Cosmological long wavelength perturbations

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Abstract

This paper presents an exact solution to the long wavelength perturbations for the scalar modes and for a scalar field theory with arbitrary potential. Locally these modes are coordinate transformations of the homogeneous background solutions (although non-locally they are not). These solutions are then used to discuss a couple of recent papers in which such perturbations play a role. Abramo, Brandenberger, and Mukhanov have recently argued that long wavelength perturbations have the effect of driving the cosmological constant to zero if the higher order perturbation equation are examined. I argue that this effect is invisible to any local observer, and thus does not constitute a relaxation of the cosmological constant in the normal sense of the term.

Grishchuk has argued that the standard lore on the strength of the perturbations at the end of inflation is wrong. I discuss the disagreement in light of the exact long wavelength solutions, and emphasize the importance of the initial conditions in resolving the disagreement.

In the inflationary models for the growth of the universe, the physics of long wavelength perturbations (wavelengths which are much longer the the Hubble radius) play an important role. It is the behaviour of the perturbations during this time period which determine the effect of the perturbations on the present structure of the universe. Recently, a couple of papers have discussed these long wavelength perturbations in different contexts. Abramo, Brandenberger, and Mukhanov [1] have argued that higher order corrections to Einstein’s equations for long wavelength perturbations have the effect of creating a negative cosmological constant which reduces the effective cosmological constant for the space-time. They also speculate that this could be a mechanism to drive the actual cosmological constant to near zero (or in particular to near the current mass density of the universe). On the other hand, Grishchuk [5,6] has claimed that the standard calculations for the size of the density perturbations at the end of inflation are wrong. Various authors [8,7] have claimed that his treatment of the behaviour of the long wavelength perturbations is wrong.

In this paper, I derive an exact expression for these long wavelength perturbations in a system in which the dominant gravitating matter is a scalar field $\Psi$ with an arbitrary potential $V(\Psi)$. I note in passing that these solutions therefor apply without much change to the evolution with an arbitrary perfect isotropic fluid, under the usual identification that

$$\rho + p = \frac{1}{2} g^{\mu\nu} \Psi_{,\mu} \Psi_{,\nu}$$  \hspace{1cm} (1)
\begin{align}
  u_\mu &= \frac{\Psi_\mu}{\sqrt{\rho + p}} \\
  \rho - p &= V
\end{align}

I. GAUGE INVARINACE VS GAUGE FIXING

Let me first discuss the ABM paper. I was confused by their result that the long wavelengths could renormalise the cosmological constant since the effect is caused by the averaged energy momentum tensor of modes whose wavelength is much larger than the Hubble’s radius at any time of interest. “How could such long wavelength modes affect the cosmological constant as seen by an observer who can measure the cosmological constant only over a region smaller than his own Hubble radius?” I would have expected modes, whose wavelength is much much larger than the Hubble radius, to look like a homogeneous universe to a local observer.

In addition, as they point out, the effective stress energy tensor of lower order perturbations can be coordinate dependent (called gauge dependent if one is looking at small coordinate transformations). Their approach to this problem was to write the equations in terms of ”gauge independent” variables. However, as I will argue in the following, there is no difference between such a ”gauge invariant” approach, and an approach which fixes the gauge in some way. Any variable in a gauge fixed formalism is completely equivalent to some gauge independent variable, and vice-versa. In particular, this means that, even though the effective stress energy tensor is not independent on gauge transformations in the gauge invariant approach, it is dependent on which set of gauge invariant variables one chooses. In the gauge fixing approach, the same problem arises in that the effective stress energy tensor is dependent on which gauge fixing choice one makes.

Let me review their approach. Instead of writing Einstein’s equations in terms of the free metric and matter perturbations, $\delta g_{\mu\nu}$, $\delta \Psi$, they choose a set of vector fields $X^\mu$ created out of the metric and field variables, (i.e., $X^\mu = X^\mu(\delta g_{\lambda\sigma}, \delta \Psi)$ ), and defined new gauge invariant perturbations

\[ \delta \bar{g}_{\mu\nu} = e^{L_X}(g_{(0)\mu\nu} + \delta g_{\mu\nu}) - g_{(0)\mu\nu} \] (4)

where $L_X$ is the Lee derivative with respect to $X$. A similar definition applies to $\delta \bar{\psi}$. $X$ is chosen so that the barred quantities are invariant under a (first order) gauge transformation.

Given the free first order metric perturbations written as

\[ \delta g_{\mu\nu} = \begin{pmatrix} \Phi + S_i^i & S_i + B, i \\ S_i + B, i & \phi g_{(0)ij} + Q_{ij} + R_{ij} + h_{ij} \end{pmatrix} \] (5)

where $\phi, B$, and $Q$ represent the scalar perturbations, the transverse vector fields, $S_i$, $R_i$ represent the vector perturbations, and the transverse-traceless tensor $h_{ij}$ represents the gravitational wave perturbations. $g_{(0)\mu\nu}$, $\Psi_0$ are assumed to be a background flat-space homogeneous solution. In the following, I will concentrate on the scalar perturbations alone ($S_i, R_i$, and $h_{ij}$ will be neglected.
The vectors $X^\mu$ are now chosen to be the infinitesimal coordinate transformations required to reduce the above general metric to some fixed form. For example, one choice (used by ABM) is

$$X_\mu = [\dot{Q} - 2HQ - B, - \partial_i Q]$$

(6)

