Long-range interactions from $U(1)$ gauge fields via dimensional mismatch

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We show how certain long-range models of interacting fermions in $d + 1$ dimensions are equivalent to $U(1)$ gauge theories in $D + 1$ dimensions, with the dimension $D$ in which gauge fields are defined larger than the dimension $d$ of the fermionic theory to be simulated. For $d = 1$ it is possible to obtain an exact mapping, providing an expression of the fermionic interaction potential in terms of half-integer powers of the Laplacian. An analogous mapping can be applied to the kinetic term of the bosonized theory. A diagrammatic representation of the theories obtained by dimensional mismatch is presented, and consequences and applications of the established duality are discussed. Finally, by using a perturbative approach, we address the canonical quantization of fermionic theories presenting non-locality in the interaction term to construct the Hamiltonians for the effective theories found by dimensional reduction. We conclude by showing that one can engineer the gauge fields and the dimensional mismatch in order to obtain long-range effective Hamiltonians with $1/r$ potentials.
I. INTRODUCTION

In the last decade, there has been significant progress in the realization of synthetic quantum matter [1] via the control and manipulation of AMO systems such as trapped ions, Rydberg atoms, and quantum gases of magnetic atoms and polar molecules [2–5]. Such advancements made possible the implementation of a variety of quantum long-range (LR) models [6–17]. Recent highlights in this direction include the physical realization of Ising and XY quantum spin chains with tunable LR interactions with ions in a Penning trap [6], neutral atoms in a cavity [14, 16, 17], trapped ions [11–13], and Rydberg atoms [7]. The resulting interactions decay algebraically with the distance $r$, and the decay exponent can be experimentally tuned [11–13]. As an example of the many and promising possibilities opened by the control of LR interactions in spin chains we may refer to the recent emulation of the one-dimensional (1D) version of quantum electrodynamics, the Schwinger model [18].

Such experimental progresses found a counterpart in the intense theoretical activity on the properties of quantum LR systems [19–41]. Among others, we mention the study of the effect of non-local interactions on the dynamics of excitations [42–49] and on the equilibrium properties and phase diagram [50–54]. This recent activity aims in perspective also to provide a quantum counterpart of the well established results for classical LR systems [55].

Usually, power-law interactions are written as $V(r) \propto r^{-d-\sigma}$, where $r$ is the distance between the particles or spins, $d$ is the physical space dimension, and $\sigma$ is the decay exponent. For $\sigma < 0$, the internal energy of the system typically diverges in the thermodynamic limit, calling for a redefinition of the interaction strength [56]. For fast enough decaying interactions, $\sigma > 0$, the system is additive and thermodynamics is well defined. For $\sigma$ belonging to a certain range, the system may present a phase transition with spontaneous symmetry breaking at a critical temperature $T_c > 0$ [57]. For $2\sigma \leq d$ one has mean field properties. However one can define a critical value $\sigma^*$ such that for $\sigma > \sigma^*$ the system has the same critical behavior of its short-range (SR) analogue. In the intermediate region $d/2 < \sigma \leq \sigma^*$ the LR interactions are found to be relevant and the determination $\sigma^*$ has been the subject of a long-lasting theoretical activity [50–54].

The main point of this analysis is that increasing $\sigma$ one is effectively passing from the upper critical dimension to the lower critical one. Therefore to each $\sigma$ there is, at least effectively, an effective dimensionality $d_{\text{eff}}$ so that the LR system is equivalent to a SR model living in $d_{\text{eff}}$. In concrete, this provides one to apply well established results for local many-body systems, such as the Mermin-Wagner-Hohenberg theorem [58, 59] and Lieb-Robinson bounds [60] on the propagation of quantum correlations, to non-local models. However, it has been challenging to cast this intuitive picture into a rigorous theoretical framework, mostly due to the complexity of the mapping directly at the operator (i.e., Hamiltonian) level.

Here, we propose a new approach to this problem, based on an exact mapping between LR interacting systems in $d$ dimensions, and particles coupled to dynamical gauge fields in a higher dimension $D > d$. The central aspect of our work is the consideration that, when fermionic degrees of freedom are confined to move on a reduced dimensionality with respect to the gauge fields, the latter can mediate different types of interactions thanks to the extra dimension(s) available, effectively providing a knob to tune the interparticle potential. Since the initial theory is fully local in $D$ spatial dimensions, this mapping allows one to exploit the full predictive power of general results for local field theories to the non-local one in $D$ dimensions. As an illustrative case sample, we exploit this strategy to discuss fermionic (and spin) models in $D = 1$, and we show that the latter systems in the presence of Coulomb interactions can be exactly mapped to Abelian gauge theories living in higher dimensions.

Our work takes inspiration from graphene experiments [61], where the system is two-dimensional, but the electromagnetic field acting on it is living in three dimensions. Specifically, in such settings, the electrons are confined in a $2D$ (i.e., $2 + 1$) plane while interacting with the electromagnetic field that lives in the full $3D$ space (i.e., $3 + 1$). The formalism of Pseudo QED, introduced in [62], provides a full dynamic way, from first principles, to deal with specific problem. For a system confined in a two-dimensional space, with $(x, y, z = 0)$ coordinates, this is done by taking the fermion kinetic part of the two-dimensional space and the electromagnetic kinetic term in three-dimensional space. The two fields are then coupled through the standard minimal coupling with the additional requirement that no fermionic current exists or flows outside $z = 0$. In standard QED, one writes the 4-current of the fermions in $3 + 1$ dimensions in the form $j_\mu^f = \bar{\psi}\gamma_\mu\psi$. Here, however, there is a dimensional mismatch and the gamma matrices indices do not run through the same set of numbers. This is overcome by considering

$$
j_\mu^f(\tau, x, y) = \begin{cases}
  j_\mu^f(\tau, x, y) \delta(z) & \text{if } \mu = 0, 1, 2 \\
  0 & \text{otherwise}
\end{cases}
$$

(1)

where $j_3^f = \bar{\psi}\gamma_\alpha\psi$ is the 3-current of the fermions in $2 + 1$ dimensions. Eq. (1) states that no fermionic current exists or flows outside the plane. By integrating out the gauge field and applying the above condition (1) the resulting Lagrangian consists on an effective $2 + 1$ dimensional Lagrangian containing a LR interaction [62–68]. This LR term is fundamentally different from the LR interaction obtained when the original electromagnetic field is also living in the same dimensionality as the fermions ($2 + 1$) and is at the basis of several peculiar properties of Pseudo QED, such as the dynamical generation of a mass term [65, 67]. The dynamical chiral symmetry breaking in reduced QED theories was studied as well [69], and the procedure of dimensional reduction was applied to the edge modes of two-dimensional topological insulators [70].

In the present work we start by generalizing the Pseudo QED construction to general dimensions. The dimensional reduction giving a LR term in $2 + 1$ dimension starting from a theory in which the electromagnetic (fermionic) field lives in $3 (2)$ spatial
dimensions can be indeed seen from the opposite point of view, i.e. i) determine what is the gauge field living in higher dimensions giving rise to a target LR interaction to be implemented via the dimensional mismatch, and ii) explore what kind of LR interactions can be realized. We then consider the general scenario where the electromagnetic field and general $U(1)$ gauge fields live in $D + 1$ dimensions and the fermions in $d + 1$ with $D \geq d$. The case $D = d$ is of course trivial in the sense that it corresponds to QED$d$, even though it is useful to keep in mind that the known results for this case should be recovered whenever we take $D = d$.

