$L^p$ solutions of backward stochastic Volterra integral equations *

Tianxiao Wang †
School of Mathematics, Shandong University, Jinan 250100, China

December 14 2009

Abstract

This paper is devoted to the unique solvability of backward stochastic Volterra integral equations (BSVIEs for short), in terms of both M-solution introduced in [15] and the adapted solutions in [6], [11]. We prove the existence and uniqueness of M-solutions of BSVIEs in $L^p$ ($1 < p < 2$), which extends the results in [15]. The unique solvability of adapted solutions of BSVIEs in $L^p$ ($p > 1$) is also considered, which also generalize the results in [6] and [11].

Keywords: Backward stochastic Volterra integral equations, M-solutions, $L^p$ solutions, adapted solutions

1 Introduction

In this paper, we are concerned with backward stochastic Volterra integral equation (BSVIE for short) of the form, $t \in [0, T]$,

$$Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t))ds - \int_t^T Z(t, s)dW(s), \quad (1)$$

where $W$ is a standard Brownian motion with values in $R^d$ defined on some complete probability space $(\Omega, \mathcal{F}, P)$, $\psi(\cdot)$ is the terminal condition and $g$ is

*This work is supported by National Natural Science Foundation of China Grant 10771122, Natural Science Foundation of Shandong Province of China Grant Y2006A08 and National Basic Research Program of China (973 Program, No. 2007CB814900).
†Corresponding author, E-mail:xiaotian2008001@gmail.com
the coefficient (also called the generator). The unknowns are the processes \((Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^p[0,T]\) (defined below), for which \((Y(\cdot), Z(t,\cdot))\) is \(\mathbb{F}\)-adapted for all \(t \in [0,T]\). Here \(\{\mathcal{F}_t\}_{t \geq 0}\) is the augmented natural filtration of \(W\) which satisfies the usual conditions.

Lin \[6\] firstly considered the solvability of the adapted solution of the form

\[
Y(t) = \xi + \int_t^T g(t,s,Y(s),Z(t,s))ds - \int_t^T Z(t,s)dW(s), \quad t \in [0,T].
\]

As to the general form (1), Yong (\[13\] and \[15\]) firstly studied them and gave the application in optimal control. One can also see \[11\] for more detailed accounts on BSVIEs (1). Both of them are natural generalization of backward stochastic differential equation (BSDE for short) of the form

\[
Y(t) = \xi + \int_t^T g(s,Y(s),Z(s))ds - \int_t^T Z(s)dW(s), \quad t \in [0,T],
\]

which was firstly introduced by Pardoux and Peng \[7\]. In order to obtain a stochastic maximum principle for optimal control of stochastic Volterra integral equation, Yong \[15\] introduced the notion of M-solution and proved the existence and uniqueness of M-solution in \(\mathcal{H}^2[0,T]\). On the other hand, Yong \[14\] adopted the so-called BSVIEs to construct a class of dynamic convex and coherent risk measures, which is different from the result in the case of BSDE in Rosazza \[9\]. We would like to mention that Wang and Zhang \[12\] gave existence and uniqueness of the adapted solution of BSVIE (2) with jumps under non-Lipschitz condition and Ren considered the similar result in hilbert space in \[8\]. See Aman \[1\] for result of BSVIE (2) under local Lipschitz condition. All of them considered the unique solvability of BSVIEs when \(p = 2\).

