1-String $B_2$-VPG Representation of Planar Graphs*

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Abstract
In this paper, we prove that every planar graph has a 1-string $B_2$-VPG representation—a string representation using paths in a rectangular grid that contain at most two bends. Furthermore, two paths representing vertices $u, v$ intersect precisely once whenever there is an edge between $u$ and $v$.

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1 Preliminaries

One way of representing graphs is to assign to every vertex a curve so that two curves cross if and only if there is an edge between the respective vertices. Here, two curves $u, v$ cross means that they share a point $s$ internal to both of them and the boundary of a sufficiently small closed disk around $s$ is crossed by $u, v, u, v$ (in this order). The representation of graphs using crossing curves is referred to as a string representation, and graphs that can be represented in this way are called string graphs.

In 1976, Ehrlich, Even and Tarjan showed that every planar graph has a string representation [8]. It is only natural to ask if this result holds if one is restricted to using only some “nice” types of curves. In 1984, Scheinerman conjectured that all planar graphs can be represented as intersection graphs of line segments [11]. This was proved first for bipartite graphs [10, 7] with the strengthening that every segment is vertical or horizontal. The result was extended to triangle-free graphs, which can be represented by line segments with at most three distinct slopes [6]. Since Scheinerman’s conjecture seemed difficult to prove for all planar graphs, interest arose in possible relaxations. Note that any two line segments can generally intersect at most once. Define 1-STRING to be the class of graphs that are intersection graphs of curves (of arbitrary shape) that intersect at most once. We also say that graphs in this class have 1-string representation. The original construction of string representations for planar graphs given in [8] requires curves to cross multiple times. In 2007, Chalopin, Gonçalves and Ochem showed that every planar graph is in 1-STRING [2, 4]. With respect to Scheinerman’s conjecture, while the argument of [2, 4] shows that the prescribed number of intersections can be achieved, it provides no idea on the complexity of curves that is required.

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Another way of restricting curves in string representations is to require them to be orthogonal, i.e., to be paths in a grid. Call a graph a VPG-graph (as in “Vertex-intersection graph of Paths in a Grid”) if it has a string representation with orthogonal curves. It is easy to see that all planar graphs are VPG-graphs (e.g. by generalizing the construction of Ehrlich, Even and Tarjan). For bipartite planar graphs, curves can even be required to have no bends [10, 7]. For arbitrary planar graphs, bends are required in orthogonal curves. Recently, Chaplick and Ueckerdt showed that 2 bends per curve always suffice [5]. Let $B_2$-VPG be the graphs that have a string representation where curves are orthogonal and have at most 2 bends; the result in [5] then states that planar graphs are in $B_2$-VPG. Unfortunately, in Chaplick and Ueckerdt’s construction, curves may cross each other repeatedly, and so it does not prove that planar graphs are in 1-STRING.

The conjecture of Scheinerman remained open until 2009 when it was proved true by Chalopin and Gonçalves [2].

1.1 Our Results

In this paper, we show that every planar graph has a string representation that simultaneously satisfies the requirements for 1-STRING (any two curves cross at most once) and the requirements for $B_2$-VPG (any curve is orthogonal and has at most two bends). Our result hence re-proves, in one construction, the results by Chalopin et al. [3] and the result by Chaplick and Ueckerdt [5].

\textbf{Theorem 1.} Every planar graph has a 1-string $B_2$-VPG representation.

Our approach is inspired by the construction of 1-string representations from 2007 [2, 4]. The authors proved the result in two steps. First, they showed that triangulations without separating triangles admit 1-string representations. By induction on the number of separating triangles, they then showed that 1-string representation exists for any planar triangulation, and consequently for any planar graph.

In order to show that triangulations without separating triangles have 1-string representation, Chalopin et al. [4] used a method inspired by Whitney’s proof that 4-connected planar graphs are Hamiltonian [12]. Asano, Saito and Kikuchi later improved Whitney’s technique and simplified his proof [1]. Our paper uses the same approach as [4], but borrows ideas from [1] and develops them further to reduce the number of cases.

2 Definitions and Basic Results

Let us begin with a formal definition of a 1-string $B_2$-VPG representation.

\textbf{Definition 2 (1-string $B_2$-VPG representation).} A graph $G$ has a 1-string $B_2$-VPG representation if every vertex $v$ of $G$ can be represented by a curve $v$ such that:

1. Curve $v$ is orthogonal, i.e., it consists of horizontal and vertical segments.
2. Curve $v$ has at most two bends.
3. Curves $u$ and $v$ intersect at most once, and $u$ intersects $v$ if and only if $(u, v)$ is an edge of $G$.

We always use $v$ to denote the curve of vertex $v$, and write $v^R$ if the representation $R$ is not clear from the context. We also often omit “1-string $B_2$-VPG” since we do not consider any other representations.
Our technique for constructing representations of a graph uses an intermediate step referred to as a “1-string $B_2$-VPG representation of a W-triangulation that satisfies the chord condition with respect to three chosen corners.” We define these terms, and related graph terms, first.

A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn so that no edges intersect except at common endpoints. All graphs in this paper are planar. We assume throughout the paper that one (abstract, hence purely combinatorial) embedding of the graph has been fixed by specifying the clockwise (CW) cyclic order of incident edges around each vertex. Subgraphs inherit this embedding, i.e., they use the induced clockwise orders. A facial region is a connected region of $G − \Gamma$ where $\Gamma$ is a planar drawing of $G$ that conforms with the abstract embedding. The circuit bounding this region can be read from the abstract embedding of $G$ and is referred to as a face. The outer-face is the one that corresponds to the unbounded region; all others are called interior faces. The outer-face cannot be read from the embedding; we assume throughout this paper that the outer-face of $G$ has been specified. Subgraphs inherit the outer-face by using as outer-face the one whose facial region contains the facial region of the outer-face of $G$. An edge of $G$ is called interior if it does not belong to the outer-face.