Lifting the restriction $D = d$ opens the door to new kind of systems with novel LR interactions but which are derived, from first principles, from a local gauge theory. Such formulations may not only be useful from a direct application point of view, as in the graphene example, but may also serve as a tool to characterize LR systems. In fact, by being able to map a LR interacting system to a local gauge theory one may be able to use tools otherwise unaccessible in LR systems like the Mermin-Wagner theorem or Lieb-Robinson bounds.

Another motivation behind the present work is that in literature it has been discussed the possibility to have tunable interactions with cold atoms, and in particular $1/r$ interactions [71]. Despite the fact that, for trapped ions, interactions of Ising-type can be made to decay algebraically with the distance $r$ with an adjustable exponent (usually in the range $\lesssim 3$), so far no experiments have been performed for a Bose or Fermi gas with an effective $1/r$ interaction also in lower dimension and proposals in this direction are certainly desirable.

With these various motivations in mind, one may consider the general scenario where there are $N_f$ fermionic flavors $\{\psi_a\}_{a=0,...,N_f}$ living in $d + 1$ dimensions coupled to $N_g$ gauge fields $\{A_{\mu}^b\}_{b=0,...,N_g}$ each one living in $D_b + 1$ dimensions (where $D_b \geq d$, $\forall b$) with coupling parameters $e_{ab}$. Furthermore, with the advent of quantum simulations of gauge theories [72–74] in which gauge symmetries are engineered in the laboratory, these systems can open the possibility to explore new kind of phenomena or provide humbler toy models to more complicated systems. A first demonstration of a quantum simulation of a gauge theory was put forward in [18]. In a near future it is expected that more complex theories may be achieved from which the above described scenario constitutes a particular example. Indeed, the fact that the fermions are restricted to a lower dimension constitutes a simplification when compared to the case where all particles live in the higher dimensional system. For reviews on the topic of quantum simulation of gauge theories see e.g. [72–74].

In this work, our attention is focused on the effective interaction between fermionic degrees of freedom. The gauge fields will be taken in three possible dimensions $D = 1, 2, 3$, minimally coupled with the fermions. This structure, of dimensionality and type of coupling, poses restrictions on the type of interactions obtained in the effective fermionic theory. Still, in the case of $d = 1$, fermionic degrees of freedom can also be integrated out paving the way to more general interactions for a given fermionic flavor (left out of the integration). Further generalizations are possible which may incorporate Higgs fields, interaction between gauge fields $(F_{\mu\nu}^a F_{\mu\nu}^b)$ or other symmetries besides $U(1)$. Such generalizations are beyond the scope of this paper where we target in detail the construction of general interacting potential between fermions and kinetic terms of bosonic theories in $d = 1$. The mapping is achieved by bosonizing all fermionic degrees of freedom in the last case, and all but one on the first. We consider also the possibility of a four fermion interaction in the form of a current-current term $j_\mu j_\mu$ where $J_\mu = \bar{\psi} \gamma_\mu \psi$.

The paper is organized as follows: In Section II we establish the general formalism for the construction of the effective fermionic theory in the lower dimensionality $(d + 1)$ and its respective relation with the gauge theory, in the same dimension, with a modified gauge kinetic term. In Section III we consider the special case $D = 2$ and $d = 1$ where we use bosonization. We show how it is possible to consider several fermionic flavors and gauge fields in such a way that, integrating all of the gauge fields and bosonizing all the fermionic flavors it is possible to construct a general bosonic kinetic term, on the Lagrangian. of the form $\phi f (−\partial_\mu^2) \phi$. Here $\phi$ is the bosonic field and $f$ is a function which can be seen as an expansion in half-integer exponents (in powers of $\alpha$ with $2\alpha \in \mathbb{Z}$). When $f$ is the identity function we have the standard kinetic term for a bosonic field. We have freedom on engineering the coefficients of these expansions by modifying the coupling of the initial theory. By following the same process bosonizing all fermionic flavors but one, it is also possible to construct a similar kind of interaction between fermions $j_\mu V (−\partial_\mu^2) j_\mu$ again with $V$ admitting a series expansion on integer and half-integer powers of the Laplacian and $j_\mu$ the fermionic current. In Section IV we provide an overview on how these kind of models can naturally fit in the class of proposals of experimental realization of quantum simulations of gauge theories available on the literature. In Section V we deal with the canonical quantization and the construction of the Hamiltonians for the models obtained by dimensional reduction, and show how to obtain non-relativistic fermions interacting via an $1/r$ potential. Our conclusions are presented in Section VI, while more technical material is in the Appendices.

II. DIMENSIONAL REDUCTION

We start by reviewing the formalism of Pseudo QED, referring to electrons confined in a plane and interacting with an electromagnetic field defined in the 3D. To make this statement explicit, one takes a matter Lagrangian $\mathcal{L}_M$ coupled with a gauge field (in Euclidean time):

$$\mathcal{L} = \mathcal{L}_M - ie j_\mu^A A_\mu + \frac{1}{4} F_{\mu\nu}^2,$$

(2)
where \( j_4^\mu = \bar{\psi} \gamma_\mu \psi \). The 4-current \( j_4 \) and the 3-current defined \( j_3 \) defined in the space \((x,y,z=0)\) are related by Eq. (1). One has then to integrate the gauge field and apply the condition (1), obtaining a 2 + 1 Lagrangian [62]. It is possible to reformulate such theory and re-write it in terms of a three-dimensional gauge field and re-storing locality in the fermionic part of the Lagrangian. The price to pay, which is actually a resource for the purposes of our paper, consists in transferring the long range term to the kinetic part of this new gauge field. Nonetheless, \( U(1) \) gauge invariance is retained.

In direct generalization of (2), we consider the Lagrangian

\[
\mathcal{L} = \mathcal{L}_M^{d+1} - ie j_{D+1}^\mu A_\mu + \frac{1}{4} F_{\mu \nu}^2 + \mathcal{L}_{GF}.
\] (3)

The term \( \mathcal{L}_{GF} \) corresponds to the Fadeev-Popov gauge fixing term. It is explicitly given by \( \mathcal{L}_{GF} = \frac{1}{2} \xi^2 (\partial_\mu A_\mu)^2 \), where different choices of \( \xi \) correspond to different gauges. Here we adopt the Feynman gauge where \( \xi = 1 \), resulting in a propagator \( G_{\mu \nu} = \frac{1}{2} \xi^2 \delta_{\mu \nu} \). The fermions are defined in the lower dimensionality \( d + 1 \), which is made explicit in the matter Lagrangian \( \mathcal{L}_M^{d+1} \). The gauge field lives in higher dimensionality \( D + 1 \). The \( D + 1 \) current is taken explicitly to be:

\[
j_{D+1}^\mu (x^\alpha) = \begin{cases} j_{D+1}^\mu (x_0, \ldots, x_d) & \text{if } \mu = 0, \ldots, d \\ 0 & \text{otherwise} \end{cases}
\] (4)

