ON THE REGULARITY OF PRODUCTS AND INTERSECTIONS OF COMPLETE INTERSECTIONS

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Abstract. This paper proves the formulae
\[ \text{reg}(IJ) \leq \text{reg}(I) + \text{reg}(J), \]
\[ \text{reg}(I \cap J) \leq \text{reg}(I) + \text{reg}(J) \]
for arbitrary monomial complete intersections \( I \) and \( J \), and provides examples showing that these inequalities do not hold for general complete intersections.

1. Introduction

Let \( I \) be a homogeneous ideal in a polynomial ring \( S \) over a field. Let
\[ 0 \rightarrow \bigoplus_j S(-b_{mj}) \rightarrow \cdots \rightarrow \bigoplus_j S(-b_{0j}) \rightarrow I \rightarrow 0 \]
be a minimal graded free resolution of \( I \). The number
\[ \text{reg}(I) := \max_j \{ b_{ij} - i \mid i = 0, \ldots, m \} \]
is called the Castelnuovo-Mumford regularity (or regularity for short) of \( I \). It is of great interest to have good bounds for the regularity [BaM].

The regularity of products of ideals was studied first by Conca and Herzog [CoH]. They found some special classes of ideals \( I \) and \( J \) for which the following formula holds:
\[ \text{reg}(IJ) \leq \text{reg}(I) + \text{reg}(J) \]
(see also [Si]). In particular, they showed that \( \text{reg}(I_1 \cdots I_d) = d \) for any set of ideals \( I_1, \ldots, I_d \) generated by linear forms. These results led them to raise the question whether the formula
\[ \text{reg}(I_1 \cdots I_d) \leq \text{reg}(I_1) + \cdots + \text{reg}(I_d) \]
holds for any set of complete intersections \( I_1, \ldots, I_d \) [CoH, Question 3.6]. Note that this formula does not hold for arbitrary monomial ideals. For instance, Terai and Sturmfels (see [St]) gave examples of monomial ideals \( I \) such that \( \text{reg}(I^2) > 2 \text{reg}(I) \).

On the other hand, Sturmfels conjectured that \( \text{reg}(I_1 \cap \ldots \cap I_d) \leq d \) for any set of ideals \( I_1, \ldots, I_d \) generated by linear forms. This conjecture was settled in the
affirmative by Derksen and Sidman [DS]. Their proof was inspired by the work of Conca and Herzog. So one might be tempted to ask whether the formula
\[ \text{reg}(I_1 \cap \cdots \cap I_d) \leq \text{reg}(I_1) + \cdots + \text{reg}(I_d) \]
holds for any set of complete intersections \( I_1, \ldots, I_d \).

The following result show that these question have positive answers in the monomial case and we shall see that there are counter-examples in the general case.

**Theorem 1.1.** Let \( I \) and \( J \) be two arbitrary monomial complete intersections. Then
\[ \text{reg}(IJ) \leq \text{reg}(I) + \text{reg}(J), \]
\[ \text{reg}(I \cap J) \leq \text{reg}(I) + \text{reg}(J). \]

Both formulae follow from a more general bound for the regularity of a larger class of ideals constructed from \( I \) and \( J \) (Theorem 3.1). The proof is a bit intricate. It is based on a bound for the regularity of a monomial ideal in terms of the degree of the least common multiple of the monomial generators and the height of the given ideal found in [HT].

We are not able to extend the first formula to more than two monomial complete intersections. But we find another proof which extends the second formula to any finite set of monomial intersections (Theorem 3.3). We would like to mention that the first formula was already proved in the case one of the ideals \( I, J \) is generated by two elements by combinatorial methods in [M].

