Emergence of skewed non-Gaussian distributions of velocity increments in isotropic turbulence

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Skewness and non-Gaussian behavior are key features of the distribution of short-scale velocity increments in isotropic turbulent flows [1][2]. Yet, the physical origin of the asymmetry and the form of the heavy tails remain elusive. Here we describe the emergence of such properties through an exactly solvable stochastic model with a hierarchy of multiple scales of energy transfer rates. By a statistical superposition of a local equilibrium distribution weighted by a background density, the increments distribution is given by a novel class of skewed heavy-tailed distributions, written as a generalization of the Meijer-G-functions. Excellent agreement in the multiscale scenario is found with numerical data of systems with different sizes and Reynolds numbers. Remarkably, the single scale limit provides poor fits to the background density, highlighting the key role of the multiple scales. Our framework can be applied to describe the challenging emergence of skewed distributions in complex systems.

The negative skewness and non-Gaussian behavior of the distribution of velocity increments between close points in an isotropic turbulent flow have long figured among the most intriguing features of the turbulence phenomenon [1][2]. Though the negative asymmetry can be derived from the Navier-Stokes equations and has been connected to the time reversal symmetry breaking [3], elucidating its physical origins and determining the form of the heavy tails persist as long-standing open questions.

Indeed, understanding the statistical properties of velocity fluctuations has always been, and remains, an essential issue in turbulence. A significant step in this direction was Kolmogorov’s theory of turbulence [1]. One of its few exact results is the so-called 4/5-law: \[ \langle \delta v_r \rangle^3 = -\frac{2}{5} \langle \epsilon \rangle r \], where \( \delta v_r = v(x+r) - v(x) \) represents the longitudinal velocity increment and \( \langle \epsilon \rangle \) is the mean energy dissipation rate. For homogeneous and isotropic turbulent flows, in which \( \langle \delta v_r \rangle = 0 \), Kolmogorov’s 4/5-law implies negative skewness and non-Gaussian statistics of velocity increments. Considerable effort has been also devoted to investigate the scaling properties of higher-order structure functions, \( \langle \delta v_r \rangle^n \sim r^{s_n}, n > 3 \), for which no exact results are known [1]. A renewed interest has arisen in the study of the increments distribution itself [4][10]. It is known that velocity increments for large separations tend to be Gaussian distributed, whereas non-Gaussian behavior is observed at short scales [1]. Non-Gaussian effects were found to appear at Reynolds numbers smaller than initially thought [11].

Here we report on a statistical approach to the distribution of short-scale velocity increments in isotropic turbulent flows that describes the emergence of both the negatively skewed asymmetry and non-Gaussian heavy tails, with very nice agreement with numerical turbulence data of systems featuring distinct sizes and Reynolds numbers. Our work is based on two key tenets of turbulence theory [1][2], namely the intermittency phenomenon and the concept of energy cascade, whereby energy is transferred from large to small eddies until dissipation by viscous forces at the shortest (Kolmogorov) scale.

Our intermittency model is built upon a hierarchy of multiple coupled scales of energy transfer rates [12][13]. The marginal distribution of short-scale velocity increments \( P(\delta v_r) \) is related to the energy transfer rate \( \epsilon(f) \) at a larger scale \( f \) through a statistical superposition of the conditional distribution \( P(\delta v_r|\epsilon(f)) \), weighted by a background distribution \( f(\epsilon(f)) \) obtained in exact closed form from our model. By considering \( P(\delta v_r|\epsilon(f)) \) as a Gaussian with nonzero mean characterized by an asymmetry parameter \( \mu \), we obtain an exact \( P(\delta v_r) \) in the form of a novel class of skewed functions with stretched exponential heavy tails. These newly defined functions constitute a generalization of the Meijer-G functions and, to our knowledge, have never been considered in the literature.

The theoretical predictions emerging from this multiscale scenario are found to be in excellent agreement with turbulence data from two extensive and independent numerical simulations of the Navier-Stokes equations. Remarkably, a poor agreement is found if only a single scale is considered. The stretched exponential heavy tails are also shown to be related to the multiple scales. Our framework can be applied to describe the emergence of skewed distributions in other complex systems, such as financial markets [14] and biological systems [15].

We work under the formalism of a unified hierarchical approach to describe the statistics of fluctuations in multiscale complex systems [12][13]. This framework is an extension to multiscale systems of the compounding [4][6] or superstatistics [16] approaches to describe complex
fluctuating phenomena. In this formalism, the probability distribution of the relevant signal—say, the velocity increments—at short scales is given by a statistical superposition of a large-scale conditional distribution weighted by the distribution of internal degrees of freedom related to the slowly fluctuating environment,

$$P(\delta v_r) = \int_0^\infty P(\delta v_r|\varepsilon_\ell)f(\varepsilon_\ell)d\varepsilon_\ell,$$

(1)

where the variable $\varepsilon_\ell$ characterizes the local equilibrium at scale $r$. In the turbulence context, $\varepsilon_\ell$ can be associated with the energy transfer rate from scale $\ell$ towards smaller scales, with $\ell \gg r$ if scales are assumed to be largely separated (see Methods). The complex statistical properties of the turbulent state are thus captured by the weighting density $f(\varepsilon_\ell)$, which incorporates the effect of the fluctuating energy flux (intermittency). Following Refs. [7, 17] we consider the variance of $\delta v_r$ over a region of size $\ell$ as a proxy for the energy transfer rate $\varepsilon_\ell$ [15]. We note, however, that in our approach the scale $\ell$ is not initially known and must be determined from the velocity data. A distinctive feature of our formalism is that the distribution $f(\varepsilon_\ell)$ in (S25) is not prescribed a priori—as in previous approaches [4, 6, 7, 16, 17, 19]—, but rather is calculated below from a hierarchical intermittency model.