From this point one can integrate out the gauge degrees of freedom obtaining then a non-local interaction between the fermions of the form

\[
\frac{e^2}{2} \int j_{D}^\mu (z) \left[ (\partial^2)^{-1} \right]_{z,z'} j_{D}^\mu (z') d^{D+1}z'.
\]

We shall denote this long range potential by \( G_D (z - z') \) exhibiting explicitly the dependence on the dimension where the gauge fields lived. The nature of these interactions is encoded in the higher dimension \( D + 1 \) and it is not dependent on the dimensionality where the fermions live which did not enter yet. We observe that the fact that integrating degrees of freedom one obtains LR terms (and possible multi-body interactions) is ubiquitous in RG treatments of models, where typically one takes a model and integrate over a sub-class of the original degrees of freedom remaining in the same dimensional space (see, for example, [75]). The difference with the models considered here is then that one performs a dimensional reduction while making the integration of gauge degrees of freedom.

An explicit expression for \( G_D \) can be obtained as

\[
G_D (z - z') = \int \frac{d^{D+1}k}{(2\pi)^{D+1}} \frac{e^{ik(z-z')}}{k^2}
\] (5)

The resulting fermionic Lagrangian can now be written exclusively in terms of the degrees of freedom in lower dimension \( d + 1 \):

\[
\mathcal{L} = \mathcal{L}_M^{d+1} + \frac{e^2}{2} \int d^{d+1}x' j_{D}^\mu (x) j_{D}^\mu (x') \times G_D (z) |_{(z_0, \ldots, z_d) \rightarrow (x_0 - x'_0, \ldots, x_d - x'_d)}
\] (6)

One can also represent the LR interaction in an operator form which will be useful later. It consists in taking the operator \( \left( - \partial_\mu^{2 \mid \mu = 0, \ldots, D} \right)^{-1} \) and integrate out the extra dimensions keeping the Laplacian for the lower dimensions. Here we adopt the notation \( \partial^2 = \partial_0^2 + \ldots + \partial_d^2 \) and represent the interaction as \( \frac{e^2}{2} j_\mu \hat{G}_{D \rightarrow d} (\partial^2) j_\mu \), where

\[
\hat{G}_{D \rightarrow d} \equiv G_{D \rightarrow d} \left( - \partial_\mu^{2 \mid \mu = 0, \ldots, d} \right) = \int \frac{d^{D-d}k}{(2\pi)^{D-d}} \frac{1}{-\partial_0^2 - \ldots - \partial_{d+1}^2 + k_1^2 + \ldots + k_{D-d}^2}.
\] (7)

The two forms of presenting the resulting Lagrangian emphasize two different aspects. When writing, as in the first case, the interaction in terms of a space-time function \( G_D (z) \), we see that the current-current interaction does not depend on the lower dimension and only on the upper dimension. In turn, when writing as above in terms of a modified dispersion relation, we see that the formal structure of the function \( G_{D \rightarrow d} \), which will have as argument the Laplacian, only depends on how many dimensions we are integrating out. Of course the two approaches are equivalent and, in fact while the function \( G_{D \rightarrow d} \) only depends on the difference between the dimensions, the operator \( \hat{G}_{D \rightarrow d} \equiv G_{D \rightarrow d} (\partial^2) \) does not. The interplay of the two ways of looking at the theory are equally useful.
As mentioned before, it is possible to transfer this LR interaction into the kinetic part of gauge fields now living in \( d + 1 \) dimensions as well. The goal is to identify the effective theory with:

\[
\mathcal{L}_d = \mathcal{L}^{d+1}_M - i \epsilon_{d+1} A_\mu + \frac{1}{4} F_{\mu\nu} \tilde{M}_{D \rightarrow d} F_{\mu\nu}
\]  

(8)

where, unlike Eq. (3), all the fields live in \( d + 1 \) dimensions and the operator \( \tilde{M} \) is to be fixed in such a way that Eq. (6) is recovered. Note that this theory is also gauge invariant in lower dimension \( (A_\mu \rightarrow A_\mu + \frac{1}{2} \partial_\alpha A_\mu) \) and \( \psi \rightarrow \psi e^{-i\alpha} \). In order to integrate the gauge fields, a gauge fixing is therefore necessary. Standard gauge fixing as it was done before is possible, but it is not ideal for our purposes. A different gauge fixing function which takes into account the non-locality of the Lagrangian turns out to be more adequate – for more details see Appendix A. The analogous of the Feynman gauge then cancels the off-diagonal terms. For this choice, the propagator is given by \( G_{\mu\nu} = \frac{1}{-\partial^2 \tilde{G}_{D \rightarrow d}} \delta_{\mu\nu} \). By the subsequent integration of the gauge fields and identification with the Lagrangian (6) in the operator form when (7) is used, we conclude that both theories coincide for

\[
\tilde{M}_{D \rightarrow d} = \left( -\partial^2 \tilde{G}_{D \rightarrow d} \right)^{-1},
\]  

(9)

which concludes the mapping for the modified gauge theory exclusively in \( d + 1 \) dimensions.

In the following we denote the dimensional reduction from a \( D + 1 \) theory to a \( d + 1 \) one by the shortcut notation “\( D \rightarrow d \)”. To conclude this Section we observe that, although the previous treatment starts from the relativistic Lagrangian (3), one can as well perform in a similar way the dimensional reduction for a non-relativistic system coupled to a gauge fields leaving in higher dimensions.

III. \( 2 \rightarrow 1 \) DIMENSIONAL REDUCTION: CONSTRUCTION OF DISPERSION RELATIONS

In this Section we consider the \( d = 1, \ D = 2 \) case, i.e., the \( 2 \rightarrow 1 \) dimensional reduction. Apart from being the simplest example of the general dimensional reduction discussed in the previous Section, we have two main reasons for focusing in detail on such case: i) the possibility of using bosonization considerably simplifies the treatment; ii) when dealing with LR interactions, one realizes that the dimensionality of the space on which the elementary constituents are defined is not crucial – not as much as in presence of SR interactions – since varying the type and the range of the LR interactions one is effectively changing the dimensionality of the system. As an example of the last statement we may consider the \( 1D \) LR Ising model, where passing \( \sigma \) from 0 to 1 one is spanning the effective dimensionality from 4 (which is the upper critical dimension of the SR Ising model) to 1. Therefore controlling the LR interactions one is equivalent (at least, in the renormalization group sense) to controlling the dimensionality of the system. Furthermore, due to the form of the operator \( \hat{G}_{2 \rightarrow 1} \) and the possibility of mapping the resulting theory through bosonization (only available when \( d = 1 \)), the class of LR models that can be addressed is larger once we introduce extra flavors and integrate them.

We start by analyzing the simplest case where only one flavor and a single gauge field is present. Before moving to the specific case of \( 2 \rightarrow 1 \), we keep the treatment general considering the more general case \( D \rightarrow 1 \). In this Section we take the matter Lagrangian to be the one of free Dirac fermions with a current-current local interaction. We do not consider current-current interactions between different fermionic flavors since the structure of the analysis does not change. However in Appendix B we discuss the inclusion of such terms and briefly discuss its consequence.

