COUPLING AND RELAXED COMMUTANT LIFTING

A.E. FRAZHO, S. TER HORST, AND M.A. KAASHOEK

Abstract. A Redheffer type description of the set of all contractive solutions to the relaxed commutant lifting problem is given. The description involves a set of Schur class functions which is obtained by combining the method of isometric coupling with results on isometric realizations. For a number of special cases, including the case of the classical commutant lifting theorem, the description yields a proper parameterization of the set of all contractive solutions, but examples show that, in general, the Schur class function determining the contractive lifting does not have to be unique. Also some sufficient conditions are given guaranteeing that the corresponding relaxed commutant lifting problem has only one solution.

0. Introduction

This paper is devoted to the relaxed commutant lifting theorem in [14]. This theorem is a generalization of the classical commutant lifting theorem [19], and it includes as special cases the Treil-Volberg lifting theorem [20], and its weighted version due to Biswas, Foias and Frazho [12].

To state the relaxed commutant lifting theorem, let us first recall the general setup. The starting point is a lifting data set \( \{A, T', U', R, Q\} \) consisting of five Hilbert space operators. The operator \( A \) is a contraction mapping \( H \) into \( H' \), the operator \( U' \) on \( K' \) is a minimal isometric lifting of \( T' \) on \( H' \), and \( R \) and \( Q \) are operators from \( H_0 \) to \( H \), satisfying the following constraints

\[
T'AR = AQ \quad \text{and} \quad R^*R \leq Q^*Q.
\]

Given this data set the relaxed commutant lifting theorem in [14] states that there exists a contraction \( B \) from \( H \) to \( K' \) such that

\[
(0.1) \quad \Pi_{H'} B = A \quad \text{and} \quad U'BR = BQ.
\]

Here \( \Pi_{H'} \) is the orthogonal projection from \( K' \) onto \( H' \). In fact, [14] provides an explicit construction for a contraction \( B \) satisfying (0.1). In the sequel we say that \( B \) is a contractive interpolant for \( \{A, T', U', R, Q\} \) if \( B \) is a contraction from \( H \) into \( K \) satisfying (0.1).

In this paper we present a Redheffer type formula to describe the set of all contractive interpolants for \( \{A, T', U', R, Q\} \). In order to state our main results we need some auxiliary operators. To this end, let \( D \) be the positive square root of

1991 Mathematics Subject Classification. Primary 47A20, 47A57; Secondary 47A48.

Key words and phrases. commutant lifting, isometric coupling, isometric realization, parameterization.
that there exists a unique unitary operator

\[ (0.2) \quad F = D_A Q \mathcal{H}_0 \quad \text{and} \quad F' = \begin{bmatrix} D_o \\ D_{T'} A R \\ D_A R \end{bmatrix} \mathcal{H}_0. \]

Notice that \( F \) is a subspace of \( \mathcal{D}_A \) and \( F' \) is a subspace of \( \mathcal{D}_o \oplus \mathcal{D}_{T'} \oplus \mathcal{D}_A \). Here we follow the convention that for a contraction \( C \), the symbol \( D_C \) denotes the positive square root of \( I - C^* C \) and \( \mathcal{D}_C \) stands for the closure of the range of \( D_C \). Furthermore, \( \mathcal{D}_o = \mathcal{D}_c \mathcal{H}_o \). Since \( T' A R = AQ \), we know from formula (4.11) in [14] that there exists a unique unitary operator \( \omega \) mapping \( F \) onto \( F' \) such that

\[ (0.3) \quad \omega(D_A Q h) = \begin{bmatrix} D_o \\ D_{T'} A R \\ D_A R \end{bmatrix} h, \quad h \in \mathcal{H}_0. \]

We also need the projections \( \Pi_{T'} \) and \( \Pi_A \) defined by

\[ (0.4) \quad \Pi_{T'} = \begin{bmatrix} 0 & I & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{D}_o \\ \mathcal{D}_{T'} \\ \mathcal{D}_A \end{bmatrix} \rightarrow \mathcal{D}_{T'}, \quad \Pi_A = \begin{bmatrix} 0 & 0 & I \end{bmatrix} : \begin{bmatrix} \mathcal{D}_o \\ \mathcal{D}_{T'} \\ \mathcal{D}_A \end{bmatrix} \rightarrow \mathcal{D}_A. \]

Notice that the previous definitions only relied upon the operators \( A, T', R \) and \( Q \). The minimal isometric lifting \( U' \) did not play a role. Recall that all minimal isometric liftings of the same contraction are isomorphic. So without loss of generality, in our main theorem, we can assume that \( U' = V \) is the Sz.-Nagy-Schäffer minimal isometric lifting of \( T' \) which acts on \( \mathcal{H}' \oplus H^2(\mathcal{D}_{T'}) \). The definitions of a minimal isometric lifting and the Sz.-Nagy-Schäffer lifting are presented in the next section. Finally, given Hilbert spaces \( \mathcal{U} \) and \( \mathcal{Y} \), we write \( S(\mathcal{U}, \mathcal{Y}) \) for the set of all operator-valued functions which are analytic on the open unit disk \( \mathbb{D} \) and whose values are contractions from \( \mathcal{U} \) to \( \mathcal{Y} \). We refer to \( S(\mathcal{U}, \mathcal{Y}) \) as the Schur class associated with \( \mathcal{U} \) and \( \mathcal{Y} \). We are now ready to state our first main result.

**Theorem 0.1.** Let \( \{A, T', V, R, Q\} \) be a lifting data set, where \( V \) on \( \mathcal{H}' \oplus H^2(\mathcal{D}_{T'}) \) is the Sz.-Nagy-Schäffer minimal isometric lifting of \( T' \). Then all contractive interpolants for this data set are given by

\[ (0.5) \quad B h = \Pi_{T'} F(\lambda)|(I_{\mathcal{D}_A} - \Lambda_{\Pi_A} F(\lambda))^{-1}D_A h, \quad h \in \mathcal{H}, \]

where \( F \) is any function from the Schur class \( S(\mathcal{D}_A; \mathcal{D}_o \oplus \mathcal{D}_{T'} \oplus \mathcal{D}_A) \) satisfying \( F(0)|_F = \omega \).

In general, formula (0.5) does not establish a one to one correspondence between \( B \) and the parameter \( F \). It can happen that different \( F \)'s yield the same \( B \). For instance, assume \( \mathcal{H}_0, \mathcal{H} \) and \( \mathcal{H}' \) to be equal to \( \mathbb{C} \), let \( A, R \) and \( Q \) be the zero operator on \( \mathbb{C} \), and take for \( T' \) the identity operator on \( \mathbb{C} \). Since \( T' \) is an isometry, the Sz.-Nagy-Schäffer minimal isometric lifting \( V \) of \( T' \) is equal to \( T' \). The latter implies that there is only one contractive interpolant \( B \) for the data set \( \{A, T', V, R, Q\} \), namely \( B = A \). The fact that \( R \) and \( Q \) are the zero operators on \( \mathbb{C} \) implies that \( F = \{0\} \) and \( F' = \{0\} \). It follows that for this data set \( \{A, T', V, R, Q\} \) the only contractive interpolant \( B \) is given by formula (0.5) where for \( F \) we can take any function in the Schur class \( S(\mathbb{C}, \mathbb{C}) \). The previous example can be seen as a special case of our second main theorem.
Theorem 0.2. Let $B$ be a contractive interpolant for the data set $\{A,T',V,R,Q\}$ where $V$ is the Sz.-Nagy-Schäffer minimal isometric lifting of $T'$. Then there is a one to one mapping from the set of all $F$ in $S(\mathcal{D}_A, \mathcal{D}_o \oplus \mathcal{D}_T \oplus \mathcal{D}_A)$ with $F(0)\|F = \omega$ such that $B$ is given by (0.6) onto the set $S(\mathcal{G}_B, \mathcal{G}'_B)$, with $\mathcal{G}_B$ and $\mathcal{G}'_B$ being given by

$$
\mathcal{G}_B = \mathcal{D}_B \oplus \mathcal{D}_B Q \mathcal{H}_0 \quad \text{and} \quad \mathcal{G}'_B = (\mathcal{D}_o \oplus \mathcal{D}_B) \oplus \begin{bmatrix} D_0 & D_B R \end{bmatrix} \mathcal{H}_0.
$$

Our proof of the above theorem also provides a procedure to obtain a mapping of the type referred to in the theorem.

It is interesting to specify Theorems 0.1 and 0.2 for the case when in the lifting data set $\{A,T',V,R,Q\}$ the operators $A, T', R$ and $Q$ are zero operators. In this case the intertwining condition $V BR = B Q$, where $V$ is the Sz.-Nagy-Schäffer minimal isometric lifting of $T' = 0$, is trivially fulfilled, and hence $B$ is a contractive interpolant if and only if

$$
Bh = \begin{bmatrix} 0 & \Theta(\cdot)h \end{bmatrix}, \quad h \in \mathcal{H},
$$

where $\Theta$ is any function in $H^2_{ball}(\mathcal{L}(\mathcal{H}, \mathcal{H}'))$. The latter means that $\Theta$ is a $\mathcal{L}(\mathcal{H}, \mathcal{H}')$-valued analytic function on $\mathbb{D}$ such that for each $h \in \mathcal{H}$ the function $\Theta(\cdot)h$ belongs to the Hardy space $H^2(\mathcal{H}')$, and $\|\Theta(\cdot)h\|_{H^2(\mathcal{H}')} \leq \|h\|$. It follows that Theorems 0.1 and 0.2 have the following corollaries.

Corollary 0.3. Let $F$ be any function in the Schur class $S(\mathcal{H}, \mathcal{H}' \oplus \mathcal{H})$, and let $\Pi$ and $\Pi'$ be the orthogonal projections of $\mathcal{H}' \oplus \mathcal{H}$ on $\mathcal{H}$ and $\mathcal{H}'$, respectively. Then the function $\Theta$ defined by

$$
\Theta(\lambda) = \Pi' F(\lambda)(I_{\mathcal{H}} - \lambda \Pi F(\lambda))^{-1}
$$

belongs to $H^2_{ball}(\mathcal{L}(\mathcal{H}, \mathcal{H}'))$, and any function in $H^2_{ball}(\mathcal{L}(\mathcal{H}, \mathcal{H}'))$ is obtained in this way

Corollary 0.4. Let $\Theta \in H^2_{ball}(\mathcal{L}(\mathcal{H}, \mathcal{H}'))$. Then there is a one to one mapping from the set of all $F$ in $S(\mathcal{H}, \mathcal{H}' \oplus \mathcal{H})$ such that (0.7) holds onto the set $S(\mathcal{D}_T, \mathcal{D}_T)$, where $\Gamma$ is the contraction from $\mathcal{H}$ into $H^2(\mathcal{H}')$ defined by

$$
(\Gamma h)(\lambda) = \Theta(\lambda)h, \quad h \in \mathcal{H}, \lambda \in \mathbb{D}.
$$

When $\mathcal{H} = \mathcal{H}' = \mathbb{C}$, and hence $\Theta$ is a scalar function, Corollary 0.3 can be found in [13], page 490, provided $\Theta$ is of unit $H^2$ norm. For $p \times q$ matrix functions $\Theta$, when $\mathcal{H} = \mathbb{C}^q$ and $\mathcal{H}' = \mathbb{C}^p$, Corollary 0.3 is Theorem 2.2 in [3]. For the general operator valued case Corollary 0.3 seems to be new. Corollary 0.4 seems to be new even in the scalar case. Notice that in the scalar case the space $\mathcal{D}_T$ in Corollary 0.4 consists of the zero element only if $\Theta$ is of unit $H^2$ norm, and $\mathcal{D}_T = \mathbb{C}$ otherwise.

Another case of special interest is the classical commutant lifting problem. As we know from [14] the commutant lifting theorem can be obtained by applying the relaxed commutant lifting theorem to the data set $\{A,T',U',I_{\mathcal{H}}, Q\}$ where $\mathcal{H}_0 = \mathcal{H}$, the operator $R$ is the identity operator on $\mathcal{H}$ and $Q$ is an isometry; see [14]. In this case, the space $\mathcal{G}'_B$ in Theorem 0.2 consists of the zero element for any choice of the contractive interpolant $B$. In other words, for the case of the classical commutant lifting formula (0.3) provides a proper parameterization, that is, for every contractive interpolant $B$ for $\{A,T',V,R,Q\}$ there exists a unique $F$. 
in $S(D_A, D_0 \oplus D_T \oplus D_A)$ with $F(0)\lambda = \omega$ such that $B$ is given by (0.5). Finally, it is noted that this formula also yields the Redheffer type parameterization for the commutant lifting theorem presented in Section XIV of [13].

If in Theorem 0.1 we take $F(\lambda) = \omega \Pi_F$, where $\Pi_F$ is the orthogonal projection of $D_A$ onto $B$, then the contractive interpolant $B$ in (0.5) is precisely the central solution presented in [14].

From Theorem 0.1 we see that $F = D_A$ implies that there is a unique contractive interpolant (which is known from Theorem 3.1 in [14]). Other conditions of uniqueness will be given in the final section of the paper.

We shall prove Theorems 0.1 and 0.2 by combining the method of isometric coupling with some aspects of isometric realization theory. The theory of isometric couplings originates from [1], [2], and was used to study the commutan t lifting problem for the first time in [5]–[9]; see also, Section VII.7 in [13].

The paper consist of six sections not counting this introduction. The first two sections have a preliminary character, and review the notions of an isometric lifting (Section 1), and an isometric realization (Section 2). In the third section we develop the notion of an isometric coupling of a pair of contractions which provides the main tool in this paper. In Section 4 we prove Theorem 0.1 for the case when $R^*R = Q^*Q$, and in Section 5 we prove Theorem 0.1 in its full generality. In the final section we prove Theorem 0.2, and we present a few sufficient conditions for the case when (0.5) provides a proper parameterization, and also conditions for uniqueness of the solution.

We conclude this introduction with a few words about notation and terminology. Throughout capital calligraphic letters denote Hilbert spaces. The Hilbert space direct sum of $\mathcal{U}$ and $\mathcal{Y}$ is denoted by

$$\mathcal{U} \oplus \mathcal{Y} \quad \text{or by} \quad \begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix}.$$ 

The set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}'$ is denoted by $L(\mathcal{H}, \mathcal{H}')$. The identity operator on the space $\mathcal{H}$ is denoted by $I_\mathcal{H}$ or just by $I$, when the underlying space is clear from the context. By definition, a subspace is a closed linear manifold. If $\mathcal{M}$ is a subspace of $\mathcal{H}$, then $\mathcal{H} \ominus \mathcal{M}$ stands for the orthogonal complement of $\mathcal{M}$ in $\mathcal{H}$. Given a subspace $\mathcal{M}$ of $\mathcal{H}$, the symbol $\Pi_\mathcal{M}$ will denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$ viewed as an operator from $\mathcal{H}$ to $\mathcal{M}$, and $P_\mathcal{M}$ will denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$ viewed as an operator on $\mathcal{H}$. Note that $\Pi_\mathcal{M}$ is the canonical embedding from $\mathcal{M}$ into $\mathcal{H}$, and hence $P_\mathcal{M} = \Pi_\mathcal{M}^* \Pi_\mathcal{M}$. Instead of $\Pi_\mathcal{M}$ we shall sometimes write $E_\mathcal{M}$, where the capital $E$ refers to embedding. A subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be cyclic for an operator $T$ on $\mathcal{H}$ whenever

$$\mathcal{H} = \bigvee_{n=0}^\infty T^n \mathcal{M} = \text{span} \{ T^n \mathcal{M} \mid n = 0, 1, 2, \ldots \}.$$ 

Finally, by definition, a $L(\mathcal{H}, \mathcal{H}')$-valued Schur class function is a function in $S(\mathcal{H}, \mathcal{H}')$, i.e., an operator-valued function which is analytic on the open unit disk $D$ and whose values are contractions from $\mathcal{H}$ to $\mathcal{H}'$. 
1. ISOMETRIC LIFTINGS

In this section we review some facts concerning isometric liftings that are used throughout this paper. For a more complete account we refer to the book [16] (see also Chapter VI in [13], and Section 11.3 in [14]).

Let \( T' \) on \( \mathcal{H}' \) be a contraction. Recall that an operator \( U \) on \( K \) is an isometric lifting of \( T' \) if \( \mathcal{H}' \) is a subspace of \( K \) and \( U \) is an isometry satisfying \( \Pi_{\mathcal{H}'} U = T' \Pi_{\mathcal{H}'} \).

Isometric liftings exist. In fact, the Sz.-Nagy-Schäffer isometric lifting \( V \) of \( T' \) is given by

\[
V = \begin{bmatrix} T' & 0 \\ E_D T' & S \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix}.
\]

Here \( S \) is the unilateral shift on the Hardy space \( H^2(\mathcal{D}_{T'}) \) and \( E \) is the canonical embedding of \( \mathcal{D}_{T'} \) onto the space of constant functions in \( H^2(\mathcal{D}_{T'}) \). To see that \( V \) in (1.1) is an isometric lifting of \( T' \) note that any operator \( U \) on \( K = \mathcal{H}' \oplus \mathcal{M} \) is an isometric lifting of \( T' \) if and only if \( U \) admits an operator matrix representation of the form

\[
U = \begin{bmatrix} T' & 0 \\ Y_1 D_T' & Y_2 \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} \mathcal{H}' \\ \mathcal{M} \end{bmatrix},
\]

where \( Y = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} : \mathcal{D}_{T'} \rightarrow \mathcal{M} \) is an isometry.

