Bounds on fluctuations for ensembles of quantum thermal machines

Matthew Gerry\textsuperscript{1,3,*}, Na'im Kalantar\textsuperscript{2,3} and Dvira Segal\textsuperscript{1,2,†}

\textsuperscript{1} Department of Physics, University of Toronto, 60 Saint George St., Toronto, Ontario, M5S 1A7, Canada
\textsuperscript{2} Department of Chemistry and Centre for Quantum Information and Quantum Control, University of Toronto, 80 Saint George St., Toronto, Ontario, M5S 3H6, Canada
E-mail: dvira.segal@utoronto.ca

Received 8 September 2021, revised 11 January 2022
Accepted for publication 17 January 2022
Published 17 February 2022

Abstract
We study universal aspects of fluctuations in an ensemble of noninteracting continuous quantum thermal machines in the steady-state limit, beyond linear response. Considering an individual machine, such as a refrigerator, in which relative fluctuations (and high order cumulants) of the cooling heat current to the absorbed heat current, \( \eta^{(n)} \), are upper-bounded, \( \eta^{(n)} \leq \eta_C^{(n)} \) with \( n \geq 2 \) and \( \eta_C \) the Carnot efficiency, we prove that an ensemble of \( N \) distinct machines similarly satisfies this upper bound on the relative fluctuations of the ensemble, \( \eta_N^{(n)} \leq \eta_C^{(n)} \).

For an ensemble of distinct quantum refrigerators with components operating in the tight-coupling limit we further prove the existence of a lower bound on \( \eta_N^{(2)} \) in specific cases, exemplified on three-level quantum absorption refrigerators and resonant-energy thermoelectric junctions. Beyond special cases, the existence of a lower bound on \( \eta_N^{(2)} \) for an ensemble of quantum refrigerators is demonstrated by numerical simulations. We further show that the lower bound on \( \eta_N^{(2)} \) may, however, be violated for an ensemble of quantum engines.

Keywords: bounds, fluctuations, ensembles, quantum, thermal, machines

1. Introduction

Significant efforts in stochastic and quantum thermodynamics \([1–5]\) are currently devoted to understanding trade-off relations in nanoscale thermal machines by weighting currents, their

\*Author to whom any correspondence should be addressed.
\textsuperscript{†}These authors contributed equally to this work.
fluctuations, entropy production, and efficiency. As an example, the ‘thermodynamic uncertainty relation’ (TUR) describes a trade-off between precision (relative fluctuations) and cost (entropy production). References [6–19] constitute representative examples in this broad and active field. The TUR further constrains the performance of thermal engines, balancing output power, power fluctuations and the engine’s efficiency [7].

A separate class of bounds, which are independent of the TUR, concerns ratios between fluctuations of different currents: the output power and input heat current [20–24]. For the process of refrigeration, for instance, one considers the ratio

$$\eta^{(n)} \equiv \frac{\langle q^n \rangle}{\langle w^n \rangle},$$

with $q$ as the extracted stochastic cooling heat current and $w$ the corresponding input heat current; $\langle A^n \rangle$, $n = 1, 2, \ldots$ is the $n$th cumulant of a stochastic variable $A$. For $n = 2$ this ratio is lower and upper-bounded in linear response for continuous machines in steady state under time reversal symmetry [22],

$$\eta^2 \leq \eta^{(2)} \leq \eta^2_C.$$  

$\eta_C$ is the Carnot bound and $\eta$ (our short notation for $\eta^{(1)}$) stands for the efficiency of the machine. Based on the upper bound on $\eta^{(2)}$, a tighter-than-Carnot efficiency bound has been derived [20–23] and shown to be tighter than bounds received from the TUR [24].

In this paper, we focus on an ensemble of distinct, continuous, steady-state thermal machines, and inquire on universal aspects of their fluctuations—beyond linear response. The machines operate between the same affinities (temperatures, chemical potentials), but are different in their working parameters such as internal energies and system-bath coupling strength. We then ask the following basic question: assuming the relative fluctuations of an individual machine (operating beyond the linear response limit) are upper and lower bounded according to equation (2), do these bounds hold for an ensemble of distinct, uncorrelated machines?

This question is not trivial, as we show below. It is particularly difficult to address the lower bound: while all our machines are upper-bounded by the same Carnot limit, their efficiencies $\eta$ (dictating the lower bound) are distinct. To make progress, we consider here individual machines operating in the so-called tight-coupling (TC) limit [25, 26]: in each machine, the stochastic input and output currents are proportional to each other, $w \propto q$. Under the TC limit, one can readily prove, beyond linear response, the validity of the upper bound in equation (2), and show that the lower bound is saturated—for an individual machine [22]. However, TC does not necessarily hold when studying a collection of subsystems, even when the constituent elements follow it. Therefore, as we discuss here, proving in general the lower bound on $\eta^{(2)}$ is a nontrivial task.

We now pose the problem to be addressed in this study. We consider individual TC machines (e.g. refrigerators) whose relative fluctuations $\eta^{(n)}_k$ [see equation (1)] for each member $k$ are bounded according to

$$\eta^{(n)}_k \equiv \eta^{(n)}_k \leq \eta^C_k, \ n = 1, 2, \ldots$$

with the efficiency $\eta_k$ possibly different for each member in the ensemble. Our objective is to find whether an ensemble of $N > 1$ distinct, independent machines satisfies the relations

$$\eta^{(n)}_N \leq \eta^{(n)}_N \leq \eta^C_N, \ n = 2, 3, \ldots$$

arbitrarily far from equilibrium; the validity of the upper and lower bounds in the linear response regime was proved in reference [22]. The currents considered in devising the ratios
(1) are the total ones, \( w = \sum_{k=1}^{N} w_k \) and \( q = \sum_{k=1}^{N} q_k \). That is, e.g. \( \eta_n^{(n)} \) is the efficiency (\( n = 1 \)) or ratio of fluctuations (\( n = 2 \)) of a system made of \( N \) constituents. Expressing equation (4) in words: considering ratios of cumulants of output to input currents in a collection of thermal machines, we would like to show that this ratio is upper bounded—by the \( n \)th power of the Carnot efficiency. Further, we interrogate whether a lower bound holds for the ensemble, given by the ensemble-averaged efficiency to the power \( n \).

As we show in this paper, the upper bound in equation (4) can be readily proved for any order \( n \). As for the lower bound, we mostly restrict ourselves to the behavior of fluctuations, \( n = 2 \), and we prove it in specific limits. Beyond those cases, we establish the lower bound more broadly for refrigerators based on numerical simulations of three-level quantum absorption refrigerators (3LQARs) and thermoelectric junctions. In contrast, we find examples of pairs (\( N = 2 \)) of thermoelectric engines that violate the lower bound in equation (4) in the far from equilibrium regime.

The question posed here, on the validity of bounds for an ensemble of machines, is critical to our ability to experimentally confirm fundamental theoretical results. In some experiments, quantum machines are constructed from a collection of systems, such as trapped ions [27], quantum dots (see review [28]), and molecules [29]. Practically, even when aiming for homogeneity, these components cannot be made precisely identical in their energies and couplings to the environment. Moreover, even for machines made from an individual ‘particle’ [30–34], experiments inevitably suffer from a certain degree of uncertainty, requiring an ensemble average. Considering e.g. quantum absorption refrigerators based on superconducting circuits [35] or nitrogen vacancy centers in diamond [36], internal energies defining the refrigerator, as well as system-bath coupling parameters cannot be precisely-repeatedly realized, thus measurements necessarily rely on averaging.

We highlight that the ratio of cumulants as defined in equation (1) is unrelated to the concept of efficiency fluctuations explored e.g. in references [37–41]. In these studies, one defines the stochastic efficiency and studies its probability distribution function. In contrast, here we focus on the currents as the stochastic variables, we construct their cumulants, then look at their ratios to define \( \eta_n^{(n)} \).

As a final introductory comment, while our examples concern quantum thermal machines, their quantum nature only involves the discreteness of their energy levels and the quantum statistics of the baths (bosonic or fermionic). As such, bounds discussed in this work are directly applicable to classical systems.

The paper is organized as follows. In section 2 we briefly review preexisting results derived in the regime of linear response. The upper bound for an ensemble of TC machines is proved in section 3. In section 4, we discuss the lower bound and arrange it in alternative forms. We investigate the validity of the lower bound in different limits with two models for refrigerators: in section 5 we focus on 3LQARs, while in section 6 we examine thermoelectric refrigerators. In section 7 we demonstrate that the lower bound can be violated for thermoelectric engines operating far from equilibrium. Appendices A–C provide additional proofs and supporting simulations for the lower bound in different limits.

2. Model-independent bounds on the ratio of fluctuations in linear response

Continuous thermal machines operating close to equilibrium, such that the mean currents, \( \langle q \rangle \) and \( \langle w \rangle \), exhibit only linear dependence on thermodynamic affinities (differences in temperature, chemical potential), are said to operate in the regime of linear response. With two affinities, these currents are expressed in terms of the Onsager response matrix as \( \langle I_0 \rangle = \sum_{j=1}^{2} L_j A_j \),
where $\langle I_i \rangle$ is either mean current ($\langle q \rangle$ or $\langle w \rangle$), $A_i$ is the corresponding affinity, and the coefficients $L_{ij} = \partial A_i / \partial I_j |_{A_k=0}$ are the Onsager matrix elements. Under the time reversibility of the microscopic dynamics, a reciprocal relation holds with $L_{ij} = L_{ji}$. The fluctuation–dissipation relation gives the variance (fluctuations) in these currents as $\langle (I_i^2) \rangle = 2L_{ii}$, while the positivity of entropy production demands that $\det L \geq 0$ [26].

The relative fluctuations for each current $I_i$ may be defined as $\langle (I_i^2) \rangle / \langle I_i \rangle^2$. Under microscopic reversibility, and provided that the system is in the appropriate operational regime (i.e. that it is indeed behaving as a refrigerator or engine, rather than undergoing only spontaneous processes), the properties of the Onsager matrix have been used to bound the ratio, $Q$ of relative fluctuations of the output to input currents as $Q \geq 1$. This inequality gives a result equivalent to the lower bound on $\eta(2)$ expressed in equation (2). The same assumptions have been used to prove the upper bound as well, giving the full hierarchy of equation (2) [22]. Since these linear response results are model-independent, they apply to the ensembles of TC machines that are the main topic of this paper.

There are, however, inherent limitations in the use of the Onsager formalism for deriving bounds on higher-order cumulants of currents in thermal machines. It cannot be applied to obtain any results for $\eta(n)$ if $n \geq 3$. Furthermore, it is not valid farther away from equilibrium, as the affinities grow in magnitude. Continuous thermal machines must operate far from equilibrium in order to generate a substantial power output, highlighting the importance of this regime for practical applications. While the level of model-independence achieved for results in the linear response regime cannot easily be obtained far from equilibrium, the following sections represent one approach to building an understanding of bounds on $\eta(n)$ under more general circumstances, albeit for a more restricted class of machines.

