We investigate the superconductivity in the \(-J-U\) model within a slave-spin formalism. We show that the BCS mean-field theory implemented with the slave-spin formalism naturally predicts two distinct gaps which are the pairing gap of the spinons \(\Delta_f\) and the Cooper pairing gap of the electrons \(\Delta_{SC} = Z \Delta_f\), where \(Z\) is the quasiparticle weight. Furthermore, we find that the nature of the superconducting state depends crucially on the interaction providing the pairing mechanism. For spin interactions, the bandwidth of the spinon hopping term is renormalized by \(Z\) but the its pairing term is not. As a result, if \(U\) exceeds the critical value for the Mott insulating state at half-filling, \(Z\) develops a strong doping dependence, leading to a doping-driven crossover from strong to weak pairing states. In the strong pairing state, while \(\Delta_f\) is enhanced as \(x \to 0\), \(\Delta_{SC}\) \(\sim x\) due to the renormalization of \(Z\). In the weak pairing state, \(Z\) does not change with \(x\) significantly. Therefore, \(\Delta_{SC}\) is mainly controlled by \(\Delta_f\), and both of them go to zero at larger doping. The crossover from strong to weak pairing states is well captured by the slave-spin formalism within reasonable range of parameters just at the mean-field level, indicating the slave-spin formalism is a powerful tool to study correlated materials. For charge interactions, we find that the bandwidth of the spinon hopping term and its pairing term are renormalized by \(Z\) and \(Z^2\) respectively. Consequently, both \(\Delta_f\) and \(\Delta_{SC}\) are suppressed at small \(x\) in large \(U\), and no crossover will occur. The implication of our results for the superconducting states in correlated materials will be discussed.

PACS numbers: 74.20.-z, 74.20.Mn, 71.10.Fd
pears as a direct consequence of the crossover. On the other hand, if the pairing is induced by charge interactions, e.g., nearest-neighbor attractive Coulomb interaction, no crossover would occur. All the results are obtained at the mean-field level within reasonable ranges of the model parameters, which demonstrates that the slave spin formalism is a powerful technique to study physics related to transition from strong to weak coupling states within the same framework.

Model and Formalism – We consider a generic single band $t - J - U$ model $H = H_t + H_J + H_U$, where

$$H_t = \sum_{ij,\sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma}, \quad H_U = U \sum_i n_{i\uparrow} n_{i\downarrow},$$

$$H_J = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j. \quad (1)$$

$J/t = 0.4$ is used for every calculation. We employ the $U(1)$ version of the slave spin formalism [10] to treat the Hubbard $U$ term, thus the electron creation operator is represented by

$$c_{i\sigma}^\dagger = S_{i\sigma}^z f_{i\sigma}^\dagger, \quad (2)$$

where $f_{i\sigma}^\dagger$ creates a physical spin $\sigma$ at site $i$ and $S_{i\sigma}^z$ creates a charge $-e$ at site $i$. The constraint to project out the unphysical state is $S_{i\sigma}^z = f_{i\sigma}^\dagger f_{i\sigma} - 1/2$. $H_J$ term is decomposed into the $d$-wave superconducting channel by the same Hubbard-Stratonovich fields proposed by Ubbens and Lee [24]. Because $H_J$ describes interactions between physical spins, the terms generated from $H_J$ will not involve slave spin degrees of freedom at the mean-field level. The resulting mean-field pairing term is

$$H_{J}\text{MF} = -\frac{J'}{2} \sum_{\langle ij \rangle} \Delta_{ij}^{d\downarrow} f_{i\downarrow} f_{j\uparrow} - f_{i\downarrow} f_{j\uparrow} + h.c., \quad (3)$$

