Almost Finite Speed of Propagation for Linear Peridynamics

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1 Main Results

We start with what is essentially a trivial observation about solutions to the wave equation in one space dimension.

**Theorem 1** Suppose

\[ u_{tt} - c^2 u_{xx} = 0 \]

and \( u \) and \( \partial_t u \) belong to the Schwartz class \( \mathcal{S}(\mathbb{R}) \) when \( t = 0 \). For any \( v \) with \( |v| \neq c \), any \( x_0 \in \mathbb{R} \) and any non-negative integer \( l \) there is a constant \( C \) such that

\[ |\partial_t u(t, x_0 + vt)|^2 + c^2 |u_x(t, x_0 + vt)|^2 \leq Ct^{-2l} \]

for all \( t > 0 \).

We will prove the following lemma.

**Lemma 1** Suppose \( J \in L^1(\mathbb{R}) \) is non-negative and even and that its moments \( \mu_k \)

\[ \mu_k = \int_{\mathbb{R}} x^k J(x) \, dx \]

exist for \( k \leq l + 2 \), with \( l \geq 0 \). There is a unique \( c > 0 \) and a unique bounded translation-invariant operator \( D : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) such that

1. \( J * v - \mu_0 v = c^2 D^2 v \)

for all \( v \) in \( L^2(\mathbb{R}) \).

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1 All integrals are to be understood as Lebesgue integrals throughout the paper.
2. $-iHD$ is a positive operator, where $H: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is the Hilbert transform,

$$(Hv)(x) = \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \frac{v(y)}{x-y} \, dy.$$ 

3. Letting $S_\lambda: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ denote scaling by a factor $\lambda$,

$$(S_\lambda v)(x) = v(\lambda^{-1}x),$$

we have

$$\lim_{h \to 0^+} h^{-1}S_hDS_h^{-1}v = v_x$$

weakly for any $v \in \text{Dom}(\partial_x) \subset L^2(\mathbb{R})$.

This $D$ is of the form

$$(Dv)(x) = \int_{\xi \in \mathbb{R}} \int_{y \in \mathbb{R}} 2\pi i \psi(\xi)e(\xi(x-y))v(y) \, d\xi \, dy,$$

where

$$e(z) = \exp(2\pi iz),$$

and $\psi$ is a function uniformly bounded, together with its derivatives up through order $l$, on $\mathbb{R}$.

This suggests the following theorem, which we will also prove.

**Theorem 2** Suppose that $u$ satisfies the integrodifferential equation\textsuperscript{2}

$$u_{tt} + \mu_0 u = J * u$$

with $J$ as in the lemma. Suppose that $u$ and $\partial_t u$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$ when $t = 0$. For any $v$ with $|v| > c$, and $x_0 \in \mathbb{R}$ and any non-negative integer $n < N$ there is a constant $C$ such that

$$|\partial_t u(t, x_0 + vt)|^2 + c^2|(Du)(t, x_0 + vt)|^2 \leq Ct^{-2l}$$

for all $t > 0$. The constant $c$ and operator $D$ are those of the lemma.

\textsuperscript{2}The convolution of $J$ with $u$ is, of course, to be understood as a convolution in the spatial variable only, for each value of the temporal variable.
Note that this is not an exact analogue of the previous theorem, because we assume $|v| > c$ rather than $|v| \neq c$. It is not clear if this difference is essential or if it is an artifact of the proof below.

The interest in this integrodifferential equation from peridynamics, a non-local theory of elasticity introduced by Silling [1]. The equation of motion in peridynamics is

$$\rho(x)u_{tt}(t, x) = \int f(u(t, y) - u(t, x), y - x, x)\, dy$$

For homogeneous materials $\rho$ is constant and $f$ is independent of its third argument. If we assume further that $f$ is linear in its first argument, which is true approximately for small displacements in all models and true exactly for small displacements in some models, then this equation reduces to the one of theorem. One consequence of the theorem is that small initial displacements remain small for all time, so we do not leave the domain of validity of the approximation.

Theorem 1 is, of course a simple consequence of the explicit solution formula

$$u(t, x) = \frac{1}{2}u(0, x + ct) + \frac{1}{2}u(0, x - ct) + \frac{1}{2c} \int_{x-ct<y<x+ct} \partial_t u(0, y)\, dy.$$  

The precise details of this formula are unimportant. It suffices to observe that

$$u(t) = k_0(t) \ast u(0) + k_{-1}(t) \ast \partial_t u(0)$$

where $k_0(t)$ and $k_{-1}(t)$ are distributions supported in the interval $[-ct, ct]$. This observation also leads to finite speed of propagation for the wave equation. It is therefore of interest that the equation

$$u_{tt} + \mu_0u = J \ast u$$

does not exhibit finite speed of propagation. More precisely, one has the following theorem.

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$^3$ $u$ may be thought of either as a function of two arguments or as a function of its first argument, taking values in functions of the second. It is in this sense that expressions like $u(t)$ are to be interpreted. Logically the two points of view are equivalent. No ambiguity is possible because one can always count arguments to determine what is meant. The same, of course, applies to derivatives of $u$ as well.
**Theorem 3** Suppose $\nu$ belongs to the Schwartz class $S'(\mathbb{R})$ and that convolution with $\nu$ is positive and self-adjoint in the sense that

$$\langle \nu \star \chi, \chi \rangle \geq 0$$

and

$$\langle \nu \star \chi_1, \chi_2 \rangle = \langle \chi_1, \nu \star \chi_2 \rangle.$$ 

for $\chi, \chi_1, \chi_2 \in \mathcal{S}(\mathbb{R})$. Then there exist families of tempered distribution $k_0(t), k_{-1}(t) \in \mathcal{S}'(\mathbb{R})$ such that the solution to the initial value problem for the integrodifferential equation

