Coriolis terms in Skyrmion Quantization

J. I. Rawlinson

Department of Applied Mathematics and Theoretical Physics, 
University of Cambridge, 
Wilberforce Road, Cambridge CB3 0WA, U.K.

Abstract

We consider the problem of quantizing a Skyrmion which is allowed to vibrate, rotate and isorotate. Previous approaches have neglected the interactions between vibrations and zero modes (analogous to so-called Coriolis terms in the molecular physics literature). A new formalism incorporating these interactions is introduced, inspired by a principal bundle approach to deformable-body dynamics. We quantize the $B=4$ and $B=7$ Skyrmions and compare the results to observed nuclear properties of Helium-4 and the Lithium-7/Beryllium-7 isospin doublet.

1 Introduction

In the Skyrme model approach to nuclear physics, atomic nuclei appear as topological solitons in a nonlinear field theory of pions. These topological solitons are known as Skyrmions. Given a Skyrmion, one may use insights from the field theory dynamics to identify a small number of collective coordinates which are relevant at low energies. The collective coordinates chosen will typically include both rotations and isorotations of the Skyrmion together with other, shape-deforming, degrees of freedom: the nucleus is viewed as a deformable body which is free to rotate in space as well as isorotate in isospace. These degrees of freedom of the Skyrmion are then quantized, hopefully giving a reasonable description of the corresponding nucleus.

Naively one may hope to separate out the zero modes (rotations and isorotations, corresponding to the action of the symmetry group $SU(2)_{\text{spin}} \times SU(2)_{\text{isospin}}$ on the Skyrmion) when quantizing this system. In some cases a complete factorisation is possible, but generally one has to live with interactions between zero modes and shape-deforming degrees of freedom.

\footnotemark[1]\email{jir25@damtp.cam.ac.uk}

\footnotetext[1]{email: jir25@damtp.cam.ac.uk}
freedom. This kind of interaction is already understood in the molecular physics literature, where Coriolis effects are known to play an important role in rovibrational spectra. Our situation is a slight generalisation: a molecule can rotate and vibrate, but a Skyrmion can additionally isorotate. Mathematically it is not difficult to incorporate isorotations, provided we have a clear understanding of the usual Coriolis effects. The framework introduced in [1] (amongst others), which formulates the problem in terms of principal bundles, is our preferred approach and easily extends to include isorotations.

In section 2 we set up the general formalism before exploring various applications in the following sections. In section 3 we consider small vibrations of a Skyrmion and show how the problem simplifies in this case. Using these ideas, we compute the quantum spectrum of a vibrating and rotating $B = 4$ Skyrmion (with cubic symmetry) in section 4, finding good agreement with the observed excited states of the $\alpha$-particle. Finally we study the lowest-frequency vibration of the $B = 7$ Skyrmion, leading to a suggestion that the surprisingly low energy of the Lithium-7/Beryllium-7 spin $\frac{3}{2}$ ground state may be in part due to an isospin Coriolis effect.

## 2 Quantization of Skyrmions

### Skyrme Lagrangian

Pion fields $\pi(x, t)$ are combined into an $SU(2)$-valued field $U : \mathbb{R}^4 \rightarrow SU(2)$

$$U(x, t) = \sigma(x, t)\mathbb{1}_2 + i\pi(x, t) \cdot \tau$$  \hspace{1cm} (1)

and the Lagrangian defining the classical field theory is (in Skyrme units)

$$L = \int d^3x \left[ \frac{1}{2} \text{Tr}(L_\mu L^\mu) + \frac{1}{16} \text{Tr}([L_\mu, L_\nu][L^\mu, L^\nu]) + m^2 \text{Tr}(U - \mathbb{1}_2) \right]$$  \hspace{1cm} (2)

with $L_\mu = U^\dagger \partial_\mu U$. Isospin symmetry corresponds to transformations $U \rightarrow A^\dagger UA$ for any constant matrix $A \in SU(2)$. Static soliton solutions are known as Skyrmions. They are classified by a topological degree $B \in \mathbb{Z}$ which is identified with the baryon number of the nucleus.
Restricted configuration space

Given a Skyrmion, we are often interested in constructing a restricted configuration space \( \mathcal{C} \) of deformations. \( \mathcal{C} \) should in principle capture the field configurations which are relevant at low energies. A natural first choice is given by the rigid-body approximation: only rotations and isorotations of the Skyrmion are included. One then quantizes geodesic motion on the corresponding submanifold \( \mathcal{C} \simeq SU(2) \times SU(2) \) with respect to the induced metric coming from the full field theory. The resulting problem is equivalent to a (generalised) rigid rotor, with quantum states classified by spin and isospin. Comparisons to nuclear data have been promising in many cases, but recent work suggests that to model real nuclei it is necessary to take additional deformations of the Skyrmion into account: we need to include more than just the zero modes. One can study vibrations of Skyrmions and find their normal modes \[2\]. Then a natural next step beyond rigid-body quantization is to include those modes with the lowest non-zero frequency (the first \( N \) of them, say). Within a harmonic approximation we can think of the resulting configuration space as \( \mathcal{C} \simeq SU(2) \times SU(2) \times \mathbb{R}^N \). More generally we may be interested in larger collective motions (not just small vibrations). Recent work on Carbon-12 \[3\] involved a configuration space of the form \( \mathcal{C} \simeq SU(2) \times SU(2) \times \Gamma \) where \( \Gamma \) has the structure of a graph, while in \[4\] Oxygen-16 was modelled by motion on \( \mathcal{C} \simeq SU(2) \times SU(2) \times M \) with \( M \) a quotient of a six-punctured sphere.

All of the examples given so far have the product structure \( \mathcal{C} \simeq SU(2) \times SU(2) \times \mathcal{C}_{\text{shapes}} \), but one could imagine a restricted configuration space \( \mathcal{C} \) which includes zero modes (generated by \( SU(2) \times SU(2) \)) but is not globally a product. \( \mathcal{C} \) should really be thought of as a principal \( SU(2) \times SU(2) \)-bundle, with rotations and isorotations generating the fibres. Locally it will be a product but this might not be true globally.

Quantum Hamiltonian

For clarity, we will at first ignore isorotations. Our configuration space comes with an action of the rotational symmetry group \( SU(2) \), and we can think of the configuration space \( \mathcal{C} \) as a principal \( SU(2) \)-bundle \( \pi : \mathcal{C} \to \mathcal{C}_{\text{shapes}} \) with rotations generating the fibres. For every point in shape space \( \mathcal{C}_{\text{shapes}} \) there is an open neighbourhood \( V \subseteq \mathcal{C}_{\text{shapes}} \) containing the point such that \( \pi^{-1}(V) \) can be identified with \( SU(2) \times V \) (one should think of this as making a particular choice of reference orientation for each fibre). Working locally, we think of a point in configuration space as a pair \((\theta_i, s_j)\) with \( s_j \), the coordinates on \( V \subseteq \mathcal{C}_{\text{shapes}} \), specifying the shape of the field configuration and with \( \theta_i \) Euler angles parametrising its orientation in
space. Our configuration space inherits a metric $\tilde{g}$ from the full Skyrme field theory. $SU(2)$ symmetry implies that this inherited metric must be symmetric under (left) translations in the $SU(2)$ factor. Thus the most general form of the inherited metric is

$$\tilde{g} = (\sigma \ ds_i) \left( \frac{\Lambda}{\mathbf{A}_i^T \Lambda \ g_{ij} + \mathbf{A}_i \cdot \Lambda \cdot \mathbf{A}_j} \right) \left( \sigma \ ds_j \right)$$

