WAVELET COORBIT SPACES VIEWED AS DECOMPOSITION SPACES

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Abstract. In this paper we show that the Fourier transform induces an isomorphism between the coorbit spaces defined by Feichtinger and Gröchenig of the mixed, weighted Lebesgue spaces $L^p_v$ with respect to the quasi-regular representation of a semi-direct product $\mathbb{R}^d \rtimes H$ with suitably chosen dilation group $H$, and certain decomposition spaces $D(Q,L^p_\ell^q)$ (essentially as introduced by Feichtinger and Gröbner) where the localized "parts" of a function are measured in the $FL^p$-norm.

This equivalence is useful in several ways: It provides access to a Fourier-analytic understanding of wavelet coorbit spaces, and it allows to discuss coorbit spaces associated to different dilation groups in a common framework. As an illustration of these points, we include a short discussion of dilation invariance properties of coorbit spaces associated to different types of dilation groups.

1. Introduction

There exist several methods in the literature for the construction of higher-dimensional wavelet systems. A rather general class of constructions follows the initial inception of the continuous wavelet transform in [20] and uses the language of group representations [25, 1, 15, 23]: Picking a suitable matrix group $H \leq \text{GL}(\mathbb{R}^d)$, the so-called dilation group, one defines the associated semidirect product $G = \mathbb{R}^d \rtimes H$. This group acts on $L^2(\mathbb{R}^d)$ via the (unitary) quasi-regular representation $\pi$ defined by

$$(\pi(x,h)f)(y) = |\det(h)|^{-1/2} f(h^{-1}(y-x)), (x,h) \in \mathbb{R}^d \times H.$$ 

The associated continuous wavelet transform of a signal $f \in L^2(\mathbb{R}^d)$ is then obtained by picking a suitable mother wavelet $\psi \in L^2(\mathbb{R}^d)$, and letting

$$W_\psi f : G \to \mathbb{C}, \quad (x,h) \mapsto \langle f, \pi(x,h)\psi \rangle. \quad (1.1)$$

A wavelet $\psi$ is called admissible if the operator $W_\psi$ is (a multiple of) an isometry as a map into $L^2(G, \mu_G)$, where $\mu_G$ denotes a left Haar measure on $G$. By definition we thus have for admissible vectors $\psi$ that

$$\forall f \in L^2(\mathbb{R}^d) : \|f\|_2^2 = \frac{1}{C_\psi} \cdot \int_H \int_{\mathbb{R}^d} |W_\psi f(x,h)|^2 \, dx \, dh|\det(h)|,$$

alternatively expressed in the weak-sense inversion formula

$$f = \frac{1}{C_\psi} \cdot \int_H \int_{\mathbb{R}^d} W_\psi f(x,h) \cdot \pi(x,h)\psi \, dx \, dh|\det(h)|.$$

An alternative, with somewhat less structure but higher design flexibility, is the semi-discrete approach described as follows: Pick a discretely labelled quadratic partition of unity $(\tilde{\psi}_i)_{i \in I}$ in
frequency domain, i.e. a family of functions satisfying

\[ \forall a.e. \xi \in \mathbb{R}^d : \sum_{i \in I} \left| \hat{\psi}_i(\xi) \right|^2 = 1 \]  

(1.2)

and consider the system of all translates of the inverse Fourier transforms \( \psi_i = \mathcal{F}^{-1}(\hat{\psi}_i) \). This system is a (continuously labelled) tight frame, i.e.

\[ \forall f \in L^2(\mathbb{R}^d) : \|f\|_2^2 = \sum_{i \in I} \int_{\mathbb{R}^d} |\langle f, L_x \psi_i \rangle|^2 \, dx, \]

where \( L_x \) denotes translation by \( x \in \mathbb{R}^d \). This norm equality can also be expressed in the weak-sense inversion formula

\[ f = \sum_{i \in I} f \ast \psi_i^* \ast \psi_i, \]

with \( \psi_i^*(x) = \overline{\psi_i(-x)} \). For compactly supported \( \hat{\psi}_i \), the translation variable can be discretized as well, yielding a tight frame, and an associated unconditionally converging frame expansion for all \( f \in L^2(\mathbb{R}^d) \).

First and second generation curvelets \([28, 3]\) are special examples of this type of generalized wavelets, as well as discrete shearlet systems (see \([21, \text{Chapter 1}]\) for an overview). In all these constructions, the desired degree of isotropy, directional selectivity, etc. in the generalized wavelet system is achieved by suitably prescribing the supports of the functions \( \hat{\psi}_i \).

The similarity between the two approaches is best realized by noticing that the admissible functions in the sense of the group-theoretic wavelet transforms are characterized by the condition

\[ \int_H |\hat{\psi}(h^T \xi)|^2 \, dh = C_\psi \]

for almost all \( \xi \in \mathbb{R}^d \), showing that the wavelet inversion formula associated to the continuous wavelet transform is also closely related to a quadratic partition of unity on the Fourier transform side, this time indexed by the dilation group \( H \).

For applications of these transforms, mathematical or otherwise, it is important to realize that each class of generalized wavelet transforms comes with a natural scale of related smoothness spaces, which are defined by norms measuring wavelet coefficient decay. In the group-related case, these are the so-called coorbit spaces introduced by Feichtinger and Gröchenig \([9, 10, 11]\). In the semi-discrete case, it has been realized recently that the decomposition spaces and their associated norms, as introduced by Feichtinger and Gröbner \([8, 7]\), provide a similarly convenient framework for the treatment of approximation-theoretic properties of anisotropic (mostly shearlet-like) wavelet systems, see e.g. \([2, 22]\).

For a long time, the prime examples of coorbit theory were provided by the modulation spaces, arising as coorbit spaces associated to the Schrödinger representation of the Heisenberg group, and the Besov spaces, which are coorbit spaces associated to the quasi-regular representation of the \( ax + b \) group (and their isotropic counterparts in higher dimensions). More recently, the introduction of shearlets (at least the group-theoretic version) triggered the systematic study of the associated coorbit spaces \([4, 5]\); coorbit spaces over the Blaschke group and their connection to complex analysis are discussed in \([12]\). The recent papers \([17, 18]\) pointed out that the study of wavelet coorbit spaces could be considerably extended to cover a multitude of group-theoretically defined wavelet systems in a unified approach that allows to prove the existence of easily constructed, nice wavelet systems and atomic decompositions in a large variety of settings.
However, with the introduction of ever larger classes of function spaces comes the necessity of developing conceptual tools helping to understand these spaces and the relationships between them. It is the chief aim of this paper to provide a bridge between the two types of generalized wavelet systems, by clarifying how wavelet coorbit spaces arising from a group action can be understood as decomposition spaces. There are several motives for this question. The first one is provided by pre-existing results in the literature pointing in this direction: In [9, Section 7.2] it was shown that (homogenous) Besov spaces arise as certain coorbit spaces of weighted, (mixed) Lebesgue spaces with respect to the quasi-regular representation of the $ax + b$ group. On the other hand, these spaces can be defined by localizing the Fourier transform of $f$ using a dyadic partition of the frequency space $\mathbb{R}^d \setminus \{0\}$ and summing the $L^p$-norms of the localized “pieces” in a certain weighted $\ell^q$-space (cf. [19, Definition 6.5.1]).

In this paper we will show that this phenomenon is no coincidence, but merely a manifestation of the general principle that every coorbit space of a (suitably) weighted mixed Lebesgue space with respect to the quasi-regular representation of the semidirect product $\mathbb{R}^d \rtimes H$ (with a closed subgroup $H \leq \text{GL}(\mathbb{R}^d)$) arises as (the inverse image under the Fourier transform of) a certain decomposition space. This means that membership of $f$ in the coorbit space can be decided by localizing the Fourier transform $\hat{f}$ with respect to a certain covering (called the covering induced by $H$) of the dual orbit $\mathcal{O} = H^T \xi_0$ and summing the $L^p$-norms of the individual pieces in a suitable weighted $\ell^q$-space.

Thus, wavelet coorbit theory becomes a branch of decomposition space theory. To some extent this was to be expected, because the structures underlying decomposition spaces – i.e., certain coverings of (subsets of) $\mathbb{R}^d$ and subordinate partitions of unity – are much more flexible than the group structure of the dilation group associated to coorbit spaces. In some sense, passing from the dilation group and its associated scale of coorbit spaces to a suitable covering and its associated scale of decomposition spaces amounts to a loss of structure, as the group is replaced by a suitably chosen index set of a discrete covering. This passage is important from a technical point of view, because by (largely) discarding the dilation group, we become free to discuss coorbit spaces associated to different dilation groups in a common framework. This observation provides a second reason for studying the connection to decomposition spaces.

Possibly the most fundamental motivation for studying this connection is that it allows to discuss the approximation-theoretic properties of a wavelet system in terms of the frequency content of the different wavelets. To elaborate on this point, let us recall the well-understood case of wavelet ONB’s in dimension one: The typical vanishing moment and smoothness conditions on the wavelets can be understood as a measure of frequency concentration. Conceptually speaking, different scales of the wavelet system correspond to different frequency bands, and increasing the degrees of smoothness and vanishing moments amounts to improving the separation between the different frequency bands, which in turn allows larger classes of homogeneous Besov spaces to be characterized in terms of the wavelet coefficients with respect to a single wavelet ONB. The papers [17, 18] extend this type of reasoning to (possibly anisotropic) higher dimensional wavelet systems and their associated coorbit spaces; here the key concept was provided by the dual action and in particular the so-called “blind spot” of the wavelet transform.

However, in the study of wavelet systems in higher dimensions, the description of frequency content poses an increasingly difficult challenge: Different wavelet systems can be understood as prescribing different ways of partitioning the frequency space into (possibly oriented) “frequency bands”; we argue that their approximation-theoretic properties are describable in terms of this behaviour. It is important to note that precisely this intuition was also used in the inception
of curvelets by Candès and Donoho \cite{Candès/Donoho}, and the results of that paper provide further evidence that the frequency partition determines the approximation-theoretic properties. However, what is needed to systematically turn this intuition into provable theorems is a suitable language describing these partitions, and allowing to assess which properties of a partition are relevant for the approximation-theoretic properties of the corresponding wavelet systems. Our paper makes a strong case that this language is provided by the decomposition spaces introduced by Feichtinger and Gröbner in \cite{Feichtinger/Grobner}, and studied more recently in \cite{Dahlke/Herrmann/Koldobsky/Pelzer/Samorodnitsky}.

To illustrate these points, we have included a discussion of dilation invariance properties of certain coorbit spaces in Section 9. Given a suitable coorbit space $\mathcal{C}_Y$ associated to a dilation group $H$, we would like to identify those invertible matrices $g$ such that $\mathcal{C}_Y$ is invariant under dilation by $g$. It is clear that the set of these matrices contains $H$: this follows from the fact that the wavelet transform intertwines (a suitably normalized) dilation by $g \in H$ with left translation by $(0, g) \in G$, and from left invariance of the Banach function space $Y$ entering the definition of the coorbit space. It is much less clear whether there are further invertible matrices $g \not\in H$ which leave $\mathcal{C}_Y$ invariant. It will be seen in Section 9 that this property depends on $H$: If $H$ is the similitude group in dimension two and the associated coorbit spaces are the isotropic Besov spaces, they are in fact invariant under arbitrary dilations. By contrast, there are shearlet coorbit spaces that are not invariant under dilation by a ninety degree rotation.

While these observations are of some independent interest (for example, the lack of rotation invariance for shearlet coorbit spaces seems to be a new observation), we have included this discussion mostly because of the way it highlights the role of the decomposition space viewpoint in understanding the different coorbit spaces. It also illustrates the importance of being able to compare coorbit spaces associated to different dilation groups: One quickly realizes that dilation by $g$ is an isomorphism $\mathcal{C}_Y \rightarrow \mathcal{C}_Y'$, where $Y'$ is a suitably chosen Banach function space over the group $G' = \mathbb{R}^d \rtimes g^{-1} H g$. Thus invariance of $\mathcal{C}_Y$ under dilation by $g$ is equivalent to an embedding statement $\mathcal{C}_Y \hookrightarrow \mathcal{C}_Y'$.

2. Notation and Preliminaries

In this paper we will always be working in the following setting: We assume that $H \leq \text{GL}(\mathbb{R}^d)$ is a closed subgroup for some $d \in \mathbb{N}$ and we consider the semidirect product $G = \mathbb{R}^d \rtimes H$ with multiplication $(x, h)(y, g) = (x + hy, hg)$. For the convenience of the reader we recall that a (left) Haar integral on the locally compact group $G$ is then given by

$$f \mapsto \int_G f(x, h) \, d(x, h) := \int_H \int_{\mathbb{R}^d} f(x, h) \, dx \frac{dh}{|\text{det}(h)|}, \quad (2.1)$$

where $dh$ denotes integration against left Haar measure on $H$. The modular function on $G$ is given by

$$\Delta_G(x, h) = \Delta_H(h) \cdot |\text{det}(h)|^{-1}. \quad (2.2)$$

We then consider the so-called quasi-regular representation $\pi$ of $G$ acting unitarily on $L^2(\mathbb{R}^d)$ by

$$\pi(x, h)f = L_x \Delta_h f = |\text{det}(h)|^{-1/2} \cdot L_x D_{h^{-1}} f, \quad (2.3)$$
where we use the operators \( L_x, \Delta_h, D_h \) (and later on also) \( M_\omega \) defined by
\[
\begin{align*}
(L_x f)(y) &= f(y - x), \\
(\Delta_h f)(y) &= |\det (h)|^{-1/2} \cdot f(h^{-1} y), \\
(D_h f)(y) &= f(h^T y), \\
(M_\omega f)(y) &= e^{2\pi i(y, \omega)} \cdot f(y)
\end{align*}
\]
for \( h \in \text{GL}(\mathbb{R}^d) \), \( x, y, \omega \in \mathbb{R}^d \) and \( f : \mathbb{R}^d \to \mathbb{C} \).

We use the following version of the Fourier transform:
\[
(\mathcal{F} f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) \cdot e^{-2\pi i(x, \xi)} \, dx \quad \text{for} \quad \xi \in \mathbb{R}^d,
\]
and consequently
\[
(\mathcal{F}^{-1} f)(x) = \int_{\mathbb{R}^d} f(\xi) \cdot e^{2\pi i(x, \xi)} \, d\xi \quad \text{for} \quad x \in \mathbb{R}^d
\]
for the (inverse) Fourier transform of \( f \in L^1(\mathbb{R}^d) \).

Using this convention, we note that on the Fourier side the quasi-regular representation is given by
\[
\mathcal{F} (\pi(x, h) f) = |\det (h)|^{-1/2} \cdot \mathcal{F} (L_x (f \circ h^{-1})),
\]
\[
= |\det (h)|^{-1/2} \cdot M_{-x} (\mathcal{F} (f \circ h^{-1})))
\]
\[
= |\det (h)|^{1/2} \cdot M_{-x} D_h \hat{f}.
\]

The results in [16 [15] show that the quasi-regular representation is irreducible and square-integrable (in short: admissible), if and only if the following conditions hold:

1. There is a \( \xi_0 \in \mathbb{R}^d \) such that the dual orbit \( \mathcal{O} := H^T \xi_0 = \{ h^T \xi_0 \mid h \in H \} \subseteq \mathbb{R}^d \) is an open set of full measure (i.e. \( \lambda(\mathcal{O}^c) = 0 \), where \( \lambda \) denotes Lebesgue measure on \( \mathbb{R}^d \)) and
2. the isotropy group \( H_{\xi_0} := \{ h \in H \mid h^T \xi_0 = \xi_0 \} \) of \( \xi_0 \) with respect to the dual action of \( H \) is compact. In this case, the isotropy group \( H_{h^T \xi_0} = h^{-1} H_{\xi_0} h \) is a compact subgroup of \( H \) for every \( h \in \mathbb{R}^d \).

In the following, we will always assume that these conditions are met. We will then see below (cf. Theorem [3]) that \( \pi \) is indeed an integrable representation, i.e. there exists \( \psi \in L^2(\mathbb{R}^d) \setminus \{0\} \) with \( W_\psi \psi \in L^1(G) \), where the Wavelet transform \( W_g f \) of \( f \) with respect to \( g \) is defined by
\[
W_g f : G \to \mathbb{C}, (x, h) \mapsto \langle f, \pi(x, h) g \rangle_{L^2(\mathbb{R}^d)}
\]
for \( f, g \in L^2(\mathbb{R}^d) \). It should be noted that this definition of the Wavelet transform coincides with the voice transform \( V_g f \) as defined in [10], because
\[
(V_g f)(x, h) = \langle \pi(x, h) g, f \rangle_{\text{anti}} = \langle f, \pi(x, h) g \rangle_{L^2(\mathbb{R}^d)},
\]
as Feichtinger uses a scalar-product that is antilinear in the first component, i.e.
\[
\langle f, g \rangle_{\text{anti}} = \int_{\mathbb{R}^d} f(x) \cdot g(x) \, dx,
\]
whereas we adopt the convention that the scalar-product \( \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d)} \) is antilinear in the second component.

The paper is organized as follows: In section 3 we clarify the exact definitions of the mixed Lebesgue spaces \( L^{p,q}_v(G) \) and the requirements on the weight \( v : G \to (0, \infty) \) for which we
will later prove the isomorphism of $\text{Co}(L^p_{v,q})$ to a suitable decomposition space. Furthermore, we show that the coorbit theory is indeed applicable in this setting. The only point for the applicability of coorbit theory that we do not cover in section 3 is the existence of analyzing vectors.

This gap is closed in the ensuing section 4 in which we recall the most important definitions from coorbit theory and show that any Schwartz function $\psi$ whose Fourier transform is compactly supported in the dual orbit $O$ is admissible as a so-called analyzing vector (even as a “better vector”). Furthermore, we show that the “reservoir” $(H^1_w)$ from which the elements of the coorbit space are taken can naturally be identified with a subspace of the space of distributions $\mathcal{D}'(O)$ on the dual orbit $O = H^T \xi_0$ determined by $H$, as well as with a subspace of $(\mathcal{F}(\mathcal{D}(O)))'$, where we use the notation $\mathcal{D}(U) = C^\infty_c(U)$ for the space of smooth functions with compact support in the open set $U \subset \mathbb{R}^d$. This notation will also be used in the remainder of the paper.

In section 5 we define the concept of an “induced covering” $Q$ of the dual orbit $O = H^T \xi_0$ and we give a precise definition of the decomposition spaces $\mathcal{D}(Q, L^p, \ell^q_w)$ for which we will later show that the Fourier transform induces an isomorphism $\mathcal{F} : \text{Co}(L^p_{v,q}) \rightarrow \mathcal{D}(Q, L^p, \ell^q_w)$. Furthermore, we recall the essential definitions for the theory of decomposition spaces (i.e. the concepts of admissible coverings, BAPUs, etc.) and show that the induced covering $Q$ is a structured admissible covering of $O$ (cf. Definition 13).

In section 6 we construct a specific partition of unity subordinate to the induced covering $Q$. In principle, one could use any partition of unity $(\varphi_i)_{i \in I}$ subordinate to $Q$ for which $\|\mathcal{F}^{-1}\varphi_i\|_{L^1(\mathbb{R}^d)}$ is uniformly bounded, but our construction has the advantage that the localizations $\mathcal{F}^{-1}(\varphi_i \cdot f)$ can be explicitly expressed in terms of the Wavelet transform $W_\varphi f$ of $f$.

This explicit formula will be prominently exploited in section 7 where we prove that the Fourier transform extends to a bounded linear map $\mathcal{F} : \text{Co}(L^p_{v,q}) \rightarrow \mathcal{D}(Q, L^p, \ell^q_w)$ for a suitable weight $\nu : \mathcal{O} \rightarrow (0, \infty)$.

In section 8 we show that the inverse Fourier transform $\mathcal{F}^{-1} : \mathcal{D}(Q, L^p, \ell^q_w) \rightarrow \text{Co}(L^p_{v,q})$ is continuous. In this section we also show that instead of the reservoir $(H^1_w)^{-1}$ for the elements of the coorbit space, one can use the more invariant reservoir $(\mathcal{F}(\mathcal{D}(O)))'$.

In section 9 we apply the established isomorphism between the coorbit space $\text{Co}(L^p_{v,q})$ and the decomposition space $\mathcal{D}(Q, L^p, \ell^q_w)$ to investigate the invariance of certain specific coorbit spaces under conjugation of the group. Using the decomposition space view, we show that all coorbit spaces $\text{Co}(L^p_{v,q})$ with respect to the similitude group are invariant under conjugation, whereas the same is in general not true for the coorbit spaces with respect to the shearlet group.

We close the technical preliminaries by noting that while the most important results of the present paper are Theorems 14 and 15 which establish the continuity of the (inverse) Fourier transform as a map from the coorbit space into the associated decomposition space (and vice versa), the most important parts of the paper in terms of ideas for the proof are the definition of the specific partition of unity (cf. equation (6.1)) and the calculation of the localization $\mathcal{F}^{-1}(\varphi \cdot f)$ in terms of the wavelet transform $W_\varphi f$ (cf. Lemma 16), as well as Lemma 14 where we show $\|W_\varphi (f \circ F^{-1})\|_{L^p_{v,q}} \leq C \cdot \|f\|_{\mathcal{D}(Q, L^p, \ell^q_w)}$ for $f \in \mathcal{D}(Q, L^p, \ell^q_w)$.

3. Applicability of Coorbit-theory for the spaces $L^p_{v,q}$

As the quasi-regular representation is irreducible and square-integrable by our standing assumptions, the main requirement for the coorbit-theory as developed by Feichtinger and
Gröchenig in [10, 11] is fulfilled. In the remainder of this paper, we will routinely abuse notation by identifying weights \( v : H \to (0, \infty) \) with their trivial extension \( G \to (0, \infty) \), \((x, h) \mapsto v(h)\). We will exclusively consider the coorbit spaces \( \text{Co}(L^p_w) \) where \( v \) only depends on the second factor.

Nevertheless, we will sometimes have occasion to consider the Banach function space \( L^p_w(G) \), where \( w : G \to (0, \infty) \) is arbitrary measurable (but we will not consider the coorbit space \( \text{Co}(L^p_w) \)) in this case. This space is defined by

\[
L^p_w(G) := \{ f : G \to \mathbb{C} \mid f \text{ measurable and } \| f \|_{L^p_w} < \infty \},
\]

with

\[
\| f \|_{L^p_w} := \left( \int_H \left( \| w(\cdot, h) \cdot f(\cdot, h) \|_{L^p(\mathbb{R}^d)} \right)^q \frac{dh}{|\det(h)|} \right)^{1/q}
\]

for \( p \in [1, \infty] \) and \( q \in [1, \infty) \) and with

\[
\| f \|_{L^p_w} := \sup_{h \in H} \left( \| w(\cdot, h) \cdot f(\cdot, h) \|_{L^p(\mathbb{R}^d)} \right).
\]

The weight \( v : H \to (0, \infty) \) need not be submultiplicative itself, but we will assume that \( v \) is \( v_0 \)-moderate for a (measurable, locally bounded) submultiplicative weight \( v_0 : H \to (0, \infty) \), i.e. we assume

\[
v(ghk) \leq v_0(g) v(h) v_0(k) \quad \forall g, h, k \in H.
\]

In this case, \( L^p_w(G) \) is invariant under left- and right translations. More precisely, we have the following:

**Lemma 1.** \( L^p_w(G) \) is invariant under left- and right translations and we have the estimates

\[
\| L_{(x, h)} \|_{L^p_w \to L^p_w} \leq v_0(1_H) \cdot v_0(h) \cdot |\det(h)|^{\frac{1}{q} - \frac{1}{p}}
\]

and

\[
\| R_{(x, h)} \|_{L^p_w \to L^p_w} \leq v_0(1_H) \cdot v_0(h^{-1}) \cdot |\det(h)|^{\frac{1}{q}} \cdot (\Delta_H(h))^{-1/q}
\]

for all \((x, h) \in G\).

**Proof.** Let \((x, h) \in G\) and \( f \in L^p_w(G) \). For \((y, g) \in G\) we have

\[
\left( L_{(x, h)}^{-1} f \right)(y, g) = f((x, h)(y, g)) = f(x + hy, hg),
\]

\[
\left( R_{(x, h)} f \right)(y, g) = f((y, g)(x, h)) = f(y + gx, gh).
\]

We first consider the left translation. For \( g \in H \) we get, using the above formula

\[
\| \left( L_{(x, h)}^{-1} f \right)(\cdot, g) \|_{L^p(\mathbb{R}^d)} = \| f(x + h \cdot, hg) \|_{L^p(\mathbb{R}^d)} = |\det(h)|^{-1/p} \cdot \| f(\cdot, hg) \|_{L^p(\mathbb{R}^d)},
\]

as can be seen using the change-of-variables formula for \( p \in [1, \infty) \) and for \( p = \infty \) using the fact that \( \mathbb{R}^d \to \mathbb{R}^d, z \mapsto x + hz \) and its inverse map both map null-sets to null-sets.
Thus, we arrive at

\[
\frac{v(g)}{|\det(g)|^{1/q}} \cdot \left\| (L_{x,h}^{-1} f) (\cdot, g) \right\|_{L^p(\mathbb{R}^d)} \leq v_0(h^{-1}) v_0(1_H) |\det(h)|^{1/q} \cdot \frac{v(hg)}{|\det(hg)|^{1/q}} \cdot \left\| f (\cdot, hg) \right\|_{L^p(\mathbb{R}^d)}.
\]

Using the (isometric) invariance of \( \| \cdot \|_{L^q(H)} \) under left-translations, we obtain

\[
\left\| L_{x,h}^{-1} f \right\|_{L^q(H)} = \left\| g \mapsto \frac{v(g)}{|\det(g)|^{1/q}} \cdot \left\| (L_{x,h}^{-1} f) (\cdot, g) \right\|_{L^p(\mathbb{R}^d)} \right\|_{L^q(H)} = v_0(h^{-1}) v_0(1_H) |\det(h)|^{1/q} \cdot \left\| f (\cdot, hg) \right\|_{L^p(\mathbb{R}^d)} < \infty.
\]

Applying this to \((x, h)^{-1} = (-h^{-1}x, h^{-1})\) instead of \((x, h)\), we finally see

\[
\left\| L_{(x,h)} f \right\|_{L^q(H)} = \left\| L_{(-h^{-1}x,h^{-1})}^{-1} f \right\|_{L^q(H)} \leq v_0(h) \cdot v_0(1_H) \cdot |\det(h)|^{1/q} \cdot \left\| f \right\|_{L^q(H)}.
\]

We now turn to the right translations. Using the translation-invariance of \( \| \cdot \|_{L^p(\mathbb{R}^d)} \), we derive

\[
\left\| (R_{(x,h)} f) (\cdot, g) \right\|_{L^p(\mathbb{R}^d)} = \left\| f (\cdot + gx, gh) \right\|_{L^p(\mathbb{R}^d)} = \left\| f (\cdot, gh) \right\|_{L^p(\mathbb{R}^d)}.
\]

This implies

\[
\frac{v(g)}{|\det(g)|^{1/q}} \cdot \left\| (R_{(x,h)} f) (\cdot, g) \right\|_{L^p(\mathbb{R}^d)} = \frac{v(1_H gh h^{-1})}{|\det(gh h^{-1})|^{1/q}} \cdot \left\| f (\cdot, gh) \right\|_{L^p(\mathbb{R}^d)} \leq v_0(1_H) \cdot v_0(h^{-1}) \cdot |\det(h)|^{1/q} \cdot \frac{v(gh)}{|\det(gh)|^{1/q}} \cdot \left\| f (\cdot, gh) \right\|_{L^p(\mathbb{R}^d)}.
\]

Now, for \( q \in [1, \infty) \) and (measurable) \( \psi : H \rightarrow \mathbb{C} \), formula (2.26) of [13] implies

\[
\left\| \psi \right\|_{L^q(H)} = \left( \Delta_H (h) \right)^{-1/q} \cdot \| \psi \|_{L^q(H)}.
\]

The same is true for \( q = \infty \), as right-translations map (left) null-sets to (left) null-sets.
Therefore, we arrive at
\[
\| R_{(x,h)} f \|_{L^p_w} = \left\| g \mapsto \left\| \frac{v(g)}{|\det (g)|^{1/q}} \cdot \left\| (R_{(x,h)} f) \left( \cdot , g \right) \right\|_{L^p(R^d)} \right\|_{L^p(H)} \\
\leq v_0 (1_H) v_0 (h^{-1}) \cdot |\det (h)|^{1/q} \cdot \left\| g \mapsto \frac{v(gh)}{|\det (gh)|^{1/q}} \cdot \left\| (f \cdot gh) \right\|_{L^p(R^d)} \right\|_{L^p(H)} \\
= v_0 (1_H) v_0 (h^{-1}) \cdot |\det (h)|^{1/q} \cdot (\Delta_H (h))^{-1/q} \cdot \| f \|_{L^p_w}.
\]

Using the \( v_0 \)-moderateness of \( v \) and boundedness of \( v_0 \) on compact sets (from below and above) one can easily establish the same properties for \( v \). These properties (as stated in the next lemma) will be frequently used in the rest of the paper.

**Lemma 2.** Let \( K \subset H \) be compact. There is a constant \( C = C(K, v_0) > 0 \) such that
\[
\frac{v(g)}{v(h)} \leq C \quad \text{holds for all } g, h \in K.
\]

Furthermore, there are \( \alpha, \beta \in (0, \infty) \) (only dependent on \( v_0 \) and \( K \)) with
\[
\alpha \leq v(h) \leq \beta \quad \text{for all } h \in K.
\]

Additionally, we note some easy closure-properties of submultiplicative functions that will be used below:

**Lemma 3.** Let \( w_1, w_2 : G \to (0, \infty) \) be submultiplicative. Then the same holds for \( w_1 \cdot w_2 \) and \( \max \{w_1, w_2\} \) as well as for \( w_1^\vee : G \to (0, \infty), x \mapsto w_1 (x^{-1}) \).

With these preparations, we can now show that the space \( L_{v, w}^p (G) \) satisfies all requirements of coorbit theory (with the exception of the requirement \( A_w \neq \{0\} \neq B_w \) which we will establish in Theorem 9 below).

**Lemma 4.** For \( u : H \to (0, \infty) \) we set
\[
\hat{u} : H \to (0, \infty), h \mapsto \max \{u(h), u(h^{-1})\}.
\]

Let
\[
w : H \to (0, \infty), h \mapsto v_0 (1_H) \cdot v_0^\vee (h) \cdot (\Delta_H^- (h)) \cdot |\det (\cdot)|^+ (h) \cdot (\Delta_H^- (h)).
\]

Then \( L_{v, w}^p (G) \) is a solid Banach function space, \( w \) is a (locally bounded, measurable) submultiplicative weight that satisfies
\[
\max \left\{ \left\| R_{(x,h)} \right\|, \left\| R_{(x,h)}^{-1} \right\|, \left\| R_{(x,h)}^{-1} \right\|, \left\| R_{(x,h)} \right\| \cdot \left\| \Delta_G ((x,h)^{-1}) \right\| \right\} \leq w(h)
\]
for all \((x,h) \in G\), where we have written \( \|T\| := \|T\|_{L_{v, w}^p \to L_{v, w}^p} \).

