Local encoding of classical information onto quantum states

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In this article we investigate the possibility of encoding classical information onto multipartite quantum states in the distant laboratory framework. We show that for all states generated by Clifford operations there always exists such an encoding, this includes all stabilizer states such as cluster states and all graph states. We also show local encoding for classes of symmetric states (which cannot be generated by Clifford operations). We generalise our approach using group theoretic methods introducing the unifying notion of Pseudo Clifford operations. All states generated by Pseudo Clifford operations are locally encodable (unifying all our examples), and we give a general method for generating sets of many such locally encodable states.

I. INTRODUCTION

In quantum information we are often operating in a framework of separate distant laboratories, and the questions of what is possible or impossible under these local restrictions is crucial for understanding how we can use quantum resources in the best way. Through these considerations we have come to see entanglement as a resource for example for quantum cryptography [1, 2], teleportation [3], dense coding [4] and measurement based quantum computing [5]. Beyond this however, when considering local access of information encoded onto quantum states, we have also seen non-local features without the presence of entanglement [6].

We now turn to consider the complement of this problem, that of local encoding of information. As well as being interesting in its own right as a local restricted task, and as a complement to local access of information (and associated notions of locality), the ability to locally encode information is in fact an important part of many quantum information protocols. Indeed the first stage of dense coding (in fact, the encoding part) is precisely local encoding of classical information. It is also strongly connected to the problem of local unitary equivalence which plays a large role in entanglement theory (as we will see below).

In this paper we raise the question “can we locally encode on all states?”. That is, given an arbitrary state, can we use this as a quantum resource and encode the maximum classical information possible. Although this is a simple question, the answer is surprisingly difficult to find, and we find that we cannot give a positive or negative answer, other than to give a large set of examples where we can encode, and show how to encode and we are unable to find any example that we cannot locally encode.

Since classical information is completely distinguishable, the problem of encoding classical information on a state, becomes the problem of generating an orthogonal bases from that state. Then, our question becomes “is it possible to locally generate a complete basis from all states?”. In this way local encoding is related to local unitary equivalence of states. Though we may naively expect this to be simple to answer, there are hints that it is a hard question, explaining why we have been unable to find the solution. The strongest such hint comes from the existence of unextendible product bases [5]. This is a set of orthogonal product states who’s compliment must be entangled - that is, it is impossible to find another state orthogonal this set which is product. This is an example where if we do not choose appropriate encoding operators, full local encoding is not possible (even though there may exist another set of local operations to encode full classical information).

We begin in section II by considering the local encoding of general product states. Our approach is then to take this encoding and extend it to sets of entangled states generated by unitaries which obey certain commutation relations. We concentrate on using Pauli operations to encode on the states, where we develop the notion of Pseudo Clifford operations whose properties allow us to give a general sufficient condition for the ability to locally encode on a state. In particular this gives a method for locally encoding on all stabiliser states, including cluster states used in measurement based quantum computation [5], CSS error correction code states and all graph states [8]. In section III we show local encoding for sets of symmetric states as examples of non-Clifford but Pseudo Clifford states. In section IV we use group analysis to investigate what states can be locally encoded by the methods we have introduced. We note that although in our methods we are restrictive on the allowed encoding operations (i.e. Pauli and derived from the product state case), our approach manages to cover all the states we consider here, and we have no example of states which we can show cannot be encoded by our methods.
II. LOCAL ENCODING ON PSEUDO CLIFFORD STATES

The problem of local encoding of classical information is equivalent to that of generating a basis by local operations. We begin by giving a formal definition of local encoding:

Definition II.1 A \( n \)-qubit quantum state \( |\psi\rangle \) is said to be locally encodable if there exists a set \( \{v_i \mid v_i \in SU(2)^{\otimes n}\}_{i=0}^{2^n-1} \) of local unitary operations such that \( \langle \psi | v_i^\dagger v_j | \psi \rangle = \delta_{ij} \) for all \( i, j \). We call such \( v_i \) local encoders, and the set \( \{v_i\} \) the local encoder set.

By this definition, we can always ignore the global phase of the encoded state \( v_i |\psi\rangle \), since we only require orthogonality between the encoded states.

The difficulty of finding the local encoder set lies in that the condition \( \langle \psi | v_i^\dagger v_j | \psi \rangle = \delta_{ij} \) in Definition II.1 is a weak condition, it only restricts the property of a local encoder set \( \{v_i\} \) applied to the given state \( |\psi\rangle \). Given a state \( |\psi\rangle \) it is difficult to check if there exists an encoding set, since we must search over all possible local unitaries. In this sense it is too weak to be easily checkable.

