LYAPUNOV EXPONENT RIGIDITY FOR GEODESIC FLOWS

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Abstract. We study the relationship between the Lyapunov exponents of the geodesic flow of a closed negatively curved manifold and the geometry of the manifold. We show that if the derivative action of the geodesic flow on the unstable bundle has equal extremal Lyapunov exponents with respect to every invariant measure supported on a periodic orbit then the manifold is homothetic to a real hyperbolic manifold. Under the assumption that the manifold is homotopy equivalent to the appropriate locally symmetric space, we characterize the negatively curved locally symmetric spaces by the values of their Lyapunov exponents on their invariant measures supported on periodic orbits. We also show how, for real and complex hyperbolic space, the assumptions on the Lyapunov exponents of periodic orbits can be replaced with assumptions on the Lyapunov exponents of a single invariant Gibbs measure. The proofs use new results from hyperbolic dynamics including the nonlinear invariance principle of Avila and Viana and the approximation of Lyapunov exponents of invariant measures by Lyapunov exponents associated to periodic orbits which was developed by Kalinin in his proof of the Livsic theorem for matrix cocycles.

1. Introduction

Our goal is to characterize negatively curved locally symmetric spaces by the behavior of their geodesic flow around periodic orbits. A central question in geometric rigidity theory is the following: Suppose that a negatively curved Riemannian manifold $M$ shares some property $P$ with a negatively curved locally symmetric space $N$. Is $M$ isometric to $N$? The most famous example of such a rigidity theorem is the Mostow rigidity theorem: if $M$ and $N$ are real hyperbolic manifolds with isomorphic fundamental groups, then $M$ and $N$ are isometric.

We can ask a different, more dynamical rigidity question: Suppose that the geodesic flow of a negatively curved Riemannian manifold $M$ shares some property $P$ with the geodesic flow of a negatively curved locally symmetric space $N$. Is there a $C^1$ time-preserving conjugacy between the geodesic flows of $M$ and $N$? A remarkable consequence of the minimal entropy rigidity theorem of Besson, Courtois, and Gallot [6] is that dynamical rigidity implies geometric rigidity, in the sense that if there is a $C^1$ time preserving conjugacy between the geodesic flows of $M$ and $N$, then $M$ and $N$ are homothetic. Recall that two Riemannian manifolds $(M, d)$ and $(N, \rho)$ with distances $d$ and $\rho$ respectively are homothetic if there is a constant $c > 0$ such that $(M, d)$ is isometric to $(N, c\rho)$. This implies that it is possible to characterize the geometry of a locally symmetric space $N$ purely by the dynamics of its geodesic flow.
flow. For some examples of the numerous rigidity problems to which this has been applied, see [5],[4], and [14] as well as the survey articles [7],[33].

Before proceeding further, we fix some notation. Throughout this paper $M$ will denote an $m$-dimensional closed Riemannian manifold of negative curvature with universal cover $\tilde{M}$. We will always assume $m \geq 3$. We write $SM$ for the unit tangent bundle of $M$. The time-$t$ map of the geodesic flow on $SM$ will be denoted by $g^t$. We endow $SM$ with the Sasaki metric, giving $SM$ the structure of a closed Riemannian manifold with norm $\| \cdot \|$ and distance $d$. We let $\theta$ be the canonical contact 1-form on $SM$ preserved by $g^t$. The letter $C$ will be used freely as a multiplicative constant which is independent of whatever parameters are under consideration.

Since $M$ is negatively curved, $g^t$ is an Anosov flow. There is a $Dg^t$-invariant splitting $TSM = E^u \oplus E^c \oplus E^s$ with $E^c$ tangent to the vector field generating $g^t$, and there exist $0 < \nu < 1$, $C > 0$ such that for $v^u \in E^u$, $v^s \in E^s$, and $t > 0$,

$$\|Dg^t(v^u)\| \leq C\nu^t\|v^u\|, \quad \|Dg^{-t}(v^u)\| \leq C\nu^t\|v^u\|$$

$E^u$, $E^s$, and $E^c$ are called the unstable, stable, and center subbundles respectively. We write $E^{cu} := E^u \oplus E^c$ and $E^{cs} := E^c \oplus E^s$ for the center unstable and center stable subbundles respectively. Each of the distributions $E^u, E^s, E^c, E^{cu}, E^{cs}$ is uniquely integrable; we denote the corresponding foliations by $W^r, r = u, c, s, cu, cs$, with $W^r(x)$ being the leaf containing $x$. We consider each leaf of these foliations to carry the induced Riemannian metric from the Sasaki metric on $SM$. We let $W^r_+(x)$ be a ball of radius $r$ centered at $x$ in the leaf $W^r(x)$. For a $g^t$-invariant subbundle $E$ of $TSM$, we write $Dg^t|E$ for the restriction of $Dg^t$ to $E$.

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$ be the division algebra of real numbers, complex numbers, quaternions, and octonions respectively. Associated to each of these are the complex, quaternionic, and Cayley hyperbolic spaces $H_k^{\mathbb{K}}$ of dimension $k$, $2k$, $4k$, and $8$ respectively. These give the complete list of negatively curved symmetric spaces. We normalize the metrics of these spaces so that they have maximal curvature $-1$. Define $r_{\mathbb{K}} := \dim_{\mathbb{R}} \mathbb{K}$.

We say that $Dg^t|E^u$ is uniformly quasiconformal if there is a constant $C > 0$ independent of the point $p \in SM$ and $t$ such that for any pair of unit vectors $v, w \in E^u_p$,

$$\|Dg^t(v)\| \cdot \|Dg^t(w)\|^{-1} \leq C.$$

For closed negatively curved Riemannian manifolds of dimension at least $3$, combined work of Gromov, Kanai, Sullivan and Tukia shows that if the sectional curvatures $K$ of $M$ satisfy $-4 < K \leq -1$ and the action of the geodesic flow on $E^u$ is uniformly quasiconformal, then $M$ is homotopy equivalent to a real hyperbolic manifold $N$, and there is a $C^1$ time preserving conjugacy between the geodesic flows of $M$ and $N$ ([13],[25],[34],[36]). When combined with the minimal entropy rigidity theorem of Besson, Courtois, and Gallot [6], this implies that $M$ is homothetic to $N$.

Our first theorem is an improvement of this result. For a periodic point $p$ of $g^t$, let $\ell(p)$ be the period of $p$. Let $\chi^{(p)}_1, \ldots, \chi^{(p)}_{m-1}$ be the complex eigenvalues of $Dg^{\ell(p)}_p : E^u_p \to E^u_p$, counted with the multiplicity of their generalized eigenspaces.
Theorem 1.1. Let $M$ be an $m$-dimensional closed negatively curved Riemannian manifold. Suppose that
\[ |\chi_i^{(p)}| = |\chi_j^{(p)}|, \quad 1 \leq i, j \leq m - 1, \]
for every periodic point $p$ of the geodesic flow on $SM$. Then $M$ is homothetic to a compact quotient of $H^m_K$.

Theorem 1.1 implies that we can characterize a real hyperbolic manifold by the behavior of a countable collection of linear maps $Dg_p^{\ell(p)} : E^u_p \to E^u_p$ associated to its geodesic flow. Furthermore, we do not even require these linear maps to be conformal; we only require that for each map $Dg_p^{\ell(p)}$, all of its eigenvalues have the same absolute value. We’ve also removed the curvature assumption on $M$.

Our next theorem partially generalizes Theorem 1.1 to characterize the closed locally symmetric spaces of variable negative curvature.

Theorem 1.2. Let $M$ be an $m$-dimensional closed negatively curved Riemannian manifold that is homotopy equivalent to a compact quotient of $H^{m/r_K}_K$, $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. Suppose that for each periodic point $p$ of the geodesic flow on $SM$, there is an indexing of the complex eigenvalues $\chi_1^{(p)}, \ldots, \chi_{m-1}^{(p)}$ of $Dg_p^{\ell(p)} : E^u_p \to E^u_p$ such that
\[ |\chi_i^{(p)}| = \ell(p), \quad 1 \leq i \leq m - r_K \]
\[ |\chi_i^{(p)}| = \ell(p)^2, \quad m - r_K + 1 \leq i \leq m - 1 \]

Then $M$ is isometric to a compact quotient of $H^{m/r_K}_K$.

For $K = \mathbb{R}$, Theorem 1.2 is implied by Theorem 1.1, but the proof of Theorem 1.2 uses different techniques. The additional assumption that $M$ is homotopy equivalent to a compact quotient of $H^{m/r_K}_K$ is necessary only at the conclusion of the proof of the theorem where it is used to apply a theorem of Connell [10] to upgrade this homotopy equivalence to an isometry.

The hypothesis on the eigenvalues of $Dg_p^{\ell(p)}$ for periodic points $p$ in Theorem 1.2 implies by a standard argument that the Liouville measure is the measure of maximal entropy for the geodesic flow $g^t$ and hence the Liouville measure and Bowen-Margulis measure of maximal entropy coincide. This argument is reviewed in the proof of Theorem 1.2 in Section 5. A conjecture due to Katok, Sullivan, and Kaimanovich states that if $M$ is a closed manifold of negative curvature then the Liouville measure and Bowen-Margulis measure coincide if and only if $M$ is locally symmetric (see [33] for this and related conjectures). Theorem 1.2 can be viewed as giving additional evidence for this conjecture; in the language of the hypotheses of the theorem, the Liouville measure and Bowen-Margulis measure coincide if and only if for every periodic point $p$,
\[ \left| \prod_{i=1}^{m-1} \chi_i^{(p)} \right| = h \cdot \ell(p) \]
where $h > 0$ is the topological entropy of $g'$. Hence Theorem 1.2 states that if we have additional information about the absolute values of the eigenvalues $|\chi_i|$ as well as their product, then we can deduce that $M$ is locally symmetric.

We next state a more ergodic theoretic formulation of Theorem 1.1. Let $E$ be a vector bundle over $SM$ carrying a norm $\| \cdot \|$ and let $\pi : E \to SM$ be the projection map. A linear cocycle over $g^t$ is a map $A : E \times \mathbb{R} \to E$ satisfying for every $t \in \mathbb{R}$,

$$\pi(A(v,t)) = g^t(\pi(v)),$$

and for any $t, s \in \mathbb{R}$,

$$A(A(v,s),t) = A(v,t+s)$$

We adopt the notation $A'$ for the map $A(-,t)$. We will principally be concerned with the linear cocycles obtained by restricting $Dg^t$ to invariant subbundles $E$ of $SM$. For a linear cocycle $A'$ over $g^t$ and an ergodic $g^t$-invariant measure $\mu$, we define the extremal Lyapunov exponents of $A'$ with respect to $\mu$ to be

$$\lambda_+(A',\mu) := \inf_{t > 0} \frac{1}{t} \int \log \| A' \| d\mu(x)$$

$$\lambda_-(A',\mu) = \sup_{t > 0} \frac{1}{t} \int \log \| A^{-t} \|^{-1} d\mu(x)$$

A negatively curved Riemannian manifold $M$ has relatively $1/4$-pinched sectional curvatures if for each $p \in M$ and each quadruple of tangent vectors $X,Y,W,Z \in T_pM$ such that $X$ and $Y$ are linearly independent and $W$ and $Z$ are linearly independent, we have

$$K(X,Y) > 4K(W,Z),$$

where $K(X,Y)$ is the sectional curvature of the plane spanned by $X$ and $Y$.

There is a rich family of ergodic invariant measures called Gibbs measures for the geodesic flow; these include the Liouville volume on $SM$, the Bowen-Margulis measure of maximal entropy for the geodesic flow, and the harmonic measure corresponding to the hitting probability of Brownian motion inside the universal cover $\tilde{M}$ of $M$ on the visual boundary $\partial \tilde{M}$.

**Theorem 1.3.** Let $M$ be a closed negatively curved Riemannian manifold with relatively $1/4$-pinched sectional curvatures. Let $\mu$ be a Gibbs measure for $g^t$. If $\lambda_+(Dg^t|E^u,\mu) = \lambda_-(Dg^t|E^u,\mu)$, then $M$ is homothetic to a compact quotient of $H^n$.

A result similar to Theorem 1.3 was claimed by Yue in [38], however the proof appears to be incomplete. This is discussed in Remark 3.2 at the end of Section 3. This raises the following question,

**Question:** Does Theorem 1.3 hold without the relative $1/4$-pinching assumption on the curvature of $M$?

For the statement of our final result, we consider $T(SM)$ to be endowed with the Sasaki metric obtained by considering $SM$ as a Riemannian manifold. A subbundle of $E$ of $T(SM)$ is $\beta$-Hölder continuous if it is locally spanned by vector fields $V_i : U \to T(SM), U$ an open subset of $SM$, which are Hölder continuous with exponent
A cocycle $A^t : \mathcal{E} \to \mathcal{E}$ over $g^t$ is $\beta$-Hölder if $\mathcal{E}$ is a $\beta$-Hölder vector bundle and $A^t$ is $\beta$-Hölder in the induced metric on $\mathcal{E}$ from $T(SM)$.

**Definition 1.4.** A $\beta$-Hölder continuous cocycle $A^t$ is $\alpha$-fiber bunched if $\alpha \leq \beta$ and there is some $T > 0$ such that
\[
\|A^t_p\| \cdot \|A^{-t}_p\| \cdot \|Dg^t|E^u\|^\alpha < 1, \quad p \in SM, \quad t \geq T
\]
\[
\|A^t_p\| \cdot \|A^{-t}_p\| \cdot \|Dg^{-t}_p|E^s\|^\alpha < 1, \quad p \in SM, \quad t \geq T
\]

Fiber bunching guarantees the existence of $A^t$-equivariant identifications of the fibers of $\mathcal{E}$ along stable and unstable manifolds called holonomies. This is discussed further at the beginning of Section 2.

A dominated splitting for $A$ is an $A$-invariant direct sum splitting $\mathcal{E} = E^1 \oplus E^2$ such that there is some norm $\| \cdot \|$ on $\mathcal{E}$, some $C > 0$ and some $0 < \lambda < 1$ satisfying for every $p \in SM$,
\[
\|A^t|E^2_p\| \leq C\lambda^t\|A^{-t}_p|E^1_p\|^{-1}, \quad t > 0
\]

For each periodic point $p$ of $g^t$, there is a unique $g^t$-invariant probability measure $\mu_p$ supported on the orbit of $p$ which is given by the normalized pushforward of Lebesgue measure by the map $t \to g^t(p)$ from $\mathbb{R}$ to $SM$.

**Theorem 1.5.** Let $M$ be an $m$-dimensional closed negatively curved Riemannian manifold. Suppose

1. There is a dominated splitting $E^u = H^u \oplus V^u$ for the unstable bundle of the geodesic flow of $M$ with strongest expanded direction $V^u$ satisfying $\dim V^u = 1$.

2. There is some $0 < \alpha < 1$ such that $H^u$ and $E^u$ are $\alpha$-Hölder continuous and $Dg^t|H^u$ is $\alpha$-fiber bunched.

3. $\lambda_+(Dg^t|H^u, \mu) = \lambda_-(Dg^t|H^u, \mu)$ for some Gibbs measure $\mu$ for $g^t$ and for every periodic point $p$, $2\lambda_+(Dg^t|V^u, \mu_p) = \lambda_+(Dg^t|V^u, \mu_p)$.

4. The restriction of $H^u$ to the unstable foliation $W^u$ is $C^1$.

Then $m$ is even and $M$ is homothetic to a compact quotient of $H^m_C/2$.

Assumptions (1) and (2) are the natural analogues of the relative 1/4-pinching assumption in Theorem 1.3. For a closed complex hyperbolic manifold $N$ equipped with the complex hyperbolic metric $d_C$, it is not hard to show that there is an open neighborhood $U$ of $d_C$ such that assumptions (1) and (2) hold for the geodesic flow of any metric in this neighborhood. This is because the geodesic flow of $d_C$ admits a dominated splitting as in assumption (1), dominated splittings are stable under $C^1$ perturbations, and one can explicitly compute that there is an $\alpha$ such that assumption (2) holds for metrics near $d_C$.

Assumption (3) plays the same role as the extremal exponent assumption in Theorem 1.3. It would be interesting to remove Assumption (4), which restricts possible applications of Theorem 1.5. On the open neighborhood $U$ of the complex hyperbolic metric above, $H^u$ will typically be no better than Hölder continuous along $W^u$ with exponent $1/2 - \varepsilon$ for some $\varepsilon > 0$. 

\[\beta.\]
The proofs of these theorems make use of two powerful tools recently developed in smooth dynamics. The first is the method of approximation of Lyapunov exponents of invariant measures over a system by Lyapunov exponents of periodic points developed by Kalinin in his recent solution of the Livsic problem for $GL(n, \mathbb{R})$-cocycles over hyperbolic systems [21]. We use this to transfer information about the periodic exponents of $g^t$ to exponents of any invariant measure for $g^t$.

The second is a far-reaching nonlinear generalization of Furstenberg’s theorem on nonvanishing Lyapunov exponents for random $GL(n, \mathbb{R})$-cocycles which characterizes when the Lyapunov exponents of a cocycle over a partially hyperbolic system vanish under suitable hypotheses. Inspired by an alternative proof by Ledrappier [27] of Furstenberg’s theorem, Avila and Viana proved a nonlinear generalization [2], and then later with Santamaria showed how this nonlinear generalization could be applied to cocycles over partially hyperbolic systems [1]. We apply a further distillation of this tool by Kalinin and Sadovskaya in [24] which is adapted to the study of cocycles which are close to being conformal. They have applied this to the study of linear cocycles with uniformly quasiconformal behavior and asymptotically conformal Anosov diffeomorphisms [23],[22].

In Section 2 we adapt the main results of Kalinin and Sadovskaya [24] regarding conformal structures for linear cocycles to our setting. We also review the concepts of fiber bunching and stable and unstable holonomies from partially hyperbolic dynamics. In Section 3 we prove Theorem 1.1 and Theorem 1.3. In Section 4 we analyze the case of a dominated splitting $E^u = H^u \oplus V^u$ and develop the dynamical tools needed for the proofs of the remaining results. In Section 5 we use these tools to prove Theorem 1.2. In Section 6 we prove Theorem 1.5.

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2. Background on Linear Cocycles over Anosov Flows

2.1. Holonomies for linear cocycles. These identifications are essential for what follows. We define here unstable holonomies. Stable holonomies are defined similarly. Stable holonomies could also be defined as the unstable holonomies of the inverse cocycle $A^{-t}$ over $g^{-t}$. The definition below is from [24], adapted to the setting of flows.

**Definition 2.1.** An unstable holonomy for a linear cocycle $A : \mathcal{E} \times \mathbb{R} \to \mathcal{E}$ over $g^t$ is a continuous map $h^u : (x, y) \to h^u_{xy}$, where $x \in SM$, $y \in W^u(x)$, such that

1. $h^u_{xy}$ is a linear map from $\mathcal{E}_x$ to $\mathcal{E}_y$;
2. $h^u_{yx} = Id$ and $h^u_{yz} \circ h^u_{xy} = h^u_{xz}$;
3. $h^u_{xy} = (A^t_y)^{-1} \circ h^u_{yg^t} \circ A^t_x$ for every $t \in \mathbb{R}$.
The next proposition gives a sufficient condition for the existence of holonomies. For a $\beta$-Hölder vector bundle $\mathcal{E}$, it is always possible to find a $\beta$-Hölder continuous system of linear identifications $I_{xy}: \mathcal{E}_x \to \mathcal{E}_y$ with $I_{xx} = Id_{\mathcal{E}_x}$ and $d(x, y) \leq r$ for some constant $r > 0$ [24].

**Proposition 2.2.** Suppose that $\mathcal{A}$ is $\beta$-Hölder and fiber bunched. Then there is an unstable holonomy $h^u$ for $\mathcal{A}$ which satisfies

$$\|h^u_{xy} - I_{xy}\| \leq C d(x, y)^\beta$$

for $x \in SM$, $y \in W^u_{\tau}(x)$, and some $C > 0$. Furthermore, the unstable holonomy satisfying (1) for some $C > 0$ is unique.

The proof of Proposition 2.2 for fiber bunched cocycles over partially hyperbolic flows is given in [24]; an identical proof works for fiber bunched cocycles over Anosov flows instead.

