New solvable Matrix Integrals

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Abstract

We generalize the Harish-Chandra-Itzykson-Zuber and certain other integrals (Gross-Witten integral and integrals over complex matrices) using the notion of tau function of matrix argument. In this case one can reduce multi-matrix integrals to the integral over eigenvalues, which in turn is a certain tau function. We also consider a generalization of the Kontsevich integral.

1 Introduction

The study of different models of random matrices is one of the most interesting topic in mathematical physics (see [1],[2],[3],[4] for a review). Models of random matrices are applied to various topics starting with quantum field theory and condensed matter physics ending with combinatorics, algebraic topology and number theory. The purpose of the paper is to generalize some known solvable matrix integrals. By the solvable matrix integrals we mean those which can be presented as tau functions and which can be evaluated by the method of orthogonal polynomials. Let us mark that in papers [5],[6] by solvability of matrix model the author means a weaker condition - a model considered to be solvable if it admits a reduction of number of integrations (from the order $\sim N^2$ to the order $N$) via a character expansion method without any relation to the tau function topic.

Today it is not surprising that the notion of tau function, introduced by Sato school (see a list of references in [7]), is widely used in statistical physics and quantum field theory as a partition function of different models (as an example see the review [8]). The special class of tau functions, which we shall exploit in the present paper and which we call tau functions of hypergeometric type, is not an exclusion. Tau functions of hypergeometric type are parameterized by a function $r$ on the lattice and are denoted by $\tau_r$. (The name hypergeometric tau function [9] is due to the fact that the simplest specialization, $r$ is rational, yields the hypergeometric function of matrix argument [10],[11].) This is a type of tau functions one may meet in many different problems, and special examples of these tau functions one can find in [12],[13],[14],[15], all these examples one may obtain by a specialization of $r$. The most of well known matrix integrals actually belong to this class of tau-functions [16],[17],[18]. (To be precise we deal with asymptotic expansion of matrix integrals and tau functions).

In the present paper we show that the integrals of tau functions of hypergeometric type of matrix argument are again tau functions of hypergeometric type. We notice that all known matrix integrals such as one-matrix model [19], Itzykson-Zuber integral [20],[21], two-matrix matrix model [21], Gross-Witten one plaquette model [22] may be considered as integrals of the simplest tau function (so-called vacuum tau function). The determinant representation of the tau functions of hypergeometric type allows to obtain generalizations of solvable multi-matrix models and a certain generalization of Kontsevich integral.
In the last part of this introduction we review some things from soliton theory.

**Soliton theory.** KP hierarchy of integrable equations [23],[7], which is the most popular example, consists of semi-infinite set of nonlinear partial differential equations

\[
\partial_{t_m} u = K_m[u], \quad m = 1, 2, \ldots ,
\]

which are commuting flows: \([\partial_{t_k}, \partial_{t_m}] u = 0\). The first nontrivial one is Kadomtsev-Petviashvili equation

\[
\partial_{t_3} u = \frac{1}{4} \partial^3_{t_1} u + \frac{3}{4} \partial^{-1}_{t_1} \partial^2_{t_2} u + \frac{3}{4} \partial_{t_1} u^2 ,
\]

which originally served in plasma physics [23], now plays a very important role both, in physics and in mathematics. Another very important equation is the equation of two-dimensional Toda lattice (TL) carefully studied in [24]

\[
\partial_{t_1} \partial_{t^*_1} \phi_n = e^{\phi_{n+1}} - e^{\phi_n} - e^{\phi_{n+1}} \tau_n(t, t^*) (3)
\]

This equation gives rise to TL hierarchy which contains derivatives with respect to the higher times \(t_1, t_2, \ldots \) and \(t^*_1, t^*_2, \ldots \).

The key point of the soliton theory is the notion of tau function, introduced by Sato (for KP tau-function see [7]). The tau function is a sort of a potential which gives rise both to TL hierarchy and KP hierarchy. It depends on two semi-infinite sets of higher times \(t_1, t_2, \ldots \) and \(t^*_1, t^*_2, \ldots \), and on discrete variable \(n\): \(\tau = \tau(n, t, t^*)\). More explicitly we have [7],[24]:

\[
u = 2\partial_{t_1}^2 \log \tau(n, t, t^*) , \quad \phi_n(t, t^*) = -\log \frac{\tau(n+1, t, t^*)}{\tau(n, t, t^*)} \quad (4)
\]

In the soliton theory so-called Hirota-Miwa variables \(x, y\), which are related to the higher times as

\[
mt_m = \sum_i x_i^m , \quad mt_m^* = \sum_i y_i^m ,
\]

are well-known. Any tau function is a symmetric function in Hirota-Miwa variables, the higher times \(mt_m, mt_m^*\) are so-called [25] power sums.

It is known fact that a typical tau function may be presented in the form of double series over partitions in Schur functions [26]:

\[
\tau(n, t, t^*) = \sum_{\lambda, \mu} K_{\lambda \mu} s_\lambda(t)s_{\mu}(t^*) ,
\]

where the coefficients \(K_{\lambda \mu}\) solve special bilinear equations [7].

**\(\tau\) functions of hypergeometric type.** Let us consider a function \(r\) which depends on a single variable \(n\), the \(n\) is an integer. We suppose \(r(n)\) to be finite for integer \(n\). Given partition \(\lambda\), we define

\[
r_\lambda(x) = \prod_{i,j \in \lambda} r(x + j - i) \quad (7)
\]

Namely, \(r_\lambda(n)\) is a product of \(r\) over all nodes of the Young diagram of the partition \(\lambda\) where argument of \(r\) is defined by entries \(i, j\) of a node [25]. The value of \(j - i\) is zero on the main diagonal; the value \(j - i\) is called the content of the node. For zero partition one puts \(r_0 \equiv 1\).

It was shown [27] that

\[
\tau_r(n, t, t^*) = \sum_{\lambda} r_\lambda(n)s_\lambda(t)s_{\lambda}(t^*) \quad (8)
\]
Throughout the text \( s_\lambda(t) = \det h_{\lambda_i - i + j(t)} \) is called the elementary Schur function, or, the same, the complete symmetric function, see [25]. We should say more about the case \( t = t(x^n) \). Mark that \( s_\lambda(t(x^n)) = 0 \) if the length of the partition \( l(\lambda) \) exceeds \( n \), therefore, in this case, the sum over \( \lambda \) (8) is restricted by \( l(\lambda) \leq n \). Due to (7), it means that \( r(k), k \leq 0 \) are not involved to the series (8) if \( t = t(x^n) \).

