On the incremental form of dissipativity

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Abstract: Following the seminal work of Zames, the input-output theory of the 70s acknowledged that incremental properties (e.g. incremental gain) are the relevant quantities to study in nonlinear feedback system analysis. Yet, non-incremental analysis has dominated the use of dissipativity theory in nonlinear control from the 80s. Results connecting dissipativity theory and incremental analysis are scattered and progress has been limited. This abstract investigates whether this theoretical gap is of fundamental nature and considers new avenues to circumvent it.

Keywords: incremental dissipativity, incremental passivity, contraction, monotonicity, circuit theory, passivity.

1. INTRODUCTION

Research goes in circles. Incremental analysis (such as incremental passivity or bounded incremental gain) dominated the landscape of feedback system analysis in the 60s and 70s (see e.g. the landmark textbook of Desoer and Vidyasagar (1975)). That line of work was pioneered in the 1960 dissertation of Zames. It is an input-output (or external) theory, and it originated from circuit theory.

In contrast, non-incremental analysis (such as Lyapunov stability of an equilibrium state) dominated the nonlinear control landscape in the 80s and 90s (see e.g. the classical textbooks of Khalil (2002), Isidori (1995), and van der Schaft (1999)). It is a state-space (or internal) theory, and it originated from mechanics.

The dissipativity theory of Willems (1972a,b) is like a trait d’union between those two periods. It connects input-output analysis and state-space analysis, circuit theory and mechanics, internal and external properties. Yet, in its original formulation, dissipativity theory is non-incremental, that is, it focuses on deviations from a particular equilibrium state. This state minimizes the internal storage, when no external energy is supplied to the system.

This paper explores the incremental form of dissipativity, which connects the incremental input-output properties studied in the 70s to the incremental form of Lyapunov stability. Widespread interest in the latter only emerged in the late 90s, after Lohmiller and Slotine (1998) advocated the value of approaching stability analysis from an incremental perspective, and to study global stability properties as integral forms of local contraction properties.

One would expect that the incremental form of dissipativity connects incremental external properties to incremental Lyapunov stability in the same way as the original theory connects non-incremental external properties to non-incremental Lyapunov stability. Yet, the development of incremental dissipativity theory over the last two decades has been scattered (see e.g. (Stan et al., 2007; Forni and Sepulchre, 2013; Forni et al., 2013; van der Schaft, 2013; Verhoek et al., 2020)). Those limited developments contrast with the growing interest in contraction analysis, on the one hand, and on the renewed interest in incremental external properties, on the other. The latter is for instance illustrated by today’s importance of maximal monotonicity, the incremental form of passivity, or Lipschitz continuity, the incremental form of contractivity, in convex optimisation (Ryu and Yin, 2022; Bauschke and Combettes, 2011) and machine learning (Szegedy et al., 2014)). The gap between those internal and external concepts calls for a bridge, which is the topic of incremental dissipativity.

After briefly reviewing the definitions and main developments of incremental analysis, both internal and external, we provide examples that point out to difficulties in bridging them with the current theory. We argue that those limitations are of fundamental nature and call for new and promising research avenues in nonlinear control.

2. THE INCREMENTAL FORM OF DISSIPATIVITY

Dissipativity was introduced by Willems (1972a). A state-space model with supply rate $w$ is said dissipative if there exists a nonnegative storage $S$ such that for any $(t_1, t_0) \in \mathbb{R}_+^2$, and any input signal $u(\cdot)$,

$$S(x_0) + \int_{t_0}^{t_1} w(t) \geq S(x_1)$$

where $x_1$ is the state solution at time $t_1$ corresponding to the initial condition $x_0$ at time $t_0$ and the input signal $u(\cdot)$.

The usual form of the state-space model is the set of ordinary differential equations $\dot{x} = f(x,u)$, $y = h(x,u)$, and the two important examples of supply are the passivity supply $w = (u|y) = u^T y$ and the gain supply $\|u\|^2 - \|y\|^2$.

The incremental form of dissipativity replaces states and inputs in the original definition by increments, that is differences of states and inputs:

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\( S(\Delta x_0) + \int_{t_0}^{t_1} \Delta w(t) \geq S(\Delta x_1) \),

where \( \Delta x_1 \) is the difference of state solutions at time \( t_1 \) corresponding to a difference of initial conditions \( \Delta x_0 \) at time \( t_0 \) and a difference of input signals \( \Delta u(\cdot) \). The incremental supply \( \Delta w \) is the supply evaluated at the difference of inputs \( \Delta u \) and difference of outputs \( \Delta y \).

