Renormalization Group in $2 + \epsilon$ Dimensions and $\epsilon \rightarrow 2$:
A simple model analysis

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**ABSTRACT**

Using a simple solvable model, i.e., Higgs–Yukawa system with an infinite number of flavors, we explicitly demonstrate how a dimensional continuation of the $\beta$ function in two dimensional MS scheme **fails** to reproduce the correct behavior of the $\beta$ function in four dimensions. The mapping between coupling constants in two dimensional MS scheme and a conventional scheme in the cutoff regularization, in which the dimensional continuation of the $\beta$ function is smooth, becomes singular when the dimension of spacetime approaches to four. The existence of a non-trivial fixed point in $2 + \epsilon$ dimensions continued to four dimensions $\epsilon \rightarrow 2$ in the two dimensional MS scheme is spurious and the asymptotic safety cannot be imposed to this model in four dimensions.
Much interest on quantum gravity in $2+\epsilon$ dimensions has recently been revived [1,2]. This dimensional continuation approach to four dimensional quantum gravity was originally proposed by Weinberg [3], in a connection with a possible way to give a predictive power on non-renormalizable theories, called “asymptotic safety.”

Einstein gravity in four dimensions is power counting non-renormalizable and thus one needs an infinite number of counter terms (and coupling constants) to remove the ultraviolet (UV) divergences. By the asymptotic safety, one requires all the renormalized coupling constants flow into (or remain at) a certain fixed point in the UV limit by the renormalization group (RG). Imposing this condition, the Landau singularity or the UV renormalon is avoided. The condition moreover puts the renormalized parameters on a finite dimensional surface, called UV critical surface [3], and gives a predictive power to the theory; all the infinite coupling constants are parameterized by a finite number of coupling constants on the surface.

The possibility whether the asymptotic safety can be imposed or not hence depends on the existence of the fixed point of RG. Since the mass dimension of the gravitational constant is negative for $D > 2$, to have a non-trivial theory (namely $G_R \neq 0$), we have to find a non-trivial fixed point of RG.

It is of course in practice impossible to determine the full structure of the RG flow in the infinite dimensional coupling constant space. The idea of Weinberg is, however, that if one uses MS scheme in the dimensional regularization, the task to find a non-trivial fixed point is drastically simplified. Namely if one could find some non-trivial zero of the $\beta$ function in this scheme by setting all the non-renormalizable (as well as super renormalizable) interactions zero, then it is a true fixed point in the infinite dimensional coupling constant space. This is a peculiarity of the dimensional regularization [3].

From this viewpoint, it is natural to take the two dimensional MS scheme (MS2), because Einstein gravity is power counting renormalizable in $D = 2$ and we are interested in a non-trivial theory $G_R \neq 0$. The result can then be continued to $2 + \epsilon$ dimensions as long as $\epsilon$ is an irrational number, because the dimensional
regularization puts the Feynman integral in a form that is singular only at rational dimensions; MS2 gives a renormalization in $2 + \epsilon$ dimensions.

Now according to the actual one loop calculation of the $\beta$ function of the gravitational constant [4] in this scheme, there exists a non-trivial zero. Therefore at least for $\epsilon$ small enough (in the spirit of the $\epsilon$ expansion) the asymptotic safety may be imposed to Einstein gravity.

Although it has now been realized [5,6,7,8] that the original program of Weinberg as it stands does not work due to a peculiarity of the Einstein action (i.e., the kinematical pole), this fact may even force us to modify the Einstein action [6,7,8].

Besides this problem of the Einstein action, really to reach four dimensions in this approach, it is a crucial if certain properties obtained in $2 + \epsilon$ dimensions can smoothly be continued to $\epsilon \to 2$. Among of them, the existence of a non-trivial fixed point and the eigenvalue of RG flow at the fixed point (the critical exponents) should smoothly be continued. This is crucial to conclude the possibility to impose the asymptotic safety in four dimensions (in which we are interested).

The aim of this article is to address the above point: Suppose MS2 shows a fixed point in $2 + \epsilon$ dimensions. They can be continued to $\epsilon \to 2$?

