Discrete exponential type systems on a quad graph, corresponding to the affine Lie algebras $A_{N-1}^{(1)}$

I T Habibullin$^{1,2}$ and A R Khakimova$^1$

$^1$ Institute of Mathematics, Ufa Federal Research Centre, Russian Academy of Sciences, 112, Chernyshevsky Street, Ufa 450008, Russia
$^2$ Bashkir State University, 32, Validy Street, Ufa 450076, Russia

E-mail: habibullinismagil@gmail.com and aigul.khakimova@mail.ru

Received 11 January 2019, revised 30 May 2019
Accepted for publication 9 July 2019
Published 13 August 2019

Abstract
The article deals with the problem of the integrable discretization of the well-known Drinfeld–Sokolov hierarchies related to the Kac–Moody algebras. A class of discrete exponential systems connected with the Cartan matrices has been suggested earlier in Garifullin et al (2012 SIGMA 8 33) which coincide with the corresponding Drinfeld–Sokolov systems in the continuum limit. It was conjectured that the systems in this class are all integrable and the conjecture has been proven by numerous examples. In the present article we study those systems from this class which are related to the algebras $A_{N-1}^{(1)}$. We found the Lax pair for arbitrary $N$, briefly discussed the possibility of using the method of formal diagonalization of Lax operators for describing a series of local conservation laws and illustrated the technique using the example of $N = 3$. Higher symmetries of the system $A_{N-1}^{(1)}$ are presented in both characteristic directions. The recursion operator for the case $N = 3$ is found. It is interesting to note that this operator is not weakly nonlocal.

Keywords: discretization, quad graph equations, formal diagonalization, conservation laws, higher symmetries, Lax pairs

1. Introduction

Exponential type systems of hyperbolic equations in partial derivatives

$$v_{x,y}^i = \exp(a_{i1}v^1 + a_{i2}v^2 + \cdots + a_{iN}v^N), \quad 1 \leq i \leq N$$

(1.1)
are actively discussed in literature due to their applications in field theory and other areas of physics, as well as in geometry, integrability theory etc.

An exponential system in the form of the two dimensional Toda lattice

\[ v^i_{x,y} = \exp(v^{i+1} - 2v^i + v^{i-1}) \]  

(1.2)
appeared many years ago within the framework of the Laplace cascade integration method (see [2]). In the context of the soliton theory this equation has been rediscovered in [3, 4]. Due to the works by Mikhailov, Olshanetsky, Perelomov, Leznov, Savel’ev, Shabat, Smirnov, Wilson, Yamilov, Drinfeld, Sokolov and many others, the mathematical theory of the exponential systems has been developed and currently (see [4–11]) it is well-known that in the case when the coefficient matrix \( A = \{a_{ij}\} \) coincides with the Cartan matrix of a semi-simple Lie algebra, then (1.1) admits a complete set of non-trivial integrals in both characteristic directions and therefore is integrable in the sense of Darboux. Similarly, if \( A \) is the generalized Cartan matrix of an affine Lie algebra then the system (1.1) can be studied by means of the inverse scattering transform method [4–11].

Inspired by Ward (see [12]) the problem of finding integrable discrete analogs of equation (1.1), which are discrete in both independent variables, has been intensively studying since the middle of 1990s. We remark that the particular 1 + 1-dimensional case, obtained by setting \( x = y \), was successfully investigated in [13]. An effective way to construct the discrete version of (1.2) based on the discretization of the bilinear equation of the Toda lattice was suggested by Hirota [14] and Miwa [15]. The Hirota–Miwa equation is a universal soliton equation from which a large variety of integrable discrete models can be derived by symmetry constraints (see [16]). An alternative approach to discretize the two dimensional Toda lattice is used by Fordy and Gibbons [17, 18], where the discrete equation is derived as the superposition formula of the Bäcklund transformations for the equation (1.2). A large class of the quad systems and their applications in physics are studied in [19].

In the article [1] a class of exponential type systems of discrete equations

\[ a e^{-u^i_{1,1} + u^i_{1,0} - u^i_{0,0}} - 1 = b \exp \left( \sum_{j=1}^{i-1} a_{ij} u^i_{j,0} + \sum_{j=i+1}^{N} a_{ij} u^i_{0,j} + a_{i0} \frac{u^i_{0,1} + u^i_{1,0}}{2} \right) \]  

(1.3)

has been suggested for the particular case \( a = b = 1 \). In the present article we assume that \( a \) and \( b \) are arbitrary nonzero constants. Here the upper index \( j \) is a natural index ranging from 1 to \( N \). For the shifts of the sought functions \( u^i_{j,m} \) in (1.3) we use the abbreviated notations \( u^i_{j+k} = u^i_{j,k} \) such that \( u^i_{j,0} \) means \( u^i_{n+1,0} \) and so on.

