Statistical mechanics of 2D turbulence with a prior vorticity distribution

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We adapt the formalism of the statistical theory of 2D turbulence in the case where the Casimir constraints are replaced by the specification of a prior vorticity distribution. A phenomenological relaxation equation is obtained for the evolution of the coarse-grained vorticity. This equation monotonically increases a generalized entropic functional (determined by the prior) while conserving circulation and energy. It can be used as a thermodynamical parametrization of forced 2D turbulence, or as a numerical algorithm to construct (i) arbitrary statistical equilibrium states in the sense of Ellis-Haven-Turkington (ii) particular statistical equilibrium states in the sense of Miller-Robert-Sommeria (iii) arbitrary stationary solutions of the 2D Euler equation that are formally nonlinearly dynamically stable according to the Ellis-Haven-Turkington stability criterion refining the Arnold theorems.

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I. INTRODUCTION

Two-dimensional incompressible and inviscid flows are described by the 2D Euler equations

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0, \quad \omega = -\Delta \psi, \quad \mathbf{u} = -\mathbf{z} \times \nabla \psi, \quad (1)$$

where $\omega$ is the vorticity and $\psi$ the streamfunction. The 2D Euler equations are known to develop a complicated mixing process which ultimately leads to the emergence of large-scale coherent structures like jets and vortices. Jovian atmosphere shows a wide diversity of structures: Jupiter’s great red spot, white ovals, brown barges,...

One question of fundamental interest is to understand and predict the structure and the stability of these quasi stationary states (QSS). To that purpose, Miller [1] and Robert & Sommeria [2] have proposed a statistical mechanics of the 2D Euler equation (a similar statistical theory had been developed earlier by Lynden-Bell [3] to describe the violent relaxation of collisionless stellar systems governed by the Vlasov equation; see [4] for a description of this analogy). The key idea is to replace the deterministic description of the flow $\omega(r,t)$ by a probabilistic description where $\rho(r,\sigma,t)$ gives the density probability of finding the vorticity level $\omega = \sigma$ in $r$ at time $t$. The observed (coarse-grained) vorticity field is then expressed as $\overline{\omega}(r,t) = \int \rho \sigma dr$. To apply the statistical theory, one must first specify the constraints attached to the 2D Euler equation. The circulation $\Gamma = \int \overline{\omega} dr$ and the energy $E = \frac{1}{2} \int \overline{\psi} dr$ will be called robust constraints because they can be expressed in terms of the coarse-grained field $\overline{\omega}$ (the energy of the fluctuations can be neglected). These integrals can be calculated at any time from the coarse-grained field $\overline{\omega}(r,t)$ and they are conserved by the dynamics. By contrast, the Casimir invariants $I_n = \int f_n(\omega) dr$, or equivalently the fine-grained moments of the vorticity $\Gamma_{n>1} = \int \overline{\omega}_n dr$, where $\overline{\omega}_n = \int \rho \sigma^n dr$, will be called fragile constraints because they must be expressed in terms of the coarse-grained vorticity. Indeed, the moments of the coarse-grained vorticity $\Gamma_{n>1} = \int \overline{\omega}_n dr$ are not conserved since $\overline{\psi} \neq \overline{\omega}$ (part of the coarse-grained moments goes into fine-grained fluctuations). Therefore, the moments $\Gamma_{n>1}$ must be calculated from the fine-grained field $\omega(r,t)$ or from the initial conditions, i.e. before the vorticity has mixed. Since we often do not know the initial conditions nor the fine-grained field, the Casimir invariants often appear as "hidden constraints".

The statistical theory of Miller-Robert-Sommeria (MRS) is based on three assumptions: (i) it is assumed that the evolution of the flow is strictly described by the 2D Euler equation (no forcing and no dissipation); (ii) it is assumed that we know the initial conditions (or equivalently the value of all the Casimirs) in detail; (iii) it is assumed that mixing is efficient and that the evolution is ergodic so that the system will reach, at statistical equilibrium, the most probable (most mixed) state. Within these assumptions [17], the statistical equilibrium state of the 2D Euler equation is obtained by maximizing the mixing entropy

$$S[\rho] = -\int \rho \ln \rho d\sigma dr, \quad (2)$$

at fixed energy $E$ and circulation $\Gamma$ (robust constraints) and fixed fine-grained moments $\Gamma_{n>1}$ (fragile constraints). We must also account for the normalization condition $\int \rho dr = 1$. This optimization principle is solved by introducing Lagrange multipliers, writing the first order variations as [2, 3]:

$$\delta S - \beta \delta E - \alpha \delta \Gamma - \sum_{n>1} \alpha_n \delta \Gamma_{n>1} - \int \zeta(r) \delta \rho dr = 0. \quad (3)$$