A triangulated disk is a planar graph $G$ for which the outer-face is a simple cycle and every interior face is a triangle. A separating triangle is a cycle $C$ of length 3 such that $G$ has vertices both inside and outside the region bounded by $C$ (with respect to some fixed embedding of $G$). Following the notation of [4], a W-triangulation is a triangulated disk which does not contain a separating triangle. A chord of a triangulated disk is an interior edge for which both endpoints are on the outer-face.

For two vertices $X, Y$ on the outer-face of a connected planar graph, define $P_{XY}$ to be the counter-clockwise (CCW) path on the outer-face from $X$ to $Y$ ($X$ and $Y$ inclusive). We often study triangulated disks with three specified distinct vertices $A, B, C$ called the corners. $A, B, C$ must appear on the outer-face in CCW order. We denote $P_{AB} = (a_1, a_2, \ldots, a_r)$, $P_{BC} = (b_1, b_2, \ldots, b_s)$ and $P_{CA} = (c_1, c_2, \ldots, c_t)$, where $c_1 = a_1 = A$, $a_r = b_1 = B$ and $b_s = c_1 = C$.

Definition 3 (Chord condition). A W-triangulation $G$ satisfies the chord condition with respect to the corners $A, B, C$ if $G$ has no chord within $P_{AB}, P_{BC}$ or $P_{CA}$, i.e., no interior edge of $G$ has both ends on $P_{AB}$, or both ends on $P_{BC}$, or both ends on $P_{CA}$.

Definition 4 (Partial 1-string $B_2$-VPG representation). Let $G$ be a connected planar graph and $E' \subseteq E(G)$ be a set of edges. An $(E')$-$B_2$-VPG representation of $G$ is a 1-string $B_2$-VPG representation of the subgraph $(V(G), E')$, i.e., curves $u, v$ cross if and only if $(u, v)$ is an edge in $E'$. If $E'$ consists of all interior edges of $G$ as well as some set of edges $F$ on the outer-face, then we write $(\text{int} \cup F)$ representation instead.

In our constructions, we have $F = \emptyset$ or $F = e$, where $e$ is an outer-face edge incident to corner $C$ of a W-triangulation. Edge $e$ is called the special edge, and we sometimes write $(\text{int} \cup e)$ representation, rather than $(\text{int} \cup \{e\})$ representation.

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For readers familiar with [4] or [1]: A W-triangulation that satisfies the chord condition with respect to corners $A, B, C$ is called a W-triangulation with 3-boundary $P_{AB}, P_{BC}, P_{CA}$ in [4], and the chord condition is the same as Condition (W2b) in [1].
2.1 2-Sided, 3-Sided and Reverse 3-Sided Layouts

To create representations where vertex-curves have few bends, we need to impose geometric restrictions on representations of subgraphs. Unfortunately, no one type of layout seems sufficient for all cases, and we will hence have three different layout types whose existence we will prove in parallel.

Definition 5 (2-sided layout). Let $G$ be a connected planar graph and $A, B$ be two distinct outer-face vertices. An $(int \cup F)$ $B_2$-VPG representation of $G$ has a 2-sided layout (with respect to corners $A, B$) if:

1. There exists a rectangle $\Theta$ that contains all intersections of curves and such that the top of $\Theta$ is intersected, from right to left in order, by the curves of the vertices of $P_{AB}$, and the bottom of $\Theta$ is intersected, from left to right in order, by the curves of the vertices of $P_{BA}$.
2. The curve $v$ of an outer-face vertex $v$ has at most one bend. (By (1), this implies that $A$ and $B$ have no bends.)

Definition 6 (3-sided layout). Let $G$ be a connected planar graph and $A, B, C$ be three distinct vertices in CCW order on the outer-face of $G$. Let $F$ be a set of exactly one outer-face edge incident to $C$. An $(int \cup F)$ $B_2$-VPG representation of $G$ has a 3-sided layout (with respect to corners $A, B, C$) if:

1. There exists a rectangle $\Theta$ containing all intersections of curves so that
   
   (i) the top of $\Theta$ is intersected, from right to left in order, by the curves of the vertices on $P_{AB}$;
   
   (ii) the left side of $\Theta$ is intersected, from top to bottom in order, by the curves of the vertices on $P_{Bb_{s-1}}$, possibly followed by $C$;
   
   (iii) the bottom of $\Theta$ is intersected, from right to left in order, by the curves of vertices on $P_{c_2A}$ in reversed order, possibly followed by $C$;
   
   (iv) curve $b_s = C = c_1$ intersects the boundary of $\Theta$ exactly once; it is the bottommost curve to intersect the left side of $\Theta$ if the special edge in $F$ is $(C, c_2)$, and $C$ is the leftmost curve to intersect the bottom of $\Theta$ if the special edge in $F$ is $(C, b_{s-1})$.

2. The curve $v$ of an outer-face vertex $v$ has at most one bend. (By (1), this implies that $B$ has precisely one bend.)
3. $A$ and $C$ have no bends.

We sometimes refer to the rectangle $\Theta$ for both 2- and 3-sided representation as a bounding box. Figure 1 (which will serve as base case later) shows such layouts for a triangle and varying choices of $F$. We also need the concept of a reverse 3-sided layout, which is similar to the 3-sided layout except that $B$ is straight and $A$ has a bend. Formally, it satisfies conditions 1(ii-iv) and (2). 1(i) is replaced by “the right side of $\Theta$ is intersected, from bottom to top in order, by the curves of the vertices on $P_{AB}$” and (3) is replaced by “$B$ and $C$ have no bends.”

2.2 The Tangling Technique

The following technique for creating edge intersections will frequently be used in our constructions. Consider a set of $k$ vertical downward rays $s_1, s_2, s_3, \ldots, s_k$ placed beside each other in left to right order. The operation of bottom-tangling from $s_1$ to $s_k$ rightwards stands for the following (see also Figure 2):
Figure 1 \((\text{int} \cup F)\) representations of a triangle: (Top) 2-sided representations for 
\(F \in \{\{(A, C)\}, \{(B, C)\}, \emptyset\}\). (Bottom) 3-sided and reverse 3-sided representations for 
\(F \in \{\{(A, C)\}, \{(B, C)\}\}\). Private regions are shaded in grey.