For the small values of \( N \), the system (1.3) was obtained in [1] from integrable cases of (1.1) by applying the method of discretization preserving integrals and symmetries. For (1.3) with \( A \) being the generalized Cartan matrix of the algebra \( D^{(2)} \) the Lax pair has been found. Later in [20] Smirnov proved that the quad systems (1.3) corresponding to simple Lie algebras \( A_N \) and \( B_N \) are Darboux integrable. These facts partially confirm our hypothesis from [1] that the quad system inherits the integrability property of the system (1.1)) and is integrable for the Cartan matrices of both simple and affine Lie algebras.

The main goal of this paper is to present the Lax pair for the quad system (1.3) associated with the algebra \( A^{(1)}_{N-1} \). We outlined a method for applying the formal diagonalization technique for finding local conservation laws and higher symmetries. In other words we approve the aforementioned hypothesis for one more series of affine Lie algebras. Unfortunately, our
Lax pairs for $A_{N-1}^{(1)}$ and $D_N^{(2)}$ are not given in terms of the Cartan–Weyl basis of the algebra as it was in the case of (1.1) (see [10]), so there are problems with generalization to other algebras. Note that alternative examples of discretizations of the Toda lattices related with the algebra $A_{N-1}^{(1)}$ are investigated in [21–24].

Quadrilateral lattices studied in the article are very close to those suggested in [25] as discretizations of the Gelfand–Dikii hierarchy. However these two classes of the discrete equations differ from each other. For instance, the first members are the discrete d’Alembert equation and respectively the discrete potential KdV equation. The second members are the couplet system (4.4) and the discrete Boussinesq equation. They are not connected by the point transformations.

An interesting integrable system of partial difference equations is suggested in [26] with an arbitrarily large number of independent variables. In a particular case when the number of independent variables is two this system looks very similar to (4.1) and (2.17), but the systems are not equivalent (see proposition 3 in section 4).

Let us briefly discuss the content of the article. In section 2 we study the quasi-periodic reduction $t_{j,n,m}^{+N} = t_{n+1,m-1}^j$ of the Hirota–Miwa equation which leads to a quadrilateral system of the form (1.3) corresponding to the algebra $A_{N-1}^{(1)}$. We show that the constraint $\psi_{j,n,m}^{+N} = \lambda \psi_{n+1,m-1}^j$ imposed on the system (2.4) associated with the Hirota–Miwa equation successfully creates a Lax pair for the resulting quad system written in the form (2.1).

We also give an example of a system obtained using the purely periodic condition $t_{j,n,m}^{+2} = t_{n,m}^j$. Note that various types of the periodic conditions for the Hirota–Miwa equation are studied in [27]. For example, it is shown that the restriction $t_{n,m}^{+2} = t_{n,m}^j$ leads to a discrete version of the sine-Gordon equation. However, to our knowledge, examples of quasi-periodic reduction of the form $t_{j,n,m}^{+N} = t_{n+1,m-1}^j$ are not considered earlier.

In section 3 we used the Lax pair derived in the previous section for describing the local conservation laws for the system (2.1). To this end we converted the Lax equations (2.12) and (2.13) around each singular value of the spectral parameter $\lambda$ to a special form allowing us to determine the asymptotic representation of the eigenfunctions. Finding this special form usually causes the main problem. In section 3 we found triangular transformations reducing the Lax equations to the required form. The case $N = 3$ is studied in more detail. We note that for constructing the asymptotic representation of the Lax eigenfunctions we used a method which slightly differs from known ones (see [28, 29]).

In the fourth section higher symmetries are given for the quad system (2.1), and finally in section 5 the recursion operators are found from the asymptotic representations of the Lax eigenfunctions for $N = 2$ and $N = 3$ in the particular values of the parameters $a = b = 1$. For the case $N = 3$ the found recursion operator is not weakly nonlocal.

