A Coupled Oscillator Model for Grover’s Quantum Database Search Algorithm

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Grover’s database search algorithm is the optimal algorithm for finding a desired object from an unsorted collection of items. Although it was discovered in the context of quantum computation, it is simple and versatile enough to be implemented using any physical system that allows superposition of states, and several proposals have been made in the literature. I study a mechanical realisation of the algorithm using coupled simple harmonic oscillators, and construct its physical model for the simplest case of four identical oscillators. The identification oracle is implemented as an elastic reflection of the desired oscillator, and the overrelaxation operation is realised as evolution of the system by half an oscillation period. I derive the equations of motion, and solve them both analytically and by computer simulation. I extend the idea of the algorithm to changes in the initial conditions, masses of springs and damping. The amplitude amplification provided by the algorithm enhances the energy of the desired oscillator, while running the algorithm backwards spreads out the energy of the perturbed oscillator among its partners.

The former (efficient focusing of energy into a specific oscillator) can have interesting applications in processes that need crossing of an energy threshold for completion, and can be useful in nanotechnological devices and catalysis. The latter (efficient redistribution of energy) can be useful in processes requiring rapid dissipation of energy, such as shock-absorbers and vibrational shielding. I present some tentative proposals.

I. GROVER’S ALGORITHM

Database search is an elementary computational task with wide-ranging applications. Its efficiency is measured in terms of the number of queries one has to make to the database in order to find the desired item. In the conventional formulation of the problem, the query is a binary oracle (i.e. a Yes/No question). For an unsorted database of \( N \) items, starting from an unbiased state and using classical Boolean logic, one requires on the average \( Q = O(N) \) queries to locate the desired item.

Lov Grover discovered a search algorithm that, using superposition of states, reduces the number of required queries to \( Q = O(\sqrt{N}) \) [1]. The algorithm was originally proposed in the context of quantum computation, but its applicability has since been widely expanded by realising that the algorithm is an amplitude amplification process that can be executed by a coupled set of wave modes. It has also been proved that the algorithm is optimal for unsorted database search [2].

Grover’s algorithm starts with a superposition state, where each item has an equal probability to get picked, and evolves it to a target state where only the desired item can get picked. Following Dirac’s notation, the starting and target state satisfy (index \( i \) labels the items),

\[
|\langle i | s \rangle|^2 = 1/N , \quad |\langle i | t \rangle|^2 = \delta_{it} .
\]  

The algorithm evolves \( |s\rangle \) towards \( |t\rangle \), by discrete rotations in the two-dimensional space formed by \( |s\rangle \) and \( |t\rangle \). The rotations are performed as an alternating sequence of the two reflection operators,

\[
U_t = 1 - 2|t\rangle\langle t| , \quad U_s = 1 - 2|s\rangle\langle s| ,
\]

\[
(-U_s U_t)^Q |s\rangle = |t\rangle .
\]

\( U_t \) is the binary oracle which flips the sign of the target state amplitude, while \( -U_s \) performs the reflection-in-the-average operation. Solution to Eq.(3) determines the number of queries as

\[
(2Q + 1) \sin^{-1}(1/\sqrt{N}) = \pi/2 .
\]

(In practice, \( Q \) must be an integer, while Eq.(4) may not have an integer solution. In such cases, the algorithm is stopped when the state has evolved sufficiently close to, although not exactly equal to, \( |t\rangle \). Then one finds the desired item with a high probability.)

The steps of the algorithm for the simplest case, \( Q = 1 \) and \( N = 4 \), are illustrated in Fig.1.

The algorithm relies on superposition and interference amongst a set of states, which are generic features of wave dynamics. It can be executed by any system of coupled wave modes, provided:

1. The superposition of modes maintains phase coherence during evolution.
2. The two reflection operations (phase changes of \( \pi \) for...
The quantum version of the algorithm involves highly fragile entanglement, and hence very short coherence times. It also needs to be implemented at the atomic scale, which is not at all easy. On the other hand, the classical wave version uses only superposition, which is much more stable, and hence it is straightforward to design demonstration models \[3,4\]. In the following, I describe implementation of Grover’s algorithm in a simple mechanical setting, using four harmonic oscillators coupled via the centre-of-mass mode.