In this paper, for mathematical interest, we try to get the existence and uniqueness of \(L^p\) (\(1 < p < 2\)) solutions for BSVIEs. In BSDEs case, we have to mention that El Karoui et al. \[4\] obtained the existence and uniqueness of the solution for BSDEs when the generator is uniformly Lipschitz, the data \(\xi\) and \(\{g(s,0,0)\}\) are in \(L^p\) (\(1 < p < 2\)). Briand and Carmona \[2\] considered the \(L^p\) solution for BSDEs with polynomial growth generators and in Briand et al. \[3\] generalized the result. Recently, Wang et al. \[10\] also studied the \(L^p\) solutions when the uniform Lipschitz condition was replaced by nonnegative adapted pro-
cess. However, neither of the above can be applied in BSVIEs case, and we have to deal with this problem with new method. Inspired by the four steps to solve the existence and uniqueness of M-solution in \(\mathcal{H}^2[0,T]\) in \[15\], we will use the similar method to deal with the situation for M-solution in \(\mathcal{H}^p[0,T]\). Similarly we can also get the result of adapted solutions for (1) when \(g\) is independent of \(Z(s,t)\).
The paper is organized as follows. In Section 2, we will present some notations, definition and some propositions. In Section 3, we will study the existence and uniqueness of M-solution of (1) and adapted solution of (1) (the generator is independent of $Z(s, t)$) respectively.

2 Preliminary

In this section, we will present some necessary notations, definitions and some propositions needed in the sequel. In the following we denote $\Delta^c = \Delta^c(0, T]$ and $\Delta^c[R, S] = \{(t, s) \in [R, S]^2, t \leq s\}$ where $R, S \in [0, T]$. Let $L^p_{\mathcal{F}_T}[R, S]$ be the set of $\mathcal{B}[R, S] \times \mathcal{F}_T$-measurable processes $\psi : \Omega \times [R, S] \rightarrow \mathbb{R}^m$ such that $E \int_S^R |\psi(t)|^p dt < \infty$. $L^p_{\mathcal{F}}[R, S]$ is set of adapted processes $X : \Omega \times [R, S] \rightarrow \mathbb{R}^m$ such that $E \int_R^S |X(t)|^p dt < \infty$.

We denote

$$\mathcal{H}^p[R, S] = L^p_{\mathcal{F}}(\Omega; C[R, S]) \times L^p_{\mathcal{F}}[R, S],$$

which is a Banach space under the norm

$$\|(y(\cdot), z(\cdot))\|_{\mathcal{H}^p[R, S]} = \left[ E \sup_{t \in [R, S]} |y(t)|^p + E \left( \int_R^S |z(t)|^2 dt \right)^{\frac{p}{2}} \right]^{\frac{1}{p}},$$

where $L^p_{\mathcal{F}}(\Omega; C[R, S])$ is set of all continuous adapted processes $X : [R, S] \times \Omega \rightarrow \mathbb{R}^m$ such that $E \left[ \sup_{t \in [R, S]} |X(t)|^p \right] < \infty$.

$L^p(R, S; L^2_{\mathcal{F}}[R, S])$ the set of processes $Z : \Omega \times [R, S] \times [R, S] \rightarrow \mathbb{R}^{m \times d}$ such that for almost every $t \in [R, S]$, $Z(t, \cdot)$ is $\mathcal{F}$-progressively measurable and

$$E \int_R^S \left( \int_R^S |Z(t, s)|^2 ds \right)^{\frac{p}{2}} dt < \infty.$$

$L^p(R, S; L^2_{\mathcal{F}}[t, T])$ the set of processes $Z : \Omega \times \Delta^c \rightarrow \mathbb{R}^{m \times d}$ such that for almost every $t \in [R, S]$, $Z(t, \cdot)$ is $\mathcal{F}$-progressively measurable and

$$E \int_R^S \left( \int_t^T |Z(t, s)|^2 ds \right)^{\frac{p}{2}} dt < \infty.$$

We denote

$$\mathcal{H}^p_0[R, S] = L^p_{\mathcal{F}}[R, S] \times L^p(R, S; L^2_{\mathcal{F}}[R, S]),$$

$$\mathcal{H}_0^p[R, S] = L^p[R, S] \times L^p(R, S; L^2_{\mathcal{F}}[t, S]).$$

We also need the following two definitions.
Definition 2.1 Let $S \in [0,T]$. A pair of $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^p[S,T]$ is called an adapted $M$-solution of BSVIE (1) on $[S,T]$ if (1) holds in the usual Itô’s sense for almost all $t \in [S,T]$ and, in addition, the following holds:

$$Y(t) = E^{\mathcal{F}_S} Y(t) + \int_S^t Z(t,s)dW(s), \quad t \in [S,T].$$

Definition 2.2 A pair of $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^p_0[0,T]$ is called an adapted solution of the following simple BSVIE (3) if (3) holds in the usual Itô’s sense,

$$Y(t) = \psi(t) + \int_t^T g(t,s,Y(s),Z(t,s))ds - \int_t^T Z(t,s)dW(s), \quad t \in [0,T]. \quad (3)$$