This is exactly the infinitesimal coordinate transformation required to make the $B$ and $Q$ terms in the perturbation metric equal to zero, leaving only the diagonal $\phi$, $\Phi$ terms non-zero. But the equations of motion for these terms are also exactly the equations of motion for the gauge fixed metric, where the gauge fixed $B$ and $Q$ are set equal to zero. Although ABM mention a few such possible choices for the vectors $X^\mu$, there are a large number which they do not mention because they restricted themselves to use only the local metric variables in defining $X^\mu$, while the use of the matter variables and of non-local metric variables would allow an expanded set of possibilities. In fact, for any gauge fixing, one can find a vector $X^\mu$ which would implement that gauge fixing. For example, if we choose

$$X^\mu = \left[-\frac{\delta \Psi}{\Psi_0}, -\partial_i \left( B - \frac{\delta \psi}{\Psi_0} \right) dt \right]$$

(7)

the gauge invariant scalar field perturbations, $\delta \bar{\psi}$, will be zero, as will the off diagonal temporal terms in the metric, $\bar{B}$. This choice of vector corresponds to the gauge fixing for the first order perturbations of the scalar field and scalar metric of $\delta \psi = 0$, and $\delta g_{0i} = 0$. The choice

$$X^\mu = \left[-\int \phi dt, -\partial_i \left( \int \frac{1}{a^2} \left( \int \phi dt \right) - B \right) dt \right]$$

(8)

on the other hand puts the gauge invariant metric into the synchronous gauge. One problem with using such non-local gauge transformation is that because of the integration “constants” (actually functions of the spatial variables) in $X$, these choices still retain some residual gauge freedom, corresponding to those integration “constants”.

The procedure which they follow is to transform their original general metric by means of these gauge transformations into what they call a gauge-invariant metric. However, this is simply the procedure of making an appropriate coordinate (gauge) transformation so as to put the metric into the appropriate gauge fixed form. They then calculate the second order fluctuations in this gauge fixed form, and obtain a result which they then claim to gauge invariant. It is, in the sense that the results have been calculated in a specific gauge, and thus that specific gauge allows no further gauge transformations. (or to put it in another language, the gauge transformation is undone by their reduction of the metric).

The gauge invariant approach is thus identical to the gauge fixing approach and any problems or advantages of one are also problems and advantages of the other.

**II. LONG WAVELENGTH SOLUTIONS**

To further the discussion, let me now solve the perturbation equations in the long wavelength limit. I will, with ABM, choose the longitudinal gauge fixing, which gives the following equations. The linear metric in this gauge is given by
\[
\begin{align*}
\text{where } \phi(t, x, y, z), \Phi(t, x, y, z) \text{ are the first order perturbation of the metric, and } \psi \text{ is the first order perturbation of the scalar field } \Psi \text{ (called } \delta \Psi \text{ above). The first order Einstein's equation } G_{ij} - T_{ij} = 0 \text{ for } i \neq j \text{ gives }
\partial_i \partial_j (\phi - \Phi) = 0, \\
\text{which by rotational invariance gives }
(\partial_i \partial_j - \frac{1}{3} \nabla^2 \delta_{ij})(\phi - \Phi) = 0 \\
\text{for all } i, j \text{ The solution is }
\Phi = \phi + \kappa_i(t)x^i + \sigma(t)(x^2 + y^2 + z^2) + \nu(t) \\
\text{The term proportional to } \sigma \text{ corresponded to a perturbation toward one of the spherical or hyperbolic homogeneous space-times. The } \nu \text{ and } \kappa_i \text{ terms can be removed by infinitesimal coordinate transformations. I will ignore these possibilities here.}
\end{align*}
\]

Einstein's equations are
\[
E_{\mu\nu} = G_{\mu\nu} - T_{\mu\nu} = 0
\]
Selecting the two constraints, we have
\[
0 = E_{tt}^{(1)}(\phi, \psi) = -6H\dot{\phi} - V\phi - \frac{1}{2}V'\psi - \frac{1}{2}\dot{\Psi}_0\dot{\psi} + \frac{2}{a^2}\nabla^2\phi
\]
\[
0 = E_{i\dot{i}} = -\frac{1}{a}\partial_i \left(2\partial_i(a\phi) + \frac{1}{2}a\partial_i(\Psi_0)\psi\right)
\]

The remaining (spatial diagonal) terms, \(E_{ii}^{1}\), are obtainable from these via the conservation equations \(E_{\mu\nu}^{\mu\nu} = 0\).

In addition the linearized equation for the scalar field is
\[
P^1 = \ddot{\psi} + 3H\dot{\psi} - \nabla^2\psi + V''(\Psi_0)\psi - 4\dot{\Psi}_0\dot{\phi} - 6H\dot{\Psi}_0\phi - 2\ddot{\Psi}_0\phi = 0
\]
In the above, \(\Psi_0(t)\) is the lowest order solution for the scalar field, \(a(t)\) is the lowest order scale factor, \(V(\Psi)\) is the potential for the scalar field, and \(H = \dot{a}/a\) is the background Hubble constant. These variables obey
\[
6H^2 = \frac{1}{2}\dot{\Psi}_0^2 + V
\]
\[
\frac{1}{2}\dot{\Psi}_0^2 = -2H
\]

In the long wavelength limit, where we neglect the term \(\nabla^2\phi\), the equation \(E_{tt}^1 = 0\) and the linearized equation for the scalar field \(P^1 = 0\) are also linearized equations for homogeneous
perturbations of the universe. But at least two of those homogeneous solutions are simply coordinate transformations. These two correspond to multiplying the spatial variables by a constant $\beta$ and the translating the time by an amount $\lambda(t)$. I.e, the new coordinates are

$$
\bar{t} = t - \lambda(t) \\
\bar{x}^i = (1 + \beta)x^i
$$

(19)

The metric now becomes

$$
ds^2 = (1 + 2\dot{\lambda})d\bar{t}^2 - a(\bar{t})^2(1 - 2\beta + 2H\lambda)\delta_{ij}d\bar{x}^id\bar{x}^j
$$

(21)

Demanding that these transformations leave the metric in the longitudinal gauge with $\phi = \Phi$ we get

$$
\dot{\lambda} = \beta - H\lambda
$$

(22)

which we can easily solve to give

$$
\lambda(t) = \frac{\beta \int a dt + \epsilon}{a}
$$

(23)

Thus,

$$
\phi = \partial_t \left( \frac{\beta \int a dt + \epsilon}{a} \right) f(x^i) = (-H\frac{\beta \int a dt + \epsilon}{a} + \beta)f(x^i)
$$

(24)

$$
\psi = \Psi_0\frac{\beta \int a dt + \epsilon}{a} f(x^i)
$$

(25)

must be solutions to $E^1_{tt} = 0$ and $P^1 = 0$ if we neglect the terms proportional to $\nabla^2$ (ie in the long wavelength limit). The only remaining equation is the $E^1_{ti} = 0$ equation. But again, substituting our trial solution into that equation clearly satisfies it.