Experimental and theoretical studies on isotropic turbulent flows indicate [4, 6, 7, 10, 17, 19, 20] that the conditional distribution $P(\delta v_r|\varepsilon_\ell)$ in (S25) is given by a Gaussian with variance $\varepsilon_\ell$. For the sake of simplicity, a Gaussian with zero mean is often considered in theoretical turbulence models [12, 13, 16, 21], leading to symmetric (i.e., non-skewed) distributions $P(\delta v_r)$.

Here we introduce a model for $P(\delta v_r|\varepsilon_\ell)$ that yields an asymmetric (skewed) distribution $P(\delta v_r)$ which can be written in exact closed form in terms of certain special functions, see below. More specifically, we consider

$$P(\delta v_r|\varepsilon_\ell) = \frac{1}{\sqrt{2\pi\varepsilon_\ell}} \exp \left[ -\frac{(\delta v_r - \langle \delta v_r|\varepsilon_\ell \rangle)^2}{2\varepsilon_\ell} \right],$$

(2)

where $\langle \delta v_r|\varepsilon_\ell \rangle$ is the conditional mean velocity. We make the choice $\langle \delta v_r|\varepsilon_\ell \rangle = \mu(\varepsilon_\ell - \langle \varepsilon_\ell \rangle)$, where $\mu$ is a flow related asymmetry parameter. (It is possible to introduce a dimensionless parameter $b = |\mu|\sqrt{\langle \varepsilon_\ell \rangle}$, but for our purposes here it is more convenient to work with $\mu$ itself; see below.) This dependence of $\langle \delta v_r|\varepsilon_\ell \rangle$ on $\varepsilon_\ell$ ensures $\langle \delta v_r \rangle = 0$ for any $\mu$, as required for isotropic turbulence. The possibility of producing asymmetric distributions by compounding distributions with nonzero mean has also been discussed in Refs. [7, 17, 22], though no specific model for the resulting distribution was presented.

We now turn to the calculation of the background distribution $f(\varepsilon_\ell)$ in Eq. (S25). The scale $\ell$ is assigned to the $N$-th level of the turbulence hierarchy ($\varepsilon_\ell \leftrightarrow \varepsilon_N$), that is, $\ell = L/2^N$, where $L$ is the integral scale and $N$ is the number of levels in the cascade down from $L$ to $\ell$. Our hierarchical intermittency model is defined by the following set of $N$ stochastic differential equations:

$$d\varepsilon_i = -\gamma_i (\varepsilon_i - \varepsilon_{i-1}) \left( 1 + \alpha^2 \varepsilon_i - \varepsilon_{i-1} \right) dt + \kappa_i \sqrt{\varepsilon_i \varepsilon_{i-1}} dW_i,$$

(3)

for $i = 1, \ldots, N$, where $\varepsilon_i \geq 0$ represents the energy transfer rate from the hierarchy level $i$ to smaller scales, $\gamma_i > 0$ is a relaxation rate, $\kappa_i > 0$ gives the strength of the multiplicative noise (and hence of the intermittency) in the hierarchical level $i$, and $W_i$ denotes a Wiener process. The parameter $\alpha > 0$ can be associated with a residual dissipation in the inertial range (see below), which is usually neglected in phenomenological cascade models.

Physically, the deterministic term in Eq. (3) represents the coupling between adjacent scales, whereas the stochastic term emerges from the complex interactions among all scales and is necessary for intermittency [13]. We further observe that a rescaling of variables $\varepsilon_i \rightarrow \varepsilon_i$ properly leaves the model dynamics unchanged, which is a required property for a multiplicative cascade model [13]. Moreover, one can verify that if $\alpha = 0$ then $\langle \varepsilon_i \rangle = \varepsilon_0$ for $t \rightarrow \infty$, whereas for $\alpha \neq 0$ it can be shown (see Supplementary Information) that $\langle \varepsilon_i \rangle/\langle \varepsilon_{i-1} \rangle = 1 - \alpha^2$, as $\alpha \rightarrow 0$, thus showing that the energy flux leaving the scale $i$ is actually smaller than that entering it. The model above is perhaps the simplest stochastic dynamical model of intermittency that allows for an analytic solution and incorporates a small degree of dissipation in the cascade, so that it can describe intermittency even at not so high Reynolds numbers where residual dissipation might be relevant. (Higher-order terms could in principle be added in [3], but they should not affect our findings significantly and, besides, destroy the exact solvability of the model.)

By denoting $f(\varepsilon_N) \equiv f(\varepsilon_i)$ in (S25), the stationary solution of (3) (see Methods) leads to the background density

$$f(\varepsilon_N) = \frac{1}{\varepsilon_0} \beta_{N,0} R_{0, N}^{N,0} \left( \frac{p - 1}{\omega/2} \right) \left( \frac{\beta_{N,0}}{\varepsilon_0} \right),$$

(4)

where $p = \beta(1 - \alpha^2)$, $\omega = 2\alpha\beta$, $\beta \equiv 2\gamma_i/\kappa_i^2$, $p \equiv (p, \ldots, p)$, $\omega \equiv (\omega, \ldots, \omega)$, $K_p(x)$ is the modified Bessel function of second kind, and $R_{0, N}^{m,n}$ is a new special function defined in the Supplementary Information. The function $R_{0, N}^{m,n}$ can be viewed as a generalization of the Meijer $G$-function $G_{p,q}^{m,n}$, with the gamma functions $\Gamma(\nu)$ essentially replaced by $K_p(x)$ in the Mellin transform [2].