If \( x^n = (x_1, \ldots, x_n) \) are eigenvalues of a \( n \times n \) matrix \( X \), and the variables \( t \) and \( x^n \) are related by (5), we write \( s_\lambda(X) := s_\lambda(t(x^n)) \). It is suitable to introduce the notion of \textit{tau function of matrix argument} by analogy with the hypergeometric function of matrix argument (see (18),(20) below):

\[
\tau(n, X, t^*) := \tau(n, t(x^n), t^*) \quad \tau(n, X, Y) := \tau(n, t(x^n), t^*(y^n)) ,
\]

where Hirota-Miwa variables \( x_1, \ldots, x_n \) are just eigenvalues of a matrix \( X \). We shall use large letters for this matrix argument. As in the case of the hypergeometric functions the tau function by definition depends only on eigenvalues of the matrix.

## 2 Useful formulae

We shall exploit the following formulae for integrations of Schur functions over the unitary group [25]. Let \( d_sU \) be the normalized Haar measure on the \( U(n) \) and let \( \delta_{\mu, \lambda} \) be the Kronecker symbol. Then

\[
\int_{U(n)} s_\lambda(AUBU^{-1})d_sU = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(I_n)} ,
\]

(11)

\[
\int_{U(n)} s_\mu(AU)s_\lambda(U^{-1}B)d_sU = \frac{s_\lambda(AB)}{s_\lambda(I_n)} \delta_{\mu, \lambda}
\]

(12)

Let \( t_\infty = (1, 0, 0, \ldots) \). For integration over complex matrices \( Z \) there are the formulae [25]

\[
\int_{C^{n^2}} s_\lambda(AZB^*) e^{-\text{Tr}ZZ^+} \prod_{i,j=1}^n d^2Z = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(t_\infty)}
\]

(13)

and

\[
\int_{C^{n^2}} s_\mu(AZ)s_\lambda(Z^*B) e^{-\text{Tr}ZZ^+} \prod_{i,j=1}^n d^2Z = \frac{s_\lambda(AB)}{s_\lambda(t_\infty)} \delta_{\mu, \lambda}
\]

(14)

Throughout the text \( d^2Z = \pi^{-n^2} \prod_{i,j=1}^n d\Re Z_{ij} d\Im Z_{ij} \); at sequel we shall omit \( C^{n^2} \). Then we need [25]

\[
s_\lambda(t_\infty) = \frac{1}{H_\lambda} , \quad H_\lambda = \prod_{i,j} (\lambda_i + \lambda'_j - i - j + 1)
\]

(15)

where \( H_\lambda \) is the hook product. The \textit{Pochhammer’s symbol related to a partition} \( \lambda = (\lambda_1, \ldots, \lambda_k) \) is the following product of the Pochhammer’s symbols \( (a)_\lambda = (a)_{\lambda_1}(a - 1)_{\lambda_2} \cdots (a - k + 1)_{\lambda_k} \), \( (a)_{\lambda_i} = \Gamma(a + \lambda_i)/\Gamma(a) \). Let \( t(a) = (a, a, a, \ldots) \). Then

\[
(a)_\lambda = H_\lambda s_\lambda(t(a)) = \frac{s_\lambda(t(a))}{s_\lambda(t_\infty)} , \quad (n)_\lambda = H_\lambda s_\lambda(I_n) = \frac{s_\lambda(I_n)}{s_\lambda(t_\infty)}
\]

(16)
In addition we have a simple relation (bosonic (l.h.s) and fermionic (r.h.s) representation of the vacuum TL tau function, since we did not find it in literature we proved it in [28]):

\[
\exp \sum_{m=1}^{\infty} mt_m t_m^* = \sum_{\lambda} s_\lambda(t) s_\lambda(t^*),
\]

which is a generalized version of Cauchy-Littlewood identity [25].

Let \( r(k) = q^k \). One gets tau function used in [29] as a generating function for double Hurwitz numbers.

Let \( r(k) = \prod_{i=1}^p (k + a_i) \prod_{i=1}^q (k + b_i)^{-1} \), and let \( l(\lambda) \) be the number of non-vanishing parts of \( \lambda \). Let us write down the formulae for the \textit{hypergeometric function of a matrix argument} \( X \) [11],[27] (throughout the paper all sums include zero partition)

\[
pF_s \left( \begin{array}{c} a_1 + M, \ldots, a_p + M \\ b_1 + M, \ldots, b_q + M \end{array} \right) X = \tau_r(M, t(x^n), t(x^n)) = \\
\sum_{\lambda \atop l(\lambda) \leq n} \frac{\prod_{k=1}^p s_\lambda(t(a_k + M))}{\prod_{k=1}^q s_\lambda(t(b_k + M))} (s_\lambda(t(x^n)))^{s-p+1} s_\lambda(X)
\]

\[ (18) \]

\textbf{Examples:}

\[
_0F_0(X) = e^{Tr X}, \quad _1F_0(a|X) = e^{\sum_{m=1}^\infty \frac{\lambda^m}{m!}Tr X^m} = \det(1 - X)^{-a}
\]

Now let \( r(k) = \prod_{i=1}^p (k + a_i)/(k + n - M)^{-1} \prod_{i=1}^q (k + b_i)^{-1} \). The \textit{hypergeometric functions of two matrix arguments} \( X, Y \) is [11],[27]

\[
pF_s \left( \begin{array}{c} a_1 + M, \ldots, a_p + M \\ b_1 + M, \ldots, b_q + M \end{array} \right) X, Y = \tau_r(M, t(x^n), t^*(y^n)) = \\
\sum_{\lambda \atop l(\lambda) \leq n} \frac{\prod_{k=1}^p s_\lambda(t(a_k + M))}{\prod_{k=1}^q s_\lambda(t(b_k + M))} (s_\lambda(t(x^n)))^{s-p+1} s_\lambda(X)s_\lambda(Y)
\]

\[ (20) \]

Here \( x^n = (x_1, \ldots, x_n), y^n = (y_1, \ldots, y_n) \) are eigenvalues of the matrices \( X, Y \). Matrices \( X, Y \) are supposed to be diagonalizable (i.e. normal) matrices.

