Quantitative aspects of the Beurling–Helson theorem: Phase functions of a special form

by

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Abstract. We consider the space $A(T^d)$ of absolutely convergent Fourier series on the torus $T^d$. The norm on $A(T^d)$ is naturally defined by $\|f\|_A = \|\hat{f}\|_{l^1}$, where $\hat{f}$ is the Fourier transform of a function $f$. For real functions $\phi$ of a certain special form on $T^d$, $d \geq 2$, we obtain lower bounds for the norms $\|e^{i\lambda \phi}\|_{A(T^d)}$ as $\lambda \to \infty$. In particular, we show that if $\phi(x, y) = a(x)|y|$ for $|y| \leq \pi$, where $a \in A(T)$ is an arbitrary nonconstant real function, then $\|e^{i\lambda \phi}\|_{A(T^2)} \gtrsim |\lambda|$.

1. Introduction

Let $A(T^d)$ be the space of all continuous functions $f$ on the torus $T^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ such that the sequence $\hat{f} = \{\hat{f}(n), n \in \mathbb{Z}^d\}$ of Fourier coefficients of $f$ is in $l^1(\mathbb{Z}^d)$. The space $A(T^d)$ is a Banach space with respect to the natural norm $\|f\|_{A(T^d)} = \|\hat{f}\|_{l^1(\mathbb{Z}^d)}$. It is well known that $A(T^d)$ is a Banach algebra (with pointwise multiplication of functions). Here, as usual,

$$\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{T^d} f(t)e^{-i(n,t)}dt, \quad n \in \mathbb{Z}^d,$$

$\mathbb{R}$ and $\mathbb{Z}$ are the additive groups of reals and integers, respectively, and $(\cdot, \cdot)$ is the inner product.

Let $\varphi$ be a continuous mapping of the circle $\mathbb{T}$ into itself, that is, a continuous function $\varphi: \mathbb{R} \to \mathbb{R}$ satisfying the condition $\varphi(t + 2\pi) = \varphi(t) (\text{mod } 2\pi)$. By the well-known Beurling–Helson theorem [1] (see also [6 Sec. VI. 9] and [7]), if $\|e^{i n \varphi}\|_{A(\mathbb{T})} = O(1)$, $n \in \mathbb{Z}$, then $\varphi$ is affine. In other words, in this case we have $\varphi(t) = \nu t + \varphi(0)$, where $\nu \in \mathbb{Z}$. This theorem readily gives the solution to

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1This version of the Beurling–Helson theorem is due to Kahane.
the Levy problem on the description of endomorphisms of the algebra $A(\mathbb{T})$. All these endomorphisms are trivial; i.e., they have the form $f(t) \to f(\nu t + t_0)$. This implies that only trivial changes of variable are admissible in $A(\mathbb{T})$.

Another version of the Beurling–Helson theorem concerns power-bounded operators on $l^1$: if $U$ is a bounded translation invariant operator from $l^1(\mathbb{Z})$ to itself such that $\|U^n\|_{l^1 \to l^1} = O(1)$, $n \in \mathbb{Z}$, then $U = \xi S$, where $\xi$ is a complex number, $|\xi| = 1$, and $S$ is a translation. Note also the following multidimensional version of the Beurling–Helson theorem. (The multidimensional case easily reduces to the one-dimensional one.) Let $\varphi: \mathbb{T}^d \to \mathbb{T}$ be a continuous mapping such that $\|e^{in\varphi}\|_{A(\mathbb{T}^d)} = O(1)$; then $\varphi(t) = (\nu, t) + \varphi(0)$ (where $\nu \in \mathbb{Z}^d$).

At the same time, although the Beurling–Helson theorem establishes an unbounded growth of the norms $\|e^{in\varphi}\|_A$ for non-affine mappings $\varphi$, the character of growth of these norms is unclear.

We note, that instead of non-affine mappings $\varphi: \mathbb{T}^d \to \mathbb{T}$, one can consider real nonconstant functions $\varphi$ on $\mathbb{T}^d$ and study the behavior of the norms $\|e^{i\lambda \varphi}\|_{A(\mathbb{T}^d)}$ for large frequencies $\lambda \in \mathbb{R}$ (without assuming the frequencies to be integer)\(^2\)

It is easy to show that if $\varphi$ is a real $C^1$-smooth function on the circle $\mathbb{T}$, then $\|e^{i\lambda \varphi}\|_{A(\mathbb{T})} \lesssim |\lambda|^{1/2}$ as $\lambda \to \infty$ (see, e.g., [6, Sec. VI. 3]). On the other hand, lower bound for $\varphi \in C^2(\mathbb{T})$ has long been known; namely, if $\varphi \in C^2(\mathbb{T})$ is a real nonconstant function, then $\|e^{i\lambda \varphi}\|_{A(\mathbb{T})} \gtrsim |\lambda|^{1/2}$. This estimate is contained implicitly in the work by Leibenson [15] and in explicit form was obtained by Kahane [3] (see also [6, Sec. VI. 3]). The proof is based on the van der Corput lemma and essentially uses nondegeneration of the curvature of a certain arc of the graph of $\varphi$.

