Within the context of a string-theory dual to $\mathcal{N} = 1$ gauge theories with gauge group $SU(N_c)$ and large $N_c$, we identify a class of solutions of the background equations for which a suitably defined dual of the gauge coupling exhibits the features of a walking theory. We find evidence for three distinct, dynamically generated scales, characterizing walking, symmetry breaking and confinement, and we put them in correspondence with field theory by an analysis of the operators driving the flow.

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INTRODUCTION AND GENERAL IDEAS

In this paper we propose to study the features of walking dynamics within the context of the string theory construction of background geometries which are conjectured to be a dual description of strongly coupled four-dimensional systems. We dispense with the model-building details of a complete technicolor model, in particular with electro-weak symmetry-breaking itself. Instead, we focus completely on the properties of walking dynamics in isolation. We find an explicit solution to the supergravity equations yielding a background which admits an interpretation in terms of a gauge theory, the coupling of which exhibits the qualitative behavior of a walking model, and study the properties of this solution.

The advantages of following this program are multiple. First of all, there is a set of well-defined and controllable expansions allowing for a systematic calculation to be performed. The running of the gauge coupling is defined in terms of the geometry. It can be tracked all the way into the strongly coupled region where traditional perturbative techniques cannot be trusted. As a result, it is possible to identify the transition in the far UV below which the running flattens. It is also possible to follow the dynamics in the far IR, where due to the appearance of a non-trivial condensate (in these specific models usually identified with the gaugino condensate), which breaks spontaneously some of the global symmetry, the running reappears, ultimately leading to confinement. It is possible to show that several separate scales are dynamically generated, without introducing UV-singularities (fine-tuning) in the background. Interestingly, we find that the existence of this background is completely independent of the presence of flavors, indicating that its dynamical origin does not necessarily arise from the interplay of $N_c$ and $N_f$ in the beta-functions. This suggests that walking dynamics does not necessarily require the presence of large numbers of fundamental fermions, which is a very welcome feature for model-building.

This is the first step in a relatively unchartered territory. In principle, besides studying the RG evolution of the underlying model, which is the main focus of this paper, it should be possible to determine all the symmetry and symmetry-breaking pattern of the model, its low-energy spectrum of composite states, including vector mesons, dilaton and pseudo-Nambu-Goldstone bosons (PNGBs), together with the field content of the dual gauge theory and all the anomalous dimensions in the IR. In practice, this is a quite extensive and challenging research program which will be completed elsewhere.

The paper is structured as follows. We start by reviewing the motivations for walking dynamics and the open questions that arise in this context. This section is intended for the reader who is not familiar with dynamical electro-weak symmetry breaking. We then review the basics of the string-theory construction of a particular class of models that are believed to be dual to a broad class of strongly coupled gauge theories, we write the equations that are going to be used in the body of the paper, and we fix the notation. This is mostly a summary of previous results that can be found in the literature, and is intended for the reader who is not acquainted with this framework. We then introduce a new class of solutions to the background equations, and develop the study of the properties of this solution in the body of the paper. We construct the solution via a systematic expansion, discuss its properties, perform an analysis of scaling dimension of the operators deforming the dual field theory, and of the symmetries of the resulting background. In particular, we show that the running of a suitably defined four-dimensional gauge coupling has, within this class of solutions, the basic properties expected in a walking theory. We conclude by summarizing a tentative research program based on these results.

ASPECTS OF WALKING TECHNICOLOR

The problems of QCD-like technicolor models

Spontaneous electro-weak symmetry breaking might be induced by the dynamical formation of a condensate in a new (strongly-coupled) sector of the complete Lagrangian extending the Standard Model. In its original form, this idea is implemented by assuming that a new non-abelian technicolor gauge symmetry be present, typically $SU(N_c)$, with a generic number of technicolors $N_c$, that a new set of $N_f$ fermions transforming according to the fundamental representation of $SU(N_c)$ are present, and that a $SU(2) \times U(1)$ subgroup of the global symmetry group of the new technicolor sector be (weakly) gauged in order to reproduce the electro-weak gauge group of the Standard Model. If the dynamics of the technicolor sector is similar to the one of QCD (in which case we refer to this scenario as QCD-like technicolor), at high scales the new interaction is asymptotically free, but quantum effects dynamically produce a physical scale $\Lambda_{TC}$ at which the coupling becomes strong, the theory confines, and the global (chiral) symmetry is spontaneously broken by the formation of a condensate of techniquarks. If $\Lambda_{TC}$ is the electro-weak scale, the induced breaking of $SU(2) \times U(1)$ would produce the physical masses of the $W$ and $Z$ gauge bosons.
This idea is particularly appealing because it provides a completely natural solution to the hierarchy problem. Three major, correlated difficulties arise when trying to build a realistic model implementing this idea. The first obstacle is that, by definition, such a scenario requires the model to be strongly coupled at the electro-weak scale, and hence the phenomenology of electro-weak interactions cannot be analyzed with standard perturbative techniques. One completely model-independent way of dealing with this relies on the idea of constructing a low-energy effective field theory (EFT) description of the interactions of the electro-weak gauge bosons, hence encoding in the coefficients of the electro-weak chiral Lagrangian all the information about the (strong) technicolor dynamics. While this approach is systematic and elegant, it has a somewhat limited predictive power, in particular because it treats the precision electro-weak parameters (such as the $T$ and $S$ parameters) as free parameters to be extracted from the data, rather than deriving them from first principles. It is only possible, within this approach, to construct arguments that yield an order-of-magnitude estimate of the expected size of the coefficients, based on the power-counting that arises from the perturbative expansion of the chiral Lagrangian itself. We call this naive dimensional analysis (NDA).

In QCD, the analog construction yields an EFT in which the NDA expectations are in substantial agreement both with the experimental data and with first principle lattice calculations. It is hence reasonable to assume that, provided the underlying dynamics be QCD-like, the NDA expectations, corrected by appropriately rescaling (in energy and in $N_c$) the established results of QCD, should give an acceptable prediction for the coefficients of the electro-weak chiral Lagrangian, and in particular for the precision parameters. However, this is in sharp contrast with the results from the combined fit of the experimental data on precision electro-weak physics, which seem to indicate that the upper limits on the precision parameters are quite tight, generically one order of magnitude below the NDA expectations. Unless one speculates that the $N_c$ and $N_f$ scalings of the QCD-like theory can be extrapolated all the way into the small-$N_c$ region, and is hence going to conclude that $N_c = 2$ is allowed by the data.

By itself, the fact that NDA estimates are in excess of the experimental data might just indicate that the expansion parameter of the electro-weak chiral Lagrangian be somewhat smaller than expected on the basis of NDA, which by itself might even be seen as a welcome, though unexpected and puzzling, feature. But a third difficulty arises when considering the generation of the standard-model masses and mixing. In order to understand how this arises, one has to remember that in the absence of a Higgs, the standard model masses are introduced via extended technicolor (ETC). Effectively, ETC consists of adding to the Lagrangian a set of irrelevant operators coupling the standard-model fermions to the new technifermions, in such a way that the formation of technifermion condensates produces, via dimensional transmutation, a low-energy description in which mass terms for the standard-model fermions are generated. This has to be done in such a way as to avoid introducing significant new sources of flavor changing neutral current (FCNC) interactions, which typically requires to assume that some family symmetry is broken (sequentially) at scales $\Lambda_{ETC} \gg \Lambda_{TC}$. The resulting mass is going to be proportional to the electro-weak scale, via a coefficient that results from the matching of the appropriate higher-order operator, which typically is a dimension-6 four-quark interaction, with coefficient $\mathcal{O}(1/\Lambda_{ETC})^2$, onto the mass term, and is hence going to be suppressed by $\Lambda_{TC}^2/\Lambda_{ETC}^2$. The fact that the mass of the top quark be as large as the electro-weak scale itself is hard to reconcile with this scenario, and introduces a further tension on power-counting arguments within the electro-weak chiral Lagrangian.

To summarize: in the absence of a systematic calculational tool, the predictive power of a technicolor model is very limited, and relies on NDA estimates of the coefficients of the low-energy EFT that are supported only by the experience with QCD. On the one hand, precision electro-weak data can be reconciled with the EFT treatment only at the price of assuming that NDA provides a systematic overestimate of the effects of dynamical electro-weak symmetry breaking (DEWSB) on the electro-weak gauge bosons. On the other hand the measured large top mass can be reconciled with the suppression of flavor changing neutral current (FCNC) interactions only at the price of assuming that NDA underestimates the effects of DEWSB in the generation of the fermion masses. All of this constitutes strong, though not definitive, arguments disfavoring this scenario.