In the MRS approach, the conservation of all the Casimirs has to be taken into account. However, in geophysical situations, the flows are forced and dissipated at small scales (due to convection in the jovian atmosphere)
so that the conservation of the Casimirs is destroyed. Ellis-Haven-Turkington [6] have proposed to treat these situations by fixing the conjugate variables \( \alpha_{n>1} \) instead of the fragile moments \( \Gamma_n^{f.g.} \) (this is essentially a suggestion that has to be tested in practice). If we view the vorticity levels as species of particles, this is similar to fixing the chemical potentials instead of the total number of particles in each species. Therefore, the idea is to treat the fragile constraints canonically, whereas the robust constraints are still treated microcanonically. A rigorous mathematical formalism has been developed in 7 and a more physical presentation has been given in 8. In the EHT approach, the relevant thermodynamical potential (grand entropy) is obtained from the mixing entropy \( S_\chi \) by using a Legendre transform with respect to the fragile moments \( \delta S \):\[
abla \chi \alpha_n \Gamma_n^{f.g.}.
\]

Expliciting the fine-grained moments, we obtain the relative (or grand) entropy \( \chi(\sigma) \equiv \exp\left\{ \sum_{n>1} \alpha_n \sigma^2 \right\} \). We shall assume that this function is imposed by the small-scale forcing so it has to be given a priori as an input in the theory 7.\[ S_\chi = S - \sum_{n>1} \alpha_n \Gamma_n^{f.g.}. \tag{4} \]

Writing the first order variations as \( S - \beta \delta E - \alpha \delta \Gamma = 0 \), leading to \( C'(\sigma) = -\beta \psi - \alpha \), and with \( \sigma \equiv \exp\left\{ \sum_{n>1} \alpha_n \sigma^2 \right\} \). We have \( C(\sigma) = \chi(\sigma) \exp\left\{ \sum_{n>1} \alpha_n \sigma^2 \right\} \). Comparing with Eq. (7), we find that \( C'(\sigma) = -\beta \psi - \alpha \),\[ C' = \chi(\sigma) \exp\left\{ \sum_{n>1} \alpha_n \sigma^2 \right\} \]

where \( \exp\left\{ \sum_{n>1} \alpha_n \sigma^2 \right\} \). We have \( S_{\sigma} = \beta \chi(\sigma) \exp\left\{ \sum_{n>1} \alpha_n \sigma^2 \right\} \). Comparing with Eq. (7), we find that \( S_{\sigma} = \beta \chi(\sigma) \exp\left\{ \sum_{n>1} \alpha_n \sigma^2 \right\} \). We have \( \beta \chi(\sigma) \exp\left\{ \sum_{n>1} \alpha_n \sigma^2 \right\} \)

which links the centered variance of the vorticity to the coarse-grained vorticity and the generalized entropy. It also clearly establishes that \( C'' > 0 \). On the other hand, the equilibrium coarse-grained vorticity \( \nabla(\sigma) \) maximizes the generalized entropy \( C(\sigma) \) at fixed circulation and energy \( \chi(\sigma) \). Comparing with Eq. (7), we find that \( S_{\sigma} = \beta \chi(\sigma) \exp\left\{ \sum_{n>1} \alpha_n \sigma^2 \right\} \). Comparing with Eq. (7), we find that \( S_{\sigma} = \beta \chi(\sigma) \exp\left\{ \sum_{n>1} \alpha_n \sigma^2 \right\} \).

II. EQUILIBRIUM STATISTICAL MECHANICS WITH A PRIOR VORTICITY DISTRIBUTION

When a prior vorticity distribution is given, the statistical equilibrium state is obtained by maximizing the relative (or grand) entropy \( S_\chi \) at fixed energy \( E \), circulation \( \Gamma \) and normalizing condition \( \int \rho \sigma = 1 \) (grand microcanonical ensemble). The conservation of the Casimirs has been replaced by the specification of the prior \( \chi(\sigma) \). Writing the first order variations as \( \delta S_\chi - \beta \delta E - \alpha \delta \Gamma - \int \zeta(\psi) \delta \rho \sigma \sigma dx = 0 \), we get the Gibbs state \[ \rho(\sigma, \sigma) = \frac{1}{Z(\sigma)} \chi(\sigma) \exp\left\{ \beta \psi + \alpha \right\}, \tag{6} \]

