Asymptotically noise decoupling for Markovian open quantum systems

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The noise decoupling problem is investigated for general N-level Markovian open quantum systems. Firstly, the concept of Cartan decomposition of the Lie algebra su(N) is introduced as a tool of designing control Hamiltonians. Next, under certain assumptions, it is shown that a part of variables of the coherence vector of the system density matrix can be asymptotically decoupled from the environmental noises. The resulting noise decoupling scheme is applied to one-qubit, qutrit and two-qubit quantum systems, by which the coherence evolution of the one-qubit and qutrit systems can always be asymptotically preserved, while, for two-qubit systems, our findings indicate that evolution of some variables can be preserved only for some initial states.

I. INTRODUCTION

In recent years, quantum information science has been a growing field which interests many researchers for potential high speed quantum computation and high security quantum communication. Decoherence is commonly recognized as the main bottleneck. Various schemes have been proposed to reduce such unexpected effects. In principle, there are two classes of schemes—open loop and closed loop strategies depending on the use of measurement and feedback. Open loop strategies include quantum error-avoiding codes, Bang-Bang control, open loop optimal control, and open loop coherent control, while closed loop strategies include quantum error-correction codes and quantum feedback control.

Though various strategies have been proposed, none of these strategies is satisfying to suppress decoherence for N-level Markovian open quantum systems; quantum error-correction codes and error-avoiding codes use several physical qubits to encode one logical qubit, which is too luxurious under existing conditions; BangBang control strategy is inapplicable in the fully Markovian regime as pointed out by Lidar; open-loop optimal control strategy can only partially decouple quantum systems from the environmental noises; quantum feedback control strategy requires complex feedback control apparatus and it is valid only for special physical systems. Thus, for the decoherence suppression problem of N-level Markovian open quantum systems, more system analysis and control methods should be introduced.

The closest work to ours can be found in Ref. 19, where open loop coherent control is applied to decoherence suppression for single-qubit Markovian systems. It is shown that the x-axis and y-axis variables of the Bloch vector can be exactly decoupled from the environmental noises, which means that the coherence of the quantum state can be well preserved. However, this scheme requires solving a time-variant linear ordinary differential equation (ODE) to obtain the open loop control laws, by which analytic control laws can be obtained only for phase damping decoherence under a strong assumption that the x-axis and y-axis variables of the Bloch vector keep constant. For more general cases, only numerical control laws can be obtained. Furthermore, divergence of the control fields may occur in this strategy as pointed out by the authors. In this paper, we propose a more general noise decoupling strategy for N-level Markovian open quantum systems based on the Cartan decomposition of the Lie algebra su(N). Open loop controls are designed to asymptotically decouple the state variables from the environmental noises. The strategy loses some precision but is easier to be fulfilled.

The paper is organized as follows. In section II Markovian open quantum systems are formulated in the coherence vector representation and the concept of the Cartan decomposition of su(N), together with three important assumptions, is introduced. Our main results are presented in section III where the scheme are also applied to one-qubit, qutrit and two-qubit systems. Further discussion and conclusion are drawn in section IV.

II. PRELIMINARIES

Consider an N-level Markovian open quantum control system in the following master equation form:

\[ \dot{\rho} = -i[H_0 + \sum_{i=1}^{n} u_i H_i, \rho] + \sum_{j=1}^{m} \Gamma_j D[L_j] \rho, \]

\[ \rho(t_0) = \rho_0, \] (1)
where the Planck constant $\hbar$ is assigned to be 1; $\rho$ refers to the system density matrix; $H_0$ and $H_i$, $i = 1, \ldots, n$ represent, respectively, the free Hamiltonian and the control Hamiltonians adjusted by the control parameters $u_i$, $i = 1, \ldots, n$. The Lindblad super operators

$$\mathcal{D}[L_j] \rho = L_j \rho L_j^\dagger - \frac{1}{2} L_j^\dagger L_j \rho - \frac{1}{2} \rho L_j L_j^\dagger,$$

characterize the damping channels and the positive constants $\Gamma_j$ denote the damping rates of the corresponding channels.

The differential equation (1) is actually a complex matrix differential equation which is hard to be analyzed. Therefore, we will convert it into a real vector differential equation. For this purpose, an orthonormal basis \{ $\Omega_0 = \frac{1}{\sqrt{N}} I, \Omega_j \}$ $j = 1, \ldots, N^2 - 1$ with respect to the matrix inner product $(X, Y) = \text{tr}(X Y^\dagger)$ should be introduced first, where $I$ is the $N \times N$ identity matrix and $\Omega_j$s are $N \times N$ Hermitian traceless matrices. The system density matrix $\rho$ can then be expressed as:

$$\rho = \frac{1}{N} I + \sum_{i=1}^{N^2-1} m_i \Omega_i = \frac{1}{N} I + m \cdot \Omega,$$

where $m \in \mathbb{R}^{N^2-1}$ is the so-called coherence vector of $\rho$. In this case, the quantum control system (1) can be reexpressed by a differential equation on $\mathbb{R}^{N^2-1}$:

$$\dot{m}(t) = O_0 m(t) + \sum_{i=1}^{n} u_i O_i m(t) + Dm(t) + g,$$  

$$m(t_0) = m_0,$$  

where $O_0, O_i \in \mathfrak{so}(N^2 - 1)$ are, respectively, the adjoint representation matrices of $-iH_0, -iH_i$ and $m_0$ is the coherence vector of $\rho_0$. The term "$Dm + g$" comes from the decohering process represented by the Lindblad terms in (1) in which the dissipative matrix $D$ is semi-negative defined, i.e., $D \leq 0$.

