On quasi-continuous approximation in classical statistical mechanics

S.M. Petrenko\textsuperscript{1}, A.L. Rebenko\textsuperscript{2}, M.V. Tertychnyi \textsuperscript{2}

\textsuperscript{1}Lviv's'kyi NLTU, Lviv, UKRAINE
petrenko2003kiev@inbox.ru
\textsuperscript{2}Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv, Ukraine
rebenko@voliacable.com ; maksym.tertychnyi@gmail.com

Abstract

A continuous infinite system of point particles with strong superstable interaction is considered in the framework of classical statistical mechanics. The family of approximated correlation functions is determined in such a way, that they take into account only such configurations of particles in $\mathbb{R}^d$ which for a given partition of the configuration space $\mathbb{R}^d$ into nonintersecting hyper cubes with a volume $a^d$ contain no more than one particle in every cube. We prove that these functions converge to the proper correlation functions of the initial system if the parameter of approximation $a \to 0$ for any positive values of an inverse temperature $\beta$ and a fugacity $z$. This result is proven both for two-body interaction potentials and for many-body case.

Keywords: Classical statistical mechanics, strong superstable potential, many-body potential, correlation functions

Mathematics Subject Classification: 82B05; 82B21

1 Introduction

The procedure of lattice approximation is very often used to study continuous systems. There is a well-known example of the lattice approximation in Euclidean quantum field theory for the model $\lambda : \phi^4 :$ in the two-dimensional space-time which transforms the
system to Ising model with unbounded continuous spin. In contrast to Euclidean quantum field theory, where lattice systems play the role of approximation, in statistical mechanics they represent part of the Nature, such as ferromagnetics, quantum oscillators etc. The theory of such systems is well developed, unlike continuous systems such as dense gases and liquids. The main difficulties in the mathematical description of continuous systems in statistical mechanics are accumulation of many number of particles in small volumes. To avoid this problems such systems as lattice gas were invented to describe some general characteristics of real gases. But in majority of works there was no parameter which in some sense restored systems to continuous gases.

In this work we propose some intermediate approximation of continuous gases, which is very close to lattice gases and all main characteristics of continuous gases can be obtained with help of limit transition.

Quasi-continuous approximation of the Equilibrium Classical Statistical Mechanics was proposed in the article [9] for the investigation of infinite systems of interacting point particles with two-body strong superstable potentials. The matter of this approximation is that in integrals which are in the definitions of the main characteristics such as partition function and correlation functions integrations are realized over such configuration which for a given partition of the configuration space $\mathbb{R}^d$ into nonintersecting hyper cubes with a volume $a^d$ contain no more than one point in every cube. Correlation functions and pressure of systems defined in such a way though have a proper limit at $a \to 0$ even for infinite volume systems if the interaction potential is sufficiently singular at the origin, more exactly if the potential is locally nonintegrable in any bounded region of $\mathbb{R}^d$ which contains an origin. This fact though is predictable from the physical point of view but from mathematical point of view it is a little bit unexpected as the Poisson measure (and Gibbs measure too) of the set of such configurations is zero.

At the same time, such defined system can be approximated by the lattice gas, an investigation of which is considerably simplified. This transition from continuous to lattice systems and vice versa is particularly important in the investigation of critical behavior of infinite systems near phase transition points.

It was proved in the article [9] that for any positive values of temperature $T$ (or inverse temperature $\beta = 1/kT$) and fugacity $z$ of infinite classical systems the approximated pressure $p(\cdot)(z, \beta; a)$, where $a$ is the parameter of approximation, tends to the proper value of the pressure $p(z, \beta)$ of the considered statistical system as $a \to 0$. In the article [5] this result was generalized for the systems with many-body interactions. Later, in the article [11] the same result was obtained for family of the correlation functions, but only for sufficient small values of fugacity $z$, the values of which were bounded by the radius of
convergence of the Kirkwood-Salsburg expansion for the correlation functions.

In this article we are going to generalize this result for the case of arbitrary positive values of fugacity \( z \) and temperature \( T \). Using an expansion in so-called dense configurations, which was proposed in [3] for finite range interaction and in the article [6] for infinite range potentials, we prove that the family of approximated correlation functions \( \rho^{c}(z, \beta; a) \) of the finite volume \( (\Lambda \subseteq \mathbb{R}^{d}) \) are uniformly bounded by a constant which does not depend on the parameter of approximation \( a \) and volume \( \Lambda \) and have pointwise limit \( \rho(z, \beta) \) as \( \Lambda \uparrow \mathbb{R}^{d} \) and \( a \to 0 \) for arbitrary values of fugacity \( z \) and temperature \( T \). This result will be proved both for two-body interaction potentials and for many-body potentials of general superstable type.

2 Configuration spaces

2.1 The main configuration spaces

Let \( \mathbb{R}^{d} \) be a d-dimensional Euclidean space. The set of positions \( \{x_{i}\}_{i \in \mathbb{N}} \) of identical particles is considered to be a locally finite subset in \( \mathbb{R}^{d} \) and the set of all such subsets creates the configuration space:

\[
\Gamma = \Gamma_{\mathbb{R}^{d}} := \left\{ \gamma \subseteq \mathbb{R}^{d} \mid |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_{c}(\mathbb{R}^{d}) \right\},
\]

where \( |A| \) denotes the cardinality of the set \( A \) and \( \mathcal{B}_{c}(\mathbb{R}^{d}) \) denote the systems of all bounded Borel sets in \( \mathbb{R}^{d} \). We also need to define the space of finite configurations \( \Gamma_{0} \):

\[
\Gamma_{0} = \bigsqcup_{n \in \mathbb{N}_{0}} \Gamma^{(n)}, \quad \Gamma^{(n)} := \left\{ \eta \subseteq \mathbb{R}^{d} \mid |\eta| = n, \ n \in \mathbb{N}_{0} \right\}, \quad \mathbb{N}_{0} = \mathbb{N} \cup \{0\}.
\]

For every \( \Lambda \in \mathcal{B}_{c}(\mathbb{R}^{d}) \) one can define a mapping \( N_{\Lambda} : \Gamma \to \mathbb{N}_{0} \) of the form

\[
N_{\Lambda}(\eta) := |\eta \cap \Lambda|.
\]

The Borel \( \sigma \)-algebra \( \mathcal{B}(\Gamma) \) is equal to \( \sigma(N_{\Lambda} \mid \Lambda \in \mathcal{B}_{c}(\mathbb{R}^{d})) \). See [3], [4] for details.

We need also to define

\[
\Gamma_{\Lambda} := \left\{ \gamma \in \Gamma_{0} \mid \gamma \subseteq \Lambda, \ \Lambda \in \mathcal{B}_{c}(\mathbb{R}^{d}) \right\},
\]

By \( \mathcal{B}(\Gamma_{\Lambda}) \) we denote the corresponding \( \sigma \)-algebras on \( \Gamma_{\Lambda} \) and \( \Gamma_{0,\Lambda} \).
2.2 Lebesgue-Poisson measure

Let $\sigma$ be Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$ and for any $n \in \mathbb{N}$ the product measure $\sigma^\otimes n$ can be considered as a measure on $\mathcal{B}(\mathbb{R}^d)^n = \{(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \mid x_k \neq x_l \text{ if } k \neq l\}$ and hence as a measure $\sigma^{(n)}$ on $\Gamma^{(n)}$ through the map \[ sym_n : (\mathbb{R}^d)^n \ni (x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\} \in \Gamma^{(n)}. \]

Define the Lebesgue-Poisson measure $\lambda_{z\sigma}$ on $\mathcal{B}(\Gamma_0)$ by the formula:
\[
\lambda_{z\sigma} := \sum_{n \geq 0} \frac{z^n}{n!} \sigma^{(n)}.
\] (2.1)

The restriction of $\lambda_{\sigma}$ to $\mathcal{B}(\Gamma_{\Lambda})$ we also denote by $\lambda_{\sigma}$. For more detailed structure and analysis of the configuration spaces $\Gamma, \Gamma_0, \Gamma_{\Lambda}$ see [1].

