A refined version of the integro-local Stone theorem

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Abstract
Let $X, X_1, X_2, \ldots$ be a sequence of non-lattice i.i.d. random variables with $E X = 0$, $E X^2 = 1$, and let $S_n := X_1 + \cdots + X_n$, $n \geq 1$. We refine Stone’s integro-local theorem by deriving the first term in the asymptotic expansion for the probability $P(S_n \in [x, x + \Delta])$ as $n \to \infty$ and establishing uniform bounds for the remainder term, under the assumption that the distribution of $X$ satisfies Cramér’s strong non-lattice condition and $E |X|^r < \infty$ for some $r \geq 3$.

Key words and phrases: integro-local Stone theorem, asymptotic expansion, random walk, central limit theorem, independent identically distributed random variables.

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1 Introduction and the main result
In the present note, we establish a refinement of the following remarkable integro-local version of the central limit theorem due to C. Stone [18, 19]. Let $X, X_1, X_2, \ldots$ be a sequence of non-lattice independent identically distributed (i.i.d.) random variables (r.v.’s) following a common distribution $F$ such that $E X = 0$, $E X^2 = 1$, and let $S_n := X_1 + \cdots + X_n$, $n \geq 1$. For $x \in \mathbb{R}$ and $\Delta > 0$, set

$$\Delta[x] := [x, x + \Delta].$$

Then, as $n \to \infty$,

$$\frac{1}{\Delta} P(S_n \in \Delta[x]) = n^{-1/2} \phi(xn^{-1/2}) + o(n^{-1/2}), \quad (1)$$

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where \( \phi(t) := (2\pi)^{-1/2}e^{-t^2/2} \) is the standard normal density and the remainder term is uniform in \( x \in \mathbb{R} \) and in \( \Delta \in [\Delta_0, \Delta_1] \) for any fixed \( 0 < \Delta_0 < \Delta_1 < \infty \) (in fact, C. Stone established more general versions of the above result, including convergence to stable laws, the multivariate case and large deviations).

It is quite appropriate to call relations of the form (1) the integro-local theorems, to distinguish them from the integral theorems (which refer to approximating probabilities of the form \( P(S_n < x), x \in \mathbb{R} \)) and the local ones (which deal with approximating the densities of \( S_n \) in the “smooth” case; note that in the arithmetic case, the integro-local theorems are in fact the local ones: they concern approximating probabilities \( P(S_n = x) \) for \( x \in \mathbb{Z} \)).

The integro-local theorem is perhaps the most perfect and precise version of the classical central limit theorem. Indeed, it does not assume any additional conditions on top of the standard requirement of finite second moments (except for distinguishing between the lattice and non-lattice cases), but has basically got the same accuracy as the local limit theorems (note that, for small \( \Delta \), the left-hand side of (1) is “almost the density” of \( S_n \)), without making any assumptions about existence of the densities. That means that the integro-local theorems are much more precise than the integral ones, and it is easy to see that one can derive the assertions of the latter from the former, but not the other way around.

The integro-local theorems are rather effective and often the most adequate technical tools in a number of problems in probability theory. For instance, they are used for computing the exact asymptotics of large deviation probabilities for sums of independent r.v.’s (cf. Chapter 9 in [4]). They also proved instrumental for studying the distribution of the first passage time of a curvilinear boundary by a random walk [5], establishing integro-local theorems for compound renewal processes (see Chapter 10 in [4]) and in a number of other problems.

Concerning the history of the problem, note that a special case of relation (1) (when \( x \) is fixed) was first established by L.A. Shepp [17]. A textbook exposition of the proof of (1) can be found in Section 8.7 of [4]. Under additional Cramér’s conditions (the moment generating function is finite in a neighborhood of zero and the strong non-lattice condition on the characteristic function of \( X \) is met, see (2) below), relation (1) was extended in [1, 2] in the multivariate setting to an asymptotic expansion in the powers of \( n^{-1/2} \) and also to the case where \( \Delta_0 \) can be vanishing. Extensions of Stone’s theorem to the case of non-identically distributed independent r.v.’s in the triangular array scheme (covering the large deviations zone as well) were established in [3].