Let $r > 0$ be small enough that for every $x \in SM$, all of the foliations $W^s_r$ are trivial on the ball of radius $r$. Given an unstable holonomy $h^u$ for a linear cocycle $\mathcal{A}$ over $g^t$, we can locally extend it to a center unstable holonomy $h^{cu}$ by defining for $y \in W^{cu}_r(x)$,

$$h^{cu}_{xy} = A^t_{g^u_{xy}} \circ h^{u}_{xy} = A^t_{g^u_{xy}} \circ A^t_{xy}$$

where $\tau = \tau(x, y)$ is the unique real number such that $g^{-\tau}y \in W^u_{\tau}(x)$. It is easily checked using the properties in Definition 2.1 that $h^{cu}$ satisfies properties analogous to those of $h^u$ on a ball of radius $r$. In particular, for $y \in W^{cu}_{\tau/2}(x)$ and $z \in W^{cu}_{\tau/2}(y)$, we have that $h^{cu}_{xy} \circ h^{cu}_{yz} = h^{cu}_{xz}$. We also see that if $y \in W^{cu}_{\tau}(x)$ and $d(g^t y, g^t x) \leq r$, then $h^{cu}_{xy} = (A^t_{y})^{-1} \circ h^{cu}_{g^t x g^t y} \circ A^t_{x}$.

It is not immediately clear from the formula that the center unstable holonomies extend to be globally defined on a center unstable leaf $W^{cu}(x)$; to prove this we use some of the special structure of the geodesic flow. Since $m \geq 3$, the universal cover of $SM$ is the unit tangent tangent bundle $\tilde{SM}$ of the universal cover $\tilde{M}$ of $M$, and $\pi_1(SM)$ is canonically isomorphic to $\pi_1(M)$ by the projection $SM \to M$. The foliations $W^s$ lift to foliations $\tilde{W}^s$ of $\tilde{SM}$ which have global product structure: for each $x, y, z \in \tilde{SM}$, the leaves $\tilde{W}^s(x), \tilde{W}^u(y), \tilde{W}^u(z)$ intersect in exactly one point.

Let $\tilde{\mathcal{E}}$ be the lift of the vector bundle $\mathcal{E}$ to a vector bundle over $\tilde{SM}$. For two points $x \in \tilde{SM}$, $y \in \tilde{W}^{cu}(x)$, the center unstable holonomy map $h^{cu}_{xy} : \tilde{\mathcal{E}}_x \to \tilde{\mathcal{E}}_y$ is defined by the formula

$$h^{cu}_{xy} = \tilde{A}^t_{g^u_{xy}} \circ h^{u}_{xy} = \tilde{A}^t_{g^u_{xy}} \circ \tilde{A}^t_{xy}$$

where $\tau = \tau(x, y)$ is the unique time $\tau \in \mathbb{R}$ such that $\tilde{g}^{-\tau}y \in \tilde{W}^u(x)$. It’s easy to check that this locally agrees with the previously defined center unstable holonomy, and gives a global extension of $h^{cu}$ satisfying the analogous properties in Definition 2.1.

Let $\partial \tilde{M}$ be the visual boundary of $\tilde{M}$. This global product structure corresponds to the Hopf parametrization,

$$\tilde{SM} = \mathbb{R} \times \partial \tilde{M} \times \partial \tilde{M} \setminus \Delta(\partial \tilde{M} \times \partial \tilde{M})$$
given as follows: Fix a basepoint \( x \in \tilde{M} \). Let \( v \in SM \). \( v \) is tangent to a geodesic \( \gamma_v \) which has endpoints \( v_+, v_- \in \partial\tilde{M} \), where \( v_+ \) corresponds to the forward endpoint of \( \gamma_v \), and \( v_- \) the backward endpoint. Let \( x_v \) be the orthogonal projection of \( x \) onto \( \gamma_v \), and let \( s \) be the distance from \( x_v \) to \( P(v) \), where \( P : SM \to \tilde{M} \) is projection. Then the identification is given by \( v \to (s, v_+, v_-) \). In this identification, the action of the geodesic flow is given by translation in the \( \mathbb{R} \)-coordinate. This parametrization of \( SM \) will be important in Sections 3 and 4.

2.2. Continuous Amenable Reduction. We now adapt the main results of [24] to our setting. Let \( E \) be a \( d \)-dimensional Hölder continuous vector bundle over \( SM \). We let \( \mu \) be a fully supported ergodic \( g^t \)-invariant measure with local product structure. This means that each point \( x \in SM \) has a neighborhood \( U \) on which the measure \( \mu \) decomposes as a product \( \mu_u \times \mu_s \times \mu_c \) corresponding to the product decomposition \( U = W^u \times W^s \times W^c \). All Gibbs measures for \( g^t \) satisfy these properties, see for instance Chapter 20 of [26]. We will usually use the Liouville volume when we need to choose a specific Gibbs measure.

Two Riemannian metrics \( \tau \) and \( \sigma \) on \( E \) are conformally equivalent if there is a function \( a : SM \to \mathbb{R} \) such that \( \tau_p = a(p)\sigma_p \). A conformal structure on \( E \) is a conformal equivalence class of Riemannian metrics on \( E \). \( A^t \) transforms a conformal structure by pulling back the associated Riemannian metric. A conformal structure represented by a Riemannian metric \( \tau \) is invariant under \( A \) if for each \( t \in \mathbb{R} \) there is a map \( \psi_t : SM \to \mathbb{R} \) satisfying

\[
(A^t)^* \tau = \psi_t \tau
\]

In this case we say that \( \psi_t \) is the multiplicative cocycle associated to the invariant conformal structure \( \tau \). \( \psi_t \) satisfies the cocycle property

\[
\psi^{t+s}(p) = \psi^t(p)\psi^s(g^t(p))
\]

for any \( t, s \in \mathbb{R} \).

Two multiplicative cocycles \( \psi_t \) and \( \varphi_t \) are cohomologous if there is a map \( \zeta : SM \to \mathbb{R} \) such that

\[
\frac{\psi_t}{\varphi_t} = \zeta \circ g^t
\]

for every \( t \in \mathbb{R} \).

If a cocycle \( A \) over \( SM \) admits stable and unstable holonomies, we say that a subbundle \( V \subset E \) is holonomy invariant if for \( y \in W^s(x) \) we have \( h^s_{xy}(V_x) = V_y \) for \( * = u \) or \( s \). Similarly we say that a conformal structure is holonomy invariant if it is invariant under pulling back by stable and unstable holonomies.

**Lemma 2.3.** Let \( A \) be a fiber bunched cocycle over \( g^t \). Suppose that

\[
\lambda_+(A, \mu) = \lambda_-(A, \mu)
\]

Then any measurable \( A \)-invariant subbundle \( V \subset E \) coincides \( \mu \)-a.e. with a \( A \)-invariant holonomy invariant continuous subbundle. Under the same hypotheses, any \( A \)-invariant measurable conformal structure \( \tau \) on \( E \) coincides \( \mu \)-a.e. with a \( A \)-invariant holonomy invariant continuous conformal structure.
Proof. The cocycle generated by $A^1$ is a fiber bunched cocycle over the partially hyperbolic diffeomorphism $g^1$. Since $g^1$ is a contact Anosov flow, $g^1$ is accessible. In the first case $V$ is a measurable invariant subbundle for $A^1$; in the second case, $\tau$ is an invariant measurable conformal structure for $A^1$. Theorem 3.3 and Theorem 3.1 respectively from [24] then apply to give the desired result. \hfill $\Box$

Lemma 2.4. Let $A$ be a fiber bunched cocycle over $g^t$. Suppose that

$$\lambda_+(A, \mu) = \lambda_-(A, \mu).$$

Then there is a finite cover $M$ of $M$ and a flag

$$0 \subseteq E^1 \subseteq E^2 \subseteq \cdots \subseteq E^k = \hat{E}$$

of continuous holonomy-invariant subbundles $E^i$, which are invariant under the action of the lifted cocycle $\hat{A}$, on the lifted bundle $\hat{E}$ over $M$. Furthermore the induced action of the cocycle $\hat{A}_i$ on $E^i/E^{i-1}$ preserves a continuous holonomy invariant conformal structure.

Proof. The vector bundle $E$ admits a measurable trivialization on a set of full $\mu$-measure by Proposition 2.12 in [3]. Since $\mu$ is fully supported on $SM$, this implies that there is a measurable map $P : E \to SM \times \mathbb{R}^d$ commuting with the projections onto $SM$ and which is linear on the fibers. $B = PAP^{-1}$ is a measurable linear cocycle over $g^t$ on the trivial vector bundle $SM \times \mathbb{R}^d$. We can apply Zimmer’s amenable reduction theorem [39] for $R$-cocycles to conclude that there is a measurable map $C : SM \to GL(d, \mathbb{R})$ such that the cocycle $\mathcal{F} = C B C^{-1}$ takes values in an amenable subgroup $G$ of $GL(d, \mathbb{R})$.

The maximal amenable subgroups of $GL(d, \mathbb{R})$ are classified in [28]. Any such group $G$ contains a finite index subgroup $K$ which is conjugate to a subgroup of a group of the form

$$H(d_1, \ldots, d_k) = \begin{bmatrix} A_1 & * & * & * \\ 0 & A_2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & A_k \end{bmatrix}$$

where $\sum_{i=1}^k d_i = d$ and $A_i \in \mathbb{R} \cdot SO(d_i, \mathbb{R})$. Thus, by conjugating the cocycle $\mathcal{F}$ if necessary, we may assume that $\mathcal{F}$ takes values in a group $G$ which contains a finite index subgroup $K$ that is contained in one of the groups $H(d_1, \ldots, d_k)$. Let $G_*$ be the stabilizer in $G$ of the flag $V^1 \subset V^2 \subset \cdots \subset V^k = \mathbb{R}^d$ corresponding to the group $H(d_1, \ldots, d_k)$ containing $K$. Thus $V^j$ is the span of the first $\sum_{i=1}^j d_i$ coordinate axes in $\mathbb{R}^d$. Let $\ell$ be the index of $G_*$ in $G$, which is finite since $K$ has finite index in $G$ and $K \subset G_*$. Let $V^{i,j}$, $j = 1, \ldots, \ell$ be the at most $\ell$ distinct images of the subspace $V^i$ under the action of $G$. Let $U^i = \bigcup_{j=1}^\ell V^{i,j}$. Then let $\hat{E}_x^{i,j} = (C \circ P)^{-1}(x) \cdot V^{i,j}$, $\hat{U}^i = (C \circ P)^{-1}(x) \cdot U^i$. The proof of Theorem 3.4 in [24] shows that if the union of measurable subbundles $\hat{U}^i$ is invariant under a fiber bunched cocycle with equal extremal exponents over an accessible partially hyperbolic system (which we can take to be the time 1 map $A^1$ of the cocycle $A$ over $g^1$), then there is a finite cover $M$ of $M$ such that the individual subbundles $\hat{E}_x^{i,j}$ lift to subbundles $E^{i,j}$ of the lifted
bundle \( \tilde{E} \) over \( \mathcal{M} \) which agree \( \mu \)-a.e. with continuous subbundles which we will also denote \( E^{i,j} \). By construction the lifts \( \mathcal{U}^t \) are invariant \( \mu \)-a.e. under the action of the lift \( \tilde{\mathcal{A}} \) of the cocycle \( \mathcal{A} \). This is because we constructed these unions of subbundles using amenable reduction over the \( \mathbb{R} \) action given by \( \mathcal{A} \), and under our measurable trivialization \( \mathcal{A} \) takes values in the group \( G \). Since \( \mathcal{A} \) is continuous and the lifts \( \mathcal{U}^t \) are continuous after modification on a \( \mu \)-null set, we conclude that each \( \mathcal{U}^t \) is everywhere invariant under \( \mathcal{A} \).

For each \( i \in \{1, \ldots, k\} \), \( x \in \mathcal{M} \), \( t \in \mathbb{R} \), and \( j \in \{1, \ldots, \ell\} \), there is thus an integer \( S_i(x,t,j) \) such that \( A^t(E^{i,j}_x) = E^{i,S_i(x,t,j)}_x \). For a fixed \( i \) and \( j \), \( S_i(x,t,j) \) depends continuously on \( x \) and \( t \) since both \( \tilde{A}^t \) and all of the subbundles \( E^{i,j} \) are continuous. Since for a fixed \( i \) and \( j \) we have that \( S_i(x,t,j) \) is continuous, integer valued, and has connected domain \( \mathcal{M} \times \mathbb{R} \), we conclude that \( S_i(x,t,j) := S_i(j) \) is constant in \( x \) and \( t \). Furthermore, since \( S_i(x,0,j) = j \), we conclude that \( S_i(j) = j \). Hence all of the subbundles \( E^{i,j} \) are invariant under \( \mathcal{A} \) as well. In particular \( \mathcal{A} \) preserves the flag \( E^1 \subset \cdots \subset E^k \) which arises as the continuous extension of the lift of the flag coming from the standard flag \( V^1 \subset V^2 \subset \cdots \subset V^k \).

To prove the second claim, note that for any \( r \geq 1 \), the induced action of the cocycle \( \mathcal{F} \) on \( V^r \Sigma_{i=1}^d / V^r \Sigma_{i=1}^{r-1} d_i = \mathbb{R}^{dr} \) preserves the standard Euclidean conformal structure on \( \mathbb{R}^{dr} \). This immediately implies that \( \tilde{\mathcal{A}} \) preserves a measurable conformal structure on the corresponding quotient bundle \( \tilde{E}^j / \mathcal{E}^j_{-1} \). By Lemma 2.3, this measurable conformal structure coincides \( \mu \)-a.e. with a holonomy invariant continuous conformal structure. \( \square \)

**Lemma 2.5.** Suppose that there is a finite cover \( \mathcal{M} \) of \( M \) such that the lifted cocycle \( \tilde{\mathcal{A}} \) on the lifted bundle \( \tilde{E} \) preserves a continuous holonomy-invariant conformal structure. Then \( \mathcal{A} \) also preserves a continuous holonomy-invariant conformal structure.

**Proof.** Let \( \tilde{C}_x \) be the space of conformal structures on the vector space \( \tilde{E}_x \). \( \tilde{C}_x \) can be identified with the Riemannian symmetric space \( SL(d, \mathbb{R}) / SO(d, \mathbb{R}) \) and in fact carries a canonical Riemannian metric of nonpositive curvature for which the induced map \( \tilde{C}_x \to \tilde{C}_y \) over the cocycle \( \tilde{\mathcal{A}} \) is an isometry [24]. In particular, for compact subsets \( K \subset \tilde{C}_x \) there is a natural barycenter map \( K \to \text{bar}(K) \) mapping \( K \) to its center of mass.

Let \( \tau \) be the continuous holonomy-invariant conformal structure preserved by \( \tilde{\mathcal{A}} \). Let \( H \) be the group of covering transformations for \( \mathcal{M} \) over \( M \), which also acts as the group of covering transformations for \( \tilde{E} \) over \( \mathcal{E} \). Let \( K_x = \bigcup_{\rho \in H} \{ \rho \cdot \tau_{\rho^{-1}(x)} \} \subset \tilde{C}_x \). The collection of compact subsets \( K_x \) depends continuously on \( x \), is holonomy-invariant, and is invariant under \( \mathcal{A} \). Hence all of the same is true of the family of barycenters \( \sigma_x := \text{bar}(K_x) \). We thus get a conformal structure \( \sigma \) that is continuous, holonomy-invariant, invariant under \( \tilde{\mathcal{A}} \), and also invariant under the action of the deck group \( H \). \( \sigma \) then descends to the desired conformal structure on \( \mathcal{E} \). \( \square \)

In subsequent sections we will use Lemmas 2.4 and 2.5 together to construct invariant conformal structures for our cocycles of interest. We will first use Lemma 2.4
to construct an invariant flag on a finite cover, then we will show this flag must be trivial, then lastly we will use Lemma 2.5 to push the invariant conformal structure back down to our original bundle.

**Remark 2.6.** Lemmas 2.3 and 2.4 are true under the more general assumption that $g^t$ is a contact Anosov flow, as this implies that the time 1 map $g^1$ for the flow is accessible. It would be interesting to prove these lemmas only under the assumption that $g^t$ is an Anosov flow, i.e., without using the accessibility of the time 1 map.

### 2.3. Lyapunov exponents and periodic approximation.

For a cocycle $\mathcal{A} : E \times \mathbb{R} \to E$ over $g^t$ and an ergodic $g^t$-invariant measure $\mu$, the multiplicative ergodic theorem [3] implies that there is a $g^t$-invariant subset $\Lambda \subset SM$ with $\mu(\Lambda) = 1$ such that over $\Lambda$ there is a measurable $g^t$-invariant splitting $E = E_1 \oplus E_2 \oplus \cdots E_k$ and numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ such that
\[
\lim_{t \to \infty} \frac{1}{t} \log \|A^t(v)\| = \lambda_i, \; v \in E_i.
\]
The numbers $\lambda_i$ are the **Lyapunov exponents** of $\mathcal{A}$. The extremal Lyapunov exponents $\lambda_+$ and $\lambda_-$ of $\mathcal{A}$ with respect to $\mu$ correspond to the top and bottom exponents $\lambda_k$ and $\lambda_1$ respectively.

For each periodic point $p$, we let $\mu_p$ denote the unique $g^t$-invariant probability measure supported on the orbit of $p$, which may be obtained as the normalized push-forward of Lebesgue measure on $\mathbb{R}$ by the map $t \to g^t(p)$. The following theorem of Kalinin enables us to approximate the Lyapunov exponents of any $g^t$-invariant measure by the Lyapunov exponents of measures concentrated on a periodic orbit. This theorem is the essential new tool needed for the proof of the Livsic theorem in the case of matrix cocycles. The fact that $g^t$ satisfies the closing property necessary in the hypothesis of the theorem as stated in [21] is the well known Anosov closing lemma for flows which can be found in Chapter 18 of [26]. The statement of Theorem 2.7 in [21] assumes that the vector bundle $E$ over $SM$ is trivial, but as remarked by Kalinin in the paper, this hypothesis is easily removed since the proof of the theorem only uses local comparisons between fibers. When we say that the Lyapunov exponents are counted with multiplicity, we mean that each exponent appears a number of times equal to the dimension $\dim E_i$ of its corresponding measurable invariant subbundle.

**Theorem 2.7.** ([21]) Let $E$ be a $d$-dimensional Hölder continuous vector bundle over $SM$, and $\mathcal{A}$ a cocycle on $E$ over $g^t$. Let $\mu$ be an ergodic $g^t$-invariant measure, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$ be the Lyapunov exponents of $\mathcal{A}$ with respect to $\mu$, counted with multiplicity. Then for every $\varepsilon > 0$, there is a periodic point $p$ of $g^t$ such that the Lyapunov exponents $\lambda_1^{(p)} \leq \lambda_2^{(p)} \leq \cdots \leq \lambda_d^{(p)}$ of $\mathcal{A}$ with respect to $\mu_p$ satisfy
\[
|\lambda_i - \lambda_i^{(p)}| < \varepsilon
\]
for each $1 \leq i \leq d$.

For a periodic point $p$ there is a simple relationship between the Lyapunov exponents $\lambda_i^{(p)}$ associated to $\mu_p$ and the complex eigenvalues $\chi_i^{(p)}$ of the map $A_p^{t(p)}$.
Let $\mathcal{E}_p \rightarrow \mathcal{E}_p$. Let $\mathcal{E}_p = \mathcal{E}_p^1 \oplus \cdots \oplus \mathcal{E}_p^k$ be the direct sum decomposition of $\mathcal{E}_p$ from the multiplicative ergodic theorem and let $\mathcal{E}_p = \mathcal{V}^1 \oplus \cdots \mathcal{V}^r$ be the primary decomposition of the linear transformation $A_p^{\ell(p)} : \mathcal{E}_p \rightarrow \mathcal{E}_p$, where each $\mathcal{V}^i$ corresponds to an irreducible factor of the minimal polynomial $A_p^{\ell(p)}$. An easy linear algebra exercise shows that the primary decomposition is subordinate to the Oseledec decomposition, i.e., for each $1 \le i \le k$,

$$\mathcal{E}^i = \mathcal{V}^{i_1} \oplus \cdots \oplus \mathcal{V}^{i_n}$$

for some integers $1 \le i_1, \ldots, i_n \le r$. Furthermore, we have the relationship

$$\frac{1}{\ell(p)} \log |\lambda_j^{(p)}| = \lambda_j^{(p)}, \quad 1 \le j \le n$$

for the real eigenvalues (or conjugate pairs of complex eigenvalues) corresponding to the subspaces $\mathcal{V}^{i_j}$. Thus the Lyapunov exponents of $\mu_p$ are given by the logarithms of the absolute values of the eigenvalues of $A_p^{\ell(p)}$, normalized by the period of $p$.

