Spin-Triplet Pairing State of Sr$_2$RuO$_4$ in the c-Axis Magnetic Field

Shuhei TAKAMATSU$^{1,3}$ and Youichi YANASE$^{1,2}$

$^1$Graduate School of Science and Technology, Niigata University, Niigata, 950-2181, Japan  
$^2$Department of Physics, Niigata University, Niigata, 950-2181, Japan

We investigate the spin-triplet superconducting state of Sr$_2$RuO$_4$ in the magnetic field along the c-axis on the basis of the four-component Ginzburg-Landau (GL) model with a weak spin-orbit coupling. We consider superconducting states described by the d-vector parallel to the ab-plane ($d \parallel ab$), and find that three spin-triplet pairing states are stabilized in the magnetic field-temperature ($H-T$) phase diagram. Although a helical state is stable at low magnetic fields, a chiral II state is stabilized at high magnetic fields. A non-unitary spin-triplet pairing state appears near the transition temperature owing to the coupling of magnetic field and chirality. We elucidate synergistic and/or competing roles of the magnetic field, chirality, and spin-orbit coupling. It is shown that a fractional vortex lattice is stabilized in the chiral II phase owing to the spin-orbit coupling.

KEYWORDS: Spin-triplet superconductivity, spin-orbit coupling, fractional vortex, Sr$_2$RuO$_4$

Since the discovery of superconductivity in Sr$_2$RuO$_4$ by Maeno et al.'s extensive studies have been carried out to clarify the symmetry of superconductivity in Sr$_2$RuO$_4$. It has been shown that Sr$_2$RuO$_4$ is a spin-triplet superconductor from both theoretical and experimental points of view. For instance, the Knight shift measurement by nuclear magnetic resonance (NMR) experiments is one of the convincing pieces of evidence of the spin-triplet pairing state. The NMR measurements showed the temperature-independent Knight shift through the superconducting transition in both field directions along the ab-plane and along the c-axis. These results indicate a spin-triplet superconducting state with a d-vector perpendicular to the magnetic field. This means that the “spin-orbit coupling” of spin-triplet Cooper pairs is very weak in Sr$_2$RuO$_4$. According to the experiments of muon spin rotation ($\mu$SR) and the Kerr effect, the time reversal symmetry is spontaneously broken in the superconducting state. These findings indicated that the chiral spin-triplet pairing state described by the d-vector $d = (p_x \pm ip_y)\hat{z}$ is stabilized at zero magnetic field.

When the spin-orbit coupling is weak as indicated by NMR measurement and by theoretical estimation, the d-vector is rotated by a magnetic field along the c-axis so as to be parallel to the ab-plane. However, most theoretical studies focused on the chiral state $d = (p_x \pm ip_y)p_{z\parallel}$ assuming that a strong spin-orbit coupling pins the direction of the d-vector. It is highly desired to investigate pairing states in the c-axis magnetic field by taking a weak spin-orbit coupling into account, but no such theoretical study has yet been conducted. The purpose of our study is to determine the spin-triplet superconducting state for $H \parallel c$ in the presence of a weak spin-orbit coupling.

Several issues regarding the superconducting state in Sr$_2$RuO$_4$ remain unsettled. For instance, the chiral edge state has not been observed, and the first-order superconducting transition at high magnetic fields along the ab-plane is incompatible with the weak-coupling theory for spin-triplet superconductors. To resolve these unresolved issues, our verification of superconducting phases for $H \parallel c$ will provide a basis for discussion. Our study would also give an insight into the spin-triplet superconducting state of UPt$_3$, in which a weak spin-orbit coupling has been indicated.

We here construct a four-component GL model assuming a weak spin-orbit coupling. Although the chiral state $d = (p_x \pm ip_y)\hat{z}$ may be stabilized at zero magnetic field, we assume that the d-vector is parallel to the ab-plane in the magnetic field along the c-axis so as to gain the magnetic energy. Focusing on this high-field superconducting phase, we take into account two spin components of order parameters, namely, $d \parallel \hat{x}$ and $d \parallel \hat{y}$. The D$_{4h}$ symmetry of the crystal structure of Sr$_2$RuO$_4$ allows two orbital components of p-wave order parameters, namely, $p_x\hat{x}$ and $p_y\hat{y}$, which are equivalently denoted by the chirality, that is, $p_x \pm ip_y$. These 2 × 2 = 4 component order parameters correspond to four irreducible representations in the D$_{4h}$ symmetry, $d = p_x\hat{x} \pm p_y\hat{y}$ and $d = p_x\hat{y} \pm p_y\hat{x}$. These pairing states are nearly degenerate when we assume a weak spin-orbit coupling. For convenience, we define the order parameters as $\Delta_{\sigma\sigma}(r, k) = \Delta_{\sigma\sigma}(r, k)\phi_0(k) + \Delta_{\sigma\sigma}(r, k)\phi_1(k)$ using four component order parameters. $\phi_0(k)$ and $\phi_1(k)$ stand for pairing functions with the same symmetry as $k_x$ and $k_y$ under the D$_{4h}$ point group, respectively.

