Finite Size Scaling Analysis of Exact Ground States for $\pm J$ Spin Glass Models in Two Dimensions

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March 23, 2022

PACS. 75.10.Nr, 75.50.Lk, 05.50.+q — Spin Glass, Random Systems, Ising models

Abstract. — With the help of exact ground states obtained by a polynomial algorithm we compute the domain wall energy $\Delta$ at zero-temperature for the bond-random and the site-random Ising spin glass model in two dimensions. We find that in both models the stability of the ferromagnetic and the spin glass order ceases to exist at a unique concentration $p_c$ for the ferromagnetic bonds. In the vicinity of this critical point, the size and concentration dependency of the first and second moment of the domain wall energy are, for both models, described by a common finite size scaling form. Moreover, below this concentration the stiffness exponent turns out to be slightly negative $\theta_S = -0.056(6)$ indicating the absence of any intermediate spin glass phase at non-zero temperature.

Since Edwards and Anderson (EA) proposed the model for spin glasses, it has been discussed not only among researchers specialized for the subject, but in a rather wide community of physicists working on random systems in general. Perhaps, this is largely because the EA model is the simplest possible model with short-ranged interaction for which we might expect a spin glass phase transition similar to that in the mean-field model, i.e. the Sherrington-Kirkpatrick model. However, a number of fundamental questions failed to be answered conclusively [1]. Even the very existence of the phase transition in three dimensions was questioned [2] recently. Whether or not the spin glass phase exists in the presence of a uniform magnetic field is even less clear [3, 4].

Another unanswered question is whether a spin glass phase (at non-zero temperature) can exist in two dimensions with an asymmetric bond distribution. It has been argued [5, 6, 7] that an intermediate spin glass phase might be present in the $p$-$T$ phase diagram between the ferromagnetic phase and the paramagnetic phase. In Fig. 1, such a $p$-$T$ phase diagram is shown including the proposed spin-glass transition line represented by the dash-dotted line. For the site-random model the evidence for the existence of a spin-glass phase seems to be even stronger than for the bond-random model [8, 9].

On the other hand, in the case of the bond-random $\pm J$ model with $p = 1/2$, arguments for the absence of a spin glass phase in two dimensions were mainly based on results from Monte Carlo simulations [10, 11] and the estimates of the domain wall energy [12, 13]. The data from Monte Carlo simulations, however, are not available at very low temperature, e.g., below $T = 0.4J$ in Bhatt and Young’s simulation [11], which naturally made it difficult to exclude the possibility of the transition at a temperature smaller than $0.4J$. Furthermore it is not clear, whether the “stiffness” exponent $\theta_S$ is really negative because the data in
Figure 1: The schematic phase diagram with the previously proposed spin-glass transition lines. $T_c^{(f)}$ stands for the critical temperature of the Ising model on the square lattice. $P$, $F$, and $SG$ stand for the paramagnetic, ferromagnetic and spin glass phases, respectively.

Cieplak and Banavar’s paper[14] clearly show a systematic positive curvature in a log-log plot of the domain wall energy versus system size for systems without vacancy. More recently results of Monte Carlo simulation at lower temperatures have been reported [10] indicating a transition at $T \simeq 0.24J$. In addition, $\theta_S = 0$ was suggested [7] based on estimates of the domain-wall energy. These results are consistent with a finite temperature phase transition for which the low-temperature phase is only marginally or weakly ordered, meaning that the two-point spin-spin correlation function decreases algebraically as a function of the distance.