Taking \( b_1 = -c + \epsilon \) where \( c \geq -n \) is an integer, we obtain [18]

\[
\lim_{\epsilon \to 0} \quad _pF_q \left( \begin{array}{c} a_1, \ldots, a_p \\ -c + \epsilon, b_2, \ldots, b_q \end{array} \right) X = \frac{(-c + \epsilon)_\sigma (b_1)_\sigma \cdots (b_q)_\sigma}{(a_1)_\sigma \cdots (a_p)_\sigma}(n)_\sigma
\]

\[ (21) \]

\[
= \det X^{n+c} \quad _pF_q \left( \begin{array}{c} a_1 + n + c, \ldots, a_p + n + c \\ 2n + c, b_2 + n + c, \ldots, b_q + n + c \end{array} \right) X
\]

\[ (22) \]

where \( \sigma \) is the partition \( (n + c, \ldots, n + c) \) with the length \( l(\sigma) = n \). Meanwhile

\[
\lim_{\epsilon \to 0} \quad _pF_q \left( \begin{array}{c} a_1, \ldots, a_p \\ -c + \epsilon, b_2, \ldots, b_q \end{array} \right) X, Y = \frac{(-c + \epsilon)_\sigma (b_1)_\sigma \cdots (b_q)_\sigma}{(a_1)_\sigma \cdots (a_p)_\sigma}(n)_\sigma
\]

\[ (23) \]

\[
= \det X^{n+c} \det Y^{n+c} \quad _pF_q \left( \begin{array}{c} a_1 + n + c, \ldots, a_p + n + c \\ 2n + c, b_2 + n + c, \ldots, b_q + n + c \end{array} \right) X, Y
\]

In particular

\[
\lim_{\epsilon \to 0} \quad _1F_1 \left( \begin{array}{c} n \\ -c + \epsilon \end{array} \right) X = _0F_0(X) \quad \det X^{n+c} = e^{TrX} \det X^{n+c}
\]
\[
\lim_{\epsilon \to 0} \binom{a, n}{-c + \epsilon} X \binom{-c + \epsilon}{a} = \det X^{n+c} F_0 (a + n + c | X) = \det X^{n+c} \det (1 - X)^{-a-n-c} \tag{26}
\]

These relations are particular cases of the limit of \( \tau_r \), where we put \( r \) to be singular: \( r \) is evaluated at the vicinity of zero as \( r(\epsilon) = \frac{1}{\epsilon} + O(\epsilon) \). Below \( c \geq -n \) is an integer and \( \sigma \) is a partition \((n + c, \ldots, n + c)\) of the length \( l(\sigma) = n \). Then we have [18]

\[
\lim_{\epsilon \to 0} \frac{\tau_r (-c + \epsilon, t(x^n), t_{\infty})}{r_{\sigma} (-c + \epsilon)} = \frac{\det X^{n+c}}{(n)_{\sigma} \tau_r (n, t(x^n), t_{\infty})}, \quad \tilde{r}(k) = r(k) \frac{k}{k + n + c}, \tag{27}
\]

\[
\lim_{\epsilon \to 0} \frac{\tau_r (-c + \epsilon, t(x^n), t^*(y^n))}{r_{\sigma} (-c + \epsilon)} = \det X^{n+c} \det Y^{n+c} \tau_r (n, t(x^n), t^*(y^n)) \tag{28}
\]

Let us mark also the determinant representations of (18) and of (20) which are of importance in applications of the method of orthogonal polynomials to the new matrix models, which we shall consider below.

The determinant representations have forms (the reader may find details in [16],[27]):

\[
\tau_r (M, t(x^n), t^*) = \frac{\det x_i^{n-k} \tau_r (M - k + 1, t(x_i), t^* |_{i,k=1})}{\Delta (x^n)}, \tag{29}
\]

(which in particular yields the determinant representation of \( p F_s \) of (18)) and

\[
\tau_r (M, t(x^n), t^*(y^n)) = c_n (M) \frac{\det \tau_r (M - n + 1, x_i, y_j |_{i,j=1})}{\Delta (x^n) \Delta (y^n)}, \tag{30}
\]

(which yields the determinant representation of \( p F_s \), see (35) below), where

\[
\Delta (x^n) = \prod_{1 \leq i < j \leq n} (x_i - x_j) = \det (x_i^{n-k}), \tag{31}
\]

\[
c_1 = 1, \quad c_n (M) = \prod_{k=1}^{n-1} (r(M - n + k))^{k-n}, \quad n > 1, \tag{32}
\]

\[
\tau_r (M - n + 1, x_i, y_j) = 1 + r(M - n + 1)x_iy_j + r(M - n + 1)r(M - n + 2)x_i^2y_j^2 + \cdots, \tag{33}
\]

where \( c_n^{-1} (M) \Delta (x^n) \Delta (y^n) \) is supposed to be non-vanishing; otherwise the formula (30) should be modified, see [18]. Most often we need the cases \( M = 0, n \). Examples of (29),(30):

\[
p F_s \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_s \end{array} \right) X = \frac{1}{\Delta (x^n)} \det x_i^{n-k} p F_s \left( \begin{array}{c} a_1 - k + 1, \ldots, a_p - k + 1 \\ b_1 - k + 1, \ldots, b_s - k + 1 \end{array} \right) |_{i,k=1} \tag{34}
\]

\[
p F_s \left( \begin{array}{c} a_1 + a_n, \ldots, a_p + a_n \\ b_1 + b_n, \ldots, b_s + b_n \end{array} \right) X, Y = \frac{c_n (n)}{\Delta (x^n) \Delta (y^n)} \det p F_s \left( \begin{array}{c} a_1 + 1, \ldots, a_p + 1 \\ b_1 + 1, \ldots, b_s + 1 \end{array} \right) |_{i,j=1} \tag{35}
\]

If \( r(0) = 0 \) there is the following formula for the open Toda lattice (for instance see [42],[16]):

\[
\tau_r (n, t, t^*) = c_n \det \left( \frac{\partial^{k-2}}{\partial t_1^{k-2} \partial t_{k-1}^{k-1}} \tau_r (1, t, t^*) \right) |_{i,k=1} \tag{36}
\]

\[
c_n = \prod_{k=1}^{n-1} (r(k))^{k-n} \tag{37}
\]

At last let us notice that \( \tau_r (n, t, t^*) \) enjoys the following symmetry relations:

\[
\tau_r (n, t, t^*) = \tau_r (n, t^*, t), \quad \tau_r (n + m, t, t^*) = \tau_r (n, t^*, t), \tag{38}
\]

\[
\tau_r (n, t_{\infty}, t^*) = \tau_{r_1} (n, t(a), t^*) = \tau_{r_2} (n, I_n, t^*), \tag{39}
\]

where \( I_n \) is \( n \) by \( n \) unit matrix, and where \( \tilde{r}(k) = r(k + m), \quad r_1(k) = (k + a)^{-1} r(k), \quad r_2(k) = (k + n)^{-1} r(k) \), the relations (39) are derived from (16), for details see [27].
3 Tau functions of a matrix argument and angle integration. The integration of tau functions over complex matrices

**Theorem 1** Let \( x_i \) and \( y_i \), where \( i = 1, \ldots, n \), are eigenvalues of matrices \( X \) and \( Y \) respectively. If for \( n > 1 \)

\[
c_n(n) = \prod_{k=1}^{n-1} (r(k))^{k-n} \neq 0 ,
\]

then there is the following generalization of HCIZ integral

\[
\int_{U(n)} \tau_r\left(n, XUYU^+, I_n\right) d_s U = \tau_r\left(n, X, Y\right) = c_n(n) \frac{\det(\tau_r(1, x_i, y_j))_{i,j=1,\ldots,n}}{\Delta(x^n)\Delta(y^n)} ,
\]

where \( \Delta(x^n), \Delta(y^n) \) are the Vandermond determinants.

