RESOLUTION FOR SHEAF OF DIFFERENTIAL OPERATORS ON SMOOTH FREE GEOMETRIC QUOTIENT OF LINEAR ACTION OF ALGEBRAIC GROUP

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In [R], Z. Ran gave a canonical construction for the universal deformation of a simple vector bundle using the Jacobi complex of an appropriate differential graded Lie algebra. Independently, H. Esnault and E. Viehweg made a similar construction. Using these tools, we obtain a resolution for the sheaf of differential operators on smooth geometric quotients of free linear actions of algebraic groups. Unlike previous applications of these methods, we are able to obtain global results.

The terms of our resolution involve symmetric and alternating powers of vector bundles easily constructed geometrically from the algebraic group and the vector space on which it acts. Our resolution is particularly simple for projective space. For $\mathbb{P}(V)$, we obtain the following sequence—

$$0 \longrightarrow \mathcal{O} \longrightarrow S^r(V^*(1)) \longrightarrow \text{Diff}^r_{\mathbb{P}(V)}/\mathcal{O} \longrightarrow 0,$$

in which the first map is symmetrization of the map in the Euler sequence. We are able to conclude that the sheaf of differential operators on $\mathbb{P}^n$, with $n > 1$, has no non-scalar global endomorphisms.

This paper forms part of the author’s UCR Ph. D. dissertation. We work over the complex numbers.

1. Schur Functor of a Complex of Vector Spaces

We define the Schur functor of a complex of vector spaces. We provide sign computations because such seems to be rare, though we will not need to use them in much generality. Signs were actually computed from scratch, so the signs contained here can be considered to be independent experimental data. Mostly this parallels the description of Schur functors given in [F].

1.1. How to Get a Symmetrizer from a Partition.

We describe how one constructs an element of the group ring, $\mathbb{C}[\text{Sym}(n)]$, from a partition of $n$. This is exactly as in [F].
**Definition.** A partition of an integer, \( n \), is a non-increasing sequence of positive integers, summing to \( n \).

We fix a partition of \( n \), \( l = (l_i) \), for the remainder of this section.

Given our partition we draw a picture of boxes, called the Young diagram. We put \( l_i \) boxes in the \( i \)th row, and we label them with integers from one to \( n \). For example, for the partition \((4, 2, 1)\), we draw the following picture—

```
1 2 3 4
5 6
7
```

Let \( S < \text{Sym}(n) \) be the subgroup of the symmetric group consisting of all permutations preserving the rows of the Young diagram, and let \( L \) be the subgroup preserving columns. We define the Young symmetrizer to be the following element of \( \mathbb{C}[	ext{Sym}(n)] \)—

\[
\Sigma(l) = \left( \sum_{\sigma \in S} \sigma \right) \left( \sum_{\sigma \in L} \text{sign}(\sigma) \sigma \right).
\]

For example, for the diagram, \[
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\]
we have \( \Sigma(2, 1) = (1 + (12))(1 - (13)) = (1 + (12) - (13) - (123)) \).

Notice that if we have a \( \text{Sym}(n) \) action on a \( \mathbb{C} \)-vector space, we get a \( \mathbb{C}[	ext{Sym}(n)] \) action.

### 1.2. Schur Functor for Complexes of Vector Spaces.

We first define the Schur functor of a graded vector space. Let \( l \) be a partition of \( n \). Let \( A = \bigoplus A_i \) be a graded vector space. We define an action of a permutation, \( \sigma \in \text{Sym}(n) \), on \( A \otimes^n \) by the following formula—

\[
a_1 \otimes \ldots \otimes a_n \mapsto \text{sign}(\sigma, \text{deg} a_1, \ldots, \text{deg} a_n) a_{\sigma_1} \otimes \ldots \otimes a_{\sigma_n},
\]

in which

\[
\text{sign}(\sigma, d_1, \ldots, d_n) = \prod_{\sigma_j < \sigma_i} (-1)^{d_j d_i}.
\]

If \( o \) of the \( d_i \) are odd, we obtain a homomorphism from \( \text{Sym}(n) \) to \( \text{Sym}(o) \) by forgetting the even things. By viewing \( \text{sign}(\sigma, d_1, \ldots, d_n) \) as the sign of the image of \( \sigma \) under this homomorphism, we see that \( \text{sign}(\cdot, d_1, \ldots, d_n) \) is a homomorphism, thus we do indeed obtain action.