3. Proof of Theorems 1.1 and 1.3

We are now ready to prove Theorem 1.1 and Theorem 1.3. A subbundle $\mathcal{V} \subset \mathcal{E}$ is proper if $0 < \dim \mathcal{V} < \dim \mathcal{E}$. For $0 < \alpha < 2$ we say that $g^t$ is $\alpha$-bunched if there is some $T > 0$ such that for $t \ge T$

$$\|Dg_p^t|E_p^u\|^2 \cdot \|Dg_p^t|E_p^c\| \cdot \|(Dg_p^t)^{-1}|E_p^s\| < 1, \quad t \ge T, p \in SM$$

For an in-depth discussion of the relationship between $\alpha$-bunching and the regularity of the Anosov splitting $T(SM) = E^u \oplus E^c \oplus E^s$, see [18].

**Lemma 3.1.** Let $g^t$ be the geodesic flow on the unit tangent bundle of a closed negatively curved manifold. Suppose that $g^t$ is 1-bunched and that there is a Gibbs measure $\mu$ such that $\lambda_+(Dg^t|E^u, \mu) = \lambda_-(Dg^t|E^u, \mu)$. Then $E^u$ has no proper measurable $g^t$-invariant subbundles.

**Proof.** Let $V \subset E^u$ be a $k$-dimensional measurable invariant subbundle. Since $g^t$ is 1-bunched the Anosov splitting of $g^t$ is $C^1$ [18] and thus $E^u$ is a $C^1$-subbundle of $T(SM)$. 1-bunching of $g^t$ also implies that the cocycle $Dg^t|E^u$ is fiber bunched. By Theorem 2.3, $V$ thus coincides $\mu$-a.e. with a continuous holonomy invariant subbundle which we will still denote by $V$.

We now describe an alternative realization of the holonomies for $Dg^t|E^u$. Recall that $\theta$ denotes the the invariant contact form for $g^t$. Since the Anosov splitting of $g^t$ is $C^1$, and $g^t$ preserves $\theta$, there is a unique $g^t$-invariant connection $\nabla$ on $SM$ such that the torsion of $\nabla$ is given by $\theta \otimes \dot{g}$, where $\dot{g}$ is the vector field generating $g^t$ on $SM$. This connection is called the Kanai connection and was constructed for contact Anosov flows with $C^1$ Anosov splitting in [25].

In Lemma 1.1 of [25], it is shown that the unstable foliation $W^u$ is totally geodesic for $\nabla$, and that $\nabla$ is $C^1$ when restricted to the leaves of the unstable foliation, and further that $\nabla$ is flat when restricted to $W^u$ leaves. The parallel transport induced by $\nabla$ on unstable leaves is thus a $C^1$ unstable holonomy for $g^t$. From the uniqueness clause of Proposition 2.2, parallel transport by $\nabla$ coincides with
the unstable holonomy constructed in Proposition 2.2, and thus $V$ is parallel with respect to $\nabla$ along unstable leaves.

For a given unstable leaf $W^u(p)$ we can then find parallel vector fields $X_1, \ldots, X_k$ spanning the restriction of $V$ to $W^u(p)$. These vector fields are $C^1$ since $\nabla|W^u(p)$ is $C^1$. The restriction of $\nabla$ to $W^u(p)$ is torsion-free since $\eta$ vanishes on $W^u$. For $C^1$ vector fields, there is still a well-defined Lie bracket, and the Frobenius theorem characterizing integrability of a distribution remains true [32]. Since

$$0 = \nabla_{X_i} X_j - \nabla_{X_j} X_i = [X_i, X_j], \quad 1 \leq i, j \leq k$$

we then conclude via the $C^1$ Frobenius theorem that $V$ is a uniquely integrable subbundle of $TW^u$. Hence there is a $C^2$ foliation $\mathcal{V}$ of $SM$ which is tangent to $V$, such that each of the leaves $\mathcal{V}(p)$ is contained within the corresponding unstable leaf $W^u(p)$.

Then $\mathcal{V}$ lifts to a foliation $\tilde{\mathcal{V}}$ of $\tilde{SM}$ which is invariant under the lifted action of $g'$ and the action of $\pi_1(M)$. We adopt the notation of the Hopf parametrization described in Section 2. For each $x \in SM$ there is a homeomorphism $\pi_x : \tilde{W}^u(x) \to \partial\tilde{M}\setminus\{x_\pm\}$ given by projection, where $x_-$ is the negative endpoint of the geodesic through $x$ on $\partial M$. Then for a pair of points $x, y \in \tilde{SM}$ we consider the homeomorphism

$$\pi_y^{-1} \circ \pi_x : \tilde{W}^u(x) \setminus \{\pi_x^{-1}(y_-)\} \to \tilde{W}^u(y) \setminus \{\pi_y^{-1}(x_-)\}$$

The homeomorphism $\pi_y^{-1} \circ \pi_x$ is easily described in terms of the global product structure of $SM$: for a point $z \in \tilde{W}^u(x) \setminus \{\pi_x^{-1}(y_-)\}$, $\pi_y^{-1}(\pi_x(z))$ is the unique intersection point of $\tilde{W}^cs(z)$ and $\tilde{W}^u(y)$.

Since the Anosov splitting of $g'$ is $C^1$, the map $\pi_y^{-1} \circ \pi_x$ is $C^1$ and the derivative is given by parallel transport with respect to the Kanai connection $\nabla$, which coincides with the global center stable holonomy map $h^cs$ for $Dg'|E^u$ by the uniqueness statement in Proposition 2.2. Since $\tilde{V}$ is invariant under the action of $g'$ and stable holonomy, $\tilde{V}$ is invariant under center stable holonomy and therefore $D(\pi_y^{-1} \circ \pi_x)(\tilde{V}) = \tilde{V}$. Since $\tilde{V}$ is uniquely integrable, this implies for every $z \in \tilde{W}^u(x)$ that $\pi_y^{-1}(\pi_x(\tilde{V}(z))) = \tilde{V}(\pi_y^{-1}(\pi_x(z)))$.

The homeomorphisms $\{\pi_x : x \in \tilde{SM}\}$ form a system of charts for $\partial\tilde{M}$ which give $\partial\tilde{M}$ the structure of a $C^1$ manifold. The equivariance property of the foliation $\tilde{V}$ with respect to these charts implies that $\tilde{V}$ descends to a $C^1$ foliation $\mathcal{F}$ of $\partial M$. Furthermore, since $\tilde{V}$ is equivariant under the action of $\pi_1(M)$ (as it was lifted from a foliation $\mathcal{V}$ on $SM$), the foliation $\mathcal{F}$ is invariant under the action of $\pi_1(M)$ on $\partial M$.

But every $\pi_1(M)$-invariant continuous foliation of $\partial M$ must be trivial, i.e., either for every $\xi \in \partial M$ we have $\mathcal{F}(\xi) = \{\xi\}$ or for every $\xi \in \partial M$ we have $\mathcal{F}(\xi) = \partial M$. This is proved in Section 4 of [15]; see also [12]. This implies that $V = \{0\}$ or $V = E^u$, which completes the proof.

**Proof of Theorem 1.1.** Since $g'$ is a contact Anosov flow preserving the contact form $\theta$ with $\ker \theta = E^u \oplus E^s$ and $d\theta|\ker \theta$ being nondegenerate, the hypotheses of Theorem 1.1 imply that for any periodic point $p$, the eigenvalues of $Dg_p' : E^s_p \to E^u_p$
$E^u_p$ are all equal in absolute value, and their common absolute value is the reciprocal of the absolute value of the eigenvalues of $Dg^T_p : E^u_p \rightarrow E^u_p$. As a consequence, for any periodic point $p$, $g^t$ is $\alpha$-bunched along the orbit of $p$ for any $\alpha < 2$. The main result of Hasselblatt in [17] then implies that $g^t$ is $1$-bunched, so that the Anosov splitting of $g^t$ is $C^1$.

Theorem 2.7 implies that for every ergodic $g^t$-invariant measure $\mu$, $\lambda_+ (Dg^t|E^u, \mu) = \lambda_- (Dg^t|E^u, \mu)$. In particular, this holds when $\mu$ is the Liouville measure on $SM$, which as remarked earlier, is a Gibbs measure for $g^t$. As remarked in Lemma 3.1, $Dg^t|E^u$ is fiber bunched and so we can apply Lemma 2.4: there is a finite cover $SM$ of $SM$ for which the conclusions of Lemma 2.4 are satisfied. Since any lift of $g^t$ to a finite cover of $SM$ is itself the geodesic flow of a closed negatively curved manifold $M$, we see that by Lemma 3.1, the invariant flag constructed in Lemma 2.4 must be trivial, and thus by Lemma 2.5 there must be a continuous holonomy invariant conformal structure on $E^u$ preserved by $Dg^t$. By Theorem 1 of [25], if $Dg^t|E^u$ preserves a continuous conformal structure then $M$ is homotopy equivalent to a real hyperbolic manifold $N$ and there is a $C^1$ time-preserving conjugacy of the geodesic flow of $M$ to the geodesic flow of $N$. The minimal entropy rigidity theorem from [6] then implies that $M$ is homothetic to $N$.

**Proof of Theorem 1.3.** Hasselblatt [18] proved that if the sectional curvatures of $M$ are relatively $1$-pinched, then $g^t$ is $1$-bunched and so the Anosov splitting of $g^t$ is $C^1$. The hypothesis that $\lambda_+ (Dg^t|E^u, \mu) = \lambda_- (Dg^t|E^u, \mu)$ for the Gibbs measure $\mu$ together with Lemma 3.1 then implies that $Dg^t$ preserves a conformal structure on $E^u$. The proof then concludes in the same manner as the proof of Theorem 1.1 above.

**Remark 3.2.** In this remark we explain the gap in [38] mentioned in the introduction. First we recall the setting of the paper. The claim is that if $M$ is a closed $m$-dimensional negatively curved manifold and $Dg^t|E^u$ is measurably irreducible in the sense that there are no $Dg^t$-invariant measurable subbundles of $E^u$, then $Dg^t|E^u$ preserves a continuous conformal structure and therefore $M$ is homothetic to a real hyperbolic manifold by the same proof as given in Theorem 1.1. In the first part of the remark we explain some flaws in the definition of boundedness for a conformal structure that is given in [38], and in the second part we explain how, even after correcting these flaws in the definition, the proof still appears to have a gap in proving boundedness at a critical step.

As mentioned in Section 2.2, the conformal structures on $E^u$ can be topologized as a fiber bundle $\mathcal{C}$ over $SM$. Each fiber $\mathcal{C}_x$ may be identified with the nonpositively curved symmetric space $S(n) := SL(n, \mathbb{R})/SO(n, \mathbb{R})$, where $n = \dim E^u = m - 1$. If we take this identification to be induced by a linear trivialization $E^u_x \rightarrow \mathbb{R}^n$, then it is unique up to an isometry of $S(n)$ and therefore $\mathcal{C}_x$ carries a canonical metric $\rho_x$ of nonpositive curvature. The bundle $\mathcal{C}$ over $SM$ has no distinguished section $SM \rightarrow \mathcal{C}$ and therefore in order to say that a conformal structure $\tau$ is “bounded” we thus have to compare it to a specific conformal structure $\tau_0 : SM \rightarrow \mathcal{C}$ which we have chosen beforehand. This is handled properly in [23], in which a conformal structure $\tau$ is defined to be bounded if there is a **continuous** conformal structure
\( \tau_0 : SM \to C \) and a constant \( C > 0 \) such that
\[
\rho_x(\tau(x), \tau_0(x)) < C \text{ for every } x \in SM.
\]

In [38], a measurable trivialization \( E^u \to SM \times \mathbb{R}^n \) is fixed and a conformal structure on \( E^u \) is then defined to be a measurable map \( \tau : SM \to S(n) \). A measurable conformal structure is defined to be "bounded" if there is a constant \( C > 0 \) such that \( \rho(\tau(x), I) < C \), where \( \rho \) is the nonpositively curved metric on \( S(n) \) and \( I \) is the image of the identity matrix in \( S(n) \). In the definition of boundedness in [23], this corresponds to taking \( \tau_0 \) to be the section defined by pulling back the standard Euclidean metric on \( \mathbb{R}^n \) via the measurable trivialization \( E^u \to SM \to \mathbb{R}^n \). This is problematic because on page 747 of [38] it is claimed that boundedness of the invariant conformal structure implies that \( Dg^t|E^u \) is uniformly quasiconformal with respect to the continuous conformal structure on \( E^u \) defined by restricting the Riemannian metric on \( T(SM) \) to \( E^u \). But \( \tau \) being a bounded distance from the measurable section \( \tau_0 \equiv I \) does not imply it is a bounded distance from any continuous conformal structure.

Even after repairing this issue with the definition of boundedness, there is still an apparent gap in the argument which occurs on page 747 of [38]. At this point measurable \( g^t \)-invariant affine connections \( D^s \) and \( D^u \) along the \( W^s \) and \( W^u \) foliations respectively have been constructed which are continuous when restricted to an individual \( W^s \) and \( W^u \) leaf respectively, but are only measurable in the transverse direction. We let \( \mu \) denote the Liouville measure on \( SM \). For \( y \in W^u(x) \) we let \( P^u_{xy} : E^u_x \to E^u_y \) be the parallel transport map with respect to \( D^u \), and let \( P^s_{xy} : E^s_x \to E^s_y \) be the analogous parallel transport map for \( D^s \) with \( z \in W^s(x) \) instead. A measurable conformal structure \( \sigma^u \) has also been constructed on the unstable bundle \( E^u \) which is \( \mu \)-a.e. parallel with respect to \( D^s \) and \( D^u \) in the following sense: for \( \mu \)-a.e. pair \( x, y \in SM \) with \( y \in W^u(x) \), there is a constant \( \xi^u(x, y) \) such that for every \( v, w \in E^u_x \)
\[
\xi^u(x, y)\sigma^u(v, w) = \sigma^s(P^s_{xy}(v), P^s_{xy}(w))
\]
An analogous statement is true for parallel transport of \( \sigma^u \) with respect to \( P^u_{xy} \). It is claimed that this data implies that \( \sigma^u \) is "locally essentially bounded" which as we've seen must be interpreted to mean that for each \( p \in SM \), there is a neighborhood \( U \) of \( p \), a constant \( C > 0 \), and a continuous section \( \tau_0 : U \to C|U \) such that
\[
\rho_x(\sigma^u(x), \tau_0(x)) < C \text{ for } \mu \text{-a.e. } x \in U.
\]
The invariance of \( \sigma^u \) under \( D^u \) and \( D^s \) together with the fact that \( P^u \) and \( P^s \) induce isometries between the fibers of \( C \) gives, for \( y \in W^u(x) \), \( z \in W^s(x) \),
\[
\rho_y(\sigma^u(y), (P^u_{yx})^*\tau_0(x)) = \rho_x(\sigma^u(x), \tau_0(x)) = \rho_x(\sigma^u(z), P^s_{xz})^*\tau_0(x)).
\]
This does not allow us to compare \( \rho_x(\sigma^u(x), \tau_0(x)) \) to \( \rho_y(\sigma^u(y), \tau_0(y)) \) unless we also have uniform bounds on \( \rho_y(\tau_0(y), (P^u_{yx})^*\tau_0(x)) \). But the parallel transport maps \( P^u_{xy} \) and \( P^s_{xz} \) depend only measurably on \( x, y, z \) and so, for instance, \( \rho_y(\tau_0(y), (P^u_{yx})^*\tau_0(x)) \) could grow arbitrarily large as \( x, y \) vary through the neighborhood \( U \) of \( p \). In particular, there is no reason for \( P^u \) and \( P^s \) to behave nicely with respect to some continuous conformal structure on \( E^u \) over \( U \). This point is not addressed in [38] and the proof appears incomplete as a result.
4. Horizontal Subbundles

In this section we assume the existence of a dominated splitting $E^u = H^u \oplus V^u$ for which $V^u$ is the most expanding bundle. We will refer to $H^u$ as the horizontal unstable bundle and $V^u$ as the vertical unstable bundle. Propositions 4.1 and 4.2 as well as Lemmas 4.3 and 4.4 of this section do not use any of the special structure of $g^t$ as a geodesic flow and therefore apply more generally when $g^t$ is any Anosov flow with a dominated splitting $E^u = H^u \oplus V^u$ of the unstable bundle.

**Proposition 4.1.** $V^u$ is uniquely integrable with smooth leaves. The resulting foliation $W^{vu}$ is smooth when restricted to $W^u$ leaves.

**Proof.** Consider $f := g^1$ as a partially hyperbolic map with invariant splitting $E^u_f = E^u_f \oplus E^c_f$, where $E^u_f = V^u$, $E^c_f = H^u \oplus E^c$, and $E^s_f = E^s$. The statements of the proposition then follow from standard results in the theory of partially hyperbolic diffeomorphisms [19].

For each $p \in SM$, we define an equivalence relation $\sim$ on points $x, y \in W^u(p)$ by $x \sim y$ if $x \in W^u(y)$. We let $Q^u(p)$ be the quotient of $W^u(p)$ by this equivalence relation, which can be identified with the space of $W^u$ leaves inside of $W^u(p)$, and we let $\Pi : W^u \to Q^u$ be the projection map. The next proposition verifies that the leaves of the $W^{vu}$ foliation are properly embedded in $W^u$, which implies that $Q^u(p)$ is a smooth manifold diffeomorphic to $\mathbb{R}^k$, $k = \dim H^u$.

**Proposition 4.2.** For each $p \in SM$, there is a smooth embedding $\iota_p : \mathbb{R}^k \to W^u(p)$ with $\iota_p(0) = p$ such that $\iota_p(\mathbb{R}^k)$ meets each $W^{vu}$ leaf inside of $W^u(p)$ in exactly one point.

**Proof.** Let $f = g^1$ and consider this as a partially hyperbolic map as in Proposition 4.1. The theory of partially hyperbolic diffeomorphisms then tells us that there is some $r > 0$ such that on any ball of radius $r$ in $SM$, the foliation tangent to $E^u_f$ is trivial [19]. Furthermore, since there is a foliation tangent to $E^u_f \oplus E^c_f$, and the unstable foliation $W^u_f$ tangent to $E^u_f$ always smoothly subfoliates $E^u_f \oplus E^c_f$, we can choose this trivialization to be smooth along $W^u$ leaves. Choose a sequence of times $t_n \to \infty$ such that $g^{-t_n}(p) \to p$ in $SM$. For each $n \in \mathbb{N}$, let $D_{n,r}$ be the disk of radius $r$ centered at $g^{-t_n}(p)$ in $W^u(g^{-t_n}(p))$.

By shrinking $r$ if necessary, we can assume that $g^{-t}$ is a contracting map on $D_{n,r}$ for each $n$ in the induced Riemannian metric on $W^u$, which implies that $g^{t_n-t_s}(D_{n,r}) \subset D_{s,r}$ for $s > n$. For each $n$, choose a compact transversal submanifold $K_n \subset D_{n,r}$ to the $W^{vu}$ foliation which contains $g^{-t_n}(p)$ and is tangent to $H^{u-t_n}(p)$ at $g^{-t_n}(p)$. $K_n$ meets each leaf of the induced foliation of $D_{n,r}$ by $W^{vu}$ in exactly one point.

Consider the collection of $k$-dimensional submanifolds $g^{t_n}(K_n)$ of $W^u(p)$. We make three claims. First we claim that if a $W^{vu}$ leaf intersects $g^{t_n}(K_n)$, then it intersects $g^{t_s}(K_n)$ for any $s > n$. Second, we claim that each $W^{vu}$ leaf meets each submanifold $g^{t_n}(K_n)$ in at most one point. Lastly, we claim that for each $W^{vu}$ leaf in $W^u(p)$, there is an $n \in \mathbb{N}$ such that $g^{t_n}(K_n)$ intersects this leaf.
For the first claim, if \( s > n \), then \( g^{-ts}(g^{tn}(K_n)) \subset D_{s,r} \) by construction. Since \( K_s \) is a full transversal inside of \( D_{s,r} \), any \( W^{vu} \) leaf intersecting \( g^{-ts}(g^{tn}(K_n)) \) also intersects \( K_s \). By \( g^t \)-invariance of the \( W^{vu} \) foliation, any \( W^{vu} \) leaf intersecting \( g^{tn}(K_n) \) thus also intersects \( g^{ts}(K_s) \).

