Computing the Roughening Transition of Ising and Solid-On-Solid Models by BCSOS Model Matching

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Abstract

We study the roughening transition of the dual of the 2D XY model, of the Discrete Gaussian model, of the Absolute Value Solid-On-Solid model and of the interface in an Ising model on a 3D simple cubic lattice. The investigation relies on a renormalization group finite size scaling method that was proposed and successfully tested a few years ago. The basic idea is to match the renormalization group flow of the interface observables with that of the exactly solvable BCSOS model. Our estimates for the critical couplings are $\beta_{\text{XY}} = 1.1199(1)$, $K_{\text{DG}} = 0.6653(2)$ and $K_{\text{ASOS}} = 0.80608(2)$ for the XY-model, the Discrete Gaussian model and the Absolute Value Solid-On-Solid model, respectively. For the inverse roughening temperature of the Ising interface we find $K_{\text{Ising}} = 0.40758(1)$. To the best of our knowledge, these are the most precise estimates for these parameters published so far.
1 Introduction

Among the phase transitions that occur in 2D or effectively 2D statistical systems, those of the so called Kosterlitz-Thouless (KT) type [1] belong to the most challenging. The KT phase transition is of infinite order: The free energy and all its derivatives stay finite at the transition point. Despite the relatively simple arguments that suggest the existence of such a transition in a variety of systems, a rigorous proof of the KT nature of the phase transition in many physically interesting systems is still lacking. Also most of the numerical studies (many of them Monte Carlo studies of the 2D XY model), could not provide an unambiguous confirmation of the KT scenario. A number of references will be given in section 5.

The reason for the problem is the appearance of corrections to scaling that vanish only logarithmically with the system size. Most of the investigations based on simulations of KT models on finite lattices suffer from these corrections.

Note, however, that there exists at least one 2D lattice model, which has been proven to undergo a KT transition by exact solution. This is the Body Centered Solid-On-Solid (BCSOS) model [2], the configurations of which are, up to boundary conditions, in one-to-one correspondence to those of a special 6-vertex model [3, 4, 5], the F-model.

A few years ago we proposed a method that allows to investigate the KT transition of a given model by comparing its block spin renormalization group (RG) flow (on finite lattices) with that of the BCSOS model [6, 7]. A matching of the two RG flows at long distance (large blocks) demonstrates that the models belong to the same universality class, which is the KT class here.

In contrast to the usual approaches, the method introduces systematic errors that decay like \( L^{-2} \), where \( L \) is the size of the lattices involved in the computations.

Our approach has been successfully applied to the Absolute Value Solid-On-Solid (ASOS) model, the Discrete Gaussian (DG) model and the dual of the standard XY model in two dimensions [6]. Successful applications to the interface of the Ising model were performed in [7], and, recently, in [8].

In the present paper we improve on the results of ref. [6] by using larger lattice sizes and increasing the statistics by a factor of about 100. This became affordable both by the availability of faster computers and the use of more efficient program code.

This paper is organized as follows: In section 2 we define the models and state the exact results for the BCSOS model relevant to our study. We briefly discuss the KT flow equations. Section 3 is devoted to a description of the matching method. In section 4 we present and discuss our numerical results. A comparison with previous estimates of the critical couplings and non-universal parameters is presented in section 5. Conclusions and an outlook follow.

In the preprint version of this paper, part of the tables is presented in a section after the Bibliography.

2 Ising Model Interfaces and Solid-On-Solid Models
2.1 Ising Model Interfaces

We consider the 3D Ising model on the simple cubic lattice, with Hamiltonian

\[ H = - \sum_{\langle x, y \rangle} s_x s_y, \quad s_x = \pm 1. \]  

(1)

The sites of the lattice are labelled by integer coordinates \( x = (x_1, x_2, x_3) \). The sum in eq. (1) is over all (unordered) nearest neighbour pairs of sites in the lattice. The partition function is

\[ Z = \sum_{\{s\}} \exp \left( -K^I H \right). \]  

(2)

Here, the summation is over all possible configurations of the Ising spins. The pair interaction is normalized such that \( K^I = 1/(k_B T) \), where \( k_B \) denotes Boltzmann’s constant, and \( T \) is the temperature.

At a critical coupling \( K^I_c \) (the estimate of a recent study \cite{9} is \( K^I_c = 0.2216546(10) \)) the infinite volume limit of the model undergoes a second order phase transition. For \( K^I > K^I_c \), the system shows spontaneous breaking of the reflection symmetry.

In order to study interfaces separating extended domains of different magnetization, we consider lattices with extension \( L \) in the \( x_1 \)- and \( x_2 \)-directions and with extension \( D \) in the \( x_3 \)-direction. We generalize eq. (1) to

\[ H = - \sum_{\langle x, y \rangle} k_{xy} s_x s_y. \]  

(3)

The lattice becomes a torus by regarding the opposite boundary planes as neighbour planes. For the Ising spins \( s \) we will apply antiperiodic boundary conditions in \( x_3 \)-direction, by letting \( k_{xy} = -1 \) for the links that connect the uppermost with the lowermost plane. For the other links we set \( k_{xy} = 1 \).

For sufficiently large \( K^I \) and large enough \( L \), the imposition of antiperiodic boundary conditions forces the system to develop exactly one interface, a region where the magnetization rapidly changes sign. This interface is parallel to a \((001)\) lattice plane.

The Ising \((001)\) interface undergoes a roughening transition at an inverse temperature \( K^I_R = 1/(k_B T_R) \) that is nearly twice as large as the bulk transition coupling \( K^I_c \) given above\cite{10}. In this work, we shall determine a new estimate for \( K^I_R \), and also for other parameters of the roughening transition. For a collection of previous estimates, see section \( \text{footnote} \).

At the roughening transition, the large scale interface behavior changes from being rigid or smooth at low temperature to being rough at high temperature. The transition shows up in a characteristic behaviour of various quantities. For example, in the smooth phase, the interfacial width stays finite when \( L \) tends to infinity, while it diverges logarithmically with the system size in the rough phase \cite{11, 12}. For general introductions to roughening, see \cite{13, 14, 15}. For comparisons of real life experiments with theory see, e.g., refs. \cite{16}.

\footnote{A pioneering work on this issue is ref. \cite{10}}
2.2 Solid-On-Solid Models

A fairly good approximation of the Ising interface is given by Solid-On-Solid (SOS) models to be introduced in this section. The SOS approximation amounts to ignoring overhangs of the Ising interface and bubbles in the two phases separated by the interface. For a review of exact results on SOS type of models, see, e.g. [13]. By duality [17] and other exact transformations (see, e.g., [18]), SOS models have been shown to be equivalent to a variety of other statistical models.

All SOS models that we shall consider have in common that they are 2D lattice spin models. Our first example of an SOS model is the Absolute-Value-Solid-On-Solid (ASOS) model. It can be considered as the SOS approximation of an (001) lattice plane interface of an Ising model on a simple cubic lattice. The model is defined by the Hamiltonian

\[ H_{ASOS} = K_{ASOS} |h_x - h_y|. \]

The spin variables \( h_x \) take integer values. Here and in the following, the Boltzmannian will always be \( \exp(-H) \). A factor \( 1/(k_B T) \), where \( k_B \) denotes Boltzmann’s constant and \( T \) the temperature, is absorbed in the definition of the Hamiltonian.