The GL free-energy density is given by

$$ f = \sum_{\sigma=\uparrow,\downarrow} (f_0^\sigma + f_{SO}^\sigma) + f_{SO2}, $$

where $f_0^\sigma$ is the same as the GL free-energy density for the $E_u$ representation of the tetragonal point group,

$$ f_0^\sigma = a(\Delta_{\sigma\sigma,1}^2 + |\Delta_{\sigma\sigma,2}|^2) + \beta_1(\Delta_{\sigma\sigma,1}^2 + |\Delta_{\sigma\sigma,2}|^2)^2 $$

$$ + \beta_2(\Delta_{\sigma\sigma,1}\Delta_{\sigma\sigma,2} - c.c.)^2 + \beta_3|\Delta_{\sigma\sigma,1}|^2|\Delta_{\sigma\sigma,2}|^2 $$

$$ + \xi_0^2(\Delta_{\sigma\sigma,1}^2 + |\Delta_{\sigma\sigma,2}|^2) $$

$$ + \xi_1^2|\Delta_{\sigma\sigma,1}|^2 + |\Delta_{\sigma\sigma,2}|^2 $$

$$ + \xi_2^2(\Delta_{\sigma\sigma,1}\Delta_{\sigma\sigma,2} + c.c.) $$

$$ + [(\Delta_{\sigma\sigma,1}\Delta_{\sigma\sigma,2})^* + c.c.]. $$

$^1$E-mail: takamatsu@phys.sc.niigata-u.ac.jp
We adopt \( \alpha = \alpha_0 (T - T^c_0) \), where \( T^c_0 \) is the transition temperature at zero magnetic field in the absence of spin-orbit coupling. We use the conventional notation \( D_j = \sqrt{j} / (2 \pi / \Phi_0) A_j \) and \( \Phi_0 = \hbar c / 2 e l \). In the weak coupling parameters, satisfy the relations following \( \beta_2 / \beta_1 = \langle | \theta_1 |^2 / | \theta_2 |^2 \rangle \), \( \xi_x^2 / \xi_y^2 = \langle \theta_1^2 / \theta_2^2 \rangle \), and \( \xi^2 / \xi_y^2 = \langle \phi \theta y v_x v_y / \langle \theta_y^2 \rangle \) where \( v_x \) and \( v_y \) are the Fermi velocity and brackets \( \langle \) denote the average over the Fermi surface. When we assume \( (\phi_x(k), \phi_y(k)) = (v_x(k), v_y(k)) \), we obtain \( \xi_1 > \xi_2 = \xi_3 \). On the other hand, we find \( \xi_1 < \xi_2 \) for the pairing functions obtained by a perturbation theory for the Hubbard model.\(^{25}\)

A weak spin-orbit coupling is taken into account in the quadratic terms \( f_{SO1}^2 \) and \( f_{SO2}^2 \), which are given by

\[
 f_{SO1}^2 = \sigma \epsilon (i \Delta_{\sigma \sigma \sigma} \Delta_{\sigma \sigma \sigma}^* + c.c.),
\]

\[
 f_{SO2}^2 = \delta \langle (\Delta_{\uparrow \downarrow \downarrow} \Delta_{\downarrow \uparrow \downarrow}^* - \Delta_{\downarrow \uparrow \downarrow} \Delta_{\downarrow \uparrow \downarrow}^*) + c.c. \rangle.
\]

According to a theoretical analysis based on the three-orbital Hubbard model for \( Sr_2 RuO_4 \), the \( f_{SO1}^2 \) term arises from the \((\alpha, \beta)\)-bands, while the \( f_{SO2}^2 \) term is due to the coupling between the \((\alpha, \beta)\)-bands and the \( \gamma \)-band.

