Maximum entropy freezeout of hydrodynamic fluctuations

Maneesha Sushama Pradeep\(^1\) and Mikhail Stephanov\(^1\)

\(^1\)Department of Physics, University of Illinois, Chicago, IL 60607, USA

(Dated: May 17, 2023)

We propose a general approach to freezing out fluctuations in heavy-ion collisions using the principle of maximum entropy. We find the results naturally expressed as a direct relationship between the irreducible relative correlators quantifying the deviations of hydrodynamic as well as hadron gas fluctuations from the ideal hadron gas baseline. The method also allows us to determine heretofore unknown parameters crucial for the freezeout of fluctuations near the QCD critical point in terms of the QCD equation of state.

\textit{Introduction} — Mapping the phase diagram of QCD is the primary goal of heavy-ion collision experiments \cite{1, 2}. Fluctuations carry important information about the phase diagram. In particular, non-monotonic behavior of fluctuation measures is a signature of the QCD critical point \cite{3–7}. Fluctuations are also important for the study of hydrodynamics in small systems as well as understanding of initial conditions \cite{8}. Nonequilibrium dynamics of fluctuations is important in general, and of particular importance is the dynamics of the non-Gaussian fluctutations near the critical point. For many of these reasons evolution of fluctuations in hydrodynamics has been intensively studied recently \cite{9–24}.

Freezeout (particilization) is the crucial step in translating hydrodynamic fluctuations into fluctuations and correlations of particles observed in experiment. There have been attempts to develop appropriate procedure for fluctuation freezeout, starting from Ref.\cite{9}. One of the known problems is the proper separation of the trivial (single-particle) contribution from the multi-particle fluctuations induced by hydrodynamic correlations. This problem was addressed in \cite{10, 25} for one particular case of charge diffusion, but a general solution is still lacking, especially for non-Gaussian fluctuations. Indeed, freezeout of non-Gaussian out-of-equilibrium fluctuations have not yet been adequately addressed, except for a proposal in Ref. \cite{24}, where the procedure introduced earlier for equilibrium fluctuations was straightforwardly generalized in a somewhat ad hoc manner. Even in this case, the proposal is limited to leading (most singular) critical contribution to fluctuations. While fluctuating hydrodynamics itself obeys conservation laws, implementing these laws at freezeout is a nontrivial issue \cite{26}.

In this Letter we introduce a very general method of freezing out fluctuations based on the principle of maximum entropy. In this approach the fluctuations of conserved quantities are matched exactly (i.e., not only the leading critical contribution) between the hydrodynamic and the kinetic (particilized) side of the freezeout transition as dictated by conservation laws. The distribution of fluctuations across particle momenta is then determined by maximizing the entropy of fluctuations – an object we introduce in this paper, which is mathematically similar to the n-PI action of the quantum field theory. The entropy of fluctuations has also a lot in common with the concept of relative entropy, which, in this case, measures the amount of entropy deficit of a system with known correlations (relative to the most agnostic state corresponding to complete thermal equilibrium).

Of course, freezeout procedure, being essential for translating hydrodynamics into particle observables, has been known for a long time in the form introduced by Cooper and Frye \cite{27}. This procedure, however, could only deal with mean hydrodynamic quantities, i.e., single-particle observables. It does not address the question of how to freeze out the fluctuating hydrodynamics.

In this Letter we apply the general principle of maximum entropy to the long-standing problem of the fluctuation freezeout and obtain several novel results, which non-trivially match some of the existing approaches to freezeout in the literature, while augmenting or correcting others. Most importantly, the new approach allows us to tackle the problem of the freezeout of non-Gaussian fluctuations, in and out of equilibrium. Furthermore we are now able to determine some of the thus far unknown parameters crucial for the freezeout of the fluctuations near the critical point.

\textit{Setup and main result} — The principle of maximum entropy has been applied recently to implement freezeout of mean hydrodynamic variables into single-particle observables in Ref. \cite{28}. The procedure reproduces the Cooper-Frye procedure when non-equilibrium effects are ignored, but allows systematic incorporation of non-equilibrium, dissipative effects. Let us, therefore, begin by briefly reviewing the maximum entropy freezeout in the simplest case of mean quantities in equilibrium.