$$u_{tt} + \nu \star u = 0$$

has a unique solution, given by

$$u(t) = k_0(t) \star u(0) + k_{-1}(t) \star \partial_t u(0).$$

For no $t > 0$ are these distributions of compact support unless $\nu = -(\alpha \delta'' + i\beta \delta' + \gamma)$ with either $\alpha = \beta = 0$ and $\gamma \geq 0$ or $\alpha > 0$ and $4\alpha \gamma + \beta^2 \geq 0$. That is the fundamental solution is of compact support only if the integrodifferential equation above is really a differential equation

$$u_{tt} - \alpha u_{xx} - i\beta u_x - \gamma u = 0.$$

We should therefore think of Theorem 2 as a substitute for finite speed of propagation, hence the phrase “almost finite speed of propagation” in the title of this paper.

Theorem 2 does not provide the sharpest estimates obtainable for this problem. With more effort it is possible to obtain the following result.

**Theorem 4** Suppose that $u$ satisfies

$$u_{tt} + \mu_0 u = J \star u,$$

where the integrals

$$\int_{x \in \mathbb{R}} x^k J(x) \, dx$$

exist for $m \leq l + 3$ and the integrals

$$\int_{x \in \mathbb{R}} x^m \partial_x^n u(0, x) \, dx$$
and
\[ \int_{x \in \mathbb{R}} x^m \partial_t \partial_x^n u(0, x) \, dx \]
exist for \( m \leq l \) and \( n \leq K + 2 \). Then there is a constant \( C_{j,k,l} \) such that
\[ |\partial_t^j \partial_x^k u(t, x)| \leq C \max(1, 1 + |x| - c|t|)^{-2l} \]
for all \( 1 \leq j \leq l \), \( k \leq K \) and all \( t, x \in \mathbb{R} \).
\[ |\partial_x^k u(t, x)| \leq C \max(1, 1 + |x| - c|t|)^{-2l-2} \]
for all \( j \leq l \), \( k \leq K \) and all \( t, x \in \mathbb{R} \).

2 Proof of Theorem

Fourier transforming the integrodifferential equation
\[ u_{tt} + \nu u = 0 \]
gives
\[ Fu_{tt} + \varphi Fu = 0, \]
where
\[ \varphi = F\nu. \]

By hypothesis,
\[ \langle \nu \ast \chi, \chi \rangle \geq 0 \]
and
\[ \langle \nu \ast \chi_1, \chi_2 \rangle = \langle \chi_1, \nu \ast \chi_2 \rangle. \]

By Plancherel,
\[ \langle \varphi F\chi, F\chi \rangle \geq 0 \]
and
\[ \langle \varphi \chi_1, \chi_2 \rangle = \langle \chi_1, \varphi \chi_2 \rangle. \]

It follows that \( \varphi(\xi) \) is real and non-negative for all \( \xi \in \mathbb{R} \).

In vector form the differential equation above is
\[ \partial_t \begin{bmatrix} Fu \\ F\partial_t u \end{bmatrix}(t, \xi) = \begin{bmatrix} 0 & 1 \\ -\varphi(\xi) & 1 \end{bmatrix} \begin{bmatrix} Fu \\ F\partial_t u \end{bmatrix}. \]
Proceeding one frequency at a time,
\[
\begin{bmatrix}
Fu \\
F∂tu
\end{bmatrix}(t, ξ) = M(\sqrt{ϕ(ξ)}, t)
\begin{bmatrix}
Fu \\
F∂tu
\end{bmatrix}(0, ξ)
\]
is the unique solution, where the matrix valued function \(M\) is given by
\[
M(ω, t) = \exp\left(\begin{bmatrix}
0 & 1 \\
-ω^2 & 0
\end{bmatrix}t\right)
= \begin{bmatrix}
\cos(ωt) & \omega^{-1}\sin(ωt) \\
-\omega\sin(ωt) & \cos(ωt)
\end{bmatrix}.
\]
Since \(M\) is even in \(ω\) the choice of square root is irrelevant.
\[
\begin{bmatrix}
Fu \\
F∂tu
\end{bmatrix}(t, ξ) = M(\sqrt{ϕ(ξ)}, t)
\begin{bmatrix}
Fu \\
F∂tu
\end{bmatrix}(0, ξ).
\]
Taking an inverse Fourier transform and a Fourier transform on the other, we obtain an integral representation for the solution \(u\) and its time derivative in terms of the initial data,
\[
\begin{bmatrix}
u \\
∂tu
\end{bmatrix}(t, x) = \int_{ξ∈R} \int_{y∈R} \kappa(t, x-y, ξ) \begin{bmatrix}
u \\
∂tu
\end{bmatrix}(0, y) dy dξ,
\]
where
\[
\kappa(t, z, ξ) = e(ξz)M(\sqrt{ϕ(ξ)}, t).
\]
It is tempting to reverse the order of integration, obtaining
\[
\begin{bmatrix}
u \\
∂tu
\end{bmatrix}(t, x) = \int_{y∈R} K(t, x-y) \begin{bmatrix}
u \\
∂tu
\end{bmatrix}(0, y) dy,
\]
where
\[
K(t, z) = \int_{ξ∈R} \kappa(t, z, ξ) dξ,
\]
but these integrals do not, in general, converge.
To get around this failure of convergence we split off a factor of
\[
α(ξ) = 1 + 4π^2Aξ^2
\]
where \(A > 0\). Defining
\[
L = 1 - A∂_x^2,
\]
we have

\[ FL\left[ \frac{u}{\partial t} \right] = \alpha(\xi)F\left[ \frac{u}{\partial t} \right] \]

