Immersions of spheres and algebraically constructible functions

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Abstract

Let $\Lambda$ be an algebraic set and let $g : \mathbb{R}^{n+1} \times \Lambda \rightarrow \mathbb{R}^2$ ($n$ is even) be a polynomial mapping such that for each $\lambda \in \Lambda$ there is $r(\lambda) > 0$ such that the mapping $g_\lambda = g(\cdot, \lambda)$ restricted to the sphere $S^n(r)$ is an immersion for every $0 < r < r(\lambda)$, so that the intersection number $I(g_\lambda|S^n(r))$ is defined. Then $\Lambda \ni \lambda \mapsto I(g_\lambda|S^n(r)) \in \mathbb{Z}$ is an algebraically constructible function.

Keywords: immersions of spheres, algebraically constructible functions, real algebraic sets.

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1 Introduction

McCwory and Parusiński [6] have introduced algebraically constructible functions in order to study the topology of real algebraic sets.

Let $\Lambda$ be a real algebraic set. An integer valued function $\phi : \Lambda \rightarrow \mathbb{Z}$ is algebraically constructible if there exist an algebraic set $W$ and a proper regular map $p : W \rightarrow \Lambda$ such that $\phi(\lambda)$ equals the Euler characteristic $\chi(p^{-1}(\lambda))$.

If that is the case then $\phi$ is semialgebraically constructible, that is there exists a semialgebraic stratification $S$ of $\Lambda$ such that $\phi$ is constant on strata of $S$. 

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S. If Λ is irreducible then φ has to be generically constant modulo 2, see for instance \([1, \text{ Proposition 2.3.2}]\), and there exist a real polynomial \(g: \Lambda \rightarrow \mathbb{R}\) and a constant \(\mu\) such that generically on \(\Lambda\): \(\phi \equiv \mu + \text{sgn} \; g \mod 4\) \([3]\).

In fact, algebraically constructible functions are precisely those constructible functions which are sums of signs of polynomials \([9], [10]\).

Let \(f: \mathbb{R}^n \times \Lambda \rightarrow \mathbb{R}^n\) be a polynomial mapping such that \(0 \in \mathbb{R}^n\) is isolated in \(f(\cdot, \lambda)^{-1}(0)\) for all \(\lambda \in \Lambda\), so that the local topological degree \(\text{deg}_0 f(\cdot, \lambda)\) of \(f(\cdot, \lambda)\) at the origin is well defined. Then \(\Lambda \ni \lambda \mapsto \text{deg}_0 f(\cdot, \lambda) \in \mathbb{Z}\) is algebraically constructible \([9, \text{ Theorem 3.4}]\).

For a recent account of the theory we refer the reader to \([2], [7], [8]\).

Whitney \([13]\) has introduced an intersection number \(I(g)\) for an immersion \(g: M^n \rightarrow \mathbb{R}^{2n}\). If \(n\) is even then \(I(g) \in \mathbb{Z}\), if \(n\) is odd then \(I(g) \in \mathbb{Z}^2\). Smale \([11]\) proved, that two immersions \(f, g: S^n \rightarrow \mathbb{R}^{2n}\) are regularly homotopic if and only if \(I(f) = I(g)\).

In this paper we investigate how does the “local” intersection number change when there is an algebraic family of immersions. Let \(g = (g_1, \ldots, g_{2n}): \mathbb{R}^{n+1} \times \Lambda \rightarrow \mathbb{R}^{2n}\) be a polynomial mapping. Assume that for each \(\lambda \in \Lambda\) there exists \(r(\lambda) > 0\) such that the mapping \(g_{\lambda} = g(\cdot, \lambda)\) restricted to the sphere \(S^n(r)\) is an immersion for \(0 < r < r(\lambda)\). If that is the case then the intersection number \(I(g_{\lambda}|S^n(r))\) is the same for all \(0 < r < r(\lambda)\). We shall prove (Theorem \(6.1\)) that the function

\[
\Lambda \ni \lambda \mapsto I(g_{\lambda}|S^n(r)) \in \mathbb{Z}
\]

is algebraically constructible.

### 2 Preliminaries

Suppose that \(L\) is a \(p\)-dimensional oriented manifold, \(H: L \rightarrow \mathbb{R}^p\) is a smooth mapping, and \(U\) is an open subset of \(L\) such that \(H^{-1}(0) \cap U\) is compact.

There exists \((N, \partial N)\) — a compact \(p\)-dimensional oriented manifold with boundary such that \(N \subset U\) and \(H^{-1}(0) \cap U \subset N \setminus \partial N\).

By the topological degree of the mapping

\[
(U, U \setminus H^{-1}(0)) \ni x \mapsto H(x) \in (\mathbb{R}^p, \mathbb{R}^p \setminus \{0\})
\]

we mean the topological degree of

\[
(N, \partial N) \ni x \mapsto H(x) \in (\mathbb{R}^p, \mathbb{R}^p \setminus \{0\})
\]

which equals the degree of the mapping

\[
\partial N \ni x \mapsto \frac{H(x)}{\|H(x)\|} \in S^{p-1}.
\]

Of course, the degree does not depend on the choice of \(N\).
Let \( F_1, \ldots, F_k; G_1, \ldots, G_{n-k} : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) be analytic functions defined in a neighbourhood of the origin.

Denote
\[
F = (F_1, \ldots, F_k) : \mathbb{R}^n, 0 \to \mathbb{R}^k, 0
\]
\[
G = (G_1, \ldots, G_{n-k}) : \mathbb{R}^n, 0 \to \mathbb{R}^{n-k}, 0
\]
\[
S^{n-1}(r) = \{ x \in \mathbb{R}^n \mid \|x\| = r \}
\]
\[
S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}
\]
\[
B^n(r) = \{ x \in \mathbb{R}^n \mid \|x\| \leq r \}
\]

Suppose that \( F^{-1}(0) \) has an isolated singularity at the origin, i.e., for \( \|x\| \) small enough, if \( F(x) = 0 \) and \( \text{rank}[DF(x)] < k \) then \( x = 0 \). If \( r > 0 \) is small enough then \( S^{n-1}(r) \) cuts \( F^{-1}(0) \) transversally, so \( M(r) = S^{n-1}(r) \cap F^{-1}(0) \) is either void or a compact \((n-k-1)\)-dimensional manifold.

We shall say that vectors \( v_1, \ldots, v_{n-k-1} \) in the tangent space \( T_x M(r) \) are \textit{well oriented} if \( \nabla F_1(x), \ldots, \nabla F_k(x), v_1, \ldots, v_{n-k-1} \) are well oriented in \( \mathbb{R}^n \). This way \( M(r) \) is oriented.

