Classical-quantum correspondence for shape-invariant systems

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Abstract
A quantization procedure, which has recently been introduced for the analysis
of Painlevé equations, is applied to a general time-independent potential of a
Newton equation. This analysis shows that the quantization procedure pre-
serves the exact solvability property for the class of shape-invariant potentials.
When a general potential is considered the quantization procedure involves
the solution of a Gambier XXVII transcendental equation. Explicit examples
involving classical and exceptional orthogonal Laguerre and Jacobi poly-
nomials are discussed.

Keywords: shape-invariant potentials, exceptional orthogonal polynomials,
Painlevé equations in Calogero form, integrable systems

1. Introduction
The linearization of an integrable system nonlinear partial differential equations (PDEs) as the
compatibility condition of an overdetermined system (i.e. the zero curvature condition (ZCC))
of the linear differential equations
\begin{align}
\begin{cases}
\partial_x \Phi &= U \Phi, \\
\partial_t \Phi &= V \Phi,
\end{cases}
\rightarrow \partial_x U - \partial_t V + [U, V] \equiv (x, t) = 0,
\end{align}

is a well-known technique which, over the years, has allowed for a systematic investigation of
many important integrable nonlinear PDEs such as the KdV, nonlinear Schroedinger and sine-
Gordon equations (see e.g. [1, 2]). This technique can also be used to study nonlinear ordinary
differential equations (ODEs). In this case, the second independent variable, say the $x$ of (1), is

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replaced by the spectral parameter $\lambda$. A remarkable example in this class of nonlinear ODEs is represented by the Painlevé equations [3]. Painlevé equations arise in many contexts and they can also be defined as particular reductions of some integrable PDEs solvable by the inverse scattering transform (see e.g. [4]–[7]). The idea of regarding each Painlevé equation as the compatibility condition of a set of linear differential equations goes back to the work of R Fuchs [8]. In some recent papers by Suleimanov [9] and by Zabrodin and Zotov [10–12] such a mechanism has been used to define the Painlevé equations in the so-called ‘Calogero form’. The study of Painlevé equations as a Hamiltonian system (also known as ‘Calogero form’) has a long history (for some relevant references see e.g. [13, 14] and references therein). In particular, in [15] it is shown that for all Painlevé equations $P(\ddot{y}, \dot{y}, y, t) = 0$ it is possible to find a transformation $(y, T)\rightarrow(\dot{u}(u, t), \quad \ddot{y} = y(u, t), \quad T = T(t)$ which maps the Painlevé equation to the Newton differential equation
\[
\dddot{u} = -\partial_u V(u, t),
\] (where the dot denotes the derivative with respect to $t$). Equation (2) can be regarded as the equation of motion for a time-dependent Hamiltonian system
\[
H(p, u, t) = \frac{p^2}{2} + V(u, t).
\] (3)
The main result of Zabrodin and Zotov [10] is the fact that equation (2) can be regarded as the compatibility condition of a linear spectral problem (LSP) which turns out to be mathematically equivalent to a time-dependent Schroedinger equation
\[
-i\partial_t \psi(x, t) = -\frac{1}{2}\partial_x^2 \psi + V_q(x, t)\psi,
\] (4)
where $x$ plays the role of the spectral parameter and the quantum potential $V_q(\cdot, t)$ turns out to be identical to the potential $V_q(\cdot, t)$ up to some renormalization in the parameters contained in $V_q(\cdot, t)$. For this reason, we will refer to equation (4) as the quantization of equation (3) in the sense of the classical-quantum correspondence as introduced by Suleimanov, Zabrodin and Zotov (SZZ) (see [9, 10]). For this paper to be self-contained, let us explicitly recall the notion of the classical-quantum correspondence for the Painlevé equations PIV and PV.

We first consider the LSP for Painlevé equation PIV [10]

\[
\begin{align*}
\partial_x \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &= \begin{pmatrix}
\frac{x^3}{2} + tx + \frac{Q + 1}{x} \\
\frac{Q^2 + \beta}{u^2x^3} - Q - \alpha - 1 - \frac{x^3}{2} - tx - \frac{Q + 1}{x}
\end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \\
\partial_t \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &= \begin{pmatrix}
\frac{x^2}{2} + \frac{u^2}{x} + t \\
\frac{-Q + a + 1}{x} - \frac{x^2}{2} + \frac{u^2}{x} - t
\end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},
\end{align*}
\] (5)

where
\[
Q = uu_t - \frac{u^4}{2} - u^2.
\]
The compatibility condition associated with the LSP (5) corresponds to the Painlevé PIV equation in the Calogero form [11]
Moreover, from the LSP (5) we observe that the function \( \psi = e^{\int (\Delta _{PV}^U) \phi_1} \) satisfies the following non-stationary ‘real’ Schrödinger equation (4)

\[
\partial_t \psi = \frac{1}{2} \partial_x^2 \psi + \left( -\frac{x^6}{8} - \frac{tx^4}{2} - \frac{1}{2} (t^2 - a)x^2 + \frac{\beta}{2} + \frac{1}{2} \right) \psi,
\]

where the potential in the Schrödinger equation (8) turns out to be the same as that of the classical equation (6) up to a shift in the parameter \( \beta \). An analogous analysis can be performed for the Painlevé equation PV.