### II. HARMONIC OSCILLATOR IMPLEMENTATION

A system of coupled harmonic oscillators is frequently studied in physics. It involves only quadratic forms, and can be solved exactly in both classical and quantum domains. Let the items in the database be represented by \( N \) identical harmonic oscillators. While they are oscillating in a specific manner, one of them is “tapped” (i.e. elastically reflected). The task is to identify which of the oscillators has been tapped, without looking at the tapping. The optimisation criterion is to design the system of oscillators, and their initial state, so as to make the identification as quickly as possible.

Grover’s algorithm requires identical coupling between any pair of oscillators. That is arranged by coupling all the \( N \) small oscillators to a big oscillator, as shown in Fig.2. The big oscillator then is coupled to the centre-of-mass mode, and becomes an intermediary between any pair of oscillators with the same coupling. Indeed, the centre-of-mass mode plays the role of the “average state”. In this setting, elastic reflection of an oscillator implements the binary oracle in velocity space (see Fig.3), and time evolution of the whole system by half an oscillation period carries out the reflection about average operation.

#### A. Dynamics

The equations of motion are:

\[
M \ddot{X} = -KX + k \sum_{i} (x_i - X),
\]

\[
m \ddot{x}_i = -k(x_i - X).
\]

The non-target \((x_i \neq t)\) oscillators influence the dynamics of the target oscillator \((x_t)\) only through the centre-of-mass position \(\bar{x} = \sum_{i=1}^{N} x_i/N\). They make up \(N - 2\) linearly independent modes, of the form \(x_{j \neq t} - x_{k \neq t}\), which decouple from \(x_t\) and \(\bar{x}\). Effective dynamics is thus in the 3-variable space \(\{X, \bar{x}, x_t\}\), with the equations of motion:

\[
\ddot{X} = -\frac{K + Nk}{M} X + \frac{Nk}{M} \bar{x},
\]

\[
\ddot{\bar{x}} = -\frac{k}{m}(\bar{x} - X),
\]

\[
\ddot{x}_t - \bar{x} = -\frac{k}{m}(x_t - \bar{x}).
\]
The last equation is easily solved, with the angular frequency \( \omega_t = \sqrt{k/m} \). The first two equations are coupled, and their eigenmodes are of the form \( X + \lambda \bar{r} \). We can find them by requiring that

\[
\ddot{X} + \lambda \bar{r} = X(-\frac{K + Nk}{M} + \frac{\lambda k}{m}) + \bar{r}(\frac{Nk}{M} - \frac{\lambda k}{m})
\]

\[= -\omega^2(X + \lambda \bar{r}). \tag{10}\]

Therefore,

\[
\lambda = \left(\frac{Nk}{M} - \frac{\lambda k}{m}\right) / \left(-\frac{K + Nk}{M} + \frac{\lambda k}{m}\right). \tag{11}\]

Let the dimensionless ratios for the spring constants and the masses be, \( R_k = K/k \) and \( R_m = M/m \). Then the sinusoidal solutions have the angular frequency

\[
\omega = \omega_t \sqrt{\frac{N + R_k}{R_m}} - \lambda. \tag{12}\]

There are two solutions to these equations, coefficients \( \lambda_\pm \) and the corresponding frequencies \( \omega_\pm \). They satisfy

\[
\omega_+^2 + \omega_-^2 = \omega_t^2 \left(1 + \frac{N + R_k}{R_m}\right), \quad \omega_+^2 \omega_-^2 = \omega_t^4 \frac{R_k}{R_m}. \tag{13}\]

The general solution to the dynamical equations is:

\[
X + \lambda_\pm \bar{r} = A_\pm \sin(\omega_\pm t + \phi_\pm), \tag{14}\]

\[
x_t - \bar{r} = A_t \sin(\omega t + \phi_t) . \tag{15}\]