with \( Z = \int_\epsilon^\infty \chi(\sigma) \exp\left\{ \beta \psi + \alpha \right\} \sigma dx \). This is the product of a universal Boltzmann factor by a non-universal function \( \chi(\sigma) \) fixed by the forcing. The coarse-grained vorticity is given by \[ \nabla = \int_\epsilon^\infty \chi(\sigma) \exp\left\{ \beta \psi + \alpha \right\} \sigma dx = F(\beta \psi + \alpha), \tag{7} \]

with \( F(\Phi) = \int \chi(\sigma) \exp\left\{ \beta \psi + \alpha \right\} \sigma dx \). It is easy to show that \( F'(\Phi) = -\omega_2(\Phi) \leq 0 \), where \( \omega_2 = \nabla^2 - \nabla^2 \geq 0 \) is the local centered variance of the vorticity. Therefore, \( F(\Phi) \) is a decreasing function. Since \( \nabla = f(\psi) \), the statistical theory predicts that the coarse-grained vorticity \( \nabla(\sigma) \) is a stationary solution of the 2D Euler equation and that the \( \nabla - \psi \) relationship is a monotonic function which is increasing at negative temperatures \( \beta < 0 \) and decreasing at positive temperatures \( \beta > 0 \). We have \( \nabla(\psi) = -\beta \omega_2 \). We note that the \( \nabla - \psi \) relationship predicted by the statistical theory can take a wide diversity of forms (usually non-Boltzmannian) depending on the prior \( \chi(\sigma) \). Furthermore, the coarse-grained distribution \( \nabla(\sigma) \) extremizes a generalized entropy in \( \sigma \) space of the form 8: \[ S_\sigma = -\int F(\nabla) dx, \tag{8} \]

at fixed circulation and energy (robust constraints). Writing the first order variations as \( \delta S - \beta \delta E - \alpha \delta \Gamma = 0 \), leading to \[ C'(\sigma) = -\beta \psi - \alpha \], and comparing with Eq. (7), we find that \( C'(x) = -F^{-1}(x) \). Therefore, \( C \) is a convex function \( C''(\sigma) > 0 \) determined by the prior \( \chi(\sigma) \) encoding the small-scale forcing according to the relation 

\[ C(\sigma) = -\int \nabla F^{-1}(x) dx = -\int [\ln \chi]'^{-1}(x) dx. \tag{10} \]

We have \( \nabla(\psi) = -\beta/C''(\psi) \). Comparing with Eq. (7), we find that, at statistical equilibrium 

\[ \omega_2 = 1/C''(\psi), \tag{11} \]

which links the centered variance of the vorticity to the coarse-grained vorticity and the generalized entropy. It also clearly establishes that \( C'' > 0 \). On the other hand, the equilibrium coarse-grained vorticity \( \nabla(\sigma) \) maximizes the generalized entropy 8. \( C(\sigma) \) at fixed circulation and energy \( \chi(\sigma) \). Comparing with Eq. (7), we find that \( S_{\sigma} = \beta \chi(\sigma) \exp\left\{ \sum_{n>1} \alpha_n \sigma^2 \right\} \). Comparing with Eq. (7), we find that \( S_{\sigma} = \beta \chi(\sigma) \exp\left\{ \sum_{n>1} \alpha_n \sigma^2 \right\} \). Comparing with Eq. (7), we find that \( S_{\sigma} = \beta \chi(\sigma) \exp\left\{ \sum_{n>1} \alpha_n \sigma^2 \right\} \).

The preceding relations are also valid in the MRS approach except that \( \chi(\sigma) \) is determined a posteriori from the initial conditions by relating the Lagrange multipliers \( \alpha_{n>1} \) to the Casimir constraints \( \Gamma_n^{f.g.} \). In this case of freely evolving flows, the generalized entropy 8.10 depends on the initial conditions, while in the case of forced flows considered here, it is intrinsically fixed by the prior vorticity distribution. On the other hand, a maximum of \( S_\chi[\rho] \) at fixed \( E \) and \( \Gamma \) is always a maximum of \( S[\rho] \) at fixed \( E \), \( \Gamma \) and \( \Gamma_n^{f.g.} \). Therefore, a maximum of the generalized entropy \( S_\sigma(\sigma) \) at fixed \( E \) and \( \Gamma \) determines a statistical equilibrium state in the MRS viewpoint 10. However, the converse is wrong in case of "ensemble inequivalence" 11.12 with respect to the conjugate variables \( \Gamma_n^{f.g.} \). Therefore, the maximization of \( S_\sigma(\sigma) \) at fixed \( E \) and \( \Gamma \) is a sufficient (but not necessary) condition of MRS thermodynamical stability.
III. RELAXATION TOWARDS EQUILIBRIUM