Since the equation $P^1 = 0$ is a second order temporal equation in $\psi$, and the $E_{tt} = 0$ is a first temporal order equation in $\phi$, we expect three linearly independent solutions to these two coupled linear equations. The above are two of them. However, solving $E_{ti} = 0$ for $\psi$ and substituting into $E^1_{ti} = 0$, we obtain a single second order equation in $\phi$. Since the $E^1_{ti} = 0$ equation uniquely determines $\psi$ if $\phi$ is known, the linearized long wavelength equations have only two solutions, and we already know two of them. The above two solutions are the only solutions to the long wavelength equations.

Note that without the $E^1_{ti} = 0$ equation, there would have been three solutions. In the infinite wavelength limit, the $E^1_{ti} = 0$ equation is satisfied identically (because of the spatial derivative in that equation), and thus there exist three linearly independent solutions to the linearized equations. As I demonstrated above, two of them correspond to coordinate transformations of the original background solution, while the third would correspond to true a physical mode. It is thus interesting that both physical long wavelength solutions go to coordinate transformations in the infinite wavelength limit, while the actual physical infinite wavelength solution has no correspondence for large but finite wavelengths.

Of course, having solved the problem in one gauge, one can determine the solution in any gauge. For interests sake, and because Grishchuk uses that gauge, let us look at the solutions
in synchronous gauge. Since we have the solutions in longitudinal gauge, the problem in
synchronous gauge can be obtained by simply making the appropriate gauge transformation.
The gauge transformation vector to go from longitudinal to synchronous gauge is given by

\[ V^\mu = \left[ -\int \phi(t, x, y, z) dt, \partial_i \int \frac{\phi dt}{a^2} dt \right] \]  

(26)

The transformed metric is

\[ \delta g^{(1)}_{tt} = \delta g^{(1)}_{ti} = 0 \]  

(27)

\[ \delta g^{(1)}_{ij} = 2a^2 \left( (\phi + H \int \phi dt) \delta_{ij} + \partial_i \partial_j \int \frac{\phi dt}{a^2} \right) \]  

(28)

Inserting the known solution for \( \phi \), i.e.,

\[ \phi = f(x, y, z) \partial_i \left( \frac{\epsilon + \beta \int a dt}{a} \right) \]  

(29)

we obtain

\[ \delta g^{(1)}_{ij} = 2a^2 \left( (\beta f(x, y, z) + \kappa(x, y, z)H) \delta_{ij} + \partial_i \partial_j f(x, y, z) \int \frac{\epsilon + \beta \int a dt}{a^3} dt \right) \]  

(30)

\[ + \partial_i \partial_j (\kappa(x, y, z) \int a^{-2} dt + \gamma(x, y, z)) \]  

(31)

\( \kappa \) and \( \gamma \) are temporal integration “constant” spatial functions. Both correspond to
gauge transformations, illustrating the well known feature of the synchronous gauge that it does
not completely specify the gauge. These terms could be removed by a gauge transformation
which leaves the system in synchronous gauge. Note that the part of the metric which is
the \( \bar{\phi} \) term (whose contribution to the metric in synchronous gauge is proportional to \( \delta_{ij} \))
contains only the \( \beta \) parameter of the two parameter family of physical solutions. The other
physical solution, given by the terms which depend on \( \epsilon \), occurs only in the spatial derivative
\( \bar{Q} \) terms. Any second order equation for \( \bar{\phi} \) would therefore pick up only one of the physical
modes. Fortunately the solution which depends on \( \epsilon \) is one which dies out at long times
in an expanding universe, and as a result their neglect would not be of importance in the
late stages. Note that \( \bar{\phi} \) is physically constant (up to a gauge transformation) for all times,
agreeing with the contention of Grishchuk (where this term is given the name of \( h \)).

Examining the long-wavelength perturbation in the scalar field, we find in synchronous
gauge that

\[ \psi = \dot{\Psi}_0(t) \kappa(x, y, z) \]  

(32)

The only term remaining in the scalar field perturbation is a pure gauge term. Ie, syn-
chronous gauge in the long wavelength limit is also a gauge in which the first order scalar
field perturbations are zero, modulo a gauge transformation.
A. Grishchuk’s Criticism

Let me now comment on the controversy between Grishchuk and the rest of the community. As has been emphasized by Grishchuk, who operates in synchronous gauge, the "growing mode" in synchronous gauge is constant for long wavelength modes. In the above, this is just the constant $\beta$. He then argues that this indicates that there is no amplification of the scalar modes in inflation while the modes are outside the Hubble radius (i.e., long wavelength). Since in this regime the scalar field (matter) perturbations are zero, do not display any “amplification” either. His conclusion is that this indicates that the scalar modes are of the same size at the end of inflation (and in fact when they reenter the Hubble radius after reheating) as they were at earlier times when those modes left the Hubble radius (when the physical wavelength first became larger than the Hubble radius). Since the gravity wave modes also do not grow during inflation, this suggests to Grishchuk that both scalar and gravity wave modes should have roughly the same size at the end of inflation. This is in direct contrast with the standard lore, that in general the gravity wave modes are much smaller than the scalar modes.