In the last section, we give a geometric approach for constructing examples of complete intersection ideals for which the inequalities \( \text{reg}(IJ) \leq \text{reg}(I) + \text{reg}(J) \) and/or \( \text{reg}(I \cap J) \leq \text{reg}(I) + \text{reg}(J) \) fails. We show for instance the following:

**Theorem 1.2.** Let \( Y \) in \( \mathbb{P}^3 \) be a curve which is defined by at most 4 equations at the generic points of its irreducible components. Consider 4 elements in \( I_Y, f_1, f_2, g_1, g_2 \) such that \( I := (f_1, f_2) \) and \( J := (g_1, g_2) \) are complete intersection ideals and \( I_Y \) is the unmixed part of \( I + J \). Then, if \( -\eta := \min\{\mu \ | \ H^0(Y, O_Y(\mu)) \neq 0\} < 0 \), one has
\[ \text{reg}(IJ) = \text{reg}(I) + \text{reg}(J) + \eta - 1. \]

A similar construction is explained for \( I \cap J \). As a consequence, many families of curves with sections in negative degrees gives rise to counter-examples for the considered inequalities. In the examples we give, \( I \) is a monomial ideal and \( J \) is either generated by one binomial or by one monomial and one binomial.

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2. Preliminaries

Let us first introduce some conventions. For any monomial ideal we can always find a minimal basis consisting of monomials. These monomials will be called the
monomial generators of the given ideal. Moreover, for a finite set of monomials $A_i = x_i^{a_{i1}} \cdots x_i^{a_{in}}$, we call the monomials $x_i^{\max\{a_{i1}\}} \cdots x_i^{\max\{a_{in}\}}$ the least common multiple of the monomials $A_i$.

The key point of our approach is the following bound for the regularity of arbitrary monomial ideals.

**Lemma 2.1.** [HT, Lemma 3.1] Let $I$ be a monomial ideal. Let $F$ denote the least common multiple of the monomial generators of $I$. Then

$$\text{reg}(I) \leq \deg F - \text{ht}(I) + 1.$$ 

This bound is an improvement of the bound $\text{reg}(I) \leq \deg F - 1$ given by Bruns and Herzog in [BrH, Theorem 3.1(a)].

If we apply Lemma 2.1 to the product and the intersection of monomial ideals, we get

$$\text{reg}(I_1 \cdots I_d) \leq \sum_{j=1}^d \deg F_j - \text{ht}(I_1 \cdots I_d) + 1,$$

$$\text{reg}(I_1 \cap \cdots \cap I_d) \leq \sum_{j=1}^d \deg F_j - \text{ht}(I_1 \cap \cdots \cap I_d) + 1,$$

where $F_j$ denotes the least common multiple of the monomial generators of $I_j$. If $I_1, \ldots, I_d$ are complete intersections, then $\text{reg}(I_j) = \deg F_j - \text{ht}(I_j) + 1$, whence

$$\text{reg}(I_1 \cdots I_d) \leq \sum_{j=1}^d \text{reg}(I_j) + \sum_{j=1}^d \text{ht}(I_j) - \text{ht}(I_1 \cdots I_d) - d + 1,$$

$$\text{reg}(I_1 \cap \cdots \cap I_d) \leq \sum_{j=1}^d \text{reg}(I_j) + \sum_{j=1}^d \text{ht}(I_j) - \text{ht}(I_1 \cap \cdots \cap I_d) - d + 1.$$

These bounds are worse the bounds in the aforementioned questions. However, the difference is not so big.

To get rid of the difference in the case $d = 2$ we need the following consequence of Lemma 2.1.

**Corollary 2.2.** Let $I$ be a monomial complete intersection and $Q$ an arbitrary monomial ideal (not necessarily a proper ideal of the polynomial ring $S$). Then

$$\text{reg}(I : Q) \leq \text{reg}(I).$$

**Proof.** Let $F$ denote the product of the monomial generators of $I$. Since every monomial generator of $I : Q$ divides a monomial generator of $I$, the least common multiple of the monomial generators of $I : Q$ divides $F$. Applying Lemma 2.1 we get

$$\text{reg}(I : Q) \leq \deg F - \text{ht}(I : Q) + 1 \leq \deg F - \text{ht} I + 1 = \text{reg}(I).$$

We will decompose the product and the intersection of two monomial ideals as a sum of smaller ideals and apply the following lemma to estimate the regularity.
Lemma 2.3. Let $I$ and $J$ be two arbitrary homogeneous ideals. Then
\[ \text{reg}(I + J) \leq \max\{\text{reg}(I), \text{reg}(J), \text{reg}(I \cap J) - 1\}. \]
Moreover, \( \text{reg}(I \cap J) = \text{reg}(I + J) + 1 \) if \( \text{reg}(I + J) \geq \max\{\text{reg}(I), \text{reg}(J)\} \) or if \( \text{reg}(I \cap J) \geq \max\{\text{reg}(I), \text{reg}(J)\} + 1 \).

