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Abstract: In the paper, the authors introduce the notion “logarithmically $h$-preinvex functions”, reveal that the class of $h$-preinvex functions include several new and known classes of preinvex functions, and establish several integral inequalities of Hermite–Hadamard type.

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1. Introduction

Due to extensive applications of convex functions in different fields of pure and applied sciences, many researchers have paid much attention to study and investigate the theory of convex functions. As a result, the concepts of classical convex functions have been extended and generalized in several directions using various innovative approaches (see, e.g. Bai, Qi, & Xi, 2013; Breckner, 1978; Cristescu & Lupsa, 2002; Dragomir, Pečarić, & Persson, 1995; Godunova & Levin, 1985; Jiang, Niu, & Qi, 2014; Noor, Awan, & Noor, 2013; Noor, Noor, & Awan, 2014; Varošanec, 2007; Wang & Qi, 2014; Wang, Wang, & Qi, 2013; Wang, Xi, & Qi, 2014; Weir & Mond, 1988).

PUBLIC INTEREST STATEMENT

The Hermite–Hadamard type inequalities for convex functions and sequences are a milestone of the theory of convex analysis. The concept of convexity for functions has been generalized and extended in many directions and in diverse forms. In the paper, the authors introduce a new notion “logarithmically $h$-preinvex functions”, reveal that the class of $h$-preinvex functions include the logarithmically $s$-preinvex functions, logarithmically $P$-preinvex functions, and logarithmically $Q$-preinvex functions, and establish several integral inequalities of Hermite–Hadamard type for these convex functions.
Motivated by this ongoing research, we now introduce a new class of preinvex functions, which are called logarithmically $h$-preinvex functions, and derive several new integral inequalities of Hermite–Hadamard type for logarithmically $h$-preinvex functions.

2. Definitions and a lemma

Let $K$ be a nonempty closed set in $\mathbb{R}^n$, let $f : K \rightarrow \mathbb{R}$ be a continuous function, and let $\eta(\cdot, \cdot) : K \times K \rightarrow \mathbb{R}^n$ be a continuous bi-function.

**Definition 2.1** (Weir & Mond, 1988) A set $K$ is said to be invex with respect to $\eta(\cdot, \cdot)$, if $a + t\eta(b, a) \in K$ for $a, b \in K$ and $t \in [0, 1]$. The invex set $K$ is also called an $\eta$-connected set.

**Remark 2.1** (Antczak, 2005) The above Definition 2.1 has a geometric interpretation. This definition essentially says that there is a path starting from a point $a$ which is contained in $K$. The point $b$ may not be one of the end points of the path. This observation plays an important role in our analysis. If $b$ is an end point of the path for every pair of points $a, b \in K$, then $\eta(b, a) = b - a$ and, consequently, invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\eta(b, a) = b - a$, but not conversely (see Mohan & Neogy, 1995; Weir & Mond, 1988 and related references therein). For the sake of simplicity, we always assume that $K = [a, a + \eta(b, a)]$ unless otherwise specified.

**Definition 2.2** (Weir & Mond, 1988) A function $f$ is said to be preinvex with respect to an arbitrary bi-function $\eta(\cdot, \cdot)$, if

$$f(a + t\eta(b, a)) \leq (1 - t)f(a) + tf(b)$$

is valid for $a, b \in K$ and $t \in [0, 1]$.

A function $f$ is said to be preconcave if and only if its negative $-f$ is preinvex. For different aspects and applications of the preinvex functions in variational inequalities (see Antczak, 2005; Barani, Ghazanfari, & Dragomir, 2012; Farajzadeh, Noor, & Noor, 2009; Jiang, Niu, Hua, & Qi, 2012; Matloka, 2013; Mishra & Noor, 2005; Mohan & Neogy, 1995; Noor, 2005, 2007a, 2007b, 1994; Noor, Qi, & Awan, 2013; Sarikaya, Alp, & Bzkurt, 2013; Sarikaya, Saglam, & Yildirim, 2008; Wang & Qi, 2014; Wang et al., 2013, 2014; Weir & Mond, 1988; Yang, Yang, & Teo, 2003).

For $\eta(b, a) = b - a$ in Equation 2.1, the preinvex function becomes a convex function in the classical sense.