Figure 2 Bottom-tangling from \(s_1\) to \(s_k\) rightwards.

1. For \(1 < i \leq k\), stretch \(s_i\) downwards so that it ends below \(s_{i-1}\).
2. For \(1 \leq i < k\), bend \(s_i\) rightwards and stretch it so that it crosses \(s_{i+1}\), but so that it 
does not cross \(s_{i+2}\).

We similarly define right-tangling upwards, top-tangling leftwards and left-tangling 
downwards as rotation of bottom-tangling rightwards by \(90^\circ\), \(180^\circ\) and \(270^\circ\) CCW. We 
define bottom-tangling leftwards as a horizontal flip of bottom-tangling rightwards, and 
right-tangling downwards, top-tangling rightwards and left-tangling upwards as \(90^\circ\), \(180^\circ\) 
and \(270^\circ\) CCW rotations of bottom-tangling leftwards.

2.3 Private Regions

Our proof starts with constructing representation for triangulations without separating 
triangles. The construction is then extended to all triangulations by merging representations 
of subgraphs obtained by splitting at separating triangles. To permit the merge, we apply 
the technique used in [4] (and re-discovered in [9]): With every triangular face, create a 
region that intersects the curves of vertices of the face in a predefined way and does not 
intersect anything else. Following the notation of [9], we call this a \textit{private region} (but we 
use a different shape).

\textbf{Definition 7} (Chair-shape). A \textit{chair-shaped area} is a region bounded by a 10-sided ortho-
gonal polygon with CW (clockwise) or CCW (counter-clockwise) sequence of interior angles 
\(90^\circ, 90^\circ, 270^\circ, 270^\circ, 90^\circ, 90^\circ, 270^\circ, 90^\circ, 90^\circ, 270^\circ\) \textit{CCW}. See also Figure 3.

\textbf{Definition 8} (Private region). Let \(G\) be a planar graph with a 1-string \(B_2\)-VPG represent-
ation \(R\) and let \(f\) be a facial triangle in \(G\). A \textit{private region} of \(f\) is a chair-shaped area \(\Phi\) 
inside \(R\) such that:
1. $\Phi$ is intersected by no curves except for the ones representing vertices on $f$.
2. All the intersections of $R$ are located outside of $\Phi$.
3. For a suitable enumeration of vertices of $f$ as \{a, b, c\}, $\Phi$ is intersected by two segments of $a$ and one segment of $b$ and $c$. The intersections between these segments and $\Phi$ occur at the edges of $\Phi$ as depicted in Figure 3.

3 Constructions for W-Triangulations

Our key tool for proving Theorem 1 is following lemma:

Lemma 9. Let $G$ be a W-triangulation that satisfies the chord condition with respect to corners $A, B, C$. For any $e \in \{(C, b_{s-1}), (C, c_2)\}$, $G$ has an $(\text{int} \cup e)$ 1-string $B_2$-VPG representation with 3-sided layout and an $(\text{int} \cup e)$ 1-string $B_2$-VPG representation with reverse 3-sided layout. Both representations have a chair-shaped private region for every interior face.

The proof of Lemma 9 will use induction on the number of vertices. To combine the representations of subgraphs, we sometimes need them to have a 2-sided layout, and hence prove the following result:

Lemma 10. Let $G$ be a W-triangulation that satisfies the chord condition with respect to corners $A, B, C$. Then $G$ has an $(\text{int} \cup F)$ 1-string $B_2$-VPG representation with 2-sided layout with respect to $A, B$ and for any set $F$ of at most one outer-face edge incident to $C$. Furthermore, this representation has a chair-shaped private region for every interior face of $G$.

Notice that for Lemma 9 the special edge must exist (this is needed in Case 1 to find private regions), while for Lemma 10, $F$ is allowed to be empty.

We will prove both lemmas simultaneously by induction on the number of vertices. First, let us make an observation that will greatly help to reduce the number of cases. Define $G^{rev}$ to be the graph obtained from graph $G$ by reversing the abstract embedding. This effectively switches corners $A$ and $B$, and replaces special edge $(C, c_2)$ by $(C, b_{s-1})$ and vice versa. If $G$ satisfies the chord condition with respect to corners $(A, B, C)$, then $G^{rev}$ satisfies the chord condition with respect to corners $(B, A, C)$. (Recall that corners must be listed in clockwise order on the outer-face.)
Presume we have a 2-sided/3-sided/reverse 3-sided representation of \( G^{\text{rev}} \). Then we can obtain a 2-sided representation of \( G \) by flipping the 2-sided one of \( G^{\text{rev}} \) horizontally, i.e., along the \( y \)-axis. We can obtain a 3-sided/reverse 3-sided representation of \( G \) by flipping the reverse 3-sided/3-sided representation of \( G^{\text{rev}} \) diagonally (i.e., along the line defined by \( (x = y) \)). Hence for all the following cases, we may (after possibly applying the above reversing operation) either make a restriction on which edge the special edge is, or we only need give the construction for 3-sided, but not for reverse 3-sided layout.

Now we begin the induction. In the base case, \( n = 3 \), so \( G \) is a triangle, and the three corners \( A, B, C \) must be the three vertices of this triangle. The desired \( (\text{int} \cup F) \) representations for all possible choices of \( F \) are depicted in Figure 1. The induction step for \( n \geq 4 \) is divided into three cases which we describe in separate subsections.

### 3.1 \( C \) has degree 2

Since \( G \) is a triangulated disk with \( n \geq 4 \), \( (b_{s-1}, c_2) \) is an edge. Define \( G' := G - C \) and \( F' := \{ (b_{s-1}, c_2) \} \). We claim that \( G' \) satisfies the chord condition for corners \( A', B' := B \) and a suitable choice of \( C' \in \{ b_{s-1}, c_2 \} \), and argue this as follows. If \( c_2 \) is not incident to a chord, then set \( C' := c_2 \); clearly the chord condition holds for \( G' \). If \( c_2 \) is incident to a chord, then by the chord condition for \( G \), the chord must end on \( P_{BC} \). Then \( b_{s-1} \) cannot be incident to a chord by planarity and the chord condition for \( G \). So, in this case with the choice \( C' := b_{s-1} \) the chord condition holds for \( G' \). Thus in either case, we can apply induction to \( G' \).