Proof. The statements follow from the exact sequence
\[ 0 \rightarrow I \cap J \rightarrow I \oplus J \rightarrow I + J \rightarrow 0 \]
and the well-known relationship between regularities of modules of an exact sequence (see e.g. [E, Corollary 20.19]). \( \square \)

3. Main results

We will prove the following general result.

Theorem 3.1. Let $I$ and $J$ be two arbitrary monomial complete intersections. Let $f_1, \ldots, f_r$ be the monomial generators of $I$. Let $Q_1,\ldots,Q_r$ be arbitrary monomial ideals. Then
\[ \text{reg}\left(f_1(J : Q_1) + \cdots + f_r(J : Q_r)\right) \leq \text{reg}(I) + \text{reg}(J). \]

The formulae of Theorem 1.1 follow from the above result because
\[ IJ = f_1J + \cdots + f_rJ = f_1(J : S) + \cdots + f_r(J : S), \]
\[ I \cap J = (f_1) \cap J + \cdots + (f_r) \cap J = f_1(J : f_1) + \cdots + f_r(J : f_r). \]

Proof. If \( r = 1 \), we have to prove that
\[ \text{reg}\left(f_1(J : Q_1)\right) \leq \deg f_1 + \text{reg}(J). \]
It is obvious that
\[ \text{reg}\left(f_1(J : Q_1)\right) \leq \deg f_1 + \text{reg}(J : Q_1). \]
By Corollary 2.2 we have \( \text{reg}(J : Q_1) \leq \text{reg}(J) \), which implies the assertion.

If \( r > 1 \), using induction we may assume that
\[ \text{reg}\left(\sum_{i=1}^{r-1} f_i(J : Q_i)\right) \leq \text{reg}(f_1,\ldots,f_{r-1}), \quad (1) \]
\[ \text{reg}\left(f_r(J : Q_r)\right) \leq \deg f_r + \text{reg}(J). \quad (2) \]
Since \( f_1(J : Q_1),\ldots,f_{r-1}(J : Q_{r-1}) \) are monomial ideals, we have
\[ \left(\sum_{i=1}^{r-1} f_i(J : Q_i)\right) : f_r = \sum_{i=1}^{r-1} \left(f_i(J : Q_i) : f_r\right). \]
Since \( f_1, \ldots, f_r \) is a regular sequence, \( f_i(J : Q_i) : f_r = f_i(J : f_rQ_i) \). Therefore,
\[
\Big( \sum_{i=1}^{r-1} f_i(J : Q_i) \Big) \cap f_r(J : Q_r) = f_r \left[ \left( \sum_{i=1}^{r-1} f_i(J : Q_i) : f_r \right) \cap (J : Q_r) \right] \\
= f_r \left[ \left( \sum_{i=1}^{r-1} f_i(J : f_rQ_i) \right) \cap (J : Q_r) \right] \\
= f_r \left[ \sum_{i=1}^{r-1} f_i \left( (J : f_rQ_i) \cap (J : f_iQ_i) \right) \right] \\
= f_r \left[ \sum_{i=1}^{r-1} f_i \left( f_rQ_i + f_iQ_r \right) \right].
\]

From this it follows that
\[
\text{reg} \left( \sum_{i=1}^{r-1} f_i(J : Q_i) \right) \leq \deg f_r + \text{reg} \left( \sum_{i=1}^{r-1} f_i \left( J : (f_rQ_i + f_iQ_r) \right) \right).
\]

Using induction we may assume that
\[
\text{reg} \left( \sum_{i=1}^{r-1} f_i \left( J : (f_rQ_i + f_iQ_r) \right) \right) \leq \text{reg}(f_1, \ldots, f_{r-1}) + \text{reg}(J).
\]

Since \( \text{reg}(I) = \text{reg}(f_1, \ldots, f_{r-1}) + \deg f_r - 1 \), this implies
\[
\text{reg} \left( \sum_{i=1}^{r-1} f_i(J : Q_i) \right) \cap f_r(J : Q_r) \leq \text{reg}(I) + \text{reg}(J) + 1. \tag{3}
\]