The aim is to find the phase space distribution function \(f_A \equiv f_A (x_A)\), where by \(A\) we denote a composite index describing discrete quantum numbers of particles such as spin or baryon number \((q_A)\) as well their momenta \(p_A\) (collectively \(A\)) and the coordinate \(x_A\). In other words, \(A\) labels a “cell” in the phase space. In position space this cell matches a hydrodynamic cell, which has a small but macroscopic size. The function \(f_A\) is the mean occupation number of the available single-particle states in the cell \(A\). It must be such that the conserved hy-
hydrodynamic quantities, such as energy-momentum and conserved charge (such as baryon number) densities in the rest frame of the fluid, $T^\mu\nu u_\nu = e u^\mu$ and $J \cdot u = n$, are matched locally at each point $x$ (i.e., in each cell) on the freezeout hypersurface by the gas of non-interacting hadrons (resonance gas), i.e.,

$$
e u^\mu(x) = \int_A p_A^\mu f_A(x), \quad n(x) = \int_A q_A f_A(x),$$

(1)

where $\int_A$ denotes the summation over particle species $A$ as well as the integration over their momenta (with Lorentz invariant measure). Obviously, there are infinitely many solutions to the constraints (1) on $f_A$. We expect that the most likely solution to describe the distributions of fluctuations, i.e., of the fluctuations will be matched by fluctuations in the hadron natural solution to the problem of freezing out the fluctuations. We shall organize hydrodynamic variables in a vector $\Psi$ of the freezeout hypersurface by the gas of non-interacting hadrons (resonance gas), i.e.,

$$
h_{\mu\nu} = \int_A \Psi_{\mu\nu} f_A,$$

(2)

where $\int_A$ are matched locally at each point $x$.

We approach can be naturally extended to incorporate matching to viscous stress and diffusive current in hydrodynamics against the non-ideal corrections to particle distribution function $f_A$, i.e., imposing additional constraints, but using the same entropy $S[f]$.

In this Letter we show that a more general application of the principle of maximum entropy can also provide a natural solution to the problem of freezing out the fluctuating hydrodynamics. In this case hydrodynamic fluctuations will be matched by fluctuations in the hadron gas, and the entropy must be a functional of the measures of fluctuations, i.e., of the correlators of particle distributions $f_A$.

To simplify notations and to make them more general, we shall organize hydrodynamic variables in a vector $\Psi$, where, e.g., $\Psi_{\mu} = e u^\mu$ and $\Psi^3 = n$. We also denote by $P_A^\mu$ the contribution of a single particle to the phase space cell $A$ to the hydrodynamic variable $\Psi^a$ in the hydrodynamic cell at point $x_a$, i.e., $P_A^\mu = p_A^\mu \delta^3(x_a - x_A)$ and $P_A^\mu = q_A \delta^3(x_a - x_A)$, so that Eqs. (1) can be written as

$$
\Psi = \int_A P_A^a f_A$$

and, correspondingly, $\delta \Psi^a = \int_A P_A^a \delta f_A$,

which translates into a relationship between the (connected) correlators in the hadron gas given by $G_{AB...} = \langle \delta f_A \delta f_B \ldots \rangle_c$ and (connected) correlators of hydrodynamic variables $H_{AB...} = \langle \delta \Psi^a \delta \Psi^b \ldots \rangle_c$.

$$
H_{AB...} = \int_A G_{AB...} P_A^{a_1} P_B^{b_1} \ldots$$

(4)

where $\int_A = \int_A \int_A$. These n-point hydrodynamic correlators ($H_n$ in shorthand) are related to n-point Wigner functions $W_n$ by the generalized Wigner transform introduced in Ref. [22].

We treat Eqs. (4) as constraints to be obeyed by $G_{AB...}$ ($G_n$ in shorthand). The maximum entropy principle will then determine $G_n$ by maximizing the entropy $S[f, G, G_3, G_4 \ldots]$ of the state of the hadron gas with given fluctuations characterized by connected correlators $G_n$. We determine this entropy functional by a calculation similar to Ref. [11] but for higher-order correlators.