Let us introduce the following LSP

\[
\partial_t \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right) = U \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right), \quad \partial_t \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right) = V \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right),
\]

where the entries of \( U, V \) are given by

\[
U_{11} = \frac{u^2 \sinh 2u}{\sinh 2x} - \frac{2\sigma}{\sinh 2x} (\cosh 2x - \cosh 2u)
+ \frac{e^{2\delta}}{4 \sinh 2x} (\cosh 4x - \cosh 4u) + \coth(2x),
\]

\[
U_{12} = e^\delta (\cosh 2x - \cosh 2u),
\]

\[
U_{21} = u^2 \frac{e^{-\delta}}{\sinh^2 2x} (\cosh 2u + \cosh 2x)
+ \frac{u}{\sinh^2 2x} (4\sigma e^{-\delta} - e^\delta (\cosh 2u + \cosh 2x))
+ 8\sigma^2 e^{-\delta} \coth^2 u \left( \frac{\sinh^2 u - \cosh^2 x}{\sinh^2 2x} \right) - 2\sigma e^\delta \frac{\sinh^2 2u}{\sinh^2 2x}
- 2e^{-\delta} \frac{e^{2\delta}}{\sinh^2 2x} \sinh^2 u \sinh^2 x
+ \frac{e^{3\delta} \sinh^2 2u}{4 \sinh^2 2x} (\cosh 2u + \cosh 2x),
\]

\[
U_{22} = -U_{11},
\]

and

\[
V_{11} = \frac{1}{2} e^{2\delta} (\cosh 2x + \cosh 2u) - 2\sigma + \frac{1}{2},
\]

\[
V_{12} = e^\delta \sinh 2x,
\]
\[
V_{21} = \frac{e^{-t}}{\sinh 2x} \left( \left( \dot{u} - \frac{e^{2i} \sinh 2u}{2} \right)^2 + \frac{4\xi^2}{\cosh^2 u} - \frac{4\xi^2 + 8\sigma^2}{\sinh^2 u} - 4\sigma^2 \coth^2 u \right),
\]
\[
V_{22} = -V_{11},
\]
where \(\sigma, \xi, \zeta\) are free parameters. The compatibility condition reduces to the following Newton equation for the Painlevé equation PV
\[
\Delta_{PV} = \ddot{u} + \dot{u} V_\xi = 0,
\]
\[
V_\xi = \frac{4\xi^2}{2 \cosh^2 u} - \frac{4(\xi + \sigma)^2}{2 \sinh^2 u} - \frac{e^{4u}}{16} \cosh 4u + \left( \sigma - \frac{1}{2} \right)e^{2u} \cosh 2u.
\]
The function \(\phi_1\) satisfies a ‘real’ Schrödinger equation for the function
\[
\psi = \phi_1 e^{\int \left( \frac{\dot{\phi}_1}{\phi_1} \right) du},
\]
\[
\partial_{\psi}^2 \psi = \frac{1}{2} \partial_x^2 \psi + \left( \frac{4\xi^2 - \frac{1}{4}}{2 \cosh^2 x} - \frac{4(\xi + \sigma)^2 - \frac{1}{4}}{2 \sinh^2 x} - \frac{e^{4u}}{16} \cosh 4x + \left( \sigma - \frac{1}{2} \right)e^{2u} \cosh 2x \right) \psi,
\]
which corresponds to the quantization of equation (18) up to a redefinition of the parameters \(\zeta, \xi, \text{and } \sigma\).

We remark that both the Painlevé PIV and Painlevé PV equations can be regarded as time-dependent integrable deformations for the potential of a harmonic oscillator with centrifugal barrier and the Poschl–Teller potential, respectively
\[
V_{\text{PIV}} \rightarrow \frac{1}{2} \alpha x^2 + \frac{\beta}{x^2},
\]
\[
V_{\text{PV}} \rightarrow \frac{4\xi^2}{2 \cosh^2 x} - \frac{4(\xi + \sigma)^2}{2 \sinh^2 x}.
\]
These potentials (22) and (23) are well-known for being ‘shape-invariant’. Shape- invariant potentials were implicitly introduced by Schrödinger in [17] and then generalized by Infeld and Hull in [18] as a mechanism to solve algebraically the bounded spectrum of a quantum mechanical system (for a more recent review on the topic see e.g. [19]).

Let us briefly recall the definition of a shape-invariant potential as a potential whose Hamiltonian operator can be factorized through two ladder operators \(a_\lambda, a_\lambda^\dagger\)
\[
\hat{H} = a_\lambda^\dagger a_\lambda = -\partial_x^2 + \hat{V}_x(x),
\]
\[
a_\lambda = -i\partial_x + i\hat{W}_x(x),
\]
\[
a_\lambda^\dagger = -i\partial_x - i\hat{W}_x^\dagger(x),
\]
having the following property
\[
a_\lambda a_\lambda^\dagger = a_{\lambda + \delta}^\dagger a_{\lambda + \delta} + \text{const},
\]
where \(\lambda, \delta\) are in general parameter vectors. It is straightforward to verify by direct computation that property (27) holds for the potentials (22) and (23), if the ladder operators
take the form
\[ a_l = -i\partial_x + i\left(-ox + \frac{l}{x}\right), \]  
(28)
\[ a_l^\dagger a_l = -\partial^2_x + \frac{l(l-1)}{x^2} + \omega^2 x^2 - 2\omega l - \omega, \]  
(29)
\[ a_l a_l^\dagger = -\partial^2_x + \frac{l(l+1)}{x^2} + \omega^2 x^2 - 2\omega l + \omega, \]  
(30)
and
\[ a_{l,g} = -i\partial_x + i(g \coth x + l \tanh x), \]  
(31)
\[ a_{l,g}^\dagger a_{l,g} = -\partial^2_x + \frac{g(g-1)}{\sinh^2 x} - \frac{l(l-1)}{\cosh^2 x} + (g + l)^2, \]  
(32)
\[ a_{l,g} a_{l,g}^\dagger = -\partial^2_x + \frac{g(g+1)}{\sinh^2 x} - \frac{l(l+1)}{\cosh^2 x} + (g + l)^2. \]  
(33)