On the other hand, if one works in the longitudinal gauge, the growing mode of the gravitational metric parameter $\phi$ goes as $\beta H \int dt/a - \beta$, which is (modulo decaying modes) identically zero if the expansion is exactly exponential, and is $\beta/\mu$ if the expansion goes as a power law $a(t) \propto t^\mu$. Furthermore, the scalar field modes go as $\dot{\Phi}(\beta(t-t_0)/(\mu + 1))$ for a power law increase in $a$. Since both $\Psi$ and $t$ increase as inflation continues, these scalar field modes grow from the start of inflation (where $\dot{\Psi}$ and $t-t_0$ are small) till the end (where $\dot{\Psi}$ is large.).

But of course neither of these statements mean anything in themselves. It is clear that statements about the growth of perturbations are highly coordinate dependent statements, and thus contain no physics per se. The key point is that arguing whether or not a perturbation grows or does not grow is irrelevant if no statement is also made as to how large the perturbations were at the beginning of inflation. The whole argument between Grishchuk and others can be reduced to the contention on the one hand that it is $\dot{\phi}$ in the longitudinal gauge which has a roughly vacuum quantum amplitude $1/\sqrt{H}$ (in units where $\hbar = 1$) at the time when the modes cross the Hubble radius, while the other contends that it is the $h$ mode, the diagonal term in the synchronous perturbation,

$$ds^2 = dt^2 - (a^2(1-2h)\delta_{ij} + Q_{i,j})dx^i dx^j$$

which has that amplitude. (This view of the debate is weakened by noting that in [3], Grishchuk seems to adopt the same initial condition conventions as others). In short, it is crucial to decide how one will quantize the modes at the earliest times. By an appropriate choice any final result can be obtained.

In also seems to an outsider that energy has been wasted about the details of the long wavelength calculations—Grishchuk contending the favourite “gauge independent variable” of the one group $\zeta = \dot{\phi}/H + \Phi$ + $\phi$ is zero in the long wavelength limit (it is not—it is just $\beta$ in the above notation—see also [4], [5]), while the others contend that he has not done his matching properly as the equation of state of the background changes (his $h$, as shown above can be chosen to be constant at all times, no matter what the potential $V$ in scalar
field terms, or no matter what the equation of state in perfect fluid terms. It is however not clear that he has always chosen this gauge. \[8\]

Trying to decide what the initial conditions are for the perturbations in the very early universe is a thorny issue. One tactic is to declare that these perturbations must be “vacuum” perturbations. However the justification for this stance is somewhat weak, since by assumption (crucial for inflation to work at all) at least a part of the system is very far from its lowest energy state. The scalar field must have a very large value, and a large non-zero energy for inflation to proceed. However, to assume that it is far from equilibrium, while all other degrees of freedom at at their minimum energy is worrisome. But this is not the place to try to examine this issue in any detail. Thus I will use the assumption that the fluctuations are in some sort of minimum energy state. Energy is however a coordinate dependent quantity, and, because of the large non-equilibrium background field, there is a tight coupling between the gravitational degrees of freedom and the matter. One must therefore adopt a formalism which takes this into account. Fortunately, because of the background field, and the special coordinate system chosen for that field, one can define an energy for the fluctuations to lowest order in the those fluctuation.

III. REDUCED HAMILTONIAN ACTION

Let me therefore derive the reduced Hamiltonian for the scalar field perturbations in a flat FRW space-time (see also Mukhanov and Anderegg \[3\] and Garriga et. al \[4\]). The Hamiltonian action for metric perturbations is

\[
S = \int (\pi^{ab} \dot{\gamma}_{ab} + \pi_{\Psi} \dot{\Psi} - NH_0 - N_b H^b) d^3 x dt
\]

where

\[
H_0 = \frac{1}{\sqrt{\gamma}} \pi^{ab} \pi^{cd} (\gamma_{ac} \gamma_{cd} - 1/2 \gamma_{ab} \gamma_{cd}) - \sqrt{\gamma} R + \frac{1}{\sqrt{\gamma}} \pi_{\Psi}^2 + \gamma^{ab} \pi_{,a} \pi_{,b} + V(\Psi)
\]

\[
H^a = \sqrt{\gamma} D_b \pi^{ab} + \pi_{\Psi} \Psi_b \gamma^{ab}
\]

where \(\gamma = det(\gamma_{ab})\) and \(D_a\) is the covariant derivative with respect to the spatial metric \(\gamma_{ab}\). I will now write this action in terms of the first order perturbations for the scalar modes.

Writing

\[
\Psi = \Psi_0(t) + e^{\psi}
\]

\[
\pi_{\Psi} = a^3 \dot{\Psi}_0 + \Phi e^{\pi_{\psi}}
\]

\[
\gamma_{ab} = a^2 (1 + 2 \phi) \delta_{ab} + Q_{,a,b}
\]

\[
\pi^{ab} = -2 H \Phi \delta^{ab} + e \left( \frac{1}{4a^2} P_\phi + \frac{1}{2} \nabla^{-2} Q \right) \delta^{ab} + \frac{1}{a^2} \left( -3 \nabla^{-2} P - \frac{1}{a^2} P_\phi \right) \delta^{ac} \delta^{bd}
\]

\(P_\phi, P_Q\) have been chosen so as to be the true conjugate momenta to \(\phi, Q\). Ie, at second order

\[
\int \pi^{ab} \gamma_{ab} = e^2 \int (P_\phi \dot{\phi} + P_Q \dot{Q}) d^3 x dt
\]

8
In addition, I choose

\[ N = 1 + e^{\alpha}; \quad N_a = e^{\beta_a} \]  \hspace{1cm} (42)

