Decoupling gravitational sources by MGD approach in Rastall gravity

S. K. Maurya\textsuperscript{1,*} and Francisco Tello-Ortiz\textsuperscript{2,†}

\textsuperscript{1}Department of Mathematical and Physical Sciences, College of Arts and Science, University of Nizwa, Nizwa, Sultanate of Oman
\textsuperscript{2}Departamento de Física, Facultad de ciencias básicas, Universidad de Antofagasta, Casilla 170, Antofagasta, Chile.

In the present work, we investigate the possibility of obtaining stellar interiors for static self-gravitating systems describing an anisotropic matter distribution in the framework of Rastall gravity through gravitational decoupling by means of minimal geometric deformation approach. Due to Rastall gravity breaks down the minimal coupling matter principle, we have provided an exhaustive explanation about how Israel-Darmois junction conditions work in this scenario. Furthermore, to obtain the deformed space-time, the mimic constraint procedure has been used. In order to check the viability of this proposal, we have applied it to the well known Tolman IV solution. A complete thermodynamic description of the effects introduced by the additional source is given. Additionally, the results have been compared with their similes in the picture of pure general relativity, pure Rastall gravity and within the framework of general relativity including gravitational decoupling. To perform the mathematical and graphical analysis we have taken the gravitational decoupling constant $\alpha$ and the Rastall’s parameter $\lambda$ as free parameters and the compactness factor $u$ to be 0.2. Applications to study neutron or quark stars are suggested by using this methodology.

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I. INTRODUCTION

Put forward for P. Rastall in 1972 \cite{1}, the so-called Rastall gravity theory can be seen as a modified gravity theory or as a generalization of Einstein gravity theory i.e General Relativity (GR from now on). Rastall’s proposal is based on the argument that the energy-momentum tensor, which fulfills the conservation law (null divergence) in a flat space-time (Minkowskian space-time), does not necessarily fulfill it in a curved background. The violation of Bianchi’s identity occurs through the introduction of a covariant term in the Einstein field equations through a dimensionless coupling constant $\lambda$. Specifically, this term corresponds to the Ricci’s scalar curvature $R$. Although the field equations given by Rastall do not have an associated Lagrangian density from which they can be obtained, these as a generalization of the GR equations respect the symmetries of the theory i.e, the group of all diffeomorphisms.

Despite that the new term is introduced by hand, its prevalence modifies not only the field equations but also the way of coupling material fields to the gravitational interaction. Clearly, the principle of minimal coupling matter breaks down. However, this brings with it new and intriguing contributions which can be useful to understand certain commonly studied phenomena such as cosmological issues, collapsed structures (black holes), stellar structures, gravitational waves, etc. In this direction, Rastall gravity is as competitive as other modified gravity theories such as f(R) and f(R,T) theories.

As it is well-known f(R) gravity theory was developed to address inflationary cosmological problems \cite{2,3}. Nowadays, f(R) gravity has been used in a wide context by addressing cosmological problems such as the existence of dark components (dark energy and dark matter), stellar structures among others. An incomplete but recent works concerning these issues can be found at the following references \cite{4}-\cite{22} (and references contained therein). Furthermore, to face related open questions in the cosmological scenario f(R,T) gravity is also a promising approach \cite{23}. In this respect f(R,T) gravity theory can be seen as an extension of f(R) gravity. Where $T$ stands for the trace of the energy-momentum tensor. The corrections coming from the trace of the energy-momentum tensor can be attributed to quantum effects \cite{23}. A wide variety of works available in the literature devoted to tackle the accelerated expansion of the Universe, energy conditions, stellar interiors and so on, it is found at the following references \cite{24,48}. In comparing f(R,T) gravity with Rastall theory, both theories break the minimal coupling matter scheme. However, the former smashes the minimal coupling matter principle by introducing matter and geometric terms (curvature invariants) while the second one by inserting only geometric objects, precisely the Ricci’s scalar. The effects introduced by the additional term have been studied extensively on different fronts to test at least theoretically how close the results are in comparison with the broad support that GR has. In this respect, the well known Tolman solutions have been extended into the Rastall gravity arena \cite{49} in order to contrast the behavior of the main salient features with the corresponding GR ones. Furthermore, to reinforce the study of stellar interiors in the Rastall scenario an anisotropic model in the background of Krori-Barua (KB) space-time was done in \cite{50} (in \cite{51} the same
authors considered KB space-time supplemented by an equation of state, specifically a quintessence model) and an isotropic compact object using conformal Killing vector technique was reported in [52]. Regarding the strong gravitational regime, remarkable solutions of black holes were presented in [53, 54]. These works motivate the investigation of the most exciting features and properties of these fascinating objects. For example the thermodynamic was studied in [55, 56] and rotating black holes were addressed at references [57, 58]. It is worth mentioning that in the context of black holes solutions GR and Rastall gravity share the same vacuum solution [1]. This is so because when the Rastall parameter goes to zero ($\lambda \to 0$), the Rastall contribution disappears. Moreover, in the limit $\lambda \to 0$ all GR results are recovered.

Based on these good antecedents, in the present work, we want to investigate the possibility of obtaining compact structures which serve to describe stellar interiors in the light of Rastall gravity theory. To accomplish it we employ gravitational decoupling by means of minimal geometric deformation (MGD) approach [60, 61]. This methodology was developed to deform Schwarzschild space-time [62, 63] in the Randall-Sundrum framework [64, 65]. In few words, gravitational decoupling through MGD algorithm consists in to extent well behaved isotropic solutions (however is not necessary to consider the input solution to be isotropic, it can be anisotropic, charged, etc.) of Einstein gravity theory to anisotropic domains. To do so one needs two main ingredients: i) add an extra source $\theta_{\mu\nu}$ to the energy-momentum tensor of the seed solution via a dimensionless coupling constant $\alpha$. The presence of this extra piece in conjunction with a spherically symmetric and static space-time (in Schwarzschild like-coordinates) leads to an intricate system of equations with seven unknown functions (if the seed solution is taking to be isotropic). To solve this complicated system and translate the isotropic fluid distribution to an anisotropic scenario ii) perform the MGD on one of the metric potentials (usually on the radial metric component $e^{\lambda}$). With this deformed potential in hand, the tangled system is separated into two simpler sets. The first one is the usual Einstein system and the second one contains the $\theta$-sector and the decoupler function $f(r)$ introduced in the MGD to split the system of equations. However, the latter one contains four unknown and only three equations. Then this system must be supplemented with extra information in order to close the problem. At this stage, a couple of comments are in order. First, after decoupling the systems, the resulting ones fulfill Bianchi’s identities (the conservation law of its corresponding energy-momentum tensors), meaning that the original source and the extra one only interact gravitationally. Second, the additional term could represent a scalar, vector or tensor field. Moreover, in principle, this new sector could not necessarily be described by GR. So, for a more detailed discussion of how this machinery works see sections [11] and [15] and the following references [66–72].

This last two years gravitational decoupling using MGD has attracted many adepts. In this respect some well known solutions (uncharged, charged) to Einstein field equations have been extended by using MGD [73–78]. Also, Black holes in $3 + 1$ (Schwarzschild outer space-time) [84] and $2 + 1$ (BTZ black hole) [85] dimensions were extended, the De (anti) Sitter space-time was worked in [80] and also the inverse problem was addressed in $3 + 1$ dimensions [83] and $2 + 1$ dimensions including cosmological constant [88]. Regarding another branches the method has been employed in cloud of strings [59], Klein-Gordon scalar fields as an extra matter content [90], extended to isotropic coordinates [91] and ultra compact Schwarzschild star, or gravastar [92]. Moreover, the method was widespread to include deformation on both metric potentials, it was called the extended-MGD [93, 94]. More recently gravitational decoupling was used to investigate higher dimensional compact structures [95] and spread out to the context of Lovelock [96] and $f(R, T)$ [97] gravity theories and in the cosmological scenario [98].

