Using a $q$-shuffle algebra to describe the basic module $V(\Lambda_0)$ for the quantized enveloping algebra $U_q(\widehat{sl}_2)$

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Abstract

We consider the quantized enveloping algebra $U_q(\widehat{sl}_2)$ and its basic module $V(\Lambda_0)$. This module is infinite-dimensional, irreducible, integrable, and highest-weight. We describe $V(\Lambda_0)$ using a $q$-shuffle algebra in the following way. Start with the free associative algebra $V$ on two generators $x, y$. The standard basis for $V$ consists of the words in $x, y$. In 1995 M. Rosso introduced an associative algebra structure on $V$, called a $q$-shuffle algebra. For $u, v \in \{x, y\}$ their $q$-shuffle product is $u \star v = uv + q^{(u,v)}vu$, where $(u,v) = 2$ (resp. $(u,v) = -2$) if $u = v$ (resp. $u \neq v$). Let $U$ denote the subalgebra of the $q$-shuffle algebra $V$ that is generated by $x, y$. Rosso showed that the algebra $U$ is isomorphic to the positive part of $U_q(\widehat{sl}_2)$. In our first main result, we turn $U$ into a $U_q(\widehat{sl}_2)$-module. Let $V$ denote the $U_q(\widehat{sl}_2)$-submodule of $U$ generated by the empty word. In our second main result, we show that the $U_q(\widehat{sl}_2)$-modules $U$ and $V(\Lambda_0)$ are isomorphic. Let $V$ denote the subspace of $V$ spanned by the words that do not begin with $y$ or $xx$. In our third main result, we show that $U = \cap U \cap V$.

Keywords. Quantized enveloping algebra, $q$-Serre relations, basic module, $q$-shuffle algebra.

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1 Introduction

The quantized enveloping algebra $U_q(\widehat{sl}_2)$ is associative, noncommutative, and infinite dimensional. A presentation by generators and relations is given in Appendix B below. The algebra $U_q(\widehat{sl}_2)$ has a subalgebra $U_q^+$, called the positive part. The algebra $U_q^+$ has a presentation involving two generators $A, B$ and two relations, called the $q$-Serre relations:

$$[A, [A, [A, B]_q]_{q^2} = 0, \quad [B, [B, [B, A]_q]_{q^2} = 0.$$

Both $U_q^+$ and $U_q(\widehat{sl}_2)$ are well known in algebraic combinatorics [4][21][26], representation theory [11][12][15][47], and mathematical physics [6][7][17][27]. In the following paragraphs, we describe a few situations in which $U_q^+$ and $U_q(\widehat{sl}_2)$ play a role.
There is an object in algebraic combinatorics called a tridiagonal pair [21]. Roughly speaking, this is a pair of diagonalizable linear maps on a finite-dimensional vector space, that each act on the eigenspaces of the other one in a block-tridiagonal fashion. According to [21] Example 1.7, for a finite-dimensional irreducible $U_q^+$-module $V$ on which the generators $A, B$ are not nilpotent, the pair $A, B$ act on $V$ as a tridiagonal pair. The resulting tridiagonal pair is said to have $q$-geometric type or $q$-Serre type. This type of tridiagonal pair is described in [13, 22, 26, 36, 48].

Another object in algebraic combinatorics is a partially ordered set $Y$ called the Young lattice [35, p. 288]. The elements of $Y$ are the Young diagrams (partitions), and the partial order is given by diagram inclusion. Define a vector space $V$ consisting of the formal linear combinations of $Y$. In the Hayashi realization [4, Theorem 10.6] the vector space $V$ becomes an integrable $U_q(\mathfrak{sl}_2)$-module with the following features. Each Young diagram $\lambda$ is a weight vector. The $U_q(\mathfrak{sl}_2)$-generators $E_0, E_1, F_0, F_1$ act on $\lambda$ as follows. Color the boxes of $\lambda$ alternating blue and red, with the top left box colored blue. The generator $E_0$ (resp. $E_1$) sends $\lambda$ to a linear combination of the Young diagrams $\mu$ obtained from $\lambda$ by removing a blue box (resp. red box). In this linear combination the $\mu$-coefficient is a power of $q$ that depends on the location of the box $\lambda/\mu$. Similarly, $F_0$ (resp. $F_1$) sends $\lambda$ to a linear combination of the Young diagrams $\mu$ obtained from $\lambda$ by adding a blue box (resp. red box). In this linear combination the $\mu$-coefficient is a power of $q$ that depends on the location of the box $\mu/\lambda$. The $U_q(\mathfrak{sl}_2)$-submodule of $V$ generated by the empty Young diagram is denoted $V(\Lambda_0)$ and called the basic representation. The $U_q(\mathfrak{sl}_2)$-module $V(\Lambda_0)$ is infinite-dimensional, irreducible, integrable, and highest-weight. For more detail about $V(\Lambda_0)$ see [4, Chapter 10], [20, Section 9], [27, Chapter 5].

Next we recall an embedding, due to M. Rosso [33, 34] of $U_q^+$ into a $q$-shuffle algebra. Start with a free associative algebra $V$ on two generators $x, y$. These generators are called letters. For $n \geq 0$, a word of length $n$ in $V$ is a product of letters $\ell_1 \ell_2 \cdots \ell_n$. We interpret the word of length 0 to be the multiplicative identity in $V$; this word is called trivial and denoted by 1. The words in $V$ form a basis for the vector space $V$; this basis is called standard. In [33, 34] M. Rosso introduced an associative algebra structure on $V$, called a $q$-shuffle algebra. For letters $u, v$ their $q$-shuffle product is $u \cdot v = uv + q(u,v)vu$, where $(u, v) = 2$ (resp. $(u, v) = -2$) if $u = v$ (resp. $u \neq v$). In [34, Theorem 15] Rosso gave an injective algebra homomorphism $\natural$ from $U_q^+$ into the $q$-shuffle algebra $V$, that sends $A \mapsto x$ and $B \mapsto y$.

We mention some applications of the map $\natural : U_q^+ \to V$. In [16] I. Damiani obtained a Poincaré-Birkhoff-Witt (or PBW) basis for $U_q^+$ whose elements are defined recursively. In [11, Proposition 6.1] J. Beck obtained another PBW basis for $U_q^+$ by adjusting some of the elements in the Damiani PBW basis. In [38] (resp. [33]) we applied the map $\natural$ to the Damiani (resp. Beck) PBW basis, and expressed the images in the standard basis for $V$. We gave the images in closed form [38, Theorem 1.7]. [43, Theorem 7.1]. The map $\natural$ is used in [39] to define the alternating elements of $U_q^+$. In [39, Theorem 10.1] a set of alternating elements is shown to form a PBW basis for $U_q^+$. This PBW basis is said to be alternating [39, Definition 10.3]. In [40] we used the alternating elements to obtain a central extension $U_q^+$ of $U_q^+$. The algebra $U_q^+$ is defined by generators and relations. These generators, said to be alternating, are in bijection with the alternating
elements of $U_q^+$. By [40] Lemma 3.3 there exists a surjective algebra homomorphism $U_q^+ \to U_q^+$ that sends each alternating generator of $U_q^+$ to the corresponding alternating element in $U_q^+$. In [40] Lemma 3.6 this homomorphism is adjusted to obtain an algebra isomorphism $U_q^+ \to U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$ where $\mathbb{F}$ is the ground field and $\{z_n\}_{n=1}^\infty$ are mutually commuting indeterminates. By [40] Theorem 10.2 the alternating generators form a PBW basis for $U_q^+$. The algebra $U_q^+$ is called the alternating central extension of $U_q^+$ [40,11]. We remark that $U_q^+$ is related to the work of Baseilhac, Koizumi, Shigechi concerning the $q$-Onsager algebra and integrable lattice models [8,10]. See [5,7,9,37,41,42,48] for related work.

Turning to the present paper, our goal is to describe the basic $U_q(\widehat{\mathfrak{s}_2})$-module $V(\Lambda_0)$ using the $q$-shuffle algebra $\mathbb{V}$. We have three main results, which are summarized below. Let $\text{End}(\mathbb{V})$ denote the algebra consisting of the linear maps from $\mathbb{V}$ to $\mathbb{V}$. We now define some maps $X, Y, K$ in $\text{End}(\mathbb{V})$. The map $X$ (resp. $Y$) is the automorphism of the free algebra $\mathbb{V}$ that sends $x \mapsto qx$ and $y \mapsto qy$ (resp. $x \mapsto x$ and $y \mapsto qy$). Define $K = X^2Y^{-2}$. Define the maps $A_L^*, B_L^*, A_R^*, B_R^*$ in $\text{End}(\mathbb{V})$ that send $1 \mapsto 0$ and for a nontrivial word $w = \ell_1\ell_2\cdots\ell_n$ in $\mathbb{V}$,

$$
A_L^*w = \ell_2\cdots\ell_n\delta_{\ell_1,x}, \quad B_L^*w = \ell_2\cdots\ell_n\delta_{\ell_1,y},
A_R^*w = \ell_1\cdots\ell_n-1\delta_{\ell_n,x}, \quad B_R^*w = \ell_1\cdots\ell_n-1\delta_{\ell_n,y}.
$$

Here $\delta_{x,s}$ is the Kronecker delta. Define the maps $A_\ell, B_\ell, A_r, B_r$ in $\text{End}(\mathbb{V})$ such that for $v \in \mathbb{V}$,

$$
A_\ell v = x \ast v, \quad B_\ell v = y \ast v, \quad A_r v = v \ast x, \quad B_r v = v \ast y.
$$

Let $\mathbb{U}$ denote the subalgebra of the $q$-shuffle algebra $\mathbb{V}$ that is generated by $x, y$. By construction the map $\sharp : U_q^+ \to \mathbb{U}$ is an algebra isomorphism. Our first main result is that $\mathbb{U}$ becomes a $U_q(\widehat{\mathfrak{s}_2})$-module on which the $U_q(\widehat{\mathfrak{s}_2})$-generators act as follows:

$$
\begin{array}{c|cccccccc}
\text{generator} & E_0 & F_0 & K_0^{\pm 1} & E_1 & F_1 & K_1^{\pm 1} & D^{\pm 1} \\
\text{action on } \mathbb{U} & A_R^* & qA_L^* & K_0^{\pm 1} & B_R^* & B_L^* & K_1^{\pm 1} & X^{\pm 1} \\
\end{array}
$$

Let $\mathbb{U}$ denote the submodule of the $U_q(\widehat{\mathfrak{s}_2})$-module $\mathbb{U}$ that is generated by the vector $1$. Our second main result is that the $U_q(\widehat{\mathfrak{s}_2})$-modules $\mathbb{U}$ and $V(\Lambda_0)$ are isomorphic. Let $\mathbb{V}$ denote the intersection of the kernel of $B_L^*$ and the kernel of $(A_L^*)^2$. The vector space $\mathbb{V}$ has a basis consisting of the words in $\mathbb{V}$ that do not begin with $y$ or $xx$. Note that the sum $\mathbb{V} = \mathbb{F}1 + \mathbb{F}x + xy\mathbb{V}$ is direct. Our third main result is that $\mathbb{U} = \mathbb{U} \cap \mathbb{V}$.

The paper is organized as follows. Section 2 contains some preliminaries. In Section 3 we recall the algebra $U_q^+$ and discuss its basic properties. In Section 4 we describe the free algebra $\mathbb{V}$. In Section 5 we describe the maps $X, Y, K$ in $\text{End}(\mathbb{V})$. In Section 6 we describe the maps $A_L^*, B_L^*, A_R^*, B_R^*$ in $\text{End}(\mathbb{V})$. In Section 7 we describe the $q$-shuffle algebra $\mathbb{V}$. In Section 8 we describe the maps $A_\ell, B_\ell, A_r, B_r$ in $\text{End}(\mathbb{V})$. In Section 9 we describe the subalgebra $\mathbb{U}$ of the $q$-shuffle algebra $\mathbb{V}$. In Sections 10, 11 we give our main results, which are Theorems 10.1, 11.1. In Section 12 we describe some variations on Theorem 10.1. In Appendix A we display some relations that are satisfied by the maps from the main body.
of the paper. In Appendix B we give a presentation of $U_q(\hat{\mathfrak{sl}}_2)$. In Appendix C we give a basis for some of the weight spaces of the $U_q(\hat{\mathfrak{sl}}_2)$-module $U$. In Appendix D we show how the $U_q(\hat{\mathfrak{sl}}_2)$-generators act on the bases in Appendix C. In Appendix E we discuss a linear algebraic situation that comes up in Section 11.

