The Semi-Quantum Computer

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We discuss the performance of the Search and Fourier Transform algorithms on a hybrid computer constituted of classical and quantum processors working together. We show that this semi-quantum computer would be an improvement over a pure classical architecture, no matter how few qubits are available and, therefore, it suggests an easier implementable technology than a pure quantum computer with arbitrary number of qubits.

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I. INTRODUCTION

In the 1980’s, Feynman suggested that computers based on the laws of Quantum Mechanics could be more efficient than the classical ones. In the 1990’s, the discovery of Shor’s and Grover’s algorithms stimulated the search for the physical implementation of a quantum computer. Diverse architectures have been tried and, among the most successfull, we can cite Nuclear Magnetic Resonance (NMR), Ion-traps, Optics. So far, the architecture that has achieved the largest number of qubits, namely seven, is NMR. The difficulty of implementing an arbitrary number of qubits has motivated alternative approaches to quantum computing. It has been demonstrated that classical optics supports quantum computing, but the number of optical elements needed grows exponentially with the number of qubits. Semi-classical architectures of quantum computing have also been suggested, with experimental implementations of the Fourier Transform of Shor’s algorithm and the Grover’s search algorithm. The kernel of these proposals is to simulate the two-qubit gates by use of classical communication among quantum gates. A quite different approach, which is a kind of quantum/classical hybrid distributed computation, dubbed type II quantum computation, has been adopted by Yepez. A different kind of distributed quantum computation has also been suggested by other authors.

The semi-quantum computer, or type II quantum computer according to Yepez, is in part quantum and in part classical. The quantum part is a set of individual genuine quantum computers with $n_q$ qubits each. These quantum nodes communicate classically with a classical computer, which may include several classical processors in parallel, as well. The key point here is that the quantum nodes are not intended to solve a whole algorithmic problem, but just a part, a subroutine, consistent with their supported number of qubits and coherence time. This hybrid architecture explores both quantum and classical parallelisms simultaneously. The quantum nodes send their partial results to the classical computer which, by its turn, performs some data processing and feeds back the quantum nodes with new tasks, which could be the same operation as before but on a new quantum state.

Our intention here is to look for evidences that tell us if this hybrid architecture could result in a computer that is faster than the classical ones. In order to do so, we are going to discuss how the Search and Fourier Transform algorithms could be adapted to the semi-quantum computer and investigate, through a complexity analysis, how they perform as compared to their pure classical and pure quantum counterparts. In section II, we discuss the Search problem. Section III is dedicated to the Fourier Transform. We conclude in section IV.

II. THE SEARCH PROBLEM

Supose that a list of $N$ elements has to be searched, in order to find the $M$ solutions ($M < N$) of a given problem, and that each solution can be identified by an appropriate oracle. Classically, all the $N$ elements have to be tested and, therefore, this procedure costs $O(N)$ operations. Grover showed that searching quantum mechanically costs just $O(\sqrt{N})$, a quadratic improvement over the classical algorithm.

The quantum algorithm is as follows. Associating an orthonormal basis $\{\ket{x}, \ x = 0, 1, \ldots, N-1\}$ to the indices of the elements in the list, define a state vector formed...
by the equal superposition of these basis states,

\[ |\psi\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle. \quad (1) \]

Consider a bi-dimensional space spanned by the states \(|\alpha\rangle\) and \(|\beta\rangle\), which are the superposition of all the non-solutions,

\[ |\alpha\rangle = \frac{1}{\sqrt{M}} \sum_{x=\text{non-solution}} |x\rangle \quad (2) \]

and the superposition of all the solutions,

\[ |\beta\rangle = \frac{1}{\sqrt{M}} \sum_{x=\text{solution}} |x\rangle. \quad (3) \]

Therefore \(|\psi\rangle\) can be rewritten as

\[ |\psi\rangle = |\alpha\rangle \langle \alpha | \psi \rangle + |\beta\rangle \langle \beta | \psi \rangle. \quad (4) \]

Now let’s check out what we gain with this procedure. Assuming \(n_q\) qubits in the quantum node, the list can be divided into \(2^n / 2^{n_q}\) sublists, where \(n = \log_2 N\). The quantum node searches the sublist with cost \(O(2^{n_q}/2)\) and it is needed \(2^n\) quantum nodes, or \(2^n\) accesses to the same quantum node, to search the entire list. Therefore the cost of this semi-quantum approach is

\[ O\left(\frac{2^n}{2^{n_q}} \times 2^{n_q/2}\right) = O\left(\frac{2^n}{2^{n_q/2}}\right). \]

Note that if \(n = n_q\), \(i.e.,\) if there is enough qubits to search the entire list at once, the cost is that of the Grover’s quantum search, namely, \(O(2^{n/2})\). On the other hand, if there are no qubits at all, the cost reduces to the classical one, \(O(2^n)\). Finally, with \(n_q\) qubits available, the cost \(O\left(\frac{2^n}{2^{n_q}}\right)\) is somewhere in between the classical and quantum searches.