So, as we pointed out above our main goal is to introduce gravitational decoupling by means of MGD in the Rastall gravity picture. This will bring new insights on how compact objects behave by the inclusion of local anisotropies in the light of Rastall theory. Moreover, the possibility of comparing with other modified gravity theories and GR results in to order to check the viability from both theoretical and experimental point of view. To do so we have followed the same procedure given in [61]. The approach proposed in [61] in order to solve the $\theta$-sector is to impose some suitable restrictions on the thermodynamic quantities that characterize the seed solution (the isotropic pressure $p$ and the energy-density $\rho$) and the components of the extra source $\theta_{\mu\nu}$. These restraints are referred as mimic constraints. These mimic constraints yield to an algebraic or differential equation that allows to obtain the deformation function $f(r)$. Each mimic constraint leads to different anisotropic solution, for example the two common mimic constraints worked in the literature are: i) $p = \rho t^\alpha$ and ii) $\rho = \theta^\alpha_t$. This means that the radial component of the $\theta$-sector mimics the isotropic pressure and that the temporal one mimics the energy-density of the seed solution. Although there has not been a physical foundation to support the aforementioned choices, until now they have not presented any physical or mathematical inconsistency or any behavior that is detrimental to what is reported within the framework of general relativity. Moreover, studies conducted in general relativity concerning to interior solutions with anisotropic component have been favored (or reinforced) in a certain way when the described mechanism has been considered. However, in favor of these considerations, it should be noted that the mimic constraints have been imposed at the level of the equations of motion, which ensures the closure of the system of equations to be solved and also a correct physical and mathematical behavior of the $\theta$-sector components. Therefore it ends with a well-behaved solution. In addition, the virtues of each
In the present work, we have considered both mimic constraints. Due to Rastall gravity departs from GR only by the presence of the Ricci scalar coupled to the theory through the so-called Rastall’s parameter $\lambda$, then there are not higher-order derivative terms of the metric potentials (no more than two spatial derivatives) in the theory. This feature facilitates the gravitational decoupling and we have done it in such a way that the deformation function and the resulting components of $\theta$-sector contain the effects of Rastall gravity. This allows the new contributions to be compared exhaustively with respect to what is obtained in pure Rastall gravity and the results of RG and RG + MGD. In addition, to contrast our results we have taken the geometry of the inner space-time to be Tolman IV. This solution which describes a spherically symmetric and static object whose material content respects a perfect fluid distribution has already been extended to anisotropic domains by MGD \cite{61} and has also been studied in the Rastall gravity picture \cite{19}. One of the most notable features of Rastall’s theory of gravity is that any perfect fluid solution of Einstein’s equations is also a solution of it. Obviously, this is from the geometrical point of view because the material content is different due to the Rastall contribution. Furthermore, to handle with the numerical part we have taking as free parameters the gravitational decoupling constant $\alpha$ and the Rastall’s parameter $\lambda$ and considering a compactness factor within the allowed range for compact stars to be $u = M_0/R = 0.2$. In this respect, it is worth mentioning that the coupling constant $\alpha$ plays an important role in the behavior on the main salient features and has a relevant incidence on the mass of the compact structure. Although we have only considered positive $\alpha$ values, in order to obtain a physically acceptable solution. The negative values of parameter $\alpha$ are not completely discarded. Particularly the choice $\rho = \theta^0_0$ for some solutions such as Heintzmann IIa \cite{73} and Durgapal-Fuloria \cite{73}, require that $\alpha$ takes negative values to avoid non-physical behavior. In our case negative $\alpha$ values are forbidden in considering both mimic constraints. So, the final deformed Tolman IV space-time is completely agree with what is expected in this research field. Finally, we want to mention that it is the first time that gravitational decoupling by MGD is used in the Rastall gravity scenario.

So, the article is organized as follows: Sec. \ref{sec:II} we revisited in brief the main ingredients of Rastall gravity theory. In Sec. \ref{sec:III} the field equations for multiple sources are presented. Sec. \ref{sec:IV} discuss the gravitational decoupling via minimal geometric deformation scheme in the Rastall gravity framework. Next, in Sec. \ref{sec:V} the matching conditions are analyzed and extensively discussed in this area. In Sec. \ref{sec:VI} Tolman IV space-time is projected into the anisotropic domain using the methodology previously discussed. Furthermore, the problem is faced by employing the mimic constraint approach, in order to determine the decoupler function $f(r)$ and the new sector, the $\theta_{\mu\nu}$ one. In the following section, Sec. \ref{sec:VII} the principal physical, observational and mathematical implications of the obtained model are addressed. Finally, Sec. \ref{sec:VIII} provides some remarks of the study reported in this article.

II. \textbf{REVISITING IN SHORT: RASTALL GRAVITY THEORY}

The main idea behind Rastall’s proposal \cite{1} is to abandon the free divergence of energy-momentum tensor in a curved space-time. Explicitly it reads

$$\nabla_\mu T^\mu_\nu \neq 0. \quad (1)$$

So, this non-conserved stress-energy tensor introduces an unusual non-minimal coupling between matter and geometry. Specifically, this non-minimal coupling is carried out into the theory by the following assumption on the divergence of the energy-momentum tensor

$$\nabla_\mu T^\mu_\nu = \lambda \nabla_\nu R. \quad (2)$$

In the above expression $\lambda \equiv g^\mu\nu R_{\mu\nu}$ stands for the Ricci’s scalar and $\lambda$ is the so-called Rastall’s parameter which is used to depict the distraction from GR and measures the affinity of the space-time geometry to couple with matter field in a non-minimal fashion. The assumption given by Eq. \ref{eq:2} is completely consistent with the following field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa_R (T_{\mu\nu} - \lambda R g_{\mu\nu}), \quad (3)$$

where $\kappa_R$ is the Rastall gravitational constant. It is worth mentioning that in the limit $\lambda \rightarrow 0$ Einstein’s field equations are recovered, then the energy-momentum tensor \ref{eq:2} is conserved. However, the above field equations \ref{eq:3} can be expressed in the following form

$$G_{\mu\nu} = \kappa_R T^{(\text{eff})}_{\mu\nu}, \quad (4)$$

and in some sense one recast the standard result $\nabla_\nu T^{(\text{eff})}_{\mu\nu} = 0$. Now, by taking the trace of Eqs. \ref{eq:4}
the Ricci scalar is writing as
\[ R = \frac{\kappa_R T}{4\lambda \kappa_R - 1}, \] (5)
then the effective stress-energy tensor reads
\[ T_{\mu\nu}^{(\text{eff})} = T_{\mu\nu} - \frac{\gamma T}{4\gamma - 1} g_{\mu\nu}, \] (6)
where \( \gamma = \kappa_R \lambda. \) As it is more commonly we have assumed \( \kappa_R = 1 \) from now on, then \( \gamma = \lambda. \) From Eq. (6) one can infer several constraints. Again taking \( \lambda = 0 \) the Rastall sector disappears and one recast GR. If a traceless energy-momentum tensor is considered, such as the electromagnetic one, Rastall contribution is totally vanished because \( T = 0. \) Additionally, \( \lambda = 1/4 \) represents non-physical situation. So, this value must be excluded in order to avoid inconsistencies. From now on we will consider \( T_{\mu\nu} \) to be a perfect fluid matter distribution, which is given by
\[ T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - g_{\mu\nu}. \] (7)

We utilize a comoving fluid 4-velocity \( u^{\sigma} = e^{-\nu/2} \delta^{\sigma}_{\mu} \), and \( \rho \) and \( p \) are representing the energy-density and the isotropic pressure respectively. So, the components of \( T_{\mu\nu}^{(\text{eff})} \) are given by
\[ T_{\mu\nu}^{(0)\text{eff}} = \rho^{(\text{eff})} = \frac{(3\lambda - 1) \rho + 3\lambda p}{4\lambda - 1}, \] (8)
\[ T_{\mu\nu}^{(1)\text{eff}} = T_{\mu\nu}^{(2)\text{eff}} = T_{\mu\nu}^{(3)\text{eff}} = -p^{(\text{eff})} = -\frac{(\lambda - 1) p + \lambda \rho}{4\lambda - 1}. \] (9)

In obtaining the expressions (8) and (9) we have used \( T = \rho - 3p. \) Moreover, as before setting \( \lambda = 0 \) yields us to Eq. (7). As we will see later the isotropic quantities \( \rho \) and \( p \) will be separated carrying out in their expressions the corresponding Rastall contributions.

