Abstract. In this paper, we study the Feldman-Katok metric in random dynamical systems and establish corresponding fiber topological entropy formula, Brin-Katok local entropy formula and fiber Katok entropy formula by replacing Bowen metric with Feldman-Katok metric. It turns out that the Feldman-Katok metric is also the weakest metric that makes the entropy formulae valid on random dynamical systems.

1. Introduction

The setup consists of a probability space \((\Omega, F, \mathcal{P}, \vartheta)\) together with an invertible ergodic system \(\vartheta\) of a compact metric space \((X, B_X)\) together with the distance function \(d\) and the Borel \(\sigma\)-algebra \(B_X\). We assume \(F\) is complete, countably generated, and separates points, and so \((\Omega, F, \mathcal{P})\) is a Lebesgue space. Let \(\mathcal{E} = \Omega \times X\) be measurable with respect to the product \(\sigma\)-algebra \(\mathcal{F} \times B\), so the fibers \(\mathcal{E}_\omega = \{x \in X: (\omega, x) \in \mathcal{E}\} = X\), \(\forall \omega \in \Omega\). A continuous bundle random dynamical system (abbreviated as \(RDS\)) \(T = (T_\omega)\) over \((\Omega, F, \mathcal{P}, \vartheta)\) is generated by mappings \(T_\omega: X \to X\) with iterates

\[
T^n_\omega = \begin{cases} 
\text{id,} & n = 0, \\
T_{\vartheta^{n-1}\omega} \circ \cdots \circ T_{\vartheta\omega} \circ T_\omega & n \geq 1,
\end{cases}
\]

where \(\text{id}\) is an identity mapping, such that such that \((\omega, x) \mapsto f_\omega(x)\) is measurable and the map \(x \mapsto f_\omega(x)\) is continuous for \(P\text{-a.e.}\ \omega \in \Omega\).

The map \(\Theta: \Omega \times X \to \Omega \times X\) is defined by \(\Theta(\omega, x) = (\vartheta\omega, T_\omega x)\) which is called the skew product transformation. For \(\omega \in \Omega, n \in \mathbb{N}\), the \(n\)-th
Bowen metric $d^n_\omega$ on $X$ is defined by

$$d^n_\omega(x, y) = \max_{0 \leq i \leq n-1} d(T^i_\omega(x), T^i_\omega(y)).$$

For $\forall \varepsilon > 0$, $B_n(\omega, x, \varepsilon)$ denote the open ball with centre $x$ and radius $\varepsilon$ in the metric $d^n_\omega$, i.e.,

$$B_n(\omega, x, \varepsilon) = \{y \in X : d^n_\omega(x, y) < \varepsilon\}.$$

In [2], Bogenschütz applied the Bowen metric to RDS, similarly gave the definition of the corresponding spanning and separated sets and proved the topological entropy of RDS. For a RDS, Zhu [26, 27] established the Brin-Katok entropy formula and Katok entropy formula on the Bowen metric.

By a pair $(X, T)$ we mean a topological dynamical system (abbreviated as TDS) where $X$ is a compact metric with the metric $d$ and $T : X \to X$ is a continuous mapping. In ergodic theory, a fundamental problem is to classify the measure-perserving systems (abbreviated as MPSs) up to isomorphism. In 1958, Kolmogorov [16] introduced the concept of entropy in ergodic theory based on Shannon [24] entropy, and proved that entropy is an isomorphic invariant of MPSs, then Sinai [25] generalized it to the general case. In ergodic theory, a remarkable result about isomorphic problem is Ornstein’s theory [22], which proves that any two Bernoulli processes of isentropic are isomorphic. The concept of a finitely determined process plays an important role in his theory, which is based on the hamming distance $\bar{d}_n$:

$$\bar{d}_n(x_0x_1\ldots x_{n-1}, y_0y_1\ldots y_{n-1}) = \left|\left\{0 \leq i \leq n-1 : x_i \neq y_i\right\}\right| / n.$$

In 1943, Kakutani [15] proposed the equivalent concept between ergodic systems, which is called Kakutani equivalence. In 1976, by changing the Hamming distance $\bar{d}_n$ in Ornsten’s theory to the edit distance $\tilde{f}_n$:

$$\tilde{f}_n(x_0x_1\ldots x_{n-1}, y_0y_1\ldots y_{n-1}) = 1 - \frac{k}{n},$$

where $k$ is the largest integer such that there exists

$$0 \leq i_1 < \cdots < i_k \leq n - 1, 0 \leq j_1 < \cdots < j_k \leq n - 1.$$
and $x_{i_m} = y_{j_m}$ for $m = 1, \ldots, k$, Feldman [8] defined loose Bernoulli systems, which brings new ideas for the classification of guarantee systems. In 2017, Kwietniak and Lacka [20] introduced the Feldman-Katok (FK) metric as the topological counterpart of edit distance. In 2020, Garcí-Ramos and Kwietniak [9] used the FK metric to describe the zero entropy Bernoulli guarantee system. Recently Cai-Li [5] established the topological entropy formula of FK metric, Brin-Katok local entropy formula and Katok entropy formula in the case of invariant measures and ergodic measures, and proved that FK metric was the weakest metric to make the topological entropy formula valid. Nie and Huang [21] further studied the restricted-sensitivity of FK metric and mean metric and obtained conditional entropy formulae.

Our purpose in this paper is to extend the results of [5] to RDS. In section 2, we introduce some basic notions needed in the paper and give entropy formulae for Bowen metric of RDS. In section 3, for a continuous bundle RDS, we first prove that the topological entropy defined with FK metric is equal with Bowen metric.

**Theorem 1.1.** Let $T$ be a continuous bundle RDS over $(\Omega, \mathcal{F}, P, \vartheta)$. Then for $P$-a.e. $\omega \in \Omega$

\[ h_{\text{top}}^{(r)}(\omega, X, T) = h_{FK}^{(r)}(\omega, X, T) = h_{FK}^{(r)}(\omega, X, T), \]

where $h_{\text{top}}^{(r)}(\omega, X, T)$ is the fiber topological entropy of $T$ with respect to $\omega \in \Omega$.

Next, we establish the Brin-Katok local entropy formula and Katok entropy formula for FK metric.