We note that in general the norms $\|e^{i\lambda \varphi}\|_{A(\mathbb{T})}$ can grow rather slowly. It was shown by Kahane [3] (see also [6, Sec. VI. 2]) that if $\varphi$ is a piecewise linear nonconstant continuous real function on $\mathbb{T}$, then $\|e^{i\lambda \varphi}\|_{A(\mathbb{T})} \simeq \log |\lambda|$. In this respect, let us recall Kahane’s conjecture about the possible essential improvement of the Beurling–Helson theorem. Namely, Kahane conjectured that the conclusion of the Beurling–Helson theorem holds under much weaker assumption that $\|e^{in\varphi}\|_{A(\mathbb{T})} = o(\log |n|)$. This conjecture, proposed at the ICM’1962 [4] and later discussed in [6, 7], is still unproved. The first strengthening of the Beurling–Helson theorem in this direction was obtained by the present author [14] by means of methods of number theory and additive combinatorics. Later, Konyagin and Shkredov [10] improved the result by combining the author’s approach with a finer technique.

In the multidimensional case it is easy to show that if $\varphi: \mathbb{T}^d \to \mathbb{R}$, $d \geq 2$, is of class $C^s$ with $s > d/2$, then $\|e^{i\lambda \varphi}\|_{A(\mathbb{T}^d)} \lesssim |\lambda|^{d/2}$, see [2] and [13, Theorem 3]. The Leibenson–Kahane’s result was extended to the multidimensional

\footnote{If $\varphi: \mathbb{T}^d \to \mathbb{T}$ is continuous, then for some $k \in \mathbb{Z}^d$ the function $\psi(t) = \varphi(t) + (k, t)$ is a (real) continuous function on $\mathbb{T}^d$ and $\|e^{in\varphi}\|_{A(\mathbb{T}^d)} = \|e^{in\psi}\|_{A(\mathbb{T}^d)}$.}
case by Hedstrom [2], who showed, that if \( \varphi \in C^2 \cap A(\mathbb{T}^d) \) and the determinant of the matrix of the second derivatives of \( \varphi \) is not identically equal to zero, then \( \|e^{i\lambda \varphi}\|_{A(\mathbb{T}^d)} \gtrsim |\lambda|^{d/2} \). This is proved by reduction to the one-dimensional case.

The estimates for the norms of \( e^{i\lambda \varphi} \) for \( C^1 \) phase functions \( \varphi \), including those in the multidimensional case, were obtained by the author in [12, 13]. Certainly, in general, the approach that uses curvature considerations fails in \( C^1 \) case. Note also that for \( C^1 \) phase functions \( \varphi \) the norms of \( e^{i\lambda \varphi} \) can grow nearly as slowly as those for the piecewise linear functions, namely the author showed [11] (see also [12]) that if \( \gamma(\lambda) \) is an arbitrary positive function on \( [0, +\infty) \) with \( \gamma(\lambda) \to \infty \) as \( \lambda \to +\infty \), then there exists a nowhere linear (i.e., not linear on any interval) function \( \varphi \in C^1(\mathbb{T}) \) such that \( \|e^{i\lambda \varphi}\|_{A(\mathbb{T})} = O(\gamma(|\lambda| \log |\lambda|)) \).

In the present paper we consider phase functions on \( \mathbb{T}^d, d \geq 2 \), of the following form. Let \( a: \mathbb{T}^k \to \mathbb{R}^m \) and \( b: \mathbb{T}^m \to \mathbb{R}^m \) be two mappings. Define a function \( \varphi \) on \( \mathbb{T}^{k+m} \) as the inner product

\[
\varphi(x, y) = (a(x), b(y)), \quad x \in \mathbb{T}^k, \; y \in \mathbb{T}^m.
\]

We require \( a \) and \( b \) to be of class \( A \) (see the next section), which guarantees that \( e^{i\lambda \varphi} \in A(\mathbb{T}^{k+m}) \) for all \( \lambda \in \mathbb{R} \). Assuming that \( b \) coincides with a non-degenerate affine mapping in some domain \( J \subseteq [-\pi, \pi]^m \), we obtain lower bounds for the norms \( \|e^{i\lambda \varphi}\|_{A(\mathbb{T}^{k+m})} \). In particular the class of the phase functions we consider includes those of the form

\[
\varphi(x_1, \ldots, x_k, y_1, \ldots, y_m) = \sum_{j=1}^{m} a_j(x_1, \ldots, x_k)|y_j|, \quad (y_1, \ldots, y_m) \in [-\pi, \pi]^m,
\]

