SOME ALGEBRAIC AND GEOMETRIC PROPERTIES OF
HYPER-KÄHLER FOURFOLD SYMMETRIES

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Abstract. We complete the classification of order 5 nonsymplectic automor-
phisms on hyper-Kähler fourfolds deformation equivalent to the Hilbert square
of a K3 surface. We then compute the topological Lefschetz number of natural
automorphisms of generalized Kummer fourfolds and we describe the geometry
of their fix loci.

1. Introduction

Hyper-Kähler manifolds, or equivalently irreducible holomorphic symplectic (IHS)
manifolds, are traditionally labelled by their deformation type, which encodes most
of the significant features of the manifold. Currently, four deformation types have
been exhibited, named after a representative of each class: the Hilbert scheme of
$n$ points on a K3 surface, for any $n \geq 1$; the $n$-th generalized Kummer variety
of an abelian surface, for any $n \geq 2$; the O’Grady variety OG6; the O’Grady variety
OG10. The present paper focuses on hyper-Kähler fourfolds of Hilbert and of Kum-
mer type, and the topic of interest concerns the classification of the biholomorphic
automorphisms of these manifolds.

As for K3 surfaces, the approach for the classification of automorphisms of IHS
manifolds is twofold: first a lattice-theoretical classification of the invariant sub-
lattice and of its orthogonal complement inside the second cohomology space with
integer coefficients, endowed with the Beauville–Bogomolov–Fujiki quadratic form;
then a description of the fix locus of the automorphism which, whenever it is
nonempty, is smooth and has usually several connected components of different
dimensions.

In the last years, many authors have contributed to the classification of prime
order automorphisms on the two families of IHS manifolds in issue here. We refer
to [6, §1],[14, §1] and references therein for a more detailed introduction of the
contribution of each author. So far the classification of order 5 nonsymplectic
automorphisms on the family of the Hilbert square of a K3 surface had still remained
incomplete because of the lack of an algebraic ingredient, which is the first main
result of this note (we refer to §2 for the definition of $S_\varphi$ and $m_\varphi$):

Theorem (Theorem 2.2). Let $L$ be an integral lattice and $\varphi \in O(L)$ an isometry
of prime order $p$. The integer $p^{m_\varphi} \mathrm{disc}(S_\varphi)$ is a square.
Using this result, in Theorem 3.1 we finish the classification of order 5 non-symplectic automorphisms on the deformation class of the Hilbert square of a K3 surface.

In the second part of this note, we describe the fix locus of natural automorphisms on generalized Kummer fourfolds, which are the automorphisms coming from the underlying abelian surface. We first give in Proposition 4.1 a generating formula for the topological Lefschetz number of these automorphisms. A similar formula already appeared in [8] but the formula given there is not correct when the automorphism contains a nontrivial translation. We correct the formula here and we give a complete proof, which requires substantial work to take care of the missing factor.

As an application, we discuss in Section 4.3 the fix loci of natural automorphisms on generalized Kummer fourfolds, whose action on cohomology has prime order.

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2. Prime order isometries of integral lattices

A lattice $L$ is a free $\mathbb{Z}$-module equipped with a nondegenerate symmetric bilinear form $(\cdot, \cdot)_L$ with integer values. Its dual lattice is $L^\vee := \text{Hom}_\mathbb{Z}(L, \mathbb{Z})$. Clearly $L$ is a sublattice of $L^\vee$ of equal rank, so the discriminant group $D_L := L^\vee/L$ is a finite abelian group, whose order $\text{discr}(L)$ is called the discriminant of $L$. The lattice $L$ is called unimodular if $\text{discr}(L) = 1$. A sublattice $M \subset L$ is called primitive if the quotient $L/M$ is a free $\mathbb{Z}$-module; it is called $p$-elementary for some prime number $p$ if $D_M \cong (\mathbb{Z}/p\mathbb{Z})^\oplus a$ for some positive integer $a$.

Let $\varphi \in O(L)$ be an isometry of $L$, of prime order $p$. The invariant subspace:

$$T_\varphi := \ker(\varphi - \text{id}_L),$$

is a primitive sublattice of $L$, since the restriction of the bilinear form to $T_\varphi$ is nondegenerate. Its orthogonal complement:

$$S_\varphi := T_\varphi^\perp,$$

is also a primitive sublattice of $L$. Denote by $\xi_p$ a primitive $p$-th root of unity, by $K := \mathbb{Q}(\xi_p)$ the $p$-th cyclotomic field and by $\mathcal{O}_K = \mathbb{Z}[\xi_p]$ its ring of algebraic integers. It is easy to check that $S_\varphi = \ker(\Phi_p(\varphi))$, where $\Phi_p$ is the $p$-th cyclotomic polynomial, so $S_\varphi \otimes \mathbb{Z} \mathbb{Q}$ is endowed with the structure of a finite-dimensional $K$-vector space. As a consequence, there exists a nonnegative integer $m_\varphi$ such that:

$$\text{rank}_\mathbb{Z} S_\varphi = \dim_\mathbb{Q} S_\varphi \otimes \mathbb{Q} = (p - 1)m_\varphi.$$

Proposition 2.1. Let $L$ be an integral lattice and $\varphi \in O(L)$ an isometry of prime order $p$. There exists a nonnegative integer $a_\varphi$ such that $\frac{L}{T_\varphi \oplus S_\varphi} \cong \bigoplus \frac{\mathbb{Z}}{(p^a \mathbb{Z})^\oplus a_\varphi}$ and we have $a_\varphi \leq m_\varphi$.

It is not difficult to see that $\frac{L}{T_\varphi \oplus S_\varphi}$ is a $p$-torsion module (see [5, Lemma 3.1], [20, Lemme 2.9]). The property $a_\varphi \leq m_\varphi$ was first proved in [1, Theorem 2.1(c)] in the context of isometries of K3 lattices and then in [6, Corollary 3.7] in the context of isometries of hyper-Kähler manifolds of K3$^{[2]}$-type, but only for $p \notin \{5, 23\}$. The above statement is a very useful generalization of these properties, which
first appeared in the third author’s PhD thesis [20, Théorème 2.18], see also [14, Lemma 8.1].

Proof. Denote by $G \subset O(L)$ the group generated by $\varphi$. By the decomposition theorem of Diederichsen–Reiner [12, Theorem 74.3], the $\mathbb{Z}[G]$-module $L$ decomposes in a direct sum of $\mathbb{Z}[G]$-modules:

$$L \cong \bigoplus_{i=1}^{s} (A_i, a_i) \oplus \bigoplus_{i=s+1}^{s+t} A_i \oplus \mathbb{Z}^{\oplus u},$$

where $A_i \subset K$ is a fractional ideal, i.e. there exists $\alpha_i \in \mathcal{O}_K$ such that $\alpha_i A_i$ is an ideal of $\mathcal{O}_K$, for $i = 1, \ldots, s + t$. The $G$-action on each summand is as follows:

- $\mathbb{Z}^{\oplus u}$ is a trivial $G$-module;
- $\varphi$ acts on each summand $A_i$ by multiplication by $\xi_p$;
- $(A_i, a_i)$ is isomorphic to $A_i \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module, $a_i \in A_i$ and $a_i \not\in (\xi_p - 1)A_i$, and $\varphi \cdot (x, k) = (\xi_p x + a_i k, k)$ for any $(x, k) \in A_i \oplus \mathbb{Z}$.