**B. The model**

We constructed the simplest system, with \( N = 4 \). The binary oracle (i.e. \( U_i \)) is the elastic reflection \( \dot{x}_t \rightarrow -\dot{x}_t \) when \( x_t = 0 \). The reflection about average operation imposes the constraint that the whole system must evolve by half an oscillation period between successive oracles. From the many possibilities, as our design parameters, we selected the convenient frequency ratios

\[
\omega_+ = \frac{3}{2} \omega_t, \quad \omega_- = \frac{1}{2} \omega_t . \tag{16}\]

Then the time period \( T = 4\pi/\omega_t \) for the whole system. Time evolution for half the period reverses \( \bar{r} \), while leaving \( \dot{x}_t - \bar{r} \) unchanged, i.e. it implements the operator \( U_x \) in the velocity space. Thus Grover’s algorithm is realised by “tapping” the target oscillator at every time interval \( \Delta t = 2\pi/\omega_t \).

The above resonance criterion corresponds to:

\[
R_k = \frac{12}{5}, \quad R_m = \frac{64}{15}, \quad \lambda_+ = -\frac{3}{4}, \quad \lambda_- = \frac{5}{4}. \quad \tag{17}\]

In a situation where all the non-target oscillators move uniformly together (i.e. all \( x_{i\neq t} \) equal, all \( \dot{x}_{i\neq t} \) equal), the displacements are

\[
x_t = -\frac{1}{2} A_+ \sin(\omega t + \phi_+) + \frac{1}{2} A_- \sin(\omega t + \phi_-) + A_t \sin(\omega t + \phi_t), \tag{18}\]

\[
x_{i\neq t} = -\frac{1}{2} A_+ \sin(\omega t + \phi_+) + \frac{1}{2} A_- \sin(\omega t + \phi_-) - \frac{1}{k} A_t \sin(\omega t + \phi_t). \tag{19}\]

Our experimental parameters differed slightly from these ideal values because of various imperfections discussed later.

**C. Results**

The uniform superposition state corresponds to the initial conditions:

\[
X = 0, \quad \dot{X} = 0, \quad x_i = 0, \quad \dot{x}_i = V . \tag{20}\]

In this case, the big oscillator returns to its initial rest state after every half a period, and the first binary oracle is applied at \( t = 0 \). The starting phases of the solution vanish, \( \phi_\pm = \phi_t = 0 \). The amplitudes are

\[
A_+ = \frac{V}{2\omega_t}, \quad A_- = \frac{5V}{2\omega_t}, \quad A_t = 0, \tag{21}\]

before the binary oracle, and

\[
A_+ = -\frac{V}{4\omega_t}, \quad A_- = \frac{5V}{4\omega_t}, \quad A_t = -\frac{3V}{4\omega_t}, \tag{22}\]

after the binary oracle. The resultant time evolution of the oscillators in the position and velocity spaces is illustrated in Fig.4. It is observed that Grover’s algorithm provides position amplification of 1.87 and velocity amplification of 2.

In actual experiment, it is much easier to start with the initial conditions:

\[
X = A, \quad \dot{X} = 0, \quad x_i = A, \quad \dot{x}_i = 0. \tag{23}\]

In this case, the velocities reach their maximum value after a quarter period, and the first binary oracle is applied at \( t = T/4 \). The starting phases of the solution are, \( \phi_\pm = \phi_t = \pi/2 \). The amplitudes are

\[
A_+ = \frac{A}{4}, \quad A_- = \frac{9A}{4}, \quad A_t = 0 . \tag{24}\]
FIG. 4: Time evolution of the oscillators in the position (top) and the velocity (bottom) spaces, when the initial state is: $X = 0$, $\dot{X} = 0$, $x_i = 0$, $\dot{x}_i = V$. Red, green and blue curves respectively denote the uniform superposition state (without any oracle), the target oscillator after the oracle at $t = 0$, and the non-target oscillators after the oracle at $t = 0$.

FIG. 5: Time evolution of the oscillators in the position (top) and the velocity (bottom) spaces, when the initial state is: $X = A$, $\dot{X} = 0$, $x_i = A$, $\dot{x}_i = 0$. Only the behaviour after the oracle is shown, $t' \equiv t - T/4$. Red, green and blue curves respectively denote the uniform superposition state (without any oracle), the target oscillator after the oracle at $t = T/4$, and the non-target oscillators after the oracle at $t = T/4$.