In this paper, we take another approach to understand the properties of local encoding: First we pose a restriction on the local encoder to be tensor products of the Pauli operations \( \{I, X, Y, Z\} \) represented by

\[
X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

in the computational basis \( \{|0\rangle, |1\rangle\} \) and obtain sufficient conditions for locally encodable states. Then we consider gradually relaxing the restriction to find wider classes of states which are locally encodable and construction of the local encoder sets.

Our restriction of the local encoder set to consist of the Pauli operations allows group theoretical analysis. The set of \( 4^n \) \( n \)-tensor product operations consisting of the Pauli operators \( \{I, X, Y, Z\} \), together with their overall phase of \( \pm 1 \) or \( \pm i \) forms a group, which is called the Pauli group, denoted by \( P \) (here we say the Pauli group on \( n \)-qubits is given by the tensor products of all qubit Pauli operators). Including the phase factor, the number of elements of the \( n \)-qubit Pauli group is \( 4 \cdot 4^n \). For local encoding, we ignore the global phase factor and only care about \( 4^n \) elements of Pauli group operators.

To investigate local encoding, we use the properties of the Pauli group and another group, the Clifford group. The Clifford group is a group consisting of all operators which leave \( P \) fixed under conjugation, and is denoted by \( C \). Formally, we write it as the set of operators \( \{C \in SU(2^n)|CPC^\dagger \in P, \forall p \in P\} \). The Clifford group is generated by combinations of the Hadamard gate \( H \), the Phase gate \( S \) and the control-NOT gate \( U_{CNOT} \) represented in the computational basis by

\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

in addition to the Pauli operations.

We first consider encoding onto product states. We denote a local encoder for a zero state, which is \( n \)-tensor products of zero states defined by \(|0\rangle \equiv |0\rangle \otimes_n = |0\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle \), by \( \{w_i\} \). For a simple product state such as \(|0\rangle \), it is apparent that a local encoder set is given by \( \{w_0 = I \otimes \ldots \otimes I \otimes I, w_1 = I \otimes \ldots \otimes I \otimes X, w_2 = I \otimes \ldots \otimes X \otimes X, \ldots, w_{2n-1} = X \otimes \ldots \otimes X \otimes X\} \). This construction is based on the fact that we can “flip” each single qubit state \(|0\rangle \) to \(|1\rangle \) by performing a Pauli operation \( X \). All combinations of \( \{I, X\} \) for \( n \)-qubits acting on \(|0\rangle \) give a set of states, which is a complete orthonormal set of the states denoted by \( \{|\tilde{i}\rangle\} \). For convenience, we express this local encoder of the zero state by

\[
\{w_i^0 \equiv w_i = X^{m_1} \otimes \ldots \otimes X^{m_n}\},
\]

where a set of indices \( \{m_1, m_2, \ldots, m_n\} \) is a binary representation of \( i \).

We now generalize this way of local encoding to any product state. A general \( n \)-qubit product state \( |\phi_{prod}\rangle \) is represented by \( n \)-tensor products of single qubit states \( |\phi_k\rangle = \cos(\theta_k/2) |0\rangle + e^{i\phi_k} \sin(\theta_k/2) |1\rangle \) where \( \theta_k \) and \( \phi_k \) are positive parameters for \( k \)th qubit satisfying \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi \leq 2\pi \). These parameters represent the angles of the state vector \( |\phi_k\rangle \) on the Bloch sphere. We consider how to perform a flip operation using a minimum number of parameters for a general \( |\phi_k\rangle \) to make it as simple and general as possible. It is known that there is no universal
Clifford group states represented by this condition as $n$ Pauli operations and also impose are also Pauli. We know that a local encoder set of the zero state $U$ this relation, the unitary operation $\{v_i\}$ constructing the local encoders of entangled states. For this purpose, we introduce a representation of a group-like properties of the Pseudo Clifford set is investigated further in section IV. By definition, all Pseudo Clifford

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Theorem II.1 *Decomposition condition*

A state $|\psi\rangle$ of $n$ qubits can be locally encoded if there exist unitaries $V_L$ and $U_{PC}$ such that $|\psi\rangle = V_L \cdot U_{PC} |\bar{0}\rangle$, $V_L$ is local, and $U_{PC}$ is Pseudo Clifford (such that it conjugates a set $\{w_i\}$ encoding $|\bar{0}\rangle$ to local Pauli operators). The local encoder set is then given by

$$\{v_i = U_{PC} w_i U^\dagger_{PC} v_i^T \}.$$  \hspace{1cm} (6)

A set of such $\{w_i\}$ is given by $\{w_i^0\}$ of Eq. (3), or similar sets replacing $X$ with $Y$, and/or replacing $I$ with $Z$.

