ON THE $q$-ANALOGUE OF TWO-VARIABLE $p$-ADIC $L$-FUNCTION

Taekyun Kim

Institute of Science Education,
Kongju National University, Kongju 314-701, S. Korea
e-mail: tkim64@hanmail.net (or tkim@kongju.ac.kr)

Abstract. We construct the two-variable $p$-adic $q$-$L$-function which interpolates the generalized $q$-Bernoulli polynomials associated with primitive Dirichlet character $\chi$. Indeed, this function is the $q$-extension of two-variable $p$-adic $L$-function due to Fox, corresponding to the case $q = 1$. Finally, we give some $p$-adic integral representation for this two-variable $p$-adic $q$-$L$-function and derive to $q$-extension of the generalized formula of Diamond and Ferro and Greenberg for the two-variable $p$-adic $L$-function in terms of the $p$-adic gamma and log gamma function.

§1. Introduction

Let $p$ be a fixed prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q - 1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Kubota and Leopoldt proved the existence of meromorphic functions, $L_p(s, \chi)$, defined over the $p$-adic number field, that serve as $p$-adic equivalents of the Dirichlet $L$-series, cf.[9, 11].

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These $p$-adic $L$-functions interpolate the values

$$L_p(1 - n, \chi) = -\frac{1}{n} (1 - \chi_n(p)p^{n-1}) B_{n,\chi_n}, \text{ for } n \in \mathbb{N} = \{1, 2, \cdots, \} ,$$

where $B_{n,\chi}$ denote the $n$th generalized Bernoulli numbers associated with the primitive Dirichlet character $\chi$, and $\chi_n = \chi w^{-n}$, with $w$ the Teichmüller character, cf.[1-37]. In this paper, we use the notation:

$$[x]_q = [x : q] = \frac{1 - q^x}{1 - q}.$$

Hence, $\lim_{q \to 1}[x] = x$ for any $x$ with $|x|_p \leq 1$ in the present $p$-adic case. Let $d$ be a fixed integer and let $p$ be a fixed prime number. We set

$$X = X_d = \lim_{N \to \infty} (\mathbb{Z}/dp^N\mathbb{Z}), \ X_1 = \mathbb{Z}_p ,$$

$$X^* = \bigcup_{0 < a < dp} \ a + dp\mathbb{Z}_p ,$$

$$a + dp^N\mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \} ,$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, cf. [11-30].

For any positive integer $N$, we set

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]} , \text{ cf.}[18] ,$$

and this can be extended to a distribution on $X$. This distribution yields an integral for each non-negative integer $m$:

$$\beta^*_{m,q} = \int_{\mathbb{Z}_p} [x]^m d\mu_q(x) = \int_X [a]^m d\mu_q(a) = \frac{1}{(1 - q)^m} \sum_{i=0}^{m} \binom{m}{i} (-1)^i \frac{i + 1}{[i + 1]} ,$$

where $\beta^*_{m,q}$ are the $m$th Carlitz’s $q$-Bernoulli numbers, cf. [3, 4, 5]. In a recent paper [10], Fox defined a two-variable $p$-adic $L$-function $L_p(s, t|\chi)$ with the property that

$$L_p(1 - m, t|\chi) = -\frac{B_{m,\chi_m}(p^*t) - \chi_m(p)p^{m-1}B_{m,\chi_m}(p^{-1}p^*t)}{m} ,$$
for positive integer $m$ and $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, where $p^* = p$ if $p > 2$ and $p^* = 4$ if $p = 2$, and $B_{m,\chi_n}(x)$ are the $m$th generalized Bernoulli polynomials attached to $\chi$. In [12], we gave the interesting results on the $p$-adic $q$-$L$-functions, a subject initiated by Neal Koblitz [24], in the beginning of the 1980’s in which the author made some contributions, cf.[11-23]. For positive integer $n$, these functions satisfy

$$L_{p,q}(1 - n, \chi) = -\frac{\beta_{n,q,\chi_n} - [p]_q^{n-1}\chi_n(p)\beta_{n,q^*,\chi_n}}{n},$$

where $\beta_{n,q,\chi_n}$ are the $n$th generalized $q$-Bernoulli numbers attached to $\chi$ which are defined by author, cf.[20]. The purpose of this paper is to construct a two-variable $p$-adic $q$-$L$-function $L_{p,q}(s,t|\chi)$ for the Dirichlet character $\chi$ with the property that

$$L_{p,q}(1 - m, t|\chi) = -\frac{\beta_{n,q,\chi_n}(p^*t) - \chi_n(p)[p]_q^{n-1}\beta_{n,q^*,\chi_n}(p^{-1}p^*t)}{n}, \quad n \in \mathbb{Z}_+, \quad n \in \mathbb{Z}_+,$$

where $\beta_{n,q,\chi_n}(x)$ are the $n$th generalized $q$-Bernoulli polynomials attached to $\chi$. This function is actually a $q$-extension of the two-variable $p$-adic $L$-function of Fox, corresponding to the case $q = 1$. For a prime number $p$ and for a Dirichlet character defined modulo some integer, the $p$-adic $L$-function was constructed by interpolating the values of complex analytic $L$-function at non-positive integers. Diamond [7, 8] obtained formulas which express the values of $p$-adic $L$-function at positive integers in terms of the $p$-adic log gamma function. In this paper, we give the $q$-extension of his results to the case of the two-variable $p$-adic $q$-$L$-function and obtain the formulas which express the values of $\frac{\partial}{\partial s}L_{p,q}(0, t|\chi)$ in terms of the $q$-extension of Diamond $p$-adic log gamma function. Finally, we give the values of $L_{p,q}(s,t|\chi)$ at $s = 1$.