The solutions provided by walking technicolor

The line of thinking leading to this (premature) conclusion is, however, flawed at its core: the NDA estimates of the coefficients of the EFT describing a strongly-coupled system are based on non-rigorous arguments, that yield acceptable results for QCD, or for QCD-like theories, only. In particular, QCD-like theories are characterized by having only one dynamical scale, and by the fact that the spectrum of anomalous dimensions of the theory is, almost at all scales, perturbative, due to the specific shape of the QCD renormalization group (RG) flow. One can think of justifying the NDA counting rules yielding the results summarized so far with arguments that rely on these two properties of QCD. But if DEWSB results from a model which is not QCD-like, none of these arguments is justified. This is the case for walking technicolor.
Walking dynamics is an essential ingredient in the modern construction of models of DEWSB. The basic assumption is that the condensate inducing electro-weak symmetry breaking emerges from a strongly coupled sector that, rather than being QCD-like, is quasi conformal and strongly coupled over a significant range of energy above the electro-weak scale. This naturally leads to the coexistence of parametrically separated dynamical scales, and to the appearance of large anomalous dimensions with important phenomenological implications. As a result, at the EFT level, very large departures from the estimates of NDA cannot be excluded, but are a natural expectation. The experimental results on electro-weak precision parameters, flavor-changing neutral currents and quark masses can hence be reconciled with this framework. The assumption of walking allows to construct realistic models that are testable at the LHC [13].

To be more specific, let us remind the reader about what are the properties of a putative walking technicolor model. While a QCD-like model is characterized by only one dynamically generated scale, the structure of a walking theory implies the existence of four distinct dynamical scales. A generic model might be asymptotically free in the far UV, because the RG flow is assumed to have a trivial UV fixed point. Due to the fact that the gauge coupling is marginally relevant, following the RG flow to lower energies the gauge coupling grows, until a first dynamical scale \( \Lambda_s \) appears. The RG equations are assumed to possess a (approximate) fixed point in the IR at strong coupling. Below \( \Lambda_s \), the running of the gauge coupling is almost flat, because the theory, while approaching its IR fixed point, is approximately conformal. The fact that the IR fixed point is strongly coupled implies that the spectrum of anomalous dimensions of the operators in the theory is expected to be radically different from the perturbative results obtained in proximity of the trivial UV fixed point. Examples of this dynamical behavior exist, for instance in the well understood context of supersymmetric QCD [8][9].

The fixed point is only approximate, and the flow, after spending some time (walking) in its vicinity, will drift away from it. At this point, the approximate scale invariance is broken. Below this scale \( \Lambda_{IR} \) the gauge coupling will start running again, and ultimately become big enough to induce confinement at a scale \( \Lambda_0 < \Lambda_{IR} \). The condensate that spontaneously breaks the global symmetries of the model must also form at the electro-weak scale \( \Lambda_{TC} \), in order to induce DEWSB. However, while in QCD this condensate arises at the confinement scale (temperature), in general its formation might take place at a higher scale (temperature), in the range \( \Lambda_0 < \Lambda_{TC} < \Lambda_{IR} \). While it is hard to believe that these three scales (temperatures) can be separated by arbitrarily large factors, the idea that chiral symmetry might break at a scale (temperature) somewhat higher than the confinement scale has been discussed in the past [10], and recently been revived both in the context of lattice studies [11] and of string-theory inspired models [12].

If this is the underlying dynamics, the NDA estimates of the precision parameters can very plausibly be modified in a quite substantial way. An early study in this direction [13] highlighted that if the anomalous dimension of the techniquarks is large enough, so that the chiral condensate is effectively dimension-2, the Weinberg sum rules are going to be modified, and as a consequence the predictions of the precision parameter \( S \) based on dispersion relations must be revised. This fact, together with the fact that several distinct scales are dynamically generated in the UV, has been the subject of several recent EFT studies inspired by the ideas of the AdS/CFT correspondence [14], leading to the conclusion that in this context large regions of the parameter space of the models are compatible with the precision electro-weak studies and are potentially testable at the LHC. At the same time, the large anomalous dimension of the chiral condensate might also provide a natural solution to the problem of the mass of the top, without introducing new significant sources of FCNC [12].

Open questions from walking dynamics

The conclusion is that if DEWSB is realized in nature, walking is very likely to be its crucial dynamical feature. But the fact that the dynamics be strongly coupled, and so different from QCD, is a major obstacle for the search of viable candidate theories in which walking emerges as a dynamical feature, rather than being put in by hand as a working assumption in the EFT.

Very little is known about the implications of walking dynamics by itself. There are a set of well-defined field theory questions which are of utmost importance from the phenomenological perspective and which cannot be addressed within the low-energy EFT, but require to have a dynamical model. For the most part, these are questions that are independent of the details of how the strongly coupled sector is coupled to the standard model fermions and gauge bosons, and of electro-weak symmetry breaking. In view of this, it is very useful to have a complete model in which walking emerges, such that the dynamical implications of walking can be studied in isolation, factoring if out for the complicated structure of a complete model of dynamical electro-weak symmetry breaking.

One set of such questions has been anticipated in the previous discussion, and includes the identification of the (four) dynamical scales in the system, the study of the mechanism leading to their formation and their separation. We also anticipated the importance of calculating of the spectrum of anomalous dimensions of the underlying theory.
in the walking regime.

Besides these, an even more fundamental set of questions is related to the study of the approximate global symmetries of the model. If the underlying dynamics is approximately conformal, (or if it possesses approximate internal global symmetries not related to the electro-weak symmetry) and the formation of condensates breaks the dilatation invariance (or any of the global internal symmetries) spontaneously, in principle a light dilaton (or a set of light pseudo-Nambu-Goldstone bosons) might be present in the model. The presence of any light scalar might change in a radical way the low-energy phenomenology, and it is important to understand under what dynamical conditions they arise. In particular, because of the dilaton quantum numbers, and of the quantum numbers of the Standard Model Higgs, distinguishing the two at a hadronic machine such as the LHC is going to be a major challenge. It is hence important to know if walking dynamics predicts the existence of such a dilaton, and what its mass and couplings are going to be. This has been investigated for a long time, but is still is a very open problem [10].

**Going for a walk.**

The potential of walking dynamics did not go unnoticed. A variety of studies exists in the literature, within the context of four-dimensional non-abelian gauge theories, looking for models in which a (approximate) fixed point of the RG equations exists in the IR, as sensible candidate for a walking technicolor model.

This is a very challenging problem for analytical calculations. Within the regime in which perturbation theory can be fully trusted, it has been established long ago [17] that due to the different dependence on $N_c$ and $N_f$ of the 1-loop and 2-loop beta function, such fixed points exist for certain regimes of $N_f/N_c$, a fact that has been studied systematically in the context of supersymmetric theories [8]. Studies based on approximate techniques attempted to generalize this beyond the regime of perturbation theory for non-supersymmetric theories, and seem to indicate that there exist models in which strongly coupled fixed points and large anomalous dimensions emerge dynamically [18]. More recently, similar results have been obtained by analyzing the RG flow with the use of of a conjectured generalization of the NSVZ [19] beta-function to non-supersymmetric set-ups [20]. However, besides being based on ad hoc approximations that require independent testing, none of these studies deal quantitatively with the more realistic scenario in which the fixed point is only approximate, and hence, after a period of walking, the theory ultimately confines.

In very recent years, progress on the lattice allowed to perform numerical studies looking for some evidence of the existence of fixed points in the IR, exact or approximate, in a variety of models. One such study [21] seems to confirm that fixed points in the spirit of [17] exist even beyond perturbation theory. More elaborate studies confirm this result also for models with matter in higher-dimensional representation [22]. A number of collaborations has been testing these same ideas with complementary techniques [23]. However, also in this case it is not yet possible to discuss in quantitative detail the full set of transitions that makes the models first enter the walking regime, then (at lower energies) leave it, and ultimately confine.

In this paper we propose an alternative approach, based on gauge/string correspondence, within which to carry on the program of looking for models that exhibit the dynamical features of a walking theory. This approach might help shedding some light over model-independent phenomenological and theoretical features of a large class of models with walking dynamics. Within this context, it should be possible to address the set of well posed theoretical questions summarized in the previous subsections, and the results might be helpful even in guiding the data analysis of studies performed using more traditional approaches.