Let \( y \in \mathbb{R}^k \) be a regular value of \( F \). Then \( F^{-1}(y) \) is either void or an \((n-k)\)-dimensional manifold. We shall say that \( w_1, \ldots, w_{n-k} \in T_x F^{-1}(y) \) are \textit{well oriented} if \( \nabla F_1(x), \ldots, \nabla F_k(x), w_1, \ldots, w_{n-k} \) are well oriented in \( \mathbb{R}^n \). This way \( F^{-1}(y) \) is also oriented.

Fix small \( r > 0 \). If \( y \) lies sufficiently close to the origin then \( S^{n-1}(r) \) cuts \( F^{-1}(y) \) transversally, so \( \overline{M(r)} = S^{n-1}(r) \cap F^{-1}(y) \) is either void or a compact \((n-k-1)\)-dimensional manifold. Moreover, \( B^n(r) \cap F^{-1}(y) \) is a compact oriented manifold with boundary \( \partial (B^n(r) \cap F^{-1}(y)) = \overline{M(r)} \). The orientation of the boundary induced from \( B^n(r) \cap F^{-1}(y) \) may be described the same way as the orientation of \( M(r) \). Manifolds \( M(r) \) and \( \overline{M(r)} \) are isotopic.

Suppose that \( F^{-1}(0) \cap G^{-1}(0) = \{0\} \). Then \( M(r) \cap G^{-1}(0) = \emptyset \) and \( \overline{M(r)} \cap G^{-1}(0) = \emptyset \).

If \( \|y\| \) is small enough then mappings
\[
M(x) \ni x \mapsto \frac{G(x)}{\|G(x)\|} \in S^{n-k-1}
\]
\[
\overline{M}(x) \ni x \mapsto \frac{G(x)}{\|G(x)\|} \in S^{n-k-1}
\]

have the same topological degree. Denote it by \( \deg(M(r)) \). It equals the degree \( \rho \) of \( \overline{G} \), where \( \overline{G} \) is the restricted mapping
\[
(F^{-1}(y) \cap B^n(r), \overline{M(r)}) \ni x \mapsto G(x) \in (\mathbb{R}^{n-k}, \mathbb{R}^{n-k} \setminus \{0\})
\]

Choose a regular value \( z \in \mathbb{R}^{n-k} \) near the origin. Then
\[
\rho = \sum \text{sgn det}[D \overline{G}(x)] \quad (\text{where } x \in \overline{G}^{-1}(z)).
\]
Using the concept of the Gram determinant the reader may check that
\[ \rho = \sum \sgn \det[D(F,G)(x)], \]
where \( x \in F^{-1}(y) \cap B^n(r) \cap G^{-1}(z) = (F,G)^{-1}(y,z) \cap B^n(r) \). If \( 0 < \|y\| \ll r \ll 1 \) then \( \rho \) equals the local topological degree \( \deg_0(F,G) \) at the origin, i.e. the topological degree of the mapping
\[ S^{n-1}(r) \ni x \mapsto \frac{(F(x),G(x))}{\|(F(x),G(x))\|} \in S^{n-1}. \]

Set
\[ N(r) = \{ x \in S^{n-1}(r) \mid F_1(x) = \ldots = F_{k-1}(x) = 0, F_k(x) \geq 0 \}. \]
Suppose that \( (F_1,\ldots,F_{k-1})^{-1}(0) \) has an isolated singularity at the origin. Since \( F^{-1}(0) \) has an isolated singularity, \( N(r) \) is either void or an \((n-k)\)-dimensional compact oriented manifold with boundary, and \( \partial N(r) = S^{n-1}(r) \cap F^{-1}(0) = M(r) \). There is the restricted function
\[ (N(r),\partial N(r)) \ni x \mapsto G(x) \in (\mathbb{R}^{n-k},\mathbb{R}^{n-k} \setminus \{0\}), \]
which will be denoted by \( \overline{G}_r \).

The topological degree of \( \overline{G}_r \) equals the topological degree of
\[ \partial N(r) \ni x \mapsto \frac{G(x)}{\|G(x)\|} \in S^{n-k-1}, \]
which is the same as \( \deg(M(r)) = \rho = \deg_0(F,G) \). We have proved:

**Proposition 2.1** Let \( F_1,\ldots,F_k;G_1,\ldots,G_{n-k} : \mathbb{R}^n,0 \rightarrow \mathbb{R},0 \) be analytic functions defined in a neighbourhood of the origin. Suppose that \((F_1,\ldots,F_{k-1})^{-1}(0)\), as well as \((F_1,\ldots,F_k)^{-1}(0)\), has an isolated singularity at the origin, and \( \{0\} \) is isolated in \((F_1,\ldots,F_k;G_1,\ldots,G_{n-k})^{-1}(0)\).

If \( r > 0 \) is small enough then the topological degree of the mapping \( \overline{G}_r \), i.e. of
\[ (N(r),\partial N(r)) \ni x \mapsto G(x) \in (\mathbb{R}^{n-k},\mathbb{R}^{n-k} \setminus \{0\}), \]
does not depend on \( r \) and is equal to the local topological degree \( \deg_0(F,G) \).

Let \( \delta : \mathbb{R}^n,0 \rightarrow \mathbb{R},0 \) be an analytic function. In [12] there is proven that if \( \alpha \) is a sufficiently large positive even integer and \( t \neq 0 \) then \( \delta - t\|x\|^\alpha \) has an isolated critical point at the origin. Using the same arguments one may prove:

**Proposition 2.2** Suppose that \((F_1,\ldots,F_{k-1})^{-1}(0)\) has an isolated singularity at the origin.

Then there exists \( \alpha_0 > 0 \) such that for any even integer \( \alpha > \alpha_0 \) and \( t \neq 0 \)
\[ (F_1,\ldots,F_{k-1},\delta - t\|x\|^\alpha)^{-1}(0) \]
has an isolated singularity at the origin.
Suppose that \( \delta \geq 0 \) and
\[
X := \{0\} \cup ((F_1, \ldots, F_{k-1}, G_1, \ldots, G_{n-k})^{-1}(0) \setminus \delta^{-1}(0))
\]
is closed. Then \( \delta^{-1}(0) \cap X = \{0\} \). Since \( X \) is closed semianalytic, there exists a \( \text{Lojasiewicz exponent } \alpha_1 \), such that \( \delta(x) > t \|x\|^{\alpha_1} \) for \( t > 0 \) and \( x \in X \setminus \{0\} \) sufficiently close to the origin.

Set
\[
L(r) = \{ x \in S^{n-1}(r) \mid F_1(x) = \ldots = F_{k-1}(x) = 0 \}.
\]
Then \( U(r) = L(r) \setminus \delta^{-1}(0) \) is an open subset of \( L(r) \), and \( U(r) \cap G^{-1}(0) = S^{n-1}(r) \cap X \) is a compact subset of \( U(r) \).