Solving the variational problem for $G_n$ with constraints in Eq. (4) we find relationships between the hydrodynamic, $H_n$, and particle, $G_n$, correlators. These relationships are especially simple and intuitive to linear order in relative correlators $\Delta G_n = G_n - \bar{G}_n$ and $\Delta H_n = H_n - \bar{H}_n$, expressing correlations relative to the baseline $\bar{G}_n$ and $\bar{H}_n$ given by the ideal hadron gas in equilibrium. In order to express these relationships we define (see Eq. (15)) a somewhat novel kind of correlation measures $\Delta G_n$ (and similarly, $\Delta H_n$) which we refer to as irreducible relative correlators (IRC). These measures, similar to $\Delta G_n$, quantify correlations relative to the ideal resonance gas, but only correlations not reducible to lower-order correlations. For a classical gas ($\theta_A = 0$) $\bar{G}_n$ are similar to irreducible correlators described in Ref. [29] or “correlation functions” in Ref. [30]. In terms of the IRCs the relationships translating hydrodynamic fluctuations into particle fluctuations take the following form:

$$
\Delta G_{AB...} = \sum_{aA... bB...} \int_{x_axB...} \Delta H_{AB...}(\tilde{H}^{-1} P G)^a_{AB...}(\tilde{H}^{-1} P G)^b_{AB...}$$

(5)

where $(\tilde{H}^{-1} P G)^a_{AB...} = \sum_{ap} \int_{A'p} \tilde{H}^{-1}_{pp'} P^p_{A'p} G_{A'p}$.

In practice, the integrals over $x_{A'}$ and $A'$ are trivial since $P^p_{A'}$ and $G_{A'p}$ are both delta-functions of their spatial coordinates and $G_{A'p}$ is also a delta-function of the momenta (and other particle quantum numbers), i.e. $G_{A'p} = f^a_{A'} \delta_{A'p} \sim \delta^3(x_{A'} - x_A) \delta^3(p_{A'} - p_A)$, where $f^a_{A'} = f_A^a (1 + \theta_A f_A)$.

In what follows we do not write explicitly, but imply, the summation/integration corresponding to repeated indices labeling either hydrodynamic variables (and cells) or hadron gas variables (and phase-space cells).

**Entropy of fluctuations** — Recall that the exponential of the entropy $S$ in Eq. (2) is proportional to the number of microstates of the system with given values
of occupation numbers $f_A$ of the hydrodynamic cells. In the thermodynamic (large volume) limit there is a large number of single-particle quantum states in each hydrodynamic cell and the number of possible ways to occupy these elementary quantum states is exponentially large, of order $e^S$. A macroscopic state is an ensemble of this exponentially large number of microscopic states with occupation numbers close to mean $f_A$. The values $f_A$ are not the same in all microscopic states, but fluctuate. The magnitude of fluctuations is suppressed in the thermodynamic limit. The probability distribution of these fluctuations is given by the exponential of $S$ in Eq. (2) with additional constrains given by Eq. (1). As usual, in thermodynamic limit, one can implement these constrains using Lagrange multipliers, i.e., using the probability distribution $\exp(S + J_a \Psi_a)$ and choosing $J_a$ to satisfy the constrains on $\Psi_a$. Using this probability distribution one can calculate the expectation values $\bar{G}_{AB\ldots}$ of the fluctuation correlators in equilibrium which will depend on $\Psi^a$. We can also consider states with (some of) the correlators $G_{AB\ldots}$ having specific values, not necessarily equal to $\bar{G}_{AB\ldots}$. These states must have lower entropy since more information is available about these states. To find their entropy we can consider the probability distribution perturbed by additional factor $\exp(K_{AB\ldots}f_{AB\ldots})$, where $K_{AB\ldots}$ play the role similar to Lagrange multipliers. Integrating over fluctuations of $f_A$ we can then obtain $G_{AB\ldots}$ which will depend on $K_{AB\ldots}$. Solving for $K$ and substituting back into the probability distribution we find the probability distribution for $f_A$ with given correlators $G_{AB\ldots}$. The Gibbs entropy of this probability distribution is the key quantity, which we would then maximize subject to constraints on $G_{AB\ldots}$ from Eq. (4).