To study this point without any ad hoc approximation, we will consider a simple exactly calculable model, i.e., Higgs–Yukawa system with an infinite number of flavors:

$$\mathcal{L} = \frac{N}{2} Z \partial_\mu \sigma \partial^\mu \sigma - \frac{N}{4G} \sigma^2 - \frac{N}{4!} g \sigma^4 + \bar{\psi} i \gamma^\mu \partial_\mu \psi - \bar{\psi} \sigma \psi,$$  \hspace{1cm} (1)

where $\psi$ is an $N$ component column vector, $\psi = (\psi_1, \cdots, \psi_N)^T$ and $Z$, $G$ and $g$ are bare coupling constants. We shall consider $N \to \infty$ limit of this system (the leading order of $1/N$ expansion [10,11]) in $D$ dimensional spacetime.

* Of course this model, or its analogue, i.e., the four fermi interaction in $D$ dimensions, have been studied countless times in the literature. For relatively recent articles, see [9].
In what follows we shall find that the answer to the above question is negative: The fixed point in $2+\epsilon$ dimensions continued to $\epsilon \to 2$, obtained in MS2 is spurious. The asymptotic safety cannot be imposed on this model.\footnote{One should be curious on this statement because the lagrangian (1) is renormalizable for $D \leq 4$ and it is not really necessary to rely on the asymptotic safety. Our point is, however, to demonstrate how the dimensional continuation of MS2 to four dimensions fails using this renormalizable model.}

Now in (1) radiative corrections due to the scalar field $\sigma$ is trivial because $N$ is just the inverse of the Plank constant and the classical contribution dominates for $N \to \infty$ (tree level exact). On the other hand for the fermion field $\psi$, the one loop correction is exact. So we can compute any Green functions in a closed form (even implicit) in this large $N$ limit \cite{10,11}. Note that in (1) we have normalized the Yukawa coupling unity and instead introduced the wave function normalization factor $Z$. Throughout this article, we will use a convention $\text{tr} 1 = 2^{D/2}$.

The central physics of the model (1) is of course the dynamical chiral symmetry ($\psi \to \gamma_5 \psi$, $\sigma \to -\sigma$ in even dimensions) breaking \cite{11,12}. To detect this, one may shift the scalar field $\sigma \to \sigma + m$ and impose the vanishing of the tadpole diagram \cite{12}, which is calculated in the dimensional regularization as:

$$\Gamma^{(1)}_{\sigma}/N = -\frac{m^2}{2G} - \frac{g m^3}{3!} + \frac{\Gamma(1 - D/2)}{(2\pi)^{D/2}} m^{D-1} = 0.$$

One may discuss the dynamical symmetry breaking by taking $m \neq 0$ solution of (2). In this article, however, we will only consider the symmetric solution $m = 0$ for a simplicity of presentation, because we are interested in the UV behavior of the system. Actually, by taking an appropriate renormalization condition it can be checked that the same counter terms in the symmetric phase can remove the UV divergences in the breaking phase.

In the dimensional regularization, the two point 1PI function of $\sigma$ is given by

$$\Gamma^{(2)}_{\sigma}(p)/N = Z p^2 - \frac{1}{2G} + \frac{\Gamma(2D/2)\Gamma(1 - D/2)}{2(2\pi)^{D/2}\Gamma(D - 1)} (-p^2)^{D/2-1},$$

(3)
and the four point function is given by
\[
\Gamma^{(4)}(p_1, p_2, p_3)/N = -g - \frac{D(D + 2)\Gamma(2 - D/2)}{4(2\pi)^{D/2}} \int_0^1 dz z^2 \int_0^1 dy y \int_0^1 dx [f(p_1, p_2, p_3; x, y, z)]^{D/2 - 2} + \text{(five permutations on } p_1, p_2, p_3) + \text{(no pole part)},
\]
where
\[
f(p_1, p_2, p_3; x, y, z) = -(1 - xyz)p_1^2 - (1 - yz)p_2^2 - (1 - z)p_3^2 - 2(1 - yz)p_1 \cdot p_2 - 2(1 - z)p_1 \cdot p_3 - 2(1 - z)p_2 \cdot p_3.
\]

Higher point 1PI functions \(\Gamma^{(n)}(p_1, \cdots, p_{n-1})\) with \(n \geq 6\) have no pole for \(D \leq 4\) and so are irrelevant for the following discussion.