On the basis of the above considerations the principal objective of this paper is to show that whenever the potential \( V_u(t, x) \) of the Newton equation (2) does not depend explicitly on time it is possible to define a LSP whose compatibility condition involves the solution of the equation (2) and of a nonlinear differential equation which can be reduced (under some specific assumptions) to the Gambier equation XXVII (GXXVII). In particular we will show that the exact solvability of GXXVII is connected with the exact solvability of the ‘quantization’ of (2). In fact it turns out that if the potential \( V \) has the shape-invariant property then it is possible to provide an exact solution of GXXVII in terms of orthogonal polynomials either classical or exceptional.

The present paper is organized as follows. In section 2, we recall the basic concepts necessary for an understanding of the quantization in the sense of the SZZ. In particular we will provide the master equation which allows us to connect any Newton equation (2) to its corresponding Schroedinger equation (4). Particular solutions will be provided in sections 2.1 and 2.2 for quantum Schroedinger equations characterized by potentials which are shape-invariant. In section 3 we will discuss in detail the classical quantum correspondence for the harmonic oscillator system with centrifugal barrier providing the exact solution of the LSP in terms of exceptional orthogonal Laguerre and Jacobi polynomials. Section 4 contains final remarks and possible future developments.

2. LSP and non-stationary Schroedinger equation

We start by considering a completely general LSP defined by two potential matrices \( \hat{U}, \hat{V} \in \mathfrak{sl}(2) \).
\[
\begin{align*}
\partial_x \Phi & = \hat{U} \Phi, \\
\partial_t \Phi & = \hat{V} \Phi,
\end{align*}
\]  
(34)
where \( \hat{U}, \hat{V} \) are given by the traceless matrices
\[
\hat{U} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad \hat{V} = \begin{pmatrix} \hat{A} & B \\ C & -\hat{A} \end{pmatrix}.
\]  
(35)
We reduce the number of undetermined functions in the entries of matrices $\tilde{U}$ and $\tilde{V}$ by considering the following gauge transformation

$$\tilde{\Phi} = T\Phi, \quad T = \begin{pmatrix} 1 & 0 \\ -a/b & 1 \end{pmatrix},$$

(36)

$$U = T^{-1}\tilde{U}T - T^{-1}\partial_x T = \begin{pmatrix} 0 & b \\ \alpha & 0 \end{pmatrix},$$

(37)

$$V = T^{-1}\tilde{V}T - T^{-1}\partial_T T = \begin{pmatrix} A & B \\ \beta & -A \end{pmatrix},$$

(38)

where $A$, $\alpha$ and $\beta$ are given by

$$A = \tilde{A} - \frac{AB}{b},$$

(39)

$$\alpha = \frac{1}{b} \left( -\text{det}(\tilde{U}) + a_x \right) - \frac{ab_x}{b^2},$$

(40)

$$\beta = \frac{a^2B}{b^2} + \frac{bC + a_i}{b} + \frac{a}{b^2}(2Ab - b_i).$$

(41)

The LSP

$$\left\{ \begin{array}{l}
\partial_x \Phi = U\Phi, \\
\partial_T \Phi = V\Phi,
\end{array} \right. \quad \Phi = \left( \phi_{ij} \right), \quad i, j = \{1, 2\}$$

(42)

can be rewritten as follows

$$\phi_{1,ij} = b\phi_{2,i} \rightarrow \phi_{1,ij} = b_x \phi_{2,i} + b\phi_{2,j},$$

(43)

$$\phi_{2,ij} = a\phi_{1,i},$$

(44)

where $\alpha$, $\beta$ are functions to be determined. The equation (43) defines a Sturm–Liouville problem if we replace

$$\phi_{2i} = \frac{\phi_{1i}}{b},$$

(45)

$$\phi_{2,i} = a\phi_{1i}.$$  

(46)

In addition, the entry $\phi_{1,i}$ has to satisfy another linear PDE which can be defined using the second equation in the LSP (42)

$$\partial_x \Phi = V\Phi \rightarrow \phi_{1,ij} = A\phi_{1,i} + B\phi_{2,i} \rightarrow \phi_{2,i} = \frac{1}{B} \phi_{1,ij} - \frac{A}{B} \phi_{1,i}.$$ 

(47)

Replacing (47) in (43) we obtain

$$\phi_{1,ij} = \frac{b_x}{B} \phi_{1,i} + \left( b\alpha - \frac{b_xA}{B} \right) \phi_{1,i}.$$ 

(48)
This equation turns into the Schrödinger equation (4) if we set

\[
\begin{align*}
B &= \frac{ib_x}{2}, \\
\alpha &= \frac{2}{b} \left( V_q(x) - iA \right),
\end{align*}
\]  