In the lattice case, an analog of (1) was obtained by B.V. Gnedenko in the univariate case (see Chapter 9 in [13]) and by E.L. Ryacheva [16] in the multivariate case.

It is most natural to ask if the remainder term in (1) can be sharpened under minimal additional assumptions. The first step in that direction was made in [6],
where it was shown that, under Cramér’s strong non-lattice condition

\[
\limsup_{|\lambda| \to \infty} |\varphi(\lambda)| < 1
\]  

on the characteristic function (ch.f.) \( \varphi(\lambda) := E e^{i \lambda X}, \lambda \in \mathbb{R}, \) of \( X, \) and the moment condition \( E |X|^r < \infty \) for some \( r \in (2, 3], \) relation (1) holds with the reminder term replaced by \( O(\Delta n^{-(r-1)/2}) \) uniformly in \( x \in \mathbb{R} \) and in \( \Delta \in (q^n, cn^{(3-r)/2}) \) for some \( q \in (0, 1) \) and every fixed \( c > 0. \) In fact, [3] actually establishes a multivariate version of that result.

In the present note, we further develop the approach from [6] to derive the first term of the asymptotic expansion for \( P(S_n \in \Delta(x)) \) with uniform bounds for the remainder term in the case when condition (2) is met and \( E |X|^r < \infty \) for some \( r \in [3, 4]. \) It will be seen from the proofs that, under appropriate moment conditions, one can extend these results to asymptotic expansions with more terms. However, since the very form of such expansions and their derivations are getting quite cumbersome, while technically they are not much different from the one-term case, we will restrict ourselves to presenting the latter only.

To formally state our main results, we will need some further notations. For \( r \in (2, \infty) \) and \( b \in (1, \infty], \) introduce the class \( F_{r,b} \) of distributions \( F \) on \( \mathbb{R} \) satisfying the following moment conditions: for \( X \sim F, \) one has \( E X = 0, E X^2 = 1 \) and

\[
E |X|^r < b.
\]

In particular, \( F_{r,\infty} \) is the class of all zero mean unit variance distributions with a finite \( r \)th absolute moment. For \( F \in F_{3,\infty}, \) we set

\[
\mu_3 := E X^3.
\]

Further, for \( \rho \in (0, 1] \) and \( b < \infty, \) we denote by \( F_{r,b}^\rho \) the totality of distributions from \( F_{r,b} \) that satisfy

\[
\sup_{|\lambda| > 1/b} |\varphi(\lambda)| < \rho.
\]  

(3)

When \( b = \infty, \) we will understand by \( F_{r,\infty}^1 \) just the totality of distributions from \( F_{r,\infty} \) that satisfy Cramér’s strong non-lattice condition (2) (or, equivalently, \( \sup_{|\lambda| > \varepsilon} |\varphi(\lambda)| < 1 \) for any \( \varepsilon > 0). \)

**Theorem 1** (i) For any distribution \( F \in F_{3,\infty}^1, \) one has

\[
\frac{1}{\Delta} P(S_n \in \Delta(x)) = \frac{1}{n^{1/2}} \phi\left(\frac{x}{n^{1/2}}\right) \left(1 + \frac{\mu_3 x}{6n} \left(\frac{x^2}{n} - 3\right) - \frac{\Delta x}{2n}\right) + \frac{R_n}{n},
\]  

(4)
where for the remainder term $R_n = R_n(x, \Delta)$ the following holds true: there exists a $q \in (0, 1)$ such that, for any fixed $\Delta_1 > 0$,

$$\lim_{n \to \infty} \sup_{q^n \in \Delta \leq \Delta_1} \sup_{x \in \mathbb{R}} |R_n(x, \Delta)| = 0.$$ 