Next we will give some propositions, which can be seen in [15]. For any $R,S \in [0,T]$, let us consider the following stochastic integral equation, $r \in [S,T], t \in [R,T],$

$$\lambda(t,r) = \psi(t) + \int_r^T h(t,s,\mu(t,s))ds - \int_r^T \mu(t,s)dW(s), \quad (4)$$

where $h : [R,T] \times [S,T] \times R^{m\times d} \times \Omega \rightarrow R^m$ is given. The unknown processes are $(\lambda(t,\cdot), \mu(t,\cdot))$, for which $(\lambda(t,\cdot), \mu(t,\cdot))$ are $\mathbb{F}$-adapted for all $t \in [R,T]$. We can regard (4) as a family of BSDEs on $[S,T]$, parameterized by $t \in [R,T]$. Next we introduce the following assumption of $h$ in (4).

(H1) Let $R,S \in [0,T]$ and $h : [R,T] \times [S,T] \times R^{m\times d} \times \Omega \rightarrow R^m$ be $\mathcal{B}([R,T] \times [S,T] \times R^{m\times d}) \otimes \mathcal{F}_T$-measurable such that $s \mapsto h(t,s,z)$ is $\mathbb{F}$-progressively measurable for all $(t,z) \in [R,T] \times R^{m\times d}$ and

$$E \int_R^T \left( \int_S^T |h(t,s,0)|ds \right)^p dt < \infty. \quad (5)$$

Moreover, the following holds:

$$|h(t,s,z_1) - h(t,s,z_2)| \leq L(t,s)|z_1 - z_2|, \quad (t,s) \in [R,T] \times [S,T], \quad z_1, z_2 \in R^{m\times d}, \quad (6)$$

where $L : [R,T] \times [S,T] \rightarrow [0,\infty)$ is a deterministic function such that for some $\varepsilon > 0$,

$$\sup_{t \in [R,T]} \int_S^T L(t,s)^{2+\varepsilon} ds < \infty.$$

Proposition 2.1 Let (H1) hold, then for any $\psi(\cdot) \in L^p_{\mathcal{F}_T}[R,T]$, (4) admits a unique adapted solution $(\lambda(t,\cdot), \mu(t,\cdot)) \in \mathbb{H}^p[S,T]$ for almost all $t \in [R,T].$
Now we look at one special case of (4). Let \( R = S \) and define
\[
\begin{align*}
Y(t) &= \lambda(t,t), \quad t \in [S,T], \\
Z(t,s) &= \mu(t,s), \quad (t,s) \in \Delta^c[S,T].
\end{align*}
\] (7)

Then the above (4) reads:
\[
Y(t) = \psi(t) + \int_t^T h(t,s,Z(t,s))ds - \int_t^T Z(t,s)dW(s), \quad t \in [S,T].
\] (8)

Here we define \( Z(t,s) \) for \((t,s) \in \Delta[S,T]\) by
\[
Y(t) = EY(t) + \int_0^t Z(t,s)dW(s).
\]