**Definition 2.3** (Noor, Noor, Awan, & Li, 2015) Let $h : J \rightarrow \mathbb{R}$, where $(0, 1) \subseteq J$ and $h \not\equiv 0$, be an interval in $\mathbb{R}$ and let $K$ be an invex set with respect to $\eta(\cdot, \cdot)$. A nonnegative function $f : K \rightarrow \mathbb{R}$ is called $h$-preinvex with respect to $\eta(\cdot, \cdot)$, if

$$f(a + t\eta(b, a)) \leq h(1 - t)f(a) + h(t)f(b)$$

holds for $a, b \in K$ and $t \in (0, 1)$.

In Noor et al. (2015), it was shown that the class of $h$-preinvex functions generalizes several other classes of convex functions. For example, if we take $h(t) = t, h(t) = \frac{1}{2}, h(t) = t^2$, and $h(t) = 1$ in (2.2), then the $h$-preinvex function reduces to the preinvex function in Weir and Mond (1988), the $Q$-preinvex function, the $s$-preinvex function, and the $P$-preinvex function, respectively. If we take $\eta(b, a) = b - a$, then the definition of $h$-preinvex functions reduces to the definition of $h$-convex functions, which was introduced in Varošanec (2007). Noor (2007a) showed that a function $f$ is preinvex if and only if...
The double inequality Equation 2.3 is known as the Hermite–Hadamard–Noor inequality for preinvex functions. If \( \eta(b, a) = b - a \), then the double inequality Equation 2.3 reduces to the classical Hermite–Hadamard inequality for convex functions. For recent developments and applications (see Sarıkaya et al., 2013).

**Definition 2.4** A function \( f : K \to (0, \infty) \) is said to be logarithmically \( h \)-preinvex with respect to \( \eta(\cdot, \cdot) \), if

\[
 f(a + t\eta(b, a)) \leq [f(a)]^{h^{1-t}}[f(b)]^{ht^t} \leq f(a) + f(b)
\]

for \( a, b \in I \) and \( t \in (0, 1) \).

**Remark 2.2** From Definition 2.4, we may obtain

\[
\ln f(a + t\eta(b, a)) \leq \ln[f(a)]^{h^{1-t}}[f(b)]^{ht^t} = \ln[f(a)]^{h^{1-t}} + \ln[f(b)]^{ht^t} = h(1-t)\ln f(a) + h(t)\ln f(b)
\]

**Remark 2.3** If \( h(t) = t^t \), then the definition of logarithmically \( h \)-prinvex function reduces to the definition of logarithmically \( s \)-preinvex function.

**Definition 2.5** A function \( f : K \to (0, \infty) \) is said to be logarithmically \( s \)-preinvex, where \( s \in (0, 1) \), with respect to \( \eta(\cdot, \cdot) \), if

\[
 f(a + t\eta(b, a)) \leq [f(a)]^{1-s} [f(b)]^s
\]

for \( a, b \in I \) and \( t \in [0, 1) \).

**Remark 2.4** If \( h(t) = 1 \), then the definition of logarithmically \( h \)-preinvex function reduces to the definition of logarithmically \( P \)-preinvex function.

**Definition 2.6** A function \( f : K \to (0, \infty) \) is said to be logarithmically \( P \)-preinvex with respect to \( \eta(\cdot, \cdot) \), if

\[
 f(a + t\eta(b, a)) \leq [f(a)]^t[f(b)]
\]

for \( a, b \in I \) and \( t \in [0, 1) \).

**Remark 2.5** If \( h(t) = \frac{1}{t} \), then the definition of logarithmically \( h \)-preinvex function reduces to the definition of logarithmically \( Q \)-preinvex function.

**Definition 2.7** A function \( f : K \to (0, \infty) \) is said to be logarithmically \( Q \)-preinvex with respect to \( \eta(\cdot, \cdot) \), if

\[
 f(a + t\eta(b, a)) \leq [f(a)]^{1/t} [f(b)]^{1/t}
\]

for \( a, b \in I \) and \( t \in (0, 1) \).

To prove some results in this paper, we need the following well-known Condition C introduced by Mohan and Neogy.
\textbf{Condition C \ (Mohan & Neogy, 1995)} Let $K \subset \mathbb{R}$ be an invex set with respect to the bi-function $\eta(\cdot, \cdot)$. Then for any $a, b \in K$ and $t \in [0, 1]$, we have

$$\eta(b, b + t\eta(a, b)) = -t\eta(a, b) \quad \text{and} \quad \eta(a, b + t\eta(a, b)) = (1 - t)\eta(a, b)$$

From Condition C, it follows that

$$\eta(b + t_2\eta(a, b), b + t_1\eta(a, b)) = (t_2 - t_1)\eta(a, b)$$

for every $a, b \in K$ and $t_1, t_2 \in [0, 1]$.