Let \( \alpha > \max(\alpha_0, \alpha_1) \) be an even positive integer. Set \( F_k = \delta - t \|x\|^\alpha \) for arbitrary \( t > 0 \). Then \( F_k(x) > 0 \) for \( x \in L(r) \cap G^{-1}(0) \setminus \delta^{-1}(0) \), and \( F_k(x) < 0 \) for \( x \in L(r) \cap G^{-1}(0) \cap \delta^{-1}(0) \).

**Proposition 2.3** Suppose that

(a) \( (F_1, \ldots, F_{k-1})^{-1}(0) \) has an isolated singularity at the origin, so that \( L(r) \) is either void or a compact oriented \((n-k)\)-dimensional manifold for small \( r > 0 \),

(b) \( \delta \geq 0 \) and \( U(r) \cap G^{-1}(0) = L(r) \cap G^{-1}(0) \setminus \delta^{-1}(0) \) is a compact subset of \( U(r) \).

Then there is a mapping \( G_r \) given by
\[
(U(r), U(r) \setminus G^{-1}(0)) \ni x \mapsto G(x) \in (\mathbb{R}^{n-k}, \mathbb{R}^{n-k} \setminus \{0\})
\]
such that if \( \alpha > \max(\alpha_0, \alpha_1) \) is an even integer, \( t > 0 \), and \( F_k = \delta - t \|x\|^\alpha \), then
\[
\mathbb{R}^n \ni x \mapsto (F(x), G(x)) \in \mathbb{R}^n
\]
has an isolated zero at the origin, and for each \( r > 0 \) small enough the topological degree of \( G_r \) equals the local topological degree \( \deg_0(F,G) \).

**Proof.** Since \( F_k(x) > 0 \) for \( x \in U(r) \cap G^{-1}(0) = L(r) \cap G^{-1}(0) \setminus \delta^{-1}(0) \), then \( U(r) \cap G^{-1}(0) \subset N(r) \setminus \partial N(r) \subset U(r) \). So the topological degree of \( G_r \) equals the topological degree of \( G_r \), and by Proposition 2.1 equals \( \deg_0(F,G) \).

\( \square \)

3 The intersection number of an immersion

Let \( M \) be an \( n \)-dimensional manifold. A \( C^1 \) map \( g : M \to \mathbb{R}^m \) is called an immersion if for each \( p \in M \) the rank of \( Dg(p) \) equals \( n \).

A homotopy \( h_t : M \to \mathbb{R}^m \) is called a regular homotopy, if at each stage it is an immersion and the induced homotopy of the tangent bundle is continuous.
Theorem 3.1 [11, Theorem B]) Two $C^\infty$ immersions from an $n$–dimensional sphere $S^n$ to $\mathbb{R}^m$ are regularly homotopic when $m \geq 2n + 1$.

As in [13] we say that an immersion $g : M \rightarrow \mathbb{R}^{2n}$ has a regular self–intersection at the point $g(p) = g(q)$ if

$$Dg(p)T_pM + Dg(q)T_qM = \mathbb{R}^{2n}.$$ 

An immersion $g : M \rightarrow \mathbb{R}^{2n}$ is called completely regular if it has only regular self–intersections and no triple points.

Assume that $n$ is even and $M$ is compact and oriented.

Let $g : M \rightarrow \mathbb{R}^{2n}$ be a completely regular immersion and have a regular self–intersection at the point $g(p) = g(q)$.

Let $u_1, \ldots, u_n \in T_pM$, $v_1, \ldots, v_n \in T_qM$ be sets of well–oriented, independent vectors in respective tangent spaces of $M$. Then the vectors $Dg(p)u_1, \ldots, Dg(p)u_n$, $Dg(q)v_1, \ldots, Dg(q)v_n$ form a basis in $\mathbb{R}^{2n}$. As in [13] we will say that the self–intersection at the point $g(p) = g(q)$ is positive or negative according to whether this basis determines the positive or negative orientation of $\mathbb{R}^{2n}$.

The intersection number of a completely regular immersion $g$ is the algebraic number of its self–intersections.

Let for a given (not necessarily completely regular) immersion $g : M \rightarrow \mathbb{R}^{2n}$ define $G : M \times M \rightarrow \mathbb{R}^{2n}$ as $G(x, y) = g(x) - g(y)$. Set

$$\Delta = \{(p, p) \mid p \in M\} \subset M \times M.$$ 

Since $g$ is an immersion, $\Delta$ is isolated in $G^{-1}(0)$, and so $G^{-1}(0) \setminus \Delta$ is a compact subset of $M \times M \setminus \Delta$. Of course, $M \times M \setminus \Delta$ is an open subset of $M \times M$, and

$$(M \times M \setminus \Delta) \setminus G^{-1}(0) = (M \times M \setminus (G^{-1}(0) \cup \Delta)).$$

The topological degree $d(g)$ of the mapping

$$(M \times M \setminus \Delta, M \times M \setminus (G^{-1}(0) \cup \Delta)) \ni (x, y) \mapsto G(x, y) \in (\mathbb{R}^{2n}, \mathbb{R}^{2n} \setminus \{0\})$$

is always an even integer. Let us denote

$$I(g) = \frac{1}{2}d(g).$$

By [5, Theorem 3.1] we have

Theorem 3.2 If $g$ is a completely regular immersion then its intersection number is equal to $I(g)$. 

Assume that either \( n \) is odd or \( M \) is non-orientable.

In this case one can also define the intersection number of a completely regular immersion \( g : M \rightarrow \mathbb{R}^{2n} \) as the number of its self-intersections modulo 2.

By [13, Theorem 2] if \( M \) is closed then the intersection number is invariant under regular homotopies. As in [14], if \( M \) is closed, any immersion \( g : M \rightarrow \mathbb{R}^{2n} \) can be made completely regular by a regular homotopy. So in this case we can define the intersection number also for \( g \) that is not completely regular.

We have a characterisation of regularly homotopic immersions due to Smale [11]:

**Theorem 3.3** [11, Theorem C] Two \( C^\infty \) immersions \( f, g \) from an \( n \)-dimensional sphere \( S^n \) to \( \mathbb{R}^{2n} \) are regularly homotopic if and only if \( I(f) = I(g) \).

