Improved Adaptive Algorithm for Scalable Active Learning with Weak Labeler

Yifang Chen∗
Paul G. Allen School of Computer Science & Engineering
University of Washington

Karthik Sankararaman, Alessandro Lazaric, Matteo Pirotta, Dmytro Karamshuk,
Qifan Wang, Karishma Mandyam, Sinong Wang, Han Fang
Meta

Abstract
Active learning with strong and weak labelers considers a practical setting where we have access to both costly but accurate strong labelers and inaccurate but cheap predictions provided by weak labelers. We study this problem in the streaming setting, where decisions must be taken online. We design a novel algorithmic template, Weak Labeler Active Cover (WL-AC), that is able to robustly leverage the lower quality weak labelers to reduce the query complexity while retaining the desired level of accuracy. Prior active learning algorithms with access to weak labelers learn a difference classifier which predicts where the weak labels differ from strong labelers; this requires the strong assumption of realizability of the difference classifier (Zhang and Chaudhuri, 2015). WL-AC bypasses this realizability assumption and thus is applicable to many real-world scenarios such as random corrupted weak labels and high dimensional family of difference classifiers (e.g., deep neural nets). Moreover, WL-AC cleverly trades off evaluating the quality with full exploitation of weak labelers, which allows to convert any active learning strategy to one that can leverage weak labelers. We provide an instantiation of this template that achieves the optimal query complexity for any given weak labeler, without knowing its accuracy a-priori. Empirically, we propose an instantiation of the WL-AC template that can be efficiently implemented for large-scale models (e.g., deep neural nets) and show its effectiveness on the corrupted-MNIST dataset by significantly reducing the number of labels while keeping the same accuracy as in passive learning.

1 Introduction
An active learning algorithm for classification aims to obtain an ǫ-optimal hypothesis (classifier) from some given hypothesis set while requesting as few labels as possible. Under some favorable conditions, active learning algorithms can require exponentially fewer labels than passive, random sampling (Hanneke et al., 2014). Here we consider the streaming setting where the unlabeled i.i.d. data comes in sequence and an active learning algorithm must make the real-time decision on whether to request the corresponding label or not.

While traditional active learning assumes access to costly but accurate labeler, recent works (e.g., Malago et al., 2014; Urner et al., 2012; Mozannar and Sontag, 2020; Huang et al., 2017) consider multiple labelers whose cost and label quality varies from each other. This scenario is commonly encountered in practical human-in-the-loop systems. In crowdsourcing, each sample may receive labels from more than one labeler but the reliability of each labeler is unknown – there might even exist malicious labelers (Kovashka et al., 2016) – and the cost of each annotation is related to their quality. In many modern content-review applications (e.g., Garcelon et al., 2022 and references therein) it is possible to automatically assign a label (possibly incorrect) by leveraging pre-trained models (e.g., trained out-of-distribution or on a different task) instead of asking to a human reviewer to provide a costly but accurate assessment of the piece (Lee et al., 2021). To recover an ǫ-optimal hypothesis with the minimum query cost, it is thus important to carefully trade-off the usage of a cheap but potentially imprecise labeler (weak labeler W) and a costly
but accurate labeler (strong labeler $O$). We will discuss more related settings in Appendix 2.

Authors in (Zhang and Chaudhuri, 2015) first gave a provable algorithm on this problem. Their approach, however, heavily relies on the existence of a difference-classifier set $H_{\text{diff}} \in H_{\text{all}}$ to predict where the weak labeler differs from the strong labeler. Therefore, it saves query cost by only querying the strong labeler when it disagrees with the weak one. They assume (a) $H_{\text{diff}}$ is known to the learner in advance and (b) a nontrivial realizable low false negative rate difference-classifier in the set $H_{\text{diff}}$ exists (e.g., a classifier that always outputs positive labels is trivial). This difference-classifier identification procedure yields an extra $\text{VCdim}(H_{\text{diff}})$-dependent term in the query complexity (i.e., calls to the strong labeler) that becomes dominating when $\text{VCdim}(H_{\text{diff}}) \gtrapprox \frac{1}{\varepsilon}$, where $\varepsilon$ is the target accuracy of output model.

In adapting this method to real-world systems, we encounter two main limitations. First, it is not always guaranteed that a low complexity realizable $H_{\text{diff}}$ exists. Second, their deterministic algorithm is vulnerable to random corrupted labelers (e.g., Patrini et al., 2017; Miller et al., 2014; Awasthi et al., 2017). In this work, we propose an algorithm called WeakLabeler-ActiveCover (WL-AC) which removes both these assumptions by incorporating doubly robust loss estimator inspired by the regret minimization algorithm with hints in (Wei et al., 2020), and develop a corresponding semi-randomized algorithm. As a trade-off, the theoretical analysis for our algorithm requires that the conditional accuracy of weak labels has relative low variance compared to bias within each region, which is milder than the knowledge of exact $H_{\text{diff}}$ and can be usually relaxed in practice. To the best of our knowledge, this is the first time this technique has been used in active learning.

Besides the realizable difference-classifier assumption, the framework proposed by (Zhang and Chaudhuri, 2015) also lacks flexibility to even more parsimonious active query strategy. To be specific, the traditional disagreement based active learning, including the one (Zhang and Chaudhuri, 2015) is based on, will query every point inside the disagreement region. This deterministic approach, although being worst-case optimal, has been shown in (Huang et al., 2015) that can be improved in some benign cases by assigning more refined query probabilities for samples inside the disagreement region. Therefore, to preserve such advantage in WL-AC, we need a corresponding weak label-leverage strategy. Here we adapt the idea of Active Cover (Huang et al., 2015) to our setting by designing a more adaptive weak label evaluation phase that keeps an automatic trades-off that preserve the benefits of the AC algorithm.

Theoretically, (1) our algorithm WL-AC retains the consistent excess risk guarantees of the AC algorithm up to a constant term, which also implies the consistent shrinking speed of the sample disagreement region as shown in Section 4.1. To achieve this, we exploit a double robust estimator and design a refined localized weak label evaluation strategy. (2) Even confronting a malicious/totally misleading weak labeler, WL-AC can guarantees the same order query complexity as the aggressive result from original AC algorithm as shown in Section 4.2. To achieve this, we develop a weak label evaluation strategy that automatically balances exploration to estimate the quality of a weak labeler and exploitation of the weak labeler to reduce the overall query complexity. (3) Our result formally characterizes when we can save query complexity w.r.t. the strong labelers in the non-realizable setting (Section 4.3).

While many works with or without theory guarantees has been wildly applied in deep learning (e.g., Gal et al., 2017; Yoo and Kweon, 2019; Sener and Savarese, 2017; Zhidanov, 2019), disagreement-based active learning is known to be impractical in large-scale model like neural net (Settles, 2009). Here we show that the proposed algorithmic framework not only has theoretical guarantees when using AC as the base query strategy but it is also highly scalable in real-world settings. We design a practical version that takes any heuristic streaming-based sample query methods and leverage it with weak labels and demonstrate its effect on a standard computer vision dataset (i.e., corrupted MNIST (Mu and Gilmer, 2019)). Specifically, (1) for the weak label generation methods, we first test on the synthetic data to demonstrate the robustness of our algorithm against different levels and distributions of noise. Then we use the label generated from some classifier trained on non-target domain to further show its practical usage; (2) For the heuristic active learning our algorithm is build-upon, we test the uniform sampling as baseline and a preliminary entropy-based uncertainty sampling as a classical active query strategy.

## 2 Preliminaries

A hypothesis class $H$ is given to the learner such that for each $h \in H$ we have $h : \mathcal{X} \rightarrow \mathcal{Y}$. Samples $(x, y) \in \mathcal{X} \times \mathcal{Y}$ are drawn independently from an underlying distribution $\mathcal{D}_*$. In deriving our theoretical contributions, we assume binary classification for convenience, i.e., $\mathcal{Y} = \{0, 1\}$, but our algorithmic framework can be easily extend to the multi-class setting as shown in the experiments. We denote the expected risk of a classifier $h \in H$ under the distribution $\mathcal{D}_*$ as $\text{err}_{\mathcal{D}_*}(h) = \mathbb{E}_{x, y \sim \mathcal{D}_*} [\frac{1}{2} h(x) \neq y]$ and the corresponding best hypothesis as $h^* = \arg\min_h \{ \text{err}_{\mathcal{D}_*}(h) \}$.

**Weak labeling oracle** Classical active learning assumes access to a strong labeling oracle $O$ that always outputs the true label with a nontrivial cost. In this paper, we additionally assume the access to a weak labeling oracle $W$, from which we can query labels, denoted as $y^{\text{WL}}$ with a neglig-
The main goal of the learner is to identify a near-optimal classifier while using as lower query cost as possible under a hypothesis class $H$. That is, we want to minimize both the query variance and weak label error.

Suppose there exists an hint oracle that, at each time $t$, gives an hint for which there is disagreement across $H$, defined as $\kappa(D) = \max_{x,x' \in D} \mathbb{P}_{y,y_{WL}}[y_{WL} \neq y|x]$. It is easy to see that $\kappa(D)$ is decreasing when $D$ shrinks.

Correspondingly, we define

$$\text{WLerr}(D) = \max_{x \in D} \mathbb{P}_{y,y_{WL}}[y_{WL} \neq y|x].$$

Assumption 2.1 implies that we want the conditional errors of WL to have low variance locally within some informative region $D$. Moreover, from practical perspective, we show in our experiment that even some underestimation of this $\kappa(D)$ can still yield improved results compared to the algorithm without WL.

### Disagreement coefficient

For some hypothesis class $H$ and subset $V \subset H$, the region of disagreement is defined as $\text{Dis}(V) = \{x \in X : \exists h, h' \in V \text{ s.t. } h(x) \neq h'(x)\}$, which is the set of unlabeled examples $x$ for which there are hypotheses in $V$ that disagree on how to label $x$. Correspondingly, the disagreement coefficient of $h^* \in H$ with respect to a hypothesis class $H$ and distribution $\nu^*$ is defined as

$$\theta(r_0) = \sup_{r \geq r_0} \mathbb{P}_{x \sim \nu^*, r} \{X \in \text{Dis}(B(h^*, r))\},$$

and $B(h^*, r) = \{h \in H | \mathbb{P}_x(h(x) \neq h^*(x)) \leq r\}$.

### Protocol and goal of the problem

We consider the streaming-based active learning problem. At each time $t \in \{1, \ldots, n\}$, nature draws $(x_t, y_t)$ from $D_*$. The learner observes just $x_t$ and chooses whether to request $y_t$ from a strong learner incurring a cost of $c_t = 1$, or to request $y_{t, WL}$ from a weak learner incurring a cost of $c_t = 0$.

The main goal of the learner is to identify a near-optimal classifier $h_{out}$ of desired accuracy with high probability while using as lower query cost as possible under $n$ total unlabeled samples. That is, we want to minimize both

$$\text{err}(h_{out}) - \text{err}(h^*) \quad \text{and} \quad \sum_{t=1}^n 1[y_t \text{ is queried}] \quad (1)$$

In the rest of the paper, we simply use “label query complexity” to refer queries to strong labeler.
While the theoretical instantiation is still implementable for simple models, this algorithm is impractical for modern large-scale applications such as deep neural nets. In this case, we can leverage the flexibility of our framework to overcome this limitation. For example, WL-AC can leverage any active learning oracle \( \mathcal{O}_{\text{baseAL}} \) to approximately compute both the sampling distribution and the disagreement region (see Section 3.1 for more details).

We now explain the key ideas of WL-AC in more details.

### Shifted doubly robust loss estimator in Phase 3 and the corresponding query strategy in Phase 1

To be robust against random corruptions while leveraging the “good” weak labels, inspired by [Wei et al. (2020)], we introduce the loss estimator

\[
\ell_{\text{shifted}}(h(x), y, y_{WL}, w) = \left( 1 \mathbb{I}[h(x) \neq y] - \mathbb{I}[h(x) \neq y_{WL}] \right) w + \mathbb{I}[h(x) \neq y_{WL}]
\]

where \( w \) is a positive weight. It is easy to see that, this estimator is unbiased and has low variance when the weak label is correct since \( \mathbb{I}[y_{WL} \neq y] \leq \mathbb{I}[y_{WL} \neq y] w \). By adopting such estimator, we can upper bound the performance difference between \((h, h')\) in the active set \( A_m \) as

\[
\mathbb{E} \left[ \frac{1 \mathbb{I}[h(x) \neq h'(x) \land x \in D_m \land y_{WL} \neq y]}{P_m(x)} \right],
\]

This suggests that, to achieve the same accuracy as the original AL strategy without access to weak labels, \( P_m(x) \) can be reduced to at most \( \mathbb{E}[y_{WL} \neq y|x] P_m(x) \), where \( P_m \) is the query probability function.

Nevertheless, without further modification, the query complexity would scale with \( \mathbb{E}[\ell_{\text{shifted}}(h, \tilde{Z}_m)] \), which is the biased estimate of excess risk of \( h^* \) (as it happens in AC). This is undesired in our setting where the variance of \( \mathbb{E}[\ell_{\text{shifted}}(h^*)] \) may be larger than in the standard AL setting without WL. Indeed, in the case that most weak labels are wrong and the excess risk of \( h^* \) is low, we have

\[
\mathbb{E}\left[ 1 \mathbb{I}[h^*(x) \neq y] - 1 \mathbb{I}[h^*(x) \neq y_{WL}] \right] w \leq 1 \mathbb{I}[y_{WL} \neq y|x] w \Rightarrow 1 \mathbb{I}[h^*(x) \neq y]|w
\]

### Adaptive/localized evaluation of the weak label performance in Phase 1

To address this problem, we evaluate the weak labelers at the beginning of each block. This deviates from the approach in [Wei et al. (2020)] where a one-time pure exploration phase is used to evaluate the quality of the hint providers before starting to exploit them. This addresses two issues: i) we are adaptive to the changes in the disagreement region \( D_m \); ii) we set a transition to NOWL rule by adapting to the generalized error of best hypothesis \( \mathbb{E}[\ell_{\text{shifted}}(h^*)] \), which is estimated via \( \ell(h_m, \tilde{Z}_{m-1}) \) (\( \mathbb{E}[\ell_{\text{shifted}}(h^*)] \) will be formally defined in the next section). Similarly to [Zhang and Chaudhuri (2015)], being adaptive to \( D_m \) means that, instead of caring about the overall quality of the weak labeler, we only care about the quality for
those samples within $D_m$ (or in the other words, close to the decision boundary).

**Trade-off between weak label evaluation in Phase 1 and training data collection in Phase 2.** The previous two techniques guarantee that the excess risk of WL-AC is consistent with the one of AC when the weak labeler quality is known. In practice, this quality is unknown and a pessimistic estimator of $\mathbb{E}[\mathbb{1}[y^{WL} \neq y \land x \in D_m]]$ is used. However, building this estimator is not trivial. Consider the extreme case where $\mathbb{E}[y^{WL} \neq y] \to 0$, then in order to get a pessimistic estimation close to 0 to fully adopt the advantage of the perfect weak labels, an infinite number of samples in phase 1 is needed, which diminish the label saving efforts in phase 2. WL-AC (see full version in Alg. 3 in appendix) uses a smart trades-off strategy that automatically balance query complexities between the weak label evaluation and training data collection phases.

### 3.1 Practical Extension

The original AC our framework is based on is not scalable for large-scale models (e.g., when $h$ is a deep neural network). First, although [Dasgupta et al. 2007] observed that $1[x \in D_m]$ can be efficiently determined using a single call to the ERM oracle, in practice, even this “oracle-efficient” definition is undesired because it requires the model to be retrained too often. Second, it is hard to explicitly calculate $D_m$, as well as estimate the sample distribution (i.e., explicitly solving optimization problem used in OP, see eq. 9).

While our theoretical instantiation inherits the same limitations of AC, we show how to design a more practical version overcoming these issues. First, we propose to use any active learning strategy $O_{baseAL}$, to estimate the active sample region $D_m$. For example, the simplest oracle can do uniform sampling under a given budget. Another classical oracle is an entropy-based uncertainty sampling strategy which calculates the cross-entropy for any given $x$ and select those with high entropy.

**Definition 3.1.** $O_{baseAL}$ takes in (1) a target context set $X'$, which theoretically is the full current active context set $D_m$, but can also practically be the observed unlabeled context set in the next block in the batched sample setting $\{x_i\}_{i=\tau_{m-1}+1}^{\tau_m}$, (2) the current trained model $h_m$ to estimate the informativeness of each sample; (3) and other relevant parameters such as decision threshold. Then $O_{baseAL}$ outputs a subset of $X'' \subset X'$.

Second, instead of solving the complicate OP problem for $P_m$, we only find a feasible solution, which is usually a uniform distribution over samples within $D_m$. Finally, our algorithm requires the knowledge of $\kappa_m$ (see Assumption 2.1). We choose $\kappa(D) = 1$, which is shown to be effective in our experiments. Besides these key relaxations, there are other subtleties one needs to take care such as using pseudo loss and a small validation set when updating the model $h_m$. We defer all the details in Appendix 13.2 for the exact implemented algorithm.

### 4 Theoretical guarantees

In this section we show that the theoretical instantiation of WL-AC preserves the advantages of AC while leveraging the informative weak labeler to improve the query complexity. Refer to Appendix 9 for a detailed description of the algorithm.