**Theorem 1.2.** Let $T$ be a continuous bundle RDS over $(\Omega, \mathcal{F}, P, \vartheta)$, $\mu \in M_1^1(\Omega \times X, T)$. Then for $\mu$-a.e. $(\omega, x) \in \Omega \times X$,

\[ h_{\mu}^{(r)}(T, \omega, x) = \lim_{\delta \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \mu(\mathcal{B}_{FK_n}(\omega, x, \delta)) \]

\[ = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \mu(\mathcal{B}_{FK_n}(\omega, x, \delta)). \]

**Theorem 1.3.** Let $T$ be a continuous bundle RDS over $(\Omega, \mathcal{F}, P, \vartheta)$, $\mu \in M_1^1(\Omega \times X, T)$. Then for $P$-a.e. $\omega \in \Omega$,

\[ h_{\mu}^{(r)}(T) \leq \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \text{sp}_{FK}(\omega, \mu, n, \varepsilon). \]
If \( \mu \in E^1_P(\Omega \times X, T) \) and \( h^{(r)}_\mu(T) < \infty \), then

\[
h^{(r)}_\mu(T) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log sp_{FK}(\omega, \mu, n, \varepsilon)
= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log sp_{FK}(\omega, \mu, n, \varepsilon),
\]

where \( sp_{FK}(\omega, \mu, n, \varepsilon) \) denotes the smallest number of any \( FK \)-dynamical balls (i.e. the balls have radius \( \varepsilon \) in the metric \( d^FK_n \)) whose union has \( \mu_\omega \)-measure larger than \( 1 - \varepsilon \).

2. Preliminaries

In this section we will introduce some notions necessary for this paper and give fiber topological entropy and measure-theoretic entropy formulae with Bowen metric of RDSs.

2.1. Entropies for RDSs. For RDSs, the definitions of the fiber topological entropy and fiber measure-theoretic entropies were introduced by [3, 14, 26, 27].

Let \( T \) be a continuous bundle RDS over \((\Omega, \mathcal{F}, P, \theta)\). For \( n \in \mathbb{N}, \omega \in \Omega, \varepsilon > 0 \), the Feldman-Katok(FK) metric \( d^FK_n \) on \( X \) is defined as follows: for \( x, y \in X \), we define an \((\omega, n, \delta)\)-match of \( x \) and \( y \) to be an order preserving (i.e. \( \pi(i) < \pi(j) \) whenever \( i < j \)) bijection \( \pi : D(\pi) \to R(\pi) \) such that \( D(\pi), R(\pi) \) are subsets of \( \{0, 1, \cdots, n - 1\} \) and for every \( i \in D(\pi) \), it holds that \( d(T^i_\omega(x), T^\pi(i)_\omega(y)) < \varepsilon \). Let \( |\pi| \) denotes the cardinality of \( D(\pi) \). Set

\[
\bar{f}_{\omega, n, \varepsilon}(x, y) = 1 - \frac{1}{n} \max \{|\pi| : \pi \text{ is an } (\omega, n, \delta) \text{-match of } x \text{ and } y\}.
\]

FK metric of RDSs is

\[
d^FK_n(x, y) = \inf\{\varepsilon > 0 : \bar{f}_{\omega, n, \varepsilon}(x, y) < \varepsilon\}. \tag{2·1}
\]

Definition 2.1. Let \( Z \) be a compact subset of \( X \). A subset \( E \subset Z \) is said to be a \( FK-(\omega, n, \varepsilon) \) spanning set of \( Z \) if \( \forall x \in Z, \exists y \in E \) with \( d^FK_n(x, y) \leq \varepsilon \). Let \( sp_{FK}(\omega, n, Z, \varepsilon) \) denote the smallest cardinality of any \( FK-(\omega, n, \varepsilon) \) spanning set for \( Z \). A set \( F \subset Z \) is said to be a \( FK-(\omega, n, \varepsilon) \) separated set of \( Z \) if \( \forall x, y \in F, x \neq y \) implies \( d^FK_n(x, y) > \varepsilon \). Let \( sr_{FK}(\omega, n, Z, \varepsilon) \) denote the largest cardinality of any \( FK-(\omega, n, \varepsilon) \) separated set for \( Z \). When \( Z = X \) we omit the restriction of \( Z \).
It is clear that \( \text{sp}_{FK}(\omega, n, \varepsilon) \leq \text{sr}_{FK}(\omega, n, \varepsilon) \leq \text{sp}_{FK}(\omega, n, \frac{\varepsilon}{2}) \). Then the following limits exist:

\[
\begin{align*}
\overline{h}^{(r)}_{FK}(\omega, X, T) &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \text{sp}_{FK}(\omega, n, \varepsilon) \\
&= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \text{sr}_{FK}(\omega, n, \varepsilon), \\
\underline{h}^{(r)}_{FK}(\omega, X, T) &= \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \text{sp}_{FK}(\omega, n, \varepsilon) \\
&= \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \text{sr}_{FK}(\omega, n, \varepsilon).
\end{align*}
\]

(2.2)

Replacing \( d_{FK}^{\varepsilon} \) by \( d_{\omega}^{n} \), the corresponding notions are \( r(\omega, n, Z, \varepsilon) \) and \( s(\omega, n, Z, \varepsilon) \). Then the fiber topological entropy of \( T \) with respect to \( \omega \) is given by

\[
\begin{align*}
\overline{h}^{(r)}_{\omega}(\omega, X, T) &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(\omega, n, \varepsilon) \\
&= \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log s(\omega, n, \varepsilon).
\end{align*}
\]

And the fiber topological entropy of \( T \) is given by

\[
h^{(r)}_{\omega}(T) = \int h^{(r)}_{\omega}(\omega, X, T) dP(\omega).
\]

2.2. Local Entropy of RDSs. For a continuous bundle \( RDS T \), we denote by \( \mathcal{P}_{P}(\Omega \times X) \) the space of probability measures on \( \Omega \times X \) having the marginal \( P \) on \( \Omega \). We always assume that \( \mu \) disintegrates with respect to \( P \), i.e. there is a family of conditional probabilities \( \{ \mu_{\omega} \} \) such that \( d\mu(\omega, x) = d\mu_{\omega}(x) dP(\omega) \).

A probability measure \( \mu \) on \( (\Omega \times X, F \times B) \) is said to be \( T \)-invariant if it is invariant under \( \Theta \) and has marginal \( P \) on \( \Omega \). Furthermore, \( \mu \) is said to be \( T \)-ergodic if it is ergodic with respect to \( \Theta \). Denote by \( M^{1}_{P}(\Omega \times X, T) \) the set of all invariant measures of \( \Omega \times X \) and by \( E^{1}_{P}(\Omega \times X, T) \) the set of all ergodic measures of \( \Omega \times X \). By Bogenschütz [3], \( \mu \) is \( \Theta \)-invariant if and only if \( T_{\omega} \mu = \mu_{\vartheta_{\omega}} P \)-a.e..