2. $q$-extension of two-variable Dirichlet’s $L$-series

In this section we assume that $q \in \mathbb{C}$ with $|q| < 1$. The $q$-Bernoulli numbers are usually defined by

$$\beta_{0,q} = \frac{q - 1}{\log q}, \quad (q\beta_q + 1)^n - \beta_{n,q} = \delta_{n,1},$$

where $\delta_{n,1}$ is the Kronecker symbol and we use the usual convention about replacing $\beta_q^i$ by $\beta_{i,q}$, cf.[13, 14]. Note that $\lim_{q \to 1} \beta_{k,q} = B_k$, where $B_k$ are the $k$th ordinary Bernoulli numbers. In [14] the $q$-Bernoulli polynomials are defined by

$$\beta_{n,q}(x) = \sum_{i=0}^{n} \binom{n}{i} q^{\delta_{i,q}} (x)^{n-i} = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} i (-q^x i)^i.$$
From the Eq.(1), we note that

$$\beta_{n,q} = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \frac{i}{[i]_q},$$

where $\binom{n}{i}$ is the binomial coefficient.

Thus, we have the generating function of $q$-Bernoulli numbers as follows:

$$F_q(t) = e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \frac{j}{[j]_q} \left( \frac{1}{q-1} \right)^j t^j = \sum_{j=0}^{\infty} \frac{\beta_{j,q} t^j}{j!}, \text{ for } |t| < 1.$$  

By (3), we easily see that the $q$-Bernoulli numbers are the unique solutions of the following $q$-difference equation in the complex plane:

$$F_q(t) = \frac{q-1}{\log q} e^{\frac{t}{1-q}} - t \sum_{n=0}^{\infty} q^n e^{[n]_q t}, \text{ for } |t| < 1.$$  

In the Eq.(2), we consider the generating function of $q$-Bernoulli polynomials as follows:

$$\sum_{n=0}^{\infty} \beta_{n,q} (x) \frac{t^n}{n!} = F_q(x, t) = e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \frac{j}{[j]_q} \left( \frac{1}{q-1} \right)^j q^{jx} \frac{t^j}{j!}, \text{ for } |t| < 1.$$  

Thus, we obtain the following $q$-difference equation for the generating function of $q$-Bernoulli polynomials in the complex plane:

$$F_q(x, t) = \frac{q-1}{\log q} e^{\frac{t}{1-q}} - t \sum_{n=0}^{\infty} q^{n+x} e^{[n+x]_q t}, \text{ for } |t| < 1.$$  

Let $\chi$ be the Dirichlet character with conductor $f = f_\chi \in \mathbb{N}$. Then the generalized $q$-Bernoulli numbers attached to $\chi$, $\beta_{n,\chi,q}$, are defined by

$$F_{q,\chi}(t) = -t \sum_{n=1}^{\infty} \chi(n) q^n e^{[n]_q t} = \sum_{n=0}^{\infty} \beta_{n,\chi,q} \frac{t^n}{n!}, \text{ for } |t| < 1.$$  

Remark. From the Eq.(7), we note that

$$\lim_{q \to 1} F_{q,\chi}(t) = \sum_{a=1}^{f} \frac{\chi(a) e^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_n x \frac{t^n}{n!}.$$
By the Eq.(7), we easily see that

\[
\beta_{n,\chi,q} = [f]^k q^{[\frac{a}{f}] - 1} \sum_{a=1}^{f} \chi(a) \beta_{k,q,f}(\frac{a}{f}).
\]

We now also define the generalized \( q \)-Bernoulli polynomials attached to \( \chi \) as follows:

\[
F_{q,\chi}(x, t) = -t \sum_{n=1}^{\infty} \chi(n) q^{n+x} e^{[n+x]q^t} = \sum_{n=0}^{\infty} \beta_{n,\chi,q}(x) \frac{t^n}{n!}.
\]

Thus, we obtain the below formula:

\[
F_{q,\chi}(x, t) = -t \sum_{a=1}^{f} \chi(a) \sum_{n=0}^{\infty} q^{n+f+a+x} e^{[nf+a+x]q^t}.
\]

From this, we note that

\[
\beta_{n,\chi,q}(x) = \sum_{k=0}^{n} \binom{n}{k} q^{kx} \beta_{k,\chi}[x]_q^{n-k} = (q^x \beta_{\chi} + [x]_q)^n,
\]

with the usual convention about replacing \( \beta_{\chi}^n \) by \( \beta_{n,\chi,q} \).

Let \( g \) be a positive integral multiple of \( f = f_\chi \). Then for each \( n \in \mathbb{Z}, n \geq 0 \), we have

\[
\sum_{n=0}^{\infty} \beta_{n,\chi,q}(x) \frac{t^n}{n!}
= -t \sum_{a=0}^{g-1} \chi(a) \sum_{n=0}^{\infty} q^{gn+a+x} e^{[gn+a+x]q^t} = -t \sum_{a=0}^{g-1} \chi(-a) \sum_{n=1}^{\infty} q^{g(n+\frac{x-a}{g})} e^{[g\frac{x-a}{g}]q^t}
= \sum_{n=0}^{\infty} \left([g]^n q^{-1} \sum_{a=0}^{g-1} \chi(-a) \beta_{n,q^g}(\frac{x-a}{g})\right) \frac{t^n}{n!}.
\]

Therefore, we obtain the following lemma:
Lemma 1. Let $g$ be a positive integral multiple of $f = f\chi$. Then for each $n \in \mathbb{N} \cup \{0\}$, we have

$$
\beta_{n,\chi,q}(x) = [g]_q^{n-1} \sum_{a=0}^{g-1} \chi(a)\beta_{n,q}(x + \frac{a}{g}) = [g]_q^{n-1} \sum_{a=0}^{g-1} \chi(-a)\beta_{n,q}(x - \frac{a}{g}).
$$

