Images of Galois representations in mod $p$ Hecke algebras

Laia Amorós

Abstract

Let $(T_f, m_f)$ denote the mod $p$ local Hecke algebra attached to a normalised Hecke eigenform $f$, which is a commutative algebra over some finite field $\mathbb{F}_q$ of characteristic $p$ and with residue field $\mathbb{F}_q$. By a result of Carayol we know that, if the residual Galois representation $\rho_f : G_\mathbb{Q} \to \text{GL}_2(\mathbb{F}_p)$ is absolutely irreducible, then one can attach to this algebra a Galois representation $\rho_f : G_\mathbb{Q} \to \text{GL}_2(T_f)$ that is a lift of $\rho_f$. We will show how one can determine the image of $\rho_f$ under the assumptions that (i) the image of the residual representation contains $\text{SL}_2(\mathbb{F}_q)$, (ii) that $m_f^2 = 0$ and (iii) that the coefficient ring is generated by the traces. As an application we will see that the methods that we use allow us to deduce the existence of certain $p$-elementary abelian extensions of big non-solvable number fields.

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1 Introduction

Let $f(z) = \sum_{n=0}^{\infty} a_n(f)q^n \in S_k(N; \mathbb{C})$ be a normalised Hecke eigenform, whose coefficients lie in some ring of integers of a number field, and let $\overline{\rho}_f : G_\mathbb{Q} \to \text{GL}_2(\mathbb{F}_p)$ denote the attached mod $p$ Galois representation, which is semisimple and uniquely determined by the coefficients of $f$ mod $p$, for primes $\ell \nmid Np$. Let $\mathbb{F}_q$ be the finite field generated by the $a_\ell(f)$ mod $p$. Consider the ring homomorphism

$$\lambda_f : \mathbb{T} \to \mathbb{F}_q$$

and let $m_f := \ker(\lambda_f)$, which is a maximal ideal of $\mathbb{T}$. We denote by $\mathbb{T}_{m_f}$ the localisation of $\mathbb{T}$ at $m_f$. Then $\mathbb{T}_{m_f}$ is a commutative local finite-dimensional $\mathbb{F}_q$-algebra and, if moreover we assume that the residual Galois representation $\overline{\rho}_f$ is absolutely irreducible, by Theorem 3 in [2], we have a continuous Galois representation

$$\rho_f : G_\mathbb{Q} \to \text{GL}_2(\mathbb{T}_{m_f})$$

such that $\overline{\rho}_f = \pi \circ \rho_f$, where $\pi$ extends the natural projection $\pi : \mathbb{T}_{m_f} \to \mathbb{F}_q$. In this setting, one is naturally interested in the image of $\rho_f$.

We will be study the case $\text{Im}(\overline{\rho}_f) = \text{GL}_2^D(\mathbb{F}_q)$, where $D \subseteq \mathbb{F}_q^\times$ indicates where the determinants of $\overline{\rho}_f$ lie. Let $\mathbb{M}_2^0(\mathbb{F}_q)$ denote the trace-0 matrices with coefficients in $\mathbb{F}_q$. In the present paper we will prove the following result.
Theorem. Let $\mathbb{F}_q$ denote a finite field of characteristic $p$ and $q = p^d$ elements, and suppose that $q \neq 2, 3, 5$. Let $(\mathbb{T}, m_{\mathbb{T}})$ be a finite-dimensional local commutative $\mathbb{F}_q$-algebra equipped with the discrete topology, and with residue field $\mathbb{T}/m_{\mathbb{T}} \cong \mathbb{F}_q$. Suppose that $m_{\mathbb{T}}^2 = 0$. Let $\Gamma$ be a profinite group and let $\rho : \Gamma \to \text{GL}_2(\mathbb{T})$ be a continuous representation such that

(a) $\text{Im}(\rho) \subseteq \text{GL}_2^D(\mathbb{T})$, where $D \subseteq \mathbb{F}_q^* \times \mathbb{F}_q$ is a subgroup.

(b) $\text{Im}(\overline{\rho}) = \text{GL}_2^D(\mathbb{F}_q)$, where $\overline{\rho}$ denotes the reduction $\rho \bmod m_{\mathbb{T}}$.

(c) $\mathbb{T}$ is generated as $\mathbb{F}_q$-algebra by the set of traces of $\rho$.

Let $m := \dim_{\mathbb{F}_q} m_{\mathbb{T}}$ and let $t$ be the number of different traces in $\text{Im}(\rho)$.

(i) If $p \neq 2$, then $t = q^{m+1}$ and

$$\text{Im}(\rho) \cong (M_2^0(\mathbb{F}_q) \oplus \ldots \oplus M_2^0(\mathbb{F}_q))^m \times \text{GL}_2^D(\mathbb{F}_q) \cong \text{GL}_2^D(\mathbb{T}).$$

(ii) If $p = 2$, then $t = q^\alpha \cdot ((q-1)2^\beta + 1)$, for some unique $0 \leq \alpha \leq m$ and $0 \leq \beta \leq d(m-\alpha)$, and in this case $\text{Im}(\rho) \cong M \rtimes \text{SL}_2(\mathbb{F}_q)$, where $M$ is an $\mathbb{F}_2[\text{SL}_2(\mathbb{F}_q)]$-submodule of $M_2^0(m_{\mathbb{T}})$ of the form

$$M \cong \bigoplus_{\alpha} M_2^0(\mathbb{F}_q) \oplus \bigoplus_{\beta} M_2^0(\mathbb{F}_q) \oplus C_2 \oplus \ldots \oplus C_2,$$

where $C_2 \subseteq S$ is a subgroup with 2 elements of the scalar matrices. Moreover, $M$ is determined uniquely by $t$ up to isomorphism.

An important application of the theorems is that they can give us information on some abelian extensions of number fields. The Galois representations that we investigate correspond to abelian extensions of very big non-solvable number fields that standard methods do not allow to treat computationally.

The structure of this paper is as follows. In section 2 we prove the image-splitting theorem (theorem 2.1), a generalisation of Manoharmayum’s main theorem in [3] that will allow us to express $\text{Im}(\rho_f)$ as a semidirect product of certain submodule $M \subseteq M_2^0(\mathbb{m}_f)$ and $\text{GL}_2^D(\mathbb{F}_q)$. In section 3 we give a complete classification of the possible images of 2-dimensional Galois representations with coefficients in local algebras over finite fields under the hypothesis previously mentioned (theorem 3.1). In section 4 we use this result in the situation where the Galois representation takes values in mod $p$ Hecke algebras coming from modular forms. The classification of the images appearing in this setting is completely solved in odd characteristic. In even characteristic we can only reduce the problem to a finite number of possibilities. In section 5 we see how one can compute explicit examples in characteristic 2 and show how in each case one can have a conjectural image. We then state some questions that arise after contrasting many examples. Finally in section 6 we will see that the methods on group extensions that we develop allow us to deduce the existence of $p$-elementary abelian extensions of big number fields, which are not computationally approachable so far.

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2 Im($\rho$) as a semidirect product

We will start by proving the image-splitting theorem, a result that generalises the Main Theorem in [6]. As a consequence of this result, we will be able to express, under certain conditions, the image of $\rho_f$ as a semidirect product, a first step to completely determine Im($\rho_f$).

In order to do this, we need to work in a more general situation. Let $(T, m_T)$ denote a finite-dimensional commutative local $\mathbb{F}_q$-algebra with residue field $T/m_T \simeq \mathbb{F}_q$ of characteristic $p$, and denote by $\pi : T \to T/m_T$ the natural projection. Suppose that $m_T^2 = 0$. Consider a closed subgroup $G \subseteq \text{GL}_q^D(T)$, where $D \subseteq \mathbb{F}_q^\times$ denotes a subgroup. Suppose that $\pi(G) = \text{GL}_q^D(\mathbb{F}_q)$. This gives us a short exact sequence

$$1 \to H \to G \to \text{GL}_q^D(\mathbb{F}_q) \to 1,$$

where $H$ is an $\mathbb{F}_p(\text{GL}_q^D(\mathbb{F}_q))$-module inside the abelian group of trace-0 matrices $M_2^0(m_T)$ with coefficients in $m_T \simeq \mathbb{F}_q$. The image-splitting theorem will allow us to describe the group $G$ as a semidirect product $H \rtimes \text{GL}_q^D(\mathbb{F}_q)$. Equivalently, it tells us that the previous short exact sequence admits a splits.

Let us introduce some notation necessary to state the image-splitting theorem. Let $k$ be a finite field of characteristic $p$. Let $W(k)$ denote its ring of Witt vectors and denote by $T : k \to W(k) \subset \mathbb{Q}_p$ the Teichmüller lift. Consider a complete local ring $(A, m_A)$ with residue field containing $k$ and consider the inclusion $\iota : k \to A/m_A$. Since $W(k)$ is a $p$-ring with residue field $k$, by the structure theorem for complete local rings ([7], Theorem 29.2) we have that there exists a local homomorphism $\iota : W(k) \to A$ which induces $\iota_0$ on the residue fields. Thus we have a commutative diagram

$$\begin{array}{ccc}
W(k) & \xrightarrow{\iota} & A \\
\downarrow T & & \downarrow \iota_0 \\
k & \xrightarrow{\iota_0} & A/m_A
\end{array}$$

Denote by $W(k)_A$ the image of $\iota : W(k) \to A$. Consider a subgroup $D \subseteq k^\times$ and define the following group

$$\text{GL}_n^D(W(k)) := \{ g \in \text{GL}_n(W(k)) \mid \det(g) \in T(D) \}.$$

For any subgroup $X \subseteq A$, we define the group

$$\text{GL}_n^D(X) := \{ g \in \text{GL}_n(X) \mid \det(g) \in \iota(T(D)) \}.$$

Then we will prove the following result.

Theorem 2.1 (Image-splitting theorem). Let $(A, m_A)$ be a complete local noetherian ring with maximal ideal $m_A$ and finite residue field $A/m_A$ of characteristic $p$. Let $\pi : A \to A/m_A$ denote the natural projection. Suppose we are given a subfield $k$ of $A/m_A$ and a closed subgroup $G$ of $\text{GL}_n(A)$. Assume that the cardinality of $k$ is at least 4 and that $k \neq \mathbb{F}_5$ if $n = 2$ and $k \neq \mathbb{F}_4$ if $n = 3$. Suppose that $\pi(G) \supseteq \text{GL}_n^D(k)$. Then $G$ contains a conjugate of $\text{GL}_n^D(W(k)_A)$.

Let $M_n^0(k)$ denote the trace-zero matrices in the group of matrices $M_n(k)$ and denote by $S$ the subspace of scalar matrices in $M_n^0(k)$. Then $S = \{0\}$ when $p \nmid n$, and $S = \{ \lambda I_n : \lambda \in k \}$ otherwise. Define $V := M_n^0(k)/S$. For any subgroup $D \subseteq k^\times$, we will consider the group $\text{GL}_n^D(k) := \{ g \in \text{GL}_n(k) : \det(g) \in D \}$.

Lemma 2.2. Assume that $k \neq \mathbb{F}_2$ if $n = 2$. Let $M \subseteq M_n^0(k)$ be an $\mathbb{F}_p[\text{GL}_n^D(k)]$-submodule for the conjugation action. Then, either $M$ is a subspace of $S$ over $\mathbb{F}_p$ or $M = M_n^0(k)$. Thus $M_n^0(k)/S$ is a simple $\mathbb{F}_p[\text{GL}_n^D(k)]$-module, and the sequence

$$0 \to S \to M_n^0(k) \to V \to 0$$

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does not split when $p \mid n$.