**ASPECTS OF THE STRING MODEL AND THE DUAL QFT**

In this section we will specify the type of string duals to strongly coupled field theories that we will be studying. There are various ways of constructing string duals to four dimensional theories. We will focus on the models that use wrapped branes. To make things concrete, suppose that we consider at first $N_c$ D5 branes. The dynamics of these five-branes is well described at low energies by a $(5 + 1)$ field theory with sixteen SUSY’s. The Lagrangian is the dimensional reduction of ten dimensional Super-Yang-Mills to six dimensions. The field content can be seen to be: a gauge field $A_M$, two sets of spinors $\lambda_i$, $\tilde{\lambda}_i$ (with four components each) and four real scalars $\phi_i$, all in the adjoint representation of $SU(N_c)$. The presence of the branes breaks the $SO(1, 9)$ symmetry group of Type IIB supergravity into $SO(1,5) \times SO(4) \approx SU(4) \times SU(2)_A \times SU(2)_B$. We summarize the field content and transformation laws in Table I.

The dual description of this field theory at strong coupling is given by the background generated by $N_c$ D5 branes
after the decoupling limit
\[ g_s \to \infty, \quad \alpha' \to 0, \quad N_c \to \infty \quad g_s \alpha' N_c = \text{fixed} \]

is taken \([24]\). This background consists of a metric, a RR three form and a dilaton and reads,
\[ ds^2 = e^{\phi} \left[ dx_{1,5}^2 + \frac{1}{4} \sum_{i=1}^{3} \tilde{\omega}_i^2 \right], \quad F(3) = \frac{N_c}{4} \tilde{\omega}_1 \wedge \tilde{\omega}_2 \wedge \tilde{\omega}_3, \quad e^{\phi} = e^{\phi_0 + \tau}. \]  

(2)

We define the \(SU(2)\) left-invariant one forms as,
\[ \tilde{\omega}_1 = \cos \psi d\tilde{\theta} + \sin \psi \sin \tilde{\theta} d\tilde{\varphi}, \]
\[ \tilde{\omega}_2 = -\sin \psi d\tilde{\theta} + \cos \psi \sin \tilde{\theta} d\tilde{\varphi}, \]
\[ \tilde{\omega}_3 = d\psi + \cos \tilde{\theta} d\tilde{\varphi}. \]

(3)

Hence,
\[ \sum_{i=1}^{3} \tilde{\omega}_i^2 = d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2 + (d\psi + \cos \tilde{\theta} d\tilde{\varphi})^2, \]

(4)

is a nice compact way or writing a three-sphere. The ranges of the three angles are \(0 \leq \tilde{\varphi} < 2\pi, 0 \leq \tilde{\theta} \leq \pi, 0 \leq \psi < 4\pi\).

Now, to produce a four dimensional effective theory out of the previous six dimensional one, we can imagine separating two directions, say \((x_4, x_5)\), and wrapping these \(N_c\) D5 branes on a small two-sphere (in other words, compactifying the \((x_4, x_5)\) space), so that low energy modes will explore only the non-compact \(3 + 1\) directions. There are different ways of choosing the two-dimensional space. All of them will lead to field theories describing the excitations of the wrapped five branes. For different technical reasons it is convenient to choose a two-dimensional space that preserves some fraction of the \(16\) SUSY’s. (For example, a torus preserves the sixteen supercharges.) We will be interested in effective 4-d theories that preserve a minimal amount of SUSY (four supercharges in 4-d). This example was worked out in \([25]\). (See also the paper \([26]\) for interesting details.) One can see that this system preserves (or partially breaks) a fraction of the original SUSY via a quite general ‘twisting procedure’ nicely explained by Witten in \([30]\). One effect of this ‘unconventional’ way of breaking SUSY is that even at high energies, the background preserves only four supercharges, as can be seen, by studying the weakly coupled spectrum \([26]\).

Let us be more precise about the twisting procedure. If we take the fields for the \(N_c\) flat D5 branes, they transform as explained under \(SO(1,5) \times SO(4)\), see the Table I above. Now, when we separate two directions, for example \((x_4, x_5)\) and compactify them on a two-sphere \(S^2\), we are breaking the global transformation group into
\[ SU(4) \times SU(2)_A \times SU(2)_B \rightarrow SU(2)_L \times SU(2)_R \times U(1)_{45} \times SU(2)_A \times SU(2)_B \]

(5)

where we made \(SO(1,3) \approx SU(4) = SU(2)_L \times SU(2)_R\). Now, we need to decompose the fields under the ‘branching’ described in eq.\((5)\). This was carefully done in Section 3 of the paper \([26]\). The next step is the ‘twisting’ (or mixing) between the global symmetries of the \(S^2\) and a \(U(1)\) factor inside \(SU(2)_A\), which produces a set of 4-d fields with or without transformation rule under the new combined \(U(1)\). This twisting procedure allows branes wrapping cycles to preserve some amount of SUSY, in spite of the cycle not necessarily admitting massless spinors \([27]\).

The ‘twisted-Kaluza-Klein’ decomposition is explained in detail in Section 4 of \([26]\). We will just need to state that the spectrum -at weak coupling- consists of a massless vector multiplet \(W_\alpha\) whose action is the usual N=1 SYM.
one and (after a mass gap related to the inverse size of the sphere) an infinite tower of massive chiral multiplets and massive vector multiplets. The degeneracies and masses are given in [26] where the field theory was proven to be equivalent to $N = 1^*$ SYM when expanded around a particular Higgs vacuum. Also, the fact that there is an infinite tower of states reminds us that the field theory is higher dimensional at high energies. Let us stress that these are all weakly coupled results.

The natural question is if there is a non-perturbative description of the field theory living on the wrapped five branes described in the previous paragraph. This is indeed the case. One has to construct a background that represents the physical situation described above, by compactifying on a sphere and twisting the background of eq.(2). The background (in string-frame) reads

$$d s^2 = \alpha' g_s e^{\phi(r)} \left[ \frac{d\hat{\tau}^2}{\alpha' g_s} + e^{2k(r)} d\rho^2 + e^{2h(r)} (d\theta^2 + \sin^2 \theta d\varphi^2) + \frac{e^{2g(r)}}{4} \left( (\tilde{\omega}_1 + a(\rho)d\theta)^2 + (\tilde{\omega}_2 - a(\rho) \sin \theta d\varphi)^2 \right) + \frac{e^{2k(r)}}{4} (\tilde{\omega}_3 + \cos \theta d\varphi)^2 \right],$$

$$F_3 = \frac{N_c}{4} \left[ - (\tilde{\omega}_1 + b(\rho)d\theta) \wedge (\tilde{\omega}_2 - b(\rho) \sin \theta d\varphi) \wedge (\tilde{\omega}_3 + \cos \theta d\varphi) + b' d\rho \wedge (-d\theta \wedge \tilde{\omega}_1 + \sin \theta d\varphi \wedge \tilde{\omega}_2) + (1 - b(\rho)^2) \sin \theta d\theta \wedge d\varphi \wedge \tilde{\omega}_3 \right].$$  (6)

In the following, we will set units so that $\alpha' = g_s = 1$. Notice that the following ‘deformations’ with respect to the ‘flat’ five-brane metric in eq.(2) have been implemented,

i) Two of the directions in $R^{1,5}$ have been renamed as $(\theta, \varphi)$ and we have compactified them on a two-sphere.

ii) There is a mixing (fibration) between the $(\theta, \tilde{\varphi}, \psi)$ coordinates and the $(\theta, \varphi)$ ones. This mixing is encoded by the presence of the functions $a(r), b(r)$ and in the fact that the last component of the metric is $(d\psi + \cos \theta d\tilde{\varphi} + \cos \theta d\varphi)$.

iii) For a given set of conditions on the functions (see below), the background in eq. (6) was shown to preserve four supercharges, hence being dual to a four dimensional $N = 1$ theory.

iv) The functions in the background above must solve the equations of motion coming from the action

$$S_{11B} = \frac{1}{G_{10}} \int d^{10} x \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{e^\phi}{12} F_3^2 \right].$$  (7)

namely,

$$R_{\mu \nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{12} e^\phi \left( 3 F_{\rho_1 \rho_2} F^{\rho_1 \rho_2 \nu} - \frac{1}{4} F_3^2 g_{\mu \nu} \right),$$

$$\nabla_\mu (e^\phi F^{\mu \rho_1 \rho_2}) = 0, \quad \partial_\mu F_{\rho_1 \rho_2 \rho_3} = 0,$$

$$\Box g\phi - \frac{1}{12} e^\phi F_3^2 = 0,$$  (8)

where we defined

$$\nabla_\mu X = \partial_\mu (\sqrt{-g} X),$$

$$\Box_g X = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu X).$$

These are the Einstein, Maxwell, and Klein-Gordon equations in the curved background, and the Bianchi identity. From here one derives the equations satisfied by the functions $[\phi(r), h(r), g(r), k(r), a(r), b(r)]$ appearing in the background. Each solution to these equations is conjectured to capture the non-perturbative dynamics of the field theory discussed above, either in a different vacuum or after some operator is inserted in the Lagrangian, deforming the theory (also in a given vacuum).