Note that the series on the right hand side of (6) and (9) are uniformly convergent. Hence, we easily see that

$$
\beta_{k,q}(x) = \frac{d^k}{dt^k} F_q(x, t)|_{t=0} = \frac{q-1}{\log q} \frac{1}{(1-q)^k} - k \sum_{n=0}^{\infty} q^{n+x}[n+x]_q^{k-1},
$$

and,

$$
\beta_{k,\chi,q}(x) = \frac{d^k}{dt^k} F_q,\chi(x, t)|_{t=0} = -k \sum_{n=1}^{\infty} [n+x]_q^{k-1} q^{n+x} \chi(n),\ k \geq 1.
$$

Therefore, we obtain the following theorem:

Theorem 2. Let $\chi$ be the primitive character with conductor $f = f\chi$. For $k \geq 1$, $q \in \mathbb{C}$ with $|q| < 1$, we have

$$
\beta_{k,q}(x) = \frac{q-1}{\log q} \frac{1}{(1-q)^k} - k \sum_{n=0}^{\infty} q^{n+x}[n+x]_q^{k-1},
$$

and,

$$
\beta_{k,\chi,q}(x) = -k \sum_{n=1}^{\infty} [n+x]_q^{k-1} q^{n+x} \chi(n).
$$

In [], the $q$-analogue of the Hurwitz’s zeta function was defined by

$$
\zeta_q(s, x) = \sum_{n=0}^{\infty} \frac{q^{n+x}}{[n+x]_q^s} - \frac{1}{(s-1)} \frac{(1-q)^s}{\log q},\ s \in \mathbb{C}.
$$

This function is meromorphic for $s \in \mathbb{C}$ with simple pole at $s = 1$. By using Theorem 2, we easily see that $\zeta_q(1-n, x) = -\frac{\beta_{n,q}(x)}{n}$, $n \in \mathbb{N}$. From the results of the above Theorem 2, we can consider the $q$-analogue of two-variable Dirichlet’s $L$-series as follows:
Definition 3. For $s \in \mathbb{C}$, we define the $q$-analogue of two-variable Dirichlet $L$-series as

$$L_q(s, x|\chi) = \sum_{n=1}^{\infty} q^{n+x} \frac{\chi(n)}{[n+x]^s_q} = (q-1) \sum_{n=1}^{\infty} \frac{\chi(n)}{[n+x]^s_{q^{-1}}} + \sum_{n=1}^{\infty} \frac{\chi(n)}{[n+x]^s_q}.$$

In [12] one variable $q$-$L$-series are defined by $L_q(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{[n]^s_q}, \ s \in \mathbb{C}$. Thus, we see that $L_q(s, 0|\chi) = L_q(s, \chi)$. In the previous paper [23] we also defined the two-variable Dirichlet $L$-series as $L(s, x|\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{(n+x)^s}, \ s \in \mathbb{C}$. From this definition, we note that $\lim_{q \to 1} L_q(s, x|\chi) = L(s, x|\chi)$. By Theorem 2 and Definition 3, we obtain the following corollary:

Corollary 4. If $k \geq 1$, $0 \leq x \leq 1$, then we have

$$L_q(1-k, x|\chi) = -\frac{\beta_{k, x, q}(x)}{k}.$$ 

Remark. Let $\chi$ be a Dirichlet character with conductor $f = f_\chi$. By (7) and (8), we easily see that

$$\sum_{k=0}^{nf-1} \frac{\chi(k)q^{k|l|}}{[k]_q} = \frac{1}{l+1} \left( \beta_{l, x, q}(nf) - \beta_{l+1, x, q} \right), \ n, l \in \mathbb{N}. \tag{11}$$

In [31] M. Schlosser investigated the $q$-analogaues of the sums of consecutive integers, squares, cubes, quarts and quint. That is, his $q$-analogue of $\sum_{k=1}^{n} k^m$, for $m = 1, 2, 3, 4, 5$, gave by employing specific identities for very-well-posed basic hypergeometric series, in conjunction with suitable specialization of the parameters. In the final page of his paper he did guess that reasonable continuation involving higher integer powers will follow the same his pattern. The Eq.(11) is the generalization for the problem which was guessed by Schlosser in [31]. Indeed if we take trivial character in Eq.(11), then Eq.(11) becomes the $q$-analogue of the sums of powers of consecutive integers involving higher order. By using the Eq.(10) and Definition 3, we obtain the below identity:

$$L_q(s, x|\chi) = [f]_q^{-s} \sum_{a=1}^{f} \chi(a)\zeta_q(f, \frac{a+x}{f}). \tag{12}$$
Let $\Gamma(s)$ be the gamma function. Then we can readily see that

\[
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} F_q(x, -t) dt
\]

\[
= \frac{q - 1}{\log q} \frac{1}{\Gamma(s)} \int_0^\infty (1 - q)^{s-1} x^{s-1} e^{-x} dx + \sum_{n=0}^{\infty} \frac{q^{n+x}}{[n+x]_q^s} \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt} t^{s-1} dt
\]

\[
= -\frac{1}{s-1} \frac{(1-q)^s}{\log q} + \sum_{n=0}^{\infty} \frac{q^{n+x}}{[n+x]_q^s} = \zeta_q(s, x).
\]

Therefore, we obtain the followings lemma:

**Lemma 5.** For $s \in \mathbb{C}$, we have

\[
\zeta_q(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} \left( \frac{q - 1}{\log q} e^{-\frac{t}{1-q}} + t \sum_{n=0}^{\infty} q^{n+x} e^{-t[n+x]_q} \right) dt.
\]

By applying Mellin transforms and residue theorem to Lemma 5, we can derive the below identity:

\[
\zeta_q(1-n, x) = \frac{(-1)^n}{n!} \beta_{n,q}(x)(2\pi i) \left( \frac{(n-1)!}{2\pi i} (-1)^{n-1} \right), \quad n \in \mathbb{N}.
\]