Substituting (2) and (4) into (S25), with some properties of the $R$-functions (Supplementary Information), we find

$$P_N(\delta v_r) = ce^{\mu y} R_{0, N+1}^{N+1,0} \left( \left\{ \frac{p - 1}{2}, \frac{\beta_{N,0}}{2\varepsilon_0} \right\} \right),$$

(5)
with \( y = \delta v_r + \mu(\varepsilon_N) \) and \( c = (2/\pi \varepsilon_0 \alpha^N)^{1/2}/[K_p(\omega)]^N \).

For a given \( N \), the distribution above has four parameters, namely: \( \alpha, \beta, \varepsilon_0, \) and \( \mu \), with the physical meaning of \( \alpha, \varepsilon_0 \) and \( \mu \) discussed above. The dimensionless constant \( \beta \) together with \( \varepsilon_0 \) define a typical scale \( (2\varepsilon_0/\beta^N)^{1/2} \) for the fluctuations of the velocity increments \( \delta v_r \), so that a larger relative noise (intermittency) strength and/or a lower relaxation rate (implying a smaller \( \beta \)) consistently yields a broader \( P_N(\delta v_r) \).

We note that the single scale case, \( N = 1 \) in Eq. (S16), corresponds to the generalized hyperbolic distribution, which has been applied [24] in the analysis of turbulent velocity increments. It seems, however, that the \( N > 1 \) multiscale scenario has not been considered before. We also highlight that \( P_N(\delta v_r) \) given by Eq. (S16) is negatively (positively) skewed for \( \mu < 0 \) (\( \mu > 0 \)), whereas for \( \mu = 0 \) a symmetric (non-skewed) distribution arises.

The large-\( |\delta v_r| \) behavior of \( P_N(\delta v_r) \) evidences the non-Gaussian tails. Indeed, for \( N > 1 \) and \( \mu < 0 \) we obtain

\[
P_N(\delta v_r) \sim |y|^\theta \exp \left[ -\beta N \left( \frac{y}{\varepsilon_0|\mu|} \right)^{1/N} \right] g(\delta v_r), \tag{6}
\]

where \( \theta = p + 1/(2N) - 3/2 \) and \( g(\delta v_r) = 1 \) for \( \delta v_r \to -\infty \) and \( g(\delta v_r) = e^{-2|\mu|y} \) for \( \delta v_r \to +\infty \). The negatively skewed marginal distribution displays an asymptotic behavior to the right \( (\delta v_r \to +\infty) \) with exponential decay, while the left tail is heavier, in the form of a modified stretched exponential. In contrast, for \( N = 1 \) modified exponential tails emerge on both sides: \( P_{N=1}(\delta v_r) \sim z^{p-1}e^{\eta y - \kappa z} \), where \( \kappa = \sqrt{\mu^2 + 2\beta/\varepsilon_0} \) and \( z = \sqrt{y^2 + 2\alpha^2\varepsilon_0} \) for \( \delta v_r \to \pm \infty \). Stretched exponentials have for long been used to fit turbulence data [5] despite the lack of a theoretical basis for this. Our model thus provides a reasonable physical framework for the emergence of such heavy tailed distributions.

We now apply the above formalism to the study of turbulent flows. We start with the analysis of an isotropic turbulence dataset [25] generated by the extensive direct numerical simulation (DNS) of the Navier-Stokes equations for a system with \( 1024^3 \) lattice points in a periodic cube and Taylor-based Reynolds number \( \text{Re}_\lambda \approx 433 \). The dataset was obtained from the Johns Hopkins University turbulence research group’s database [25]. To test our intermittency model and show that it applies well to turbulence data, we shall analyze here the velocity increments \( \delta v_r \) computed at the smallest resolved scale \( r \), which lies in the near dissipation range as \( r \approx 2.14\eta \) [25], where \( \eta \) is the Kolmogorov scale. A more complete analysis including other scales \( r \) is left for future studies.
FIG. 3: Distributions of velocity increments and background density: model results and DNS data with 4096³ lattice points. Distribution of velocity increments (main panel) and background density (inset) for a system with 4096³ points and Reynolds number Reλ ≈ 600 [26]. The nice fit of the DNS data (circles) to the theoretical model (red lines) occurs for N = 5 scales. A poor fit is noticed in green lines for N = 1.

As described in Methods, we first need to determine the asymmetry parameter μ and the optimal window size M of the large dataset {δv_i(t)} over which the variance of δv_i(t) is supposed to remain approximately constant. In this DNS dataset, the optimization procedure provides μ = −1.82 and M = 19 (yielding ⟨ε_N⟩ = 1.09 × 10^{-3}). In Fig. 1 we show that the empirical conditional distributions P(δv_j|ε_N) for M = 19 are indeed well described by Gaussians, thus validating the assumption [2]. Also, the inset of Fig. 2(b) displays the nice agreement between the numerical compounding (solid line) of the Gaussian [2] with the empirical f(ε_N), see Eq. (S25), and the velocity increments distribution from the DNS data (circles). We note furthermore that the scale ℓ = Mr belongs to the inertial range, since ℓ ≈ 40.7η nearly coincides with the Taylor scale λ ≈ 41.1η [25], thus confirming the separation of scales in Eq. (225) discussed in Methods. Using that the integral scale in this case [25] is L = 104.7η = 224r, we estimate the number of scales in the model hierarchy: N = log_2(L/ℓ) = log_2(224r/19r) ≈ 4.