The proof follows directly from (8),(11),(30). In particular we have (see (18),(20))

\[
\int_{U(n)} \mathcal{F}_s\left(a_1, \ldots, a_p | XUYU^+\right) d_s U = \mathcal{F}_s\left(a_1, \ldots, a_p | X, Y\right)
\]

Let us consider the simplest examples of (42), which, at the same time, are the solvable cases of

\[
\int_{U(n)} e^{\sum_{m=1}^{\infty} t_m \text{Tr}(XUYU^+)^m} d_s U = \sum_{\lambda, l(\lambda) \leq n} \frac{s_{\lambda}(X) s_{\lambda}(Y)}{s_{\lambda}(I_n)}
\]

Similar series for matrix integrals were considered in the papers [5],[6] without connections with classical integrable systems.

1. HCIZ integral, [20],[21],[1],[30],[3], which we obtain choosing \( p = s = 0 \)

\[
\int_{U(n)} e^{\text{Tr}(XUYU^+)} d_s U = \sum_{\lambda, l(\lambda) \leq n} \frac{s_{\lambda}(X) s_{\lambda}(Y)}{H_{\lambda s_{\lambda}(I_n)}} = \mathcal{F}_0\left(X, Y\right) = c_n \frac{\det(e^{x_i y_j})}{\Delta(x)\Delta(y)} ,
\]

\[
c_1 = 1 , \quad c_n = c_n(n) = \prod_{k=1}^{n-1} (r(k))^{k-n} = \prod_{k=1}^{n-1} \left(\frac{1}{k}\right)^{k-n} , \quad n > 1 , \quad r(k) = \frac{1}{k}
\]

2. Choosing \( p = s + 1 = 1 \) we get

\[
\int_{U(n)} \text{det}(1 - XUYU^+)^{-a} d_s U = \sum_{\lambda, l(\lambda) \leq a} (a)_{\lambda} \frac{s_{\lambda}(X) s_{\lambda}(Y)}{H_{\lambda s_{\lambda}(I_n)}} = \tau_r\left(n, X, Y\right)
\]

\[
= \mathcal{F}_0\left(a; X, Y\right) = c_n \frac{\det(1-x_i y_j)^{n-1-a}}{\Delta(x^n)\Delta(y^n)} ,
\]

\[
c_n = c_n(n) = \prod_{k=1}^{n-1} (r(k))^{k-n} , \quad n > 1 , \quad r(k) = \frac{a - n + k}{k}
\]

Now let us consider integrals over complex matrices. Different integrals over complex matrices were considered in the papers of I.Kostov, see [31],[32] as an example.
Theorem 2 Under the conditions of the Theorem 1 we have
\[ \int \tau_r (n, t_\infty, XZY Z^+) e^{-Trzz^+} d^2Z = \tau_r (n, X, Y) = c_n(n) \frac{\det (\tau_r (1, x_i, y_j))_{i,j=1,...,n}}{\Delta (x^n) \Delta (y^n)} \] (49)
where \( d^2Z = \pi^{-n^2} \prod_{i,j=1}^n dR \zeta d\zeta Z_{ij} \). In particular we have
\[ \int \frac{F_s (a_1, \ldots, a_p)}{b_1, \ldots, b_s} XZY Z^+ e^{-Trzz^+} d^2Z = \frac{p+1}{1} \frac{F_s (n, a_1, \ldots, a_p)}{b_1, \ldots, b_s \mid X, Y} \] (50)
The proof follows from (8),(13),(30),(18), (20). Examples of solvable models provided by (50): (1) \( p = s = 0 \), then
\[ \int e^{\sum_{m=1}^{\infty} \text{Tr}(xzyz^+)} e^{-Trzz^+} d^2Z = \sum_{\lambda, \ell(\lambda) \leq n} s_\lambda(X)s_\lambda(Y) = \prod_{i,j} (1 - x_i y_j)^{-1} = \mathcal{F}_0 (n \mid X, Y) \] (51)
(2) \( p = s + 1 = 1 \):
\[ \int e^{\sum_{m=1}^{\infty} \frac{\text{a}}{m} \text{Tr}(xzyz^+)} e^{-Trzz^+} d^2Z = \int \text{det} \left( 1 - \frac{1}{2} \text{XZY} Z^+ \right)^{-a} e^{-Trzz^+} d^2Z \] (52)
\[ = \sum_{\lambda, \ell(\lambda) \leq n} \left( a \right)_\lambda s_\lambda(X)s_\lambda(Y) = 2 \mathcal{F}_0 (a \mid n \mid X, Y) \] (53)
which in general is a divergent series (until \(-a\) is a non-negative integer).

Next theorem is a generalization of Gross-Witten integral:

**Theorem 3** Let \( z_i, i = 1, \ldots, n \) be eigenvalues of the matrix \( XY \)
\[ \int_{U(n)} \tau_r (n, t, XU) \tau_r (n, U^{-1}Y, I_n) d_s U = \tau_{r \tilde{r}} (n, t, XY) \] (54)
\[ = \frac{\text{det} (z_i^{n-k} \tau_{r \tilde{r}} (n - k + 1, t, t^*(z_i)))_{i,k=1,\ldots,n}}{\Delta (z^n)} \] (55)
where \( \Delta (z^n) \) is the Vandermonde determinant. Also
\[ \int_{U(n)} \tau_r (n, t, XU) \tau_r (n, U^{-1}X^{-1}, t^*) d_s U = \tau_{r_1} (n, t, t^*) \] (56)
where \( r_1 \) is the following step function:
\[ r_1(k) = r(k) \tilde{r}(k) \text{, } k > 0 \text{, } r(k) = 0 \text{, } k \leq 0 \] (57)
For the proof we use (12),(29). The simplest example of (56), \( X = 1 \), \( r = \tilde{r} = 1 \) is the well-known model of unitary matrices, see [33] about this model. We obtain
\[ \int_{U(n)} e^{\sum_i \gamma_i TrU^i + \sum_i \gamma_i TrU^{-i}} d_s U = \sum_{\lambda, \ell(\lambda) \leq n} s_\lambda(t)s_\lambda(t^*) = \tau_r (n, t, t^*) \] (58)
In (58) \( r \) is the following step function: \( r(k) = 1, k > 0 \); \( r(k) = 0, k \leq 0 \).