As an example of this result, we show that any two qubit state is locally encodable. All two qubit states can be written in the Schmidt decomposition as $|\psi\rangle = \cos(\theta/2) |a_0 \rangle \otimes |b_0 \rangle + \sin(\theta/2) |a_1 \rangle \otimes |b_1 \rangle$. The Schmidt decomposition state can be represented by

$$u_1 \otimes u_2 \cdot U_{CNOT} \cdot (s(\pi) \cdot (r(\theta)) \otimes I) |\bar{0}\rangle,$$

where $u_1$ and $u_2$ map a computational basis $\{|i\}\}$ into $\{|a_i\}\}$ and $\{|b_i\}\}$, respectively, and

$$r(\theta) = \begin{pmatrix} \cos(\theta/2) & -i \sin(\theta/2) \\ i \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$  \hspace{1cm} (8)

(See Fig. 3). Note that $s(\pi) = S \cdot Z$ therefore it is a Clifford operation. Although $r(\theta)$ is not a Clifford operation, it is a Pseudo Clifford operation. In fact, it also commutes with $w_0^0$, thus this state is always locally encodable by $\{v_i = (s(\pi) \otimes I) \cdot U_{CNOT} \cdot w_i^0 \cdot U_{CNOT} \cdot (s(\pi) \otimes I) \cdot u_i^T \otimes u_2^T\}$.

Further, Theorem II.1 also means that all stabiliser states can be locally encoded. This includes cluster states used in measurement based quantum computation [3], CSS error correction code states and all graph states [8].

A natural question now is, how large is the class of states covered by Theorem II.1? We do not at present have a good answer, except to say that there are many interesting states covered, and that it is not trivial to classify in a simple way those covered. The difficulty arises from the fact that for any state $|\psi\rangle$, there will be many unitaries $U$ such that $|\psi\rangle = U |\bar{0}\rangle$ and we only need one to satisfy the conditions in Theorem II.1 for it to be locally encodable. A simple example is given by the two qubit control phase operation (for arbitrary phase) $CP(\theta)$, this is not in the form of $V_L \cdot U_{PC}$, however as we see above, any state of two qubits can be written in this form, i.e. there exist a $V_L$ and $U_{PC}$ such that $CP(\theta)|\bar{0}\rangle = V_L U_{PC} |\bar{0}\rangle$.

In the remainder of this paper, we will look at explicit sets of examples where Theorem II.1 can be applied, and use group theory to construct local encoding sets.

**III. LOCAL ENCODING ON SYMMETRIC BASIS STATES (A LARGE CLASS OF EXAMPLES)**

We now focus on the Pseudo Clifford operation part and investigate local encoders for the states generated by the non-Clifford but Pseudo Clifford operations $U \in \mathcal{PC}$ by imposing an additional condition $w_i = v_i$ to the definition of

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{All states $|\psi\rangle$ generated by the above circuit can be locally encoded (c.f. Theorem II.1).}
\end{figure}
the Pseudo Clifford operation $v_i = U_w U^\dagger$ for $2^n$ elements of Pauli operations $\{w_i\}$. That is, we ask that our encoders remain the same as for the state $|0\rangle$. This is satisfied if the $U$ commutes with the $\{w_i\}$, in which case $V$ becomes $U$ and $\{v_i\} \rightarrow \{w_i\}$ in Fig. 1 which now is commuting.

Our idea for searching the non-Clifford but Pseudo Clifford operation $U$ is that we investigate unitary operations which are represented by a sum of several Pauli group operations $\{p_i\}$, namely, $U = \sum_i a_i p_i$, where $a_i$ is a normalized coefficient ($\sum_i |a_i|^2 = 1$) and the set $\{p_i\}$ should be carefully chosen such that the sum represent a unitary operations and we choose $U$ such that it commutes with our chosen $\{w_i\}$. In this section, we present constructions of the local encoders for a class of states called symmetric basis states by finding the sum representations of the Pseudo Clifford operation generating the states. A general investigation is given in Section 4.