To create a 2-sided representation of \( G \), we use a 2-sided \( (\text{int} \cup F') \) representation \( R' \) of \( G' \). We introduce a new vertical curve \( C \) placed between \( b_{s-1} \) and \( c_2 \) below \( R' \). Add a bend at the upper end of \( C \) and extend it leftwards or rightwards. If the special edge \( e \) exists, then extend \( C \) until it hits the curve of the other endpoint of \( e \); else extend it only far enough to allow for the creation of the private region.

To create a 3-sided representation of \( G \), we use a 3-sided \( (\text{int} \cup F') \) representation \( R' \) of \( G' \). Note that regardless of which vertex is \( C' \), we have \( b_{s-1} \) as bottommost curve on the left and \( c_2 \) as leftmost curve on the bottom. Introduce a new horizontal segment representing \( C \) which intersects \( c_2 \) if \( F' = \{ (C, c_2) \} \), or a vertical segment which intersects \( b_{s-1} \) if \( F' = \{ (C, b_{s-1}) \} \).

In both constructions, after suitable lengthening, the curves intersect the bounding box in the required order. One can find the chair-shaped private region for the only new face \( \{ C, c_2, b_{s-1} \} \) as shown in Figure 1. Observe that no bends were added to the curves of \( R' \) and that \( C \) has the required number of bends in both representations.

Since we have given the constructions for both possible special edges, we can obtain the reverse 3-sided representation by diagonally flipping a 3-sided representation of \( G^{\text{rev}} \).

### 3.2 \( G \) has a chord incident to \( C \)

By the chord condition, this chord has the form \( (C, a_i) \) for some \( 1 < i < r \). Select the chord that minimizes \( i \). The graph \( G \) can be split along the chord \( (C, a_i) \) into two graphs \( G_1 \) and \( G_2 \). Both \( G_1 \) and \( G_2 \) are bounded by simple cycles, hence triangulated disks. No edges were added, so neither \( G_1 \) nor \( G_2 \) contains a separating triangle. So, both of them are W-triangulations.

We select \((A, a_i, C)\) as corners for \( G_1 \) and \((B, C, a_i)\) as corners for \( G_2 \) and can easily verify that \( G_1 \) and \( G_2 \) satisfy the chord condition with respect to those corners:

- \( G_1 \) has no chords on \( P_{Aa_i} \) or \( P_{CA} \) as they would violate the chord condition in \( G \). There is no chord on \( P_{a_iC} \) as it is a single edge.
Case 1: $C$ has degree 2. (Top) 2-sided representation. (Bottom) 3-sided representation.

Figure 4

Case 2(a): Constructing an \((\text{int} \cup (C, b_{s-1}))\) representation when $C$ is incident with a chord, in 2-sided (middle) and 3-sided (right) layout.

Figure 5

$G_2$ has no chords on $P_{a,B}$ or $P_{B,C}$ as they would violate the chord condition in $G$. There is no chord on $P_{a,C}$ as it is a single edge.

So we can apply induction to both $G_1$ and $G_2$, obtain representations $R_1$ and $R_2$ for them, and combine them suitably. In the 3-sided case, we will do so for all possible choices of special edge, and hence need not give the constructions for reverse 3-sided layout as explained earlier.

Case 2(a): $F \neq \{ (C, c_2) \}$.

Inductively construct a 2-sided \((\text{int} \cup (C, a_i))\) representation $R_1$ of $G_1$. Inductively, construct an \((\text{int} \cup F)\) representation $R_2$ of $G_2$, which should be 2-sided if we want the result to be 2-sided and 3-sided if we want the result to be 3-sided. Note that either way $C^{R_2}$ and $a_i^{R_2}$ are the leftmost and rightmost segment on the bottom side of $R_2$.

Rotate $R_1$ by $180^\circ$, and translate it so that it is below $R_2$ with $a_1^{R_1}$ in the same column as $a_1^{R_2}$. Stretch $R_1$ horizontally as needed until $C^{R_1}$ is in the same column as $C^{R_2}$. Then $a_i^{R_1}$ and $C^{R_1}$ for $R \in \{ R_1, R_2 \}$ can each be unified without adding bends by adding vertical segments. The curves of outer-face vertices of $G$ then cross (after suitable lengthening) the bounding box in the required order. See also Figure 5.

Every face $f$ of $G$ is contained in $G_1$ or $G_2$ and hence has a private region in $R_1$ or $R_2$. As our construction does not make any changes inside the bounding boxes of $R_1$ and $R_2$, the
Figure 6 Case 2(b): $C$ is incident with a chord, $F = (C, c_2)$, and $c_2 \neq A$. Private regions for newly created faces are shaded grey.

private region of $f$ is contained in $R$ as well.

Case 2(b): $F = \{(C, c_2)\}$.

For the 2-sided construction, we apply the reversal trick: Construct a 2-sided representation of $G^\text{rev}$ (here Case 2(a) then applies) and flip it horizontally.

For the 3-sided construction, we need a different approach, which is quite similar to Case 1 in [1] Proof of Lemma 2. Let $G_Q = G_1 - C$, and observe that it is bounded by $P_{c_2A}$, $P_{A,a_i}$, and the path formed by the neighbours $c_2 = u_1, u_2, \ldots, u_q = a_i$ of $C$ in CCW order. We must have $x \geq 2$, but possibly $G_1$ is a triangle $\{C, A, a_i\}$ and $G_Q$ then degenerates into an edge. If $G_Q$ contains at least three vertices, then none of $u_2, \ldots, u_{q-1}$ belongs to $P_{AB}$ since chord $(C, a_i)$ was chosen closest to $A$, and so $G_Q$ is a W-triangulation.

Case 2(b)1: $A \neq c_2$.

