FILIFORM $\mathbb{Z}_2 \times \mathbb{Z}_2$-COLOR LIE SUPERALGEBRAS

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Abstract. We continue the study of the filiform $\mathbb{Z}_2 \times \mathbb{Z}_2$-color Lie superalgebras. All of them can be obtained by using infinitesimal deformations, i.e. cocycles. In this work we give the total dimension of such cocycles (for any dimensions $n$, $m$, $p$ and $t$ of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-color Lie superalgebras). Also, we give a basis of such cocycles in some generic and concrete cases.

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1. Introduction

In the understanding of the properties of physical systems is central the concept of symmetry and its associated algebraic structures. In particular, that a better comprehension of the laws of physics may be achieved by an identification of the possible mathematical structures as well as their classification. Thus, for instance, the properties of elementary particles and their interactions are very well understood within Lie algebras. Furthermore, the discovery of supersymmetry gave rise to the concept of Lie superalgebras which becomes central in theoretical physics and mathematics. Color Lie superalgebras can be considered as one of the possible generalizations of Lie superalgebras with physical applications ([3], [15], [16] and [12]).

In this work, we shall consider color Lie superalgebras with a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading vector space due to the great amount of physical applications of this vector space, see e.g. [4], [5], [6] and [7]. In particular, we will focus our study in a very important type of nilpotent $\mathbb{Z}_2 \times \mathbb{Z}_2$-color Lie superalgebra, i.e. filiform $\mathbb{Z}_2 \times \mathbb{Z}_2$-color Lie superalgebras.

Filiform Lie algebras was firstly introduced in [17] by Vergne. This type of nilpotent Lie algebra has important properties; in particular, every filiform Lie algebra can be obtained by a deformation of the model filiform algebra $L_n$. In the same way as filiform Lie algebras, all filiform Lie superalgebras can be obtained by infinitesimal deformations of the model Lie superalgebra $L_{n,m}$ [1], [8], [10] and [11]. In [12] we generalized this concept obtaining filiform $(G, \beta)$-color Lie superalgebras and the model filiform $(G, \beta)$-color Lie superalgebra as well as the existence of “adapted” basis for these color Lie superalgebras. We too proved that in order to obtain all the class of filiform $(G, \beta)$-color Lie superalgebras it is only necessary to determine some infinitesimal deformations of the model filiform $(G, \beta)$-color Lie superalgebra.

In [13] we have studied these infinitesimal deformations, i.e. cocycles, for the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ obtaining in particular a decomposition into 10 subspaces...
that depend on the election of the commutation factor $\beta$. In the present work we continue the study of the filiform $\mathbb{Z}_2 \times \mathbb{Z}_2$-color Lie superalgebras. In particular, we give the total dimension of such cocycles (for any dimensions $n$, $m$, $p$ and $t$ of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-color Lie superalgebras). Also, we give a basis of such cocycles in some generic and concrete cases.

2. Preliminaries

It is considered the abelian group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, i.e. $G = \{e, a, b, c\}$ with identity element $e$ and $a + a = b + b = c + c = e$, $a + b = c$, $a + c = b$, $b + c = a$.

The vector space $V$ is said to be $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded if it admits a decomposition in direct sum, $V = V_a \oplus V_b \oplus V_c$. An element $X$ of $V$ is called homogeneous of degree $\gamma$ ($deg(X) = d(X) = \gamma$), $\gamma \in \mathbb{Z}_2 \times \mathbb{Z}_2$, if it is an element of $V_\gamma$. Let $V = V_a \oplus V_b \oplus V_c$ and $W = W_a \oplus W_b \oplus W_c$ be two $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded vector spaces. A linear mapping $f : V \rightarrow W$ is said to be homogeneous of degree $\gamma$ ($deg(f) = d(f) = \gamma$), $\gamma \in \mathbb{Z}_2 \times \mathbb{Z}_2$, if $f(V_\alpha) \subset W_{\alpha + \gamma}$ for all $\alpha \in \mathbb{Z}_2 \times \mathbb{Z}_2$. The mapping $f$ is called a homomorphism of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded vector space $V$ into the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded vector space $W$ if $f$ is homogeneous of degree $e$. Now it is evident how we define an isomorphism or an automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded vector spaces.

A superalgebra $\mathfrak{g}$ is, in particular, a $\mathbb{Z}_2$-graded algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then if we denote by $[\ , \ ]$ the bracket product of $\mathfrak{g}$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha + \beta(\text{mod} 2)}$ for all $\alpha, \beta \in \mathbb{Z}_2$. Thus, we call $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ a Lie superalgebra if the bracket product satisfies the following identities:

1. $[X, Y] = -(1)^{\alpha \beta} [Y, X] \quad \forall X \in \mathfrak{g}_\alpha, \forall Y \in \mathfrak{g}_\beta$.
2. $(-1)^\alpha [X, [Y, Z]] + (-1)^\beta [Y, [Z, X]] + (-1)^\gamma [Z, [X, Y]] = 0$

for all $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta, Z \in \mathfrak{g}_\gamma$ with $\alpha, \beta, \gamma \in \mathbb{Z}_2$.

In particular, if $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra then we have that $\mathfrak{g}_0$ is a Lie algebra and $\mathfrak{g}_1$ has structure of $\mathfrak{g}_0$-module.

Definition 2.1. Let $G$ be an abelian group. A commutation factor $\beta$ is a map $\beta : G \times G \rightarrow \mathbb{F} \setminus \{0\}$, ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$), satisfying the following constraints:

1. $\beta(g, h)\beta(h, g) = 1$ for all $g, h \in G$
2. $\beta(g, h + k) = \beta(g, h)\beta(g, k)$ for all $g, h, k \in G$
3. $\beta(g + h, k) = \beta(g, k)\beta(h, k)$ for all $g, h, k \in G$

The definition above implies, in particular, the following relations:

$\beta(e, g) = \beta(g, e) = 1, \quad \beta(g, h) = \beta(-h, g), \quad \beta(g, g) = \pm 1 \quad \forall g, h \in G$

where $e$ denotes the identity element of $G$. In particular, fixing $g$ one element of $G$, the induced mapping $\beta_g : G \rightarrow \mathbb{F} \setminus \{0\}$ defines a homomorphism of groups.

Definition 2.2. Let $G$ be an abelian group and $\beta$ a commutation factor. The (complex or real) $G$-graded algebra

$L = \bigoplus_{g \in G} L_g$

with bracket product $[\ , \ ]$, is called a $(G, \beta)$-color Lie superalgebra if for any $X \in L_g, Y \in L_h,$ and $Z \in L$ we have

1. $[X, Y] = -\beta(g, h)[Y, X]$ (anticommutative identity)
Corollary 2.2.1. Let \( L = \bigoplus_{g \in G} Lg \) be a \((G, \beta)\)-color Lie superalgebra. Then we have

1. \( L_e \) is a (complex or real) Lie algebra where \( e \) denotes the identity element of \( G \).
2. For all \( g \in G \setminus \{e\} \), \( L_g \) is a representation of \( L_e \). If \( X \in L_g \) and \( Y \in L_h \), then \([X,Y] \) denotes the action of \( X \) on \( Y \).

Examples. (1) For the particular case \( G = \{e\} \), \( L = L_e \) reduces to a Lie algebra.

(2) If \( G = \mathbb{Z}_2 = \{0, 1\} \) and \( \beta(1, 1) = -1 \) we have ordinary Lie superalgebras, i.e. a Lie superalgebra is a \((\mathbb{Z}_2, \beta)\)-color Lie superalgebra where \( \beta(i,j) = (-1)^{ij} \) for all \( i, j \in \mathbb{Z}_2 \).

(3) If \( A = \bigoplus_{g \in G} A_g \) is a \( G \)-graded associative algebra, then setting

\[
[X,Y] = XY - \beta(g,h)YX
\]

for \( X \in A_g, Y \in A_h \), we make \( A \) into \((G, \beta)\)-color Lie superalgebra \([A]_\beta\).

Definition 2.3. A representation of a \((G, \beta)\)-color Lie superalgebra is a mapping \( \rho : L \rightarrow \text{End}(V) \), where \( V = \bigoplus_{g \in G} V_g \) is a graded vector space such that

\[
[\rho(X), \rho(Y)] = \rho(X)\rho(Y) - \beta(g,h)\rho(Y)\rho(X)
\]

for all \( X \in L_g, Y \in L_h \).

As for all \( g, h \in G \) we have \( \rho(L_g)V_h \subseteq V_{g+h} \), then any \( V_g \) has the structure of a \( L_e \)-module. In particular if we consider the adjoint representation \( \text{ad}_L \) we have that every \( L_g \) has structure of \( L_e \)-module.

Let \( L = \bigoplus_{g \in G} Lg \) be a \((G, \beta)\)-color Lie superalgebra. The descending central sequence of \( L \) [13] is defined by

\[
C^0(L) = L, \quad C^{k+1}(L) = [C^k(L), L] \quad \forall k \geq 0
\]

If \( C^k(L) = \{0\} \) for some \( k \), the \((G, \beta)\)-color Lie superalgebra is called nilpotent. The smallest integer \( k \) such as \( C^k(L) = \{0\} \) is called the nilindex of \( L \).

Also, we are going to define some new descending sequences of ideals.

Definition 2.4. [12] Let \( L = \bigoplus_{g \in G} Lg \) be a \((G, \beta)\)-color Lie superalgebra. Then, we define the new descending sequences of ideals \( C^k(L_e) \) (where \( e \) denotes the identity element of \( G \)) and \( C^k(L_g) \) with \( g \in G \setminus \{e\} \), as follows:

\[
C^0(L_e) = L_e, \quad C^{k+1}(L_e) = [C^k(L_e), L_e], \quad k \geq 0
\]

and

\[
C^0(L_g) = L_g, \quad C^{k+1}(L_g) = [L_e, C^k(L_g)], \quad k \geq 0, \quad g \in G \setminus \{e\}
\]
Using the descending sequences of ideals defined above we give an invariant of color Lie superalgebras called color-nilindex. We are going to particularize this definition for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$.

**Definition 2.5.** [13] If $L = L_e \oplus L_a \oplus L_b \oplus L_c$ is a nilpotent $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta)$-color Lie superalgebra, then $L$ has color-nilindex $(p_e, p_a, p_b, p_c)$, if the following conditions hold:

\[(C^{p_e - 1}(L_e)) (C^{p_a - 1}(L_a)) (C^{p_b - 1}(L_b)) (C^{p_c - 1}(L_c)) \neq 0\]

and

\[C^{p_e}(L_e) = C^{p_a}(L_a) = C^{p_b}(L_b) = C^{p_c}(L_c) = 0\]

**Definition 2.6.** [12] Let $L = \bigoplus_{g \in G} L_g$ be a $(G, \beta)$-color Lie superalgebra. $L_g$ is called a $L_e$-filiform module if there exists a decreasing subsequence of vectorial subspaces in its underlying vectorial space $V$, $V = V_m \supset \cdots \supset V_1 \supset V_0$, with dimensions $m, m - 1, \ldots, 0$, respectively, $m > 0$, and such that $[L_e, V_{i+1}] = V_i$.

**Definition 2.7.** [12] Let $L = \bigoplus_{g \in G} L_g$ be a $(G, \beta)$-color Lie superalgebra. Then $L$ is a **filiform color Lie superalgebra** if the following conditions hold:

1. $L_e$ is a filiform Lie algebra where $e$ denotes the identity element of $G$.
2. $L_g$ has structure of $L_e$-filiform module, for all $g \in G \setminus \{e\}$

For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ we give another equivalent definition for filiform color Lie superalgebras using the invariant called color-nilindex.

**Definition 2.8.** [13] Any $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta)$-color Lie superalgebra $L = L_e \oplus L_a \oplus L_b \oplus L_c$ is a **filiform color Lie superalgebra** if its color-nilindex is exactly

\[(\dim L_e - 1, \dim L_a, \dim L_b, \dim L_c)\]

It is not difficult to see that for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, there are, up to symmetries, four possible commutation factors $\beta$ non-degenerated, i.e. $\beta_1$, $\beta_2$, $\beta_3$ and $\beta_4$ with:

1. $\beta_1(a, a) = \beta_1(b, b) = \beta_1(a, c) = \beta_1(b, c) = 1$
2. $\beta_2(a, a) = \beta_2(b, b) = \beta_2(a, b) = -1$
3. $\beta_3(a, b) = \beta_3(a, c) = \beta_3(b, c) = -1$

in all other cases $\beta_i(-, -, -) = 1$ with $i \in \{1, 2, 3, 4\}$, thus $\beta_4 = 1$.