In the case where a small-scale forcing imposes a prior vorticity distribution $\chi(\sigma)$, it is possible to propose a thermodynamical parametrization of the turbulent flow in a relaxation equation that conserves the circulation and the energy (robust constraints) and that increases the generalized entropy (8)-(10) fixed by the prior. This equation can be obtained from a generalized Maximum Entropy Production principle (MEPP) in $\omega$-space [3]. We write $\omega = \tilde{\omega} + \omega$ and take the local average of the 2D Euler equation (11). This yields $D\tilde{\omega}/Dt = -\nabla \cdot \nabla \omega = 0$ where $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the material derivative and $\mathbf{u}$ is the turbulent current. Then, we determine the optimal current $\mathbf{J}$ which maximizes the rate of entropy production $\dot{S} = -\int C''(\omega) \mathbf{J} \cdot \nabla \omega d\mathbf{r}$ at fixed energy $\dot{E} = \int \mathbf{J} \cdot \nabla \psi d\mathbf{r} = 0$, assuming that the energy of the fluctuations $\mathbf{J}^2/2\tilde{\omega}$ is bounded. According to this phenomenological principle, we find that the coarse-grained vorticity evolves according to [3, 9]:

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nabla \cdot \left\{ D \left[ \nabla \omega + \beta(t) \frac{C''(\omega)}{C''(\omega)} \nabla \psi \right] \right\}, \quad (12)$$

$$\beta(t) = -\frac{\int D \nabla \omega \cdot \nabla \psi d\mathbf{r}}{\int \frac{D}{C''(\omega)} \nabla \psi d\mathbf{r}}, \quad \omega = -\Delta \psi, \quad (13)$$

where $\beta(t)$ is a Lagrange multiplier enforcing the energy constraint $\dot{E} = 0$ at any time. It is shown in [3] that these equations increase monotonically the entropy (H-theorem $\dot{S} \geq 0$) provided that $D > 0$. Furthermore, a steady state of (12) is linearly dynamically stable if it is a (local) entropy maximum at fixed circulation and energy (minima or saddle points of entropy are linearly unstable). Therefore, the relaxation equations (12)-(13) generically converge towards a (local) entropy maximum (if there is no entropy maximum the solutions of the relaxation equations can have a singular behaviour). If there exists several local entropy maxima the selection will depend on a complicated notion of basin of attraction. The diffusion coefficient $D$ is not determined by the MEPP but it can be obtained from a Taylor’s type argument leading to $D = K\epsilon^2 \omega_2^{1/2}$ where $\epsilon$ is the coarse-graining mesh size and $K$ is a constant of order unity [8]. Assuming that the relation (12) remains valid out-of-equilibrium (see Appendix C of [3]), we get the closed expression $D = K\epsilon^2/\sqrt{C''(\omega)}$. This position dependant diffusion coefficient, related to the strength of the fluctuations, can “freeze” the system in a sub-region of space (“bubble”) and account for incomplete relaxation and lack of ergodicity [4, 13]. The relaxation equation (12) belongs to the class of nonlinear mean field Fokker-Planck equations introduced in [3]. This relaxation equation conserves only the robust constraints (circulation and energy) and increases the generalized entropy (8)-(10) fixed by the prior vorticity distribution $\chi(\sigma)$. It differs from the relaxation equations proposed by Robert & Sommervia [14] for freely evolving flows which conserve all the constraints of the 2D Euler equation ($E$, $\Gamma$ and all the Casimirs) and monotonically increase the mixing entropy $S$. In Eqs. (12)-(13), the specification of the prior $\chi(\sigma)$ (determined by the small-scale forcing) replaces the specification of the Casimirs (determined by the initial conditions). However, in both models, the robust constraints $E$ and $\Gamma$ are treated microcanonically (i.e. they are rigorously conserved). The relaxation equations of Robert & Sommervia [14] and Chavanis [2] are essentially phenomenological in nature but they can serve as numerical algorithms to compute maximum entropy states. In that context, since we are only interested by the stationary state (not by the dynamics), we can take $D = Cst$. and drop the advective term in the relaxation equation. Then, Eq. (12) can be used to construct (i) arbitrary EHT statistical equilibria (ii) a subset of MRS statistical equilibria (see the last paragraph of Sec. III).

