Harmonic-Number Summation Identities, Symmetric Functions, and Multiple Zeta Values

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Abstract
We show how infinite series of a certain type involving generalized harmonic numbers can be computed using a knowledge of symmetric functions and multiple zeta values. In particular, we prove and generalize some identities recently conjectured by J. Choi, and give several more families of identities of a similar nature.

1 Introduction
Let $H_n^{(r)}$ denote the generalized harmonic number $\sum_{j=1}^n \frac{1}{n^r}$; if $r = 1$ we omit the superscript. This paper is concerned with series of the form
\[
\sum_{n=1}^{\infty} \frac{F(H_n, H_n^{(2)}, \ldots, H_n^{(j)})}{n^{s_1}(n+1)^{s_2} \cdots (n+k-1)^{s_k}},
\]
where $F(x_1, \ldots, x_j) \in \mathbb{Q}[x_1, \ldots, x_j]$ and $s_1, \ldots, s_k$ are nonnegative integers with $s_1 + \cdots + s_k \geq 2$. There are many interesting identities giving closed
forms for such sums, starting with the formulas

\[ \sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4} \zeta(4), \]  

both due to Euler [9]. Many similar formulas have been established since, and there is an extensive literature; see, e.g., [1, 3, 5, 7, 10, 19, 21, 20, 23].

In the 1990’s, multiple zeta values were introduced by the author [12] and D. Zagier [22]. These are defined by

\[ \zeta(i_1, i_2, \ldots, i_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}} \]  

for positive integers \( i_1, i_2, \ldots, i_k \) with \( i_1 > 1 \). For integer \( s \geq 2 \), any sum of the form

\[ \sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^s}, \]

which includes Euler’s examples (2), is readily expressible in terms of multiple zeta values as \( \zeta(r + s) + \zeta(s, r) \). In recent decades intensive study has led to the development of an extensive theory of multiple zeta values, which we summarize in §3 below. The point of this paper is that sums of form (1) can be expressed in terms of multiple zeta values, and using some facts about symmetric functions and multiple zeta values often allows such expressions to be put in a particularly simple form.

Recently J. Choi [4, Corollary 3] proved a sequence of identities:

\[ \sum_{n=1}^{\infty} \frac{H_n}{(n+1)(n+2)} = 1 \]  

(4)

\[ \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{(n+1)(n+2)} = 1 \]  

(5)

\[ \frac{1}{6} \sum_{n=1}^{\infty} \frac{H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}}{(n+1)(n+2)} = 1 \]  

(6)

\[ \frac{1}{24} \sum_{n=1}^{\infty} \frac{H_n^4 - 6H_n^2 H_n^{(2)} + 8H_n H_n^{(3)} + 3(H_n^{(2)})^2 - 6H_n^{(4)}}{(n+1)(n+2)} = 1. \]

(7)
The sequence $P_k$ of multivariate polynomials in the numerators, which starts

\[ P_1(x_1) = x_1, \quad P_2(x_1, x_2) = \frac{1}{2}(x_1^2 - x_2), \quad P_3(x_1, x_2, x_3) = \frac{1}{6}(x_1^3 - 3x_1x_2 + 2x_3), \ldots \]

turns out to be well-known in the theory of symmetric functions; in fact

\[ P_k(p_1, p_2, \ldots, p_k) = e_k, \]

where $p_i$ is the $i$th power sum and $e_i$ is the $i$th elementary symmetric function. We discuss the $P_k$ in §2 below. We also discuss another sequence of polynomials $Q_k$, which are simply the $P_k$ without signs, i.e.,

\[ Q_1(x_1) = x_1, \quad Q_2(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2), \quad Q_3(x_1, x_2, x_3) = \frac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3), \ldots \]

(The $Q_k$ express the complete symmetric functions in terms of power sums.) Not only is the identity

\[ \sum_{n=1}^{\infty} \frac{P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{(n+1)(n+2)} = 1 \] (8)

true for all $k$, but in fact it has a generalization involving the $Q_k$, of which Choi proved [4, Corollary 5] the first few cases:

\[ \sum_{n=1}^{\infty} \frac{H_n^2}{(n+1)(n+2)} = 1 + \zeta(2) \] (9)

\[ \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n(H_n^2 - H_n^{(2)})}{(n+1)(n+2)} = 1 + \zeta(2) + \zeta(3) \] (10)

\[ \frac{1}{6} \sum_{n=1}^{\infty} \frac{H_n(H_n^3 - 3H_nH_n^{(2)} + 2H_n^{(3)})}{(n+1)(n+2)} = 1 + \zeta(2) + \zeta(3) + \zeta(4) \] (11)

(in these identities there is an erroneous factor of 2 in [4] which we have removed). The general result as follows. (We make the convention that $H_0^{(r)} = 0$ for all $r$, so that the result holds if $k = l = 0$.)
Theorem 1. If $P_k, Q_k$ are the polynomials discussed above, $k, l$ nonnegative integers, then

$$
\sum_{n=0}^{\infty} \frac{Q_l(H_n, H_n^{(2)}, \ldots, H_n^{(l)}) P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{(n+1)(n+2)}
= \begin{cases} 
\sum_{j=0}^{k} \binom{k+l-j}{k+1-j} \zeta(k+l-j) - \zeta(l), & l \geq 2, \\
\sum_{j=0}^{k-1} \zeta(k+1-j) + 1, & l = 1, \\
1, & l = 0.
\end{cases}
$$

The denominator $(n+1)(n+2)$ appearing in Theorem 1 can be replaced with other polynomials. The analogue of Theorem 1 for the denominator $n(n+1)$ is especially simple.

Theorem 2. For $k, l$ nonnegative integers with $k+l \geq 1$,

$$
\sum_{n=1}^{\infty} \frac{Q_l(H_n, H_n^{(2)}, \ldots, H_n^{(l)}) P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n(n+1)} = \binom{k+l+1}{k+1} \zeta(k+l+1).
$$

When the denominator is $n^2$ we have the following result.

Theorem 3. For the polynomials $P_k, Q_k$ discussed above, $k$ nonnegative,

$$
\sum_{n=1}^{\infty} \frac{Q_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n^2} = (k+1) \zeta(k+2) \quad (12)
$$

$$
\sum_{n=1}^{\infty} \frac{P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n^2} = \frac{k+3}{2} \zeta(k+2) + \frac{1}{2} \sum_{j=2}^{k} \zeta(j) \zeta(k+2-j) \quad (13)
$$

Remark 1. For $k = 1$, both equations give the first of Euler’s formulas (2) mentioned above. Equation (12) can be deduced from [8, Corollary 1]; the special cases $k = 2$ and $k = 3$ appear as [5, eqn. (1.5a)] and [23, eqn. (2.3f)] respectively, and $k = 4$ appears in [7]. The sum and difference of equations (12) and (13) for $k = 3$ can be recognized as [23, eqn. (2.5e)] and [23, eqn. (2.5d)] respectively.

In fact we have a general formula for

$$
\sum_{n=1}^{\infty} \frac{Q_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)}) P_l(H_n, H_n^{(2)}, \ldots, H_n^{(l)})}{n^2}
$$
in terms of multiple zeta values (Theorem 9 below), but it can only be reduced to ordinary zeta values in certain cases.

We are able to obtain results for other denominators as well, including the following.

**Theorem 4.** For nonnegative integers $k, l$,

$$
\sum_{n=0}^{\infty} \frac{Q_l(H_{n+1}, H_{n+1}^{(2)}, \ldots, H_{n+1}^{(l)})P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{(n+1)^2} = \binom{l+k+1}{l} \zeta(l+k+2).
$$

**Remark 2.** Several special cases of Theorem 4 occur in the literature. The case $k = l = 1$ is

$$
\sum_{n=1}^{\infty} \frac{H_n H_{n+1}}{(n+1)^2} = 3 \zeta(4),
$$

which appears as [5, eqn. (1.2a)]. The cases $(k, l) = (2, 1)$ and $(k, l) = (1, 2)$ appear in [23] as equations (2.3c) and (2.3e) respectively.