For the second claim, suppose that \( W^{vu}(q) \) intersects \( g^{tn}(K_n) \) in the points \( q \) and \( q' \), for \( q \neq q' \). \( W^u(p) \) is exponentially contracted under \( g^{-t} \), so for large enough \( s \), there will be a curve contained entirely in \( g^{-ts}(W^{vu}(q)) \cap D_{s,r} \) which joins \( g^{-ts}(q) \) to \( g^{-ts}(q') \). On the other hand, since the splitting \( E^u = V^u \oplus H^u \) is dominated, as \( s \to \infty \), the tangent spaces to \( g^{ts-n}(K_n) \) are uniformly asymptotic to the sequence of planes \( H'_{g^{ts-n}}(p) \). Thus for large enough \( s \), \( g^{ts-n}(K_n) \) will be a small disk that is almost parallel to \( H'_{g^{ts-n}}(p) \); in particular it will meet each leaf of \( W^{vu} \cap D_{s,r} \) in at most one point. But this contradicts the existence of the segment joining \( g^{-ts}(q) \) to \( g^{-ts}(q') \) inside of \( g^{-ts}(W^{vu}(q)) \cap D_{s,r} \).

For the last claim, recall that \( W^u(p) \) is defined as the set of points in \( SM \) asymptotic to the orbit of \( p \) under \( g^{-t} \). Since \( g^{-tn}(p) \to p \), it follows that for any \( q \in W^u(p) \), there is some \( n > 0 \) such that \( g^{-tn}(q) \in D_{n,r} \); the last claim follows.

Having proven those three claims, we now construct the desired embedding inductively. Set \( U_1 := g^n(K_1) \). To construct \( U_n \) from \( U_{n-1} \), take the submanifold \( g^n(K_n) \) of \( W^u(p) \) and use the smoothness of the \( W^{vu} \) foliation of \( W^u(p) \) to map \( g^n(K_n) \) smoothly onto a submanifold of \( W^u(p) \) which contains \( q \in U_{n-1} \) for each \( q \) such that \( W^{vu}(q) \cap g^n(K_n) \) is nonempty. By the first claim \( U_n \subset U_s \) for \( s \geq n \). By the second and third claim, the submanifold \( U := \bigcup_{n=1}^{\infty} U_n \) meets each \( W^{vu} \) leaf in exactly one point. Properness of the embedding follows from the fact that the \( W^{vu} \) foliation is locally trivial and that \( U \) meets each \( W^{vu} \) leaf in only one point. \( \Box \)

Next we build a \( C^1 \) \( g^t \)-invariant connection \( \nabla \) on the tangent bundle \( TQ^u \) to \( Q^u \) which will correspond to a \( g^t \)-invariant connection on the bundle \( E^u/V^u \) over \( SM \). \( \nabla \) will play the same role in the proof of Lemma 4.5 below as the Kanai connection in the proof of Lemma 3.1.

**Lemma 4.3.** Suppose that \( Dg^t|H^u \) is fiber bunched. Then there is a \( C^1 \), flat, torsion-free \( g^t \)-invariant connection \( \nabla \) on \( Q^u \). For points \( p, q \in W^u(p) \) and \( w \in H^u_p \),

\[
D\Pi_q^{-1} \circ P_{\Pi_q(p)\Pi_q(q)} \circ D\Pi_p = h^u_{pq}(w)
\]

where \( P \) is parallel transport with respect to \( \nabla \).

**Proof.** Since \( V^u \) is a smooth subbundle of \( E^u \) when restricted to \( W^u \), the quotient bundle \( E^u/V^u \) over \( W^u \) is smooth. The projection \( E^u \to E^u/V^u \) induces a bundle isomorphism \( H^u \to E^u/V^u \) which is equivariant with respect to the action of \( Dg^t \) on \( H^u \) and the induced action of \( Dg^t \) on \( E^u/V^u \). We push forward the Riemannian metric on \( H^u \) to a Riemannian metric on \( E^u/V^u \), with respect to which the induced action of \( Dg^t \) is fiber bunched. The isomorphism \( H^u \to E^u/V^u \) also induces an unstable holonomy \( \tilde{h}^u \) for the action of \( Dg^t \) on \( E^u/V^u \). Since \( E^u/V^u \) has a smooth structure along \( W^u \) leaves with respect to which \( Dg^t \) is smooth and the action of \( Dg^t \) on \( E^u/V^u \) is 1-fiber bunched, the unstable holonomy \( \tilde{h}^u \) is \( C^1 \) along \( W^u \) leaves.
By the uniqueness of this unstable holonomy, we have the following alternative construction of $\bar{h}^u$. Take two compact transversals $K_1$ and $K_2$ to the $W^{vu}$ foliation which meet the same collection of $W^{vu}$ leaves (or equivalently, they have the same projection to $Q^u(p)$). The projection $E^u \to E^u/V^u$ induces natural bundle isomorphisms $TK_1 \to E^u/V^u$ over each of these transversals. Then the derivative of the chart transition map $(\Pi|_{K_2})^{-1} \circ \Pi|_{K_1}$ is the unstable holonomy $\bar{h}^u$ when we make the identifications $TK_1 \cong E^u/V^u$

The projection $\Pi : W^u(p) \to Q^u(p)$ is smooth and hence induces a derivative map $\Pi^* : E^u \to TQ^u$ with $V^u = \ker \Pi$. Hence for each $x \in W^u(p)$ the induced map $\Pi^* : E^u_x/V^u_x \to TQ^u_{\Pi(x)}$ is an isomorphism. For $w, z \in Q^u(p)$ which are the image of $x$ and $y \in W^u(p)$ respectively, we define $P_{wz} : TQ^u_w \to TQ^u_z$ by $P_{wz} = \Pi^*_w \circ \bar{h}_{xz} \circ \Pi^*_x^{-1}$. We claim that $P_{wz}$ does not depend on the preimages $w$ and $z$ which were chosen. Suppose that $x'$ and $y'$ are two other points projecting to $w$ and $z$ respectively. Then

$$\Pi^*_y \circ \bar{h}_{x'y'} \circ \Pi^*_x^{-1} = \Pi^*_y \circ (\Pi^*_y^{-1} \circ \Pi^*_y) \circ \bar{h}_{x'y'} \circ (\Pi^*_x^{-1} \circ \Pi^*_x) \circ \Pi^*_x^{-1}$$

$$= \Pi^*_y \circ \bar{h}_{x'y'} \circ \Pi^*_x^{-1}$$

where we have used the observation that the derivatives of the transition maps for $\Pi$ are given by the unstable holonomy $\bar{h}^u$, and also the properties of the unstable holonomy $\bar{h}^u$ itself.

It’s straightforward to check that $P_{wz}$ is equivariant with respect to the induced derivative action $Dg^t : TQ^u(p) \to TQ^u(g^t(p))$, using the equivariance property of $\bar{h}^u$. $P_{wz}$ is also $C^1$ in the variables $w$ and $z$ and has the property that for $x, y, z \in Q^u(p)$, $P_{yz} \circ P_{xy} = P_{xz}$. This implies that for each $X \in TQ^u(p)$,

$$P(X) = \{Y \in TQ^u : P_{xy}(X) = Y \text{ for some } x, y \in Q^u(p)\}$$

is a $C^1$ submanifold of $TQ^u$ which is transverse to the tangent spaces $TQ^u_x$. The tangent spaces to the foliation of $TQ^u$ by these subfoliations define an Ehresmann connection on $Q^u(p)$ which we can then use to define a connection $\nabla$ on $Q^u(p)$.

It only remains to show that $\nabla$ is torsion-free. Let $T$ be the torsion tensor of $\nabla$. $T$ is a mixed tensor of type $(2, 1)$ on $TQ^u$ which is invariant under $g^t$. But the fact that $Dg^t|H^u$ is fiber bunched implies that $Dg^t$ acts by exponential contraction on tensors of type $(2, 1)$ on $TQ^u$. This forces $T \equiv 0$ so that $\nabla$ is torsion-free. □

The following lemma is fundamental to everything that follows in this paper. Recall that in the proof of Theorem 3.1, one of the critical steps was to establish that the stable holonomy of the cocycle $Dg^t|E^u$ could be represented as the derivative of the holonomy map between unstable leaves induced by the center stable foliation. Lemma 4.4 establishes the analogous property in our situation.
Let \( r > 0 \) be small enough that all of the foliations \( W^s \) under consideration are trivial on a ball of radius \( r \). Given two points \( x, y \in SM \) with \( y \in W^s_r(x) \), there is then a well-defined \( W^c \)-holonomy map \( L_{xy} : W^u_r(x) \to W^u_r(y) \). For \( z \in W^u_r(x) \), \( L_{xy}(z) \) is defined to be the unique point in the intersection \( W^s_r(z) \cap W^u_r(y) \). In general this holonomy map is only Hölder continuous [31]. We establish under proper fiber bunching assumptions on \( Dg^t|H^u \) that \( L_{xy} \) is differentiable when restricted to curves tangent to \( H^u \). If we think of \( g^t \) as a partially hyperbolic diffeomorphism as in Proposition 4.1 with center bundle \( H^u \oplus E^c \), Lemma 4.4 can be viewed as an extension of Theorem B in [31] to the case in which there may not be a foliation tangent to the center distribution.

**Lemma 4.4.** Suppose that \( E^u \) and \( H^u \) are \( \beta \)-Hölder continuous and that \( Dg^t|H^u \) is \( \beta \)-fiber bunched. Then the \( W^c \)-holonomy map \( L_{xy} : W^u_r(x) \to W^u_r(y) \) maps \( C^1 \) curves tangent to \( H^u \) to \( C^1 \) curves tangent to \( H^u \) and therefore \( L_{xy} \) is differentiable along \( H^u \). For \( z \in W^u_r(x) \), the derivative of \( L_{xy} \) along \( H^u \) is given by

\[
D_z L_{xy}|H^u = h^c_{xy} \]

**Proof.** Let \( x, y \) be two points in \( SM \) such that \( x \in W^s_r(y) \). Set \( x_n = g^n x \) and \( y_n = g^n y \). For each \( n \geq 0 \), choose a hypersurface \( S_n \) of uniform size and biLipschitz to an open subset of \( \mathbb{R}^{2m} \) with Lipschitz constants independent of \( n \) that is transverse to the direction of the flow \( E^c \), and contains \( W^u_r(x_n) \) and \( W^u_r(y_n) \). Let \( f_n : S_{n-1} \to S_n \) be the smooth map defined by \( f_n(r) = g^n(r) \), where \( t(r) \) is the unique time, smoothly depending on \( r \), with \( t(x_n-1) = 1 \) and such that \( g^n(r) \in S_n \). \( f_n \) is defined on a neighborhood of \( x_{n-1} \) of uniform size, independent of \( n \). Further, \( f_n \) is uniformly hyperbolic on the interior of this neighborhood with the same contraction and expansion estimates (up to multiplicative constants) as \( g^t \) on the stable and unstable bundles \( E^u \) and \( E^s \). Set \( F^n = f_n \circ f_{n-1} \circ \cdots \circ f_1 \). Note that \( F^n \) is defined on increasingly small neighborhoods of \( x \) as \( n \to \infty \); the only points for which \( F^n \) is defined for all \( n \geq 1 \) are the points on the intersection of \( W^c_r(x) \) with \( S := S_0 \).

Let \( \beta \) be the minimum of the Hölder exponents of \( H^u \) and \( E^u \) viewed as subbundles of \( T(SM) \). As remarked in the Introduction, there is a \( \beta \)-Hölder system of linear identifications \( I_{pq} : E^u_p \to E^u_q \) defined for \( p \) near \( q \) with \( I_{pp} \) being the identity on \( E^u_p \). We can choose these identifications so that \( I_{pq}(H^u_p) = H^u_q \). For each \( n \), let \( A_n : W^u_r(x_n) \to W^u_r(y_n) \) be a diffeomorphism with \( A_n(x_n) = y_n \). Since the unstable foliation is Hölder continuous in the \( C^1 \) topology with Hölder exponent \( \beta \), we can choose \( A_n \) such that

\[
\|I_{qp} \circ DA_n - Id\| \leq C d(p, q)^\beta \\
\|DA_n \circ I_{pq} - Id\| \leq C d(p, q)^\beta
\]

for some constant \( C > 0 \) and \( p \in W^u_r(x_n) \), \( q \in W^u_r(y_n) \). For \( z \in S_n \), let \( \hat{W}^s(z) \) denote the smooth projection of \( W^s_r(z) \) onto \( S_n \) along the orbit foliation \( E^c \), given by using \( g^t \) to flow these leaves onto \( S_n \). Let \( \hat{H}^u \), \( \hat{E}^u \), and \( \hat{E}^s \) denote the projection of these subbundles onto \( TS_n \) by flowing along the orbit foliation.

Let \( \varphi \) be the holonomy map between \( W^u_r(x) \) and \( W^u_r(y) \) induced by the projected stable foliation \( \hat{W}^s \). Let \( \varphi_n = F^{-n} \circ A_n \circ F^n \), which is defined on a neighborhood of \( x \) (dependent on \( n \)) inside of \( W^u_r(x) \). Let \( \gamma : [-1, 1] \) be a \( C^1 \) curve tangent to \( H^u \)
inside of $W^u(x)$ with $\gamma(0) = x$. Our first goal is to prove that $\varphi \circ \gamma$ is differentiable at 0, i.e., that the image of the curve $\gamma$ under $\hat{W}^s$-holonomy along the transversal $S$ is differentiable at $p$.

We first claim that the sequence of linear maps $\{(DF_y^{-n} \circ DA_n \circ DF_x^n)|H^u_x : n \in \mathbb{N}\}$ is Cauchy (note that we have restricted the domain of these maps to $H^u_x$). We closely follow the proof of Proposition 2.2 given in [24]. We begin with the formula

$$(DF_y^n)^{-1} \circ DA_n \circ DF_x^n = DA_0 + \sum_{i=0}^{n-1} (DF_y^n)^{-1} \circ R_i \circ DF_x^n$$

where $R_i = (Dy_{f_i+1})^{-1} \circ DA_{i+1} \circ Dx_{f_i+1} - DA_i$. For the rest of the proof we will consider all linear maps as restricted to $\hat{E}^u$ for the purpose of calculating norms. We want to estimate the product

$$\| (DF_y^n)^{-1} \| \cdot \| DF_x^n \| \leq \prod_{i=0}^{n-1} \| (DF_y^n)^{-1} \| \cdot \| Dx_{f_i} \| \| \hat{H}^u \|$$

To bound the first factor, we observe that $\| (Dx_{f_i}^{-1}) \| \| \hat{H}^u \| = \| (Dx_{f_i})^{-1} \|$ since $\hat{H}^u$ is the less expanded term of the dominated splitting $\hat{E}^u = \hat{V}^u \oplus \hat{H}^u$. We then use the estimate

$$\frac{\| (Dy_{f_i}^{-1}) \|}{\| (Dx_{f_i}^{-1}) \|} \leq \frac{\| (Dy_{f_i}^{-1}) - I_{x_i y_i} \circ (Dx_{f_i}^{-1}) \circ I_{x_i+1 y_{i+1}}^{-1} \|}{\| (Dx_{f_i}^{-1}) \|} + \frac{\| I_{x_i y_i} \circ (Dx_{f_i}^{-1}) \circ I_{x_i+1 y_{i+1}}^{-1} \|}{\| (Dx_{f_i}^{-1}) \|}$$

for some constant $C'$. Here we use the fact that $\| I_{pq} \|$ is uniformly bounded when $p$ and $q$ are close (say $d(p,q) \leq \epsilon$), and that the derivative $D_p f_i$ is smooth as a function of $p$, hence when we use the identifications $I_{pq}$, it becomes Hölder with Hölder exponent $\beta$.

To bound the second factor, we note that $D_p f_i |\hat{H}^s$ is fiber bunched since the cocycle $Dg|\hat{H}^u$ we derived it from was fiber bunched. Hence there is a constant $\delta < 1$ such that

$$\| (D_p f_i)^{-1} \| \cdot \| D_p f_i \| \leq \| D_p f_i \| \| \hat{E}^s \|^{-\delta}$$

for all $p \in S_i$, where $\delta$ is independent of $i$.

Putting these two bounds together, we obtain

$$\| (DF_y^n)^{-1} \| \cdot \| DF_x^n \| \leq \prod_{i=0}^{n-1} (C' d(x_i, y_i)^\beta + 1) \prod_{i=0}^{n-1} \delta \| D_i f_i \| \| \hat{E}^s \|^{-\delta}$$
Now we can also estimate
\[
\| (DF_y^n)^{-1} \cdot DF_x^n | \hat{H}^u_+ \| \leq C'' \delta^n \prod_{i=0}^{n-1} \| D_{x,i} f_i | \hat{E}^s_+ \|^{-\beta}
\]

Now we can also estimate
\[
\| R_i \| \leq \| (D_{y,i} f_{i+1})^{-1} \circ DA_{i+1} \| \cdot \| D_{x,i} f_{i+1} - DA_{i+1}^{-1} \circ D_{y,i} f_{i+1} \circ DA_i \|
\]
\[
\leq Cd(x,y)^{\delta^i} \prod_{j=0}^{i-1} \| D_{x,j} f_j | \hat{E}^s_+ \|^{\beta}
\]

for some constant $C$. In the first inequality we used the Hölder closeness of $DA_i$ to the identity, together with uniform bounds on the norms of all of the linear maps involved. In the second inequality we use the fact that $x$ and $y$ lie on the same stable manifold in $S$. We have the basic bound
\[
\| (DF_y^n)^{-1} \circ DA_n \circ DF_x^n - DA_0 | \hat{H}^u_+ \| \leq \sum_{i=0}^{n-1} \| (DF_y^n)^{-1} \circ R_i \circ DF_x^n \| \cdot \| R_i \|
\]
\[
\leq \sum_{i=0}^{n-1} \| (DF_y^n)^{-1} \| \cdot \| DF_x^n \| \cdot \| R_i \|
\]

We replace the right side with the previously obtained bounds on the factors $\| (DF_y^n)^{-1} \| \cdot \| DF_x^n \| $ and $\| R_i \|$. This gives an upper bound of
\[
\sum_{i=0}^{n-1} \left( C'' \delta^i \prod_{j=0}^{i-1} \| D_{x,j} f_j | \hat{E}^s_+ \|^{-\beta} \cdot Cd(x,y)^{\delta^i} \prod_{j=0}^{i-1} \| D_{x,j} f_j | \hat{E}^s_+ \|^{\beta} \right) \leq C^* d(x,y)^{\beta}
\]

for some constant $C^*$. Also note that
\[
\| (DF_y^{n+1})^{-1} \circ DA_{n+1} \circ DF_x^{n+1} - (DF_y^n)^{-1} \circ DA_n \circ DF_x^n | \hat{H}^u_+ \|
\]
\[
= \| (DF_y^n)^{-1} \circ R_n \circ DF_x^n | \hat{H}^u_+ \|
\]
\[
\leq C^* \delta^n d(x,y)^{\beta}
\]

This second inequality immediately implies that the sequence of linear maps
\[
\{ (DF_y^n)^{-1} \circ DA_n \circ DF_x^n | \hat{H}^u_+ : n \in \mathbb{N} \}
\]
is Cauchy. Hence this sequence converges to a linear map $T_{xy} : \hat{H}^u_x \to \hat{E}^u_y$. However, for any given vector $v \in \hat{H}^u_x$, $DA_n \circ DF_x^n(v)$ is a vector which makes an angle $\theta_n$ with $\hat{H}^u_y$, where $\theta_n$ is uniformly bounded away from $\pi/2$, independent of $n$. Applying $DF_y^{-n}$ exponentially contracts this angle since the splitting $\hat{E}^u = \hat{V}^u \oplus \hat{H}^u$ is dominated, so letting $n \to \infty$, we conclude that $T_{xy}$ must have image in $\hat{H}^u$. For each $j \geq 0$, we can also consider the sequence of linear maps
\[
\{ (DF_y^{n+j} \circ (DF_y^j)^{-1})^{-1} \circ DA_{n+j} \circ DF_x^{n+j} \circ (DF_x^j)^{-1} | \hat{H}^u_x : n \in \mathbb{N} \}
\]
For the same reasons as for the original sequence, this sequence is Cauchy and converges to a limit that we denote $T_{x,y}$ which is a linear map from $\hat{H}^u_x$ to $\hat{H}^u_y$. It is straightforward to check that for each $n$ we have $(DF_y^n)^{-1} \circ T_{x,y} \circ DF_x^n = T_{xy}$.
by writing out the limiting expression for $T_{x_ny_n}$. Since we chose the transversal $S$ to contain $W^n_\omega(x)$ and $W^n_\nu(y)$, we have $\tilde{H}_x^u = H_x^u$, and the same for $y$. We now consider the center stable holonomy map $h^{cs}_{xy} : H_x^u \to H_y^u$. This is equivariant with respect to $DF^n$ as well and also depends in a $\beta$-Hölder manner on the points $x$ and $y$. Then

\[
\|h^{cs}_{xy} - P_{xy}\| = \|(DF^n)^{-1} \circ (h^{cs}_{x_ny_n} - T_{x_ny_n}) \circ DF^n|H_x^u\| \\
\leq \|(DF^n)_y^{-1}\| \cdot \|DF^n|H_x^u\| \cdot \|h^{cs}_{x_ny_n} - T_{x_ny_n}\| \\
\leq C\delta^n \prod_{i=0}^{n-1} \|D_{x_n}f_i|E^s\|^{-\beta}d(x_n, y_n)^\beta \\
\leq C^*\delta^n
\]

for some constant $C^*$. As $n \to \infty$, $\delta^n \to 0$, so $h^{cs}_{xy} = T_{xy}$.