It is worth mentioning that Condition C plays a crucial and significant role in the development of the variational-like inequalities and optimization problems (see Farajzadeh et al., 2009; Mohan & Neogy, 1995; Noor, 1994; Noor et al., 2013 and related references therein).

The following lemma is also necessary for us.

\textbf{Lemma 2.1 \ (Barani et al., 2012)} Let $f : K \to (0, \infty)$ be a differentiable mapping on $[a, a + \eta(b, a)] \subseteq K$ with $\eta(b, a) > 0$. If $f' \in L_1[a, a + \eta(b, a)]$, then

$$\frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x)dx - \frac{f(a) + f(a + \eta(b, a))}{2} = \frac{\eta(b, a)}{2} \int_0^1 (1 - 2t)f'(a + t\eta(b, a))dt$$

\section{Main results}

We now start out to establish several new integral inequalities of Hermite–Hadamard type for logarithmically $h$-preinvex functions.

\textbf{Theorem 3.1} \ Let $f$ be a logarithmically $h$-preinvex function such that $h(\frac{1}{2}) \neq 0$. Also suppose that Condition C holds for $\eta$, then, for $\eta(b, a) > 0$, we have

$$\ln f\left(\frac{2a + \eta(b, a)}{2}\right)^{1/2h(1/2)} \leq \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} \ln f(x)dx$$

$$\leq \left[\ln f(a) + \ln f(b)\right] \int_0^1 h(t)dt$$

Consequently,

$$f\left(\frac{2a + \eta(b, a)}{2}\right)^{1/2h(1/2)} \leq \exp\left[\frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} \ln f(x)dx\right] \leq \left|f(a)f(b)\right|^{1/h}\int_0^1 h(t)dt$$

\textbf{Proof} \ Since $f$ is logarithmically $h$-preinvex, using Condition C, we have

$$f\left(\frac{2a + \eta(b, a)}{2}\right) = f\left(a + (1 - t)\eta(b, a) + \frac{\eta(a + t\eta(b, a), a + (1 - t)\eta(b, a))}{2}\right)$$

$$\leq \left|f(a + t\eta(b, a))\right|^{h^{1/2}}\left|f(a + (1 - t)\eta(b, a))\right|^{h^{1/2}}$$

$$= \left[\left|f(a + t\eta(b, a))\right|f(a + (1 - t)\eta(b, a))\right]^{h^{1/2}}$$

Taking the logarithm on both sides of the above inequality yields

$$\ln f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \ln f(a + t\eta(b, a))f(a + (1 - t)\eta(b, a))^{h^{1/2}}$$

$$= h\left(\frac{1}{2}\right) \ln f(a + t\eta(b, a))f(a + (1 - t)\eta(b, a))$$
which implies that

\[
\frac{1}{h(1/2)} \ln f \left( \frac{2a + \eta(b, a)}{2} \right) \leq \ln [f(a + t\eta(b, a))f(a + (1 - t)\eta(b, a))] \\
= \ln f(a + t\eta(b, a)) + \ln f(a + (1 - t)\eta(b, a))
\]

Integrating on both sides of the above inequality with respect to \( t \in [0, 1] \) gives

\[
\frac{1}{h(1/2)} \ln f \left( \frac{2a + \eta(b, a)}{2} \right) = \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(x) dx \\
+ \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(x) dx = \frac{2}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(x) dx
\]

which means that

\[
\frac{1}{2h(1/2)} \ln f \left( \frac{2a + \eta(b, a)}{2} \right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(x) dx \tag{3.1}
\]

Integrating on both sides of

\[
\ln f(a + t\eta(b, a)) \leq h(1 - t) \ln f(a) + \ln h(t)f(b)
\]

with respect to \( t \in [0, 1] \) shows

\[
\frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(x) dx \leq [\ln f(a) + \ln f(b)] \int_0^1 h(t) dt \tag{3.2}
\]

Combining Equations 3.1 and 3.2 reveals that

\[
\ln f \left( \frac{2a + \eta(b, a)}{2} \right)^{1/2h(1/2)} \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(x) dx \\
\leq [\ln f(a) + \ln f(b)] \int_0^1 h(t) dt
\]

which is equivalent to

\[
f \left( \frac{2a + \eta(b, a)}{2} \right)^{1/2h(1/2)} \leq \exp \left[ \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(x) dx \right] \\
\leq \exp^{\ln f(a) + \ln f(b)} \int_0^1 h(t) dt \\
= \exp^{[f(a)f(b)]^{1/h(b,a)}} \\
= [f(a)f(b)]^{1/h(b,a)}
\]

The proof of Theorem 3.1 is complete.