Select the corners of $G_Q$ as $(A_Q := c_2, B_Q := A, C_Q := u_q)$, and observe that it satisfies the chord conditions since the three corners are distinct and the three outer-face paths are subpaths of $P_{CA}$ and $P_{AB}$ or in the neighbourhood of $C$, respectively. Inductively construct a 2-sided $\text{int} \cup \{u_q, u_{q-1}\}$ representation $R_Q$ of $G_Q$ with private region for every interior face. Inductively, construct a 3-sided $\text{int} \cup \{C, a_i\}$ representation $R_2$ of $G_2$.

To combine $R_Q$ with $R_2$, rotate $R_Q$ by $180^\circ$. Appropriately stretch $R_Q$ and translate it so that it is below $R_2$ with $a_i^{R_2}$ and $a_i^{R_Q}$ in the same column, and so that none of the curves $u_{q-1}, \ldots, u_1 = c_2$ crosses the bounding box of $R_2$. Then $a_i^{R_Q}$ and $a_i^{R_2}$ can be unified without adding bends by adding a vertical segment. Curves $u_{q-1}, \ldots, u_1 = c_2$ in the rotated $R_Q$ can be appropriately stretched upwards, intersected by $C^{R_2}$ after stretching it leftwards, and they can be top-tangled leftwards. All the curves of outer-face vertices of $G$ then cross (after suitable lengthening) a bounding box in the required order.

All faces in $G$ that are not interior to $G_Q$ or $G_2$ are bounded by $(C, u_k, u_{k+1})$, $1 \leq k < q$. The chair-shaped private regions for such faces can be found as shown in Figure 6.
Case 2(b)2: \( A = c_2 \)

In this case the previous construction cannot be applied since the corners for \( G_Q \) would not be distinct. We give an entirely different construction.

If \( G_Q \) has at least 3 vertices, then \( q \geq 3 \) since otherwise by \( A = c_2 = u_1 \) edge \( (A, u_q) \) would be a chord on \( P_{AB} \). Choose as corners for \( G_Q \) the vertices \( A_Q := A, B_Q := a_i = u_q \) and \( C_Q := u_{q-1} \) and observe that the chord condition holds since all three paths on the outer-face belong to \( P_{AB} \) or are in the neighbourhood of \( C \). By induction \( G_Q \) has a 2-sided \((\text{int} \cup \langle u_q, u_{q-1} \rangle)\) representation \( R_Q \) with the respective corners and private region for every interior face of \( G_Q \). If \( G_Q \) has at most 2 vertices, then \( G_Q \) consists of edge \( (A, a_2) \) only, and we use as representation \( R_Q \) two parallel vertical segments \( a_2 \) and \( A \).

We combine \( R_Q \) with a representation \( R_1 \) of \( G_1 \) that is different from the one used in the previous cases; in particular we rotate corners. Set \( C_2 := a_i, A_2 := B \) and \( B_2 := C \), and construct a reverse 3-sided layout \( R_2 \) of \( G_2 \). Rotate \( R_2 \) by 180°, and translate it so that it is situated below \( R_1 \) with \( a_i^{R_1} \) and \( a_i^{R_2} \) in the same column. Then, extend \( C^{R_2} \) until it crosses \( u_{q-1}, \ldots, u_1 \) (after suitable lengthening), and then bottom-tangle \( u_{q-1}, \ldots, u_1 \) rightwards. This creates intersections for all edges in path \( u_q, u_{q-1}, \ldots, u_1 \), except for \( (u_q, u_{q-1}) \), which is either on the outer-face or had an intersection in \( R_Q \). One easily verifies that the result is a 3-sided layout, and private regions can be found for the new interior faces as shown in Figure 7.

3.3 \( G \) has no chords incident with \( C \) and \( \text{deg}(C) \geq 3 \)

In this case, we will give explicit constructions for 2-sided, 3-sided and reverse 3-sided layout, and may hence (after applying the reversal trick) assume that the special edge, if it exists, is \((C, c_2)\).

Let \( u_1, \ldots, u_q \) be the neighbours of vertex \( C \) in clockwise order, starting with \( b_{q-1} \) and ending with \( c_2 \). We know that \( q = \text{deg}(C) \geq 3 \) and that \( u_2, \ldots, u_{q-1} \) are not on the outer-face,
since $C$ is not incident to a chord.

Let $u_j$ be a neighbour of $C$ that has at least one other neighbour on $P_{CA}$, and among all those, choose $j$ to be minimal. Such a $j$ exists because $G$ is triangulated and therefore $u_{q-1}$ is adjacent to both $C$ and $u_q$. We need two subcases.

**Case 3(a):** $j \neq 1$

Denote the neighbours of $u_j$ on $P_{CA}$ by $t_1, \ldots, t_x$ in the order in which they appear on $P_{CA}$. Separate $G$ into subgraphs as follows (see also Figure 8):

- The right graph $G_R$ is bounded by $(A, P_{BA}, B, P_{BC}, u_1, u_2, \ldots, u_j, t_x, P_{CA}, A)$.
- The bottom graph $G_B$ is bounded by $(u_j, t_1, P_{BC}, t_x, u_j)$. We are chiefly interested in its subgraph $G_Q := G_B - u_j$.
- The left graph $G_L$ is bounded by $(C, P_{CA}, t_1, u_j, C)$. We are chiefly interested in its subgraph $G_0 := G_L - \{u_j, C\}$.

The idea is to obtain representations of these subgraphs and then to combine them suitably. We first explain how to obtain the representation $R_R$ used for $G_R$. Clearly $G_R$ is a W-triangulation, since $u_2, \ldots, u_j$ are interior vertices of $G$, and hence the outer-face of $G_R$ is a simple cycle. Set $A_R := A$ and $B_R := B$. If $B \neq u_1$ then set $C_R := u_1$ and observe that $G_R$ satisfies the chord condition with respect to these corners:

- $G_R$ does not have any chords with both ends on $P_{AR} \cup B_R = P_{AB}, P_{BR} \subseteq P_{BC}$, or $P_{tx} \subseteq P_{CA}$ since $G$ satisfies the chord condition.
- If there were any chords between $u_1, \ldots, u_j$ and $P_{CA}$, then by $C_R = u_1$ the chord would either connect two neighbours of $C$ (hence give a separating triangle of $G$), or connect some $u_i$ for $i < j$ to $P_{CA}$ (contradicting minimality of $j$), or connect $u_j$ to some other vertex on $P_{CA}$ (contradicting that $t_x$ is the last neighbour of $u_j$ on $P_{CA}$). Hence no such chord can exist either.