(49)

\[i\Phi_{i,i} = -\frac{1}{2} \Phi_{i,xxx} + V_q(x) \Phi_{i,i}.
\]  

(50)

The final step consists in making the system (42) a compatible system, namely to impose the ZCC.

\[ZCC \equiv \partial_t U - \partial_x V + [U, V] = 0.
\]  

(51)

From the components of $ZCC_{1,2}$ and $ZCC_{1,1} = ZCC_{2,2}$ we can determine the functions $A$ and $\beta$

\[
\begin{align*}
A &= \frac{1}{2b} \left( b_t - \frac{i}{2} b_{xx} \right), \\
\beta &= \frac{1}{b} \left( A_t + \frac{i}{2} \alpha b_x \right).
\end{align*}
\]  

(52)

The last condition $ZCC_{2,1} = 0$ can be used to fix the function $b(x, t)$. However, since the goal is to connect the quantum linear problem to classical mechanics we require that the function $b(x, t)$ be dependent on time through the time-dependent variable $u(t)$, $b = b(x, u(t))$. Moreover we set the function $u(t)$ such that it satisfies the conservation of energy for a classical system

\[\frac{\dot{u}^2}{2} + V_c(u(t)) = 0.
\]  

(53)

With this assumption $ZCC_{2,1} = 0$ turns into the following PDE

\[
4i\dot{u} \left( bb_{xxx} - b_x b_{xx} \right) - 8V_q b_x^2 + 8V_q b_x^2 + b_x^2 - 2b_x b_{xxx}
\]
\[+ b \left( 4V_q b_{xx} + 8V_{qq} b_{xx} - 4V_q b_{xx} - 8V_q b_{xxx} + b_{xxxx} \right) = 0,
\]  

(54)

where we have denoted $V_c' \equiv V_{c,uu}$ and $V_q' \equiv V_{qq,xx}$. Considering our purposes we can simplify the nonlinear PDE (54) with non-constant coefficients to two nonlinear ODEs by assuming the following ansatz on the function $b(x, u(t))$

\[b = b_1(x) - b_2(u(t)).
\]  

(55)

In particular equation (54) turns out to be satisfied whenever the two ODEs

\[V_c b_{22}^2 = k_1 b_2^2 + k_2 b_2 + k_3, \quad k_1, k_2, k_3 \in \mathbb{R},
\]  

(56)

\[V_q b_{12}^2 = k_1 b_2^2 + k_2 b_1 + k_3 + \frac{b_{12} b_{1xxx}}{4} - \frac{b_{12}^2}{8},
\]  

(57)

hold. The function $b_2$ can easily be determined for any potential $V_c$ by integrating the first-order elliptic ODE (56).
On the other hand we have to solve a nonlinear third-order ODE (57) in order to obtain the function \( b_1(x) \) which produces a given quantum potential \( V_q(x) \). If we set the free parameters \( k_2 = k_3 = 0 \) we can reduce (57) to the second-order ODE

\[
b_1(x) = e^{\int f(x) dx} \to f_{xx} + 4k_1 - 4V_q(x)f^2 + 2f^2f_x - \frac{f^2}{2} + \frac{f^4}{4} = 0. \tag{59}
\]

The equation (59) turns out to be the Gambier equation GXXVII ([16]). Such an equation can be linearized to a linear 4th-order ODE. However, it is possible to express (57) through a system of two Sturm–Liouville problems for the functions \( \rho \) and \( b_1 \)

\[
\begin{cases}
\rho_{xx} = \left( 2V_q - e^{\sqrt{2k_1}} \right) \rho, \\
b_{1xx} = 2\rho b_1 + 2e^{\sqrt{2k_1}} b_1, \quad \epsilon^2 = 1,
\end{cases} \tag{60}
\]

which is equivalent to the 4th order ODE

\[
V_q b_{1x} + 2V_q b_{1xx} = 2k_1 b_1 + \frac{b_{1xxxx}}{4}. \tag{61}
\]

The parameter \( k_2 \) can be introduced with the shift \( b \to b + \frac{k_2}{k_1} \) and \( k_3 \) is a constant of integration.

In particular if we assume that the solution for \( \rho \) in (60) can be expressed as \( \rho = e^{W(x)} \) with prepotential \( W(x) \), then the quantum potential \( V_q \) can be expressed as \( 2V_q = e^{\sqrt{2k_1}} + W_{xx} + W_x^2 \). Under these assumptions we can recast the Sturm–Liouville problem involving \( b_1 \) as a Schroedinger equation by introducing the following gauge transformation \( b_1 = e^W \psi \)

\[
\psi_{xx} = \left( W_x^2 - W_{xx} + 2e^{\sqrt{2k_1}} \right) \psi. \tag{62}
\]

The solution of (57) reduces to the solution of a couple of Schroedinger equations (60) involving \( V_q \) as a potential. In order to provide some explicit example, let us consider the potential \( V_q \) characterized by the shape-invariant property.

\[
\begin{cases}
a = -\partial_x + W_x, \\
a^\dagger = \partial_x + W_x,
\end{cases} \tag{63}
\]

such that

\[
W_x^2 + W_{xx} = 2V_q - e^{\sqrt{2k_1}}. \tag{64}
\]

The system (60) turns into the Schroedinger equations

\[
\begin{cases}
a^\dagger a \rho = 0, \\
\left( aa^\dagger + 2e^{\sqrt{2k_1}} \right) \psi = 0.
\end{cases} \tag{65}
\]

It is straightforward to verify that, given a basis of eigenfunctions for the operator \( a^\dagger a \phi = \lambda \phi \), we obtain
where $k_1$ depends on the eigenvalue $\lambda$ of the equation (62).

