{k,\ell,0}^{(s)} A_{2,s+4k+2\ell-2}^{-1} \) is in \( \chi_q(T_{k,\ell,0}^{(s)}) \) which is impossible by the FM algorithm for \( T_{k,\ell,0}^{(s)} \). Similarly, \( n_i \in \chi_q(M_i), i = 2, \ldots, \ell-1 \), but \( n_2, \ldots, n_{\ell-1} \) are not in \( \chi_q(T_{k,\ell,0}^{(s)}) \chi_q(T_{k-1,\ell-1,0}^{(s+4)}) \).

By definition,
\[
M_\ell = (3,3s+4 \cdots 3s+4k-4) (3,3s+4 \cdots 3s+4k-8) 2s+4k-3 \times (1s+4k+2s+4k+43s+4k-2\ell-2) (3s+4k+3s+4k+4 \cdots 3s+4k-2\ell-2).
\]

Let \( U = \{(2, aq^{s+4k}), (3, aq^{s+4k})\} \subseteq I \times \mathbb{C}^\times \). Let \( \mathcal{M} \) be the finite set consisting of the following monomials
\[
m_0 = M_\ell, \quad m_1 = m_0 A_{3,s+4k-2}^{-1}, \quad m_2 = m_1 A_{2,s+4k}^{-1}.
\]

It is clear that \( \mathcal{M} \) satisfies the conditions in Theorem 2.2 Therefore
\[
\text{trunc}_{M_\ell} Q_U^\ell(\chi_q(M)) = \sum_{m \in \mathcal{M}} m
\]
and hence \( n_\ell = M_\ell A_{3,s+4k-2}^{-1} A_{2,s+4k}^{-1} \) is in \( \chi_q(M_\ell) \).

If \( n_\ell \) is in \( \chi_q(T_{k,\ell,0}^{(s)}) \chi_q(T_{k-1,\ell-1,0}^{(s+4)}) \), then \( T_{k,\ell,0}^{(s)} A_{3,s+4k-2}^{-1} A_{2,s+4k}^{-1} \) is in \( \chi_q(T_{k,\ell,0}^{(s)}) \) which is impossible by the FM algorithm for \( T_{k,\ell,0}^{(s)} \).

Similarly, for \( i = \ell+1, \ldots, k+\ell-2 \), we have \( n_i \in \chi_q(M_i) \) and \( n_i \not\in \chi_q(T_{k,\ell,0}^{(s)}) \chi_q(T_{k-1,\ell-1,0}^{(s+4)}) \).\( \square \)
7. Proof Proposition 3.5

Let $A^{(s)}$ be a module in the system I. By (2.3), $\chi_q(A^{(s)})$ is obtained from $\chi_q(A^{(0)})$ by a shift of indices. For simplicity, we do not write the upper-subscripts "(s)" in the proof. The $q$-characters of Kirillov-Reshetikhin modules can be computed from $\chi_q(1_0), \chi_q(2_0), \chi_q(3_0)$, see [Her06].

It suffices to prove the following results.

(1) By using (3.9), we can compute $\chi(T_{0,\ell,r}), r \in \{0,1,2\}, \ell \in \mathbb{Z}_{\geq 0}$.
(2) By using (3.6), (3.7), and (3.8), we can compute $\chi(T_{k,0,m}), \chi(T_{k,0,m}), \chi(S_{k,m}), k, m \in \mathbb{Z}_{\geq 0}$.
(3) By using (3.5), (3.10)–(3.13) and the $q$-characters computed in (1), (2), we can compute $\chi_q(R_{k,2\ell+i,\ell}), \chi_q(T_{k,\ell,0}), i = 0,1,2, k, \ell \in \mathbb{Z}_{\geq 0}$.

Proof of (1). We use induction on $\ell, r$. If $\ell \leq 1, r = 0$, then (1) is clearly true. Suppose that (1) is true for $\ell = \ell_1 + 1, r = r_1$ and $\ell = \ell_1, r = r_1 + 1$ respectively.