If $B = u_1$, then set $C_R := u_2$ (which exists by $q \geq 3$) and similarly verify that it satisfies the chord condition as $P_{BR} \subseteq P_{CA}$ is the edge $(B, u_2)$. Since $C_R \subseteq \{u_1, u_2\}$ in both cases, we can apply induction on $G_R$ and obtain an $(\text{int} \cup \{u_1, u_2\})$ representation $R_R$. We use as layout for $R_R$ the type that we want for $G$, i.e., use a 2-sided/3-sided/reverse 3-sided layout if we want $G$ to have a 2-sided/3-sided/reverse 3-sided representation.

Next consider the graph $G_0$, which is bounded by $u_{j+1}, \ldots, u_q$, $P_{CA}, t_1$, and the neighbours of $u_j$ in CCW order between $t_1$ and $u_{j+1}$. We have cases:

1. $j = q - 1$, and hence $t_1 = u_q = c_2$ and $G_0$ consists of only $c_2$. In this case we use a single vertical line segment $c_2$ as representation $R_0$.
The boundary of the chain graph is shown bold. Graphs $G_1, \ldots, G_4$ are the blocks of chain. Right: Merging $(\text{int} \cup F_i)$ representations with 2-sided layouts of chain blocks $G_i, 1 \leq i \leq 4$, into an $(\text{int} \cup \emptyset)$ representation of the chain graph $G_Q$.

(2) $j < q - 1$, so $G_0$ contains at least three vertices $u_{q-1}, u_q$ and $t_1$. Then $G_0$ is a W-triangulation since $C$ is not incident to a chord and by choice of $t_1$. Also, it satisfies the chord condition with respect to corners $A_0 := t_1, B_0 := c_2$ and $C_0 := u_{j+1}$ since the three paths on its outer-face are subpaths of $P_{CA}$ or contained in the neighbourhood of $C$ or $u_j$. In this case, construct a 2-sided $(\text{int} \cup (u_{j+1}, u_{j+2}))$ representation $R_0$ of $G_0$ with respect to these corners inductively.

Finally, we create a representation $R_Q$ of $G_Q = G_L - u_j$. If $G_Q$ is a single vertex or a single edge, then simply use vertical segments for the curves of its vertices. Otherwise, we can show:

\textbf{Claim 11.} $G_Q$ has a 2-sided $(\text{int} \cup \emptyset)$ 1-string $B_2$-VPG representation with respect to corners $t_1$ and $t_2$.

\textbf{Proof.} $G_Q$ is not necessarily 2-connected, so we cannot apply induction directly. Instead we break it into $x - 1$ graphs $G_1, \ldots, G_{x-1}$, where for $i = 1, \ldots, x - 1$ graph $G_i$ is bounded by $P_{t_i, t_{i+1}}$ as well as the neighbours of $u_i$ between $t_i$ and $t_{i+1}$ in CCW order. Note that $G_i$ is either a single edge, or it is bounded by a simple cycle since $u_j$ has no neighbours on $P_{CA}$ between $t_i$ and $t_{i+1}$. In the latter case, use $B_i := t_i, A_i := t_{i+1},$ and $C_i$ an arbitrary third vertex on the outer-face of $G_i$, which exists since the outer-face is a simple cycle. Observe that $G_i$ satisfies the chord condition since all paths on the outer-face of $G_i$ are either part of $P_{BC}$ or in the neighbourhood of $u_j$. Hence by induction there exists a 2-sided $(\text{int} \cup \emptyset)$ representation $R_i$ of $G_i$. If $G_i$ is a single edge $\left(t_i, t_{i+1}\right)$, then let $R_i$ consists of two vertical segments $t_i$ and $t_{i+1}$.

Since each representation $R_i$ has at its leftmost end a vertical segment $t_i$ and at its rightmost end a vertical segment $t_{i+1}$, we can combine all these representations by aligning $t_i^{R_i}$ and $t_{i+1}^{R_i}$ horizontally and filling in the missing segment. See also Figure 9. One easily verifies that the result is a 2-sided $(\text{int} \cup \emptyset)$ representation of $G_Q$.

We now explain how to combine these three representations $R_R, R_Q$ and $R_0$; see also Figure 10. Rotate $R_Q$ by $180^\circ$ and translate it so that it is below $R_R$ with $t_{2}^{R_R}$ and $t_{x}^{R_Q}$ in the same column; then connect these two curves with a vertical segment. Rotate $R_0$ by $180^\circ$ and translate it so that it is below $R_R$ and to the left and above $R_Q$, and $t_{1}^{R_R}$ and $t_{1}^{R_Q}$ are in the same row; then connect these two curves with a vertical segment. Horizontally stretch $R_0$ and/or $R_R$ so that the vertical segments of $u_2^{R_R}, \ldots, u_j^{R_R}$ that are at the bottom
Figure 10 Case 3: Combining subgraphs when $\text{deg}(3) \geq 3$, there is no chord incident with $C$, and $F \subseteq \{(C, c_2)\}$. (Top left) 3-sided and (top right) reverse 3-sided construction. (Bottom) 2-sided construction for the case $F = \{(C, c_2)\}$ and $F = \emptyset$. The private regions for the new faces are shaded grey. The construction match the graph depicted in Figure 8 left.

