Gauge Theory on a Four Sphere

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Abstract

In this paper we will analyse the quantization of a gauge theory on a four sphere. This will be done by mode expanding all the fields in the theory in terms of harmonic modes. We will also analyse the BRST symmetry of this theory.

1 Introduction

Scalar field theory is quantized by imposing a canonical quantization relations or by summing over all field configurations. If we try to impose the canonical quantization relations on a theory with gauge degrees of freedom, we will get constraints; and if we try to sum over the field configurations, we will get divergence in the partition function. One way to deal with this problem is the Wheeler-DeWitt approach [1]-[9]. Another way this problem can be dealt with is by fixing a gauge. This is achieved at a quantum level by adding a gauge fixing term and a ghost term. The resultant theory thus obtained has negative norm states called the ghost states. These states can be removed by using a symmetry called the BRST symmetry [10]-[14]. The BRST symmetry been studied for various theories with a gauge symmetry [15]-[38]. In the BRST formalism the effective Lagrangian is given by a sum of the original classical Lagrangian, the gauge fixing term and the ghost term. The charge corresponding to the invariance of the effective Lagrangian under BRST symmetry is nilpotent. It is thus used to factor out the physical states. In this paper we will analyse a gauge theory on a four sphere and apply the BRST formalism to it.

2 Associated Legendre Function

In this section we will review some properties of associated Legendre function. These properties will be used to construct various spherical harmonics. In general an associated Legendre function \( P^{-\mu}_\nu(x) \) is given by

\[
P^{-\mu}_\nu(x) = \frac{1}{\Gamma(1+\mu)} \left( \frac{1-x}{1+x} \right)^{\mu/2} F(-\nu, \nu+1, \mu+1, \frac{1-x}{2}).
\]  

(1)

Now \( \Gamma(1+\mu) \) is the Gamma function and \( F(a, b, c, x) \) is the hypergeometric function. The hypergeometric function \( F(a, b, c, x) \) in general is given
by
\[ F(a, b, c, x) = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{2b(c+1)}x^2 + \cdots. \quad (2) \]

We will need the lowering and raising operators for \( \nu \) to find relations between various spherical harmonics.

The lowering operator for \( \nu \) is given by
\[ \left( 1 - x^2 \right) \frac{d}{dx} + \nu x \right) P_{\nu}^{-\mu}(x) = (\nu - \mu) P_{\nu-1}^{-\mu}(x). \quad (3) \]

The raising operator for \( \nu \) is given by
\[ \left( 1 - x^2 \right) \frac{d}{dx} - (\nu + 1)x \right) P_{\nu}^{-\mu}(x) = -(\nu + \mu + 1) P_{\nu+1}^{-\mu}(x). \quad (4) \]

It will be useful to define \( D_m \) as follows:
\[ D_m = \frac{d}{dx} + m \cot \chi. \quad (5) \]

Then as
\[ \left[ \frac{d}{dx} + m \cot \chi \right] (\sin \chi)^n f(\chi) = \\
\sin^n \chi \left[ \frac{d}{dx} + (m + n) \cot \chi \right] f(\chi), \quad (6) \]
we can write
\[ D_m \sin^n \chi f(\chi) = \sin^n \chi D_{m+n} f(\chi). \quad (7) \]

We also have
\[ - \sin \chi D_n = \left( 1 - \cos^2 \chi \right) \frac{d}{d \cos \chi} - n \cos \chi \right). \quad (8) \]

## 3 Spherical Harmonics

In this section we will review spherical harmonics on \( S^4 \). We will first see how we can construct scalar spherical harmonics on \( S^n \), if we are given the scalar spherical harmonics on \( S^{n-1} \). Then we will explicitly construct the scalar spherical harmonics on \( S^4 \). The metric on \( S^n \) can be written as follows:
\[ ds^2 = d\chi^2 + \sin^2 \chi ds_{n-1}. \quad (9) \]

Here \( ds_{n-1} \) is the metric on \( S^{n-1} \). We can define a function \( _nP_L^\ell \) as
\[ _nP_L^\ell (\chi) = c_n (\sin \chi)^m P_{L-m}^{-l+m} (\cos \chi), \quad (10) \]
where \( m = (2 - n)/2 \) and \( c_n \) is a normalization constant given by
\[ c_n = \left[ \frac{(2L + n - 1)(L + l + n - 2)!}{2(L + l)!} \right]^{1/2}. \quad (11) \]
Now we define scalar spherical harmonics in $n$ dimensions as

\[ Y_{LLp...m}(\chi, \theta, \phi) = n P_L^l(\chi) Y_{lp...m}(\theta, \phi) \]  