The above approach is a generalization of the well-known Bloch vector representation for two-level quantum systems. Physically, the length of the coherence vector represents the amount of coherence in the quantum state. Conceptually, it is unit for the pure state and shorter for the mixed state. Notice that the control $\sum_i u_i H_i$ drives the quantum system along a sphere on which coherence is conserved, while the decohering operators pull the vector towards the equilibrium state, e.g. the ground state in a spontaneous emission.

To simplify the equation (3) and facilitate our discussions, it is useful to discuss the choice of the matrix basis \{ $\frac{1}{\sqrt{N}} I, \Omega_j \}$ $j = 1, \ldots, N^2 - 1$. In this regard, we introduce the so-called Cartan decomposition of the Lie algebra $\mathfrak{su}(N)$ as follows:

$$\mathfrak{su}(N) = p \oplus \mathfrak{c}, \ [\mathfrak{c}, \mathfrak{c}] \subset \mathfrak{c}, \ [p, p] \subset \mathfrak{c}, \ [p, \mathfrak{c}] \subset p.$$  

Notice that $\mathfrak{su}(N)$ is a $N^2 - 1$ dimensional Lie algebra of all traceless skew-Hermitian $N \times N$ matrices, hence the basis matrices of $p$ and $\epsilon$ can be expressed as \{ $-i\Omega_i^p \}_i=1, \ldots, m$ and \{ $-i\Omega_i^\epsilon \}_i=m+1, \ldots, N^2-1$ where $\Omega_i^p$ and $\Omega_i^\epsilon$ are traceless Hermitian matrices. Further, every traceless Hermitian matrix is a linear combination of \{ $\Omega_i^p, \Omega_i^\epsilon$ \}. Correspondingly, the density matrix $\rho$ can be represented as:

$$\rho = \frac{1}{N} I + \sum_{i=1}^{m} m_i^p \Omega_i^p + \sum_{i=m+1}^{N^2-1} m_i^\epsilon \Omega_i^\epsilon.$$  

As will be shown later, the matrices \{ $\Omega_i^p \}_i=1, \ldots, m$ play central roles in our control strategy. In fact, we will choose the control Hamiltonians $H_i$ from these matrices.

It should be pointed out that the Cartan decomposition always exists. In fact, we can obtain a trivial decomposition if we let $\epsilon = \mathfrak{su}(N)$. The decomposition is also not unique.

Before proceeding our discussions, we introduce three assumptions for the equation (3):

(H1) Complete decoherence condition:

$$[O_0, D] = 0, \quad O_0 g = 0;$$

(H2) Convergence condition: $D < 0$;

(H3) $-iH_0 \in \epsilon$.

(H1) has an important physical interpretation that the stationary distribution $\rho_\infty$ of the uncontrolled system satisfies $[H_0, \rho_\infty] = 0$ (see the appendix of Ref. [17] for a rigorous proof). In other words, in the energy representation, the off-diagonal entries of the stationary system density matrix $\rho_\infty$ disappear as a result of decoherence. (H2) is introduced to guarantee the existence of the convergent solution of (3). In fact, we can always make $D < 0$ with the aid of the feedback control modification (11). (H3) is easy to be satisfied under special choices of the Cartan decomposition.

III. ASYMPOTICALLY NOISE DECOUPLING STRATEGY

The term "$Dm + g$" in equation (3) is the environment-induced dissipative term which destroys coherence of the quantum states. Our target is to select the control Hamiltonians $H_i$ and design the corresponding controls $u_i$ to force the trajectory $m(t)$ of the equation (3) as close as possible to the target trajectory $m^\epsilon(t)$ which is the solution of the following unperturbed system:

$$\dot{m}(t) = O_0 m(t), \quad m(t_0) = m_0.$$  

It has been demonstrated that the Markovian open control system (3) is always uncontrollable [22]. Therefore, one can never track the target trajectory precisely.
However, we will show that certain variables of $m(t)$ may asymptotically tend to the corresponding variables of $m^0(t)$ by properly designed control laws. That is to say, these variables can be asymptotically decoupled from the environmental noises. In fact, according to (4), the coherence vector $m$ can be divided into two parts: $m = (m^1, m^2)^T$, where

$$m^1 = (m^1_1, \ldots, m^1_{m^1})^T,$$

$$m^2 = (m^2_{m+1}, \ldots, m^2_{N^2-1})^T.$$

From lemma A.2, we have

$$O_0 = \text{diag}(O_0^{11}, O_0^{22}),$$

where $O_0^{11}$, $O_0^{22}$ are respectively, $m$ and $N^2 - m - 1$ dimensional square anti-symmetric matrices, then the target trajectory can be written as:

$$m^0(t) = e^{O_0(t-t_0)}m_0 = (m^{10}(t), m^{20}(t))^T,$$

where

$$m^{10}(t) = e^{O_0^{11}(t-t_0)}m_0^{1},
\quad m^{20}(t) = e^{O_0^{22}(t-t_0)}m_0^{2},$$

and $m_0 = (m_0^1, m_0^2)^T$. In this case, the vector $m^1(t)$ can be driven to the corresponding target trajectory $m^{10}(t)$. In fact, we have the following theorem:

**Theorem 1** Suppose the assumptions (H1), (H2) and (H3) are satisfied and the control Hamiltonians $H_i$ in (7) are the basis matrices $\{O_i\}$ corresponding to $p$. The following control law

$$u = (u_1, \ldots, u_m)^T = e^{-O_0^{22}(t-t_0)}\xi$$

steers the control trajectory $m(t)$ of the equation (3) asymptotically to the stationary solution:

$$m^\infty(t) = (e^{O_0^{11}(t-t_0)}m_0^{1}, e^{O_0^{22}(t-t_0)}\eta)^T,$$

where the constant vectors $\xi \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^{N^2-m-1}$ are the solutions of the following nonlinear algebraic equation:

$$F_1(\xi, \eta) = \sum_{i=1}^{m} \xi_i O_1^{12}\eta + D_{11} m_0^{1} + D_{12}\eta + g_1 = 0,$$

$$F_2(\xi, \eta) = -\sum_{i=1}^{m} \xi_i (O_1^{12})^T m_0^{1} + D_{21} m_0^{1}
+ D_{22}\eta + g_2 = 0,$$

where

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

Rigorous proof of the theorem is left to appendix A. Theorem 1 shows that the state variables of the coherence vector of $\rho$ corresponding to $p$ can be asymptotically decoupled from the environmental noises. Note that $O_0^{11}$ is an $m$-dimensional anti-symmetric matrix which has only zero or pure imaginary eigenvalues, the control laws presented in (7) are in fact sinusoidal signals which are usual for electro-magnetic fields used in the laboratory.