2. Derivation of the Lax pair

2.1. Quasi-periodic reduction

In the article we deal with the quad system (1.3) corresponding to the algebra $A_{N-1}^{(1)}$ which can be written in terms of the variable $t_{n,m}^j = \exp \left\{-u_{j,n,m}^j \right\}$ as follows:
\[ \begin{align*}
\frac{a_{1,0}^j t_{1,1}^j - t_{1,0}^j t_{0,1}^j}{t_{1,0}^j + t_{0,1}^j} &= b_{1,1}^j t_{1,0}^j, \\
\frac{a_{1,0}^j t_{1,1}^j - t_{1,0}^j t_{0,1}^j}{t_{1,0}^j + t_{0,1}^j} &= b_{1,1}^j t_{1,0}^j, \\
\frac{a_{0,0}^N t_{1,1}^N - t_{1,0}^N t_{0,1}^N}{t_{1,0}^N + t_{0,1}^N} &= b_{1,1}^N t_{1,0}^N.
\end{align*} \tag{2.1}\]

Evidently system (2.1) can be obtained from the Hirota–Miwa equation
\[ \frac{a_{1,0}^j t_{1,1}^j - t_{1,0}^j t_{0,1}^j}{t_{1,0}^j + t_{0,1}^j} = b_{1,1}^j t_{1,0}^j, \quad -\infty < j < +\infty \tag{2.2} \]

by imposing the quasi-periodicity closure constraint
\[ t_{1,0}^{j+N} = t_{1,0}^{j-1}, \quad N \geq 2. \tag{2.3} \]

Note that for the simplest case \( N = 1 \) (2.2) implies the discrete version of the d’Alembert equation.

Our aim is to derive the Lax pair for (2.1) from the overdetermined system of the linear equations
\[ \begin{align*}
\psi_{j,1}^{j+1} &= \frac{t_{1,0}^{j+1}}{t_{0,1}^{j+1}} \psi_{j,0}^{j+1} - \psi_{j,0}^{j+1}, \\
\psi_{j,0}^{j+1} &= \psi_{j,0}^j + b_{j,0} t_{1,0}^{j-1} \psi_{j,0}^{j-1} \\
\psi_{j,0}^{j} &= \psi_{j,0}^{j+1} + a_{j,0} t_{0,0}^j \\
\psi_{j,0}^{j+1} &= \psi_{j,0}^{j+1} + a_{j,0} t_{0,0}^j.
\end{align*} \tag{2.4} \]

associated with the equation (2.2). More precisely, when \( t_{1,0}^j \) solves equation (2.1) then the system (2.4) is compatible. However the converse is not true: the compatibility of the system does not imply (2.2). Nevertheless (2.4) can be used effectively for constructing the true Lax pair for the quad system (2.1). Evidently (2.4) implies the hyperbolic type discrete linear equation
\[ \begin{align*}
\frac{\psi_{j,1}^{j+1} - \psi_{j,1}^{j}}{t_{1,0}^{j+1}} &= \frac{a_{j,0} t_{0,0}^{j+1} \psi_{j,0}^{j+1}}{t_{1,0}^{j+1}} - \frac{a_{j,0} t_{0,0}^{j} \psi_{j,0}^{j}}{t_{1,0}^{j}} = 0.
\end{align*} \tag{2.5} \]

It is widely known that the Laplace invariants are important characteristics of the hyperbolic type discrete and continuous equations. Recall that for an equation of the form
\[ f_{1,1} + b_{0,0} f_{1,0} + c_{0,0} f_{0,1} + d_{0,0} f_{0,0} = 0 \tag{2.6} \]

the Laplace invariants \( K_1 \) and \( K_2 \) are determined due to the rules (see [30, 31])
\[ K_1 = \frac{b_{0,0} c_{1,0}}{d_{1,0}}, \quad K_2 = \frac{b_{0,1} c_{0,1}}{d_{0,1}}. \]

Further we will use theorem (see [31]) claiming that equation (2.6) and the equation
\[ f_{1,1} + \tilde{b}_{0,0} f_{1,0} + \tilde{c}_{0,0} f_{0,1} + \tilde{d}_{0,0} f_{0,0} = 0 \]

are related with one another by the multiplicative transformation \( f = \lambda \tilde{f} \) if and only if they have the same pair of the Laplace invariants, i.e. \( K_1 = \tilde{K}_1 \) and \( K_2 = \tilde{K}_2 \).