where $J' = 3J/4$, and $\Delta_{ij}^{d\downarrow} \equiv \langle f_{i\downarrow} f_{j\uparrow} - f_{i\uparrow} f_{j\downarrow} \rangle$ is the spinon pairing order parameter which has the $d$-wave symmetry of $\Delta_\ell = \Delta_\ell^{d\uparrow\downarrow} = -\Delta_\ell^{d\downarrow\uparrow}$. Next we follow the $U(1)$ slave spin formalism to treat $H_t + H_U$ terms [19], and the procedure is described briefly below. We start by performing the standard Hubbard-Stratonovich transformation to decouple the spinon and the slave spin degrees of freedom so that we can write down their Hamiltonians separately. Next, we assume that the dynamics of the slave spins is the same for each site, thus the mean-field slave-spin Hamiltonian is reduced to a single site problem which can be diagonalized exactly. Finally, we solve the corresponding mean-field equations subject to the constraint which is satisfied on the average at the mean-field level.

The final mean-field Hamiltonians are

$$H^{\text{MF}} = \sum_\hat{k} \left[ \sum_\sigma (Z_\sigma \epsilon(\hat{k}) - \mu) f_{\hat{k}\sigma}^\dagger f_{\hat{k}\sigma}^\dagger + [\Delta(\hat{k}) f_{\hat{k}\uparrow}^\dagger f_{\hat{k}\downarrow} + h.c.],ight.$$

$$
\left. H^{\text{MF}} = \sum_\sigma \left( \epsilon_\sigma(\hat{z}_\sigma)^2 + h.c. \right) + \lambda_\sigma S_{\sigma}^z \right] + U (S_{\uparrow}^z + \frac{1}{2}) (S_{\downarrow}^z + \frac{1}{2}), \quad (4)$$

where $Z_\sigma \equiv \langle |\langle \hat{z}_\sigma \rangle|^2 \rangle$ is the quasiparticle weight, $\epsilon(\hat{k}) = 2t(\cos k_x + \cos k_y)$, $\Delta(\hat{k}) = J' \Delta f(\cos k_x - \cos k_y)$, and $\hat{z}_\sigma = Z_{\sigma} - \sqrt{1 + \delta} / \sqrt{2}$ where $\delta$ is an infinitesimal value to regularize $\langle \hat{z}_\sigma \rangle$ for the case of $\langle S_{\sigma}^z \rangle = 1/2$. $\epsilon_\sigma(\hat{z}_\sigma) = \frac{1}{2} \sum_\hat{k} \epsilon(\hat{k}) (f_{\hat{k}\sigma}^\dagger f_{\hat{k}\sigma})$ is the average kinetic energy of the spinon with spin $\sigma$. The parameters of $\Delta_f, Z_\sigma, \mu$, and $\lambda_\sigma$ are obtained by solving the corresponding mean-field equations subject to the following constraint

$$1 - \frac{x}{\xi} \sum_{\hat{k},\sigma} \langle f_{\hat{k}\sigma}^\dagger f_{\hat{k},\sigma} \rangle = 1 + \sum_\sigma \langle S_{\sigma}^z \rangle, \quad (5)$$

where $x$ is the doping away from the half-filling.

The derivation of the mean-field equations can be found in Supplementary Materials. In the present work, we only focus on spin singlet pairing state, and consequently all the parameters are spin-independent. We will drop the $\sigma$ index from now on.

Once the mean-field equations are solved, we can obtain the electron pairing order parameter, which is also the true superconducting order parameter, by

$$\Delta_{SC} = \langle S_{\uparrow}^z S_{\downarrow}^z \rangle \Delta_f = Z \Delta_f \quad (6)$$

