Equivalence of Wilson Actions

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Abstract

We introduce the concept of equivalence among Wilson actions. Applying the concept to a real scalar theory on a euclidean space, we derive the exact renormalization group transformation of K. G. Wilson, and give a simple proof of universality of the critical exponents at any fixed point of the exact renormalization group transformation. We also show how to reduce the original formalism of Wilson to the simplified formalism by J. Polchinski.

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I. INTRODUCTION

The purpose of this paper is to introduce the concept of equivalence among Wilson actions. We consider a generic real scalar theory in $D$-dimensional euclidean space, and denote the Fourier component of the scalar field with momentum $p$ by $\phi(p)$.

A Wilson action $S[\phi]$ is a real functional of $\phi(p)$. A momentum cutoff is incorporated so that the exponentiated action $e^{S[\phi]}$ can be integrated with no ultraviolet divergences. An example is given by

$$S[\phi] = -\int p^2 K\left(\frac{\Lambda^2}{\Lambda^2 - p^2}\right) \frac{1}{2} \phi(p)\phi(-p) + S_I[\phi]$$

(1)

where the cutoff function $K(p^2)$ is a positive function of $p^2$ that is 1 at $p^2 = 0$, and decreases toward 0 rapidly for $p^2 \gg 1$. The first term of the action can suppress the modes with momenta higher than $\Lambda$ sufficiently that $e^{S[\phi]}$ can be integrated over $\phi(p)$ of all momenta. The second term consists of local interaction terms. In the continuum approach adopted here, correlation functions are defined for $\phi(p)$ of all momenta, even those above $\Lambda$.

A Wilson action is meant to describe low momentum (energy) physics accurately, but not physics at or above the cutoff scale. If two Wilson actions describe the same low energy physics, we regard them as equivalent. It is the purpose of this paper to provide a concrete definition of equivalence using the continuum approach.

The paper is organized as follows. In sect. II we introduce modified correlation functions, and then define equivalence of Wilson actions as the equality of the modified correlation functions. Basically, two Wilson actions are equivalent if their differences can be removed if we give them a massage at their respective cutoff scales. We derive two versions of explicit formulas that relate two equivalent Wilson actions. The concept of equivalence is applied in the rest of the paper.

In sect. III we derive the exact renormalization group (ERG) transformation of Wilson by considering a particular type of equivalence. We derive the ERG differential equation from our equivalence, which amounts to an integral solution to the differential equation. In sect. IV we discuss the relation between the original formulation of ERG transformation by Wilson and the formulation by J. Polchinski which is more convenient for perturbation theory. In the previous literature only a passing remark has been given on this relation. Our short discussion of their relation is complete and hopefully illuminating.

Sect. V prepares us for the discussion of universality in sect. VI. We generalize the definition
of equivalence so that the exact renormalization group transformation can have fixed points. In sect. VI, we assume a fixed point of the ERG transformation, and show that the critical exponents defined at the fixed point are independent of the choice of cutoff functions. This is what we mean by universality. We conclude the paper in sect. VII.

Throughout the paper we work in $D$-dimensional euclidean momentum space. We use the following abbreviated notation

$$\int_p \equiv \int \frac{d^D p}{(2\pi)^D}, \quad \delta(p) \equiv (2\pi)^D \delta^{(D)}(p)$$

II. EQUIVALENCE

Given a Wilson action $S[\phi]$, we denote the correlation functions by

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_S \equiv \int [d\phi] \phi(p_1) \cdots \phi(p_n) e^{S[\phi]}$$

We consider modifying the correlation functions for high momenta without touching them for small momenta. We define modified correlation functions by

$$\langle \langle \phi(p_1) \cdots \phi(p_n) \rangle \rangle^K_K \equiv \prod_{i=1}^n \frac{1}{K(p_i)} \cdot \left\langle \exp \left( - \int_p \frac{k(p)}{p^2} \frac{\delta^2}{\delta \phi(p) \delta \phi(p) \delta \phi(-p) \delta \phi(-p)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle_S$$

where $K(p)$ and $k(p)$ are non-negative functions of $p^2$. (We will call them cutoff functions.) As $p^2 \to 0$, we must find

$$K(p) \to 1, \quad k(p) \to 0$$

so that the correlation functions are not modified at small momenta. In addition we constrain $K(p)$ by

$$K(p) \to 0$$

In other words $K(p)$ is small for $p^2$ larger than the squared cutoff momentum of the Wilson action. The fluctuations of $\phi(p)$ with $p$ larger than the cutoff are suppressed, and we enhance their correlations by the large factor $1/K(p)$ in (4). The exponential on the right-hand side of (4) amounts to mixing a free scalar with the propagator $-k(p)/p^2$ to the original scalar field $\phi$. Since $k(0) = 0$, the free scalar has no dynamics of its own.

For example, we may take

$$K(p) = e^{-\frac{p^2}{\Lambda^2}}$$
and

\[ k(p) = \frac{p^2}{\Lambda^2} \quad \text{or} \quad e^{-\frac{p^2}{\Lambda^2}} \left(1 - e^{-\frac{p^2}{\Lambda^2}}\right) \quad (8) \]

where \( \Lambda \) is the momentum cutoff \( \Lambda \) of the Wilson action \( S \).