**Proposition 1.** Under the quasi-periodicity condition (2.3) equation (2.5) and the equation
\[ \psi_{j,1}^{j+1} - \psi_{j,1}^{j} = \frac{a_{j,0} t_{0,0}^{j+1} \psi_{j,0}^{j+1}}{t_{1,0}^{j+1}} - \frac{a_{j,0} t_{0,0}^{j} \psi_{j,0}^{j}}{t_{1,0}^{j}} = 0 \tag{2.7} \]

are related by the multiplicative transformation, or more precisely, by the equation
\[ \psi_{j,0}^{j} = \Lambda^j \psi_{j,0}^{j+N}. \tag{2.8} \]
Proof. Let us first give the Laplace invariants in enlarged (not abbreviated!) form
\[ K_1(n, m, j) = \frac{t_{j+2}^n t_{j+1}^n a}{t_{j+1}^m t_{j+2}^m a}, \quad K_2(n, m, j) = \frac{t_{j+1}^{n+1} t_{j+2}^{n+1} a}{t_{j+1}^{m+1} t_{j+2}^{m+1} a}. \]

Now it is easily seen that constraint (2.3) implies \( K_1(n + 1, m, j) = K_1(n, m + 1, j + N) \) and \( K_2(n + 1, m, j) = K_2(n, m + 1, j + N) \). These two relations due to the above-mentioned theorem allow one to complete the proof.

Proposition 2. The coefficient \( \Lambda' \) in the relation (2.8) does not depend on any of the variables \( j, n, m \).

Proof. By applying the shift operator \( D_m \), acting according to the rule \( D_m y_m = y_{m+1} \) to equation (2.8), we evidently obtain \( \psi_{1,1}^j = A_{0,1}^j \psi_{0,2}^{j+1} \). Next we replace \( \psi_{1,1}^j \) due to the hyperbolic type equation (2.5) and find
\[
\psi_{1,0}^j + \frac{t_{j+1}^0 t_{j+1}^0}{t_{j+1}^1 t_{j+1}^1} \psi_{0,1}^j = a t_{j+1}^0 t_{j+1}^0 \psi_{0,0}^j = A_{0,1}^j \psi_{0,2}^{j+1}.
\]

Then we get rid the function \( \psi^j \) by virtue of the relation (2.8). As a result we arrive at the equation
\[
\psi_{1,1}^{j+1} = \frac{A_{1,-2}^j}{A_{1,0}^j} \psi_{1,0}^{j+1} = A_{0,0}^j t_{j+1}^0 t_{j+1}^0 \psi_{0,1}^{j+1} + A_{1,0}^j t_{j+1}^0 t_{j+1}^0 \psi_{0,1}^{j+1} = 0,
\]
which should coincide with (2.7). Now we compare the corresponding coefficients of these two equations and by means of (2.8) we get three equations for \( \Lambda' \)
\[
A_{1,-2}^j = A_{1,0}^j, \quad A_{1,0}^j = A_{0,0}^j, \quad A_{0,-1}^j = A_{1,0}^j,
\]
which approve that \( \Lambda' \) does not depend on \( n \) and \( m \).

It obviously follows from (2.8) that
\[
\psi_{1,0}^j = \frac{t_{j+1}^0 t_{j+1}^0}{t_{j+1}^1 t_{j+1}^1} \psi_{0,0}^j = \Lambda' \left( \psi_{0,1}^j - \frac{t_{j+1}^0 t_{j+1}^0}{t_{j+1}^0 t_{j+1}^0} \psi_{0,1}^{j+1} \right).
\]

The left hand side of (2.9) coincides with \( -\psi_{0,0}^j \). Let us replace the fraction by means of the relation (2.3) and find that due to (2.4) the right hand side in (2.9) coincides with \( -\Lambda' \psi_{0,0}^{j+1} \). Thus we have relation \( \psi_{0,1}^{j+1} = \Lambda' \psi_{0,1}^{j+N+1} \), which together with (2.8) gives \( \Lambda' = A^{j+1} \).

Proposition 2 is proved.

Let us consider quad system (2.1). It is easily verified that (2.1) can be quasi-periodically prolonged to the infinite interval \( -\infty < j < +\infty \) by setting \( t_{j,0}^1 = t_{j,0}^{j+N} \) such that the prolonged function \( t_{j,m}^1 \) will solve equation (2.2). Then the propositions 1 and 2 provide the following gluing conditions
where $\lambda$ is an arbitrary parameter.

