On convergence and growth of sums $\sum c_k f(kx)$

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Abstract

We investigate the almost everywhere convergence of $\sum_{k=1}^{\infty} c_k f(kx)$ where $f$ is a mean zero periodic function with bounded variation. The classical approach, going back to the 1940’s, depends on estimates for GCD sums $\sum_{k,\ell=1}^{N} (n_k,n_\ell)^2/(n_kn_\ell)$, a connection leading to the a.e. convergence condition $\sum_{k=1}^{\infty} c_k^2 (\log k)^{2+\varepsilon} < \infty$. Using a recent profound GCD estimate, Aistleitner and Seip \cite{2} weakened the convergence condition to $\sum_{k=1}^{\infty} c_k^2 (\log \log k)^{\gamma} < \infty$, $\varepsilon > 0$. In this paper we show that $\sum_{k=1}^{\infty} c_k^2 (\log \log k)^{\gamma} < \infty$ suffices for $\gamma > 4$ and not for $\gamma < 2$, settling the convergence problem except the unknown critical exponent of the loglog. Our method yields also new information on the growth of sums $\sum_{k=1}^{N} f(n_kx)$, a problem closely connected with metric discrepancy estimates for $\{n_kx\}$. In analogy with the previous result we show that optimal bounds for $\sum_{k=1}^{N} f(n_kx)$ for $f \in BV$ differ from the analogous (known) trigonometric results by a loglog factor.

1 Introduction

Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying

$$f(x+1) = f(x), \quad \int_0^1 f(x)dx = 0, \quad \int_0^1 f^2(x)dx < \infty.$$ \hspace{1cm} (1)

The almost everywhere convergence of series

$$\sum_{k=1}^{\infty} c_k f(kx)$$ \hspace{1cm} (2)

has been a much investigated problem of analysis. By Carleson’s theorem \cite{12}, in the case $f(x) = \sin 2\pi x$, $f(x) = \cos 2\pi x$ the series (2) converges a.e. provided

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\[ \sum_{k=1}^{\infty} c_k^2 \leq \infty. \] Gaposhkin [17] showed that this remains valid if the Fourier series of \( f \) converges absolutely; in particular this holds if \( f \) belongs to the Lip (\( \alpha \)) class for some \( \alpha > 1/2 \). However, Nikishin [19] showed that the analogue of Carleson’s theorem fails for \( f(x) = \text{sgn} \sin 2\pi x \) and fails also for some continuous \( f \). Berkes [6] showed that there is a counterexample in the Lip (1/2) class, and thus Gaposhkin’s result is sharp. For the class Lip (\( \alpha \)), \( 0 < \alpha \leq 1/2 \) and other classical function classes like \( C(0,1) \), \( L^p(0,1) \), \( BV(0,1) \) or function classes defined by the order of magnitude of their Fourier coefficients, the results are much less complete: there are several sufficient criteria (see e.g. [1], [9], [10], [11], [15], [20]; see also [16] for the history of the subject until 1966) and necessary criteria (see [6]), but there are large gaps between the sufficient and necessary conditions. Very recently, Aistleitner and Seip [2] proved that for \( f \in BV \) the series \( \sum_{k=1}^{\infty} c_k f(kx) \) converges a.e. if \( \sum_{k=1}^{\infty} c_k^2 (\log \log k)^\gamma < \infty \) for some \( \gamma > 0 \). The purpose of this paper is to improve this result further and to provide a near optimal a.e. convergence condition for \( \sum_{k=1}^{\infty} c_k f(kx) \) in the case \( f \in BV \).

**Theorem 1.** Let \( f : \mathbb{R} \to \mathbb{R} \) satisfy (7) and have bounded variation on \([0,1]\). Let \( (c_k) \) be a real sequence satisfying

\[ \sum_{k=1}^{\infty} c_k^2 (\log \log k)^\gamma < \infty \]

for some \( \gamma > 4 \). Then for any increasing sequence \( (n_k) \) of positive integers the series \( \sum_{k=1}^{\infty} c_k f(n_k x) \) converges a.e. On the other hand, for any \( 0 < \gamma < 2 \) there exists an increasing sequence \( (n_k) \) of positive integers and a real sequence \( (c_k) \) such that (3) holds, but \( \sum_{k=1}^{\infty} c_k f(n_k x) \) is a.e. divergent for \( f(x) = x - \lfloor x \rfloor - 1/2 \).

