Plasma Perturbations and Cosmic Microwave Background Anisotropy in the Linearly Expanding Milne-like Universe

S.L. Cherkas and V.L. Kalashnikov

Abstract We expose the scenarios of primordial baryon-photon plasma evolution within the framework of the Milne-like universe models. Recently, such models find a second wind and promise an inflation-free solution of a lot of cosmological puzzles including the cosmological constant one. Metric tensor perturbations are considered using the five-vectors theory of gravity admitting the Friedmann equation satisfied up to some constant. The Cosmic Microwave Background (CMB) spectrum is calculated qualitatively.

1 Introduction

Present universe is transparent for photons, but it was not the same before the hydrogen recombination at the red-shifts of \( z \approx 1100 \) when it was filled with the photon-baryon plasma. Protons and electrons were coupled to the radiation through the Compton scattering by electrons which in turn are coupled to the baryons by Coulomb interaction \([1,3]\). Such primordial plasma perturbations were widely considered in cosmology, and their fingerprints depend on a law of the universe expansion that is the crucial point for our further analysis.

Recently, the Milne-like cosmologies considering the linearly expanding (in cosmic time) universe models \([4,5]\) again attract an attention \([6-15]\). Instead of the

---

1 z is the red-shift parameter used as a measure of cosmological time and distance: \( z + 1 = a_0 / a(\eta) \), where \( a_0 \) is the present scale factor value, and \( a(\eta) \) is the scale factor at some earlier photon emission time \( \tau \) \([12]\).
original open and empty Milne universe model [4, 5]. The flat universes filled with some exotic matter are considered. It seems reasonable to associate such “a primordial matter fluid” with the vacuum [16].

We will consider the perturbations of plasma consisting of photons, baryons, and electrons in a linearly expanding (Milne-like) universe with taking into account the metric tensor and vacuum perturbations. Here, we will use the oversimplified model of plasma as a pure radiation, i.e., a substance with the equation of state $w = 1/3$ [3] to obtain an analytical solution. This approximation is admissible because initially, the temperature is sufficiently large to consider all the particles as a relativistic fluid. Then, the particles decay eventually to the photons, electrons, and baryons. According to observations, the number of photons is of $10^9$ times larger than that of nucleons and electrons. Thus, the nucleons contribute at only the late stage of the universe evolution. We will base our analysis of the metric tensor perturbations, which contribute to the primordial plasma formation, on the five-vectors theory of gravity [18]. The quantization of this model could resolve the problem of huge vacuum energy [19] and allow omitting its main part [3].

2 Perturbations of Plasma and Vacuum

We expose the perturbation theory for primordial photon-baryon plasma, vacuum and metric tensor. Vacuum issues the well-known challenge for quantum or, at least, semiclassical theory [16, 19, 22]. Here, we will consider a vacuum purely classically, that is as a substance producing the linear expansion of the universe in the framework of the developed theory [18] which admits adding or extracting some constant to the energy density.

2.1 Underlying Gravity Theory

The conventional theory of the CMB spectrum is the General Relativity theory (GR) (e.g., see [2]). In the case of the Milne-like cosmology, the issue is more complicated, because an origin of linear universe expansion is not clear. As was shown, such linear expansion could arise from the residual vacuum fluctuations of quantized fields including the scalar and gravitational ones after omitting the main part of huge vacuum energy [16]. As was mentioned above, the mystery of cosmological vacuum is among the critical issues of modern physics [19, 21, 22]. Below we

---

2 The universe proposed initially by Milne describes an open and empty (i.e., Minkowski) spacetime which expands linearly with time [1, 4, 5]. It is negatively-curved spatially (i.e., hyperbolical in 3-dimensions) but is “flat” in 4 (i.e., spacetime)-dimensions.

3 We use a classical definition for the equation of state parameter $w$ corresponding to a perfect fluid, that is the ratio of pressure to density [17].

4 Below, the system of units $\hbar = c = 1$ will be used, and we define the present scale factor as $a_0 = 1$. 

---
will use the theory which validates omitting the vacuum extra-energy and, besides, provides obtaining the analytical solutions.

Let’s start from the Einstein-Hilbert action for GR in the form of (23):

$$ S = - \frac{M_p^2}{12} \int \mathcal{G} \sqrt{-g} d^4x, $$

(1)

where $\mathcal{G} = g^{\alpha \beta} \left( \Gamma^\rho_{\alpha \nu} \Gamma^\nu_{\beta \rho} - \Gamma^\rho_{\alpha \beta} \Gamma^\nu_{\nu \rho} \right)$, and $M_p$ is the Planck mass, which is chosen as $M_p = \sqrt{\frac{3}{4\pi}}$.

The next step is a violation of the general coordinate covariance principle in (1) according to the Milne’s perception of the principally different concept of time in GR and quantum mechanics [24–26], so that we will consider the restricted class of metrics $g_{\mu \nu}$ in the form of

$$ ds^2 \equiv g_{\mu \nu} dx^\mu dx^\nu = a^2 (1 - \partial_m P^m) d\eta^2 - \gamma_{ij}(dx^i + N^i d\eta)(dx^j + N^j d\eta), $$

(2)

where $\gamma_{ij}$ is the induced three metric, $a = \gamma^{1/6}$ is the scale factor defined locally, and $\gamma = \det \gamma_{ij}$. A spatial part of the interval (2) can be written as

$$ dl^2 \equiv \gamma_{ij} dx^i dx^j = a^2 (d\eta, x) \tilde{\gamma}_{ij} dx^i dx^j, $$

(3)

where $\tilde{\gamma}_{ij} = \gamma_{ij}/a^2$ is a matrix with the unit determinant. The interval (2) is analogous to the ADM one [27], but the expression $1 - \partial_m P^m$ is used instead of a lapse function, where $\partial_m$ is a partial derivative and $P^m$ is a three-vector. Varying the action over vectors $P, N$ and three metric $\gamma_{ij}$ leads to the equations of the five-vectors theory (FVT) [18]:

$$ \frac{\partial g^{\mu \nu}}{\partial \gamma_{ij}} \left( \frac{\partial (\mathcal{G} \sqrt{-g})}{\partial g^{\mu \nu}} - \frac{\partial}{\partial x^\lambda} \left( \frac{\partial (\mathcal{G} \sqrt{-g})}{\partial (\partial_\lambda g^{\mu \nu})} \right) - \frac{6}{M_p^2} T^{\mu \nu} \sqrt{-g} \right) = 0, $$

$$ \frac{\partial g^{\mu \nu}}{\partial N^r} \left( \frac{\partial (\mathcal{G} \sqrt{-g})}{\partial g^{\mu \nu}} - \frac{\partial}{\partial x^\lambda} \left( \frac{\partial (\mathcal{G} \sqrt{-g})}{\partial (\partial_\lambda g^{\mu \nu})} \right) - \frac{6}{M_p^2} T^{\mu \nu} \sqrt{-g} \right) = 0, $$

$$ \frac{\partial g^{\mu \nu}}{\partial (\partial_j P^m)} \frac{\partial}{\partial x^j} \left( \frac{\partial (\mathcal{G} \sqrt{-g})}{\partial g^{\mu \nu}} - \frac{\partial}{\partial x^\lambda} \left( \frac{\partial (\mathcal{G} \sqrt{-g})}{\partial (\partial_\lambda g^{\mu \nu})} \right) - \frac{6}{M_p^2} T^{\mu \nu} \sqrt{-g} \right) = 0. $$

(4)

Eqs. (4) are weaker than the GR ones. At the same time, the restrictions $\nabla (\nabla \cdot P) = 0$ and $\nabla (\nabla \cdot N) = 0$ on the Lagrange multipliers arise [18]. In the particular case of $\nabla \cdot N = 0$, the Hamiltonian constraint is satisfied up to some constant.