An isometric lifting \( U \) of \( T' \) is called minimal when \( \mathcal{H}' \) is cyclic for \( U \). The Sz.-Nagy-Schäffer isometric lifting of \( T' \) is minimal. If the isometric lifting \( U \) is given by (1.2), then the lifting is minimal if and only if the space \( Y_1 D_T' \) is cyclic for \( Y_2 \).

Two isometric liftings \( U_1 \) on \( K_1 \) and \( U_2 \) on \( K_2 \) of \( T' \) are said to be isomorphic if there exists a unitary operator \( \Phi \) from \( K_1 \) onto \( K_2 \) such that

\[
\Phi U_1 = U_2 \Phi \quad \text{and} \quad \Phi h = h \quad \text{for all} \quad h \in \mathcal{H}'.
\]

Minimality of an isometric lifting is preserved under an isomorphism, and two minimal isometric liftings of \( T' \) are isomorphic.

Finally, when \( U \) on \( K \) is a isometric lifting of \( T' \), then the subspace \( K' \), given by

\[
K' = \bigvee_{n=0}^{\infty} U^n \mathcal{H}',
\]

is reducing for \( U \), that is, both \( K' \) and its orthogonal complement \( \tilde{K} = K \ominus K' \) are invariant under \( U \). Furthermore, in that case the operator \( U' = \Pi_K U | K' \) on \( K' \) is a minimal isometric lifting of \( T' \), and the operator \( U \) admits a operator matrix decomposition of the form

\[
U = \begin{bmatrix} U' & 0 \\ 0 & \tilde{U} \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} K' \\ \tilde{K} \end{bmatrix},
\]

where \( \tilde{U} \) is an isometry on \( \tilde{K} \). We shall call \( U' \) in (1.3) the minimal isometric lifting of \( T' \) associated with \( U \).

The following proposition summarizes the results referred to above in a form that will be convenient for this paper. For details we refer to Section 11.3 in [14].

**Theorem 1.1.** Let \( T' \) be a contraction on \( \mathcal{H}' \), let \( V \) on \( \mathcal{H}' \oplus H^2(\mathcal{D}_{T'}) \) be the Sz.-Nagy-Schäffer (minimal) isometric lifting of \( T' \), and let \( U \) on \( \mathcal{H}' \oplus \mathcal{M} \) be an arbitrary isometric lifting of \( T' \) given by (1.2). Then there exists a unique isometry \( \Phi \) from
\( \mathcal{H} \oplus H^2(D_T) \) into \( \mathcal{H} \oplus \mathcal{M} \) such that \( U\Phi = \Phi V \) and \( \Phi|\mathcal{H} = I_{\mathcal{H}} \). In fact, \( \Phi \) is given by

\[
\Phi = \begin{bmatrix}
I_{\mathcal{H}} & 0 \\
0 & \Lambda
\end{bmatrix} : \begin{bmatrix}
\mathcal{H}' \\
H^2(D_T)
\end{bmatrix} \to \begin{bmatrix}
\mathcal{H}' \\
\mathcal{M}
\end{bmatrix},
\]

where \( \Lambda \) is defined by

\[
\Lambda h = \sum_{n=0}^{\infty} Y_2^n Y_1 h_n, \quad h(\lambda) = \sum_{n=0}^{\infty} \lambda^n h_n \in H^2(D_T),
\]

with \( Y_1 \) and \( Y_2 \) as in \( \text{[2.2]} \). Moreover, \( (\Lambda^* m)(\lambda) = Y_1^* (I - \lambda Y_2^*)^{-1} m \) for each \( m \in \mathcal{M} \). Finally, \( \Phi \) is unitary if and only if \( U \) is a minimal isometric lifting of \( T' \), and in that case the isometric liftings \( V \) and \( U \) of \( T' \) are isomorphic.

The isometry \( \Phi \) introduced in the above theorem will be referred to as the unique isometry associated with \( T' \) that intertwines \( V \) with \( U \). Since \( V \) is uniquely determined by \( T' \), we shall denote this isometry simply by \( \Phi_{U, T'} \). When \( U \) on \( \mathcal{K} \) is an isometric lifting of \( T' \) and \( U' \) on \( \mathcal{K}' \) is the minimal isometric lifting of \( T' \) associated with \( U \), then the operator \( \Pi_{\mathcal{K}, \mathcal{K}'} \Phi_{U, T'} \) is the unique isometry associated with \( T' \) that intertwines \( V \) with \( U' \), that is, \( \Phi_{U', T'} = \Pi_{\mathcal{K}, \mathcal{K}'} \Phi_{U, T'} \) or, equivalently, \( \Pi_{\mathcal{K}', \mathcal{K}} \Phi_{U', T'} = \Phi_{U, T'} \).

2. Isometric realizations

In this section we review some of the classical results on controllable isometric realizations, and we prove a few additional results that will be useful in the later sections.

We say that \( \{Z, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}\} \) (or simply \( \{Z, B, C, D\} \)) is a realization of a \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \)-valued function \( G \) if

\[
G(\lambda) = D + \lambda C (I_X - \lambda Z)^{-1} B
\]

for all \( \lambda \) in some open neighborhood of the origin in the complex plane. Here \( Z \) is an operator on \( \mathcal{X} \) and \( B \) is an operator from \( \mathcal{U} \) into \( \mathcal{X} \) while \( C \) is an operator mapping \( \mathcal{X} \) into \( \mathcal{Y} \) and \( D \) is an operator from \( \mathcal{U} \) into \( \mathcal{Y} \) (where \( \mathcal{X}, \mathcal{U} \) and \( \mathcal{Y} \) are all Hilbert spaces). In this case, we refer to the function defined by the right hand side of \( \text{[2.1]} \) as the associated transfer function. A realization \( \{Z, B, C, D\} \) is called isometric if the operator

\[
M = \begin{bmatrix}
D & C \\
B & Z
\end{bmatrix} : \begin{bmatrix}
\mathcal{U} \\
\mathcal{X}
\end{bmatrix} \to \begin{bmatrix}
\mathcal{Y} \\
\mathcal{X}
\end{bmatrix}
\]

is an isometry. The \( 2 \times 2 \) operator matrix in \( \text{[2.2]} \) is called the system matrix associated with the realization \( \{Z, B, C, D\} \). The transfer function of an isometric realization belongs to the Schur class \( \mathcal{S}(\mathcal{U}, \mathcal{Y}) \), that is, if \( \{Z, B, C, D\} \) is an isometric realization, then the function \( G \) defined by \( \text{[2.1]} \) is a contractive analytic \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \)-valued function on \( \mathbb{D} \). Conversely, if \( G \in \mathcal{S}(\mathcal{U}, \mathcal{Y}) \), then there is an isometric realization \( \{Z, B, C, D\} \) such that \( \text{[2.1]} \) holds for all \( \lambda \) in \( \mathbb{D} \).

The transfer function of a realization can also be expressed in terms of the system matrix \( M \). In fact, if \( \{Z, B, C, D\} \) is a realization and \( M \) is the associated system matrix, then in a neighborhood of the origin the transfer function \( G \) is also given by

\[
G(\lambda) = \Pi_{\mathcal{Y}} M (I_{\mathcal{U} \oplus \mathcal{X}} - \lambda J_{\mathcal{X}} M)^{-1} \Pi_{\mathcal{U}},
\]
where $J_X$ is the partial isometry from $\mathcal{Y} \oplus \mathcal{X}$ to $\mathcal{U} \oplus \mathcal{X}$ given by

$$J_X = \begin{bmatrix} 0 & 0 \\ 0 & I_X \end{bmatrix} : \begin{bmatrix} \mathcal{Y} \\ \mathcal{X} \end{bmatrix} \to \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \end{bmatrix}.$$

Indeed, for $\lambda$ sufficiently close to zero we have

$$G(\lambda) = D + \lambda C(I_X - \lambda Z)^{-1} B = \begin{bmatrix} D & C \\ \lambda(I_X - \lambda Z)^{-1} B & 0 \end{bmatrix} \begin{bmatrix} I_U \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} D & C \\ \lambda(I_X - \lambda Z)^{-1} B & 0 \end{bmatrix}^{-1} \begin{bmatrix} I_U \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} D & C \end{bmatrix} \begin{bmatrix} I_U \lambda(I_X - \lambda Z)^{-1} B & 0 \\ 0 & I_X \end{bmatrix}^{-1} \begin{bmatrix} I_U \\ 0 \end{bmatrix}$$

$$= \Pi_Y M(I_{U \oplus X} - \lambda \lambda Z M)^{-1} \Pi^*_U.$$

Since the right side of (2.1) is a Schur class function if $M$ in (2.2) is an isometry, the same holds true for the right hand side of (2.3). Notice that the function $G$ defined by (2.3) can also be written in the form:

$$G(\lambda) = \Pi_Y (I_{U \oplus X} - \lambda \lambda Z M)^{-1} \Pi^*_U.$$

If for a realization $\{Z, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}\}$ the space $\bigvee_{n=0}^{\infty} Z^n B \mathcal{U}$ is equal to $\mathcal{X}$, then the realization or the pair $\{Z, B\}$ is called controllable. In other words, a realization is controllable if and only if the space $B \mathcal{U}$ is cyclic for $Z$. In terms of the system matrix $M$ in (2.2) the realization $\{Z, B, C, D\}$ is controllable if and only if

$$\mathcal{X} = \Pi_X \bigvee_{n=0}^{\infty} (J_X)^n \mathcal{U} \{0\}.$$

The above condition (2.1) is also equivalent to the requirement that $\{J_X M, \Pi^*_U\}$ is a controllable pair. In the particular case when $\mathcal{U} = \mathcal{Y}$ in (2.2), condition (2.1) can be written in an even simpler form. This is the contents of the next lemma.

**Lemma 2.1.** Let $M$ be as in (2.2), and assume $\mathcal{U} = \mathcal{Y}$. Then $\{Z, B\}$ is controllable if and only if $\mathcal{U} \oplus \{0\}$ is cyclic for $M$, that is,

$$\bigvee_{n=0}^{\infty} M^n \mathcal{U} \{0\} = \mathcal{U} \mathcal{X}.$$

**Proof.** Let $E_{\mathcal{U}}$ be the canonical embedding of $\mathcal{U}$ into $\mathcal{U} \oplus \mathcal{X}$, and define $M_0$ to be the operator

$$M_0 = \begin{bmatrix} 0 & 0 \\ B & Z \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \end{bmatrix} \to \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \end{bmatrix}.$$

Then $M_0 = M - E_{\mathcal{U}} \begin{bmatrix} C & D \end{bmatrix}$. This feedback relation implies that the pair $\{M_0, E_{\mathcal{U}}\}$ is controllable if and only if the pair $\{M, E_{\mathcal{U}}\}$ is controllable. Thus (2.1) holds if and only if $\{M_0, E_{\mathcal{U}}\}$ is controllable. Now notice that for all integers $n \geq 1$, we have

$$M_0^n E_{\mathcal{U}} = \begin{bmatrix} 0 & 0 \\ Z^{n-1} B & Z^n \end{bmatrix} \begin{bmatrix} I_U \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ Z^{n-1} B \end{bmatrix} : \mathcal{U} \to \mathcal{U} \mathcal{X}.$$

It follows that
\[
\bigoplus_{n=0}^{\infty} M_n^0 E_{iu} = \left[ \begin{array}{c} \mathcal{U} \\ \{0\} \end{array} \right] \oplus \bigoplus_{n=1}^{\infty} M_n^0 E_{iu} = \mathcal{U} \oplus \bigoplus_{n=1}^{\infty} Z^{n-1} B \mathcal{U}.
\]
We conclude that (2.3) holds if and only if the pair \( \{Z, B\} \) is controllable. \(\Box\)

A realization \( \{Z, B, C, D\} \) or the pair \( \{C, Z\} \) is called observable if \( CZ^n x = 0 \) for all integers \( n \geq 0 \) implies that the vector \( x \) is equal to zero. Since the orthogonal complement of \( \ker CZ^n \) is equal to the closure of \( \text{Im} (Z^\ast)^n C^\ast \), we see that observability of the realization \( \{Z, B, C, D\} \) is equivalent to the controllability of the dual realization \( \{Z^\ast, C^\ast, B^\ast, D^\ast\} \).

Two realizations \( \{Z_1, B_1, C_1, D_1\} \) and \( \{Z_2, B_2, C_2, D_2\} \) are said to be unitarily equivalent if \( D_1 = D_2 \) and there exists a unitary operator \( W \) mapping \( X_1 \) onto \( X_2 \) such that
\[
WZ_1 = Z_2W, \quad WB_1 = B_2 \quad \text{and} \quad C_2W = C_1.
\]
Unitary equivalence does not change the transfer function. More precisely, when two realizations are unitarily equivalent, then their transfer functions coincide in a neighborhood of zero. For isometric controllable realizations the converse is also true. In fact we have the following theorem.

**Theorem 2.2.** Let \( G \) be a \( \mathbb{L}(\mathcal{U}, \mathcal{Y}) \)-valued function. Then \( G \in \mathbb{S}(\mathcal{U}, \mathcal{Y}) \) if and only if \( G \) admits an isometric realization. In this case, \( G \) admits a controllable isometric realization and all controllable isometric realizations of \( G \) are unitarily equivalent. In particular, formula (2.1) provides a one to one correspondence between the \( \mathbb{L}(\mathcal{U}, \mathcal{Y}) \)-contractive analytic functions on \( \mathbb{D} \) and (up to unitary equivalence) the controllable isometric realizations of \( \mathbb{L}(\mathcal{U}, \mathcal{Y}) \)-valued functions.

The above result appears in a somewhat different form in [19] as a theorem representing a Schur class function as a characteristic operator function. A full proof, with isometric systems replaced by their dual ones, can be found in [4] which also gives additional references. In Section 1.3 of [15] the theorem is proved using the Naimark dilation theory.

We conclude this section with a proposition that will be useful in the later sections. The starting point is an isometry \( Y \) of the type appearing in (1.2). More precisely,
\[
Y = \left[ \begin{array}{c} Y_1 \\ Y_2 \end{array} \right] : \begin{bmatrix} \mathcal{D}' \\ \mathcal{M} \end{bmatrix} \rightarrow \mathcal{M}.
\]

**Proposition 2.3.** Let \( Y \) in (2.6) be an isometry. Assume \( \mathcal{M} = \mathcal{D} \oplus \mathcal{X} \), and let \( \Pi_\mathcal{D} \) and \( \Pi_\mathcal{X} \) be the orthogonal projections of \( \mathcal{M} \) onto \( \mathcal{D} \) and \( \mathcal{X} \), respectively. Put
\[
F(\lambda) = \Pi_\mathcal{D} Y^\ast (I_{\mathcal{M}} - \lambda J_\mathcal{X} Y^\ast)^{-1} \Pi_\mathcal{D}, \quad \lambda \in \mathbb{D},
\]
where \( \Pi_\mathcal{D} \oplus \mathcal{D} \) is the orthogonal projection of \( \mathcal{D}' \oplus \mathcal{M} \) onto \( \mathcal{D}' \oplus \mathcal{D} \), and
\[
J_\mathcal{X} : \mathcal{D}' \oplus \mathcal{M} \rightarrow \mathcal{M}, \quad J_\mathcal{X}(d' + m) = \Pi_\mathcal{X}m.
\]
Then \( F \) belongs to the Schur class \( \mathbb{S}(\mathcal{D}, \mathcal{D}' \oplus \mathcal{D}) \) and
\[
Y_1^\ast (I_{\mathcal{M}} - \lambda Y_2^\ast)^{-1} \Pi_\mathcal{D} = \Pi' F(\lambda) \left( I_{\mathcal{D}} - \lambda \Pi F(\lambda) \right)^{-1}, \quad \lambda \in \mathbb{D},
\]
where \( \Pi \) and \( \Pi' \) are the orthogonal projections of \( \mathcal{D}' \oplus \mathcal{D} \) onto \( \mathcal{D} \) and \( \mathcal{D}' \), respectively.
It will be convenient first to prove a lemma. Let \( \Gamma \) be a contraction from \( \mathcal{M} \) into \( \mathcal{E}_1 \oplus \mathcal{M} \). Partition \( \Gamma \) as a \( 2 \times 1 \) operator matrix, as follows

\[
\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} : \mathcal{M} \to \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{M} \end{bmatrix}.
\]

Furthermore, let \( \mathcal{E}_2 \) be a subspace of \( \mathcal{M} \), and consider the function

\[
\Xi(\lambda) = \Gamma_1(I - \lambda \Gamma_2)^{-1} \Pi^*_2, \quad \lambda \in \mathbb{D},
\]

Here \( \Pi_2 \) is the orthogonal projection of \( \mathcal{M} \) onto \( \mathcal{E}_2 \). Since \( \Gamma \) is a contraction, the same holds true for \( \Gamma_2 \), and hence \( I - \lambda \Gamma_2 \) is invertible for each \( \lambda \in \mathbb{D} \). Thus \( \Xi \) is well-defined on \( \mathbb{D} \). Next, let \( \mathcal{X} \) be the orthogonal complement of \( \mathcal{E}_2 \) in \( \mathcal{M} \), and thus \( \mathcal{M} = \mathcal{E}_2 \oplus \mathcal{X} \). Then \( \Gamma \) also admits a \( 3 \times 2 \) operator matrix representation, namely

\[
\Gamma = \begin{bmatrix} D_1 & C_1 \\ D_2 & C_2 \\ B & Z \end{bmatrix} : \begin{bmatrix} \mathcal{E}_2 \\ \mathcal{X} \end{bmatrix} \to \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{X} \end{bmatrix}.
\]

Put

\[
F(\lambda) = \begin{bmatrix} D_1 & C_1 \\ D_2 & C_2 \end{bmatrix} + \lambda \begin{bmatrix} C_1 & C_2 \end{bmatrix} (I_X - \lambda Z)^{-1} B, \quad \lambda \in \mathbb{D}.
\]

Again, since \( \Gamma \) is a contraction, the operator \( Z \) is a contraction, and hence \( F \) is well-defined on \( \mathbb{D} \).