**Definition.** Given \( A \) and \( l \) as above, the Schur module, \( S^l A \), is the image of the Young symmetrizer, \( \Sigma(l) : A \otimes^n \rightarrow A \otimes^n \).

Since \( S^l A \) is naturally a direct summand of \( A \otimes^n \), we see that \( S^l(\cdot) \) is a functor, which we call the Schur functor.

We now extend our definition to case of a complex. If \( A \) is a complex with differential \( d \), we make \( A \otimes^n \) into a complex with differential being

\[
a_1 \otimes \ldots \otimes a_n \mapsto \sum_i \left( \prod_{j < i} (-1)^{\text{deg} a_j} \right) a_1 \otimes \ldots \otimes da_i \otimes \ldots \otimes a_n.
\]
Proposition 1. $\Sigma$ and $d$ commute.

Proof. We prove this with direct computation. Let $\Sigma = \sum_{\sigma \in \text{Sym}(n)} c(\sigma)\sigma$, with $c(\sigma) \in \mathbb{C}$. The image of $a_1 \otimes \ldots \otimes a_n$ under $d \circ \Sigma$ is

$$
\sum_i \left( \prod_{j < i} (-1)^{\deg a_j} \right) \sum_{\sigma \in \text{Sym}(n)} c(\sigma) \sign(\sigma, \deg a_1, \ldots, \deg a_i + 1, \ldots, \deg a_n) a_{\sigma 1} \otimes \ldots \otimes da_i \otimes \ldots \otimes a_{\sigma n}.
$$

We change the order of the sums and sum over $\sigma i$ in place of $i$ and obtain the following—

$$
\sum_{\sigma \in \text{Sym}(n)} \sum_i \left( \prod_{j < \sigma i} (-1)^{\deg a_j} \right) c(\sigma) \sign(\sigma, \deg a_1, \ldots, \deg a_{\sigma i} + 1, \ldots, \deg a_n) a_{\sigma 1} \otimes \ldots \otimes da_{\sigma i} \otimes \ldots \otimes a_{\sigma n}.
$$

The image of $a_1 \otimes \ldots \otimes a_n$ under $\Sigma \circ d$ is

$$
\sum_{\sigma \in \text{Sym}(n)} c(\sigma) \sign(\sigma, \deg a_1, \ldots, \deg a_n) \sum_i \left( \prod_{j < i} (-1)^{\deg a_{\sigma j}} \right) a_{\sigma 1} \otimes \ldots \otimes da_{\sigma i} \otimes \ldots \otimes a_{\sigma n}.
$$

Thus we need to compare, for fixed $\sigma$ and $i$,

$$
\sign(\sigma, \deg a_1, \ldots, \deg a_{\sigma i} + 1, \ldots, \deg a_n) \left( \prod_{j < i} (-1)^{\deg a_j} \right)
$$

with

$$
\sign(\sigma, \deg a_1, \ldots, \deg a_{\sigma i} + 1, \ldots, \deg a_n) \left( \prod_{j < \sigma i} (-1)^{\deg a_j} \right).
$$