(3)

where the $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are left-invariant one forms on $SU(2)$ and where $\Lambda, \mathbf{A}_i$ and $g_{ij}$ only depend on the shape coordinates $s_i$. Note that we have suppressed the index on $\sigma$, that $\Lambda$ is a $3 \times 3$ matrix, and that a bold font is used to indicate that $\mathbf{A}_i$ is a 3-component vector for each $i$. The suggestive notation $\mathbf{A}_j$ has been used as it will turn out that this corresponds to a particular connection on the principal bundle $\mathcal{C}$. We now construct a quantum Hamiltonian by computing the Laplace-Beltrami operator on $\mathcal{C}$. Recall that the Laplace-Beltrami operator $\Delta$ corresponding to a metric $G$ has an expression in local coordinates

$$\Delta f = \frac{1}{\sqrt{|G|}} \partial_i \left( \sqrt{|G|} G^{ij} \partial_j f \right).$$

(4)

For the calculation of $\Delta$ it is useful to note that $\tilde{g}$ can be rewritten as

$$\tilde{g} = \left( d\theta \ ds_i \right) G \left( \frac{d\theta}{ds_j} \right)$$

(5)

(here we closely follow [1]) where

$$G = \begin{pmatrix} \lambda^T & 0 \\ \mathbf{A}_i^T & I \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & g_{ij} \end{pmatrix} \begin{pmatrix} \lambda \\ \mathbf{A}_j \end{pmatrix}. $$

(6)

$\lambda$ is the matrix which captures the relationship between the left-invariant one forms $\sigma$ and the (coordinate) one forms $d\theta$. Thus we can compute $|G| = |\lambda|^2 |\Lambda| |g_{ij}|$ and then use the expression (4) to obtain a quantum Hamiltonian

$$\mathcal{H} = \frac{1}{2} \mathbf{L} \cdot \Lambda^{-1} \cdot \mathbf{L} + \frac{1}{2} (p_i - \mathbf{L} \cdot \mathbf{A}_i) g_{ij}^{-1} (p_j - \mathbf{L} \cdot \mathbf{A}_j) + V_2(s) + V(s)$$

(7)

where we have included both the kinetic term $-\hbar^2 \Delta$ and a potential $V(s)$ on configuration space. $\mathbf{L}$ is the (usual) body-fixed angular momentum operator familiar from rigid-body theory ($\mathbf{J}$ will denote the space-fixed angular momentum operator) and $p_i = -i\hbar \frac{\partial}{\partial s_i}$. Also
appearing in the kinetic term is

\[ V_2(s) = \frac{\hbar^2}{2} \left( \frac{1}{|A| |g_{ij}|} \right)^{\frac{1}{4}} \partial_i \left( g_{ij}^{-1} \partial_j \left( \frac{1}{|A| |g_{ij}|^{\frac{1}{4}}} \right) \right). \]  \\

(8)

**Effective problem on \( C_{\text{shapes}} \)**

Exploiting rotational symmetry, we can classify the energy eigenstates of (7) by \( J \) (where \( J (J + 1) \) is the eigenvalue of \( J^2 \) in the usual way) and \( J_3 \). Recall from rigid-body theory that a complete set of commuting operators for the rotational part of the problem is given by \( J^2, J_3, L_3 \) and so within a particular \((J, J_3)\) sector we can expand the total wavefunction

\[ \Psi = \sum_{L_3 = -J}^{+J} \chi_{L_3}(s) |J J_3 L_3\rangle. \]  \\

(9)

Within this sector, we see that \( \Psi \) can be thought of as a complex vector-valued function

\[
\begin{pmatrix}
\chi_{-J}(s) \\
\vdots \\
\chi_{J}(s)
\end{pmatrix}
\]

on \( V \subseteq C_{\text{shapes}} \). Of course, we have only been working locally, i.e. in some patch of \( C \) which looks like a product \( SU(2) \times V \). The total wavefunction, defined on all of \( C \), isn’t a vector-valued function on the base space but is more precisely a section of a (complex) vector bundle of rank \( 2J + 1 \). These two notions coincide for the case of trivial bundles. In the more general case, we would work with functions in separate patches and then impose appropriate conditions on the overlaps to ensure they give a genuine section.

Substituting the expansion for \( \Psi \) above into the Hamiltonian, we obtain the Schrodinger equation

\[
\frac{1}{2} \mathbf{L} \cdot \Lambda^{-1} \cdot \mathbf{L} \begin{pmatrix}
\chi_{-J}(s) \\
\vdots \\
\chi_{J}(s)
\end{pmatrix} + \frac{1}{2} (p_i - \mathbf{L} \cdot \mathbf{A}_i) \, g_{ij}^{-1} (p_j - \mathbf{L} \cdot \mathbf{A}_j) \begin{pmatrix}
\chi_{-J}(s) \\
\vdots \\
\chi_{J}(s)
\end{pmatrix} + (V_2(s) + V(s) - E) \begin{pmatrix}
\chi_{-J}(s) \\
\vdots \\
\chi_{J}(s)
\end{pmatrix} = 0
\]

(10)

where now the operators \( \mathbf{L} \) act by matrix multiplication. This is the effective problem on \( C_{\text{shapes}} \). It is equivalent to the motion of a particle on \( C_{\text{shapes}} \) coupled to an \( SU(2) \) gauge
field, with the particle transforming in the $(2J + 1)$-dimensional irrep of the gauge group and with the gauge field (or connection) corresponding to $A_i$. Gauge transformations are equivalent to redefining our choice of reference orientation for each $s \in C_{\text{shapes}}$. Note that the rotational motion influences the motion on $C_{\text{shapes}}$ through the familiar minimal coupling $p_j - L \cdot A_j$ of the momentum $p_j$ to the gauge field. To completely separate out rotational motion would require us to find a gauge where $A_i$ vanishes. $A_i$, while gauge dependent, has gauge-invariant properties such as (possibly non-vanishing) curvature. The curvature of $A_i$ can therefore be viewed as an obstruction to complete separation of rotational motion from the other degrees of freedom.

### Including isospin

The above derivation is easily modified to include the possibility of isospin. Once again the metric must take the form

$$
\tilde{g} = \begin{pmatrix} \sigma & ds_i \\ A_i^T \Lambda & g_{ij} + A_i \cdot \Lambda \cdot A_j \end{pmatrix} \begin{pmatrix} \sigma \\ ds_j \end{pmatrix}
$$

where now $\sigma = (\sigma_1^l, \sigma_2^l, \sigma_3^l, \sigma_1^t, \sigma_2^t, \sigma_3^t)$ includes both left-invariant one forms $(\sigma_1^l, \sigma_2^l, \sigma_3^l)$ associated with rotations and $(\sigma_1^t, \sigma_2^t, \sigma_3^t)$ associated with isorotations. $\Lambda$ and $A_i$ have now become a $6 \times 6$ matrix and (for each $i$) a 6-component vector respectively. One ends up with the Hamiltonian

$$
\mathcal{H} = \frac{1}{2} \left( \begin{pmatrix} L \\ K \end{pmatrix} \cdot \Lambda^{-1} \cdot \begin{pmatrix} L \\ K \end{pmatrix} \right) + \frac{1}{2} \left( p_i - \left( \begin{pmatrix} L \\ K \end{pmatrix} \cdot A_i \right) g^{-1}_{ij} \left( p_j - \left( \begin{pmatrix} L \\ K \end{pmatrix} \cdot A_j \right) \right) + V_2(s) + V(s)
$$

where

$$
V_2 = \frac{\hbar^2}{2} \left( |\Lambda| \cdot |g_{ij}| \right)^{-\frac{1}{2}} \partial_i \left( g_{i}^{-1} \partial_j (|\Lambda| \cdot |g_{ij}|)^{\frac{1}{2}} \right).
$$

We will make use of this Hamiltonian later, but for now we will go back to only including rotations.