Figure 2(a) and the main panel of Fig. 2(b) display, respectively, the marginal distribution P_N(δv_r) and background density f(ε_N) for N = 4. The theoretical results are shown in solid lines and the empirical data are depicted in circles, with excellent agreement observed in both P_N(δv_r) and f(ε_N). The best fit parameters are α = 0.17 and β = 2.72. For comparison, we also plot the best fit using N = 1 (single scale), which clearly does not perform as well as the one with N = 4. This result evidences that this DNS dataset cannot be properly described with only a single scale. (We also confirmed that the N = 4 case produces a better fit than N = 2, 3, 5.)

We now turn to analyze more recent turbulence data [26] from the DNS of the Navier-Stokes equations for a larger system with 4096³ points and higher Reλ ≈ 600. In this case, the smallest resolved scale is r ≈ 1.11η and the integral scale L = 907r [26]. (Again we only analyze velocity increments at this smallest scale.)

The theoretical results (solid lines) and DNS data (circles) for P_N(δv_r) (main panel) and f(ε_N) (inset) are shown in Fig. 3. Here we find μ = −1.50 and M = 27 (yielding ⟨ε_N⟩ = 9.06 × 10^{-4}), whereas α ≈ 0 and β = 2.55. From the data in [26] we obtain a larger number of scales N = log_2(907r/27r) ≈ 5. Indeed, for N = 5 a remarkable agreement with the empirical data is observed for both f(ε_N) and P_N(δv_r), as seen in the inset and main panel (red curve of Fig. 3) respectively. As in the previous analysis, the fit (green curve) using only a single scale (N = 1) is not as accurate as that with N = 5. Accordingly, we found that the cases N = 2, 3, 4, 6 also led to poorer fits when compared to N = 5. The larger N obtained for the second dataset, which has a higher Reλ, is consistent with the fact that L/η increases with Reλ, and so we expect more steps in the cascade (hence larger N) as Reλ enhances. Further, the fact that α ≈ 0 for N = 5 also agrees with the suggested interpretation that the α-term in [3] represents a residual dissipation in the inertial range, which is expected to become negligible for very large Reλ.

Comparing the inset of Fig. 3 with Fig. 2(b), we see that the distribution f(ε_N) is broader in the former case, which is consistent with the expected “amplification of intermittency” [9] as r decreases. (Recall that r/L decreases by a factor of four in the second dataset.) Our model is thus rather versatile in that it can describe intermittency down to the near dissipation range, as the changing behavior of the distribution of velocity increments with varying r can be accommodated in the distribution f(ε_N). Further investigation of the model for larger separation scales and the parameters’ dependence on the Reynolds number are left for future studies.

In conclusion, we have developed a hierarchical model to investigate the emergence of the negative skewness and non-Gaussian behavior of the distribution of short-scale velocity increments in isotropic turbulence, with a remarkable agreement with data from two independent numerical datasets in the multiscale scenario.

The general character and plasticity of our formalism make it readily adaptable to investigate the emergence of skewed (non-Gaussian) statistics in other complex systems, such as financial markets [14] and biological systems [15]. For example, the symmetric version of our theory has been successfully applied to explain the emergence of turbulence in a photonic random laser, as recently reported in [27].

Methods.

Theoretical background. One important physical assumption built into Eq. [S25] is the separation of time
and length scales, as the background variable $\varepsilon_\ell$ is supposed to vary more slowly (in time and space) than the signal $\delta v_\ell$, thus allowing it to reach a quasi-equilibrium distribution $P(\delta v_\ell|\varepsilon_\ell)$. (In the statistical mechanics language, structures of size $\ell$ act as a ‘heat bath’ for the fast fluctuating quantity $\delta v_\ell$.)

Indeed, in the hierarchical model (5) we assume that the time scales within the cascade are largely separated, with faster dynamics at smaller scales, i.e., $\gamma_N \gg \gamma_{N-1} \gg \cdots \gg \gamma_1$. We consider furthermore that $\kappa_N \gg \kappa_{N-1} \gg \cdots \gg \kappa_1$, which is reasonable since one expects stronger intermittency at smaller scales, in such a way that the dimensionless ratio $\beta \equiv 2\gamma_1/\kappa_1^2$ remains invariant across scales, see Eq. (5). Under these assumptions, the stationary solution of the Fokker-Planck equation associated with (5) under Itô prescription is

$$f(\varepsilon_i|\varepsilon_{i-1}) = \frac{(\varepsilon_i/\varepsilon_{i-1})^{p-1}}{2\varepsilon_i-1}_0 \int \frac{\beta \varepsilon_i}{\varepsilon_{i-1}} - \frac{\beta \alpha^2 \varepsilon_{i-1}}{\varepsilon_i} \) \exp(p \frac{\beta \varepsilon_i}{\varepsilon_{i-1}} - \frac{\beta \alpha^2 \varepsilon_{i-1}}{\varepsilon_i} \), \tag{7}$$

with $p = \beta(1-\alpha^2)$, $\omega = 2\alpha\beta$ and $K_p(x)$ is the modified Bessel function of second kind. Equation (7) has the form of a generalized inverse Gaussian (GIG) distribution.

By denoting $f(\varepsilon_N) \equiv f(\varepsilon_i)$ in Eq. (S25), we write

$$f(\varepsilon_N) = \int_0^\infty \cdots \int_0^\infty f(\varepsilon_N|\varepsilon_{N-1}) \prod_{i=1}^{N-1} [f(\varepsilon_i|\varepsilon_{i-1})] d\varepsilon_i]. \tag{8}$$

Notably, the integrals can be performed exactly yielding Eq. (4) (see Supplementary Information for details).