A. One gauge field and one fermionic flavor with a Thirring term in \( D \rightarrow 1 \)

The Lagrangian for the massless case is:

\[
\mathcal{L} = -\bar{\psi} (\gamma_\mu \partial_\mu + i e \gamma_\mu A_\mu) \psi + \frac{g}{2} (\bar{\psi} \gamma_\mu \psi)^2 + \frac{1}{4} F_{\mu\nu} \tilde{M}_{D \rightarrow 1} F_{\mu\nu}
\]  

(10)

The goal of this Section is to obtain an effective Lagrangian for the bosonic field resulting from bosonization. The procedure is closely related to the bosonization of the Thirring model \([76]\) and of the Schwinger model \([77]\). A possible path involves a Hubbard-Stratonovich transformation to replace the four fermion coupling by introducing a vector field \( B_\mu \), which is taken here to be such that

\[
\frac{g}{2} (\bar{\psi} \gamma_\mu \psi)^2 \rightarrow -ie B_\mu (\bar{\psi} \gamma_\mu \psi) + \frac{e^2}{2g} B_\mu^2.
\]  

(11)

Each vector field can, in two dimensions, be parameterized by two scalar fields as follows:

\[ A_\mu = \partial_\mu \chi - i \epsilon_{\mu\nu} \partial_\nu \varphi \]

and

\[ C_\mu = \partial_\mu \chi' - i \epsilon_{\mu\nu} \partial_\nu \varphi'. \]

The coupling between the vector fields and the fermions can be eliminated by a phase and chiral
transforms. It appears then an extra term present due to the chiral anomaly and both $\chi$ and $\chi'$ decouple from the rest of the Lagrangian. The new fermionic variables are then mapped to a free bosonic theory. After a translation of the bosonic field, the current of the initial fermionic field can be written in terms of the new bosonic variable $j_\mu \equiv \pi^{-1/2} \varepsilon_{\mu\nu} \partial_\nu \phi$. Details of this procedure and of the generalizations discussed below can be found in Appendix B.

Finally, integrating the field $\phi'$ the effective Lagrangian is:

$$\mathcal{L} = \frac{1}{2} \left( 1 + \frac{g}{\pi} \right) (\partial_\mu \phi)^2 - \frac{e}{\sqrt{\pi}} \partial_\mu \phi \partial_\mu \varphi - \frac{1}{2} \partial^2 \varphi \hat{M}_{D\rightarrow 1} \partial^2 \varphi$$  \hspace{1cm} \text{(12)}$$

The integration of $\varphi$ yields the final Lagrangian:

$$\mathcal{L} = \frac{1}{2} \left( 1 + \frac{g}{\pi} \right) (\partial_\mu \phi)^2 + \frac{e^2}{2\pi} \hat{M}_{D\rightarrow 1} \phi.$$  

Of course, when $D = 1$, then $\hat{M}_{D\rightarrow 1} = 1$ and the known result is recovered (see for example [77] for $g = 0$).

Since we wish to explore more complicated theories, it is useful to introduce a diagrammatic representation of the theories we are working on. In Fig. 1, we represent the different fields on the theory and connect them, if they are coupled to each other, by straight lines. Straight lines connecting fermionic flavors (including self coupling) correspond to current-current interaction, and finally, bosons are connected by as many straight lines as there are derivatives present in their coupling. In the case of the vector fields, we put as many bars on top of the field as the original dimension that the field lives in. This means that if the kinetic term is $F_{\mu\nu} \hat{M}_{D\rightarrow 1} F^{\mu\nu}$, there will be $D$ bars on top of the respective vector field. The diagrams do not specify the actual value of the coupling, even though it can be associated with each line making the diagrams much more heavy. We plot the initial and final theories (10) and (12) in Fig. 1. In Appendix B we use the diagrammatic representation to represent the intermediate mappings that allow us to establish the relation between the initial and final theory. We do this systematically for all the theories represented in the main text.

**B. Controlling the kinetic term of bosonic theories**

The Lagrangians (10) and (12) are easily generalized to arbitrary number of flavors $N_f$ by introducing a flavor index $\psi \rightarrow \psi_\alpha$, $g \rightarrow g_\alpha$ and consequently $\phi \rightarrow \phi_\alpha$. Furthermore, we allow the coupling between the different fermionic flavors and the gauge field to be different: $e \rightarrow e_\alpha$. In terms of the bosonization procedure this amounts to introduce $N_f$ auxiliary fields $B_\alpha^\mu$ and define a set of new variables $C_\alpha^\mu = A_\mu + B_\alpha^\mu$. All the rest is analogous to what was described before. Interactions between different flavors are obtained once the gauge field is integrated out: $\mathcal{L} = \frac{1}{2} \left( 1 + \frac{g_\alpha}{\pi} \right) (\partial_\mu \phi_\alpha)^2 + \frac{e_\alpha^2}{2\pi} \phi_\alpha \hat{M}_{D\rightarrow 1} \phi_\alpha$ with implicit sum over flavors. Fig. 2 represents diagrammatically the case of two flavors. Integrating $\phi_2$ (corresponding to one of the flavors) in the final theory results in:

$$\mathcal{L} = \frac{1}{2} \phi_1 \left[ \left( 1 + \frac{g_1}{\pi} \right) (-\partial^2) + \frac{e_1^2}{2\pi} \hat{M}_{D\rightarrow 1} \phi_1 \right]$$  \hspace{1cm} \text{(13)}$$

We now apply the previous results for the case $D = 2$. From Eq. (7) we see that $\hat{G}_{2\rightarrow 1} = \left[ 2\sqrt{-\partial^2} \right]^{-1}$ and therefore $\hat{M}_{D\rightarrow 1}^{-1} = \sqrt{-\partial^2}/2$ from Eq. (9). For large distances (small momentum) this is the dominant term of the denominator. The
relevant scales for this limit can be controlled via $g_2$ and $e_2$. Expanding the denominator one gets
\[
\frac{1}{(1 + \frac{g_2}{\pi}) (-\partial^2) + \frac{e_2^2}{\pi} M_{2 \to 1}^{-1}} = \frac{\pi}{e_2^2} M_{2 \to 1}^{-1} \sum_n \left[ -\left(1 + \frac{g_2}{\pi}\right) \frac{2\pi}{e_2^2} \sqrt{-\partial^2}\right]^n
\]
(14)

Substituting back into the Lagrangian generates a series expansion in $\sqrt{-\partial^2}$ and therefore we get all integer and half integer powers $\sqrt{-\partial^2}$ in the form:
\[
\mathcal{L} = -\frac{1}{2} \phi \left[ \sum_{n=1}^\infty \sigma_{n/2} \left(-\partial^2\right)^n \right] \psi,
\]
(15)
where we dropped the index of the remaining field $\phi$. The first terms of these expansion are $\sigma_{1/2} = e_2^2 \left(1 - (8\pi)^{-1}\right)/2\pi$, $\sigma_1 = 1 + \frac{g_2}{\pi} + \frac{e_2^2}{\pi} \left(1 + \frac{g_2}{\pi}\right)$ and for higher terms we have $\sigma_{n/2} = -\frac{e_2^2}{16\pi} \left[ -\left(1 + \frac{g_2}{\pi}\right) \frac{2\pi}{e_2^2}\right]^n$.