We interpret the \( h_x \) as heights with respect to a certain base. For finite positive \( K_{ASOS} \) the Hamiltonian will favour that neighbouring spins take similar values. When \( K_{ASOS} \) is large enough, the surface will not fluctuate too wildly (smooth phase). On the other hand, if \( K_{ASOS} \) is below a certain critical value, the surface becomes “rough”, and, e.g., the surface thickness diverges when the system size goes to infinity.

Let us now turn to the Discrete Gaussian (DG) model. The Hamiltonian is

\[ H_{DG} = K_{DG} (h_x - h_y)^2. \]

The spin variables \( h_x \) take integer values. Note that the Hamiltonian looks exactly like that of a continuous Gaussian model. However, the restriction of the \( h_x \) to integer values introduces a nontrivial interaction. The Discrete Gaussian model is dual to the XY model with Villain action [17]. This model is defined by the partition function

\[ Z_V = \int_{-\pi}^{\pi} \prod_x d\Theta_x \prod_{<x,y>} B(\Theta_x - \Theta_y), \]

with

\[ B(\Theta) = \sum_{p=-\infty}^{\infty} \exp \left( -\frac{1}{2} \beta_V (\Theta - 2\pi p)^2 \right) \]

and

\[ \frac{1}{2\beta_V} = K_{DG}. \]

\[ ^2 \text{What is called Hamiltonian in the language of Statistical Mechanics is called action in the framework of Euclidean Quantum Field Theory.} \]
The index “V” here refers to “Villain”.

The XY model with “standard (cosine) action” has the partition function

\[ Z_{XY} = \int_{-\pi}^{\pi} \prod_{x} d\Theta_{x} \exp \left( \beta_{XY} \sum_{<x,y>} \cos(\Theta_{x} - \Theta_{y}) \right). \] (9)

The standard action is the mostly discussed action for an XY model. The dual of this model is given by the partition function

\[ Z_{XY}^{SOS} = \sum_{\{h\}} \prod_{<x,y>} I_{|h_{x} - h_{y}|}(\beta_{XY}), \] (10)

where the \( I_{n} \) are modified Bessel functions. Again \( h_{x} \) is integer.

We finally introduce the Body Centered Solid-On-Solid (BCSOS) or F-model. The BCSOS model was introduced by van Beijeren \[2\] as an SOS approximation of an interface in an Ising model on a body centered cubic lattice on a (001) lattice plane. For a detailed analysis of this model with respect to roughening and surface structure, see \[14, 19, 15\]. The effective 2D lattice splits in two sublattices like a checker board. In the original formulation, on one of the sublattices the spins take integer values, whereas the spins on the other sublattice take half-integer values. We adopt a different convention: spins on “odd” lattice sites take values of the form \( 2n + \frac{1}{2} \), and spins on “even” sites are of the form \( 2n - \frac{1}{2}, n \) integer. The Hamiltonian of the BCSOS model can be expressed as

\[ H_{BCSOS} = K_{BCSOS} \sum_{[x,y]} |h_{x} - h_{y}|. \] (11)

The sum is over next-to-nearest neighbour pairs \([x,y]\), and nearest neighbour spins \( h_{x} \) and \( h_{y} \) obey the constraint \( |h_{x} - h_{y}| = 1 \). Van Beijeren \[3\] showed that the BCSOS model can be transformed into the F-model, which is a special six vertex model. The F-model can be solved exactly with transfer matrix methods \[3, 4, 5\]. The roughening transition occurs at

\[ K_{R}^{BCSOS} = \frac{1}{2} \ln 2. \] (12)

For \( K \lesssim K_{R} \), the correlation length behaves like

\[ \xi_{BCSOS} \simeq \frac{1}{4} \exp \left( \frac{\pi^{2}}{8 \sqrt{\frac{1}{2} \ln 2}} \kappa^{-\frac{1}{2}} \right), \quad \kappa = \frac{K - K_{R}}{K_{R}}. \] (13)

### 2.3 Renormalization Group Flow of Interface Models

It is believed (though not proven rigorously) that, in the vicinity of the fixed point relevant for the KT transition, the RG flow of SOS models and also of the 3D Ising model interface is well described by two parameters \( \beta \) and \( z \) \[1\]. The two parameters are the inverse temperature and a fugacity \( z \).
The 2D Sine Gordon model is especially suited to discuss the flow of these parameters with the length scale, since this model contains $\beta$ and $z$ as bare parameters in its Hamiltonian:

$$H^{SG} = \frac{1}{2\beta} \sum_{<x,y>} (\phi_x - \phi_y)^2 - z \sum_x \cos(2\pi \phi_x),$$

where the $\phi_x$ are real numbers. For the continuum version of the model, with a momentum cutoff, one can derive the parameter flow under infinitesimal RG transformations. It is given by

$$\dot{x} = -z^2, \quad \dot{z} = -xz,$$

where $z = const \cdot z$ and $x = \pi \beta - 2$. $const$ depends on the particular cutoff scheme used. The derivative is taken with respect to the logarithm of the cutoff scale.

For large $x$ the fugacity $z$ flows towards $z = 0$. The large distance behaviour of the model is therefore that of a Gaussian model (without a mass term). For small $x$, $z$ grows with increasing length scale. The theory is therefore massive, i.e., has finite correlation length. The critical trajectory separates these two regions in the coupling constant space. It ends at a Gaussian fixed point characterized by $x = 0$ or $\beta = \frac{2}{\pi}$. On the critical trajectory the fugacity vanishes as

$$z(t) = \frac{1}{z_0^{-1} + t},$$

where $t$ is the logarithm of the cutoff scale. Eqs. (15) are the basis for KT theory. Its immediate consequences are derived in statistical mechanics textbooks, see e.g. [20]. E.g., the correlation length in the smooth phase of an SOS model should diverge like

$$\xi \simeq A \exp \left( C\kappa^{-1/2} \right), \quad \kappa = \frac{K - K_R}{K_R},$$

when $K \to K_R$. We would like to emphasize another important consequence of the KT equations that becomes apparent from the solution eq. (16): At criticality, the fugacity, which parametrizes the deviation of the theory from a massless Gaussian model, decays with increasing scale $t = \ln L$, only like $(\ln L)^{-1}$. In lattice studies, $L$ is more or less the lattice extension. Therefore, any method that is based on an observation of the Gaussian behaviour at long distance, suffers strongly from finite fugacity corrections even on very large lattices.

### 3 The Matching Method

The method of ref. [3] is closely related to the finite size scaling methods proposed by Nightingale [21] and Binder [22]. No attempt is made to compute the RG flow of the couplings explicitly, but rather the RG flow is monitored by evaluating quantities that are primarily sensitive to the lowest frequency fluctuations on a finite lattice. One should stress that the method does not use any of the quantitative results of KT theory. Merely the qualitative result that there are two important coupling parameters in the flow is used.
In order to separate the low frequency modes of the field a block spin transformation \([23, 24]\) is used. Blocked systems of size \(l \times l\) are considered. The size \(B\) of a block (measured in units of the original lattice spacing) is then given by \(B = L/l\), where \(L\) is the linear size of the original lattice. The linear blocking procedure defined by

\[
\phi_X = B^{-2} \sum_{x \in X} h_x ,
\]

where \(X\) labels square blocks of a linear extension \(B\), is used. This linear blocking rule has the half-group property that the successive application of two transformations with a scale-factor of \(B\) have exactly the same effect as a single transformation with a scale-factor of \(B^2\).