### 3 Equilateral triangle in $\mathbb{R}^3$

We will be interested in small vibrations of Skyrmions, ultimately applying the above ideas to quantization of the $B = 4$ and $B = 7$ Skyrmions. But let us start with a simpler problem which illustrates the main ideas: as a model for a Skyrmion, consider an equilateral triangular
arrangement of point particles in $\mathbb{R}^3$, with particle $i$ having mass $m$ and position vector $\mathbf{r}_i$. We imagine these are attached by identical springs. This (equilateral) arrangement has symmetry group $D_{3h}$, and so its vibrations can be classified by irreducible representations (irreps) of this group. There are three normal modes (not including zero modes), which split into the irreps $A \oplus E'$ under the action of $D_{3h}$. For the spring model, the $E'$ vibration has lowest frequency with $\frac{\omega_{E'}}{\omega_A} = \frac{1}{\sqrt{2}}$. Suppose we are interested in a configuration space $\mathcal{C} \cong SO(3) \times \mathbb{R}^2$ which includes only this doubly-degenerate vibration ($E'$) together with rotations. This is clearly a trivial bundle. We will use coordinates $s = (s_1, s_2)$ on $C_{\text{shapes}}$, and Euler angles $\theta^i$ to specify orientation. Let $d$ be the distance from each particle to the centre of mass in the equilibrium configuration, and work in units where $\hbar = 1, m = 1, d = 1$. Let the coordinates $(\theta^i = 0, s)$ correspond to the configuration

$$
\mathbf{r}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \end{pmatrix}
$$

$$
\mathbf{r}_2 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2\sqrt{3}} \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} -\frac{1}{2\sqrt{3}} \\ \frac{1}{2} \\ 0 \end{pmatrix}
$$

$$
\mathbf{r}_3 = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2\sqrt{3}} \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2} \\ 0 \end{pmatrix}.
$$

This is our gauge choice. A general configuration $(\theta^i, s)$ with $\theta^i \neq 0$ can be deduced from a rotation of the corresponding reference configuration $(0, s)$. We will assume $V(s) = \frac{1}{2} \omega^2 s^2$. We can compute the metric induced from the Euclidean metric on $\mathbb{R}^9$ (three point particles) which leads, by comparison to the expression

$$
\bar{g} = \begin{pmatrix} \sigma & ds_i \\ ds_i & \Lambda \end{pmatrix} \begin{pmatrix} \Lambda \mathbf{A}_j \\ \mathbf{A}_i^T \mathbf{A}_j + \mathbf{A}_i \cdot \Lambda \mathbf{A}_j \end{pmatrix} \begin{pmatrix} \sigma \\ ds_j \end{pmatrix},
$$

$$
\Lambda = \begin{pmatrix} \frac{3}{2} + \sqrt{3}s_1 + \frac{1}{2} (s_1^2 + s_2^2) & -\sqrt{3}s_2 & 0 \\ -\sqrt{3}s_2 & \frac{3}{2} - \sqrt{3}s_1 + \frac{1}{2} (s_1^2 + s_2^2) & 0 \\ 0 & 0 & 3 + s_1^2 + s_2^2 \end{pmatrix},
$$

10
\[ g_{ij} = \frac{1}{3 + s_1^2 + s_2^2} \begin{pmatrix} 3 + s_1^2 & s_1 s_2 \\ s_1 s_2 & 3 + s_2^2 \end{pmatrix}, \]  

and

\[ A_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

We can already see that the gauge field takes a familiar form for small \((s_1, s_2)\): we have

\[ A_1 \sim \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad A_2 \sim \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

and so the effective motion on \(C_{\text{shapes}}\) will appear as if coupled to a constant magnetic field (of strength \(\frac{1}{3} L_3\)) pointing out of the \((s_1, s_2)\)-plane. For this example it is also simple to compute

\[ V_2(s) = \frac{1}{2} \left( |\Lambda| |g_{ij}| \right)^{-\frac{1}{2}} \partial_i \left( g_{ij}^{-1} \partial_j \left( |\Lambda| |g_{ij}| \right)^{\frac{3}{2}} \right) \]

\[ = \frac{1}{2} - 6 + s_1^2 + s_2^2 \]

\[ 2 \left( 3 - s_1^2 - s_2^2 \right)^2. \]

Recall that the full quantum Hamiltonian is

\[ \mathcal{H} = \frac{1}{2} L \cdot \Lambda^{-1} \cdot L + \frac{1}{2} \left( p_i - L \cdot A_i \right) g_{ij}^{-1} \left( p_j - L \cdot A_j \right) + V_2(s) + V(s). \]

We now make the following approximation: assume that the vibrational frequency \(\omega\) is large so that the most important terms in the above Hamiltonian give a harmonic oscillator

\[ \mathcal{H}_0 = \frac{1}{2} \left( p_1^2 + p_2^2 \right) + \frac{1}{2} \omega^2 \left( s_1^2 + s_2^2 \right). \]

We will expand the full Hamiltonian in \(\frac{1}{\omega}\), keeping the leading corrections to the \(\mathcal{H}_0\) system. Note that \(\mathcal{H}_0\) has eigenvalues \(\sim \omega\). Also note that, schematically, \(s^2 \sim \frac{1}{\omega}\) and \(p^2 \sim \omega\) for the harmonic oscillator from which the orders of other terms in \(\mathcal{H}\) can be deduced. Expanding out the full Hamiltonian, we have

\[ \mathcal{H} = \mathcal{H}_0 + \frac{1}{3} \left( L^2 - \frac{1}{2} L_3^2 \right) + \frac{1}{3} L_3 J_s + \frac{1}{6} J_s^2 - \frac{1}{3} + \ldots \]

\[ \sim \omega^1 \quad \sim \omega^0 \quad \sim \text{higher} \]
where $J_s = s_1 p_2 - s_2 p_1$ is an operator which will be referred to as the vibrational angular momentum. To this order, the only effect of $V_2(s)$ is to contribute an additive constant (here, $-\frac{1}{3}$) to the Hamiltonian. This will be the case more generally and so we will neglect $V_2(s)$ in later examples. So the terms that remain are $H_0$ (a harmonic oscillator corresponding to vibrations), $\frac{1}{3} (L^2 - \frac{1}{2} L_3^2)$ (the familiar rigid-body Hamiltonian, corresponding to rotations) and finally the term $\frac{1}{3} L_3 J_s + \frac{1}{6} J_s^2$ which comes from the gauge field. It gives the leading correction due to rotation-vibration coupling. This is referred to as a Coriolis term in the molecular physics literature.

