On a class of Semi-Elliptic Diffusion Models.
Part I: a constructive analytical approach for global solutions

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Abstract

Semi-elliptic stochastic differential equations (SDEs) are common models among practitioners. However, the value functions and sensitivities of such models are described by degenerate parabolic partial differential equations (PDEs) where the existence of regular global solutions is not trivial, and where densities do not exist in spaces of measurable functions but only in a distributional sense in general. In this paper, we show that for a related class of such equations regular global solutions can be constructed. Moreover, the solution scheme has a probabilistic interpretation where the existence of regular densities on certain subspaces of the state space can be exploited. Prominent examples of models of practical interest belonging to this class include factor reduced LIBOR market models and Cheyette models. Moreover, factor reduced SDEs originating from a full factor model are in the class to which our theorem applies.

The result is also of interest for the theory of degenerate parabolic equations. A more detailed analysis of numerical and computational issues, as well as quantitative experiments will be found in the second part.

Keywords: Degenerate parabolic equations, financial derivatives, sensitivities, Monte-Carlo methods.

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1 Introduction

In this paper, we show that for a class of semi-elliptic equations with relatively rough initial data on a linear subspace (where a hypo-ellipticity condition holds), and regular data on the complementary linear subspace, regular global solutions can be constructed. Moreover, we derive a solution scheme which has a probabilistic interpretation such that it can be used to build a simulation scheme of the associated SDE. The class of equations considered is particularly relevant for valuation models derived in finance.

1.1 Main Theorem and its Assumptions

The prototypical application of the main theorem of this paper is the valuation of a financial derivative: put simply, the valuation of a financial derivative consists of a model of some underlyings (usually given by a stochastic differential equation (SDE)) and a value function representing the value of the financial derivative as a function of the modeled underlyings. In its simplest case, the value function is known at some future time as a function of the underlyings, the payoff function, and the value function of the financial derivative at earlier times can then be represented as some conditional expectation of the payoff function.¹

Our main theorem makes an assumption about the relation of the model SDE and the payoff function, namely

- the model SDE is semi-elliptic, i.e. may have a degenerated diffusion matrix, and
- the payoff function may be rough (i.e. in $L^p$) in some state variables, and
- the subspace of smoothness of the payoff function fits a subspace where the semi-elliptic operator corresponding to the SDE is not smoothing, i.e. is not even micro-hypoelliptic.

In Section 2.1.1 we illustrate that this assumption is a natural situation in derivative pricing models.

1.2 Layout of the Paper, Proof of the Main Theorem

Our main theorem is motivated by stochastic differential equations and payoff functions encountered in the realm of valuing financial derivatives. We consider semi-elliptic models which satisfy a hypo-ellipticity condition on a linear subspace and have adapted payoffs with mixed regularity.

Our main theorem is an extension of Hörmander’s well-known result concerning a sufficient condition for the hypoellipticity of linear differential operators with variable coefficients (the so called Hörmander condition, cf. [12]). Smoothness (i.e. $C^\infty$-regularity) of the density of the associated SDEs is a subject of Malliavin calculus. Indeed a Malliavin-type estimate obtained in [23] is crucial for our main theorem.

Moreover, our proof consists of a constructive PDE-scheme. This scheme can be turned into a Monte-Carlo scheme for the original associated SDE.

¹The more general case where intermediate payoffs depend on future values of the derivative is usually a straight forward extension, defining a backward-induction of value functions.
The paper is structured as follows: Section 2 gives a thorough motivation for our considered class of SDEs and payoff functions and discusses their relationships to PDEs, transition densities and the calculation of sensitivities in the context of semi-elliptic models. Section 3 gives a list of some models where our main theorem readily applies. These models are common among practitioners. Section 4 extends our considerations to general reduced diffusion models. Section 5 reviews the existence of global weak solutions from the stochastic perspective and motivates the main theorem from a mathematical perspective. In Section 6 we state and prove our main theorem formulated as a global existence result of the associated PDE. Section 8 briefly sketches the application of our constructive proof from Section 6 to the Monte-Carlo simulation of the corresponding SDE. Section 9 concludes the paper with some further remarks.

2 Motivation

2.1 Semi-Elliptic Stochastic Differential Equations in Finance

Semi-elliptic stochastic differential equations (SDEs) are common models among practitioners. One approach to obtaining such a model is to start from a model SDE with full rank diffusion matrix (i.e., full factor model) and perform a principal component analysis (PCA) to build a factor reduced model with a low-rank diffusion matrix. The result is a ‘reduced model’ which uses only a reduced set of, e.g., Brownian drivers while retaining the full state space dimension. The convection term, i.e. drift, is not affected by the reduction. A prominent example of a model where a factor reduction is usually applied is the LIBOR market model. A factor reduction can be advantageous in terms of calculation speed, see [14] and of course noise reduction. Often the model is reduced to factors (i.e., modes) which are of particular importance to the application and PCA is just one way of determining important factors. If, for example, a model is used as a pricing model, one might determine important factors from product properties. Practitioners usually attribute importance to factors like level, slope and curvature, see [19].

Another approach to obtaining a factor-reduced SDE is to use a specific modeling assumption leading to a low factor model with, e.g., low Markov dimension. A prominent example here is the Cheyette model: starting from an infinite dimensional HJM model a specific (separable) instantaneous volatility structure will lead to an, e.g., two dimensional Markovian one factor model. These examples belong to a general class of stochastic processes $X_t$ with values in $\mathbb{R}^n$ and satisfying a stochastic differential equation

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \quad 0 \leq t \leq T, \ X(0) = x.$$  \hspace{1cm} (1)

Elementary SDE theory tells us that the global existence of Levy-continuous solutions can be derived if the coefficient functions satisfy a global Lipschitz condition (cf. discussion in Section (5) below). The Feynman-Kac formalism shows that for a considerable class of data functions $f$ (related to the payoffs in finance) and stochastic processes $X$ the function

$$(t,x) \rightarrow E(f(X_t))$$  \hspace{1cm} (2)
and its derivatives (related to derivative prices and Greeks in finance) are determined by the solution of a Cauchy problem on $[0, T] \times \mathbb{R}^n$ (where $T > 0$ is an arbitrary finite horizon)

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{1}{2} \sum_{i=1}^{m} V_i^2 u + V_0 u \\
u(0, x) &= f(x),
\end{aligned}$$

(3)

where

$$V_i = \sum_{j=1}^{n} v_{ji}(x) \frac{\partial}{\partial x_j},$$

(4)

are vector fields with $0 \leq i \leq m$. Hörmander provided a sufficient condition for the existence of regular densities. However, in the case of (1) this Hörmander condition is satisfied only on a certain subspace of $\mathbb{R}^n$ in general. In most applications in finance, reduced models are of a special form where the vector fields (4) live in a fixed subspace such that the Cauchy problem (3) has a certain block structure (we list a set of the most popular models below). For this reason we emphasize this case because it is of particular interest for the practitioner.

It turns out that our main result can be extended to a more general result which does not depend on the block structure and which is an extension of Hörmander’s result. We state this generalization (the proof method is similar).

2.1.1 Relation of the Payoff and the Model

A factor reduction, i.e. the reduction of the brownian drivers, is often motivated by computationally feasibility: the model should be computational feasible and still provide an accurate representation of the perceived underlying dynamic (e.g., historically observed “factors”). However, in addition the modeling has to consider the application. For the valuation of a payoff, the degree of regularity of the payoff function will give weight to certain aspects of the underlying’s dynamics. This weighting of “factors” is often different from the distribution of historically observed factor dynamics. It is exactly this weighting which leads to the fact that a meaningful semi-elliptic model for financial derivative pricing will most likely fall into the class of models considered by our main theorem.\(^2\)

**Example** A simple example of the situation described is the modeling of an interest rate curve and the valuation of an (interest rate) spread option. The interest rate curve can be formalized as an infinite dimensional space. One approach to modeling it is to focus on a discrete set of interest rates (e.g., forward rates with fixed period length) and define the interest rate curve using appropriate interpolations. Let us consider the model

$$\begin{aligned}
dL_k(t) &= \mu_k(t)dt + \sigma(t)dW_1(t) + \gamma(t)dW_2(t), \\
dL_{k+1}(t) &= \mu_{k+1}(t)dt + \sigma(t)dW_1(t) - \gamma(t)dW_2(t)
\end{aligned}$$

for two adjacent forward rates of fixed period length $L_k$ and $L_{k+1}$. Assume that $W_1$ and $W_2$ are independent Brownian drivers. In this model $\sigma$ is the instantaneous volatility of a joint interest rate movement while $\gamma$ is the instantaneous

\(^2\)We will give a selection of some common models in Section 3.
2.2 Relation to PDEs, Existence of Densities

While a factor reduction is straight-forward on the SDE level and easy to implement, it may severely affect the sense in which the solution of the corresponding PDE, i.e., the density exists. The most important situation we consider in this paper (at least from the point of view of the practitioner) is the following: Consider a matrix function \( x \rightarrow (a_{ji})^{d,m}(x) \), \( 1 \leq j \leq d \) on \( \mathbb{R}^n \) with \( n \geq d \), and \( m \) smooth vector fields of dimension \( d \)

\[
A_i = \sum_{j=1}^{d} a_{ji} \frac{\partial}{\partial x_j}, \quad (5)
\]
where \( 0 \leq i \leq m \). Consider an additional vector field of dimension \( n - d \)

\[
B := \sum_{j=d+1}^n a_{j0} \frac{\partial}{\partial x_j}.
\]  

Consider the Cauchy problem on \([0, T] \times \mathbb{R}^n\) (where \( T > 0 \) is an arbitrary finite horizon)

\[
\begin{cases}
\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i=1}^m A_i^2 u + (A_0 + B)u \\
u(0, x) = f(x).
\end{cases}
\]  

(6)

**Remark 1.** The Cauchy problem (7) is written in the block structure form which we have in most cases of practical interest. Generalisations which are coordinate-independent are considered in section 4 and in section 7.

**Remark 2.** Note that the Cauchy problem (7) may be written in the more familiar form

\[
\begin{cases}
\frac{\partial u}{\partial t} \frac{1}{2} \sum_{i,j=1}^k a_{ij}^* \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^n \mu_i \frac{\partial u}{\partial x_i} = 0, \\
u(0, x) = f(x),
\end{cases}
\]  

(8)

for some \( k \leq n \), and where \((a_{ij}^*)\) is a diffusion matrix and \(\mu_i\) are drift terms defined in terms of coefficients of the vector fields \(A_i\). Note that we have \(k \leq d\). If \((a_{ij}^*)\) is uniformly elliptic on the subspace \(\mathbb{R}^k\), then we are back in the situation of [9].

Assume that (7) satisfies the Hörmander condition with respect to the subspace \(\mathbb{R}^d\), i.e. assume that

\[
\{A_i, [A_j, A_k], [[A_j, A_k], A_l], \ldots | 1 \leq i \leq m, \ 0 \leq j, k, l \ldots \leq m \}
\]  

(9)

spans \(\mathbb{R}^d\) at each point \(x\). Here \(\ldots\) denotes the Lie bracket of vector fields. If \(n = d\) we are back in the situation of Hörmander’s class of hypoellipticity. The existence of regular solutions of the Cauchy problem in this case is well known (cf. [12]). Indeed Hörmander’s result shows us that a family of smooth transition densities exists if (9) holds for every \(x \in \mathbb{R}^n\) (where \(n = d\)). This is no longer true if \(n > d\). In this paper we show that in this case we still have regular solutions if the Hörmander condition holds on a subspace where data are locally measurable (i.e. \(L^\infty_{\text{loc}}\)) and here data are smooth (and satisfying some growth condition) on the complementary space where the Hörmander condition does not hold.

Note that a consequence of a factor reduction is that the density may lose regularity. Often a classical (regular) density will no longer exist. Consider the simple example in the following section 2.3.

### 2.3 Example: A simple two dimensional toy model

Let us consider the following Cauchy problem on \([0, T] \times \mathbb{R}^2\):

\[
\begin{cases}
\frac{\partial u}{\partial t} = \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x_1^2} + \mu \frac{\partial u}{\partial x_1}, \\
u(0, x) = f(x_1) + g(x_2).
\end{cases}
\]  

(10)
The solution of this equation is

$$\int_{\mathbb{R}} f(y_1) \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{(x_1 - y_1)^2}{2\sigma^2 t} \right) dy_1 + g(x_2 + \mu t). \quad (11)$$

This leads us to a ‘distributional density’ of the form

$$p(t, x, y) := \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{(x_1 - y_1)^2}{2\sigma^2 t} \right) \delta(x_2 + \mu t - y_2). \quad (12)$$

Indeed, formally we have (let us denote $dy = dy_1 dy_2$)

$$\int (f(y_1) + g(y_2)) p(t, x, y) dy =$$

$$\int_{\mathbb{R}} f(y_1) \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{(x_1 - y_1)^2}{2\sigma^2 t} \right) \delta(x_2 + \mu t - y_2) dy_1 dy_2 + \int_{\mathbb{R}} g(y_2) \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{(x_1 - y_1)^2}{2\sigma^2 t} \right) \delta(x_2 + \mu t - y_2) dy_1 dy_2 \quad (13)$$

$$= \int_{\mathbb{R}} f(y_1) \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{(x_1 - y_1)^2}{2\sigma^2 t} \right) dy_1 + g(x_2 + \mu t - y_2)$$

$$= u(t, x),$$

assuming that some ‘kind of Fubini theorem’ can be applied. We see from this example that no regular density exists except in a distributional sense. This simple example shows that no regular density exists in a situation of degenerate diffusion models. More particularly, the regularity theory of densities in the context of Malliavin calculus does not apply. Note that the associated process is $X = (X_1, X_2)$ with

$$dX_1 = \sigma dW$$

$$dX_2 = \mu dt. \quad (14)$$

Such a process exists as is well-known in the context of stochastic analysis (cf. citation of theorem 1 below). The simple example shows that no regular density exists in a situation of degenerate diffusion models with $n > d$. On the other hand, (14) (and thus (10)) could just be the result of a factor reduction of the process

$$dX_1 = \sigma dW$$

$$dX_2 = \sigma_2 dW_2 + \mu dt. \quad (15)$$

where the factor reduction (i.e., dropping $dW_2$) may have been applied because $\sigma_2$ was considered to be small.