We first introduce symmetric basis states. Symmetric states are the states which are invariant under exchange of arbitrary two qubits. Symmetric basis states are special symmetric states. We define a $n$-qubit symmetric basis state $|n, m\rangle$ as a symmetric state consisting of $n - m$ qubits in $|0\rangle$ states and $m$ qubits in $|1\rangle$ states, namely,

$$|n, m\rangle := \sqrt{\frac{m!(n-m)!}{n!}} \sum_{\pi} \prod_{i} \left( |0\rangle \otimes \ldots |0\rangle \otimes |1\rangle \otimes \ldots |1\rangle \right),$$

where $\sum_{\pi}$ is taken by all permutations $\pi$ of the tensor products of $(n - m)$ $|0\rangle$ states and $m$ $|1\rangle$ states. A set of symmetric basis states of $\{|n, m\rangle\}_{m=0}^{n}$ forms a complete basis of the $(n+1)$-dimensional symmetric subspace of the $n$-qubit Hilbert space $\mathcal{H}^\otimes n$.

We consider the representation of $|n, m\rangle = U |0\rangle$ using a non-local unitary operation $U$. The unitary operation $U$ cannot be Pauli operations and Clifford operations for the case of $n > 2$ or the case of product states $m \neq 0$ or $m \neq n$. It requires non-Clifford unitary operators to generate $|n, m\rangle$ from $|0\rangle$, since the coefficient of the symmetric basis state $\sqrt{m!(n-m)!/n!}$ cannot be obtained by the Clifford group operations which can only give coefficients of the form $1/\sqrt{2^k}$, where $k = 0, 1, \ldots, n$. In this section, we show that this $U$ represented by a sum of Pauli group operators becomes the Pseudo Clifford operation for $|n, 1\rangle$ and $|n, n-1\rangle$ symmetric basis states by choosing the appropriate local encoder and the Pauli group operations in the sum representation of $U$. We extend this method to show that some other symmetric basis states and related states are also locally encodable.

We first show how to represent $U$ in the Pauli sum representation for 3-qubit symmetric basis states $|3, 1\rangle$, which is alternatively called a $W$ state. We consider the following set of Pauli operations:

$$\{v_0 = I \otimes I \otimes I, \ v_1 = X \otimes I \otimes I, \ v_2 = Z \otimes X \otimes I, \ v_3 = Z \otimes Z \otimes X, \ v_4 = v_1 v_2, \ v_5 = v_2 v_3, \ v_6 = v_3 v_1, \ v_7 = v_1 v_2 v_3\}$$

$$\{v_i\}_{i=0}^{7}$$

Since $v_1$, $v_2$ and $v_3$ forms generators of the above set $\{v_i\}_{i=0}^{7}$, we denote the set of operators generated by the generators by $\langle\{v_1, v_2, v_3\}\rangle = \{v_i\}_{i=0}^{7}$. It is easy to check that the set $\{v_i\}$ are also local encoders for $|3, 0\rangle$ and $|3, 3\rangle = X \otimes X \otimes X |0\rangle$.

We choose the unitary operation $U_W$ generating the symmetric basis state $|3, 1\rangle = U_W |0\rangle$ in the Pauli sum representation as

$$U_W = \frac{1}{\sqrt{3}} (p_0 + p_1 + p_2) = \frac{1}{\sqrt{3}} (X \otimes Z \otimes Z + I \otimes X \otimes Z + I \otimes I \otimes X).$$
Note that $U_W$ is carefully chosen such that the operators $\{p_i\}$ in the Pauli sum representation anti-commute to ensure the unitarity of $U_W$, and that $U_W$ commutes with $v_i$. Therefore, we have

$$\langle 3, 1 | v_i^+ v_j | 3, 1 \rangle = \langle 0 | U_W v_i^+ v_j U_W | 0 \rangle = \langle 0 | v_i^+ v_j | 0 \rangle = \delta_{ij},\quad (12)$$

and the set of operators $\{v_i\}_{i=0}^7$ given by Eq. (10) are the local encoder for $|3, 1\rangle$. Since $|3, 2\rangle = X \otimes X \otimes X \cdot U_W | 0 \rangle$ and $X$ operations applied after $U_W$ do not change the local encoder due to Theorem 11.1 $|3, 2\rangle$ can also be locally encodable by the same $\{v_i\}_{i=0}^7$.

By generalizing the construction of the 3-qubit symmetric basis states, we show that the $n$-qubit symmetric basis states $|n, 0\rangle$, $|n, 1\rangle$, $|n, n-1\rangle$ and $|n, n\rangle$ are also locally encodable.