**Corollary 3.1** Let \( f \) be a logarithmically \( s \)-preinvex function. Also suppose that Condition C holds for \( \eta \), then, for \( \eta(b, a) > 0 \), we have

\[
\ln f \left( \frac{2a + \eta(b, a)}{2} \right)^{2^{s-1}} \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(x) dx \leq \frac{\ln f(a) + \ln f(b)}{s + 1}
\]

Consequently,

\[
f \left( \frac{2a + \eta(b, a)}{2} \right)^{2^{s-1}} \leq \exp \left[ \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(x) dx \right] \leq [f(a)f(b)]^{1/(s+1)}
\]
Proof This follows from taking \( h(t) = t^s \) for \( s \in (0, 1) \) in Theorem 3.1.

Corollary 3.2  Let \( f \) be a logarithmically \( P \)-preinvex function. Also suppose that Condition C holds for \( \eta \), then, for \( \eta(b, a) > 0 \), we have

\[
\ln f\left( \frac{2a + \eta(b, a)}{2} \right) \leq \frac{2}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(x) dx \leq 2[\ln f(a) + \ln f(b)]
\]

Consequently,

\[
f\left( \frac{2a + \eta(b, a)}{2} \right) \leq \exp \left[ \frac{2}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(x) dx \right] \leq [f(a)f(b)]^2
\]

Proof This follows from letting \( h(t) = 1 \) in Theorem 3.1.

Corollary 3.3  Let \( f \) be a logarithmically \( Q \)-preinvex function. Also suppose that Condition C holds for \( \eta \), then, for \( \eta(b, a) > 0 \), we have

\[
\frac{1}{4} \ln f\left( \frac{2a + \eta(b, a)}{2} \right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(x) dx
\]

Consequently,

\[
f\left( \frac{2a + \eta(b, a)}{2} \right)^{1/4} \leq \exp \left[ \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(x) dx \right]
\]

Proof This follows from setting \( h(t) = \frac{1}{4} \) in Theorem 3.1.

Remark 3.1  When \( \eta(b, a) = b - a \), the above results reduce to ones for classical logarithmically \( h \)-convex functions, logarithmic \( s \)-convex functions, logarithmic \( P \)-convex functions, and logarithmic \( Q \)-convex functions, respectively (see Noor et al., 2013).

Theorem 3.2  Let \( f, g : \mathbb{K} \rightarrow \mathbb{R} \) be logarithmically \( h \)-preinvex functions and \( a, a + \eta(b, a) \in \mathbb{K} \) with \( \eta(b, a) > 0 \). Then

\[
\int_a^{a+\eta(b, a)} f(x)g(x)dx \leq a \int_0^1 \{f(a)\}^{h1-t}/\alpha \{f(b)\}^{h1-t}/\beta dt + \beta \int_0^1 \{g(a)\}^{h1-t}/\alpha \{g(b)\}^{h1-t}/\beta dt
\]

Proof Using Young’s inequality \( ab \leq a^{1/s} + b^{1/\beta} \) for \( a, \beta > 0 \) and \( a + \beta = 1 \) produces

\[
\int_a^{a+\eta(b, a)} f(x)g(x)dx = \int_0^1 f(a + t\eta(b, a))g(a + t\eta(b, a))dt
\]

\[
\leq \int_0^1 \{a[f(a + t\eta(b, a))]^{1/s} + \beta[g(a + t\eta(b, a))]^{1/\beta}\} dt
\]

\[
\leq \int_0^1 \{a[(f(a))^{h1-t}f(b)]^{h1-t}/\alpha + \beta[(g(a))^{h1-t}g(b)]^{h1-t}/\beta\} dt
\]

\[
= a \int_0^1 [(f(a))^{h1-t}/a[f(b)]^{h1-t}/\beta] dt + \beta \int_0^1 [(g(a))^{h1-t}/\alpha [(g(b))]^{h1-t}/\beta] dt
\]

The proof of Theorem 3.2 is complete.