Under smoothness assumptions, incremental dissipativity is equivalent to differential dissipativity. The differential form of dissipativity is obtained by letting increments become infinitesimal, leading to the definition

\[ S(\delta x_0) + \int_{t_0}^{t_1} \delta w(t) \geq S(\delta x_1). \]

Differential solutions of the state-space model are the solutions of the (linear) variational systems determined by the linearisation of the state-space model around solutions.

Incremental dissipativity is considered by Stan et al. (2007) and also Pavlov and Marconi (2008), as a methodology to study synchronization of nonlinear oscillators. See also (Verhoek et al., 2020) for a more recent development. Differential dissipativity is proposed by Forni and Sepulchre (2013) as the generalization of differential stability to open systems. It is studied by Manchester and Slotine (2013), and also by Forni and Sepulchre (2018) in the context of dominance analysis. Differential passivity is studied by van der Schaft (2013); Forni et al. (2013).

It is usual in dissipativity theory to assume that the storage is minimum at the equilibrium state \( x = 0 \) and that this equilibrium trajectory corresponds to the zero input \( u(\cdot) \equiv 0 \). In that case, dissipativity is the special form of incremental dissipativity when considering only increments or deviations from the equilibrium trajectory. A generalization of that non-incremental definition is equilibrium-independent dissipativity (Hines et al., 2011), which again is a special form of incremental dissipativity, with increments only considered from equilibrium trajectories.

Quadratic storage functions of the type \( S(x) = x^T P x \) occupy a special place in the theory of dissipativity, as they lead to algorithms of solutions of many control questions via Linear Matrix Inequalities (Boyd et al., 1994). Likewise, the majority of applications of incremental and differential dissipativity assume quadratic storages \( S(\Delta x) = (\Delta x)^T P (\Delta x) \), where \( \Delta x = x_1 - x_2 \), or \( S(\delta x) = (\delta x)^T P (\delta x) \). Beyond linear time-invariant systems, it is of course a restriction to assume that the storage is quadratic and time-invariant, and even more so to assume that the storage is trajectory independent. For a time-invariant state-space model, a general form of incremental storage is a nonnegative function of two arguments \( S(x_1, x_2) \). Likewise, a general form of differential storage is a nonnegative function of two arguments \( S(x, \delta x) \), with the infinitesimal increment \( \delta x \) an element of the tangent space at \( x \). A practical question, however, is how to construct such general storages in applications.

3. INTERNAL INCREMENTAL PROPERTIES

Willems considered dissipativity theory to be the Lyapunov theory of open systems. For closed systems, the dissipation inequality reduces to \( S(x_1) \leq S(x_0) \), and the storage becomes the Lyapunov function used to study the stability of, say, the dynamical system \( \dot{x} = f(x) \). In other words, the internal property associated to dissipativity is the classical concept of Lyapunov stability. Like dissipativity, Lyapunov stability has historically received far more attention in its non-incremental form: the focus is on deviations with respect to the equilibrium solution \( x = 0 \). The 1998 paper of Lohmiller and Slotine is seminal in that it highlights the value of studying Lyapunov stability as an integral form of differential stability, rather than via the conventional non-incremental Lyapunov analysis. The authors focus on exponential stability of the linearized dynamics along any trajectory, a property that they name contraction. Contraction is also studied by Russo et al. (2010); Pavlov et al. (2005). It is the differential form of incremental Lyapunov stability, first studied by Angeli (2002). The concept of a differential Lyapunov function as a Lyapunov function constructed in the tangent bundle is developed by Forni and Sepulchre (2014).

The study of incremental rather than non-incremental Lyapunov stability has gained popularity in the last two decades, suggesting the relevance of incremental stability properties in all nonlinear control applications that address questions beyond the stability of a specific equilibrium solution. It is worth noting that Lohmiller and Slotine consider trajectory dependent Lyapunov functions (also called contraction metrics) \( S(x, t) = \delta x^T P(t, x) \delta x \). Likewise, Angeli considers incremental Lyapunov functions of the general form \( V(x_1, x_2) \), and Forni and Sepulchre consider trajectory dependent differential Lyapunov functions (also called Finsler metrics) of the general form \( V(x, \delta x) \). Yet, the majority of applications only use trajectory independent quadratic Lyapunov functions.

4. EXTERNAL INCREMENTAL PROPERTIES

Desoer and Vidyasagar make a comprehensive study of the external properties of nonlinear systems in their classic treatise, Feedback Systems: Input/Output Properties (Desoer and Vidyasagar, 1975). This line of research originated in the early work of Zames (1960, 1966); Sandberg (1964) and others.