**Proof.** Since any $\mathbb{F}_p[GL_n^D(k)]$-submodule of $M_0^r(k)$ is also an $\mathbb{F}_p[SL_n(k)]$-submodule, we can apply Lemma 3.3 of [6]. \hfill \Box

**Lemma 2.3.** Let $R$ be a ring and $M$ a semisimple left $R$-module which decomposes as $M = M_1 \oplus \ldots \oplus M_t$, with $M_i \subseteq M$ simple modules. Let $N \subseteq M$ be a submodule of $M$. Then $N$ is semisimple and is isomorphic to a direct sum of a subset of the modules $M_1, \ldots, M_t$.

**Proof.** By Proposition 3.12 in [4], for every submodule $N \subseteq M$ there exists a submodule $N' \subseteq M$ such that $M = N \oplus N'$. Now by Corollary 14.6 in [3] we have that, in this case, $N$ is isomorphic to a direct sum of a subset of the modules $M_1, \ldots, M_t$. \hfill \Box

Let $G$ be a finite group. Given an $\mathbb{F}_p[G]$-module $M$ and a normalised 2-cocycle $x : G \times G \to M$ (where normalised means that $x(1, g) = x(g, 1) = 0$, for every $g \in G$), one can define the twisted semidirect product $M \rtimes_x G$. It consists of elements $(m, g)$ with $m \in M$, $g \in G$ and composition law given by

$$(m_1, g_1)(m_2, g_2) := (x(g_1, g_2) + m_1 + g_1 m_2, g_1 g_2).$$

The cohomology class of $x$ in $H^2(G, M)$ represents the extension

$$0 \to M \to M \rtimes_x G \to G \to 1$$

$$m \mapsto (m, 1) \quad (m, g) \mapsto g.$$

Assume that $M$ has finite cardinality, and let $N \subseteq M$ be an $\mathbb{F}_p[G]$-submodule. Just like Manoharmayum’s theorem, the image-splitting theorem relies heavily on Proposition 2.2 in [6]. To use this, we basically need to prove two conditions: (i) that the map $H^2(G, N) \to H^2(G, M)$ is injective, and that (ii) $H^1(G, M) = 0$.

More concretely, let $N \subseteq M$ be two $\mathbb{F}_p[GL_n^D(k)]$-submodules of $M_0^r(k)^r$, for some $r \geq 1$. We will prove that, after some small assumption on $k$, we have that $$H^1(GL_n^D(W_m), k) = 0$$ and that there is an injection $$H^2(GL_n^D(W_m), N) \hookrightarrow H^2(GL_n^D(W_m), M).$$

**Assumption 2.4.** The cardinality of $k$ is at least 4, $k \neq \mathbb{F}_5$ if $n = 2$ and $k \neq \mathbb{F}_4$ if $n = 3$.

**Lemma 2.5.** Let $C$ be a finite group such that $(\# C, p) = 1$. Let $V$ be an $\mathbb{F}_p$-vector space. Then $$H^i(C, V) = 0, \quad \forall i \geq 1.$$

**Proof.** Consider the restriction-corestriction map

$$H^i(C, V) \xrightarrow{\text{res}} H^i(1, V) \xrightarrow{\text{cor}} H^i(C, V),$$

which corresponds to multiplication by $\# C = c$. For $x \in H^i(C, V)$ we have that $cx = 0$. So since $V$ is an $\mathbb{F}_p$-vector space and $c$ is invertible in $\mathbb{F}_p$, we must have $x = 0$. Thus $H^i(C, V) = 0$ for all $i > 0$. \hfill \Box
Lemma 2.6. Let $k$ be a finite field of characteristic $p$ satisfying Assumption 2.4. Let $D$ be a subgroup of $k^\times$. Then
(a) $H^1(\text{GL}_n^D(W_m), k) = 0$, for all $m \geq 1$.
(b) If $p \nmid n$ then $H^1(\text{GL}_n^D(W_m), M_n^0(k)) = 0$, for all $m \geq 1$.

Proof. (a) Consider the short exact sequence
$$0 \rightarrow \text{SL}_n(W_m) \rightarrow \text{GL}_n^D(W_m) \overset{\text{det}}{\rightarrow} T(D) \rightarrow 1.$$We have the inflation-restriction exact sequence
$$0 \rightarrow H^1(T(D), k^{\text{SL}_n(W_m)}) \rightarrow H^1(\text{GL}_n^D(W_m), k) \rightarrow H^1(\text{SL}_n(W_m), k)^{T(D)}.$$Since $D \subseteq k^\times$, in particular $(\#D, p) = 1$, so $H^1(T(D), k^{\text{SL}_n(W_m)}) = 0$ by Lemma 2.5. Moreover, by [6] Theorem 3.5, we have that $H^1(\text{SL}_n(W_m), k) = 0$, so we conclude that $H^1(\text{GL}_n^D(W_m), k) = 0$ from the inflation-restriction sequence.

(b) Consider the short exact sequence
$$0 \rightarrow \text{SL}_n(W_m) \rightarrow \text{GL}_n^D(W_m) \overset{\text{det}}{\rightarrow} T(D) \rightarrow 1.$$We have the inflation-restriction exact sequence
$$0 \rightarrow H^1(T(D), M_n^0(k)^{\text{SL}_n(W_m)}) \rightarrow H^1(\text{GL}_n^D(W_m), M_n^0(k)) \rightarrow H^1(\text{SL}_n(W_m), M_n^0(k))^{T(D)}.$$By [6] (Theorem 3.2 and Proposition 3.6), $H^1(\text{SL}_n(W_m), M_n^0(k)) = 0$. Since $(\#D, p) = 1$, by Lemma 2.5 we have that $H^1(T(D), M_n^0(k)^{\text{SL}_n(W_m)}) = 0$. Thus, from the inflation-restriction sequence, we have that $H^1(\text{GL}_n^D(W_m), M_n^0(k)) = 0$. □

Lemma 2.7. Let $k$ be a finite field of characteristic $p$ and cardinality at least $4$. Let $D \subseteq k^\times$ be a subgroup. Let $N \subseteq M$ be finite $\mathbb{F}_p[\text{GL}_n^D(W_m)]$-submodules of $M_n^0(k)^r$, for some $r \geq 1$. Then there is an injection
$$H^2(\text{GL}_n^D(W_m), N) \hookrightarrow H^2(\text{GL}_n^D(W_m), M), \quad \forall m \geq 1.$$Proof. Put $G := \text{GL}_n^D(W_m)$. Since taking cohomology is functorial, it is enough to prove that the map
$$H^2(G, M) \rightarrow H^2(G, M_n^0(k)^r)$$is injective for any $\mathbb{F}_p[G]$-submodule $M$ of $M_n^0(k)^r$. Put $Q := M_n^0(k)^r / M$, $C := \det(G) \subseteq T(k^\times)$ and $\#C = c$. To simplify notation, let $M_0$ denote $M_n^0(k)$. By Lemma 2.5, for any $\mathbb{F}_p$-vector space $V$, we have that $H^i(C, V) = 0$, for all $i \geq 1$. Thus, taking $C$-invariants is exact, and we have the following commutative diagram with exact rows and columns:
By [6] Theorem 3.1, we know that \( H^2(\text{SL}_n(W_m), M) \to H^2(\text{SL}_n(W_m), M^n_0(k)^r) \) is injective. This implies that (1) is surjective. Arrows (a) and (b) are isomorphisms, and with the surjectivity of (1) we also have that (3) is surjective. Finally, this implies that arrow (4) is injective too. \( \square \)

**Corollary 2.8.** Let \( k \) be a finite field of characteristic \( p \) and cardinality at least 4. Let \( M, N \) be two \( \mathbb{F}_p[\text{GL}_n^D(W_m)] \)-submodules of \( M^n_0(k)^r \) for some integer \( r \geq 1 \). Suppose that we are given \( x \in H^2(\text{GL}_n^D(W_m), M) \) and \( y \in H^2(\text{GL}_n^D(W_m), N) \) such that \( x \) and \( y \) represent the same cohomology class in \( H^2(\text{GL}_n^D(W_m), M^n_0(k)^r) \). Then there exists \( z \in H^2(\text{GL}_n^D(W_m), M \cap N) \) such that \( x = z \) in \( H^2(\text{GL}_n^D(W_m), M) \) and \( y = z \) in \( H^2(\text{GL}_n^D(W_m), N) \).

**Proof.** Consider the short exact sequence

\[
0 \to M \cap N \xrightarrow{m \mapsto m \cap m} M \oplus N \xrightarrow{m \mapsto m - m \cap m} M + N \to 0.
\]

For the sake of exposition, let us denote \( H^2(\text{GL}_n^D(W_B), X) \) only by \( H^2(X) \). By the above lemma, we obtain an exact sequence

\[
0 \to H^2(M \cap N) \xrightarrow{s} H^2(M) \oplus H^2(N) \xrightarrow{\beta} H^2(M + N).
\]

We have \( x \oplus y \in H^2(M) \oplus H^2(N) \). Since \( H^2(M + N) \to H^2(M^n_0(k)^r) \) is injective, it follows that \( x \oplus y \in \ker \beta \), and thus \( x \oplus y \in \text{Im}(\alpha) \). \( \square \)

Let \( \Gamma \) denote the kernel of the mod \( p^m \)-reduction map \( \text{GL}_n^D(W_{m+1}) \to \text{GL}_n^D(W_m) \). Then for any \( m \geq 1 \) we have

\[
\Gamma = \{ 1 + p^m M \mod p^{m+1} \mid M \in M^n_0(W(k)) \} \cong M^n_0(k),
\]

where \( \phi(1 + p^m M) := M \mod p \), and this isomorphism is compatible with the \( \text{GL}_n^D(W_m) \)-action by conjugation. Consider the subgroup \( Z := \{(1 + p^m \lambda)\text{id}_n \mid \lambda \in k\} \subseteq \Gamma \). If \( S \) denotes the subspace of scalar matrices in \( M^n_0(k) \) and we set \( V := M^n_0(k)/S \), we then have \( \phi(Z) = S \) and \( \phi(\Gamma/Z) = V \).

**Lemma 2.9.** Let \( k \) be a finite field of characteristic \( p \) and cardinality at least 4.

(a) Suppose that \( k \neq \mathbb{F}_4 \) if \( n = 3 \). Then the short exact sequence

\[
1 \to \Gamma \to \text{GL}_n^D(W_{m+1}) \to \text{GL}_n^D(W_m) \to 1
\]

does not split for any integer \( m \geq 1 \).

(b) If \( p \mid n \), then the short exact sequence

\[
1 \to \Gamma/Z \to \text{GL}_n^D(W_{m+1})/Z \to \text{GL}_n^D(W_m) \to 1
\]

does not split for any \( m \geq 1 \).

**Proof.** (a) Consider the following short exact sequences

\[
1 \to \Gamma \to \text{GL}_n^D(W_{m+1}) \xrightarrow{\pi} \text{GL}_n^D(W_m) \to 1
\]

\[
1 \to \Gamma \to \text{SL}_n(W_{m+1}) \to \text{SL}_n(W_m) \to 1.
\]

By [6] Proposition 3.7, we know that the second exact sequence does not split. Suppose that the first one does, and denote by \( s : \text{GL}_n^D(W_m) \to \text{GL}_n^D(W_{m+1}) \) a splitting\(^1\) such that \( \pi \circ s = \text{id} \). Let

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\(^1\)Given a short exact sequence of groups \( 1 \to M \to E \to G \to 1 \) we say that a mapping of sets \( s : G \to E \) is a *section* if \( p \circ s = \text{id}_G \). If the section is a group homomorphism we call it a *splitting*.  