### 4 Immersions on small spheres

Let

\[ h = (h_1, \ldots, h_l) : \mathbb{R}^n \rightarrow \mathbb{R}^l \]

\[ g = (g_1, \ldots, g_k) : \mathbb{R}^n \rightarrow \mathbb{R}^k \]

be \( C^1 \) mappings. Put \( M := h^{-1}(0) \). Suppose that each point \( p \in M \) is a regular point of \( h \), i.e. the rank of the derivative matrix \( Dh(p) \):

\[
\begin{bmatrix}
\frac{\partial h_1}{\partial x_1}(p) & \frac{\partial h_1}{\partial x_2}(p) & \cdots & \frac{\partial h_1}{\partial x_n}(p) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial h_l}{\partial x_1}(p) & \frac{\partial h_l}{\partial x_2}(p) & \cdots & \frac{\partial h_l}{\partial x_n}(p)
\end{bmatrix}
\]

equals \( l \) at each point \( p \in M \). If that is the case then \( M \) is a \( C^1 (n - l) \)-manifold, and there is the restricted mapping \( g|M : M \rightarrow \mathbb{R}^k \).

**Proposition 4.1**

\[
\text{rank } \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(p) & \cdots & \frac{\partial g_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial x_1}(p) & \cdots & \frac{\partial g_k}{\partial x_n}(p) \end{bmatrix} = \text{rank } \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(p) & \cdots & \frac{\partial g_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial x_1}(p) & \cdots & \frac{\partial g_k}{\partial x_n}(p) \end{bmatrix} - l
\]

at each point \( p \in M \).
Corollary 4.2 The mapping $g|M : M \rightarrow \mathbb{R}^k$ is an immersion (i.e. $\text{rank } Dg|M \equiv n - 1$) if and only if at each $p \in M$

$$\begin{bmatrix} \frac{\partial g_1}{\partial x_1}(p) & \cdots & \frac{\partial g_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1}(p) & \cdots & \frac{\partial g_n}{\partial x_n}(p) \end{bmatrix}$$

i.e. this matrix has a non-zero $(n \times n)$–minor.

Corollary 4.3 Let $g = (g_1, g_2, \ldots, g_{2n}) : \mathbb{R}^{n+1}, 0 \rightarrow \mathbb{R}^{2n}, 0$ be a $C^1$ function. Let us denote $\omega(x) := x_1^2 + x_2^2 + \ldots + x_{n+1}^2$ for $x \in \mathbb{R}^{n+1}$. Of course, $S^n(r) = \{x \mid \omega(x) - r^2 = 0\}$.

Then the following conditions are equivalent:

(a) there exists $r_0 > 0$ such that $g$ restricted to each sphere of a radius $0 < r \leq r_0$ is an immersion;

(b) if $M_1(x), \ldots, M_N(x)$ are all the $(n + 1) \times (n + 1)$–minors of the matrix

$$\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_{n+1}} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial g_{2n}}{\partial x_1} & \frac{\partial g_{2n}}{\partial x_2} & \cdots & \frac{\partial g_{2n}}{\partial x_{n+1}} \end{bmatrix},$$

then there exists $r_0 > 0$ such that for each $x \in \mathbb{R}^{n+1}$ with $0 < \|x\| < r_0$ there exists $i \in \{1, \ldots, N\}$ such that $M_i(x) \neq 0$.

Let $g : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{2n}$ be a $C^2$ mapping. Assume that for each $r \in [r_1, r_2]$, where $0 < r_1 < r_2$, $g|S^n(r)$ is an immersion. We can define a regular homotopy

$$H : [0, 1] \times S^n(r_1) \rightarrow \mathbb{R}^{2n},$$

$$(t, x) \mapsto g \left( \left( 1 - t + \frac{tr_2}{r_1} \right) x \right)$$

between $H(0, x) = g|S^n(r_1)(x)$ and $H(1, x) = g \left( \frac{tr_2}{r_1} x \right)$. By Theorem 3.3 $I(g|S^n(r_1)) = I(g(\frac{tr_2}{r_1} x)|S^n(r_1)) = I(g|S^n(r_2))$.

If there exists $r_0 > 0$ such that $g$ restricted to a sphere of a radius $0 < r \leq r_0$ is an immersion then the intersection number of each $g|S^n(r)$ is defined and does not depend on $r$.

Let $n$ be an even positive integer and let

$$g = (g_1, \ldots, g_{2n}) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{2n}$$
be a $C^2$ mapping. Suppose that $g$ restricted to each sphere of a radius small enough is a completely regular immersion.

Let $s \in \mathbb{R}$, $s \neq 0$, and consider a new mapping $g_s = (sg_1, g_2, \ldots, g_{2n})$. Then $g_s$ restricted to a sphere of a radius small enough is an immersion.

Indeed, let $M_1, \ldots, M_N$ be all the $(n + 1) \times (n + 1)$-minors of the matrix:

$$
\begin{bmatrix}
\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_{n+1}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g_{2n}}{\partial x_1} & \frac{\partial g_{2n}}{\partial x_2} & \cdots & \frac{\partial g_{2n}}{\partial x_{n+1}}
\end{bmatrix}.
$$

Then either $\tilde{M}_j = sM_j$ or $\tilde{M}_j = M_j$, where $M_j$ is the appropriate minor corresponding to the function $g$. By Corollary 4.3 there exists $r_0 > 0$ such that for each $0 < \|x\| < r_0$ there exists such $i$ that $M_i(x) \neq 0$, and so $\tilde{M}_i(x) \neq 0$. It means that for each $s \in \mathbb{R} \setminus \{0\}$ the mapping $g_s$ restricted to a sphere of a radius small enough is an immersion.

Let $r > 0$ be small and let $p, q \in S^n(r)$. Then $g(p) = g(q)$ if and only if $g_s(p) = g_s(q)$. Since $g$ restricted to each sphere of a small radius is completely regular, so is $g_s$.

**Corollary 4.4** $I(g_s|S^n(r)) = \text{sgn}(s)I(g|S^n(r))$.

**Proof.** Take $p, q \in S^n(r)$ such that $p \neq q$ and $g(p) = g(q)$. Suppose that $v_1, \ldots, v_n$ form a well-oriented basis in $T_pS^n(r)$, and $w_1, \ldots, w_n$ form a well-oriented basis in $T_qS^n(r)$.

For any vector $w$, the first coordinate of $Dg_s(p)w$ is equal to the first coordinate of $Dg(p)w$ multiplied by $s$, the other coordinates of $Dg_s(p)w$ are the same as the appropriate coordinates of $Dg(p)w$. Consider two matrices:

$$
[Dg_s(p)v_1, \ldots, Dg_s(p)v_n, Dg_s(q)w_1, \ldots, Dg_s(q)w_n]
$$

and

$$
[Dg(p)v_1, \ldots, Dg(p)v_n, Dg(q)w_1, \ldots, Dg(q)w_n].
$$

The first row in the first matrix is equal to the first row in the second matrix multiplied by $s$, other respective rows are identical. Then

$$
\det[Dg_s(p)v_1, \ldots, Dg_s(p)v_n, Dg_s(q)w_1, \ldots, Dg_s(q)w_n] = s \cdot \det[Dg(p)v_1, \ldots, Dg(p)v_n, Dg(q)w_1, \ldots, Dg(q)w_n].
$$

Hence

$$
I(g_s|S^n(r)) = \frac{1}{2} \sum \text{sgn}(\det[Dg_s(p)v_1, \ldots, Dg_s(p)v_n, Dg_s(q)w_1, \ldots, Dg_s(q)w_n]) = \text{sgn}(s)I(g|S^n(r)),
$$
where \( p, q \in S^n(r) \), such that \( p \neq q \) and \( g_s(p) = g_s(q) \).