**Theorem III.1 Constructive method**

Symmetric basis states $|n, 0\rangle$, $|n, 1\rangle$, $|n, n-1\rangle$ and $|n, n\rangle$ are locally encodable by the Pauli operators given by

$$\langle (g_i = Z^{(i-1)} \otimes X \otimes I^{(n-i)}) | 1 \rangle_i = 1\rangle.$$

**Proof III.1** The product state cases $|n, 0\rangle$ and $|n, n\rangle$ are trivial. We show the proof for the case of $|n, 1\rangle$. Defining a unitary operation in the Pauli sum representation

$$U_{n,1} = \frac{1}{\sqrt{n}} \sum_{i=1}^n I^{(i-1)} \otimes X \otimes Z^{(i-1)},\quad (14)$$

the symmetric basis state $|n, 1\rangle$ is represented by $|n, 1\rangle = U_{n,1} | 0 \rangle$. This $U_{n,1}$ satisfies commutation relation $[U_{n,1}, v_i] = 0$ for all $i$. From the relationship

$$\langle n, 1 | v_i^+ v_j | n, 1 \rangle = \langle 0 | U_{n,1} v_i^+ v_j U_{n,1} | 0 \rangle = \langle 0 | v_i^+ v_j | 0 \rangle = \delta_{ij},\quad (15)$$

the set $\langle (g_i = Z^{(i-1)} \otimes X \otimes I^{(n-i)}) | 1 \rangle_i = 1\rangle$ are the local encoder of $|n, 1\rangle$. For $|n, n-1\rangle$, it is also locally encodable by the same local encoder of $|n, 1\rangle$, due to the relationship $|n, n-1\rangle = U_{n,1} | X \otimes X \otimes 0 \rangle$ and Theorem 11.1.

In fact this encoding was used to construct a basis of $W$ states in 11.1 in the context of multipartite entanglement distillation. Here we see if holds for other states as well.

We can also extend the locally encodable states beyond $|n, 1\rangle$ and $|n, n-1\rangle$ states using the constructive method above. By replacing $U_{n,1}$ by $U_{n,1}^{\pm} = \sum_{i=1}^n a_i I^{(i-1)} \otimes X \otimes Z^{(n-i)}$, $(a_i \in \mathbb{R})$ where $\sum a_i^2 = 1$, a state with non-even real weights on permutations can be written in the Pauli sum representation $|\Xi_{n,1}\rangle = U_{n,1}^{\pm} | 0 \rangle$. The state $|\Xi_{n,1}\rangle$ is also locally encodable by the local encoders of $|n, 1\rangle$, since our construction does not depend on the coefficient $a_i$.

Next, we try to find the local encoders of other symmetric basis states by using induction.

**Lemma III.1 Inductive method**

If $|n, k\rangle$ and $|n, k-1\rangle$ are locally encodable by the same local encoder set $\{v_i\}_{i=1}^n$, then $|n+1, k\rangle$ is locally encodable by a new local encoder set $\{I \otimes v_i, Z \otimes v_i\}$.

**Proof III.2** Since we have $|n, m\rangle = \frac{1}{2}(|0\rangle |n-1, m\rangle + |1\rangle |n-1, m-1\rangle)$, we see that

$$\langle n+1, k | Z^i \otimes v_j^i I \otimes v_k | n+1, k \rangle = \frac{1}{2} \left( \langle 0 | Z^i | 0 \rangle \langle n, k | v_j^i v_k | n, k \rangle + \langle 1 | Z^i | 1 \rangle \langle n, k-1 | v_j^i v_k | n, k-1 \rangle \right) = \delta_{ij} \delta_{jk}.\quad (16)$$

Hence, two states encoded by any two different encoding operators taken from the set $\{I \otimes v_i, Z \otimes v_i\}_{i=1}^n$ are orthogonal. Thus, $|n+1, k\rangle$ is locally encoded by the set $\{I \otimes v_i, Z \otimes v_i\}$.

All the 3-qubit symmetric basis states can be locally encoded by the same local encoders given by Eq. (10), we can see that the symmetric basis states of $|4, 1\rangle$, $|4, 2\rangle$, $|4, 3\rangle$, $|5, 2\rangle$, $|5, 3\rangle$ and $|6, 3\rangle$ are locally encodable from this lemma of the inductive method. For $|4, 2\rangle$, we find that there is another local encoder given by

$$\{I \otimes v_0, I \otimes v_1, Z \otimes v_2, I \otimes v_3, Z \otimes v_4, Z \otimes v_5, I \otimes v_6, Z \otimes v_7, \}
\begin{align*}
X \otimes Z \otimes Z \otimes (X \cdot Z) \otimes v_1, X \otimes v_2, (X \cdot Z) \otimes v_3, X \otimes v_4, X \otimes v_5, (X \cdot Z)v_6, X \otimes v_7. \quad (17)
\end{align*}$$