Next, we will apply our control strategy to three typical systems in quantum information science: one-qubit, qutrit and two-qubit systems.

### A. One-qubit quantum systems

Consider the one-qubit systems which are fundamental in the quantum information science. Assume that the free Hamiltonian $H_0 = \omega \sigma_z$ where $\sigma_z$ is the z-axis Pauli matrix, then the Cartan decomposition of the corresponding Lie algebra $su(2)$ can be chosen as $su(2) = p + \epsilon$, where the basis of $p$ and $\epsilon$ are $\{-i\frac{\sqrt{2}}{2}\sigma_x, -i\frac{\sqrt{2}}{2}\sigma_y\}$ and $\{-i\frac{\sqrt{2}}{2}\sigma_z\}$ respectively.

Now, for the coherence vector representation of $\rho$, i.e.,

$$\rho = \frac{1}{2} I + m_x \frac{1}{\sqrt{2}} \sigma_x + m_y \frac{1}{\sqrt{2}} \sigma_y + m_z \frac{1}{\sqrt{2}} \sigma_z,$$

we can conclude that the variables $m_x = \frac{1}{\sqrt{2}} tr(\sigma_x \rho)$ and $m_y = \frac{1}{\sqrt{2}} tr(\sigma_y \rho)$ can be asymptotically decoupled from the environmental noises if the assumptions (H1), (H2) and (H3) are satisfied.

As an example, we study the one-qubit amplitude damping decoherence model [1] which can be used to describe spontaneous emissions of the two-level atoms. In this case, the control system can be expressed as the following master equation:

$$\dot{\rho} = -i[\omega \sigma_z + u_x \sigma_x + u_y \sigma_y, \rho] + \Gamma D [\sigma_-, \rho],$$

where $u_x, u_y$ are the controls and $\omega \in \mathbb{R}$ denotes the Rabi frequency; $\sigma_- = \sigma_x - i \sigma_y$ is the lowering operator of the two-level system and $\Gamma > 0$ denotes the decoherence rate, e.g., the damping rate of the spontaneous emission process. To simplify the calculations, we use the well-known Bloch vector representation:

$$\rho = \frac{1}{2}(I + m_x \sigma_x + m_y \sigma_y + m_z \sigma_z),$$

where $m_x = tr(\sigma_x \rho)$, $m_y = tr(\sigma_y \rho)$ and $m_z = tr(\sigma_z \rho)$. Note that this representation is different from the coherence vector representation only by a trivial constant multiplicative factor.

The master equation (10) can be converted into the following equation in the Bloch vector representation:

$$\dot{m} = \omega O_x m + u_x O_x m + u_y O_y m + Dm + g,$$

where $m = (m_x, m_y, m_z)^T \in \mathbb{R}^3$ and

$$O_x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad O_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$O_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\Gamma D [\sigma_-, \rho] = \frac{1}{2} \Gamma \left[ \begin{array}{c} m_x \\ m_y \\ m_z \end{array} \right] \left[ \begin{array}{c} m_x \\ m_y \\ m_z \end{array} \right] \frac{1}{\sqrt{2}}.$$
\[ O_z = \begin{pmatrix} -1 & \quad 0 \\ 1 & \quad 0 \end{pmatrix} \]

are the basis elements of the Lie algebra \( so(3) \) of the 3 dimensional orthogonal group; \( D = \text{diag}(-\frac{1}{2}, \frac{1}{2}, -\Gamma) \) and \( g = (0, 0, -\Gamma)^T \) come from the decoherence process. It can be verified from [10] and [11] that the assumptions (H1), (H2) and (H3) are all satisfied.

From the equation \( [O_{0i}, O_{0j}] = \sum_{i} (O_{0i}^{\dagger})_{ij} O_{0j}^{0} \) and

\[ [O_z, O_z] = O_y, \quad [O_z, O_y] = -O_x, \]

it can be easily computed that \( O_{0i}^{\dagger} = \begin{pmatrix} -\omega & \quad \xi \\ \chi & \quad \omega \end{pmatrix} \). According to theorem 4, the controls can be designed as:

\[
\begin{pmatrix} u_x \\ u_y \end{pmatrix} = e^{-O_{0i}^{\dagger}(t-t_0)} \xi
= \begin{pmatrix} \cos \omega(t-t_0)\xi_1 - \sin \omega(t-t_0)\xi_2 \\ \sin \omega(t-t_0)\xi_1 + \cos \omega(t-t_0)\xi_2 \end{pmatrix}
= \begin{pmatrix} A \cos(\omega(t-t_0) + \phi) \\ A \sin(\omega(t-t_0) + \phi) \end{pmatrix},
\]

under which \( m_x, m_y \) tend to the corresponding variables of the unperturbed system \( m = \omega O_z m \). The constant \( \xi = (\xi_1, \xi_2)^T \) can be solved by [9] as follows:

\[
\begin{align*}
\xi_1 &= \frac{\Gamma m_{0y}}{2C_0^2} \left( 1 \pm \sqrt{1 - 2C_0^2} \right), \\
\xi_2 &= -\frac{\Gamma m_{0x}}{2C_0^2} \left( 1 \pm \sqrt{1 - 2C_0^2} \right),
\end{align*}
\]

where \( m_0 = (m_{0x}, m_{0y}, m_{0z})^T \) is the initial state and \( C_0^2 = m_{0x}^2 + m_{0y}^2 \) represents the initial coherence in the quantum system. It can be observed that \( C_0^2 \) should be no larger than 1 to guarantee the existence of the solution of (9). Amplitude and phase of the control fields can then be calculated as:

\[
A = \sqrt{\xi_1^2 + \xi_2^2} = \frac{\Gamma}{2C_0^2} \left| 1 \pm \sqrt{1 - 2C_0^2} \right|, \\
\phi = \arctan \frac{\xi_2}{\xi_1} = \arctan \left( \frac{m_{0x}}{m_{0y}} \right).
\]

Let \( \omega = 3/\tau_0, t_0 = 0 \) and the initial state

\[
\rho_0 = \frac{1}{2} I + \frac{\sqrt{2}}{4} \sigma_x + \frac{\sqrt{2}}{4} \sigma_z,
\]

where \( \tau_0 \) is a time constant which is introduced to obtain dimensionless evolution time. For concrete systems, \( \tau_0 \) can be determined by the system time scale, e.g. relaxing time of systems. With simple calculations, it can be shown that \( u_x = \frac{\sqrt{2}}{2} \Gamma \sin(3t/\tau_0), \quad u_y = -\frac{\sqrt{2}}{2} \Gamma \cos(3t/\tau_0) \). Simulation results of the variables \( m_x, m_y \) and controls \( u_x, u_y \) are shown in Figure 1 and 2.

Since the coherence \( C^2 = m_x^2 + m_y^2 \) in \( \rho \) is determined by \( m_x \) and \( m_y \), the coherence of the state will vanish completely without control. The figure shows that the controlled trajectory tends asymptotically to the target trajectory where the two trajectories are so close that they almost coincide together, which implies that the coherence of the state is asymptotically preserved with our control strategy.

In the study of Lidar et al on the same problem in Ref. 10, the state variables \( m_x \) and \( m_y \) are exactly decoupled from the environmental noises under the following feedback-like control laws:

\[
\begin{align*}
u_x &= -\frac{\Gamma}{2m_x} m_y, \\
u_y &= \frac{\Gamma}{2m_x} m_x,
\end{align*}
\]

which are to be substituted into (11) to get explicit open-
loop control laws. This leads to the following equations:

\[
\begin{align*}
    m_x(t) &= m_x^0(t) = m_{0x} \cos \omega t + m_{0y} \sin \omega t, \\
    m_y(t) &= m_y^0(t) = -m_{0x} \sin \omega t + m_{0y} \cos \omega t, \\
    m_z m_z &= -\Gamma m_z^2 - \Gamma m_z - C_0^2,
\end{align*}
\]

where \( m_0 = (m_{0x, m_{0y}, m_{0z}})^T \) and \( C_0^2 = m_{0x}^2 + m_{0y}^2 \).

Generally speaking, the last equation of (13) has no analytic solutions and we can only obtain numerical solutions. With the same parameters in the above example, we can obtain plots of controls \( u_x, u_y \) in Figure 3.

Compare Figure 2 and 3, it can be shown that the control laws in Figure 3 are more complex, and they approach to our laws asymptotically. Furthermore, a divergent solution of the last equation of (13) may lead to divergent control fields. With simple calculations, it can be shown that solution of the last equation of (13) is convergent if and only if

\[
m_{z_0} < \frac{-1 + \sqrt{1 - 2C_0^2}}{2}.
\]

Figure 4 shows the divergent control fields when the initial state is chosen as \( m_0 = (\sqrt{\frac{3}{2}}, 0, \sqrt{\frac{1}{2}})^T \).

**FIG. 4:** Divergent control fields for the initial state \( m_0 = (\sqrt{\frac{3}{2}}, 0, \sqrt{\frac{1}{2}})^T \). The solid line is \( u_x \) and the plus-sign line is \( u_y \). Here, to obtain dimensionless quantities, \( u_x \) and \( u_y \) are divided by the decoherence intensity \( \Gamma \).

**FIG. 5:** Plot of \( \Delta m_{xy} = \sqrt{(m_x^1 - m_x^0)^2 + (m_y^1 - m_y^0)^2} \) where \( m_x^1 \) and \( m_y^1 \) are respectively \( x, y \) entries of the trajectory obtained by our strategy and \( m_x^0 \) and \( m_y^0 \) are \( x, y \) entries of the trajectory obtained by Lidar’s.

It should be pointed out that, though the control laws obtained by our strategy are simple and the divergence problem does not occur, our control laws can only asymptotically, not exactly, decouple the state variables from the environmental noises. The plot of the difference between the trajectories obtained by our strategy and Lidar’s strategy (ideal trajectories) are shown in Figure 4.

**FIG. 6:** Three-level atoms with ∨ configuration as shown in Figure 6 where \( |0\rangle, |1\rangle \) and \( |2\rangle \)

\[
\begin{align*}
|1\rangle &\quad |2\rangle \\
|0\rangle &\quad \quad \quad \\
\end{align*}
\]

B. Qutrit quantum systems

Consider the three-level atoms \([33, 34, 35, 36]\) with ∨ configuration as shown in Figure 6 where \( |0\rangle, |1\rangle \) and \( |2\rangle \) are the eigenstates of the free Hamiltonian of three-level atoms with eigenvalues \( E_0 < E_1 < E_2 \) respectively. The two excited states \( |1\rangle \) and \( |2\rangle \) are coupled to the ground state \( |0\rangle \).