Now, we apply Lemma 2.3 to the decomposition
\[
f_1(J : Q_1) + \cdots + f_r(J : Q_r) = \left( \sum_{i=1}^{r-1} f_i(J : Q_i) \right) + f_r(J : Q_r).
\]

and obtain
\[
\text{reg} \left( f_1(J : Q_1) + \cdots + f_r(J : Q_r) \right) \leq \max \left\{ \text{reg} \left( \sum_{i=1}^{r-1} f_i(J : Q_i) \right), \text{reg} \left( f_r(J : Q_r) \right), \text{reg} \left( \sum_{i=1}^{r-1} f_i(J : Q_i) \right) \cap f_r(J : Q_r) - 1 \right\} \leq \text{reg}(I) + \text{reg}(J)
\]
by using (1), (2), (3).

**Remark 3.2.** The above proof would work in the case of more than two monomial complete intersections if we have a similar result as Lemma 2.2. For instance, if we can prove
\[
\text{reg}(IJ : Q) \leq \text{reg}(I) + \text{reg}(J)
\]
for two monomial complete intersections \( I, J \) and an arbitrary monomial ideal \( Q \), then we can give a positive answer to the question of Conca and Herzog in the case \( d = 3 \) for monomial ideals. We are unable to verify the above formula though computations in concrete cases suggest its validity.
Now we will extend the second formula of Theorem 1.1 for any set of monomial complete intersections.

**Theorem 3.3.** Let \( I_1, \ldots, I_d \) be arbitrary monomial complete intersections. Then

\[
\text{reg}(I_1 \cap \cdots \cap I_d) \leq \text{reg}(I_1) + \cdots + \text{reg}(I_d).
\]

**Proof.** We will use induction on the number \( n \) of variables and the number

\[
s := \text{reg}(I_1) + \cdots + \text{reg}(I_d).
\]

First, we note that the cases \( n = 1 \) and \( s = 1 \) are trivial.

Assume that \( n \geq 2 \) and \( r \geq 2 \). Let \( x \) be an arbitrary variable of the polynomial ring \( S \). It is easy to see that \((I_1, x), \ldots, (I_d, x)\) are monomial complete intersections and

\[
(I_1 \cap \cdots \cap I_d, x) = (I_1, x) \cap \cdots \cap (I_d, x).
\]

Therefore, using induction on \( n \) we may assume that

\[
\text{reg}(I_1 \cap \cdots \cap I_d, x) \leq \text{reg}(I_1, x) + \cdots + \text{reg}(I_d, x).
\]

If \( x \) is a non-zerodivisor on \( I_1 \cap \cdots \cap I_d \) and if we assume that the intersection is irredundant, then \( I_j : x = I_j \) and hence \( \text{reg}(I_j, x) = \text{reg}(I_j) \) for all \( j = 1, \ldots, d \). In this case,

\[
\text{reg}(I_1 \cap \cdots \cap I_d) = \text{reg}(I_1 \cap \cdots \cap I_d, x) \leq \text{reg}(I_1) + \cdots + \text{reg}(I_d).
\]

If \( x \) is a zerodivisor on \( I_1 \cap \cdots \cap I_d \), we involve the ideal

\[
(I_1 \cap \cdots \cap I_d) : x = (I_1 : x) \cap \cdots \cap (I_d : x).
\]

If \( I_j : x \neq I_j \), either \( I_j : x = S (x \in I_j : x) \) or \( I_j : x \) is a monomial complete intersection generated by the monomials obtained from the generators of \( I_j \) by replacing the monomial divisible by \( x \) by its quotient by \( x \). In the latter case, we have \( \text{reg}(I_j : x) = \text{reg}(I_j) - 1 \). Since there exists at least an ideal \( I_j \) with \( I_j : x \neq I_j \), the ideal \( (I_1 \cap \cdots \cap I_d) : x \) is an intersection of monomial complete intersections such that the sum of their regularities is less than \( s \). Using induction on \( s \) we may assume that

\[
\text{reg}((I_1 \cap \cdots \cap I_d) : x) \leq \text{reg}(I_1 : x) + \cdots + \text{reg}(I_d : x)
\]

\[
\leq \text{reg}(I_1) + \cdots + \text{reg}(I_d) - 1.
\]

