THE MODULI SPACE OF STABLE VECTOR BUNDLES OVER A REAL ALGEBRAIC CURVE

INDRANIL BISWAS, JOHANNES HUISMAN, AND JACQUES HURTUBISE

Abstract. We study the spaces of stable real and quaternionic vector bundles on a real algebraic curve. The basic relationship is established with unitary representations of an extension of $\mathbb{Z}/2$ by the fundamental group. By comparison with the space of real or quaternionic connections, some of the basic topological invariants of these spaces are calculated.

1. Introduction

The moduli space of vector bundles on a compact Riemann surface can be understood from many points of view. Weil, in an early paper [We], launched their study in terms of “matrix divisors”. The next major work was the classification of vector bundles over elliptic curves by Atiyah [At1]. In the middle of the nineteen sixties, Narasimhan and Seshadri [NS], picking up on some ideas of Weil and Atiyah, showed in a fundamental paper how the moduli space of stable vector bundles on Riemann surface could be understood in terms of unitary representations of the fundamental group of the surface. This point of view was vastly expanded upon by Atiyah and Bott [AB] in 1982, who showed how the moduli space of stable vector bundles on Riemann surface could be understood in terms of unitary representations of the fundamental group of the surface. This point of view was vastly expanded upon by Atiyah and Bott [AB] in 1982, who showed how the moduli space was naturally realised inside the space of all connections as the minima of the Yang–Mills functional; they then used this idea to analyse the topology of the moduli space in Morse theoretical terms, showing that the Morse function of energy on the connections was perfect as an equivariant Morse function, and that in consequence one could obtain the cohomology of the moduli space in terms of the equivariant cohomology of the space of connections (a computation that involves fairly standard ingredients) and the cohomology of higher order critical points, which can be obtained inductively from the moduli spaces of vector bundles of lower rank.

This work [AB], with its ties to symplectic geometry, equivariant Morse theory, and more generally, ideas and techniques from physics, opened up a whole series of perspectives in the study of moduli: a complete review would take up this whole paper, but let us mention on the side of symplectic geometry and Morse theory the efforts of Jeffrey and Kirwan, whose techniques gave a complete computation of the cohomology rings (see, e.g., [JK]); on the physical side, the work of E. Verlinde [V], giving Riemann–Roch numbers for the moduli, and Witten [Wi], who gave formulae for symplectic of their volumes. For these physical insights, complete mathematical proofs have occupied a large number of mathematicians.

From the point of view of representation theory, one natural question once one knows the representation ring is to compute the real and quaternionic representations. In our context, this means that we should restrict our attention to real algebraic curves, and
study the real and quaternionic moduli on these spaces. Our point of view on real geometry follows that advocated by, e.g., Atiyah [At2], in which real or quaternionic objects are complex objects invariant under an anti–holomorphic involution.

Our purpose in this paper is to consider some of the foundations of this theory. We begin by settling some rather basic questions such as the topological classification of real and quaternionic bundles: equivariant topology, as usual, reserves a few surprises. We then examine the appropriate group whose representations we will study, the orbifold group of the surface under the real structure, and define the real and quaternionic representations of this group. These representations are shown to correspond to flat real or quaternionic bundles, and a real and quaternionic version of the Narasimhan–Seshadri theorem is given.

The rest of the paper is devoted to developing some understanding of the topology of the moduli space, in the spirit of Atiyah and Bott. A simple echoing of their ideas in this context presents difficulties: the spaces involved have torsion in their cohomology and normal bundles are not necessarily orientable, for example. We do describe the spaces of real and quaternionic connections, and compute some of their invariants such as the fundamental groups and second homotopy groups: as we have a lower bound on the indices of the higher order critical points, this gives us the same information for the moduli spaces.

Much thus remains to be done. The real and quaternionic fixed point spaces are fixed point sets under an antisymplectic involution, and the whole arsenal of symplectic techniques used to compute cohomology has mod 2 variants over the fixed point set; see e.g. the work of Duistermaat [Du] and that of Biss, Guillemin and Holm [BGH]. The mod 2 cohomology should then be computable following the ideas of Atiyah and Bott, Jeffrey and Kirwan. In a similar vein, Ho and Jeffrey [HJ] have a computation of volumes on representation spaces on a non–orientable surface which should be adaptable to this context.

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2. Real curves, and real and quaternionic vector bundles

Let $Y$ be a smooth geometrically connected projective real algebraic curve of genus $g$. The Riemann surface associated to $Y$ is the connected complex curve

$$X = Y(\mathbb{C}) = Y \times_{\mathbb{R}} \mathbb{C}$$

obtained by field extension. Since $Y$ is a real curve, the Galois group $\text{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2$ acts on $Y(\mathbb{C})$; this action is an anti–holomorphic involution

$$\sigma : Y(\mathbb{C}) \longrightarrow Y(\mathbb{C}).$$

In particular, $\sigma$ reverses orientation. The real points $Y(\mathbb{R})$ of $Y$ are the fixed points of $\sigma$. It follows that the quotient $Y(\mathbb{C})/\text{Gal}(\mathbb{C}/\mathbb{R}) = X/\sigma$ is an unoriented, and possibly nonorientable, compact connected topological surface with, possibly empty, boundary.

Our development uses this point of view of a real curve as a complex curve equipped with an involution, and from now on, we will let $X$ denote an irreducible complex curve
equipped with an anti–holomorphic involution $\sigma$. Let $X/\sigma$ denote the quotient. The fixed points of $\sigma$ will be denoted by $X(\mathbb{R})$.

We will distinguish three types of real curve:

The curve $X$ is said to be of

**Type 0:** if $X(\mathbb{R}) = \emptyset$, in which case $X/\sigma$ is a surface without boundary and is not orientable,

**Type I:** if $X(\mathbb{R})$ is non-empty, and $X \setminus X(\mathbb{R})$ is not connected, in which case $X/\sigma$ has a nonempty boundary and is orientable, and

**Type II:** if $X(\mathbb{R})$ is non-empty, and $X \setminus X(\mathbb{R})$ is connected, in which case $X/\sigma$ has a nonempty boundary and is not orientable.

Details of the cell structure of these surfaces are given below.

**Type 0.**

We consider first curves $X$ of even genus $g = 2\hat{g}$. The quotient $X/\sigma$ can then be obtained by taking a surface $S$ of genus $\hat{g}$, cutting out a disk to obtain a surface with boundary $S_0$, then gluing the boundary circle to itself by identifying antipodal points. The full surface $X$ is obtained by doubling the surface $S_0$ to a disjoint union of $S_0$ and $\sigma(S_0)$ (with the opposite orientation) and gluing along the boundary by sending any point $z$ of $S_0$ in one copy of the surface to the antipodal point of $z$ in the other copy. Let us choose the base point $x_0$ on the boundary of $S_0$; its antipodal point $\sigma(x_0)$ also lies on the boundary; let $\gamma$ denote a half circle on the boundary joining these two points. One has standard generators $\alpha_i, \beta_i$, where $i = 1, \ldots, \hat{g}$, and $(\sigma(\gamma) \circ \gamma)$ for the fundamental group of $S_0$, satisfying

$$\prod_{i=1}^{\hat{g}} [\alpha_i, \beta_i](\sigma(\gamma) \circ \gamma) = 1;$$

likewise there are generators $\alpha_i, \beta_i, i = \hat{g} + 1, \ldots, 2\hat{g}$, and $(\sigma(\gamma) \circ \gamma)$ for the fundamental group of $\sigma(S_0)$, satisfying

$$\prod_{i=\hat{g}+1}^{2\hat{g}} [\alpha_i, \beta_i](\sigma(\gamma) \circ \gamma)^{-1} = 1.$$

The fundamental group of the whole curve $X$ then has $\alpha_i, \beta_i$, where $i = 1, \ldots, 2\hat{g}$, as standard generators.

Corresponding to this description of the fundamental group, one has a cell decomposition of $S_0$, with two zero–cells $\{x_0, \sigma(x_0)\}$, one–cells $\alpha_i, \beta_i, i = 1, \ldots, g, \gamma, \sigma(\gamma)$, and a single two–cell, glued to the one–cells via the relation (2.1); this lifts to a cell decomposition of $X$ with one–cells $\alpha_i, \sigma(\alpha_i), \beta_i, \sigma(\beta_i), \gamma, \sigma(\gamma)$, and two two–cells interchanged by $\sigma$.

In a similar fashion, we can consider curves of odd genus $g = 2\hat{g} + 1$. One can construct the quotient $X/\sigma$ of a surface of genus $g = 2\hat{g} + 1$ by taking a Riemann surface $S$ of genus $\hat{g}$, removing two disks to obtain a surface with boundary $S_0$, and gluing the two boundaries using an orientation preserving diffeomorphism (as in the construction of the Klein bottle from a cylinder). The surface $X$ in turn is constructed by taking two copies $S_0^1$ and $S_0^2 = \sigma(S_0^1)$ of $S_0$ with opposite orientations and gluing one component of the boundary of $S_0^1$ to the other component of the boundary of $S_0^2$ using the orientation
preserving diffeomorphism. The involution \(\sigma\) interchanges the two boundary circles of any copy of \(S_0\), and it takes any interior point in one component to the corresponding point in the other component. Let \(\gamma\) be a boundary circle of a copy of \(S_0\). Choose a base point \(x_0\) on \(\gamma\), and join it in \(S_0\) to its image \(\sigma(x_0)\) on \(\sigma(\gamma)\) by a curve \(\delta\). One then has that the fundamental group of \(S_0\) is generated by \(\{\alpha_i, \beta_i\}_{i=1}^{2g+1}, \gamma, \delta^{-1} \circ \sigma(\gamma) \circ \delta\) with the relation

\[
\left(\prod_{i=1}^{g} [\alpha_i, \beta_i]\right)\gamma(\delta^{-1} \circ \sigma(\gamma) \circ \delta) = 1.
\]

Likewise, one can choose generators \(\{\alpha_i, \beta_i\}_{i=1}^{2g+1}, \gamma, \delta^{-1} \circ \sigma(\gamma) \circ \delta\) for the fundamental group of \(\sigma(S_0)\), such that setting \(\alpha_{g+1} = \sigma(\delta) \circ \delta, \beta_{g+1} = \gamma\), we have that \(\alpha_i, \beta_i\), where \(i = 1, \ldots, 2g + 1\), is a standard set of generators for the fundamental group of \(X\). Corresponding to this description of the fundamental group, one has a cell decomposition of \(S_0\), with two zero–cells \(x_0, \sigma(x_0)\), one–cells \(\{\alpha_i, \beta_i\}_{i=1}^{2g}, \gamma, \sigma(\gamma), \delta\), and a single two–cell; this lifts to a cell decomposition of \(X\) with one–cells \(\alpha_i, \sigma(\alpha_i), \beta_i, \sigma(\beta_i), \gamma, \sigma(\gamma), \delta, \sigma(\delta)\), and two–two–cells interchanged by \(\sigma\).