where \( a_j, j = 1, \ldots, m \), are real functions in \( A(\mathbb{T}) \). Though these phase functions are of a very special kind, they, as we will see, provide examples of \( \varphi \)'s for which the growth of \( \|e^{i\lambda \varphi}\|_{A(\mathbb{T}^{k+m})} \) is very fast. Note that we do not assume a smoothness of the mappings \( a \) and \( b \) in (1). The fast growth in the case we consider is a consequence of the geometric structure of \( \varphi \) and not that of its smoothness, namely, the key role is played by the massiveness of the image of the torus \( \mathbb{T}^k \) under \( a \). The simplest application of the results of the present paper pertains to the two-dimensional case. If \( \varphi(x, y) = a(x)|y|, \; |x| \leq \pi, \; |y| \leq \pi, \) where \( a \in A(\mathbb{T}) \) is an arbitrary nonconstant real function, then \( \|e^{i\lambda \varphi}\|_{A(\mathbb{T}^2)} \gtrsim |\lambda| \). When \( d = k+m \geq 3 \) our results show that the growth that corresponds to the phase functions of the form (2) can be even faster then that in the smooth case; for example, if a mapping \( a(x) = (a_1(x), a_2(x)) \) of \( \mathbb{T} \) into \( \mathbb{R}^2 \) is space-filling (see the next section), then, setting \( \varphi(x, y_1, y_2) = a_1(x)|y_1| + a_2(x)|y_2| \), we have \( \|e^{i\lambda \varphi}\|_{A(\mathbb{T}^3)} \gtrsim |\lambda|^2 \). Note, that, for \( \varphi \in C^2(\mathbb{T}^3) \) we have \( \|e^{i\lambda \varphi}\|_{A(\mathbb{T}^3)} \lesssim |\lambda|^{3/2} \), according to the upper bound indicated above.

\[\text{3These papers treat the general case of the spaces } A_p, 1 \leq p < 2, \text{ of functions } f \text{ with } \hat{f} \in L^p.\]
2. Statement of the theorem. Corollaries for space-filling mappings

We measure the massiveness of a set in terms of its covering number. Recall that the covering number $N_F(\varepsilon)$ of a bounded set $F \subset \mathbb{R}^m$ is the smallest number of balls of radius $\varepsilon > 0$ needed to cover $F$.

Let $s: \mathbb{T}^p \rightarrow \mathbb{R}^q$ be some mapping. Thus,

$$s(x) = (s_1(x), s_2(x), \ldots, s_q(x)), \quad x \in \mathbb{T}^p.$$  

We say that $s$ is a mapping of class $A$ if all coordinate functions $s_j : \mathbb{T}^p \rightarrow \mathbb{R}$, $j = 1, 2, \ldots, q$, are in $A(\mathbb{T}^p)$. Clearly, if $a: \mathbb{T}^k \rightarrow \mathbb{R}^m$ and $b: \mathbb{T}^m \rightarrow \mathbb{R}^m$ are mappings of class $A$, then the corresponding function $\varphi$ defined in (1) belongs to $A(\mathbb{T}^{k+m})$, and hence $e^{i\lambda \varphi} \in A(\mathbb{T}^{k+m})$ for all $\lambda \in \mathbb{R}$.

We write $\xi(\lambda) \gtrsim \eta(\lambda)$ if there exists a constant $c > 0$ independent of $\lambda \in \mathbb{R}$ such that $\xi(\lambda) \geq c\eta(\lambda)$ whenever $|\lambda|$ is sufficiently large.

**Theorem.** Let $a: \mathbb{T}^k \rightarrow \mathbb{R}^m$ and $b: \mathbb{T}^m \rightarrow \mathbb{R}^m$ be mappings of class $A$. Assume that $b$ coincides with a non-degenerate affine mapping in some domain $J \subseteq [-\pi, \pi]^m$. Let $\varphi$ be the function on $\mathbb{T}^{k+m}$ defined as the inner product $\varphi(x,y) = (a(x), b(y))$ for $x \in \mathbb{T}^k$ and $y \in \mathbb{T}^m$. Then

$$\|e^{i\lambda \varphi}\|_{A(\mathbb{T}^{k+m})} \gtrsim N_W(1/|\lambda|), \quad \lambda \in \mathbb{R}, \quad |\lambda| \rightarrow \infty,$$

where $W = a(\mathbb{T}^k)$ is the image of the torus $\mathbb{T}^k$ under $a$.

The proof of the theorem is given in Section 3. Mappings of class $A$ and some related open problems are discussed in Remarks at the end of the paper. For now, let us indicate two immediate corollaries of the theorem.

