Differential description and irreversibility of depolarizing light–matter interactions

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Abstract
The widely used Jones and Mueller differential polarization calculi allow non-depolarizing deterministic polarization interactions to be described in an efficient way. Recently, the differential Mueller formalism has been successfully extended to the case of depolarizing transformations. In this article, a stochastic differential Jones formalism is shown to provide a clear physical insight on light depolarization, which arises when the interaction of polarized light with a medium involves randomized anisotropic properties. Based on this formalism, several intrinsic depolarization metrics which naturally arise to efficiently characterize light depolarization in a random medium are presented, and an irreversibility property of depolarizing transformations is finally established.

Keywords: polarimetry, depolarization, irreversibility, differential polarization formalisms

1. Introduction

In the field of polarimetry, Jones and Stokes/Mueller formalisms have always appeared as dual and often exclusive approaches, whose specific characteristics have been exploited for diverse applications. On the one hand, the description of field coherence in the Jones calculus, which relates the input and output two-dimensional complex electric field through \( \mathbf{E}_{\text{out}} = \mathbf{J} \mathbf{E}_{\text{in}} \), justifies its use in ellipsometry [1, 2], optical design [3–6], spectroscopy [6], astronomy [7] or radar (PolSar) [8]. On the other hand, Mueller calculus is widely used in applications such as biophotonics [9, 10], material characterization [11, 12] or teledetection [13], as it is based on optical field observables (intensity measurements), relating the input and output four-dimensional real Stokes vector through \( \mathbf{s}_{\text{out}} = \mathbf{M} \mathbf{s}_{\text{in}} \). As a consequence, these approaches fundamentally differ in their capacity to characterize depolarizing light–matter interactions (i.e., non-deterministic polarization transformations yielding a partial randomization of the input electric field). As Jones already pointed out in one of his seminal papers [14], deterministic Jones matrices are unable to directly describe depolarizing interactions, which can however be handled in the Mueller formalism via depolarizing Mueller matrices. This discrepancy between both standpoints takes part in the debate, still topical in the scientific community, about the physical origin of light depolarization [15–23].

In this article, after recalling some well-known properties of non-depolarizing Mueller matrices and the differential Jones and Mueller formalisms in section 2, we show in section 3 that these differential polarization formalisms, which naturally arise from group theory, provide new physical insight on depolarizing light–matter interactions. This approach allows us to define new intrinsic depolarization metrics which are described in section 4, before an irreversibility property for depolarizing transformations is demonstrated in section 5, as a counterpart to the well-known invariance property verified by deterministic interactions, which is recalled below.

2. Properties of non-depolarizing light–matter interactions and differential polarization formalisms

Let us first review some well-known properties of deterministic polarization transformations, which can be described by a \( 2 \times 2 \) complex Jones matrix \( \mathbf{J} \), or equivalently, by a \( 4 \times 4 \) real-valued non-depolarizing Mueller matrix denoted \( \mathbf{M}_{\text{nd}} \).
where the subscript nd indicates a non-depolarizing transformation. In that case, there is a clear one-to-one relationship between $\mathbf{j}$ and the so-called Mueller–Jones matrix $\mathbf{M}_{nd}$ through [24, 25]

$$\mathbf{M}_{nd} = \mathbf{A}(\mathbf{j} \otimes \mathbf{j}^\dagger)\mathbf{A}'$$

with $\mathbf{A} = \frac{1}{\sqrt{2}}
\begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 \\
0 & i & -i & 0
\end{bmatrix},$ (1)

where $\otimes$ denotes Kronecker product. The matrix $\mathbf{A}$ verifies $\mathbf{A}^{-1} = \mathbf{A}'$, and can also be rewritten in compact form as $\mathbf{A} = \text{vec}([\vec{\sigma}_0, \vec{\sigma}_1, \vec{\sigma}_2, \vec{\sigma}_3])\vec{p}'/\sqrt{2}$, where vec($\vec{\sigma}_i$) denotes the $i$th Pauli matrix $\vec{\sigma}_i$, written in (column) vector form. In the remainder of this article, superscripts $*, T$ and $\dagger$ respectively denote complex conjugation, standard and Hermitian matrix transposition. Interestingly, when one considers normalized unit-determinant matrices, respectively denoted $\bar{j}$ and $\bar{M}_{nd}$, both descriptions appear to be isomorphic representations of the same six-dimensional group, namely the proper orthochronous Lorentz group $SO^\ast(1,3)$ for unit-determinant Mueller matrices $\bar{M}_{nd}$ and the special linear group $SL(2, \mathbb{C})$ for unit-determinant Jones matrices $\bar{j}$ [26, 27]. As a result, there is a well-known analogy between deterministic polarization transformations and special relativity [27–33]. In particular, non-depolarizing interactions correspond to Lorentz transformations and must therefore preserve the Minkowski metric $\|s\|^2_0 =\quad (s^0)^2 - (s^1)^2 - (s^2)^2 - (s^3)^2$ of the input Stokes vector $s_\text{in}$. This metric is defined by $\|s\|^2_0 = s^0G_0 = I^2(1 - T^2)$, with $G = \text{diag}\{1, -1, -1, -1\}$ and $T = \sqrt{s^1^2 + s^2^2 + s^3^2}/s_0$ denotes the light degree of polarization (DOP), and $I = s_0$ denotes the light intensity [30]. By extension, this invariance property can be related to the preservation of the Shannon entropy of the field, which is an informational measure of the disorder of the two-dimensional transverse electric field, as

$$H(s) = -\int P_E(E)\ln P_E(E) = \ln \pi e^2 \|s\|^2_0 / 4,$$ (2)

under the assumption of complex Gaussian circular fluctuations [34]. As pointed out in [33], such an invariance property is neither verified by the light intensity, $I$, nor by its DOP, $T$.