### III. The Discrete Fourier Transform

Shor\cite{2} discovered that doing a Fourier Transform quantum mechanically is exponentially faster than classically. This quantum Fourier Transform (QFT) is the kernel of his algorithm to factorize numbers, with the notorious implication of possibly breaking, with polynomial cost, the most used public key cryptographic protocol, namely, the RSA\cite{12}, that is based on the product of any two huge prime numbers. It has to be noted that, in the Shor’s factorization algorithm, rather than Fourier transforming an arbitrary quantum state, the QFT is applied to a particular state as part of the process of factorization. Therefore, this subtle application of the QFT does not face two potentialy difficult problems, namely, preparing an arbitrary quantum state\cite{13} and measuring the complex phases\cite{14} of its Fourier Transform, which comes to be the problem we will treat here.

Classically, the best known algorithm to calculate the Discrete Fourier Transform of a real function at \(N\) points (\(DFT_N\)) is the well known Fast Fourier Transform (\(FFT\)), which costs \(O(N \log N)\) operations. Our point here is to use the \(QFT\), which costs \(O(N \log_2 N)\) quantum operations, to improve the \(FFT\) algorithm. This preamble done, now the semi-quantum approach to discrete Fourier transform a function is described.

A real function tabulated at \(N\) points can be substituted by its interpolating polynomial

\[ X(t) = \sum_{j=0}^{N-1} x_j t^j, \quad (5) \]

which can be represented by the real vector

\[ \tilde{X} = (x_0, x_1, \cdots, x_{N-1}) , \quad (6) \]

whose Fourier transform is the complex vector

\[ \tilde{y} = (y_0, y_1, \cdots, y_{N-1}) , \quad (7) \]
where

\[ y_k \equiv X(\omega_N^k) = \sum_{j=0}^{N-1} x_j \omega_N^{kj} = \sum_{j=0}^{N-1} x_j \exp \left( \frac{2\pi i k j}{N} \right). \quad (8) \]

The \( \{\omega_N^k\} \) are the \( N \) complex roots of the unity.

The quantum version of the DFT is as follows. Given the real vector \( \vec{X} \), prepare the associated quantum state

\[ |X\rangle = C_x \sum_{j=0}^{N-1} x_j |j\rangle , \quad (9) \]

where \( C_x \) is a normalization factor and \( \{|j\rangle, j = 0,1,\ldots,N-1\} \) is an orthonormal basis.

The DFT of \( \vec{X} \) is the vector \( \vec{Y} \), whose associated quantum state, written in the same basis as \( |X\rangle \), is

\[ |Y\rangle = C_y \sum_{k=0}^{N-1} y_k |k\rangle , \quad (10) \]

where \( C_y \) is a normalization factor.

\(|X\rangle\) and \(|Y\rangle\) are related by the unitary operator \( U_{QFT} \),

\[ U_{QFT}|X\rangle = |Y\rangle. \quad (11) \]

The action of \( U_{QFT} \) on the basis states is

\[ U_{QFT}|j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp \left( \frac{2\pi i kj}{N} \right) |k\rangle. \quad (12) \]

The FFT algorithm is based on the observation that squaring the \( N \) complex roots of the unity \( \{\omega_N^k\} \) produces just \( \frac{N}{2} \) distinct complex numbers \( \{\omega_N^k/2\} \). Therefore, the DFT problem reduces to a recursion of identical subproblems, with the \( y_k \) evaluated as [Fig. 3]

\[ y_k = X(\omega_N^k) = X_e[(\omega_N^k)^2] + w_N^k X_o[(\omega_N^k)^2], \quad (13) \]

where

\[ X_e(t) = x_0 + x_2 t + x_4 t^2 + \cdots, \quad (14) \]

\[ X_o(t) = x_1 + x_3 t + x_5 t^2 + \cdots. \quad (15) \]

Now assuming a quantum computer with \((n_q+1)\) qubits, let’s see how to calculate the DFT\(_N\) \((N = 2^n)\). For the sake of clarity, the case \( n_q = 2 \) is explicitly considered, but the generalization for arbitrary \( n_q \) is straightforward.