Fixing a $\beta_i$ ($1 \leq i \leq 4$), we will note by $\mathcal{L}^{n,m,p,t}_{g,r,s,u}$ the variety of all $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_i)$-color Lie superalgebras $L = L_e \oplus L_a \oplus L_b \oplus L_c$ with $\dim(L_e) = n + 1$, $\dim(L_a) = m$, $\dim(L_b) = p$ and $\dim(L_c) = t$.

Thus, $\mathcal{N}^{n,m,p,t}_{g,r,s,u}$ is the subset of $\mathcal{L}^{n,m,p,t}_{g,r,s,u}$ formed by all $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_i)$-color Lie superalgebras with color-nilindex $(t_0, t_1, t_2, t_3)$ where $t_0 \leq q$, $t_1 \leq r$, $t_2 \leq s$ and $t_3 \leq u$. We observe that the set $\mathcal{N}^{n,m,p,t}_{g,r,s,u}$ is the variety of all nilpotent $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_i)$-color Lie superalgebras. For simplicity we write $\mathcal{N}^{n,m,p,t}$ instead of $\mathcal{N}^{n,m,p,t}_{g,r,s,u}$.

We denote by $\mathcal{F}^{n,m,p,t}$ the subset of $\mathcal{N}^{n,m,p,t}$ composed of all filiform color Lie superalgebras.

In the particular case of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ the theorem of adapted basis rests as follows for $L = L_e \oplus L_a \oplus L_b \oplus L_c \in \mathcal{F}^{n,m,p,t}$.
\[ [X_0, X_i] = X_{i+1}, \quad 1 \leq i \leq n - 1, \quad [X_0, X_{n-1}] = 0, \]
\[ [X_0, Y_j] = Y_{j+1}, \quad 1 \leq j \leq m - 1, \quad [X_0, Y_m] = 0 \]
\[ [X_0, Z_k] = Z_{k+1}, \quad 1 \leq k \leq p - 1, \quad [X_0, Z_p] = 0, \]
\[ [X_0, W_s] = W_{s+1}, \quad 1 \leq s \leq t - 1, \quad [X_0, W_t] = 0. \]

with \{X_0, X_1, \ldots, X_n\} a basis of \(L_e\), \{Y_1, \ldots, Y_m\} a basis of \(L_a\), \{Z_1, \ldots, Z_p\} a basis of \(L_b\) and \{W_1, \ldots, W_t\} a basis of \(L_c\). The others bracket products are open (they only have to verify the conditions for be a filiform color Lie superalgebra).

The model filiform \((\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)\)-color Lie superalgebra, \(L^{n,m,p,t}\), is the simplest filiform \((\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)\)-color Lie superalgebra and it is defined in an adapted basis \{\(X_0, X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z_1, \ldots, Z_p, W_1, \ldots, W_t\}\) by the following bracket products non-null

\[
L^{n,m,p,t} = \begin{cases} 
[X_0, X_i] = X_{i+1}, & 1 \leq i \leq n - 1 \\
[X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m - 1 \\
[X_0, Z_k] = Z_{k+1}, & 1 \leq k \leq p - 1 \\
[X_0, W_s] = W_{s+1}, & 1 \leq s \leq t - 1 
\end{cases}
\]

We observe that this definition does not depend on the election of the commutation factor \(\beta_1\).

Note that from now on we will restrict our study to \(\beta_1\) as commutation factor because this is the most complex commutation factor.

We observe that a *module* \(V = V_e \oplus V_a \oplus V_b \oplus V_c\) of the \((\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)\)-color Lie superalgebra \(L\) is a bilinear map of degree \(e\) \(L \times V \rightarrow V\) satisfying

\[
\forall X \in L_g, \ Y \in L_h \ v \in V : \ X(Yv) - \beta_i(g, h)Y(Xv) = [X, Y]v
\]

color Lie superalgebra cohomology (see e.g. [14]) is defined as follows: in particular, the superspace of \(q\)-dimensional cocycles of the \((\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)\)-color Lie superalgebra \(L = L_e \oplus L_a \oplus L_b \oplus L_c\) with coefficients in the \(L\)-module \(V_e \oplus V_a \oplus V_b \oplus V_c\) will be given, up to symmetry, by

\[
C^q(L; V) = \bigoplus_{q_e + q_a + q_b + q_c = q} \text{Hom}(\wedge^q L_e \otimes S^{q_a} L_a \otimes S^{q_b} L_b \otimes \wedge^q L_c, V)
\]

This space will be graded by \(C^q(L; V) = C^q_0(L; V) \oplus C^q_3(L; V) \oplus C^q_6(L; V) \oplus C^q_9(L; V)\) with

\[
C^q_p(L; V) = \bigoplus_{q_a + q_a + q_b + q_c = q \ \text{and} \ \{q_a + q_a + p_1 \equiv r_1 \mod 2 \} \ \text{and} \ \{q_a + q_b + p_2 \equiv r_2 \mod 2 \} \ \text{with} \ p = (p_1, p_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ and } r = (r_1, r_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2.
\]

The *coboundary operator* \(\delta^q : C^q(L; V) \rightarrow C^{q+1}(L; V)\), with \(\delta^{q+1} \circ \delta^q = 0\) and \(q \geq 1\), is defined in general as follows
with $L$ a $2$-cocycle rests
module via the adjoint representation.

A color Lie superalgebra $L$ that satisfies the following two relations (for more details in general see [12]):

\[ \text{Theorem 2.8.1.} \]

We will consider in our study the $2$-cocycles of degree $\alpha$, and $A_0, A_1, \ldots, A_q \in L$ are homogeneous with
degrees $\alpha_0, \alpha_1, \ldots, \alpha_q$, respectively.

Let $Z^q(L; V)$ be the kernel of $\delta^q$ and let $B^q(L; V)$ be the image of $\delta^q$, thus we have that $B^q(L; V) \subset Z^q(L; V)$. The elements of $Z^q(L; V)$ are the $q$-cocycles, the elements of $B^q(L; V)$ are the $q$-coboundaries. Thus, we have the cohomology groups

\[ H^q(L; V) = Z^q(L; V) / B^q(L; V) \]
\[ H^q_p(L; V) = Z^q_p(L; V) / B^q_p(L; V) , \text{ if } G = \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ then } p = e, a, b, c \]

We will consider in our study the $2$-cocycles of degree $e$, $Z^2_e(L^{n,m,p,t}; L^{n,m,p,t})$ with $L^{n,m,p,t}$ the model filiform $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)$-color Lie superalgebra. Taking into account the law of $L^{n,m,p,t}$ the condition that have to verify $\psi \in C^2_e(L^{n,m,p,t}; L^{n,m,p,t})$
to be a $2$-cocycle rests

\[ (\delta^2 \psi)(A_0, A_1, A_2) = [A_0, \psi(A_1, A_2)] - \beta_1(\alpha_0, \alpha_1) [A_1, \psi(A_0, A_2)] + \beta_1(\alpha_0 + \alpha_1, \alpha_2) [A_2, \psi(A_0, A_1)] - \psi([A_0, A_1], A_2) + \beta_1(\alpha_1, \alpha_2) \psi([A_0, A_2], A_1) + \psi(A_0, [A_1, A_2]) = 0 \]

for all $A_0, A_1, A_2 \in L^{n,m,p,t}$. We note that $L^{n,m,p,t}$ has the structure of a $L^{n,m,p,t}$
module via the adjoint representation.

On the other hand, recall that an infinitesimal deformation $\varphi$ of the $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)$-
color Lie superalgebra $L$, $L = L_e \oplus L_{e_0} \oplus L_{h} \oplus L_{c}$, it is a bilinear map $\varphi : L \times L \longrightarrow L$
that satisfies the following two relations (for more details in general see [12]):

1. $\varphi(X, Y) = -\beta_1(g, h) \varphi(Y, X)$, for all $X \in L_g$ and $Y \in L_h$.
2. $\mu \circ \varphi + \varphi \circ \mu = 0$, with $\mu$ representing the law of $L$, i.e. for all $X \in L_g$, $Y \in L_h$ and $Z \in L_k$ we have

\[ (\mu \circ \varphi + \varphi \circ \mu)(X, Y, Z) = \]
\[ \beta_1(k, g) \mu(X, \varphi(Y, Z)) + \beta_1(h, k) \mu(Z, \varphi(X, Y)) + \beta_1(g, h) \mu(Y, \varphi(Z, X)) + \beta_1(k, g) \varphi(X, \mu(Y, Z)) + \beta_1(h, k) \varphi(Z, \mu(X, Y)) + \beta_1(g, h) \varphi(Y, \mu(Z, X)) = 0 \]

For an arbitrary group grading $G$ and an admissible commutation factor $\beta$ we have the following result

Theorem 2.8.1. [12] (1) Any filiform $(G, \beta)$-color Lie superalgebra law $\mu$ is isomorphic to $\mu_0 + \varphi$ where $\mu_0$ is the law of the model filiform $(G, \beta)$-color Lie superalgebra and $\varphi$ is an infinitesimal deformation of $\mu_0$ verifying that $\varphi(X_0, X) = 0$ for
all $X \in L$, with $X_0$ the characteristic vector of model one.
(2) Conversely, if $\varphi$ is an infinitesimal deformation of a model filiform $(G, \beta)$-color Lie superalgebra law $\mu_{0}$ with $\varphi(X_{0}, X) = 0$ for all $X \in L$, then the law $\mu_{0} + \varphi$ is a filiform $(G, \beta)$-color Lie superalgebra law iff $\varphi \circ \varphi = 0$.

$$\varphi \circ \varphi(X, Y, Z) = \beta(k, g)\varphi(X, \varphi(Y, Z)) + \beta(h, k)\varphi(Z, \varphi(X, Y)) + \beta(g, h)\varphi(Y, \varphi(Z, X))$$

for all $X \in L_{g}$, $Y \in L_{h}$ and $Z \in L_{k}$.

So, any filiform $(\mathbb{Z}_{2} \times \mathbb{Z}_{2}, \beta_{1})$-color Lie superalgebra is a linear deformation of the model filiform $(\mathbb{Z}_{2} \times \mathbb{Z}_{2}, \beta_{1})$-color Lie superalgebra. We observe too that, in particular the most interesting deformations are the infinitesimal deformations $\varphi$ that verify $\varphi \circ \varphi = 0$ (in the sense of Jacobi), i.e. the infinitesimal deformations that are linearly integrable.

Next, we present the correspondence between 2-cocycles and infinitesimal deformations.

**Proposition 2.9.** [13] $\psi$ is an infinitesimal deformation of $L^{n, m, p, t}$ iff $\psi$ is a 2-cocycle of degree $e$, $\psi \in Z_{2}^{2}(L^{n, m, p, t}; L^{n, m, p, t})$.

So, in order to determine all the filiform $(\mathbb{Z}_{2} \times \mathbb{Z}_{2}, \beta_{1})$-color Lie superalgebras it is only necessary to compute the infinitesimal deformations or so called 2-cocycles of degree $e$, that vanish on the characteristic vector $X_{0}$.

**Lemma 2.10.** [13] Let $Z^{2}(L; L)$ be the 2-cocycles $Z_{2}^{2}(L^{n, m, p, t}; L^{n, m, p, t})$ that vanish on the characteristic vector $X_{0}$ with $L^{n, m, p, t} = L = L_{e} \oplus L_{a} \oplus L_{b} \oplus L_{c}$. Then, $Z^{2}(L; L)$ can be divided into ten subspaces, i.e.