In particular, for \( n = 2 \) and \( n = 4 \), (4) gives

\[ \langle \phi(p_1)\phi(p_2) \rangle_S^{K,k} = \frac{1}{K(p_1)^2} \left( \langle \phi(p_1)\phi(p_2) \rangle_S - \frac{k(p_1)}{p_1^2} \delta(p_1 + p_2) \right) \quad (9) \]

\[ \langle \phi(p_1)\ldots\phi(p_4) \rangle_S^{K,k} = \prod_{i=1}^{4} \frac{1}{K(p_i)} \left[ \langle \phi(p_1)\ldots\phi(p_4) \rangle_S ight. \\
- \langle \phi(p_1)\phi(p_2) \rangle_S \frac{k(p_3)}{p_3^2} \delta(p_3 + p_4) - \langle \phi(p_3)\phi(p_4) \rangle_S \frac{k(p_1)}{p_1^2} \delta(p_1 + p_2) \\
\left. + \frac{k(p_1)}{p_1^2} \delta(p_1 + p_2) \frac{k(p_3)}{p_3^2} \delta(p_3 + p_4) \right] \quad (10) \]

For small momenta, the modified correlation functions (4) reduce to the ordinary correlation functions (3).

Now, we would like to introduce the concept of equivalence among Wilson actions. Let us regard two Wilson actions \( S_1, S_2 \) as equivalent if, with an appropriate choice of \( K_{1,2} \) and \( k_{1,2} \), their modified correlation functions become identical for any \( n \) and momenta:

\[ \langle \phi(p_1)\ldots\phi(p_n) \rangle_{S_1}^{K_{1},k_1} = \langle \phi(p_1)\ldots\phi(p_n) \rangle_{S_2}^{K_{2},k_2} \quad (11) \]

Since the functions \( K_{1,2}, k_{1,2} \) keep the low energy physics intact, \( S_1 \) and \( S_2 \) describe the same low energy physics. In the following we solve (11) to obtain an explicit relation between the two actions.

We first rewrite (11) as

\[ \left\langle \exp \left( - \int \frac{k_2(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) \phi(p_1)\ldots\phi(p_n) \right\rangle_{S_2} = \prod_{i=1}^{n} \frac{K_2(p_i)}{K_1(p_i)} \cdot \left\langle \exp \left( - \int \frac{k_1(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) \phi(p_1)\ldots\phi(p_n) \right\rangle_{S_1} \quad (12) \]

Since functional integration by parts gives

\[ \left\langle \exp \left( - \int \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) \phi(p_1)\ldots\phi(p_n) \right\rangle_{S} = \int [d\phi] e^{S[\phi]} \exp \left( - \int \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) \phi(p_1)\ldots\phi(p_n) \]

\[ = \int [d\phi] \phi(p_1)\ldots\phi(p_n) \exp \left( - \int \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) e^{S[\phi]} \quad (13) \]
we obtain
\[
\int [d\phi] \left( \prod_{i=1}^{n} \phi(p_i) \right) \cdot \exp \left( - \int \frac{k_2(p)}{p^2} \frac{1}{2} \delta^2 \frac{k_1(p)}{\rho_2(p)} \delta \phi(p) \delta \phi(-p) \right) e^{S_2[\phi]}
\]
\[
= \int [d\phi] \left( \prod_{i=1}^{n} \frac{K_2(p_i)}{K_1(p_i)} \phi(p_i) \right) \cdot \exp \left( - \int \frac{k_1(p)}{p^2} \frac{1}{2} \delta^2 \frac{k_1(p)}{\rho_2(p)} \delta \phi(p) \delta \phi(-p) \right) e^{S_1[\phi]}
\] (14)
This implies that
\[
\exp \left( - \int \frac{k_2(p)}{p^2} \frac{1}{2} \delta^2 \frac{k_1(p)}{\rho_2(p)} \delta \phi(p) \delta \phi(-p) \right) e^{S_2[\phi]} = \left[ \exp \left( - \int \frac{k_1(p)}{p^2} \frac{1}{2} \delta^2 \frac{k_1(p)}{\rho_2(p)} \delta \phi(p) \delta \phi(-p) \right) e^{S_1[\phi]} \right]_{\text{subst}}
\] (15)
where the suffix “subst” denotes the substitution of
\[
\frac{K_1(p)}{K_2(p)} \phi(p)
\] (16)
for \( \phi(p) \) on the right-hand side. Hence, we obtain an intermediate result
\[
e^{S_2[\phi]} = \exp \left( \int \frac{k_2(p)}{p^2} \frac{1}{2} \delta^2 \frac{k_1(p)}{\rho_2(p)} \delta \phi(p) \delta \phi(-p) \right) \left[ \exp \left( - \int \frac{k_1(p)}{p^2} \frac{1}{2} \delta^2 \frac{k_1(p)}{\rho_2(p)} \delta \phi(p) \delta \phi(-p) \right) e^{S_1[\phi]} \right]_{\text{subst}}
\] (17)
We can rewrite this in two ways. First, noting that under the substitution (16), we must substitute
\[
\frac{K_2(p)}{K_1(p)} \frac{\delta}{\delta \phi(p)}
\] (18)
for \( \frac{\delta}{\delta \phi(p)} \), we obtain the first relation
\[
e^{S_2[\phi]} = \exp \left[ \int \frac{1}{p^2} \left\{ k_2(p) - k_1(p) \left( \frac{K_2(p)}{K_1(p)} \right)^2 \right\} \frac{1}{2} \delta^2 \frac{\delta}{\delta \phi(p)} \delta \phi(-p) \right] \exp \left( S_1 \left[ \frac{K_1(p)}{K_2(p)} \phi \right] \right)
\] (19)
Note that the two actions are the same for \( \phi(p) \) with small \( p \), since the function of \( p^2 \) in the curly bracket above is negligible for small \( p^2 \). In this sense the two actions differ only by local terms.
Alternatively, we rewrite (17) as
\[
e^{S_2[\phi]} = \left[ \exp \left[ \int \frac{1}{p^2} \left\{ k_2(p) \left( \frac{K_1(p)}{K_2(p)} \right)^2 - k_1(p) \right\} \frac{1}{2} \delta^2 \frac{\delta}{\delta \phi(p)} \delta \phi(-p) \right] \right] e^{S_1[\phi]}_{\text{subst}}
\] (20)
Using the gaussian formula
\[
\exp \left( \int A(p) \frac{1}{2} \delta^2 \frac{\delta}{\delta \phi(p)} \delta \phi(-p) \right) \exp (S[\phi])
\]
\[
= \int [d\phi'] \exp \left( - \int \frac{1}{2A(p)} \left( \phi'(-p) - \phi(-p) \right) \left( \phi'(p) - \phi(p) \right) + S[\phi'] \right)
\] (21)
(proven in Appendix A), we obtain the second relation

