Hypergraph Colouring and Degeneracy

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Abstract

A hypergraph is \( d \)-degenerate if every subhypergraph has a vertex of degree at most \( d \). A greedy algorithm colours every such hypergraph with at most \( d + 1 \) colours. We show that this bound is tight, by constructing an \( r \)-uniform \( d \)-degenerate hypergraph with chromatic number \( d + 1 \) for all \( r \geq 2 \) and \( d \geq 1 \). Moreover, the hypergraph is triangle-free, where a triangle in an \( r \)-uniform hypergraph consists of three edges whose union is a set of \( r + 1 \) vertices.

1 Introduction

Erdős and Lovász [7] proved the following fundamental result about colouring hypergraphs

**Theorem 1** ([7]). For fixed \( r \), every \( r \)-uniform hypergraph with maximum degree \( \Delta \) has chromatic number at most \( O(\Delta^{1/(r-1)}) \).

Theorem 1 implies that every \( r \)-uniform hypergraph with maximum degree \( \Delta \) has an independent set of size at least \( \Omega(n/\Delta^{1/(r-1)}) \). Spencer [10] proved the following stronger bound.

**Theorem 2** ([10]). For fixed \( r \), every \( r \)-uniform hypergraph with \( n \) vertices and average degree \( d \) has an independent set of size at least \( \Omega(n/d^{1/(r-1)}) \).

A hypergraph is \( d \)-degenerate if every subhypergraph has a vertex of degree at most \( d \). A minimum-degree-greedy algorithm colours every \( d \)-degenerate

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1 A hypergraph \( G \) consists of a set \( V(G) \) of vertices and a set \( E(G) \) of subsets of \( V(G) \) called edges. A hypergraph is \( r \)-uniform if every edge has size \( r \). A graph is a 2-uniform hypergraph. A hypergraph \( H \) is a subhypergraph of a hypergraph \( G \) if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). A colouring of a hypergraph \( G \) assigns one colour to each vertex in \( V(G) \) such that no edge in \( E(G) \) is monochromatic. The chromatic number of \( G \), denoted by \( \chi(G) \), is the minimum number of colours in a colouring of \( G \). A colouring of \( G \) can be thought of as a partition of \( V(G) \) into independent sets, each containing no edge. The degree of a vertex \( v \) is the number of edges that contain \( v \). See the textbook of Berge [3] for other notions of degree in a hypergraph.
hypergraph with at most \( d + 1 \) colours. This bound is tight for graphs \((r = 2)\) since the complete graph on \( d + 1 \) vertices is \( d \)-degenerate, and of course, has chromatic number \( d + 1 \). However, this observation does not generalise for \( r \geq 3 \).

In particular, for the complete \( r \)-uniform hypergraph on \( n \) vertices, every vertex has degree \((n - 1)^{r-1}\), yet the chromatic number is \( \left\lceil \frac{n}{r-1} \right\rceil \). Thus for \( r \geq 3 \), the degeneracy is much greater than the chromatic number.

Given Theorems 1 and 2, it seems plausible that for \( r \geq 3 \), every \( r \)-uniform \( d \)-degenerate hypergraph is \( o(d) \)-colourable. It even seems possible that every \( r \)-uniform \( d \)-degenerate hypergraph is \( O(d^{1/(r-1)}) \)-colourable. This natural strengthening of Theorems 1 and 2 would (roughly) say that \( G \) can be partitioned into independent sets, whose average size is that guaranteed by Theorem 2.

This note rules out these possibilities, by showing that the naive upper bound \( \chi \leq d + 1 \) is tight for all \( r \). This is the main conclusion of this paper. Moreover, we prove it for triangle-free hypergraphs, where a triangle in an \( r \)-uniform hypergraph consists of three edges whose union is a set of \( r + 1 \) vertices.

Observe that this definition with \( r = 2 \) is equivalent to the standard notion of a triangle in a graph (although there are other notions of a triangle in a hypergraph \([4]\)).

**Theorem 3.** For all \( r \geq 2 \) and \( d \geq 1 \) there is a triangle-free \( d \)-degenerate \( r \)-uniform hypergraph with chromatic number \( d + 1 \).

Theorem 3 and its proof is a generalisation of a result of Alon et al. \([2]\) who proved it for graphs \((r = 2)\). Of course, the complete graph \( K_{d+1} \) is \( d \)-degenerate with chromatic number \( d + 1 \). The triangle-free property was the main conclusion of their result. See \([1, 9]\) for other related results.

## 2 Proof

Theorem 3 is a corollary of the following:

**Lemma 4.** Fix \( r \geq 2 \). For all \( d \geq 1 \) there is a triangle-free \( d \)-degenerate \( r \)-uniform hypergraph \( G_d \) with chromatic number \( d + 1 \), such that in every \((d+1)\)-colouring of \( G_d \) each colour is assigned to at least \( r - 1 \) vertices.