To prove differentiability of $\varphi \circ \gamma$, take a coordinate chart on $S$ (as well as each of the transversals $S_n$) so that we can work with the linear structure on $\mathbb{R}^{2m}$. Let $y$ correspond to the origin. We will not change the notation for the maps, so they should be understood in this chart. Let $v = \gamma'(0)$. We need to show that $\varphi(\gamma(s))$ agrees with its claimed linearization $s \cdot h^{cs}_{xy}(v)$ to first order at the origin. First observe that the calculations above are valid if we replace $x$ and $y$ by any two points $x'$, $y'$ in $S$ such that $y' \in \tilde{W}^u_{loc}(x')$, whenever $n$ is small enough (relative to $x'$ and $y'$) that the iterates $F,F^2,\ldots,F^n$ are all defined on a neighborhood of $x'$ and $y'$. This implies that

\[
\|(DF^n)^{-1} \circ DA_n \circ DF^n_{\gamma(s)}(\gamma'(s)) - DA_0(\gamma'(s))\| \leq C|s|^{\beta}
\]

whenever $s$ is small enough that $F^n$ is defined on a neighborhood of $\gamma(s)$ and $A_n(F^n(\gamma(s)))$ lies in the image of $F^n$. The constant $C$ is independent of $n$, so $(DF^n)^{-1} \circ DA_n \circ DF^n_{\gamma(s)}(\gamma'(s))$ is a Hölder continuous function of $s$ with Hölder exponent and constant independent of $n$ for $|s|$ small. Note that $A_n(F^n(\gamma(s)))$ will not necessarily lie on $\tilde{W}^s(F^n(\gamma(s)))$, but it will be $\beta$-Hölder close to the intersection of $\tilde{W}^s(F^n(\gamma(s)))$ with $W^u_{\nu}(y_n)$, so our estimates remain valid. By the mean value inequality, we thus obtain

\[
\|\varphi_n(\gamma(s)) - s \cdot D\varphi_n(\gamma'(0))\| \leq C|s|^{1+\beta}
\]

for a constant $C$.

We next estimate the difference between $\varphi$ and $\varphi_n$ near $\gamma(0)$. Observe that $\varphi = (F^n)^{-1} \circ \psi_n \circ F^n$, where $\psi_n$ is the $\tilde{W}^s$-holonomy map from $\tilde{W}_{\nu}(x_n)$ to $W^u_{\nu}(y_n)$. Hence for $s$ small enough that $\gamma(s)$ is in the domain of definition of the expressions below,

\[
\|\varphi_n(\gamma(s)) - \varphi(\gamma(s))\| = \|(F^n)^{-1} \circ A_n \circ F^n - (F^n)^{-1} \circ \psi_n \circ F^n(\gamma(s))\| \\
\leq C\|(DF^n)^{-1}|E^u\| \cdot \|A_n \circ F^n(\gamma(s)) - \psi_n \circ F^n(\gamma(s))\|
\]

since $F^{-n}$ exponentially contracts distances on unstable leaves. Next we note that $\psi_n$ and $A_n$ are $\beta$-Hölder close in the $C^0$ topology. As a consequence, since they
both map $x$ to $y$,

$$C\|(DF^n)^{-1}|E^u|| \cdot \|A_n \circ F^n(\gamma(s)) - \psi_n \circ F^n(\gamma(s))\|
\leq C\|(DF^n)^{-1}|E^u||d(F^n(x), F^n(y))\| \beta \|F^n(\gamma(s))\|
\leq C\|(DF^n)^{-1}|E^u|| \cdot \|DF^n|E^s|\| \beta \cdot \|F^n(\gamma(s))\|
\leq C\|(DF^n)^{-1}|E^u|| \cdot \|DF^n|E^s|\| \beta \cdot \|DF^n|\hat{H}^u|| \cdot |s|
\leq C\delta^n|s|$$

where we have not paid much attention to the constant $C$ in front (which will change from line to line). In the third line we use the exponential contraction of stable leaves by $F^n$, and in the fourth line we use the fiber bunching property on $H^u$ transferred to the induced bundle $\hat{H}^u$, noting that $\|(DF^n)^{-1}|E^u|| = \|(DF^n)^{-1}|\hat{H}^u||$.

We now compare $\varphi \circ \gamma$ to the linearization $h_{xy}^{cs}(v) \cdot s$ at 0. Fix $n \in \mathbb{N}$. For $|s|$ small enough that all of the expressions above are defined for this $n$, we obtain

$$\|\varphi(\gamma(s)) - h_{xy}^{cs}(v) \cdot s\| \leq \|\varphi(\gamma(s)) - \varphi_n(\gamma(s))\| + \|\varphi_n(\gamma(s)) - s \cdot D\varphi_n(v)\|
+ |s| \cdot \|D\varphi_n(v) - h_{xy}^{cs}(v)\|
\leq C(\delta^n|s| + |s|^{1+\beta} + |s| \cdot \|D\varphi_n(\gamma'(0)) - h_{xy}^{cs}(v)\|)$$

Dividing through by $|s|$, we obtain

$$\frac{\|\varphi(\gamma(s)) - h_{xy}^{cs}(v) \cdot s\|}{|s|} \leq C(\delta^n + |s|^{\beta} + \|D\varphi_n(v) - h_{xy}^{cs}(v)\|)$$

We can consider $n := n(s)$ as an integer function of $s$ such that $n(s) \to \infty$ as $s \to 0$. Then as $s \to 0$, the right side converges to 0. We thus obtain that $\varphi \circ \gamma$ agrees to first order with its linearization at 0, i.e., $\varphi \circ \gamma$ is differentiable at 0, and furthermore, $(\varphi \circ \gamma)'(0) = h_{xy}^{cs}(\gamma(0))\gamma'(0))$.

Now observe that holonomy from $W^u_r(x)$ to $W^u_r(y)$ along the projected stable foliation $\hat{W}^s$ corresponds precisely to $W^{cs}$-holonomy in $SM$. Hence the curve $\varphi \circ \gamma$ is also the image of $\gamma$ under the $W^{cs}$-holonomy $L_{xy}$. We can apply our calculations to the other points of $\gamma$ by recentering at each pair of points $x'$, $y'$ lying on $\gamma$ and $\varphi \circ \gamma$ respectively with $y' \in W^r_{xy}(x')$. This proves that $\varphi \circ \gamma$ is differentiable for every $t \in [-1, 1]$, and furthermore we have the derivative formula

$$(\varphi \circ \gamma)'(t) = h_{xy}^{cs}(\gamma(t))\gamma'(t))$$

which completes the proof. \qed

We conclude this section with the proof of the major irreducibility result we will need for Sections 5 and 6. Unlike the previous results of this section, the proof of Lemma 4.5 makes extensive use of the structure of $g^t$ as a geodesic flow. For a continuous subbundle $\mathcal{E}$ of $T(SM)$ and a point $p \in SM$, we define the $\mathcal{E}$-accessibility class $\mathcal{A}(p; \mathcal{E})$ of $p$ to be the set of all points $q \in SM$ which can be joined to $p$ by a piecewise $C^1$ curve $\gamma$ tangent to $\mathcal{E}$.

**Lemma 4.5.** Suppose $E^u$ and $H^u$ are $\beta$-Hölder continuous and that $Dg^t|H^u$ is $\beta$-fiber bunched. Suppose that there is a Gibbs measure $\mu$ such that $\lambda_+(Dg^t|H^u, \mu) = \lambda_-(Dg^t|H^u, \mu)$. Let $\mathcal{E} \subset H^u$ be a nonzero measurable $g^t$-invariant subbundle. Then $\mathcal{E} = H^u$. 

Proof. By Lemma 2.3, \( E \) coincides \( \mu \)-a.e. with a continuous \( g^t \)-invariant, holonomy invariant subbundle of \( H^u \), which we will also denote by \( E \). We claim that \( S(p; E) \) is dense in \( W^u(p) \) for every \( p \in SM \). Since \( E \) is \( g^t \)-invariant, \( S(g^t p; E) = g^t(S(p; E)) \) for every \( t \in \mathbb{R} \). Pass to the universal cover \( \tilde{SM} \) and note that for every \( \gamma \in \Gamma := \pi_1(M) \), we also have \( S(D\gamma(p); E) = D\gamma(S(p; E)) \) for every \( p \in SM \), since the lifted bundle \( \tilde{E} \) is invariant under \( \Gamma \). As in the proof of Lemma 3.1, we let \( \pi_x : \tilde{W}^u(x) \to \partial \tilde{M} \setminus \{x_\gamma \} \) be the projection homeomorphism onto the boundary. For \( x, y \in \tilde{SM} \), consider as before the transition homeomorphism
\[
\pi_y^{-1} \circ \pi_x : \tilde{W}^u(x) \setminus \{\pi_x^{-1}(y)\} \to \tilde{W}^u(y) \setminus \{\pi_y^{-1}(x)\}
\]
Lemma 4.4 implies that \( \pi_y^{-1} \circ \pi_x \) is differentiable when restricted to \( C^1 \) curves tangent to \( H^u \), and that the derivative is given by the global center stable holonomy map \( h^{cs} \) for \( H^u \). Since \( E \) is invariant under \( h^{cs} \), this implies that
\[
(\pi_y^{-1} \circ \pi_x)(S(p; E)) = S(\pi_y^{-1}(\pi_x(p)); E).
\]
for any \( p \in \tilde{W}^u(x) \). We thus conclude that for each \( \xi \in \partial \tilde{M} \), there is a well defined subset \( S(\xi) \) of \( \partial \tilde{M} \) consisting of all points \( \zeta \in \tilde{M} \) which can be joined to \( \xi \) by a curve \( \gamma \) in \( \partial \tilde{M} \) which is piecewise \( C^1 \) and tangent to \( E \) in some \( \pi_x \) coordinate chart (and therefore is tangent to \( E \) in any such coordinate chart). Furthermore the \( \Gamma \)-equivariance of \( E \) accessibility classes translates into the relation
\[
\gamma(S(\xi)) = S(\gamma(\xi)).
\]
We would like to show that \( S(\xi) \) is dense in \( \partial \tilde{M} \). Let \( U \) be an open set in \( \partial \tilde{M} \) which does not contain \( \xi \). Let \( x \) be the image of \( \xi \) in an unstable leaf \( \tilde{W}^u(x) \). Take a small open neighborhood \( A \) of \( x \) which is disjoint from the image of \( U \) in \( \tilde{W}^u(x) \). We claim that if \( A \) is small enough, then \( S(\gamma; E) \) intersects the topological boundary \( \partial A \) of \( A \) for every \( y \in A \). We begin by reducing this to an equivalent 2-dimensional problem. Take a coordinate chart on \( \tilde{W}^u(x) \) mapping \( x \) to the origin of \( \mathbb{R}^{m-1} \) and \( E_x \) to the coordinate plane corresponding to the first \( k \) coordinates, where \( k = \dim E \). Let \( C_\varepsilon \) be the cube \( [-\varepsilon, \varepsilon]^{m-1} \) centered at \( p \). Take some \( q \in C_\varepsilon \) and consider the projected image \( \tilde{q} \) of \( q \) in \( \mathbb{R}^k = E_x \), the first \( k \) coordinates. Let \( L_q \) be the line through \( \tilde{q} \) parallel to the first coordinate axis. Choose a direction among the last \( m - k - 1 \) coordinates (for definiteness, the \((k+1)\)st coordinate). Let \( P_q \) be the plane spanned by \( L_q \) and the \((k+1)\)st coordinate. As long as \( \varepsilon \) is small enough (uniformly in \( q \)), for every \( p \in C_\varepsilon \) the intersection of \( E_p \) with \( P_q \) will be a line. Fix this \( \varepsilon \) from now on. Identify \( P_q \) with \( \mathbb{R}^2 \). We see then that it suffices to solve the following equivalent problem: Given an ODE \( y' = f(x) \) with \( f \) continuous and \( |f| \leq K \) everywhere on \([-\varepsilon, \varepsilon]^2 \) (note this \( K > 0 \) is uniform in \( q \)) show that there is a \( C^1 \) solution \( \sigma \) with \( \sigma(0) = 0 \) such that either \( \sigma(t) \) is defined on \([0, \varepsilon]\) or else there is some \( t \in [0, \varepsilon] \) such that \( |\sigma(t)| > \varepsilon \).

By the Cauchy-Peano existence theorem for ODEs with continuous coefficients the uniform bound \( |f| \leq K \) ensures that there is a uniform \( \delta > 0 \) such that a solution to the initial value problem \( \sigma(t_0) = y_0, \sigma'(t) = f(\sigma(t)) \) exists on \([t_0, t_0 + \delta] \) provided \(|\sigma(t_0)| \leq \varepsilon \) on \([t_0, t_0 + \delta] \) and \( t \leq \varepsilon \) (Theorem 2.19 of [35]). Thus, starting at \( \sigma(0) = 0 \), construct a solution existing on \([0, \delta] \), then concatenate this with a solution existing on \([\delta, 2\delta] \) and so on. This process ends when either \( k\delta > \varepsilon \) (which happens after a
finite number $\varepsilon/\delta$ of steps) or when a solution exceeds $\varepsilon$ in absolute value. In either case, we are done.

Thus $\mathcal{A}(\xi)$ intersects $\partial A$ for $A$ small enough. The pairs of endpoints of axes of the isometries $\gamma \in \Gamma$ of $\widetilde{M}$ are dense in $\partial \widetilde{M} \times \partial \widetilde{M}$, hence we can find an isometry $\gamma$ with the forward endpoint $\gamma_+ \in U$ of the axis lying in $U$, and the backward endpoint $\gamma_- \in A$. Since $\gamma$ gives rise to north-south dynamics on the sphere $\partial \widetilde{M}$ there is some $k > 0$ such that $A \subset \gamma^k A$ and $\gamma^k(\partial A) \subset U$. There is thus some $\xi \in A$ such that $\gamma^k\xi = \xi$. But we know that $\mathcal{A}(\xi)$ intersects $\partial A$ and thus $\gamma^k(\mathcal{A}(\xi)) = \mathcal{A}(\xi)$ intersects $U$. This implies the desired conclusion.

Fix a periodic point $p \in SM$ of period $T$. Since the bundle $E$ is a holonomy-invariant subbundle of $H_u$, it descends to a subbundle $DII(E)$ of $TQ^u(p)$. $DII(E)$ is parallel with respect to the connection $\nabla$ constructed in Lemma 4.3, hence since $\nabla$ is torsion-free (as in Lemma 3.1), $DII(E)$ is uniquely integrable and there is thus a foliation $\mathcal{F}$ tangent to $DII(E)$ inside of $Q^u$. Let $\widetilde{p}$ be the projection of $p$ in $Q^u(p)$ and let $\widetilde{F}(p)$ be the inverse images of all points in the leaf $\mathcal{F}(\widetilde{p})$ through $\widetilde{p}$ of $\mathcal{F}$ inside of $Q^u(p)$. It is clear that $\mathcal{A}(p; E) \subset \widetilde{F}(p)$, since any piecewise $C^1$ curve tangent to $E$ and passing through $p$ must project to a piecewise $C^1$ curve contained entirely inside of $\mathcal{F}(p)$. On the other hand, as shown above, $\mathcal{A}(p; E)$ must be dense in $E^u(p)$.

We thus conclude that $\widetilde{F}(p)$ is dense in $E^u(p)$, and therefore $\mathcal{F}(\widetilde{p})$ is dense in $Q^u(p)$. But this is absurd unless $\mathcal{F}(\widetilde{p}) = Q^u$: let $U$ be a neighborhood of $\widetilde{p}$ on which $g^{-T}$ acts as an exponential contraction and such that the foliation $\mathcal{F}$ can be trivialized as slices $\mathbb{R}^k \times \{a\}$. Two different slices of the foliation in $U$ have the property that they cannot be connected by a $C^1$ curve $\sigma$ lying entirely in $U$. If $\mathcal{F}(\widetilde{p})$ is dense in $Q^u(p)$, we can find some slice of $\mathcal{F}$ in $U$ that does not pass through $\widetilde{p}$ and a unit speed curve $\sigma : [0, \ell] \to Q^u(p)$ with $\sigma(0) = \widetilde{p}$ and $\sigma(\ell) = q \in U$ lying in a different slice. Consider $g^{-kT} \circ \sigma$ for $k > 0$ large. By the definition of the unstable leaf, for $k$ large enough the entire curve $g^{-kT} \circ \sigma$ is contained inside of $U$, and by the $g^t$-invariance of $\mathcal{F}$, $g^{-kT} \circ \sigma$ is always contained inside of $\mathcal{F}(\widetilde{p})$. This implies that $g^{-kT}(\sigma(\ell))$ lies in the slice through $\widetilde{p}$ of $\mathcal{F}$ in $U$. The contraction property of $g^{-T}$ on $U$ implies that $g^{-kT}(U) \subset U$ is an open connected subset containing $\widetilde{p}$, and the foliation can be trivialized on this open subset, so we can join $g^{-kT}(\sigma(\ell))$ to $\widetilde{p}$ by a curve $\tau$ lying entirely inside of $g^{-kT}(U)$. But then $g^{kT} \circ \tau$ is a curve joining $\widetilde{p}$ to $\sigma(\ell)$ lying entirely inside of the slice of $\mathcal{F}$ through $\widetilde{p}$ inside of $U$, which is the contradiction that completes the proof.

We isolate a corollary of the arguments in the proof of Lemma 4.5 which is of independent interest, obtained by taking $\mathcal{E} = H_u$ and ignoring the assumptions on extremal Lyapunov exponents in the above argument.

**Corollary 4.6.** Let $g^t$ be the geodesic flow of a closed negatively curved manifold $M$. Suppose that there is a dominated splitting $E^u = H_u \oplus V^u$, that $E^u$ and $H_u$ are $\beta$-Hölder continuous, and $Dg^t|H^u$ is $\beta$-fiber bunched. Then $\mathcal{A}(p; H^u)$ is dense in $W^u(p)$ for every $p \in SM$.

**Remark 4.7.** Given two small disjoint open sets $U$ and $V$ in $W^u(p)$, the piecewise $C^1$ curve $\gamma$ constructed in Corollary 4.6 which starts in $U$ and ends in $V$ will typically
take a long, winding route through \( W^u(p) \) which increases exponentially in length as \( U \) and \( V \) shrink in size, regardless of how close \( U \) and \( V \) are in \( W^u(p) \). Thus it is not immediately clear whether the conclusion in Corollary 4.6 can be improved to \( A(p; H^u) = W^u(p) \).