The next step is to develop a theory for the scalar perturbations in the gauge of $P = 0, N = 0$: ...
ds^2 = a(\eta)^2 (1 + 2A) \left( d\eta^2 - \left( \left( 1 + \frac{1}{3} \sum_{m=1}^{3} \frac{\partial^2 F}{\partial x^m} \right) \delta_{ij} - \partial_i \partial_j F \right) dx^i dx^j \right). \quad (5)

An interval (5) is a particular form of the interval (2) up to the higher order terms in \( F(\eta, x) \) by virtue of

\[
\ln \left[ \det \left( \left( 1 + \frac{1}{3} \sum_{m=1}^{3} \frac{\partial^2 F}{\partial x^m} \right) \delta_{ij} - \partial_i \partial_j F \right) \right] \approx \text{tr} \left( \left( \frac{1}{3} \sum_{m=1}^{3} \frac{\partial^2 F}{\partial x^m} \right) \delta_{ij} - \partial_i \partial_j F \right) = 0.
\]

Writing Eqs. (4) up to the first order relatively \( A(x, \eta) \) and \( F(x, \eta) \) leads to the required perturbation theory.

### 2.2 Energy-Momentum Tensor

As was above mentioned, we violate eventually the general coordinates’ transformation invariance by the restriction of the metrics’ class by representing them in the form of (2). To build the energy-momentum tensor in the field theory, one should write the corresponding special relativistic expression and then change the partial derivatives to covariant ones. Using a hydrodynamic approximation is more convenient. In this framework the energy-momentum tensor is

\[
T_{\mu \nu} = (p + \rho)u_\mu u_\nu - p g_{\mu \nu}.
\] (6)

The equations of motion for some fluid in the GR can be obtained from both the equations of motion of the fluid point-like components and the conservation of the energy-momentum tensor \( D_\mu T^{\mu \nu} = 0 \), where \( D_\mu \) is a covariant derivative. In FVT, the energy-momentum tensor conserves only in the Minkowski space-time. However, one can deduce the equation of motion for fluid from the conservation of energy-momentum tensor by virtue of the Eqs. (4) self-consistency in the particular gauge (5). Below, we will consider the scalar perturbations of a fluid \( c \) (the index \( c \) denotes a kind of fluid) in the form of \( \rho_c(\eta, x) = \rho_c(\eta) + \delta \rho_c(\eta, x) \), \( p_c(\eta, x) = p_c(\eta) + \delta p_c(\eta, x) \) and represent the 4-velocity in the form of

\[
u^\mu_c = \frac{1}{a(\eta)} \{ (1 - A), \nabla \nu_c(\eta, x) \}, \quad (7)
\]

where \( \nu_c(\eta, x) \) is a scalar function.

### 2.3 Zero-Order Equations

The zero-order evolution equation for logarithm of the scale factor \( \alpha(\eta) = \ln a(\eta) \) takes the form
\[ \alpha'' + \alpha^2 = M_p^{-2} e^{2\alpha} (\rho - 3p), \] (8)

where \( \rho = \sum \rho_c \) and \( p = \sum p_c \) are uniform energy density and pressure, respectively. Summation is performed over all the kinds of matter, but here we will consider only vacuum \( c = v \) and radiation \( c = r \). For every component of a substance, the equation of motion is:

\[ \rho'_c + 3\alpha'(\rho_c + p_c) = 0. \] (9)

Pressure of a fluid is connected with the energy density as \( p_c = w_c \rho_c \) (see the footnote 3 above and Ref. [17]). It is worth mentioning that the Friedmann equation is satisfied only up to some constant in the framework of the model considered:

\[ M_p^{-2} e^{4\alpha} \rho(\eta) - \frac{1}{2} e^{2\alpha} \alpha'^2 = \text{const}, \] (10)

that is the integral of motion of Eqs. (8), (9).

As was shown [28], the residual vacuum fluctuations can explain a nearly-linear universe expansion. Here, for simplicity, we will use an empirical consideration. Let us analyze a linear universe expansion that means \( a(\eta) = B \exp(\mathcal{H} \eta) \) in conformal time, and find the corresponding empirical equation for the vacuum state. The very simple equation of state arises if we set a constant in the Friedmann equation (10) so that

\[ M_p^{-2} e^{4\alpha} \rho_v - \frac{1}{2} e^{2\alpha} \alpha'^2 = 0. \] (11)

It is possible because \( \rho_v e^{4\alpha} \) is also constant. Under such choice of a constant, the equation of the vacuum state will be \( w_v = -1/3 \). This equation of state is widely discussed earlier [9, 10, 14]. One may obtain from Eq. (9) \( \rho_v e^{2\alpha} = \text{const} \) for the vacuum, that results in (see Eq. (11)):

\[ a(\eta) = \exp(\alpha(\eta)) = B \exp(\mathcal{H} \eta), \] (12)

where \( B \) is some constant. In the cosmic time \( dt = a(\eta) d\eta \)

\[ a(t) = \mathcal{H} t, \] (13)

i.e., it is a linear expansion of the universe.

### 2.4 Perturbations

Introducing the quantity \( V_c = (\rho_c + p_c) v_c \) for every fluid \( c \) and expanding all perturbations into the Fourier series \( \delta \rho_c(x) = \sum_k \delta \rho_k e^{ikx} \) etc. result in the equations for perturbations:
for the radiation

\[ -6A_k + 6A_k'\alpha' + k^2F_k' + \frac{18}{M_p^2} e^{2\alpha} \sum_c V_{c,k} = 0, \quad (14) \]

\[ -18\alpha' A_k' - 18A_k\alpha'' - 6k^2A_k + k^4F_k + \frac{18}{M_p^2} e^{2\alpha} \sum_c \delta \rho_{c,k} + 4A_k\rho_c = 0, \quad (15) \]

\[ -12A_k - 3 (F_k'' + 2\alpha' F_k') + k^2F_k = 0, \quad (16) \]

\[ -9 (A_k'' + 2\alpha A_k') - 18A_k\alpha'' - 18A_k\alpha' - 9k^2A_k + k^4F_k \]

\[ - \frac{9}{M_p^2} e^{2\alpha} \sum_c 4A_k (3\rho_c - \rho_c) + 3\delta \rho_{c,k} - \delta \rho_{c,k} = 0, \quad (17) \]

\[ -3\alpha' (\delta \rho_{c,k} + \delta \rho_{c,k}) - 3A_k' (\rho_c + \rho_c) - \delta \rho_{c,k}' + k^2V_{c,k} = 0, \quad (18) \]

\[ (\rho_c + \rho_c)A_k + 4V_{c,k} \alpha' + \delta \rho_{c,k} + V_{c,k}' = 0. \quad (19) \]

The last two equations, obtained from the energy-momentum conservation, are assumed to be valid for every \( c \)-substance under consideration. The choice of the constant in Eq. (10) is arbitrary. The constraint equations (14) and (15) are consistent with the other equation under this arbitrary choice. It is not true in a perturbation theory within the framework of GR, where a perturbation of the constraint equations is consistent with other equations only if a sum of the mean densities of all fluids equals the critical density (for the flat universe). Here we consider the flat universe in a mean, but the sum of the mean densities is determined up to some constant, and nevertheless, all the equations for perturbations are self-consistent. With that chosen constant in Eq. (10), the radiation does not affect the universe expansion and the equation of state \( w_r = -1/3 \) for the vacuum results in linear expansion of the universe. Thus, the equations of state are \( w_r = -1/3 \) for the vacuum and \( w_r = 1/3 \) for the radiation.