Looking for configurations preserving some SUSY is in practice easier than looking for generic solutions of the (second order) equations of motion [8] for the background of eq. (6), since supersymmetric solutions are obtained by solving first order “BPS” equations. It is usually the case, and it has been shown explicitly for the background (6), that solutions of the first order BPS equations automatically solve the equations of motion.
A useful result: The BPS equations

The first order equations have been carefully derived in [28, 29]. Let us quote a result from [29]: if we “change basis” and write the functions of the background in terms of a set of functions $P(\rho), Q(\rho), Y(\rho), \tau(\rho), \sigma(\rho)$ as

$$4e^{2h} = \frac{P^2 - Q^2}{P \cosh \tau - Q}, \quad e^{2g} = P \cosh \tau - Q, \quad e^{2k} = 4Y, \quad a = \frac{P \sinh \tau}{P \cosh \tau - Q}, \quad N_c b = \sigma. \quad (9)$$

Using these new variables, one can manipulate the BPS equations to obtain a single decoupled second order equation for $P(\rho)$, while all other functions are simply obtained from $P(\rho)$ as follows:

$$Q(\rho) = (Q_o + N_c) \cosh \tau + N_c(2\rho \cosh \tau - 1),$$
$$\sinh \tau(\rho) = \frac{1}{\sinh(2\rho - 2\rho_o)}, \quad \cosh \tau(\rho) = \coth(2\rho - 2\rho_o),$$
$$Y(\rho) = \frac{P'}{8},$$
$$e^{4\phi} = \frac{e^{4\rho_o} \cosh(2\rho_o)^2}{(P^2 - Q^2)Y \sinh^4 \tau},$$
$$\sigma = \tanh(\tau + N_c) = \frac{(2N_c \rho + Q_o + N_c)}{\sinh(2\rho - 2\rho_o)}. \quad (10)$$

The second order equation mentioned above reads,

$$P'' + P'(\frac{P'' + Q'}{P - Q} + \frac{P' - Q'}{P + Q} - 4\cosh(2\rho - 2\rho_o)) = 0. \quad (11)$$

So, to summarize, any solution of the eq. (11) will generate via the expressions in eq. (10) a set of functions $[h, g, k, \phi, a, b]$ as obtained using eq. (9), such that when plugged back into eq. (8) gives a string background solving eq. (8), preserving four supercharges, that is dual to a minimally SUSY 4-d field theory.

THE APPROXIMATE SOLUTION

In this section we present a solution to the above BPS equations which has the property that a suitably defined gauge coupling -see below- possesses a plateau in some intermediate energy scale. Since the solution is not known analytically in closed form, we proceed by first exhibiting an approximate solution with this property and then, in the next section we will show how one can systematically compute the corrections to this solution in the form of an expansion in the coupling of a dimension-six operator.

Let us start with a few comments on the constants, $\rho_o$ and $Q_o$, appearing in the equations (10) of the previous section. The constant $\rho_o$ sets the minimum possible endpoint of the geometry in the IR. Without loss of generality we will set $\rho_o = 0$ in the following. The value of the constant $Q_o$ determines whether the solution in fact extends up to $\rho_o$ in the IR. Unless $Q_o = -N_c$, all solutions will end in the IR at some $\rho_{IR} > \rho_o$ [29]. Here we will consider only solutions with $Q_o = -N_c$ such that the solution extends all the way to $\rho_o = 0$ in the IR.

In order to exhibit the approximate solution we are interested in, we write the second order equation (11) in the form

$$\partial_{\rho} (s(P^2 - Q^2)P') + 4sP'QQ' = 0, \quad (12)$$

where

$$s(\rho) = \sinh^2 \tau = \frac{1}{\sinh^2(2\rho)}. \quad (13)$$

Integrating eq. (12) twice we obtain

$$P^3 - 3Q^2 P + 6 \int_{\rho_2}^{\rho} d\rho' QQ' P + 12 \int_{\rho_2}^{\rho} d\rho' s^{-1} \int_{\rho_1}^{\rho'} d\rho'' s P' QQ' = e^3 R(\rho)^3, \quad (14)$$
where

\[ R(\rho) \equiv (\cos^3 \alpha + \sin^3 \alpha(\sinh(4\rho) - 4\rho))^{1/3}, \]  
(15)

and \( c, \alpha \) are arbitrary integration constants, and we will fix the values of \( 0 \leq \rho_1, \rho_2 \leq \infty \) later. The approximate solution we are looking for is obtained by taking \( c \) large compared to \( N_c \). In this limit one obtains the approximate solution

\[ P \approx cR(\rho). \]  
(16)

An interesting property of this solution is that it makes the dilaton constant (to leading order in \( N_c/c \)), namely

\[ e^{4(\phi - \phi_0)} = \frac{3}{c^3 \sin^3 \alpha} + \mathcal{O}((N_c/c)^2). \]  
(17)

However, for this to be a well defined solution we need to ensure that \( P(\rho) > Q(\rho) \) for all \( \rho \geq 0 \), since any \( \rho \) where \( P = Q \) is a singular point of eq. (11) or eq. (12). Now, for small \( \rho \) we have \( Q(\rho) = \mathcal{O}(\rho^2) \), while \( P(\rho) \approx c \cos \alpha + \mathcal{O}(\rho^3) \) and so \( P > Q \) is ensured by requiring \( \cos \alpha > 0 \). For large \( \rho \) we have \( Q(\rho) \sim 2N_c\rho \), while \( P(\rho) \sim 2^{-1/3}c \sin \alpha e^{3\rho/3} \). So, requiring that \( \sin \alpha > 0 \) is again sufficient to ensure that \( P > Q \) for large \( \rho \). To ensure that \( P > Q \) for all \( \rho \), though, we need to look closer at the solution. Clearly, in the approximation of eq. (16) \( P \geq c \cos \alpha \) for all \( \rho \). At some value of \( \rho = \rho_* \), we have \( \sinh(4\rho_*) - 4\rho_* \approx c \cos^3 \alpha \) and \( P \) starts deviating from the constant value \( c \cos \alpha \). As we shall see below, to allow for a large region of walking behavior we will need to take \( \cot \alpha \gg 1 \), in which case

\[ \rho_* \approx \frac{1}{4}(\log 2 + 3 \log \cot \alpha) \gg 1. \]  
(18)

In order to ensure that \( P > Q \) everywhere, it is sufficient to require that \( P(\rho_*) > Q(\rho_*) \), which puts an upper bound on \( \cot \alpha \), namely

\[ 1 \ll \cot \alpha \lesssim \exp \left( \frac{2^{4/3}c}{3N_c} \right). \]  
(19)

This relation defines the approximation we will be working with in the rest of the paper. In this approximation \( P(\rho) \) remains almost constant for \( 0 \leq \rho \lesssim \rho_* \). As we will see below, the fact that \( P \) is almost constant up to very large scales \( \rho_* \) produces an intermediate energy region over which the four-dimensional gauge coupling is almost constant. The larger \( \cot \alpha \), the wider this region is, with the only limitation provided by the upper bound on the value of \( \cot \alpha \), which depends on the ratio \( c/N_c \).

### A Systematic Expansion

In the limit \( c/N_c \rightarrow \infty \) the solution presented in the previous section is an exact solution of the BPS equations. However, it is instructive to determine the subleading corrections to this solution in an expansion in powers of \( N_c/c \). Such a systematic expansion was constructed in [29], where the constant \( c \) was identified with the coupling of a dimension-6 operator. The leading term in the expansion for large \( c \) presented in Appendix B of [29] is the solution in eq. (16) presented in the previous section with \( \cos \alpha = 0 \). Generalizing this expansion to the case \( \cos \alpha \neq 0 \) is straightforward, as one merely needs to replace eq. (16) as the leading solution in the recursion relations that determine the subleading corrections.