Motivated by the perturbation theory of the Sine Gordon model two types of observables are chosen: such that are “sensitive” to the flow of the kinetic term (flow of \(K\)), and such that are sensitive to the fugacity (periodic perturbation of a massless Gaussian model). For the first type of observables

\[
A_1 = \langle (\phi_X - \phi_Y)^2 \rangle ,
\]

where \(X\) and \(Y\) are nearest neighbours on the block lattice, and

\[
A_2 = \langle (\phi_X - \phi_Z)^2 \rangle ,
\]

where \(X\) and \(Z\) are next to nearest neighbours, are chosen. Note that these quantities are only defined for \(l > 1\). As a monitor for the fugacity the following set of quantities (defined for \(l = 1, 2, 4\) and \(8\)) is taken:

\[
A_3 = \langle \cos(1 \cdot 2\pi \phi_X) \rangle ,
\]

\[
A_4 = \langle \cos(2 \cdot 2\pi \phi_X) \rangle .
\]

### 3.1 Determination of the Roughening Coupling

There are two parameters which have to be adjusted in order to match the RG flow of an SOS model or of the Ising interface with that of the critical BCSOS model: The coupling \(K_S\) of the Solid-On-Solid or Ising model and in addition the ratio \(b = B^S/B^B = L^S/L^B\) of the lattice sizes (and hence the block sizes) of the SOS or Ising model and the BCSOS model. In general a \(b \neq 1\) is necessary to compensate for the different starting points of the two models on the critical RG-trajectory \([4]\). For the proper values of the roughening coupling \(K^S_R\) and the matching constant \(b\) observables of the SOS and the BCSOS model match like

\[
A^{S}_{i,l}(b, K^S_R) = A^{B}_{i,l}(B, K^B_R) + O(B^{-\omega}) ,
\]

where \(i\) labels the observable and \(l\) the size of the blocked lattice. The \(O(B^{-\omega})\) corrections are due to irrelevant operators. \(\omega\) is the correction to scaling exponent. The perturbation theory of the Sine Gordon model suggests \(\omega = 2\).

In order to obtain numerical estimates for the roughening coupling \(K^S_R\) and the matching factor \(b\) for a given lattice size \(L^B\) of the BCSOS model we require that eq. \([22]\) is exactly fulfilled for two block observables.
We solve the system of two equations for the two observables $A_{i,l}$ and $A_{j,l}$ numerically by first computing the $K_{i,l}^{S}(b)$ and $K_{j,l}^{S}(b)$ that solve the single equations for a given value of $b$. The intersection of the two curves $K_{i,l}^{S}(b)$ and $K_{j,l}^{S}(b)$ gives us then the solution of the system of two equations. For an illustration of this method see figures 5 and 6 of ref. [6].

In [6] we demonstrated, that the corrections to scaling for the observables $A_1$ and $A_2$ for SOS models are similar to those in the massless continuous Gaussian model. Therefore we considered the “improved” observable $D_1$ which is defined as follows:

$$D_1(L) = \frac{A_1^{(0)}(\infty)}{A_1^{(0)}(L)} A_1(L).$$

(23)

$A_1^{(0)}$ is computed for the massless Gaussian model defined by

$$H_0 = \frac{1}{2} \sum_{<x,y>} (\psi_x - \psi_y)^2.$$

(24)

An improved quantity $D_2$ is defined analogously. Explicit results for $A_1^{(0)}$ and $A_2^{(0)}$ are given in table 8.

Obviously this modification does not affect the large $L$ behaviour since $A_1^{(0)}(L) = A_1^{(0)}(\infty) + O(L^{-2})$. It turns out that the results for our largest lattice sizes are virtually unaffected by this kind of improvement.

### 3.2 Determination of Non-Universal Constants

The matching programme also allows to determine the non-universal constants appearing in formulae describing the divergence of observables near the roughening transition. In [6] we showed that the two non-universal parameters $A$ and $C$ determining the critical behaviour of the correlation length, cf. eq. (17), can be determined from information of the matching procedure. For one of the models matched with the BCSOS model, one finds

$$A^{SOS} = b_m A^{BCSOS}$$

(25)

$$C^{SOS} = q^{-1/2} C^{BCSOS}$$

(26)

where the parameters $A^{BCSOS}$ and $C^{BCSOS}$ can easily be extracted from eq. (13). If

$$R = \frac{\partial A^{BCSOS}_{i,l}}{\partial K^{BCSOS}} / \frac{\partial A_{i,l}}{\partial K},$$

(27)

where quantities have to be taken at the roughening couplings, is the same for all observables, which is the case for our data, then $q$ is given by

$$q = \frac{K^{SOS}_R}{K^{BCSOS}_R} R.$$

(28)

For a more detailed discussion see ref. [6].
4 Numerical Results

We simulated the BCSOS model at its critical coupling $K_{BCSOS}^{R} = \frac{1}{2} \ln 2$ using the loop-algorithm of Evertz, Marcu and Lana [25]. One has to note that periodic boundary conditions of the F-model do not correspond to periodic boundary conditions of the BCSOS model. Therefore updates of loops that wind around the lattice are forbidden.

We performed $10^7$ measurements for all lattice sizes considered. We have chosen the number of loop-updates between two successive measurements such that the autocorrelation times were about 1.

In addition to the observables $A_{i,l}$ we measured the interface thickness, the total energy $E$ and $A_{i,l} \times E$, which is needed to compute derivatives of the observables with respect to the coupling. In order to save disc space we accumulated 1000 measurements before writing to the file. The statistical errors were computed by jacknifing the $10^4$ prebinned data. As random number generator we used a combination of three shift register generators.

We checked the reliability of the updating program by comparing the estimates from $10^8$ measurements for $L = 4$ with the exact results obtained by explicitly averaging over all BCSOS configurations. Our data are also consistent with those of ref. [3]. In [3] lattices of size up to $L = 96$ were used, $4 \times 10^6$ measurements were performed, and the G05CAF random number generator of the NAGLIB was used. Note also that the computer programs of [3] were written independently of the programs used in the present study.

The results for the BCSOS observables are summarized in tables 9 and 10. Our estimates for the slopes of the observables are given in tables 11 and 12. With slope we here mean the derivative of the observables with respect to the coupling $K_{BCSOS}^{R}$, taken at the critical value.

We then performed the simulations for the ASOS, the DG and the dual of the XY model. The simulations of the ASOS and DG model were done using a demon version [26] of the Valleys-to-Mountains reflection (VMR) algorithm [27]. The simulation of the dual XY model was done using the standard version of the VMR algorithm. In both cases we used the G05CAF routine of the NAGLIB as random number generator.