1. If $\beta_{i} = \beta_{1}$, then

$$Z^{2}(L; L) = Z^{2}(L; L) \cap \text{Hom}(L_{a} \wedge L_{e}, L_{c}) \oplus Z^{2}(L; L) \cap \text{Hom}(L_{e} \wedge L_{a}, L_{a}) \oplus Z^{2}(L; L) \cap \text{Hom}(L_{a} \wedge L_{b}, L_{b}) \oplus Z^{2}(L; L) \cap \text{Hom}(L_{e} \wedge L_{c}, L_{e}) \oplus Z^{2}(L; L) \cap \text{Hom}(L_{a} \wedge L_{e}, L_{c}) \oplus Z^{2}(L; L) \cap \text{Hom}(L_{e} \wedge L_{a}, L_{a}) \oplus Z^{2}(L; L) \cap \text{Hom}(L_{a} \wedge L_{b}, L_{b}) \oplus Z^{2}(L; L) \cap \text{Hom}(L_{b} \wedge L_{b}, L_{b}) \oplus Z^{2}(L; L) \cap \text{Hom}(L_{e} \wedge L_{c}, L_{e}) \oplus Z^{2}(L; L) \cap \text{Hom}(L_{a} \wedge L_{e}, L_{c})$$

$$= H_{1} \oplus H_{2} \oplus H_{3} \oplus H_{4} \oplus H_{5} \oplus H_{6} \oplus H_{7} \oplus H_{8} \oplus H_{9} \oplus H_{10}$$

3. Dimension of $Z^{2}(L; L)$ with $\beta_{1}$ as commutation factor

As $Z^{2}(L; L) = H_{1} \oplus H_{2} \oplus H_{3} \oplus H_{3} \oplus H_{5} \oplus H_{6} \oplus H_{7} \oplus H_{8} \oplus H_{9} \oplus H_{10}$ then it will be enough calculate the dimensions of such subspaces.

We can decompose $\varphi \in Z^{2}(L; L)$ as $\varphi = h_{1} + h_{2} + h_{3} + h_{4} + h_{5} + h_{6} + h_{7} + h_{8} + h_{9} + h_{10}$ with $h_{i} \in H_{i}$. Thus, the condition that $h_{1} \in \text{Hom}(L_{e} \wedge L_{e}, L_{e})$ has to verify to be a cocycle is exactly (for more details see [13])

1. $[X_{i}, h_{1}(X_{j}, X_{k})] - [X_{j}, h_{1}(X_{i}, X_{k})] + [X_{k}, h_{1}(X_{i}, X_{j})] - h_{1}([X_{i}, X_{j}], X_{k}) + h_{1}([X_{i}, X_{k}], X_{j}) + h_{1}(X_{i}, [X_{j}, X_{k}]) = 0$

and also $X_{0} \notin \text{Im} \ h_{1}$ (analogously $X_{0} \notin \text{Im} \ h_{5}$, $X_{0} \notin \text{Im} \ h_{6}$ and $X_{0} \notin \text{Im} \ h_{7}$).

But taking into account the expression of $L$, the above equation can be simplified to

1. $[X_{0}, h_{1}(X_{j}, X_{k})] - h_{1}(X_{j+1}, X_{k}) - h_{1}(X_{j}, X_{k+1}) = 0$

Similarly, we obtain the following final equations for the remaining $h_{i} \in H_{i}$:
Recall the basic properties of the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) and its finite-dimensional modules, see e.g. \([2], [9]\):

\[ \mathfrak{sl}(2, \mathbb{C}) = \langle X_-, H, X_+ \rangle \]

with the commutation relations that follow:

\[
\begin{aligned}
[X_+, X_-] &= H \\
[H, X_+] &= 2X_+ \\
[H, X_-] &= -2X_-
\end{aligned}
\]

If we consider \( V = \langle e_1, \ldots, e_n \rangle \) as a \( n \)-dimensional \( \mathfrak{sl}(2, \mathbb{C}) \)-module, then there will be a unique structure of an irreducible \( \mathfrak{sl}(2, \mathbb{C}) \)-module in \( V \) \([2]\):

\[
\begin{aligned}
X_+ \cdot e_i &= e_{i+1}, & 1 \leq i \leq n - 1, \\
X_+ \cdot e_n &= 0, \\
H \cdot e_i &= (-n + 2i - 1)e_i, & 1 \leq i \leq n.
\end{aligned}
\]

Note that \( e_n \) is the maximal vector of \( V \) and its weight (the highest weight of \( V \)) is equal to \( n - 1 \).

In general, if we consider \( W_0, W_1, \ldots, W_k \) \( \mathfrak{sl}(2, \mathbb{C}) \)-modules, then the space \( \text{Hom}(\otimes_{i=1}^k W_i, W_0) \) is too a \( \mathfrak{sl}(2, \mathbb{C}) \)-module:

\[
(\xi \cdot \varphi)(x_1, \ldots, x_k) = \xi \cdot \varphi(x_1, \ldots, x_k) - \sum_{i=1}^{i=k} \varphi(x_1, \ldots, \xi \cdot x_i, x_{i+1}, \ldots, x_n)
\]

with \( \xi \in \mathfrak{sl}(2, \mathbb{C}) \) and \( \varphi \in \text{Hom}(\otimes_{i=1}^k W_i, W_0) \). For our study we will consider \( k = 2 \) and \( W_0 = W_1 = W_2 = V_c \), obtaining then

\[
(\xi \cdot \varphi)(x_1, x_2) = \xi \cdot \varphi(x_1, x_2) - \varphi(\xi \cdot x_1, x_2) - \varphi(x_1, \xi \cdot x_2).
\]

Thus, an element \( \varphi \in \text{Hom}(V_c \otimes V_c, V_c) \) is said to be invariant or maximal vector if \( X_+ \cdot \varphi = 0 \), that is

\[
\begin{aligned}
(3.1) \quad X_+ \cdot \varphi(x_1, x_2) - \varphi(X_+ \cdot x_1, x_2) - \varphi(x_1, X_+ \cdot x_2) &= 0, & \forall x_1, x_2 \in V_c.
\end{aligned}
\]

Next, we are going to consider the structure of irreducible \( \mathfrak{sl}(2, \mathbb{C}) \)-module in \( V_c = \langle X_1, \ldots, X_n \rangle = L_c/\langle CX_0 \rangle \) obtaining:

\[
\begin{aligned}
X_+ \cdot X_i &= X_{i+1}, & 1 \leq i \leq n - 1, \\
X_+ \cdot X_n &= 0,
\end{aligned}
\]
If we identify the multiplication of $X_+$ and $X_i$ in the $\mathfrak{sl}(2,\mathbb{C})$-module $V_e = \langle X_1, \ldots, X_n \rangle$ with the bracket $\{X_0, X_i\}$ in $L_e$, then the expressions (1) and (3.1) are equivalent. Thus, any skew-symmetric bilinear map $\varphi : V_e \otimes V_e \to V_e$ will be an element of the space of cocycles $H_1$ if and only if $\varphi$ is a maximal vector of the $\mathfrak{sl}(2,\mathbb{C})$-module $\text{Hom}(V_e \otimes V_e, V_e)$, with $V_e = \langle X_1, \ldots, X_n \rangle$. But as each irreducible $\mathfrak{sl}(2,\mathbb{C})$-module has (up to nonzero scalar multiples) a unique maximal vector, then the dimension of the space of cocycles $H_1$ is equal to the number of summands of any decomposition of $\text{Hom}(V_e \otimes V_e, V_e)$ into the direct sum of irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules.

Thanks to the symmetric structure of the weights, instead of to sum the maximal vectors it is possible, and easier, to sum the vectors of weight 0 or 1.

At this point, we are going to apply this $\mathfrak{sl}(2,\mathbb{C})$-module method for to obtain the dimension of the space of cocycles $H_1$.

We will consider a natural basis $B$ of $\text{Hom}(V_e \otimes V_e, V_e)$:

$$\varphi_{i,j}^s(X_k, X_l) = \begin{cases} X_s & \text{if } (i,j) = (k,l) \\ 0 & \text{in all other cases} \end{cases}$$

where $1 \leq i, j, k, l, s \leq n$, with $i \neq j$ and $\varphi_{i,j}^s = -\varphi_{j,i}^s$. So it is enough to find the basis vectors $\varphi_{i,j}^s$ with weight 0 or 1. The weight of an element $\varphi_{i,j}^s$ is $n + 2(s - i - j) + 1$, that is

$$(H \cdot \varphi_{i,j}^s)(X_i, X_j) = H \cdot \varphi_{i,j}^s(X_i, X_j) - \varphi_{i,j}^s(H \cdot X_i, X_j) - \varphi_{i,j}^s(X_i, H \cdot X_j)$$

$$= H \cdot X_s - \varphi_{i,j}^s((n_1 - 2s)X_i, X_j) - \varphi_{i,j}^s(X_i, (n_1 - 2s)X_j)$$

$$= (n_1 - 2s)X_s - (n_1 - 2s)X_s = (n_1 - 1 + 2s)X_s$$

Thus, if $n$ is even we will find the elements $\varphi_{i,j}^s$ with weight 1 and if $n$ is odd we will find those of them with weight 0 (note that if $n$ is even then $\lambda(\varphi)$ is odd, and if $n$ is odd then $\lambda(\varphi)$ is even).

We can consider three times the sequence that corresponds with the weights of $V_e = \langle X_1, X_2, \ldots, X_{n-1}, X_n \rangle$ i.e. $-n + 1, -n + 3, \ldots, n - 3, n - 1$. Then, we have to count the number of all possibilities to obtain 1 (if $n$ is even) or 0 (if $n$ is odd). Note that $\lambda(\varphi_{i,j}^s) = \lambda(X_s) - \lambda(X_i) - \lambda(X_j)$, where $\lambda(X_s)$ belongs to the last sequence, and $\lambda(X_i)$, $\lambda(X_j)$ belong to the first and second sequences respectively. Taking into account the skew-symmetry of $\varphi_{i,j}^s$, that is $\varphi_{i,j}^s = -\varphi_{j,i}^s$ and $i \neq j$ we obtain the following theorem:

**Theorem 1.** Let $Z^2(L; L)$ be the 2-cocycles $Z^2(L^{n,m,p,t}; L^{n,m,p,t})$ that vanish on the characteristic vector $X_0$. Then, if $H_1 = Z^2(L; L) \cap \text{Hom}(L_e \otimes L_e, L_e)$ we have that
in particular:
\[
\begin{align*}
\dim H_1 &= \begin{cases} 
n\frac{(3n-2)}{8} & \text{if } n \text{ is even} \\
\frac{3n^2 - 4n + 1}{8} + \left\lfloor \frac{n+1}{4} \right\rfloor & \text{if } n \text{ is odd}
\end{cases}
\end{align*}
\]

**Proof.** It can be considered 4 cases where the reasoning is not very difficult, i.e. \( n \equiv i \pmod{4} \), with \( 1 \leq i \leq 3 \).

Similarly, we can apply the above method in order to calculate the dimension of \( H_2 \). Thus, if we consider the structure of \( \mathfrak{sl}(2, \mathbb{C}) \)-module in \( \text{Hom}(V_c \otimes V_a, V_a) \) then we would have that an element \( \varphi \in \text{Hom}(V_c \otimes V_a, V_a) \) is said to be invariant or maximal vector if \( X_+ \cdot \varphi = 0 \), that is
\[
(3.2) \quad X_+ \cdot \varphi(x_1, x_2) - \varphi(X_+ \cdot x_1, x_2) - \varphi(x_1, X_+ \cdot x_2) = 0, \quad \forall x_1 \in V_c, \forall x_2 \in V_a.
\]

In this case we are going to consider the structure of irreducible \( \mathfrak{sl}(2, \mathbb{C}) \)-module in \( V_c = \langle X_1, \ldots, X_n \rangle = L_c/\mathbb{C} X_0 \) and in \( V_a = \langle Y_1, \ldots, Y_n \rangle = L_a \), thus in particular:
\[
\begin{align*}
X_+ \cdot X_i &= X_{i+1}, \quad 1 \leq i \leq n - 1 \\
X_+ \cdot X_n &= 0 \\
X_+ \cdot Y_j &= Y_{j+1}, \quad 1 \leq j \leq m - 1 \\
X_+ \cdot Y_m &= 0
\end{align*}
\]

We identify the multiplication of \( X_+ \) and \( X_i \) in the \( \mathfrak{sl}(2, \mathbb{C}) \)-module \( V_c = \langle X_1, \ldots, X_n \rangle \), with the bracket product \([X_0, X_i]\) in \( L_c \). Analogously with \( X_+ \cdot Y_j \) and \([X_0, Y_j]\). Thanks to these identifications, the expression to be a cocycle of \( H_2 \):
\[
(2) \quad [X_0, h_2(X_j, Y_k)] - h_2(X_{j+1}, Y_k) - h_2(X_j, Y_{k+1}) = 0
\]
and (3.2) are equivalent, so any skew-symmetric bilinear map \( \varphi: V_c \wedge V_a \rightarrow V_a \) will be an element of \( H_2 \) if and only if \( \varphi \) is a maximal vector of the \( \mathfrak{sl}(2, \mathbb{C}) \)-module \( \text{Hom}(V_c \wedge V_a, V_a) \), with \( V_c = \langle X_1, \ldots, X_n \rangle \) and \( V_a = L_a \). But, as each \( \mathfrak{sl}(2, \mathbb{C}) \)-module has (up to nonzero scalar multiples) a unique maximal vector, then the dimension of \( H_2 \) is equal to the number of summands of any decomposition of \( \text{Hom}(V_c \wedge V_a, V_a) \) into direct sum of irreducible \( \mathfrak{sl}(2, \mathbb{C}) \)-modules.