In order to derive the Lax pair for the quad system (2.1) we impose the gluing conditions on the linear system (2.4) and find

\begin{align*}
\psi_{1,0}^j &= t_{1,0}^1 t_{0,0}^j \psi_{0,0}^j - \psi_{0,0}^{j+1}, \quad 1 \leq j \leq N - 1, \\
\psi_{1,0}^N &= t_{1,0}^N t_{0,0}^N \psi_{0,0}^N - \lambda \psi_{1,0}^{j-1}
\end{align*}

(2.10)

and

\begin{align*}
\psi_{0,1}^1 &= \psi_{0,0}^1 + b \lambda^{-1} t_{0,0}^{j+1} t_{0,0}^j \psi_{1,0}^{j-1}, \\
\psi_{0,1}^j &= \psi_{0,0}^j + b \lambda^{-1} t_{0,0}^{j+1} t_{0,0}^j \psi_{0,0}^{j-1}, \quad 2 \leq j \leq N.
\end{align*}

(2.11)

Let us apply now the shift operator $D_m$ to the last equation in (2.10):

\begin{align*}
\psi_{1,0}^N &= t_{1,0}^N t_{0,0}^N \psi_{0,0}^N - \lambda \psi_{1,0}^{j-1}.
\end{align*}

Due to the equation (2.5) taken at the value $j = N$ we obtain after substitution $t_{1,0}^{N+1} = t_{2,0}^1$, $t_{0,0}^{N+1} = t_{1,0}^1$ that

\begin{align*}
\psi_{1,0}^N &= -a t_{0,0}^{N+1} t_{1,0}^1 \psi_{0,0}^1 - \lambda \psi_{1,0}^{j-1}.
\end{align*}

By replacing $\psi_{1,0}^1$ from (2.10) we bring the equation to the suitable form

\begin{align*}
\psi_{1,0}^N &= a \frac{t_{0,0}^{N+1} t_{1,0}^1}{t_{1,0}^1 t_{0,0}^1} \psi_{0,0}^1 + \lambda \psi_{0,0}^2 + a \frac{t_{0,0}^{N+1} t_{1,0}^1}{t_{1,0}^1 t_{0,0}^1} \psi_{0,0}^1.
\end{align*}

In the same way we can rewrite in the appropriate form the first equation in (2.11). Since the reasonings are very similar to that used above we omit them and give only the final result

\begin{align*}
\psi_{0,1}^1 &= a \frac{t_{0,0}^{N+1} t_{1,0}^1}{t_{1,0}^1 t_{0,0}^1} \psi_{0,0}^1 + b \lambda^{-1} \frac{t_{0,0}^{N+1} t_{1,0}^1}{t_{1,0}^1 t_{0,0}^1} \psi_{0,0}^1 + b \lambda^{-1} \frac{t_{0,0}^{N+1} t_{1,0}^1}{t_{1,0}^1 t_{0,0}^1} \psi_{0,0}^1.
\end{align*}

Let us summarize the computations above and present the desired Lax pair:

\begin{align*}
\psi_{1,0}^j &= t_{1,0}^{j+1} t_{0,0}^j \psi_{0,0}^j - \psi_{0,0}^{j+1}, \quad 1 \leq j \leq N - 1, \\
\psi_{1,0}^N &= -a \frac{t_{0,0}^{N+1} t_{1,0}^1}{t_{1,0}^1 t_{0,0}^1} \psi_{0,0}^1 + \lambda \psi_{0,0}^2 + a \frac{t_{0,0}^{N+1} t_{1,0}^1}{t_{1,0}^1 t_{0,0}^1} \psi_{0,0}^1
\end{align*}

(2.12)

and

\begin{align*}
\psi_{0,1}^1 &= \psi_{0,0}^1 + b \lambda^{-1} \frac{t_{0,0}^{N+1} t_{1,0}^1}{t_{1,0}^1 t_{0,0}^1} \psi_{0,0}^1, \\
\psi_{0,1}^j &= \psi_{0,0}^j + b \lambda^{-1} \frac{t_{0,0}^{N+1} t_{1,0}^1}{t_{1,0}^1 t_{0,0}^1} \psi_{0,0}^{j-1}, \quad 2 \leq j \leq N.
\end{align*}
\[\psi_{0,1}^1 = a_{t,0}^{0,1} \psi_{0,0}^1 + b_0^2 \lambda^{-1} \frac{t_{0,0}^{N-1,2}}{t_{0,1}^{N,1}} \psi_{0,0}^{N-1} + b_\lambda t_{0,0}^{N,1} \psi_{0,0}^N,\]

\[\psi_{0,1}^f = \psi_{0,0}^f + b_{0,1}^{j+1} t_{0,1}^{j-1} \psi_{0,0}^{-1}. \quad 2 \leq j \leq N. \quad (2.13)\]

Surprisingly the Lax pair (2.12) and (2.13) is obtained from (2.4) by changing only two equations.