The total Hamiltonian \( H \) is expressed as \( H = H_0 + H_d \) where \( H_0 \) is the free Hamiltonian and \( H_d \) is the control Hamiltonian. The free Hamiltonian \( H_0 \) can be written as:

\[
H_0 = E_0 |0\rangle \langle 0| + E_1 |1\rangle \langle 1| + E_2 |2\rangle \langle 2|.
\]

The control Hamiltonian

\[
H_d = g_{10} |1\rangle \langle 0| + g_{10}^* |0\rangle \langle 1| + g_{20} |2\rangle \langle 0| + g_{20}^* |0\rangle \langle 2|
\]

represents the interaction between the atoms and the driving electromagnetic fields. The complex coefficients \( g_{10}, g_{20} \) can be adjusted by the amplitudes and phases of the driving fields.

Consider the open system in which only amplitude damping decoherence channels induced by spontaneous emissions are introduced. The master equation model is expressed as:

\[
\dot{\rho} = -i[H_0 + H_d, \rho] + \Gamma_1 D[\sigma_{01}^+] \rho + \Gamma_2 D[\sigma_{20}^+] \rho,
\]

where \( \sigma_{01}^+ = |0\rangle \langle i| \) is the lowering operator from the excited state \( |i\rangle \) to the ground state \( |0\rangle \). The two Lindblad terms \( \Gamma_i D[\sigma_{01}^+] \rho \) represent the transition from \( |i\rangle \) to \( |0\rangle \) caused by the spontaneous emission process.

For three-level systems, the matrix basis of the corresponding Lie algebra \( su(3) = \{-i\Omega_k\}_{k=1,...,8} \) can be
The control Hamiltonian can be written as:

\[ H_0 = \omega_3 \Omega_3 + \omega_8 \Omega_8 + \frac{E_0 + E_1 + E_2}{3} I, \]

where

\[ \omega_3 = \frac{\sqrt{7}}{2}(E_1 - E_2), \quad \omega_8 = \frac{\sqrt{6}}{6}(E_1 + E_2 - 2E_0). \]

Since the constant energy \( (E_0 + E_1 + E_2)/3 \) only contributes a global phase to the system state, it is sufficient to consider the following traceless free Hamiltonian

\[ H_0 = \omega_3 \Omega_3 + \omega_8 \Omega_8. \]

Furthermore, let \( g_{10} = \frac{1}{\sqrt{7}}(u_4 + i u_5) \), \( g_{20} = \frac{1}{\sqrt{6}}(u_6 + i u_7) \). The control Hamiltonian can be written as:

\[ H_d = u_4 \Omega_4 + u_5 \Omega_5 + u_6 \Omega_6 + u_7 \Omega_7, \]

where \( u_i \)'s are the controls to be designed.

Now, from the coherence vector representation of \( \rho \), i.e.,

\[ \rho = \frac{1}{3} I + \sum_{i=1}^{8} m_i \Omega_i, \quad m_i = \text{tr}(\Omega_i \rho), \]

we obtain the following coherence vector representation of the master equation (13):

\[ \dot{m} = \omega(\omega_3 O_3 + \omega_8 O_8) m + \sum_{i=4}^{7} u_i O_i m + Dm + g, \]

where \( O_i = ad(-i\Omega_i) \). It can be verified from (14) and (10) that the assumptions (H1), (H2) and (H3) are all satisfied. According to theorem [1] the variables \( m_i, i = 4, 5, 6, 7 \) in the equation (15) can be asymptotically decoupled from the environmental noises under the following control law:

\[ u = (u_4, u_5, u_6, u_7)^T = e^{-O^{11}_0 (t-t_0)} \xi, \]

where

\[ O^{11}_0 = \begin{pmatrix} 0 & \frac{\sqrt{7}}{2} & 0 & 0 \\ -\frac{\sqrt{7}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{7}}{2} \\ 0 & 0 & \frac{\sqrt{7}}{2} & 0 \end{pmatrix}, \]

\[ + \omega_8 \begin{pmatrix} 0 & 0 & \frac{\sqrt{6}}{2} & 0 \\ -\frac{\sqrt{6}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{6}}{2} \\ 0 & 0 & \frac{\sqrt{6}}{2} & 0 \end{pmatrix}, \]

and \( \xi \) can be numerically solved from the equation (9). Unlike one-qubit quantum systems, the algebraic equation (9) has no analytic solution in this case.

Let \( \Gamma_1 = \Gamma_2 = \Gamma \), \( t_0 = 0 \), \( E_0 = -\frac{136}{90} \), \( E_1 = -\frac{136}{90} \), \( E_2 = -\frac{136}{90} \) and the initial state be the mixed state:

\[ \rho_0 = \frac{1}{3} I + \frac{\sqrt{7}}{4} \Omega_4 + \frac{\sqrt{6}}{4} \Omega_6 - \frac{\sqrt{6}}{12} \Omega_8, \]

where \( \tau_0 \) is a time constant which is introduced to obtain dimensionless evolution time. For concrete systems, \( \tau_0 \) can be determined by the system time scale, e.g. relaxation time of systems. With simple calculations, it can be shown that \( u_4 = -0.7063 \Gamma \sin(4.4 t/\tau_0) \), \( u_5 = -0.7063 \Gamma \cos(4.4 t/\tau_0) \), \( u_6 = -0.7063 \Gamma \sin(2.5111 t/\tau_0) \), \( u_7 = -0.7063 \Gamma \cos(2.5111 t/\tau_0) \). Here, the amplitudes and initial phases of the control fields are obtained by numerically solving the equations (9). Simulation results of the variables \( m_i \) and controls \( u_i, i = 4, 5, 6, 7 \) are shown in Figure [7] and [8].