**Data analysis.** We describe in the following how to apply our formalism to the data analysis of turbulent flows.

Consider a large dataset $\{\delta v_\ell(j)\}$ of longitudinal velocity increments, with $j = 1, ..., N_\ell$. As a first step, we need to determine the optimal window size $M$ over which the variance of $\delta v_\ell(j)$ is supposed to remain approximately constant. By dividing the original series into overlapping intervals of size $M$, we define (13) an estimator of the local variance for each interval as $\varepsilon(k) = \sum_{j=1}^M (\delta v(k-j) - \overline{\delta v}(k))^2/M$, where $\overline{\delta v}(k) = \sum_{j=1}^M \delta v(k-j)/M$, with $k = M, ..., N_\ell$. We take the variance over a region of size $\ell = Mr$ as a proxy measure for the energy transfer rate (7) (17) (18). For various choices of $M$ and varying the asymmetry parameter $\mu$ for each $M$, we numerically compound the empirical distribution of the variance series $\{\varepsilon(k)\}$ with the Gaussian given in (2), and select the optimal parameters $\alpha$ and $\beta$ for which the compounding integral (S25) best fits the distribution of velocity increments computed from the original data. (See, e.g., (28) (30) for other methods to estimate $M$ for the variance series in the case of superposition of Gaussians with zero mean, thus not corresponding to our context.) The knowledge of $M$ allows to express $\varepsilon_0$ in terms of the mean $\langle \varepsilon_N \rangle$ of the variance series and the parameters $\alpha$ and $\beta$ (see Supplementary Information), leaving only $\alpha$ and $\beta$ to be determined.

Once $M$ is set, we estimate the number $N$ of scales in the cascade by $N = \log_2(L/\ell) = \log_2(L/Mr)$, where $L$ is the integral scale; see discussion preceding Eq. (3).

After obtaining the series $\{\varepsilon_N(k)\}$, we fit the empirical $f(\varepsilon_N)$ to Eq. (4) to determine $\alpha$ and $\beta$. Finally, the theoretical $P_N(\delta v_\ell)$ is computed by inserting the parameters in (S16).

In this work we analyze two large datasets of isotropic turbulence generated by the extensive DNS of the Navier-Stokes equations. The first dataset corresponds to a system with 1024$^3$ lattice points in a periodic cube and Taylor-based Reynolds number $Re_\lambda \approx 433$, whose simulation spans five large eddy turnover times from which we considered $\approx 3 \times 10^8$ points for our statistics. The second one (26) regards a larger system with 4096$^3$ points and higher $Re_\lambda \approx 600$, with only one snapshot in time from which we also took $\approx 3 \times 10^8$ points.

In each dataset we start by analyzing the conditional distribution $P(\delta v_\ell|\varepsilon_N)$, which requires computing first the joint distribution $P(\delta v_\ell, \varepsilon_N)$. To this end, we adopt here the following ad-hoc prescription: for each window of size $M$ we compute the corresponding variance $\varepsilon_N(k)$ and associate it with the velocity increment $\delta v_\ell$ at the center of the respective window. The variance series $\{\varepsilon_N(k)\}$ thus generated is then ‘binarized’ and for each bin we compute the respective histogram $P(\delta v_\ell|\varepsilon_N)$ of velocity increments. In Fig. 4 we show that the empirical conditional distributions $P(\delta v_\ell|\varepsilon_N)$ for the first dataset, obtained for $M = 19$, are indeed well described by Gaussians, thus validating the assumption (2).

We also stress that, although our model has four free parameters, they are determined in pairs—first $\varepsilon_0$ and $\mu$, then $\alpha$ and $\beta$—in a two-step procedure involving $f(\varepsilon_N)$, which is a more stringent constraint than a direct fit of $P_N(\delta v_\ell)$. Indeed, the fact that $f(\varepsilon_N)$ is available to fit the empirical data in an unambiguous way (see Fig. 2(b) and the inset of Fig. 3) actually proves to be an important feature of our method, since it is known that the distribution of velocity increments $P_N(\delta v_\ell)$ can be almost equally well fitted by different theoretical expressions, thus making it difficult to select between competing models (13).

We further stress that no curve fitting was performed in Fig. 2(a) or in the main panel of Fig. 3; the fits were done only in the main panel of Fig. 2(b) and inset of Fig. 3 to obtain the parameters $\alpha$ and $\beta$ entering the density $f(\varepsilon_N)$. Once $\alpha$ and $\beta$ were known, we simply plotted $P_N(\delta v_\ell)$ using Eq. (S16) and superimposed it with the empirical histogram for the velocity increments $\delta v_\ell$. Thus, the nice agreement in $P_N(\delta v_\ell)$ exhibited in Fig. 2(a) and main panel of Fig. 3 using the parameters determined from $f(\varepsilon_N)$ attests to the method’s self-consistency.

**Data availability.** All relevant data are available from
the authors.

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Supplemental Materials:
Emergence of skewed non-Gaussian distributions of velocity increments in isotropic turbulence

THEORETICAL FORMALISM: EXACT CALCULATION OF THE BACKGROUND DENSITY

We start by providing additional details on the exact calculation of the background density, \( f(\varepsilon_N) \equiv f(\varepsilon_r) \), which incorporates the crucial effect of the fluctuating energy flux (intermittency) on the turbulence properties. As indicated in Eq. (1) of the manuscript, in this scenario the marginal distribution of short-scale velocity increments, \( P_N(\delta v_r) \equiv P(\delta v_r) \), is obtained by compounding \( f(\varepsilon_r) \) with the Gaussian conditional distribution \( P(\delta v_r | \varepsilon_r) \).