2.3 Partition of $\mathbb{R}^d$

Following Ruelle [13] define the partition of the Euclidean space $\mathbb{R}^d$ into elementary cubes. Let $a > 0$ be arbitrary. For each $r \in \mathbb{Z}^d$ we define an elementary cube with an edge $a$ and a center $ar$:
\[
\Delta_a(r) := \{x \in \mathbb{R}^d \mid a(r^i - 1/2) \leq x^i < a(r^i + 1/2)\}\] (2.2)

We will write $\Delta$ instead of $\Delta_a(r)$, if a cube $\Delta$ is considered to be arbitrary and there is no reason to emphasize that it is centered at the concrete point $ar$. Let $\overline{\Delta}_a$ be the partition of $\mathbb{R}^d$ into cubes $\Delta_a(r)$. Define, also, the notion of compatible partitions.

**Definition 2.1.** Two partitions $\overline{\Delta}_a$ and $\overline{\Delta}_{a'}$ with $a' < a$ are compatible if $a/a' \in \mathbb{N}$ and partition $\overline{\Delta}_a$ can be obtained from the partition $\overline{\Delta}_{a'}$ removing all edges of its cubes which do not lie on the edges of the partition $\overline{\Delta}_a$.

To avoid some confusion we work in this article only with compatible partitions.

2.4 Additional configuration spaces

Define two additional configuration spaces: $\Gamma_{\Lambda}^{dil}$ we call a space of dilute configurations and $\Gamma_{\Lambda}^{den}$ a space of dense configurations.
Without any restriction of general case, we consider only that \( \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \) which is union of cubes \( \Delta_a(r) \) with some fixed \( a \), which depends on the interaction potential. In the cases where this particular partition will be important we denote by \( \Lambda(a) \) the union of such cubs. Then

\[
\Gamma^{\text{dir}}: = \{ \gamma \in \Gamma_\Lambda | \mid \gamma \mid = 0 \lor 1 \text{ for all } \Delta \subset \Lambda \} \tag{2.3}
\]

and

\[
\Gamma^{\text{den}}: = \{ \gamma \in \Gamma_\Lambda | \mid \gamma \mid \geq 2 \text{ for all } \Delta \subset \Lambda \}. \tag{2.4}
\]

For any \( \Delta \in \overline{\Delta}_a \) and any fixed configuration \( \eta \in \Gamma_\Lambda \) we split the space of dense configurations \( \Gamma^{\text{den}}_\Delta \) into two subspaces:

\[
\Gamma^{(\geq)}_\Delta(\eta) = \Gamma^{(\geq)}_\Delta : = \{ \gamma \in \Gamma^{\text{den}}_\Delta | \mid \gamma \mid \geq d_\eta^\varepsilon(\Delta) \}
\]

and

\[
\Gamma^{(\leq)}_\Delta(\eta) = \Gamma^{(\leq)}_\Delta : = \{ \gamma \in \Gamma^{\text{den}}_\Delta | \mid \gamma \mid \leq d_\eta^\varepsilon(\Delta) \}, \tag{2.6}
\]

where \( \Delta \equiv \Delta_a(r), \ 0 < \varepsilon \leq 1 \) and

\[
d_\eta(\Delta) = \text{dist}(\eta, \Delta), \ d_\eta^\varepsilon(\Delta) = (d_\eta(\Delta))^\varepsilon. \tag{2.7}
\]

where \( \Delta \) is the closure of the cube \( \Delta \). It’s obviously that \( \Gamma^{\text{den}}_\Delta = \Gamma^{(\geq)}_\Delta \cup \Gamma^{(\leq)}_\Delta \). And finally for \( X_k = \bigcup_{i=1}^k \Delta_a(r_i) \)

\[
\Gamma^{(\geq)}_{X_k}(\eta) = \Gamma^{(\geq)}_{X_k} : = \{ \gamma \subset X_k | \mid \gamma \mid > d_\eta^\varepsilon(\Delta) \text{ for all } \Delta \subset X_k \} \tag{2.8}
\]

and

\[
\Gamma^{(\leq)}_{X_k}(\eta) = \Gamma^{(\leq)}_{X_k} : = \{ \gamma \subset X_k | \mid \gamma \mid \leq d_\eta^\varepsilon(\Delta) \text{ for all } \Delta \subset X_k \}. \tag{2.9}
\]

3 Interaction

For the general case interaction between particles is realized by infinite sequence of interaction potentials:

\[
V = (0, 0, V_2(x_1, x_2), V_3(x_1, x_2, x_3), ..., V_p(x_1, ..., x_p), ...) \tag{3.1}
\]

In case of two-body interaction, which is the most popular among physicists components of the sequence \((3.1)\) look like:

\[
V_2(x_1, x_2) = \phi(|x_1 - x_2|), \ V_p \equiv 0, \ p \geq 3, \tag{3.2}
\]
The energy of any configuration $\gamma \in \Gamma_0$ is defined by the following formula:

$$U(\gamma) = U_V(\gamma) = \sum_{p=2}^{\vert \gamma \vert} \sum_{\{x_1, \ldots, x_p\} \subset \gamma} V_p(x_1, \ldots, x_p) = \sum_{\eta \subseteq \gamma : \vert \eta \vert \geq 2} V(\eta), \quad (3.3)$$

and interaction energy between two configurations $\eta, \gamma \in \Gamma_0$ by

$$W(\eta; \gamma) = W_V(\eta; \gamma) = U(\eta \cup \gamma) - U(\eta) - U(\gamma) = \sum_{p=2}^{\vert \eta, \gamma \vert} \sum_{\{x_1, \ldots, x_i\} \subset \eta} \sum_{i+j=p} \sum_{\{y_1, \ldots, y_j\} \subset \gamma} V_p(x_1, \ldots, x_i, y_1, \ldots, y_j). \quad (3.4)$$

The correspondent formulas for two-body interaction are:

$$U(\gamma) = U_\phi(\gamma) = \sum_{\{x_1, x_2\} \subset \gamma} \phi(|x_1 - x_2|), \quad (3.5)$$

$$W(\eta; \gamma) = W_\phi(\eta; \gamma) = \sum_{x \in \eta \atop y \in \gamma} \phi(|x - y|). \quad (3.6)$$

We introduce 3 kinds of interactions, which will be used in this article:

**Definition 3.1.** Interaction $U$ is called:

a) stable (S), if there exists $B>0$ such that:

$$U(\gamma) \geq -B|\gamma|, \quad \text{for any } \gamma \in \Gamma_0; \quad (3.7)$$

b) superstable (SS), if there exist $A > 0$, $B \geq 0$ and partition $\Delta_a$ such that:

$$U(\gamma) \geq A \sum_{\Delta \in \Delta_a} |\gamma \Delta|^2 - B|\gamma|, \quad \text{for any } \gamma \in \Gamma_0; \quad (3.8)$$

c) strong superstable (SSS), if there exist $m \geq 2$, $a_0 > 0$ s.t. for any $0 < a \leq a_0$ there exist $A(a) > 0$, $B(a) \geq 0$ s.t.

$$U(\gamma) \geq A(a) \sum_{\Delta \in \Delta_a : |\gamma \Delta| \geq 2} |\gamma \Delta|^m - B(a)|\gamma|, \quad \text{for any } \gamma \in \Gamma_0. \quad (3.9)$$

In accordance with these definitions there is a problem to describe conditions on potentials, which ensure stability, superstability or strong superstability of an infinite statistical system. This problem has a long story. A short review of this problem and some new results one can find in [10] and [14].
Remark 3.1. It is clear that if the equation (3.8) holds for some partition $\Delta_a$ with the constants $A$ and $B$ then it holds with the same constants $A$ and $B$ for any partition $\Delta_{a'}$ for which $a' < a$ and they are compatible.