The proof of Theorem 1 also provides crucial new information on the growth of sums

\[ \sum_{k=1}^{N} f(n_k x) \]

where \( f \in BV \) satisfies (11). The order of magnitude of such sums is closely related to the classical problem of estimating the discrepancy of \( \{n_k x\} \) in the theory of Diophantine approximation. The problem goes back to Weyl [21] and the strongest result for general \( (n_k) \) is due to Baker [4], who proved that the discrepancy \( D_N(\{n_k x\}) \) of the sequence \( \{n_k x\} \) satisfies

\[ D_N(\{n_k x\}) \ll N^{-1/2} (\log N)^{3/2 + \varepsilon} \quad \text{a.e.} \]

for any \( \varepsilon > 0 \) and any increasing sequence \( (n_k) \) of integers. On the other hand, Berkes and Philipp [5] constructed an increasing \( (n_k) \) such that

\[ D_N(\{n_k x\}) \ll N^{-1/2} (\log N)^{1/2} \quad \text{a.e.} \]
is not valid. The optimal exponent of the logarithm remains open, in fact we do not even know if \( \varepsilon = 0 \). There has been, however, considerable progress in a closely related, slightly easier problem. Relation (5) and Koksma’s inequality (see e.g. [18], p. 143) imply that for any \( f \in BV \) satisfying (1) we have

\[
\left| \sum_{k=1}^{N} f(n_k x) \right| = O(\sqrt{N} (\log N)^{3/2+\varepsilon}) \quad \text{a.e.}
\]

Aistleitner, Mayer and Ziegler [3] improved here the upper bound to

\[
O(\sqrt{N} (\log N)^{3/2} (\log \log N)^{-1/2+\varepsilon})
\]

and in an unpublished manuscript Berkes [7] showed that for polynomially increasing \((n_k)\) the upper bound can be improved to \(O(\sqrt{N} (\log N)^{1/2+\varepsilon})\). In [2] Aistleitner and Seip removed the restriction of polynomial growth of \((n_k)\), obtaining the result for all \((n_k)\). On the other hand, the sequence \((n_k)\) in Berkes and Philipp [8] actually satisfies

\[
\limsup_{N \to \infty} \frac{1}{\sqrt{N} (\log N)^{1/2}} = \infty \quad \text{a.e.}
\]

for \( f(x) = x - [x] - 1/2 \). In this paper we will prove the following result.

**Theorem 2.** Let \( f \) satisfy (1) and have bounded variation on \([0,1]\). Let \( \varphi \) be a non-decreasing function satisfying

\[
\sum_{k=1}^{\infty} \frac{1}{k \varphi(k)^2} < \infty,
\]

(6)

Then for any increasing sequence \((n_k)\) of positive integers we have

\[
\left| \sum_{k=1}^{N} f(n_k x) \right| = O(\sqrt{N} \varphi(N) (\log \log N)^2) \quad \text{a.e.}
\]

(7)

To clarify the meaning of this theorem, let us note that Carleson’s theorem combined with the Kronecker lemma yields that under (3) we have

\[
\lim_{N \to \infty} \frac{\sum_{k=1}^{N} \cos 2\pi n_k x}{\sqrt{N} \varphi(N)} = 0 \quad \text{a.e.}
\]

Berkes and Philipp [8] proved that this result is best possible in the sense that if \( \varphi \) is a non-decreasing function with \( \sup_{k \geq 1} \varphi(k^2)/\varphi(k) < \infty \) and

\[
\sum_{k=1}^{\infty} \frac{1}{k \varphi(k)^2} = \infty,
\]

3
then there exists an increasing sequence \((n_k)\) of integers such that
\[
\limsup_{N \to \infty} \frac{\sum_{k=1}^{N} \cos 2\pi n_k x}{\sqrt{N \varphi(N)}} = \infty \quad \text{a.e.}
\]
This result describes precisely the growth of sums (4) in the trigonometric case \(f(x) = \cos 2\pi x\). Theorem 2 shows that, in analogy with Theorem 1, optimal bounds for sums (4) in the trigonometric case and for \(f \in \text{BV}\) differ only in a loglog power.

2 Proofs

For the proof of Theorem 1 we need the following variant of Lemma 2 in [6].