2.4 Discussion of the Cauchy-problem (7) from the perspective of Malliavin calculus, microhypoellipticity, and PDE-methods

The example above shows that operators of the form (7) are not hypoelliptic for $B \neq 0$ in general. Recall that a differential operator $L$ with $C^\infty$-coefficients
is called hypoelliptic on an open set $\Omega \subseteq \mathbb{R}^n$ if for any distribution $u$ on $\Omega$ is in $C^\infty$ if $Lu \in C^\infty$. Now, for $u$ with

$$u(t, x) = \int_\mathbb{R} f(y_1) \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{(x_1 - y_1)^2}{2\sigma^2 t} \right) dy_1 + g(x_2 + \mu t). \quad (16)$$

we have

$$\frac{\partial u}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x_1^2} - \mu \frac{\partial u}{\partial x_2} = 0, \quad (17)$$

where $g$ may be $C^1$, and this shows that the operator in example 1 is not hypoelliptic. This provides us with a rigorous argument that there is no regular density for the Cauchy problem in (10), because it is known that for linear operators with constant coefficients hypoellipticity is equivalent to the existence of a fundamental solution in $C^\infty$ on $\mathbb{R}^n \setminus \{0\}$.

It is of interest here to note that operators with constant coefficients may not be hypoelliptic even if they are obtained by freezing coefficients of hypoelliptic operators with variable coefficients. For example the hypoelliptic operator on $\mathbb{R}^3$ of the form

$$L_x \equiv \frac{\partial^2}{\partial x_1^2} + \left( \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} \right)^2 \quad (18)$$

has a counterpart ($c = \text{some constant}$)

$$L_c \equiv \frac{\partial^2}{\partial x^2} + \left( \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} \right)^2 \quad (19)$$

which is not hypoelliptic, because $h(cy - z)$ is a solution of $L_c h = 0$ for a $C^1$ function $h$. Note that Hörmander’s result is sensitive to such differences, since it is concerned with microhypoellipticity which implies hypoellipticity. Consider equation (7) with $B = 0$, i.e. consider

$$\begin{cases}
Lu \equiv \frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i=1}^m A_i^2 u + A_0 u \\
u(0, x) = f(x).
\end{cases} \quad (20)$$

If Hörmander’s condition is satisfied (20), then the operator of (20) is microhypoelliptic, i.e.

$$\text{WF}(u) = \text{WF}(Lu), \quad (21)$$

where $\text{WF}(u)$ denotes the wave front set of a distribution $u$, i.e. the intersection of the characteristic varieties of pseudo-differential operators $P$ of order zero which satisfy $Pu \in C^\infty$. Indeed, it is clear that the operators of (7) are not microhypoelliptic for $B \neq 0$ in general. This is reflected by the fact for pseudodifferential operators of negative order the characteristic variety equals the whole cotangential bundle $T^\ast \Omega$ of the domain $\Omega$. Hence, the microhypoelliptic theory is designed for pseudodifferential operators of order 0 (at least), and this does not apply in our case.

From the probabilistic point of view the Cauchy problem (7) has a Levy-continuous solution (cf. the discussion in section 2 below). For smooth data we immediately get regularity of the first order time derivative and a classical solution (cf. remark 3 below). However, especially in finance, we are interested
The regularity theory of densities in the context of Malliavin calculus does not apply directly (as we remarked in [9]). However, it may still be useful, especially for the analysis of the subspace of dimension $d$ where the Hörmander condition holds. Let us consider (7) from a probabilistic perspective. The associated Stratonovic integral of a process starting at $x \in \mathbb{R}^n$ is

$$X_t = x + \int_0^t A_0(X_s)ds + \sum_{k=1}^m \int_0^t A_k(X_s) \circ dW_k(s)$$

(22)

where $W_i$ denote Brownian motions and $\circ$ indicates that the integral is in the Stratonovic sense. If $n = d$ and the Hörmander condition holds, then the associated covariance matrix process $\sigma_t$ is defined by

$$\sigma_t = Z_t^{-1} \left[ \int_0^t Z_s A_s(X_s) A_s^T(X_s) Z_s^T \right] Z_t^{-1,T}$$

(23)

where $Z$ is a matrix-valued invertible process defined by

$$Z_t = I_d - \int_0^t Z_s D A_0(X_s) ds - \sum_{i=1}^n \int_0^t Z_s D A_i(X_s) d \circ W_i(s),$$

(24)

and $I_d$ denotes the $d$-dimensional identity matrix. From the perspective of financial applications it is interesting to note that $\sigma_t$ is almost surely invertible, i.e.

$$\sigma_t^{-1} \in L^p$$

(25)

for every real number $p$ and $t$ in some arbitrary finite time horizon $[0, T]$, because this property is shared by many stochastic volatility diffusion models. The reduced models where it does not hold but holds only on a linear subspace are considered in this paper.

The theory of partial differential equations has investigated systems of equations of hyperbolic-parabolic type for a long time. One of the crucial conditions for such systems is the so-called Kawashima-condition. Different equivalent forms of this condition are considered in [24]. Several equivalent conditions of global existence and regularity have been developed for systems of the form

$$D^0 w_t + \sum_{j=1}^n D^j w_{x_j} - \sum_{j,k=1}^n E^{j,k} w_{x_j,x_k} + F w,$$

(26)

where $D_t$, $E^{j,k}$, and $F$ are are $m \times m$-matrices. For general results these matrices are considered to be constant. In this case, for initial data

$$w_0 \in H^s (\mathbb{R}^n) \cap L^p (\mathbb{R}^n),$$

(27)

and if $D^j$, $1 \leq j \leq n$ and $E^{j,k}$, $1 \leq j, k \leq n$ are real symmetric matrices, and

$$B(\xi) := \sum_{j,k} B^{j,k} \xi_j \xi_k \geq 0$$

(28)

for $\xi \in S^{n-1}$ (the $n - 1$-dimensional sphere in $\mathbb{R}^n$), then for dissipative systems there are constants $\delta, C > 0$ where we have

$$\|D^j_t w(t)\|_{L^r} \leq C \left[ e^{-\delta t} \|D^j_x w_0\| + (1 + t)^{\gamma + l/2} \|w_0\|_{L^p} \right]$$

(29)
for all integers \( l \leq s \). For variable coefficient matrices the existence results are weaker (additional conditions, small data or less regularity). Observe that such results are derived for data with uniform regularity assumptions. Our theorem is different and special in this respect.

2.5 Relation to Sensitivities (Greeks)

In our main theorem we will construct a regular solution to the associated (backward) PDE of a factor reduced SDE, given that

- the regularity of the initial condition of the PDE, i.e. the regularity of the associated payoff function, is (in a certain sense) compatible with the diffusion matrix, namely that we have smoothness of the initial condition where we lack diffusion.

Note also that this assumption is important, or even a requirement, when computing Greeks. If, for example, the payoff function is non-differentiable in a coordinate for which the underlying SDE lacks diffusion the corresponding (second order) partial derivative, i.e. the sensitivity or Greek, will be not unique (or unbounded). In this situation, the “ignorant” numerical calculation of path-wise Monte-Carlo Greeks will not converge and the numerical calculation of a Greek involving likelihood ratios cannot be performed due to the lack of a (regular) density. The constructive scheme involves regular densities on the subspace where the diffusion lives and these densities can be used when calculating Greeks numerically. Thus, the constructive iteration scheme used in the proof of our main theorem may itself result in a numerical (Monte-Carlo) scheme to calculate robust Monte-Carlo Greeks in the setup of a reduced factor model.

3 Applications from Finance

3.1 Factor Reduced Lognormal LIBOR Market Models

Reduced factor (LIBOR market) models are common among practitioners. In [10] a numerical scheme of a reduced factor LIBOR market model is given by considering the full proxy scheme on a linear subspace; in [7] by considering the partial proxy scheme. Both schemes require knowledge of transition probabilities. In [10] it was possible to give the transition density for the full factor model (WKB expansion), while for the reduced factor model only the transition density of an approximating scheme was considered. As an important example, let us look at the LIBOR market model with tenor structure \( 0 < T_1 \ldots < T_{n+1} \) in terminal measure \( P_{n+1} \) (induced by the terminal zero coupon bond \( B_{n+1}(t) \)). The dynamics of the forward LIBORs \( L_i(t) \), defined in the interval \([0, T_i]\) for \( 1 \leq i \leq n \), are described by

\[
dL_i = \sum_{j=i+1}^{n} \frac{\delta_j L_i \gamma_i^\top \gamma_j}{1 + \delta_j L_j} \, dt + L_i \gamma_i^\top \, dW^{(d)} =: \mu_i^{(d)}(L)L_i + L_i \gamma_i^\top \, dW^{(d)},
\]

where \( \delta_i = T_{i+1} - T_i \) are day count fractions and \( t \rightarrow \gamma_i(t) = (\gamma_{i,1}(t), \ldots, \gamma_{i,d}(t)) \), \( 0 \leq t \leq T_i \), are deterministic, bounded and smooth volatility vector functions.
We denote the matrix with rows $\gamma_i^\top$ by $\Gamma$ and assume that $\Gamma$ is invertible. For simplicity, assume that in the following $\Gamma(t) = \Gamma$ does not depend on $t$ (it is just a technical matter to generalize to time-dependent coefficients). In (30), $(W^{(n+1)}(t) \mid 0 \leq t \leq T_n)$ is a standard $d$-dimensional Wiener process under the measure $P_{n+1}$ with $d$, $1 \leq d \leq n$, being the number of driving factors. First, we consider the full-factor LIBOR model with $d = n$ in the time interval $[0, T_1)$. The standard log-transformation leads to $K_i := \ln L_i$, $1 \leq i \leq n$, and

$$dK_i = \frac{1}{2}dL_i - \frac{1}{2x_i}d(L_i) = \left(-\frac{\gamma_i^\top \gamma_i}{2} + \mu_i^L(t, e^{K_1}, \ldots, e^{K_n})\right) dt + \gamma_i^\top dW^{(n+1)}$$

(31)

where $\mu^L$ is given in (30). In these coordinates the associated infinitesimal generator is related to a strictly parabolic operator, while in the original coordinates parabolic degeneracies appear. Nevertheless, parabolic degeneracies appear even in logarithmic coordinates $K$ for reduced factor models. In such models one considers $d$ driving factors for some $d$, $0 < d < n$ instead of $n$ driving factors up to time $T_1$. Then, instead of an invertible matrix $\Gamma$, we have a matrix $F$ typically of rank $d$, e.g. the factor-reduced LIBOR market model typically used by practitioners takes the form

$$dK = \mu^K(t, K)dt + FdW,$$

(32)

where $F$ is a $n \times d$ matrix with $d \leq n$ (e.g. $40 \times 5$ matrix). Reduced factor models are popular among practitioners, because they are computationally parsimonious, while the essential dynamical features of the LIBOR market model can be preserved if $d$ is not too small. Note that

$$FF^T$$

is a $n \times n$ matrix of rank $d$ while $F^TF$ is a $d \times d$ matrix of rank $d$. If $G = (f_{d+1}, \cdots, f_n)$ is the matrix consisting of the eigenvectors $f_i$ of $\ker (FF^T)$, then the system (32) is equivalent to

$$\begin{align*}
&d \left(F^TK\right) = F^T\mu^K(t, K)dt + F^TFdW^{(n+1)} \\
&d(G^TK) = G^T\mu(t, K)dt,
\end{align*}$$

(34)

because $G^TF = 0$. The evaluation problems for options and Greeks in this type of model lead us directly to degenerate parabolic Cauchy problems (cf. (35) below). The interest in regular global solutions to projective degenerate equations is not limited to reduced factor LIBOR market models, of course. So let us formulate it in a fashion which looks less specific, i.e. without reference to lognormal coordinates, and as the more familiar initial-value problem. Instead of specific loading factor matrix $\Gamma$ we have a volatility matrix $\sigma\sigma^T$, which may depend on time and assets. In logarithmic coordinates this leads to the Cauchy problem

$$\begin{align*}
&\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{i,j=1}^k \left(\sigma\sigma^T\right)_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n \mu(t, x) \frac{\partial u}{\partial x_i} = 0, \\
u(0, x) = f(x).
\end{align*}$$

(35)
with \(0 < k < n\) to have global regular solutions. For simplicity of notation, we use the abbreviation

\[
a_{ij}(t, x) := \frac{1}{2} (\sigma \sigma^T)_{ij} (t, x),
\]

where we keep in mind the fact that we assume that for each \((t, x)\) in the domain of (35) the matrix \(a_{ij}(t, x)\) has a decomposition as in (36). Note that condition (36) is not trivial; a sufficient condition is that all \(a_{ij} \in C^2\) are bounded and have bounded first and second order derivatives. This assumption is important if we want to make the link to probabilistic schemes and to Monte-Carlo methods, as we do in the second part. However, for the statement of the main theorem we do not need these assumptions. From a practical point of view this seems not to be a serious restriction.