**Corollary 1.** Let $a \in A(\mathbb{T})$ and $b \in A(\mathbb{T})$ be real nonconstant functions. Assume that $b$ is linear on some interval. Let $\varphi$ be the function on $\mathbb{T}^2$ given by $\varphi(x,y) = a(x)b(y)$, $x \in \mathbb{T}$, $y \in \mathbb{T}$. Then

$$\|e^{i\lambda \varphi}\|_{A(\mathbb{T}^2)} \gtrsim |\lambda|, \quad \lambda \in \mathbb{R}, \quad |\lambda| \rightarrow \infty.$$

In particular, this corollary implies the estimate mentioned at the end of Introduction: if a function $\varphi$ on $\mathbb{T}^2$ has the form $\varphi(x,y) = a(x)|y|$, $-\pi \leq y \leq \pi$, where $a \in A(\mathbb{T})$ is an arbitrary nonconstant real function, then $\|e^{i\lambda \varphi}\|_{A(\mathbb{T}^2)} \gtrsim |\lambda|$.

The second corollary deals with the general case in which the range of the mapping $a$ is maximally massive. We say that a mapping $a: \mathbb{T}^k \rightarrow \mathbb{R}^m$ is space-filling if the image $a(\mathbb{T}^k)$ has non-empty interior. A weaker condition is that the Lebesgue measure of the set $a(\mathbb{T}^k)$ is positive. It is not difficult to see that there exist space-filling mappings $a$ of class $A$ for any $k$ and $m$. Let us verify this. Clearly, only the case of increasing dimension, that is, the case of $m > k$, is non-trivial. Note that if $f \in A(\mathbb{T})$, then the function $F$ on $\mathbb{T}^k$ defined by $F(x_1, x_2, \ldots, x_k) = f(x_1)$ satisfies $F \in A(\mathbb{T}^k)$. Hence it suffices to consider the case of $k = 1$; i.e., it suffices to show that there exist space-filling curves of
class \( A \) in \( \mathbb{R}^m \). Recall that a closed set \( E \subseteq T \) is called a *Helson set* if every function continuous on \( E \) is the restriction to \( E \) of some function in \( A(T) \). Fix a perfect nowhere dense set \( E \) that is a Helson set. (For the existence of such sets see, e.g., [17, Theorems 5.2.2 and 5.6.6].) Let \( K \) be a closed cube in \( \mathbb{R}^m \). Since \( E \) is perfect and nowhere dense, it follows (by the classical Hausdorff–Alexandroff theorem) that there exists a continuous mapping \( \alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_m(t)) \) of \( E \) onto \( K \). Since \( E \) is a Helson set, we see that for each \( j = 1, 2, \ldots, m \) there exists a function \( a_j \in A(T) \) coinciding with \( \alpha_j \) on \( E \). By taking the real parts, we can assume that all \( a_j \) are real. Let \( a(t) = (a_1(t), a_2(t), \ldots, a_m(t)), t \in T \). We obtain

\[
A(T) \supseteq A(E) = \alpha(E) = K,
\]

as desired.

Clearly, if \( W \subseteq \mathbb{R}^m \) is an arbitrary set of positive measure, then

\[
\inf_{\varepsilon>0} N_W(\varepsilon) \varepsilon^m > 0;
\]

hence we obtain the following corollary of the theorem.

**Corollary 2.** In addition to the assumptions of the theorem, let the mapping \( a: T^k \rightarrow \mathbb{R}^m \) be space-filling at least in the weak sense; i.e., let \( a(T^k) \) have positive Lebesgue measure. Then

\[
\|e^{i\lambda \varphi}\|_{A(T^{k+m})} \gtrsim |\lambda|^m, \quad \lambda \in \mathbb{R}, \quad |\lambda| \to \infty.
\]

We note that if \( \varphi \) is a sufficiently smooth real function on \( T^{k+m} \), then, according to the upper bound indicated in the introduction, we have \( \|e^{i\lambda \varphi}\|_{A(T^{k+m})} \lesssim |\lambda|^{(k+m)/2} \). On the other hand, by taking a space-filling mapping

\[
(a_1(x), \ldots, a_m(x)): T^k \rightarrow \mathbb{R}^m
\]

of class \( A \) and taking \( \varphi \) to be of the form (2), we find in the non-trivial case of dimension-raising \( a \), that is, for \( m > k \), that a substantially faster growth of the \( A \)-norms of \( e^{i\lambda \varphi} \) occurs. Namely, we have \( \|e^{i\lambda \varphi}\|_{A(T^{k+m})} \gtrsim |\lambda|^m \) by Corollary 2. (Certainly this effect is possible only when \( k + m \geq 3 \).)