Following [29] we write \( P \) in a formal expansion in inverse powers of \( c \) as

\[ P = \sum_{n=0}^{\infty} c^{1-n} P_{-n}. \]  
(20)

Inserting this expansion in eq. (14) we obtain recursively

\[ P_1 = R, \]
\[ P_0 = 0, \]
\[ P_{-1} = -\frac{1}{3}P_1^{-2} \left( -3Q^2 P_1 + 6 \int_{\rho_2}^{\rho} d\rho' QQ' P_1 + 12 \int_{\rho_2}^{\rho} d\rho' s^{-1} \int_{\rho_1}^{\rho'} d\rho'' sQQ' P_1' \right), \]
\[ P_{-2} = 0, \]
\[ P_{-n-2} = -\frac{1}{3} P_1^{-2} \left\{ \sum_{m=1}^{n+2} \left( 2P_1 P_{-m}P_{-n-2} + \sum_{k=1}^{n-m+3} P_{-m}P_{-k}P_{-n-k-2} \right) - 3Q^2 P_{-n} \right\} + 6 \int_{\rho_2}^0 d\rho' QQ' P_{-n} + 12 \int_{\rho_2}^0 d\rho' s^{-1} \int_{\rho_1}^{\rho'} d\rho'' sQQ' P_{-n} \right\}, \quad n \geq 1. \] (21)

It follows by induction that \( P_k = 0 \) for all even \( k \).

At this point we have to make a choice for the values of \( \rho_1 \) and \( \rho_2 \). Given that \( P_1 \sim e^{4\rho/3} \) as \( \rho \to \infty \), requiring that \( P_k \) for \( k < 1 \) are all subleading with respect to \( P_1 \) as \( \rho \to \infty \) sets \( \rho_1 = \infty \). Moreover, provided \( \cos \alpha \neq 0 \), as \( \rho \to 0 \),
\[ Q = 2N_c \left( \frac{2}{3} \rho^2 + O(\rho^4) \right), \]
\[ P_1 = \cos \alpha \left( 1 + \frac{32}{9} \tan^2 \alpha \rho^3 + O(\rho^5) \right). \] (22)

Requiring then that \( P_k \) for \( k < 1 \) are all subleading with respect to \( P_1 \) as \( \rho \to 0 \) sets \( \rho_2 = 0 \). With these choices, for odd \( k \), \( P_k \sim e^{4k\rho/3}\rho^m(k) \) as \( \rho \to \infty \), where \( m(k) \) is a \( k \)-dependent positive integer, while as \( \rho \to 0 \), \( P_k = O(\rho^3) \) for \( k < 1 \). In particular, provided \( \cos \alpha \neq 0 \), for small \( \rho \)
\[ P = c \cos \alpha + \mu(c, \alpha)\rho^3 + O(\rho^4), \] (23)
which shows that, provided \( \cos \alpha \neq 0 \), the solution described by eq. [29] has type I IR asymptotics (see eq. (4.24) in [29]). The corresponding behavior in the IR (for \( \rho \to 0 \)) of the functions in eq. [29] is
\[ e^{2h} = \frac{1}{2} c \cos \alpha \rho \left( 1 - \frac{4\rho^2}{3} + \frac{\mu + 8N_c/3}{c \cos \alpha} \rho^3 + O(\rho^4) \right), \]
\[ e^{2g} = \frac{c \cos \alpha}{2\rho} \left( 1 + \frac{4\rho^2}{3} + \frac{\mu - 8N_c/3}{c \cos \alpha} \rho^3 + O(\rho^4) \right), \]
\[ e^{2k} = \frac{3}{2} \mu \rho^2 + O(\rho^3), \]
\[ e^{4(\phi-\phi_0)} = \frac{32}{3\mu^2 \cos^2 \alpha} + O(\rho), \]
\[ a = 1 - 2\rho^2 + \frac{8N_c}{3c \cos \alpha} \rho^3 + O(\rho^4), \]
\[ b = 1 - \frac{2}{3} \rho^2 + O(\rho^4). \] (24)

The constant \( \mu \) is a very non-trivial function of \( c \) and \( \alpha \) given by
\[ \mu(c, \alpha) = \frac{16}{3 \cos^2 \alpha} \left( \frac{2}{3} \sin^3 \alpha + \sum_{k=0}^{\infty} c^{-2k-1} \int_0^\infty d\rho' sQQ' P_{-2k+1} \right). \] (25)

In the approximation eq. (13) the first correction can be evaluated approximately to obtain
\[ \mu(c, \alpha) \approx \frac{16}{3 \cos^2 \alpha} \left( \frac{2}{3} \sin^3 \alpha + \frac{N_c^2 \sin^3 \alpha}{3c \cos^2 \alpha} \left( \log 2 + 3 \log \cot \alpha \right)^2 + O(N_c^4/c^3) \right). \] (26)

It is easy to see from eq. (12) and the BPS equations eq. (10) that \( \mu(c, \alpha) \) is related to the value of the dilaton at \( \rho = 0 \) (see also Fig. 11), namely
\[ e^{4(\phi(0) - \phi_0)} = \frac{32}{3\mu(c, \alpha)c^2 \cos^2 \alpha} \approx \frac{3}{c^3 \sin^3 \alpha} \left( 1 - \frac{1}{2} \left( \log 2 + 3 \log \cot \alpha \right)^2 \left( \frac{N_c}{c} \right)^2 + O \left( \left( \frac{N_c}{c} \right)^4 \right) \right). \] (27)

Numerical calculations show that this is indeed a very good approximation. For the UV (large \( \rho \)) expansions, we refer the reader to the Class II asymptotics in Section 4.3.1 of the paper [29]. Plots of the background functions as functions of \( \rho \) are shown in Figs. 11 12.
A short discussion on the singularity

The aim of this short section is to briefly summarize the properties of the singularity that our IIB solution presents in the IR (that is, for small values of the radial coordinate). This paper is mostly concerned with the properties of the metric away from the singularity (for $\rho > 1$), where walking dynamics seems to emerge, yet it is worth reminding the reader about the characterization of the singularity as good, and hence not problematic.

The Ricci scalar (and other invariants) diverge at $\rho = 0$. Accepting such a singularity in the background requires to specify in what sense and to what extent the AdS/CFT ideas apply. This problem did not go unnoticed and since the early days different criteria were developed, in order to select when a singular space-time can be accepted or must be rejected as the dual to a field theory.

The outcome of these studies is a set of criteria, which if satisfied ensure that the strongly-coupled dynamics of the field theory is faithfully reproduced by the string solution. We call good singularities those that satisfy these criteria. One of these criteria requires that the potential of the supergravity theory (after reduction to five-dimensions) be bounded from above [33]. A second criterion requires that the component $g_{tt}$ of the metric be bounded [35]. In many examples, these two criteria give the same result, accepting or rejecting the same singular space-times, though no general proof is known about the equivalence of these two criteria. In our case, $|g_{tt}| = e^\phi$ is bounded and hence our IR singularity is good, satisfying the criterion in [35].

One characteristic of good singularities is the fact that the calculation of many invariants involving the singular region yields finite results. For instance, in computing the invariant $\sqrt{-g}$ the singularities at $\rho \to 0$ cancel. Other examples of this cancellation involve certain brane probe actions, and this fact is believed to be robust enough that the presence of a good singularity should not be a reason of concern.

ABOUT THE GAUGE COUPLING

The next step is to identify the 4-d physical quantities and to relate them to the background. For example, defining the gauge coupling and its running with energy. In this section we comment on one aspect of this problem.

It will be convenient in this section to reinstate the factors of $\alpha'$ and $g_s$. Let us start by reminding the reader that the six-dimensional theory on the D5 branes has a ’t Hooft coupling given by $\lambda_6 = g_s \alpha' N_c$. This coupling has units of length squared as a 6-d coupling should. Now, when we wrap the branes on a small two-cycle $\Sigma_2$ and explore the theory at low energies, we will effectively observe a 4-d theory. A natural way of defining the (dimensionless) gauge coupling of this theory is by combining,

$$g_4^2 = \frac{g_s^2}{\alpha' \text{Vol} \Sigma_2}$$

(28)
FIG. 2: The background functions $h$, $g$, $k$ and $a$ as a function of $\rho$. Here we plot the first three orders in the expansion (20) and we compare it with the numerical solution. It is clear that the expansion (20) converges sufficiently fast.

Then, the question is how to select the cycle $\Sigma_2$. It can be shown that the two-manifold defined by the surface

$$\Sigma_2 = \{ \theta = \tilde{\theta}, \varphi = 2\pi - \tilde{\varphi}, \psi = \pi \}, \quad (29)$$

with the other coordinates $x^\mu, \rho$ taken to be constant, defines the only two-cycle in the geometry of eq.(6)\(^1\). The characteristic of the submanifold in eq.(29) is that it gets rid of the ‘fibration terms’. In turn, this means that this is the only dimension-2 manifold (without boundaries) that the branes can wrap.

A good definition of the 4-d coupling can be obtained following the papers [31] that considered a “probe” five-brane extended along the Minkowski directions and the two-cycle of eq.(29) $R^{1,3} \times \Sigma_2$ in the background eq. (6).