In our analysis, we leverage an instantiation of WL-AC, called NOWL-AC$(\{L_m\}_{m=1}^M, \delta)$, that in each block $m$

- waits without querying during $[\tau_{m-1} + 1, \tau_m]$ and then calculates $P_m$ in the NOWL mode as the original AC;
- queries labels according to the planned $P_m$ during $[\tau_m + 1, \tau_m]$ and then updates the training dataset;
- calculates the empirical loss and updates the model in NOWL mode.

Overall, this can be regarded as a weak version of original AC with extra unlabeled samples during $\tau_{m-1} + 1$ to $\tau_m$. Fortunately, it is easy to show that for this version we have $\tau_m - \tau_{m-1} \leq O(\tau_m - \tau_{m})$ and thus, the total unlabeled sample complexity is still of the same order.

In addition, we define $N_m$ as the expected query number within block $m$ if we run NOWL-AC with the same inputs,

$$N_m = E_m[E_{x \sim \mathcal{D}_m}[P_m(x)]]$$

where $E_m$ is the expectation condition on history before block $m$. Later we will use this $N_m$ as a base term in label complexity analysis.

### 4.1 Generalization guarantees

Our results depend on a generalization error defined as

$$\mathfrak{err}_m(h) = \frac{1}{\sum_{j=1}^{L_m} \sum_{j=1}^{L_j} \mathbb{E}_{D_j}[\mathbb{1}[h(x) \neq y \land x \in D_j]]}$$

Note that the results in the AL literature are often in terms of $\mathfrak{err}_D(h)$, which is an upper bound of $\mathfrak{err}_m(h)$. Therefore, using the generalization error helps to characterize a tighter bound. Moreover, $\mathfrak{err}_m(h)$ seems to be algorithm-and-empirical-distribution dependent due to the $D_j$ term but we argue that the same issues occur in the original AC analysis and can be easily reduced to some meaningful cases as shown in their paper.

**Theorem 4.1.** Pick any $0 < \delta < 1/\epsilon$ such that $|\mathcal{H}|/\delta > \sqrt{192}$. By running the theoretical version (Algorithm 2 in appendix), we have for all epochs $m = 1, 2, \ldots, M$, with probability at least $1 - \delta$

$$\mathfrak{err}_D(h) - \mathfrak{err}_D(h^*) \leq O(\Delta_m^* \epsilon) \quad \text{for all} \ h \in A_{m+1}$$

Chen, Sankararaman, Lazaric, Pirotta, Karamshuk, Wang, Mandayam, Wang, Fang
where $\Delta_0^* = \Delta_0$ and $\Delta_m^* := c_1 \sqrt{\epsilon_m \tau_m (h^*)} + c_2 \epsilon_m \log \tau_m$ for $m \geq 1$. And this directly gives the final output guarantee

$$
\text{err}(h) - \text{err}(h^*) \leq O \left( \frac{\log (|H|/\delta)}{n} \cdot \frac{\text{err}_M(h^*)}{\epsilon} + \frac{\log (|H|^2/\delta)}{n} \right)
$$

This generalization guarantees is consistent with the one in NOWL-AC with same inputs for every block, and therefore optimal. We want to emphasize that, to get such consistency with added WL leverage strategy, some significant modification in analysis is required and has been postponed into Appendix 11.

4.2 Label complexity analysis within each block

Before giving label complexity upper bounds, we define a useful parameter $\phi_m := \frac{\mathbb{E}[1|x \in D_m]|\phi_m}{\epsilon_m (h^*) + \log (\sum_{j=1}^m L_j) \epsilon_m}$ which intuitively describes how fast the error of best hypothesis decreases with the shrinking disagreement region, with some additional regularized term. In the following theorem, we show its connection to some standard AL parameters.

**Theorem 4.2** (Upper bound of $\phi_m$). For any block $m$, we always have $\phi_m \leq 1 + \frac{1}{\epsilon_m (h^*) + \log (\sum_{j=1}^m L_j) \epsilon_m}$.

In the rest of the section, we will directly use $\phi_m$ because we believe this term well characterized the case where weak labelers help.

**Theorem 4.3** (Expected label complexity of block $m$). Let

$$
T_{1, m} = N_m \kappa(D_m) \text{WLerr}_m
$$

$$
T_{2, m} = \sqrt{\ln(2 M \log (n)/\delta) \kappa(D_m) N_m} + \sqrt{\frac{L_m^2}{\sum_{j=1}^m L_j} \log^2 \left( \frac{|H|}{\delta} \right) \mathbb{E}[1|x \in D_m] |\phi_m|}
$$

$$
+ \ln(2 M \log(\sum_{j=1}^m L_j)) \log(n) |\phi_m|
$$

$$
T_{3, m} = \mathbb{P} \left[ \text{WLerr}_m > \frac{1}{\log (\sum_{j=1}^m L_j) \phi_m} \right] N_m
$$

where $\text{WLerr}_m = \text{WLerr}(D_m)$. Then, with probability at least $1 - \delta$, for any block $m$,

$$
\mathbb{E}_{\tau_m} \left[ \sum_{t=\tau_{m-1}+1}^{\tau_m} \mathbb{E}_t \left[ 1[y_t \text{ is queried}] \right] \right] \leq O \left( \min \{ T_{1, m} + T_{2, m} + T_{3, m}, N_m \} \right)
$$

**Discussion** Theorem 4.3 shows that WL-AC preserves the advantage of original algorithm under the low quality weak labeler case. Indeed the expected query complexity in block $m$ is never larger than $N_m$, i.e., the expected number of queries of NOWL-AC.

Ignoring $N_m$, the query complexity is characterized by the terms $T_i$. The term $T_1$ characterizes the number of queries we could save from using weak labels if their accuracy upper bound is known. In reality, however, the WLerr itself needs to be evaluated through samples. Therefore, the $T_2$ and $T_3$ capture the trades-off between the accurate WL evaluation and the WL-leveraged training sample collection.

While the first term in $T_2$ captures the required number of samples to evaluate the weak labels if no $P_m, \min$ lower bound exists, the second term explicitly write out the minimum label complexity of the original AC.

The third term of $T_2$ captures the number of samples required to decide whether we need to switch to NOWL mode which we give more detailed discussion in the next paragraph.

Finally, $T_3$ characterizes the label complexity in NOWL-mode, which depends on whether the conditional WLerror is worse than the true error of the best hypothesis restricted to the disagreement region. Note that WLerr is always non-increasing with the shrinking of the disagreement region. This implies that the WL is helpful as long as it gives high accuracy on those informative samples, which aligns with the intuition in [Zhang and Chaudhuri, 2015] called Localized Difference Classifier Training. Other than the theoretical implication, we want to remark that an explicit rule on whether we should stop using weak labeler is also good for practice.

4.3 More discussion on the total label complexity

Although our algorithm is guaranteed to always preserve the advantage of original AC in each block, the total complexity depends on the total number $M$ of blocks. Indeed, the total complexity of WL-AC is obtained by accumulating the query complexity of each block $m$. Several strategies can be used to control the number of blocks. For example, we can use a linear schedule or the classical block length doubling techniques. For ease of exposure, we consider $L_m = \sum_{j=1}^{m-1} L_j = 2^m$, which leads to $M = \mathcal{O}(\log n)$.

In general, our results is not directly comparable with previous work in [Huang et al., 2015] because we are not in the realizable setting and it is hard to give an explicit form of $P_m(\cdot)$ as discussed in the original AC analysis. Nevertheless, here we give an intuitive discussion on their relation based on a relaxed upper bound.

**Theorem 4.4** (Worst-case total label complexity under benign setting, informal). Choose $L_m = 2^m$. Suppose there

Note that this lower bound $P_m, \min$ comes from the heavy tail term in concentration inequality. We conjecture that it can be removed by using more robust estimators like Catoni estimator as in [Wei et al., 2020]. Nevertheless, it is not clear how to simultaneously achieve efficiency so we left that as an open problem.
exists an generalized WL error upper bound $\overline{WLerr} \in [0, 1]$ that

$$\sum_{m=1}^{M} T_{1,m} + T_{3,m} \leq \overline{WLerr} \sum_{m=1}^{M} N_{m},$$

Then we have with probability at least $1 - \delta$,

$$\sum_{t=1}^{n} \mathbb{I}[y_t \text{ is queried}] \leq O\left(\tilde{\theta} \sqrt{\text{err}_M(h^*)} n \log(|H|/n/\delta) + \log(|H|/n/\delta)^2 + \theta^* \overline{WLerr}(\text{err}_M(h^*)n + \log(|H|/\delta))\right)$$

where

$$\tilde{\theta} = \frac{\theta^* \log(n)}{\sqrt{\theta^* \left(\sum_{m=1}^{M} \text{max}\{\kappa(D_m), \frac{\ln(2M \log(n)/\delta)}{\ln(\log(n)^2/\delta)}\}\frac{\ln(\log(n)^2/\delta)}{\log(|H|/n/\delta)}\}}.$$ 

Remark 4.1. Note that in this bound we treat each $P_m$ in NOWL-AC as 1 and relax $\phi$ to its upper bound $\theta^*$.

Comparison with the label complexity in difference-classifier-based algorithm (Huang et al., 2015) The first term can be upper bounded as $O\left(\frac{\theta^* \log(|H|)}{\text{err}_M(h^*) + 1}\right)$, where $\epsilon$ is the target excess risk. This corresponds to the term $O\left(\theta^* \text{VCdim}(H) \left(\frac{\text{err}_M(h^*)}{\epsilon} + 1\right)\right)$ in bound of Huang et al. (2015) - both describe the required samples to evaluate the weak labels. Therefore, when the accuracy of weak labels has low variance ($\kappa(D_m)$ is small for most blocks), our algorithm can save up to $\frac{\text{VCdim}(H)}{\log(|H|)}$ in this term. Intuitively, this suggests that, when a lower complexity $H$ is given and a proper $h^* \in H$ exists, the previous one can give better result. Otherwise, our result is more preferable.

The second term can be upper bounded as $O\left(\frac{\theta^* \overline{WLerr} \log(|H|)}{\epsilon} \text{err}_M(h^* \epsilon)^2\right)$, which corresponds to the term $O\left(\frac{\theta^* \text{WLerr} \log(|H|)}{\epsilon} \text{err}_M(h^* \epsilon)^2\right)$ in the bound of Zhang and Chaudhuri (2015). Both describe the label complexity we could save by knowing weak label quality, but how these two WL quality characterization term related to each other remains unclear.

5 Experiments

In this section, we show the effectiveness of the proposed practical version of WL-AC discussed in Section 3.1 through a set of experiments on corrupted MNIST. The MNIST-C (Mu and Gilmer, 2019) is a comprehensive suite of 16 different types of corruptions applied to the MNIST dataset (LeCun et al., 2010). Each contains 60000 training samples and 10000 test samples. We consider a 2-layer convolutional neural network as hypothesis class in all the experiments.

- **Synthetic weak labels** We choose one type of corruption (“impulse-noise” corruption in main paper) as target task and generate synthetic weak labels purely based on the true labels. We further consider two processes for the generation of the synthetic weak labels. In the first case denoted as noisy annotators (NA), we add uniform random noise across all the samples. That is, each sample has a certain probability to get the true label, otherwise it will get a random label from the other 9 classes. We show the robustness of our algorithm by testing on various levels of noise. In the second case denoted as localized classifiers (LC), we first do passive learning on the target task and then give correct weak labels to those samples only with high entropy, so the classifier is good on this local region. Thus we show the adaptiveness of our algorithm in the case where, although the overall quality of weak labelers are mediocre, the weak labeler is informative for those sample close to the decision boundary.

- **Biased Pre-Trained Labelers** We train classifiers using different corruptions (“identity”, “motion-blur” and “dotted-line” in main paper) than the target task (“impulse noise”) and use these pre-trained classifiers to generate weak labels.

The overall objective of these experiments is to show the effectiveness of WL-AC compared to standard AL algorithms without weak labels and passive learning in realistic setting. The practical instantiation of WL-AC is described in Algorithm 4 and can be used with any heuristic NOWL selection strategy. In our experiment, we first use uniform random sampling as baseline. Then we choose most commonly used query framework – entropy-based (ET) uncertainty sampling, which is known to generalizing easily to probabilistic multi-label classifiers (Settles, 2009). Specifically, here we choose a hard-threshold for uncertainty sampling and only query samples above that threshold. To conduct a thorough empirical study on how our algorithm performs with various AL methods remains to be a future direction.

We use ET(x) to denote entropy based uncertainty sampling with a threshold x and NA/LC(x) to denote the weak label setting where x is the noise level. Furthermore, we use “passive” to denote uniform sampling with the base NOWL-AL.

In each experiment, we average the results over 5 different random seeds. In each independent repetition, we randomly shuffle the whole training dataset, choosing 600 samples (1% of the dataset) as validation set and the rest as
5.1 Results

We start considering experiment ❶. Figure 1(top) shows that our algorithm is able to leverage the weak labels when the noise level is low (below 0.5) and greatly outperforms both the baseline (passive and ET(0)) learners. The gain in query complexity is significant, up to a 80% save in queries in the best case. In addition, this figure also show that our algorithm preserves the advantage of uncertainty sampling except the 0.5 noisy case. One possible reason is that 0.5 is in the most ambiguous case – if the noise ratio is low then we can easily take advantage of weak labels, if it is high then we can quickly detect that and switch to NOWL mode. Finally, it is also worth to notice that Figure 1(top) shows that our algorithm is robust to high levels of noise where it achieves similar performance as passive learning.

The uniform label noise fails to consider the case that the weak labels are more accuracy on those sample close to decision boundary. In fact, you may notice that accuracy increase rate becomes as flat as the passive learning in the later horizon, since the $\text{err}_m(f^*)$ decreases faster then WLerr$_m$ and the algorithm ultimately switch to NOWL mode. Therefore, in Figure 1(bottom), we test on this carefully designed informative classifier and further demonstrate the advantage of our algorithm in more structured weak labelers. Without applying uncertainty sampling techniques, the advantage gained by leveraging weak labels will diminish when the active sampling set is shrinking. Only by incorporating this base AL strategy, the algorithm can get full advantage of the property of weak labeler.

Figure 2 reports the results in scenario ❷. Even in this more realistic experiment, similarly to the uniform noise experiment, WL-AC is able to leverage the weak labelers and reduce the query complexity to about 40% of the one of the passive learner and 56% of the one of the active learner. This implies not only the theoretical soundness of WL-AC, but the potential practical usage of our algorithm. These experiments confirm the theoretical findings and show that WL-AC is effective in leveraging “good” hints while being robust to wrong hints.

Accuracy regarding to unlabeled sample complexity

We postpone the plots of the accuracy regarding to the number of observed unlabeled samples in Appendix 13.3, all of which show that our algorithm guarantee the similar accuracy as the passive one in term of the unlabeled samples.
6 Conclusions

We introduced a novel algorithm, called WL-AC, for active learning with weak labelers that has both strong theoretical guarantees and good empirical performance. From a theoretical perspective, we showed that the query complexity of WL-AC is no worse than AC and, in certain problem instance, the access to weak labelers can lead to a significant reduction of the query complexity. Furthermore, we showed that WL-AC can be easily extended to practical scenarios where complex models (e.g., neural networks) are required. Our experiments on corrupted MNIST support the theoretical findings and show a significant improvement (more than 30%) in query complexity compared to standard active learning methods (i.e., not using weak labels).

There are still many open questions. From the theoretical perspective, a natural combine our methods with the difference-classifier-based methods or find a better WL quality characterization that can establish a clear definition between these two. From the practical perspective, a more comprehensive numerical evaluation on incorporating our templates with various up-to-date AL in deep learning methods can further improve our understanding.

References

Awasthi, P., Balcan, M. F., and Long, P. M. (2017). The power of localization for efficiently learning linear separators with noise. Journal of the ACM (JACM), 63(6):1–27.

Dasgupta, S., Hsu, D. J., and Monteleoni, C. (2007). A general agnostic active learning algorithm. Advances in neural information processing systems, 20.

Fang, M., Yin, J., and Tao, D. (2014). Active learning for crowdsourcing using knowledge transfer. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 28.

Gal, Y., Islam, R., and Ghahramani, Z. (2017). Deep bayesian active learning with image data. In International Conference on Machine Learning, pages 1183–1192. PMLR.

Garcelon, E., Avadhanula, V., Lazaric, A., and Pirotta, M. (2022). Top K ranking for multi-armed bandit with noisy evaluations. In AISTATS, volume 151 of Proceedings of Machine Learning Research, pages 6242–6269. PMLR.

Hanneke, S. et al. (2014). Theory of disagreement-based active learning. Foundations and Trends® in Machine Learning, 7(2-3):131–309.

Huang, S.-J., Chen, J.-L., Mu, X., and Zhou, Z.-H. (2017). Cost-effective active learning from diverse labelers. In IJCAI, pages 1879–1885.

Huang, T.-K., Agarwal, A., Hsu, D. J., Langford, J., and Schapire, R. E. (2015). Efficient and parsimonious agnostic active learning.

Kakade, S. M. and Tewari, A. (2008). On the generalization ability of online strongly convex programming algorithms. Advances in Neural Information Processing Systems, 21.

Kovashka, A., Russakovsky, O., Fei-Fei, L., and Grauman, K. (2016). Crowdsourcing in computer vision. Foundations and Trends® in Computer Graphics and Vision, 10(3):177–243.

LeCun, Y., Cortes, C., and Burges, C. (2010). Mnist handwritten digit database. ATT Labs [Online]. Available: http://yann.lecun.com/exdb/mnist, 2.

