Comment on a relativistic model for coalescing neutron star binaries

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Abstract

The Wilson approximate dynamics and the Einstein dynamics are compared for binary systems. At the second post-Newtonian approximation, genuine two-body aspects are found to differ by up to 114%. In the regime of a formal innermost stable circular orbit (ISCO) the both dynamics differ by up to 7%.

Recently Wilson and Mathews [1] proposed a truncated version of the Einstein field equations to treat the coalescence of binary neutron stars in a much simplified but still sufficiently precise manner. The main idea of this approximation is to neglect the independent (“true”) degrees of freedom of the gravitational field, i.e. in particular, the full gravitational radiation content. For spherically symmetric processes the proposed scheme is identical with the Einstein equations, in non-spherically symmetric dynamical situations, even stationary ones, the proposed scheme and the Einstein theory only coincide at the first post-Newtonian level of approximation. It is perhaps worth mentioning that, in contrast to the Einstein theory, the proposed scheme should allow post-Newtonian series expansions to all orders in (integer) powers of $1/c^2$.

Most recently, Wilson et al. [2] applied the Wilson scheme to the question of instabilities in close neutron star binaries and found the remarkable result that general relativity may cause otherwise stable stars to collapse prior to merging. In another recent paper Cook et al. [3] tested the Wilson scheme for isolated, rapidly rotating relativistic stars. They found a deviation from the Einstein theory of at most 5% which they interpreted as very encouraging for a better understanding of binary star evolution.

In this paper we apply the Wilson scheme to point-like binary systems at the second post-Newtonian approximation and calculate the periastron advance as well as the orbital period. For circular motion, also the dependence of the angular momentum on the orbital angular frequency is given. In addition, we use the Hamiltonian of the Wilson scheme to calculate a formal innermost stable circular orbit (ISCO) for binary systems. The obtained results are confronted with the corresponding results of the Einstein dynamics.

In the Einstein theory the periastron advance and the orbital period have been calculated by Damour and Schäfer [4] starting from a second post-Newtonian Hamilton function. In isotropic coordinates and in the center of mass system ($\mathbf{P}_1 = -\mathbf{P}_2 = \mathbf{P}$), in reduced varibales ($\mathbf{p} = \mathbf{P}/\mu$, $r = \mathbf{R}/GM$) the reduced Hamilton function $\hat{H} = H/\mu$ reads

$$\hat{H}(\mathbf{r}, \mathbf{p}, \nu) = \frac{1}{2}\mathbf{p}^2 - \frac{1}{r} - \frac{1}{8c^2}(1 - 3\nu)\mathbf{p}^4 - \frac{1}{2rc^2}[(3 + \nu)\mathbf{p}^2 + \nu(\mathbf{n} \cdot \mathbf{p})^2]$$
where \( \nu = \mu/M \) with \( \mu = M_1M_2/M \) and \( M = M_1 + M_2 \). \( M_1 \) and \( M_2 \) denote the masses of the two bodies. The linear momenta of the bodies are \( P_1 \) and \( P_2 \), and \( R \) denotes a difference of their coordinate position vectors, \( R = R_1 - R_2 \); \( c \) is the velocity of light.

The fractional periastron advance per orbital revolution, \( k \), and the orbital period, \( P \), were found to be

\[
k = \frac{3}{h^2c^2} \left[ 1 + \frac{1}{2}(5 - 2\nu) \frac{E}{c^2} + \frac{5}{4}(7 - 2\nu) \frac{1}{h^2c^2} \right],
\]

\[
P = \frac{2\pi GM}{\sqrt{-2E^3}} \left[ 1 - \frac{1}{4}(15 - \nu) \frac{E}{c^2} - \frac{3}{32}(35 + 30\nu + 3\nu^2) \frac{E^2}{c^4} \right]
+ \frac{3}{2}(5 - 2\nu) \frac{\sqrt{-2E^3}}{hc^3},
\]

where \( E \) is the total center-of-mass energy (numerical value of \( \hat{H} \)) and \( h \) is the absolute value of the reduced angular momentum \( J/GM\mu \).

The Hamilton function in the Wilson scheme is easily obtained as the two–body special case of the n-body matter Hamilton function \( \hat{H}_{mat} \) of Schäfer [5] (eq. (3.14)). In reduced form this Hamiltonian reads (isotropic coordinates),

\[
\hat{H}_{mat}(r, p, \nu) = \frac{1}{2}p^2 - \frac{1}{r} - \frac{1}{8c^2}(1 - 3\nu)p^4 - \frac{1}{2r^6c^2} + \frac{5}{4}(5 + \nu)p^2 + \nu(n \cdot p)^2
\]

\[+ \frac{1}{2r^2c^2} + \frac{1}{16c^2}(1 - 5\nu + 5\nu^2)p^6
\]

\[+ \frac{5}{8rc^4}(1 - 4\nu)p^4 + \frac{1}{4r^2c^4}[(10 + 19\nu)p^2 - 3\nu(n \cdot p)^2] - \frac{1}{4r^3c^4}(1 + \nu). \tag{4}
\]

In the test–body limit, \( \nu = 0 \), the two Hamilton functions \( \hat{H} \) and \( \hat{H}_{mat} \) are identical as they should be on reasons of spherical symmetry of the central body.