Results – It is instructive to discuss the limits of $U \to 0$ and $U \to \infty$. For $U \to 0$, $Z \approx 1$ and $\Delta_f \approx \Delta_{SC}$. As a result, we recover the standard mean-field results on $t - J$ model without no-double occupation constraint, which only has single gap and the largest gap is found at the half-filling. On the other hand, for $U \to \infty$, $Z$ reduces to the Gutzwiller factor $g_t = 2x/(1 + x)$ and $\Delta_f$ and $\Delta_{SC}$ become two distinct gaps. From $H^{\text{MF}}$ in Eq. 4 because the kinetic energy term is renormalized by $Z$ while the pairing term is not, the spinon pairing $\Delta_f$ is greatly enhanced at small $x$, and the effect of $Z$ becomes insignificant at larger $x$. In contrast, because the electron pairing gap $\Delta_{SC}$ is given by Eq. 6 $\Delta_{SC} \sim x$ at small $x$ and $\sim \Delta_f$ at large $x$. As a result, a crossover from strong to weak pairing states driven by the doping $x$ is expected in the large $U$ limit, which leads to a superconducting dome naturally. The mean-field results demonstrating the crossover are plotted in Fig. 4. Clearly, the crossover occurs around $U_c/t \approx 11$, and the doping dependences of $\Delta_f$ and $\Delta_{SC}$ change dramatically, consistent with our discussion given above. Fig. 4 presents the results with different $U/t$ as a function of $x$. It is remarkable to see a dome-like shape naturally appear in $\Delta_{SC}(x)$ for $U/t \geq 12$, and the ‘optimal doping’ (the doping with largest $\Delta_{SC}$) is pushed to higher doping as $U$ increases, as expected. We also plot out $Z$ and compare it with
the Gutzwiller approximation \( g_t = 2x/(1 + x) \). The slave spin method indeed matches the Gutzwiller approximation very well at small \( x \), and the high order corrections due to the finite \( U \) become significant at larger \( x \).

We want to emphasize several advantages of the slave spin formalism compared to the traditional slave boson approach. In the traditional slave boson approach, the charge degrees of freedom are represented by the slave bosons which are usually assumed to have Bose-Einstein condensate at mean-field level. As a result, the bosonic degrees of freedom are in fact treated classically. In the slave spin formalism, both the slave spin and the spinon are treated quantum mechanically at the mean-field level already, which is the main reason why it can capture various exotic strongly correlated states at the mean-field level. Second, although the slave boson approach can qualitatively obtain a quasiparticle weight \( Z \sim x \), it is not straightforward to obtain the Gutzwiller factor \( g_t \) directly from the Hubbard model in the large \( U \) limit. Recently, a new slave boson scheme that can correctly capture \( g_t \) in the large \( U \) limit has been proposed, but it has a serious drawback that the non-interacting limit can not be obtained. \[29\] In contrast, the mean-field equations based on the slave spin method are tailored to yield \( g_t \) in the large \( U \) limit, and the \( U(1) \) version can obtain the non-interacting limit correctly. As a result, it is crucial to use the slave spin formalism to study the crossover in the \( t - J - U \) model discussed above, since it is necessary to treat both the strongly and the weakly interacting limits within the same framework.

\begin{equation}
H_C = -V \sum_{<i,j>} n_i n_j \approx -V Z^2 \sum_{<i,j>} n_i^f n_j^f, \quad (7)
\end{equation}

where \( n_i^f = \sum_\sigma f_{i\sigma}^f f_{i\sigma} \). If we introduce the same Hubbard-Stratonovich fields proposed by Ubbens and Lee\[24\] to decouple Eq. 7 in \( d \)-wave pairing channel, the mean-field pairing term is:

\begin{equation}
H_C^{MF} = - \sum_{\vec{k}} Z^2 \Delta'(\vec{k}) f_{\vec{k}\uparrow}^f f_{-\vec{k}\downarrow}^f + h.c., \quad (8)
\end{equation}

where \( \Delta'(\vec{k}) = V \Delta f (\cos k_x - \cos k_y) \). It can be seen that \( H_C^{MF} \) has a renormalization of \( Z^2 \). Because the bandwidth of the spinon hopping term given in Eq. 4 is renormalized by \( Z \) only, the pairing term is more suppressed than the kinetic energy by a factor of \( Z \), indicating that both \( \Delta_f \) and \( \Delta_{SC} \) would go to zero at small \( x \) in large \( U \). As a result, the crossover observed in the case of the pairing driven by spin interactions will not happen in this case.