It follows that the sublattices $T_{\varphi}$ and $S_{\varphi}$ of $L$ are the direct summands of their respective counterparts on each summand of the above decomposition of $L$. Clearly $\mathbb{Z}^{\oplus u} \subset T_{\varphi}$, $A_i \subset S_{\varphi}$ and $S_{\varphi,i} := (A_i, a_i) \cap S_{\varphi} = A_i \oplus \{0\}$ for each $i$. To compute $T_{\varphi,i} := (A_i, a_i) \cap T_{\varphi}$ we proceed as follows. Since $\frac{p}{\xi_p - 1} = \sum_{j=1}^{p-1} j \xi_p^j$, we have that $\frac{1}{\xi_p - 1} a_i \in A_i$ and we compute that $y_i := (\frac{p}{\xi_p - 1} a_i, 1) \in T_{\varphi,i}$. Observe that $(A_i, a_i)$ is not isomorphic to $S_{\varphi,i} \oplus T_{\varphi,i}$. Otherwise we would have a decomposition $(0, 1) = (x, 0) + (-x, 1)$ with $(-x, 1) \in T_{\varphi,i}$, but then:

$$(0, 0) = (\varphi - \text{id})(-x, 1) = ((1 - \xi_p)x + a_i, 0),$$

which is not possible since $a_i \not\in (\xi_p - 1)A_i$. It follows that $\frac{(A_i, a_i)}{S_{\varphi,i} \oplus T_{\varphi,i}} \cong \mathbb{Z}/p\mathbb{Z}$ and $T_{\varphi,i} = \mathbb{Z}y_i$. Summing up, we get:

$$\frac{L}{S_{\varphi} \oplus T_{\varphi}} \cong \bigoplus_{i=1}^{s} (A_i, a_i) \oplus \mathbb{Z}^{\oplus u} \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus s}.$$

We thus put $a_{\varphi} := s$, and since $S_{\varphi} \cong \bigoplus_{i=1}^{s+t} A_i$, we have $m_{\varphi} = \frac{\text{rank } S_{\varphi}}{p-1} = s + t \geq a_{\varphi}$. \hfill $\square$

**Theorem 2.2.** Let $L$ be an integral lattice and $\varphi \in O(L)$ an isometry of prime order $p$. The integer $p^{m_{\varphi}} \text{disc}(S_{\varphi})$ is a square.

This result first appeared in this context in [5, Equation (2)], in the special case $m_{\varphi} = 1$, and was generalized in the third author’s PhD thesis [20, Théorème 2.23].

Proof. As observed during the proof of Proposition 2.1, there is a $\varphi$-equivariant decomposition $S_{\varphi} \cong \bigoplus_{i=1}^{m_{\varphi}} A_i$, where $A_i \subset K$ are fractional ideals on which $\varphi$ acts by multiplication by $\xi_p$. This induces a $\varphi$-equivariant isomorphism:

$$S_{\varphi} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{i=1}^{m_{\varphi}} \mathbb{Q}(\xi_p) =: E,$$

which endows the $\mathbb{Q}$-vector space $E$ with a rational valued, nondegenerate, symmetric bilinear form $(-, -)_E$ for which the endomorphism $\psi$ of $E$ acting by multiplication by $\xi_p$ on each summand is an isometry. The statement is clearly equivalent
to proving that the Gram matrix $B$ of $\langle -, - \rangle_E$, computed in any $\mathbb{Q}$-basis of $E$, has the property that $p^{m_\varphi} \cdot |\det(B)|$ is a rational square. We proceed by induction on $m_\varphi \geq 1$. The case $m_\varphi = 1$ follows from results of Bayer–Fliessiger [3], see [5, Equation (2)]. Assuming $m_\varphi \geq 2$, we denote $e^i_j := \xi^i_j$ for $i = 1, \ldots, m_\varphi$ and $j = 0, \ldots, p - 2$. For each $i$, the family $(e^i_j)_j$ is a basis of the $i$-th summand $E_i$ of $E$. In this basis, the linear map $\psi$ has matrix $M_{m_\varphi} := \text{diag}(C_p, \ldots, C_p)$ where $C_p$ is the companion matrix of the cyclotomic polynomial $\Phi_p$. Denote by $B$ the Gram matrix of $\langle -, - \rangle_E$, decomposed into square blocks $B_{i,j}$ corresponding to the restrictions of the bilinear form to $E_i \times E_j$:

$$
(B_{i,j})_{k,\ell} = \langle e^k_i, e^\ell_j \rangle_E = B_{(p-1)(i-1) + k, (p-1)(j-1) + \ell} \quad \forall k, \ell \in \{0, \ldots, p - 2\}.
$$

Note that $B_{i,i}$ is symmetric and that $B^i_{i,j} = B_{j,i}$ for all indices $i, j \in \{1, \ldots, m_\varphi\}$. Observe that for each indices $i, j$, the block $B_{i,j}$ is either invertible or zero. Indeed, in case $B_{i,j}$ is not invertible, there exists a nonzero element $y_j \in E_j$ such that $B_{i,j}y_j = 0$. Then for every $x_i \in E_i$, we have $\langle x_i, y_j \rangle_E = 0$, and since $\psi$ is an isometry we obtain:

$$
0 = \langle \psi^{-k}(x_i), y_j \rangle_E = \langle x_i, \xi^k_j \rangle_E \quad \forall k \in \{0, \ldots, p - 2\},
$$

so $B_{i,j}(\xi^k_j) = 0$ for all $k$. But the family $(\xi^k_j)_{j}$ is a basis of $E_j$, so $B_{i,j} = 0$. We consider three cases.

**First case:** at least one diagonal block of $B$ is nonzero. Without loss of generality, we may assume that $B_{1,1} \neq 0$. Write $B = \begin{pmatrix} B_{1,1} & V \\ V^\top & W \end{pmatrix}$ and denote:

$$
T := W - V^\top B_{1,1}^{-1} V,
$$

the Schur complement of $B_{1,1}$ in $B$. Then $\det(B) = \det(B_{1,1}) \cdot \det(T)$. Since $\varphi$ restricts to an isometry of $E_1$, we know that $p \cdot |\det(B_{1,1})|$ is a square. Since $\varphi$ is an isometry of $E$, using the matrix relation $M_{m_\varphi}^\top B M_{m_\varphi} = B$ we get:

$$
C_p^\top B_{1,1} C_p = B_{1,1}, \quad C_p^\top V M_{m_\varphi-1} = V, \quad M_{m_\varphi-1}^\top W M_{m_\varphi-1} = W,
$$

from which we obtain $T = M_{m_\varphi-1}^\top T M_{m_\varphi-1}$. This means that, by restriction, $\varphi$ is an isometry of $\bigoplus_{i=2}^\varphi E_i$ endowed with the symmetric bilinear form with Gram matrix $T$. By induction, $p^{m_\varphi - 1} \cdot |\det(T)|$ is a square, hence $p^{m_\varphi} |\det(B)|$ is a square.