Lee, N., Li, B. Z., Wang, S., Fung, P., Ma, H., Yih, W.-t., and Khabsa, M. (2021). On unifying misinformation detection. In Proceedings of the 2021 Conference of the North American Chapter of the Association for Computational Linguistics: Human Language Technologies, pages 5479–5485, Online. Association for Computational Linguistics.

Malago, L., Cesa-Bianchi, N., and Renders, J. (2014). Online active learning with strong and weak annotators. In NIPS Workshop on Learning from the Wisdom of Crowds.

Miller, B., Kanthelian, A., Afroz, S., Bachwani, R., Dauber, E., Huang, L., Tschantz, M. C., Joseph, A. D., and Tygar, J. D. (2014). Adversarial active learning. In Proceedings of the 2014 Workshop on Artificial Intelligent and Security Workshop, pages 3–14.

Mozannar, H. and Sontag, D. (2020). Consistent estimators for learning to defer to an expert. In International Conference on Machine Learning, pages 7076–7087. PMLR.

Mu, N. and Gilmer, J. (2019). MNIST-C: A robustness benchmark for computer vision. CoRR, abs/1906.02337.

Patrini, G., Rozza, A., Krishna Menon, A., Nock, R., and Qu, L. (2017). Making deep neural networks robust to label noise: A loss correction approach. In Proceedings of the IEEE conference on computer vision and pattern recognition, pages 1944–1952.

Rakhlin, A. and Sridharan, K. (2013). Online learning with predictable sequences. In Conference on Learning Theory, pages 993–1019. PMLR.

Sener, O. and Savarese, S. (2017). Active learning for convolutional neural networks: A core-set approach. arXiv preprint arXiv:1708.00489.

Settles, B. (2009). Active learning literature survey.

Steinhardt, J. and Liang, P. (2014). Adaptivity and optimism: An improved exponentiated gradient algorithm. In International Conference on Machine Learning, pages 1593–1601. PMLR.
Improved Adaptive Algorithm for Scalable Active Learning with Weak Labeler

Urner, R., David, S. B., and Shamir, O. (2012). Learning from weak teachers. In *Artificial intelligence and statistics*, pages 1252–1260. PMLR.

Wei, C.-Y. and Luo, H. (2018). More adaptive algorithms for adversarial bandits. In *Conference On Learning Theory*, pages 1263–1291. PMLR.

Wei, C.-Y., Luo, H., and Agarwal, A. (2020). Taking a hint: How to leverage loss predictors in contextual bandits?

Yan, S., Chaudhuri, K., and Javidi, T. (2016). Active learning from imperfect labelers. *Advances in Neural Information Processing Systems*, 29.

Yan, Y., Rosales, R., Fung, G., Farooq, F., Rao, B., and Dy, J. (2012). Active learning from multiple knowledge sources. In *Artificial Intelligence and Statistics*, pages 1350–1357. PMLR.

Yan, Y., Rosales, R., Fung, G., Subramanian, R., and Dy, J. (2014). Learning from multiple annotators with varying expertise. *Machine learning*, 95(3):291–327.

Yoo, D. and Kweon, I. S. (2019). Learning loss for active learning. In *Proceedings of the IEEE/CVF conference on computer vision and pattern recognition*, pages 93–102.

Zhang, C. and Chaudhuri, K. (2015). Active learning from weak and strong labelers. *Advances in Neural Information Processing Systems*, 28.

Zhdanov, F. (2019). Diverse mini-batch active learning. *arXiv preprint arXiv:1901.05954*. 
## Appendix

### Contents

1 Introduction

2 Preliminaries

3 WL-AC: A Template for AL with Weak Labelers
   3.1 Practical Extension

4 Theoretical guarantees
   4.1 Generalization guarantees
   4.2 Label complexity analysis within each block
   4.3 More discussion on the total label complexity

5 Experiments
   5.1 Results

6 Conclusions

7 Appendix: Related works

8 Appendix: Notations for analysis

9 Appendix: WL-AC (Theoretical version)
   9.1 A more detailed explanation on term $N_m$ and its relationship with NOWL-AC

10 Appendix: Guarantees of pessimistic estimation WLerror

11 Appendix: Generalization guarantees and its analysis
   11.1 Relation with the original proofs without WL evaluation in Phase 1 (Read before going to analysis)
   11.2 Deviation bounds and the benefits of using shifted double robust estimator
   11.3 Concentrations under the WL leveraging strategy
   11.4 Main results and its analysis
   11.5 Auxiliary lemma
12 Appendix: Label complexity and its analysis

12.1 Analysis for OP[9] ................................................................. 26
12.2 Analysis for the unlabeled sample complexity .................................. 27
12.3 Analysis for Algo8 on the label complexity ....................................... 27
12.4 Analysis on number of hint mode blocks .......................................... 29
12.5 Main results in Section 4.2 and the analysis ...................................... 29
12.6 Main results for Section 4.3 and its analysis ...................................... 31
12.7 Auxiliary lemma ........................................................................... 33

13 Appendix: Practical WL-AC and Experiments

13.1 A summary of algorithm modification .............................................. 34
13.2 Algorithm (practical version) .......................................................... 36
13.3 Performance regarding to the number of observed unlabeled samples .......... 37
7 Appendix: Related works

Active learning with multiple labeling sources has been studied in various settings. One common direction is to assume explicit structural labelers with various label cost and localized expertise, and then to adaptively select the proper labelers by modeling them during active learning. (Yan et al., 2012, 2014; Fang et al., 2014; Huang et al., 2017). But most of those works fail in providing rigorous statistically guarantees on their results and their strategy are mostly labeler-wise, which is too coarse compared to instance-wise studies as in Zhang and Chaudhuri (2015). Besides those empirical results, there are another line of theoretical works, which assumes the non-parametric settings. That is, those labelers are generally adhere to the rule that “similar samples have similar labels”. (Urner et al., 2012; Yan et al., 2016). Note that this notion can be viewed as one variants of our “low local variance” WL assumption. Finally, all those works except the Zhang and Chaudhuri (2015) we mentioned in main paper consider the pool-based setting, which is different from our streaming-based setting.

Online learning and bandits is another major topic in interactive learning literature which aims to minimize the regret regarding to best policy over time. As an counterpart of weak labeler setting in active learning, online learning with the help of loss predictors has been widely studied over the past decades. Many results have been achieved with full information feedback (Rakhlin and Sridharan, 2013; Steinhardt and Liang, 2014), as well as partial information feedback such as multi-armed bandits (Rakhlin and Sridharan, 2013; Wei and Lud, 2018). Our techniques are mainly inspired by the stochastic contextual bandits setting in the recent state-of-the-art result in Wei et al. (2020), specifically, we use the same doubly-robust estimator to reduce the loss estimator variance, and we borrow the high-level idea of starting an adaptive pure exploration phase for unknown hint quality (which is weak label quality in our setting).

8 Appendix: Notations for analysis

We define more notations here that will be used in the following analysis.

- Let $D_s$ and $D_{WL}$ be the underlying joint sampling distribution where each $(x, y)$, $(x, y^{WL})$ drawn from. And denote the corresponding conditional probability distribution as $P_s(\cdot|x)$, $P_{WL}(\cdot|x)$. Most times, we will neglect the subscription in $E$ and use that to represent the expectation of all random variables inside the bracket.

- The regret of a classifier $h \in \mathcal{H}$ relative to another $h' \in \mathcal{H}$ is $\text{reg}(h, h') := \text{err}(h) - \text{err}(h')$, and the analogous empirical regret on $S$ is $\text{reg}(h, h', S) := \text{err}(h, S) - \text{err}(h', S)$. When the second classifier $h'$ in (empirical) regret is omitted, it is taken to be the (empirical) error minimizer in $\mathcal{H}$.

- Define the expected regret associated to $\text{err}(h, \tilde{Z}_m)$ and $\text{reg}(h, h', \tilde{Z}_m)$.

$$\text{reg}_m(h, h') := E_N \left[ |\mathbb{1}(h(X) \neq h_m(X)) - \mathbb{1}(h'(X) \neq h_m(X))| \mathbb{1}(X \notin D_m) \right] + E_{X,Y} \left[ (\mathbb{1}(h(X) \neq Y) - \mathbb{1}(h'(X) \neq Y)) \mathbb{1}(X \in D_m) \right],$$

$$\tilde{\text{reg}}_m(h, h') := \frac{1}{\sum_{j=1}^{m} L_j} \sum_{j=1}^{m} L_j \text{reg}_j(h, h').$$

- Let $Z_m$ be the subset of $\tilde{Z}_m$ that are actually being queried.

- In some proof step, we use $\approx$ to hide the constant factor since we only care about the order.

- For convenience, we use $\sum_Z$ to denote $\sum_{(x,y,y^{WL},w) \in Z}$ for any collection dataset $Z$.

- We use $E_x[\cdot]$ to denote the expectation at time $t$ (or sample $t$) condition on the past.

- We use $m(i)$ to denote the block any $i$-th samples located.

- Let $\hat{N}_m = E \left[ \sum_{i=m}^{\hat{r}_m} \mathbb{1}[y_i \text{ is queried}] \right]$, which is the expected query number in Phase 1 (Algo.3) in any block $m$. 

9 Appendix: WL-AC (Theoretical version)

Here we present a detailed theoretical version of WL-AC under the templates in Algo. 1

Algorithm 2 WL-AC (theoretical version)

1: **Input:** Constants \( c_1, c_2, c_3 \), confidence \( \delta \), error radius \( \gamma \), parameters \( \alpha, \beta, \xi \) for (OP). Data collection scheduled block length \( L_1, L_2, \ldots \) satisfying \( L_{m+1} \leq \sum_{j=1}^{m} L_j \) and \( L_1 = 3 \). Strong labeler \( O \) and weak labeler \( W \). Candidate hypothesis set \( \mathcal{H} \).

2: **initialize:** epoch \( m = 0 \), \( \hat{Z}_0 := \emptyset \), \( Z_0 := \emptyset \), \( \Delta_0 := c_1 \sqrt{t} + c_2 e_1 \log 3 \), where

\[
\epsilon_m := \frac{32 \left( \log(|\mathcal{H}|/\delta) + \log(\sum_{j=1}^{m} L_j) \right)}{\sum_{j=1}^{m} L_j}.
\]

3: for \( m = 1, 2, \ldots, M \) do

4: **Phase 1:** WL evaluation and query probability assignment

5: Compute the NOWL-leveraged query probability \( P_{m, \text{NOWL}} = \text{OPT}(\text{USE-WL} = \text{FALSE}) \) in [9]

6: Call WL-EVAL in Algo. 3 and obtain the current WL evaluation dataset \( \tilde{Z}_m \), the flag USE-WL, which decides whether we should transit to NO-WL mode or not and the pessimistically estimated conditional WL error \( \text{WLerr}_m \) across \( D_m \).

7: Compute the WL leveraged query probability \( \hat{P}_m(x) = \text{OPT}(\text{USE-WL}_m, \text{WLerr}_m) \)

8: Set the stopping time of Phase 1 as \( \tau_m \) and the corresponding length as \( \bar{L}_m \)

9: **Phase 2:** Train data collection based on planned query probability

10: Set \( S = \emptyset \)

11: for \( t = \tau_m + 1, \ldots, \tau_m + L_m \) do

12: if \( x \in D_m \) then

13: Draw \( Q_t \sim \text{Bernoulli}(\hat{P}_m(x)) \) and query label \( y_t \) if \( Q_t = 1 \)

14: Update the set of examples:

\[
S := \begin{cases} S \cup \left\{ x_t, y_t, h_{m}(x_t), h_{m}(x_t), 1 \right\}, & Q_t = 1 \\ S \cup \left\{ x_t, 1, 1, 0 \right\}, & \text{otherwise.} \end{cases}
\]

15: end if

16: end for

17: Set \( \tilde{Z}_m = \tilde{Z}_{m-1} \cup S \)

18: **Phase 3:** Model and disagreement region updates using collected train data

19: Calculate the estimated error for all \( h \in \mathcal{H} \) using shifted double robust estimator

\[
\text{err}(h, \tilde{Z}_m) = \frac{\sum_{(x,y,y_{\text{WL}},w) \in \tilde{Z}_m} [\mathbb{I}[h(x) \neq y] - \mathbb{I}[h(x) \neq y_{\text{WL}}] w + \mathbb{I}[h(x) \neq y_{\text{WL}}] w]}{\sum_{(x,y,y_{\text{WL}},w) \in \tilde{Z}_m} \mathbb{I}[h(x) \neq y] w}
\]

20: USE-WL \( m = \text{TRUE} \)

21: Calculate the empirical best hypothesis \( h_{m+1} := \arg\min_{h \in \mathcal{H}} \text{err}(h, \tilde{Z}_m) \)

22: Update the active hypothesis set (hypothesis that always include the best hypothesis)

\[
\Delta_m := c_1 \sqrt{\epsilon_m \text{err}(h, \tilde{Z}_m) + c_2 e_m \log(\sum_{j=1}^{m} L_j)}
\]

\[
A_{m+1} := \left\{ \text{err}(h, \tilde{Z}_m) - \text{err}(h_{m+1}, \tilde{Z}_m) \leq \gamma \Delta_m \right\}
\]

\[
D_{m+1} := \text{DIS}(A_{m+1}) = \{ x \in \chi | \exists h, h' \in A_{m+1}, h(x) \neq h'(x) \}
\]

23: Set the end time of the current block as \( \tau_m = t = \tau_m + L_m \)

24: end for

25: return \( h_M \)

Note that we write out the Line [6] in the templates into more details, instead of merging into the single WL-EVAL. The various constants in Alg.2 must satisfy:
Algorithm 3 WL evaluation for block $m$ (WL-EVAL, theoretical version)

1: **Input:** $D_m$, $\text{err}(h_m, \tilde{Z}_{m-1}), \Delta_{m-1}$, $L_m$, the current collected WL evaluation samples $\tilde{Z}_{m-1}$ and the query probability without WL denoted as $P_m$
2: Set $\text{cr}(h_m, \tilde{Z}_{m-1}) = \text{err}(h_m, \tilde{Z}_{m-1}) + \Delta_{m-1}$
3: **Step 1:** Check if the current estimated best error is already good enough 
4: Set $N_m = \mathbb{E}[P_m(x)]$, which is the expected query number of block $m$ without leveraging the WL 
5: if $N_m \leq \left(\frac{2^{2}\ln(2M \log(n)/\delta)}{\text{cr}(h_m, \tilde{Z}_{m-1})} - |\tilde{Z}_{m-1}|\right) \mathbb{E}[|x \in D_m|]$ then 
6: **return** Use-WL = False, $W_{Err_{m}} = 1$, $\tilde{Z}_{m} = \tilde{Z}_{m-1}$
7: **end if**
8: **Step 2:** Check if the WL performance is worse than $\text{cr}(h_m, \tilde{Z}_{m-1})$
9: Initialize $k = K_m = \lceil\log\left(\frac{6}{\text{cr}(h_m, \tilde{Z}_{m-1})}\right)\rceil$ and initialize $\hat{S} = \emptyset$
10: Get $\kappa_m = \kappa(D_m)$ from $\mathcal{W}$ as defined in [2.1]
11: Draw $\max\{(2^{k+1} \ln(2M \log(n)/\delta) - |\tilde{Z}_{m-1}|), 0\}$ number of new unlabeled samples and query labels for those inside $D_m$. For each sample at $t$, add them as 

$$\hat{S} := \begin{cases} \hat{S} \cup \{x_t, y_t, y^\text{WL}_t\}, & \text{if } Q_t = 1 \\ \hat{S} \cup \{x_t, 1, 1\}, & \text{otherwise.} \end{cases}$$

and update $\tilde{Z}_{m,k} = \tilde{Z}_{m-1} \cup \hat{S}$
12: Calculate the empirical mean $W_{Err_{m,k}} = \frac{1}{|\tilde{Z}_{m,k}|} \sum_{(x,y,y^\text{WL}) \in \tilde{Z}_{m,k}} 1[y^\text{WL} \neq y \wedge x \in D_m]$
13: Calculate the pessimistic overall estimation $W_{Err_{m,k}} = \min\{W_{Err_{k}} + 2^{-k}, \mathbb{E}[x \in D_m]\}$
14: if $W_{Err_{m,k}} \geq \text{cr}(h_m, \tilde{Z}_{m-1})$ then 
15: **return** Use-WL = False, $W_{Err_{m}} = 1$, $\tilde{Z}_{m} = \tilde{Z}_{m,k}$
16: **end if**
17: **Step 3:** Get a more precise estimation on WL performance
18: if $\max\{4N_m \min\left\{\frac{\text{err}(h_m, \tilde{Z}_{m,k})}{\mathbb{E}[x \in D_m]}, 1\right\}, L_m P_{m,\text{min}} \mathbb{E}[|x \in D_m|]\} \leq \mathbb{E}[x \in D_m] 2^{k+1} \ln(2M \log(n)/\delta)$ then 
19: **return** Use-WL = True, $W_{Err_{m}} = \min\left\{\frac{\text{err}(h_m, \tilde{Z}_{m,k})}{\mathbb{E}[x \in D_m]}, 1\right\}$, $\tilde{Z}_{m,k}$
20: **end if**
21: for $k = K_m + 1, \ldots$ do
22: Initialize $\hat{S} = \emptyset$
23: Draw $\max\{(2^{k+1} \ln(2M \log(n)/\delta) - |\tilde{Z}_{m,k-1}|), 0\}$ number of new unlabeled samples and query labels for those inside $D_m$. For each sample at $t$, add them as 

$$\hat{S} := \begin{cases} \hat{S} \cup \{x_t, y_t, y^\text{WL}_t\}, & \text{if } Q_t = 1 \\ \hat{S} \cup \{x_t, 1, 1\}, & \text{otherwise.} \end{cases}$$

and update $\tilde{Z}_{m,k} = \tilde{Z}_{m,k-1} \cup \hat{S}$
24: Calculate the optimistic estimation $W_{Err_{m,k}}$ using $\tilde{Z}_{m,k}$ as before
25: if $\max\{4N_m \min\left\{\frac{\text{err}(h_m, \tilde{Z}_{m,k})}{\mathbb{E}[x \in D_m]}, 1\right\}, L_m P_{m,\text{min}} \mathbb{E}[|x \in D_m|]\} \leq \mathbb{E}[x \in D_m] 2^{k+1} \ln(2M \log(n)/\delta)$ then 
26: **return** Use-WL = True, $W_{Err_{m}} = \min\left\{\frac{\text{err}(h_m, \tilde{Z}_{m,k})}{\mathbb{E}[x \in D_m]}, 1\right\}$, $\tilde{Z}_{m,k}$
27: **end if**
28: **end for**

