Effective field equation on m-brane embedded in n-dimensional bulk of Einstein and f(R) gravity

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Abstract

We have derived effective gravitational field equation on a lower dimensional hypersurface (known as a brane), placed in a higher dimensional bulk spacetime for both Einstein and f(R) gravity theories. We have started our analysis on n-dimensional bulk from which the effective field equation on a (n − 1)-dimensional brane has been obtained by imposing $Z_2$ symmetry. Subsequently, we have arrived at the effective equation in (n − 2)-dimensions starting from the effective equation for (n − 1) dimensional brane. This analysis has been carried forward and is used to obtain the effective field equation in (n − m)-dimensional brane, embedded in a n-dimensional bulk. Having obtained the effective field equation in Einstein gravity, we have subsequently generalized the effective field equation in (n − m)-dimensional brane which is embedded the n-dimensional bulk spacetime endowed with $f(R)$ gravity. Finally we have presented an application of our result in cosmological context.

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# Contents

1 Introduction ............................................. 3

2 Effective Field Equation In Einstein Gravity: Background and Formulation 4  
   2.1 Effective Field Equation on \((n - 1)\)-dimensional Brane .................. 5  
   2.2 Effective Field Equation on \((n - 2)\)-dimensional Brane .................. 7  
   2.3 Generalization to \((n - m)\)-dimensional Brane ............................ 10

3 Generalization to bulk \(f(R)\) gravity 11  
   3.1 Effective Field Equation on the \((n - 1)\)-dimensional Brane ................. 11  
   3.2 Generalization to \((n - m)\)-dimensional Brane ............................ 12

4 An Application ............................................ 13

5 Discussion .................................................. 16

A Appendix: Detailed Calculations 17  
   A.1 Identities for the Derivation of \((n - 1)\) Dimensional Effective Field Equation ... 17  
   A.2 Identities for the Derivation of \((n - 2)\) Dimensional Effective Field Equation ... 19
1 Introduction

One of the fundamental problems in theoretical physics amounts to unifying all known interactions. A consistent unification of gravity with the other fundamental interactions has been an active area of research over many years. One of the leading contenders along this direction is superstring theory. In superstring theory the universe is assumed to be 11-dimensional of which seven are compactified leaving four non-compact dimensions, which we observe. Among various candidates, the 10-dimensional $E_8 \times E_8$ heterotic string theory is a strong candidate as the theory is able to accommodate the standard model within it. Recently it has been shown that the 10-dimensional $E_8 \times E_8$ heterotic string theory is related to the 11-dimensional M theory (with an orbifold symmetry $S^1/Z_2$). This also confines the standard model particles to standard non-compact 4-dimensional spacetime while gravity can probe the higher dimensional spacetime $[1–3]$.

A simplistic model which captures most of the key notions is a 5-dimensional model in which matter fields are confined to 4-dimensional spacetime while gravity exists in all the five spacetime dimensions. The first step along this direction with this simplified picture presented above was taken by Randall and Sundrum by introducing two 4-dimensional sub-manifolds (known as brane) in a 5-dimensional anti-de Sitter bulk $[4]$. This arrangement of two positive and negative tension branes separated by a finite distance (known as radion field $[5–7]$) can address the hierarchy problem by lowering the Higgs mass to electroweak scale on the visible brane. Subsequently, they have introduced another model, which consists of a positive tension brane with an infinite extra dimension $[8]$. Due to the presence of an extra spacetime dimension it is expected that there would be deviation of various physical results from Einstein gravity, which would become more significant in the high energy limit. The brane world model subsequently, was applied in the context of particle phenomenology $[9–13]$, black hole physics $[14–18]$ and cosmology $[7,19–22]$.

A natural extension of the Randall-Sundrum (RS) scenario to models with more than one warped extra dimension have been proposed, where several independent $S^1/Z_2$ orbifold is considered along with the 4-dimensional non-compact manifold $M_4$ $[23–27]$. These multiply warped models are mainly motivated from various considerations, which include: radion stabilization, fermion mass hierarchy in Standard Model etc. There have been significant number of follow-up works in these multiply warped models which involves: inflation, cosmic acceleration, matter field localization, Kaluza-Klein modes of graviton, gauge and scalar fields $[28–33]$. However all these results depend on certain assumptions of bulk metric elements and bulk Einstein equation under some simplified conditions, for example in presence of homogeneity and isotropy.

There exist an elegant way in which effective field equation for gravity on a lower dimensional spacetime can be obtained. This involves use of geometric quantities like metric, curvature induced on the brane from bulk. Using these induced geometric quantities it is possible to arrive at a relation between the bulk curvature tensor and the brane curvature tensor, known as Gauss-Codazzi equation $[34,35]$. From various contractions of the Gauss-Codazzi equation we can determine the effective field equation on the brane by relating bulk Einstein tensor to that on the brane.

This approach was applied in order to obtain the gravitational field equation on the brane from a 5-dimensional bulk in $[36]$. The effective equation contains non-local terms inherited from the bulk. Subsequently starting from the effective field equation derived in $[36]$ a class of vacuum solutions has been obtained in $[17]$, which has been further generalized in $[37]$.
It is strongly believed that general relativity is only a low energy approximation of some underlying fundamental high energy theory \([38, 39]\). This suggests that Einstein-Hilbert action in high energy limit, should be modified by introduction of higher curvature terms. Such an alternative theory is the \(f(R)\) theory, where \(R\) is the bulk Ricci scalar. This theory has been studied extensively in the context of solar system tests, inflationary paradigm, late time cosmic acceleration along with possibility of detecting gravitational waves \([40–44]\). Moreover two-brane models, due to the presence of \(f(R)\) term in the bulk has also been a topic of discussion recently in collider physics \([45–47]\) relating to the absence of graviton Kaluza-Klein Modes in LHC.

The above approach of obtaining effective gravitational field equation is usually confined to Einstein gravity on the 3-brane alone. This subsequently has been generalized to obtain effective field equation on the 3-brane when the bulk contains \(f(R)\) gravity. This effective field equations has been used in \([48]\) to obtain vacuum solution in \(f(R)\) bulk.

In this work, we generalize the above setup by working in an \(n\)-dimensional bulk and then obtaining effective equation on an \((n - m)\)-dimensional brane where \(m\) can take any values. This seems natural from a string theory view point, where the universe is 11-dimensional, while our universe is a 3-brane with 4-spacetime dimensions. Following this argument we have derived effective gravitational field equation on a \((n - m)\)-dimensional brane embedded in a \(n\)-dimensional bulk. We start with a \(n\)-dimensional bulk spacetime and then derive the effective field equation on a \((n - 1)\)-dimensional brane, which can then be used as the starting point to derive effective field equation on a \((n - 2)\)-dimensional brane. Having obtained the effective field equation on the \((n - 2)\)-dimensional brane one can easily recognize the pattern of obtaining the effective field equation in any arbitrary lower dimensional surfaces. Thus following the road from \(n\)-dimensional bulk to \((n - 2)\)-dimensional brane, we can obtain the effective field equation on \((n - m)\) dimensional brane as well. The above analysis can be easily generalized to \(f(R)\) gravity model in the bulk following the analysis presented in \([48]\).

The paper is organized as follows: In Sec. 2 we have elaborated the derivation of effective field equation for Einstein gravity. First, in Sec. 2.1 we present the derivation of effective field equation in \((n - 1)\)-dimensional brane, which is subsequently generalized in Sec. 2.2 to \((n - 2)\)-dimension. Following the pattern we obtain the effective gravitational field equation in \((n - m)\)-dimensional brane in Sec. 2.3. This result has been generalized for \(f(R)\) gravity model in Sec. 3. Finally we provide an application of our results in Sec. 4 before concluding with a discussion.

Throughout the paper, we maintain the following convention: all Latin letters \(a, b, \ldots\) run over all the \(n\) spacetime indices, Greek letters \(\alpha, \mu, \ldots\) run over the \((n - 1)\) spacetime indices. Finally capitalized Latin letters \(A, B, \ldots\) run over the \((n - 2)\) spacetime indices.

## 2 Effective Field Equation In Einstein Gravity: Background and Formulation

Hamiltonian formulation in general relativity comes up with the \((1 + 3)\) splitting of the spacetime, where a time direction is singled out. For this purpose we introduce a time coordinate \(t = t(x^a)\) and foliate the spacetime with such \(t = \) constant surfaces. Introducing a vector field \(t^a = \partial / \partial t\) we can move out of the spacetime by moving along the integral curves for \(t^a\). This involves a change in the \(t\) coordinate introducing the shift function \(N\) and change in the orientation captured by the lapse functions \(N^\alpha\). Then \(N, N^\alpha\) yield 4 components of the Einstein tensor,
while the remaining six components are being determined by induced metric on the surface \( h_{\alpha\beta} \). The induced metric can be obtained from the bulk metric \( g_{ab} \) in two ways. First, we can introduce normalized normals to the surface given by:
\[
 n^a = -N \nabla_a t
\]
and then define the induced metric as
\[
 h_{ab} = g_{ab} - n^a n^b
\]
(This formulation does not work for null surfaces, for which a complete treatment has been provided in [49]). Secondly, we can introduce coordinates \( y^\alpha \) on the \( t = \text{constant} \) hypersurface on which we can introduce surface tetrads \( e_\alpha^a = \partial x^a / \partial y^\alpha \) and obtain induced metric as:
\[
 h_{\alpha\beta} = g_{ab} e_\alpha^a e_\beta^b
\]
With the induced metric we can define covariant derivative on the surface and then using vectors on the \( t = \text{constant} \) surface we can introduce curvature tensor on the spatial hypersurface. The relation between 4-dimensional and 3-dimensional curvature tensor is obtained through Gauss-Codazzi equation. This can be manipulated in various ways in order to arrive at the effective Einstein’s equation on the \( t = \text{constant} \) hypersurface. This procedure can be extended easily for timelike surfaces (i.e. spacelike normals) as well.