We determine the pairing state by minimizing the above GL free-energy density using the variational method. We take into account variational wave functions of Cooper pairs so that the solution of the linearized GL equation is obtained. Writing \( \Delta_{\sigma \sigma \sigma} = \langle \Delta_{\sigma \sigma \sigma} \rangle + i \epsilon_{\sigma \sigma \sigma} \), and \( \Delta_{\sigma \sigma \sigma} = \langle \Delta_{\sigma \sigma \sigma} \rangle + i \epsilon_{\sigma \sigma \sigma} \) and differentiating the quadratic terms of eq. (2) with respect to \( \Delta_{\sigma \sigma \sigma} \) and \( \Delta_{\sigma \sigma \sigma} \), we obtain the linearized GL equation in the absence of spin-orbit coupling as

\[
 \frac{2 \pi \hbar}{\Phi_0} \begin{pmatrix}
 \kappa (1 + 2 \Pi_1, \Pi_1) & - \rho_1 \Pi_2 & - \rho_1 \Pi_2 & \mu_12 \Pi_2 & \mu_22 \Pi_2 \\
 - \rho_1 \Pi_2 & \kappa (1 + 2 \Pi_1, \Pi_1) & - \rho_1 \Pi_2 & \mu_12 \Pi_2 & \mu_22 \Pi_2 \\
 - \rho_1 \Pi_2 & - \rho_1 \Pi_2 & \kappa (1 + 2 \Pi_1, \Pi_1) & - \rho_1 \Pi_2 & - \rho_1 \Pi_2 \\
 - \rho_1 \Pi_2 & - \rho_1 \Pi_2 & - \rho_1 \Pi_2 & \kappa (1 + 2 \Pi_1, \Pi_1) & - \rho_1 \Pi_2 \\
 - \rho_1 \Pi_2 & - \rho_1 \Pi_2 & - \rho_1 \Pi_2 & - \rho_1 \Pi_2 & \kappa (1 + 2 \Pi_1, \Pi_1)
\end{pmatrix} \Delta_{\sigma \sigma \sigma} = 2 \lambda \begin{pmatrix}
 \Delta_{\sigma \sigma \sigma}1 \\
 \Delta_{\sigma \sigma \sigma}2 \\
 \Delta_{\sigma \sigma \sigma}1 \\
 \Delta_{\sigma \sigma \sigma}2 \\
 \Delta_{\sigma \sigma \sigma}1
\end{pmatrix},
\]

where \( \kappa = \xi_x^2 + \xi_y^2 \), \( \rho_1 = \xi_x^2 - \xi_y^2 \pm 2 \xi_2 \), and \( \Pi_1 = - \langle \Phi_0 / 4 \pi \hbar \rangle / (D_1 \pm D_2) \). Note that this equation is independent of the spin label \( \sigma \).

For \( \xi_1 > \xi_2 \), the minimum eigenvalue \( \lambda = \lambda_{min} \) is obtained for the pairing state with a positive chirality, in which the order parameters \( \langle \Delta_{\sigma \sigma \sigma} \rangle \) and \( \langle \Delta_{\sigma \sigma \sigma} \rangle \) are described using the Landau level expansion

\[
 \begin{pmatrix}
 \psi_{\uparrow \uparrow \uparrow}(r, \delta) \\
 \psi_{\downarrow \downarrow \downarrow}(r, \delta)
\end{pmatrix} = \sum_{n \geq 0} \begin{pmatrix}
 a_{n0} \varphi_{n}(r, \delta) \\
 a_{n+2} \varphi_{n+2}(r, \delta)
\end{pmatrix}.
\]

The superconducting transition temperature in the absence of spin-orbit coupling is obtained as \( T^c_0(H) = T^c_0 (1 - \lambda_{min}) \). The pairing state with a negative chirality is described by another eigenstate of the linearized GL equation as

\[
 \begin{pmatrix}
 \psi_{\uparrow \uparrow \uparrow}(r, \delta) \\
 \psi_{\downarrow \downarrow \downarrow}(r, \delta)
\end{pmatrix} = \sum_{n \geq 0} \begin{pmatrix}
 b_{n0} \varphi_{n}(r, \delta) \\
 b_{n+2} \varphi_{n+2}(r, \delta)
\end{pmatrix}.
\]