We see that they are equal because

$$
\sign(\sigma, \deg a_1, \ldots, \deg a_{\sigma i} + 1, \ldots, \deg a_n) \cdot \sign(\sigma, \deg a_1, \ldots, \deg a_{\sigma i} + 1, \ldots, \deg a_n)
$$

$$
= \sign(\sigma^{-1}, \deg a_1, \ldots, \deg a_{\sigma i} + 1, \ldots, \deg a_n) \cdot \sign(\sigma^{-1}, \deg a_1, \ldots, \deg a_{\sigma i} + 1, \ldots, \deg a_n)
$$

$$
= \left( \prod_{\sigma i < \sigma j} (-1)^{\deg a_{\sigma j}} \right) \left( \prod_{\sigma j < \sigma i} (-1)^{\deg a_{\sigma j}} \right) = \left( \prod_{j < \sigma i} (-1)^{\deg a_j} \right) \left( \prod_{j < i} (-1)^{\deg a_{\sigma j}} \right).
$$

□

Commutativity of $d$ and $\Sigma$ tells us that $S^l(A)$ is a direct summand of the complex $A^{\otimes n}$ as a complex. In particular, $S^l(\cdot)$ takes quasi–isomorphic complexes to quasi–isomorphic complexes.

The dual of a partition is the partition corresponding the the transpose of its Young diagram. For example, the dual of $(3, 1)$ is $(2, 1, 1)$. If $a$ and $b$ are dual partitions, we write $\wedge b = S^a$. 

Differential Operators on Quotient 3
2. Definitions of DGLA and Jacobi Complex

Given a complex, $L$, with a map $[,] : \wedge^2 L \to L$, we obtain a map $L \otimes [,] : L \otimes \wedge^2 L \to L \otimes L$. Symmetrizing we obtain a map $\wedge^3 L \to \wedge^2 L$. Similarly we obtain maps $\wedge^i L \to \wedge^{i-1} L$. Given another complex, $M$, with a map $\langle, \rangle : L \otimes M \to M$, we obtain a map $L \otimes \langle, \rangle : L \otimes L \otimes M \to L \otimes M$. Symmetrizing we obtain a map $\wedge^2 L \otimes M \to L \otimes M$.

**Definition.** A differential graded Lie algebra (DGLA) is a complex, $L$, together with a map $[,] : \wedge^2 L \to L$ such that the composition $\wedge^3 L \to \wedge^2 L \to L$ is zero. A module of $L$ is a complex, $M$, with a map $\langle, \rangle : L \otimes M \to M$ such that the following square commutes—

$$
\begin{array}{ccc}
\wedge^2 L \otimes M & \xrightarrow{[ , ] \otimes M} & L \otimes M \\
\downarrow & & \downarrow \\
L \otimes \langle, \rangle & \xrightarrow{\langle, \rangle} & M \\
\end{array}
$$

**Definition.** Let $L$ be a DGLA. Jacobi complex, $J^r(L)$, is the following complex—

$$
\begin{array}{ccccccc}
& \wedge^r L & \longrightarrow & \ldots & \longrightarrow & \wedge^3 L & \longrightarrow & \wedge^2 L & \longrightarrow & L \\
\end{array}
$$

with $\wedge^i L$ in degree $-i$ for $1 \leq i \leq r$ and zeros elsewhere. The stupid filtration of $J^r(L)$ is the filtration, $F^i J^r(L) = J^i(L)$ for $i \leq r$.

Notice that if $L$ is a DGLA and $M$ an $L$–module, then the complex $L \longrightarrow M$, with the zero map, naturally forms a DGLA as follows—

$$
\begin{array}{ccc}
\wedge^2 L & \xrightarrow{[ , ]} & L \\
\downarrow & & \downarrow \\
S^2 M & \xrightarrow{0} & M \\
\end{array}
$$

$J^r(L \to M)$ splits up as a direct sum of complexes, with a summand for each degree in $M$.

**Definition.** Let $M$ be a $L$–module. The other Jacobi complex, $J^r(L, M)$, is the summand of $J^r(L \longrightarrow M)$ having degree one in $M$.

**Definition.** If $V$ is a finite dimensional vector space, we call $V$ a coalgebra if $V^*$ is an algebra. In this case, if $W$ is another a finite dimensional vector space, we call $W$ a $V$ module if $W$ is a $V^*$ module.