Now we present some examples in order to illustrate the above theoretical considerations.

2.1. The harmonic oscillator potential

As a simple example we consider the harmonic oscillator potential

$$2V_q + \sqrt{2k_1} = \omega^2 x^2 + \frac{l(l-1)}{x^2} + \omega(2l + 1 + 4N),$$

(66)

Such a quantum potential $V_q$ can be obtained by setting the prepotential $W$ as follows

$$W = \frac{\omega}{2} x^2 + l \ln(x) + \ln\left(L_N^{l+\frac{1}{2}}\left(-\omega x^2\right)\right),$$

(67)

where $L_N^l$ are the Laguerre polynomials. The potential $V_q$ in (66) doesn’t depend on the sign of $\omega$. We assume from now on that $\omega$ is a positive real number in order to avoid singularities in the Schroedinger equations (65). Under these assumptions it is easy to determine the function $\psi$ from the eigenfunctions of the Harmonic oscillator

$$\psi_{n,l} = e^{-\frac{\omega^2}{4} x^2} L_n^{l+\frac{1}{2}}(\omega x^2),$$

(68)

which satisfy the equation

$$a^+a \phi_{n,l} = \omega(4l + 2 + 4N + 4n)\phi_{n,l}.$$  (69)

From equation (69) we obtain that $\psi = a\phi_{n,l}$ satisfies

$$\left(aa^+ - 2\sqrt{2k_1}\right)\psi = 0, \quad k_1 = \frac{\omega^2}{2}(2l + 1 + 2N + 2n)^2,$$

(70)

and the quantum potential $V_q$ turns out to be

$$V_q = \frac{\omega^2}{2} x^2 + \frac{l(l-1)}{2x^2} + \omega(N - n).$$

(71)

Also, the function $b_1$ takes the form

$$b_1 = e^W\psi = 2\omega x^{2l+1}P(N, n, l, \omega x^2),$$

(72)

$$P(N, n, l, \omega x^2) = L_{N}^{l+\frac{1}{2}}\left(-\omega x^2\right) L_{n}^{l+\frac{1}{2}}\left(-\omega x^2\right) - L_{N-1}^{l+\frac{1}{2}}\left(-\omega x^2\right) L_{n-1}^{l+\frac{1}{2}}\left(-\omega x^2\right),$$

(73)

where $P(N, n, l, \omega x^2)$ are the exceptional Laguerre orthogonal polynomials. Exceptional orthogonal polynomials have been introduced quite recently by Gomez-Ullate et al in [20].

The introduction of these new orthogonal polynomials led to new families of shape-invariant potentials (see e.g. the papers of Quesne [21], Odake and Sasaki [22]).

2.2. The Poschl–Teller potential

Another interesting example is provided by the solution of the ODE (57) when the potential $V_q$ is given by the Poschl–Teller potential
2V_q + \sqrt{2K_1} = \frac{(g + N)(g + N + 1)}{\sin^2 x} + \frac{(h + N - 1)(h + N - 2)}{\cos^2 x} - (2N + h - g - 1)^2.

(74)

In this case the prepotential $W$ takes the form

$$W = -(g + N)\ln(\sin(x)) + (h + N - 1)\ln(\cos(x)) + \ln\left(P_{N}^{g-N-\frac{1}{2},h+N-\frac{1}{2}}(\cos(2x))\right).$$

(75)

where $P_{N}^{g,h}$ are the Jacobi polynomials. We introduce the following set of eigenfunctions for the Hamiltonian operator defined by $a^\dagger a$

$$\phi_{n,g+N,h}(x) = \sin^{g+N+1}(x)\cos^{h+N-1}(x)P_{n}^{g+N+\frac{1}{2},h+N-\frac{1}{2}}(\cos(2x)),

(76)

which satisfies the following equation

$$a^\dagger a\phi_{n,g+N,h} = (2n + 1 + 2g)(4N + 2n + 2h - 1)\phi_{n,g+N,h}.

(77)

From (76) we obtain the wavefunction in the form $\psi = a\phi_{n,g+N,h}$ which satisfies

$$(aa^\dagger - 2\sqrt{2K_1})\psi = 0; \ k_1 = \frac{1}{8}(2n + 1 + 2g)^2(2n + 2h + 4N - 1)^2,$$

(78)

and the potential $V_q$ turns out to be

$$V_q = \frac{(g + N)(g + N + 1)}{2 \sin^2 x} + \frac{(h + N - 1)(h + N - 2)}{2 \cos^2 x} - \frac{(2N + h - g - 1)^2 + (2n + g + h + 2N)^2}{4}.