The authors of [31] turn on a gauge field $F_{\mu\nu}$ on the Minkowski directions of this five-brane probe (with tension $T_{D5}$) and compute its action (given by the Born-Infeld-Wess-Zumino action);

$$S_{\text{probe}} = -T_{D5} \int d^4x d\Sigma_2 e^{-\phi} \sqrt{-\det[g_{\mu\nu, \text{ind}} + 2\pi\alpha'F_{\mu\nu}]} + T_{D5} \int C_{2, \text{ind}} \wedge F_2 \wedge F_2 \quad (30)$$

The induced 6-d configuration on the probe brane extends along $R^{1,3} \times \Sigma_2$ is

$$ds^2_{\text{ind}} = e^\phi \left[ dx^2_{1,3} + \alpha' g_s \left( e^{2h} + \frac{e^{2g}}{4} (a - 1)^2 \right) (d\theta^2 + \sin^2 \theta d\varphi^2) \right]$$

$$C_{2, \text{ind}} = \frac{N_c}{2} (\psi - \psi_0) \sin \theta d\theta \wedge d\varphi,$$

$$F_2 = F_{\mu\nu} dx^\mu \wedge dx^\nu. \quad (31)$$

\(^1\) Actually, there is another possible two cycle, given by $\Sigma'_2 = \{ \theta = \pi - \tilde{\theta}, \varphi = \tilde{\varphi}, \psi = \pi \}$, but this one will give the same results for all the observables we consider here. Probably it is the same two-cycle, with a different orientation.
From here we can compute the determinant\(^2\)

\[
\sqrt{-\det[g_{ab,ind} + 2\pi a' F_{ab}]} = e^{3\phi}(a' g_s)[e^{2h} + \frac{e^{2g}}{4}(a - 1)^2] \sqrt{1 - 4\pi^2 a'^2 F_{\mu\nu}^2 \sin \theta}. \tag{32}
\]

Plugging this result into eq.(30) and expanding for small values of the field strength (or equivalently, for \(\alpha' \to 0\)), we get

\[
S_{probe} \approx -T_{D5} \int d^4x \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \times \\
\left[ e^{2\phi}(a' g_s)[e^{2h} + \frac{e^{2g}}{4}(a - 1)^2] \left(1 - 4\pi^2 a'^2 e^{-2\tilde{\phi}} \eta^\alpha \eta^{\rho\beta} F_{\mu\nu} F_{\alpha\beta}\right)\right], \tag{33}
\]

where we used that \(g^{\mu\nu} = e^{-\phi} \eta^{\mu\nu}\). Now, performing the integral over the angles, the term with the Yang-Mills action reads,

\[
S_{probe} \approx 4\pi T_{D5}(2\pi^2 a'^2)(a' g_s)[e^{2h} + \frac{e^{2g}}{4}(a - 1)^2] \int d^4x F_{\mu\nu} F^{\mu\nu} = \frac{1}{4g^2} \int d^4x F_{\mu\nu} F^{\mu\nu}. \tag{34}
\]

From here, and using the fact that the tension of a five brane satisfies \(32\pi^2 g_s a'^3 T_{D5} = 1\), we read that the 4-d gauge coupling is

\[
\frac{8\pi^2}{g^2} = 2[e^{2h} + \frac{e^{2g}}{4}(a - 1)^2] = Pe^{-\tau}. \tag{35}
\]

Before we evaluate this quantity on our solution, a few comments are in order.

i) It was shown in the paper \([32]\), that the five branes on the submanifold \(R^{1,3} \times \Sigma_2\) preserve SUSY only if the probe brane is at infinite radial distance from the end of the space (\(\rho \to \infty\)). This result is valid for the particular solution considered there and does not extend to the present background. Indeed, a five brane in the configuration described above does not preserve the same spinors as the background, hence, it is not SUSY, rendering our definition of gauge coupling non SUSY preserving. This will not concern us and we will take the definition of eq.(65) as a good estimation, valid over all the range of the radial coordinate.

ii) If we consider this definition, we can see that the gauge coupling diverges in the IR (signaling confinement). Also, this gauge coupling vanishes in the far UV. The fact that it becomes small should not be a reason of worry. This 4-d gauge being small is not indicating that the ten dimensional geometry is highly curved. Indeed, the 4-d coupling becomes small when the geometry is smooth and well approximated by IIB supergravity.

iii) Using a particular radius-energy relation it was shown in \([31]\) that one particular solution to the eq. of motion \([11]\) reproduces the NSVZ beta function. For the class of solutions we are interested in, the paper \([29]\) shows that the beta function is characteristic of a 6-d field theory.

In Figure 3 we plot the 4-d gauge coupling as defined in eq.(65) on the solutions we found above.

**APPROXIMATE SYMMETRIES OF THE SOLUTION**

As we have seen above, the gauge coupling exhibits three qualitatively different regimes. Namely, the IR regime for \(0 < \rho < 1\), an intermediate plateau for \(1 < \rho < \rho_s\) where the gauge coupling is almost constant, and a UV regime for \(\rho > \rho_s\). We study the (approximate) symmetries of the background in these three different regimes.

To leading order in \(c/N_c\), i.e. in the approximation of eq. \([10]\), the background of eq. \([10]\)-using the definitions in eq.\([10]\)-takes the form

\[
ds^2 = \frac{\sqrt{3}}{c^3/2 \sin^{3/2} \alpha} \left\{ dx_{1,3}^2 + \frac{cP'_1}{8} \left(4d\rho^2 + (\omega_3 + \tilde{\omega}_3)^2\right) \right. \\
+ \left. \frac{cP'_1 \cosh \tau}{4} \left( d\tilde{\rho}_2^2 + d\tilde{\omega}_2^2 + 2 \tanh \tau (\omega_1 \tilde{\omega}_1 - \omega_2 \tilde{\omega}_2) \right) \right\} + O\left(N_c^2/c\right), \tag{36}
\]

\(^2\) Notice that the term \(\int C_{2,ind} \wedge F_2 \wedge F_2\) will generate the theta-term \(\Theta \int d^4x F_{\mu\nu}^* F^{\mu\nu}\). We will omit this term in the following and concentrate on the definition of the gauge coupling.
FIG. 3: The 't Hooft coupling $g^2 N_c/(8\pi^2)$ as a function of $\rho$ for various values of the parameters $c$, $\alpha$. All three curves are for $N_c = 10$, while $c = 60$, $\alpha = 0.01$ for (i), $c = 90$, $\alpha = 0.002$ for (ii) and $c = 100$, $\alpha = 0.0005$ for (iii). The red (long dashes) curves are the $O(c)$ approximation in the expansion (20), the blue (medium dashes) lines are the $O(1/c)$ approximation, the green (short dashes) lines are the $O(1/c^3)$ approximation, and the black (dotted) lines are the numerical solutions.

\begin{align}
F(3) &= N_c \left\{ -d \left[ \frac{2\rho}{\sinh(2\rho)} (\omega_1 \wedge \tilde{\omega}_2 - \omega_2 \wedge \tilde{\omega}_1) \right] + \frac{1}{4} (\omega_1 \wedge \omega_2 - \tilde{\omega}_1 \wedge \tilde{\omega}_2) \wedge (\omega_3 + \tilde{\omega}_3) \right\},
\end{align}

where $d\Omega_2^2 = \omega_2^2 + \omega_3^2 = d\theta^2 + \sin^2 \theta d\phi^2$ and $d\tilde{\Omega}_2^2 = \tilde{\omega}_2^2 + \tilde{\omega}_3^2 = d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2$ are the metrics on two 2-spheres, and we have introduced the one-forms

\begin{align}
\omega_1 &= d\theta, \quad \omega_2 = \sin \theta d\phi, \quad \omega_3 = \cos \theta d\phi,
\end{align}

in addition to the left-invariant $SU(2)$ forms of eq. (3). Let us now look at the behavior of this background in the three different dynamical regimes.

Starting with the far IR, $\rho \lesssim 1$, we see from Table IIII that no further simplifications occur in the above background and so the only isometries are the obvious $ISO(1, 3) \times U(1) \times U(1) \times \mathbb{Z}_2$, being the two $U(1)$s the translations in $\varphi, \tilde{\varphi}$, while $\mathbb{Z}_2$ comes from the change $\psi \rightarrow \psi + 2\pi$.