**Second case:** all diagonal blocks of $B$ are zero and at least one nondiagonal block is not skew-symmetric. Without loss of generality, we may assume that $B_{1,2} + B^T_{1,2} \neq 0$. On the summand $E_1 \oplus E_2$ we base-change with the matrix $\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$. The Gram matrix $B$ now takes the form $\begin{pmatrix} B_{1,2} + B^T_{1,2} & V \\ V^\top & W \end{pmatrix}$, so we turn back to the first case.

**Third case:** all diagonal blocks of $B$ are zero and all nondiagonal blocks are skew-symmetric. Without loss of generality we may assume that $B_{1,2}$ is invertible. Write $B = \begin{pmatrix} A & V \\ V^\top & W \end{pmatrix}$, with $A = \begin{pmatrix} 0 & B_{1,2} \\ -B_{1,2} & 0 \end{pmatrix}$, and denote:

$$
T := W - V^\top A^{-1} V,
$$

and denote:
the Schur complement of $A$ in $B$. Then $\det(B) = \det(A) \det(T) = \det(B_{1,2})^2 \det(T)$. As above, we check that, by restriction, $\varphi$ is an isometry of $\bigoplus_{i=3}^m E_i$ endowed with the symmetric bilinear form with Gram matrix $T$. By induction, $p^{m_\varphi - 2} \cdot |\det(T)|$ is a square, hence $p^{m_\varphi} |\det(B)|$ is a square.

**Corollary 2.3.** Under the assumptions of Theorem 2.2, if furthermore the lattice $L$ is unimodular, then $\text{discr}(S_{\varphi}) = p^{a_\varphi}$ and $a_\varphi \equiv m_\varphi \mod (2)$.

**Proof.** If $L$ is unimodular, then $D_{S_{\varphi}} \cong D_{T_{\varphi}} \cong \left( \frac{\mathbb{Z}}{p\mathbb{Z}} \right)^{\oplus a_\varphi}$, so $\text{discr}(S_{\varphi}) = p^{a_\varphi}$ (see [2, §1.2]), hence by Theorem 2.2, the number $a_\varphi + m_\varphi$ is even. \qed

3. Order five symmetries of the Hilbert square of a K3 surface

Let $X$ be a projective irreducible holomorphic symplectic manifold, deformation equivalent to the Hilbert square of a K3 surface. We recall that the second cohomology group with integer coefficients $H^2(X, \mathbb{Z})$ is a lattice for the Beauville–Bogomolov–Fujiki quadratic form. Let $f \in \text{Aut}(X)$ be a nonsymplectic biholomorphic automorphism of prime order $p$. It is easy to see that $2 \leq p \leq 23$ (see [9, §5.4]). We consider the isometry $\varphi := f^* \text{ induced by } f$ on the lattice $H^2(X, \mathbb{Z})$.

In the sequel we write $m, a, S, T$ instead of $m_\varphi, a_\varphi, S_\varphi, T_\varphi$. Since the representation $\text{Aut}(X) \to O(H^2(X, \mathbb{Z}))$, $f \mapsto (f^*)^{-1}$ is faithful, nonsymplectic prime order automorphisms are classified by the data of the invariant lattice $T$ of $\varphi$ and its orthogonal complement $S$. For $2 \leq p \leq 19$ and $p \neq 5$ the classification is given in [6], for $p = 23$ it is given in [5]. The case $p = 5$ was missing: it requires the tools developed in the previous section.

**Theorem 3.1.** Let $f$ be an order five nonsymplectic automorphism acting on an irreducible holomorphic symplectic manifold, deformation equivalent to the Hilbert square of a K3 surface. Then its invariant lattice $T$ and its orthogonal complement $S$ are one of the following:

| $m$ | $a$ | $S$ | $T$ |
|-----|-----|-----|-----|
| 1   | 1   | $U \oplus H_{5}$ | $E_8(-1)^{\oplus 2} \oplus H_{5} \oplus (-2)$ |
| 2   | 2   | $U \oplus H_{5} \oplus A_4(-1)$ | $E_8(-1) \oplus H_{5} \oplus A_4(-1) \oplus (-2)$ |
| 3   | 1   | $U \oplus E_6(-1) \oplus H_{5}$ | $E_8(-1) \oplus H_{5} \oplus (-2)$ |
| 3   | 3   | $U \oplus H_{5} \oplus A_4(-1)^{\oplus 2}$ | $H_{5} \oplus A_4(-1)^{\oplus 2} \oplus (-2)$ |
| 4   | 2   | $U \oplus E_6(-1) \oplus H_{5} \oplus A_4(-1)$ | $H_{5} \oplus A_4(-1) \oplus (-2)$ |
| 4   | 4   | $U(5) \oplus E_8(-1) \oplus H_{5} \oplus A_4(-1)$ | $H_{5} \oplus A_4(-5) \oplus (-2)$ |
| 5   | 1   | $U \oplus E_8(-1)^{\oplus 2} \oplus H_{5}$ | $H_{5} \oplus (-2)$ |

In this table, the lattices $E_8(-1)$ and $A_4(-1)$ are the opposites of the usual positive definite root lattices, $U$ is the hyperbolic plane, $H_5$ is the rank two lattice with Gram matrix $\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, $A_4^*(-5)$ is the dual lattice of $A_4$ with intersection form multiplied by $-5$, and $(−2)$ is the rank one lattice generated by a vector of square $−2$. This result first appeared in the third author’s PhD thesis [20, Théorème 3.24] (note that there is a typo there, and that in case $(4, 4)$ we have $U(5) \oplus A_4(-1) \cong U \oplus A_4(-5)$; the lattices reproduced here are those of [6, Table 2]).

**Proof.** By [9, Lemma 5.5], we have $D_5 \cong (\mathbb{Z}/5\mathbb{Z})^{\oplus a}$, so Theorem 2.2 gives $a \equiv m \mod (2)$. By Proposition 2.1, we have $a \leq m$ and $4m = \text{rank}_{\mathbb{Z}} S \leq 23$, so $1 \leq m \leq 5$. Without loss of generality, we may suppose $m = 2$. We have $\varphi(X) = X$, so $\varphi^* \in \text{Sp}(H^2(X, \mathbb{Z}))$. Let $\theta = \varphi^* - 1$ be the symplectic automorphism of order $5$. Then $\theta^5 = 1$ and $\theta \neq -1$. Since $\theta$ is nonsymplectic, it must be that $\theta^2 \neq -1$. Therefore $\theta^4 \neq -1$.