The normalized input state is

\[ |X_4\rangle = x_0|00\rangle + x_1|01\rangle + x_2|10\rangle + x_3|11\rangle, \quad (16) \]

and the normalized output state is

\[ |Y_4\rangle = U_{QFT}|X_4\rangle = \]

\[ \sum_{j=0}^{3} y_j |j\rangle = \sum_{j=0}^{3} \left( y_{00} + (y_{1a} + iy_{1b})|01\rangle + y_{210} + (y_{1a} - iy_{1b})|11\rangle \right), \quad (17) \]

where the \( x_k \) and \( y_k \) are real numbers.

In Fig. 4 we propose a quantum circuit for this problem. The circuit is a slight modification of the original QFT one and includes an ancillary qubit, which is necessary for the proper determination of the Fourier phases.

In Tab. II we show how the Fourier phases can be measured. It involves four projections followed by eight measurements. Then some classical operations, summarized in Tab. III, are necessary to rebuild the phases. Note that the role of the ancillary qubit is to determine the sign (+ or −) of the real figures.

Now let’s see what we have gained with this semi-quantum approach to perform the DFT\(_N\). The classical part of the algorithm can still be represented by a tree, but now with just \((n - n_q)\) levels and the operations to be done are those summarized in Tab. IV and Eq. 15. The polynomial evaluations of Eq. 14 and Eq. 15 are avoided in the semi-quantum algorithm. Each level of the tree [Fig. 4] involves \( O(2^n) \) classical operations and, therefore, the total cost is \( O((n - n_q)2^n) \). The quantum part of the algorithm involves \( \frac{2^n}{n_q} \) quantum nodes, or accesses to the same quantum node, and each quantum node performs a QFT, which involves \( O(n_q^2) \) quantum operations, resulting in a total cost of \( O(n_q^22^{n-n_q}) \).

The access to the quantum node also includes an arbitrary state preparation that, according to Long et. al.
TABLE I: Measurement procedure to determine the Fourier phases.

| Projection in the basis | Result in the basis | Measurement in the basis | Result |
|-------------------------|---------------------|--------------------------|--------|
| | $\frac{2\cos(x)+y}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | $m_1 \equiv \frac{|x|^2}{2}$, $m_2 \equiv \frac{|x|^2+y|^2}{2}$ |
| $|0\rangle$ | | | |
| $|1\rangle$ | | | |
| $|0\rangle+|1\rangle$ | | | |
| $|0\rangle-|1\rangle$ | | | |
| $|0\rangle+|1\rangle$ | | | |
| $|0\rangle-|1\rangle$ | | | |

TABLE II: Classical operations to rebuild the Fourier phases.