### 3. Universal upper bound on ratio of fluctuations

In this section, we prove that arbitrarily far from equilibrium, ratios of cumulants of order $n$ of an ensemble with $N$ noninteracting and uncorrelated distinct heat machines, operating under the same affinities, are bounded by

$$\eta_N^{(n)} \leq \eta_C^n,$$

so long as the inequality is satisfied at the level of the individual machine, $\eta_k^{(n)} \leq \eta_C^n$. $\eta_C$ is the Carnot bound dictated by the temperatures common to all machines.

The proof holds for engines and refrigerators; we describe it in the language of refrigerators. We consider small, possibly quantum, refrigerators with $w = \sum_{k=1}^N w_k$ the total stochastic heat current absorbed in the cooling process from the so-called work bath and $q = \sum_{k=1}^N q_k$ the total extracted heat current from the cold bath. The components operate independently and are not correlated. We assume that each individual member of the ensemble satisfies

$$\eta_k \equiv \frac{\langle q_k \rangle}{\langle w_k \rangle} \leq \eta_C,$$

$$\eta_k^{(2)} \equiv \frac{\langle (q_k^2) \rangle}{\langle (w_k^2) \rangle} \leq \eta_C^2,$$

$$\eta_k^{(n)} \equiv \frac{\langle (q_k^n) \rangle}{\langle (w_k^n) \rangle} \leq \eta_C^n, \quad n > 2.$$  

To assist readers, we explicitly included the first two definitions. It can be shown that equation (6) holds in the TC limit even far from equilibrium—for individual machines [22].
Concrete systems operating in the TC limit are 3LQARs and resonant-level thermoelectric junctions [22].

For clarity, we begin with the case \( N = 2 \) and consider two refrigerators \( A \) and \( B \). It is not difficult to prove that the ratio of their total fluctuations are bounded by the Carnot-efficiency squared,

\[
\eta_{N=2}^{(n)} = \frac{\langle \langle q_A + q_B \rangle \rangle}{\langle \langle w_A + w_B \rangle \rangle} = \frac{\langle \langle q_A^0 \rangle \rangle}{\langle \langle w_A^0 \rangle \rangle} + \frac{\langle \langle q_B^0 \rangle \rangle}{\langle \langle w_B^0 \rangle \rangle} \\
\leq \eta_C^0 \left( \frac{\langle \langle w_A^0 \rangle \rangle}{\langle \langle w_A^0 \rangle \rangle + \langle \langle w_B^0 \rangle \rangle} \right) + \eta_C^0 \left( \frac{\langle \langle w_B^0 \rangle \rangle}{\langle \langle w_A^0 \rangle \rangle + \langle \langle w_B^0 \rangle \rangle} \right) = \eta_C^0. \tag{7}
\]

In the first line, we use the fact that the machines are uncorrelated. The last line is arrived at based on bounds for the individual machines.

Next, along the same principle we prove by induction the \((N + 1)\)th inequality based on the validity of an upper bound for an \( N \)-member ensemble. We denote by \( q_k \) and \( w_k \) the stochastic current of the \( k \)th machine. \( q_{N+1} \) and \( w_{N+1} \) are the stochastic currents of the \((N + 1)\)th member of the ensemble; \( \eta_N \) (and similarly for higher \( n \)) is the efficiency of an \( N \)-sized ensemble. We now write

\[
\eta_{N+1}^{(n)} = \frac{\sum_{k=1}^{N+1} \langle \langle q_k \rangle \rangle}{\sum_{k=1}^{N+1} \langle \langle w_k \rangle \rangle} \\
= \frac{\sum_{k=1}^{N} \langle \langle q_k \rangle \rangle + \langle \langle q_{N+1} \rangle \rangle}{\sum_{k=1}^{N} \langle \langle w_k \rangle \rangle + \langle \langle w_{N+1} \rangle \rangle} \\
= \frac{\sum_{k=1}^{N} \langle \langle q_k \rangle \rangle}{\sum_{k=1}^{N} \langle \langle w_k \rangle \rangle} \left( \frac{\sum_{k=1}^{N} \langle \langle w_k \rangle \rangle}{\sum_{k=1}^{N} \langle \langle w_k \rangle \rangle + \langle \langle w_{N+1} \rangle \rangle} \right) + \frac{\langle \langle q_{N+1} \rangle \rangle}{\langle \langle w_{N+1} \rangle \rangle} \left( \frac{\langle \langle w_{N+1} \rangle \rangle}{\sum_{k=1}^{N} \langle \langle w_k \rangle \rangle + \langle \langle w_{N+1} \rangle \rangle} \right) \\
\leq \eta_C^0. \tag{8}
\]

In the last step we used \( \frac{\sum_{k=1}^{N} \langle \langle q_k \rangle \rangle}{\sum_{k=1}^{N} \langle \langle w_k \rangle \rangle} \leq \eta_C^0 \) per our assumption of the validity of the upper bound for an ensemble with \( N \) elements. We also made use of \( \frac{\langle \langle q_{N+1} \rangle \rangle}{\langle \langle w_{N+1} \rangle \rangle} \leq \eta_C^0 \), valid for every individual machine.

Summing up, we proved that if the inequality \( \eta_n \leq \eta_C^0 \) holds for an individual machine, it also holds for a machine made of a collection of \( N > 1 \) distinct, uncorrelated systems—as long as they operate between the same temperatures, thus bounded by the same Carnot bound. The working elements of our machine are made distinct in their internal parameters and their coupling to the surroundings. For example, in figure 1 we depict a refrigerator with its working fluid including multiple three-level systems that are distinct in their energy spacings. In figure 4, we illustrate a thermoelectric device that comprises an array of independent junctions.
Figure 1. (a) Quantum absorption refrigerators operating with (b) an individual three-level system as its working fluid, (c) an ensemble of distinct three-level systems, e.g. of different energy spacing and distinct system-bath coupling parameters. The temperatures of the heat baths are marked by $T_w > T_h > T_q$. The transition between the ground level to the intermediate one, of energy gap $\theta_q$, is coupled to the cold bath. The transition between the intermediate level to the top one, of gap $\theta_w$, is enacted by the work bath. In a cooling operation, heat absorbed by the three-level system from the cold and work baths is emitted to the hot bath.

4. Lower bound on ratio of fluctuations

Unlike the upper bound that we proved in section 3, we are not able to prove the lower bound in general, but only in specific cases (sections 5 and 6). Before discussing these cases, in this section, we limit ourselves to a pair of systems ($N = 2$) and organize the lower bound in a compatible, curious form.

We consider two machines, labelled $A$ and $B$, both operating in the TC limit thus satisfying the relation $\eta_2^2 = \eta_2^{(2)}$. We focus on the relation

$$\eta_{N=2}^2 \leq \eta_{N=2}^{(2)}$$

for the combined $N = 2$ system. The ratio of fluctuations for the pair is given by

$$\eta_{N=2}^{(2)} = \frac{\langle \langle (q_A + q_B)^2 \rangle \rangle}{\langle \langle (w_A + w_B)^2 \rangle \rangle} = \frac{\langle \langle q_A^2 \rangle \rangle + \langle \langle q_B^2 \rangle \rangle}{\langle \langle w_A^2 \rangle \rangle + \langle \langle w_B^2 \rangle \rangle}$$

$$= \eta_A^{(2)} \frac{\langle \langle w_A^2 \rangle \rangle}{\langle \langle w_A^2 \rangle \rangle} + \eta_B^{(2)} \frac{\langle \langle w_B^2 \rangle \rangle}{\langle \langle w_B^2 \rangle \rangle}$$

$$= \lambda \eta_A^{(2)} + (1 - \lambda) \eta_B^{(2)},$$

(10)
where \( \lambda \equiv \langle \frac{(w^2_A)}{(w^2)} \rangle \). Crucially, \( 0 \leq \lambda \leq 1 \). If we assume without loss of generality that \( \eta_A^{(2)} \geq \eta_B^{(2)} \), we have that \( \eta_B^{(2)} \leq \eta_{N=2}^{(2)} \leq \eta_A^{(2)} \).

We may similarly expand the square of the efficiency itself,

\[
\eta_{N=2}^2 = \frac{(q_A + q_B)^2}{\langle w \rangle^2}
= \eta_A^2 \frac{(w_A)^2}{\langle w \rangle^2} + \eta_B^2 \frac{(w_B)^2}{\langle w \rangle^2} + 2 \eta_A \eta_B \frac{(w_A)(w_B)}{\langle w \rangle^2},
\]

where in the last line we have used the strict equality for individual machines due to the TC limit. Because of the TC limit, the relation \( \eta_A^{(2)} \geq \eta_B^{(2)} \) further implies that \( \eta_A \geq \eta_B \), and \( \eta_A \geq \sqrt{\eta_B^{(2)}} \). With these relations, we get

\[
\eta_{N=2}^2 \leq \eta_A^{(2)} \left[ \frac{(w_A)^2}{\langle w \rangle^2} + \frac{(w_B)^2}{\langle w \rangle^2} + 2 \frac{(w_A)(w_B)}{\langle w \rangle^2} \right] = \eta_A^{(2)},
\]

\[
\eta_{N=2}^2 \geq \eta_B^{(2)} \left[ \frac{(w_A)^2}{\langle w \rangle^2} + \frac{(w_B)^2}{\langle w \rangle^2} + 2 \frac{(w_A)(w_B)}{\langle w \rangle^2} \right] = \eta_B^{(2)},
\]

thus \( \eta_B^{(2)} \leq \eta_{N=2}^2 \leq \eta_A^{(2)} \). It is immediately clear that in the special case that \( \eta_A^{(2)} = \eta_B^{(2)} \), a strict equality, \( \eta_{N=2}^2 = \eta_A^{(2)} \), is obtained. More generally, there exists some \( 0 \leq \kappa \leq 1 \) such that \( \eta_{N=2}^2 = \kappa \eta_A^{(2)} + (1 - \kappa) \eta_B^{(2)} \). Solving for this coefficient gives

\[
\kappa = \frac{\langle w_A \rangle^2}{\langle w \rangle^2} + 2 \frac{\eta_B}{\eta_A + \eta_B} \frac{\langle w_A \rangle\langle w_B \rangle}{\langle w \rangle^2}.
\]

The relation on the pair of machines, \( \eta_{N=2}^2 \leq \eta_{N=2}^{(2)} \), then, is satisfied exactly when \( \kappa \leq \lambda \), or,

\[
\frac{\langle w_A \rangle^2}{\langle w \rangle^2} + 2 \frac{\eta_B}{\eta_A + \eta_B} \frac{\langle w_A \rangle\langle w_B \rangle}{\langle w \rangle^2} \leq \frac{\langle (w_A^2) \rangle}{\langle (w^2) \rangle}.
\]

Since \( 2\eta_B/(\eta_A + \eta_B) \leq 1 \), this is always the case for a pair of machines meeting the stronger condition, \( \langle (w_A^2) \rangle/\langle (w^2) \rangle \geq \langle w_A \rangle/\langle w \rangle \), or, equivalently,

\[
\frac{\langle (w_A^2) \rangle}{\langle (w_B^2) \rangle} \geq \frac{\langle w_A \rangle}{\langle w_B \rangle}, \text{ for } \eta_A \geq \eta_B.
\]

Equation (12) is equivalent to the lower bound (9). In contrast, equation (13) provides a stronger condition: satisfying equation (13) necessarily means obeying the original relation, (9). However, we may violate the inequality (13) yet still satisfy the lower bound (9).