### III. FIELD EQUATIONS: MULTIPLE SOURCES

In this section we describe the field equations for multiple matter sources. So, the standard field equations are given by
\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}^{(\text{tot})}, \] (10)
where \( T_{\mu\nu}^{(\text{tot})} \) stands for
\[ T_{\mu\nu}^{(\text{tot})} = T_{\mu\nu}^{(\text{eff})} + \alpha \theta_{\mu\nu}. \] (11)

The new sector \( \theta_{\mu\nu} \) always can be seen as corrections to the theory and be consolidated as part of an effective energy-momentum tensor. This extra source can be a scalar, vector or tensor fields and will produce anisotropy in self-gravitating systems. It is coupled to the matter sector through a dimensionless constant parameter i.e, \( \alpha. \) On the other hand, \( T_{\mu\nu}^{(\text{eff})} \) represents the usual matter sector, that is isotropic, anisotropic, or charged distributions, among others. In the present case \( T_{\mu\nu}^{(\text{eff})} \) is given by Eqs. (8)- (9). As we are interested in the study of spherically symmetric and static fluid spheres, next we regard the most general line element in the standard Schwarzschild like coordinates \( \{t, r, \phi, \theta\} \) to be
\[ ds^2 = e^{\nu(r)} dt^2 - e^{\eta(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \] (12)
The staticity of this space-time is ensured by considering \( \nu \) and \( \eta \) as functions of the radial coordinate \( r \) only. Putting together equations (8), (9), (13) and (12) one arrives at the following set of equations
\[ e^{-\eta} \left( \frac{\eta'}{r} - \frac{1}{r^2} \right) \frac{1}{r^2} = \rho^{(\text{eff})} + \alpha \theta_{t}^{2}, \] (13)
\[ e^{-\eta} \left( \frac{\eta'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = p^{(\text{eff})} - \alpha \theta_{r}^{2}, \] (14)
\[ \frac{e^{-\eta}}{4} \left( 2\nu'' + \nu' + 2\theta'' - \frac{\eta'}{r} - \nu' \theta' \right) = p^{(\text{eff})} - \alpha \theta_{\phi}^{2}. \] (15)

where
\[ \rho^{(\text{eff})} = \frac{(3\lambda - 1) \rho + 3\lambda p}{4\lambda - 1}, \] (16)
\[ -p^{(\text{eff})} = -\frac{(\lambda - 1) p + \lambda \rho}{4\lambda - 1}. \] (17)

The corresponding conservation law \( \nabla^{\mu} T_{\mu\nu}^{(\text{tot})} = 0 \) associated with the system (13)-(15) reads
\[ - \frac{dp^{(\text{eff})}}{dr} - \alpha \left[ \frac{\nu'}{2} \left( \theta_{t}^{2} - \theta_{r}^{2} \right) - \frac{d\theta_{r}^{2}}{dr} + \frac{2}{r} (\theta_{r}^{2} - \theta_{t}^{2}) \right] \] (18)
\[ - \frac{\nu'}{2} (p^{(\text{eff})} + \rho^{(\text{eff})}) = 0. \]

It is found that the system of non-linear differential equations (13)-(15) consists of seven unknown functions, the metric potentials \( \{\eta, \nu\} \), the thermodynamic observables \( \{\rho^{(\text{eff})}, p^{(\text{eff})}\} \) and the components of the extra source \( \{\theta_{t}^{2}, \theta_{r}^{2}, \theta_{\phi}^{2}\} \). In order to find these unknown we adopt a systematic approach. Furthermore, for the system (13)-(15), the matter content (total energy density, total radial pressure and total tangential pressure) can be identified as
\[ \rho^{(\text{tot})} = \rho^{(\text{eff})} + \alpha \theta_{t}^{2}, \] (19)
\[ p_{t}^{(\text{tot})} = p^{(\text{eff})} - \alpha \theta_{r}^{2}, \] (20)
\[ p_{r}^{(\text{tot})} = p^{(\text{eff})} - \alpha \theta_{\phi}^{2}. \] (21)

It is clear that an anisotropic behaviour arises into the system due to the present of the \( \theta \)-sector if \( \theta_{t}^{2} \neq \theta_{r}^{2}. \) So, in order to measure the anisotropy behaviour we defined the anisotropy factor as follows
\[ \Delta = \bar{p}_{t}^{(\text{tot})} - \bar{p}_{r}^{(\text{tot})} = \alpha (\theta_{r}^{2} - \theta_{\phi}^{2}). \] (22)
At this stage the system of Eqs. [13]-[15] could indeed be treated as an anisotropic fluid, which would require one to consider five unknown functions, namely, the two metric functions \( \nu \) and \( \eta \), and the total functions in Eqs. [19] - [21]. However, we are going to implement a different approach, as explained below.

**IV. GRAVITATIONAL DECOUPLING: A MGD APPROACH**

As said before, gravitational decoupling by MGD scheme becomes a simple and powerful tool to extend spherically and static isotropic fluid solutions to anisotropic domains [61]. To see how this approach works let us start by turning off the coupling \( \alpha \), so we are describing a perfect fluid solution given by \( \{ \xi, \mu, \rho^{(\text{eff})}, p^{(\text{eff})} \} \), being \( \xi \) and \( \mu \) the corresponding metric functions. The metric [12] now reads

\[
\text{ds}^2 = e^{\xi(r)} dt^2 - \frac{dr^2}{\mu(r)} - r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( \mu(r) = 1 - \frac{2m}{r} \) is the standard GR expression containing the mass function of the fluid configuration. Next to see the effects of the \( \theta \)-sector on the perfect fluid distribution we turn on the coupling \( \alpha \). These effects can be encoded in the geometric deformation undergone by the perfect fluid geometry \( \{ \xi, \mu \} \) in Eq. [23], namely

\[
\xi \rightarrow \nu = \xi + \alpha h, \quad \mu \rightarrow e^{-\eta} = \mu + \alpha f,
\]

where \( h \) and \( f \) are the deformations introduced in the temporal and radial metric component, respectively. It’s worth mentioning that the foregoing deformations are purely radial functions, this feature ensures the spherical symmetry of the solution. The MGD scheme consists in set off either \( h \) or \( f \). In this opportunity we will set \( h = 0 \), it means that the temporal component remains unchanged and the anisotropy lies on the radial component [61]. So, we have

\[
\mu(r) \rightarrow e^{-\eta(r)} = \mu(r) + \alpha f(r).
\]

Upon replacing Eq. [26] in the equations [13]-[15], the system splits into two sets of equations. The first one corresponds to \( \alpha = 0 \) it means perfect fluid matter distribution

\[
-\frac{\mu'}{r} - \frac{\mu}{r^2} + \frac{1}{r^2} = \rho^{(\text{eff})}
\]

\[
\mu \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = p^{(\text{eff})}
\]

\[
\frac{\mu}{4} \left( 2\nu'' + \nu'^2 + 2\frac{\nu'}{r} \right) + \frac{\mu'}{4} \left( \nu' + \frac{2}{r} \right) = p^{(\text{eff})}.
\]

From now on we shall call the above system of equations the Einstein-Rastall system. It can be solved for \( \rho \) and \( p \) by using Eqs. [16] and [17], in order to express these quantities as functions of the metric potentials only [49]. Explicitly we have

\[
-\frac{\mu'}{r} - \frac{\mu}{r^2} + \frac{1}{r^2} - \lambda \left[ -\mu \left( \frac{4}{r^2} + \frac{3\nu'}{r} \right) + \frac{4}{r^2} - \frac{\mu'}{r} \right] = \rho,
\]

\[
\mu \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} + \lambda \left[ -\mu \left( \frac{4}{r^2} + \frac{3\nu'}{r} \right) + \frac{4}{r^2} - \frac{\mu'}{r} \right] = p.
\]

As was pointed out earlier, both \( \rho \) and \( p \) after some algebraic manipulations in their own expressions contain the Rastall information as it was expected. Besides by putting \( \lambda = 0 \) in Eqs. [30]-[31] one recovers the original GR field equations for isotropic matter distributions. Furthermore, by adding [30] to [31] one regains the usual inertial mass density \( \rho + p \) which is given by

\[
\rho + p = \frac{\mu \nu' - \mu'}{r}.
\]

Another interesting point to be noted here, is that the isotropic condition is exactly the same like in GR i.e

\[
4(1 - \mu) + 2\nu (\mu' - \mu \nu') + r^2 (2\mu \nu'' + \mu \nu'^2 + \mu' \nu') = 0.
\]

Equation [34] says that any solution describing a perfect fluid matter distribution in GR is also a solution in the arena of Rastall gravity theory. Obviously there is a subtlety, since both GR and Rastall theory share only the geometrical content but not the material one, is in this sense that “any” solution to Einstein theory of gravity can be seen as a solution in the gravitational Rastall framework. So, the other set of equations corresponding to the factor \( \theta \) are given as,

\[
-\frac{f'}{r} - \frac{f}{r^2} = \theta_i^r,
\]

\[
-f \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) = \theta_r^\nu.
\]

\[
-\frac{f}{4} \left( 2\nu'' + \nu'^2 + 2\frac{\nu'}{r} \right) - \frac{f'}{4} \left( \nu' + \frac{2}{r} \right) = \theta_r^\nu.
\]