2.2. Periodic reduction

Let us briefly discuss the periodic closure condition (see also [27])

\[t_{0,0}^{j+N} = t_{0,0}^j \quad (2.14)\]

for the Hirota–Miwa equation (2.2). It is easily checked that (2.14) generates a closure constraint for \(\psi\):

\[\psi_{0,0}^{j+N} = \xi \psi_{0,0}^j.\]

As a result we get a quad system

\[at_{0,0}^{j+1,1} - t_{1,0}^{j,0,1} = b t_{0,0}^N,\]

\[at_{0,0}^{j+1,1} - t_{1,0}^{j+1,1} = b t_{0,0}^{N-1,j+1} \quad 2 \leq j \leq N - 1, \quad (2.15)\]

In this case, the corresponding reduction of linear equation (2.4) is found immediately

\[\Phi_{1,0} = F \Phi, \quad \Phi_{0,1} = G \Phi, \quad (2.16)\]

where \(\Phi = (\psi^1, \psi^2, \ldots, \psi^N)^T\) and

\[F = \begin{pmatrix}
\frac{t_{0,0}^{1,0}}{t_{0,0}^{0,1}} & -1 & 0 & \cdots & 0 \\
0 & \frac{t_{0,0}^{1,0}}{t_{0,0}^{0,1}} & -1 & \cdots & 0 \\
0 & 0 & \frac{t_{0,0}^{1,0}}{t_{0,0}^{0,1}} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-\xi & 0 & \cdots & 0 & \frac{t_{0,0}^{1,0}}{t_{0,0}^{0,1}}
\end{pmatrix},\]

\[G = \begin{pmatrix}
1 & 0 & \cdots & 0 & \xi^{-1} \frac{b_{0,1}^N}{b_{0,0}^{N+1}} \\
b_{0,1}^{N-1} & 1 & \cdots & 0 & 0 \\
0 & b_{0,1}^{N-1} & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & b_{0,1}^{N-1} & 1
\end{pmatrix}.\]

It can be checked by a direct computation that system (2.16) does not define a Lax pair for (2.15) (in contrast to the quasi-periodic case (2.3)). More precisely, the consistency of (2.16)
does not imply (2.15). However the situation is changed if we pass in (2.15) to the potential variables \( r_{j,0}^j = \frac{r_{j,0}'}{\xi_{j,0}} \):

\[
\begin{align*}
    a r_{1,1}^1 &= r_{1,0}^1 + r_{0,1}^1 r_{1,0}^N \left( \frac{\xi_{1,0}}{\xi_{1,0}} - \frac{1}{\xi_{0,1}} \right), \\
    a r_{1,1}^j &= r_{1,0}^j + r_{j-1,0}^{j-1} r_{1,0}^j \left( \frac{\xi_{1,0}}{\xi_{1,0}} - \frac{1}{\xi_{0,1}} \right), \quad 2 \leq j \leq N - 1, \\
    a r_{1,1}^N &= r_{1,0}^N + r_{N-1,0}^{N-1} r_{1,0}^N \left( \frac{\xi_{1,0}}{\xi_{1,0}} - \frac{1}{\xi_{0,1}} \right).
\end{align*}
\] (2.17)

Now the system (2.16) with the potentials

\[
\begin{align*}
    F &= \begin{pmatrix}
         r_{0,0}^1 & -1 & 0 & \ldots & 0 \\
         0 & r_{0,0}^2 & -1 & \ldots & 0 \\
         0 & 0 & r_{0,0}^3 & \ldots & 0 \\
         \vdots & \vdots & \vdots & \ddots & \vdots \\
         -\xi & 0 & \ldots & 0 & r_{0,0}^N
\end{pmatrix}, \\
    G &= \begin{pmatrix}
         1 & 0 & \ldots & 0 & \xi^{-1} \frac{a r_{1,0}^1 - r_{0,0}^1}{\xi_{0,0}} \\
         \frac{a r_{1,0}^2 - r_{0,0}^2}{\xi_{0,0}} & 1 & \ldots & 0 & 0 \\
         0 & \frac{a r_{1,0}^3 - r_{0,0}^3}{\xi_{0,0}} & \ldots & 0 & 0 \\
         \vdots & \vdots & \ddots & \ddots & \vdots \\
         0 & 0 & \ldots & \frac{a r_{1,0}^N - r_{0,0}^N}{\xi_{0,0}} & 1
\end{pmatrix}
\end{align*}
\]

provides the Lax pair for (2.17).