Due to the restriction \( l(\lambda) \leq n \) it is not the l.h.s of Cauchy-Littlewood formula (17). Let us mark that the right hand side of (58) is the subject of the so-called Gessel theorem if one
take \( t = t(x^m), \quad t^* = t^*(y^m), \quad n < m \). This case was considered in [4]. There exist different
determinant representations of (58), which are due to (29), a modification of (30) and due to
(36), see [18].

An example of (54) is

\[
\int_{U(n)} e^{\sum_m t_m \text{Tr}(XU)^m} \det(1 - U^{-1}Y)^{-b} \, d_s U = \sum_{\lambda, \ell(\lambda) \leq n} (b_\lambda s_\lambda(t)s_\lambda(XY)) \frac{(n)_\lambda}{(n)} (59)
\]

\[
\det(z_i - \tau_r(n - k + 1, t(z_i), t))_{i,k=1}^n = \frac{\Delta(z^n)}{\Delta(z^n)}
\]

where \( \tau_r(n - k + 1, t(z_i), t) = \sum_{j=0}^{\infty} \frac{(b - k + 1)_j}{(n - k + 1)_j} z_i^j h_j(t) \) (60)

where \( z_i, i = 1, \ldots, n \) are eigenvalues of the matrix \( XY \). Here \( h_k(t) \) is the elementary Schur
function, see (9).

**Consequence 1** Let \( A, X, Y \) are \( n \) by \( n \) normal matrices. For integer \( m \geq 0 \) consider the
partition \( \sigma = (m, \ldots, m) \) of the length \( l(\sigma) = n \). We have

\[
\int_{U(n)} \tau_r(n, A, XU) \tau_1(n, U^{-1}Y, I_n) \, dU = \det A^m \det X^m r_\sigma(n) \tau_r_1(n, A, XY), \quad (61)
\]

where \( r_1(k) = r(k + m) \bar{r}(k) \). Also

\[
\int_{U(n)} \tau_r(n, A, XU) \tau_1(n, U^{-1}Y, I_n) \, dU = \det Y^m \bar{r}_\sigma(n) \tau_r_1(n, A, XY), \quad (62)
\]

where \( r_1(k) = r(k + m) \bar{r}(k + m) \). In particular

\[
\int_{U(n)} \det U^{-m} \mathcal{F}_s(a_1, \ldots, a_p | A, XU) \mathcal{F}_s(\tilde{a}_1, \ldots, \tilde{a}_q | U^{-1}Y) \, dU = \prod_{i=1}^p \frac{(a_i)_\sigma}{(n)_\sigma} \det A^m \det X^m \, \prod_{s=1}^q \frac{1}{(b_i)_\sigma} \det A^m \det X^m \quad \text{(63)}
\]

\[
\int_{U(n)} \det U^{-m} \mathcal{F}_s(a_1, \ldots, a_p | A, XU) \mathcal{F}_s(\tilde{a}_1, \ldots, \tilde{a}_q | U^{-1}Y) \, dU = \prod_{i=1}^p \frac{1}{(n)_\sigma} \det Y^m \bar{r}_\sigma(n) \tau_r_1(n, A, XY), \quad (64)
\]

\[
\int_{U(n)} \det U^{-m} \mathcal{F}_s(a_1, \ldots, a_p | A, XU) \mathcal{F}_s(\tilde{a}_1, \ldots, \tilde{a}_q | U^{-1}Y) \, dU = \prod_{i=1}^p \frac{1}{(n)_\sigma} \det Y^m \bar{r}_\sigma(n) \tau_r_1(n, A, XY), \quad (65)
\]

Let us notice that putting \( A = I_n \) in \( \mathcal{F}_s(\cdot | A, X) \) one gets \( \mathcal{F}_s(\cdot | X) \) with the same sets of
indices \( \{a_i\}, \{b_i\} \). Choosing \( F_0 \) of (19) we obtain the simplest example of (63)-(66):

\[
\int_{U(n)} \det U^{-m} \text{Tr}(XU + U^{-1}Y) \, dU = \frac{1}{(n)_\sigma} \det \left( \begin{array}{c} X^m \\ Y^m \end{array} \right) \quad \text{(67)}
\]

For the determinant representation see (29). At first this integral was evaluated in [34] via
methods of [35]. The matrix integral in case \( m = 0 \) was considered in [36] by a different
method (by the method of orthogonal polynomials), and links with the KP equation and with
the so-called generalized Kontsevich model were established. For \( m = 0 \) and \( X = Y = \frac{1}{g} I_n \)
the integral is used for the study of two-dimensional QCD [22] and called Gross-Witten one
plaquette model (about this integral see also [37]):

\[
I_{GW}(g) = \int_{U(n)} e^{-\frac{1}{g} \text{Tr}(U + U^*)} \, dU 
\]

(68)
In \( n \to \infty \) limit the Gross-Witten model enjoys a third order phase transition [22].

The other example of (63)-(66) is \((\sigma \text{ is the partition } (m, \ldots, m), l(\sigma) = n)\)

\[
\int_{U(n)} \det U^{\pm m} \det(1 - XU)^{-a} \det(1 - U^{-1}Y)^{-\tilde{a}} d_U = \frac{1}{(n)_\sigma} \left\{ \left( a \right)_\sigma \det X^m \left( \tilde{a} \right)_\sigma \det Y^m \right\} _2 F_1 \left( a, \tilde{a} + m \mid n + m \right) \tag{69}
\]

The determinant representation of the Gauss hypergeometric function see (29). Let us mark that \(_2 F_1(a, b; c|X)\) with integer \(a, b, c\) solves Painleve V equation [38].