Our starting point is the multiple integral representation of the background density, given by Eq. (8) of the manuscript,

\[
f(\varepsilon_N) = \int_0^\infty \cdots \int_0^\infty f(\varepsilon_{N-1} | \varepsilon_{N-2}) \prod_{i=1}^{N-1} [f(\varepsilon_i | \varepsilon_{i-1}) d\varepsilon_i], \tag{S1}
\]

in which the generalized inverse Gaussian (GIG) distribution,

\[
f(\varepsilon_i | \varepsilon_{i-1}) = \frac{(\varepsilon_i / \varepsilon_{i-1})^{p-1}}{2\varepsilon_{i-1}^{p/2}K_p(\omega)} \exp \left( -\frac{\beta \varepsilon_i}{\varepsilon_{i-1}} - \frac{\beta \alpha^2 \varepsilon_{i-1}}{\varepsilon_i} \right), \tag{S2}
\]

arises as the solution of the system of stochastic differential equations, Eq. (3), with \( \kappa_i = \sqrt{2\gamma_i / \beta}, p = \beta(1 - \alpha^2), \omega = 2\alpha\beta, \) and \( K_p(x) \) as the modified Bessel function of second kind. Introducing the new variable \( x_i = \varepsilon_i / \varepsilon_{i-1} \), we write

\[
f(\varepsilon_i | \varepsilon_{i-1}) d\varepsilon_i = g_i(x_i) dx_i, \tag{S3}
\]

where

\[
g_i(x_i) = \frac{x_i^{p-1}}{2\alpha^p K_p(\omega)} \exp \left( -\beta x_i - \frac{\beta \alpha^2}{x_i} \right). \tag{S4}
\]

We proceed by observing that

\[
\varepsilon_N = \frac{\varepsilon_N}{\varepsilon_{N-1}} \frac{\varepsilon_{N-1}}{\varepsilon_{N-2}} \cdots \frac{\varepsilon_1}{\varepsilon_0} = \varepsilon_0 \prod_{j=1}^{N} x_j. \tag{S5}
\]

Next we recall that the Mellin transform [S1] of a function \( f(x) \) is defined by

\[
\tilde{f}(s) = \int_0^\infty x^{s-1} f(x) dx, \tag{S6}
\]

which implies the following relation between the Mellin transforms of \( f(\varepsilon_N) \) and \( g_i(x) \):

\[
\tilde{f}(s) = \int_0^\infty \varepsilon_N^{s-1} f(\varepsilon_N) d\varepsilon_N
= \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{N} [f(\varepsilon_i | \varepsilon_{i-1})] d\varepsilon_N \cdots d\varepsilon_1
= \varepsilon_0^{s-1} \prod_{i=1}^{N} \int_0^\infty x_i^{s-1} g_i(x_i) dx_i
= \varepsilon_0^{s-1} \prod_{i=1}^{N} \tilde{g}_i(s). \tag{S7}
\]
The Mellin transform of Eq. (S4) is

\[ \tilde{g}(s) = \alpha^{s-1} \frac{K_{s+p-1}(\omega)}{K_p(\omega)}. \]  

(S8)

Inserting Eq. (S8) into Eq. (S7), we see that the Mellin transform of \( f(\varepsilon_N) \) is

\[ \tilde{f}(s) = \varepsilon_0^{-s-1} \left[ \alpha^{s-1} \frac{K_{s+p-1}(\omega)}{K_p(\omega)} \right]^N \]

\[ = (\varepsilon_0 \alpha^N)^{-s-1} \left[ \frac{K_{s+p-1}(\omega)}{K_p(\omega)} \right]^N. \]

(S9)

Now, using Eq. (S9) and the formula of the inverse Mellin transform, we can write \( f(\varepsilon_N) \) as the contour integral

\[ f(\varepsilon_N) = \frac{1}{\varepsilon_0 [\alpha K_p(\omega)]^N} \frac{1}{2\pi i} \int_{\Gamma} \left( \frac{\varepsilon_N}{\varepsilon_0 \alpha^N} \right)^{-s} [K_{s+p-1}(\omega)]^N ds. \]

(S10)

Further progress can be made by introducing a generalization of the Meijer-G function \( \mathbb{S}_2 \), which we shall refer to as the \( R \)-function, in terms of the following Mellin-Barnes integral,

\[ R_{p,q}^{m,n}(a, A \mid x) = \frac{1}{2\pi i} \int_{\Gamma} x^{-s} \tilde{R}_{p,q}^{m,n}(a, A \mid s) ds, \]

(S11)

where

\[ \tilde{R}_{p,q}^{m,n}(a, A \mid s) = \prod_{j=1}^{m} B_j^s K_{b_j+s}(2B_j) \prod_{k=1}^{n} A_k^{-s} K_{1-a_k-s}(2A_k) \]

\[ \frac{1}{\prod_{k=m+1}^{p} A_k^s K_{a_k+s}(2A_k) \prod_{j=m+1}^{q} B_j^{-s} K_{1-b_j-s}(2B_j)}, \]

(S12)

and \( a = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_p) \), \( A = (A_1, \ldots, A_n, A_{n+1}, \ldots, A_p) \), \( b = (b_1, \ldots, b_m, b_{m+1}, \ldots, b_q) \), \( B = (B_1, \ldots, B_m, B_{m+1}, \ldots, B_q) \). The contour path \( \Gamma \) is chosen so that the conditions for the existence of the inverse Mellin transform are satisfied \( \mathbb{S}_1 \), since \( R \) and \( \tilde{R} \) are Mellin pairs (see also below). From Eq. (S12) we see that

\[ R_{0,N}^{N,0}(b, B \mid x) = \frac{1}{2\pi i} \int_{\Gamma} x^{-s} \prod_{j=1}^{N} B_j^s K_{b_j+s}(2B_j) ds, \]

(S13)

where \( b = (b_1, \ldots, b_N) \) and \( B = (B_1, \ldots, B_N) \).