As each irreducible module contains either a unique (up to scalar multiples) vector of weight 0 or a unique vector of weight 1, then we will sum them.

Next, we consider a natural basis of \( \text{Hom}(V_c \wedge V_a, V_a) \) consisting of the following maps where \( 1 \leq s, j, l \leq m \) and \( 1 \leq i, k \leq n \):
\[
\varphi_{i,j}^s(X_k, Y_l) = \begin{cases} 
Y_s & \text{if } (i, j) = (k, l) \\
0 & \text{in all other cases}
\end{cases}
\]

It will be enough to find the basis vectors \( \varphi_{i,j}^s \) with weight 0 or 1. It is not difficult to see that the weight of an element \( \varphi_{i,j}^s \) (with respect to \( H \)) is
\[
\lambda(\varphi_{i,j}^s) = \lambda(Y_s) - \lambda(X_i) - \lambda(Y_j) = n + 2(s - i - j) + 1.
\]

Thus, if \( n \) is even then \( \lambda(\varphi) \) is odd, and if \( n \) is odd then \( \lambda(\varphi) \) is odd. So, if \( n \) is even it will be sufficient to find the elements \( \varphi_{i,j}^s \) with weight 1 and if \( n \) is
odd it will be sufficient to find those with weight 0. To do that we consider the three sequences that correspond with the weights of \( V_e = \langle X_1, \ldots, X_n \rangle \), \( V_a = \langle Y_1, Y_2, \ldots, Y_m \rangle \) and \( V_0 = \langle Y_1, Y_2, \ldots, Y_m \rangle \), i.e.: \(-n+1, \ldots, n-3, n-1; -m+1, -m+3, \ldots, m-3, m-1\).

We shall have to count the number of all possibilities to obtain 1 (if \( n \) is even) or 0 (if \( n \) is odd). Note that \( \lambda(Y_0) = \lambda(Y_2) - \lambda(X_1) - \lambda(Y_2) \), where \( \lambda(Y_0) \) belongs to the last sequence, and \( \lambda(X_1), \lambda(Y_2) \) belong to the first and second sequences respectively. Thus, we obtain the following theorem.

**Theorem 2.** Let \( Z^2(L; L) \) be the 2-cocycles \( Z^2(L^{n,m,p,t}; L^{n,m,p,t}) \) that vanish on the characteristic vector \( X_0 \). Then, if \( H_2 = Z^2(L; L) \cap \text{Hom}(L_e \wedge L_a, L_a) \) we have

\[
\dim H_2 = \begin{cases} 
\frac{4nm - n^2 + 1}{4} & \text{if } n \text{ is odd, } n < 2m + 1 \\
\frac{4nm - n^2}{4} & \text{if } n \text{ is even, } n < 2m + 1 \\
m^2 & \text{if } n \geq 2m + 1
\end{cases}
\]

**Proof.** It can be considered 4 cases where the reasoning is not very difficult, i.e. \( n \equiv i (\mod 4) \), with \( 1 \leq i \leq 3 \).

Similarly to the previous reasoning we can obtain the equivalent result for \( H_3 \) and \( H_4 \) (see the dimensions in the Main Theorem 1).

Next, we are going to apply the \( \mathfrak{sl}(2, \mathbb{C}) \)-module method in order to compute the dimension of \( H_5 = Z^2(L; L) \cap \text{Hom}(S^2L_a, L_e) \).

Thus, we consider the structure of \( \mathfrak{sl}(2, \mathbb{C}) \)-module in \( \text{Hom}(V_a \otimes V_a, V_e) \) and then we have that an element \( \varphi \in \text{Hom}(V_a \otimes V_a, V_e) \) is said to be invariant or maximal vector if \( X_+ \cdot \varphi = 0 \), that is

\[
(3.3) \quad X_+ \cdot \varphi(x_1, x_2) - \varphi(X_+ \cdot x_1, x_2) = 0, \quad x_1, x_2 \in V_a.
\]

On the other hand, a cocycle \( \varphi \) belongs to \( H_5 \) it will be a symmetric bilinear map \( \varphi : S^2L_a \to L_e \), such that

\[
(5) \quad [X_0, h_5(Y_j, Y_k)] - h_5(Y_{j+1}, Y_k) - h_5(Y_j, Y_{k+1}) = 0.
\]

Then, we consider the structure of an irreducible \( \mathfrak{sl}(2, \mathbb{C}) \)-module in \( V_e = \langle X_1, \ldots, X_n \rangle = L_e / \mathbb{C}X_0 \) and in \( V_a = \langle Y_1, \ldots, Y_m \rangle = L_a \), thus in particular:

\[
\begin{cases} 
X_+ \cdot X_i = X_{i+1}, & 1 \leq i \leq n - 1 \\
X_+ \cdot X_n = 0 \\
X_+ \cdot Y_j = Y_{j+1}, & 1 \leq j \leq m - 1 \\
X_+ \cdot Y_m = 0
\end{cases}
\]

We identify the multiplication of \( X_+ \) and \( X_i \) in the \( \mathfrak{sl}(2, \mathbb{C}) \)-module \( V_e = \langle X_1, \ldots, X_n \rangle \), with the bracket product \([X_0, X_i]\) in \( L_e \). Analogously with \( X_+ \cdot Y_j \) and \([X_0, Y_j]\). Thanks to these identifications the equation (5) to be a cocycle of \( H_5 \) and the equation (3.3) are equivalent, so any symmetric bilinear map \( \varphi \), \( \varphi : S^2V_a \to V_e \) will be an element of \( H_5 \) if and only if \( \varphi \) is a maximal vector.
of the $\mathfrak{sl}(2, \mathbb{C})$-module $\text{Hom}(S^2 V_a, V_c)$, with $V_c = \langle X_1, \ldots, X_n \rangle$ and $V_a = L_a$. But as each $\mathfrak{sl}(2, \mathbb{C})$-module has (up to nonzero scalar multiples) a unique maximal vector, then the dimension of $H_5$ is equal to the number of summands of any decomposition of $\text{Hom}(S^2 V_a, V_c)$ into direct sum of irreducible $\mathfrak{sl}(2, \mathbb{C})$-modules.

As each irreducible module contains either a unique (up to scalar multiples) vector of weight 0 or a unique vector of weight 1, then we we will sum them.

Next, we consider a natural basis of $\text{Hom}(S^2 V_a, V_c)$ consisting of the following maps where $1 \leq s \leq n$ and $1 \leq i, j, k, l \leq m$:

$$\varphi_{i,j}^s(Y_k, Y_l) = \begin{cases} X_s & \text{if } (i, j) = (k, l) \\ 0 & \text{in all other cases} \end{cases}$$

It will be enough to find the basis vectors $\varphi_{i,j}^s$ with weight 0 or 1. It is not difficult to see that the weight of an element $\varphi_{i,j}^s$ is

$$\lambda(\varphi_{i,j}^s) = \lambda(X_s) - \lambda(Y_i) - \lambda(Y_j) = 2m - n + 2(s - i - j) + 1.$$  

**Remark 3.1.** If $n$ is even then $\lambda(\varphi)$ is odd, and if $n$ is odd then $\lambda(\varphi)$ is even. So, if $n$ is even it will be sufficient to find the elements $\varphi_{i,j}^s$ with weight 1 and if $n$ is odd it will be sufficient to find those of them with weight 0.

In order to find the elements with weight 0 or 1, we can consider the three sequences that correspond with the weights of $V_a = \langle Y_1, Y_2, \ldots, Y_{m-1}, Y_m \rangle$, $V_c = \langle X_1, X_2, \ldots, X_{n-1}, X_n \rangle$, i.e.: $-m + 1, -m + 3, \ldots, m - 3, m - 1; -m + 1, -m + 3, \ldots, m - 3, m - 1$ and $-n + 1, -n + 3, \ldots, n - 3, n - 1$.

We shall have to count the number of all possibilities to obtain 1 (if $n$ is even) or 0 (if $n$ is odd). Note that $\lambda(\varphi_{i,j}^s) = \lambda(X_s) - \lambda(Y_i) - \lambda(Y_j)$, where $\lambda(X_s)$ belongs to the last sequence, and $\lambda(Y_i)$, $\lambda(Y_j)$ belong to the first and second sequences respectively. Taking into account the symmetry of $\varphi_{i,j}^s$ we obtain the following theorem:

**Theorem 5.** If $H_5 = Z^2(L; L) \cap \text{Hom}(S^2 L_a, L_c)$, then we have the following values for the dimension of $H_5$

$$\dim H_5 = \begin{cases} \frac{m(m+1)}{2} & \text{if } n \geq 2m - 1 \\ \frac{1}{8}(4mn - n^2 + 2n + 3) & \text{if } n < 2m - 1, \quad n \equiv 1(\text{mod } 4) \text{ and } m \text{ odd, or } n \equiv 3(\text{mod } 4) \text{ and } m \text{ even} \\ \frac{1}{8}(4mn - n^2 + 2n - 1) & \text{if } n < 2m - 1, \quad n \equiv 3(\text{mod } 4) \text{ and } m \text{ odd, or } n \equiv 1(\text{mod } 4) \text{ and } m \text{ even} \\ \frac{1}{8}(4mn - n^2 + 2n) & \text{if } n < 2m - 1 \text{ and } n \text{ even} \end{cases}$$

**Proof.** It is convenient to distinguish the following six cases where the reasoning for each case is not hard:
(1). \( n \geq 2m - 1 \)
(2). \( n \leq 2m - 1, n \text{ even.} \)
(3). \( n \leq 2m - 1, n \equiv 1 \pmod{4}, m \text{ odd.} \)
(4). \( n \leq 2m - 1, n \equiv 3 \pmod{4}, m \text{ even.} \)
(5). \( n \leq 2m - 1, n \equiv 1 \pmod{4}, m \text{ even.} \)
(6). \( n \leq 2m - 1, n \equiv 3 \pmod{4}, m \text{ odd.} \)

\[ \square \]

Similarly to the previous reasoning we can obtain the equivalent result for \( H_6, H_7, H_8, H_9 \) and \( H_{19} \) (see the dimensions in the Main Theorem 1).