Such choice of the constant in (10) is an invention expired by the existence of the analytical solution in this case. The above system of equations can be reduced to a single linear equation with the constant coefficients under the assumption of \( a(\eta) = B \exp(\mathcal{H}\eta) \) and \( \rho_c = \frac{\rho_{c0}}{a^3(\eta)} \), where \( \rho_{c0} \) is a density of radiation at the present time:

\[ 9\delta \rho_{c,k}^{(4)} + 6 (30\mathcal{H}\delta \rho_{c,k}^{(5)} + (222\mathcal{H}^2 + k^2) \delta \rho_{c,k}^{(3)} + 10\mathcal{H} (72\mathcal{H}^2 + k^2) \delta \rho_{c,k}^{(1)}) 

+ (48\mathcal{H}^2 + k^2) (108\mathcal{H}^2 + k^2) \delta \rho_{c,k} = 0. \quad (20) \]

That allows obtaining the solution for the perturbation of radiation density:

\[ \delta \rho_{c,k} = e^{-6\mathcal{H}\eta} \left( C_1 e^{-i\mathcal{H}k} + C_2 e^{i\mathcal{H}k} \right) + e^{-4\mathcal{H}\eta} \left( C_3 e^{-i\mathcal{H}k} + C_4 e^{i\mathcal{H}k} \right). \quad (21) \]

For a “flux” of the radiation fluid \( V_{c,k} \), we have

\[ V_{c,k} = \frac{B^2 \mathcal{H} M_p^2}{6k\rho_{c0}} e^{-4\mathcal{H}\eta} \left( C_1 (k - i\sqrt{3}\mathcal{H}) e^{-i\mathcal{H}k} + C_2 (k + i\sqrt{3}\mathcal{H}) e^{i\mathcal{H}k} \right). \quad (22) \]

\[ \delta \rho_{c,k} = w_r \delta \rho_{c,k} \text{ is assumed, as well.} \]
Other functions $A_k, F_k, \delta \rho_{v,k}, V_{v,k}$ found from the system (14)-(19) are presented in Appendix.

The constants $C_1, C_2, C_3, C_4$ have to be determined from the initial conditions. The constants $Z_1, Z_2$ (see Appendix) do not contribute to the radiation density perturbations. Thus, we will equal them to zero. Indeed, it is reasonable to assume that an empty universe (i.e., filled by the only vacuum) has no any rising physical perturbation, and only perturbations connected with the radiation over the vacuum have a physical meaning. For simplicity, we assume that the only perturbations of radiation density $\delta \rho_{r,k}(\eta_{in})$ are non-zero initially, where $\eta_{in}$ is an initial moment in conformal time.

Then, the solutions of the perturbation theory equations take the form:

$$\delta \rho_{r,k}(\eta) = e^{4\mathcal{H}(\eta_{in} - \eta)} \left( 4\sqrt{3}\mathcal{H} \sin \left( \frac{k(\eta - \eta_{in})}{\sqrt{3}} \right) \right)$$

$$+ k \cos \left( \frac{k(\eta - \eta_{in})}{\sqrt{3}} \right) \delta \rho_{r,k}(\eta_{in}) / k, \quad \delta \rho_{r,k}(\eta_{in}) / k$$

$$V_{v,k}(\eta) = 0, \quad A_k(\eta) = -\frac{B^4 e^{4\mathcal{H}}}{4\rho_0} \delta \rho_{r,k}(\eta), \quad F_k(\eta) = -\frac{3B^4 e^{4\mathcal{H} \eta_{in}}}{2k^2 \rho_0 (3\mathcal{H}^2 + k^2)} \left( (12\mathcal{H}^2 + k^2) \cos \left( \frac{k(\eta - \eta_{in})}{\sqrt{3}} \right) \right)$$

$$+ 3\sqrt{3}\mathcal{H} k \sin \left( \frac{k(\eta - \eta_{in})}{\sqrt{3}} \right) \delta \rho_{r,k}(\eta_{in}), \quad V_{v,k}(\eta) = \frac{B^2 \mathcal{H}^2 M_P^2 e^{4\mathcal{H} \eta_{in} - 2\eta \mathcal{H}}}{12k\rho_0 (3\mathcal{H}^2 + k^2)} \left( \sqrt{3} (12\mathcal{H}^2 + k^2) \sin \left( \frac{k(\eta - \eta_{in})}{\sqrt{3}} \right) \right)$$

$$- 9\mathcal{H} k \cos \left( \frac{k(\eta - \eta_{in})}{\sqrt{3}} \right) \delta \rho_{r,k}(\eta_{in}), \quad \delta \rho_{r,k}(\eta_{in}) = 3\mathcal{H} V_{v,k}(\eta). \quad (27)$$

The quantities $V_{v,k}(\eta)$ and $\delta \rho_{r,k}(\eta)$ will not be needed for the CMB spectrum calculations and will not be considered further.

### 2.5 “Gauge Invariant” Variables

The issue is that the metric \[5\] has not a typical form of

$$ds^2 = a^2(\eta) \left( (1 + 2\Phi(\eta, x)) d\eta^2 - (1 - 2\Psi(\eta, x)) \delta_{ij} dx^i dx^j \right),$$

which appears in the conventional perturbation theory [3] of GR. The comparability of previous results with those of the GR conventional perturbation theory can be provided by the “gauge invariant” densities, velocities and potentials [1]:

$$ds^2 = a^2(\eta) \left( (1 + 2\Phi(\eta, x)) d\eta^2 - (1 - 2\Psi(\eta, x)) \delta_{ij} dx^i dx^j \right), \quad (28)$$
\[ \delta_{r,k}(\eta) = \frac{\delta \rho_{r,k}(\eta)}{\rho_r(\eta)} - 2a'(\eta)F_k'(\eta), \quad \tilde{v}_{r,k} = \frac{V_{r,k}(\eta)}{\rho_r(\eta) + p_r(\eta)} - \frac{F_k'(\eta)}{2}, \]
\[ \Phi_k(\eta) = A_k(\eta) + \frac{a'(\eta)F_k'(\eta) + a(\eta)F_k''(\eta)}{2a(\eta)}, \]
\[ \Psi_k(\eta) = -\frac{a'(\eta)F_k'(\eta)}{2a(\eta)} - A_k(\eta) + \frac{1}{6}k^2F_k(\eta). \quad (29) \]

We could not work with the “invariant” potentials initially because the metric (28) has not the form (2) and does not admit obtaining the consistent system of the equations when the zero-order Friedmann equation is violated, i.e., satisfied up to some constant (10). For our simplified approach, when only initial value \( \delta \rho_{r,k} \) is nonzero, the calculated “invariant quantities” are
\[ \tilde{\delta}_{r,k}(\eta) = \frac{1}{(3\mathcal{H}^2 + k^2)} \left( (12\mathcal{H}^2 + k^2) \cos \left( \frac{k(\eta - \eta_{\text{in}})}{\sqrt{3}} \right) + 3\sqrt{3}k \sin \left( \frac{k(\eta - \eta_{\text{in}})}{\sqrt{3}} \right) \right) \delta_{r,k}(\eta_{\text{in}}), \]
\[ \tilde{v}_{r,k}(\eta) = \frac{1}{4k(3\mathcal{H}^2 + k^2)} \left( 9\mathcal{H}k \cos \left( \frac{k(\eta - \eta_{\text{in}})}{\sqrt{3}} \right) - \sqrt{3}(12\mathcal{H}^2 + k^2) \sin \left( \frac{k(\eta - \eta_{\text{in}})}{\sqrt{3}} \right) \right) \delta_{r,k}(\eta_{\text{in}}), \]
\[ \Phi_k(\eta) = 0, \quad \Psi_k(\eta) = 0. \quad (30) \]

where we take into account that \( \frac{\rho_0}{\Theta^4 \exp(4\mathcal{H} \eta_{\text{in}})} = \rho_r(\eta_{\text{in}}) \) and \( \delta_{r,k}(\eta_{\text{in}}) = \frac{\delta \rho_{r,k}(\eta_{\text{in}})}{\rho_r(\eta_{\text{in}})} \).