Historically, Jones [35] and Azzam [36] respectively introduced the so-called differential Jones and Mueller calculus, with a corresponding differential Jones matrix (dJm) $\mathbf{j}$ and a differential Mueller matrix (dMm) $\mathbf{m}_{nd}$, where the subscript nd still indicates a non-depolarizing transformation. Both approaches describe the local evolution of a transversally polarized wave along direction $\mathbf{n}$ through the respective differential equations $d\mathbf{E}/dh = \mathbf{j}E$, and $d\mathbf{s}/dh = \mathbf{m}_{nd}s$. According to group theory, these differential descriptions lead to a representation of deterministic polarization transformations, either in group $SL(2, \mathbb{C})$ for $\mathbf{j}$ or in $SO^\ast(1, 3)$ for $\mathbf{m}_{nd}$, by their counterpart in the corresponding Lie algebra $\mathfrak{s}(2, \mathbb{C})$ for $\mathbf{j}$, or $\mathfrak{so}^\ast(1, 3)$ for $\mathbf{m}_{nd}$, the latter verifying Minkowski $G$-antisymmetry, i.e., $\mathbf{m}_{nd} + \mathbf{G} \mathbf{m}_{nd}' \mathbf{G} = 0$ [37]. There is a clear equivalence between these four representations, which are linked through the following commutative diagram

$$\begin{align*}
\mathbf{j} & \in \mathfrak{s}(2, \mathbb{C}) \quad \mathbf{A}(\mathbf{j} \otimes \mathbf{j}^\dagger)\mathbf{A}' & \mathbf{m}_{nd} & \in \mathfrak{so}^\ast(1,3) \\
\exp & \quad \exp, \\
\mathbf{j} & \in SL(2, \mathbb{C}) \quad \mathbf{A}(\mathbf{j} \otimes \mathbf{j}^\dagger)\mathbf{A}' & \mathbf{M}_{nd} & \in SO^\ast(1,3) \\
\end{align*}$$

the macroscopic and differential matrices being related by the exponential map, through $\mathbf{j} = \exp(\mathbf{J}\Delta z)$ and $\mathbf{M}_{nd} = \exp(\mathbf{m}_{nd}\Delta z)$ when propagation over $\Delta z$ through a homogeneous medium is assumed.

As Lie algebras can be viewed as the tangent spaces to the corresponding Lie groups at the identity element [37, 38], the differential Jones or Mueller formalisms allow polarization properties of a sample or a material to be described in a linearized geometry. One of the powerful consequences of such a linearization lies in the simple linear parameterizations of the differential matrices in terms of anisotropic optical properties of the sample. Indeed, the dMm $\mathbf{m}_{nd}$, that both characterize the polarimetric properties of an infinitesimal plane-parallel slab of a deterministic linear optical medium, read

$$\mathbf{j} = \frac{1}{2}
\begin{bmatrix}
2\kappa_i + \kappa_q & -i(2\eta_i + \eta_q) & \kappa_u - \eta_v - i(\eta_u + \kappa_v) \\
\kappa_u + \eta_v - i(\eta_u - \kappa_v) & 2\kappa_i - \kappa_q - i(2\eta_i - \eta_q)
\end{bmatrix},$$ (4)

and, $\mathbf{m}_{nd} =
\begin{bmatrix}
2\kappa_i + \kappa_q & \kappa_u - \eta_v & \kappa_v \\
\kappa_q & 2\kappa_i - \eta_v & \eta_u \\
-\kappa_v & \eta_u & 2\kappa_i - \eta_q
\end{bmatrix}. $ (5)

This choice of notation conventions implies that the considered monochromatic plane wave can be written as $\mathbf{E}(z, t) = E_0\exp(i\omega t + pz)$ where $p = \kappa - i\eta$, with propagation constants $\kappa = 2\pi f / \lambda$ and $\eta = 2\pi n / \lambda$ respectively related to the imaginary and real part of the complex refractive index $n = n + ik$ [35, 39]. By analogy, in the above matrices parameter $\kappa$ denotes the isotropic extinction coefficient (in amplitude) induced by the sample or material. Parameter $\eta$ stands for the anisotropic (absolute) optical phase incurred by the interaction with the sample, and whose information is lost in the Mueller description [14]. As for the other terms, the subscripts $q, u, v$ refer to linear $x$–$y$, linear $\pm 45\degree$ and circular left/right optical anisotropies, through $\chi_{q, u, v} = \chi_{45\degree, \text{exp}} = \chi_{45\degree, \text{exp}}$, with $x = \kappa$ when describing absorption anisotropy (diattenuation), or $x = \eta$ when describing phase anisotropy (birefringence).