### 3. Proof of the theorem

Let \( A(\mathbb{R}^d) \) be the space of all functions \( f \) of the form

\[
f(t) = \int_{\mathbb{R}^d} g(\xi)e^{i(x,t)}dx,
\]

where \( g \in L^1(\mathbb{R}^d) \). This space is closely related to \( A(T^d) \). It is a Banach algebra with respect to the norm \( \|f\|_{A(\mathbb{R}^d)} = \|g\|_{L^1(\mathbb{R}^d)} \) and the usual multiplication of functions. It is convenient to give an equivalent definition by saying that \( A(\mathbb{R}^d) \) contains every continuous bounded function on \( \mathbb{R}^d \) whose Fourier transform \( \widehat{f} \) in the sense of tempered distributions belongs to \( L^1(\mathbb{R}^d) \). Naturally we
let \( \|f\|_{A(\mathbb{R}^d)} = \|\hat{f}\|_{L^1(\mathbb{R}^d)} \). Here the normalizing factor in the Fourier transform is chosen so that

\[
\hat{f}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(t) e^{-i(x,t)} dt
\]

for \( f \in L^1(\mathbb{R}^d) \). Technically, it is more convenient to use the \( A(\mathbb{R}^d) \)-norm, so the symbol \( \hat{\ } \) will denote the Fourier transform in \( \mathbb{R}^d \) everywhere in the proof of the theorem.

We will use the following notation. If \( \xi \in \mathbb{R}^d \), then \( |\xi| \) stands for the length of the vector \( \xi \). Let \( F \subseteq \mathbb{R}^d \) be an arbitrary set. By \( (F)_\varepsilon \) we denote the \( \varepsilon \)-neighbourhood of \( F \); i.e., \( (F)_\varepsilon = \{ t \in \mathbb{R}^d : \inf_{x \in F} |t - x| \leq \varepsilon \} \). For \( \lambda \in \mathbb{R} \), let \( \lambda F = \{ \lambda x, x \in F \} \). If \( F \) is measurable, then \( |F| \) stands for its (Lebesgue) measure. The symbol \( * \) denotes the convolution of functions in \( L^1(\mathbb{R}^d) \) or the convolution of measures. This notation is used for various \( d \), but this will not lead to a misunderstanding.

The proof of the theorem is based on a modification of our method, that we used in [12, 13] for \( C^1 \)-phase functions (it can be called the concentration of Fourier transform large values method).

We will need some preliminary constructions and lemmas.

For \( \delta > 0 \), let \( \Delta_\delta \) be the “triangle function” supported by the interval \( (-\delta, \delta) \); that is,

\[
\Delta_\delta(t) = \max(0, 1 - |t|/\delta), \quad t \in \mathbb{R}.
\]

It is well known (and easy to verify) that

\[
\hat{\Delta_\delta}(u) = \frac{2 \sin^2(\delta u/2)}{\delta u^2}, \quad u \in \mathbb{R} \setminus \{0\}; \quad \hat{\Delta_\delta}(0) = \frac{\delta}{2\pi}.
\]

Note that since \( \hat{\Delta_\delta}(u) \geq 0 \) for all \( u \in \mathbb{R} \), it follows that

\[
\|\Delta_\delta\|_{A(\mathbb{R})} = \Delta(0) = 1.
\]

For an arbitrary interval \( Q \subseteq \mathbb{R} \), let \( \Delta_Q \) be the triangle function supported by \( Q \); that is, \( \Delta_Q(t) = \Delta_{|Q|/2}(t - c_Q) \), where \( c_Q \) is the center of the interval \( Q \). One has \( \hat{\Delta_Q} = \hat{\Delta}_{|Q|/2} \) and hence

\[
\|\Delta_Q\|_{A(\mathbb{R})} = 1.
\]

Let \( Q_1, Q_2, \ldots, Q_d \) be intervals in \( \mathbb{R} \). For the parallelepiped \( Q = Q_1 \times Q_2 \times \ldots \times Q_d \subseteq \mathbb{R}^d \), set

\[
\Delta_Q(t) = \Delta_{Q_1}(t_1) \Delta_{Q_2}(t_2) \ldots \Delta_{Q_d}(t_d), \quad t = (t_1, t_2, \ldots, t_d) \in \mathbb{R}^d.
\]

Obviously (see [11]),

\[
\|\Delta_Q\|_{A(\mathbb{R}^d)} = 1.
\]

In addition, if \( Q \) is centered at the origin, then \( \hat{\Delta_Q} \geq 0 \).

The following simple lemma is of a technical nature.
Lemma 1. Let $Q \subseteq \mathbb{R}^d$ be a cube with edges of length $2\delta$ parallel to the coordinate axes. Then $|\widehat{\Delta_Q}(u)| \geq 4^{-d}(\delta/2\pi)^d$ for all $u \in (-1/\delta, 1/\delta)^d$.

Proof. Since $|\sin \alpha| \geq |\alpha|/2$ for $|\alpha| \leq 1$, from (3) it follows that

$$\widehat{\Delta}(u) \geq \frac{\delta}{8\pi}$$

for $|u| \leq 1/\delta$.