**Lemma 2.4.** Let \( \Xi \) and \( F \) be the functions defined by (2.10) and (2.12), respectively. Then \( F \) belongs to the Schur class \( S(\mathcal{E}_2, \mathcal{E}_1 \oplus \mathcal{E}_2) \) and

\[
\Xi(\lambda) = \Pi_1 F(\lambda)(I - \Pi_2 F(\lambda))^{-1}, \quad \lambda \in \mathbb{D},
\]

where \( \Pi_1 \) and \( \Pi_2 \) are the orthogonal projections of \( \mathcal{E}_1 \oplus \mathcal{E}_2 \) onto \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \), respectively.

**Proof.** The function \( F \) is the transfer function of the system

\[
\{ Z, B, \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}; \mathcal{X}, \mathcal{E}_2, \mathcal{E}_1 \oplus \mathcal{E}_2 \}.
\]

By (2.11) the system matrix corresponding to this system is equal to \( \Gamma \), and hence it is a contraction. This implies that \( F \) belongs to the Schur class \( S(\mathcal{E}_2, \mathcal{E}_1 \oplus \mathcal{E}_2) \); cf., Theorem 4.1 in [10] where this is proved for time-variant systems.

To prove (2.13) fix \( \lambda \in \mathbb{D} \). Using the partitioning of \( \Gamma \) in (2.11) we see that for each \( e \in \mathcal{E}_2 \) we have

\[
\Xi(\lambda)e = \begin{bmatrix} D_1 & C_1 \end{bmatrix} \left( I - \lambda \begin{bmatrix} D_2 & C_2 \\ B & Z \end{bmatrix} \right)^{-1} \begin{bmatrix} e \\ 0 \end{bmatrix}.
\]

To find \( \Xi(\lambda)e \) we have to compute the first column of the inverse of the \( 2 \times 2 \) operator matrix

\[
\left[ \begin{array}{cc} I - \lambda D_2 & -\lambda C_2 \\ -\lambda B & I - \lambda Z \end{array} \right].
\]

Since \( I - \lambda Z \) is invertible, the Schur complement \( \Delta(\lambda) \) of \( I - \lambda Z \) in (2.14) is well-defined and is given by

\[
\Delta(\lambda) := I - \lambda D_2 - \lambda^2 C_2 (I - \lambda Z)^{-1} B = I - \lambda \Pi_2 F(\lambda).
\]
It follows (cf., Remark 1.2 in [11]) that
\[
\begin{bmatrix}
I - \lambda D_2 & -\lambda C_2 \\
-\lambda B & I - \lambda Z
\end{bmatrix}^{-1} = \begin{bmatrix}
(I - \lambda \Pi_2 F(\lambda))^{-1} & * \\
\lambda (I - \lambda Z)^{-1} B (I - \lambda \Pi_2 F(\lambda))^{-1} & * 
\end{bmatrix}.
\]
Thus
\[
\Xi(\lambda)e = \begin{bmatrix}
D_1 & C_1
\end{bmatrix} \begin{bmatrix}
(I - \lambda \Pi_2 F(\lambda))^{-1} e \\
\lambda (I - \lambda Z)^{-1} B (I - \lambda \Pi_2 F(\lambda))^{-1} e
\end{bmatrix}
= (D_1 - \lambda C_1 (I - \lambda Z)^{-1} B) (I - \lambda \Pi_2 F(\lambda))^{-1} e
= \Pi_1 F(\lambda) (I - \lambda \Pi_2 F(\lambda))^{-1} e.
\]
Since \(e\) is an arbitrary element of \(E_2\), this proves (2.13). \(\square\)

**Proof of Proposition 2.3.** Since \(Y\) is assumed to be an isometry, \(Y^*\) is a contraction. Now apply Lemma 2.4 with \(D'\) in place of \(E_1\), with \(Y^*\) in place of the contraction \(\Gamma\) in (2.9), and with \(D\) in place of \(E_2\). With these choices the function \(\Xi\) in (2.10) coincides with the function defined by the left hand side of (2.8). Thus in order to finish the proof it remains to show that with \(\Gamma = Y^*, \ E_1 = D'\), and \(E_2 = D\) the function \(F\) in (2.12) is also given by (2.7). But this follows by applying to \(F\) in place of \(G\) that the function \(G\) in (2.11) is also given by (2.8). Indeed, since \(F\) is the transfer function of the system
\[
\{Z, B, \begin{bmatrix}
C_1 \\
0
\end{bmatrix}, \begin{bmatrix}
D_1 & D_2
\end{bmatrix} : \mathcal{X}, D, D' \oplus D',
\]
and the system matrix of this system is equal to \(Y^*\), the equivalence between (2.11) and (2.12) yields in a straightforward way that \(F\) in (2.12) is also given by (2.7). \(\square\)

3. ISOMETRIC COUPLINGS

Throughout this section \(\{T', A\}\) is a pair of contractions, \(T'\) on a Hilbert space \(\mathcal{H}'\) and \(A\) from a Hilbert space \(\mathcal{H}\) to \(\mathcal{H}'\).

An *isometric coupling* of \(\{T', A\}\) is a pair \(\{U \in \mathcal{K}, \tau\}\) of operators such that \(U\) is an isometric lifting of \(T'\), acting on \(\mathcal{K}\) (and thus \(\mathcal{H}' \subset \mathcal{K}\)), and \(\tau\) is an isometry from \(\mathcal{H}\) to \(\mathcal{K}\) with \(\Pi_\mathcal{H}' \tau = A\). If the space \(\mathcal{K}\) is of no interest, then we will just write \(\{U, \tau\}\). An isometric coupling \(\{U \in \mathcal{K}, \tau\}\) of \(\{T', A\}\) is called minimal if, in addition, the space \(\mathcal{H}' \vee \tau \mathcal{H}\) is cyclic for \(U\), that is,
\[
\mathcal{K} = \bigvee_{n=0}^{\infty} U^n (\mathcal{H}' \vee \tau \mathcal{H}).
\]

There exist minimal isometric couplings of \(\{T', A\}\). To see this, let \(U\) be the operator on \(\mathcal{H}' \oplus H^2(D_{T'}) \oplus H^2(D_A)\) given by the following operator matrix representation
\[
U = \begin{bmatrix}
T' & 0 & 0 \\
E_{D_{T'}} & S_{D_{T'}} & 0 \\
0 & 0 & S_{D_A}
\end{bmatrix}.
\]
Here \(E_{D_{T'}}\) is the canonical embedding of \(D_{T'}\) onto the space of constant functions of \(H^2(D_{T'})\), and \(S_{D_{T'}}\) and \(S_{D_A}\) are the unilateral shifts on \(H^2(D_{T'})\) and \(H^2(D_A)\), respectively. Notice that the operator defined by the 2 \(\times\) 2 operator matrix in the left upper corner of the matrix for \(U\) is the Sz.-Nagy-Schröfer minimal isometric
lifting of $T'$. Since $S_{D_A}$ is an isometry, we conclude that $U$ is also an isometric lifting of $T'$. Now let $\tau$ be the isometry defined by

$$\tau = \begin{bmatrix} A \\ 0 \\ E_{D_A}D_A \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} H' \\ H^2(D_{T'}) \\ H^2(D_A) \end{bmatrix}$$

where $E_{D_A}$ is the canonical embedding of $D_A$ onto the space of constant functions of $H^2(D_A)$. Then $\{U, \tau\}$ is a minimal isometric coupling of $\{T', A\}$.

Two isometric couplings $\{U_1 \text{ on } K_1, \tau_1\}$ and $\{U_2 \text{ on } K_2, \tau_2\}$ of $\{T', A\}$ are said to be isomorphic if there exists a unitary operator $\Psi$ from $K_1$ to $K_2$ such that

$$\Psi U_1 = U_2 \Psi, \quad \Psi \tau_1 = \tau_2 \quad \text{and} \quad \Psi h = h \text{ for all } h \in H'.$$

In this case

$$(3.1) \quad \Psi U_1 \tau_1 = U_2 \tau_2.$$ 

Minimality is preserved under isomorphic equivalence. Indeed, when the pairs $\{U_1 \text{ on } K_1, \tau_1\}$ and $\{U_2 \text{ on } K_2, \tau_2\}$ are isomorphic isometric couplings of $\{T', A\}$, and $\Psi$ from $K_1$ to $K_2$ is an isomorphism between the two isometric couplings, then

$$\bigvee_{n=0}^{\infty} U_2^n (\mathcal{H}' \vee \tau_2 \mathcal{H}) = \bigvee_{n=0}^{\infty} (\Psi U_1 \Psi^*)^n (\mathcal{H}' \vee \Psi \tau_1 \mathcal{H}) = \bigvee_{n=0}^{\infty} U_1^n (\Psi^* \mathcal{H}' \vee \Psi^* \Psi \tau_1 \mathcal{H}) = \bigvee_{n=0}^{\infty} U_1^n (\mathcal{H}' \vee \tau_1 \mathcal{H}).$$

We say that an isometric coupling $\{U \text{ on } K, \tau\}$ of $\{T', A\}$ is special if $K$ is a Hilbert direct sum of the space $\mathcal{H}'$, the space $D_A$ and some Hilbert space $X$, that is, $K = \mathcal{H}' \oplus D_A \oplus X$, and the action of $\tau$ is given by $\tau h = Ah \oplus D_A h \oplus 0$, where 0 is the zero vector in $X$. In other words, an isometric coupling $\{U \text{ on } K, \tau\}$ of $\{T', A\}$ is special if, in addition, $D_A$ is a subspace of $M$, where $M = K \ominus \mathcal{H}'$, and $\tau$ admits a matrix representation of the form

$$\tau = \begin{bmatrix} A \\ \Pi_{D_A}D_A \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ M \end{bmatrix}.$$ 

The importance of special isometric couplings follows from Theorem 3.4 below. To prove this theorem we need a few auxiliary propositions. The first also settles the question of existence of special isometric couplings.

**Proposition 3.1.** Every isometric coupling is isomorphic to a special isometric coupling.

**Proof.** Let $\{U \text{ on } K, \tau\}$ be an isometric coupling of $\{T', A\}$, and put $M = K \ominus \mathcal{H}'$. Since $\tau$ is an isometry and $\Pi_{\mathcal{H}'} \tau = A$, the operator $\tau$ admits a matrix representation of the form:

$$(3.2) \quad \tau = \begin{bmatrix} A \\ \Gamma D_A \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ M \end{bmatrix} \text{ where } \Gamma : D_A \to M \text{ is an isometry;}$$

see Section IV.1 of [13] or Section XXVII.5 of [17]. Now let $D = \text{Im} \Gamma$, and put $X = M \ominus D$. Then $D$ is closed, and we can view $\Gamma$ as a unitary operator from $D_A$.
onto $\mathcal{D}$. Define the unitary operator $\sigma$ by
\[
\sigma = \begin{bmatrix} I_{\mathcal{H}'} & 0 & 0 \\ 0 & \Gamma & 0 \\ 0 & 0 & I_\mathcal{X} \end{bmatrix} : \begin{bmatrix} \mathcal{H}' \\ \mathcal{D}_A \\ \mathcal{X} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}' \\ \mathcal{D} \\ \mathcal{X} \end{bmatrix}.
\]

Also define $U_0 = \sigma^* U \sigma$ and $\tau_0 = \sigma^* \tau$. Then $\{U_0, \tau_0\}$ is a special isometric coupling of $\{T', A\}$ which is isomorphic to $\{U, \tau\}$. \hfill \Box

Since minimality of isometric couplings is preserved under isomorphisms, and isometric couplings do exist (see the third paragraph of this section), the above proposition shows that any $\{T', A\}$ admits a special minimal isometric coupling.

Recall that an isometric lifting $U$ of $T'$ can always be represented (see (1.2)) in the following form:
\[
(3.3) \quad U = \begin{bmatrix} T' & 0 \\ Y_1 D_{T'} & Y_2 \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} \mathcal{H}' \\ \mathcal{M} \end{bmatrix} \quad \text{where} \quad Y = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} : \begin{bmatrix} \mathcal{D}_{T'} \\ \mathcal{M} \end{bmatrix} \rightarrow \mathcal{M}
\]
is an isometry. According to (3.3) this $U$ also admits a matrix representation of the form:
\[
U = \begin{bmatrix} U' & 0 \\ 0 & \tilde{U} \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} \mathcal{K}' \\ \tilde{\mathcal{K}} \end{bmatrix} \quad \text{where} \quad \mathcal{K}' = \bigoplus_{n=0}^{\infty} U^n \mathcal{H}', \quad \tilde{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}'.
\]

Here $U'$ on $\mathcal{K}'$ is the minimal isometric lifting of $T'$ associated with $U$ (see Section 1), and $\tilde{U}$ is an isometry on $\tilde{\mathcal{K}}$. We can now state the next proposition.

**Proposition 3.2.** Let $\{U, \tau\}$ be an isometric coupling of $\{T', A\}$, where $U$ is determined by (3.3) and $\tau$ by (3.2). Set $\mathcal{D} = \text{Im} \Gamma$, where $\Gamma$ is given by (3.2), and for $Y$ in (3.3) consider the following operator matrix representation:
\[
(3.4) \quad Y = \begin{bmatrix} D & C \\ B & Z \end{bmatrix} : \begin{bmatrix} \mathcal{D}_{T'} \oplus \mathcal{D} \\ \mathcal{X} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{D} \\ \mathcal{X} \end{bmatrix} \quad \text{where} \quad \mathcal{X} = \mathcal{M} \oplus \mathcal{D}.
\]
Then $\{U, \tau\}$ is a minimal isometric coupling of $\{T', A\}$ if and only if the pair $\{Z, B\}$ is controllable.

**Proof.** Since $\tau$ is given by (3.2), the space $\mathcal{H}' \oplus \tau \mathcal{H}$ is equal to $\mathcal{H}' \oplus \mathcal{D}$. Thus we have to show that $\mathcal{H}' \oplus \mathcal{D}$ is cyclic for $U$ if and only if the pair $\{Z, B\}$ is controllable. To do this we associate with $U$ two auxiliary operators, namely
\[
\tilde{U} = \begin{bmatrix} 0 & 0 \\ Y_1 D_{T'} & Y_2 \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} \mathcal{H}' \\ \mathcal{M} \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ Y_1 & Y_2 \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} \mathcal{D}_{T'} \\ \mathcal{M} \end{bmatrix}.
\]

Notice that the range of $U - \tilde{U}$ belongs to $\mathcal{H}'$. Since $\mathcal{H'} \subset \mathcal{H}' \oplus \mathcal{D}$, this implies that $\mathcal{H}' \oplus \mathcal{D}$ is cyclic for $U$ if and only if $\mathcal{H}' \oplus \mathcal{D}$ is cyclic for $\tilde{U}$. By induction one proves that for $n = 1, 2, 3, \ldots$ we have
\[
\tilde{U}^n = \begin{bmatrix} 0 & 0 \\ Y_2^{n-1} Y_1 D_{T'} & Y_2^n \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} \mathcal{H}' \\ \mathcal{M} \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ Y_2^{n-1} Y_1 & Y_2^n \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} \mathcal{D}_{T'} \\ \mathcal{M} \end{bmatrix}.
\]

On the other hand
\[
\begin{bmatrix} Y_2^{n-1} Y_1 D_{T'} & Y_2^n \end{bmatrix} \begin{bmatrix} \mathcal{H}' \\ \mathcal{D} \end{bmatrix} = \begin{bmatrix} Y_2^{n-1} Y_1 & Y_2^n \end{bmatrix} \begin{bmatrix} \mathcal{D}_{T'} \\ \mathcal{D} \end{bmatrix}, \quad n = 1, 2, 3, \ldots
\]
and hence $\mathcal{H}' \oplus \mathcal{D}$ is cyclic for $\tilde{U}$ if and only if $\mathcal{D}_{T'} \oplus \mathcal{D}$ is cyclic for $M$. 

It remains to prove that $\mathcal{D}_T \oplus \mathcal{D}$ is cyclic for $M$ if and only if the pair $\{Z, B\}$ is controllable. Using $Y = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}$ and (3.4), we see that

$$M = \begin{bmatrix} \hat{D} & \hat{C} \\ B & Z \end{bmatrix},$$

where $\hat{D} = \Pi_D^* D$ and $\hat{C} = \Pi_D^* C$ with $\Pi_D$ equal to the orthogonal projection of $\mathcal{D}_T \oplus \mathcal{D}$ onto $\mathcal{D}$. By employing Lemma 2.1 with $\mathcal{U} = \mathcal{Y} = \mathcal{D}_T \oplus \mathcal{D}$, we see that $\mathcal{D}_T \oplus \mathcal{D}$ is cyclic for $M$ if and only if the pair $\{Z, B\}$ is controllable, which completes the proof. □

**Proposition 3.3.** Let $\{U_1 \text{ on } \mathcal{K}_1, \tau_1\}$ and $\{U_2 \text{ on } \mathcal{K}_2, \tau_2\}$ be special isometric couplings of $\{T', A\}$. For $j = 1, 2$ set $\mathcal{X}_j = \mathcal{X}_j \oplus (\mathcal{H}' \oplus \mathcal{D}_A)$, and let $Y(j)$ be the isometry from $\mathcal{D}_T \oplus \mathcal{D}_A \oplus \mathcal{X}_j$ into $\mathcal{D}_A \oplus \mathcal{X}_j$ corresponding to $U_j$ via (3.3). Consider the following operator matrix representation

$$Y(j) = \begin{bmatrix} D_j & C_j \\ B_j & Z_j \end{bmatrix} : \begin{bmatrix} \mathcal{D}_T \oplus \mathcal{D}_A \\ \mathcal{X}_j \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{D}_A \\ \mathcal{X}_j \end{bmatrix} \text{ for } j = 1, 2.$$

Then $\{U_1, \tau_1\}$ and $\{U_2, \tau_2\}$ are isomorphic if and only if $\{Z_1, B_1, C_1, D_1\}$ and $\{Z_2, B_2, C_2, D_2\}$ are unitarily equivalent realizations.