Let \( U \) (and \( V \)) be \( N \) by \( N \) (and respectively \( n \) by \( n \)) unitary matrices, \( N \geq n \), and \( A, X \) (and \( B, Y \)) are \( n \) by \( N \) (and respectively \( N \) by \( n \)) **rectangle matrices**. One combines results of Theorems 3 and 1 to get

\[
\int_{U(n)} \int_{U(N)} e^{\text{Tr} \ XUYV^+ + \text{Tr} VAU^+ B} d_U d_V = _0 F_1 (n|BX, YA) \tag{70}
\]

At first the similar result was obtained in [34] (see also [39]), where the answer was given as a determinant of Bessel functions, by (35) it is the same as \(_0 F_1\) of (70). Other examples of combining of results of Theorems 1,3:

\[
\int_{U(n)} \int_{U(N)} \det(1 - XUYV^+)^{-a} e^{\text{Tr} VAU^+ B} d_U d_V = _1 F_1 \left( a \mid n \right) \tag{71}
\]

\[
\int_{U(n)} \int_{U(N)} \det(1 - XUYV^+)^{-a} \det(1 - VXU^+ Y)^{-b} d_U d_V = _2 F_1 \left( a, b \mid n \right) \tag{72}
\]

**Theorem 4** Let \( z_i, i = 1, \ldots, n \) be eigenvalues of the matrix \( XY \)

\[
\int \tau_r (n, t, XZ) \tau_{\tilde{r}} (n, Z^+ Y, t_\infty) e^{-\text{Tr}ZZ^+} d^2 Z = \frac{\det \left( z_i^{n-k} r_{\tilde{r}} (n - k + 1, t, t^* (z_i)) \right)}{\Delta (x^n)} \tag{73}
\]

Also

\[
\int \tau_r (n, t, XZ) \tau_{\tilde{r}} (n, Z^+ X^{-1}, t^*) e^{-\text{Tr}ZZ^+} d^2 Z = \tau_r (n, t, t^*) ,
\]

where \( r_1 \) is the following step function:

\[
r_1 (k) = kr(k) \tilde{r}(k) , \; k > 0 , \; r(k) = 0 , \; k \leq 0 \tag{74}
\]

The proof follows from (29). Examples:

\[
\int e^{\sum_{m=1}^\infty t_m \text{Tr}(XZ)^m} e^{\text{Tr}(Z^+ Y)} e^{-\text{Tr}ZZ^+} d^2 Z = e^{\sum_{m=1}^\infty t_m \text{Tr}(XY)^m} \tag{75}
\]

\[
\int e^{\sum_{m=1}^\infty t_m \text{Tr}Z^m} e^{\sum_{m=1}^\infty t_m \text{Tr}(Z^+)^m} e^{-\text{Tr}ZZ^+} d^2 Z = \sum_{\lambda} (n)_\lambda s_\lambda (t) s_\lambda (t^*) \tag{76}
\]

Let us notice that series in the r.h.s. of (78) coincide with the series for hermitian-antihemitian matrix model and with the series for the model of normal matrices [17], which in \( n \to \infty \) limit describes the interface dynamics of a spot of water inside oil [40].

Applying (27) and (28) we get
Consequence 2 Let $A, X, Y$ are $n$ by $n$ normal matrices. For integers $m \geq 0$ we have
\[
\int \tau_r(n, A, XZ) \tau_r(n, Z^+Y, t_\infty) \det(Z^+)^m e^{-TrZZ^+} d^2Z
\]
\[(79)\]
\[
= (n)_\sigma r_\sigma(n) \det A^m \det X^m \tau_{r_1}(n, A, XY),
\]
\[(80)\]
where the partition $\sigma = (m, \ldots, m)$ and are of the length $l(\sigma) = n$. In (80) $r_1(k) = \frac{k+m}{k} r(k + m)\bar{r}(k)$. Also
\[
\int \tau_r(n, A, XZ) \tau_r(n, Z^+Y, t_\infty) \det Z^m e^{-TrZZ^+} d^2Z = \bar{r}_\sigma(n) \det Y^m \tau_{r_1}(n, A, XY),
\]
\[(81)\]
where the partition $\sigma = (m, \ldots, m)$ and are of the length $l(\sigma) = n$. In (81) $r_1(k) = r(k)\bar{r}(k+m)$.

In particular
\[
\int_{\mathcal{F}_s} \left( \begin{array}{c|c} a_1, \ldots, a_p & A, XZ \\ \hline b_1, \ldots, b_s & \end{array} \right) \frac{pF_s}{\prod_{i=1}^{p} (a_i)_{\sigma}} \det A^m \det X^m \det Z^m e^{-TrZZ^+} d^2Z
\]
\[(82)\]
\[
= \frac{\prod_{i=1}^{p} (a_i)_{\sigma}}{\prod_{i=1}^{s} (b_i)_{\sigma}} \det A^m \det X^m \det Z^m e^{-TrZZ^+} d^2Z
\]
\[(83)\]
\[
= \frac{\prod_{i=1}^{\tilde{p}} (\tilde{a}_i)_{\sigma}}{\prod_{i=1}^{\tilde{s}} (\tilde{b}_i)_{\sigma}} \det Y^m \det Z^m e^{-TrZZ^+} d^2Z
\]
\[(84)\]
For instance
\[
\int \det(Z^+)^m e^{TrZ^+Y} e^{TrXZ-TrZZ^+} d^2Z = \det X^m e^{TrXY},
\]
\[(85)\]
\[
\int \det(Z^+)^m \det(1-Z^+Y)^{-a} e^{TrXZ-TrZZ^+} d^2Z = \det X^m \det(1-XY)^{-a},
\]
\[(86)\]
\[
\int \det(Z^+)^m \det(1-XZ)^{-a} e^{TrZ^+Y} d^2Z = X^m 2F_0(-k,a|XZY)
\]
\[(87)\]
\[
\int \det(Z^+)^m \det(1-XZ)^{-a} e^{TrZ^+Y} d^2Z = X^m 2F_0(-k,a|XZY)
\]
\[(88)\]

Let $Z_1$ (and $Z_2$) be $N$ by $N$ (and respectively $n$ by $n$) unitary matrices, $N \geq n$, and $A, X$ (and $B, Y$) are $n$ by $N$ (and respectively $N$ by $n$) \textbf{rectangle matrices}. Theorems 4,2 yields the following examples:
\[
\int_{CN^2} \int_{CN^2} e^{TrXZ_1YZ_2^+} + e^{TrXZ_1YZ_2^+} + e^{TrZ_2AZ_1^+} - e^{TrZ_2AZ_1^+} - e^{TrZ_2AZ_1^+} - e^{TrZ_2AZ_1^+} d^2Z_1 d^2Z_2 = 0F_0(BX, YA)
\]
\[(89)\]
\[
\int_{CN^2} \int_{CN^2} \det(1-XZ_1YZ_2^+) d^2Z_1 d^2Z_2 = 1F_0(a|BX, YA)
\]
\[(90)\]

4 Kontsevich-type integrals

Lemma If $f_1, \ldots, f_n$ is a set of functions of one variable, and $g(x^n)$ is an antisymmetric function of $x_1, \ldots, x_n$, then
\[
\int \cdots \int g(x^n) \det(f_j(x_i)) \prod_{i=1}^{n} dx_1 \cdots dx_n = n! \int \cdots \int g(x^n) \prod_{i=1}^{n} f_i(x_i) dx_1 \cdots dx_n
\]
\[(91)\]
(below we put $g(x^n) = \Delta(x^n)$).