3. Formal asymptotic solutions of the direct scattering problem and the local conservation laws

The method of the formal diagonalization of the Lax pairs given by differential operators is suggested in [10]. It is based on the ideas and technique applied earlier [32] in order to construct asymptotic solutions to the systems of differential equations with a parameter, when the parameter goes to its singular value. Let us recall that the formal diagonalization provides a main step in solving the direct scattering problem and allows us to describe local conservation laws and higher symmetries.

For the system of the linear discrete equations with a parameter two different tools to construct formal asymptotics were suggested in [28, 29]. However we do not see how to apply either of them to the case of the systems (2.12) and (2.13) since the method from [28] needs a special form of the potential and that of [29] can be applied only in the case when the corresponding quad equation admits evolutionary type higher symmetries. That is why we are forced to suggest an alternative way for formal diagonalization which slightly differs from the above-mentioned ones. Below we briefly discuss the scheme.

Let us consider a system of the discrete linear equations
\[ Y_{n+1} = f_n Y_n, \quad f_n = \sum_{j=-1}^{\infty} f_{n}^{(j)} \lambda^{-j}, \]  
(3.1)

where \( f_{n}^{(j)} \in \mathbb{C}^{k \times k} \) for \( j \geq -1 \) are matrix valued functions. In order to identify the matrix structure of the potential we divide the matrices into blocks as

\[ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \]

where the blocks \( A_{11}, A_{22} \) are square matrices. Here we assume that in (3.1) the coefficient \( f_{n}^{(-1)} \) is of one of the forms

\[ f_{n}^{(-1)} = \begin{pmatrix} 0 & 0 \\ 0 & A_{22} \end{pmatrix}, \quad \det A_{22} \neq 0 \]  
(3.2)

or

\[ f_{n}^{(-1)} = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad \det A_{11} \neq 0. \]  
(3.3)

Now our goal is to bring (3.1) to a block-diagonal form

\[ \varphi_{n+1} = h_n \varphi_n, \]  
(3.4)

where \( h_n \) is a formal series

\[ h_n = h_{n}^{(-1)} \lambda + h_{n}^{(0)} + h_{n}^{(1)} \lambda^{-1} + h_{n}^{(2)} \lambda^{-2} + \cdots \]  
(3.5)

with the coefficients having the block structure

\[ h_{n}^{(j)} = \begin{pmatrix} h_{11}^{(j)} & 0 \\ 0 & h_{22}^{(j)} \end{pmatrix}. \]  
(3.6)

To this end we use the linear transformation \( Y_n = T_n \varphi_n \) assuming that \( T_n \) is also a formal series

\[ T_n = E + T_{n}^{(1)} \lambda^{-1} + T_{n}^{(2)} \lambda^{-2} + \cdots, \]

where \( E \) is the unity matrix and \( T_{n}^{(j)} \) is a matrix with vanishing block-diagonal part:

\[ T_{n}^{(j)} = \begin{pmatrix} 0 & T_{12}^{(j)} \\ T_{21}^{(j)} & 0 \end{pmatrix}. \]

After replacing \( Y_n = T_n \varphi_n \) in (3.1) we get

\[ T_{n+1} h_n = \left( \sum_{j=-1}^{\infty} f_{n}^{(j)} \lambda^{-j} \right) T_n, \]  
(3.7)

where \( h_n = \varphi_{n+1} \varphi_n^{-1} \). Let us replace in (3.7) the factors by their formal expansions:

\[ (E + T_{n+1}^{(1)} \lambda^{-1} + \cdots)(h_{n}^{(-1)} \lambda + h_{n}^{(0)} + \cdots) = (f_{n}^{(-1)} \lambda + f_{n}^{(0)} + \cdots)(E + T_{n}^{(1)} \lambda^{-1} + \cdots). \]

By comparing coefficients at the powers of \( \lambda \) we derive a sequence of equations:

\[ h_{n}^{(-1)} = f_{n}^{(-1)}, \]  
(3.8)
T^{(k)}_{n+1} h^{(-1)}_{n} + h^{(-1)}_{n} - f^{(1)}_{n} T^{(k)}_{n} = R^{k}_{n}, \quad k \geq 1. \tag{3.9}

Here $R^{k}_{n}$ denotes terms that have already been found in the previous steps.