In the large \(N\) limit, any Green function among \(\psi\) is given by tree diagrams made from vertices \(\Gamma^{(n)}(p_1, \cdots, p_{n-1})\) and the propagator of \(\sigma\) field (and putting \(\bar{\psi}\psi\) to the external lines). All the information of the UV divergence is thus contained in the above two functions.

1. **MS scheme in \(D = 2\)**

Let us start our discussion on RG from the two dimensional MS scheme (MS2). Setting \(D = 2 + \epsilon\), only pole terms \(1/\epsilon^n\) are subtracted in this scheme [3]. We define the renormalization constants (note the canonical dimension of \(\sigma\) is 1 irrespective of the spacetime dimension \(D\)) as
\[
Z = \mu^{-2+\epsilon}Z \mu^{Z \mu}R, \quad G = \mu^{-\epsilon}Z \mu^{G \mu}R, \quad g = \mu^{-2+\epsilon}Z \mu^{g \mu}R,
\]
from (3) and (4), we see \(Z_Z = Z_g = 1\) and
\[
Z_G = \frac{1}{1 - \frac{2}{\pi}G \mu} = \sum_{n=0}^{\infty} \left(\frac{2}{\pi}\right)^n \frac{1}{\epsilon^n} G^n \mu R.
\]
We stress that a pole singularity at \(D = 4\) in (3) and (4) is not subtracted in this scheme.
From (7), the $\beta$ functions are given by

$$
\beta_Z \equiv \mu \frac{\partial Z_R}{\partial \mu} = (2 - \epsilon)Z_R, \quad \beta_G \equiv \mu \frac{\partial G_R}{\partial \mu} = \epsilon G_R - \frac{2}{\pi} G_R^2,
$$

$$
\beta_g \equiv \mu \frac{\partial g_R}{\partial \mu} = (2 - \epsilon)g_R.
$$

(8)

The $\beta$ function of $Z_R$ and $g_R$ is of course trivial in this scheme. On the other hand the $\beta$ function of $G_R$ has non-trivial zero \([10,11]\) at $G_R^* = \pi \epsilon / 2$ and so for $\epsilon \neq 0$ we have two fixed points in the full coupling constant space:

$$
Z_R = 0, \quad G_R = 0, \quad g_R = 0, \quad (A),
$$

$$
Z_R = 0, \quad G_R = G_R^* = \frac{\pi \epsilon}{2}, \quad g_R = 0, \quad (B).
$$

(9)

For $\epsilon \leq 2$ the point (A) is infrared (IR) stable. For $\epsilon = 2$ the direction of $Z_R$ and $g_R$ becomes scale invariant and an arbitrary value of $Z_R$ and $g_R$ (with $G_R = 0$ or $G_R^*$) gives the fixed point.

In general a fixed point is characterized by the eigenvalue of a matrix $\partial \beta_i / \partial g_j$ at the fixed point (critical exponents) \([13]\):

$$
2 - \epsilon = 4 - D, \quad \epsilon = D - 2, \quad 2 - \epsilon = 4 - D, \quad \text{for (A)},
$$

$$
2 - \epsilon = 4 - D, \quad - \epsilon = 2 - D, \quad 2 - \epsilon = 4 - D, \quad \text{for (B)},
$$

(10)

and should be the same under the change of the renormalization scheme \([13]\). Therefore they can be used to identify the corresponding fixed points between different schemes.