As described in the templates, there are two purposes for WL-EVAL: (1) Deciding whether to transit to NOWL mode by comparing the WL error with estimated $\text{cr}(h_m, \tilde{Z}_{m-1})$; (2) Pessimistically estimating the conditional WL error if USE-WL. To achieve this, we have three steps. In Step 1, we compute the required number of estimate $\mathbb{E}[|y^\text{WL} \neq y| | x \in D_m|$ with sufficient accuracy to compare with $\text{cr}(h_m, \tilde{Z}_{m-1})$. If such number is too large, which indicates $\text{cr}(h_m, \tilde{Z}_{m-1})$ has already been sufficiently small, then we transit to NOWL mode without collecting any samples in WL-EVAL. Otherwise, we collect the required number of samples and do the comparison with $\text{cr}(h_m, \tilde{Z}_{m-1})$ in Step 2.
If we pass the comparison, then we further estimate the conditional WL error in Step 3.
We further give an example raised from the original AC to demonstrate how we keep the parsimoniousness property in some benign cases. Classifiers are defined jointly:

\[ \text{optimize WL performance upper bound WLerr}_m, \text{ USE-WL} \]

\[
\min_{P} \mathbb{E}_X \left[ \frac{1}{1 - P(X)} \right] \\
\text{s.t. } \forall h \in \mathcal{H}, \begin{cases} 
\mathbb{E}_X \left[ \frac{1}{P(X)} \right] & \text{WLerr}_m \leq b_{m}^\text{WL} (h) \quad \text{USE-WL} \\
\mathbb{E}_X \left[ \frac{1}{P(X)} \right] & \leq b_{m}^\text{NOWL} (h) \quad \text{otherwise}
\end{cases}
\]

where \(\mathbb{1}_m^n (x) = 1 \text{ if } h(x) \neq h_m (x) \land x \in D_m\),

\[
b_{m}^\text{NOWL} (h) = 2\alpha^2 \mathbb{E}_X [\mathbb{1}_m^n (x)] + 2\beta^2 \gamma \text{reg} \left( h, h_m, \hat{Z}_{m-1} \right) \tau_{m-1} \Delta_{m-1} + \xi \tau_{m-1} \Delta_{m-1}^2,
\]

\[
b_{m}^\text{WL} (h) = \frac{1}{2} (b_{m}^\text{NOWL} (h) - \mathbb{E}_X [\mathbb{1}_m^n (x)])
\]

\[
P_{\text{min}, m} = \min \left( \sqrt{\frac{c_3}{\sum_{i=1}^{m-1} L_m} \text{err} (h_m, \hat{Z}_{m-1})} + \log (\sum_{i=1}^{m-1} L_m) \right) \frac{1}{2}
\]

Remark 9.1 (Efficient implementation). As discussed in [Huang et al., 2015], this theoretical version of WL-AC algorithm itself can actually be implemented efficiently by solving \(\mathbb{1}_{[x \in D_m]}\) with ERM oracle and estimating the expected distribution with empirical distribution under the small scale models.

9.1 A more detailed explanation on term \(N_m\) and its relationship with NOWL-AC

Readers may notice that we abuse the notation \(N_m\) a little bit. Here we define \(N_m = \mathbb{E}[P_m(x) \text{ past history in WL-AC}]\) in Line 3 while in Section 2 we define \(N_m = \mathbb{E}[P_m(x) \text{ past history in NOWL-AC}]\). It is hard to argue that how exactly these two are closed to each other because each OP at block \(m\) depends on \(\text{reg} (h, h_m, \hat{Z}_{m-1})\) which is affected by the randomness during querying in Phase 2. Nevertheless, by the generalization guarantees shown in Theorem 11.1 we can guarantee that these two are roughly in the same order. Specifically, from the theorem we have for any \(h\),

\[
\text{reg} (h, h_m, \hat{Z}_{m-1}) \leq \frac{3}{2} \tilde{\text{reg}}_m (h, h^*) + \frac{\eta}{2} \Delta_{m-1}
\]

\[
\text{reg} (h, h_m, \hat{Z}_{m-1}) \geq \frac{1}{2} \tilde{\text{reg}}_m (h, h^*) - \frac{\eta}{4} \Delta_{m-1},
\]

no matter whether this \(\hat{Z}_{m-1}\) comes from NOWL-AC or WL-AC. Therefore, replace this upper bound and lower bound into the NOWL mode of WL-AC and the corresponding constraints in NOWL-AC, we have roughly

\[
b_{m}^\text{NOWL} = \tilde{\Theta} (\mathbb{E}_X [\mathbb{1}_m^n (X)] + \tilde{\text{reg}}_m (h, h^*) \tau_{m-1} \Delta_{m-1} + \tau_{m-1} \Delta_{m-1}^2),
\]

regardless of the randomness in query, in both WL-AC version and its NOWL-AC counterparts. Thus this \(N_m\) in algorithm is in the same order as we define in main paper, and therefore preserves benefits in original AC.

We further give an example raised from the original AC to demonstrate how we keep the parsimoniousness property in some benign cases.

Example restatement Let \(U\) denote the uniform distribution on \([-1, 1]\). The data distribution \(\mathcal{D}(\mathcal{X}, \mathcal{Y})\) and the classifiers are defined jointly:

- With probability \(\epsilon\), \(y = h^*(x), \ h(x) \sim U \{-1, 1\}, \ \forall h \neq h^*\).
- With probability \(1 - \epsilon\),
  \[
  y \sim U \{-1, 1\}, \ h^*(x) \sim U \{-1, 1\}, \ h_r (x) = -h^*(x) \quad \text{for some } h_r \text{ drawn uniformly at random from } \mathcal{H} \backslash h^*,
  \]
  \[
  h(x) = h^*(x), \ \forall h \neq h^* \land h \neq h_r.
  \]

Under this setting, the best classifier has \(\text{err}(h^*) = \frac{\epsilon}{1 + \epsilon}\) and others has \(\text{err}(h) = \frac{1}{2}\). It is easy to see that only \(\epsilon\) case is informative for us to distinguish \(h^*\). The traditional disagreement based algorithm will waste a lost of queries on uninformative sample due to the slow shrinking disagreement region.
Now in [Huang et al., 2015], they show that \( P_m(x) = P_{\min,m}, \forall x \in D_m \) is a feasible solution when \(|\mathcal{H}|\) is large because this choice of \( P(x) \) satisfies
\[
\mathbb{E}_X \left[ \mathbb{I}(h(x) \neq h_m(x) \land x \in D_m) \right] \leq O(\xi) \leq O(\xi \tau_{m-1} \Delta^2_{m-1}). \quad \text{(see details in original paper)}
\]

The same argument works also in our case, because \( \tau_{m-1}, \Delta_{m-1} \) does not affected by the randomness of sampling. That is,
\[
\mathbb{E}_X \left[ \mathbb{I}(h(x) \neq h_m(x) \land x \in D_m) \right] \leq O(\xi) \leq \frac{1}{2} \xi \tau_{m-1} \Delta^2_{m-1} \leq \hat{\delta}_m^W(h)
\]

10 Appendix: Guarantees of pessimistic estimation WLerror

Define the event that that the error WLhas been estimated within proper confidence region
\[
\mathcal{E}_{WL} := \left\{ \text{WLerr}_{m,k} \in \left[ \frac{1}{2} \mathbb{E}_{x,y,y^W} \mathbb{I}[x \in D_m \land y^W \neq y] - 2^{\frac{-k}{2}}, 2 \mathbb{E}_{x,y,y^W} \mathbb{I}[x \in D_m \land y^W \neq y] + 2^{-k} \right], \forall k \leq K^e, \forall m \in [M] \right\}
\]

**Lemma 10.1.** By running Algorithm in each block, \( \Pr[\mathcal{E}_{WL}] \geq 1 - \delta \).

**Proof.** By using Bernstein inequality and the union bound over all blocks, we have with probability \( 1 - \delta/2 \), for all block \( m \) and \( k \in [K^e_m, K^e] \),
\[
\frac{1}{|Z_{m,k}|} \sum_{(x,y,y^W) \in \hat{Z}_{m,k}} \mathbb{I}[y^W \neq y \land x \in D_m] - \mathbb{E}[\mathbb{I}[y^W \neq y \land x \in D_m]] \leq \sqrt{\frac{2}{|Z_{m,k}|} \sum_{(x,y,y^W) \in \hat{Z}_{m,k}} \mathbb{I}[y^W \neq y \land x \in D_m]} \ln(4M \log(n)/\delta) + 3 \ln(2M \log(n)/\delta)
\]

On the other hand, by applying empirical Bernstein inequality and the union bound, we have with probability \( 1 - \delta/2 \), for all \( k, m \)
\[
\mathbb{E}[\mathbb{I}[y^W \neq y \land x \in D_m]] - \frac{1}{|Z_{m,k}|} \sum_{(x,y,y^W) \in \hat{Z}_{m,k}} \mathbb{I}[y^W \neq y \land x \in D_m] \leq \sqrt{\frac{2}{|Z_{m,k}|} \sum_{(x,y,y^W) \in \hat{Z}_{m,k}} \mathbb{I}[y^W \neq y \land x \in D_m]} \ln(4M \log(n)/\delta) + 3 \ln(2M \log(n)/\delta)
\]

Now by the choice \( |\hat{Z}_{m,k}| = 2^{k+1} \ln(2M \log(n)/\delta) \), we get the desired bound. \( \square \)

**Corollary 10.1.** When \( \mathcal{E}_{WL} \) holds, for any \( m \) and \( k \leq K^e_m \), \( \text{WLerr}_{m,k} = \min\{2\hat{\delta}_m^W + 2^{-k}, 1\} \) is an pessimistic estimation (upper bound) of \( \mathbb{E}[\mathbb{I}[x \in D_m \land y^W \neq y]] \).

By using this assumption, we get the following WL quality estimation guarantees.

**Lemma 10.2** (Guarantee on the estimated conditional error). Under the assumption \( \mathcal{E}_{WL} \) when \( \mathcal{E}_{WL} \) holds, we have for any \( m \) and \( k \in [K^e_m, K^e] \),
\[
\text{WLerr}(D_m) \leq \min\left\{ \kappa(D_m) \frac{2\hat{\delta}_m^W + 2^{-k}}{\mathbb{E}[\mathbb{I}[x \in D_m]]}, 1 \right\} \leq \min\left\{ \kappa(D_m) \left( 4\hat{\delta}_m^W(D_m) + 3 \frac{2^{-k}}{\mathbb{E}[\mathbb{I}[x \in D_m]]} \right), 1 \right\}
\]

Notice here \( \min\left\{ \kappa(D_m) \frac{\text{WLerr}_{m,k}}{\mathbb{I}[x \in D_m]}, 1 \right\} \) is the pessimistic estimation of the conditional WL error.
Proof. Recall by definition in Assumption 2.1, WLerr\(D_m) = \max_{x \in D} \mathbb{E}[\mathbb{I}[y^{WL} \neq y]|x].\) Therefore we have,
\[
\max_{x \in D} \mathbb{E}[\mathbb{I}[y^{WL} \neq y]|x] \leq \kappa(D_m) \min_{x \in D_m} \mathbb{E}[\mathbb{I}[y^{WL} \neq y]|x] \\
\leq \kappa(D_m) \frac{\mathbb{E}[\mathbb{I}[y^{WL} \neq y \land x \in D_m]]}{\mathbb{E}[\mathbb{I}[x \in D_m]]} \\
\leq \kappa(D_m) \frac{2WLerr_m + 2^k}{\mathbb{E}[\mathbb{I}[x \in D_m]]} \\
\leq \kappa(D_m) \frac{4E[\mathbb{I}[y^{WL} \neq y \land x \in D_m]]}{\mathbb{E}[\mathbb{I}[x \in D_m]]} + 3 \cdot 2^{-k} \\
\leq \kappa(D_m) \frac{4E[\mathbb{I}[x \in D_m]] \max_{x \in D_m} \mathbb{E}[\mathbb{I}[y^{WL} \neq y]|x] + 3 \cdot 2^{-k}}{\mathbb{E}[\mathbb{I}[x \in D_m]]} \\
\leq 4 \kappa(D_m) \max_{x \in D} \mathbb{E}[\mathbb{I}[y^{WL} \neq y]|x] + 3 \kappa(D_m) \frac{2^{-k}}{\mathbb{E}[\mathbb{I}[x \in D_m]]}
\]
where the third and forth inequality comes from \(\mathcal{E}_{WL}.\)

\[\square\]

11 Appendix: Generalization guarantees and its analysis

11.1 Relation with the original proofs without WL evaluation in Phase 1 (Read before going into analysis)

Before start proving the generalization guarantees, we want to remark that our analysis is built on the original proof of 
Huang et al. (2015), so we will skip most of the proofs that are exactly the same as original one and only give analysis on those significant modifications. In order to make readers easy to connect our proofs with the original one, without loss of generality, we will ignore the samples collected in Phase 1, that is \(\hat{\tau}_m = \tau_m\) in the rest of sections, and therefore, \(L_m = \tau_m - \tau_{m-1}\). In the other word, within this section, readers can regard the pessimistic estimated WLerr directly given by some oracle.

One subtility to make such simplification is that, our final generalization guarantees will only depend on the number of total unlabeled data in Phase 2, that is \(\{L_m\}_{m=1}^M\), but does not include the number of unlabeled data in phase 1, that is \(\tau_m - \tau_{m-1}\). In order to make our excess risk result depends on the whole unlabeled sample complexity \(n\) so that we can compare it with the original AC, we show that \(L_m\) and \(\tau_m - \tau_{m-1}\) are roughly as the same order. Therefore, for all the \(\tau_m\) notation in the rest of analysis, after considering the Phase 1, we need to replace \(\tau_m\) with \(O(\tau_m)\) (see Lemma 12.2 which does not effect the order of those number-of-unlabeled-samples dependent results.)

11.2 Deviation bounds and the benefits of using shifted double robust estimator

We first show the deviation bound with or without the shifted double robust estimator. This bound is different than the original one in Huang et al. (2015) but later we will show that we can control the deviation of the empirical regret and error terms by combining similar techniques in original paper and our carefully designed hint leveraging strategies.

For convenience, let’s let’s first define the instantaneous variance of regret and error in expectation for any fixed block \(m\) and any pair of hypothesis \(h, h’\) (or a single hypothesis \(h\)) as follows
\[
\text{regVar}_{WL}(h, h’, m) = \mathbb{E} \left[ \frac{2}{P_m(x)} \mathbb{I}[y^{WL} \neq y] + 1 \right] \mathbb{I}[h(x) \neq h’(x) \land x \in D_m] + \mathbb{I}[h(x) \neq h’(x) \land x \notin D_m],
\]
\[
\text{regVar}_{NOWL}(h, h’, m) = \mathbb{E} \left[ \frac{1}{P_m(x)} \mathbb{I}[h(x) \neq h’(x) \land x \in D_m] + \mathbb{I}[h(x) \neq h’(x) \land x \notin D_m] \right],
\]
\[
\text{errVar}_{WL}(h, m) = \mathbb{E} \left[ \frac{\mathbb{1}(x \in D_m \land y^{WL} \neq y)}{P_m(x)} + \mathbb{1}(x \in D_m \land h(x) \neq y) \right],
\]
\[
\text{errVar}_{NOWL}(h, m) = \mathbb{E} \left[ \frac{\mathbb{1}(x \in D_m \land h(x) \neq y)}{P_m(x)} \right].
\]

Note that subscript WL denotes the USE-WL mode where double robust estimator is used, otherwise the normal inverse weighted estimator is used. Again as we discussed in the first contribution in Sections 3 that, the weak labeler will never
hurt estimating difference between two policies (up to some constant error) since \( \mathbb{1}[y^{WL} \neq y] \leq 1 \). But it will hurt estimating the error for a single classifier. Specifically, we have,

**Lemma 11.1** (Deviation bounds). Pick 0 < \( \delta < 1/e \) such that \( |\mathcal{H}|/\delta > \sqrt{192} \). With probability at least 1 − \( \delta \) the following holds for all \( m \geq 1 \).

\[
\begin{align*}
\left| \text{reg} \left( h, h', \hat{Z}_m \right) - \text{reg}_m(h, h') \right| \\
\leq \frac{\epsilon_m}{\tau_m} \left( \sum_{j \in M_{m, WL}} (\tau_j - \tau_{j-1}) \text{regVar}_{WL}(h, h', j) + \sum_{j \in M_{m, NOWL}} (\tau_j - \tau_{j-1}) \text{regVar}_{NOWL}(h, h', j) \right) + \frac{2\epsilon_m}{P_{\min, m}} \\
\left| \text{err} \left( h, Z_m \right) - \text{err}_m(h) \right| \\
\leq \frac{\epsilon_m}{\tau_m} \left( \sum_{j \in M_{m, WL}} (\tau_j - \tau_{j-1}) \text{errVar}_{WL}(h, j) + \sum_{j \in M_{m, NOWL}} (\tau_j - \tau_{j-1}) \text{errVar}_{NOWL}(h, j) \right) + \frac{\epsilon_m}{P_{\min, m}}
\end{align*}
\]

where \( M_{m, WL} = \{ j | j \leq m, \text{USE-WL}_j = \text{True} \} \) and vice versa.