2.1 Effective Field Equation on \((n-1)\)-dimensional Brane

Having described the general setup let us now consider the effective field equation on a brane embedded in a higher dimensional bulk, which is the main motivation of this work. The procedure sketched above can then be applied for a generic situation where a single \((n-1)\)-dimensional brane is embedded in a \( n \)-dimensional bulk spacetime with Einstein gravity. For this set up the effective gravitational field equation on the \((n-1)\) dimensional brane can be written as (see App. A.1):
\[
 (n-1) G_{\alpha\beta} = \kappa^2 \left[ T_{ab} e_\alpha^a e_\beta^b + h_{\alpha\beta} \left\{ \epsilon T_{ab} n^a n^b - \frac{1}{n-1} T \right\} \right] - \epsilon E_{\alpha\beta} + \epsilon \left[ K K_{\alpha\mu} - K_{\alpha\beta} K_{\beta}^\mu - \frac{1}{2} h_{\alpha\beta} (K^2 - K_{\mu\nu} K_{\mu\nu}) \right]
\]

The above expression connects bulk energy momentum tensor and the extrinsic curvature to the \((n-1)\) dimensional Einstein tensor. Given \( T_{ab} \), along with surface contribution we can solve the above equation to derive \((n-1)\) dimensional solution.

The above effective equation is applicable to both spacelike and timelike surfaces. However for our purpose, we will need only timelike surfaces, which are the branes. Thus the basic assumption to be followed in this section is that the \( n \) dimensional spacetime can be projected on a \((n-1)\) dimensional brane with spacelike normal. The line element then takes the form:
\[
 ds^2 = d\chi^2 + h_{\alpha\beta} dx^\alpha dx^\beta
\]

The brane is located at \( \chi = 0 \) on this \( n \) dimensional spacetime. Using the standard assumption that normal matter can exist only on the brane, the bulk energy momentum tensor can be written as:
\[
 T_{ab} = - \Lambda_n g_{ab} + S_{\alpha\beta} e_\alpha^a e_\beta^b \delta(\chi)
\]
where, \( S_{\alpha\beta} \) represents the brane energy-momentum tensor and \( \Lambda_n \) is the bulk cosmological constant. As the surface \( \chi = 0 \) contain surface energy momentum tensor, the extrinsic curvature is not continuous. The discontinuity is related to the surface stress energy tensor through the
following relation:

\[ S_{\alpha\beta} = \frac{1}{\kappa_n^2} ([K_{\alpha\beta}] - h_{\alpha\beta} [K]) \]  (4)

where \([K_{\alpha\beta}]\) and \([K]\) denote jump in \(K_{\alpha\beta}\) and \(K\) across the \(\chi = 0\) surface. Thus we obtain the jump in the extrinsic curvature which after imposing \(Z_2\) symmetry, has the expression

\[ K^+_{\alpha\beta} = -K^-_{\alpha\beta} = \frac{1}{2} \kappa_n^2 \left( S_{\alpha\beta} - \frac{1}{n-2} h_{\alpha\beta} S \right) \]  (5)

From the extrinsic curvature we can contract both the indices to obtain a scalar. This scalar \(K\) corresponding to each side of the \(\chi = 0\) hypersurface is represented by \(K_{\pm}\). Therefore from Eq. (5) it has the expression:

\[ K_{\alpha\beta} = -K^+_{\alpha\beta} = -\left( \kappa_n^2 S/2(n-2) \right) \]

Using both the expressions for \(K_{\alpha\beta}\) and \(K\) we finally arrive at the following identities:

\[ KK_{\alpha\beta} = \frac{\kappa_n^4}{4} \left( \frac{1}{(n-2)^2} h_{\alpha\beta} S^2 - \frac{1}{(n-2)} S S_{\alpha\beta} \right) \]  (6)

\[ K_{\alpha\mu} K^\mu_{\beta} = \frac{\kappa_n^4}{4} \left( S_{\alpha\mu} S_{\beta}^\mu - \frac{2}{(n-2)} S S_{\alpha\beta} + \frac{1}{(n-2)^2} h_{\alpha\beta} S^2 \right) \]  (7)

\[ K^2 = \frac{\kappa_n^4}{4} \frac{1}{(n-2)^2} S^2 \]  (8)

\[ K_{\alpha\beta} K_{\alpha\beta} = \frac{\kappa_n^4}{4} \left( S_{\alpha\beta} S_{\alpha\beta} - \frac{2}{n-2} S^2 + \frac{n-1}{(n-2)^2} S^2 \right) \]  (9)

In order to proceed further we need to decompose the brane stress-energy tensor in two parts. One of them involves the brane cosmological constant while the other provides matter stress-energy tensor on the brane. Implementation of this separation leads to,

\[ S_{\alpha\beta} = -\sigma h_{\alpha\beta} + \tau_{\alpha\beta} \]  (10)

where, \(\sigma\) represents the brane tension. Next we substitute this decomposition of \(S_{\alpha\beta}\) into the above expressions involving extrinsic curvature. This ultimately leads to,

\[ KK_{\alpha\beta} - K_{\alpha\mu} K^\mu_{\beta} - \frac{1}{2} h_{\alpha\beta} \left( K^2 - K_{\mu\nu} K^{\mu\nu} \right) = \kappa_n^4 \left[ -\frac{1}{4} \tau_{\alpha\mu} \tau^\mu_{\beta} + \frac{1}{8} h_{\alpha\beta} \tau_{\mu\nu} \tau^{\mu\nu} + \frac{1}{4(n-2)} \tau_{\alpha\beta} \right] - \frac{1}{8(n-2)} h_{\alpha\beta} \tau^2 + \kappa_n^4 \left[ \frac{(n-3)}{4(n-2)} \sigma \tau_{\alpha\beta} - \frac{(n-3)}{8(n-2)} \sigma^2 h_{\alpha\beta} \right] \]  (11)

Eqs. (3) and (11) now need to be substituted in Eq. (1) in order to obtain the final form of the effective equation on the brane. After the substitution the effective Einstein equation on the brane takes the form

\[ (n-1) G_{\alpha\beta} = -\left( \frac{n-3}{n-1} \right) \kappa_n^2 \left( \Lambda_n + \frac{n-1}{8(n-2)} \kappa_n^2 \sigma^2 \right) h_{\alpha\beta} \]

\[ + 8\pi \left( \sigma \kappa_n^4 \frac{n-3}{32\pi(n-2)} \right) \tau_{\alpha\beta} + \kappa_n^4 \tau_{\alpha\beta} - E_{\alpha\beta} \]  (12)
where we have introduced the second rank tensor $\Pi_{\alpha\beta}$ as:

$$\pi_{\alpha\beta} = \left[ -\frac{1}{4} \tau_{\alpha\mu} \tau_{\beta} + \frac{1}{8} h_{\alpha\beta} \tau_{\mu\nu} \tau^{\mu\nu} + \frac{1}{4(n-2)} \tau_{\alpha\beta} - \frac{1}{8(n-2)} h_{\alpha\beta} \tau^2 \right]$$

(13)

After rewriting Eq. (12) can also be presented in a more compact form, which can be expressed as:

$$^{(n-1)}G_{\alpha\beta} = \kappa^2_{n-1} T_{\alpha\beta}$$

(14)

where the energy momentum tensor $T_{\alpha\beta}$ has the expression:

$$T_{\alpha\beta} = -\Lambda_{n-1} h_{\alpha\beta} + \frac{\kappa^4_{n}}{\kappa^2_{n-1}} \pi_{\alpha\beta} - \frac{1}{\kappa^2_{n-1}} E_{\alpha\beta}$$

(15)

In the above expression $\Lambda_{n-1}$ represents the $(n-1)$-dimensional cosmological constant which has the expression,

$$\Lambda_{n-1} = \frac{(n-3)}{n} \frac{\kappa^2_{n}}{\kappa^2_{n-1}} \left( \Lambda_{n} + \frac{n-1}{8(n-2)} \kappa^2_{n} \sigma^2 \right)$$

(16)

This generalizes the expression for the effective cosmological constant on a lower dimensional hypersurface embedded in a higher dimensional bulk. For $n = 5$ we retrieve fine balancing relation of the Randall Sundrum single brane model [36].

Thus this formalism, with a negative $n$-dimensional cosmological constant can provide a solution to the cosmological constant problem by tuning the brane tension such that $\Lambda_{n-1} = 0$. In which case the brane tension can be obtained as:

$$\sigma^2 = \frac{8(n-2)\left|\Lambda_{n}\right|}{(n-1)\kappa^2_{n}}$$

(17)

Note that once the $(n-1)$-dimensional cosmological constant has been set to zero it will remain zero in all the lower dimensional branes. Also $G_{n-1}$ is the $(n-1)$ dimensional gravitational constant having the expression,

$$\kappa^2_{n-1} = 8\pi G_{n-1} = \sigma \kappa^4_{n} \frac{n-3}{4(n-2)}$$

(18)

Having discussed the effective gravitational field equation in $(n-1)$-dimension, we will now move one more step by deriving effective field equation in $(n-2)$-dimensional spacetime.