In the case $x = 1$ we note that $\zeta_q(1-n, 1) = -\frac{\beta_{n,q}(1)}{n} = -\frac{(q\beta_q+1)^n}{n} = -\frac{\beta_{n,q}}{n}, \quad n > 1$. By (9), we easily see that

\[
\frac{1}{\Gamma(s)} \int_0^\infty F_q,\chi(x, -t)t^{s-2} dt
\]

\[
= \sum_{n=1}^{\infty} \chi(n)q^{n+x} \frac{1}{\Gamma(s)} \int_0^\infty e^{-[n+x]_q t} t^{s-1} dt = L_q(s, x|\chi), \quad s \in \mathbb{C}.
\]

Thus, we have

\[
L_q(1-n, x|\chi) = -\frac{\beta_{n,x,q}(x)}{n}, \quad n \in \mathbb{N}.
\]
Let
\[
H_q(s, a, F) = \sum_{m \equiv a \pmod{m \geq 0}} q^m \left\lfloor \frac{m}{q} \right\rfloor s + \frac{1}{F} (1 - q)^s (1 - s) \log q
\]
(14)
\[
= \sum_{n=0}^{\infty} \frac{q^{a+nF}}{[a+nF]^s} + \frac{1}{F} (1 - q)^s (1 - s) \log q = [F]_q^{-s} \zeta_q^F(s, a/F),
\]
where \(a\) and \(F\) are positive integers with \(0 < a < F\). Let \(\chi(\neq 1)\) be the Dirichlet character with conductor \(F\). Then the \(q\)-analogue of Dirichlet \(L\)-function can be expressed as the sum
\[
L_q(s, \chi) = \sum_{a=1}^{F} \chi(a) H_q(s, a, F), \quad s \in \mathbb{C}, \text{ cf.}[17].
\]
(15)
The function \(H_q(s, a, F)\) is a meromorphic for \(s \in \mathbb{C}\) with simple pole at \(s = 1\), having residue \(\frac{q^{-1}}{F \log q}\), and it interpolates the values
\[
H_q(1-n, a, F) = - \frac{[F]_q^{n-1}}{n} \beta_{n,q,F}(a/F), \quad \text{where } n \in \mathbb{Z}, \ n \geq 1.
\]
(16)
We now modify the partial \(q\)-zeta function as follows:
\[
H_q(s, a, F) = \frac{1}{s-1} \frac{1}{[F]_q} [a]_q^{1-s} \sum_{j=0}^{\infty} \left(1 - \frac{s}{j}\right) q^{aj} \beta_{j,q,F} \left(\frac{F}{a}\right)_q^j, \quad \text{for } s \in \mathbb{C}.
\]
(17)
By (14), (15), (16) and (17), we easily see that
\[
L_q(s, \chi) = \frac{1}{s-1} \frac{1}{[F]_q} \sum_{a=1}^{F} \chi(a) [a]_q^{1-s} \sum_{m=0}^{\infty} \left(1 - \frac{s}{m}\right) q^{am} \beta_{m,q,F} \left(\frac{F}{a}\right)_q^m.
\]
(18)
From the Definition 3, we note that \(L_q(s, x|\chi)\) is analytic for \(s \in \mathbb{C}\), except \(s \neq 1\) when \(\chi \neq 1\). Using Eq.(17) to define \(H_q(s, a + x, F)\) for all \(a \in \mathbb{Z}\) with \(0 < a < F\), \(x \in \mathbb{R}\) with \(0 \leq x \leq 1\), we obtain
\[
L_q(s, x|\chi) = \sum_{a=1}^{F} \chi(a) H_q(s, a + x, F).
\]
(19)
Let $F$ and $a$ be positive integers with $0 < a < F$, and let
\begin{equation}
L_q(s, x|\chi) = \frac{1}{s-1} [F]_q \sum_{a=1}^{F} \chi(a) [a + x]_q^{1-s} \sum_{m=0}^{\infty} \left(1 - \frac{s}{m}\right) q^{(a+x)m} \beta_{m,q,F} \left[\frac{F}{a + x}\right]_q^{m}.
\end{equation}

Then, $L_q(s, x|\chi)$ is analytic for $s \in \mathbb{C}$, except $s \neq 1$ when $\chi = 1$. Furthermore, for each $n \in \mathbb{Z}$ with $n \geq 1$, we have
\begin{equation}
L_q(1-n, x|\chi) = -\frac{\beta_{n,x,q}(x)}{n}.
\end{equation}

In this section we introduced some of the basic facts about one-variable $q$-L-series and two-variable $q$-L-series in complex plane. Then their values at negative integers are given in terms of generalized $q$-Bernoulli numbers and polynomials attached to $\chi$. We also evaluate $L_q(1, x|\chi)$ and give some relation with $q$-Bernoulli numbers and polynomials. By the definition of $L_q(s, x|\chi)$, we easily see that
\begin{equation}
L_q(s, x|\chi) = \frac{1}{s-1} [F]_q \sum_{a=1}^{F} \chi(a) [a + x]_q^{1-s} + [a + x]_q^{1-s} \sum_{m=1}^{\infty} \left(1 - \frac{s}{m}\right) q^{(a+x)m} \beta_{m,q,F} \left[\frac{F}{a + x}\right]_q^{m}.
\end{equation}

We now give the below Taylor expansion of $[a + x]_q^{1-s}$ at $s = 1$:
\begin{equation}
[a + x]_q^{1-s} = 1 - (s-1) \log[a + x]_q + \cdots.
\end{equation}

Thus, we see that
\begin{equation}
L_q(1, x|\chi) = \frac{1}{[F]_q} \sum_{a=1}^{F} \chi(a) \left\{ -\log([a + x]_q) + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} q^{(a+x)m} \beta_{m,q,F} \left[\frac{F}{a + x}\right]_q^{m} \right\}.
\end{equation}