In both cases, using the cellular decompositions as given above, the quotient surface \(X/\sigma\) is a cofibration

\[
\vee_{i=1}^{g+1} S^1 \longrightarrow X/\sigma \longrightarrow S^2.
\]

**Type I.**

One now has a quotient surface \(X/\sigma\) which is an orientable surface \(S_0\) with \(r\) boundary circles and genus \(\hat{g} = (1 + g - r)/2\). Choose a base point on each boundary curve; let \(\gamma_i, i = 1, \ldots, r\), denote the boundary circles, and let \(\delta_i, i = 2, \ldots, r\), be paths joining the base point on \(\gamma_1\) to the base point on \(\gamma_i\). The surface \(X/\sigma\) is the one obtained by glueing a disk to the one skeleton \(\alpha_1, \beta_1, \ldots, \alpha_{\hat{g}}, \beta_{\hat{g}}, \gamma_1, \ldots, \gamma_r, \delta_2, \ldots, \delta_r\) by the boundary circle to \(\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_{\hat{g}} \beta_{\hat{g}} \alpha_{\hat{g}}^{-1} \beta_{\hat{g}}^{-1} \gamma_1 \delta_2 \gamma_2 \delta_2^{-1} \cdots \gamma_r \delta_r^{-1}\). This gives a cell decomposition for \(X/\sigma\) with \(r\) zero–cells, one–cells \(\alpha_1, \beta_1, \ldots, \alpha_{\hat{g}}, \beta_{\hat{g}}, \gamma_1, \ldots, \gamma_r, \delta_2, \ldots, \delta_r\), and one two–cell. The cofibration for \(X/\sigma\) is

\[
\vee_{i=1}^{g+r} S^1 \longrightarrow X/\sigma \longrightarrow S^2,
\]

**Type II.**

One now has a quotient surface \(X/\sigma\) which is an unoriented surface with \(r\) boundary circles. It can be obtained from an oriented surface \(S\) of genus \(\hat{g} = (g - r)/2\) from which one removes \((r + 1)\) open disks obtaining a surface \(S_0\) with boundaries \(\gamma_0, \ldots, \gamma_r\); the quotient \(X/\sigma\) is obtained from \(S_0\) by glueing \(\gamma_0\) to itself, identifying each point to its antipodal point. Choose a base point on each boundary curve of \(S_0\); let \(\delta_i, i = 1, \ldots, r\), be paths joining the base point on \(\gamma_0\) to the base point on \(\gamma_i\). The surface \(X/\sigma\) is the one obtained by glueing a disk to the one–skeleton spanned by \(\alpha_1, \beta_1, \ldots, \alpha_{\hat{g}}, \beta_{\hat{g}}, \gamma_0, \ldots, \gamma_r, \delta_1, \ldots, \delta_r\) by the boundary circle to \(\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_{\hat{g}} \beta_{\hat{g}} \alpha_{\hat{g}}^{-1} \beta_{\hat{g}}^{-1} \gamma_0 \gamma_0 \delta_1 \gamma_1 \delta_1^{-1} \cdots \delta_r \gamma_r \delta_r^{-1}\). This gives a cell decomposition for \(X/\sigma\) with \(r + 1\) zero cells, one–cells \(\alpha_1, \beta_1, \ldots, \alpha_{\hat{g}}, \beta_{\hat{g}}, \gamma_0, \ldots, \gamma_r, \delta_1, \ldots, \delta_r\), and one two–cell.

The cofibration is

\[
\vee_{i=1}^{g+r+1} S^1 \longrightarrow X/\sigma \longrightarrow S^2,
\]
3. Real and quaternionic bundles

Let $\pi : E \to X$ be a holomorphic vector bundle over $X$. Let $\overline{E}$ be the same real vector bundle, with the conjugate complex structure. Note that $\sigma^* \overline{E}$ is also a holomorphic vector bundle over $X$. By a lift of $\sigma$ to $E$ we mean an anti-holomorphic isomorphism, antilinear on the fibers

$$\widetilde{\sigma} : E \to E,$$

making the diagram

$$\begin{array}{ccc}
E & \xrightarrow{\widetilde{\sigma}} & E \\
\downarrow \pi & & \downarrow \pi \\
X & \xrightarrow{\sigma} & X
\end{array}$$

commute. Note that this is equivalent to a holomorphic isomorphism of vector bundles $E \cong \sigma^* (\overline{E})$. We will be studying real and quaternionic bundles over real curves:

Definition 3.3. A pair $(E, \widetilde{\sigma})$ as in (3.1) is said to be real if the composition

$$E \xrightarrow{\widetilde{\sigma}} E \xrightarrow{\widetilde{\sigma}} E$$

is the identity map of $E$.

A pair $(E, \widetilde{\sigma})$ as in (3.1) is said to be quaternionic if the composition

$$E \xrightarrow{\widetilde{\sigma}} E \xrightarrow{\widetilde{\sigma}} E$$

coincides with $-\text{Id}_E$.

A simple vector bundle $E$ is one such that there are no holomorphic endomorphisms $E \to E$ apart from the constant scalar multiplications.

Proposition 3.1. Let $E \to X$ be simple of positive rank. Suppose that $E \cong \sigma^* (\overline{E})$; then $E$ admits either a real or or a quaternionic structure, but $E$ cannot have both.

Proof. Take an anti-holomorphic isomorphism

$$\widetilde{\sigma} : E \to E,$$

lifting $\sigma$. Since $\widetilde{\sigma}^2 : E \to E,$

$$\widetilde{\sigma}^2 = c \cdot \text{Id}_E,$$

where $c \in \mathbb{C}^*$. Since $\widetilde{\sigma}^2$ commutes with $\widetilde{\sigma}$, it follows that $c \in \mathbb{R} \setminus \{0\}$. Note that if $\widetilde{\sigma}$ is replaced by $a \cdot \widetilde{\sigma}$, where $a \in \mathbb{R} \setminus \{0\}$, then $(\widetilde{\sigma})^2$ gets replaced by $a^2(\widetilde{\sigma})^2$. Therefore, $E$ admits either a real structure or a quaternionic structure.

If one has a real structure $\widetilde{\sigma}$ and a quaternionic structure $\hat{\sigma}$ on $E$, then $\widetilde{\sigma}^2 = 1$, $\hat{\sigma}^2 = -1$, and $\widetilde{\sigma} \circ \hat{\sigma} = c$ for some constant $c$. Therefore,

$$-1 = \widetilde{\sigma}^2 \hat{\sigma}^2 = \widetilde{\sigma} (\widetilde{\sigma} \hat{\sigma}) \hat{\sigma} = \widetilde{\sigma} (c \cdot \hat{\sigma}) \hat{\sigma} = c \overline{\sigma} \hat{\sigma} = \overline{c} \hat{\sigma},$$

a contradiction. \qed

One can easily find a decomposable bundle which is both real and quaternionic. For example, take $E = \mathcal{O}_X \oplus \mathcal{O}_X$. Let $\sigma_0$ be the real structure on $\mathcal{O}_X$ defined by $f \mapsto f \circ \sigma$. So $E$ has the real structure $\sigma_0 \oplus \sigma_0$. On the other hand, for any $J \in \text{GL}(2, \mathbb{C})$ with $J^2 = -\text{Id}$, composing $\sigma_0 \oplus \sigma_0$ with $J$ we get a quaternionic structure on $E$. 
Proposition 3.2. Consider two triples \((E, h, \hat{\sigma})\) and \((E', h', \hat{\sigma}')\), where
- \((E, h)\) and \((E', h')\) are flat holomorphic hermitian vector bundles over \(X\) of rank \(n\),
- \(\hat{\sigma}\) and \(\hat{\sigma}'\) are either both real structures or both quaternionic structures, and
- \(\hat{\sigma}\) and \(\hat{\sigma}'\) are unitary with respect to \(h\) and \(h'\) respectively.

Assume that the holomorphic vector bundle \(E\) is isomorphic to \(E'\). Then there is a holomorphic isomorphism 
\[ f : E \rightarrow E' \]
that transports \(h\) to \(h'\) and \(\hat{\sigma}\) to \(\hat{\sigma}'\).

Proof. We will first show that there is a holomorphic isomorphism 
\[ f : E \rightarrow E' \]
that transports \(\hat{\sigma}\) to \(\hat{\sigma}'\). For this, consider the complex vector space 
\[ \mathcal{V} := H^0(X, \text{Hom}(E, E')) = H^0(X, E' \otimes E^*) \].
We have a conjugate linear isomorphism 
\[ \varphi : \mathcal{V} \rightarrow \mathcal{V} \]
defined by \(T \mapsto - \hat{\sigma}' \circ T \circ \hat{\sigma}\). Let 
\[ \mathcal{V}_\varphi \subset \mathcal{V} \]
be the real subspace fixed by \(\varphi\). Since both \(\hat{\sigma}\) and \(\hat{\sigma}'\) are involutions, it follows that \(\varphi\) is also an involution. Therefore, the natural map 
\[ \mathcal{V}_\varphi \oplus \sqrt{-1} \mathcal{V}_\varphi \rightarrow \mathcal{V} \]
is an isomorphism; the automorphism \(\varphi\) acts on \(\sqrt{-1} \mathcal{V}_\varphi\) as multiplication by \(-1\).

Since \(E\) is holomorphically isomorphic to \(E'\), there is a nonempty Zariski open subset of \(\mathcal{V}\) consisting of isomorphisms from \(E\) to \(E'\). Any nonempty Zariski open subset of \(\mathcal{V}\) must intersect \(\mathcal{V}_\varphi\). Any isomorphism \(E \rightarrow E'\) lying in \(\mathcal{V}_\varphi\) takes \(\hat{\sigma}\) to \(\hat{\sigma}'\).

This reduces us to the situation of a bundle \(E\), with real structure \(\hat{\sigma}\), and two Hermitian metrics \(h, h'\), with \(\sigma\) unitary for both of them. Since the Hermitian metrics \(h, h'\) on \(E\) give rise to flat connections, \(E\) is polystable of degree zero, that is a sum \(\bigoplus_j V_j\) of stable bundles, with all summands of degree zero; let us write this as \(\bigoplus_{\alpha} V^{\oplus n_{\alpha}}\), with the \(V_{\alpha}\) distinct bundles. Each summand \(V^{\oplus n_{\alpha}}\) is orthogonal to the others with respect to both \(h\) and \(h'\); on these summands, the two metrics are related by an element of \(Gl(n_{\alpha})\). Let \(V \subset E\) be a subbundle of \(E\) of the form \(V = \bigoplus_{\alpha} V^{\oplus k_{\alpha}}\), then, for either of our two metrics \(h, h'\):
- \(V\) is preserved by the Chern connections associated to the metric,
- the orthogonal complement \(V^{\perp} = \bigoplus_{\alpha} V^{\oplus (n_{\alpha} - k_{\alpha})}\) is also preserved, and
- since the map \(\hat{\sigma}\) is unitary, if \(V\) is preserved by \(\hat{\sigma}\), then \(V^{\perp}\) is also preserved by \(\hat{\sigma}\).

These facts allow us to proceed inductively: it will be enough to prove the proposition under the assumption that \((E, \hat{\sigma})\) is irreducible; this means that the only holomorphic subbundles of \(E\) preserved by \(\hat{\sigma}\) are the zero subbundle and \(E\) (respectively, the zero subbundle and \(E'\)).

If \((E, \hat{\sigma})\) is irreducible, then there are exactly two possibilities:
• $H^0(X, \text{Hom}(E,E)) = \mathbb{C}$, in particular, $E$ is indecomposable.
• $H^0(X, \text{Hom}(E,E)) = \mathbb{C} \oplus \mathbb{C}$; in this case, $E = F \oplus \sigma^* F$,
  \[ H^0(X, \text{Hom}(F,F)) = \mathbb{C}, \]
  and $F \neq \sigma^* F$.

In the second case, where $\mathcal{V} := H^0(X, \text{Hom}(E,E)) = \mathbb{C} \oplus \mathbb{C}$, the hermitian metrics on $E = F \oplus \sigma^* F$

is induced by hermitian metrics on $F$. Also, the real subspace

$\mathcal{V}^ρ \subset \mathcal{V} = \mathbb{C} \oplus \mathbb{C}$

coincides with all homomorphisms of the form $(\lambda, \lambda)$, where $\lambda \in \mathbb{C}$.

In both cases, scaling a holomorphic isomorphism from $E$ to $E'$ by a suitable real number, the required isomorphism is obtained. \[\square\]

4. The topology of real and quaternionic bundles

As above, let $X$ be a Riemann surface of genus $g$ with an anti–holomorphic involution $\sigma$. We are interested in the classification of holomorphic vector bundles with real or quaternionic structures, i.e., lifts $\tilde{\sigma}$ of the involution $\sigma$ to anti–holomorphic maps on the bundles, antilinear on the fibers, satisfying $\tilde{\sigma}^2 = 1$ for the real structures, and $\tilde{\sigma}^2 = -1$ for the quaternionic structures.

A first step lies in understanding how these bundles are classified topologically. Rank $n$ vector bundles on any manifold are classified by homotopy classes of maps into the classifying space $B\mathcal{U}_n$. This last space is a Grassmannian of $n$–planes in infinite dimensional complex space, obtained as a limit of the Grassmannians of $n$–planes in $N$–dimensional space. We note that these Grassmannians, and the universal bundles over them, can be given real and quaternionic structures, by considering the natural actions of the involutions on $\mathbb{C}^N$:

• Real case: $\rho(x_1, \cdots, x_n) = (\overline{x}_1, \cdots, \overline{x}_N)$
• Quaternionic case: $\rho(x_1, \cdots, x_{2k}) = (\overline{x}_2, \overline{x}_1, \cdots, \overline{x}_{2k}, \overline{x}_{2k-1})$, where $N = 2k$.