By the classification of automorphisms of order five on $H^2(X, \mathbb{Z})$, the only possibilities are $\theta^2 = \lambda \cdot 1$, where $\lambda \in \mathbb{Z}$, and $\theta = \lambda \cdot 1$. Thus $\theta^2 = \lambda^2 \cdot 1$ and $\theta^4 = \lambda^4 \cdot 1$.

In particular, $\theta^2 \neq -1$ implies that $\lambda^2 \neq -1$. Therefore $\lambda = \pm 1$. Hence $\theta^2 \neq -1$ implies that $\theta^2 = 1$. In other words, $\theta$ is a symplectic isometry of order 5.

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and moreover, by [9, Proposition 4.13] we have \( a \leq 23 - 4m \). This gives a first list of possible pairs \((m,a)\):

\[
(1, 1), (2, 0), (2, 2), (3, 1), (3, 3), (4, 0), (4, 2), (4, 4), (5, 1), (5, 3).
\]

Again by [9, Lemma 5.5], the lattice \( S \) has signature \((2, 4m - 2)\), the lattice \( T \) is hyperbolic and \( D_T \cong \mathbb{Z}/2\mathbb{Z} \oplus D_S \). Since \( \text{rank}_\mathbb{Z} S = 4m \geq 3 + m \geq 3 + a \) we can apply [17, Corollary 1.13.5]: the lattice \( S \) decomposes as \( S = U \oplus S' \) for some 5-elementary lattice \( S' \) of signature \((1, 4m - 3)\) and discriminant group \((\mathbb{Z}/5\mathbb{Z})^{\oplus a}\). Applying [19, Section 1] we get that such a lattice \( S' \) exists for each value of \((m, a)\) listed above, except \((2, 0)\) and \((4, 0)\), and that \( S' \) is unique with these invariants if \( m \geq 2 \). For \( m = 1 \) the unicity comes from [11, Table 15.2a, p. 362]. The genus of the lattice \( T \), which is the orthogonal complement of \( S \) for its unique embedding in \( H^2(X, \mathbb{Z}) \), is characterized by [17, Proposition 1.15.1]. In each case but one, the isometry class of \( T \) is determined by [11, Corollary 22, p. 395].

The only missing case is \((m, a) = (5, 3)\). The lattice \( S \) is isometric to:

\[
U^{\oplus 3} \oplus E_8(-1) \oplus U(5) \oplus A_4(-1),
\]

so the discriminant form of \( S \) is isomorphic to:

\[
\left( \frac{\mathbb{Z}}{5\mathbb{Z}} \right)^{\oplus 2} \left( \begin{array}{cc} 0 & 1/5 \\ 1/5 & 0 \end{array} \right) \oplus \frac{\mathbb{Z}}{5\mathbb{Z}} \left( -\frac{4}{5} \right).
\]

We now show that this lattice admits no primitive embedding in \( H^2(X, \mathbb{Z}) \). For this we assume the converse. Its orthogonal complement \( T \) would have signature \((1, 2)\) and discriminant form isomorphic to:

\[
\left( \frac{\mathbb{Z}}{5\mathbb{Z}} \right)^{\oplus 2} \left( \begin{array}{cc} 0 & 1/5 \\ 1/5 & 0 \end{array} \right) \oplus \frac{\mathbb{Z}}{5\mathbb{Z}} \left( \frac{4}{5} \right) \oplus \frac{\mathbb{Z}}{5\mathbb{Z}} \left( -\frac{1}{2} \right).
\]

Assume that such a lattice does exist. Consider its 5-adic completion \( T_5 = T \otimes \mathbb{Z}_5 \), whose discriminant \( q_5 \) form is \( \left( \frac{\mathbb{Z}}{5\mathbb{Z}} \right)^{\oplus 2} \left( \begin{array}{cc} 0 & 1/5 \\ 1/5 & 0 \end{array} \right) \oplus \frac{\mathbb{Z}}{5\mathbb{Z}} \left( \frac{4}{5} \right) \). The rank of \( T_5 \) is then equal to 3, which is the length of \( D_{T_5} \). By [17, Theorem 1.9.1], there exists a unique 5-adic lattice \( K \) of rank 3 and discriminant form \( q_5 \). Thus the determinants of \( T_3 \) and \( K \) differ by an invertible square in \( \mathbb{Z}_5^* \). One has:

\[
\det T_5 = \det T = (-1)^3 |D_T| = -2 \cdot 5^3,
\]

\[
\det K = \det q_5 = -\frac{4}{5^3},
\]

so this amounts to saying that 2 is a square in \( \mathbb{Z}_5^* \), which is clearly not true. So such a lattice \( T \) does not exist. Finally, we easily get, as stated, a representative of the isometry classes of \( S \) and \( T \) in each case of the classification. \( \square \)

Using [1, Theorem 5.3] we see that each case of the classification is geometrically realized by an automorphism of the Hilbert square of a K3 surface, induced by an order five nonsymplectic automorphism of the underlying surface. Following the terminology of [6, Definition 4.1], for \( X \) deformation equivalent to the Hilbert square \( \Sigma^{[2]} \) of a K3 surface \( \Sigma \), an automorphism \( f \) of \( X \) is called natural if the pair \((X, f)\) deforms to a pair \((\Sigma^{[2]}, \sigma^{[2]}))\), where \( \sigma \) is an automorphism of \( \Sigma \). The following result answers the open question [6, Remark 6.1]:
Corollary 3.2. Every order five nonsymplectic automorphism acting on an irreducible holomorphic symplectic manifold, deformation equivalent to the Hilbert square of a K3 surface, is natural.

Proof. Assume first that $(m,a) \neq (1,1)$. By [6, Corollary 5.7], in each case of the classification above, the action of the automorphism on $H^2(X,\mathbb{Z})$, hence $f$ itself, is uniquely determined by the lattices $S$ and $T$. Since the moduli spaces of lattice-polarized nonsymplectic automorphisms considered here are connected (see [7, Theorem 4.5, Section 5.3, Theorem 5.6, Section 7.1]), and since each case is already realized by an automorphism coming from a K3 surface, this shows that nonsymplectic order five automorphisms are natural. If $(m,a) = (1,1)$, the moduli space is zero-dimensional so the result holds trivially. □

4. Natural automorphisms of Kummer fourfolds

4.1. Generalized Kummer varieties. Let $A$ be a two-dimensional complex torus with origin $0 \in A$. Let $A^{(n)}$ be the $n$-th symmetric power of $A$, $s: A^{(n)} \to A$ the summation morphism and $\pi: A^n \to A^{(n)}$ the quotient morphism. Let $A^{[n]}$ be the Hilbert scheme of $n$ points on $A$, $\rho: A^{[n]} \to A^{(n)}$ the Hilbert–Chow morphism and put $\sigma := s \circ \rho$.