The potentials \( \Phi_k, \Psi_k \) are zero only because we use the simplified initial condition, where \( \delta \rho_{r,k} \) is nonzero initially.

### 2.6 Silk Dumping

Electrons scatter the photons before the time of the last scattering surface. Although we consider photon-electron-baryon plasma as some perfect medium with the equation of state \( w = 1/3 \), the photon diffusion due to the Thompson scattering contributes to the electron-photon scattering process [2]. To estimate this (so-called Silk dumping) contribution to the perturbations, we follow the methodology of Refs. [1,3] suggesting the suppression of the expressions (23), (25), (26), (27) and (29) by the factor \( \exp \left( -\frac{k^2}{k_D^2} \right) \), where \( k_D \) is written as
\[ k_D(\eta_r) \approx \left( \frac{2}{15} \int_0^{\eta_r} \frac{d\eta}{\sigma_{\text{eff}} a} \right)^{-1/2} = \left( \frac{2}{15\sigma_{\text{eff}} \eta_{\text{in}}} \int_0^{\eta_r} a^2 d\eta \right)^{-1/2}. \quad (31) \]
and $\sigma_T = 6.65 \times 10^{-25}$ cm$^2$ is the Thompson cross section. The free electron density $n_e$ before recombination equals to the baryon density and scales as $n_e = n_{b0} a^{-3}$, where $n_{b0}$ is the baryon present density

$$n_{b0} = \Omega_b M_p^2 \mathcal{H}^2 / 2m_p$$

expressed through a dimensionless quantity $\Omega_b$, a proton mass $m_p$ and a critical density $M_p^2 \mathcal{H}^2 / 2$. Formally, for the dependence given by (12), an integration in (31) has to begin from $\eta = -\infty$. However, as was shown in [28], the universe started from a power-law expansion changed by (12) afterward. It was also shown, that $B$ is of the order of $10^{-30}$. Under this condition, $B$ does not play a role if the lower limits of $\eta$ equal $-\infty$ or zero (the results are approximately the same in both cases).

Substituting the dependence (12) and the conformal time of the last scattering surface $\eta_r = \frac{1}{H} \ln \frac{10^{-3}}{B}$, that corresponds to the scale factor $a_r \approx 10^{-3}$, into (31) results in

$$k_D(\eta_r) = \sqrt{\frac{15\sigma_T n_{b0} \mathcal{H}}{2}} \times 10^3 \approx 10^3 \sqrt{\Omega_b \mathcal{H}}.$$  

(33)

As one may see, plasma is closer to an ideal fluid for greater matter density. For instance, the conventional value of $\Omega_b = 0.03$ results in the damping scale of $k_D \sim 170$ in the units of $\mathcal{H}$.

### 3 CMB Spectrum

In the previous section, we have considered the perturbation theory which describes the evolution of the plasma (radiation) in the presence of the vacuum perturbations. This evolution extends up to the “last scattering surface”, i.e., up to a moment when the universe becomes transparent for radiation. Conformal time of the last scattering surface $\eta_r$ corresponds to the temperatures $T_r \sim 3000$ K and the redshift $z_r \approx 1100$. Describing the photons’ propagation from the last scattering surface to an observer is insufficient to use hydrodynamic approximation so that the Boltzmann equation is needed, which can be written in the form of

$$\frac{\partial f}{\partial \eta} + \frac{dx^i}{d\eta} \frac{\partial f}{\partial x^i} + \frac{dp_i}{d\eta} \frac{\partial f}{\partial p_i} = St[f],$$

(34)

where the right hand side $St[f]$ represents the collision integral. If the distribution function $f$ is assumed to be a scalar, it would depend on $x^i$ and $p_i$ because the photon number $dN = f(x^i, p_j, \eta)dx^1 dx^2 dx^3 dp_1 dp_2 dp_3$ is scalar according to the Liouville theorem and the quantity $dx^1 dx^2 dx^3 dp_1 dp_2 dp_3$ is scalar. The expressions describing the photon propagation are

$$\frac{dp_\alpha}{d\lambda} = -\Gamma_{\beta\gamma}^{\alpha} p_\beta p_\gamma = -\Gamma_{\beta\gamma}^{\alpha} g^\sigma g^{\rho\delta} p_\sigma p_\delta,$$
\[ \frac{dx^\alpha}{d\lambda} = p^\alpha = g^{\alpha\beta} p_\beta, \tag{35} \]

where \( \lambda \) is an affine parameter along the photon trajectory. Using the last equation for the zero component \( \frac{dx^0}{d\xi} = \frac{d\eta}{d\xi} = p^0 \) of derivatives with respect to \( \lambda \) allows rewriting it in the terms of derivatives with respect to \( \eta \).

Then, the Boltzmann equation can be reduced to the equation for a temperature perturbation by substitution

\[ f(x^i, p_j, \eta) = \frac{1}{\exp \left( \frac{p_0(\eta)}{p_0(\eta)\sqrt{g_{00}}(1+\Theta_{n,n,\eta})} \right) - 1}, \tag{36} \]

where \( \Theta_{n,x,\eta} \) is a temperature contrast and a unit vector \( n_i = p_i / (\sum_{n=1}^{3} p_n^2) \).

Finally, for the coefficients of the Fourier transform \( \Theta_{n,n,\eta} = \sum_{k} \Theta_{k}(\eta, n) e^{\imath k x} \) calculations with the metric (5) give

\[ \frac{\partial \Theta_k}{\partial \eta} - i k \mu \Theta_k - i k \mu A_k + A'_k + \frac{k^2}{6} (3 \mu^2 - 1) F_k' = \tau'(\Theta_k - \Theta_{0k} - v_{bk} \mu), \tag{37} \]

where \( \mu = n \cdot k / k \) is the cosine of the angle between \( n \) and \( k \), \( \Theta_{0k}(\eta) \) is the component \( l = 0 \) of \( \Theta_k(n, \eta) \) in the expansion of the Legendre polynomials

\[ \Theta_{0k} = l \int_{-1}^{1} P_l(\mu) \Theta_k(\mu) \frac{d\mu}{2}, \tag{38} \]

and \( v_{bk} \) is the Fourier transform of the function determining baryon velocity. The function \( \tau(\eta) \) describes the photon Compton scattering by electrons: \( \tau' = -\sigma_T n_e a \), where \( \sigma_T \) is a cross section of the Thomson scattering and \( n_e \) is a free electron density. Before the last scattering surface, the photons are tightly coupled with electrons and protons by the Thomson scattering, and the electrons, in turn, are tightly coupled with baryons by the Coulomb interaction. As a consequence, any bulk motion of the photons must be shared by the baryons. Although we do not consider baryons explicitly, one may assume roughly that baryons and photons are in equilibrium and thus

\[ v_{bk} = -3i \Theta_{0k}(\eta). \tag{39} \]

Further, the monopole \( \Theta_{0k} \) and dipole \( \Theta_{1k} \) components of the temperature perturbations can be connected with the perturbations of density and velocity. From one hand side, the 00-component of the energy-momentum tensor in line with (4) is

\[ T^0_{0k} = \delta \rho_k(\eta). \tag{40} \]

On the other hand, it can be expressed via a temperature perturbation [11]:

\[ T^0_{0k} = 4\rho \int \Theta_k(n, \eta) \frac{d^2 n}{4\pi}. \tag{41} \]
Comparison of (40) and (41) gives \( \Theta_{k}(\eta) = \frac{1}{4\pi} \delta \rho_{k}(\eta) = \frac{1}{4} \delta_{k} \). Analogously, in the first order of the perturbation theory, the components \( T_{0j} \) take the form

\[
T_{0j} = -a^2(\eta)(p_r(\eta) + p_v(\eta)) \partial_j v_r(\eta, x),
\]

or

\[
T_{0k}^{j} = \frac{4}{3} \rho_v(\eta) j_k v_k(\eta).
\]

At the same time (11)

\[
T_{ik}^{j} = -4\rho_v \int n^j \Theta_{k}(n, \eta) \frac{d^2 n}{4\pi}
\]

As consequence of (38), (39), (43) and (44), one has \( v_{ik} = -3i \theta_{ik} = -ik v_{ik} \), and Eq. (37) can be rewritten in the form of

\[
\Theta'_{k} - (ik \mu + \tau') \Theta_{k} = e^{ik \mu \eta + \tau} \frac{d}{d\eta} \left( \Theta_{k} e^{-ik \mu \eta - \tau} \right) = S_{k},
\]

where \( S_{k} = -\tau' \delta_{k} + \tau' ik \mu v_{ik} + ik \mu A_{k} - A'_{k} = \frac{k^2}{6} (3\mu^2 - 1) F_{k}' \).