The set of equations [30]-[31] and [35]-[37] satisfies the following conservation equations,

\[
\frac{\nu'}{2} \left( \rho + p \right) + \frac{dp}{dr} - \frac{\lambda}{4\nu - 1} \frac{d}{dr} \left( \rho - 3p \right) = 0,
\]

\[
-\frac{\nu'}{2} \left( \theta_i^r - \theta_r^\nu \right) + \frac{d\theta_i^r}{dr} - \frac{2}{r} \left( \theta_r^\nu - \theta_r^\nu \right) = 0.
\]
We note that the linear combination of conservation equations (38) and (39) via the coupling constant \( \alpha \) provides the conservation equation for the energy momentum tensor \( T^{\mu}_{\nu}(\text{tot}) = T^{\mu}_{\nu}(\text{eff}) + \alpha \theta^{\mu}_{\nu} \), which is given as,

\[
- \frac{dp}{dr} + \alpha \left[ \frac{\nu'}{2} \left( \theta^t_{\nu} - \theta^r_{\nu} \right) - \frac{d\theta^r}{dr} + \frac{2}{r} \left( \theta^\phi_{\nu} - \theta^\theta_{\nu} \right) \right] - \frac{\nu'}{2} \left( \rho + p \right) + \frac{\lambda}{4\lambda - 1} \frac{d}{dr} \left( \rho - 3p \right) = 0.
\]

The Eq. (40) is the same expression as Eq. (18) but in an explicit form. Furthermore, as can be seen there is an extra term (the last one) in (40), the so called Rastall force (or simple the Rastall contribution). This additional term could in principle be attractive or repulsive in nature, due to its behaviour depends on the sign of the Rastall parameter \( \lambda \).

At this point it is necessary to comment that from now on the total energy-energy tensor \( T^{\mu\nu}_{\text{tot}} \) will be defined by the following components

\[
T^{\mu\nu}_{\text{tot}}(r) = \rho(r) + \alpha \theta^t_{\nu}(r), \quad T^{\mu}_{\nu}(r) = p(r) - \alpha \theta^r_{\nu}(r), \quad p^i_{\text{tot}}(r) = p(r) - \alpha \theta^\phi_{\nu}(r),
\]

where \( \rho \) and \( p \) are given by Eqs. (30) and (31), respectively. These equations contain additional the geometric terms provided by the Rastall contribution. In this way, as we will see in the following sections, there will be a full affect of the Rastall sector in the decoupler function \( f(r) \) and consequently in the \( \theta \)-sector, as expected.

V. EXTERIOR SPACE-TIME: JUNCTION CONDITIONS

A crucial aspect in the study of stellar distributions is the junction conditions. These provide smooth matching of interior \( \mathcal{M}^- \) and exterior \( \mathcal{M}^+ \) geometries at the surface \( \Sigma \) (defined by \( r = R \)) of the stellar object to investigate some significant characteristics of their evolution. To study how the junction conditions work in this context we will assume that the inner stellar geometry \( \mathcal{M}^- \) is given by the MGD metric,

\[
ds^2 = e^{\nu(r)} dt^2 - \left( 1 - 2 \frac{\bar{m}(r)}{r} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where the interior mass function is given by

\[
\bar{m}(r) = m(r) - \frac{\alpha}{2} f(r).
\]

Now the internal manifold \( \mathcal{M}^- \) should be joined in a smoothly way with outer space-time. This exterior manifold in principle could contains some contributions coming from the \( \theta \)-sector. So, it means that the exterior space-time surrounding the compact structure is no longer a vacuum space-time. The most general outer manifold is described by

\[
ds^2 = e^{\nu^+(r)} dt^2 - e^{\eta^+(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]

To match the internal configuration (44) with the exterior one (10) we employ the well known Israel-Darmois (ID hereinafter) junction conditions (92) (for a recent and more clear discussion of how these conditions work see (101) and (103)). These conditions are the most general ones. The ID matching conditions involve the first and second fundamental forms. The first fundamental form express the continuity of the metric potentials across the boundary \( \Sigma \). More specifically, the metric potentials describe the intrinsic geometry of the manifolds. So, the first fundamental form reads

\[
\left[ ds^2 \right]_\Sigma = 0,
\]

concisely

\[
e^{\nu^-(r)}|_{r=R} = e^{\nu^+(r)}|_{r=R},
\]

and

\[
1 - \frac{2M_0}{R} + \alpha f(R) = e^{\eta^+(r)}|_{r=R},
\]

being \( M_0 = \bar{m}(R) \) the total gravitational mass contained by the fluid sphere. The second fundamental form is related with the continuity of the extrinsic curvature \( K_{\mu\nu} \) induced by \( \mathcal{M}^- \) and \( \mathcal{M}^+ \) on \( \Sigma \). The continuity of \( K_{rr} \) component across \( \Sigma \) yields to

\[
\left[ p^{(\text{tot})}_{r}(r) \right]_\Sigma = [p(r) - \alpha \theta^r_{\nu}(r)]_\Sigma = 0.
\]

At this stage some comments are in order. First, \( p^{(\text{tot})}_{r} \) has a little different fashion in comparison with the expression (16) since \( \rho \) and \( p \) were separated implying that the Rastall terms are no longer contained in \( p^{(\text{eff})}_{r} \) through \( \lambda \) as shown Eq. (9). Now the terms coming from the Rastall sector are encoded in separate expressions for \( \rho \) and \( p \) given by Eqs. (30) and (31). From this point of view it is clear how Rastall contribution comes into the field equations. Moreover, from now on we shall denote the Rastall input as follows

\[
F_\lambda = \lambda \left[ -\mu - \frac{4\nu'}{r^2} + \frac{3\nu'^2}{r^2} \right].
\]

It should be noted that the form of \( F_\lambda \) depends on the choice of \( T_{\mu\nu} \), which in our case is given by Eq. (7) describing a perfect fluid matter distribution. Hence, \( p(r) \) in Eq. (50) is given by Eq. (31). Second, in this way the Rastall sector will come into the \( \theta \)-sector through the decoupler function \( f(r) \) (as we will see later) in order to see the effects on it. So Eq. (50) reads

\[
[p(r) - \alpha \theta^r_{\nu}(r)]_{r=R^-} = [-F_\lambda(r) - \alpha \theta^r_{\nu}(r)]_{r=R^+}.
\]
Equation (52) tells us that the outer space-time receives contributions from the $\theta$-sector, as well as from the Rastall non-minimal coupling matter assumption. In this respect, in the study of compact structures within the framework of modified gravity theories such as $f(R)$, the exterior space-time receives contributions from the inclusion of higher-order derivative terms coming from the Ricci scalar, Ricci tensor and invariant constructed with the Riemann tensor. In principle, these contributions can alter or introduce some modifications on the usual junction conditions. Moreover, the outer manifold could be different from the usual ones i.e Schwarzschild vacuum solution, Reissner-Nordström, for example. At this stage and based on the previous discussion, a couple of comments are pertinent in order to clarify how to proceed in modified type gravity theories. In this direction, Capozziello et al. have argued that in the $f(R)$ domain the mass-radius profile undergoes modifications due to the presence of high order curvature terms such as $R^2$, $R^3$, etc. Besides, in it was discussed the well-known Israel-Darmois matching conditions in the framework of $f(R)$ gravity in considering both the isotropic and the anisotropic compact matter distributions. They conclude that Israel-Darmois matching conditions are not satisfied at all in the $f(R)$ gravity arena. However, in the present situation, one could be dropped out the Rastall contribution $F_\alpha^\nu$ from the external space-time. To do so, one needs to consider an outer geometry free from material content i.e, a vanishing energy-momentum tensor $T_{\mu\nu} = 0$. Then from Eqs. (4) and (5) one arrives to

$$G_{\mu\nu} = 0. \quad (53)$$

The above expression implies that both Einstein and Rastall gravity theories share exactly the same vacuum solution i.e, the exterior Schwarzschild solution. If one wants to consider contributions provided from the Rastall sector, the outer space-time is no longer vacuum since it is filled by an effective cosmological constant describing a De-Sitter (anti) space-time. So, Eq. (52) becomes to

$$\frac{1}{R^2} + \frac{\nu'(R)}{R} = \alpha g(R) \left( \frac{1}{R^2} + \frac{2M}{R^3} \left( 1 - \frac{2M}{R} \right) \right). \quad (57)$$

It should be noted that if the geometric deformation function $g(r)$ of the outer manifold is taken to be null, then one recovers the original Schwarzschild exterior solution. In consequence Eq. (57) leads to the condition

$$p_{\nu}^{(\text{tot})}(R) = p(R) + \alpha f(R) \left( \frac{1}{R^2} + \frac{\nu'(R)}{R} \right) = 0. \quad (58)$$

Equation (58) is an important result since the compact object will therefore be in equilibrium in a true exterior space-time without material content (vacuum) only if the total radial pressure at the surface vanishes. The obtained condition (58) determines the size of the object i.e the radius $R$ which means that the material content is confined within the region $0 < r < R$. Furthermore the continuity of the remaining components $K_{\theta\theta}$ and $K_{\phi\phi}$ leads to

$$\tilde{m}(R) = M_0. \quad (59)$$

Equation (59) is the total effective mass contained in the sphere.