\[
e^{S_2[\phi]} = \int [d\phi'] \exp \left[ - \int_p \frac{p^2}{2(k_2(p)(K_1(p)/K_2(p))^2 - k_1(p))} \times \left( \phi'(p) - \frac{K_1(p)}{K_2(p)} \phi(p) \right) \left( \phi'(-p) - \frac{K_1(p)}{K_2(p)} \phi(-p) \right) + S_1[\phi'] \right] \tag{22}
\]

We have thus obtained two explicit formulas (19, 22) relating two equivalent Wilson actions. The remaining sections give applications of these formulas. In Appendix B we give corresponding results for a Dirac fermion field.

### III. EXACT RENORMALIZATION GROUP

Let us apply the results of the previous section to derive the exact renormalization group transformation of K. G. Wilson. (sect. 11 of [1]) We choose

\[
\begin{aligned}
K_1(p) &= K \left( \frac{p}{\Lambda} \right), \quad k_1(p) = k \left( \frac{p}{\Lambda} \right) \\
K_2(p) &= K \left( \frac{p}{\Lambda e^{-t}} \right), \quad k_2(p) = k \left( \frac{p}{\Lambda e^{-t}} \right)
\end{aligned}
\tag{23}
\]

so that the two sets of cutoff functions differ only by the choice of a momentum cutoff. We demand that the modified correlation functions (4) be independent of the momentum cutoff:

\[
\langle \langle \phi(p_1) \cdots \phi(p_n) \rangle \rangle_{S_2^{K_2,k_2}} = \langle \langle \phi(p_1) \cdots \phi(p_n) \rangle \rangle_{S_1^{K_1,k_1}} \tag{24}
\]

Now, using the first formula (19), we obtain

\[
e^{S_2[\phi]} = \exp \left[ \int_p \frac{1}{p^2} \left\{ k \left( \frac{p}{\Lambda} \right) e^t - k \left( \frac{p}{\Lambda} \right) \left( \frac{K \left( \frac{p}{\Lambda} e^t \right)}{K \left( \frac{p}{\Lambda} \right)} \right)^2 \right\} \frac{1}{2} \delta \frac{\delta}{\delta \phi(p)} \right] \times \exp \left( S_1 \left[ k \left( \frac{p}{\Lambda} \right) \phi(p) \right] \right) \tag{25}
\]

By denoting \( S_1 \) as \( S_\Lambda \) and \( S_2 \) as \( S_{\Lambda e^{-t}} \), and taking \( t \) infinitesimal, we obtain the exact renormalization group (ERG) differential equation

\[
- \Lambda \frac{\partial}{\partial \Lambda} e^{S_\Lambda[\phi]} = \int_p \left[ \frac{\Delta \left( \frac{p}{\Lambda} \right)}{K \left( \frac{p}{\Lambda} \right)} \phi(p) \frac{\delta}{\delta \phi(p)} \right] + \frac{1}{p^2} \left( 2 \frac{\Delta \left( \frac{p}{\Lambda} \right)}{K \left( \frac{p}{\Lambda} \right)} k \left( \frac{p}{\Lambda} \right) - \Lambda \frac{\partial}{\partial \Lambda} k \left( \frac{p}{\Lambda} \right) \right) \frac{1}{2} \delta \frac{\delta}{\delta \phi(p)} \delta \phi(-p) \right] e^{S_\Lambda[\phi]} \tag{26}
\]
where we define
\[ \Delta \left( \frac{p}{\Lambda} \right) \equiv \Lambda \frac{\partial}{\partial \Lambda} K \left( \frac{p}{\Lambda} \right) \] (27)
This amounts to (11.8) of ref. [1]. For the particular choice
\[ k(p) = K(p) \left( 1 - K(p) \right) \] (28)
gets simplified to
\[ -\Lambda \frac{\partial}{\partial \Lambda} e^{S_{\Lambda}[\phi]} = \int_p \left[ \frac{\Delta \left( \frac{p}{\Lambda} \right)}{K \left( \frac{p}{\Lambda} \right)} \phi(p) \frac{\delta}{\delta \phi(p)} + \frac{\Delta \left( \frac{p}{\Lambda} \right)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] e^{S_{\Lambda}[\phi]} \] (29)
This was introduced first by J. Polchinski.[2]
Alternatively, we can use the second formula (22) which gives
\[ e^{S_{\Lambda_{-t}}[\phi]} = \int [d\phi'] \exp \left[ -\int_p \frac{p^2}{k \left( \frac{p}{\Lambda} e^t \right)} \frac{k \left( \frac{p}{\Lambda} e^t \right)^2}{K \left( \frac{p}{\Lambda} e^t \right)} - k \left( \frac{p}{\Lambda} \right) \right. \\
\times \left. \frac{1}{2} \left( \phi'(p) - \frac{K \left( \frac{p}{\Lambda} e^t \right)}{K \left( \frac{p}{\Lambda} \right)} \phi(p) \right) \left( \phi'(-p) - \frac{K \left( \frac{p}{\Lambda} e^t \right)}{K \left( \frac{p}{\Lambda} \right)} \phi(-p) \right) + S_{\Lambda}[\phi'] \right] \] (30)
This is a well known integral solution of the ERG differential equation (26). (This is discussed in details, for example, in [4].) Though mathematically equivalent, our starting point (24) of this section is easier to understand than the differential equation (26) or its integral solution (30).