5. Model I: ensemble of absorption refrigerators

Fundamental results on autonomous-continuous thermal machines are often illustrated and examined within simple models for quantum absorption refrigerators [42–44]; recent experiments realized a QAR using trapped ions [27, 32]. In a QAR, heat is extracted from a cold \((q)\) bath and released into a hot \((h)\) bath by absorbing heat from the so-called work \((w)\) reservoir.
The reversed operation realizes a heat engine. We identify three temperatures, $T_w > T_h > T_q$ in a QAR, and three stochastic heat currents, $w, h, q$, defined positive when flowing towards the system.

The performance of QARs has been investigated in different models for elucidating principles in quantum thermodynamics. For example, QARs have been analyzed from weak to strong couplings to the baths [45–48], in models supporting multiple competing cycles [49, 50], and when quantum coherences between eigenstates survive in the steady-state limit [51, 52]; these are illustrative examples out of a rich literature. In this paper, we utilize the three-level model of Scovil and Schulz-DuBois [53] to illustrate bounds on relative fluctuations of currents for an ensemble of QARs. We limit our discussion to the weak system-bath coupling limit, which can be handled with a perturbative quantum Master equation, providing the full counting statistics of the model [52, 54].

A schematic diagram of our model is displayed in figure 1(a). An individual 3LQAR, illustrated in figure 1(b), has particularly served to elucidate concepts in quantum thermodynamics, since at weak system-bath coupling it can be solved analytically. In this model, the three baths are coupled selectively to the different transitions: the cold ($q$) bath allows the transitions across $\theta_q$, from the ground state to the intermediate level. The work ($w$) bath is coupled to a transition of energy gap $\theta_w$, from the intermediate level to the top one. In a cooling process, the hot bath ($h$) extracts the heat, $\theta_q + \theta_w$.

5.1. Single refrigerator, $N = 1$

Consider an individual 3LQAR of spacings $\theta_q$ and $\theta_w$. The cooling efficiency of the engine is defined as $\eta = \frac{\langle q \rangle}{\langle w \rangle}$, with $q$ ($w$) the stochastic cooling (work) heat currents. It can be shown that in the weak system-bath coupling limit, an individual 3LQAR operates in the TC limit: for every quanta $\theta_q$ absorbed from $T_q$, a quanta $\theta_w$ must be absorbed from the work bath. The efficiency and $\eta^{(n)}$ therefore obey the following relations [54]:

$$\eta \equiv \frac{\langle q \rangle}{\langle w \rangle} = \frac{\theta_q}{\theta_w},$$

$$\eta^{(n)} \equiv \frac{\langle \langle q^n \rangle \rangle}{\langle \langle w^n \rangle \rangle} = \left( \frac{\theta_q}{\theta_w} \right)^n = \eta^n \quad n = 2, 3, \ldots$$

(14)

Furthermore, the cooling condition is [54]

$$\langle q \rangle \propto n_q(\theta_q)n_w(\theta_w)[n_h(\theta_h) + 1] - [n_q(\theta_q) + 1][n_w(\theta_w) + 1][n_h(\theta_h) \geq 0,$$

(15)

with $n_i(\theta_i) = \left[ e^{\beta_i \theta_i} - 1 \right]^{-1}$ the Bose Einstein distribution function of the bath $i = h, w, q$ with transition $\theta_i$. The cooling condition can be equivalently written as

$$\frac{\theta_q}{\theta_w} \leq \frac{\beta_h - \beta_w}{\beta_q - \beta_h},$$

(16)

with $\beta_i = 1/T_i$ the inverse temperature. We identify the left-hand side by the efficiency, $\eta$, and the right-hand side by the Carnot efficiency, thus

$$\eta \leq \eta_C = \frac{\beta_h - \beta_w}{\beta_q - \beta_h}.$$  

(17)

Altogether, an individual 3LQAR operates in the TC limit and it satisfies

$$\eta^n = \eta^{(n)} \leq \eta^n_C.$$  

(18)
5.2. Ensemble of $N > 1$ distinct refrigerators

An ensemble of distinct, uncorrelated refrigerators, with possibly different spacings $\theta_q$ and $\theta_w$ and different system-bath coupling energies, provides a nontrivial setting for exemplifying the lower bound on ratios of fluctuations. We represent such an ensemble in figure 1(c). We assume that all our systems are operating between the same heat baths at temperatures $T_i$, $i = w, h, q$, and we fix $\theta_h$.

First, given the validity of the upper bound, equation (18), for each individual machine, we conclude that a similar upper bound holds for the ensemble, as we proved in section 3. In what follows we therefore focus on establishing a lower bound on $\eta_N^{(n)}$.

5.2.1. QARs with different system-bath couplings. We consider an ensemble of three-level systems, each with different coupling strength to the baths but with the same gaps $\theta_q$ and $\theta_w$. Members of this ensemble sit vertically (at the same $\theta_q$) along different curves, as exemplified in figure 2(a) with asterisks. While the heat currents depend on both the coupling parameters and the energy spacings, notably for any individual refrigerator the ratio $\eta^{(n)}$ depends only on the latter. Next we prove the saturation of the lower bound,

$$\eta_N^{(n)} = \eta_N^{(n)}.$$  \hspace{1cm} (19)

For each 3LQAR, $\langle w_k \rangle / \langle q_k \rangle = \theta_w / \theta_q \equiv \alpha$, see equation (14). Therefore, the efficiency of the ensemble of refrigerators to the power $n$ is

$$\eta_N^{(n)} = \left( \frac{\sum_{k=1}^{N} \langle q_k \rangle}{\sum_{k=1}^{N} \langle w_k \rangle} \right)^n = \left( \frac{\sum_{j} \langle q_j \rangle}{\sum_{j} \langle w_j \rangle} \frac{\sum_{k} \langle w_k \rangle}{\sum_{k} \langle w_k \rangle} \right)^n = \frac{1}{\alpha^n}. \hspace{1cm} (20)$$

Similarly, based on equation (14),

$$\eta_N^{(n)} = \left( \frac{\sum_{k=1}^{N} \langle q_k \rangle}{\sum_{k=1}^{N} \langle w_k \rangle} \right)^n = \frac{1}{\alpha^n}, \hspace{1cm} (21)$$

and we confirm equation (19).

5.2.2. QARs with distinct gaps. We now prove the lower bound for an ensemble of 3LQARs characterized by distinct energy gaps, but coupled in the same manner to the different baths; we highlight that each three-level system may couple asymmetrically to the three different baths, but all the three-level systems follow the same coupling parameters. The proof described in this section holds in the low-cooling regime identified as region I in figure 2(a); points marked by circles exemplify members of this ensemble. In appendix A we describe a complementary proof (for $N = 2$ and $n = 2$) that holds in region II, at the edge of the cooling window marked in figure 2(a).

Henceforth, for simplicity and without loss of generality we set the total gap at $\theta_h = 1$. Assuming that $\theta_q \ll 1$, we Taylor-expand the cooling current (note that the expansion coefficients, $\beta$ and $\gamma$, in the following are not to be confused with inverse temperature and coupling strength, respectively),

$$\langle q \rangle \approx \alpha \theta_q + \beta \theta_q^2 + \cdots \hspace{1cm} (22)$$
Figure 2. Exemplifying the performance of 3LQARs operating between $\beta_q = 0.4$, $\beta_h = 0.2$, and $\beta_w = 0.1$. (a) Cooling current, (b) current extracted from the work bath, (c) cooling efficiency, (d) fluctuations of cooling current, (e) fluctuations of the so-called work current, and (f) the relative noise $\eta(2)$. Other parameters are $\theta_h = 1$, and system-bath couplings $\gamma_q = \gamma_h = \gamma_w$ (dashed), $\gamma_q = 2\gamma_h = 2\gamma_w$ (full). In section 5.2.2, we prove the lower bound $\eta_n^L \leq \eta(2)$ in region I. In appendix A, it is shown to hold in region II, albeit only for $n = 2$ and $N = 2$.

Plugging this expansion into $\eta = \frac{\langle q \rangle}{\langle w \rangle} = \frac{\theta_q}{\theta_w}$ provides a consistent expansion for the input work,

$$\langle w \rangle \approx \alpha - (\alpha - \beta)\theta_q + \cdots$$

(23)

Similarly, we write a Taylor expansion for the fluctuations of the cooling current,

$$\langle \langle q^2 \rangle \rangle \approx \gamma \theta_q^2 + \delta \theta_q^3 + \cdots$$

(24)

and using $\eta(2) = \frac{\langle q^2 \rangle}{\langle w \rangle^2} = \frac{\langle q^2 \rangle}{\langle w \rangle^2}$ put together the consistent expansion for the fluctuations of the work current,

$$\langle \langle w^2 \rangle \rangle \approx \gamma + \theta_q(\delta - 2\gamma) + \cdots$$

(25)

Since we assume small cooling current and correspondingly small fluctuations (see figure 2), $\theta_q \ll 1$, $\beta \theta_q \ll \alpha$, and $\delta \theta_q \ll \gamma$, the lowest order expansions for the currents and their fluctuations are,

$$\langle q \rangle \approx \alpha \theta_q, \quad \langle w \rangle \approx \alpha.$$

(26)

$$\langle \langle q^2 \rangle \rangle \approx \gamma \theta_q^2, \quad \langle \langle w^2 \rangle \rangle \approx \gamma.$$

These expressions are consistent with equation (14); recall that we set here the total gap at $\theta_h = 1$. We are now ready to test the lower bound. We begin with two refrigerators $A$ and $B$ of spacings $\theta_{qA}$ and $\theta_{qB}$, that are coupled in the same manner to the baths (for example, $A$ and $B$ are marked by circles in figure 2(a)). Since these systems lie on the same curve, $\langle q_A \rangle \approx \alpha \theta_{qA}, \langle q_B \rangle \approx \alpha \theta_{qB}, \langle w_A \rangle \approx \langle w_B \rangle \approx \alpha \theta_{qA}, \langle q_A^2 \rangle \approx \langle q_B^2 \rangle \approx \gamma \theta_{qA}^2, \langle w_A^2 \rangle \approx \langle w_B^2 \rangle \approx \gamma \theta_{qA}^2$. 


\[ \langle w_B^2 \rangle \approx \gamma. \] We now test the inequality

\[ \eta_{N=2}^2 = \left( \frac{\langle q_A^2 \rangle + \langle q_B^2 \rangle}{\langle w_A^2 \rangle + \langle w_B^2 \rangle} \right)^2 \leq \left( \frac{\langle q_A^2 \rangle + \langle q_B^2 \rangle}{\langle w_A^2 \rangle + \langle w_B^2 \rangle} \right) = \eta_{N=2}^{(2)}, \tag{27} \]

by substituting the currents and fluctuations. It reduces to

\[ \frac{(\theta_{qA} + \theta_{qB})^2}{4} \leq \frac{\theta_{qA}^2 + \theta_{qB}^2}{2}, \tag{28} \]

which is true since \((\theta_{qA} - \theta_{qB})^2 \geq 0.\)

This proof can be generalized to the bound on \(\eta^{(n)}\) for an ensemble of \(N\) refrigerators. Given the small-\(\theta_q\) expansions of the \(n\)th cumulant \([54]\), \(\langle q^n \rangle \approx \gamma_n \theta_q^n\), \(\langle w^n \rangle \approx \gamma_n\), the lower-bound inequality reads

\[ \eta_{N}^n = \left( \frac{\sum_k \theta_{qk} n_k}{N} \right)^n \leq \frac{\sum_k \theta_{qk}^n n_k}{N} = \eta_{N}^{(n)} n, \tag{29} \]

Taking the \(n\)th root gives the desired result by the power mean inequality \([55]\), with power 1 on the left-hand side and \(n\) on the right,

\[ \eta_{N} = \left( \frac{1}{N} \sum_k \theta_{qk} \right)^1 \leq \left( \frac{1}{N} \sum_k \theta_{qk}^n \right)^{1/n} = \left[ \eta_{N}^{(n)} \right]^{1/n}. \tag{30} \]

The power mean inequality also implies that \(n = 2\) provides the tightest bound on \(\eta_{N}\).