In the case $x = 0$, we have
\begin{equation}
L_q(1, 0|\chi) = L_q(1, \chi) = \frac{1}{[F]_q} \sum_{a=1}^{F} \chi(a) \left\{ -\log([a]_q) + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} q^{am} \beta_{m,q,F} \left[\frac{F}{a}\right]_q^{m} \right\}.
\end{equation}

The values of $L_q(s, x|\chi)$ at negative integers are algebraic, hence may be regarded as lying in an extension of $\mathbb{Q}_p$. We therefore look for a $p$-adic function which agrees with $L_q(s, x|\chi)$ at the negative integers in the next section.
3. Two-variable $p$-adic $q$-$L$-functions

In this section we shall consider the $p$-adic analogs of the two-variable $q$-$L$-functions which were introduced in the previous section. Indeed this functions are the $q$-analogs of the $p$-adic functions due to Fox, corresponding to the case $q = 1$. Let $w$ denote the Teichmüller character, having conductor $f_w = p^*$. For an arbitrary character $\chi$, we define $\chi_n = \chi w^{-n}$, where $n \in \mathbb{Z}$, in the sense of the product of characters. Throughout this section, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < p^{-\frac{1}{p-1}}$. Let $< a : q > = w^{-1}(a)[a]_q = \frac{|a|_q}{w(a)}$. Then, we note that $< a : q > = 1$ (mod $p^*p^{-\frac{1}{p-1}}$). By the definition of $< a : q >$, we easily see that $< a + p^*t : q > = w^{-1}(a + p^*t)[a + p^*t]_q = w^{-1}(a)[a]_q + w^{-1}(a)q^a[p^*t]_q \equiv 1$ (mod $p^*p^{-\frac{1}{p-1}}$), where $t \in \mathbb{C}_p$ with $|t|_p \leq 1$. The $p$-adic logarithm function, $\log_p$, is the unique function $\mathbb{C}^\times_p \to \mathbb{C}_p$ that satisfy (1) $\log_p(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$, $|x|_p < 1$, (2) $\log_p(xy) = \log_p x + \log_p y$, $\forall x, y \in \mathbb{C}^\times_p$, and $\log p = 0$. Let $A_j(x) = \sum_{n=0}^{\infty} a_{n,j} x^n$, $a_{n,j} \in \mathbb{C}$, $j = 0, 1, 2, \ldots$ be a sequence of power series, each of which converges in a fixed subset $D = \{ s \in \mathbb{C}_p ||s|_p \leq |p^*|^{-1}p^{-\frac{1}{p-1}} \}$ of $\mathbb{C}_p$ such that (1) $a_{n,j} \to a_{n,0}$ as $j \to \infty$ for $\forall n$; (2) for each $s \in D$ and $\epsilon > 0$, there exists $n_0 = n_0(s, \epsilon)$ such that $\left| \sum_{n \geq n_0} a_{n,j} s^n \right|_p < \epsilon$ for $\forall j$. Then $\lim_{j \to \infty} A_j(s) = A_0(s)$ for all $s \in D$. This is used by Washington [36] to show that each of the function $w^{-s}(a)a^s$ and $\sum_{m=0}^{\infty} \left( \frac{s}{m} \right) \left( \frac{F}{a} \right)^m B_m$, where $F$ is the multiple of $p^*$ and $f = f_\chi$, is analytic in $D$. Let $F$ be a positive integral multiple of $p^*$ and $f = f_\chi$, and let

$$
L_{p,q}(s, t|\chi) = \frac{1}{s - 1} \left[ \frac{F}{q} \right] q \sum_{a=1}^{F} \chi(a) < a + p^*t > 1-s \sum_{m=0}^{\infty} \left( \frac{1-s}{m} \right) \beta_{m,q^p} q^{(a+p^*)m} \frac{F}{a + p^*t} \right] ^m q^{a+p^*t}.
$$

Then $L_{p,q}(s, t|\chi)$ is analytic for $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, provided $s \in D$, except $s \neq 1$ when $\chi \neq 1$. For $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, we see that $\sum_{j=0}^{\infty} \left( \frac{s}{m} \right) \beta_{j,q^p} q^{(a+p^*)j} \left[ \frac{F}{a + p^*t} \right] ^j q^{a+p^*t}$ is analytic for $s \in D$. It readily follows that $< a + p^*t > s = w^{-s}(a)[a + p^*t]_q s = < a > s \sum_{m=0}^{\infty} \left( \frac{s}{m} \right) \left( q^a[a]_q^{-1}[p^*t]_q \right) ^m$ is analytic for $t \in \mathbb{C}_p$ with $|t|_p \leq 1$ when $s \in D$. Thus, since $(s-1)L_{p,q}(s, t|\chi)$ is a finite sum of products of these two functions, it must also be analytic for $t \in \mathbb{C}_p$, $|t|_p \leq 1$, whenever $s \in D$. Note that

$$
\lim_{s \to 1} (s-1)L_{p,q}(s, t|\chi) = \begin{cases} 
\frac{1}{[F]_q} \frac{q^{p-1}}{\log q} (1 - \frac{1}{p}), & \text{if } \chi = 1, \\
0, & \text{if } \chi \neq 1.
\end{cases}
$$
We now let \( n \in \mathbb{Z}, n \geq 1 \), and fix \( t \in \mathbb{C}_p \) with \(|t|_p \leq 1\). Since \( F \) must be a multiple of \( f = f_{\chi_n} \), Lemma 1 implies that

\[
\beta_{n,\chi_n,q}(p^*t) = [F]_{q}^{n-1} \sum_{a=0}^{F-1} \chi_n(a) \beta_{n,q,F}(\frac{a + p^*t}{F}).
\]

If \( \chi_n(p) = 0 \), then \((p, f_{\chi_n}) = 1\), so that \( \frac{F}{p} \) is a multiple of \( f_{\chi_n} \). Therefore, we obtain

\[
\chi_n(p)[p]_{q}^{n-1} \beta_{n,\chi_n,q^p}(p^{-1}p^*t) = [F]_{q}^{n-1} \sum_{a=0}^{F-1} \chi_n(a) \beta_{n,q,F}(\frac{a + p^*t}{F}).
\]