The newly defined special function \( R_{p,q}^{m,n} \) can be viewed as a generalization of the Meijer-G function \( G_{p,q}^{m,n} \), in which the gamma functions \( \Gamma(\nu) \) are essentially replaced by \( K_p(x) \) Bessel functions in the Mellin-Barnes integral above, Eqs. (S12) and (S13). Indeed, by using the limit form \( K_p(x) \to \Gamma(\nu)2^{1-\nu}x^{-\nu} \), \( \nu > 0 \), \( x \to 0 \), we observe that \( R_{p,q}^{m,n} \to aG_{p,q}^{m,n} \), where \( a \) is a constant, when the argument of the Bessel functions tends to zero.

Finally, by comparing Eq. (S10) with Eq. (S13), we obtain the expression for the background density, Eq. (4) of the manuscript:

\[ f(\varepsilon_N) = \frac{1}{\varepsilon_0 [\alpha K_p(\omega)]^N} R_{0,N}^{N,0} \left( \begin{array}{c} \varepsilon_0 \\ \varepsilon_N \end{array} \right) \]

(S14)

where \( p = (\beta, \ldots, \beta) \) and \( \omega = (\omega, \ldots, \omega) \).

The last step consists in compounding Eq. (S14) with the Gaussian conditional distribution via Eq. (1) to obtain exactly the marginal distribution of short-scale velocity increments, \( P_N(\delta v_r) \), Eq. (5) of the manuscript, which is also given in terms of an \( R \)-function. To see this, note that the Gaussian distribution in Eq. (2) can be written as

\[ P(\delta v_r \mid \varepsilon) = \frac{e^{\mu_z y}}{\sqrt{\pi}} \left( \frac{2\mu}{y} \right)^{1/2} R_{1,0}^{1,1} \left( \begin{array}{c} 1, \frac{|\mu y|}{2} \\ \frac{1}{2}, \frac{|2y|}{2} \end{array} \right) (2\varepsilon_r)^{1/2}, \]

(S15)
with \( y = \delta v_r + \mu \langle \varepsilon \rangle \). Thus, the statistical composition of Eqs. (S14) and (S15) is performed using the integral involving the product of two \( R \)-functions (see property (S24) below). We thus find

\[
P_N(\delta v_r) = c e^{\mu y} R_{0,N+1}^{N+1,0} \left[ \left( 0, p - \frac{1}{2}, \left[ (\mu y^2, \frac{\omega}{2}) \right] \right) \right],
\]

with \( y = \delta v_r + \mu \langle \varepsilon \rangle \) and \( c = (2/\pi \varepsilon_0 \alpha N)^{1/2} / [K_p(\omega)]^N \).

It follows from Eq. (S7) that the mean of \( f_N(\varepsilon_N) \) is obtained by setting \( s = 2 \) in (S9):

\[
\langle \varepsilon_N \rangle = \varepsilon_0 \left[ \frac{\alpha K_{p+1}(\omega)}{K_p(\omega)} \right]^N,
\]

which implies

\[
\frac{\langle \varepsilon_N \rangle}{\langle \varepsilon_{N-1} \rangle} = \alpha \frac{K_{p+1}(\omega)}{K_p(\omega)}.
\]

Now, using \( K_{\nu}(z) \approx \Gamma(\nu)2^{\nu-1}z^{-\nu} \), for \( z \to 0, \nu > 0 \), it then implies

\[
\frac{\langle \varepsilon_N \rangle}{\langle \varepsilon_{N-1} \rangle} \approx 2 \alpha \frac{\Gamma(p+1)}{\omega \Gamma(p)} = 2 \alpha p = 1 - \alpha^2, \quad \alpha \to 0.
\]

Recursive application of this relation yields

\[
\langle \varepsilon_N \rangle \approx (1 - \alpha^2)^N \varepsilon_0 \approx (1 - N\alpha^2) \varepsilon_0.
\]

We lastly remark that the novel transcendent \( R \)-function, which emerges from our \( N \)-scale intermittency model, seems to have never been previously considered in the literature.

**Properties of the \( R \)-Function**

The general usefulness of the \( R \)-function representation arises from a number of identities that can be derived from extensions of related identities of the Meijer-\( G \) function. Therefore, we give below a short list of some general properties of the \( R \)-function.