We note that in general, the assumption of the cooling current being linear in the gap, \(\langle q \rangle = \alpha \theta_q\), does not necessarily correspond to a linear response limit, \(\langle q \rangle \propto (T_w - T_i)\). However, for the 3LQARs, this is in fact the case \([54]\): a linear dependence of \(\langle q \rangle\) with \(\theta_q\) develops hand in hand with the current becoming linear in the temperature difference, though the temperature difference could be large, \(\Delta T/T_i \gg 1\).

In appendix A, we prove the lower bound in region II as marked in figure 2(a), albeit limited to \(n = 2\) and \(N = 2\).

5.2.3. Simulations. We exemplify in figure 2 the behavior of individual 3LQARs. The population dynamics, heat currents and their fluctuations are calculated as described in reference \([54]\) using kinetic-like quantum master equations. The three-level system is coupled to the heat baths with an Ohmic spectral density function, with the excitation rate constant, e.g. between the lowest state and the intermediate one, \(k_q = \Gamma_q (\theta_q n_q \frac{\theta_q}{\theta_w})\) where we use an Ohmic model, \(\Gamma_q (\theta_q) = \gamma_q \theta_q^{-1}\). \(n_q (\theta_q)\) is the Bose–Einstein distribution function. The relaxation rate constant follows from local detailed balance. \(\gamma_i\) are dimensionless coefficients that control the coupling strength. Simulations in figure 2 agree with theoretical results for the efficiency \(\eta = \theta_q / \theta_w\) and the ratio of fluctuations, \(\eta^{(2)} = (\theta_q / \theta_w)^2\).

In figure 3(a), we present simulation results for \(\eta_{N=2}^{(2)}\). We generate a randomized set of 300 refrigerators by sampling uniformly over different values of \(\theta_q\) within the cooling domain. As for the coupling strength, they are sampled from a uniform distribution \(0 \leq \gamma_i \leq 1\), and they can be distinct and unequal at each realization \((\gamma_h \neq \gamma_q \neq \gamma_w)\). We then consider every possible pair of refrigerators and calculate for this pair the total cooling \((\langle q_A \rangle + \langle q_B \rangle)\) and work currents \((\langle w_A \rangle + \langle w_B \rangle)\), as well as their fluctuations. \((\langle q_A^2 \rangle + \langle q_B^2 \rangle)\) and \((\langle w_A^2 \rangle + \langle w_B^2 \rangle)\).

This allows us to calculate the efficiency for each pair, \(\eta_{N=2} = (\langle q_A \rangle + \langle q_B \rangle) / (\langle w_A \rangle + \langle w_B \rangle)\).
Figure 3. Demonstration that the lower bound $\eta_2 N \leq \eta_2^{(2)}$ is valid beyond regions I and II of the cooling window. (a) Each dot corresponds to a pair of three-level systems with random values for $\theta_q$ and system-bath couplings, $\gamma_i$. For this pair, we calculate the total cooling and work currents, as well as their fluctuations. In the inset we test the inequality (13), which is stronger than the lower bound: we present the difference $\langle \langle w^2_A \rangle \rangle / \langle \langle w^2_B \rangle \rangle - \langle w_A \rangle / \langle w_B \rangle$ for pairs with $\eta_A \geq \eta_B$ and show that it is always positive. (b) Same calculations as in (a), but for a triplet of 3LQARs. Temperatures are the same as in figure 2. Coupling strengths are uniformly sampled from $0 \leq \gamma_i \leq 1$, $\theta_q$ is sampled from the cooling region.

and the ratio of their fluctuations, $\eta_2^{(2)}$. Though most of our refrigerators lie outside regions I and II, we do not observe violations to the lower bound, $\eta_2^2 N \leq \eta_2^{(2)}$, note that parameters are outside the linear response regime [22].

In figure 3(b), we select 90,000 samples of triplet 3LQARs and calculate their total current and fluctuations; the total cooling current is given by $\langle q_A \rangle + \langle q_B \rangle + \langle q_C \rangle$ and the associated fluctuations by $\langle \langle q^2_A \rangle \rangle + \langle \langle q^2_B \rangle \rangle + \langle \langle q^2_C \rangle \rangle$, with analogous expressions giving the work current and its fluctuations. We then obtain the efficiency and the ratio of fluctuations for these triplet 3LQARs. Again we confirm based on simulations that the lower bound is satisfied.

5.2.4. Discussion. We organize our observations up to this point on the validity of the bounds (4) for an ensemble of TC refrigerators: (i) the upper bound holds without additional assumptions (section 3). As for a lower bound on $\eta_2^{\text{opt}}$, for the 3LQARs as discussed in this section, we proved that: (ii) the lower bound saturates for an ensemble of systems with identical spacings, $\theta_q$, but distinct couplings to the baths. (iii) The lower bound holds for an ensemble of systems operating with equal coupling schemes. This proof holds in the limit of small cooling currents, in region I, characterized by a vanishing $\theta_q$. We were also able to prove (appendix A) the lower bound in region II, characterized by a vanishing input work current, albeit limited to $n = 2$ and $N = 2$. (iv) Extensive simulations confirmed the validity of the lower bound more broadly beyond linear response. Even more so, our simulations showed that an inequality more general than the lower bound holds, namely equation (13). (v) In appendix B we discuss the validity of the lower bound for a three-level model operating as an engine.
Figure 4. An ensemble of thermoelectric junctions, made of a collection of noninteracting and uncorrelated quantum dots, each characterized by a resonant level, $\epsilon_k$, and coupling strengths, $\Gamma_L$ and $\Gamma_R$, to the left and right lead, respectively. The junctions all operate between the same pair of metallic leads.

6. Model II: ensemble of tight-coupling thermoelectric refrigerators

A thermoelectric junction comprises a left (L) and right (R) metal lead, between which charge and energy transport may be mediated through an embedded quantum system. The leads serve as thermal baths, differing in temperature and chemical potential such that the associated affinities oppose one another [26]; we suppose that $T_L < T_R$ and $\mu_L > \mu_R$. In what follows, we assume resonant electron transport through a discrete level [31], and discuss the existence of the lower bound on $\eta^{(n)}_N$ for an ensemble with $N$ uncorrelated thermoelectric junctions as depicted in figure 4.

6.1. Single junction, $N = 1$

We identify a stochastic charge current $j_k$, for a junction labelled by $k$, focusing here on the TC limit [25, 26], wherein the mean energy current is $\epsilon_k \langle j_k \rangle$-proportional to the charge current, where $\epsilon_k$ is the resonant level of a single quantum dot characterizing the system through which transport is mediated. In this limit, the charge current and fluctuations are given by [56, 57]

$$
\langle j_k \rangle = T_1 [f_L(\epsilon_k) - f_R(\epsilon_k)],
$$

$$
\langle \langle j_k^2 \rangle \rangle = T_1 [f_L(\epsilon_k)(1 - f_R(\epsilon_k)) + f_R(\epsilon_k)(1 - f_L(\epsilon_k))] - T_2 [f_L(\epsilon_k) - f_R(\epsilon_k)]^2.
$$

(31)

The coefficients $T_1$ and $T_2$ are determined by the coupling strengths $\Gamma_L$ and $\Gamma_R$ between the system and the two leads. For the resonant level model, $T_1 = \frac{\Gamma_L \Gamma_R}{\Gamma_L + \Gamma_R}$ and $T_2 = \frac{2\Gamma_L^2 \Gamma_R^2}{(\Gamma_L + \Gamma_R)^3}$.
see references [17, 18]. \( f_L(\epsilon) \) and \( f_R(\epsilon) \) are the Fermi–Dirac distributions for the two leads. A thermoelectric device may act as a refrigerator or an engine in various operational regimes, as determined by the directions of power and heat currents, which are proportional to the charge current in the TC limit. The mean currents thus obey \( \langle w_L \rangle = \Delta \mu (\langle j_L \rangle) \) (\( \nu = L \) for a refrigerator, \( R \) for an engine) with \( \Delta \mu = \mu_L - \mu_R \). The fluctuations in these quantities are similarly determined by those for the charge current: \( \langle \langle w_L^2 \rangle \rangle = \Delta \mu^2 \langle \langle j_L^2 \rangle \rangle \), \( \langle \langle q_L^2 \rangle \rangle = (\epsilon_k - \mu_L)^2 \langle \langle j_L^2 \rangle \rangle \).

An individual TC thermoelectric junction has been shown to satisfy the equality \( \eta_k^n = \eta_k^{(n)} \) for any \( n = 1, 2, 3, \ldots \) [22], where, with \( \epsilon_k \) the dot energy for the machine \( k \),

\[
\eta_R = \frac{\Delta \mu}{\epsilon_k - \mu_R} \quad \text{(engine)}
\]
\[
\eta_L = \frac{\epsilon_k - \mu_L}{\Delta \mu} \quad \text{(refrigerator)}.
\]

Immediately, the upper bound, \( \eta_k^n \leq \eta_k^{(n)} \), also holds [22], where \( \eta_C \) represents the operational Carnot bound.

6.2. Ensemble of \( N > 1 \) distinct junctions

We now consider an ensemble of distinct TC thermoelectric junctions, labelled by indices \( k \), whose internal parameters such as \( \epsilon_k \) and system–lead coupling strengths may differ, but which operate between leads with the same set of bath parameters. For example, we envision a thermoelectric device with parallel quantum dots each conducting resonantly with their own energy and coupling strengths to the metal electrodes; see figure 4.

The validity of the upper bound for a single junction, along with the general result expressed in equation (8), leads to the analogous upper bound for an ensemble of such junctions as discussed in section 3. The following discussion will therefore focus on when the lower bound, \( \eta^n \leq \eta^{(n)} \), holds for such an ensemble, in the \( n = 2 \) case.