**Symmetry arguments**

Before moving on, let’s reflect on what we have done in this example. The coordinates $(\theta^i, s)$ were actually carefully chosen so that the metric took the form

$$\tilde{g} = \left( \begin{array}{cc} \Lambda_0 & \Lambda_0 \Lambda_j \\ A^T_i \Lambda_0 & \delta_{ij} \end{array} \right) \left( \begin{array}{c} \sigma \\ ds_j \end{array} \right)$$

with, to the order we are interested in, $\Lambda = \Lambda_0$ a constant matrix (the moment of inertia tensor for the equilibrium configuration) and bottom-right entry $\delta_{ij}$ (normal coordinates for the vibration) and with the off-diagonal entry $\Lambda A_i$ vanishing at the equilibrium configuration (this says that rotations and vibrations are orthogonal at the equilibrium configuration). Then, to the order we are interested in, $A_i$ is linear in the shape coordinates. Now recall that the equilibrium configuration has $D_{3h}$ symmetry and that the vibration we are interested in transforms in the $E'$ representation of $D_{3h}$, $\rho_{\text{vib}} \cong E'$. In particular, the metric $\tilde{g}$ enjoys a $D_{3h}$ symmetry and so the gauge field $A_i$ is not just an arbitrary linear function of the shape coordinates but corresponds to a singlet of $D_{3h}$ under an action of $D_{3h}$ isomorphic to $\rho_{\text{vib}} \otimes \rho_{\text{vib}} \otimes \rho_{\text{rot}}$ where $\rho_{\text{rot}}$ denotes the representation in which rotations $(R_x, R_y, R_z)$ transform under $D_{3h}$ (for our example $\rho_{\text{rot}} = A'_2 \oplus E''$). This observation is equivalent to Jahn’s rule, which is known in molecular physics as a necessary condition for the existence of non-trivial first-order Coriolis terms [14]. In the present case, a simple character theory calculation shows that $\rho_{\text{vib}} \otimes \rho_{\text{vib}} \otimes \rho_{\text{rot}}$ contains precisely one copy of the trivial irrep of $D_{3h}$. So the gauge field $A_i$ is determined by a single constant $\eta$. It has to transform trivially under $\rho_{\text{vib}} \otimes \rho_{\text{vib}} \otimes \rho_{\text{rot}}$, which in this case means that the $A_i$ must satisfy

$$\forall g \in D_{3h} : \rho_{\text{rot}} (g) (\rho_{\text{vib}} (g))_{ik} A_k \left( (\rho_{\text{vib}} (g))_{jl}^{-1} s_l \right) = A_i (s_j).$$
so that

$$A_1 = \eta \begin{pmatrix} 0 & 0 \\ s_2 & 0 \end{pmatrix}, \quad A_2 = \eta \begin{pmatrix} 0 & 0 \\ -s_1 & 0 \end{pmatrix}. \quad (25)$$

The only reason to do the explicit calculation of the previous section was to determine that $\eta = \frac{1}{3}$. We might more generally take $\eta$ to be a free parameter. This insight will prove useful in situations where it is not so easy to compute the gauge field explicitly, and all we have is knowledge of the relevant symmetry group together with the transformation properties of the vibration.

4 $B = 4$ Skyrmion and the $\alpha$-particle

![B=4 Skyrmion with $O_h$ symmetry. Figure courtesy of Dankrad Feist.](image)

Figure 1: $B=4$ Skyrmion with $O_h$ symmetry. Figure courtesy of Dankrad Feist.

We now apply our insights from the previous section to the problem of a vibrating and rotating Skyrmion. The minimal energy $B = 4$ Skyrmion has $O_h$ symmetry and is illustrated in Figure 1. The isospin 0 quantum states of this Skyrmion correspond to the $\alpha$-particle. In [13] the authors performed rigid-body quantization of the $O_h$-symmetric $B = 4$ Skyrmion, finding a ground state with spin $J = 0$ and a first excited state with $J = 4$. The spin 4 excitation (at roughly 40 MeV) has not yet been experimentally observed, however there are numerous observed excited states with lower spin in the $20-30$ MeV range [15] which are not captured by the rigid-body picture. The $O_h$ symmetry group of the rigid $B = 4$ Skyrmion
is too large to allow such excitations (which have spins 0, 1 and 2) and so the data suggests
that vibrations must be included if we want to describe these states.

**Vibrations of the $B = 4$ Skyrmion**

The lowest four vibrational modes [2] of the $B = 4$ $O_h$-symmetric Skyrmion are listed in
Table 1. The associated frequencies are those calculated in [2] for a dimensionless pion mass
of $m = 1$. These vibrations have been classified using $O_h$ representation theory. As a group,
$O_h$ is generated by a 3-fold rotation $C_3$, 4-fold rotation $C_4$ together with an inversion element
$-I$. $O_h$ has 10 irreps with the corresponding character table given in Table 2. Following a

| Frequency | Irrep of $O_h$ | Description |
|-----------|---------------|-------------|
| 0.46      | $E^+$         | Two opposite faces pull away from each other to form two $B = 2$ tori. In the other direction, four edges pull away to become four $B = 1$ Skyrmions. |
| 0.48      | $F_2^+$       | An opposing pair of square-symmetric faces deform to become rhombus-shaped. |
| 0.52      | $A_2^-$       | Four vertices of the cube pull away, retaining tetrahedral symmetry. These then come in again and the other four vertices pull away to form the dual tetrahedron. |
| 0.62      | $F_2^-$       | Two opposite edges from the same face pull away from the origin. On the opposite face, the perpendicular edges also pull away. |

Table 1: Vibrations of $B = 4$ Skyrmion. Frequencies and descriptions from [2].

similar approximation scheme to the previous section, our aim is to compute the quantum
spectrum of a vibrating and rotating $B = 4$ Skyrmion. We assume that the vibrations in
Table 1 are the most important and neglect any other degrees of freedom. We will ignore
isorotations as we are interested in isospin 0 states corresponding to the $\alpha$-particle. (We will
include isorotations when we look at the $B = 7$ Skyrmion in the next section).

**The $F_2^-$ vibration**

To start off, we consider just the triply degenerate $F_2^-$ vibration of the $B = 4$ Skyrmion
along with rotations, $C \simeq SU(2) \times \mathbb{R}^3$. As in the case of the equilateral triangle, we assume
the vibrations are small. The equilibrium configuration has symmetry group $O_h$, which acts
on physical space as follows:

$$C_4 : (x, y, z) \rightarrow (-y, x, z)$$
Table 2: $O_h$ character table [16].

\[
\begin{array}{cccccccccc}
O_h & E & 8C_3 & 6C_2 & 6C_4 & 3C_2 = (C_4)^2 & i & 6S_4 & 8S_6 & 3\sigma_h & 6\sigma_d \\
\hline
A_{1g}^+ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
A_{2g}^+ & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\
E^+ & 2 & -1 & 0 & 0 & 2 & 2 & 0 & -1 & 2 & 0 \\
F_{1g}^+ & 3 & 0 & -1 & 1 & -1 & 3 & 1 & 0 & -1 & -1 \\
F_{2g}^+ & 3 & 0 & 1 & -1 & -1 & 3 & -1 & 0 & -1 & 1 \\
A_{1u}^- & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
A_{2u}^- & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\
E^- & 2 & -1 & 0 & 0 & 2 & -2 & 0 & 1 & -2 & 0 \\
F_{1u}^- & 3 & 0 & -1 & 1 & -1 & -3 & -1 & 0 & 1 & 1 \\
F_{2u}^- & 3 & 0 & 1 & -1 & -1 & -3 & 1 & 0 & 1 & -1 \\
\end{array}
\]