The difference of these quantities yields

\[
\beta_{n,\chi_n,q}(p^*t) - \chi_n(p)[p]_{q}^{n-1} \beta_{n,\chi_n,q^p}(p^{-1}p^*t) = [F]_{q}^{n-1} \sum_{a=0}^{F-1} \chi_n(a) \beta_{n,q,F}(\frac{a + p^*t}{F}).
\]

By using the distribution of \( q \)-Bernoulli polynomials, we easily see that

\[
\beta_{n,q,F}(\frac{a + p^*t}{F}) = [F]_{q}^{-n} [a + p^*t]_{q}^{n} \sum_{m=0}^{n} \binom{n}{m} q^{(a+p^*t)m} \left[ \frac{F}{a + p^*t} \right]_{q^a+p^*t}^{m} \beta_{m,q,F}.
\]

Since \( \chi_n(a) = \chi(a)w^{-n}(a) \) and for \((a, p) = 1\), and \( t \in \mathbb{C}_p \) with \(|t|_p \leq 1\), we have

\[
\beta_{n,\chi_n,q}(p^*t) - \chi_n(p)[p]_{q}^{n-1} \beta_{n,\chi_n,q^p}(p^{-1}p^*t)
\]

\[
= \frac{1}{[F]_{q}} \sum_{a=1}^{F} \chi(a) < a + p^*t \geq n \sum_{m=0}^{\infty} \binom{n}{m} q^{(a+p^*t)m} \left[ \frac{F}{a + p^*t} \right]_{q^a+p^*t}^{m} \beta_{m,q,F}
\]

Thus, we see that

\[
-\frac{1}{n} \left( \beta_{n,\chi_n,q}(p^*t) - \chi_n(p)[p]_{q}^{n-1} \beta_{n,\chi_n,q^p}(p^{-1}p^*t) \right) = L_{p,q}(1 - n, t|\chi), \text{ for } n \in \mathbb{N}.
\]

Therefore we obtain the following theorem:
Theorem 6. Let $F$ be a positive integral multiple of $p^*$ and $f = f_\chi$, and let

\begin{equation}
L_{p,q}(s,t| \chi) = \frac{1}{s-1} \frac{1}{[F]_q} \sum_{a=1}^{F} \chi(a) < a + p^*t > 1-s \sum_{m=0}^{\infty} \left(1 - s\right) q^{(a + p^*t)m} \beta_{m,q} F^{m} \left(\frac{a + p^*t}{a + p^*t}\right). 
\end{equation}

Then, $L_{p,q}(s,t| \chi)$ is analytic for $t \in \mathbb{C}_p$, $|t|_p \leq 1$, provided $s \in D$, except $s \neq 1$ when $\chi \neq 1$. Also, if $t \in \mathbb{C}_p$, $|t|_p \leq 1$, this function is analytic for $s \in D$ when $\chi \neq 1$, and meromorphic for $s \in D$, with simple pole at $s = 1$ having residue $\frac{1}{[F]_q} q^{p-1} \left(1 - \frac{1}{p}\right)$ when $\chi = 1$. Furthermore, for each $n \in \mathbb{Z}$, $n \geq 1$, we have

\begin{equation}
L_{p,q}(1-n,t| \chi) = -\frac{1}{n} \left(\beta_{n,\chi,n,q}(p^*t) - \chi_n(p)[p]_q^{n-1} \beta_{n,\chi,n,q}(p^{-1}p^*)\right).
\end{equation}

Remark. (1) Note that $L_{p,q}(s,0| \chi) = L_{p,q}(s, \chi)$ for $s \in D$ with $s \neq 1$ if $\chi = 1$, where $L_{p,q}(s, \chi)$ is $p$-adic $q$-L-function, cf. [12].

(2) Let $L_p(s,t| \chi)$ be the two-variable $p$-adic $L$-functions of Fox. Then we see that $\lim_{q \to 1} L_{p,q}(s,t| \chi) = L_p(s,t| \chi)$.

By means of a method provided by Washington [36], we now generalize to two-variable $p$-adic $q$-L-function, $L_{p,q}(s,t| \chi)$, by modifying $L_{p,q}(s, \chi)$, which was first defined by the function

\begin{equation}
H_{p,q}(s,a,F) = \frac{1}{s-1} \frac{1}{[F]_q} \sum_{j=0}^{\infty} \left(1 - s\right) \beta_{j,q_F} q^{aj} \left[\frac{F}{a}\right]_q^j,
\end{equation}

where $s \in D$, $s \neq 1$, $a \in \mathbb{Z}$ with $(a,p) = 1$, and $F$ is a multiple of $p^*$, cf. [17].

The function $L_{p,q}(s, \chi)$ can be rewritten as the sum

\begin{equation}
L_{p,q}(s, \chi) = \sum_{a=1}^{F} \chi(a) H_{p,q}(s,a,F), \text{ cf. [17]},
\end{equation}

provided $F$ is a multiple of both $p^*$ and $f = f_\chi$. The function $H_{p,q}(s,a,F)$ is a meromorphic for $s \in D$ with a simple pole at $s = 1$, having residue $\frac{1}{[F]_q} q^{p-1} \log q$, and it
interpolates the values

\[ H_{p,q}(1-n,a,F) = -\frac{1}{n} w^{-n}(a) \beta_{n,q,F}(\frac{a}{F}), \]

where \( n \in \mathbb{Z}, \ n \geq 1, \) cf. [17, 12].