Solution of Eq. (45) takes the form of

\[
\Theta_{k}(\eta_0) = \Theta_{k}(\eta_{in}) e^{-\mu k (\eta_0 - \eta_{in}) - \tau (\eta_{in}) + \tau (\eta_0)} + \int_{\eta_{in}}^{\eta_0} S_{k} e^{-\mu k (\eta - \eta_{in}) - \tau (\eta_{in}) + \tau (\eta)} d\eta \approx
\]

\[
\int_{\eta_{in}}^{\eta_0} e^{-\tau (\eta)} \left( -\tau' \delta_{k} - \tau' v_{ik} \frac{d}{d\eta} - A'_{k} - A_{k} \frac{d}{d\eta} \right) \frac{F_{k}'}{6} \left( -3 \frac{d^2}{d\eta^2} - k^2 \right) e^{-ik \mu (\eta - \eta_{in})} d\eta,
\]

where \( \eta_0 \) is the present day conformal time, \( \eta_{in} \) is some initial moment of time before the last scattering surface, when the universe was not transparent for light. The terms containing \( e^{-\tau (\eta_{in})} \) are omitted because the function \( e^{-\tau (\eta)} \) vanishes quickly if \( \eta < \eta_{r} \).

Using the integral (38) and the integral

\[
\int_{-1}^{1} \frac{d\mu}{2} P_{j}(\mu) e^{-ik \mu (\eta - \eta_{0})} = \frac{1}{\pi} j_{j}(k (\eta - \eta_{0}))
\]

leads to

\[
\Theta_{k}(\eta_0) = \int_{\eta_{in}}^{\eta_0} e^{-\tau (\eta)} \left( -\tau' \frac{1}{4} \delta_{k} - A'_{k} + F_{k}^{l} k^{2} \right) j_{j}(k (\eta - \eta_{0}))
\]

\[
- (\tau' v_{ik} + A_{k}) k_{j} j_{k}(k (\eta - \eta_{0})) + \frac{F_{k}^{l}}{2} k^{2} j_{j}'(k (\eta - \eta_{0})) \right) d\eta.
\]

One may rewrite Eq. (38) in the terms of invariant potentials, densities and velocities (29).
\[ \Theta_{\mathbf{k}}(\eta_0) = \int_{\eta_0}^{\eta_0} e^{-\tau(\eta)} \left( -\tau' \left( \frac{\delta_k}{4} + \Phi_k \right) j_l(k(\eta - \eta_0)) 
- \tau' \tilde{\nu}_k k j_l(k(\eta - \eta_0)) + \left( \Phi'_k + \Psi'_k \right) j_l(k(\eta - \eta_0)) \right) d\eta. \] (49)

The integrand expressions in (48) and (50) differ by a total derivative, which does not contribute to the integral because \( e^{-\tau(\eta)} \approx 0 \) at the lower limit, and the Bessel function \( j_l(0) = 0 \) for \( l > 0 \) at the upper limit.

According to (30), the invariant potentials \( \Psi \) and \( \Phi \) equal zero in our simplified consideration when only \( \delta_{\mathbf{k}}(\eta_0) \) is nonzero. Thus, there is no the Sachs-Wolff effect and the expression (50) is reducible to

\[ \Theta_{\mathbf{k}}(\eta_0) = \int_{\eta_0}^{\eta_0} (-\tau') e^{-\tau(\eta)} \left( \frac{\delta_k}{4} j_l(k(\eta - \eta_0)) + \tilde{\nu}_k k j'_l(k(\eta - \eta_0)) \right) d\eta 
= \frac{\delta_{\mathbf{k}}(\eta_0)}{4} j_l(k(\eta_0 - \eta_0)) + \tilde{\nu}_k \eta_0 j'_l(k(\eta_0 - \eta_0)), \] (50)

where the fact is used that the visibility function \( g(\eta) = -\tau' e^{-\tau(\eta)} \) is peaked near last scattering surface \( \eta_0 \). On the other hand, the integral \( \int g(\eta) d\eta = 1 \), and thereby, it is like the Dirac delta-function \( g(\eta) = \delta(\eta - \eta_0) \).

Using the expressions for \( \delta_{\mathbf{k}} \) and \( \tilde{\nu}_k \) from (30), we obtain the expressions for the coefficients

\[ C_l = \frac{2}{\pi} \int_{0}^{\infty} \left< \Theta_{\mathbf{k}}(\eta_0) \right> > k^2 dk 
= \frac{2}{\pi} \int_{0}^{\infty} \left| \left( 12 \mathcal{H}^2 + k^2 \right) \cos \left( \frac{k(\eta_0 - \eta_0)}{\sqrt{3}} \right) + 3 \sqrt{3} \mathcal{H} k \sin \left( \frac{k(\eta_0 - \eta_0)}{\sqrt{3}} \right) 
- k j_l(k(\eta_0 - \eta_0)) \right|^2 \mathcal{P}(k, \eta_0) \frac{dk}{k}, \] (51)

where \( \mathcal{P}(k, \eta_0) = k^3 < \delta_{\mathbf{k}}(\eta_0) \delta_{\mathbf{k}}(\eta_0) > \) is a primordial fluid spectrum which serves as an initial condition for the plasma perturbations considered in the previous section.

---

6 The visibility function gives the probability of a CMB photon scattering out of the line of sight within of a \( d\eta \)—layer on the last scattering surface \( \mathbf{1} \).
3.1 Effect of the Finite Thickness of the Last Scattering Surface

A real-world visibility function \( g(\eta) \) is not exactly the Dirac delta-function, but it is smeared over a finite region of \( \eta \). One may approximately assume that it has the Gaussian form

\[
g(\eta) = -\tau'(\eta) \exp(-\tau) = \frac{1}{\Delta \eta_r \sqrt{2\pi}} \exp \left( -\frac{(\eta - \eta_r)^2}{2\Delta \eta_r^2} \right),
\]

where \( \Delta \eta_r \) is a width of the last scattering surface. That corresponds to

\[
\tau(\eta) = -\ln \left( \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{\eta - \eta_r}{\sqrt{2} \Delta \eta_r} \right) \right).
\]

Let us consider the exact integral

\[
\int_{-\infty}^{\infty} g(\eta) e^{ik(\eta - \eta^*)} d\eta = \exp \left( -k^2 \frac{\Delta \eta_r^2}{2} \right) e^{ik(\eta_r - \eta^*)}.
\]

As it is seen (Eq. 54), the variable \( \eta \) is changed by \( \eta_r \) in the expression \( e^{ik(\eta - \eta^*)} \) after integration, and besides a suppression factor appears.