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$g \in \text{SL}_n(W_\text{m}) \subseteq \text{GL}_n^D(W_\text{m})$. We have $1 = \det(g) = \det(\pi(s(g))) = \pi \det(s(g))$, so $\det(s(g)) = 1 + p^m \lambda$, for some $\lambda \in W_{m+1}$. But this contradicts the fact that $\det(s(g)) \in T(D) \subseteq T(k^\times)$ unless $\lambda \equiv 0 \mod p$. In this case we obtain that $\det(s(g)) = 1$, so $g \in \text{SL}_n(W_{m+1})$, which contradicts the non-splitting of the second exact sequence.

(b) Consider the following short exact sequences

$$
1 \rightarrow \Gamma/Z \rightarrow \text{GL}_n^D(W_{m+1})/Z \xrightarrow{\pi} \text{GL}_n^D(W_\text{m}) \rightarrow 1
$$

$$
1 \rightarrow \Gamma/Z \rightarrow \text{SL}_n(W_{m+1})/Z \rightarrow \text{SL}_n(W_\text{m}) \rightarrow 1.
$$

By [6] Corollary 3.11 we know that the second exact sequence does not split. The same argument of (a) works to show that the first exact sequence does not split.

From now on assume that we are given fields $k \subseteq k'$ of characteristic $p$. Let $\mathcal{C}$ be the category of complete local noetherian rings $(A, m_A)$ with residue field $A/m_A = k'$ and with morphisms that are the identity on $k'$. For an object $A$ in $\mathcal{C}$ we will write $W$ and $W_A$ for $W(k)$ and $W(k)_A$, respectively. The image-splitting theorem follows from the next result.

**Proposition 2.10.** Let $\pi : (A, m_A) \rightarrow (B, m_B)$ be a surjection of artinian local rings in $\mathcal{C}$ with $m_A \ker \pi = 0$ and $B \neq 0$. Fix a subgroup $D \subseteq k^\times$. Let $H$ be a subgroup of $\text{GL}_n^D(A)$ such that $\pi(H) = \text{GL}_n^D(W_B)$. Assume that $k$ satisfies Assumption [24]. Then there exists an element $u \in \text{GL}_n(A)$ such that

$$
\pi(u) = 1 \quad \text{and} \quad uHv^{-1} \supseteq \text{GL}_n^D(W_A).
$$

**Proof.** Consider the induced map $\pi : \text{GL}_n(A) \rightarrow \text{GL}_n(B)$. We have the following short exact sequence:

$$
0 \rightarrow M_n(\ker \pi) \xrightarrow{j} \pi^{-1}(\text{GL}_n^D(W_B)) \xrightarrow{\pi} \text{GL}_n^D(W_B) \rightarrow 1
$$

Let $G := \pi^{-1}(\text{GL}_n^D(W_B)) \cap \text{GL}_n^D(A)$, and consider the restriction $\pi|_G : G \rightarrow \text{GL}_n^D(W_B)$, which is still surjective because we are assuming that there is a subgroup $H \subseteq \text{GL}_n^D(A)$ such that $\pi(H) = \text{GL}_n^D(W_B)$. Then $\ker(\pi|_G) = \{v \in M_n(\ker \pi) \mid \det(1 + v) \in T(D)\}$, and for $v \in \ker \pi$ we have

$$
\det(1 + v) = \det \begin{pmatrix}
1 + v_{11} & \cdots & v_{1n} \\
\vdots & \ddots & \vdots \\
v_{n1} & \cdots & 1 + v_{nn}
\end{pmatrix} = 1 + (v_{11} + v_{22} + \ldots + v_{nn}) + \{\text{ things in } \ker(\pi)^2\}.
$$

Since $\ker(\pi) \subseteq m_A$, we have $\ker(\pi)^2 \subseteq m_A \cdot \ker(\pi) = 0$. So $\det(1 + v) = 1 + \sum_{i=1}^n v_{ii} = 1 + \text{tr}(v) \in T(D) \subseteq T(k^\times)$. Thus we have $\det(1 + v) \in (1 + m_A) \cap T(D) \subseteq (1 + m_A) \cap T(k^\times) = \{1\}$, so $\text{tr}(v) = 0$ and $\ker(\pi|_G) = M_n^0(\ker \pi)$, and we have a second short exact sequence:

$$
0 \rightarrow M_n(\ker \pi) \xrightarrow{j} \pi^{-1}(\text{GL}_n^D(W_B)) \xrightarrow{\pi} \text{GL}_n^D(W_B) \rightarrow 1
$$

$$
0 \rightarrow M_n^0(\ker \pi) \xrightarrow{j} G \xrightarrow{\pi} \text{GL}_n^D(W_B) \rightarrow 1
$$

For a subring $X \subseteq \text{GL}_n^D(A)$, let $M_n^0(X) := \{v \in M_n^0(\ker \pi) : j(v) \in X\}$. Then $\ker(\pi|_{\text{GL}_n^D(W_A)}) = M_n^0(\text{GL}_n^D(W_A))$ and we have the short exact sequence

$$
0 \rightarrow M_n^0(\text{GL}_n^D(W_A)) \xrightarrow{j} \text{GL}_n^D(W_A) \xrightarrow{\pi} \text{GL}_n^D(W_B) \rightarrow 1.
$$
Fix a section \( s : \text{GL}^D_n(W_B) \to \text{GL}^D_n(W_A) \subseteq G \) that sends the identity to the identity and let \( x : \text{GL}^D_n(W_B) \times \text{GL}^D_n(W_B) \to M^0_n(\text{GL}^D_n(W_A)) \) be the 2-cocycle representing the previous extension. Since in particular \( x \in H^2(\text{GL}^D_n(W_B), M_n(\ker \pi)) \), the section \( s \) and the cocycle \( x \) give an identification

\[
\varphi : \pi^{-1}(\text{GL}^D_n(W_B)) \xrightarrow{\sim} M_n(\ker \pi) \times_x \text{GL}^D_n(W_B) \quad \text{GL}^D_n(W_A) \hookrightarrow M^0_n(\text{GL}^D_n(W_A)) \times_x \text{GL}^D_n(W_B).
\]

We now want to apply Proposition 2.2 in [6]. We have the following commutative diagram:

\[
\begin{array}{ccc}
0 & \to & M^0_n(H) \to M^0_n(H) \times_x \text{GL}^D_n(W_B) \to \text{GL}^D_n(W_B) \to 1 \\
\| & & \| & \downarrow \varphi \\
0 & \to & M^0_n(H) \to \varphi(H) \to \text{GL}^D_n(W_B) \to 1.
\end{array}
\]

First suppose that \( p \nmid n \). By the assumptions on \( k \), we have that \( H^1(\text{GL}^D_n(W_B), M^0_n(k)) = 0 \) by Lemma 2.6. Thus we have \( H^1(\text{GL}^D_n(W_B), M^0_n(\ker \pi)) = H^1(\text{GL}^D_n(W_B), M^0_n(\ker \pi)) \) and consequently \( H^1(\text{GL}^D_n(W_B), M^0_n(k) \otimes_k \ker \pi) = 0 \). Furthermore, by Lemma 2.7, we have an injection

\[
H^2(\text{GL}^D_n(W_B), M^0_n(H)) \hookrightarrow H^2(\text{GL}^D_n(W_B), M^0_n(\ker \pi)).
\]

Hence we can apply Proposition 2.2 in [6] and conclude that

\[
M^0_n(H) \times_x GL^D_n(W_B) = v \varphi(H) v^{-1}
\]

for some \( (v, e) \in M^0_n(\ker \pi) \times_x \text{GL}^D_n(W_B) \). Take \( u \in G \subseteq \pi(\text{GL}^D_n(W_B)) \) such that \( \varphi(u) = v \). Then

\[
M^0_n(H) \times_x \text{GL}^D_n(W_B) = \varphi(uHv^{-1}), \quad \text{where} \quad u \in G \text{ such that } \pi(u) = 1.
\]

Now suppose that \( p \mid n \). In this case the role of \( M \) is played by \( M_n(\ker \pi) \). From the short exact sequence

\[
1 \to \text{SL}_n(W_B) \to \text{GL}_n^D(W_B) \to T(D) \to 1
\]

we obtain the following exact sequence

\[
H^1(T(D), M_n(\ker \pi)) \to H^1(\text{GL}^D_n(W_B), M_n(\ker \pi)) \to H^1(\text{SL}_n(W_B), M_n(\ker \pi)),
\]

with \( H^1(T(D), M_n(\ker \pi)) = 0 \) by Lemma 2.3 and \( H^1(\text{SL}_n(W_B), M_n(\ker \pi)) = 0 \) by [6], Proposition 4.2. Thus we also have that \( H^1(\text{GL}^D_n(W_B), M_n(\ker \pi)) = 0 \). On the other hand, since \( H^1(\text{GL}^D_n(W_B), \ker \pi) = 0 \) by Lemma 2.6 from the exact sequence \( 0 \to M^0_n(k) \to M_n(k) \to k \to 0 \) we get the following exact sequence:

\[
H^1(\text{GL}^D_n(W_B), M^0_n(k)) \to H^1(\text{GL}^D_n(W_B), M_n(k)) \to 0 \to
\]

\[
H^2(\text{GL}^D_n(W_B), M^0_n(k)) \to H^2(\text{GL}^D_n(W_B), M_n(k)).
\]

Using the injection \( H^2(\text{GL}^D_n(W_B), M^0_n(k)) \hookrightarrow H^2(\text{GL}^D_n(W_B), M_n(k)) \) from the previous exact sequence and the fact that \( M_n(\ker \pi) \simeq M_n(k) \otimes_k \ker \pi \), we have an injection

\[
H^2(\text{GL}^D_n(W_B), M^0_n(k)) \hookrightarrow H^2(\text{GL}^D_n(W_B), M_n(k)).
\]

By Lemma 2.7, we also have an injection

\[
H^2(\text{GL}^D_n(W_B), M^0_n(H)) \hookrightarrow H^2(\text{GL}^D_n(W_B), M^0_n(\ker \pi)),
\]

so we can conclude that

\[
H^2(\text{GL}^D_n(W_B), M^0_n(H)) \to H^2(\text{GL}^D_n(W_B), M_n(\ker \pi))
\]
is an injection. Hence, we can apply Proposition 2.2 from [6] and obtain
\[ M^0_n(H) \times_x \text{GL}^D_n(W_B) = \psi(H)v^{-1}, \]
for some \((v, e) \in M_n(\ker \pi) \times_x \text{GL}^D_n(W_B)\). Take \(u \in \pi^{-1}(\text{GL}^D_n(W_B))\) such that \(\varphi(u) = v\). Then
\[ M^0_n(H) \times_x \text{GL}^D_n(W_B) = \varphi(uHu^{-1}), \quad \text{where } u \in \pi^{-1}(\text{GL}^D_n(W_B)) \text{ with } \pi(u) = 1. \]

In both cases we have an element \(u \in \text{GL}_n(A)\) such that \(\varphi(uHu^{-1}) = M^0_n(H) \times_x \text{GL}^D_n(W_B)\) and \(\pi(u) = 1\).

**Claim:** \(M^0_n(\text{GL}^D_n(W_A)) \subseteq M^0_n(H)\).

Suppose the previous claim is true. Then
\[ \varphi(\text{GL}^D_n(W_A)) = M^0_n(\text{GL}^D_n(W_A)) \times_x \text{GL}^D_n(W_B) \subseteq M^0_n(H) \times_x \text{GL}^D_n(W_B) = \varphi(uHu^{-1}), \]
so \(\text{GL}^D_n(W_A) \subseteq uHu^{-1}\), and we are done.