**Theorem 5.** For nonnegative integers $k, l$ with $k + l \geq 1$,

$$
\sum_{n=1}^{\infty} \frac{Q_l(H_n, H_n^{(2)}, \ldots, H_n^{(l)})P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n(n+1)(n+2)} = \begin{cases} 
\frac{1}{2} \left[ \binom{k+l+1}{k} \zeta(k+l+1) - \sum_{j=0}^{k} \binom{k+l-j}{l+1-j} \zeta(k+l-j) - \zeta(l) \right], & l \geq 2, \\
\frac{1}{2} \left[ (k+2) \zeta(k+2) - \sum_{j=0}^{k-1} \zeta(k+1-j) - 1 \right], & l = 1, \\
\frac{1}{2} (\zeta(k+1) - 1), & l = 0.
\end{cases}
$$

Equation (8) can be generalized in another direction.

**Theorem 6.** For integers $k \geq 0$ and $q \geq 2$,

$$
\sum_{n=0}^{\infty} \frac{P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{(n+1)(n+2) \cdots (n+q)} = \frac{1}{(q-1)!} \frac{1}{(q-1)^{k+1}}.
$$

This actually follows from a result of J. Spieß [21], but we prove it by our own methods. As a corollary we get the formula

$$
\sum_{n=1}^{\infty} \frac{P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n(n+1) \cdots (n+q-1)} = \frac{1}{(q-1)!} \left[ \zeta(k+1) - \sum_{j=1}^{q-2} \frac{1}{j^{k+1}} \right] \quad (14)
$$
for integers $k > 0$ and $q \geq 2$, which generalizes the case $l = 0$ of Theorems 2 and 5. We note that the case $k = 1$ of identity (14) coincides with Theorem 1 of [2].

Our main technical tool is the introduction of a class of functions $\eta_{s_1,\ldots,s_k}$, which we call $H$-functions, from the quasi-symmetric functions (a superalgebra of the symmetric functions) to the reals such that

$$\eta_{s_1,s_2,\ldots,s_k}(p_1) = \sum_{n=1}^{\infty} \frac{H_n}{\eta^{s_1(n+1)s_2\cdots(n+k-1)s_k}}.$$ 

Here $(s_1, s_2, \ldots, s_k)$ is a sequence of nonnegative integers whose sum is 2 or more. We are able to express $\eta_{s_1,\ldots,s_k}(u)$, for any quasi-symmetric function $u$, in terms of multiple zeta values. Furthermore, for three particular choices of the sequence $(s_1,\ldots,s_k)$, namely $(2), (1,1)$ and $(0,1,1)$, we are able to write simple formulas for $\eta_{s_1,\ldots,s_k}(u)$ when $u$ is a product of elementary and complete symmetric functions. The proofs rely on a certain class of symmetric functions that came up in earlier work of the author [15], along with some results about multiple zeta values. By taking linear combinations of $\eta_2, \eta_{1,1}$, and $\eta_{0,1,1}$, we are able to prove many more summation formulas, as discussed in §5.

2 Symmetric and quasi-symmetric functions

Let $x_1, x_2, \ldots$ be a countable set of indeterminates, each of which has degree 1. Let $P$ be the set of formal power series in the $x_i$ of bounded degree. A symmetric “function” is an element of $u \in P$ such that the coefficient of any monomial $x_1^{a_1} \cdots x_k^{a_k}$ (with the $i_j$ distinct) in $u$ is the same as the coefficient of the monomial $x_1^{a_1} \cdots x_k^{a_k}$ in $u$. The symmetric functions form a ring $\text{Sym}$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of $n$, the monomial symmetric function $m_\lambda$ is the “smallest” symmetric function that contains the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$. A symmetric function of degree $n$ is a linear combination of the monomial symmetric functions $m_\lambda$ with $\lambda$ running over partitions of $n$. Some important symmetric functions are the power sums

$$p_k = m_{(k)} = x_1^k + x_2^k + \cdots,$$

the elementary symmetric functions

$$e_k = m_{(1,\ldots,1)} = x_1 x_2 \cdots x_k + x_2 x_3 \cdots x_{k+1} + \cdots.$$
and the complete symmetric functions

\[ h_k = \sum_{|\lambda| = n} m_\lambda. \]

An element \( u \in \mathcal{P} \) such the the coefficient in \( u \) of any monomial \( x_{i_1}^{a_1} \cdots x_{i_k}^{a_k} \) with \( i_1 < i_2 < \cdots < i_k \) is the same as that of \( x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} \) is called quasi-symmetric. This is a weaker condition than being symmetric: every symmetric function is quasi-symmetric, but there are quasi-symmetric functions like

\[ \sum_{i<j} x_i^2 x_j \]

(15) that are not symmetric. There is a ring \( \text{QSym} \supset \text{Sym} \) of quasi-symmetric functions. For any composition (ordered partition) \( (a_1, \ldots, a_k) \) of \( n \), the monomial symmetric function \( M_{(a_1, \ldots, a_k)} \) is the “smallest” quasi-symmetric function containing \( x_1^{a_1} \cdots x_k^{a_k} \); for example, the formal power series (15) is \( M_{(2,1)} \). Any quasi-symmetric function of degree \( n \) is a linear combination of monomial quasi-symmetric functions of the same degree. Any monomial symmetric function is a sum of monomial quasi-symmetric functions, e.g.,

\[ m_{(1,1)} = M_{(1,1)}, \quad m_{(2,1)} = M_{(2,1)} + M_{(1,2)}. \]

The ring \( \text{Sym} \) of symmetric functions is a polynomial ring in the \( e_k \), and also in the \( h_k \), and also in the \( p_k \). In particular, there are polynomials \( P_n \) and \( Q_n \) so that

\[ e_n = P_n(p_1, p_2, \ldots, p_n) \quad \text{and} \quad h_n = Q_n(p_1, p_2, \ldots, p_n). \]

In fact, these are exactly the polynomials that appear in the Introduction. Explicit formulas are well-known.

**Proposition 1.** For \( n \geq 1 \),

\[ P_n(y_1, \ldots, y_n) = \sum_{m_1+2m_2+\cdots=n} \frac{(-1)^{m_2+m_4+\cdots}}{m_1!m_2!\cdots} \left( \frac{y_1}{1} \right)^{m_1} \left( \frac{y_2}{2} \right)^{m_2} \cdots \]  

(16)

\[ Q_n(y_1, \ldots, y_n) = \sum_{m_1+2m_2+\cdots=n} \frac{1}{m_1!m_2!\cdots} \left( \frac{y_1}{1} \right)^{m_1} \left( \frac{y_2}{2} \right)^{m_2} \cdots. \]  

(17)
Proof. If

\[ E(t) = \sum_{n=0}^{\infty} e_n t^n = \prod_{i \geq 1} (1 + t x_i), \quad H(t) = \sum_{n=0}^{\infty} h_n t^n = \prod_{i \geq 1} \frac{1}{1 - t x_i}, \]

\[ P(t) = \sum_{n=1}^{\infty} p_n t^{n-1} = \sum_{i \geq 1} \frac{x_i}{1 - t x_i} \]

are the respective generating functions of the elementary, complete, and power-sum symmetric functions, then evidently \( E(t) = H(-t)^{-1} \) and \( H'(t) = P(t)H(t) \). It follows that

\[ H(t) = \exp \left( \int_0^t P(s) ds \right) \quad \text{and} \quad E(-t) = \exp \left( - \int_0^t P(s) ds \right), \]

which can be expanded out to give the conclusion. \( \square \)

**Remark 3.** The polynomials \( P_k \) appear in [21], where they are denoted \( P_k/k! \). The \( Q_k \) appear in [3], where they are denoted \( \Omega_k \), and in [8], where they are denoted \( P_k \).