Let \( \mu \in M^{1}_{P}(\Omega \times X, T) \) and \( \zeta \) be a finite measurable partition of \( \Omega \times X \), then the limit

\[
h^{(r)}_{\mu}(T, \zeta) := \lim_{n \to \infty} \frac{1}{n} \int H^{(r)}_{\mu}(T_i^{\omega}(\omega))^{-1} \zeta_{\omega}(\omega) dP(\omega)
\]

(2.4)
exists, where $\zeta_{\vartheta, \omega}$ is the $\vartheta^\omega$-section of $\zeta$ and
\[
H_{\mu}(\bigvee_{i=0}^{n-1}(T_i^\omega)^{-1}\zeta_{\vartheta, \omega}) = -\sum_{A \in \bigvee_{i=0}^{n-1}(T_i^\omega)^{-1}\xi} \mu_\omega(A) \log \mu_\omega(A).
\]
The number
\[
h_{\mu}^{(r)}(T) := \sup \{ h_{\mu}^{(r)}(T, \zeta) \mid \zeta \text{ is a finite measurable partition of } \Omega \times X \}
\]
is called the measure-theoretic entropy of $(T, \mu)$.

The classical Brin-Katok entropy formula and Katok entropy formula of the TDS were established by Brin and Katok \cite{4, 17}. Zhu \cite{26, 27} gave a random version of Brin-Katok entropy formula and Katok entropy formula. Let $T$ be a continuous bundle RDS and $\mu \in M_P^1(\Omega \times X, T)$, it can be proved that for $\mu$-a.e. $(\omega, x) \in \Omega \times X$, the following equation holds:
\[
h_{\mu}^{(r)}(T, \omega, x) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \mu_\omega(B_n(\omega, x, \delta))
\]
(2.5)
\[
= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \mu_\omega(B_n(\omega, x, \delta)),
\]
where we call $h_{\mu}^{(r)}(T, \omega, x)$ the local entropy at $(\omega, x)$ with respect to $\mu$. And
\[
h_{\mu}^{(r)}(T) = \int h_{\mu}^{(r)}(T, \omega, x) d\mu(\omega, x).
\]
(2.6)
Particularly, if $\mu \in E^1_P(\Omega \times X, T)$, $h_{\mu}^{(r)}(T) = h_{\mu}^{(r)}(T, \omega, x)$ for $\mu$-a.e. $(\omega, x) \in \Omega \times X$.

For $\varepsilon > 0, \delta > 0, \omega \in \Omega$ and $\mu \in E^1_P(\Omega \times X, T)$, denote
\[
r(\omega, \mu, n, \varepsilon, \delta) = \min \{ r(\omega, n, Z, \varepsilon) : Z \subset X, \mu_\omega(Z) \geq 1 - \delta \}.
\]
Note that for any fixed $n, \delta$, the map $\varepsilon \mapsto r(\omega, \mu, n, \varepsilon, \delta)$ is monotone decreasing and for any fixed $n, \varepsilon$, the map $\varepsilon \mapsto r(\omega, \mu, n, \varepsilon, \delta)$ is monotone increasing. Zhu \cite{27} proved that for any $\delta \in (0, 1)$, if $h_{\mu}^{(r)}(T) < \infty$, then for $P$-a.e. $\omega \in \Omega$,
\[
h_{\mu}^{(r)}(T) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(\omega, \mu, n, \varepsilon, \delta)
\]
(2.7)
\[
= \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log r(\omega, \mu, n, \varepsilon, \delta).
\]

3. Entropy formulae for FK metric on RDSs

In this section, we shall extend the results of \cite{5} to RDSs.
3.1. Topological entropy formula.

**Lemma 3.1.** Let $T$ be a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathcal{P}, \vartheta)$, $n \in \mathbb{N}$, and $x, y \in X$. Then for $\mathcal{P}$-a.e. $\omega \in \Omega$, we have $d^{FK}_{\omega}(x, y) \leq d^{n}_{\omega}(x, y)$.

**Proof.** Fix $x, y \in X$. For any $\varepsilon > d^{n}_{\omega}(x, y)$, then

$\pi: \{0, 1, \ldots, n - 1\} \rightarrow \{0, 1, \ldots, n - 1\}$

is an $(\omega, n, \varepsilon)$-match of $x$ and $y$, where $\pi = id$. Clearly $\bar{f}_{\omega,n,\varepsilon}(x, y) = 0$, then

$d^{FK}_{\omega}(x, y) \leq \varepsilon$.

Let $\varepsilon \rightarrow d^{n}_{\omega}(x, y)$.

**Theorem 3.2.** Let $T$ be a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathcal{P}, \vartheta)$. For $\mathcal{P}$-a.e. $\omega \in \Omega$,

$\begin{align*}
3.1 & \quad h^{(r)}_{\text{top}}(\omega, X, T) = h^{(r)}_{FK}(\omega, X, T) = h^{(r)}_{FK}(\omega, X, T).
\end{align*}$

**Proof.** Let $\mathcal{U}$ be a finite open cover of $X$ with the Lebesgue number $2\varepsilon_0$. Fix $\omega \in \Omega, n \in \mathbb{N}$. For any $0 < \varepsilon < \varepsilon_0$, Let $E$ be a $FK$-$(\omega, n, \varepsilon)$ spanning set with $|E| = sp_{FK}(\omega, n, \varepsilon)$. By the definitions of $FK$-$(\omega, n, \varepsilon)$ spanning and $d^{FK}_{\omega}$, we can get the following relationship:

$X = \bigcup_{x \in E} \bigcup_{k = \lfloor (1 - \varepsilon)n \rfloor}^{n} \bigcup_{\pi: |\pi| = k}^{\pi \text{ is order preserving}} (T^{i}_{\omega})^{-1} B(T^{(i)}_{\omega}x, \varepsilon)$.

It is obvious that $B(T^{(i)}_{\omega}x, \varepsilon)$ is contained in some element of $\mathcal{U}$. Hence $\bigcap_{i \in D(\pi)} (T^{i}_{\omega})^{-1} \mathcal{U}$ is contained in some element of $\bigvee_{i \in D(\pi)} (T^{i}_{\omega})^{-1} \mathcal{U}$.

We can see that

$\begin{align*}
\bigcap_{i \in D(\pi)} (T^{i}_{\omega})^{-1} \mathcal{U} &= (\bigvee_{i \in D(\pi)} (T^{i}_{\omega})^{-1} \mathcal{U}) \bigvee (\bigvee_{i \in D(\pi)} (T^{i}_{\omega})^{-1} \mathcal{U}),
\end{align*}$

and

$| \bigvee_{i \in D(\pi)} (T^{i}_{\omega})^{-1} \mathcal{U}| \leq |\mathcal{U}|^{n - |\pi|}$.

Hence $\bigcap_{i \in D(\pi)} (T^{i}_{\omega})^{-1} \mathcal{U}$ can be covered by $|\mathcal{U}|^{n - |\pi|}$ elements of $\bigvee_{i = 0}^{n - 1} (T^{i}_{\omega})^{-1} \mathcal{U}$.

Since the number of order preserving bijection $\pi$ with $|\pi| = k$ is $(C^{k}_{n})^{2}$, it is easy to see that $X$ can be covered by
\[ |E| \sum_{k=[(1-\varepsilon)n]}^{n} (C_n^k)^2|U|^{n-k} \]
elements of \( \bigcup_{i=0}^{n-1} (T_\omega^i)^{-1}U \).