We here denote the \( n \)-th Landau level wave functions \( \varphi_n(r, \delta, 0) = \sum c_m e^{i m \phi} (2^{n - 1} \sqrt{\pi})^{-1/2} H_n(x + mv) e^{-i (x + mv)^2 / 2} \) and \( \varphi_n(r, \delta) = e^{-i \delta} \varphi_n(r - \delta, 0) \) using the unit of length \( l = \sqrt{\Phi_0 / 2 \pi \hbar} \) and the Hermite polynomials \( H_m(x) \). We here assume the square lattice structure of the vortex in accordance with the small-angle neutron scattering experiment.\(^{20}\) Thus, we take \( c_m = 1 \) for all integers \( m \) and \( q = \sqrt{2 \pi} \).
non-unitary state” by looking at the chirality in each spin component of Cooper pairs. For $d = p_x \hat{x} - p_y \hat{y}$, the Cooper pairs formed by quasiparticles with up spin have the positive chirality as $\Delta_{\uparrow\uparrow} = p_x + i p_y$, while the chirality is negative for down spin as $\Delta_{\downarrow\downarrow} = -(p_x - i p_y)$. When the magnetic field is switched on, the positive chirality is enhanced by the coupling of chirality and magnetic field. Thus, the dominant order parameter is $\Delta_{\uparrow\uparrow} = p_x + i p_y$ in the non-unitary state, although a subdominant order parameter $\Delta_{\downarrow\downarrow} = -(p_x - i p_y)$ is induced by the spin-orbit coupling. Note that the cooperation of the chirality, magnetic field, and spin-orbit coupling plays an essential role in stabilizing this orbital-induced non-unitary state.

II phase. This is because the magnetic field favors the positive chirality of Cooper pairs with down spin $\Delta_{\downarrow\downarrow} = p_x + i p_y$. This coupling of chirality and magnetic field competes with the spin-orbit coupling, which stabilizes the negative chirality of $\Delta_{\downarrow\downarrow}$. The former is more important than the latter at high fields, and therefore stabilizes the chiral II phase. On the other hand, the negative chirality component $\Delta_{\downarrow\downarrow} = -(p_x - i p_y)$, which is represented by $C_4$, is suppressed in the chiral II state. Taking the $C_1$ and $C_3$ terms into account, the chiral II state is described by the d-vector $d \sim (p_x + i p_y)\hat{x}$ or $d \sim (p_x + i p_y)\hat{y}$. These two pairing states are degenerate because the spin-orbit coupling terms between $C_1$ and $C_3$ cancel out. In other words, Cooper pairs with up spin $\Delta_{\uparrow\uparrow} = p_x + i p_y$ are almost decoupled from those with down spin $\Delta_{\downarrow\downarrow} = p_x + i p_y$ in the chiral II phase. This unusual decoupling leads to the fractional vortex lattice, which we discuss below.

We now turn to the vortex lattice structure. Interestingly, we find that the fractional vortex lattice appears in the chiral II phase in contrast to the conventional full-quantum vortex lattices in the non-unitary and helical phases. We schematically illustrate these vortex lattices in Fig. 4. It is shown that the vortex cores of order parameters for $\Delta_{\uparrow\uparrow} = p_x - i p_y$ (C2) and for $\Delta_{\downarrow\downarrow} = p_x + i p_y$ (C4) shift from those of $\Delta_{\downarrow\downarrow} = p_x + i p_y$ (C1) and $\Delta_{\downarrow\downarrow} = p_x - i p_y$ (C4). Thus, a core-less vortex is stabilized in the chiral II phase, which is regarded as a fractional vortex lattice state.\(^\text{28}\) Note that this fractional vortex lattice is stabilized by the spin-orbit coupling, in sharp contrast to the usual role of spin-orbit coupling. When we assume a strong spin-orbit coupling, the fractional vortex lattice is destabilized so that the spin-orbit coupling energy is gained.\(^\text{28}\) Contrary to this common knowledge, the role of spin-orbit coupling is altered in the chiral II phase because of the chirality arising from two orbital components. We will show details of the fractional vortex lattice structure in another paper.

Here, we investigate the parameter dependences of the phase diagram. We would like to stress again that the structure of the phase diagram is not altered by the signs of the spin-orbit couplings $\epsilon$ and $\delta$ or by the magnitude relation $\xi_1 > \xi_2$. When we change the sign of these parameters, the d-vector in each phase changes as summarized in Tables I and II. The magnitude relation between $\xi_1$ and $\xi_2$ determines the chirality (Table I). On the other hand, the signs of the spin-orbit couplings $\epsilon$ and $\delta$ fluctuate between positive and negative in contrast to the unidirectional behavior of the chirality.