**Lemma.** Let \(1 \leq p_1 < q_1 < p_2 < q_2 < \ldots\) be integers such that \(p_{k+1} \geq 4q_k\); let \(I_1, I_2, \ldots\) be sets of integers such that \(I_k \subset [2^{p_k}, 2^{q_k}]\) and each element of \(I_k\) is divisible by \(2^{p_k}\). Let \(f(x) = x - \lfloor x \rfloor - \frac{1}{2}\) and
\[
X_k = X_k(\omega) = \sum_{j \in I_k} f(j \omega) \quad (k = 1, 2, \ldots, \omega \in (0, 1)).
\]
Then there exist independent r.v.’s \(Y_1, Y_2, \ldots\) on the probability space \(((0, 1), \mathcal{B}, \lambda)\) such that \(|Y_k| \leq \text{card} I_k\), \(EY_k = 0\) and
\[
\|X_k - Y_k\| \leq 2^{-k} \quad (k \geq k_0),
\]
where \(\| \cdot \|\) denotes the \(L^2(0, 1)\) norm.

**Proof.** Let \(\mathcal{F}_k\) denote the \(\sigma\)-field generated by the dyadic intervals
\[
U_\nu = \left[\nu 2^{-4q_k}, (\nu + 1)2^{-4q_k}\right] \quad 0 \leq \nu < 2^{4q_k}
\]
and set
\[
\xi_j = \xi_j(\cdot) = E(f(j \cdot) | \mathcal{F}_k), \quad j \in I_k
\]
\[
Y_k = Y_k(\omega) = \sum_{j \in I_k} \xi_j(\omega).
\]
Since \(|f| \leq 1\), we have \(|\xi_j| \leq 1\) and thus \(|Y_k| \leq \text{card} I_k\). Further, by \(f \in \text{BV}\) the Fourier coefficients of \(f\) are \(O(1/k)\) and thus from Lemma 3.1 of [5] it follows that
\[
\|\xi_j(\cdot) - f(j \cdot)\| \leq C_1 j 2^{-4q_k} \quad j \in I_k
\]
and since \(I_k\) has at most \(2^{q_k}\) elements, we get
\[
\|X_k - Y_k\| \leq C_2 \cdot 2^{-q_k} \leq 2^{-k} \quad \text{for} \quad k \geq k_0.
\]
Since \( p_{k+1} \geq 4q_k \) and since each \( j \in I_{k+1} \) is a multiple of \( 2p_{k+1} \), each interval \( U_\nu \) in (9) is a period interval for all \( f(jx), j \in I_{k+1} \) and thus also for \( \xi_j, j \in I_{k+1} \). Hence \( Y_{k+1} \) is independent of the \( \sigma \)-field \( \mathcal{F}_k \) and since \( \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \) and \( Y_k \) is \( \mathcal{F}_k \) measurable, the r.v.’s \( Y_1, Y_2, \ldots \) are independent. Finally \( E\xi_j = 0 \) and thus \( EY_k = 0 \).

**Proof of Theorem 1.** We start with the second statement and actually prove a little more: we show that for any positive sequence \( \varepsilon_k \to 0 \) there exists an increasing sequence \( (n_k) \) of integers and a real sequence \( (c_k) \) such that
\[
\sum_{k=1}^{\infty} c_k^2 (\log \log k)^2 \varepsilon_k < \infty
\]
and \( \sum_{k=1}^{\infty} c_k f(n_k x) \) diverges a.e. for \( f(x) = x - [x] - 1/2 \). Put \( \varepsilon_k^* = \sup_{j \geq k} \varepsilon_j \) and let \( \psi(k) \) be a sequence of positive integers growing so rapidly that \( \psi(k+1)/\psi(k) \geq 2 \) \((k = 1, 2, \ldots)\) and setting
\[
r_k = \psi(k)^3, \quad M_k = \sum_{j \leq k} r_j \psi(j) = \sum_{j \leq k} \psi(j)^4,
\]
we have
\[
\sum_{k=1}^{\infty} \varepsilon_k^* M_{k-1} < \infty.
\]