3. Stochastic Jones and Mueller differential formalisms for depolarizing interactions

After having recalled the polarization differential formalisms and these well-established properties of deterministic non
depolarizing transformations, it is naturally motivating to question whether such differential formalisms can deepen the physical understanding of depolarizing light–matter interactions. In fact, it has recently been proposed to extend the differential Mueller formalism to the more intricate case of depolarizing transformations, by the introduction of depolarizing dMm’s. This approach has paved the way for a number of interesting results obtained on depolarizing transformations [40–48]. We propose below to shed new light on the recent developments on depolarizing dMm’s, by using an alternative description involving stochastic differential Jones matrices. For that purpose, let us consider a stochastic dMm J = J0 + ∆J, modeling a random depolarizing local transformation of the field, where J0 = (J) is the deterministic average polarization transformation, whose form has been recalled in equation (4), and where the fluctuations matrix verifies (ΔJ) = 0. Assuming infinitesimal propagation over Δz in the considered medium, the Jones matrix for such a transformation can be written J = exp(ΔJΔz) ≈ 1 + ΔJΔz at first order in Δz. From equation (4), this relation can be conveniently rewritten in a vector form in the Pauli matrices basis {σj}j=0...3 as Vfi ≈ (1 + Δp(0) Δz Δp)T, with p(0) = p0 + Δp, p = p0 + Δp. In these expressions, the deterministic average values read

\[ p_0 = 2\kappa_i - 2\eta_i, \quad p = p_0 + \Delta p. \tag{6} \]

whereas Δp(0) and Δp denote zero mean random variables describing the fluctuations of the anisotropy parameters.

From Vfi, one can derive the Cloude’s coherency matrix (CCM) of the polarimetric transformation, defined as the second-order moment matrix of vector Vfi, i.e., C(J) = <VfiV† fi>. The CCM can also be directly computed from a Mueller matrix M through

\[ C(M) = \frac{1}{2} \sum_{j,k=0}^{3} MA(\sigma_j \otimes \sigma_k^*) A^†, \tag{7} \]

with σj denoting the jth Pauli matrix [26, 49]. The CCM is widely used in the field of Mueller polarimetry as it provides relevant criteria to assess the physical realizability of macroscopic Mueller matrices [26, 49]. With the above notations, the CCM can be decomposed into a sum of two terms C(J) = Cnd + Cd, where Cnd = Vh,iV† hi with Vh,i ≈ (1 + Δp(0) Δz Δp)T, and where the covariance matrix Cn = (<Vfi − Vh,i)(Vfi − Vh,i)> reads

\[ Cn = \left( \frac{\Delta p(0)}{2} \right)^T c_0 e^T e c_0. \tag{8} \]

with c0 = ⟨Δp(0)⟩2, c = ⟨Δp(0)Δp⟩ and the 3 × 3 sub-matrix C = ⟨ΔpΔp⟩. From such a decomposition, it is clear that a deterministic transformation (with Δp(0) = 0 and Δp = 0, hence Cnd = 0) results in a CCM of rank one, Cnd being the matrix of a projector. A CCM of rank one is precisely known as the Cloude’s condition for a polarimetric transformation to be non-depolarizing [26, 49]. Conversely, as soon as C is a non-null matrix, the rank of C(J) is greater than one, hence the corresponding transformation is depolarizing according to Cloude’s criterion [26, 49]. As a result, the depolarizing nature of a transformation appears to be completely comprehended by the 3 × 3 positive semi-definite Hermitian (PSDH) submatrix C, i.e., by 9 independent real parameters. As will be seen below, this submatrix also allows one to define interesting intrinsic depolarization metrics.

It is now quite straightforward to identify the two terms of the CCM, namely Cnd and Cd, with the CCM’s of, respectively, the non-depolarizing and the depolarizing dMm’s introduced in earlier works [50–52]. Indeed, in the dMm formalism, the Mueller matrix for the considered local transformation reads M = exp(M Δz) ≈ Id + M Δz at first order in Δz. As suggested in [52], the dMm M can be decomposed into a G-antisymmetric part, namely Mnd, and a G-symmetric part M = Mnd + MΔz, with the 9 parameters G-symmetric part reading [50–52]

\[ M = \left[ \begin{array}{ccc} 0 & d_{ii} & d_{ii} & d_{ii} \\ -d_{ii} & -d_{ii} & d_{ii} & d_{ii} \\ -d_{ii} & d_{ii} & -d_{ii} & d_{ii} \\ -d_{ii} & d_{ii} & d_{ii} & -d_{ii} \end{array} \right]. \tag{9} \]

The proposed decomposition of the dMm M has an important physical meaning: the depolarization properties of light propagating in a sample must pile up quadratically with Δz, whereas deterministic anisotropy parameters classically evolve linearly with propagation distance. Such a decomposition of M involving a quadratic behavior of the depolarizing dMm has been recently proposed in [53], but without a clear physical justification that is brought by the above approach. This interesting property of depolarization in samples has been recently verified experimentally on controlled test samples [48], and it may have crucial implications in the analysis of depolarizing media in experimental polarimetry [2, 54].