Now, from the exact sequence

\[
0 \rightarrow S/(I_1 \cap \cdots \cap I_d) : x \rightarrow x \rightarrow S/I_1 \cap \cdots \cap I_d \rightarrow S/(I_1 \cap \cdots \cap I_d, x) \rightarrow 0
\]

we can deduce that

\[
\text{reg}(I_1 \cap \cdots \cap I_d) \leq \max \{ \text{reg}((I_1 \cap \cdots \cap I_d) : x) + 1, \text{reg}(I_1 \cap \cdots \cap I_d, x) \}
\]

\[
\leq \text{reg}(I_1) + \cdots + \text{reg}(I_d).
\]

\( \square \)
4. Counter-examples

We will explain a geometric approach, using projective curves, for constructing families of counter-examples to the inequalities \( \operatorname{reg}(IJ) \leq \operatorname{reg}(I) + \operatorname{reg}(J) \) and \( \operatorname{reg}(I \cap J) \leq \operatorname{reg}(I) + \operatorname{reg}(J) \). We then give a specific family of such examples, based on the example [CD, 2.3].

For simplicity, we will work with curves \( \mathbb{P}^3 \), although this technique may be easily extended to curves in any projective space.

Recall that setting, for a finitely generated graded \( S \)-module \( M \),
\[
a_i(M) := \max\{\mu \mid H^i_m(M)_\mu \neq 0\}
\]
if \( H^i_m(M) \neq 0 \) and \( a_i(M) := -\infty \) else (\( m \) is the graded maximal ideal of \( S \), one has
\[
\operatorname{reg}(M) = \max_i\{a_i(M) + i\}.
\]

In the special case were \( M = S/I \) is of dimension two (hence defines a projective scheme of dimension one), one has
\[
\operatorname{reg}(S/I) = \max\{a_0(S/I), a_1(S/I) + 1, a_2(S/I) + 2\},
\]
and if furthermore \( I \) is saturated (in other words is the defining ideal of the corresponding embedded projective scheme), one has \( a_0(S/I) := -\infty \).

From now on we set \( S := k[x, y, z, t] \) for the homogeneous coordinate ring of \( \mathbb{P}^3 \).

**Step 1.** Construct a curve in \( \mathbb{P}^3 \) with sections in negative degrees.

This is equivalent to constructing a graded unmixed ideal \( I \) with \( \dim(S/I) = 2 \), such that \( H^i_m(S/I) \) has elements in negative degrees.

One way to construct such a curve is to start from another curve \( X \) and two elements in its defining ideal of degrees \( d_1, d_2 \) that form a complete intersection strictly containing the curve, and such that the regularity of the ideal of the curve is at least \( d_1 + d_2 - 1 \). A good choice is to take a reduced irreducible curve \( X \) whose regularity is at least equal to the sum of the two smallest degree \( d_1, d_2 \) of minimal generators of its defining ideal. The monomial curves offers a good collection of curves of that type.

By liaison, the residual of \( X \) in the complete intersection of degrees \( d_1, d_2 \) is a non reduced curve \( Y \) that satisfies:
\[
H^0(Y, \mathcal{O}_Y(\mu)) \neq 0 \iff \mu \geq d_1 + d_2 - \operatorname{reg}(I_X) - 2
\]
by [CU, 4.2]. In particular \( H^i_m(S/I_Y)_\mu \neq 0 \) for \( d_1 + d_2 - \operatorname{reg}(I_X) - 2 \leq \mu < 0 \) and
\[
\operatorname{indeg}(H^i_m(S/I_Y)) = d_1 + d_2 - \operatorname{reg}(I_X) - 2.
\]

Also by liaison, \( \operatorname{reg}(I_Y) \leq d_1 + d_2 - 2 \) if \( X \) is reduced. In particular \( I_Y \) is generated in degrees at most \( d_1 + d_2 - 2 \) in this case.

**Step 2.** To obtain counter-examples to the inequality \( \operatorname{reg}(I \cap J) \leq \operatorname{reg}(I) + \operatorname{reg}(J) \).