Since the cube $Q$ is obtained by a shift of the cube $(-\delta, \delta)^d$, we have $|\widehat{\Delta_Q}(u)| = |\widehat{\Delta_{(-\delta,\delta)^d}}(u)|$. It remains to use relation (5). The proof of the lemma is complete. \(\square\)

Let $\omega$ be the modulus of continuity of the mapping $a$; i.e.,

$$\omega(\delta) = \sup_{|x_1-x_2| \leq \delta} |a(x_1) - a(x_2)|, \quad \delta \geq 0.$$  

The function $\omega(\delta)$ is non-decreasing and continuous on $[0, +\infty)$, and $\omega(0) = 0$. We will assume that the mapping $a$ is nonconstant; otherwise the assertion of the theorem is obvious. Thus, $\omega(\delta) > 0$ for all $\delta > 0$.

By the assumption of the theorem, $b(y) = Py + y_0$ in some domain $J \subseteq [-\pi, \pi]^m$, where $y_0 \in \mathbb{R}^m$ and $P$ is a real $m \times m$ matrix with $\det P \neq 0$. Without loss of generality, we can assume that $J = (-l, l)^m$, where $0 < l < \pi$. By $P^*$ we denote the transpose of $P$.

Set

$$\rho = \sup_{y \in J} |Py + y_0|.$$ 

Fix a constant $\varepsilon_0$ such that

$$(7) \quad 0 < \varepsilon_0 \leq 1/2, \quad 2\rho \varepsilon_0 \leq \frac{1}{2} 4^{-k}.$$ 

For each sufficiently large $\lambda > 0$, define $\delta_\lambda$ by the condition

$$(8) \quad \lambda \omega(\sqrt{k}2\delta_\lambda) = \varepsilon_0.$$ 

The following lemma is the key claim for the proof of the theorem.

Lemma 2. Let $\lambda > 0$ be sufficiently large. Then for each $v \in (\lambda W)_{\varepsilon_0}$ there exists a cube $I_{\lambda,v} \subseteq [-\pi, \pi]^k$ with edges of length $2\delta_\lambda$ parallel to coordinate axes such that

$$|\langle (\Delta_{I_{\lambda,v}} \times I) e^{i\lambda_v} \rangle (u, P^*v) | \geq c \delta_\lambda^k$$

for all $u \in (-1/\delta_\lambda, 1/\delta_\lambda)^k$. The constant $c > 0$ is independent of $u, v,$ and $\lambda$.

Proof. Let $v \in (\lambda W)_{\varepsilon_0}$. Then one can find a point $x_{\lambda,v} \in [-\pi, \pi]^k$ such that

$$(9) \quad |v - \lambda a(x_{\lambda,v})| \leq \varepsilon_0.$$ 

Let us assume that $\lambda > 0$ is so large that $2\delta_\lambda < 2\pi$. Then we can find a (closed) cube $I_{\lambda,v} \subseteq [-\pi, \pi]^k$, with edges of length $2\delta_\lambda$ parallel to coordinate axes, such
that \( x_{\lambda,v} \in I_{\lambda,v} \). Let us verify that the conclusion of the lemma holds for this cube.

Define a function \( \varphi_{\lambda,v} \) by setting
\[
\varphi_{\lambda,v}(x, y) = \left( \frac{1}{\lambda} v, P y + y_0 \right), \quad x \in \mathbb{R}^k, \quad y \in \mathbb{R}^m.
\]
Let \( x \in I_{\lambda,v} \) and \( y \in J \). Then (see (8), (9))
\[
|\varphi(x, y) - \varphi_{\lambda,v}(x, y)| = |(a(x), Py + y_0) - (\lambda^{-1}v, Py + y_0)| \leq \rho |a(x) - \lambda^{-1}v| \leq \rho |a(x) - a(x_{\lambda,v})| + \rho |a(x_{\lambda,v}) - \lambda^{-1}v| \leq \rho \omega \left( \sqrt{k} \delta \lambda \right) + \rho \varepsilon_0 / \lambda = 2 \rho \varepsilon_0 / \lambda.
\]
So,
\[
|e^{i\lambda \varphi(x,y)} - e^{i\lambda \varphi_{\lambda,v}(x,y)}| \leq 2 \rho \varepsilon_0.
\]
Hence for all \( u \in \mathbb{R}^k \), in view of (7), we obtain
\[
\left| (\Delta_{I_{\lambda,v}} \times J) e^{i\lambda \varphi}(u, P^*v) - (\Delta_{I_{\lambda,v}} \times J) e^{i\lambda \varphi_{\lambda,v}}(u, P^*v) \right|
\leq \frac{1}{(2\pi)^{k+m}} \int_{\mathbb{R}^k \times \mathbb{R}^m} \Delta_{I_{\lambda,v}} \times J(x, y) |e^{i\lambda \varphi(x,y)} - e^{i\lambda \varphi_{\lambda,v}(x,y)}| dxdy
\]
\[
\leq \left| \Delta_{I_{\lambda,v}} \times J(0) \right| 2 \rho \varepsilon_0 = 2 \rho \varepsilon_0 \left| \Delta_{J}(0) \right| \left| \Delta_{I_{\lambda,v}}(0) \right|
\]
\[
\leq \frac{1}{2} 4^{-k} \left| \Delta_{J}(0) \right| \left( \frac{\delta \lambda}{2\pi} \right)^k.
\]
At the same time,
\[
(\Delta_{I_{\lambda,v}} \times J) e^{i\lambda \varphi_{\lambda,v}}(u, P^*v)
\]
\[
= \frac{1}{(2\pi)^{k+m}} \int_{x \in I_{\lambda,v}} \int_{y \in J} \Delta_{I_{\lambda,v}}(x) \Delta_{J}(y) e^{it(y, Py + y_0)} e^{-i(u, x)} e^{-i(P^*v, y)} dxdy
\]
\[
= e^{i(v, y_0)} \widehat{\Delta_{J}}(0) \widehat{\Delta_{I_{\lambda,v}}}(u),
\]
and hence it follows from Lemma 1 that
\[
\left| (\Delta_{I_{\lambda,v}} \times J) e^{i\lambda \varphi_{\lambda,v}}(u, P^*v) \right| \geq \left| \Delta_{J}(0) \right| 4^{-k} \left( \frac{\delta \lambda}{2\pi} \right)^k
\]
for \( u \in (-1/\delta \lambda, 1/\delta \lambda)^k \). Taking (10) into account, we complete the proof of the lemma.