Introduce a new horizontal segment $C$ and place it so that it intersects curves $u_{q}, \ldots, u_{j+2}$, $u_2, \ldots, u_{j}$, $u_{j+1}$ (after lengthening them, if needed). For a 2-sided layout also attach a vertical segment to $C$. If $j < q - 1$ then top-tangle $u_q, \ldots, u_{j+2}$ leftwards. Bottom-tangle $u_2, \ldots, u_j$ rightwards. The construction hence creates intersections for all edges in the path $u_1, \ldots, u_q$, except for $(u_{j+2}, u_{j+1})$ (which was represented in $R_0$) and $(u_2, u_1)$ (which was represented in $R_R$).

Bend and stretch $u_j^{R_R}$ rightwards so that it crosses the curves of all its neighbours in $G_0 \cup G_Q$. Finally create intersections for all edges between these neighbours of $u_j$ that are interior, by top-tangling their curves rightwards.

One verifies that the curves intersect the bounding boxes as desired. The constructed representations contains a private region for all interior faces of $G_R$, $G_Q$ and $G_0$ by induction. The remaining faces are of the form $(C, u_i, u_{i+1})$, $1 \leq i < q$, and $(u_j, w_k, w_{k+1})$ where $w_k$ and $w_{k+1}$ are two consecutive neighbours of $u_j$ on the outer-face of $G_0$ or $G_Q$. Private regions for those faces are shown in Figure 10.
Case 3(b): \( j = 1 \), i.e., there exists a chord \((b_{s-1}, c_i)\)

In this case we cannot use the above construction directly since \(b_{s-1}\) ends on the left (in the 3-sided construction) while we need \(u_j\) to end at the bottom and not to be on the outer-face. However, if we use a different vertex as \(u_j\) (and argue carefully that the chord condition then holds), then the same construction works.

Recall that \(u_1, \ldots, u_q\) are the neighbours of corner \(C\) in CW order starting with \(b_{s-1}\) and ending with \(c_2\). We know that \(q \geq 3\) and \(u_2, \ldots, u_{q-1}\) are not on the outer-face. Now define \(j'\) as follows: Let \(u_j, j' > 1\) be a neighbour of \(C\) that has at least one another neighbour on \(P_{CA}\), and choose \(u_j\) so that \(j'\) is minimal while satisfying \(j' > 1\). Such a \(j'\) exists since \(u_{q-1}\) has another neighbour on \(P_{CA}\), and by \(q \geq 3\) we have \(q - 1 > 1\). Now, separate \(G\) as in the previous case, except use \(j'\) in place of \(j\). Thus, define \(t_1, \ldots, t_x\) to be the neighbours of \(u_j\) on \(P_{CA}\), in order, and separate \(G\) into three graphs as follows:

- The right graph \(G_R\) bounded by \((A, P_{CA}, B, P_{BA}, u_1, u_2, \ldots, u_j, t_x, t_{x+1}, A)\).
- The bottom graph \(G_B\) is bounded by \((u_{j'}, t_1, P_{BA}, t_x, u_j')\). and define \(G_Q := G_B - u_{j'}\).
- The left graph \(G_L\) is bounded by \((C, P_{CA}, t_1, u_j, C)\) and define \(G_0 := G_L - \{u_j, C\}\).

Observe that the boundaries of all the graphs are simple cycles, and thus they are W-triangulations. Select \((A_R := A, B_R := B, C_R := u_2)\) to be the corners of \(G_R\) and argue the chord condition as follows:

- \(G_R\) does not have any chords on \(P_{CA}\) as such chords would either contradict minimality of \(j'\), or violate the chord condition in \(G\).
- \(G_R\) does not have any chords of \(P_{BA} b_R = P_{AB} b\).
- \(G_R\) does not have any chords on \(P_{BA}\) as it is a subpath of \(P_{BC}\) and they would violate the chord condition in \(G\). It also does not have any chords in the form \((G_R = u_2, b_i), 1 \leq \ell < s - 1\) as they would have to intersect the chord \((b_{s-1}, a_i)\), violating the planarity of \(G\). Hence, \(G_R\) does not have any chords on \(P_{BA}\).
- Notice in particular that the chord \((u_1, a_i)\) of \(G_R\) is not a violation of the chord condition since we chose \(u_2\) as a corner, and hence the ends of this corner are on different sub-paths of the outer-face of \(G_R\).

Hence, we can obtain a representation \(R_R\) of \(G_R\) with 2-sided, 3-sided and reverse 3-sided layout and special edge \((u_1 = b_{s-1}, u_2)\). For graphs \(G_Q\) and \(G_0\) the corners are chosen, the chord condition is verified, and the representations are obtained exactly as in Case 3a. Since the special edge of \(G_R\) is \((u_1, u_2)\) as before, curves \(u_1\) and \(u_2\) are situated precisely as in Case 3a. Hence, the representations can be merged and private regions can be found exactly as in Case 3a.

This ends the description of the construction in all cases, and hence proves Lemma 10 and Lemma 9.

4 From 4-Connected Triangulations to All Planar Graphs

In this section, we prove Theorem 1. Observe that Lemma 9 essentially proves it for 4-connected triangulations. As in [4] we extend it to all graphs by induction on the number of separating triangles and utilize their lemma:
Lemma 12 (Chalopin, Gonçalves, 2010). Let $G$ be a planar triangulated graph and let $\Delta = (a, b, c)$ be an inclusion-wise minimal separating triangle in $G$. The subgraph $G'$ induced by the vertices strictly inside $\Delta$ is either an isolated vertex, or a $W$-triangulation with some corners $(A, B, C)$ such that vertices on $P_{AB}$ are adjacent to $a$, vertices on $P_{BC}$ are adjacent to $b$, and vertices to $P_{CA}$ are adjacent to $c$. Furthermore, $G'$ satisfies the chord condition with respect to $(A, B, C)$.

Theorem 13. Let $G$ be triangulation with outer-face $(A, B, C)$. $G$ has a $1$-string $B_2$-VPG representation with a chair-shaped private region for every interior face $f$ of $G$.