In the following, I will assume that all spatial dependence is of the form \( e^{\pm ik \cdot x} \) so \( \nabla^2 \) becomes \(-k^2\). Substituting these into the action, and examining only terms at second order in \( e \), one gets

\[ S = \int P_\phi \dot{\phi} + P_Q \dot{Q} + P_\psi \dot{\psi} - \alpha H_{0(1)} + \beta H_{a(1)} \cdot a - H_{0(2)} d^3x dt \]  \hspace{1cm} (43)

where the () index indicates the order (in powers of \( e \)) of the term in question. Although I could write out the detailed form of the various terms, it would not be illuminating at this stage. To mimic the longitudinal gauge choice, I will choose my coordinates so that \( Q = \dot{Q} = \beta = 0 \). From the variation equation of the action with respect to \( P_Q \), this results in the relation \( P_Q = \frac{1}{3} k^2 P_\phi \). Using the Hamiltonian equations for \( \dot{P}_Q \) and \( \dot{P}_\phi \), derived by varying the action with respect to \( Q \) and \( \phi \), one finds that this choice of gauge leads to the relation \( \alpha = -\phi \).

The resulting constraint equations are

\[ 0 = H_{0(1)} = \frac{1}{2H} (\dot{\Psi}_0 (a^3 \ddot{H} + 6a^3 H \dot{H} P_\psi) \psi + 8a^3 (3H^2 + 3H^2 \dot{H} - \dot{H} k^2) \phi + 2H \dot{H} P_\phi + 2\dot{\Psi}_0 \dot{H}) \]  \hspace{1cm} (44)

\[ 0 = H_{a(1), a} = -k^2 \left( \frac{1}{3a^2} P_\phi + 4H a \phi - a \dot{\Psi}_0 \psi \right) \]  \hspace{1cm} (45)

I then solved these for \( P_\psi \) and \( \psi \), and substituted the results back into the action. Clearly, the terms (the constraints) multiplying \( \alpha \) and \( \beta \) vanish, and the action arises solely from the symplectic form and \( H_{0(2)} \) terms. The result

\[ S = \int \left[ \frac{4k^2}{16H a^2} P_\phi \dot{\phi} + \frac{1}{72H a^5} k^2 (\dot{H} P_\phi^2 + (48a^3 H \dot{H} + 24a^3 \dddot{H}) + (288a^6 H^2 \dot{H} + 144a^6 H \dddot{H} + 144k^2 a^4 \dddot{H}) \phi^2) \right] dt \]

was still somewhat of a mess, so I defined the new variable \( w \) by

\[ \phi = \frac{wH}{a} \]  \hspace{1cm} (47)

Also, let I had to change define the momentum, \( P_\phi \), (which we note is no longer conjugate to \( \phi \)) to a new one, \( P_w \), which was conjugate to \( w \) (ie, the momentum is conjugate if the action contains the term \( P_w \dot{w} \) with unit coefficient). The necessary transformation is

\[ P_w = -\frac{H}{3a^3 H} k^2 P_\phi \]  \hspace{1cm} (48)

The action finally reduces to

\[ S = \int \left[ P_w \dot{w} + \frac{\dot{H} a}{8H^2} P_w^2 + \frac{\dot{H}}{H} P_w w + 2H^2 k^2 \left( \frac{2aH^2 \dot{H} + a^2 H \dddot{H} + k^2 \dddot{H}}{a^3 H^2} \right) w^2 \right] dt \]  \hspace{1cm} (49)
Making another series of transformations

\[ w = W \sqrt{\frac{-\dot{H}}{2kH}} \]  

(50)

\[ P_w = \left( (P_W - w \left[ \frac{\dot{H}}{2H} + \frac{3}{2} \frac{\dot{H}^2}{H} \right]) \frac{kH}{\sqrt{-\dot{H}a}} \right) \]  

(51)

and performing the appropriate temporal integrations by parts (in order to remove terms containing \( W \dot{P}_W \) or \( WW \)) we finally get

\[ S = P_W \dot{W} - \frac{1}{2} \left( P_w^2 + \left( \frac{k^2}{a^2} - \frac{\sqrt{-\dot{H}}}{H} \partial^2 \frac{H}{\sqrt{-\dot{H}}} \right) W^2 \right) \]  

(52)

Note that this is just the Hamiltonian for a Harmonic oscillator with time dependent frequency. I could further reduce the action by changing time to conformal time \( \tau \), and defining a new variable \( \mathcal{W} = W/a \). This would remove the \( \frac{1}{a^2} \) dependence in the term proportional to \( k^2 \), but the system is simple enough as it stands.

This Hamiltonian action can be quantized in the usual manner. Assuming that at very early times the state for this variable is in the ground state for this Hamiltonian, one obtains that it is be the variable \( W \) which will have the vacuum quantum amplitude of fluctuations, which, using the WKB approximation at early times, gives \( |W| \approx \sqrt{\frac{a^2}{k}} \) at those times.

Furthermore one can solve the equation for \( w \) exactly if one makes certain assumptions about the background solution. Varying the action with respect to \( w \) and \( P_w \) and eliminating \( P_w \) gives an equation for \( w \):

\[ \ddot{w} - \left( \frac{\dot{H}}{H} - 2 \frac{\dot{H}^2}{H^2} \right) \dot{w} + \frac{k^2}{a^2} w = 0 \]  

(53)

For long wavelengths, using our known solutions for \( \phi \), the solutions for \( w \) is \( -\epsilon - \beta \int a(t) dt + \beta a/H \). (Substituting this into this equation for \( w \) verifies it as the general solution.) We now need to match this to the solutions for large \( k \) (or small time), using the now known amplitude for \( W \) at early times. The usual method is to match assuming that one is in De Sitter space-time at early times. However, I find this an uncomfortable procedure, as the equation for \( w \), and the relation between \( w \) and \( W \), the simple quantum variable, is singular as \( H \) goes to a constant, since both depend on \( \dot{H} \) and \( \ddot{H} \). Furthermore, both the equations for \( w \) and \( W \) are potentially singular in this limit. Instead I will assume that

\[ a(t) = t^\mu, \]  