Therefore
\[ N\left( \bigcup_{i=0}^{n-1} (T_\omega^i)^{-1}U \right) \leq \left| E \right| \sum_{k=[(1-\varepsilon)n]}^{n} (C_n^k)^2|U|^{n-k} \]
\[ \leq sp_{FK}(\omega, n, \varepsilon)|U|^{n\varepsilon+1}(n\varepsilon + 1)(C_n^{[n\varepsilon]+1})^2. \]

Then
\[ \frac{1}{n} N\left( \bigcup_{i=0}^{n-1} (T_\omega^i)^{-1}U \right) \leq \frac{\log sp_{FK}(\omega, n, \varepsilon)}{n} + \frac{\log(n\varepsilon + 1)}{n} \]
\[ + \frac{2 \log C_n^{[n\varepsilon]+1}}{n} + (\varepsilon + \frac{1}{n}) \log |U|. \]

By Stirling’s formula, \( \lim_{n \to \infty} \frac{1}{n} \log C_n^{[n\varepsilon]+1} = -(1-\varepsilon) \log(1-\varepsilon) - \varepsilon \log \varepsilon. \)

We have
\[ h_{top}^{(r)}(\omega, X, U) \leq \lim \inf_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log sp_{FK}(\omega, n, \varepsilon). \]
i.e.
\[ (3.2) \quad h_{top}^{(r)}(\omega, X, T) \leq h_{FK}^{(r)}(\omega, X, T). \]

On the other hand, since
\[ d_{\omega}^{FK}(x, y) \leq d_{\omega}^{n}(x, y), \]
we have
\[ sp_{FK}(\omega, n, \varepsilon) \leq r_{n}(\omega, n, \varepsilon), \]

By (2.3), it can be obtained that
\[ (3.3) \quad h_{top}^{(r)}(\omega, X, T) \geq \lim \sup_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log sp_{FK}(\omega, n, \varepsilon) = \overline{h}_{FK}^{(r)}(\omega, X, T). \]

Combining with the fact that \( \underline{h}_{FK}^{(r)}(\omega, X, T) \leq \overline{h}_{FK}^{(r)}(\omega, X, T), \) we finish the proof. \( \square \)
3.2. Measure-theoretical local entropy formulæ of FK metric.

Now we consider the measurable case. First we need some preparation. Recall the definition of edit distance $\bar{f}_n$:

$$\bar{f}_n(x_0x_1 \cdots x_{n-1}; y_0y_1 \cdots y_{n-1}) = 1 - \frac{k}{n},$$

where $k$ is the largest integer such that there exists

$$0 \leq i_1 < \cdots < i_k \leq n - 1, 0 \leq j_1 < \cdots < j_k \leq n - 1$$

and $x_{i_m} = y_{j_m}$ for $m = 1, \ldots, k$.

Let $\mu \in M^1_P(\Omega \times X, T)$ and $\xi = \{A_1, A_2, \ldots, A_m\}$ be a finite partition of $X$. For any fixed $\omega \in \Omega$, we can identify the elements in $\bigvee_{i=0}^{n-1} (T^i_\omega)^{-1}\xi$ and $\{1, 2, \ldots, m\}^n$ by

$$(\cdot \cdot \cdot) \quad \bigcap_{i=0}^{n-1} (T^i_\omega)^{-1}A_t_i = (t_0, t_1, \ldots, t_{n-1}).$$

Hence when $t \in \{1, 2, \ldots, m\}^n$ and $A, B \in \bigvee_{i=0}^{n-1} (T^i_\omega)^{-1}\xi$, we can respectively talk about $\mu_\omega(t)$ and $\bar{f}_n(A, B)$.

Next we give the notion of $\pi_X$ which is the projection from $\Omega \times X$ onto $X$:

$$\pi_X : \Omega \times X \to X$$

$$(\omega, x) \mapsto x.$$ 

It is obvious that $\pi_X$ is measurable. For $B \in \mathcal{B}$, we have

$$\pi_X \mu(B) = \mu \circ \pi_X^{-1}(B) = \int \mu_\omega(B) dP(\omega).$$

By the compactness of $X$, we know that $\pi_X \mu$ is a regular measure and we can construct a finite measurable partition $\eta$ with $\pi_X \mu(\partial \eta) = 0$, i.e. $\mu_\omega(\partial \eta) = 0$ for $P$-a.e. $\omega \in \Omega$.

Recall that $B_{\eta}(\omega, x, \delta) = \{y \in X : d^\eta_\omega(x, y) < \delta\}$ is the Bowen ball of RDS. Then we replace $d^\eta_\omega$ by $d^{FK_\omega}$ and $\bar{f}_n$ and denote

$$B_{FK_\omega}(\omega, x, \delta) = \{y \in X : d^{FK_\omega}_\omega(x, y) < \delta\},$$

$$B_{\bar{f}_n}(A, \kappa) = \{B \in \bigvee_{i=0}^{n-1} (T^i_\omega)^{-1}\xi : \bar{f}_n(A, B) < \kappa\}.$$

To prove the the Brin-Katok entropy formula for $d^{FK_\omega}$, Shannon-McMillan-Breiman theorem of RDS needs to be used as a tool, which had been proved in [26].
Lemma 3.3. Let \((X, d)\) be a compact metric space, \(T\) a continuous bundle RDS on \((X, d)\) over \((\Omega, \mathcal{F}, P, \vartheta)\). For any finite partition \(\xi\) of \(X\), if \(\mu \in M^1_P(\Omega \times X, T)\), then we have
\[
\lim_{n \to \infty} -\frac{1}{n} \log \mu_\omega(\xi^n(x)) = h_\mu(\xi, \omega, x)
\]
where \(\xi^n(x)\) is the member of the partition \(\bigvee_{i=0}^{n-1} (T_i^\omega)^{-1}\xi\) to which \(x\) belongs.

Then we can get the following theorem.

Theorem 3.4. Let \(T\) be a continuous bundle RDS over \((\Omega, \mathcal{F}, P, \vartheta)\), \(\mu \in M^1_P(\Omega \times X, T)\), then for \(\mu\)-a.e. \((\omega, x) \in \Omega \times X\),
\[
h_\mu(T, \omega, x) = \lim_{\delta \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_\omega(B_{FK_n}(\omega, x, \delta))
\]
(3.6)
\[
= \lim_{\delta \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_\omega(B_{FK_n}(\omega, x, \delta)).
\]

Proof. We first prove the inequality
\[
h_\mu(T, \omega, x) \geq \lim_{\delta \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_\omega(B_{FK_n}(\omega, x, \delta)).\]

Since \(d_{FK_n}(x, y) \leq d_{n}(x, y)\), by (2.5), we can obtain that
\[
h_\mu(T, \omega, x) \geq \lim_{\delta \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_\omega(B_{FK_n}(\omega, x, \delta)).\]

Now we proceed to prove the inequality
\[
h_\mu(T, \omega, x) \leq \lim_{\delta \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_\omega(B_{FK_n}(\omega, x, \delta)).\]

Let
\[
\mathcal{E}_0 = \{ (\omega, x) \in \Omega \times X : h_\mu(T, \omega, x) < \infty \},
\]
\[
\mathcal{E}_\infty = \{ (\omega, x) \in \Omega \times X : h_\mu(T, \omega, x) = \infty \}.
\]

Without loss of generality, we assume that \(\mu(\mathcal{E}_0) > 0, \mu(\mathcal{E}_\infty) > 0\).