The $n$-th generalized Kummer variety of $A$ is the fibre $A^{[n]} := \sigma^{-1}(0)$. It fits into the following cartesian diagram:

\[
\begin{array}{ccc}
A \times A^{[n]} & \xrightarrow{\nu} & A^{[n]} \\
p & & \downarrow \sigma \\
A & & A
\end{array}
\]

where $p$ is the projection onto the first factor, $n: A \to A$ is multiplication by $n$ and $\nu: A \times A^{[n]} \to A^{[n]}, (a,\xi) \mapsto a + \xi$ is a Galois covering with Galois group $G_n = T_n(A)$, the finite abelian group of $n$-torsion points on $A$. Here, $G_n$ acts on $A \times A^{[n]}$ by $g \cdot (a,\xi) = (a - g \cdot a, g \cdot \xi)$. The variety $A^{[n]}$ is a projective irreducible holomorphic symplectic manifold of dimension $2n - 2$.

4.2. Topological Lefschetz numbers. Let $\psi: A \to A$ be an automorphism of the complex manifold $A$. It decomposes as $\psi(x) = b + h(x) = (t_b \circ h)(x)$ for some $b \in A$, where $t_b$ denotes the translation by $b$ and $h: A \to A$ is an isomorphism of complex Lie groups, both uniquely determined. It naturally induces an isomorphism $\psi^{[n]}: A^{[n]} \to A^{[n]}$. If we further assume that $b \in T_n(A)$, the restriction of $\psi^{[n]}$ induces an automorphism $\psi^{[n]}: A^{[n]} \to A^{[n]}$.

We are interested in the topological Lefschetz number:

\[L(\psi^{[n]}) = \sum_k (-1)^k \text{tr} \left( (\psi^{[n]})^*|_{H^k(A^{[n]},\mathbb{C})} \right).\]

The rough idea (see [15] for a similar idea) is to compute it by the relation:

\[L(\psi)L(\psi^{[n]}) = L(\psi \times \psi^{[n]}).\]

Since $L(\psi)$ may be zero, we replace the Lefschetz number $L(f)$ of an automorphism $f: X \to X$ of a projective variety $X$ by the polynomial trace:

\[L(f,q) := \sum_k (-1)^k \text{tr} \left( f^*|_{H^k(X,\mathbb{C})} \right) q^k,\]
which is always invertible in $\mathbb{C}[q]$. Evaluating at $q = 1$, we recover the usual topological Lefschetz number.

In our situation, since translations on $A$ are homotopic to the identity, they act trivially on the cohomology groups of $A$, hence $L(\psi, q) = L(h, q)$. But for $n \geq 2$, in general $L(\psi^{[n]}) \neq L(h^{[n]})$, since although translations by $n$-torsion points do act trivially on $H^2(A^{[n]}, \mathbb{C})$ (see [8, Corollary 5]) they act nontrivially on the whole cohomology space of $A^{[n]}$, as observed in [18, Theorem 1.3].

The third author noticed that there is a mistake in the proof of [8, Proposition 7] giving a formula for the Lefschetz number of natural automorphisms of generalized Kummer varieties. Actually, the formula itself is not always true when the automorphism contains a nontrivial translation. In the sequel we state the correct formula and give a complete proof.

**Proposition 4.1.** Let $\psi \in \text{Aut}(A)$, decomposed as $\psi = t_b \circ h$ with $b \in T_n(A)$, and $\Psi := \psi^* : H^1(A, \mathbb{C}) \to H^1(A, \mathbb{C})$. Then:

$$
\sum_{n \geq 0} L(\psi^{[n]}, q) q^{-2n} t^n = \frac{1}{L(\psi, q)} \sum_{\chi \in (G_n)^h} \chi(b) \prod_{i \geq 1} \prod_{i=0}^4 \left( \det(1 - (\lambda^i \Psi) q^{-2i} t^i \chi) \right)^{(-1)^{i+1}}.
$$

In particular,

$$
L(\psi^{[n]}, q) = \frac{q^{2n}}{L(\psi, q)} \left. \frac{d^n}{n! dt^n} \right|_{t=0} \sum_{\chi \in (G_n)^h} \chi(b) \prod_{i \geq 1} \prod_{i=0}^4 \left( \det(1 - (\lambda^i \Psi) q^{-2i} t^i \chi) \right)^{(-1)^{i+1}}.
$$

**Proof.** The morphism $\nu$ is not natural with respect to the actions induced by $\psi$ since $\nu \circ (\psi \times \psi^{[n]}) \neq \psi^{[n]} \circ \nu$, but it commutes with the action of $h \times \psi^{[n]}$ on $A \times A^{[n]}$ and of $\psi^{[n]}$ on $A^{[n]}$; this is enough for our purpose since $L(\psi, q) = L(h, q)$.

Since $\nu$ is a finite morphism, its higher direct images vanish, so the Leray–Serre spectral sequence yields a canonical isomorphism:

$$
H^*(A \times A^{[n]}, \mathbb{C}) \cong H^*(A^{[n]}, \nu_* \mathbb{C}),
$$

which is compatible with the actions of $h \times \psi^{[n]}$ and $\psi^{[n]}$, where the action of $\psi^{[n]}$ on $\nu_* \mathbb{C}$ is given at each fiber $\zeta \in A^{[n]}$ by:

$$
(\psi^{[n]})_\zeta : (\nu_* \mathbb{C})(\zeta) \to (\nu_* \mathbb{C})(\psi^{[n]}(\zeta)), \quad f \mapsto f \circ (h \times \psi^{[n]}).
$$

Since $\nu$ is a Galois covering, it is a classical result that $\nu_* \mathbb{C}$ decomposes as a direct sum of character sheaves over the character group $G'_n$ of $G_n = T_n(A)$:

$$
\nu_* \mathbb{C} = \bigoplus_{\chi \in G'_n} L_{A^{[n]}, \chi}
$$

where $L_{A^{[n]}, \chi}$ is the locally constant sheaf on $A^{[n]}$ with fibre $\mathbb{C}$, associated to the character $\chi$, whose sections over an open set $U \subset A^{[n]}$ are the continuous functions $f : \nu^{-1}(U) \to \mathbb{C}$ (where $\mathbb{C}$ is given the discrete topology) such that $f(g \cdot (a, \xi)) = \chi(\xi) f(a, \xi)$ for all $(a, \xi) \in \nu^{-1}(U)$ and $g \in G_n$. It follows (see also [10, Proposition 18]) that:

$$
H^*(A \times A^{[n]}, \mathbb{C}) \cong \bigoplus_{\chi \in G'_n} H^*(A^{[n]}, L_{A^{[n]}, \chi}).
$$
The direct image $n_\mathcal{L}$ decomposes similarly as a direct sum of character sheaves over the character group $G_n^\vee$ of $G_n = T_n(A)$:

$$n_\mathcal{L} = \bigoplus_{\chi \in G_n^\vee} L_{A,\chi}$$

where $L_{A,\chi}$ is the locally constant sheaf on $A$ with fibre $\mathbb{C}$, associated to the character $\chi$. Here we make the group $G_n$ act by subtraction on $A$, so the morphism $p$ is $G_n$-equivariant. With this convention, the sections of $L_{A,\chi}$ over an open set $U \subset A$ are the continuous functions $f : n^{-1}(U) \to \mathbb{C}$ such that $f(g \cdot x) = f(x-g) = \chi(g)f(x)$ for all $x \in n^{-1}(U)$ and $g \in G_n$.