To compare, the original bounds only have NOWL terms.

**Proof.** We will only prove the WL mode since the proof for NOWL is exactly the same as the original proof.

Again our proof follows the similar steps as in the proof of Lemma 2 in the original paper with a specialized adaption to our shifted loss estimator. First we look at the concentration of the empirical regret on \( \hat{Z}_m \). To avoid clutter, we overload our notation so that \( D_i = D_{m(i)}, h_i = h_{m(i)} \) and \( P_i = P_{m(i)} \) when \( i \) is the index of an example rather than a round.

For any pair of classifier \( h \) and \( h' \), we define the random variables for the instantaneous regrets:

\[
\hat{R}_i := \mathbb{1}[x_i \in D_i \left( \frac{\mathbb{1}[h(x_i) \neq y_i] - \mathbb{1}[h(x_i) \neq y_i^{WL}]}{P_i(x_i)} Q_i + \mathbb{1}[h(x_i) \neq y_i^{WL}] \right) - \left( \frac{\mathbb{1}[h'(x_i) \neq y_i] - \mathbb{1}[h'(x_i) \neq y_i^{WL}]}{P_i(x_i)} Q_i + \mathbb{1}[h'(x_i) \neq y_i^{WL}] \right)] + \mathbb{1}[x_i \notin D_i \left( \mathbb{1}[h(x_i) \neq h_i(x_i)] - \mathbb{1}[h'(x_i) \neq h_i(x_i)] \right) + \mathbb{1}[x_i \notin D_i \left( \mathbb{1}[h(x_i) \neq h_i(x_i)] - \mathbb{1}[h'(x_i) \neq h_i(x_i)] \right)]
\]

and the associated \( \sigma \)-fields \( \mathcal{F}_i := \sigma \left( \{ X_j, Y_j, Q_j \}_{j=1}^i \right) \). We have that \( \hat{R}_i \) is measurable with respect to \( \mathcal{F}_i \). Therefore \( \hat{R}_i - \mathbb{E} \left[ \hat{R}_i | \mathcal{F}_{i-1} \right] \) forms a martingale difference sequence adapted to the filtrations \( \mathcal{F}_i, i \geq 1 \), and \( x_i, y_i, Q_i \) are independent from the past. Now we want to state several properties of the instantaneous regrets.

First of all, we can simply the instantaneous regrets as

\[
\hat{R}_i = \mathbb{1}[x_i \in D_i \mathbb{1}[y_i = y_i^{WL}](\mathbb{1}[h(x_i) \neq y_i^{WL}] - \mathbb{1}[h'(x_i) \neq y_i^{WL}]) + \mathbb{1}[x_i \in D_i \mathbb{1}[y_i \neq y_i^{WL}] \frac{Q_i}{P_i(x_i)} (\mathbb{1}[h(x_i) \neq y_i] - \mathbb{1}[h'(x_i) \neq y_i]) - (\mathbb{1}[h(x_i) \neq y_i^{WL}] - \mathbb{1}[h'(x_i) \neq y_i^{WL}])]
\]

Therefore, we can get the upper bound of magnitude, expectation and variance as

\[
\left| \hat{R}_i - \mathbb{E}[\hat{R}_i | \mathcal{F}_{i-1}] \right| \leq \frac{2}{P_i(x_i)} \leq \frac{2}{P_{\min, m}} \text{ for all } m \geq m(i)
\]

\[
\mathbb{E}[\hat{R}_i | \mathcal{F}_{i-1}] = \mathbb{1}[x_i \in D_i \mathbb{1}[h(x_i) \neq y_i] - \mathbb{1}[h'(x_i) \neq y_i]) + \mathbb{1}[x_i \notin D_i \mathbb{1}[h(x_i) \neq h_i(x_i)] - \mathbb{1}[h'(x_i) \neq h_i(x_i)]]
\]

\[
\mathbb{E} \left[ \left( \hat{R}_i - \mathbb{E}[\hat{R}_i | \mathcal{F}_{i-1}] \right)^2 | \mathcal{F}_{i-1} \right] \leq \text{regVar}_{WL}(h, h', m(i))
\]

Therefore, by applying a variant of freedman inequality (Lemma 6 in original paper) we get the desired bound. (For more details please refer to the explanation on eqn.(51) in the original paper.)
Then we consider the concentration of the empirical error on the importance-weighted examples. Define the random examples for the empirical errors:

$$E_i := \left( \frac{\mathbb{1}[h(x_i) \neq y_i] - \mathbb{1}[h(x_i) \neq y_i^{WL}]}{P_j(x_i)} Q_i + \mathbb{1}[h(x_i) \neq y_i^{WL}] \right) \mathbb{1}[x_i \in D_i]$$

and the associated $\sigma$-fields $F_i := \sigma\left(\{X_j, Y_j, Q_j\}_{j=1}^\infty\right)$. By the same analysis of the sequence of instantaneous regrets, we have $E_i - \mathbb{E}[E_i | F_{i-1}]$ is a martingale difference sequence adapted to the filtrations $F_i, i \geq 1$, with the following properties:

$$\mathbb{E}[E_i | F_{i-1}] = \mathbb{E}\left[\mathbb{1}(X_i \in D_i \land h(X_i) \neq Y_i) | F_{i-1}\right] = \text{err}_{m(i)}(h)$$

$$|E_i - \mathbb{E}[E_i | F_{i-1}]| \leq \frac{1}{P_{\min,m(i)}} \leq \frac{1}{P_{\min,m}}$$

for all $m \geq m(i)$

$$\mathbb{E}\left[(E_i - \mathbb{E}[E_i | F_{i-1}])^2 | F_{i-1}\right] \leq \text{errVar}_{WL}(h, m(i))$$

Again, by applying the variant of Freedman inequality and other sublte steps in the original paper, we get the desired bound.

\[\square\]

11.3 Concentrations under the WL leveraging strategy

Let $E_{\text{train}}$ denote the event that the assertions of Lemma 11.1 and we know that $\mathbb{P}[E_{\text{train}}] \leq 1 - \delta$, we obtain the following propositions for the concentration of empirical regret and error terms. As we shown below, our results is very similar to the original one. We provides some key proof steps to illustrate in details.

**Proposition 11.1 (Regret concentration – modified prop 1).** Fix an epoch $m \geq 1$, suppose the events $E_{WL}$ and $E_{\text{train}}$ holds and assume $h^* \in A_j$ for all epoch $j \leq m$,

$$|\text{reg}(h, h', Z_m) - \tilde{\text{reg}}_{m}(h, h')| \leq \frac{1}{4} \tilde{\text{reg}}_m(h) + 2\alpha \sqrt{\frac{\epsilon_m}{\tau_m}} \sum_{j=1}^m (\tau_j - \tau_{j-1}) \text{reg}_j(h) + 2\alpha \sqrt{3\text{err}_m(h^*)\epsilon_m}$$

$$+ \beta \sqrt{2\gamma\epsilon_m \Delta_m \sum_{j=1}^m (\tau_j - \tau_{j-1}) (\text{reg}(h, Z_{j-1}) + \text{reg}(h^*, Z_{j-1}) + 5\Delta_m)}$$

To compare, the original bound is almost same except the slightly different constant requirements on $c_1, c_2$ inside the definition of $\Delta_m$

**Proof.** One key step in the original proof is to show that both $\text{regVar}_{WL}(h, h^*, j)$ and $\text{regVar}_{NOWL}(h, h^*, j)$ for any fixed $j$ is upper bounded by the follows (see the second equation block in **Proof of Proposition 1 in original paper**).

$$\mathbb{E}[2\alpha^2 \mathbb{1}(x \in D_j) (\mathbb{1}(h(x) \neq h_1(x)) + \mathbb{1}(h^*(x) \neq h_1(x)))$$

$$+ 2\beta^2 \gamma \tau_{j-1} \Delta_{j-1} (\text{reg}(h, Z_{j-1}) + \text{reg}(h^*, Z_{j-1})) + 2\xi \tau_{j-1} \Delta_{j-1}^2$$

$$+ \mathbb{1}(h(x) \neq h^*(x) \land x \notin D_j)]$$

Note the this upper bound for $\text{regVar}_{NOWL}(h, h^*, j)$ has been proved in the original paper, and here we will focus on showing the upper bound of $\text{regVar}_{WL}(h, h^*, j)$. Firstly, we have

$$\mathbb{E} \left[ \left( \frac{2}{P_j(x)} \mathbb{1}[y^{WL} \neq y] + 1 \right) \mathbb{1}[x \in D_j] \mathbb{1}[x \notin D_j] \mathbb{1}[h(x) \neq h^*(x)] \right]$$

$$\leq \mathbb{E} \left[ \left( \frac{2}{P_j(x)} \mathbb{1}[y^{WL} \neq y] + 1 \right) \mathbb{1}[x \in D_j] (\mathbb{1}[h(x) \neq h_m(x)] + \mathbb{1}[h^*(x) \neq h_m(x)]) + \mathbb{1}[x \notin D_j] \mathbb{1}[h(x) \neq h^*(x)] \right]$$
Next we show that the output of OP\textsuperscript{9} can yield the target upper bound on the first term. That is, for any $h$, we have

\[
\mathbb{E}
\left[
\frac{2}{p(x)} \left( h(x) \neq h_j(x) \wedge x \in D_j \wedge y^{WL} \neq y \right) + \mathbb{I}[h(x) \neq h_j(x) \wedge x \in D_j]
\right]
\leq \mathbb{E}
\left[
\frac{2}{p(x)} \left( h(x) \neq h_j(x) \wedge x \in D_j \right) \mathbb{E}[\mathbb{I}[y^{WL} \neq y|x]]
\right] + \mathbb{E} \left[ \mathbb{I}[h(x) \neq h_j(x) \wedge x \in D_j] \right]
\leq \mathbb{E}
\left[
\frac{2}{p(x)} \left( h(x) \neq h_j(x) \wedge x \in D_j \right)
\right] V_W(D_j) + \mathbb{E} \left[ \mathbb{I}[h(x) \neq h_j(x) \wedge x \in D_j] \right]
\leq b_j^{NOWL}(h)
\]

where the first inequality comes from Assumption\textsuperscript{2.1} the second inequality comes from Lemma\textsuperscript{10.2} and the last inequality comes directly from optimization constrains in OP\textsuperscript{9}.

By choosing the $h = h$ or $h^*$ and combine these two, we have

\[
\mathbb{E} \left[ \left( \frac{2}{p_j(x)} \mathbb{I}[y^{WL} \neq y] + 1 \right) \mathbb{I}[x \in D_j] + \mathbb{I}[x \notin D_j] \right] \mathbb{I}[h(x) \neq h^*(x)] \leq 2b_j^{NOWL}(h) + \mathbb{I}[x \notin D_j] \mathbb{I}[h(x) \neq h^*(x)],
\]

which is our desired results by replacing our definition of $b_j^{NOWL}$. Then all the following proofs are as the same as the original proofs and we will skip here. In the end we have

\[
|\text{reg}(f, f', Z_m) - \tilde{\text{reg}}_m(f, f')| \\
\leq \frac{1}{4} \tilde{\text{reg}}_m(h) + 2\alpha^2 \epsilon_m + 2\alpha \sqrt{\frac{\epsilon_m}{\tau_m} \sum_{j=1}^{m} (\tau_j - \tau_{j-1}) \text{reg}_j(h) + 2\alpha \sqrt{3\text{err}_m(h^*)\epsilon_m}} \\
+ \beta \sqrt{2\gamma \epsilon_m \Delta_m \sum_{j=1}^{m} (\tau_j - \tau_{j-1})(\text{reg}_j(h, \hat{Z}_{j-1}) + \text{reg}(h^*, \hat{Z}_{j-1}) + \Delta_m + \frac{2\epsilon_m}{\tau_m}} \\
\leq \frac{1}{4} \tilde{\text{reg}}_m(h) + 2\alpha \sqrt{\frac{\epsilon_m}{\tau_m} \sum_{j=1}^{m} (\tau_j - \tau_{j-1}) \text{reg}_j(h) + 2\alpha \sqrt{3\text{err}_m(h^*)\epsilon_m}} \\
+ \beta \sqrt{2\gamma \epsilon_m \Delta_m \sum_{j=1}^{m} (\tau_j - \tau_{j-1})(\text{reg}_j(h, \hat{Z}_{j-1}) + \text{reg}(h^*, \hat{Z}_{j-1}) + 5\Delta_m}
\]

Proposition 11.2 (Error concentration – modified prop 2). Fix an epoch $m \geq 1$, suppose the events $\mathcal{E}_{WL}$ and $\mathcal{E}_{main}$ holds and assume $h^* \in A_j$ for all epoch $j \leq m$,

\[
|\text{err}_m(h^*) - \text{err}(h_{m+1}, \hat{Z}_m)| \leq \frac{5}{2} + 4 \log \tau_m \Delta_m + \frac{\text{err}_m(h^*)}{2} + \frac{\text{err}(h_{m+1}, \hat{Z}_m)}{2} + \text{reg}\left(h^*, h_{m+1}, \hat{Z}_m\right).
\]

To compare, the original bound without WL is

\[
|\text{err}_m(h^*) - \text{err}(h_{m+1}, \hat{Z}_m)| \leq \frac{\text{err}_m(h^*)}{2} + \frac{3\Delta_m}{2} + \text{reg}\left(h^*, h_{m+1}, \hat{Z}_m\right).
\]

Later we will show that $\text{err}(h_{m+1}, \hat{Z}_m)$ in our result can be combined with other terms. So our upper bound is slightly enlarged compared to previous one but still in the same order.
Proof. Again we follow the similar proof steps as in the Proof of Proposition. Thus, based on Lemma 11.1 the key step is to show the following upper bounds

\[
\frac{\epsilon_m}{\tau_m} \left( \sum_{j \in M_m, WL} (\tau_j - \tau_{j-1}) \text{errVar}_{WL} (h, j) + \sum_{j \in M_m, NOWL} (\tau_j - \tau_{j-1}) \text{errVar}_{NOWL} (h, j) \right) + \frac{\epsilon_m}{P_{\min, m}}
\]

First of all, by using the same approach as in the original proofs, we can easily upper bound the NOWL term

\[
\frac{\epsilon_m}{\tau_m} \left( \sum_{j \in M_m, NOWL} (\tau_j - \tau_{j-1}) \text{errVar}_{NOWL} (h, j) \right) \leq \frac{\epsilon_m}{P_{\min, m}} \sum_{j \in M_m, WL} (\tau_j - \tau_{j-1}) \text{errVar}_{WL} (h, j)
\]

Now we will focus on upper bound the WL term as follows,

\ [
\frac{\epsilon_m}{\tau_m} \sum_{j \in M_m, WL} (\tau_j - \tau_{j-1}) \text{errVar}_{WL} (h, j)
\]

\[
\leq \frac{\epsilon_m}{\tau_m} \sum_{j \in M_m, WL} (\tau_j - \tau_{j-1}) \mathbb{E} \left[ \frac{1}{P_{\min, m}} \left( I \left( x \in D_j \land y_{WL} \neq y \right) \right) \right]
\]

\[
\leq \frac{\epsilon_m}{\tau_m} \sum_{j \in M_m, WL} (\tau_j - \tau_{j-1}) \frac{\text{err} (h_j, \hat{Z}_{j-1}) + \Delta_{j-1}}{P_{\min, m}}
\]

\[
\leq \frac{\epsilon_m}{\tau_m} \sum_{j \in M_m, WL} \frac{\tau_j - \tau_{j-1}}{\tau_{j-1}} \frac{\text{err} (h_{m+1}, \hat{Z}_m)}{P_{\min, m}} + \frac{\epsilon_m}{\tau_m} \sum_{j \in M_m, WL} (\tau_j - \tau_{j-1}) \frac{\Delta_{j-1}}{P_{\min, m}}
\]

\[
\leq \frac{\epsilon_m}{P_{\min, m}} \sum_{j \in M_m, WL} (\tau_j - \tau_{j-1}) \frac{\text{err} (h_{m+1}, \hat{Z}_m)}{P_{\min, m}} + \frac{\epsilon_m}{\tau_m} \sum_{j \in M_m, WL} (\tau_j - \tau_{j-1}) \frac{\Delta_{j-1}}{P_{\min, m}}
\]

where the second inequality comes from the optimism of $\text{WLerr}_j$ when $\mathcal{E}_{WL}$ holds and the condition $\text{WLerr}_k \leq \text{errVar}_{WL} (h, j)$ in Step 2 of Alg. 3 the third inequality comes from Lemma 11.2 and the last inequality comes from Lemma 8 in the original paper. Combine these two terms, we have the target bound

\[
|\text{err} (h^*, Z_m) - \text{err}_{m} (h^*)| \leq \frac{\epsilon_m \text{err}_{m} (h^*)}{P_{\min, m}} + 2 \log \tau_m \frac{\epsilon_m \text{err} (h_{m+1}, Z_m)}{P_{\min, m}} + 2 \log \tau_m \frac{\epsilon_m \Delta_{m}}{P_{\min, m}} + \frac{\epsilon_m}{P_{\min, m}}
\]

\[
\leq \left( \frac{3}{2} + 4 \log \tau_m \right) \frac{\epsilon_m}{P_{\min, m}} + \frac{\epsilon_m \text{err}_{m} (h^*)}{2} + \frac{\text{err} (h_{m+1}, \hat{Z}_m)}{2} + \Delta_{m}
\]

Therefore, by using the same observation as in the original paper, we have

\[
\left| \text{err}_{m} (h^*) - \text{err} \left( h_{m+1}, \hat{Z}_m \right) \right| \leq \left| \text{err} (h^*, Z_m) - \text{err}_{m} (h^*) \right| + \text{reg} (h^*, h_{m+1}, \hat{Z}_m)
\]

\[
\leq \text{target bound}
\]

Corollary 11.1. Based on proposition 11.2, we can get the following error estimation guarantees,

\[
\text{err}_{m} (h^*) \leq 5 + 8 \log \tau_m \Delta_{m} + 3 \text{err} (h_{m+1}, \hat{Z}_m) + 2 \text{reg} (h^*, h_{m+1}, \hat{Z}_m)
\]

\[
\text{err} (h_{m+1}, \hat{Z}_m) \leq 5 + 8 \log \tau_m \Delta_{m} + 3 \text{err}_{m} (h^*) + 2 \text{reg} (h^*, h_{m+1}, \hat{Z}_m)
\]
11.4 Main results and its analysis

Based on these two propositions, we are ready to prove this general version of the theorem.