Suppose that $\ell = \ell_1 + 1, r = r_1$. If $r = 0$, then $T_{0,\ell,r}$ is a Kirillov-Reshetikhin modules and hence (1) is true. If $r = 1$, then by using (3.9), $T_{0,\ell,r}$ can be computed by using the $q$-characters for Kirillov-Reshetikhin modules and induction hypothesis. If $r = 2$, then by using (3.9), $T_{0,\ell,r}$ can be computed by using the $q$-characters for Kirillov-Reshetikhin modules, $\chi_q(0,k,1), k \in \mathbb{Z}_{\geq 0}$, and induction hypothesis. Similarly, we can show that (1) is true for $\ell = \ell_1, r = r_1 + 1$.

Proof of (2). It suffices to show the following result.

(2') By using (3.6), (3.7), and (3.8), we can compute

$$\chi(T_{k,0,m}), k \leq k_1, m \leq m_1, \quad \chi(T_{m,0,k}), k \leq k_1, m \leq m_1,$$
$$\chi(S_{k,m}), k \leq 2k_1, m \leq m_1 - 1, \quad \chi(S_{m,k}), m \leq m_1 - 1, k \leq 2k_1.$$

If $t_1 < 0$ or $t_2 < 0$, then we do not need to compute $S_{t_1,t_2}$.

We will prove (2') by using induction on $k_1, m_1$. If $k_1 = 1, m_1 = 1$, then (2') is clearly true. Suppose that (2') is true for $k_1 = k_2, m_1 = m_2, k_2, m_2 \geq 1$. We will show that (2') is true for $k_1 = k_2 + 1, m_1 = m_2$ and $k_1 = k_2, m_1 = m_2 + 1$ respectively.

Suppose that $k_1 = k_2 + 1, m_1 = m_2$. We only need to show that the following $q$-characters

$$\chi(S_{2k_2+1,m}), m \leq m_2 - 1, \quad \chi(S_{2k_2+2,m}), m \leq m_2 - 1, \quad \chi(T_{k_2+1,0,m}), m \leq m_2,$$
$$\chi(T_{m,0,k_2+1}), m \leq m_2, \quad \chi(S_{m,2k_2+1}), m \leq m_2 - 1, \quad \chi(S_{m,2k_2+2}), m \leq m_2 - 1$$
can be computed. We compute the following $q$-characters

\[
\chi_q(S_{2k,2m-1}), \ m \leq \left\lfloor \frac{m_2 - 1}{2} \right\rfloor, \ \chi_q(T_{2k+1,0,m}), \ m \leq \left\lfloor \frac{m_2 - 1}{2} \right\rfloor,
\]

\[
\chi_q(T_{k,0,m}), \ m \leq m_2 - 1, \ \chi_q(S_{2k+1,m}), \ m \leq m_2 - 1, \ \chi_q(T_{k+1,0,m}), \ m \leq m_2,
\]

\[
\chi_q(S_{2k+1,m}), \ m \leq \left\lfloor \frac{m_2 - 1}{2} \right\rfloor, \ \chi_q(T_{2k+2,0,m}), \ m \leq \left\lfloor \frac{m_2 - 1}{2} \right\rfloor,
\]

\[
\chi_q(S_{2k+2,m}), \ m \leq m_2 - 1,
\]

in the order as shown. At each step, we consider the module that we want to compute as a top module and use the corresponding relation in Theorem 3.3 and known $q$-characters. Therefore $\chi_q(S_{2k+1,m}), \ m \leq m_2 - 1$, $\chi_q(S_{2k+2,m}), \ m \leq m_2 - 1$, $\chi_q(T_{k+1,0,m}), \ m \leq m_2$, can be computed. The fact that the $q$-characters in (7.2) can be computed can be proved similarly. Therefore (2') is true for $k_1 = k_2 + 1, m_1 = m_2$.

Similarly, we can show that (2') is true for $k_1 = k_2, m_1 = m_2 + 1$.

Proof of (3). It suffices to prove the following result.