**Main Theorem 1.** Let \( L^{n,m,p,t} = L \) be the model filiform \((\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)\)-color Lie superalgebra. Let \( Z^2(L;L) \) be the 2-cocycles \( Z^2_2(L;L) \) that vanish on the characteristic vector \( X_0 \). Then, \( \dim(Z^2(L;L)) = \sum_{i=1}^{10} \dim H_i \) with

\[
\begin{align*}
\dim H_1 &= \begin{cases} 
\frac{n(3n-2)}{8} & \text{if } n \text{ is even} \\
\frac{3n^2-4n+1}{8} + \left\lceil \frac{n+1}{4} \right\rceil & \text{if } n \text{ is odd} \\
\frac{4nm-n^2+1}{4} & \text{if } n \text{ is odd, } n < 2m + 1
\end{cases} \\
\dim H_2 &= \begin{cases} 
\frac{4nm-n^2}{4} & \text{if } n \text{ is even, } n < 2m + 1 \\
m^2 & \text{if } n \geq 2m + 1 \\
\frac{4np-n^2+1}{4} & \text{if } n \text{ is odd, } n < 2p + 1
\end{cases} \\
\dim H_3 &= \begin{cases} 
\frac{4np-n^2}{4} & \text{if } n \text{ is even, } n < 2p + 1 \\
p^2 & \text{if } n \geq 2p + 1 \\
\frac{4nt-n^2+1}{4} & \text{if } n \text{ is odd, } n < 2t + 1
\end{cases} \\
\dim H_4 &= \begin{cases} 
\frac{4nt-n^2}{4} & \text{if } n \text{ is even, } n < 2t + 1 \\
t^2 & \text{if } n \geq 2t + 1
\end{cases}
\end{align*}
\]
\[ \dim H_5 = \begin{cases} \frac{m(m+1)}{2} & \text{if } n \geq 2m - 1 \\ \frac{1}{8}(4mn - n^2 + 2n + 3) & \text{if } n < 2m - 1, \ n \equiv 1(\mod 4) \text{ and } m \text{ odd, or} \\
\frac{1}{8}(4mn - n^2 + 2n - 1) & \text{if } n < 2m - 1, \ n \equiv 3(\mod 4) \text{ and } m \text{ even} \end{cases} \]

\[ \dim H_6 = \begin{cases} \frac{1}{8}(4pm - n^2 + 2n + 3) & \text{if } n < 2p - 1, \ n \equiv 1(\mod 4) \text{ and } p \text{ odd, or} \\
\frac{1}{8}(4pm - n^2 + 2n - 1) & \text{if } n < 2p - 1, \ n \equiv 3(\mod 4) \text{ and } p \text{ even} \end{cases} \]

\[ \dim H_7 = \begin{cases} \frac{1}{8}(4tn - n^2 - 2n - 1) & \text{if } n < 2t - 1, \ n \equiv 1(\mod 4) \text{ and } t \text{ odd, or} \\
\frac{1}{8}(4tn - n^2 - 2n + 3) & \text{if } n < 2t - 1, \ n \equiv 3(\mod 4) \text{ and } t \text{ even} \end{cases} \]

(1). If \( m + p - t \) is even
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\[ \dim H_8 = \begin{cases} 
mt & \text{if } p \geq m + t \\
 tp - 1 & \text{if } p < m + t, p = m - t + 2 \\
 tp & \text{if } p < m + t, p < m - t + 2 \\
\frac{1}{4}(-m^2 - t^2 - p^2 + 2tp + 2mt + 2mp) & \text{if } p < m + t, p > m - t + 2, p \geq t - m + 2 \\
 mp & \text{if } p < m + t, p > m - t + 2, p < t - m + 2 
\end{cases} \]

(2). If $m + p - t$ is odd

\[ \dim H_8 = \begin{cases} 
mt & \text{if } p \geq m + t - 1 \\
 tp & \text{if } p < m + t - 1, p \leq m - t + 1 \\
\frac{1}{4}(-m^2 - t^2 - p^2 + 2tp + 2mt + 2mp + 1) & \text{if } p < m + t - 1, p > m - t + 1, p \geq t - m + 1 \\
 mp & \text{if } p < m + t - 1, p > m - t + 1, p < t - m + 1 
\end{cases} \]

(1). If $m + t - p$ is even

\[ \dim H_9 = \begin{cases} 
mp & \text{if } t \geq m + p \\
 tp - 1 & \text{if } t < m + p, t = m - p + 2 \\
 tp & \text{if } t < m + p, t < m - p + 2 \\
\frac{1}{4}(-m^2 - t^2 - p^2 + 2tp + 2mt + 2mp) & \text{if } t < m + p, t > m - p + 2, t \geq p - m + 2 \\
 mt & \text{if } t < m + p, t > m - p + 2, t < p - m + 2 
\end{cases} \]

(2). If $m + t - p$ is odd
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\[ \dim H_9 = \begin{cases} 
mp & \text{if } t \geq m + p - 1 \\
 tp & \text{if } t < m + p - 1, t \leq m + p + 1 \\
\frac{1}{4}(-m^2 - t^2 - p^2 + 2tp + 2mt + 2mp + 1) & \text{if } t < m + p - 1, t > m - p + 1, \\
 mt & \text{if } t < m + p - 1, t > m - p + 1, t \leq m - p + 1.
\end{cases} \]

(1) If \( t + p - m \) is even

\[ \dim H_{10} = \begin{cases} 
mt & \text{if } p \geq m + t \\
mp - 1 & \text{if } p < m + t, p = t - m + 2 \\
mp & \text{if } p < m + t, p < t - m + 2
\end{cases} \]

(2) If \( t + p - m \) is odd

\[ \dim H_{10} = \begin{cases} 
mt & \text{if } p \geq m + t - 1 \\
mp & \text{if } p < m + t - 1, p \leq t - m + 1
\end{cases} \]

4. Computing a basis of \( Z^2(L; L) \cap \text{Hom}(S^2L_a, L_e) \)

In this section we are going to develop a method that permit us to compute a basis of \( H_5 = Z^2(L; L) \cap \text{Hom}(S^2L_a, L_e) \) in each case, and with \( \beta_1 \) as commutation factor. We will consider the subspace \( H_5 \) because of it is completely composed by linearly integrable cocycles (it is not difficult to see that \( h_5 \circ h_5 = 0 \)). Thus, if we have \( h_5 \in H_5 \) and \( L^{n,m,p,t} \) is the model filiform color Lie superalgebra, then \( L^{n,m,p,t} + h_5 \) will be too a filiform color Lie superalgebra.

Recall the weight of an element \( \varphi^{s}_{i,j}, \varphi^{s}_{i,j} \in \text{Hom}(S^2V_a, V_e) \), defined by
with $1 \leq s \leq n$ and $1 \leq i, j, k, l \leq m$. Then the weight $\lambda(\varphi_{i,j}^s)$ is

$$
\lambda(\varphi_{i,j}^s) = \lambda(X_s) - \lambda(Y_i) - \lambda(Y_j) = 2m - n + 2(s - i - j) + 1.
$$

We have the following relationships between the two weights:

$$
d(X_0) = X_0, \quad d(X_i) = iX_i, \quad d(Y_j) = jY_j, \quad d(Z_k) = kZ_k, \quad d(W_s) = sW_s;
$$

with $1 \leq i \leq n$, $1 \leq j \leq m$, $1 \leq k \leq p$, $1 \leq s \leq t$.

This weight will be denoted by $p(\varphi)$. We have that

$$
p(\varphi_{i,j}^s) = s - i - j.
$$

We have the following relationships between the two weights:

$$
\lambda(\varphi) = 2p(\varphi) + 2m - n + 1,
$$

$$
p(\varphi) = \frac{1}{2}(\lambda(\varphi) - 2m + n - 1).
$$

Let $\varphi$ be an element of $H_5$, with weight $\lambda(\varphi)$. As $\varphi$ is a maximal vector of the $\mathfrak{sl}(2, \mathbb{C})$-module Hom($S^2V_0, V_0$), its weight $\lambda(\varphi)$ is always a nonnegative integer, $\lambda(\varphi) \geq 0$.

On the other hand, $p(\varphi)$ is always less or equal than $n - 2$, $p(\varphi) \leq n - 2$. In fact, $\varphi_{1,1}^n$ is an element with maximal weight $p(\varphi)$, $p(\varphi_{1,1}^n) = n - 2$. So, we have the following estimates for $p(\varphi)$:

$$
\frac{n - 2m - 1}{2} \leq p(\varphi) \leq n - 2.
$$

In order to get a basis of $H_5$ it is enough to obtain the basis for each subspace $H_5(p)$ of $H_5$, spanned by all the elements with weight $p$ such that $p$ satisfies (4.1). Let $\psi_{k,s}$ be an element of Hom($S^2V_0, V_0$) with weight $p$, $p(\psi_{k,s}) = s - 2k$, and defined by

$$
\psi_{k,s}(Y_i, Y_j) = \begin{cases} X_s & \text{if } i = k \\ 0 & \text{in the other case} \end{cases}
$$

with $1 \leq s \leq n$, $1 \leq k \leq m$ and satisfying the equations

$$
[\psi_{k,s}(Y_i, Y_j)] - \psi_{k,s}(Y_{i+1}, Y_j) - \psi_{k,s}(Y_i, Y_{j+1}) = 0, \quad 1 \leq i, j \leq m - 1.
$$

We observe that $\psi_{k,s}$ is not always a cocycle of $H_5$. In particular, $\psi_{k,s}$ will be a cocycle of $H_5$ if and only if it satisfies the equations

$$
[\psi_{k,s}(Y_i, Y_m)] - \psi_{k,s}(Y_{i+1}, Y_m) = 0, \quad 1 \leq i \leq m.
$$

By induction the following formula for $\psi_{k,s}$ can be proved:

$$
\psi_{k,s}(Y_i, Y_j) = (-1)^{k - i} [C_{j-k}^{k-i} X_{i+j+s-2k}]
$$

with $1 \leq i < j \leq m$, $k \leq \frac{i+j}{2}$ and to simplify the expressions we have denoted by $C_{j-k}^{k-i}$ the expression $C_{j-k}^{k-i} - \frac{1}{2} C_{j-k-1}^{k-i}$. We suppose that $C_0^q = 0$ if $q < 0$ or $t < 0$ or $q > t$, and $C_0^0 = C_0^1 = 1$ with $t > 0$. In the remaining cases we have $C_t^q = \frac{t}{q(t-q)!}$. 

Proof. We only have to check whether \( \psi_{k,s} \) satisfies or not the equations

\[
[X_0, \psi_{k,s}(Y_i, Y_m)] = \psi_{k,s}(Y_{i+1}, Y_m), \text{ with } 1 \leq i \leq m.
\]

If \( p(\psi_{k,s}) = n - m - 1 \), then

\[
\psi_{k,s}(Y_1, Y_m) = (-1)^{k-1}C_{m-k}^{-1}X_n
\]

and \( \psi_{k,s}(Y_2, Y_m) = \cdots = \psi_{k,s}(Y_m, Y_m) = 0 \) which clearly satisfy the above equations. If \( p(\psi_{k,s}) > n - m - 1 \), then \( \psi_{k,s}(Y_1, Y_m) = \cdots = \psi_{k,s}(Y_m, Y_m) = 0 \) and also satisfies the cocycle equations of \( H_5 \).

If \( p(\psi_{k,s}) < n - m - 1 \), then

\[
\psi_{k,s}(Y_1, Y_m) = (-1)^{k-1}C_{m-k}^{-1}X_t
\]

with \( t < n \). If we apply the cocycle equations we have

\[
[X_0, \psi_{k,s}(Y_1, Y_m)] = \psi_{k,s}(Y_2, Y_m) = (-1)^{k-1}C_{m-k}^{k-2}X_{t+1},
\]

but

\[
[X_0, \psi_{k,s}(Y_1, Y_m)] = [X_0, (-1)^{k-1}C_{m-k}^{-1}X_t] = (-1)^{k-1}C_{m-k}^{k-1}X_{t+1}
\]

and then

\[
C_{m-k}^{k-2} - \frac{1}{2}C_{m-k-1}^{k-2} = -C_{m-k}^{k-1} + \frac{1}{2}C_{m-k-1}^{k-1},
\]

which is a contradiction. \( \square \)

**Proposition 4.2.** Let \( \psi \in H_5 \) be a cocycle with weight \( p = p(\psi) \leq n - m - 2 \). Then

\[
\psi = \sum_{s-2k=p} a_k \psi_{k,s}
\]

for some numbers \( a_k \).

Proof. Let \( \psi \in H_5 \) a cocycle with weight \( p \). Then \( \psi(Y_i, Y_i) = a_i X_{2i+p} \). We are going to consider the difference \( \Psi = \psi - \sum_{s-2k=p} a_k \psi_{k,s} \).

It is easy to check that \( \Psi \) is a symmetric bilinear map such that \( \Psi(Y_i, Y_1) = \Psi(Y_2, Y_2) = \cdots = \Psi(Y_m, Y_m) = 0 \).

In fact, as \( \psi_{k,s} \) satisfies the equations \( \text{[12]} \), \( \Psi \) satisfies them too, which implies that \( \Psi \) vanishes. That is, if we fix \( i \) with \( 1 \leq i \leq m - 1 \), we can prove by induction that \( \Psi(Y_i, Y_{i+k}) = 0 \), for all \( k \geq 0 \): We suppose that the relation is true up to \( k \), thanks to \( \text{[12]} \), for \( k + 1 \) we have

\[
[X_0, \Psi(Y_i, Y_{i+k})] = 0 = \Psi(Y_{i+1}, Y_{(i+1)+(k-1)}) + \Psi(Y_i, Y_{i+k+1}) = 0 + \Psi(Y_i, Y_{i+k+1})
\]

which proves the result. \( \square \)
Proposition 4.3. Let $\psi$ be a non-zero cocycle of weight $p = p(\psi) \leq n - m - 2$

$$\psi = \sum_{s-2k=p} a_k \psi_{k,s}. $$

Then $p \geq n - 2m$.