(54)

with \( \mu \) a large constant. (The limit, \( \mu \to \infty \), corresponds to De Sitter space.) Under this assumption for \( a \), the equation for \( w \) becomes

\[ \ddot{w} - \frac{\mu}{t} \dot{w} + \frac{k^2}{t^\mu} w = 0 \]  

(55)
Defining the new variable \( \tau = -\int \frac{dt}{a} = \frac{1}{(\mu-1)\mu-\tau} \), the conformal time, the equation for \( w \) becomes

\[
\partial_\tau^2 w + 2\frac{\mu}{(\mu - 1)\tau} \partial_\tau w + k^2 w = 0 \tag{56}
\]

which has as solutions Bessel functions

\[
w = (k\tau)^{-\frac{\mu+1}{2(\mu-1)}} \left( A J\left(\frac{\mu+1}{2(\mu-1)}, k\tau\right) + B Y\left(\frac{\mu+1}{2(\mu-1)}, k\tau\right) \right) \tag{57}
\]

The solution for \( W \) is then

\[
W = k\sqrt{\frac{\mu}{a}} w = k\sqrt{\mu}((\mu - 1)\tau)^{-\frac{\mu}{2(\mu-1)}} w \tag{58}
\]

Now, as argued above, \( W \) will have amplitude of order \( \sqrt{a/2k} \) at early times. This gives the equation for \( A, B \) of

\[
A = B = \frac{1}{\sqrt{2\mu}} k^{\frac{\mu-3}{2(\mu-1)}} (\mu - 1)^{-\frac{\mu}{\mu-1}} \tag{59}
\]

(\( A, B \) are actually quantum operators, and these expressions are shorthand for \( \sqrt{\langle A^2 \rangle} \) and \( \sqrt{\langle B^2 \rangle} \).) Matching to the solution for long wavelengths (large \( t \) or small \( \tau \)), namely

\[
w = -\epsilon - \beta t^{\mu+1} \left( \frac{1}{(\mu+1)} - \frac{1}{\mu} \right) = -\epsilon + \frac{\beta}{\mu(\mu+1)} t^{\mu+1} \tag{60}
\]

\[
= A \frac{k^{\frac{\mu+1}{2(\mu-1)}}}{\Gamma(1+\frac{\mu+1}{2(\mu-1)})} + B \frac{\Gamma\left(\frac{\mu+1}{2(\mu-1)}\right)}{\pi \Gamma\left(\frac{1}{2}\right)} \tag{61}
\]

Thus

\[
\beta = \frac{\Gamma(1/2)\sqrt{\mu}}{k^{\frac{3}{2}}\pi} \left[ \frac{2^{\frac{1}{2(\mu-1)}} \Gamma\left(\frac{\mu+1}{2(\mu-1)}\right) (1 + \frac{1}{\mu})}{k^{\frac{1}{2(\mu-1)}} (\mu + 1)^{2(\mu-1)} \Gamma\left(\frac{1}{2}\right)} \right] \tag{62}
\]

Since the term proportional to \( \epsilon \) dies out rapidly, I will not bother with giving its value. This gives the amplitude for the quantum fluctuations during the period while the fluctuations are outside the Hubble radius. Note that they depend on \( \mu \). We can write \( \sqrt{\mu} = \frac{H}{\sqrt{-H}} = 2\frac{H}{\Psi_0} \).

The term in square brackets is a function of \( \mu \) which is almost unity for large \( \mu \), but will give the slow change in the spectrum (\( k \) dependence) as \( \Psi_0 \) changes slowly during the course of inflation.

In conclusion, if one accepts the above procedure for determining the initial quantum fluctuations (an assumption which is not altogether beyond question) and if one accepts the validity of the linear theory for studying the determination of the initial amplitude for the fluctuations, then the answer which Grishchuk obtains for the final density fluctuations is wrong.
A. Comparison to other Reductions

As mentioned, such reductions have been carried out previously. It may however be of interest to relate my original approach to this problem. Mukhanov, Feldman and Brandenberger [2] report on such a reduction of the Lagrangian action. They derive a reduced Lagrangian action in terms of the longitudinal gauge variable

\[ v = a(\psi - (\dot{\Psi}_0/H)\phi) \] (63)

Using the constraint, \( E^{(1)}_{ii} = 0 \), this reduces to

\[ v = -\frac{4H}{a\Psi_0}\partial_t \left( \frac{a}{H}\dot{\phi} \right) \] (64)

or, in terms of the reduced variable \( w \),

\[ v = -\frac{2H}{a\sqrt{-\dot{H}}}\dot{w} \] (65)

My first reaction to this was to notice that in the long wavelength limit, \( \dot{w} \) eliminates the long wavelength solution proportional to \( \epsilon \) and thus depends only on \( \beta \). Thus it would appear that the variable \( v \) does not capture the full set of physical solutions to the perturbation equations in the long wavelength limit, making it an unsuitable candidate for quantization. However, the relation is more subtle, and in particular it demonstrates another case where the \( k \rightarrow 0 \) limit is not the same as the \( k = 0 \) case.

Examining equation 53, we notice that it can be written as a relation between \( w \) and \( v \). In particular, we note that

\[ k^2 w = \frac{aH^2}{H} \partial_t \frac{a\dot{H}}{H^2} v \] (66)

Thus, although the zeroth order \( \epsilon \) dependent solution for \( w \) does not contribute to \( v \), the next \( k^2 \) order part of the solution does. There is thus a direct relation between the two linearly independent solutions for \( w \) and the two for \( v \), despite the apparent contradiction when \( k = 0 \).