There are two main constraints on producing this expansion. For one side there are only 4 parameters ($e_a$ and $g_a$) which means we cannot control an arbitrary number of terms $\sigma_{n/2}$. On the other side, the sign of $\sigma_{n/2}$ is well defined with $n$ even giving a negative coefficient (except for $n = 2$) and $n$ odd giving positive coefficients. One can increase the freedom of choice of the absolute value of the coefficients observing that we can enter a third flavor $\psi_3$ with a Thirring interaction and a new gauge field $A_{\mu}^3$ which is only coupled to flavors 1 and 3. Following the same procedure of bosonization and integration of the degrees of freedom of the third flavor, we get a similar expression with new coefficients $\sigma_{n/2}^\text{new} \equiv \sigma_{n/2}^{(12)} + \sigma_{n/2}^{(13)}$. Here we denoted the previous coefficient by $\sigma_{n/2}^{(12)}$ putting in evidence that it results from an interaction between flavors 1 and 2. Analogously the new contribution is denoted with indices (13). This procedure can be followed to an arbitrary number of flavors and one gets an effective coefficient $\sigma_{n/2} = (1 + \frac{g_2}{\pi}) \delta_{1,n/2} + \sum_{i=2}^{N_f} \sigma_{n/2}^{(i)}$ where $\sigma_{n/2}^{(i)} = -\frac{e_2^2}{16\pi} \left[ -\left(1 + \frac{g_2}{\pi}\right) \frac{2\pi}{e_2^2}\right]^n$ for $n \geq 1$. The coefficient $\sigma_{1/2}$ does not change and it is controlled exclusively by $e_1$. By considering more and more number of flavors in this scheme we are able to control the coefficients $\sigma_{n/2}$ to an arbitrary order with some constraints. The general structure of such theories with $N_f$ flavors is illustrated in Figure 3.

C. Controlling fermionic interactions

The integration of the gauge fields naturally leads to the introduction of non local terms in the fermionic action. In order to obtain the expansion on half integer powers (15), it was crucial to integrate the fermionic degrees of freedom. This enabled us to overcome the paradigmatic extra quadratic term of the form $\frac{e_2^2}{2\pi} \phi M_{2 \to 1}^{-1} \phi$. Here we apply the same procedure. We want to retain explicitly one fermionic field while integrating the remaining ones. To this end we bosonize all but one fermionic degree of freedom. In order to avoid unnecessary complications, here we restrict ourselves to the case of two fermionic fields and no Thirring coupling. Considering then the two flavors $\psi$ and $\psi'$ and bosonizing the former leads to:
\[
\mathcal{L} = -\bar{\psi} \gamma_\mu \partial_\mu \psi + ej_\mu \epsilon_{\mu \nu} \partial_\nu \psi - \frac{e_f'}{\sqrt{\pi}} \phi' (-\partial^2) \phi + \frac{1}{2} (\partial \phi')^2 - \frac{1}{2} \phi (-\partial^2) \tilde{M}_{D-1} (-\partial^2) \phi.
\]
(16)
Integrating the degrees of freedom of $\phi'$ (in the form of $\phi'$) one gets:
\[
\mathcal{L} = -\bar{\psi} \gamma_\mu \partial_\mu \psi + ej_\mu \epsilon_{\mu \nu} \partial_\nu \psi - \frac{e_a^2}{2\pi} \phi (-\partial^2) \phi - \frac{1}{2} \phi (-\partial^2) \tilde{M}_{D-1} (-\partial^2) \phi.
\]
(17)
The final form of the LR fermionic theory is obtained by integrating out the gauge field, for which it is useful to re-introduce $A_\mu = -i\epsilon_{\mu\nu}\partial_\nu\phi$. The result is:

$$\mathcal{L} = -\bar{\psi}\gamma_\mu \partial_\mu \psi + \frac{1}{2}e^2 j_\mu \frac{1}{\pi} M_{D\rightarrow 1} \left(-\partial^2\right)^{\frac{n}{2}} j_\mu.$$  \hspace{1cm} (18)

Analogously to the case of the bosonic kinetic term, we focus now on the case $D = 2$ and consider the dominant term for large distances. In this case the denominator is of the form $1 + (2\pi/e')^2 \sqrt{-\partial^2}$. The expansion in Taylor series will give then a series of the form $j_\mu \left(-\partial^2\right)^{\frac{n}{2}} j_\mu$. As in the previous subsection, the coefficients of the expansion can be chosen with a certain freedom. This is again done by considering more gauge fields exactly as in Figure 3 where we can choose to include the Thirring terms or not. The final theory obtained is then:

$$\mathcal{L} = -\bar{\psi}\gamma_\mu \partial_\mu \psi + \frac{1}{2}e^2 j_\mu \left[\sum_{n=1}^{N_f} \frac{\lambda_n/2}{2^n}\right] \left(-\partial^2\right)^{\frac{n}{2}} j_\mu.$$  \hspace{1cm} (19)

where the coefficients $\lambda$ are given by

$$\lambda_{n/2} = \sum_{i=2}^{N_f} \frac{\pi}{e_i^2} \left(-\frac{2\pi}{e_i^2}\right)^n$$  \hspace{1cm} (20)

in the absence of the Thirring term.

IV. EXPERIMENTAL IMPLEMENTATIONS

In this Section we briefly discuss the recent advances in the simulation of gauge fields with cold atoms and trapped ion systems in connection with the formalism discussed in Sec. II-III. As mentioned in Sec. I, the first experimental demonstration was reported last year using trapped ions [18]. Several proposals have discussed possible implementation schemes of the lattice version of different gauge theories from $1 + 1$ to $3 + 1$ dimensions (see Refs.[73, 74] for a review). These proposals either focus on pure gauge systems, or on gauge fields coupled to matter, but always with matching dimensions. The extension to include a possible mismatch is however straightforward since the relevant terms are already present in Hamiltonian and it is only needed to suppress fermion hopping in the relevant direction(s).

As examples we take two different proposals [78, 79] which implement gauge symmetry in two different ways using ultracold atoms loaded in optical lattices. The dimensional reduction of the space spanned by the fermions consists on substituting the periodic lattice potential by a confining one in the dimensions to be fixed. This poses no threat to the implementation of gauge symmetries.

More in detail, in the first scheme, gauge symmetry is obtained as a low energy effective symmetry by introducing an energy punishment for states that violate gauge symmetry. In perturbation theory, the terms that arise are correlated hoppings of bosons and fermions, corresponding to matter-gauge coupling, and pure bosonic terms, corresponding to the pure gauge contribution for the Hamiltonian. If there are no fermions in a given part of the system no correlated hopping would be obtained but the pure gauge terms would still be present in perturbation theory.