### 2.2 Effective Field Equation on $(n-2)$-dimensional Brane

Similar analysis can be performed while obtaining effective field equation on the $(n-2)$ dimensional brane starting from the $(n-1)$-dimensional spacetime. The field equation in the $(n-1)$ dimensional spacetime can be used which ultimately leads to the following expression for effective field equation in $(n-2)$ dimensions as (see App. A.2):

$$^{(n-2)}G_{AB} = \frac{(n-4)}{(n-3)} \kappa^2_{n-1} \left[ T_{\alpha\beta} e^\alpha_A e^\beta_B + \left\{ T_{\alpha\beta} s^\alpha s^\beta - \frac{1}{(n-2)} T \right\} q_{AB} \right]$$

$$- \mathcal{E}_{AB} + \left[ \kappa_{AB} \mathcal{K} - \kappa_{AC} \mathcal{K}^C_B - \frac{1}{2} q_{AB} (\kappa^2 - \kappa_{AB} \kappa^{CA}) \right]$$

(19)
This expression represents the effective Einstein’s equation in an \((n-2)\) dimensional space-time. Our next task is to rewrite the above equation in terms of various components of energy momentum tensor, especially the extrinsic curvature components.

Let us now try to match our solutions to that of \((n-2)\) dimensional brane energy momentum tensor. As usual we assume the following form of metric ansatz,

\[
d s^2 = d\chi^2 + h_{\alpha\beta} dx^\alpha dx^\beta
\]

\[
= d\chi^2 + h_{\alpha\beta} \epsilon_\alpha^\beta \epsilon_\chi^\delta d\chi^2 + q_{AB} dx^A dx^B
\]

where \(q_{AB}\) is the induced metric on the \((n-2)\)-dimensional brane. Following the previous derivation of effective field equation here also we divide the energy momentum tensor, \(\tau_{\alpha\beta}\) in two parts, \(t_{\alpha\beta}\) pertaining to \((n-1)\) dimensional brane and \(\xi_{\alpha\beta}\) as energy momentum tensor in the \((n-2)\) dimensional brane. Thus we have the following relation,

\[
\tau_{\alpha\beta} = t_{\alpha\beta} + \delta (\zeta) \xi_{AB} e_\alpha^A e_\beta^B
\]

\[
= t_{\alpha\beta} + \delta (\zeta) e_\alpha^A e_\beta^B (-\Sigma q_{AB} + \nu_{AB})
\]

where \(\Sigma\) is the brane tension and \(\nu_{AB}\) is the brane energy momentum tensor. Then the tensor \(\xi_{AB}\) can be obtained from the discontinuity of the extrinsic curvature from the relation,

\[
\xi_{AB} = \frac{1}{\kappa_{n-1}^2} ([K_{AB}] - q_{AB} [K])
\]

The jump in the extrinsic curvature by imposing \(Z_2\) symmetry can be obtained in terms of \(\xi_{AB}\) as,

\[
K^+_{AB} = -K^-_{AB} = \frac{1}{2} \kappa_{n-1}^2 \left( \xi_{AB} - \frac{1}{n-3} q_{AB} \xi \right)
\]

Having obtained the extrinsic curvature, the scalar \(K\) can be obtained by contraction with \(q_{AB}\). This has the expression: \(K^+ = -K^- = (\kappa_{n-1}^2 \xi/2(n-3))\). We therefore arrive at the following expressions for various combinations of \(\xi_{AB}\) as they appear in effective field equation:

\[
\xi_{AB} = (n-2)\Sigma^2 q_{AB} - \nu \Sigma q_{AB} - (n-2)\Sigma \nu_{AB} + \nu \nu_{AB}
\]

\[
\xi_{AC} \xi^C_B = \Sigma^2 q_{AB} - 2\Sigma \nu_{AB} + \nu_{AC} \nu^C_B
\]

\[
\xi^2 = (n-2)^2 \Sigma^2 - 2(n-2)\Sigma \nu + \nu^2
\]

\[
\xi_{AB} \xi^{AB} = (n-2)\Sigma^2 - 2\Sigma \nu + \nu_{AB} \nu^{AB}
\]

This enables us to write an expression for the extrinsic curvature contractions in terms of \(\xi_{AB}\), which can further be simplified in terms of \(\nu_{AB}\) following the above identities as:

\[
K_{AB} K - K_{AC} K^C_B - \frac{1}{2} q_{AB} (K^2 - K_{AB} K^{AB})
\]

\[
= \frac{\kappa_{n-1}^4}{4} \left[ \frac{1}{n-3} \xi_{AB} - \xi_{AC} \xi^C_B - \frac{1}{2(n-3)} \xi^2 q_{AB} + \frac{1}{2} q_{AB} \xi^{CD} \xi_{CD} \right]
\]

\[
= \kappa_{n-1}^4 \left[ -\frac{1}{4} \nu_{AC} \nu^C_B + \frac{1}{8} q_{AB} \nu_{CD} \nu^{CD} + \frac{1}{4(n-3)} \nu_{AB} + \frac{1}{8(n-3)} q_{AB} \nu^2 \right]
\]

\[
- \kappa_{n-1}^4 \Sigma^2 q_{AB} + \kappa_{n-1}^2 \Sigma (n-4) q_{AB}
\]

\[
= \frac{\kappa_{n-1}^4}{8(n-3)} q_{AB} + \kappa_{n-1}^2 \Sigma (n-4) q_{AB}
\]
It now just needs some rearrangement to write down the effective field equation on the \((n-2)\) dimensional brane. After inserting all relevant expressions and then rearranging the effective Einstein’s equation on this \((n-2)\) dimensional brane we obtain,

\[
(n-2)G_{AB} = -\left(\frac{n-4}{n-2}\right)\Lambda_{n-1} q_{AB} + \frac{(n-4)}{(n-3)} \pi G_{n-1} \left[ t_{\alpha\beta} e^\alpha_A e^\beta_B + q_{AB} \left( t_{\alpha\beta} s^\alpha s^\beta - \frac{1}{2(n-2)} t \right) \right] + \frac{(n-4)}{(n-3)} \kappa_n^4 \left[ \Pi_{\alpha\beta} e^\alpha_A e^\beta_B + q_{AB} \left( \Pi_{\alpha\beta} s^\alpha s^\beta - \frac{1}{2} \Pi \right) \right] - \frac{(n-4)}{(n-3)} \kappa_n^4 \Sigma \left( n-4 \right) (n-3) \nu_{AB} \]

(29)

where, we have introduced, the following quantities,

\[
\Pi_{\alpha\beta} = \left[ \frac{1}{4} t_{\alpha\mu} t^\mu_{\beta} + \frac{1}{8} h_{\alpha\beta} t^\mu_{\mu} + \frac{1}{4(n-2)} t t_{\alpha\beta} - \frac{1}{8(n-2)} h_{\alpha\beta} t^2 \right]
\]

(30)

\[
\Upsilon_{AB} = \left[ \frac{1}{4} \nu_{AC} \nu^C_B + \frac{1}{8} q_{AB} \nu_{CD} \nu^{CD} + \frac{1}{4(n-3)} \nu_{AB} - \frac{1}{8(n-3)} q_{AB} \nu^2 \right]
\]

(31)

The effective equation can be written in conventional form after some algebraic manipulations such that:

\[
n^{-1} G_{AB} = \kappa_{n-2}^2 T_{AB}
\]

(32)

where energy momentum tensor \(T_{AB}\) has the expression:

\[
T_{AB} = -\Lambda_{n-2} q_{AB} + \nu_{AB} + \frac{\kappa_{n-1}}{\kappa_{n-2}} \Upsilon_{AB} - \frac{1}{\kappa_{n-2}} \Sigma_{AB}
\]

(33)

Here, we have introduced the following two objects for notational simplicity,

\[
\Lambda_{n-2} = \left( \frac{n-4}{n-2} \right) \kappa_{n-1} \left[ \Lambda_{n-1} + \frac{(n-2)}{8(n-3)} \kappa_{n-1} \Sigma \right]
\]

(34)

\[
\kappa_{n-2}^2 = 8\pi G_{n-2} = \kappa_{n-1} \Sigma \frac{(n-4)}{4(n-3)}
\]

(35)
where $\Lambda_n-2$ and $\kappa_n-2$ can be interpreted as the $(n-2)$-dimensional cosmological constant and $(n-2)$ dimensional gravitational constant respectively. However for the case, in which $t_{\alpha\beta} = 0$, i.e., the bulk energy momentum tensor vanishes, the above equation gets simplified to,

$$(n-2)G_{AB} = \kappa_n^{-2}T_{AB}$$

(36)

where $T_{AB}$ has the simplified form:

$$T_{AB} = -\Lambda_n-2q_{AB} + \nu_{AB} + \frac{\kappa_n^{4}}{\kappa_n^{2}-2} \Gamma_{AB} - \frac{1}{\kappa_n^{2}-2} \mathcal{E}_{AB}$$

$$- \left( \frac{n-4}{n-3} \right) \frac{1}{\kappa_n^{2}-2} \left[ E_{\alpha\beta} e_{A}^{\alpha} e_{B}^{\beta} + q_{AB} \left( E_{\alpha\beta} s^{\alpha} s^{\beta} - \frac{E}{n-2} \right) \right]$$

(37)

Thus the structure of effective field equation in $(n-2)$-dimensional brane for vanishing bulk energy momentum tensor is mostly identical to the $(n-1)$-dimensional counterpart except for the Weyl curvature parts. The structure of the effective equation should be quite evident at this stage, which we are going to exploit later to obtain effective field equation in $(n-m)$-dimensional brane.