Now according to (9) and (10) it seems that we may impose the asymptotic safety by setting $Z_R = 0$ and $g_R = 0$. Note that (9) is not $\epsilon$ expansion but is exact \([10]\). After imposing the asymptotic safety, the theory becomes the Gross–Neveu model \([11]\) in $2 + \epsilon$ dimensions. But there is no obstruction in (9) and (10) to take a limit $\epsilon \to 2$ and to go to the four dimensions. Then $G_R \to \pi$ in the UV limit and we have a non-trivial theory in four dimensions. This may be taken as a possible definition of a four dimensional Gross–Neveu model.
As will be shown shortly (or as is easily expected) this is not the case. The fixed point \((B)\) for \(\epsilon \rightarrow 2\) is an artifact of the present scheme. We will show this fact by finding a mapping between the renormalized couplings in MS2 and a conventional scheme in the cutoff regularization. However before going into this, let us summarize what is actually happening in four dimensions.

2. MS scheme in \(D = 4\)

To see the situation in four dimensions, it is most convenient to use the four dimensional MS scheme (MS4). Setting \(D = 4 - 2\epsilon'\) and

\[
Z = \mu^{-2\epsilon'} Z_Z Z_R, \quad G = \mu^{-2+2\epsilon'} Z_G Z_R, \quad g = \mu^{-2\epsilon'} Z_g g_R,
\]

we see the following choice removes the pole \(1/\epsilon'\) in (3) and (4):

\[
Z_Z = 1 - \frac{1}{8\pi^2} \frac{1}{\epsilon' Z_R}, \quad Z_G = 1, \quad Z_g = 1 - \frac{3}{2\pi^2} \frac{1}{\epsilon' g_R}.
\]

(12)

Note that \(G\) receives no radiative correction. From this we have the \(\beta\) functions in this scheme

\[
\beta_Z = 2\epsilon' Z_R - \frac{1}{4\pi^2}, \quad \beta_G = (2 - 2\epsilon') G_R, \quad \beta_g = 2\epsilon' g_R - \frac{3}{\pi^2}.
\]

(13)

The \(\beta\) functions for \(Z_R\) and \(g_R\) seem peculiar but are consistent. Remind that we have taken the Yukawa coupling in (1) unity. If we put the Yukawa coupling \(\lambda\) instead, we would have \(\lambda^2\) and \(\lambda^4\) in the second terms; or if one prefers a standard form of the \(\beta\) function, the inverse coupling may be considered

\[
\frac{\partial Z_R^{-1}}{\partial \mu} = -2\epsilon' Z_R^{-1} + \frac{1}{4\pi^2} Z_R^{-2}.
\]

(14)

Clearly for \(\epsilon' = 0\), \(Z_R \rightarrow \infty\) as \(\mu \rightarrow \infty\) (note \(Z_R^{-1} \rightarrow +\infty\) is connected to \(Z_R^{-1} \rightarrow -\infty\)).
From the form of the $\beta$ function (13) we see there exists a unique fixed point,
\[
Z_R = \frac{1}{8\pi^2} \frac{1}{\epsilon'}, \quad G_R = 0, \quad g_R = \frac{3}{2\pi^2} \frac{1}{\epsilon'},
\]
which is IR stable for $\epsilon' > 0$. The corresponding critical exponents read,
\[
2\epsilon' = 4 - D, \quad 2 - 2\epsilon' = D - 2, \quad 2\epsilon' = 4 - D.
\]
Comparing the critical exponents (16) and (10), we realize that the fixed point in (15) corresponds to (A) in (9) in MS2. Where is the another fixed point (B)?

By comparing the $\beta$ functions in two schemes (8) and (13), (identifying both of the renormalization scales $\mu$), we find a mapping between $G_R^{\text{MS2}}$ and $G_R^{\text{MS4}}$ in $2 < D < 4$, (for which both of schemes give a renormalization):
\[
G_R^{\text{MS4}} = \frac{G_R^{\text{MS2}}}{1 - \frac{2}{\pi\epsilon} G_R^{\text{MS2}}}
\]
We realize the fixed point (B) in (9), $G_R^{\text{MS2}} = \pi\epsilon/2$ is mapped to the infinity of $G_R^{\text{MS4}}$ for arbitrary dimension $2 < D < 4$. The mapping from MS2 to MS4 is therefore singular at the fixed point (B) in MS2.