In the second, gauge symmetry arises from internal symmetries of the system. The principle, however, is exactly the same. Due to conservation of total hyperfine angular momentum only certain scattering processes are selected. With a judicious choice of atomic species only the fermion-boson correlated hopping and pure gauge terms will be selected by angular momentum conservation. Again the absence of fermions will retain the former processes corresponding exactly to the kind of theories we explore here. Finally, another technique which is applicable to both schemes is to render fermionic tunneling off-resonant in the transverse directions.

V. LONG-RANGE EFFECTIVE HAMILTONIANS

The Hamiltonians for the effective theories described in the previous sections are, in general, highly non-trivial. This is the result of non-locality in time of the Lagrangian. Due to the presence of arbitrarily high powers of time derivatives, the Euler-Lagrange equations are modified. The Hamiltonian formulation of such theories can be achieved within the Ostrogradsky’s construction [80]. The canonical quantization of theories with non-local kinetic terms, like Pseudo QED, was adressed in [81–83].

Here, however, we would like to address the canonical quantization of fermionic theories presenting non-locality in the interaction term. It has been shown that, under certain circumstances, and in a perturbative setting, it is possible to use the
free equations of motion in order to eliminate the non-locality in time [84, 85]. Specifically, such procedure is possible when the non-local terms are governed by a small coupling parameter. It can then be shown that there exists a field transformation equivalent to the application of the free equations of motion (by consistently disregarding higher orders of the coupling). The fact that the non-locality is obtained from the integration of degrees of freedom of a renormalizable theory plays a fundamental role [84]. In that case, if there are no unphysical effects at first order approximation due to non-locality, we expect that such unphysical effect cancel as well at higher orders. This is due to the fact that the original theory - that is, the one before integration of degrees of freedom - is well defined. Consequently, systematically disregarding the higher powers of the coupling parameter should be consistent and the approximation well defined. We follow this procedure here in first order perturbation theory.

In previous Sections we have worked with the imaginary time formulation which is suited to establish the connection with statistical mechanics. In this Section we construct a quantum Hamiltonian and work in real time.

We focus on the case of non-relativistic fermions, as the case of Dirac fermions raises different kinds of questions not to be adressed here. In particular, for the case of massless Dirac fermions, the free equations of motion imply $\Box \psi = 0$. Application of them in the non-local term will yield, at first order on the coupling, $\mathcal{O} (0)$ which is in general divergent. Therefore, and according to the previous discussion, perturbation theory is not well defined. For non-relativistic fermions this problem is not present and, furthermore, in the limit of large mass we can truncate the temporal derivatives up to some order. In the following we illustrate this procedure and compute the Hamiltonian for the lowest order.

The Lagrangian for the non-relativistic fermion is then given by:

$$\mathcal{L} = \psi^\dagger \left( i\partial_0 - eA_0 + \frac{\hbar^2}{2m} (\partial_i + ieA_i)^2 + \mu \right) \psi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \tag{21}$$

Notice that in Eq. (21) there is the extra term proportional to $e^2 A_0^2$. This term would give rise to higher order terms: since our treatment is perturbative on $e$, those will be discarded.

The current is given by $j_0 = \psi^\dagger \psi$ and $j_i = i\frac{\hbar^2}{2m} \partial_i \psi^\dagger \psi$. The procedure of variable substitution can be done by considering the current-current interaction in position space:

$$j^\mu (t, x) \mathcal{O} (t - t', x - x') j_\mu (t', x').$$

Then we consider an expansion of the current of the type

$$j_\mu (t', x') = \sum_{n=0}^{\infty} \frac{(t' - t)^n}{n!} \partial^\mu_0 j_\mu (t, x').$$

Now one can use the equations of motion to replace the time derivatives. In particular from $\left( i\partial_0 + \left( \frac{\hbar^2}{2m} \partial_i^2 - \mu \right) \right) \psi = 0$ one can make the replacement (up to $e^2$ order):

$$\partial^\mu_0 j_0 \to \sum_{l=0}^{n} \binom{n}{l} \left( -i \frac{\hbar^2}{2m} \nabla^2 + i\mu \right)^l \psi^\dagger \left( i \frac{\hbar^2}{2m} \nabla^2 - i\mu \right)^{2n-l} \psi$$

with analogous expressions for the other components of $j_\mu$. This term at $x'$ is coupled to $j_\mu (t, x)$ by the function:

$$\int dt' \mathcal{O} (t - t', x - x') (t - t')^n$$

In this notation $\mathcal{O} (t - t', x - x')$ are the matrix elements of the operator $G_{3\rightarrow1} (-\partial^2)$, in its real time form, presented in Appendix C.

With such replacements the theory becomes local in time at the cost of having (generally complicated and non local) spatial interactions. We observe that, in general, since $\mathcal{O} (t, x)$ is an even function on time, the terms with $n$ odd will not contribute.

For illustrative purposes we compute the Hamiltonian density for the case of gauge fields living in $3 + 1$ dimensions, chemical potential set to zero and large mass limit. Each $n$-th derivative above will give rise to a prefactor $\left( \frac{\hbar^2}{2m} \right)^n$ and therefore at lowest order of the large mass limit we can disregard all terms but $n = 0$. Furthermore, as $j_i$ is proportional to $\left( \frac{\hbar^2}{2m} \right)$, also these terms are of higher order so they are dropped. We are left with the interaction term:

$$- \frac{e^2}{2} j_0 (t, x) \left[ \int dt' \mathcal{O} (t - t', x - x') \right] j_0 (t, x') \tag{24}$$

For the case of gauge fields in 3 dimensions the effective interaction $\int dt' \mathcal{O} (t - t', x - x')$ is:

$$- \frac{e^2}{8\pi} \int \frac{d^2 q}{(2\pi)^2} dt' \log \left( 1 + \frac{\Lambda^2}{q^2 - q_0^2} \right) e^{-i(t-t')q_0 + (x-x')q_1} - \frac{e^2}{16\pi^2 |x-x'|} \left( 1 - e^{-|x-x'|\Lambda} \right) \tag{25}$$
Then in the limit of large cut-off $\Lambda$ we obtain the effective Lagrangian:

$$\mathcal{L} = \psi^\dagger \left( i\partial_0 + \frac{\hbar^2}{2m} \partial_x^2 \right) \psi (t, x) - \frac{e^2}{16\pi^2} \int dx' \psi^\dagger (t, x) \psi (t, x) \frac{1}{|x - x'|} \psi^\dagger (t, x') \psi (t, x')$$

(26)

This generates an effective Hamiltonian of fermions interacting with a $1/x$ potential which is the Coulomb potential expected on this limit. Namely the limit of large massive non-relativistic fermions weakly coupled to a three-dimensional gauge field is given by:

$$H = \int dx \left[ -\frac{\hbar^2}{2m} \psi^\dagger (t, x) \partial_x^2 \psi (t, x) + \frac{e^2}{16\pi^2} \int dx' \psi^\dagger (t, x) \psi (t, x) \frac{1}{|x - x'|} \psi^\dagger (t, x') \psi (t, x') \right]$$

(27)

The inclusion of the next leading order on the mass will give rise to two new kind of terms: one given by the other current component $j_1(t, x) j_1(t, x') / |x - x'|$ and the other being a density-density interaction coming from Eq. (23) with $n = 2$. The later will scale as the inverse square of the cut-off and therefore can be dropped in the large cut-off limit. By other side the the first term comming from $j_1$ interactions can be interpreted, in the lattice language, as a correlated hopping between two fermions at a distance $|x - x'|$. This also allows a better understanding of the initial approximation: for large masses the particles are slow enough that in lowest order the interaction is simply a density-density interaction.