Remark 3.2. It is clear that if the potential is strong superstable then it is simply superstable with $A = A(a_0), B = B(a_0)$.

3.1 Definition of the system with two-body interaction

(A): Assumption on the interaction potential. Consider a general type of potentials $\phi$, which are continuous on $\mathbb{R}_+ \setminus \{0\}$ and for which there exist $r_0 > 0, R > r_0, \varphi_0 > 0, \varphi_1 > 0, \varepsilon_0 > 0$ such that:

1) $\phi(|x|) \equiv -\phi^{-}(|x|) \geq -\frac{\varphi_1}{|x|^{d+\varepsilon_0}}$ for $|x| \geq R$, \hspace{1cm} (3.10)

2) $\phi(|x|) \equiv \phi^+ (|x|) \geq \frac{\varphi_0}{|x|^s}, \; s \geq d \; \text{for} \; |x| \leq r_0$, \hspace{1cm} (3.11)

where

$\phi^+ (|x|) := \max \{0, \phi(|x|)\}, \; \phi^-(|x|) := -\min \{0, \phi(|x|)\}$. \hspace{1cm} (3.12)

Note that in the definition 3.1, c)(SSS) the constant $a_0 \leq r_0$. For the interaction potentials which satisfy the assumptions (A) define two important characteristics (for any $\Delta \in \Delta_a$ with $a \leq a_0$):

1) $v_\varepsilon(a) := \sum_{\Delta' \subseteq \Delta} \sup_{x \in \Delta} \sup_{y \in \Delta'} \phi^{-}(|x-y|)|x-y|^\varepsilon$, for any $\varepsilon < \varepsilon_0$; \hspace{1cm} (3.13)

2) $b(a) := \inf_{\{x,y\} \subset \Delta} \phi^+(|x-y|)$. \hspace{1cm} (3.14)

Due to the translation invariance of the 2-body potential the values $v_0$ and $b$ do not depend on the position of $\Delta$. The following statement is true.

Proposition 3.1. Let potential $\phi$ satisfy the assumption (A). Then the interaction is strong superstable and the energy $U$ satisfies the inequality (3.9) with some $0 < a_0 < r_0$ and if $s > d$ then

$m = 2, \; A(a) = \frac{b(a) - 2v_0(a)}{4} > 0, \; B(a) = \frac{v_0(a)}{2}$ \hspace{1cm} (3.15)

for $a \leq a_0$. 

7
See the proof in [11]. More powerful result was obtained in the article [10], but for our goals it is sufficient to apply the inequalities (3.15).

Following [6] we introduce the following notations, which will be used in our future estimates:

\[
\phi^+(|x|) := (1 - \delta) \phi^+(|x|), \quad U^+_\delta := U_{\phi^+_\delta},
\]

\[
\phi^{st}_\delta := \delta \phi^+(|x|) - \phi^{-}(|x|), \quad U^{st}_\delta := U_{\phi^{st}_\delta}, \quad \delta \in (0, 1).
\]

One can deduce from (3.16), (3.17), that:

\[
\phi(|x|) = \phi^+_\delta (|x|) + \phi^{st}_\delta (|x|), \quad U(\gamma) = U^+_\delta (\gamma) + U^{st}_\delta (\gamma).
\]

**Proposition 3.2.** Let potential \( \phi \) satisfy the assumption (A). Then there exist \( 0 < a_* < r_0 \) such that for any constant \( \delta \in (0, 1/2) \)

\[
(1 - \delta)b(a) > 2v_0(a), \quad \text{for } a \leq a_*
\]

and the potential \( \phi^{st}_\delta \) is stable: \( U^{st}_\delta := U_{\phi^{st}_\delta}(\gamma) \geq -B_\delta|\gamma| \), \( \gamma \in \Gamma_0 \) with

\[
B_\delta = \frac{1}{2}v_0(a_*) = \frac{\delta}{4}b(a_*).
\]

**Proof.** The inequality (3.19) follows from the assumption (A) and the definitions (3.13) and (3.14) as for small \( a \) they behave as:

\[
b(a) \sim \frac{\varphi_0}{a^s} \quad \text{and} \quad v_\varepsilon(a) \sim \frac{\phi_\varepsilon}{a^d},
\]

and for \( s > d \) we can choose sufficiently small \( a = a_* \) or \( \varphi_0 \gg \phi_\varepsilon \) for \( s = d \), where

\[
\phi_\varepsilon = \int_{\mathbb{R}^d} \varphi^{-}(|x|)|x|^sdx.
\]

As in [11] (see Proposition 2.1) one can calculate that

\[
U_{\phi^{st}_\delta}(\gamma) \geq \sum_{\Delta \in \Sigma_0: |\gamma\Delta| \geq 2} |\gamma\Delta|^2 \left( \frac{\delta b(a)}{4} - \frac{v_0(a)}{2} \right) - \frac{v_0(a)}{2}|\gamma|.
\]

Let us chose \( a_* \) as a root of equation

\[
\delta \frac{b(a)}{4} - \frac{v_0(a)}{2} = 0.
\]

Then to satisfy (3.19) we have to choose \( \delta > 1/2 \) and the constant \( B_\delta \) in (3.20) can be expressed in terms of parameters of the interaction potential \( \varphi_0, \phi_0, s \) and dimension of the space \( d \) (see Proposition 2.2 in [11]).
3.2 Definition of the system with many-body interaction

In this section we consider a general type of many-body interaction specified by a family of $p$-body potentials $V_p : \mathbb{R}^{dp} \to \mathbb{R}, p \geq 2$. About the family of potentials $V := \{V_p\}_{p \geq 2}$ we will assume:

A1. Continuity.

$$V_p \in C((\mathbb{R}^d)^p), \quad p \geq 2,$$

where

$$((\mathbb{R}^d)^n = \{(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n | x_k \neq x_l \text{ при } k \neq l\}.$$

A2. Symmetry. For any $p \geq 2$, any $(x)_p = (x_1, \ldots, x_p) \in (\mathbb{R}^d)^p$, and any permutation $\pi$ of numbers $\{1, \ldots, p\}$

$$V_p(x_1, \ldots, x_p) = V_p(x_{\pi(1)}, \ldots, x_{\pi(p)}).$$

A3. Translation invariance. For any $p \geq 2$, any $(x_1, \ldots, x_p) \in (\mathbb{R}^d)^p$, and any $x_0 \in \mathbb{R}^d$

$$V_p(x_1, \ldots, x_p) = V_p(x_1 + x_0, \ldots, x_p + x_0).$$

A4. Superstability. For any $p \geq 2$ the potentials $V_p$ can be represented as

$$V_p = \tilde{V}_p^+ + V_p^{(st)}, \quad V_p^{(st)} = \nabla_p^+ + V_p^-,$$

(3.25)

where $\tilde{V}_p^+ + \nabla_p^+ = V_p^+, V_p^\pm$ are defined in the same way as in (3.12) and $V_p^{(st)}, p \geq 2$ provides the stability of the corresponding energy $U$, i.e. there exists a constant $B \geq 0$ such that for any configuration $\eta \in \Gamma_0$

$$U_{V^{(st)}}(\eta) \geq -B|\eta|.$$

(3.26)

The corresponding decomposition for the energy:

$$U(\gamma) = U^+(\gamma) + U^{st}(\gamma).$$

(3.27)

Sufficient conditions on the potentials $V_p$ providing superstability inequality were obtained in [14].