The polynomials \( P_n \) and \( Q_n \) can be written as determinants [18, Ch. I §2]:

\[
n! P_n(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & 1 & 0 & \ldots & 0 \\ y_2 & y_1 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{n-1} & y_{n-2} & y_{n-3} & \ldots & n-1 \\ y_n & y_{n-1} & y_{n-2} & \ldots & y_1 \end{vmatrix}
\]

and

\[
n! Q_n(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & -1 & 0 & \ldots & 0 \\ y_2 & y_1 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{n-1} & y_{n-2} & y_{n-3} & \ldots & -n+1 \\ y_n & x_{n-1} & y_{n-2} & \ldots & y_1 \end{vmatrix}
\]

From these formulas can be deduced some properties of the polynomials \( P_n \) and \( Q_n \).

**Proposition 2.** Let \( P_n, Q_n \) be as defined above. Then
1. All the coefficients of \( n!P_n \) and \( n!Q_n \) are integers.

2. \( n!P_n(a, a, \ldots, a) = a(a - 1) \cdots (a - n + 1) \) and \( n!Q_n(a, a, \ldots, a) = a(a + 1) \cdots (a + n - 1) \). Hence the coefficients of \( P_n \) sum to 0 for \( n \geq 2 \), and the coefficients of \( Q_n \) sum to 1.

Let \( a_1, \ldots, a_n \) be a finite sequence of real constants. There is a homomorphism \( \text{QSym} \to \mathbb{R} \) sending each \( x_i \) with \( i \leq n \) to \( a_i \), and each \( x_i \) with \( i > n \) to 0. We denote the image of \( u \in \text{QSym} \) under this homomorphism by \( u(a_1, \ldots, a_n) \).

**Lemma 1.** For positive integers \( n \) and \( k \),

\[
h_k(a_1, \ldots, a_n, a_{n+1}) = \sum_{j=0}^{k} h_{k-j}(a_1, \ldots, a_n) a_j^{n+1}.
\]

**Proof.** From the generating function \( H(t) \) we have

\[
\sum_{j=0}^{\infty} t^j h_j(a_1, \ldots, a_{n+1}) = \prod_{i=1}^{n+1} \frac{1}{1-a_it} = (1 + a_{n+1}t + a_{n+1}^2 t^2 + \cdots) \prod_{i=1}^{n} \frac{1}{1-a_it}
\]

\[
= (1 + a_{n+1}t + a_{n+1}^2 t^2 + \cdots) \sum_{j=0}^{\infty} t^j h_j(a_1, \ldots, a_n),
\]

and the conclusion follows by considering the coefficient of \( t^k \).

As in [15], let

\[
N_{n,m} = \sum_{\text{partitions } \lambda \text{ of } n \text{ with } m \text{ parts}} m_{\lambda} = \sum_{\text{compositions } I \text{ of } n \text{ with } m \text{ parts}} M_I
\]

for \( n \geq m \). The \( N_{n,m} \) also appear in [18, Ch. I, §2, ex. 19], where they are denoted \( p_n^{(m)} \). We shall need the following result in §4.

**Lemma 2.** For \( 0 \leq j \leq k \),

\[
e_j h_{k-j} = \sum_{p=j}^{k} \binom{p}{j} N_{k,p}.
\]
Proof. Let

\[ F(t, s) = 1 + \sum_{n \geq m \geq 1} N_{n,m} t^n s^m \in \text{Sym}[[t, s]] \]

be the generating function of the \( N_{m,n} \). Then as in [15, Lemma 1], we have

\[ F(t, s) = \prod_{i \geq 1} \left( 1 + \frac{stx_i}{1-tx_i} \right) = \prod_{i \geq 1} \frac{1 + (s-1)tx_i}{1-tx_i} = H(t)E((s-1)t). \]

From this follows (cf. [15, Lemma 2], [18, loc. cit.])

\[ N_{k,j} = \sum_{p=0}^{k-j} \binom{j+p}{j} (-1)^{j} e_{j+p} h_{k-j-p}, \]

and the system of equations for fixed \( k \) can be backsolved to give the conclusion. \( \square \)

3 Multiple zeta values

Multiple zeta values \( \zeta(i_1, i_2, \ldots, i_k) \) are defined by equation (3) above. We refer to \( i_1 + \cdots + i_k \) as the weight of this multiple zeta value, and \( k \) as its depth. Multiple zeta values of arbitrary depth were introduced by the author [12] and D. Zagier [22], though the depth 2 case was already studied by Euler [9].

It is evident that multiple zeta values are related to quasi-symmetric functions. In the notation introduced in the last section,

\[ \zeta(i_k, i_{k-1}, \ldots, i_1) = \lim_{n \to \infty} M_{(i_1, \ldots, i_k)}(1, \frac{1}{2}, \ldots, \frac{1}{n}) \]  \hspace{1cm} (18)

for any composition \( (i_1, \ldots, i_k) \) with \( i_k \geq 2 \). There is a subalgebra \( \text{QSym}^0 \) of \( \text{QSym} \) generated by all the monomial quasi-symmetric functions \( M_{(i_1, \ldots, i_k)} \) with \( i_k > 1 \), and we can define a homomorphism \( \zeta : \text{QSym}^0 \to \mathbb{R} \) that sends \( 1 \in \text{QSym}^0 \) to \( 1 \in \mathbb{R} \) and \( M_{(i_1, \ldots, i_k)} \), \( i_k > 1 \), to \( [18] \) (See [14] for a detailed discussion). We write \( \text{Sym}^0 \) for the subalgebra \( \text{QSym}^0 \cap \text{Sym} \) of Sym; if we think of Sym as the polynomial algebra on the \( p_i \), then \( \text{Sym}^0 \) is the subalgebra generated by \( p_2, p_3, \ldots \).

One of the first major results in the modern theory of multiple zeta values is the following “sum theorem”, conjectured by C. Moen (see [12]) and proved by A. Granville [11].
Theorem (Sum Theorem). The sum of all multiple zeta values of weight $n$ and depth $d \leq n - 1$ is $\zeta(n)$.

Call a composition $(i_1, \ldots, i_k)$ “admissible” if $i_1 > 1$. The following result was proved in [12].

Theorem (Derivation Theorem). For any admissible composition $(i_1, \ldots, i_k)$,

$$\sum_{j=1}^{k} \zeta(i_1, \ldots, i_{j-1}, i_j + 1, i_{j+1}, \ldots, i_k) = \sum_{j=1}^{k} \sum_{p=1}^{i_j-1} \zeta(i_1, \ldots, i_{j-1}, i_j - p + 1, p, i_{j+1}, \ldots, i_k).$$

Another important result on multiple zeta values is the duality theorem. It was actually conjectured in [12], but the proof comes easily from a description of multiple zeta values as iterated integrals (see [22] or [13]). To describe it requires some definitions. Let $\Sigma$ be the function that takes a composition to its sequence of partial sums:

$$\Sigma(i_1, \ldots, i_k) = (i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_k).$$

On the set $I_n$ of increasing integer sequences chosen from the set $\{1, \ldots, n\}$, there are functions $R_n : I_n \to I_n$ and $C_n : I_n \to I_n$ given by

$$R_n(s_1, \ldots, s_k) = (n + 1 - s_k, n + 1 - s_{k-1}, \ldots, n + 1 - s_1)$$
$$C_n(s_1, \ldots, s_k) = \text{complement of } \{s_1, \ldots, s_k\} \text{ in } \{1, \ldots, n\}$$

For a composition $(i_1, \ldots, i_k)$ of weight $n$, let

$$\tau(i_1, \ldots, i_k) = \Sigma^{-1} R_n C_n \Sigma(i_1, \ldots, i_k).$$

Then $\tau(I)$ is admissible if $I$ is, and we have the following result.

Theorem (Duality Theorem). For any admissible composition $I$, $\zeta(\tau(I)) = \zeta(I)$.