Proof. Our approach is exactly the same as in [4], except that we must be careful not to add too many bends when merging subgraphs at separating triangles, and hence must use $3$-sided layouts. Formally, we proceed by induction on the number of separating triangles. In the base case, $G$ has no separating triangle, i.e., it is $4$-connected. As the outer-face is a triangle, $G$ clearly satisfies the chord condition. Thus, by Lemma 9, it has a $3$-sided $(\text{int} \cup (B, C))$ representation $R$ with private region for every face. $R$ has an intersection for every edge except for $(A, B)$ and $(A, C)$. These intersections can be created by top-tangling $B, A$ rightwards and bottom-tangling $C, A$ leftwards. Recall that $A$ initially did not have any bends, so it has $2$ bends in the constructed representation of $G$. The existence of private regions is guaranteed by Lemma 9. See Figure 11 for an illustration.

Now assume for induction that $G$ has $k + 1$ separating triangles. Let $\Delta = (a, b, c)$ be an inclusion-wise minimal separating triangle of $G$. By Lemma 12, the subgraph $G_1$ induced by the vertices inside $\Delta$ is either an isolated vertex, or a $W$-triangulation $(A, B, C)$ such that the vertices on $P_{AB}$ are adjacent to $a$, the vertices on $P_{BC}$ are adjacent to $b$, and the vertices on $P_{CA}$ are adjacent to $c$. Furthermore, $G_1$ satisfies the chord condition. Also, graph $G_2 = G - G_1$ is a $W$-triangulation that satisfies the chord condition and has $k$ separating triangles. By induction, $G_2$ has a representation $R_2$ with a chair-shaped private region for every interior face $f$. Let $\Phi$ be the region for face $\Delta$. Permute $a, b, c$, if needed, so that the naming corresponds to the one needed for the private region.

Case 1: $G_2$ is a single vertex $v$.

Represent $v$ by $2$-bend orthogonal curve $v$ inserted into $\Phi$ that intersects $a, b$ and $c$.

The construction, together with private regions for the newly created faces $(a, b, v)$, $(a, c, v)$ and $(b, c, v)$, is shown in Figure 12 (left).
Case 2: $G_2$ is a W-triangulation.

Recall that $G_2$ satisfies the chord condition with respect to corners $(A, B, C)$. Apply Lemma\[9\] to construct a 3-sided $(\text{int} \cup (C, b_{s-1}))$ representation $R_2$ of $G_2$. Let us assume that (after possible rotation) $\Phi$ has the orientation shown in Figure 12 (right); if it had the symmetric orientation then we would do a similar construction using a reverse 3-sided representation of $G_2$. Place $R_2$ inside $\Phi$ as shown in Figure 12 (right). Stretch the curves representing vertices on $P_{CA}$, $P_{AB}$ and $P_{Bb_{s-1}}$ downwards, upwards and leftwards respectively so that they intersect $a, b$ and $c$. Top-tangle leftwards the curves $A = a_1, a_2, \ldots, a_r = B$. Left-tangle downwards the curves $B = b_1, b_2, \ldots, b_{s-1}$ and bend and stretch $C$ downwards so that it intersects $a$. Bottom-tangle leftwards the curves $C = c_1, \ldots, c_t = A$. It is easy to verify that the construction creates intersection for all the edges between vertices of $\Delta$ and the outer-face of $G_2$. The tangling operation then creates intersections for all the outer-face edges of $G_2$ except edge $(C, b_{s-1})$, which is already represented in $R_2$.

Every curve that receives a new bend represents a vertex on the outer-face of $G_2$, which means that it initially had at most 1 bend. Curve $A$ is the only curve which receives 2 new bends, but this is allowed as $A$ does not have any bends in $R_2$. Hence, the number of bends for every curve does not exceed 2.

Private regions for faces formed by vertices $a, b, c$ and vertices on the outer-face of $G_2$ can be found as shown in Figure 12 right.

With Theorem 13 in hand, we can show our main result: every planar graph has a 1-string $B_2$-VPG representation.

Proof of Theorem 11. If $G$ is a planar triangulated graph, the claim holds by Theorem 13.
So, assume that $G$ is a planar graph. Then *stellate* the graph, i.e., insert a vertex into each non-triangulated face and connect it to all vertices on that face. It is well known that after at most 3 repetitions, the construction produces a 3-connected triangulated graph $G'$ such that $G$ is an induced subgraph of $G'$. Apply Theorem 13 to construct a 1-string $B_2$-VPG representation $R'$ of $G'$. By removing curves representing vertices that are not in $G$, we obtain a 1-string $B_2$-VPG representation of $G$.

5 Conclusions and Outlook

We showed that every planar graph has a 1-string $B_2$-VPG representation, i.e., a representation as intersection graph of strings where string cross at most once and each string is orthogonal with at most two bends. One advantage of this is that the coordinates to describe such a representation are small, since orthogonal drawings can be deformed easily such that all bends are at integer coordinates. Every vertex curve has at most two bends and hence at most 3 segments, so the representation can be made to have coordinates in an $O(n) \times O(n)$-grid with perimeter at most $3n$. Note that none of the previous results provided an intuition of the required size of the grid.

Following the steps of our proof, it is not hard to see that our representation can be found in linear time, since the only non-local operation is to test whether a vertex has a neighbour on the outer-face. This can be tested by marking such neighbours whenever they become part of the outer-face. Since no vertex ever is removed from the outer-face, this takes overall linear time.

The representation constructed in this paper uses curves of 8 possible shapes for planar graphs. One can in fact verify that the 2-sided layout (which only uses 2-sided layouts in its recursions) uses only 4 possible shapes: C, Z and their horizontal mirror images. Hence for triangulations without separating triangles (and, after stellating, all 4-connected planar graphs) 4 shapes suffice. A natural question is if one can restrict the number of shapes required to represent all planar graphs.

Bringing this effort further, is it possible to restrict the curves even more? Felsner et al. [9] asked the question whether every planar graph is the intersection graph of only two shapes, namely $\{L, \Gamma\}$. Note that this would also provide a different proof of Scheinerman’s conjecture. Somewhat in between: is every planar graph the intersection graph of $xy$-monotone orthogonal curves, preferably in the 1-string model?

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