\textbf{Discussion} – At the first sight, it seems that the crossover discussed above could be understood as a version of BEC-BCS crossover\[30, 32\] due to the reduction of the bandwidth. However, to reach the BEC regime, the chemical potential \( \mu \) has to be smaller than the minimum of the band energy so that no Fermi surface is left and the system becomes 'bosonic'. In our calculations, although the Fermi energy is reduced by \( Z \), \( \mu \) remains higher than the minimum of the band energy in every result, indicating that the fermionic pairing is still dominating. Physically, this can be understood as follows. The main effect of the Hubbard \( U \) is to suppress charges...
from moving instead of providing extra glues for the pairing. Moreover, the Hubbard $U$ tends to eliminate local pairing since it costs a large energy to put two electrons on the same site. Therefore, the pairing remains non-local despite of the kinetic energy being reduced by $U$, and the BEC picture does not work here.

To account for the phase transition at finite temperature in cuprates, an improvement beyond the mean-field level is necessary. In the present mean-field theory, $\Delta_f$ and $\Delta_{SC}$ have the same transition temperature as implied in Eq. [6]. This is due to the assumption of the slave spin dynamics being the same for each site. Generally speaking, both the quantum and the thermal fluctuations could invalidate this assumption. As a result, $(S_{ij}^+ S_{jk}^-)$ could deviate from $Z$ beyond the mean-field level. If such a formalism beyond mean-field level is available, the transition temperatures of $\Delta_f$ and $\Delta_{SC}$ could be different. In this case, the superconducting phase is characterized by $Z << 1$, $\Delta_f \neq 0$, and $(S_{ij}^+ S_{jk}^-) \neq 0$, the pseudogap phase is characterized by $Z << 1$, $\Delta_f \neq 0$, and $(S_{ij}^+ S_{jk}^-) = 0$, and the strange metal phase is characterized by $Z << 1$, $\Delta_f = 0$, and $(S_{ij}^+ S_{jk}^-) = 0$. The present theory could be extended to the pairing state intertwined with other orders like charge density wave state, stripe state, etc.

Another drawback of the present slave spin formalism is that the renormalization on the spinon is not directly considered through the mean-field equations. In addition to $g_t$, the Gutzwiller approximation also leads to a renormalization factor $g_s = 4/(1 + x)^2$ to the nearest neighbor antiferromagnetic Heisenberg interaction and $\Delta_{SC}$ is limited by $Z$ so that $\Delta_{SC} \sim x$. At larger $x$, $\Delta_{SC}$ is mainly determined by the spinon pairing $\Delta_f$, thus $\Delta_{SC}$ starts to decrease after a critical doping. It is interesting to note that a superconducting dome appears naturally. The corresponding quasiparticle weight $Z$ as a function of $x$. The black line represents the Gutzwiller factor $g_t = 2x/(1 + x)$. The slave spin matches $g_t$ at small doping very well.

**Conclusion**—We have developed a BCS mean-field theory implemented with the slave-spin formalism and have investigated the superconducting states in the $t - J - U$ model. We have found that this formalism naturally predicts two distinct gaps, the spinon pairing gap $\Delta_f$ and the superconducting gap $\Delta_{SC} = Z \Delta_f$. For the case of the pairing arising from the spin interaction $H_f$, we have found that a crossover from strong to weak pairing states driven by the doping $x$ could occur as $U$ exceeds the critical value $U_c$ for the Mott insulating state at half-filling. In the strong pairing regime, $\Delta_f$ is largely enhanced due to the reduction of the spinon bandwidth, but the superconducting gap $\Delta_{SC} \sim x$ at small $x$. In the weak pairing regime, both $\Delta_f$ and $\Delta_{SC}$ behave similarly. On the other hand, if the pairing is induced by charge interactions, both $\Delta_f$ and $\Delta_{SC}$ would go to zero at small $x$, and consequently no crossover would occur. We have obtained a superconducting dome in the phase diagram of $\Delta_{SC}$ vs $x$ with reasonable parameters, and all the results are obtained at the mean-field level. Our results have demonstrated that the slave spin formalism is a powerful technique to study study physics related to transition from strong to weak coupling states within the same framework.
Acknowledgement—We appreciate fruitful discussions with T. K. Lee. This work is supported by a start up fund from Binghamton University.