(ii) Moreover, for any fixed $r \in (3, 4], b < \infty$ and $\rho \in (0, 1)$, representation (4) holds true with the following uniform remainder bound over the distribution class $\mathcal{F}_{r, b}$: there exists a $q \in (0, 1)$ such that, for any fixed $\Delta_1 > 0$,

$$\limsup_{n \to \infty} \sup_{F \in \mathcal{F}_{r, b}} \sup_{q^n \in \Delta \leq \Delta_1} \sup_{x \in \mathbb{R}} n^{(r-3)/2} |R_n(x, \Delta)| < \infty.$$ 

Remark 1 Note that the lower bound $q^n$ for the range of $\Delta$ values in the theorem cannot be “qualitatively” improved. Indeed, assume that $F$ satisfies the conditions of part (i) and has an atom of size $q_0 \in (0, 1)$ at zero. Then $\mathbb{P}(S_n = 0) \geq q_0^n$. Therefore, for $\Delta \in (0, q_0^n)$, the left-hand side of (1) will be at least one, whereas the right-hand side of that relation will be $o(1)$, so that (1) cannot hold true for such values of $\Delta$.

Remark 2 In the general case of non-standardized r.v.’s $X$ with some $\mu := \mathbb{E}X$ and $\sigma^2 > 0$, expansion (4) will hold with $x$, $\Delta$ and $\mu_3$ on its right-hand side replaced by $\sigma^{-1}(x-\mu)$, $\sigma^{-1}\Delta$ and $\sigma^{-3}\mathbb{E}(X-\mu)^3$, respectively. The uniformity assertion in part (ii) will have to be reformulated in that case as well.

Remark 3 Note that our Theorem 1 can be viewed as an integro-local version of the famous Chebyshev–Cramér asymptotic expansion (a.k.a. the Edgeworth expansion, due to the contribution made in [9]) for the distribution function of $S_n$, which was introduced in [7] and formally proved in [8, 10]. Assume for a moment that $\mathbb{E}X^4 < \infty$ and suppose that condition (2) is met. Then, denoting by $\Phi$ the standard normal distribution function, by $\text{He}_k(x) := e^{-D^2/2}x^k$, where $D := \frac{d}{dx}$, the $k$-th Chebyshev–Hermite polynomial, and by $\gamma_k$ the $k$-th cumulant of $X$, $k = 1, 2, \ldots$, one has the following asymptotic expansion (see e.g. Section 5.7 in [14]): as $n \to \infty$,

$$\mathbb{P}\left(\frac{S_n}{n^{1/2}} < v \right) = \Phi(v) - \phi(v)\left[\frac{\gamma_3}{6n^{1/2}}\text{He}_2(v) + \frac{1}{n}\left(\frac{\gamma_3^2}{72}\text{He}_5(v) + \frac{\gamma_4}{24}\text{He}_3(v)\right)\right] + o(n^{-1})$$

uniformly in $v \in \mathbb{R}$. Taking a fixed $\Delta > 0$, setting $x := vn^{1/2}$ and expanding the terms in the expression on the right-hand side with $v$ substituted by $v + \Delta n^{-1/2}$, we obtain

$$\mathbb{P}(S_n \in \Delta[x]) = \mathbb{P}\left(\frac{S_n}{n^{1/2}} < v + \frac{\Delta}{n^{1/2}}\right) - \mathbb{P}\left(\frac{S_n}{n^{1/2}} < v\right)$$

$$= \frac{\Delta}{n^{1/2}} \phi(v) \left(1 + \frac{\gamma_3}{6n^{1/2}}\text{He}_3(v) - \frac{\Delta}{2n^{1/2}}\text{He}_1(v)\right) + o(n^{-1})$$

$$= \frac{\Delta}{n^{1/2}} \phi(v) \left(1 + \frac{\mu_3}{6n^{1/2}}(v^3 - 3v) - \frac{\Delta v}{2n^{1/2}}\right) + o(n^{-1}),$$

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where we replaced $\gamma_3$ with $\mu_3$ as these two quantities coincide for standardized r.v.’s. In this note, we prove that the above relation holds already when $\mu_3$ is finite. Moreover, we establish that the above asymptotic relation holds uniformly in $\Delta \in [q^n, \Delta_1]$ for some $q \in (0, 1)$ and any fixed $\Delta_1 > 0$. That result cannot be derived from any asymptotic expansions for distribution functions. In addition, we give uniform bounds for the remainder term in the case where $E|X|^r < b < \infty$ for some $r \in (3, 4]$.