Fig. 1. (Color online) The $H$-$T$ phase diagram of four-component GL model [eq. (1)] for $\beta_1 = 1.0$, $\beta_2 = \beta_3 = 0.5$, $\xi_1/\xi = 1.2$, $\xi_2/\xi = 0.83$, $\kappa/\xi = 0.3$, $\epsilon = -1.5 \times 10^{-2}$, and $\delta = 3.75 \times 10^{-3}$. The unit of magnetic field is $H_0^c = \hbar / 2 \xi^2$. The red solid line shows the superconducting transition temperature $T_c$, while the green dot-dashed line shows the second superconducting transition temperature $T_{c2}$. The blue double-dot-dashed line and black dashed line depict the first-order transition and crossover, respectively. We define the crossover temperatures as $|C_1| = 2|C_4|$. The non-unitary phase, helical phase, and chiral II phase depicted in the phase diagram are approximately described by the d-vector $d = (\xi + i \eta)(p_x + i p_y)$, $d = p_x \hat{x} - p_y \hat{y}$, and $d = (p_x + i p_y)\hat{x}$, respectively, for our choice of parameters $\xi_1 > \xi_2$, $\epsilon < 0$, and $\delta > 0$. The other cases are summarized in Tables I and II.

Fig. 2. (Color online) Temperature dependence of the absolute values of variational parameters $|C_1|$ (red solid line) and $|C_4|$ (blue dashed line), since $C_2 = C_3 = 0$. We define the crossover temperature as $|C_1| = 2|C_4|$.

Fig. 3 shows the variational parameters $|C_1|$ at a rather high magnetic field $H/H_{c2}^0 = 0.8$. We see that the parameter $C_3$ has a large magnitude in the low-temperature chiral II phase. This is because the magnetic field favors the positive chirality of Cooper pairs with down spin $\Delta_{\downarrow\downarrow} = p_x + i p_y$. This coupling of chirality and magnetic field competes with the spin-orbit coupling, which stabilizes the negative chirality of $\Delta_{\downarrow\downarrow}$. The former is more important than the latter at high fields, and therefore stabilizes the chiral II phase. On the other hand, the negative chirality component $\Delta_{\downarrow\downarrow} = -(p_x - i p_y)$, which is represented by $C_4$, is suppressed in the chiral II state. Taking the $C_1$ and $C_3$ terms into account, the chiral II state is described by the d-vector $d \sim (p_x + i p_y)\hat{x}$ or $d \sim (p_x + i p_y)\hat{y}$. These two pairing states are degenerate because the spin-orbit coupling terms between $C_1$ and $C_3$ cancel out. In other words, Cooper pairs with up spin $\Delta_{\uparrow\uparrow} = p_x + i p_y$ are almost decoupled from those with down spin $\Delta_{\downarrow\downarrow} = p_x + i p_y$ in the chiral II phase. This unusual decoupling leads to the fractional vortex lattice, which we discuss below.

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couplings $\epsilon$ and $\delta$ determine the d-vector of the helical phase (Table II). The spin component in the non-unitary phase is also determined by the sign of $\epsilon$ (Table I).

When the magnitude of spin-orbit couplings is decreased, the chiral II phase is stabilized. Figure 5 shows that the chiral II phase is stable in almost the entire parameter range of the $H$-$T$ phase diagram for tiny spin-orbit couplings, $\epsilon = -1.5 \times 10^{-3}$ and $\delta = 3.75 \times 10^{-4}$. On the other hand, moderate spin-orbit couplings, $\epsilon = -0.15$ and $\delta = 0.0375$, stabilize the helical phase in the entire parameter range, as shown in Fig. 6. In both cases, the vortex lattice structure is not altered from Fig. 4. A microscopic calculation based on the three-orbital Hubbard model has estimated these parameters as $\epsilon, \delta \leq 0.01$.$^{50}$ Thus, the moderate spin-orbit couplings assumed in Fig. 6 are not likely realized in Sr$_2$RuO$_4$.