**Proof.** Assume that $\{Z_1, B_1, C_1, D_1\}$ and $\{Z_2, B_2, C_2, D_2\}$ are unitarily equivalent, that is, $D_1 = D_2$ and there exists a unitary operator $W$ from $\mathcal{X}_1$ onto $\mathcal{X}_2$ such that

$$WZ_1 = Z_2W, \quad WB_1 = B_2, \quad \text{and} \quad C_1 = C_2W.$$

Now let $\Phi$ be the unitary operator from $\mathcal{K}_1$ onto $\mathcal{K}_2$ defined by

$$\Phi = \begin{bmatrix} I_{\mathcal{H}'} & 0 & 0 \\ 0 & I_{\mathcal{D}_A} & 0 \\ 0 & 0 & W \end{bmatrix} : \begin{bmatrix} \mathcal{H}' \\ \mathcal{D}_A \\ \mathcal{X}_1 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}' \\ \mathcal{D}_A \\ \mathcal{X}_2 \end{bmatrix}.$$

Then $\Phi h = h$ for all $h$ in $\mathcal{H}'$. Because $\{U_1, \tau_1\}$ and $\{U_2, \tau_2\}$ are special, we see that

$$\tau_j = \begin{bmatrix} A \\ D_A \\ 0 \end{bmatrix} : \mathcal{H} \rightarrow \begin{bmatrix} \mathcal{H}' \\ \mathcal{D}_A \\ \mathcal{X}_j \end{bmatrix} \text{ for } j = 1, 2.$$

Hence $\Phi \tau_1 = \tau_2$. Using the appropriate operator matrix decomposition we arrive at

$$\Phi U_1 = \begin{bmatrix} I_{\mathcal{H}'} & 0 & 0 \\ 0 & I_{\mathcal{D}_A} & 0 \\ 0 & 0 & W \end{bmatrix} \begin{bmatrix} T' \\ D_1 \Pi_{D_T} D_T' \\ B_1 \Pi_{D_T} D_T' \end{bmatrix} \begin{bmatrix} 0 & 0 \\ D_1 \Pi_{D_A} D_T' \\ B_1 \Pi_{D_A} D_T' \end{bmatrix} \begin{bmatrix} C_1 \\ Z_1 \end{bmatrix}.$$

A similar calculation shows that

$$U_2 \Phi = \begin{bmatrix} T' \\ D_2 \Pi_{D_T} D_T' \\ B_2 \Pi_{D_T} D_T' \end{bmatrix} \begin{bmatrix} 0 & 0 \\ D_2 \Pi_{D_A} D_T' \\ B_2 \Pi_{D_A} D_T' \end{bmatrix} \begin{bmatrix} C_2 \\ Z_2 \end{bmatrix}.$$

$$\text{for } j = 1, 2.$$
Because \( D_1 = D_2 \) and (3.5) holds, we see that \( \Phi U_1 = U_2 \Phi \). In other words, \( \{ U_1, \tau_1 \} \) and \( \{ U_2, \tau_2 \} \) are isomorphic.

Conversely assume that \( \{ U_1, \tau_1 \} \) and \( \{ U_2, \tau_2 \} \) are isomorphic. Then there exists a unitary operator \( \Phi \) from \( K_1 \) onto \( K_2 \) such that \( \Phi h = h \) for all \( h \) in \( H' \) and \( \Phi \tau_1 = \tau_2 \) and \( \Phi U_1 = U_2 \Phi \). Because \( \tau_1 \) and \( \tau_2 \) admit matrix representations of the form presented in (3.7) and \( \text{Im} \, D_A \) is dense in \( D_A \), we see that \( \Phi h = h \) for all \( h \in D_A \). So \( \Phi \) admits a matrix representation as in (3.4) where \( W \) is a unitary operator from \( X_1 \) onto \( X_2 \). By combining \( \Phi U_1 = U_2 \Phi \) with the matrix representations for \( \Phi U_1 \) in (3.8) and \( U_2 \Phi \) in (3.4), we see that

\[
D_1 = D_2, \quad W Z_1 = Z_2 W, \quad WB_1 = B_2 \quad \text{and} \quad C_1 = C_2 W.
\]

Hence \( \{ Z_1, B_1, C_1, D_1 \} \) and \( \{ Z_2, B_2, C_2, D_2 \} \) are unitarily equivalent realizations. \( \square \)

**Theorem 3.4.** Let \( \{ T', A \} \) be a pair of contractions, \( T' \) acting on \( H' \) and \( A \) from \( H \) into \( H' \). Then there is a one to one map from the set of minimal isometric couplings of \( \{ T', A \} \), with isomorphic ones being identified, onto the Schur class \( S(D_A, D_T \oplus D_A) \). This map is defined as follows. Let \( \{ U, \tau \} \) be a minimal isometric coupling of \( \{ T', A \} \), which may be assumed to be special, by Proposition 3.1. Define

\[
F_{U, \tau}(\lambda) = \Pi_{D_T \oplus D_A} Y^* (I_M - \lambda J_X^* Y*)^{-1} \Pi_{D_A},
\]

where \( Y \) is the isometry uniquely determined by \( U \) via (3.3), \( X = M \oplus D_A \), and \( J_X^* \) is the partial isometry from \( D_T \oplus D_A \oplus X \) to \( D_A \oplus X \) given by

\[
J_X^* = \begin{bmatrix} 0 & 0 & D_T \oplus D_A \\ 0 & I_X & \end{bmatrix} : \begin{bmatrix} D_T \oplus D_A \\ X \end{bmatrix} \rightarrow \begin{bmatrix} D_A \\ X \end{bmatrix}.
\]

Then \( \{ U, \tau \} \mapsto F_{U, \tau} \) is the desired map.

**Proof.** We know from Proposition 3.1 that every isometric coupling is isomorphic to a special one. So without loss of generality we can assume the isometric couplings to be special.

From Proposition 3.2 and Section 1 it is clear that there is a one to one correspondence between the special minimal isometric couplings of \( \{ T', A \} \) and the isometries \( Y \) mapping the space \( D_T \oplus D_A \oplus X \) into \( D_A \oplus X \), where \( X \) is some Hilbert space and the pair \( \{ \Pi_X Y \Pi_Y^*, \Pi_X Y \Pi_{D_T} \oplus D_A \} \) is controllable. In fact, this one to one correspondence is provided by (5.3). Furthermore, formula (3.4) establishes a one to one correspondence between the isometries \( Y \) mapping the space \( D_T \oplus D_A \oplus X \) into \( D_A \oplus X \) and the isometric realizations

\[
\{ Z, B, C, D; X, D_T \oplus D_A, D_A \},
\]

and in this one to one correspondence \( Z = \Pi_X Y \Pi_Y^* \) and \( B = \Pi_X Y \Pi_{D_T} \oplus D_A \).

From Theorem 3.2 we know that there is a one to one correspondence between the controllable isometric realizations, with the unitarily equivalent ones being identified, and the \( S(D_T \oplus D_A, D_A) \) Schur class functions. Next, note that the map \( G \mapsto F \), where \( F(\lambda) = G(\lambda)^* \), is a one to one map from \( S(D_T \oplus D_A, D_A) \) onto \( S(D_A, D_T \oplus D_A) \). Following up all these one to one correspondences and using the results of Section 2 we see that the map from a special minimal isometric coupling \( \{ U, \tau \} \) to \( F \) is given by \( F = F_{U, \tau} \). To complete the proof, it remains to apply Proposition 3.3. \( \square \)

We conclude this section with a lemma that will be useful in the next section.
Lemma 3.5. Let \( \{U_1 \text{ on } K_1, \tau_1\} \) and \( \{U_2 \text{ on } K_2, \tau_2\} \) be isomorphic isometric couplings of \( \{T', A\} \), and let \( V \) on \( \mathcal{H}' \oplus H^2(\mathcal{D}_{T'}) \) be the Sz.-Nagy-Sch"afer minimal isometric lifting of \( T' \). For \( j = 1, 2 \) let \( \Phi_j \) be the unique isometry associated with \( T' \) intertwining \( V \) and \( U_j \). Then

\[
\Phi_1^* \tau_1 = \Phi_2^* \tau_2.
\]

Proof. Let \( \Psi \) from \( K_1 \) to \( K_2 \) be an isomorphism from \( \{U_1, \tau_1\} \) to \( \{U_2, \tau_2\} \). Define \( \Theta \) from \( \mathcal{H}' \oplus H^2(\mathcal{D}_{T'}) \) into \( K_1 \) by setting \( \Theta = \Psi^* \Phi_2 \). Then \( \Theta \) is an isometry, \( \Theta h = \Psi^* \Phi_2 h = \Psi^* h = h \) for all \( h \in \mathcal{H}' \), and

\[
\Theta V = \Psi^* \Phi_2 V = \Psi^* U_2 \Phi_2 = U_1 \Psi^* \Phi_2 = U_1 \Theta.
\]

So, by Theorem 1.1 (see also the last paragraph of Section 1), the operator \( \Theta \) is the unique isometry associated with \( T' \) intertwining \( V \) and \( U_1 \), that is, \( \Theta = \Phi_1 \). It follows that \( \Phi_1 = \Psi^* \Phi_2 \), and hence \( \Phi_1^* \tau_1 = \Phi_2^* \Psi \tau_1 = \Phi_2^* \tau_2 \), which completes the proof. \( \square \)

4. Main theorem for the case when \( R^* R = Q^* Q \)

In this section \( \{A, T', V, R, Q\} \) is a lifting data set, with \( V \) on \( \mathcal{H}' \oplus H^2(\mathcal{D}_{T'}) \) being the Sz.-Nagy-Sch"afer minimal isometric lifting of \( T' \). In particular, \( T'AR = AQ \) and \( R^* R \leq Q^* Q \). Recall that \( B \) from \( \mathcal{H} \) into \( \mathcal{H}' \oplus H^2(\mathcal{D}_{T'}) \) is a contractive interpolant for \( \{A, T', V, R, Q\} \) if \( B \) is a contraction satisfying \( \Pi_{\mathcal{H}'} B = A \) and \( VBR = BQ \).

Our aim is to prove Theorem 0.1 assuming that \( R^* R = Q^* Q \). First let us reformulate Theorem 0.1 for this case. For this purpose note that for \( R^* R = Q^* Q \) the spaces \( \mathcal{F} \) and \( \mathcal{F}' \) defined by (0.2) are given by

\[
\mathcal{F} = \overline{D_A Q \mathcal{H}_0} \quad \text{and} \quad \mathcal{F}' = \overline{\begin{bmatrix} D_{T'} AR \\ D_A R \end{bmatrix} \mathcal{H}_0}.
\]

Observe that \( \mathcal{F} \subset \mathcal{D}_A \) and \( \mathcal{F}' \subset \mathcal{D}_{T'} \oplus \mathcal{D}_A \). Furthermore, the unitary operator \( \omega \) mapping \( \mathcal{F} \) onto \( \mathcal{F}' \) in (0.3) is now determined by

\[
(4.1) \quad \omega(D_A Q h) = \begin{bmatrix} D_{T'} AR \\ D_A R \end{bmatrix} h, \quad h \in \mathcal{H}_0.
\]

The following is the main result of this section.

Theorem 4.1. Let \( \{A, T', V, R, Q\} \) be a lifting data set, where \( V \) on \( \mathcal{H}' \oplus H^2(\mathcal{D}_{T'}) \) is the Sz.-Nagy-Sch"afer minimal isometric lifting of \( T' \), and assume that \( R^* R = Q^* Q \). Then all contractive interpolants \( B \) for \( \{A, T', V, R, Q\} \) are given by

\[
(4.2) \quad B h = \begin{bmatrix} A h \\ \Pi_{T'} F(\lambda) (I - \lambda \Pi_A F(\lambda))^{-1} D_A h \end{bmatrix}, \quad h \in \mathcal{H},
\]

where \( F \) is any function in \( \mathcal{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A) \) satisfying \( F(0) | \mathcal{F} = \omega \). Here \( \omega \) is the unitary operator defined in (4.1) while \( \Pi_{T'} \) and \( \Pi_A \) are the projections given by

\[
\Pi_{T'} = \begin{bmatrix} I & 0 \end{bmatrix} : \overline{\mathcal{D}_{T'} \mathcal{D}_A} \rightarrow \mathcal{D}_{T'} \quad \text{and} \quad \Pi_A = \begin{bmatrix} 0 & I \end{bmatrix} : \overline{\mathcal{D}_{T'} \mathcal{D}_A} \rightarrow \mathcal{D}_A.
\]

The proof of the above theorem will be based on a further refinement (which we present in two propositions) of the theory of isometric couplings presented in the previous section. In fact, to obtain contractive interpolants for \( \{A, T', V, R, Q\} \) we
shall need isometric couplings \( \{U, \tau\} \) of \( \{T', A\} \) satisfying the additional intertwining relation \( U\tau R = \tau Q \). This is the contents of the first proposition (Proposition 4.2 below). The existence of such couplings is guaranteed by the second proposition (Proposition 4.3 below), which is based on Theorem 3.4. In the sequel, for simplicity, we shall write \( \mathcal{V} \) for the space \( \mathcal{H}' \oplus H^2(\mathcal{D}_{T'}) \).

**Proposition 4.2.** Let \( \{A, T', V, R, Q\} \) be a lifting data set, with \( V \) on \( \mathcal{V} \) being the Sz.-Nagy-Schäffer minimal isometric lifting of \( T' \), and assume that \( R^* R = Q^* Q \). Let \( \{U, K, \tau\} \) be an isometric coupling of \( \{T', A\} \) satisfying \( U\tau R = \tau Q \), and let \( \Phi \) be the unique isometry from \( \mathcal{V} \) into \( \mathcal{K} \) associated with \( T' \) intertwining \( V \) with \( U \). Then

\[
B = \Phi^* \tau
\]

is a contractive interpolant for \( \{A, T', V, R, Q\} \), and all contractive interpolants for this data set are obtained in this way. More precisely, if \( B \) is a contractive interpolant for \( \{A, T', V, R, Q\} \), then there exists a minimal special isometric coupling \( \{U, \tau\} \) and \( \Phi \) such that \( B = \Phi^* \tau \) and \( U\tau R = \tau Q \).

**Proof.** First let us show that \( B \) defined by (4.3) is a contractive interpolant for the data set \( \{A, T', V, R, Q\} \). Obviously, \( B \) is a contraction. Put \( \mathcal{K}' = \text{Im} \Phi \). Recall (see Section 1) that \( \Phi \Phi^* \) is the orthogonal projection of \( \mathcal{K} \) onto \( \mathcal{K}' \). From Theorem 4.1 we know that \( \mathcal{K}' = \bigvee_{n \geq 0} U^n \mathcal{H}' \) is a reducing subspace for \( U \). It follows that \( U \) commutes with \( \Phi \Phi^* \). Since \( \Phi \Phi^* \) is the identity operator on \( \mathcal{V} = \mathcal{H}' \oplus H^2(\mathcal{D}_{T'}) \), we obtain

\[
VBR = \Phi^* U \Phi(\Phi^* \tau) R = \Phi^* (U \Phi \Phi^*) \tau R = \Phi^* \Phi \Phi^* U \tau R = \Phi^* \tau Q = BQ.
\]

Thus \( B \) is a contractive interpolant for \( \{A, T', V, R, Q\} \).