2 Preliminaries

We now begin our formal argument. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Let $\mathbb{F}$ denote a field with characteristic zero. Throughout this paper, every vector space we discuss is understood to be over $\mathbb{F}$. Every algebra we discuss is understood to be associative, over $\mathbb{F}$, and have a multiplicative identity. A subalgebra has the same multiplicative identity as the parent algebra. Let $A$ denote an algebra. An automorphism of $A$ is an algebra isomorphism $A \to A$. The opposite algebra $A^{\text{opp}}$ consists of the vector space $A$ and the multiplication map $A \times A \to A$, $(a, b) \mapsto ba$. An antiautomorphism of $A$ is an algebra isomorphism $A \to A^{\text{opp}}$.

We recall a few concepts from linear algebra. Let $V$ denote a vector space, and consider an $\mathbb{F}$-linear map $T : V \to V$. The map $T$ is said to be nilpotent whenever there exists a positive integer $n$ such that $T^n = 0$. The map $T$ is said to be locally nilpotent whenever for all $v \in V$ there exists a positive integer $n$ such that $T^nv = 0$. If $T$ is nilpotent then $T$ is locally nilpotent. If $T$ is locally nilpotent and the dimension of $V$ is finite, then $T$ is nilpotent.

Throughout the paper, fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. Recall the notation $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, $n \in \mathbb{N}$.

3 The positive part of $U_q(\hat{\mathfrak{sl}}_2)$

Later in the paper, we will discuss the quantized enveloping algebra $U_q(\hat{\mathfrak{sl}}_2)$. For now, we consider a subalgebra $U_q^+$ of $U_q(\hat{\mathfrak{sl}}_2)$, called the positive part. Shortly we will give a presentation of $U_q^+$ by generators and relations.

For elements $\mathcal{X}, \mathcal{Y}$ in any algebra, define their commutator and $q$-commutator by

$$[\mathcal{X}, \mathcal{Y}] = \mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}, \quad [\mathcal{X}, \mathcal{Y}]_q = q\mathcal{X}\mathcal{Y} - q^{-1}\mathcal{Y}\mathcal{X}.$$  

Note that

$$[\mathcal{X}, [\mathcal{X}, [\mathcal{X}, \mathcal{Y}]_q]_q]_q = \mathcal{X}^3\mathcal{Y} - [3]_q\mathcal{X}^2\mathcal{Y}\mathcal{X} + [3]_q\mathcal{X}\mathcal{Y}\mathcal{X}^2 - \mathcal{Y}\mathcal{X}^3.$$  

Definition 3.1. (See [30 Corollary 3.2.6].) Define the algebra $U_q^+$ by generators $A, B$ and relations

$$[A, [A, [A, B]_q]_q]_q = 0, \quad [B, [B, [B, A]_q]_q]_q = 0. \quad (2)$$

We call $U_q^+$ the positive part of $U_q(\hat{\mathfrak{sl}}_2)$. The relations (2) are called the $q$-Serre relations.
We mention some symmetries of $U_q^+$.

**Lemma 3.2.** There exists an automorphism $\sigma$ of $U_q^+$ that sends $A \leftrightarrow B$. Moreover $\sigma^2 = \text{id}$, where $\text{id}$ denotes the identity map.

**Lemma 3.3.** (See [38, Lemma 2.2].) There exists an antiautomorphism $\dagger$ of $U_q^+$ that fixes each of $A$, $B$. Moreover $\dagger^2 = \text{id}$.

**Lemma 3.4.** (See [42, Lemma 3.4].) The maps $\sigma$, $\dagger$ commute.

**Definition 3.5.** Let $\tau$ denote the composition of $\sigma$ and $\dagger$. Note that $\tau$ is an antiautomorphism of $U_q^+$ that sends $A \leftrightarrow B$. We have $\tau^2 = \text{id}$.

Next we describe a grading for the algebra $U_q^+$. The $q$-Serre relations are homogeneous in both $A$ and $B$. Therefore, the algebra $U_q^+$ has a $\mathbb{N}^2$-grading for which $A$ and $B$ are homogeneous, with degrees $(1, 0)$ and $(0, 1)$ respectively. For $(r, s) \in \mathbb{N}^2$ let $U_q^+(r, s)$ denote the $(r, s)$-homogeneous component of the grading. The dimension of $U_q^+(r, s)$ is described by a generating function, as we now discuss. Let $t$ and $u$ denote commuting indeterminates.

**Definition 3.6.** Define the generating function

$$\Phi(t, u) = \prod_{n=1}^{\infty} \frac{1}{1 - t^n u^{n-1}} \frac{1}{1 - t^n u^n} \frac{1}{1 - t^{n-1} u^n}.$$  

Using $(1 - z)^{-1} = 1 + z + z^2 + \cdots$ we expand the above generating function as a power series:

$$\Phi(t, u) = \sum_{(r,s) \in \mathbb{N}^2} d_{r,s} t^r u^s, \quad d_{r,s} \in \mathbb{N}.$$  

For notational convenience, define $d_{r,-1} = 0$ and $d_{-1,s} = 0$ for $r, s \in \mathbb{N}$.

**Example 3.7.** (See [39, Example 3.4].) For $0 \leq r, s \leq 6$ we display $d_{r,s}$ in the $(r, s)$-entry of the matrix below:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 3 & 3 & 3 & 3 \\
1 & 3 & 6 & 8 & 9 & 9 & 9 \\
1 & 3 & 8 & 14 & 19 & 21 & 22 \\
1 & 3 & 9 & 19 & 32 & 42 & 48 \\
1 & 3 & 9 & 21 & 42 & 66 & 87 \\
1 & 3 & 9 & 22 & 48 & 87 & 134
\end{pmatrix}$$

We have $\Phi(t, u) = \Phi(u, t)$. Moreover $d_{r,s} = d_{s,r}$ for $(r, s) \in \mathbb{N}^2$.

**Lemma 3.8.** (See [39, Definition 3.2, Corollary 3.7].) For $(r, s) \in \mathbb{N}^2$ we have

$$d_{r,s} = \dim U_q^+(r, s).$$

Our next goal is to show that $d_{r,s-1} \leq d_{r,s}$ and $d_{r-1,s} \leq d_{r,s}$ for $(r, s) \in \mathbb{N}^2$. To reach the goal, we modify the generating function $\Phi(t, u)$ in the following way.
Definition 3.9. Define the generating function
\[
\Delta(t, u) = \prod_{n=1}^{\infty} \frac{1}{1 - t^n u^{n-1}} \frac{1}{1 - t^n u^n} \frac{1}{1 - t^n u^{n+1}}.
\] (3)

Lemma 3.10. We have \( \Delta(t, u) = \Phi(t, u)(1 - u) \) and \( \Delta(u, t) = \Phi(t, u)(1 - t) \). Moreover
\[
\Delta(t, u) = \sum_{(r, s) \in \mathbb{N}^2} (d_{r, s} - d_{r, s - 1}) t^r u^s, \quad \Delta(u, t) = \sum_{(r, s) \in \mathbb{N}^2} (d_{r, s} - d_{r - 1, s}) t^r u^s.
\]

Proof. Use Definitions 3.6, 3.9.
\]

Lemma 3.11. For \((r, s) \in \mathbb{N}^2\) we have \(d_{r, s - 1} \leq d_{r, s}\) and \(d_{r - 1, s} \leq d_{r, s}\).

Proof. Expand the right-hand side of (3) as a power series. In this power series, the coefficient of \(t^r u^s\) is nonnegative for \((r, s) \in \mathbb{N}^2\). The result follows in view of Lemma 3.10.
\]

Our next general goal is to compute \(\max\{d_{r, s} | s \in \mathbb{N}\}\) for \(r \in \mathbb{N}\), and \(\max\{d_{r, s} | r \in \mathbb{N}\}\) for \(s \in \mathbb{N}\). To reach the goal, we will use the concept of a partition.

For \(n \in \mathbb{N}\), a partition of \(n\) is a sequence \(\lambda = \{\lambda_i\}_{i=1}^{\infty}\) of natural numbers such that \(\lambda_i \geq \lambda_{i+1}\) for \(i \geq 1\) and \(n = \sum_{i=1}^{\infty} \lambda_i\). Let \(p_n\) denote the number of partitions of \(n\). For example,
\[
\begin{array}{c|cccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
p_n & 1 & 1 & 2 & 3 & 5 & 7 & 11 \\
\end{array}
\]

Define the generating function for partitions:
\[
p(t) = \sum_{n \in \mathbb{N}} p_n t^n.
\] (4)

The following result is well known; see for example [13, Theorem 8.3.4].
\[
p(t) = \prod_{n=1}^{\infty} \frac{1}{1 - t^n}.
\] (5)

We expand the generating function \((p(t))^3\) as a power series:
\[
(p(t))^3 = \sum_{n \in \mathbb{N}} \mu_n t^n, \quad \mu_n \in \mathbb{N}.
\] (6)

Consider the coefficients \(\{\mu_n\}_{n \in \mathbb{N}}\). For example,
\[
\begin{array}{c|cccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\mu_n & 1 & 3 & 9 & 22 & 51 & 108 & 221 \\
\end{array}
\]

Proposition 3.12. For \(r \in \mathbb{N}\) we have
\[
\mu_r = \max\{d_{r, s} | s \in \mathbb{N}\}.
\] (7)

For \(s \in \mathbb{N}\) we have
\[
\mu_s = \max\{d_{r, s} | r \in \mathbb{N}\}.
\] (8)
Proof. First, for \( r \in \mathbb{N} \) we verify (7). Let \( \mu'_r \) denote right-hand side of (7). We show that \( \mu_r = \mu'_r \). By Lemma 3.11 we may view
\[
\mu'_r = \sum_{s \in \mathbb{N}} (d_{r,s} - d_{r,s-1}).
\]
By this and Lemma 3.10
\[
\Delta(t, 1) = \sum_{(r,s) \in \mathbb{N}^2} (d_{r,s} - d_{r,s-1})t^r = \sum_{r \in \mathbb{N}} \mu'_rt^r.
\] (9)
Set \( u = 1 \) in (3), and evaluate the result using (5), (6). This yields
\[
\Delta(t, 1) = \prod_{n=1}^{\infty} \frac{1}{(1 - t^n)^3} = (p(t))^3 = \sum_{r \in \mathbb{N}} \mu_rt^r.
\] (10)
Comparing (9), (10) we obtain \( \mu_r = \mu'_r \) for \( r \in \mathbb{N} \). We have verified (7). The second assertion in the proposition statement follows from the first assertion in the proposition statement and the comment above Lemma 3.8. \( \square \)

Our next general goal is to embed \( U_q^+ \) into a \( q \)-shuffle algebra. For this \( q \)-shuffle algebra the underlying vector space is a free algebra on two generators. This free algebra is described in the next section.

4 The free algebra \( \mathbb{V} \)

Let \( x, y \) denote noncommuting indeterminates. Let \( \mathbb{V} \) denote the free algebra with generators \( x, y \). By a letter in \( \mathbb{V} \) we mean \( x \) or \( y \). For \( n \in \mathbb{N} \), a word of length \( n \) in \( \mathbb{V} \) is a product of letters \( \ell_1 \ell_2 \cdots \ell_n \). We interpret the word of length 0 to be the multiplicative identity in \( \mathbb{V} \); this word is called trivial and denoted by 1. The vector space \( \mathbb{V} \) has a basis consisting of its words; this basis is called standard.

We mention some symmetries of the free algebra \( \mathbb{V} \). For the next four lemmas, the proofs are routine and omitted.

Lemma 4.1. There exists an automorphism \( \sigma \) of the free algebra \( \mathbb{V} \) that sends \( x \leftrightarrow y \). Moreover \( \sigma^2 = \text{id} \).

Lemma 4.2. There exists an antiautomorphism \( \dagger \) of the free algebra \( \mathbb{V} \) that fixes each of \( x, y \). Moreover \( \dagger^2 = \text{id} \).

Lemma 4.3. The map \( \sigma \) from Lemma 4.1 commutes with the map \( \dagger \) from Lemma 4.2.