6.2.1. Thermoelectric junctions with different system-bath couplings. Consider an ensemble of \( N \) thermoelectric junctions, labelled by \( k \), with the same resonant level, \( \epsilon \), characterizing the quantum system, as represented in figure 5(a) by the pair of asterisks. The cooling efficiency, \( \langle q_L \rangle / \langle w_L \rangle = (\epsilon - \mu_L) / \Delta \mu \) is equal for all such refrigerators since it does not depend on the system-bath coupling. As such, one can readily prove, via an argument mirroring that in section 5.2.1, the strict equality \( \eta_k^n = \eta_k^{(n)} \) for the ensemble.

6.2.2. Thermoelectric junctions with distinct energy levels. Now, we will focus specifically on the case of a pair (\( N = 2 \)) of TC thermoelectric junctions, \( A \) and \( B \), operating as refrigerators, with cooling currents flowing from the left (cold) lead, \( \langle q_L \rangle = (\epsilon_k - \mu_L) \langle j_L \rangle \), \( k = A, B \). The cooling and work currents are taken, by convention, to be positive. We will show that for this pair taken as a single device, the lower bound, \( \eta_{N=2}^R \leq \eta_{N=2}^{(R)} \) holds, provided we restrict ourselves to a specific range of values for \( \epsilon_A \) and \( \epsilon_B \), namely, region II, near the edge of the cooling window such that \( \langle j_L \rangle \) is small. For instance, such an ensemble may be represented by the pair of circles in figure 5(a). Furthermore, we will suppose that one refrigerator, say \( B \), has a significantly larger charge current than the other (\( \langle j_L \rangle / \langle j_B \rangle \ll 1 \)). This requires that \( \epsilon_A > \epsilon_B \), so \( \eta_A > \eta_B \). The complementary proof for region I is given in appendix C.

We express the dot energy for each thermoelectric refrigerator as \( \epsilon_k = \mu_k + \Delta \mu \beta_k / \Delta \beta - u_k \) with \( \Delta \beta = \beta_A - \beta_B, \Delta \mu = \mu_A - \mu_B \). Note that the charge current vanishes when \( u_k = 0 \) since \( f_L(\epsilon_k) = f_R(\epsilon_k) \) at this point [58], and initially grows linearly with \( u_k \): \( \langle j_L \rangle \approx au_k \), where
Figure 5. Exemplifying the performance of individual thermoelectric refrigerators between $\beta_L = 2$, $\beta_R = 1$, $\mu_L = 1$, and $\mu_R = -1$. (a) Cooling current, (b) work current, (c) cooling efficiency, (d) fluctuations of cooling current and (e) work current, and (f) the relative noise $\eta^2$. In section 6.2.2 we prove the lower bound in region II under certain conditions. Region I is explored in appendix C.

$\alpha$ is some constant coefficient. Then we have that $u_A/u_B \ll 1$, and we may write out the full efficiency of the pair, $\eta_{N=2} = [(\epsilon_A - \mu_L)u_A + (\epsilon_B - \mu_L)u_B]/[(u_A + u_B)\Delta\mu]$. Squaring and truncating after first order in $u_A/u_B$, we get

$$\eta_{N=2}^2 \approx \eta_B^2 \left[ 1 + 2 \left( \frac{\epsilon_A - \mu_L}{\epsilon_B - \mu_L} - 1 \right) \frac{u_A}{u_B} \right].$$  \hspace{1cm} (33)

Choosing sufficiently large $\epsilon_k$ leads to a small value for $f_L(\epsilon_k) - f_R(\epsilon_k)$. We therefore ignore the contribution to the fluctuations, as given by equation (31), proportional to $(f_L(\epsilon_k) - f_R(\epsilon_k))^2$. This corresponds to the assumption that cotunneling processes may be neglected. We write

$$\langle \langle j_k^2 \rangle \rangle = T_1 [f_L(\epsilon_k)(1 - f_R(\epsilon_k)) + f_R(\epsilon_k)(1 - f_L(\epsilon_k))].$$  \hspace{1cm} (34)

This can be shown to be proportional to the charge current,

$$\langle \langle j_k^2 \rangle \rangle = \langle j_k \rangle \coth \left( \frac{\Delta\beta u_k}{2} \right) \equiv c_k \langle j_k \rangle,$$  \hspace{1cm} (35)

with $c_k$ defined from this relation. The ratio of fluctuations of the cooling current to work current is given in terms of $u_A$ and $u_B$ as

$$\eta_{N=2}^{(2)} = \eta_B^{(2)} \frac{1 + \frac{(\epsilon_A - \mu_L)^2}{(\epsilon_B - \mu_L)^2} \frac{c_A u_A}{c_B u_B}}{1 + \frac{c_A u_A}{c_B u_B}}.$$  \hspace{1cm} (36)

As $u_k$ approaches zero, $c_k$ goes to infinity, so we cannot assume $c_A u_A/c_B u_B$ is small. However, we can still compare $\eta_{N=2}^{(2)}$ and $\eta_B^{(2)}$, finding, after keeping only terms up to first order in $u_A/u_B$. 


Figure 6. Demonstration that the lower bound $\eta_{(2)}^2 \leq \eta_{(2)}^{(2)}$ is valid beyond the linear regime of the charge current. (a) Each dot corresponds to a pair of TC thermoelectric refrigerators with random values for $\epsilon_k$ as well as coupling strengths. For this pair, we calculate the total cooling and work currents as well as their fluctuations. (b) Same calculations as in (a), but for a triplet of thermoelectrics. $\beta_L = 2$, $\beta_R = 1$, $\mu_L = 1$, and $\mu_R = -1$. Coupling strengths are uniformly sampled from $0 \leq \Gamma \leq 1/2$, $\epsilon_k$ is sampled from the cooling region, $1 < \epsilon_k < 3$. The operational Carnot bound for refrigeration is $\eta_C = \beta_R / (\beta_L - \beta_R)$.

that the lower bound is equivalent to the inequality

$$2 \left( \frac{\epsilon_A - \mu_L}{\epsilon_B - \mu_L} - 1 \right) \leq \frac{c_A}{c_B} \left[ \frac{(\epsilon_A - \mu_L)^2}{(\epsilon_B - \mu_L)^2} - 1 \right].$$

Since $\epsilon_A > \epsilon_B$, this holds as long as $c_A / c_B \geq 2$. We note that this is certainly the case since, by assumption, $u_A$ and $u_B$ are small, so $c_A / c_B \approx u_B / u_A \gg 1$. We may conclude that in this regime, $\eta_{(2)}^{(2)} \leq \eta_{(2)}^{(2)}$.

The proof outlined in this section is quite limited. It holds for a pair of refrigerators, $N = 2$, and only for the second cumulant, $n = 2$. Complementing this proof, in appendix C, we prove the lower bound in region I, now for an ensemble of arbitrary size $N$ and to the order $n$. Furthermore, figure 6 demonstrates, via the results of simulations, that the upper and lower bounds (for $n = 2$) appear to hold for pairs and triplets of thermoelectric refrigerators outside regions I and II.

7. Violation of the lower bound for a pair of tight-coupling thermoelectric engines

The validity of the lower bound has been proven for three-level refrigerators and thermoelectric refrigerators in corresponding cases, in regions I and II. We now consider an analogous set of assumptions as discussed in section 6.2, but applied to ensembles of thermoelectric junctions operating instead as engines. We find that there exist circumstances under which the lower bound must be violated, identifying a limitation to the applicability of the lower bound on $\eta_{(2)}^{(2)}$. 

16
for ensembles of thermal machines, and highlighting an interesting distinction between the behaviours of thermoelectric engines and refrigerators.

Specifically, we consider a pair of thermoelectric engines in the TC limit, far from equilibrium, operating in the regime of high $\epsilon_k$, for both $k = A, B$, and thus small charge currents. Note that, due to the assumption of TC, this operational regime has no upper boundary with respect to $\epsilon_k$. For simplicity, we take the two junctions to have identical coupling strengths to the leads. Furthermore, we suppose that the magnitude of the charge current is considerably greater for engine $A$ than for $B$. Since, in this regime, charge current decreases with $\epsilon_k$, $\epsilon_B > \epsilon_A$, so, referring to equation (32), $\eta_A > \eta_B$. We will show that, under these assumptions, the lower bound on the ratio of fluctuations, $\eta_{N=2}^{(2)} \leq \eta_{N=2}^{(2)}$, is violated.

In contrast to the refrigerator case, we now define the heat current with respect to the right (hot) lead $\langle q_k \rangle = (\epsilon_k - \mu_R) \langle j_k \rangle$. We first write out the efficiency of the pair of engines,

$$\eta_{N=2} = \frac{\langle w_A \rangle + \langle w_B \rangle}{\langle q_A \rangle + \langle q_B \rangle} = \frac{\Delta \mu (\langle j_A \rangle + \langle j_B \rangle)}{(\epsilon_A - \mu_R) \langle j_A \rangle + (\epsilon_B - \mu_R) \langle j_B \rangle} = \eta_A \left[ 1 + \frac{\langle j_B \rangle}{\langle j_A \rangle} \right].$$ \hspace{1cm} (38)

Noting that $\langle j_B \rangle / \langle j_A \rangle \ll 1$, we expand to first order in this ratio. After squaring and using that $\eta_A^2 = \eta_A^{(2)}$, we have

$$\eta_{N=2}^{(2)} \approx \eta_A^{(2)} \left[ 1 + 2 \left( 1 - \frac{\epsilon_B - \mu_R}{\epsilon_A - \mu_R} \right) \frac{\langle j_B \rangle}{\langle j_A \rangle} \right].$$ \hspace{1cm} (39)

Next, we consider the fluctuations in the work and heat currents. The assumption of small currents allows us to neglect cotunneling processes—that is, contributions proportional to $(f_L(\epsilon_k) - f_R(\epsilon_k))^2$. Furthermore, we use the identity $f_L(\epsilon_k) [1 - f_R(\epsilon_k)] + f_R(\epsilon_k) [1 - f_L(\epsilon_k)] = [f_L(\epsilon_k) - f_R(\epsilon_k)] \coth(-\Delta \beta \epsilon_k + \beta \Delta \mu)/2 \equiv c_L [f_L(\epsilon_k) - f_R(\epsilon_k)]$ (note that $\Delta \beta = \beta_L - \beta_R > 0$, $\beta = (\beta_L + \beta_R)/2$) to express the charge current fluctuations for each engine in terms of the charge current itself:

$$\langle \langle J_k^2 \rangle \rangle = c_k \langle j_k \rangle.$$ \hspace{1cm} (40)

Then, arguments similar to those used to derive equation (39) give

$$\eta_{N=2}^{(2)} \approx \eta_A^{(2)} \left[ 1 + \left( 1 - \frac{\epsilon_B - \mu_R}{\epsilon_A - \mu_R} \right)^2 \frac{c_B \langle j_B \rangle}{c_A \langle j_A \rangle} \right].$$ \hspace{1cm} (41)