Interpreting \( w \) as a submultiplicative weight on \( G \), we have
\[
\| f \ast g \|_{L_{v, w}^p} \leq \| f \|_{L_{v, w}^p} \cdot \| g \|_{L_{v, w}^p} \quad \text{for all } f \in L_{v, w}^p (G) \text{ and } g \in (L_{v, w}^1 (G))^\vee,
\]
where we used the notation \( g^\vee : G \to \mathbb{C}, x \mapsto g(x^{-1}) \).

**Remark.** In summary, this shows that \( w \) is a suitable control weight for \( L_{v, w}^p (G) \) in the sense of [10] (cf. [10] equations (3.1) and (4.10)) and note that any weight dominating a control weight is again an admissible control weight by [10] Theorem 4.2(iii)).
Proof. We first note that $v_0 (1_H) = v_0 (1_H 1_H) \leq v_0 (1_H) \cdot v_0 (1_H)$ implies that the constant map $h \mapsto v_0 (1_H)$ is submultiplicative. Now Lemma 3 easily shows (with the multiplicativity of $|\det (\cdot)|$ and $\Delta_H$) that $w$ is submultiplicative. As $|\det (\cdot)|$ and $\Delta_H$ are continuous and $v_0$ is locally bounded and measurable, the same is true of $w$.

We first prove inequality (3.1). To this end, we notice that for $\alpha \in (0, \infty)$ and $r \in [-1, 1]$ the inequality $\max \{\alpha, \alpha^{-1}\} \geq 1$ yields the estimate

$$\alpha^r \leq \max \{\alpha, \alpha^{-1}\}, \quad (3.3)$$

In the following we will apply this for $\alpha = |\det (h)|$ or $\alpha = \Delta_H (h)$.

Lemma 1 together with $-\frac{1}{q} \leq - \frac{1}{p} - \frac{1}{q} \leq \frac{1}{p} \leq 1$ shows

$$\|L(x,h)\|_{L^p \to L^q} \leq v_0 (h) \cdot v_0 (1_H) \cdot |\det (h)|^{\frac{1}{p} - \frac{1}{q}}$$

$$\geq v_0 (1_H) \cdot \max \{v_0 (h), v_0 (h^{-1})\} \cdot \max \{|\det (h)|, |\det (h^{-1})|\}$$

$$\leq w (h),$$

where we used $\max \{\Delta_H (h), \Delta_H (h^{-1})\} \geq 1$.

In the same way, equation (3.1) and Lemma 1 imply $\|R(x,h)\|_{L^p \to L^q} \leq w (x, h)$. By symmetry of $w$, we also get $\|L(x,h)^{-1}\|_{L^p \to L^q} \leq w ((x, h)^{-1}) = w (x, h)$. Finally, by Lemma 1 and because of $1 - \frac{1}{q} \in [-1, 1]$, we arrive at

$$\|R(x,h)^{-1}\|_{L^p \to L^q} \cdot \Delta_G ((x, h)^{-1})$$

$$\leq v_0 (1_H) \cdot v_0 (h) \cdot |\det (h)|^{\frac{1}{q}} \cdot (\Delta_H (h))^{1/q} \cdot (\Delta_G (x, h))^{-1}$$

$$= v_0 (1_H) \cdot v_0 (h) \cdot |\det (h)|^{\frac{1}{q}} \cdot (\Delta_H (h))^{\frac{1}{q}} \cdot (\Delta_G (x, h))^{-1} \cdot |\det (h)|$$

$$\leq w (h).$$

The Banach function space properties of $L^p \, \nu$ are routinely checked. Finally, we establish the convolution relation (3.2). Here we first observe the identity

$$F(x,h) := \int_G f (y, k) \cdot g \left((y, k)^{-1} (x, h)\right) \, d (y, k)$$

$$= \int_G |f ((x, h) (y, k)) \cdot g' (y, k)| \, d (y, k) \in [0, \infty)$$

which is valid by left invariance. Now Minkowski’s inequality for integrals (cf. [14] Theorem 6.19) with

$$d \nu := \frac{v(h)}{|\det (h)|^{1/q}} \cdot dh$$
yields (together with the solidity of $L^q(\nu)$) the estimate

$$\|F\|_{L^p(v)} \leq \left\|h \mapsto \left( x \mapsto \int_G |f((x,h)(y,k)) \cdot g^\vee(y,k)| \, d(y,k) \right) \right\|_{L^p(\nu)}$$

In particular, we conclude $F(x,h) < \infty$ for almost every (depending on $h$) $x \in \mathbb{R}^d$ for $\nu$-almost every $h \in H$. Since we have $\frac{1}{|\det(h)|^{1/2}} > 0$ for all $h \in H$ and because $F$ is measurable (which is implied by Fubini’s theorem), we see $F(x,h) < \infty$ for almost every $(x,h) \in G$. Thus, the convolution-defining integral

$$(f * g)(x,h) = \int_G f(y,k) \cdot g\left((y,k)^{-1}(x,h)\right) \, d(y,k)$$

converges absolutely for almost every $(x,h) \in G$ with $|(f * g)(x,h)| \leq F(x,h)$. By solidity of $L^p(v)$, this implies $f * g \in L^p(v)(G)$ and

$$\|f \|_{L^p(v)(G)} \cdot \|g\|^\vee_{L^p(v)(G)} \leq \|F\|_{L^p(v)} \leq \|f\|_{L^p(v)} \cdot \|g\|^\vee_{L^p(v)} < \infty.$$

(3.4)

By solidity of $L^p(v)$, this implies $f * g \in L^p(v)(G)$ and

$$\|f \|_{L^p(v)(G)} \cdot \|g\|^\vee_{L^p(v)(G)} \leq \|F\|_{L^p(v)} \leq \|f\|_{L^p(v)} \cdot \|g\|^\vee_{L^p(v)} < \infty.$$

4. Admissibility of $F^{-1}(\mathcal{D}(\Omega))$ as analyzing vectors and identification of $(\mathcal{H}_w^1)^{\sim}$ with a subspace of $\mathcal{D}'(\Omega)$

In this section we show that any Schwartz function $\psi \in \mathcal{S}(\mathbb{R}^d)$ whose Fourier transform $\hat{\psi}$ is compactly supported in the dual orbit $\Omega$ is admissible as an analyzing vector. This will also allow us to identify the “reservoir” $(\mathcal{H}_w^1)^{\sim}$ that is used in the definition of coorbit spaces with (a subspace of) the space of distributions $\mathcal{D}'(\Omega)$ on the dual orbit $\Omega$ as well as with a subspace of $(\mathcal{F}(\mathcal{D}(\Omega)))'$. Before we go into the details of the proof, we recall some important definitions related to coorbit theory. First of all we recall the definition of the set of **analyzing vectors**

$$\mathcal{A}_w := \{ \psi \in L^2(\mathbb{R}^d) \mid W\psi \in L^1_w(G) \}$$

and of the set of **better vectors**

$$\mathcal{B}_w := \{ \psi \in L^2(\mathbb{R}^d) \mid W\psi \in W^R(L^\infty,L^1_w) \}$$

from [10] pages 317 and 321].

Here, we use the notion of the so-called **Wiener amalgam space** $W^R(L^\infty,Y)$ for a solid Banach function space $Y$ (cf. [10] pages 312 and 315]). For the definition of this space, let
$U \subset G$ be an open, precompact unit-neighborhood. For $f : G \to \mathbb{C}$ we define the (right sided) control function $K_U f$ of $f$ with respect to $U$ by

$$K_U f : G \to [0, \infty), \ x \mapsto \|\chi_U x f\|_{L^\infty(G)}.$$  \hspace{1cm} (4.1)

The (right sided) Wiener amalgam space with local component $L^\infty$ and global component $Y$ is then defined by

$$W^R (L^\infty, Y) := \{ f \in L^\infty_{\text{loc}}(G) \mid K_U f \in Y \}$$

with norm $\|f\|_{W^R (L^\infty, Y)} := \|K_U f\|_Y$. Here one should note that (as long as $G$ is first countable) $K_U f$ is a lower semicontinuous (and hence measurable) function. Then $W^R (L^\infty, Y)$ is a Banach space that is independent of the actual choice of $U$ and is continuously embedded in $Y$. These properties are shown in [26, Lemma 2.1 and Lemma 2.2 together with Theorem 2.3] for the left sided Wiener amalgam space

$$W (L^\infty, Y) = \{ f \in L^\infty_{\text{loc}}(G) \mid K'_U f \in Y \},$$

where the (left sided) control function $K'_U f$ of $f$ with respect to $U$ is defined by

$$K'_U f : G \to [0, \infty), \ x \mapsto \|\chi_U x f\|_{L^\infty(G)}.$$  \hspace{1cm} (4.2)

Note that we have

$$(K_U f)(x) = \|\chi_{Ux^{-1}} \cdot f^\vee\|_{L^\infty(G)} = (K'_U (f^\vee))(x^{-1}) = \|K'(U^{-1} f^\vee)\|_Y.$$

and hence

$$\|f\|_{W^R (L^\infty, Y)} = \|(K'_U (f^\vee))\|_Y = \|K'_U (f^\vee)\|_Y = \|f^\vee\|_{W (L^\infty, Y^\vee)} = \|f\|_{W (L^\infty, Y^\vee)}.$$

Thus, one can easily derive the analogous properties for the right sided amalgam spaces.

We mention that in [10], Feichtinger uses continuous functions $k \in C_c(G)$ as a cut-off for localization instead of the simple characteristic function $\chi_U$ that we use. As we use $L^\infty(G)$ as the local component, this makes no difference.

Below, we will show that any Schwartz function $\psi$ with Fourier transform $\hat{\psi} \in \mathcal{D}(\mathcal{O})$ already satisfies $\psi \in \mathcal{B}_w \subset \mathcal{A}_w$ for every (locally bounded, submultiplicative) weight $w : G \to (0, \infty)$ that only depends on the second component, i.e. which satisfies $w(x, h) = w(h)$ for $(x, h) \in G$.

This shows in particular that $\mathcal{B}_w$ is nontrivial, which closes the gap for the applicability of coorbit theory that was left open in section 3 (cf. Lemma [1]).

Moreover, we show that the map

$$\mathcal{F}^{-1} : \mathcal{D}(\mathcal{O}) \to \mathcal{H}_w^1, g \mapsto \mathcal{F}^{-1} g$$

is well-defined and continuous, where $\mathcal{H}_w^1$ is defined by

$$\mathcal{H}_w^1 := \{ f \in L^2(\mathbb{R}^d) \mid W_{\psi} f \in L^1_w(G) \}$$

for some fixed analyzing vector $\psi \in \mathcal{A}_w \setminus \{0\}$ with norm $\|f\|_{\mathcal{H}_w^1} := \|W_{\psi} f\|_{L^1_w(G)}$, cf. [10] page 317. This will allow us to show that for $f \in (\mathcal{H}_w^1)^\vee$ (where $(\mathcal{H}_w^1)^\vee$ denotes the space of bounded, antilinear functionals on $\mathcal{H}_w^1$) the map

$$\mathcal{F} f : \mathcal{D}(\mathcal{O}) \to \mathbb{C}, g \mapsto f(\mathcal{F}^{-1} g)$$  \hspace{1cm} (4.3)

is well-defined, linear and continuous, i.e. an element of the space of distributions $\mathcal{D}'(\mathcal{O})$. This definition of $\mathcal{F} f$ may seem awkward at first, but it is natural; see Remark [4] below.
Recall that the coorbit space \( \text{Co} (L^p_w) \) is defined by

\[
\text{Co} (L^p_w) = \left\{ f \in (H^1_w)^\sim \mid W_\psi f \in L^p_w (G) \right\}
\]

for some fixed \( \psi \in A_w \setminus \{0\} \) and a control weight \( w : G \to (0, \infty) \) for \( L^p_w (G) \). Thus, the map

\[
\mathcal{F} : \text{Co} (L^p_w) \to \mathcal{D}' (O), \ f \mapsto \mathcal{F} f \text{ with } \mathcal{F} f \text{ as defined in equation (1.7)}
\]

is well-defined. In section 7 we will show that it is indeed well-defined and bounded as a map into the decomposition space \( \mathcal{D} (Q, L^p_w, \ell^q_u) \), where \( Q \) is a suitable covering of \( O \) induced by \( H \) and where \( w : O \to (0, \infty) \) is a suitably chosen weight.

**Remark.** The definition of \( \mathcal{F} f \) in equation (1.8) (denoted as \( \mathcal{F} (H^1_w)^\sim f \) in this remark to distinguish it from the “ordinary” Fourier transform) is natural, because we have the embedding

\[
\mathcal{H}^1_w = \left\{ f \in L^2 (\mathbb{R}^d) \mid W_\psi f \in L^1_w (G) \right\}
\]

\[
\overset{\text{Eq. (4.3)}}{=} \quad \overset{\text{Eq. (4.3)}}{=} \quad \overset{\text{Eq. (4.3)}}{=} \quad \overset{\text{Eq. (4.3)}}{=}
\]

where \( (\cdot, \cdot)_{S', S} \) denotes the bilinear(!) pairing between \( S' (\mathbb{R}^d) \) and \( S (\mathbb{R}^d) \).

This shows that \( \mathcal{F} (H^1_w)^\sim \) and the “ordinary” Fourier transform agree on \( L^2 (\mathbb{R}^d) \subset (H^1_w)^\sim \).

In order to show that every Schwartz function \( \psi \) whose Fourier transform is compactly supported in \( O \) is admissible as an analyzing vector, we will need the (well known) fact that the orbit map

\[
p_{\xi_0} : H \to O, h \mapsto h^T \xi_0
\]

is a proper map, i.e. for \( K \subset O \) compact the preimage \( p_{\xi_0}^{-1} (K) \subset H \) is also compact. We will see that this is a consequence of our admissibility assumptions on \( H \), more precisely of the compactness of the isotropy group \( H_{\xi_0} \leq H \) and of the fact that the orbit \( O \) is an open subset of \( \mathbb{R}^d \). This is the first point (apart from the applicability of coorbit theory), where we actually use these assumptions.

**Lemma 6.** For a compact set \( K \subset O \) the inverse image \( p_{\xi_0}^{-1} (K) \subset H \) is also compact.

**Proof.** By the closed subgroup theorem, \( H \) is a Lie group. As a second countable, locally compact space it admits an exhaustion by precompact open sets, i.e. \( H = \bigcup_{n \in \mathbb{N}} U_n \), where the \( U_n \subset H \) are open precompact sets satisfying \( U_n \subset U_{n+1} \) for all \( n \in \mathbb{N} \). This implies that \( C \subset H \) is relatively compact iff \( C \subset U_n \) holds for some \( n \in \mathbb{N} \).

By general properties of the orbit maps of smooth Lie group actions (cf. [24, Proposition 7.26]), \( p_{\xi_0} \) has constant rank. As \( p_{\xi_0} : H \to O \) is surjective, the global rank theorem (cf. [24, Theorem 4.14]) shows that \( p_{\xi_0} \) is a smooth submersion and hence an open map.
This implies that the sets $V_n := p_{\xi_0}(U_n) \subset \mathcal{O}$ form an increasing cover of $\mathcal{O}$ by open sets. Let $K \subset \mathcal{O}$ be compact. By the same reasoning as before, we get $K \subset V_n$ for some $n \in \mathbb{N}$. But this implies $p_{\xi_0}^{-1}(K) \subset H_{\xi_0}U_n \subset H_{\xi_0}U_{\overline{n}}$, where $H_{\xi_0} \leq H$ is the compact(!) stabilizer of $\xi_0$. As $\bigcup_{n} \subset H$ is compact, this shows that $p_{\xi_0}^{-1}(K)$ is compact as a closed subset of the compact set $H_{\xi_0}U_{\overline{n}}$. \hfill $\square$

Before we can prove the main result of this section, we need some additional results on the continuity of the maps $H \to \mathcal{S}(\mathbb{R}^d), h \mapsto Dh\psi$ and $H \to \mathcal{S}(\mathbb{R}^d)\big|_{\gamma}h \mapsto (W_\psi (F^{-1}f))(\cdot,h)$. These results will then be used to show the continuity of the map

$$\mathcal{D}(\mathcal{O}) \to W^r (L^\infty, L^{p,q}_w(G)), g \mapsto W_\psi (F^{-1}g).$$

**Lemma 7.** Let $\varnothing \neq U \subset \mathbb{R}^d$ be open and let $\gamma : U \times \mathbb{R}^d \to \mathbb{R}^d$ be smooth with the additional property that for all compact sets $L \subset U$ and $K \subset \mathbb{R}^d$ the set

$$\bigcup_{p \in L} (\gamma (p, \cdot))^{-1} (K) \subset \mathbb{R}^d$$

is bounded. Furthermore, let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ be arbitrary.

Then the map

$$\Phi : U \to \mathcal{D}(\mathbb{R}^d), p \mapsto \varphi (\gamma (p, \cdot))$$

is well-defined and continuous. In particular, $\Phi : U \to \mathcal{S}(\mathbb{R}^d)$ is continuous.

**Remark.** The stated requirements for $\gamma$ are (under the identification $\mathbb{R}^{d \times d} \cong \mathbb{R}^{d^2}$) fulfilled for the choice

$$\gamma : \text{GL}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d, (h, \xi) \mapsto h^T \xi.$$ 

**Proof of the remark.** Let $L \subset \text{GL}(\mathbb{R}^d)$ and $K \subset \mathbb{R}^d$ be compact. Then $L^{-1} \subset \text{GL}(\mathbb{R}^d)$ is also compact, which implies $\|h^{-T}\| \leq C_1$ for some $C_1 > 0$ and all $h \in L$. Furthermore, there is some $C_2 > 0$ such that $|\xi| \leq C_2$ holds for all $\xi \in K$.

For $y \in \bigcup_{h \in L} (\gamma (h, \cdot))^{-1} (K)$ we then have $\xi := h^T y = \gamma (h, y) \in K$ for some $h \in L$. This implies

$$|y| = |h^T \xi| \leq \|h^{-T}\| \cdot |\xi| \leq C_1 C_2.$$ \hfill $\square$

**Proof of Lemma [7].** As $\gamma$ (and hence also $\gamma (p, \cdot)$ for every $p \in U$) is smooth, we see that $\varphi (\gamma (p, \cdot)) \in C^\infty (\mathbb{R}^d)$ is also smooth. Now let $K := \text{supp} (\varphi)$. As $\gamma (p, \cdot)$ is continuous, $(\gamma (p, \cdot))^{-1} (K) \subset \mathbb{R}^d$ is closed. It is thus easy to see that

$$\text{supp} (\Phi (p)) = \text{supp} (\varphi (\gamma (p, \cdot))) \subset (\gamma (p, \cdot))^{-1} (K) \quad (4.5)$$

holds for every $p \in U$. Now the assumption (with $L = \{p\}$) yields that the right-hand side is a bounded subset of $\mathbb{R}^d$. Thus, $\varphi (\gamma (p, \cdot)) \in \mathcal{D}(\mathbb{R}^d)$ is compactly supported, so that $\Phi$ is well-defined.

To prove the continuity of $\Phi$, let $(p_n)_{n \in \mathbb{N}} \in U^N$ with $p_n \xrightarrow{n \to \infty} p_0$ for some $p_0 \in U$. Then $L := \{p_n \mid n \in \mathbb{N}\} \cup \{p_0\}$ is a compact subset of $U$. The assumption (and Heine-Borel) thus yield the compactness of

$$M := \bigcup_{p \in L} (\gamma (p, \cdot))^{-1} (K) \subset \mathbb{R}^d.$$ 

By equation [4.5] we see

$$\text{supp} (\Phi (p_0)) \subset M \quad \text{and} \quad \text{supp} (\Phi (p_n)) \subset M \quad \text{for all } n \in \mathbb{N}. \quad (4.6)$$
Now for every multi-index $\beta \in \mathbb{N}_0^d$ the map
\[
\Psi_\beta : U \times \mathbb{R}^d \to \mathbb{C}, (p, x) \mapsto \left(\frac{\partial^{|\beta|}((\varphi \circ \gamma)(q, y))}{\partial y_1^{\beta_1} \cdots \partial y_d^{\beta_d}}\right)_{(q, y) = (p, x)}
\]
is smooth, hence continuous. In particular, $\Psi_\beta$ is uniformly continuous on the compact set $L \times M \subset U \times \mathbb{R}^d$. This yields, for arbitrary $\varepsilon > 0$, some $\delta > 0$ such that $|\Psi_\beta(p, x) - \Psi_\beta(q, y)| < \varepsilon$ holds for all $(p, x), (q, y) \in L \times M$ with $|(p, x) - (q, y)| < \delta$. Let $n_0 \in \mathbb{N}$ with $|p_n - p_0| < \delta$ for $n \geq n_0$. For $n \geq n_0$ and $x \in \mathbb{R}^d$ there are two cases:

1. $x \notin M$. By equation (4.10) this means $x \notin \text{supp} \,(\Phi(p_n))$ and $x \notin \text{supp} \,(\Phi(p_0))$. Hence, we conclude $\Phi(p_0)|_V \equiv 0 \equiv \Phi(p_n)|_V$ for the neighborhood $V := M^c$ of $x$. In particular
\[
|\left(\frac{\partial^{|\beta|}((\varphi \circ \gamma)(q, y))}{\partial y_1^{\beta_1} \cdots \partial y_d^{\beta_d}}\right)_{(q, y) = (p, x)}| = 0 < \varepsilon.
\]

2. $x \in M$. Then $(p_n, x), (p_0, x) \in L \times M$ with $|(p_n, x) - (p_0, x)| = |p_n - p_0| < \delta$. By choice of $\delta$ this implies
\[
|\left(\frac{\partial^{|\beta|}((\varphi \circ \gamma)(q, y))}{\partial y_1^{\beta_1} \cdots \partial y_d^{\beta_d}}\right)_{(q, y) = (p, x)}| = |\Psi_\beta(p_n, x) - \Psi_\beta(p_0, x)| < \varepsilon.
\]

Now [27, Theorem 6.5] (and the associated remark) show $\Phi(p_n) \xrightarrow{\mathcal{D}(\mathbb{R}^d)} \Phi(p_0)$ (recall that the supports of $\Phi(p_n)$ are “uniformly compact” by equation (4.10)).

Since the inclusion $\mathcal{D}(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d)$ is continuous by [27, Theorem 7.10], we are done. □

Using this lemma, we can now show that $h \mapsto (W_\psi f)(\cdot, h)$ is continuous with compact support as a map of $H$ into the space of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ as long as we have $\hat{\psi}, \hat{f} \in \mathcal{D}(\mathcal{O})$:

**Lemma 8.** For $f, \psi \in L^2(\mathbb{R}^d)$ we have the identity
\[
(W_\psi f)(x, h) = |\text{det} (h)|^{1/2} \cdot \left(\mathcal{F}^{-1} \left(\hat{f} \cdot D_h \hat{\psi}\right)\right)(x) \quad \text{for all } (x, h) \in G. \tag{4.7}
\]

Now let $f, \psi \in \mathcal{S}(\mathbb{R}^d)$ with $\hat{f}, \hat{\psi} \in \mathcal{D}(\mathcal{O})$. Then
\[
\Gamma : H \to \mathcal{S}(\mathbb{R}^d), h \mapsto \mathcal{F}^{-1} \left(\hat{f} \cdot D_h \hat{\psi}\right)
\]
is well-defined and continuous with compact support
\[
\text{supp} (\Gamma) \subset \left(p_{\xi_0}^{-1}(\text{supp} (\hat{f}))\right)^{-1} \cdot H_{\xi_0} \cdot p_{\xi_0}^{-1}(\text{supp} (\hat{\psi})) , \tag{4.8}
\]
where we used $p_{\xi_0} : H \to \mathcal{O}, h \mapsto h^T \xi_0$.

**Proof.** Equation (4.7) is an easy consequence of the Plancherel theorem, equation (2.20) and the definitions.

We now show that $\Gamma$ is well-defined and continuous under the assumptions $\hat{f}, \hat{\psi} \in \mathcal{D}(\mathcal{O})$. To this end we note that the multiplication map
\[
\mu_{\hat{f}} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d), g \mapsto \hat{f} \cdot g
\]
is (well-defined and) continuous by [27, Theorem 7.4(b)], and the same holds for the inverse Fourier transform. Finally, Lemma 7 and the corresponding remark show that the map
\[
\Phi : H \to \mathcal{S}(\mathbb{R}^d), h \mapsto D_h \hat{\psi}
\]
is well-defined and continuous. Here we used the assumption $\hat{\psi} \in \mathcal{D}(\mathbb{R}^d)$. In summary, this shows that $\Gamma = \mathcal{F}^{-1} \circ \mu_{\hat{f}} \circ \Phi$ is well-defined and continuous.
Now let \( h \in H \) with \( 0 \neq \Gamma (h) = F^{-1} \left( \hat{f} \cdot D_{h} \psi \right) \). This yields \( \hat{f} : D_{h} \psi \neq 0 \) and thus there is some \( \xi \in \text{supp} (\hat{f}) \cap \text{supp} \left( D_{h} \psi \right) \subset \text{supp} (\hat{f}) \cap h^{-T} (\text{supp} (\hat{\psi})) \).

The inclusions \( \text{supp} (\hat{f}) \subset O = HT \xi_{0} \) and \( \text{supp} (\hat{\psi}) \subset O = HT \xi_{0} \) yield \( g_{1}, g_{2} \in H \) satisfying \( g_{1}^{T} \xi_{0} = \xi = h^{-T} g_{2} \xi_{0} \).

This implies \( \left( g_{1} h g_{2}^{-1} \right)^{T} \xi_{0} = \xi_{0} \), i.e. \( g_{1} h g_{2}^{-1} \in H_{\xi_{0}} \) and thus \( h \in g_{1}^{-1} H_{\xi_{0}}, g_{2} \). The inclusions \( g_{1} \in p^{-1}_{\xi_{0}} \left( \text{supp} (\hat{f}) \right) \) and \( g_{2} \in p^{-1}_{\xi_{0}} \left( \text{supp} (\hat{\psi}) \right) \) establish equation \([13]\). Now Lemma \([13]\) shows that \( \Gamma \) indeed has compact support.

Using the lemmata \([10, 11]\) and \([11, 12]\) we now show the announced admissibility of every \( \psi \in \mathcal{S} (\mathbb{R}^{d}) \) whose Fourier transform is compactly supported in the dual orbit \( O \). The following result extends \([17, \text{Lemma} \ 2.7]\), by including a continuity statement that will be useful for the following.

**Theorem 9.** Let \( w_{0} : H \to (0, \infty) \) be measurable and locally bounded and let \( N \in \mathbb{N}_{0} \). Define 

\[
 w : G \to (0, \infty), (x, h) \mapsto (1 + |x|)^{N} \cdot w_{0} (h).
\]

Fix \( \psi \in \mathcal{S} (\mathbb{R}^{d}) \) with \( \hat{\psi} \in \mathcal{D} (O) \). Then the map

\[
 g : \mathcal{D} (O) \to W^{R} (L^{\infty}, L^{p,q}_{w} (G)) : g \mapsto W_{\psi} (F^{-1} g)
\]

is well-defined and continuous.

**Remark.** This implies in particular that the map

\[
 \mathcal{D} (O) \to W^{R} (L^{\infty}, L^{1,1}_{w} (G)) \ni L^{1,1}_{w} (G) \ni F^{-1} \psi = W_{\psi} (F^{-1} \psi) \ni W^{R} (L^{\infty}, L^{1}_{w} (G))
\]

is well-defined and continuous. Furthermore it shows \( W_{\psi} \psi = W_{\psi} (F^{-1} \psi) \in W^{R} (L^{\infty}, L^{1}_{w} (G)) \) which means \( \psi \in \mathcal{B}_{w} \subset \mathcal{A}_{w} \subset \mathcal{H}_{w} \) (with the notation of \([10, \text{Page} \ 321]\)). As \( \psi \in F^{-1} (\mathcal{D} (O)) \) was arbitrary, we get \( F^{-1} (\mathcal{D} (O)) \subset \mathcal{B}_{w} \).

Finally, the above theorem implies that the map

\[
 F^{-1} : \mathcal{D} (O) \to \mathcal{H}^{1}_{w}, g \mapsto F^{-1} g
\]

is well-defined and continuous.

**Proof.** For \( \kappa \in \mathbb{N}_{0} \) and \( g \in \mathcal{S} (\mathbb{R}^{d}) \), let

\[
 |g|_{\kappa} := \max_{\alpha \in \mathbb{N}_{0}^{d}} \sup_{x \in \mathbb{R}^{d}} |\partial^{\alpha} g (x)|.
\]

Then the topology on \( \mathcal{S} (\mathbb{R}^{d}) \) is induced by the family of norms \( (|.|_{\kappa})_{\kappa \in \mathbb{N}_{0}} \). Choose \( N_{0} \in \mathbb{N} \) satisfying \( N_{0} > \frac{d}{p} + N \). By continuity of the (inverse) Fourier transform \( F^{-1} : \mathcal{S} (\mathbb{R}^{d}) \to \mathcal{S} (\mathbb{R}^{d}) \), there is some \( N_{1} \in \mathbb{N} \) and a constant \( C_{1} > 0 \) such that

\[
 |F^{-1} g|_{N_{0}} \leq C_{1} \cdot |g|_{N_{1}}
\]

holds for all \( g \in \mathcal{S} (\mathbb{R}^{d}) \). Here we used that the norms \( |.|_{\ell} \) are ordered, i.e. we have \( |.|_{\ell} \leq |.|_{m} \) for \( \ell \leq m \).

Let \( K \subset O \) be an arbitrary compact subset. For \( K_{2} := \text{supp} (\hat{\psi}) \) we define

\[
 L := \left( p^{-1}_{\xi_{0}} (K) \right)^{-1} \cdot H_{\xi_{0}} \cdot p^{-1}_{\xi_{0}} (K_{2}) \subset H.
\]
By Lemma 6, $L$ is a compact subset of $H$. For

$$g \in D_K(O) := \{ f \in D(O) \mid \text{supp}(f) \subset K \},$$

Lemma 8 shows that

$$\Gamma_g : H \rightarrow S(\mathbb{R}^d), h \mapsto \mathcal{F}^{-1} \left( g \cdot D_h \hat{\psi} \right) = \mathcal{F}^{-1} \left( \mathcal{F}^{-1} g \cdot D_h \hat{\psi} \right)$$

is well-defined and continuous with compact support $\text{supp}(\Gamma_g) \subset L$.