Let $X$ be $n$ by $n$ Hermitian matrix with eigenvalues $x_1, \ldots, x_n$, let and $Y$ be $n$ by $n$ matrix with eigenvalues $y_1, \ldots, y_n$. Let $dX = \prod_{i<k} dX_{ik} dX_{ki} \prod_{i=1}^{n} dX_{ii} = dU \Delta(x^n)^2 \prod_{i=1}^{n} dx_i$. The determinant representation (29), the lemma and theorems 1,2 result in
Theorem 5 The following integrals over Hermitian matrix $X$ and over complex matrix $Z$
\[
\int \tau_r(n, t, X)\tau_r(n, XY, I_n) dX = \int \tau_r(n, t, X)\tau_r(n, ZXZ^+Y, t_{\infty}) e^{-Trzz^+} d^2Z dX =
\]
is equal to
\[
\frac{n!C}{\Delta(y^n)} \int \det(x_i^{n-k}\tau_r(n-k+1, t(x_i), t^*)\tau_r(1, x_i, y_i))_{i,k=1,...,n} \, dx_1 \cdots dx_n
\]
\[
= \frac{n!C}{\Delta(y^n)} \det \left( \int x^{n-k}\tau_r(n-k+1, t(x), t^*)\tau_r(1, x, y_i) dx \right)_{i,k=1,...,n}
\]
For the proof we use (41) and the Lemma (91) and (29). One obtains the Kontsevich integral\[
\int e^{tX} e^{-TrXY} dX [41] if he choose $r = 1$, $t_k = \delta_{k,3}$ and $\hat{r}(k) = 1/k.$
Also we have

Theorem 6 Let $a_1, \ldots, a_n (y_1, \ldots, y_n)$ are eigenvalues of matrix $A$ (of matrix $Y$)
\[
\int \tau_r(n, A, X)\tau_r(n, XY, I_n)\, dX = \frac{n!C}{\Delta(y^n)} \det \left( \int \tau_r(1, a_j, x)\tau_r(1, x, y_i) dx \right)_{i,k=1,...,n}
\]

5 New multi-matrix models which can be solved by the method of orthogonal polynomials and the Schur function expansion

Here we consider in short new solvable multi-matrix integrals which we obtain with the help of the previous consideration, details will appear in the separately paper.

Let us remind that the multi-matrix model of Hermitian matrices\[
I = \int e^{\text{Tr}V_1(M_1) + \cdots + \text{Tr}V_N(M_N)} e^{\text{Tr}M_1M_2} e^{\text{Tr}M_2M_3} \cdots e^{\text{Tr}M_{N-1}M_N} \, dM_1 \cdots dM_N,
\]
where $M_1, \ldots, M_N$ are Hermitian $n$ by $n$ matrices and
\[
V_k(M_k) = \sum_{m=1}^{\infty} i_m(k)M_k^m, \quad k = 1, \ldots, N
\]
was shown to be multi-component KP tau function and solved by the method of orthogonal polynomials.

Now we consider more general integral over matrices $M_1, \ldots, M_N$
\[
I = \int e^{\text{Tr}V_1(M_1) + \cdots + \text{Tr}V_N(M_N)} R_1(M_1M_2) R_2(M_2M_3) \cdots R_{N-1}(M_{N-1}M_N) \, dM_1 \cdots dM_N
\]
If we choose the interaction term as hypergeometric function of matrix argument (see (10))
\[
R_k(M_kM_{k+1}) = \tau_{rk}(n, t, M_kM_{k+1}), \quad t = (1, 0, 0, 0, \ldots)
\]
then due to (41) it is possible to perform the angle integration over each $U_k(n)$ where $M_k = U_kX^{(k)}U_k^+$, $X^{(k)} = \text{diag}(x_1^{(k)}, \ldots, x_n^{(k)})$. Really, (41) is a tau function which has a determinant representation (30). Finely we obtain the following integral over eigenvalues $x_i^{(k)}$ ($i = 1, \ldots, n; k = 1, \ldots, N)$:
\[
I = c \prod_{i=1}^{n} \rho_k(x_1^{(1)}x_2^{(2)} \cdots x_i^{(2)}x_3^{(3)} \cdots \rho_{N-1}(x_i^{(N-1)}x_i^{(N)} \rho_N) \prod_{k=1}^{N} e^{M(x_i^{(k)}m)} m x_i^{(k)}, \quad (100)
\]
where

\[ \rho_k(x^{(k)}_i, x^{(k+1)}_i) = \tau_k(1, t, x^{(k)}_i, x^{(k+1)}_i), \quad t = (1, 0, 0, \ldots) \]  

(101)

(In the case of the familiar multi-matrix model (96) each \( \rho_k(xy), k = 1, \ldots, N \) is \( e^{-xy} \)).

The integral (100) may be evaluated by the method of orthogonal polynomials \( \{p_m(t^N, x)\}, \{\pi_m(t^N, x)\} \) which depend on the collection of times \( t^N = (t^{(1)}, t^{(2)}, \ldots, t^{(N)}) \)

\[
\int p_m(t^N, x)\pi_n(t^N, y)\omega(t^N, x, y)dx dy = \delta_{nm}e^{\phi_n(t^N)}, \quad n, m = 0, 1, 2, \ldots
\]  

(102)

Each of polynomials is of a form

\[ p_m(t^N, x) = \sum_{n \leq m} p_{nm}(t^N) x^n, \quad \pi_m(t^N, x) = \sum_{n \leq m} \pi_{nm}(t^N) x^n, \]

and the weight function is

\[ \omega(t^N, x, y) = e^{V_1(x)+V_N(y)} \int \rho_k(x^{(2)}_x)\rho_2(x^{(2)}_x)x^{(3)} \cdots \rho_{N-1}(x^{(N-1)}y) \prod_{k=2}^{N-1} e^{V_k(x^{(k)})}dx^{(k)} \]  

(103)

We obtain

\[ I = c \prod_{k=1}^n e^{\phi_k(t^N)} \]  

(104)

Let us notice that the orthogonal polynomials are Baker-Akhiezer functions of TL hierarchy. Gauss-Zakharov-Shabat factorization problem for TL equation [24] directly results from the relation (102)

\[ K_+(t^N) = K_-(t^N)e^{\Phi}G(t^N), \quad \Phi = diag(\phi_1(t^N), \phi_2(t^N), \ldots), \]  

(105)

where

\[ (K_+)_nm = (\pi^{-1})_nm, \quad (K_-)_nm = p_{nm}, \quad G_{nm} = \int x^n y^m \omega(t^N, x, y)dx dy \]  

(106)

(as it was in the case of two-matrix model [42]).