With such parameterizations of Mnd and M, respectively described through equations (5) and (9), the CCM of M = Id + MΔz can be obtained using the relationship recalled in equation (7). A direct calculus yields

\[ C(M) = \left[ \begin{array}{c} \Delta p(0) \\ \frac{\Delta z}{2} p_0 \end{array} \right] \Sigma, \tag{10} \]

with

\[ \Sigma = 2 \left[ \begin{array}{cccc} -d_{ii} + d_{ii} + d_{ii} & d_{ii} + id_{ii} & d_{ii} - id_{ii} \\ d_{ii} - id_{ii} & d_{ii} - d_{ii} + d_{ii} & d_{ii} + id_{ii} \\ d_{ii} + id_{ii} & d_{ii} - id_{ii} \end{array} \right]. \tag{11} \]
It can be observed that $\mathbf{C}(\mathbf{M})$ corresponds to an approximation of the CCM obtained above with the Jones formalism, where each element has been truncated to the first non-null term of its Taylor expansion in $\Delta z$, and where all information about the absolute phase delay $\eta$ has been lost. However, the lower-right $3 \times 3$ submatrix $\Sigma$ and its previous expression $\Sigma = (\Delta p \Delta P)$ given in equation (8) can be identified, as they correspond to the same order of approximation (in $\Delta z^2$) yielding the following set of equations:

$$
2d_{\rho_{quv}} = \{[(\Delta \kappa)^2 + (\Delta \kappa)^2 \eta_{v,qu}] + [(\Delta \eta)^2 + (\Delta \eta)^2 \eta_{v,qu}] \},$
$$
2d_{\rho_{quv}} = \{[(\Delta \kappa_{u,v,qu})^{2} + (\Delta \kappa_{v,qu})^{2}] + [(\Delta \eta_{u,v,qu})^{2} + (\Delta \eta_{v,qu})^{2}] \},$
$$
2d_{\rho_{quv}} = \{[(\Delta \kappa_{u,v,qu})^{2} + (\Delta \kappa_{v,qu})^{2}] - (\Delta \eta_{u,v,qu})^{2} - (\Delta \eta_{v,qu})^{2} \}.  
$$

This clearly shows that the nine depolarization parameters $d_{\rho_{quv}}$, $d_{\rho_{vuq}}$, and $d_{\rho_{uvq}}$ of the dMm $\mathbf{m}_d$ are physically related to the second-order statistical properties (covariance) of the anisotropy parameters, hence implying specific relationships between their respective values. Such an observation was reported for the first time in [23] through a somewhat intricate calculus involving stochastic dMm’s. In addition, these specific relationships imply necessary conditions on the elements of $\mathbf{M}_f$ so that it is physically admissible. In that perspective, it can be shown [45, 53] that a physically admissible depolarizing dMm $\mathbf{m}_d$ must belong (up to double cosetting by Lorentz transformations) to one of the two canonical forms derived in [55, 56]:

$$
\mathbf{m}_d^{(0)} = \text{diag}(d_1 + d_2 + d_3; -d_1 - d_2 + d_3; d_1 - d_2 - d_3),$$
$$
\mathbf{m}_d^{(I)} = \begin{bmatrix}
    0 & 0 & 0 \\
    -d_2 & d_1 - d_2 & 0 \\
    0 & 0 & -d_1 \\
    0 & 0 & 0 - d_1
\end{bmatrix},
$$

with the following conditions on the canonical depolarization parameters: $d_i \geq 0$, for $i \in \{1, 3\}$.

4. Intrinsic depolarization metrics

These previous results evidence the fact that the depolarization properties of light propagating in a medium at an infinitesimal level are intrinsically described by the matrix $\Sigma$ (or equivalently $\mathbf{C}$), which contains the 9 depolarization parameters described above. However, usual depolarization metrics are defined either on the macroscopic Mueller matrix of the medium, or on its CCM. For instance, the standardly used depolarization index is defined as $P = \{\text{tr}[(\mathbf{M}_d^{(0)} - \mathbf{M}_0^{(0)})/3\mathbf{M}_0^{(0)}]\}^{1/2}$ [57], and can vary between 0 (totally depolarizing) to 1 (non depolarizing). Another description of depolarization uses the so-called Cloude entropy $S_{\text{C}} = -\sum_{i=1}^{4} \lambda_i \log \lambda_i$ with $\lambda_i = \lambda_i/\sum_{j=1}^{4} \lambda_j$ or, written in compact form, $S = -\text{tr}[(\mathbf{M}/||\mathbf{M}||) \log (\mathbf{M}/||\mathbf{M}||)]$, where $||\mathbf{M}|| = \text{tr}\sqrt{\mathbf{M}^T \mathbf{M}}$ denotes the trace norm [59], with $\sqrt{\mathbf{X}}$ and $\log_{\mathbf{X}} = \ln(\mathbf{X})/\ln(4)$ respectively referring to square root and logarithm of a matrix $\mathbf{X}$. Fundamentally differing from the Shannon entropy $H(s)$ of a Stokes vector $s$ introduced in equation (2), the Cloude entropy (denoted by $S$ to avoid any confusion) is essentially a measure of the dispersion of the relative magnitude of the CCM eigenvalues, ranging from 0 (non depolarizing sample) to 1 (perfect depolarizer). Though often useful, such depolarization metrics can nevertheless be unsatisfactory in some situations. Indeed, two interactions sharing identical fluctuations properties of the optical anisotropy parameters (i.e., same matrix $\Sigma$) but with distinct principal polarization transformation vector $p_0$ can have different depolarization indices or Cloude entropies in the general case. This is due to the fact that the depolarization index is calculated from the entire Mueller matrix, and that the Cloude entropy depends on the four eigenvalues of the CCM, i.e., both metrics simultaneously depend on the deterministic polarization transformation and on the fluctuating parameters.