Proceeding nearly as before,

\[ \left[ \frac{Fu}{F\partial t} \right](t,\xi) = \alpha(\xi)^{-1}M(\sqrt{\varphi(\xi)},t)\left[ FL\frac{u}{\partial t} \right](0,\xi) \]

leads to

\[ \left[ \frac{u}{\partial t} \right](t,x) = \int\int_{\xi \in \mathbb{R}, y \in \mathbb{R}} \beta(t,x-y,\xi) \left[ \frac{Lu}{L\partial t} \right](0,y) \, dy \, d\xi, \]

where

\[ \beta(t, z, \xi) = \alpha(\xi)^{-1}e(\xi z)M(\sqrt{\varphi(\xi)},t). \]

Splitting \( e(\xi z) \) into \( \cos(2\pi \xi z) \) and \( i\sin(2\pi \xi z) \), we see that the latter is odd and does not contribute to the integral. Thus we can equally well write

\[ \left[ \frac{u}{\partial t} \right](t,x) = \int\int_{\xi \in \mathbb{R}, y \in \mathbb{R}} \lambda(t,x-y,\xi) \left[ \frac{Lu}{L\partial t} \right](0,y) \, dy \, d\xi, \]

where

\[ \lambda(t, z, \xi) = \alpha(\xi)^{-1} \cos(2\pi \xi z)M(\sqrt{\varphi(\xi)},t). \]

This time we can reverse the order of integration, obtaining

\[ \left[ \frac{u}{\partial t} \right](t,x) = \int_{y \in \mathbb{R}} B(t, x-y) \left[ \frac{Lu}{L\partial t} \right](0,y) \, dy, \]

where

\[ B(t, z) = \int_{\xi \in \mathbb{R}} \lambda(t, z, \xi) \, d\xi, \]

because the integrand is bounded by a constant times the integrable factor \( \alpha(\xi)^{-1} \), and hence the Lebesgue Dominated Convergence Theorem applies. This gives us a representation of the form

\[ \left[ \frac{u}{\partial t} \right](t) = B(t) \ast \left[ \frac{Lu}{L\partial t} \right](0). \]

The convolution is to be interpreted in the usual sense, but it can be equally well considered as the convolution of a tempered distribution with a test..
function. Taking this latter interpretation, we may move the constant coefficient differential operator $L$, 
\[
\left[ \frac{u}{\partial_t u} \right](t) = K(t) * \left[ \frac{u}{\partial_t u} \right](0),
\]
where $K = LB$

provided that we interpret the differentiations and convolutions in the sense of distributions.

In terms of components,
\[
u(t) = k_0(t) * u(0) + k_{-1}(t) * \partial_t u(0),
\]
\[
\partial_t u(t) = k_1(t) * u(0) + k_0(t) * \partial_t u(0),
\]
\[
k_j = Lb_j,
\]
\[
b_j(t, z) = \int_{\xi \in \mathbb{R}} \lambda_j(t, z, \xi) \, d\xi,
\]
and
\[
\lambda_j(t, z, \xi) = \alpha(\xi)^{-1} \cos(2\pi \xi z)(\varphi(\xi))^{j/2} \lambda_j(t \sqrt{\varphi(\xi)})
\]

where
\[
\theta_j(\zeta) = \begin{cases} 
(-1)^{j/2} \cos \zeta & \text{if } \zeta \text{ is even,} \\
(-1)^{(j-1)/2} \sin \zeta & \text{if } \zeta \text{ is odd.}
\end{cases}
\]

In fact, differentiation under the integral sign shows that we have, for all $j \geq 0$,
\[
\partial^j_t u(0) = b_j(t) * (Lu)(0) + b_{j-1}(t) * (L\partial_t u)(0)
\]
and
\[
\partial^j_t u(0) = k_j(t) * u(0) + k_{j-1}(t) * \partial_t u(0).
\]

The various $b$'s are locally integrable functions, so the convolutions in the first of this pair of equations may be understood in the usual sense, provided the initial data are integrable. The $k$'s are distributions of order at most 2, and the convolutions in the second equation are to be understood in the sense of convolutions. Even more trivially, we have
\[
\partial^j_t \partial^k_x u(t) = b_j(t) * (L\partial^k_x u)(0) + b_{j-1}(t) * (L\partial^k_x \partial_t u)(0)
\]
and
\[
\partial^j_t \partial^k_x u(t) = k_j(t) * \partial^k_x u(0) + k_{j-1}(t) * \partial^k_x \partial_t u(0).
\]
It remains to determine when \( k_0(t) \) and \( k_{-1}(t) \) are of compact support. Throughout the remainder of this section \( t \) is fixed and positive.

If \( k_j(t) \) is a distribution of order \( n \) supported in the interval \([-X, X]\) then there is a constant \( C_j \) such that

\[
|\langle k_j, \theta \rangle| \leq C_j \sum_{m=0}^{n} \max_{-X \leq x \leq X} |\theta^{(m)}(x)|
\]

for all smooth functions \( \theta \). There is no loss of generality in assuming \( C_j \geq 1 \).