Earlier in ref. [5] it was observed that renormalized correlation functions of QED can be constructed out of correlation functions of its Wilson action: the modified correlation functions (4) coincide with renormalized correlation functions. Cutoff independent correlation functions have been also discussed by O. J. Rosten.[6]

IV. POLCHINSKI VS. WILSON

As a second application, we consider
\[ \begin{cases} 
K_1(p) = K \left( \frac{p}{\Lambda} \right), & k_1(p) = k \left( \frac{p}{\Lambda} \right) \\
K_2(p) = K' \left( \frac{p}{\Lambda} \right), & k_2(p) = k' \left( \frac{p}{\Lambda} \right) \equiv K' \left( \frac{p}{\Lambda} \right) \left( 1 - K' \left( \frac{p}{\Lambda} \right) \right) 
\end{cases} \] (31)
Note that \( k_2 \) follows Polchinski’s convention (28), which is convenient for perturbative applications. Given a solution \( S_1 \) of the ERG differential equation (26) with \( K_1, k_1 \), we wish to construct an equivalent \( S_2 \) that solves (26) with \( K_2, k_2 \).
Since the modified correlation functions are independent of $\Lambda$, we only need to demand equivalence at an arbitrary $\Lambda$. Using (19) and denoting $S_1$ as $S$ and $S_2$ as $S'$, we obtain
\[
e^{S'[\phi]} = \exp \left[ \int \frac{1}{p^2} \left( k'(p) - k(p) \left( \frac{K'(p)}{K(p)} \right)^2 \right) \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] \exp \left[ S \left( \frac{K}{K'} \phi \right) \right] \tag{32}\]
where division by $\Lambda$ for the variables of the cutoff functions is implied but omitted. We obtain a particularly simple result if we choose $K'$ satisfying
\[
k'(p) = k(p) \left( \frac{K'(p)}{K(p)} \right)^2 \tag{33}\]
This gives
\[
K'(p) = \frac{K(p)}{K(p)^2 + k(p)} \cdot K(p) \tag{34}\]
With this choice, we obtain
\[
S'[\phi] = S \left[ K^2 + \frac{k}{K} \phi \right] \tag{35}\]
For the particular choice of Wilson
\[
K(p) = e^{-p^2}, \quad k(p) = p^2 \tag{36}\]
we obtain
\[
K'(p) = \frac{1}{1 + p^2 e^{2p^2}} \tag{37}\]

V. ERG FOR FIXED POINTS

We now apply the results of sect. III to show the universality of critical exponents at a fixed point of the ERG transformation, by which we mean the independence of critical exponents on the choice of cutoff functions $K, k$. For the ERG transformation to have a fixed point, we must change the transformation given in sect. III in two ways: [1] first by adopting a dimensionless notation, and second by introducing an anomalous dimension to the scalar field. After these changes, the Wilson action $S_t$ depends on $t$ such that
\[
\langle \phi(p_1 e^{\Delta t}) \cdots \phi(p_n e^{\Delta t}) \rangle_{S_t + \Delta t}^{K, k} = e^{\Delta t \cdot n \left( -\frac{D+2}{2} + \gamma \right)} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_t}^{K, k} \tag{38}\]
for the same cutoff functions $K(p), k(p)$ independent of $t$. This is the new form of equivalence between $S_t$ and $S_{t+\Delta t}$: their modified correlation functions are the same up to a scale transformation. On the right-hand side, $-\frac{D+2}{2}$ gives the canonical mass dimension of the
field \( \phi(p) \) (since this is the Fourier transform, we obtain \( \frac{D-2}{2} - D = -\frac{D+2}{2} \)), and \( \gamma \) is the anomalous dimension, taken for simplicity as a \( t \)-independent constant.