Lemma 4.4. There exists an antiautomorphism \( \tau \) of the free algebra \( \mathbb{V} \) that sends \( x \leftrightarrow y \). The map \( \tau \) is the composition the map \( \sigma \) from Lemma 4.1 and the map \( \dagger \) from Lemma 4.2. We have \( \tau^2 = \text{id} \).
Example 4.5. The automorphism $\sigma$ sends
\[
xxx \leftrightarrow yyy, \quad xyy \leftrightarrow yxy, \quad xyxxyy \leftrightarrow yyyxx.
\]
The antiautomorphism $\dagger$ sends
\[
xxx \leftrightarrow xxx, \quad xyy \leftrightarrow yyx, \quad xyxxyy \leftrightarrow yyxyx.
\]
The antiautomorphism $\tau$ sends
\[
xxx \leftrightarrow yyy, \quad xyy \leftrightarrow xxy, \quad xyxxyy \leftrightarrow xxyxy.
\]
The free algebra $\mathbb{V}$ has a $\mathbb{N}^2$-grading for which $x$ and $y$ are homogeneous, with degrees $(1, 0)$ and $(0, 1)$ respectively. For $(r, s) \in \mathbb{N}^2$ let $\mathbb{V}(r, s)$ denote the $(r, s)$-homogeneous component of the grading. These homogeneous components are described as follows. Let $w = \ell_1\ell_2\cdots\ell_n$ denote a word in $\mathbb{V}$. The $x$-degree of $w$ is the cardinality of the set $\{i | 1 \leq i \leq n, \ell_i = x\}$. The $y$-degree of $w$ is the cardinality of the set $\{i | 1 \leq i \leq n, \ell_i = y\}$. For $(r, s) \in \mathbb{N}^2$ the subspace $\mathbb{V}(r, s)$ has a basis consisting of the words in $\mathbb{V}$ that have $x$-degree $r$ and $y$-degree $s$. The dimension of $\mathbb{V}(r, s)$ is equal to the binomial coefficient $\binom{r + s}{r}$. By construction $\mathbb{V}(0, 0) = \mathbb{F}1$. By construction, the sum $\mathbb{V} = \sum_{(r, s) \in \mathbb{N}^2} \mathbb{V}(r, s)$ is direct.

Example 4.6. The following is a basis for the vector space $\mathbb{V}(2, 3)$:
\[
\begin{align*}
xyyxy, & \quad yxyxy, \quad xyyxy, \quad yxxyy, \\
yxyxy, & \quad yxxyy, \quad yyxyx, \quad yyyxx.
\end{align*}
\]
Let $\text{End}(\mathbb{V})$ denote the algebra consisting of the $\mathbb{F}$-linear maps from $\mathbb{V}$ to $\mathbb{V}$. Let $I$ denote the identity in $\text{End}(\mathbb{V})$.

5 The maps $X$, $Y$, $K$

In this section we describe some maps $X$, $Y$, $K$ in $\text{End}(\mathbb{V})$ that will be used in our main results.

Definition 5.1. Let $X$ denote the automorphism of the free algebra $\mathbb{V}$ that sends $x \mapsto qx$ and $y \mapsto y$. Let $Y$ denote the automorphism of the free algebra $\mathbb{V}$ that sends $x \mapsto x$ and $y \mapsto qy$.

Example 5.2. The map $X$ sends
\[
xxx \mapsto q^3xxx, \quad xyy \mapsto q^2xyy, \quad xyxxyy \mapsto q^3xyxyy.
\]
The map $Y$ sends
\[
xxx \mapsto xxx, \quad xyy \mapsto q^2xyy, \quad xyxxyy \mapsto q^2xyxyy.
\]

Lemma 5.3. For $(r, s) \in \mathbb{N}^2$ the maps $X$ and $Y$ act on $\mathbb{V}(r, s)$ as $q^rI$ and $q^sI$, respectively.
Proof. By the description of \( V(r, s) \) above Example \([4,6]\)

By construction the maps \( X, Y \) are invertible, and they commute.

**Definition 5.4.** Define \( K = X^2Y^{-2} \). Thus \( K \) is the automorphism of the free algebra \( V \) that sends \( x \mapsto q^2x \) and \( y \mapsto q^{-2}y \).

**Example 5.5.** The map \( K \) sends
- \( xxx \mapsto q^6xxx \), \( xyy \mapsto xyy \), \( xyxyy \mapsto xyxyy \).

**Lemma 5.6.** For \( (r, s) \in \mathbb{N}^2 \) the map \( K \) acts on \( V(r, s) \) as \( q^{2r-2s}I \).

**Proof.** By Lemma \([5,3]\) and Definition \([5,4]\)\.

**Lemma 5.7.** The following diagrams commute:

- \( \begin{array}{ccc}
\mathbb{V} & \xrightarrow{X^\pm} & \mathbb{V} \\
\downarrow \sigma & & \downarrow \sigma \\
\mathbb{V} & \xrightarrow{Y^\pm} & \mathbb{V} \\
\downarrow & & \downarrow \\
\mathbb{V} & \xrightarrow{K^\pm} & \mathbb{V}
\end{array} \)
- \( \begin{array}{ccc}
\mathbb{V} & \xrightarrow{Y^\pm} & \mathbb{V} \\
\downarrow \sigma & & \downarrow \sigma \\
\mathbb{V} & \xrightarrow{X^\pm} & \mathbb{V} \\
\downarrow & & \downarrow \\
\mathbb{V} & \xrightarrow{K^\pm} & \mathbb{V}
\end{array} \)
- \( \begin{array}{ccc}
\mathbb{V} & \xrightarrow{X^\pm} & \mathbb{V} \\
\downarrow \uparrow & & \downarrow \uparrow \\
\mathbb{V} & \xrightarrow{Y^\pm} & \mathbb{V} \\
\downarrow & & \downarrow \\
\mathbb{V} & \xrightarrow{K^\pm} & \mathbb{V}
\end{array} \)
- \( \begin{array}{ccc}
\mathbb{V} & \xrightarrow{Y^\pm} & \mathbb{V} \\
\downarrow \uparrow & & \downarrow \uparrow \\
\mathbb{V} & \xrightarrow{X^\pm} & \mathbb{V} \\
\downarrow & & \downarrow \\
\mathbb{V} & \xrightarrow{K^\pm} & \mathbb{V}
\end{array} \)
- \( \begin{array}{ccc}
\mathbb{V} & \xrightarrow{Y^\pm} & \mathbb{V} \\
\downarrow \tau & & \downarrow \tau \\
\mathbb{V} & \xrightarrow{X^\pm} & \mathbb{V} \\
\downarrow & & \downarrow \\
\mathbb{V} & \xrightarrow{K^\pm} & \mathbb{V}
\end{array} \)
- \( \begin{array}{ccc}
\mathbb{V} & \xrightarrow{X^\pm} & \mathbb{V} \\
\downarrow \tau & & \downarrow \tau \\
\mathbb{V} & \xrightarrow{Y^\pm} & \mathbb{V} \\
\downarrow & & \downarrow \\
\mathbb{V} & \xrightarrow{K^\pm} & \mathbb{V}
\end{array} \)

**Proof.** Routine. \( \square \)

### 6 The maps \( A_L^*, B_L^*, A_R^*, B_R^* \)

In this section we recall from \([32]\) some maps \( A_L^*, B_L^*, A_R^*, B_R^* \) in \( \text{End}(V) \) that will be used in our main results. First we mention some notation. The Kronecker delta \( \delta_{r,s} \) is equal to 1 if \( r = s \), and 0 if \( r \neq s \).

**Definition 6.1.** (See \([32\text{, Lemma 4.3}]\).) Define the maps \( A_L^*, B_L^*, A_R^*, B_R^* \) in \( \text{End}(V) \) as follows. For a nontrivial word \( w = \ell_1\ell_2\cdots\ell_n \) in \( V \),

- \( A_L^* w = \ell_2\cdots\ell_n\delta_{\ell_1,x} \), \( B_L^* w = \ell_2\cdots\ell_n\delta_{\ell_1,y} \),
- \( A_R^* w = \ell_1\cdots\ell_{n-1}\delta_{\ell_n,x} \), \( B_R^* w = \ell_1\cdots\ell_{n-1}\delta_{\ell_n,y} \).

Moreover

\[
A_L^* 1 = 0, \quad B_L^* 1 = 0, \quad A_R^* 1 = 0, \quad B_R^* 1 = 0. \quad (11)
\]
Example 6.2. The maps $A^*_L, B^*_L, A^*_R, B^*_R$ are illustrated in the table below.

| $w$ | $x$ | $y$ | $xx$ | $xy$ | $yx$ | $yy$ |
|-----|-----|-----|------|------|------|------|
| $A^*_L w$ | 1 | 0 | $x$ | $y$ | 0 | 0 |
| $B^*_L w$ | 0 | 1 | 0 | 0 | $x$ | $y$ |
| $A^*_R w$ | 1 | 0 | $x$ | 0 | $y$ | 0 |
| $B^*_R w$ | 0 | 1 | 0 | $x$ | 0 | $y$ |

Lemma 6.3. For $v \in \mathbb{V}$,

$A^*_L(vx) = v, \quad A^*_L(yv) = 0, \quad B^*_L(xv) = 0, \quad B^*_L(yv) = v,$

$A^*_R(vx) = v, \quad A^*_R(vy) = 0, \quad B^*_R(vx) = 0, \quad B^*_R(vy) = v.$

Proof. Use Definition 6.1.

For notational convenience, define $\mathbb{V}(r, -1) = 0$ and $\mathbb{V}(-1, s) = 0$ for $r, s \in \mathbb{N}$.

Lemma 6.4. For $(r, s) \in \mathbb{N}^2$ we have

$A^*_L \mathbb{V}(r, s) \subseteq \mathbb{V}(r - 1, s), \quad B^*_L \mathbb{V}(r, s) \subseteq \mathbb{V}(r, s - 1),$

$A^*_R \mathbb{V}(r, s) \subseteq \mathbb{V}(r - 1, s), \quad B^*_R \mathbb{V}(r, s) \subseteq \mathbb{V}(r, s - 1).$

Proof. By Definition 6.1 or Lemma 6.3.

Lemma 6.5. The maps $A^*_L, B^*_L, A^*_R, B^*_R$ are locally nilpotent on the vector space $\mathbb{V}$.

Proof. We mentioned above Example 4.6 that the sum $\mathbb{V} = \sum_{(r, s) \in \mathbb{N}^2} \mathbb{V}(r, s)$ is direct. The result follows from this and Lemma 6.4.

Next we describe how the maps $X, Y$ are related to the maps $A^*_L, B^*_L, A^*_R, B^*_R$.

Lemma 6.6. We have

$X A^*_L = q^{-1} A^*_L X, \quad X B^*_L = B^*_L X, \quad X A^*_R = q^{-1} A^*_R X, \quad X B^*_R = B^*_R X,$

$Y A^*_L = A^*_L Y, \quad Y B^*_L = q^{-1} B^*_L Y, \quad Y A^*_R = A^*_R Y, \quad Y B^*_R = q^{-1} B^*_R Y.$

Proof. By Lemmas 5.3, 6.3.

The next result is about $A^*_L$ and $B^*_L$; a similar result holds for $A^*_R$ and $B^*_R$. Observe that the sum $\mathbb{V} = \mathbb{F}1 + x\mathbb{V} + y\mathbb{V}$ is direct.

Lemma 6.7. The following (i)–(v) hold:

(i) $\ker A^*_L$ has a basis consisting of the words in $\mathbb{V}$ that do not begin with $x$;

(ii) $\ker A^*_L = \mathbb{F}1 + y\mathbb{V}$;

(iii) $\ker B^*_L$ has a basis consisting of the words in $\mathbb{V}$ that do not begin with $y$;

(iv) $\ker B^*_L = \mathbb{F}1 + x\mathbb{V}$;
(v) \( \ker A_L^* \cap \ker B_L^* = \mathbb{F}1. \)

Proof. Use Definition 6.1 and the observation above the lemma statement.

The following result appears in [32]; we give a short proof for the sake of completeness.

Lemma 6.8. (See [32 Lemma 4.6].) Let \( W \) denote a nonzero subspace of \( V \) that is closed under \( A_L^* \) and \( B_L^* \). Then \( 1 \in W \).

Proof. For \( n \in \mathbb{N} \) define \( V_n = \sum_{r+s \leq n} V(r,s) \). Note that \( V_0 = \mathbb{F}1 \). We have \( V_{n-1} \subseteq V_n \) for \( n \geq 1 \), and \( V = \bigcup_{n \in \mathbb{N}} V_n \). For \( n \geq 1 \) we have \( A_L^* V_n \subseteq V_{n-1} \) and \( B_L^* V_n \subseteq V_{n-1} \), in view of Lemma 6.4. Since \( W \neq 0 \), there exists \( n \in \mathbb{N} \) such that \( W \cap V_n \neq 0 \). Assume for the moment that \( n = 0 \). Then \( 1 \in W \) and we are done. Next assume that \( n \geq 1 \). Without loss, we may assume that \( W \cap V_{n-1} = 0 \). Pick \( 0 \neq v \in W \cap V_n \). We have \( A_L^* v \in W \cap V_{n-1} = 0 \) and \( B_L^* v \in W \cap V_{n-1} = 0 \), so \( v \in \mathbb{F}1 \) in view of Lemma 6.7(v). By construction \( 0 \neq v \in W \), so \( 1 \in W \).