Following [16], we denote by $\mathcal{L}_\chi$ the locally constant sheaf on $A^{(n)}$ such that $\pi^{-1}\mathcal{L}_\chi \cong L_{A,\chi} \boxtimes \cdots \boxtimes L_{A,\chi}$, and we put $L^{[n]}_\chi := \rho^{-1}\mathcal{L}_\chi$. We construct, for any $\chi \in G_n^\vee$, an isomorphism of local systems $L^{[n]}_\chi \xrightarrow{\sim} L_{A^{[n]}_\chi}$ as follows. Given a point $\zeta = (\zeta_1, \ldots, \zeta_n) \in A^{[n]}$ and functions $f_i \in L_{A,\chi}(\zeta_i, i = 1, \ldots, n$ defining a section $f_1 \boxtimes \cdots \boxtimes f_n \in L^{[n]}_\chi(\zeta)$, we associate a function $f : \nu^{-1}(\zeta) \to \mathbb{C}$ defining a section $f \in L_{A^{[n]}_\chi}(\zeta)$. At a point $(a, \xi) \in \nu^{-1}(\zeta)$, with $\xi = (x_1, \ldots, x_n)$, we have $\zeta = a + \xi$, $\zeta_i = a + x_i$ for all $i$ and $\sum_{i=1}^n x_i = 0$. Choose elements $\bar{a}, \bar{x}_1, \ldots, \bar{x}_{n-1} \in A$ such that $n\bar{a} = a$, $n\bar{x}_i = x_i$ for $i = 1, \ldots, n-1$ and put $\bar{x}_n := -\sum_{i=1}^{n-1} \bar{x}_i \in n^{-1}(x_n)$. We then define:

$$f(a, \xi) := \prod_{i=1}^n f_i(\bar{a} + \bar{x}_i).$$

It is easy to check that this definition does not depend on the choice of $\bar{a}, \bar{x}_1, \ldots, \bar{x}_{n-1}$, that $f \in L_{A^{[n]}_\chi}(\zeta)$, i.e.: $f \cdot (a, \xi) = \chi(g)f(a, \xi) = f(a-g, g+\xi) \quad \forall g \in G_n$,

and that this construction defines an isomorphism $L^{[n]}_\chi(\zeta) \xrightarrow{\sim} L_{A^{[n]}_\chi}(\zeta)$.

Recall that $\psi[\cdot]$ acts on $\nu_*\mathcal{L}$. It is easy to check that if $f \in L_{A^{[n]}_\chi}$ then $(\psi[\cdot])^*(f) = f \circ (h \times \psi[\cdot]) \in L_{A^{[n]}_\chi \circ h\text{b}}$. Let us compute the action induced on $\bigoplus_{\chi \in G_n^\vee} L^{[n]}_\chi$ by the isomorphisms $L^{[n]}_\chi \xrightarrow{\sim} L_{A^{[n]}_\chi}$. At a point $(a, \xi) \in A \times A^{[n]}$, using the same notation as above, we compute:

$$((\psi[\cdot])^*f)(a, \xi) = f(h(a), b + h(\xi))$$

$$= f(h(a), b + h(x_1), \ldots, b + h(x_n))$$

$$= \prod_{i=1}^{n-1} f_i(h(a) + \bar{b} + h(\bar{x}_i)) \cdot f_n(h(a) + \bar{b} + h(\bar{x}_n) - b)$$

where the element $\bar{b} + h(\bar{x}_n) - b \in n^{-1}(b + h(x_n))$ is chosen according to the construction explained above. Thus:

$$((\psi[\cdot])^*f)(a, \xi) = \chi(b) \prod_{i=1}^n (f_i \circ \psi)(\bar{a} + \bar{x}_i).$$

This computation implies the following. Consider the action of the automorphism $\psi$ on $L_{A,\chi}$ defined at any point $x \in A$, and for any element $f \in L_{A,\chi}(x)$, by $\psi^* f = \chi(b)$
f \circ \psi. It is easy to check that, if f \in L_{A,\chi}(x), then \psi^* f \in L_{A,\chi}(h^{-1}(x)). Let us denote by (\psi^*)^{[n]} the action of \psi induced on L^{[n]}_{\chi}, such that:

\((\psi^*)^{[n]}(f_1 \otimes \cdots \otimes f_n) = \psi^* f_1 \otimes \cdots \otimes \psi^* f_n\).

Equation (2) then shows:

\((\psi^{[n]})(f) = \chi(b)(\psi^*)^{[n]}(f) \in L_{A^{[n]},\chi}(h^{-1}) \forall f \in L_{A^{[n]},\chi}^{[n]}\).

In other words, the action \((\psi^{[n]})^* : L_{A^{[n]},\chi} \rightarrow L_{A^{[n]},\chi}(h)\) corresponds through the isomorphisms \(L^{[n]}_{\chi} \cong L_{A^{[n]},\chi}\) to the action \(\chi(b)(\psi^*)^{[n]} : L^{[n]}_{\chi} \rightarrow L^{[n]}_{\chi}\).

Using [16, Theorem 1.2], we get linear isomorphisms:

\[
H^*(A \times A^{[n]}, \mathbb{C}[2n]) \cong \bigoplus_{\chi \in G^{\chi}_{n}} H^*(A^{[n]}, L_{A^{[n]},\chi}(2n))
\]

\[
\cong \bigoplus_{\chi \in G^{\chi}_{n}} H^*(A^{[n]}, L^{[n]}_{\chi}(2n))
\]

\[
\cong \bigoplus_{\chi \in G^{\chi}_{n}} \mathfrak{S}^* \left( \bigoplus_{\nu \geq 1} H^*(A, L^\otimes_{\chi}(2\nu)) \right),
\]

such that the action of \(h \times \psi^{[n]}\) on the left hand side corresponds to the action of \(\chi(b)(\psi^*)^{[n]}\) on the second line, and acts on the last line by \(\psi^*\) on each cohomological group, followed by a multiplication by \(\chi(b)\) on each super-symmetric tensor.