### 3.2 Low Dimensional Markovian HJM Models, Cheyette Model

The LIBOR market model can be interpreted as a finite dimensional Markovian Heath-Jarrow-Morton (HJM) model with a (usually) high Markov dimension \(n\). We may observe the situation of semi-elliptic equations in lower dimensional Markovian HJM models, e.g. as for the Cheyette model. The Cheyette model, cf. [3], can be derived from an HJM model by assuming a specific structure of the instantaneous volatility structure. The volatility structures were first considered by Ritchken and Sankarasubramanian (1995), cf. [22]. For an introduction to the HJM framework see, e.g., [6].

Given an HJM dynamic for the instantaneous forward rate

\[
df(t, T) = \alpha(t, T) \, dt + \sigma(t, T) \, dW(t),
\]

\[
f(0, T) = f_0(T)
\]

endowed with a special volatility structure

\[
\sigma(t, T) := g(t) \, h(T),
\]

where \(g : [0, T^*] \to \mathbb{R} \setminus \{0\}\) denotes a deterministic function and \(h : [0, T^*] \times \Omega \to \mathbb{R}^m\) an \(m\)-dimensional Markov process. Then the short-rate process is given by

\[
r(t) = f(0, t) + X(t),
\]

where

\[
dX(t) = \left( Y(t) - \kappa(t)X(t) \right) dt + \eta(t) \, dW(t), \quad X(0) = 0,
\]

\[
dY(t) = \left( \eta^2(t) - 2\kappa(t)Y(t) \right) dt, \quad Y(0) = 0,
\]

and

\[
\eta(t) = \frac{\sigma(t, t)}{g(t)}, \quad \kappa(t) = \frac{g'(t)}{g(t)}.
\]

Here, the factor reduced model (40) results from a modeling assumption and not from a PCA factor reduction.
3.3 Heston Model

The Heston stochastic volatility model is given by

\[
\begin{align*}
    dS(t) &= \mu S(t) dt + \sqrt{\nu(t)} S(t) dW_1(t) \\
    d\nu &= \kappa (\theta - \nu(t)) dt + \xi \sqrt{\nu(t)} dW_2(t) \\
    dW_1(t) dW_2(t) &= \rho dt
\end{align*}
\]

The numerical simulation of the Heston model is known as a challenging problem. Exact simulation is possible, but requires computationally expensive Fourier transforms, see [1]. Here, a proxy simulation scheme [8] can be an alternative, using a simple Euler scheme for path generation and to correct the probability densities.

3.4 SABR Model

The SABR model [11] is given by

\[
\begin{align*}
    dL(t) &= \sigma(t) L(t)^\beta dW_1(t) \\
    d\sigma &= \alpha(t) \sigma(t) dW_2(t) \\
    dW_1(t) dW_2(t) &= \rho dt
\end{align*}
\]

where \(L\) is some forward rate. The model may be used to endow a reduced factor LIBOR market model with smile modeling.

3.5 Stochastic Volatility CEV LIBOR Market Model / LMM-SABR Model

A popular extension of the LIBOR Market Model is to endow the model with SABR-like stochastic volatility CEV dynamics, see [21]. One such extension is

\[
\begin{align*}
    dL_i &= \mu^L_i (t, L) dt + L_i^\beta_i \lambda \gamma_i \top dW^{(d)}, \\
    d\lambda &= \alpha \lambda dW^{d+1}
\end{align*}
\]

Using the transformation

\[
K_i := \begin{cases}
    L_i^{1-\beta_i} & \text{for } \beta_i \neq 1 \\
    \ln(L_i) & \text{for } \beta_i = 1
\end{cases}
\]

we can write the model with a block diffusion structure as

\[
\begin{align*}
    d\lambda &= \alpha \lambda dW^{d+1}, \\
    d(F^\top K) &= F^\top \mu^K(t, L) dt + \lambda F^\top F dW^{(d)}, \\
    d(G^\top K) &= G^\top \mu^K(t, L) dt,
\end{align*}
\]

where \(F, G\) are as in Section 3.1.

\[\text{The model may also be used to model other underlyings.}\]
4 General reduced Diffusion models

Although the examples above show that there is a considerable class of reduced market models which can be put into the form (7), a more general form of a reduced market model can be defined if we consider Cauchy problems where a Hörmander condition is satisfied at each state space point \( x \) with respect to a \( d \)-dimensional subspace where this subspace and its dimension \( d \equiv d(x) \) may depend on that state space point. It is worth considering this general case and its relation to the case where the subspace and its dimension are fixed. We shall state a generalization of our main theorem in this case. It has the advantage of having a coordinate-invariant formulation. Consider a matrix function \((t, x) \rightarrow (v_{ji})^{n \times m}(t, x), \ 1 \leq j \leq n, \ 0 \leq i \leq m\) on \( \mathbb{R}^n \), and \( m \) smooth vector fields

\[
V_i = \sum_{j=1}^{n} v_{ji}(t, x) \frac{\partial}{\partial x_j}, \quad (41)
\]

where \( 0 \leq i \leq m \). Note that we allow for time-dependent coordinates now, but this is not the main point. Consider the Cauchy problem on \([0, T] \times \mathbb{R}^n\) (where \( T > 0 \) is an arbitrary finite horizon)

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{1}{2} \sum_{i=1}^{m} V_i^2 u + V_0 u \\
u(0, x) &= f(x).
\end{aligned} \quad (42)
\]

This may be rewritten in the form

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{1}{2} \sum_{i,j=1}^{n} v_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} u + \sum_{j=1}^{n} v_{j0}(t, x) \frac{\partial u}{\partial x_j} \\
u(0, x) &= f(x),
\end{aligned} \quad (43)
\]

where

\[
(v_{ij}^*) (t, x) = \sum_{k=1}^{m} (V_i)^{\otimes^2}_k. \quad (44)
\]

A general reduced Cauchy problem is defined by the condition that for all \( t \in [0, T] \) and \( x \in \mathbb{R}^n \)

\[
(v_{ij}^*) (t, x) \quad (45)
\]

has rank \( d \equiv d(t, x) \leq n \), where for each \((t, x) \in [0, T] \times \mathbb{R}^n\) the number \( d(t, x) \) is determined by the Hörmander condition at \((t, x) \in [0, T] \times \mathbb{R}^n\), i.e.

\[
\{V_i(t, x), [V_j, V_k](t, x), [[V_j, V_k], V_l](t, x), \ldots | 1 \leq i \leq m, \ 0 \leq j, k, l \cdots \leq m\} \quad (46)
\]

spans a linear subspace \( W_{(t,x)} \) of dimension \( d(t, x) \). Note that at each point \((t, x)\) the linear subspace is now isomorphic to \( \mathbb{R}^d \) but varies in \( \mathbb{R}^n \), i.e. the components of the vectors in \( W_{(t,x)} \) are not fixed components of \( \mathbb{R}^n \) in general. Note that the situation is different from the situation in [9] and all the examples above where the family \((V_i)_{1 \leq i \leq n}\) spans an invariant \( d \)-dimensional subspace of invariant dimension as \((t, x)\) varies. The same holds for the usual stochastic volatility extensions (e.g. the SABR-model listed above). Let us consider a non-linear transformation which diagonalizes the diffusion matrix \((\omega_{ij}^*) \geq 0\) locally. Consider a function \( w \) in transformed coordinates

\[
w(t, y) = u(t, x). \quad (47)
\]
Then
\[
\sum_{ij=1}^{n} a^*_{ij} \partial^2_{u} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} = \sum_{ij=1}^{n} a^*_{ij} \left( \sum_{k,l=1}^{n} \frac{\partial^2 w}{\partial y_k \partial y_l} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{k} \frac{\partial w}{\partial y_k} \frac{\partial^2 y_k}{\partial x_i \partial x_j} \right),
\]

hence the leading order term of the PDE becomes
\[
\text{Tr} (J_x(y) A^* J_x(y)^T D_y^2 v),
\]

where \( A^* \) is the diffusion matrix with respect to the new coordinates, i.e. \( A^*(t, y) = (a^*_{ij})(t, x(y)) \) and \( J_x(y) \) denotes the Jacobian. Let \( \text{diag}(\lambda_i_1, \ldots, \lambda_i_d) \) denote the matrix with \( d \) diagonal elements which are not equal to zero in the rows \( i_1, i_2, \ldots, i_d \), and zero elsewhere. If we have a diagonalization of \( (a^*_{ij}) \) at a fixed point \( (t, x) \) with \( d \) eigenvalues strictly larger than zero (i.e. a point of nondegeneracy in a linear subspace of dimension \( d \)), then the implicit function theorem applied to
\[
(t, x) \rightarrow (t, x, J_x(y) A^* J_x(y)^T) = (x, \text{diag}(\lambda_i_1, \ldots, \lambda_i_d))
\]
shows that there is a transformation locally around that point \( (t, x) \) where we get a partition of the space
\[
\mathbb{R}^n = \{U_{i_1, i_2, \ldots, i_d} | 1 \leq i_l \leq d\} \cup C,
\]
where the open sets
\[
U_{i_1, i_2, \ldots, i_d} = \{x | J_x(y) A^* J_x(y)^T = \text{diag}(\lambda_i_1, \ldots, \lambda_i_d)\},
\]
have \( \lambda_{i_l} \neq 0 \) and \( C \) is the complements set in \( \mathbb{R}^n \), a set of points where elliptic degeneracy appears in a subspace of dimension \( d \). Note that a uniform ellipticity condition can play the role of a homotopy invariant and therefore it may be natural to write the equation in block structure in such cases. If there are elliptic degeneracies as in the multivariate SABR-models considered above we may use the additional structure to provide the reduction to the block structure (cf. the discussion above). If we consider time-discretization, then we may consider different block structures at different time steps such that the subspace may move in time. This leads to quite flexible schemes which cover a considerable class of stochastic volatility models. However, if the reduced model is obtained from a full factor model by a PCA (dependent on the state space point \( x \) or \( (t, x) \)) we are in the general situation of a Cauchy problem \((42)\) (and/or \((43)\)) which cannot be reduced to a block structure. Let us consider the essential time-homogenous case (i.e. coefficients depend on the spatial variables). In this case we may consider the intersection of all \( x \)-dependent subspaces which are spanned by the local Hörmander condition at \( x \), i.e.
\[
I_H := \cap_{x \in \mathbb{R}^n} W_x
\]
where \( W_x \) is the vector subspace of dimension \( d(x) \) spanned by \((46)\) above at \( x \). Here the notation \( I_H \) indicates that we are considering an intersection of spaces defined by local Hörmander conditions. Our most general theorem will show that regular global solutions of \((42)\) (resp. \((43)\)) exist if the data are rough (i.e. in \( L^p(\mathbb{R}^n) \) only in \( I_H \) and are smooth on the complementary
vector subspace $\mathbb{R}^n \setminus I_H$. In the following section we look at the situation from the perspective of elementary stochastic analysis. We observe that from this perspective regularity is closely linked to regularity of the data. We shall see then that a constructive analytic scheme together with Malliavin type estimates lead us to stronger results.

5 Existence of global weak solutions and regularity from the stochastic perspective

First let us note that we use the term ‘weak solution’ here as it is used in the context of partial differential equations, i.e. equations can be interpreted in a weak sense and solutions can be constructed in weak function spaces. This use of the term ‘weak solution’ is quite different from its use in the context of stochastic differential equations. Here, a solution where the versions of the stochastic terms are given in advance and the constructed solution is adapted to a given filtration is often called a ‘strong solution’. In the latter sense the solution of the cited theorem is a strong solution but, as we observe, the solution exists only in a weak function space, hence is ‘weak’ from an analytical perspective. The existence of stochastic processes related to degenerate parabolic equations is not unfamiliar in the context of ordinary stochastic differential equations or in the context of more advanced stochastic analysis (such as Malliavin calculus). A standard theorem of ordinary stochastic differential equations (for statement and proof cf. [20]) is the following.