Applying this to the function \( \theta(x) = e^{t\sqrt{-\xi x}} \) we see that \( Fk_{j(t)}(\xi) \) is an entire function satisfying.

\[
|(Fk_{j(t)})(\xi)| \leq C_j \sum_{m=0}^{n} (2\pi |\xi|)^m e^{2\pi X |\Im \xi|}.
\]

For \( \xi \in \mathbb{R} \) we have

\[
(Fk_0(t))(\xi) = \cos(t \sqrt{\varphi(\xi)}),
\]

\[
(Fk_{-1}(t))(\xi) = \frac{\sin(t \sqrt{\varphi(\xi)})}{\sqrt{\varphi(\xi)}},
\]

and

\[
\varphi(\xi) = \frac{(Fk_0)(\xi)^2}{1 - (Fk_{-1})(\xi)^2}.
\]

The last of these equations can be used to extend \( \varphi \) to a meromorphic function, which must then satisfy the preceding two equations except at poles of \( \varphi \). There are, however, no such poles, because if \( \eta \in \mathbb{C} \) is a pole of \( \varphi \) then, by the equation \( (Fk_0(t))(\xi) = \cos(t \sqrt{\varphi(\xi)}) \), \( \sqrt{\eta} \) is an essential singularity of \( Fk_0(t) \), which we already saw was entire. Thus \( \varphi \) is an entire function, but the preceding argument gives no bounds for its size. We will need such bounds on circles of large radius.

If

\[
|\xi| = R
\]

where \( R \geq \frac{1}{\pi} \) then

\[
|(Fk_0(t))(\xi)| \leq 2^{n+1} \pi^n C_j R^n e^{2\pi X R}.
\]

Let \( q(\xi) = e(t \sqrt{\varphi(\xi)}) \). There is no reason, at this stage, to believe that the square root can be taken in a continuous manner, so we simply choose a square root arbitrarily at each point. If \( |q| \geq \sqrt{2} \) then

\[
|q| \leq 2(|q| - |q|^{-1}) \leq 2|q + q^{-1}| \\
\leq 4|(Fk_0(t))| \leq 2^{n+3} \pi^n C_j R^n e^{2\pi X R}.
\]
The same estimate holds trivially if $|q| < \sqrt{2}$. The argument above is equally valid if we replace $q$ by $q^{-1}$ everywhere, so

$$e^{2\pi|\text{Im} \sqrt{\varphi}(\xi)||t|} = \max(|q|, |q^{-1}|) \leq 4|(F_{k_0}(t))| \leq 2^{n+3}\pi^n C_j R^n e^{2\pi X R},$$

from which

$$|\text{Im} \sqrt{\varphi}(\xi)| \leq g(R),$$

where

$$g(R) = \frac{2XR}{|t|} + \frac{n}{2\pi|t|} \log R + \frac{1}{2\pi|t|} \log(2^{n+3}\pi^n C_j).$$

Now

$$\text{Re} \varphi(\xi) = (\text{Re} \sqrt{\varphi(\xi)})^2 - (\text{Im} \sqrt{\varphi(\xi)})^2 \geq - (\text{Im} \sqrt{\varphi(\xi)})^2 \geq g(R)^2.$$ 

Define

$$v(\xi) = \text{Re} \varphi(\xi),$$

$$v_+(\xi) = \max(v(\xi), 0), \quad v_-(\xi) = \max(-v(\xi), 0),$$

so that

$$v(\xi) = v_+(\xi) - v_-(\xi), \quad |v(\xi)| = v_+(\xi) - v_-(\xi).$$

$v$ is the real part of a holomorphic function, hence harmonic. By the Mean Value Property

$$0 = v(0) = \frac{1}{2\pi R} \int_{|\xi|=R} v(\xi) \, ds,$$

from which we obtain

$$\frac{1}{2\pi R} \int_{|\xi|=R} v_+(\xi) \, ds = \frac{1}{2\pi R} \int_{|\xi|=R} v_-(\xi) \, ds$$

and

$$\frac{1}{2\pi R} \int_{|\xi|=R} |v(\xi)| \, ds = \frac{2}{2\pi R} \int_{|\xi|=R} v_-(\xi) \, ds$$

Our earlier estimate on $\text{Re} \sqrt{\varphi(\xi)}$ gives

$$v_-(\xi) \leq g(R)^2$$

and hence

$$\frac{1}{2\pi R} \int_{|\xi|=R} |v(\xi)| \, ds \leq 2g(R)^2.$$
Differentiating the Poisson Formula repeatedly gives

\[ \partial_x^m \partial_y^n v(x + iy) = \frac{1}{2\pi R} \int_{|\xi|=R} P_{m,n}(x, y, \text{Re} \xi, \text{Im} \xi) v(\xi) \, ds \]

where

\[ P_{m,n}(x, y, x', y') = \frac{p_{m,n}(x, y, x', y')}{q(x, y, x', y')} \]

\( p_{m,n} \) is a polynomial of degree \( m + n + 2 \), and

\[ q(x, y, x', y') = (x - x')^2 + (y - y')^2. \]

We therefore have

\[ |\partial_x^m \partial_y^n v(x + iy)| \leq \frac{2g(R)^2 \max_{|\xi|=R} p_{m,n}(x, y, \text{Re} \xi, \text{Im} \xi)}{(R^2 - x^2 - y^2)^{m+n+1}} \]

The numerator is \( O(R^{m+n+4}) \), so the right hand side tends to zero as \( R \) tends to infinity if \( m + n > 2 \). Thus \( v \) is a quadratic polynomial, from which it follows that \( \varphi \) is a quadratic polynomial. We have already seen that it is real on the real line, so

\[ \varphi(\xi) = \hat{\alpha} \xi^2 + \hat{\beta} \xi + \hat{\gamma}. \]

Positivity implies that either \( \varphi \) is a non-negative constant or \( \hat{\alpha} > 0 \) and \( 4\hat{\alpha}\hat{\gamma} - \hat{\beta}^2 \geq 0 \). Writing

\[ \alpha = \frac{\hat{\alpha}}{4\pi^2}, \quad \beta = \frac{\hat{\beta}}{2\pi}, \quad \gamma = -\hat{\gamma}, \]

we have

\[ \varphi(\xi) = -\alpha (2\pi i \xi)^2 - i\beta (2\pi i \xi) - c \]

or

\[ \nu = -(\alpha \delta'' + i\beta \delta' + \gamma \delta). \]