Recall that the coherence \( C^{01}_{01} = m_4^2 + m_5^2 \) between \( |0 \rangle \) and \( |1 \rangle \) is determined by \( m_4 \) and \( m_5 \); the coherence \( C^{02}_{02} = m_6^2 + m_7^2 \) between \( |0 \rangle \) and \( |2 \rangle \) is determined by \( m_6 \), \( m_7 \). The coherence in \( \rho \) will vanish completely without control according to Figure [7]. The controlled trajectory is driven so closely to the target trajectory that they almost coincide together, i.e., the coherence between the excited state \( |i \rangle \) and the ground state \( |0 \rangle \) can be asymptotically preserved with our control strategy.

Just like one-qubit systems, if we want to exactly decouple the corresponding variables from environmental noises, only numerical control laws can be obtained. However, with the same parameters in the above example, we find that the corresponding time-variant ordinary
differential equation to obtain the numerical control laws has no solutions. In fact, in order to solve the equation, we must calculate an inverse matrix and this matrix is singular for the initial state in the example. It means that exactly decoupling strategy fails for this example.

Finally, the results obtained by our control strategy can be directly extended to the $\wedge$-type three level atoms and other kinds of decoherence channels including phase damping decoherence channels and depolarizing decoherence channels, as long as the assumptions (H1), (H2) and (H3) are satisfied.

C. Two-qubit quantum systems

For two-qubit systems, the corresponding Lie algebra $su(4)$ has a Cartan decomposition $su(4) = p + \epsilon$, where

$$p = \left\{ \frac{1}{2} \sigma_i \otimes \sigma_j | i, j = x, y, z \right\},$$

$$\epsilon = \left\{ \frac{1}{2} I \otimes \sigma_j, \frac{1}{2} \sigma_i \otimes I | i, j = x, y, z \right\}.$$

Now the coherence vector representation of $\rho$ can be written as:

$$\rho = \frac{1}{4} I \otimes I + \frac{1}{2} \sum_i m_i^1 \sigma_i \otimes I + \frac{1}{2} \sum_j m_j^2 I \otimes \sigma_j + \frac{1}{4} \sum_{i,j} m_{ij}^2 \sigma_i \otimes \sigma_j,$$

where

$$m_i^1 = \frac{1}{2} \text{tr}(\sigma_i \otimes I) \rho, \quad m_j^2 = \frac{1}{2} \text{tr}(I \otimes \sigma_j) \rho, \quad m_{ij}^2 = \frac{1}{4} \text{tr}(\sigma_i \otimes \sigma_j) \rho.$$ 

It is shown that the two-qubit variables $\{m_{ij}^2\}$ can be asymptotically decoupled from the environmental noises if the assumptions (H1), (H2) and (H3) are all satisfied.

As an example, consider the two-qubit independent amplitude damping decoherence model which describes two atoms that simultaneously undergo spontaneous emissions. The control system is described by the following master equation:

$$\dot{\rho} = -i[H_0 + \sum_{i,j} u_{ij} H_{ij}, \rho] + \Gamma_1 D[\frac{1}{2} \sigma^1_i \otimes I] \rho + \Gamma_2 D[\frac{1}{2} I \otimes \sigma^2_j] \rho,$$

where $H_0 = \omega_1 \frac{1}{2} \sigma^1_x \otimes I + \omega_2 \frac{1}{2} I \otimes \sigma^2_z$ and $H_{ij} = \frac{1}{2} \sigma_{ij} \otimes \sigma_{ij}^\dagger$, $i, j = x, y, z$, are the free and control Hamiltonians respectively; $\sigma^1_x = \sigma^1_z - i \sigma^1_y$, $i = 1, 2$ is the lowering operator of the $i^{th}$ subsystem and the positive coefficients $\Gamma_i$ denote the corresponding decoherence rates. The two Lindblad terms represent the amplitude damping decoherence channels of the two subsystems.

Rewriting the master equation (18) in the coherence vector representation, one can verify that the system satisfies the assumptions (H1), (H2) and (H3). Therefore, our control strategy (17) can be applied. Unlike the one-qubit case, the quadratic algebraic equation (17) does not have analytic solutions, however, we can obtain numerical solutions of $\xi$.

Let $\omega_1 = \omega_2 = 1/\tau_0$, $t_0 = 0$, $\Gamma_1 = \Gamma_2 = \Gamma$, where $\tau_0$ is a time constant which is introduced to obtain dimensionless evolution time, and the initial state be the mixed state:

$$\rho_0 = \frac{1}{2} \cdot \frac{1}{4} I + \frac{1}{2} |\phi_0 \rangle \langle \phi_0|,$$

where $|\phi_0 \rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is the maximally entangled Bell state. In this case, among the two-qubit variables $\{m_{ij}^2\}$ in the equation (17), only $m_{12y}^2$, $m_{21y}^2$, $m_{12x}^2$, $m_{21x}^2$ and $m_{12z}^2$ are non-zero. The simulation results of these variables are shown in Figure 9. It is shown that the uncontrolled trajectories of $m_{12y}^2$, $m_{21y}^2$, $m_{12x}^2$ and $m_{12y}^2$ and $m_{12z}^2$ evolve away from anticipated values. With controls plugged in, the controlled trajectory tracks asymptotically the target trajectory so close that they almost coincide together.
Further, with simple calculations, it can be shown that only $u_{xz} = -\frac{1}{2}\Gamma \cos(2t/\tau_0)$, $u_{yz} = -\frac{1}{2}\Gamma \cos(2t/\tau_0)$, $u_{xx} = -\frac{3}{2}\Gamma \sin(2t/\tau_0)$, $u_{yy} = -\frac{3}{2}\Gamma \sin(2t/\tau_0)$, and $u_{zz} = -\frac{1}{2}\Gamma$ are non-zero. Plots of the controls are shown in Figure 10.