**Theorem 3.** Let $T > 0$ and let $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, and $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ be measurable functions, where

$$|b(t, x)| + |\sigma(t, x)| \leq C(t + |x|); \quad x \in \mathbb{R}^n, \ t \in [0, T],$$

for some constant generic $C > 0$ and with $|\sigma(t, x)| = \sqrt{\sum_{ij} |\sigma_{ij}|^2}$ ($|.|$ denoting the Euclidean norm), and such that

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq C|x - y|; \quad x \in \mathbb{R}^n, \ t \in [0, T].$$

Let $Z$ be a random variable independent of the $\sigma$-algebra $\mathcal{F}_\infty$ generated by $W(s), \ s \geq 0$ and such that $E(|Z|^2) < \infty$. Then the stochastic differential equation

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \ 0 \leq t \leq T, \ X(0) = Z$$

has a unique $t$-continuous solution $(t, \omega) \to X(t, \omega)$, where each component of $X(t, \omega)$ belongs to the space

$$\mathcal{V}(0, T) := \{h(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R} | h \text{ satisfies (i),(ii), (iii)}\},$$

along with the conditions

(i) $(t, \omega) \to h(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel $\sigma$-algebra on $[0, \infty)$,

(ii) $h(t, \omega)$ is $\mathcal{F}_t$-adapted,
(iii) $E \left[ \int_0^T h(t, \omega)^2 dt \right] < \infty.$

Note that there is no strict ellipticity condition involved. Since we are interested in Greeks, our main concern is the regularity of expectations

$$u(t, x) = E^x (f(X_t)),$$  \hfill (57)

where $X_t$ is the solution of (56) and $f$ is some function. Regularity of $u$ clearly depends on the regularity of $f$. What do we know from the perspective of stochastic analysis and what do we know from an analytic perspective? Let us first consider regularity from the perspective of stochastic analysis. Examination of the proof of theorem 3 above shows that the function $(t, x) \mapsto u(t, x)$ is Feller-continuous, i.e. is a bounded continuous function for $t \geq 0$ if $f$ is bounded and continuous. If $f$ is non-negative and continuous, then the function $u$ is lower semi-continuous, because in this case $u$ can be obtained as a limit of an increasing sequence of continuous functions. However, if data are smooth then we have what we may call 'hypoellipticity from a stochastic perspective'. Consider the situation of theorem (3). If in addition the data are smooth, i.e. $f \in C^\infty$, and $\sigma_{ij}, b_i \in C^\infty$, then the solution $(t, x) \mapsto E (f(X_t))$ (58)

is smooth, i.e. in $C^\infty$. The proof is by iterative use of Dynkin’s formula.

Advanced regularity issues are discussed in the literature in the context of Malliavin calculus (examples of results may be found in [2] and references therein). Malliavin calculus in particular provides sufficient conditions for the existence of densities. However, as our simple example in section 1.2 shows, in general densities exist only in a distributional sense and the results concerning the existence of regular densities do not apply. Note that theorem 3 provides us with a solution $X_t$ with initial data $x \in \mathbb{R}^n$ and with associated probability measure $P$ such that for each $t \geq 0$ we have a distribution

$$F_t^x (z) := P(X_t \leq z)$$  \hfill (59)

for all $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$. Let us consider the time-homogeneous case. A density $(t, x; y) \mapsto p(t, x; y)$ (if it exists) satisfies

$$F_t^x (z) = \int_{-\infty}^{z_1} \cdots \int_{-\infty}^{z_n} p(t, x, y)dy$$  \hfill (60)

for $t \geq 0$ and $x \in \mathbb{R}^n$ given. Such a ‘function’ $p$ may be found in a generalized sense in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. More precisely, we may measure it in a space

$$H^s (\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \mid \hat{f} \text{ is a function, and } \|f\|_s^2 < \infty \right\},$$  \hfill (61)

where

$$\|f\|_s^2 = \int |\hat{f}(\eta)|^2 (1 + |\eta|^2)^s d\eta,$$  \hfill (62)

$\hat{f}$ denotes the Fourier transform of $f$, and $s \in \mathbb{R}$. From this perspective we surely have

$$y \mapsto p(t, x, y) \in H^{-n} (\mathbb{R}^n),$$  \hfill (63)
but not much more. In analytical terms random variables are measurable functions ($L^\infty$), and the theorem 3 says essentially that global Lipschitz conditions of drift and volatility coefficient functions and initial conditions with finite second moments imply the existence of a global solution with finite second moments. Again, in analytical terms this means that the solution is a measurable $L^2$-function. From an analytical perspective, regularity of degenerate operators with regular coefficients (which is the ‘practical perspective’ from our point of view) is considered in the context of sufficient conditions for hypoellipticity (cf. [12]).

6 Global regularity for a class of degenerate parabolic equations

Practical incentives have led us to a class of semi-elliptic problems where the computation of Greeks seems to be difficult. In a situation of bounded continuous payoffs the standard theory of stochastic differential equations provides Feller-continuous value functions which solve the associated Cauchy problems. If the payoffs are only continuous (but nonnegative) then the standard theory provides only lower semi-continuous solutions. For the computation of Greeks it is desirable to have more regular solutions. Note that in the situation of financial applications the restriction to bounded continuous payoffs (or, equivalently, initial data) is a limitation. We would like to have at least Lipschitz continuous payoffs where the growth of the payoff has an exponential bound (note that Cauchy problems in finance are formulated in logarithmic coordinates).

Furthermore, we have observed that there are some limitations concerning the regularity of the initial data. In an extreme case, where the initial data are $\delta$-distributions, the solution would be a density. However, in general there is no density which exists in a space of $L^\infty$-functions in general. The reason is quite obvious: on a certain subspace the degenerate operator of the factor-reduced problem operates as a vector-field, and the vector-field merely transports irregularities and even distributions in a certain sense as our example in section 1.2 shows. A way out is to have slightly different and additional assumptions tailored by the practical needs and by the needs of our constructive analytical method. We allow the initial data to be of exponential growth at infinity. On the other hand, they are allowed to be $L^p$ on a subspace where the diffusion lives. In this way we avoid transportation of the kinks by the vector field in our analytic AD-scheme. Next we state and prove an extension of [9]. The proof method is similar, but in addition we use certain estimates obtained in [23]. We formulate the problem in a coordinate-dependent way and state a coordinate-independent formulation below. We state the theorem in the essential case of space-dependent coefficients.

**Theorem 4.** Let $0 < d \leq n$, $T > 0$ some real number, and $1 \leq p \leq \infty$. Consider the Cauchy problem (7) on $[0, T] \times \mathbb{R}^n$. Assume that the Hörmander condition holds on the the subspace $\mathbb{R}^d$ of the first $d$ coordinates. Assume that

---

*Originally AD(I) scheme refers to an alternate direction implicit scheme for solving PDEs. Since the iteration scheme used in theorem 4 closely resembles this technique, we sometimes refer to it as an analytic AD(I) scheme.*
the initial data function \( f : \mathbb{R}^n \to \mathbb{R} \) satisfies

\[(i) \quad \text{for all } x_{d+1}, \ldots, x_n \text{ fixed the function} \]
\[ (x_1, \ldots, x_d) \to f(x_1, \ldots, x_d, x_{d+1}, \ldots, x_n) \]
\[ \text{is } L^p_{\text{loc}}, \quad 1 \leq p \leq \infty \text{ on } \mathbb{R}^d, \]

\[(ii) \quad \text{for all } x_1, \ldots, x_d \text{ fixed the function} \]
\[ (x_{d+1}, \ldots, x_n) \to f(x_1, \ldots, x_d, x_{d+1}, \ldots, x_n) \]
\[ \text{is } C^\infty (\mathbb{R}^{n-d}), \]

\[(iii) \quad \text{for all } x \in \mathbb{R}^n \]
\[ |f(x)| \leq C \exp(C|x|) \quad \text{for some constant } C > 0. \]

Assume that the coefficients are smooth (i.e. \( C^\infty \)) of linear growth with bounded derivatives, i.e.

\[ a_{ij} \in C^\infty_{l,b}(\mathbb{R}^n) \]  \hspace{1cm} (65)

for \( i = 0 \) and \( 1 \leq j \leq n \), or \( 1 \leq i \leq m \) and \( 1 \leq j \leq d \). Then the Cauchy problem (8) has a global classical solution \( u \), where

\[ u \in C^\infty ([0, T] \times \mathbb{R}^n), \]  \hspace{1cm} (66)

where for data \( f \) the singular behaviour in \( t = 0 \) is determined by the Malliavin-type estimate in [23] as follows: for given natural numbers \( m \) and \( N \) there is a number \( q \) such that the solution \( u \) and its time derivatives up to order \( m \) and its spatial derivatives up to order \( N \) are located in the space

\[ C^{q,\text{loc}}_{m,N} (\mathbb{R}^{n-d}) := \{ v \mid |v|^q v \in C_{m,N,\text{loc}} ([0, T] \times \mathbb{R}^n) \}, \]  \hspace{1cm} (67)

where

\[ C_{m,N,\text{loc}} ([0, T] \times \mathbb{R}^n) := \left\{ g \mid g \neq 0 \text{ and } \sum_{l \leq m} \| D^l g \|_{\text{loc}} + \sum_{|\alpha| \leq N} \| D^\alpha g \|_{\text{loc}} < \infty \right\}, \]  \hspace{1cm} (68)

with \( \| \cdot \|_{\text{loc}} \) denoting the local supremum norm. Moreover, a lower bound for \( q \) is given by \( q = \max |\alpha| \leq N (n_{m,a}, 0) + \frac{a}{2} \) where \( n_{m,a} \) is determined by the estimate in [23] of the singular behavior of the density (cf. theorem 13 below).

Remark 5. Note that the space in (67) is just introduced in order to consider the singular behavior at \( t = 0 \). One may sharpen this estimate (considering subspace dimension \( d \)).

In the proof of the theorem we construct a functional series which converges to the solution of the Cauchy problem. The convergence is proved by applying Hörmander estimates for subproblems related to the functional series.

Remark 6. Note that the exponential growth of data typically appears in the context of finance—cf. a call option in the logarithmic coordinates.

Next we state the corollary, which can be obtained directly from the proof of theorem 4. However there are some special issues like the computation of the time step size, where the step size may depend on the regularity of the data such that schemes with increasing time step size seem possible. However, these are special computational issues which will be considered in the second part of this paper.
Corollary 7. The solution of the Cauchy problem (8) can be obtained by computation of the functional series

\[ u(t, x) = u^1(t, x) + \sum_{l=1}^{\infty} \delta u^{2l+1}(t, x), \]  

(69)

where the terms of the right side of (69) are defined below. The series is computed with respect to some time discretization of the form

\[ \rho T_0, 2\rho T_0, \ldots, m\rho T_0, \ldots \]

for some \( T_0 > 0 \), where \( \rho \) can be computed a priori and depends only on the diffusion dimension \( k \), the dimension \( n \), and on the coefficient data. Assume that \( u \) has been computed up to time \( m\rho T_0 \) for some \( m \geq 0 \). Then we compute \( u \) on the extended time horizon \([0, (m+1)\rho T_0]\) as follows. We first compute the solution of the first order Cauchy problem

\[ \frac{\partial u}{\partial t} - \sum_{i=1}^{n} \mu_i(x) \frac{\partial u}{\partial x_i} = 0, \]

(70)

with initial data \( u(m\rho T_0, x_1, \ldots, x_d, \cdot) \) for fixed \( (x_1, \ldots, x_d) \). We may compute this solution by solving associated ODEs along characteristic curves (which are also solutions of ODEs). Then we compute the solution of the parabolic Cauchy problem

\[ \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \mu_i(x) \frac{\partial u}{\partial x_i} = 0, \]

(71)

of the equation

\[ \delta u^{2l} = u^{2l} - u^{2l-2} \]

(72)

and the solution

\[ \delta u^{2l+1} = u^{2l+1} - u^{2l-1} \]

(73)

where for \( m \geq 1 \) \( \delta u^{m, m} \) has zero initial conditions, i.e. \( \delta u^{m, m}(0, x) = 0 \). Here for \( m = 1, 3, \ldots (x_{d+1}, \ldots, x_n) \) is fixed, and for \( m = 0, 2, \ldots (x_1, \ldots, x_d) \) is fixed.
Proof. (Theorem 3). Note that for $d < n$ systems of type (8) are equations of mixed type: one part of the equation acting on a $d$-dimensional subspace is hypoelliptic while the other part may be termed hyperbolic (a very simple form of scalar first order type). However, fixing $x^d = (x_1, \ldots, x_d)$ we may interpret

$$\sum_{i=d+1}^{n} \mu_i(x^d, x^{n-d}) \frac{\partial}{\partial x_i}$$

as a vector field acting on the subspace $\mathbb{R}^{n-d}$. Similarly, if we fix $x^{n-d}$, then on the subspace $\mathbb{R}^d$ the operator in (8) acts essentially as in the parabolic equation on subspace $\mathbb{R}^d$ of form

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{k} a_i^j(x^k, x^{n-k}) \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \mu_i(x^d, x^{n-d}) \frac{\partial u}{\partial x_i} = 0.$$  \hfill (76)

A standard AD-scheme may be used to join these heterogeneous parts of the whole space operator together. However, in order to get a convergent scheme we first need two basic assumptions: i) the existence of global solutions to certain vector field problems associated with the vector field (75), and ii) the existence of global solutions for the parabolic equations of type (76). If both assumptions are satisfied then we are able to present an AD-scheme which proves to be convergent locally with respect to time. Iteration of the scheme in time using the semigroup property of the operator then leads to a global scheme. Let us start with the first assumption, the global existence of certain vector fields. Fixing $x^d = (x_1, \ldots, x_d)$ we may interpret

$$\sum_{i=d+1}^{n} \mu_i(x^d, x^{n-d}) \frac{\partial}{\partial x_i}$$

as a vector field acting on the subspace $\mathbb{R}^{n-d}$. In the scheme we have to deal with inhomogenous vector fields.