Note that this action of $O_h$ is isomorphic to $F_1^-$. Introduce coordinates $(\theta^i, s)$ such that the total metric takes the form (to the order we are interested in)

\[
\tilde{g} = \left( \sigma \quad ds_i \right) \left( \begin{array}{cc}
\Lambda_0 & \Lambda_0 A_i \\
A_i^T \Lambda_0 & \delta_{ij}
\end{array} \right) \left( \sigma \quad ds_j \right)
\]

with

\[
\Lambda_0 = \begin{pmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix},
\]

$A_i$ linear and vanishing at the equilibrium configuration $s_1 = s_2 = s_3 = 0$. We still have some freedom in which vibrational coordinates $s = (s_1, s_2, s_3)$ we choose, and we will choose them so that they transform under $O_h$ as follows:

\[
C_4 : (s_1, s_2, s_3) \rightarrow (s_2, -s_1, -s_3)
\]

\[
C_3 : (s_1, s_2, s_3) \rightarrow (s_2, s_3, s_1)
\]

\[
-I : (s_1, s_2, s_3) \rightarrow (-s_1, -s_2, -s_3).
\]

Note at this point that, unlike in the point particle example of the previous section, we do not have explicit expressions for the Skyrme field configurations corresponding to each
\( s = (s_1, s_2, s_3) \). However, it is always possible to pick coordinates so that the action of \( O_h \) is realised exactly as above (since the representation \( \rho_{\text{vib}} \) of \( O_h \) given in (29) is indeed isomorphic to \( F_2^- \)). As in the point particle model of the previous section, it turns out that 
\[
\rho_{\text{vib}} \otimes \rho_{\text{vib}} \otimes \rho_{\text{rot}} = F_2^- \otimes F_2^- \otimes F_1^+ \]
contains precisely one singlet, and that the gauge field is therefore determined up to a single multiplicative scalar \( \eta_- \):

\[
\begin{align*}
A_1 &= \frac{\eta_-}{2T} \begin{pmatrix} 0 \\ -s_3 \\ s_2 \end{pmatrix} \\
A_2 &= \frac{\eta_-}{2T} \begin{pmatrix} s_3 \\ 0 \\ -s_1 \end{pmatrix} \\
A_3 &= \frac{\eta_-}{2T} \begin{pmatrix} -s_2 \\ s_1 \\ 0 \end{pmatrix}.
\end{align*}
\] (30)

Substituting this into the general expression in (20), we arrive at the Hamiltonian

\[
\mathcal{H} \approx \frac{1}{2} \mathbf{p}^2 + \frac{1}{2} \omega^2 F_2^- s^2 + \frac{1}{2T} \mathbf{L}^2 - \frac{\eta_-}{2T} \mathbf{L} \cdot \mathbf{J}_s + \frac{\eta_-^2}{8T} \mathbf{J}_s^2
\] (31)

where \( p_i = -i\hbar \frac{\partial}{\partial s_i} \) and \( \mathbf{J}_s = \mathbf{s} \times \mathbf{p} \). A similar picture to that in equation (22) emerges: we have a harmonic oscillator system and a rigid-body system which are coupled through the additional term \(-\frac{\eta_-}{2T} \mathbf{L} \cdot \mathbf{J}_s + \frac{\eta_-^2}{8T} \mathbf{J}_s^2\) involving the usual body-fixed angular momentum \( \mathbf{L} \) and a vibrational angular momentum \( \mathbf{J}_s \). In principle \( \eta_- \) could be calculated from the Skyrme model given explicit Skyrme field configurations, much like how we calculated \( \eta = \frac{1}{3} \) in the preceding (point particle) example. We will take it to be a free parameter.

### Computing the spectrum

We are interested in the Hamiltonian

\[
\mathcal{H} = \frac{1}{2} \mathbf{p}^2 + \frac{1}{2} \omega^2 F_2^- s^2 + \frac{1}{2T} \mathbf{L}^2 - \frac{\eta_-}{2T} \mathbf{L} \cdot \mathbf{J}_s + \frac{\eta_-^2}{8T} \mathbf{J}_s^2.
\] (32)

It will help to rewrite the Hamiltonian using the fact that, as \( \mathbf{J}_s \) and \( \mathbf{L} \) commute,

\[
\mathbf{L} \cdot \mathbf{J}_s = \frac{1}{2} \mathbf{L}^2 + \frac{1}{2} \mathbf{J}_s^2 - \frac{1}{2} M^2
\] (33)

where we have introduced a new angular momentum operator \( \mathbf{M} = \mathbf{J}_s - \mathbf{L} \). (Note that \(-\mathbf{L}, \) not \( +\mathbf{L}, \) obeys the usual angular momentum commutation relations: \( \mathbf{L} \) is the vector of body-fixed angular momentum operators so its commutation relations differ by a minus sign.
compared to *space-fixed* angular momentum operators). Then

$$\mathcal{H} = \frac{1}{2} \mathbf{p}^2 + \frac{1}{2} \omega_{F_2}^2 s^2 + \left( \frac{1}{2\mathcal{I}} - \frac{\eta}{4\mathcal{I}} \right) L^2 + \frac{\eta}{4\mathcal{I}} M^2 + \left( \frac{\eta^2}{8\mathcal{I}} - \frac{\eta}{4\mathcal{I}} \right) J_s^2. \quad (34)$$

Energy eigenstates $\Psi$ can be classified by $M^2, J_s^2, L^2$ and the vibrational phonon-number $N_{F_2}^-$, and additionally by their transformation under the $O_h$ symmetry group, where $O_h$ acts on a state by transforming the vibrational coordinates, and then performing a compensating rotation:

$$\Psi \rightarrow \rho_{\text{rot}}(g) \otimes \rho_{\text{vib}}(g) \Psi. \quad (35)$$

Explicitly, this action is generated by

$$C_4 : \Psi \rightarrow Pe^{-\frac{2\pi i}{4} n_4 \cdot M} \Psi$$
$$C_3 : \Psi \rightarrow e^{-\frac{2\pi i}{3} n_3 \cdot M} \Psi$$
$$-I : \Psi \rightarrow P \Psi \quad (36)$$

where $n_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $n_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ and where $P$ is the parity operation on the vibrational coordinates $s \rightarrow -s$. The Finkelstein-Rubinstein (F-R) constraints tell us that physical states should be taken to transform trivially under the action of $\langle C_4, C_3 \rangle \cong O \leq O_h$, the subgroup consisting of rotations. Recall $O_h \cong O \times \mathbb{Z}_2$, a direct product of groups, with $\mathbb{Z}_2$ the subgroup generated by the parity operation $-I \in O_h$. So such representations fall into two classes, $A_1^+$ or $A_1^-$, depending on their transformation under the $\mathbb{Z}_2$. This determines the parity of the state as $+$ or $-$. Within each fixed $M^2, J_s^2, L^2, N_{F_2}^-$ sector, we compute the character of the action of $O_h$ and then look for representations of type $A_1$. For example: suppose we are interested in one-phonon states (i.e. states with one quantum of vibrational energy). Such states have $J_s = 1$. We might look for states with $J = L = 2$. Adding these angular momenta, we have several possibilities for the total angular momentum $M = J_s - L = 3, 2, 1$. So, if we are interested say in $M = 3$, we have narrowed down to a 7-dimensional subspace. We now look at how this 7-dimensional subspace transforms under the action above, computing the associated character $\chi$. We then find that $\langle \chi, A_1^+ \rangle = 0$ and $\langle \chi, A_1^- \rangle = 1$ giving a single negative parity $2^-$ state.
Other vibrations