By using \( H_{p,q}(s,a+p^{*}t,F) \), we can express \( L_{p,q}(s,t|\chi) \) for all \( a \in \mathbb{Z}, \ (a,p) = 1, \) and \( t \in \mathbb{C}_p \) with \( |t|_p \leq 1, \) as follows:

\[ (25) \quad L_{p,q}(s,t|\chi) = \sum_{a=1}^{F} \frac{\chi(a)}{(a,p)=1} H_{p,q}(s,a+p^{*}t,F). \]

From the proof of Theorem 6, we note that \( H_{p,q}(s,a+p^{*}t,F) \) is analytic for \( t \in \mathbb{C}_p, \ |t|_p \leq 1, \) where \( s \in D, \ s \neq 1, \) and meromorphic for \( s \in D, \) with a simple pole at \( s = 1, \) when \( t \in \mathbb{C}_p, \ |t|_p \leq 1. \) Let us consider the first partial derivative of the function \( L_{p,q}(s,t|\chi) \) at \( s = 0. \) It is easy to see that

\[ \frac{\partial^n}{\partial t^n} L_{p,q}(s,t|\chi) = \left( \frac{-s}{n} \right) n! \left( \frac{p^{*} \log q}{q-1} \right)^n L_{p,q}(s+n,t|\chi_n), \]

for all \( s \in D, \ s \neq 1 \) if \( \chi = 1, \) and \( t \in \mathbb{C}_p \) with \( |t|_p \leq 1. \)

Furthermore, we note that

\[
\lim_{s \to 1-n} \left( \frac{-s}{n} \right) L_{p,q}(s+n,t|\chi_n) = -\frac{(n-1)!}{n!} \lim_{s \to 1-n} (s+n-1)L_{p,q}(s+n,t|\chi_n)
\]

\[ = -\frac{1}{n} \sum_{(a,p)=1}^{F} \chi_n(a) \beta_{0,q,F} = -\frac{1}{n} \left( \beta_{0,\chi_n,q} - \chi_n(p)[p]_{q}^{-1} \beta_{0,\chi_n,q,p} \right). \]

Thus, we have

\[ \frac{\partial^n}{\partial t^n} L_{p,q}(1-n,t|\chi) = -\frac{n!}{n} \left( \frac{p^{*} \log q}{q-1} \right)^n \left( \beta_{0,\chi_n,q} - \chi_n(p)[p]_{q}^{-1} \beta_{0,\chi_n,q,p} \right). \]

Since \( \beta_{0,\chi,q} = 0 \) if \( \chi \neq 1, \) this become
\[
\frac{\partial^n}{\partial t^n} L_{p,q}(1-n,t|\chi) = \begin{cases} 
-(n-1)! \left( p^* \log q \right)^n (1 - \frac{1}{p}) \frac{q^F - 1}{[F] \log q}, & \text{if } \chi = 1, \\
0, & \text{if } \chi \neq 1. 
\end{cases}
\]

In the case \( n = 1 \), we easily see that
\[
\frac{\partial}{\partial t} L_{p,q}(0,t|\chi) = \begin{cases} 
-p^* \log q \frac{q^F - 1}{[F] \log q}, & \text{if } \chi = 1, \\
0, & \text{if } \chi \neq 1. 
\end{cases}
\]

The value of \( \frac{\partial}{\partial s} L_{p,q}(0,t|\chi) \) is the coefficient of \( s \) in the expansion of \( L_{p,q}(s,t|\chi) \) at \( s = 0 \). By using Taylor expansion at \( s = 0 \), we see that
\[
\frac{1}{1-s} = 1 + s + \cdots,
\]
\[
< a + p^* t >^{1-s} = (1 - s \log_p < a + p^* t > + \cdots),
\]
\[
\left( \frac{1-s}{m} \right) = \frac{(-1)^{m+1}}{m(m-1)} s + \cdots.
\]

By employing these expansion, along with some algebraic manipulation, we evaluate \( \frac{\partial}{\partial s} L_{p,q}(0,t|\chi) \). From the definition of \( L_{p,q}(s,t|\chi) \), we note that
\[
L_{p,q}(s,t|\chi) = \frac{1}{s-1} [F]_q \sum_{a=1}^{F} \chi(a) < a + p^* t >^{1-s} \sum_{m=0}^{\infty} \left( \frac{1-s}{m} \right) \beta_{m,q,F} q^{(a+p^* t) m} \frac{F}{a + p^* t} [q^{a+p^* t}]^m.
\]

Thus, we have
\[
\frac{\partial}{\partial s} L_{p,q}(s,t|\chi) \big|_{s=0} = \sum_{a=1}^{F} \chi_1(a) \left\{ \left( \frac{[a+p^* t]_q}{[F]_q} \beta_{0,q,F} + \beta_{1,q,F} \right) \log_p < a + p^* t > 
\right.
\]
\[
- \frac{[a+p^* t]_q}{[F]_q} \beta_{0,q,F} + \sum_{m=2}^{\infty} \frac{(-1)^{m}}{m(m-1)} \frac{[F]_q}{[a+p^* t]_q} \beta_{m,q,F} \frac{[a+p^* t]_q}{[F]_q} \bigg] 
\]
\[
+ (q-1) \sum_{a=1}^{F} \chi_1(a) \left\{ \left[ a + p^* t \right]_q \beta_{1,q,F} \left( -1 + \log_p < a + p^* t > \right) 
\right.
\]
\[
+ \sum_{m=2}^{\infty} \sum_{l=1}^{m} \left( \frac{m}{l} \right) (q-1)^{l-1} \left[ a + p^* t \right]_q^{l-m+1} [F]_q^{m-1} \beta_{m,q,F} \bigg].
\]
We now define the Daehee $q$-operator, $D_{q,F}(x,y)$, as follows:

\begin{equation}
D_{q,F}(x,y) = (\log_p x - 1) x^{\beta_1,q} + \sum_{m=0}^{\infty} \sum_{l=1}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) (y-1)^{l-m} (y^F - 1)^{m-1} x^{l-m+1} \beta_{m,q}.
\end{equation}