So we have,

**Proposition 2.2** Let \( (H1) \) hold, then for any \( S \in [0,T], \psi(\cdot) \in L^p_{\mathcal{F}_T}[S,T], \) (8) admits a unique adapted M-solution \((Y(\cdot),Z(\cdot,\cdot)) \in \mathcal{H}^p[S,T]\) (adapted solution \((Y(\cdot),Z(\cdot,\cdot)) \in \mathcal{H}^p_0[S,T], \) respectively). If \( h \) also satisfies \((H1), \psi(\cdot) \in L^p_{\mathcal{F}_T}[S,T], \)
and \((Y(\cdot),Z(\cdot,\cdot)) \in \mathcal{H}^p[S,T] \) is the unique adapted M-solution of BSVIE (8) with \((h,\psi) \) replaced by \((h,\psi), \) then \( \forall t \in [S,T], \)
\[
E \left\{ \left| Y(t) - \overline{Y}(t) \right|^p + \left( \int_t^T |Z(t,s) - \overline{Z}(t,s)|^2ds \right)^{p/2} \right\} \leq CE \left[ \left| \Psi(t) - \overline{\Psi}(t) \right|^p + \left( \int_t^T |h(t,s,Z(t,s)) - \overline{h}(t,s,Z(t,s))|ds \right)^p \right].
\] (9)
Hereafter \( C \) is a generic positive constant which may be different from line to line.

Let’s give another special case. Let \( r = S \in [R,T] \) be fixed. Define
\[
\psi^S(t) = \lambda(t,S), \quad Z(t,s) = \mu(t,s), \quad t \in [R,S], s \in [S,T].
\]
Then (4) becomes: \( t \in [R,S], \)
\[
\psi^S(t) = \psi(t) + \int_S^T h(t,s,Z(t,s))ds - \int_S^T Z(t,s)dW(s), \quad (10)
\]
and we have the following result.

**Proposition 2.3** Let \( (H1) \) hold, then for any \( \psi(\cdot) \in L^p_{\mathcal{F}_T}[R,S], \) (10) admits a unique adapted solution \((\psi^S(\cdot),Z(\cdot,\cdot)) \in L^p_{\mathcal{F}_S}[R,S] \times L^p(R,S;L^2_{\mathcal{F}_T}[S,T]). \)
3 \( L^p \) solution for BSVIE

In this section, we will make use of the above propositions to give the unique existence of M-solution of (1) and adapted solutions of (3). First we assume,

(H2) Let \( g : \Delta^c \times R^m \times R^{m\times d} \times R^{m\times d} \times \Omega \rightarrow R^m \) be \( B(\Delta^c \times R^m \times R^{m\times d} \times R^{m\times d}) \otimes \mathcal{F}_T \)-measurable such that the following holds:

\[
E \int_0^T \left( \int_t^T |g_0(t,s)|ds \right)^p dt < \infty,
\]

where \( g_0(t,s) = g(t,s,0,0,0) \). Moreover, it holds

\[
|g(t,s,y,z,\zeta) - g(t,s,\bar{y},\bar{z},\bar{\zeta})| \leq L_1(t,s)|y - \bar{y}| + L_2(t,s)|z - \bar{z}| + L_3(t,s)|\zeta - \bar{\zeta}|
\]

\( \forall (t,s) \in \Delta^c \), \( y, \bar{y} \in R^m \), \( z, \bar{z}, \zeta, \bar{\zeta} \in R^{m\times d} \), a.s. where \( L_i : \Delta^c \rightarrow R \) is a deterministic function such that the following holds:

\[
\sup_{t \in [0,T]} \int_0^t L_p^p(s,t)ds < \infty, \quad \sup_{t \in [0,T]} \int_t^T L_2(t,s)^{2+\varepsilon}ds < \infty, \quad \sup_{t \in [0,T]} \int_0^t L_3^{2p}(s,t)ds < \infty,
\]

where \( \varepsilon \) is a positive constant, and \( p \in (1,2) \).

**Theorem 3.1** Let (H2) hold, then for any \( \psi(\cdot) \in L^p_{\mathcal{F}_T}[0,T] \), (1) admits a unique adapted M-solution in \( \mathcal{H}^p[0,T] \), where \( p \in (1,2) \).