An element of \(H^k(A \times A^{[n]}, \mathbb{C}[2n])\) has cohomological degree \(k - 2n\) and conformal weight \(n\), an element of \(H^\nu(A, L^\otimes_{\chi}(2\nu))\) has degree \(k - 2\) and weight \(\nu\). Notation \(\mathfrak{S}^*\) denotes the super-symmetric algebra, where the super-structure concerns only the cohomological degree: the weighting does not interfere with the super-structure. The above isomorphisms respect these bigradings.

As explained in [16, p. 768], one has \(H^*(A, L^\otimes_{\chi}(2\nu)) = 0\) unless \(L^\otimes_{\chi} = \mathbb{C}_\nu\). In order to keep track of the weighting, we work in the space of formal series in the parameter \(t\) encoding the weight, with coefficients in the space of finitely dimensional graded super-vector spaces. We thus have:

\[
\bigoplus_{\nu \geq 1} H^*(A, L^\otimes_{\chi}(2\nu)) t^\nu \cong \bigoplus_{\nu \geq 1} H^*(A, \mathbb{C}[2]) t^\nu |\chi|,
\]

where \(|\chi|\) denotes the order of \(\chi\) in \(G^{\chi}_{n}\). We get finally:

\[
\bigoplus_{n \geq 0} H^*(A \times A^{[n]}, \mathbb{C}[2n]) t^n \cong \bigoplus_{\chi \in G^{\chi}_{n}} \mathfrak{S}^* \left( \bigoplus_{\nu \geq 1} H^*(A, \mathbb{C}[2]) t^\nu |\chi| \right).
\]

Let us now compute the formal series of polynomial traces \(\sum_{n \geq 0} L(h \times \psi^{[n]}, q)t^n\) using this isomorphism. First we observe that the action on the right hand side is induced by the natural action of \(\psi^*\), but it sends the block indexed by \(\chi\) to the block indexed by \(\chi \circ h\). So in the computation of the trace, only those characters which are invariant by \(h\) contribute to the trace, we denote them \(\chi \in (G^{\chi}_{n})^h\). Once the trace of \(\psi^*\) is computed, it is multiplied by \(\chi(b)\) on each block \(\chi\) to take into account our previous computation of the action. Now the trace of \(\psi^*\) on each super-symmetric block can be computed using standard techniques for traces of natural operators on bigraded super-vector spaces, for which we refer to [4, Section 3] and [9,
Lemma 6. Denoting by $\Psi$ the action of $\psi^*$ on $H^1(A, \mathbb{C})$, and using the fact that $H^*(A, \mathbb{C}) = \bigwedge^* H^1(A, \mathbb{C})$, we obtain:

$$
\sum_{n \geq 0} L(h \times \psi^n, q) q^{-2n} t^n = \sum_{\chi \in (G_n)^h} \chi(b) \prod_{v \geq 1} \prod_{i=0}^4 \left( \det(1 - (\wedge^i \Psi) q^{i-2} t^{v|\chi|}) \right)^{(-1)^{i+1}}.
$$

Since $L(h \times \psi^n, q) = L(h, q) \cdot L(\psi^n, q) = L(\psi, q) \cdot L(\psi^n)$ we get the expected formula. 

\begin{remark}
The isomorphisms of local systems $L_{\chi}^{[n]} \xrightarrow{\sim} L_{A[n], \chi}$ constructed in the proof for any $\chi \in G^\vee_n$ can be understood more conceptually as follows. Since diagram (1) is $G_n$-equivariant, cartesian, and since $n$ is finite and unramified, we have an isomorphism of $G_n$-local systems (see also [10, Proposition 18]):

$$
\nu_s \equiv \nu_s^{-1} \equiv \sigma^{-1} n_s \sigma.
$$

Following these isomorphisms, it is easy to see that $L_{A[n], \chi} \cong \sigma^{-1} L_{A, \chi}$ for all $\chi \in G^\vee_n$. We have $\sigma^{-1} L_{A, \chi} = \rho^{-1} s^{-1} L_{A, \chi}$, and it is easy to see that:

$$
\sigma^{-1} L_{A, \chi} \cong L_{\chi},
$$

so finally $L_{A[n], \chi} \cong \sigma^{-1} L_{A, \chi} \cong \rho^{-1} L_{\chi} = L_{\chi}^{[n]}$.

As a consequence of Proposition 4.1, by evaluating to $q = 1$ we recover the formula stated in [8, Proposition 7], this time with the missing factors:

\begin{corollary}
Let $\psi \in \text{Aut}(A)$, decomposed as $\psi = t_b \circ h$ with $b \in T_n(A)$, and $\Psi := \psi^* : H^1(A, \mathbb{C}) \rightarrow H^1(A, \mathbb{C})$. Then:

$$
L(\psi)L(\psi^n) = \frac{d^n}{n! \, d t^n} \sum_{t=0}^{\infty} \chi(b) \prod_{v \geq 1} \exp \left( \sum_{s \geq 1} \frac{\det(1 - \Psi^s) q^{s|\chi|}}{s} t^{v|\chi|} \right).
$$

\end{corollary}

4.3. Application. Automorphisms of a two dimensional complex torus $A$ were classified by Fujiki [13]. From this we extract a list of those automorphisms whose action on the second cohomology space $H^2(A, \mathbb{C})$ has prime order, and we consider the automorphisms induced on the generalized Kummer fourfold $A^{[3]}$. To get a complete picture, we consider all automorphisms $\psi^{[3]}$ with $\psi = t_b \circ (\pm h)$, where $h$ is a group automorphism of prime order on $A$ and $t_b$ is the translation by a 3-torsion point $b \in A$. As an application of our formula in Proposition 4.1, we give in each case the topological Lefschetz number and we briefly discuss the fix locus. Details on the geometric study can be found in [20, Section 1.2]. We refer to [20, Lemma 1.13] for a discussion of the fix loci of the iterates of the automorphisms listed below. In complement, and in relation to the first part of this note, we refer to [14] for a lattice-theoretical classification of these automorphisms.

4.3.1. Type 0. Assume that $h = \text{id}$. Clearly $L(\text{id}^{[3]}) = 108$. For $b \neq 0$ we get $L(t_b^{[3]}) = 27$, the fix locus consists of 27 points.

We have $L((- \text{id})^{[3]}) = 60$, the fix locus consists of a copy of $A^{[2]}$ and 36 isolated points; the same result holds for $(-t_b)^{[3]}$ if $b \neq 0$. 

4.3.2. **Type 1.** The torus $A = E \times E'$ is a product of two elliptic curves and $h = \text{id}_E \times (-\text{id}_{E'})$. We have $L(h^{[3]}) = 12$, the fix locus consists of a copy of $E \times \mathbb{P}^1$ blown up in 9 points, three copies of $E^{(2)}$, one copy of $E^{[3]}$ and one copy of $E \times E$.

Let $b = (u, v)$. If $u = 0$ then $L(\psi^{[3]}) = 12$ and the fix locus is the same as above. If $\lambda \neq 0$, we have $L(\psi^{[3]}) = 3$, the fix locus consists of 3 points.