Again we performed $10^7$ measurements and accumulated 1000 measurements before writing to the file. In order to get estimates for the observables in a neighbourhood of the simulation point, we employed a second order Taylor expansion (note that the prebinning forbids the use of a reweighting technique). We thus computed the first and second derivatives of the observables,

$$\frac{dA}{dK} = \langle H \rangle \langle A \rangle - \langle HA \rangle ,$$

and

$$\frac{d^2 A}{dK^2} = \langle H^2 A \rangle - 2\langle H \rangle \langle HA \rangle + 2\langle H \rangle^2 \langle A \rangle - \langle H^2 \rangle \langle A \rangle ,$$

for the ASOS and the DG model. In the XY case analogous formulae were derived. We carefully checked by comparing with results obtained from simulations at shifted couplings that the Taylor expansion of the $A_{i,l}$ to second order was sufficiently precise.
We performed the simulations at the previously best known estimates for the roughening couplings, namely $K_{DG}^0 = 0.6645$, $K_{ASOS}^0 = 0.8061$ and $\beta_{XY}^0 = 1.1197$.

In order to keep the length of this paper within reasonable bounds, we present the numerical results only for the $XY$ model, see tables 13 and 14. The tables for the other models are available from the authors upon request.

Finally we performed the simulations for the Ising model. The simulations were done using the (VMR) algorithm\[27\] adapted to the Ising interface as discussed in refs.\[28, 12\]. Following the proposal of ref.\[29\] we encoded all $D$ spins with the same $x_1$ and $x_2$ in a single word. We found no way to execute operations in parallel on the spins in one word. The advantage of this particular coding merely lies in considerably reduced memory access time.

The local update of the system was speeded up by replacing the standard Metropolis update by a demon update implemented in multispin coding technique. Also the elimination of the bubbles in the bulk phases was done using a multispin coding technique.

In total these improvements led to a reduction of the CPU time required for a given statistic by a factor of about 4 compared with the code used in refs.\[28, 12, 8\].

We performed the simulations at the previously best known estimate for the roughening couplings, $K_I^0 = 0.40754$.

We performed $3 \times 10^6$ to $8 \times 10^6$ measurements for lattice sizes ranging from $32 \times 32 \times 31$ to $192 \times 192 \times 31$. Again we pre-binned the results of 1000 measurements before writing to disc.

For the Ising interface we also computed the Taylor expansion of the observables to second order.

In the case of the Ising interface we performed, in addition of the matching with the BCSOS model, the matching with the ASOS model at $K_{ASOS}^R = 0.80608$, which is our present estimate of the roughening coupling for the ASOS model. For this purpose we performed additional simulations of the ASOS model at $K_{ASOS}^R = 0.80608$ for the lattice sizes 24, 48, 56, 80, 112 and 160. The idea behind the matching with the ASOS model is that corrections to scaling in the ASOS model are similar to those of the Ising interface. Therefore it should be possible to obtain reliable estimates for the roughening coupling from smaller lattice sizes this way than from the matching with the BCSOS model.

The total CPU requirement for all our simulations accumulates to nearly 4 years on typical modern workstations. The computers used include all types ranging from Pentium PCs to Alpha AXP-3000/400 and the parallel computer Hitachi SR2001 with 64 processors. For an overview of the lattice sizes employed and the CPU resources needed for the various models, see table 1.

We extracted all our estimates from the matching of the two observables $D_2$ and $A_3$ (the last column in the tables). Here the convergence seems optimal.

To obtain estimates for the roughening couplings and the matching $b_m$ we employed the following

**RULE:** Start with the largest block lattice size, i.e., $l = 8$ (the statistical errors are the smallest here). As a first estimate $E_1$ take the value for the largest lattice size $L$ available. Then check whether the estimate is $2\sigma$-compatible with the results (also for $l = 8$) for the next two smaller
Table 1: The lattice sizes $L$ employed in our simulations of the various models, together with the CPU resources needed on an “average modern workstation”.

| model  | lattice sizes                                             | CPU   |
|--------|-----------------------------------------------------------|-------|
| BCSOS  | 16,24,32,40,48,56,64,80,96,112,128,160,192,224,256       | 200 d |
| XY     | 32,48,64,96,192                                           | 70 d  |
| ASOS   | 32,48,64,96,128,192,256,384,512                           | 440 d |
| DG     | 16,24,32,48,64                                           | 15 d  |
| Ising  | 32,48,64,96,128,192                                       | 650 d |

Table 2: Our results for the roughening couplings and the matching $b_m$, together with the $l, L$ values that were used (see RULE). In the Ising model case, the index “a” refers to the matching with the BCSOS model, the index “b” refers to the matching with the ASOS model.

| model   | $K_R$          | from $l, L$ | $b_m$          | from $l, L$ |
|---------|----------------|-------------|----------------|-------------|
| XY      | 1.1199(1)      | 4,192       | 0.93(1)        | 4,192       |
| ASOS    | 0.80608(2)     | 8,512       | 2.78(3)        | 4,512       |
| DG      | 0.6653(2)      | 2,64        | 0.32(1)        | 2,64        |
| Ising,a | 0.40759(2)     | 2,192       | 3.20(4)        | 2,192       |
| Ising,b | 0.40758(1)     | 8,192       | 3.21(3)        | 8,192       |

Given that the corrections die out like $L^{-2}$ our RULE ensures that systematic errors in the determination of the roughening coupling are smaller than the statistical errors quoted.

We invite the careful reader to go through this procedure in the case of the XY-model (tables [3] and [4]). Our final estimates for the critical couplings and the matching $b_m$, together with the values of $l$ and $L$ where the decision procedure stopped, are given in table [2].

In order to determine the non-universal constants $A$ and $C$ we need estimates for the ratios of slopes $R$ defined in eq. (27). These ratios for the different observables and for the different block/lattice sizes are presented (for the XY model as an example) in table [5]. From this table, and from the corresponding tables for the other SOS models and the Ising model, we extracted by applying again our RULE a final estimate for the ratio of slopes $R$. Our estimates

lattice sizes. $2\sigma$-compatibility of two estimates $m_1$, $m_2$ with statistical errors $e_1$, $e_2$ here means that $|m_1 - m_2| < 2 \left[ e_1^2 + e_2^2 \right]^{1/2}$. Then also check the $2\sigma$-consistency of $E_1$ with the estimates for the three largest available $L$-values for $l = 4$ and $l = 2$. If the estimates are consistent, take $E_1$ as the final estimate. Otherwise restart the whole procedure with $l = 4$, i.e. take as the first estimate $E_1$ the value from the largest $L$ and $l = 4$. If there is again a failure, restart at $l = 2$. 

We invite the careful reader to go through this procedure in the case of the XY-model (tables [3] and [4]). Our final estimates for the critical couplings and the matching $b_m$, together with the values of $l$ and $L$ where the decision procedure stopped, are given in table [2].