In the article [13] uniform (in volumes $\Lambda_n$) bounds for the family of correlation functions were obtained for potentials which guarantee superstability (SS) and low regularity condition (LR) (see [13]). For 2-body potentials which satisfy the assumptions (A) both
of these conditions are fulfilled. But for many-body potentials which are not positive for
$p \geq 3$ LR-condition is not satisfied. So, as in the articles [2] and [7] we formulate so called
attraction-repulsion relations (instead of LR-condition) which gives a possibility to obtain
uniform bounds.

To formulate these assumption for potentials $V_p$, consider some auxiliary constructions.
Let $p \geq 2$ and $N \in \mathbb{N}$. For any union $X_N := \bigcup_{j=1}^{N} \Delta_j$ of cubes $\Delta$ from the partition $\Delta_a$
(див. (2.2)) and any $\varepsilon \geq 0$ define values:

$$I_{p}^{k_1, \ldots, k_N} (\Delta_1; \ldots; \Delta_N) := \sup_{\substack{x_1 \in \Delta_1, \ldots, x_N \in \Delta_N \\ i_1 = 1, k_1, \ldots, i_N = 1, k_N}} \ V_p^- (x_1^{(1)}, \ldots, x_N^{(N)}),$$

where $k_1 + \cdots + k_N = p, j_1, j_2, \ldots, j_N$ and

$$I_{p}^{k_1, \ldots, k_M} (\Delta_1; \ldots; \Delta_M | \varepsilon; (\Delta)_\pi) :=$$

$$= \sum_{\Delta'_1, \ldots, \Delta'_M \in \Delta_a} I_{p}^{k_1, \ldots, k_M, 1, 1, \ldots, 1} (\Delta_1; \ldots; \Delta_M; \Delta'_1; \ldots; \Delta'_M) \prod_{i=1}^{k} \left(1 + d^\varepsilon_{\Delta'_i, \Delta_\pi (i)}\right),$$

where $d^\varepsilon_{\Delta'_i, \Delta_\pi (i)} = (\text{dist}(\Delta'_i, \Delta_\pi (i)))^\varepsilon$, $\pi$ is the mapping of indices $\{1, \ldots, K\}$ into the set of
indices $\{1, \ldots, M\}$, $(\Delta)_\pi := \{\Delta_{\pi (1)}, \ldots, \Delta_{\pi (K)}\}$ i.e. $k_1 + \cdots + k_M + K = p$. The distance between
cubes is the distance between their closures.

Note that because of translation invariance of interaction potentials for $M = 1$ all
indices $\pi (i) = 1$ and

$$I_{p}^{k_1} (\Delta_1; \varepsilon; \Delta_1) = I_{p}^{k_1} (a; \varepsilon),$$

i.e. it depends on the size of cube $\Delta_1$, but it does not depend on positions of $\Delta_1$. For a
positive part $\tilde{V}_p^+$ of interaction potentials define the following values:

$$u_{p}^{k_1, \ldots, k_N} (\Delta_1, \ldots, \Delta_N) := \inf_{\substack{x_1 \in \Delta_1, \ldots, x_N \in \Delta_N \\ i_1 = 1, k_1, \ldots, i_N = 1, k_N}} \tilde{V}_p^+ (x_1^{(1)}, \ldots, x_N^{(N)}).$$

**A5. Attraction-repulsion relations.** There exist $a_0 > 0$, such that for any $N \in \mathbb{N}$,
any set $X_N := \bigcup_{j=1}^{N} \Delta_j, \Delta \in \Delta_a$ with $a \leq a_0$ the following inequalities are true:
(i) for any $\Delta \in \Delta_a$ and any $p \geq 2$

$$V_p (x_1, \ldots, x_p) \geq 0, \text{ if } \{x_1, \ldots, x_p\} \subset \Delta.$$


(ii) for any \( p \geq 2, 1 \leq N < p \), and \( \pi : \{1, \ldots, n\} \mapsto \{1, \ldots, N\} \)

\[
v_p^{k_1, \ldots, k_N}(\Delta_1, \ldots, \Delta_N) \geq 2 \sum_{l=0}^{\infty} \sum_{m_i \geq 1, i=1, \ldots, N, n \geq 1 \atop m_1 + \cdots + m_N + n = p + l} C_{k_1}^{m_1} \cdots C_{k_N}^{m_N} (2p)^{n} I_{p+l}^{m_1, \ldots, m_N | n}(\Delta_1, \ldots, \Delta_N; \varepsilon, (\Delta) \pi),
\]

where \( k_1 + \cdots + k_N = p, \ C_k^m = k! / m! (k - m)! \), if \( k \geq m \) \( \text{ or } C_k^m = 0 \) if \( m > k \).

**Remark 3.1.** Inequality (3.33) is a consequence of the combinatorial arguments, which is relevant to control the negative part of interaction potentials. From the physical point of view it means that for the case when there are at least two particles in some cube (just this situation takes place in case \( N < p \)), then for sufficiently small size of a cube edge their \( p \)-body repulsion energy has to be greater than the attraction energy of these two particles for all particles of a system and for all \( l \geq p \)-body interactions.

**Lemma 3.1.** Let the sequence of potentials \( V = \{V_p\}_{p \geq 2} \) satisfy \( A1 - A5 \). Then the interaction is strong superstable (SSS), i.e. there exist \( m \geq 2, a_0 > 0 \) s.t. for any \( 0 < a \leq a_0 \) there exist \( A(a) > 0, B(a) \geq 0 \) s.t.

\[
U(\gamma) \geq A(a) \sum_{\Delta \in \Xi_n, |\gamma| \Delta \geq 2} |\gamma| \Delta|^m - B(a)|\gamma|, \text{ for any } \gamma \in \Gamma_0.
\]

with

\[
A(a) = v_2^2(a) - 2 \sum_{p \geq 2} 4pI_p^{1|p-1}(a; 0), \quad B(a) = \sum_{p \geq 2} I_p^{1|p-1}(a; 0), \quad m = 2,
\]

and for any \( \gamma \in \eta \cup \Gamma_{X'}^{(>)} \) and \( \bar{\tau} \in \Gamma_{X'}^{(<<)} \cup \Gamma_{A\backslash(X \cup X')}^{(dil)} \), \( X' \cap X = \emptyset \),

\[
-\beta W(\gamma | \bar{\tau}) - \frac{1}{2} \beta U_{\bar{\tau}+}(\gamma) \leq \beta I|\eta|,
\]

where \( I(a) := \sum_{p \geq 2} 2pI_p^{1|p-1}(a, 0) \) (see (3.29)-(3.30)).