The conditions on \( \hat{\alpha}, \hat{\beta}, \) and \( \hat{\gamma} \) above are equivalent to

\[ \alpha > 0 \quad 4\alpha \gamma + \beta^2 \geq 0. \]
3 Proof of the Lemma

We start by showing the existence of such $c$ and $D$. Because $J$ is even any moments of odd order are zero, including the first moment. The second moment

$$\mu_2 = \int_{x \in \mathbb{R}} x^2 J(x) \, dx$$

exists and is positive. Let $c$ be the positive solution of $c$,

$$c^2 = \frac{\mu_2}{2}.$$ 

The Fourier transform

$$(FJ)(\xi) = \int_{x \in \mathbb{R}} J(x)e(-\xi x) \, dx,$$

is then twice continuously differentiable. The usual properties of the Fourier transform show that

$$\varphi = \mu_0 - FJ$$

vanishes at the origin along with its derivative and that it is positive everywhere else. Its second derivative is

$$\varphi''(\xi) = -(FJ)'(\xi) = 4\pi^2 \int_{x \in \mathbb{R}} J(x) e(-\xi x) \, dx.$$

Using the elementary relation

$$e(-\xi x) = \cos(2\pi \xi x) - i \sin(2\pi \xi x)$$

and the evenness of $J$,

$$\varphi''(\xi) = 4\pi^2 \int_{x \in \mathbb{R}} x^2 J(x) \cos(2\pi \xi x) \, dx$$

In particular,

$$\varphi''(0) = 4\pi^2 \mu_2 > 0.$$ 

Since $|\cos(2\pi \xi x)| \leq 1$ everywhere we have

$$|\varphi''(\xi)| \leq 4\pi^2 \mu_2$$

for all $\xi \in \mathbb{R}$. 

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We define $\psi$ by

$$\psi(\xi)^2 = 2 \varphi(\xi) / \varphi''(0),$$

taking the positive square root for positive values of $\xi$ and the negative square root for negative values. This choice clearly makes $\psi$ an odd function.

A simple calculation shows that

$$\psi'(0) = 1$$

and

$$\psi'(\xi)^2 = \frac{\varphi'(\xi)^2}{2 \varphi''(0) \varphi(\xi)}$$

for $\xi \neq 0$. From $\varphi''(\xi) \leq \varphi''(0)$ it follows that

$$\varphi(\eta) \leq \varphi(\xi) + \varphi'(\xi)(\eta - \xi) + \frac{1}{2} \varphi''(0)(\eta - \xi)^2$$

for all $\xi, \eta \in \mathbb{R}$. Since $\varphi(\eta) \geq 0$,

$$\varphi(\xi) + \varphi'(\xi)(\eta - \xi) + \frac{1}{2} \varphi''(0)(\eta - \xi)^2 \geq 0.$$

Taking

$$\eta = \xi + \frac{\varphi'(\xi)}{\varphi''(0)}$$

gives

$$\varphi(\xi) - \frac{1}{2} \frac{\varphi'(\xi)^2}{\varphi''(0)} \geq 0$$

or

$$\psi'(\xi)^2 \leq 1.$$

In fact, all the derivatives of $\psi$ through order $l$ are bounded. By hypothesis $J$ has moments of order up to $l + 2$. In other words, the integrals

$$\mu_k = \int_{x \in \mathbb{R}} x^m J(x) \, dx$$

exist for $k \leq l + 2$. This is equivalent to the existence of the integrals

$$\nu_k \int_{x \in \mathbb{R}} |x^m J(x)| \, dx.$$
Since $J$ is non-negative $\nu_k = \mu_k$ if $k$ is even, but if $k$ is odd then $\nu_k$ is positive while $\mu_k$ is zero. In any case

$$\varphi^{(k)}(\xi) = -(2\pi i)^k \int_{x \in \mathbb{R}} x^k J(x) e(-\xi x) \, dx$$

for $1 \leq k \leq l + 2$, so

$$|\varphi^{(k)}(\xi)| \leq (2\pi)^k \nu_k.$$  

Now

$$\lim_{\xi \to \pm \infty} = \mu_0$$

by the Riemann-Lebesgue Lemma. There is therefore a $\Xi$ such that

$$|\varphi(\xi) - \mu_0| \leq \frac{\mu_0}{2}$$

for $|\xi| \geq \Xi$.

$$\psi(\xi) = \pm \sqrt{\frac{2\mu_0}{\varphi''(0)}} \sqrt{\frac{\varphi(\xi)}{\mu_0}}.$$  

The first factor is constant. The second is the composition of the square root function, which is smooth on the compact set $[1/2, 3/2]$, with $\varphi$, which is uniformly bounded along with its derivatives up through order $l$ everywhere and takes values in $(1/2, 3/2)$ for $|\xi| \geq \Xi$. Thus $\psi$ is uniformly bounded along with its derivatives through order $l + 2$ for $|\xi| \geq \Xi$.

Near zero we have to proceed differently. Integration by parts twice yields the relation

$$\int_{0}^{\xi} (\xi - \eta) \varphi''(\eta) \, d\eta = \varphi(\xi).$$

Making the change of variable $\eta = \tau \xi$,

$$\varphi(\xi) = \xi^2 \int_{0}^{1} (1 - \tau) \varphi''(\tau \xi) \, d\tau,$$

from which we obtain

$$\psi(\xi) = \xi \sqrt{2 \int_{0}^{1} (1 - \tau) \frac{\varphi''(\tau \xi)}{\varphi''(0)} \, d\tau}.$$  

Since $\varphi''$ is continuous we can find an $\epsilon > 0$ such that

$$|\varphi''(\eta) - \varphi''(0)| < \frac{1}{2} |\varphi''(0)|$$
for all $|\xi| < \epsilon$. For such $\xi$ we have

$$\frac{1}{2} < 2 \int_0^1 (1 - \tau) \frac{\varphi''(\tau \xi)}{\varphi''(0)} \, d\tau < \frac{3}{2}.$$  

Repeated differentiation under the integral sign shows that this quantity and its derivatives up through order $l$ are uniformly bounded. As before, this is true of the square root as well. Multiplication by $\xi$ does not change this, since $\xi$ and all its derivatives are uniformly bounded in $[-\epsilon, \epsilon]$.