VI. STELLAR INTERIOR: TOLMAN IV MODEL

In this section we solve the set of equations (59)-(61) by imposing some suitable constraints on the $\theta_{\mu\nu}$ components in order to obtain the deformation function $f(r)$ and then compute the full energy-momentum-tensor $T_{\mu\nu}^{(\text{tot})}$. Among all the possibilities, to tackle the system of equations (55)-(57) we follow the same procedure as Ovalle. The imposition of some extra information is necessary in order to close the system of equations (59)-(61). Furthermore, to obtain the deformation function $f(r)$ also is necessary provide a seed solution satisfying equations (59)-(61). To illustrate how gravitational decoupling by means of MGD works in the Rastall gravity scenario we apply it to the well known Tolman IV solution. This space-time was already studied in the context of MDG in and in the framework of Rastall theory. So, it allows us to compare the resulting deformed solution immersed in an anisotropic scenario with previous results already obtained and therefore establish if it is plausible the study of the compact structure containing anisotropic matter distributions in the arena of Rastall gravity + gravitational decoupling by MGD approach. Before to proceed we present the well known Tolman IV space-time within the Rastall framework which
is described by the following metric potentials
\[ e^{\nu(r)} = B^2 \left( 1 + \frac{r^2}{A^2} \right), \quad \mu(r) = \frac{(1 - \frac{r^2}{A^2}) (1 + \frac{r^2}{A^2})}{(1 + \frac{2r}{A})}, \]
and characterized by the following thermodynamic observables (in the Rastall context)
\[ \rho = \frac{1}{C^2 (A^2 + 2r^2)^2} \left\{ 2\lambda \left[ r^2 \left( 2C^2 - 11A^2 \right) + 3 \left( A^4 + 4r^4 \right) \right] + 3A^4 + A^2 \left( 3C^2 + 7r^2 \right) + 2\nu \left( C^2 + 3r^2 \right) \right\}, \]
\[ p = \frac{1}{C^2 (A^2 + 2r^2)^2} \left\{ 2\lambda \left[ r^2 \left( 11A^2 - 2C^2 \right) + 3 \left( A^4 + 4r^4 \right) \right] + A^2 + 2r^2 \left( C^2 - A^2 - 3r^2 \right) \right\}. \]

This constraint implies that the stress-energy tensor for the seed solution coincides with the anisotropy in the radial direction. Consequently Eq. (61) and Eq. (60) are equal. Thus, this yields to an algebraic general expression for the deformation function
\[ f(r) = -\mu(r) + \left[ \frac{1}{r^2} - F_{\lambda(r)} \right] \left[ \frac{\nu'}{r} + \frac{1}{r^2} \right]^{-1}, \]
where \( F_{\lambda(r)} \) and \( \mu(r) \) are given by Eq. (61) and Eq. (60), respectively and \( \nu' \) can be obtained from Eq. (60). Then, the general minimally deformed radial metric potential is expressed as
\[ e^{-\eta} = (1 - \alpha) \mu(r) + \alpha \left[ \frac{1}{r^2} - F_{\lambda(r)} \right] \left[ \frac{\nu'}{r} + \frac{1}{r^2} \right]^{-1}. \]

The resulting expression of \( f(r) \) after inserting the corresponding elements in Eq. (65), is given by
\[ f(r) = \frac{-r^2}{C^2 A^2 (2r^2)^2 (A^2 + 3r^2)} \left[ \left( A^2 + r^2 \right) \left( 6A^4 \lambda + 22A^2 r^2 \lambda - 4C^2 r^2 \lambda + 24r^4 \lambda - 5A^2 r^2 + 2C^2 r^2 - 6r^4 - A^4 + A^2 C^2 \right) \right]. \]

In general, the deformed Tolman IV solution by virtue of \( \uparrow \) can be expressed as
\[ ds^2 = B^2 \left( 1 + \frac{r^2}{A^2} \right) dt^2 - \left[ (1 - \alpha) \left( 1 - \frac{r^2}{A^2} \right) \left( 1 + \frac{r^2}{A^2} \right) \right] \left[ \frac{\nu'}{r} + \frac{1}{r^2} \right]^{-1} dr^2 - r^2 d\Omega^2. \]

A. \( \theta \)-effects: Mimicking the pressure for anisotropy

The so called mimic constraints are some restrictions imposed at the level of the field equations \( \uparrow \) and \( \uparrow \) after introduce the decoupler mechanism \( \uparrow \). In principle, these choices lead to well-behaved solutions, that is, free of undesired physical and mathematical behaviors such as singularities, non-decreasing thermodynamical functions, violation of the condition of causality, among others. However, other options can be considered, for example a direct and adequate representation for the geometric deformation function \( f(r) \) \( \uparrow \) which satisfies the basic requirements of physical and mathematical admissibility, or relate only \( \theta \)-sector the components through a barotropic, polytropic or linear equation of state. In this opportunity an acceptable interior solution is deduced when forcing the associated radial pressure \( \theta_r \) to mimic the isotropic pressure \( p(r) \). Explicitly
\[ \theta_r(r) = p(r). \]

This constraint implies that the stress-energy tensor for the seed solution coincides with the anisotropy in the radial direction. Consequently Eq. (61) and Eq. (60) are equal. Thus, this yields to an algebraic general expression for the deformation function
\[ f(r) = -\mu(r) + \left[ \frac{1}{r^2} - F_{\lambda(r)} \right] \left[ \frac{\nu'}{r} + \frac{1}{r^2} \right]^{-1}, \]
where \( F_{\lambda(r)} \) and \( \mu(r) \) are given by Eq. (61) and Eq. (60), respectively and \( \nu' \) can be obtained from Eq. (60). Then, the general minimally deformed radial metric potential is expressed as
\[ e^{-\eta} = (1 - \alpha) \mu(r) + \alpha \left[ \frac{1}{r^2} - F_{\lambda(r)} \right] \left[ \frac{\nu'}{r} + \frac{1}{r^2} \right]^{-1}. \]

The resulting expression of \( f(r) \) after inserting the corresponding elements in Eq. (65), is given by
\[ f(r) = \frac{-r^2}{C^2 A^2 (2r^2)^2 (A^2 + 3r^2)} \left[ \left( A^2 + r^2 \right) \left( 6A^4 \lambda + 22A^2 r^2 \lambda - 4C^2 r^2 \lambda + 24r^4 \lambda - 5A^2 r^2 + 2C^2 r^2 - 6r^4 - A^4 + A^2 C^2 \right) \right]. \]

In general, the deformed Tolman IV solution by virtue of \( \uparrow \) can be expressed as
\[ ds^2 = B^2 \left( 1 + \frac{r^2}{A^2} \right) dt^2 - \left[ (1 - \alpha) \left( 1 - \frac{r^2}{A^2} \right) \left( 1 + \frac{r^2}{A^2} \right) \right] \left[ \frac{\nu'}{r} + \frac{1}{r^2} \right]^{-1} dr^2 - r^2 d\Omega^2. \]

Next, following the discussion in section \( \uparrow \) the constant parameters, namely \( A, B \) and \( C \) defining the interior solution can be obtained from Eqs. (61) and (64) leading to
\[ e^{\nu(r)} |_{r=R^-} = [\mu(r) + \alpha f(r)] |_{r=R^-} = 1 - \frac{2M}{R^+}. \]

where the Schwarzschild mass \( \bar{M} \) coincides with total mass \( M_0 \) contained in the spheres at the boundary \( \Sigma \). Furthermore,
\[ (1 - \alpha) p(r) |_{r=R^-} = 0. \]