Proof. Let $\psi = \sum_{s-2k=p} a_k \psi_{k,s}$ be a cocycle with $p < n - 2m$. If $\psi$ is non-zero, then there exists $i$ such that $a_i \neq 0$, and thus $\psi(Y_i,Y_i) = a_i X_{2i+p} \neq 0$. As $\psi$ is a cocycle it can be seen that

$$(adX_0)^{2(m-i)}(\psi(Y_i,Y_i)) = \alpha \psi(Y_m,Y_m)$$

Then $0 \neq a_i X_{m+2p} = \alpha \psi(Y_m,Y_m)$. Thus $\psi(Y_m,Y_m) = a_m X_{m+2p}$ with $a_m \neq 0$ and $m + 2p < n$, so $[X_0, \psi(Y_m,Y_m)] = a_m X_{m+2p+1} \neq 0$ which is a contradiction with the equations that verifies for being a cocycle:

$$[X_0, \psi(Y_m,Y_m)] = \psi([X_0,Y_m],Y_m) + \psi(Y_m,[X_0,Y_m]) = 0.$$

Proposition 4.4. Let $\psi$ be a cocycle of $H_5$

$$\psi = \sum_{s-2k=p} a_k \psi_{k,s}$$

with $\max \{ \frac{n-2m-1}{2}, n - 2m \} \leq p \leq n - m - 2$. Then $\psi$ is a cocycle iff

$$(adX_0)^{-1}(\psi(Y_1,Y_m)) = \cdots = (adX_0)(\psi(Y_{r-1},Y_m)) = \psi(Y_r,Y_m)$$

with $r = n - m - p$.

Proof. As each $\psi_{k,s}$ satisfies the equations (4.12), $\psi$ satisfies them, too. Thus, we have that $\psi$ will be a cocycle of $H_5$ iff $\psi$ satisfies the equations

$$[X_0, \psi(Y_i,Y_m)] - \psi(Y_{i+1},Y_m) = 0, \text{ with } 1 \leq i \leq m$$

which proves the result. □

The above proposition give us a method to construct all the cocycles with weight $p$, $\max \{ \frac{n-2m-1}{2}, n - 2m \} \leq p \leq n - m - 2$, and together with Proposition 4.1 the complete description of $H_5$ can be obtained. An explicit description of $H_5$ with $n \geq 2m - 1$ will be done in the following section.

5. Basis of $H_5$ for $n \geq 2m - 1$

In this section we are going to apply the method described in the above section in order to construct an explicit basis of $H_5$. In particular, we shall give a basis of $H_5$ in the case with more cocycles: $\frac{m(m+1)}{2}$ with $n \geq 2m - 1$ (see Main Theorem 1).

Main Theorem 2. A basis of the space of cocycles $Z^2(L;L) \cap \text{Hom}(S^2L_a,L_c)$, with $n \geq 2m - 1$, will be given by the following cocycles

- For each $p$ such that $n - m - 1 \leq p \leq n - 2$, there are $\left\lfloor \frac{n-p}{2} \right\rfloor$ cocycles of weight $p$ in the basis, that is

$$\psi_{1,p+2}, \psi_{2,p+4}, \cdots, \psi_{\left\lfloor \frac{n-p}{2} \right\rfloor, p+2\left\lfloor \frac{n-p}{2} \right\rfloor}$$
\( p = n - 2m \), there is only one cocycle of weight \( p \) in the basis, that is
\[
\psi_{m,n} = a_1 \psi_{1,n-2m+2} + a_2 \psi_{2,n-2m+4} + \cdots + a_{m-1} \psi_{m-1,n-2} + \psi_{m,n}
\]
with
\[
a_i = 2 + \sum_{q=2}^{m-i} 2^q \left[ \prod_{k=1}^{q-1} \sum_{j_k=1}^{m-i-1-\sum_{t=1}^{k-1} j_t} (-1)^{j_k+1} c_{m-i-\sum_{t=1}^{k-1} j_t} \right], \quad 1 \leq i \leq m - 1
\]

• For each \( p \) such that \( n - 2m < p \leq n - m - 2 \), there are \( l - r + 1 = \lfloor \frac{n-p}{2} \rfloor - (n - m - p) + 1 \) cocycles of weight \( p \) in the basis, that is
\[
\psi_{r,p+2r} = a_1 \psi_{1,p+2} + a_2 \psi_{2,p+4} + \cdots + a_{r-1} \psi_{r-1,p+2(r-1)} + \psi_{r,p+2r}
\]
and
\[
\psi_{h,p+2h} = a_1^h \psi_{1,p+2} + a_2^h \psi_{2,p+4} + \cdots + a_{r-1}^h \psi_{r-1,p+2(r-1)} + \psi_{h,p+2h}
\]
for \( r < h \leq l, 1 \leq i \leq r - 1 \), with
\[
a_i = 1 + 2(-1)^{r-i+1} c_{m-r}^{-i} + \sum_{q=2}^{r-i} 2^q \left[ \prod_{k=1}^{q-1} \sum_{j_k=1}^{m-i-1-\sum_{t=1}^{k-1} j_t} (-1)^{j_k+1} c_{m-i-\sum_{t=1}^{k-1} j_t} \right] + \sum_{q=2}^{r-i} 2^q (-1)^{r-i+q} \left( \prod_{k=1}^{q-1} \sum_{j_k=1}^{m-i-1-\sum_{t=1}^{k-1} j_t} c_{m-i-\sum_{t=1}^{k-1} j_t} \right)
\]
and
\[
a_i^h = 2(-1)^{h-i} \psi_{m-h} + (-1)^{h-r} c_{m-h}^{-i} \sum_{q=2}^{r-i} 2^q \left[ \prod_{k=1}^{q-1} \sum_{j_k=1}^{m-i-1-\sum_{t=1}^{k-1} j_t} (-1)^{j_k+1} c_{m-i-\sum_{t=1}^{k-1} j_t} \right] + 2(-1)^{h-i+1} c_{m-h}^{-i} \sum_{q=2}^{r-i} 2^q (-1)^{h-i+q} \left( \prod_{k=1}^{q-1} \sum_{j_k=1}^{m-i-1-\sum_{t=1}^{k-1} j_t} c_{m-i-\sum_{t=1}^{k-1} j_t} \right) \psi_{h,m-h}
\]

**Proof.** Thanks to the condition \( n \geq 2m - 1 \), the weight \( p \) can only be contained in the interval \(-1 \leq p \leq n - 2\). As Proposition 1.1 gives us the description of the cocycles with \( p \geq n - m - 1 \), it remains to describe a basis of the cocycles of \( H_5 \) such that:
\[
n - 2m \leq p \leq n - m - 2.
\]
If we fix \( p \) satisfying \( n - 2m \leq p \leq n - m - 2 \), then all the mappings \( \psi_{k,0} \) with weight \( p \) will be
\[
\psi_{1,p+2}, \psi_{2,p+4}, \ldots, \psi_{l,p+2l}
\]
with \( l = \lfloor \frac{n-p}{2} \rfloor \). In fact, as \( p \geq n - 2m \), then \( l = \lfloor \frac{n-p}{2} \rfloor \leq m \) and \( \min (\lfloor \frac{n-p}{2} \rfloor, m) = \lfloor \frac{n-p}{2} \rfloor \). Let \( \psi \) be
\[
\psi = a_1 \psi_{1,p+2} + a_2 \psi_{2,p+4} + \cdots + a_l \psi_{l,p+2l}.
\]
Recall that to simplify the expressions, we denote by \( C_j^k \) the expression \( C_j^k - \frac{1}{2} C_j^{k-1} \).
Proposition 4.4 gives us \( r - 1 = n - m - p - 1 \) linear equations in \( a_1, \ldots, a_l \):

\[
(adX_0)^i(\psi(Y_{r-i}, Y_m)) = \psi(Y_r, Y_m), \quad 1 \leq i \leq r - 1.
\]

That is, if \( p > n - 2m \) the resulting system is

\[
\begin{align*}
\frac{1}{2}a_1 - a_2C_{m-2}^1 + a_3C_{m-3}^2 + \cdots + a_l(-1)^{l-1}C_{m-1}^{l-1} & = \frac{1}{2}a_r + \cdots + a_l(-1)^{l-r}C_{m-1}^{l-r} \\
\frac{1}{2}a_2 - a_3C_{m-3}^1 + \cdots + a_l(-1)^{l-1}C_{m-2}^{l-2} & = \frac{1}{2}a_r + \cdots + a_l(-1)^{l-r}C_{m-2}^{l-r} \\
& \vdots \\
\frac{1}{2}a_{r-1} + \cdots + a_l(-1)^{l-r+1}C_{m-1}^{l-r+1} & = \frac{1}{2}a_r + \cdots + a_l(-1)^{l-r}C_{m-1}^{l-r},
\end{align*}
\]

and if \( p = n - 2m \), then \( r = m \), and thus the coefficient of \( a_r = a_m \) will be 1 instead of \( \frac{1}{2} \). In this case the resulting system in \( a_1, \ldots, a_{m-1} \) can be expressed by the following matrix

\[
\begin{pmatrix}
\frac{1}{2} & -C_{m-2}^1 & C_{m-3}^2 & \cdots & (-1)^{m-2}C_{m-1}^{m-2} \\
0 & \frac{1}{2} & -C_{m-3}^1 & \cdots & (-1)^{m-3}C_{m-2}^{m-3} \\
& \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \frac{1}{2}
\end{pmatrix}
\]

The basis of the set of solutions of this last system can be obtained by induction in the following way \( a_{m-1} = 2; \ a_{m-2} = 2 + 2C_{1}^1(a_{m-1}); \ a_{m-3} = 2 + 2C_{2}^1(a_{m-2}) - 2C_{1}^2(a_{m-1}) \), that is

\[
a_i = 2 + 2 \sum_{j=1}^{m-i-1} (-1)^{i+j}C_{m-i-j}^{i}(a_{i+j}), \quad i = m - 1, m - 2, \ldots, 1.
\]

The recursion formula is easy to apply for specific values of \( m \) and \( n \). Developing the recursion formula we obtain an explicit expression for \( a_i \):

\[
a_i = 2 + \sum_{q=2}^{m-i} 2^q \prod_{k=1}^{q-1} \sum_{j_k=1}^{m-i-1-\sum_{l=1}^{k-1} j_l} (-1)^{j_k + 1}C_{m-i-\sum_{l=1}^{k} j_l}^{j_k}, \quad 1 \leq i \leq m - 1
\]

For these values of \( a_i \) we obtain the following cocycle that corresponds to the value 1 of the coefficient of \( \psi_{m,n} \). Thus, we shall call it \( \psi_{m,n} \)

\[
\psi_{m,n} = a_1\psi_{1,n-2m+2} + a_2\psi_{2,n-2m+4} + \cdots + a_{m-1}\psi_{m-1,n-2} + \psi_{m,n}.
\]

In the case \( p > n - 2m \) the system admits \( l - r + 1 = \left\lceil \frac{n-p}{2} \right\rceil - n + m + p + 1 \) linearly independent solutions that correspond to the following possibilities for the vector \((a_r, a_{r+1}, \ldots, a_l)\):

\[
(1, 0, \ldots, 0), \ (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)
\]

• If \((a_r, a_{r+1}, \ldots, a_l) = (1, 0, \ldots, 0)\), the resulting system in \( a_1, \ldots, a_{r-1} \) is given by the matrix
In the case \( n \) is even, \( a_{r-1} = 1 + 2C_{m-r}; \ a_{r-2} = 1 - 2C_{m-r} + 2C_{m-r+1}(a_{r-1})..., \) that is
\[
a_i = 1 + 2(-1)^{r-i+1}C_{m-r}^{r-i} + 2 \sum_{j=1}^{r-i-1} (-1)^{j+1}C_{m-i-j}^{r}(a_{i+j}), \quad i = r - 1, r - 2, \ldots, 1.
\]

Developing the recursion formula we obtain the explicit expression for \( a_i \) (enunciated in the Theorem) with \( 1 \leq i \leq r - 1 \). Hence for these values of \( a_i \) (\( 1 \leq i \leq r - 1 \)) and \( a_r = 1, a_{r+1} = \cdots = a_1 = 0 \), we obtain the cocycle \( \bar{\psi}_{r,p+2r} \) given by
\[
\bar{\psi}_{r,p+2r} = a_1\psi_{1,p+2} + a_2\psi_{2,p+4} + \cdots + a_{r-1}\psi_{r-1,p+2(r-1)} + \psi_{r,p+2r}.
\]

- In the remaining cases, it can be seen that for each \( h \), \( r + 1 \leq h \leq t \), such that \( a_{h+1} = 1 \) and \( a_k = 0 \) with \( r \leq k \leq l \) and \( k \neq h \), we obtain the cocycle \( \bar{\psi}_{h,p+2h} \),
\[
\bar{\psi}_{h,p+2h} = a_1^h\psi_{1,p+2} + a_2^h\psi_{2,p+4} + \cdots + a_{r-1}^h\psi_{r-1,p+2(r-1)} + \psi_{h,p+2h}
\]
for \( 1 \leq i \leq r - 1 \), and with \( a_i^h \) as in the enunciated of the Theorem.