To prove the reverse implication, assume that \( B \) is a contractive interpolant. We have to construct a minimal special isometric coupling \( \{U, \tau\} \) of \( \{T', A\} \) satisfying \( U\tau R = \tau Q \) such that \( B \) is given by (4.3). Since \( B \) is a contraction, we may consider the subspaces

\[
\tilde{\mathcal{F}} = D_B \mathcal{H}_0 \quad \text{and} \quad \tilde{\mathcal{F}}' = D_B \tilde{\mathcal{H}}_0.
\]

Using \( VBR = BQ \) with \( R^* R = Q^* Q \), and the fact that \( V \) is an isometry, we see that for each \( h \in \mathcal{H}_0 \) we have

\[
\|D_B Qh\|^2 = \|Qh\|^2 - \|BQh\|^2 = \|Rh\|^2 - \|VBRh\|^2 = \|Rh\|^2 - \|BRh\|^2 = \|D_B Rh\|^2.
\]

Hence there exists a unique unitary operator \( \tilde{\omega} \) from \( \tilde{\mathcal{F}} \) onto \( \tilde{\mathcal{F}}' \) such that \( \tilde{\omega} D_B R = D_B Q \). Next, define the subspaces

\[
\tilde{\mathcal{G}} = D_B \oplus \tilde{\mathcal{F}} \quad \text{and} \quad \tilde{\mathcal{G}}' = D_B \oplus \tilde{\mathcal{F}}'.
\]

Notice that \( D_B = \tilde{\mathcal{F}} \oplus \tilde{\mathcal{G}} \) and \( D_B = \tilde{\mathcal{F}}' \oplus \tilde{\mathcal{G}}' \). Thus \( \tilde{\omega} \) defines a partial isometry \( \Omega \) on \( D_B \) as follows:

\[
\Omega = \begin{bmatrix} \tilde{\omega} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \tilde{\mathcal{F}} \\ \tilde{\mathcal{G}} \end{bmatrix} \to \begin{bmatrix} \tilde{\mathcal{F}}' \\ \tilde{\mathcal{G}}' \end{bmatrix}.
\]

Observe that \( D_B \) coincides with \( \tilde{\mathcal{G}} \). Define \( V_\Omega \) to be the Sz.-Nagy-Schäffer minimal isometric lifting of \( \Omega \) on \( V_\Omega = D_B \oplus H^2(\tilde{\mathcal{G}}) \). Thus \( V_\Omega \) has the following operator...
matrix representation

\[
V_\Omega = \begin{bmatrix}
\tilde{\omega} & 0 & 0 \\
0 & 0 & 0 \\
0 & E_\tilde{\omega} & S_\tilde{\omega}
\end{bmatrix} : \begin{bmatrix}
\tilde{\mathcal{F}} \\
\tilde{\mathcal{G}} \\
H^2(\tilde{\mathcal{G}})
\end{bmatrix} \rightarrow \begin{bmatrix}
\tilde{\mathcal{F}}' \\
\tilde{\mathcal{G}}' \\
H^2(\tilde{\mathcal{G}})
\end{bmatrix}.
\]

Here \( E_\tilde{\omega} \) is the canonical embedding of \( \tilde{\mathcal{G}} \) onto the space of constant functions in \( H^2(\tilde{\mathcal{G}}) \), and \( S_\tilde{\omega} \) is the unilateral shift on the Hardy space \( H^2(\tilde{\mathcal{G}}) \). Since \( V_\Omega \) is a minimal isometric lifting of \( \Omega \), we have \( V_\Omega = \mathcal{D}_B \oplus H^2(\tilde{\mathcal{G}}) = \bigcup_{n=0}^{\infty} V_{\Omega}^n \mathcal{D}_B \). Now, put

\[
U_\Omega = \begin{bmatrix}
V & 0 \\
0 & V_\Omega
\end{bmatrix} \text{ on } \begin{bmatrix}
\mathcal{V} \\
V_\Omega
\end{bmatrix},
\]

(4.4)

\[
\tau_\Omega = \begin{bmatrix}
B \\
\Pi_{\mathcal{D}_B} \mathcal{D}_B
\end{bmatrix} : \mathcal{H} \rightarrow \begin{bmatrix}
\mathcal{V} \\
V_\Omega
\end{bmatrix}.
\]

(4.5)

Since \( V \) on \( \mathcal{V} = \mathcal{H}' \oplus H^2(\mathcal{D}_\mathcal{F}') \) is an isometric lifting of \( T' \), the operator \( U_\Omega \) is an isometric lifting of \( T' \), and \( A = \Pi_{\mathcal{D}_B'} \mathcal{D}_B \). It follows that \( \{ U_\Omega, \tau_\Omega \} \) is an isometric coupling of \( \{ T', A \} \). Notice that \( \mathcal{V} \oplus \tau_\Omega \mathcal{H} = \mathcal{V} \oplus \mathcal{D}_B \). Because \( \mathcal{H}' \) is cyclic for \( V \) and \( \mathcal{D}_B \) is cyclic for \( V_\Omega \), the reducing decomposition of \( U_\Omega \) in (1.3) shows that \( \mathcal{H}' \oplus \tau_\Omega \mathcal{H} \) is cyclic for \( U_\Omega \). In other words, the isometric coupling \( \{ U_\Omega, \tau_\Omega \} \) is minimal. Since \( V \mathcal{B} \mathcal{R} = B \mathcal{Q} \), the construction of \( U_\Omega \) and \( \tau_\Omega \) implies that

\[
U_\Omega \tau_\Omega = \tau_\Omega Q \quad \text{and} \quad B = \Pi_{\mathcal{D}_S} \tau_\Omega.
\]

Indeed, for \( h \in \mathcal{H}_0 \) we have

\[
U_\Omega \tau_\Omega R h = \begin{bmatrix}
V & 0 \\
0 & V_\Omega
\end{bmatrix} \begin{bmatrix}
B R h \\
\mathcal{D}_B R h
\end{bmatrix} = \begin{bmatrix}
V B R h \\
V_\Omega D_B R h
\end{bmatrix}.
\]

However, \( D_B R h \in \tilde{\mathcal{F}} \), and hence \( V_\Omega D_B R h = \tilde{\omega} D_B R h = \mathcal{D}_B Q h \), which follows from the definition of \( \tilde{\omega} \). Since, by assumption, \( V \mathcal{B} \mathcal{R} = B \mathcal{Q} h \), we see that

\[
U_\Omega \tau_\Omega R h = \begin{bmatrix}
B \mathcal{Q} h \\
\mathcal{D}_B \mathcal{Q} h
\end{bmatrix} = \tau_\Omega \mathcal{Q} h,
\]

which proves the first identity in (4.4). The second is clear from the definition of \( \tau_\Omega \).

From the construction of \( U_\Omega \) it follows that the unique isometry \( \Phi_\Omega \) associated with \( T' \) that intertwines \( V \) with \( U_\Omega \) is equal to \( \Pi_{\mathcal{D}_S} \), where \( \Pi_{\mathcal{D}_S} \) is the orthogonal projection of \( \mathcal{V} \oplus V_\Omega \) onto \( \mathcal{V} \). This together with the second identity in (4.4) yields

\[
B = \Pi_{\mathcal{D}_S} \tau_\Omega = \Phi_\Omega^* \tau_\Omega.
\]

By Proposition 3.4.4 and the fact that minimality of isometric couplings is preserved under isomorphisms, there exists a minimal special isometric coupling \( \{ U, \tau \} \) of \( \{ T', A \} \) which is isomorphic to \( \{ U_\Omega, \tau_\Omega \} \). Using Lemma 3.3 and formula (1.17) we obtain \( B = \Phi^* \tau \), where \( \Phi \) is the unique isometry associated with \( T' \) that intertwines \( V \) with \( U \).

It remains to prove that \( U \tau R = \tau Q \). Let \( \Psi \) be the isomorphism that transforms \( \{ U_\Omega, \tau_\Omega \} \) into \( \{ U, \tau \} \). In particular, \( \Psi \tau_\Omega = \tau \). Moreover formula (3.1) yields \( \Psi U_\Omega \tau_\Omega = U \tau \). Since \( U_\Omega \tau_\Omega R = \tau_\Omega Q \), it follows that \( U \tau R = \Psi U_\Omega \tau_\Omega R = \Psi \tau_\Omega Q = \tau Q \). \[ \square \]
Proposition 4.3. Let \( \{A, T', V, R, Q\} \) be a lifting data set. Assume that \( R^*R = Q^*Q \), and let \( \omega \) be the unitary operator defined in (4.3). Consider a minimal special isometric coupling \( \{U, \tau\} \) of \( \{T', A\} \), and let \( F_{\{U, \tau\}} \) be the function in the Schur class \( S(D_A, D_T' \oplus D_A) \) defined by (3.10). Then \( U \tau R = \tau Q \) if and only if \( F_{\{U, \tau\}}(0)|F' = \omega^* \). In particular, there exists a special isometric coupling \( \{U, \tau\} \) of \( \{T', A\} \) satisfying \( U \tau R = \tau Q \).

It will be convenient first to prove the following lemma.

Lemma 4.4. Let \( \{A, T', U', R, Q\} \) be a lifting data set satisfying \( R^*R = Q^*Q \). Let \( \{U, \tau\} \) be a special isometric coupling of \( \{T', A\} \), and consider its operator matrix representation of the form

\[
U = \begin{bmatrix} T' & 0 \\ Y_1 D_{T'} & Y_2 \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} H' \\ \mathcal{M} \end{bmatrix} \quad \text{where} \quad Y = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} : \begin{bmatrix} D_{T'} \\ \mathcal{M} \end{bmatrix} \to \mathcal{M}
\]

is an isometry. Then \( U \tau R = \tau Q \) if and only if \( Y|F' = \omega^* \).

Proof. Since the coupling is special, the space \( D_A \) is a subspace of \( \mathcal{M} \) and

\[
\tau = \begin{bmatrix} A \\ \Pi_{D_A} D_A \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} H' \\ \mathcal{M} \end{bmatrix}.
\]

It follows that for \( h \) in \( \mathcal{H}_0 \), we have

\[
U \tau Rh = \begin{bmatrix} T' & 0 \\ Y_1 D_{T'} & Y_2 \end{bmatrix} \begin{bmatrix} ARh \\ D_A Rh \end{bmatrix} = \begin{bmatrix} T' ARh \\ Y_1 D_{T'} ARh + Y_2 D_A Rh \end{bmatrix}.
\]

Thus

\[
U \tau R = \tau Q \iff Y \begin{bmatrix} D_{T'} ARh \\ D_A Rh \end{bmatrix} = D_A Qh, \quad h \in \mathcal{H}_0.
\]

Since \( Y \) is an isometry, we see that \( U \tau R = \tau Q \) if and only if \( Y|F' \) is a unitary operator from \( F' \) onto \( F \) with the same action as \( \omega^* \). Because of the uniqueness of \( \omega \), this proves the lemma.

Proof of Proposition 4.3. Let \( Y \) be the isometry determined by the operator matrix representation for \( U \) in (4.8), and set \( F = F_{\{U, \tau\}} \). From Lemma 4.4 we know that \( U \tau R = \tau Q \) if and only if \( Y|F' = \omega^* \). Thus we have to show that

\[
F(0)|F = \omega \iff Y|F' = \omega^*.
\]

By consulting (3.10) we see that \( F(0) = \Pi_{D_{T'} \oplus D_A} Y^* \Pi_{D_A} \). The fact that \( F' \subset D_{T'} \oplus D_A \) allows us to view \( \omega \) as an isometry from \( F \) into \( D_{T'} \oplus D_A \). Since \( F \subset D_A \), it follows that the first condition in (4.9) is equivalent to

\[
Y^*|F = \begin{bmatrix} \omega \\ \gamma \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} D_{T'} \oplus D_A \\ \mathcal{X} \end{bmatrix},
\]

where \( \gamma \) is some operator from \( F \) into \( \mathcal{X} \). However, \( \omega \) is an isometry and \( Y^*|F \) is a contraction. This implies that \( \gamma = 0 \). We conclude that the first condition in (4.9) is equivalent to \( Y^*|F = \omega \). By taking adjoints, and using that \( \omega \) is a unitary operator from \( F \) onto \( F' \), we see that the same holds true for the second condition in (4.9).
Finally, for each \( \lambda \in \mathbb{D} \) define the operator \( F(\lambda) \) from \( \mathcal{D}_A \) into \( \mathcal{D}_T \oplus \mathcal{D}_A \) by setting 
\[ F(\lambda)d = \omega \Pi_{\mathcal{F}}d \] for \( d \in \mathcal{D}_A \). Here \( \Pi_{\mathcal{F}} \) is the orthogonal projection of \( \mathcal{D}_A \) onto \( \mathcal{F} \). Then \( F \) belongs to the Schur class \( \mathcal{S}(\mathcal{D}_A, \mathcal{D}_T \oplus \mathcal{D}_A) \). Hence, by Theorem 3.3, there exists a minimal special isometric coupling \( \{U, \tau\} \) of \( \{T', A\} \) such \( F = F(U, \tau) \). Since \( F(0)|\mathcal{F} = \omega \), we conclude that \( U\tau R = \tau Q \).

**Proof of Theorem 4.1** We split the proof into two parts.

**Part 1.** Assume that \( B \) is a contractive interpolant for the data set \( \{A, T', V, R, Q\} \). Since \( R^*R = Q^*Q \), we know from Proposition 4.2 that there exists a (minimal) special isometric coupling \( \{U, \tau\} \) of \( \{T', A\} \) such that \( U\tau R = \tau Q \) and \( B = \Phi^*\tau \), where \( \Phi \) is the unique isometry from \( \mathcal{V} = \mathcal{H}' \oplus H^2(\mathcal{D}_T) \) into \( \mathcal{K} \) associated with \( T' \) intertwining \( V \) with \( U \). Now write
\[ U = \begin{bmatrix} T' & 0 \\ Y_1 D_{T'} & Y_2 \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{H}' \\ \mathcal{M} \end{bmatrix}, \]
where \( \mathcal{M} = \mathcal{K} \oplus \mathcal{H}' \). Since \( \{U, \tau\} \) is special, we have \( \mathcal{D}_A \subset \mathcal{M} \), and
\[ \tau = \begin{bmatrix} A \\ \Pi_{\mathcal{D}_A} D_A \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ \mathcal{M} \end{bmatrix}. \]
The identity \( B = \Phi^*\tau \) and the formula for \( \Phi \) in Theorem 1.1 show that \( B = A \oplus \Lambda^* \Pi_{\mathcal{D}_A}^* D_A \), where
\[ (\Lambda^* m)(\lambda) = Y_1^*(I - \lambda Y_2^*)^{-1}, \quad m \in \mathcal{M} \quad \text{and} \quad \lambda \in \mathbb{D}. \]
It follows that
\[ Bh = \begin{bmatrix} Ah \\ Y_1^*(I - \lambda Y_2^*)^{-1} \Pi_{\mathcal{D}_A} D_A h \end{bmatrix}, \quad h \in \mathcal{H}. \]
To obtain the expression for \( B \) given in (1.2) we apply Proposition 2.3 with \( \mathcal{D} = \mathcal{D}_A \) and \( \mathcal{D}' = \mathcal{D}_T \). It follows that (1.2) holds with \( F \in \mathcal{S}(\mathcal{D}_A, \mathcal{D}_T \oplus \mathcal{D}_A) \) given by
\[ F(\lambda) = \Pi_{\mathcal{D}_T \oplus \mathcal{D}_A} Y^*(I - \lambda J_{\mathcal{X}} Y^*)^{-1}, \quad \lambda \in \mathbb{D}. \]
Here \( \mathcal{X} = \mathcal{M} \oplus \mathcal{D}_A \), the operator \( \Pi_{\mathcal{D}_T \oplus \mathcal{D}_A} \) is the orthogonal projection of \( \mathcal{D}_T \oplus \mathcal{M} \) onto \( \mathcal{D}_T \oplus \mathcal{D}_A \), and
\[ J_{\mathcal{X}} = \begin{bmatrix} 0 & P_{\mathcal{X}} \end{bmatrix} : \begin{bmatrix} \mathcal{D}_T \oplus \mathcal{M} \end{bmatrix} \to \mathcal{M}, \quad Y = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} : \begin{bmatrix} \mathcal{D}_T \oplus \mathcal{M} \end{bmatrix} \to \mathcal{M}. \]
In other words, using the terminology introduced in Theorem 3.3, we have \( F = F(U, \tau) \). Since \( U\tau R = \tau Q \), Proposition 4.3 shows that \( F(0)|\mathcal{F} = \omega \), which completes the first part of the proof.

**Part 2.** Let \( F \) be any function in \( \mathcal{S}(\mathcal{D}_A, \mathcal{D}_T \oplus \mathcal{D}_A) \) satisfying \( F(0)|\mathcal{F} = \omega \). We have to show that \( B \) defined by (1.2) is a contractive interpolant for the given data set. According to Theorem 3.3 there is a minimal special isometric coupling \( \{U, \tau\} \) of \( \{T', A\} \) such \( F = F(U, \tau) \), where \( F(U, \tau) \) is defined by (3.10). The fact that \( F(0)|\mathcal{F} = \omega \) yields \( U\tau R = \tau Q \), by Proposition 4.3.

Since \( B \) is given by (1.2), we can use Proposition 2.3 (with \( \mathcal{D} = \mathcal{D}_A \) and \( \mathcal{D}' = \mathcal{D}_T \)) and Theorem 1.1 to show that \( B = \Phi^*\tau \), where \( \Phi \) is the unique isometry associated with \( T' \) intertwining \( V \) (the Sz.-Nagy-Schäffer minimal isometric lifting of \( T' \)) with \( U \). This allows us to apply Proposition 4.3 to show that \( B \) is a contractive interpolant. \( \square \)
5. Proof of the First Main Theorem

In this section we shall prove Theorem 5.1. The proof will be based on the analogous result for the case when \( R^*R = Q^*Q \), which was proved in the preceding section, and on Proposition 5.1 below, which allows us to reduce the general case to the case when \( R^*R = Q^*Q \).