Hence if \( g \) restricted to a sphere of a radius small enough is a completely regular immersion, then we have a family \( \{g_s\} \) of \( C^2 \) mappings and the intersection number of their restrictions to a sphere of a small radius satisfies:

\[
\mathbb{R} \setminus \{0\} \ni s \mapsto I(g_s|S^n(r)) = \text{sgn}(s) I(g|S^n(r)) \in \mathbb{Z},
\]

which is of course determined by a sign of a polynomial. In particular the function \( s \mapsto I(g_s|S^n(r)) \) is algebraically constructible.

Now let us consider the mapping

\[
g = (g_1, g_2, g_3, g_4) : \mathbb{R}^3 \rightarrow \mathbb{R}^4
\]

\[
g(x, y, z) = (x, y, xz, yz).
\]

Then for each \( r > 0 \), \( g|S^2(r) \) is an immersion because the matrix:

\[
\begin{bmatrix}
\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\
\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\
\frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \\
\frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
z & 0 & x \\
0 & z & y \\
2x & 2y & 2z
\end{bmatrix}
\]

has a non-zero \((3 \times 3)\)–minor at each point \( p \in \mathbb{R}^3 \setminus \{0\} \). It is easy to verify that \( g|S^2(r)(p) = g|S^2(r)(q) \) if and only if \( p = (0, 0, r) \) and \( q = (0, 0, -r) \).

At \( p = (0, 0, r) \)

\[
Dg(p) =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
r & 0 & 0 \\
0 & r & 0
\end{bmatrix}
\]

and vectors \( v_1 = (1, 0, 0) \) and \( v_2 = (0, 1, 0) \) form a well-oriented basis in \( T_pS^2(r) \).

At \( q = (0, 0, -r) \)

\[
Dg(q) =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-r & 0 & 0 \\
0 & -r & 0
\end{bmatrix}
\]

and vectors \( w_1 = (1, 0, 0) \) and \( w_2 = (0, -1, 0) \) form a well-oriented basis in \( T_qS^2(r) \).

Since

\[
\det[Dg(p)v_1, Dg(p)v_2, Dg(q)w_1, Dg(q)w_2] = \det
det
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 \\
r & 0 & -r & 0 \\
0 & r & 0 & r
\end{bmatrix}
= -4r^2,
\]
\( g \mid S^2(r) \) is a completely regular immersion and

\[
I(g \mid S^2(r)) = -1.
\]

Then for each \( s \in \mathbb{R} \setminus \{0\} \), the mapping \( g_s = (sx, y, xz, yz) \) restricted to \( S^2(r) \) is a completely regular immersion, and

\[
I(g_s \mid S^2(r)) = - \operatorname{sgn}(s).
\]

So \( \mathbb{R} \setminus \{0\} \ni s \mapsto I(g_s \mid S^2(r)) = - \operatorname{sgn} s \in \{-1, 1\} \) is an algebraically constructible function, which is of course nontrivial and constant modulo 2, but not constant modulo 4.

### 5 Families of analytic mappings

Let \( \Lambda \subset \mathbb{R}^p \) be an analytic set. Let

\[
g = (g_1, \ldots, g_{2n}) : \mathbb{R}^{n+1} \times \Lambda \longrightarrow \mathbb{R}^{2n},
\]

where \( g_1, \ldots, g_{2n} \) are analytic functions.

For fixed \( \lambda \in \Lambda \) we will denote by \( g_{\lambda} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{2n} \) the mapping defined by \( g_{\lambda}(x) = g(x, \lambda) \).

Let \( g_{2n+1}(x) = \omega(x) := x_1^2 + x_2^2 + \ldots + x_{n+1}^2 \), and let

\[
G_1(x, y, \lambda) := g_1(x, \lambda) - g_1(y, \lambda), \quad (1 \leq i \leq 2n)
\]

\[
G_{2n+1}(x, y) := g_{2n+1}(x) - g_{2n+1}(y) = \|x\|^2 - \|y\|^2,
\]

\[
G = (G_1, \ldots, G_{2n}) : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \Lambda \longrightarrow \mathbb{R}^{2n}.
\]

Then \( G_1, \ldots, G_{2n+1} \) are analytic, and there exist analytic functions \( h_{ij} \) such that

\[
G_i(x, y, \lambda) = h_{i1}(x, y, \lambda)(x_1 - y_1) + \ldots + h_{i(n+1)}(x, y, \lambda)(x_{n+1} - y_{n+1}).
\]

The functions \( h_{ij} \) are not uniquely determined.

We fix such functions \( h_{ij} \), and for \( 1 \leq i_1 < i_2 < \ldots < i_{n+1} \leq 2n + 1 \) we define

\[
W_{i_1 \ldots i_{n+1}} = \begin{vmatrix}
  h_{i_1 1} & h_{i_1 2} & \cdots & h_{i_1 (n+1)} \\
  h_{i_2 1} & h_{i_2 2} & \cdots & h_{i_2 (n+1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  h_{i_{n+1} 1} & h_{i_{n+1} 2} & \cdots & h_{i_{n+1} (n+1)} 
\end{vmatrix}
\]

By Cramer’s rule

\[
(x_1 - y_1)W_{i_1 \ldots i_{n+1}} = \begin{vmatrix}
  G_{i_1} & h_{i_1 2} & \cdots & h_{i_1 (n+1)} \\
  G_{i_2} & h_{i_2 2} & \cdots & h_{i_2 (n+1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  G_{i_{n+1}} & h_{i_{n+1} 2} & \cdots & h_{i_{n+1} (n+1)} 
\end{vmatrix}
\]
(x_{n+1} - y_{n+1}) W_{i_1 \ldots i_{n+1}} = \begin{vmatrix}
h_{i_11} & h_{i_12} & \cdots & G_{i_1} 
h_{i_21} & h_{i_22} & \cdots & G_{i_2} 
\vdots & \vdots & \ddots & \vdots 
h_{i_{n+1}1} & h_{i_{n+1}2} & \cdots & G_{i_{n+1}}
\end{vmatrix}

Points \( x \) and \( y \) lie on the same sphere in \( \mathbb{R}^{n+1} \) centered at the origin if and only if \( \|x\|^2 = \|y\|^2 \), i.e. if \( G_{2n+1}(x, y) = 0 \). If that is the case, then \( g(x, \lambda) = g(y, \lambda) \) if and only if \( G_1(x, y, \lambda) = \ldots = G_{2n}(x, y, \lambda) = 0 \).