IV. EXPLICIT EXAMPLES

Let us consider, for illustration, the prior vorticity distribution $\chi(\sigma)$ introduced by Ellis-Haven-Turkington [6] in their model of jovian vortices. It corresponds to a de-centered Gamma distribution

$$\chi(\sigma) = \frac{1}{\Omega_2|\lambda|} R \left[ \frac{1}{\Omega_2\lambda} \left( \sigma + \frac{1}{\lambda} \right) ; \frac{1}{\Omega_2\lambda^2} \right], \quad (14)$$

where $R(\sigma; a) = \Gamma(a)^{-1}e^{a-1}e^{-\sigma}$ for $\sigma \geq 0$ and $R = 0$ otherwise. The scaling of $\chi(\sigma)$ is chosen such that $\langle \sigma \rangle = 0$, $\var(\sigma^2) = \Omega_2$ and $\text{skew}(\sigma) \equiv \langle \sigma^3 \rangle / (\sigma^2)^{3/2} = 2\Omega_2^{1/2}$. We get

$$Z(\Phi) = \chi(\Phi) = \frac{e^{\Phi/\lambda}}{(1 + \Omega_2\Phi)^1/(1 + \Omega_2\Phi^2)}, \quad (15)$$

$$C(\omega) = Z(\Phi) = -\left( \ln \chi \right)'(\Phi) = \frac{-\Omega_2\Phi}{1 + \Omega_2\Phi}. \quad (16)$$

Inversing the relation (16), we obtain

$$-\Phi = \frac{1}{\Omega_2} \frac{\omega}{1 + \lambda\omega} = C'(\omega). \quad (17)$$

After integration, we obtain the generalized entropy

$$C(\omega) = \frac{1}{\Omega_2} \left[ \omega - \frac{1}{\lambda} \ln(1 + \lambda\omega) \right]. \quad (18)$$

In the limit $\lambda \to 0$, the prior is the Gaussian distribution

$$\chi(\sigma) = \frac{1}{\sqrt{2\pi\Omega_2}} e^{-\omega^2/(2\Omega_2)}, \quad (19)$$

and we have $Z(\Phi) = e^{\Phi} \Omega_2^{1/2}, \ C(\omega) = -\Omega_2\Phi, \ C(\omega) = -\omega^2/(2\Omega_2)$. The generalized entropy $S = -\frac{1}{2\Omega_2} \int \omega^2 d\mathbf{r}$ associated with a Gaussian prior is proportional (with the
opposite sign) to the coarse-grained enstrophy: \( S = -\Gamma^{\omega}_{-\partial^{2}\omega}/(2\Omega_{2}) \) \cite{9}. This gaussian prior leads to Fononoff flows \cite{13} that have oceanic applications.

When the prior is given by Eq. (14), the generalized entropy satisfies \( C''(\varpi) = 1/(\Omega_{2}(1 + \lambda\varpi)^{2}) \) and we obtain a parametrization of the form

\[
\frac{\partial \varpi}{\partial t} + \mathbf{u} \cdot \nabla \varpi = \nabla \cdot \left\{ D \left( \nabla \varpi + \beta(t)\Omega_{2}(1 + \lambda\varpi)^{2}\nabla \psi \right) \right\},
\]

(20)

\[
\beta(t) = -\int D\nabla \cdot \nabla \psi d\mathbf{r} = \int D\Omega_{2}(1 + \lambda\varpi)^{2}(\nabla \psi)^{2} d\mathbf{r} = K\epsilon^{2}\Omega_{2}^{1/2}\left[ 1 + \lambda\varpi \right].
\]

(21)

For \( \lambda = 0 \) (Gaussian limit), we get

\[
\frac{\partial \varpi}{\partial t} + \mathbf{u} \cdot \nabla \varpi = \nabla \cdot \left\{ D \left( \nabla \varpi + \beta(t)\Omega_{2}\nabla \psi \right) \right\},
\]

(22)

\[
\beta(t) = -\int D\nabla \cdot \nabla \psi d\mathbf{r} = \int D\Omega_{2}(\nabla \psi)^{2} d\mathbf{r} = K\epsilon^{2}\Omega_{2}^{1/2}.
\]

(23)

Since \( D \) and \( \Omega_{2} \) are uniform, we have \( D\varpi/\partial t = D\left( \Delta \varpi - \beta(t)\Omega_{2}\varpi \right) \) with \( \beta(t) = -\Gamma^{\omega}_{-\partial^{2}\omega}(t)/(2\Omega_{2}E) = S(t)/E \) (to arrive at this result, we have used integration by parts in Eq. (23)).