At $D = 4$ there is no fixed point from (13) and it is definitely impossible to impose the asymptotic safety. Moreover in UV limit $Z_R$ and $g_R$ diverge for any choice of the bare parameter, so the theory is pathological in four dimensions.*

In conclusion, the non-trivial fixed point in four dimensions, that is detected by a dimensional continuation of MS2 is spurious. To see this point much clearer, we shall consider one more another scheme in the next section, in which the $\beta$ function is continuous in the whole range of the dimension $2 \leq D \leq 4$. In some sense it interpolates the two dimensional MS scheme and the four dimensional MS scheme.

* This depends on what is called the “coupling constant.” By rescaling $\sigma \to \sigma/\sqrt{Z}$, the Yukawa coupling becomes $1/\sqrt{Z}$. Then RG tells that the theory becomes weakly interacting massless boson and fermion, which is not pathological at all. Our definition of the coupling constant is motivated by the Gross–Neveu model, in which the fermion has a hard contact interaction. See also a discussion in Conclusion.
3. CUTOFF REGULARIZATION

We apply the Euclidean momentum cutoff regularization in this section \((k^4 \equiv k^0 / i)\), putting the momentum cutoff \(\Lambda\) after a symmetrization of a denominator of the Feynman integral. The two point 1PI function in \(D\) dimensions in this regularization reads

\[
\Gamma^{(2)}(p)/N = Zp^2 - \frac{1}{2G} - \frac{1}{(2\pi)^{D/2} \Gamma(D/2)} \int_0^\Lambda dx \int_0 ds s^{D/2-1} \frac{-p^2 x(1-x) - s}{[-p^2 x(1-x) + s]^2}.
\]

The four point function at the symmetric point reads

\[
\Gamma^{(4)}(p_1, p_2, p_3)/N \bigg|_{p_i \cdot p_j = -\mu^2 \delta_{ij} + s} = -g - \frac{36}{(2\pi)^{D/2} \Gamma(D/2)} \int_0^1 dx \int_0^1 dy \int_0^1 dz s^{D/2+1} \left[s + \mu^2 g(x, y, z)\right]^{-4} + (\text{finite part}),
\]

where

\[
g(x, y, z) = z(1 - z + \frac{1}{3} y - \frac{1}{3} x y + \frac{2}{3} x y z + \frac{2}{3} x y z - y^2 z - x^2 y^2 z).
\]

Under this regularization, we take the following Gell-Mann–Low type renormalization condition:

\[
\Gamma^{(2)}(p)/N \bigg|_{p^2 = -\mu^2} = -\mu^{D-2} Z_R - \frac{\mu^{D-2}}{2G_R},
\]

\[
(\partial / \partial p^2) \Gamma^{(2)}(p)/N \bigg|_{p^2 = -\mu^2} = \mu^{D-4} Z_R;
\]

\[
\Gamma^{(4)}(p_1, p_2, p_3)/N \bigg|_{p_i \cdot p_j = -\mu^2 \delta_{ij} + s} = -\mu^{D-4} g_R.
\]

Now the functions in (19) and (18) have different divergent behavior depending on the spacetime dimension. Therefore \(2 < D < 4\) case and \(D = 4\) case should separately be treated.
For $2 < D < 4$, we first take a derivative $\partial/\partial p^2$ of the both sides of (18) and compare it with the second of (21). We see (for $\Lambda \to \infty$)

$$Z = \mu^{D-4} \left[ Z_R - \frac{(D - 1) \Gamma(D/2) \Gamma(2 - D/2)}{(2\pi)^{D/2} \Gamma(D)} \right].$$

(22)

Of course there is no divergence here and this gives the $\beta$ function of $Z_R$ in $2 < D < 4$,

$$\beta_Z = (4 - D)Z_R - \frac{2(D - 1) \Gamma(D/2) \Gamma(3 - D/2)}{(2\pi)^{D/2} \Gamma(D)}.$$

(23)