Contrarily, the new insight brought by the differential Jones and Mueller calculi allows one to naturally define intrinsic depolarization metrics, which only depend on the fluctuations of the anisotropy parameters of the sample. One can first define the intrinsic differential depolarization metric as $P_{\delta} = ||\Sigma ||_{F} \cdot \text{tr}[(\mathbf{M}/||\mathbf{M}||) \log_{\mathbf{M}} (\mathbf{M}/||\mathbf{M}||)]$, where $||\mathbf{M}||_{F} = \sqrt{\text{tr}[(\mathbf{M}^T \mathbf{M})]}$ denotes the Frobenius matrix norm [46]. Such a quantity can vary from 0 (for non-depolarizing interactions) to (potentially) infinity and can be efficient in situations where standard approaches fail to correctly describe the depolarizing nature of a light–matter interaction. This property is emphasized with an illustrative example in [46]. In addition, one can gain further physical insight into the depolarization properties of such interaction by analyzing other quantities on the submatrix $\Sigma$. For instance, the determinant of $\Sigma$ can be interpreted as a depolarization volume $V_{\delta} = \text{det}(\Sigma)$. This quantity is equal to zero as soon as one polarimetric direction has null fluctuations, indicating perfect correlation between at least two polarization directions. Another interesting approach is to analyze the Cloude entropy of the submatrix $\Sigma$ itself, i.e., $S_{\Sigma} = -\text{tr}[(\Sigma)/||\Sigma||] \log_{\Sigma} (\Sigma/||\Sigma||)$, where the subscript $\delta$ indicates that the Cloude entropy is computed under the differential approach. This Cloude entropy varies between 0 and 1 with the relative distribution of the eigenvalues of $\Sigma$, thus revealing possible depolarization anisotropy.

It is interesting to note that the quantities $P_{\delta}$, $V_{\delta}$ and $S_{\Sigma}$ are defined irrespective of the propagation distance, and are invariant by deterministic unitary transformations, thus justifying their qualification of intrinsic metrics. This has the strong physical meaning that the sample or the light–matter interaction studied must keep the same depolarization properties regardless of its deterministic anisotropic properties. Moreover, combining these three depolarization parameters, one can get direct information on the number and degeneracy of non-null canonical parameters, as evidenced in table 1. Such a procedure, which does not require reducing $\mathbf{m}_d$ to its canonical form, also allows one to identify type-(I) canonical family from type-(II) when all three canonical parameters are non-null. It is important to note that, in practice, the intrinsic
depolarization metrics proposed above can be computed directly from the Mueller matrix \( \mathbf{M} \) using the following procedure: (i) compute the matrix logarithm of \( \mathbf{M} \); (ii) identify \( \mathbf{m}_I \) by taking the G-symmetric part of \( \log(\mathbf{M}) \) and normalizing by the square sample thickness \( \Delta z \); (iii) compute the CCM of \( \mathbf{m}_I \) to identify the submatrix \( \Sigma_1 \); (iv) and finally compute the above metrics \( P_\alpha \), \( \mathcal{V}_\beta \) and \( \mathcal{S}_\gamma \).

### 5. Irreversibility property of depolarizing light–matter interactions

These considerations provide a fundamental insight on the origin of depolarization as a randomization of light polarization due to statistical fluctuations of the anisotropy parameters, giving access to meaningful intrinsic depolarization metrics. Finally they also allow us to demonstrate the following irreversibility property of depolarizing light–matter interactions:

**Property 1.** For any admissible fully or partially polarized input Stokes vector \( \mathbf{s}_m \), a physically realizable depolarizing non-singular and unit determinant Mueller matrix \( \mathbf{M} \) verifies \( \| \mathbf{s}_m \|_3^2 = \| \mathbf{M} \mathbf{s}_m \|_3^2 \geq \| \mathbf{s}_m \|_3^2 \).

The demonstration of this property in the general case of a standard Mueller matrix has never been reported before to our best knowledge, and is provided in a more general form in appendix (property A.1). For the sake of conciseness, we provide the demonstration of an equivalent ‘local’ property for a depolarizing \( \mathbf{d}\mathbf{M} \mathbf{m} = \mathbf{m}_0 + \mathbf{m}_I \) with null trace \( \langle \kappa_i \rangle = 0 \), which reads

\[
\frac{d\|\mathbf{s}\|_3^2}{dz} = s^T \left[ \mathbf{m}_0^T \mathbf{G} + \mathbf{G} \mathbf{m}_0 \right] s \geq 0
\]