D. Perturbations and stability

Gravity: A uniform gravitational field shifts the equilibrium positions of the vertically hanging springs, but apart from that it has no effect on their dynamics. This is obvious from the expression for the potential energy,

$$\frac{1}{2} k x^2 - mgx = \frac{1}{2} k \left( x - \frac{mg}{k} \right)^2 - \frac{m^2 g^2}{2k} .$$

Imperfect synchronization of the initial state: When the initial velocities are arbitrary instead of uniform, the energy amplification of the target oscillator provided by the algorithm is not four-fold. It is instead limited to the initial energy present in the $\{X, \pi, x_1\}$ modes,

$$\left[ \left( \dot{x}_1^2 + \frac{4}{3} (\ddot{x}_1 - \pi)^2 \right)/\dot{x}_1^2 \right]_{t=0}$$
which is still substantial for the generic situation where the initial $\hat{x}_t$ and $\overline{x}$ do not differ by a large amount.

**Imprecise reflection operations:** In practice, the reflection operations may not exactly implement phase changes of $\pi$. Also, the measurement operation terminating the algorithm may not take place at the precise instant of maximum amplification. The energy amplification depends only quadratically on such phase errors from the ideal values, e.g. if the reflection phase change is $\pi + \epsilon$, then the loss in energy amplification is $O(\sin^2(\epsilon/2))$.

**Inelastic reflections:** When the binary oracle produces an inelastic reflection of the target oscillator, with a coefficient of restitution $1 - \epsilon$ (i.e. $v_t \rightarrow (\epsilon-1)v_t$), the energy amplification decreases by the factor $(1-\epsilon/4)^2$. Even in the extreme case of $\epsilon = 1$, the energy amplification is a sizeable $(9/4)$-fold.

**Spring masses:** Real springs are not massless. To a good approximation, the energy taken up by the springs can be estimated by assuming a constant velocity gradient along the springs, and then absorbing the spring energy by altering the masses of the objects attached to the springs. The dominant correction is to add one-third of the spring mass to the objects at either end. The remainder, in our set up and on the average, amounts to adding one-twelfth of the mass of the small springs to the big mass. This prescription allows tuning of the masses of the oscillators after measuring the masses of the springs, in order to maximise the energy amplification.

**Damping:** For a weakly damped oscillator (damping force $-2\gamma mv$), its amplitude changes linearly with the damping coefficient $\gamma$, while its frequency changes quadratically.

\[ \dot{x} + 2\gamma \dot{x} + \omega_0^2 x = 0 \implies x = Ae^{-\gamma t} \cos\left(\sqrt{\omega_0^2 - \gamma^2} t + \phi\right) \]  

(28)

Thus small external disturbances reduce the energy amplification, by a factor $\exp(-4\pi\gamma/\omega)$, but have little effect on the all important phase coherence amongst the oscillators that governs interference of the modes.

Overall, we find that most deviations of the parameters from their ideal values affect the performance of the algorithm only quadratically, and can be easily taken care of. Damping provides the only linear perturbation, which should be controlled to the best possible extent.

### III. POSSIBLE APPLICATIONS

A variety of coupled vibrational systems with small damping can be made easily. They can provide either fast focusing or fast dispersal of energy, and can therefore be an important component in processes sensitive to energy availability.

**A. Focusing of energy**

Efficient concentration of the total energy of a coupled oscillator system into a specific oscillator can be used as a trigger or a sensor, where an external disturbance becomes the cause for the reflection oracle. The focusing of energy could also be useful in nanomechanical systems where the component concerned cannot be directly controlled, and a possibility of using coupled cantilever beams as a switch is pointed out in Fig.6.

There exist many processes that need crossing of an energy threshold for completion. Their rates are typically governed by the Boltzmann factor for the energy barrier, $\exp(-E_{\text{barrier}}/kT)$. Energy amplification can speed up the rates of such processes by large factors, leading to catalysis.