Note that \(X\) is compact, we can construct a family of increasing finite Borel partitions \(\{\xi_i\}_{i=1}^\infty\) of \(X\) with \(\mu_\omega(\partial\xi_i) = 0, \forall i\) and \(\text{diam}(\xi_i) \to 0, i \to \infty\).

Given \(\varepsilon > 0\). Since
\[
h_\mu(\xi, \omega, x) \to h_\mu(T, \omega, x), \text{a.e.}(\omega, x) \in \mathcal{E}_0,
\]
we can find \(\xi \in \{\xi_i\}_{i=1}^\infty\) with \(\mu(A) > \mu(\mathcal{E}_0) - \frac{\varepsilon}{2}\), where
\[
A=\{(\omega, x) \in \mathcal{E}_0 : |h_\mu(\xi, \omega, x) - h_\mu(T, \omega, x)| < \frac{\varepsilon}{2}\}.
\]
For $\delta > 0$, we define

$$U_\delta(\xi) = \{(\omega, x) \in E : B_{F_{K_n}}(\omega, x, \delta) \setminus \xi(x) \neq \emptyset\}.$$  

It is clear that for $(\omega, x) \in U_\delta(\xi)$, the $\delta$-ball about $x$ on the metric $d_{F_{K_n}}$ is not contained in the element of $\xi$ which $x$ belongs. On the other hand, we can find that

$$U_\delta(\xi) = \bigcup_{\omega \in \Omega} (U_\delta(\xi))_\omega$$

where

$$(U_\delta(\xi))_\omega = \{x \in X : (\omega, x) \in U_\delta(\xi)\}, \omega \in \Omega.$$

By the definitions above, for $P$-a.e. $\omega \in \Omega$, $\bigcap_{\delta > 0} (U_\delta(\xi))_\omega = \partial \xi$. Then we can get that

$$\lim_{\delta \to 0} \mu_\omega(U_\delta(\xi))_\omega = 0.$$

For simplicity, we assume that this convergence is uniform in $\omega$ (otherwise, for $\forall n > 0$, by the Egorov theorem, we can choose $\Omega_n \subset \Omega$, with $P(\Omega_n) > 1 - \frac{1}{n}$, such that the above convergence is uniform in $\omega \in \Omega_n$). Choose $0 < \kappa < \varepsilon$ with

$$2\kappa \log|\xi| - 4\kappa \log \kappa - 4(1 - \kappa) \log(1 - \kappa) < \frac{\varepsilon}{2}.$$

Therefore we can find $\delta \in (0, \frac{\kappa}{2})$ such that $\forall \delta_0 \leq \delta$, for $P$-a.e. $\omega \in \Omega$,

$$\mu_\omega(U_{\delta_0}(\xi))_\omega < \left(\frac{\kappa}{4}\right)^2$$

and

$$\mu(U_{\delta_0}(\xi)) = \int_\Omega (U_{\delta_0}(\xi))_\omega \mu(\omega) < \left(\frac{\kappa}{4}\right)^2.$$

By Birkhoff theorem, $\exists \chi_{U_{\delta_0}(\xi)}^* \in L^1(\Omega \times X, F \times B, \mu)$ such that for $\mu$-a.e. $(\omega, x) \in \Omega \times X$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{U_{\delta_0}(\xi)}(\Theta^i(\omega, x)) = \chi_{U_{\delta_0}(\xi)}(\omega, x).$$
By Chebyshev inequality, we have
\[
\mu\left(\left\{ (\omega, x) : \chi_{U_{\delta_0}(\xi)}(\omega, x) < 1 - \frac{\kappa}{4} \right\}\right) \geq 1 - \int \chi_{U_{\delta_0}(\xi)}(\omega, x) d\mu(\omega, x) = 1 - \frac{\mu(U_{\delta_0}(\xi))}{\kappa/4} > 1 - \frac{\kappa}{4}.
\]

For simplicity as before, we assume that the convergence in (3.8) is uniform in \((\omega, x)\). Therefore, when for \(n\) large enough, we can obtain that
\[
\mu\left(\left\{ (\omega, x) : \frac{1}{L} \sum_{i=0}^{L-1} \chi_{U_{\delta_0}(\xi)}(\Theta^i(\omega, x)) < 1 - \frac{\kappa}{2}, \forall L \geq n \right\}\right) > 1 - \frac{\kappa}{2}.
\]

Denote
\[
E_L = \left\{ (\omega, x) : \frac{1}{L} \sum_{i=0}^{L-1} \chi_{U_{\delta_0}(\xi)}(\Theta^i(\omega, x)) < 1 - \frac{\kappa}{2}, \forall L \geq n \right\}.
\]

Then we can find that \(\mu(E_L) > 1 - \frac{\varepsilon}{2}\). Hence \(A \cap E_L \subset E_0\) and \(\mu(A \cap E_L) > \mu(E_0) - \varepsilon\). Let
\[
A_k = \left\{ (\omega, x) \in A \cap E_L : k\varepsilon \leq h_\mu(T, \omega, x) < (k + 1)\varepsilon \right\}.
\]

We have \(\bigcup_{k=0}^{\infty} A_k = A \cap E_L\). Hence there exists some \(N_1\) such that \(\mu(\bigcup_{k=0}^{N_1} A_k) > \mu(E_0) - \varepsilon\). By Lemma 3.5, we have
\[
\lim_{n \to \infty} \frac{-1}{n} \log \mu_\omega(\xi^n(x)) = h^{(r)}_\mu(\xi, \omega, x), a.e.(\omega, x) \in \Omega \times X.
\]

For simplicity, we assume that the convergence in (3.9) is uniform in \((\omega, x)\). For \(\left(\frac{\varepsilon}{2}\right)^2\), \(\exists N_2 \in \mathbb{N}\), then \(\forall n > N_2\), we have
\[
\frac{-1}{n} \log \mu_\omega(\xi^n(x)) - h^{(r)}_\mu(\xi, \omega, x) > \left(\frac{\varepsilon}{2}\right)^2.
\]