Lemma 6.9. Let \( W \) denote a nonzero subspace of \( V \) that is closed under \( A_R^* \) and \( B_R^* \). Then \( 1 \in W \).

Proof. The \( \dagger \)-image \( W^\dagger \) is a subspace of \( V \) that is invariant under \( A_L^* \) and \( B_L^* \). We have \( 1 \in W^\dagger \) by Lemma 6.8 and \( 1^\dagger = 1 \) by construction, so \( 1 \in W \).

Lemma 6.10. The following diagrams commute:

\[
\begin{array}{ccccccc}
V & \xrightarrow{A_L^*} & V & \xrightarrow{B_L^*} & V & \xrightarrow{A_R^*} & V & \xrightarrow{B_R^*} & V \\
\downarrow \sigma & & \downarrow \sigma & & \downarrow \sigma & & \downarrow \sigma & & \downarrow \sigma \\
V & \xrightarrow{B_L^*} & V & \xrightarrow{A_L^*} & V & \xrightarrow{B_R^*} & V & \xrightarrow{A_R^*} & V \\
\downarrow \dagger & & \downarrow \dagger & & \downarrow \dagger & & \downarrow \dagger & & \downarrow \dagger \\
V & \xrightarrow{A_R^*} & V & \xrightarrow{B_R^*} & V & \xrightarrow{A_L^*} & V & \xrightarrow{B_L^*} & V \\
\downarrow \tau & & \downarrow \tau & & \downarrow \tau & & \downarrow \tau & & \downarrow \tau \\
V & \xrightarrow{B_R^*} & V & \xrightarrow{A_R^*} & V & \xrightarrow{B_L^*} & V & \xrightarrow{A_L^*} & V \\
\end{array}
\]

Proof. Routine.
7 The \( q \)-shuffle algebra \( \mathbb{V} \)

In the previous sections we discussed the free algebra \( \mathbb{V} \). There is another algebra structure on \( \mathbb{V} \), called the \( q \)-shuffle algebra. This algebra was introduced by Rosso \[33, 34\] and described further by Green \[18\]. We will adopt the approach of \[18\], which is suited to our purpose. The \( q \)-shuffle product is denoted by \( \star \). To describe this product, we first consider some special cases. We have \( 1 \star v = v \star 1 = v \) for \( v \in \mathbb{V} \). For letters \( u, v \) we have \( u \star v = uv + vuq^{(u,v)} \), where

\[
\begin{array}{c|cc}
\text{ } & x & y \\
\hline
x & 2 & -2 \\
y & -2 & 2 \\
\end{array}
\]

Thus

\[
\begin{align*}
x \star x &= (1 + q^2)xx, & x \star y &= xy + q^{-2}yx, \\
y \star x &= yx + q^{-2}yx, & y \star y &= (1 + q^2)yy.
\end{align*}
\]

For a letter \( u \) and a nontrivial word \( v = v_1v_2 \cdots v_n \) in \( \mathbb{V} \),

\[
\begin{align*}
u \star v &= \sum_{i=0}^{n} v_1 \cdots v_i uv_{i+1} \cdots v_n q^{(v_1,u)+(v_2,u)+\cdots+(v_i,u)}, \\
v \star u &= \sum_{i=0}^{n} v_1 \cdots v_i uv_{i+1} \cdots v_n q^{(v_n,u)+(v_{n-1},u)+\cdots+(v_{i+1},u)}.
\end{align*}
\]  

(12) (13)

For example

\[
\begin{align*}
y \star (xxx) &= yxxx + q^{-2}xyxx + q^{-4}xxyx + q^{-6}xxxy, \\
(xxx) \star y &= q^{-6}yxxx + q^{-4}xxyx + q^{-2}xxxy + xxxy.
\end{align*}
\]

For nontrivial words \( u = u_1u_2 \cdots u_r \) and \( v = v_1v_2 \cdots v_s \) in \( \mathbb{V} \),

\[
\begin{align*}
u \star v &= u_1((u_2 \cdots u_r) \star v) + v_1(u \star (v_2 \cdots v_s))q^{(u_1,v_1)+(u_2,v_1)+\cdots+(u_r,v_1)}, \\
u \star v &= (u \star (v_1 \cdots v_{s-1}))v_s + ((u_1 \cdots u_{r-1}) \star v)u_r q^{(u_r,v_1)+(u_r,v_2)+\cdots+(u_r,v_s)}.
\end{align*}
\]

For example, assume \( r = 2 \) and \( s = 2 \). Then

\[
u \star v = u_1u_2v_1v_2 + u_1v_1u_2v_2q^{(u_2,v_1)} + u_1v_1v_2u_2q^{(u_2,v_1)+(u_2,v_2)} + v_1u_1u_2v_2q^{(u_1,v_1)+(u_2,v_1)} + v_1v_2u_1u_2q^{(u_1,v_1)+(u_2,v_1)} + v_1u_1v_2u_2q^{(u_1,v_1)+(u_2,v_2)} + v_1v_2u_1u_2q^{(u_1,v_1)+(u_2,v_1)+(u_2,v_2)}.
\]

The map \( \sigma \) from Lemma 4.1 is an automorphism of the \( q \)-shuffle algebra \( \mathbb{V} \). The map \( \dagger \) from Lemma 4.2 is an antiautomorphism of the \( q \)-shuffle algebra \( \mathbb{V} \). The map \( \tau \) from Lemma 4.3 is an antiautomorphism of the \( q \)-shuffle algebra \( \mathbb{V} \). Above Example 4.6 we mentioned an \( \mathbb{N}^2 \)-grading of the free algebra \( \mathbb{V} \). This is also an \( \mathbb{N}^2 \)-grading for the \( q \)-shuffle algebra \( \mathbb{V} \).

See \[19, 29, 31, 32, 38, 40\] for more information about the \( q \)-shuffle algebra \( \mathbb{V} \).
8 The maps $A_\ell$, $B_\ell$, $A_r$, $B_r$

In this section we recall from [32] some maps $A_\ell$, $B_\ell$, $A_r$, $B_r$ in $\text{End}(\mathbb{V})$ that will be used in our main results.

**Definition 8.1.** (See [32] Definition 7.1.) Define the maps $A_\ell$, $B_\ell$, $A_r$, $B_r$ in $\text{End}(\mathbb{V})$ as follows. For $v \in \mathbb{V}$,

$$A_\ell v = x \star v, \quad B_\ell v = y \star v, \quad A_r v = v \star x, \quad B_r v = v \star y.$$ 

**Example 8.2.** The maps $A_\ell$, $B_\ell$, $A_r$, $B_r$ are illustrated in the table below.

|   | $v$ | $x$ | $y$ | $xy$ |
|---|-----|-----|-----|-----|
| $A_\ell v$ | $x$ | $q[2]qxx$ | $xy + q^{-2}yx$ | $q[2]qxy + xyx$ |
| $B_\ell v$ | $y$ | $q^{-2}xy + yx$ | $q[2]qyy$ | $q^{-1}[2]qxy + yxy$ |
| $A_r v$ | $x$ | $q[2]qxx$ | $q^{-2}xy + yx$ | $q^{-1}[2]qxy + xyx$ |
| $B_r v$ | $y$ | $xy + q^{-2}yx$ | $q[2]qyy$ | $q[2]qxy + yxy$ |

**Lemma 8.3.** For $(r, s) \in \mathbb{N}^2$ we have

$$A_\ell \mathbb{V}(r, s) \subseteq \mathbb{V}(r + 1, s), \quad B_\ell \mathbb{V}(r, s) \subseteq \mathbb{V}(r, s + 1),$$

$$A_r \mathbb{V}(r, s) \subseteq \mathbb{V}(r + 1, s), \quad B_r \mathbb{V}(r, s) \subseteq \mathbb{V}(r, s + 1).$$

**Proof.** By Definition 8.1 and the description of $\mathbb{V}(r, s)$ above Example 4.6.

Next we describe how the maps $X$, $Y$ are related to the maps $A_\ell$, $B_\ell$, $A_r$, $B_r$.

**Lemma 8.4.** We have

$$XA_\ell = qA_\ell X, \quad XB_\ell = B_\ell X, \quad XA_r = qA_r X, \quad XB_r = B_r X,$$

$$YA_\ell = A_\ell Y, \quad YB_\ell = qB_\ell Y, \quad YA_r = A_r Y, \quad YB_r = qB_r Y.$$ 

**Proof.** By Lemmas 5.3, 8.3.

**Lemma 8.5.** The following diagrams commute:

\[ \begin{array}{ccccccc}
\mathbb{V} & \xrightarrow{A_\ell} & \mathbb{V} \\
\sigma & \downarrow & \sigma \\
\mathbb{V} & \xrightarrow{B_\ell} & \mathbb{V} \\
\end{array} \quad \begin{array}{ccccccc}
\mathbb{V} & \xrightarrow{A_r} & \mathbb{V} \\
\sigma & \downarrow & \sigma \\
\mathbb{V} & \xrightarrow{B_r} & \mathbb{V} \\
\end{array} \quad \begin{array}{ccccccc}
\mathbb{V} & \xrightarrow{A_\ell} & \mathbb{V} \\
\sigma & \downarrow & \sigma \\
\mathbb{V} & \xrightarrow{A_r} & \mathbb{V} \\
\end{array} \quad \begin{array}{ccccccc}
\mathbb{V} & \xrightarrow{B_\ell} & \mathbb{V} \\
\sigma & \downarrow & \sigma \\
\mathbb{V} & \xrightarrow{B_r} & \mathbb{V} \\
\end{array} \]

**Proof.** Routine.
9 The subspace $\mathbb{U}$

In this section we discuss a subspace $\mathbb{U} \subseteq \mathbb{V}$ that will be used in our main results.

**Definition 9.1.** Let $\mathbb{U}$ denote the subalgebra of the $q$-shuffle algebra $\mathbb{V}$ that is generated by $x, y$.

The algebra $\mathbb{U}$ is described as follows. By [33, Theorem 13] or [18, p. 10],

\[
x * x * x * y - [3]_q x * y * x * x + [3]_q y * x * y - y * x * x * x = 0,
\]

\[
y * y * y * x - [3]_q y * y * x * y + [3]_q y * y * y * x - x * y * y * y = 0.
\]

So in the $q$-shuffle algebra $\mathbb{V}$ the elements $x, y$ satisfy the $\mathbb{q}$-Serre relations. Therefore, there exists an algebra homomorphism $\natural$ from $\mathbb{U}^+$ to the $q$-shuffle algebra $\mathbb{V}$, that sends

\[ A \mapsto x \quad \text{and} \quad B \mapsto y. \]

The map $\natural$ has image $\mathbb{U}$ by Definition 9.1, and is injective by [34, Theorem 15]. Consequently $\natural: \mathbb{U}^+ \rightarrow \mathbb{U}$ is an algebra isomorphism. By construction the following diagrams commute:

\[
\begin{array}{ccc}
U_q^+ & \xrightarrow{\natural} & \mathbb{V} \\
\sigma \downarrow & & \downarrow \sigma \\
U_q^+ & \xrightarrow{\natural} & \mathbb{V}
\end{array}
\]

\[
\begin{array}{ccc}
U_q^+ & \xrightarrow{\tau} & \mathbb{V} \\
\downarrow \tau & & \downarrow \tau \\
U_q^+ & \xrightarrow{\natural} & \mathbb{V}
\end{array}
\]

Consequently $\mathbb{U}$ is invariant under each of $\sigma, \dagger, \tau$. Earlier we mentioned an $\mathbb{N}^2$-grading for both the algebra $U_q^+$ and the $q$-shuffle algebra $\mathbb{V}$. These gradings are related as follows. The algebra $\mathbb{U}$ has an $\mathbb{N}^2$-grading inherited from $U_q^+$ via $\natural$. With respect to this grading, for $(r, s) \in \mathbb{N}^2$ the $(r, s)$-homogeneous component of $\mathbb{U}$ is the $\natural$-image of the $(r, s)$-homogeneous component of $U_q^+$. We denote this homogeneous component by $\mathbb{U}(r, s)$. By construction,

\[
\mathbb{U}(r, s) = \mathbb{V}(r, s) \cap \mathbb{U}, \quad (r, s) \in \mathbb{N}^2.
\]

By construction $\mathbb{U}(0, 0) = F_1$. By construction, the sum $\mathbb{U} = \sum_{(r, s) \in \mathbb{N}^2} \mathbb{U}(r, s)$ is direct. By Lemma 3.8 and the construction,

\[
d_{r,s} = \dim \mathbb{U}(r, s), \quad (r, s) \in \mathbb{N}^2.
\]

**Lemma 9.2.** For $(r, s) \in \mathbb{N}^2$ the following hold on $\mathbb{U}(r, s)$:

\[ X = q^r I, \quad Y = q^s I, \quad K = q^{2r-2s} I. \]

**Proof.** By Lemmas 5.3, 5.6 and since $\mathbb{U}(r, s) \subseteq \mathbb{V}(r, s)$. □

**Lemma 9.3.** The vector space $\mathbb{U}$ is invariant under each of

\[ X^{\pm 1}, \quad Y^{\pm 1}, \quad K^{\pm 1}. \]

**Proof.** By Lemma 9.2 and since $\mathbb{U} = \sum_{(r, s) \in \mathbb{N}^2} \mathbb{U}(r, s)$. □
Lemma 9.4. (See [32, Proposition 9.1].) The vector space $\mathbb{U}$ is invariant under each of

$$A^*_L, \quad B^*_L, \quad A^*_R, \quad B^*_R.$$  

For notational convenience, define $\mathbb{U}(r, -1) = 0$ and $\mathbb{U}(-1, s) = 0$ for $r, s \in \mathbb{N}$.