The natural map $J^r(L) \to S^2 J^r(L)$,

$$
\begin{array}{ccccccc}
\ldots & \longrightarrow & \wedge^3 L \otimes L \oplus S^2 \wedge^2 L & \longrightarrow & \wedge^2 L \otimes L & \longrightarrow & \wedge^2 L \\
\uparrow & & \uparrow & & \uparrow \\
\ldots & \longrightarrow & \wedge^4 L & \longrightarrow & \wedge^3 L & \longrightarrow & \wedge^2 L & \longrightarrow & L, \\
\end{array}
$$

gives a map $h^0(J^r(L)) \to S^2 h^0(J^r(L))$ and the map $J^r(L, M) \to J^r(L) \otimes J^r(L, M)$ gives a map $h^0(J^r(L, M)) \to h^0(J^r(L)) \otimes h^0(J^r(L, M))$. We call these maps of cohomology groups comultiplication. This comultiplication turns $h^0(J^r(L)) \oplus \mathbb{C}$ into a coalgebra and $h^0(J^r(L, M))$ into a module.
**Definition.** Let $V$ and $W$ be vector bundles on a smooth algebraic variety, $X$. Let $X \times X$ be the product with projections $p_i$, and diagonal $\Delta$. The **Sheaf of differential operators of order $r$ from $V$ to $W$**, $\text{Diff}^r(V, W)$, is $\mathcal{H}\text{om}(p_1\ast(p_2^r(V) \otimes \mathcal{O}_{X \times X}/I_\Delta^{r+1}), W)$. We write $\text{Diff}^r$ for $\text{Diff}^r(\mathcal{O}, \mathcal{O})$.

The ring structure on $\mathcal{O}_{X \times X}$ makes $\text{Diff}^r$ into a sheaf of $\mathcal{O}_X$–coalgebras, and $\text{Diff}^r(V, W)$ into a $\text{Diff}^r$–module.

### 3. The Result

Let $X$ be an algebraic variety and $G$ be an algebraic group bundle over $X$. Let $\mathfrak{g}$ be the corresponding Lie algebra bundle. Let $V$ be a vector bundle and let $A : G \times X V \to V$ be a linear action. Let $s \in \Gamma(V)$ be a section. From this section, we obtain a map $G \to V$. Let $a : \mathfrak{g} \to V$ be the (relative) tangent of the map, $G \to V, g \mapsto g(s)$, and let $b : \mathfrak{g} \otimes V \to V$ be map one obtains from the (relative) tangent of the map $G \to GL(V)$.

We wish to understand something of the map $G \times V \to V$ in terms of $a$ and $b$.

**Lemma 2.** Using notation as above, let $\{1\}_1 = \text{Spec}(\mathcal{O}_X \oplus \mathfrak{g}^*)$ be the first infinitesimal neighborhood of 1 in $G$ and $\{s\}_r$ be the $r$th infinitesimal neighborhood of $s$ in $V$. The map

$$f : \bigoplus_{i=0}^{r+1} S^iV^* \to \bigoplus_{i=0}^{r} S^iV^* \bigoplus \bigoplus_{i=0}^{r} \mathfrak{g}^* \otimes S^iV^*,$$

corresponding to $\{1\}_1 \times \{s\}_r \to \{s\}_{r+1}$, is given by

$$\Pi_{i=1}^{d}v_i \mapsto \Pi_{i=1}^{d}v_i \bigoplus a^*(v_1) \otimes \Pi_{i=2}^{d}v_i + b^*(v_1) \otimes \Pi_{i=2}^{d}v_i. \quad (*)$$