As shown in [12], if $I$ and $J$ are admissible compositions then their juxtaposition $IJ$ has the property that $\tau(IJ) = \tau(J)\tau(I)$. Since the composition $I = (2)$ is self-dual, $\tau(I)$ ends in 1 if and only if $I$ doesn’t begin with 2.
We note that multiple zeta values of depth greater than 1 cannot in general be written as rational polynomials in the depth 1 (ordinary) zeta values: for example, there is no such expression known for $\zeta(2,6)$. It is true, however, that all multiple zeta values of weight 7 or less can be so expressed. Euler [9] gave the formula

$$\zeta(n, 1) = \frac{n}{2} \zeta(n + 1) + \frac{1}{2} \sum_{i=1}^{n-2} \zeta(n - i) \zeta(i + 1)$$

valid for $n \geq 2$, and if $a + b$ is odd the double zeta value $\zeta(a, b)$ can be written as a rational polynomial in the $\zeta(i)$. Multiple zeta values of “height one”, i.e., those of the form $\zeta(n, 1, \ldots, 1)$, are also rational polynomials in the $\zeta(i)$, as can be seen from the generating function [13]

$$\sum_{m,n \geq 1} s^m t^n \zeta(m + 1, 1, \ldots, 1) = 1 - \exp \left( \sum_{j \geq 2} \frac{\zeta(j)}{j} (s^j + t^j - (s + t)^j) \right).$$

We also note that all known identities of multiple zeta values preserve weight.

4 $H$-functions and summation formulas

Now we show how to obtain families of summation formulas like those given in the Introduction. For $u \in \text{QSym}$ and nonnegative integers $s_1, \ldots, s_k$ with $s_1 + \cdots + s_k \geq 2$, define the $H$-function $\eta_{s_1,\ldots,s_k} : \text{QSym} \to \mathbb{R}$ by

$$\eta_{s_1,\ldots,s_k}(u) = \sum_{n=1}^{\infty} \frac{u(1, \frac{1}{2}, \ldots, \frac{1}{n})}{n^{s_1}(n + 1)^{s_2} \cdots (n + k - 1)^{s_k}}. \tag{20}$$

Then we have the following result.

**Theorem 7.** $\eta_{s_1,\ldots,s_k}(u)$ converges for any $u \in \text{QSym}$.

**Proof.** It suffices to show that $\eta_{s_1,\ldots,s_k}(M_I)$ converges for any composition $I$. Writing $I = (i_1, \ldots, i_j)$, we have

$$\eta_{s_1,\ldots,s_k}(M_I) = \sum_{n=1}^{\infty} \frac{1}{n^{s_1}(n + 1)^{s_2} \cdots (n + k - 1)^{s_k}} \sum_{1 \leq n_1 < \cdots < n_j \leq n} \frac{1}{n_1^{i_1} \cdots n_j^{i_j}}$$

$$= \sum_{1 \leq n_1 < \cdots < n_j} \frac{1}{n_1^{i_1} \cdots n_j^{i_j}} \sum_{m=n_j}^{\infty} \frac{1}{m^{s_1}(m + 1)^{s_2} \cdots (m + k - 1)^{s_k}}.$$
Now the terms in the latter sum are evidently bounded above by those for $\eta_2(M_I)$. But

$$\eta_2(M_I) = \sum_{1 \leq n_1 < \ldots < n_j} \frac{1}{n_1^{i_1} \ldots n_j^{i_j}} \sum_{m=n_j}^{\infty} \frac{1}{m^2} = \zeta(i_j + 2, i_{j-1}, \ldots, i_1) + \zeta(2, i_j, \ldots, i_1), \quad (21)$$

which converges. (In the case $I = \emptyset$, equation (21) should be interpreted as $\eta_2(1) = \zeta(2)$. In general $\eta_{s_1,\ldots,s_k}(1)$ is the “$H$-series” $H(s_1,\ldots,s_k)$ discussed in [16].)

In this section we shall be concerned with the examples $\eta_2$, $\eta_{1,1}$, and $\eta_{0,1,1}$. We already have equation (21) for $\eta_2$. For the other two functions we have the following result.

**Theorem 8.** For any composition $I = (i_1,\ldots,i_j)$,

$$\eta_{1,1}(M_I) = \zeta(i_{j} + 1, i_{j-1}, \ldots, i_1), \quad (22)$$

$$\eta_{0,1,1}(M_I) = \begin{cases} 1, & \text{if } I = (1), \\ \eta_{0,1,1}(M_{(i_1,\ldots,i_{j-1})}), & \text{if } i_j = 1 \text{ and } j \geq 2, \\ \zeta(i_j,\ldots,i_1) - \eta_{0,1,1}(M_{(i_1,\ldots,i_{j-1},i_{j-1})}), & \text{otherwise}. \end{cases} \quad (23)$$

**Proof.** Since

$$\sum_{m=n}^{\infty} \frac{1}{m(m+1)} = \frac{1}{n}$$

it follows that

$$\eta_{1,1}(M_I) = \sum_{1 \leq n_1 < \ldots < n_j} \frac{1}{n_1^{i_1} \ldots n_j^{i_j}} \sum_{m=n_j}^{\infty} \frac{1}{m(m+1)} = \zeta(i_j + 1, i_{j-1}, \ldots, i_1).$$
We have also
\[
\eta_{0,1,1}(M_I) = \sum_{1 \leq n_1 < \cdots < n_j} \frac{1}{n_1^{i_1} \cdots n_j^{i_j}} \sum_{m=n_j+1}^{\infty} \frac{1}{m(m+1)}
\]
\[
= \sum_{1 \leq n_1 < \cdots < n_j} \frac{1}{n_1^{i_1} \cdots n_j^{i_j} (n_j + 1)}.
\]

If \( I = (1) \), this is
\[
\eta_{0,1,1}(p_1) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.
\]

Now suppose \( I \neq (1) \). If \( i_j = 1 \), we have
\[
\eta_{0,1,1}(M_I) = \sum_{1 \leq n_1 < \cdots < n_{j-1}} \frac{1}{n_1^{i_1} \cdots n_{j-1}^{i_{j-1}} n_j (n_j + 1)}
\]
\[
= \sum_{1 \leq n_1 < \cdots < n_{j-1}} \frac{1}{n_1^{i_1} \cdots n_{j-1}^{i_{j-1}}} \sum_{m=n_{j-1}+1}^{\infty} \frac{1}{m(m+1)}
\]
\[
= \sum_{1 \leq n_1 < \cdots < n_{j-1}} \frac{1}{n_1^{i_1} \cdots n_{j-1}^{i_{j-1}} (n_{j-1} + 1)} = \eta_{0,1,1}(M(i_1,\ldots,i_{j-1})).
\]

On the other hand, if \( i_j > 1 \) we have
\[
\eta_{0,1,1}(M_I) = \sum_{1 \leq n_1 < n_2 < \cdots < n_j} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_{j-1}^{i_{j-1}} n_j^{i_j-1}} \left( \frac{1}{n_j} - \frac{1}{n_j + 1} \right)
\]
\[
= \zeta(i_j, i_{j-1}, \ldots, i_1) - \eta_{0,1,1}(M(i_1,\ldots,i_{j-1},i_j-1)),
\]
and the conclusion follows.

Now we consider the images under \( \eta_2 \), \( \eta_1 \) and \( \eta_{0,1,1} \) of the symmetric functions \( N_{n,k} \) introduced in §2 above. For integers \( n \geq 2 \) and \( 1 \leq k < n \), let \( S_{n,k} \) be the sum of all multiple zeta values of weight \( n \) and depth \( k \): by the sum theorem for multiple zeta values, \( S_{n,k} = \zeta(n) \). We can also write \( S_{n,k} = S_{n,k}^{[2]} + S_{n,k}^{r} \), where
\[
S_{n,k}^{[2]} = \sum_{2+i_2+\cdots+i_k=n} \zeta(2, i_2, \ldots, i_k).
\]

Note that \( S_{n,n-1}^{[2]} = S_{n,n-1} = \zeta(n) \), and that \( S_{n,1}^{r} = S_{n,1} = \zeta(n) \) for \( n > 2 \). The following fact is an immediate consequence of equation (21).
Proposition 3. \( \eta_2(N_{n,k}) = S_{n+2,k} + S_{n+2,k+1}'\).