Let us solve (38) to obtain \( S_{t+\Delta t} \) in terms of \( S_t \). Following the same line of arguments given in sect. II, we obtain

\[
e^{S_{t+\Delta t}[\phi]} = \exp \left( \int_p \frac{k(p)}{p^2} \left( \frac{\Delta(p)}{K(p)} + \frac{D+2}{2} - \gamma \right) \phi(p) + \frac{\partial \phi(p)}{\partial p} \right) \frac{\delta}{\delta \phi(p)} e^{S_t[\phi]} \]  

(39)  

where “subst” stands for the substitution of

\[
e^{\Delta t\left(\frac{D+2}{2} - \gamma\right)} \frac{K(p)}{K(p e^{\Delta t})} e^{\Delta t \gamma} \frac{\delta}{\delta \phi(p e^{\Delta t})} \]  

(40)  

for \( \phi(p) \). Since this substitution implies the substitution of

\[
e^{\Delta t\left(D - \frac{D+2}{2} + \gamma\right)} K(p e^{\Delta t}) \frac{\delta}{\delta \phi(p e^{\Delta t})} \]  

(41)  

for \( \frac{\delta}{\delta \phi(p)} \), we obtain

\[
e^{S_{t+\Delta t}[\phi]} = \exp \left[ \int_p \frac{1}{p^2} \left( k(p) - k(p e^{-\Delta t}) \right) \frac{K(p)^2}{K(p e^{-\Delta t})^2} e^{\Delta t \gamma} \frac{\delta}{\delta \phi(p)} e^{S_t[\phi]} \right] \]  

(42)  

Taking \( \Delta t \) infinitesimal, we obtain the ERG differential equation

\[
\partial_t e^{S_t[\phi]} = \int_p \left[ \left( \frac{\Delta(p)}{K(p)} + \frac{D+2}{2} - \gamma \right) \phi(p) + p_\mu \frac{\partial \phi(p)}{\partial p_\mu} \right] \frac{\delta}{\delta \phi(p)} e^{S_t[\phi]} \\
+ \int_p \frac{1}{p^2} \left( 2 \frac{\Delta(p)}{K(p)} k(p) + 2 p^2 \frac{dk(p)}{dp^2} - 2 \gamma k(p) \right) \frac{\delta}{\delta \phi(p)} e^{S_t[\phi]}  

(43)  

For Polchinski’s choice

\[
k(p) = K(p) (1 - K(p))  

(44)  

this gets simplified to

\[
\partial_t e^{S_t[\phi]} = \int_p \left[ \left( \frac{\Delta(p)}{K(p)} + \frac{D+2}{2} - \gamma \right) \phi(p) + p_\mu \frac{\partial \phi(p)}{\partial p_\mu} \right] \frac{\delta}{\delta \phi(p)} e^{S_t[\phi]} \\
+ \int_p \frac{1}{p^2} \left( \Delta(p) - 2 \gamma K(p) (1 - K(p)) \right) \frac{\delta}{\delta \phi(p)} e^{S_t[\phi]}  

(45)  

which is given in [4]. For Wilson’s choice

\[
K(p) = e^{-p^2}, \quad k(p) = p^2  

(46)  

[43] gives

\[
\partial_t e^{S_t[\phi]} = \int_p \left[ \frac{D}{2} \phi(p) + p_\mu \frac{\partial \phi(p)}{\partial p_\mu} \right] \frac{\delta}{\delta \phi(p)} e^{S_t[\phi]} \\
+ \int_p \left( 1 - \gamma + 2 p^2 \right) \left( \phi(p) \frac{\delta}{\delta \phi(p)} + \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) e^{S_t[\phi]}  

(47)  


which reproduces (11.17) of [1] under the identification
\[
\frac{d\rho(t)}{dt} = 1 - \gamma
\] (48)

Now, the anomalous dimension \(\gamma\) is chosen for the existence of a fixed point action \(S^*\) that satisfies
\[
\int P \left[ \left( \frac{\Delta(p)}{K(p)} + \frac{D + 2}{2} - \gamma \right) \phi(p) + p_\mu \frac{\partial \phi(p)}{\partial p_\mu} \right] \frac{\delta}{\delta \phi(p)} e^{S^*[\phi]}
+ \frac{1}{p^2} \left( 2 \frac{\Delta(p)}{K(p)} k(p) + 2 p^2 \frac{dk(p)}{dp^2} - 2 \gamma k(p) \right) \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} e^{S^*[\phi]} = 0
\] (49)
At the fixed point, the correlation functions obey the scaling law:
\[
\langle \langle \phi(p_1 e^t) \cdots \phi(p_n e^t) \rangle \rangle_{S^*}^{K,k} = e^{tn \left( \frac{D+2}{2} + \gamma \right)} \langle \langle \phi(p_1) \cdots \phi(p_n) \rangle \rangle_{S^*}^{K,k}
\] (50)
Only for specific choices of \(\gamma\), (49) has an acceptable solution. For example, if we assume \(S^*\) to be quadratic in \(\phi\), the solution becomes non-local unless \(\gamma = 0, -1, -2, \cdots\). We then obtain
\[
S^* = -\frac{1}{2} \int P \frac{p^{2(1-\gamma)}}{ZK(p)^2 + k(p)p^{2(-\gamma)}} \phi(p)\phi(-p)
\] (51)
which gives
\[
\langle \langle \phi(p)\phi(q) \rangle \rangle_{S^*}^{K,k} = \frac{Z}{p^{2(1-\gamma)} \delta(p+q)}
\] (52)
where \(Z\) is an arbitrary positive constant. (This is discussed in Appendix of [1].)

VI. UNIVERSALITY OF CRITICAL EXPONENTS

We now discuss universality of critical exponents at an arbitrary fixed point \(S^*\) of the ERG transformation, reviewed in the previous section. Universality within the ERG formalism has been shown in ref. [7]; our discussion below has the merit of conciseness.