Then by the Chebyshev inequality, for \(n\) large enough we have
\[
\mu\left(\left\{ (\omega, x) \in \bigcup_{k=0}^{N_1} A_k : \frac{-1}{n} \log \mu_\omega(\xi^n(x)) > h^{(r)}_\mu(\xi, \omega, x) - \frac{\varepsilon}{2} \right\}\right) > \mu\left(\bigcup_{k=0}^{N_1} A_k\right) - \frac{\varepsilon}{2}.
\]
By the definition of $A_K$, we have
$$
\mu\left( \bigcup_{k=0}^{N_1} \left\{ (\omega, x) \in A_k : -\frac{1}{n} \log \mu_\omega(\xi^n(x)) > h^{(r)}_\mu(\xi, \omega, x) - \frac{\varepsilon}{2} \right\} \right) > \mu\left( \bigcup_{k=0}^{N_1} A_k \right) - \frac{\varepsilon}{2}.
$$

Hence we can find $B_k \subset A_k$, $0 \leq k \leq N_1$ such that
$$
\mu\left( \bigcup_{k=0}^{N_1} B_k \right) > \mu(E_0) - \frac{3\varepsilon}{2},
$$

Denote the filbers
$$
E_{L,\omega} = \left\{ x : \frac{1}{L} \sum_{i=0}^{L-1} \chi_{U_{\delta_0}(\xi)}(T^i_\omega(x)) < 1 - \frac{\kappa}{2}, \forall L \geq n \right\},
$$

$$
A_{k,\omega} = \{ x : (\omega, x) \in A_k \}, \quad B_{k,\omega} = \{ x : (\omega, x) \in B_k \}.
$$

For $\forall \ n > N_2$, $\forall x \in B_{k,\omega}$, we have

\begin{equation}
-\frac{1}{n} \log \mu_\omega(\xi^n(x)) > h^{(r)}_\mu(\xi, \omega, x) - \frac{\varepsilon}{2} > h^{(r)}_\mu(T, \omega, x) - \varepsilon \geq (k-1)\varepsilon.
\end{equation}

The second inequality holds because $x \in A$ and the last inequality holds because $x \in A_k$.

Let $N_0 = \max\{\{N_2, L\}, n > N_0$.

**Claim:** Fix $\in \Omega$. For $x \in B_{k,\omega}$, we have

$$
B_{FKn}(\omega, x, \delta_0) \subset \bigcup_{t : \tilde{f}_n(t, \xi^n(x)) < \kappa} t.
$$

**Proof of claim:** Let $y \in B_{FKn}(\omega, x, \delta_0)$. We only need to prove that there exists $t = (t_0, \ldots, t_{n-1})$ such that $y \in \bigcap_{i=0}^{n-1} (T^i_\omega)^{-1} A_t$ and $\tilde{f}_n(t, \xi^n(x)) < \kappa$.

By the definition of $d_{FKn}^\omega$, for fixed $\omega \in \Omega, n \in \mathbb{N}, x, y \in X$, there exists an $(\omega, n, \delta_0)$-match of $x$ and $y$ with

$$
|\pi| = |D(\pi)| > n(1 - \delta_0).
$$

Since $x \in E_{L,\omega}$, we have

$$
|\{ 0 \leq i \leq n - 1 : T^i_\omega(x) \notin (U_{\delta_0}(\xi))_\omega \} | > n(1 - \frac{\kappa}{2}).
$$

Note that

$$
|D(\pi)| > n(1 - \delta_0) > n(1 - \frac{\kappa}{2}),
$$

we have

$$
|\{ i \in D(\pi) : T^i_\omega(x) \notin (U_{\delta_0}(\xi))_\omega \} | > n(1 - \frac{\kappa}{2}).
$$
Denote
\[ D_1 = \{ i \in D(\pi) : T_i^j(x) \notin (U_{\delta_0}(\xi))_\omega \}. \]
Clearly \( D_1 \subset D(\pi) \) and for \( \forall j \in D_1 \),
\[ T_j^j(x) \notin (U_{\delta_0}(\xi))_\omega, \]
by \((3.7)\) we can obtain that
\[ B(T_j^j, \delta_0) \subset \xi(T_j^j x). \]
For \( \forall j \in D_1 \), \( d(T_j^j(x), T_{\omega(j)}(y)) < \delta_0 \), we have
\[ T_{\omega(j)}(y) \in B(T_j^j, \delta_0). \]
Hence \( T_{\omega(j)}(y) \in \xi(T_j^j x) \). It follows that
\[ \bar{f}_n(\xi^n(y), \xi^n(x)) \leq 1 - \frac{|D_1|}{n} < \kappa. \]
The claim is proved.