* Electronic address: wlee@binghamton.edu

[1] P. A. Lee, N. Nagaosa, and X.-G. Wen, Rev. Mod. Phys. 78, 17 (2006).
[2] P. Phillips, Rev. Mod. Phys. 82, 1719 (2010).
[3] T. K. Lee, C.-M. Ho, and N. Nagaosa, Phys. Rev. Lett. 90, 067001 (2003).
[4] W.-C. Lee, T. K. Lee, C.-M. Ho, and P. W. Leung, Phys. Rev. Lett. 91, 057001 (2003).
[5] M. Lugus, L. Spanu, F. Becca, and S. Sorella, Phys. Rev. B 74, 165122 (2006).
[6] F. Tan and Q.-H. Wang, Phys. Rev. Lett. 100, 117004 (2008).
[7] H. Ma, T. K. Lee, and Y. Chen, New Journal of Physics 15, 043045 (2013).
[8] S. Daul, D. J. Scalapino, and S. R. White, Phys. Rev. Lett. 84, 4188 (2000).
[9] S. Basu, R. J. Gooding, and P. W. Leung, Phys. Rev. B 63, 100506 (2001).
[10] F. C. Zhang, Phys. Rev. Lett. 90, 207002 (2003).
[11] F. Yuan, Q. Yuan, and C. S. Ting, Phys. Rev. B 71, 104505 (2005).
[12] M. Abram, J. Kaczmarczyk, J. Jedrak, and J. Spa/lek, Phys. Rev. B 88, 094502 (2013).
[13] A. Farrell and T. Pereg-Barnea, Phys. Rev. B 89, 035112 (2014).
[14] L. de’Medici, A. Georges, and S. Biermann, Phys. Rev. B 72, 205124 (2005).
[15] L. de’ Medici, S. R. Hassan, M. Capone, and X. Dai, Phys. Rev. Lett. 102, 126401 (2009).
[16] S. R. Hassan and L. de’ Medici, Phys. Rev. B 81, 035106 (2010).
[17] L. de’ Medici, Phys. Rev. B 83, 205112 (2011).
[18] R. Yu and Q. Si, Phys. Rev. B 84, 235115 (2011).
[19] R. Yu and Q. Si, Phys. Rev. B 86, 085104 (2012).
[20] R. Yu and Q. Si, Phys. Rev. Lett. 110, 146402 (2013).
[21] L. de’ Medici, G. Giovannetti, and M. Capone, Phys. Rev. Lett. 112, 177001 (2014).
[22] G. Giovannetti, L. de’ Medici, M. Aichhorn, and M. Capone, Phys. Rev. B 91, 085124 (2015).
[23] S. Mukherjee, N. F. Quackenbush, H. Paik, C. Schluter, T.-L. Lee, D. G. Schlom, L. F. J. Piper, and W.-C. Lee, ArXiv e-prints (2016), 1603.00485.
[24] M. U. Ubbens and P. A. Lee, Phys. Rev. B 46, 8434 (1992).
[25] G. Kotliar and A. E. Ruckenstein, Phys. Rev. Lett. 57, 1362 (1986).
[26] T. Li, P. Wölfle, and P. J. Hirschfeld, Phys. Rev. B 40, 6817 (1989).
[27] M. C. Gutzwiller, Phys. Rev. Lett. 10, 159 (1963).
[28] F. Gebhard, Phys. Rev. B 41, 9452 (1990).
[29] F. Lechermann, A. Georges, G. Kotliar, and O. Parcollet, Phys. Rev. B 76, 155102 (2007).
[30] J. Ranninger and J. M. Robin, Phys. Rev. B 53, R11961 (1996).
[31] Q. Chen, I. Kosztin, B. Jankó, and K. Levin, Phys. Rev. B 59, 7083 (1999).
[32] P. Pieri and G. C. Strinati, Phys. Rev. B 61, 15370 (2000).
[33] Q. Chen, J. Stajic, S. Tan, and K. Levin, Physics Reports 412, 1 (2005).
SUPPLEMENTARY MATERIALS