Noting now that one can average sections $((s + \tilde{\sigma}(s))/2)$ to make them real or quaternionic, the usual arguments (see Milnor and Stasheff [MS, Section 5]) using partitions of unity give us continuous maps of the Riemann surfaces into the Grassmannians in such a way that the bundles with their real or quaternionic structures are given by pull–backs; also, involution–equivariant homotopy classes of involution–equivariant maps correspond to isomorphism classes of real or quaternionic bundles.

4.1. Real case. Subcase 1: $X$ has real points.

Let $X(\mathbb{R})$, the curve of points fixed by $\sigma$, have $r$ components; the quotient $X/\sigma$ is then a surface with boundary $X(\mathbb{R})$, that is $r$ boundary circles. We have given above a cell decomposition of $X/\sigma$ with 0–cells on the boundary, one–cells that are either on the boundary, or only have ends on the boundary, and with one two–cell. This lifts to a cell decomposition of $X$ such that the cells $C$ are either on the real components, or are such that the pair $C, \sigma(C)$ is distinct. In particular, we have two 2–cells.
The real subspace of $BU_n$ is simply $BO_n$. We want to classify $\sigma$–invariant maps into $BU_n$; $X(\mathbb{R})$ is then mapped to $BO_n$. We map the 0–skeleton to a point, and then consider the images of the boundary circles. The image of each boundary circle gives a homotopy class in $\pi_1(BO_n) = \mathbb{Z}/2$; this corresponds to the first Stiefel–Whitney class of the real $\mathbb{R}^n$–bundle over the boundary circle.

Having fixed these homotopy classes, we want to see what degrees of freedom remain. The rest of the one–skeleton can be contracted to a point: one simply chooses one cell $C, \sigma(C)$, contracts it ($BU_n$ is simply connected), and applies the involution in $BU_n$ to obtain the homotopy for $\sigma(C)$. The remaining degree of freedom then lies in the two two–cells $c_2, \sigma(c_2)$. The homotopy classes of attachings of $c_2$ with fixed boundary are classified by $\pi_2(BU_n) = \mathbb{Z}$; as what one does to $\sigma(c_2)$ is determined by what one does to $c_2$, we are done.

It is useful however, to be a bit more explicit. One can realise the 2–dimensional class generating $H_2(BU_n)$ corresponding to the first Chern class as a complex projective line $S$, whose equator, in turn, is a one dimensional real projective line representing the 1–dimensional generator of $H_1(BO_n)$. The map of the surface into the classifying space can be chosen to lie in $S$, as follows. We start with the one-skeleton. Recall that the real components of the curve are a certain number of circles; on some of them, say $s$ of them, the Stiefel–Whitney classes of the subbundle $E_\mathbb{R}$ over $X(\mathbb{R})$ consisting of elements fixed by $\tilde{\sigma}$ is non-zero; these circles can each get mapped homeomorphically, with the same orientation, to the equator of our sphere; the other boundary circles, and the other one-cells, can simply be mapped to the base point. Once one has this, the disk $c_2$ can be homotoped while fixing the boundary ($BU_n$ is the union of $S$ and cells of dimension at least four) to $s$ times the northern hemisphere of $S$, plus a sum of $k$ copies of $S$, all with a positive orientation. The disk $\sigma(c_2)$ is then mapped to $s$ times the southern hemisphere of $S$, plus a sum of $k$ copies of $S$, again all with a positive orientation (the changes of orientations on the surface under $\sigma$ and on $S$ under reflection in the equator cancel). Thus our surface $X$ is mapped to $2k+s$ times the 2–sphere $S$, and so:

**Proposition 4.1.** Real rank $n$ vector bundles $E$ on a real surface $X$ with real points are classified topologically by the first Stiefel–Whitney classes

$$w_1(E_{i,\mathbb{R}}) \in H^1(X_{i,\mathbb{R}}, \mathbb{Z}/2) = \mathbb{Z}/2$$

of $E_{\mathbb{R}}$ over the components $X_{i,\mathbb{R}}$ of $X_{\mathbb{R}}$, and the first Chern class $c_1(E) \in \mathbb{Z}$, subject to the restriction $c_1(E) \equiv \sum_i w_1(E_{i,\mathbb{R}}) \mod (2)$. All such combinations of Stiefel–Whitney classes and Chern classes do occur.

Subcase 2: $X$ has no real points. This case is somewhat simpler. Here one takes the cell decomposition of $S_0$ given in Section 2, and lifts it to $X$, so that one has a cell decomposition consisting of pairs $C, \sigma(C)$. Choosing one cell from each pair, one can homotope the 0–skeleton and one–skeleton in turn to a single point in $BO_n$. Then the two two–cells $c_2, \sigma(c_2)$ are each homotopic to $k$ copies of the standard two–spheres, so that the total degree over all of $X$ is $2k$.

**Proposition 4.2.** Real rank $n$ vector bundles $E$ on a real surface $X$ with no real points are classified topologically by their first Chern class $c_1(E)$, which must be even. All such degrees do occur.
4.2. **Quaternionic case.** Subcase 1: Odd rank. Here, as there is no odd-dimensional quaternionic vector space, the curve $X$ must have no real points.

We consider first curves $X$ of even genus $g = 2\hat{g}$, with the cell decomposition given in Section 2. Now consider mappings of this to $BU_n$, on which the involution $\rho$ acts without fixed points. Choosing base points $q, \rho(q)$ on the equator of our standard sphere $S$, on which $\rho$ acts as the antipodal map, we can assume that $x_0$ is mapped to $q$, and so $\sigma(x_0)$ to $\sigma(q)$. Going on to the one–skeleton, we can then move the cells $\gamma, \sigma(\gamma)$ to the equator of $S$, and homotope all the other 1–cells to points. The two–cell $c_2$ can then be homotoped to a copy of the northern hemisphere of $S$, plus a certain number $k$ of copies of $S$; the cell $\sigma(c_2)$ becomes the southern hemisphere of $S$, plus the same number $k$ of copies of $S$. The total degree becomes $2k + 1$, telling us that the first Chern of the resulting bundle must be odd, and that this classifies the bundles.

In turn, we consider curves of odd genus $g = 2\hat{g} + 1$, again with the cell decomposition given in Section 2. For our mappings of $X$ to $BU_n$, we can again assume that $x_0$ is mapped to $q$, and so $\sigma(x_0)$ to $\sigma(q)$. The one–skeleton, apart from $\delta, \sigma(\delta)$ now contracts completely to the base points. One then has that the image of the closure of the union of $\delta$ and the 2–cell $c_2$, spans a multiple $k$ of the standard two–sphere in $BU_n$, and as before, the same must be true of their image under $\sigma$. This forces the total degree now to be even, and so the first Chern of the resulting bundle must be even, and this classifies the bundles.

Subcase 2: Even rank $n = 2k$. In this case, one can have fixed points for the real structure; they get mapped under the classifying map to the fixed point set of $\rho$, which is $BSp_k$ in $BU_n$. The classification now is quite simple: taking any of the cell decompositions given above, we can homotope their one–skeletons to a single point in $BSp_k$, and then the two two–cells each give a multiple $k$ of the standard sphere $S$; this gives degree $2k$ in total, so that the degree of the bundle must be even.

Summarising:

**Proposition 4.3.** The quaternionic bundles of rank $n$ over a real Riemann surface of genus $g$ are classified by their degree $k$, with all degrees satisfying $k + n(g - 1) \equiv 0 \mod (2)$ occurring.

An alternative proof of the necessity of the condition when the bundle is holomorphic is given as follows: assume that we have a quaternionic bundle $E$ of rank $n$, degree $k$; if one tensors this bundle by the real line bundle $L = d(O(p + \sigma(p)))$ of degree $2d$, then for high enough $d$ the first cohomology group of $E \otimes L$ vanishes, and the dimension of the zeroth cohomology is given by the Riemann–Roch formula: $h^0 = k + n(2d) + n(-g + 1)$. The bundle $E \otimes L$ is still quaternionic, so the space of sections has a quaternionic structure and must be of even dimension.

5. **The fundamental group, its unitary representations, and flat bundles**

5.1. **Fundamental groups.** If the involution $\sigma$ is free, the fundamental group of $X$ and the fundamental group of the quotient $X/\sigma$ fit into an exact sequence:

\[
(5.1) \quad 0 \longrightarrow \pi_1(X) \longrightarrow \pi_1(X/\sigma) \longrightarrow \mathbb{Z}/2 \longrightarrow 0,
\]
If the involution $\sigma$ has fixed points, the above no longer holds; for example, if one takes the Riemann sphere with the involution induced by conjugation, the quotient is a disk. Whether $\sigma$ does have fixed points or not, however, the surface $X/\sigma$ has a natural structure of a $\mathbb{Z}/2$–orbifold, and we can still have the diagram \((5.1)\), provided we replace $\pi_1(X/\sigma)$ by the orbifold fundamental group $\Gamma$:

\[
0 \longrightarrow \pi_1(X) \longrightarrow \Gamma \longrightarrow \mathbb{Z}/2 \longrightarrow 0,
\]

The orbifold fundamental group of $X/\sigma$ is also known as the equivariant fundamental group of $X$. Explicitly, we can define it as follows.

We choose a base point $x_0 \in X$ such that $\sigma(x_0) \neq x_0$. Consider the space of all homotopy classes of continuous paths $\gamma : [0, 1] \to X$, where homotopies fix the end points, such that

- $\gamma(0) = x_0$, and
- $\gamma(1) \in \{x_0, \sigma(x_0)\}$.

Thus the homotopy classes split into two disjoint sets, depending on the image $\gamma(1)$: the first is simply the fundamental group $\pi_1(X) = \pi_1(X, x_0)$, and the second is the set $\text{Path}(X) = \text{Path}(X, x_0)$ of all homotopy classes of paths from $x_0$ to $\sigma(x_0)$. The disjoint union of $\pi_1(X)$ and $\text{Path}(X)$ will be our group $\Gamma$, with the following composition rule: let $\gamma_1, \gamma_2 \in \Gamma$. If $\gamma_1(1) = x_0$, then define

$$\gamma_2 \gamma_1 = \gamma_2 \circ \gamma_1,$$

where $\circ$ is the usual composition of paths. If $\gamma_1(1) = \sigma(x_0)$, then define

$$\gamma_2 \gamma_1 := \sigma(\gamma_2) \circ \gamma_1.$$

Note that for $\gamma \in \text{Path}(X)$, we have

\[
\gamma^{-1}(t) = (\sigma(\gamma))(1 - t)
\]

It follows that for any two paths $\gamma_1, \gamma_2 \in \text{Path}(X)$

\[
\gamma_2^{-1} \gamma_1 = \gamma_2^{-1} \circ \gamma_1
\]

Finally, mapping $\pi_1(X)$ to zero and $\text{Path}(X)$ to one, we have the exact sequence \((5.2)\), as desired.

One can give explicit presentations of $\Gamma$. These are computed in \([\text{Hu}]\), which we recall; we will also give some alternate presentations, more compatible with the standard ones of $X$.

**Case 0:** $X$ is of type 0. The results of \([\text{Hu}]\) give:

$$\Gamma = \langle \delta_1, \ldots, \delta_{g+1} | \delta_1^2 \cdots \delta_{g+1}^2 = 1 \rangle,$$

where $g$ is the genus of the curve. All the generators $\delta_i$ belong to $\text{Path}(X)$.