In order to determine the non-universal constants $A$ and $C$ we need estimates for the ratios of slopes $R$ defined in eq. (27). These ratios for the different observables and for the different block/lattice sizes are presented (for the XY model as an example) in table [5]. From this table, and from the corresponding tables for the other SOS models and the Ising model, we extracted by applying again our RULE a final estimate for the ratio of slopes $R$. Our estimates
Table 3: Our results for the non-universal constants $A$ and $C$. In the Ising model case, the index “a” refers to the matching with the BCSOS model, the index “b” refers to the matching with the ASOS model.

$$\begin{array}{|c|c|c|} \hline \text{model} & A & C \\ \hline \text{XY} & 0.233(3) & 1.776(4) \\ \text{ASOS} & 0.695(8) & 1.099(4) \\ \text{DG} & 0.080(3) & 2.438(6) \\ \text{Ising,a} & 0.80(1) & 1.03(2) \\ \text{Ising,b} & 0.80(1) & 1.01(1) \\ \hline \end{array}$$

Table 4: Comparison of our results for the DG model with previous estimates.

| Authors                      | year | $K_R^{DG}$ | $A$      | $C$      |
|------------------------------|------|------------|----------|----------|
| Swendsen [30]                | 1977 | 0.77(6)    |          |          |
| Shugard et al. [31]          | 1978 | 0.68       |          |          |
| Janke and Nather [32]        | 1991 | 0.665(5)   | 0.1204(18) | 2.370(11) |
| dito, fit1                   |      | 0.6657(3)  | 0.0287(7) | 2.812(14) |
| dito, fit2                   |      | 0.6595(3)  |          |          |
| Evertz et al. [33]           | 1993 | 0.662(3)   |          |          |
| Hasenbusch et al. [7, 6]     | 1992/94 | 0.6645(6)  | 0.078(5) | 2.44(3) |
| Hasenbusch and Pinn, this work | 1996 | 0.6653(2)  | 0.080(3) | 2.438(6) |

for the non-universal constants $A$ and $C$ are given in table 3.

5 Comparison with Previous Studies

In this section we present a comparison of our present results with some previous estimates on the critical couplings $K_R$ and non-universal parameters $A$ and $C$.

Let us start with the DG model. See table 4 for two estimates from the seventies and some more modern results that can be compared with the present estimates. A comparison of the findings in [32] and [33] with the estimates of [6] was presented in ref. [6]. We would like to comment just at the apparent 1σ-incompatibility of the present estimate for $K_R^{DG}$ with that of [6]. A closer look at our data reveals that this is most likely a statistical fluctuation: Discarding the $L = 48$ and $L = 64$ lattices from the analysis does not move our present estimate towards the result in [6], which was obtained with the same method and with lattices of size up to $L = 32$.

We now turn to the XY model. A table of previous estimates in comparison with previously published results is given in table 5. We find our present estimates consistent with our previous results in [6]. Most of the results of the other authors, also the estimates from series analysis
by Campostrini et al. [40] are inconsistent with the present estimate. We conclude that in all these cases the systematic errors due to corrections to scaling are underestimated.

In the case of the ASOS model, we only compare with our previous estimate [6] and with an estimate by Adler from a ninth-order low temperature series. The series estimate has a quite large error, but is consistent with our result.

We conclude this section with a comparison of the Ising interface estimates. Here we find that all the cited estimates of the roughening couplings are consistent with each other. Note, however, the large errors in the estimates that were obtained with techniques other than the matching method. The estimate of Mon et al. [42] for $A$ seems to be the result of a wrong method.

### 6 Conclusions and Outlook

By increasing the statistics by a factor of about 100 and by also using larger lattices compared to ref. [6], we obtained the most accurate estimates for the roughening couplings of the Ising interface, the ASOS model, the DG model and the 2D XY model published so far. In contrast to other methods systematical errors are under control. The matching procedure converges like $L^{-2}$ while other methods that rely on analytic results derived from KT theory are plagued by corrections logarithmic in the lattice size. In addition to the precise numbers for the roughening coupling and other non-universal constants the matching provides an unambiguous confirmation.
of the KT nature of the phase transition of the models considered. It is interesting to compute the observables used for the matching method for the Sine Gordon model in perturbation theory. This will allow to rederive the KT flow equations from finite size scaling. Furthermore, it will provide quantitative information about the RG flow, in particular about the critical trajectory, which can be compared with the numerical results given in the present paper.

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| $L$ | $A_{1,2}^{(0)}$ | $A_{2,2}^{(0)}$ | $A_{1,4}^{(0)}$ | $A_{2,4}^{(0)}$ | $A_{1,8}^{(0)}$ | $A_{2,8}^{(0)}$ |
|-----|---------------|---------------|---------------|---------------|---------------|---------------|
| 16  | 0.1231474     | 0.1718750     | 0.2439180     | 0.3260905     | 0.3177271     | 0.4230146     |
| 24  | 0.1205651     | 0.1689815     | 0.2341466     | 0.3156629     | 0.2815115     | 0.3834000     |
| 32  | 0.1196545     | 0.1679687     | 0.2306619     | 0.3119780     | 0.2682336     | 0.3691650     |
| 40  | 0.1192317     | 0.1675000     | 0.2290354     | 0.3102652     | 0.2619443     | 0.3624936     |
| 48  | 0.1190016     | 0.1672454     | 0.2281480     | 0.3093327     | 0.2584828     | 0.3588444     |
| 56  | 0.1188628     | 0.1670918     | 0.2276114     | 0.3087697     | 0.2563784     | 0.3566345     |
| 64  | 0.1187726     | 0.1669922     | 0.2272625     | 0.3084039     | 0.2550051     | 0.3551961     |
| 80  | 0.1186664     | 0.1668750     | 0.2268516     | 0.3079734     | 0.2533820     | 0.3535002     |
| 96  | 0.1186088     | 0.1668113     | 0.2266280     | 0.3077394     | 0.2524963     | 0.3525768     |
| 112 | 0.1185740     | 0.1667730     | 0.2264931     | 0.3075982     | 0.2519608     | 0.3520192     |
| 128 | 0.1185514     | 0.1667480     | 0.2264055     | 0.3075066     | 0.2516126     | 0.3516570     |
| 160 | 0.1185248     | 0.1667187     | 0.2263024     | 0.3073988     | 0.2512025     | 0.3512307     |
| 192 | 0.1185104     | 0.1667028     | 0.2262464     | 0.3073402     | 0.2509794     | 0.3509989     |
| 224 | 0.1185017     | 0.1666932     | 0.2262126     | 0.3073049     | 0.2508448     | 0.3508592     |
| 256 | 0.1184960     | 0.1666870     | 0.2261907     | 0.3072820     | 0.2507573     | 0.3507684     |
| 384 | 0.1184858     | 0.1666757     | 0.2261509     | 0.3072403     | 0.2505986     | 0.3506036     |
| 512 | 0.1184822     | 0.1666718     | 0.2261370     | 0.3072258     | 0.2505429     | 0.3505459     |
| $\infty$ | 0.118478     | 0.166667    | 0.2261199     | 0.307207     | 0.250471     | 0.350472     |

Table 8: Exact results for $A_{1,2}^{(0)}$ and $A_{2,2}^{(0)}$ as functions of the size of the fundamental lattice ($L$) and the size of the blocked lattice ($l$). The last row contains values extrapolated to $L = \infty$. 
Table 9: Monte Carlo results for the $A_i$ obtained at the critical coupling of the BCSOS model. The block-observables $A_i$ are defined in the text. $L$ is the original lattice size, and $l$ is the size of the blocked system.