Similarly a comparison of (19) and (21) gives

$$g = \mu^{D-4} \left[ g_R - \frac{6 \Gamma(2 + D/2) \Gamma(2 - D/2)}{(2\pi)^{D/2} \Gamma(D/2)} \int_0^1 dx \int_0^1 dy \int_0^1 dz \, z^2 h(x, y, z)^{D/2-2}$$

$$+ O((4 - D)^0) \right],$$

(24)

since the finite part has no singularity at $D = 4$. The $\beta$ function of $g_R$ is therefore given by

$$\beta_g = (4 - D)g_R - \frac{3}{\pi^2} + O((4 - D)).$$

(25)

To obtain the $\beta$ function of $G_R$, we directly take a derivative $\mu \partial/\partial \mu$ of the both sides of the first of (21) and (18). After noting (22) and (23), we have for $2 < D < 4$,*

$$\beta_G = (D - 2)G_R - \frac{4(D - 1) \Gamma(D/2) \Gamma(3 - D/2)}{(2\pi)^{D/2} \Gamma(D)} G_R^2.$$

(26)

* This is not the same as RG function in Gross–Neveu model in $2 < D < 4$ dimensions [10,11,9] because the introduction of $Z$ modifies the divergent structure of $G$. If we start $Z = g = 0$ in (1) instead (the Gross–Neveu model) we will have $1/(D - 4)$ pole in the second term.
Repeating all the above steps in $D = 4$, we have

$$Z = Z_R - \frac{1}{8\pi^2} \ln \Lambda^2/\mu^2, \quad g = g_R - \frac{3}{2\pi^2} \ln \Lambda^2/\mu^2 + \text{const.},$$

(27)

and thus

$$\beta_Z = -\frac{1}{4\pi^2}, \quad \beta_g = -\frac{3}{\pi^2}. \quad (28)$$

On the other hand,

$$\beta_G = 2G_R - \frac{1}{2\pi^2}G_R^2. \quad (29)$$

We note that although the divergence structure is different for $2 < D < 4$ and $D = 4$ (for example $Z$ is finite in $2 < D < 4$ but is logarithmically divergent for $D = 4$), the $\beta$ functions themselves are continuous in this scheme. Namely the $\beta$ functions in (23), (25) and (26) have a correct $D \to 4$ limit, (28) and (29) respectively. This is the advantage of this scheme and the expressions of $\beta$ functions (23), (25) and (26) can continuously be used throughout $2 \leq D \leq 4$ (one can check they also hold for $D = 2$).

In $D = 4$ (23) and (25) precisely coincide with the result of MS4, the first and the third of (13). On the other hand, the form of $\beta_G$ in $D = 4$ in both scheme are completely different; (26) and the second of (13). The non-trivial zero of (26) is mapped to infinity in MS4. The mapping between the present scheme and MS4 is thus somewhat singular even in $D = 4$. In any case, whatever one prefers MS4 or the conventional scheme in the cutoff regularization, the very fact that there is no fixed point in the whole coupling constant space in $D = 4$ does not change. In this sense both of them are consistent.

Let us now consider the relation between MS2 and the present scheme. According to (23), (25) and (26) in $2 < D < 4$ there exist two fixed points. It is easy to see that the critical exponents at those fixed points are precisely given by the table (10) and thus they are nothing but the fixed points (A) and (B) observed in MS2. When $D \to 4$, due to the constant term in (23) and (25), the fixed points
are lost (or go to infinity), while they survive in (8). Note that there is no general guarantee that both of them give a consistent answer for \( D \to 4 \), because MS2 is not a renormalization in \( D = 4 \) in the sense that Green functions are not made finite.

The relation between both of schemes should therefore be singular at \( D = 4 \). Actually, by identifying \( \mu \) in the both schemes, it is easy to see that

\[
Z_R = Z_R^{\text{MS2}} + \frac{2(D - 1)\Gamma(D/2)^2 \Gamma(3 - D/2)}{(2\pi)^{D/2} \Gamma(D)(4 - D)} ,
\]

\[
G_R = \frac{G_R^{\text{MS2}}}{1 - \frac{1}{(D - 2)} \left[ \frac{2}{\pi} - \frac{4(D - 1)\Gamma(D/2)^2 \Gamma(3 - D/2)}{(2\pi)^{D/2} \Gamma(D)} \right]} G_R^{\text{MS2}} ,
\]

\[
g_R = g_R^{\text{MS2}} + \frac{3}{\pi^2(4 - D)} + O((4 - D)^0) .
\]

This is the main result of this article: * Although the dimensional continuation of two dimensional MS scheme gives a non-trivial fixed point at \( D = 4 \), it is an artifact of the dimensional continuation. In terms of the conventional renormalization scheme in \( D = 4 \) the fixed point corresponds to the infinity and has no physical relevance.