Due to the decomposition of $|4, 2\rangle$ into $|4, 2\rangle = |0\rangle \otimes |3, 2\rangle / \sqrt{2} + |1\rangle \otimes |3, 1\rangle / \sqrt{2}$, and using the relations

$$\langle 3, 1 | v_1 | 3, 2 \rangle = -2, \langle 3, 2 | v_1 | 3, 1 \rangle = -2, \langle 3, 1 | v_3 | 3, 2 \rangle = 2, \langle 3, 2 | v_3 | 3, 1 \rangle = 2, \langle 3, 1 | v_3 | 3, 2 \rangle = 2, \langle 3, 2 | v_3 | 3, 1 \rangle = -1, \quad (18)$$

we can directly check the orthogonality of the encoded quantum states given by Eq. (17). We still do not have a construction of local encoders for $|6, 2\rangle$ and $|6, 4\rangle$ states, thus, it is not proven that they are locally encodable or not. The summary of locally encodable symmetric basis states and their encoders are given in Fig. 4.
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In this section, we consider a reverse engineering of this problem and present a way to derive a class of locally encodable states for a given Pauli local encoder set \( \{v_i\} \). We first investigate general group theoretical properties of a unitary operator \( U \). For local encoding, we require \( U \) such that \( U |0\rangle = |0\rangle \). Hence Pseudo Clifford sets are defined with respect to particular Pauli subgroups \( P \). For a subset \( S \) of the Pauli group, since for arbitrary \( s_1, s_2 \in S \), \( U_{s_1}U_{s_2}U^\dagger = U_{s_1}U_{s_2}U_{s_1}U^\dagger \in P \). Thus, we can choose \( S \) as a subgroup of the Pauli group \( P \). Further, we can prove that the cardinal number of \( S \) equals that of \( U_{s}U_{s}^\dagger \) as in the following lemma:

**Lemma IV.1** A map \( f : s \in S \mapsto U_{s}U_{s}^\dagger \in S' \subseteq SU(2)^\otimes n \) is a group isomorphic mapping.

**Proof IV.1** From the definition, the map \( f \) is surjective and homomorphism. For arbitrary \( u_1, u_2 \in S' \), there exist \( s_1, s_2 \in S \) such that \( U_{s_1}U_{s_2}U^\dagger = U_{s_1}U_{s_2}U_{s_2}U^\dagger \) since the map \( f \) is surjective. Then, \( s_1 = s_2 \) if \( u_1 = u_2 \).

From Lemma IV.1 we give a formal definition of the Pseudo Clifford set.

**Definition IV.2 (Pseudo Clifford set)** For subgroups \( P_1 \) and \( P_2 \) (\(|P_1| = |P_2|\)) of the Pauli group \( P \), the corresponding Pseudo Clifford set \( C(P_1, P_2) \) is defined by

\[
C(P_1, P_2) := \{ U \in SU(2^n) \mid U_{P_1}U^\dagger = P_2, P_1, P_2 \subseteq P \}.
\]

Hence Pseudo Clifford sets are defined with respect to particular Pauli subgroups \( P_1 \) and \( P_2 \) – and they conjugate one to the other. The set \( C(P_1, P_2) \) is not necessarily a group and equals to the Clifford group \( C \) if \( P_1 = P_2 = P \).

For local encoding, we require \(|P_1| = 2^n \). In Theorem IV.1 we choose the set \( P_1 = \{w_i\} \) which locally encodes \( |0\rangle \) (for example \( \{w_i = w^0_i\} \)). From Theorem IV.1 the state \( U |0\rangle \) can be locally encoded if \( U \in C(P_1 = \{w_i\}, P_2) \), and by definition \( P_2 \) is the set of local encoders, given by \( P_2 = \{v_i = Uw_j^0 U^\dagger\} \), as can easily be seen

\[
|0\rangle U^\dagger v_i^0 v_j U |0\rangle = |0\rangle U^\dagger v_i^0 U^\dagger U v_j U |0\rangle = |0\rangle w_j^0 w_j |0\rangle = \delta_{ij}.
\]