\(S^*\) depends on \(K, k\), but we know from sect. III that for any choice of \(K, k\) there is an equivalent action that gives the same modified correlation functions. (19) gives the equivalent action \(S'^*\) for \(K', k'\) as
\[
e^{S'^*[\phi]} = \exp \left[ \int P \frac{1}{p^2} \left( k'(p) K(p)^2 - k(p) \right) \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \exp \left( S^* \left[ \frac{K}{K'} \phi \right] \right) \right]
\] (53)
Since the integrand of the exponent vanishes at \(p^2 = 0\), \(S^*\) and \(S'^*\) differ by local terms. Since
\[
\langle \langle \phi(p_1) \cdots \phi(p_n) \rangle \rangle_{S'^*}^{K',k'} = \langle \langle \phi(p_1) \cdots \phi(p_n) \rangle \rangle_{S^*}^{K,k}
\] (54)
the anomalous dimension $\gamma$ is independent of the choice of $K, k$.

Now, the anomalous dimension $\gamma$ is not the only critical exponent defined at the fixed point $S^*$. The other exponents appear as scale dimensions of local composite operators. A composite operator $O_y(p)$ with momentum $p$ is a functional of $\phi$ satisfying

$$\left\langle \frac{O_y(p e^t) \phi(p_1 e^t) \cdots \phi(p_n e^t)}{K,k} \right\rangle_{S^*}^{K,k} = e^{t \{-y + n(-\frac{D}{2} + \gamma)\}} \left\langle \frac{O_y(p) \phi(p_1) \cdots \phi(p_n)}{K,k} \right\rangle_{S^*}^{K,k}$$

(55)

where the modified correlation functions are defined by

$$\left\langle \frac{O_y(p) \phi(p_1) \cdots \phi(p_n)}{K,k} \right\rangle_{S^*}^{K,k} = \prod_{i=1}^{n} \frac{1}{K(p_i)} \left\langle O_y(p) \exp \left( - \int_p \frac{k(p)}{2} \frac{1}{p^2} \delta \phi(p) \delta \phi(-p) \right) \phi(p_1) \cdots \phi(p_n) \right\rangle_{S^*}$$

(56)

and $-y$ is the scale dimension of $O_y$. (The scale dimension of $O_y(x) = \int_p e^{ipx} O_y(p)$ in coordinate space is $D - y$.) For the equivalent fixed point action $S''^*$ with $K', k'$, the corresponding composite operator has the same modified correlation functions:

$$\left\langle \frac{O_y(p) \phi(p_1) \cdots \phi(p_n)}{K,k} \right\rangle_{S''^*}^{K,k'} = \left\langle \frac{O_y'(p) \phi(p_1) \cdots \phi(p_n)}{K',k'} \right\rangle_{S''^*}$$

(57)

This gives $O_y'(p)$ as

$$O_y'(p) e^{S''^*[\phi]} = \exp \left( \int_p \frac{1}{p^2} \left( k'(p) \frac{K(p)^2}{K'(p)^2} - k(p) \right) \frac{1}{\delta^2} \delta \phi(p) \delta \phi(-p) \right) \left[ O_y(p) e^{S^*[\phi]} \right]_{\text{subst}}$$

(58)

where “subst” implies substitution of

$$\frac{K(p)}{K'(p)} \phi(p)$$

(59)

into $\phi(p)$. The scale dimension $y$ is thus independent of the choice of $K, k$. We conclude that all the critical exponents are independent of $K, k$.

Before closing this section, we would like to discuss two issues related to the fixed point action $S^*$.

A. Ambiguity of the fixed point action

Given $K, k$, and an appropriate choice of $\gamma$, the fixed point solution $S^*$ of the ERG differential equation is still not unique. This is because normalization of the scalar field can be arbitrary.
Given $S^*$, we can construct $S^*_Z$ satisfying
\[ \{ \phi(p_1) \cdots \phi(p_n) \}^K_{S^*_Z} = Z^{2n} \{ \phi(p_1) \cdots \phi(p_n) \}^K_{S^*} \] (60)
To obtain $S^*_Z$, we set $K_2 = \sqrt{Z} K_1$ and $k_2 = k_1$ in (19). We then get
\[ e^{S^*_Z[\phi]} = \exp \left( - (Z - 1) \int p \frac{k(p)}{p^2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) e^{S^* \left[ \frac{\phi}{\sqrt{Z}} \right]} \] (61)
For example, the $Z$-dependence of the gaussian fixed point ($\gamma = 0$) is given by
\[ S_{G,Z}^*[\phi] = - \frac{1}{2} \int p \frac{k(p)}{Z K(p)^2 + k(p)} \phi(p) \phi(-p) \] (62)
Taking $Z = 1 + 2\epsilon$, where $\epsilon$ is infinitesimal, we obtain
\[ S^*_{1+2\epsilon}[\phi] - S^*[\phi] = \epsilon \mathcal{N}^*[\phi] \] (63)
where
\[ \mathcal{N}^*[\phi] \equiv - \int p \left\{ \phi(p) \frac{\delta S^*}{\delta \phi(p)} + \frac{k(p)}{p^2} \left( \frac{\delta S^*}{\delta \phi(p)} \frac{\delta S^*}{\delta \phi(-p)} + \frac{\delta^2 S^*}{\delta \phi(p) \delta \phi(-p)} \right) \right\} \] (64)
is a local composite operator satisfying
\[ \{ \mathcal{N}^*[\phi] \phi(p_1) \cdots \phi(p_n) \}^K_{S^*} = n \{ \phi(p_1) \cdots \phi(p_n) \}^K_{S^*} \] (65)
Obviously, $\mathcal{N}^*$ has scale dimension 0. $\mathcal{N}^*[\phi]$ is often called an equation-of-motion operator. [4]