**Theorem 11.1.** For all epochs $m = 1, 2, \ldots, M$ and all $h \in \mathcal{H}$, the following holds with probability at least $1 - 2\delta$,

\[
\left| \text{reg}(h, h^*, \bar{Z}_m) - \tilde{\text{reg}}_m(h, h^*) \right| \leq \frac{1}{2} \tilde{\text{reg}}_m(h, h^*) + \frac{\eta}{4} \Delta_m
\]

\[
\text{reg}(h^*, h_{m+1}, \bar{Z}_m) \leq \frac{\eta \Delta_m}{4} \text{ and } h^* \in A_m
\]

\[
\text{err}(h_{m+1}, \bar{Z}_m) \leq (5 + 8 \log \sum_{j=1}^{m} L_j + \frac{\eta}{2}) \Delta_m + 3 \tilde{\text{err}}_m(h^*)
\]

\[
\tilde{\text{err}}_m(h^*) \leq (5 + 8 \log \sum_{j=1}^{m} L_j + \frac{\eta}{2}) \Delta_m + 3 \text{err}(f_{m+1}, \bar{Z}_m)
\]

**Proof.** First of all, assume $\mathcal{E}_{\text{WL}}$ and $\mathcal{E}_{\text{train}}$ holds, so all the previous results holds.

Now this theorem is proved inductively and follows the similar step as in Section 7.2.1 in the original proof. Firstly because our proposition [11.1] gives the same result as in original proof, so we have for any block $m > 1$,

\[
\left| \text{reg}(h, h^*, Z_m) - \tilde{\text{reg}}_m(h, h^*) \right| \\
\leq \frac{1}{4} \tilde{\text{reg}}_m(f) + 2\alpha \sqrt{\frac{\epsilon_m \sum_{j=1}^{m} (\tau_j - \tau_{j-1}) \text{reg}_j(f_j) + 2\alpha \sqrt{3\tilde{\text{err}}_m(h^*)\epsilon_m}}{\tau_1}} \\
+ \beta \sqrt{2\gamma \epsilon_m \Delta_m \sum_{j=1}^{m} (\tau_j - \tau_{j-1}) \text{reg}(f, \bar{Z}_j - 1 + \text{reg}(h^*, \bar{Z}_j) + 5\Delta_m, \tau_3)}
\]

Now we are ready to upper bound these three terms. In order to prove inductively, we use $\mathcal{E}_{\text{train}, m}$ to state that the upper bounds in theorem hold for block $m$. We can easily bound $\mathcal{T}_1$ and $\mathcal{T}_3$ using the fact $\mathcal{E}_{\text{train}, m-1}$ holds, the analysis for this part is exactly the same as original proof so we will skip the details and directly stated the result.

\[
\mathcal{T}_1 \leq \frac{\eta \Delta_m}{12} + 24 \alpha^2 \epsilon_m \log \tau_m
\]

\[
\mathcal{T}_3 \leq \frac{1}{4} \tilde{\text{reg}}_m(h, h^*) + \frac{7\eta \Delta_m}{72}
\]

So here our main focus is to show the upper bound of $\mathcal{T}_2$ which is effected by our modified algorithm. By applying the first inequality in Corollary [11.1], we get simplify $\mathcal{T}_2$ as

\[
\mathcal{T}_2 = 2\alpha \sqrt{3\epsilon_m \tilde{\text{err}}_m(h^*)} \leq 2\alpha \sqrt{3\epsilon_m \left( (5 + 8 \log \tau_m) \Delta_m + 3 \text{err}(h_{m+1}, \bar{Z}_m) + 2 \text{reg}(h^*, h_{m+1}, \bar{Z}_m) \right)}
\]

\[
\leq 2\alpha \sqrt{15\epsilon_m \text{err}(h_{m+1}, \bar{Z}_m) + 2\alpha \sqrt{15 + 24 \log \tau_m} \epsilon_m \Delta_m + 2\alpha \sqrt{6\epsilon_m \text{reg}(h^*, h_{m+1}, \bar{Z}_m)}
\]

\[
\leq 2\alpha \sqrt{15\epsilon_m \text{err}(h_{m+1}, \bar{Z}_m) + \Delta_m + (30 + 48 \log \tau_m) \alpha^2 \epsilon_m + \frac{1}{4} \text{reg}(h^*, h_{m+1}, \bar{Z}_m)}
\]
Therefore combine the upper bound of $\mathcal{T}_2$ with bounds of $\mathcal{T}_1, \mathcal{T}_3$, we have

\[
|\operatorname{reg}(h, h^*, Z_m) - \tilde{\operatorname{reg}}_m(h, h^*)| \\
\leq \frac{1}{2} \tilde{\operatorname{reg}}_m(h) + \frac{\eta \Delta_m}{12} + 24 \alpha^2 \epsilon_m \log \tau_m \\
+ 2 \alpha \sqrt{9 \epsilon_m \operatorname{err}(h_{m+1}, \bar{Z}_m) + \Delta_m} + \frac{1}{4} \tilde{\operatorname{reg}}(h^*, h_{m+1}, \bar{Z}_m) + (30 + 48 \log \tau_m) \alpha^2 \epsilon_m \\
+ \frac{1}{4} \tilde{\operatorname{reg}}_m(h, h^*) + \frac{7 \eta \Delta_m}{72} + 5 \Delta_m \\
\leq \frac{1}{2} \tilde{\operatorname{reg}}_m(f) + 102 \alpha^2 \epsilon_m \log \tau_m + \frac{13 \eta \Delta_m}{72} + 2 \alpha \sqrt{9 \epsilon_m \operatorname{err}(h_{m+1}, \bar{Z}_m) + 6 \Delta_m} + \frac{1}{4} \tilde{\operatorname{reg}}(h^*, h_{m+1}, \bar{Z}_m)
\]

Further recalling that $c_1 \geq 6 \alpha$ and $c_2 \geq 102 \alpha^2$ by our assumptions on constants, we obtain

\[
|\operatorname{reg}(h, h^*, Z_m) - \tilde{\operatorname{reg}}_m(h, h^*)| \\
\leq \frac{1}{2} \tilde{\operatorname{reg}}_m(h) + \frac{13 \eta \Delta_m}{72} + 7 \Delta_m + \frac{1}{4} \tilde{\operatorname{reg}}(h^*, h_{m+1}, \bar{Z}_m)
\]

To complete the proof of the bound (36), we now substitute $h = h_{m+1}$ in the above bound, which yields

\[
\frac{1}{2} \tilde{\operatorname{reg}}_m(h_{m+1}, h^*) - \frac{5}{4} \tilde{\operatorname{reg}}(h, h^*, \bar{Z}_m) \leq \frac{13 \eta}{72} \Delta_m + 7 \Delta_m
\]

Since $h^* \in A_i$ for all epochs $i \leq m$, we have $\tilde{\operatorname{reg}}(h, h^*) \geq \operatorname{reg}(h, h^*) \geq 0$ for all classifiers $h \in \mathcal{H}$. Consequently, we see that

\[
\operatorname{reg}(h^*, h_{m+1}, \bar{Z}_m) = -\operatorname{reg}(h_{m+1}, h^*, \bar{Z}_m) \leq \frac{52 \eta}{360} \Delta_m + \frac{28}{5} \Delta_m \leq \frac{\eta}{4} \Delta_m
\]

Finally, from Corollary [11] and the previous upper bound result on $\operatorname{reg}(h^*, h_{m+1}, \bar{Z}_m)$, we can get the third and the forth inequality. That is,

\[
\operatorname{err}_m(h^*) \leq (5 + 8 \log \tau_m) \Delta_m + 3 \operatorname{err}(h_{m+1}, \bar{Z}_m) + 2 \operatorname{reg}(h^*, h_{m+1}, \bar{Z}_m) \\
\leq (5 + 8 \log \tau_m) \Delta_m + 3 \operatorname{err}(h_{m+1}, \bar{Z}_m) + \frac{\eta \Delta_m}{2} \\
\operatorname{err}(h_{m+1}, \bar{Z}_m) \leq (5 + 8 \log \tau_m) \Delta_m + 3 \operatorname{err}_m(h^*) + 2 \operatorname{reg}(h^*, h_{m+1}, \bar{Z}_m) \\
\leq (5 + 8 \log \tau_m) \Delta_m + 3 \operatorname{err}_m(h^*) + \frac{\eta \Delta_m}{2}
\]

Recall that we simplify the notation in proof and the $\tau_m$ is actually $\sum_{j=1}^m L_j$. So we get the target bound.

Now we proof the Theorem 4.1 as a direct follow-up.

**Theorem 11.2** (Main theorem, Restate). Pick any $0 < \delta < 1/e$ such that $|\mathcal{H}|/\delta > \sqrt{192}$. Then recalling that $h^* = \arg \min_{h \in \mathcal{H}} \operatorname{err}(h)$, we have for all epochs $m = 1, 2, \ldots, M$, with probability at least $1 - \delta$

\[
\operatorname{reg}(h, h^*) \leq O(\Delta_m^*) \quad \text{for all } h \in A_{m+1}
\]

where $\Delta_m^* = \Delta_0$ and $\Delta_m^* := c_1 \sqrt{\epsilon_m \operatorname{err}_m(h^*)} + c_2 \epsilon_m \log(\sum_{j=1}^m L_j)$ for $m \geq 1$

**Proof.** By using the exact same proof in the original paper we have easily get

\[
\operatorname{reg}(h) \leq 4 \gamma \Delta_m.
\]
We skip the proof here. Now we show our modified version can again leads to \( \Delta_m \leq 4\Delta^*_m \). It is trivial true for \( m = 1 \) because \( \Delta^*_1 = \Delta_1 \). For \( m \geq 2 \), we have

\[
\Delta_m \leq c_1 \sqrt{\epsilon_m \left( \text{err}(h_{m+1}, \tilde{Z}_m) \right)} + c_2 \epsilon_m \log \tau_m \\
\leq c_1 \sqrt{\epsilon_m \left( 5 + 8 \log \tau_m \right) \Delta_m + 3 \text{err}_m(h^*) + 2 \text{reg} \left( h^*, h_{m+1}, \tilde{Z}_m \right)} + c_2 \epsilon_m \log \tau_m \\
\leq c_1 \sqrt{\epsilon_m \left( \frac{\eta}{2} + 13 \log \tau_m \right) \Delta_m + 3 \text{err}_m(h^*) + 2 \text{reg} \left( h^*, h_{m+1}, \tilde{Z}_m \right)} + c_2 \epsilon_m \log \tau_m \\
\leq c_1 \sqrt{3 \text{err}_m(h^*)} + c_1 \left( \frac{\eta}{2} + 13 \log \tau_m \right) \Delta_m + c_2 \epsilon_m \log \tau_m \\
\leq c_1 \sqrt{3 \text{err}_m(h^*)} + c_2 \epsilon_m \left( \frac{\eta}{2} + 13 \log \tau_m \right) + \frac{\Delta_m}{2} + c_2 \epsilon_m \log \tau_m \\
\leq 2\Delta^*_m + \frac{\Delta_m}{2}
\]

where the last inequality uses our choice of constants \( c_1 \left( \frac{\eta}{2} + 13 \right) \leq c_2 \). Rearrange terms and again recall that we simplify the notation in proof and the \( \tau_m \) is actually \( \sum_{j=1}^m L_j \), so we complete the proof.  

\[
12.1 \text{ Auxiliary lemma}
\]

**Lemma 11.2.** For any fixed block number \( m \), we have

\[
\sum_{i=1}^m (\tau_i - \tau_{i-1}) \text{err}(h_i, \tilde{Z}_{i-1}) \leq 4\tau_m \log \tau_{m+1} \text{err}(h_{m+1}, \tilde{Z}_m)
\]

**Proof.** It is easy to see that, for any \( m \geq i \)

\[
\tau_{i-1} \text{err}(h_i, \tilde{Z}_{i-1}) \leq \tau_i \text{err}(h_{i+1}, \tilde{Z}_{i-1}) \\
\leq \tau_i \text{err}(h_{i+1}, \tilde{Z}_i) \\
\leq \ldots \leq \tau_{m-1} \text{err}(f_m, \tilde{Z}_{m-1})
\]

Now by the fact \( \sum_{i=1}^m \frac{\tau_{i+1} - \tau_i}{\tau_i} \leq 4 \log \tau_{m+1} \) in [Huang et al. 2015], Lemma 8, we have

\[
\sum_{i=1}^m (\tau_i - \tau_{i-1}) \text{err}(h_i, \tilde{Z}_{i-1}) = \sum_{i=1}^m \frac{(\tau_i - \tau_{i-1})}{\tau_{i-1}} \tau_{i-1} \text{err}(h_i, \tilde{Z}_{i-1}) \\
\leq \sum_{i=1}^m \frac{(\tau_i - \tau_{i-1})}{\tau_{i-1}} \tau_m \text{err}(h_{m+1}, \tilde{Z}_m) \\
\leq 4\tau_m \log \tau_{m+1} \text{err}(h_{m+1}, \tilde{Z}_m) 
\]

\[
\]

12 Appendix: Label complexity and its analysis

12.1 Analysis for OP²

The label complexity in Phase 1 and Phase 2 in block \( m \) can be explicitly written as follows.

**Lemma 12.1.** When \( \mathcal{E}_{WL} \) holds, for any fixed block \( m \) with \( \text{USE-WL} = True \),

\[
\mathbb{E} \left[ \sum_{t=\tau_{m-1}+1}^{\tau_m} \mathbb{1}[y_t \text{ is queried}] \right] \leq (\tau_m - \tau_{m-1}) \mathbb{E}[x \in D_m] \\
\mathbb{E} \left[ \sum_{t=\tau_m}^{\tau_{m+1}} \mathbb{1}[y_t \text{ is queried}] \right] \leq \max \{ 4N_m \text{WLerr}_m, L_m P_{m, \text{min}} \mathbb{E}[\mathbb{1}[x \in D_m]] \}.
\]


Proof. The first one comes from the fact that we only query samples inside \( D_m \) in the Phase 1. For the second result, notice that by choosing \( P_m(x) = 4P_m(x)WL_{err_m} \), we get a feasible solution of \( \text{OPT} \) without considering the \( P_{m,\text{min}} \), as shown below

\[
\mathbb{E}_x \left[ \frac{\mathbb{1}(h(x) \neq h_m(x) \land x \in D_m)}{P_m(x)} \right] WL_{err_m} = \mathbb{E}_x \left[ \frac{\mathbb{1}(h(x) \neq h_m(x) \land x \in D_m)}{4P_m(x)WL_{err_m}} \right] WL_{err_m} = \frac{1}{4} \mathbb{E}_x \left[ \frac{\mathbb{1}(h(x) \neq h_m(x) \land x \in D_m)}{P_m(x)} \right] \leq \frac{1}{4} \eta_{m,\text{NLW}}(h) \leq \frac{1}{2} (\eta_{m,\text{NLW}}(h) - \mathbb{E}_x[\mathbb{1}_h(x)])
\]

Now combine this with the minimum query probability requirement, we get the desired result. \( \square \)

12.2 Analysis for the unlabeled sample complexity

First we show that, the number of unlabeled sample we draw from Phase 1 is at most the same order as the number in Phase 2, which means out results in terms of \( n \) remains in the same order as before.

Lemma 12.2. For any fixed block \( m \), \( \tau_m - \tau_{m-1} \leq 8L_m \).