The aim of the present letter is to reinvestigate this issue by studying the domain wall energy at zero temperature via the determination of exact ground states for large system sizes and huge sample numbers. This can be done very efficiently with the help of a polynomial algorithm described by Barahona et al. [5], which amounts to finding a minimum-weight-perfect-matching in a weighted graph with $N = L^2$ nodes and has computational complexity $\mathcal{O}(N^3)$. The model that we consider is the two-dimensional Ising spin glass with binary couplings defined by the Hamiltonian

$$\mathcal{H} \equiv - \sum_{\langle ij \rangle} J_{ij} S_i S_j \quad (1)$$

where $S_i = \pm 1$ are Ising spins, $(ij)$ are nearest neighbor sites on an $L \times L$-square lattice and the interactions strengths $J_{ij}$ are quenched random variables taking on one of the two values, $+J$ and $-J$. We consider two different cases: In the bond-random model all interactions living on the bonds are independently distributed with a concentration $p \in [0, 1]$ of ferromagnetic bonds ($J_{ij} = +1$). For the site-random model one first generates independently distributed random variables for all sites, $\epsilon_i = \pm 1$. The concentration of type-A-sites, i.e. those with $\epsilon = +1$, is $c$, the type-B-sites ($\epsilon = -1$) occur with probability $1 - c$. Then, $J_{ij}$ is set to be $-J$ if and only if $\epsilon_i = \epsilon_j = -1$, and it is set to be $+J$, otherwise. In this case the ferromagnetic bond concentration is given by $p = (2 - c)c$.

We calculate the domain wall energy $\Delta$ defined by $\Delta \equiv E_p - E_a$ where $E_p$ and $E_a$ are the ground state energies with the periodic and the anti-periodic boundary conditions in the $x$-direction, respectively. Free boundary conditions are imposed in the $y$-direction. Of crucial importance are the exponents $\rho$ and $\theta_S$ that characterize the system size dependence of the moments of the domain wall energy:

$$[\Delta] \propto L^\rho \quad \text{and} \quad [\Delta^2]^{1/2} \propto L^{\theta_S} \quad (2)$$

A positive value for $\rho$ indicates the stability of a ferromagnetic ground state even in the presence of thermal fluctuations and thus the existence of the ferromagnetic long range order
Figure 2: The scaling plot of $[\Delta]$ for the bond-random model. $p_c^{(1)} = 0.896$, $\phi_1 = 0.77$ and $\psi_1 = -0.19$ are assumed. The inset is the view focused on the region near $p = p_c^{(1)}$.

at finite temperature \[13\]. On the other hand, a positive value for the stiffness exponent, $\theta_S$, with $\rho$ being negative at the same time, still indicates the stability of the ground state, which possesses now long range order different from a ferromagnetic one. Thus a positive $\theta_S$ is interpreted as a sign for a spin glass phase at non-zero temperature. We define $p_c^{(1)}$ and $p_c^{(2)}$ as the critical concentrations of ferromagnetic bonds at which the asymptotic $L$ dependences of $[\Delta]$ and $[\Delta^2]^{1/2}$, respectively, change from increasing to decreasing, i.e., the concentrations where a ferromagnetic phase and a spin glass phase, respectively, cease to exist at finite temperature.

We computed $[\Delta]$ and $[\Delta^2]^{1/2}$ for $L = 4, 6, 8, 12, 16, 24$ and 32 at various values of $p$ ranging from 0.50 up to 0.95. While the number of bond samples depends on $L$ and $p$, it is 32768 for one of the most time consuming data points, such as the one for $L = 32$ and $p = 0.5$. We hypothesize the following finite size scaling form for $[\Delta]$:

$$[\Delta] L^{\psi_1} = f_1((p - p_c^{(1)}) L^{\phi_1}).$$  \hspace{1cm} (3)  

The three parameters $p_c^{(1)}$, $\phi_1$ and $\psi_1$ have to be chosen such that a good data collapse for all data is obtained. To quantify the “goodness” of this fit, we used an appropriate cost function $S(p_c, \phi, \psi)$ \[4\] whose minimum value should be close to unity when the fit is statistically acceptable. When we use all the obtained data points, the best fit is achieved with $S(p_c, \phi, \psi) \sim 2$, which indicates that there is a non-negligible systematic error, i.e., a correction to scaling. Therefore, we have tried a similar analysis on a restricted set of data, omitting data which are presumably outside the asymptotic scaling regime, namely, data with $p$ far from $p_c^{(1)}$ and data for small system sizes. For instance, the goodness of the fit can be significantly improved to $S = 1.12$ by omitting the data for $p = 0.95$, which yields the estimates

$$p_c^{(1)} = 0.896(1), \quad \phi_1 = 0.77(1), \quad \psi_1 = -0.19(2).$$  \hspace{1cm} (4)  