Let us proceed directly to the proof of the theorem. Without loss of generality, we can assume that \( \lambda > 0 \) is sufficiently large. Recall that \( J = (-l, l)^m \), where \( 0 < l < \pi \).
Consider the expansion
\[ e^{i\lambda \varphi(t)} = \sum_{n \in \mathbb{Z}^{k+m}} c_\lambda(n) e^{i(n,t)}, \quad t \in \mathbb{T}^{k+m}, \]
and define measures \( \mu_\lambda \) and \( \sigma_\lambda \) by setting
\[ \mu_\lambda = \sum_{n \in \mathbb{Z}^{k+m}} c_\lambda(n) \delta_n, \quad \sigma_\lambda = \sum_{n \in \mathbb{Z}^{k+m}} |c_\lambda(n)| \delta_n, \]
where \( \delta_t \) stands for the unit mass concentrated at a point \( t \in \mathbb{R}^{k+m} \). Let \( I_{\lambda,v} \) be the cube whose existence has been established in Lemma 2. Since the parallelepiped \( I_{\lambda,v} \times J \) is obtained by a shift of the parallelepiped \( (-\delta_\lambda, \delta_\lambda)^k \times (-l, l)^m \), we have
\[ |(\Delta I_{\lambda,v} \times J)\wedge| = |(\Delta(-\delta_\lambda, \delta_\lambda)^k \times (-l, l)^m)\wedge|. \]
Therefore,
\[ |(\Delta I_{\lambda,v} \times J)e^{i\lambda \varphi})\wedge(u, P^*v)| = |(\Delta I_{\lambda,v} \times J)\wedge \mu_\lambda(u, P^*v)| \leq |(\Delta I_{\lambda,v} \times J)\wedge \sigma_\lambda(u, P^*v) = |(\Delta(-\delta_\lambda, \delta_\lambda)^k \times (-l, l)^m)\wedge \sigma_\lambda(u, P^*v) \]
for all \( u \in \mathbb{R}^k \) and \( v \in \mathbb{R}^m \). Thus, using Lemma 2 we see that the estimate
\[ c\delta_\lambda^k \leq |(\Delta(-\delta_\lambda, \delta_\lambda)^k \times (-l, l)^m)\wedge \sigma_\lambda(u, P^*v) \]
holds for all
\[ (u, v) \in \left( -\frac{1}{\delta_\lambda}, \frac{1}{\delta_\lambda} \right)^k \times (\lambda W)_{\varepsilon_0}. \]
Hence (see (6))
\[ c|\lambda W|_{\varepsilon_0} \leq c\delta_\lambda^k \left| \left( -\frac{1}{\delta_\lambda}, \frac{1}{\delta_\lambda} \right)^k \times (\lambda W)_{\varepsilon_0} \right| \]
\[ \leq \int_{\mathbb{R}^k \times \mathbb{R}^m} |(\Delta(-\delta_\lambda, \delta_\lambda)^k \times (-l, l)^m)\wedge \sigma_\lambda(u, P^*v)| dudv \]
\[ \leq \frac{1}{|\det P|} \int_{\mathbb{R}^k \times \mathbb{R}^m} |(\Delta(-\delta_\lambda, \delta_\lambda)^k \times (-l, l)^m)\wedge \sigma_\lambda(u, v)| dudv \]
\[ = \frac{1}{|\det P|} \sum_{n} |c_\lambda(n)| = \frac{\|e^{i\lambda \varphi}\|_{A(\mathbb{T}^{k+m})},}{|\det P|}. \]
At the same time, it is well known (e.g., see [16 Secs. 5.4 and 5.6]) that
\[ v_d N_F(2\varepsilon)e^d \leq |(F)\varepsilon| \]
for an arbitrary bounded set $F \subseteq \mathbb{R}^d$, where $v_d$ is the volume of the unit ball in $\mathbb{R}^d$. Thus, since $\varepsilon_0 \leq 1/2$ (see (7)), for the left-hand side in (11) we obtain

$$|(\lambda W)_{\varepsilon_0}| \gtrsim N_{MW}(2\varepsilon_0) \geq N_{MW}(1) = N_W(1/\lambda).$$