(79)

As previously noted the function $b_1$ turns out to be $b_1 = \psi e^W$. In this case the function $b$ given by (55), can be expressed in terms of exceptional orthogonal polynomials

$$e^W\psi = -(2n + 1 + 2g)\cos(x)^2 2^{N-1}P_{e_{N}^{g,h}}(\cos(2x)),

(80)

where the $P_{e_{N}^{g,h}}(\cos(2x))$ are the exceptional Jacobi orthogonal polynomials [22]

$$P_{e_{N}^{g,h}}(\cos(2x)) = a_{e_{N}^{g,h}}(\cos(2x))P_{n}^{g+N-\frac{1}{2},h+N-\frac{1}{2}}(\cos(2x)) + b_{e_{N}^{g,h}}(\cos(2x))P_{n-1}^{g+N-\frac{1}{2},h+N-\frac{1}{2}}(\cos(2x)),$$

$$a_{e_{N}^{g,h}}(x) = P_{N}^{g-N-\frac{1}{2},h+N-\frac{1}{2}}(x) +$$

$$\times\frac{2n(h + N - g - 1)P_{N-1}^{g-N+\frac{1}{2},h+N-\frac{1}{2}}(x)}{(h + 2N - 2 - g)(g + h + 2n + 2N - 1)} -$$

$$\frac{n(2h + 4N - 3)P_{N-2}^{g-N+\frac{1}{2},h+N-\frac{1}{2}}(x)}{(2g + 2n + 1)(h + 2N - g - 2)},$$

$$b_{e_{N}^{g,h}}(x) = \frac{(h + N - g - 1)(2g + 2n + 2N - 1)}{(2g + 2n + 1)(g + h + 2n + 2N - 1)} \times P_{N-1}^{g-N+\frac{1}{2},h+N-\frac{1}{2}}(x).$$

(81)
2.3. The stationary hydrogen atom

Finally, we consider the case of the hydrogen atom. Let us consider as prepotential the following

\[
W = -\frac{\mu}{2(N-1)} x - l \ln x + \ln \left( L_N^{2l+1}(\frac{\mu x}{N-1}) \right),
\]

(82)

where \( L_N^l(x) \) are the Laguerre polynomials. We replace this prepotential \( W \) in equations (63) and (65)

\[
\begin{align*}
\alpha &= -\partial_x + W, \\
\alpha^\dagger &= \partial_x + W,
\end{align*}
\]

(83)

such that equation (64) takes the form

\[
W_x^2 + W_{xx} = \sqrt{2k} + 2V_q = \frac{\mu^2}{4(l-N)^2} - \frac{\mu}{x} + \frac{l(l+1)}{x^2},
\]

(84)

where the wavefunction \( \psi \) and the parameter \( k_1 \) are given by

\[
\psi = a \left( e^{-\frac{\mu}{2(l-N)^2}} L_N^{2l+1}(\frac{\mu x}{l-N}) \right), \quad k_1 = \frac{2\mu^2(N^2 - (n + 1)^2 - 2l(N + n + 1))^2}{4^2(l-N)^4(l+n+1)^2}.
\]

(85)

In particular if we set \( n = N = 0 \) we obtain an explicit form for the function \( b_1 \)

\[
b_1 = e^W \psi = e^{-\frac{\mu}{2}} \frac{2l+1}{2l} \left( 2l - \frac{\mu}{l+1} \right).
\]

(86)

from which we get for the classical potential \( V_c \)

\[
V_c = k_1 \left( b_1 \right)^2 = \frac{(2l+1)^2}{8l(l+1)} \left( \frac{l(l+1)}{u^2} - \frac{\mu}{u} + \frac{\mu^2}{4l(l+1)} \right).
\]

(87)

As already shown for the quantization of (8), (22) and (21), (23) we verify that if we chose \( b_1(\cdot) = b_2(\cdot) \) (which is the choice adopted in [10] in order to have \( b_1(0) = 0 \)) then we obtain a classic potential \( V_c(u) \) whose limit for large \( l \) coincides with \( V_q(u) \). However, we should remark that in this case the energy of the system turns out to be fixed and corresponds to that of a particle moving in a Kepler/Coulomb potential on a circular orbit.

3. The Harmonic oscillator and the exact solution of its LSP

In the previous sections we have shown that any one-dimensional Newton equation (2) can be associated with a LSP which coincides with a time-dependent Schroedinger equation. The aim of this section is to discuss thoroughly an explicit example with the goal of providing the exact solution of the LSP (which in general is a superposition of solutions for the time-independent Schroedinger equation) for any given solution of its classical counterpart (90).
Let us consider the Harmonic oscillator potential obtained from the solution of (56) with $b = x^2$ and the constant $k_1 = 2\omega^2$, $k_2 = -4E$, $k_3 = -2l^2$. This choice produces the following Lax pair (1) with potential matrices of the form

$$U = \begin{pmatrix}
0 & x^2 - u(t)^2 \\
-\frac{l^2 - \frac{1}{4} + \omega^2 x^2 - \frac{l^2}{u(t)^2} - \omega^2 u(t)^2}{x^2 - u(t)^2} & 0
\end{pmatrix}, \quad (88)$$

$$V = \begin{pmatrix}
-i - \frac{2u(t)\dot{u}}{2(x^2 - u(t)^2)} & ix \\
\frac{-4l^2 x^2 + u(t)^2 \left(1 - 4l^2 - 4x^2 \omega^2 + 4x^2 \left(\omega^2 u(t)^2 + \dot{u}(t)^2\right)\right)}{4x\dot{u}(t)\left(x^2 - u(t)^2\right)^2} & \frac{i + 2u(t)\dot{u}}{2(x^2 - u(t)^2)}
\end{pmatrix}. \quad (89)$$