Proof. According the stopping condition of the algorithm, at the end of the block \( m \), we have the number of new drawn unlabeled samples as \( 2^{K^+_{m,1}} \ln(2M \log(n) / \delta) \). When \( K^+_m = K^*_m \), then we directly have

\[
2^{K^+_{m,1}} \ln(2M \log(n) / \delta) \leq \frac{12 \times 2 \ln(2M \log(n) / \delta)}{\mathbb{E}[\mathbb{1}[x \in D_m]]} \leq \frac{N_m}{\mathbb{E}[\mathbb{1}[x \in D_m]]} \leq \frac{L_m \mathbb{E}[\mathbb{1}[x \in D_m]]}{\mathbb{E}[\mathbb{1}[x \in D_m]]} = L_m
\]

When \( K^+_m > K^*_m \),

\[
2^{K^+_{m,1}} \ln(2M \log(n) / \delta) = 2 \times 2^{(K^*_m - 1) + 1} \ln(2M \log(n) / \delta) \\
\leq 2 \times \frac{\max\{4N_m, WL_{\text{err}_{m,1}}, L_m P_{m,\text{min}} \mathbb{E}[\mathbb{1}[x \in D_m]]\}}{\mathbb{E}[\mathbb{1}[x \in D_m]]} \\
\leq 2 \times \frac{\max\{4L_m, \mathbb{E}[\mathbb{1}[x \in D_m]], L_m P_{m,\text{min}} \mathbb{E}[\mathbb{1}[x \in D_m]]\}}{\mathbb{E}[\mathbb{1}[x \in D_m]]} \\
\leq 8L_m
\]

Therefore, roughly speaking if original AC requires \( n \) number of unlabeled samples to achieve their regret guarantee, then here we at most need \( \text{const} \times n = O(n) \) unlabeled sample to achieve the similar guarantees. This is weak requirement since unlabeled samples sources are usually unlimited.

12.3 Analysis for Algo 3 on the label complexity

In the rest of this section, we show Algo 3 can automatically balance Phase 1 and Phase 2.

Lemma 12.3. When \( \mathcal{E}_{WL} \) hold, then for any fixed block \( m \), if \( K^+_m \geq K^*_m + 1 \),

\[
K_m \leq \max\{\min\{T_1, T_2\}, T_3\}
\]
On the other hand, when $U$

Finally we focus on the Step 3. By the stopping condition and Lemma 12.1, we always have

Now in the case that $K_m = K^*_m$, we directly have the upper bound

Otherwise, by replace the upper bound of $K_m^c$ from Lemma 12.3 we finish the proof.

Now before we going to the final proof, we need to upper bound the number of WL mode oracle in the next section.
12.4 Analysis on number of hint mode blocks

For each block $m$, there are two conditions that will lead to NO HINT mode,

- **Condition 1**: $N_m \leq \left(6 \frac{\kappa_m}{\eta_m(h_m, \tilde{Z}_{m-1})}\right) \mathbb{E}[\mathbb{I}[x \in D_m]]$. This suggests the biased estimated error for best hypothesis is too small that, even evaluating whether the hint performs better than this can cost too much samples in Phase 1.

- **Condition 2**: $WLerr_{m,K^m} \geq \text{err}(h_m, \tilde{Z}_{m-1})$. This suggests the estimated hint performance is worse than the biased estimated error for best hypothesis, therefore may deteriorate the active learning strategy.

Firstly, we will show later in the proof of main theorem that the occurrence of Condition 1 is not important because it can be implicitly upper bounded in the end. So here we will only focus on condition 2.

**Lemma 12.5.** For any fixed block $m$, as long as $\mathbb{E}[y_{WL} \neq y | x \in D_m] \leq \frac{1}{8} \text{err}(h_m, \tilde{Z}_{m-1})$, we have $WLerr_{m,K^m} \leq \frac{1}{2} \text{err}(h_m, \tilde{Z}_{m-1})$.

**Proof.** When $\mathcal{E}_{WL}$ holds, we have

$$WLerr_{m,K^m} \leq 4\mathbb{E}[y_{WL} \neq y \land x \in D_m] + 3 \cdot 2^{-K^m}$$

$$\leq \frac{1}{2} \text{err}(h_m, \tilde{Z}_{m-1}) + \frac{1}{2} \text{err}(h_m, \tilde{Z}_{m-1})$$

$$= \text{err}(h_m, \tilde{Z}_{m-1})$$

where the first term of last inequality comes from the assumption and second term of the last inequality comes from our choice of $K^m$ in the algorithm.

12.5 Main results in Section 4.2 and the analysis

**Theorem 12.1** (label complexity of block $m$ (Restate)). Let

$$\mathcal{T}_{1,m} = N_m \kappa_m WLerr_m$$

$$\mathcal{T}_{2,m} = \sqrt{\ln(2M \log(n)/\delta)\kappa_m N_m}$$

$$+ \sqrt{\frac{L_m}{\tau_{m-1}} \log(|\mathcal{H}|n/\delta) L_m \left(\log \sum_{j=1}^{m} L_j\right) \mathbb{E}[\mathbb{I}[x \in D_m]] \phi_m}$$

$$+ \ln(2M \log(n)/\delta) \left(\log \sum_{j=1}^{m} L_j\right) \phi_m$$

$$\mathcal{T}_{3,m} = 1 \left[WLerr_m > \frac{1}{\left(\log \sum_{j=1}^{m} L_j\right) \phi_m}\right] N_m$$

where $WLerr_m = WLerr(D_m)$. So the expected label complexity within block $m$ is upper bound by

$$\mathcal{O}(\min \{\mathcal{T}_{1,m} + \mathcal{T}_{2,m} + \mathcal{T}_{3,m}, N_m\})$$

**Proof.** Therefore, we can decompose the expected sample complexity within block $m$ as

$$\mathbb{E}_{\tau_m} \left[\sum_{t=\tau_{m-1}+1}^{\tau_m} \mathbb{E}_t [\mathbb{I}[y_t \text{ is queried}]]\right]$$

$$= \mathbb{I}[K^m \geq K^{m^*} + 1] \#m + \mathbb{I}[K^m = K^{m^*} \land \text{subalgo stop as step 3}] \#m$$

$$\mathcal{T}_1$$

$$+ \mathbb{I}[[\text{subalgo stop as step 1}] \#m + \mathbb{I}[[\text{subalgo stop as step 2}]] \#m$$

$$\mathcal{T}_3$$

$$+ \mathbb{I}[[\text{subalgo stop as step 2}]] \#m$$

$$\mathcal{T}_4$$
Now we are ready to separately bound these three terms by using Lemma 12.4.

For $\mathcal{T}_{1,m}$, we have,

$$\mathcal{T}_1 \leq \mathbb{1}[K^c_m \geq K^*_m + 1]$$

* $\max \left\{ \min \left\{ 32\kappa_m N_m \text{WLerr}_m + \sqrt{24N_m\kappa_m \ln(2M \log(n)/\delta)}, N_m \right\}, L_m P_{m,\min} \mathbb{E}[\mathbb{1}[x \in D_m]] \right\}$

$$\leq N_m \min \{ 32\kappa_m \text{WLerr}_m, 1 \} + 5 \sqrt{\ln(2M \log(n)/\delta)} \kappa_m N_m + P_{m,\min} L_m \mathbb{E}[\mathbb{1}[x \in D_m]]$$

By the definition of $P_{m,\min}$, we can further upper bound the third term as

$$P_{m,\min} L_m \mathbb{E}[\mathbb{1}[x \in D_m]] \leq \frac{L_m}{\sum_{j=1}^{m-1} L_j} \sum_{j=1}^{m-1} L_j \sqrt{\frac{c_3}{n\kappa_M \sum_{j=1}^{m-1} L_j \text{err}(f_m, \hat{Z}_{m-1})}} \mathbb{E}[\mathbb{1}[x \in D_m]]$$

$$= c_3 \frac{L_m}{\sum_{j=1}^{m-1} L_j} \sqrt{\frac{n\kappa_M \sum_{j=1}^{m-1} L_j \text{err}(f_m, \hat{Z}_{m-1})}{\text{err}(f_m, \hat{Z}_{m-1})}} \mathbb{E}[\mathbb{1}[x \in D_m]]$$

$$\approx c_3 \frac{L_m}{\sum_{j=1}^{m-1} L_j} \sqrt{\frac{\tau M \sum_{j=1}^{m-1} L_j}{\text{err}(f_m, \hat{Z}_{m-1})}} \mathbb{E}[\mathbb{1}[x \in D_m]]$$

$$\approx \frac{L_m}{\sum_{j=1}^{m-1} L_j} \sqrt{\frac{\sum_{j=1}^{m-1} L_j \log(|\mathcal{H}|n/\delta)}{\text{err}(f_m, \hat{Z}_{m-1})}} \mathbb{E}[\mathbb{1}[x \in D_m]]$$

$$\approx \frac{L_m}{\sum_{j=1}^{m-1} L_j} \sqrt{\frac{\sum_{j=1}^{m} L_j \log(|\mathcal{H}|n/\delta)}{\text{err}(f_m, \hat{Z}_{m-1})}} \mathbb{E}[\mathbb{1}[x \in D_m]]$$

where the first inequality comes from Lemma 12.7 and the last inequality comes from Lemma 12.6.

For $\mathcal{T}_{2,m}$ and $\mathcal{T}_{3,m}$, we can combine them as

$$\mathcal{T}_2 + \mathcal{T}_3 \leq \min \left\{ \mathbb{E}[x \in D_m] \left[ \frac{12 \ln(2M \log(n)/\delta)}{\text{err}(h_m, \hat{Z}_{m-1})} \right], N_m \right\}$$

$$= \min \left\{ \frac{12 \ln(2M \log(n)/\delta)}{\text{log}(\sum_{j=1}^{m} L_j) \phi_m), N_m \right\}$$

Again the last inequality comes from Lemma 12.6.

Finally for $\mathcal{T}_{4,m}$, by using Lemma 12.5 we have

$$\mathcal{T}_4 \leq \mathbb{1}[\text{WLerr}_{K^*_m} > \text{err}(h_m, \hat{Z}_{m-1})] N_m$$

$$\leq \mathbb{1} \left[ \mathbb{E}[\mathbb{1}[h(x) \neq y \wedge x \in D_m]] > \text{err}(h_m, \hat{Z}_{m-1}) \right] N_m$$

$$\leq \mathbb{1} \left[ \text{WLerr}_m \mathbb{E}[\mathbb{1}[x \in D_m]] > \text{err}(h_m, \hat{Z}_{m-1}) + \Delta_m \right] N_m$$

$$= \mathbb{1} \left[ \text{WLerr}_m > \frac{1}{\log(\sum_{j=1}^{m} L_j) \phi_m} \right] N_m$$
Theorem 12.2 (Upper bound of $\phi_m$). For any block $m$, with probability at least $1 - \delta$, we have always upper bound $\phi_m \leq \min\{\theta^*, \frac{\Theta_m(h^*)}{\epsilon_m} + \log(\sum_{j=1}^m L_j) \epsilon_m\}.$

Proof. When $\mathcal{E}_{\text{train}}$ holds, we can upper bound the $\mathbb{E}[1 | x \in D_m]$ as

$$
\mathbb{E}[1 | x \in D_m] \leq \theta^* \left(16 \gamma \Delta_{m-1}^* + 2 \epsilon_m(h^*)\right)
\leq \theta^* \left(16 \gamma \Delta_{m-1}^* + 2 \epsilon_m(h^*)\right)
= \theta^* \left(16 \gamma \left(c_1 \sqrt{\epsilon_{m-1}(h^*)} + c_2 \epsilon_m \log \left(\sum_{j=1}^m L_j \right)\right) + 2 \epsilon_m(h^*)\right)
\leq \theta^* \left(16 \gamma (2c_1 + c_2 \log \left(\sum_{j=1}^m L_j \right)) \epsilon_m + 8 \gamma \epsilon_m + 2 \epsilon_m(h^*)\right)
$$

where the first inequality comes from eqn.(62) in the original paper. Therefore, by definition of $\phi_m$ we directly have

$$
\mathbb{E}[1 | x \in D_m] \leq O(\theta^*)
$$

Alternatively, we can upper bound it as

$$
\mathbb{E}[1 | x \in D_m] \leq \frac{1}{\epsilon_m(h^*) + \log(\sum_{j=1}^m L_j) \epsilon_m}
$$

\hfill \Box

12.6 Main results for Section 4.3 and its analysis

Theorem 12.3 (Total label complexity under benign setting). Suppose we end up dividing $n$ incoming labels into $M$ block, then with probability at least $1 - \delta$, we have the total label complexity upper bounded by

$$
\min \left\{ T_1^{\text{total}}, T_2^{\text{total}}, T_3^{\text{total}}, N_{\text{total}} \right\},
$$

where,

$$
T_1^{\text{total}} = \sum_{m=1}^M N_m \max \{ \kappa_m W\mathcal{L}_{err_m}, 1 \}
$$

$$
T_2^{\text{total}} = \ln(2M \log(n)/\delta) \left(\sum_{m=1}^M \max \{ \kappa_m, \frac{N_m}{\ln(2M \log(n)/\delta)} \} N_{\text{total}} \right)
+ \ln(|\mathcal{H}|n/\delta) \left(\log \left(\sum_{j=1}^m L_j \right) \sum_{m=1}^M L_m \mathbb{E}[1 | x \in D_m] \sum_{m=1}^M \frac{L_m}{\sum_{j=1}^{m-1} L_j} \phi_m \right)
+ \sum_{m=1}^M \ln(2M \log(n)/\delta) \left(\log \left(\sum_{j=1}^m L_j \right) \phi_m \right)
$$

$$
T_3^{\text{total}} = \sum_{m=1}^M \mathbb{I} \left[ W\mathcal{L}_{err_m} > \frac{1}{\left(\log \sum_{j=1}^m L_j \right) \phi_m} \right] N_m
$$

Proof. First of all, by applying Lemma 3 of \cite{kakade2008}, we get that with probability at least $1 - \delta$

$$
\forall n \geq 3, \sum_{i=1}^n \mathbb{I}[y_i \text{is queried}] \leq 2 \sum_{i=1}^n \mathbb{E}_i \left[ \mathbb{I}[y_i \text{is queried}] \right] + 4 \log(4 \log(n)/\delta).
$$
Therefore, to get the actual label complexity w.h.p, it is enough to upper bound the expected one. By summing over $T_{2,m}$ over all blocks and applying Cauchy-swartz, we have

$$
\sum_{m=1}^{M} \sqrt{\ln(2M \log(n)/\delta) \max\{\kappa_m, \frac{N_m}{\ln(2M \log(n)/\delta)}\}} N_m
$$

$$
+ \sum_{m=1}^{M} \left[ \log(|\mathcal{H}|/\delta) \left( \log \sum_{j=1}^{m} L_j \right) \left( L_m \mathbb{E}[\mathbb{I}[x \in D_m]] \phi_m \right) \phi_m \right]
$$

$$
+ \sum_{m=1}^{M} \ln(2M \log(n)/\delta) \left( \log \sum_{j=1}^{m} L_j \right) \phi_m
$$

$$
\leq \sqrt{\ln(2M \log(n)/\delta) \left( \sum_{m=1}^{M} \max\{\kappa_m, \frac{N_m}{\ln(2M \log(n)/\delta)}\} \right)} \sum_{m=1}^{M} N_m
$$

$$
+ \log(|\mathcal{H}|/\delta) \left( \log \sum_{j=1}^{m} L_j \right) \sum_{m=1}^{M} L_m \mathbb{E}[\mathbb{I}[x \in D_m]] \sum_{m=1}^{M} \frac{L_m}{\sum_{j=1}^{m-1} L_j} \phi_m
$$

$$
+ \sum_{m=1}^{M} \ln(2M \log(n)/\delta) \left( \log \sum_{j=1}^{m} L_j \right) \phi_m
$$

\[\Box\]

**Theorem 12.4** (Worst-case total label complexity under benign setting (A detailed version)). Choose $L_m = 2^m$. Suppose there exists a generalized WL error upper bound $\tilde{\text{WLerr}} \in [0, 1]$ that

$$
\sum_{m=1}^{M} T_{1,m} + T_{3,m} \leq \tilde{\text{WLerr}} \sum_{m=1}^{M} N_m,
$$

Then we have with probability at least $1 - \delta$,

$$
\sum_{t=1}^{n} \mathbb{I}[y_t \text{ is queried}] \leq O\left( \left( \tilde{\theta} \sqrt{\text{err}_M(h^*) n \log(|\mathcal{H}|/\delta)} + \sqrt{\text{err}_M(h^*) n \log(|\mathcal{H}|/\delta)} \right)^2 + \tilde{\theta} \log(|\mathcal{H}|/\delta) \tilde{\theta} + \log(|\mathcal{H}|/\delta) \log(n)^2 \tilde{\theta}^* \right)
$$

$$
+ O\left( \theta^* \text{WLerr} \sqrt{\text{err}_M(h^*) n} + \theta^* \text{WLerr} \sqrt{\text{err}_M(h^*) n \log(|\mathcal{H}|/\delta)} \right)
$$

where $\tilde{\theta} = \theta^* \log(n) + \sqrt{\theta^* \left( \sum_{m=1}^{M} L_m \mathbb{E}[\mathbb{I}[x \in D_m]] \frac{N_m}{\ln(2M \log(n)/\delta)} \right) \frac{\ln(\log(n)^2/\delta)}{\log(|\mathcal{H}|/\delta)}}$.