Choose three elements in \( I_Y \) such that they generate an ideal \( K := (f_1, f_2, f_3) \) whose unmixed part is \( I_Y \). This is always possible if \( Y \) is generically defined by
at most 3 equations - in the context of step 1, this is for instance the case if the multiplicities of the irreducible components of $X$ in the complete intersection are at most 3, or if the supports of $X$ and $Y$ are distinct. Recall that $I_Y$ is generated in degrees at most $d_1 + d_2 - 2$ if $X$ is reduced. Therefore if further the supports of $X$ and $Y$ are distinct (in other words if $Y$ is a geometric link of the reduced curve $X$ by a complete intersection ideal $b$) then one may choose $f_1$ and $f_2$ to be the generators of $b$ and $f_3 \in I_Y - \cup_{p \in \text{Ass}(I_X)} p$ may be chosen of degree at most $d_1 + d_2 - 2$. Then, by [Ch, 0.6],

$$a_0(S/K) = - \text{indeg}(H^1_m(S/I_Y)) + \sigma - 4,$$

$$a_1(S/K) = - \text{indeg}(I_Y/K) + \sigma - 4,$$

$$a_2(S/K) \leq - \text{indeg}(K) + \sigma - 5,$$

where $\sigma$ is the sum of the degrees of the three forms. Since $\text{indeg}(H^1_m(S/I_Y))$ is negative, it follows that

$$\text{reg}(S/K) = a_0(S/K) = - \text{indeg}(H^1_m(S/I_Y)) + \sigma - 4.$$

Modifying the generators of $K$, if needed, we may assume that $I = (f_1, f_2)$ is a complete intersection ideal and then Lemma 2.3 shows that

$$\text{reg}(I \cap (f_3)) = \text{reg}(K) + 1 = - \text{indeg}(H^1_m(S/I_Y)) + \sigma - 2 > \sigma - 1 = \text{reg}(I) + \text{reg}((f_3))$$

if (and only if) $\text{indeg}(H^1_m(S/I_Y)) < -1$.

**Step 3.** To obtain counter-examples to the inequality $\text{reg}(IJ) \leq \text{reg}(I) + \text{reg}(J)$.

Choose four elements elements $f_1, f_2, g_1, g_2$ in $I_Y$ such that:

- $I := (f_1, f_2)$ and $J := (g_1, g_2)$ are complete intersection ideals,
- $I_Y$ is the unmixed part of $I + J$.

Using the example in step 2, one may take the same ideal for $I$, $g_1 := f_3$ and for $g_2$ any element in $I_Y$ which is prime to $f_3$ (for instance, modifying the generators of $I$, if needed, one may take $g_2 := f_2$).

It follows from the isomorphism $\text{Tor}^S(I/S, J) \simeq (I \cap J)/IJ$ and the fact that $\text{depth}(S/(I \cap J)) > 0$ that $H^0_{m}(S/IJ) \simeq H^0_{m}(\text{Tor}^S(I/S, J))$. By [Ch, 5.9], $H^0_{m}(\text{Tor}^S(I/S, J))$ is the graded $k$-dual of $H^1_m(S/I_Y)$ up to a shift in degrees by $\sigma' - 4$, where $\sigma'$ is the sum of the degrees of the 4 forms. We therefore have $H^0_{m}(S/IJ)_\mu \simeq H^1_m(S/I_Y)_{\sigma' - 4 - \mu}$. This implies

$$a_0(S/IJ) = - \text{indeg}(H^1_m(S/I_Y)) + \sigma' - 4.$$

It also follows from the estimates of [Ch, 3.1 (i), (ii) and (iii)] on $a_1(\text{Tor}^S(I/S, J))$, $a_2(\text{Tor}^S(I/S, J))$ and $\text{reg}(S/(I + J))$ that

$$\text{reg}(IJ) = a_0(S/IJ) + 1 = - \text{indeg}(H^1_m(S/I_Y)) + \sigma' - 3 > \sigma' - 2 = \text{reg}(I) + \text{reg}(J)$$

if (and only if) $\text{indeg}(H^1_m(S/I_Y)) < -1$.