Let us define

\[
A := \{(x, y, \lambda) \mid G_1(x, y, \lambda) = \ldots = G_{2n}(x, y, \lambda) = G_{2n+1}(x, y) = 0 \}
\]

\[
= \{(x, y, \lambda) \mid \exists r > 0 \ x, y \in S^n(r), \ g(x, \lambda) = g(y, \lambda) \} \cup \{0\} \times \{0\} \times \Lambda,
\]

\[A_{\lambda} := \{(x, y) \mid (x, y, \lambda) \in A\}.\]

Then \( A \) is closed in \( \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \Lambda\).

Let \( \Delta := \{(x, x) \mid x \in \mathbb{R}^{n+1}\}. \) Then \( \Delta \subset A_{\lambda} \) and \( \Delta \times \Lambda \) is closed in \( \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \Lambda\). Moreover \( (x, y, \lambda) \in \Delta \times \Lambda \) if and only if \( x_1 - y_1 = x_2 - y_2 = \ldots = x_{n+1} - y_{n+1} = 0 \).

If \( (x, y) \in A_{\lambda} \setminus \Delta \) then \( W_{i_1 \ldots i_{n+1}}(x, y, \lambda) = 0 \), for every \( 1 \leq i_1 < \ldots < i_{n+1} \leq 2n + 1 \).

Let us define

\[
B := A \cap \bigcap \{ (x, y, \lambda) \mid W_{i_1 \ldots i_{n+1}}(x, y, \lambda) = 0 \},
\]

\[
B_{\lambda} := A_{\lambda} \cap \bigcap \{ (x, y) \mid W_{i_1 \ldots i_{n+1}}(x, y, \lambda) = 0 \}
\]

\[= \{(x, y) \mid (x, y, \lambda) \in B\}.\]

Then \( B \) is closed in \( \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \Lambda \). If \( (x, y) \in A_{\lambda} \) and \( x \neq y \), then \( (x, y) \in \bigcap \{ (x, y) \mid W_{i_1 \ldots i_{n+1}}(x, y, \lambda) = 0 \} \), so

\[
(5.1) \quad A_{\lambda} \setminus \Delta = B_{\lambda} \setminus \Delta.
\]

For \( 1 \leq i, r \leq n + 1 \), if

\[
\frac{\partial}{\partial z_i} := \frac{1}{2} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial y_i} \right),
\]

then

\[
\frac{\partial}{\partial z_i}(x_r - y_r) = \begin{cases} 0, & i \neq r \\ 1, & i = r \end{cases}.
\]
For $1 \leq j \leq 2n+1$,

$$\frac{\partial G_j}{\partial z_i} = \left( \sum_{r=1}^{n+1} \frac{\partial h_{jr}}{\partial z_i} (x_r - y_r) \right) + h_{ji};$$

$$\frac{\partial G_j}{\partial z_i}(x, x, \lambda) = h_{ji}(x, x, \lambda).$$

On the other hand we have

$$\frac{\partial G_j}{\partial z_i}(x, y, \lambda) = \frac{1}{2} \left( \frac{\partial g_j}{\partial x_i}(x, \lambda) + \frac{\partial g_j}{\partial y_i}(y, \lambda) \right);$$

$$\frac{\partial G_j}{\partial z_i}(x, x, \lambda) = \frac{\partial g_j}{\partial x_i}(x, \lambda).$$

Thus

$$h_{ji}(x, x, \lambda) = \frac{\partial g_j}{\partial x_i}(x, \lambda).$$

**Lemma 5.1** Let $\lambda \in \Lambda$. Then $B_\lambda \cap \Delta = \{0\}$ in some neighbourhood of the origin if and only if there exists $r(\lambda) > 0$ such that for all $0 < r < r(\lambda)$ the mapping $g_\lambda = (g_1(\cdot, \lambda), \ldots, g_{2n}(\cdot, \lambda))$ restricted to a sphere of the radius $r$ is an immersion.

**Proof.** We have $h_{ji}(x, x, \lambda) = \frac{\partial g_j}{\partial x_i}(x, \lambda)$, so for $(x, x) \in \Delta$ the determinants $W_{i_1\ldots i_{n+1}}(x, x)$ are the $(n+1) \times (n+1)$–minors of the matrix

$$\begin{bmatrix}
\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_{n+1}} \\
\frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_{n+1}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g_{2n}}{\partial x_1} & \frac{\partial g_{2n}}{\partial x_2} & \cdots & \frac{\partial g_{2n}}{\partial x_{n+1}}
\end{bmatrix}.$$  

The function $g_\lambda$ satisfies the condition $(b)$ of Corollary 4.3 if and only if there exists $r(\lambda) > 0$ such that for all $0 < \|x\| < r(\lambda)$ we have $(x, x) \notin B_\lambda$, i.e. $B_\lambda \cap \Delta = \{0\}$ in some neighbourhood of the origin. 

Let us define

$$\delta : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R},$$

$$\delta(x, y) = \|x - y\|^2.$$  

Then $\delta \geq 0$ and $\delta^{-1}(0) = \Delta$.

**Proposition 5.2** If $\lambda \in \Lambda$ then the following conditions are equivalent:

(a) there exists $r(\lambda) > 0$ such that for all $0 < r < r(\lambda)$ the mapping $g_\lambda$ restricted to a sphere of the radius $r$ is an immersion;
(b) \( \{0\} \) is isolated in the set
\[ \Delta \cap \bigcap_{1 \leq i_1 < \ldots < i_{n+1} \leq 2n+1} W_{i_1 \ldots i_{n+1}}(\cdot, \lambda)^{-1}(0) = \Delta \cap \bigcap_{1 \leq i_1 < \ldots < i_{n+1} \leq 2n+1} W_{i_1 \ldots i_{n+1}}(\cdot, \lambda)^{-1}(0) \cap A_{\lambda}, \]
where \( 1 \leq i_1 < \ldots < i_{n+1} \leq 2n+1; \)
(c) \( B_\lambda \cap \Delta = \{0\} \) in a neighbourhood of the origin.
If that is the case, then
(d) in a neighbourhood of the origin \( \overline{A_{\lambda} \setminus \Delta} = B_\lambda \), and \( (A_{\lambda} \setminus \Delta) \cup \{0\} \) is closed;
(e) if \( r > 0 \) is small enough then \( S^n(r/\sqrt{2}) \times S^n(r/\sqrt{2}) \cap (G(\cdot, \cdot, \lambda))^{-1}(0) \setminus \Delta \) is a compact subset of \( S^n(r/\sqrt{2}) \times S^n(r/\sqrt{2}) \setminus \Delta \).