The calculation of the entropy of fluctuations along these lines for a two-point correlator can be found in Ref. [11], where it is also pointed out that the result mathematically resembles the 2-PI action in quantum field theory [31–35]:

$$S_2 = S + \frac{1}{2} \text{Tr} \left[ \log(-CG) + CG + 1 \right],$$

where $C_{AB} = \delta^2 S / (\delta f_A \delta f_B)$ and $G = G_2$. The difference $S_2 - S$ vanishes when $G$ equals $-C^{-1} = \bar{G}$ and can be viewed as the additional (negative) entropy of the state with additional constraints on correlators relative to the entropy of the state with correlations given simply by $\bar{G}$.

We want now to determine the correlator $G_{AB}$ satisfying the constraints in Eq. (4). The most likely value of $G$ is given by the maximum of the entropy $S_2$ subject to these constraints. Introducing Lagrange multiplier matrix $\Lambda_{ab}$ we find, solving the constrained variational problem, that

$$G_{AB}^{-1} = \bar{G}_{AB}^{-1} + \Lambda_{ab} P_A^a P_B^b. \tag{7}$$

We can then determine the Lagrange multipliers by substituting (7) into (4) and we find

$$\Lambda = H^{-1} - \bar{H}^{-1}. \tag{8}$$

Substituting into Eq. (7) we obtain

$$G_{AB}^{-1} = \bar{G}_{AB}^{-1} + (H^{-1} - \bar{H}^{-1})_{ab} P_A^a P_B^b. \tag{9}$$

Non-Gaussian fluctuations — Extending this calculation to higher-order correlators, specifically to $n = 3, 4$ relevant for non-Gaussian fluctuations in experiments [1, 2, 36, 37], we obtain the entropy $S_4[f, G, G_3, G_4]$ as

$$S_4 = S_3 + \frac{1}{24} C_{ABCD} G_{ABCD} - \frac{1}{48} G^{-1}_{AC} G^{-1}_{BD} G^{-1}_{EF} G^{-1}_{HI} G_{ABEH} C_{CDFI}$$

$$+ \frac{1}{8} G_{ABC} G_{DEF} G_{HIJK} G_{AB}^{-1} G_{CD}^{-1} G_{EF}^{-1} G_{HI}^{-1} G_{LM}^{-1} G_{AC} G_{EF} G_{HI} G_{IKL} G_{BDM}$$

$$- \frac{1}{12} G_{AB}^{-1} G_{CD}^{-1} G_{EF}^{-1} G_{AC} G_{EF} G_{HI} G_{IKL} G_{BDM} \tag{11}$$

where $:X: \equiv X - \bar{X}$. The result is mathematically similar to the $n$-PI action in QFT, [38] as was the case for $n = 2$, with $S_n$ corresponding to truncation at $n - 1$ loops. (As discussed in Ref. [22] the loop expansion corresponds to expansion in the magnitude of fluctuations.) We can now maximize the entropy $S_4$ with respect to $G_3$ and $G_4$ subject to constraints from hydrodynamic correlators $H_3$ and $H_4$ in Eq. (4) and find

$$G_{ABC} = \left[ C_{QRS} + \left\{ H_{abc}^k - (PG)^a_k (PG)^b_k (PG)^c_k \right\} C_{TUV} \right] G_{QA} G_{RB} G_{SC}, \tag{12}$$

where $H_{abc} = \left\{ H_{abc}^k - (PG)^a_k (PG)^b_k (PG)^c_k \right\} C_{TUV}$.
\[ G_{ABCD} = \left( C_{QRST} + \left[ 3G_{YX}C_{YQR}C_{XST} \right]_{QRST} \right) \\
+ \left( H^{abcd} - P^{a}_{i} P^{b}_{j} P^{c}_{k} P^{d}_{l} \right) \left( G_{1M}G_{JN}G_{KOG_{LP}C_{MNOP}} \right) \\
+ 3G_{XY} \left( G_{1}^{IJ}G_{XKL} \right)_{IJKL} \right) \\
\times (H^{-1}P)_{a}^{n}(H^{-1}P)_{b}^{m}(H^{-1}P)_{c}^{n}(H^{-1}P)_{T}^{n} \\
\times G_{QAG_{RD}G}_{SCG_{TD}}, \quad (13) \]

where \( C_{AB\ldots} = \delta^{n}S/(\delta f_{A}\delta f_{B}\ldots) \) and we used the notation \([\ldots]_{ABC\ldots} \) for average over the permutations of indices.