**Proof of the claim:** If \(W_A \to W_B\) is an injection, then \(M^0_n(\text{GL}^D_n(W_A))\) is \((0)\) and the claim is true. Otherwise, we identify \(W_A\) with \(W_n\) for some \(n \geq 1\) and \(W_B\) with \(W_m\) for some \(1 \leq m \leq n\). Since we have a filtration \(\ldots \subseteq W_{m-1} \subseteq W_m \subseteq W_{m+1} \subseteq \ldots\), without loss of generality we may assume that \(n = m + 1\). This gives a natural identification of \(\Gamma := \ker \pi = M^0_n(\text{GL}^D_n(W_A))\) with \(M^0_n(k)\). We will use this identification in what follows.

Let \(x \in H^2(\text{GL}^D_n(W_B), M^0_n(k))\) represent the extension
\[ 0 \to M^0_n(k) \overset{j}{\to} \text{GL}^D_n(W_A) \to \text{GL}^D_n(W_B) \to 1, \tag{2} \]
and let \(y \in H^2(\text{GL}^D_n(W_B), M^0_n(H))\) represent the extension
\[ 0 \to M^0_n(H) \overset{j}{\to} H \to \text{GL}^D_n(W_B) \to 1. \]

We have a commutative diagram
\[
\begin{array}{ccc}
0 & \to & M^0_n(k) \\
\cap & \cap & \cap \\
0 & \to & M^0_n(\ker \pi) \\
\cup & \cup & \\
0 & \to & M^0_n(H) \\
\end{array}
\begin{array}{ccc}
\to & \text{GL}^D_n(W_A) & \overset{\pi}{\to} \text{GL}^D_n(W_B) & \to 1 \\
\| & \| & \| \\
\to & G & \overset{\pi}{\to} \text{GL}^D_n(W_B) & \to 1 \\
\end{array}
\begin{array}{ccc}
\to & H & \overset{\pi}{\to} \text{GL}^D_n(W_B) & \to 1 \\
\end{array}
\]
so \(x\) and \(y\) represent the same cohomology class in \(H^2(\text{GL}^D_n(W_B), M^0_n(\ker \pi))\), namely
\[ 0 \to M^0_n(\ker \pi) \to G \to \text{GL}^D_n(W_B) \to 1. \]

By Corollary 2.8 there exists \(z \in H^2(\text{GL}^D_n(W_B), M^0_n(k) \cap M^0_n(H))\) such that \(x\) and \(z\) (respectively, \(y\) and \(z\)) represent the same cohomology class in \(H^2(\text{GL}^D_n(W_B), M^0_n(k))\) (respectively, in \(H^2(\text{GL}^D_n(W_B), M^0_n(H))\)).

Suppose that \(M^0_n(k)\) is not contained in \(M^0_n(H)\). Then we have that \(M^0_n(k) \cap M^0_n(H) \subseteq S\) by Lemma 6.3. If \(M^0_n(k) \cap M^0_n(H) = (0)\), then \(x = 0\), which contradicts the non-splitting of the extension (2) (cf. Lemma 2.9). Thus \(M^0_n(k) \cap M^0_n(H) \subseteq S\) is a submodule of \(S\) different from \((0)\), so we must be in the situation where \(p \mid n\). Now the image of \(x\) in \(H^2(\text{GL}^D_n(W_B), M^0_n(k) \otimes S)\) represents the extension
\[ 0 \to M^0_n(k) \otimes S \overset{j}{\to} \text{GL}^D_n(W_{m+1})/\text{mod} p^m \to \text{GL}^D_n(W_m) \to 1. \]
(we use that \(Z \cong S\)). Since this is non-split by Lemma 2.9 and \(V = M^0_n(k) / S\), the image of \(x\) in \(H^2(\text{GL}^D_n(W_B), V)\) is not \(0\). This contradicts the fact that \(x\) is in the image of the map \(H^2(\text{GL}^D_n(W_B), S) \to H^2(\text{GL}^D_n(W_B), M^0_n(k))\).
Now the only thing that we are left to do is to see how the image-splitting theorem follows from Proposition 2.10. Suppose that $k$ is a finite field of characteristic $p$ that satisfies Assumption 2.4. Let $(A, m_A)$ be a local ring in $C$ and $\pi : A \to A/m_A$ denote the natural projection, and fix some subgroup $D \subseteq \text{GL}_n(A)$ such that $\pi(G) \supseteq \text{GL}_n^D(k)$. We want to see that then $G$ contains a conjugate of $\text{GL}_n^D(W_A)$. We may assume that $\pi(G) = \text{GL}_n^D(k)$ without loss of generality (indeed, consider $\overline{G} := G \cap \pi^{-1}(\text{GL}_n^D(k))$. Then $\pi(\overline{G}) = \text{GL}_n^D(k)$. If we see that $\overline{G}$ contains a conjugate of $\text{GL}_n^D(W_A)$, then this conjugate also lies in $G$). The next step is to see that actually we can assume that $G \subseteq \text{GL}_n^D(A)$. We will need the following two lemmas.

**Lemma 2.11.** The group $G \cap \text{GL}_n^D(A)$ is a closed normal subgroup of $G$ and the quotient group $G/(G \cap \text{GL}_n^D(A))$ is pro-$p$.

*Proof.* Consider the group homomorphism $G \to \text{det}(G) \to \text{det}(G)/(T(D) \cap \text{det}(G))$, whose kernel is $G \cap \text{GL}_n^D(A)$. By the first isomorphism theorem, we have an isomorphism

$$G/(G \cap \text{GL}_n^D(A)) \cong \text{det}(G)/(G \cap T(D)).$$

Let us show that $\text{det}(G)/(T(D) \cap \text{det}(G))$ is a pro-$p$ group. We have $D \equiv \text{det}(G) \mod m_A$, and if we consider this congruence multiplicatively, we have $D \equiv \text{det}(G) \mod 1 + m_A$. Since $T(D) \equiv D \mod 1 + m_A$, we have

$$T(D) \equiv \text{det}(G) \mod 1 + m_A \quad \Rightarrow \quad \text{det}(G) \subseteq (1 + m_A) \cdot T(D)$$

$$\Rightarrow \text{det}(G)/(T(D) \cap \text{det}(G)) \subseteq (1 + m_A)/(T(D) \cap (1 + m_A)) = 1 + m_A.$$

Since the 1-units are a pro-$p$ group, any subgroup of them will also be a pro-$p$ group and we are done. \[ \square \]

Since $G/(G \cap \text{GL}_n^D(A))$ is pro-$p$, this implies that $\pi(G/(G \cap \text{GL}_n^D(A)))$ is a $p$-group. Thus $\pi(G)/\pi(G \cap \text{GL}_n^D(A)) = \text{GL}_n^D(k)/\pi(G \cap \text{GL}_n^D(A))$ is a $p$-group, so $\pi(G \cap \text{GL}_n^D(A))$ is normal in $\text{GL}_n^D(k)$ and has index a power of $p$.

**Lemma 2.12.** Let $k$ be a finite field of cardinality at least 4. Let $J$ be a normal subgroup of $\text{GL}_n^D(k)$ not contained in the scalar matrices $S$ and with $\text{det}(J) = D$. Denote by $\overline{J}$ its image in $\text{PGL}_n^D(k)$. Then $\overline{J} = \text{PGL}_n^D(k)$.

*Proof.* Let $K := J \cap \text{SL}_n(k)$. We have the following diagram

$$\begin{array}{cccc}
K & \subseteq & J & \subseteq \text{GL}_n^D(k) & \subseteq \text{GL}_n(k) \\
\downarrow & & \downarrow & & \downarrow \\
\overline{K} & \subseteq & \overline{J} & \subseteq \text{PGL}_n^D(k) & \subseteq \text{PGL}_n(k) \\
\cap & & \cap & & \cap \\
\text{PSL}_n(k) & & & & \\
\end{array}$$

Since $J \subseteq \text{GL}_n^D(k)$ and $\text{SL}_n(k) \subseteq \text{GL}_n^D(k)$, we also have that $K \subseteq \text{SL}_n(k)$. Thus $\overline{K} \leq \text{PSL}_n(k)$. Now $\text{PSL}_n(k)$ is simple since $\#k \geq 4$, and since $K \not\subseteq k^\times$, we must have $\overline{K} = \text{PSL}_n(k)$. Thus we have

$$\text{PSL}_n(k) \subseteq \overline{J} \subseteq \text{PGL}_n^D(k) \subseteq \text{PGL}_n(k).$$

Since there is an isomorphism $\text{PGL}_n(k)/\text{PSL}_n(k) \cong k^\times/(k^\times)^n$ given by the determinant map $\text{det} : \text{PGL}_n \to k^\times/(k^\times)^n$, and since $\text{det}(J) = D$, we conclude that $\overline{J} = \text{PGL}_n^D(k)$. \[ \square \]
Let \( H := G \cap \text{GL}_n^D(A) \) and \( J := \pi(H) \subseteq \text{GL}_n^D(k) \). Since \( \pi(G) = \text{GL}_n^D(k) \), we have that \( J \) is not contained in \( S \). We thus can apply the previous lemma to \( J \) and we obtain that \( \mathcal{T} = \text{PGL}_n^D(k) \). Thus since \( J \not\subseteq k^\times \), we have \( J = \text{GL}_n^D(k) \). Finally, since \( \pi(H) = \text{GL}_n^D(k) \), if we prove that \( H \) contains a conjugate of \( \text{GL}_n^D(W_A) \), this will also lie in \( G \). So we can assume, without loss of generality, that \( G \subseteq \text{GL}_n^D(A) \).

Finally, suppose that the assumptions of Theorem 2.1 are satisfied. Then, without loss of generality, we can assume the following:

(T1) \( G \subseteq \text{GL}_n^D(A) \),

(T2) \( G \equiv \text{GL}_n^D(k) \bmod m_A \),

(T3) \( k \) satisfies Assumption 2.3.

In order to apply Proposition 2.10, first note that the ring \((A,m_A)\) is the projective limit of the artinian quotients \( A/m_i^A \), i.e. we have that \( A = \varprojlim A/m_i^A \). Fix \( i \geq 1 \) and let \((A_i,m_i) := (A/m_i^A,m_A/m_i^A)\). Consider the projection \( \pi_i : A_i \rightarrow A_{i-1} \). Let \( G_i := G/m_i^A \subseteq \text{GL}_n^D(A_i) \). To apply the proposition, the following two conditions have to be satisfied:

(P1) \( m_i \ker \pi_i = 0 \),

(P2) \( \pi_i(G_i) = \text{GL}_n^D(W_{A_{i-1}}) \).

Condition (P1) follows from the easy computation

\[
\ker \pi_i = \ker(A/m_i^A \rightarrow A/m_i^{i-1}) = m_i^{i-1}/m_i^A
\]

so, in particular, \( m_i \ker \pi_i = 0 \).

We will use induction to prove that if condition (P2) is satisfied by a certain conjugate of the group \( G_i \), for some \( i \geq 2 \), then it is also satisfied by a conjugate of \( G_{i+1} \). This will be enough to conclude the image-splitting theorem. For \( i = 2 \) we have \( \pi_2 : \text{GL}_n(A_2) \rightarrow \text{GL}_n(A_1) \) and

\[
\pi_2(G_2) = G_1 = G/m_A \overset{(T2)}{=} \text{GL}_n^D(k) = \text{GL}_n^D(W_A) .
\]

Thus we can apply Proposition 2.10 and obtain that there exists \( u_2 \in \text{GL}_n(A_2) \) such that \( \pi_2(u_2) = 1 \) and

\[
G_2 \supseteq u_2^{-1} \text{GL}_n^D(W_{A_2}) u_2 .
\]

We have \( \pi_3(G_3) = G_2 \supseteq u_2^{-1} \text{GL}_n^D(W_{A_2}) u_2 \Rightarrow \pi_3(w_3 G_3 w_3^{-1}) \supseteq \text{GL}_n^D(W_{A_2}) \), for some \( w_3 \in \text{GL}_n(A_3) \) with \( \pi_3(w_3) = u_2 \). Let \( H_3 := w_3 G_3 w_3^{-1} \). We can assume without loss of generality that \( \pi_3(H_3) = \text{GL}_n^D(W_{A_2}) \) (indeed, if we put \( \mathcal{T}_3 := H_3 \cap \pi_3^{-1}(\text{GL}_n^D(W_{A_2})) \), then \( \mathcal{T}_3 \subseteq \text{GL}_n^D(A_3) \) and \( \pi_3(\mathcal{T}_3) = \text{GL}_n^D(W_{A_2}) \)). So \( H_3 \) satisfies condition (P2). Let us continue with the induction.