Now, comparing equations (39) and (41), we find that the lower bound $\eta_{N=2}^{(2)} \leq \eta_{N=2}^{(2)}$ is equivalent to the inequality

$$2 \left( 1 - \frac{\epsilon_B - \mu_R}{\epsilon_A - \mu_R} \right) \leq \frac{c_B}{c_A} \left[ 1 - \left( \frac{\epsilon_B - \mu_R}{\epsilon_A - \mu_R} \right)^2 \right].$$ \hspace{1cm} (42)

Recall that $c_k = \coth(-\Delta \beta \epsilon_k + \beta \Delta \mu)/2$; working far from equilibrium, we suppose that $\Delta \beta$ is sufficiently large that $c_k$ and $c_B$ meeting our above assumptions may be chosen such that both $c_A \approx -1$ and $c_B \approx -1$. Thus, $c_B/c_A \approx 1$. This simplifies the necessary and sufficient condition for the lower bound to

$$2 \left( 1 - \frac{\epsilon_B - \mu_R}{\epsilon_A - \mu_R} \right) \leq \left[ 1 - \left( \frac{\epsilon_B - \mu_R}{\epsilon_A - \mu_R} \right)^2 \right],$$ \hspace{1cm} (43)
Figure 7. Violations of the lower bound observed for pairs of tight-coupling thermoelectric engines with sufficiently high $\epsilon_A$ and $\epsilon_B$, shown as a function of $\epsilon_B > \epsilon_A$ for two choices of $\epsilon_A$. Quantities are calculated using the full expressions for currents and fluctuations given in equation (31). Parameters are $\beta_L = 2, \beta_R = 1, \mu_L = 1, \mu_R = -1$, and $\Gamma_L = \Gamma_R = 1$.

or, equivalently,

$$\left(\frac{\epsilon_B - \mu_R}{\epsilon_A - \mu_R} - 1\right)^2 \leq 0.$$  \hfill (44)

Our assumption that $\epsilon_B > \epsilon_A$ renders equation (44) a clear contradiction. Therefore, the set of assumptions considered here describes a pair of thermoelectric engines that must violate the lower bound on the ratio of work to heat current fluctuations, $\eta^{(2)}_{N=2}$. We emphasize that this derivation relies on the thermoelectric engine operating far from equilibrium. In linear response, a small value for $\Delta \beta$ would conflict with the ability to achieve $c_B/c_A \approx 1$ for any choice of $\epsilon_A$ and $\epsilon_B$ satisfying the rest of the assumptions. Thus, there is no contradiction with reference [22].

In figure 7 we provide supporting numerical evidence for the violation of the lower bound for thermoelectric engines operating in region II. While we use approximate expressions for the fluctuations in our analysis, the numerical simulations use the full expressions of equation (31), demonstrating that these violations are not merely a consequence of the assumptions made.

8. Summary

We studied universal bounds on ratios of fluctuations, as well as higher order cumulants, in an ensemble of distinct, uncorrelated thermal machines operating in steady state and arbitrarily far from equilibrium. We built our bounds for the ensemble of machines from bounds on the individual machine, $\eta^n_{n} = \eta^{(n)} \leq \eta_C^n$. These relations hold, e.g. for systems obeying the TC limit, such as three-level absorption refrigerators and resonant-level thermoelectric junctions. $\eta_C$ is the Carnot efficiency and $\eta^{(n)}$ is ratio of the nth cumulants of the output current to input current. We proved that:
(a) An upper bound holds for an ensemble of $N$ distinct machines, engines or refrigerators, arbitrarily far from equilibrium, $\eta_n^{(N)} \leq \eta_C$.

(b) The lower bound on the ensemble, $\eta_n^N \leq \eta_n^{(N)}$, is more limited in scope. While it seems to hold for refrigerators, based on analytic and numerical work, we demonstrated that thermoelectric engines may violate the lower bound for $N \geq 2$.

(c) Focusing on 3LQARs and thermoelectric junctions, we proved the validity of the lower bound, $\eta_n^N \leq \eta_n^{(N)}$, in different cases: when the cooling current was small, and for an ensemble of distinct machines with identical efficiencies.

(d) While analytic proofs of the lower bound for refrigerators were limited in scope (e.g. to $n = 2$ and $N = 2$ in region II near the edge of the cooling window), simulations beyond these strict regimes support its validity more broadly.

The significance of our study is in developing bounds for ensembles of machines. While bounds on the individual system may be trivial, as in the TC limit, a machine that collects input and output from several tightly-coupled components (recall figures 1 and 4) may behave nontrivially in this respect, even under classical laws and in the absence of interactions between components. Adding interactions to the working fluid, e.g. by using an interacting atomic gas or introducing Coulomb interactions between quantum dots in a thermoelectric device, opens up the door to new effects, such as many-particle enhancement of performance [59–64]. Furthermore, quantum statistical effects combined with interactions or collective unitary operations on the components can enhance performance, as predicted in reference [65].

An important result of our work is in tightening bounds on efficiency: as an outcome of the lower bound, equation (30), one immediately finds that $\eta \leq [\eta_n^{(n)}]^{1/n} \leq \eta_C$, and that the case with $n = 2$ provides the tightest upper bound on the efficiency. Whether this is a general result, or only valid for refrigerators in the small-$\theta$ small-cooling domain, remains an open question.

In future work we will focus on understanding the validity of the lower bound on $\eta^{(0)}$ in general settings, and on exploring the fundamental differences between engines and refrigerators as they pertain to bounds on fluctuations.

Acknowledgments

DS acknowledges support from an NSERC Discovery Grant and the Canada Research Chair program. The work of NK was supported by the Ontario Graduate Scholarship (OGS). The research of MG was supported by the NSERC Canada Graduate Scholarship-Master’s and the OGS. Na’im Kalantar and Matthew Gerry contributed equally to this work and are joint ‘first authors’.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. Lower bound on $\eta_n^{(2)}$ for 3LQARs at the boundary of the cooling region (region II)

Here, we present a proof of the validity of a lower bound for two 3LQARs, applicable in the limit of small currents. This proof holds at the boundary of the cooling domain (see region II in
The proof presented here is limited to two QARs, and its extensions to \( N \) refrigerators is not obvious.

For simplicity, we define the scaled currents \( q/\theta_q \rightarrow q \) and \( w/\theta_w \rightarrow w \). We also assume that \( T_w \rightarrow \infty \). The cooling current and its noise are given in terms of the bath-induced transition rate constants \( \Gamma_{q,h,w} \), see text in section 5.2.3 and reference [54]. We also use the short notation \( \tilde{\Gamma}_q = \Gamma_q(\theta_q) \). The cooling current and its fluctuations are given by [54]

\[
\langle q \rangle = \frac{\tilde{\Gamma}_h \tilde{\Gamma}_q \tilde{\Gamma}_w}{M} (e^{q/T_h} - e^{q/T_w}),
\]

(A1)

\[
\langle \langle q^2 \rangle \rangle = \frac{2}{M} \left( \tilde{\Gamma}_h + \tilde{\Gamma}_q + \tilde{\Gamma}_w + \tilde{\Gamma}_q e^{q/T_h} + \tilde{\Gamma}_w + \tilde{\Gamma}_h e^{q/T_w} \right) \langle q \rangle^2
\]

\[
+ \frac{\tilde{\Gamma}_q \tilde{\Gamma}_h \tilde{\Gamma}_w}{M} (e^{q/T_h} + e^{q/T_w})
\]

with \( M = (\tilde{\Gamma}_h e^{q/T_h} + \tilde{\Gamma}_w) \left( \tilde{\Gamma}_w + \tilde{\Gamma}_h e^{q/T_w} \right) + \left( \tilde{\Gamma}_q + \tilde{\Gamma}_h \right) \left( \tilde{\Gamma}_w + \tilde{\Gamma}_h e^{q/T_w} \right)
\]

\[
+ \left( \tilde{\Gamma}_q + \tilde{\Gamma}_h \right) \left( \tilde{\Gamma}_q e^{q/T_h} + \tilde{\Gamma}_w \right) - \tilde{\Gamma}_q^2 e^{q/T_h} - \tilde{\Gamma}_w^2 - \tilde{\Gamma}_q e^{q/T_w}. \quad \text{(A2)}
\]

The cooling window is defined by the condition \( 0 < \theta_q < \theta_h \frac{T_h}{T_q} \). When \( \langle q \rangle \ll \Pi_{i=h,q,w} \tilde{\Gamma}_i \), the second term in the noise dominates, thus it is given by

\[
\langle \langle q^2 \rangle \rangle \approx \frac{\theta_h \langle q \rangle}{e^{q/T_h} - e^{q/T_w}} \left( \frac{\theta_h}{2 T_h} - \frac{\theta_q}{2 T_q} \right) \langle q \rangle \equiv F(\theta_q) \langle q \rangle. \quad \text{(A3)}
\]

We now consider two refrigerators, A and B, with identical total gaps \( \theta_h \), but different internal energies, \( \theta_{qa} \neq \theta_{qb} \). Correspondingly, each QAR supports the cooling current \( \langle q_a \rangle \) with the associated noise \( \langle \langle q_a^2 \rangle \rangle \). As for the lower bound, we would like to show that (recovering the energy gaps in the expressions for the currents and noises):

\[
\eta^2 = \frac{\theta_{qa}^2 \langle q_a \rangle^2}{(\theta_{qa} \langle q_a \rangle + \theta_{qa} \langle q_b \rangle)^2} \leq \frac{\theta_{qa}^2 \langle \langle q_a^2 \rangle \rangle + \theta_{qa}^2 \langle \langle q_b^2 \rangle \rangle}{\theta_{qa}^2 \langle q_a^2 \rangle + \theta_{qa}^2 \langle q_b^2 \rangle} = \eta^2. \quad \text{(A4)}
\]

Using equation (A3) and that \( \theta_w = \theta_h - \theta_q \), we get

\[
\frac{[\theta_{qa} \langle q_a \rangle + \theta_{qa} \langle q_b \rangle]^2}{[(\theta_h - \theta_{qa}) \langle q_a \rangle + (\theta_h - \theta_{qa}) \langle q_b \rangle]^2}
\]

\[
\leq \frac{\theta_{qa}^2 F(\theta_{qa}) \langle q_a \rangle + \theta_{qa}^2 F(\theta_{qa}) \langle q_b \rangle}{(\theta_h - \theta_{qa})^2 F(\theta_{qa}) \langle q_a \rangle + (\theta_h - \theta_{qa})^2 F(\theta_{qa}) \langle q_b \rangle}. \quad \text{(A5)}
\]