(12)

for any physical Stokes vector \( \mathbf{s} \), with equality if and only if the dMM is non depolarizing \( \mathbf{m}_I = 0 \). The above expression of \( d\|\mathbf{s}\|_3^2/dz \) is easily obtained by computing the first order Taylor expansion of \( \|\mathbf{s}(z + \Delta z)\|_3^2 = \|\mathbf{G} \mathbf{s}(z) + (\mathbf{I} + \mathbf{m}_I \Delta z)\mathbf{s}(z)\|_3^2 \) and using the \( G \)-antisymmetry of \( \mathbf{m}_I \) when \( \kappa_i = 0 \). The positivity can then be easily shown on the two canonical forms of \( \mathbf{M} \) recalled above. One indeed has \( d\|\mathbf{s}\|_3^2/dz = \Delta z \left[ 2s_0^2 (d_1 + d_2 + d_3) - s_1^2 (d_1 - d_2 - d_3) - s_2^2 (d_2 - d_3 - d_1) - s_3^2 (d_3 - d_1 - d_2) \right] \) for type-(I) depolarizing \( \mathbf{d}\mathbf{M} \mathbf{m} \), whereas, for type-(II), \( d\|\mathbf{s}\|_3^2/dz = \Delta z \left[ 2d_2 (s_2^2 + s_3^2) + d_1 (s_1^2 - s_2^2 + s_3^2) \right] \). These two quantities are obviously non negative under the physicality conditions recalled above (i.e., \( \forall i \in \{1, 3\}, \langle d_i \rangle \geq 0 \) and for admissible Stokes vectors (i.e., \( \|\mathbf{s}\|_3 \geq 0 \)).

Property 1 and its ‘local’ counterpart have a strong physical meaning since they reveal the irreversible effect of a depolarizing transformation on the propagating field. This irreversibility clearly appears through the necessary increase of the Minkowski metric of the Stokes vector defining the field polarization state. Interestingly, this irreversibility property has an informational or thermodynamical counterpart, under the hypothesis of complex Gaussian circular random fluctuations of the field. Indeed, as a consequence of equation (12), the Shannon entropy of the bidimensional electrical field vector must obey an irreversible evolution with depolarizing transformations, as \( \frac{dH(s)}{dz} = -\frac{2}{\|s\|_3^2} \frac{d\|s\|_3^2}{dz} \geq 0 \). Such an irreversible behavior of the Minkowski metric \( \|s\|_3 \) (or equivalently of the Shannon entropy \( H(s) \)) confirms that these quantities are best adapted to describe the polarimetric randomization (depolarization) of a propagating beam. Indeed, contrarily to the field intensity or the standard DOP \( P \), the quantities \( \|s\|_3 \) and \( H(s) \) are preserved through non-singular deterministic (and reversible) transformations as \( \frac{dH(s)}{dz} = \frac{d\|s\|_3^2}{dz} = 0 \) in that case, but must necessarily grow with irreversible depolarizing transformations. This is schematically illustrated in figure 1 where it can be seen that, depending on the input Stokes vector \( \mathbf{s}_m \), a non-depolarizing \( \mathbf{M}_\text{nd} \) or a depolarizing \( \mathbf{M}_\text{d} \) polarimetric transformation can lead to an increase or a decrease of \( P \).

This can be also quantitatively illustrated with the following simple example. Let us consider the Mueller matrix \( \mathbf{M} = \mathbf{M}_0 \mathbf{M}_I \) formed by the sequential combination of a (unit-determinant) diagonal depolarizer \( \mathbf{M}_I = \text{diag}[1, \beta, \beta, \beta]/\beta^3/4 \) with \( \beta \in [0, 1] \), followed by a partial polarizer along the X axis with (unit-determinant) Mueller matrix:

\[
\mathbf{M}_\text{d} = \frac{1}{\sqrt{1 - \alpha^2}} \begin{bmatrix}
1 & \alpha & 0 & 0 \\
\alpha & 1 & 0 & 0 \\
0 & 0 & \sqrt{1 - \alpha^2} & 0 \\
0 & 0 & 0 & \sqrt{1 - \alpha^2}
\end{bmatrix},
\]

with \( \alpha \in [0, 1] \).

Let us now consider for instance that the fully polarized Stokes vector \( \mathbf{s}_{m,1} = [1, 1, 0, 0]^T \), with DOP \( P(\mathbf{s}_{m,1}) = 1 \) and Minkowski metric \( \|\mathbf{s}_{m,1}\|_3 = 0 \), interacts with a sample described by \( \mathbf{M} \). In that case, the DOP of the output Stokes vector \( \mathbf{s}_{m,1} = \mathbf{M}_\text{d} \mathbf{s}_{m,1} \) reads \( P(\mathbf{s}_{m,1}) = (\alpha + \beta)/\beta \), which is clearly lower than the unit DOP of the input light, seemingly indicating a depolarizing interaction. This is confirmed by the evolution of the Minkowski metric which verifies property 1 as \( \|\mathbf{s}_{m,2}\|_3 = (1 - \beta^2)/\beta^3/2 \geq 0 \). This is also the case with the Stokes vector \( \mathbf{s}_{m,2} = [1, 0, 0, 0]^T \) of a totally unpolarized light, as
Figure 1. Schematic representation of the effect of a non-depolarizing ($\tilde{M}_{\text{nd}}$) or a depolarizing ($\tilde{M}_d$) transformation on (a) the standard DOP $\mathcal{P}$, and (b) the Minkowski metric $||s||_{1,3}$ or Shannon entropy $H(s)$, for all possible input Stokes vectors $s_{in}$ (represented around a chromatic disk for the sake of simplicity).