The Schur function expansion for these matrix matrix models is given by (6), where \( K_{\lambda\mu} \) are expressed via products of skew Schur functions, will be presented in the forthcoming paper.

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References

[1] Mehta,M.L.: Random Matrices Academic Press, Inc, 1991

[2] Tracy,C.A. and Widom,H., Correlation functions, cluster functions, and spacing distributions for random matrices, J. Statist. Phys. 92 809–835 (1998)

[3] Zinn-Justion,P. and Zuber,J.-B.: On some integrals over the U(N) unitary group and their large limit, math-ph/0209019

[4] van Moerbeke,P.: Integrable Lattices: Random Matrices and Random Permutations, CO/0010135
[5] Kazakov, V.: Solvable matrix models, *hep-th/003064*

[6] Kazakov, V., Staudacher, M. and Wynter, T.: Character Expansion Method for Matrix Models of Dually Weighted Graphs, *hep-th/9502132 v 2*

[7] Jimbo, M and Miwa, T.: Solitons and Infinite Dimensional Lie Algebras, *Publ. RIMS Kyoto Univ.** 19* (1983) 943-1001

[8] Morozov, A.Yu.: Integrability and Matrix Models, *Uspehi Fizicheskikh Nauk** 164* (1994) 3-62

[9] Orlov, A.Yu. and Scherbin, D.M.: Multivariate hypergeometric functions as tau functions of Toda lattice and Kadomtsev-Petviashvili equation, *Physics D* (2001)

[10] Gross, K.I. and Richards, D.S.: Special functions of matrix arguments I: Algebraic induction, zonal polynomials, and hypergeometric functions, *Transactions Amer Math Soc*, **301** (1987) 781-811

[11] Vilenkin, N.Ya. and Klimyk, A.U.: *Representation of Lie Groups and Special Functions. Volume 3: Classical and Quantum Groups and Special Functions*, Kluwer Academic Publishers, 1992

[12] Nakatsu, T., Takasaki, K. and Tsujimaru, S.: Quantum and Classical Aspects of Deformed $c = 1$ Strings, *Nucl. Phys. B** 443* (1995) 155; *hep-th/9501038*

[13] Takasaki, T.: The Toda Lattice Hierarchy and Generalized String Equation, *Comm. Math. Phys.** 181* (1996) 131; *hep-th/9506089*

[14] A. Okounkov, Toda equations for Hurwitz numbers, *math.AG/0004128*

[15] Nekrasov, N.: Seiberg-Witten Prepotential From Instanton Counting, *hep-th/0206161*

[16] Orlov, A.Yu.: Soliton theory, Symmetric Functions and Matrix Integrals, *SI/0207030*

[17] Harnad, J. and Orlov, A.Yu.: Matrix Integrals as Borel sums of Schur Function Expansions, *nlin.SI/0209035*

[18] Orlov, A.Yu.: Tau functions and Matrix Integrals, *math-ph/0210012 v4*

[19] Kazakov, V.: *Phys Lett B** 150* (1985) 282

[20] Harish-Chandra: *Am. J. Math** 79* (1958) 87-120

[21] Itzykson, C. and Zuber, J.B.: *J.Math.Phys.* 21 (1980) 411

[22] Gross, D.J. and Witten, E.: Possible Third Order Phase Tranzition in the Large N Lattice Gauge Theory, *Phys Rev D** 21* (1980) 446

[23] Zakharov, V.E. and Shabat, A.B.: *J. Funct. Anal. Appl.* 8 (1974) 226

[24] Ueno, K. and Takasaki, K.: *Adv. Stud. Pure Math.* 4 (1984) 1-95

[25] Macdonald, I.G.: *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1995
[26] Takasaki, K.: Initial value problem for the Toda lattice hierarchy, *Adv. Stud. Pure Math.* 4 (1984) 139-163

[27] Orlov, A.Yu. and Scherbin, D.M.: Hypergeometric solutions of soliton equations, *Theor Math Phys* 128 (2001) 84-108

[28] Harnad, J. and Orlov, A.Yu.: Scalar products of symmetric functions and matrix integrals, *nlin.SI/0211051* (To appear in proceedings of NEEDS2002, eds., A. Gonzales, World Scientific, 2002.)

[29] Okounkov, A. and Pandaripande, R.: The equivariant Gromov-Witten theory of $\mathbb{P}^1$, *math.AG/0207233 v1*

[30] Zinn-Justin, P.: HCIZ integral and 2D Toda lattice hierarchy, *math-ph/0202045*

[31] Kostov, I.K., Staudacher, M. and Wynter, T.: Complex Matrix Models and Statistics of Branched Covering of 2D Surfaces, *Commun. Math. Phys.* 191 (1998) 283-298; *hep-th/9703189*

[32] Kostov, I.K.: Exact Solution of the Six-Vertex Model on a Random Lattice, *Nucl. Phys. B* 575 (2000) 513-534; *hep/9911023*

[33] Zabrodin, A., Kharchev, S., Mironov, A., Marshakov, A. and Orlov, A.: Matrix Models among Integrable Theories: Forced Hierarchies and Operator Formalism, *Nuclear Physics B* 366 (1991) 569-601

[34] Schlittgen, B and Wetting, T.: Generalizations of some integrals over the unitary group”, *math-ph/0209030 v1*

[35] Balantekin, A.B.: Character expensions, Itzykson-Zuber integrals, and the QCD partition function”, *Phys. Rev. D* 62 (2000) 085017-1 - 085017-8

[36] Mironov, A., Morozov, A. and Semenoff, G.: Unitary Matrix Integrals in the Framework of the Generalized Kontsevich Model, *Intern J Mod Phys A* 11 (1996) 5031-5080

[37] Kostov, I.K.: $U(N)$ Gauge Theory and Lattice Strings, *Nucl. Phys. B* 415 (1994) 29-70; *hep-th/9306110*

[38] Adler, M. and van Moerbeke, P.: Integrals over Grassmannians and Random permutations, *math.CO/0110281*

[39] Fyodorov, Y. and Strahov, E

[40] Mineev-Weinstein, M., Wiegmann, P. and Zabrodin, A.: Integrable structure of interface dynamics, *LAUR-99-0703*

[41] Kontsevich, M. *Funk. Anal. i ego Prilozh.* 25 (1991) 50-57

[42] Gerasimov, A., Marshakov, A., Mironov, A., Morozov, A. and Orlov, A.: Matrix Models of 2D Gravity and Toda Theory, *Nuclear Physics B* 357 (1991) 565-618