5. Proof of Theorem 1.2

In this section we prove Theorem 1.2. We use Lemma 5.2 to construct a dominated splitting \( E^u = H^u \oplus V^u \) to which we can apply the results of Section 4.

Let \( X \) be a topological space and let \( f : X \to X \) be a continuous map. A sequence of functions \( a_n : X \to \mathbb{R} \), \( n \in \mathbb{N} \) is subadditive if
\[
a_{n+m}(x) \leq a_n(x) + a_m(f^m(x)) \quad \forall m, n \in \mathbb{N} \text{ and } \forall x \in X.
\]
Here \( f^m \) denotes the \( m \)th iterate of \( f \). We need the following proposition whose proof can be found in [24].

**Proposition 5.1.** Let \( f \) be a homeomorphism of a compact metric space \( X \) and \( a_n : X \to \mathbb{R} \) a subadditive sequence of functions. Suppose that
\[
\inf_{n \geq 1} \frac{1}{n} \int_X a_n \, d\mu < 0
\]
for every \( f \)-invariant ergodic Borel probability measure \( \mu \). Then there exists \( N > 0 \) such that \( a_N(x) < 0 \) for every \( x \in X \).

We will prove a result which is somewhat more general than what we need.

**Lemma 5.2.** Let \( A \) be a cocycle on a vector bundle \( E \) over \( g \). Suppose there exist \( c_-, c_+ \in \mathbb{R} \), positive integers \( r, s \), and some \( k \), \( 1 \leq k \leq d - 1 \), where \( d = \dim E \), such that for every periodic point \( p \in SM \),
\[
\lambda_i^{(p)} = c_-, \quad 1 \leq i \leq k
\]
\[
\lambda_i^{(p)} = c_+ + 1 \leq i \leq d
\]
\[
r(d - k)c_+ - skc_- = 0
\]
Then there is a dominated splitting \( E = H \oplus V \) for the cocycle \( A \). The restriction of \( H \oplus V \) to the orbit of a periodic point \( p \) coincides with the Oseledets splitting for the measure \( \mu_p \).

**Proof.** The hypotheses imply that for every ergodic invariant measure \( \mu \), we have \( \lambda_i(\mu) = c_- \) for \( 1 \leq i \leq k \) and \( \lambda_i(\mu) = c_+ \) for \( k + 1 \leq i \leq d \) since by Theorem 2.7 there is a sequence of periodic points \((p_j)_j\) such that \( \lambda_i^{(p_j)} \to \lambda_i(\mu) \) for each \( 1 \leq i \leq d \), and these equalities hold for each \( j \).

Fix a Hölder continuous Riemannian metric on \( E \) and use this to identify \( E \) with its dual bundle \( E^* \). Let \( B^t \) be the inverse adjoint of the cocycle \( A^t \). \( B^t \) is the cocycle over \( g^t \) given by \( B^t = [(A^t)^*]^{-1} : E_x \to E_{g^tx} \), where \( (A^t)^* \) is the adjoint of the map \( A^t_x \). The Lyapunov exponents of \( B^t \) are given by the negatives of the Lyapunov exponents of \( A^t \), see [11]. We use \( \Lambda^t A^t \) to denote the induced action of \( A^t \) on the exterior power bundle \( \Lambda^t E \) of \( E \).
Fix $\varepsilon > 0$ smaller than $c_+ - c_-$. Consider the subadditive sequence
\[ b_n(x) = \log \| B_x^n \| + (c_- - \varepsilon)n \]
For every ergodic invariant measure $\mu$ we have
\[ \lim_{n \to \infty} \frac{1}{n} \int_{SM} b_n d\mu = -\lambda_-(\mu) + (c_- - \varepsilon) < 0 \]
by hypothesis. Hence by Proposition 5.1 there is some $N_1 > 0$ such that $b_{N_1}(x) < 0$ for every $x \in SM$. Applying this same reasoning to the subadditive sequences
\[ a_n = \log \| A_x^n \| - (c_+ + \varepsilon)n \]
we obtain $N_2 > 0$, such that $a_{N_2}(x) < 0$ for every $x \in SM$.

Observe that for a subadditive sequence of functions $f_n$, if $f_N(x) < 0$ for every $x$, then $f_{kN}(x) < 0$ for every $x$, since
\[ f_{kN}(x) = \sum_{j=0}^{k-1} f_N(g_j^N(x)) \leq kf_N(x) < 0 \]
Let $N$ be the least common multiple of $N_1$ and $N_2$. Then $N$ is a common integer for which the functions $a_N$ and $b_N$ are both negative.

Thus by the above inequalities,
\[ \| B_x^N \| \leq e^{-(c_- - \varepsilon)N} \]
and,
\[ \| A_x^N \| \leq e^{(c_+ + \varepsilon)N} \]
Let $S = r(d-k) + sk$. Consider the action of $R^t = \Lambda^\alpha \Lambda^k B^t \wedge \Lambda^\beta \Lambda^{d-k} A^t$ on the top exterior bundle $\Lambda^S \mathcal{E}^S$. $\Lambda^S \mathcal{E}^S$ is one dimensional, so $\phi^t(x) = \log \| R^t_x \|$ is an additive cocycle over $g^t$. We claim $\phi^t$ is a coboundary. Fix a periodic point $p \in SM$ of period $\ell(p)$. Then for every $n$ we have $\phi^{n\ell(p)}(p) = n\phi^{\ell(p)}(p)$. Hence
\[ \phi^{\ell(p)}(p) = \lim_{n \to \infty} \frac{1}{n} \log \| R^{n\ell(p)}_p \| = -s \sum_{i=1}^{k} \lambda_i^{(p)}(x) + r \sum_{i=k+1}^{d} \lambda_i^{(p)}(x) = 0 \]
by the resonance condition on the periodic orbits. Thus by the Livsic theorem for Anosov flows we conclude that $\phi^t$ is a coboundary. In particular, there is some constant $C > 1$ such that
\[ C^{-1} \leq e^{\phi^t(x)} \leq C \]
for every $t \in \mathbb{R}$ and $x \in SM$.

Let $\sigma_i(x)$, $1 \leq i \leq d$, denote the singular values of $A_x^N$, listed in nondecreasing order. We know that
\[ e^{(c_- - \varepsilon)N} \leq \sigma_i(x) \leq e^{(c_+ + \varepsilon)N} \]
for each $i$ and $x$. Further, using the bound on $\phi^t$ obtained above we get that
\[ C^{-1} \leq \left( \prod_{i=1}^{k} \sigma_i(x) \right)^{-s} \left( \prod_{i=k+1}^{d} \sigma_i(x) \right)^{r} \leq C \]
which implies, upon moving one $\sigma_k(x)$ to the left side,
\[ \sigma_k(x) \leq C e^{(ks-1)(-((c_- - \varepsilon)N - r(d-k)(c_+ + \varepsilon)N)} = C e^{(c_+ + (S-1)\varepsilon)N} \]
and also implies, upon moving everything except \( \sigma_{k+1}(x) \) to the left side,
\[
\sigma_{k+1}(x) \geq C^{-1} e^{k((c_+ - \varepsilon)c)N} e^{-(r(d-k)-1)(c_+ - \varepsilon)N} = C^{-1} e^{(c_+ - (S-1)\varepsilon)N}
\]
where both inequalities hold for every \( x \in SM \).

The arguments from here on are adapted from Chapter 2 of [37]. Take \( \varepsilon \) small enough that \( c_+ - c_- - 2(S-1)\varepsilon > 0 \) and \( c_+ - (2(S-1)+1)\varepsilon > c_- + (2(S-1)+1)\varepsilon \). Next choose an integer \( \ell \) large enough that
\[
C e^{\ell(c_- + (S-1)\varepsilon)N} < C^{-1} e^{\ell(c_+ - (S-1)\varepsilon)N}
\]
Observe that all of the bounds above work if we replace \( A^N_x \) by an iterate \( A^N_{x+} \), since the functions \( a_{\ell N} \) and \( b_{\ell N} \) remain everywhere nonnegative on \( SM \). Set \( T := \ell N \). For each \( n \in \mathbb{N} \), let \( H^n(x) \) be the subspace of \( \mathcal{E}_n \) corresponding to the first \( k \) singular values of \( A^N_{x+T} \) and \( V^n(x) \) the subspace corresponding to the last \( d-k \) singular values. These subspaces are well defined since we have shown that
\[
\sigma_{k,n}(x) \leq C e^{(c_- + (S-1)\varepsilon)nT} < C^{-1} e^{(c_+ - (S-1)\varepsilon)nT} \leq \sigma_{k+1,nT}(x)
\]
where \( \sigma_{k,nT} \) are the singular values of \( A^N_{x+T} \). Further, \( \mathcal{E} = H^n \oplus V^n \) is a continuous decomposition of \( \mathcal{E} \) into orthogonal subspaces. For subspaces \( U_1, U_2 \), we write \( \angle(U_1, U_2) \) for the maximal angle made by a vector in \( U_1 \) with a vector in \( U_2 \). This determines a metric on the Grassmannian of subspaces of a particular dimension in \( \mathbb{R}^d \). We claim that there is some \( C' > 0 \) and \( \chi < 1 \) such that
\[
|\sin(\angle(H^n(x), H^{n+1}(x)))| \leq C' \chi^n
\]
for every \( n \) and every \( x \in SM \). Fix unit vectors \( w_n \in H^n_x \) and \( w_{n+1} \in H^{n+1}_x \). Set \( \alpha_n = \angle(w_{n+1}, v_n) \). We can write
\[
w_n = \cos(\alpha_n) w_{n+1} + \sin(\alpha_n) v_{n+1}
\]
for some unit vector \( v_{n+1} \in V^{n+1} \). Then \( v_{n+1} \) and \( w_{n+1} \) remain orthogonal upon applying \( A^N_{x+T} \). It follows that
\[
\|A^{(n+1)T} w_n\| \geq |\sin(\alpha_n)| \cdot \|A^{(n+1)T} v_{n+1}\| \geq C^{-1} e^{(c_+ - (S-1)\varepsilon)(n+1)T} |\sin(\alpha_n)|
\]
But we also have
\[
\|A^{(n+1)T} w_n\| \leq \|A^{(n+1)T} T w_n\| \leq \|A^{(n+1)T} T e^{-e^{(c_+ - (S-1)\varepsilon)nT}}\|
\]
Putting all of this together we obtain that
\[
|\sin(\alpha_n)| \leq C^2 \|A^{(n+1)T} e^{-(c_+ - (S-1)\varepsilon)nT} \|
\]
with \( \chi = e^{-(c_+ + c_- + 2(S-1)\varepsilon)T} < 1 \) and \( C' = C^2 \sup_{x \in SM} \|A^{(n+1)T} e^{-(c_+ - (S-1)\varepsilon)T} \|
\)

The claim implies that \( \{H^n(x)\}_{n \in \mathbb{N}} \) is a Cauchy sequence in the Grassmannian of \( k \)-planes in \( \mathbb{R}^d \) for each \( x \), hence converges to a unique \( k \)-plane \( H(x) \) as \( n \to \infty \).

We claim that \( H \) is invariant under \( A^T \). We have
\[
|\sin(\angle(A^T(H^n(x)), H(g^T x)))| = \lim_{n \to \infty} |\sin(\angle(A^T(H^n(x)), H^{n+1}(g^T x)))|
\]
so we must estimate \( \angle(A^T(H^n(x)), H^{n+1}(g^T x)) \). As above, let \( w_n \in A^T(H^n(x)) \) and \( u_n \in H^{n+1}(g^T x) \) be unit vectors, and write
\[
w_n = \cos(\beta_n) u_n + \sin(\beta_n) v_n
\]
for some unit vector \( v_n \in V^{n+1}(g^T x) \), where \( \beta_n = \angle(w_n, u_n) \). Then \( u_n \) and \( v_n \) remain orthogonal upon applying \( A^{(n+1)T} \). Hence an identical calculation to the one done above gives \(|\sin(\beta_n)| \leq C^* \chi^{n+1} \) for some constant \( C^* \) and the same \( \gamma \) as before. Hence as \( n \to \infty \), \( \sin \left( \angle \left( A^T(H^n(x)), H^{n+1}(g^T x) \right) \right) \to 0 \). This implies that \( H \) is invariant under \( A^T \).

Lastly, we need to obtain an estimate on \( \|A^{nT}H\| \). From convergence of the geometric series we obtain the bound
\[
\sin \left( \angle(H(x), H^n(x)) \right) \leq \sum_{j=n}^{\infty} \sin \left( \angle(H^{j+1}(x), H^j(x)) \right)
\]
\[
\leq C' \sum_{j=n}^{\infty} \chi^j
\]
\[
= \frac{C' \chi^n}{1 - \chi}
\]

We calculated above that
\[
\|A^{(n+1)T}H^n\| \leq \|A^T\|Ce^c(S-1)^cnT
\]
Set \( \gamma_n = \angle(H(x), H^n(x)) \). If we combine the above estimate with the angle estimate, we obtain
\[
\|A^{(n+1)T}H\| \leq \|A^{(n+1)T}H^n\| + |\sin(\gamma_n)|\|A^{(n+1)T}V^n\|
\]
\[
\leq \|A^{(n+1)T}H^n\| + C'' \chi^n \|A^{(n+1)T}V^n\|
\]
for some constant \( C'' \). But
\[
\|A^{(n+1)T}V^n\| \leq \|A^{(n+1)T}\| \leq e^{(c+\varepsilon)(n+1)T}
\]
Combining this with the formula \( \chi = e^{(-c_+ c_- 2(S-1)c)T} \) and using our earlier derived bound for \( \|A^{(n+1)T}H^n\| \), we at last have
\[
\|A^{(n+1)T}H\| \leq \|A^T\|Ce^{(-c_+ c_- + 2(S-1)c)nT} + C'' \chi^{-1} e^{(c_+ + 2(S-1)+1)c(n+1)T}
\]
\[
\leq C^* e^{(c_+ + 2(S-1)+1)c(n+1)T}
\]
\( C^* \) being some constant, as usual.

Passing to the inverse cocycle \( A^{-T} \) instead switches the role of \( c_+ \) and \( c_- \), \( V \) and \( H \), and so on. So we draw analogous conclusions: there is a \( d - k \)-dimensional \( A^{-T} \)-invariant subspace \( V \) with an estimate
\[
\|A^{-(n+1)T}V\| \leq C^* e^{(-c_+ + 2(S-1)+1)c(n+1)T}
\]
\( \varepsilon \) was chosen small enough that \( c_+ - (2(S-1)+1) \varepsilon > c_- + (2(S-1)+1) \varepsilon \). Hence there is some \( 0 < \lambda < 1 \) such that
\[
\|A^{(n+1)T}H\| \cdot \|A^{-(n+1)T}V\| \leq C\lambda^{(n+1)T}
\]
This implies that \( H \) and \( V \) are everywhere transverse, as for \( n \) large the minimal expansion on \( V \) of \( A^{(n+1)T} \) is strictly greater than the maximal expansion on \( H \). Therefore we have a direct sum splitting \( E = H \oplus V \). This is a dominated splitting for \( A^{(n+1)T} \) when \( n \) is sufficiently large, by the above inequality. Hence some large iterate of \( A^T \) admits a dominated splitting \( E = H \oplus V \). This is equivalent to \( A^T \) itself admitting a dominated splitting.
Lemma 5.3. Under the hypotheses in the statement of Theorem 1.2, there is a dominated splitting $E^u = H^u \oplus V^u$ for $Dg^t|E^u$, with $H^u$ being the weakly expanding subspace. The cocycle $Dg^t|H^u$ preserves a conformal structure on $H^u$.

Proof. By the correspondence between Lyapunov exponents of measures $\mu_p$ supported on periodic orbits and the normalized absolute values of the eigenvalues of the map $Dg_p^{\ell(p)} : E^u_p \to E^u_p$ discussed at the end of Section 2, we see that under the hypotheses of Theorem 1.2 the Lyapunov exponents $\lambda_i^{(p)}$ of $g^t$ along periodic orbits satisfy

$$\lambda_i^{(p)} = 1, \ 1 \leq i \leq m - r_K$$

$$\lambda_i^{(p)} = 2, \ m - r_K + 1 \leq i \leq m - 1$$

We can thus apply Lemma 5.2 with $c_- = 1, c_+ = 2, k = m - r_K$ and choosing any pair of positive integers $r$ and $s$ such that

$$\frac{r}{s} = \frac{m - r_K}{2(r_K - 1)}$$

Looking back at the proof of Lemma 5.2, we see that for any $\varepsilon > 0$ we have estimates for $t > 0$ and some constant $C > 1$,

$$C^{-1}e^{(1-\varepsilon)t} \leq \|Dg^t|H^u\| \leq Ce^{(1+\varepsilon)t}$$

$$C^{-1}e^{(2-\varepsilon)t} \leq \|Dg^t|V^u\| \leq Ce^{(2+\varepsilon)t}$$

where the constant $C$ will depend on $\varepsilon$. Since the splitting $E^u = V^u \oplus H^u$ is dominated and $E^u$ is Hölder continuous with some Hölder exponent $\alpha$, $H^u$ is also Hölder continuous with some exponent $\beta$ because the subbundles in a dominated splitting are Hölder continuous. Let $\gamma = \min(\alpha, \beta)$. Take $\varepsilon$ small enough that $(2 + \gamma)\varepsilon - \gamma < 0$

By applying everything above to the stable bundle $E^s$ instead, we obtain the estimate $\|Dg^t|E^s\| \leq Ce^{-(-1-\varepsilon)t}$ for $t > 0$, increasing the constant $C$ if necessary. Then for $t > 0$,

$$\|Dg^t|E^s\|^{\gamma} \cdot \|Dg^{-t}|H^u\| \cdot \|Dg^t|H^u\| \leq C^3e^{((2+\gamma)\varepsilon - \gamma)t}$$

For $t$ large enough, $C^3e^{((2+\gamma)\varepsilon - \gamma)t} < 1$, and so we conclude that $Dg^t|H^u$ is $\gamma$-fiber bunched.

Let $\mu$ be the Liouville volume on $SM$. By Theorem 2.7,

$$\lambda_+(Dg^t|H^u, \mu) = \lambda_-(Dg^t|H^u, \mu) = 1$$

since the same statement is true for all Lyapunov exponents of invariant measures concentrated on periodic orbits. By combining Lemmas 2.4 and 2.5 we see that either $Dg^t|H^u$ preserves a conformal structure on $H^u$ or after passing to a finite cover there is a proper continuous $g^t$-invariant subbundle $E \subset H^u$. But by Lemma 4.5, any finite cover of $H^u$ has no proper invariant subbundles, since $H^u$ and $E^u$ are $\gamma$-Hölder and $Dg^t|H^u$ is $\gamma$-fiber bunched. Hence $Dg^t|H^u$ preserves a conformal structure. \(\Box\)

Proof of Theorem 1.2. By Lemma 5.3 there is a dominated splitting $E^u = H^u \oplus V^u$ and there is a Hölder continuous Riemannian metric $\tau$ on $H^u$ for which $Dg^t|H^u$ is
conformal. Let $\psi^t$ be the multiplicative cocycle associated to the invariant conformal structure $\tau$. Since for each periodic point $p \in SM$ we have $\lambda_i(p) = 1$ for $1 \leq i \leq m - r_K - 1$, we conclude that for each periodic point $p$ we have $\psi^t(p) = e^{2t(i)}$. By the Livsic theorem for Anosov flows (Chapter 19 of [26]) this implies that $\psi^t$ is cohomologous to the constant functions $e^{2t}$ on $SM$. More precisely, there is a continuous function $\zeta : SM \to \mathbb{R}$ such that

$$\frac{\psi^t}{e^{2t}} = \zeta \circ g^t$$

Define a new Riemannian metric $\sigma$ on $H^u$ by setting $\sigma = \zeta^{-1} \tau$. Observe that

$$(g^t)^* \sigma = (\zeta \circ g^t)^{-1} \psi^t \tau = e^{2t} \zeta^{-1} \tau = e^{2t} \sigma$$

The construction of the dominated splitting in Lemma 5.2 shows that there is some constant $C$ such that

$$\| Dg^t v \| \geq C e^{(2-\varepsilon)t} \| v \|, \; v \in V^u$$

for some $0 < \varepsilon < 1$, in our standard Riemannian metric on $E^u$. This metric on $V^u$ can be modified into a new Riemannian metric to eliminate the constant $C$ [16], which is called a Lyapunov metric. Extend $\sigma$ to a Riemannian metric on $E^u$ by declaring $H^u$ and $V^u$ to be orthogonal, and choosing some adapted Lyapunov metric on $V^u$ which satisfies

$$e^{(2-\varepsilon)t} |v| \leq |Dg^t v|$$

for some $0 < \varepsilon < 1$ and every $v \in V^u$. Let $| \cdot |$ be the norm associated to $\sigma$. We have constructed $\sigma$ such that for any vector $v \in E^u$,

$$|Dg^t v| \geq e^t |v|$$

for every $t \in \mathbb{R}$. We extend $\sigma$ to a Riemannian metric on $T(SM)$ by declaring the Anosov splitting $T(SM) = E^u \oplus E^c \oplus E^s$ to be an orthogonal splitting, taking the generator of $g^t$ in $E^c$ to have unit length, and using the flip map $p \to -p$ (which maps $E^u$ to $E^s$) to extend $\sigma$ to be defined on $E^s$, with analogous bounds. Let $\rho$ be the distance on $SM$ associated to $\sigma$, and for a point $p \in SM$, let $\rho_p$ be the induced Riemannian distance on $W^u(p)$.