In the intermediate regime, $1 \lesssim \rho \lesssim \rho_*$, we see that the background simplifies to

\begin{align}
ds^2 &\approx \frac{\sqrt{3}}{c^{3/2} \sin^{3/2} \alpha} \left\{ dx_{1,3}^2 + \frac{c \cos \alpha}{4} \left[ \left( \frac{\tan^3 \alpha e^{4\rho}}{3} \right) (4d\rho^2 + (\omega_3 + \tilde{\omega}_3)^2) + d\omega_3^2 + d\tilde{\omega}_3^2 \right] \right\}, 
\end{align}

\begin{align}
F(3) &\approx \frac{N_c}{4} (\omega_1 \wedge \omega_2 - \tilde{\omega}_1 \wedge \tilde{\omega}_2) \wedge (\omega_3 + \tilde{\omega}_3).
\end{align}

Note that since $\rho < \rho_*$ in this regime we have $\tan^3 \alpha e^{4\rho} < 2 < 1$. If this was equal to 1 then the compact part of the metric would be that of $T_{11}$, the base of the conifold. This indeed happens in the UV region, $\rho \gtrsim \rho_*$, where the background takes the form

\begin{align}
ds^2 &\approx \frac{\sqrt{3}}{c^{3/2} \sin^{3/2} \alpha} \left\{ dx_{1,3}^2 + \frac{2^{-1/3} c \sin \alpha}{4} e^{4\rho/3} \left[ \frac{2}{3} (4d\rho^2 + (\omega_3 + \tilde{\omega}_3)^2) + d\omega_3^2 + d\tilde{\omega}_3^2 \right] \right\},
\end{align}
\[ F_{(3)} \approx \frac{N_c}{4} (\omega_1 \wedge \omega_2 - \tilde{\omega}_1 \wedge \tilde{\omega}_2) \wedge (\omega_3 + \tilde{\omega}_3). \]  

(42)

It can be shown that the metric

\[ ds^2 = d\Omega_2^2 + d\tilde{\Omega}_2^2 + \zeta (\omega_3 + \tilde{\omega}_3)^2, \]  

(43)

where \( \zeta \) is a constant, and the 3-form given above possess an \( SU(2) \times SU(2) \times U(1) \) isometry for any non-zero value of \( \zeta \). In particular, irrespective of the value of \( \zeta \), the seven Killing vectors take the form

\[
\begin{align*}
\xi_1 &= \sin \varphi \partial_\theta + \cot \theta \cos \varphi \partial_\varphi - \csc \theta \cos \varphi \partial_\psi \\
\tilde{\xi}_1 &= \sin \tilde{\varphi} \partial_\tilde{\theta} + \cot \tilde{\theta} \cos \tilde{\varphi} \partial_\tilde{\varphi} - \csc \tilde{\theta} \cos \tilde{\varphi} \partial_\tilde{\psi} \\
\xi_2 &= - \cos \varphi \partial_\theta + \cot \theta \sin \varphi \partial_\varphi - \csc \theta \sin \varphi \partial_\psi \\
\tilde{\xi}_2 &= - \cos \tilde{\varphi} \partial_\tilde{\theta} + \cot \tilde{\theta} \sin \tilde{\varphi} \partial_\tilde{\varphi} - \csc \tilde{\theta} \sin \tilde{\varphi} \partial_\tilde{\psi} \\
\xi_3 &= \partial_\varphi \\
\tilde{\xi}_3 &= \partial_\tilde{\varphi} \\
\xi &= \partial_\psi
\end{align*}
\]

It follows that both in the intermediate regime, \( 1 \lesssim \rho \lesssim \rho_* \), and in the far UV regime, \( \rho \gtrsim \rho_* \), the background possess an approximate \( ISO(1,3) \times SU(2) \times SU(2) \times \mathbb{Z}_{2N_c} \) isometry, where the \( U(1) \) generated by \( \partial_\psi \) is broken to \( \mathbb{Z}_{2N_c} \) by the R-symmetry anomaly (see for example [34]).

The Killing vectors are the same for the metrics in the intermediate region \((1 < \rho < \rho_*)\) given in eq. (39) and that in the “UV-region” \((\rho > \rho_*)\) written in eq. (41). Nevertheless, there are differences between the metrics in eq. (39) and in eq. (41). Indeed, ignoring the Minkowski part, the metric in eq. (39) can be intuitively pictured as a cigar-shaped geometry, with a radial direction \( \rho \), a ‘circle’ \((\omega_3 + \tilde{\omega}_3)\) all fibered over the \( S^2 \times S^2 \) manifold. On the other hand, the metric of eq. (41) is the conifold, that can be intuitively thought of as a cone over \( S^2 \times S^3 \). This in turn is responsible for an important distinction. In the metric of eq. (41) we can rescale the coordinates

\[ x_i \rightarrow \delta x_i, \quad \rho \rightarrow \rho + \frac{3}{2} \log \delta \]  

(44)

and the metric of eq. (41) is just conformal to itself (that is it gets multiplied by a factor of \( \delta^2 \)). This may suggest that the theory gains some kind of scale-invariance when approaching the far UV region. This is reminiscent of the picture proposed in [36]. Notice also that due to subleading corrections in \( N_c/c \) the dilaton changes at \( \rho_* \), as can be seen in Fig. 1.

To close this section, let us mention that the fact that the metric and the background in the three regions have a different geometric interpretation, and different symmetries, implies that there exist three distinct dynamical phases in the dual field theory, and that the scales separating them have a physical meaning that goes beyond the analysis of the gauge coupling we focused upon in this paper.

In order to better understand what the intermediate phase is, it would be interesting to study the model in limit in which the gaugino condensate is switched off, to learn about the phase structure of the underlying six-dimensional gauge theory, also in relation to the compactification of the internal space. Finally to study the spectrum of the theory and how the dynamical scales determine it. All of this is going to be explored elsewhere.
In order to gain some insight on the meaning of this class of solutions for the dual field theory, it is useful to examine the behavior of the leading-order approximation $P \simeq P_1 = cR$ in the far UV, for $\rho \gg \rho_*$. We can change variables, $\rho = -\frac{3}{2} \log z$, where $z$ is now proportional to a length scale. The physical scales in the problem can be rewritten in terms of $0 < z < 1$. The far IR scale at which the space ends $\rho = \rho_0 = 0$ corresponds to $z_0 = 1$. The scale $\rho \sim 1$ corresponds to $z_1 = \exp[-2/3] \simeq 1/2$. The scale $\rho_* \gg 1$ corresponds to $z_* = \exp[-2/3\rho_*] \ll 1$, and finally the far UV in which the background approaches the conifold is $z \to 0$.

Expanding around $z \to 0$:

$$P \simeq c \sin \alpha \left( \frac{1}{2^{1/3}} \frac{1}{z^2} + \frac{2^{2/3} c \cos^3 \alpha}{3 \sin^2 \alpha} z^4 + 2^{5/3} c \sin \alpha z^4 \log z + O \left( \frac{N_c}{c} \right) \right) \quad (45)$$

$$a \simeq 2z^3 + O \left( \frac{N_c}{c} \right). \quad (46)$$

The expressions for the other functions appearing in the background are similar.

In this expansion, one can identify the presence of three independent dynamical quantities. The $z^3$ term signals the presence of a dimension-3 condensate (usually interpreted as gaugino condensate), responsible for the behavior in the IR at the scale $z_0 = 1$. The coefficient $2^{-1/3} c \sin \alpha$ appears in front of the $z^{-2}$ term, which can be thought of as the effect of deforming the theory with the insertion of an operator of dimension-6, with non-vanishing coupling. The presence of the term that, up to logarithmic corrections, scales as $z^4$ can be thought of as the VEV of an operator that is dimension-4 in the limit in which $\cot^3 \alpha \gg 1$. The size of the VEV is given by $2 \cot^3 \alpha / 3$.

The interpretation of the running of the gauge coupling is hence that at very high energy the model is dominated by the insertion of the dimension-6 operator. At the scale $z_*$ (equivalently, $\rho_*$) the dimension-4 condensate appears and dominates the dynamics. This operator is marginally irrelevant, and hence its influence is superseded at low energies.
FIG. 4: The three qualitatively different energy regimes and the corresponding operators that dominate the solution in each of these regions.

(below $\rho \lesssim 1$, or at distance $z > z_1$) by the dimension-3 operator. The operators of dimension 3 and 6 dominate the dynamics at small and large energies, respectively. The existence of a finite intermediate range governed by the dimension 4 operator results from its coefficient being large.

Notice that we could approximate the expression for $P$ by reabsorbing the $\ln z$ correction in the scaling dimension of the VEV. In doing so, using the expression for $\rho_*$ one sees that the VEV scales as $z^d$, where

$$d = 4 + 12 e^{-4 \rho_*}. \quad (48)$$

This is interesting for two reasons. First, because it shows that not taking $\cot \alpha \gg 1$, would correspond to giving a VEV to a highly irrelevant operator. As a result, the background would be effectively governed just by two effects: the dimension-6 coupling and the dimension-3 VEV. This is the case already studied in the literature [29], and we will not comment any further on it. The second interesting fact is that in the intermediate energy region $1 \lesssim \rho \lesssim \rho_*$ the running is governed by a quasi-marginal operator with scaling dimension $d \approx 4$: the length of the intermediate region is governed by the difference $d - 4$, as reasonable to expect.