Define the truncation operator \( P_T \) by

\[ P_T u(t) = \begin{cases} u(t) & \text{if } t < T \\ 0 & \text{otherwise.} \end{cases} \]

Given a Hilbert space \( \mathcal{L} \), the extension of \( \mathcal{L} \), denoted \( \mathcal{L}_c \), is the space of signals \( u \) such that \( P_T u \in \mathcal{L} \) for all \( T \). The incremental gain of an operator \( H \) on \( \mathcal{L}_c \) is

\[ \sup_{u_1, u_2 \in \mathcal{L}_c, u_1 \neq u_2, T > 0} \frac{\| P_T H u_1 - P_T H u_2 \|}{\| u_1 - u_2 \|}. \]

An operator \( H \) on \( \mathcal{L}_c \) is said to be incrementally passive if, for all \( u_1, u_2 \in \mathcal{L}_c, T > 0 \),

\[ (P_T u_1 - P_T u_2) (P_T H u_1 - P_T H u_2) \geq 0. \]

These two properties are equivalent under the scattering transform, which originates in electrical circuit theory. The equivalent properties for operators on a Hilbert space \( \mathcal{L} \) are the Lipschitz constant, defined as

\[ L = \sup_{u_1, u_2 \in \mathcal{L}, u_1 \neq u_2} \frac{\| H u_1 - H u_2 \|}{\| u_1 - u_2 \|}. \]
and incremental positivity, or monotonicity, defined as
\[ \langle u_1 - u_2 | Hu_1 - Hu_2 \rangle \geq 0. \]
for all \( u_1, u_2 \in \mathcal{L} \).

These two properties are fundamental in the theory of convex optimization, and indeed the property of monotonicity forms the basis of an ever expanding literature on algorithms for large scale and nonsmooth problems (Bertsekas, 2011; Bauschke and Combettes, 2011; Ryu and Yin, 2022; Ryu and Boyd, 2016; Parikh and Boyd, 2013).

5. NONLINEAR CIRCUITS

Passivity is the backbone of linear circuit theory, originating in questions about the synthesis of dynamic behaviors in one-port circuits (Cauer, 1926; Brune, 1931; Foster, 1924). A fundamental result of circuit theory is that any passive LTI system can be realised by interconnecting elementary circuit elements (resistors, capacitors, inductors, and transformers).

To an extent, this key connection between circuit theory and control theory has been lost in the nonlinear theory of passivity. It can be argued that this is the result of focusing on the non-incremental form of passivity.

Chua realised early that passive nonlinear circuit elements could generate rich nonlinear behaviors when connected to batteries. For instance, a nonlinear resistor is passive provided that its voltage-current characteristic lies in the first and third quadrant (making the instantaneous power always non-negative), but this does not prevent the device to have negative resistance when linearised away from zero. Such negative resistance devices are key elements of bistable memories, nonlinear oscillators, chaotic circuits, and spiking circuits. They lose however a key property of the linear theory of passivity, which is that the inverse of a passive transfer function is passive. As a consequence, the non-incremental form of passivity is of little use in nonlinear circuit analysis beyond the stability analysis of the zero equilibrium. This limitation perhaps explains why nonlinear circuits are marginal with respect to mechanical systems in the nonlinear textbooks of the 80s and 90s.

The situation is different with the concept of incremental passivity, or maximal monotonicity. This concept precisely originated from attempts to extend the algorithmic tractability of passive linear circuit theory to networks of nonlinear resistors (Duffin, 1946; Minty, 1960, 1961a,b; Desoer and Wu, 1974). Consistent with their focus on non-incremental analysis, monotonicity is absent from the nonlinear textbooks of the 80s and 90s.

Our recent work (Chaffey and Sepulchre, 2021) argues that as the incremental form of passivity, monotonicity retains many of the desirable properties of linear passivity theory. Yet, rather surprisingly, we observe in that paper that the modern state-space theory of nonlinear circuits is built from non-monotone elements! Chua defines circuit elements from two fundamental laws of electricity: the derivative of charge equates current \( (\dot{q} = i) \) and the derivative of flux equates voltage \( (\dot{\phi} = v) \). Chua defines four key circuit elements by adding to those two laws one monotone relation between any two of the four variables. A monotone relation \( R(i, v) = 0 \) defines a resistor; \( C(v, q) = 0 \) defines a capacitor; \( L(\phi, i) = 0 \) defines an inductor; and \( M(q, \phi) = 0 \) defines a memristor. But we show in (Chaffey and Sepulchre, 2021) that out of those four elements, only the resistor defines a monotone relationship between current and voltage. As one-port circuits, such nonlinear capacitors, inductors, and memristors all fail to define monotone (or incrementally passive) voltage-current relationships.