Lemma 9.5. For $(r, s) \in \mathbb{N}^2$ we have

$$A^*_L \mathbb{U}(r, s) \subseteq \mathbb{U}(r - 1, s), \quad B^*_L \mathbb{U}(r, s) \subseteq \mathbb{U}(r, s - 1),$$

$$A^*_R \mathbb{U}(r, s) \subseteq \mathbb{U}(r - 1, s), \quad B^*_R \mathbb{U}(r, s) \subseteq \mathbb{U}(r, s - 1).$$

Proof. By (15) and Lemmas 6.4, 9.4.

Lemma 9.6. The subspace $\mathbb{U}$ is invariant under each of

$$A_\ell, \quad B_\ell, \quad A_r, \quad B_r.$$  

Proof. By Definitions 9.1, 8.1.

Lemma 9.7. For $(r, s) \in \mathbb{N}^2$ we have

$$A_\ell \mathbb{U}(r, s) \subseteq \mathbb{U}(r + 1, s), \quad B_\ell \mathbb{U}(r, s) \subseteq \mathbb{U}(r, s + 1),$$

$$A_r \mathbb{U}(r, s) \subseteq \mathbb{U}(r + 1, s), \quad B_r \mathbb{U}(r, s) \subseteq \mathbb{U}(r, s + 1).$$

Proof. By (15) and Lemmas 8.3, 9.6.

In [32, Propositions 9.1, 9.3] there are many relations satisfied by the maps

$$K, \quad K^{-1}, \quad A^*_L, \quad B^*_L, \quad A^*_R, \quad B^*_R, \quad A_\ell, \quad B_\ell, \quad A_r, \quad B_r.$$  

For convenience we reproduce these relations in Appendix A. These relations will be used in our main results.

10 The $U_q(\widehat{sl}_2)$-module $\mathbb{U}$ and its submodule $\mathbb{U}$

We now bring in the algebra $U_q(\widehat{sl}_2)$. The definition of this algebra can be found in Appendix B. In the present section, we turn the vector space $\mathbb{U}$ into a $U_q(\widehat{sl}_2)$-module, and describe the submodule $\mathbb{U}$ generated by the vector $1$.

The following is our first main result.

**Theorem 10.1.** The vector space $\mathbb{U}$ becomes a $U_q(\widehat{sl}_2)$-module on which the $U_q(\widehat{sl}_2)$-generators act as follows:

| generator | action on $\mathbb{U}$ | $E_0$ | $F_0$ | $A^*_L$ | $q^\frac{1}{2}K^\pm_1$ | $E_1$ | $F_1$ | $B^*_R$ | $K^\pm_1$ | $D^\pm_1$ |
|-----------|------------------------|-------|-------|----------|-----------------|-------|-------|----------|----------|----------|
| $A^*_R$   | $qA_rK^{-1}q^{-1}A_L$   | $q^\frac{1}{2}K^\pm_1$ |       | $B^*_L$  | $B^*_R$         |       |       | $K^\pm_1$ | $K^\pm_1$ | $X^\pm_1$ |
Proof. This is routinely checked using Lemmas 9.3, 9.4, 9.6 along with the relations in Lemmas 6.6, 8.4 and Appendix A. Among the things to check, is that \( qA, K^{-1} - q^{-1}A_\ell \) and \( B, K - B_\ell \) satisfy the \( q \)-Serre relations. This can be checked easily using [32, Lemma 10.3, Corollary 10.4].

Consider the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \) from Theorem 10.1. Recall the \( \mathbb{N}^2 \)-grading of \( \mathbb{U} \) from around (15). Next we describe how the \( U_q(\widehat{sl}_2) \)-generators act on the homogeneous components of this grading.

**Lemma 10.2.** For \((r, s) \in \mathbb{N}^2\) the following hold on \( \mathbb{U}(r, s) \):

\[
K_0 = q^{2s-2r+1}I, \\
K_1 = q^{2r-2s}I, \\
D = q^{-r}I.
\]

Moreover

\[
E \mathbb{U}(r, s) \subseteq \mathbb{U}(r - 1, s), \\
F_0 \mathbb{U}(r, s) \subseteq \mathbb{U}(r + 1, s), \\
E_1 \mathbb{U}(r, s) \subseteq \mathbb{U}(r, s - 1), \\
F_1 \mathbb{U}(r, s) \subseteq \mathbb{U}(r, s + 1).
\]

**Proof.** By Lemmas 9.2, 9.5, 9.7 and the data in Theorem 10.1.

In this paragraph we recall a few concepts about \( U_q(\widehat{sl}_2) \)-modules; see for example [20, Section 3.2]. Let \( W \) denote a \( U_q(\widehat{sl}_2) \)-module. A weight space for \( W \) is a common eigenspace for the action of \( K_0, K_1, D \) on \( W \). The sum of these weight spaces is direct. We call \( W \) a weight module whenever \( W \) is equal to the sum of its weight spaces. If \( W \) is a weight module, then every submodule of \( W \) is a weight module [20, Proposition 3.2.1].

We return our attention to the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \) from Theorem 10.1. We mentioned above Lemma 9.2 that the sum \( \mathbb{U} = \sum_{(r, s) \in \mathbb{N}^2} \mathbb{U}(r, s) \) is direct. By this and (17), the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \) is a weight module, and its weight spaces are the nonzero subspaces among \( \mathbb{U}(r, s) \) \((r, s \in \mathbb{N})\). Note that these weight spaces have finite dimension.

It turns out that the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \) is not irreducible. Next we consider its submodules.

**Definition 10.3.** Let \( \mathbb{U} \) denote the submodule of the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \) that is generated by the vector \( 1 \).

**Lemma 10.4.** For the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \),

(i) \( \mathbb{U} \) is contained in every nonzero submodule of the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \);

(ii) \( \mathbb{U} \) is the unique irreducible submodule of the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \).

**Proof.** (i) Let \( W \) denote a nonzero submodule of the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \). The vector space \( W \) is invariant under \( A_R^*, B_R^* \) by Theorem 10.1 so \( 1 \in W \) by Lemma 6.9. The \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \) is generated by \( 1 \), so \( \mathbb{U} \subseteq W \).

(ii) By (i) above.

Next we consider how the \( U_q(\widehat{sl}_2) \)-generators act on the vector \( 1 \).
Lemma 10.5. For the $U_q(\hat{\mathfrak{sl}}_2)$-module $U,$
\[ K_0 1 = q 1, \quad K_1 1 = 1, \quad D 1 = 1, \]
\[ E_0 1 = 0, \quad F_0^2 1 = 0, \quad E_1 1 = 0, \quad F_1 1 = 0. \]

Proof. This is routinely checked using the data in Theorem 10.1 and Lemma 10.2. \qed

There is a well known $U_q(\hat{\mathfrak{sl}}_2)$-module $V(\Lambda_0)$ that is said to be basic; see [20, p. 221]. The module $V(\Lambda_0)$ is highest weight, integrable, and level one; see [4, Chapter 10] and [27, Chapter 5]. The module $V(\Lambda_0)$ is characterized as follows.

Lemma 10.6. (See [27, pp. 63, 64].) There exists a $U_q(\hat{\mathfrak{sl}}_2)$-module $V(\Lambda_0)$ with the following property: $V(\Lambda_0)$ is generated by a nonzero vector $v$ such that
\[ K_0 v = q v, \quad K_1 v = v, \quad D v = v, \]
\[ E_0 v = 0, \quad F_0^2 v = 0, \quad E_1 v = 0, \quad F_1 v = 0. \]

Moreover $V(\Lambda_0)$ is irreducible, infinite-dimensional, and unique up to isomorphism of $U_q(\hat{\mathfrak{sl}}_2)$-modules.

The following is our second main result.

Theorem 10.7. The $U_q(\hat{\mathfrak{sl}}_2)$-modules $U$ and $V(\Lambda_0)$ are isomorphic.

Proof. By Definition 10.3 and Lemmas 10.5, 10.6. \qed

Descriptions of $V(\Lambda_0)$ can be found in [4, Chapter 10] and [20, Section 9] and [27, Chapter 5]; see also [14, Section 20.4] and [28, Chapter 14]. Our next general goal is to describe $V(\Lambda_0)$ from the point of view of $U.$

Definition 10.8. For $(r, s) \in \mathbb{N}^2$ define $U(r, s) = U \cap U(r, s).$

The $U_q(\hat{\mathfrak{sl}}_2)$-module $U$ is a weight module, and its weight spaces are the nonzero subspaces among $U(r, s)$ ($r, s \in \mathbb{N}$). More detail is given in the next result.

Lemma 10.9. The following (i)–(iv) hold for the $U_q(\hat{\mathfrak{sl}}_2)$-module $U.$

(i) $U(0, 0) = F 1.$

(ii) The sum $U = \sum_{(r, s) \in \mathbb{N}^2} U(r, s)$ is direct.

(iii) For $(r, s) \in \mathbb{N}^2$ the following hold on $U(r, s):$
\[ K_0 = q^{2s - 2r + 1} I, \quad K_1 = q^{2r - 2s} I, \quad D = q^{-r} I. \]

(iv) For $(r, s) \in \mathbb{N}^2,$
\[ E_0 U(r, s) \subseteq U(r - 1, s), \quad F_0 U(r, s) \subseteq U(r + 1, s), \]
\[ E_1 U(r, s) \subseteq U(r, s - 1), \quad F_1 U(r, s) \subseteq U(r, s + 1), \]
where $U(r, -1) = 0$ and $U(-1, s) = 0.$
Proof. (i) By Definition 10.8 and since \( \mathbb{U}(0, 0) = \mathbb{F}1 \).

(ii) Since \( \mathbb{U} \) is a weight module.

(iii), (iv) By Lemma 10.2 and Definition 10.8.

Our next goal is to describe the weight space dimensions for the \( U_q(\hat{\mathfrak{sl}}_2) \)-module \( \mathbb{U} \). Recall the partition numbers \( \{p_n\}_{n \in \mathbb{N}} \) from Section 3.

**Proposition 10.10.** For \((r, s) \in \mathbb{N}^2\) the vector space \( \mathbb{U}(r, s) \neq 0 \) if and only if \( r \geq (r - s)^2 \).

In this case \( \dim \mathbb{U}(r, s) = p_n, \) where \( n = r - (r - s)^2 \).

Proof. For the \( U_q(\hat{\mathfrak{sl}}_2) \)-module \( V(\Lambda_0) \) the weight space dimensions are described in [20, pp. 221, 222]. The result follows from that description and Theorem 10.7 above.

**Example 10.11.** For \(0 \leq r, s \leq 6\) the dimension of \( \mathbb{U}(r, s) \) is given in the \((r, s)\)-entry of the matrix below:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 3 & 2 & 0 & 0 \\
0 & 0 & 1 & 3 & 5 & 3 & 1 \\
0 & 0 & 0 & 1 & 5 & 7 & 5 \\
0 & 0 & 0 & 0 & 2 & 7 & 11 \\
\end{pmatrix}
\]

Compare the above matrix with the one in Example 3.7.

Next we describe the generating function \( \sum_{(r, s) \in \mathbb{N}^2} \dim \mathbb{U}(r, s)t^ru^s \).

Define the generating function

\[
\phi(t, u) = \sum_{n \in \mathbb{Z}} t^{n^2} u^{n^2-n}.
\]

(20)

Note that

\[
\phi(t, u) = 1 + t + tu^2 + t^4u^2 + t^4u^6 + \cdots.
\]

**Proposition 10.12.** We have

\[
\sum_{(r, s) \in \mathbb{N}^2} \dim \mathbb{U}(r, s)t^ru^s = p(tu)\phi(t, u),
\]

where \( p(t) \) is from (4) and \( \phi(t, u) \) is from (20).