Throughout this section \( \{A, T', V, R, Q\} \) is a lifting data set with \( V \) being the Sz.-Nagy-Schäffer minimal isometric lifting of \( T' \). As before, put \( D_o = \overline{D_oH} \), where \( D_o \) is the positive square root of \( Q^*Q - R^*R \). Introduce the following operators:

\[
A_o = \begin{bmatrix} A & 0 \\ 0 & I_{D_o} \end{bmatrix} : \mathcal{H} \oplus D_o \rightarrow \mathcal{H}' \oplus D_o, \quad T_o' = \begin{bmatrix} T' & 0 \\ 0 & 0 \end{bmatrix} \text{ on } \mathcal{H}' \oplus D_o,
\]

\[
R_o = \begin{bmatrix} R \\ D_o \end{bmatrix} : \mathcal{H}_o \rightarrow \mathcal{H}' \oplus D_o, \quad Q_o = \begin{bmatrix} Q \\ 0 \end{bmatrix} : \mathcal{H}_o \rightarrow \mathcal{H}' \oplus D_o,
\]

\[
\tilde{V} = \begin{bmatrix} T' & 0 & 0 & 0 \\ 0 & S_{D_o} & 0 & 0 \\ E_{D_o} & 0 & S_{D_o} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{H}' \\ D_o \\ H^2(D_o) \end{bmatrix}.
\]

Here \( E_{D_o}, E_{D_o}' \) and \( D_o \) are the canonical embeddings of \( D_o \) and \( D_o \) onto the spaces of constant functions of \( H^2(D_o) \) and \( H^2(D_o) \), respectively, and \( S_{D_o} \) and \( S_{D_o}' \) are the forward shifts on \( H^2(D_o) \) and \( H^2(D_o) \), respectively. Identifying \( H^2(D_o) \oplus H^2(D_o) \) it is straightforward to check that \( \tilde{V} \) is the Sz.-Nagy-Schäffer minimal isometric lifting of \( T' \), and that the quintet

(5.1) \[ \{A_o, T_o', \tilde{V}, R_o, Q_o\} \]

is a lifting data set satisfying \( R_o^*R_o = Q_o^*Q_o \).

**Proposition 5.1.** If \( \tilde{B} \) from \( \mathcal{H} \oplus D_o \) to \( \mathcal{H}' \oplus D_o \oplus H^2(D_o) \) is a contractive interpolant for the data set (5.1), then the operator \( B \) from \( \mathcal{H} \) to \( \mathcal{H}' \oplus H^2(D_o) \), defined by

(5.2) \[ B = \Pi_{\mathcal{H}' \oplus H^2(D_o)} \tilde{B} \Pi_{\mathcal{H}'}, \]

is a contractive interpolant for the data set \( \{A, T, V, R, Q\} \), and all contractive interpolants for \( \{A, T, V, R, Q\} \) are obtained in this way.

**Proof.** Let \( \tilde{B} \) be a contractive interpolant for the data set (5.1). Then \( \tilde{B} \) is of the following form

(5.3) \[ \tilde{B} = \begin{bmatrix} A & 0 \\ 0 & I_{D_o} \\ \Gamma_1 D_A & 0 \\ \Gamma_2 D_A & 0 \end{bmatrix} : \mathcal{H} \oplus D_o \rightarrow \mathcal{H}' \oplus H^2(D_o), \]

where

\[ \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} : D_A \rightarrow \begin{bmatrix} H^2(D_o) \\ H^2(D_o) \end{bmatrix} \]

is a contraction.

Moreover, \( \tilde{V} \tilde{B} R_o = \tilde{B} Q_o \). Now, using this \( \tilde{B} \), let \( B \) be the operator defined by (5.2). In other words

\[ B = \begin{bmatrix} A \\ \Gamma_1 D_A \end{bmatrix} : \mathcal{H} \rightarrow \mathcal{H}' \oplus H^2(D_o). \]
By virtue of \( \hat{V} \hat{B} R_o = \hat{B} Q_o \) it follows that
\[
\begin{bmatrix}
T' & 0 \\
E_{D_{T'}}, D_{T'} & S_{D_{T'}}
\end{bmatrix}
\begin{bmatrix}
AR \\
\Gamma_1 D_A R
\end{bmatrix}
= \begin{bmatrix}
AQ \\
\Gamma_1 D_A Q
\end{bmatrix}.
\]
Thus \( B \) is a contraction, \( A = \Pi_{H_o} B \) and \( V B R = B Q \), that is, \( B \) is a contractive interpolant for the data set \( \{A, T', V, R, Q\} \).

Next, let \( B \) from \( \mathcal{H} \) to \( \mathcal{V} = \mathcal{H}' \oplus H^2(D_{T'}) \) be an arbitrary contractive interpolant for the data set \( \{A, T', V, R, Q\} \). We have to show that \( B \) is given by (5.2), where \( \hat{B} \) is some contractive interpolant for the data set (5.1). In fact, from (5.3) we see that it suffices to find a contraction \( \Gamma \) from \( D_{B} \) into \( H^2(D_o) \) such that the operator \( \hat{B} \), given by
\[
(5.4) \quad \hat{B} = \begin{bmatrix}
B & 0 \\
0 & I_{D_{o}} \\
\Gamma D_B & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{H} \\
D_{o}
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{V} \\
D_{o} \oplus H^2(D_o)
\end{bmatrix},
\]
satisfies the intertwining relation \( \hat{W} B R_o = \hat{B} Q_o \), where \( \hat{W} \) is the operator which one obtains by interchanging the second and the third column and the second and third row in the operator matrix for \( \hat{V} \). Put
\[
(5.5) \quad B_o = \begin{bmatrix}
B & 0 \\
0 & I_{D_{o}} \\
\Gamma D_B & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{H} \\
D_{o}
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{V} \\
D_{o} \oplus H^2(D_o)
\end{bmatrix}, \quad V_o = \begin{bmatrix}
V & 0 \\
0 & 0
\end{bmatrix} \quad \text{on} \quad \begin{bmatrix}
\mathcal{V} \\
D_{o}
\end{bmatrix}.
\]
Since \( V B R = B Q \), we have \( V_o B_o R_o = B_o Q_o \). Now, notice that \( B_o = B \oplus I_{D_{o}} \) is a contraction. Furthermore, \( V_o \) is a partial isometry, and the Sz.-Nagy-Schäffer minimal isometric lifting of \( V_o \) is equal to \( \hat{W} \). Thus
\[
(5.6) \quad \{B_o, V_o, \hat{W}, R_o, Q_o\}
\]
is a lifting data set. Since \( R_o^* R_o = Q_o^* Q_o \), we know from Theorem 4.1 that the data set (5.6) has a contractive interpolant \( \hat{B} \). By identifying the spaces
\[
\mathcal{H}' \oplus D_{o} \oplus H^2(D_{T'}) \oplus H^2(D_o) \quad \text{and} \quad \mathcal{H} \oplus H^2(D_{T'}) \oplus D_{o} \oplus H^2(D_o)
\]
one sees that this operator \( \hat{B} \) is also a contractive interpolant for the data set (5.1), and from (5.4) it follows that with this choice of \( \hat{B} \) the identity (5.2) holds. \( \square \)

**Proof of Theorem 0.1.** We split the proof into two parts.

**Part 1.** Let \( B \) be a contractive interpolant for the data set \( \{A, T', V, R, Q\} \). Then \( B \) is of the form (5.2) for some contractive interpolant \( \hat{B} \) for \( \{A_o, T'_o, \hat{V}, R_o, Q_o\} \). Since \( R_o^* R_o = Q_o^* Q_o \), we can use Theorem 4.1 to find a formula for \( \hat{B} \). To write this formula, we need the subspaces
\[
\mathcal{F}_o = \overline{D_{A_o} Q_o \mathcal{H}_o} \quad \text{and} \quad \mathcal{F}_o' = \overline{\begin{bmatrix} D_{T'_o} A_o R_o \\ D_{A_o} R_o \end{bmatrix} \mathcal{H}_o},
\]
and the unitary operator \( \omega_o \) from \( \mathcal{F}_o \) onto \( \mathcal{F}_o' \) given by
\[
\omega_o D_{A_o} Q_o = \begin{bmatrix} D_{T'_o} A_o R_o \\ D_{A_o} R_o \end{bmatrix}.
\]
In this setting,
\[
(5.7) \quad D_{A_o} = \begin{bmatrix} D_A & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D_{T'_o} = \begin{bmatrix} D_{T'} & 0 \\ 0 & I_{D_o} \end{bmatrix}.
\]
A straightforward computation shows that
\[ D_{A_0}Q_0 = \begin{bmatrix} D_A Q_0 \\ 0 \end{bmatrix} : H_0 \to \begin{bmatrix} \mathcal{H} \\ \mathcal{D}_0 \end{bmatrix} \]
and
\[ \begin{bmatrix} D_{T_2}A_0R_0 \\ D_{A_0}R_0 \end{bmatrix} = \begin{bmatrix} D_{T_1}AR \\ D_0 \\ D_A R \\ 0 \end{bmatrix} : H_0 \to \begin{bmatrix} \mathcal{H}' \\ \mathcal{D}_0 \\ \mathcal{H} \\ \mathcal{D}_0 \end{bmatrix}. \]

By interchanging in the last column the first two coordinate spaces and identifying the vector \( x \oplus 0 \) with the vector \( x \), we see that
\[(5.8) \quad \mathcal{F}_0 = \mathcal{F}, \quad \mathcal{F}_0' = \mathcal{F}' \quad \text{and} \quad \omega_0 = \omega, \]
where the subspaces \( \mathcal{F} \) and \( \mathcal{F}' \) and the unitary operator \( \omega \) are defined in Section 4.1. Let us now apply Theorem 4.1 to \( \tilde{\mathcal{F}} \). It follows that
\[(5.9) \quad \tilde{\mathcal{B}}x = \begin{bmatrix} A_0x \\ \Pi_{T_2'}F_0(\lambda)(I - \lambda \Pi_A F_0(\lambda))^{-1}D_{A_0}x \end{bmatrix}, \quad x \in \begin{bmatrix} \mathcal{H} \\ \mathcal{D}_0 \end{bmatrix}, \]
where \( F_0 \in \mathbf{S}(\mathcal{D}_{A_0}, \mathcal{D}_{T_2} \oplus \mathcal{D}_{A_0}) \) satisfies \( F_0(0)|\mathcal{F} = \omega_0 \). Here
\[ \Pi_{T_2'} = \begin{bmatrix} I & 0 \end{bmatrix} \quad \text{and} \quad \Pi_{A_0} = \begin{bmatrix} 0 & I \end{bmatrix} \quad : \begin{bmatrix} \mathcal{D}_{T_2} \\ \mathcal{D}_{A_0} \end{bmatrix} \to \mathcal{D}_{A_0}. \]

From (5.7) we see that we can identify in a canonical way \( \mathcal{D}_{A_0} \) with \( \mathcal{D}_A \), and \( \mathcal{D}_{T_2} \) with \( \mathcal{D}_0 \oplus \mathcal{D}_{T_2'} \). This together with (5.8) shows that we can view \( F_0 \) as a function \( F \) from the Schur class \( \mathbf{S}(\mathcal{D}_{A_0}, \mathcal{D}_0 \oplus \mathcal{D}_{T_2'} \oplus \mathcal{D}_A) \) satisfying \( F(0)|\mathcal{F} = \omega \) and
\[(5.10) \quad \hat{\mathcal{B}} \begin{bmatrix} h \\ d_0 \end{bmatrix} = \begin{bmatrix} Ah \\ 0 \\ \Pi_{T_2'}F_0(\lambda)(I - \lambda \Pi_A F_0(\lambda))^{-1}D_{A_0}h \\ 0 \\ \Pi_{D_0}F_0(\lambda)(I - \lambda \Pi_A F_0(\lambda))^{-1}D_{A_0}h \end{bmatrix}, \quad \begin{bmatrix} h \\ d_0 \end{bmatrix} \in \begin{bmatrix} \mathcal{H} \\ \mathcal{D}_0 \end{bmatrix}. \]

Here \( \Pi_{T_2'} \) and \( \Pi_A \) are the projections given by (0.4) and
\[ \Pi_{D_0} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{D}_0 \\ \mathcal{D}_{T_2} \\ \mathcal{D}_A \end{bmatrix} \to \mathcal{D}_0. \]

Since \( B \) is obtained from \( \hat{B} \) via (5.2), we conclude that \( B \) has the desired form (0.5).

**Part 2.** The reverse implication is proved in a similar way. Indeed, assume that \( B \) is given by (15.5), where \( F \in \mathbf{S}(\mathcal{D}_A, \mathcal{D}_0 \oplus \mathcal{D}_{T'} \oplus \mathcal{D}_A) \) satisfies \( F(0)|\mathcal{F} = \omega \). Using the identifications made in the first part of the proof, we can view \( F \) as a function \( F_0 \in \mathbf{S}(\mathcal{D}_{A_0}, \mathcal{D}_{T_2} \oplus \mathcal{D}_{A_0}) \) satisfying \( F_0(0)|\mathcal{F}_0 = \omega_0 \). But then we can use Theorem 4.1 to show that \( \hat{B} \) defined by (5.9) is a contractive interpolant for the data set \( \{A_0, T'_2, \check{V}, R_0, Q_0\} \). Since \( \hat{B} \) is also given by (5.10), we conclude that \( B \) and \( \hat{B} \) are related as in (5.2). Thus Proposition 5.1 implies that \( B \) is a contractive interpolant for \( \{A, T', V, R, Q\} \).
6. Parameterization and uniqueness of solutions

In this section we prove the second main theorem (Theorem 6.1). As a consequence of this theorem we obtain conditions on the lifting data set \{A, T', V, R, Q\} guaranteeing that the parameterization in Theorem 0.1 is proper, that is, conditions on \{A, T', V, R, Q\} implying that for every contractive interpolant \(B\) for \{A, T', V, R, Q\} there exists a unique \(F\) in \(S(D_A, D_o \oplus D_T \oplus D_A)\) with \(F(0) = \omega\) such that \(B = B_F\). Here \(B_F\) is the contractive interpolant for \{A, T', V, R, Q\} produced by the Schur class function \(B\) from \(S(D_A, D_o \oplus D_T \oplus D_A)\) with \(F(0) = \omega\) as in Theorem 0.1 that is,

\[
B_F h = \begin{bmatrix}
\Pi_T F(\lambda) (I_{D_A} - \lambda \Pi_A F(\lambda))^{-1} D_A h
\end{bmatrix}, \quad h \in \mathcal{H}.
\]

We shall also present conditions on \{A, T', V, R, Q\} implying the existence of a unique interpolant for \{A, T', V, R, Q\}.

To shorten the notation in this section we define

\[
\mathcal{V} = \mathcal{H}' \oplus H^2(D_T) \quad \text{and} \quad \mathcal{V}' = \mathcal{H}' \oplus \mathcal{D}_o \oplus H^2(D_T) \oplus H^2(D_o).
\]

Also, for a given contractive interpolant \(B\) for \{A, T', V, R, Q\} we define the spaces \(\mathcal{F}_B\) and \(\mathcal{F}'_B\) by

\[
\mathcal{F}_B = \mathcal{D}_B Q_{\mathcal{H}_0} \quad \text{and} \quad \mathcal{F}'_B = \begin{bmatrix}
D_o
D_BR
\end{bmatrix}_{\mathcal{H}_0}.
\]

Notice that \(\mathcal{G}_B\) and \(\mathcal{G}'_B\) in (6.1) are then given by

\[
\mathcal{G}_B = \mathcal{D}_B \oplus \mathcal{F}_B \quad \text{and} \quad \mathcal{G}'_B = (\mathcal{D}_o \oplus \mathcal{D}_B) \ominus \mathcal{F}_B.
\]

With the above notation and definitions we can reformulate Theorem 0.2 as follows.

**Theorem 6.1.** Let \{A, T', V, R, Q\} be a lifting data set with \(V\) the Sz.-Nagy-Schäffer minimal isometric lifting of \(T'\), and let \(B\) be a contractive interpolant for the data set \{A, T', V, R, Q\}. Then there exists a one to one mapping from the set of all \(F\) in \(S(D_A, D_o \oplus D_T \oplus D_A)\) with \(F(0) = \omega\) such that \(B = B_F\) onto the Schur class \(S(\mathcal{G}_B, \mathcal{G}'_B)\), with \(\mathcal{G}_B\) and \(\mathcal{G}'_B\) as in (6.4).

For the proof of Theorem 6.1 it will be convenient to first prove two lemmas. Let \{A_o, T_o', \tilde{V}, R_o, Q_o\} be as defined in Section 5. Given a contractive interpolant \(B\) for the data set \{A, T', V, R, Q\}, we let \(B_o\) and \(V_o\) be the operators defined by (5.5). Furthermore, as in the previous section, we define \(\tilde{W}\) to be the operator which one obtains by interchanging the second and the third column and the second and third row in the operator matrix for \(\tilde{V}\). Recall that both \{A_o, T_o', \tilde{V}, R_o, Q_o\} and \{B_o, V_o, \tilde{W}, R_o, Q_o\} are lifting data sets. From the construction of \(\tilde{W}\) from \(\tilde{V}\) we see that both \(\tilde{W}\) and \(\tilde{V}\) are minimal isometric liftings of both \(T_o'\) and \(V_o\).

**Lemma 6.2.** Let \(B\) be a contractive interpolant for \{A, T', V, R, Q\} and let the pair \{U on \(\mathcal{K}, \tau\)\} be an isometric coupling of \{V_o, B_o\}. Then

(i) the pair \{U, \tau\} is an isometric coupling of \{T_o', A_o\};

(ii) the pair \{U, \tau\} is minimal as an isometric coupling of \{V_o, B_o\} if and only if \(U, \tau\) is minimal as an isometric coupling of \{T_o', A_o\};

(iii) the operator \(U\) is an isometric lifting of both \(T'\) and \(T_o'\); moreover, \(\Phi_{U, T'}\) is the canonical embedding of \(V\) into \(\mathcal{K}\), and \(\Phi_{U, T_o'} v = v\) for all \(v \in V\);

(iv) the contractive interpolant \(B = \Pi_Y^* \Phi_{U, T_o'} \tau \Pi_{\mathcal{H}}^*\).