For two points $x, y \in W^u(p)$, define

$$\alpha_p(x, y) = \sup \{ t \in \mathbb{R} : \rho_p(g^t x, g^t y) \leq 1 \}$$

and let $\eta_p(x, y) = e^{-\alpha_p(x, y)}$. Hasselblatt [16] proved that $\eta_p$ defines a metric on $W^u(p)$ and that the Hausdorff measure associated to $\eta_p$ is the Bowen-Margulis measure of maximal entropy for $g^t$. Furthermore the Hausdorff dimension of $\eta_p$ is given by the topological entropy of $g^t$.

We claim that the multiplicative cocycle given by the Jacobian of the restriction of $g^t$ to unstable manifolds in the Riemannian metric $\kappa$ is cohomologous to the constant function $e^{(m+2r-2)t}$. First recall how the unstable Jacobian $J^{u,t}$ is defined: the metric $\sigma$ locally defines a distinguished section $X$ of the line bundle $\bigwedge^{m-1} E^u$ over $SM$, given by taking the wedge product of vectors in $E^u$ forming an orthonormal frame. Then $J^{u,t}$ is defined by the equation $(g^t)^* X = J^{u,t} X$. For every periodic point $p$, the assumption on Lyapunov exponents implies that the Lyapunov exponent of the induced action of $g^t$ on $\bigwedge^{m-1} E^u$ along any periodic
orbit is $m + r_K - 2$. This immediately implies that $J_{u,t}(p) = e^{(m + r_K - 2)t}(p)$, which gives the desired statement by the Livsic theorem. Since the unstable Jacobian cocycle is cohomologous to a constant cocycle, the Riemannian volume with respect to $\sigma$ is the measure of maximal entropy, and therefore the topological entropy of $g^t$ is $m + r_K - 2$, see for instance Chapter 20 of [26].

By assumption there is a compact quotient $N$ of $H_K^{m/r_K}$ and a homotopy equivalence $M \to N$. This homotopy equivalence induces a quasiconformal homeomorphism $\partial \widehat{M} \to \partial H_K^{m/r_K}$ [8]. The conformal dimension of $H_K^{m/r_K}$ is $m + r_K - 2$ and therefore the Hausdorff dimension of $\partial \widehat{M}$ is at least $m + r_K - 2$ [29]. $\eta_p$ induces a visual metric on $\partial \widehat{M}$ and thus the Hausdorff dimension of $\eta_p$ is an upper bound on the Hausdorff dimension of $\partial \widehat{M}$ [29]. We conclude that $\partial \widehat{M}$ has Hausdorff dimension exactly $m + r_K - 2$. Since there is a quasiconformal homeomorphism $\partial M \to \partial H_K^{m/r_K}$, $H_K^{m/r_K}$ is a symmetric space, and $\partial \widehat{M}$ has the same Hausdorff dimension as $\partial H_K^{m/r_K}$, we conclude by Theorem 1.2 of [10] that $M$ and $N$ are isometric.  

6. Proof of Theorem 1.5

In this section we prove Theorem 1.5. We let $M$ be a closed negatively curved manifold with geodesic flow $g^t$ and assume that $g^t$ admits a dominated splitting $E^u = H^u \oplus V^u$ with $\dim V^u = 1$. We assume there is some $0 < \alpha \leq 1$ such that $H^u$ and $E^u$ are $\alpha$-Hölder continuous and $Dg^t|H^u$ is $\alpha$-fiber bunched. By assumption there is a Gibbs measure $\mu$ for $g^t$ such that $\lambda_+(Dg^t|H^u, \mu) = \lambda_-(Dg^t|H^u, \mu)$. Lastly we assume that the restriction of $H^u$ to the $W^u$ foliation is a $C^1$ subbundle of $TW^u$ and that for every periodic point $p$, $2\lambda_+(Dg^t|H^u, \mu_p) = \lambda_+(Dg^t|V^u, \mu_p)$.

By combining Lemma 2.4, Lemma 2.5, and Lemma 4.5 as in the proof of Lemma 5.3, we see that $Dg^t|H^u$ preserves a conformal structure which we represent by a Riemannian metric $\tau$ with associated multiplicative cocycle $\psi^t$. The $C^1$ regularity of $H^u$ along $W^u$ implies $C^1$ regularity for the unstable holonomies of $W^u$:

**Lemma 6.1.** On a fixed unstable leaf $W^u(p)$, the unstable holonomy $(x, y) \to h_{x,y}^u$, $x, y \in W^u(p)$ is $C^1$ as a function of $x$ and $y$. The derivative of the unstable holonomy is uniformly Hölder continuous in $x$ and $y$.

**Proof.** Since $H^u$ is $C^1$ along $W^u$ leaves and $E^u/V^u$ is smooth along these leaves, the projection $H^u \to E^u/V^u$ is $C^1$. As remarked in Lemma 4.3, the unstable holonomy of the action of $Dg^t$ on $E^u/V^u$ is induced by the unstable holonomy of $Dg^t|H^u$ by this projection. Since the unstable holonomy on $E^u/V^u$ is $C^1$, this implies that the unstable holonomy for $H^u$ is $C^1$.  

By Lemma 6.1, we can choose a representative of the conformal class of $\tau$ which is $C^1$ along $W^u$. We will fix this representative from now on.

We claim we may assume that the line bundle $V^u$ is orientable. If $V^u$ is not orientable, pass to a double cover of $SM$ in which $V^u$ is orientable. This corresponds to the geodesic flow on a Riemannian double cover $M^*$ of $M$. If we prove that $M^*$
is isometric to a complex hyperbolic manifold, it then follows immediately that $M$ is as well. Hence it suffices to assume $V^u$ is orientable.

Extend $\tau$ to a Riemannian metric on $E^u$ by setting $V^u$ and $H^u$ to be orthogonal and taking some continuous section $Z$ of $V^u$ which is smooth along $W^u$ leaves to define the unit length on $V^u$. $\tau$ remains $C^1$ since $H^u$ is a $C^1$ subbundle of $E^u$. Let $\alpha$ be the 1-form on $W^u$ defined by taking the inner product using $\tau$ with $Z$. It’s clear that $\ker \alpha \cap E^u = H^u$ and that $\alpha$ is $C^1$. Since $V^u$ and $H^u$ are $Dg^t$-invariant, for each $t \in \mathbb{R}$ there is some $C^1$ (along $W^u$ leaves) function $\varphi^t$ such that $(g^t)^* \alpha = \varphi^t \alpha$. $\varphi^t$ is a multiplicative $\mathbb{R}$-cocycle.

The assumption that $2\lambda_+(Dg^t|H^u, \mu_p) = \lambda_+(Dg^t|V^u, \mu_p)$ for every periodic point $p$ implies that $\varphi^t$ is cohomologous to $\psi^t$, since the right side is the logarithm of the factor $\alpha$ is multiplied by along a periodic orbit, and the left hand side is the periodic data associated to the conformal structure on $H^u$. Thus let $\zeta : SM \to \mathbb{R}$ be the $C^1$ function such that

$$\frac{\psi^t}{\varphi^t} = \frac{\zeta \circ g^t}{\zeta}$$

for every $t \in \mathbb{R}$. If we then replace $\tau$ by the Riemannian metric $\sigma = \zeta^{-1} \tau$ on $H^u$, we have $(g^t)^* \sigma = \varphi^t \sigma$.

**Lemma 6.2.** $d\alpha|H^u$ is nondegenerate on all of $SM$.

**Proof.** Observe first that for every $t \in \mathbb{R}$,

$$(g^t)^*(d\alpha) = d((g^t)^*(\alpha)) = d(\varphi^t \alpha) = d\varphi^t \wedge \alpha + \varphi^t d\alpha$$

and hence $(g^t)^* d\alpha|H^u = \psi^t d\alpha|H^u$. It follows that

$$R_k = \{ p \in SM : \dim \{ v \in H^u_p : \iota_v d\alpha = 0 \} = k \}$$

is a $g^t$-invariant subset of $SM$. Since $\mu$ is ergodic with respect to the action of $g^t$ and $\bigcup_{k=0}^{m-2} R_k = SM, \mu(R_k) = 1$ for some $k$. If $0 < k < m - 2$, then $p \to \{ v \in H^u_p : \iota_v d\alpha = 0 \}$ is a proper measurable $g^t$-invariant subbundle of $H^u$, which is impossible by Lemma 4.5.

Suppose now that $\mu(R_0) = 1$. Then $d\alpha|H^u = 0$ for a dense set of points in $SM$ since $\mu$ has full support in $SM$, hence $d\alpha|H^u = 0$ for every $p \in SM$ since $d\alpha$ is continuous. This implies by the $C^1$ Frobenius theorem that $H^u$ is uniquely integrable as a subbundle of $E^u$ over $W^u$. We claim that this contradicts Corollary 4.6. Fix a periodic point $p \in SM$ of period $T$ and let $\mathcal{F}$ be the $C^1$ foliation tangent to $H^u$ inside $W^u(p)$. By Corollary 4.6, each leaf of $\mathcal{F}$ is dense in $W^u(p)$.

But this is absurd, for reasons similar to those given in the proof of Lemma 4.5. Let $U$ be a neighborhood of $p$ on which $g^{-T}$ acts as an exponential contraction and such that the foliation $\mathcal{F}$ can be trivialized as plaques $\mathbb{R}^{m-2} \times \{a\}$. Two different plaques of the foliation in $U$ have the property that they cannot be connected by a $C^1$ curve $\sigma$ lying entirely in $U$. If $\mathcal{F}(p)$ is dense in $W^u(p)$, we can find some slice of $\mathcal{F}$ in $U$ that does not pass through $p$ and a unit speed curve $\gamma : [0, T] \to W^u(p)$ with $\gamma(0) = p$ and $\gamma(T) = q \in U$ lying in a different plaque. Consider $g^{-kT} \circ \gamma$ for $k > 0$ large. By the definition of the unstable leaf, for $k$ large enough the entire curve $g^{-kT} \circ \sigma$ is contained inside of $U$, and by the $g^T$-invariance of $\mathcal{F}, g^{-kT} \circ \sigma$ is always
contained inside of $\mathcal{F}(p)$. This implies that $g^{-kT}(\sigma(\ell))$ lies in the local leaf through $p$ of $\mathcal{F}$ in $U$. The contraction property of $g^{-T}$ on $U$ implies that $g^{-kT}(U) \subset U$ is an open connected subset containing $p$, and the foliation can be trivialized on this open subset, so we can join $g^{-kT}(\sigma(\ell))$ to $p$ by a curve $\sigma$ lying entirely inside of $g^{-kT}(U)$. But then $g^{kT} \circ \sigma$ is a curve joining $p$ to $\gamma(\ell)$ lying entirely inside of the local leaf of $\mathcal{F}$ through $p$ inside of $U$, which is a contradiction.

The only remaining possibility is $\mu(R_{m-2}) = 1$. Hence $da_p|H^u_p$ is nondegenerate for $\mu$-a.e. $p \in SM$. The bundle $H^u$ over $SM$ is measurably trivial, so as a measure space with measure $\mu$ the bundle $H^u$ over $SM$ is isomorphic to $SM \times \mathbb{R}^{m-2}$. Set $\omega$ to be $da|H^u$ in these measurable coordinates. Let $p \in SM$ be a point at which $\omega_p$ is nondegenerate. In the fiber over $p$ we will construct a linear isomorphism $J : \mathbb{R}^{m-2} \to \mathbb{R}^{m-2}$ with $J^2 = -I$, where $I$ is the identity on the fibers. Further, $J$ will depend measurably on $p$. Since $\omega_p$ is nondegenerate, there is a unique matrix $A_p$ satisfying

$$\omega_p(x, y) = \sigma_p(A_p x, y)$$

for every $x, y \in \mathbb{R}^{m-2}$, and $A_p$ depends measurably on $p$, being given by solving linear equations. It’s easy to check that $A_p$ is skew adjoint with respect to $\sigma$, and therefore that $A \cdot A^T = -A^2$ is positive definite and symmetric, where $A^T$ denotes the transpose with respect to the Riemannian metric $\sigma$. This matrix thus has a unique positive definite square root which also depends measurably on $p$. Define $J = (\sqrt{-A^2})^{-1}A$. Then $J^2 = -I$. Further, the adjoint of $J$ with respect to $\tau$ is given by $-J$, since $\sqrt{-A^2}$ is also self-adjoint. We also note that $\omega$ is compatible with $J$:

$$\omega(Jx, Jy) = \sigma(AJx, Jy) = \sigma(JAx, Jy) = \sigma(Ax, -J^2y) = \omega(x, y)$$

This gives our desired measurable section $J$. Observe also that

$$\omega(x, Jx) = \sigma(Ax, Jx) = -\sigma(JAx, x) = \sigma(\sqrt{-A^2}x, x) > 0$$

since the matrix $\sqrt{-A^2}$ is positive definite. We also have

$$\omega(x, Jy) = -\omega(Jx, y) = \omega(y, Jx)$$

Hence $(x, y) \to \omega(x, Jy)$ is a positive definite symmetric bilinear form.

We next claim $J$ is $\mu$-a.e. invariant under $g^t$. For this, first observe that

$$\varphi^t \omega(x, y) = \omega(Dg^t x, Dg^t y) = \sigma(A \cdot Dg^t x, Dg^t y)$$

$$\sigma(Dg^t \cdot Ax, Dg^t y) = \varphi^t \sigma(Ax, y) = \varphi^t \omega(x, y)$$

both equalities occurring $\mu$-a.e. Hence we conclude that

$$\sigma(A \cdot Dg^t x, Dg^t y) = \sigma(Dg^t \cdot Ax, Dg^t y)$$

for every $x, y \in \mathbb{R}^{m-2}$. Since $y$ is arbitrary and $Dg^t$ is an isomorphism, this implies

$$\tau(A \cdot Dg^t x, v) = \tau(Dg^t \cdot Ax, v)$$

for every $x, v \in \mathbb{R}^{m-2}$, which implies that $Dg^t A = ADg^t$ at $\mu$-a.e. $p \in SM$. It’s easy to see that if a matrix $P$ commutes with a positive definite symmetric matrix $Q$, then $P$ commutes with $\sqrt{Q}$; since $P$ commutes with $Q$, $P$ and $Q$ are simultaneously diagonalizable, but then it is obvious that $P$ commutes with $\sqrt{Q}$, since in the new basis both $P$ and $\sqrt{Q}$ are diagonal. Hence $Dg^t$ also commutes with $J$. 
Let $H^u_p = H^u \otimes \mathbb{C}$ be the complexification of the bundle $H^u$, and take the natural extension by scalars of the linear map $Dg^t$. Since $Dg^t$ commutes with $J$, the bundles

$$E_p^{1,0} = \{ v \in H^u_p \otimes \mathbb{C} : Jv = iv \}$$

$$E_p^{0,1} = \{ v \in H^u_p \otimes \mathbb{C} : Jv = -iv \}$$

are invariant under the extended action of $Dg^t$. However, the bundle $H^u \otimes \mathbb{C}$ retains the property that the action of $g^t$ on $H^u$ has equal extremal exponents with respect to $\mu$. Thus Lemma 2.3 implies, since $E^{1,0}$ and $E^{0,1}$ are measurable $Dg^t$ invariant subbundles of $H^u_\mathbb{C}$, that $E^{1,0}$ and $E^{0,1}$ coincide $\mu$-a.e. with continuous, holonomy invariant subbundles of $H^u_\mathbb{C}$. This immediately implies that $J$ is H"older continuous and invariant under stable and unstable holonomy, since we can define $J$ on $H^u$ by $J(v + w) = iv - iw$ for $v \in E^{1,0}$, $w \in E^{0,1}$, and this decomposition is continuous and holonomy invariant.

Hence $\eta(X,Y) := \varphi(X, JY)$ is a continuous symmetric bilinear form on $H^u$ which is positive definite $\mu$-a.e. Therefore $\eta$ defines a measurable conformal structure on $H^u$. Since at $\mu$-a.e. point of $SM$ we have

$$\eta(Dg^t X, Dg^t Y) = \varphi(Dg^t X, J(Dg^t Y)) = \varphi \eta(X,Y)$$

we conclude that $\eta$ is $g^t$-invariant and hence by Lemma 2.3 coincides $\mu$-a.e. with a continuous conformal structure on $H^u$. Since $\eta$ itself is continuous, this implies that $\eta$ defines a continuous conformal structure on $H^u$. In particular $\eta$ is positive definite on all of $SM$, from which it follows that $\varphi | H^u$ is nondegenerate on $SM$. \qed

In particular there is at least one point $p \in SM$ for which $\varphi | H^u_p$ is nondegenerate. This means that the vector space $H^u_p$ carries a nondegenerate 2-form, which implies that it is even dimensional. Thus $dim H^u = m - 2$ is even, so $m$ is even.

Put coordinates $(x_1, \ldots, x_k, y_1, \ldots, y_k, z)$ on $\mathbb{R}^{m-1}$, where $k = (m-2)/2$. Let

$$\nu = dz + \frac{1}{2} \sum_{i=1}^k x_i dy_i - y_i dx_i$$

be the standard contact 1-form on $\mathbb{R}^{m-1}$. Let $\mathcal{H} = \text{ker} \nu$.

**Lemma 6.3.** For each $p \in SM$ such that $\varphi | H^u_p$ is nondegenerate, there is an $r > 0$ such that there is a $C^1$ function $\xi_p : W^u_r(p) \to (0, \infty)$ and a $C^2$ diffeomorphism $\Psi_p : W^u_r(p) \to U$, $U$ a neighborhood of the identity in $\mathbb{R}^{m-1}$, with $\Psi_p^*(\nu) = \xi_p \alpha$.

**Proof.** Since the almost complex structure $J$ on $H^u$ and the conformal class of $\sigma$ are both holonomy invariant, it follows that the line bundle spanned by $\varphi | H^u$ inside of $\Lambda^2 H^u$ is holonomy invariant and therefore for every point $p \in SM$, there is a $C^1$ function $\xi_p : W^u_r(p) \to (0, \infty)$ defined by

$$(h^u_{pq})^* \alpha = \xi_p(q) \alpha$$

where $\alpha$ is restricted to $H^u$. The holonomy relation $Dg^t_q \circ h^u_{pq} = h^u_{pg^t q} \circ Dg^t_q$ implies that $\varphi^t(q) \xi_p(q) = \xi_{g^t(q)}(g^t(q)) \varphi^t(p)$, since $\varphi | H^u$ scales by $\varphi^t$ when acted on by $Dg^t$. 