**Proof.** According to [Huang et al. (2015)], we have

$$
N_{\text{total}} \leq \sum_{L_m} \mathbb{E}[\mathbb{I}[x \in D_m]] \leq \tilde{O}\left( \theta^* \sqrt{\text{err}_M(h^*) n} + \theta^* \sqrt{\text{err}_M(h^*) n \log(|\mathcal{H}|/\delta)} + \log(|\mathcal{H}|/\delta) \right)
$$
Therefore, by the definition of $\tilde{\theta}$, we have

$$\tau_2^{\text{total}} \lesssim \sqrt{\ln(\log(n)^2/\delta) \left( \sum_{m=1}^{M} \max\{\kappa_m, \frac{N_m}{\ln(2M \log(n)/\delta)}\} \right) N_{\text{total}}} + \sqrt{\log(|\mathcal{H}|n/\delta) \log(n) \sum_{L_m} E[\|x \in D_m\|] \sum_{m=1}^{M} \phi_m}$$

+ \sum_{m=1}^{M} \ln(\log(n)^2/\delta) \log(n) \phi_m

\lesssim \sqrt{\theta^* \text{err}_m(h^*)n} + \theta^* \sqrt{\text{err}_m(h^*)n} \log(|\mathcal{H}|/\delta) + \log(|\mathcal{H}|/\delta)

\times \left( \sqrt{\ln(\log(n)^2/\delta) \left( \sum_{m=1}^{M} \max\{\kappa_m, \frac{N_m}{\ln(2M \log(n)/\delta)}\} \right) + \log(|\mathcal{H}|n/\delta) \log(n)^2\theta^*} \right)

+ \log(|\mathcal{H}|n/\delta) \log(n)^2\theta^*

\leq \tilde{\theta} \sqrt{\text{err}_m(h^*)n} + \sqrt{\text{err}_m(h^*)n} \log(|\mathcal{H}|/\delta) + \tilde{\theta} \log(|\mathcal{H}|/\delta) + \log(|\mathcal{H}|n/\delta) \log(n)^2\theta^*

\square

12.7 Auxiliary lemma

Lemma 12.6. When $\mathcal{E}_{\text{WL}}$ and $\mathcal{E}_{\text{main}}$ holds, for any block $m$, we have

$$\frac{1}{\text{err}(h_m, Z_{m-1})} \leq \frac{5 + 8 \log \left( \sum_{j=1}^{m} L_j \right) + \frac{\eta}{2}}{\text{err}_m(h^*) + \log \left( \sum_{j=1}^{m} L_j \right) \epsilon_m} \leq \log(\tau_m) \phi_m$$

which is equivalent to

$$\mathbb{E}[x \in D_m] \leq \frac{5 + 8 \log \left( \sum_{j=1}^{m} L_j \right) + \frac{\eta}{2}}{\text{err}_m(h^*) + \log \left( \sum_{j=1}^{m} L_j \right) \epsilon_m} \mathbb{E}[x \in D_m] \lesssim \log(\tau_m) \phi_m$$

Proof:

$$\text{err}(h_m, Z_{m-1}) + \Delta_{m-1}$$

$$\geq \frac{3}{2(5 + 8 \log \left( \sum_{j=1}^{m} L_j \right) + \frac{\eta}{2})} \text{err}(h_m, Z_{m-1}) + \Delta_{m-1}$$

$$= \frac{1}{2(5 + 8 \log \left( \sum_{j=1}^{m} L_j \right) + \frac{\eta}{2})} \left( 3 \text{err}(h_m, Z_{m-1}) + 2(5 + 8 \log \left( \sum_{j=1}^{m} L_j \right) + \frac{\eta}{2}) \Delta_{m-1} \right)$$

$$\geq \frac{1}{2(5 + 8 \log \left( \sum_{j=1}^{m} L_j \right) + \frac{\eta}{2})} \left( 3 \text{err}(h_m, Z_{m-1}) + (5 + 8 \log \left( \sum_{j=1}^{m} L_j \right) + \frac{\eta}{2}) \Delta_{m-1} \log \left( \sum_{j=1}^{m} L_j \right) \epsilon_{m-1} \right)$$

$$\geq \frac{\text{err}_m(h^*) + \log \left( \sum_{j=1}^{m} L_j \right) \epsilon_m}{2(5 + 8 \log \left( \sum_{j=1}^{m} L_j \right) + \frac{\eta}{2})}$$

where the last inequality comes from Theorem 11.1

Lemma 12.7. For any block $m$, we have

$$P_{m, \text{min}} \lesssim \frac{c_3}{\sqrt{\left( \sum_{j=1}^{m-1} L_j \right) \text{err}(h_m, Z_{m-1})}}$$
Proof. First, it is easy to see that, when \( \text{err}(f_m, \tilde{Z}_{m-1}) \geq \log \left( \sum_{j=1}^{m} L_j \right) \epsilon_{m-1} \), we have
\[
\left( \sum_{j=1}^{m-1} L_j \right) \left( \text{err}(h_m, \tilde{Z}_{m-1}) + \Delta_{m-1} \right) \leq \left( \sum_{j=1}^{m-1} L_j \right) \left( \text{err}(h_m, \tilde{Z}_{m-1}) \right)
\]
On the other hand, when \( \text{err}(f_m, \tilde{Z}_{m-1}) < \left( \log \sum_{j=1}^{m} L_j \right) \epsilon_{m-1} \), we have
\[
\left( \sum_{j=1}^{m-1} L_j \right) \left( \text{err}(h_m, \tilde{Z}_{m-1}) + \Delta_{m-1} \right) \leq \left( \sum_{j=1}^{m-1} L_j \right) \log \left( \sum_{j=1}^{m} L_j \right) \epsilon_{m-1} \leq \log \left( \sum_{j=1}^{m} L_j \right).
\]
Combine these two inequalities we have
\[
\sqrt{\left( \sum_{j=1}^{m-1} L_j \right) \text{err}(h_m, \tilde{Z}_{m-1})} \leq \sqrt{\left( \sum_{j=1}^{m-1} L_j \right) \left( \text{err}(h_m, \tilde{Z}_{m-1}) + \Delta_{m-1} \right)} \leq \sqrt{\left( \sum_{j=1}^{m-1} L_j \right) \left( \text{err}(h_m, \tilde{Z}_{m-1}) \right)} + \log \left( \sum_{j=1}^{m} L_j \right).
\]
Therefore, by definition of \( P_{m,\min} \), we get the target bound.

13 Appendix: Practical WL-AC and Experiments

13.1 A summary of algorithm modification

The main difference between the practical and the theoretical version of WL-AC, as stated in Section 3, is the choice of plugable base AL strategy used to calculate the disagreement region as well as its corresponding query probability planning. Other than this, we explain more subtle modifications here, which should not effect the high level picture but help the practical implementation.

A more deterministic block schedule In the original templates, only the length of Phase 2 inside each block is fixed as \( L_m \) and the length of Phase 1 is a random variable based on the algorithm and data distribution. Such setting is easy for analysis but inconvenient for implementation. Therefore, here we fix total length of each block, including Phase 1 and Phase 2, as \( L_m \). (So in Algo 2 we use “scheduled training data collection length” and here in Algo 4, we use “scheduled block length”). Therefore, instead of comparing the newly queried number of samples in Phase 1 with the expected query number in the fixed Phase 2, we will compare that with the remaining expected query number in Phase 2 while keep the total block length the same. In the following Algo 5 we use subscript “+” to denote the newly added samples in Phase 1 (WL-EVAL) and use subscript “−” to refer to the remaining samples in Phase 2.

Estimate the sample distribution The real \( \mathbb{E}[\mathbb{I}[x \in D_m]] \) is not accessible, so we instead assume each data is uniformly distributed and estimate the distribution by the past observed data as shown in Line 24.

A relaxed protocol Instead of adhering to the strict streaming setting where each time only the current \( x_t \) is observable, we relax our experiment to the "batched" streaming setting where all the unlabeled context for next block is available at the beginning of the block as shown in Line 22. Therefore, instead of calculating the expect number of remaining samples within \( D_m \), we can get the exact remaining samples in the block \( m \), denoted as \( \bar{D}_m \). We conjectured that, calculating the expected one using empirical distribution based the past data should yield similar result without this batched assumption. But here we want to rule out the error caused in distribution estimation and only focus on verifying our main strategy.

Validation data and pseudo loss for neural net Several common modifications are required when running neural net. First, instead of using the true loss \( \mathbb{I}[y \neq y'] \), we use the pseudo loss denoted as \( \ell \). Here we choose cross-entropy for multi-class classification. Furthermore, instead of using training loss \( \text{err}(h, Z_m) \), we keep track of validation loss, which is usually a better estimates of the true classification error. Note [Huang et al. 2015] also uses the validation loss when
implementing the original AC. The required size of validation set is much smaller than the training set, so its addition to total sample complexity is negligible.

In addition, this validation set has also been used for early stopping in model training/updating. In the algorithm, we use

\[ h_{m+1}, \text{err}(h_{m+1}, Z_{\text{val}}) = O_{\text{train}}(Z_{\text{val}}, \tilde{Z}_m \cup \dot{Z}_m) \]

to denote this process. That is, the oracle \( O_{\text{train}} \) takes in the validation dataset, the training dataset and output the updated model as well as the validation loss.

**Less greedy query in Step 3** In Step 3 of Algo 3, we double the evaluation length in each epoch. In practice, we can increase the length in a milder way to further refine the sample complexity. (Theoretical they have the similar order.) Here we increase a constant number \( L_x \) in each iteration as shown in Line 20. Note that each iteration only requires an inference step in neural net, so the computational cost is neglectable.

**Other parameter choices** It is well known that the model complexity of neural net is hard to estimate. Therefore, the \(|\mathcal{H}|\)-dependent parameters in our algorithms can only be heuristically chosen, indicated by \( \approx \) in the algorithm below. We also heuristically choose \( \kappa_m = 1 \). A more comprehensive experiments might be conduct in the future. But the preliminary experiments we conduct here have already shown some positive results.
13.2 Algorithm (practical version)

**Algorithm 4** WL-AC (Practical version)

1: **Input:** Scheduled block length $L_1, L_2, \ldots$ satisfying $L_{m+1} \leq \sum_{j=1}^{m} L_j$. Strong labeler $O$ and weak labeler $W$. Candidate hypothesis set $\mathcal{H}$ (a neural net model).
2: a known small validation dataset $Z_{\text{val}}$ that $|Z_{\text{val}}| \ll n$ used for hyper-parameter tuning in model training
3: a based AL strategy $O_{\text{baseAL}}$
4: a training oracle fine-tuned by $Z_{\text{val}}$ denoted as $O_{\text{train}} : \mathcal{X} \times \mathcal{Y} \times Z_{\text{val}} \rightarrow \mathcal{H}$, and its corresponding pseudo loss $\ell$

5: **initialize:** epoch $m = 0$, $Z_0 := \emptyset$, $Z_0 := \emptyset$
6: for $m = 1, 2, \ldots, M$ do
7:   Set the stopping time of Phase 1 as $\tau_m$ and the corresponding length as $\hat{L}_m$
8:   **Phase 1:** WL evaluation and query probability assignment
9:   **Phase 2:** Train data collection based on planned query probability
10: Set $S = \emptyset$
11: for $t = \tau_m + 1, \ldots, \tau_m + L_m$ do
12:   if $x_t \in D_m$ then
13:     Draw $Q_t \sim \text{Bernoulli}(\hat{P}_m)$
14:     Update the set of examples:
15:       $S := \left\{ \begin{array}{l} S \cup \left\{ (x_t, h_m(x_t), h_m(x_t), 1) \right\}, \quad Q_t = 1 \\ S \cup \{ x_t, 1, 1, 0 \}, \quad \text{otherwise.} \end{array} \right.$
16:   end if
17: end for
18: Set $\tilde{Z}_m = \tilde{Z}_{m-1} \cup S$
19: **Phase 3:** Model and disagreement region updates using collected train data
20: Calculate the estimated error for all $h \in \mathcal{H}$ using shifted double robust estimator and pseudo loss $\ell$

$$\text{err}(h, \tilde{Z}_m) = \frac{\sum_{(x,y,y_{WL}^m) \in \tilde{Z}_m} \left( \ell(h(x), y) - \ell(h(x), y_{WL}^m) \right) |w + \ell(h(x), y_{WL}^m)|}{\sum_{(x,y,y_{WL}^m) \in \tilde{Z}_m} \ell(h(x), y)w}$$

21: Retrain the model using all collected data, get updated model and corresponding validation error $h_{m+1}, \text{err}(h_{m+1}, Z_{\text{val}}) = O_{\text{train}}(Z_{\text{val}}, \tilde{Z}_m \cup \tilde{Z}_m)$
22: Observe the next $L$ context $\{x_t\}_{t=\tau_m+L}$
23: Set empirical disagreement region $\tilde{D}_{m+1} = O_{\text{baseAL}}(\{x_t\}_{t=\tau_m+L}, h_{m+1}, \text{other params})$
24: Calculate empirical disagreement probability $\tilde{E}[D_m] = O_{\text{baseAL}}(\{x_t\}_{t=\tau_m+L}, h_{m+1}, \text{other params})$
25: Set the beginning time of the next block as $\tau_m = t = \tau_m + L_m$
26: end for
27: return $h_M$
Algorithm 5 WL evaluation for block $m$ (WL-EVAL, practical version)

1: **Input:** Current disagreement region $\hat{D}_m$, unlabeled samples in this block $\{x_t\}_{t=m+1}^r$ the current optimistic biased estimation of the best error $\hat{h}_{m, Z_{val}}$ Incremental length in Step3 denoted as $L_i$

2: **Step1:** Check if the current estimated best error is already good enough

3: Initialize $N_{m, -, \text{unlabel}} = |\hat{D}_m|$, which is the deterministic query number of block $m$ used for training without leveraging WL

4: Set $N_{m, +, \text{unlabel}} \approx \{\hat{h}_{m, Z_{val}} \omega - \hat{Z}_{m-1}\}$, which is the deterministic UNLABELLED number we need to newly draw in order to do WL evaluation.

5: Set $N_{m, +, \text{label}} = |\mathcal{O}_{\text{baseAL}}(h_m, \{x_t\}_{t=m}^r N_{m, +, \text{unlabel}})|$, which is the deterministic LABELLED number we need to newly draw in order to do WL evaluation

6: if $N_{m, -, \text{label}} \leq N_{m, +, \text{label}}$ then

7: **return** Use-WL = False, $\hat{\text{WLerr}}_m = 1$, $\hat{Z}_m = \hat{Z}_{m-1}$

8: **end if**

9: **Step2:** Check if the WL performance is worse than $\hat{err}(h_m, Z_{val})$

10: Set $\kappa_m = 1, k = 1$

11: Draw $N_{m, +, \text{unlabel}}$ number of new unlabeled samples denoted as $\hat{S}$ and query labels for those inside $D_m$. For each sample at $t$, add them as

$\hat{S} := \begin{cases} \hat{S} \cup \{x_t, y_t, y_t^{WL}\}, & Q_t = 1 \\ \hat{S} \cup \{x_t, 1, 1\}, & \text{otherwise.} \end{cases}$

and update $\hat{Z}_{m-1} = \hat{Z}_{m-1} \cup \hat{S}$

12: Calculate the empirical mean $\hat{\text{WLerr}}_{m, k} = \frac{1}{|\hat{Z}_{m, k}|} \sum_{(x,y) \in \hat{Z}_{m, k}} 1[y^{WL} \neq y \land x \in D_m]$

13: Calculate the optimistic estimation $\text{WLerr}_{m, k} = \min\{(\text{WLerr}_{m, k} + \frac{1}{|\hat{Z}_{m, k}|} \hat{E}[x \in D_m])\}$

14: if $\hat{\text{WLerr}}_{m, k} \leq \hat{\text{err}}(h_m, Z_{val})$ then

15: **return** Use-WL = False, $\hat{\text{WLerr}}_m = 1$, $\hat{Z}_m = \hat{Z}_{m, k}$

16: **end if**

17: **Step3:** Get a more precise estimation on WL performance

18: while $N_{m, -, \text{unlabel}} \leq N_{m, +, \text{label}}$ do

19: $k \leftarrow k + 1$

20: Draw $L_k$ new unlabeled samples and query labels for those inside $D_m$. For each sample at $t$, add them as

$\hat{S} := \begin{cases} \hat{S} \cup \{x_t, y_t, y_t^{WL}\}, & Q_t = 1 \\ \hat{S} \cup \{x_t, 1, 1\}, & \text{otherwise.} \end{cases}$

and update $\hat{Z}_{m, k} = \hat{Z}_{m, k-1} \cup \hat{S}$

21: Update the total query at the current block $N_{m, +, \text{label}}$

22: Calculate the the optimistic estimation $\hat{\text{WLerr}}_{m, k}$ as before

23: Update the remaining number of samples in this block that are planed to be queried, denoted as $N_{m, -, \text{label}}$

24: **end while**

25: **return** Use-WL = True, $\hat{\text{WLerr}}_m = \min\{\kappa_m \hat{\text{WLerr}}_{m, k} \hat{E}[x \in D_m] + 1\}$, $\hat{Z}_m = \hat{Z}_{m, k}$

---

**Optimization Problem** (OP, practical version) to compute $P_m$

**Input:** Estimated hints performance upper bound $\hat{\text{WLerr}}_m$

$$P_m = \max\{\text{WLerr}_m, P_m, \text{min}\}$$

13.3 Performance regarding to the number of observed unlabeled samples
Figure 3: ❶ - Performance comparison under uniform noisy (left) and informative classifier (right) weak labeler, regarding to the number of unlabeled samples.

Figure 4: ❷ - Performance comparison under pre-trained classifier weak labeler, regarding to the number of unlabeled samples.