**B. Universal fixed point action?**

We have shown that the modified correlation functions are universal up to normalization of the scalar field. We now ask if there is a universal Wilson action $S^*_{\text{univ}}$ that gives the universal modified correlation functions as its unmodified correlation functions:
\[ \{ \phi(p_1) \cdots \phi(p_n) \}^K_{S^*_{\text{univ}}} = \{ \phi(p_1) \cdots \phi(p_n) \}^K_{S^*} \] (66)
This implies
\[ e^{S^*_{\text{univ))[\phi]} = \left[ \exp \left( - \frac{1}{2} \int p \frac{k(p)}{p^2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) e^{S^*[\phi]} \right]_{\text{sub}} \] (67)
where $S^*$ is the fixed point action for $K, k$, and “sub” denotes substitution of
\[ K(p) \phi(p) \] (68)
for $\phi(p)$. The above result is obtained from (19) by setting

$$K_2 = 1, \quad k_2 = 0 \quad (69)$$

We expect that the right-hand side is independent of $K, k$; i.e., $S^*_{\text{univ}}$ has no cutoff. But we know that the use of a cutoff is essential for Wilson actions, and there must be something wrong with $S^*_{\text{univ}}$.

Let us first consider the example of the gaussian fixed point given by

$$S_G[\phi] = -\frac{1}{2} \int_p \frac{p^2}{K(p)^2 + k(p)^2} \phi(p)\phi(-p) \quad (70)$$

This gives the modified two-point function

$$\langle\langle \phi(p)\phi(q)\rangle\rangle^{K,k}_{S_G} = \frac{1}{p^2} \delta(p + q) \quad (71)$$

(67) indeed gives an action free from a cutoff:

$$S_G,\text{univ}[\phi] = -\frac{1}{2} \int_p p^2 \phi(p)\phi(-p) \quad (72)$$

For interacting theories, though, we expect (67) makes no sense. Let us look at this a little more closely. As $K_2, k_2$, we choose

$$\begin{cases} K_2(p) = K_{-t}(p) \equiv K(pe^{-t}) \\ k_2(p) = k_{-t}(p) \equiv k(pe^{-t}) \end{cases} \quad (73)$$

In the limit $t \to +\infty$, we obtain (69):

$$\lim_{t \to +\infty} K_{-t}(p) = 1, \quad \lim_{t \to +\infty} k_{-t}(p) = 0 \quad (74)$$

We then define $S^*_{-t}$ so that

$$\langle\langle \phi(p_1)\cdots\phi(p_n)\rangle\rangle^{K_{-t},k_{-t}}_{S^*_{-t}} = \langle\langle \phi(p_1)\cdots\phi(p_n)\rangle\rangle^{K,k}_{S^*} \quad (75)$$

$S^*_{-t}$ is related to $S^*$ by the ERG transformation of sect. III. Since the momentum cutoff of $S^*$ is of order 1 (we are using the dimensionless convention), that of $S^*_{-t}$ is of order $e^t$. Hence, $S^*_{\text{univ}}$ has the infinite momentum cutoff. We then expect that the terms of $S^*_{\text{univ}}$ to have divergent coefficients.

Thus, there is no fixed point action $S^*_{\text{univ}}$ that gives the correlation functions without modification.
VII. CONCLUDING REMARKS

We have introduced the concept of equivalence among Wilson actions. Our equivalence is physically more transparent than the other formulations of the exact renormalization group via differential equations or integral formulas. In particular we have applied our equivalence to obtain a simple proof of universality of critical exponents within the ERG formalism.

Appendix A: Gaussian Formula

In this appendix we prove the formula

\[
\exp \left[ \int A(p) \frac{1}{2} \delta \phi(p) \delta \phi(-p) \right] \exp [S[\phi]]
= \int [d\phi'] \exp \left[ - \int A(p) \phi'(p) \phi'(p) + S[\phi + \phi'] \right]
\]

(A1)

Though the following proof requires the positivity of \( A(p) \), we expect the formula to remain valid as long as both hand sides make sense. The left-hand side makes sense for any \( A(p) \), and the right-hand side makes sense even if \( A(p) < 0 \) for some \( p \) as long as convergence of functional integration is provided by the Wilson action \( S \).

It is easy to understand this formula in terms of Feynman graphs. The right-hand side implies the coupling of a scalar field \( \phi' \) whose propagator is \( A(p) \). Contracting the pairs of \( \phi' \), we obtain the left-hand side. More formally, we can prove the equality by comparing the generating functionals of both hand sides for arbitrary source \( J(p) \). Let us first compute the generating functional of the left-hand side:

\[
e^{W_L[J]} \equiv \int [d\phi] \exp \left[ \int J(-p) \phi(p) \right] \exp \left[ \int A(p) \frac{1}{2} \delta \phi(p) \delta \phi(-p) \right] \exp [S[\phi]]
\]

(A2)

Integrating this by parts, we obtain

\[
e^{W_L[J]} = \int [d\phi] \exp [S[\phi]] \exp \left[ \int A(p) \frac{1}{2} \delta \phi(p) \delta \phi(-p) \right] \exp \left[ \int J(-p) \phi(p) \right]
= \int [d\phi] \exp [S[\phi]] + \int \left( J(-p) \phi(p) + \frac{1}{2} J(-p) A(p) J(p) \right)
\]

(A3)

We next compute the generating functional of the right-hand side:

\[
e^{W_R[J]} \equiv \int [d\phi] \exp \left[ \int J(-p) \phi(p) \right] \\
\times \int [d\phi'] \exp \left[ - \frac{1}{2} \int A(p) \phi'(p) \phi'(p) + S[\phi + \phi'] \right]
\]