We now include the vibrations transforming as \( F^{-2}, F^{+2}, A^{-2} \), treating the vibrational frequencies as free parameters, and fit the resulting spectrum to data in the \(< 30 \text{ MeV}\) range. We could also include the \( E^{+} \) vibration but it turns out that including it gives no improvement to the fit to experimental data. In fact it will turn out that almost all of the data can be explained solely in terms of \( F^{-2} \) and \( F^{+2} \) modes, with a higher frequency \( A^{-2} \) mode important for a couple of higher energy (\( \sim 28 \text{ MeV}\)) states. The \( F^{-2} \) and \( F^{+2} \) can have Coriolis terms whereas symmetry considerations exclude any non-trivial Coriolis term for the \( A^{-2} \). This leads to the Hamiltonian

\[
\mathcal{H} = \frac{1}{2} p_s^2 + \frac{1}{2} \omega_{F^{-2}}^2 s^2 + \frac{1}{2} p_t^2 + \frac{1}{2} \omega_{F^{+2}}^2 t^2 + \frac{1}{2} p_u^2 + \frac{1}{2} \omega_{A^{-2}}^2 u^2 
+ \frac{1}{2L} L^2 - \frac{\eta_{F^{-2}}}{2L} J_s + \frac{\eta_{F^{+2}}}{8L} J_s^2 - \frac{\eta_{A^{-2}}}{2L} L \cdot J_t + \frac{\eta_{F^{+2}}}{8L} J_t^2.
\]

where coordinates \( s, t, u \) correspond to the vibrations \( F^{-2}, F^{+2}, A^{-2} \) respectively. As in our analysis of (32), it will be useful to introduce a total angular momentum operator \( M = J_s + J_t - L \) combining vibrational angular momentum operators \( J_s, J_t \) with body-fixed angular momentum \( L \). Energy eigenstates \( \Psi \) can be classified by \( M^2 = (J_s + J_t - L)^2, J_s^2, J_t^2, L^2 \) and vibrational phonon-numbers \( N_{F^{-2}}, N_{F^{+2}}, N_{A^{-2}} \), and additionally by their transformation under the \( O_h \) symmetry group, where \( O_h \) acts on a state by transforming the vibrational coordinates and then performing a compensating rotation of the state:

\[
\Psi \rightarrow \rho_{\text{rot}} (g) \otimes \rho_{\text{vib}} (g) \Psi.
\]

Explicitly, this action is generated by

\[
C_4 : \Psi \rightarrow P_u P_t P_s e^{-\frac{2\pi i}{3} n_t \cdot M} \Psi \\
C_3 : \Psi \rightarrow e^{-\frac{2\pi i}{3} n_3 \cdot M} \Psi \\
-I : \Psi \rightarrow P_u P_s \Psi
\]

where \( n_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, n_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \) and where \( P_s, P_t, P_u \) are parity operators on the vibrational coordinates. We demand that states transform trivially under the subgroup \( \langle C_4, C_3 \rangle \cong O \) consisting of rotations. Within a fixed \( M^2 = (J_s + J_t - L)^2, J_s^2, J_t^2, L^2, N_{F^{-2}}, N_{F^{+2}}, N_{A^{-2}} \)
sector, we perform character theory calculations and determine $A_1$ summands as before. Calculating the resulting spectrum, and then fitting the frequencies, Coriolis parameters $\eta_{+/-}$ and moment of inertia $\Lambda$ of the $B = 4$ to nuclear data, we obtain the best fit (in a least-squares sense) for the values

\[
\begin{align*}
\hbar \omega_{F_2^+} &\approx 9.7 \text{MeV} & \frac{\hbar^2}{I} &\approx 4 \text{MeV} \\
\hbar \omega_{F_2^-} &\approx 11.7 \text{MeV} & \eta_+ &\approx 0.71 \\
\hbar \omega_{A_2^-} &\approx 15.1 \text{MeV} & \eta_- &\approx 0.13
\end{align*}
\]  

(40)

In Table 3 we list all allowed states up to 30 MeV for the parameter values in (40). The states in this energy range consist of both 1-phonon and 2-phonon excitations. With these 6 parameters we are able to describe 11 of the 12 experimentally observed Helium-4 states below 30 MeV complete with the correct spin and parity assignments, and we predict one further $0^+$ state at 23.4 MeV. A column $E_{\eta=0}$ is included to show the spectrum when Coriolis effects are neglected: the Coriolis corrections have a particularly large effect on the $2^+$ states, raising the energy of the lowest $2^+$ excitation by as much as 3.3 MeV (comparing favourably to experiment). Note that the ordering of the fitted frequencies does not agree with that of the Skyrme model values in Table 1, which put the $E^+$ as the lowest-energy vibration. This discrepancy can perhaps be understood by considering behaviour beyond small vibrations: recall that the $E^+$ vibration is associated with the breakup of the $B = 4$ Skyrmion into two $B = 2$ or four $B = 1$ Skyrmions. Physically, the breakup energy for $^4\text{He} \rightarrow ^2\text{H} + ^2\text{H}$ is 23.8 MeV, higher than the breakup energy for $^4\text{He} \rightarrow ^2\text{H} + \text{p}$ (which should be associated with the $F_2$ modes) at 20.3 MeV.

Our picture suggests that the 20.2 MeV $0^+$ state should be identified with a two-phonon excitation of the $F_2^-$ mode of the cube. Promisingly, electron scattering measurements [6] of the transition form factor for the $0^+$ suggest collective behaviour, as noted by the authors of [7]. More recent work based on an ab initio study gives further evidence for the collective interpretation of this state, suggesting a breathing mode [8]. We agree on the collective nature of this state but, based on the $B = 4$ cube, suggest that the breathing mode should be assigned a higher frequency than our $F_2^-$ mode. To compare these two interpretations it would be worthwhile computing transition form factors from our model. This would require the explicit form of the Skyrme fields at each point in our configuration space which, while possible in principle, is beyond the scope of this paper. There have also been studies of the negative parity excited states making use of Wigner’s theory based on approximate $SU(4)$ symmetry [9]. Our novel picture has the advantage of giving a unified understanding of
almost all observed excited states, both positive and negative parity, in terms of simple vibrations of the $B = 4$ cube.

\[
\begin{array}{cccccccccc}
J^P & N_{F^-} & N_{F^+} & N_{A^-} & J^2_s & J^2_t & L^2 & M^2 & E & E_{exp} & E_{q=0} \\
0^+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0^+ & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 19.4 & 20.2 & 19.4 \\
0^- & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 21.9 & 21.0 & 21.4 \\
2^- & 1 & 0 & 0 & 2 & 0 & 6 & 12 & 22.2 & 21.8 & 21.7 \\
0^+ & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 23.4 & - & 23.4 \\
1^- & 1 & 1 & 0 & 2 & 2 & 2 & 0 & 24.2 & 24.3 & 25.4 \\
2^+ & 0 & 1 & 0 & 2 & 6 & 12 & 27.0 & 27.4 & 23.7 \\
1^+ & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 28.3 & 28.3 & 28.8 \\
1^- & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 28.5 & 28.4 & 30.8 \\
2^- & 1 & 1 & 0 & 2 & 2 & 6 & 0 & 28.9 & 28.4 & 33.4 \\
0^- & 0 & 2 & 0 & 0 & 0 & 6 & 6 & 0 & 28.6 & \\
2^+ & 0 & 2 & 0 & 0 & 6 & 6 & 0 & 28.4 & 28.7 & 35.4 \\
2^+ & 2 & 0 & 0 & 6 & 0 & 6 & 0 & 29.9 & 29.9 & 31.4 \\
\end{array}
\]

Table 3: Vibrating $B = 4$ spectrum up to 30 MeV.