Proof. This is a reformulation of Proposition 10.10.

In Appendix C, we give a basis for each nonzero \( \mathbb{U}(r, s) \) such that \( r + s \leq 10 \).
11 A characterization of the $U_q(\hat{\mathfrak{sl}}_2)$-module $U$

In order to motivate this section, we glance at the basis vectors displayed in Appendix C. Each displayed vector is a linear combination of some words in $V$ that do not begin with $y$ or $xx$. Consequently, each displayed vector is contained in the kernel of $B_L^*$ and the kernel of $(A_L^*)^2$. Using this observation, we will characterize the $U_q(\hat{\mathfrak{sl}}_2)$-module $U$.

**Definition 11.1.** Let $V$ denote the intersection of the kernel of $B_L^*$ and the kernel of $(A_L^*)^2$. Note that $V$ is a subspace of the vector space $V$.

We have several comments about $V$.

**Lemma 11.2.** The vector space $V$ has a basis consisting of the words in $V$ that do not begin with $y$ or $xx$.

*Proof.* By Definitions 6.1 11.1.

**Lemma 11.3.** The sum $V = F1 + Fx + xyV$ is direct.

*Proof.* By Lemma 11.2.

We are going to show that $U = U \cap V$. We will do this in several steps. In the first step, we show that $U \cap V$ is a submodule of the $U_q(\hat{\mathfrak{sl}}_2)$-module $U$.

**Lemma 11.4.** The vector space $V$ is invariant under each of $X^{\pm1}$, $Y^{\pm1}$, $K^{\pm1}$, $A_R^*$, $B_R^*$.

*Proof.* Use Definitions 5.1 5.4 and Lemma 11.3.

**Lemma 11.5.** The vector space $V$ is invariant under each of

$$qA_rK^{-1} - q^{-1}A_\ell, \quad B_rK - B_\ell.$$

*Proof.* We will use Lemma 11.3. The map $qA_rK^{-1} - q^{-1}A_\ell$ sends $1 \mapsto (q - q^{-1})x$ and $x \mapsto 0$. The map $B_rK - B_\ell$ sends $1 \mapsto 0$ and

$$x \mapsto q^2x*y - y*x = q^2(xy + q^{-2}yx) - (yx + q^{-2}xy) = (q^2 - q^{-2})xy.$$

Pick $(r, s) \in \mathbb{N}^2$ and a word $w \in V(r, s)$. The map $qA_rK^{-1} - q^{-1}A_\ell$ sends

$$xyw \mapsto q^{1+2s-2r}(xyw)\ast x - q^{-1}x\ast(xyw). \quad (21)$$

Using (13) we obtain

$$(xyw)\ast x = xy(w\ast x) + [2]q^{2r-2s-1}xwy. \quad (22)$$

Using (12) we obtain

$$x\ast(xyw) = q[2]_q xwy + xy(x\ast w). \quad (23)$$
By (21)–(23) the map \( qA, K^{-1} - q^{-1}A \ell \) sends
\[
xyw \mapsto xy(q^{1+2s-2r}w \ast x - q^{-1}x \ast w).
\]
The map \( B_r K - B_\ell \) sends
\[
xyw \mapsto q^{2r-2s}(xyw) \ast y - y \ast (xyw).
\]
(24)

Using (13) we obtain
\[
(xyw) \ast y = xy(w \ast y) + q^{2s-2r+2}xyyw + q^{2s-2r}xyw.
\]
(25)

Using (12) we obtain
\[
y \ast (xyw) = yxyw + q^{-2}xyyw + xy(y \ast w).
\]
(26)

By (24)–(26) the map \( B_r K - B_\ell \) sends
\[
xyw \mapsto xy((q^{2} - q^{-2})yw + q^{2r-2s}w \ast y - y \ast w).
\]
The result follows from the above comments.

Lemma 11.6. The vector space \( \mathbb{U} \cap \mathbb{V} \) is a submodule of the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \).

Proof. By Theorem 10.1 and Lemmas 11.4, 11.5.

Lemma 11.7. The vector space \( \mathbb{U} \) is a submodule of the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \cap \mathbb{V} \).

Proof. We have \( 1 \in \mathbb{U} \) by the comment below (15). We have \( 1 \in \mathbb{V} \) by Lemma 11.3. So \( 1 \in \mathbb{U} \cap \mathbb{V} \). The result follows in view of Definition 10.3 and Lemma 11.6.

The \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \cap \mathbb{V} \) is a weight module, and its weight spaces are the nonzero subspaces among \( \mathbb{U}(r, s) \cap \mathbb{V} \ (r, s \in \mathbb{N}) \). We will return to the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \cap \mathbb{V} \) after some comments about \( \mathbb{U} \).

Lemma 11.8. The \( U_q(\widehat{sl}_2) \)-generators \( E_0, E_1 \) are locally nilpotent on the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \).

Proof. By Lemma 6.5 and Theorem 10.1.

Lemma 11.9. The \( U_q(\widehat{sl}_2) \)-generators \( F_0, F_1 \) are not locally nilpotent on the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \).

Proof. The words \( xx \) and \( y \) are contained in \( \mathbb{U} \), but \( F^n_0(xx) \neq 0 \) and \( F^n_1y \neq 0 \) for all \( n \in \mathbb{N} \).

Lemma 11.10. The \( U_q(\widehat{sl}_2) \)-generators \( F_0, F_1 \) are locally nilpotent on the \( U_q(\widehat{sl}_2) \)-module \( \mathbb{U} \cap \mathbb{V} \).
Proof. First consider \( F_0 \). Assume that there exists \( v \in U \cap V \) such that \( F_0^{m+1}v \neq 0 \) for all \( m \in \mathbb{N} \). We will get a contradiction. By our comments below Lemma 11.7 we may assume without loss of generality that \( v \in U(r, s) \) for some \( (r, s) \in \mathbb{N}^2 \). We have \( r \geq 1 \) and \( s \geq 1 \), by Lemma 11.3 and \( F_0^31 = 0 \) and \( F_0^{r+s} = 0 \). Let \( n \in \mathbb{N} \). By (8) and Lemma 11.6 we have \( F_0^{m+1}v \in U(r + n, s) \cap V \). In particular \( F_0^{m+1}v \in V \), so \( (A^*_L)^2F_0^{m+1}v = 0 \) and \( B_0^L F_0^{m+1}v = 0 \) in view of Definition 11.4. We have \( A_L^* F_0^{m+1}v \neq 0 \), by Lemma 6.7(v) and since \( B_0^L F_0^{m+1}v = 0 \). We have \( A_L^* F_0^{m+1}v \in U(r + n, s) \) by Lemma 9.5. By these comments \( v \neq 0 \). By this and 0 \( \neq \ker(A_L^*) \cap (r + n, s) \). The map \( A_L^* \) is locally nilpotent by Lemma 6.5. In Appendix A we find \( A_L^* A_L - \hat{q}^2 A_L A_L^* = I \). The vector space \( U \) is invariant under \( A_L^* \) and \( A_L \). By these comments, we may apply Appendix E with \( S = A_L^* \) and \( T = A_L \) and \( V = U \). By Lemma 11.8 the map \( A_L^* \) is surjective on \( U \). By this and Lemma 9.5 \( A_L^* U(r + n, s) = U(r + n - 1, s) \). By this and \( 0 \neq \ker(A_L^*) \cap (r + n, s) \), we obtain \( \dim U(r + n - 1, s) < \dim U(r + n, s) \). Since \( n \in \mathbb{N} \) is arbitrary,

\[
\dim U(r - 1, s) < \dim U(r, s) < \dim U(r + 1, s) < \dim U(r + 2, s) < \cdots
\]

This contradicts (8) and (16), so \( F_0 \) is locally nilpotent.

Next we consider \( F_1 \). Assume that there exists \( v \in U \cap V \) such that \( F_1^{m+1}v \neq 0 \) for all \( m \in \mathbb{N} \). We will get a contradiction. By our comments below Lemma 11.7 we may assume without loss of generality that \( v \in U(r, s) \) for some \( (r, s) \in \mathbb{N}^2 \). We have \( r \geq 1 \) and \( s \geq 1 \), by Lemma 11.3 and \( F_11 = 0 \) and \( F_1^{m+1}v = 0 \). Let \( m \in \mathbb{N} \). By (19) and Lemma 11.6 we have \( F_1^{m+1}v \in U(r, s + m + 1) \cap V \). In particular \( F_1^{m+1}v \in V \), so \( B_1^L F_1^{m+1}v = 0 \) in view of Definition 11.4. By these comments \( v \neq 0 \). By this and \( 0 \neq \ker(B_1^*) \cap U(r, s + m + 1) \). Therefore \( 0 \neq \ker(B_1^*) \cap U(r, s + m + 1) \). The map \( B_1^* \) is locally nilpotent by Lemma 6.5. In Appendix A we find \( B_1^* B_1 \hat{q}^2 B_1 B_1^* = I \). The vector space \( U \) is invariant under \( B_1^* \) and \( B_1 \). By these comments, we may apply Appendix E with \( S = B_1^* \) and \( T = B_1 \) and \( V = U \). By Lemma 11.8 the map \( B_1^* \) is surjective on \( U \). By this and Lemma 9.5 \( B_1^* U(r, s + m + 1) = U(r, s + m) \). By this and \( 0 \neq \ker(B_1^*) \cap U(r, s + m + 1) \), we obtain \( \dim U(r, s + m) < \dim U(r, s + m + 1) \). Since \( m \in \mathbb{N} \) is arbitrary,

\[
\dim U(r, s) < \dim U(r, s + 1) < \dim U(r, s + 2) < \dim U(r, s + 3) < \cdots
\]

This contradicts (7) and (16), so \( F_1 \) is locally nilpotent. \Halmos

The following is our third main result.

**Theorem 11.11.** We have \( U = U \cap V \).

**Proof.** By Lemmas 11.8 11.10 the \( U_q(\hat{sl}_2) \)-generators \( E_0, E_1, F_0, F_1 \) are locally nilpotent on the \( U_q(\hat{sl}_2) \)-module \( U \cap V \). Therefore, the \( U_q(\hat{sl}_2) \)-module \( U \cap V \) is integrable in the sense of [4] Definition 4.2. By this and [20] Theorem 3.5.4, the \( U_q(\hat{sl}_2) \)-module \( U \cap V \) is completely reducible. By Lemma 11.7 \( U \) is a submodule of the \( U_q(\hat{sl}_2) \)-module \( U \cap V \). By these comments, there exists a submodule \( W \) of the \( U_q(\hat{sl}_2) \)-module \( U \cap V \) such that the sum \( U \cap V = U + W \) is direct. Assume for the moment that \( W \neq 0 \). Then \( U \subseteq W \) by Lemma 10.4(i), for a contradiction. Consequently \( W = 0 \), so \( U = U \cap V \). \Halmos

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12 Variations on the theme

In Theorem 10.1 we turned the vector space \( \mathbb{U} \) into a \( U_q(\widehat{\mathfrak{sl}_2}) \)-module. In this section we describe three more ways to do this. Each way yields a \( U_q(\widehat{\mathfrak{sl}_2}) \)-module \( \mathbb{U} \) that is isomorphic to the one in Theorem 10.1.

**Proposition 12.1.** For each row in the table below, the vector space \( \mathbb{U} \) becomes a \( U_q(\widehat{\mathfrak{sl}_2}) \)-module on which the \( U_q(\widehat{\mathfrak{sl}_2}) \)-generators act as indicated:

| generator | action on \( \mathbb{U} \) | \( E_0 \) | \( F_0 \) | \( K_0^{\pm 1} \) | \( E_1 \) | \( F_1 \) | \( K_1^{\pm 1} \) | \( D^{\pm 1} \) |
|-----------|-----------------|------|------|-----------------|------|------|-----------------|------|
| \( B_R^* \) | \( \frac{qB_rK-q^{-1}B_r}{q-q^{-1}} \) | \( q^{\pm 1}K^{\pm 1} \) | \( A_r^* \) | \( \frac{A_rK^{-1}-A_r}{q-q^{-1}} \) | \( K^{\pm 1} \) | \( Y^{\pm 1} \) |
| \( A_L^* \) | \( \frac{qA_rK^{-1}-q^{-1}A_r}{q-q^{-1}} \) | \( q^{\pm 1}K^{\pm 1} \) | \( B_L^* \) | \( \frac{B_rK-B_r}{q-q^{-1}} \) | \( K^{\pm 1} \) | \( X^{\mp 1} \) |
| \( D^{\pm 1} \) | \( \frac{qB_rK-q^{-1}B_r}{q-q^{-1}} \) | \( q^{\pm 1}K^{\pm 1} \) | \( A_r^* \) | \( \frac{A_rK^{-1}-A_r}{q-q^{-1}} \) | \( K^{\pm 1} \) | \( Y^{\pm 1} \) |

The above three \( U_q(\widehat{\mathfrak{sl}_2}) \)-modules \( \mathbb{U} \) are isomorphic to the \( U_q(\widehat{\mathfrak{sl}_2}) \)-module \( \mathbb{U} \) in Theorem 10.1. For row 1 (resp. row 2) (resp. row 3), a \( U_q(\widehat{\mathfrak{sl}_2}) \)-module isomorphism is given by the restriction of \( \sigma \) (resp. \( \dagger \)) (resp. \( \tau \)) to \( \mathbb{U} \).