**Remark 4** One might also mention here the nonuniform error bounds for asymptotic expansions for probabilities $P(n^{-1/2}S_n \in A)$ for convex sets $A$ obtained in the multivariate case in [12]. However, for the purposes of solving the problem addressed in the present paper, these bounds are not better than the standard approximation rates from the classical asymptotic expansions mentioned in the previous remark.

## 2 The proof of the main result

The proof uses the method of characteristic functions with smoothing. We will only prove part (ii), since the proof of part (i) follows exactly the same scheme and is somewhat simpler as it requires fewer explicit bounds. So we will assume throughout this section that $F \in F^\rho_{r,b}$ for some fixed $\rho \in (0, 1)$, $r \in (3, 4]$ and $b < \infty$.

First we will derive fine asymptotics for the “smoothed” distributions of $S_n$, namely, for the distributions of

$$\tilde{S}_n := S_n - \delta U, \quad n \geq 1,$$

where $\delta = \delta(\Delta, n)$ will be chosen later and $U$ is an r.v. uniformly distributed over $(0, 1)$ and independent of $\{X_n\}_{n \geq 1}$. Denote by

$$\psi(\lambda) := \mathbb{E} e^{-i\lambda U} = \frac{1 - e^{-i\lambda}}{i\lambda}, \quad \lambda \in \mathbb{R},$$

the ch.f. of $-U$. Since the ch.f. of $\tilde{S}_n$ is clearly equal to $\varphi^n(\lambda)\psi(\delta\lambda)$ and the function

$$|\varphi^n(\lambda)\psi(\delta\lambda)\psi(\Delta\lambda)| \leq \min\{1, 4\delta^{-1}\Delta^{-1}\lambda^{-2}\}, \quad \lambda \in \mathbb{R},$$

is integrable on $\mathbb{R}$, we have from the standard inversion formula for ch.f.’s that yields the increments of the respective distribution functions (see e.g. (3.11) in Section XV.4 of [11])

that

$$P(\tilde{S}_n \in \Delta]\) = \frac{\Delta}{2\pi} \int e^{-i\lambda x} \varphi^n(\lambda)\psi(\delta\lambda)\psi(\Delta\lambda)d\lambda = \frac{\Delta}{2\pi}(I_1 + I_2 + I_3), \quad (5)$$

where we defined $I_j := \int_{B_j} e^{-i\lambda x} \varphi^n(\lambda)\psi(\delta\lambda)\psi(\Delta\lambda)d\lambda, \quad j = 1, 2, 3$, as the integrals over the respective regions

$$B_1 := \{\lambda \in \mathbb{R} : |\lambda| < h_1 n^{-1/3}\},$$

$$B_2 := \{\lambda \in \mathbb{R} : h_1 n^{-1/3} \leq |\lambda| < h_2\},$$

$$B_3 := \{\lambda \in \mathbb{R} : |\lambda| \geq h_2\}$$
for fixed $h_j > 0$, $j = 1, 2$, to be chosen later.