**Proof.** The main line of the proof is the same as the proof of Lemma 3.2 in the article [2] and as the proof of Lemma 3.1 in the article [7]. The main difference is in the fact that for obtaining the inequality (3.35) we use a little bit cumbersome but weaker condition (3.33) than in [2], [7].
3.3 Partition functions and correlation functions

We introduce an important function, which will be used for the approximation of statistical systems:

\[ \chi^\Delta(\gamma) = \begin{cases} 1, & \text{for } |\gamma| = 0 \lor 1, \\ 0, & \text{otherwise.} \end{cases} \]  \hspace{1cm} (3.36)

Let us write an expression for the statistical sum, that includes all possible configurations from \( \Gamma \) and an expression for the statistical sum, that includes only dilute configurations from \( \Gamma_{dil} \):

\[ Z(\Lambda)(z, \beta) := \int_{\Gamma_{dil}} e^{-\beta U(\gamma)} \lambda z\sigma(d\gamma), \]  \hspace{1cm} (3.37)

\[ Z^-(\Lambda)(z, \beta, a) := \int_{\Gamma_{dil}} e^{-\beta U(\gamma)} \prod_{\Delta \in \Delta a \cap \Lambda} \chi^\Delta(\gamma) \lambda z\sigma(d\gamma) := \int_{\Gamma_{dil}} e^{-\beta U(\gamma)} \lambda a z\sigma(d\gamma). \]  \hspace{1cm} (3.38)

Let us define correlation function \( \rho(\eta; z, \beta) \) in the case of grand canonical ensemble:

\[ \rho(\eta; z, \beta) := \frac{1}{Z(\Lambda)(z, \beta)} \int_{\Gamma} e^{-\beta U(\eta \cup \gamma)} \lambda z\sigma(d\gamma), \quad \eta \in \Gamma. \]  \hspace{1cm} (3.39)

and corresponding correlation functions of quasi-continuous approximation \( \rho^-(\eta; z, \beta, a) \) are defined as:

\[ \rho^-(\eta; z, \beta, a) := \frac{1}{Z^-(\Lambda)(z, \beta, a)} \int_{\Gamma} e^{-\beta U(\eta \cup \gamma)} \lambda a z\sigma(d\gamma), \]  \hspace{1cm} (4.2)

where according to (3.38)

\[ \lambda a z\sigma(\eta \cup d\gamma) := \prod_{\Delta \in \Delta a \cap \Lambda} \chi^\Delta(\eta \cup \gamma) \lambda z\sigma(d\gamma). \]  \hspace{1cm} (3.41)

4 Main results

We prove the results for the infinite volume characteristics, so let \( (\Lambda_l) \) be a sequence of bounded Lebesgue measurable regions of \( \mathbb{R}^d \):

\[ \Lambda_1 \subset \Lambda_2 \subset \ldots \subset \Lambda_n \subset \ldots, \quad \bigcup_l \Lambda_l = \mathbb{R}^d, \]  \hspace{1cm} (4.1)

and the sequence \( (\Lambda_l) \) tends to \( \mathbb{R}^d \) in the sense of Fisher (see \[12\], Ch.2, S. 2.1).

It is well-known that for any configuration \( \eta \in \Gamma_0 \) and any sequence \( (\Lambda_l) \), such that \( \eta \subset \Lambda_1 \) there exists subsequence \( (\Lambda_k) \) of \( (\Lambda_l) \), such that

\[ \lim_{k \to \infty} \rho_{\Lambda_k}(\eta; z, \beta) = \rho(\eta; z, \beta) < \infty \]  \hspace{1cm} (4.2)
for all positive $z, \beta$ uniformly on $B(\Gamma_0)$. This result follows from the uniform bounds of the family $\{\rho_{\Lambda_l} : \Lambda_l \in B_c(\mathbb{R}^d)\}$:

$$\rho_{\Lambda_l}(\eta; z, \beta) \leq \xi |\eta| e^{-\beta U^+_k}$$

with some positive $\xi$, independent of $\Lambda_l, \eta$.

The inequality (4.3) without exponent in r.h.s. was obtained for the first time in the article [13]. In the work [8] a new proof (much easier) was presented with exponent $e^{-\beta U^+_1/2}$ and in the articles [2] and [6] it was proved for many-body interactions for finite range and infinite range cases respectively.

In the next section we give a sketch of proof of the following lemma

**Lemma 4.1.** Let the interaction potential $V$ satisfy the assumptions (A) for two-body and $A1 - A5$ for many-body interactions. Then there exist some $0 < a_* \leq a_0 < r_0$ and a positive constant $\xi_- = \xi_-(a_*)$, which does not depend on $\Lambda_l, a$ and $\eta$, s.t.

$$\rho_{\Lambda_l}^{-} (\eta; z, \beta, a) \leq \xi_{\eta}^{-|\eta|} e^{-\beta U^+_k},$$

holds for any $a < a_*$ such that $a_*/a \in \mathbb{N}$.

So, as in the previous case, there exists subsequence $(\Lambda''_m)$ of the sequence $(\Lambda_l)$ such that one can define

$$\rho^{-}(\eta; z, \beta, a) = \lim_{m \to \infty} \rho_{\Lambda''_m}^{-} (\eta; z, \beta, a) < \infty.$$  

(4.5)

**Remark 4.1.** The limit functions $\rho(\eta; z, \beta)$ and $\rho^{-}(\eta; z, \beta, a)$ in (4.2) and (4.5) can be different for different subsequences $\Lambda'_k$ and $\Lambda''_m$. So, in order to make the function $\rho^{-}(\eta; z, \beta, a)$ be approximation of the function $\rho(\eta; z, \beta)$ we have to take the subsequence $\Lambda''_m$ in the limit (4.5) as some subsequence $\Lambda'_k$.

Then we can formulate the following result.

**Theorem 4.1.** Let the interaction potential $V$ satisfy the assumptions (A) for two-body and $A1 - A5$ for many-body interactions. Then for any $\varepsilon > 0$, any positive $z$ and $\beta$ and any configuration $\eta \in \Gamma_0$ there exists $a = a(z, \beta, \varepsilon) > 0$ such that:

$$|\rho(\eta; z, \beta) - \rho^{-}(\eta; z, \beta, a)| < \varepsilon,$$

(4.6)

where $\rho(\eta; z, \beta)$ and $\rho^{-}(\eta; z, \beta, a)$ are the limits of $\rho_{\Lambda''_m}(\eta; z, \beta)$ and $\rho_{\Lambda''_m}^{-}(\eta; z, \beta, a)$ respectively with the same subsequence of the sequence $(\Lambda_l)$ (see Remark 4.1).
Proof. The proof is based on the existence of the limits (4.2), (4.3) and the following lemma.

**Lemma 4.2.** Let the interaction potential $V$ satisfy the assumptions (A) for two-body and $A_1 - A_5$ for many-body interactions. Then for any sequence $\Lambda_l$ of the type (4.1)

$$\lim_{a \to 0} \rho_{\Lambda_l}^{-}(\eta; z, \beta, a) = \rho_{\Lambda_l}(\eta; z, \beta).$$

and hence for any $\varepsilon > 0$ there exists $a < a_*$, s.t. the following inequality holds:

$$|\rho_{\Lambda_l}^{-}(\eta; z, \beta, a) - \rho_{\Lambda_l}(\eta; z, \beta)| \leq \frac{\varepsilon}{3}. \quad (4.8)$$

From the existence of the limits (4.2) and (4.3) for any $\varepsilon > 0 \exists K_1 \in \mathbb{N}$, s.t. for any $k \geq K_1$ the following inequality holds:

$$|\rho_{\Lambda_m}^{-}(\eta; z, \beta) - \rho(\eta; z, \beta)| \leq \frac{\varepsilon}{3}. \quad (4.9)$$

and $\exists K_2 \in \mathbb{N}$, s.t. for any $k \geq K_2$ the following inequality holds:

$$|\rho_{\Lambda_m}^{-}(\eta; z, \beta, a) - \rho^{-}(\eta; z, \beta, a)| \leq \frac{\varepsilon}{3}. \quad (4.10)$$

Then the statement of the theorem 4.1 follows from (4.8) with $\Lambda_l \equiv \Lambda_m^0$ and (4.9), (4.10):

$$|\rho(\eta; z, \beta) - \rho^{-}(\eta; z, \beta, a)| =$$

$$= |\rho(\eta; z, \beta) - \rho_{\Lambda_m}^{-}(\eta; z, \beta)| +$$

$$+ |\rho_{\Lambda_m}^{-}(\eta; z, \beta) - \rho^{-}(\eta; z, \beta, a)| +$$

$$+ |\rho^{-}(\eta; z, \beta, a) - \rho^{-}(\eta; z, \beta, a)| \leq$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

\[ \square \]

**Corollary 4.1.** The inequality (4.6) ensures existence of the limit:

$$\lim_{a \to 0} \rho_{\Lambda}^{-}(\eta; z, \beta, a) = \rho(\eta; z, \beta). \quad (4.11)$$

for any positive $z$, any $\beta > 0$ and $\eta \in \Gamma_0$.