IV. HIGHER ORDER PERTURBATIONS

Let us now return to the question of whether or not the higher order long wavelength perturbations act to renormalise the cosmological constant, as discussed in ABM. Since to first order, the solutions locally (when spatial derivatives of the perturbations are neglected) look like simple coordinate transformations of the homogeneous solutions, their effect on the metric to higher order will again be that of simple coordinate transformations. I.e., in the long wavelength limit, the local effect of the perturbations will just be identical to a simple coordinate transformation of the background equations. As far as the local physics is concerned, the evolution of the universe is identical to that of a homogeneous universe,
with unrenormalised coupling constants. I.e., the effective cosmological constant will not be renormalized as far as local, sub-Hubble radius physics, is concerned.

However, their analysis is not concerned with local physics. Rather what ABM argue is that, if we examine the average evolution of the universe in the large, these long wavelength perturbations act to alter the effective long wavelength evolution of the universe as a negative cosmological constant would. To examine this let us, again in the longitudinal gauge, examine the behaviour of the long wavelength perturbations. However, I will not follow their technique. In their approach, they wish to regard the first order perturbations as contributing an extra stress energy tensor to the zeroth order equations of motion (i.e., the effective stress energy tensor of the first order equations renormalizes the zeroth order stress energy tensor.) I do not wish to follow this procedure as it raises delicate problems in consistency. Since by hypothesis, the zeroth order equations do not obey the zeroth order Einstein equations, a consistent derivation of the equations of motion becomes difficult. Instead I will follow the procedure of consistently (though probably not convergent) expanding in a series of small perturbations. I.e., the metric will be assumed to have the form of

$$g_{\mu\nu} = g_{0\mu\nu} + e g_{1\mu\nu} + e^2 g_{2\mu\nu} + ...$$  \hspace{1cm} (67)

$$\Psi = \Psi_0 + e\psi_1 + e^2\psi_2 + ...$$  \hspace{1cm} (68)

where the equations of motion are then consistently expanded as a function of $e$. The effective stress energy tensor of the first order perturbations will act as the source for the second order perturbations in the metric and field.

The biggest problem is that contribution to the second order terms from the first order perturbations (i.e., the effective energy momentum tensor for the second order perturbations) depend on the gauge chosen for the first order terms. Assume that we have two different gauge invariant formulations, determined by two separate vectors $X^\mu$ and $\bar{X}^\mu$.

If in one system, the second order perturbations are given by $\delta g^{(2)\mu\nu}$, then in the other gauge, the solutions will be (see appendix A)

$$\delta\tilde{g}^{(2)\mu\nu} = \delta g^{(2)\mu\nu} + 2\mathcal{L}_\Delta g^{(1)\mu\nu} + \mathcal{L}_\Delta \mathcal{L}_\Delta g^{(0)\mu\nu} + \mathcal{L}_{\Delta_2} g^{(0)\mu\nu}$$  \hspace{1cm} (69)

where $\Delta$ is the (first order) vector field which transforms the first gauge fixing to the second. $\Delta_2$ is the second order part of this fixing. (note that in most of the work in ABM, was only concerned with the first order gauge fixing vectors. Since the second order equations were never solved, the need for the second order components of the $X^\mu$ was not there.)

As I show in the appendix, by an appropriate choice of the first and second order gauge fixing, I can set the second order homogeneous perturbation in the metric to whatever value I wish.

Let me be more definite. Let us first work in the ABM gauge, the longitudinal gauge.

Going back to the observation that the first order solutions are, locally, just coordinate transformations, the reason for the apparent re-normalization of the cosmological constant is thus also clear. At each point the universe acts identically to the way it acts at each other point, except displaced in time by an amount which varies from place to place. If one now averages the universe over a fixed time slice, the averaged value of the expansion at fixed time will not be the same as the expansion rate at the averaged value of the time because of the non-linear nature of the expansion with time. Naively one would expect,
\[ <a(\tau(t,x)> = a(<\tau(t,x)> + a''(<\tau(t,x>)(<\tau(t,x)^2) - <\tau(t,x)^2), \text{where } \tau \text{ is the uniform time (naively the time defined such that } \delta\psi \text{ is zero along the } \tau \text{ constant surfaces). The second term is thus present because of the non-linear relation between } a \text{ and } t. \]

These conclusions about the also seem to be in agreement with the analysis of Salopek [11] who uses the Hamilton Jacobi methods to exactly solve for the evolution of the universe in the long wavelength limit, for restricted choices of the matter fields. His results seem to also suggest that the long wavelength evolution does not renormalise the cosmological constant.

After completion of this work, the paper by Kodama and Hamazaki [12] was brought to my attention where they have found the solution to the long wavelength equations for a general multicomponent system. They also find that the long wavelength limit does not correspond to the homogeneous solutions for the scalar perturbations.

**APPENDIX A**

To derive the equation for the perturbations, let us first define what we mean by the expansion of the metric in the various orders. Consider the space of all solutions of Einstein’s equations. This space can be defined by the set of tensor components \( g_{\mu\nu}(x) \) and fields \( \phi(x) \). Now consider a path through this space of all solutions parameterized by an arbitrary parameter \( e \). Thus we will have a set of functions \( g_{\mu\nu}(x,e) \), and we will choose the path and \( e \) such that at \( e = 0 \), the metric is the given background metric. For each value of \( e \) these are solutions of the equations, and thus we have

\[
G_{\mu\nu}(g_{\alpha\beta}(x,e)) = T_{\mu\nu}(g_{\alpha\beta}(x,e),\phi(x,e)).
\]  

(70)

Since this equation is valid by assumption for all \( e \), derivatives of these equations with respect to \( e \) will also be satisfied. Furthermore, assuming that these equations are analytic in \( e \) near \( e = 0 \), we can also create a power series expansion of the set of solutions along the path in \( e \).