Similarly, Theorem 8 gives the following.

Proposition 4. \( \eta_{1,1}(N_{n,k}) = S_{n+1,k} = \zeta(n+1) \).

Proposition 5. \( \eta_{0,1,1}(N_{n,n}) = 1 \), and if \( n \geq 2 \),

\[
\eta_{0,1,1}(N_{n,k}) = \begin{cases} 
\zeta(n) - \eta_{0,1,1}(N_{n-1,1}), & k = 1, \\
\zeta(n) + \eta_{0,1,1}(N_{n-1,k-1} - N_{n-1,k}), & 1 < k < n.
\end{cases}
\]

Proof. Note that \( N_{n,n} = e_n \), and by applying equation (23) repeatedly, we have

\[
\eta_{0,1,1}(e_n) = \eta_{0,1,1}(e_{n-1}) = \cdots = \eta_{0,1,1}(e_1) = 1.
\] (24)

The second statement is immediate from (23). \( \square \)

Note that equation (8) follows from the case \( k = n \) of the preceding result.

In the case \( k = 1 \), the result is

\[
\eta_{0,1,1}(p_n) = \zeta(n) - \eta_{0,1,1}(N_{n-1,1}) = \cdots = \\
\zeta(n) - \zeta(n-1) + \cdots + (-1)^n \zeta(2) + (-1)^{n+1}.
\] (25)

Equation (25) implies the following result, which appears in the remark following [20, Theorem 2.1].

Corollary 1. For \( k \geq 2 \),

\[
\sum_{n=1}^{\infty} \frac{H_n^{(k)}}{(n+1)(n+2)} = \sum_{i=0}^{k-2} (-1)^i \zeta(k - i) + (-1)^{k-1}.
\]

Now we in a position to prove Theorems 1 through 3 of the Introduction by finding the images under the three \( H \)-functions \( \eta_2, \eta_{1,1}, \) and \( \eta_{0,1,1} \) of the symmetric function \( e_j h_{n-j} \). First we consider \( \eta_2(e_j h_{n-j}) \).

Theorem 9. Let \( 0 \leq j \leq n \). Then

\[
\eta_2(e_j h_{n-j}) = \begin{cases} 
(n+1)\zeta(n+2), & j = 0, \\
\sum_{p=j}^{n} \binom{p-1}{j-1} S_{n+2,p} + \binom{n+1}{j+1} \zeta(n+2), & j \geq 1.
\end{cases}
\]
Proof. If \( j = 0 \) we have
\[
\eta_2(h_n) = \eta_2 \left( \sum_{k=1}^{n} N_{n,k} \right) = \sum_{k=1}^{n} (S^T_{n+2,k} + S^{[2]}_{n+2,k+1})
\]
\[
= S_{n+2,1} + \sum_{k=2}^{n} (S^{[2]}_{n+2,k} + S^T_{n+2,k}) + S_{n+2,n+1} = (n+1)\zeta(n+2).
\]
Now suppose \( j \geq 1 \). Then using Lemma \( 2 \)
\[
\eta_2(e_j h_{n-j}) = \sum_{k=j}^{n} \binom{k}{j} \eta_2(N_{n,k})
\]
\[
= \sum_{k=j}^{n} \binom{k}{j} (S^T_{n+2,k} + S^{[2]}_{n+2,k+1})
\]
\[
= S^T_{n+2,j} + \sum_{k=j+1}^{n} \left[ \binom{k-1}{j} S^{[2]}_{n+2,k} + \binom{k}{j} S^T_{n+2,k} \right] + \binom{n}{j} S_{n+2,n+1}
\]
\[
= S^T_{n+2,j} + \sum_{k=j+1}^{n} \left[ \binom{k-1}{j} \zeta(n+2) + \binom{k-1}{j-1} S^T_{n+2,k} \right] + \binom{n}{j} \zeta(n+2)
\]
\[
= \sum_{k=j}^{n} \binom{k-1}{j-1} S^T_{n+2,k} + \binom{n+1}{j+1} \zeta(n+2).
\]

Proof of Theorem \( 3 \). Equation (12) follows from the case \( j = 0 \) of the preceding result. By duality of multiple zeta values, \( S^T_{n,k} = S^R_{n,n-k} \), where \( S^R_{n,k} \) is the sum of all weight-\( n \), depth-\( k \) multiple zeta values whose exponent string ends in 1. From the preceding result we have
\[
\eta_2(e_n) = S^T_{n+2,n} + \zeta(n+2) = S^R_{n+2,2} + \zeta(n+2),
\]
from which follows
\[
\eta_2(e_n) = \zeta(n+1,1) + \zeta(n+2) = \frac{n+3}{2} \zeta(n+2) + \frac{1}{2} \sum_{j=2}^{n} \zeta(j) \zeta(n+2-j).
\]
using Euler’s formula (19), and thus equation (13). \( \square \)
We also have the following result.

**Corollary 2.** For \( n \geq 2 \),

\[
\eta_2(e_{n-1}h_1) = \zeta(n, 2) + n\zeta(n + 1, 1) + (n + 1)\zeta(n + 2).
\]

**Proof.** We have

\[
\eta_2(e_{n-1}h_1) = S^T_{n+2,n-1} + (n - 1)S^T_{n+2,n} + (n + 1)\zeta(n + 2)
\]

\[
= S^R_{n+2,3} + (n - 1)S^R_{n+2,2} + (n + 1)\zeta(n + 2)
\]

\[
= \sum_{j=2}^{n} \zeta(j, n + 1 - j, 1) + (n - 1)\zeta(n + 1, 1) + (n + 1)\zeta(n + 2)
\]

\[
= \zeta(n, 2) + \zeta(n + 1, 1) + (n - 1)\zeta(n + 1, 1) + (n + 1)\zeta(n + 2),
\]

where we have used the derivation theorem for multiple zeta values in the last step.

In general the formula for \( \eta_2(e_jh_{n-j}) \) given by Theorem 9 cannot be reduced to ordinary zeta values if \( 1 \leq j < n \), unless \( n \leq 5 \). For example,

\[
\eta_2(e_2h_2) = 10\zeta(6) + S^T_{6,2} + 2S^T_{6,3} + 3S^T_{6,4}
\]

\[
= 11\zeta(6) + 3S^R_{6,2} - S^R_{6,2} + 2S^R_{6,3}
\]

\[
= 11\zeta(6) + 3\zeta(5, 1) - \zeta(2, 4) + 2(\zeta(4, 1, 1) + \zeta(3, 2, 1) + \zeta(2, 3, 1))
\]

\[
= 11\zeta(6) + 3\zeta(5, 1) - \zeta(2, 4) + 2(\zeta(5, 1) + \zeta(4, 2))
\]

\[
= 11\zeta(6) + 5\zeta(5, 1) + 2\zeta(4, 2) - \zeta(2, 4)
\]

\[
= 10\zeta(6) + \frac{1}{2}\zeta(3)^2,
\]

but

\[
\zeta(e_5h_1) = \zeta(6, 2) + 6\zeta(7, 1) + 7\zeta(8) = \zeta(6, 2) + 6\zeta(3)\zeta(5) + \frac{83}{2}\zeta(8)
\]

has no known expression as a rational polynomial in the \( \zeta(i) \) since \( \zeta(6, 2) \) doesn’t.