| $L$ | $l$ | $A_1$     | $A_2$     | $A_3$     | $A_4$     |
|-----|-----|-----------|-----------|-----------|-----------|
| 16  | 1   | 0.26511(30) | 0.06737(26)|           |           |
| 24  | 1   | 0.24004(31) | 0.05437(25)|           |           |
| 32  | 1   | 0.22572(32) | 0.04772(25)|           |           |
| 40  | 1   | 0.21526(31) | 0.04341(24)|           |           |
| 48  | 1   | 0.20778(30) | 0.03959(24)|           |           |
| 56  | 1   | 0.20157(31) | 0.03704(24)|           |           |
| 64  | 1   | 0.19656(33) | 0.03571(24)|           |           |
| 80  | 1   | 0.18884(33) | 0.03252(25)|           |           |
| 96  | 1   | 0.18209(34) | 0.03032(25)|           |           |
| 112 | 1   | 0.17797(35) | 0.02880(25)|           |           |
| 128 | 1   | 0.17399(33) | 0.02750(23)|           |           |
| 160 | 1   | 0.16704(37) | 0.02529(26)|           |           |
| 192 | 1   | 0.16341(40) | 0.02419(26)|           |           |
| 224 | 1   | 0.15935(41) | 0.02265(26)|           |           |
| 256 | 1   | 0.15616(42) | 0.02188(26)|           |           |

| $L$ | $l$ | $A_1$     | $A_2$     | $A_3$     | $A_4$     |
|-----|-----|-----------|-----------|-----------|-----------|
| 16  | 2   | 0.085740(27)| 0.117435(45)| 0.22382(18)| 0.05597(13)|
| 24  | 2   | 0.083135(26)| 0.115240(43)| 0.20229(18)| 0.04436(12)|
| 32  | 2   | 0.082161(25)| 0.114434(42)| 0.18981(19)| 0.03851(12)|
| 40  | 2   | 0.081661(24)| 0.113976(41)| 0.18079(19)| 0.03444(12)|
| 48  | 2   | 0.081351(24)| 0.113660(41)| 0.17409(18)| 0.03174(12)|
| 56  | 2   | 0.081129(24)| 0.113497(41)| 0.16888(19)| 0.02975(12)|
| 64  | 2   | 0.080932(24)| 0.113249(41)| 0.16449(20)| 0.02812(12)|
| 80  | 2   | 0.080719(24)| 0.113064(41)| 0.15767(20)| 0.02567(12)|
| 96  | 2   | 0.080541(24)| 0.112855(41)| 0.15218(21)| 0.02401(12)|
| 112 | 2   | 0.080422(24)| 0.112743(41)| 0.14834(21)| 0.02275(12)|
| 128 | 2   | 0.080306(22)| 0.112617(38)| 0.14485(20)| 0.02145(11)|
| 160 | 2   | 0.080165(25)| 0.112424(42)| 0.13916(23)| 0.01983(12)|
| 192 | 2   | 0.080055(25)| 0.112278(43)| 0.13567(24)| 0.01848(12)|
| 224 | 2   | 0.080008(26)| 0.112258(44)| 0.13225(25)| 0.01763(12)|
| 256 | 2   | 0.079953(26)| 0.112235(45)| 0.12936(26)| 0.01699(13)|
| L  | l  | $A_1$          | $A_2$          | $A_3$          | $A_4$          |
|----|----|----------------|----------------|----------------|----------------|
| 16 | 4  | 0.176651(27)   | 0.230216(40)   | 0.221936(80)   | 0.077948(65)   |
| 24 | 4  | 0.165308(24)   | 0.219807(37)   | 0.194147(82)   | 0.052961(63)   |
| 32 | 4  | 0.161166(23)   | 0.215927(35)   | 0.179384(82)   | 0.043583(61)   |
| 40 | 4  | 0.159116(22)   | 0.213952(35)   | 0.169635(82)   | 0.038175(60)   |
| 48 | 4  | 0.157915(22)   | 0.212736(34)   | 0.162291(81)   | 0.034469(59)   |
| 56 | 4  | 0.157067(22)   | 0.211902(34)   | 0.156869(82)   | 0.031988(59)   |
| 64 | 4  | 0.156463(21)   | 0.211246(34)   | 0.152173(85)   | 0.029793(59)   |
| 80 | 4  | 0.155714(21)   | 0.210486(34)   | 0.145120(86)   | 0.026922(58)   |
| 96 | 4  | 0.155185(22)   | 0.209912(34)   | 0.139701(89)   | 0.024779(59)   |
| 112 | 4 | 0.154811(21)   | 0.209486(34)   | 0.135546(90)   | 0.023253(58)   |
| 128 | 4 | 0.154514(20)   | 0.209164(31)   | 0.131951(85)   | 0.021861(53)   |
| 160 | 4 | 0.154099(22)   | 0.208674(35)   | 0.126456(96)   | 0.019967(59)   |
| 192 | 4 | 0.153818(22)   | 0.208360(35)   | 0.12257(10)    | 0.018579(58)   |
| 224 | 4 | 0.153583(22)   | 0.208080(35)   | 0.11912(10)    | 0.017534(58)   |
| 256 | 4 | 0.153391(23)   | 0.207869(36)   | 0.11647(11)    | 0.016720(59)   |
| 16  | 8  | 0.265665(22)   | 0.325828(30)   | 0.345213(51)   | 1.000000       |
| 24  | 8  | 0.216030(16)   | 0.280733(24)   | 0.227866(37)   | 0.105810(34)   |
| 32  | 8  | 0.196646(14)   | 0.264669(22)   | 0.205656(38)   | 0.072017(32)   |
| 40  | 8  | 0.188911(14)   | 0.257187(21)   | 0.189140(37)   | 0.055770(31)   |
| 48  | 8  | 0.184346(13)   | 0.252912(20)   | 0.179067(36)   | 0.048696(30)   |
| 56  | 8  | 0.181606(13)   | 0.250139(20)   | 0.171361(37)   | 0.043493(30)   |
| 64  | 8  | 0.179710(13)   | 0.248261(20)   | 0.165313(38)   | 0.039956(30)   |
| 80  | 8  | 0.177383(12)   | 0.245856(20)   | 0.156139(38)   | 0.034926(29)   |
| 96  | 8  | 0.175951(12)   | 0.244337(20)   | 0.149442(40)   | 0.031601(30)   |
| 112 | 8  | 0.174995(12)   | 0.243293(20)   | 0.144178(39)   | 0.029200(29)   |
| 128 | 8  | 0.174309(11)   | 0.242532(18)   | 0.139959(37)   | 0.027326(27)   |
| 160 | 8  | 0.173355(12)   | 0.241458(20)   | 0.133345(41)   | 0.024585(29)   |
| 192 | 8  | 0.172733(12)   | 0.240736(20)   | 0.128466(43)   | 0.022659(29)   |
| 224 | 8  | 0.172270(12)   | 0.240186(20)   | 0.124527(44)   | 0.021187(29)   |
| 256 | 8  | 0.171898(13)   | 0.239742(20)   | 0.121338(46)   | 0.020028(29)   |