Suppose that \( \pi_i(H_i) = \text{GL}_n^D(W_{A_{i-1}}) \) holds for some \( i \geq 3 \), where \( H_i := w_i G_i w_i^{-1} \) for some \( w_i \in \text{GL}_n(A_i) \) with \( \pi_4 \circ \pi_5 \circ \ldots \circ \pi_i(w_i) = w_3 \). Then by Proposition 2.10 there exists \( u_i \in \text{GL}_n(A_i) \) with \( \pi_i(u_i) \in 1 + m_i \) such that \( H_i \supseteq u_i^{-1} \text{GL}_n^D(W_{A_i}) u_i \), so

\[
\text{GL}_n^D(W_{A_i}) \subseteq u_i H_i u_i^{-1} = u_i w_i G_i w_i^{-1} u_i^{-1} = \pi_{i+1}(w_{i+1} G_{i+1} w_{i+1}^{-1}),
\]

for some \( w_{i+1} \in \text{GL}_n(A_{i+1}) \) with \( \pi_{i+1}(w_{i+1}) = u_i w_i \). Thus if we put \( H_{i+1} := w_{i+1} G_{i+1} w_{i+1}^{-1} \), we can assume (again without loss of generality) that \( \pi_{i+1}(H_{i+1}) = \text{GL}_n^D(W_{A_i}) \). Thus \( H_{i+1} \) satisfies condition (P2).

Finally, from this we obtain that \( G \) contains a conjugate of \( \text{GL}_n^D(W(k)_A) \) taking the projective limit over \( i \):

\[
G = \varprojlim_i G_i = \varprojlim_i w_i^{-1} H_i w_i \supseteq \varprojlim_i w_i^{-1} \text{GL}_n^D(W(k)_{A_i}) w_i = w^{-1} \text{GL}_n^D(W(k)_A) w,
\]

where \( w := \varprojlim_i w_i \in \text{GL}_n^D(A) \). This finishes the proof of Theorem 2.1.
Corollary 2.13. Let $k$ be a finite field of characteristic $p$ with cardinality at least 4, $k \neq \mathbb{F}_5$ if $n = 2$ and $k \neq \mathbb{F}_4$ if $n = 3$. Let $(A, m_A)$ be a finite-dimensional commutative local ring with residue field $k$ and $m_A^2 = 0$. Let $G \subseteq GL^D_n(A)$ be a subgroup. Suppose that $G \mod m_A = GL^D_n(k)$. Then there exists an $\mathbb{F}_p[GL^D_n(k)]$-submodule $M \subseteq M^0_n(m_A)$ such that $G$, up to conjugation by an element $u \in GL_n(A)$ with $\pi(u) = 1$, is a (non-twisted) semidirect product of the form

$$G \cong M \rtimes GL^D_n(k).$$

Proof. We are in the hypothesis of the image-splitting theorem, and since moreover we are assuming that $A$ is a $k$-algebra, we have that $W(k)_A = k$. So in this situation there exists $u \in GL_n(A)$ such that $\pi(u) = 1$ and $G \supseteq u^{-1}GL^D_n(k)u$. Let $G' := uGu^{-1} \subseteq GL^D_n(A)$. Denote by $\pi : A \to A/m_A \simeq k$ the natural projection and consider the following split short exact sequence:

$$0 \to M_n(m_A) \to GL_n(A) \xrightarrow{\pi} GL_n(k) \to 1$$

Take $m = (m_{ij})_{1 \leq i, j \leq n} \in M_n(m_A)$. An easy computation shows that $1 + m \in GL^D_n(A)$ if and only if $\text{tr}(m) = 0$:

$$\text{det}(1 + m) = \text{det} \begin{pmatrix} 1 + m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & 1 + m_{nn} \end{pmatrix} = 1 + \text{tr}(m) + m_A^2,$$

and $1 + \text{tr}(m) + m_A^2 = 1 + \text{tr}(m) \in k$ if and only if $\text{tr}(m) = 0$. This gives a split short exact sequence

$$0 \to M^0_n(m_A) \to GL^D_n(A) \xrightarrow{\pi} GL^D_n(k) \to 1$$

and we have a commutative diagram

$$
\begin{array}{ccc}
0 & \to & M_n^0(\ker \pi) \\
\cup & & \cup \\
0 & \to & M \\
\end{array} \xrightarrow{\pi} \begin{array}{ccc}
GL_n^D(A) & \to & GL_n^D(k) \\
\cup & & \cup \\
G' & \to & GL_n^D(k) \\
\end{array} \to 1.
$$

with $G' \supseteq GL^D_n(k)$ and $M \subseteq M_n^0(\ker \pi)$ a $GL^D_n(k)$-submodule. So in particular, the second short exact sequence splits and $G'$ is of the form $G' = M \rtimes GL^D_n(k)$. \qed

3 Explicit description of $\text{Im}(\rho)$

We will study continuous odd Galois representations

$$\rho : G_Q \to GL_2(\mathbb{T})$$

where $(\mathbb{T}, m_\mathbb{T})$ denotes a finite-dimensional local commutative algebra over a finite field $\mathbb{F}_q$ of characteristic $p$, equipped with the discrete topology and with $\mathbb{F}_q$ as residue field. We assume that $\mathbb{T}$ is generated by the traces of the image of $\rho$ and that $m_\mathbb{T}^2 = 0$.

We use the image-splitting theorem to determine, under the hypothesis that $\rho$ has big residual image, the image of $\rho$. More concretely, if $\overline{\rho} : G_Q \to GL_2(\mathbb{F}_p)$ denotes the corresponding residual Galois representation, we will assume that $\text{Im}(\overline{\rho}) = GL^D_2(\mathbb{F}_q)$, where $D = \text{Im}(\det(\overline{\rho})) \subseteq \mathbb{F}_q^\times$. Then we will show that the number $t$ of different traces in $\text{Im}(\rho)$ and the dimension $m$ of $m_\mathbb{T}$ determine uniquely, up to isomorphism, the group $\text{Im}(\rho)$. 

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Theorem 3.1. Let $\mathbb{F}_q$ denote a finite field of characteristic $p$ and $q = p^d$ elements, and suppose that $q \neq 2, 3, 5$. Let $(\mathbb{T}, m_\mathbb{T})$ be a finite-dimensional local commutative $\mathbb{F}_q$-algebra equipped with the discrete topology, and with residue field $\mathbb{T}/m_\mathbb{T} \simeq \mathbb{F}_q$. Suppose that $m_\mathbb{T}^2 = 0$. Let $\Gamma$ be a profinite group and let $\rho : \Gamma \to GL_2(\mathbb{T})$ be a continuous representation such that

(a) $\text{Im}(\rho) = GL_2^D(\mathbb{F}_q)$, where $\rho$ denotes the reduction $\rho \mod m_\mathbb{T}$ and $D := \text{Im} (\det \circ \rho)$,

(b) $\text{Im}(\rho) \subsetneq GL_2^D(\mathbb{T})$,

(c) $\mathbb{T}$ is generated as $\mathbb{F}_q$-algebra by the set of traces of $\rho$.

Let $m := \text{dim}_q m_\mathbb{T}$ and let $t$ be the number of different traces in $\text{Im}(\rho)$.

If $p \neq 2$, then $t = q^{m+1}$ and

$$\text{Im}(\rho) \simeq \left( \bigoplus_{0 \leq \alpha \leq m} M^0_2(\mathbb{F}_q) \right) \times GL_2^D(\mathbb{F}_q) \simeq GL_2^D(\mathbb{T}).$$

If $p = 2$, then $t = q^a \cdot ((q - 1)2^\beta + 1)$, for some $0 \leq \alpha \leq m$ and $0 \leq \beta \leq d(m - \alpha)$, and in this case $\text{Im}(\rho) \simeq M \times GL_2^D(\mathbb{F}_q)$, where $M$ is an $\mathbb{F}_2[GL_2^D(\mathbb{F}_q)]$-submodule of $M^0_2(m_\mathbb{T})$ of the form

$$M \simeq \bigoplus_{\alpha} M^0_2(\mathbb{F}_q) \oplus \bigoplus_{\beta} \bigoplus_{m} C_2,$$

where $C_2 \subseteq S$ is a subgroup of order 2 of the scalar matrices. Moreover, $M$ is determined uniquely by $t$ up to isomorphism.

Proof. By Cohen Structure Theorem (cf. [3], Theorem 7.7) and the assumption that $m_\mathbb{T}^2 = 0$, we have that $\mathbb{T} \simeq \mathbb{F}_q[X_1, \ldots, X_m]/(X_iX_j)_{1 \leq i, j \leq m}$. Let $\pi : \mathbb{T} \to \mathbb{T}/m_\mathbb{T} \simeq \mathbb{F}_q$. Then $\text{ker} \pi = m_\mathbb{T}$ is a $\mathbb{F}_q$-vector space of dimension $m$. Put $G := \text{Im}(\rho) \subseteq GL_2^D(\mathbb{T})$. We will separate the cases $p \neq 2$ and $p = 2$.

By the assumptions on $\mathbb{F}_q$ we know (after Corollary 2.13) that, if $p \neq 2$, then $G \simeq M \times GL_2^D(\mathbb{F}_q)$ for some $GL_2^D(\mathbb{F}_q)$-submodule $M \subseteq M^0_2(\ker \pi) \simeq \bigoplus_{m} M^0_2(\mathbb{F}_q)$. Since $p \neq 2$, we are in the case where $M^0_2(\mathbb{F}_q)$ is a simple $\mathbb{F}_p[GL_2(\mathbb{F}_q)]$-module, so by Lemma 6.1 we know that $M$ is isomorphic to a direct sum of, a priori, $\alpha \leq m$ copies of $M^0_2(\mathbb{F}_q)$. Let us count the number of traces of $G$ (depending on $\alpha$). Consider the following split exact sequence

$$0 \rightarrow \bigoplus_{m} M^0_2(\mathbb{F}_q) \rightarrow \bigoplus_{m} GL_2^D(\mathbb{F}_q) \rightarrow G \rightarrow 1$$

After a possible reordering of the variables $X_i$, for an element $\mu \in M \subseteq M^0_2(\ker \pi)$ we have $i(\mu) = 1 + A_1X_1 + \ldots + A_\alpha X_\alpha$, where $A_i \in M^0_2(\mathbb{F}_q)$ for $1 \leq i \leq \alpha$. Then for an element $g \in G$, we have $g = (1 + \mu)h$, with $\mu \in M$ and $h \in GL_2^D(\mathbb{F}_q)$, and

$$\text{tr}(g) = \text{tr}(1 + \mu)h = \text{tr}(h) + \text{tr}(A_1h)X_1 + \ldots + \text{tr}(A_\alpha h)X_\alpha.$$

Let $t$ denote the number of different traces in $G$. Then

$$t = \# \left\{ \text{tr}(h) + \sum_{i=1}^{\alpha} \text{tr}(A_ih)X_i : A_i \in M^0_2(\mathbb{F}_q), h \in GL_2^D(\mathbb{F}_q) \right\}$$

13
\[
= \# \{ \text{tr}(h) : h \in \GL^D_2(\F_q) \} \cdot \prod_{i=1}^\alpha \# \{ \text{tr}(A_i h) X_i : A_i \in \M^0_2(\F_q), h \in \GL^D_2(\F_q) \} \\
= q \cdot \prod_{i=1}^\alpha \# \{ 0, X_i, a X_i, \ldots, a^{\alpha-1} X_i \} = q^{\alpha+1},
\]
where \( a \) is some element in \( \F_q \) of order \( q - 1 \).