**Proof of Theorem 2.** By Lemma 2 and Proposition 4

\[
\eta_{1,1}(e_jh_{n-j}) = \sum_{k=j}^{n} \binom{k}{j} \eta_{1,1}(N_{n,k}) = \sum_{k=j}^{n} \binom{k}{j} \zeta(n + 1),
\]

and the conclusion follows.
Proof of Theorem 1. It is enough to show that

\[ \eta_{0,1,1}(e_j h_{n-j}) = \begin{cases} \sum_{k=0}^{j} \binom{n-k}{j+1-k} \zeta(n-k) - \zeta(n-j), & j \leq n-2, \\ \sum_{k=0}^{n-2} \zeta(n-k) + 1, & j = n-1, \\ 1, & j = n, \end{cases} \]

for \( n \geq 2 \). Recall from equation (24) that \( \eta_{0,1,1}(\epsilon_n) = 1 \), so the result is true for \( j = n \). If \( j = 0 \), we have

\[ \eta_{0,1,1}(h_n) = \sum_{k=1}^{n} \eta_{0,1,1}(N_{n,k}) = S_{n,1} - \eta_{0,1,1}(N_{n-1,1}) + \eta_{0,1,1}(N_{n-1,1}) + S_{n,2} - \eta_{0,1,1}(N_{n-1,1}) + \cdots + \eta_{0,1,1}(N_{n-1,n-2}) + S_{n,n-1} - \eta_{0,1,1}(N_{n-1,n-1}) + 1 \]

and again the result holds. Now let \( 0 < j < n \). Then

\[ \eta_{0,1,1}(e_j h_{n-j}) = \sum_{k=j}^{n} \binom{k}{j} \eta_{0,1,1}(N_{n,k}) \]

\[ = \sum_{k=j}^{n-1} \binom{k}{j} [\zeta(n) + \eta_{0,1,1}(N_{n-1,k-1} - N_{n-1,k})] + \binom{k}{j} \]

\[ = \sum_{k=j}^{n} \binom{k}{j} \zeta(n) + \eta_{0,1,1}(N_{n-1,j-1}) + \sum_{k=j}^{n-2} \left[ \binom{k+1}{j} - \binom{k}{j} \right] \eta_{0,1,1}(N_{n-1,k}) - \binom{n-1}{j} + \binom{n}{j} \]

\[ = \binom{n}{j+1} \zeta(n) + \sum_{k=j}^{n-1} \binom{k}{j-1} \eta_{0,1,1}(N_{n-1,k}) \]

\[ = \binom{n}{j+1} \zeta(n) + \eta_{0,1,1}(e_{j-1} h_{n-1}), \]

and the result follows by induction on \( j \).

5 Further summation formulas

We return to the general \( H \)-functions \( \eta_{s_1,...,s_k} \) defined by equation (20). If \( s_k > 0 \), we call \( k \) the length of \( \eta_{s_1,...,s_k} \). We have the following result (cf. [16].
Lemma 1)

**Proposition 6.** Let \( s_1, \ldots, s_k \) be a nonnegative integer sequence with \( s_i, s_j \geq 1 \) for \( 1 \leq i < j \leq k \). If \( s_1 + \cdots + s_k \geq 3 \), then

\[
\eta_{s_1, \ldots, s_i, \ldots, s_j, \ldots, s_k} = \frac{1}{j - i} (\eta_{s_1, \ldots, s_i, \ldots, s_j - 1, \ldots, s_k} - \eta_{s_1, \ldots, s_i - 1, \ldots, s_j, \ldots, s_k}).
\]

**Proof.** This follows immediately from the definition and

\[
\frac{1}{(n + i - 1)(n + j - 1)} = \frac{1}{j - i} \left[ \frac{1}{n + i - 1} - \frac{1}{n + j - 1} \right].
\]

\( \square \)

5.1 Length 2

The following result is immediate from Proposition 6.

**Proposition 7.** Any \( H \)-function of length 2 can be written as a rational linear combination of the functions \( \eta_p, \eta_{0,p}, p \geq 2 \), and \( \eta_{1,1} \).

For example, since \( \eta_{2,1} = \eta_2 - \eta_{1,1} \) we have

\[
\sum_{n=1}^{\infty} \frac{H_n}{n^2(n+1)} = \eta_2(p_1^2) - \eta_{1,1}(p_1^2) = \eta_2(p_2) + 2\eta_2(e_2) - \eta_{1,1}(p_2) - 2\eta_{1,1}(e_2)
\]

\[
= \zeta(2,2) + \zeta(4) + 2\zeta(4) + 2\zeta(3,1) - \zeta(3) - 2\zeta(3) = \frac{17}{4}\zeta(4) - 3\zeta(3).
\]

In the preceding section we gave formulas for the values of \( \eta_2 \) and \( \eta_{1,1} \) on monomial quasi-symmetric functions \( M_I \). For \( \eta_p \) and \( \eta_{0,p} \) we have the following result.

**Proposition 8.** Let \( I = (i_1, \ldots, i_j) \) be a composition. If \( p \geq 2 \), then

\[
\eta_p(M_I) = \zeta(p, i_j, \ldots, i_1) + \zeta(p + i_j, i_{j-1}, \ldots, i_1)
\]

\[
\eta_{0,p}(M_I) = \zeta(p, i_j, \ldots, i_1).
\]

**Proof.** This is immediate from the equation

\[
\eta_{s_1, \ldots, s_k}(M_{(i_1, \ldots, i_j)}) = \sum_{1 \leq n_1 < n_2 < \cdots < n_j} \sum_{m=n_j}^{\infty} \frac{1}{n_1^{s_1} \cdots n_j^{s_j}} \sum_{m=n_j}^{\infty} \frac{1}{m^{s_1}(m+1)^{s_2} \cdots (m+k-1)^{s_k}}
\]

appearing in the proof of Theorem [7].

\( \square \)
This result has the following corollary, special cases of which have appeared in the literature. Special cases of the first equation appear many places, the case $k = 2$ of the second appears as [23, eqn. (2.5c)], and the third equation can be deduced from [8, Corollary 1].

**Corollary 3.** For $k \geq 1$,

$$\sum_{n=1}^{\infty} \frac{H_n^{(k)}}{n^3} = \zeta(k + 3) + \zeta(3, k)$$

$$\sum_{n=1}^{\infty} \frac{P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n^3} = \zeta(k + 2, 1) + \zeta(k + 1, 1, 1)$$

$$\sum_{n=1}^{\infty} \frac{Q_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n^3} = \zeta(k + 3) + \sum_{j=2}^{k+1} S_{k+3,j}. T_k^{(3,j)}.$$

**Proof.** In each case, apply the first part of Proposition 8 with $p = 3$. □

Another corollary with many special cases in the literature is the following.

**Corollary 4.** For $k \geq 1$,

$$\sum_{n=1}^{\infty} \frac{H_n^{(k)}}{(n + 1)^2} = \zeta(2, k) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{(n + 1)^2} = \zeta(k + 2).$$

**Proof.** Similar to that for the preceding corollary, using the second part of Proposition 8. □

**Remark 4.** In the case $k = 1$, both equations give

$$\sum_{n=1}^{\infty} \frac{H_n}{(n + 1)^2} = \zeta(3),$$

which appears as [5, eqn. (1.1a)]. In the case $k = 2$, the two equations give

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(n + 1)^2} = \zeta(2, 2) = \frac{3}{4} \zeta(4) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{(n + 1)^2} = 2 \zeta(4),$$

20
from which follows
\[ \sum_{n=1}^{\infty} \frac{H_n^2}{(n+1)^2} = \frac{11}{4} \zeta(4). \]
Cf. [11 eqn. (2)], [5] eqns. (1.2b),(1.4a),(1.5b), and [3] eqns. (24a),(24b).
In the case \( k = 3 \), the second equation gives
\[ \sum_{n=1}^{\infty} \frac{H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}}{(n+1)^2} = 6 \zeta(5), \]
which is [23 eqn. (2.3a)].

There is not in general a nice formula for \( \eta_{0,2}(h_k) \) or for \( \eta_3(h_k) \), but we do have the following result.