Thus, for each \( p \) all the cocycles described above, and so all the cocycles described in the theorem, are linearly independent. It remains to count them.

In the case \( n - m - 1 \leq p \leq n - 2 \) the cocycles obtained are all of the form \( \psi_{k,s} \) (see Proposition 4.1). In particular there are \( \lfloor \frac{n-p}{2} \rfloor \) cocycles for each \( p \), thus in total for this case we have
\[
\sum_{p=n-m-1}^{n-2} \lfloor \frac{n-p}{2} \rfloor = \begin{cases} 
\frac{m^2 + 2m}{4} & \text{if } m \text{ is even} \\
\frac{m^2 + 2m + 1}{4} & \text{if } m \text{ is odd}
\end{cases}
\]

In the case \( n - 2m \leq p \leq n - m - 2 \), we have
\[
\sum_{p=n-2m}^{n-m-2} \lfloor \frac{n-p}{2} \rfloor - n + m + p + 1 = \begin{cases} 
\frac{m^2}{4} & \text{if } m \text{ is even} \\
\frac{m^2 - 1}{4} & \text{if } m \text{ is odd}
\end{cases}
\]

If we sum, we obtain in total \( \frac{m(m+1)}{2} \) cocycles of \( Z^2(L; L) \cap \text{Hom}(S^2L_\gamma, L_e) \) that are linearly independent. As the dimension of the space is \( \frac{m(m+1)}{2} \) the proof is finished. \( \square \)

**Corollary.** For each \( \psi \) of the theorem above, with \( n \geq 2m - 1 \), we will have that
\[
L^{n,m,p,t} + \psi
\]
is a filiform \((\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)\)-color Lie superalgebra

6. Basis of \(H_5\) for \(n < 2m - 1\)

Now, for \(n < 2m - 1\) the construction of a cocycle basis is more complicated. In these cases \(p\) can be less than \(-1\), for example if \(p = -2\) there are not \(\lfloor \frac{n-p}{2} \rfloor\) mappings \(\psi_{k,s}\) with weight \(p\) because of \(\psi_{1,-2+2}\) does not exist. Thus, in general, for each \(p\) the mappings \(\psi_{k,s}\) with weight \(p\) there will be \(\psi_{l,p+2l}\) with \(1 \leq p+2l \leq n\), that is

\[\psi_{l,p+2l}, \text{ with } \max \{1, \lfloor \frac{l-p}{2} \rfloor \} \leq l \leq \lfloor \frac{n-p}{2} \rfloor\]

So, for \(n < 2m - 1\) we have that \(\max \{\frac{n-2m-1}{2}, n-2m\} = \frac{n-2m-1}{2}\) and thus it can be considered in two cases:

**Lemma 6.1.** Let \(n\) and \(m\) be such that \(n < 2m - 1\), and \(C = \bigcup_p C_p\), where \(C_p\) represents the set of cocycles of weight \(p\). To obtain basis cocycles of \(C_p\) we will have firstly to distinguish two cases:

- **Case 1.** If \(p\) verifies \(n - m - 1 \leq p \leq n - 2\). For each one of these \(p\) we obtain \(\lfloor \frac{n-p}{2} \rfloor - \max \{1, \lfloor \frac{l-p}{2} \rfloor \} + 1\) cocycles of weight \(p\) in the basis, that is
  
  \[\psi_{l,p+2l}, \text{ with } \max \{1, \lfloor \frac{l-p}{2} \rfloor \} \leq l \leq \lfloor \frac{n-p}{2} \rfloor\]

- **Case 2.** If \(p\) verifies \(\lfloor \frac{n-2m-1}{2} \rfloor \leq p \leq n - m - 2\). For each one of these \(p\) we obtained \(\lfloor \frac{n-p}{2} \rfloor - \max \{1, \lfloor \frac{l-p}{2} \rfloor \} + 1 = (n - m - 1 - p)\) of weight \(p\) in the basis. \(\lfloor \frac{n-p}{2} \rfloor - \max \{1, \lfloor \frac{l-p}{2} \rfloor \} + 1\) corresponds to the number of mappings \(\psi_{k,s}\) with weight \(p\), and \(n - m - 1 - p\) corresponds to the number of equations that a linear combination \(\psi = \sum_{s-2k=p} a_k \psi_{k,s}\) has to satisfy to be a cocycle:

\[(adX_0)^{-1}(\psi(Y_1, Y_m)) = \ldots (adX_0)(\psi(Y_{r-1}, Y_m)) = \psi(Y_r, Y_m)\]

with \(r = n - m - p\).

This last case can be subdivided into 4 subcases depending on the possibilities of \(n\) and \(m\):

- **Case 2.1** If \(n\) is even. In this case the interval for \(p\) is \(\frac{n-2m}{2} \leq p \leq n - m - 2\).
- **Case 2.2** If \(n \equiv 1\)(mod 4) and \(m\) is even. In this case the interval for \(p\) is \(\frac{n-2m+1}{2} \leq p \leq n - m - 2\).
- **Case 2.3** If \(n \equiv 3\)(mod 4) and \(m\) is odd. The interval for \(p\) is \(\frac{n-2m+1}{2} \leq p \leq n - m - 2\).
- **Case 2.4** If \(n \equiv 1\)(mod 4) and \(m\) is odd. The interval for \(p\) is \(\frac{n-2m-1}{2} \leq p \leq n - m - 2\).
- **Case 2.5** If \(n \equiv 3\)(mod 4) and \(m\) is even. The interval for \(p\) is \(\frac{n-2m-1}{2} \leq p \leq n - m - 2\).

**Proof.** If \(n \equiv 1\)(mod 4) and \(m\) is even, although \(\lfloor \frac{n-2m-1}{2} \rfloor = \frac{n-2m-1}{2}\) this is not the correct lower bound for \(p\) because if \(p = \frac{n-2m-1}{2}\) then the number of cocycles with this weight will be 0. In fact, if we put \(n = 4k + 1\) and \(m = 2k'\) then
Let $n = 2k$ with $\lfloor \frac{n}{2} \rfloor \leq k \leq m - 1$ will be given by the following cocycles
$$\psi_{l,p+2l}, \quad \text{with} \quad 1 \leq l \leq \lfloor \frac{n-p}{2} \rfloor$$

- $n = 2k$ with $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ will be given by the following cocycles
$$\psi_{l,p+2l}, \quad \text{with} \quad \lceil \frac{1-p}{2} \rceil \leq l \leq \lfloor \frac{n-p}{2} \rfloor$$

Proof. In general (as we have said in Case 1.), for each $p$ such that $n-m-1 \leq p \leq n-2$ we obtain $\lfloor \frac{n-p}{2} \rfloor - \max \{1, \lceil \frac{1-p}{2} \rceil \} + 1$ cocycles of weight $p$ in the basis, that is
$$\psi_{l,p+2l}, \quad \text{with} \quad \max \{1, \lceil \frac{1-p}{2} \rceil \} \leq l \leq \lfloor \frac{n-p}{2} \rfloor$$

In our case $n = 2k$ with $\lfloor \frac{n}{2} \rfloor \leq k \leq m - 1$ we have that
$$\lfloor \frac{1-p}{2} \rfloor \leq \lceil \frac{1-2k+m+1}{2} \rceil \leq \lfloor 1 + \frac{m}{2} - \lfloor \frac{m}{2} \rfloor \rfloor = 1$$

If $k$ is such that $1 \leq k \leq \lfloor \frac{m}{2} \rfloor - 1$, then as $n-m-1 \leq p \leq -2$ and $n = 2k$ we have that $2k - m - 1 \leq p \leq -2$. So $\lceil \frac{1-p}{2} \rceil \geq \lceil \frac{1+2}{2} \rceil = 2$ and $\max \{1, \lfloor \frac{1-p}{2} \rfloor \} = \lceil \frac{1-p}{2} \rceil$ which concludes the proof.

Proposition 6.3. A basis of $C_p$ with $-1 \leq p \leq n-2$ for the dimensions $n = 2k$ with $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ will be given by the following cocycles
$$\psi_{l,p+2l}, \quad \text{with} \quad 1 \leq l \leq \lfloor \frac{n-p}{2} \rfloor$$

Proof. As $-1 \leq p$, then $\lfloor \frac{1-p}{2} \rfloor \leq 1$. So, $\max \{1, \lfloor \frac{1-p}{2} \rfloor \} = 1$. □

Proposition 6.4. The dimension of the space of cocycles $\bigcup_{p=n-m-1}^{n-2} C_p$ with $n = 2k$, $1 \leq k \leq m - 1$, is as follows:
\[
\begin{cases}
\frac{m^2 + 2m}{4} & \text{if } m \text{ is even and } \left\lfloor \frac{m}{2} \right\rfloor \leq k \leq m - 1 \\
\frac{m^2 + 2m + 1}{4} & \text{if } m \text{ is odd and } \left\lfloor \frac{m}{2} \right\rfloor \leq k \leq m - 1 \\
\frac{2n + 2nm - n^2}{4} & \text{if } 1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor - 1
\end{cases}
\]

Proof. If \( \left\lfloor \frac{m}{2} \right\rfloor \leq k \leq m - 1 \), there are \( \frac{m - p}{2} \) cocycles for each \( p \), thus in total for this case we have

\[
\sum_{p=n-m-1}^{n-2} \left\lfloor \frac{n-p}{2} \right\rfloor = \begin{cases}
\frac{m^2 + 2m}{4} & \text{if } m \text{ is even} \\
\frac{m^2 + 2m + 1}{4} & \text{if } m \text{ is odd}
\end{cases}
\]

If \( 1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor - 1 \), then we are going to do separately the cases, when \( m \) is even and \( m \) is odd. When \( m \) is even the number of cocycles can be resumed as follows

\[
\sum_{p=n-m-1}^{n-2} \left\lfloor \frac{n-p}{2} \right\rfloor - 2[1+2+\cdots+(\frac{m}{2}-k)] = \frac{m^2 + 2m}{4} - 2[1+2+\cdots+(\frac{m}{2})] = \frac{2mn - n^2 + 2n}{4}
\]

And for an odd \( m \) we have that

\[
\sum_{p=n-m-1}^{n-2} \left\lfloor \frac{n-p}{2} \right\rfloor - \left( \left\lfloor \frac{m}{2} \right\rfloor - k \right) - 2[1 + 2 + \cdots + (\left\lfloor \frac{m}{2} \right\rfloor - k - 1)] =
\]

\[
\frac{m^2 + 2m + 1}{4} - \left( \frac{m + 1 - n}{2} \right) - \left( \left( \frac{m - n - 1}{2} \right) \left( \frac{m - n + 1}{2} \right) \right) = \frac{2mn - n^2 + 2n}{4}
\]

\( \Box \)

**Theorem 6.4.1.** A basis of the space of cocycles \( C = \bigcup_{p} C_{p} \), with \( n = 2m - 2 \), will be given by the following \( \frac{n}{2}(4mn - n^2 + 2n) \) cocycles

- For each \( p \) such that \( n - m - 1 \leq p \leq n - 2 \), there are \( \left\lfloor \frac{n-p}{2} \right\rfloor \) cocycles of weight \( p \) in the basis, that is

\[
\psi_{1,p+2}, \psi_{2,p+4}, \ldots, \psi_{\left\lfloor \frac{n-p}{2} \right\rfloor, p+2\left\lfloor \frac{n-p}{2} \right\rfloor}
\]

- For each \( p \) such that \( \frac{n-2m}{2} \leq p \leq n - m - 2 \), there are \( l-r+1 = \left\lfloor \frac{n-p}{2} \right\rfloor - (n-m-p) + 1 \) cocycles of weight \( p \) in the basis, that is

\[
\psi_{r,p+2r} = a_{1} \psi_{1,p+2} + a_{2} \psi_{2,p+4} + \cdots + a_{r-1} \psi_{r-1,p+2(r-1)} + \psi_{r,p+2r}
\]

and

\[
\psi_{h,p+2h} = a_{1}^{h} \psi_{1,p+2} + a_{2}^{h} \psi_{2,p+4} + \cdots + a_{r-1}^{h} \psi_{r-1,p+2(r-1)} + \psi_{h,p+2h}
\]

for \( r < h \leq l, 1 \leq i \leq r - 1 \). Here \( a_{i} \) and \( a_{i}^{h} \) have the same expression as in Main Theorem 2.
Proposition 6.2 with \( k \).