When the prior has two intense peaks \( \chi(\sigma) = \delta(\sigma - \sigma_{0}) + \delta(\sigma - \sigma_{1}) \), the equilibrium coarse-grained vorticity is

\[
\varpi = \sigma_{1} + \frac{\sigma_{0} - \sigma_{1}}{1 + \epsilon(\sigma_{0} - \sigma_{1})(\beta\psi + \alpha)}.
\]

(24)

This is similar to the Fermi-Dirac statistics. Inverting this relation to express \( \Phi = \beta\psi + \alpha \) as a function of \( \varpi \) and integrating the resulting expression, we obtain the generalized entropy

\[
S[\varpi] = -\int [p \ln p + (1 - p) \ln(1 - p)] d\mathbf{r},
\]

(25)

where \( \varpi = p\sigma_{0} + (1 - p)\sigma_{1} \). At equilibrium, we have \( \omega_{2} = 1/C''(\varpi) = (\sigma_{0} - \varpi)/(\varpi - \sigma_{1}) \). For the two-peaks distribution, we get a parametrization of the form

\[
\frac{\partial \varpi}{\partial t} + \mathbf{u} \cdot \nabla \varpi = \nabla \cdot \left\{ D \left( \nabla \varpi + \beta(t)(\sigma_{0} - \varpi)/(\varpi - \sigma_{1})\nabla \psi \right) \right\},
\]

(26)

\[
\beta(t) = -\int D\nabla \cdot \nabla \psi d\mathbf{r} = \int D(\sigma_{0} - \varpi)/(\varpi - \sigma_{1})(\nabla \psi)^{2} d\mathbf{r} = K\epsilon^{2}\omega_{2}^{1/2}.
\]

(27)

These are the same equations as in the MRS theory in the two levels case \( \omega \in \{ \sigma_{0}, \sigma_{1} \} \). They amount to maximizing the Fermi-Dirac-like entropy \cite{20} at fixed circulation and energy. This entropy has been used by Bouclon & Sommeria to model jovian vortices. In the MRS viewpoint, this entropy describes the free merging of a system with two levels of vorticity \( \sigma_{0} \) and \( \sigma_{1} \) while in the viewpoint developed here, it describes the evolution of a forced system where the forcing has two intense peaks described by the prior \( \chi(\sigma) = \delta(\sigma - \sigma_{0}) + \delta(\sigma - \sigma_{1}) \)

(28)

Other examples of prior vorticity distributions and associated generalized entropies are collected in \cite{9}.

V. NONLINEAR DYNAMICAL STABILITY

Let us consider the Casimir functionals \( S[\omega] = -\int C(\omega)d\mathbf{r} \) where \( C \) is any convex function \( (C'' > 0) \). Since \( S, E \) and \( \Gamma \) are individually conserved by the 2D Euler equation, the maximization problem

\[
\max_{\omega} \{ S[\omega] \mid E[\omega] = E, \Gamma[\omega] = \Gamma \},
\]

(28)
determines a steady state of the 2D Euler equation that is formally nonlinearly dynamically stable \cite{1}. Writing the first variations as \( \delta S - \beta E - \alpha \Gamma = 0 \), the steady state is characterized by a monotonic relation \( \omega = F(\beta\psi + \alpha) = f(\psi) \) where \( F(x) = (C')^{-1}(-x) \). Let us introduce the Legendre transform \( J = S - \beta E \) and consider the maximization problem

\[
\max_{\omega} \{ J[\omega] = S[\omega] - \beta E[\omega] \mid \Gamma[\omega] = \Gamma \}.
\]

(29)