Equations (9), (12) and (13) can be solved for correlators \( G_{n} \) iteratively. However, the structure of these equations is somewhat easier to appreciate in the linearized limit, applicable when the correlations relative to hadron gas, i.e., \( G_{n} - \bar{G}_{n} \equiv \Delta G_{n} \), are sufficiently small (a reasonable approximation for heavy-ion collisions). In this limit the solution can be expressed compactly by Eq. (9), or, with summation/integration implied, as

\[ \Delta AB \ldots = \bar{\Delta} H_{a b \ldots}(H^{-1}\bar{P}G)_{A}^{a}(H^{-1}\bar{P}G)_{B}^{b} \ldots, \quad (14) \]

in terms of the correlators \( \bar{\Delta}G_{n} \) and \( \bar{\Delta}H_{n} \), which could be termed irreducible relative (connected) correlators (IRC). These correlators quantify “genuine” (i.e., not reducible to lower-order correlations) \( n \)-point correlations in \( G_{n} \) relative to the ideal hadron gas \( \bar{G}_{n} \). This is achieved by recursively subtracting these lower-order correlations:

\[ \Delta G_{AB} \equiv \Delta G_{AB}; \]
\[ \Delta G_{ABC} \equiv \left[ \Delta G_{ABC} - 3\Delta G_{D}(\bar{G}^{-1}\bar{G}_{3})_{DBC} \right]_{ABC}; \]
\[ \Delta G_{ABCD} \equiv \left[ \Delta G_{ABCD} - 6\Delta G_{ABF}(\bar{G}^{-1}\bar{G}_{3})_{FCD} \right] - 4\Delta G_{ABF}(\bar{G}^{-1}\bar{G}_{4})_{FBCD}; \]
\[ - 3\Delta G_{EF}(\bar{G}^{-1}\bar{G}_{3})_{EAB}(\bar{G}^{-1}\bar{G}_{3})_{FCD}]_{ABCD}. \quad (15) \]

Similar relations define IRCs \( \bar{\Delta}H \) of hydrodynamic variables, with \( H \) instead of \( G \) and indices \( ab\ldots \) instead of \( AB\ldots \).

Note that the factors \( (\bar{G}^{-1}\bar{G}_{n})_{AB\ldots} \equiv \Delta_{AX} \bar{G}_{XBC\ldots} \) in Eq. (15), in the case of negligible quantum statistics effects (or \( \theta_{A} = 0 \) in Eq. (3)), are equal to \( \delta_{AB\ldots} \). Thus, in this case, the IRCs \( \bar{\Delta}G_{n} \) coincide with correlators \( C_{ab\ldots} \) described in Ref. [29], whose phase space integrals give factorial cumulants. Such correlators and factorial cumulants play important role in the acceptance dependence of the fluctuation measures [29, 30].

Comparison with existing methods — We can now compare the results of the maximum entropy approach with other freezeout procedures used in the literature to implement freezeout of fluctuations.

Ref. [9] considered fluctuations of \( f_{A} \) caused by fluctuations of hydrodynamic parameters such as temperature and chemical potential, \( J_{a} \) in our notations, i.e., \( \delta f_{A} = (\delta f_{A}/\delta J_{a})\delta J_{a} = (\bar{P}G)_{a}^{a}\delta J_{a} \), where, as before, \( \bar{G}_{AB} = f_{A}^{a}\delta_{AB} \). Using hydrodynamic correlators \( \langle \delta J_{a} \delta J_{b} \rangle = H^{-1}_{ab} \) one then finds:

\[ \Delta G_{AB} = H^{-1}_{a b}(\bar{P}G)_{A}^{a}(\bar{P}G)_{B}^{b}, \quad (16) \]
as opposed to our Eq. (9). We see that the problem with Eq. (16) is in the absence of the separate contribution of the ideal gas fluctuations, \( \bar{G}_{AB} = f_{A}^{a}\delta_{AB} \), which matches \( \bar{H} \) in hydrodynamics, but does not describe correlations between two different particles [10, 25]. While the approach of Ref. [9] could satisfy the constraints (4), it does so, in part, via spurious two-particle correlations. This problem was addressed in Ref. [10, 25] for charge fluctuations, where the ideal gas (Poisson) contribution to \( \bar{H} \) was subtracted before applying “freezeout (thermal) smearing” to the remainder, \( \bar{H} - \bar{H} \) in our notations. Thus, maximum entropy approach reproduces, in Eq. (14), the procedure in Ref. [10, 25] for two-point correlators. The subtractions of lower order terms in Eqs. (15) generalize this procedure to higher-order correlators.

Fluctuations near the QCD critical point in equilibrium have been described by considering a fluctuating critical mode \( \sigma \) coupled to the observed particles via their \( \sigma \)-dependent masses Refs. [3–7, 39]. This approach was further generalized in Ref. [24] to non-equilibrium critical fluctuations by mapping the correlators of \( \sigma \) to correlators of the specific entropy \( m = s/n - \) the critical field in Hydro+ [11]. We can now compare this approach to the result of the maximum entropy method by considering only the matrix element \( H_{mm} \) of hydrodynamic correlator \( H \) corresponding to the fluctuations of the specific entropy \( m \).

Furthermore, since this approach only considers the leading (most singular) critical contribution, for our comparison, we can neglect lower-order correlations, which contribute subleading behavior in terms of the dependence on the correlation length near the critical point [5]. In practice this means \( \bar{\Delta}G_{n} = \Delta G_{n} \) up to subleading (less critical) terms.

Translating the freezeout prescription of Ref. [24] into our notations we find:

\[ \Delta G_{AB} = \frac{g_{A B} m_{A} m_{B}}{Z T^{2}} \frac{E_{A}}{E_{B}} \Delta H_{mm} f_{A}^{a} f_{B}^{b}, \quad (17) \]

where \( g_{A B} \) and \( Z \) are parameters describing the coupling of \( \sigma \) to particles \( A, B \) (see Ref. [24]). The maximum entropy freezeout gives:

\[ \Delta G_{AB} = \Delta H_{mm}(\bar{H}^{-1}\bar{P}G)_{m A}(\bar{H}^{-1}\bar{P}G)_{m B}. \quad (18) \]

Comparing Eqs. (17) and (18) we find that they could be reconciled if \( g_{A} \) had particle energy dependence given
by the factor \( P_A^n = (E_A - wq_A/n)/(nT)\delta^3(x_m - x_A) \) – the contribution of particle A to the fluctuation of \( m = s/n \). The absence of the energy dependence of \( g_A \) in Eq. (17) is a consequence of the simplifying assumption that the field \( \sigma \) couples to mass term. Maximum entropy method allows us to relax this assumption and determine the “coupling” \( g_A \) together with its energy dependence from the equation of state (EOS) of QCD:

\[
g_A = \sqrt{\frac{E_A}{m_A}} \left( \frac{w_c}{w_c - q_A} \right) \left( \frac{E_A - q_A}{w_c} \right),
\]

where \( (H^{-1})_{mm} \) is the hadron gas contribution to the fluctuations of specific entropy \( m \), which can be also found from the non-singular contribution to the EOS [40] as \( (H^{-1})_{mm} = n^2/c_p \).

Since the QCD EOS is not known (yet), we shall demonstrate how to estimate \( g_A \) using the parametric EOS introduced in Ref. [40]. First, following Ref. [24], we find \( Z \) by matching the leading singularity in the QCD EOS to that in the Ising model:

\[
Z = \lim_{T,\mu \to T_c,\mu_c} \frac{e_p T}{n^2(T) \eta^2(T_0)^{-\eta}} = \frac{M_0 T_c^4}{h_0 n_c^2(T_c \xi_0)^{2-\eta}} \times \left( \cot \alpha_1 - \frac{s_c}{n_c} \right)^2 \left[ \frac{\sin \alpha_1}{w \sin(\alpha_1 - \alpha_2)} \right]^2,
\]

where \( w, \alpha_1, \) and \( \xi_0 \) are parameters, defined in Refs. [40, 41], which control the orientation and strength of the critical point singularity located at \( T = T_c \) and \( \mu = \mu_c \), with enthalpy given by \( w_c = n_c \mu_c + s_c T_c \). The same expression as in the square brackets determines the width of the critical region [42]. The values of \( M_0 \) and \( h_0 \) are fixed in Ref. [40].