By a well known result of Gál [14] there exists, for each \( k \geq 1 \), a sequence \( m_1^{(k)} < m_2^{(k)} < \ldots < m_{\psi(k)}^{(k)} \) of positive integers such that
\[
\int_0^1 \left( \sum_{j=1}^{\psi(k)} f(m_j^{(k)} x) \right)^2 \, dx \geq \text{const} \cdot \psi(k)(\log \log \psi(k))^2.
\]

We define sets
\[
(10) \quad I_1^{(1)}, I_2^{(1)}, \ldots, I_{r_1}^{(1)}, I_1^{(2)}, \ldots, I_{r_2}^{(2)}, \ldots, I_1^{(k)}, \ldots, I_{r_k}^{(k)}, \ldots
\]
of positive integers by
\[
I_j^{(k)} = 2^j \{ m_1^{(k)}, \ldots, m_{\psi(k)}^{(k)} \}, \quad 1 \leq j \leq r_k, \, k \geq 1
\]
where \( c_j^{(k)} \) are suitable positive integers. (Here for any set \( \{a, b, \ldots \} \subset R \) and \( \lambda \in R \), \( \lambda \{a, b, \ldots \} \) denotes the set \( \{\lambda a, \lambda b, \ldots \} \).) Clearly we can choose the integers \( c_j^{(k)} \) inductively so that the sets \( I_j^{(k)} \) in (10) satisfy the conditions assumed in the Lemma for the sets \( I_j^k \); Since the left hand side of (9) does not change if we replace every \( m_j^{(k)} \) with \( am_j^{(k)} \) for some integer \( a \geq 1 \), setting
\[
X_j^{(k)}(\omega) = \sum_{\nu \in I_j^{(k)}} f(\nu \omega)
\]
we have

\[ E \left( X_j^{(k)} \right)^2 \geq \text{const} \cdot \psi(k)(\log \log \psi(k))^2. \]

By the Lemma, there exist independent r.v.'s \( Y_j^{(k)} \) (\( 1 \leq j \leq r_k \), \( k = 1, 2, \ldots \)) such that \( |Y_j^{(k)}| \leq \psi(k), \ EY_j^{(k)} = 0 \) and

\[ \sum_{k,j} \|X_j^{(k)} - Y_j^{(k)}\| < \infty \]

whence

\[ \sum_{k,j} |X_j^{(k)} - Y_j^{(k)}| < \infty \quad \text{a.e.} \]

By (11) and (12) we have

\[ E(Y_j^{(k)})^2 \geq \text{const} \cdot \psi(k)(\log \log \psi(k))^2. \]

Hence setting

\[
Z_k = \frac{1}{\sqrt{r_k} \psi(k) \log \log \psi(k)} \sum_{j=1}^{r_k} Y_j^{(k)}, \quad \sigma_k^2 = E \left( \sum_{j=1}^{r_k} Y_j^{(k)} \right)^2,
\]

we get from the central limit theorem with Berry-Esseen remainder term (see e.g. Feller [13], p. 544), (7), and \( r_k = \psi(k)^3 \),

\[
P(Z_k \geq 1) \geq P \left( \sum_{j=1}^{r_k} Y_j^{(k)} \geq c \sigma_k \right) \geq \]

\[
\geq (1 - \Phi(c)) - \text{const} \cdot \frac{r_k \psi(k)^3}{(r_k \psi(k)(\log \log \psi(k))^2)^{3/2}} \geq \]

\[
\geq 1 - \Phi(c) - o(1) \geq c' > 0 \quad (k \geq k_0)
\]

where \( \Phi \) denotes the Gaussian distribution function and \( c \) and \( c' \) are positive absolute constants. Since the r.v.’s \( Z_k \) are independent, the Borel–Cantelli lemma implies that \( P(Z_k \geq 1 \text{ i.o.}) = 1 \), i.e. \( \sum_{k=1}^{\infty} Z_k \) is a.e. divergent, which, in view of (13), yields that

\[ \sum_{k=1}^{\infty} \frac{1}{\sqrt{r_k} \psi(k) \log \log \psi(k)} \sum_{j=1}^{r_k} X_j^{(k)} \quad \text{is a.e. divergent.} \]