Next for \( \omega \in \Omega \), we estimate the \( \mu_\omega \)-measure of the set
\[ \{ x \in B_{k,\omega}: \mu_\omega(B_{FK_n}(\omega, x, \delta_0)) > e^{-n(k-2)\varepsilon} \}. \]
Note that the number of elements in \( B_{\bar{f}_n}(\xi^n(x), \kappa) \) is not more than \( (C_n^{[\kappa n]})^2|\xi|^{n\kappa} \), then we can see that
\[ \{ x \in B_{k,\omega}: \mu_\omega(B_{FK_n}(\omega, x, \delta_0)) > e^{-n(k-2)\varepsilon} \} \]
\[ \subset \{ x \in B_{k,\omega}: \mu_\omega(\bigcup_{t: \bar{f}_n(t,\xi^n(x)) < \kappa} t) > e^{-n(k-2)\varepsilon} \} \]
\[ \subset \{ x \in B_{k,\omega}: \exists t \in B_{\bar{f}_n}(\xi^n(x), \kappa), s.t. \mu_\omega(t) > \frac{e^{-n(k-2)\varepsilon}}{(C_n^{[\kappa n]})^2|\xi|^{n\kappa}} \} \]
\[ \subset \bigcup_{t \cap B_{k,\omega} \neq \emptyset} \{ t: \exists t' \in B_{\bar{f}_n}(t, \kappa), s.t. \mu_\omega(t') > \frac{e^{-n(k-2)\varepsilon}}{(C_n^{[\kappa n]})^2|\xi|^{n\kappa}} \} \]
For the last set, the number of such \( t \) is not more than
\[ e^{n(k-2)\varepsilon}((C_n^{[\kappa n]})^2|\xi|^{n\kappa})^2. \]
We only need to estimate the \( \mu_\omega \) of a single \( t \). Let \( x \in t \cap B_{k,\omega} \). Since
\[ t \in \bigvee_{i=1}^{n-1} (T_i^j)^{-1} \xi, \] it follows that \( t \in \xi^n(x) \), hence \( t = \xi^n(x) \). By \((3.10)\) we have
\[ -\frac{1}{n} \log \mu_\omega(\xi^n(x)) > (k-1)\varepsilon. \]
That is $\mu_\omega(t) < e^{-n(k-1)\epsilon}$. Hence
\[
\mu_\omega \{ x \in B_{k,\omega} : \mu_\omega(B_{FK_n}(\omega, x, \delta_0)) > e^{-n(k-2)\epsilon} \} < e^{-n(k-1)\epsilon}e^{n(k-2)\epsilon}((C_n^{[n\epsilon]})^2|\xi|^{n\epsilon})^2
\]
\[
= e^{-n\epsilon}((C_n^{[n\epsilon]})^2|\xi|^{n\epsilon})^2.
\]
By Stirling’s formula,
\[
\lim_{n \to \infty} \frac{1}{n} \log C_n^{[n\epsilon]} = -(1 - \epsilon) \log (1 - \epsilon) - \epsilon \log \epsilon.
\]
Clearly
\[
\lim_{n \to \infty} \frac{1}{n} \log (C_n^{[n\epsilon]})^4|\xi|^{2n\epsilon+2} = e^{2n\epsilon \log |\xi| - 4n\epsilon \log \kappa - (1-\kappa) \log (1-\kappa)} < e^{\frac{\kappa}{2}}.
\]
Then $\sum_n e^{-n\epsilon}((C_n^{[n\epsilon]})^4|\xi|^{2n\epsilon+2}$ is convergent. By the Borel-Cantelli lemma, we have
\[
\liminf_{n \to \infty} \frac{\log \mu_\omega(B_{FK_n}(x, \delta_0))}{n} \geq (k-2)\epsilon > h^{(r)}_{\mu}(T, \omega, x) - 3\epsilon, \text{ a.e.} (\omega, x) \in B_k.
\]
Hence we have
\[
\liminf_{n \to \infty} \frac{\log \mu_\omega(B_{FK_n}(\omega, x, \delta_0))}{n} > h^{(r)}_{\mu}(T, \omega, x) - 3\epsilon, \text{ a.e.} (\omega, x) \in \bigcup_{k=0}^{N} B_k.
\]
Note that $\mu(\bigcup_{k=0}^{N} B_k) > \mu(\mathcal{E}_0) - \frac{3\epsilon}{2}$, then by the arbitrariness of $\epsilon$, we can get that
\[
\lim \liminf_{\delta \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_\omega(B_{FK_n}(\omega, x, \delta)) \geq h^{(r)}_{\mu}(T, \omega, x), \text{ a.e.} (\omega, x) \in \mathcal{E}_0.
\]
Similarly we can prove
\[
\lim \liminf_{\delta \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_\omega(B_{FK_n}(\omega, x, \delta)) \geq h^{(r)}_{\mu}(T, \omega, x), \text{ a.e.} (\omega, x) \in \mathcal{E}_\infty.
\]
Therefore
\[
\lim \liminf_{\delta \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_\omega(B_{FK_n}(\omega, x, \delta)) \geq h^{(r)}_{\mu}(T, \omega, x), \text{ a.e.} (\omega, x) \in \Omega \times X.
\]
The proof of Theorem 3.4 has been finished. □

Next we similarly introduce the measure-theoretic version of $sp_{FK}(\omega, n, \epsilon)$.

**Lemma 3.5.** If $0 \leq p_i \leq 1, \forall 0 \leq i \leq 1$, and $\sum_{i=1}^{n} p_i = 1$, then
\[
\sum_{i=1}^{n} (-p_i \log p_i) \leq \log n.
\]

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Definition 3.6. Let $T$ be a continuous bundle RDS over $(\Omega, \mathcal{F}, P, \vartheta)$, $\mu \in M^1_P(\Omega \times X, T)$. Denote

$$sp_{FK}(\omega, \mu, n, \varepsilon) = \min\{sp_{FK}(\omega, n, Z, \varepsilon) : Z \subset X, \mu_\omega(Z) > 1 - \varepsilon\}.$$  

Theorem 3.7. Let $T$ be a continuous bundle RDS over $(\Omega, \mathcal{F}, P, \vartheta)$, $\mu \in M^1_P(\Omega \times X, T)$. Then for $P$-a.e. $\omega \in \Omega$, we have

$$h_\mu^{(r)}(T) \leq \liminf_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log sp_{FK}(\omega, \mu, n, \varepsilon).$$

If $\mu \in E^1_P(\Omega \times X, T)$ and $h_\mu^{(r)}(T) < \infty$, then

$$h_\mu^{(r)}(T) = \liminf_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log sp_{FK}(\omega, \mu, n, \varepsilon)$$

$$= \limsup_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log sp_{FK}(\omega, \mu, n, \varepsilon).$$

Proof. Note that $d^K_{\omega}(x, y) \leq d^0_{\omega}(x, y)$ and when $\mu \in E^1_P(\Omega \times X, T)$, by (2.7), for $P$-a.e. $\omega \in \Omega$, we have

$$h_\mu^{(r)}(T) \geq \limsup_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log sp_{FK}(\omega, \mu, n, \varepsilon).$$

Then we only need to prove that when $\mu \in M^1_P(\Omega \times X, T)$, we have

$$h_\mu^{(r)}(T) \leq \liminf_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log sp_{FK}(\omega, \mu, n, \varepsilon).$$

Given a finite Borel partition $\eta$ of $X$ and $\delta > 0$. Let $\eta = \{A_1, \cdots, A_k\}$. Note that $\{\Omega \times A_1, \cdots, \Omega \times A_k\}$ is a special finite partition of $\Omega \times X$, thus we need to show

$$h_\mu^{(r)}(T, \eta) \leq \liminf_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log sp_{FK}(\omega, \mu, n, \varepsilon) + 2\delta.$$  

Take $0 < \kappa < \frac{1}{2}$ with

$$-4\kappa \log 2\kappa - 2(1 - 2\kappa) \log(1 - 2\kappa) + (\kappa^2 + 3\kappa) \log(k + 1) < \delta.$$

Note that $\pi_X \mu$ is normal, then for each $A_i$ we can find a close set $B_i$ with $\pi_X \mu(A_i \setminus B_i) < \frac{\kappa^2}{k}$. Hence we can construct a finite Borel partition $\xi = \{B_1, \cdots, B_k, B_{k+1}\}$ of $X$ such that $B_i$ is closed for $1 \leq i \leq k$, $\pi_X \mu(B_{k+1}) < \kappa^2$, and

$$h_\mu^{(r)}(T, \eta) < h_\mu^{(r)}(T, \xi) + \delta.$$  

Let $B = \bigcup_{i=1}^k B_i$, $b = \min_{1 \leq i < j \leq k} d(B_i, B_j)$. Denote

$$B_\Omega = \Omega \times B = \{\Omega \times B_1, \cdots, \Omega \times B_k\}.$$  

It is easily to see that $b > 0$ and $\mu(B_\Omega) > 1 - \kappa^2$.  