Considering the very small errors accompanying the data points, it is remarkable that all the data even including those for $L = 4$ can be expressed by the finite size scaling form \[3\]. The resulting scaling plot is shown in Fig. 4. We have confirmed that other choices of data points producing a value of $S$ close to unity result in estimates of $p_c^{(1)}$, $\phi_1$, and $\psi_1$ consistent with the estimates quoted above. The value of $p_c^{(1)}$ is consistent with most of previous estimate such as 0.88(2) \[15\], 0.89(2) \[5\] and 0.89(1) \[16\] while inconsistent with 0.885(1) \[7\].

In Fig. 3, $[\Delta^2]^{1/2}$ is plotted against $L$. The lowest curve with crosses, which is almost straight with negative but very small slope, corresponds to $p = 1/2$. In other words, the
Figure 3: The domain-wall energy \([\Delta^2]^{1/2}\) of the bond-random model plotted against the system size \(L\) for various ferromagnetic-bond concentration \(p\). The inset is the view focused on the data points for \(p = 1/2\). The straight line in the inset is obtained by the fitting to the data points excluding the two leftmost ones.

Figure 4: The scaling plot of \([\Delta^2]^{1/2}\) of the bond-random model.

Domain-wall energy decreases systematically but it does so very slowly. The method of least squares using all the data points yields \(\theta_S = -0.052(1)\) whereas all but the first two points (for \(L = 4\) and 6) results in \(\theta_S = -0.060(2)\). Therefore, we quote here the value \(\theta_S = -0.056(6)\) as our estimate.

Our results disagree with the suggestion \(\theta_S = -0.25\) by Cieplak and Banavar\[14\]. Considering the size of actual reduction in \([\Delta^2]^{1/2}\) as \(L\) grows from 4 to 32, we cannot rule out the possibility that the exact value of this exponent is 0, which means \(0 < \lim_{L \to \infty} \Delta < \infty\). Such a scenario would be consistent with a suggestion by Ozeki\[7\]. In this case one has a marginal situation and we cannot decide whether the long-range order persists at a low but finite temperature based solely on a calculation of the stiffness exponent. We may, however, say that the low-temperature phase is only weakly ordered even if the phase transition takes place at a finite temperature.

Similarly to the above-mentioned procedure employed for \([\Delta]\), we have tried a finite size scaling analysis for the data of \([\Delta^2]^{1/2}\),

\[ [\Delta^2]^{1/2} L^{\psi_2} = f_2((p - p_c^{(2)}) L^{\phi_2}) \]  

(5)

Now we have to omit more data to get an acceptable fit with the value of \(S\) close to unity, indicating that the correction to scaling is larger for \([\Delta^2]^{1/2}\) than for \([\Delta]\). However, a good data collapse is obtained when we use only data for \(L \geq 12\) and \(0.85 \leq p \leq 0.91\). The best
Figure 5: The domain-wall energy $[\Delta^2]^{1/2}$ of the site-random model plotted against the system size $L$ for various ‘A-site’ concentration $c$.

The fit yields

$$p_c^{(2)} = 0.894(2), \quad \phi_2 = 0.79(6), \quad \psi_2 = -0.16(4)$$

with $S = 0.99$. The resulting scaling plot is shown in Fig. 4. The present estimate of $p_c^{(2)}$ is larger than but marginally consistent with all the previous estimates such as 0.86(2) [17], 0.85 [5] and 0.870 [18], while it is clearly inconsistent with 0.854(2) [7].