Finally, we discuss the pairing state in Sr$_2$RuO$_4$ on the basis of our results in Figs. 1, 5, and 6. Until now, double superconducting transitions have not been observed in Sr$_2$RuO$_4$ in the magnetic field along the c-axis.$^{2,3}$ This experimental status implies tiny spin-orbit couplings as in Fig. 5 or moderate spin-orbit couplings as in Fig. 6. The chiral II state is realized in the former case, while the helical state is realized in the latter case. The latter interpretation is, however, incompatible with the results of the NMR experiment, which did not show any temperature dependence of Knight shift through $T_c$.$^{4,5}$ On the other hand, the case of tiny spin-orbit couplings in Fig. 5 seems to be consistent with the experiments. Although the fractional vortex lattice in Fig. 4(b) has not been observed in the small-angle neutron scattering experiment,$^{26}$ this discrepancy is not a serious issue because the fractional vortex lattice is affected by the strong coupling effect as well as by the Meissner effect. In our future study, we will investigate these effects on the vortex lattice structure. In contrast to these cases, the multiple-phase diagram as shown in Fig. 1 should appear in Sr$_2$RuO$_4$ when spin-orbit couplings are small but not tiny. No clear experimental observation has been reported for the multiple-phase diagram. However, weak evidence has been obtained by a small-angle neutron scattering experiment.$^{26}$ When the fractional vortex lattice is realized in the chiral II phase, the spatial distribution of the magnetic field is smeared, consistent with the unclear field distribution obtained in an experiment at high magnetic fields.$^{26}$ Further experimental study is desired to determine the superconducting state of Sr$_2$RuO$_4$ in the c-axis magnetic field.

In summary, we studied the spin-triplet superconducting state of Sr$_2$RuO$_4$ in the magnetic field along the c-axis. We assumed a weak spin-orbit coupling of Cooper pairs and considered the d-vector parallel to the ab-plane. We determined multiple phases against magnetic fields and temperatures by analyzing the four-component GL model using the variational method. Interestingly, we showed three spin-triplet superconducting phases, which are orbital-induced non-unitary phase, chiral II phase, and helical phase. We clarified the synergistic and competing roles of the chirality, magnetic field, and spin-orbit coupling for these pairing states. We would like to stress that these intriguing superconducting phases are obtained because we take into account a weak but finite spin-orbit coupling and allow the mixing of irreducible representations in the order parameters. We also found that the fractional vortex lattice is stabilized by a weak spin-orbit coupling in the chiral II phase. This is in sharp contrast to the theory of non-chiral spin-triplet superconductors.$^{28}$ This finding pave the way for realizing a novel topological defect with Majorana fermions$^{29}$ in spin-triplet superconductors with spin and orbital degrees of freedom.

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Chiral II phase

\[
\begin{align*}
\xi_1 > \xi_2 & : (p_x + i p_y)\hat{x}, (p_x + i p_y)\hat{y} \\
\xi_1 < \xi_2 & : (p_x - i p_y)\hat{x}, (p_x - i p_y)\hat{y}
\end{align*}
\]

\[
\begin{align*}
\epsilon > 0 & : \hat{x}, \hat{y} \\
\epsilon < 0 & : \hat{x}, \hat{y}
\end{align*}
\]

Non-unitary phase

\[
\begin{align*}
\delta > 0 & : (\hat{x} - i\hat{y})(p_x + i p_y) \\
\delta < 0 & : (\hat{x} + i\hat{y})(p_x + i p_y) \\
\delta > 0 & : (\hat{x} - i\hat{y})(p_x - i p_y) \\
\delta < 0 & : (\hat{x} + i\hat{y})(p_x - i p_y)
\end{align*}
\]

Table I. Summary of d-vector in the non-unitary phase and chiral II phase. The magnitude relation between \(\xi_1\) and \(\xi_2\) determines the chirality. The spin component in the non-unitary phase is determined by the sign of spin-orbit coupling \(\epsilon\).

| Helical phase | \(p_x\hat{x} + p_y\hat{y}\) | \(p_x\hat{x} - p_y\hat{y}\) | \(p_x\hat{x} - p_y\hat{y}\) | \(p_x\hat{x} + p_y\hat{y}\) |
|---------------|----------------|----------------|----------------|----------------|
| \(\epsilon > 0\) | \(\delta > 0\) | \(\delta < 0\) | \(\delta > 0\) | \(\delta < 0\) |
| \(\epsilon < 0\) | \(\delta < 0\) | \(\delta > 0\) | \(\delta < 0\) | \(\delta > 0\) |

Table II. Summary of d-vector in the helical phase. The sign of spin-orbit couplings \(\epsilon\) and \(\delta\) determines the d-vector.