The proof thus consists of three parts. The first part will consider the solution of the vector field for a finite time step. The second part will consider the solution to the parabolic subproblem. In the following sections the coupling of the two steps is represented by a function $g$ below. The third part of the proof combines the two steps and shows that the scheme converges, constructing a solution on $[0, T] \times \mathbb{R}^n$. However, before we start the argument, we should mention that it is sufficient to prove the theorem under the stronger assumption of bounded payoff $f$, because we can transform the problem (8) for $u$ to a problem for $\tilde{u} := e^{-d(x)} u := e^{-\sqrt{a+q|x|^2}} u$ for some $a > 0$, $q > C^2$, and where $|x|$ denotes the Euclidean norm. Then $\tilde{u}$ solves an equivalent problem with identical diffusion term but transformed drift vector $\tilde{b} := b - \frac{1}{2} \nabla d \cdot \sigma \sigma^T$ and an additional potential term $\tilde{c} := c + b \cdot \nabla d - \frac{1}{4} \text{tr} (\sigma \sigma^T) D^2 d - \frac{1}{2} |\nabla d|^2$. Here $D^2 d$ denotes the Hessian of the function $d$ and $\text{tr}$ denotes the trace of a matrix. The following argument then holds for bounded payoffs, and applying the transformation above we see that it can be extended easily to the case where exponential growth of payoffs is allowed.

Remark 8. Note that we may choose $q > C^2 > 0$ such that the initial data decay exponentially as $|x| \uparrow \infty$. From now on we assume that $q$ is chosen in this way.
6.1 Existence of the Vector Field

First we have

**Proposition 9.** Fix $x^d \in \mathbb{R}^d$. Assume that the conditions of theorem 1 are satisfied. Assume that $g \in C^1([0, T] \times \mathbb{R}^n)$. Then there exists a smooth global flow $F^t$ generated by the vector field (75) on $[0, T] \times \mathbb{R}^{n-d}$ such that the first order equation problem

$$\frac{\partial u}{\partial t} = \sum_{i=d+1}^{n} \mu_i(x^d, x^{n-d}) \frac{\partial}{\partial x_i} u + g(x^d, x^{n-d}),$$

$$u(0, x^d, x^{n-d}) = f(x^d, x^{n-d}),$$

has the solution

$$u(t, x^d, x^{n-d}) = f(x^d, x^{n-d}) + \int_0^t g(x^d, F^s x^{n-d}) ds.$$  

**Proof.** Note that the variables $x^d$ are fixed and serve as parameters. Consider the characteristic form

$$\chi(z, \xi) = \xi_0 - \sum_{i=k+1}^{n} \mu_i \xi_i$$  

of the operator $L \equiv \frac{\partial}{\partial t} - \sum_{i=d+1}^{n} \mu_i \frac{\partial}{\partial x_i}$, where $\xi = (\xi_0, \xi_{d+1}, \ldots, \xi_n)$. The surface $S := \{ t = 0 \}$ has a constant normal vector $(1, 0, \cdots, 0)$, hence is non-characteristic for the surface $S$, i.e. at any point $z = (t, x)$ we have

$$(1, 0, \cdots, 0) \notin \text{char}_z(L) := \left\{ \xi \neq 0 | \xi_0 - \sum_{i=d+1}^{n} \mu_i \xi_i = 0 \right\}.$$  

Hence, basic PDE-theory tells us that the first order Cauchy problem has a unique local solution in a sufficiently small neighborhood of the surface $S$ and is given in the form of solutions of associated ODEs along its characteristic curves. This leads to a solution up to a time $T_1$. Next we may iterate the argument in time. Assume that this does not lead to a global solution but to a limit $T_\infty > 0$. Then on the time horizon $[0, T_\infty]$ we have a classical solution. Moreover the solution has a representation on this horizon as a family of ODE-solutions along characteristic curves, and where the assumptions on the coefficients (65), (55) imply that this family of solutions is uniformly bounded up to time $T_\infty$. Hence we may apply the first order PDE argument above again for the Cauchy problem with initial data $S_{T_\infty} := \{ t = T_\infty \}$ and extend the solution beyond the horizon $[0, T_\infty]$. Hence there is a unique global solution. For each $x_0^{n-d} \in \mathbb{R}^{n-d}$ the flow $F_t$ of the vector field $\sum_i \mu_i \frac{\partial}{\partial x_i}$ defines a characteristic curve $x_0^{n-d}(t) := F_t x_0^{n-d}$. Note that

$$F^t x^{n-d}$$  

is a solution of the homogeneous Cauchy problem

$$\frac{\partial u}{\partial t} = \sum_{i=d+1}^{n} \mu_i(x^d, x^{n-d}) \frac{\partial}{\partial x_i} u,$$

$$u(0, x^d, x^{n-d}) = f(x^d, x^{n-d}),$$

and then the form of the solution (80) of the inhomogenous equation follows from Duhamel’s principle. ■
6.2 Construction of the solution via an AD-Scheme

The notation above, which indicates that some coordinates are fixed \((x^d, x^{n-d})\), is a little cumbersome and so from here on we shall sometimes just write \(x\) instead of \((x^d, x^{n-d})\) when it is quite clear from the context which components of \(x\) should be considered as fixed. Next we define a local iteration scheme involving global flows of the type discussed in Proposition 9 above and solutions of parabolic equations of the form

\[
\frac{\partial u}{\partial t} = \sum_{i,j=1}^{k} a_{ij}^*(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \mu_i(x) \frac{\partial u}{\partial x_i}
\]

with \(x^{n-d}\) fixed. The natural ansatz is an AD-scheme of the following form: we define

**Vector Field Step:** \((l \geq 0)\)

\[
\frac{\partial u_{2l}}{\partial t} - \sum_{i=d+1}^{n} \mu_i(x) \frac{\partial u_{2l}}{\partial x_i} = \begin{cases} 
0 & \text{if } l = 0 \\
\sum_{i,j=1}^{k} a_{ij}^*(x) \frac{\partial^2 u_{2l-1}}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \mu_i(x) \frac{\partial u_{2l-1}}{\partial x_i} & \text{if } l > 0
\end{cases}
\]

**Diffusion Step:** \((l \geq 0)\)

\[
\frac{\partial u_{2l+1}}{\partial t} - \sum_{i,j=1}^{k} a_{ij}^*(x) \frac{\partial^2 u_{2l+1}}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \mu_i(x) \frac{\partial u_{2l+1}}{\partial x_i} = \sum_{i=d+1}^{n} \mu_i(x) \frac{\partial u_{2l+1}}{\partial x_i}
\]

For each \(m\) we define \(u_{m}(0, \cdot) = f(\cdot)\) and \(u_{m+1}(0, \cdot) = f(\cdot)\). Here, in equation (87) we understand \((x_{d+1}, \cdots, x_n)\) to be fixed, and in (86) we understand \((x_1, \cdots, x_k)\) to be fixed. In order to prove convergence time step by time step we rewrite the scheme in time-dilatation coordinates \((\rho \text{ will be small})\)

\[
t : [0, \infty) \to [0, \infty),
\]

\[
t(\tau) = \rho \tau.
\]

Then we get an equation in \(\tau\) equivalent to (8) where the coefficients of the symbol of the operator become small if \(\rho\) is small. We have

\[
\frac{dt}{d\tau} = \rho,
\]

and

\[
\begin{cases}
\frac{\partial u}{\partial \tau} - \rho \sum_{i,j=1}^{k} a_{ij}^*(x) \frac{\partial^2 u}{\partial x_i \partial x_j} - \rho \sum_{i=1}^{n} \mu_i(x) \frac{\partial u}{\partial x_i} = 0, \\
u(0, x) = f(x).
\end{cases}
\]

An iteration step of the scheme considered in transformed time \(\tau\) for some
time horizon \([0, T_0]\) is then given by
\[
\frac{\partial u^{\rho, 2l}}{\partial \tau} = \sum_{i=d+1}^{n} \rho \mu_i(x) \frac{\partial u^{\rho, 2l}}{\partial x_i} \]
for \(l = 0\)

\[
= \left\{ \begin{array}{ll}
0 & \text{for } l = 0 \\
\sum_{i,j=1}^{k} \rho a_{ij}^+(x) \frac{\partial^2 u^{\rho, 2l-1}}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \rho \mu_i(x) \frac{\partial u^{\rho, 2l-1}}{\partial x_i}, & \text{for } l > 0
\end{array} \right.
\]
and
\[
\frac{\partial u^{\rho, 2l+1}}{\partial \tau} - \rho \sum_{i,j=1}^{k} a_{ij}^+(x) \frac{\partial^2 u^{\rho, 2l+1}}{\partial x_i \partial x_j} = \sum_{i=1}^{d} \rho \mu_i(x) \frac{\partial u^{\rho, 2l+1}}{\partial x_i}
\]
for \(l \geq 0\). The initial conditions are
\[
u^{\rho,m}(0, x) = f(x), \quad m \geq 0,
\]
where for \(m = 1, 3, \ldots (x_{d+1}, \ldots, x_n)\) is fixed, and for \(m = 0, 2, \ldots (x_1, \ldots, x_d)\)
is fixed. We construct the solution in the form
\[
u^\rho(\tau, x) = u^{\rho, 1}(\tau, x) + \sum_{l \geq 1} \delta u^{\rho, 2l+1}(\tau, x),
\]
where for \(l \geq 1\)
\[
\delta u^{\rho, 2l+1} = u^{\rho, 2l+1} - u^{\rho, 2l-1}
\]
satisfies
\[
\frac{\partial \delta u^{\rho, 2l+1}}{\partial \tau} - \rho \sum_{i,j=1}^{k} a_{ij}^+(x) \frac{\partial^2 \delta u^{\rho, 2l+1}}{\partial x_i \partial x_j} = \sum_{i=1}^{d} \rho \mu_i(x) \frac{\partial \delta u^{\rho, 2l+1}}{\partial x_i},
\]
and in each substep where the right side in (95)
\[
\delta u^{\rho, 2l} = u^{\rho, 2l} - u^{\rho, 2l-2}
\]
satisfies
\[
\frac{\partial \delta u^{\rho, 2l}}{\partial \tau} - \sum_{i=d+1}^{n} \rho \mu_i(x) \frac{\partial \delta u^{\rho, 2l}}{\partial x_i} = \sum_{i,j=1}^{k} \rho a_{ij}^+(x) \frac{\partial^2 \delta u^{\rho, 2l-1}}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \rho \mu_i(x) \frac{\partial \delta u^{\rho, 2l-1}}{\partial x_i}.
\]
Moreover, for \(m \geq 1\), \(\delta u^{\rho,m}\) has zero initial conditions, i.e. \(\delta u^{\rho,m}(0, x) = 0\).