Letting

$$u := \lambda_n^{1/2}, \quad v := xn^{-1/2}, \quad \hat{\psi}(\lambda) := \psi(\delta \lambda)\psi(\Delta \lambda),$$

we can re-write $I_1$ as

$$I_1 = n^{-1/2} \int_{A_n} e^{-iuv} \varphi^n(un^{-1/2}) \hat{\psi}(un^{-1/2}) du, \quad A_n := (-h_1n^{1/6}, h_1n^{1/6}).$$

(6)

Our first step in evaluating $I_1$ consists in deriving representation (14) below for the second factor in the integrand. Make use of following expansion for the ch.f. $\varphi$:

$$1 - \varphi(\lambda) = \frac{1}{2} \lambda^2 + \frac{i\mu_3}{6} \lambda^3 + \theta(\lambda)\lambda^3,$$

(7)

where

$$|\theta(\lambda)| \leq \frac{2^{4-r}E|X|^r|\lambda|^{r-3}}{r(r-1)(r-2)} \leq \frac{b}{3} |\lambda|^{r-3}$$

(8)

(see e.g. Section 12.4 in [14]). Note that, by Lyapunov’s inequality, for $|\lambda| \leq b^{-1}$ one has

$$\left| \frac{i\mu_3}{6} \lambda^3 + \theta(\lambda)\lambda^3 \right| \leq \frac{b^{3/r}}{6} |\lambda|^3 + \frac{b}{3} |\lambda|^r = \frac{|b^{1/r} \lambda|^3}{6} + \frac{|b^{1/r} \lambda|^r}{3} \leq \frac{b^{3/r-1}}{2} \lambda^2.$$ 

(9)

Hence, for $\lambda$ from that range,

$$|1 - \varphi(\lambda)| \leq \frac{\lambda^2}{2} + \frac{b^{3/r-1}}{2} \lambda^2 \leq \lambda^2 \leq \frac{b^{-2}}{2} < 1.$$ 

(10)

Since by the Taylor formula with remainder in Lagrange form one has

$$|\ln(1 - z) + z| \leq \frac{|z|^2}{2(1 - c)^2}, \quad z \in \mathbb{C}, \quad |z| \leq c < 1,$$

(11)

we conclude that, in the domain $|u| < b^{-1}n^{1/2}$,

$$n \ln \varphi(un^{-1/2}) = n \ln(1 - (1 - \varphi(un^{-1/2}))
\quad = n \ln\left(1 - \frac{u^2}{2n} - \frac{i\mu_3 u^3}{6n^{3/2}} + \frac{u^3}{n^{3/2}} \theta(un^{-1/2}) \right) = -\frac{u^2}{2} + w,$$

(12)

where $w := -\frac{i\mu_3 u^3}{6n^{3/2}} + \frac{u^3}{n^{3/2}} \theta_1(un^{-1/2})$ and, in view of (8), (10) and (11), for $|\lambda| \leq b^{-1}$ one has

$$|\theta_1(\lambda)| \leq |\theta(\lambda)| + \frac{|1 - \varphi(\lambda)|^2}{2(1 - b^{-2})^2|\lambda|^3} \leq \frac{b}{3} |\lambda|^{r-3} + \frac{|\lambda|}{2(1 - b^{-2})^2}.$$ 