| $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|-------|-------|-------|-------|-------|
| If $\frac{|x_0+y|^2}{2} \neq m_2$ Then $y_0 = -\sqrt{2m_1}$ Else $y_0 = +\sqrt{2m_1}$ End If |
| $y_1$ | $y_2$ | $y_3$ | $y_4$ |
| If $\frac{|x_1+y|^2}{2} \neq m_3$ Then $y_1 = -\sqrt{2m_3}$ Else $y_1 = +\sqrt{2m_3}$ End If |
| If $\frac{|x_2+y|^2}{2} \neq m_4$ Then $y_2 = -\sqrt{2m_4}$ Else $y_2 = +\sqrt{2m_4}$ End If |
| If $\frac{|x_3+y|^2}{2} \neq m_5$ Then $y_3 = -\sqrt{2m_5}$ Else $y_3 = +\sqrt{2m_5}$ End If |
| $y_4$ | $y_5$ | $y_6$ | $y_7$ |
| If $\frac{|x_4+y|^2}{2} \neq m_6$ Then $y_4 = -\sqrt{2m_6}$ Else $y_4 = +\sqrt{2m_6}$ End If |
| $y_5$ | $y_6$ | $y_7$ | $y_8$ |
| If $\frac{|x_5+y|^2}{2} \neq m_7$ Then $y_5 = -\sqrt{2m_7}$ Else $y_5 = +\sqrt{2m_7}$ End If |
| $y_6$ | $y_7$ | $y_8$ | $y_9$ |
| If $\frac{|x_6+y|^2}{2} \neq m_8$ Then $y_6 = -\sqrt{2m_8}$ Else $y_6 = +\sqrt{2m_8}$ End If |
| $y_7$ | $y_8$ | $y_9$ | $y_{10}$ |
| If $\frac{|x_7+y|^2}{2} \neq m_9$ Then $y_7 = -\sqrt{2m_9}$ Else $y_7 = +\sqrt{2m_9}$ End If |
| $y_8$ | $y_9$ | $y_{10}$ | $y_{11}$ |
| If $\frac{|x_8+y|^2}{2} \neq m_{10}$ Then $y_8 = -\sqrt{2m_{10}}$ Else $y_8 = +\sqrt{2m_{10}}$ End If |
| $y_{10}$ | $y_{11}$ | $y_{12}$ | $y_{13}$ |
| If $\frac{|x_9+y|^2}{2} \neq m_{11}$ Then $y_9 = -\sqrt{2m_{11}}$ Else $y_9 = +\sqrt{2m_{11}}$ End If |
| $y_{11}$ | $y_{12}$ | $y_{13}$ | $y_{14}$ |
| If $\frac{|x_{10}+y|^2}{2} \neq m_{12}$ Then $y_{10} = -\sqrt{2m_{12}}$ Else $y_{10} = +\sqrt{2m_{12}}$ End If |
| $y_{12}$ | $y_{13}$ | $y_{14}$ | $y_{15}$ |
| If $\frac{|x_{11}+y|^2}{2} \neq m_{13}$ Then $y_{11} = -\sqrt{2m_{13}}$ Else $y_{11} = +\sqrt{2m_{13}}$ End If |
| $y_{13}$ | $y_{14}$ | $y_{15}$ | $y_{16}$ |
| If $\frac{|x_{12}+y|^2}{2} \neq m_{14}$ Then $y_{12} = -\sqrt{2m_{14}}$ Else $y_{12} = +\sqrt{2m_{14}}$ End If |
| $y_{14}$ | $y_{15}$ | $y_{16}$ | $y_{17}$ |
| If $\frac{|x_{13}+y|^2}{2} \neq m_{15}$ Then $y_{13} = -\sqrt{2m_{15}}$ Else $y_{13} = +\sqrt{2m_{15}}$ End If |
| $y_{15}$ | $y_{16}$ | $y_{17}$ | $y_{18}$ |
| If $\frac{|x_{14}+y|^2}{2} \neq m_{16}$ Then $y_{14} = -\sqrt{2m_{16}}$ Else $y_{14} = +\sqrt{2m_{16}}$ End If |
| $y_{16}$ | $y_{17}$ | $y_{18}$ | $y_{19}$ |
| If $\frac{|x_{15}+y|^2}{2} \neq m_{17}$ Then $y_{15} = -\sqrt{2m_{17}}$ Else $y_{15} = +\sqrt{2m_{17}}$ End If |
| $y_{17}$ | $y_{18}$ | $y_{19}$ | $y_{20}$ |
| If $\frac{|x_{16}+y|^2}{2} \neq m_{18}$ Then $y_{16} = -\sqrt{2m_{18}}$ Else $y_{16} = +\sqrt{2m_{18}}$ End If |

 costs $O(n^2 2^{n_q})$. Therefore, the total cost of the quantum part of the algorithm is (preparation + QFT): $O(n^2 2^{n_q}) + O((n^2 2^{n_q} - n_q) 2^n)$. Summarizing, performing a Discrete Fourier Transform on the semi-quantum computer costs (state preparation + QFT + classical operations): $O(n^2 2^{n_q}) + O((n^2 2^{n_q} - n_q) 2^n) + O(n - n_q) 2^n$. For a fixed number of qubits ($n_q$ constant), the total cost can be rewritten as (quantum operations + classical operations): $O(2^n) + O(n 2^n - 2^n)$. This expression shows that $O(2^n)$ classical operations are being substituted by $O(2^n)$ quantum operations. Therefore, this semi-quantum approach is profitable only in the case that the quantum operations are faster than the classical ones. It is reasonable to speculate that the quantum operations would be faster than the classical ones, if we remember that to determine the $2^n$ phases, we handle just $n$ qubits and obtain real figures with the (hopefully high) precision of the detectors. On the other hand, in the classical procedure, we have to handle $2^n \times \text{precision}$ bits, where $\text{precision}$ is the number of bits necessary to represent the real figures with the desired precision. Therefore, for a given precision, the quantum procedure is always exponentially smaller in space (i.e., number of bits or memory) than the classical one. It is worth noticing that, ignoring the state preparation, the cost of the $\text{DFT}$ as a function of the number of qubits ($n_q$) goes from the classical $O(n 2^n)$, when $n_q = 0$, to the quantum $O((n^2 2^n)$, when $n_q = n$, and for an arbitrary $n_q$, the cost is somewhere in between the both. Therefore, if the complexity of the state preparation stage could be improved, the semi-quantum approach would be certainly profitable.