The periastron advance of the dynamics eq. (4) is easily obtained following the path way of Ref. [4]. It comes out in the form

\[
k_{mat} = \frac{3}{h^2c^2} \left[ 1 + \frac{1}{2}(5 - \frac{3}{2}\nu - \frac{41\nu^2}{12}) \frac{E}{c^2} + \frac{1}{4}(35 - \frac{27}{2}\nu - \frac{31\nu^2}{4}) \frac{1}{h^2c^2} \right]. \tag{5}
\]

As one can see from equations (2) and (5) the two different periastron advances have the following structure

\[
k = \frac{1}{c^2}k^{1pN} + \frac{1}{c^4}(k^{2pN}_o + k^{2pN}_\nu), \tag{6}
\]

where \( k^{npN}_o \) denotes the \( \nu \)–independent terms and \( k^{npN}_\nu \) the \( \nu \)–dependent terms of \( k \). \( k \) and \( k_{mat} \) are different in the \( \nu \)–dependent terms of the second post–Newtonian order only. Using the Newtonian relation between energy and angular momentum, \( E = (e^2 - 1)/2h^2 \), where \( e \) denotes the eccentricity of the binary orbit, for the same energy and angular momentum, the fractional difference between the two periastron advances at the genuine two–body 2pN level reads

\[
\Delta k = (k - k_{mat})/|k^{2pN}_\nu| = \frac{1}{24(4 + c^2)} \left[ 4(12 + 13\nu) - (6 - 41\nu)e^2 \right]. \tag{7}
\]
The application of this expression to the case of equal-mass binaries ($\nu = 1/4$) in circular motion orbit ($e = 0$) gives a fractional difference between $k$ and $k_{\text{mat}}$ at the genuine two-body 2pN level of about 63%.

The orbital period in the Wilson scheme is obtained in the form, again following the route of Ref. [4],

$$P_{\text{mat}} = \frac{2\pi GM}{\sqrt{-2E}} \left[ 1 - \frac{1}{4}(15 - \nu) \frac{E}{c^2} - \frac{15}{32}(7 + 6\nu - 25\nu^2) \frac{E^2}{c^4} \right. \right. \left. + \frac{1}{2}(15 - \frac{9}{2}\nu - \frac{41}{4}\nu^2) \sqrt{-2E} \right].$$ (8)

Analogously to the periastron advance, the two orbital periods have the following structure

$$P = P^N + \frac{1}{c^2}(P^{1pN}_o + P^{1pN}_\nu) + \frac{1}{c^4}(P^{2pN}_o + P^{2pN}_\nu).$$ (9)

Also they are different in the $\nu$–dependent terms of the second post–Newtonian order only. For the same energy and angular momentum, the fractional difference between the two orbital periods at the genuine two–body 2pN level reads

$$\Delta P = \frac{(P - P_{\text{mat}})}{|P^{2pN}_\nu|} = \frac{16}{3} \frac{6 - \nu \left(41 - 24\sqrt{1 - e^2}\right)}{128 + 3(10 + \nu)\sqrt{1 - e^2}}.$$ (10)

where we have used again the Newtonian relation between energy and angular momentum. Compared to $\Delta k$ in the case of $e = 0$ and $\nu = 1/4$, the expression (10) reaches 6% only.

For circular orbits the angular frequency, $\omega$, is defined through the expression $\Phi/P$, where $\Phi$, the angle advance fore one orbital period, is given by $\Phi = 2\pi(1 + k)$. Taking into account the relation between the energy and the angular momentum for circular orbits, for our two dynamical situations, Einstein and Wilson respectively, the relation between angular momentum and orbital angular frequency turn out to be

$$h = \frac{1}{\omega^{1/3}} \left[ 1 + \frac{1}{2}(3 + \frac{1}{3}\nu) \frac{\omega^{2/3}}{c^2} + \frac{1}{8}(27 - 19\nu + \frac{17}{3}\nu^2) \frac{\omega^{4/3}}{c^4} \right].$$ (11)

$$h_{\text{mat}} = \frac{1}{\omega^{1/3}} \left[ 1 + \frac{1}{2}(3 + \frac{1}{3}\nu) \frac{\omega^{2/3}}{c^2} + \frac{1}{8}(27 - 39\nu - \frac{17}{3}\nu^2) \frac{\omega^{4/3}}{c^4} \right].$$ (12)

The fractional difference between $h$ and $h_{\text{mat}}$ at the genuine two–body 2pN level reads, applying the same frequency in both cases,

$$\Delta h = \frac{(h - h_{\text{mat}})}{|h^{2pN}_\nu|} = \frac{16}{57} + \frac{3\nu}{57}.$$ (13)

For the case of equal masses, $\nu = 1/4$, $\Delta h$ results in the value of 113.7%.