(13)
Now from (12) we have
\[ \varphi_n(u n^{-1/2}) = e^{-u^2/2}e^w = e^{-u^2/2}[1 + w + (e^w - (1 + w))] , \]
where, again from the Taylor formula with remainder in Lagrange form,
\[ |e^w - (1 + w)| \leq \frac{|w|^2}{2} e^{|w|} . \]
Setting \( h_1 := \left( \frac{b^3 r}{6} + \frac{b}{3} + \frac{1}{2(1-b^{-2})r} \right)^{-1/3} \), we have from (13) that, for \( u \in A_n \),
\[ |w| = \left| -\frac{i \mu_3 u^3}{6 n^{1/2}} + \frac{u^3}{n^{1/2}} \theta_1(u n^{-1/2}) \right| \leq h_1^{-3} \frac{|u|^3}{n^{1/2}} < 1 . \]
From here and the previous two displayed formulae we see that, for \( u \in A_n \) with the chosen \( h_1 \), one has
\[ \varphi_n(u n^{-1/2}) = e^{-u^2/2} \left( 1 - \frac{i \mu_3 u^3}{6 n^{1/2}} + \frac{u^3}{n^{1/2}} \theta_2(u n^{-1/2}) \right) , \]
where
\[ |\theta_2(u n^{-1/2})| \leq |\theta_1(u n^{-1/2})| + \frac{e |u|^3}{2 h_1^{-3} n^{1/2}} . \]
Furthermore, since \( \psi(\lambda) = 1 - i/(2\lambda) + O(\lambda^2) \) as \( \lambda \to 0 \), one has
\[ \hat{\psi}(u n^{-1/2}) = \frac{1 - e^{-iu n^{-1/2}}}{iu n^{-1/2}} \cdot \frac{1 - e^{-iu \Delta n^{-1/2}}}{iu \Delta n^{-1/2}} = 1 - \frac{iu (\Delta + \delta)}{2n^{1/2}} + O\left( u^2 \Delta^2 n^{-1} \right) , \]
provided that \( \delta \leq \Delta \). Substituting the obtained representations for \( \varphi^n \) and \( \hat{\psi} \) into (6) yields
\[ I_1 = n^{-1/2} \int_{A_n} e^{-iuv-u^2/2} du = \frac{i \mu_3}{6n} \int_{A_n} u^3 e^{-iuv-u^2/2} du \]
\[ - \frac{i(\Delta + \delta)}{2n} \int_{A_n} u e^{-iuv-u^2/2} du + R_1^{(1)} , \]
where it is not hard to show that, for \( \Delta \leq \Delta_1 \) with a fixed \( \Delta_1 > 0 \), one has the uniform bound
\[ |R_1^{(1)}| \leq c(r, b) n^{-(r-1)/2} ; \]
here and in what follows, by \( c(r, b) \in (0, \infty) \) we denote constants that can depend on the values of \( r \) and \( b \), and may be different even within one and the same formula. To indicate how the bound (16) was obtained, just note that

\[
\left| n^{-1/2} \int_{A_n} e^{-iuw - u^2/2} \frac{u^3}{n^{1/2}} \theta_2(un^{-1/2}) du \right| \leq n^{-1} \int_{A_n} e^{-u^2/2} |u|^3 |\theta_2(un^{-1/2})| du \\
\leq n^{-1} \int_{A_n} e^{-u^2/2} |u|^3 \left( |\theta_1(un^{-1/2})| + \frac{e|u|^3}{2h_1^6 n^{1/2}} \right) du \\
\leq n^{-1} c(r, b) \int e^{-u^2/2} |u|^3 \left( \frac{|u|^{r-3}}{n^{(r-3)/2}} + \frac{|u|^3}{n^{1/2}} \right) du \leq c(r, b)n^{-(r-1)/2}.
\]

The contributions of other cross-products in the expression for \( \varphi^n\hat{\psi} \) are bounded in a similar way, replacing \( |\mu_3| \) with its upper bound \( b^{3/r} \) due to Lyapunov’s moment inequality.

Clearly, replacing in (15) the integrals \( \int_{A_n} \) with \( \int_{\mathbb{R}} \) will introduce an error of the order \( o(n^{-2}) \) uniform over the class \( F_{r,b} \). Therefore, making that change and then replacing the integrals over the whole line with their explicit values, and setting \( \delta := \Delta n^{-1} \), we conclude that

\[
I_1 = 2\pi \phi(v) \left( n^{-1/2} + \frac{\mu_3}{6n} (v^3 - 3v) - \frac{\Delta}{2n} v \right) + R_n^{(2)},
\]

where for \( R_n^{(2)} \) holds true the same bound (16) as for \( R_n^{(1)} \).