The expression (50) contains the exponents \( e^{(k\pm k/\sqrt{3})\eta} \) originating from both Bessel functions and \( \tilde{\delta}_k \). Thus, the suppression factor \( e^{-(k\pm k/\sqrt{3})\Delta \eta_r^2/2} \) appears in (50) as a result of integration, which has to be introduced into the integrand of (51). The overall damping factor originates from both Silk dumping and finite width of the last scattering surface, but the last gives the main contribution. The calculation of the last scattering surface width has to take into account the process of hydrogen recombination. In the standard \( \Lambda \)CDM model, one needs using the kinetic equations involving at least three levels of the hydrogen atom. The Milne-like universe
$D(k) = \left( \int_0^{\eta_b} g(\eta) \left( \bar{\delta}_4(\eta) j_i(k(\eta - \eta_0)) + \bar{v}_4(\eta) j'_i(k(\eta - \eta_0)) \right) d\eta \right)^2 / \left( \frac{\bar{\delta}_4(\eta)}{4} j_i(k(\eta - \eta_0)) \right)^2 \text{ for } \Delta \eta_0 = 0.03, l = 300. \text{ Dashed and solid curves correspond to } D(k) = \exp \left( -\frac{(k - k/\sqrt{3})^2}{2} \Delta \eta_0^2 \right) \text{ and } D(k) = \exp \left( -\frac{(k + k/\sqrt{3})^2}{2} \Delta \eta_0^2 \right) \text{ respectively.}$

Fig. 2 Calculated damping factor due to finite width of the last scattering surface.

$\frac{n_p n_e}{n_H} = \frac{X_e^2}{1 - X_e} \eta_b = \left( \frac{T m_e}{2 \pi} \right)^{3/2} \exp \left( -\frac{B_H}{T} \right)$

is a good estimation, where $n_p$ is a proton density, and $n_H$ is a density of neutral atoms.

Eq. (55) allows obtaining the hydrogen ionization degree $X_e = n_p / n_b$, where $n_b = n_p + n_H$. An optical depth is calculated as

$$\tau(\eta) = \sigma T \int_\eta^{\eta_b} n_b(\eta') X_e(\eta') a(\eta') d\eta'$$

where $n_b$ scales as $n_b(\eta) = n_{id0}/a^3(\eta)$ and $n_{id0}$ is given by (32). The visibility function for different values of the matter density is shown in Fig. 1. As one can see the width $\Delta \eta_0$ of the Gaussian approximation is 0.05 for $\Omega_b = 0.03$ and 0.03 for $\Omega_b = 0.3$. In the last case, the visibility function has non-Gaussian shape. However, the initial stage of recombination affects mainly the “left front” of the visibility function which becomes “sharper” and can be approximated by a Gaussian function shown in Fig. 1b.

The expression (54) is exact only for averaging of the exponent, however it is approximately valid and for more complicated expressions like the integrand of (50). As one can see from Fig. 2 the lowest suppression factor $e^{-(k - k/\sqrt{3})^2/2}$ should be taken for the calculations.
4 Results and discussion

A distance from the last scattering surface to the present time observer is \( \eta_0 - \eta_r \). For the Milne-like universe \( \{12\} \) these distances are \( \eta_0 = \frac{1}{H} \ln \frac{1}{B} \) and \( \eta_r = \frac{1}{H} \ln \frac{a_r}{B} \) respectively. Thus, one has \( \eta_0 - \eta_r \sim H^{-1} \ln z_r \sim 7H^{-1} \) independent of \( B \).

![Diagram of time scales](image)

**Fig. 3** Schematic representation of the time scales in the a) standard \( \Lambda \)CDM and b) linear cosmologies respectively.

To calculate the spectrum according to \( (51) \), one needs knowing the initial spectrum. The standard model of cosmological inflation gives almost flat spectrum, i.e., \( P(k) \approx \text{const} \) and the oscillations in the observed CMB anisotropy spectrum are interpreted as a result of acoustic oscillation of the photon-baryon plasma. There is a principled difference between the standard model and the linear cosmology considered here. In the standard model, the typical angular scale is \( \theta \sim \frac{\eta_0 - \eta_r}{\eta_0} \sim \frac{\ln B}{\eta_0} \). As a consequence of \( \eta_r < \eta_0 \) in the \( \Lambda \)CDM model, one may obtain the angular scale of \( \theta \sim 1^\circ \) coinciding with the experimental one. In the linear cosmology \( \eta_r \sim \eta_0 \) (see scheme in Fig. 4) and the spectrum oscillations should have another origin. In particular, they could originate from the oscillations of the initial spectrum \( P(k, \eta_{\text{in}}) \), which can be taken in the form

\[
P(k, \eta_{\text{in}}) = 3 \times 10^{-7}\sin k\eta_{\text{in}}^2.
\]

For the dependence \( \{12\} \), one has to take \( \eta_{\text{in}} \sim 0.06 \) to obtain experimentally observed angular scale, that gives \( \theta \sim \frac{\eta_{\text{in}}}{\eta_0 - \eta_r} \sim 0.4^\circ \).

It is easy to calculate cosmic (i.e. physical) time \( t_{\text{in}} \) corresponding to the conformal time \( \eta_{\text{in}} \). Integrating with \( \{12\} \) gives \( t_{\text{in}} = \int_0^{\eta_{\text{in}}} a(\eta)d\eta \approx B \eta_{\text{in}} \), where it is taken into account that \( \dot{H} \eta_{\text{in}} \ll 1 \). For instance, taking \( \eta_{\text{in}} = 0.06 / \dot{H} \) and \( B = 3.8 \times 10^{-38} \) gives \( t_{\text{in}} = 5.9 \times 10^{-22} \) s, which corresponds to the lifetime of
the Higgs boson $t_H = 2\pi / \Gamma_H$, where $\Gamma_H = 7\ MeV$. Here, it is implied that Higgs bosons are created initially [28], then decay into another particles and, finally, into the baryons and photons. Taking another value of $B$ one requires connecting $\eta_{in}$ with another physical process.

The initial spectrum (57) has to be multiplied by the damping factor

$$D(k) \approx \exp \left( -\left( k - k / \sqrt{3} \right)^2 \Delta \eta_r^2 \right) \approx \exp \left( -k^2 / 80 \right).$$

and substituted into Eq. (51). We do not predict absolute values, and the coefficient in (57) is taken to reproduce only highest first CMB peak. The result, shown in Fig. 5 (a) demonstrates a too strong suppression of higher harmonics in comparison with the observational data. To improve the agreement, one may take a rising initial spectrum

$$\mathcal{P}(k, \eta_{in}) = 3 \times 10^{-7} \sin k \eta_{in}^2 \exp \left( k^2 / \kappa_{in}^2 \right).$$

with $\kappa_{in} = 87$ in order to obtain the overall damping factor about of $\exp \left( -k^2 / 200^2 \right)$, because $80^2 - 87^2 \approx 200^2$.

The result of calculation with this formula is shown in Fig. 5 (b). The Planck-satellite data give a very precise measurement of the CMB anisotropy [2][29][31]. One can see the qualitative coincidence with the spectrum observed by the Planck-satellite. The positions of the peaks are shifted relatively observed ones. However, it is no wonder because the model considered is rough and requires further development. At least, the model needs taking into account the baryonic content explicitly. Of course, no analytic solutions for perturbations could be found with this complication. The Silk dumping and the finite width of the last scattering surface have to be taken into account more accurately. Besides, more complicated models of the initial spectrum have to be considered.

---

7 The case of the best agreement with the observational data is considered: $\Omega_m = 0.3$, $\Delta \eta_r = 0.03$. 

---

Fig. 4 a) Initial spectrum multiplied by the all the damping factors, i.e., the resulting spectrum $\mathcal{P}(k) = 3 \times 10^{-7} \sin 0.06k^2 \exp \left(-k^2 / 200^2\right)$, which reproduces the observational data qualitatively. b) Rising initial spectrum $\mathcal{P}(k) = 3 \times 10^{-7} \sin 0.06k^2 \exp\left(k^2 / 87^2\right)$. It is seen, that the perturbations with $k > 350\ H$ lie in the nonlinear region, because $\mathcal{P}(k) > 1$. 