After cross-multiplying, all the \( \langle q^2 \rangle \) terms vanish and we are left with

\[
\sum_{i \neq k} \left( \theta_{qi}^2 (\theta_h - \theta_q) F(\theta_{qi}) + 2 \theta_{qi} \theta_{qk} (\theta_h - \theta_q) F(\theta_{qi}) \right) \langle q_i \rangle
\]

\[
\leq \sum_{i \neq k} \left( \theta_{qi}^2 (\theta_h - \theta_q) F(\theta_{qi}) + 2 \theta_{qi}^2 (\theta_h - \theta_q) (\theta_h - \theta_{qi}) F(\theta_{qi}) \right) \langle q_i \rangle. \quad \text{(A6)}
\]
Since each current \(\langle q \rangle\) can be set freely by setting the transition rates, the inequality is true if and only if each \(\langle q \rangle\) part obeys the inequality:

\[
\theta_h^2(\theta_h - \theta_{qa})^2F(\theta_{qa}) + 2\theta_{qa}\theta_{qg}(\theta_h - \theta_{qa})^2F(\theta_{qa}) \\
\leq \theta_{qb}^2(\theta_h - \theta_{qa})^2F(\theta_{qa}) + 2\theta_{qa}^2(\theta_h - \theta_{qb})(\theta_h - \theta_{qb})F(\theta_{qa}).
\] (A7)

After additional manipulations we get

\[
[\theta_h^2(\theta_{qa}^2 - \theta_{qb}^2) - 2\theta_h\theta_{qa}\theta_{qg}(\theta_{qa} - \theta_{qb})] F(\theta_{qb}) \\
+ [2\theta_{qa}\theta_h(\theta_h - \theta_{qa})(\theta_{qb} - \theta_{qa})] F(\theta_{qa}) \leq 0.
\] (A8)

The function \(F(\theta)\) is positive, and it grows exponentially as \(\theta\) approaches the asymptote at the edge of the cooling window, \(\theta_q = \theta_h^\frac{T_A}{T_h}\), from below. Therefore, if \(\theta_{qa} < \theta_{qb} < \theta_h^\frac{T_A}{T_h}\), the first term in equation (A8) dominates. It is given by

\[
\theta_{qa}(\theta_{qa} + \theta_{qb})(\theta_{qa} - \theta_{qb}) - 2\theta_{qa}\theta_{qg}(\theta_{qa} - \theta_{qb}) \leq 0,
\] (A9)

which is reduced to

\[
\theta_{qa}(\theta_{qa} + \theta_{qb}) - 2\theta_{qa}\theta_{qb} \geq 0.
\] (A10)

This is true since \(\theta_h > \theta_{qa}\) and \(\theta_h > \theta_{qb}\).

Likewise, the second term in equation (A8) dominates in the opposite case, if \(\theta_{qb} < \theta_{qa} < \theta_h^\frac{T_A}{T_h}\). We then check whether

\[
2\theta_{qa}\theta_h(\theta_h - \theta_{qa})(\theta_{qb} - \theta_{qa}) \leq 0,
\] (A11)

which is true, since the last term is negative and the rest are positive.

### Appendix B. Lower bound on \(\eta_2^{(2)}\) for three-level engines

We consider here the performance of the three-level model as an engine, rather than a refrigerator. We show that in the limit of small currents, at the edge of the engine’s window, the lower bound for \(\eta^{(2)}\) holds. This is to be contrasted with behavior observed for a thermoelectric engine, section 7, which shows violations to the lower bound in a corresponding domain.

The system acts as an engine when \(\theta_h^\frac{T_A}{T_h} < \theta_q < \theta_h\). Heat is absorbed from the hot \((h)\) bath and emitted at the \(w\) bath as useful work, with leftover heat released at the \(q\) bath. In this engine’s regime, assuming the thermal noise dominates (since the current is small), we write

\[
\langle \langle q^2 \rangle \rangle = -F(\theta_q)\langle q \rangle,
\] (B1)

where was defined in equation (A3). Recall that by our conventions, \(\langle q \rangle\) is positive when flowing into the system, and thus it is negative when the system acts as an engine. Furthermore, when the system operates as an engine, \(F\) is positive and decreasing with \(\theta_q\).

For an engine, the efficiency is the work current over the heat input from the hot bath. As usual, we set the total gap size \(\theta_h\) the same for both systems. Rewriting equation (A4) for the two engines gives

\[
\eta_2^2 = \frac{(\theta_h^2\langle q_{fa} \rangle + \theta_{qg}^2\langle q_{fb} \rangle)^2}{(\theta_{fa}\langle q_{fa} \rangle + \theta_{hb}\langle q_{fb} \rangle)^2} \leq \frac{\theta_{qa}^2\langle q_{fa}^2 \rangle + \theta_{qg}^2\langle q_{fb}^2 \rangle}{\theta_{fa}^2\langle q_{fa}^2 \rangle + \theta_{qb}^2\langle q_{fb}^2 \rangle} = \eta_2^{(2)}.
\] (B2)
Figure 8. Demonstration that the lower bound $\eta_2^N \leq \eta_2^{(2)}$ is valid for the three-level absorption engine. (a) Each dot corresponds to a pair of engines with random values for $\theta_q$ as well as coupling energies. For this pair, we calculate the total work and absorbed heat currents as well as their fluctuations. (b) Same calculations as in (a), but for a triplet of engines. We used $\beta_q = 0.4$, $\beta_h = 0.2$, and $\beta_w = 0.1$. The system-bath coupling strengths are uniformly sampled from $0 \leq \gamma_i \leq 1$, $\theta_q$ is sampled from the engine region.

Recall that, according to our convention used in appendices A and B $\langle q \rangle$ is a scaled measure, i.e. it counts the number of quanta exchanged, rather than the heat current itself. Following the same steps as in appendix A, we end with equation (A7), but with $\theta_q \rightarrow \theta_w$, except in the $F$ functions, and $\theta_w \rightarrow \theta_h$. As before, we chose without loss of generality only the $\langle q_A \rangle$ terms.

The lower bound is true if

$$2(\theta_{qA} - \theta_{qB})(\theta_h - \theta_{qA})F(\theta_{qA}) \leq (\theta_{qA} - \theta_{qB})(2\theta_h - \theta_{qA} - \theta_{qB})F(\theta_{qB}).$$

When $\theta_{qA} > \theta_{qB}$,

$$2(\theta_h - \theta_{qB})F(\theta_{qA}) \leq (2\theta_h - \theta_{qA} - \theta_{qB})F(\theta_{qB})$$

which is true, using that $F$ is a decreasing function of $\theta_q$. When $\theta_{qA} < \theta_{qB}$,

$$2(\theta_h - \theta_{qA})F(\theta_{qB}) \geq (2\theta_h - \theta_{qA} - \theta_{qB})F(\theta_{qB}),$$

which also holds.

In figure 8, we search numerically for violations of the lower bound for an ensemble of three-level absorption engines in a broad parameter space. As we are not able to identify such violations, we hypothesize that the three-level system acting as either a refrigerator or an engine satisfies the lower bound on $\eta_2^{(2)}$.

Appendix C. Lower bound on $\eta_n^{(n)}$ for thermoelectric refrigerators in region I

We now turn our attention to a pair of TC thermoelectric refrigerators, $A$ and $B$, with both $\epsilon_A$ and $\epsilon_B$ at the opposite end of the cooling window from those discussed in section 6. We refer to this as "region I", as depicted in figure 5. In this region, $\langle q_k \rangle, k = A, B$, is small, vanishing
Figure 9. Comparison of the approximate expressions for (a) \( \eta_{N=2}^2 \) and (b) \( \eta_{N=2}^{(2)} \), given by equations (C1) and (C2), respectively, with their exact values. Here, \( u_k = \epsilon_k - \mu_L \).

We focus on the case \( u_B < u_A \) to avoid redundancy. The good correspondence between approximate and exact values suggests that the approximations made are valid. Indeed, upon comparing panels (a) and (b), the lower bound is seen to be satisfied.

When \( \epsilon_k = \mu_L \), however, the charge current itself \( \langle j_k \rangle \) remains finite and nonzero. We suppose that \( \langle j_A \rangle \approx \langle j_B \rangle \approx \langle j_m \rangle |_{\epsilon_m = \mu_L} \equiv J \). Defining \( u_k \equiv \epsilon_k - \mu_L \), we have that \( \langle q_k \rangle = \langle j_k \rangle u_k \approx J u_k \).

The cooling efficiency of a single refrigerator is \( \eta_k = u_k / \Delta \mu \), and the TC limit gives \( \eta_k^{(2)} = (u_k / \Delta \mu)^2 \).

Writing out the efficiency of the pair, we have

\[
\eta_{N=2} \approx \frac{\langle q_A \rangle + \langle q_B \rangle}{2 \Delta \mu J} \approx \frac{u_A + u_B}{2 \Delta \mu} = \frac{\eta_A}{2} \left( 1 + \frac{u_B}{u_A} \right) .
\]

(C1)

Since region I is not characterized by a small charge current, we cannot assume that the fluctuations in the charge or heat currents are given predominantly by thermal noise. However, we may echo our assumption above by supposing that \( \langle j_A^2 \rangle \approx \langle j_B^2 \rangle \approx \langle j_m^2 \rangle |_{\epsilon_m = \mu_L} \equiv S \).

Then, fluctuations in the cooling currents are given by \( \langle q_k^2 \rangle \approx Su_k^2 \), and the ratio of cooling to work current fluctuations is given by

\[
\eta_{N=2}^{(2)} \approx \frac{u_A^2 + u_B^2}{2 \Delta \mu J^2} \equiv \frac{\eta_A^{(2)}}{2} \left[ 1 + \left( \frac{u_B}{u_A} \right)^2 \right] .
\]

(C2)

In figure 9 we demonstrate that equations (C1) and (C2) indeed provide a good approximation for the efficiency and the ratio of fluctuations for refrigerators in region I.

Squaring equation (C1), utilizing that \( \eta_A^2 = \eta_A^{(2)} \), and comparing it to equation (C2), we see that the lower bound, \( \eta_{N=2}^2 \leq \eta_{N=2}^{(2)} \), is equivalent to the inequality

\[
\frac{1}{4} \left( 1 + \frac{u_B}{u_A} \right)^2 \leq \frac{1}{2} \left[ 1 + \left( \frac{u_B}{u_A} \right)^2 \right] .
\]

(C3)

or, rearranging,

\[
\left( \frac{u_B}{u_A} - 1 \right)^2 \geq 0 .
\]

(C4)

This is clearly satisfied for any choice of \( u_A \) and \( u_B \), so we conclude that the lower bound \( \eta_{N=2}^2 \leq \eta_{N=2}^{(2)} \) is satisfied for a pair of refrigerators in this regime.
We now extend this argument more generally to $n > 2$ and $N > 2$, by getting expressions for $\eta_N$ and $\eta_N^{(n)}$ of the ensemble analogous to equations (C1) and (C2),

\[
\eta_N = \frac{\sum_{k=1}^{N} u_k}{N \Delta \mu},
\eta_N^{(n)} = \frac{\sum_{k=1}^{N} u_k^n}{N \Delta \mu^n}.
\]

(C5)

Taking the $n$th root of $\eta_N^{(n)}$, we see that the lower bound, $\eta_n^N \leq \eta_N^{(n)}$, is equivalent to

\[
\frac{1}{N} \sum_{k=1}^{N} u_k \leq \left( \frac{1}{N} \sum_{k=1}^{N} u_k^n \right)^{\frac{1}{n}},
\]

(C6)

which is always true as a result of the power mean inequality [55].