VI. CONCLUSIONS

In this paper we explored a class of models of fermions coupled to gauge fields living in higher dimensions. These models are found to have direct application in physical systems, like in graphene, but can have a wider range of applicabilities. Here we focused on the possibility of mapping long-range (LR) models of statistical mechanics to local gauge theories with a dimensional mismatch. Such mappings allow one to apply tools that are only available in local theories to non-local theories, providing immediate access to insights on the dynamics of the latter. Moreover, the described mapping can be used as a tool to engineer desidered LR interactions by a properly engineering the gauge fields in the larger dimensionality space(s).

By introducing more fermionic flavors we were able, in the context of bosonization, to obtain a kinetic term which consists of an expansion in half integer powers of the Laplacian $-\partial^2$. More general expansions in arbitrary powers are likely non achievable from this mechanism, since it is expected that they would break break unitarity. In fact it was showed in Ref. [63] that the only unitary theories with the pure gauge term modified to be $\sim F_{\mu\nu} \frac{1}{(\partial^2)^{\frac{\alpha}{2}}} F_{\mu\nu}$ in 2 + 1 dimensions are for $\alpha = 0$ and $\alpha = 1/2$ [notice however that (19) is unitary].

The coefficients of these expansions display some freedom of choice by changing the parameters of the initial local theory. They are, however, still bounded by certain conditions, even though we showed that the freedom of choice can be increased by adding more flavours. An interesting question is what kind of non-locality can be obtained by a local theory as the ones considered here. It would be particularly interesting to investigate if the conditions obtained on these coefficients are a consequence of the mechanism considered (i.e., a local theory in $d + 1$ dimensions with minimal coupling between matter and gauge fields) and/or if they constitute a physical condition provided by unitarity. We also provided an overview on the implementation of this kind of gauge theories. In general existing proposals admit a straightforward generalization for the realization of artificial gauge theories with dimensional mismatch.

Our procedure can further be generalized by considering additional couplings to Higgs fields, interaction between gauge fields or other gauge symmetries besides $U(1)$. The integration of bosonic gauge fields or general gauge fields may also enlarge the space of LR models obtained after the dimensional reduction and it would be interesting to investigate if one can obtain fermionic interaction expansions like (19) in higher dimensions.

From the application side, our results can be applicable to a series of long-range interacting problems, where a mapping to a local higher dimensional field theory would allow to apply generic results for local field theories. This includes the characterization of topological order (for example towards the extension of the 10-fold classification to LR hopping free fermion theories [31]), the spreading of quantum information, and the study of localization mechanisms in the presence of LR hopping in one-dimensional systems [86]. From a different perspective, our approach can be potentially applicable to fracton models, as the latter, in some specific cases, can be understood as physical systems where gauge and matter degrees of freedom effectively live in different dimensionality [87].

Acknowledgements: Discussions with N. Defenu, G. Gori, C. Morais Smith, G. Palumbo, S. Ruffo, N. Westerberg and U.-J. Wiese are gratefully acknowledged. The authors acknowledge the hospitality of the Galileo Galilei Institute during the workshop ‘From Static to Dynamical Gauge Fields with Ultracold Atoms’, where part of this work was performed.

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and integrating over \( \omega \) of the off diagonal terms require a gauge fixing depending on \( \hat{\omega} \). Since the kinetic term of this modified gauge theory is affected by \( \hat{\omega} \) proceed with an integration over \( \omega \). It is worth noting that for the case of coupling between different flavors \( b \) can be chosen arbitrarily according to convenience. The scale \( \omega \) lies in the fact that the Gaussian weighting function, one finds that the resulting propagator is given by:

\[
G_{\mu\nu} = \left[ \frac{1}{\partial^2} \delta_{\mu\nu} + \left( 1 - \frac{1}{\xi} \right) \frac{\partial^\mu \partial^\nu}{(-\partial^2)^2} \right] M_{d \rightarrow d}^{-1} \tag{A1}
\]

Alternatively one can consider the same gauge fixing function as before, \( G(A^\alpha) = \partial_\mu A_\mu - \omega \), changing the weighting factor to \( e^{-\omega M^{-1} \partial_\mu A_\mu - \omega / 2 \xi} \). Both of the approaches are related by a simple variable transformation.

Appendix B: General procedure for arbitrary number of fields and diagrammatic

In this Appendix we present the bosonization procedure used on these theories accompanied by diagrammatic illustrations. Even though the diagrams do not replace the calculations they become useful to understand the structure of the procedure. The general strategy consists on four steps:

1) Eliminate quartic fermion interaction terms by an Hubbard-Stratonovich transformation. Here we adopt the notation, for the fictitious field, of \( B^a_{\hat{b}} \) or \( B^a_\mu \) in the case \( a = b \). It is worth noting that for the case of coupling between different flavors a decoupling can be achieved by replacing the fermionic interacting term between flavors \( a \) and \( b \) as follows:

\[
\hat{\omega}_{ab,j_\mu} \rightarrow -ie B^a_{\hat{b}} (j_{a}^\mu + j_{b}^\mu) + \frac{\xi^2}{2g_{ab}} (B^b_{\mu})^2. \tag{11}
\]

The scale \( e \) can be chosen arbitrarily according to convenience. The integration of \( B^a_{\hat{b}} \) generates not only the correct coupling between different flavors but also self energy couplings. For this reason it is necessary, in general, to introduce another field \( B^a_\mu \) in order to compensate this, even when self flavor coupling is absent in the original fermionic Lagrangian.

Diagramatically: Eliminate any line connecting fermionic flavors (possibly self coupled) and substitute by a vector field connecting the two flavors \( a \) and \( b \). In case of self coupling there is only a line connecting the vector field to the fermion.
2) This point is divided in 3 main parts:

2a) Take the vector fields and parameterize them by two bosonic fields: $\bar{A}_\mu = \partial_\mu \chi_a - i\epsilon_{\mu\nu} \partial_\nu \varphi_a$ and $B_{\mu}^{ab} = \partial_\mu \chi_{ab}' - i\epsilon_{\mu\nu} \partial_\nu \varphi_{ab}'$. Note that the indices without bars run through the different flavors and with bars through the gauge fields: $a, b \in \{1, . . . , N_f \}$ and $\bar{a} \in \{1, . . . , N_g \}$.