### 2.3 Generalization to $(n-m)$-dimensional Brane

Having discussed the effective gravitational field equation in $(n-2)$-dimension starting from $n$-dimension through a two step decomposition, it is now time to consider the generalization of the effective field equation for gravity to $(n-m)$-dimension. This can be obtained following the analogy of previous results. The final expression for effective field equation in $(n-m)$ dimension is:

$$(n-m)G_{AB} = \kappa_n^{2}T^{(n-m)}_{AB}$$

(38)

where the energy momentum tensor has the following expression:

$$T^{(n-m)}_{AB} = -\Lambda_{n-m}q^{(n-m)}_{AB} + \nu_{AB} + \frac{\kappa_n^{4}}{\kappa_n^{2}-m} \Gamma^{(n-m)}_{AB} - \frac{1}{\kappa_n^{2}-m} E^{(n-m)}_{AB}$$

$$+ \sum_{i=2}^{m} \left( \frac{n-i-2}{n-i-1} \right) \left[ \kappa_n^{2} \left( \frac{\kappa_n^{2}-i+1}{\kappa_n^{2}-i} \right) \left( E^{(n-i+1)}_{\alpha\beta} e_{A}^{\alpha} e_{B}^{\beta} + q_{AB} \left( E^{(n-i+1)}_{\alpha\beta} s^{\alpha} s^{\beta} - \frac{1}{n-2} \left( E^{(n-i+1)}_{\alpha\beta} \right) \right) \right) \right]$$

$$+ \frac{\kappa_n^{4-i+2}}{\kappa_n^{i-1}} \left( \Pi^{(n-i+1)}_{\alpha\beta} e_{A}^{\alpha} e_{B}^{\beta} + q_{AB} \left( \Pi^{(n-i+1)}_{\alpha\beta} s^{\alpha} s^{\beta} - \frac{1}{n-2} \left( \Pi^{(n-i+1)}_{\alpha\beta} \right) \right) \right)$$

$$- \frac{1}{\kappa_n^{2-i}} \left( E^{(n-i+1)}_{\alpha\beta} e_{A}^{\alpha} e_{B}^{\beta} + q_{AB} \left( E^{(n-i+1)}_{\alpha\beta} s^{\alpha} s^{\beta} - \frac{E^{(n-i+1)}}{n-2} \right) \right)$$

(39)
with the following identifications:

\[
\Lambda_{n-m} = \left(\frac{n-m-2}{n-m}\right) \kappa_{n-m+1}^2 \Lambda_{n-m+1} + \frac{(n-m)}{8(n-m-1)} \kappa_{n-m+1} \Sigma^2
\]  

(40)

\[
\kappa_{n-m}^2 = 8\pi G_{n-m} = \frac{(n-m-2)}{4(n-m-1)} \Sigma^{(n-m)} \kappa_{n-m+1}^4
\]  

(41)

\[
\Pi_{\alpha\beta} = -\frac{1}{4} t_{\alpha\mu} t^\mu_{\beta} + \left[ \frac{1}{8} h_{\alpha\beta} t_{\mu\nu} t^{\mu\nu} + \frac{1}{4(n-m)} t_{\alpha\beta} - \frac{1}{8(n-m)} h_{\alpha\beta} t^2 \right]
\]  

(42)

\[
\Upsilon_{AB} = -\frac{1}{4} \tau^{AC}_B \tau^C_D + \frac{1}{8(n-m-1)} \tau^{AB} - \frac{1}{8(n-m-1)} q_{AB} \tau^2
\]  

(43)

However it may be observed that in the situation where there is only matter on the \((n-m)\)-dimensional brane but not on the other higher dimensional branes, all the additional terms vanishes except the Weyl tensor part. If the bulk spacetime is assumed to be AdS then the Weyl tensor can taken to be zero, in which case the effective field equation takes a simplified form:

\[
(n-m)G_{\mu\nu} = \kappa_{n-m}^2 \left( \Lambda_{n-m} q_{AB}^{(n-m)} + \frac{\kappa_{n-m+1}^4}{\kappa_{n-m}^2} \Upsilon_{AB}^{(n-m)} \right)
\]  

(44)

Thus in this simple situation the effective field equation has contribution from \((n-m)\) dimensional cosmological constant, energy momentum tensor on the brane and higher order correction terms originating from \(T_{AB}\). Having derived the effective field equation in Einstein gravity let us now consider the identical situation in \(f(R)\) gravity as well.

### 3 Generalization to bulk \(f(R)\) gravity

In recent years modifications of Einstein-Hilbert action by higher curvature terms is a subject of great interest. A very promising candidate among such modifications is the \(f(R)\) gravity theory. The modifications introduced by the \(f(R)\) term in the Lagrangian can address variety of problems. This theory also has the potential to survive all known tests of general relativity. It is therefore worthwhile to explore the nature of the effective gravitational field equation on a brane where the bulk is endowed with \(f(R)\) gravity.

#### 3.1 Effective Field Equation on the \((n-1)\)-dimensional Brane

We consider a \(n\)-dimension bulk spacetime endowed with \(f(R)\) gravity. Following the standard procedure i.e., starting from the Gauss-Codazzi equation, subsequently using the \(Z_2\) symmetry, we can obtain effective gravitational field equation on the brane. This has already been derived earlier in the context of a 4-dimensional brane embedded in a 5-dimensional bulk (see [48] and the references therein). The effective gravitational field equation on the \((n-1)\)-dimensional brane turns out to be:

\[
(n-1)G_{\mu\nu} = \kappa_{n-1}^2 T_{\mu\nu}^{(n-1)}
\]  

(45)

where the stress-energy tensor \(T_{\mu\nu}^{(n-1)}\) appearing on the right hand side can be decomposed into several small pieces. These include effective \((n-1)\)-dimensional cosmological constant induced
from the n-dimensional cosmological constant, brane energy momentum tensor and its higher order contribution. Also due to the presence of \( f(\mathcal{R}) \) gravity in the bulk there is an extra term and all these terms along with the non-local bulk Weyl tensor turns out to be:

\[
T^{(n-1)}_{\mu\nu} = -\Lambda_{n-1} h_{\alpha\beta} + \tau_{\alpha\beta} + \frac{\kappa_4^n}{\kappa_{n-1}^2} \pi_{\alpha\beta} - \frac{1}{\kappa_{n-1}^2} E_{\alpha\beta} + \frac{1}{\kappa_{n-1}^2} Q_{\alpha\beta} \tag{46}
\]

where the additional term due to \( f(\mathcal{R}) \) gravity is \( Q_{\mu\nu} \). This additional term \( Q_{\mu\nu} \) has the following expression:

\[
Q_{\mu\nu} = \left[ \frac{1}{4} \frac{f(\mathcal{R})}{f'(\mathcal{R})} - \frac{1}{4} \mathcal{R} - \frac{2}{3} \frac{\square f'(\mathcal{R})}{f'(\mathcal{R})} + \frac{2}{3} \frac{\nabla_a \nabla_b f'(\mathcal{R})}{f'(\mathcal{R})} \right] h_{\mu\nu} + \frac{2}{3} \frac{\nabla_a \nabla_b f'(\mathcal{R})}{f'(\mathcal{R})} \epsilon^a_{\mu} \epsilon^b_{\nu} \tag{47}
\]

Note that in the Einstein-Hilbert limit i.e. \( f(\mathcal{R}) \to \mathcal{R} \) the above term identically vanishes and the effective equation reduces to Eqs. (14) and (15). In the expression of the energy momentum tensor, \( \Lambda_{n-1} \) is the \((n-1)\)-dimensional brane cosmological constant (for explicit expression see Eq. (16)), \( \tau_{\mu\nu} \) is the brane energy momentum tensor, \( \pi_{\alpha\beta} \) contain higher order terms like \( \tau_\alpha^\mu \tau_\beta_\mu \) etc. (see Eq. (13) for detailed expression). Finally \( E_{\mu\nu} \) represents the non-local bulk effect.

In order to simplify the expression for \( Q_{\mu\nu} \), we can impose some assumptions. The simplification can be significant with the choice: \( \partial_y R = 0 \). Then using Taylor series expansion of bulk curvature \( \mathcal{R} \) around \( y = 0 \) hypersurface leads to: \( \mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1 y + \mathcal{R}_2 y^2/2 + \mathcal{O}(y^3) \). As we have assumed that the bulk curvature depends only on the extra dimension \( y \), all the coefficients appearing in the Taylor series expansion are constants. Hence we can conclude that all the derivatives calculated at \( y = 0 \) yield a constant contribution which does not depend on any of the brane coordinates. In this case we can rewrite the stress-energy tensor as:

\[
T^{(n-1)}_{\mu\nu} = -\Lambda_{\text{eff}} h_{\alpha\beta} + \tau_{\alpha\beta} + \frac{\kappa_4^n}{\kappa_{n-1}^2} \pi_{\alpha\beta} - \frac{1}{\kappa_{n-1}^2} E_{\alpha\beta} \tag{48}
\]

where we have an effective cosmological constant,

\[
\Lambda_{\text{eff}} = \Lambda_{n-1} - \frac{F(\mathcal{R})}{\kappa_{n-1}^2} \tag{49}
\]

Thus the cosmological constant gets modified due to \( f(\mathcal{R}) \) gravity in the bulk leading to an additional constant term \( F(\mathcal{R}) \) to the induced cosmological constant \( \Lambda_{n-1} \). This can provide a possible explanation to the cosmological constant problem by fine tuning \( \Lambda_{n-1} \) with \( F(\mathcal{R}) \) term where the quantity \( F(\mathcal{R}) \) has the following expression:

\[
F(\mathcal{R}) = \left( g(\mathcal{R}) + \frac{2}{3} \frac{\nabla A \nabla B f'(\mathcal{R})}{f'(\mathcal{R})} n^A n^B \right)_{y=0} \tag{50}
\]

Having derived the effective gravitational field equation in \((n-1)\)-dimensional brane we will now write down the counterpart of this relation in \((n-m)\)-dimensional brane as well.