$\|s_{\text{out},2}\|_{1,3}^2 = 1/\beta^{3/2} \geq \|s_{\text{in},2}\|_{1,3}^2 = 1$. However in that case, the polarimetric transformation described by $\tilde{M}$ leads to an increase of the DOP since $\mathcal{P}(s_{\text{out},2}) = \alpha$, whereas $\mathcal{P}(s_{\text{in},2}) = 0$, which could be apparently in contradiction with the depolarizing nature of $\tilde{M}$. This simple example thus confirms that the evolution of the standard DOP does not share the same irreversibility behavior, described in property 1, that is followed by the Minkowski metric or the field Shannon entropy.

6. Conclusion

As a conclusion, we have illustrated in this article how differential polarimetric formalisms could provide a clear physical picture of the origins of depolarization of light when interacting with a medium. The derivation conducted from the differential Jones formalism approach provides a straightforward derivation of the CCM of a depolarizing polarimetric transformation, from which we have been able to conclude that 9 independent parameters are required to describe depolarization. Moreover, the statistical analysis presented has evidenced the fact that depolarization properties locally grow up quadratically with propagation distance, contrarily to deterministic anisotropy properties which pile up linearly. We have also proposed to define new intrinsic depolarization metrics, which are able to describe how a polarization transformation will randomize the input field components, independently of the main deterministic polarization transformation. Lastly, a physically meaningful implication regarding an irreversibility property of the Minkowski metric of the Stokes vector under depolarizing transformations has been established, with an insightful interpretation in terms of growth of the beam Shannon entropy under the hypothesis of complex Gaussian circular fluctuations.

Appendix

In this appendix, we provide the demonstration of the following property, which is a generalized version of property 1 for non unit-determinant Mueller matrices.

**Property A.1.** For any admissible fully or partially polarized input Stokes vector $s_{in}$, a physically realizable non-singular Mueller matrix $M$ with $\det(M) > 0$ verifies

$$\|s_{\text{out}}\|_{1,3} = \|M s_{in}\|_{1,3}^2 \geq |\det(M)|^{1/2} \|s_{in}\|_{1,3}^2.$$  \hspace{1cm} (A.1)

It can be readily checked that when $\det(M) = 1$, one obtains the formulation of the property 1 given in the article.

Before providing the demonstration of this property, let us recall the definitions of Stokes realizability and Cloude realizability for general Mueller matrices [60, 61]:

**Stokes realizability:** A Mueller matrix $M$ is said *Stokes realizable* if, for any fully polarized Stokes vector $s_{in}$ verifying $\|s_{in}\|_{1,3} = 0$ and $s_0 > 0$, one has $\|s_{\text{out}}\|_{1,3} = \|M s_{in}\|_{1,3} > 0$. It must be noted that this last inequality is a particular case of the condition of equation (A.1). In that sense, property A.1 can be seen as a generalized version of the Stokes realizability condition.

**Cloude realizability:** A Mueller matrix $M$ is said *Cloude realizable* (or Jones realizable in [60, 61]) if and only if its CCM $\mathcal{C}(M)$ is PSDH [26, 49, 61]. The Cloude realizability implies the Stokes realizability whereas the converse is not true.

The demonstration of the above property A.1, which has never been reported so far to our best knowledge, is largely based on the work conducted by Rao et al [60, 61], which has permitted to exhibit two canonical forms of Stokes realizable non-singular Mueller matrices, under double cosetting by $SO^+(1, 3)$ elements. Their main result can be summarized following the through theorem:

**Theorem 1.** (Rao et al [60]) If a non-singular Mueller matrix $M$ verifying $\det(M) > 0$ is Stokes realizable then $\exists L_{\rho}, L_R \in SO^+(1, 3)$ such that $M = L_{\rho} M^{(1)} L_R$ with

$$M^{(1)} = \text{diag} [\sqrt{\rho_0}, \sqrt{\rho_1}, \sqrt{\rho_2}, \sqrt{\rho_3}]$$

with $\rho_0 \geq \rho_1 \geq \rho_2 \geq \rho_3 > 0$, or $M = L_{\rho} M^{(1)} L_R$ with

$$M^{(11)} = \begin{bmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & \rho_0 - n_0 / \sqrt{\rho_0} & 0 & 0 \\ 0 & 0 & \sqrt{\rho_2} & 0 \\ 0 & 0 & 0 & \sqrt{\rho_3} \end{bmatrix}$$

with $n_0 > \rho_0 > \rho_2 > \rho_3 > 0$. 


It can be noted that additional constraints on the canonical parameters apply in order to ensure Cloude realizability [61], leading to the same canonical forms and constraints derived by Bolshakov et al in [62], where the authors analyzed the physical realizability of Mueller matrices from the viewpoint of the so-called G-polar decomposition. However, these additional constraints are not required here for the following demonstration.