**Proof.** Below (14) we mentioned that \( \mathbb{U} \) is invariant under each of \( \sigma \), \( \dagger \), \( \tau \). The result follows from this along with Lemmas 9.3, 9.4, 9.6 and Lemmas 5.7, 6.10, 8.5.

13 Acknowledgements

The author thanks Pascal Baseilhac for many conversations about \( U_q(\widehat{\mathfrak{sl}_2}) \) and \( U_q^+ \).
14 Appendix A: Some relations

In this appendix we list some relations satisfied by the maps

\[ K, \quad K^{-1}, \quad A_L^*, \quad B_L^*, \quad A_R^*, \quad B_R^*, \quad A_\ell, \quad B_\ell, \quad A_r, \quad B_r. \]

**Proposition 14.1.** (See [32, Proposition 9.1].) We have

\[
KA_L^* = q^{-2}A_L^*K, \quad KB_L^* = q^2B_L^*K, \\
KA_R^* = q^{-2}A_R^*K, \quad KB_R^* = q^2B_R^*K, \\
KA_\ell = q^2A_\ell K, \quad KB_\ell = q^2B_\ell K, \\
KA_r = q^2A_r K, \quad KB_r = q^2B_r K, \\
A_L^*A_R^* = A_R^*A_L^*, \quad B_L^*B_R^* = B_R^*B_L^*, \\
A_L^*B_R^* = B_R^*A_L^*, \quad B_L^*A_R^* = A_R^*B_L^*, \\
A_\ell A_r = A_r A_\ell, \quad B_\ell B_r = B_r B_\ell, \\
A_\ell B_r = B_r A_\ell, \quad B_\ell A_r = A_r B_\ell, \\
A_L^* B_\ell = B_\ell A_L^*, \quad B_L^* A_\ell = q^{-2}A_L^* B_\ell, \\
A_R^* B_\ell = B_\ell A_R^*, \quad B_R^* A_\ell = q^{-2}A_R^* B_\ell, \\
A_L^* A_\ell - q^2 A_\ell A_L^* = I, \quad A_R^* A_r - q^2 A_r A_R^* = I, \\
B_L^* B_\ell - q^2 B_\ell B_L^* = I, \quad B_R^* B_r - q^2 B_r B_R^* = I, \\
A_L^* A_r - A_r A_L^* = K, \quad B_L^* B_r - B_r B_L^* = K^{-1}, \\
A_R^* A_\ell - A_\ell A_R^* = K, \quad B_R^* B_\ell - B_\ell B_R^* = K^{-1}, \\
\]

\[
A_L^3 B_\ell - [3]q A_L^2 B_\ell A_\ell + [3]q A_\ell B_\ell A_L^2 - B_\ell A_L^3 = 0, \\
B_L^3 A_\ell - [3]q B_L^2 A_\ell B_\ell + [3]q B_\ell A_\ell B_L^2 - A_\ell B_L^3 = 0, \\
A_L^3 B_r - [3]q A_L^2 B_r A_r + [3]q A_r B_r A_L^2 - B_r A_L^3 = 0, \\
B_L^3 A_r - [3]q B_L^2 A_r B_r + [3]q B_r A_r B_L^2 - A_r B_L^3 = 0. \\
\]

**Proposition 14.2.** (See [32, Proposition 9.3].) The following relations hold on \(U:\)

\[
(A_L^*)^3 B_L^* - [3]q(A_L^*)^2 B_L^* A_L^* + [3]q A_L^* B_L^*(A_L^*)^2 - B_L^*(A_L^*)^3 = 0, \\
(B_L^*)^3 A_L^* - [3]q(B_L^*)^2 A_L^* B_L^* + [3]q B_L^* A_L^*(B_L^*)^2 - A_L^*(B_L^*)^3 = 0, \\
(A_R^*)^3 B_R^* - [3]q(A_R^*)^2 B_R^* A_R^* + [3]q A_R^* B_R^*(A_R^*)^2 - B_R^*(A_R^*)^3 = 0, \\
(B_R^*)^3 A_R^* - [3]q(B_R^*)^2 A_R^* B_R^* + [3]q B_R^* A_R^*(B_R^*)^2 - A_R^*(B_R^*)^3 = 0. \\
\]

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15 Appendix B: The algebra $U_q(\widehat{sl}_2)$

In this appendix we recall the quantized enveloping algebra $U_q(\widehat{sl}_2)$. We will generally follow the approach of Ariki [4, Section 3.3]. We will refer to the matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

We index the rows and columns of $A$ by 0, 1.

**Definition 15.1.** (See [4, Definition 3.16].) Define the algebra $U_q(\widehat{sl}_2)$ by generators $K_i^{\pm 1}$, $D^{\pm 1}$, $E_i$, $F_i$, $i \in \{0, 1\}$

and the following relations. For $i, j \in \{0, 1\}$,

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad DD^{-1} = D^{-1} D = 1,$$

$$[K_i, K_j] = 0, \quad [D, K_i] = 0,$$

$$K_i E_j K_i^{-1} = q^{A_{i,j}} E_j, \quad K_i F_j K_i^{-1} = q^{-A_{i,j}} F_j,$$

$$DE_0 D^{-1} = q E_0, \quad DF_0 D^{-1} = q^{-1} F_0,$$

$$[D, E_i] = 0, \quad [D, F_i] = 0,$$

$$[E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$[E_i, [E_i, E_j]_{q^{-1}}] = 0, \quad [F_i, [F_i, F_j]_{q^{-1}}] = 0, \quad i \neq j.$$

**Note 15.2.** The Ariki notation is related to our notation as follows.

| Ariki notation | our notation |
|---------------|-------------|
| $v$           | $q$         |
| $t_i$         | $K_i$       |
| $q^d$         | $D$         |
| $\alpha_j(h_i)$ | $A_{i,j}$ |

**Note 15.3.** (See [4, Section 3.3].) The algebra $U_q(\widehat{sl}_2)$ is sometimes called the quantum algebra of type $A_1^{(1)}$.

16 Appendix C: The subspaces $U(r, s)$

In this appendix, we give a basis for each nonzero $U(r, s)$ such that $r + s \leq 10$.

| $r$ | $s$ | basis for $U(r, s)$ |
|-----|-----|---------------------|
| 0   | 0   | 1                   |
| 1   | 0   | $x$                 |

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| $r$ | $s$ | Basis for $U(r,s)$ |
|-----|-----|---------------------|
| 1   | 1   | $xy$               |
| 2   | 1   | $xyx, xyx$         |
| 1   | 2   | $xxy, yxy$         |
| 3   | 2   | $xyxy, xyyx, xyyx + xyxxxy$ |
| 3   | 3   | $xyyxy, xyyxy + xyxxy + xyxyy$ |
| 4   | 2   | $xxyxx + [3]qxyyxxx$ |
| 3   | 3   | $xyxyxy + [3]qxyyxxx + xyyxyxy$ |
| 5   | 3   | $[3]qxyyxxxy + [3]qxyyxxxx + [2]qxyyxxxy$ |
|     |     | $+2xyxyxx + xxyyxx + [3]qxyyxxxx$ |
| 4   | 4   | $xyxxxy, xyyxxxy, xyyxyxy + xyyxyxy + xyyxyxy$, $xyxxxy + [3]qxyyxxxx$ |
|     |     | $xxyxyx + xxyxxxy + xxyxyx + xxyxyx + xxyxyx$, $xxyxyx + [3]qxyyxxxy + xxyxxxy$ |
| 5   | 4   | $[3]qxyxyxxxy + [3]qxyyyyyxy + [2]qxyyyyyxy$ |
|     |     | $+2xyxyxx + xxyyxx + [3]qxyyyyyxy$ |
|     |     | $+xyxyxx + xxyyxx + xyxyxx + xyxyxx + xyxyxx$, $xxyxyx + [3]qxyyyyyxy$ |
| 4   | 5   | $xyxyxxxy + xxyyyyyy + xyyxyxyx + xyyxyxy + xyyxyxy$, $xxyxyx + xxyyyyyy + xyyxyxyx + xyyxyxy + xyyxyxy$, $[3]qxyxyxxy + [3]qxyyyyyyy + [3]qxyxxyxy + xxyxyxy$ |
| $r$ | $s$ | basis for $\mathbf{U}(r, s)$ |
|-----|-----|-----------------|
| 6   | 4   | $[3]_qxyxyxyxyx + [3]_qxyryrrrxy + [2]_q^2xyywwxyxyx$  
      |      | $+ 2xyywxyxyx + xwyxywxxy + [3]_qxyyyxyxyx$  
      |      | $+ xyyxyxyxx + xyyxyxyxx + xyyxyxyxx$  
      |      | $+ 3xxyxyxyxx + [3]_qxyyxyxyxx + [3]_qxyxyxyxx$  
      |      | $+ [3]_qxyyxyxyxx + [3]_qxyyyxyxyxx + [3]_qxyyyxyxyxx$  
      | 5   | $[3]_qxyyxyxyxx + xyyxyxyxx + [3]_qxyyyxyxyxx + [3]_qxyyyxyxyxx + xyyxyxyxx$  
      |      | $xxyyxyxy$  
      |      | $xxyyxyxy + xyyxyxy + xyyxyxy + xyyxyxy + xyyxyxy + xyyxyxy$  
      |      | $+ xyyxyxy + xyyxyxy + xyyxyxy$  
      |      | $+ xyyxyxy + xyyxyxy + xyyxyxy + xyyxyxy + xyyxyxy + xyyxyxy$  
      |      | $+ [3]_qxyyxyxyxx + xyyxyxyxx + [3]_qxyyyxyxyxx + [3]_qxyyyxyxyxx$  
      |      | $+ [3]_qxyyxyxyxx + xyyxyxyxx + [3]_qxyyyxyxyxx + [3]_qxyyyxyxyxx$  
      | 4   | $[3]_qxyyxyxyxx + xyyxyxyxx + [3]_qxyyxyxyxx + [3]_qxyyxyxyxx$  
      |      | $+ [3]_qxyyxyxyxx + xyyxyxyxx + [3]_qxyyxyxyxx + [3]_qxyyxyxyxx$  
      |      | $+ [3]_qxyyxyxyxx + xyyxyxyxx + [3]_qxyyxyxyxx + [3]_qxyyxyxyxx$  
      |      | $+ [3]_qxyyxyxyxx + xyyxyxyxx + [3]_qxyyxyxyxx + [3]_qxyyxyxyxx$  
      |      | $+ [3]_qxyyxyxyxx + xyyxyxyxx + [3]_qxyyxyxyxx + [3]_qxyyxyxyxx$  

### 17 Appendix D: Some matrix representations

In this appendix we consider the $U_q(\widehat{\mathfrak{sl}_2})$-module $\mathbf{U}$ from Definition 10.3. We display the matrices that represent the actions of $E_0, F_0, K_0, E_1, F_1, K_1, D$ on the bases in Appendix C.