### 3.2 Generalization to \((n-m)\)-dimensional Brane

In this section we will generalize the above setup to an arbitrary lower dimensional hypersurface, namely, the \((n-m)\)-dimensional brane. This can be done easily by taking a cue from the
discussion in Einstein gravity. The effective field equation in the \((n-m)\)-dimensional brane leads to the following expression:

\[
(n-m)G_{AB} = \kappa_{n-m}^2 T_{AB}^{(n-m)}
\]

with the following expression for the energy-momentum tensor as:

\[
T_{AB}^{(n-m)} = -\Lambda_{n-m}q_{AB}^{(n-m)} + \tau_{AB}^{(n-m)} + \frac{\kappa_{n-m+1}^4}{\kappa_{n-m}^2} \Upsilon_{AB}^{(n-m)} + \frac{1}{\kappa_{n-m}^2} Q_{AB}^{(n-m)} - \frac{1}{\kappa_{n-m}^2} E_{AB}^{(n-m)}
\]

\[
= \sum_{i=2}^{m} \left( \frac{n-i-2}{n-i-1} \right) \left( \frac{\kappa_{n-i+1}^2}{\kappa_{n-i}^2} \right) \left[ \Pi^{(n-i+1)} e^{\alpha \beta}_{AB} + q_{AB} \left( \frac{\Pi^{(n-i+1)} s^{\alpha} s^{\beta} - \frac{1}{n-2} \Pi^{(n-i+1)}}{n-2} \right) \right] + \frac{\kappa_{n-i+2}^4}{\kappa_{n-i}^2} \left( \Pi^{(n-i+1)} e^{\alpha \beta}_{AB} + q_{AB} \left( \frac{\Pi^{(n-i+1)} s^{\alpha} s^{\beta} - \frac{Q^{(n-i+1)}}{n-2}}{n-2} \right) \right)
\]

\[
= \frac{1}{\kappa_{n-i}^2} \left( E^{(n-i+1)} e^{\alpha \beta}_{AB} + q_{AB} \left( E^{(n-i+1)} s^{\alpha} s^{\beta} - \frac{E^{(n-i+1)}}{n-2} \right) \right)
\]

Here \(\Lambda_{n-m}\) is the induced brane cosmological constant, \(\tau_{AB}^{(n-m)}\) represents the brane energy momentum tensor and \(\Upsilon_{AB}^{(n-m)}\) is the higher order term. All the other extra terms originate from the effect of higher dimensional energy momentum tensor, \(f(R)\) gravity and non-local effects from the bulk. However just as in the previous section in this case also we can simplify the expression further by assuming that bulk curvature depends only on the extra dimension \(y\). Also if we assume that there is no energy momentum tensor on the higher dimensional branes and the bulk is AdS, then Weyl tensor vanishes identically. The effective gravitational field equation, then takes a particularly simple form. In this case following [48], we arrive at:

\[
(n-m)G_{AB} = \kappa_{n-m}^2 \left( -\Lambda_{n-m}q_{AB}^{(n-m)} + \tau_{AB}^{(n-m)} + \frac{\kappa_{n-m+1}^4}{\kappa_{n-m}^2} \Upsilon_{AB}^{(n-m)} \right)
\]

where \(\Lambda_{n-m}\) has to be constructed from \(\Lambda_{\text{eff}} = \Lambda_{n-1} - F(R)\) by standard procedure illustrated for Einstein gravity. Thus in this case the effective equation would be identical to that obtained from general relativity, except for a modified cosmological constant. Also \(\kappa_{n-m}^2\) is related to \((n-m)\)-dimensional gravitational constant, \(\tau_{AB}^{(n-m)}\) is the matter energy-momentum tensor on the \((n-m)\)-dimensional brane, with \(\Upsilon_{AB}^{(n-m)}\) being its higher order extension.

This completes our discussion on effective equation. However as an illustration we will consider a simple application of our work in a cosmological context in the next section.

### 4 An Application

Having discussed effective field equation in both Einstein and \(f(R)\) gravity theory on a lower dimensional hypersurface we apply this in a cosmological context. As an illustration we start with 6-dimensional bulk spacetime and try to obtain a cosmological solution from effective field
equation on a 4-dimensional brane for Einstein’s gravity. In order to obtain analytical solution for simplicity we assume $E_{ab} = 0$ i.e. contribution from electric part of Weyl tensor is vanishing. Then effective field equation in 4-dimension takes the following form:

$$
G_{AB} = \kappa^2 \Lambda^2 q_{AB} + \kappa^2 \tau_{AB} + \kappa^4 \Upsilon_{AB}
$$

$$
+ \frac{2}{3} \kappa^2 \left( t_{\alpha \beta} e^A_{\alpha} e^B_{\beta} + q_{AB} \left( t_{\alpha \beta} s^\alpha_{\alpha} s^\beta_{\beta} - \frac{1}{4} \right) \right)
$$

$$
+ \frac{2}{3} \kappa^4 \left( \Pi_{\alpha \beta} e^A_{\alpha} e^B_{\beta} + q_{AB} \left( \Pi_{\alpha \beta} s^\alpha_{\alpha} s^\beta_{\beta} - \frac{1}{4} \Pi \right) \right)
$$

(54)

In this expression along with brane energy momentum tensor $\tau_{AB}$ and its higher order term $\Upsilon_{AB}$, two additional contribution from 5-dimensional matter energy-momentum tensor is present. We assume that the brane is filled with perfect fluid with the following form for the energy-momentum tensor on the 4-dimension as:

$$
\tau^A_B = \text{diag} (-\rho, p, p, p)
$$

(55)

where $\rho$ represents energy density of the matter fields and $p$ yields its pressure. The energy-momentum tensor in 5-dimension is taken to be pressure free dust, such that its energy momentum tensor has the simple form:

$$
\tau^5_\beta = \text{diag} (-\rho_5, 0, 0, 0, 0)
$$

(56)

where the subscript ‘5’ indicates that it is matter from 5-dimensional brane. Thus we arrive at the following expression for the higher order components:

$$
\Pi_{tt} = \frac{3}{32} \rho^2_5; \quad \Pi = -\frac{3}{32} \rho^2_5
$$

(57)

Having obtained all the components in the above effective field equation, we can obtain the effective field equation itself. Both the time-time and spatial components of effective equation lead to:

$$
3H^2 = \kappa^2 \Lambda^4 + \kappa^2 \rho + \frac{1}{12} \kappa^4 \rho^2 + \frac{1}{2} \kappa^2 \rho_5 + \frac{3}{64} \kappa^4 \rho^2_5
$$

$$
H^2 + \frac{\dot{a}}{a} = \kappa^2 \Lambda^4 - \kappa^2 \rho - \frac{1}{12} \kappa^4 \left( \rho^2 + 2 p \rho \right) - \frac{1}{6} \kappa^2 \rho_5 - \frac{1}{64} \kappa^4 \rho^2_5
$$

(58)

For notational convenience we will write $\kappa^2 \Lambda^4 = \Lambda_{\text{eff}}$. Also for the bulk matter we have $\rho_5 = (\rho_{5,0}/\ell)(1/a^3)$, where $\ell$ represents the finite length of the 5-dimensional brane coordinate. Also if we assume that the matter in the 4-dimensional brane is also pressureless then we have the following expression for the Hubble parameter:

$$
H^2 = \frac{\Lambda_{\text{eff}}}{3} + \frac{8 \pi G_N}{3} \rho_0 + \frac{1}{36} \kappa^4 \rho^2_0 + \frac{1}{6} \kappa^2 \frac{\rho_5,0}{\ell} + \frac{1}{36} \kappa^4 \rho^2_5,0 \frac{1}{\ell^2} a^6
$$

(59)

where $\rho_0$ represents the value of energy density in the present epoch and $\rho_{5,0}$ represents the same for 5-dimensional matter. The above expression for Hubble parameter can be simplified and arranged properly leading to:

$$
H^2 = \frac{\Lambda_{\text{eff}}}{3} + \left\{ \frac{8 \pi G_N}{3} \rho_0 + \frac{1}{6} \kappa^2 \frac{\rho_5,0}{\ell} \right\} \frac{1}{a^3} + \left\{ \frac{1}{36} \kappa^4 \rho^2_0 + \frac{1}{64} \kappa^4 \rho^2_5,0 \frac{1}{\ell^2} \right\} \frac{1}{a^6}
$$