Let $0 < \varepsilon < \frac{b}{2}$, $n \in \mathbb{N}$. For all $\omega \in \Omega$, by the definition of $sp_{FK}(\omega, \mu, n, \varepsilon)$, there exists $Z \in X$ such that $\mu(\bigcup_{i=1}^{m(n)} B_{FK_n}(\omega, x_i, \varepsilon)) > 1 - \varepsilon$, where $m(n) = sp_{FK}(\omega, \mu, n, \varepsilon)$. For simplicity, we denote

$$F_n = \{(\omega, x) : \omega \in \Omega, x \in \bigcup_{i=1}^{m(n)} B_{FK_n}(\omega, x_i, \varepsilon)\}$$

and

$$E_n = \{(\omega, x) : \frac{1}{n} \sum_{i=0}^{n-1} \chi_B(T_\omega^i(x)) \leq 1 - \varepsilon\}.$$  

We can get that $\mu(E_n) < \kappa$. Put $W_n = B_\Omega \cap F_n \cap E_n^c$ and denote the fiber

$$W_{n,\omega} = \{x : (\omega, x) \in W_n\}.$$  

then we can get that $\mu(\omega, W_{n,\omega}) > 1 - \kappa^2 - \kappa - \varepsilon$. For $z \in W_{n,\omega}$, we have

$$\frac{1}{n} \sum_{i=0}^{n-1} \chi_B(T_\omega^i z) > 1 - \kappa.$$  

**Claim:** For any fixed $\omega \in \Omega$, $1 \leq i \leq k$,

$$|\{A \in \bigvee_{j=0}^{n-1} (T_\omega^i)^{-1} : A \cap B_{FK_n}(\omega, x_i, \varepsilon) \cap W_{n,\omega} \neq \emptyset\}| \leq (C_n^{n(n(2\kappa+2\varepsilon))})^2 |\xi|^{n(2\kappa+2\varepsilon)}.$$  

**Proof of claim.** Let $A_1, A_2 \in \bigvee_{j=0}^{n-1} (T_\omega^i)^{-1} \xi$ with

$$A_1 \cap B_{FK_n}(\omega, x_i, \varepsilon) \cap W_{n,\omega} \neq \emptyset, A_2 \cap B_{FK_n}(\omega, x_i, \varepsilon) \cap W_{n,\omega} \neq \emptyset.$$  

For all $x \in A_1, y \in A_2$, we have $d_{FK_n}(x, y) < 2\varepsilon$. Then there exists an $\omega, (n, 2\varepsilon)$-match $\pi$ of $x$ and $y$ with

$$|D(\pi)| > n(1 - 2\varepsilon).$$  

Denote

$$D_x = \{0 \leq j \leq n - 1 : T_\omega^j x \in B\}, D_y = \{0 \leq j \leq n - 1 : T_\omega^j y \in B\}.$$  

Let

$$D' = \pi^{-1}(\pi(D(\pi) \cap D_x) \cap D_y).$$  

For simplicity, we assume that $D'$ is not empty. It is easy to see that $D' \subset D(\pi)$ and $|D'| > n(1 - 2\kappa - 2\varepsilon)$. For every $j \in D'$,

$$d(T_\omega^j x, T_\omega^j y) < 2\varepsilon < b$$
and $T^{j\omega} x, T^{\pi(j)} y \in B$. Hence $T^{j\omega} x, T^{\pi(j)} y$ must lie one of the the same set in $\{B_1, \cdots, B_k\}$. It follows that
\[
\bar{f}_n(A_1, A_2) < 2\kappa + 2\varepsilon.
\]
Note that the number of $A$ satisfying
\[
\bar{f}_n(A_1, A) < 2\kappa + 2\varepsilon
\]
is not more than
\[
(C_n^{(2\kappa+2\varepsilon)})^2 |\xi|^{n(2\kappa+2\varepsilon)}.
\]
The proof of the claim is finished.

Now we can estimate $h^{(r)}_\mu (T, \xi)$. First we estimate $H^{(r)}_{\mu\omega}(\bigvee_{i=0}^{n-1}(T^i_\omega)^{-1}\xi)$.
\[
H^{(r)}_{\mu\omega}(\bigvee_{i=0}^{n-1}(T^i_\omega)^{-1}\xi)
\leq H^{(r)}_{\mu\omega}(\bigvee_{i=0}^{n-1}(T^i_\omega)^{-1}\xi \cup \{W_{n,\omega}, X \setminus W_{n,\omega}\})
\leq \mu(\omega(W_{n,\omega}) \log(|\{A: A \in \bigvee_{i=0}^{n-1}(T^i_\omega)^{-1}\xi, A \cap W_{n,\omega} \neq \emptyset\} |) - \mu(\omega(W_{n,\omega}) \log \mu(\omega(W_{n,\omega})
+ \mu(\omega(W_{n,\omega}) \log(|\{A: A \in \bigvee_{i=0}^{n-1}(T^i_\omega)^{-1}\xi, A \cap W_{n,\omega} \neq \emptyset\} |) - \mu(\omega(W_{n,\omega}) \log \mu(\omega(W_{n,\omega})
\leq \mu(\omega(W_{n,\omega}) \log(\sum_{i=1}^{m(n)} |\{A: A \in \bigvee_{i=0}^{n-1}(T^i_\omega)^{-1}\xi, A \cap B_{\mathcal{F}K_n}(\omega, x_i, \varepsilon) \cap W_{n,\omega} \neq \emptyset\} |)
+ (\kappa^2 + \kappa + \varepsilon)n \log(k+1) + \log 2.
\]
Thus
\[
h^{(r)}_\mu (T, \xi) = \lim_{n \to \infty} \frac{1}{n} \int H^{(r)}_{\mu\omega}(\bigvee_{i=0}^{n-1}(T^i_\omega)^{-1}\xi)dP(\omega)
\leq \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log s_{\mathcal{F}K}(\omega, \mu, n, \varepsilon) + 2 \sup_{n \to \infty} \frac{1}{n} \log C_n^{(n(2\kappa+2\varepsilon))}
+ (\kappa^2 + 3\kappa + 3\varepsilon) \log(k+1).
\]
By Stirling’s formula, let $\varepsilon \to 0$ , we can obtain
\[
h^{(r)}_\mu (T, \xi) \leq \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log s_{\mathcal{F}K}(\omega, \mu, n, \varepsilon)
- 4\kappa \log 2\kappa - 2(1 - 2\kappa) \log(1 - 2\kappa) + (\kappa^2 + 3\kappa) \log(k+1)
< \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log s_{\mathcal{F}K}(\omega, \mu, n, \varepsilon) + \delta.
\]
Hence
\[ h^{(r)}_{\mu}(T, \eta) < h^{(r)}_{\mu}(T, \xi) + \delta < \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log sp_{FK}(\omega, \mu, n, \varepsilon) + 2\delta. \]

By the arbitrariness of \( \delta \), we have
\[ (3.13) \quad h^{(r)}_{\mu}(T) \leq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log sp_{FK}(\omega, \mu, n, \varepsilon). \]

\[ \Box \]

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