Furthermore, two isometric couplings \( \{U_1, \tau_1\} \) and \( \{U_2, \tau_2\} \) of \( \{V_o, B_o\} \) are isomorphic as isometric couplings of \( \{V_o, B_o\} \) if and only if they are isomorphic as isometric couplings of \( \{T'_{o}, A_{o}\} \).

**Proof.** First remark that \( K \) can be decomposed as \( V \oplus D_o \oplus M \) for some Hilbert space \( M \). Relative to this direct sum decomposition the operators \( U \) and \( \tau \) admit operator matrix representations of the form

(6.5)

\[
U = \begin{bmatrix} V & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} V \\ D_o \\ M \end{bmatrix} \quad \text{and} \quad \tau = \begin{bmatrix} B & 0 \\ 0 & I_{D_o} \end{bmatrix} : \begin{bmatrix} H \\ D_o \\ M \end{bmatrix} \rightarrow \begin{bmatrix} V \\ D_o \\ M \end{bmatrix},
\]

where * represents operators which are not specified any further.

(i) Because \( V \) is an isometric lifting of \( T' \) and \( U \) is an isometric lifting of \( V \), as we can see from (6.5), we obtain that \( U \) is an isometric lifting of \( T' \). From (6.6) we can immediately see that \( U \) also is an isometric lifting of the zero operator on \( D_o \). Hence \( U \) is an isometric lifting of \( T'_o \). Since \( \tau \) is as in (6.5) and \( \Pi_{H_{o}}B = A \), we see that \( \Pi_{H_{o}}\tau = A \). So \( \{U, \tau\} \) is an isometric coupling of \( \{T'_o, A_o\} \).

(ii) Assume that \( \{U, \tau\} \) is minimal as an isometric coupling of \( \{V_o, B_o\} \), and thus that the space \( (V \oplus D_o) \vee (H \oplus D_o) \) is cyclic for \( U \).

Notice that in general we have for every operator \( W \) on a Hilbert space \( \mathcal{L} \) with \( U \) and \( V \) subspaces \( \mathcal{L} \) that

\[
\bigwedge_{n=0}^{\infty} W^n(U \vee V) = (\bigwedge_{n=0}^{\infty} W^nU) \vee (\bigwedge_{n=0}^{\infty} W^nV).
\]

Applying this with \( U \) in (6.5), the fact that \( V \) on \( V \) is a minimal isometric lifting of \( T' \) and the fact that \( (V \oplus D_o) \vee (H \oplus D_o) \) is cyclic for \( U \) yield

\[
K = \bigwedge_{n=0}^{\infty} U^n((V \oplus D_o) \vee (H \oplus D_o))
\]

\[
= (\bigwedge_{n=0}^{\infty} U^nV) \vee (\bigwedge_{n=0}^{\infty} U^n(D_o \vee (H \oplus D_o)))
\]

\[
= V \vee (\bigwedge_{n=0}^{\infty} U^n(D_o \vee (H \oplus D_o)))
\]

\[
= (\bigwedge_{n=0}^{\infty} \Pi_{H} V^nH') \vee (\bigwedge_{n=0}^{\infty} U^n(D_o \vee (H \oplus D_o)))
\]

\[
= (\bigwedge_{n=0}^{\infty} U^nH') \vee (\bigwedge_{n=0}^{\infty} U^n(D_o \vee (H \oplus D_o)))
\]

\[
= \bigwedge_{n=0}^{\infty} U^n((H' \oplus D_o) \vee (H \oplus D_o)).
\]

Hence \( (H' \oplus D_o) \vee (H \oplus D_o) \) is cyclic for \( U \), and thus \( \{U, \tau\} \) is minimal as an isometric coupling of \( \{T'_o, A_o\} \).

Conversely, assume that \( \{U, \tau\} \) is minimal as an isometric coupling of \( \{T'_o, A_o\} \). In other words, \( (H' \oplus D_o) \vee (H \oplus D_o) \) is cyclic for \( U \). Note that \( H' \) is a subspace of \( V \) and hence \( (H' \oplus D_o) \vee (H \oplus D_o) \) is a subspace of \( (V \oplus D_o) \vee (H \oplus D_o) \).
This implies that $(V \oplus D_o) \vee \tau(H \oplus D_o)$ is cyclic for $U$ as well. Thus $\{U, \tau\}$ is minimal as an isometric coupling of $\{V_o, B_o\}$. 

(iii) We already showed, in (i), that $U$ is an isometric lifting of both $T'$ and $T'_o$. From (6.5) we see that $V$ is the minimal isometric lifting of $T'$ associated with $U$ and thus, using the remark in the final paragraph of Section 1, we obtain that

$$\Phi_{U,T'} = \Pi_V^* \Phi_{V,T'} = \Pi_V^* I_V = \Pi_V^*.$$ 

Because $\Phi_{U,T'}$ is the unique isometry associated with $T'_o$ that intertwines $\hat{V}$ with $U$, we see that the isometry $\Phi_{U,T'_o} \Pi_V^*$ satisfies $\Phi_{U,T'_o} \Pi_V^* h = h$ for all $h \in H'$ and

$$U \Phi_{U,T'_o} \Pi_V^* = \Phi_{U,T'_o} \hat{V} \Pi_V^* = \Phi_{U,T'_o} \Pi_V^* V.$$ 

Hence $\Phi_{U,T'_o} \Pi_V^*$ is the unique isometry associated with $T'$ that intertwines $V$ and $U$. Thus for all $v \in V$ we have

$$\Phi_{U,T'_o} v = \Phi_{U,T'_o} \Pi_V^* v = \Phi_{U,T'} v = v.$$ 

(iv) In the proof of (iii) we saw that $\Phi_{U,T'_o} \Pi_V^* = \Phi_{U,T'_o} = \Pi_V^*$. Hence from (6.5) we obtain that

$$B = \Pi_V \tau \Pi_H^* = \Pi_V \Phi_{U,T'_o} \tau \Pi_H^*.$$ 

It remains to prove the final statement of the lemma. For this purpose, let $\{U_1\}$ on $K_1, \tau_1$ and $\{U_2\}$ on $K_2, \tau_2$ be isometric couplings of $\{V_o, B_o\}$. 

First assume that $\{U_1, \tau_1\}$ and $\{U_2, \tau_2\}$ are isomorphic as isometric couplings of $\{V_o, B_o\}$. We can immediately see from the definition of an isomorphism and the fact that $H' \oplus D_o$ is a subspace of $V \oplus D_o$, that every isomorphism from $\{U_1, \tau_1\}$ to $\{U_2, \tau_2\}$ as isometric couplings of $\{V_o, B_o\}$ also is an isomorphism from $\{U_1, \tau_1\}$ to $\{U_2, \tau_2\}$ as isometric couplings of $\{T'_o, A_o\}$. Hence $\{U_1, \tau_1\}$ and $\{U_2, \tau_2\}$ are isomorphic as isometric couplings of $\{T'_o, A_o\}$. 

Conversely, assume that $\{U_1, \tau_1\}$ and $\{U_2, \tau_2\}$ are isomorphic as isometric couplings of $\{T'_o, A_o\}$ and that $\Psi$ is an isomorphism from $\{U_1, \tau_1\}$ to $\{U_2, \tau_2\}$. Then $\Psi \Pi_V^*$ is an isometry from $V$ to $K_2$ with $\Psi \Pi_V^* h = h$ for each $h \in H'$. Since $V$ is the minimal isometric lifting of $T'$ associated with $U_1$, we obtain

$$U_2 \Psi \Pi_V^* = \Psi U_1 \Pi_V^* = \Psi \Pi_V^* V.$$ 

Thus with (iii) we see that

$$\Psi \Pi_V^* = \Phi_{U_2,T'_o} = \Pi_V^*.$$ 

Since $\Psi$ is an isomorphism between isometric couplings of $\{T'_o, A_o\}$, the operator $\Psi$ is the identity on $H' \oplus D_o$. In particular, $\Psi d = d$ for each $d \in D_o$. Hence $\Psi$ also is an isomorphism from $\{U_1, \tau_1\}$ to $\{U_2, \tau_2\}$ as isometric couplings of $\{V_o, B_o\}$. 

**Lemma 6.3.** Let $B$ be a contractive interpolant for the data set $\{A, T', V, R, Q\}$, and let $\{U$ on $K, \tau\}$ be an isometric coupling of $\{T'_o, A_o\}$ such that

$$B = \Pi_V \Phi_{U,T'_o} \tau \Pi_H^*.$$ 

Then there exists an isometric coupling $\{\hat{U}, \hat{\tau}\}$ of $\{T'_o, A_o\}$, isomorphic to $\{U, \tau\}$, such that $\{\hat{U}, \hat{\tau}\}$ also is an isometric coupling of $\{V_o, B_o\}$. 

**Proof.** From the remark in the last paragraph of Section 1 we can conclude that $\Phi_{U,T'_o} = \Pi_V^* \Phi_{U,T'_o}$, where $U'$ on $K'$ is the minimal isometric lifting of $T'_o$ associated with $U$. Since $U'$ is minimal and $\hat{V}$ is the Sz.-Nagy-Sch"afer minimal isometric lifting
of $T'_o$, we see that the unique isometry $\Phi_{U', T'_o}$ associated with $T'_o$ that intertwines $\tilde{V}$ with $U'$ is unitary. Define $\tilde{\Psi}$ from $\tilde{V} \oplus \mathcal{W}$ to $\mathcal{K}$, where $\mathcal{W} = \mathcal{K} \ominus \mathcal{K}'$, by

$$\tilde{\Psi} = \begin{bmatrix} \Phi_{U', T'_o} & 0 \\ 0 & I_\mathcal{W} \end{bmatrix} : \begin{bmatrix} \tilde{V} \\ \mathcal{W} \end{bmatrix} \rightarrow \mathcal{K}' \mathcal{W}.$$ 

Then $\tilde{\Psi}$ is a unitary operator with $\tilde{\Psi} x = x$ for all $x \in \mathcal{H}' \oplus \mathcal{D}_o$. So the operators $\tilde{U} = \tilde{\Psi}^* U \tilde{\Psi}$ and $\tilde{\tau} = \tilde{\Psi}^* \tau$ form an isometric coupling $\{\tilde{U}, \tilde{\tau}\}$ of $\{T'_o, A_o\}$ that is isomorphic to $\{U, \tau\}$.

Since $\Phi_{U', T'_o}$ intertwines $\tilde{V}$ and $U'$, we obtain that $\tilde{V}$ is the minimal isometric lifting of $T'_o$ associated with $\tilde{U}$. Recall that $\tilde{V}$ also is an isometric lifting of $V_o$. Hence $\tilde{U}$ is an isometric lifting of $V_o$.

Because $V$ is the minimal isometric lifting of $T'_o$ associated with $\tilde{U}$, we have $\Phi_{\tilde{U}, T'_o} = \Pi^*_\mathcal{K}$. Since $\{U, \tau\}$ and $\{U, \tau\}$ are isomorphic, Lemma 3.5 implies that

$$B = \Pi V \Phi_{U, T'} \tilde{\tau} \Pi^*_\mathcal{H} = \Pi V \Phi_{U, T'} \tilde{\tau} \Pi^*_\mathcal{H} = \Pi V \tilde{\tau} \Pi^*_\mathcal{H}.$$

Note that $\Pi V \oplus \mathcal{D}_o \tilde{\tau} = A_o$, and thus, since both $\tilde{\tau}$ and $A_o/\mathcal{D}_o$ are isometries, we get that $\Pi V \oplus \mathcal{D}_o \tilde{\tau} = B_o$. Hence $\{\tilde{U}, \tilde{\tau}\}$ is an isometric coupling of $\{V_o, B_o\}$. □

**Proof of Theorem 6.1.** Let $S_B$ be the set defined by

$$S_B = \{ F \in S(D_A, D_o \oplus D_{T'} \oplus D_A) \mid F(0) = \omega \text{ and } B = B_F \}.$$ 

We have to show that there exists a one to one mapping from $S_B$ onto $S(G_B, G'_B)$.

By applying Theorem 3.4 to the pair $\{T'_o, A_o\}$ and Proposition 4.3 to the lifting data set $\{A_o, T'_o, \tilde{V}, R_o, Q_o\}$, and using the identities in (5.8), we obtain that the mapping

$$\{U, \tau\} \mapsto F_{U, \tau}$$

given by Theorem 3.4 is a one to one mapping from the set of (equivalence classes of) minimal isometric couplings $\{U, \tau\}$ of $\{T'_o, A_o\}$ satisfying $U \tau R_o = \tau Q_o$ onto the set of all functions $F \in S(D_A, D_o \oplus D_{T'} \oplus D_A)$ satisfying $F(0) = \omega$. Moreover, from Proposition 1.2 and Proposition 5.1 applied to $\{A_o, T'_o, \tilde{V}, R_o, Q_o\}$, we obtain that the mapping (6.7) maps the set of (equivalence classes of) minimal isometric couplings $\{U, \tau\}$ of $\{T'_o, A_o\}$ satisfying $U \tau R_o = \tau Q_o$ and $B = \Pi V \Phi_{U, T'} \tau \Pi^*_\mathcal{H}$ onto the set $S_B$ defined by (6.6).

Then, using Lemma 6.2 and Lemma 6.3 we obtain that there exists a one to one mapping from $S_B$ onto the set of (equivalence classes of) minimal isometric couplings $\{U, \tau\}$ of $\{V_o, B_o\}$ satisfying $U \tau R_o = \tau Q_o$.

Note that, because $B$ is a contractive interpolant for $\{A, T', V, R, Q\}$ and thus $VBR = BQ$, we have that $B, V, V, R, Q$ is a lifting data set, and that the lifting data set $\{B_o, V_o, \tilde{W}, R_o, Q_o\}$ is constructed from $\{B, V, V, R, Q\}$ in the same way as we constructed $\{A_o, T'_o, \tilde{V}, R_o, Q_o\}$ from $\{A, T', V, R, Q\}$ in Section 4. Moreover, since $V$ is an isometry and thus $D_V = \{0\}$, we get that $F_B$ and $F'_B$ in (5.3) correspond to $\{B, V, V, R, Q\}$ as $\mathcal{F}$ and $\mathcal{F}'$ correspond to $\{A, T', V, R, Q\}$. Hence there exists a unique unitary operator $\omega_B$ from $\mathcal{F}_B$ to $\mathcal{F}'_B$ defined by

$$\omega_B D_B Q = \begin{bmatrix} D_o \\ D_B R \end{bmatrix}.$$ 

By again applying Theorem 3.4, Proposition 4.3 and the identities in (5.8), but now to the pair $\{V_o, B_o\}$ and the lifting data set $\{B_o, V_o, \tilde{W}, R_o, Q_o\}$, we obtain that
there exists a one to one mapping from the set of (equivalence classes of) minimal isometric couplings \{U, \tau\} of \{B_0, V_0, W, R_0, Q_0\} satisfying \(U\tau R_0 = \tau Q_0\) onto the set of all functions \(H \in S(D_B, D_0 \oplus D_B)\) satisfying \(H(0)|F_B = \omega_B\). Thus there exists a one to one mapping from \(S_B\) onto the set of all functions \(H \in S(D_B, D_0 \oplus D_B)\) satisfying \(H(0)|F_B = \omega_B\).

For each \(H \in S(D_B, D_0 \oplus D_B)\) we have that \(H(0)|F_B = \omega_B\) if and only if there exists a (unique) \(G\) in \(S(G_B, G_B')\), with \(G_B\) and \(G_B'\) as in (6.8), such that

\[
H(\lambda) = \begin{bmatrix} \omega_B & 0 \\ 0 & G(\lambda) \end{bmatrix} \begin{bmatrix} F_B \\ G_B \end{bmatrix} \rightarrow \begin{bmatrix} F_B' \\ G_B' \end{bmatrix}, \quad \lambda \in \mathbb{D}.
\]

Hence there exists a one to one mapping from the set \(S_B\) onto \(S(G_B, G_B')\).

In fact, in the proof of Theorem 6.1 we do not only show that there exists a one to one mapping from the set \(F\) in \(S(D_A, D_0 \oplus D_T, \oplus D_A)\) with \(F(0)|F = \omega\) such that \(B = B_F\) onto \(S(G_B, G_B')\), but we actually indicate how such a mapping can be constructed. To be more specific, the construction in the reverse way goes as follows.