To bound \( I_2 \), note that it follows from (17) and (2) that

\[
|\varphi(\lambda) - \lambda^2/2| \leq \frac{1}{2} b^{3/r-1} \lambda^2 \quad \text{for} \quad |\lambda| < h_2 := b^{-1}.
\]

Therefore, setting \( g := \frac{1}{2}(1 - b^{3/r-1}) \in (0, \frac{1}{2}) \), we have

\[
|\varphi(\lambda)| = |1 - (1 - \varphi(\lambda) - \lambda^2/2) - \lambda^2/2| \\
\leq |1 - g\lambda^2| \leq e^{-g\lambda^2}, \quad |\lambda| < h_2,
\]

implying that

\[
|I_2| \leq \int_{B_2} |e^{-\lambda x} \varphi^n(\lambda)\hat{\psi}(\lambda)| d\lambda \leq \int_{B_2} |\varphi(\lambda)|^n d\lambda \leq \int_{|\lambda| > h_1 n^{-1/3}} e^{-ng\lambda^2} d\lambda \\
= n^{-1/2} \int_{|u| > h_1 n^{1/6}} e^{-gu^2} du \leq n^{-1/2} \left( 1 + \frac{1}{2gh_1^2 n^{1/3}} \right) e^{-gh_1^2 n^{1/3}}.
\]

It remains to bound \( I_3 \). Since \( |\hat{\psi}(\lambda)| \leq 2^2/(\Delta \delta \lambda^2) \), \( \delta = \Delta n^{-1} \) and \( h_2 = b^{-1} \), one has

\[
|I_3| \leq \rho^n \int_{|\lambda| > h_2} |\hat{\psi}(\lambda)| d\lambda \leq \frac{4\rho^n n}{\Delta^2} \int_{|\lambda| > h_1} \frac{d\lambda}{\lambda^2} = 8b\Delta^{-2} \rho^n n.
\]
Choosing an arbitrary fixed $q \in (\rho^{1/2}, 1)$ and setting $\eta := \rho q^{-2} \in (0, 1)$, we will have

$$|I_3| \leq 8b\eta n \quad \text{for all} \quad \Delta \geq q^n. \quad (19)$$

Now returning to (5) and using representation (17) for $I_1$ and the bounds (18), (19) for $I_2$ and $I_3$, we obtain that, for $\Delta \geq q^n$, one has

$$\mathbb{P}(\tilde{S}_n \in \Delta[x]) = \Delta \phi(v) \left( n^{-1/2} + \frac{\mu_3}{6n} (v^3 - 3v) - \frac{\Delta}{2n} v \right) + \Delta R_n^{(3)}, \quad (20)$$

where for $R_n^{(3)}$ holds true the same bound (16) as for $R_n^{(1)}$.

Now set $(\Delta - \delta)[x] := [x, x + \Delta - \delta)$. Clearly, $\{\tilde{S}_n \in (\Delta - \delta)[x]\} \subset \{S_n \in \Delta[x]\}$, so that from (20) we see that

$$\mathbb{P}(S_n \in \Delta[x]) \geq \mathbb{P}(\tilde{S}_n \in (\Delta - \delta)[x])$$

$$= \Delta (1 - n^{-1}) \phi(v) \left( n^{-1/2} + \frac{\mu_3}{6n} (v^3 - 3v) - \frac{\Delta (1 - n^{-1})}{2n} v \right) + \Delta R_n^{(3)}$$

$$= \Delta \phi(v) \left( n^{-1/2} + \frac{\mu_3}{6n} (v^3 - 3v) - \frac{\Delta}{2n} v \right) + \Delta R_n^{(4)}, \quad (21)$$

where for $R_n^{(4)}$ holds true the same upper bound (16) as for $R_n^{(1)}$.

Setting $\tilde{S}_n := S_n + \delta U$, repeating the above calculations and using the obvious inclusion $\{S_n \in \Delta[x]\} \subset \{\tilde{S}_n \in (\Delta + \delta)[x]\}$, we obtain an upper bound for the probability $\mathbb{P}(S_n \in \Delta[x])$ which is of the same form as the right-hand side of (21).

Since clearly $n^{v-3/2} \cdot n R_n^{(4)} < c(r, b) < \infty$ according to the bound (16), that completes the proof of part (ii) of our theorem.

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