By Lemma 7 the map

$$\Phi : H \rightarrow S(\mathbb{R}^d), h \mapsto D_h \hat{\psi} = \hat{D_h \psi}$$

is continuous, so that the continuous function

$$H \rightarrow \mathbb{R}_+, h \mapsto \max_{\alpha \in \mathbb{N}_0^d \atop |\alpha| \leq N_1} \sum_{\beta \leq \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \cdot |D_h \hat{\psi}|_{N_1}$$

attains its maximum $C_2 \geq 0$ on the compact set $L \subset H$. Now the Leibniz rule shows, for $h \in L$, $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq N_1$:

$$(1 + |x|)^{N_1} \cdot \left| \left( \partial^\alpha \left( g \cdot D_h \hat{\psi} \right) \right) (x) \right| \leq \sum_{\beta \leq \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \cdot \left| (\partial^\beta g) (x) \right| \cdot (1 + |x|)^{N_1} \cdot \left| \left( \partial^{\alpha-\beta} D_h \hat{\psi} \right) (x) \right| \leq |g|_{N_1} \cdot \sum_{\beta \leq \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \cdot |D_h \hat{\psi}|_{N_1} \leq C_2 \cdot |g|_{N_1},$$

which implies the estimate $\left| g \cdot D_h \hat{\psi} \right|_{N_1} \leq C_2 \cdot |g|_{N_1}$. Thus, for $h \in L$ we derive

$$\left| \Gamma_g (h) \right|_{N_0} = \left| \mathcal{F}^{-1} \left( g \cdot D_h \hat{\psi} \right) \right|_{N_0} \leq \max_{\alpha \in \mathbb{N}_0^d \atop |\alpha| \leq N_1} \sum_{\beta \leq \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \cdot |D_h \hat{\psi}|_{N_1} \leq C_1 \cdot |g|_{N_1}.$$

For $g \in D_K$ and $h \in H \setminus L \subset H \setminus \text{supp}(\Gamma_g)$ we have $|\Gamma_g (h)|_{N_0} = 0$. Together, this shows our first intermediate estimate

$$|\Gamma_g (h)|_{N_0} \leq C_1 C_2 \cdot |g|_{N_1} \cdot \chi_L (h) \quad \text{for } h \in H \text{ and } g \in D_K. \quad (4.10)$$

Let $V \subset H$ be an arbitrary open, precompact unit neighborhood. In the following, we will use $U := B_1(0) \times V \subset G$ for the control function $K_U$ of the Wiener amalgam space. Let $C_3 := \max_{k \in V} \| k^{-1} \| > 0$. For $(y, v) \in U$ and $(x, h) \in G$ we then have

$$|x| = \| v^{-1} vx \| \leq \| v^{-1} \| \cdot |vx| \leq C_3 \cdot |vx|$$

and thus

$$|y + vx| \geq |vx| - |y| \geq \frac{|x|}{C_3} - 1.$$

This implies

$$1 + |x| = 1 + C_3 \frac{|x|}{C_3} \leq 1 + C_3 (1 + |y + vx|) \leq (1 + C_3) \cdot (1 + |y + vx|)$$
and hence

\[ |(W_\psi (F^{-1}g)) ((y, v) (x, h))| \]

\[ = |(W_\psi (F^{-1}g)) ((y + vx, vh))| \]  

\[ \leq |\det (vh)|^{1/2} \cdot \left| F^{-1} \left( g \cdot D_{vh} \psi \right) \right| (y + vx) \]  

\[ \leq |\det (vh)|^{1/2} \cdot (1 + |y + vx|)^{-N_0} \cdot \left| F^{-1} \left( g \cdot D_{vh} \psi \right) \right|_{N_0} \]

\[ \leq (1 + C_4)^{N_0} \cdot |\det (vh)|^{1/2} \cdot (1 + |x|)^{-N_0} \cdot |\Gamma_g (vh)|_{N_0} \]  

\[ \leq C_1 C_2 (1 + C_3)^{N_0} \cdot |g|_{N_1} \cdot \chi_{L} (vh) \cdot |\det (vh)|^{1/2} \cdot (1 + |x|)^{-N_0} . \]

Note that \( \chi_{L} (vh) \neq 0 \) implies \( vh \in L \) and thus \( h \in v^{-1} L \subseteq \mathcal{V}^{-1} L \). With \( C_4 := \max_{h \in L} |\det (k)|^{1/2} \) and \( C_5 := C_1 C_2 (1 + C_3)^{N_0} C_4 \), we thus derive

\[ (K_U (W_\psi (F^{-1}g))) (x, h) = \|X_{U(x,h)} \cdot W_\psi (F^{-1}g)\|_{L^\infty (G)} \]

\[ \leq \sup_{(y, v) \in U} |(W_\psi (F^{-1}g)) ((y, v) (x, h))| \]

\[ \leq C_5 \cdot |g|_{N_1} \cdot \chi_{L^{-1}L} (h) \cdot (1 + |x|)^{-N_0} . \]

Because of \( N_0 > \frac{d}{p} + N \), the constant \( C_6 := \| (1 + |x|)^{-N_0} \|_{L^p (\mathbb{R}^d)} \) is finite. This shows (cf. the definition of \( w \) in the statement of the theorem)

\[ \| g \|_{W^{(L^\infty, L^p, q)} (\mathcal{O})} = \| K_U (W_\psi (F^{-1}g)) \|_{L^p (\mathbb{R}^d)} \]

\[ \leq C_5 |g|_{N_1} \cdot \| \det (h^{-1}) \|^{1/q} \cdot \| w_0 (h) \cdot \chi_{L^{-1}L} (h) \cdot (1 + |x|)^{-N_0} \|_{L^p (\mathbb{R}^d)} \]

\[ =: C_7 |g|_{N_1} < \infty \]

for all \( g \in \mathcal{D}_K (\mathcal{O}) \), where the constant \( C_7 \) does not depend upon \( g \). Here, we used compactness of \( \mathcal{V}^{-1} L \) and local boundedness of \( w_0 \).

As the norm \( |.|_{N_1} \) is easily seen to be continuous on \( \mathcal{D}_K (\mathcal{O}) \), where the topology on \( \mathcal{D}_K \) is given by uniform convergence of all derivatives (cf. [27, Section 6.2]), we see that the map \( \varphi: \mathcal{D}_K (\mathcal{O}) \rightarrow W^{(L^\infty, L^p, q)} (\mathcal{O}) \) is well-defined and continuous. Now [27, Theorem 6.6] yields continuity of \( \varphi \).

Using the above theorem, we now show that the reservoir \( \mathcal{H}_w^1 (\mathcal{O}) \) can be identified with a subspace of the space of all distributions \( \mathcal{D}' (\mathcal{O}) \) on the dual orbit \( \mathcal{O} \). This is a more convenient reservoir than \( \mathcal{H}_w^1 (\mathcal{O}) \) for two reasons:

1. As long as the group \( H \) is fixed, the space \( \mathcal{D}' (\mathcal{O}) \) is independent of the parameters \( p, q, v \) of the space \( L^p_q (\mathcal{G}) \).

   The same is not true for \( \mathcal{H}_w^1 (\mathcal{O}) \), as different choices of \( v \) lead to different control weights \( w \) (cf. Lemma 4 and thus to different spaces \( \mathcal{H}_w^1 \). Note though, that this is not a serious issue, as [10, Theorem 4.2] shows that the resulting coorbit space is (with certain restrictions) independent of the choice of \( w \).
(2) Even if two different groups $H, H'$ are considered, the spaces $D'(O)$ and $D'(O')$ (where $O'$ is the open dual orbit of $H'$) can be compared with each other.

If for example the dual orbits $O, O'$ of $H$ and $H'$ coincide, it is possible to make sense of the statement that the coorbit space $Co(Y,H)$ embeds into $Co(Y',H')$ if each $f \in Co(Y,H) \subset (H^1_w)^{\sim} \subset D'(O) = D'(O')$ is also an element of $Co(Y',H') \subset D'(O')$ (and if the map thus defined is bounded). Here we have already used the identification of $(H^1_w)^{\sim}$ with a subspace of $D'(O)$.

One can even do this if the orbits do not coincide, but are merely ordered (i.e. $O \subset O'$ or vice versa).

**Corollary 10.** Let $w : H \to (0,\infty)$ be locally bounded. Then the map
\[
\mathcal{F} : (H^1_w)^{\sim} \to D'(O), \, f \mapsto \mathcal{F}f
\]
with
\[
\mathcal{F}f : D(O) \to \mathbb{C}, \, g \mapsto f(\mathcal{F}^{-1}g)
\]
as defined in equation (4.3) is well-defined, injective and continuous with respect to the weak-$\ast$-topology on $(H^1_w)^{\sim}$.

**Proof.** Let $f \in (H^1_w)^{\sim}$ and $g \in D(O)$. Then we also have $\overline{g} \in D(O)$ and the conjugation map
\[
eq c : D(O) \to D(O), \, g \mapsto \overline{g}
\]
is easily seen to be antilinear and continuous. Theorem 9 (and the ensuing remark) show $\mathcal{F}^{-1}\overline{\overline{g}} \in H^1_w$ as well as the continuity of $\mathcal{F}^{-1} : D(O) \to H^1_w$.

The expression
\[
(\mathcal{F}f)(g) \quad \text{Eq. (4.3)} \quad f(\mathcal{F}^{-1}g) \in \mathbb{C}
\]
is well-defined because of $\mathcal{F}^{-1}\overline{\overline{g}} \in H^1_w$. The antilinearity of $f$ and $c$ show that the map
\[
\mathcal{F}f = f \circ \mathcal{F}^{-1} \circ c : D(O) \to \mathbb{C}
\]
is linear and continuous as a composition of continuous maps, i.e. $\mathcal{F}f \in D'(O)$. This shows that $\mathcal{F}$ is well-defined.

In order to show continuity of $\mathcal{F}$, let $\iota_g : D'(O) \to \mathbb{C}, \, \varphi \mapsto \varphi(g)$ be the evaluation map (for some $g \in D(O)$). Then we have
\[
(\iota_g \circ \mathcal{F})(f) = \iota_g(\mathcal{F}(f)) = (\mathcal{F}(f))(g) = f(\mathcal{F}^{-1}g) = \iota_{\mathcal{F}^{-1}g}(f) \quad \text{for all} \quad f \in (H^1_w)^{\sim},
\]
where $\iota_{\mathcal{F}^{-1}g}$ denotes the evaluation map on $(H^1_w)^{\sim}$. This map is continuous on $(H^1_w)^{\sim}$ by the definition of the weak-$\ast$-topology. Hence we see that $\iota_g \circ \mathcal{F} = \iota_{\mathcal{F}^{-1}g}$ is continuous. As the topology on $D'(O)$ is induced by the family of evaluation maps, this shows the continuity of $\mathcal{F} : (H^1_w)^{\sim} \to D'(O)$ with respect to the weak-$\ast$-topology on $(H^1_w)^{\sim}$.

In order to show the injectivity of $\mathcal{F}$, let $\psi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ with $\widehat{\psi} \in D(O)$ be arbitrary and let $f \in (H^1_w)^{\sim}$ with $\mathcal{F}f = 0$. Note that $\pi(x,h)\psi \in \mathcal{S}(\mathbb{R}^d)$ is a Schwartz function whose Fourier transform has compact support
\[
supp(\mathcal{F}(\pi(x,h)\psi)) \quad \text{Eq. (4.3)} \quad \supp\left(M_{-x}D_h\widehat{\psi}\right) \subset h^{-T}(\text{supp}(\widehat{\psi})) \subset O.
\]
This shows $\hat{\mathcal{F}}(\pi(x,h)\psi) \in \mathcal{D}(\mathcal{O})$ and thus
\[
(W_\psi f)(x,h) = \langle \pi(x,h)\psi, f_{\text{anti}} \rangle = \langle \pi(x,h)\psi, f \rangle = f \left( \mathcal{F}^{-1}(\pi(x,h)\psi) \right)
\]
\[
\text{Eq. } 10 \quad (\mathcal{F} f) \left( \mathcal{F}(\pi(x,h)\psi) \right)
\]
\[
f \equiv 0
\]
i.e. $W_\psi f \equiv 0$. But [10, Theorem 4.1] shows that $W_\psi : (\mathcal{H}_w^1)\sim \rightarrow L^\infty_{w_0}(G)$ is injective (note that we have $\psi \in \mathcal{B}_w \setminus \{0\} \subset A_w \setminus \{0\}$ by Theorem 9), which implies $f \equiv 0$. \hfill $\Box$

Instead of applying the Fourier transform to $f \in (\mathcal{H}_w^1)\sim$ in order to yield $\mathcal{F}f \in \mathcal{D}'(\mathcal{O})$, we can also “pass on” the application of the Fourier transform to the space on which $f$ is defined. This is described in the next corollary. We will see in section 8 that the reservoir $(\mathcal{F}(\mathcal{D}(\mathcal{O})))'$ that is used in this corollary is a very natural alternative “reservoir” for the definition of coorbit spaces.

**Corollary 11.** Let $w : H \rightarrow (0, \infty)$ be locally bounded. Then the map

$\Theta : (\mathcal{H}_w^1)\sim \rightarrow (\mathcal{F}(\mathcal{D}(\mathcal{O})))'$, $f \mapsto (\varphi \mapsto f(\bar{\varphi}))$

is a well-defined, injective linear map that is continuous with respect to the weak-$\ast$-topology on $(\mathcal{H}_w^1)\sim$.

Here, the space $\mathcal{F}(\mathcal{D}(\mathcal{O}))$ is endowed with the unique topology that makes the Fourier transform $\mathcal{F} : \mathcal{D}(\mathcal{O}) \rightarrow \mathcal{F}(\mathcal{D}(\mathcal{O})) \leq \mathcal{S}(\mathbb{R}^d)$ a homeomorphism and the dual space $(\mathcal{F}(\mathcal{D}(\mathcal{O})))'$ is equipped with the weak-$\ast$-topology.

With the definition of the Fourier transform on $(\mathcal{H}_w^1)\sim$ of corollary 10, we have

$$\mathcal{F}f = (\Theta f) \circ \mathcal{F} \quad \text{for all } f \in (\mathcal{H}_w^1)\sim.$$  \text{(4.11)}

**Proof.** First note that we have $\overline{\mathcal{F}\varphi} = \mathcal{F}^{-1}\overline{\varphi}$ for $\varphi \in \mathcal{D}(\mathcal{O})$. For $\psi = \mathcal{F}\varphi \in \mathcal{F}(\mathcal{D}(\mathcal{O}))$, this shows $\overline{\psi} = \mathcal{F}^{-1}\overline{\varphi} \in \mathcal{H}_w^1$ by Theorem 9. Here, we used that $\overline{\varphi} \in \mathcal{D}(\mathcal{O})$ holds as well, i.e. that $\mathcal{D}(\mathcal{O})$ is invariant under conjugation. In summary, this entails that

$$\Theta f : \mathcal{F}(\mathcal{D}(\mathcal{O})) \rightarrow \mathbb{C}$$

is a well-defined linear map.

For $\varphi \in \mathcal{D}(\mathcal{O})$ we have

$$((\Theta f) \circ \mathcal{F})(\varphi) = (\Theta f)(\overline{\varphi}) = f(\overline{f(\overline{\varphi})}) = (\mathcal{F}^{-1}\overline{\varphi}) = f(\mathcal{F}^{-1}\overline{\varphi}) \text{ Eq. } 10 \quad (\mathcal{F} f)(\varphi),$$

which proves equation (4.11). Furthermore, corollary 10 implies that the right hand side is a continuous linear function of $\varphi \in \mathcal{D}(\mathcal{O})$. The definition of the topology on $\mathcal{F}(\mathcal{D}(\mathcal{O}))$ thus implies $\Theta f \in (\mathcal{F}(\mathcal{D}(\mathcal{O})))'$.

In order to show the injectivity of $\Theta$, assume $\Theta f = 0$ for some $f \in (\mathcal{H}_w^1)\sim$. Equation (4.11) then yields $\mathcal{F}f = 0$, which implies $f \equiv 0$ by Corollary 10.

Finally, let $\psi = \mathcal{F}\varphi \in \mathcal{F}(\mathcal{D}(\mathcal{O}))$ be arbitrary. As in the proof of corollary 10 we see that the evaluation map $\iota_\psi : (\mathcal{F}(\mathcal{D}(\mathcal{O})))' \rightarrow \mathbb{C}, f \mapsto f(\psi)$ satisfies

$$\iota_\psi(\Theta f) = (\Theta f)(\overline{\varphi}) = (\mathcal{F} f)(\overline{\varphi}) = f(\mathcal{F}^{-1}\overline{\varphi}) = \iota_{\mathcal{F}^{-1}\overline{\varphi}}(f),$$

where the right hand side is a continuous function of $f \in (\mathcal{H}_w^1)\sim$ with respect to the weak-$\ast$-topology. This proves the claimed continuity of $\Theta$. \hfill $\Box$
5. Construction of an induced covering and definition of the corresponding decomposition space

In this section we will show how to obtain the induced covering \( Q \) mentioned in the introduction and we will prove that our construction indeed yields an admissible covering (cf. Definition 12 below). The idea for the construction of \( Q \) is the following: Choose a (necessarily countable) well-spread family \((h_i)_{i \in I}\) in \( H \). For precompact \( Q \subset \mathcal{O} \) with \( \overline{Q} \subset \mathcal{O} \) and \( \mathcal{O} = \bigcup_{i \in I} h_i^{-1}Q \) we then define \( Q := (Q_i)_{i \in I} := (h_i^{-1}Q)_{i \in I} \). It is worth noting that this induced covering is of a very simple form in which every set \( Q_i \) is a linear image of a fixed set \( Q \). We will also see that the covering is well behaved in the sense that there is a constant \( C > 0 \) such that \( \|h_i^{-1}h_j\| \leq C \) holds for all \( i, j \in I \) with \( Q_i \cap Q_j \neq \emptyset \).

Finally, we will state the exact definition of the space \( \mathcal{D}(Q, L^p, \ell^q) \) as considered in this paper. Our definition is slightly different than the one in [2, Definition 3].

Before we show that our construction of the induced covering indeed yields an admissible covering, we recall the following fundamental definitions from [8, Definition 2.1 and Definition 2.3].

**Definition 12.** (cf. [8 Definition 2.1 and Definition 2.3])

Let \( X \neq \emptyset \) be a set and assume that \( Q = (Q_i)_{i \in I} \) is a family of subsets of \( X \). For a subset \( J \subset I \) we then define the **(index)-cluster of \( J \)** as

\[
J^* := \{ i \in I \mid \exists j \in J : Q_i \cap Q_j \neq \emptyset \}.
\]

Inductively, we define \( J^{(n)} := J \) and \( J^{(n+1)} := (J^{(n)})^* \) for \( n \in \mathbb{N}_0 \). For convenience, we also set \( i^{(n)} := \{i\}^{(n)} \) and \( i^* := \{i\}^* \) for \( i \in I \). Furthermore, for any subset \( J \subset I \) we define \( Q_J := \bigcup_{i \in J} Q_i \). With this notation we introduce the convenient shortcuts \( Q_{i^*} := Q_{i^{(n)}} \) and \( Q_{i} := Q_{i^{(n)}} \) for \( i \in I \) and \( k \in \mathbb{N}_0 \).

We say that \( Q \) is an **admissible covering** of \( X \), if the following holds

1. \( X = \bigcup_{i \in I} Q_i \) (i.e. \( Q \) is a covering of \( X \)) and
2. There exists \( n_0 \in \mathbb{N} \) with \( |i^*| \leq n_0 \) for all \( i \in I \).

In [2, Definition 7], Borup and Nielsen specialized this notion to the concept of a so-called **structured admissible covering**. They only considered coverings of the whole euclidean space \( \mathbb{R}^d \). In the next definition we generalize this to coverings of arbitrary open subsets \( \emptyset \neq U \subset \mathbb{R}^d \).

**Definition 13.** (based upon [2 Definition 7])

Let \( \emptyset \neq U \subset \mathbb{R}^d \) be open and let \( I \neq \emptyset \) be a countable index-set. Furthermore assume that \((T_i)_{i \in I}\) and \((b_i)_{i \in I}\) are families of invertible linear transformations \( T_i \in \text{GL}(\mathbb{R}^d) \) and of translations \( b_i \in \mathbb{R}^d \), respectively.

Let \( P, Q \subset \mathbb{R}^d \) be precompact open subsets with \( \overline{P} \subset Q \). We then say that the family \( Q := (Q_i)_{i \in I} := (T_iQ + b_i)_{i \in I} \) is a **structured admissible covering** (of \( U \)), if

1. \( Q \) and \( P := (T_iP + b_i)_{i \in I} \) are admissible coverings of \( U \) and
2. there is a constant \( C > 0 \) such that \( \|T_i^{-1}T_j\| \leq C \) holds for all \( i, j \in I \) satisfying \( Q_i \cap Q_j \neq \emptyset \).

Borup and Nielsen then showed (cf. [2, Proposition 1]) that every structured admissible covering admits a so-called **bounded admissible partition of unity (BAPU)** which can

\[\text{This implies in particular that we have } Q_i \subset U \text{ for all } i \in I.\]
then be used in order to define the decomposition spaces $\mathcal{D}(Q, L^p, \ell_q)$.

More precisely, the conditions for a BAPU as used in this paper are as follows:

**Definition 14.** (cf. [8 Definition 2.2] and [2 Definition 2])

Let $\emptyset \neq U \subset \mathbb{R}^d$ be open and let $Q = (Q_i)_{i \in I}$ be an admissible covering of $U$. A family $(\varphi_i)_{i \in I}$ of functions is called a **bounded admissible partition of unity** (BAPU) subordinate to $Q$, if

1. $\varphi_i \in \mathcal{D}(U)$ for all $i \in I$,
2. $\sum_{i=1}^{\infty} \varphi_i(x) = 1$ for all $x \in U$ (i.e. $(\varphi_i)_{i \in I}$ is a partition of unity on $U$),
3. $\varphi_i(x) = 0$ for all $x \in U \setminus Q_i$ for all $i \in I$,
4. $\sup_{i \in I} \|\mathcal{F}^{-1} \varphi_i\|_{L^1(\mathbb{R}^d)} < \infty$.

Note that Borup and Nielsen even require $\sup_{i \in I} |\det(T_i)|^{\frac{1}{d}-1} \|\mathcal{F}^{-1} \varphi_i\|_{L^p(\mathbb{R}^d)} < \infty$ for all $p \in (0, 1]$ where each $Q_i$ is given by $Q_i = T_iQ + b_i$. This stronger condition is necessary to ensure well-definedness of the decomposition space $\mathcal{D}(Q, L^p, \ell_q)$ in the Quasi-Banach regime $p \in (0, 1)$. In this paper we will only consider the range $p \in [1, \infty]$.

Mainly as a simplification of notation, we introduce the term of a **decomposition covering**.

**Definition 15.** Let $\emptyset \neq U \subset \mathbb{R}^d$ be an open set. A family $Q = (Q_i)_{i \in I}$ of subsets of $U$ is called a **decomposition covering** if

1. $Q$ is an admissible covering of $U$,
2. $Q_i \neq \emptyset$ for all $i \in I$ and
3. there exists a BAPU $(\varphi_i)_{i \in I}$ subordinate to $Q$.

An easy adaptation of the proof of [2 Proposition 1] (we allow any open set $\emptyset \neq U \subset \mathbb{R}^d$, whereas Borup and Nielsen only consider coverings of the whole euclidean space $\mathbb{R}^d$) then yields the following:

**Theorem 16.** Let $\emptyset \neq U \subset \mathbb{R}^d$ be an open set. Then every structured admissible covering $Q = (Q_i)_{i \in I}$ of $U$ is a decomposition covering of $U$.

We next recall density and discreteness properties for subsets of topological groups.

**Definition 17.** Let $G$ denote an arbitrary locally compact group, and $(g_j)_{j \in J} \subset G$ a family of group elements. Let $U \subset G$ denote a neighborhood of the identity. The family $(g_j)_{j \in J}$ is called **$U$-dense** if we have $G = \bigcup_{j \in J} g_jU$ and **$U$-discrete** if for all distinct $j, j' \in J$ the equality $g_jU \cap g_{j'}U = \emptyset$ holds. Furthermore, $(g_j)_{j \in J}$ is called **separated** if it is $U$-discrete with respect to some neighborhood $U$ of the identity. The family is called **well-spread**, if it is separated and $U$-dense, for a relatively compact neighborhood $U$ of the identity. Finally, we call $(g_j)_{j \in J}$ **relatively separated** if it is the union of finitely many separated sets.

Using the fact that $p_{t_0}$ is a proper map (cf. Lemma 6), we can prove the following fundamental result which shows that elements of $H$ that are projected “close to each other” by $p_{t_0}$ are already close in $H$. This property will ensure that the induced covering is admissible.

**Lemma 18.** Let $\emptyset \neq K_1, K_2 \subset \mathcal{O}$ be compact. Then there is a compact set $L \subset H$ such that $g \in hL$ holds for all $g, h \in H$ satisfying $h^{-T}K_1 \cap g^{-T}K_2 \neq \emptyset$.

Now choose a relatively compact unit-neighborhood $V \subset H$ and let $(h_i)_{i \in I} \subset H$ denote a $V$-separated family. Given $h \in H$, let

$$I_h := \{ i \in I \mid h^{-T}K_1 \cap h_i^{-T}K_2 \neq \emptyset \}.$$
Then there exists a constant \( C = C(K_1, K_2, V) \) such that
\[ |I_h| \leq C \quad \text{holds for all } h \in H. \]

**Proof.** Choose
\[ L = L(K_1, K_2) := \left( p_{\xi_0}^{-1}(K_1) \right)^{-1} \cdot H_{\xi_0} \cdot p_{\xi_0}^{-1}(K_2). \]

Lemma \( \circ \) shows that \( L \subset H \) is compact.

Let \( g, h \in G \) with \( h^{-T}K_1 \cap g^{-T}K_2 \neq \emptyset \). This yields suitable \( k_1 = h_1^T \xi_0 \in K_1 \subset O \) and \( k_2 = h_2^T \xi_0 \in K_2 \subset O \) satisfying \( h^{-T}h_1^T \xi_0 = g^{-T}h_2^T \xi_0 \). We conclude
\[ (h_1h^{-1}g)h_2^{-1} = \xi_0. \]

In other words, we have \( h_1h^{-1}g^{-1}h_2 \in H_{\xi_0} \) and thus
\[ g \in hh_1^{-1}H_{\xi_0}h_2 \subset h \cdot \left( p_{\xi_0}^{-1}(K_1) \right)^{-1} \cdot H_{\xi_0} \cdot p_{\xi_0}^{-1}(K_2) = h \cdot L. \]

Now let \( (h_i)_{i \in I} \) be \( V \)-separated. For \( i \in I \) we have \( h_i^{-T}K_1 \cap h_i^{-T}K_2 \neq \emptyset \) and thus \( h_i \in hL \) as shown above. But this implies \( h_iV^\circ \subset h_iV \subset hL \). Thus, \( (h_iV^\circ)_{i \in I} \) is a pairwise disjoint collection of subsets of \( hL \). We thus get, for every finite subset \( J \subset I_h \),
\[ |J| = \frac{\sum_{i \in J} \mu_H(h_iV^\circ)}{\mu_H(V^\circ)} = \mu_H \left( \bigcup_{i \in J} h_iV^\circ \right) / \mu_H(V^\circ) \leq \frac{\mu_H(hLV)}{\mu_H(V^\circ)} = \frac{\mu_H(LV)}{\mu_H(V^\circ)}, \]
where we used the left-invariance of \( \mu_H \) in the first and last step. Noting that the right hand side is independent of \( h \) completes the proof. \( \square \)

As a last preparation, we need the following well-known existence result for countable, “well-spread” families in \( H \). It follows by choosing (using Zorn’s Lemma) a \( V \)-discrete subset of \( H \) that is maximal with respect to inclusion.

**Lemma 19.** Let \( U, V \subset H \) be neighborhoods of the identity such that \( VV^{-1} \subset U \) holds. Then there exists a family \( (h_i)_{i \in I} \) of elements of \( H \) such that we have \( G = \bigcup_{i \in I} h_iU \) and so that the family of sets \( (h_iV)_{i \in I} \) is pairwise disjoint. Every such family is necessarily countably infinite. In particular, there exists a countably infinite well-spread family \( (h_i)_{i \in I} \) in \( H \).

We now give the construction of the induced covering. It is worth noting that there is still some freedom in choosing that covering. As we will see below, the coorbit space and the decomposition space will be isomorphic under the Fourier transformation for every such choice.

**Theorem 20.** Let \( (h_i)_{i \in I} \) be well-spread in \( H \). For every precompact set \( Q \subset \mathbb{R}^d \) that satisfies \( Q \subset O \) and \( O = \bigcup_{i \in I} h_i^{-T}Q \) we say that the covering \( Q = \{ h_i^{-T}Q \}_{i \in I} \) of \( O \) is a covering of \( O \) induced by \( H \). Such a set \( Q \) exists for every choice of the family \( (h_i)_{i \in I} \). In particular, one can choose \( Q = U^{-T} \xi_0 \), where \( U \subset H \) is a precompact set satisfying \( H = \bigcup_{i \in I} h_iU \).

Every such covering is admissible. Furthermore there is a constant \( C = C \left( (h_i)_{i \in I}, Q \right) > 0 \) such that \( \| h_i^T h_j^{-T} \| \leq C \) holds for all \( i \in I \) and \( j \in I^* \), where the cluster is formed with respect to \( Q \).

If \( Q \subset \mathbb{R}^d \) is open and precompact with \( \overline{Q} \subset O \) and if there is some open set \( P \subset \mathbb{R}^d \) satisfying \( P \subset Q \) and \( O = \bigcup_{i \in I} h_i^{-T}P \) then \( Q \) is a structured admissible covering of \( O \) and in particular a decomposition covering.
Proof. We first show the existence of $Q \subset \mathbb{R}^d$ with the stated properties. By assumption on $(h_i)_{i \in I}$, there is some precompact set $U \subset H$ such that $H = \bigcup_{i \in I} h_iU$. This means

$$\mathcal{O} = H^T \xi_0 = H^{-T} \xi_0 = \bigcup_{i \in I} h_i^{-T} U^{-T} \xi_0.$$  

This implies that the choice $Q = U^{-T} \xi_0 \subset \mathcal{O}$ guarantees the compact stated properties. Here we note that $U^{-T} \xi_0 \subset \mathcal{O}$ is compact (as a continuous image of the compact set $\overline{U} \subset H$).