On $\mathbf{U}(0, 0)$:

$$K_0 : (q), \quad K_1 : (1), \quad D : (1).$$

From $\mathbf{U}(1, 0)$ to $\mathbf{U}(0, 0)$:

$$E_0 : (1), \quad E_1 : (0)$$

From $\mathbf{U}(0, 0)$ to $\mathbf{U}(1, 0)$:

$$F_0 : (1), \quad F_1 : (0)$$

On $\mathbf{U}(1, 0)$:

$$K_0 : (q^{-1}), \quad K_1 : (q^2), \quad D : (q^{-1})$$

From $\mathbf{U}(1, 1)$ to $\mathbf{U}(1, 0)$:

$$E_0 : (0), \quad E_1 : (1)$$
From $U(1,0)$ to $U(1,1)$:

$$ F_0 : (0), \quad F_1 : ([2]_{q}) $$

On $U(1,1)$:

$$ K_0 : (q), \quad K_1 : (1), \quad D : (q^{-1}). $$

From $U(2,1) + U(1,2)$ to $U(1,1)$:

$$ E_0 : (1 \ 0), \quad E_1 : (0 \ 1) $$

From $U(1,1)$ to $U(2,1) + U(1,2)$:

$$ F_0 : \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad F_1 : \begin{pmatrix} 0 \\ [2]_{q} \end{pmatrix} $$

On $U(2,1) + U(1,2)$:

$$ K_0 : \text{diag}(q^{-1}, q^{3}), \quad K_1 : \text{diag}(q^{2}, q^{-2}), \quad D : \text{diag}(q^{-2}, q^{-1}). $$

From $U(2,2)$ to $U(2,1) + U(1,2)$:

$$ E_0 : \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} $$

From $U(2,1) + U(1,2)$ to $U(2,2)$:

$$ F_0 : \begin{pmatrix} 0 & 1 \\ 0 & [3]_{q} \end{pmatrix}, \quad F_1 : \begin{pmatrix} [2]_{q} & 0 \\ [2]_{q} & 0 \end{pmatrix} $$

On $U(2,2)$:

$$ K_0 : \text{diag}(q, q), \quad K_1 : \text{diag}(1, 1), \quad D : \text{diag}(q^{-2}, q^{-2}). $$

From $U(3,2) + U(2,3)$ to $U(2,2)$:

$$ E_0 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_1 : \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} $$

From $U(2,2)$ to $U(3,2) + U(2,3)$:

$$ F_0 : \begin{pmatrix} 1 & 1 \\ 0 & [2]_{q} \end{pmatrix}, \quad F_1 : \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} $$

On $U(3,2) + U(2,3)$:

$$ K_0 : \text{diag}(q^{-1}, q^{-1}, q^{3}), \quad K_1 : \text{diag}(q^{2}, q^{2}, q^{-2}), \quad D : \text{diag}(q^{-3}, q^{-3}, q^{-2}). $$
From $U(4, 2) + U(3, 3)$ to $U(3, 2) + U(2, 3)$:

$$
E_0 : \begin{pmatrix}
1 & 0 & 0 & 0 \\
[3]_q & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
E_1 : \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

From $U(3, 2) + U(2, 3)$ to $U(4, 2) + U(3, 3)$:

$$
F_0 : \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 2 \\
0 & 0 & [3]_q
\end{pmatrix},
F_1 : \begin{pmatrix}
[2]_q & 0 & 0 \\
0 & [2]_q & 0 \\
[2]_q & 0 & 0
\end{pmatrix}
$$

On $U(4, 2) + U(3, 3)$:

$$K_0 : \text{diag}(q^{-3}, q, q, q), \quad K_1 : \text{diag}(q^4, 1, 1, 1), \quad D : \text{diag}(q^{-4}, q^{-3}, q^{-3}, q^{-3}).$$

From $U(4, 3) + U(3, 4)$ to $U(4, 2) + U(3, 3)$:

$$
E_0 : \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix},
E_1 : \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

From $U(4, 2) + U(3, 3)$ to $U(4, 3) + U(3, 4)$:

$$
F_0 : \begin{pmatrix}
0 & 1 & 0 & 2 \\
0 & 0 & 0 & [2]_q^2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
F_1 : \begin{pmatrix}
[2]_q & 0 & 0 & 0 \\
[2]_q & 0 & 0 & 0 \\
[4]_q & 0 & 0 & 0 \\
0 & 0 & [2]_q & 0
\end{pmatrix}
$$

On $U(4, 3) + U(3, 4)$:

$$K_0 : \text{diag}(q^{-1}, q^{-1}, q^{-1}, q^3, q^3), \quad K_1 : \text{diag}(q^2, q^2, q^2, q^{-2}, q^{-2}), \quad D : \text{diag}(q^{-4}, q^{-4}, q^{-4}, q^{-3}, q^{-3}).$$

From $U(5, 3) + U(4, 4)$ to $U(4, 3) + U(3, 4)$:

$$
E_0 : \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 \\
[3]_q & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
E_1 : \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

From $U(4, 3) + U(3, 4)$ to $U(5, 3) + U(4, 4)$:

$$
F_0 : \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & [3]_q & 0 & 0 \\
0 & 0 & 0 & 0 & [2]_q^2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & [3]_q & 0
\end{pmatrix},
F_1 : \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
[2]_q & 0 & [2]_q & 0 & 0 & 0 \\
0 & [2]_q & [2]_q & 0 & 0 & 0 \\
0 & 0 & [2]_q & [2]_q & 0 & 0 \\
0 & 0 & 0 & [2]_q [3]_q & 0 & 0 \\
[2]_q & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$
On $\mathbf{U}(5, 3) + \mathbf{U}(4, 4)$:

$$K_0 : \text{diag}(q^{-3}, q, q, q, q), \quad K_1 : \text{diag}(q^4, 1, 1, 1, 1),$$
$$D : \text{diag}(q^{-5}, q^{-4}, q^{-4}, q^{-4}, q^{-4}).$$

From $\mathbf{U}(5, 4) + \mathbf{U}(4, 5)$ to $\mathbf{U}(5, 3) + \mathbf{U}(4, 4)$:

$$E_0 : \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_1 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From $\mathbf{U}(5, 3) + \mathbf{U}(4, 4)$ to $\mathbf{U}(5, 4) + \mathbf{U}(4, 5)$:

$$F_0 : \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & \left[2\right]_q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad F_1 : \begin{pmatrix} \left[4\right]_q & 0 & 0 & 0 & 0 & 0 \\ 3\left[2\right]_q & 0 & 0 & 0 & 0 & 0 \\ \left[2\right]_q & 0 & 0 & 0 & 0 & 0 \\ \left[2\right]_q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \left[2\right]_q & \left[2\right]_q & 0 \\ 0 & \left[2\right]_q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left[2\right]_q \end{pmatrix}.$$

On $\mathbf{U}(5, 4) + \mathbf{U}(4, 5)$:

$$K_0 : \text{diag}(q^{-1}, q^{-1}, q^{-1}, q^{-1}, q^{-1}, q^3, q^3), \quad K_1 : \text{diag}(q^2, q^2, q^2, q^2, q^{-2}, q^{-2}, q^{-2}),$$
$$D : \text{diag}(q^{-5}, q^{-5}, q^{-5}, q^{-5}, q^{-4}, q^{-4}, q^{-4}).$$

From $\mathbf{U}(6, 4) + \mathbf{U}(5, 5) + \mathbf{U}(4, 6)$ to $\mathbf{U}(5, 4) + \mathbf{U}(4, 5)$:

$$E_0 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & \left[3\right]_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \left[3\right]_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \quad E_1 : \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
From $\textbf{U}(5,4) + \textbf{U}(4,5)$ to $\textbf{U}(6,4) + \textbf{U}(5,5) + \textbf{U}(4,6)$:

\[
F_0 : \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad F_1 : \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3[2]_q & [2]_q & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & [2]_q & 0 & 0 & 0 & 0 & 0 & 0 \\
2[2]_q & 0 & 0 & 0 & [2]_q & 0 & 0 & 0 \\
[2]_q & 0 & 0 & [2]_q & [2]_q & 0 & 0 & 0 \\
[2]_q & 0 & 0 & [2]_q & 0 & 0 & 0 & 0 \\
[2]_q[3]_q & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

On $\textbf{U}(6,4) + \textbf{U}(5,5) + \textbf{U}(4,6)$:

\[
K_0 : \text{diag}(q^{-3}, q^{-3}, q, q, q, q, q, q, q, q, q, q), \quad K_1 : \text{diag}(q^4, q^4, 1, 1, 1, 1, 1, 1, q^{-4}),
\]

\[
D : \text{diag}(q^{-6}, q^{-6}, q^{-5}, q^{-5}, q^{-5}, q^{-5}, q^{-5}, q^{-5}, q^{-5}, q^{-4}).
\]

18 Appendix E: Some linear algebra

In this appendix we consider the following situation. Let $V$ denote an infinite-dimensional vector space. Let $S : V \to V$ and $T : V \to V$ denote $\mathbb{F}$-linear maps. Assume that $S$ is locally nilpotent and

\[
ST - q^2TS = I. \quad (27)
\]

We will show that $S$ is surjective and $T$ is injective. We remark that the surjectivity of $S$ is used in the proof of Lemma [11,10] and the injectivity of $T$ is used to obtain the surjectivity of $S$.

**Lemma 18.1.** The map $T$ is injective.

**Proof.** Let $v \in V$ such that $Tv = 0$. We show that $v = 0$. For $n \geq 1$, use (27) and induction on $n$ to obtain

\[
TS^n v = -q^{-n-1}[n]_q S^{n-1} v. \quad (28)
\]

Since $S$ is locally nilpotent, there exists $n \geq 1$ such that $S^n v = 0$. By applying (28) repeatedly, we see that each of $S^n v, S^{n-1} v, \ldots Sv, v$ is equal to 0. In particular $v = 0$. \qed

For $n \in \mathbb{N}$, we adjust (27) to obtain

\[
ST - q^n[n+1]_q I = q^2(TS - q^{n-1}[n]_q I). \quad (29)
\]

Therefore, the kernel of $ST - q^n[n+1]_q I$ is equal to the kernel of $TS - q^{n-1}[n]_q I$. Let $V_n$ denote this common kernel. By construction

\[
(ST - q^n[n+1]_q I) V_n = 0, \quad (TS - q^{n-1}[n]_q I) V_n = 0. \quad (30)
\]

Note that the sum $\sum_{n \in \mathbb{N}} V_n$ is direct. For notational convenience define $V_{-1} = 0$. 30
Lemma 18.2. We have \( \ker(S) = V_0 \).

Proof. By the discussion below (29), we obtain \( \ker(TS) = V_0 \). The map \( T \) is injective by Lemma 18.1, so \( \ker(S) = \ker(TS) \). Therefore \( \ker(S) = V_0 \). \( \square \)

Lemma 18.3. For \( n \in \mathbb{N} \) we have

\[
SV_n \subseteq V_{n-1}, \quad TV_n \subseteq V_{n+1}.
\]

Proof. First we verify \( SV_n \subseteq V_{n-1} \). For \( n = 0 \) this holds by Lemma 18.2. For \( n \geq 1 \) we use (30) to obtain

\[
(ST - q^n)[n]qI)SV_n = S(TS - q^n)[n]qI)V_n = S0 = 0,
\]

so \( SV_n \subseteq V_{n-1} \). Next we verify \( TV_n \subseteq V_{n+1} \). For \( n \geq 0 \) we have

\[
(TS - q^n[n + 1]qI)TV_n = T(ST - q^n[n + 1]qI)W_n = T0 = 0,
\]

so \( TV_n \subseteq V_{n+1} \). \( \square \)

Lemma 18.4. For \( n \geq 1 \) the following maps are inverses:

\[
S : V_n \to V_{n-1}, \quad q^{1-n}[n]^{-1}T : V_{n-1} \to V_n.
\]

Proof. By (30) we have \((ST_n - I)V_{n-1} = 0 \) and \((T_nS - I)V_n = 0 \), where \( T_n = q^{1-n}[n]^{-1}T \). \( \square \)

Lemma 18.5. For \( n \geq 1 \) the maps

\[
S : V_n \to V_{n-1}, \quad T : V_{n-1} \to V_n
\]

are bijections.

Proof. By Lemma 18.4 \( \square \)

Lemma 18.6. For \( n \in \mathbb{N} \),

\[
\ker(S^{n+1}) = V_0 + V_1 + \cdots + V_n.
\]

Proof. We use induction on \( n \). First assume that \( n = 0 \). Then (32) holds by Lemma 18.2. Next assume that \( n \geq 1 \). The inclusion \( \supseteq \) in (32) holds by Lemma 18.3. We next obtain the inclusion \( \subseteq \) in (32). Let \( v \in \ker(S^{n+1}) \). We will show that \( v \in V_0 + V_1 + \cdots + V_n \). We have \( 0 = S^{n+1}v = S^nSv \), so by induction \( Sv \in V_0 + V_1 + \cdots + V_{n-1} \). By Lemma 18.5 there exists \( w \in V_1 + V_2 + \cdots + V_n \) such that \( Sw = Sv \). Therefore \( S(w - v) = 0 \), so \( w - v \in V_0 \). By these comments \( v \in V_0 + V_1 + \cdots + V_n \). \( \square \)

Lemma 18.7. We have \( V = \sum_{n \in \mathbb{N}} V_n \).

Proof. Since \( S \) is locally nilpotent, we have \( V = \cup_{n \in \mathbb{N}} \ker(S^{n+1}) \). The result follows from this and Lemma 18.6 \( \square \)

Lemma 18.8. The map \( S \) is surjective.

Proof. By Lemma 18.7 and since \( V_n = SV_{n+1} \) for \( n \in \mathbb{N} \). \( \square \)
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