The proof of the theorem is complete.

Remarks. 1. Recall that, as we have verified in Section 2, for each $m$ there exists a space-filling curve of class $A$ in $\mathbb{R}^m$. Actually some explicit examples of space-filling curves of class $A$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ are well known. For example, the corresponding mappings can be obtained as sums of lacunary power series. The first results of this kind go back to Salem and Zygmund. Their approach was later developed in [8, Theorem II], where it was shown that if $\sigma > 1$ and $\inf_k n_{k+1}/n_k > 1$, then the mapping

$$a(t) = \sum_{k=1}^{\infty} \frac{1}{k^\sigma} e^{i n_k t}$$

is space-filling in the complex plane. Another example in $\mathbb{R}^2$ is the Shoenberg curve (see [18, Chap. 7]); it is easily seen that the Shoenberg curve is of class $A$. As to the other classical space-filling curves that can be found in [18] the author does not know if, being properly modified to obtain continuous mappings of the circle, they are of class $A$. In particular, consider the classical Peano mapping $p: [0,1] \to \mathbb{R}^2$. Define a mapping $\tilde{p}: \mathbb{T} \to \mathbb{R}^2$ by setting $\tilde{p}(t) = p(|t|/\pi)$ for $t \in [-\pi, \pi]$. Is it true that $\tilde{p}$ is of class $A$?

2. It is well known that the covering number of a set is closely related to its dimension (see, e.g., [16, Chaps. 4–5]). Assuming that Hausdorff dimension $\dim_H W$ or Minkowski dimension $\dim_M W$ of a set $W$ equals $s_0$, we have $N_W(\varepsilon) \gtrsim 1/\varepsilon^s$ for every $s$ with $s < s_0$. Thus, if $a: \mathbb{T} \to \mathbb{R}^m$ is a mapping of class $A$ such that the image $W = a(\mathbb{T})$ is of Hausdorff or Minkowski dimension $s_0$, then for the corresponding phase function $\varphi$ the theorem of the present work yields $\|e^{i\lambda \varphi}\|_{A(\mathbb{T}^{k+m})} \gtrsim |\lambda|^s$ for every $s$ with $s < s_0$. The fact that for each $s_0$ with $1 \leq s_0 \leq m$, there exists a mapping $a: \mathbb{T} \to \mathbb{R}^m$ of class $A$ such that the image $W = a(\mathbb{T})$ has Hausdorff dimension $s_0$ readily follows from the well known theorem on the range of the sum of a Gaussian trigonometric series [5, Sec. XIV.4, Th. 1]. Namely, let $X_n$ and $Y_n$ be similar independent Gaussian random variables in $\mathbb{R}^m$. Consider the Gaussian trigonometric series of the form

$$a(t) = \sum_{n=0}^{\infty} 2^{-n/s_0} (X_n \cos 2^nt + Y_n \sin 2^nt).$$

We have $\dim_H a(\mathbb{T}) = s_0$ almost surely. We also note that explicit examples of planar curves of class $A$ of a given dimension can be obtained by using the
Weierstrass function
\[ w(t) = \sum_{n \geq 0} \frac{1}{2^{(2-s_0)n}} \cos 2^n t, \]
where \( 1 < s_0 < 2 \). It was proved in [9] that the Minkowski dimension of the graph of \( w \) on \([-\pi, \pi]\) is \( s_0 \). By setting \( a(t) = (|t|, w(t)) \) for \( t \in [-\pi, \pi] \), we obtain a mapping \( a : \mathbb{T} \to \mathbb{R}^2 \) of class \( A \) such that \( \dim_M a(\mathbb{T}) = s_0 \).

3. In general it is not clear what sets in \( \mathbb{R}^m, m \geq 2 \), can be obtained as images of mappings \( a : \mathbb{T}^k \to \mathbb{R}^m \) of class \( A \) (for \( m > k \)). Let \( W \) be a compact set in \( \mathbb{R}^m \). Assume that \( W \) is a continuous image of the circle \( \mathbb{T} \). Is it true that \( W \) is the image of \( \mathbb{T} \) under some mapping of class \( A \)? The author does not know the answer even for \( m = 2 \).

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