Now let $Q = (h_i^{-T} Q)_{i \in I}$ be a covering of $\mathcal{O}$ induced by $H$. Let $V \subset H$ be a precompact unit neighborhood such that $(h_i)_{i \in I}$ is $V$-discrete. Set $K := K_1 := K_2 := \overline{Q} \subset \mathcal{O}$ and choose the compact set $L = L(K_1, K_2) = L(Q) \subset H$ and the constant $C = C(K_1, K_2, V) > 0$ as in Lemma \[\text{18}\]. For $i \in I$ and $j \in i^*$ we then have $\emptyset \neq Q_i \cap Q_j \subset h_i^{-T} \overline{Q} \cap h_j^{-T} \overline{Q}$ so that Lemma \[\text{18}\] implies $h_j \in h_iL$ and thus $\|h_i^T h_j^{-T}\| = h_j^{-1} \|h_i\| \leq \max_{g \in L^{-1}} \|g\|$. Furthermore, we have, in the notation of Lemma \[\text{18}\],

$$i^* = \{ j \in I \mid h_j^{-T} Q \cap h_i^{-T} Q \neq \emptyset \} = I_h,$$

and thus $|i^*| = |I_h| \leq C$. This shows that $Q$ is admissible.

Finally, assume that $Q \subset \mathbb{R}^d$ is open and precompact with $Q \subset \mathcal{O}$ and that there is an open set $P \subset \mathbb{R}^d$ satisfying $P \subset Q$ and $\mathcal{O} = \bigcup_{i \in I} h_i^{-T} P$. This yields $\mathcal{O} = \bigcup_{i \in I} h_i^{-T} Q \subset \bigcup_{i \in I} h_i^{-T} P \subset \bigcup_{i \in I} h_i^{-T} Q \subset \mathcal{O}$. Hence the above implies that $Q$ and $P = (h_i^{-T} P)_{i \in I}$ are both admissible coverings of $\mathcal{O}$. The estimate $\|h_i^T h_j^{-T}\| = \|(h_i^{-T})^{-1} h_j^{-T}\| \leq C$ for all $i \in I$ and $j \in I$ with $Q_i \cap Q_j \neq \emptyset$ shown above then completes the proof of the fact that $Q$ is a structured admissible covering. Theorem \[\text{16}\] then shows that $Q$ is a decomposition covering. 

We next introduce our notion of decomposition spaces. Note that we somewhat extend the definition of Borup and Nielsen in \[\text{2}\]; they only consider coverings of all of $\mathbb{R}^d$, whereas the decomposition spaces that we consider arise from a covering of the open dual orbit, which is always a proper subset of $\mathbb{R}^d$.

**Definition 21.** (cf. \[\text{3}\] Definitions 2.4 and 3.1 and \[\text{2}\] Definition 3)

Let $\emptyset \neq U \subset \mathbb{R}^d$ be an open subset. Furthermore assume that $Q = (Q_i)_{i \in I}$ is a decomposition covering of $U$ with BAPU $(\varphi_i)_{i \in I}$. Let $u : I \to (0, \infty)$ be a $Q$-moderate weight, that is there exists some $C > 0$ satisfying $u(i) \leq C \cdot u(j)$ for all $i \in I$ and all $j \in i^*$.

Let $p, q \in [1, \infty]$. For $f \in \mathcal{D}'(U)$ we define

$$\|f\|_{\mathcal{D}(Q, L^p, \ell^q)} := \left\| u(i) \cdot \|F^{-1} (\varphi_i f)\|_{L^p(\mathbb{R}^d)} \right\|_{\ell^q(I)} \in [0, \infty],$$

where we use the convention that for a family $(c_i)_{i \in I}$ with $c_i \in [0, \infty)$, the $\ell^q$-norm is infinite as soon as one of the $c_i$ equals infinity.

Finally, we define the decomposition space with respect to the covering $Q$ and the weight $u$ with integrability exponents $p, q$ as

$$\mathcal{D}(Q, L^p, \ell^q) := \left\{ f \in \mathcal{D}'(U) \mid \|f\|_{\mathcal{D}(Q, L^p, \ell^q)} < \infty \right\}.$$

**Remark.** A few remarks pertaining this definition are in order:

1. We first note that $\varphi_i f \in \mathcal{D}'(U)$ is a distribution with compact support, because of $\varphi_i \in \mathcal{D}(U)$. By \[\text{27}\] Example 7.12(a)], $\varphi_i f$ is thus a tempered distribution so that $F^{-1} (\varphi_i f) \in \mathcal{S}'(\mathbb{R}^d)$ makes sense. Furthermore, \[\text{27}\] Theorem 7.23 shows that $F(\varphi_i f)$
(and hence also $F^{-1}(\varphi_i f)$) is a smooth function of polynomial growth. In particular, $F^{-1}(\varphi_i f)$ is pointwise defined by
\[
(F^{-1}(\varphi_i f))(x) = (\varphi_i f)(e_x) = f(\varphi_i e_x)
\]
with $e_x : \mathbb{R}^d \to \mathbb{C}, \xi \mapsto e^{2\pi i (x, \xi)}$. Thus the expression $\|F^{-1}(\varphi_i f)\|_{L_p(\mathbb{R}^d)} \in [0, \infty]$ makes sense.

(2) In the following we will use the notations $u_i := u(i)$ and $\|(x_i)_{i \in I}\|_{\ell_q^u} := \|(u_i \cdot x_i)_{i \in I}\|_{\ell_q^u}$. As $\ell_q^u$ is permutation-invariant and because $u$ is $Q$-moderate, \cite{S} Lemma 3.2 shows that $\ell_q^u$ is a $Q$-regular BK-space (see \cite{S} Definition 2.5). The proof of \cite{S} Theorem 2.3(B) then shows that $D(Q, L_p, \ell_q^u)$ is independent of the particular choice of the BAPU $(\varphi_i)_{i \in I}$ with equivalent norms for each choice.

For the convenience of the reader (and because our definition differs slightly from the one in \cite{S}), we give a sketch of the argument:
(a) If $(\psi_i)_{i \in I}$ is another BAPU for $Q$, we have $\psi_i = \psi_i \varphi_i^*$ for all $i \in I$. Indeed, for $j \in I \setminus i*$, we have $\psi_j \equiv 0$ on $Q_i$, so that $\sum_{i \in I} \varphi_i \equiv 1$ on $Q \supset Q_i$ forces $\varphi_i^* \equiv 1$ on $Q_i$. Now $\psi_i \equiv 0$ on $Q \supset Q_i$ implies $\psi_i = \psi_i \varphi_i^*$.
(b) Using Young’s inequality, this implies
\[
\left\|F^{-1}(\psi_i f)\right\|_{L_p(\mathbb{R}^d)} = \left\|F^{-1}(\psi_i \varphi_i^* f)\right\|_{L_p(\mathbb{R}^d)}
= \left\|(F^{-1} \psi_i) * F^{-1}(\varphi_i^* f)\right\|_{L_p(\mathbb{R}^d)}
\leq \left\|F^{-1} \psi_i\right\|_{L^1(\mathbb{R}^d)} \left\|F^{-1}(\varphi_i^* f)\right\|_{L_p(\mathbb{R}^d)}
\leq C \cdot \sum_{j \in i*} \left\|F^{-1}(\varphi_j f)\right\|_{L_p(\mathbb{R}^d)}
\]
where we used the property $\sup_{i \in I} \left\|F^{-1} \psi_i\right\|_{L^1(\mathbb{R}^d)} < \infty$ of a $Q$-BAPU in the last step.
(c) The fact that $\ell_q^u$ is $Q$-regular means by definition that the map
\[
\Gamma : \ell_q^u(I) \to \ell_q^u(I), (x_i)_{i \in I} \mapsto \left(\sum_{j \in i*} x_j\right)_{i \in I}
\]
is well-defined and bounded. Using the $Q$-moderateness of $u$ and the fact that $|i*| \leq C$ is uniformly bounded, this is also easy to see directly.
(d) Putting everything together, we arrive at
\[
\left\|\left(\left\|F^{-1}(\psi_i f)\right\|_{L_p(\mathbb{R}^d)}\right)_i\right\|_{\ell_q^u} \leq C \cdot \left\|\Gamma \left(\left(\left\|F^{-1}(\varphi_i f)\right\|_{L_p(\mathbb{R}^d)}\right)_i\right)\right\|_{\ell_q^u}
\leq C \cdot \left\|\Gamma\right\| : \left\|\left(\left\|F^{-1}(\varphi_i f)\right\|_{L_p(\mathbb{R}^d)}\right)_i\right\|_{\ell_q^u}
\]
By symmetry, we also get the reverse inequality.
(3) It is worth noting that the exact construction used in \cite{S} would be to take $B = FL^p\left(\mathbb{R}^d\right)$ and $A = FL^1\left(\mathbb{R}^d\right)$ and then to define the decomposition space as
\[
D(Q, L_p, \ell_q) = \left\{f \in \left(FL^1\left(\mathbb{R}^d\right) \cap C_c(\mathbb{R}^d)\right)' \left\|f\right\|_{D(Q, L_p, \ell_q)} < \infty \right\}.
\]
Thus we are not exactly in this setting, as we use $D'(U)$ as our reservoir instead of $(FL^1\left(\mathbb{R}^d\right) \cap C_c(\mathbb{R}^d))'$. As seen above, the arguments used in \cite{S} nevertheless carry over to the present situation.
(4) Analogous to the proof of [8, Theorem 2.2(A)] we can also show that $D$ (Q, $L^p, \ell_q^n$) is a Banach space that embeds continuously into $D'(U)$. For the convenience of the reader we again give a sketch of the proof:
(a) The identity $\sum_{i \in I} \varphi_i \equiv 1$ on $U$ shows that $(\varphi_i (C^*))_{i \in I}$ covers $U$. Because of $\varphi_i (C^*) \subset Q^*_i$, this shows that $(Q^*_i)_{i \in I}$ is an open cover of $U$, where $Q^*_i$ denotes the topological interior of $Q_i$.
(b) Let $K \subset U$ be compact and note that we have $K \subset \bigcup_{i \in I_K} Q^*_i$ for some finite set $I_K \subset I$. This easily entails $\varphi_{I_K} \equiv 1$ on $K$.
(c) For $f \in D(Q, L^p, \ell_q^n)$ and $\gamma \in D_K (U)$ (i.e. $\gamma \in D(U)$ with $\text{supp} (\gamma) \subset K$), we thus get
\[
|f (\gamma)| = \left| f \left( \sum_{i \in I_K} \varphi_i \gamma \right) \right| \leq \sum_{i \in I_K} |(\varphi_i f) (\gamma)|
= \sum_{i \in I_K} \left| \langle F^{-1} (\varphi_i f), \gamma \rangle \right|_{S', S}
\leq \|\gamma\|_{L^{p'}(\mathbb{R}^d)} \cdot \sum_{i \in I_K} \left[ \frac{1}{u_i} \cdot u_i \right] \|F^{-1} (\varphi_i f)\|_{L^p(\mathbb{R}^d)}
\leq \|\gamma\|_{L^{p'}(\mathbb{R}^d)} \cdot \left( \sum_{i \in I_K} \frac{1}{u_i} \right) \cdot \|f\|_{D(Q, L^p, \ell_q^n)},
\]
where $p' \in [1, \infty]$ is conjugate to $p$.
(d) The estimate (5.2) easily yields $f_n (\gamma) \xrightarrow{n \to \infty} f (\gamma)$ for $f_n \xrightarrow{D(Q, L^p, \ell_q^n)} f$ and all $\gamma \in D(U)$, i.e. $f_n \xrightarrow{n \to \infty} f$ in the weak-* topology on $D'(U)$.
(e) If $(f_n)_{n \in \mathbb{N}}$ is Cauchy in $D(Q, L^p, \ell_q^n)$, equation (5.2) shows that $(f_n (\gamma))_{n \in \mathbb{N}}$ is Cauchy and hence convergent to some $f(\gamma) \in C$ for every $\gamma \in D(U)$. Now [27, Theorem 6.17] implies that this yields a distribution $f \in D'(U)$. Using equation (5.1), we derive (with $e_x : \mathbb{R}^d \to \mathbb{C}, \xi \mapsto e^{2\pi i (x, \xi)}$)
\[
(F^{-1} (\varphi_i f)) (x) = f(\varphi_i e_x) = \lim_{n \to \infty} f_n (\varphi_i e_x) = \lim_{n \to \infty} (F^{-1} (\varphi_i f_n)) (x).
\]
It is easy to see that $(F^{-1} (\varphi_i f_n))_{n \in \mathbb{N}}$ is Cauchy in $L^p (\mathbb{R}^d)$ for all $i \in I$. Together, this yields
\[
c^{(n)}_i := \|F^{-1} (\varphi_i (f - f_n))\|_{L^p(\mathbb{R}^d)} \xrightarrow{n \to \infty} 0.
\]
(f) The (reversed) triangle inequality yields
\[
\left| c^{(n)}_i - c^{(m)}_i \right| \leq \| F^{-1} (\varphi_i (f_n - f_m))\|_{L^p(\mathbb{R}^d)}
\]
and thus
\[
\left\| \left( c^{(n)}_i - c^{(m)}_i \right) \right\|_{\ell_q^n} \leq \left\| \left( \| F^{-1} (\varphi_i (f_n - f_m))\|_{L^p(\mathbb{R}^d)} \right)_{i \in I} \right\|_{\ell_q^n}
= \| f_n - f_m \|_{D(Q, L^p, \ell_q^n)} \xrightarrow{n, m \to \infty} 0,
\]
where $p' \in [1, \infty]$ is conjugate to $p$. 

Furthermore, let \( Q \) be a specific example showing that \( n \) is then easy to see that \( \bigcup \) is valid for all \( i \) in the previous remark is in general not complete, as the following example shows.

(5) The completeness of \( D(Q, L^p, \ell^n_0) \) is in contrast to [2], where the authors only consider the case \( U = \mathbb{R}^d \) and then define the decomposition space as

\[
D^c(Q, L^p, \ell^n_0) := \left\{ f \in S' (\mathbb{R}^d) \left| \| f \|_{D(Q, L^p, \ell^n_0)} < \infty \right. \right\} .
\]

With this definition, the decomposition space is in general not complete, as the following example shows.

**Example.** In the following, we provide a specific example showing that \( D^c(Q, L^p, \ell^n_0) \) as defined in the previous remark is in general not complete. Let \( I := \mathbb{Z}, T_i := \text{id}_\mathbb{R} \) and \( b_i := i \) for \( i \in \mathbb{Z} \). Furthermore, let \( Q := (-\frac{2}{3}, \frac{2}{3}) \) and \( P := (-\frac{2}{5}, \frac{2}{5}) \), as well as \( Q_i := T_i Q + b_i = (i - \frac{2}{5}, i + \frac{2}{5}) \). It is then easy to see that \( \bigcup_{i \in I} (T_i P + b_i) = \mathbb{R} \) and that \( x \in Q_i \cap Q_j \neq \emptyset \) implies

\[
i - \frac{3}{4} < x < j + \frac{3}{4}
\]

and hence \( i - j < \frac{6}{4} = 2 \). Because of \( i - j \in \mathbb{Z} \) we conclude \( i - j \leq 1 \). By symmetry we get \( |i - j| \leq 1 \) and thus \( i^* \subset \{ i - 1, i, i + 1 \} \). This shows that \( Q = (Q_i)_{i \in I} \) is a structured admissible covering of \( \mathbb{R} \). Now consider the weight \( u_i := 10^{-i} \) for \( i \in \mathbb{Z} \) and note that because of the estimate

\[
\frac{u_i}{u_j} = 10^{j-i} \leq 10^{j-i} \leq 10
\]

which is valid for all \( i \in I \) and \( j \in i^* \subset \{ i - 1, i, i + 1 \} \), the weight \( u \) is \( Q \)-moderate.

Theorem [10] guarantees the existence of a BAPU \( (\varphi_i)_{i \in I} \) subordinate to \( Q \). Note that for \( i \in I \) we have

\[
\bigcup_{j \notin \{i\}} Q_j \subset \left( -\infty, (i - 1) + \frac{3}{4} \right) \cup (i + 1) - \frac{3}{4}, \infty) = \left( -\infty, i - \frac{1}{4} \right) \cup (i + \frac{1}{4}, \infty) .
\]

This implies, together with \( \varphi_j(x) = 0 \) for \( x \in \mathbb{R} \setminus Q_j \) and \( \sum_{i \in I} \varphi_i(x) = 1 \) for all \( x \in \mathbb{R} \) that we have \( \varphi_i(x) = 1 \) for \( x \in \left[ i - \frac{2}{5}, i + \frac{2}{5} \right] \) for all \( i \in I \).

Now choose a nonnegative function \( \psi \in D((-\frac{1}{3}, \frac{1}{3})) \setminus \{0\} \) and define \( f_n := \sum_{j=1}^{n} 4^n \cdot L_n \psi \) for \( n \in \mathbb{N} \). Because of

\[
\text{supp} (L_n \psi) \subset \left( n - \frac{1}{4}, n + \frac{1}{4} \right) \subset \left( \bigcup_{j \notin \{n\}} Q_j \right)^c
\]

it is then easy to see that

\[
\varphi_i \cdot L_n \psi = \begin{cases} 0, & i \neq n, \\ L_i \psi, & i = n \end{cases}
\]
holds for $i, n \in \mathbb{Z}$. For $n \geq m \geq m_0$ we thus get

$$
\|f_n - f_m\|_{\mathcal{D}(Q, L^1, \ell_u^1)} = \sum_{i \in \mathbb{Z}} 10^{-i} \left\| F^{-1} \left( \phi_i \cdot \sum_{j=m+1}^{n} 4^j \cdot L_j \psi \right) \right\|_{L^1(\mathbb{R}^d)}
= \sum_{i=m+1}^{n} 10^{-i} 4^i \cdot \left\| F^{-1} (L_i \psi) \right\|_{L^1(\mathbb{R}^d)}
\leq \left\| F^{-1} \psi \right\|_{L^1(\mathbb{R}^d)} \cdot \sum_{i=m_0+1}^{\infty} \frac{4^i}{10^i} \to 0,
$$

so that $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{D} S_r (Q, L^1, \ell_u^1)$.

If there was some $f \in \mathcal{D} S_r (Q, L^1, \ell_u^1) \subset \mathcal{S}' (\mathbb{R}^d)$ satisfying $f_n \xrightarrow{n \to \infty} f$, the continuous embeddings

$$
\mathcal{D} S_r (Q, L^1, \ell_u^1) \hookrightarrow \mathcal{S} (Q, L^1, \ell_u^1) \hookrightarrow \mathcal{D}' (\mathbb{R}^d)
$$

would imply

$$
\langle f, g \rangle_{\mathcal{D}', \mathcal{D}} = \lim_{n \to \infty} \langle f_n, g \rangle_{\mathcal{D}', \mathcal{D}} \quad \text{for all } g \in \mathcal{D} (\mathbb{R}^d).
$$

Because of the definition of the topology on $\mathcal{S} (\mathbb{R}^d)$, by [14, Proposition 5.15] and because of $f \in \mathcal{S}' (\mathbb{R}^d)$ there are suitable $N \in \mathbb{N}$ and $C > 0$ such that

$$
\left| \langle f, g \rangle_{\mathcal{S}', \mathcal{S}} \right| \leq C \sup_{\alpha \in \mathbb{N}_0^d, x \in \mathbb{R}^d} (1 + |x|)^N \cdot |(\partial^\alpha g) (x)|
$$

holds for all $g \in \mathcal{S} (\mathbb{R}^d)$.

For $n \in \mathbb{N}$ and $g := T_n \psi$ we have $\text{supp} (g) \subset (n - \frac{1}{4}, n + \frac{1}{4}) \subset [0, n + 1]$ and thus

$$
\sup_{\alpha \in \mathbb{N}_0^d, x \in \mathbb{R}^d} (1 + |x|)^N \cdot |(\partial^\alpha g) (x)| \leq (n + 2)^N \cdot \sup_{\alpha \in \mathbb{N}_0^d, x \in \mathbb{R}^d} |(\partial^\alpha \psi) (x)| = (n + 2)^N \cdot C_{\psi, N}
$$

for some constant $C_{\psi, N} \in (0, \infty)$. But because of $\text{supp} (L_n \psi) \cap \text{supp} (L_i \psi) = \emptyset$ for $i, n \in \mathbb{Z}$ with $i \neq n$ we have, for $m \geq n$, the identity

$$
\langle f_m, g \rangle_{\mathcal{S}', \mathcal{S}} = \sum_{j=1}^{m} 4^j \langle L_j \psi, L_n \psi \rangle_{\mathcal{S}', \mathcal{S}} = 4^n \cdot \langle \psi, \psi \rangle_{\mathcal{S}', \mathcal{S}}
$$

and thus

$$
4^n \cdot \|\psi\|_{L^2(\mathbb{R}^d)}^2 = \lim_{m \to \infty} \left| \langle f_m, g \rangle_{\mathcal{S}', \mathcal{S}} \right| = \left| \langle f, g \rangle_{\mathcal{S}', \mathcal{S}} \right|
\leq C \sup_{\alpha \in \mathbb{N}_0^d, x \in \mathbb{R}^d} (1 + |x|)^N \cdot |(\partial^\alpha g) (x)|
\leq CC_{\psi, N} \cdot (n + 2)^N
$$

for all $n \in \mathbb{N}$, a contradiction.

This shows that there is no $f \in \mathcal{D} S_r (Q, L^1, \ell_u^1)$ with $f_n - f\|_{\mathcal{D}(Q, L^1, \ell_u^1)} \xrightarrow{n \to \infty} 0$, so that $\mathcal{D} S_r (Q, L^1, \ell_u^1)$ is not complete.
In the next lemma, we indicate the way in which we will choose the $Q$-moderate weight $u : I \to (0, \infty)$.

**Lemma 22.** Let $\emptyset \neq U \subset \mathbb{R}^d$ be open and let $Q = (Q_i)_{i \in I}$ be a cover of $U$. Finally, let $u : U \to (0, \infty)$ be $Q$-moderate in the sense that there is some constant $C > 0$ such that
\[
\frac{u(x)}{u(y)} \leq C \quad \text{holds for all } i \in I \text{ and all } x, y \in Q_i.
\]
We then say that $u' : I \to (0, \infty)$ is a discretization of $u$ if for every $i \in I$ there is some $x_i \in Q_i$ so that $u'(i) = u(x_i)$ holds. In that case, $u'$ is also $Q$-moderate.

Furthermore, any two discretizations $u', u''$ of $u$ are equivalent in the sense that the estimate $C^{-1} \cdot u'' \leq u' \leq C \cdot u''$ holds for all $i \in I$. In particular we have $\ell_u^q = \ell_{u''}^q$ with equivalent norms and thus also $D(Q, L^p, \ell_u^q) = D(Q, L^p, \ell_{u''}^q)$ (if $Q$ is a decomposition covering of $U$).

**Remark.** Employing the independence of $\ell_u^q$ of the chosen discretization, we write $D(Q, L^p, \ell_u^q)$ for $D(Q, L^p, \ell_{u'}^q)$ and $\ell_u^q(I)$ for $\ell_{u'}^q(I)$ for every discretization $u'$ of $u$. The chosen discretization $u'$ of $u$ will often be denoted by $u$ again.

It is important to note that the notation $\ell_u^q(I)$ is a slight abuse of notation, as the discretization of $u$ (heavily) depends upon the chosen covering $Q$. Hence, different coverings with the same index set can lead to very different spaces $\ell_u^q(I)$.

It is thus important to remember that for a weight $u : U \to (0, \infty)$ the notation $\ell_u^q(I)$ should (and will) only be used as long it is clearly understood which covering is used to form the discretization.

**Proof of Lemma** Let $i \in I$ and $j \in i^*$. Thus there is some $x \in Q_i \cap Q_j \neq \emptyset$. The $Q$-moderateness of $u$ yields
\[
\frac{u'(i)}{u'(j)} = \frac{u(x)}{u(y)} \leq C \cdot \frac{u(x)}{u(y)} \leq C^2,
\]
which shows that $u'$ is $Q$-moderate.

If $u', u''$ are both discretizations of $u$, let $i \in I$, choose $x_i, x'_i \in Q_i$ satisfying $u'_i = u(x_i)$ and $u''_i = u(x'_i)$ and derive
\[
u'_i = u(x_i) \leq C \cdot u(x'_i) = C \cdot u''_i.
\]
The reverse estimate follows by symmetry. \[\square\]

Finally, we transplant a weight $v : H \to (0, \infty)$ onto the dual orbit $O$ by choice of a cross-section. The resulting function on $O$ will be called a transplant of $v$ from $H$ onto $O$. The main observation of the following lemma is that for any two such transplants $u_1, u_2$ of a moderate weight $v$, the quotient $u_1/u_2$ is bounded from above and away from zero, i.e. the two transplants are equivalent.

**Lemma 23.** Let $v : H \to (0, \infty)$ be $v_0$-moderate for some locally bounded, submultiplicative weight $v_0 : H \to (0, \infty)$.

For each $\xi \in O$ choose some $h_\xi \in H$ satisfying $h_\xi^2 \xi_0 = \xi$ and define
\[
u : O \to (0, \infty), \xi \mapsto v(h_\xi).
\]

\[\text{Note that } u' \text{ is a weight on the discrete index set } I, \text{ so that } Q\text{-moderateness of } u' \text{ means that there is a constant } C > 0 \text{ such that we have } u_i \leq C \cdot u_j \text{ for all } i \in I \text{ and } j \in i^* \text{ (cf. Definition 21).}\]
Then $u$ is a Q-moderate function for every covering $Q$ of $\mathcal{O}$ induced by $H$.

Furthermore, any two choices $h_\xi, h_\xi' \in H$ satisfying $h_\xi^T \xi_0 = \xi = (h_\xi')^T \xi_0$ yield equivalent weights $u, u'$ in the sense that $C^{-1} \cdot u' \leq u \leq C \cdot u'$ holds for some constant $C \in (0, \infty)$. The same is true for any two choices of $\xi_0$.

**Remark 24.** The induced covering is of the form $Q = (Q_i)_{i \in I} = (h_i^{-T}Q)_{i \in I}$ for some well-spread family $(h_i)_{i \in I}$ and a suitable set $Q \subset \mathcal{O}$. As long as we have $\xi_0 \in Q$, we can choose $h_i^{-T} \xi_0 = h_i^{-1}$ and discretize $u$ by $u'_i = u (h_i^{-T} \xi_0) = v (h_i^{-T} \xi_0) = v (h_i^{-1})$, where we used $h_i^{-T} \xi_0 \in Q_i$.

It is worth noting that $\xi_0 \in \mathcal{O}$ can be selected arbitrarily, so that it is always possible to choose $\xi_0 \in Q$.

**Proof.** Let $Q = (h_i^{-T}Q)_{i \in I}$ be a covering of $\mathcal{O}$ induced by $H$. This means that $(h_i)_{i \in I}$ is well-spread in $H$, that $Q \subset \mathbb{R}^d$ is precompact with $\overline{Q} \subset \mathcal{O}$ and that $Q$ is a covering of $\mathcal{O}$. Let $i \in I$ and $x, y \in Q_i = h_i^{-T}Q$ be arbitrary. Then we have

$$(h_x h_i)^T \xi_0 = (h_i^{-T}Q) \xi_0 = h_i^{-T}x \in Q \subset \overline{Q} \quad \text{and similarly} \quad (h_y h_i)^T \xi_0 \in \overline{Q}.$$ 

This means $h_x h_i \in p_{\xi_0^{-1}}(\overline{Q}) =: K$ and $h_i h_i \in K$. Note that $K \subset H$ is compact by Lemma [6].

Using this, we can estimate

$$u(x) = v(h_x) = v(h_x h_i^{-1} h_i^{-1} h_y 1_H) \leq v_0(h_x h_i^{-1} h_i^{-1} h_y) \cdot v(h_y) \cdot (1_H) = v_0(1_H) \cdot v_0(h) \cdot v(h_y) \cdot v(h_y) \leq \left[ v_0(1_H) \cdot \sup_{h \in K} v_0(h) \right] \cdot u(y) = C \cdot u(y),$$

where the value $C = v_0(1_H) \cdot \sup_{h \in K} v_0(h)$ is independent of $x, y \in Q_i$ and of $i \in I$ and finite because $v_0$ is locally bounded. Thus, $u$ is $Q$-moderate.

In order to show the independence of the choice of $h_\xi$, let $\xi \in \mathcal{O}$ and choose $h_\xi, h_\xi' \in H$ such that $h_\xi^T \xi_0 = \xi = (h_\xi')^T \xi_0$. Then $h_\xi \cdot (h_\xi')^{-1} \in H_{\xi_0}$ which implies

$$v(h_\xi) = v(h_\xi (h_\xi')^{-1} \cdot h_\xi') \leq v(h_\xi') \cdot \left[ v_0(1_H) \cdot \sup_{h \in H_{\xi_0}} v_0(h) \right],$$

where the expression in brackets is an absolute constant which is finite by local boundedness of $v_0$ and compactness of $H_{\xi_0}$.

The proof of independence of $\xi_0$ runs along similar lines, and is omitted. \hfill \Box

Finally, we want to show that all induced coverings (with respect to the same well-spread family $(h_i)_{i \in I}$) yield the same decomposition spaces, at least as long as the weight $u$ is obtained by transplanting a weight on $H$ onto $\mathcal{O}$. This will be an easy consequence of the following more general lemma.

**Lemma 25.** Let $\emptyset \not= U \subset \mathbb{R}^d$ be an open set and assume that $Q = (Q_i)_{i \in I}$ and $Q' = (Q'_i)_{i \in I}$ are two admissible coverings of $U$ that are indexed by the same set $I$. Furthermore, assume that $Q$ is a decomposition covering of $U$ that satisfies $Q_i \subset Q'_i$ for all $i \in I$.

Then $Q'$ is also a decomposition covering of $U$. More precisely, any Q-BAPU $(\varphi_i)_{i \in I}$ is also a BAPU for $Q'$. Finally, if $u : U \rightarrow (0, \infty)$ is $Q'$-moderate, it is also $Q$-moderate and for
$p, q \in [1, \infty]$ we have
$$\mathcal{D}(Q, L^p, \ell_u^q) = \mathcal{D}(Q', L^p, \ell_u^q)$$
with equivalent norms.

**Proof.** The only property of a BAPU $(\varphi_i)_{i \in I}$ that is specific to the covering $Q$ (or $Q'$) is the requirement $\varphi_i(x) = 0$ for all $x \in U \setminus Q_i$. Because of $U \setminus Q'_i \subset U \setminus Q_i$, it is clear that this condition is also fulfilled for the covering $Q'$ instead of $Q$.