Finally, we are left with the intervals $[\epsilon, \Xi]$ and $[-\Xi, -\epsilon]$. The $k$'th derivative is bounded by induction on $k$. Differentiating

$$\xi^3 \int_0^1 \frac{\varphi'''(\tau \xi)}{2} (1 - \tau)^2 \, d\tau$$

$k$ times gives

$$\psi(\xi) \psi^{(k)}(\xi) = \frac{\varphi^{(k)}(\xi)}{\varphi''(0)} - \frac{1}{2} \sum_{j=1}^{k-1} \psi^{(k)}(\xi) \psi^{(n-j)}(\xi)$$

As long as $k \leq l+1$ the right hand side is uniformly bounded by the induction hypothesis. $\psi$ is non-zero in these intervals and, because the intervals are compact, is bounded away from zero. Therefore $\psi^{(k)}$ is uniformly bounded, thus recovering our inductive hypothesis.

We define the operator $D$ to be multiplication of the Fourier transform by $\psi$, multiplied by a factor of $2\pi i$ to make the result real,

$$(FDv)(\xi) = 2\pi i \psi(\xi)(Fv)(\xi)$$

or

$$(Dv)(x) = \int_{\xi \in \mathbb{R}} \int_{y \in \mathbb{R}} 2\pi i \psi(\xi) e(\xi(x-y)) v(y) \, d\xi \, dy.$$  

Note that since $D$ is a Fourier multiplier it is automatically translation invariant. It is tempting reverse the order of the integrals and write

$$Dv = Q \ast v$$

where

$$Q(x) = 2\pi i \int_{\xi \in \mathbb{R}} \psi(\xi) e(\xi x) \, dx.$$  

This integral, however, does not converge. It can be given a meaning in the sense of distributions. The resulting distribution $Q$ is regular and integrable.
away from zero, but has a singularity like \(1/x\) there. Convolution of \(Q\) can therefore be given a meaning through principal value integrals, as is done with the Hilbert transform. This, however, does not seem to be worth the effort.

Next we note that
\[
(FS_{h^{-1}}v)(\xi) = h^{-1}(Fv)(h^{-1}\xi),
\]
\[
(FDS_{h^{-1}}v)(\xi) = 2\pi i h^{-1}(\psi(\xi))(Fv)(h^{-1}\xi),
\]
\[
(FS_{h}DS_{h^{-1}}v)(\xi) = 2\pi i \psi(h\xi)(Fv)(\xi),
\]
and
\[
(FS_{h}DS_{h^{-1}}v)(\xi) = 2\pi i h^{-1}\psi(h\xi)(Fv)(\xi).
\]
Because \(\psi(0) = 0\) and \(\psi'(0)\) we have
\[
\lim_{h \to 0^+} h^{-1}\psi(h\xi) = \xi.
\]
From \(|\psi'(\xi)| \leq 1\) it follows that
\[
|h^{-1}\psi(h\xi)| \leq |\xi|.
\]
If then \(w \in L^2(\mathbb{R})\) and \(v \in \text{Dom}(\partial_x)\) then
\[
\lim_{h \to 0^+} \int_{\xi \in \mathbb{R}} h^{-1}(FS_{h}DS_{h^{-1}}v)(\xi)\overline{(Fw)(\xi)} d\xi
\]
is
\[
\int_{\xi \in \mathbb{R}} 2\pi i \xi(Fv)(\xi)\overline{(Fw)(\xi)} d\xi,
\]
by Lebesgue Dominated Convergence. Since the Fourier transform is an isometry this is equivalent to the statement that
\[
\lim_{h \to 0^+} \left<h^{-1}S_{h}DS_{h^{-1}}v, w\right> = \left<v_x, w\right>,
\]
which is the weak convergence promised in the lemma.

Finally we observe that
\[
-i(FHDv)(\xi) = \pi \text{ sign } \xi \psi(\xi)(FV)(\xi).
\]
The Fourier multiplier \(\pi \text{ sign } \xi \psi(\xi)\) is positive almost everywhere, so \(-iHD\) is a positive operator. This concludes the proof of the existence of \(c\) and \(D\).
4 Proof of Theorem \[2\]

To prove this, first note that, by the usual formula for the Fourier transform of a convolution,
\[
\mu_0 u - J \ast u = -c^2 D^2 u.
\]

Our integrodifferential equation can therefore be rewritten as
\[
u_{tt} - c^2 D^2 u = 0.
\]

Introducing the quantities
\[
p = \partial_t u, \quad q = cD u,
\]

our second order equation is equivalent to the first order system
\[
p_t = cD q, \quad q_t = cD p.
\]

Defining,
\[
w^\pm = p \pm q,
\]

we find
\[
w_t^\pm = \pm cD w^\pm.
\]

Taking Fourier transforms,
\[
Fw_t^\pm = \pm 2\pi i c\psi Fw^\pm,
\]

with solution
\[
(Fw^\pm)(t, \xi) = e(\pm c\psi(\xi)t)(Fw^\pm)(0, \xi)
\]