Here we derive the mean-field equations from Eq. 4 in the main text. First, we derive the BCS gap equation in the spinon sector. Using the Bogoliubov transformation, we obtain

\[ f_{\vec{k} \uparrow} = \cos \theta_{\vec{k}} \alpha_{\vec{k},+} + \sin \theta_{\vec{k}} \alpha_{\vec{k},-}, \]
\[ f_{\vec{k} \downarrow} = \sin \theta_{\vec{k}} \alpha_{\vec{k},+} - \cos \theta_{\vec{k}} \alpha_{\vec{k},-}, \]
\[ \cos \theta_{\vec{k}} = \frac{|\langle \tilde{z}_\sigma \rangle|^2 \epsilon(\vec{k}) - \mu}{E(\vec{k})}, \quad \sin \theta_{\vec{k}} = \frac{\Delta(\vec{k})}{E(\vec{k})}. \]
\[ E(\vec{k}) = \sqrt{(|\langle \tilde{z}_\sigma \rangle|^2 \epsilon(\vec{k}) - \mu)^2 + \Delta^2(\vec{k})}. \]

(9)

\( \alpha_{\vec{k},\pm} \) are the Bogoliubov quasiparticles with eigen energies \( \pm E(\vec{k}) \). The self-consistent equations are:

\[ \Delta_f = \frac{1}{2N} \sum_{\vec{k}} d(\vec{k}) \sin \theta_{\vec{k}} \left[ n_f(-E(\vec{k})) - n_f(E(\vec{k})) \right], \]
\[ n = 1 - x \]
\[ x = \frac{1}{N} \sum_{\vec{k}} \cos \theta_{\vec{k}} \left[ n_f(-E(\vec{k})) - n_f(E(\vec{k})) \right], \]

(10)

where \( n_f(E) \) is the Fermi-Dirac function, and \( x \) is the doping away from the half-filling.

The iteration procedure is as follows. Starting from an initial guess of \( Z \), we first solve the BCS gap equation in the spinon sector subject to the fixed doping \( x \). Then, we can compute the average kinetic energy of the spinon

\[ \epsilon^{a\sigma} = \frac{1}{\Omega} \sum_{\vec{k}} \epsilon(\vec{k}) \cos \theta_{\vec{k}} \left[ n_f(-E(\vec{k})) - n_f(E(\vec{k})) \right], \]

(11)

which will be used in the mean-field Hamiltonian for the slave spin \( H^{s,MF} \).

The constraint to remove the enlarged Hilbert space is

\[ S_{\sigma}^z = f_{\sigma}^\dagger f_{\sigma} - \frac{1}{2}. \]

(12)

This constraint will be taken into account after the slave spin mean-field Hamiltonian is diagonalized. Since the slave spin sector is effectively a single site problem in the mean-field theory, it can be diagonalized exactly. Then we determine \( \lambda \) by satisfying the constraint in Eq. 12 ‘on the average’ by

\[ M_\sigma \equiv \langle S_{\sigma}^z \rangle = \frac{1}{\Omega} \sum_{\vec{k}} \langle f_{\sigma}^\dagger f_{\sigma} \rangle - \frac{1}{2} \]

(13)

We can further simplify the above equation to

\[ 1 - x = M_\uparrow + M_\downarrow + 1 \rightarrow M_\uparrow + M_\downarrow = -x. \]

(14)

Since we are only interested in the spin singlet pairing states, we have \( M_\uparrow = M_\downarrow = -x/2 \).

After \( \lambda \) is determined, we can compute \( Z \) which will be used for the next cycle of the calculation. We repeat the procedure until the self-consistency is reached with an error less than 10\(^{-6}\).