IV. CONCLUSION

In this paper, we have searched for evidences that told us if a semi-quantum computer, which has an architecture that includes both classical and quantum processors communicating classically, would have some advantage over a pure classical architecture. We have shown how to perform a search in a list and a Fourier Transform in this semi-quantum computer. In the former case, we have shown that the Grover algorithm is trivially adaptable to the semi-quantum computer and has a performance ($O(2^n)$) that is always superior to the classical one ($O(n^2)$) and inferior to the pure quantum one ($O(2^n)$). In the latter case, we have used the Quantum Fourier Transform (QFT) algorithm ($O(n^2)$) to improve the classical Fast Fourier Transform algorithm ($O(n 2^n)$). We have shown that using a single ancillary qubit to control the QFT transformation allows us to measure the Fourier phases. Due to the costly state preparation stage, we have concluded that what we profit in the semi-quantum approach is to save $O(2^n)$ classical operations, that involves $O(2^n)$ bits, in favor of $O(2^n)$ quantum operations, that involves just $n$ qubits. The expression we have obtained for the cost of this semi-quantum approach as a function of the number of qubits in the quantum node, namely, (state preparation + QFT + classical operations): $O(n^2 2^{n_q}) + O((n^2 2^{n_q} - n_q) 2^n) + O(n - n_q) 2^n$, shows that, ignoring the state preparation stage, the cost, as the number of qubits grows, ranges from the classical $O(n 2^n)$ to the quantum $O(n^2)$. On the other hand, the least cost one can hope for the arbitrary state preparation stage is $O(2^n)$ and, therefore, it will allways dominate the complexity of the semi-quantum Fourier Transform. Finally, our results suggest that a semi-quantum computer could be an improvement over a pure classical
architecture, even in the case of a small number of qubits. Implementations of the two algorithms discussed in this paper on an semi-quantum computer based on optics, be quantum or classical, seems to be relatively easy and we intend to perform such experiments to test these ideas.

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[1] R.P. Feynman, Int. J. Theor. Phys. 21(1982)467.
[2] P.W. Shor, Algorithms for quantum computation: discrete logarithms and factoring. In Proceedings, 35th Annual Symposium of Foundations of Computer Science, IEEE Press, Los Alamitos, CA, 1994.
[3] L.K. Grover, Phys. Rev. Lett. 79, 2(1997)325; 79, 23(1997)4709; 80,19(1998)4329.
[4] (for a review see, for example, this especial issue:) Fort. Phys. 48(2000)766-1138.
[5] (see, for example, this paper and the references therein:) E. Knill, R. Laflamme, G.J. Milburn, NATURE, 409(2001)46.
[6] L.M.K. Vandersypen, M. Steffen, G. Breyta, C.S. Yannoni, M.H. Sherwood, I.L. Chuang, NATURE, 414(2001)883.
[7] (see, for example:) N. Bhattacharya, H.B. van den Heuvel, R.J.C. Spreeuw, Phys. Rev. Lett. 88,13(2002)137901; R.J.C. Spreeuw, Phys. Rev. A, 63(2001)062302. N.J. Cerf, C. Adami, P.G. Kwiat, Phys. Rev. A, 57,3(1998)R1477.
[8] R.B. Griffiths, C. Niu, Phys. Rev. Lett. 76,17(1996)3228.
[9] J.C. Howell, J.A. Yeazell, Phys. Rev. Lett. 85,1(2000)198.
[10] J. Yepez, Comp. Phys. Comm. 146(2002)277-279.
[11] J.I. Cirac, A.K. Ekert, S.F. Huelga, C. Macchiavello, Phys. Rev. A, 59,6(1999)4249.
[12] R.L. Rivest, A. Shamir, L.M. Adleman, Comm. ACM, 21, 2(1978)120-126.
[13] Gui-Lu Long, Y. Sun, Phys. Rev. A, 64(2001)014303.
[14] D.F.V. James, P.G. Kwiat, W.J. Munro, A.G. White, Phys. Rev. A, 64(2001)052312.