Theorem 3.3  Let \( f : \mathbb{K} \rightarrow (0, \infty) \) be a differentiable function such that \( f' \in L_1[a, a + \eta(b, a)] \). If \( |f'|^q \) is logarithmically \( h \)-preinvex on \( K \) for \( q > 1 \), \( h(t) + h(1-t) = 1 \) and \( \eta(b, a) > 0 \), then

\[
\int_a^{a+\eta(b, a)} f(x)g(x)dx \leq a \int_0^1 \{f(a)\}^{h1-t}/\alpha \{f(b)\}^{h1-t}/\beta dt + \beta \int_0^1 \{g(a)\}^{h1-t}/\alpha \{g(b)\}^{h1-t}/\beta dt
\]
\[
\frac{1}{\eta(a, b)} \int_{a}^{a+\eta(b, a)} f(x)dx - \frac{f(a) + f(a + \eta(b, a))}{2} \\
\leq \frac{(b - a)|f'(a)|}{2^q} \left( \int_{0}^{1} |1 - 2t| f'(b)\left(\frac{h(t)}{f'(a)}\right)^q dt \right)^{1/q}
\]

Proof Using Lemma 2.1, the well-known power mean inequality, and the condition that \( |f'|^q \) is logarithmically \( h \)-preinvex gives

\[
\frac{1}{\eta(a, b)} \int_{a}^{a+\eta(b, a)} f(x)dx - \frac{f(a) + f(a + \eta(b, a))}{2} \\
= \frac{\eta(b, a)}{2} \left( \int_{0}^{1} |1 - 2t| f'(a + t\eta(b, a))dt \right) \\
\leq \frac{\eta(b, a)}{2} \left( \int_{0}^{1} |1 - 2t| dt \right)^{1-1/q} \left( \int_{0}^{1} |1 - 2t| f'(a + t\eta(b, a))|^{q(1-1/t)} dt \right)^{1/q} \\
\leq \frac{\eta(b, a)}{2} \left( \frac{1}{2} \right)^{1-1/q} \left( \int_{0}^{1} |1 - 2t| f'(a + t\eta(b, a))|^{qh(t)} dt \right)^{1/q} \\
= \frac{\eta(b, a)|f'(a)|}{2^{2^{q-1}/q}} \left( \int_{0}^{1} |1 - 2t| f'(b)\left(\frac{h(t)}{f'(a)}\right)^q dt \right)^{1/q}
\]

This completes the proof of Theorem 3.3

Remark 3.2 For different suitable choices of \( h \), we can obtain corresponding results for logarithmically \( s \)-preinvex functions, and logarithmically \( P \)-preinvex functions.

Corollary 3.4 Let \( f : K \to \mathbb{R} \) be a differentiable function such that \( f' \in L_1[a, a + \eta(b, a)] \). If \( |f'|^q \) is logarithmically \( h \)-preinvex on \( K \), then, for \( \eta(b, a) > 0 \), we have

\[
\frac{1}{\eta(a, b)} \int_{a}^{a+\eta(b, a)} f(x)dx - \frac{f(a) + f(a + \eta(b, a))}{2} \\
\leq \frac{\eta(b, a)|f'(a)|}{2} \left( \int_{0}^{1} |1 - 2t| f'(b)\left(\frac{h(t)}{f'(a)}\right)^q dt \right)^{1/q}
\]

Proof This is a direct consequence of Theorem 3.3 for \( q = 1 \).

Theorem 3.4 Let \( f : K \to (0, \infty) \) be a differentiable function such that \( f' \in L_1[a, a + \eta(b, a)] \). If \( |f'|^q \) is logarithmically \( h \)-preinvex on \( K \) for \( q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \) and if \( h(t) + h(1-t) = 1 \), then, for \( \eta(b, a) > 0 \), we have

\[
\frac{1}{\eta(a, b)} \int_{a}^{a+\eta(b, a)} f(x)dx - \frac{f(a) + f(a + \eta(b, a))}{2} \\
\leq \frac{\eta(b, a)|f'(a)|}{2} \left( \frac{1}{p+1} \right)^{1/p} \left( \int_{0}^{1} |1 - 2t| f'(b)\left(\frac{h(t)}{f'(a)}\right)^q dt \right)^{1/q}
\]

Proof This directly follows from the proof of Theorem 3.3.
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