---

The initial spectrum (57) has to be multiplied by the damping factor

$$D(k) \approx \exp \left( -(k - k / \sqrt{3})^2 \Delta \eta_r^2 \right) \approx \exp \left( -k^2 / 80 \right).$$

and substituted into Eq. (51). We do not predict absolute values, and the coefficient in (57) is taken to reproduce only highest first CMB peak. The result, shown in Fig. 5 (a) demonstrates a too strong suppression of higher harmonics in comparison with the observational data. To improve the agreement, one may take a rising initial spectrum

$$\mathcal{P}(k, \eta_{in}) = 3 \times 10^{-7} \sin k \eta_{in}^2 \exp \left( k^2 / \kappa_{in}^2 \right).$$

with $\kappa_{in} = 87$ in order to obtain the overall damping factor about of $\exp \left( -k^2 / 200^2 \right)$, because $80^2 - 87^2 \approx 200^2$.

The result of calculation with this formula is shown in Fig. 5 (b). The Planck-satellite data give a very precise measurement of the CMB anisotropy [2][29][31]. One can see the qualitative coincidence with the spectrum observed by the Planck-satellite. The positions of the peaks are shifted relatively observed ones. However, it is no wonder because the model considered is rough and requires further development. At least, the model needs taking into account the baryonic content explicitly. Of course, no analytic solutions for perturbations could be found with this complication. The Silk dumping and the finite width of the last scattering surface have to be taken into account more accurately. Besides, more complicated models of the initial spectrum have to be considered.

---

7 The case of the best agreement with the observational data is considered: $\Omega_m = 0.3$, $\Delta \eta_r = 0.03$. 

---
From a fundamental point of view, it could imagine some breathtaking physics like the inflation theory. However, it could be quite different, because the inflation cannot produce, a “violet”, i.e., rising with $k$, initial spectrum (59). In principle, the linear cosmology needs no inflation, because the scales of perturbations modes always remain within the horizon and there is no need in any model like inflation for the superhorizon spectral modes. Thus, the linear universe seems in some sense simpler compared to the standard $\Lambda$CDM model. However, the most fundamental problem of the linear cosmology is a requirement of more accurate consideration of vacuum perturbations with taking into account the quantum properties of the vacuum. The above simple model of vacuum as a fluid with the equation of state $w = -1/3$ is an only very rough heuristic approximation.

Unfortunately, well-known software packages such as CAMB [32] and CMB-FAST [33] are absolutely useless for the calculation of CMB spectrum in the linear cosmology because they assume a quite different formation mechanism for the CMB spectrum peaks. It seems that the tools for the ionization history analysis, such as RECFAST [34], also have to be modified to take into account more than three levels of the hydrogen atom. It results from the fact that partially ionized hydrogen plasma is closer to thermal equilibrium due to the slower expansion of the Milne-like universe and, thereby, more hydrogen levels are populated. It seems that the pure equilibrium Saha formula used above gives a sufficiently good approximation in this case.

It should also to do some notes about distortion of the CMB spectrum from blackbody one [35]. The expected distortion of the spectrum caused by hydrogen recombination should be much smaller than that in the $\Lambda$CDM model.
Appendix

The expression for the perturbation of the vacuum density is given by

$$
\delta \rho_{v,k} = \frac{B^2 \mathcal{H}^3 M_p^2 e^{-4\eta \mathcal{H}}}{8\rho_{v0}^2 (3, \mathcal{H}^2 + k^2)} \left( -C_1 (B^2 \mathcal{H} M_p^2 e^{2\eta \mathcal{H}} (3, \mathcal{H}^2 + 2i\sqrt{3}, \mathcal{H} k - k^2) \right.
$$

$$
-6 \mathcal{H} \rho_{v0} + 2i\sqrt{3} \rho_{v0} e^{\frac{ni}{\sqrt{3}}} + C_2 ( -B^2 \mathcal{H} M_p^2 e^{2\eta \mathcal{H}} (3, \mathcal{H}^2 - 2i\sqrt{3}, \mathcal{H} k - k^2) \right)
$$

$$
+6 \mathcal{H} \rho_{v0} + 2i\sqrt{3} \rho_{v0} e^{\frac{ni}{\sqrt{3}}} \left) + \frac{B^2 \mathcal{H}^3 M_p^2 e^{-2\eta \mathcal{H}}}{4\rho_{v0} (3, \mathcal{H}^2 + k^2)} \left( C_3 (3, \mathcal{H} + i\sqrt{3}) e^{-\frac{ni}{\sqrt{3}}} \right.
$$

$$
+ C_4 (3, \mathcal{H} - i\sqrt{3}) e^{\frac{ni}{\sqrt{3}}} \left) - \frac{k^2 M_p^2 e^{-3\eta \mathcal{H}}}{18B^2} \left( Z_1 e^{-\eta \sqrt{\mathcal{H}^2 + k^2}} + Z_2 e^{\eta \sqrt{\mathcal{H}^2 + k^2}} \right),
$$

Then

$$
V_{ik} = -\frac{B^2 \mathcal{H}^2 M_p^2 e^{-4\eta \mathcal{H}}}{24\rho_{v0}^2 (3, \mathcal{H}^2 + k^2)} \left( C_1 (B^2 \mathcal{H} k M_p^2 e^{2\eta \mathcal{H}} (3, \mathcal{H}^2 + 2i\sqrt{3}, \mathcal{H} k - k^2) \right.
$$

$$
-2i\rho_{v0} (6i\sqrt{3}, \mathcal{H}^2 - 3i, \mathcal{H} k + \sqrt{3} k^2) e^{\frac{ni}{\sqrt{3}}} + C_2 (B^2 \mathcal{H} k M_p^2 e^{2\eta \mathcal{H}} (3, \mathcal{H}^2
$$

$$
-2i\sqrt{3}, \mathcal{H} k - k^2) + 2i\rho_{v0} (6i\sqrt{3}, \mathcal{H}^2 + 3i, \mathcal{H} k + \sqrt{3} k^2) e^{\frac{ni}{\sqrt{3}}} \left) + \frac{B^2 \mathcal{H}^2 M_p^2 e^{-2\eta \mathcal{H}}}{12\rho_{v0} (3, \mathcal{H}^2 + k^2)} \left( C_3 (3, \mathcal{H} + i\sqrt{3}) e^{-\frac{ni}{\sqrt{3}}} + C_4 (3, \mathcal{H} - i\sqrt{3}) e^{\frac{ni}{\sqrt{3}}} \right)
$$

$$
+ \frac{k^2 M_p^2 e^{-3\eta \mathcal{H}}}{34B^2} \left( Z_1 (\sqrt{3}, \mathcal{H}^2 + k^2 + 3, \mathcal{H}) e^{-\eta \sqrt{\mathcal{H}^2 + k^2}} \right.
$$

$$
+ Z_2 (\sqrt{3}, \mathcal{H}^2 + k^2 - 3, \mathcal{H}) e^{\eta \sqrt{\mathcal{H}^2 + k^2}} \right),
$$

$$
F_k = \frac{B^4 e^{-2\eta \mathcal{H}}}{4\rho_{v0}^2 (3, \mathcal{H}^2 + k^2)} \left( C_1 (B^2 \mathcal{H} M_p^2 e^{2\eta \mathcal{H}} (-3i\sqrt{3}, \mathcal{H}^2 + 6, \mathcal{H} k
$$