Alternatively, one can take the standard basis for $\pi_1(X)$ produced in Section 2; the group $\Gamma$ has $\gamma$ as an extra generator. In the even genus ($g = 2\tilde{g}$) case:

$$\Gamma = \langle \{\alpha_i, \beta_i\}_{i=1}^{2\tilde{g}}, \gamma | \prod_{i=1}^{\tilde{g}}[\alpha_i, \beta_i] \gamma^2 = 1, \prod_{i=1}^{\tilde{g}}[\alpha_i, \beta_i] = 1 \rangle,$$
and in the odd genus \((g = 2\hat{g} + 1)\) case:

\[
\Gamma = \langle \{\alpha_i, \beta_i\}_{i=1}^{2\hat{g}+1}, \gamma \mid \gamma^2 = \alpha_{\hat{g}+1}, \prod_{i=1}^{2\hat{g}+1} [\alpha_i, \beta_i] = 1 \rangle,
\]

**Case I: \(X\) is of type I.** Let \(r\) be the number of connected components of \(X(\mathbb{R})\). Equivalently, \(r\) is the number of boundary components of \(X/\sigma\). Let \(\hat{g}\) be the genus of \(X/\sigma\). Since the “double” of \(X/\sigma\) is a compact connected topological surface without boundary of genus \(g\), one has \(2\hat{g} + r = g + 1\). The orbifold fundamental group \(\Gamma\) is generated by \(\delta_1, \ldots, \delta_{r+\hat{g}}, \eta_1, \ldots, \eta_{r+\hat{g}}\) subject to the relations (Hu)

\[
\eta_i^2 = 1\text{ and } [\delta_i, \eta_i] = 1, \text{ for } i = 1, \ldots, r, \text{ and } \\
\delta_1 \cdots \delta_r \cdot [\delta_{r+1}, \eta_{r+1}] \cdots [\delta_{r+\hat{g}}, \eta_{r+\hat{g}}] = 1.
\]

The generators \(\eta_1, \ldots, \eta_r\) belong to \(\text{Path}(X)\). The other generators are contained in the subgroup \(\pi_1(X)\).

Alternately, one can choose the base point \(x_0\) on the fixed point set, and let \(\gamma\) be the constant path from \(x_0\) to \(\sigma(x_0)\). Then, in \(\Gamma = \pi_1(X) \cup \text{Path}(X)\), \(\gamma^2 = 1\) and \(\gamma\alpha = \sigma(\alpha)\gamma\) for any element \(\alpha\) of \(\pi_1(X)\). The generators are then given by standard generators for the fundamental group \(\pi_1(X)\), plus \(\gamma\):

\[
\Gamma = \langle \{\alpha_i, \beta_i\}_{i=1}^{g}, \gamma \mid \gamma^2 = 1, \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1, \gamma\alpha_i = \sigma(\alpha_i)\gamma, \gamma\beta_i = \sigma(\beta_i)\gamma \rangle.
\]

**Case II: \(X\) is of type II.** Let \(r\) be the number of connected components of \(X(\mathbb{R})\). Again, \(r\) is also equal to the number of boundary components of \(X/\sigma\). Let \(k\) be the genus of \(X/\sigma\). It is understood here that the genus of a nonorientable surfaces is equal to the number of cross–caps. In any case, it is easily seen that \(k + r = g + 1\). Then the orbifold fundamental group \(\Gamma\) is generated by \(\delta_1, \ldots, \delta_r, \eta_1, \ldots, \eta_{r+k}\) subject to the relations

\[
\eta_i^2 = 1\text{ and } [\delta_i, \eta_i] = 1, \text{ for } i = 1, \ldots, r, \text{ and } \\
\delta_1 \cdots \delta_r \cdot \eta_{r+1}^2 \cdots \eta_{r+k}^2 = 1.
\]

The generators \(\eta_1, \ldots, \eta_{r+k}\) belong to \(\text{Path}(X)\). The other generators are contained in the subgroup \(\pi_1(X)(\text{Hu})\).

Alternately, one can give the same description as in case I:

\[
\Gamma = \langle \{\alpha_i, \beta_i\}_{i=1}^{g}, \gamma \mid \gamma^2 = 1, \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1, \gamma\alpha_i = \sigma(\alpha_i)\gamma, \gamma\beta_i = \sigma(\beta_i)\gamma \rangle.
\]

Note that these are not quite presentations, as we have not specified the action of \(\sigma\).

5.2. **Real and quaternionic unitary representations.** We are interested in unitary representations of \(\pi_1(X, x_0)\) which extend to \(\Gamma\) in an appropriate way.

Let the Galois group \(\text{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\) act naturally on the unitary group \(U_n\). Such an action gives rise to a semi–direct product of \(U_n\) and \(\text{Gal}(\mathbb{C}/\mathbb{R})\), which we will denote by \(\tilde{U}_n\). We call \(\tilde{U}_n\) the extended unitary group. It acts naturally on \(\mathbb{C}^n\) by complex
linear and conjugate linear unitary automorphisms. If we denote complex conjugation
\((z_1, \ldots, z_n) \rightarrow (\overline{z}_1, \ldots, \overline{z}_n)\) of \(\mathbb{C}^n\) by \(\tau\), then \(\tilde{U}_n\) decomposes into two components
\[\tilde{U}_n = U_n \cup U_n \tau.\]
Since \(U_n\) is a normal subgroup in \(\tilde{U}_n\), one has a short exact sequence
\[(5.5)\]
\[e \rightarrow U_n \rightarrow \tilde{U}_n \xrightarrow{\xi} \mathbb{Z}/2 \rightarrow e,\]
**Definition 5.6.** Let \(\rho \in \text{Hom}(\pi_1(X), U_n)\) be a representation of \(\pi_1(X)\).

We say that the representation \(\rho\) has a real extension \(\hat{\rho}\) if it extends as a representation \(\hat{\rho}\) from \(\pi_1(X)\) to \(\Gamma\) in such a way that the diagram
\[
\begin{array}{ccc}
e & \longrightarrow & \pi_1(X) \\
\downarrow & & \downarrow \hat{\rho} \\
e & \longrightarrow & \Gamma \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow e
\end{array}
\]
is commutative. The extension \(\hat{\rho}\) will be referred to as a real unitary representation.

We say that the representation \(\rho\) has a quaternionic extension \(\hat{\rho}\) if it extends as a map from \(\Gamma\) to \(\tilde{U}(n)\) in such a way that the diagram
\[
\begin{array}{ccc}
e & \longrightarrow & \pi_1(X) \\
\downarrow & & \downarrow \hat{\rho} \\
e & \longrightarrow & U(n) \longrightarrow \tilde{U}(n) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow e
\end{array}
\]
commutes, and that \(\rho(a) \rho(b) = (-1)^{\xi(a)\xi(b)} \rho(ab)\), where one takes now \(\mathbb{Z}/2\) to be the set \(\{0, 1\}\). The extension \(\hat{\rho}\) will be referred to as a quaternionic unitary representation.

**Proposition 5.1.** For two extensions \(\hat{\rho}\) and \(\hat{\rho}\) consider for \(x \in \text{Path}(X)\) the unitary matrix \(C(x) = \hat{\rho}(x)^{-1} \overline{\hat{\rho}(x)}\). This \(C(x) = C\) is independent of \(x\) in \(\text{Path}(X)\). Furthermore, \(C\) commutes with the representation \(\rho\) on \(\pi_1(X, x_0)\).

If \(\rho : \pi_1(X, x_0) \rightarrow U(n)\) is irreducible, it cannot simultaneously have a real and a quaternionic extension.

**Proof.** For \(x, y \in \text{Path}(X)\), we have
\[
\hat{\rho}(x)^{-1} \hat{\rho}(x)(\hat{\rho}(y)^{-1} \hat{\rho}(y))^{-1} = \hat{\rho}(x)^{-1} \hat{\rho}(xy^{-1}) \hat{\rho}(y) = \hat{\rho}(x)^{-1} \hat{\rho}(xy^{-1}) \hat{\rho}(y) = 1
\]
(note that in the quaternionic case, \(\hat{\rho}(x^{-1}) = -\hat{\rho}(x)^{-1}\)). We have, for \(z\) in \(\pi_1(X)\), that
\[
\hat{\rho}(x)^{-1} \hat{\rho}(x) \rho(z)(\hat{\rho}(x)^{-1} \hat{\rho}(x))^{-1} = \hat{\rho}(x)^{-1} \hat{\rho}(xzx^{-1}) \hat{\rho}(x)
\]
\[
= \hat{\rho}(x)^{-1} \hat{\rho}(xzx^{-1}) \hat{\rho}(x) = \hat{\rho}(z) = \rho(z),
\]
so that \(C\) commutes with the representation.

Let \(\hat{\rho}\) be real and let \(\hat{\rho}\) be quaternionic. For \(x \in \text{Path}(X)\), we have \(x^2 \in \pi_1(X, x_0)\). Hence if \(\hat{\rho}(x)^{-1} \circ \hat{\rho}(x) = C \cdot \text{Id}\),
\[
1 = \hat{\rho}(x^{-2}) \hat{\rho}(x^2) = -\hat{\rho}(x)^{-1} (C^\ast) \hat{\rho}(x) = -\overline{C} C,
\]
a contradiction. \(\square\)
5.3. **Unitary representations and flat bundles.** Now let

\[ \text{Rep}_R(X, n) \]

denote the family of real representations of \( \Gamma \), and

\[ \text{Rep}_H(X, n) \]

denote the family of quaternionic representations of \( \Gamma \).

Two homomorphisms \( \rho, \rho' \in \text{Rep}_R(X, n) \) (or both in \( \text{Rep}_H(X, n) \)) are called *equivalent* if there is an element \( A \in U_n \) such that

\[ \rho'(g) = A \rho(g) A^{-1}, \]

for all \( g \in \Gamma \). The set of equivalence classes of elements of \( \text{Rep}_R(X, n) \), (respectively, \( \text{Rep}_H(X, n) \)) will be denoted by \( \overline{\text{Rep}_R}(X, n) \) (respectively, \( \overline{\text{Rep}_H}(X, n) \)).

**Theorem 5.2.** There is a natural bijective correspondence between the equivalence classes of elements of \( \overline{\text{Rep}_R}(X, n) \) (respectively, \( \overline{\text{Rep}_H}(X, n) \)) and the isomorphism classes of triples \( (V, \nabla^V \sigma) \), where \( (V, \nabla^V) \) is a unitary flat vector bundle over \( X \) of rank \( n \) and

\[ \sigma : V \longrightarrow V \]

is a real (respectively, quaternionic) structure on \( V \) such that \( \nabla^V \) is \( \sigma \)-equivariant:

\[ \nabla^V(x) = (\tilde{\sigma}^{-1})^* \nabla^V(\sigma(x)) \]

**Proof.** 1. *From representation to bundle.* Let us start with a representation \( \rho \) of \( \Gamma \). Its restriction to \( \pi_1(X) \) defines in the usual fashion a flat bundle over \( X \): one starts with the trivial flat unitary bundle \( \tilde{X} \times \mathbb{C}^n \) over the universal cover \( \tilde{X} \), and if \( g \in \pi_1(X) \) is represented by a deck transformation, one identifies \( p \times \mathbb{C}^n \) with \( g(p) \times \mathbb{C}^n \) in such a way that the trivial flat bundle descends to a bundle \( E \) on \( X \) with monodromy representation given by \( \rho \).

If \( \gamma \) is any homotopy class of paths from \( x \) to \( y \), let \( T_\gamma \) represent the parallel transport from \( x \) to \( y \). If \( x \) and \( y \) are both the base point \( x_0 \), one can identify

\[ T_\gamma = \rho(\gamma) : E_{x_0} \longrightarrow E_{x_0}. \]

We define the lift \( \tilde{\sigma} \) of \( \sigma \) to \( E \) as follows. At \( x_0 \), we choose a path \( \gamma \) from \( x_0 \) to \( \sigma(x_0) \), and set:

\[
\begin{align*}
(5.7) & \quad \tilde{\sigma}_{x_0} : E_{x_0} \longrightarrow E_{\sigma(x_0)} \\
(5.8) & \quad v \longmapsto T_\gamma \rho(\gamma)^{-1}(v);
\end{align*}
\]

**Lemma 5.3.** The isomorphism \( \tilde{\sigma}_{x_0} \) is independent of the choice of the path \( \gamma \) connecting \( x_0 \) to \( \sigma(x_0) \).

**Proof.** Take any other path \( \delta : [0, 1] \longrightarrow X \) such that \( \delta(0) = x_0 \) and \( \delta(1) = \sigma(x_0) \). Consider the element \( \delta^{-1}\gamma = \delta^{-1} \circ \gamma \in \pi_1 \). Since \( \rho(\delta^{-1}\gamma) = \rho(\delta)^{-1}\rho(\gamma) \), and \( T_{\delta^{-1}\gamma} = T_{\delta^{-1}} \circ T_\gamma \), the result follows. \( \square \)

For other \( x \in X \), choose any path \( \delta \) from \( x \) to \( x_0 \), and set

\[
\begin{align*}
(5.9) & \quad \tilde{\sigma}_x : E_x \longrightarrow E_{\sigma(x)} \\
(5.10) & \quad v \longmapsto T_{\rho(\delta)^{-1}\sigma(x_0)^{-1}} T_{\delta}
\end{align*}
\]
Lemma 5.4. The isomorphism \( \tilde{\sigma}_x \) is independent of the choice of the path \( \delta \) connecting \( x \) to \( x_0 \).