This last expression (70) imposes a natural constraint on the free parameter \( \alpha \) given by
\[ \alpha < 1, \]
in order to preserve \( p_t^{(\text{tot})} > p_r^{(\text{tot})} \) at all points inside the collapsed structure which ensures in addition \( \Delta > 0, \)
what prevents the system from unwanted behavior such as instabilities. Moreover, from \(22\) we obtain
\[
p_{t}^{(\text{tot})}(r) = p_{r}^{(\text{tot})}(r) + \frac{\alpha r^2}{C^2(A^2 + 2r^2)^2(A^2 + 3r^2)^2}\left[8A^6\lambda + 4A^4C^2\lambda + 24A^4r^2\lambda + 12A^2r^4\lambda - 24C^2r^4\lambda + 12A^2C^2r^2 + 12C^2r^2 + 3A^4C^2\right],
\]
 remembering that \(p_{r}^{(\text{tot})}(0) = (1 - \alpha)p\). As it is observed \(p_{t}^{(\text{tot})}(0)\) imposes a lower bound over \(\alpha\) i.e., \(\alpha > 0\). Thus, the positiveness of the total tangential pressure throughout the compact object is ensured. Therefore we have
\[
0 < \alpha < 1,
\]
in order to get a well behaved stellar interior. Another interesting point to be noted here is that the condition \(70\) leads to
\[
C = \frac{1}{2R^2 + A^2 - 4R^2\lambda}\left[\left(4R^2\lambda - 2R^2 - A^24R^2\right)\left(6A^4\lambda + 22A^2R^2\lambda + 24R^2\lambda - A^4 - 5A^2R^2 - 6R^4\right)\right]^{1/2},
\]
which shows that \(C\) is \(\alpha\) independent. Moreover a detailed computation from \(70\) shows that the remaining parameters namely \(A\) and \(B\) also are \(\alpha\) independent when one chooses the constant \(64\) (these expressions are to long to be displayed here, for this reason we only give the appropriate comments). So, expressions \(69\) and \(71\) are the sufficient and necessary conditions to obtain the full set of constant parameters \(A, B\) and \(C\) describing the interior solution. On the other hand, the remaining thermodynamic observable \(\rho^{(\text{tot})}\) can be obtained as follows
\[
\rho^{(\text{tot})}(r) = \rho(r) + \alpha\theta_{t}^{(r)},
\]
where \(\rho(r)\) is by Eq. \(60\) and \(\theta_{t}^{(r)}\) has the following expression
\[
\theta_{t}^{(r)} = \frac{1}{C^2(A^2 + 2r^2)^3(A^2 + 3r^2)^2}\left[18A_{10}^8\alpha + 146A^8ar^2 - 20A^6C^2ar^2 + 522A^6ar^4 - 72A^4C^2ar^4 + 1014A^4ar^6 - 60A^2C^2ar^6 + 1020A^2ar^8 - 24C^2ar^8 + 432ar^{10} - 3A^{10} + 3A^6C^2 - 31A^6r^2 + 16A^6C^2r^2 - 125A^6r^4 + 29A^4C^2r^4 - 249A^4r^6 + 24A^2C^2r^6 - 252A^2r^8 + 12C^2r^8 - 108r^{10}\right],
\]
The anisotropy factor is given by the following expression
\[
\Delta(r) = \frac{\alpha r^2}{C^2(A^2 + 2r^2)^2(A^2 + 3r^2)^2}\left[8A^6\lambda + 4A^4C^2\lambda + 24A^4r^2\lambda + 12A^2r^4\lambda - 24C^2r^4\lambda + 12A^2C^2r^2 + 12C^2r^2 + 3A^4C^2\right].
\]
It should be noted that at the center of the compact configuration \(\Delta(0) = 0\). This is so because at the center of the star \(p_{t}^{(\text{tot})}(0) = p_{r}^{(\text{tot})}(0)\). Besides, \(p_{t}^{(\text{tot})} > p_{r}^{(\text{tot})}\) and \(p_{t}^{(\text{tot})} > 0\) everywhere inside the configuration implying \(\Delta > 0\) at all points in the inner solution. Fig. \(4\) illustrates the behaviour of the main salient physical quantities that characterize the system. It is worth mentioning that the anisotropic behavior will occur if both pressures must are decreasing in nature in the inner region. In fact, the incompatibility compared to the isotropic pressure causes the energy-density to be altered. Obviously, the equilibrium of the system under the action of the gravitational gravity and hydrostatic repulsion is modified. To close this section we would like to mention that this anisotropic solution is not unique. As was pointed out before the different election and relations on the \(\theta\) components and the decoupler function \(f(r)\) can be assumed. In the next section, a different constrains is considered yielding to a different anisotropic solution.

B. \(\theta\)-effects: Mimicking the density for anisotropy

Another way to close system \(35\) and obtain a physically and mathematically admissible solution, is to consider that the isotropic density given by \(30\) mimics its ”simile” of the anisotropic sector given by \(35\). Then we have
\[
\theta_{t}^{(r)}(r) = \rho(r).
\]
So, by equating Eqs. \(35\) and \(35\) we arrive to a general expression for the decoupler function \(f(r)\) given by
\[
f(r) = \mu(r) - 1 + \frac{1}{r} \int F_{\lambda}(r) r^2 dr + D,
\]
being \(D\) an integration constant. To avoid divergent behavior in the stellar interior we set \(D = 0\). Thus \(74\) becomes
\[
f(r) = \mu(r) - 1 + \frac{1}{r} \int F_{\lambda}(r) r^2 dr.
\]
Thus the deformed radial metric potential \(e^{-\eta}\) is given by
\[
e^{-\eta} = (1 + \alpha)\mu(r) + \alpha \left(\frac{1}{r} \int F_{\lambda}(r) r^2 dr - 1\right).
\]
FIG. 1: **Mimic Constraint** \( p(r) = \theta' \). To obtain the trend of the principal thermodynamic observables we have considered throughout the study the following mass-radius ratio \( M_0/R = 0.2 \). Moreover, the red curve (dashed) representing Rastall + MGD corresponds to \( \alpha = 0.2 \) and \( \lambda = -0.4 \), for the blue (dashed-dotted) one corresponding to pure Rastall gravity \( \alpha \) and \( \lambda \) are 0.0 and -0.4 respectively. Next, the green (short-dashed) line corresponding to GR + MGD takes \( \alpha = 0.2 \) and \( \lambda = -0.4 \). Finally, the black curve (solid) representing GR theory \( \alpha = \lambda = 0 \).

Upper row: **Left panel** illustrates the monotonic behaviour from the center to the boundary of the total radial pressure at all points inside the structure. As it is observed this quantity vanishes at surface. The **Right panel** shows the trend of the total tangential pressure everywhere inside the compact object.

Lower row: **Left panel** exhibits the behaviour of the total energy-density. Finally, the **Right panel** displays a comparison between the total radial and total tangential pressure. It should be noted that the presence of anisotropies cause the pressures values to drift apart.

Therefore the general deformed Tolman IV solution is written as

\[
\begin{align*}
\text{ds}^2 &= B^2 \left( 1 + \frac{r^2}{A^2} \right) \text{d}t^2 - \left[ (1 + \alpha) \mu(r) + \alpha \left( \frac{1}{r} \int F(r) r^2 \text{d}r - 1 \right) \right] \text{d}r^2 - r^2 \text{d}\Omega^2. \\
&= B^2 \left( 1 + \frac{r^2}{A^2} \right) \text{d}t^2 - \left[ (1 + \alpha) \mu(r) + \alpha \left( \frac{1}{r} \int F(r) r^2 \text{d}r - 1 \right) \right] \text{d}r^2 - r^2 \text{d}\Omega^2.
\end{align*}
\] (82)

As before, the parameters \( A \), \( B \) and \( C \) are obtained from the junction conditions. However, the imposition of constraint \[78\] slightly changes the information obtained from condition [50]. Now from [60] one gets an expression for constant \( C \) in terms of the radius \( R \), the constant \( A \), and the free parameters \( \alpha \) and \( \lambda \). The consequences of the dependence of the constant \( C \) with respect to \( \alpha \) and \( \lambda \) will be a matter of the following section.