**Remark 10.** The construction of the solution is designed to be used as a numerical scheme in the following sense

1. the suitable time step size \(\rho T_0\) may be determined explicitly,
2. the irregularity of the payoff affects only the first summand \(u^{\rho, 1}\) of (93).
We claim that for small \( \rho \) the scheme just described is locally convergent with respect to time. Then iteration of the scheme in time using the semigroup property leads to a convergent scheme of a global solution to the Cauchy problem

\[
\begin{aligned}
\frac{\partial u}{\partial \tau} - \frac{1}{2} \sum_{i,j=1}^{k} \rho a_{ij}^* (x) \frac{\partial^2 u}{\partial x^i \partial x^j} - \sum_{i=1}^{n} \rho \mu_i (x) \frac{\partial u}{\partial x^i} &= 0, \\
u(0, x) &= f(x).
\end{aligned}
\] (97)

Note that iteration in time means that we start the next time step with the initial data \( u^\rho(T_0,.) \), and after repeating the scheme above we get the next initial data \( u^\rho(2T_0,.) \) and so on. The choice of \( \rho \) depends on certain a priori estimates which we cite below. Since the constant of the a priori estimate depends only on time-independent parameters we can set up a uniform time discretization where the original problem is computed on subintervals \([\rho T_0, (i+1)\rho T_0]\), \( 0 \leq i \leq M \) for some integer \( M \) with \( T = MT_0 \) and some small \( \rho \). At each time step \( i \) this is equivalent to computing the transformed problem on subintervals \([iT_0, (i+1)T_0]\), \( 0 \leq i \leq M \), where the series

\[
u^\rho(\tau, x) = \nu^{\rho,1}(\tau, x) + \sum_{l \geq 1} \delta \nu^{\rho,2l+1}(\tau, x),
\] (98)

converges pointwise to a classical solution of (8). Since the equations are only hypoelliptic on the subspace \( \mathbb{R}^d \) we cannot apply Schauder estimates (at least not directly). However estimates in [23] based on the Malliavin calculus lead to estimates of densities at each iteration step on the subspace \( \mathbb{R}^d \). This leads to an estimate of the solution where classical estimates of distributional solutions of heat equations with source terms are involved. The form of (98) allows us to prove the regularity result for data which are only \( L^p \) on some subspace. It is convenient that the higher order corrections \( \delta u^m \), \( m \geq 1 \) have zero initial data as we shall see below. For \( u^{\rho,1} \) we obtain interior estimates which are Hölder up to the boundary using a simple trick. Consider the series

\[
u^{\rho,2m+1}(\tau, x) = \nu^{\rho,1}(\tau, x) + \sum_{l=1}^{m} \delta \nu^{\rho,2l+1}(\tau, x).
\] (99)

The local convergence of the series (98) in time is such that the time derivative, and the spatial derivatives up to second order of \( \nu^{\rho,2m+1} \) \( m \) have an absolutely convergent majorant. Hence we may check that (98) is a solution by applying the operator of (100) to (98) and evaluating it term by term. By a simple induction argument you check that

\[
\begin{aligned}
\frac{\partial \nu^{\rho,2m+1}}{\partial \tau} - \frac{1}{2} \sum_{i,j=1}^{k} \rho a_{ij}^*(x) \frac{\partial^2 \nu^{\rho,2m+1}}{\partial x^i \partial x^j} &- \sum_{i=1}^{n} \rho \mu_i (x) \frac{\partial \nu^{\rho,2m+1}}{\partial x^i} = \\
- \sum_{i=d+1}^{n} \rho \mu_i (x) \frac{\partial^2 \nu^{\rho,2m+1}}{\partial x^i} &+ \sum_{i=d+1}^{n} \rho \mu_i (x) \frac{\partial \nu^{\rho,2m}}{\partial x^i} \\
&= - \sum_{i=d+1}^{n} \rho \mu_i (x) \frac{\partial \nu^{\rho,2m+1}}{\partial x^i}.
\end{aligned}
\] (100)

The question now is in which classical space we have a convergence of the functional series (99) (note that we need a space such that the derivatives of the
functional series up to first by member). The guide is the density estimation in ([23]). Accordingly we define for each positive real number $q$

$$C_{1,2}^{q,\alpha,loc}([0, T] \times \mathbb{R}^n) := \left\{ v | t^q v \in C_{1,2}^{\alpha,loc}([0, T] \times \mathbb{R}^n) \right\},$$

(101)

where

$$C_{1,2}^{\alpha,loc}([0, T] \times \mathbb{R}^n) := \left\{ f | \| f \|_{\alpha} + \| f_1 \|_{\alpha} + \sum_{|\alpha| \leq 2} \| D_x^\alpha f \|_{\alpha} < \infty \right\}$$

(102)

along with $\| \cdot \|_{\alpha}$ denoting a Hölder norm. In general (in order to prove higher regularity) we may consider the spaces

$$C_{m,N}^{q,\alpha,loc}([0, T] \times \mathbb{R}^n) := \left\{ v | t^q v \in C_{m,N}^{\alpha,loc}([0, T] \times \mathbb{R}^n) \right\},$$

(103)

where

$$C_{m,N}^{\alpha,loc}([0, T] \times \mathbb{R}^n) := \left\{ f | \| f \|_{\alpha} + \sum_{i \leq m} \| D^i f \|_{\alpha} + \sum_{|\alpha| \leq N} \| D^\alpha f \|_{\alpha} < \infty \right\}.$$  

(104)

It is easy to check that for each $q > 0$ the space $C_{1,2}^{q,\alpha,loc}([0, T] \times \mathbb{R}^n)$ is locally a Banach space where the norm is obvious from its definition and denoted by $\| \cdot \|_{1,2}$. The considerations above imply that proof of convergence $\| \delta u^{m+1/q} \|_{1,2} \downarrow 0$ does indeed imply that the functional series (98) is a solution of the Cauchy problem. Since the equation is only hypoelliptic on a subspace we cannot apply Schauder estimates directly. However, we have the following Malliavin estimate of the density which is proved in [23].

**Theorem 11.** Consider a $d$-dimensional diffusion process of the form

$$dX_t = \sum_{i=1}^d b_i(X_t) dt + \sum_{j=1}^d \sigma_{ij}(X_t) dW_t^j$$

(105)

with $X(0) = x \in \mathbb{R}^d$ with values in $\mathbb{R}^d$ and on a time interval $[0, T]$. Assume that $b_i, \sigma_{ij} \in C^\infty_b$. Then the law of the process $X$ is absolutely continuous with respect to the Lebesgue measure, and the density $p$ exists and is smooth, i.e.

$$p : (0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \in C^\infty \left((0, T] \times \mathbb{R}^d \times \mathbb{R}^d\right).$$

(106)

Moreover, for each nonnegative natural number $j$, and multiindices $\alpha, \beta$ there are increasing functions of time

$$A_{j,\alpha,\beta}, B_{j,\alpha,\beta} : [0, T] \to \mathbb{R},$$

(107)

and functions

$$n_{j,\alpha,\beta}, m_{j,\alpha,\beta} : \mathbb{N} \times \mathbb{N} \to \mathbb{N},$$

(108)

such that

$$\left| \frac{\partial^{[\alpha]} \delta^{[\beta]} q(t, x, y)}{\partial t^\alpha \partial x^{\beta} \partial y^\gamma} p(t, x, y) \right| \leq \frac{A_{j,\alpha,\beta}(t)(1 + x)^{m_{j,\alpha,\beta}}}{t^{n_{j,\alpha,\beta}}} \exp \left(-B_{j,\alpha,\beta}(t) \frac{(x - y)^2}{t} \right).$$

(109)

Moreover, all functions (107) and (108) depend on the level of iteration of Lie-bracket iteration at which the Hörmander condition becomes true.
We cannot apply the theorem (11) directly, but it has a probabilistic side. We note

**Corollary 12.** In the situation of (11) above, solution \(X^x_t\) starting at \(x\) is in the standard Malliavin space \(D^\infty\), and there are constants \(C_{l,q}\) depending on the derivatives of the drift and dispersion coefficients such that for some constant \(\gamma_{l,q}\)

\[
|X^x_t|_{l,q} \leq C_{l,q}(1 + |x|)^{\gamma_{l,q}}. \tag{110}
\]

Here \(\|\|_{l,q}\) denotes the norm where derivatives up to order \(l\) are in \(L^q\) (in the Malliavin sense).

For a proof consider \([23]\). Using this corollary we may show that (97) and the right sides (95) exponentially decay as the modulus of the spatial variables go to infinity (recall that we consider a scheme for the transformed equation according to (78). We have

**Lemma 13.** There is a constant \(c > 0\) such that for \(l \geq 1\)

\[
|u^{l,1}|_0, \ |\delta u^{l,2}|_0 \leq \exp(-c|x|) \text{ as } |x| \uparrow \infty. \tag{111}
\]

Here, \(|\|_0\) denotes the supremum (recall that we transformed the problem along with its initial data.)

**Proof.** First we consider the Cauchy problem for \(u^{l,1}\) in (97). (for each \(l, q\)) rewrite the probabilistic representation of the Cauchy problem in terms of the related Markov family \(Y^{1,x}_t\) with \((1 + |x|)^{\gamma_{l,q}}Y^x_t = X^{1,x}_t\), where \(X^{1,x}_t\) is the Markov family related to the Cauchy problem (97). Then we have a probabilistic representation of the solution \(u^{l,1}\) of (97) with

\[
u^{l,1}(t, x) := E\left(f(X^x_t)\right) = E\left(g(Y^x_t)\right), \tag{112}
\]

and where \(g(.) := h((1 + |x|)^{\gamma_{l,q}}.)\) is bounded and decays exponentially for \(|x| \uparrow \infty\). Similarly for equations with a source term. The densities related to the Markov family \(Y^x\) have densities which can be estimated by densities which satisfy an equation which is equivalent to the heat equation with some initial data or some source term. Hence there is a constant \(c > 0\) such that

\[
|u^{l,1}|_0 \leq \exp(-c|x|) \text{ as } |x| \uparrow \infty. \tag{113}
\]

Next \(|\delta u^{l,2}|_0 \leq \exp(-c|x|)\) as \(|x| \uparrow \infty\) follows from proposition 9. Similarly, for \(l \geq 1\) we use the probabilistic representation

\[
\delta u^{l,2l+1}(t, x) := \int_0^t E\left(\delta u^{l,2l}(s, X^x_s)\right) ds = \int_0^t E\left(\delta u^{l,2l}(s, (1 + |x|)^{\gamma_{l,q}}Y^x_s)\right) ds, \tag{114}
\]

and proposition 9 and get \(|\delta u^{l,2l}|_0, |\delta u^{l,2l+1}|_0 \leq \exp(-c|x|)\) as \(|x| \uparrow \infty\) inductively. 

Now we may write the members of (99) in terms of the family of densities of the Markov family \((Y^x_t)_{x \in \mathbb{R}^n}\). First we define the family of densities \((t, x, y) \rightarrow p^x(t, x, y)\) via

\[
P^x(Y^x_t \in dy) = p^x(t, x, y)dy, \tag{115}
\]
where \( P^x \) denotes the law of the process \( Y_t^x \) which starts at \( x \in \mathbb{R}^n \). Then we get the representations

\[
u^{ρ,1}(t, x) := E(f(X_t^x)) = E(g(Y_t^x)) = \int g(y)p_Y(t, x, y)dy, \tag{116}
\]

and

\[
\delta u^{ρ,2l+1}(t, x) := \int_0^t E(\delta u^{ρ,2l}(s, X_s^x)) \, ds = \int_0^t E(\delta u^{ρ,2l}(s, (1 + |x|)^{n_2} Y_s^x)) \, ds
\]

\[
= \int_0^t \delta u^{ρ,2l}(s, (1 + |y|)^{n_2} p_Y(s, x, y)) \, dyds. \tag{117}
\]

From the estimates \((110)\) (proved in in \([23]\)) we get

\[
|Y_t^x|_{t, q} \leq Ct_q \tag{118}
\]

for some constants \(0 < C_{t,q} < \infty\), and for the choice \(γ_{t,q} \geq m_{0,0,0}\)

\[
|p_Y(t, x, y)| \leq \frac{A_{0,0,0}(t)}{t^{0.0,0}} \exp \left( -B_{0,0,0}(t) \frac{(x - y)^2}{t} \right) \tag{119}
\]

Since \(A_{0,0,0}(t)\) and \(B_{0,0,0}(t)\) are increasing functions of \(t\) (cf. \([23]\)) we may choose \(A^* := A_{0,0,0}(T)\) and \(B^* = B_{0,0,0}(0)\) in order to get the estimate

\[
|p_Y(t, x, y)| \leq \frac{A^*}{t^{0.0,0}} \exp \left( -B^* \frac{(x - y)^2}{t} \right) \tag{120}
\]

Similar estimates hold for higher time and higher spatial derivatives of the density \(p_Y\). Of course, the only difference is that the singular behavior with respect to \(t\) as \(t \downarrow 0\) and the constants \(A^*, B^*\) change according to the estimate \((109)\). More precisely, for each nonnegative natural number \(j\), and multiindices \(α, β\) we get constant

\[
A_{j,α,β}^* = A_{j,α,β}(T), \quad \text{and} \quad B_{j,α,β}^* = B_{j,α,β}(0),
\]

and functions

\[
n_{j,α,β}, m_{j,α,β} : \mathbb{N} \times \mathbb{N}^d \times \mathbb{N}^d \to \mathbb{N},
\]

as in \([23]\) such that

\[
\left| \frac{∂^{α}}{∂x^{α}} \frac{∂^{β}}{∂y^{β}} p_Y(t, x, y) \right| \leq \frac{A_{j,α,β}^*}{t^{0.0,0}} \exp \left( -B_{j,α,β}^* \frac{(x - y)^2}{t} \right) =: q_{j,α,β}^y(t, x, y). \tag{121}
\]

Now by Young's inequality, for \(f \in L\) and \(g \in L^p\), \(0 \leq p \leq \infty\), we have \(f \ast g \in L^p\) \((\ast\) denoting convolution\), where

\[
\|f \ast g\| \leq \|f\|_1 \|g\|_p. \tag{122}
\]