Our goal in this section is to derive a set of Pauli encodable states for a given Pauli local encoder \( \{v_i\} \in P_2 \).
By denoting a generator of the local encoder set \( \{ w_i \} \in P_1 \) by \( \{ g_i \}_{i=1}^n \), and a generator of the local encoder set \( \{ v_i = U w_i U^\dagger \} \in P_2 \) by \( \{ g_i \}_{i=1}^n \), we reduce the relationship for the local encoders \( \{ w_i \} \) and \( \{ v_i \} \) to the relationship for the generators \( \{ g_i \} \) and \( \{ g_i' \} \). Further, since Clifford operations give a way to map a generating set of Pauli operations to another generating set, there always exists a Clifford operation \( C \) such that
\[
C^\dagger U g_i U^\dagger C = C^\dagger g'_i C = g_i, \quad \text{for arbitrary } g_i,
\]
for an arbitrary \( g_i \), where \( Z = C^\dagger U \). Therefore, we can assume that \( Z \) maps all the elements of \( P_1 \) to themselves. This result shows that multiplication of appropriate Clifford operation \( C^\dagger \) reduces our problem to a simple case of \( g_i' = g_i \) (i.e. \( \{ v_i \} = \{ w_i \} \)), which we have investigated in Section 3.

To obtain a non-Clifford but Pseudo Clifford operation \( Z \), we define an hermitian commutative set \( L \) by
\[
L = \{ p \in P \mid [p, g_i] = 0, \quad p : \text{Hermitian, } \forall g_i \in \{ g_i \}_{i=1}^n \}. \tag{23}
\]
Thus, we can construct a subgroup \( SU_L \) of \( SU(2^n) \), because \( L \) is a subalgebra of the Lie algebra of \( SU(2^n) \). From the definition of \( L \), all the elements of \( SU_L \) commute with all the elements of \( S \). Therefore, we obtain a Pauli encodable state given by \( CZ|0\rangle (C \in C, \; Z \in SU_L) \).

Our method for obtaining the Pauli encoders is summarized in the following steps.

1. Choose a set of Pauli generators \( \{ g_i \}_{i=1}^n \) for the zero state \( |0\rangle \).
2. Construct a hermitian commutative set \( L \) defined by Eq. (23) associated with the generators \( \{ g_i \}_{i=1}^n \).
3. Construct a Lie group \( SU_L \) from the Lie subalgebra \( L \). We call the Lie group \( SU_L \) a Pauli encoder group.
4. For an arbitrary Clifford operation \( C \) and an arbitrary \( Z \in SU_L \), we obtain a Pauli encodable state \( CZ|0\rangle \) and a Pauli encoder \( \langle \{ Cg_i C^\dagger \}_{i=1}^n \rangle \).

In general the difficulties arise in step ii) – not all generator sets will allow for a nice construction of \( L \) (23). We show two concrete constructions for the Pauli encodable states for given Pauli encoders in the followings. If the generator set is given by \( g_i = I^\otimes (i-1) \otimes X \otimes I^\otimes (n-i) \), we have the hermitian commutative set \( L \) defined by
\[
L = \{ w_i^0 \equiv X^{i_1} \otimes X^{i_2} \otimes \cdots \otimes X^{i_n}, \; i := i_1i_2\cdots i_n \in \mathbb{Z}_2^n \}, \tag{24}
\]
as used earlier. Thus, an element \( Z \in SU_L \) is given by
\[
Z = \exp [i \sum_{i \in \mathbb{Z}_2^n} c_i w_i^0], \; (c_i \in \mathbb{C}). \tag{25}
\]
Therefore, with an arbitrary Clifford operation \( C \), a Pauli encodable state is given by
\[
C \exp [i \sum_{i \in \mathbb{Z}_2^n} c_i w_i^0] |0\rangle, \tag{26}
\]
and the corresponding Pauli encoder set is given by
\[
\langle \{ C \cdot (I^\otimes (i-1) \otimes X \otimes I^\otimes (n-i)) \cdot C^\dagger \}_{i=1}^n \rangle. \tag{27}
\]
If the generator is given by \( g_i = Z^\otimes (i-1) \otimes X \otimes Z^\otimes (n-i) \) \( i=1, \ldots, n \), we have the hermitian commutative set \( L \) defined by
\[
L = \langle \{ g_i \equiv I^\otimes (i-1) \otimes X \otimes Z^\otimes (n-i) \}_{i=1}^n \rangle. \tag{28}
\]
Thus, an element \( Z \in SU_L \) is given by
\[
Z = \exp [i \sum_{i=0}^n c_i q_i] |0\rangle, \; (c_i \in \mathbb{C}), \tag{29}
\]
and, with an arbitrary Clifford operation \( C \), a Pauli encodable state is given by
\[
C \exp [i \sum_{i=0}^n c_i q_i] |0\rangle \tag{30}
\]
where we take \( q_0 = I^\otimes n \). The corresponding Pauli encoder is given by
\[
\langle \{ C \cdot (Z^\otimes (i-1) \otimes X \otimes Z^\otimes (n-i)) \cdot C^\dagger \}_{i=1}^n \rangle. \tag{31}
\]
We see that the symmetric basis state represented by \( U_{n,1} \) of Eq. (14) is a special case of Eq. (30).
V. CONCLUSION AND DISCUSSION