Table 10: Continuation of table 9.
| $L$ | $l$ | $A_1$     | $A_2$     | $A_3$     | $A_4$     |
|-----|-----|-----------|-----------|-----------|-----------|
| 16  | 1   |           |           | 4.9725(59)| 2.2881(60)|
| 24  | 1   |           |           | 5.5222(88)| 2.2951(89)|
| 32  | 1   |           |           | 5.888(12) | 2.285(12) |
| 40  | 1   |           |           | 6.216(14) | 2.304(15) |
| 48  | 1   |           |           | 6.463(17) | 2.310(17) |
| 56  | 1   |           |           | 6.659(20) | 2.326(20) |
| 64  | 1   |           |           | 6.839(23) | 2.289(23) |
| 80  | 1   |           |           | 7.131(29) | 2.308(28) |
| 96  | 1   |           |           | 7.340(34) | 2.319(34) |
| 112 | 1   |           |           | 7.526(40) | 2.260(40) |
| 128 | 1   |           |           | 7.714(42) | 2.308(42) |
| 160 | 1   |           |           | 8.079(59) | 2.286(57) |
| 192 | 1   |           |           | 8.202(71) | 2.358(69) |
| 224 | 1   |           |           | 8.478(81) | 2.432(79) |
| 256 | 1   |           |           | 8.719(95) | 2.348(91) |
| 16  | 2   | -0.54495(72)| -0.7764(12)| 3.9804(37)| 1.6759(32)|
| 24  | 2   | -0.56652(96)| -0.8146(16)| 4.4235(56)| 1.6717(47)|
| 32  | 2   | -0.5876(12) | -0.8476(20)| 4.7222(73)| 1.6559(59)|
| 40  | 2   | -0.6055(15) | -0.8725(25)| 4.9708(89)| 1.6655(74)|
| 48  | 2   | -0.6208(17) | -0.8961(29)| 5.172(11) | 1.6574(88)|
| 56  | 2   | -0.6331(20) | -0.9126(34)| 5.345(13) | 1.659(10) |
| 64  | 2   | -0.6469(23) | -0.9316(38)| 5.484(14) | 1.642(12) |
| 80  | 2   | -0.6644(28) | -0.9587(47)| 5.744(18) | 1.651(14) |
| 96  | 2   | -0.6806(33) | -0.9807(56)| 5.907(21) | 1.620(17) |
| 112 | 2   | -0.6822(39) | -0.9882(67)| 6.056(25) | 1.659(20) |
| 128 | 2   | -0.7079(40) | -1.0227(68)| 6.189(26) | 1.639(21) |
| 160 | 2   | -0.7199(55) | -1.0320(94)| 6.474(36) | 1.641(29) |
| 192 | 2   | -0.7412(64) | -1.072(11) | 6.571(44) | 1.642(35) |
| 224 | 2   | -0.7613(75) | -1.103(13) | 6.734(50) | 1.655(40) |
| 256 | 2   | -0.7748(88) | -1.121(15) | 6.939(58) | 1.592(46) |

Table 11: Monte Carlo results for the derivatives of the $A_i$ with respect to the coupling, taken for the BCSOS model at the critical coupling. The block-observables $A_i$ are defined in the text. $L$ is the original lattice size, and $l$ is the size of the blocked system.
| $L$ | $l$ | $A_1$         | $A_2$         | $A_3$         | $A_4$         |
|-----|-----|---------------|---------------|---------------|---------------|
| 16  | 4   | -1.02443(81)  | -1.3944(12)   | 2.9483(20)    | 1.5854(17)    |
| 24  | 4   | -1.0343(10)   | -1.4308(15)   | 3.2837(29)    | 1.4291(24)    |
| 32  | 4   | -1.0626(12)   | -1.4790(18)   | 3.5337(36)    | 1.3832(30)    |
| 40  | 4   | -1.0909(14)   | -1.5208(22)   | 3.7292(44)    | 1.3796(37)    |
| 48  | 4   | -1.1165(17)   | -1.5573(26)   | 3.8885(51)    | 1.3631(44)    |
| 56  | 4   | -1.1395(19)   | -1.5901(30)   | 4.0296(61)    | 1.3619(51)    |
| 64  | 4   | -1.1596(21)   | -1.6176(33)   | 4.1442(67)    | 1.3456(58)    |
| 80  | 4   | -1.1943(26)   | -1.6676(41)   | 4.3474(85)    | 1.3524(72)    |
| 96  | 4   | -1.2182(31)   | -1.7022(48)   | 4.491(10)     | 1.3459(87)    |
| 112 | 4   | -1.2410(36)   | -1.7339(56)   | 4.627(12)     | 1.3395(99)    |
| 128 | 4   | -1.2668(37)   | -1.7680(58)   | 4.744(12)     | 1.329(10)     |
| 160 | 4   | -1.2935(50)   | -1.7997(78)   | 4.942(17)     | 1.361(14)     |
| 192 | 4   | -1.3372(59)   | -1.8660(92)   | 5.055(20)     | 1.372(17)     |
| 224 | 4   | -1.3603(68)   | -1.897(11)    | 5.186(23)     | 1.352(20)     |
| 256 | 4   | -1.3771(79)   | -1.921(12)    | 5.351(27)     | 1.330(23)     |
| 16  | 8   | -1.27254(71)  | -1.65259(97)  | 2.6102(15)    | 0.00000       |
| 24  | 8   | -1.09526(73)  | -1.5480(11)   | 2.4655(15)    | 1.4653(13)    |
| 32  | 8   | -1.09968(81)  | -1.5481(12)   | 2.7517(19)    | 1.4710(16)    |
| 40  | 8   | -1.10306(92)  | -1.5727(14)   | 2.9006(22)    | 1.3380(19)    |
| 48  | 8   | -1.1188(10)   | -1.6001(16)   | 3.0496(26)    | 1.3139(22)    |
| 56  | 8   | -1.1344(12)   | -1.6265(18)   | 3.1721(30)    | 1.2878(26)    |
| 64  | 8   | -1.1514(13)   | -1.6537(20)   | 3.2715(34)    | 1.2758(30)    |
| 80  | 8   | -1.1828(16)   | -1.7011(25)   | 3.4528(41)    | 1.2607(36)    |
| 96  | 8   | -1.2067(18)   | -1.7356(29)   | 3.5904(50)    | 1.2524(43)    |
| 112 | 8   | -1.2338(21)   | -1.7768(33)   | 3.7300(58)    | 1.2482(50)    |
| 128 | 8   | -1.2553(21)   | -1.8070(34)   | 3.8241(59)    | 1.2377(52)    |
| 160 | 8   | -1.2906(29)   | -1.8574(46)   | 4.0041(81)    | 1.2439(71)    |
| 192 | 8   | -1.3261(34)   | -1.9101(54)   | 4.1256(99)    | 1.2387(85)    |
| 224 | 8   | -1.3503(39)   | -1.9428(62)   | 4.252(11)     | 1.240(10)     |
| 256 | 8   | -1.3741(45)   | -1.9792(72)   | 4.382(13)     | 1.230(11)     |

Table 12: Continuation of table 11.
Table 13: XY results for the matching factor obtained in the way described after eq. (22).