We now come to the demonstration of property A.1. The condition of equation (A.1) is obviously a sufficient condition for Stokes realizability since it implies the usual Stokes realizability condition. To demonstrate that it is a necessary condition, it suffices to show that the two canonical forms of Stokes realizable non-singular Mueller matrices (recalled in theorem 1) verify the condition of equation (A.1). The generalization to any Stokes realizable non-singular Mueller matrix is then straightforward since the double cosetting by SO+(1, 3) elements does not modify the Minkowski metric.

Without loss of generality, consider an input Stokes vector with unit intensity $s_m = [1 \ s']^T$, with $s' = [s_1 \ s_2 \ s_3]^T$ and $|s'|^2 = P^2$. In that case, one has $|s_m|^2 = 1 = p^2$ where $\mathcal{P} \in [0, 1]$ denotes the DOP of the input light.

Canonical form $\mathbf{M}^{(i)}$: In that case, one has
\[
\det(\mathbf{M}^{(i)}) = [\rho_0 \rho_1 \rho_2 \rho_3]^3/2 \quad \text{and} \quad |s_m|^2 = s_m^T (\mathbf{M}^{(i)})^T \mathbf{G} \mathbf{M}^{(i)} s_m = \rho_0 - [s_1^2 \rho_1 + s_2^2 \rho_2 + s_3^2 \rho_3],
\]
where $s_i \leq \rho_0, \ \forall \ i \in \{1, 3\}$.

Canonical form $\mathbf{M}^{(II)}$: The demonstration for the second canonical form is a little more involved. In that case, one has
\[
\det(\mathbf{M}^{(II)}) = \rho_0 [\rho_2 \rho_3]^3/2 \quad \text{and}
\]
\[
\mathcal{D}_{\mathbf{II}}(s_m) = |s_m|^2 = |s_m|^2 = n_0 (1 - s_1^2) + 2 \rho_0 s_1 (1 - s_1) - \rho_1 \rho_2 \rho_3 [s_2^2 + s_3^2] - \rho_0 (1 - P^2) = 0,
\]
\[
\rho_2 \geq \rho_3 \geq 0,
\]
and noticing that $s_2^2 + s_3^2 = P^2 - s_1^2$, it is easy to show that
\[
\mathcal{D}_{\mathbf{II}}(s_m) \geq |n_0 (1 - s_1^2) + 2 \rho_0 s_1 (1 - s_1) - \rho_2 (P^2 - s_1^2) - [\rho_0 \rho_2]^3/2 (1 - P^2).
\]
As a result, it now suffices to prove the positivity of the term at the right-hand side of the above inequality, for any input Stokes vector $s_m$.

For that purpose, we first notice that this quantity is a polynomial function of order 2 in $s_1$, which we shall rewrite $f(s_1)$. For the input Stokes vector to be physically admissible, one must have $s_1 \in [-\mathcal{P}, \mathcal{P}]$. Let us first show that $f(s_1) \geq 0$ at its boundaries:
\[
f(-\mathcal{P}) = (1 + \mathcal{P}) \times [n_0 (1 + \mathcal{P}) - 2 \rho_0 \mathcal{P} - [\rho_0 \rho_2]^3/2 (1 - \mathcal{P})] \geq (1 + \mathcal{P})^2 (n_0 - \rho_0), \quad \text{since} \ \rho_0 > \rho_2 \geq 0, \ \text{since} \ n_0 > \rho_0 \text{and} \ \mathcal{P} \geq 0.
\]
\[
f(\mathcal{P}) = (1 - \mathcal{P}) \times [n_0 (1 - \mathcal{P}) + 2 \rho_0 \mathcal{P} - [\rho_0 \rho_2]^3/2 (1 + \mathcal{P})] \geq (1 - \mathcal{P})^2 (n_0 - \rho_0), \quad \text{since} \ \rho_0 > \rho_2 \geq 0, \ \text{since} \ n_0 > \rho_0 \text{and} \ \mathcal{P} \in [0, 1].
\]
Lastly, a parabolic function verifying $f(\mathcal{P}) \geq 0$ and $f(\mathcal{P}) \leq 0$ is necessarily non-negative over the interval $[-\mathcal{P}, \mathcal{P}]$ if the function is concave, i.e., if
\[
\frac{df(s_1)}{ds_1^2} = 2 [(n_0 - \rho_0) - (\rho_0 - \rho_2)] = 2(U - V) \leq 0 \quad \Rightarrow \quad U \leq V
\]
with $U = (n_0 - \rho_0) > 0$ and $V = (\rho_0 - \rho_2) > 0$.

On the other hand, if the function is convex, one has $U \geq V$ and $f(s_1)$ can be negative over $[-\mathcal{P}, \mathcal{P}]$ only if its minimum $s_1^k \in [-\mathcal{P}, \mathcal{P}]$. However, it is easy to show that solving $df(s_1)/ds_1 = 0$ leads to $s_1^k = U/(U - V) \geq 1 \geq \mathcal{P}$ since $U > 0$, $V > 0$ and $U - V > 0$, thus implying that the function $f(s_1)$ cannot admit a minimum value in the interval $[-\mathcal{P}, \mathcal{P}]$. This finally proves the positivity of $f(s_1)$ for $s_1 \in [-\mathcal{P}, \mathcal{P}]$, and thus of $\mathcal{D}_{\mathbf{II}}(s_m)$ for any admissible input Stokes vector $s_m$, hence achieving the proof of property A.1.

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