If we interpret \( J \) as an energy-Casimir functional, the maximization problem \cite{20} corresponds to the Arnold criterion of formal nonlinear dynamical stability. The variational problems \cite{28} and \cite{29} have the same critical points (cancelling the first variations) but not necessarily the same maxima (regarding the second variations). A solution of \cite{29} is always a solution of the more constrained problem \cite{28}. However, the reciprocal is wrong. A solution of \cite{28} is not necessarily a solution of \cite{29}. The maximization problem \cite{20}, and the associated Arnold theorems, provide just a sufficient condition of nonlinear dynamical stability. The criterion \cite{25} of Ellis-Haven-Turkington is more refined and allows to construct a larger class of nonlinearly stable steady states. For example, important equilibrium states in the weather layer of Jupiter are nonlinearly dynamically stable according to the refined stability criterion \cite{28} while they do not satisfy the Arnold theorems \cite{6}. The maximization problem \cite{20} determines a subclass of solutions of the maximization problem \cite{28}. This is similar to a situation of “ensemble inequivalence” with respect to the conjugate variables \( (E, \beta) \) in thermodynamics \cite{11} \cite{12}. Indeed, \cite{28} is similar to a criterion of “microcanonical stability” while \cite{20} is similar to a criterion of “canonical stability” in thermodynamics, where \( S \) is similar to an entropy and \( J \) is similar to a free energy \cite{9}. Canonical stability implies microcanonical stability but the converse is wrong in case of ensemble inequivalence \cite{18}. Since the relaxation equations \cite{12} \cite{13} solve the maximization problem \cite{28}, they can serve as numerical algorithms to compute nonlinearly dynamically stable stationary solutions of the 2D Euler equation according to the criterion of Ellis-Haven-Turkington. Note that if we fix \( \beta \), the relaxation equation \cite{12} increases monotonically the “free energy” \( J = S - \beta E \) (H-theorem, \( \dot{J} \geq 0 \)) until a (local) maximum of \( J \) at fixed \( \Gamma \) is reached \cite{9}. Therefore, we obtain a numerical algorithm that solves the maximization problem \cite{20} and determines a subclass of nonlinearly dynamically stable stationary solutions of the 2D Euler equation corresponding to the Arnold criterion.
VI. CONCLUSION

In this paper, we have shown that the maximization of a functional \( S[\omega] \) at fixed circulation \( \Gamma \) and energy \( E \) in 2D turbulence can have several interpretations. When \( S \) is given by \( \mathbb{S}_1 \), this maximization problem determines: (i) The whole class of stable EHT statistical equilibria for a given prior vorticity distribution \( \chi(\sigma) \) fixed by the small-scale forcing. (ii) A subclass of stable MRS statistical equilibria for initial conditions leading to a vorticity distribution \( \chi(\sigma) \) at statistical equilibrium. When \( S \) is given by \( \mathbb{S}_2 \) where \( C \) is an arbitrary convex function, this maximization problem determines a non-linearly dynamically stable stationary solution of the 2D Euler equation according to the refined EHT criterion. The next step is to determine whether particular forms of generalized entropies are better adapted than others to describe specific flows and whether they can be re-grouped in “classes of equivalence” [3]. For example, the entrophy functional turns out to be relevant for certain oceanic situations [15] and the Fermi-Dirac-like entropy for jovian flows [16]. Working with a suitable generalized entropy \( S[\omega] \) with only two constraints (\( \Gamma, E \)) is more convenient than working with an infinite set of Casimirs. This reduced maximization problem is still very rich because, for any considered form of generalized entropy \( S[\omega] \), many bifurcations can take place in the parameter space \( (E, \Gamma) \) [3, 6, 16].

APPENDIX A: GENERALIZED ENTROPY

We can introduce the generalized entropy \( S[\varpi] \) in the following manner. Initially, we want to determine the vorticity distribution \( \rho_1(\mathbf{r}, \sigma) \) which maximizes \( S_\chi[\rho] \) with the robust constraints \( E[\varpi] = E, \Gamma[\varpi] = \Gamma \), and the normalization condition \( \int \rho \, d\sigma = 1 \). To solve this maximization problem, we can proceed in two steps. First step: we determine the distribution \( \rho_1(\mathbf{r}, \sigma) \) which maximizes \( S_\chi[\rho] \) with the constraints \( \int \rho \, d\sigma = 1 \) and a fixed vorticity profile \( \int \rho \, d\sigma = \varpi(\mathbf{r}) \) (note that fixing \( \varpi \) automatically determines \( \Gamma \) and \( E \)). This gives a distribution \( \rho_1[\varpi(\mathbf{r}), \sigma] \) depending on \( \varpi(\mathbf{r}) \) and \( \sigma \). Substituting this distribution in the functional \( S_\chi[\rho] \), we obtain a functional \( S[\varpi] = S_\chi[\rho_1] \) of the vorticity \( \varpi \). Second step: we determine the vorticity field \( \varpi(\mathbf{r}) \) which maximizes \( S[\varpi] \) with the constraints \( E[\varpi] = E \) and \( \Gamma[\varpi] = \Gamma \). Finally, we have \( \rho_2(\mathbf{r}, \sigma) = \rho_1[\varpi(\mathbf{r}), \sigma] \). Let us be more explicit. The distribution \( \rho_1(\mathbf{r}, \sigma) \) that extremizes \( S_\chi[\rho] \) with the constraints \( \int \rho \, d\sigma = 1 \) and \( \int \rho \, d\sigma = \varpi(\mathbf{r}) \) satisfies the first order variations \( \delta S_\chi - \int \Phi(\mathbf{r}) \delta (\int \rho \, d\sigma) \, d\mathbf{r} - \int \zeta(\mathbf{r}) \delta (\int \rho \, d\sigma) \, d\mathbf{r} = 0 \), where \( \Phi(\mathbf{r}) \) and \( \zeta(\mathbf{r}) \) are Lagrange multipliers. This yields