This can be used to estimate general functions \(s \in L^1(\mathbb{R} \times \mathbb{R}^n)\)

\[
\sup_t \| \int_{-∞}^t |q_{j,α,β}^y(s, x, y)| dsdy \leq C\|s\|_1 \tag{123}
\]
for \( q \geq n_{j,\alpha,\beta} - n/2 \) where \( C \) is an upper bound of the \( L_1 \) norm of the kernel \( p_Y \). Now we may apply such estimates for source terms \( \delta u^{p,2l+1} \) and their derivatives for \( l \geq 1 \) (an extension which is zero for all \( t < 0 \) can be defined in order to have them in the form of \( s \) above), and similarly for \( u^{p,1} \). More precisely in order to get convergence in

\[
C_{m,N}^{q,\alpha,\text{loc}} ([0,T] \times \mathbb{R}^n) := \left\{ v | t^q v \in C_{m,N}^{\alpha,\text{loc}} ([0,T] \times \mathbb{R}^n) \right\},
\]

of the functional series (99) and its time derivatives up to order \( m \) and its spatial derivatives up to order \( N \) we act as follows. For given \( T_0 > 0 \) and any nonnegative numbers \( m, N \) we find \( \rho \) small enough and \( q \) such that iterative application of the estimate leads to a absolutely convergent geometric series majorant for the solution approximations \( u^{p,m} = u^{p,1} + \sum_{l=1}^{m} \delta u^{p,2l+1} \) and for their time derivative and their spatial derivatives up to order \( m \) and \( N \) respectively. We now have for some \( c \in (0,1) \)

\[
|\delta u^{p,2l+1}|_{m,N}^{q,\alpha,\text{loc}} \leq c^{2l+1}, \quad |\delta u^{p,2l}|_{m,N}^{q,\alpha,\text{loc}} \leq c^2 \quad \text{as } l \uparrow \infty.
\]

which implies convergence on the domain with horizon \( T_0 \) for the appropriate choice of \( \rho \). The proof of convergence for one time step uses a priori estimates of the Malliavin type for the equation in transformed coordinates. This corresponds to one time step on the subinterval \([0, \rho T_0]\) in original coordinates (here \( T_0 > 0 \) is arbitrary while \( \rho \) has to be chosen). It is clear then that the same argument can be applied to all successive subintervals \([i \rho T_0, (i+1)\rho T_0]\), \( 0 \leq i \leq M \) resp. \((i \rho T_0, (i+1)\rho T_0]\), \( 0 \leq i \leq M \) with respective initial data obtained from the previous time step (we use the semi-group property of the evolution equation). Iteration of this argument in time proves global convergence of the scheme. Note that for each time step the functional series and its derivatives up to first order with respect to time and up to second order with respect to the spatial variables have an absolutely convergent majorant. Hence componentwise differentiation of the functional series is justified, and the convergent functional series is indeed a solution of the Cauchy problem. \( \blacksquare \)

**Remark 14.** For numerical purposes (in order to avoid dealing with time singularities numerically) you may define a corresponding series for the function

\[
(t, x) \rightarrow v^p_q(t, x) := \frac{1}{q} t^q u^p(t, x).
\]

Indeed, in order to prove that \((t, x) \rightarrow v^p_q(t, x)\) is \( j \)-times differentiable with respect to \( t \) and \( \alpha \)-times differentiable with respect to \( x \) we may choose

\[
q := n_{j,\alpha,0} + 1.
\]

The iteration scheme is then of the form

\[
\frac{\partial \delta v^{p,2l+1}}{\partial t} = \rho \sum_{i,j=1}^{k} \alpha_{ij} u(x) \frac{\partial^2 \delta v^{p,2l+1}}{\partial x_i \partial x_j} - \sum_{i=1}^{k} \mu_i(x) \frac{\partial^2 \delta v^{p,2l+1}}{\partial x_i^2} = \sum_{i=k+1}^{n} \rho \mu_i(x) \frac{\partial v^{p,2l}}{\partial x_i} + \rho t^{q-1} \delta v^{p,2l}(t, x),
\]

with zero initial conditions, i.e. \( \delta v^{p,2l+1}(0, x) = 0 \).
7 A coordinate independent generalization of Hörmander’s result

We state our most general theorem in the essential time-homogeneous case. Consider a matrix-valued function $x \rightarrow (v_{ji})^{n,m}(x)$, $1 \leq j \leq n$, $0 \leq i \leq m$ on $\mathbb{R}^n$, and $m$ smooth vector fields

$$V_i = \sum_{j=1}^{n} v_{ji}(x) \frac{\partial}{\partial x_j},$$

(129)

where $0 \leq i \leq m$. Note that we now allow for time-dependent coordinates, but this is not the main point. Consider the Cauchy problem on $[0,T] \times \mathbb{R}^n$ (where $T > 0$ is an arbitrary finite horizon)

$$\begin{cases}
\frac{du}{dt} = \frac{1}{2} \sum_{i=1}^{m} V_i^2 u + V_0 u \\
u(0,x) = f(x).
\end{cases}$$

(130)

Define for all $x \in \mathbb{R}^n$

$$W_x := \text{span}\left\{ V_i(x), [V_j, V_k](x), \ldots | 1 \leq i \leq m, 0 \leq j, k, \ldots \leq m \right\}.$$  

(131)

Furthermore let $I_H = \cap_{x \in \mathbb{R}^n} W_x$. The following general theorem assumes that the roughness of the data is located in $I_H$. Note that in the case $I_H = \{0\}$ the following theorem collapses to proposition (??) above. We have

**Theorem 15.** Let $1 \leq p \leq \infty$. Consider the Cauchy problem (130) on $[0,T] \times \mathbb{R}^n$. Assume that the initial data function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies

(i) the function $f$ is $L^p_{\text{loc}}, 1 \leq p \leq \infty$ on $I_H$,

(ii) the function $f$ is $C^\infty$ on $\mathbb{R}^n \setminus I_H$,

(iii) for all $x \in \mathbb{R}^n$

$$|f(x)| \leq C \exp(C|x|)$$

for some constant $C > 0$.

Assume that the coefficients are smooth (i.e. $C^\infty$) and of linear growth with bounded derivatives, i.e.

$$v_{ji} \in C^\infty_{1,\delta}(\mathbb{R}^n)$$

(133)

for $i = 0$ and $1 \leq j \leq n$, or $1 \leq i \leq m$ and $1 \leq j \leq n$. Then the Cauchy problem (8) has a global classical solution $u$, where

$$u \in C^\infty([0,T] \times \mathbb{R}^n),$$

(134)

where for sufficiently regular data $f$ the singular behaviour at $t = 0$ is determined by the Malliavin-type estimate in [23] as follows: for given natural numbers $m$ and $N$ there is a number $q$ such that the solution $u$ and its time derivatives up
to order \(m\) and its spatial derivatives up to order \(N\) are located in the space (we do not consider a Banach space here, cf. remark to 4 above)

\[
C^{q,\alpha, loc}_{m,N} ([0,T] \times R^n) := \{ v | v^q \in C^{\alpha, loc}_{m,N} ([0,T] \times R^n) \}.
\]

Moreover, \(q = \max |\alpha| \leq N(n_{m,\alpha,0} - n/2)\) where \(n_{m,\alpha,0}\) is determined by the estimate in [23] of the singular behavior of the density (cf. theorem 13 below).

The proof for this theorem is a quite similar to the proof in the preceding section. The difference is that instead of the flow of a vector field we now have to consider a flow (defined by the semigroup) of a semi-elliptic SDE in a subspace of smooth data.

8 Higher order probabilistic weighted Monte Carlo schemes and comparison with proxy schemes and partial proxy schemes

The abstract scheme above does not show us how to compute. It especially does not show us how to compute in a probabilistic manner, i.e. with Monte-Carlo schemes. In this section we present weak higher order weighted Monte-Carlo-schemes in a way that allows us to extend the scheme of [10] and related schemes in [7] and [18]. Why do weak higher order schemes matter in finance (compared to Euler-schemes, Milstein schemes and other classical schemes)?

In order to answer the question let us recall some basic facts. Consider an ordinary stochastic differential equation of the form

\[
dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x_0 \in R^n
\]

on a time interval \([0,T]\) and a time discretization \(k\Delta T, 0 \leq k \leq N\) with \(N\Delta T = T\). We assume that the assumptions of theorem 3 above are satisfied. An Euler scheme is described by the recursion

\[
Y_{k+1} = Y_k + b(k\Delta t, Y_k)\Delta T + \sigma(k\Delta t, Y_k) (W_{(k+1)\Delta T} - W_{k\Delta T}).
\]

The latter scheme is a (simple) example of strongly convergent scheme, there is some \(\gamma > 0\) such that

\[
\epsilon(\delta) = E(|X_T - Y(T)|) \leq C\Delta T^\gamma
\]

as \(\Delta T\) becomes small. Indeed, in the case of an Euler scheme we have convergence with \(\gamma = 0.5\). More elaborate schemes like the Milstein scheme and trapezoidal methods also converge in a strong sense. Why should we consider higher order weak schemes then? What is the relation between strong convergence and weak convergence? Weak convergence is defined with respect to a function class. We say that an approximation scheme \(Y\) converges weakly with order \(\gamma > 0\) to \(X\) as \(\Delta t \downarrow 0\) and with respect to a function class \(C\), if for all \(g \in C\)

\[
|E(g(X_T)) - E(g(Y_T))| \leq C\Delta T^\gamma
\]

as \(\Delta T \downarrow 0\). It depends on the regularity of functions in \(C\) whether (139) is a strong condition. However, especially in finance, we may have low regularity of
payoffs, e.g. in the case of digital payoffs. If we compute $\Delta$s near maturity then the situation is even worse. i.e. we get distributional behavior at maturity. In particular, the asymptotic behavior of the density of $X$ is not correctly approximated by an Euler scheme, because the Euler scheme leads to an Euclidean limit while

$$
\lim_{\Delta T \downarrow 0} \Delta T \ln p(\Delta T, x, y) = d^2(x, y),
$$

(140)

where with $(\sigma \sigma^T)^{ij}$ being the entries of the inverse of $\sigma \sigma^T$, the function $d^2$ is the Riemannian metric induced by the line element $\sum_{i,j=1}^n (\sigma \sigma^T)^{ij} dx_i dx_j$.

However, (140) is also interesting because it is the leading order term of analytic higher order expansions of the density. Numerically stable higher order density approximations of the form

$$
p(t, x; 0, y) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{d^2_R(x, y)}{4t} \right) \left( \sum_{k=0}^\infty d_k(x, y) t^k \right),
$$

(141)

have been developed in [17] recently.

**Remark 16.** Note the difference to the WKB-expansion

$$
p(t, x; 0, y) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{d^2_R(x, y)}{4t} + \sum_{k=0}^\infty c_k(x, y) t^k \right),
$$

(142)

used in [18]; in the latter article the expansion was applied to classical LIBOR market models and is computed up to $c_1$.

In [18] standard interest rate options with a maturity of ten years could be computed in the standard full-factor Libor market model framework in just one time step. However, there are disadvantages to simulating the full factor model. First, computational experience shows that more than 7 or 8 Brownian motion factors lead to numerical noise and do not improve the calibration of models with respect to the market data. Furthermore, there is a trade-off between the efficiency gained by the analytic higher order expansions and the loss of efficiency caused by the number of Brownian motion factors of the full factor model. Naturally questions arise. Is it possible to extend the use of higher order analytic expansions in the framework of the proxy scheme in [10] and/or in the partial proxy scheme in [7]? Second, can the latter schemes be extended by a probabilistic scheme defined along the constructive solution scheme of semi-elliptic diffusions defined in this paper. Clearly, the regularity result obtained in this paper is an advantage: it allows us to get error estimates for weak higher order schemes for Greeks which cannot be obtained without such a result. Furthermore, it allows us to derive estimators which are of bounded variance even for small time-to-maturity and/or small volatilities. In this respect the result of this paper allows the extension of the result of bounded variance in [18]. Let us go deeper, and derive a probabilistic scheme from the series of parabolic Cauchy problems which define the solution of the semi-elliptic equation of theorem 4 above. For its implementation we have to select a cutoff and approximate on a finite domain $\Omega \subset \mathbb{R}^n$, but this is not essential for the following discussion. Furthermore we consider only the initial time step. It is clear then how to proceed in time with respect to a time discretization. For each
time step we assume that the diffusion coefficients $a_{ij}$ and the drift coefficients $\rho_i$ depend only on the spatial variables. Let $k \leq d$ be fixed and fix the coordinates $x^{n-k} = (x_{k+1}, \ldots, x_n)$. Let $p_{x^{n-k}}$ be the fundamental solution of

$$\frac{\partial u^{n+1}}{\partial \tau} - \rho \sum_{i,j=1}^k a_{ij}(x^{k}, x^{n-k}) \frac{\partial^2 u^{n+1}}{\partial x_i \partial x_j} - \sum_{i=1}^k \rho \mu_i(x^{k}, x^{n-k}) \frac{\partial u^{n+1}}{\partial x_i} = 0. \tag{143}$$

Then according to our results above we get the probabilistic representation

$$u^{n+1}(\tau, \cdot) = \int_{\mathbb{R}^k} f(y^{k}, x^{n-k}) p_{x^{n-k}}(\tau, x^{k}; 0, y^{k}) dy^{k}$$

$$+ \int_0^\tau \int_{\mathbb{R}^k} \sum_{i=k+1}^n \rho \mu_i(t(s), y^{k}, x^{n-k}) \times$$

$$\frac{\partial f}{\partial x_i}(y^{k}, F^\tau x^{n-k}) p_{x^{n-k}}(\tau, x^{k}; s, y^{k}) dy^{k} ds. \tag{144}$$