In this work we have looked at the possibility of encoding classical information onto quantum states by local unitary operations. We have presented explicit encodings for large sets of states including all stabiliser and various symmetric basis states. We have introduced the notion of Pseudo Clifford which unifies these states under one general local encoding method. Finally, by resorting to group theoretic analysis we have given a method to find large sets of states with the same local encodings.

Although the methods used for local encoding presented here are not as general as possible, they do cover many interesting sets of states. It seems to be a difficult problem to describe the extent to which states are covered by our encoding strategies. It remains an open problem whether all states are locally encodable at all.

We may also be interested in different ways of local encoding for other reasons and applications. For example in dense coding, we wish to encode the information by acting on only a subset of the parties (the idea being that then by sending that same subset through a quantum channel, we can communicate more information than the that allowed by the Holevo bound). The local encoding presented here could not be used for such a protocol. We can then ask can we extend these results to consider such protocols, or what can our results say about when we can or cannot.

For example, we know that for optimum dense coding, we must have a maximally entangled state between the senders and receivers. Imagine we try to dense code using the state \( |\psi\rangle = \frac{1}{\sqrt{2^{n-1}}} \sum_{i=0}^{2^{n-1}} U|i\rangle_s \otimes |i\rangle_r \in \mathcal{H}^\otimes n \otimes \mathcal{H}^\otimes n \), where \( |i\rangle_x \) are the product states of the computational basis over the senders’ and receivers’ spaces for subscripts \( s \) and \( r \) respectively, and \( U \) acts on \( S \) only. The Schmidt basis on the side of the senders (subscript \( s \)) is in general entangled across the set of senders by unitary \( U \). If \( U \) is just identity, then we can encode simply using the Pauli operators [11]. Surprisingly we can see that the senders can still encode the full basis locally, independent of \( U \). This can be easily shown, since for optimal dense coding, we would require \( \langle \psi | v^\dag_i \otimes I \cdot v^\dag_j \otimes I |\psi\rangle = \delta_{ij} \) for all \( i, j \). It is easy to see that the \( U \) drops out and the condition is equivalent to simply \( Tr[v^\dag_i v^\dag_j] = 0 \). This is satisfied for the local Pauli, hence they allow dense coding for these states. Thus maximum entangled states can be always optimally locally dense coded independent of the Schmidt basis. The same result can be obtained by using the Choi-Jamiolkowski isomorphism [12] by bringing the unitary over to the receiver’s side \( |\psi\rangle = \frac{1}{\sqrt{2^{n-1}}} \sum_{i=0}^{2^{n-1}} U|i\rangle_s \otimes V|i\rangle_r = \frac{1}{\sqrt{2^{n-1}}} \sum_{i=0}^{2^{n-1}} |i\rangle_s \otimes U^T V |i\rangle_r \). In this way it is clear that the standard Pauli approach will work from [11].

We can also note that some of the states considered here have mirror results in local decoding. It is known that the ability to decode such encoded classical information is bounded by the entanglement [12], and explicit bounds are given for \( W \)-states and large sets of graph states (which, in the case of graph states, can be made tight [13]). We can thus compare the amount of information we can encode to that we can decode \( \Delta I_{local} = I_{local, encode} - I_{local, decode} \). For graph states this gives \( \Delta I_{local} = E(|\psi\rangle) = n/2 \) (where \( E \) is the geometric measure of entanglement [14]). Indeed, from [14] for all states where we can locally encode we have \( \Delta I_{local} \geq E(|\psi\rangle) \). This allows us to talk about a kind of irreversibility of local information - we can encode much easier than decode locally, and the difference is bounded by the entanglement. It is interesting to consider what such a quantity would mean in relation to other tasks such as measurement based quantum computing, error correction e.t.c.

We see then that there are many open questions remaining, and that these results may have potential interest in various areas of quantum information processing and studies of locality. In addition we may also consider the usefulness of the task directly in many-party quantum cryptographic scenarios where we have distributed encoders and decoders. These will be the topics of ongoing study.

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