In other words, \( \sum_{i=1}^{\infty} c_i f(n_i x) \) is a.e. divergent, where

\[ (n_i)_{i \geq 1} = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{r_k} I_j^{(k)} \]
and
\[
c_i^2 = \frac{1}{r_k \psi(k)(\log \log \psi(k))^2} \quad \text{for} \quad M_{k-1} < i \leq M_k.
\]
Now for \(M_{k-1} < i \leq M_k\) we have by the exponential growth of \(\psi(j)\),
\[
i \leq 2\psi(k)^4, \quad \log \log i \leq 2 \log \log \psi(k) \quad (k \geq k_0)
\]
and consequently for \(M_{k-1} < i \leq M_k\)
\[
c_i^2(\log \log i)^2 \epsilon_i \leq \text{const} \cdot \frac{1}{r_k \psi(k)} \epsilon_{M_{k-1}}.
\]
Hence
\[
\sum_{i=1}^{\infty} c_i^2(\log \log i)^2 \epsilon_i \leq \text{const} \cdot \sum_{k=1}^{\infty} \epsilon_{M_{k-1}} < +\infty,
\]
completing the proof of the second half of Theorem 1.

We prove now the sufficiency of (3) in Theorem 1 for \(\gamma > 4\). In [2], a slightly weaker result is proved, namely the convergence of \(\sum_{k=1}^{\infty} c_k f(kx)\) under the assumption \(\sum_{k=1}^{\infty} c_k^2 (\log k)^\epsilon < \infty\) for some \(\epsilon > 0\). To get the present result, a slight improvement of the argument in [2] is needed. Let
\[
\Gamma(N) = \sup_{n_1, \ldots, n_N} \frac{1}{N} \sum_{k, \ell=1}^{N} \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha},
\]
where the supremum is taken over all distinct positive integers \(n_1, \ldots, n_N\). We start out of the formula
\[
(17) \quad \Gamma(N) \leq \prod_{j=1}^{r_N} (1 - v_j)^{-1}(1 - v_j^{-1} t_j^{-1})^{-1} \prod_{k=\tau_N+1}^{N-1} (1 - v_j^{-1} r_N^{-1})^{-1} + \exp \left( C \sum_{\ell=1}^{N-1} t_\ell^2 \right)
\]
on page 7 of [2]. We have chosen here \(\xi = 2\), \(C\) is an absolute constant and letting \(p_j\) denote the \(j\)-th prime, for \(\alpha \in (\log 2/\log 3, 1) = I\) we have \(p_j^{-\alpha} < 1/2\) for \(j \geq 2\) and thus with the notations of [2] we have for \(j \geq 2\)
\[
t_j = p_j^{-\alpha}, \quad \tau_j = 2p_j^{-\alpha}, \quad v_j = \max(\tau_j, (2\alpha - 1)^{-1/2} \tau_N), \quad r_N = [2 \log N] + 1.
\]
We estimate the right hand side of (17), just as in [2], by using the prime number theorem. Let
\[
s_N = \max\{1 \leq j \leq r_N : \tau_j \geq (2\alpha - 1)^{-1/2} \tau_N\}
\]
for \(N \geq N_0\) and split the first product on the right hand side of (17) into two subproducts \(P_1\) and \(P_2\), extended for the indices \(1 \leq j \leq s_N\) and \(s_N + 1 \leq j \leq r_N\). Let further \(P_3\) denote the second product on the right hand side of (17). We estimate
Letting $C_1, C_2, \ldots$ denote positive absolute constants, let us observe that by $p_k \sim k \log k$ we have for any $\alpha \in I$

\[
\prod_{j=4}^{s_N} (1 - 2p_j^{-\alpha})^{-2} \leq \prod_{j=4}^{r_N} (1 - 2p_j^{-\alpha})^{-2} \leq \exp \left( 4 \sum_{j=4}^{r_N} p_j^{-\alpha} \right) \leq \exp \left( C_1 \sum_{j=4}^{r_N} (j \log j)^{-\alpha} \right) 
\leq \exp \left( C_2 \frac{1}{1 - \alpha} \right)
\]

Thus we have

\[
P_1 \leq C_3 \exp \left( \frac{C_2}{1 - \alpha} (\log N)^{1 - \alpha} \right) \quad \text{for } \alpha \in I.
\]

Similar estimates hold for $P_2$ and $P_3$ (which do not involve a singularity at $\alpha = 1$ and hence the factor $1/(1 - \alpha)$) and we arrive at

\[
(18) \quad \Gamma(N) \leq \exp \left( \frac{C_4}{1 - \alpha} (\log N)^{1 - \alpha} \right) \quad \text{for } \alpha \in I.
\]

We thus see that if in the first estimate of Theorem 1 of [2] we drop the factor $(\log \log N)^{-\alpha}$, the resulting estimate holds uniformly for $\alpha \in I$.