So, we have in total for this case we have

\[ \text{Proof.} \]

For each \( r \) with \( (1,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,\ldots,0,1) \)

Corollary. For each \( \psi \) of the theorem above we will have that

\[ L^{2m-2,m,p,t+\psi} \]

is a filiform \((\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)\)-color Lie superalgebra

If we try to compute a basis of the space of cocycles \( Z^2(L;L) \cap \text{Hom}(S^2L_n, L_c) \), with even \( n \) and \( n < 2m - 2 \) and weight \( p \) such that \( \frac{n-2m}{2} \leq p \leq n - m - 2 \), it is necessary to distinguish two cases: \(-1 \leq p \leq n - m - 2\) and \( \frac{n-2m}{2} \leq p \leq -2\).
Proposition 6.5. A basis of $C_p$ with $-1 \leq p \leq n - m - 2$, for the dimensions $n = 2k$, $1 \leq k \leq m - 2$ will be given by the following cocycles where $r = n - m - p$

$$\psi_{r,p+2r} = a_1 \psi_{1,p+2} + a_2 \psi_{2,p+4} + \cdots + a_{r-1} \psi_{r-1,p+2(r-1)} + \psi_{r,p+2r}$$

and

$$\psi_{h,p+2h} = a_1^h \psi_{1,p+2} + a_2^h \psi_{2,p+4} + \cdots + a_{r-1}^h \psi_{r-1,p+2(r-1)} + \psi_{h,p+2h}$$

for $r < h \leq l$, and $a_i$ and $a_i^j$ have the same expression as in Main Theorem 2.

Proof. If $-1 \leq p \leq n - m - 2$, for each $p$ there will be $\lfloor \frac{n-p}{2} \rfloor - (n - m - 1 - p)$ cocycles of weight $p$ in the basis. $\lfloor \frac{n-p}{2} \rfloor$ corresponds to the number of mappings $\psi_{k,s}$ with weight $p$: $\psi_{l,p+2l}$ with $1 \leq l \leq \lfloor \frac{n-p}{2} \rfloor$. $n - m - 1 - p$ corresponds to the number of equations that a linear combination $\psi = \sum_{s=-2k=p} a_k \psi_{k,s}$ has to satisfy to be a cocycle. Thus, the systems that result are similar to ones of the Theorem 6.4.1 case $\frac{n-2m}{2} \leq p \leq n - m - 2$.

If $\frac{n-2m}{2} \leq p \leq -2$, for each $p$ there will be $\lfloor \frac{n-p}{2} \rfloor - \lfloor \frac{n-p+2}{2} \rfloor + 1 - (n - m - 1 - p)$ cocycles of weight $p$ in the basis. $\lfloor \frac{n-p}{2} \rfloor - \lfloor \frac{n-p+2}{2} \rfloor + 1$ corresponds to the number of mappings $\psi_{k,s}$ with weight $p$: $\psi_{l,p+2l}$ with $1 \leq l \leq \lfloor \frac{n-p}{2} \rfloor$. $n - m - 1 - p$ corresponds to the number of equations that a linear combination $\psi = \sum_{s=-2k=p} a_k \psi_{k,s}$ has to satisfy to be a cocycle:

$$(adX_0)^{-1}(\psi(Y_1,Y_m)) = \ldots (adX_0)(\psi(Y_{r-1},Y_m)) = \psi(Y_r,Y_m)$$

with $r = n - m - p$. The main problem is that the matrices of the resulting systems are not reduced, so it will be necessary to fix concrete dimensions to solve the problem. Thus, we have the following result

Proposition 6.6. For the following concrete dimensions a basis of $C = \bigcup_p C_p$ is constituted by:

- If $n = 2$ and $m = 3$, then we have $\{\psi_{1,1}, \psi_{1,2}, \psi_{2,2}\}$ as a basis of $C$.

- If $n = 2$ and $m = 4$, then we have $\{\psi_{1,1}, \psi_{1,2}, \psi_{2,1}, \psi_{2,2}\}$ as a basis of $C$.

- If $n = 4$ and $m = 4$, then we have $\{\psi_{1,1}, \psi_{1,2}, \psi_{1,3}, \psi_{1,4}, \psi_{2,3}, \psi_{2,4}, \psi_{2,2}\}$ as a basis of $C$, with $\overline{\psi_{2,2}} = \psi_{2,2} + 2\psi_{3,4}$.

- If $n = 4$ and $m = 5$, then we have $\{\psi_{1,1}, \psi_{1,2}, \psi_{1,3}, \psi_{1,4}, \psi_{2,2}, \psi_{2,3}, \psi_{2,4}, \psi_{3,4}, \psi_{2,1}\}$ as a basis of $C$, with $\overline{\psi_{2,1}} = \psi_{2,1} + \psi_{3,3}$.

- If $n = 6$ and $m = 5$, then we have $\{\psi_{1,2}, \psi_{1,3}, \psi_{1,4}, \psi_{1,5}, \psi_{1,6}, \psi_{2,4}, \psi_{2,5}, \psi_{2,6}, \psi_{3,6}, \psi_{1,1}, \psi_{2,3}, \psi_{2,4}, \psi_{2,5}\}$ as a basis of $C$, with $\overline{\psi_{1,1}} = \psi_{1,1} - \frac{1}{3}\psi_{3,5}$, $\overline{\psi_{2,3}} = \psi_{2,3} + \psi_{3,5}$, $\overline{\psi_{4,6}} = \frac{2}{3}\psi_{2,2} + \frac{1}{3}\psi_{3,4} + \psi_{4,6}$.

- If $n = 6$ and $m = 6$, then we have $\{\psi_{1,1}, \psi_{1,2}, \psi_{1,3}, \psi_{1,4}, \psi_{1,5}, \psi_{1,6}, \psi_{2,3}, \psi_{2,4}, \psi_{2,5}, \psi_{2,6}, \psi_{3,5}, \psi_{3,6}, \overline{\psi_{2,2}}, \overline{\psi_{3,4}}, \overline{\psi_{4,5}}\}$ as a basis of $C$, with $\overline{\psi_{2,2}} = \psi_{2,2} - 3\psi_{3,4}$, $\overline{\psi_{3,4}} = \psi_{3,4} + \frac{2}{3}\psi_{4,6}$, $\overline{\psi_{4,5}} = \frac{5}{3}\psi_{2,1} + \frac{4}{3}\psi_{3,3} + \psi_{4,5}$.
• If $n = 8$ and $m = 6$, then we have
\[ \{ \psi_{1,3}, \psi_{1,4}, \psi_{1,5}, \psi_{1,6}, \psi_{1,7}, \psi_{1,8}, \psi_{2,5}, \psi_{2,6}, \psi_{2,7}, \psi_{2,8}, \psi_{3,7}, \psi_{3,8}, \psi_{1,2}, \psi_{2,4}, \psi_{3,5}, \psi_{3,6}, \psi_{4,7}, \psi_{5,8} \} \] as a basis of $C$, with $\psi_{1,2} = \psi_{1,2} + \frac{1}{2} \psi_{4,8}$, $\psi_{2,4} = \psi_{2,4} - 3 \psi_{4,8}$, $\psi_{3,5} = 21 \psi_{1,1} + 5 \psi_{2,3} + \psi_{3,5}$, $\psi_{3,6} = \psi_{3,6} + \frac{9}{2} \psi_{4,8}$, $\psi_{4,7} = -28 \psi_{1,1} - 5 \psi_{2,3} + 4 \psi_{4,7}$, $\psi_{5,8} = \frac{5}{4} \psi_{2,2} + \psi_{3,4} + \frac{3}{4} \psi_{4,6} + \psi_{5,8}$.

• If $n = 8$ and $m = 7$, then we have
\[ \{ \psi_{1,2}, \psi_{1,3}, \psi_{1,4}, \psi_{1,5}, \psi_{1,6}, \psi_{1,7}, \psi_{1,8}, \psi_{2,4}, \psi_{2,5}, \psi_{2,6}, \psi_{2,7}, \psi_{2,8}, \psi_{3,6}, \psi_{3,7}, \psi_{3,8}, \psi_{4,8}, \psi_{1,1}, \psi_{2,3}, \psi_{3,5}, \psi_{4,6}, \psi_{5,7}, \psi_{5,8} \} \] as a basis of $C$, with $\psi_{1,1} = \psi_{1,1} + \frac{1}{2} \psi_{4,7}$, $\psi_{2,3} = \psi_{2,3} - \psi_{4,7}$, $\psi_{3,5} = \psi_{3,5} + 2 \psi_{4,7}$, $\psi_{4,6} = 3 \psi_{2,2} + 2 \psi_{3,4} + \psi_{4,6}$, $\psi_{5,7} = \frac{7}{2} \psi_{2,1} + \frac{5}{2} \psi_{3,3} + \frac{5}{2} \psi_{4,5} + \psi_{5,7}$, $\psi_{5,8} = -\psi_{2,2} - \frac{1}{2} \psi_{3,4} + \psi_{5,8}$.

Proof. We are going to summarize along 6 steps of the procedure to obtain a cocycle basis for the given concrete dimensions $n$ and $m$ ($n < 2m - 1$ and $n = 2k$):

Step 1. If $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$, then go to Step 2. If not go to step 4.

Step 2. From $p = -1$ to $n - 2$ calculate all the basis cocycles of the form
\[ \psi_{l,p+2l}, \quad \text{with} \quad 1 \leq l \leq \lfloor \frac{n-p}{2} \rfloor \]
Go to Step 3.

Step 3. From $p = n - m - 1$ to $-2$ calculate all the basis cocycles of the form
\[ \psi_{l,p+2l}, \quad \text{with} \quad \lceil \frac{n-p}{2} \rceil \leq l \leq \lfloor \frac{n-p}{2} \rfloor \]
Go to Step 5.

Step 4. From $p = n - m - 1$ to $n - 2$ calculate all the basis cocycles of the form
\[ \psi_{l,p+2l}, \quad \text{with} \quad 1 \leq l \leq \lfloor \frac{n-p}{2} \rfloor \]
Go to Step 5.

Step 5. From $p = -1$ to $n - m - 2$ calculate all the basis cocycles of the form
\[ \psi_{r,p+2r} = a_1 \psi_{r+1,p+2} + a_2 \psi_{r+2,p+4} + \cdots + a_{r-1} \psi_{r-1,p+2(r-1)} + \psi_{r,p+2r} \]
where $r = n - m - p$, and
\[ \psi_{h,p+2h} = a_1^h \psi_{h+1,p+2} + a_2^h \psi_{h+2,p+4} + \cdots + a_{r-1}^h \psi_{r-1,p+2(r-1)} + \psi_{h,p+2h} \]
for $r < h \leq l$, and $a_i$ and $a_i^h$ have the same expression as in Main Theorem 2.

Go to Step 6.

Step 6. From $p = \frac{n-2m}{2}$ to $-2$ calculate all the mappings $\psi_{l,p+2l}$ with $\lceil \frac{n-p}{2} \rceil \leq l \leq \lfloor \frac{n-p}{2} \rfloor$. Put the mappings into a linear combination $\psi = \sum_{k=-2k=p} a_k \psi_{k,s}$ and impose that have to verify the equations:
\[ (adX_0)^{r-1}(\psi(Y_1,Y_m)) = \ldots (adX_0)(\psi(Y_{r-1},Y_m)) = \psi(Y_r,Y_m) \]
with $r = n - m - p$. From solving the above equations we obtain a basis of solutions that constitute the basis cocycles that rest.

\[ \square \]

Corollary. For each $\psi$ of the proposition above, with $n$ and $m$ given in each case, we will have that
\[ L^{n,m,p,t} + \psi \]
is a filiform $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)$-color Lie superalgebra
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