The rest of the principal variables are obtained after inserting the following decoupler function \( f(r) \)

\[
\begin{align*}
f(r) &= \frac{1}{8rC^2(A^2 + 2r^2)} \left[ 3\sqrt{2}\lambda A \left( A^4 + 2A^2C^2 + 2A^2r^2 \\
&+ 4C^2r^2 \right) \arctan \left( \frac{r\sqrt{2}}{A} \right) - 6A^4r\lambda - 12A^2C^2r\lambda \\
&+ 8A^2r^3\lambda - 16C^2r^3\lambda + 32r^5\lambda - 8A^2r^3 - 8C^2r^3 - 8r^3 \right].
\end{align*}
\] (83)
and by virtue of (78), the thermodynamic observables can be computed as follows

\[ p_r^{(\text{tot})} = p(r) - \alpha \theta_r', \quad (84) \]

\[ p_t^{(\text{tot})} = p(r) - \alpha \theta_\varphi', \quad (85) \]

and by virtue of (78)

\[ \rho^{(\text{tot})} = (1 + \alpha) \rho(r), \quad (86) \]

where \( \rho(r) \) and \( p(r) \) are given by Eqs. (30) and (31), respectively. The final expressions are too long to be displayed here, for this reason we have omitted them. At this stage we have completed our mathematical and graphical analysis with Fig. 2. In the next section, we are going to discuss in details the mathematical and physical implications of \( \theta \)-sector in the Rastall gravity scenario.

**VII. ANALYSIS AND DISCUSSION**

In this section we analyze the physical consequences of the results obtained in [VIA] and [VIIB]. So, the pertinent comments for the resulting interior solution obtained in VIA are

1. Regarding the junction conditions. The first fundamental form carries an interesting conclusion about the extra piece introduced by the decoupler function in the radial metric potential \( e^{-\lambda} \). In performing the matching condition with the vacuum exterior space-time described by Schwarzschild solution (see Eq. (89)), the geometric sector describing the deformed part has totally vanished. This means that the total mass contained in the sphere, at the boundary \( \Sigma \) exactly coincides with the Schwarzschild mass i.e., \( \tilde{m}(R) = m(R) \rightarrow M_0 = \tilde{M} \). So, from Eq. (40) one recovers the usual expression for the gravitational mass \( e^{-\lambda} = 1 - 2m(r)/r \) (because \( \alpha r f(r)/2 = 0 \)). However, \( m(r) \) is given by

\[ m(r) = 4\pi \int \rho^{(\text{tot})} r^2 dr = 4\pi \int (\rho + \alpha \theta_r') r^2 dr. \quad (87) \]

As it is observed the energy-density has an extra contribution coming from the \( \theta \)-sector. Therefore, the seed energy-density \( \rho \) is altered by the presence of this additional contribution. Nonetheless, the mass remains the same with or without \( \theta \)-sector. From the physical point of view, as it is shown by the left panel of Fig 1 (lower row) in the framework of Rastall+MGD (red curve) the total energy-density dominates the other scenarios. The object is denser at the center in the Rastall+MGD background, but towards the surface, the total energy-density is dominated by the other scenarios. This behavior suggests that the same mass is redistributed around the core of the star and diminishing towards the surface of the object.

2. From the second fundamental form, explicitly given by Eq. (70) one gets and expression for the constant \( C \) independent of the free parameter \( \alpha \). This implies that \( C \) takes the same values in considering pure Rastall gravity and Rastall+MGD approach. Moreover, as \( p_t^{(\text{tot})} = (1 - \alpha) p(r) \), the resulting central pressure is below the central pressure in Rastall gravity as illustrates Fig. 1 in the upper row (left and right panels). This \( \alpha \)-independence is a feature of the mimic constraint (63). In addition as \( \alpha \) is restricted to belong to \((0, 1)\) in order to obtain an admissible interior solution, by imposing (64) the resulting object’s core will be denser when adding MGD approach to other theories and the central pressure will be less than the seed central pressure.

3. Local anisotropies arising in the system due to the presence of the extra source \( \theta_{\mu\nu} \) introduce a positive anisotropy factor \( \Delta(r) \) at all points inside the compact structure. This is a very important issue in the study of compact configurations because a positive anisotropy factor introduces a repulsive force that counteracts the gravitational gradient. Therefore preventing the compact object from collapsing below its Schwarzschild radius. In addition stability and balance mechanisms are enhanced [105, 106]. Besides, as was pointed out by Gokhroo and Mehra [107], a positive anisotropy factor allows to build more compact objects. In the present study, this feature is depicted in the right panel (lower row) of Fig. 1. It is observed that both the total radial and total tangential pressure coincide at the center and then drift apart towards the boundary of the object.

4. Another relevant point is the macro information of the compact structure i.e., the mass and radius, obtained from astrophysical observations. Both quantities are related by mean of the compactness factor \( u \), which is related to the surface gravitational redshift \( z_s \). Explicitly

\[ z_s = \frac{1}{\sqrt{1 - 2u}} - 1. \quad (88) \]

In considering isotropic fluid spheres \( u \) has and upper bound known as Buchdahl’s limit [108] given
FIG. 2: Mimic Constraint $\rho(r) = \theta'_t$. To obtain the trend of the principal thermodynamic observables we have considered throughout the study the following mass-radius ratio $M_0/R = 0.2$. Moreover, the red curve (dashed) representing Rastall + MGD corresponds to $\alpha = 0.2$ and $\lambda = -0.4$, for the blue (dashed-dotted) one corresponding to pure Rastall gravity $\alpha$ and $\lambda$ are 0.0 and −0.4 respectively. Next, the green (short-dashed) line corresponding to GR + MGD takes $\alpha = 0.2$ and $\lambda = -0.4$. Finally, the black curve (solid) representing GR theory $\alpha = \lambda = 0$. Upper row: Left panel illustrates the monotonic behaviour from the center to the boundary of the total radial pressure at all points inside the structure. As it is observed this quantity vanishes at surface. The Right panel: shows the trend of the total tangential pressure everywhere inside the compact object. Lower row: Left panel exhibits the behaviour of the total energy-density. Finally, the Right panel displays a comparison between the total radial and total tangential pressure. It should be noted that the presence of anisotropies cause the pressures values to drift apart.

by $u \leq 4/9$. By taking the equality the maximum allowed gravitational surface redshift for an isotropic spherical matter distribution is $z_s = 2$. Nevertheless, Ivanov studies [109] suggested that when anisotropies are included in the stellar interior, the surface gravitational redshift increases its maximum value in comparison with its isotropic counterpart. Obviously, $z_s$ can not be arbitrarily large and its maximum value depends on the mechanism to introduce anisotropies into the system. In this respect as we discussed above, the mimic constraint (64) does not alter the total mass (keeping the same radius) of the configuration, only is redistributed within the object. So, in this case, the observed $z_s$ does not change despite the system contains local anisotropies [61, 77].

Now we proceed with the appropriated comments for the results obtained in VI B. In this respect, the mimic constraint (78) gives more interesting results than the mimic constraint (64).

1. By imposing the mimic constraint (78), the resulting junction conditions provide new insights in considering the total mass of the compact object. This time the extra piece $f(r)$ contributes to the matching conditions. This means that the coupling con-
stant $\alpha$ has an active role. So, the observed mass is no longer the same. This is so because

$$\dot{m}(r) = 4\pi \int \rho^{(\text{tot})} r^2 dr = 4\pi \int (1 + \alpha) \rho r^2 dr,$$

(89)

then

$$\dot{m}(r) = (1 + \alpha) m(r).$$

(90)

Therefore, by virtue of (88) the mass function $\dot{m}(r)$ mimics the mass $m(r)$ of the seed solution. However, the maximum value that the mass and energy density of the compact structure can take is strongly constrained by the values taken by the dimensionless constant $C$. This is because parameter $C$ now depends on $\alpha$ and $\lambda$. Thus it is evident from the condition of null pressure on the surface of the object, which is different from the case previously considered where the constant $C$ only depended on the Rastall parameter $\lambda$. The behavior of the constant $C$, in this case, determines the energy-density behavior in the center of the object and consequently the value of its mass. So, to obtain a denser object, we need to go in the direction of increasing energy-density and mass. This is possible by assigning small values to parameter $\alpha$, which implies that the constant $C$ decreases in the module. Nevertheless, the value of parameter $\alpha$ cannot be arbitrarily small since the effects of anisotropy on the stellar interior would be negligible. In addition, negative values of $\alpha$ would introduce instabilities in the system since the total tangential pressure would be less than the total radial pressure, which represents a physically inadmissible situation.

2. With respect to the central pressure, it increases if the magnitude of $\alpha$ decreases. If $\alpha$ is very small (close to zero) the anisotropy from $\theta$-sector will be negligible. If $\alpha$ is negative then the anisotropy factor $\Delta$ will be too, introducing into the system a force attractive in nature. In conclusion, $\alpha$ is bound to be positive definite. However, it should be noted that the restrictions imposed by choice (78) on the parameter $\alpha$ depend on the chosen seed solution. For example, for the Heintzman IIA [73] and Durgapal-Fuloria [74] isotropic models, studied in the framework of GR+MGD, the mimic constraint [72] only allows negative values for $\alpha$, which ensures a physically acceptable solution.