It is remarkable that not only $p_c^{(2)}$ but also $\phi_2$ and $\psi_2$ agree with the corresponding values in (4) within the statistical errors. While the agreement in $p_c$ already suggests the absence of the intermediate phase, we consider the agreement in the critical indices as another evidence for the absence of the intermediate spin-glass phase, since it is hardly possible that the first and the second moment of $\Delta$ show the same critical behavior at different values of $p_c$.

We now focus on the site-random model. In Fig. 5, $[\Delta^2]^{1/2}$ is plotted as a function of the system size. A significant correction to scaling can be seen in Fig. 5. The same remark applies to the first moment, $[\Delta]$. We have performed a finite-size-scaling analysis similar to what has been done for the bond-random model. As for $[\Delta]$, in order to reduce the cost function, $S$, down to unity, the smallest system sizes $L = 4$ and $L = 6$ have to be excluded from the scaling plot. The data out of the range, $0.60 \leq c \leq 0.68$, are also excluded in the quantitative estimation of $c_c$ and the indices $\phi_1$ and $\psi_1$. As for $[\Delta^2]^{1/2}$, an even stronger correction to scaling is present as shown in Fig. 5, making an additional system size ($L = 8$) unavailable for the quantitative estimation of $c_c$. The range of $c$ from which the data are chosen is again $0.60 \leq c \leq 0.68$.

The critical concentration, $c_{c}^{(i)}$, and the critical indices, $\phi_i$ and $\psi_i$, are defined in a similar fashion to (4) and (5), resulting in

$$c_{c}^{(1)} = 0.658(3), \quad \phi_1 = 0.78(2), \quad \psi_1 = -0.18(3)$$
$$c_{c}^{(2)} = 0.661(4), \quad \phi_2 = 0.79(2), \quad \psi_2 = -0.23(3)$$

Obviously $c_{c}^{(1)} = c_{c}^{(2)}$ within the error bars, from which also we here conclude that no intermediate spin glass phase exists. In addition, again, the critical indices for $[\Delta]$ agree with those for $[\Delta^2]^{1/2}$ within the statistical errors. These critical indices agree with those for the bond random model ((4) and (5)), implying that both models belong to the same universality class.

To summarize, we have performed a systematic calculation of the domain wall energy at zero temperature with systems larger than previous calculations for both bond-random and site-random models. We observed a significant cross-over effect or a correction-to-scaling especially for the site random model, while no indication for a finite temperature spin glass
phase could be detected. Some of the previous evidences for the positive stiffness exponents were based on systems smaller than those studied in the present paper and therefore may be attributed to this cross-over effect. The critical concentration for the ferromagnetic bonds and critical indices estimated from $[\Delta]$ agree with those from $[\Delta^2]^{1/2}$, again indicating the absence of an intermediate phase. Moreover, the critical indices for the site-random model agree with those for the bond-random model, which suggests that both models indeed show the same universal critical behavior and have qualitatively identical features away from $p_c$.

We have also seen that the domain wall energy seems almost independent of the system size below $p_c$ for the bond-random model at $p = 1/2$. Our result of a negative stiffness exponent $\theta_S = -0.056(6)$ is statistically significant, although it is difficult to exclude the possibility of $\theta_S = 0$ because of the very small change in $[\Delta^2]^{1/2}$ actually observed.

Several other calculations are still in progress. We have calculated the spin glass susceptibility at very low temperature down to $T = 0.05J$, and again found no evidence for the phase transition at a finite temperature. We have also computed the domain wall energy with a replica boundary condition similar to the one used in Ozeki’s work [18]. These results will be published elsewhere [19]. As a future perspective it might be promising to perform a similar study of the two-dimensional Ising spin glass model with next-nearest neighbor interactions and for which indications of a spin glass transition have been found [20].

This work was mainly done while one of the authors (N.K.) was at the supercomputer center HLRZ c/o Forschungszentrum Jülich, Germany. He would like to thank the HLRZ for its hospitality and financial support.

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