For two-body interaction this result in the region of sufficiently small values of a parameter $z$ is obtained in the article [11].
5 Proof of the Lemmas 4.1 and 4.2

5.1 Proof of the Lemma 4.1

The proof of the lemma 4.1 is based on the expansion of correlation functions into dense configurations which was proposed in [6] (see, also, [7]) and actually coincides with the proof of the theorem 2.2 of the article [6] for two-body interaction and with the proof of the theorem 2.1 of the article [7] for many-body interaction. The main difference in proving the lemma 4.1 is that in the definition of the correlation functions $\rho(-,\eta;z,\beta,a)$ the integrals are w.r.t. the measure $\lambda^a$ (see (3.40), (3.41)), unlike in the definition of the correlation functions $\rho(\eta;z,\beta)$ where the integrals are w.r.t. the measure $\lambda$ (see (3.37)–(3.39)) which takes into account all possible configurations. So, the main goal of this lemma is to show that the constant $\xi_-$ in the inequality (4.4) does not depend on the parameter $a$. So, in this section we give only main point in the construction of expansion and estimate some value which did not appear in the previous proofs.

In order to arrange this expansion let us define also an indicator of a dense configuration in any cube $\Delta \in \Delta_a$ as $\chi_{\Delta}(\gamma) = 1 - \chi_{\Delta}(\gamma)$.

Then we use the following partition of the unity for any $\gamma \in \Gamma_a$ with $a = a_*$, i.e. $\Delta_{a_*}$:

$$1 = \prod_{\Delta \subseteq \Lambda(a_*)} [\chi_{\Delta}(\gamma) + \chi_{\Delta}(\gamma)] = \sum_{n=0}^{N_{\Lambda(a_*)}} \sum_{\{\Delta_1, ... , \Delta_n\} \subseteq \Lambda(a_*)} \prod_{i=1}^{n} \chi_{\Delta_i}^{(\gamma)} \prod_{\Delta \subseteq \Lambda(a_*) \setminus \cup_{i=1}^{n} \Delta_i} \chi_{\Delta}(\gamma) := \sum_{\emptyset \subseteq X \subseteq \Lambda(a_*)} \chi_{\Delta_{a_*}}(\gamma) \chi_{\Lambda(a_*) \setminus X}(\gamma),$$

(5.1)

where $N_{\Lambda} = |\Lambda|/a_*^d$ (here the symbol $|\Lambda|$ means Lebesgue measure of the set $\Lambda(a_*)$) is the number of cubes $\Delta$ in the volume $\Lambda = \Lambda(a_*)$(see subsection 2.4), and

$$\chi_{\Delta_{a_*}}(\gamma) = \prod_{\Delta \subseteq X} \chi_{\Delta}(\gamma).$$

(5.2)

Inserting (5.1) with $a = a_*$ into the definition (3.40) of correlation functions $\rho(\eta;z,\beta,a)$ with $a < a_*$ s.t. $a_* \in N$ we obtain:

$$\rho(\cdot)(\eta;z,\beta,a) = \frac{1}{Z^(-)(z,\beta,a)} \sum_{\emptyset \subseteq X \subseteq \Lambda(a_*)} \int_{\Gamma_a} e^{-\beta U(\eta,\gamma)} \chi_{\Delta_{a_*}}(\gamma) \chi_{\Lambda(a_*) \setminus X}(\gamma) \lambda^a(\eta \cup d\gamma).$$

(5.3)

**Remark 5.1.** We want to stress that the sets $X$ in (5.3) are the unions of cubes $\Delta \in \Delta_{a_*}$, but in the product of the definition $\chi_{\Lambda(a_*) \setminus X}(\gamma)$ (see (3.41)) $\Delta \in \Delta_a$ with $a < a_*$ and $a_*/a \in N$. 

15
The next steps in the construction of expansion and estimates are completely the same as in the proof of the theorem 2.2 of the article [6] for two-body interaction and the theorem 2.1 of the article [7] for many-body interaction. It is necessary only to note that to change the integration w.r.t measure \( \lambda_a z_{\sigma} (\eta \cup d\gamma \Delta') \) for the integration w.r.t. measure \( \lambda_a (d\gamma \Delta') \) (see (3.41)) we use the following inequality:

\[
\chi_{\Delta'} (\eta \cup \gamma) \leq \chi_{\Delta'} (\gamma),
\]

which follows from the definition (3.36) for any \( \Delta' \in \Delta_a \) and any \( \gamma \in \Gamma \).

\section*{5.2 Proof of the Lemmas 4.2}

Let us insert now the unity (5.1) (but with partition \( \Lambda \) into cubes with edges \( a \) instead of \( a_* \) and the argument \( \eta \cup \gamma \) in each function \( \chi_{\Delta'} \)) in (3.39). Then we obtain the following expansion:

\[
\rho_{\Lambda}(\eta; z, \beta) = z_{|\eta|} \sum_{X \subseteq \Lambda} \int_{\Gamma_{\Lambda}} e^{-\beta U(\eta, \gamma)} \tilde{\chi}_+^X (\eta \cup \gamma) \tilde{\chi}_-^{\Lambda \setminus X} (\eta \cup \gamma) \lambda_{\sigma} (d\gamma).
\]

Extracting the first term at \( X = \emptyset \) and using the definitions (3.37)-(3.40) we can rewrite (5.4) in the following form:

\[
\rho_{\Lambda}(\eta; z, \beta) = \frac{Z_{\Lambda}^-(z, \beta, a)}{Z_{\Lambda}(z, \beta)} \rho_{\Lambda}^-(\eta; z, \beta, a) + R_{\Lambda}(\eta; z, \beta, a),
\]

where

\[
R_{\Lambda}(\eta; z, \beta, a) = \frac{z_{|\eta|}}{Z_{\Lambda}(z, \beta)} \sum_{\emptyset \neq X \subseteq \Lambda} \int_{\Gamma_{\Lambda}} e^{-\beta U(\eta, \gamma)} \tilde{\chi}_+^X (\eta \cup \gamma) \tilde{\chi}_-^{\Lambda \setminus X} (\eta \cup \gamma) \lambda_{\sigma} (d\gamma).
\]

The proof of the lemma 4.2 is based on two technical lemmas.