Finally, note that if we had \( t = q^{\alpha+1} \), with \( \alpha < m \), then we can assume, without loss of generality, that the set of traces of \( G \) is \( T := \{ a_0 + a_1 X_1 + \ldots + a_\alpha X_\alpha \mid a_0, \ldots, a_\alpha \in \F_q \} \) and by assumption, \( T = \langle T \rangle \). On the other hand, we have an isomorphism
\[
\GL^D_2(T) \cong (\M^0_2(\F_q) \oplus \ldots \oplus \M^0_2(\F_q)) \times \GL^D_2(\F_q)
\]
given by the previous split short exact sequence. In particular, we have that \( X_{\alpha+1} \in T \), so \( X_{\alpha+1} \) is a linear combination of \( X_1, \ldots, X_\alpha \), which is a contradiction.

Now suppose that \( p = 2 \). By the assumptions on \( \F_q \) we can use Corollary 2.14 and obtain that \( G \cong M \times \GL^D_2(\F_q) \) for some \( \F_2[\GL^D_2(\F_q)] \)-submodule \( M \subseteq \M^0_2(\F_q)^m \). We are in the case where \( \M^0_2(\F_q) \) is indecomposable, it contains the semisimple \( \F_2[\GL^D_2(\F_q)] \)-module \( S = \{ \lambda \text{Id} : \lambda \in \F_q \} \), and \( \M^0_2(\F_q)/S \) is simple (cf. Lemma 6.3). Thus we can apply Lemma 6.5 and we obtain that
\[
\begin{align*}
M &\cong \bigoplus_{\alpha} \M^0_2(\F_q) + \cdots + \M^0_2(\F_q) + \bigoplus_{\beta} C_2 + \cdots + C_2, & 0 \leq \alpha \leq m, & 0 \leq \beta \leq d(m - \alpha).
\end{align*}
\]
Consider the following split short exact sequence:
\[
0 \rightarrow \M^0_2(\ker \pi) \overset{\pi}{\rightarrow} \GL^D_2(T) \overset{\|}{\rightarrow} \GL^D_2(\F_q) \rightarrow 1, \quad \text{and} \quad \M^0_2(\F_q) \oplus \ldots \oplus \M^0_2(\F_q) \rightarrow \GL^D_2(\F_q[X_1, \ldots, X_m]/(X_i X_j)_{1 \leq i, j \leq m}) \rightarrow \GL^D_2(\F_q) \rightarrow 1.
\]
\[
\begin{align*}
0 &\rightarrow \M^0_2(\F_q) + \ldots + \M^0_2(\F_q) \quad \rightarrow \quad \GL^D_2(\F_q[X_1, \ldots, X_m]/(X_i X_j)_{1 \leq i, j \leq m}) \quad \rightarrow \quad \GL^D_2(\F_q) \quad \rightarrow \quad 1 \\
\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \ldots \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix} &\rightarrow & \begin{pmatrix} 0 & 1 \\ b_1 & a_1 \end{pmatrix} X_1 + \ldots + \begin{pmatrix} 0 & a_m \\ c_m & d_m \end{pmatrix} X_m & - & \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}
\end{align*}
\]
In this setting, after a possible reordering of the variables \( X_i \), for an element \( \mu \in M \) we have
\[
\iota(\mu) = 1 + \mu = 1 + A_1 X_1 + \ldots + A_\alpha X_\alpha + B_1 X_{\alpha+1} + \ldots + B_s X_m,
\]
where
\[
A_k \in N_k \cong \M^0_2(\F_q), \quad \text{for} \quad 1 \leq k \leq \alpha,
\]
\[
B_k = \begin{pmatrix} b_k & 0 \\ 0 & b_k \end{pmatrix} \in N_{\alpha+k} \cong C_2 + \ldots + C_2, \quad \text{for} \quad 1 \leq k \leq s = m - \alpha, \quad \text{and} \quad \sum_{k=1}^s e_k = \beta.
\]
We want to compute the number of different traces in \( G \) depending on the module \( M \). For an element \( g \in G \) we have \( g = (1 + \mu) h \) with \( \mu \in M \) and \( h \in \GL^D_2(\F_q) \), and
\[
\text{tr}(g) = \text{tr}(1 + \mu) h)
\]
= \text{tr}(h) + \text{tr}(A_1 h)X_1 + \ldots + \text{tr}(A_\alpha h)X_\alpha + \text{tr}(B_1 h)X_{\alpha+1} + \ldots + \text{tr}(B_s h)X_m \\
= \text{tr}(h) + \text{tr}(A_1 h)X_1 + \ldots + \text{tr}(A_\alpha h)X_\alpha + \text{tr}(h)b_1 X_{\alpha+1} + \ldots + \text{tr}(h)b_s X_m \\
= \text{tr}(A_1 h)X_1 + \ldots + \text{tr}(A_\alpha h)X_\alpha + \text{tr}(h)(1 + b_1 X_{\alpha+1} + \ldots + b_s X_m).

Let \( t \) denote the number of different traces in \( G \). Then

\[
\begin{align*}
t &= \# \left\{ \sum_{k=1}^\alpha \text{tr}(A_k h)X_k + \text{tr}(h)(1 + \sum_{k=1}^s b_k X_{\alpha+k}) : A_k \in M_2^0(\mathbb{F}_q), b_k \in C^{\varepsilon_k}, h \in GL_2^D(\mathbb{F}_q) \right\} \\
&= \# \left\{ \sum_{k=1}^\alpha a_k X_k + \text{tr}(h)(1 + \sum_{k=1}^s b_k X_{\alpha+k}) : a_k \in \mathbb{F}_q, b_k \in C^{\varepsilon_k}, h \in GL_2^D(\mathbb{F}_q) \right\} \\
&= \left\{ \sum_{k=1}^\alpha a_k X_k : a_k \in \mathbb{F}_q \right\} \cdot \# \left\{ \text{tr}(h)(1 + \sum_{k=1}^s b_k X_{\alpha+k}) : b_k \in C^{\varepsilon_k}, h \in GL_2^D(\mathbb{F}_q) \right\},
\end{align*}
\]

where \( t_1 = q^\alpha \) and

\[
\begin{align*}
t_2 &= \# \{ 0 \} + \# \{ x \cdot (1 + b_1 j_1 X_{\alpha+1} + \ldots + b_{\alpha j_s} X_m) : b_{\alpha j_k} \in C^{\varepsilon_k}, x \in \mathbb{F}_q^* \} \\
&= 1 + (q - 1) \cdot 2^{e_1} \cdot \ldots \cdot 2^{e_s} = 1 + (q - 1) \cdot 2^{\sum e_k} = 1 + (q - 1)2^\beta.
\end{align*}
\]

So we obtain the formula \( t = q^\alpha \cdot (1 + (q - 1)2^\beta) \).

Finally, let us prove that the number of traces determines the module \( M \) up to isomorphism. Let \( t' = q^{\alpha'} \cdot (1 + (q - 1)2^{\beta'}) \). Let us check that if \( t = t' \) then one has \( \alpha = \alpha' \) and \( \beta = \beta' \). Recall that \( q = 2^d \). Suppose that \( \alpha \geq \alpha' \). Then

\[
q^\alpha(1 + (q - 1)2^\beta) = q^{\alpha'}(1 + (q - 1)2^{\beta'}) \iff q^{\alpha - \alpha'}(1 + (q - 1)2^{\beta'}) = 1 + (q - 1)2^{\beta'}.
\]

If \( \beta' > 0 \) then \( (R) \equiv 1 \mod 2 \), so \( \alpha = \alpha' \) and then \( \beta = \beta' \). If \( \beta' = 0 \) then \( (R) = q \). Since \( 1 + (q - 1)2^\beta \geq q \) and \( q^{\alpha - \alpha'} \geq 1 \), we necessarily have \( \beta = 0 = \beta' \) and \( \alpha = \alpha' \).

\[ \square \]

**Remark 3.2.** Theorem 3.1 shows that if the field cut out by \( \mathfrak{p} \) admits some abelian extension (of the type we are considering), then it is a “big one”. So the cases \( m \geq 2 \) will occur rarely.

### 4 Application: Computation of images of Galois representations with values in mod \( p \) Hecke algebras

In order to apply Theorem 3.1 to Galois representations coming from modular forms, let us recall some notation. Fix a level \( N \geq 1 \), a weight \( k \geq 2 \), a prime \( p \nmid N \) and a Dirichlet character \( \varepsilon : (\mathbb{Z}/N)\times \rightarrow \mathbb{F}_p \). Let \( \mathbb{T} \) denote the Hecke algebra of \( S_k(N, \varepsilon; \mathbb{F}_q) \) generated by the **good** operators \( T_\ell \), i.e. those with \( \ell \nmid Np \). For a normalised Hecke eigenform \( f(z) = \sum_{n=0}^\infty a_n(f)q^n \in S_k(N, \varepsilon; \mathbb{F}_q) \), whose coefficients generate the field \( \mathbb{F}_q \), let \( m_f := \ker \lambda_f \) denote the maximal ideal of \( \mathbb{T} \) given by the ring homomorphism \( \lambda_f : \mathbb{T} \rightarrow \mathbb{F}_q \), \( T_n \mapsto a_n(f) \). Consider the local algebra \( \mathbb{T}_f := \mathbb{T}_{m_f} \), which we will refer to as the **local mod \( p \) Hecke algebra** associated to the mod \( p \) modular form \( f \). Let \( \rho_f : G_Q \rightarrow \text{GL}_2(\mathbb{T}_f) \) be the Galois representation attached to \( \mathbb{T}_f \) and let \( \mathfrak{p}_f : G_Q \rightarrow \text{GL}_2(\mathbb{F}_p) \) denote its reduction modulo the maximal ideal. Assume that \( \text{Im}(\mathfrak{p}_f) = \text{GL}_2^D(\mathbb{F}_p) \), where \( D = \text{Im}((\det \rho_f)) \). Let \( m := \dim_{\mathbb{F}_p} m_f \) and let \( t \) denote the number
of different traces in \( \text{Im}(\rho_f) \). Then using the previous theorem one can easily deduce that, if \( p \neq 2 \), then \( t = q^{m+1} \) and

\[
\text{Im}(\rho_f) \simeq (M_2^0(F_q) \oplus \ldots \oplus M_2^0(F_q)) \rtimes \text{GL}_2^D(F_q).
\]

If \( p = 2 \), then \( t = q^\alpha \cdot ((q - 1)2^\beta + 1) \), for some \( 0 \leq \alpha \leq m \) and \( 0 \leq \beta \leq d(m - \alpha) \), and in this case

\[
\text{Im}(\rho_f) \simeq (M_2^0(F_q) \oplus \ldots \oplus M_2^0(F_q) \oplus C_2 \oplus \ldots \oplus C_2) \rtimes \text{GL}_2^D(F_q),
\]

where \( C_2 \subseteq S \) is a subgroup of 2 elements.