4.3.3. **Type 2.** The torus $A = \frac{E \times E'}{(\mathbb{Z}/2\mathbb{Z})^2}$ is the quotient of $E \times E'$ by the translation by a 2-torsion point $(a, a')$ with $a \in T_2(E), a' \in T_2(E')$, both nonzero, and $h = \text{id}_E \times (-\text{id}_{E'})$. We have $L(h^{[3]}) = 12$, the fix locus consists of a copy of $A/\langle h \rangle$ blown up in 9 points, one copy of $E^{(2)}$ and one copy of $E^{[3]}$.

Let $b = (u, v)$. If $u = 0$ then $L(\psi^{[3]}) = 12$ and the fix locus is the same as above. If $u \neq 0$, we have $L(\psi^{[3]}) = 3$ and the fix locus consists of 3 points.

4.3.4. **Type 3.** The torus is $A = \frac{E \times E'}{(\mathbb{Z}/2\mathbb{Z})^2}$ where $(\mathbb{Z}/2\mathbb{Z})^2 \rightarrow E \times E'$ is any group monomorphism, and $h = \text{id}_E \times (-\text{id}_{E'})$. We have $L(h^{[3]}) = 12$, the fix locus consists of a copy of $A/\langle h \rangle$ blown up in 9 points and one copy of $E^{[3]}$.

Let $b = (u, v)$. If $u = 0$ then $L(\psi^{[3]}) = 12$ and the fix locus is the same as above. If $u \neq 0$, we have $L(\psi^{[3]}) = 3$ and the fix locus consists of 3 points.

4.3.5. **Type 4.** The torus is $A = E_4 \times E_4$ where $E_4 = \frac{C_{24}}{\mathbb{Z}[i]}$, and $h = \left( \begin{array}{cc} 1 & \delta \\ 0 & \zeta_4 \end{array} \right)$. We have $L(h^{[3]}) = 16$, the fix locus consists of four copies of $\mathbb{P}^1$ and 8 isolated points. The same result holds when $h$ is composed by a nonzero translation.

4.3.6. **Type 5.** The torus is $A = E \times E_6$ where $E_6 = \frac{C_{12}}{\mathbb{Z}[i]}$, with $\zeta_n$ is a primitive $n$-th root of unity, and $h = \left( \begin{array}{cc} 1 & 0 \\ 0 & \zeta_3 \end{array} \right)$. We have $L(h^{[3]}) = 27$, the fix locus consists of nine copies of $\mathbb{P}^1$, one copy of $E^2$, three copies of $E^{[3]}$ and three copies of $E$.

Let $b = (u, v) \in T_3(A)$. Denote $\Delta_6$ the image in $E_6$ of the real line $\mathbb{R} \frac{1 + \zeta_6}{3}$. If $u = 0$ and $v \not\in \Delta_6$, then $L(\psi^{[3]}) = 27$, the fix locus consists of nine copies of $\mathbb{P}^1$, one copy of $E^2$, three copies of $E^{[3]}$ and three copies of $E$. Otherwise $L(\psi^{[3]}) = 0$ but the fix locus depends on the translation. If $u = 0$ and $v \not\in \Delta_6$, it consists of three copies of $E^{(2)}$ and three copies of $E$; if $u \neq 0$ and $v \not\in \Delta_6$, it consists of three copies of $E_6$; if $u \neq 0$ and $v \in \Delta_6$ the fix locus is empty: this case already appeared in [8, Proposition 8] in the context of the construction of Enriques fourfolds.

We have $L((-h)^{[3]}) = 9$, the fix locus consists of one copy of $E$, one copy of $\mathbb{P}^1$ and 7 isolated points. The same result holds when $(-h)$ is composed by a nonzero translation.

4.3.7. **Type 6.** The torus $A = \frac{E \times E_6}{\mathbb{Z}/3\mathbb{Z}}$ is the quotient of $E \times E_6$ by the translation by a 3-torsion point $(a, a')$ with $a \in T_3(E) \setminus \{0\}$ and $a' = \frac{1 + \zeta_6}{3}$, and $h = \left( \begin{array}{cc} 1 & 0 \\ 0 & \zeta_3 \end{array} \right)$.

We have $L(h^{[3]}) = 9$, the fix locus consists of three copies of $\mathbb{P}^1$, one copy of $E^{[3]}$ and one copy of $E$.

Let $b = (u, v) \in T_3(A)$, it can be written as $b = (u, v) + \frac{2}{3}(a, a')$ with $u \in T_3(E)$, $v \in T_3(E_6)$ and $t \in \{0, 1, 2\}$. If $t = 0$ and $u \in Za$ then $L(\psi^{[3]}) = 9$, the fix locus consists of one copy of $E^{[3]}$, three copies of $\mathbb{P}^1$ and one copy of $E$. Otherwise $L(\psi^{[3]}) = 0$ but the fix locus depends on the translation. If $t = 0$ and $u \notin Za$, it consists of one copy of $E_6$, if $t \neq 0$ the fix locus is empty.
We have \( L((-h)^{[3]} limb) = 9 \), the fix locus consists of one copy of \( \mathbb{P}^1 \) and 7 isolated points. The same result holds when \((-h)\) is composed by a nonzero translation.

4.3.8. Type 7. The torus is \( A = E_6 \times E_6 \) and \( h = \begin{pmatrix} \zeta_5 & 0 \\ 0 & \zeta_3 \end{pmatrix} \). We have \( L(h^{[3]}) = 36 \), the fix locus consists of one copy of the minimal resolution of \( A/\langle h \rangle \) and 21 isolated points.

Let \( b = (u, v) \). If \( b \in \Delta_6 \times \Delta_6 \), then \( L(\psi^{[3]}) = 36 \) and the fix locus is the same as above, otherwise \( L(\psi^{[3]}) = 27 \) and it consists of nine copies of \( \mathbb{P}^1 \) and 9 isolated points.

We have \( L((-h)^{[3]}) = 12 \), the fix locus consists of one copy of \( \mathbb{P}^1 \) and 10 isolated points. The same result holds when \((-h)\) is composed by a nonzero translation.

4.3.9. Type 8. The torus is \( A = \mathbb{C}^2/\Lambda \) where \( \Lambda \) is the lattice generated by the four vectors:

\( (1, 1), (\zeta_5, \zeta_5^2), (\zeta_5^3, \zeta_5^4), (\zeta_5^3, \zeta_5) \),

and \( h = \begin{pmatrix} \zeta_5 & 0 \\ 0 & \zeta_5^2 \end{pmatrix} \). We have \( L(h^{[3]}) = 13 \), the fix locus consists of one copy of \( \mathbb{P}^1 \) and 11 isolated points. The same result holds when \( h \) is composed by a nonzero translation.

We have \( L((-h)^{[3]}) = 5 \), the fix locus consists of 5 points. The same result holds when \((-h)\) is composed by a nonzero translation.

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