In [7,8] the Diamond gamma function is defined by

\begin{equation}
G_p(x) = \left( x - \frac{1}{2} \right) \log_p x - x + \sum_{j=2}^{\infty} \frac{B_j}{j(j-1)} x^{1-j}, \text{ for } |x|_p > 1.
\end{equation}

We now consider a $q$-analogue of the above Diamond gamma function as follows:

\begin{equation}
G_{p,q}(x) = \int_{\mathbb{Z}_p} \left\{ (x + [z]_q) \log_p (x + [z]_q) - (x + [z]_q) \right\} q^{-z} d\mu_q(z), \text{ for } |x|_p > 1.
\end{equation}

From the above Eq.(29), we note that $G_{p,q}(x)$ is locally analytic on $\mathbb{C}_p \setminus \mathbb{Z}_p$. By (29), we easily see that

\begin{equation}
G_{p,q}(x) = (x^{\beta_0,q} + \beta_1,q) \log_p x - x^{\beta_0,q} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \beta_{n+1,q} x^{-n}, \text{ for } |x|_p > 1.
\end{equation}

Note that $\lim_{q \to 1} G_{p,q}(x) = G_p(x)$. Since, $w(a)$ is a root of unity for $(a, p) = 1$, we see that

\begin{equation}
\log_p < a + p^*t > = \log_p(a + p^*t) + \log_p w^{-1}(a) = \log_p(a + p^*t).
\end{equation}

From the Eq.(26), Eq.(27), Eq.(30) and Eq.(31), we note that

\begin{align*}
\frac{\partial}{\partial s} L_{p,q}(0,t|\chi) &= \sum_{a=1}^{F} \sum_{(a,p)=1}^{F} \chi_1(a) \left\{ [F]_q^{-1} q^{p^*t} \log_p [F]_q \beta_{0,q} \beta_0,F[a] + G_{p,q,F} \left( \frac{[a + p^*t]_q}{[F]_q} \right) \right. \right. \\
&\left. \left. + (q - 1) D_{q,F,F}([a + p^*t]_q, q) \right\} = -q^{p^*t} L_{p,q}(0, \chi) \log_p [F] \\
+ \sum_{a=1}^{F} \sum_{(a,p)=1}^{F} \chi_1(a) G_{p,q,F} \left( \frac{[a + p^*t]_q}{[F]_q} \right) \right. \right. \\
&\left. \left. + (q - 1) \sum_{(a,p)=1}^{F} \chi_1(a) D_{q,F,F}([a + p^*t]_q, q). \right. \right. \\
\end{align*}

Therefore we obtain the following theorem:
Theorem 7. Let \( \chi \) be the primitive Dirichlet character, and let \( F \) be a positive integral multiple of \( p^* \) and \( f = f_\chi \). Then for any \( t \in \mathbb{C}_p \) with \( |t|_p \leq 1 \), we have

\[
\frac{\partial}{\partial s} L_{p,q}(0, t|\chi) = \sum_{a=1}^{F} \frac{\chi_1(a)G_{p,q^F}([a+p^*t]_q)}{[F]_q} - q^{p^*} L_{p,q}(0, \chi) \log [F]_q
\]

\[
+ (q - 1) \sum_{a=1}^{F} \chi_1(a)D_{q^F,F}([a+p^*t]_q, q).
\]

Now we give the value of \( L_{p,q}(s, t|\chi) \) at \( s = 1 \) when \( \chi \neq 1 \). From the definition of \( L_{p,q}(s, t|\chi) \), we have

\[
L_{p,q}(s, t|\chi) = \frac{1}{s - 1} \sum_{a=1}^{F} \chi(a) \left\{ < a + p^*t > ^{1-s} \beta_{0,q^F}
\right.
\]

\[
+ < a + p^*t > ^{1-s} \sum_{m=1}^{\infty} \left( \frac{1}{m} \right) \beta_{m,q^F} q^{(a+p^*t)m} \left( \frac{[F]_q}{a + p^*t|_q} \right)^m \left( \left\lceil \frac{a + p^*t}{p} \right\rceil q \right).
\]

By using Taylor expansion at \( s = 1 \), we see that

\[
\lim_{s \to 1} L_{p,q}(s, t|\chi) = - \frac{1}{[F]_q} \sum_{a=1}^{F} \chi(a) \log_p < a + p^*t >
\]

\[
- \frac{1}{[F]_q} \sum_{a=1}^{F} \chi(a) \sum_{m=1}^{\infty} (-1)^{m-1} \frac{(m-1)!}{m!} \beta_{m,q^F} q^{(a+p^*t)m} \left( \frac{[F]_q}{a + p^*t|_q} \right)^m.
\]

Therefore we obtain the following theorem:

Theorem 8. Let \( \chi \) be the Dirichlet character with conductor \( f = f_\chi \) and let \( F \) be the positive integral multiple of \( p^* \) and \( f = f_\chi \). Then we have

\[
L_{p,q}(1, t|\chi) = \frac{1}{[F]_q} \sum_{a=1}^{F} \chi(a) \left\{ - \log_p < a + p^*t >
\right.
\]

\[
+ \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \beta_{m,q^F} q^{(a+p^*t)m} \left( \frac{[f]_q}{a + p^*t|_q} \right)^m, \text{ for } t \in \mathbb{C}_p \text{ with } |t|_p \leq 1.
\]
Remark. From the above Theorem 8, we note that

\[ L_{p,q}(1,0|\chi) = L_{p,q}(1,\chi) = \frac{1}{[F]_q} \sum_{a=1}^{\infty} \frac{\chi(a)}{[a]_q^{\log_p a}} + \sum_{m=1}^{\infty} \frac{\beta_{m,q} q^m}{m} \left( \frac{F_q}{[a]_q} \right)^m, \]

and \( \lim_{q \to 1} L_{p,q}(1,\chi) = L_p(1,\chi). \)

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