**Proposition 9.** \( \eta_{0,2}(h_{k+1}) + \eta_3(h_k) = (k+2)\zeta(k+3). \)

**Proof.** From the third equation of Corollary 3 we have
\[ \eta_3(h_k) = \zeta(k+3) + S_{k+3,2}^T + \cdots + S_{k+3,k+1}^T, \]
and from Proposition 8
\[ \eta_{0,2}(h_{k+1}) = \eta_{0,2}(N_{k+1,1} + \cdots + N_{k+1,k+1}) = S_{k+3,2}^{[2]} + S_{k+3,3}^{[2]} + \cdots + S_{k+3,k+2}^{[2]}, \]
Since \( S_{k+3,k+2}^{[2]} = \zeta(k+3) \), these two equations can be added to obtain the conclusion. \( \square \)

The preceding result can be written
\[
\sum_{n=1}^{\infty} \frac{Q_{k+1}(H_n, H_n^{(2)}, \ldots, H_n^{(k+1)})(n+1)^2}{n^3} + \sum_{n=1}^{\infty} \frac{Q_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n^3} = (k+2) \zeta(k+3).
\]

The second equation of Corollary 4 can be generalized as follows.

**Theorem 10.** For nonnegative integers \( l, k \),
\[
\sum_{n=0}^{\infty} \frac{Q_l(H_n, H_n^{(2)}, \ldots, H_n^{(l)}) P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{(n+1)^2} = \binom{l + k + 1}{k + 1} \zeta(l + k + 2) - \sum_{p=k}^{l+k-1} \binom{p}{k} S_{l+k+2,l+k+1-p}^{R}.
\]
Proof. From Proposition 8 we have $\eta_{0,2}(N_{n,k}) = S_{n+2,k+1}^{[2]}$. Hence, using Lemma 2 and the sum theorem for multiple zeta values,

$$
\eta_{0,2}(e_k h_l) = \sum_{p=k}^{l+k-1} \binom{p}{k} \eta_{0,2}(N_{l+k,p}) = \sum_{p=k}^{l+k} \binom{p}{k} S_{l+k+2,p+1}^{[2]}
$$

$$
= \sum_{p=k}^{l+k-1} \binom{p}{k} S_{l+k+2,p+1}^{[2]} + \left( \binom{l+k}{k} \right) \zeta(l+k+2)
$$

$$
= \sum_{p=k}^{l+k-1} \binom{p}{k} \left( \zeta(l+k+2) - S_{l+k+2,p+1}^{T} \right) + \left( \binom{l+k}{k} \right) \zeta(l+k+2)
$$

$$
= \left( \binom{l+k}{k+1} + \binom{l+k}{k} \right) \zeta(l+k+2) - \sum_{p=k}^{l+k-1} \binom{p}{k} S_{l+k+2,l+k+1-p}^{R},
$$

and the result follows.

From this we can deduce the following result. The special case $k = 2$ of the first equation appears as [23, eqn. (2.3b)].

**Corollary 5.** For $k \geq 1$,

$$
\sum_{n=0}^{\infty} \frac{H_n P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{(n+1)^2} = (k + 2)\zeta(k+3) - \zeta(k+2,1)
$$

$$
\frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{(n+1)^2} P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)}) = \binom{k+3}{2} \zeta(k+4) - (k + 2)\zeta(k+3,1) - \zeta(k+2,2).
$$

**Remark 5.** From the case $k = 2$ of the second equation follows

$$
\sum_{n=1}^{\infty} \frac{H_n^4}{(n+1)^2} = \frac{100}{3} \zeta(6) + 4\zeta(3)^2 + \zeta(2,4) + 2\zeta(2,2,2) = \frac{859}{24} \zeta(6) + 3\zeta(3)^2,
$$

which was stated as a conjecture in [6].

**Proof of Theorem 4.** From Lemma 1 it follows that

$$
Q_l(H_{n+1}, H_{n+1}^{(2)}, \ldots, H_{n+1}^{(l)}) = \sum_{j=0}^{l} \frac{Q_j(H_n, H_n^{(2)}, \ldots, H_n^{(j)})}{(n+1)^{l-j}}
$$
and so
\[
\sum_{n=0}^{\infty} Q_l(H_{n+1}, H_{n+1}^{(2)}, \ldots, H_{n+1}^{(l)}) P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)}) / (n+1)^2 = 
\sum_{j=0}^{l} \sum_{n=1}^{\infty} Q_j(H_n, H_n^{(2)}, \ldots, H_n^{(j)}) P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)}) / (n+1)^{2+l-j}.
\]

Comparing this with Theorem 10, we see that to prove the result it suffices to show
\[
\sum_{j=0}^{l-1} \sum_{n=1}^{\infty} Q_j(H_n, H_n^{(2)}, \ldots, H_n^{(j)}) P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)}) / (n+1)^{2+l-j} = \sum_{p=k}^{l+k-1} \binom{p}{k} S_T^{l+k+2, p+1},
\]
or
\[
\sum_{j=3}^{l+2} \eta_0,j(e_k h_{2+l-j}) = \sum_{p=k}^{l+k-1} \binom{p}{k} S_T^{l+k+2, p+1}. \quad (26)
\]

Using Lemma 2 and Proposition 8, the left-hand side of (26) is
\[
\sum_{j=3}^{l+2} \sum_{p=k}^{k+l+2-j} \binom{p}{k} \eta_0,j(N_{k+l+2-j, p}) = \sum_{j=3}^{l+2} \sum_{p=k}^{k+l+2-j} \binom{p}{k} S_T^{l+k+2, p+1},
\]
where \(S_T^{n,k}\) is the sum of all weight \(n\), depth \(k\) multiple zeta values whose exponent string starts with \(j\). This can be rearranged as
\[
\sum_{j=3}^{k+l-1} \sum_{p=k}^{k+l+2-p} \binom{p}{k} S_T^{l+k+2, p+1} = \sum_{j=3}^{k+l-1} \sum_{p=k}^{k+l-1} \binom{p}{k} S_T^{l+k+2, p+1},
\]
and equation (26) follows.

5.2 Length 3

For length 3 \(H\)-functions, Proposition 6 gives the following result.

**Proposition 10.** Any \(H\)-function of length 3 can be written as a rational linear combination of the functions \(\eta_p, \eta_0,p, \eta_0,0,p, p \geq 2, \eta_{1,1}, \) and \(\eta_{0,1,1}.\)
For example, we can prove Theorem 5 by writing \( \eta_{1,1} \) as a linear combination of previously studied functions.

**Proof of Theorem 5.** From Proposition 6, \( \eta_{1,1} = \frac{1}{2}(\eta_{1,1} - \eta_{0,1,1}) \). Hence we can use Theorems 1 and 2 to get

\[
\eta_{1,1}(e^j h_{n-j})
= \begin{cases}
\frac{1}{2} \left[ (n+1) \zeta(n+1) - \sum_{k=0}^{j} (n-k) \zeta(n-k) - \zeta(n-j) \right], & j \leq n-2, \\
\frac{1}{2} \left[ (n+1) \zeta(n+1) - \sum_{k=0}^{n-2} \zeta(n-k) - 1 \right], & j = n-1, \\
\frac{1}{2} (\zeta(n+1) - 1), & j = n,
\end{cases}
\]

from which the conclusion follows.

We have developed formulas for \( \eta_{1,1} \), \( \eta_{0,1,1} \), \( \eta_p \), and \( \eta_{0,p} \) of a monomial quasi-symmetric function \( M_I \). For \( \eta_{0,0,p}(M_I) \) we have the following result.

**Proposition 11.** If \( p \geq 2 \), then

\[
(\eta_{0,p} - \eta_{0,0,p})(M_I) = \sum_{1 \leq n_1 < n_2 < \cdots < n_j} \frac{1}{n_1^{i_1} \cdots n_j^{i_j} (n_j + 1)^p}
\]

for any composition \( I = (i_1, \ldots, i_j) \).

In particular, we have the following.