Similarly for the higher order terms we get the recursive probabilistic representation

$$\delta u^{n+2l+1}(\tau, \cdot) = \int_0^\tau \int_{\mathbb{R}^k} \sum_{i=k+1}^n \rho \mu_i(y^{k}, x^{n-k}) \times$$

$$\left( \frac{\partial}{\partial x_i} \int_0^\tau \sum_{j=1}^k \rho a_{ij}(y^{k}, x^{n-k}) \frac{\partial^2 u^{n+2l-1}}{\partial x_i \partial x_j} (y^{k}, F^\tau x^{n-k}) ds \right)$$

$$+ \frac{\partial}{\partial x_i} \int_0^\tau \sum_{i=1}^k \rho \mu_i(y^{k}, x^{n-k}) \frac{\partial u^{n+2l-1}}{\partial x_i} (s, y^{k}, F^\tau x^{n-k}) \times$$

$$p_{x^{n-k}}(\tau, x^{k}; s, y^{k}) dy^{k} ds. \tag{145}$$

The representation (145) together with the initial representation (144) gives the following 'naive' weighted Monte-Carlo scheme. Let $\zeta^{x^k}$ be a random variable with normal density $\phi(t, x^k, \cdot)$.\(^5\) Then we have the probabilistic representations

$$u^{n+1}(\tau, \cdot) = E \left( \frac{f(\zeta^{x^k}, x^{n-k}) p_{x^{n-k}}(\tau, x^k; 0, x^{n-k})}{\phi(\tau, x^k, \zeta^{x^k})} \right)$$

$$+ E \left( \int_0^\tau \sum_{i=k+1}^n \rho \mu_i(\zeta^{x^k}, x^{n-k}) \times$$

$$\frac{\partial f}{\partial x_i}(\zeta^{x^k}, F^\tau x^{n-k}) p_{x^{n-k}}(\tau, x^k; s, \zeta^{x^k}) \frac{\phi(s, x^k, \zeta^{x^k})}{\phi(\tau, x^k, \zeta^{x^k})} \right). \tag{146}$$

\(^5\)Note that in general practice $\phi$ will be time-homogenous.
the scheme for the price terms. The estimators for the Greeks are easily obtained by differentiating

Note that we have the same density for both the lower and the higher order proof of bounded variance in the case of full factor models (cf. (145) consider two variations of bounded variance estimators. In one case there is a it may have unbounded variance for small time and/or small variance. We differentiated equations. We call the estimate for derivatives 'naive' because experimental evidence (cf. 

numerical evidence which supports the theorem. In the other case there is some differentiator equations. We call the estimate for derivatives 'naive' because it may have unbounded variance for small time and/or small variance. We experimental evidence (cf. [18]) for a different estimator which is simpler and

\[
\delta u^0 \delta t^2 \delta t^3 (\tau, x) = E \left( \int_0^\tau \sum_{i=1}^k \rho \mu_i (\zeta^k, x^{n-k}) \times \right.
\]

\[
\left. \left( \frac{\partial}{\partial x_i} \int_0^\tau \sum_{i,j=1}^k \rho \sigma_{ij} (\zeta^k, x^{n-k}) \frac{\partial^2 u^0 \delta t^2 \delta t^3}{\partial x_i \partial x_j} (s, \zeta^k, \mathcal{F}^{\tau-s} x^{n-k}) ds \right. \right.
\]

\[
+ \frac{\partial}{\partial x_i} \int_0^\tau \sum_{i=1}^k \rho \mu_i (\zeta^k, x^{n-k}) \frac{\partial^2 u^0 \delta t^2 \delta t^3}{\partial x_i} (s, \zeta^k, \mathcal{F}^{\tau-s} x^{n-k}) \times \right.
\]

\[
\left. \frac{\rho_{n-k} (\tau, x^{n-k}; s, \zeta^k)}{\phi(s, x^{n-k}, \zeta^k)} ds \right) \right) \right).
\]

Substitution of expectation $E$ by the sum $\frac{1}{M} \sum_{m=1}^M$ and writing for each $m$ the $m$-th realization of $\zeta^k$ as $m \zeta^k$ leads to the following ‘naive’ Monte-Carlo scheme for the price

\[
u^0 (\tau, s) \cong \frac{1}{M} \sum_{m=1}^M \left( \int_0^\tau \sum_{i=1}^k \rho \mu_i (m \zeta^k, x^{n-k}) \times \right.
\]

\[
\left. \frac{\partial F \phi (s, x^{n-k}, \zeta^k)}{\phi(s, x^{n-k}, \zeta^k)} ds \right) \right), \right)
\]

and

\[
\delta u^0 \delta t^2 \delta t^3 (\tau, x) = \frac{1}{M} \sum_{m=1}^M \left( \int_0^\tau \sum_{i=1}^k \rho \mu_i \left( \int (\zeta^k, x^{n-k}) \right) \times \right.
\]

\[
\left. \left( \frac{\partial}{\partial x_i} \int_0^\tau \sum_{i,j=1}^k \rho \sigma_{ij} (t(s), m \zeta^k, x^{n-k}) \frac{\partial^2 u^0 \delta t^2 \delta t^3}{\partial x_i \partial x_j} (s, m \zeta^k, \mathcal{F}^{\tau-s} x^{n-k}) ds \right. \right.
\]

\[
+ \frac{\partial}{\partial x_i} \int_0^\tau \sum_{i=1}^k \rho \mu_i (m \zeta^k, x^{n-k}) \frac{\partial^2 u^0 \delta t^2 \delta t^3}{\partial x_i} (s, m \zeta^k, \mathcal{F}^{\tau-s} x^{n-k}) \times \right.
\]

\[
\left. \frac{\rho_{n-k} (\tau, x^{n-k}; m \zeta^k)}{\phi(s, x^{n-k}, m \zeta^k)} ds \right) \right).
\]

Note that we have the same density for both the lower and the higher order terms. The estimators for the Greeks are easily obtained by differentiating (146) and (145) and then writing the MC approximations for the corresponding differentiated equations. We call the estimate for derivatives ‘naive’ because it may have unbounded variance for small time and/or small variance. We consider two variations of bounded variance estimators. In one case there is a proof of bounded variance in the case of full factor models (cf. ([18])), and some numerical evidence which supports the theorem. In the other case there is some experimental evidence (cf. [7] for a different estimator which is simpler and
works in many cases). In this section we shall extend the proof in the former case. The idea of the proof is as follows. We consider the construction of the solution for the Cauchy problem in the form

$$u^\rho(\tau, x) = u^{\rho,1}(\tau, x) + \sum_{l \geq 1} \delta u^{\rho,2l+1}(\tau, x),$$

(150)

where $u^{\rho,1}(\tau, x)$ is defined as in (144) and $\delta u^{\rho,2l+1}(\tau, x)$ is defined as in (145) (the probabilistic expressions of the last section). We observed that convergence is such that derivatives of the series can be taken term by term. This means that we can estimate the variance of the Greeks term by term, i.e. up to the regularity constraints for the coefficients we have for any multiindex $\beta$

$$\frac{\partial}{\partial x^\beta} u^\rho(\tau, x) = \frac{\partial}{\partial x^\beta} u^{\rho,1}(\tau, x) + \sum_{l \geq 1} \frac{\partial}{\partial x^\beta} \delta u^{\rho,2l+1}(\tau, x).$$

(151)

For simplicity let us consider the $\Delta s'$. The ‘naive’ estimator is constructed from

$$\frac{\partial}{\partial x^\beta} u^{\rho,1}(\tau,.) = E \left( \frac{\partial}{\partial x^\beta} \int_0^\tau \sum_{i=1}^n \rho \mu_i(\xi^k, x^{n-k}) \right),$$

(152)

(note that the derivative of $f$ is with respect to $x_i$ with $i \geq k+1$, where it exists in a classical sense according to our assumptions) and from

$$\frac{\partial}{\partial x^\beta} \delta u^{\rho,2l+1}(\tau, x) = E \left( \int_0^\tau \sum_{i=1}^n \rho \mu_i(\xi^k, x^{n-k}) \frac{\partial^2 u^{\rho,2l-1}}{\partial x^\beta, \partial x^\beta} (s, \xi^k, x^{n-k}) ds \right).$$

(153)

Now we can adapt the estimator for the full factor model (cf. [18]) to the reduced factor case. Note that we continue to work in logarithmic coordinates (for financial applications the following may be rewritten in lognormal coordinates if this is desirable). The idea is the following. In order to get an estimator of bounded variance even for small volatility and/or small time we consider a smooth function $g : \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ with $|\partial g(t, x, z)/\partial z| \neq 0$, and an $\mathbb{R}^k$-valued random variable $\xi$ on some probability space with density $\lambda_t$ where $\lambda_t(z) \neq 0$ for all $z$ and $t$ (for example, the standard normal density). Then
we write \( \zeta^{x^k} := g(\tau, x^k, \xi) \) where we assume that it has a density \( \phi(\tau, x^k, \cdot) \).
Hence the first order term of our constructive scheme of the estimator becomes (analytic form)

\[
\frac{\partial}{\partial x} \mu^{l-1}(\tau, \cdot) = E \left( \frac{\partial}{\partial x} f(g(\tau, x^k, \xi), x^{n-k}) p_{n-k}(\tau, x^k, 0, g(\tau, x^k, 1)) \phi(\tau, x^k, g(\tau, x^k, 1)) \right) + E \left( \frac{\partial}{\partial x} \int_0^\tau \sum_{i=k+1}^n \rho \mu_i(g(\tau, x^k, \xi), x^{n-k}) \times \frac{\partial f}{\partial x} \bigg|_{\tau} \phi(\tau, x^k, g(\tau, x^k, 1)) ds \right).
\]

It turns out that a well chosen \( g \) in (154) is sufficient to get bounded variance estimators, i.e. we could keep the structure of the ‘naive estimator’ for the higher order terms. For the higher order terms we may use the naive weighted MC algorithm (without controller function) since the data (of the source terms) are smooth and have bounded derivatives (hence we may use (153) in this case).
Again, substitution of expectation \( E \) by the sum \( \frac{1}{M} \sum_{m=1}^M \) and for each \( m \) writing the \( m \)-th realization of \( \xi \) as \( m\xi \) gives the corresponding Monte Carlo estimator

\[
\frac{\partial}{\partial x} \mu^{l-1}(\tau, \cdot) = \frac{1}{M} \sum_{m=1}^M \left( \frac{\partial}{\partial x} f(g(\tau, x^k, m\xi), x^{n-k}) p_{n-k}(\tau, x^k, 0, g(\tau, x^k, m\xi)) \phi(\tau, x^k, g(\tau, x^k, m\xi)) \right) + \frac{1}{M} \sum_{m=1}^M \left( \frac{\partial}{\partial x} \int_0^\tau \sum_{i=k+1}^n \rho \mu_i(g(s, x^k, m\xi), x^{n-k}) \times \frac{\partial f}{\partial x} \bigg|_{\tau} \phi(s, x^k, g(s, x^k, m\xi)) ds \right),
\]

and (149) for the higher order corrections (in practice it is sufficient to compute the latter sum for the lowest correction term \( l = 1 \)). The choice of the control function \( g \) is discussed in [18]. It can be proved that the latter estimator is of bounded variance even when \( \Delta s \) are to be computed in the framework of stochastic volatility models and volatility and/or time to maturity is small. It is clear that the scheme above defines an extension of the scheme in [10]. Furthermore, extension of the scheme to general reduced diffusion models is possible. The scheme can be combined with the partial proxy scheme in [7].

9 Further remarks

Though there exists no regular density for the class of semi-elliptic equations considered in this paper, the analytical scheme proposed leads to a probabilistic scheme involving regular densities living on subspaces at each state of the construction. This leads naturally to weighted Monte-Carlo schemes which improve the accuracy of the schemes in [7, 10]. Compared with the scheme considered in
[18], the advantage of such schemes is that we do not need to simulate the full-factor model. However, the results for bounded variance of the estimators in [18] can be extended to the scheme above. Note that in [18] regularity of the density was used in order to obtain a class of estimators for low volatilities and time beyond the class considered in [5]. In [18] it was shown that analytical expansions of the density may be useful in order to improve the efficiency of a scheme. Such density expansions may also be used for the present scheme. Similar remarks apply to the use of expansions of characteristic functions as in [16], of course. Note that in the context of stochastic volatility models sometimes only the characteristic function is known. This is sometimes due to additional degeneracies of the diffusion coefficients. The extension of our analytical schemes to such models and models with jump measures is of interest. It is clear that such an extension hinges on a priori estimates of related degenerate diffusion and partial integral-differential equations, and is possible in some cases at least. It may also be of interest to combine the present scheme with a front fixing scheme for American derivatives where the obstacle problem is transformed to a hypercube (cf. [15]). Numerical analysis and simulations in the context of the LIBOR market model will be considered in part II of this paper.

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