In relation (26) of [2] the arbitrary parameter $0 < \varepsilon < 1$ appears. The subsequent arguments lead to relation (27) in [2], yielding the norm bound

\[
(19) \quad cJ^{-\varepsilon/2}(\log N)^{1/2} \left( \sum_{k=M_1+1}^{M_2} c_k^2 \right)^{1/2} \exp(c(\varepsilon)(\log N)^{\varepsilon/2}(\log \log N)^{-1/2}).
\]

Using (18) instead of the estimate in the first line of Theorem 1 in [2], we get

\[
(20) \quad C_5 J^{-\varepsilon/2}(\log N)^{1/2} \left( \sum_{k=M_1+1}^{M_2} c_k^2 \right)^{1/2} \exp \left( \frac{C_6}{\varepsilon} (\log N)^{\varepsilon/2} \right)
\]

instead of (19). By increasing $C_6$ if necessary, we can assume $C_6 \geq 4$. Here $C_5$ and the further constants $C_7, C_8, \ldots$ are allowed to depend also on $f$. We choose now $J$ by

\[
J^{\varepsilon/2} = (\log N)^{1/2} \exp \left( \frac{2C_6}{\varepsilon} (\log N)^{\varepsilon/2} \right)
\]

and thus the expression (20) becomes

\[
C_5 \left( \sum_{k=M_1+1}^{M_2} c_k^2 \right)^{1/2} \exp \left( -\frac{C_6}{\varepsilon} (\log N)^{\varepsilon/2} \right).
\]
Applying the Rademacher-Mensov inequality as in [2], it follows that the norm in formula (29) of [2] can be bounded by

\[(21) \quad C_7(\log N) \exp \left( -\frac{C_6}{\varepsilon}(\log N)^{\varepsilon/2} \right) \left( \sum_{k=1}^N c_k^2 \right)^{1/2} \exp \left( -\frac{C_6\varepsilon}{2}(\log N)^{\varepsilon/2} \right) \left( \sum_{k=1}^N c_k^2 \right)^{1/2} \cdot \]

Choosing \( \varepsilon = 1/(\log \log N) \) and using \( C_6 \geq 4 \), the expression in (21) will be \( \leq C_8(\sum_{k=1}^N c_k^2)^{1/2} \). On the other hand,

\[(22) \quad \log J = \frac{1}{\varepsilon} \log \log N + \frac{4C_6}{\varepsilon^2}(\log N)^{\varepsilon/2}, \]

which becomes \( \leq C_9(\log \log N)^2 \) with the choice of \( \varepsilon \). Thus the last expression in the first displayed formula on page 16 of [2] becomes

\[\leq C_{10}(\log \log N)^2 \left( \sum_{k=1}^N c_k^2 \right)^{1/2} \cdot \]

This shows that the expression \( c(\varepsilon)(\log N)^{\varepsilon} \) on the right hand side of the maximal inequality in Lemma 4 of [2] can be replaced by \( C_{11}(\log \log N)^4 \). Arguing further as in [2], this leads to the sufficiency of the convergence condition \( \gamma > 4 \).

**Proof of Theorem 2.** Let \( S_N = S_N(x) = \sum_{k=1}^N f(n_kx) \) and let \( \varphi \) be a non-decreasing function satisfying \( \sup_{k \geq 1} \varphi(2k)/\varphi(k) < \infty \). By the just proved stronger version of Lemma 4 of [2] we have

\[\int_0^1 \max_{1 \leq k \leq N} S_k^2 \, dx \leq CN(\log \log N)^4 \]

with some constant \( C > 0 \). Thus

\[P \left( \max_{1 \leq k \leq 2N} |S_k| \geq \sqrt{2^N \varphi(2^N)(\log \log 2^N)^2} \right) \leq \frac{C}{\varphi^2(2^N)}. \]

Now [3] and the monotonicity of \( \varphi \) imply \( \sum_{k=1}^\infty \frac{1}{\varphi^2(2^k)} < \infty \) and thus Theorem 2 follows from the Borel-Cantelli lemma.

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