Taking inverse Fourier transforms,
\[
w^\pm(t, x) = \int_{\xi \in \mathbb{R}} e(\xi x \pm c\psi(\xi)t)(Fw^\pm)(0, \xi) \, d\xi.
\]

In particular,
\[
w^\pm(t, x_0 + vt) = \int_{\xi \in \mathbb{R}} e((\xi x_0 + s^\pm t)(Fw^\pm)(0, \xi) \, d\xi.
\]

where
\[
s^\pm(\xi) = v\xi \pm c\psi(\xi).
\]
The integrodifferential equation is symmetric under reflections, so we may assume without loss of generality that $v > c$. Then
\[ \partial_s s^\pm \geq v - c > 0. \]

We now integrate by parts repeatedly,
\[ w^\pm(t, x_0 + vt) = t^{-1} \int_{\xi \in \mathbb{R}} r_n(t, \xi)e(s^\pm(\xi)t) d\xi, \]
where $r_h$ is given inductively by
\[ r_0(t, \xi) = e(\xi x_0)(Fw^\pm)(0, \xi), \]
\[ r_{h+1}(t, \xi) = -\partial_\xi((2\pi i \partial_\xi s^\pm(\xi))^{-1}r_h(t, \xi)) \]
Now, by induction on $h$,
\[ \|r^\pm_h\|_{L^1(\mathbb{R})} < \infty. \]
Also $w^\pm(0, x)$ belongs to $\mathcal{S}(\mathbb{R})$, so $(Fw^\pm)(0, \xi)$ is also in $\mathcal{S}(\mathbb{R})$. It follows that
\[ |w^\pm(t, x_0 + vt)| \leq C^\pm t^{-2l}. \]

The parallelogram identity
\[ p^2 + q^2 = \frac{1}{2}[(w^+)^2 + (w^-)^2] \]
then gives the required estimate.

5 Proof of Theorem 4

Starting from the equations
\[ b_j(t, z) = \int_{\xi \in \mathbb{R}} \lambda_j(t, z, \xi) d\xi \]
and
\[ \lambda_j(t, z, \xi) = \alpha(\xi)^{-1} \cos(2\pi \xi z)(2\pi c \psi(\xi))^j \theta_j(2\pi c \psi(\xi)) \]
of the previous section and using the trigonometric identity
\[ \theta_j(x) \cos(y) = \frac{1}{2} (\theta_j(x + y) + \theta_j(x + y)) \]
to write
\[ b_j = b_j^+ + b_j^- \]
where
\[ b_j^\pm(t, z) = \frac{1}{2} \int_{\xi \in \mathbb{R}} \alpha(\xi)^{-1}(2\pi \psi(\xi)ct)^j \theta_j(2\pi \sigma^\pm(t, z, \xi)) d\xi \]
and
\[ \sigma^\pm(t, z, \xi) = \psi(\xi)ct \pm \xi z \]
In this formula we integrate by parts \( N \) times, using the relation
\[ \partial_\xi^N \theta_j - N(\zeta) = \theta_j(\zeta). \]
It is easy to see that the boundary terms all vanish. We thus find
\[ b_j^\pm(t, z) = \int_{\xi \in \mathbb{R}} q_j^\pm(t, z, \xi) \theta_j(2\pi \sigma^\pm(t, z, \xi)) d\xi \]
where the \( q \)'s are given inductively by
\[ q_j^\pm(0, z, \xi) = \frac{1}{2} \alpha(\xi)^{-1}(2\pi c\psi(\xi))^j, \]
\[ q_j^\pm,0(t, z, \xi) = -\partial_\xi \left[ \rho^\pm(t, z, \xi)^{-1} q_j^\pm(t, z, \xi) \right], \]
where
\[ \rho^\pm(t, z, \xi) = 2\pi \partial_\xi \sigma^\pm(t, z, \xi) = 2\pi(ct\psi'(\xi) \pm z). \]
If we define functions \( \gamma_{j,n}^\pm \) for \( 0 \leq h \leq n \) by
\[ \gamma_{j,0,0}^\pm(\xi) = \frac{1}{2} \alpha(\xi)^{-1}(2\pi c\psi(\xi))^j \]
and
\[ \gamma_{j,n+1,h}^\pm(\xi) = 2(n + h)\pi ct \psi'(\xi) \gamma_{j,n,h}^\pm(\xi) - \partial_\xi \gamma_{j,n,h}^\pm(\xi) \]
then a simple induction on \( n \) shows that
\[ q_{j,n}^\pm(t, z, \xi) = \sum_{h=0}^{n} \gamma_{j,n,h}^\pm(\xi) \rho^\pm(t, z, \xi)^{-n-h}. \]
\[ ^4 \text{The terms on the right where the last subscript is out of range, i.e. the first term for } k = 0 \text{ and the second for } k = n + 1, \text{ are to be interpreted as zero.} \]
By induction, $\gamma_{j,n,k}^\pm$ is a linear combination of terms consisting of derivatives of $\alpha$ multiplied by products of derivatives of $\psi$. These latter derivatives are of order at most $n + 1$. Thus, if $n \leq l - 1$,

$$C_{j,n,h} = 2\|\gamma_{j,n,h}^\pm\|_{L^1(\mathbb{R})} < \infty.$$ 

Since

$$|\theta_{j-N}(\xi)| \leq 1$$

and

$$|\rho^\pm(t,z,\xi)| \geq |z| - c|t|$$

for all $\xi$ we conclude that

$$|b_j^\pm(t,z)| \leq \frac{1}{2} \sum_{h=0}^{n} C_{j,n,h}(|z| - c|t|)^{-n-h}$$

if $|z| > c|t|$ and hence

$$|b_j(t,z)| \leq \sum_{h=0}^{n} C_{j,n,h}(|z| - c|t|)^{-n-h}.$$ 

Special care is required for $j = -1$. The prove given above fails for $j < 0$ because $\psi(\xi)^{-j}$ and its derivatives are no longer bounded. $j < -1$ is irrelevant, but we need $j = -1$ for the proof of Theorem 4. Fortunately

$$\partial_t b_{-1}(t,z) = b_0(t,z)$$

and

$$b_{-1}(0,z) = 0$$

so we can simply integrate the estimates for $j = 0$ to obtain

$$|b_{-1}(t,z)| \leq \sum_{h=0}^{n} \frac{1}{(n + h - 1)c} C_{0,n,h}(|z| - c|t|)^{-n-h+1}.$$ 