What is wrong? Is it monotonicity that is non-physical, or should we revise Chua’s definition of circuit elements? Our view is that the passivity properties of elementary circuit elements should be trajectory independent, which pleads for a definition of circuit elements that are passive when linearized along arbitrary trajectories. Several recent papers have pointed out difficulties with the energy-dissipation properties of theoretical nonlinear circuit elements (see e.g. Jeltsema and van der Schaft (2020) for a recent perspective). Our view is that all difficulties have to do with the distinction between passivity and monotonicity, that is, the non-incremental and incremental forms of passivity.

6. MORE PUZZLING EXAMPLES

According to Chua’s definition, the state-space model of a current-controlled nonlinear capacitor is
\[ \dot{q} = i, v = c(q) \]
We show in (Chaffey and Sepulchre, 2021) that this model is non-monotone from current to voltage except for the linear capacitor \( v = Cq \). In that sense, the failure of monotonicity is generic for a nonlinear state-space model whose block diagram is the series interconnection of an integrator and a static monotone nonlinearity.

One could argue that any integrator should be monotone, whether linear or nonlinear: positive increments should always integrate positively...yet, it is rather common to model the saturation of an integrating process with a nonlinear monotone function (say sigmoidal) applied at the integrator output. Unfortunately, such a state-space model defines a monotone input-output operator only when the nonlinear function is linear.

The example that originally caught our attention to the problem exposed in this paper is the potassium current model derived by Hodgkin and Huxley in their seminal paper about the biophysical mechanism of action potentials (Hodgkin and Huxley, 1952). They propose the model
\[ \dot{n} = \alpha(V)n + \beta(V)(1 - n), \quad I = gK n^4(V - V_K), \]
where the nonlinear functions \( \alpha(\cdot) \) and \( \beta(\cdot) \) are monotone, the conductance \( g_K \) is a positive constant, and over a domain where \( V \geq V_K \) and \( n \in [0, 1] \). Again, it is not difficult to show that this state-space model does not define a monotone operator from voltage to current (van Waarde and Sepulchre, 2021). Yet, Hodgkin and Huxley derived their model by fitting a set of trajectories that all exhibit maximal monotonicity.

Those basic examples are intriguing. They suggest an inherent difficulty to model the external property of incremental passivity with state-space models, even when those state-space models only interconnect LTI passive transfer functions with monotone static nonlinearities.
7. DISCUSSION

There is no difficulty in formulating the incremental form of dissipativity, as a property that connects an internal incremental property (Lyapunov incremental stability) and an external incremental property associated to a particular signal supply (say incremental passivity). Yet, the examples discussed above show that the simplest nonlinear state-space models that ought to be incrementally dissipative are not. This difficulty calls for further attention. Perhaps it explains the somewhat incremental progress of incremental dissipativity theory over the last decade. This is in contrast with the increasing popularity of incremental stability and incremental passivity in control, machine learning, and optimization.

We conclude this paper by indicating two possible avenues of research to bridge internal and external incremental properties.

A first avenue is to broaden the internal representations of input-output operators beyond conventional state-space models. The recent work (van Waarde and Sepulchre, 2021) explores a data-driven construction of kernel-based models. The recent work (van Waarde and Sepulchre, 2021) explores a data-driven construction of kernel-based operators with desirable incremental external properties. In that approach, the dissipativity property is specified by incremental integral quadratic constraints. The dissipativity property regularizes the data fitting using the most natural concept of regularized least squares, bypassing the construction of a state-space model. If one adopts the viewpoint that the state-space model is a mere construct for algorithmic purposes, one could argue that it can be dispensed with. This is in line with recent developments of data-driven control or system-level synthesis (Wang et al., 2019). Finite impulse response representations of input-output operators offer an alternative to ordinary differential equations.

A second avenue is to retain the conventional state-space models but to restrict the external signal spaces. Classical input-output theory has focused on \( L^p \) signal spaces, but we know today that those spaces are too general for algorithmic purposes. It seems plausible that all the examples discussed in the present paper become monotone in suitable reproducing kernel Hilbert spaces. A well-known example of such space is the space of band-limited signals. Such restrictions of the signal space would acknowledge that no physical model is meant to cover an infinite range of amplitudes and frequencies. We control theorists take this limitation for granted when dealing with linear models but will perhaps need to acknowledge it more prominently when considering nonlinear models. It is once again striking that Zames 1960 dissertation discusses the importance of band-limited signals at length, twenty years ahead of wavelet theory and at a time when the theory of reproducing kernel Hilbert spaces was in its infancy. Research goes in circles.

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