![FIG. 10: Temporal evolution of (a) $m_{yy}^{12}$, (b) $m_{yz}^{12}$, (c) $m_{xz}^{12}$, (d) $m_{xx}^{12}$ and (e) $m_{zz}^{12}$: the asterisk line denotes the controlled trajectory; the plus-sign line is the uncontrolled trajectory; the solid line is the target trajectory.](image)

![FIG. 10: (a) Plot of $u_{xz}$ and $u_{xx}$ where the solid line is $u_{xz}$ and the plus-sign line is $u_{xx}$; (b) Plot of $u_{yz}$ and $u_{yy}$ where the solid line is $u_{yz}$ and the plus-sign line is $u_{yy}$. Here, to obtain dimensionless quantities, $u_{ij}$ are divided by the decoherence intensity $\Gamma$.](image)

FIG. 10: (a) Plot of $u_{xy}$ and $u_{xx}$ where the solid line is $u_{xy}$ and the plus-sign line is $u_{xx}$; (b) Plot of $u_{yx}$ and $u_{yy}$ where the solid line is $u_{yx}$ and the plus-sign line is $u_{yy}$. Here, to obtain dimensionless quantities, $u_{ij}$ are divided by the decoherence intensity $\Gamma$.

IV. DISCUSSION

In subsection III C, it has been pointed out that the two-qubit variables $\{m_{ij}^{12}\}$ can be asymptotically decoupled from the noises. In Ref. [37], we proposed a multiparticle mixed-state entanglement measure modified from Jaeger’s Minkowskian norm entanglement measure [35, 36, 37], which, for two-qubit states, is defined as:

$$E(\rho) = \max \left\{ 2 \sum_{i,j} (m_{ij}^{12})^2 - \frac{1}{2}, 0 \right\}.$$  

The entanglement measure is only related to the two-qubit variables $\{m_{ij}^{12}\}$. For this reason, it is reasonable to expect that the entanglement of states can be asymptotically preserved by our control strategy.

Unfortunately, we find that, for most entangled states, the algebraic equation (9) has no solution and our control strategy cannot preserve entanglement completely under the least-squared solution. It should be further studied to what extent our control strategy may help to preserve entanglement.

Our asymptotical noise decoupling strategy applies control Hamiltonians from the Cartan decomposition of the Lie algebra $su(N)$ to decouple the systems from the environmental noises under reasonable assumptions. Such control may not be applicable in laboratory in...
present condition, especially for the two-qubit example, but it still provides useful hints in systematic design of decoherence control. The construction of the Cartan decomposition is essential in our scheme. Since the decomposition is not unique, the finding of a "good" decomposition to achieve the expected control performance is an interesting problem that needs further research.

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APPENDIX A: PROOF OF THEOREM

Before presenting the proof of the theorem, we first introduce two lemmas:

Lemma A.1 Consider the following time-variant linear system:
\[ \dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0. \] (A1)
If there exist a \(N \times N\) matrix \(Q(t)\) and positive numbers \(\eta, \mu, \nu\) such that:
\[ \eta I \leq Q(t) \leq \mu I, \]
\[ A^T(t)Q(t) + Q(t)A(t) + Q(t) \leq -\nu I, \]
we have:
\[ |x(t)|^2 \leq \frac{\mu}{\eta} e^{-\frac{\nu}{\eta}(t-t_0)}|x_0|^2. \]

The following lemma shows that, if the control Hamiltonians \(H_i\) are chosen as \(\Omega_i^p\) in [A], the control system [A] has a simple structure.

Lemma A.2 Let \(O_0 = ad(-iH_0)\), where \(ad(A)\) is the adjoint representation matrix of \(A\), then

(1) we have \(O_0 = \text{diag}(O_0^{11}, O_0^{22})\), where \(O_0^{11}, O_0^{22}\) are respectively, \(m\) and \(N^2 - m - 1\) dimensional square anti-symmetric matrices. For \(O_1^p = ad(-i\Omega_i^p)\), we have:
\[ O_1^p = \begin{pmatrix} -\Omega_i^{12} & O_i^{12} \\ -O_i^{21} & \Omega_i^{21} \end{pmatrix}. \]

(2) we have the following equation:
\[ e^{-O_0(t-t_0)}O_1^pe^{O_0(t-t_0)} = \sum_{j=1}^{m} (e^{-O_0^{11}(t-t_0)})_{ij}O_j^p, \]
where \((e^{-O_0^{11}(t-t_0)})_{ij}\) is the \(ij^{th}\) entry of the \(m\)-dimensional matrix \(e^{-O_0^{11}(t-t_0)}\).

Proof.

(1) Corresponding to the Cartan decomposition [A], we have the following decomposition of the representation space \(\mathbb{R}^{N^2-1}\):
\[ \mathbb{R}^{N^2-1} = \mathbb{R}^p \oplus \mathbb{R}^e, \] (A2)
where
\[ \mathbb{R}^p = \{(a_1, \cdots, a_m, 0, \cdots, 0) | a_i \in \mathbb{R}\}, \]
\[ \mathbb{R}^e = \{(0, \cdots, 0, a_{m+1}, \cdots, a_{N^2-1}) | a_i \in \mathbb{R}\}. \]

From the assumption (H3), \(O_0\) can be written as:
\[ O_0 = \sum_{l=m+1}^{N^2-1} \omega_l ad(-i\Omega_i^l). \] (A3)

According to the Cartan decomposition [A], the special structure of \(O_0\) and \(O_1^p\) can be easily verified from the fact that \(ad(p)\) maps \(\mathbb{R}^p\) into \(\mathbb{R}^e\), \(\mathbb{R}^e\) into \(\mathbb{R}^p\), and \(ad(c)\) maps \(\mathbb{R}^e\) into \(\mathbb{R}^e\), \(\mathbb{R}^p\) into \(\mathbb{R}^p\).