5 Vibration-isospin coupling and the $B = 7$ Skyrmion

Figure 2: B=7 Skyrmion with $I_h$ symmetry. Figure courtesy of Chris Halcrow.

The lowest-energy $B = 7$ Skyrmion is a dodecahedron with symmetry group $I_h$ and its normal modes were studied in detail in [10]. If vibrations are not included, the high degree of symmetry of the $B = 7$ means that the lowest energy isospin $\frac{1}{2}$ state has spin $\frac{7}{2}$. In reality the
observed ground state of the Lithium-7/Beryllium-7 isodoublet has spin $\frac{3}{2}$. It was suggested in [11] that to capture this state one should include a five-fold degenerate vibration which transforms in the $H^5_g$ irrep of $I_h$ and which is generated by pairs of opposite pentagonal faces pulling away from the center of the Skyrmion. In [12] this vibration was treated within a harmonic approximation and interactions between rotations and vibrations were neglected. Here we extend that analysis to include the Coriolis corrections.

We take our configuration space $C \simeq SU(2) \times SU(2) \times \mathbb{R}^5$ to include the $H^5_g$ vibration along with rotations and isorotations. Recalling (12), we should now take $A_i$ to be a 6-component vector for each $i$ as we are including isorotations. Introduce coordinates such that the total metric takes the form (to the order we are interested in)

$$\tilde{g} = \begin{pmatrix} \sigma & ds_i \\ A_i^T \Lambda_0 & \delta_{ij} \end{pmatrix} \begin{pmatrix} \sigma \\ ds_j \end{pmatrix} \quad (41)$$

with

$$\Lambda_0 = \begin{pmatrix} \Lambda_0 I_3 & 0 \\ 0 & \Lambda_K I_3 \end{pmatrix}, \quad (42)$$

$A_i$ linear and vanishing at the equilibrium configuration $s_1 = s_2 = s_3 = 0$. Recall Jahn’s rule from the end of section 3. Now that we are including isorotations, Jahn’s rule should be generalised: the gauge field $A_i$ now corresponds to a singlet of $I_h$ under an action of $I_h$ isomorphic to $\rho_{\text{vib}} \otimes \rho_{\text{vib}} \otimes \left( \rho_{\text{rot}} \oplus \rho_{\text{isorot}} \right)$ where $\rho_{\text{isorot}}$ denotes the representation in which isorotations transform under $I_h$. In the present case, rotations transform as $T^3_{1g}$ and isorotations transform as $T^3_{2g}$. An easy calculation shows that

$$H^5_g \otimes H^5_g \otimes \left( T^3_{1g} \oplus T^3_{2g} \right) \simeq 2A^1_g \oplus \cdots \quad (43)$$

so there is the possibility of non-trivial Coriolis terms coupling vibrations to spin and isospin (note that we have two copies of the trivial representation and so the coupling will be determined up to two arbitrary constants). We wish to find the symmetry-allowed form of $A_1$, and for this we need explicit coordinates: note that the usual action of $I_h$ on $\mathbb{R}^3 = \langle e_1, e_2, e_3 \rangle$ is isomorphic to $T^3_{1u}$ and that the symmetric square $T^3_{1u} \otimes_{\text{sym}} T^3_{1u} \simeq A^1_g \oplus H^5_g$ contains a copy of the $H^5_g$ irrep we are interested in. So we pick vibrational coordinates $s_1, s_2, s_3, s_4, s_5$ (and conjugate momenta $p_i$) such that the action of $I_h$ is just like the action of $I_h$ on this $H^5_g$
where $M_{ij} = s_i p_j - s_j p_i$. $J_s^L$ and $J_s^K$ generate rotations in what is now a 5-dimensional vibrational space and generalise the vibrational angular momentum $J_s$ of (31). We are interested in eigenstates of (46), which can be classified by $L^2$, $K^2$ and vibrational phonon-
number. Consider one-phonon states: with respect to a Cartesian basis \( \{ s_k \exp(-\alpha s^2) \} \) of vibrational wavefunctions, it is clear how the \( M_{ij} \) act:

\[
M_{ij} s_k \exp(-\alpha s^2) = -i (\delta_{il} \delta_{jk} - \delta_{jl} \delta_{ik}) s_l \exp(-\alpha s^2)
\]

and thus how \( J_s^L \) and \( J_s^K \) act. We diagonalise \( \mathcal{H} \) numerically. The relevant group for imposing the F-R constraints is the universal cover of the icosahedral group \( I \), namely the binary icosahedral group \( 2I \subset SU(2) \), which has presentation

\[
\langle a, b \mid (ab)^2 = a^3 = b^5 \rangle.
\]

F-R constraints tell us that physical states must transform trivially under the action of the generators \( a \) and \( b \), given in our coordinates by

\[
s : \Psi \rightarrow e^{\frac{2\pi i}{3} n^L_a L} \otimes e^{\frac{2\pi i}{5} n^K_a K} \otimes \rho_{vib}(a) \Psi \\
t : \Psi \rightarrow e^{\frac{2\pi i}{3} n^L_b L} \otimes e^{\frac{6\pi i}{5} n^K_b K} \otimes \rho_{vib}(b) \Psi
\]

where

\[
_{\text{n}}^L_a = \left( \begin{array}{cc} \sqrt{\frac{2}{15}} (5 - \sqrt{5}) & 0 \\ 0 & \sqrt{\frac{1}{15}} (5 + 2\sqrt{5}) \end{array} \right), \quad _{n}^K_a = \left( \begin{array}{cc} -\sqrt{\frac{2}{15}} (5 + \sqrt{5}) & 0 \\ 0 & -\sqrt{\frac{1}{15}} (5 - 2\sqrt{5}) \end{array} \right), \quad _{0}^L_b = \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \quad _{0}^K_b = \left( \begin{array}{c} 0 \\ 1 \end{array} \right).
\]