(12)

Here $Y_{lp...m}$ are the scalar spherical harmonics in $n$ dimensions and $Y_{pq...m}$ are the scalar spherical harmonics in $n - 1$ dimensions. Now the action of $\nabla^2$ on scalar spherical harmonic is

\[ - \nabla^2 Y_{LPq...m} = L(L + 1) Y_{LPq...m}. \]  

(13)

In this way we can construct the spherical harmonics from spherical harmonics on $S^1$. Now we define the spherical harmonics on $S^1$ as

\[ Y_m = \frac{1}{\sqrt{2\pi}} \exp(im\phi), \]  

(14)

so we can construct spherical harmonics on $S^4$ as

\[ Y_{Llpm} = \frac{1}{\sqrt{2\pi}} c_4 c_2 c_3 \cdot \cdot \cdot \cdot \]  

(15)

So for the scalar spherical harmonics on $S^4$, we have

\[ - \nabla^2 Y_{Llpm} = [L(L + 3) - 1] Y_{Llpm}. \]  

(16)

If $g$ is the metric on $S^4$, then $Y_{Llpm}$ are normalized as follows:

\[ \int d^4x \sqrt{g} Y_{Llpm} Y_{Llpm}^* = \delta_{LL} \delta_{ll} \delta_{pp} \delta_{mm}. \]  

(17)

There are two kinds of vector spherical harmonics on $S^3$. Let them be denoted by $A_{Llp...0;0}$ and $A_{Llp...0;1}$. Then the action of $\nabla^2$ operator on them is given by

\[ - \nabla^2 A_{Llp...0;0} = [L(L + n - 1) - 1] A_{Llp...0;0}, \]  

\[ - \nabla^2 A_{Llp...0;1} = [L(L + n - 1) - 1] A_{Llp...0;1}. \]  

(18)

The divergence of vector spherical harmonics vanishes, so we have

\[ \nabla^a A_{Llp...0;0} = 0, \]  

\[ \nabla^a A_{Llp...0;1} = 0. \]  

(19)

It can be shown that the vector spherical harmonics on $S^4$ are given by

\[ A_{Llp...0;1} = 0, \]  

(20)

\[ A_{Llp...0;1} = n_1 P_{\ell+1} A^p_{lpm}, \]  

(21)

\[ A_{Llp...0;0} = n_2 (\sin \chi)^{-2} P_{\ell+1} Y^lpm, \]  

(22)

\[ A_{Llp...0;0} = n_2 \cdot \frac{1}{\ell(\ell - 2)} D_1 P_{\ell+1} \nabla Y^lpm. \]  

(23)

where $Y^lpm$ and $A^lpm$ are scalar and vector spherical harmonics on $S^4$, respectively. The are normalization constants $n_1$ and $n_2$ are chosen, such that

\[ \int d^4x \sqrt{g} a^i_{Llp...0} A^a_{Llp...0} A^a_{Llp...0} = 1, \]  

(24)
\[
\int d^4x \sqrt{g} g^{ab} A_a^{L_{lpm};1} A_b^{L_{lpm};1} = 1.
\]

(25)

Now for the vector spherical harmonics on \( S^4 \), we have
\[
\begin{align*}
- \nabla^2 A_a^{L_{lpm};0} &= [L(L + 3) - 1] A_a^{L_{lpm};0}, \\
- \nabla^2 A_a^{L_{lpm};1} &= [L(L + 3) - 1] A_a^{L_{lpm};1},
\end{align*}
\]

(26)

and
\[
\begin{align*}
\nabla^a A_a^{L_{lpm};0} &= 0, \\
\nabla^a A_a^{L_{lpm};1} &= 0.
\end{align*}
\]

(27)

4 Faddeev-Popov for Yang-Mills Theories

Now we mode expand a field for the Yang-Mills theory on \( S^4 \) as
\[
T^A A_a^A = \sum a^{L_{lpm},n} A_a^{L_{lmp};n},
\]

(28)

If we take the gauge fixing condition as the Lorentz gauge
\[
F[A] = \nabla_a A^a = 0,
\]

(29)

then the gauge fixing condition for each of the modes is
\[
F[A] = \nabla^a A_a^{L_{lmp};n} = 0,
\]

(30)

where \( n = 0, 1 \). Now term \( \mathcal{L}_g \) is given by
\[
\mathcal{L}_g = -\sqrt{g} \left[ \frac{1}{2\alpha} \nabla_a A^a \nabla_b A^{ab} \right],
\]