(3') By using (3.5), (3.10)–(3.13) and the $q$-characters computed in (1), (2), we can compute the following $q$-characters:

\[
\chi_q(R_{k,2\ell}), \ k \leq 2k_1 - 3, \ell \leq \ell_1, \ \chi_q(R_{k,2\ell-1}), \ k \leq 2k_1 - 1, \ell \leq \ell_1,
\]

\[
\chi_q(R_{k,2\ell-1}), \ k \leq 2k_1 - 1, \ell \leq \ell_1, \ \chi_q(T_{k,\ell,0}), \ k \leq k_1, \ell \leq 2\ell_1.
\]

Let $k_2 = 1, \ell_2 = 1$. The $q$-characters of $R_{k,i,0} = T_{k,i,0}, i = 0, 1, 2$, are computed in (1). Therefore (3') is true in this case by using (3.10) and the $q$-characters computed in (1).

Suppose that (3') is true for $k_1 = k_2, \ell_1 = \ell_2, k_2 \geq 1, \ell_2 \geq 0$. We will show that (3') is true for $k_1 = k_2, \ell_1 = \ell_2 + 1$ and $k_1 = k_2 + 1, \ell_1 = \ell_2$ respectively. Let $k_1 = k_2, \ell_1 = \ell_2 + 1$.

We need to show that the following $q$-characters

\[
\chi_q(R_{k,2\ell+2,\ell+1}), \ k \leq 2k_2 - 1, \ \chi_q(R_{k,2\ell+2,\ell}), \ k \leq 2k_2 - 1,
\]

\[
\chi_q(R_{k,2\ell+1,\ell}), \ k \leq 2k_2 - 1, \ \chi_q(T_{k,2\ell+1,0}), \ k \leq k_2, \ \chi_q(T_{k,2\ell+2,0}), \ k \leq k_2,
\]

can be computed. We compute the following $q$-characters

\[
\chi_q(T_{k-2,2\ell,0}), \ k \leq k_2, \ \chi_q(R_{k-3,2\ell+1,\ell}), \ k \leq k_2,
\]

\[
\chi_q(T_{k-1,2\ell+1,0}), \ k \leq k_2, \ \chi_q(R_{k-1,2\ell+2,\ell}), \ k \leq k_2,
\]

\[
\chi_q(R_{k-1,2\ell+2,\ell+1}), \ k \leq k_2, \ \chi_q(T_{k,2\ell+2,0}), \ k \leq k_2,
\]

in the order as shown. At each step, we consider the module that we want to compute as a top module and use the corresponding relation in Theorem 3.3 and known $q$-characters. Therefore (3') is true for $k_1 = k_2, \ell_1 = \ell_2 + 1$.

Similarly, we can show that (3') is true for $k_1 = k_2 + 1, \ell_1 = \ell_2$.

### 8. Conjectural character formulas

Every $U_q(\mathfrak{g})$-module $V$ is also a $U_q(\mathfrak{g})$-module. We use $\text{Res}(V)$ to denote the $U_q(\mathfrak{g})$-module obtained by restricting $V$ to $U_q(\mathfrak{g})$. We use the system I to compute the characters
of the restrictions of modules in the system I. According to these computations, we obtain the following conjecture.

**Conjecture 8.1.** For \( s \in \mathbb{Z}, k, \ell, m \in \mathbb{Z}_{\geq 0}, \) \( \text{Res}(\mathcal{T}_{k,\ell,0}^{(s)}), \text{Res}(\mathcal{T}_{k,0,m}^{(s)}), \text{Res}(\tilde{\mathcal{T}}_{k,0,m}^{(s)}) \) are decomposed to direct sums of irreducible \( U_q(\mathfrak{g}) \)-modules as follows.

\[
\text{Res}(\mathcal{T}_{k,\ell,0}^{(s)}) = \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} \bigoplus_{j=0}^{\lfloor \frac{\ell}{2} \rfloor} V((2i - 2j)\omega_1 + (\ell - 2i)\omega_2 + k\omega_3),
\]

\[
\text{Res}(\mathcal{T}_{k,0,m}^{(s)}) = \bigoplus_{i=0}^{\lfloor \frac{m}{2} \rfloor} V((m - 2i)\omega_1 + k\omega_3),
\]

\[
\text{Res}(\tilde{\mathcal{T}}_{k,0,m}^{(s)}) = \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} V((k - 2i)\omega_1 + m\omega_3).
\]

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School of mathematics and statistics, Lanzhou University, Lanzhou 730000, P. R. China.

E-mail address: lijr07@gmail.com, lijr@lzu.edu.cn