**Proof.** We proceed by induction on \( d \). First consider the base case \( d = 1 \). Let \( n := r(r - 1) \). Let \( V(G_1) := \{v_1, \ldots, v_n\} \) and \( E(G_1) := \{e_i : 1 \leq i \leq n - r + 1\} \), where \( e_i := \{v_i, v_{i+1}, \ldots, v_{i+r-1}\} \). If \( S \subseteq V(G_1) \) and \( i \) is minimum such that \( v_i \in S \), then \( v_i \) has degree at most \( 1 \) in the subhypergraph induced by \( S \). Thus \( G_1 \) is \( 1 \)-degenerate. If \( e_i, e_j, e_k \) are three edges in \( G_1 \) with \( i < j < k \), then \( e_i \cup e_j \cup e_k \) includes the \( r + 2 \) distinct vertices \( v_i, v_{i+1}, \ldots, v_{i+r-1}, v_{j+r-1}, v_{k+r-1} \). Hence \( G_1 \) is triangle-free. Consider a 2-colouring of \( G_1 \). Clearly, \( G_1 \) contains \( r - 1 \) pairwise disjoint edges, each of which contains vertices of both colours. Hence each colour is assigned to at least \( r - 1 \) vertices. This completes the base case.
Now assume that $G_{d-1}$ is a triangle-free $(d-1)$-degenerate $r$-uniform hypergraph with chromatic number $d$, such that in every $d$-colouring of $G_{d-1}$ each colour is assigned to at least $r-1$ vertices.

Initialise $G_{d}$ to consist of $d+r-2$ disjoint copies $H_1, \ldots, H_{d+r-2}$ of $G_{d-1}$. Let $S$ be a set of $(r-1)d$ vertices in $H_1 \cup \cdots \cup H_{d+r-2}$ such that $|S \cap V(H_i)| \in \{0, r-1\}$ for $1 \leq i \leq d+r-2$. That is, $S$ contains exactly $r-1$ vertices from exactly $d$ of the $H_i$, and contains no vertices from the other $r-2$. Now, for each such set $S$, add $r-1$ new vertices $v_1, \ldots, v_{r-1}$ to $G_{d}$ and add the new edge $(S \cap V(H_i)) \cup \{v_j\}$ to $G_{d}$ whenever $|S \cap V(H_i)| = r-1$. Thus each new vertex has degree $d$. Since $H_1 \cup \cdots \cup H_{d+r-2}$ is $d$-degenerate, $G_{d}$ is also $d$-degenerate.

Suppose on the contrary that $G_{d}$ contains a triangle $T$. Since $G_{d-1}$ is triangle-free, at least one edge in $T$ is a new edge, which is contained in $V(H_i) \cup \{v\}$ for some $1 \leq i \leq d+r-2$ and some new vertex $v$. Each vertex in a triangle is in at least two of the edges of the triangle. However, by construction, $v$ is contained in only one edge contained in $V(H_i) \cup \{v\}$. Thus $G_{d}$ is triangle-free.

Since $H_1 \cup \cdots \cup H_{d+r-2}$ is $d$-degenerate, and no edge contains only new vertices, assigning all the new vertices a $(d+1)$-th colour produces a $(d+1)$-colouring of $G_{d}$. Thus $\chi(G_{d}) \leq d+1$.

Suppose on the contrary that $G_{d}$ has a $(d+1)$-colouring with at most $r-2$ vertices of some colour, say ‘blue’. Say the other colours are $1, \ldots, d$. At most $r-2$ copies of the $H_i$ contain blue vertices. Hence, without loss of generality, $H_1, \ldots, H_d$ contain no blue vertices. That is, $H_1, \ldots, H_d$ are $d$-coloured with colours $1, \ldots, d$. By induction, $H_i$ contains a set $S_i$ of $r-1$ vertices coloured $i$ for $1 \leq i \leq d$. By construction, there are $r-1$ vertices $v_1, \ldots, v_{r-1}$ in $G_{d}$, such that $S_i \cup \{v_j\}$ is an edge of $G_{d}$ for $1 \leq i \leq d$ and $1 \leq j \leq r-1$. Since each such edge is not monochromatic, each vertex $v_j$ is coloured blue. In particular, there are at least $r-1$ blue vertices, which is a contradiction. Therefore, in every $(d+1)$-colouring of $G_{d}$, each colour class has at least $r-1$ vertices, as claimed. (In particular, $G_{d}$ has no $d$-colouring.) \qed

### 3 An Open Problem

We conclude with an open problem. The girth of a graph (that contains some cycle) is the length of its shortest cycle. Erdős [5] proved that there exists a graph with chromatic number at least $k$ and girth at least $g$, for all $k \geq 3$ and $g \geq 4$. (Erdős and Hajnal [6] proved an analogous result for hypergraphs). Theorem 3 strengthens this result for triangle-free graphs (that is, with girth $g = 4$). This leads to the following question: Does there exist a $d$-degenerate graph with chromatic number $d+1$ and girth $g$, for all $d \geq 2$ and $g \geq 4$? Odd cycles prove the $d = 2$ case. An affirmative answer would strengthen the above result of Erdős [5]. A negative answer would also be interesting—this would provide a non-trivial upper bound on the chromatic number of $d$-degenerate graphs with girth $g$. 

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Note
After this paper was written the author discovered the beautiful paper by Kostochka and Nešetřil [8] which proves a strengthening of Theorem 3 and includes the positive solution of the above open problem.

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