The first few allowed states are listed in Table 4 along with the expectation values of the Coriolis terms. These Coriolis terms represent our corrections to the spectrum found in [12], which assumed complete separation of rotations and vibrations. That work focused on the isospin \( \frac{1}{2} \) sector: within this sector one obtains a zero-phonon state with spin \( \frac{7}{2} \) (identified with a 4.6 MeV excitation of Lithium-7) and one-phonon states with spins \( \frac{3}{2}, \frac{5}{2}, \) and \( \frac{7}{2} \) (identified with 0, 6.7 and 9.7 MeV excitations of Lithium-7). Ignoring Coriolis terms, the one-phonon states form a rotational band with energies following a simple \( J(J+1) \) pattern. The experimental data doesn’t fit this pattern particularly well: the rotational band energy ratio

\[
\frac{E(J = \frac{7}{2}) - E(J = \frac{3}{2})}{E(J = \frac{5}{2}) - E(J = \frac{3}{2})} = \frac{7 \times 9 - 3 \times 5}{5 \times 7 - 3 \times 5} = 2.4
\]

which is to be compared with the experimental result \( \frac{9.7}{6.7} \approx 1.4 \). We now consider the effect
of including Coriolis terms for these isospin $\frac{1}{2}$ states. Recall that, for the $B = 7$ Skyrmion, $\frac{\Lambda_0}{\Lambda_L} \sim 0.1$ as found in [13]. So it is reasonable to assume that, for the one-phonon states, the most important effect of the Coriolis terms is the energy splitting of size $\frac{\eta_K}{\Lambda_K}$, which (for $\eta_K > 0$) lowers the energy of the spin $\frac{3}{2}$ state while raising the energies of the spin $\frac{5}{2}$ and $\frac{7}{2}$ states. We now get

$$
E\left(J = \frac{7}{2}\right) - E\left(J = \frac{3}{2}\right) = 7 \times 9 - 3 \times 5 + \frac{8\eta_K\Lambda_L}{\Lambda_K} \\
E\left(J = \frac{5}{2}\right) - E\left(J = \frac{3}{2}\right) = 5 \times 7 - 3 \times 5 + \frac{8\eta_K\Lambda_L}{\Lambda_K}
$$

which reproduces the experimental ratio of 1.4 for a Coriolis parameter of $\eta_K \approx \frac{25}{4} \frac{\Lambda_0}{\Lambda_L} \sim 0.5$. It would be interesting to calculate $\eta_K$ explicitly from the Skyrme model and compare to this value.

We have learnt from this example that, in situations where the isospin moment of inertia is much smaller than the spin moment of inertia, it is perfectly possible for the isospin Coriolis corrections to compete with the usual $\frac{1}{2\Lambda_L} J (J + 1)$ rotational band splittings. This kind of effect is particularly important for odd $B$ Skyrmions like the $B = 7$, where non-zero isospin is inevitable (isospin taking half-integer values). This fits with the fact that the rotational band picture has been much more successful for even $B$ nuclei than for odd $B$ nuclei.

| Spin/Isospin | Energy without Coriolis terms | Coriolis terms |
|--------------|-------------------------------|----------------|
| $\left(\frac{7}{2}/\frac{1}{2}\right)_0$-phonon | $\frac{5}{2} \hbar \omega + \frac{\eta_0^2}{2\Lambda_L} \frac{7}{2} + \frac{\eta_0^2}{2\Lambda_K} \frac{13}{2}$ | $0$ |
| $\left(\frac{3}{2}/\frac{1}{2}\right)_1$-phonon | $\frac{7}{2} \hbar \omega + \frac{\eta_0^2}{2\Lambda_L} \frac{3}{2} + \frac{\eta_0^2}{2\Lambda_K} \frac{13}{2}$ | $3 \frac{\eta_0}{2\Lambda_L} - \frac{3}{2}\frac{\eta_K}{2\Lambda_K} + 6 \frac{\eta_0^2}{4\Lambda_L} + 6 \frac{\eta_0^2}{4\Lambda_K}$ |
| $\left(\frac{5}{2}/\frac{1}{2}\right)_1$-phonon | $\frac{7}{2} \hbar \omega + \frac{\eta_0^2}{2\Lambda_L} \frac{5}{2} + \frac{\eta_0^2}{2\Lambda_K} \frac{13}{2}$ | $\frac{1}{2} \frac{\eta_0}{2\Lambda_L} + \frac{\eta_0}{2\Lambda_K} + 6 \frac{\eta_0^2}{4\Lambda_L} + 6 \frac{\eta_0^2}{4\Lambda_K}$ |
| $\left(\frac{7}{2}/\frac{1}{2}\right)_1$-phonon | $\frac{7}{2} \hbar \omega + \frac{\eta_0^2}{2\Lambda_L} \frac{7}{2} + \frac{\eta_0^2}{2\Lambda_K} \frac{13}{2}$ | $-3 \frac{\eta_0}{2\Lambda_L} + \frac{\eta_0}{2\Lambda_K} + 6 \frac{\eta_0^2}{4\Lambda_L} + 6 \frac{\eta_0^2}{4\Lambda_K}$ |
| $\left(\frac{3}{2}/\frac{3}{2}\right)_0$-phonon | $\frac{5}{2} \hbar \omega + \frac{\eta_0^2}{2\Lambda_L} \frac{3}{2} + \frac{\eta_0^2}{2\Lambda_K} \frac{5}{2}$ | $0$ |
| $\left(\frac{1}{2}/\frac{3}{2}\right)_1$-phonon | $\frac{7}{2} \hbar \omega + \frac{\eta_0^2}{2\Lambda_L} \frac{1}{2} + \frac{\eta_0^2}{2\Lambda_K} \frac{3}{2}$ | $-3 \frac{\eta_0}{2\Lambda_L} + \frac{3}{2} \frac{\eta_K}{2\Lambda_K} + 6 \frac{\eta_0^2}{4\Lambda_L} + 6 \frac{\eta_0^2}{4\Lambda_K}$ |
| $\left(\frac{3}{2}/\frac{3}{2}\right)_1$-phonon | $\frac{7}{2} \hbar \omega + \frac{\eta_0^2}{2\Lambda_L} \frac{3}{2} + \frac{\eta_0^2}{2\Lambda_K} \frac{5}{2}$ | $-3 \frac{\eta_0}{2\Lambda_L} - \frac{3}{2} \frac{\eta_K}{2\Lambda_K} + 6 \frac{\eta_0^2}{4\Lambda_L} + 6 \frac{\eta_0^2}{4\Lambda_K}$ |

Table 4: Spectrum including one-phonon $H^5$ excitations of the $B = 7$ Skyrmion.

## 6 Conclusions and further work

We have developed a model of Helium-4 based on $F_2$ and $A_2$ vibrations of the cubic $B = 4$ Skyrmion. Our model includes interactions between rotations and vibrations in the form of Coriolis terms. The spectrum gives a good match to the experimental data, with the Coriolis
terms significantly improving the fit. The lowest state not captured by the model is a $0^-$ state at 28.6 MeV, and we predict one so far unobserved $0^+$ state at 23.4 MeV. We have also extended these ideas to the $B = 7$ Skyrmion, clarifying the role of isospin-vibration coupling.

The example in section 5 suggests a general feature which should occur in vibrational quantization of Skyrmions with non-zero isospin. It has been noted (e.g. in [17]) that for large $B$ the isospin moments of inertia for Skyrmions are much smaller than the spin moments of inertia, with $\Lambda_K \sim B$ and $\Lambda_L \sim B^{3/2}$. So, for large $B$, isospin Coriolis corrections can become more important than the usual $\frac{1}{2\Lambda_L} J(J+1)$ rotational band splittings. In fact the stable large nuclei all have large isospin and so these ideas are important for the Skyrme model description of many real nuclei.

It would be interesting to calculate the actual values for the Coriolis coefficients $\eta$ numerically within the Skyrme model. This requires explicit field configurations for vibrating Skyrmions but such configurations have been calculated before in e.g. [2]. It would also be interesting to study the effect of the gauge field $A_i$ for a situation in which shape space includes large deformations (not just small vibrations).

Finally it should be noted that, while our ideas have been outlined within the context of the Skyrme model, this work is very general and these ideas could be applied to other soliton systems in which one is interested in the interplay between zero and non-zero modes.

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