It is possible to verify by direct calculation that the ZCC is equivalent to the Newton equation (2) for a classical particle moving under the oscillator potential plus a centrifugal barrier

$$\partial_t U - \partial_t V + [U, V] = 0 \equiv \ddot{u} - \frac{l^2}{u(t)^3} + \omega^2 u(t) = 0, \quad (90)$$

which is satisfied for the function

$$u(t) = \sqrt{\frac{E}{\omega^2} - \frac{\sqrt{E^2 - l^2 \omega^2}}{\omega^2} \sin(2\omega t)}. \quad (91)$$

The LSP (9) for the Lax pair (88) and (89) reduces to the following system of two linear PDEs

$$i\phi_{t_1} + E\phi_1 = -\frac{1}{2}\phi_{1x} + \left(\frac{l^2 - \frac{1}{4}}{2x^2} + \frac{\omega^2}{2} x^2\right)\phi_1, \quad (92)$$

$$2i\left(x^2 - u(t)^2\right)\phi_{t_1} = \left(1 - 2iu(t)u(t)\right)\phi_1 - 2x\phi_{1x}, \quad (93)$$

$$\phi_2 = \frac{\phi_{1x}}{x^2 - u(t)^2}, \quad (94)$$

where the functions $\phi_1$ and $\phi_2$ are the two components of the wave function vector

$$\Phi = \begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}. \quad (95)$$

Let us expand the function $\phi_1$ as a series of stationary solutions of the Schroedinger equation (92)

$$\phi_1 = \sum_n e_n e^{\frac{-\sqrt{E_n}}{\hbar}} \psi_n^l, \quad e_n = \omega (2n + l + 1) - E, \quad \psi_n^l \equiv e^{\frac{-\pi}{\hbar}x^2} x^{l+\frac{1}{2}} L_n^l \left(\omega x^2\right). \quad (96)$$
Replacing (96) in (93) and taking into account the relations
\[ x\partial_x \chi_n^l = (n + 1)\chi_n^l - \frac{1}{2}\chi_n^{l-1}, \]
\[ -\omega x^2 \chi_n^l = (n + 1)\chi_n^{l+1} - (2n + l + 1)\chi_n^l + (n + l)\chi_{n-1}^l, \]
\[ u^2 \partial_{x^2} e^{-ic\gamma t} = \frac{-iEe_n e^{-ic\gamma t} + e_n \sqrt{E^2 - l^2\omega^2}}{2\omega^2} (e^{-ic\gamma t} + e^{-ic\gamma t}), \]
we arrive at the following set of equations determining the coefficients \(c_n\) of the series (96)
\[ e^{-ic\gamma t} \left( a_n\chi_n^{l+1} + b_n\chi_n^l + \gamma_n\chi_{n-1}^l \right) = 0, \quad \forall n \]
\[ a_n = ic_{n+1} \frac{\sqrt{E^2 - l^2\omega^2}}{\omega} \left( 1 + \frac{E_n}{\omega} \right) + 2c_n(n + 1) \left( \frac{E_n}{\omega} - 1 \right) = 0, \]
\[ b_n = 2c_n \left( 1 - \frac{E_n}{\omega} \right) (2n + l + 1) + \frac{E_n}{\omega^2} = 0, \]
\[ \gamma_n = 2c_n(n + l) \left( \frac{E_n}{\omega} + 1 \right) + ic_{n-1} \frac{\sqrt{E^2 - l^2\omega^2}}{\omega} \left( 1 - \frac{E_n}{\omega} \right) = 0. \]

This system of equations can be solved for \(c_n\) with a series of two terms if we set the classical energy \(E\) to the value \(E = \omega(2n + l + 2)\). With this choice we determine the coefficients \(c_n\) to be
\[ \frac{c_{n+1}}{c_n} = -i\sqrt{n + \frac{1}{l + 1}}, \quad c_i = 0 \quad (i \neq n, n + 1), \quad n \geq 0. \]

Therefore a classical particle moving with energy \(E = \omega(2n + l + 2)\), \(n \geq 0\) is associated to a \(\phi_1\) which is a superposition of two quantum states with energies \(E_1 = \omega(2n + l + 1)\) and \(E_2 = \omega(2n + l + 3)\).

\[ \phi_1 = \frac{1}{\langle \phi_1 | \phi_1 \rangle} \left( e^{-\frac{i\alpha}{\omega}} e^{-\frac{i\omega}{2} x^2} L_{n+1}^l(\omega x^2) - \frac{i\sqrt{n + l + 1}}{\sqrt{n + l} + \frac{1}{l + 1}} e^{-\frac{i\omega}{2} x^2} x^l + \frac{1}{l + 1} \right), \]
which produces a probability \(\langle \phi_1 | \phi_1 \rangle\) which oscillates with frequency \(\omega\).

To conclude let us consider the case of a stationary particle of energy \(E = \omega l\). In this case the solution for \(\phi_1\) turns out to be
\[ \phi_1 = \frac{e^{-i\alpha} e^{-\frac{i\omega}{2} x^2} x^l + \frac{1}{l + 1}}{\langle \phi_1 | \phi_1 \rangle}, \]
whose probability, as expected, turns out to be time-independent.