**proof.** We think of $\{1\}_1$ as an affine algebraic group scheme over $X$, with structure sheaf $\mathcal{O}_X \oplus \mathfrak{g}^*$. We identify $\{s\}_r$ with $\{0\}_r = \text{Spec}(\bigoplus_{i=0}^{r+1} S^iV^*)$ by translation by $s$, which we denote $t_s : V \to V$. The question is local on $X$, so we let $\{e_i\}$ be a local basis of $V$, and let $\{x_i\}$ be the dual basis. The map in question is $t_{-s} \circ A \circ (G \times t_s) : G \times V \to V$. Since $(*)$ preserves multiplication and $V^*$ generates $\text{Sym}V^*$, we only need to compute the map from $V^*$. Let $v \in \Gamma(V^*)$. Let $c : V \otimes V^* \to \mathfrak{g}^*$ be the dual of the tangent of the map $G \to GL(V)$. $t_{-s}^\#$, the map of sheaves of $\mathcal{O}_X$ algebras corresponding to $t_{-s}$, takes $v$ to $v - v(s)$. $A^\#$ takes $v - v(s)$ to $\sum_i c(v \otimes e_i) \otimes x_i + v - v(s)$. Finally $(G \times t_s)^\#$ takes this to $\sum_i c(v \otimes e_i) \otimes x_i + x_i(s) c(v \otimes e_i) + v + v(s) - v(s)$. \qed

Let $U \subset V$ be an open union of $G$ orbits. Assume that a geometric quotient $\pi : U \to Y$ exists and that $Y$ is smooth over $X$. Assume that the section, $s$, is contained in $U$. Let $p : Y \to X$. Let $I_s$ be the ideal sheaf of the image of $s$ in $Y$.

**Proposition 3.** We use notation as above. Give $L = (\mathfrak{g} \to V)$ the natural DGLA structure. Then $h^0(J^r(L))^* \oplus \mathcal{O} = p_\ast(\mathcal{O}_Y/I_s^{r+1})$. If the map, $\mathfrak{g} \to V$, is injective, then $J^r(L)$ has no other cohomology. Also the multiplication on $p_\ast(\mathcal{O}_Y/I_s^{r+1})$ coming from multiplication on $\mathcal{O}_Y$ coincides with the natural multiplication on $h^0(J^r(L))^*$.

**proof.** From the map $\pi : U \to Y$ we get a map of sheaves of $\mathcal{O}_X$ algebras—

$$\pi^\# : \mathcal{O}_X \to \mathcal{O}_Y/I_s^{r+1} \to \mathcal{O}_Y/I_s^r.$$
The action, $A : G \times V \to V$, and projection, $p_2$, give us two $O_X$ algebra maps—

$$A^#, p_2^# : \oplus_{i=0}^r S^iV^* \to \oplus_{i=0}^r S^iV^* \bigoplus \oplus_{i=1}^r S^{i-1}V^* \otimes g^*.$$ 

$p_2^#$ is identity on the right summand of the target and zero on the left summand. Lemma identifies $A^#$. Thus $A^# - p_2^#$ maps to the right summand only. Notice that $A^# - p_2^#$ coincides with the dual of the final map of $\text{Tot}(J^r(L))$.

We make a sequence, which we will show to be left exact, out of $A^# - p_2^#$ and $\pi^#$—

$$0 \to p_1^*(O_Y/I_s^{r+1}) \to \oplus_{i=0}^r S^iV^* \to \oplus_{i=1}^r S^{i-1}V^* \otimes g^*.$$ 

$(A^# - p_2^#) \circ \pi^#$ is zero because $A^#$ and $p_2^#$ agree on $G$ invariant functions. $\pi^#$ is injective because $\pi$ is surjective. Thus $p_1^*(O_Y/I_s^{r+1})$ maps injectively to $\ker(A^# - p_2^#) = h^0(J^r(L))^* \oplus O$. This map is filtered because both range and domain have the filtration induced from the filtration on $\oplus_{i=0}^r S^iV^*$.

Because $h^i(L) = T_{Y/X}$ and $L$ has no higher cohomology, we see that $h^i(\wedge^i L) = S^iT_{Y/X}$ and $\wedge^i L$ has no higher cohomology. This tells us that

$$\text{gr}(h^0(J^r(L))) = \oplus_{i=1}^r S^iT_{Y/X}.$$ 

Therefore the map from $p_1^*(O_Y/I_s^{r+1})$ to $h^0(J^r(L))^* \oplus O$ is isomorphism on all graded parts. Thus our sequence is left exact.