Now consider another path through this space of solutions defined by an \( e \) dependent coordinate transformation

\[ \tilde{x}^\alpha = \tilde{x}^\alpha(x,e). \]  

(71)

such that

\[
g_{\mu\nu}(x,e) = \frac{\partial\tilde{x}^\alpha}{\partial x^\mu} \frac{\partial\tilde{x}^\beta}{\partial x^\nu} \tilde{g}_{\mu\nu}(\tilde{x}(x,e),e)
\]  

(72)

for some other path through the space of solutions defined by \( \tilde{g}_{\mu\nu}(\tilde{x},e) \).

Now define

\[ \eta^\mu = \frac{\partial\tilde{x}^\alpha(x,e)}{\partial e} \frac{\partial X^\mu(\tilde{x}(x,e),e)}{\partial \tilde{x}^\alpha} \]  

(73)

where \( X^\mu(\tilde{x},e) \) is defined so that \( X^\mu(\tilde{x}(x,e),e) = x^\mu \) for all \( e \). Also define the second order infinitesimal coordinate transformation by
\[ \zeta^\mu = \frac{\partial \eta^\mu(x,e)}{\partial e} \]  \hspace{1cm} (74)

Taking the second derivative of the equation \[72\] with respect to \( e \) at \( e = 0 \), and assuming that \( \tilde{x}^\alpha(\mathbf{r},0) = x^\alpha \), we obtain the second order coordinate transformations. After some algebra, one finds that

\[ \frac{\partial^2 g_{\mu\nu}(x,e=0)}{e^2} = \frac{\partial \tilde{g}_{\mu\nu}(x,e=0)}{\partial e^2} + \mathcal{L}_\eta \tilde{g}_{\mu\nu}(x,e=0) + 2 \mathcal{L}_\eta \frac{\partial \tilde{g}_{\mu\nu}(x,e=0)}{\partial e} + \mathcal{L}_\eta \mathcal{L}_\eta \tilde{g}_{\mu\nu}(x,e=0) \]  \hspace{1cm} (75)

I.e., the second order metric components are not independent of the first order coordinate transformation.

Let us consider the particular case where the first order solution \( \tilde{h}_{\mu\nu} \equiv \frac{\partial \tilde{g}_{\mu\nu}(x,e=0)}{\partial e} \) is evaluated in the longitudinal gauge (i.e., is diagonal), and has spatial dependence with spatial wave-vector \( k \). Assuming that \( k \) is very small (super-horizon modes), I will neglect terms proportional to \( k \) or \( k^2 \). (In the ABM paper, the first order \( k \) dependent terms would be zero because of the assumption that the fluctuations are statistically homogeneous, and have no preferred direction). Furthermore, let me assume that only \( \eta^0 \) is non-zero, with arbitrary time dependence and with spatial dependence with wave vector \( k \). Furthermore, let us look only at the effect of this first order gauge transformation on the second order spatially homogeneous terms. Then,

\[ \mathcal{L}_\eta \tilde{g}_{\mu\nu} = 2 \eta^0 \delta_{0\nu} \delta_{0\nu} + 2a \dot{\eta}^0 \delta_{ij} \]  \hspace{1cm} (76)

and

\[ \mathcal{L}_\eta (2\tilde{h}_{00} + \mathcal{L}_\eta g_{00}) = \eta^0 (2\tilde{h}_{00} + 2\ddot{\eta}^0) + 4(\tilde{h}_{00} + \dot{\eta}^0)\dot{\eta}^0 \]  \hspace{1cm} (77)

and

\[ \mathcal{L}_\eta (2\tilde{h}_{ij} + \mathcal{L}_\eta g_{ij}) = 2\eta^0 \left( \dot{h}_{ij} + \dot{\eta}^0 (\eta^0 (a\ddot{a} + \dot{a}^2) + \dot{\eta}^0 (a\dot{a})) \right) \delta_{ij} \]  \hspace{1cm} (78)

Using the gauge choice from the main paper, such that \( h_{ij} = 2a^2 \Phi \delta_{ij}, \ h_{00} = 2\phi \dot{h}, \ \phi = \Phi \approx \text{const} \), we have, neglecting terms of order \( k^2 \),

\[ \mathcal{L}_\eta (2\tilde{h}_{00} + \mathcal{L}_\eta g_{00}) = 2\eta^0 \ddot{\eta}^0 + 4\phi \dot{\eta}^0 + 4(\dot{\eta}^0)^2 \]  \hspace{1cm} (79)

\[ \mathcal{L}_\eta (2\tilde{h}_{ij} + \mathcal{L}_\eta g_{ij}) = a^2 \delta_{ij} 2\eta^0 (2H\phi + (\dot{H} + 2H^2)\dot{\eta}^0) + H(\dot{\eta}^0)^2 \]  \hspace{1cm} (80)

Note that all of the off diagonal terms due to this first order gauge transformation are zero because of the averaging over \( k \). We know the gauge changes for the homogeneous modes, with \( \zeta^0(t) \) and \( \zeta^i = \lambda x^i \), which gives the changes

\[ \mathcal{L}_\zeta \tilde{g}_{00} = 2\dot{\zeta}^0 \mathcal{L}_\zeta \tilde{g}_{ij} = (2a\dot{a} + \lambda)\delta_{ij} \]  \hspace{1cm} (81)

The second order terms from the first order gauge transformations cannot be cast into this form. One thus has two independent functions of \( t \), namely that produced by the first order inhomogeneous gauge transformation, and that caused by the second order homogeneous gauge transformation. By making appropriate choices of both the first order gauge with spatial wavenumber \( k \) and of the homogeneous second order gauge, and neglecting terms of order \( k^2 \), one can set the two second order homogeneous terms in the metric (i.e the 00 and the diagonal spatial components) to any value one desires, including 0.
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