(60)
Thus at early universe we have a high energy regime, where $\rho$ is so high that the solutions simplifies substantially, yielding clearer physical insight. For that purpose we choose the situation $\Lambda > 0$. Though the solutions has a complicated appearance, we can impose certain conditions under which the solutions simplifies substantially, leading to the following solution for the scale factor:

$$C_\Lambda = \frac{\Lambda_{\text{eff}}}{3}; \quad C_\rho = \frac{8\pi G_N}{3} \rho_0 + \frac{1}{6} \kappa_5^2 \rho_{5,0} \ell; \quad C_{\text{Sq}} = \frac{1}{36} \kappa_5^4 \rho_0^2 + \frac{1}{64} \kappa_6^4 \rho_{5,0}^2$$ \hspace{2cm} (61)

such that we obtain the following solution to the scale factor as:

$$\exp \left[ 3 \sqrt{C_\Lambda} (t - t_0) \right] = \frac{2 \sqrt{C_\Lambda} \sqrt{C_\Lambda a^6 + C_\rho a^4 + C_{\text{Sq}} + 2C_\Lambda a^3 + C_\rho}}{2 \sqrt{C_\Lambda} \sqrt{C_\rho} + C_\rho + C_{\text{Sq}} + 2C_\Lambda + C_\rho}; \quad C_\Lambda > 0 \hspace{2cm} (62)$$

$$3 \sqrt{-C_\Lambda} (t_0 - t) = \sin^{-1} \left( \frac{2C_\Lambda a^3 + C_\rho}{\sqrt{C_\rho} - 4C_\Lambda C_{\text{Sq}}} \right) - \sin^{-1} \left( \frac{2C_\Lambda + C_\rho}{\sqrt{C_\rho} - 4C_\Lambda C_{\text{Sq}}} \right); \quad C_\Lambda < 0, \quad C_\rho > 4C_\Lambda C_{\text{Sq}} \hspace{2cm} (63)$$

Though the solutions has a complicated appearance, we can impose certain conditions under which the solutions simplifies substantially, leading to the following solution to the scale factor as:

$$a^3 = \left\{ 6\pi G \rho_0 + 3 \frac{\kappa_5^2}{8} \rho_{5,0} \ell \right\} t^2 + \left\{ \frac{12\pi G \rho_0^2}{\lambda T} \left( 1 + \frac{9 \kappa_6^4 \rho_{5,0}^2}{16 \kappa_5^4 \ell^2 \rho_0^2} \right) \right\} t \hspace{2cm} (64)$$

leading to the following solution for the scale factor:

$$H^2 = \left\{ \frac{8\pi G_N \rho_0}{3} + \frac{\kappa_5^2}{6} \rho_{5,0} \ell \right\} \frac{1}{a^3} + \frac{4\pi G \rho_0^2}{3\lambda T} \left( 1 + \frac{9 \kappa_6^4 \rho_{5,0}^2}{16 \kappa_5^4 \ell^2 \rho_0^2} \right) \frac{1}{a^6} \hspace{2cm} (65)$$

It is clear from the above expression that the universe undergoes a transition in the expansion rate. The time scale when this happened depends on whether the brane matter energy dominates over the bulk matter or not. Below we present the result for timescale of transition for both the situations:

$$t \sim \sqrt{\frac{1}{3\pi G \lambda_T}} = \frac{4}{\lambda_T \kappa_5^2} = -\frac{2}{3} \Lambda_5^{-1}; \quad (\rho_{5,0}/\ell \ll \rho_0) \hspace{2cm} (66)$$

$$t \sim \frac{\kappa_5^2}{\kappa_5^2} \left( 1 - \frac{16\pi G N \rho_0 \ell}{\kappa_5^2 \rho_{5,0}} \right) \quad (\rho_{5,0}/\ell \gg \rho_0) \hspace{2cm} (67)$$

Thus at early universe we have a high energy regime, where $a \sim t^{1/3}$; while at late time low energy regime the scale factor variation with time modifies to $a \sim t^{2/3}$, which is the standard
evolution of the matter field. Above we have illustrated a particular choice, however we could have chosen several other assumptions which can lead to different results, showing the complexity and rich structure of these brane world cosmological scenarios.

Thus with matter present in the 5-dimensional brane we can have standard cosmology with proper scaling of the scale factor with time. Moreover from the effective field equation it is clear that the 5-dimensional matter adds to the 4-dimensional one and enhances the total non-relativistic matter content of the universe (4-dimensional brane we live in). Hence matter fields present in the extra dimension can act as a viable source of dark matter (a related issue has been addressed in [51]). Also structure of angular power spectrum, perturbation in this multiple extra dimension scenario can lead to interesting and important results by providing constraints on various parameters in this models. This is a work under progress and the results will be presented elsewhere.

5 Discussion

Motivated by the success of extra dimensional models to explain various physical problems, e.g., hierarchy problem, cosmological constant problem in this work we have considered a very general higher dimensional model. Starting from Einstein’s theory in the bulk, which is taken to be $n$-dimensional we construct various induced objects on an $(n-1)$-dimensional hypersurface, called brane. Using the Gauss-Codazzi equation and by imposing $Z_2$ symmetry we have derived the effective field equation in $(n-1)$ dimensional brane. In the limit when $n \rightarrow 5$ we reproduce earlier results. Having accomplished the job of obtaining the effective field equation in $(n-1)$ dimension, we go one step further. Extending this analysis, the effective field equation in $(n-2)$-dimensional brane is derived. This in turn helps us to write down the effective field equation on any arbitrary $(n-m)$-dimensional brane. This provides a complete generalization of existing set up, by providing effective field equation in any lower dimensional hypersurface. We generalize this analysis further by incorporating $f(R)$ gravity in the bulk instead of Einstein’s gravity.

In $f(R)$ gravity also we follow the same procedure by relating geometrical objects like extrinsic curvature, Riemann curvature in the $n$-dimensional bulk with the $(n-1)$-dimensional brane. In this case as well from the structure of the effective equation we can immediately generalize the set up on an arbitrary lower dimensional hypersurface as well.

Having derived all the necessary theoretical ingredients, we consider a possible application of our results. As an illustration, we consider a 6-dimensional bulk spacetime in which a 4-dimensional brane is embedded. We have taken the 5-dimensional sub-manifold to contain matter, which changes the energy momentum tensor of the effective field equation on the 4-dimensional brane. Starting from the effective field equation we have solved for the scale factor and have shown that the standard cosmology to be an inherent and natural consequence of our model. Finally we comment on the possibility of the bulk matter as a possible candidate of dark matter in the lower dimensional brane.

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A Appendix: Detailed Calculations

In this section, we provide some detailed expressions, which have been used in the main text for arriving at various results.

A.1 Identities for the Derivation of $(n-1)$ Dimensional Effective Field Equation

The starting point of obtaining the effective field equation is the Gauss-Codazzi equation, which relates the $(n-1)$-dimensional curvature tensor to $n$-dimensional curvature tensor. For our purpose these two tensors are related through the following relation:

$$(n-1)R_{\alpha\beta\mu\nu} = (n)R_{abcd}e^a_{\alpha}e^b_{\beta}e^c_{\mu}e^d_{\nu} - \epsilon (K_{\alpha\nu}K_{\beta\mu} - K_{\alpha\mu}K_{\beta\nu})$$ (69)

where we have used the definition:

$$K_{\alpha\beta} = e^a_{\alpha}e^b_{\beta}\nabla a b$$ (70)

with $\epsilon = n_i n^i$. Contraction of Eq. (69) with the $(n-1)$ dimensional metric $h_{\alpha\beta} = e^a_{\alpha}e^b_{\beta}(g_{ab} - \epsilon n_a n_b)$ leads to the connection between $(n-1)$ dimensional Ricci tensor to $n$ dimensional curvature tensor as,

$$(n-1)R_{\alpha\mu} = h^{\beta\nu}(n-1)R_{\alpha\beta\mu\nu}$$

$$= (n)\epsilon R_{abcd}e^a_{\alpha}e^b_{\beta}e^c_{\mu}h^{\beta\nu} - \epsilon (K_{\alpha\nu}K_{\beta\mu} - K_{\alpha\mu}K_{\beta\nu}) h^{\beta\nu}$$

$$= (n)\epsilon R_{abcd}e^a_{\alpha}e^b_{\beta}e^c_{\mu} (g^{bd} - \epsilon n^b n^d) - \epsilon (K_{\alpha\nu}K_{\beta\mu} - K_{\alpha\mu}K_{\beta\nu})$$

$$= (n)\epsilon R_{ac}e^a_{\alpha}e^c_{\mu} - \epsilon (n)\epsilon R_{abcd}e^a_{\alpha}e^b_{\beta}e^c_{\mu}h^{\beta\nu} - \epsilon (K_{\alpha\nu}K_{\beta\mu} - K_{\alpha\mu}K_{\beta\nu})$$ (71)

The Ricci scalar can now be obtained as,

$$(n-1)R = h^{\alpha\mu}(n-1)R_{\alpha\mu}$$

$$= (n)\epsilon R_{ac}e^a_{\alpha}e^c_{\mu}h^{\alpha\mu} - \epsilon (n)\epsilon R_{abcd}e^a_{\alpha}e^b_{\beta}e^c_{\mu}h^{\alpha\mu} - \epsilon (K_{\alpha\nu}K_{\beta\mu} - K_{\alpha\mu}K_{\beta\nu}) h^{\alpha\mu}$$

$$= (n)\epsilon R_{ac} (g^{ac} - \epsilon n^a n^c) - \epsilon (n)\epsilon R_{abcd}n^b n^d (g^{ac} - \epsilon n^a n^c) - \epsilon (K_{\alpha\nu}K_{\beta\mu} - K_{\alpha\mu}K_{\beta\nu})$$

$$= (n) - 2\epsilon (n)\epsilon R_{ac}n^a n^c - \epsilon (K_{\alpha\nu}K_{\beta\mu} - K_{\alpha\mu}K_{\beta\nu})$$ (72)