Reinstating the proper units leads to the replacement

$$\rho = \frac{3}{2} \log \frac{\mu - \Lambda_0 + \Lambda}{\Lambda}, \quad (49)$$

$$\mu = \left( e^{2 \rho} - 1 \right) \Lambda + \Lambda_0, \quad (50)$$

where $\Lambda$ and $\Lambda_0$ have dimension of an energy, and where $\Lambda_0$ is the scale at which the singularity in the solution appears. This is the scale of confinement.

The gaugino condensate is proportional to the scale $\Lambda^3$ and it dominates the dynamics for scales below $\rho_{IR} \simeq 1$, that is for $\mu < \Lambda_0 + \Lambda$. The scale $\rho_{IR} \simeq 1$ at which the dimension-4 operator takes over in governing the running corresponds to a physical scale $\mu = \Lambda_{IR} \approx \Lambda_0 + \Lambda$. The $\rho_*$ scale at which the theory becomes controlled by the dimension-6 operator corresponds to a third, independent, dynamical energy scale

$$\Lambda_* = \Lambda_0 + \left( \exp \left[ \frac{2}{3} \rho_* \right] - 1 \right) \Lambda \gg \Lambda_{IR}. \quad (51)$$

above which the gauge theory is effectively six-dimensional. These three qualitatively distinct regimes are shown in Fig. 4.

A comment about fine-tuning

The emergence of an energy window over which the theory seems to walk is controlled by the choice $\alpha \ll 1$. This might be seen as a signal of fine-tuning, since this choice is what makes the $d \sim 4$ VEV dominate over the more relevant gaugino condensate.

In field theory, fine-tuned parameters are unnatural ones that are renormalized additively (their RG equations involve operator mixing), and for which the experimental value turns out to be parametrically smaller that the natural one dictated by the additive renormalization from other couplings in the theory. If this is the case, setting the value of the unnatural coupling in the IR to be arbitrarily small requires to precisely choose the UV boundary conditions of the RG flow so that large cancellations occur in the running. The reason why fine-tuned parameters are looked upon with suspicion is that their RG flow cannot be extrapolated to arbitrarily large scales, without running into difficulties in the interpretation of the couplings at high energies, which generically indicates that the model should be completed in the UV.
The eq. (19) automatically implies that this is not happening in our case. The upper bound on \( \cot \alpha \) ensures that the solution of the background equations, and hence all the derived quantities, can be extrapolated up to arbitrary high energies without facing singularities or pathological behaviors. Conversely, if this condition is violated by the choice of \( \alpha \) and \( c \), then in tracking back the gauge coupling from the IR up to high scales will ultimately result into running into a singularity, that signals some sort of pathology of the RG flow. The fact that we impose the upper limit of \( \cot \alpha \) avoids such singular behavior, and within the context of this paper allows us to disregard the possible problems related with fine-tuning.

However, it must be emphasized that this approach is very conservative, probably much more so that is needed in the context of dynamical electro-weak symmetry breaking, which is our original motivation. As explained in the introduction, a technicolor model is by definition incomplete, since by itself it does not yield a natural mechanism for generating the standard-model fermion masses. In any realistic model, one must assume that the technicolor dynamics is embedded in a more general dynamical theory (ETC), and the scale at which this happens might well be just few orders of magnitude above the electro-weak scale. Even if the extrapolation to large scales of the background geometry towards the UV runs into a singularity, not necessarily this implies that the solution hence constructed has to be discarded, because the embedding into a more general theory will anyhow modify the equations. In constructing a realistic model, one might hence consider relaxing the bound in eq. (19), and consequently on the four-dimensional ’t Hooft coupling.

**SUMMARY AND OUTLOOK**

We reconsidered a class of Type IIB backgrounds that have been studied extensively in the literature, found a new solution, and developed it as a systematic expansion starting from the approximation yield by a limiting case. We studied the set of operators (VEV’s and insertions) that it corresponds to in the dual field theory, and the approximate symmetries of the background in the three regions in which the background behaves differently. The main result we obtained is that an appropriately defined four-dimensional gauge coupling exhibits the features of a walking theory. This coupling can be rewritten as a function of the renormalization scale as

\[
\lambda = \frac{g^2 N_c}{8\pi^2} \approx \frac{\lambda^* \coth \rho}{(1 + 2e^{-4\rho^*} (\sinh(4\rho^*) - 4\rho^*)^{1/3}},
\]

where the constants \( \lambda^* \) and \( \rho^* \) can be written in terms of the parameters \( \alpha \) and \( c \) defined in the body of the paper as

\[
\lambda^* = \frac{N_c}{c \cos \alpha},
\]

\[
\rho^* = \frac{1}{4} \ln (2 \cot^3 \alpha) = \frac{3}{2} \ln \frac{\Lambda^* - \Lambda_0 + \Lambda}{\Lambda},
\]

where \( \rho \) is related to the renormalization scale \( \mu \) as

\[
\rho = \frac{3}{2} \ln \frac{\mu - \Lambda_0 + \Lambda}{\Lambda},
\]

with \( \Lambda_0 \) the confinement scale, \( \Lambda \) the spontaneous symmetry breaking scale, and \( \Lambda^* \) the walking scale, and where the upper bound on the ’t Hooft coupling

\[
\lambda^* \lesssim \frac{1}{2\rho^*},
\]

ensures that \( \lambda \) is well defined all the way to \( \rho \to 0 \).

We showed that for \( \mu < \Lambda \) the isometries of the internal manifold in the background, and the R-symmetries of the dual theory, are spontaneously broken by the VEV of a dimension-3 operator. We showed that there is an approximate dilatation symmetry for \( \mu \gg \Lambda^* \). In the intermediate region the dynamics is dominated by the VEV of a marginally irrelevant operator, while in the far UV by the insertion of a dimension-6 operator. It would be nice to make a dedicated study of the dual field theory in the intermediate region.

All of the above agrees with the idea that this is a model in which the dynamics is walking. What is next? There are at least four directions in which to further develop this study, all of which might yield very interesting and useful results.
The first and most urgent question concerns the spectrum of the theory. As anticipated, a very important open question in the context of walking technicolor has to do with the presence of light scalar and pseudo-scalar degrees of freedom, in connection with the spontaneous breaking of internal and dilatational approximate symmetries. Studying the spectrum of fluctuations in the background should shed some light on this problem, allowing to compute the masses of possible light degrees of freedom, and in particular allowing to identify the parametric dependence on the fundamental scales and couplings present in the background.

A second important, though more challenging, study should lead to the identification of the field theory dual to the present background. Both the field theory in the UV, and the spectrum of anomalous dimensions in the intermediate and IR regions are of utmost interest for model-building, as stressed in the introductory sections.

Third, the study of gauge invariant observables in this background, like Wilson and ‘t Hooft loops, or Domain Walls, may give further information on the dual QFT.

In the fourth place, this preliminary study opens the way to a large possible set of investigations in a broad class of models. The fact that we discovered a class of solutions yielding a background which has some of the features of a walking theory, in a model that had already been extensively analyzed in the literature, suggests that possibly large numbers of string-motivated set-ups might lead to analogous results. It would be very interesting to understand how general the occurrence of the dynamical properties discussed here is. It might even be that much simpler set-ups admit solutions that walk over some energy interval. The powerful computational techniques of the string-gauge theory duality represent a wonderful opportunity for model building, and for better understanding the mechanism leading to electro-weak symmetry breaking. A more extensive survey of the models in which this possibility is realized would be very interesting.

Finally, the original motivation of all of this is electro-weak symmetry breaking, but the model we studied does not have any electro-weak symmetry to start. It would be very interesting to understand how to effectively couple backgrounds as the one described here to a weakly-coupled sector with the gauge symmetries of the Standard Model and construct a complete realization of a dynamical mechanism for electro-weak symmetry breaking with walking behavior. A possible way to go would be to add the ‘Standard Model’ as probe branes in this background (the embedding should not preserve any SUSY). In this case, the Standard Model dynamics would feel the influence of the Technicolor sector, but not the other way around. In this ‘quenched’ approach, we could apply all the technology developed in recent years, see [37] for a review.

The work of MP is supported in part by the Wales Institute of Mathematical and Computational Sciences. Carlos Nunez thanks Nick Evans, Angel Paredes and Ed Threlfall for discussions on a related topic.

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