In the case that $g \to L$ is injective, $\wedge^i L$ has no cohomology other than $h^i$ therefore $J^r(L)$ has no cohomology other than $h^0$.

Since $\pi^#$ preserves multiplication, the multiplication on $p_1^*(O_Y/I_s^{r+1})$ is the one inherited from $\oplus_{i=0}^r S^iV^*$. This latter multiplication corresponds to the comultiplication on $h^0(J^r(V[-1]))$, in which $V[-1]$ is given the trivial DGLA structure. Since $V[-1] \to L$ is a map of DGLA, $h^i(J^r(L))$ has comultiplication inherited from $h^0(J^r(V[-1]))$. This proves that the multiplications coincide. 

Let us fix some notation. Let $G$ be an algebraic group and $V$ be a vector space. Let $A : G \times V \to V$ be a free linear action. Let $g = T_1(G)$. Let $U \subset V$ be an open union of orbits. Assume that a geometric quotient $\pi : U \to X$ exists and that $X$ is smooth.

Let $G$ act on $G \times U$ by conjugation on the first factor and $A$ on the second factor. Let $\bar{G}$ be the quotient of $G \times U$. By doing group arithmetic on the first factor and mapping to $X$ by projection to the second factor, we make $\bar{G}$ into an algebraic group bundle. Let $\bar{g}$ be the Lie algebra bundle.

Let $\bar{V}$ be the quotient of $V \times U$ by the diagonal action of $G$. Notice that $\bar{V}$ is a $\bar{G}$-representation over $X$. The diagonal map $U \to V \times U$ gives a section, $\Delta$, of $\bar{V}$. Let $\bar{A} : G \times X \bar{V} \to \bar{V}$ be the action. Let $\bar{U}$ be the quotient of $U \times U$. Notice that the quotient of $\bar{U}$ by action of $\bar{G}$ is $X \times X$ and the section, $\Delta$, maps to diagonal.

From the section $\Delta : X \to \bar{U}$, we obtain a map $G \to \bar{U}$. Let $a : \bar{g} \to \bar{V}$ be the (relative) tangent of this map. Let $b : \bar{g} \otimes \bar{V} \to \bar{V}$ be the map coming from the (relative) tangent of the map $G \to GL(V)$.

**Corollary 4.** Give $L = (\bar{g} \to \bar{V})$ the natural DGLA structure. Then $h^0(J^r(L)) \oplus O = \text{Diff}^r$ and $J^r(L)$ has no other cohomology. Also the coalgebra structures coincide.

Let $W$ be another vector bundle and $B : G \times W \to W$ be a linear action. As before, we make a $G$-representation, $\bar{W}$, with action $\bar{B} : G \times X \bar{W} \to \bar{W}$. 

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The above text is a continuation of a mathematical discussion involving various algebraic and differential geometry concepts, focusing on the properties of maps and bundles under specific group actions. The text explores the cohomology and comultiplication structures within the context of these actions, providing insights into the behavior of these structures under different algebraic and geometric operations.
Corollary 5. Give \( L = (g \to V) \) the natural DGLA structure and give \( W \) the natural \( L \)-module structure. Then \( h^0(J^r(L, W)) = \text{Diff}^r(W^*, O_X) \) and \( J^r(L, W) \) has no other cohomology. Also the module structures coincide.

proof. We apply Proposition to the \( \bar{G} \) action on \( \bar{U} \oplus \bar{W}^* \), with the section \((\Delta, 0)\). Notice that the quotient is the geometric vector bundle \( \text{gl} \), with \( \Delta \).

We proceed according to the following general formula: given the affine coordinate ring of a geometric vector bundle, we obtain the associated module by extracting the degree one component and dualizing; module structure may be obtained from the multiplication of the of degree zero component by the degree one component of the affine coordinate ring.