4. Conclusion

As has been shown above, the dimensional continuation of the result of MS2 to \( D \to 4 \) cannot be used in this model. Our model (1) seems of course almost trivial. However for \( N \to \infty \) we can eliminate \( \sigma \) using the equation of motion †

\[
\sigma = -\frac{2G}{N} \bar{\psi} \psi + \frac{4G^2 Z}{N} \Box (\bar{\psi} \psi) - \frac{8G^3 Z^2}{N} \Box^2 (\bar{\psi} \psi) + \frac{8G^4 g}{3} (\bar{\psi} \psi)^3 + \cdots ,
\]

\[\text{Note the second expression is regular at } D = 2\]

\[\text{† This may be used to compute the anomalous dimension of various composite operators, } (\bar{\psi} \psi), \Box (\bar{\psi} \psi), (\bar{\psi} \psi)^3 \text{ etc., by combining with the fact that } \sigma \text{ is not renormalized in the leading order of } 1/N \text{ expansion.}\]
and obtain
\[
\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi + \frac{G}{N} (\bar{\psi} \psi)^2 - \frac{2GZ}{N} \bar{\psi} \Box (\bar{\psi} \psi) + \frac{4G^3 Z}{N} \bar{\psi} \Box^2 (\bar{\psi} \psi) + \frac{2G^4 g}{3N^3} (\bar{\psi} \psi)^4
\]
\[
- \frac{16G^5 gZ}{3N^3} (\bar{\psi} \psi)^3 \Box (\bar{\psi} \psi) + \frac{16G^6 gZ^2}{N^3} \{ (\bar{\psi} \psi)^3 \Box^2 (\bar{\psi} \psi) + 3[ \bar{\psi} \psi (\bar{\psi} \psi)]^2 \}
\]
\[
+ O((\bar{\psi} \psi)^6, \Box^3).
\]

(32)

This lagrangian seems highly non-renormalizable, but what the lagrangian (1) tells is that the system is renormalizable even in \( D = 4 \). An Infinite type of UV divergences which appear in the calculation of (32) can be removed by the renormalization of only three parameters, \( Z, G \) and \( g \). In this sense the seemingly non-renormalizable model (32) has a predictive power. (This is of course a trivial statement in view of (1)). Note that \( Z, G \) and \( g \) appear in the coefficients with positive powers in this expansion. Thus what we considered in this article can be stated: "Is it possible to impose the asymptotic safety on (32) in \( D = 4 \)?" According to the two dimensional MS scheme, the asymptotic safety puts \( Z = g = 0 \) and the model reduces to four dimensional Gross–Neveu model. We showed this is not the case.

On the other hand, at \( D = 3 \), i.e., another physical dimension in \( 2 \leq D \leq 4 \), there is no contradiction in the above analysis. The asymptotic safety can therefore be imposed in \( D = 3 \) and the system reduces to the Gross–Neveu model. This should be so to be consistent with the fact that the Gross–Neveu model in \( D = 3 \) is renormalizable [10,13,14]. This better UV behavior in \( D = 3 \) is clearly related to the fact that there does not appear new type of UV divergence in \( D = 3 \) than \( D = 2 \).

Although we considered only a simple (even without gauge symmetry) model in this article, the fact we observed itself seems independent on the detail of the model: Though some of fixed points in the whole coupling constant space appear in a simple form in MS scheme, it seems in general quite dangerous to continue the result until the spacetime dimension in which new type of UV divergence (or new pole in the dimensional regularization) appears.
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