The above relation can be further simplified by using an identity for $(n)R_{ab}n^a n^b$, which can be written as:

$$(n)R_{ab}n^a n^b = \nabla_a (n^b \nabla_b n^a) - \nabla_a n^b \nabla_b n^a - \nabla_b (n^b \nabla_a n^a) + (\nabla_b n^b)^2$$

$$= -\nabla_i (Kn^i - a^i) - K_{\alpha\beta}K^{\alpha\beta} + K^2$$ (73)

Then we obtain $(n)R$ in terms of $(n-1)R$ as:

$$(n)R = (n-1)R - \epsilon (K_{\alpha\mu}K^{\alpha\mu} - K^2) - 2\epsilon \nabla_i (Kn^i - a^i)$$ (74)
The Riemann tensor can be decomposed into Ricci tensor, Ricci scalar and Weyl curvature tensor.

\[(n-1)G_{\alpha\mu} = (n-1)R_{\alpha\mu} - \frac{1}{2} h_{\alpha\mu}(n-1)R \]

\[= (n)R_{ac} e^a_{\alpha} e^c_{\mu} - \epsilon (n)R_{abcd} e^a_{\alpha} n^b e^c_{\mu} n^d - \epsilon (K_{\alpha\mu} K_{ab} - K_{\alpha \beta} K_{ab}) - \frac{1}{2} h_{\alpha\mu} \left[(n)R - 2\epsilon (n)R_{ac} n^a n^c - \epsilon (K_{\mu \nu} K_{\mu \nu} - K^2)\right] \]

\[= (n)R_{ac} - \frac{1}{2} g_{ac} (n)R + \epsilon (n)R_{ac} n^a n^c h_{\alpha\mu} - \epsilon (n)R_{abcd} e^a_{\alpha} n^b e^c_{\mu} n^d + \epsilon \left[KK_{\alpha\mu} - K_{\alpha \beta} K_{\beta \mu} - \frac{1}{2} h_{\alpha\mu} (K^2 - K_{\mu \nu} K_{\mu \nu})\right] \quad (75) \]

The relation between \((n-1)\) dimensional Einstein tensor and \(n\) dimensional Riemann tensor is therefore given by

\[(n)R_{abcd} = (n)C_{abcd} + \frac{1}{n-2} (g_{ac} R_{bd} - g_{ad} R_{bc} - g_{bc} R_{ad} + g_{bd} R_{ac}) \]

\[\quad - \frac{1}{(n-1)(n-2)} (g_{ac} g_{bd} - g_{ad} g_{bc}) R \quad (76)\]

which suggests,

\[(n)R_{abcd} e^a_{\alpha} n^b e^c_{\mu} n^d = (n)C_{abcd} e^a_{\alpha} e^c_{\mu} n^d + \frac{1}{n-2} (R_{ac} n^a n^c h_{\alpha\mu} + \epsilon R_{ac} e^a_{\alpha} e^c_{\mu}) \]

\[\quad - \frac{\epsilon}{(n-1)(n-2)} h_{\alpha\mu} R \quad (77)\]

Then substitution of the above equation in Eq. (75) leads to,

\[(n-1)G_{\alpha\beta} = (n)G_{\alpha\beta} e^a_{\alpha} e^b_{\beta} + \epsilon \left(\frac{n-3}{n-2}\right) (n)R_{ab} n^a n^b h_{\alpha\beta} - \frac{1}{n-2} (n)R_{ab} e^a_{\alpha} e^b_{\beta} + \frac{1}{(n-1)(n-2)} (n)R h_{\alpha\beta} - \epsilon (n)E_{\alpha\beta} \]

\[+ \epsilon \left[KK_{\alpha\mu} - K_{\alpha \beta} K_{\beta \mu} - \frac{1}{2} h_{\alpha\mu} (K^2 - K_{\mu \nu} K_{\mu \nu})\right] \quad (78)\]

where we have defined, \((n)E_{\alpha\beta} = (n)C_{abcd} e^a_{\alpha} n^b e^c_{\beta} n^d\). This yields the \(n\) dimensional Einstein equation, with the following expressions:

\[(n)R_{ab} - \frac{1}{2} g_{ab} (n)R = \kappa_n^2 T_{ab}; \quad (n)R = -\frac{2}{n-2} \kappa_n^2 T; \quad (n)R_{ab} = \kappa_n^2 \left(T_{ab} - \frac{1}{n-2} g_{ab} T\right) \quad (79)\]
We therefore have the following expression

\[(n)G_{ab}e^a_\alpha e^b_\beta + \epsilon \left( \frac{n-3}{n-2} \right) (n)R_{ab}a^a n^b h_{\alpha \beta} - \frac{1}{n-2} (n)R_{ab}e^a_\alpha e^b_\beta + \frac{1}{(n-1)(n-2)} (n)R h_{\alpha \beta} \]

\[= \kappa^2_n T_{ab} e^a_\alpha e^b_\beta + \epsilon \kappa^2_n \left( \frac{n-3}{n-2} \right) \left( T_{ab} - \frac{1}{n-2} g_{ab} T \right) n^a n^b h_{\alpha \beta} \]

\[= \kappa^2_n \frac{1}{n-2} \left( T_{ab} - \frac{1}{n-2} g_{ab} T \right) e^a_\alpha e^b_\beta - \kappa^2_n \left( \frac{2}{(n-1)(n-2)} h_{\alpha \beta} T \right) \]

\[= \kappa^2_n \frac{n-3}{n-2} T_{ab} e^a_\alpha e^b_\beta + h_{\alpha \beta} \left\{ \epsilon T_{ab} n^a n^b - \frac{1}{n-1} T \right\} \]

This is the result used in Sec. 2.1.

### A.2 Identities for the Derivation of \((n-2)\) Dimensional Effective Field Equation

We start with the connection between \((n-2)\) dimensional curvature tensor and \((n-1)\) dimensional curvature tensor as,

\[(n-2)R_{ABCD} = (n-1) R_{\alpha \beta \mu \nu} e^\alpha_A e^\beta_B e^\mu_C e^\nu_D - (K_{AD} K_{BC} - K_{AC} K_{BD}) \]

where we have the extrinsic curvature on the \((n-2)\) dimensional surface as: \(K_{AB} = e^\alpha_A e^\beta_B \nabla \alpha s_\beta\), with \(s_\alpha\) being the normal to the surface. Then the induced \((n-2)\) dimensional metric turns out to be, \(q_{AB} = e^\alpha_A e^\beta_B (h_{\alpha \beta} - s_\alpha s_\beta)\). Contracting the above equation with \(q^{BD}\) leads to,

\[(n-2)R_{AC} = (n-1) R_{\alpha \beta \mu \nu} e^\alpha_A e^\beta_B e^\mu_C e^\nu_D q^{BD} - (K_{AD} K^D_C - K_{AC} K) \]

\[= (n-1) R_{\alpha \beta \mu \nu} e^\alpha_A e^\beta_B (h^{\beta \nu} - s^{\beta} s^{\nu}) - (K_{AD} K^D_C - K_{AC} K) \]

\[= (n-1) R_{\alpha \beta \mu \nu} e^\alpha_A e^\beta_B (h^{\beta \nu} - s^{\beta} s^{\nu}) - (n-1) R_{\alpha \beta \mu \nu} e^\alpha_A e^\beta_B (s^{\beta} s^{\nu} - (K_{AD} K^D_C - K_{AC} K) \]

On further contraction we arrive at the following result,

\[(n-2)R = q^{AB} (n-2)R_{AB} \]

\[= (n-1) R - 2 (n-1) R_{\alpha \mu} s^\alpha s^{\mu} - (K_{AB} K^{AB} - K^2) \]

Then we can use Eq. (73) to obtain an identical relation for \((n-1) R_{\alpha \mu} s^\alpha s^{\mu}\). This leads to the relation:

\[(n-1) R = (n-2) R - (K_{AB} K^{AB} - K^2) - 2 D_\alpha (K s^\alpha - s^\beta D_\beta s_\alpha) \]

which on using Eq. (74) yields:

\[(n) R = (n-2) R - (K_{AB} K^{AB} - K^2) - 2 D_\alpha (K s^\alpha - s^\beta D_\beta s^\alpha) \]

\[+ (K_{\mu \nu} K^{\mu \nu} - K^2) + 2 \nabla \alpha (Kn^\alpha - a^\alpha) \]
In order to obtain the effective equation on this \((n-2)\) dimensional surface we need various contractions of the Riemann tensor. For that purpose we start with the following identity,

\[
(\alpha \beta \mu \nu)^{n-1} \alpha \beta \mu \nu = \left\{ (\alpha \beta \mu \nu)^{n-1} \alpha \beta \mu \nu + \frac{1}{n-3} \left[ h_{\alpha \beta} (\alpha \beta \mu \nu)^{n-1} \alpha \beta \mu \nu - h_{\alpha \beta} (\alpha \beta \mu \nu)^{n-1} \alpha \beta \mu \nu + R_{\alpha \beta} (n-1) R_{\alpha \beta} \right] \right\}
\]

\[
= c_{AB} + \frac{1}{n-3} \left( q_{AB} (\alpha \beta \mu \nu)^{n-1} \alpha \beta \mu \nu + (n-1) R_{\alpha \beta} e_{A}^{\alpha} e_{B}^{\beta} \right)
\]

\[
= c_{AB} + \frac{1}{n-3} \left( q_{AB} (\alpha \beta \mu \nu)^{n-1} \alpha \beta \mu \nu + (n-1) R_{\alpha \beta} e_{A}^{\alpha} e_{B}^{\beta} \right)
\]