In modifying a procedure of Kidder et al. [6] we calculate now a formal ISCO for the second post–Newtonian binary dynamics of the Wilson scheme. The idea of the method by Kidder et al. was to add to the equations of motion of a test–mass in Schwarzschild spacetime all $\nu$–dependent terms of the second post–Newtonian binary equations of motion.
We apply this method to the Hamiltonian \([4]\) which describes the Wilson approximate dynamics at the second post–Newtonian approximation.

The reduced energy \(\hat{H}_o\) of a test body in Schwarzschild spacetime, in isotropic coordinates, reads
\[
\frac{\hat{H}_o}{c^2}(r,p) = \frac{1 - 1/2c^2r}{1 + 1/2c^2r} \sqrt{1 + \left(\frac{1}{2c^2r}\right)^{-4} \frac{p^2}{c^2}} - 1.
\] (14)

Augmentation of this expression by the second post–Newtonian \(\nu\)–dependent terms, \(\hat{H}_\nu\), of the Hamiltonian \([4]\) yields the following so–called hybrid approximation
\[
\hat{H}_* = \hat{H}_o + \hat{H}_\nu.
\] (15)

For circular orbits this expression can be expressed as a function of \(r\) and the reduced angular momentum \(h\) (\(n \cdot p = 0, p^2 = h^2/\nu^2\)). The radius of the ISCO is than obtained by the aid of the equations
\[
\frac{\partial \hat{H}_*(r,h)}{\partial r} = 0 = \frac{\partial^2 \hat{H}_*(r,h)}{\partial r^2}.
\] (16)

We have solved these equations numerically. For two equal masses, \(\nu = 1/4\), we obtained for the ISCO
\[
r_{\text{mat}} = 6.82 \text{GM}/c^2.
\] (17)

Starting from the second post–Newtonian two–body Hamiltonian \([4]\) the ISCO has been calculated by Schäfer and Wex [7]. They used the same method and they obtained \(r = 7.34 \text{GM}/c^2\). The result \([17]\) obtained from the Hamiltonian \([4]\) differs by 7.1\% from this value for \(r\).

For an easy comparison of the two different dynamical situations the numerical values of the energy, \(\hat{H}\), and the angular momentum, \(h\), both coordinate independent quantities, are given in Table 1.

|       | TBSD | HWAD | HED  |
|-------|------|------|------|
| \(r\) | 4.95 | 6.82 | 7.34 |
| \(J\) | -0.057 | -0.050 | -0.047 |
| \(E\) | -0.057 | -0.050 | -0.047 |

The results obtained for the periastron advances, eqs. (2) and (5), and the orbital periods, eqs. (3) and (8), are valid in dynamical regimes where post–Newtonian approximations apply. In the test–body limit as well as in the first post–Newtonian approximation the periastron advances and the orbital periods coincide. They only differ in the genuine binary 2pN parts, this means in the \(\nu\)–dependent terms of 2pN order. The fractional difference of the 2pN \(\nu\)-dependent terms for \(h\) takes the remarkably large value of about 114\% in the case of equal–mass binaries in circular motion with the same orbital frequency.

On the other side, the estimated value for the ISCO of the hybrid 2pN Wilson scheme, eq. (14), differs by about 7\% from the previously calculated value for the 2pN binary dynamics, \([4]\), by Schäfer and Wex [7]. In Tab. 1 we have summarized the numerical values for the
ISCO radius, the energy and the angular momentum for the different models. Surely, near the last stable circular orbit post–Newtonian approximations lose their meaning. Nevertheless, it seems reasonable to conclude that in the Wilson scheme the binary system, near the ISCO, is stronger bounded (see Tab. 1). The conclusion is supported by the fact the in regimes where the 2pN approximation applies the stronger binding holds, see Fig. 1: Given the same orbital angular frequency, the Wilson approximate dynamics has smaller angular momentum, i.e. smaller moment of inertia. Another support results from the energy expression as function of \( \omega \), most easily obtained from the relation \( dE = \omega dh \), also showing stronger binding.

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Fig. 1: The value of the genuine two–body 2pN term in $h$ is plotted as function of $\omega$, the orbital angular frequency for circular motion, for the both models. The upper curve belongs to the 2pN Einstein dynamics whereas the lower curve belongs to the 2pN Wilson approximate dynamics.