As one can see, in characteristic 2 the result does not allow to completely determine the image of \( \rho_f \). Using the packages \texttt{HeckeAlgebras} and \texttt{ArtinAlgebras} (cf. \cite{8}) implemented in Magma, one can compute these algebras, and try to determine the image in concrete cases.

The idea is to compute, for a fixed level, weight and character, all local Hecke algebras on the corresponding space. For each one of them, after ensuring that the conditions of theorem 3.1 are satisfied, one just has to compute Hecke operators up to certain bound, and count the number of different operators that appear. If one computes far enough, this number can give us a candidate for the image of the corresponding Galois representation. We will see explicit examples of this procedure in the following section.

## 5 Examples in characteristic 2

In \cite{11} many examples for \( p = 2 \) are listed for \( m = 1, 2 \) and 3. In this setting, since there are several possibilities for the image of \( \rho_f \) (which we can distinguish by the number of different traces \( \tilde{t} \) computed), we can only “guess” the group that we obtain. However, we will see that the possible numbers of traces that can be obtained in each case are integers enough separated so that it is very likely that we can guess the right number of traces. Here we will summarise the results obtained in \cite{11}. We take \( 1 \leq N \leq 1500 \) and \( k = 2, 3 \).

### Examples with \( m = 1 \).

Here we have \( t = q^\alpha \cdot ((q - 1)2^\beta + 1) \), with \( 0 \leq \alpha \leq 1 \) and \( 0 \leq \beta \leq d(1 - \alpha) \). The possible numbers of traces in this setting are summarised in table 1.

| \( \alpha \) | \( \beta \) |
|-------------|-----------|
| 0           | 0 1 2 3   |
| 1           | 1 64 - -  |

Table 1: Possible number of traces when \( m = 1 \).

Let us describe a concrete example in full detail. The first example that we find with \( m = 1 \) and degree \( d = 2 \) is in level \( N = 67 \). In this case, by computing Hecke operators up to \( b = 1000 \), we find \( \tilde{t} = 7 \). As table 1 shows, we have the following possibilities for \( t \): 4, 7, 13 or 16. So we can only exclude \( t = 4 \). Let \( T^{(1)}, \ldots, T^{(7)} \) denote the seven operators that we find. If we take a closer look to them, we observe the following multiplicities for each one:

\[
21 \times T^{(1)}, \ 31 \times T^{(2)}, \ 15 \times T^{(3)}, \ 14 \times T^{(4)}, \ 39 \times T^{(5)}, \ 16 \times T^{(6)}, \ 30 \times T^{(7)}.
\]

So if the theoretical number of traces were \( t = 13 \) or 16, it would mean that there are still at least 6 Hecke operators left to appear, which seems highly unlikely.
Just to be even more confident that \( t = 7 \), if we compute up to \( b = 5000 \), we still find \( \tilde{t} = 7 \) and the following multiplicities:

\[ 58 \times T^{(1)}, 114 \times T^{(2)}, 69 \times T^{(3)}, 67 \times T^{(4)}, 185 \times T^{(5)}, 63 \times T^{(6)}, 111 \times T^{(7)}. \]

This means that the group \( G \) in this case would be \( G \simeq C_2 \rtimes SL_2(F_4) \).

**Examples with \( m = 2 \).**

Here we have \( t = q^\alpha \cdot ((q - 1)2^\beta + 1) \), with \( 0 \leq \alpha \leq 2 \) and \( 0 \leq \beta \leq d(2 - \alpha) \). The possible numbers of traces in this setting are summarised in table 2.

**Examples with \( m = 3 \).**

Here we have \( t = q^\alpha \cdot ((q - 1)2^\beta + 1) \), with \( 0 \leq \alpha \leq 3 \) and \( 0 \leq \beta \leq d(3 - \alpha) \). The possible numbers of traces in this setting are summarised in table 3.
In table 4 we summarise all the examples that we have computed. We separate the cases
by the dimension \( m \) of the maximal ideal \( m_f/m_f^2 \), the degree \( d \) of the finite field \( \mathbb{F}_q \), and the
weight \( k \). We run through levels \( 1 \leq N \leq 1500 \) and count the number of different examples that
we obtain depending on the number of different traces that we find. When we find 2 examples
that come form the same modular form (i.e. when we find two examples whose operators are
exactly the same) then we denote this in the table by \( 2 \times \).

| \( m \) | \( d \) | \( k \) | \( t \) | \( M \) | \( N = p_1 \) | \( N = p_1p_2 \) | \( N = p_1p_2p_3 \) |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 7 | \( C_2 \) | 58 | 2 \times 15 | |
| 3 | 2 | 15 | \( C_2 \) | 32 | 2 \times 9 | |
| 4 | 2 | 31 | \( C_2 \) | 42 | 2 \times 19 | |
| 2 | 2 | 2 | 13 | \( C_2 \oplus C_2 \) | 1 | 48 | 2 \times 7 |
| 3 | 2 | 29 | \( C_2 \oplus C_2 \) | 18 | 2 \times 3 | 2 \times 8 |
| 3 | 3 | 29 | \( C_2 \oplus C_2 \) | 7 | 2 \times 11 | |
| 4 | 2 | 61 | \( C_2 \oplus C_2 \) | 41 | 2 \times 2 | |
| 4 | 3 | 61 | \( C_2 \oplus C_2 \) | 1 | | |
| 2 | 2 | 3 | 25 | \( C_2 \oplus C_2 \oplus C_2 \) | 20 | 2 \times 8 | |
| 3 | 3 | 57 | \( C_2 \oplus C_2 \oplus C_2 \) | 29 | 2 \times 11 | |
| 4 | 3 | 121 | \( C_2 \oplus C_2 \oplus C_2 \) | 34 | 2 \times 7 | |
| 2 | 2 | 2 | 28 | \( M_2^0(\mathbb{F}_q) \oplus C_2 \) | 2 | | |
| 3 | 2 | 120 | \( M_2^0(\mathbb{F}_q) \oplus C_2 \) | 6 | | |
| 3 | 2 | 25 | \( C_2 \oplus C_2 \oplus C_2 \) | 6 | | |
| 2 | 3 | 25 | \( C_2 \oplus C_2 \oplus C_2 \) | 6 | | |
| 3 | 2 | 57 | \( C_2 \oplus C_2 \oplus C_2 \) | 15 | | |
| 4 | 2 | 121 | \( C_2 \oplus C_2 \oplus C_2 \) | 8 | | |
| 3 | 2 | 3 | 49 | \( C_2 \oplus C_2 \oplus C_2 \oplus C_2 \) | 22 | 2 \times 1 | |
| 3 | 3 | 113 | \( C_2 \oplus C_2 \oplus C_2 \oplus C_2 \) | 24 | | |
| 4 | 3 | 241 | \( C_2 \oplus C_2 \oplus C_2 \oplus C_2 \) | 26 | | |
| 3 | 2 | 2 | 52 | \( M_2^0(\mathbb{F}_q) \oplus C_2 \oplus C_2 \) | 5 | | |

Table 4: Number of examples found

After the previous tables, one is led to the following questions.

**Question 5.1.** If \( \dim_{\mathbb{F}_q} m_f/m_f^2 = 1 \), then \( G \simeq C_2 \times \text{SL}_2(\mathbb{F}_q) \).

**Question 5.2.** If \( \dim_{\mathbb{F}_q} m_f/m_f^2 = 2 \), then

\[
\text{Im}(\rho_f) \simeq \begin{cases}
(C_2 \oplus C_2) \times \text{SL}_2(\mathbb{F}_q), & \text{or} \\
(C_2 \oplus C_2 \oplus C_2) \times \text{SL}_2(\mathbb{F}_q), & \text{or} \\
(M_2^0(\mathbb{F}_q) \oplus C_2) \times \text{SL}_2(\mathbb{F}_q). & \text{or}
\end{cases}
\]

**Question 5.3.** If \( \dim_{\mathbb{F}_q} m_f/m_f^2 = 3 \), then

\[
\text{Im}(\rho_f) \simeq \begin{cases}
(C_2 \oplus C_2 \oplus C_2) \times \text{SL}_2(\mathbb{F}_q), & \text{or} \\
(C_2 \oplus C_2 \oplus C_2 \oplus C_2) \times \text{SL}_2(\mathbb{F}_q), & \text{or} \\
(M_2^0(\mathbb{F}_q) \oplus C_2 \oplus C_2) \times \text{SL}_2(\mathbb{F}_q). & \text{or}
\end{cases}
\]
6 Existence of big non-solvable extensions of number fields

In this section we use the previous results to predict the existence of some $p$-elementary abelian field extensions. Let $\mathbb{F}_q$ denote a finite field of characteristic $p$ and let $\rho : G_\mathbb{Q} \to \text{GL}_2(\mathbb{T})$ be a Galois representation that takes values in some finite-dimensional local commutative $\mathbb{F}_q$-algebra $\mathbb{T}$. Suppose that $\mathbb{T}$ is generated by the traces of $\rho$, and that $\rho$ has big residual image, i.e. $\text{Im}(\overline{\rho}) = \text{GL}_2^D(\mathbb{F}_q)$, where $D \subseteq \mathbb{F}_q^\times$. The explicit description that we give in section 3 of the image of $\text{Im}(\overline{\rho})$ allows us to compute a part of a certain ray class field of the field $K$ cut out by $\overline{\rho}$ (i.e. $G_K = \ker(\overline{\rho})$).

More concretely, let $m$ denote the maximal ideal of $\mathbb{T}$, assume that $m^2 = 0$ and let $m = \dim_{\mathbb{F}_q} m$. Let $\mathbb{G} := \text{Im}((\overline{\rho}))$, $G := \text{Im}(\rho)$ and $H := M^0_2(\mathbb{F}_q)^m$. By Theorem 3.1 we have that

$$G \simeq H \rtimes \mathbb{G}.$$  

This gives a short exact sequence $1 \to H \to G \to \mathbb{G} \to 1$, so $\mathbb{G}$ acts on $H$ by conjugation through preimages. Let $L := \mathbb{Q}^{\ker(\rho)}$ and $K := \mathbb{Q}^{\ker(\overline{\rho})}$. Then Galois theory tells us that we have field extensions

$$L \to K \to \mathbb{Q}$$

The group $H$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{3dm}$, where $q = p^d$. Thus it is a $p$-elementary abelian group, so $L/K$ is an abelian Galois extension, unramified outside $Np$ and of degree $p^{3dm}$. Moreover, since $H$ is a simple $\mathbb{F}_p[\text{GL}_2^D(\mathbb{F}_q)]$-module and the conjugation action of $\mathbb{G}$ on $H$ is nontrivial, we have that the extension $L/K$ cannot be defined over $\mathbb{Q}$.

Proposition 6.1. Let $\mathbb{F}_q$ be a finite field of characteristic $p \neq 2$. Assume the hypothesis of theorem 3.1 Then there exist number fields $L/K/\mathbb{Q}$ with $G_L = \ker(\rho)$ and $G_K = \ker(\overline{\rho})$ such that $\text{Gal}(K/\mathbb{Q}) = \text{GL}_2^D(\mathbb{F}_q)$ and

$$\text{Gal}(L/\mathbb{Q}) = \text{M}^0_2(\mathbb{F}_q) \oplus \ldots \oplus \text{M}^0_2(\mathbb{F}_q) \rtimes \text{Gal}(K/\mathbb{Q}),$$

with $\text{Gal}(K/\mathbb{Q})$ acting on $\text{Gal}(L/K)$ by conjugation. Moreover, the extension $L/K$ is an abelian extension of degree $p^{3dm}$ that cannot be defined over $\mathbb{Q}$ and which is unramified at all primes $\ell \mid pN$.