\[
(\alpha \beta \mu \nu)^{n-1} \alpha \beta \mu \nu = \frac{1}{(n-2)(n-3)} (n-1) R_{\alpha \beta} c_{AB} (n-1) R_{\alpha \beta} c_{AB}
\]

where we have defined the tensor \(E_{AB} = (n-1) C_{\alpha \beta \mu \nu} e_{A}^\alpha e_{B}^\beta e_{C}^\mu e_{D}^\nu\). Then the effective equation on the \((n-2)\) dimensional hypersurface turns out to be,

\[
(n-2) G_{AB} = (n-1) G_{\alpha \beta} e_{A}^{\alpha} e_{B}^{\beta} + (n-1) R_{\alpha \beta} s_{A}^{\alpha} s_{B}^{\beta} q_{AB} - 2 E_{AB} + \frac{1}{n-2}(n-3) q_{AB} (n-1) R_{\alpha \beta} c_{AB}
\]

\[
+ \left[ K_{AB} K - K_{AC} K_{B}^{C} \right]
\]

\[
- \frac{1}{2} q_{AB} \left( K_{2} - K_{AB} K_{AB} \right)
\]

\[
= (n-1) G_{\alpha \beta} e_{A}^{\alpha} e_{B}^{\beta} + \frac{n-4}{n-3} q_{AB} (n-1) R_{\alpha \beta} s_{A}^{\alpha} s_{B}^{\beta} - \frac{1}{n-3} (n-1) R_{\alpha \beta} e_{A}^{\alpha} e_{B}^{\beta}
\]

\[
+ \frac{n-4}{n-3} q_{AB} (n-1) R - 2 E_{AB}
\]

\[
\]

Now instead of \((n-1) R_{\alpha \beta}\), we can use the following relation: \((n-1) R_{\alpha \beta} = (n-1) G_{\alpha \beta} + \frac{1}{2} h_{\alpha \beta} (n-1) R\), to replace Ricci tensors in terms of Ricci scalar and Einstein tensors. This leads to,

\[
(n-2) G_{AB} = (n-1) G_{\alpha \beta} e_{A}^{\alpha} e_{B}^{\beta} + \left\{ (n-1) G_{\alpha \beta} s_{A}^{\alpha} s_{B}^{\beta} + \frac{n-3}{2(n-2)} (n-1) R \right\} q_{AB}
\]

\[
- 2 E_{AB} + \left[ K_{AB} K - K_{AC} K_{B}^{C} \right]
\]

\[
- \frac{1}{2} q_{AB} \left( K_{2} - K_{AB} K_{AB} \right)
\]

\[
= (n-1) G_{\alpha \beta} e_{A}^{\alpha} e_{B}^{\beta} + \frac{n-4}{n-3} q_{AB} (n-1) R_{\alpha \beta} s_{A}^{\alpha} s_{B}^{\beta} - \frac{1}{n-3} (n-1) R_{\alpha \beta} e_{A}^{\alpha} e_{B}^{\beta}
\]

\[
+ \frac{n-4}{n-3} q_{AB} (n-1) R - 2 E_{AB}
\]

\[
\]

Now introducing the Einstein equation in \((n-1)\) dimension we arrive at,

\[
(n-1) G_{\alpha \beta} = \kappa_{n-1}^{2} T_{\alpha \beta}; \quad (n-1) R = -\frac{2}{(n-3)^{2}} \kappa_{n-1}^{2} T
\]

Using this expression in Eq. (88) leads to the effective field equation in \((n-2)\)-dimension, which is presented in Sec. 2.2.

References

[1] J. Polchinski, (1998) String Theory Vol. I and II (Cambridge University Press, Cambridge).
[2] P. Horava and E. Witten, *Nucl. Phys.* **B475**, 94 (1996).

[3] P. Horava and E. Witten, *Nucl. Phys.* **B460**, 506 (1996).

[4] L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83**, 3370 (1999) arXiv:hep-ph/9905221.

[5] C. Csáki, M. Graesser, L. Randall and J. Terning, *Phys. Rev. D* **62**, 045015 (2000).

[6] W.D. Goldberger and M.B. Wise, *Phys. Rev. Lett.* **83**, 4922 (1999) arXiv:hep-ph/9907447.

[7] S. Chakraborty and S. SenGupta, *Eur. Phys. J. C* **74**, 3045 (2014). arXiv:1306.0805.

[8] L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83**, 4690 (1999) arXiv:hep-th/9906182.

[9] A. Djouadi, *Phys. Rept.* **457**, 1 (2008).

[10] H. Davoudiasl, J.L. Hewett and T.G. Rizzo, *Phys. Lett. B* **473**, 43 (2000).

[11] H. Davoudiasl, J.L. Hewett and T.G. Rizzo, *Phys. Rev. D* **63**, 075004 (2001).

[12] R.S. Hundi and S. SenGupta, *J. Phys. G* **40** 075002 (2013).

[13] S. Sen, B. Mukhopadhyay and S. Sengupta, *J. Phys. G* **40**, 015004 (2013).

[14] A. Chamblin, S.W. Hawking and H.S. Reall, *Phys. Rev. D* **61**, 065007 (2000).

[15] J. Garriga and T. Tanaka, *Phys. Rev. Lett.* **84**, 2778 (2000).

[16] A. Lukas, B.A. Ovrut, D. Waldram, *Phys. Rev. D* **60**, 086001 (1999).

[17] N. Dadhich, R. Maartens, P. Papadopoulos and V. Rezania, *Phys. Lett. B* **487**, 1 (2000).

[18] S. Chakraborty and T. Bandyopadhyay, *Int. J. Theo. Phys.* **47**, 2493 (2008) arXiv:0707.1182.

[19] P. Binetruy, C. Deffayet and D. Langlois, *Nucl. Phys.* **B565**, 269 (2000) arxiv:hep-th/9905210.

[20] C. Csáki, M. Graesser, C. Kold, and J. Terning, *Phys. Lett. B* **462**, 34 (1999).

[21] M. Cvetic and H. Soleng, *Phys. Rep.* **282**, 159 (1997).

[22] K. Benakli, *Int. J. Mod. Phys. D* **8**, 153 (1999).

[23] Z. Chacko and A.E. Nelson, *Phys. Rev. D* **62**, 085006 (2000).

[24] A.G. Cohen and D.B. Kaplan, *Phys. Lett. B* **470**, 52 (1999).

[25] R. Gregory, *Phys. Rev. Lett.* **84**, 2564 (2000).

[26] M. Giovannini, H. Meyer and M.E. Shaposhnikov, *Nucl. Phys.* **B619**, 615 (2001).

[27] D. Choudhury and S. SenGupta, *Phys. Rev. D* **76**, 064030 (2007).

[28] K.L. McDonald, *Class. Quantum Grav.* **24**, 79 (2007).
[29] A.A. Saharian, *Phys. Rev. D* **74**, 124009 (2006).
[30] R. Koley, J. Mitra and S. SenGupta, *Phys. Rev. D* **78**, 045005 (2008).
[31] A. Das, R.S. Hundi and S. SenGupta, *Physical Review D* **83**, 116003 (2011).
[32] B. Mukhopadhyaya, S. Sen and S. SenGupta, *J. Phys. G* **40**, 015004 (2013).
[33] S. Chakraborty and S. SenGupta, *Phys. Rev. D* **89**, 126001 (2014). arXiv:1401.3279.
[34] T. Padmanabhan, (2010) *Gravitation: Foundations and Frontiers* (Cambridge University Press).
[35] E. Poisson, (2004) *A Relativist’s Toolkit* (Cambridge University Press).
[36] T. Shromizu, K. Maeda and M. Sasaki, *Phys. Rev. D* **62**, 024012 (2000).
[37] T. Harko and M.K. Mak, *Phys. Rev. D* **69**, 064020 (2004).
[38] I.L. Buchbinder, S.D. Odinstov and I.L. Shapiro, (1992) *Effective Action in Quantum Gravity*, IOP Publishing, Bristol.
[39] D.V. Vassilevich, *Phys. Rept.* **388**, 279 (2003).
[40] S. Nojiri and S. D. Odinstov, *Phys. Rept.* **505**, 59 (2011).
[41] T. P. Sotiriou and V. Faraoni, *Rev. Mod. Phys.* **82**, 451 (2010).
[42] S. Nojiri and S. D. Odinstov *Phys. Rev. D* **68**, 123512 (2003).
[43] A. De Felice and S. Tsujikawa, *Living Rev. Relativity* **13**, 3 (2010).
[44] C. Corda, *Int. J. Mod. Phys. D*, **18**, 2275 (2009).
[45] O. Aslan and D.A. Demir *Phys. Lett. B* **635**, 343 (2006).
[46] T. R. P. Carames, M. E. X Guimaraes and J. M. Hoff da Silva, *Phys. Rev. D* **87**, 106001 (2013).
[47] S. Chakraborty and S. SenGupta, *Phys. Rev. D* **90**, 047901 (2014). arXiv:1403.3164.
[48] S. Chakraborty and S. SenGupta, *Eur. Phys. J. C* **75**, 11 (2015) arXiv:1409.4115.
[49] K. Parattu, S. Chakraborty, B.R. Majhi and T. Padmanabhan, arXiv:1501.01053.
[50] S. Chakraborty and S. SenGupta, arXiv:1412.7783.
[51] S. Chakraborty, A. Banerjee and T. Bandyopadhyay, arXiv:0707.0199.