Notice that the above considerations shows that any curve $Y$ in $\mathbb{P}^3$ which is generically defined by at most 4 equations and has sections starting in degree $-2$ or below gives rise to counter-examples:
Theorem 4.1. Let $Y$ in $\mathbb{P}^3$ be a curve which is defined by at most 4 equations at the generic points of its irreducible components. Consider 4 elements in $I$, $f_1, f_2, g_1, g_2$ such that $I := (f_1, f_2)$ and $J := (g_1, g_2)$ are complete intersection ideals and $I_Y$ is the unmixed part of $I + J$. Then, if $-\eta := \min\{\mu \mid H^0(Y, O_Y(\mu)) \neq 0\} < 0$, one has

$$\text{reg}(IJ) = \text{reg}(I) + \text{reg}(J) + \eta - 1.$$ 

A specific class of examples. We consider, as in [CD, 2.3], the monomial curve $Z_{m,n}$ parametrized on an affine chart by $(1: \theta: \theta^{mn}: \theta^{n+1})$, for $m, n \geq 2$.

The binomial $y^{mn} - x^{mn-1}z$ is a minimal generator of the defining ideal of $Z_{m,n}$, hence $\text{reg}(I_{Z_{m,n}}) \geq mn$. It follows from a theorem of Bresinski et al. [BCFH], who determines the regularity of all curves with a parametrization $(1:t: t^a: t^b)$, that $\text{reg}(I_{Z_{m,n}}) = mn$, and this may be easily checked in this special case.

The ideal of this curve contains minimal generators $x^mt - y^mz$ and $z^{n+1} - xtn$.

A component of the scheme defined by the complete intersection ideal $b_{m,n} := (x^mt - y^mz, z^{n+1} - xtn)$ is the simple line $x = z = 0$, and we take $X := Z_{m,n} \cup \{x = z = 0\}$. On one hand, $\text{reg}(I_X) \geq mn + 1$ because $xy^{mn} - x^{mn}z$ is a minimal generator of $I_X$. On the other hand, $\text{reg}(I_X) \leq \text{reg}(I_{Z_{m,n}}) + \text{reg}((x,z)) = mn + 1$ because $\dim(S/I_{Z_{m,n}} + (x,z)) \leq 1$. Therefore, $\text{reg}(I_X) = mn + 1$.

$Y$ is a geometric link of $X$ with

$$\text{indeg}(H^1_n(S/I_Y)) = m + n + 2 - (mn + 1) - 2 = -(m - 1)(n - 1).$$

One has $I_Y = (x^mt - y^mz) + (z,t)^n$. To see this, notice that $(x^mt - y^mz) + (z,t)^n$ defines a locally complete intersection scheme supported on the line $z = t = 0$, has positive depth and multiplicity (or degree) at least $n$—for instance because two such unmixed ideals differ for two different values of $n$. The containment

$$b_{m,n} = I_X \cap I_Y \subseteq I_X \cap ((x^mt - y^mz) + (z,t)^n)$$

and the fact that $\deg(I_X \cap I_Y) = \deg(I_X) + n$ forces $I_Y$ to coincide with the unmixed ideal $(x^mt - y^mz) + (z,t)^n$. It is also easy to provide a minimal free $S$-resolution of the ideal $(x^mt - y^mz) + (z,t)^n$, and show these facts along the same line as in the proof of [CD, 2.4].

For step 2, we take $I := (t^n, z^n)$ and $K := (x^mt - y^mz)$, whose saturation is $I_Y$, and therefore,

$$\text{reg}(I \cap (x^mt - y^mz)) = (m - 1)(n - 1) + m + 2n - 1 = (m + 1)n$$

which is bigger than $\text{reg}(I) + \text{reg}((x^mt - y^mz)) = m + 2n$ if and only if $mn > m + n$ (i.e. iff $(m,n) \neq (2,2)$).

For step 3, we can take $I := (t^n, z^n)$ and $J := (x^mt - y^mz, t^n)$, then $\sigma' = m + 3n + 1$. By Theorem 4.1 we have

$$\text{reg}(IJ) = (m - 1)(n - 1) + m + 3n - 2 = mn + 2n - 1.$$ 

Hence, $\text{reg}(IJ) > \text{reg}(I) + \text{reg}(J) = m + 3n - 1$ if and only if $(m,n) \neq (2,2)$.  

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