\textbf{Lemma 5.1.} Let the interaction potential \( V \) satisfy the assumptions (\( A \)) for two-body and \( A_1 - A_5 \) for many-body interactions. Then for any fixed volume \( \Lambda \in \mathfrak{B}_c(\mathbb{R}^d) \) and any configuration \( \eta \in \Gamma_0 \) the following holds:

\[
\lim_{a \to 0} R_{\Lambda}(\eta; z, \beta, a) = 0.
\]

\textit{Proof.} See Appendix. \hfill \blacksquare
Lemma 5.2. Let the interaction potential \( V \) satisfy the assumptions (A) for two-body and A1 – A5 for many-body interactions. Then for any fixed volume \( \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \) the following holds:

\[
\lim_{a \to 0} \frac{Z_{\Lambda}^{-}(z, \beta, a)}{Z_{\Lambda}(z, \beta)} = 1.
\]

(5.8)

Proof. In the articles [9] and [5] the following estimate was obtained:

\[
\lim_{a \to 0} \frac{Z_{\Lambda}^{-}(z, \beta, a)}{Z_{\Lambda}(z, \beta)} \geq 1,
\]

on which the proof of the fact that the pressure of approximated system converges to the pressure of the real system is based. From the other hand in accordance with the definitions (3.37), (3.38) it is clear that

\[
\frac{Z_{\Lambda}^{-}(z, \beta, a)}{Z_{\Lambda}(z, \beta)} \leq 1.
\]

As a result we have

\[
\lim_{a \to 0} \frac{Z_{\Lambda}^{-}(z, \beta, a)}{Z_{\Lambda}(z, \beta)} = 1.
\]

6 Appendix

Proof of the lemma 5.1

Using (3.18) for two-body potential and (3.27) for many-body interaction, one can rewrite (5.6) in such a way:

\[
R^\Lambda(\eta; z, \beta, a) = \frac{2^{z|\eta|}}{Z_{\Lambda}(z, \beta)} \sum_{\emptyset \neq X \subseteq \Lambda} \int_{\Gamma_{\Lambda}} e^{-\frac{1}{2} \tilde{U}^+(\eta \cup \gamma X) + U^+(\eta \cup \gamma X)} \tilde{\chi}_{X}^+ (\eta \cup \gamma) \times e^{-\beta W(\eta \cup \gamma X; \gamma \Lambda \setminus X)} - \frac{1}{2} \tilde{U}^+(\eta \cup \gamma X) e^{-\beta U(\gamma \Lambda \setminus X)} \tilde{\chi}_{\Lambda \setminus X}^-(\eta \cup \gamma) \lambda_{\sigma} (d\gamma).
\]

(6.1)

Using infinite divisibility property of Lebesgue-Poisson measure, the estimate:

\[
e^{-\beta W(\eta \cup \gamma X; \gamma \Lambda \setminus X)} - \frac{1}{2} \tilde{U}^+(\eta \cup \gamma X) \leq e^{\beta v_+ (a)(|\eta| + |\gamma X|)}
\]

and the fact that

\[
\tilde{\chi}_{\Lambda \setminus X}^- (\eta \cup \gamma) \leq 1,
\]
where \( v_+(a) = v_0(a) \) for two-body potential and \( v_+(a) = \tilde{I}(a) \) for many-body interaction (see \( (3.35) \)), we obtain from \( (6.1) \):

\[
R^\Lambda(\eta; z, \beta, a) \leq \frac{\left(ze^{\beta v_+(a)}\right)^{|\eta|}}{Z_\Lambda(z, \beta)} \sum_{\emptyset \neq X \subseteq \Lambda} \int_{\Gamma_X} e^{-\beta \left(\frac{1}{2}U^+(\eta \cup \gamma_X) + U^+(\eta \cup \gamma_X) + v_+(a)\right)\gamma_X} \times \bar{\chi}_+^X(\eta \cup \gamma) \lambda_{z\sigma}(d\gamma_X).
\]

(6.2)

Let us take into account that

\[
Z_{\Lambda \setminus X}(z, \beta) = \int_{\Gamma_{\Lambda \setminus X}} e^{-\beta U(\gamma_{\Lambda \setminus X})} \lambda_{z\sigma}(d\gamma_{\Lambda \setminus X})
\]

and \( Z_{\Lambda \setminus X}(z, \beta) \leq Z_\Lambda(z, \beta) \).

Then we have from \( (6.2) \):

\[
R^\Lambda(\eta; z, \beta, a) \leq \left(ze^{\beta v_+(a)+B(a)}\right)^{|\eta|} \sum_{\emptyset \neq X \subseteq \Lambda} \int_{\Gamma_X} e^{-\beta \left(\frac{1}{2}U^+(\eta \cup \gamma_X) + U^+(\eta \cup \gamma_X) + v_+(a)\right)\gamma_X} \bar{\chi}_+^X(\eta \cup \gamma) \lambda_{z\sigma}(d\gamma_X).
\]

(6.3)

Let \( \Lambda_\eta \) be a union of cubes which contain points from the configuration \( \eta \). Then using Proposition 3.1, lemma 3.1 and inequalities \( (3.9),(3.34) \) we have:

\[
R^\Lambda(\eta; z, \beta, a) \leq \left(ze^{\beta(v_+(a)+B(a))}\right)^{|\eta|} (R_1^\Lambda + R_2^\Lambda),
\]

(6.4)

where

\[
R_1^\Lambda = \sum_{\emptyset \neq X \subseteq \Lambda \setminus \Lambda_\eta} \int_{\Gamma_X} \sum_{\kappa \neq \gamma \neq \Delta} \beta \left(-\frac{1}{2}A(a)|\gamma_\Delta|^2+(B(a)+v_+(a))|\gamma_\Delta|\right) \bar{\chi}_+^X(\eta \cup \gamma) \lambda_{z\sigma}(d\gamma_X),
\]

\[
R_2^\Lambda = \sum_{\emptyset \neq X \subseteq \Lambda \setminus \Lambda_\eta} \int_{\Gamma_X} \sum_{\kappa \neq \gamma \neq \Delta} \beta \left(-\frac{1}{2}A(a)|\gamma_\Delta|+(B(a)+v_+(a))|\gamma_\Delta|\right) \bar{\chi}_+^X(\eta \cup \gamma) \lambda_{z\sigma}(d\gamma_X)
\]

with \( A(a) \) and \( B(a) \) as in \( (3.15) \) for two-body potentials and \( (3.34) \) for many-body interaction. Using again the infinite divisible property of the Lebesgue-Poisson measure and its definition one can calculate

\[
\int_{\Gamma_\Delta} e^{-\beta \frac{1}{2}A(a)|\gamma_\Delta|^2+\beta(B(a)+v_+(a))|\gamma_\Delta|} \chi_+^\Delta(\gamma_\Delta) \lambda_{z\sigma}(d\gamma_\Delta) =
\]

\[
= \sum_{n=2}^{\infty} \frac{(ad)^n}{n!} e^{-\frac{1}{2}A(a)n^2+\beta(B(a)+v_+(a))n} \leq \epsilon_1(a),
\]

(6.5)
where
\[ \epsilon_1(a) \to 0, \quad \text{якщо} \quad a \to 0. \]  

(6.6)

Then after summing w.r.t. \( X \) we obtain the following estimate:
\[ R_1^\Lambda \leq (1 + \epsilon_1(a)) \frac{|\Lambda \setminus \Lambda_0|}{a^d} - 1 \leq \epsilon_1(a) \frac{|\Lambda \setminus \Lambda_0|}{a^d} (1 + \epsilon_1(a)) \frac{|\Lambda \setminus \Lambda_0|}{a^d} - 1. \]  

(6.7)