**Proof.** We will split the proof into four steps.

**Step 1.** Choose \( S \in [0,T] \) in a manner that we can determine the unique existence of M-solution \((Y(t),Z(t,s)) \in \mathcal{H}^p[0,T]\) for \((t,s) \in [S,T]^2\). First let \( \mathcal{M}^p[0,T] \) be the space of all \((y(\cdot),z(\cdot,\cdot)) \in \mathcal{H}^p[0,T]\) such that

\[
y(t) = Ey(t) + \int_0^t z(t,s)dW(s), \quad t \in [0,T]. \tag{11}
\]

Thanks to the martingale moment inequalities in [5], we deduce that,

\[
E \int_0^T \left| \int_0^t z(t,s)dW(s) \right|^p dt \leq C_p E \int_0^T \left( \int_0^t |z(t,s)|^2 ds \right)^{\frac{p}{2}} dt, \quad p > 0, \tag{12}
\]

and

\[
E \int_0^T \left( \int_0^t |z(t,s)|^2 ds \right)^{\frac{p}{2}} dt \leq C_p E \int_0^T \left| \int_0^t z(t,s)dW(s) \right|^p dt, \quad p > 1, \tag{13}
\]
where $C_p$ is a constant depending on $p$. Thus it is easy to show that $\mathcal{M}^p[0,T]$ is a closed nonempty subspace of $\mathcal{H}^p[0,T]$. (11) and (13) imply,

$$E \int_0^T \left( \int_0^t |z(t, s)|^2 ds \right)^{\frac{p}{2}} dt \leq C_p E \int_0^T |y(t)|^p dt,$$

thus the following result holds,

$$E \int_0^T |y(t)|^p dt + E \int_0^T \left( \int_0^T |z(t, s)|^2 ds \right)^{\frac{p}{2}} dt \leq C_p E \int_0^T |y(t)|^p dt + C_p E \int_0^T \left( \int_t^T |z(t, s)|^2 ds \right)^{\frac{p}{2}} dt$$

$$+ C_p E \int_0^T \left( \int_0^t |z(t, s)|^2 ds \right)^{\frac{p}{2}} dt \leq C_p E \int_0^T |y(t)|^p dt + C_p E \int_0^T \left( \int_t^T |z(t, s)|^2 ds \right)^{\frac{p}{2}} dt.$$

Therefore, we can introduce another norm for $\mathcal{M}^p[0,T]$ as follows:

$$\|((y(\cdot), z(\cdot, \cdot)))\|_{\mathcal{M}^p[0,T]} = \left[ E \int_0^T |y(t)|^p dt + E \int_0^T \left( \int_t^T |z(t, s)|^2 ds \right)^{\frac{p}{2}} dt \right]^\frac{1}{p}.$$ 

Let us consider the following equation:

$$Y(t) = \psi(t) + \int_t^T g(t, s, y(s), Z(t, s), z(s, t)) ds - \int_t^T Z(t, s) dW(s), \quad t \in [S, T]$$

(14)

for any $\psi(\cdot) \in L^p_{\mathcal{F}_T}[S,T]$ and $(y(\cdot), z(\cdot, \cdot)) \in \mathcal{M}^p[S,T]$. By Proposition 2.2, we observe that (14) admits a unique adapted M-solution $(Y(\cdot), Z(\cdot, \cdot))$, and we can define a map $\Theta : \mathcal{M}^p[S,T] \to \mathcal{M}^p[S,T]$ by

$$\Theta(y(\cdot), z(\cdot, \cdot)) = (Y(\cdot), Z(\cdot, \cdot)), \quad \forall(y(\cdot), z(\cdot, \cdot)) \in \mathcal{M}^p[S,T].$$