Proof. Lemma 5.1 implies that \((a) \iff (c)\). Since \( \Delta \subset A_{\lambda} \)
\[ \Delta \cap \bigcap_{1 \leq i_1 < \ldots < i_{n+1} \leq 2n+1} W_{i_1 \ldots i_{n+1}}(\cdot, \lambda)^{-1}(0) \cap A_{\lambda} = \Delta \cap B_{\lambda}, \]
so \((b) \iff (c)\). By \((5.1)\) \( A_{\lambda} \setminus \Delta = B_{\lambda} \setminus \Delta \), so \( \overline{A_{\lambda} \setminus \Delta} = \overline{B_{\lambda} \setminus \Delta} \). If \( B_{\lambda} \cap \Delta = \{0\} \) in some neighbourhood of the origin, then \( \overline{B_{\lambda} \setminus \Delta} = B_{\lambda} \) in some neighbourhood of the origin, hence \((c) \implies (d)\).

By \((5.1)\), \( A_{\lambda} \setminus \Delta = B_{\lambda} \setminus \Delta \) and by \((c)\) (maybe after making \( r(\lambda) \) smaller) for \( 0 < r < r(\lambda) \) we have \( B_{\lambda} \cap S^{n+1}(r) \cap \Delta = \emptyset \), so
\[ B_{\lambda} \cap S^{n+1}(r) = (B_{\lambda} \setminus \Delta) \cap S^{n+1}(r) = (A_{\lambda} \setminus \Delta) \cap S^{n+1}(r) \]
\[ = \{ (x, y) \in S^{n+1}(r) \mid \|x\|^2, \|y\|^2, g_\lambda(x) = g_\lambda(y), x \neq y \} \]
\[ = (S^{n+1}(r) \cap G_{2n+1}^{-1}(0) \cap (G(\cdot, \cdot, \lambda))^{-1}(0)) \setminus \Delta \]
\[ = S^n(r/\sqrt{2}) \times S^n(r/\sqrt{2}) \cap (G(\cdot, \cdot, \lambda))^{-1}(0)) \setminus \Delta. \]

\( B_{\lambda} \) is closed, so \( B_{\lambda} \cap S^{n+1}(r) \) is compact and we have \((e)\).

\[ \square \]

6 Immersions of spheres — the algebraic case

Assume that \( \Lambda \subset \mathbb{R}^p \) is an algebraic set and each \( g_i \) is a polynomial. Then there exist polynomials \( h_{ij} \) such that
\[ G_i(x, y, \lambda) = h_{i1}(x, y, \lambda)(x_1 - y_1) + \ldots + h_{i(n+1)}(x, y, \lambda)(x_{n+1} - y_{n+1}). \]
Now \( W_{i_1 \ldots i_{n+1}} \) are polynomials, and the sets \( A, B, A_{\lambda} \) and \( B_{\lambda} \) are algebraic.

Let us define
\[ Z = (\delta^{-1}(0) \times \Lambda) \cap B = (\Delta \times \Lambda) \cap B, \]
\[ Z_{\lambda} = \delta^{-1}(0) \cap B_{\lambda} = \Delta \cap B_{\lambda}. \]
So \(\delta(x, y, \lambda) := \delta(x, y)\) is continuous on \(B\) and the assumptions of [4, Corollary 3.1] hold. Then there exist a finite division \(\Lambda = \bigcup S_i\) (where \(S_i\) are semialgebraic), continuous, semialgebraic functions \(h_i : B \cap (\mathbb{R}^{2n+2} \times S_i) \rightarrow \mathbb{R}\), and constants \(q_i \in \mathbb{Q}^+\) such that for \(\lambda \in S_i\), \((x, y) \in B_\lambda\)

\[
\text{dist}((x, y), Z_\lambda)^{q_i} \leq h_i(x, y, \lambda)\delta(x, y).
\]

If \(\lambda \in S_i\), then there exists such a constant \(c_\lambda > 0\) that

\[
|h_i(x, y, \lambda)| < c_\lambda
\]

for each \((x, y) \in B_\lambda\) sufficiently close to the origin.

Set \(d(x, y) = x_1^2 + \ldots + x_{n+1}^2 + y_1^2 + \ldots + y_{n+1}^2 = \|(x, y)\|^2\). From now on we assume that for each \(\lambda \in \Lambda\) there exists \(r(\lambda) > 0\) such that for each \(0 < r < r(\lambda)\) the mapping \(g_\lambda|S^n(r)\) is an immersion.

By Proposition 5.2, locally \(Z_\lambda = \Delta \cap B_\lambda = \{0\}\). Then for \((x, y) \in B_\lambda\) close to the origin

\[
d(x, y)^\frac{q_i}{2} = \text{dist}((x, y), \{0\})^{q_i} = \text{dist}((x, y), Z_\lambda)^{q_i} \leq c_\lambda\delta(x, y).
\]

If \(\alpha > \max_i\{\frac{q_i}{2}\}\) is an integer, then for each \(\lambda \in \Lambda\) and \(t > 0\)

\[
(6.1) \quad \delta(x, y) \geq td(x, y)^\alpha
\]

for \((x, y) \in B_\lambda\) lying sufficiently close to the origin. By (6.1) \(A_\lambda \setminus \Delta = B_\lambda \setminus \Delta\), so the inequality (6.1) holds on \(A_\lambda \setminus \Delta\), i.e. on

\[
\{(x, y) \mid G_1(x, y, \lambda) = \ldots = G_{2n}(x, y, \lambda) = \|x\|^2 - \|y\|^2 = 0\} \setminus \delta^{-1}(0).
\]

By Proposition 5.2 (d), \((A_\lambda \setminus \Delta) \cup \{0\}\) is closed.

A polynomial \(F_1(x, y) = \|x\|^2 - \|y\|^2\) has an isolated critical point at the origin. Then

\[
L(r) = S^{2n-1}(r) \cap F_1^{-1}(0) = S^n(r/\sqrt{2}) \times S^n(r/\sqrt{2})
\]

is a compact oriented \(2n\)-dimensional manifold. Since \(\delta(x, y) = \|x - y\|^2 \geq 0\),

\[
U(r) = L(r) \setminus \delta^{-1}(0) = S^n(r/\sqrt{2}) \times S^n(r/\sqrt{2}) \setminus \Delta.
\]

By Proposition 2.2 there exists \(\alpha_0 > 0\) such that for any integer \(\alpha > \alpha_0\) and \(t \neq 0\), \((F_1, \delta - td^\alpha)^{-1}(0)\) has an isolated singularity at the origin. (As \(F_1\), \(\delta\) and \(d\) are homogeneous of degree 2, it is easy to verify that \(\alpha_0 = 1\) would be enough.)