Defining \( \tilde{g}_A \) so that \( g_A = \tilde{g}_A \sin(\alpha_1/[w \sin(\alpha_1 - \alpha_2)]) \), we can use parameters in Refs. [40, 41] (\( \mu_c = 350 \text{ MeV}, T_c = 143.2 \text{ MeV}, \xi_0 = 1 \text{ fm} \)) to estimate the values of the couplings at zero momentum (\( p_A = 0 \)): \( \tilde{g}_{\pi,0} \approx -3.1, \tilde{g}_{p,0} \approx 0.18, \tilde{g}_{\eta,0} \approx 5.5 \).

The approach in Refs. [3–7, 24, 39] leading to Eq. (17) does not leave only the magnitude, but also the sign of \( g_A \) undetermined. While the overall sign can be changed by redefining the critical field \( \sigma \), the relative sign of \( g_A \) for different particles, or different momenta of the same particle, i.e., different \( A \), is not arbitrary and can be found in the maximum entropy approach using Eq. (19).

Thus, we find that the critical mode coupling to (low momentum) protons is opposite in sign from the coupling to either pions or antiprotons. This can be traced back to the fact that fluctuations of the number of protons contribute to the fluctuations of the ratio \( s/n \) with opposite sign from that of pions or antiprotons, since pions contribute to the numerator, while protons (mostly) to the denominator of the ratio.

Experimental implications of the changing sign of \( g_A \sim P_A^n \) could be studied by considering cross-species correlators discussed in Ref. [6] or correlations between particles with different momenta, i.e., \( A \neq B \). In both cases one would expect anticorrelation when the product \( g_A g_B \sim P_A^n P_B^n \) is negative.

**Conclusions** — Maximum entropy principle is widely used in many applications in statistics, information theory, economics, biology and bioinformatics, data science, computation, pattern recognition, etc. Of course, thermodynamics itself is based on that very principle. The thermodynamic state is, by definition, the state of maximum entropy, i.e., the most likely ensemble of microscopic states, given the known (i.e., measured) properties of the system, such as total energy. The application to freezeout could be viewed as answering the question of what is the most likely ensemble of free-streaming particles after freezeout given the information about the hydrodynamic conditions before the freezeout.

The key idea is that this information could include not only the values of mean quantities but also of the hydrodynamic fluctuations (i.e., correlators \( H_{\sigma} \)) out of equilibrium. These can be obtained, for example, from a Hydro+ calculation [11, 20, 21, 24], or by solving full hydrodynamic fluctuation equations [18, 19, 22]. Maximum entropy freezeout then determines the most likely ensemble of free-streaming final particles which matches all this available information (equation of state and the predictions of hydrodynamics with fluctuations).

Remarkably, the results are consistent with the picture, already considered in the literature, of hadron gas coupled to fluctuating fields inducing correlations. This not only corroborates the picture, but provides a non-trivial insight into the entropic origin of the correlations. Crucial for practical applications, the maximum entropy approach provides information about the couplings determining the magnitude of the correlations as well as the generalization to non-Gaussian fluctuations in or out of equilibrium.

Obviously, it would be very interesting to implement this novel approach in heavy-ion collision simulations to explore the potential implications and to compare the results with experimental data. In particular, the data from the Beam Energy Scan at RHIC, whose results are being analyzed by the STAR collaboration at this time. Such applications are beyond the scope of this Letter and we defer these investigations to future work.

We thank K. Rajagopal and Y. Yin for helpful comments. This work is supported by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics within the framework of the BEST Topical Collaboration and grant No. DE-FG0201ER41195.
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