Let $(\overline{Y}(\cdot), \overline{Z}(\cdot, \cdot)) \in \mathcal{M}^p[S,T]$ and $\Theta(\overline{Y}(\cdot), \overline{Z}(\cdot, \cdot)) = (\overline{Y}(\cdot), \overline{Z}(\cdot, \cdot))$. Consequently,
(9) gives that

$$E \int_S^T |Y(t) - \overline{Y}(t)|^p dt + E \int_S^T \left( \int_t^T |Z(t,s) - \overline{Z}(t,s)|^2 ds \right)^{\frac{p}{2}} dt \leq C \int_S^T \left( \int_t^T |g(t,s,Y(t),Z(t,s),\overline{Z}(s,t)) - g(t,s,\overline{y}(s),Z(t,s),\overline{Z}(s,t))| ds \right)^p dt$$

$$\leq C E \int_S^T \left\{ \int_t^T L_1(t,s)|y(s) - \overline{y}(s)| ds \right\}^p dt + C E \int_S^T \left\{ \int_t^T L_3(t,s)|z(s,t) - \overline{z}(s,t)| ds \right\}^p dt$$

$$\leq C E \int_S^T (T-t)^{\frac{p}{2}} E \int_t^T L_1^p(t,s)|y(s) - y(s)|^p ds dt$$

$$+ C E \int_S^T (T-t)^{\frac{p}{2}} E \int_t^T L_3^p(t,s)|z(s,t) - z(s,t)|^p ds dt$$

$$\leq C(T-S)^{\frac{p}{2}} \sup_{t \in [0,T]} \int_0^t L_1^p(s,t) ds \cdot E \int_S^T |y(s) - y(s)|^p ds$$

$$+ C(T-S)^{\frac{p}{2}} \sup_{t \in [0,T]} \left( \int_0^t L_3^{2p}(s,t) ds \right)^{\frac{2p}{2}} E \int_S^T \left( \int_0^T |z(t,s) - \overline{z}(t,s)|^2 ds \right)^{\frac{p}{2}} dt$$

$$\leq C(T-S)^{\frac{p}{2}} E \int_S^T |y(t) - \overline{y}(t)|^p dt.$$

Then we can choose a constant $\eta = T - S$ so that $C \eta^{\frac{p}{2}} = \frac{1}{2}$. Hence (14) admits a unique fixed point $(Y(\cdot), Z(\cdot, \cdot)) \in M^p[S,T]$ which is the unique adapted $M$-solution of equation (1) over $[S,T]$.

Step 2: We can apply the martingale representation theorem to determine the value of $Z(t,s)$ for $(t,s) \in [S,T] \times [R,S]$ with $0 < R < S$, i.e.,

$$E^{\mathcal{F}_S} Y(t) = E^{\mathcal{F}_R} Y(t) + \int_R^S Z(t,s)dW(s).$$

Step 3: We determine the value of $Z(t,s)$ for $(s,t) \in [S,T] \times [R,S]$ by solving a stochastic Fredholm integral equation, that is,

$$\psi^S(t) = \psi(t) + \int_S^T g^S(t,s,Z(t,s)) ds - \int_S^T Z(t,s)dW(s), \quad (15)$$

for $t \in [R,S]$, where

$$g^S(t,s,z) = g(t,s,Y(s),z,Z(s,t)).$$
with \((t, s, z) \in [R, S] \times [S, T] \times L^p(S, T; L^2_F[R, S])\). From Proposition 2.3, we know that (15) has a unique adapted solution \((\psi^S(\cdot), Z(\cdot, \cdot)) \in L^q_{\mathcal{F}_S}[R, S] \times L^p(R, S; L^2_F[S, T])\) with \(\psi^S(\cdot)\) being \(\mathcal{F}_S\)-measurable.