Let us assume that \(n\) is even. Take an integer \(\alpha > \max_i\{\frac{q_i}{2}\}, \alpha_0\). Put \(F_2 = \delta - td^\alpha\) for arbitrary \(t > 0\), and

\[
F = (F_1, F_2),
\]

\[
G_\lambda = G(\cdot, \cdot, \lambda) : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, 0 \rightarrow \mathbb{R}^{2n}, 0.
\]
By Proposition 5.2 (e), \( L(r) \cap G_{-1}(0) \setminus \delta^{-1}(0) \) is a compact subset of \( U(r) \). By Proposition 2.3 for \( r > 0 \) small enough, the topological degree of

\[
(S^n(\frac{r}{\sqrt{2}}) \times S^n(\frac{r}{\sqrt{2}}) \setminus \Delta, S^n(\frac{r}{\sqrt{2}}) \times S^n(\frac{r}{\sqrt{2}}) \setminus (G_{-1}(0) \cup \Delta)) \longrightarrow (\mathbb{R}^{2n}, \mathbb{R}^{2n} \setminus \{0\})
\]

\[
(x, y) \mapsto G_\lambda(x, y) = g_\lambda(x) - g_\lambda(y),
\]
i.e. \( 2I(g_\lambda | S^n(r/\sqrt{2})) \), equals the local topological degree \( \deg_0(F, G_\lambda) \).

Let us define a polynomial mapping \( H : \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \Lambda \longrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2n} : \)

\[
H(t, x, y, \lambda) = (F_1(x, y), \delta(x, y) - td(x, y)^\alpha, g(x, \lambda) - g(y, \lambda)).
\]

Denote \( H_\lambda^t(x, y) = H(t, x, y, \lambda) \). For any \( \lambda \in \Lambda \) and \( t > 0 \), the mapping \( H_\lambda^t \) has an isolated zero at the origin and the local topological degree \( \deg_0(H_\lambda^t) = 2I(g_\lambda | S^n(r)) \) for \( 0 < r < r(\lambda) \). If \( t < 0 \) then \( \delta - td^\alpha > 0 \), except of the origin, so that \( \deg_0(H_\lambda^t) = 0 \).

**Theorem 6.1** Let \( \Lambda \subset \mathbb{R}^n \) be an algebraic set and let \( n \) be an even integer. Let

\[
g_1(x, \lambda), \ldots, g_{2n}(x, \lambda) : \mathbb{R}^{n+1} \times \Lambda \longrightarrow \mathbb{R}^{2n}
\]

be polynomials. Assume that for each \( \lambda \in \Lambda \) there exists \( r(\lambda) > 0 \) such that for each \( 0 < r < r(\lambda) \) the mapping \( g_\lambda \) restricted to a sphere \( S^n(r) \) of the radius \( r \) centered at the origin is an immersion. Then the function \( \Lambda \ni \lambda \mapsto I(g_\lambda | S^n(r)) \in \mathbb{Z} \) is algebraically constructible.

**Proof.** Using the same arguments as in [9] Theorem 3.4] one may prove that there exist polynomials \( h_1, \ldots, h_s \) on \( \mathbb{R} \times \Lambda \), such that if \( 0 \) is isolated in \( (H_\lambda^t)^{-1}(0) \) then

\[
\deg_0(H_\lambda^t) = \sum_{i=1}^s \text{sgn} h_i(t, \lambda),
\]
i.e. this function is algebraically constructible.

For \( r > 0 \) small enough

\[
I(g_\lambda | S^n(r)) = \lim_{t \to 0} \frac{1}{2} (\deg_0(H_\lambda^t) + \deg_0(H_{-1}^t))
\]

\[
= \lim_{t \to 0} \frac{1}{2} \sum_{i=1}^s (\text{sgn} h_i(\lambda, t) + \text{sgn} h_i(\lambda, -t)) =: \psi(\lambda).
\]

According to [9] Lemma 6.5] the function \( \Lambda \ni \lambda \mapsto \psi(\lambda) \in \mathbb{Z} \) is algebraically constructible, and so is \( \Lambda \ni \lambda \mapsto I(g_\lambda | S^n(r)) \).

**Remark.** It is easy to see that if mappings \( g_\lambda \) are immersions on small spheres only for \( \lambda \in \Lambda \setminus \Sigma \), where \( \Sigma \) is a proper algebraic subset of \( \Lambda \), then the function \( \Lambda \ni \lambda \mapsto I(g_\lambda | S^n(r)) \in \mathbb{Z} \) is generically algebraically constructible, i.e. it coincides with an algebraically constructible function in \( \Lambda \setminus \Sigma \).
References

[1] Akbulut, S., King, H.: Topology of real algebraic sets. MSRI Publ. 25, New York: Springer – Verlag, 1992

[2] Coste, M.: Real Algebraic Sets, Summer School and Conference on Real Algebraic Geometry and its Applications (2003)

[3] Coste, M., Kurdyka, K.: Le discriminant d’un morphisme de variétés algébriques réelles. Topology 37 no. 2, 393–399 (1998)

[4] Fekak, A.: Exposants de Lojasiewicz pour les fonctions semi-algébriques. Annales Polonici Mathematici 56 123–131 (1992)

[5] Lashof, R., Smale, S.: On the immersion of manifolds in euclidean space. Ann. of Math. (2) 68, 562–583 (1958)

[6] McCrory, C., Parusiński, A.: Algebraically constructible functions. Ann. Sci. École Norm. Sup. (4) 30, no.4, 527–552 (1997)

[7] McCrory, C., Parusiński, A.: Topology of real algebraic sets of dimension 4: necessary conditions. Topology 39 no. 3, 495–523 (2000)

[8] McCrory, C., Parusiński, A.: Algebraically Constructible Functions: Real Algebra and Topology, preprint (2002)

[9] Parusiński, A., Szafraniec, Z.: Algebraically constructible functions and signs of polynomials. Manuscripta Math. 93, no. 4, 443–456 (1997)

[10] Parusiński, A., Szafraniec, Z.: On the Euler characteristic of fibres of real polynomial maps. Singularities Symposium—Lojasiewicz 70 (Kraków, 1996; Warsaw, 1996), Banach Center Publ. 44 Polish Acad. Sci., Warsaw (1998), 175–182 (1998)

[11] Smale, S.: The classification of immersions of spheres in Euclidean spaces. Ann. of Math. (2) 69, 327–344 (1959)

[12] Szafraniec, Z.: On the Euler characteristic of analytic and algebraic sets. Topology 25, 411–414 (1986)

[13] Whitney, H.: The self–intersections of a smooth n–manifold in 2n–space. Ann. of Math. (2) 45, 220–246 (1944)

[14] Whitney, H.: Differentiable manifolds. Ann. of Math. (2) 37, no.3, 645–680 (1936)