(2) From [A3] and the equality \(ad([A,B]) = [ad(A),ad(B)]\), it can be shown that:
\[ [O_0, O_1^p] = \sum_{j=1}^{m} (O_1^{11})_{ij} O_j^p. \] (A4)

In fact, it can be deduced that:
\[ [O_0, O_1^p] = \sum_{l=m+1}^{N^2-1} \omega_l ad(-i\Omega_i^l, ad(-i\Omega_i^p)) = \sum_{l=m+1}^{N^2-1} \sum_{j=1}^{m} \omega_l c_{lij} ad(-i\Omega_i^p) = \sum_{j=1}^{m} \left( \sum_{l=m+1}^{N^2-1} \omega_l c_{lij} \right) O_j^p = \sum_{j=1}^{m} (O_1^{11})_{ij} O_j^p, \]
where \(\{c_{ijk}\}\) is the structure coefficients of the Lie algebra \(su(N)\). From [A3], it can be easily verified by induction that
\[ (-O_0)^{(k)}O_1^p = \sum_{j=1}^{m} ((-O_1^{11})^k)_{ij} O_j^p, \]
where \([A^{(i)}, B] = [A, [A^{(i-1)}, B]]\) and \([A^{(0)}, B] = B\).

Now, from the equality
\[ e^A B e^{-A} = \sum_{i=0}^{\infty} \frac{1}{i!} [A^{(i)}, B], \]
we have:
\[
\begin{align*}
  & e^{-O_0(t-t_0)}O_1^p e^{O_0(t-t_0)} \\
  &= \sum_{k=0}^{\infty} \frac{1}{k!} (-O_0(t-t_0))^k , O_1^p \\
  &= \sum_{k=0}^{\infty} \sum_{j=1}^{m} \frac{1}{k!} (-O_0^{11}(t-t_0))^k, i_j O_1^p \\
  &= \sum_{j=1}^{m} (e^{-O_0^{11}(t-t_0)})i_j O_1^p.
\end{align*}
\]

Proof Of Theorem 1

Substitute the control laws (7) into the equation:
\[
\dot{m}(t) = O_m m(t) + \sum_{i=1}^{m} u_i O_1^p m(t) + Dm(t) + g. \quad (A5)
\]

Let $m_1(t), m_2(t)$ are two solutions of the equation (A5) with the initial values to be $m_0^1, m_0^2$ respectively, then $m_1(t) - m_2(t)$ satisfies the following time-variant linear equation:
\[
\dot{x}(t) = (D + O_0 + \sum_{i=1}^{m} u_i O_1^p) x(t) \quad (A6)
\]

with the initial value $x(t_0) = m_0^1 - m_0^2$.

Let $Q = I, \eta = \mu = 1, \nu = 2d_{\text{min}},$ where $-d_{\text{min}} < 0$ (from the assumption (H2)) is the maximal eigenvalue of $D$. From lemma A.1, it can be verified that:
\[
| m_1(t) - m_2(t) | \leq e^{-2d_{\text{min}}(t-t_0)} | m_0^1 - m_0^2 |,
\]
which means $m_1(t) - m_2(t) \to 0$ when $t$ tends to infinity.

From the above analysis, to show that the solution of equation (3) tends to (8), it is sufficient to prove that (8) satisfies the equation (A7), which is equivalent to
\[
\sum_{i=1}^{m} u_i O_1^p e^{O_0(t-t_0)} m_0^\infty + De^{O_0(t-t_0)} m_0^\infty + g = 0,
\]
where $m_0^\infty = (m_0^1, \eta)^T$. From the complete decoherence condition (H1), we have $[e^{O_0(t-t_0)}, D] = 0$ and $e^{O_0(t-t_0)}g = g$, which results in
\[
\sum_{i=1}^{m} u_i O_1^p e^{O_0(t-t_0)} m_0^\infty + e^{O_0(t-t_0)} (D m_0^\infty + g) = 0. \quad (A7)
\]

Substituting (7) into the first term in (A7) and from the lemma A.2 we have:
\[
\begin{align*}
  & \left( O_1^p m_0^\infty, \ldots, O_m^p m_0^\infty(t) \right) \\
  = & \left( e^{O_0(t-t_0)} \left( e^{-O_0(t-t_0)} O_1^p e^{O_0(t-t_0)} m_0^\infty, \ldots, e^{-O_0(t-t_0)} O_m^p m_0^\infty \right) \right) e^{-O_0^1(t-t_0)} \xi \\
  = & e^{O_0(t-t_0)} \left( \sum_{k=1}^{m} (e^{-O_0^1(t-t_0)})i_k O_1^p m_0^\infty, \ldots, \sum_{k=1}^{m} (e^{-O_0^1(t-t_0)})i_k O_m^p m_0^\infty \right) e^{-O_0^1(t-t_0)} \xi \\
  = & e^{O_0(t-t_0)} \left( O_1^p m_0^\infty, \ldots, O_m^p m_0^\infty \right) e^{-O_0^1(t-t_0)} T e^{-O_0^1(t-t_0)} \xi \\
  = & e^{O_0(t-t_0)} \left( O_1^p m_0^\infty, \ldots, O_m^p m_0^\infty \right) e^{-O_0^1(t-t_0)} e^{-O_0^1(t-t_0)} \xi \\
  = & e^{O_0(t-t_0)} \sum_{i=1}^{m} \xi_i O_i^p m_0^\infty.
\end{align*}
\]

Therefore, the equation (A7) is reduced to
\[
\begin{align*}
  & \left( \sum_{i=1}^{m} \xi_i O_i^p m_0^\infty + D m_0^\infty + g \right) = 0,
\end{align*}
\]
which is equivalent to (9).
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