Proof. If \( \delta, \delta' \) are two paths, one wants to show that
\[
\tilde{\sigma}_{x_0} = T_{\sigma(\delta')} T_{\sigma(\delta)}^{-1} \tilde{\sigma}_{x_0} T_{\delta} T_{\delta'}^{-1}
\]
This means that we want
\[
\tilde{\sigma}_{x_0} = T_{\sigma(\delta')} T_{\sigma(\delta)}^{-1} \sigma (\gamma) \rho (\delta \delta'^{-1})
\]
This however is precisely the definition of \( \tilde{\sigma}_{x_0} \) using the path \( \sigma (\delta') \circ \sigma (\delta)^{-1} \circ \gamma \)

Lemma 5.5. Parallel transport is invariant under \( \tilde{\sigma} \): if \( \eta \) joins \( x \) to \( y \), then
\[
T_{\sigma(\eta)} = \tilde{\sigma}_y T_{\eta} \tilde{\sigma}_x^{-1}
\]
This is just a consequence of the definition of \( \tilde{\sigma}_y \) by parallel transport from the base point.

Lemma 5.6. \( \tilde{\sigma}_{\sigma(x)} \tilde{\sigma}_x = 1 \), if the representation is real, and \(-1\), if it is quaternionic.

Proof. When \( x = x_0 \), one has
\[
\tilde{\sigma}_{\sigma(x_0)} \tilde{\sigma}_{x_0} = T_{\sigma(\gamma^{-1})} T_{\gamma} \rho (\gamma)^{-1} T_{\gamma^{-1}} T_{\gamma} \rho (\gamma)^{-1}
\]
(5.11)
\[
= \rho (\gamma^2) \rho (\gamma)^{-1} \rho (\gamma)^{-1}
\]
(5.12)
which is \(+1\) if the representation is real, and \(-1\) if it is quaternionic. The proof for other points follows by parallel transport.

We have now defined a flat bundle with the right properties from the representation. We now must check that the result is invariant under the equivalence of representations. This is straightforward.

2. From bundle to representation

We now turn things around, and take a real (respectively, quaternionic) bundle \((E, \tilde{\sigma})\), along with a \( \tilde{\sigma} \)-invariant connection \( \nabla \). This then has invariant parallel transport, as defined in (5.5). Choosing a basis for the fiber \( E_{x_0} \) over the base point \( x_0 \), we can, as usual, define the holonomy representation \( \rho : \pi_1(X) \to U_n : \)
\[\rho (g) = T_g\]
One must then extend this to \( \Gamma \). Inverting the procedure of part 1. of the proof gives, for a path \( \gamma \) from \( x_0 \) to \( \sigma (x_0) \):
\[\rho (\gamma) = \tilde{\sigma}_x^{-1} T_{\gamma}\]
It is immediate that this is anti–linear. Composing two elements of \( Path(X) \) gives:
\[
\rho (\gamma) \rho (\gamma') = \tilde{\sigma}_{x_0}^{-1} T_{\gamma} \tilde{\sigma}_{x_0}^{-1} T_{\gamma'}
\]
(5.13)
\[
= \tilde{\sigma}_{x_0}^{-1} \tilde{\sigma}_{\sigma(x_0)}^{-1} T_{\sigma(\gamma)} T_{\gamma'}
\]
(5.14)
\[
= \tilde{\sigma}_{x_0}^{-1} \rho (\gamma) \rho (\gamma)^{-1} \rho (\gamma)^{-1}
\]
(5.15)
\[
= \tilde{\sigma}_{x_0}^{-1} \rho (\gamma \gamma')
\]
(5.16)
which is \( \pm \rho (\gamma \gamma') \) depending on whether the bundle is real or quaternionic.
One checks in a similar fashion that the other compositions (when one of the two elements lies in $\pi_1(X)$) satisfy the right relations. Thus real bundles give real representations, and quaternionic bundles give quaternionic ones. We note that the global conjugations of $\rho$ by a unitary matrix correspond to changes in our unitary trivialisation of $E_{x_0}$; these changes give rise to equivalent representations. This completes the proof of the theorem. □

5.4. **Representations and flat connections on $S_0$.** The correspondence of real or quaternionic representations with flat connections enables (or at least makes more evident) a description of these representations in terms of representations of the fundamental group of the surface $S_0$.

- **Type 0, even genus; real bundles.** In this case, as we have seen, one has that $S_0$ is a once punctured surface of genus $g/2$, with two marked points chosen on the boundary which are interchanged by the real structure. We note that a trivialisation at one of the marked points gives a trivialisation at the other. Restricting flat connections to $S_0$, and noting that apart from some constraints on the boundary, connections on $S_0$ determine real connections $X$, we obtain, by integrating, for the moduli space of flat connections:

\[
\mathcal{M}_R = \left\{ \{A_i, B_i\}_{i=1}^{g/2}, C \in U_n| \prod_{i=1}^{g/2} [A_i, B_i] C\overline{C} = 1 \right\} / U_n
\]

The $U_n$ action is by
\[
(A_i, B_i, C) \mapsto (gA_i g^{-1}, gB_i g^{-1}, gC \overline{g}^{-1})
\]

- **Type 0, odd genus; real bundles.** Here, $S_0$ is a twice punctured surface of genus $(g−1)/2$, with one marked point chosen on each boundary, which are interchanged by the real structure. The moduli space is then

\[
\mathcal{M}_R = \left\{ \{A_i, B_i\}_{i=1}^{(g−1)/2}, C, D \in U_n| \prod_{i=1}^{g/2} [A_i, B_i] C D \overline{C} D^{-1} = 1 \right\} / U_n
\]

The $U_n$ action is by
\[
(A_i, B_i, C, D) \mapsto (gA_i g^{-1}, gB_i g^{-1}, gC g^{-1}, gD \overline{g}^{-1})
\]

- **Type I; real bundles.** Here, $S_0$ is an $r$–punctured surface of genus $\hat{g} = (g−r+1)/2$, with one marked point chosen on each boundary; one has

\[
\mathcal{M}_R = \left\{ \{A_i, B_i\}_{i=1}^{\hat{g}}, \{D_j\}_{j=2}^{r} \in U_n, \{C_j\}_{j=1}^{r} \in O_n \right\}
\]

\[
\prod_{i=1}^{\hat{g}} [A_i, B_i] C_1 \prod_{i=2}^{r} D_j C_j D_j^{-1} = 1 \right\} / (O_n)^r.
\]

The group action is by
\[
(g_1, \cdots, g_r)(A_i, B_i, C_j, D_j) \mapsto (g_1 A_i g_1^{-1}, g_1 B_i g_1^{-1}, g_j C_j g_j^{-1}, g_1 D_j g_1^{-1})
\]
• Type II; real bundles. Here, $S_0$ is an $r+1$–punctured surface of genus $\hat{g} = (g-r)/2$, with one marked point chosen on each boundary; one has

$$\mathcal{M}_R = \left\{ \{A_i, B_i\}_{i=1}^r, C_0, \{D_j\}_{j=1}^r \in U_n, \{C_j\}_{j=1}^r \in O_n \right\}$$

(5.23)

The group action is by

$$(g_0, g_1, \cdots, g_r)(A_i, B_i, C_j, D_j) \mapsto (g_0A_i g_0^{-1}, g_0B_i g_0^{-1}, g_0C_j g_0^{-1}, g_jC_j g_j^{-1}, g_0D_j g_j^{-1})$$

• Quaternionic bundles In all cases the formulae are the same, except that one replaces conjugation ($C \mapsto \overline{C}$) by minus conjugation ($C \mapsto \overline{-C}$) in the formulae, and $O_n$ by $Sp_{n/2}$, when they occur in (5.17), (5.19), (5.21), (5.23).

5.5. Central extensions. So far we have discussed flat bundles, corresponding to representations of the orbifold fundamental group; these bundles of necessity have trivial degree. For bundles of arbitrary degree $k$, one can consider projectively flat connections, or alternately connections on the complement of a point with central monodromy around that point which is given by $\exp(2\pi \sqrt{-1} k/n)$. In our case, it is in fact more convenient to choose a pair of points $p, \sigma(p)$, with $p \neq \sigma(p)$; the constraint on monodromy is then that the monodromy around each point be given by $\exp(\pi \sqrt{-1} k/n)$. (Note that if $L$ is a small loop around $p$, with monodromy $\exp(\pi \sqrt{-1} k/n)$, reality forces the monodromy along $\sigma(L)$ to be $\exp(-\pi \sqrt{-1} k/n)$; on the other hand $\sigma$ also changes orientation.)

From the point of view of representations of the fundamental group $\pi_1(X)$, one is taking a representation of a central extension of the group; real and quaternionic representations correspond to representations of a central extension of $\Gamma$: one has the diagram of central extensions

$$\begin{align*}
  e & \quad \longrightarrow \quad \mathbb{Z} \quad \longrightarrow \quad \pi_1(X)' \quad \longrightarrow \quad \pi_1(X) \quad \longrightarrow \quad e \\
  e & \quad \longrightarrow \quad \hat{\mathbb{Z}} \quad \longrightarrow \quad \hat{\Gamma}' \quad \longrightarrow \quad \hat{\Gamma} \quad \longrightarrow \quad e
\end{align*}$$

(5.25)

From the point of view of connections on $S_0$, only one of our two points, say $p$, lies in $S_0$. Our moduli spaces with non–zero degree is then given by formulae (5.17), (5.19), (5.21), (5.23), but with the $= 1$ replaced by $= \exp(\pi \sqrt{-1} k/n)$.

6. Connections and vector bundles

6.1. Real and quaternionic connections and the Yang–Mills functional. We have examined, for real and quaternionic bundles, both their topological classification and their flat or projectively flat structures. Now we want to study the moduli space of their holomorphic structures. One of the best approaches to understanding these is that pioneered by Narasimhan and Seshadri [NS], whereby the bundles are defined by the $\overline{\partial}$–operator of a unitary connection. The space of these connections has a natural energy functional, the Yang–Mills functional, given by the $L^2$–norm of the curvature of the connections. The gradient flow of this functional preserves the holomorphic structure, and allows us
to relate moduli to the set of critical points, following the approach of Atiyah and Bott [AB].

Let us first consider the case of degree zero. The theorem of Narasimhan and Seshadri [NS], in this context, says that minima of the functional, which are flat connections, correspond to stable or polystable bundles; integrating these connections gives representations of the fundamental group of the surface into the unitary group. The proof of this theorem given by Donaldson in [Do] follows precisely the variational viewpoint.

Extending to bundles of non–zero degree \( k \), the minima of the Yang–Mills functional now correspond to having a curvature tensor that is central, and a constant multiple of the Kähler form. Integrating the connection, however, only gives a representation into \( PU_n \); instead, one can, following the approach of [AB] and our definition of the central extension given above, modify the connection so that it is flat on the complement of a point, with a simple central pole and residue \( 2\pi\sqrt{-1}\frac{k}{N} \) at the point; this amounts to finding an appropriate scalar 1–form. Minima then correspond to representations of the extension of the fundamental group described above.

Now let us put in real structures. We restrict to trivialisations invariant under the real structure. The real structure \( \sigma \) acts on connections by pull–back of forms and conjugation; note that since \( \sigma \) is anti–holomorphic, the real structure interchanges the \((0, 1)\) and \((1, 0)\) components. In local coordinates near a fixed point: \( \sigma^*(A(z)dz + A'(z)d\overline{z}) = \overline{A'}(\sigma(z))dz + \overline{A}(\sigma(z))d\overline{z} \). The connection is real if it is mapped to itself under the involution.

Our involution preserves the Yang–Mills functional, and symmetric criticality gives us that a real holomorphic bundle has a constant central curvature real connection, which is invariant under the real or quaternionic structure, if and only if it is polystable as a complex bundle. (From the Morse flow point of view, if one starts at an invariant connection, one has to end up at one.) For bundles of non–zero degree, we can then modify the connection now by a scalar one–form so that the connection is flat and that at a pair of distinct points \( p, \sigma(p) \) there is a pole whose monodromy is \( \exp(\pi\sqrt{-1}\frac{k}{n}) \).