| \(L\) | \(l\) | \(A_1,A_3\)      | \(A_2,A_3\)      | \(D_1,A_3\)      | \(D_2,A_3\)      |
|-------|-------|------------------|------------------|------------------|------------------|
| 32    | 2     | 0.9176(46)       | 0.9381(65)       | 0.9007(55)       | 0.9273(74)       |
| 48    | 2     | 0.9239(62)       | 0.9307(78)       | 0.9148(69)       | 0.9235(86)       |
| 64    | 2     | 0.9206(80)       | 0.9275(92)       | 0.9148(86)       | 0.9231(97)       |
| 96    | 2     | 0.9362(97)       | 0.9386(117)      | 0.9335(101)      | 0.9365(121)      |
| 128   | 2     | 0.9376(107)      | 0.9356(118)      | 0.9362(109)      | 0.9345(120)      |
| 192   | 2     | 0.9275(136)      | 0.9331(170)      | 0.9266(138)      | 0.9324(172)      |
| 32    | 4     | 0.8896(14)       | 0.9035(17)       | 0.8601(18)       | 0.8820(20)       |
| 48    | 4     | 0.9080(20)       | 0.9162(23)       | 0.8923(24)       | 0.9045(26)       |
| 64    | 4     | 0.9172(27)       | 0.9231(31)       | 0.9082(29)       | 0.9165(33)       |
| 96    | 4     | 0.9263(36)       | 0.9300(38)       | 0.9217(38)       | 0.9266(40)       |
| 128   | 4     | 0.9364(40)       | 0.9363(43)       | 0.9340(41)       | 0.9345(45)       |
| 192   | 4     | 0.9287(50)       | 0.9271(52)       | 0.9273(51)       | 0.9260(52)       |
| 32    | 8     | 0.8421(3)        | 0.8622(4)        | 0.7731(6)        | 0.8149(6)        |
| 48    | 8     | 0.8760(6)        | 0.8903(7)        | 0.8329(8)        | 0.8610(9)        |
| 64    | 8     | 0.8890(8)        | 0.9009(10)       | 0.8629(10)       | 0.8825(11)       |
| 96    | 8     | 0.9118(11)       | 0.9199(13)       | 0.8988(13)       | 0.9109(14)       |
| 128   | 8     | 0.9185(14)       | 0.9245(16)       | 0.9109(16)       | 0.9191(17)       |
| 192   | 8     | 0.9267(19)       | 0.9297(21)       | 0.9227(20)       | 0.9269(22)       |
| $L$ | $l$ | $A_1,A_3$ | $A_2,A_3$ | $D_1,A_3$ | $D_2,A_3$ |
|-----|-----|--------|--------|--------|--------|
| 32  | 2   | 1.119982(94) | 1.119545(117) | 1.120352(105) | 1.119773(133) |
| 48  | 2   | 1.119933(90) | 1.119822(120) | 1.120084(98) | 1.119940(130) |
| 64  | 2   | 1.119757(95) | 1.119661(114) | 1.119839(100) | 1.119722(119) |
| 96  | 2   | 1.119707(86) | 1.119682(101) | 1.119735(89) | 1.119703(104) |
| 128 | 2   | 1.119831(92) | 1.119852(102) | 1.119845(94) | 1.119863(104) |
| 192 | 2   | 1.119789(92) | 1.119742(114) | 1.119797(93) | 1.119748(115) |
| 32  | 4   | 1.120176(48) | 1.119726(53) | 1.121162(52) | 1.120428(59) |
| 48  | 4   | 1.120034(46) | 1.119846(51) | 1.120403(50) | 1.120116(55) |
| 64  | 4   | 1.119868(45) | 1.119754(52) | 1.120044(47) | 1.119883(55) |
| 96  | 4   | 1.119791(41) | 1.119735(49) | 1.119860(42) | 1.119785(51) |
| 128 | 4   | 1.119798(41) | 1.119800(48) | 1.119829(42) | 1.119823(48) |
| 192 | 4   | 1.119866(42) | 1.119884(47) | 1.119881(43) | 1.119896(48) |
| 32  | 8   | 1.120078(32) | 1.118559(33) | 1.125145(34) | 1.122223(39) |
| 48  | 8   | 1.120069(29) | 1.119442(29) | 1.121941(28) | 1.120741(33) |
| 64  | 8   | 1.120095(26) | 1.119701(28) | 1.120979(28) | 1.120310(30) |
| 96  | 8   | 1.119951(26) | 1.119754(26) | 1.120271(26) | 1.119973(27) |
| 128 | 8   | 1.119906(24) | 1.119789(26) | 1.120057(25) | 1.119894(27) |
| 192 | 8   | 1.119888(22) | 1.119841(24) | 1.119949(23) | 1.119885(24) |

Table 14: XY results for the roughening coupling obtained in the way described after eq. (22).
| $L$ | $l$ | $D_1$    | $D_2$   | $A_1$    | $A_2$  |
|-----|-----|----------|---------|----------|--------|
| 16  | 1   | -0.4167(14) | -0.4137(17) |
| 24  | 1   | -0.4206(14) | -0.4195(25) |
| 32  | 1   | -0.4249(17) | -0.4227(33) |
| 48  | 1   | -0.4252(19) | -0.4267(48) |
| 64  | 1   | -0.4255(24) | -0.4270(67) |
| 96  | 1   | -0.4257(32) | -0.4410(99) |
| 128 | 1   | -0.4267(41) | -0.439(13)  |
| 192 | 1   | -0.4280(59) | -0.413(19)  |
| 16  | 2   | -0.4271(11) | -0.4302(11) | -0.4154(13) | -0.4046(12) |
| 24  | 2   | -0.4299(13) | -0.4314(13) | -0.4194(13) | -0.4112(17) |
| 32  | 2   | -0.4298(15) | -0.4304(14) | -0.4236(14) | -0.4167(23) |
| 48  | 2   | -0.4288(19) | -0.4285(19) | -0.4244(17) | -0.4195(34) |
| 64  | 2   | -0.4281(23) | -0.4285(24) | -0.4238(20) | -0.4171(46) |
| 96  | 2   | -0.4355(33) | -0.4365(33) | -0.4263(25) | -0.4326(70) |
| 128 | 2   | -0.4250(41) | -0.4252(42) | -0.4281(32) | -0.4140(92) |
| 192 | 2   | -0.4282(60) | -0.4277(62) | -0.4276(46) | -0.412(14)  |
| 16  | 4   | -0.4196(73) | -0.4250(77) | -0.4070(13) | -0.3657(12) |
| 24  | 4   | -0.4259(89) | -0.4283(90) | -0.4140(13) | -0.3962(11) |
| 32  | 4   | -0.4296(98) | -0.4303(95) | -0.4182(13) | -0.4061(14) |
| 48  | 4   | -0.4290(12) | -0.4291(11) | -0.4218(13) | -0.4148(21) |
| 64  | 4   | -0.4294(13) | -0.4296(13) | -0.4213(15) | -0.4187(28) |
| 96  | 4   | -0.4324(17) | -0.4323(17) | -0.4250(17) | -0.4236(43) |
| 128 | 4   | -0.4296(21) | -0.4290(20) | -0.4264(21) | -0.4177(57) |
| 192 | 4   | -0.4295(30) | -0.4300(29) | -0.4281(29) | -0.4050(85) |
| 24  | 8   | -0.4239(92) | -0.4230(69) | -0.4120(18) | -0.3830(53) |
| 32  | 8   | -0.4237(70) | -0.4279(81) | -0.4087(11) | -0.3694(18) |
| 48  | 8   | -0.4287(87) | -0.4301(86) | -0.4147(13) | -0.3952(12) |
| 64  | 8   | -0.4296(99) | -0.4303(95) | -0.4177(13) | -0.4035(15) |
| 96  | 8   | -0.4320(12) | -0.4328(12) | -0.4204(15) | -0.4159(23) |
| 128 | 8   | -0.4313(14) | -0.4311(13) | -0.4228(15) | -0.4192(31) |
| 192 | 8   | -0.4311(18) | -0.4307(17) | -0.4267(19) | -0.4175(48) |

Table 15: Monte Carlo estimates for the ratio of slopes for the XY and BCSOS model block observables, taken at the critical couplings, c.f. the definition in eq. (27).