3. Finally, it is important to highlight that in the present case the observational differences between isotropic and anisotropic distributions are evident. Due to the total mass varies by directly depending on $\alpha$, then compactness factor $u$ changes. This fact alters the surface gravitational redshift $z_s$ value. It follows immediately from the definition of $z_s$

$$z_s(\alpha) = \frac{1}{\sqrt{1 - 2u(\alpha)}} - 1,$$

(91)

where the $\alpha$ dependency is explicit, if $\alpha$ increases then the mass grows, in consequence $u$ increases. Then, the factor $1/\sqrt{1 - 2u(\alpha)}$ increases implying that $z_s$ grows its value as it is expected when the compact object becomes denser.

VIII. CONCLUDING REMARKS

We extended gravitational decoupling via minimal geometric deformation approach into the Rastall gravity scenario. To illustrate how this methodology works in the background of Rastall gravity, the well known Tolman IV space-time describing a spherically symmetric and static perfect fluid sphere was analyzed. This model was already studied in the light of general relativity + minimal geometric deformation scheme [61] and in the arena of pure Rastall theory [49]. In both cases, the resulting model respects the general requirements in order to describe a well-behaved solution.

Since Rastall theory of gravity contains an extra term which deviates the attention from general relativity behavior, in this work we have investigated the effects of this extra term and the possibility to obtaining compact structures which could serve to describe neutron or quark stars. Due to the presence of this additional term, the minimal coupling matter breaks down and in consequence, Bianchi’s identities are violated (the conservation law of the energy-momentum tensor). This issue could in principle modified the junction condition mechanism as happened in f(R) gravity, for example. In this respect, we have discussed extensively how Rastall contribution remains inside the compact configuration, allowing the implementation of the most general matching conditions i.e, the Israel-Darmois junction conditions [99, 100]. Moreover, as was pointed out by Rastall [1], his proposal and Einstein theory share the same vacuum solution, the outer Schwarzschild space-time.

To translate the Tolman IV solution to an anisotropic domain in the Rastall framework, we have followed the same approach given in [61]. This approach consists in imposing some suitable conditions relating the thermodynamic seed observables with the corresponding components of the new sector i.e, the $\theta$-sector. With this extra information at hand, the problem is closed because the decoupler function $f(r)$ and the full $\theta$-sector is determined. The methodology followed in this work in order to tackle the system of equations (89) is known as the mimic constraints approach. Among all the possibilities the most common ones worked in the literature are: i) $p(r) = \rho r^2$, ii) $\rho = \rho^t$, that is the $r-r$ component of the $\theta$-sector mimics the seed pressure $p(r)$ and the $t-t$ one mimics the seed energy-density $\rho(r)$. However, it should be noted that an adequate decoupler function $f(r)$ can be imposed in order to close the problem (for more details see [73, 74, 92]). The advantage of both proposals is evident. Regarding the first one, it allows obtaining the decoupler function $f(r)$ in an easy way. This is so
because, one obtains after equate the corresponding field equations for \( p(r) \) and \( \theta^r_r \) an algebraic equation (see Eq. 30). The second choice does not lead to an algebraic equation, but to a first-order differential equation (Eq. 31). At this point, it is worth mentioning that in the case of general relativity + minimal geometric deformation, the Rastall contribution \( F_\lambda \) is not there. So, obtain the decoupler function \( f(r) \) is easier than our case, due to the Rastall piece \( F_\lambda \) strongly depends on the metric potentials \( \mu(r) \) and \( \nu(r) \). So, when the \( t-t \) component of the \( \theta \)-sector mimics the seed energy-density \( \rho \) this additional term could introduce some mathematical complications.

The emergence of Rastall term after impose the mimic constraints, is due to after split the system of equations (15)-(17) by introducing the minimal geometric deformation (20), the resulting seed sector (27)-(29) was solved in order to express \( p(r) \) and \( \rho(r) \) in a separate way. The resulting expressions for \( p(r) \) and \( \rho(r) \), (30) and (31) respectively, contain the usual Einstein terms and the Rastall contribution. This additional term in coupled to the field equations via a dimensionless constant \( \lambda \), the so-called Rastall’s parameter [1]. Clearly, in the limit, \( \lambda \rightarrow 0 \) Einstein’s gravity theory is recovered. Thus, the fact to separate \( p(r) \) and \( \rho(r) \) introduces the Rastall contribution into the \( \theta \)-sector through the deformation function \( f(r) \). Therefore, the incidence of Rastall contributions is evident. Since, not only Rastall’s parameter \( \lambda \) is affecting the dynamic of the solution, but also the extra geometrical terms.

Mimic constraint methodology does not introduce new information into the problem. Because, these constraints are imposed at the level of the field equations, relating them after separate the system of equations (27)-(29) by mean of minimal geometric deformation approach. The consistency of these choices is reflected in the obtained solutions. Where in both cases the evolution of thermodynamic parameters reveals an appropriate behavior as dictated by the basic requirements associated with the study of compact structures. Furthermore, the mimic constraint grasp plays an important role in some observational parameters such as the surface gravitational redshift \( z_s \). As it is well known the surface gravitational redshift relates the macro observables features of any compact configuration i.e. the mass and radius. In this respect, Ivanov studies [109] suggests that the \( z_s \) changes in magnitude when anisotropies are present in the material content of the compact object. Moreover, Böhmer and Harko [110] discussed the effects on the compactness factor in the anisotropic matter distribution case. Notwithstanding in the present study, the mimic constraint \( p(r) = \theta^r_r \) does not modify the total mass of the compact object, it only redistributes the mass inside the stellar interior. Consequently, the compactness factor \( u \) and surface gravitational redshift \( z_s \) remain unchanged, which makes it difficult to distinguish between an object whose material content is isotropic from an anisotropic one. In distinction with the case \( p(r) = \theta^r_r \) where the total mass of the object is modified, therefore the observational implications are different.

To show how the anisotropic effects introduced by the \( \theta \)-sector works in the Rastall framework, we have revisited the behavior of the main salient features in the arena of general relativity, general relativity + gravitational decoupling minimally deformed and pure Rastall theory. In this concern we have fixed the space parameter to be \( \{ u, \alpha, \lambda \} = \{ \text{RG} \ \{ 0.2, 0, 0 \}; \text{RG} + \text{MGD} \ \{ 0.2, 0.2, 0 \}; \text{RT} \ \{ 0.2, 0, -0.4 \}; \text{RT} + \text{MGD} \ \{ 0.2, 0.2, -0.4 \} \} \) (RT means Rastall theory). From fig. 1 (these plots correspond to \( p(r) = \theta^r_r \) solution) it is clear that the PRT+MGD radial and tangential pressure dominate the corresponding ones in the picture of GR and GR+MGD. Nonetheless, PRT dominates all frames. Particularly, in comparing RT with RT+MGD, the final radial pressure in RT+MGD represents only a portion of the pressure of the RT, indeed \( p(r)_{\text{(RT+MGD)}} = (1-\alpha) p(r)_{\text{(RT)}} \). On the other hand, the final energy-density in the RT+MGD dominates all scenarios. So, by using \( p(r) = \theta^r_r \) the final configuration is denser than RG, RG+MGD and RT+MGD. However, the increase in energy-density does not reflect a change in the total mass of the object (as discussed earlier). Fig. 2 (these plots correspond to \( p(r) = \theta^r_r \) solution), the salient radial and tangential pressure in the RT+MGD picture are dominated by GR, RG+MGD and RT, only the salient energy-density dominates over GR and RG+MGD but is dominated by RT framework. Although the first solution has a central energy-density greater than the second solution, in the latter the total mass, the compactness factor and the surface gravitational redshift undergo modifications which, as discussed, have strong observational implications. Finally, both solutions present a positive anisotropy factor \( \Delta \). In fact, this characteristic avoids the system to undergo unstable behavior.

As a final remark, we want to highlight two things. First, it is possible to obtain well behaved stellar interiors in the framework of Rastall gravity by using gravitational decoupling via minimal geometric deformation approach. The two families of solutions found in this work satisfy and share all the physical and mathematical properties required in the study of compact configurations, which serve to understand the behavior of real astrophysical objects such as neutron stars, for example. Second, it was found that Rastall theory is a promising scenario to study the existence of compact structures described by an anisotropic matter distribution, which results can be contrasted with the well-posed general relativity theory.

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