To estimate \( R_2^\Lambda \) let us rewrite it in the form:
\[ R_2^\Lambda = \sum_{\emptyset \neq X \subseteq \Lambda, \ X \cap \Lambda_0 \neq \emptyset} R_0^\Lambda(\eta_{X \cap \Lambda_0}; z, \beta, a) \times \]
\[ \int_{\Gamma \setminus \Lambda_0} e^{\Delta_{X \setminus \Lambda_0}} \beta(-\frac{1}{2}A(a)\gamma_\Lambda + (B(a)+v_*(a))\gamma_\Delta) \chi_{+}^{X \setminus \Lambda_0}(\gamma_\Lambda \cap \Lambda_0) \lambda_{\sigma}(\gamma_X \setminus \Lambda_0), \]

where
\[ R_0^\Lambda = \int_{\Gamma \setminus \Lambda_0} e^{\Delta_{X \setminus \Lambda_0}} \beta(-\frac{1}{2}A(a)\gamma_\Lambda + (B(a)+v_*(a))\gamma_\Delta) \chi_{+}^{X \setminus \Lambda_0}(\gamma_\Lambda \cap \Lambda_0) \lambda_{\sigma}(\gamma_X \setminus \Lambda_0) = \]
\[ \prod_{\Delta \in X \setminus \Lambda_0} \int_{\Gamma_\Delta} e^{\beta(-\frac{1}{2}A(a)\gamma_\Lambda + (B(a)+v_*(a))\gamma_\Delta)} \chi_{+}^{X \setminus \Lambda_0}(\gamma_\Lambda \cup \gamma_\Delta) \lambda_{\sigma}(\gamma_\Delta). \]  

(6.9)

Estimating maximum of the exponent we obtain:
\[ R_0^\Lambda(\eta_{X \cap \Lambda_0}; z, \beta, a) \leq e^{-\beta(2A(a) - B(a) - v_*(a))} \prod_{\Delta \in X \setminus \Lambda_0} \int_{\Gamma_\Delta} \lambda_{\sigma}(\gamma_\Delta) \leq e^{-\beta(2A(a) - B(a) - v_*(a))} e^{z a^d |\eta|} \]

(6.10)

Using (6.8), (6.10) we can estimate \( R_2^\Lambda \) from above in the form:
\[ R_2^\Lambda \leq e^{-\beta(2A(a) - B(a) - v_*(a))} e^{z a^d |\eta|} \times \]
\[ \sum_{\emptyset \neq X \subseteq \Lambda, \ X \cap \Lambda_0 \neq \emptyset} \int_{\Gamma \setminus \Lambda_0} e^{\Delta_{X \setminus \Lambda_0}} \beta(-\frac{1}{2}A(a)\gamma_\Lambda + (B(a)+v_*(a))\gamma_\Delta) \chi_{+}^{X \setminus \Lambda_0}(\gamma_\Lambda \cap \Lambda_0) \lambda_{\sigma}(\gamma_X \setminus \Lambda_0). \]  

(6.11)

Let us take into account that for any \( \mathcal{B}(\Gamma_\Lambda) \)-measurable function \( F(\gamma) \) the following holds:
\[ \sum_{\emptyset \neq X \subseteq \Delta_0 \cap \Lambda, \ X \cap \Lambda_0 \neq \emptyset} \int_{\Gamma \setminus \Lambda_0} F(\gamma_X \setminus \Lambda_0) \lambda_{\sigma}(\gamma_X \setminus \Lambda_0) \leq \]
\[ (2^{|\eta|} - 1) \sum_{X \subseteq \Delta_0 \cap \Lambda_0} \int_{\Gamma_X} F(\gamma_X) \lambda_{\sigma}(\gamma_X). \]  

(6.12)
Using this fact and infinite divisibility property of Lebesgue-Poisson measure we obtain from (6.11):

\[
R^2_2 \leq e^{-\beta(2A(a)-B(a)+\nu_\ast(a))} e^{a^d|\eta| (2^{|\eta|} - 1)} \times \\
\sum_{X \subseteq \Lambda \setminus \Lambda_{\eta}} \prod_{\Delta \in X} \int_{\Gamma_\Delta} e^{\beta \left( -\frac{1}{2} A(a) |\gamma_\Delta|^2 + (B(a)+\nu_\ast(a)) |\gamma_\Delta| \right)} \times \\
\chi_+(\gamma_\Delta) \lambda_\sigma (d\gamma_\Delta) \leq e^{-\beta(2A(a)-B(a)+\nu_\ast(a))} e^{z a^d |\eta| (2^{|\eta|} - 1) (1 + \epsilon_1(a)) \frac{|\Lambda \setminus \Lambda_{\eta}|}{a^d}}. 
\]

(6.13)

It follows from (6.4), (6.7), (6.13) that:

\[
R^\Lambda(\eta; z, \beta, a) \leq (ze^{\beta(\nu_\ast(a)+B(a))})^{|\eta|} (1 + \epsilon_1(a)) \left( \epsilon_1(a) \frac{|\Lambda \setminus \Lambda_{\eta}|}{a^d} + \\
(2^{|\eta|} - 1) (1 + \epsilon_1(a)) e^{-\beta(2A(a)-B(a)-\nu_\ast(a))} e^{z a^d |\eta|} \right) \to 0, \quad \text{якщо} \quad a \to 0, 
\]

(6.14)

This is the end of the proof.

References

[1] S. Albeverio, Yu. G. Kondratiev, and M. Röckner, Analysis and geometry on configuration spaces, *J. Funct. Anal.* **154**(2), 444-500 (1998).

[2] O. V. Kutoviy, A. L. Rebenko, Existence of Gibbs state for continuous gas with many-body interaction, *J. Math. Phys.*, **45**(4), 1593-1605 (2004).

[3] A. Lenard, States of classical statistical mechanical systems of infinitely many particles. I, *Arch. Rational Mech. Anal.*, **59**, 219-239 (1975).

[4] A. Lenard, States of classical statistical mechanical systems of infinitely many particles. II, *Arch. Rational Mech. Anal.*, **59**, 241-256 (1975).

[5] S. N. Petrenko, Quasicontinuous Approximation of Statistical Systems with many-body interactions (in Ukraine) // Naukovyi visnyk Lvivs’kogo NLTU Ukrainy. Zbirnyk naukovo-tehnichnyh prats’ — 2008. — Vol. 18.9. — P. 287—296.
[6] S. N. Petrenko, A. L. Rebenko, Superstable criterion and superstable bounds for infinite range interaction I: two-body potentials, *Meth. Funct. Anal. and Topology*, **13**, 50–61 (2007).

[7] S. N. Petrenko, A. L. Rebenko, Superstable criterion and superstable bounds for infinite range interaction II: many-body potentials // Збірник праць Інституту математики НАН України. — 2009. — Vol. 6, № 1. — P. 191–208.

[8] A. L. Rebenko, New Proof of Ruelle’s Superstability Bounds, *J. Stat. Phys.*, **91**, 815-826 (1998).

[9] A. L. Rebenko, M. V. Tertychnyi, Quasicontinuous Approximation of Statistical Systems with Strong Superstable interactions, *Proc. Inst. Math. NASU*, **4**, No 3, 172-182 (2007).

[10] A. L. Rebenko, M.V. Tertychnyi, On stability, superstability and strong superstability of classical systems of Statistical Mechanics, *Meth. Funct. Anal. and Topology* (2008), **14**, Nu. 3, P. 287–296.

[11] A. L. Rebenko, M.V. Tertychnyi, Quasilattice approximation of statistical systems with strong superstable interactions: Correlation functions, *J.Math. Phys.* (2009), **50**, Nu. 3, P. 033301-10.

[12] D. Ruelle, Statistical Mechanics, (Rigorous results), *W.A. Benjamin, inc. N.Y.–Amsterdam* (1969).

[13] D. Ruelle, Superstable interactions in classical statistical mechanics, *Commun. Math. Phys.*, **18**, 127-159 (1970).

[14] M.V. Tertychnyi, Sufficient conditions for superstability of many-body interactions, *Meth. Funct. Anal. and Topology* (2008), **14**, Nu. 4, P. 386–396.