**Theorem 6.1.** (Real version of Narasimhan–Seshadri theorem.) The moduli space of polystable real (respectively, quaternionic) bundles of degree \( k \) and rank \( n \) is diffeomorphic to the space of equivalence classes of real (respectively, quaternionic) unitary representations of the extended orbifold fundamental group \( \Gamma' \) in \( U(n) \) which map the centre to \( \exp(\pi\sqrt{-1}\frac{k}{n}) \).

Note that stability as a complex bundle for a real bundle is the same as being stable as a real bundle. Indeed, if \( E \) is destabilised by \( F \), it is destabilised either by the sheaf \( F + \sigma^*F \) or by the sheaf \( F \cap \sigma^*F \).

6.2. The space of connections, and gauge groups. The core idea of Atiyah and Bott in developing their understanding of the topology of the space of stable bundles then is to think of it as being the space of minima of the Yang–Mills functional on connections modulo gauge, and think of this topology of the minima in a Morse theoretic way as being obtained, homologically at least, by “subtracting” from the topology of all connections the topology of the higher order critical points. The total space of connections modulo gauge is the quotient of the contractible space \( A \) of connections by the group of
gauge transformations, and this group is close to acting freely; indeed, generically, the stabilisers consist of constant, central gauge transformations. Quotienting the group by these, therefore, our space of connections modulo gauge is almost the classifying space of the quotient group. Indeed, replacing the space of connections by the classifying space of this group allows an inductive computation, at least in the equivariant setting. In our case, the space $\mathcal{A}$ of unitary connections is replaced by the space $\mathcal{A}_\mathbb{R}$ of real ($\sigma$–invariant) connections; it is again an affine space, and so contractible. We again have a group $\mathcal{G}_\mathbb{R}$ of real ($\sigma$–invariant) gauge transformations, which, up to the subgroup $\mathbb{Z}$ of constant central gauge transformations, acts generically freely; the classifying spaces of $\mathcal{G}_\mathbb{R}$ and of $\mathcal{G}_\mathbb{R}/\mathbb{Z}$ are our objects of interest.

We first consider the case of real bundles. In our calculations, we will suppose that $n$ is greater than 2, and give the results at the end for $n = 1, 2$.

Real case, type 0 curves

For curves $X$ of even genus $g = 2\hat{g}$, we consider the cell decompositions of $S_0$ given in Section 2, with in particular one–cells $\alpha_1, \beta_1, \ldots, \alpha_{\hat{g}}, \beta_{\hat{g}}, \gamma$; on $X$, these cells get doubled into pairs $c, \sigma(c)$. Similarly, in the odd genus case, one uses the description of the surface $S_0$ given above, with cells $\alpha_1, \beta_1, \ldots, \alpha_{\hat{g}}, \beta_{\hat{g}}, \gamma, \delta$; again, on $X$, these cells get doubled into pairs $c, \sigma(c)$.

Let us now consider the $\sigma$–equivariant gauge group $\mathcal{G}_\mathbb{R}$ on $X$; we begin by considering the based gauge group $\mathcal{G}_\mathbb{R}^0$ of gauge transformations which are the identity over the base point, taken to lie on the boundary of $S_0$. As a gauge group on $S_0$, $\mathcal{G}_\mathbb{R}$ corresponds to the subgroup of all unitary gauge transformations such that their restriction to the boundary satisfies $g(x) = \overline{g}(\tau(x))$, where $\tau$ is the antipodal map. For the based gauge group, if $x_0$ is the base point, $g(x_0) = g(\tau(x_0)) = Id$. From the cofibration above (2.2), and the cell decompositions we have given (with cells mostly coming in pairs $c, \sigma(c)$), one has the fibration for the based gauge group:

\begin{equation}
\mathcal{G}_\mathbb{R}^0(S^2) \longrightarrow \mathcal{G}_\mathbb{R}^0 \longrightarrow \prod_{i=1}^{g+1} \Omega(U_n),
\end{equation}

where $\mathcal{G}_\mathbb{R}^0(S^2) = \Omega^2(U_n)$ is the based gauge group on the two–sphere. For the based gauge group, the restriction map to the base point on $X$ gives the fibration

\begin{equation}
\mathcal{G}_\mathbb{R}^0 \longrightarrow \mathcal{G}_\mathbb{R} \longrightarrow U_n.
\end{equation}

Finally, we will want to quotient by the (real) centre of $U_n$:

\begin{equation}
\mathbb{Z}/2 \longrightarrow \mathcal{G}_\mathbb{R} \longrightarrow \mathcal{G}_\mathbb{R}.
\end{equation}

This then yields the corresponding fibrations on classifying spaces (we note that $B\Omega G = \Omega G_0$, the connected component of the identity):

\begin{equation}
\Omega(U_n)_0 \longrightarrow B\mathcal{G}_\mathbb{R}^0 \longrightarrow \prod_{i=1}^{g+1} U_n,
\end{equation}

\begin{equation}
B\mathcal{G}_\mathbb{R}^0 \longrightarrow B\mathcal{G}_\mathbb{R} \longrightarrow BU_n,
\end{equation}
We now want the first few homotopy groups of these spaces. We note that \( \pi_1(BG) = \pi_{i-1}(G) \), for any group \( G \).

For the fundamental groups, the homotopy sequences for the fibrations give:

\[
\begin{align*}
0 & \to \pi_1(BG) \to \mathbb{Z}^{g+1} \to 0 \\
\mathbb{Z} & \to \pi_1(BG) \to \pi_1(BG) \to 0 \\
\mathbb{Z}/2 & \to \pi_1(BG) \to \pi_1(BG) \to 0
\end{align*}
\]

The first sequence tells us that \( \pi_1(BG) = \mathbb{Z}^{g+1} \). For the middle sequence we want to know what the image of \( \mathbb{Z} \) is. We work instead with

\[
\pi_1(U_n) = \mathbb{Z} \to \pi_0(G) \to \pi_0(G) \to 0
\]

Let \( f(t) \) represent a generator of \( \pi_1(U_n) \), with \( f(0) = f(1) = 1 \). Choosing a disk around the base point, with a radial coordinate such that \( r = 1 \) is the boundary of the disk, one can lift \( f(t) \) to \( G \), by \( F(t, r) = f(1-r)t \), extending \( F \) by the identity outside the disk, with the obvious exception that near the conjugate of the base point, one has \( F(t, r) \). The question is then whether \( F(1, r) \) lies in the same connected component of \( G \) as the identity. To fix our ideas, let us look first at two cases, the case \( X = S^2 \) and \( n = 1 \), and the case \( X = T^2 \), the torus, and \( n = 1 \). Our real structures are the antipodal map for the two–sphere, the fixed point–free structure for the torus, and complex conjugation on the circle.

**Lemma 6.2.** The \( \sigma \)-equivariant based maps \( g : S^2 \to S^1 \) are classified by \( \mathbb{Z} \); the \( \sigma \)-equivariant unbased maps, by \( \mathbb{Z}/2 \).

The \( \sigma \)-equivariant based maps \( g : T^2 \to S^1 \) which are homotopically trivial as non–equivariant maps are classified by \( \mathbb{Z} \); the \( \sigma \)-equivariant, homotopically trivial as non–equivariant maps, unbased maps, by an element of \( \mathbb{Z}/2 \).

**Proof.** An equivariant map \( g : S^2 \to S^1 \) is determined by a map from the two–disk \( D^2 \) to \( S^1 \), subject to the constraint that the restriction to the boundary circle \( \partial D^2 \) is equivariant. We can lift any map to a map \( D^2 \to \mathbb{R} \), where \( \mathbb{R} \) covers \( S^1 \) in the standard way, mapping the integers to the identity. Now lift our map \( g \) to \( \hat{g} \), with the base point being mapped to \( 0 \in \mathbb{R} \). Choosing a point \( p \) on the boundary circle of \( D^2 \), let \( a = \hat{g}(p) \) be the image of \( p \). On the lifted map to the line, \( \sigma \)-equivariance manifests itself by the requirement on \( \partial D^2 \) that the image of \( \sigma(p) \) in \( \mathbb{R} \) be \( -a + m \), for some integer \( m \). This integer will be our invariant: indeed, any equivariant homotopy \( G \) will have a lift \( \hat{G} \) satisfying \( \hat{G}(t, \sigma(p)) = -\hat{G}(t, p) + m \); there have to be points in the image on both sides of \( m/2 \), if the map is non constant; on the other hand, the boundary can be homotoped to the constant map to \( m/2 \). The mapping on the rest of the disk can then be homotoped to a standard map. If one drops the basing condition, we note that the point \( a \) is only defined up to an integer \( k \); this in turn means that \( m \) is defined only up to \( 2k \). Thus one still has the parity of \( m \) as an invariant, but nothing else.

The maps of the torus follow essentially the same pattern. \( \square \)

In the case of interest to us here, the restriction of our map \( F \) to the cycles \( \alpha_i, \beta_i \) is homotopically trivial: the cycle exits, then enters the disk. One can then up to homotopy
contract these cycles of \( X \) to a point to obtain a map

\[ \hat{F} : S^2 \rightarrow U_n \]

or a map \( \hat{F} : T^2 \rightarrow U_n \), depending on the parity of the genus. One then can take the determinant, to get

\[ \hat{G} : S^2 \rightarrow U_1 \]

or \( \hat{G} : T^2 \rightarrow U_1 \); the fact that our original \( f \) was a generator of the fundamental group of \( U_n \) tells us that the relevant \( m \) in the preceding lemma is \( \pm 2 \). In short, the map \( \mathbb{Z} \rightarrow \pi_1(BG_R^0) \) is an injection.

One can ask what its image is. The isomorphism \( \pi_1(BG_R^0) = \mathbb{Z}^{g+1} \) is given by restriction to the one–skeleton. As noted, on the cycles \( \alpha_i, \beta_i \) this restriction is zero (the cycle exits, then reenters the disk); in the even genus case, on the cycle \( \gamma \), it goes to twice the generator (on \( X \), the cycle exits the disk centred at \( p \) to go to the disk centred at \( \sigma(p) \)); in the odd genus case, the restriction is trivial on \( \gamma \), and twice a generator on \( \delta \). This then gives \( \pi_1(BG_R^0) = \mathbb{Z}^g \oplus \mathbb{Z}/2 \), in both cases.

The final sequence amounts to \( \mathbb{Z}/2 \rightarrow \pi_0(G_R) \rightarrow \pi_0(\overline{G_R}) \rightarrow 0 \). The image of \( \mathbb{Z}/2 \) depends on whether one can deform \( 1 \) to \( -1 \) in \( G_R \). When \( n \) is even, this is possible: setting \( n = 2k \) and choosing a basis so that the real structure is given by \( \sigma(z_1, z_2, \cdots, z_{2k-1}, z_{2k}) = (\overline{z}_2, \cdots, \overline{z}_{2k-1}), \) one can define a real path by diag \( (e^{i\theta}, e^{-i\theta}, \cdots, e^{i\theta}, e^{-i\theta}) \). In the odd dimensional case, one can use the lifting to \( \mathbb{R} \) argument in the lemma above to show that it is not. This gives \( \pi_1(BG_R^0) = \mathbb{Z}^g \oplus \mathbb{Z}/2 \), when \( n \) is even, and \( \mathbb{Z}^g \), when \( n \) is odd.

For the second homotopy group, one then has

\[
\begin{align*}
\mathbb{Z} & \rightarrow \pi_2(BG_R^0) \rightarrow 0 \\
0 & \rightarrow \pi_2(BG_R^0) \rightarrow \pi_2(BG_R) \rightarrow 0 \\
0 & \rightarrow \pi_2(BG_R) \rightarrow \pi_2(\overline{BГ_R}) \rightarrow \mathbb{Z}/2 \rightarrow 0
\end{align*}
\]

(6.8)

One can think of these sequences in terms of classifications of bundles on \( S^2 \times X \); in these terms, the top row map \( \mathbb{Z} \rightarrow \pi_2(BG_R^0) \) in essence is the restriction of the second Chern class, which is non–zero. Thus \( \pi_2(BG_R^0) = \mathbb{Z} \), and so, from the second sequence, \( \pi_2(BG_R) = \mathbb{Z} \). In the last sequence, we have seen that the map to \( \mathbb{Z}/2 \) is onto when \( n \) is even, and zero when \( n \) is odd. When \( n \) is even, one has two classes, one related to the second Chern classes, and the other, to a global path from \( 1 \) to \( -1 \); we saw above that the latter lifted to \( B\hat{G}_R \); thus \( \pi_2(B\hat{G}_R^0) = \mathbb{Z} \oplus \mathbb{Z}/2 \) when \( n \) is even, \( \mathbb{Z} \) when \( n \) is odd.