**B. Dispersal of energy**

Grover’s algorithm is fully reversible. The reflection operators $U_x$ and $U_t$ are inverses of themselves. So the algorithm can be run backwards as $(-U_t U_x)^Q|t\rangle = |s\rangle$. That disperses large initial energy in the target oscillator to a uniform distribution among its partners. In the coupled oscillator model, the initial condition would be $\dot{x}_t = V$ and $\dot{x}_{t\neq t} = 0$. After waiting for $t = 2\pi/\omega$, and then reversing $\dot{x}_t$ produces $\dot{x}_t = V/2$.

This behaviour can be useful in quickly reducing localised perturbations by redistributing its energy throughout the system. Instead of damping a single perturbed oscillator, it is much more efficient to disperse the energy into several oscillators while damping all of them together. To illustrate the concept, consider the situation where all the oscillators have the same damping coefficient $\gamma$. The normal modes in the $\{X, \overline{X}, x_i\}$ space then separate the same way as in Eq.(10), and the relations in Eqs.(12,13) are retained. Damping shifts the oscillation frequencies according to $\omega' = \sqrt{\omega^2 - \gamma^2}$, and the general solution becomes:

\[ X + \lambda_k \overline{\overline{X}} = A_k e^{-\gamma t} \sin(\omega'_k t + \phi_k), \]  

(29)

\[ x_t - \overline{x} = A_t e^{-\gamma t} \sin(\omega'_t t + \phi_t). \]  

(30)
FIG. 7: A hierarchical system of coupled oscillators can act as a shock absorber. The initial impulse is assumed to be a local disturbance, which subsequently spreads out.

The coupled oscillator dynamics of Grover’s algorithm is maintained by keeping the frequency ratios unchanged,

\[ \omega'_{t} = \frac{3}{2} \omega'_{t}, \quad \omega'_{\varphi t} = \frac{1}{2} \omega'_{t}. \]  

(31)

If the initial conditions are chosen as

\[ X = 0, \quad \dot{X} = 0, \quad x_{i} = 0, \quad \dot{x}_{i} = V, \quad x_{i\neq t} = 0, \]  

(32)

then after half an oscillation period, \( T'/2 = 2\pi/\omega'_{t} \),

\[ \dot{x}_{t} = \frac{1}{2} V e^{-2\pi \gamma/\omega'_{t}}, \quad \dot{x}_{i\neq t} = -\frac{1}{2} V e^{-2\pi \gamma/\omega'_{t}}. \]  

(33)

This results show that the distribution of energy among the coupled oscillators suppresses the energy of the target oscillator by an extra factor of 4, in addition to the usual damping factor for a stand-alone oscillator. It is indeed the maximum possible reduction in energy, combining both the mechanisms.

The damping coefficient that maximises the rate of energy loss of the target oscillator is given by

\[ \frac{d}{d\gamma} \left[ 1 - e^{-4\pi \gamma/\omega'_{t}} \right] = 0 \implies \gamma = 0.213\omega_{t}. \]  

(34)

With this choice, in time \( T'/2 \), the energy of the target oscillator is reduced to 1.6% of its initial value. Although the energy loss is more compared to a stand-alone oscillator, it is less localized because it is distributed among the target oscillator, its partners and the big oscillator. The corresponding spring constant and mass ratios are:

\[ R_{k} = 2.91, \quad R_{m} = 4.68. \]  

(35)

A hierarchical system of coupled oscillators (see Fig.7) can be even more efficient in dispersal of energy, by implementing the above mechanism simultaneously at multiple scales. The simplest choice would be to couple four small oscillators to a big one at every level, with appropriate mass, spring and damping parameters.

C. Transfer of energy

It is also possible to combine dispersal and concentration operations to transfer energy from one oscillator to another via the centre-of-mass mode. For example, in case of four coupled oscillators, initial energy in oscillator \( |t_{1}\rangle \) can be transferred to oscillator \( |t_{2}\rangle \) by

\[ (U_{s}U_{t_{2}})(U_{t_{1}}U_{s})|t_{1}\rangle = |t_{2}\rangle. \]  

(36)

In this manner, a local signal received by a large detector can be first dispersed over the whole system and then extracted at a specific location.

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