$$+ i\sqrt{3} k^2) - 6i\sqrt{3}, \mathcal{H} \rho_{v0} - 6k \rho_{v0} e^{\frac{ni}{\sqrt{3}}} + C_2 (B^2 \mathcal{H} M_p^2 e^{2\eta \mathcal{H}} (3i\sqrt{3}, \mathcal{H}^2 + 6, \mathcal{H} k - i\sqrt{3} k^2) + 6i\sqrt{3}, \mathcal{H} \rho_{v0} - 6k \rho_{v0} e^{-\frac{ni}{\sqrt{3}}} \right) - \frac{3B^4}{2k\rho_{v0} (3, \mathcal{H}^2 + k^2)}
$$

$$
\left( C_3 (k - i\sqrt{3}, \mathcal{H}) e^{-\frac{ni}{\sqrt{3}}} + C_4 (k + i\sqrt{3}, \mathcal{H}) e^{\frac{ni}{\sqrt{3}}} \right) + e^{-\eta \mathcal{H}} \left( Z_1 e^{-\frac{\eta \sqrt{3} k^2 + 2k}{\sqrt{3}}} \right.
$$

$$
+ Z_2 e^{\frac{\eta \sqrt{3} k^2 + 2k}{\sqrt{3}}} \right),
$$

$$
A_k = \frac{e^{-2\eta \mathcal{H}}}{24\rho_{v0}^2} \left( C_1 (-6B^4 \rho_{v0} + B^6 \mathcal{H} M_p^2 e^{2\eta \mathcal{H}} (3, \mathcal{H} + i\sqrt{3}) e^{-\frac{ni}{\sqrt{3}}}
$$

$$
+ C_2 (-6B^4 \rho_{v0} + B^6 \mathcal{H} M_p^2 e^{2\eta \mathcal{H}} (3, \mathcal{H} - i\sqrt{3}) e^{\frac{ni}{\sqrt{3}}}) \left) \right).$$

\[-\frac{B^4}{4\rho_0} \left( C_3 e^{-\frac{m_1}{\sqrt{3}}} + C_4 e^{\frac{m_1}{\sqrt{3}}} \right).\]

References

1. Mukhanov, V.: Physical Foundations of Cosmology, (Cambridge University Press, Cambridge, 2005)
2. Durrer, R.: The Cosmic Microwave Background (Cambridge University Press, Cambridge, 2008)
3. Dodelson, S.: Modern Cosmology, (Elsevier, Amsterdam, 2003)
4. Milne, E. A.: Kinematic Relativity (The Clarendon Press, Oxford, 1935)
5. Milne, E.A.: Relativity, Gravitation and World-Structure (The Clarendon Press, Oxford, 1935)
6. Dev, A., Safonova, M., Jain, D. and Lohiya, D.: Cosmological Tests for a Linear Coasting Cosmology. Phys. Lett. B548, 12–18 (2002)
7. Singh, P. and Lohiya, D.: Constraints on Lepton Asymmetry from Nucleosynthesis in a Linearly Coasting Cosmology. J. Cosmol. Astropart. Phys. 05, 061 (2015)
8. Benoit-Lévy, A. and Chardin, G.: The Dirac-Milne cosmology. Int. J. Mod. Phys.: Conf. Ser. 30, 1402072 (2014)
9. Melia, F.: On recent claims concerning the $R_h = ct$ Universe. Monthly Notices of the Royal Astronomical Society 446, 1191–1194 (2015)
10. Melia, F.: The Linear Growth of Structure in the $R_h = ct$ Universe. Monthly Notices of the Royal Astronomical Society 464, 1966–1976 (2017)
11. Shafer, D.L.: Robust model comparison disfavors power law cosmology. Phys. Rev. D 91, 103516 (2015)
12. Bengochea, G.R. and Leon G.: Puzzling initial conditions in the $R_h = ct$ model. Eur. Phys. J. C 76, 626 (2016)
13. Tutusaus, I., Lamine, B., Blanchard, A., Dupays, A., Zolnierowski, Y., Cohen-Tanugi, J., Ealet, A., Escoffier, S., Le Fèvre, O., Ilić, S., Pisani, A., Plaszczynski, S., Sakr, Z., Salvatelli, V., Schücker, Th., Tilquin, A., Virey, J.-M.: Power law cosmology model comparison with CMB scale information. Phys. Rev D 94, 103511 (2016)
14. M. V. John, Realistic coasting cosmology from the Milne model. arXiv:1610.09885 [astro-ph.CO]
15. E. Ling, Milne-like spacetimes and their role in Cosmology. arXiv:1706.01408 [gr-qc]
16. Cherka S.L., Kalashnikov, V.L., Universe driven by the vacuum of scalar field: VFD model. Proc. Int. Conf. “Problems of Practical Cosmology”, St.-Petersburg, 2008, Vol. II (Russian Geographical Society, Saint Petersburg (2008)), p. 135, arXiv: astro-ph/0611795.
17. Melia, F.: The cosmic equation of state. Astrophysics and Space Science 356, 393–398 (2015)
18. Cherkas, S.L., Kalashnikov, V.L.: Theory of gravity admitting arbitrary choice of the energy density level. arXiv:1609.00811[gr-qc]
19. Weinberg, S.: The cosmological constant problem. Review of Modern Physics 61, 1–23 (1989)
20. N. D. Birrell, P. C. W. Davis, Quantum Fields in Curved Space (Cambridge Univ. Press, Cambridge, 1982)
21. Zel'dovich, Y.B.: The cosmological constant and the theory of elementary particles. Soviet Physics Uspekhi 11, 381–393 (1968)
22. Adler, R.J., Casey, B., and Jacob O.C.: Vacuum catastrophe: An elementary exposition of the cosmological constant problem. American J. Phys. 63, 620–626 (1995)
23. Landau, L.D., Lifshitz E.M., The Classical Theory of Fields (Butterworth-Heinemann, Oxford, 2000)
24. Lehmkuhl, D., Schiemann, G., Scholz E. (Eds.): Towards a Theory of Spacetime Theories (Springer, Boston, 2010)
25. Anderson, E.: The Problem of Time (Springer, Switzerland, 2017)
26. Zeh, H.D.: *The Physical Basis of the Direction of Time* (Springer, Berlin, 2007)
27. Arnowitt, R., Deser, S., Misner, C.W.: The Dynamics of General Relativity, in: *Gravitation: an introduction to current research*, Witten, L. (Ed.). (Wiley, New York, 1962), chap. 7, p. 227. arXiv: gr-qc/0405109
28. Cherkas, S.L., Kalashnikov, V.L.: Matter creation and primordial CMB spectrum in the inflationless Milne-like cosmologies. Proceeding of the National Academy of Sciences of Belarus (Physics and Mathematics series), No. 4, 88–97 (2017) https://arxiv.org/abs/1707.06073
29. ESA: Planck collaboration results. https://www.cosmos.esa.int/web/planck/publications
30. Aghanim, N., et al.: Planck 2015 results. XI. CMB power spectra, likelihoods, and robustness of parameters. A&A A11 594, (2016)
31. Hu, W., Sugiyama, N.: Small Scale Cosmological Perturbations: An Analytic Approach. Astrophys.J. 471, 542–570 (1996)
32. Lewis, A., Challinor, A., Lasenby, A.: Efficient Computation of CMB anisotropies in closed FRW models. Astrophys. J. 538, 473–476 (2000)
33. Seljak, U., Zaldarriaga, M.: A Line of Sight Approach to Cosmic Microwave Background Anisotropies. Astrophys. J. 469, 437–444 (1996)
34. Seager, S., Sasselov, D. D., Scott, D.: A New Calculation of the Recombination Epoch. Astrophys. J. 523, L1–L5 (1999)
35. Rubino-Martín, J.A., Chluba, J., Sunyaev, R.A.: Lines in the Cosmic Microwave Background Spectrum from the Epoch of Cosmological Hydrogen Recombination. Mon. Not. R. Astron. Soc. 371, 1939–1952 (2006)