- **Mellin transform**

\[
\int_0^\infty dx x^{s-1} R_{p,q}^{m,n} \left( a, A \bigg| b, B \right) x = \alpha^{-s} \bar{R}_{p,q}^{m,n} \left( a, A \bigg| b, B \right)
\]

- **Argument inversion**

\[
R_{p,q}^{m,n} \left( a, A \bigg| b, B \right) x = R_{q,p}^{n,m} \left( 1 - b, B \bigg| 1 - a, A \right) x
\]

- **Power absorption**

\[
x^{\sigma} R_{p,q}^{m,n} \left( a, A \bigg| b, B \right) x = \prod_{j=1}^{q} B_j^\sigma \prod_{k=1}^{p} A_k^\sigma R_{p,q}^{m,n} \left( \sigma 1 + a, A \bigg| 1 + b, B \right) x
\]

- **Integral involving the product of two \( R \)-functions**

\[
\int_0^\infty R_{p,q}^{m,n} \left( a, A \bigg| b, B \right) \xi x R_{r,t}^{c,d} \left( c, C \bigg| d, D \right) \eta x dx = \prod_{j=1}^{r} D_j \prod_{j=1}^{r} C_j R_{n+r,m+t}^{m-r,n+r} \left( \left( a - d, A, D \right) \xi \bigg| \left( b - c, B, C \right) \eta \right)
\]
NUMERICAL PROCEDURE

We now provide further details on the numerical procedure to apply our theoretical formalism to the analysis of general (i.e., either numerical or experimental) turbulence data.

The first step is to determine the optimal window size \( M \) to compute the background series of variance estimators \( \{\epsilon(k)\} \) built from the dataset as described in the manuscript. This is done simultaneously to the fitting of the asymmetry parameter \( \mu \).

The general idea is to search for the optimal pair \( (M, \mu) \) that yields the best agreement between the distribution computed numerically from Eqs. (1) and (2) of the manuscript, using the empirical density \( f(\epsilon) \), and the empirical distribution of velocity increments. In practice, we compute the integral in Eq. (1) of the manuscript as a Monte Carlo sum,

\[
P(\delta v_r) = \int_0^\infty P(\delta v_r|\epsilon) f(\epsilon) d\epsilon \approx \frac{1}{M} \sum_{i=1}^{N_M} \frac{1}{\sqrt{2\pi}\epsilon_i} \exp \left\{ -\frac{[\delta v_r - \mu (\epsilon_i - \langle \epsilon \rangle)]^2}{2\epsilon_i} \right\},
\]

(S25)

where \( \langle \epsilon \rangle = \sum_i \epsilon_i/N_M \) and \( N_M = N_v - M \) is the number of windows of size \( M \). If this step is successful then one guarantees that a proper modeling of the background density will lead to a good theoretical description of the increments distribution, as described below.

We therefore note that the window size \( M \) is not a free parameter in the usual sense, but it rather represents an internal length scale that needs to be obtained from the data. Other methods to estimate \( M \) for Gaussians with zero mean have been proposed, e.g., in Refs. [30-32] of the manuscript, but they do not apply to our case since our conditional Gaussians have nonzero mean, and so it was necessary to find both \( M \) and \( \mu \) simultaneously.

The next step is to compute the background distribution of the variance series for the optimal value of \( M \) and proceed to the fitting of the theoretical prediction, Eq. (S14) (Eq. (4) of the manuscript). Through the Mellin transform formula (S7) with \( s = 2 \), yielding Eq. (S17), we can relate the \( \epsilon_0 \) parameter to the first statistical moment of the distribution (S14), which is measured from the variance series, and the parameters \( \alpha \) and \( \beta \). This means that \( \epsilon_0 \) is not a free parameter, so that the only two free parameters in (S14) are \( \alpha \) and \( \beta \). These two parameters are then fitted using the value of \( N \) estimated according to the description in the manuscript. (For comparison, we also analyze fits for other values of \( N \); see manuscript.)

To perform the fit to Eq. (S14), we must calculate the \( R \)-function. We note that for \( N \) from 1 up to 6 the multiple integral (S1) may be the most efficient way. As mentioned in the manuscript, the \( N = 1 \) case is a generalized hyperbolic distribution. Interestingly, the case \( N = 2 \) also allows for an exact integration, and, in fact, for every two new hierarchy levels — and hence two additional integrals in (S1) — one integral can be executed exactly, reducing at least by half the number of integrals to be computed numerically.

On the other hand, it is also possible to compute numerically the complex integral (S13). In this sense, a striking fact is that the aforementioned generalization of the Meijer-\( G \) function through the substitution of the gamma functions in (S14) by the Bessel functions \( K_v(x) \) in the Mellin-Barnes integral, Eqs. (S11) and (S12), greatly simplifies the structure of poles of the integrand. Regarding the index \( \nu \), the Bessel function for a fixed \( x > 0 \) has a pole only at infinity, and decays to zero for \( \nu = \epsilon \pm i\infty \). Thus, any vertical contour in the complex plane satisfies the conditions of the Mellin inversion theorem and is suitable for the computation. The function grows very rapidly away from \( \nu = 0 \), developing strong oscillations in the real and imaginary parts, which led us to choose a contour that passes through \( \nu = 0 \) in the real line to attain fast numerical convergence. For a purely imaginary \( \nu \) the function \( K_\nu(x) \) is real for \( x > 0 \), so that for a single \( K \)-function the integral in Eq. (S11) is real. For a product of \( K \)-functions with different indexes, which happens for any \( N > 1 \), the contour should pass as close as possible to the zeros of these indexes to provide convergence and stability.

Lastly, with all parameters in hand, we plot the model prediction for the distribution of velocity increments, which depends on another \( R \)-function, as given by Eq. (5) of the manuscript, and compare with the one from the original empirical turbulence data.

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[S1] A. Erdelyi, ed., Tables of Integral Transforms (McGraw Hill, New York, 1954), vol. 1.
[S2] A. M. Mathai, R. K. Saxena, and H. J. Haubold, The H-Function: Theory and Applications (Springer-Verlag, New York, 2009).