(31)

and the ghost field term is given by
\[
\mathcal{L}_{gh} = -i \sqrt{g} [\nabla^a \Gamma^A D_a c^A].
\]

(32)

Here \( D_a c^A \) is the covariant derivative and is given by
\[
D_a c^A = \nabla_a c^A + f^A_{BC} A^B a c^C.
\]

(33)

Here \( c^A \) and \( \Gamma^A \) are the ghosts and anti-ghosts respectively. Now we can get the total Lagrangian as follows:
\[
\mathcal{L}_t = \mathcal{L} + \mathcal{L}_g + \mathcal{L}_{gh}.
\]

(34)

We can write the free part of the total action containing terms quadratic in \( A^A a \) as follows:
\[
S_{tf} = \int d^4x d^4x' \frac{\sqrt{g(x)} \sqrt{g(x')}}{2} A^A a(x) D(x, x')_{ab} A^A a(x') A^{A'}(x'),
\]

(35)

where
\[
D(x, x')_{ab} = \left( g_{ab} \nabla^2 + \frac{1}{\alpha} \nabla_b \nabla_a \right) \delta(x, x').
\]

(36)
\( \mathcal{D}(x, x')_{ab'} \) does not have any zero eigenvalue and so \( \mathcal{D}(x, x')_{ab'} \) can be inverted. Its inverse is given by Green’s function \( G(x, x')_{ab'} \),

\[
G(x, x')_{ab'} = \mathcal{D}^{-1}(x, x')_{ab'}.
\]  
(37)

Thus we have

\[
G(x, x')_{ab'} = \sum N^{(L, \alpha)} A^L_{\alpha \mu \nu}(x) A^L_{\beta \nu \mu}(x').
\]  
(38)

It may be noted that it is possible to choose a very special gauge, called axial gauge for Yang-Mills theories. In this gauge the ghosts decouple from the gauge fields.

## 5 BRST Symmetry

We will now discuss the BRST for gauge theory on \( S^4 \). The sum of the gauge fixing Lagrangian and the ghost Lagrangian can be written as

\[
\mathcal{L}_g + \mathcal{L}_{gh} = \sqrt{g}s[-c\nabla_a A^a + \frac{\alpha}{2} c b] = 0.
\]  
(39)

where \( b \) does not contain any derivatives and the functional integral over \( b \) can be done by completing the square. This way we will recover the original gauge fixing term. The BRST transformations are give by

\[
sA^A_a = D_a c^A,
\]
\[
sB^A = 0,
\]
\[
s\pi^A = B^A,
\]
\[
s c^A = -\frac{i}{2} f^{ABC} c^B c^C.
\]  
(40)

Here the BRST transformation of \( A^A_a \) is obtained by replacing the infinitesimal parameter \( \Lambda^A \) in the gauge transformations by the ghost field \( c^A \) and the BRST transformation of \( c^A \) is obtained by taking the commutator of two gauge transformations and then replacing all the infinitesimal parameters by the \( c^A \). The BRST transformation of \( B^A \) vanishes and BRST transformation of \( \pi^A \) is \( B^A \). These BRST transformations are nilpotent

\[
s^2 A^A_a = 0,
\]
\[
s^2 c^A = 0,
\]
\[
s^2 \pi^A = 0,
\]
\[
s^2 B^A = 0.
\]  
(41)

Due to the nilpotency of these BRST transformations the sum of the gauge fixing and ghost terms is invariant under these BRST transformations,

\[
s\mathcal{L}_g + s\mathcal{L}_{gh} = \sqrt{g}s^2[-c\nabla_a A^a + \frac{\alpha}{2} c b] = 0.
\]  
(42)

Now the original Lagrangian is invariant by itself as the action of \( s \) on it just generates a gauge transformation with ghost fields acting as the gauge parameters

\[
s\mathcal{L} = 0.
\]  
(43)

Thus, the total Lagrangian \( \mathcal{L}_t \) is invariant under the BRST transformations.

\[
s\mathcal{L}_t = 0.
\]  
(44)
6 Conclusion

In this paper we have analysed the quantization of gauge theory on $S^4$. We have also analyzed the BRST symmetry of this model. It is known that gauge theories with BRST symmetry also have another symmetry called the anti-BRST symmetry. In anti-BRST symmetry the role of the ghosts and anti-ghosts is interchanged. It will be interesting to analyse the anti-BRST symmetry of this model.

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