ORCID iDs

Matthew Gerry https://orcid.org/0000-0003-0367-3094
Dvira Segal https://orcid.org/0000-0002-8027-8920

References

[1] Seifert U 2012 Stochastic thermodynamics, fluctuation theorems, and molecular machines Rep. Prog. Phys. 75 126001
[2] Seifert U 2008 Stochastic thermodynamics: principles and perspective Eur. Phys. J. B 64 423
[3] den Broeck C V 2013 Stochastic thermodynamics: a brief introduction Physics of Complex Colloids ed C Bechinger, F Sciortino and P Ziherl (Amsterdam: IOS Press) pp 155–94
[4] Kosloff R 2019 Quantum thermodynamics and open-systems modeling J. Chem. Phys. 150 204105
[5] Vinjanampathy S and Anders J 2016 Quantum thermodynamics Contemp. Phys. 57 545
[6] Barato A C and Seifert U 2015 Thermodynamic uncertainty relation for biomolecular processes Phys. Rev. Lett. 114 158101
[7] Pietzonka P and Seifert U 2018 Universal trade-off between power, efficiency, and constancy in steady-state heat engines Phys. Rev. Lett. 120 190602
[8] Gingrich T R, Horowitz J M, Perunov N and England J L 2016 Dissipation bounds all steady-state current fluctuations Phys. Rev. Lett. 116 120601
[9] Horowitz J M and Gingrich T R 2017 Proof of the finite-time thermodynamic uncertainty relation for steady-state currents Phys. Rev. E 96 020103
[10] Falasco G, Esposito M and Delvenne J-C 2020 Unifying thermodynamic uncertainty relations New J. Phys. 22 053046
[11] Macieszczak K, Brandner K and Garrahan J P 2018 Unified thermodynamic uncertainty relations in linear response Phys. Rev. Lett. 121 130601
[12] Timpanaro A M, Guarnieri G, Goold J and Landi G T 2019 Thermodynamic uncertainty relations from exchange fluctuation theorems Phys. Rev. Lett. 123 090604
[13] Dechant A 2019 Multidimensional thermodynamic uncertainty relations J. Phys. A: Math. Theor. 52 035001
[14] Brandner K, Hanazato T and Saito K 2018 Thermodynamic bounds on precision in ballistic multiterminal transport Phys. Rev. Lett. 120 090601
[15] Hasegawa Y 2021 Thermodynamic uncertainty relation for general open quantum systems Phys. Rev. Lett. 126 010602
[16] Miller H J D, Mohammady M H, Perarnau-Llobet M and Guarnieri G 2021 Thermodynamic uncertainty relation in slowly driven quantum heat engines Phys. Rev. Lett. 126 210603
[17] Agarwalla B K and Segal D 2018 Assessing the validity of the thermodynamic uncertainty relation in quantum systems Phys. Rev. B 98 155438
[18] Liu J and Segal D 2019 Thermodynamic uncertainty relation in quantum thermoelectric junctions Phys. Rev. E 99 062141
[19] Saryal S, Sadekar O and Agarwalla B K 2021 Thermodynamic uncertainty relation for energy transport in a transient regime: a model study Phys. Rev. E 103 022141
[20] Ito K, Jiang C and Watanabe G 2019 Universal bounds for fluctuations in small heat engines (arXiv:1910.08096)
[21] Kamijima T, Otsubo S, Ashida Y and Sagawa T 2021 Higher order efficiency bound and its application to nonlinear nanothermoelectrics Phys. Rev. E 104 044115
[22] Saryal S, Gerry M, Khait I, Segal D and Agarwalla B K 2021 Universal bounds on fluctuations in continuous thermal machines Phys. Rev. Lett. 127 190603
[23] Saryal S and Agarwalla B K 2021 Bounds on fluctuations for finite-time quantum Otto cycle Phys. Rev. E 103 L060103
[24] Mohanta S, Saryal S and Agarwalla B K 2021 Universal bounds on cooling power and cooling efficiency for autonomous absorption refrigerators (arXiv:2106.12809)
[25] Kedem O and Caplan S R 1965 Degree of coupling and its relation to efficiency of energy conversion Trans. Faraday Soc. 61 1897
[26] Benenti G, Casati G, Saito K and Whitney R S 2017 Fundamental aspects of steady-state conversion of heat to work at the nanoscale Phys. Rep. 694 1
[27] Klatzow J et al 2019 Experimental demonstration of quantum effects in the operation of microscopic heat engines Phys. Rev. Lett. 122 110601
[28] Sothmann B, Sánchez R and Jordan A N 2014 Thermoelectric energy harvesting with quantum dots Nanotechnology 26 032001
[29] Peterson J P S, Batalhão T B, Herrera M, Souza A M, Sarthour R S, Oliveira I S and Serra R M 2019 Experimental characterization of a spin quantum heat engine Phys. Rev. Lett. 123 240601
[30] Röllnagel J, Dawkins S T, Tolazzi K N, Abah O, Lutz E, Schmidt-Kaler F and Singer K 2016 A single-atom heat engine Science 352 325
[31] Josefsson M, Svilans A, Burke A M, Hoffmann E A, Fahlvik S, Thelander C, Leijnse M and Linke H 2018 A quantum-dot heat engine operating close to the thermodynamic efficiency limits Nat. Nanotechnol. 13 920
[32] Maslennikov G, Ding S, Hablützel R, Gan J, Roulet A, Nimmrichter S, Dai J, Scarani V and Matsukevich D 2019 Quantum absorption refrigerator with trapped ions Nat. Commun. 10 202
[33] von Lindenfels D, Gräb O, Schmiedeberg C T, Kaushal V, Schulz J, Mitchison M T, Goold J, Schmidt-Kaler F and Poschinger U G 2019 Spin heat engine coupled to a harmonic-oscillator flywheel Phys. Rev. Lett. 123 080602
[34] Bouton Q, Nettersheim J, Burgardt S, Adam D, Lutz E and Widera A 2021 A quantum heat engine driven by atomic collisions Nat. Commun. 12 2063
[35] Hofer P P, Perarnau-Llobet M, Brask J B, Silva R, Huber M and Brunner N 2016 Autonomous quantum refrigerator in a circuit QED architecture based on a Josephson junction Phys. Rev. B 94 235420
[36] Bar-Gill N 2019 NV color centers in diamond as a platform for quantum thermodynamics in the quantum regime (Fundamental Theories of Physics) vol 195 ed F Binder, L Correa, C Gogolin, J Anders and G Adesso (Berlin: Springer)
[37] Verley G, Esposito M, Willaert T and Van den Broeck C 2014 The unlikely Carnot efficiency Nat. Commun. 5 4721
[38] Proesmans K, Driesen C, Cleuren B and Van den Broeck C 2015 Efficiency of single-particle engines Phys. Rev. E 92 032105
[39] Jiang J H, Agarwalla B K and Segal D 2015 Efficiency statistics and bounds for systems with broken time-reversal symmetry Phys. Rev. Lett. 115 040601
[40] Esposito M, Ochoa M A and Galperin M 2015 Efficiency fluctuations in quantum thermoelectric devices Phys. Rev. B 91 115417
[41] Denzler T and Lutz E 2020 Efficiency fluctuations of a quantum heat engine Phys. Rev. Res. 2 032062
[42] Kosloff R and Levy A 2014 Quantum heat engines and refrigerators: continuous devices Annu. Rev. Phys. Chem. 65 365
[43] Correa L A, Palao J P, Alonso D and Adesso G 2014 Quantum-enhanced absorption refrigerators Sci. Rep. 4 1
[44] Mitchison M T 2019 Quantum thermal absorption machines: refrigerators, engines and clocks
Contemp. Phys. 60 164
[45] Strasberg P, Schaller G, Lambert N and Brandes T 2016 Nonequilibrium thermodynamics in the
strong coupling and non-Markovian regime based on a reaction coordinate mapping New J. Phys. 18 073007
[46] Mu A, Agarwalla B K, Schaller G and Segal D 2017 Qubit absorption refrigerator at strong coupling
New J. Phys. 19 123034
[47] Friedman H M, Agarwalla B K and Segal D 2018 Quantum energy exchange and refrigeration: a
full-counting statistics approach New J. Phys. 20 083026
[48] Kato A and Tanimura Y 2016 Quantum heat current under non-perturbative and non-Markovian
conditions: applications to heat machines J. Chem. Phys. 145 224105
[49] Correa L A, Palao J P and Alonso D 2015 Internal dissipation and heat leaks in quantum
thermodynamic cycles Phys. Rev. E 92 032136
[50] Friedman H and Segal D 2019 Cooling condition for multilevel quantum absorption refrigerators
Phys. Rev. E 100 062112
[51] Kilgour M and Segal D 2018 Coherence and decoherence in quantum absorption refrigerators Phys.
Rev. E 98 012117
[52] Liu J and Segal D 2021 Coherences and the thermodynamic relation: insights from quantum
absorption refrigerators Phys. Rev. E 103 032138
[53] Scovil H E D and Schulz-DuBois E O 1959 Three-level masers as heat engines Phys. Rev. Lett. 2 262
[54] Segal D 2018 Current fluctuations in quantum absorption refrigerators Phys. Rev. E 97 052145
[55] Bullen P S 2003 Handbook of Means and Their Inequalities (Berlin: Springer)
[56] Levitov L S and Lesovik G B 1993 Charge distribution in quantum shot noise JETP Lett. 58 230
[57] Schonhammer K 2007 Full counting statistics for noninteracting fermions: exact results and the
Levitov–Lesovik formula Phys. Rev. B 75 2053229
[58] Humphrey T E and Linke H 2005 Reversible thermoelectric nanomaterials Phys. Rev. Lett. 94 096601
[59] Beau M, Jaramillo J and del Campo A 2016 Scaling-up quantum heat engines efficiently via
shortcuts to adiabaticity Entropy 18 168
[60] Jaramillo J, Beau M and Campo A D 2016 Quantum supremacy of many-particle thermal machines
New J. Phys. 18 073019
[61] Chen Y-Y, Watanabe G, Yu Y-C, Guan X-W and del Campo A 2019 An interaction-driven many-
particle quantum heat engine and its universal behavior npj Quantum Inf. 5 88
[62] Bengtsson J, Tengstrand M N, Wacker A, Samuelsson P, Ueda M, Linke H and Reimann S M 2018
Quantum Szilard engine with attractively interacting bosons Phys. Rev. Lett. 120 100601
[63] Kloc M, Cejnar P and Schaller G 2019 Collective performance of a finite-time quantum Otto cycle
Phys. Rev. E 100 042126
[64] Kloc M, Meier K, Hadjikyriakos K and Schaller G 2021 Superradiant many-qubit absorption
refrigerator Phys. Rev. Appl. 16 044061
[65] Watanabe G, Venkatesh B P, Talkner P, Hwang M-J and del Campo A 2020 Quantum statistical
enhancement of the collective performance of multiple bosonic engines Phys. Rev. Lett. 124 210603