Assume that \(G\) is a Schur class function from \(S(G_B, G_B')\), with \(B\) some contractive interpolant for \(\{A, T', V, R, Q\}\). Define \(H \in S(D_B, D_0 \oplus D_B)\) by (6.8). Then \(H\) satisfies \(H(0)|F_B = \omega_B\), and thus from Section 2 we obtain that there exists an isometry \(M\) from \(D_0 \oplus D_B \oplus Y\) to \(D_B \oplus Y\), for some Hilbert space \(Y\), such that

\[
H(\lambda) = \Pi_{D_0 \oplus D_B} M^*(I_{D_0 \oplus Y} - \lambda J^*_YM^*)^{-1}\Pi_{D_B}^*, \quad \lambda \in \mathbb{D},
\]

where \(M\) satisfies the controllability type condition

\[
\mathcal{Y} = \Pi_{\mathcal{Y}} \sqrt{n=0} (J_{\mathcal{Y}}M)^n \begin{bmatrix} D_0 + D_B \\ \{0\} \end{bmatrix} \quad \text{and} \quad M|F_B = \omega_B^*.
\]

Here \(J_{\mathcal{Y}}\) is the partial isometry given by

\[
J_{\mathcal{Y}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} : \begin{bmatrix} D_B \\ \mathcal{Y} \end{bmatrix} \rightarrow \begin{bmatrix} D_0 + D_B \\ \mathcal{Y} \end{bmatrix}.
\]

Notice that because \(V_0\) and \(B_0\) are as in (5.5) we obtain that \(DV_0 = \Pi_{D_0}, DV_0 = D_0, D_{B_0} = D_B\) and \(D_{B_0} = D_B\Pi_\mathcal{H}\). Thus we can define

\[
\tilde{U} = \begin{bmatrix} V_0 & 0 \\ M|D_0\Pi_{D_0} & M|(D_B \oplus \mathcal{Y}) \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} \mathcal{Y} \oplus D_0 \\ D_B \oplus \mathcal{Y} \end{bmatrix}
\]

and

\[
\tilde{\tau} = \begin{bmatrix} B_0 & 0 \\ \Pi_{D_0}^* D_B \Pi_\mathcal{H} \end{bmatrix} : \mathcal{H} \oplus D_0 \rightarrow \begin{bmatrix} \mathcal{Y} \oplus D_0 \\ D_B \oplus \mathcal{Y} \end{bmatrix}.
\]

Then \(\{\tilde{U}, \tilde{\tau}\}\) is a special isometric coupling of \(\{V_0, B_0\}\). Because \(M\) satisfies (6.9), the coupling \(\{\tilde{U}, \tilde{\tau}\}\) is minimal and \(\tilde{U}\tilde{\tau}R_0 = \tilde{\tau}Q_0\). Hence by Lemma 6.2 we obtain that \(\{\tilde{U}, \tilde{\tau}\}\) also is a minimal isometric coupling of \(\{T'_0, A_0\}\) with \(B = \Pi_{D_T'} \Phi_{U, T'_0} \tilde{\tau} \Pi_{D_0}^*\).

According to Proposition 4.1, the coupling \(\{\tilde{U}, \tilde{\tau}\}\) is isomorphic to a special isometric coupling \(\{U\ on K, \tau\} of \{T'_0, A_0\}\). This isometric coupling \(\{U, \tau\}\) is minimal, satisfies \(U\tau R_0 = \tau Q_0\) and, by Lemma 4.3, we have that \(B = \Pi_{\mathcal{H}} \Phi_{U', T'_0} \Pi_{\mathcal{H}}^*\).

The isometry \(U\) defines an isometry \(Y\) from \(D_T' \oplus D_0 \oplus D_A \oplus X\) to \(D_A \oplus X\), with \(X = K \ominus (\mathcal{H}' \oplus D_0 \oplus D_A)\), by (1.2), with \(T'_0\) on \(\mathcal{H}' \oplus D_0\) instead of \(T'\) on \(\mathcal{H}'\). Then
the functions $F$ in $S(D_A, D_o \oplus D_{T'} \oplus D_A)$ with $F(0)|F = \omega$ satisfying $B = B_F$
corresponding to the function $G \in S(G_B, G'_B)$ is given by

$$F(\lambda) = \Pi_{D_{T'} \oplus D_o \oplus D_A} Y^*(I - \lambda J_A Y)^{-1} \Pi_{D_A}, \quad \lambda \in \mathbb{D},$$

with $J_X$ being the partial isometry given by

$$J_X = \begin{bmatrix} 0 & 0 \\ 0 & I_X \end{bmatrix} : \left[ \begin{array}{c} D_A \\ \mathcal{X} \end{array} \right] \rightarrow \left[ \begin{array}{c} D_{T'} \oplus D_o \oplus D_A \\ \mathcal{X} \end{array} \right].$$

From Theorem 6.1 we immediately obtain the next corollary.

**Corollary 6.4.** Let $B$ be a contractive interpolant for $\{A, T', V, R, Q\}$. Then there is a unique $F$ in $S(D_A, D_o \oplus D_{T'} \oplus D_A)$ with $F(0)|F = \omega$ such that $B = B_F$ if and only if $F_B = D_B$ or $F'_B = D_o \oplus D_B$.

The next lemma gives some sufficient conditions on $\{A, T', V, R, Q\}$ under which the parameterization in Theorem 6.1 is proper. To this end, define the subspace $\mathcal{F}'_A$ of $D_o \oplus D_A$ by

$$(6.10) \quad \mathcal{F}'_A = \begin{bmatrix} D_o \\ D_A R \end{bmatrix} \mathcal{H}_0.$$

**Lemma 6.5.** Let $\{A, T', V, R, Q\}$ be a lifting data set. Then

(i) $F = D_A$ implies that there exists a unique contractive interpolant $B$ and that $F_B = D_B$;

(ii) $F'_A = D_o \oplus D_A$ implies that $F'_B = D_o \oplus D_B$ for every contractive interpolant $B$.

If either $F = D_A$ or $F'_A = D_o \oplus D_A$ holds, then the mapping $F \rightarrow B_F$ given by (6.1) is one to one from the set of all $F \in S(D_A, D_o \oplus D_{T'} \oplus D_A)$ satisfying $F(0)|F = \omega$ onto the set of all contractive interpolants.

**Proof.** Let $B$ be a contractive interpolant for $\{A, T', V, R, Q\}$. Then $\Pi_{\mathcal{H}} B = A$, hence there exists a contraction $\gamma$ from $D_A$ to $H^2(D_{T'}^\prime)$ such that

$$B = \left[ \begin{array}{c} A \\ \Gamma D_A \end{array} \right] : \mathcal{H} \rightarrow \left[ \begin{array}{c} \mathcal{H}' \\ H^2(D_{T'}^\prime) \end{array} \right].$$

From this we obtain that for all $h \in \mathcal{H}$

$$\|D_B h\|^2 = \|h\|^2 - \|Bh\|^2 = \|h\|^2 - \|Ah\|^2 - \|\Gamma D_A h\|^2$$

$$= \|D_A h\|^2 - \|\Gamma D_A H\|^2 = \|D_B D_A h\|^2.$$

Note that $D_T \subset D_A$ and thus $\overline{D_T D_A H} = D_T$. Hence there exists a unitary operator $\gamma$ from $D_T$ onto $D_B$ such that $D_B = \gamma D_T D_A$.

(i) Assume that $F = D_A$. Then there is only one $F$ in $S(D_A, D_o \oplus D_{T'} \oplus D_A)$ with $F(0)|F = \omega$. Hence, by Theorem 6.1 there can be only one contractive interpolant $B$ for $\{A, T', V, R, Q\}$, and for this contractive interpolant $B$ there can be only one $F$ in $S(D_A, D_o \oplus D_{T'} \oplus D_A)$ with $F(0)|F = \omega$ such that $B = B_F$. Moreover, we have

$$F_B = D_B Q H_0 = \gamma \overline{D_T D_A Q H_0} = \overline{\gamma D_T D_A Q H_0} = \gamma D_T F = \gamma D_T D_A = D_B.$$
(ii) Assume that $\mathcal{F}_A' = D_o \oplus D_A$. Then we have that

\[
\mathcal{F}_B' = \begin{bmatrix}
D_o \\
D_B R
\end{bmatrix} \mathcal{H}_0 = \begin{bmatrix}
I_{D_o} & 0 \\
\gamma D_T & D_A R
\end{bmatrix} \mathcal{H}_0 = \begin{bmatrix}
I_{D_o} & 0 \\
\gamma D_T & D_A R
\end{bmatrix} \mathcal{F}_A'
\]

\[
= \begin{bmatrix}
I_{D_o} & 0 \\
0 & \gamma D_T
\end{bmatrix} \begin{bmatrix}
D_o \\
D_A
\end{bmatrix} = \begin{bmatrix}
D_o \\
\gamma D_T D_A
\end{bmatrix} = \begin{bmatrix}
D_o \\
D_B
\end{bmatrix}.
\]

The final statement of the lemma follows immediately from (i), (ii) and Corollary 6.4.

For the classical commutant lifting theorem, that is, when $\mathcal{H} = \mathcal{H}_0$, $R = I_H$ and $Q$ is an isometry on $\mathcal{H}$, we have already seen in Section 0 that the parameterization in Theorem 0.1 is proper. This result also follows from Lemma 6.5 (ii). Indeed, if $\mathcal{H} = \mathcal{H}_0$, $R = I_H$ and $Q$ is an isometry on $\mathcal{H}$, then

$\mathcal{F}_A' = \{0\} \oplus D_A I_H \mathcal{H} = \{0\} \oplus D_A = D_o \oplus D_A$.

Finally we derive some sufficient conditions on $\{A, T', V, R, Q\}$ guaranteeing that there is only one contractive interpolant. From Lemma 6.3 we already know that the condition $\mathcal{F} = D_A$ is such a condition. In the same way we can see that the condition $\mathcal{F}' = D_{T'} \oplus D_o \oplus D_A$ is sufficient.

For the classical commutant lifting theorem the combination of these two conditions is also a necessary condition. That is, if $\mathcal{H} = \mathcal{H}_0$, $R = I_H$ and $Q$ is an isometry on $\mathcal{H}$, then there is only one contractive interpolant if and only if $\mathcal{F} = D_A$ or $\mathcal{F}' = D_{T'} \oplus D_o \oplus D_A$. We can see this as follows. If the parametrization in Theorem 0.1 for the lifting data set $\{A, T', V, R, Q\}$ is proper, then there is only one contractive interpolant for $\{A, T', V, R, Q\}$ if and only if there is only one $F$ in $\mathcal{S}(D_A, D_o \oplus D_{T'} \oplus D_A)$ with $F(0) | \mathcal{F} = \omega$. The latter is equivalent to the condition $\mathcal{F} = D_A$ or $\mathcal{F}' = D_{T'} \oplus D_o \oplus D_A$.

Notice that when $T'$ is an isometry, then the Sz.-Nagy-Schäffer minimal isometric lifting of $T'$ is $T'$ itself. So in that case there also is only one contractive interpolant $B$ for $\{A, T', V, R, Q\}$, namely $B = A$. In the next lemma we summarize the above, and improve the condition $\mathcal{F}' = D_{T'} \oplus D_o \oplus D_A$ a bit further.

**Proposition 6.6.** Assume that for $\{A, T', V, R, Q\}$ either $T'$ is an isometry, $\mathcal{F} = D_A$ or $\mathcal{F}' = D_{T'} \oplus D_o \oplus D_A \subset \mathcal{F}'$. Then there exists a unique contractive interpolant for $\{A, T', V, R, Q\}$.

**Proof.** We have already seen above that the requirement $T'$ is an isometry and the equality $\mathcal{F} = D_A$ are both sufficient conditions. So assume that we have $\mathcal{F}' = D_{T'} \oplus D_A \subset \mathcal{F}'$. Define for all $F \in \mathcal{S}(D_A, D_o \oplus D_{T'} \oplus D_A)$ the Schur class functions $F_o = \Pi_{D_o} F$, $F_{T'} = \Pi_{D_{T'}} F$ and $F_A = \Pi_{D_A} F$. Hence for all $\lambda \in \mathbb{D}$

\[
F(\lambda) = \begin{bmatrix}
F_o(\lambda) \\
F_{T'}(\lambda) \\
F_A(\lambda)
\end{bmatrix} : D_A \to \begin{bmatrix}
D_o \\
D_{T'} \\
D_A
\end{bmatrix}.
\]

Then we have

\[
(6.11) \quad \Pi_{T'} F(\lambda) (I_{D_A} - \lambda F_A(\lambda))^{-1} D_A = F_{T'}(\lambda) (I_{D_A} - \lambda F_A(\lambda))^{-1} D_A, \quad \lambda \in \mathbb{D}.
\]
All \( F \in S(\mathcal{D}_A, \mathcal{D}_0 \oplus \mathcal{D}_T \oplus \mathcal{D}_A) \) with \( F(0)\mathcal{F} = \omega \) admit a matrix representation of the form
\[
F(\lambda) = \begin{bmatrix} \omega & 0 \\ 0 & G(\lambda) \end{bmatrix} : \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{F}' \\ \mathcal{G}' \end{bmatrix}, \quad \lambda \in \mathbb{D},
\]
for some \( G \in S(\mathcal{G}, \mathcal{G}') \) where \( \mathcal{G} = \mathcal{D}_A \ominus \mathcal{F} \) and \( \mathcal{G}' = (\mathcal{D}_0 \oplus \mathcal{D}_T \oplus \mathcal{D}_A) \ominus \mathcal{F}' \). Hence, because \( \mathcal{D}_T \oplus \mathcal{D}_A \subseteq \mathcal{F}' \) all \( F \in S(\mathcal{D}_A, \mathcal{D}_0 \oplus \mathcal{D}_T \oplus \mathcal{D}_A) \) with \( F(0)\mathcal{F} = \omega \) have identical \( F_T \) and \( F_A \) and thus from (6.11) we see that \( B_F \) is the same operator for all \( F \in S(\mathcal{D}_A, \mathcal{D}_0 \oplus \mathcal{D}_T \oplus \mathcal{D}_A) \) with \( F(0)\mathcal{F} = \omega \). Hence by Theorem 0.1 there is only one contractive interpolant. \( \square \)

Acknowledgement. We thank Ciprian Foias for useful discussions on an earlier version of this paper.

Added in proof. At the IWOTA 2004 conference in Newcastle, when a preliminary version of this paper had been completed, the authors learned that W.S. Li and D. Timotin had a preprint ready in which the coupling method was also used to study the relaxed commutant lifting problem and the set of its solutions. Although the same method was used in the same area the two papers turned out to be quite complementary in style and results. We are happy that the editors of *Integral Equations and Operator Theory* agreed to publish the final versions of both papers, one directly after the other in this volume.

References

[1] V.M. Adamjan and D.Z. Arov, Scattering operators and contraction semigroups in Hilbert space, *Doklady* **165** (1965), 1377–1380.

[2] V.M. Adamjan and D.Z. Arov, On the unitary couplings of isometric operators, *Mat. Issled. Kisinev* **1** (1966), 3–66 (Russian).

[3] D. Alpay, V. Bolotnikov, Y. Peretz, On the tangential interpolation problem for \( H_2 \) functions, *Trans. Amer. Math. Soc.* **347** (1995), 675–686.

[4] T. Ando, *De Branges spaces and analytic functions*, Lecture notes of the division of Applied Mathematics Research Institute of Applied Electricity, Hokkaido University, Sapporo, Japan, 1990.

[5] R. Arocena, Generalized Toeplitz kernels and dilations of intertwining operators, *Integral Equations and Operator Theory*, **6** (1983), 759–778.

[6] R. Arocena, On the parameterization of Adamjan, Arov and Krein, *Publ. Math. Orsay* **83** (1983), 7–23.

[7] R. Arocena, On generalized Toeplitz kernels and their relation with a paper of Adamjan, Arov and Krein, in: *Functional Analysis Homomorphy and Approximation Theory*. Math. Studies **86**, North-Holland Amsterdam, 1984, pp. 1–22.

[8] R. Arocena, A theorem of Naimark, linear systems and scattering operators, *J. Funct. Anal.* **69** (1986), 281–288.

[9] R. Arocena, Unitary extensions of isometries and contractive intertwining operators, in: *The Gohberg Anniversary Collection II*, OT **41**, Birkhäuser Verlag Basel, 1989, pp. 13–23.

[10] D.Z. Arov, M.A. Kaashoek and D.R. Pik, Optimal time-variant systems and factorization of operators, I: minimal and optimal systems, *Integral Equations and Operator Theory*, **31** (1998), 389–420.

[11] H. Bart, I. Gohberg and M.A. Kaashoek, *Minimal factorization of matrix and operator functions*, OT **1**, Birkhäuser Verlag, Basel, 1979.

[12] A. Biswas, C. Foias and A. E. Frazho, Weighted Commutant Lifting, *Acta Sci. Math. (Szeged)*, **65** (1999), 657–686.

[13] C. Foias and A. E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, OT **44**, Birkhäuser Verlag, Basel, 1990.

[14] C. Foias, A.E. Frazho, and M.A. Kaashoek, Relaxation of metric constrained interpolation and a new lifting theorem, *Integral Equations and Operator Theory*, **42** (2002), 253–310.
[15] A.E. Frazho and M.A. Kaashoek, A Naimark dilation perspective of Nevanlinna-Pick interpolation, *Integral Equations and Operator theory*, **42** (2002), 253–310.

[16] B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North Holland Publishing Co., Amsterdam-Budapest, 1970.

[17] I. Gohberg, S. Goldberg and M. Kaashoek, *Classes of Linear Operators Vol.II*, OT **63**, Birkhäuser Verlag, Basel, 1993.

[18] D. Sarason, Exposed points in $H^1$, I, in: *The Gohberg anniversary collection, Vol. II*, OT **41**, Birkhäuser Verlag, Basel, 1989, pp. 485–496.

[19] B. Sz.-Nagy and C. Foias, Dilation des commutants d’opérateurs, *C. R. Acad. Sci. Paris, série A*, **266** (1968), 493-495.

[20] S. Treil and A. Volberg, A fixed point approach to Nehari’s problem and its applications, in: *The Harold Widom Anniversary Volume*, OT **71**, Birkhäuser Verlag, Basel, 1994, pp.165-186.

DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907, USA

E-mail address: frazho@ecn.purdue.edu

AFDELING WISKUNDE, FACULTEIT DER EXACTE WETENSCHAPPEN, VRIJE UNIVERSITEIT, DE BOELELAAN 1081A, 1081 HV AMSTERDAM, THE NETHERLANDS

E-mail address: terhorst@few.vu.nl

AFDELING WISKUNDE, FACULTEIT DER EXACTE WETENSCHAPPEN, VRIJE UNIVERSITEIT, DE BOELELAAN 1081A, 1081 HV AMSTERDAM, THE NETHERLANDS

E-mail address: ma.kaashoek@few.vu.nl