This estimate and the ones obtained previously are, of course, only useful when $|z| - c|t|$ is positive and reasonably large. For $|z| - c|t|$ small or negative it is best to return to the relation

$$b_j(t,z) = \int_{\xi \in \mathbb{R}} \lambda_j(t,z,\xi)$$
and note that the integrand $\lambda_j(t, z, \xi)$ consists of the integrable factor $\alpha(\xi)^{-1}$ multiplied by something uniformly bounded, so we have the trivial estimate

$$|b_j(t, z)| \leq C_j.$$ We can combine this with the estimates obtained previously to find

$$|b_j(t, z)| \leq C_{j, m} \min(1, |z| - c|t|)^{-m}$$

for all $t$ and $z$, where $m = n - 1$ if $j = -1$ and $m = n$ if $j \geq 0$.

Now we use the relation

$$\partial_j t \partial_k x u(t) = b_j(t) \ast (L\partial^k_x u)(0) + b_{j-1}(t) \ast (L\partial^k_x \partial_t u)(0)$$

derived in the previous section. Writing this as

$$\partial_j t \partial_k x u(t, x) = U_{j, k}(t, x) + U'_{j, k}(t, x),$$

where

$$U_{j, k}(t, x) = \int_{y \in \mathbb{R}} b_j(t, x - y)(L\partial^k_x u)(0, y) \, dy$$

and

$$U'_{j, k}(t, x) = \int_{y \in \mathbb{R}} b_{j-1}(t, x - y)(L\partial^k_x u)(0, y) \, dy,$$

we have the trivial estimates

$$|U_{j, k}(t, x)| \leq W_{k, l} Z_{j, l}$$

and

$$|U'_{j, k}(t, x)| \leq W'_{k, l} Z_{j-1, l}$$

where

$$W_{k, l} = \int_{y \in \mathbb{R}} (1 + |y|)^l |\partial^k_x u(0, y)| \, dy$$

$$W'_{k, l} = \int_{y \in \mathbb{R}} (1 + |y|)^l |\partial^k_x \partial_t u(0, y)| \, dy,$$

and

$$Z_{j, l} = \max_{y \in \mathbb{R}} |b_{j-1}(t, x - y)|(1 + |y|)^{-l}.$$ Now

$$W_{k, l} \leq \sum_{m=0}^{l} \frac{l!}{m!(l-m)!} (V_{k, m} + AV_{k+2, m}).$$

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\[ W_{k,l} \leq \sum_{m=0}^{l} \frac{l!}{m!(l-m)!} (V_{k,m} + AV_{k+2,m}), \]

where

\[ V_{k,m} = \int_{y \in \mathbb{R}} |y|^m \partial_x^k u(0,y) \, dy \]

and

\[ V'_{k,m} = \int_{y \in \mathbb{R}} |y|^m \partial_x^k \partial_t u(0,y) \, dy. \]

By hypothesis \( V_{k,m} \) and \( V'_{k,m} \) are finite for \( k \leq K + 2 \) and \( m \leq l \), so \( W_{j,l} \) and \( W'_{j,l} \) are finite.

Also

\[ Z_{j,l} \leq C_{j,l} \max_{y \in \mathbb{R}} \min(1, |x - y| - c|t|)^{-l} (1 + |y|)^{-l} \]

or, evaluating the right hand side,

\[ Z_{j,l} \leq \begin{cases} 
C_{j,l} & \text{if } |x| \leq c|t| + 1, \\
2^{2l} C_{j,l} (1 + |x| - c|t|)^{-2l} & \text{if } |x| \geq c|t| + 1.
\end{cases} \]

From this it follows that

\[ Z_{j,l} \leq 2^{2l} C_{j,l} \min(1, |x| - c|t|)^{-2l} \]

for all \( t, x \in \mathbb{R} \) and hence that

\[ |U_{j,k}(t,x)| \leq 2^{2l} C_{j,l} W_{k,l} \min(1, |x| - c|t|)^{-2l} \]

The same is true for \( U'_{j,k}(t,x) \), with \( W'_{k,l} \) in place of \( W_{k,l} \) and \( C_{j-1,l} \) in place of \( C_{j,l} \), except possibly for \( j = 0 \) where we have to substitute our weaker estimate for \( b_{-1} \), obtaining

\[ |U'_{0,k}(t,x)| \leq 2^{2l-2} C_{-1,l} W'_{k,l} \min(1, |x| - c|t|)^{-2l-2}. \]

Combining all these estimates,

\[ |\partial_t^j \partial_x^k u(t,x)| \leq C_{j,k,l} \min(1, |x| - c|t|)^{-2l} \]

if \( 0 < j \leq l \) and

\[ |\partial_x^k u(t,x)| \leq C_{0,k,l} \min(1, |x| - c|t|)^{-2l-2}. \]

This concludes the proof of Theorem 4.
References

[1] S. A. Silling, Reformulation of elasticity theory for discontinuities and long-range forces, J. of the Mech. and Phys. of Solids, 48 (2000), pp. 175–209.