Proposition tells us that the affine coordinate ring of the infinitesimal neighborhood of \((\Delta, 0)\) in \( p_r^*(\bar{W}^*) \) is given by \( h^0(J^{r+1}(L \to \bar{W}^*))^* \oplus \mathcal{O} \). The degree one component is \( h^0(J^r(L, \bar{W}^*))^* \) and the degree zero component is \( h^0(J^r(L, \mathcal{O}))^* = \mathcal{O}_{X \times X}/I^r \). We dualize the former with respect to the latter. Since

\[
\text{Hom}_{\mathcal{O}_{X \times X}}(p_r^*(\cdot), \mathcal{O}_{X \times X}) = p_r^*\text{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{O}_X),
\]

we obtain

\[
h^0(J^r(L, \bar{W}))^* = p_1^*(p_2^*(\bar{W}^*) \otimes \mathcal{O}_{X \times X}/I^r).
\]

Applying \( \text{Hom}(\cdot, \mathcal{O}_X) \) to both sides, we obtain

\[
h^0(J^r(L, \bar{W})) = \text{Diff}^r(\bar{W}^*, \mathcal{O}).
\]

We conclude that the module structures coincide by noticing that, in general, the comultiplication of the degree zero by the degree one components of \( h^0(J^r(L \to M)) \) coincides with the comultiplication \( h^0(J^r(L, M)) \to h^0(J^r(L)) \otimes h^0(J^r(L, M)) \).

4. Application to Projective Space

Let \( V \) be a \( n + 1 \) dimensional vector space, and \( Q \) be a one dimensional vector space. We view \( \mathbb{P}(V) \) as the quotient of \( \text{Hom}(V, Q) \setminus \{0\} \) by \( GL(Q) \). Let \( L = (\mathcal{O} \to V^*(1)) \). We notice that

\[
J^r(L) = \begin{pmatrix}
S^r(V^*(1)) \\
\uparrow \\
S^{r-1}(V^*(1)) \\
\sim \\
\vdots \\
\sim \\
S^2(V^*(1)) \\
\uparrow \\
V^*(1) \\
\sim \\
V^*(1) \\
\uparrow \\
\mathcal{O}
\end{pmatrix}
\]

We simplify this and obtain the following short exact sequence—

\[
0 \to \mathcal{O} \to S^r(V^*(1)) \to \text{Diff}^r/\mathcal{O} \to 0.
\]

Using this we identify

\[
h^0(\text{End}(\text{Diff}^r/\mathcal{O})) = \mathbb{P}^0(\text{End}(\mathcal{O} \to S^r(V^*(1)))).
\]
The spectral sequence for hypercohomology tells us that for \( n > 1 \), this is one dimensional. Thus \( \text{Diff}^r_{\mathcal{P}(V)}/\mathcal{O} \) has no non scalar global endomorphisms.

However, this is not the case for \( \text{Diff}^r(\mathcal{O}(k), \mathcal{O}(k)) \) for \( k \neq 0 \). Notice that

\[
J^r(L, \mathcal{O}(k)) =
\begin{align*}
S^r(V^*(1))(k) & \quad \xrightarrow{k+r-1} \\
S^{r-1}(V^*(1))(k) & \quad \xrightarrow{k+1} \\
\cdots & \\
S^2(V^*(1))(k) & \quad \xrightarrow{k+2} \\
V^*(k+1) & \quad \xrightarrow{k+1} \\
\mathcal{O}(k) & \quad \xrightarrow{k} \mathcal{O}(k)
\end{align*}
\]

We see that

\[
\text{Diff}^r(\mathcal{O}(k), \mathcal{O}(k)) = \begin{cases}
S^k(V^*(1)) \oplus S^r(V^*(1))/S^k(V^*(1)) & \text{for } 0 \leq k \leq r \\
S^r(V^*(1)) & \text{otherwise}.
\end{cases}
\]

**References**

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