\[
\rho_1(\mathbf{r}, \sigma) = \frac{1}{Z(\mathbf{r})} \chi(\sigma) e^{-\sigma \varpi(\mathbf{r})},
\]

where \( Z(\mathbf{r}) \) and \( \Phi(\mathbf{r}) \) are determined by

\[
Z(\mathbf{r}) = \int \chi(\sigma) e^{-\sigma \varpi(\mathbf{r})} \, d\sigma = \hat{\chi}(\Phi) \quad \text{and} \quad \varpi(\mathbf{r}) = \frac{1}{Z(\mathbf{r})} \int \chi(\sigma) e^{-\sigma \varpi(\mathbf{r})} \, d\sigma = -\langle \ln \hat{\chi}(\Phi) \rangle.
\]

This critical point is a maximum of \( S_\chi \) with the above-mentioned constraints since \( \delta^2 S_\chi = -\int \frac{\partial^2 Z}{\partial \varpi^2} \, d\varpi < 0 \). Then \( S_\chi[\rho] = \int \rho (\sigma \varpi + \ln \hat{\chi}(\Phi)) \, d\sigma = \int (\tilde{\varpi} \Phi + \ln \hat{\chi}(\Phi)) \, d\mathbf{r} \). Therefore \( S[\varpi] = S_\chi[\rho] \) is given by \( S[\varpi] = -\int C(\varpi) \, d\mathbf{r} \) with \( C(\varpi) = -\tilde{\varpi} \Phi - \ln \hat{\chi}(\Phi) \). Now, \( \Phi(\mathbf{r}) \) is related to \( \varpi(\mathbf{r}) \) by \( \varpi(\mathbf{r}) = -\langle \ln \hat{\chi}(\Phi) \rangle \). This implies that \( C'(\varpi) = -\Phi = -\langle \ln \hat{\chi} \rangle^{-1}(\varpi) \) so that

\[
C(\varpi) = -\int \langle \ln \hat{\chi} \rangle^{-1}(-x) \, dx.
\]

This is precisely the generalized entropy [10]. Therefore, \( \rho_2(\mathbf{r}, \sigma) = \rho_1[\varpi(\mathbf{r}), \sigma] \) is a maximum of \( S_\chi[\rho] \) at fixed \( E \) and \( \Gamma \) iff \( \varpi(\mathbf{r}) \) is a maximum of \( S[\varpi] \) at fixed \( E \) and \( \Gamma \).

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[17] Some attempts have been proposed to go beyond the assumptions of the statistical theory. For example, Chavanis & Sommeria [3] consider a strong mixing limit in which only the first moments of the vorticity are relevant instead of the whole set of Casimirs. They also introduce the concept of maximum entropy bubbles (or restricted equilibrium states) in order to account for situations where the evolution of the flow is not ergodic in the whole available domain but only in a subdomain.
[18] Since the EHT statistical equilibria (with a given prior) satisfy a maximization problem of the form \( 28 \) with \( C(\varpi) \) given by Eq. \( 10 \), they are both thermodynamically stable (with respect to fine grained perturbations \( \delta(\rho, \sigma) \)) and formally nonlinearly dynamically stable (with respect to coarse-grained perturbations \( \varpi(\mathbf{r}) \)). Note that the MRS statistical equilibrium may not satisfy the nonlinear dynamical stability criterion \( 28 \) according to the discussion at the end of Sec. 11. This intriguing observation demands further investigation.