2b) Do a chiral transformation eliminating the remaining couplings between fermions and bosons. This is given by

$$\psi_a = e^{-i \sum_{\bar{b}} e_{\bar{a},\bar{b}} \gamma_s \varphi_{\bar{b}}} \psi'_a, \quad \psi'_a = \psi_a - \frac{ie}{\gamma_s} \sum_{\bar{b}} (\sum_{\bar{b}} e_{\bar{a},\bar{b}} \gamma_s \varphi_{\bar{b}} + \sum_{\bar{b}} e_{\bar{a},\bar{b}} \gamma_s \varphi_{\bar{b}})^2.$$ 

Due to the chiral anomaly the Lagrangian acquires some extra terms in the form of $L \rightarrow L - \frac{1}{\pi} \sum_a \left( \sum_{\bar{b}} e_{\bar{a},\bar{b}} \gamma_s \varphi_{\bar{b}} + \sum_{\bar{b}} e_{\bar{a},\bar{b}} \gamma_s \varphi_{\bar{b}} \right)^2$.

2c) Map the free fermionic theory to the free boson theory $\psi'_a \rightarrow \phi'_a$. Then we transform back the bosonic field:

$$\phi'_a = \phi_a - \frac{1}{\sqrt{\pi}} \sum_{\bar{b}} (\sum_{\bar{b}} e_{\bar{a},\bar{b}} \gamma_s \varphi_{\bar{b}} + \sum_{\bar{b}} e_{\bar{a},\bar{b}} \gamma_s \varphi_{\bar{b}})^2.$$ 

This transformation cancels the term originated by the chiral anomaly. It also creates a coupling between the bosonic fields $\phi_a$ and the degrees of freedom associated with the vector fields. As in the case of Section III C one can retain, without bosonizing, the desired fermionic flavors.

*Diagramatically:* Replace fermionic variables $\psi_a$ by bosonic flavors $\phi_a$, and vector field variables by a respective bosonic field. All coupling lines become double, signaling that all interactions have the form $\partial_\mu \phi \partial_\mu \varphi$.

3) Integrate the desired fields.

*Diagramatically:* Each bosonic variable has the standard kinetic term, with exception to the one with bars on top (that originates from the original gauge field in higher dimensions). When one field is integrated out it is erased from the diagram and it establishes couplings between fields that were connected to it in the previous diagram. Furthermore, it changes the kinetic term of all the fields that were linked to it. Care is needed at this point since if the integrated field is one that originates from a fictitious vector field, it just renormalizes the original kinetic term. For example in Eq. (12) the integration of the fictitious field just renormalized the pre-factor of the kinetic term $1 \rightarrow 1 + g/\pi$.

We show this process for the two specific cases used in our calculations. We consider as well an extra case in which there is a current-current coupling between fermions. This serves to illustrate how the diagrams can be used to quickly get the structure of the theory without performing any calculation.

### 1. One flavor, self coupled, gauge field originating from $D + 1$

This process is plotted in Figure 4. The numbers on top of the arrows indicate the steps described above. In the final diagram where we only have $\phi$ and $\varphi$ we read immediately that the theory has the structure:

$$L \equiv \frac{\lambda_1}{2} (\partial_\mu \phi)^2 + \lambda_2 \partial_{\mu} \phi \partial_{\mu} \varphi - \frac{1}{2} \partial^2 \varphi M_{D \rightarrow 1} \partial^2 \varphi.$$ (B1)

The actual values of $\lambda_1$ and $\lambda_2$ are not obtained from the diagrams and one has to do the actual computation, getting, as in Section III, $\lambda_1 = 1 + g/\pi$ and $\lambda_2 = -e/\sqrt{\pi}$, which is Eq. (12) of the main text.

### 2. Two flavors, self coupled, and gauge field originating from $D + 1$

In Figure 5 we detail the process of integration of Figure 2 concerning Sec. III B of the main text.
3. Two flavors, self coupled, coupled as well to each other and to a gauge field originating from $D + 1$

The diagrammatic process of considering an initial current-current coupling between the fermions is presented in Figure 6. The resulting theory will have the form:

$$L = \frac{\lambda_1}{2} (\partial_\mu \phi_1)^2 + \frac{\lambda_2}{2} (\partial_\mu \phi_2)^2 - \frac{1}{2} \partial^2 \varphi M_{D+1} \partial^2 \varphi + \lambda_{12} \partial_\mu \phi_1 \partial_\mu \phi_2 + \lambda_1 \partial_\mu \phi_1 \partial_\mu \phi + \lambda_2 \partial_\mu \phi_2 \partial_\mu \varphi.$$  \hspace{1cm} \text{(B2)}

This new interaction, concerning the inclusion of a current-current interaction between different fermionic flavors, will not change the general expansions 15 or 19, but instead will give an extra freedom on choosing the coefficients.

Appendix C: Non-local quantities for $D$ ranging from 1 to 3

The explicit computation for the function (5) is possible for the different dimensions. Changing to hyperspherical coordinates, the integrals can be reduced to $G_D (z) = \frac{\alpha_D}{(2\pi)^{D+2}} \int_0^+ \int_0^{\alpha_D} k^{D-2} e^{ik|z|} \cos \theta \, dk \, d\theta \sin \theta (\theta^{D-1} k D-2 e^{ik|z|} \cos \theta)$, where $\alpha_D = \pi$ for any $D$ except $D = 1$, where $\alpha_1 = 2\pi$. For the case of $D = 1$ we introduce an IR cut-off $q_0$. The results are given by:

- $D = 1$:

$$G_1 (z) = -\frac{1}{2\pi} \left( \gamma + \frac{1}{2} \log \left( q_0 |z| \right) \right)$$
(where \( \gamma \) is the Euler’s constant and \( q_0 \) the IR cut-off);

- \( D = 2 \):
  
  \[
  G_2(z) = \frac{1}{4\pi |z|};
  \]

- \( D = 3 \):
  
  \[
  G_3(z) = \frac{1}{4\pi |z|^2}.
  \]

Analogously one can compute the functional form of the various operators originated from dimensional integration. As explained on the main text, the functional form only depends on the dimensionality difference \( D - d \) while the Laplacian should be the one of the lower dimensionality \( d + 1 \). We report then the cases \( D - d = 2 \) having in mind \( D = 3 \) and \( d = 1 \) and \( D - d = 1 \) corresponding to \( D = 2 \) and \( d = 1 \) or \( D = 3 \) and \( d = 2 \). For comparison we also write the case trivial case \( D - d = 0 \). The case of \( D - d = 1 \) corresponds to Pseudo QED which is already reported in literature [62]. We note that for \( D - d = 2 \) the integral is divergent and we introduce a UV cut-off \( \Lambda \). This cut-off is for the integrated dimensions so it can be thought of as a continuous system in \( d + 1 \) dimensions, but with a finite lattice spacing in the perpendicular dimensions:

- \( D - d = 2 \):
  
  \[
  G_{D\to d}(-\partial^2) = \frac{1}{4\pi} \log \left( \frac{\Lambda^2 - \partial^2}{-\partial^2} \right);
  \]

  (where \( \Lambda \) is the UV cut-off);

- \( D - d = 1 \):
  
  \[
  G_{D\to d}(-\partial^2) = \frac{1}{2} \frac{1}{\sqrt{-\partial^2}};
  \]

- \( D = d \):
  
  \[
  G_{D\to d}(-\partial^2) = \frac{1}{-\partial^2}
  \]

  (i.e., the trivial case where no extra dimensions are integrated).