If $u : U \to (0, \infty)$ is $Q'$-moderate, there is a constant $C > 0$ so that $\frac{u(x)}{u(y)} \leq C$ holds for all $i \in I$ and all $x, y \in Q'_i$. Because of $Q_i \subset Q'_i$ the same estimate also holds for all $i \in I$ and all $x, y \in Q_i$.

In order to show the equality $\mathcal{D}(Q, L^p, \ell_u^q) = \mathcal{D}(Q', L^p, \ell_u^q)$ note that for a discretization $u' : I \to (0, \infty)$ with respect to $Q$, $u'$ is also a discretization for $Q'$, as for $i \in I$ there is some $x_i \in Q_i \subset Q'_i$ that satisfies $u'_i = u(x_i)$. As seen above, any $Q$-BAPU $(\varphi_i)_{i \in I}$ is also a $Q'$-BAPU.

With these choices we see
$$\|f\|_{\mathcal{D}(Q, L^p, \ell_u^q)} = \left\| \left( u'_i \cdot \|F^{-1}(\varphi_i f)\|_{L^p(\mathbb{R}^d)} \right)_{i \in I} \right\|_{L^q(I)} = \|f\|_{\mathcal{D}(Q', L^p, \ell_u^q)}$$
for all $f \in \mathcal{D}'(U)$. This shows $\mathcal{D}(Q, L^p, \ell_u^q) = \mathcal{D}(Q', L^p, \ell_u^q)$ and because any choice of discretization of the weight $u$ (with respect to $Q$ or $Q'$) and of the BAPUs for $Q$ or $Q'$ yield equivalent norms on $\mathcal{D}(Q, L^p, \ell_u^q)$ or $\mathcal{D}(Q', L^p, \ell_u^q)$, the claim follows (for any such choice).

We can now conclude that two different induced decomposition coverings – with respect to the same well-spread family $(h_i)_{i \in I}$ – yield identical decomposition spaces. The isomorphism between $\mathcal{D}(Q, L^p, \ell_u^q)$ and $\mathcal{C}_0(L^p, \varphi)$ that we will prove below (see Theorems 37 and 43) will of course show that the same is true even if different well-spread families $(h_i)_{i \in I}$ are used to obtain the two decomposition coverings $Q, Q'$. Even so, this does not make the following result redundant, as it will allow us to switch from the covering $(h_i^{-T}Q)_{i \in I}$ to a larger covering $(h_i^{-T}Q')_{i \in I}$ in the proof of Theorem 37.

**Corollary 26.** Let $Q = (h_i^{-T}Q)_{i \in I}$ and $Q' = (h_i^{-T}Q')_{i \in I}$ be two (possibly different) decomposition coverings of $\mathcal{O}$ induced by $H$. Then we have
$$\mathcal{D}(Q, L^p, \ell_u^q) = \mathcal{D}(Q', L^p, \ell_u^q)$$
with equivalent norms for every weight $u : \mathcal{O} \to (0, \infty)$ obtained by transplanting a $v_0$-moderate weight $v : H \to (0, \infty)$ onto $\mathcal{O}$, where $v_0 : H \to (0, \infty)$ is submultiplicative and locally bounded.

**Proof.** The assumptions guarantee that $(h_i)_{i \in I}$ is well-spread in $H$ and that $\overline{\bigcup_i h_i^{-T}Q} \subset \mathcal{O}$ are compact sets that satisfy $\bigcup_i h_i^{-T}Q = \mathcal{O} = \bigcup_i h_i^{-T}Q'$. Thus, $Q'' := Q \cup Q' \subset \mathcal{O}$ also satisfies $\mathcal{O} = \bigcup_i h_i^{-T}Q''$ and $\overline{\bigcup_i h_i^{-T}Q''} \subset \mathcal{O}$ is compact.

Thus, $Q, Q'$ and $Q''$ are coverings of $\mathcal{O}$ induced by $H$. Theorem 20 then shows that these are admissible coverings and Lemma 25 shows that $u$ is moderate with respect to any of these coverings. Because $Q$ is a decomposition covering, the same is true for $Q''$ by Lemma 25. The inclusion
$$Q_i \cup Q'_i = h_i^{-T}Q \cup h_i^{-T}Q' \subset h_i^{-T}Q'' = Q''_i$$
which is valid for all $i \in I$ and Lemma 25 then imply
$$\mathcal{D}(Q, L^p, \ell_u^q) = \mathcal{D}(Q'', L^p, \ell_u^q) = \mathcal{D}(Q'', L^p, \ell_u^q)$$
with equivalent norms. \[\blacksquare\]
6. Construction of a specific BAPU for induced coverings

In this section we construct a specific BAPU for the covering of the dual orbit induced by $H$. This BAPU will allow us to prove the continuity of $F : Co (L^p, Q) \to D (Q, L^p, R^n)$, where $Q$ is an induced decomposition covering of the dual orbit. The idea of the construction is to take a Schwartz function $\psi \in S (\mathbb{R}^d)$ such that $\hat{\psi} \in D (O)$ is compactly supported in $O$ and then define

$$
\varphi_U (\xi) := \int_U |\hat{\psi} (h^T \xi) |^2 \, dh \quad \text{for } \xi \in O
$$

(6.1)

for $U \subset H$ precompact and measurable. We will then show that $\varphi_U \in D (O)$ is smooth with

$$
\text{supp} (\varphi_U) \subset \overline{U}^T \text{supp} (\hat{\psi})
$$

and that $(\varphi_U)_{i \in I}$ is a (multiple of a) partition of unity on $O$ if $(U_i)_{i \in I}$ is a partition of $H$.

We now show that the construction indicated in equation (6.1) indeed yields a test function $\varphi_U \in D (O)$. Note that we could also use “differentiation under the integral sign” instead of Lemma 7 in the following proof. But as we need Lemma 7 nonetheless, we prefer the following argument.

**Lemma 27.** Assume $\psi \in S (\mathbb{R}^d)$ with $\hat{\psi} \in D (O)$. Let $U \subset H$ be precompact and measurable. Then

$$
\varphi_U : \mathbb{R}^d \to [0, \infty), \xi \mapsto \int_U |\hat{\psi} (h^T \xi) |^2 \, dh
$$

is well-defined with $\varphi_U \in D (O)$.

More precisely, we have $\varphi_U \equiv 0$ on $\mathbb{R}^d \setminus (U^{-T} \cdot \hat{\psi} (C^*))$ and thus

$$
\text{supp} (\varphi_U) \subset \overline{U}^T \cdot \text{supp} (\hat{\psi}) \subset H^T O = O.
$$

**Proof.** Let $\varphi := |\hat{\psi}|^2 \in D (O) \subset D (\mathbb{R}^d)$. Lemma 7 shows that

$$
\Phi : \text{GL} (\mathbb{R}^d) \to D (\mathbb{R}^d), \, h \mapsto \varphi (h^T \cdot) = |\hat{\psi} (h^T \cdot) |^2
$$

is well-defined and continuous, so that $\Phi$ is in particular continuous on the compact set $\overline{U} \subset H$.

For $k \in \mathbb{N}_0$ the inclusion $\iota_k : D (\mathbb{R}^d) \hookrightarrow C^k_b (\mathbb{R}^d)$ with

$$
C^k_b (\mathbb{R}^d) := \left\{ f \in C^k (\mathbb{R}^d) \left| \| f \|_{C^k_b} := \sum_{|\alpha| \leq k} \| \partial^\alpha f \|_{\text{sup}} < \infty \right\}
$$

is continuous, so that the Bochner integral of the function $\iota_k \circ \Phi$, $\psi_k := \int_U (\iota_k \circ \Phi) (h) \, dh \in C^k_b (\mathbb{R}^d)$

is well-defined, because of $\mu_H (U) \leq \mu_H (\overline{U}) < \infty$, where $\mu_H$ is the Haar-measure on $H$.

As the evaluation mapping $\alpha_\xi : C^k_b (\mathbb{R}^d) \to C, \, f \mapsto f (\xi)$ is continuous for every $\xi \in \mathbb{R}^d$, we easily see $\psi_k (\xi) = \varphi_U (\xi)$ for each $\xi \in \mathbb{R}^d$, so that we conclude $\varphi_U = \psi_k \in C^k_b (\mathbb{R}^d)$ for all $k \in \mathbb{N}_0$ which shows that $\varphi_U$ is smooth.

Finally, if we have $0 \neq \varphi_U (\xi) = \int_U |\hat{\psi} (h^T \xi) |^2 \, dh$, there is some $h \in U$ with $h^T \xi \in \hat{\psi} (C^*)$, i.e.

$$
\xi \in h^{-T} \cdot \hat{\psi} (C^*) \subset U^{-T} \cdot \hat{\psi} (C^*). \quad \Box
$$
We now calculate the (inverse) Fourier transform of the function $\varphi_U$ as defined in the preceding lemma. The formula that we derive will be important for the proof of the continuity of the Fourier transform $F: C_0(L^p,q) \to \mathcal{D}(Q, L^p, \ell^q_0)$.

**Lemma 28.** Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\hat{\psi} \in \mathcal{D}(\mathcal{O})$. For $\gamma := F^{-1}(|\hat{\psi}|^2) \in \mathcal{S}(\mathbb{R}^d)$ and some precompact, measurable $U \subset H$ we have, with $\varphi_U$ defined as in Lemma 27, \[
(F^{-1}\varphi_U)(x) = \int_U \frac{\gamma(h^{-1}x)}{|\det(h)|} \, dh \quad \text{for all} \quad x \in \mathbb{R}^d. \tag{6.2}
\] Furthermore, the estimate \[
\|F^{-1}\varphi_U\|_{L^1(\mathbb{R}^d)} \leq \mu_H(U) \cdot \|\gamma\|_{L^1(\mathbb{R}^d)} < \infty
\] holds, where $\mu_H$ denotes the chosen (left) Haar-measure on $H$.

**Proof.** Let $\varphi := |\hat{\psi}|^2 \in \mathcal{D}(\mathcal{O})$. Then Lemma [3] shows that \[
\Phi : GL(\mathbb{R}^d) \to \mathcal{D}(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d), \quad h \mapsto \varphi(h^T \cdot) = |\hat{\psi}(h^T \cdot)|^2
\] is well-defined and continuous. Hence \[
\int_U \int_{\mathbb{R}^d} \left| \hat{\psi}(h^T \xi) \right|^2 \cdot e^{2\pi i(x, \xi)} \, d\xi \, dh = \int_U \|\Phi(h)\|_{L^1(\mathbb{R}^d)} \, dh \leq \mu_H(U) \cdot \sup_{h \in \overline{U}} \|\Phi(h)\|_{L^1(\mathbb{R}^d)} < \infty,
\] where we used that $\overline{U}$ is compact. Fubini’s theorem, the change of variables formula and Fourier inversion now yield \[
(F^{-1}\varphi_U)(x) = \int_{\mathbb{R}^d} \int_U \left| \hat{\psi}(h^T \xi) \right|^2 \, dh \cdot e^{2\pi i(x, \xi)} \, d\xi
\] \[
= \int_U \frac{1}{|\det(h)|} \cdot \int_{\mathbb{R}^d} \left| \hat{\psi}(h^T \xi) \right|^2 \cdot e^{2\pi i(h^{-1}x, h^T \xi)} \cdot |\det(h^T)| \, d\xi \, dh
\] \[
eq \int_U \gamma(h^{-1}x) \cdot \int_{\mathbb{R}^d} \hat{\gamma}(\xi) \cdot e^{2\pi i(h^{-1}x, \xi)} \, d\xi \, dh
\] \[
= \int_U \gamma(h^{-1}x) \cdot |\det(h)| \, dh.
\] A second application of Fubini’s theorem and the change of variables formula finally yields \[
\|F^{-1}\varphi_U\|_1 = \int_{\mathbb{R}^d} \left| \int_U \frac{\gamma(h^{-1}x)}{|\det(h)|} \, dh \right| \, dx
\] \[
\leq \int_U \int_{\mathbb{R}^d} \gamma(h^{-1}x) \cdot |\det(h^{-1})| \, dx \, dh
\] \[
= \int_U \int_{\mathbb{R}^d} |\gamma(h)| \cdot |\det(h)| \, d\xi \, dh
\] \[
\leq \mu_H(U) \cdot \|\gamma\|_{L^1(\mathbb{R}^d)} < \infty. \quad \square
\]

We now want to show that $(\varphi_{U_i})_{i \in I}$ yields a partition of unity on $\mathcal{O}$. In order to do so, we first need the following technical lemma. Its proof is straightforward, and therefore omitted.
Lemma 29. The map
\[ \Theta : \mathbb{R}^d \times C_0 \left( \mathbb{R}^d \right) \to C_0 \left( \mathbb{R}^d \right), (\omega, g) \mapsto M_\omega g \]
is (jointly) continuous.

Here, \( C_0 \left( \mathbb{R}^d \right) \) is the space of (complex valued) continuous functions vanishing at infinity endowed with the sup-norm.

We are now almost ready to show that \( (\varphi_{U_i})_{i \in I} \) indeed yields a (multiple of a) partition of unity on \( O \) for each (precompact, measurable) partition \( (U_i)_{i \in I} \) of \( H \). The only thing missing is the so-called wavelet inversion formula. For the validity of this formula, we again use our assumptions on \( H \), which imply (as noted above) that \( \pi \) is an irreducible, square-integrable representation on \( L^2 \left( \mathbb{R}^d \right) \). Then [6, Theorem 3] states (in our notation) the following:

**Theorem 30.** (Theorem 3) There is a self-adjoint, positive operator \( K : \text{dom} \left( K \right) \to L^2 \left( \mathbb{R}^d \right) \) (with \( \text{dom} \left( K \right) \leq L^2 \left( \mathbb{R}^d \right) \)) satisfying the following conditions:

1. For \( \psi \in L^2 \left( \mathbb{R}^d \right) \) the following are equivalent:
   a. \( W_\psi \psi \in L^2 \left( \mathbb{R}^d \right) \),
   b. \( W_\psi f \in L^2 \left( \mathbb{R}^d \right) \) for some \( f \in L^2 \left( \mathbb{R}^d \right) \setminus \{0\} \),
   c. \( \psi \in \text{dom} \left( K^{-1/2} \right) \).

2. For \( \varphi, \psi \in \text{dom} \left( K^{-1/2} \right) \) (i.e. with \( W_\psi \psi, W_\varphi \varphi \in L^2 \left( \mathbb{R}^d \right) \)) and arbitrary \( f, g \in L^2 \left( \mathbb{R}^d \right) \) we have
\[
\left\langle W_\psi f, W_\varphi g \right\rangle_{L^2(G)} = \left\langle K^{-1/2} \varphi, K^{-1/2} \psi \right\rangle_{L^2(\mathbb{R}^d)} \cdot \left\langle f, g \right\rangle_{L^2(\mathbb{R}^d)} . \tag{6.3}
\]

For \( \psi \in L^2 \left( \mathbb{R}^d \right) \setminus \{0\} \) with \( W_\psi \psi \in L^2 \left( \mathbb{R}^d \right) \) and
\[
C_\psi := \left\langle K^{-1/2} \psi, K^{-1/2} \psi \right\rangle_{L^2(\mathbb{R}^d)} = \frac{\|W_\psi \psi\|^2_{L^2(G)}}{\|\psi\|^2_{L^2(\mathbb{R}^d)}} > 0
\]
this entails the wavelet inversion formula
\[
f = \frac{1}{C_\psi} \cdot \int_G \left( W_\psi f \right)(x) \cdot \pi(x) \psi \, dx \quad \text{in the weak sense for all } f \in L^2 \left( \mathbb{R}^d \right) . \tag{6.4}
\]

The following lemma relates the constant \( C_\psi \) to a continuous partition of unity on the Fourier transform side. The discretization of this partition of unity (essentially by cutting up the integration domain \( H \) into chunks of comparable sizes) will provide the BAPU \( (\varphi_{U_i}) \).

**Lemma 31.** For \( \psi \in \mathcal{S} \left( \mathbb{R}^d \right) \setminus \{0\} \) with \( \hat{\psi} \in \mathcal{D} \left( O \right) \) we have \( W_\psi \psi \in L^2(G) \). Furthermore, we have
\[
\frac{1}{C_\psi} \cdot \int_H \left| \hat{\psi} \left( h^T \xi \right) \right|^2 \, dh = 1 \quad \text{for every } \xi \in O . \tag{6.5}
\]

**Proof.** First note that by Theorem 3 \( W_\psi \psi \in L^1(G) \). In addition, \( W_\psi \psi \in L^\infty(G) \) by the Cauchy-Schwarz inequality, whence finally \( W_\psi \psi \in L^2(G) \). Now [15, Lemma 9] yields
\[ C_\psi = \int_H \left| \hat{\psi} \left( h^T \xi \right) \right|^2 \, dh . \]

With these preparations it is now easy to show that \( (\varphi_{U_i})_{i \in I} \) indeed yields (a multiple of) a BAPU if the sets \( (U_i)_{i \in I} \) form a suitable partition of \( H \).
Theorem 32. Let \((h_i)_{i \in I}\) be well-spread in \(H\) with \(H = \bigcup_{i \in I} h_i U\) for some precompact, measurable \(U \subset H\). Furthermore, let \(\psi \in \mathcal{S}([\mathbb{R}^d] \setminus \{0\})\) satisfy \(\hat{\psi} \in \mathcal{D}(\mathcal{O})\).

Let \((i_n)_{n \in \mathbb{N}}\) be an enumeration of \(I\) (note that \(I\) is countably infinite by Lemma 24) and define \(U_{i_n} := h_{i_n} U \setminus \bigcup_{m=1}^{n-1} h_{i_m} U\) for \(n \in \mathbb{N}\). Then \((U_{i_n})_{i \in I}\) is a measurable partition of \(H\) satisfying \(U_i \subset h_i U\) for all \(i \in I\).

Define \(Q := U^{-T} \left( \hat{\psi}^{-1}(C^*) \right) \subset \mathcal{O}\). Then \(Q \subset \mathcal{O}\) is open and precompact satisfying \(\overline{Q} \subset \mathcal{O}\) and \(\mathcal{O} = \bigcup_{i \in I} h_i^{-T} Q\), so that \(Q = (h_i^{-T} Q)_{i \in I}\) is a covering of \(\mathcal{O}\) induced by \(H\). Finally, \((\varphi_i)_{i \in I} := \left(\frac{1}{\hat{\psi}} \varphi_{U_{i_n}}\right)_{i \in I}\) defines a BAPU that is subordinate to this covering.

Proof. It is easy to see that \((U_{i_n})_{i \in I}\) forms a measurable partition of \(H\). Note that

\[
Q = \bigcup_{h \in U} h^{-T} \left( \hat{\psi}^{-1}(C^*) \right) \subset \mathcal{O}
\]

is open as a union of open sets. Furthermore, we have \(\overline{Q} \subset U^{-T} \text{supp}(\hat{\psi}) \subset \mathcal{O}\), so that \(\overline{Q} \subset \mathcal{O}\) is compact. Because of \(\psi \neq 0\) we also have \(\hat{\psi} \neq 0\), so that there exists some \(\xi_1 \in \hat{\psi}^{-1}(C^*) \subset \mathcal{O}\).

We have \(\xi_1 = h\xi_0\) for some \(h \in H\). This implies

\[
\bigcup_{i \in I} h_i^{-T} Q \supset \bigcup_{i \in I} (h_i U)^{-T} h^T \xi_0 = h^T \xi_0 = \mathcal{O}.
\]

By Lemma 27, we have \(\varphi_{U_{i_n}} \in \mathcal{D}(\mathcal{O})\) with \(\varphi_{U_{i_n}}(x) = 0\) for

\[
x \in \mathcal{O} \setminus \left( U_i^{-T} \left( \hat{\psi}^{-1}(C^*) \right) \right) \supset \mathcal{O} \setminus \left( (h_i U)^{-T} \left( \hat{\psi}^{-1}(C^*) \right) \right) = \mathcal{O} \setminus (h_i^{-T} Q).
\]

Furthermore, Lemma 28 yields

\[
\|F^{-1} \varphi_i\|_{L^1(\mathbb{R}^d)} = \frac{1}{\gamma} \|F^{-1} \varphi_i|_{L^1(\mathbb{R}^d)} \leq \mu_H(U_i) \cdot \frac{\|\gamma\|_{L^1(\mathbb{R}^d)}}{\gamma} \leq \mu_H(h_i U) \cdot \frac{\|\gamma\|_{L^1(\mathbb{R}^d)}}{\gamma} =: C
\]

for \(\gamma := F^{-1}(\hat{\psi}) \in \mathcal{S}(\mathbb{R}^d)\).

Finally, Lemma 31 and the definition of \(\varphi_U\) (equation 6.1) show that for \(\xi \in \mathcal{O}\) we have

\[
\sum_{i \in I} \varphi_i(\xi) = \frac{1}{\gamma} \cdot \sum_{n=1}^{\infty} \varphi_{U_{i_n}}(\xi)
\]

\[
= \frac{1}{\gamma} \cdot \sum_{n=1}^{\infty} \int_{U_{i_n}} \left| \hat{\psi}(h^T \xi) \right|^2 dh
\]

\[
= \frac{1}{\gamma} \cdot \int_{\bigcup_{n \in \mathbb{N}} U_{i_n}} \left| \hat{\psi}(h^T \xi) \right|^2 dh
\]

\[
= 1,
\]

because \((U_{i_n})_{n \in \mathbb{N}}\) is a partition of \(H\). Thus, \((\varphi_i)_{i \in I}\) is recognized as a \(Q\)-BAPU.
7. Continuity of the Fourier transform from Coorbit spaces into Decomposition spaces

In this section we will show that the Fourier transform on $\text{Co}(L^p,q_v)$ as defined in Corollary 10 is well-defined and continuous as a map into the decomposition space $D(\mathcal{Q}, L^p, \ell^u_q)$, where $\mathcal{Q}$ is a covering of $\mathcal{O}$ induced by $H$ and $u$ is the transplant of a suitable weight on $H$.

We will first show this for $f \in \text{Co}(L^p,q_v) \cap S(\mathbb{R}^d)$ and then use a density result (namely the Atomic Decomposition in $\text{Co}(L^p,q_v)$, cf. [10, Theorem 6.1]) to establish the result in the general case.

We start by explicitly computing the localizations

$$\mathcal{F}^{-1}(\varphi_V \cdot \hat{f})$$

for an arbitrary measurable, precompact set $V \subset H$ and $f \in S(\mathbb{R}^d)$ in terms of the wavelet transform $W_\psi f$. As the norm on $\text{Co}(L^p,q_v)$ is defined in terms of $W_\psi f$, this is the essential step in our proof.

In the ensuing calculations, we will use the following elementary result:

**Lemma 33.** Let $f, g \in L^1(\mathbb{R}^d)$. For $h \in \text{GL}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we have

$$(D_h (f * g))(x) = |\det(h)| \cdot ((D_h f) * (D_h g))(x),$$

whenever either side of the equation is defined.

**Lemma 34.** Let $f, \psi \in S(\mathbb{R}^d)$ where $\hat{\psi}$ has compact support in $\mathcal{O}$. Furthermore assume that $V \subset H$ is precompact and measurable. For $x \in \mathbb{R}^d$ we have

$$\left(\mathcal{F}^{-1}(\varphi_V \cdot \hat{f})\right)(x) = \int_V |\det(h)|^{-3/2} \cdot ((W_\psi f) (\cdot , h) * D_h - T \psi)(x) \, dh,$$

with $\varphi_V$ defined in equation (6.1).

**Proof.** Choose $\gamma := \mathcal{F}^{-1}(|\hat{\psi}|^2) \in S(\mathbb{R}^d)$ as in Lemma 28. By the convolution theorem we have

$$\gamma = \mathcal{F}^{-1}(\hat{\psi} \cdot \overline{\hat{\psi}}) = \left(\mathcal{F}^{-1}\hat{\psi}\right) * \left(\mathcal{F}^{-1}\overline{\hat{\psi}}\right) = \psi \ast \psi^*$$

with $\psi^* : \mathbb{R}^d \to \mathbb{C}, x \mapsto \overline{\psi(-x)}$.

Using Lemma 38 together with $(f * D_h - T \psi^*)(x) = |\det(h)|^{1/2} \cdot (W_\psi f)(x, h)$ and basic properties of convolution products, we obtain

$$f * D_h - T \gamma = |\det(h)|^{-1/2} \cdot ((W_\psi f)(\cdot, h)) * (D_h - T \psi).$$

Now Lemma 28 yields the representation

$$\left(\mathcal{F}^{-1}\varphi_V\right)(x) = \int_V \frac{\gamma(h^{-1}x)}{|\det(h)|} \, dh \quad \text{for } x \in \mathbb{R}^d.$$
for the inverse Fourier transform of $\varphi_V$. Putting everything together and using the convolution theorem and Fubini’s theorem, we see

$$\left( F^{-1} \left( \varphi_V \ast \hat{f} \right) \right) (x) = \left( \left( F^{-1} \varphi_V \right) \ast F^{-1} \hat{f} \right) (x)$$

$$= \int_{\mathbb{R}^d} \varphi_V (y) \cdot \left( F^{-1} \varphi_V \right) (x - y) \, dy$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi_V (y) \cdot \frac{\gamma (h^{-1} (x - y))}{|\det (h)|} \, dy \, dh$$

$$= \int_{\mathbb{R}^d} |\det (h)|^{-1} (f \ast D_{h^{-1} \gamma}) (x) \, dh$$

$$= \int_{\mathbb{R}^d} |\det (h)|^{-3/2} ((W_\varphi) (\cdot, h) \ast D_{h^{-1} \gamma}) (x) \, dh.$$

Here we used Fubini’s theorem, as justified by

$$\int_{\mathbb{R}^d} |\det (h)|^{-1} \cdot \int_{\mathbb{R}^d} |\varphi_V (y)| \cdot |\gamma (h^{-1} (x - y))| \, dy \, dh$$

$$\leq \int_{\mathbb{R}^d} |\det (h)|^{-1} \cdot \|\varphi_V\|_{L^1 (\mathbb{R}^d)} \cdot \|\gamma\|_{\text{sup}} \, dh$$

$$\leq \mu_H \left( \mathbb{V} \right) \cdot \|\varphi_V\|_{L^1 (\mathbb{R}^d)} \cdot \|\gamma\|_{\text{sup}} \cdot \max_{h \in \mathbb{V}^{-1}} |\det (h)| < \infty.$$  □

With this representation of the localized “pieces” of $\hat{f}$, we are now ready to prove the continuity of $F : S \left( \mathbb{R}^d \right) \cap \text{Co} (L^p, q) \to D \left( \mathcal{Q}, L^p, \ell_q \right)$, where $u$ is a transplant of $h \mapsto |\det (h^{-1})|^{\frac{1}{2} - \frac{d}{q}} \cdot v (h^{-1})$.

**Lemma 35.** Let $p, q \in [1, \infty]$. Then

$$v' : \mathbb{R}^d \to (0, \infty), h \mapsto |\det (h^{-1})|^{\frac{1}{2} - \frac{d}{q}} \cdot v (h^{-1})$$

is moderate with respect to the measurable, locally bounded, submultiplicative weight

$$v_0' : \mathbb{R}^d \to (0, \infty), h \mapsto |\det (h^{-1})|^{\frac{1}{2} - \frac{d}{q}} \cdot v_0 (h^{-1}).$$

Choose an arbitrary $\psi \in S (\mathbb{R}^d \setminus \{0\})$ with $\hat{\psi} \in D (\mathcal{O})$ and let $(h_i)_{i \in I}$ be well-spread in $H$ with $H = \bigcup_{i \in I} h_i U$ for some precompact unit-neighborhood $U \subset H$.

Let $Q := U^{-1} \left( \hat{\psi}^{-1} (C^2) \right)$ and $Q = (h_i^{-1} Q)_{i \in I}$ be the corresponding induced covering of $\mathcal{O}$ (cf. Theorem 20). Let $u : \mathcal{O} \to (0, \infty)$ be a transplant of $v'$ onto $\mathcal{O}$.

Then there is a constant $C > 0$ satisfying

$$\left\| \hat{f} \right\|_{D (\mathcal{Q}, L^p, \ell_q)} \leq C \cdot \|\varphi_V\|_{\text{Co} (L^p, q)} < \infty$$

for all $f \in S (\mathbb{R}^d) \cap \text{Co} (L^p, q)$.

**Remark.** In the above setting, for a suitable choice of $u$, one possible discretization of $u$ with respect to $Q$ is given by

$$u_i = |\det (h_i)|^{\frac{1}{2} - \frac{d}{q}} \cdot v (h_i)$$

and any (different) choice yields a weight on $I$ that is equivalent to $(u_i)_{i \in I}$.

**Proof.** Lemma 35 shows that $v_0' : \mathbb{R}^d \to (0, \infty), h \mapsto v_0 (h^{-1})$ and hence also $v'$ are submultiplicative. It is easy to see that $v'$ is moderate with respect to $v_0'$. This implies that $v'$ is moderate with respect to $v_0$. It is clear that with $v_0$ also $v_0'$ is locally bounded and measurable.
Choose \((\bar{\varphi}_i)_{i \in I} = \left(\frac{1}{v_i} \varphi U_i\right)_{i \in I}\) as in Theorem 32. By that theorem, \((\bar{\varphi}_i)_{i \in I}\) is a Q-BAPU.

Let \(f \in \mathcal{S}(\mathbb{R}^d) \cap \text{Co}(L^{q,\infty})\). We use Lemma 34, Minkowski’s inequality for integrals (cf. Theorem 6.19]) and Young’s inequality to calculate, for arbitrary precompact and measurable \(V \subset H\):

\[
\left\| \mathcal{F}^{-1} \left( \varphi_V \cdot \hat{f} \right) \right\|_{L^p(\mathbb{R}^d)} \leq \int_V |\det(h)|^{-3/2} \| (W_V f) (\cdot, h) \|_{L^p(\mathbb{R}^d)} \, dh
\]

Young

\[
\leq \int_V |\det(h)|^{-3/2} \| D (\cdot, h) \|_{L^1(\mathbb{R}^d)} \cdot \| (W_V f) (\cdot, h) \|_{L^p(\mathbb{R}^d)} \, dh
\]

\[
= \| \psi \|_{L^1(\mathbb{R}^d)} \cdot \int_V |\det(h)|^{-1/2} \| (W_V f) (\cdot, h) \|_{L^p(\mathbb{R}^d)} \, dh
\]

(7.1)

where we assumed \(\mu_H(V) > 0\) in the last step.