4. Concluding remarks and future outlook

The main result of the paper is the application of the quantization procedure in the sense of the SZZ [9, 10] to any time-independent potential. We have shown that such a quantization
can be realized up to the solution of the Gambier XXVII equation. In particular it is shown that explicit solutions can always be computed for any shape-invariant potential. Particular examples have been analyzed for classical and exceptional orthogonal Laguerre and Jacobi polynomials. Finally the solution of the LSP associated with the quantization of the Harmonic oscillator is provided explicitly. The classical energy $E = \omega (2n + l + 2)$ turns out to be the mean value of the energy eigenvalues of the two wavefunctions in the series (96) which satisfy the LSP $E_1 = \omega (2n + l + 1) < E < E_2 = \omega (2n + l + 3)$ establishing in this way a new connection between the classical Newton equation (90) and its quantum counterpart (93). There are reasons to expect that this connection can also be found for more general quantization procedures with time-dependent potentials. An analysis of equations (56) and (57) for potentials similar to the one studied in sections 2 and 3 can provide us with an explicit form of the wavefunction satisfying the LSP (42). Since the ODE (57) constitutes a special case for any time-independent potential, it is evident that our approach can be applied to systems which describe much more diverse types of potentials. Another interesting avenue for future research could include the study of soliton surfaces based on the LSP for quantum Hamiltonian systems. These surfaces are directly expressed in terms of the wavefunction satisfying the associated LSP (1) of the considered model (see e.g. [23, 24]). A visual image of such surfaces reflecting the behaviour of solutions can be of interest, providing information about the properties of these surfaces, which otherwise would be hidden in some implicit mathematical expression. These tasks will be undertaken in a future work.

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References

[1] Calogero F and Degasperis A 1982 Spectral transform and solitons: tools to solve and investigate nonlinear evolution equations Studies in Mathematics and its Applications vols 1 and 13 (Amsterdam: North-Holland)
[2] Ablowitz M J and Clarkson P A 1991 Solitons, Nonlinear Evolution Equations and Inverse Scattering (London Mathematical Society vol 149) (Cambridge: Cambridge University Press)
[3] Bobenko A I and Eitner U 2000 Painlevé Equations in the Differential Geometry of Surfaces (Lect. Notes Math. vol 1753) (Berlin: Springer)
[4] Iwasaki K, Kimura H, Shimomura S and Yoshida M 1991 From Gauss to Painlevé a modern theory of special functions Aspects of Mathematics vol E16 (Braunschweig: Vieweg)
[5] Conte R 1999 The Painlevé property. One century later (CRM series in Mathematical Physics) (New York: Springer)
[6] Painlevé P 1900 Mémoire sur les équations différentielles dont l’intégrale général est uniforme Bull. Soc. Math. Phys. France 28 201–6
[7] Painlevé P 1902 Sur les équations différentielles du second ordre et d’ordre supérieur dont l’intégrale générale est uniforme Acta Math. 25 1–85
[8] Fuchs R 1905 Sur quelques équations différentielles linéaires du second ordre C. R. Acad. Sci., Paris 141 555–8
[9] Suleimanov B 2008 Quantization of the second Painlevé equation and the problem of the equivalence of its L–A pairs Theor. Math. Phys. 156 1280–91 (Engl. transl.) Suleimanov B 2008 Teor. Mat. Fys. 156 364–377 (in Russian)
[10] Zabrodin A and Zotov A 2012 Quantum Painlevé–Calogero correspondence J. Math. Phys. 53 073507
[11] Zabrodin A and Zotov A 2012 Quantum Painlevé–Calogero correspondence for Painlevé VI J. Math. Phys. 53 073508
[12] Zabrodin A and Zotov A 2012 Classical-quantum correspondence and functional relations for Painlevé equations arXiv:1212.5813
[13] Malmquist J 1922/23 Sur les équations différentielles du second ordre dont l’intégrale générale a ses points critiques fixes Ark. Mat. Astr Fys. 17 1–89
[14] Okamoto K 1980 Polynomial Hamiltonians associated with Painlevé equations Proc. Japan Acad. A 56 264–8
[15] Levin A and Olshanetsky M 2000 Painlevé–Calogero correspondence, Calogero–Moser–Sutherland models CRM Ser. Math. Phys. (Montreal 1997) (Berlin: Springer)
[16] Gambier B 1910 Sur les équations différentielles du second ordre et du premier degré dont l’intégrale générale est a points critiques fixes Acta Math. 33 1–55
[17] Schroedinger E 1940 A method of determining quantum mechanical eigenvalues and eigenfunctions Proc. R. Ir. Acad. A 46 9–16
[18] Infeld L and Hull T E 1951 The factorization method Rev. Mod. Phys. 23 21–68
[19] Cooper F, Khare A and Sukhatme U 1995 Supersymmetry and quantum mechanics Phys. Rep. 251 267–385
[20] Gomez-Ullate D, Kamran N and Milson R 2009 An extended class of orthogonal polynomials defined by a Sturm–Liouville operator J. Math. Anal. Appl. 359 352–67
[21] Quesne C 2008 Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry J. Phys. A: Math. Theor. 41 392001
[22] Odake S and Sasaki R 2009 Infinitely many shape-invariant potentials and new orthogonal polynomials Phys. Lett. B 679 414–7
[23] Grundland A M, Post S and Riglioni D 2014 Soliton surfaces and generalized symmetries of integrable systems J. Phys. A: Math. Theor. 47 015201
[24] Fokas A S, Gel’fand I M, Finkel F and Liu Q M 2000 A formula for constructing infinitely many surfaces on Lie algebras and integrable equations Sel. Math. New. Ser. 6 347–75