(A4)
We first shift $\phi'$ by $-\phi$, and then shift $\phi$ by $+\phi'$ to obtain
\[
e^{W_R[J]} = \int [d\phi][d\phi'] \exp \left[ \int_p J(-p) (\phi(p) + \phi'(p)) \right] \\
\times \exp \left[ -\frac{1}{2} \int_p \frac{1}{A(p)} \phi(p)\phi(-p) + S[\phi'] \right] \\
= \int [d\phi] \exp \left[ \int_p \left( -\frac{1}{2A(p)} \phi(p)\phi(-p) + J(-p)\phi(p) \right) \right] \\
\times \int [d\phi'] \exp \left[ \int_p J(-p)\phi'(p) + S[\phi'] \right]
\] (A5)

If $A(p)$ is positive, we can perform the gaussian integral over $\phi$ to obtain
\[
e^{W_R[J]} = \int [d\phi'] \exp \left[ \frac{1}{2} \int_p J(-p)A(p)J(p) + \int_p J(-p)\phi'(p) + S[\phi'] \right]
\] (A6)

We thus obtain
\[
W_L[J] = W_R[J]
\] (A7)

for arbitrary $J$. This proves the gaussian formula (A1).

Finally, shifting $\phi'$ by $-\phi$, we rewrite (A1) as
\[
\exp \left[ \int_p A(p) \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right] \exp[S[\phi]] \\
= \int [d\phi'] \exp \left[ -\int_p \frac{1}{2A(p)} (\phi'(-p) - \phi(-p))(\phi'(p) - \phi(p)) + S[\phi'] \right]
\] (A8)

This is the form used in section II.

**Appendix B: Equivalence of Fermionic Wilson Actions**

For a Dirac spinor field $\psi$ and its complex conjugate $\bar{\psi}$, we define modified correlation functions by
\[
\left\langle \psi(p_1) \cdots \psi(p_n) \bar{\psi}(q_n) \cdots \bar{\psi}(q_1) \right\rangle^K_{S,k} = \prod_{i=1}^n \frac{1}{K(p_i)K(q_i)} \\
\times \left\langle \psi(p_1) \cdots \psi(p_n) \exp \left( -\int_p \frac{\delta}{\delta\psi(p)} \frac{k(p)}{\bar{p}} \frac{\delta}{\delta\bar{\psi}(-p)} \right) \bar{\psi}(q_n) \cdots \bar{\psi}(q_1) \right\rangle_{S}
\] (B1)

so that
\[
\left\langle \psi(p) \bar{\psi}(q) \right\rangle^K_{S} = \frac{1}{K(p)^2} \left[ \left\langle \psi(p) \bar{\psi}(q) \right\rangle_{S} - \frac{k(p)}{\bar{p}} \delta(p + q) \right]
\] (B2)

Two Wilson actions $S_{1,2}$ are equivalent if $K_{1,2}$ and $k_{1,2}$ exist so that
\[
\left\langle \psi(p_1) \cdots \psi(p_n) \bar{\psi}(q_n) \cdots \bar{\psi}(q_1) \right\rangle^{K_{1,k_1}}_{S_{1}} = \left\langle \psi(p_1) \cdots \psi(p_n) \bar{\psi}(q_n) \cdots \bar{\psi}(q_1) \right\rangle^{K_{2,k_2}}_{S_{2}}
\] (B3)
The formula analogous to (22) is given by
\[
e^{S_2[\psi, \bar{\psi}]} = \int [d\psi' d\bar{\psi}'] \exp \left[ - \int_p \left( \bar{\psi}'(-p) - \frac{K_1(p)}{K_2(p)} \bar{\psi}(-p) \right) \frac{\not{p}}{k_2(p)K_1(p)^2 - k_1(p)} \times \left( \psi'(p) - \frac{K_1(p)}{K_2(p)} \psi(p) \right) + S[\psi', \bar{\psi}'] \right]
\] (B4)

The formula analogous to (19) is somewhat more complicated to write down. Denoting
\[
A_{ab}(p) \equiv \left( \frac{1}{p} \right)_{ab} \left( k_2(p) - k_1(p) \frac{K_2(p)^2}{K_1(p)^2} \right)
\] (B5)
we obtain
\[
e^{S_2[\psi, \bar{\psi}]} = \text{Tr} \left[ \exp \left( \int_p \frac{\delta}{\delta \psi(p)} A(p) \frac{\delta}{\delta \bar{\psi}(-p)} \right) \exp \left( S_1 \left[ \frac{K_1}{K_2} \psi, \frac{K_1}{K_2} \bar{\psi} \right] \right) \right]
\] (B6)
\[
\equiv \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \int_{p_1, \ldots, p_n} \prod_{i=1}^{n} A_{a_i b_i}(p_i)
\times \frac{\delta}{\delta \psi_{b_n}(-p_n)} \ldots \frac{\delta}{\delta \psi_{b_1}(-p_1)} \exp \left( S_1 \left[ \frac{K_1}{K_2} \psi, \frac{K_1}{K_2} \bar{\psi} \right] \right) \frac{\delta}{\delta \psi_{a_1}(p_1)} \ldots \frac{\delta}{\delta \psi_{a_n}(p_n)}
\] (B7)
where the spinor indices are summed over. The exponential implies contraction of \(\psi(p)\bar{\psi}(q)\) by \(A(p)\delta(p + q)\).

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