Consider the $C^1$ 1-form $\xi_p \alpha$ on $W^u(p)$. For $q \in W^u(g^{-t}p)$, $X \in E^u_q$, 
$$\xi_p(g^t(q))\alpha(Dg^tX) = \xi_p(g^t(q))\varphi^t(q)\alpha(X) = \xi_{g^{-t}p}(q)\varphi^t(g^{-t}(p))\alpha(X)$$
Hence $(g^t)^*(\xi_p \alpha) = \varphi^t(g^{-t}(p))\xi_{g^{-t}(p)} \alpha$. Taking the exterior derivative of each side, we get 
$$(g^t)^*(d(\xi_p \alpha)) = \varphi^t(g^{-t}(p))d(\xi_{g^{-t}(p)} \alpha)$$
Let $Z_p$ be the Reeb vector field for $\xi_p \alpha$ on $W^u(p)$ defined by 
$$\xi_p \alpha(Z_p) = 1$$
and thus we can write 
$$d(\xi_p \alpha)(Z_p, X) = 0 \text{ for every } X \in H^u.$$ 
From the exterior derivative relation we obtain 
$$Dg^t(Z_p) = \varphi^t(p)Z_{g^t(p)},$$
and thus we can write 
$$Z_p = \varphi^{-t}(p)Dg^t(Z_{g^{-t}(p)}).$$
We claim that there is a small $\delta > 0$ such that there is a constant $\gamma > 0$ independent of $p$ such that $Z_p$ makes an angle of at least $\gamma$ with $H^u$ on $W^u_\delta(p)$. Note that for every $p \in SM$ we have $Z_p \notin H^u$ since $d(\xi_p \alpha)|H^u = d\alpha|H^u$ is nondegenerate.

We first show that there is some $\gamma > 0$ such that $Z_p(p)$ makes an angle of at least $2\gamma$ with $H^u$. If this did not hold, then we could find a sequence of points $p_n \in SM$ converging to $p \in SM$ with $Z_{p_n}/\|Z_{p_n}\| \to Y \in H^u$. Let $\delta > 0$ be small enough that the unstable holonomy $h^u$ of $H^u$ has uniformly Hölder continuous derivative on $W^u_\delta(q)$ for every $q \in SM$. Then the unstable holonomies and their derivatives on $W^u_\delta(p_n)$ converge uniformly to the unstable holonomy and its derivative on $W^u_\delta(p)$. It follows that the $C^1$ 1-forms $\xi_p \alpha$ on $W^u_\delta(p_n)$ converge uniformly in the $C^1$ topology to the 1-form $\xi_p \alpha$ on $W^u_\delta(p)$. In particular $d(\xi_{p_n}\alpha)$ converges uniformly to $d(\xi_p \alpha)$. Hence $Z_{p_n} \to Z_p$ uniformly in $n$. But the assumption that $Z_{p_n}/\|Z_{p_n}\| \to Y \in H^u$ implies that $Z_p \in H^u$, which is impossible.

We first show that there is some $\gamma > 0$ such that $Z_p(p)$ makes an angle of at least $2\gamma$ with $H^u$. Hence, since the unstable holonomy has uniformly Hölder derivative on $W^u_\delta(p)$, there is some $r > 0$ independent of $p$ such that $Z_p$ makes an angle of at least $\gamma$ with $H^u$ on $W^u_r(p)$. Since for $t > 0$ large enough $g^{-t}(W^u_\delta(p)) \subset W^u_\delta(g^{-t}p)$, $Z_{g^{-t}(p)}$ makes an angle of at least $\gamma$ with $H^u$ on $g^{-t}(W^u_\delta(p))$. Thus the expression 
$$Z_p = \varphi^{-t}(p)Dg^t(Z_{g^{-t}(p)})$$
implies that $Z_p$ is parallel to $V^u$, since as $t \to \infty$ the angle between $V^u$ and $Dg^t(Z_{g^{-t}(p)})$ converges uniformly to zero on $W^u_\delta(p)$. Here we need that the angle of $Z_{g^{-t}(p)}$ with $H^u$ is uniformly bounded away from zero on $g^{-t}(W^u_\delta(p))$.

Thus $Z_p$ is parallel to $V^u$ and so $d(\xi_p \alpha)$ vanishes on $V^u$. Since $d(\xi_p \alpha)|H^u = \xi_p d\alpha$, we conclude that $d(\xi_p \alpha)|H^u$ is invariant under unstable holonomy. In particular it is $C^1$. Since the splitting $E^u = H^u \oplus V^u$ is $C^1$, we get that $d(\xi_p \alpha)$ itself is $C^1$.

We conclude that $\xi_p \alpha$ is a $C^1$ contact form on $W^u_{loc}(p)$ with $d(\xi_p \alpha)$ also being $C^1$. The foliation associated to the Reeb vector field $Z_p$ is the strong unstable foliation $W^u$, which is smooth. $d(\xi_p \alpha)$ descends to a $C^1$ closed 2-form $\omega$ on the quotient $Q^u(p)$ of $W^u$ by the $W^u$ foliation.
Recall that \( \Pi : W^u(p) \to Q^u(p) \) denotes projection. Darboux’s theorem for symplectic forms implies that there is a neighborhood \( U \) of \( \Pi(p) \) and a \( C^2 \) diffeomorphism \( F : U \to L \) onto a neighborhood \( L \subset \mathbb{R}^{m-2} \) of 0 which pulls back the standard symplectic form \( dv \) on \( L \) to \( \omega \) on \( U \). Choose a smooth transversal \( K \) to the \( W^{uu} \) foliation of \( W^u(p) \) with \( \Pi(K) = U \) and \( p \in K \). Let \( \tilde{U} \) be the open neighborhood of \( p \) given by taking, for each \( q \in K \), an open neighborhood of \( q \) in \( W^{uu}(q) \). Put coordinates on \( \tilde{U} \) by smoothly identifying \( K \) with a neighborhood of 0 in \( \mathbb{R}^{m-2} \), and smoothly trivializing the \( W^{uu} \) foliation to the map to the \( z \) coordinate axis (the last coordinate) in a neighborhood of 0 in \( \mathbb{R}^{m-1} \).

Using these coordinates we lift \( F \) to a \( C^1 \) diffeomorphism \( \Psi_p : \tilde{U} \to \tilde{L} \) of a neighborhood \( \tilde{U} \) of \( p \in W^u(p) \) to a neighborhood \( \tilde{L} \) of 0 in \( \mathbb{R}^{m-1} \). Since a contact form is invariant under its Reeb flow, the coordinate representation of \( \xi_{p}\alpha \) in the above defined coordinates on \( \tilde{U} \) is constant along the \( z \)-axis. It follows that \( d(\xi_{p}\alpha) \) is also constant along the \( z \)-axis. Define \( \phi : \tilde{U} \to \tilde{L} \) in these coordinates by \( \phi(x, z) = (F(x), z) \), where \( x \in U \). Then \( \phi^* d\nu = d(\xi_{p}\alpha) \) and therefore the 1-form \( \beta = \phi^* (\nu) - \xi_{p}\alpha \) is closed. Define for \( x \in U \),

\[
f(x) = \int_0^1 \beta(tx) \, dt
\]

Note that the \( z \)-component of \( \beta \) is 0 since \( \xi_{p}\alpha \) is independent of the \( z \)-coordinate and \( \phi \) is the identity on the \( z \)-coordinate. Note also that \( f \) does not depend on \( z \).

Set \( \Psi_p(x,z) = (\phi(x), z-f(x)) \). We claim that \( \Psi_p^*(\nu) = \xi_{p}\alpha \). Let \( \gamma : [0,1] \to \tilde{U} \) and let \( \sigma_t, t \in [0,1] \), be the unique radial curve (when projected to \( U \)) tangent to \( \ker \alpha \) joining a point on the \( z \)-axis to \( \gamma(1) \). We thus obtain a continuous map \( (t,s) \to \sigma_t(s) \) of \([0,1]^2\) into \( \tilde{U} \), which we will denote by \( \sigma \).

Now consider the lift of a radial curve \( \eta : s \to sx \ (x \in U) \) to a curve \( \tilde{\eta} \) tangent to \( \ker \alpha \). The \( z \)-coordinate of \( \tilde{\eta} \) is then given by \( \int_0^1 -\xi_{p}(sx)\alpha(sx) \, ds \).

Then

\[
\Psi(\tilde{\eta}(t)) = \left( \phi(tx), \int_0^t -\xi_{p}(sx)\alpha(sx) \, ds - f(tx) \right) = \left( \phi(tx), \int_0^t -\phi^* (\nu(sx)) \, ds \right)
\]

which is the lift of the curve \( s \to \phi(sx) \) in \( B(r) \) to a curve tangent to \( \ker \nu \). We thus conclude that \( \Psi \) maps lifts of radial curves tangent to \( \ker \alpha \) to curves tangent to \( \ker \nu \). We also note that we still have \( \Psi_p^* d\nu = d(\xi_{p}\alpha) \): both \( d\nu \) and \( d(\xi_{p}\alpha) \) vanish on vectors parallel to the \( z \)-axis, and \( D\Psi_p \) has the same action as \( D\phi \) on the components of the vectors lying in \( U \). As a consequence, the 1-form \( \kappa = \Psi_p^*(\nu) - \xi_{p}\alpha \) is closed. This implies that the path integral of \( \sigma^* \kappa \) around the boundary of \([0,1]^2\) is zero. We just showed that on radial curves, \( \ker \Psi_p^* \nu = \ker \alpha \), so that the path integral of \( \kappa \) over \( \sigma_0 \) and \( \sigma_1 \) is zero. The curve \( t \to \sigma_t(0) \) is tangent to the \( z \)-axis, and \( \Psi^*(\nu) - \xi_{p}\alpha \) vanishes on the \( z \)-axis, since \( \Psi \) restricts to the identity on the \( z \)-axis and \( \xi_{p}\alpha(\partial_z) = \nu(\partial_z) = 1 \), where \( \partial_z \) is the coordinate vector field parallel to the \( z \)-axis. Thus we conclude that

\[
\int_{\gamma} \kappa = 0
\]

Since this holds for every curve \( \gamma \) in \( \tilde{U} \), we conclude that \( \kappa = 0 \), so that \( \Psi_p^*(\nu) = \xi_{p}\alpha \). \qed
Let $G$ be the $m - 1$ dimensional Heisenberg group. The contact form $\nu$ is a left-invariant 1-form on $G$ and therefore $\mathcal{H} = \ker \nu$ is a left-invariant distribution on $G$.

**Lemma 6.4.** The Anosov splitting of $g^1$ is $C^1$.

**Proof.** Let $\delta > 0$ be small enough that all of the foliations $W^s(p)$ are trivial on a ball of radius $\delta$. Let $p \in SM$ and $q \in W^s_\delta(p)$. Let $r > 0$ be such that the conclusion of Lemma 6.3 holds on $W^s_r(p)$ and $W^u_r(q)$. We claim that the center stable holonomy map $\varphi : W^s_r(p) \to W^u_r(q)$ is $C^1$. Consider the map $F := \Psi_q \circ \varphi \circ \Psi_p^{-1}$ defined on a neighborhood of 0 in $\mathbb{R}^{m-1}$. By Lemma 4.4 and Lemma 6.3, $F$ maps $C^1$ curves tangent to $\mathcal{H}$ to $C^1$ curves tangent to $\mathcal{H}$. It follows that $F$ is differentiable along the contact distribution $\mathcal{H}$, with derivative given by the center stable holonomy $h^c s$ in these local coordinates.

This implies that $F$ is Pansu differentiable as a map from the Heisenberg group $G$ into itself [30]. The Pansu derivative at each point of $G$ is a homomorphism $DF^\mathcal{H}_p : G \to G$ uniquely determined by the derivative action of $F$ on $\mathcal{H}$. Furthermore, since the center stable holonomy preserves the conformal structure $\sigma$ on $F^1$, this derivative action on $\mathcal{H}$ is conformal in the induced norm of $\sigma$ on the horizontal distribution $\mathcal{H}$. This implies that $F$ is 1-quasiconformal in the Carnot-Carathéodory metric on the Heisenberg group associated to the metric $\sigma$ on the horizontal distribution. But all such 1-quasiconformal maps on any domain in $G$ are given by projective automorphisms of $G$ [9]. Thus $\Psi_q \circ \varphi \circ \Psi_p^{-1}$ is smooth and so since $\Psi_q$ and $\Psi_p$ are $C^1$, we conclude that $\varphi$ is $C^1$.

This proves that center stable holonomy between unstable leaves is $C^1$. It follows that the unstable bundle $E^u$ is $C^1$ along the $W^{cs}$ foliation, and since $E^u$ is smooth along the $W^u$ foliation, it follows from Journe’s lemma [20] that $E^u$ is $C^1$ and therefore the Anosov splitting is $C^1$. Using the flip map $p \to -p$ for the geodesic flow, we can interchange the role of $E^u$ and $E^s$ and apply all of the results of this section to $E^s$ as well. It follows that $E^s$ is $C^1$ as well and therefore the Anosov splitting of $g^1$ is $C^1$. $\square$

Let $k = m/2$. We can now complete the proof that $M$ is isometric to a cocompact quotient of $\mathbb{H}^k_C$ by constructing a time-preserving conjugacy between the geodesic flow on $M$ and that of a complex hyperbolic manifold.

First we obtain a candidate manifold by embedding the fundamental group $\Gamma$ of $M$ as a lattice in $SU(k, 1)$. As a consequence of Lemma 6.4, the boundary $\partial M$ has a natural $C^1$ structure induced by the charts given by projections of lifts of unstable leaves in $SM$. Since $H^u$ is invariant under center stable holonomy, $H^u$ descends to a continuous codimension-1 distribution $\mathcal{H}$ on $\partial M$ which is invariant under the action of $\Gamma$.

Recall that $\partial \mathbb{H}^k_C$ can be identified with the one point compactification $\hat{G}$ of the Heisenberg group $G$. The horizontal distribution $\mathcal{H}$ gives rise to a codimension one distribution $\mathcal{L}$ on $\partial \mathbb{H}^k_C$. 

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Proposition 6.5. There is a $C^1$ diffeomorphism $F : \partial \widetilde{M} \to \partial \mathbb{H}^k_C$ such that $DF(\mathcal{H}) = \mathcal{L}$.

Proof. Pick a point $\xi \in \widetilde{M}$. Let $p \in S\widetilde{M}$ be a point forward asymptotic to $\xi$. The chart $\Psi_p : W^u(p) \to G$ from Lemma 6.3 maps $H^u$ to $\mathcal{H}$. This induces a $C^1$ map $F : \partial \widetilde{M}\setminus\{\xi\} \to G$ mapping $\mathcal{H}$ to $\mathcal{L}$. $F$ has an obvious extension to a map $F : \partial \widetilde{M} \to \widetilde{G}$ given by setting $F(\xi)$ to be the point of $\widetilde{G}$ at infinity. We claim that the extended map $F$ remains $C^1$. Take some $q \in S\widetilde{M}$ not forward asymptotic to $\xi$, but to $\zeta \in \partial \widetilde{M}$ instead. The map $F$ on $\partial \widetilde{M}\setminus\zeta$ appears in the coordinate chart on $W^u(q)$ as the holonomy transition map $\Psi_p \circ \phi \circ \Psi_q^{-1} : G \to G$, where $\phi$ is center stable holonomy from $W^u(q)$ to $W^u(p)$. As shown in the proof of Lemma 6.4, this map is a projective automorphism of $G$; in particular, it is smooth. It follows that $F$ is $C^1$ on all of $\partial \widetilde{M}$, and via the natural identification of $\widetilde{G}$ with $\partial \mathbb{H}^k_C$, the proposition follows.

Proposition 6.6. $\Gamma$ is isomorphic to a cocompact lattice $\Lambda$ in $SU(k,1)$.

Proof. For each $\gamma \in \Gamma$ considered as acting on $\partial \widetilde{M}$, we have from Lemma 6.5 an induced $C^1$ action $\delta$ on $\partial \mathbb{H}^k_C$ given by $\delta = F \circ \gamma \circ F^{-1}$. Unpacking definitions, the action of $\delta$ is given in local coordinates $\Psi_p \circ \phi \circ \Psi_q^{-1} : G \to G$. Since $\gamma$ preserves the conformal class of the metric $\tau$, $\Psi_p \circ \gamma \Psi_p^{-1} : G \to G$ will be a $1$-quasiconformal self-map of $G$, hence a projective automorphism of $G$. So $\delta$ acts by a projective automorphism on $\partial \mathbb{H}^k_C$. Each such action determines a unique isometry $\eta$ of $\mathbb{H}^k_C$. The assignment $\gamma \to \eta$ is then an injective homomorphism $\Gamma \to SU(k,1)$ which we will denote by $\phi$. We claim that the image $\Lambda$ of $\Gamma$ must be a cocompact lattice in $SU(k,1)$.

We claim each $\delta \in \Lambda$ must represent a hyperbolic isometry of $\mathbb{H}^k_C$. Letting $\delta = \phi(\gamma)$, note that the action of $\gamma$ on $\partial \widetilde{M}$ has exactly two fixed points $\gamma_+$ and $\gamma_-$ which are the forward and backward limits of points on the boundary under $\gamma$. Given an open neighborhood $U$ of $\gamma_+$, note that for any compact subset $K \subset \partial \widetilde{M}\setminus\{\gamma_+, \gamma_-\}$ there is some positive integer $k$ such that $\gamma^k(K) \subset U$. This property of the action of $\gamma$ also holds for $\delta$, since the action of $\delta$ on $\partial \mathbb{H}^k_C$ is conjugate. Thus $\delta$ must be a hyperbolic isometry.

If $\Lambda$ is not discrete then there is an element $\delta \in \Lambda$ and a sequence of elements $\delta_n \in \Lambda$ converging to $\delta$ such that $\delta_n \neq \delta$ for any $n \in \mathbb{N}$. Conjugating by $F$ gives us a sequence of elements $\phi^{-1}(\delta_n) \in \Gamma$ and $\phi^{-1}(\delta) \in \Gamma$. Let $\gamma_n := \phi^{-1}(\delta_n)$ and $\gamma := \phi^{-1}(\delta) \in \Gamma$. It is easy to see that conjugation by $F$ takes the endpoints $\delta_{n,+}$ and $\delta_{n,-}$ of the axis of $\delta_n$ to the endpoints $\gamma_{n,+}$ and $\gamma_{n,-}$ of the axis of $\gamma_n$, preserving the orientation. Since $\delta_{n,+} \to \delta_+$, we must have $\gamma_{n,+} \to \gamma_+$ and $\gamma_{n,-} \to \gamma_-$. This implies that the axis of $\gamma_n$ converges to the axis of $\gamma$. But since the action of $\Gamma$ on $\widetilde{M}$ is properly discontinuous, the translation length of $\gamma_n$ along the axis must increase to $\infty$ as $n \to \infty$. Using the $C^1$ structure on $\partial \widetilde{M}$, it follows that $\|D\gamma_n\gamma_n\| \to \infty$ as $n \to \infty$. But since $F$ is $C^1$, we conclude that $\|D\delta_n\delta_n\| \to \infty$ as well, a contradiction. Thus $\Lambda$ is discrete.
Since $\Lambda$ is a discrete torsion-free subgroup of $SU(k, 1)$ consisting entirely of hyperbolic isometries, the action of $\Lambda$ on $\mathbb{H}^k_{\mathbb{C}}$ is free and properly discontinuous and therefore the quotient $N := \mathbb{H}^k_{\mathbb{C}}/\Lambda$ is a manifold. Since $N$ has a contractible universal cover, it is a $K(\Lambda, 1)$ space and therefore it is homotopy equivalent to $M$, since $\Gamma$ and $\Lambda$ are isomorphic. In particular, $H_m(N, \mathbb{Z}_2) = H_m(M, \mathbb{Z}_2) \cong \mathbb{Z}_2$ so $N$ is a closed manifold and thus $\Lambda$ is a cocompact lattice.

Let $N = \mathbb{H}^k_{\mathbb{C}}/\Lambda$. We have shown that $N$ is a closed manifold and that there is a $C^1$ diffeomorphism $F : \partial \tilde{M} \to \partial \tilde{N}$ equivariant with respect to the action of $\Gamma$ on $\partial \tilde{M}$ and the action of $\Lambda$ on $\partial \tilde{N}$. Since $F$ is a diffeomorphism it maps the Lebesgue measure class on $\partial \tilde{M}$ to the Lebesgue measure class on $\partial \tilde{N}$. Corollary 4.6 of [14] implies that there is a $C^1$ time-preserving (up to scaling) conjugacy between the geodesic flows of $M$ and $N$, and therefore by the minimal entropy rigidity theorem $M$ and $N$ are homothetic.

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