Step 4: We can complete the unique existence of adapted M-solution by induction. □

Let us consider the following BSVIE,

\[
Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s))ds - \int_t^T Z(t, s)dW(s), \quad t \in [0, T].
\]

When \(\psi(\cdot) = \xi\), Lin [6] studied the adapted solution of (16) and Wang and Zhang [12] considered BSVIE (16) with jump under non-Lipschitz coefficient. We would like to mention the work of results in Hilbert space in [8]. Recently, the authors introduced a notion of S-solution of BSVIE (1) and they considered the unique existence of adapted solution by means of S-solution in [11]. Next we will give a general result in \(\mathcal{H}^p_0[0, T]\), \(p \in (1, 2)\), which generalizes the above results. We have,

**Theorem 3.2** Let (H2) hold, we assume that,

\[
\sup_{t \in [0, T]} \int_0^T L^q_{\mathcal{F}_t}(t, s)ds < \infty, \quad p > 2,
\]

\[
\sup_{t \in [0, T]} \int_0^t L^p_{\mathcal{F}_t}(s, t)ds < \infty, \quad 1 < p < 2,
\]

where \(\frac{1}{p} + \frac{1}{q} = 1\). Then for any \(\psi(\cdot) \in L^p_{\mathcal{F}_T}[0, T]\), (16) admits a unique adapted solution in \(\mathcal{H}^p_0[0, T]\).

**Proof.** We split the proof into several steps.

Step 1: In this step, we will determine the value of \((Y(t), Z(t, s))\) for \((t, s) \in \Delta^c[S, T]\). We consider the following equation: \(t \in [S, T]\),

\[
Y(t) = \psi(t) + \int_t^T g(t, s, y(s), Z(t, s))ds - \int_t^T Z(t, s)dW(s),
\]

for any \(\psi(\cdot) \in L^p_{\mathcal{F}_T}[S, T]\) and \((y(\cdot), z(\cdot, \cdot)) \in \mathcal{H}^p_0[S, T]\). By Proposition 2.2, we know that (17) admits a unique adapted solution \((Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^p_0[S, T]\), and we can define a map \(\Theta : \mathcal{H}^p_0[S, T] \rightarrow \mathcal{H}^p_0[S, T]\) by

\[
\Theta(y(\cdot), z(\cdot, \cdot)) = (Y(\cdot), Z(\cdot, \cdot)), \quad \forall (y(\cdot), z(\cdot, \cdot)) \in \mathcal{H}^p_0[S, T].
\]
Let \((\overline{\gamma}(\cdot), \overline{\pi}(\cdot, \cdot)) \in \mathcal{H}^2_0[S, T]\) and \(\Theta(\overline{\gamma}(\cdot), \overline{\pi}(\cdot, \cdot)) = (\overline{Y}(\cdot), \overline{Z}(\cdot, \cdot))\). By (9) we see that,

\[
E \int_S^T |Y(t) - \overline{Y}(t)|^p dt + E \int_S^T \left( \int_t^T |Z(t, s) - \overline{Z}(t, s)|^2 ds \right)^{\frac{p}{2}} dt \\
\leq CE \int_S^T \left\{ \int_t^T |g(t, s, y(s), Z(t, s)) - g(t, s, \overline{y}(s), Z(t, s))| ds \right\}^p dt \\
\leq CE \int_S^T \left\{ \int_t^T L_1(t, s)|y(s) - \overline{y}(s)| ds \right\}^p dt.
\]

If \(p \in (1, 2)\), we arrive at,

\[
CE \int_S^T \left\{ \int_t^T L_1(t, s)|y(s) - \overline{y}(s)| ds \right\}^p dt \\
\leq C \int_S^T (T - t)^{\frac{p}{2}} E \int_t^T L_1^p(t, s)|y(s) - y(s)|^p ds dt \\
\leq (T - S)^{\frac{p}{2}} C \sup_{t \in [0, T]} \int_t^T L_1^p(s, t) ds E \int_S^T |y(t) - y(t)|^p dt,
\]

and if \(p > 2\),

\[
CE \int_S^T \left\{ \int_t^T L_1(t, s)|y(s) - \overline{y}(s)| ds \right\}^p dt \\
\leq CE \int_S^T \left( \int_t^T L_1^q(t, s) ds \right)^{\frac{p}{q}} \int_t^T |y(s) - \overline{y}(s)|^p ds \\
\leq C(T - S)^{\frac{p}{2}} \sup_{t \in [0, T]} \left( \int_t^T L_1^q(t, s) ds \right)^{\frac{p}{q}} E \int_S^T |y(t) - \overline{y}(t)|^p dt,
\]

where \(\frac{1}{p} + \frac{1}{q} = 1\). Then we can choose a constant \(\eta = T - S\) so that \(C \max\{\eta^\frac{p}{q}, \eta\} = \frac{1}{2}\). Hence (17) admits a unique fixed point \((Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^p_0[S, T]\) which is the unique adapted solution of equation (16) over \([S, T]\).