For \(i \in I\) choose \(V = U_i \subset h_i U\) (cf. Theorem 32) and assume \(\mu_H(U_i) > 0\). In the case \(q \in [1, \infty)\), Jensen’s inequality yields (by convexity of \(R \to \mathbb{R}_+, x \mapsto |x|^q\) the estimate

\[
\left\| \mathcal{F}^{-1} \left( \varphi_U \cdot \hat{f} \right) \right\|^q_{L^p(\mathbb{R}^d)} \leq \left( \| \psi \|_{L^1(\mathbb{R}^d)} \cdot \mu_H(U_i) \right)^q \int_{U_i} \| \det(h) \|^{-1/2} \cdot \| (W_V f) (\cdot, h) \|_{L^p(\mathbb{R}^d)} \, dh
\]

\[
\leq \| \psi \|_{L^1(\mathbb{R}^d)} \cdot \mu_H(U_i)^{q-1} \int_{U_i} \| \det(h) \|^{q - 2} \cdot \| (W_V f) (\cdot, h) \|_{L^p(\mathbb{R}^d)} \, dh
\]

(7.2)

where we used \(\mu_H(U_i) \leq \mu_H(h_i U) \leq \mu_H(U)\) and \(q - 1 \geq 0\) in the last step. Note that the above estimate is trivial in the case \(\mu_H(U_i) = 0\).

For \(h \in U_i \subset h_i U\) we now have \(h = h_i u\) for some \(u \in U\). With \(C_1 := \min_{k \in U} |\det(k)|\) and \(C_2 := \max_{k \in U} |\det(k)|\), we thus get

\[
\frac{|\det(h)|}{|\det(h_i)|} = |\det(u)| \in [C_1, C_2].
\]

As the map \(\mathbb{R}_+ \to \mathbb{R}_+, x \mapsto x^{\frac{q-2}{2}}\) is monotonic (increasing for \(q \leq 2\) and decreasing for \(q \geq 2\), we derive

\[
\frac{|\det(h)|^{\frac{q-2}{2}}}{|\det(h_i)|^{\frac{q-2}{2}}} \in \left[ \min \left\{ C_1^{\frac{q-2}{2}}, C_2^{\frac{q-2}{2}} \right\}, \max \left\{ C_1^{\frac{q-2}{2}}, C_2^{\frac{q-2}{2}} \right\} \right] =: [C_3, C_4].
\]

Now let \(C_5 := v_0(1_H) \cdot \sup_{k \in U^{-1}} v_0(k)\). Then we have

\[
v(h) = v(1_H h u^{-1}) \leq v_0(1_H) \cdot v(h) \cdot v_0(u^{-1}) \leq C_5 \cdot v(h)
\]

for all \(h = h_i u \in U_i\).
Putting all this together and setting $C_6 := \|\psi\|_{L^1(\mathbb{R}^d)} \cdot (\mu_H (\overline{U}))^{1 - \frac{q}{d}} / C_\psi$, we arrive at

\[
\sum_{i \in I} \left( \frac{1}{|\det (h_i)|^{\frac{1}{d} - \frac{q}{d}} / v (h_i)} \cdot \left\| \mathcal{F}^{-1} \left( \varphi \cdot \hat{f} \right) \right\|_{L^p(\mathbb{R}^d)} \right)^q
\]

\[
\leq C_6^q \cdot \sum_{i \in I} \left( \left( \frac{|\det (h)|}{|\det (h_i)|} \right)^{\frac{1}{d} - \frac{q}{d}} \cdot v (h) \cdot \left\| (W_{\psi} f) (\cdot, h) \right\|_{L^p(\mathbb{R}^d)} \right)^q \frac{dh}{|\det (h)|}
\]

\[
\leq C_6^q \cdot \sum_{i \in I} \left( \left( \frac{|\det (h)|}{|\det (h_i)|} \right)^{\frac{1}{d} - \frac{q}{d}} \cdot v (h) \cdot \left\| (W_{\psi} f) (\cdot, h) \right\|_{L^p(\mathbb{R}^d)} \right)^q \frac{dh}{|\det (h)|}
\]

\[
= C_6^q \cdot \int_U \left( v (h) \cdot \left\| (W_{\psi} f) (\cdot, h) \right\|_{L^p(\mathbb{R}^d)} \right)^q \frac{dh}{|\det (h)|}
\]

\[
= C_6^q \cdot \left\| W_{\psi} f \right\|_{L^p, q (U)}^q < \infty.
\]

This settles the case $q < \infty$. In the remaining case $q = \infty$, we use equation (22) to estimate

\[
\sup_{i \in I} |\det (h_i)|^{\frac{1}{d} - \frac{q}{d}} \cdot v (h_i) \cdot \left\| \mathcal{F}^{-1} \left( \varphi \cdot \hat{f} \right) \right\|_{L^p(\mathbb{R}^d)}
\]

\[
\leq C_\psi^{-1} \cdot \left\| \psi \right\|_{L^1(\mathbb{R}^d)} \cdot \sup_{i \in I} \int_{U_i} \left( \frac{|\det (h)|}{|\det (h_i)|} \right)^{-1/2} \cdot v (h) \cdot \left\| (W_{\psi} f) (\cdot, h) \right\|_{L^p(\mathbb{R}^d)} \frac{dh}{|\det (h)|}
\]

\[
\leq C_\psi^{-1} \cdot C_5^{-1/2} \cdot \left\| \psi \right\|_{L^1(\mathbb{R}^d)} \cdot \sup_{i \in I} \int_{U_i} v (h) \cdot \left\| (W_{\psi} f) (\cdot, h) \right\|_{L^p(\mathbb{R}^d)} \frac{dh}{|\det (h)|}
\]

\[
\leq C_\psi^{-1} \cdot C_5^{-1/2} \cdot \left\| \psi \right\|_{L^1(\mathbb{R}^d)} \cdot \mu_H (U_i) \cdot \left\| W_{\psi} f \right\|_{L^p, q}
\]

where we again used $\mu_H (U_i) \leq \mu_H (h_i U) \leq \mu_H (\overline{U})$.

Now note that $u_i = u (h_i^{-1} \xi_0) = v (h_i^{-1}) = |\det (h_i)|^{\frac{1}{d} - \frac{q}{d}} \cdot v (h_i)$ is a valid discretization of a suitable transplant of $v'$ onto $H$ (cf. remark 23). Thus, the above estimates show

\[
\left\| f \right\|_{D (Q, L^p, \ell^q_n)} \leq \left\{ \begin{array}{ll}
C_4 C_5 C_6 \cdot \left\| W_{\psi} f \right\|_{L^p, q (U)} = C_4 C_5 C_6 \cdot \left\| f \right\|_{C_0 (\mathbb{R}^d)}, & q < \infty, \\
C_4^{-1} C_5^{-1/2} C_5 \cdot \left\| \psi \right\|_{L^1(\mathbb{R}^d)} \cdot \mu_H (\overline{U}) \cdot \left\| f \right\|_{C_0 (\mathbb{R}^d)}, & q = \infty,
\end{array} \right.
\]

where all constants $C_1, \ldots, C_6$ are independent of $f$. By Lemma 23 and Lemma 22 any two discretizations of transplants of $v'$ yield equivalent norms on $D (Q, L^p, \ell^q_n)$. Thus, the proof is complete. \(\Box\)

In order to establish the general result, we first show that the decomposition space $D (Q, L^p, \ell^q_n)$ satisfies a form of the Fatou property.

**Lemma 36.** Let $\emptyset \neq U \subset \mathbb{R}^d$ be open and assume that $Q = (Q_i)_{i \in I}$ is a decomposition covering of $U$. Let $u : U \to (0, \infty)$ be $Q$-moderate and let $p, q \in [1, \infty]$ be arbitrary. Assume that $(f_n)_{n \in \mathbb{N}}$ is a sequence in $D (Q, L^p, \ell^q_n)$ that satisfies $f_n \xrightarrow{n \to \infty} f \in D' (U)$ where convergence is to be understood in the weak-$*$-sense, i.e. pointwise on $D (U)$.
Finally assume that \( \liminf_{n \to \infty} \| f_n \|_{D(Q, L^p, \ell^q_1)} \) is finite. Then \( f \in D(Q, L^p, \ell^q_1) \) holds with
\[
\| f \|_{D(Q, L^p, \ell^q_1)} \leq \liminf_{n \to \infty} \| f_n \|_{D(Q, L^p, \ell^q_1)} .
\]

**Proof.** Let \( (\varphi_i)_{i \in I} \) be a \( Q \)-BAPU. For \( i \in I \) and \( x \in \mathbb{R}^d \), [27] Theorem 7.23 applied to \( F^{-1} \) instead of \( F \) yields, with \( e_x : \mathbb{R}^d \to \mathbb{C}, \xi \mapsto e^{2\pi i (x, \xi)} \),
\[
(F^{-1}(\varphi_i f))(x) = (\varphi_i f)(e_x) = f(\varphi_i e_x) = \lim_{n \to \infty} f_n(\varphi_i e_x) = \lim_{n \to \infty} (\varphi_i f_n)(e_x) = \lim_{n \to \infty} (F^{-1}(\varphi_i f_n))(x),
\]
where we used the fact that \( \varphi_i f \) and \( \varphi_i f_n \) are distributions with compact support so that [27] Theorem 7.23 is applicable.

Using the Fatou property of \( L^p(\mathbb{R}^d) \), we see \( F^{-1}(\varphi_i f) \in L^p(\mathbb{R}^d) \) for all \( i \in I \) with
\[
\| F^{-1}(\varphi_i f) \|_{L^p(\mathbb{R}^d)} \leq \liminf_{n \to \infty} \| F^{-1}(\varphi_i f_n) \|_{L^p(\mathbb{R}^d)} \leq \liminf_{n \to \infty} \frac{1}{u_i} \| f_n \|_{D(Q, L^p, \ell^q_1)} < \infty,
\]
where \( (u_i)_{i \in I} \) is the chosen discretization of \( u \).

As \( \ell^q(I) \) also enjoys the Fatou property and is solid, we finally derive
\[
\| f \|_{D(Q, L^p, \ell^q_1)} = \left\| \left( u_i \cdot \| F^{-1}(\varphi_i f) \|_{L^p(\mathbb{R}^d)} \right) \right\|_{\ell^q(I)} \leq \left\| \left( u_i \cdot \liminf_{n \to \infty} \| F^{-1}(\varphi_i f_n) \|_{L^p(\mathbb{R}^d)} \right) \right\|_{\ell^q(I)} \leq \liminf_{n \to \infty} \left\| \left( u_i \cdot \| F^{-1}(\varphi_i f_n) \|_{L^p(\mathbb{R}^d)} \right) \right\|_{\ell^q(I)} = \liminf_{n \to \infty} \| f_n \|_{D(Q, L^p, \ell^q_1)} < \infty.
\]
In particular, we have \( f \in D(Q, L^p, \ell^q_1) \). \( \square \)

We now prove the first half of our claimed isomorphism between \( \text{Co}(L^p_v, q) \) and \( D(Q, L^p, \ell^q_1) \), namely the continuity of the Fourier transform from \( \text{Co}(L^p_v, q) \) to \( D(Q, L^p, \ell^q_1) \). The proof uses the density (in a suitable topology) of \( \mathcal{S}(\mathbb{R}^d) \cap \text{Co}(L^p_v, q) \) in \( \text{Co}(L^p_v, q) \) together with Lemma 55 where we use Lemma 36 to pass to the limit.

**Theorem 37.** Let \( v : H \to (0, \infty) \) be measurable and moderate with respect to the measurable, locally bounded, submultiplicative weight \( v_0 : H \to (0, \infty) \). Let \( p, q \in [1, \infty) \), choose \( v' \) as in Lemma 56 and let \( u : \mathcal{O} \to (0, \infty) \) be a transplant of \( v' \) onto \( \mathcal{O} \). Finally, assume that \( Q \) is an arbitrary decomposition covering of \( \mathcal{O} \) induced by \( H \).

Then the Fourier transform
\[
F : \text{Co}(L^p_v, q) \to D(Q, L^p, \ell^q_1), f \mapsto Ff
\]
with
\[
Ff : \mathcal{D}(\mathcal{O}) \to \mathbb{C}, g \mapsto f \left( \left( F^{-1} g \right) \right) \text{ for } f \in \text{Co}(L^p_v, q)
\]
defined as in Corollary 17 is a well-defined, continuous linear map.

**Proof.** By definition of an induced covering, there is a well-spread family \( (h_i)_{i \in I} \) in \( H \) and a precompact subset \( Q \subset \mathbb{R}^d \) that satisfies \( \mathcal{O} = \bigcup_{i \in I} h_i^T Q \) and \( Q \subset \mathcal{O} \) as well as \( \mathcal{Q} = (h_i^{-T} Q)_{i \in I} \).

As \( (h_i)_{i \in I} \) is well-spread, there exists a precompact set \( U \subset H \) that satisfies \( H = \bigcup_{i \in I} h_i U \).
Choose an arbitrary \( \psi \in S (\mathbb{R}^d) \setminus \{0\} \) with \( \hat{\psi} \in \mathcal{D} (\mathcal{O}) \). Define \( Q' := U^{-T} \left( \hat{\psi}^{-1} (\mathbb{C}^*) \right) \) and let \( Q' = (h_i^{-T} Q')_{i \in I} \) be the corresponding induced covering of \( \mathcal{O} \) (cf. Theorem 9). By Corollary 20 we have \( \mathcal{D} (Q', L^p, \ell_q^u) = \mathcal{D} (Q', L^p, \ell_u^q) \) with equivalent norms, so that it suffices to consider \( Q' \) instead of \( Q \).

Let \( w : H \to (0, \infty) \) be defined as in Lemma 14. Theorem 9 shows that \( \psi \in \mathcal{B}_w \) is a “better vector”, so that by the Atomic Decomposition Theorem [10, Theorem 6.1], there is some unit neighborhood \( V \subset G \), such that for every \( V \)-dense and relatively separated family \( X = (x_j)_{j \in J} \) in \( G \) the following are true:

1. There is a bounded linear **analysis operator** \( A : \text{Co} (L^p_v, q) \to (L^p_v, q)_d (X) \) such that for every \( f \in \text{Co} (L^p_v, q) \) we have

   \[
   f = \sum_{j \in J} [ (Af)_j \cdot \pi (x_j) \psi ]
   \]

   with convergence (at least) in the weak-\( \ast \)-topology on \( (\mathcal{H}_w^1)') \).

2. The **synthesis operator**

   \[
   S : (L^p_v, q)_d (X) \to \text{Co} (L^p_v, q) , (\lambda_j)_{j \in J} \mapsto \sum_{j \in J} [ \lambda_j \cdot \pi (x_j) \psi ]
   \]

   is well-defined and bounded.

Here, the space \( (L^p_v, q)_d \) is the so-called **associated discrete BK-space** to \( L^p_v, q \) (cf. [10, Definition 3.4]). The only property of this space that we need is that it is a solid sequence space, i.e. if \( (\lambda_j)_{j \in J} \) and \( (\gamma_j)_{j \in J} \) are sequences so that \( |\lambda_j| \leq |\gamma_j| \) holds for all \( j \in J \) and with \( (\gamma_j)_{j \in J} \in (L^p_v, q)_d \), then we have \( (\lambda_j)_{j \in J} \in (L^p_v, q)_d \) with a corresponding norm estimate \( \|(\lambda_j)_{j \in J}\|_{L^p_v, q} \leq \|(\gamma_j)_{j \in J}\|_{L^p_v, q} \).

Let \( W \subset V \) be a compact unit neighborhood that satisfies \( WW^{-1} \subset V \). By Lemma 19 it follows that there is a countably infinite family \( (g_j)_{j \in J} \) in \( G \) that is \( V \)-dense and \( W \)-separated. This family is a fortiori relatively separated, so that the above results apply to \( (x_j)_{j \in J} = (g_j)_{j \in J} \).

Let \( (J_n)_{n \in \mathbb{N}} \) be an enumeration of \( J \) and let \( f \in \text{Co} (L^p_v, q) \). For \( n \in \mathbb{N} \) define

\[
f_n := \sum_{j=1}^n [ (Af)_{j} \cdot \pi (g_j) \psi ] = S (Af \cdot \chi_{\{j_{1}, \ldots , j_{n}\}}) \in \text{Co} (L^p_v, q).
\]

Now note that for \( g_j = (x_j, k_j) \) we have \( \pi (g_j) \psi = |\det k_j|^{-1/2} \cdot L_{x_j} D_{k_j^{-1}} \psi \in S (\mathbb{R}^d) \) and thus \( f_n \in S (\mathbb{R}^d) \cap \text{Co} (L^p_v, q) \).

The convergence in equation (7.3) in the weak-\( \ast \)-topology on \( (\mathcal{H}_w^1)') \) and Corollary 10 show \( \mathcal{F} f_n \overset{n \to \infty}{\longrightarrow} \mathcal{F} f \) with convergence in the weak-\( \ast \)-topology on \( \mathcal{D}' (\mathcal{O}) \). Finally note that \( \mathcal{F} f_n \) coincides with the “ordinary” Fourier transform \( \hat{f}_n \) of \( f_n \in S (\mathbb{R}^d) \subset L^2 (\mathbb{R}^d) \) by Remark 5.
Hence, we get
\[ \|Ff\|_{\mathcal{D}(Q,L^p,\ell_q^n)} = \|\hat{f}\|_{\mathcal{D}(Q,L^p,\ell_q^n)} \]
\[
\leq C \cdot \|f\|_{\text{Co}(L^p_{v,q})} \\
= C \cdot \|S(\hat{A}f \cdot \chi_{(j_1,\ldots,j_n)})\|_{\text{Co}(L^p_{v,q})} \\
\leq C \cdot \|S\| \cdot \|\hat{A}f \cdot \chi_{(j_1,\ldots,j_n)}\|_{(L^p_{v,q})}.
\]
Theorem 38 finishes the proof. 

8. Continuity of the Inverse Fourier Transform from Decomposition Spaces into Coorbit Spaces

In this section we show that the inverse Fourier transform $F^{-1} : \mathcal{D}(Q,L^p,\ell_q^n) \to \text{Co}(L^p_{v,q})$ is well-defined and continuous. This poses the problem that an element $f \in \mathcal{D}(Q,L^p,\ell_q^n)$ is a distribution $f \in \mathcal{D}'(\mathcal{O})$ on $\mathcal{O}$ and not (necessarily) a tempered distribution (cf. the remark following Definition 21). Thus, it is not immediately clear how to define the inverse Fourier transform $F^{-1}f$ of $f$.

In order to solve this problem, we use the map
\[ \Theta : (\mathcal{H}^1_v)^{\prime} \to (\mathcal{F}(\mathcal{D}(\mathcal{O})))^{\prime}, f \mapsto (\varphi \mapsto f(\overline{\varphi})) \]
introduced in Corollary 11 to identify the coorbit space $\text{Co}(L^p_{v,q})$ with the alternative coorbit space
\[ \tilde{\text{Co}}_{\varphi}(L^p_{v,q}) := \{ f \in (\mathcal{F}(\mathcal{D}(\mathcal{O})))^{\prime} \mid W_\varphi f \in L^p_{v,q}(G) \}. \]

Here, we define the wavelet transform $W_\varphi f$ for $f \in (\mathcal{F}(\mathcal{D}(\mathcal{O})))^{\prime}$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\overline{\varphi} \in \mathcal{D}(\mathcal{O})$ by
\[ W_\varphi f : G \to \mathbb{C}, (x,h) \mapsto \overline{W_\varphi f(x,h)} := f(\overline{\pi(x,h)} \overline{\varphi}). \]

On the space $\tilde{\text{Co}}_{\varphi}(L^p_{v,q})$, the definition of the inverse fourier transform is then straightforward.

The following theorem makes the claimed identification of $\text{Co}(L^p_{v,q})$ with $\tilde{\text{Co}}_{\varphi}(L^p_{v,q})$ explicit:

Theorem 38. Let $\varphi \in \mathcal{S}(\mathbb{R}^d \setminus \{0\})$ with $\overline{\varphi} \in \mathcal{D}(\mathcal{O})$ be arbitrary. Then the restriction of the map $\Theta$ to $\text{Co}(L^p_{v,q})$ induces an isometric isomorphism
\[ \Theta : \text{Co}(L^p_{v,q}) \to \tilde{\text{Co}}_{\varphi}(L^p_{v,q}) \]
as long as $\varphi$ is used as an analyzing vector for $\text{Co}(L^p_{v,q})$, i.e. $\|f\|_{\text{Co}(L^p_{v,q})} = \|W_\varphi f\|_{L^p_{v,q}}$. Here, $\tilde{\text{Co}}_{\varphi}(L^p_{v,q})$ is endowed with the norm $\|f\|_{\tilde{\text{Co}}_{\varphi}(L^p_{v,q})} := \|W_\varphi f\|_{L^p_{v,q}}$.

In particular, $\|\cdot\|_{\tilde{\text{Co}}_{\varphi}(L^p_{v,q})}$ defines a norm on $\tilde{\text{Co}}_{\varphi}(L^p_{v,q})$, so that $W_\varphi : \tilde{\text{Co}}_{\varphi}(L^p_{v,q}) \to L^p_{v,q}(G)$ is injective.

Furthermore, the above implies that the space $\tilde{\text{Co}}_{\varphi}(L^p_{v,q})$ is independent of $\varphi$ with equivalent norms for different choices. We will thus write $\tilde{\text{Co}}(L^p_{v,q})$ in the future.
The hard part is the claimed surjectivity of the restricted map \( \Theta \). In order to prove it, an important step is to show that the extended wavelet transform \( \Psi f \) defined in equation (8.1) is a continuous function that satisfies the reproduction formula

\[
\Psi f = \Psi f \ast \frac{W_\psi f}{C_\psi}.
\]

After establishing these properties, we will give the proof of Theorem 38. First of all, we will need the following technical Lemma regarding the continuity of modulation on certain spaces. The proof is a straightforward application of the Leibniz rule together with the fact that all derivatives of the complex exponentials \( x \mapsto \exp(i(x, \xi)) \) are bounded on compact sets and is therefore omitted.

**Lemma 39.** Let \( \emptyset \neq U \subset \mathbb{R}^d \) be an open set and let \( K \subset U \) be compact and \( k \in \mathbb{N}_0 \). Then the map

\[
\Phi : C_K^k (U) \times \mathbb{R}^d \to C_K^k (U), (f, \xi) \mapsto \left( x \mapsto e^{i(x, \xi)} \cdot f (x) \right)
\]

is well-defined and continuous. Here, \( C_K^k (U) \) is the space

\[
C_K^k (U) := \{ f \in C_K (U) \mid \text{supp}(f) \subset K \}
\]

with norm

\[
\| f \|_{C_K^k (U)} := \max_{\alpha \in \mathbb{N}^d_0 \pi \in \mathbb{R}^d} \| \partial^\alpha f \| (x).
\]

Using this result on the continuity of modulation, we can now show that the extended wavelet transform as defined in equation (8.1) actually defines a continuous function that satisfies the expected reproduction formula.

**Lemma 40.** Let \( \psi \in S (\mathbb{R}^d) \) with \( \hat{\psi} \in \mathcal{D} (\mathcal{O}) \). Then we have

\[
\text{supp} \left( \mathcal{F}^{-1} \pi (x, h) \psi \right) \subset h^{-T} \text{supp} (\hat{\psi}) \subset \mathcal{O},
\]

which implies \( \mathcal{F}^{-1} \pi (x, h) \psi \in \mathcal{D} (\mathcal{O}) \), i.e. \( \pi (x, h) \psi \in \mathcal{F} (\mathcal{D} (\mathcal{O})) \) for all \( (x, h) \in G \). This shows that \( \Psi f \) as defined in equation (8.1) is well-defined.

Furthermore, for \( \psi \neq 0 \) and \( \varphi \in S (\mathbb{R}^d) \) with \( \hat{\varphi} \in \mathcal{D} (\mathcal{O}) \) we have

\[
\pi (\alpha) \varphi = \frac{1}{C_\psi} \cdot \int_G \left( \pi (\beta) \psi, \pi (\alpha) \varphi \right)_{L^2 (\mathbb{R}^d)} \cdot \pi (\beta) \psi \, d\beta,
\]

for all \( \alpha \in G \), where the integral is to be understood in the weak sense in \( \mathcal{F} (\mathcal{D} (\mathcal{O})) \).

Finally, for \( f \in (\mathcal{F} (\mathcal{D} (\mathcal{O})))' \) and \( \psi \neq 0 \), the wavelet transform \( \Psi f \) defined in equation (8.1) is a continuous function that satisfies the reproduction formula

\[
\Psi f = \Psi f \ast \frac{W_\psi f}{C_\psi}.
\]

**Proof.** We first note the general identity \( \mathcal{F}^{-1} f = \overline{\widehat{f}} \) which is valid for arbitrary \( f \in L^1 (\mathbb{R}^d) \). This yields

\[
\mathcal{F}^{-1} \pi (x, h) \psi = \overline{\mathcal{F} (\pi (x, h) \psi)} = |\det(h)|^{1/2} \cdot M_x D_h \psi = |\det(h)|^{1/2} \cdot (\psi \circ h^T) \cdot e_x
\]

(8.4)
with \(e_x : \mathbb{R}^d \to \mathbb{C}, \xi \mapsto e^{2\pi i \langle x, \xi \rangle}\). We conclude

\[
\text{supp} \left( \mathcal{F}^{-1} \pi(x, h) \psi \right) = \text{supp} \left( \widehat{\psi \circ h^T} \right) = h^{-T} (\text{supp} (\hat{\psi})) \subset h^{-T} \mathcal{O} = \mathcal{O},
\]
as claimed. As \(\widehat{\psi \circ h^T} \cdot e_x\) is smooth, we get \(\mathcal{F}^{-1} \pi(x, h) \psi \in \mathcal{D} (\mathcal{O})\).

Fix \(\alpha = (x, h) \in G\). For \(\beta = (y, g) \in G\) with \(\langle \pi (\beta) \psi, \pi (\alpha) \varphi \rangle_{L^2} \neq 0\), the Plancherel theorem yields

\[
0 \neq \langle \pi (\beta) \psi, \pi (\alpha) \varphi \rangle_{L^2} = \left\langle \pi (\beta) \psi, \pi (\alpha) \varphi \right\rangle_{L^2} \overset{\text{Eq. (2.4)}}{=} \left| \det (gh) \right|^{1/2} : \left\langle M_y D_y \hat{\psi}, M_x D_x \hat{\varphi} \right\rangle_{L^2} ,
\]
which implies

\[
\emptyset \neq \text{supp} \left( D_y \hat{\psi} \right) \cap \text{supp} (D_h \hat{\varphi}) = h^{-T} \text{supp} (\hat{\varphi}) \cap g^{-T} \text{supp} (\hat{\psi}) .
\]

For \(K_1 := \text{supp} (\hat{\varphi}), K_2 := \text{supp} (\hat{\psi}) \subset \mathcal{O}\) and \(L = L(K_1, K_2) \subset H\) compact as in Lemma \(\text{L8}\), the same lemma yields \(g \in hL\) and thus

\[
\text{supp} \left( \mathcal{F}^{-1} \pi(\beta) \psi \right) \subset g^{-T} \text{supp} (\hat{\psi}) \subset h^{-T} L^{-T} K_2 =: K_3 \subset \mathcal{O}.
\]

Note that \(K_3 = K_3(h)\) depends upon \(h\) but that \(\alpha = (x, h) \in G\) is fixed.

This shows that for \(\ell \in \mathbb{N}_0\) the map

\[
\Phi_\ell : G \to C_{K_3} (\mathcal{O}), \quad \beta = (y, g) \mapsto \langle \pi (\beta) \psi, \pi (\alpha) \varphi \rangle_{L^2} \cdot \mathcal{F}^{-1} \pi(\beta) \psi \overset{\text{Eq. (8.4)}}{=} \langle \pi (\beta) \psi, \pi (\alpha) \varphi \rangle_{L^2} \cdot \left| \det (gh) \right|^{1/2} : \left\langle M_y D_y \hat{\psi}, M_x D_x \hat{\varphi} \right\rangle_{L^2}
\]
is well-defined with \(\Phi (y, g) = 0\) for \(g \notin hL\). The strong continuity of \(\pi\) and the Lemmata \(\text{L9}\) and \(\text{L7}\) (with the ensuing remark) show that \(\Phi_\ell\) is actually continuous. This implies that \(\Phi_\ell\) is measurable and that \(\Phi_\ell (G) \subset \mathcal{C}_{K_3} (\mathcal{O})\) is \(\sigma\)-compact and hence separable.

The continuity of \(\Gamma : H \to \mathcal{D} (\mathbb{R}^d), g \mapsto D_y \hat{\psi}\), which was shown in Lemma \(\text{L1}\) and the ensuing remark imply finiteness of the constant

\[
C_\rho := \max_{\gamma \leq \rho} \max_{g \in hL} \left\| \partial^\gamma (\psi \circ g^T) \right\|_{\sup} = \max_{\gamma \leq \rho} \max_{g \in hL} \left\| \partial^\gamma (\Gamma (g)) \right\|_{\sup}
\]
for all \(\rho \in \mathbb{N}_0^d\). Using the Leibniz rule, we arrive at

\[
\left\| \partial^\rho \left( e_y \cdot (\psi \circ g^T) \right) \right\|_{\sup} = \left\| \sum_{\gamma \leq \rho} \binom{\rho}{\gamma} \cdot \partial^\gamma e_y \cdot \partial^{\rho-\gamma} (\psi \circ g^T) \right\|_{\sup} \leq \sum_{\gamma \leq \rho} \binom{\rho}{\gamma} \left\| (2 \pi i y)^\gamma e_y \right\|_{\sup} \cdot C_\rho \leq C_\rho \cdot \sum_{\gamma \leq \rho} \binom{\rho}{\gamma} |2 \pi y|^{|\gamma|} \leq C'_\rho \cdot (1 + |y|)^{|\rho|}
\]
for all \(g \in hL\) and some constant \(C'_\rho > 0\). Define \(C := \max_{g \in hL} \left| \det (g) \right|^{1/2}\) and \(\zeta := \pi (\alpha) \varphi\) and note

\[
\left| \langle \pi (\beta) \psi, \pi (\alpha) \varphi \rangle_{L^2} \right| = \left| \langle \zeta, \pi (\beta) \psi \rangle_{L^2} \right| = \left| (W \psi \zeta) (\beta) \right| .
\]
Together with $\Phi_\ell (y, g) = 0$ for $g \not\in hL$, this proves the estimate

$$
\| \partial^\rho \Phi_\ell (\beta) \|_\sup \leq C \cdot |(W_\psi \zeta) (\beta)| \cdot |\det (g)|^{1/2} \cdot \| \partial^\rho \left( \epsilon_\psi \right) \|_\sup
$$

for all $\rho \in \mathbb{N}_0^d$ and $\beta = (y, g) \in G$. In summary, we showed

$$
\| \Phi_\ell (y, g) \|_{C^\ell_{K_3}} \leq C \max_{|\rho| \leq \ell} C_\rho' \cdot (1 + |y|)^\ell \cdot |(W_\psi \zeta) (\beta)|.
$$

But equation (2.4) yields $\hat{\zeta} = |\det (h)|^{1/2} \cdot M_{-x} D_{\hat{h}} \hat{\varphi}$, showing that supp $(\hat{\zeta}) \subset h^{-T} \text{supp} (\hat{\varphi}) \subset \mathcal{O}$ is compact, so that Theorem (3) (with $w_0 \equiv 1$ and $N = \ell$) yields $W_\psi \zeta \in L^1 \mathcal{O} \overset{\ell}{\rightarrow} \mathcal{L}^p (\mathbb{R}^d)$. This shows that $\Phi_\ell$ is Bochner integrable, so that the integral

$$
\varphi_\ell := \frac{1}{C_\psi} \cdot \int_G \Phi_\ell (y, g) \ d(y, g) \in C^\ell_{K_3} (\mathcal{O}) \hookrightarrow \bigcap_{\rho \in [1, \infty]} L^p (\mathbb{R}^d)
$$

is well-defined.