**Real case, type I curves**

Let \( G_R^0 \) be the group of gauge transformations which are the identity at each base point, with one base point per boundary circle, as above. One notes that, in the one–skeleton, along the \( \gamma_i \) the gauge transformations must lie in the \( O_n \) while elsewhere they lie in \( U_n \). In particular, note that each \( \delta_i \) contributes a \( \Omega(U_n) \) to \( G_R^0 \). The corresponding fibrations are

\[
\Omega^2(U_n) \rightarrow G_R^0 \rightarrow \prod_{i=1}^g \Omega(U_n) \times \prod_{i=1}^r \Omega(O_n),
\]

(6.9)

\[
G_R^0 \rightarrow G_R \rightarrow \prod_{i=1}^r O_n
\]

(6.10)
This is, as above, trivial when $n$ is even. For the second homotopy groups, one has

$$\pi_1(BG^{0}_{\mathbb{R}}) = \pi_0(G^{0}_{\mathbb{R}}) = \mathbb{Z}^{g} \oplus (\mathbb{Z}/2)^{r+1}.$$ Finally, one can also look at the covering (6.11):

$$\mathbb{Z}/2 \longrightarrow \pi_0(G^{0}_{\mathbb{R}}) \longrightarrow \pi_0(G^{0}_{\mathbb{R}}) \longrightarrow 0.\]

This is, as above, trivial when $n$ is even, injective when $n$ is odd, and so $\pi_1(BG^{0}_{\mathbb{R}}) = \pi_0(G^{0}_{\mathbb{R}}) = \mathbb{Z}^{g} \oplus (\mathbb{Z}/2)^{r+1}$ for $n$ even, and $\mathbb{Z}^{g} \oplus (\mathbb{Z}/2)^{r}$ for $n$ odd.
In the top sequence, one has, as before, that the map \(Z \rightarrow \pi_1(\mathcal{G}_R^0)\) is injective; thus \(\pi_1(\mathcal{G}_R^0) = Z\). This can be thought of as being “localised” on the \(S^1 \times S^2\) in the cofibration \([23]\). The second sequence then tells us that there are classes in \(\pi_1(\mathcal{G}_R)\) represented by an integer, localised on \(S^1 \times S^2\), and classes in \((\mathbb{Z}/2)^r\), localised on \(S^2 \times \text{(one–skeleton)}\); more explicitly, one can represent the classes on the one–skeleton by a loop in \(O_n\) along the real curves, which when one moves into \(U_n\), deforms to the identity. If we then take our loop on the real curve, we can extend it into a neighbourhood in such a way that it is the identity on the boundary of the neighbourhood. This tells us that the sequence splits:

\[
\pi_2(\mathcal{G}_R) = \mathbb{Z} \oplus (\mathbb{Z}/2)^r. 
\]

Finally, one also has:

\[
\begin{array}{c}
0 \rightarrow \pi_1(\mathcal{G}_R^0) \rightarrow \pi_1(\mathcal{G}_R) \rightarrow \pi_1(\prod_{i=1}^{r} O_n/\pm) \rightarrow 0,
\end{array}
\]

with \(\pi_1((\prod_{i=1}^{r} O_n)/\pm)\) is \((\mathbb{Z}/2)^{r+1}\) when \(n = 0(4)\), \((\mathbb{Z}/2)^{r-1} \oplus \mathbb{Z}/4\) when \(n = 2(4)\), and \((\mathbb{Z}/2)^r\) when \(n\) is odd. Again, the sequence splits, and so \(\pi_2(\mathcal{G}_R^0) = \mathbb{Z} \oplus (\mathbb{Z}/2)^{r+1}\) when \(n = 0(4)\), \(\mathbb{Z} \oplus (\mathbb{Z}/2)^{r-1} \oplus \mathbb{Z}/4\) when \(n = 2(4)\), and \(\mathbb{Z} \oplus (\mathbb{Z}/2)^r\) when \(n\) is odd.

For \(n = 2\), we get for the fundamental groups \(\pi_1(\mathcal{G}_R^0) = \mathbb{Z}^g \oplus \mathbb{Z}^r\), \(\pi_1(\mathcal{G}_R) = \pi_1(\mathcal{G}_R^0) \oplus \mathbb{Z}^r \oplus \mathbb{Z}/2\), and \(\pi_2(\mathcal{G}_R^0) = \mathbb{Z}\), \(\pi_2(\mathcal{G}_R) = \mathbb{Z}^{r+1}\), and \(\pi_2(\mathcal{G}_R^0) = \mathbb{Z}^r\).

**Real case, type II curves**

Again, we use the cell decompositions given in Section 2. Let \(\mathcal{G}_R^0\) be the group of gauge transformations which are the identity at each base point; one notes that along the \(\gamma_i, i = 1, \ldots, r\), the gauge transformations must lie in the \(O_n\) while elsewhere they lie in \(U_n\). Note that each \(\delta_i\) contributes a \(\Omega(U_n)\) to \(\mathcal{G}_R^0\). The corresponding fibrations are

\[
\Omega^2(U_n) \rightarrow \mathcal{G}_R^0 \rightarrow \prod_{i=1}^{g+1} \Omega(U_n) \times \prod_{i=1}^{r} \Omega(O_n),
\]

\[
\mathcal{G}_R^0 \rightarrow \mathcal{G}_R \rightarrow U_n \times \prod_{i=1}^{r} O_n.
\]

\[
\mathbb{Z}/2 \rightarrow \mathcal{G}_R \rightarrow \mathcal{G}_R^0
\]

\[
\mathcal{G}_R^0 \rightarrow \mathcal{G}_R \rightarrow (U_n \times \prod_{i=1}^{r} O_n)/\pm
\]

This gives

\[
\Omega(U_n)_0 \rightarrow B\mathcal{G}_R^0 \rightarrow \prod_{i=1}^{2g+r+1} U_n \times \prod_{i=1}^{r} (O_n)_0,
\]

\[
B\mathcal{G}_R^0 \rightarrow B\mathcal{G}_R \rightarrow BU_n \times \prod_{i=1}^{r} BO_n,
\]

and

\[
B\mathbb{Z}/2 \rightarrow B\mathcal{G}_R \rightarrow B\mathcal{G}_R^0
\]
Repeating the arguments for type I and type 0 curves, one obtains the results given in the table below.

Quaternionic case, $n$ odd.

In this case, as we have seen, the curve can have no fixed points. One can still consider invariant connections, and gauge transformations which commute with the involution $\sigma$. Again, the constant central gauge transformations which commute with $\sigma$ are $\pm 1$. We then can compute, using (2.2), fibrations as in (6.1), (6.2), (6.3). The results appear below.

Quaternionic case, $n$ even, type 0 curve.

We get fibrations as in (6.1), (6.2), (6.3), and the same results.

Quaternionic case, $n$ even, type I curve.

Here one has fibrations as in (6.9), (6.10), (6.11), (6.12), but with $\text{Sp}_{n/2}$ replacing $\text{O}_n$. This simplifies things, as $\text{Sp}_{n/2}$ is connected and simply connected. Results are given below.

Quaternionic case, $n$ even, type II curve.

Here one has fibrations as in (6.18), (6.19), (6.20), (6.21), but with $\text{Sp}_{n/2}$ replacing $\text{O}_n$. Again, see below.
## Summary: Homotopy groups of the classifying spaces

| | $\pi_1(BG_0^0)$ | $\pi_1(BG_R)$ | $\pi_1(BG_0^R)$ | $\pi_2(BG_0^0)$ | $\pi_2(BG_R)$ | $\pi_2(BG_0^R)$ |
|---|---|---|---|---|---|---|
| Real, type I | $\mathbb{Z}^g \oplus (\mathbb{Z}/2)^r$ | $\mathbb{Z}^g \oplus (\mathbb{Z}/2)^{r+1}$ | $\mathbb{Z}^g \oplus (\mathbb{Z}/2)^{r+1}$ | $\mathbb{Z}$ | $\mathbb{Z} \oplus (\mathbb{Z}/2)^r$ | $\mathbb{Z} \oplus (\mathbb{Z}/2)^{r+1}$ |
| (n > 2) | (n > 2) | (n > 2, even) | (n > 1 odd) | (n > 1) | (n > 2) | (n > 0(4)) |
| $\mathbb{Z}^g(n = 2)$ | $\mathbb{Z}^g(n = 2)$ | $\mathbb{Z}^g(n = 2)$ | $\mathbb{Z}^g(n = 2)$ | $\mathbb{Z}^g(n = 2)$ | $\mathbb{Z}^g(n = 2)$ | $\mathbb{Z}^g(n = 2)$ |
| $\mathbb{Z}^g(n = 1)$ | $\mathbb{Z}^g(n = 1)$ | $\mathbb{Z}^g(n = 1)$ | $\mathbb{Z}^g(n = 1)$ | $\mathbb{Z}^g(n = 1)$ | $\mathbb{Z}^g(n = 1)$ | $\mathbb{Z}^g(n = 1)$ |
| Real, type I | $\mathbb{Z}^g+1$ | $\mathbb{Z}^g+1$ | $\mathbb{Z}^g+1$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| (n = 2) | (n = 2) | (n = 2) | (n = 1) | (n = 1) | (n = 1) | (n = 1) |
| $\mathbb{Z}^g+1(n = 1)$ | $\mathbb{Z}^g+1(n = 1)$ | $\mathbb{Z}^g+1(n = 1)$ | $\mathbb{Z}^g+1(n = 1)$ | $\mathbb{Z}^g+1(n = 1)$ | $\mathbb{Z}^g+1(n = 1)$ | $\mathbb{Z}^g+1(n = 1)$ |
| Real, type II | $\mathbb{Z}^g+1 \oplus (\mathbb{Z}/2)^r$ | $\mathbb{Z}^g \oplus (\mathbb{Z}/2)^{r+1}$ | $\mathbb{Z}^g \oplus (\mathbb{Z}/2)^{r+1}$ | $\mathbb{Z}$ | $\mathbb{Z} \oplus (\mathbb{Z}/2)^r$ | $\mathbb{Z} \oplus (\mathbb{Z}/2)^{r+1}$ |
| (n > 2) | (n > 2) | (n > 2, even) | (n > 1 odd) | (n > 1) | (n > 2) | (n > 0(4)) |
| $\mathbb{Z}^g+1(n = 2)$ | $\mathbb{Z}^g+1(n = 2)$ | $\mathbb{Z}^g+1(n = 2)$ | $\mathbb{Z}^g+1(n = 2)$ | $\mathbb{Z}^g+1(n = 2)$ | $\mathbb{Z}^g+1(n = 2)$ | $\mathbb{Z}^g+1(n = 2)$ |
| $\mathbb{Z}^g+1(n = 1)$ | $\mathbb{Z}^g+1(n = 1)$ | $\mathbb{Z}^g+1(n = 1)$ | $\mathbb{Z}^g+1(n = 1)$ | $\mathbb{Z}^g+1(n = 1)$ | $\mathbb{Z}^g+1(n = 1)$ | $\mathbb{Z}^g+1(n = 1)$ |
| Quat., type 0 | $\mathbb{Z}^g+1$ | $\mathbb{Z}^g$ | $\mathbb{Z}^g$, | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| (n = 2) | (n = 2) | (n = 2) | (n = 1) | (n = 1) | (n = 1) | (n = 1) |
| Quat., type I | $\mathbb{Z}^g$ | $\mathbb{Z}^g$ | $\mathbb{Z}^g$, | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| (n even) | (n even) | (n even) | (n = 1) | (n = 1) | (n = 1) | (n = 1) |
| Quat., type II | $\mathbb{Z}^g+1$ | $\mathbb{Z}^g+1$ | $\mathbb{Z}^g+1$, | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| (n even) | (n even) | (n even) | (n = 1) | (n = 1) | (n = 1) | (n = 1) |
Using the determinant map $U_n \rightarrow U_1$ and the various fibrations in mapping spaces that are deduced from it, we can compute the homotopy groups in the $SU_n$ case, or, more generally, for fixed determinant:

**Homotopy groups of the classifying spaces, fixed determinant ($n > 1$)**

| Type   | $\pi_1(BSG^0_{\mathbb{R}})$ | $\pi_1(BSG_{\mathbb{R}})$ | $\pi_2(BSG^0_{\mathbb{R}})$ | $\pi_2(BSG_{\mathbb{R}})$ | $\pi_3(BSG_{\mathbb{R}})$ |
|--------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| Real, type 0 | 0                           | 0                           | $\mathbb{Z}$                 | $\mathbb{Z}$                 | $\mathbb{Z} \oplus \mathbb{Z}/2$ (n even) |
| Real, type I  | $(\mathbb{Z}/2)^r$ (n > 2)                      | $(\mathbb{Z}/2)^r$ (n > 2)                      | $\mathbb{Z}$                 | $\mathbb{Z} \oplus (\mathbb{Z}/2)^r$ (n > 2) | $\mathbb{Z} \oplus (\mathbb{Z}/2)^r+1$ (n = 0(4)) |
| Real, type II | $(\mathbb{Z}/2)^r$ (n > 2)                      | $(\mathbb{Z}/2)^r$ (n > 2)                      | $\mathbb{Z}$                 | $\mathbb{Z} \oplus (\mathbb{Z}/2)^r+1$ (n = 0(4)) | $\mathbb{Z} \oplus (\mathbb{Z}/2)^r+1 \oplus \mathbb{Z}/4$ (n = 2(4), n > 2) |
| Quat., type 0 | 0                           | 0                           | $\mathbb{Z}$                 | $\mathbb{Z}$                 | $\mathbb{Z} \oplus \mathbb{Z}/2$ (n even) |
| Quat., type I (n even) | 0                           | 0                           | $\mathbb{Z}$                 | $\mathbb{Z}$                 | $\mathbb{Z} \oplus \mathbb{Z}/2$ (n even) |

6.3. **Critical points of the Yang–Mills functional.** We noted above that the Yang–Mills functional is invariant under the real involution; this then tells us that the gradient flows, which take us to critical points in the full space of connections, are preserved by the real structure: the real gradient flow takes us to a critical point in the full space. Conversely, if there is a direction $v$ in which the derivative of the Yang–Mills functional is negative, then it is negative in $\sigma^*(v)$ and so in $v + \sigma^*(v)$. Critical points for the real flows are then real critical points for the full space, and so, applying the results of Atiyah and Bott ([AB]), we have, as noted above,

**Proposition 6.3.** The critical points of the Yang–Mills functional on the space of real (resp. quaternionic) connections on our surface correspond to sums of real (resp. quaternionic) bundles, each equipped with a connection with constant central curvature.

The discussion above, and the results of Atiyah and Bott, give us a real bundle which decomposes a sum of bundles with constant central curvature; the only thing left to remark is that the real structure must respect this decomposition.
We next must compute the indices of the critical points, adapting [AB]. In the full space, (see [AB]) critical points correspond to a sum of bundles $\oplus_i V_i$ with possibly different slopes (degree/rank); the index is given by the real dimension of the space $H^1(\Sigma, \oplus_{\mu(V_i) > \mu(V_j)} V_i^* \otimes V_j)$, where $\mu$ is the slope. At a real critical point, the real index will then be of dimension one half of this, i.e., its complex dimension.

**Proposition 6.4.** For non–minimal critical points in degree $k$, rank $N$, the index is bounded below by $1 + (n-1)(g-1)$.

Thus, the inclusion of the space of minima of the Yang Mills functional (polystable real bundles) into the space of connections (modulo gauge) induces isomorphisms in homology and homotopy groups in dimensions less than $(n-1)(g-1)$, and surjections in dimension $(n-1)(g-1)$.

**Proof.** Let us suppose that a real bundle $E$ of degree $k$, rank $n$ is destabilised by a bundle $F$ of degree $\ell$, rank $m$; we have $\ell n > km$. The index is given by

$$h^1(X, \text{Hom}(F, E/F)) \geq m(n-m)(g-1) - c_1(\text{Hom}(F, E/F)) = m(n-m)(g-1) - (-\ell(n-m) + (k-\ell)m),$$

by Riemann–Roch. By the stability condition, this is bounded below by $m(n-m)(g-1) + 1$, which in turn is bounded below by $(n-1)(g-1) + 1$. \hfill \Box

Furthermore, looking at the various degrees $k$, case by case, we can improve the bound given in the proposition:

- $(n, g) = (2, 2)$: For $k = 0$, the index is bounded below by 3, instead of 2.
- $(n, g) = (3, 2)$: For $k = 0$, the index is bounded below by 5, instead of 3.
- $(n, g) = (2, 3)$: For $k = 0$, the index is bounded below by 4, instead of 3.

The same estimates apply for the case of fixed determinant.

The quotient space of real connections modulo gauge, being the quotient of an affine space, is connected. Thus, applying the information on the non-minimal critical points of the Yang-Mills functional informs us on the connectedness of the space of minima, that is the moduli space of polystable bundles:

**Theorem 6.5.** For $g \geq 2, n \geq 2$, there are connected moduli spaces of real or quaternionic polystable holomorphic bundles for each allowed topological type. The topological type is determined by the first Chern class, as well as Stiefel–Whitney classes of the restriction of the real bundles to $X(\mathbb{R})$.

We next note that we have computed the first two homotopy groups of a space very close to the space of connections, that is the spaces $B\mathcal{G}_R$. Indeed, if $A$ is the affine space of real connections, one has a map $B\mathcal{G}_R \simeq E\mathcal{G}_R \times A\mathcal{G}_R \to A/\mathcal{G}_R$, which is a fibration with trivial fibres over the set of generic connections over which $\mathcal{G}_R$ acts freely. One can pull back the Yang-Mills functional to $B\mathcal{G}_R$, and consider it there; the indices stay the same. If one is in the case when degree and rank are coprime (the “coprime case”), let us restrict to the sets $B\mathcal{G}_R \leq c$, $(A/\mathcal{G}_R) \leq c$, where the Yang-Mills functional takes values less than $c$, if $c$ is below the level of the first non-minimal critical point, there are no reducible connections, and then

$$B\mathcal{G}_R \leq c \simeq (A/\mathcal{G}_R) \leq c$$

One can then use the results on the homotopy groups that we have computed and transfer them to $(A/\mathcal{G}_R) \leq c$, and hence, via the Morse flows, to the moduli of stable bundles.
Theorem 6.6. In the coprime case, for \((n - 1)(g - 1) > 2\), or for \(k = 0\), \((n, g) = (3, 2), (2, 3)\), the fundamental groups and second homotopy groups of the moduli spaces \(\mathcal{M}_R\) of real or quaternionic bundles and \(\mathcal{M}_R^0\) of real or quaternionic bundles of fixed determinant are as follows:

| Type | \(\pi_1(\mathcal{M}_R)\) | \(\pi_2(\mathcal{M}_R)\) | \(\pi_1(\mathcal{M}_R^0)\) | \(\pi_2(\mathcal{M}_R^0)\) |
|------|----------------|----------------|----------------|----------------|
| Real, type 0 | \(\mathbb{Z}^g \oplus \mathbb{Z}/2\) \((n \text{ even})\) | \(\mathbb{Z} \oplus \mathbb{Z}/2\) \((n \text{ even})\) | 0, | \(\mathbb{Z} \oplus \mathbb{Z}/2\) \((n \text{ even})\) |
| | \(\mathbb{Z}^g\) \((n \text{ odd})\) | \(\mathbb{Z} \) \((n \text{ odd}, > 1)\) | | \(\mathbb{Z} \) \((n \text{ odd}, > 1)\) |
| Real, type I | \(\mathbb{Z}^g \oplus (\mathbb{Z}/2)^{r+1}\) \((n \text{ even})\) | \(\mathbb{Z} \oplus (\mathbb{Z}/2)^{r+1}\) \((n = 0(4))\) | \((\mathbb{Z}/2)^r\) \((n = 0(4))\) | \(\mathbb{Z} \oplus (\mathbb{Z}/2)^{r+1}\) \((n = 0(4))\) |
| | \(\mathbb{Z}^g \oplus (\mathbb{Z}/2)^r\) \((n \text{ odd})\) | \(\mathbb{Z} \oplus (\mathbb{Z}/2)^{r-1} \oplus \mathbb{Z}/4\) \((n = 2(4), n > 2)\) | \(\mathbb{Z} \oplus (\mathbb{Z}/2)^{r}\) \((n \text{ odd})\) | \(\mathbb{Z} \oplus (\mathbb{Z}/2)^{r-1} \oplus \mathbb{Z}/4\) \((n = 2(4))\) |
| | \(\mathbb{Z}^{g+r} \oplus \mathbb{Z}/2(n = 2)\) | \(\mathbb{Z} \oplus (\mathbb{Z}/2)^r\) \((n \text{ odd})\) | \(\mathbb{Z}^r(n = 2)\) | \(\mathbb{Z} \oplus (\mathbb{Z}/2)^r\) \((n \text{ odd})\) |
| Real, type II | \(\mathbb{Z}^g \oplus (\mathbb{Z}/2)^{r+1}\) \((n \text{ even})\) | \(\mathbb{Z} \oplus (\mathbb{Z}/2)^{r+1}\) \((n = 0(4))\) | \((\mathbb{Z}/2)^r\) \((n = 0(4))\) | \(\mathbb{Z} \oplus (\mathbb{Z}/2)^{r+1}\) \((n = 0(4))\) |
| | \(\mathbb{Z}^g \oplus (\mathbb{Z}/2)^{r}\) \((n \text{ odd})\) | \(\mathbb{Z} \oplus (\mathbb{Z}/2)^{r-1} \oplus \mathbb{Z}/4\) \((n = 2(4), n > 2)\) | \(\mathbb{Z} \oplus (\mathbb{Z}/2)^{r}\) \((n \text{ odd})\) | \(\mathbb{Z} \oplus (\mathbb{Z}/2)^{r-1} \oplus \mathbb{Z}/4\) \((n = 2(4))\) |
| | \(\mathbb{Z}^{g+r} \oplus \mathbb{Z}/2(n = 2)\) | \(\mathbb{Z} \oplus (\mathbb{Z}/2)^r\) \((n \text{ odd})\) | \(\mathbb{Z}^r(n = 2)\) | \(\mathbb{Z} \oplus (\mathbb{Z}/2)^r\) \((n \text{ odd})\) |
| Quat., type 0 | \(\mathbb{Z}^g \oplus \mathbb{Z}/2\) \((n \text{ even})\) | \(\mathbb{Z} \oplus \mathbb{Z}/2\) \((n \text{ even})\) | 0 | \(\mathbb{Z} \oplus \mathbb{Z}/2\) \((n \text{ even})\) |
| | \(\mathbb{Z}^g\) \((n \text{ odd})\) | \(\mathbb{Z} \) \((n \text{ odd})\) | | \(\mathbb{Z} \) \((n \text{ odd})\) |
| Quat., type I \((n \text{ even})\) | \(\mathbb{Z}^g \oplus \mathbb{Z}/2\) \((n \text{ even})\) | \(\mathbb{Z} \oplus \mathbb{Z}/2\) \((n \text{ even})\) | 0 | \(\mathbb{Z} \oplus \mathbb{Z}/2\) \((n \text{ even})\) |
| Quat., type II \((n \text{ even})\) | \(\mathbb{Z}^g \oplus \mathbb{Z}/2\) \((n \text{ even})\) | \(\mathbb{Z} \oplus \mathbb{Z}/2\) \((n \text{ even})\) | 0 | \(\mathbb{Z} \oplus \mathbb{Z}/2\) \((n \text{ even})\) |

For \((n, g) = (3, 2), (2, 3)\) the fundamental groups are as given in the table above, and the second homotopy groups are quotients of the groups given above. For \((n, g) = (2, 2)\), the fundamental groups are quotients of the groups given in the table above.

One can also give results for framed bundles, that is the moduli of pairs \((E, t)\), where \(E\) is polystable and \(t\) is a trivialisation at the base point; here one is quotenting the space of connections by the based gauge group \(\mathcal{G}_R^0\); this action is always free, and so life is simpler; one has

\[(6.26) \quad BG_R^0 \simeq \mathcal{A}/\mathcal{G}_R^0\]

and so then one can transfer our homotopy results even in the non-coprime case, reading off the first and second homotopy groups of the framed moduli spaces from those of \(BG_R^0\), and from those of \(BSG_R^0\) in the fixed determinant case, referring to the tables above.

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