**Proposition 12.** Let \( T = \eta_{0,2} - \eta_{0,0,2} \). Then \( T(p_1) = 2 - \zeta(2) \), and for a composition \( I = (i_1, \ldots, i_j) \neq (1) \),

\[
T(M_I) = \begin{cases}
\zeta(i_j, \ldots, i_1) - \eta_{0,1,1}(M_{(i_1, i_1, \ldots, i_j)}) - T(M_{(i_1, i_1, \ldots, i_j)}), & i_j > 1, \\
\eta_{0,1,1}(M_{(i_1, i_1, \ldots, i_j)}) - \zeta(2, i_j-1, \ldots, i_1) + T(M_{(i_1, i_1, \ldots, i_j)}), & i_j = 1.
\end{cases}
\]

**Proof.** The statement about \( (\eta_{0,2} - \eta_{0,0,2})(p_1) \) follows immediately from the preceding result. Also, if \( I \neq (1) \) we have

\[
(\eta_{0,2} - \eta_{0,0,2})(M_I) = \sum_{1 \leq n_1 < \cdots < n_j} \frac{1}{n_1^{i_1} \cdots n_j^{i_j} (n_j + 1)^2}
= \sum_{1 \leq n_1 < \cdots < n_j} \frac{1}{n_1^{i_1} \cdots n_j^{i_j} (n_j + 1)} - \sum_{1 \leq n_1 < \cdots < n_j} \frac{1}{n_1^{i_1} \cdots n_j^{i_j-1} (n_j + 1)^2}.
\]

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If \( n_j > 1 \), this is
\[
\eta_{0,1,1}(M_{i_1, \ldots, i_{j-1}}) = (\eta_{0,2} - \eta_{0,0,2})(M_{i_1, \ldots, i_{j-1}, i_j}) = \\
\zeta(i_j, \ldots, i_{j-1}) - \eta_{0,1,1}(M_{i_1, \ldots, i_{j-1}, i_j}) - (\eta_{0,2} - \eta_{0,0,2})(M_{i_1, \ldots, i_{j-1}, i_j})
\]
using Theorem 8. If \( n_j = 1 \), \( (\eta_{0,2} - \eta_{0,0,2})(M_I) \) is
\[
\eta_{0,1,1}(M_{i_1, \ldots, i_{j-1}}) - \sum_{1 \leq n_1 < \cdots < n_j} \frac{1}{n_1^{i_1} \cdots n_{j-1}^{i_{j-1}}(n_j + 1)^2} = \\
\eta_{0,1,1}(M_{i_1, \ldots, i_{j-1}}) - \zeta(2, i_{j-1}, \ldots, i_1) + \sum_{1 \leq n_1 < \cdots < n_{j-1}} \frac{1}{n_1^{i_1} \cdots n_{j-1}^{i_{j-1}}(n_{j-1} + 1)^2} = \\
\eta_{0,1,1}(M_{i_1, \ldots, i_{j-1}}) - \zeta(2, i_{j-1}, \ldots, i_1) + (\eta_{0,2} - \eta_{0,0,2})(M_{i_1, \ldots, i_{j-1}})
\]
again using Theorem 8.

It follows that
\[
(\eta_{0,2} - \eta_{0,0,2})(e_k) = 1 - \zeta(k + 1) + (\eta_{0,2} - \eta_{0,0,2})(e_{k-1}) = \cdots = (k + 1) - \zeta(k + 1) - \zeta(k) - \cdots - \zeta(2).
\]
Similarly,
\[
(\eta_{0,2} - \eta_{0,0,2})(p_k) = \eta_{0,1,1}(p_k) - (\eta_{0,2} - \eta_{0,0,2})(p_{k-1}),
\]
which together with equation (25) implies
\[
(\eta_{0,2} - \eta_{0,0,2})(p_k) = \sum_{j=0}^{k-3} (-1)^j (j + 1)\zeta(k - j) + (-1)^k k\zeta(2) + (-1)^{k+1}(k + 1).
\]
In view of Corollary 4, the preceding equations imply
\[
\eta_{0,0,2}(e_k) = \sum_{j=2}^{k+2} \zeta(j) - (k + 1)
\]
and
\[
\eta_{0,0,2}(p_k) = \zeta(2, k) + \sum_{j=0}^{k-3} (-1)^{j+1}(j + 1)\zeta(k - j) - (-1)^k k\zeta(2) + (-1)^k(k + 1).
\]
5.3 General length

For general length we have Theorem 6, which can be thought of as generalizing $\eta_{0,1,1}(e_k) = 1$.

Proof of Theorem 6. We use induction on $k$. For the base case $k = 0$ we must show

$$\frac{1}{(q-1)!} \frac{1}{q-1} = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2) \cdots (n+q)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1) \cdots (n+q-1)}.$$

By [16, Theorem 2] the right-hand side is

$$H(1, \ldots, 1, q) = \frac{1}{(q-1)!} \sum_{i=0}^{q-2} \binom{q-2}{i} (-1)^i$$

$$= \frac{1}{(q-1)!} \cdot \frac{1}{q-1} \sum_{i=0}^{q-2} (-1)^i \left(\frac{q-1}{i+1}\right) = \frac{1}{(q-1)!} \cdot \frac{1}{q-1}.$$

Now assume inductively that

$$\eta_{0,1,\ldots,1}(e_k) = \frac{1}{(q-1)!} \cdot \frac{1}{(q-1)^{k+1}}.$$

Then using partial fractions followed by telescoping we have

$$\eta_{0,1,\ldots,1}(e_{k+1}) = \sum_{1 \leq n_1 < \cdots < n_{k+2}} \frac{1}{n_1 \cdots n_{k+1} n_{k+2} (n_{k+2} + 1) \cdots (n_{k+2} + q - 1)}$$

$$= \frac{1}{q-1} \sum_{1 \leq n_1 < \cdots < n_{k+2}} \left[ \frac{1}{n_1 \cdots n_{k+1} n_{k+2} (n_{k+2} + 1) \cdots (n_{k+2} + q - 2)} \right.$$

$$- \frac{1}{n_1 \cdots n_{k+1} (n_{k+2} + 1) \cdots (n_{k+2} + q - 1)} \left.] \right.$$

$$= \frac{1}{q-1} \sum_{1 \leq n_1 < \cdots < n_{k+1}} \frac{1}{n_1 \cdots n_{k+1} (n_{k+1} + 1) \cdots (n_{k+1} + q - 1)}$$

$$= \frac{1}{q-1} \eta_{0,1,\ldots,1}(e_k) = \frac{1}{(q-1)!} \cdot \frac{1}{(q-1)^{k+2}},$$

where we used the induction hypothesis in the last step. \qed
Remark 6. J. Spieß [21, Theorem 15] provides a formula for the partial sum
\[
\sum_{n=0}^{m} \frac{P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{(n+1)(n+2)\cdots(n+q)}
\]
which in the limit \( m \to \infty \) gives Theorem 6.

From Theorem 6 we can obtain the following result, which generalizes the case \( k = 0 \) of Theorems 2 and 5.

Corollary 6. For positive integers \( k, q \) with \( q > 1 \),
\[
\sum_{n=1}^{\infty} \frac{P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n(n+1)\cdots(n+q-1)} = \frac{1}{(q-1)!} \left[ \zeta(k+1) - \sum_{j=1}^{q-1} \frac{1}{j^{k+1}} \right].
\]

Proof. It suffices to show that
\[
\eta_{1,\ldots,1}(e_k)_{\frac{q}{q+1}} = \frac{1}{(q-1)!} \left[ \zeta(k+1) - \sum_{j=1}^{q-2} \frac{1}{j^{k+1}} \right]. \tag{27}
\]
We prove this by induction on \( q \), the base case \( q = 2 \) being the case \( l = 0 \) of Theorem 2. Assume inductively that equation (27) holds. By Proposition 6
\[
\eta_{1,\ldots,1}(e_k)_{\frac{q}{q+1}} = \frac{1}{q} \left[ \eta_{1,\ldots,1}(e_k)_{\frac{q}{q+1}} - \eta_{0,1,\ldots,1}(e_k)_{\frac{q}{q+1}} \right].
\]
Using the induction hypothesis (27) and Theorem 6 this can be written
\[
\frac{1}{q!} \left[ \zeta(k+1) - \sum_{j=1}^{q-2} \frac{1}{j^{k+1}} - \frac{1}{(q-1)^{k+1}} \right] = \frac{1}{q!} \left[ \zeta(k+1) - \sum_{j=1}^{q-1} \frac{1}{j^{k+1}} \right].
\]

\[
\square
\]

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