Proof. Let $\overline{G} := \text{Im}(\overline{\rho})$, $G := \text{Im}(\rho)$ and $H := M^0_2(\mathbb{F}_q)^m$. By Theorem 3.1 we have that $G \simeq H \rtimes \mathbb{G}$.

This gives a short exact sequence $1 \to H \to G \to \mathbb{G} \to 1$, so $\mathbb{G}$ acts on $H$ by conjugation through preimages. Let $L := \mathbb{Q}^{\ker(\rho)}$ and $K := \mathbb{Q}^{\ker(\overline{\rho})}$. Then Galois theory tells us that we have field extensions

$$L \to K \to \mathbb{Q}$$

The group $H$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{3dm}$, where $q = p^d$. Thus it is a $p$-elementary abelian group, so $L/K$ is an abelian Galois extension, unramified outside $Np$ and of degree $p^{3dm}$. Moreover, since $H$ is a simple $\mathbb{F}_p[\text{GL}_2^D(\mathbb{F}_q)]$-module and the conjugation action of $\mathbb{G}$ on $H$ is nontrivial, we have that the extension $L/K$ cannot be defined over $\mathbb{Q}$.

Proposition 6.2. Let $\mathbb{F}_q$ be a finite field of characteristic 2 and assume the hypothesis of theorem 3.1. Then there are integers $0 \leq \alpha \leq m$ and $0 \leq \beta \leq d(m - \alpha)$ such that $t = q^\alpha \cdot ((q - 1)2^\beta + 1)$, and there exists number fields $L/K/\mathbb{Q}$ with $G_L = \ker(\rho)$ and $G_K = \ker(\overline{\rho})$ such that $\text{Gal}(K/\mathbb{Q}) = \text{GL}_2^D(\mathbb{F}_q)$ and

$$\text{Gal}(L/\mathbb{Q}) \simeq M \rtimes \text{Gal}(K/\mathbb{Q}),$$

where $M$ is an $\mathbb{F}_2[\text{GL}_2^D(\mathbb{F}_q)]$-submodule of $\text{M}^0_2(m_2)$ of the form $M := M^0_2(\mathbb{F}_q) \alpha \oplus G_2^\beta$, and the group $\text{Gal}(K/\mathbb{Q})$ acts by conjugation on $\text{Gal}(L/K)$. The extension $L/K$ is an abelian extension of degree $2^{3dm+d}$ unramified at all primes $\ell \nmid 2N$.  

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Moreover, if $\alpha \neq 0$ there is an extension $L_1$ of $K$ with $\text{Gal}(L_1/K) \simeq M^0_n(F_q)^{\alpha}$ which cannot be defined over $\mathbb{Q}$ and an extension $L_2$ of $K$ with $\text{Gal}(L_2/K) = C^\beta_2$, such that $L = L_1L_2$. 

**Proof.** Let $\overline{\rho} := \text{Im}(\rho)$, $G := \text{Im}(\rho)$ and $H := M^0_2(F_q)^{\alpha} \oplus C^\beta_2 \simeq C^{eta_2 + \alpha}_2$. By Proposition 3.1 we have that $G \simeq H \rtimes \overline{\rho}$. This gives a short exact sequence $1 \to H \to G \to \overline{G} \to 1$, so $\overline{G}$ acts on $H$ by conjugation through preimages.

Let $L := \overline{Q}^{\ker(\rho)}$ and $K := \overline{Q}^{\ker(\overline{\rho})}$. Let $H_1 := M^0_2(F_q)^{\alpha}$, $H_2 := C^\beta_2$, $L_1 := \overline{Q}^{H_1}$ and $L_2 := \overline{Q}^{H_2}$. By Galois theory, we have field extensions

![Diagram](image)

The group $H$ is an abelian group of order $2^{3\alpha + \beta}$, and the corresponding field extension is unramified outside $2N$ because $\rho$ is. Moreover, since $H_2$ is contained in the centre of $G$, the action of $\overline{G}$ on $H_2$ is trivial, so $L_1 = KE$, and the Galois group of the field extension $L_1/K$ is isomorphic to the Galois group of $E/Q$. On the other hand, $H_2$ is not contained in the centre of $G$, so the action of $\overline{G}$ on $H_2$ is nontrivial, and the extension $L_2/K$ cannot be defined over $\mathbb{Q}$. \qed

**Appendix: Auxiliary lemmas**

**Lemma 6.3.** Assume that $k \neq \mathbb{F}_2$ if $n = 2$. Let $M \subseteq M^0_n(k)$ be an $\mathbb{F}_p[\text{GL}_n^D(k)]$-submodule for the conjugation action. Then, either $M$ is a subspace of $S$ over $\mathbb{F}_p$ or $M = M^0_n(k)$. Thus $M_n^0(k)/S$ is a simple $\mathbb{F}_p[\text{GL}_n^D(k)]$-module, and the sequence

$$0 \to S \to M_n^0(k) \to \bigvee \to 0$$

does not split when $p \mid n$.

**Proof.** Since any $\mathbb{F}_p[\text{GL}_n^D(k)]$-submodule of $M_n^0(k)$ is also an $\mathbb{F}_p[\text{SL}_n(k)]$-submodule, we can apply Lemma 3.3 of [3]. \qed

**Lemma 6.4.** Let $R$ be a ring and $M$ a semisimple left $R$-module which decomposes as $M = M_1 \oplus \ldots \oplus M_t$, with $M_i \subseteq M$ simple modules. Let $N \subseteq M$ be a submodule of $M$. Then $N$ is semisimple and is isomorphic to a direct sum of a subset of the modules $M_1, \ldots, M_t$.

**Proof.** By Proposition 3.12 in [4], for every submodule $N \subseteq M$ there exists a submodule $N' \subseteq M$ such that $N = N \oplus N'$. Now by Corollary 14.6 in [3] we have that, in this case, $N$ is isomorphic to a direct sum of a subset of the modules $M_1, \ldots, M_t$. \qed

Let $\mathbb{F}_q$ denote a finite field of characteristic 2, and let $(\mathbb{T}, m_\mathbb{T})$ be a finite-dimensional local commutative $\mathbb{F}_q$-algebra equipped with the discrete topology, and with residue field $\mathbb{T}/m_\mathbb{T} \simeq \mathbb{F}_q$. In this section we will prove the analogous result of the previous section in characteristic 2. We
Note that we obtain an invertible $k$-scalar, so we can write:

$$\text{End}_k(G) \cong GL_n(F_q).$$

As we will see, the case of characteristic 2 is more complicated, due to the fact that the module $M^0_2(F_q)$ of trace-0 matrices is no longer a simple module. Thus more work needs to be done. We start with a technical lemma that will be needed afterwards.

**Lemma 6.5.** Let $k$ be a field with trivial Brauer group, $G$ a finite group and $M$ an indecomposable $k[G]$-module. Suppose that $M$ has a semisimple submodule $S \subseteq M$ such that $M/S$ is simple, and such that all the other submodules of $M$ are contained in $S$. Consider a submodule $N \subseteq \bigoplus_{i=1}^n M$. Then $N \cong N_1 \oplus \ldots \oplus N_n$, with $N_i \subseteq M$ submodule.

**Proof.** For any module $M$, denote by $M^n$ the direct sum $\bigoplus_{i=1}^n M$. Let $T$ denote the simple module $M/S$ and consider the projection $\pi : M^n \to T^n$, which gives a commutative diagram

$$\begin{array}{ccc}
0 & \to & S^n \\
\uparrow & & \uparrow \iota \\
0 & \to & N \cap S^n \\
\uparrow & & \uparrow \iota \\
0 & \to & N \\
\uparrow & & \uparrow \pi(N) \\
0 & \to & \pi(N) \\
\end{array}$$

Note that, since $T^n$ is a semisimple module, by Lemma 6.4 we have that $\pi(N) \cong T^k$, for some $0 \leq k \leq n$. Let $\alpha : T^k \to \pi(N)$ denote this isomorphism. We will see that we can assume that $\pi(N)$ maps onto the first $k$ copies of $T$ (after isomorphism). Consider the composition

$$\varphi_{ij} : T \ni \iota_j \mapsto \iota_j \pi(N) \ni \iota_j T^n \ni \iota_j T,$$

where $\iota_j$ denotes the natural inclusion of $T$ into the $j$-th component of $T^k$, for $1 \leq j \leq k$, and $\pi_i$ denotes the natural projection from $T^n$ to the $i$th component $T$, for $1 \leq i \leq n$. Now, since $\varphi_{ij}$ is an endomorphism of a simple $k[G]$-module, by Schur’s lemma (cf. [4], 3.17) we have that $\text{End}_k(G)(M)$ is a division ring. Since we are assuming that the Brauer group of $k$ is trivial, this gives that $\text{End}_k(M) = k$, so $\text{End}_k(G)(M) \subseteq k$. Thus we have that $\varphi_{ij}$ is just multiplication by a scalar, so we can write $\varphi_{ij}(t) = \alpha_{ij} t$, for $\alpha_{ij} \in k$, $t \in T$. We then can express $\varphi := \iota \circ \alpha$ as

$$\varphi : \begin{pmatrix} T^k \\
\iota_1 \\
\iota_2 \\
\vdots \\
\iota_k \end{pmatrix} \ni \begin{pmatrix} a_{11} & \ldots & a_{1k} \\
a_{21} & \ldots & a_{2k} \\
\vdots & \ddots & \vdots \\
a_{n1} & \ldots & a_{nk} \end{pmatrix} \mapsto \begin{pmatrix} t_1 \\
t_2 \\
\vdots \\
t_k \end{pmatrix}.$$

Since $\varphi$ is injective, the matrix $A := \begin{pmatrix} a_{11} & \ldots & a_{1k} \\
\vdots & \ddots & \vdots \\
a_{n1} & \ldots & a_{nk} \end{pmatrix}$ has rank $k$. Using Gauß elimination, we can find an invertible $n \times n$ matrix $C$ such that

$$CA = \begin{pmatrix} 1 & 0 & \ldots & 0 \\
0 & 1 & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 \end{pmatrix}.$$

Note that $-C : M^n \to M^n$ and $-C : T^n \to T^n$ are isomorphisms of $k[G]$-modules. By construction we obtain

$$C \cdot \iota(\pi(N)) = C \cdot \iota \cdot \alpha(T^k) = C \cdot \varphi(T^k) = T \oplus \ldots \oplus T \oplus 0 \oplus \ldots \oplus 0.$$
So, without loss of generality, we can assume that
\[ \pi(N) = T \oplus \ldots T \oplus 0 \oplus \ldots 0 \subseteq T^n. \]

Let \( T_i \) denote the \( i \)th component of \( T \) in \( T^n \). Consider the submodule \( N \cap M_i \subseteq M \), for \( 1 \leq i \leq n \). For \( 1 \leq i \leq k \), we have that \( \pi(N) \cap T_i = T \), which implies that \( S \subseteq N \cap M_i \), and by the assumptions on \( M \), we have \( N \cap M_i = M \). For \( k + 1 \leq i \leq n \), we have that \( \pi(N) \cap T_i = 0 \), and thus \( N \cap M_i \subseteq S \).

If we put \( S = S_1 \oplus \ldots \oplus S_t \), with \( S_i \subseteq S \) simple modules for \( 1 \leq i \leq t \), we obtain
\[ N \cong M \oplus \ldots M \oplus L, \]
where \( L \subseteq S^l \), for some \( 0 \leq l \leq n - k \), is isomorphic to a direct sum of a subset of the modules \( S_1, \ldots, S_t \) (after Lemma 6.3). \[ \square \]

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L. Amorós, Faculté des Sciences, de la Technologie et de la Communication. 6 rue Richard Coudenhove-Kalergi, L-1359 Luxembourg

E-mail address laia.amoros@uni.lu