For $f \in L^2 (\mathbb{R}^d)$, the map $C^\ell_{K_3} (\mathcal{O}) \rightarrow \mathbb{C}, g \mapsto \langle g, f \rangle_{L^2}$ is a bounded linear functional. Using the left invariance of the Haar measure, the weak inversion formula (6.4) and Plancherel’s theorem, we calculate

$$
\langle \varphi_\ell, f \rangle_{L^2} = \frac{1}{C_\psi} \cdot \int_G \left( \langle \pi (\beta) \psi, \pi (\alpha) \varphi \rangle_{L^2} \cdot F^{-1} \pi (\beta) \psi, f \rangle_{L^2} \right) \ d\beta
= \frac{1}{C_\psi} \cdot \int_G \left( \langle \pi (\alpha^{-1} \beta) \psi, \pi (\alpha) \varphi \rangle_{L^2} \cdot \langle \pi (\beta) \psi, f \rangle_{L^2} \right) \ d\beta
= \frac{1}{C_\psi} \cdot \int_G \left( \langle \varphi, \pi (\alpha^{-1} \beta) \psi \rangle_{L^2} \cdot \langle \pi (\alpha^{-1} \beta) \psi, \pi (\alpha^{-1} \beta) \pi (\alpha^{-1} \beta) \psi \rangle_{L^2} \right) \ d\beta
= \frac{1}{C_\psi} \cdot \int_G \langle \varphi, W_\psi \varphi (\gamma) \rangle \cdot \langle \pi (\gamma) \psi, \pi (\alpha^{-1} \beta) \psi \rangle_{L^2} \ d\gamma
\overset{\text{Eq. (6.4)}}{=} \langle \varphi, \pi (\alpha^{-1} \beta) \psi \rangle_{L^2} = \langle \pi (\alpha^{-1} \beta) \psi, \pi (\alpha^{-1} \beta) \psi \rangle_{L^2}
= \langle \pi (\alpha) \varphi, f \rangle_{L^2} = \langle F^{-1} \pi (\alpha) \varphi, f \rangle_{L^2}.
$$

As this holds for every $f \in L^2 (\mathbb{R}^d)$, we get $F^{-1} \pi (\alpha) \varphi = \varphi_\ell$ almost everywhere and then everywhere, as both sides are continuous functions. In particular, $\varphi_\ell \in \mathcal{D} (\mathcal{O})$. 
Now choose \( f \in (\mathcal{F}(\mathcal{D}(\mathcal{O})))' \). By [27, Theorem 6.8] and Hahn-Banach, \( f \circ \mathcal{F}|_{C^\ell_{K_3}(\mathcal{O})} \) admits a continuous extension \( \tilde{f} \) to \( C^\ell_{K_3}(\mathcal{O}) \) for a suitable \( \ell \in \mathbb{N}_0 \). We thus get

\[
\begin{align*}
  f (\pi(\alpha) \varphi) &= (f \circ \mathcal{F}) \left( \mathcal{F}^{-1} \pi(\alpha) \varphi \right) \\
  &= (f \circ \mathcal{F})(\varphi) = \tilde{f}(\varphi) \\
  &= \frac{1}{C_\psi} \int_G \tilde{f}(\Phi_{\ell}(\beta)) \, d\beta \\
  &= \frac{1}{C_\psi} \int_G (\pi(\beta) \psi, \pi(\alpha) \varphi)_{L^2} \cdot (f \circ \mathcal{F}) \left( \mathcal{F}^{-1} \pi(\beta) \psi \right) \, d\beta \\
  &= \frac{1}{C_\psi} \int_G (\pi(\beta) \psi, \pi(\alpha) \varphi)_{L^2} \cdot f \left( \pi(\beta) \psi \right) \, d\beta,
\end{align*}
\]

which proves equation (8.2), as \( f \in (\mathcal{F}(\mathcal{D}(\mathcal{O})))' \) was arbitrary. Additionally, the choice \( \varphi = \psi \) shows

\[
(W_\psi f)(\alpha) \overset{\text{Eq. (8.1)}}{=} \frac{1}{C_\psi} \int_G (\pi(\beta) \psi, \pi(\alpha) \varphi)_{L^2} \cdot f \left( \pi(\beta) \psi \right) \, d\beta = \frac{1}{C_\psi} \int_G (W_\psi f)(\beta) \cdot (W_\psi \psi)(\beta^{-1} \alpha) \, d\beta = \left( (W_\psi f) \ast \frac{W_\psi \psi}{C_\psi} \right)(\alpha)
\]

which is nothing else than equation (8.3).

The only thing missing is continuity of \( W_\psi f \). For this, let \( h_0 \in H \) be arbitrary and choose a compact neighborhood \( K_4 \subset H \) of \( h_0 \). For \( \alpha = (x, h) \in \mathbb{R}^d \times K_4 \) we then have \( \text{supp} \left( \mathcal{F}^{-1} \pi(x, h) \psi \right) \subset h^{-1} K_2 \subset K_4^{-T} K_2 =: K_5 \). For \( f \in (\mathcal{F}(\mathcal{D}(\mathcal{O})))' \) we can choose as above a continuous extension \( \tilde{f} \) of \( f \circ \mathcal{F}|_{\mathcal{D}(\mathcal{O}) \cap C^\ell_{K_3}(\mathcal{O})} \) to \( C^\ell_{K_5}(\mathcal{O}) \) for some \( \ell \in \mathbb{N}_0 \). We then have

\[
(W_\psi f)(x, h) = (f \circ \mathcal{F}) \left( \mathcal{F}^{-1} \pi(x, h) \psi \right) \overset{\text{Eq. (8.2)}}{=} |\det(h)|^{1/2} \cdot (f \circ \mathcal{F}) \left( \left( \bar{\psi} \circ h^T \right) \cdot e_x \right) = |\det(h)|^{1/2} \cdot \tilde{f} \left( \left( \bar{\psi} \circ h^T \right) \cdot e_x \right)
\]

for all \( (x, h) \in \mathbb{R}^d \times K_4 \). Noting that the Lemmata 39 and 7 (with the ensuing remark) show that the right-hand side defines a continuous function in \( (x, h) \) completes the proof. \( \square \)

Using this rather technical result, we can now give the proof of Theorem 38, i.e. of the identification of \( \text{Co}(L^p_\psi) \) with \( \text{Co}_\psi(L^p_\psi) \).

**Proof of Theorem 38** Let \( w : H \to (0, \infty) \) be the control weight for \( L^p_\psi(G) \) as defined in Lemma 1; we will interpret this to be a weight on \( G \) by \( w(x, h) = w(h) \) for \( (x, h) \in G \).

For \( f \in \text{Co}(L^p_\psi) \subset (\mathcal{H}_w^1)^\perp \) we have

\[
(W_\psi (\Theta f))(x, h) \overset{\text{Eq. (8.1)}}{=} \Theta(f) \left( \pi(x, h) \psi \right) = f (\pi(x, h) \psi) = (W_\psi f)(x, h). \quad (8.5)
\]
This shows that $\Theta$ is well-defined and isometric. But note that we do not yet know that $\|\|_{C_0(\mathcal{L}^{p,q})}$ is a norm. Nevertheless, $\Theta$ is injective, because it is the restriction of an injective map (cf. Corollary 44).

In the proof of the surjectivity of $\Theta$ below, we will need the fact that $W_\psi$ is injective on $C_0(\mathcal{L}^{p,q})$, which we now prove. Together with the continuity of $W_\psi f$ for $f \in (\mathcal{F}(\mathcal{D}(O)))'$ (cf. Lemma 40), this will also show that $\|\|_{C_0(\mathcal{L}^{p,q})}$ is a norm. Choose $f \in C_0(\mathcal{L}^{p,q})$ with $W_\psi f \equiv 0$ and let $\varphi \in \mathcal{F}(\mathcal{D}(O))$ be arbitrary. Note that we have $\tilde{\varphi} = \mathcal{F}^{-1}(\varphi) \in \mathcal{D}(O)$. Thus, equation (8.2) (with $\alpha = 1$ and $\psi$ instead of $\varphi$) shows

$$f(\varphi) = f\left(\frac{\varphi_1}{\|\|_\psi}\right) = \frac{1}{C_\psi} \int_G \langle \varphi, \mathcal{F}(\varphi) \rangle_{L^2(\mathbb{R}^d)} \cdot f\left(\frac{\varphi_1}{\|\|_\psi}\right) d\beta = 0.$$

As $\varphi \in \mathcal{F}(\mathcal{D}(O))$ was arbitrary, we conclude $f \equiv 0$.

It remains to show that $\Theta$ is surjective. This will in particular imply that $\tilde{C_0}(\mathcal{L}^{p,q})$ is independent of the choice of $\psi$, as the same is true of $C_0(\mathcal{L}^{p,q})$ and of $\Theta$. To this end, let $f \in \tilde{C_0}(\mathcal{L}^{p,q})$ be arbitrary.

Let $U \subset G$ be an arbitrary open, precompact neighborhood of $1_G$ and set $E := \frac{W_\psi f}{C_\psi}$. Note that Theorem 80 implies $E \in W^2(\mathcal{L}^\infty, L^1_w)$, i.e. $K_{U^{-1}} E \in L^1_w(G)$, where $K_{U^{-1}} E$ denotes the (right sided) control function of $E$ with respect to $U^{-1}$ (cf. equation 44).

The definition of $\tilde{C_0}(\mathcal{L}^{p,q})$ yields $F := \frac{W_\psi f}{C_\psi} \in L^{p,q}_w(G)$, whereas Lemma 40 implies that $F$ obeys the reproduction formula $F = F \ast E$. Observe that we have $E(z^{-1}) = E(z)$ for all $z \in G$. Using this, we get, for $\alpha \in G$ and $\beta \in U$ the estimate

$$|F(\alpha \beta)| = \left|\left((F \ast E)(\alpha \beta)\right)\right|$$

$$\leq \left|E(z^{-1})\right| \cdot \int_G |F(y)| \cdot |E(\beta^{-1} \alpha^{-1} y)| dy$$

$$= \left|\left((F \ast (K_{U^{-1}} E))(\alpha^{-1} y)\right)\right|.$$
Now \cite{26} Lemma 3.3 shows that we have the estimate

\[
r(x) := \|L^{-1}\|_{W(L^\infty, L^p)} \leq \|L^{-1}\|_{L^p} \leq \text{w}^{-1} = w(x).
\]

Finally, \cite{26} Lemma 3.2 yields \(W(L^\infty, L^p) \hookrightarrow L^1_{I^f}(G) \hookrightarrow L^\infty_{I^u}(G)\) and hence \(F \in L^\infty_{I^u}(G)\).

By Lemma \cite{10} \(F\) obeys the reproduction formula \(F = F \ast W_\psi\), so that \cite{10} Theorem 4.1(iv) guarantees the existence of \(g \in (H^1_{I^u})^\sim\) satisfying \(W_\psi g = W_\psi f \in L^p_{I^u}(G)\). This immediately entails \(g \in C_0(L^p_{I^u}(G))\) and thus \(\Theta g \in \tilde{C}_0(L^p_{I^u})\).

But equation \cite{26} shows \(W_\psi(\Theta g) = W_\psi g = W_\psi f\) which implies \(f = \Theta g\), because we have seen above that \(W_\psi\) is injective on \(C_0(L^p_{I^u})\).

\[\square\]

We now show that the map \(F^{-1} : \mathcal{D}'(O) \rightarrow (\mathcal{F}(\mathcal{D}(O)))', f \mapsto f \circ F^{-1}\) restricts to a continuous map \(F^{-1} : \mathcal{D}(Q, L^p, \ell^q_u) \rightarrow \tilde{C}_0(L^p_{I^u}, q)\).

**Lemma 41.** Assume that \(Q = (h_i^{-T}Q)_{i \in I}\) is a decomposition covering of \(O\) induced by \(H\) and choose

\[u_i = \text{det}(h_i)^{-\frac{1}{2}} \cdot v(h_i)\text{ for }i \in I.
\]

Choose \(\psi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}\) with \(\hat{\psi} \in \mathcal{D}(O)\) and let \(p, q \in [1, \infty)\). Then there is a constant \(C > 0\) such that

\[
\|W_\psi(f \circ F^{-1})\|_{L^p} \leq C \cdot \|f\|_{L^p(Q, L^p, \ell^q_u)} < \infty
\]

holds for all \(f \in \mathcal{D}(Q, L^p, \ell^q_u)\). Thus, the map

\[F^{-1} : \mathcal{D}(Q, L^p, \ell^q_u) \rightarrow \tilde{C}_0(L^p_{I^u}), f \mapsto f \circ F^{-1}\]

is well-defined and bounded.

**Remark 42.** It is worth noting that the inverse Fourier transform \(F^{-1} : \mathcal{D}(Q, L^p, \ell^q_u) \rightarrow \tilde{C}_0(L^p_{I^u})\) is well-defined on \(L^2(\mathbb{R}^d)\) and \(\mathcal{D}(Q, L^p, \ell^q_u)\) with the ordinary Fourier transform, where \(f \in L^2(\mathbb{R}^d)\) is considered as an element of \(\mathcal{D}'(O)\) by \(f(\varphi) := \int f \cdot \varphi dx = \langle f, \varphi \rangle_{L^2}\text{ for }\varphi \in \mathcal{D}(O)\).

For the proof, simply note that we have

\[
(f \circ F^{-1})(\varphi) = f(F^{-1}\varphi) = \langle f, \mathcal{F}(F^{-1}\varphi) \rangle_{L^2} = \langle f, \mathcal{F}F^{-1}\varphi \rangle_{L^2} = \langle F^{-1}f, \varphi \rangle_{L^2} = (F^{-1}f)(\varphi)
\]

for \(f \in L^2(\mathbb{R}^d)\) and \(\varphi \in \mathcal{D}(\mathcal{D}(O)) \subset L^1(\mathbb{R}^d)\).

**Proof.** Let \((\varphi_i)_{i \in I}\) be a \(Q\)-BAPU. Define \(K := \text{supp}(\hat{\psi}) \subset O\). For \(h \in H\) we define

\[I_h := \{i \in I \mid h^{-T}K \cap h_i^{-T}Q \neq \emptyset\}.
\]

Note that \(\overline{Q} \subset O\) is compact by definition of an induced covering. Therefore, Lemma \cite{15} yields a constant \(C_1 = C_1((h_i)_{i \in I}, Q, K, I) > 0\) with \(|I_h| \leq C\) for all \(h \in H\). Note that we have

\[
\sum_{i \in I_h} \varphi_i(x) = 1 \text{ for all } x \in h^{-T}K,
\]

because for \(x \in h^{-T}K\) and \(i \in I\) with \(\varphi_i(x) \neq 0\) we have \(x \in \varphi_i^{-1}(C) \subset h_i^{-T}Q\) and thus \(x \in h^{-T}K \cap h_i^{-T}Q\), i.e. \(i \in I_h\). This shows \(1 = \sum_{i \in I} \varphi_i(x) = \sum_{i \in I_h} \varphi_i(x)\).

Let \((x, h) \in G\) be arbitrary. Because of Lemma \cite{10} we know \(F_{x,h} := F^{-1}(x, h) \psi \in \mathcal{D}(O)\) with

\[
\text{supp}(F_{x,h}) \subset h^{-T}\text{supp}(\hat{\psi}) = h^{-T}K.
\]
Together with equation (8.6) this yields

\[ F_{x,h} = \sum_{i \in I_h} \varphi_i F_{x,h} . \]

Let \( f \in D(Q, L^p, \ell^q_R) \). Then the above identity yields the fundamental localization identity

\[ (W_\psi (f \circ \mathcal{F}^{-1})) (x, h) \overset{\text{Eq. (8.7)}}{=} (f \circ \mathcal{F}^{-1}) \left( \pi(x, h) \psi \right) \]
\[ = f(F_{x,h}) \]
\[ = \sum_{i \in I_h} f(\varphi_i F_{x,h}) = \sum_{i \in I_h} (\varphi_i f)(F_{x,h}) . \quad (8.7) \]

Note that \( \varphi_i f \) is a distribution with compact support (and hence a tempered distribution) and that we have \( \mathcal{F}^{-1} (\varphi_i f) \in L^p(\mathbb{R}^d) \) because of \( f \in D(Q, L^p, \ell^q_R) \). Using this and the definition of the quasi-regular representation, we calculate

\[ (\varphi_i f)(F_{x,h}) = (\varphi_i f) \left( \mathcal{F}^{-1} (\pi(x, h) \psi) \right) = (\mathcal{F}^{-1} (\varphi_i f)) \left( \pi(x, h) \psi \right) \]
\[ = |\det(h)|^{-1/2} \cdot \int_{\mathbb{R}^d} (\mathcal{F}^{-1} (\varphi_i f))(y) \cdot (D_{h^{-\tau}} \psi)(y - x) \, dy \]
\[ = |\det(h)|^{-1/2} \cdot \int_{\mathbb{R}^d} (\mathcal{F}^{-1} (\varphi_i f))(y) \cdot (D_{h^{-\tau}} \psi^*)(x - y) \, dy \]
\[ = |\det(h)|^{-1/2} \cdot \left( (\mathcal{F}^{-1} (\varphi_i f)) \ast (D_{h^{-\tau}} \psi^*) \right)(x) \]

with \( \psi^*(y) = \overline{\psi(-y)} \) for \( y \in \mathbb{R}^d \).

Using Young’s inequality, we derive

\[ \| x \mapsto (\varphi_i f)(F_{x,h}) \|_{L^p(\mathbb{R}^d)} \]
\[ \leq |\det(h)|^{-1/2} \cdot \| (\mathcal{F}^{-1} (\varphi_i f)) \ast (D_{h^{-\tau}} \psi^*) \|_{L^p(\mathbb{R}^d)} \]
\[ \leq |\det(h)|^{-1/2} \cdot \| D_{h^{-\tau}} \psi^* \|_{L^1(\mathbb{R}^d)} \cdot \| (\mathcal{F}^{-1} (\varphi_i f)) \|_{L^p(\mathbb{R}^d)} \]
\[ \leq |\det(h)|^{1/2} \cdot \| \psi^* \|_{L^1(\mathbb{R}^d)} \cdot \| (\mathcal{F}^{-1} (\varphi_i f)) \|_{L^p(\mathbb{R}^d)} . \]

Together with the localization identity (8.7) this shows

\[ \| (W_\psi (f \circ \mathcal{F}^{-1})) (\cdot, h) \|_{L^p(\mathbb{R}^d)} \leq \sum_{i \in I_h} \| x \mapsto (\varphi_i f)(F_{x,h}) \|_{L^p(\mathbb{R}^d)} \]
\[ \leq |\det(h)|^{1/2} \cdot \| \psi^* \|_{L^1(\mathbb{R}^d)} \cdot \sum_{i \in I_h} \| (\mathcal{F}^{-1} (\varphi_i f)) \|_{L^p(\mathbb{R}^d)} . \quad (8.8) \]

for all \( h \in H \).

By definition of an induced covering, \((h_i)_{i \in I}\) is well-spread in \( H \), so that there is a precompact, measurable set \( U \subset H \) with \( H = \bigcup_{i \in I} h_i U \). Choose \( K_2 := \overline{h^{-T} K \cup \overline{Q}} \) and note that the family \( Q' := (Q'_i)_{i \in I} := (h_i^{-T} K_2)_{i \in I} \) is an induced covering of \( O \). Note that for \( i \in I, h \in h_i U \) and \( j \in I_h \) we have

\[ \varnothing \neq h^{-T} K \cap h_j^{-T} \overline{Q} \subset h_i^{-T} U^{-T} K \cap h_j^{-T} \overline{Q} \subset Q'_i \cap Q'_j \]

and thus \( j \in i^* Q' \), i.e.

\[ I_h \subset i^* Q' \text{ for all } i \in I \text{ and } h \in h_i U , \quad (8.9) \]
where the cluster \( i^* \) is taken with respect to \( Q' \). Theorem 20 shows that \( Q' \) is an admissible covering of \( O \), so that

\[
C_1 := \sup_{i \in I} |i^*| \in \mathbb{N}
\]

is a finite constant. Furthermore, Lemmata 22 and 23 show that \((u_i)_{i \in I}\) is \( Q' \)-moderate, so that we have \( u_i \leq C_2 \cdot u_j \) for all \( i \in I \) and \( j \in i^* \) for some constant \( C_2 > 0 \). Finally, let \( C_3 := \max_{u \in \mathcal{F}} |\det (u)|^{\frac{1}{2} - \frac{1}{q}} \) and \( C_4 := \sup_{u \in \mathcal{F}} v_0 (1_H) v_0 (u) \). Then we have, for \( i \in I \) and \( h = h_i u \in h_i U \):

\[
|\det (h)|^{\frac{1}{2} - \frac{1}{q}} \leq C_3 \cdot |\det (h_i)|^{\frac{1}{2} - \frac{1}{q}} \quad \text{and} \quad v (h) = v (1_H h_i u) \leq v_0 (1_H) v (h_i) v_0 (u) \leq C_4 \cdot v (h_i) .
\]

We first show the claim of the lemma in the case \( q = \infty \). To this end, let \( h \in H \) be arbitrary. Then we have \( h \in h_i U \) for some \( i \in I \). We can then estimate

\[
v (h) \cdot \left\| (W_{\psi} (f \circ F^{-1})) (\cdot, h) \right\|_{L^p (\mathbb{R}^d)} \leq C_3 \cdot C_4 \cdot \left\| \psi^* \right\|_{L^1 (\mathbb{R}^d)} \cdot \sum_{j \in i^* Q'} \left[ u_i \cdot \left\| F^{-1} (\varphi_j f) \right\|_{L^p (\mathbb{R}^d)} \right]
\]

and Eq. \( 8.9 \)

\[
\frac{1}{2} - \frac{1}{q} = \frac{1}{2} - \frac{1}{q} \quad \text{and} \quad \text{Eq. } 8.9
\]

\[
\leq C_2 C_3 C_4 \cdot \left\| \psi^* \right\|_{L^1 (\mathbb{R}^d)} \cdot \sum_{j \in i^* Q'} \left[ u_j \cdot \left\| F^{-1} (\varphi_j f) \right\|_{L^p (\mathbb{R}^d)} \right]
\]

\[
q = \infty \leq C_2 C_3 C_4 \cdot \left\| \psi^* \right\|_{L^1 (\mathbb{R}^d)} \cdot \left| i^* \right| \cdot \left\| f \right\|_{D (Q, L^p, \ell_2^q)}
\]

\[
\leq C_1 C_2 C_3 C_4 \cdot \left\| \psi^* \right\|_{L^1 (\mathbb{R}^d)} \cdot \left\| f \right\|_{D (Q, L^p, \ell_2^q)} < \infty,
\]

where the constants \( C_1, \ldots, C_4 \) and \( \left\| \psi^* \right\|_{L^1 (\mathbb{R}^d)} \) are independent of \( f \).

In the case \( 1 \leq q < \infty \) we first note that equation \( 8.8 \) implies, for \( i \in I \) and \( h \in h_i U \), the estimate

\[
|\det (h)|^{-1} \cdot \left( v (h) \cdot \left\| (W_{\psi} (f \circ F^{-1})) (\cdot, h) \right\|_{L^p (\mathbb{R}^d)} \right)^q \leq \left( |\det (h)|^{-1} \cdot \left( v (h) \cdot |\det (h)|^{1/2} \cdot \left\| \psi^* \right\|_{L^1 (\mathbb{R}^d)} \cdot \sum_{j \in i^* Q'} \left\| F^{-1} (\varphi_j f) \right\|_{L^p (\mathbb{R}^d)} \right)^q \right)
\]

\[
\leq \left( \left\| \psi^* \right\|_{L^1 (\mathbb{R}^d)} \cdot |\det (h)|^{\frac{1}{2} - \frac{1}{q}} \cdot v (h) \right)^q \cdot |i^* \varphi'| \cdot \max \left\{ \left\| F^{-1} (\varphi_j f) \right\|_{L^p (\mathbb{R}^d)} \mid j \in i^* \varphi' \right\}
\]

\[
\leq \left( \left[C_1 C_3 C_4 \left\| \psi^* \right\|_{L^1 (\mathbb{R}^d)} \right)^q \cdot \sum_{j \in i^* Q'} \left[ u_j \cdot \left\| F^{-1} (\varphi_j f) \right\|_{L^p (\mathbb{R}^d)} \right]^q \right)
\]

\[
\leq \left( \left[C_2 C_3 C_4 \right)^q \cdot \sum_{j \in i^* Q'} \left[ u_j \cdot \left\| F^{-1} (\varphi_j f) \right\|_{L^p (\mathbb{R}^d)} \right]^q \right)
\]

\[
=: \left[ C_3^q \cdot \sum_{j \in i^* Q'} \left[ u_j \cdot \left\| F^{-1} (\varphi_j f) \right\|_{L^p (\mathbb{R}^d)} \right]^q \right).
\]
Because of $H = \bigcup_{i \in I} h_i U$, this yields

$$\|W_{\psi} (f \circ F^{-1})\|_{L^p_{\psi}}^q = \int_H \left( v(h) \cdot \| (W_{\psi} (f \circ F^{-1})) (\cdot, h) \|_{L^p_{\psi}} \right)^q \frac{dh}{|\det(h)|} \leq \sum_{i \in I} \int_{h_i U} \left( v(h) \cdot \| (W_{\psi} (f \circ F^{-1})) (\cdot, h) \|_{L^p_{\psi}} \right)^q \frac{dh}{|\det(h)|} \leq \sum_{i \in I} \left( \mu_H (h_i U) C_3^q \cdot \sum_{j \in i^*} [u_j \cdot \| F^{-1} (\varphi_j f) \|_{L^p_{\psi}}]^q \right) \leq (\mu_H (Q) \cdot C_3^q C_1 \cdot \sum_{j \in I} [u_j \cdot \| F^{-1} (\varphi_j f) \|_{L^p_{\psi}}]^q \right) = \mu_H (Q) \cdot C_3^q C_1 \cdot \| f \|_{D(Q, L^p, \ell^q_u)}^q < \infty$$

which proves the claim in the case $1 \leq q < \infty$. In the step marked with (*), we used the equivalence

$$j \in i^* \Leftrightarrow Q_j \cap Q'_i \neq \emptyset \Leftrightarrow i \in j^* \varrho'$$

which is valid for all $i, j \in I$. \qed

It is now easy to show that the map $\Theta^{-1} \circ F^{-1} : D(Q, L^p, \ell^q_u) \to Co(L^p_{\psi})$ (with $\Theta$ as in Theorem 38 and $F^{-1}$ as in Lemma 41 above) is a bounded inverse to the Fourier transform $F : Co(L^p_{\psi}) \to D(Q, L^p, \ell^q_u)$ as defined in Theorem 37.

**Theorem 43.** Let $p, q \in [1, \infty]$ and assume that $Q$ is a decomposition covering of $O$ induced by $H$. Finally, let $u : O \to (0, \infty)$ be a transplant of $v'$ onto $O$, where

$$v' : H \to (0, \infty), h \mapsto |\det (h^{-1})|^{\frac{1}{2}-\frac{d}{4}} \cdot v (h^{-1})$$

is defined as in Lemma 36.

Then $F : Co(L^p_{\psi}) \to D(Q, L^p, \ell^q_u)$ as defined in Theorem 27 is an isomorphism of Banach spaces with bounded inverse

$$\Theta^{-1} \circ F^{-1} : D(Q, L^p, \ell^q_u) \to Co(L^p_{\psi}).$$

**Proof.** Let $f \in Co(L^p_{\psi})$ and define $g := (\Theta^{-1} \circ F^{-1}) (F f)$. We will show $\Theta f = \Theta g$. The injectivity of $\Theta$ (cf. Theorem 38) then implies $f = g$, i.e. $(\Theta^{-1} \circ F^{-1}) \circ F = id_{Co(L^p_{\psi})}$, which in particular entails the surjectivity of $\Theta^{-1} \circ F^{-1}$.

In order to show $\Theta f = \Theta g$, choose an arbitrary $\psi \in S(\mathbb{R}^d \setminus \{0\})$ with $\hat{\psi} \in D(O)$. We have $\Theta g = F^{-1} (F f)$. Note that we cannot simply “cancel” $F^{-1}$ and $F$, as $F f$ is defined by equation 143 and $F^{-1} (F f)$ is defined as in Lemma 31.
Using these definitions, we derive
\[(W\psi(\Theta g))(x,h) \overset{\text{Eq. (5.1)}}{=} (\Theta g)\left(\overline{\pi(x,h)}\psi\right)\]
\[= (F^{-1}(F f))\left(\overline{\pi(x,h)}\psi\right)\]
\[\overset{\text{Def. of } F^{-1} \text{ in Lemma (11)}}{=} (F f)\left(F^{-1}\overline{\pi(x,h)}\psi\right)\]
\[\overset{\text{Eq. (8.1)}}{=} f\left(F^{-1}F^{-1}\overline{\pi(x,h)}\psi\right)\]
\[= f\left(F^{-1}\overline{\pi(x,h)}\psi\right)\]
\[= f\left(\overline{\pi(x,h)}\psi\right)\]
\[\overset{\text{Eq. (5.1)}}{=} (\Theta f)\left(\overline{\pi(x,h)}\psi\right)\]
\[\overset{\text{Eq. (8.1)}}{=} \left(W\psi(\Theta f)\right)(x,h),\]
where we used the easily verified identity \(F^{-1}\overline{\varphi} = \overline{\varphi}\) for \(\varphi \in S(\mathbb{R}^d)\). As \(W\psi\) is injective on \(\widetilde{\text{Co}}(L^p_w)\) (cf. Theorem 38), the above identity shows \(\Theta g = \Theta f\).