Step 2: We determine the value of \(Z(t, s)\) for \((s, t) \in [S, T] \times [R, S]\) by solving a stochastic Fredholm integral equation, that is,

\[
\psi^S(t) = \psi(t) + \int_t^T g^S(t, s, Z(t, s)) ds - \int_t^T Z(t, s) dW(s), \tag{18}
\]

for \(t \in [R, S]\), where \(g^S(t, s, z) = g(t, s, Y(s), z)\). From Proposition 2.3, we know that (18) admits a unique adapted solution \((\psi^S(\cdot), Z(\cdot, \cdot)) \in L^p_{\mathcal{F}_S}[R, S] \times L^p(R, S; L^2_0[S, T])\) with \(\psi^S(\cdot)\) being \(\mathcal{F}_S\)-measurable.

Step 3: We can complete the unique existence of adapted solution by induction. \(\square\)
References

[1] A. Aman, M. N’Zi, *Backward stochastic nonlinear Volterra integral equations with local Lipschitz drift*, Probab. Math. Stat. 25 105–127 (2005).

[2] Ph. Briand, R. Carmona, *BSDEs with polynomial growth generators*, J. Appl. Math. Stoch. Anal. 13, (2000), 207–238.

[3] Ph. Briand, B. Delyon, Y. Hu, E. Pardoux and L. Stoica, *Lp solutions of backward stochastic differential equations*, Stoch. Proc. Appl. 108, 2003, 109–129.

[4] N. El Karoui, S. Peng and M. Quenez, *Backward stochastic differential equations in finance*, Math. Finance. 7 (1997) 1–71.

[5] I. Karatzas, S. E. Shreve, *Brownian Motion and stochastic calculus*, Springer, Heidelberg, (1988).

[6] J. Lin, *Adapted solution of backward stochastic nonlinear Volterra integral equation*, Stoch. Anal. Appl. 20 (2002),165–183.

[7] E. Pardoux, S. Peng, *Adapted solution of a backward stochastic differential equation*, Syst. Control. Lett. 14 (1990), 55–61.

[8] Y. Ren, *On solutions of Backward stochastic Volterra integral equations with jumps in hilbert spaces*, J Optim Theory Appl, (2009), DOI 10.1007/s10957-009-9596-2.

[9] E. Rosazza, *Risk measures via g-expectation*, Insurance Mathematics and Economics, 39 (2006), 19–34.

[10] J. Wang, Q. Ran, Q. Chen, *Lp solutions of BSDEs with stochastic Lipschitz condition*, J. Appl. Math. Stoch. Anal. 2007 (2007), DOI:10.1155/2007/78196.

[11] T. Wang, Y. Shi, *Symmetrical solutions of backward stochastic Volterra integral equations and applications*, Discrete Contin. Dyn. Syst. B, submitted.

[12] Z. Wang, X. Zhang, *Non-Lipschitz backward stochastic Volterra integral equations with jumps*, Stoch. Dyn. 7 (2007), 479-496.

[13] J. Yong, *Backward stochastic Volterra integral equations and some related problems*, Stochastic Proc. Appl. 116 (2006), 779–795.

[14] J. Yong, *Continuous-time dynamic risk measures by backward stochastic Volterra integral equations*, Appl. Anal. 86 (2007), 1429–1442.
[15] J. Yong, *Well-posedness and regularity of backward stochastic Volterra integral equation*, Probab. Theory Relat. Fields. 142 (2008), 21-77.