In the opposite direction, we note that surjectivity of \(F^{-1} : \mathcal{F}(\mathcal{D}(\mathcal{O})) \to \mathcal{D}(\mathcal{O})\) implies that
\[F^{-1} : \mathcal{D}(\mathcal{Q}, L^p, \ell^q_w) \leq \mathcal{D}'(\mathcal{O}) \to \widetilde{\text{Co}}(L^p_w) \leq (\mathcal{F}(\mathcal{D}(\mathcal{O})))', f \mapsto f \circ F^{-1}\]
as defined in Lemma (11) is injective. As \(\Theta\) and hence also \(\Theta^{-1} : \widetilde{\text{Co}}(L^p_w) \to \text{Co}(L^p_w)\) are bijective by Theorem 38, this shows that \(\Theta^{-1} \circ F^{-1} : \mathcal{D}(\mathcal{Q}, L^p, \ell^q_w) \to \text{Co}(L^p_w)\) is injective. Above, we have already seen that this map is surjective with
\[\left(\Theta^{-1} \circ F^{-1}\right) \circ F = \text{id}_{\widetilde{\text{Co}}(L^p_w)}.\]
This shows that \(\Theta^{-1} \circ F^{-1}\) is bijective with bijective(!) inverse \(F : \text{Co}(L^p_w) \to \mathcal{D}(\mathcal{Q}, L^p, \ell^q_w). \quad \square\)

9. A sample application: Dilation invariance of certain coorbit spaces

In this section we discuss the issue of invariance of coorbit spaces under dilations by matrices that are not necessarily contained in the group. Here, we will restrict ourselves to two examples showing that this question can be fairly subtle, with the answer depending on the dilation group. As was pointed out in the introduction, the question of comparing coorbit spaces associated to different dilation groups arises rather naturally in this context, and the decomposition space view will allow (at least in one case) a rather speedy answer to it.

We start by spelling out how the wavelet transform of a function \(f\) dilated by some matrix \(g \in \text{GL}(\mathbb{R}^d)\) over the semidirect product \(\mathbb{R}^d \rtimes H\), with \(g\) not necessarily contained in \(H\), is related to a wavelet transform of \(f\) over the group \(\mathbb{R}^d \rtimes g^{-1}Hg\). For this purpose, it is beneficial to introduce the quasi-regular representation \(\sigma\) of the full affine group \(\mathbb{R}^d \rtimes \text{GL}(\mathbb{R}^d)\) acting unitarily on \(L^2(\mathbb{R}^d)\) by
\[\sigma(x,g)f = |\det(g)|^{-1/2}f\left(g^{-1}(y-x)\right), \quad (x,g) \in \mathbb{R}^d \times \text{GL}(\mathbb{R}^d),\]
thus extending the quasi-regular representations of both \(\mathbb{R}^d \rtimes H\) and \(\mathbb{R}^d \rtimes g^{-1}Hg\). The proof of the lemma consists in straightforward computations, and is therefore omitted.
Lemma 44. Let $H_1$ denote a closed matrix group fulfilling our standing admissibility assumptions, let $g \in \text{GL}(\mathbb{R}^d)$ be arbitrary, and define $H_2 = g^{-1}H_1g$.

(a) Let $O_1 = H_1^T \xi_0$ denote the open dual orbit associated to $H_1$, then the open dual orbit associated to $H_2$ is given by

$$O_2 = g^T O_1 = H_2^T g^T \xi_0.$$

(b) Assume that $\psi_1 \in S(\mathbb{R}^d)$ satisfies $\hat{\psi}_1 \in \mathcal{D}(O_1)$. Then $\psi_2 = \sigma(0, g^{-1}) \psi_1 \in S(\mathbb{R}^d)$ fulfills $\hat{\psi}_2 \in \mathcal{D}(O_2)$.

(c) Let $H_i$ and $\psi_i$ be as in the previous parts, and let $f \in L^2(\mathbb{R}^d)$. Denote by $W_{\psi_i}^1$ the associated wavelet transforms. Then we have the relation

$$(W_{\psi_1}^1(\sigma(0,g)f))(x,h) = (W_{\psi_2}^2 f)(g^{-1}x, g^{-1}hg).$$

(d) Let $v_1$ denote a moderate weight function on $H_1$, and let

$$v_2 : H_2 \to (0, \infty), h \mapsto v_1(ghg^{-1}).$$

Then we have, for $f \in L^2(\mathbb{R}^d)$, that

$$\sigma(0,g)f \in \text{Co} \left( L_{v_1}^{p,q}(\mathbb{R}^d \times H_1) \right) \iff f \in \text{Co} \left( L_{v_2}^{p,q}(\mathbb{R}^d \times H_2) \right).$$

In particular, $\text{Co} \left( L_{v_1}^{p,q}(\mathbb{R}^d \times H_1) \right) \cap L^2(\mathbb{R}^d)$ is invariant under $\sigma(0, g)$ iff

$$\text{Co} \left( L_{v_1}^{p,q}(\mathbb{R}^d \times H_1) \right) \cap L^2(\mathbb{R}^d) \subset \text{Co} \left( L_{v_2}^{p,q}(\mathbb{R}^d \times H_2) \right)$$

holds.

Note that in general, the question of embeddings between coorbit spaces with respect to different dilation groups is not even well-posed; this is one reason why the statement in part (d) is restricted to $L^2$-functions. By definition,

$$\text{Co}(Y) = \left\{ f \in (H^1_w(G))^\sim \mid W_{\psi} f \in Y \right\},$$

i.e. the elements $f \in \text{Co}(Y)$ “live” in the space $(H^1_w(G))^\sim$ of antilinear functionals on $H^1_w(G) = \{ g \in L^2(G) \mid W_{\psi} g \in L^1_w(G) \}$, where $w : G \to (0, \infty)$ is a suitable control-weight for the solid BF-space $Y$ and where $\psi \in \mathcal{A}_w \setminus \{0\}$ is fixed. Clearly, this definition depends on $G$, and thus on $H$.

Thus, it is not obvious how an element $f \in \text{Co}(Y) \subset (H^1_w(G))^\sim$ for some group $G$ can be interpreted as an element $f \in \text{Co}(Y') \subset (H^1_w(G'))^\sim$ for a different group $G'$ (and different $Y'$, $w'$). But in the setting of this paper, both groups are of the form $G = \mathbb{R}^d \rtimes H$ and $G' = \mathbb{R}^d \rtimes H'$ and operate on $L^2(\mathbb{R}^d)$ by virtue of the quasi-regular representation. Thus, we adopt the following conventions:

Definition 45. Let $p_1, p_2, q_1, q_2 \in [1, \infty]$.

(1) Let $H_1, H_2 \leq \text{GL}(\mathbb{R}^d)$ be closed subgroups that fulfill our standing assumptions and assume also that $v_i : H_i \to (0, \infty)$ obeys our standing assumptions for $i = 1, 2$. Let $G_i = \mathbb{R}^d \rtimes H_i$ for $i = 1, 2$. We then say that a bounded linear map

$$T : \text{Co} \left( L_{v_1}^{p_1,q_1}(G_1) \right) \to \text{Co} \left( L_{v_2}^{p_2,q_2}(G_2) \right)$$

is an embedding of coorbit spaces if $T f = f$ holds for all $f \in L^2(\mathbb{R}^d) \cap \text{Co} \left( L_{v_1}^{p_1,q_1}(G_1) \right)$. 53
(2) Let \( \emptyset \neq U_1, U_2 \subset \mathbb{R}^d \) be open and let \( Q^{(i)} \) be a decomposition covering of \( U_i \) for \( i = 1, 2 \). Finally, assume that \( u_i : U_i \to (0, \infty) \) is \( Q^{(i)} \)-moderate for \( i = 1, 2 \). We then say that a bounded linear map

\[
S : \mathcal{D} \left( Q, L^{p_1}, \ell^{q_1}_{u_1} \right) \to \mathcal{D} \left( Q, L^{p_2}, \ell^{q_2}_{u_2} \right)
\]

is an **embedding of decomposition spaces** if the identity \( Sf = f \) holds for all \( f \in L^2 \left( \mathbb{R}^d \right) \cap \mathcal{D} \left( Q, L^{p_1}, \ell^{q_1}_{u_1} \right) \).

**Remark.** It is worth noting that an embedding in the above sense is not required to be an injective map.

The existence of an embedding of coorbit spaces can be characterized by the existence of embeddings between the associated decomposition spaces as follows:

**Lemma 46.** Let \( H_1, H_2, G_1, G_2 \) and \( v_1, v_2 \) be as in Definition 4.4 and let \( Q^{(j)} = \left( Q_i^{(j)} \right)_{i \in I^{(j)}} \) be an induced decomposition covering of the dual orbit \( O_j \subset \mathbb{R}^d \) (with respect to \( H_j \)) for \( j = 1, 2 \).

Finally, choose \( p_1, p_2, q_1, q_2 \in [1, \infty] \), define

\[
v_j : H_j \to (0, \infty), h \mapsto \left| \det (h^{-1}) \right|^{\frac{1}{2} - \frac{1}{p_j}} v_j (h^{-1})
\]

and let \( u_i^{(j)} = \left( u_i^{(j)} \right)_{i \in I^{(j)}} \) be a transplant of \( v_j \) to \( O_j \) for \( j = 1, 2 \).

Then \( T : \text{Co} \left( L^{p_1, q_1}_{v_1} (G_1) \right) \to \text{Co} \left( L^{p_2, q_2}_{v_2} (G_2) \right) \) is an embedding of coorbit spaces if and only if

\[
\mathcal{F} \circ T \circ \mathcal{F}^{-1} : \mathcal{D} \left( Q^{(1)}, L^{p_1, \ell^{q_1}_{u_1}} \right) \to \mathcal{D} \left( Q^{(2)}, L^{p_2, \ell^{q_2}_{u_2}} \right)
\]

is an embedding of decomposition spaces. Here, \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) are defined as in Theorems 37 and 43 respectively.

**Proof.** Theorem 43 implies that \( T \) is bounded if and only if \( \mathcal{F} \circ T \circ \mathcal{F}^{-1} \) is bounded.

Now let \( T \) be an embedding of coorbit spaces and let \( f \in \mathcal{D} \left( Q^{(1)}, L^{p_1, \ell^{q_1}_{u_1}} \right) \cap L^2 \left( \mathbb{R}^d \right) \). The Remarks 5 and 42 show that \( \mathcal{F}^{-1}f \) is the “ordinary” inverse Fourier transform of \( f \in L^2 \left( \mathbb{R}^d \right) \) and that \( \mathcal{F} \mathcal{F}^{-1}f = f \) holds, because the Fourier transform \( \mathcal{F} \) as defined in Theorem 37 also coincides with the standard Fourier transform of \( \mathcal{F}^{-1}f \in L^2 \left( \mathbb{R}^d \right) \). Hence, we get

\[
\left( \mathcal{F} \circ T \circ \mathcal{F}^{-1} \right) (f) \quad \mathcal{F}^{-1}f \in L^2 \left( \mathbb{R}^d \right) \cap \text{Co} \left( L^{p_1, q_1}_{v_1} (G_1) \right) \quad \mathcal{F} \quad f,
\]

i.e. \( \mathcal{F} \circ T \circ \mathcal{F}^{-1} \) is an embedding of decomposition spaces.

The proof of the converse direction is completely analogous. \( \square \)

We will now analyze the existence of embeddings between the coorbit space \( \text{Co} \left( L^{p, q} \left( \mathbb{R}^2 \times H \right) \right) \) and the coorbit space \( \text{Co} \left( L^{p, q}_{g^{-1}v} \left( \mathbb{R}^2 \times g^{-1}Hg \right) \right) \) with respect to the conjugated group \( g^{-1}Hg \), where \( g^{-1}v \) is defined by

\[
g^{-1}v : g^{-1}Hg \to (0, \infty), h \mapsto v (ghg^{-1})
\]

We will see that both spaces coincide (up to harmless identifications) for the similitude group, whereas the same is in general not true for the shearlet group.
Our first example is the **similitude group**

\[
H_1 := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 \neq 0 \right\} = \left\{ r \cdot \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \mid r > 0 \text{ and } \varphi \in [0, 2\pi) \right\}.
\]

Here, the dual orbit is given by \(O_1 = \mathbb{R}^2 \setminus \{0\}\). In the following, we will be using the well-spread family

\[
(h_k)_{k \in \mathbb{Z}} := (2^{-k} \cdot \text{id}_{\mathbb{R}^2})_{k \in \mathbb{Z}}
\]

and the precompact, open sets

\[
P := \left\{ x \in \mathbb{R}^2 \mid \frac{3}{2} < |x| < \frac{3}{2} \right\},
\]

\[
Q := \left\{ x \in \mathbb{R}^2 \mid \frac{1}{2} < |x| < 2 \right\}
\]

that satisfy \(P \subset Q \subset \overline{Q} \subset O_1\) as well as \(O_1 = \bigcup_{k \in \mathbb{Z}} h_k P = \bigcup_{k \in \mathbb{Z}} h_k Q\). Thus, Theorem \(24\) shows that

\[
Q := (Q_k)_{k \in \mathbb{Z}} := (h_k^{-1} Q)_{k \in \mathbb{Z}} = (2^k \cdot Q)_{k \in \mathbb{Z}}
\]

is a decomposition covering of \(O_1\) induced by \(H_1\).

Now let \(g \in \text{GL} (\mathbb{R}^d)\) be arbitrary. The conjugate group \(H_1 (g) := g^{-1} H_1 g \leq \text{GL} (\mathbb{R}^d)\) then has the same open dual orbit \((O_1) (g) = \mathbb{R}^2 \setminus \{0\} = O_1\) and the family \((g^{-1} h_k g)_{k \in \mathbb{Z}} = (h_k)_{k \in \mathbb{Z}}\) is well-spread in \(H_1 (g)\). Thus, \(Q\) is also a decomposition covering of \((O_1) (g) = O_1\) induced by \(g^{-1} H_1 g\).

For a weight \(v : H_1 \to (0, \infty)\) that is \(v_0\)-moderate for some locally bounded, submultiplicative, measurable weight \(v_0 : H_1 \to (0, \infty)\) we can then define \(g^{-1} v g := v \circ \Phi_g\) and \(g^{-1} v_0 g := v_0 \circ \Phi_g\) for the isomorphism \(\Phi_g : H_1 (g) \to H_1, h \mapsto g h g^{-1}\). Then \(g^{-1} v g\) is \(g^{-1} v_0 g\)-moderate and we get

\[
D(Q, L^p, \ell^q_n) = D(Q, L^p, \ell^q_{g^{-1} v g})
\]

for all \(p, q \in [1, \infty]\), where we have chosen the discretizations

\[
u_k = |\det (h_k)|^{\frac{1}{2} - \frac{1}{q}} \cdot v (h_k)
\]

and

\[
(g^{-1} v g)_k = |\det (h_k)|^{\frac{1}{2} - \frac{1}{q}} \cdot v (g h_k g^{-1}) = u_k
\]

of the transplant of

\[
u' : H_1 \to (0, \infty), h \mapsto |\det (h^{-1})|^{\frac{1}{2} - \frac{1}{q}} \cdot v (h^{-1})
\]

or of

\[
(g^{-1} v g)' : g^{-1} H_1 g \to (0, \infty), h \mapsto |\det (h^{-1})|^{\frac{1}{2} - \frac{1}{q}} \cdot (g^{-1} v g)(h^{-1})
\]

onto \(O_1\) or onto \((O_1) (g)\), respectively (see remark \(24\) or Lemma \(35\) for the validity of this choice).

The identity map \(\text{id} : D(Q, L^p, \ell^q_n) \to D(Q, L^p, \ell^q_{g^{-1} v g})\) is thus an embedding of decomposition spaces, so that Lemma \(10\) shows that

\[
\mathcal{F}^{-1} \circ \mathcal{F} : \text{Co} \left( L^p \otimes (\mathbb{R}^2 \rtimes H_1) \right) \to \text{Co} \left( L^p_{g^{-1} v g} (\mathbb{R}^2 \rtimes g^{-1} H_1 g) \right)
\]

is an embedding of coorbit spaces. The same holds of course for the inverse map.

It should be noted that the above embedding reduces to the identity as long as \((H^1_w)\) is identified with a subspace of \(D' (\mathbb{R}^2 \setminus \{0\})\) (cf. Corollary \(14\)). It is furthermore worth noting that the same argument could be applied to any admissible group \(H\) and any \(g \in \text{GL} (\mathbb{R}^d)\) as
long as \( H \) admits a well-spread family \((h_i)_{i \in I}\) that commutes with \( g \) (i.e. \( g^{-1}h_i g = h_i \) holds for all \( i \in I \)) and as long as the dual orbits of \( g^{-1}Hg \) and \( H \) coincide.

As a corollary to these observations, we obtain that the homogeneous Besov spaces \( \dot{B}^{p,q}_s(\mathbb{R}^2) \) are invariant under arbitrary dilations. We expect that this result is well-known, although we were not able to locate a convenient source for it. Note that a proof of this fact using the standard tensor wavelet ONB’s promises to be fairly cumbersome, due to the rather poor compatibility of those bases with arbitrary dilations.

Our second example is the shearlet group

\[
H_2 := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1/2} \end{pmatrix} \mid a \in (0, \infty), b \in \mathbb{R}, \varepsilon \in \{ \pm 1 \} \right\}.
\]

Here we will show (using the standard definition of coorbit spaces instead of the characterization via decomposition spaces) that there is some \( \psi \in \mathcal{S}(\mathbb{R}^d) \) that belongs to the “conjugated” coorbit space \( \text{Co} \left( L^1_{g^{-1}}(G_2(g)) \right) \) with \( G_2(g) := \mathbb{R}^2 \times g^{-1}H_2g \) and \( v : H_2 \to (0, \infty), h \mapsto |\det(h)|^{3/2} \), where \( g := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is the rotation by \( 90^\circ \) in the counter-clockwise direction, but not to the shearlet coorbit space \( \text{Co}(L^1_p(G_2)) \) for \( G_2 = \mathbb{R}^2 \times H_2 \). We will then see that the decomposition space point of view provides useful intuition why this is true.

We first note that the dual orbit of \( H_2 \) is given by \( \mathcal{O}_2 := H^T_2 \left( \frac{1}{3} \right) = \mathbb{R}^* \times \mathbb{R} \). In contrast, the dual orbit of the conjugated group \( H_2(g) := g^{-1}H_2g \) is

\[
g^T H^T_2 g^{-T} \left( \frac{1}{3} \right) = g^T (\mathbb{R}^* \times \mathbb{R}) = g^T \mathcal{O}_2 = \mathbb{R} \times \mathbb{R}^*.
\]

For the construction of \( \psi \), choose an arbitrary \( \varphi \in \mathcal{D}(B_1(\frac{9}{5})) \) with \( \varphi \geq 0 \) and \( \varphi \equiv 1 \) on \( B_{1/2}(\frac{9}{5}) \) and define \( \psi := \mathcal{F}^{-1} \varphi \in \mathcal{S}(\mathbb{R}^d) \) and \( \psi_0 := \mathcal{F}^{-1} \left( L_{(\frac{3}{5})} \varphi \right) \in \mathcal{S}(\mathbb{R}^d) \). This choice ensures that

\[
\text{supp}(\hat{\psi}_0) = \text{supp} \left( L_{(\frac{3}{5})} \varphi \right) = \text{supp}(\varphi) + (3, 0)^T \subset B_1((3, 0)^T)
\]

is a compact subset of \( \mathbb{R}^* \times \mathbb{R} = \mathcal{O}_2 \). Thus, Theorem 9 shows \( \psi_0 \in \mathcal{B}_w \), where \( w \) is a control weight (as in Lemma 4) for \( L^1_p(G_2) \).

For \( x \in \mathbb{R}^d \) and \( h \in \text{GL}(\mathbb{R}^d) \) we can now calculate

\[
(W_{\psi_0, \psi})(x, h) = \langle \psi, \pi(x, h) \psi_0 \rangle_{L^2} = \langle \hat{\psi}, \mathcal{F}(\pi(x, h) \psi_0) \rangle_{L^2}.
\]

Using the Plancherel identity, we get

\[
|\det(h)|^{1/2} \langle \hat{\psi}, \mathcal{F}(\pi(x, h) \psi_0) \rangle_{L^2} = \langle \hat{\varphi}, M^{-\frac{1}{2}}Dh \hat{\psi}_0 \rangle_{L^2}.
\]

where \( \varphi \equiv \hat{\phi} \) and

\[
\varphi \geq 0 \quad \Rightarrow \quad |\det(h)|^{1/2} \left( \int_{\mathbb{R}^d} \varphi(y) \cdot e^{2\pi i (-x, y)} (L_{(\frac{3}{5})} \varphi)(h^T y) \, dy \right) \cdot e^{2\pi i (x, y)}.
\]

For \( \xi \in \mathbb{R} \) with \( |\xi| \leq 1/2 \), the Lipschitz continuity (with Lipschitz constant \( L = 1 \)) of the cosine implies

\[
\cos(\xi) \geq \cos(0) - |\cos(0) - \cos(\xi)| \geq 1 - |\xi| \geq \frac{1}{2}.
\]
For \( y \in \text{supp}(\varphi) \subseteq B_1(\frac{3}{4}) \) we have \(|y| \leq 1 + \|(\frac{3}{4})\| = 4\). For \( x \in \mathbb{R}^2 \) with \(|x| \leq \frac{1}{16\pi} \) this implies \(|2\pi \langle x, y \rangle| \leq \frac{1}{2} \). Using this and the estimate for the cosine above, we arrive at

\[
| (W_{\psi\beta})(x, h) | \\
\geq | \det(h)|^{1/2} \cdot \Re \left( \int_{\mathbb{R}^d} \varphi(y) \cdot \varphi(h^T y - (\frac{3}{4})) \cdot e^{2\pi i \langle x, y \rangle} \, dy \right) \\
\varphi \geq 0 \\
\geq | \det(h)|^{1/2} \cdot \frac{1}{2} \int_{\mathbb{R}^d} \varphi(y) \cdot \varphi(h^T y - (\frac{3}{4})) \cdot \cos(2\pi \langle x, y \rangle) \, dy \\
\varphi \equiv 1 \text{ on } B_{1/2}(\frac{9}{3}) \\
\geq \frac{| \det(h)|^{1/2}}{2} \cdot \lambda \left( B_{1/2}(\frac{9}{3}) \cap h^{-T} \left( B_{1/2}(\frac{3}{4}) \right) \right) \tag{9.2}
\]

for \(|x| \leq \frac{1}{16\pi}\). Here \( \lambda \) denotes the 2-dimensional Lebesgue measure.

In the step marked with \((*)\), we used \( \varphi \equiv 1 \) on \( B_{1/2}(\frac{9}{3}) \) and the fact that \( y \in h^{-T} \left( B_{1/2}(\frac{3}{4}) \right) \) implies \( h^T y - (\frac{3}{4}) \subseteq B_{1/2}(\frac{3}{4}) - (\frac{3}{4}) = B_{1/2}(\frac{9}{3}) \).

Note that we have \( M := (0, \frac{1}{2}) \times (3 - \frac{1}{2}, 3 + \frac{1}{2}) \subseteq B_{1/2}(\frac{9}{3}) \). Let \( (\frac{3}{4}) \in M \) be arbitrary. For \( h_{\alpha, \beta} := \left( \begin{array}{c} \alpha \\ \beta \\ x \end{array} \right) \in H_2 \) with \( \alpha \in (0, \infty) \) and \( \beta \in \mathbb{R} \) we then have

\[
\left( \begin{array}{c} x \\ y \end{array} \right) \in h^{-T}_{\alpha, \beta} \left( B_{1/2}(\frac{3}{4}) \right) \\
\Leftrightarrow \frac{ax}{\beta x + \alpha^{1/2} y} = h^T_{\alpha, \beta} \left( \begin{array}{c} x \\ y \end{array} \right) \in B_{1/2}(\frac{3}{4}) \\
\Leftrightarrow |x| |\alpha - \frac{3}{x}| < \frac{1}{4} \quad \text{and} \quad |x| \left| \beta - \left( \frac{3}{x} - \frac{\alpha^{1/2} y}{x} \right) \right| < \frac{1}{4} \\
\Leftrightarrow \alpha \in \left( \frac{11}{4x}, 13 \right) \quad \text{and} \quad \beta \in B_{\frac{4x}{5}}(\beta_{\alpha, x, y}), \tag{9.3}
\]

where we used the abbreviation \( \beta_{\alpha, x, y} = \frac{3}{x} - \frac{\alpha^{1/2} y}{x} \).

Recall that a (left) Haar integral on the shearlet group \( H_2 \) is given by

\[
\int_{H_2} f(h) \, dh = \int_{\mathbb{R}^*} \int_{\mathbb{R}} f( \text{sgn}(\alpha) h_{|\alpha|, \beta}) \, d\beta \, d\alpha. \tag{57}
\]
Thus, we finally arrive at

\[ \|\psi\|_{\text{Co}(L^1_{\psi}(G_2))} = \|W_{\psi_0}\psi\|_{L^1_{\psi}(G_2)} \]

\[ \geq \int_0^{1/16} \int_{H_2} |\det(h)|^{1/2} \cdot \lambda(B_{1/2}(\frac{9}{3}) \cap h^{-1}B_{1/2}(\frac{3}{3})) \cdot |\det(h)|^{7/6} \frac{dh}{|\det(h)|} \, dx \]

\[ \geq C \cdot \int_0^{\infty} |\det(h_{\alpha,\beta})|^{2/3} \cdot \lambda(B_{1/2}(\frac{9}{3}) \cap h_{\alpha,\beta}^{-1}B_{1/2}(\frac{3}{3})) \, d\beta \frac{d\alpha}{\alpha^2} \]

\[ = C \cdot \int_0^{\infty} \alpha^{-1} \cdot \lambda(B_{1/2}(\frac{9}{3}) \cap h_{\alpha,\beta}^{-1}B_{1/2}(\frac{3}{3})) \, d\beta \, d\alpha \]

\[ \geq C \cdot \int_0^{1/4} \int_{\frac{3}{4}}^{\frac{9}{4}} \int_0^{\infty} \alpha^{-1} \cdot \chi_{h_{\alpha,\beta}^{-1}(B_{1/2}(\frac{3}{3}))} \left( \frac{x}{y} \right) \, d\beta \, d\alpha \, dy \, dx \]

\[ \geq C \cdot \int_0^{1/4} \int_{\frac{3}{4}}^{\frac{9}{4}} \frac{1}{2x} \cdot \int_0^{\frac{13}{11}} \alpha^{-1} \, d\alpha \] \( \cdot \int_0^{1/4} \frac{1}{x} \, dx = \infty, \]

i.e. \( \psi \notin \text{Co}(L^1_{\psi}(G_2)) \).

Regarding the question of membership of \( \psi \) in the coorbit space of the conjugate group \( G_2(g) = \mathbb{R}^2 \times g^{-1}H_2g \), we note that \( \text{supp}(\psi) \subset B_1(\frac{9}{3}) \) is a compact subset of the dual orbit \( O_2(g) = \mathbb{R} \times \mathbb{R}^* \) of \( g^{-1}H_2g \). By Theorem \([9]\) this shows \( \psi \in B_w \), where \( w \) is a control weight (as in Lemma \([3]\) for \( L^1_{g^{-1}g}(\mathbb{R}^d \times g^{-1}H_2g) \)). Note that the „atomic decomposition“ theorem for coorbit spaces (cf. \([10]\) Theorem 6.1) implies that the inclusion

\[ B_w \subset \text{Co}(L^1_{g^{-1}g}(G_2(g))) \]

is valid. All in all, this proves that \( \psi \in \mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \) satisfies

\[ \psi \in \text{Co}(L^1_{g^{-1}g}(G_2(g))) \setminus \text{Co}(L^1_{\psi}(G_2)) \].

This shows that the coorbit spaces of the shearlet group are (in general) not invariant under dilation. It should be noted that the above reasoning could be adapted to other transformations as well; we claim that the only orthogonal transformations under which all shearlet coorbit spaces are invariant are the reflections \( (x_1, x_2)^T \mapsto (-x_1, x_2)^T \), \( (x_1, x_2)^T \mapsto (x_1, -x_2)^T \) and the rotation \( (x_1, x_2)^T \mapsto (-x_1, -x_2)^T \).

Intuitive reasons for this phenomenon (and for the choice of \( \psi \)) that are suggested by the decomposition space point of view are the following:

1. \( \psi \) does not vanish on the „blind spot“ \( \mathbb{R}^2 \setminus O_2 = \{0\} \times \mathbb{R} \) of the shearlet group.
Choose

\[ v' : H_2 \to (0, \infty), h \mapsto \left|\det (h^{-1})\right|^{\frac{1}{2}} \cdot v(h^{-1}) = \left|\det (h)\right|^{\frac{1}{2}} \cdot \left|\det (h)\right|^{-\frac{1}{2}} = \left|\det (h)\right|^{-\frac{1}{2}} \]

as in Theorem 37 and let \( \xi_0 := \left( \frac{1}{0} \right) \in \mathcal{O}_2 \).

For \( \left( \frac{x}{y} \right) \in \mathcal{O}_2 \) we then have

\[ h_{(x,y)} := \text{sgn}(x) \cdot \left( \begin{array}{c} |x| \\ \text{sgn}(x) \cdot y \\ |x|^{1/2} \end{array} \right) \in H_2. \]

This shows that

\[ u \left( \frac{x}{y} \right) := v' \left( h_{(x,y)} \right) = \left|\det (h_{(x,y)})\right|^{-\frac{1}{2}} = |x|^{-1} \]

is a valid transplant of \( v' \) onto \( \mathcal{O}_2 \).

But this (and thus every) transplant of \( v' \) onto \( \mathcal{O}_2 \) blows up near the „blind spot“ \( \mathbb{R}^2 \setminus \mathcal{O}_2 = \{0\} \times \mathbb{R} \).

Using these observations one can show \( \hat{\psi} \not\in \mathcal{D} \left( \mathcal{Q}_2, L^1, \ell^1_0 \right) \) for a suitable decomposition covering \( \mathcal{Q}_2 \) of \( \mathcal{O}_2 \) induced by \( H_2 \), which implies \( \psi \not\in \mathcal{C} \left( L^1_0 (G_2) \right) \) by Theorem 37.

10. OUTLOOK

While the discussion in the previous section is in parts somewhat ad hoc and restricted, we believe that it nicely illustrates the use that can be made of the decomposition space formalism. The systematic study of embeddings between decomposition spaces, and their application to the study of wavelet coorbit spaces, is the subject of ongoing research, and will be treated in more detail in upcoming publications. Another direction of research that is currently pursued concerns the extension of the results to include quasi-Banach spaces, in particular with the aim of treating spaces of the type \( \mathcal{C} \left( L^p_0, Q_2 \right) \) with \( p \) and/or \( q \) in \( (0, 1) \).

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