To find the unknown coefficients $T^{(i)}_{n}$, we must solve linear equations that look like difference equations. However, due to the special form of the coefficient $f^{(1)}_{n}$, these equations are linear algebraic and therefore are solved without ‘integration’. In other words $T^{(i)}_{n}$ and $h^{(i)}_{n}$ are local functions of the potential since they depend on a finite numbers of the shifts of the functions $f^{(1)}_{n}$, $f^{(0)}_{n}$, $f^{(1)}_{n}$, etc. Indeed, equation (3.9) obviously implies

$$
\begin{pmatrix}
0 & D_{n}(T^{(k)}_{12})A_{22} \\
0 & 0
\end{pmatrix}
+ \begin{pmatrix}
p & 0 \\
0 & q
\end{pmatrix}
- \begin{pmatrix}
0 & 0 \\
A_{22}T^{(k)}_{21} & 0
\end{pmatrix}
= R^{k}_{n}.
$$

Here $D_{n}$ is the operator shifting the argument $n$: $D_{n}Y_{n} = y_{n+1}$, and $p = h^{(k-1)}_{11}$, $q = h^{(k-1)}_{22}$. Evidently this equation is easily solved and the searched matrices $T^{(k)}_{n}$ and $h^{(k-1)}_{n}$ are uniquely found for any $k \geq 1$.

Suppose now that the system of equations

$$
Y_{n+1,m} = (f^{(1)}_{n,m} \lambda + f^{(0)}_{n,m} + \cdots)Y_{n,m}, \quad Y_{n,m+1} = G_{n,m}(\lambda)Y_{n,m}, \tag{3.10}
$$

where $G_{n,m}(\lambda)$ is analytic at a vicinity of $\lambda = \infty$, is the Lax pair for the nonlinear quad system

$$
Q(\mu^{(l)}_{n,m}) = 0,
$$

i.e. $Q$ depends on the variable $\mu^{(l)}_{n,m}$ and on its shifts with respect to the variables $j, n, m$.

Assume that function $f^{(1)}_{n,m}$ has the structure (3.2). Then due to the reasonings above there exists a linear transformation $Y_{n,m} \mapsto \psi_{n,m} = T^{-1}_{n,m}Y_{n,m}$ which reduces the first equation in (3.10) to a block-diagonal form (3.4)-(3.6). It can be checked that this transformation brings also the second equation of (3.2) to an equation of the same block structure

$$
\varphi_{n,m+1} = S_{n,m}\varphi_{n,m},
$$

where $S_{n,m}$ is a formal power series

$$
S_{n,m} = S^{(0)}_{n,m} + S^{(1)}_{n,m}\lambda^{-1} + S^{(2)}_{n,m}\lambda^{-2} + \cdots
$$

and

$$
S^{(j)}_{n,m} = \begin{pmatrix}
S^{(j)}_{11} & 0 \\
0 & S^{(j)}_{22}
\end{pmatrix}.
$$

Since the compatibility property of linear systems is preserved under a change of variables, we have the relation

$$
S_{n+1,m}h_{n,m} = h_{n+1,m}S_{n,m},
$$

which implies due to the block-diagonal structure that

$$(D_{n} - 1) \log \det(S_{n}) = (D_{m} - 1) \log \det(h_{n}), \quad i = 1, 2.
$$

By evaluating and comparing the coefficients at the powers of $\lambda$ we derive the sequence of the local conservation laws.

The Lax pairs considered in the article have also the second singular point $\lambda = 0$, so we briefly discuss the Lax pair represented as
\[ Y_{n+1,m} = F_{m,n} Y_{n,m}, \quad Y_{n,m+1} = (g_{n,m}^{(-1)} \lambda^{-1} + g_{n,m}^{(0)} + g_{n,m}^{(1)} \lambda + \cdots) Y_{n,m}, \]  

(3.11)

where \( F_{m,n} = F_{m,n}(\lambda) \) is analytic at a vicinity of \( \lambda = 0 \). We propose that here the term \( g_{n,m}^{(-1)} \) has the block structure (3.3). In this case the block-diagonalization is performed in a way very similar to the one recalled above for the singularity \( \lambda = \infty \).

3.1. Quad system corresponding to \( A_{N-1}^{(1)} \)

Let us apply the scheme discussed above to the Lax pair of the quad system (2.1). To this end we present the Lax pair (2.12) and (2.13) in a matrix form

\[ \Phi_{1,0} = F \Phi, \quad \Phi_{0,1} = G \Phi, \]  

(3.12)

where \( F \) and \( G \) are as follows:

\[
F = \begin{pmatrix}
\frac{\partial f_{10}^{11}}{\partial \mu_{\bar{m}0}} & -1 & 0 & \ldots & 0 & 0 \\
0 & \frac{\partial f_{10}^{11}}{\partial \mu_{\bar{n}0}} & -1 & \ldots & 0 & 0 \\
0 & 0 & \frac{\partial f_{10}^{11}}{\partial \mu_{\bar{n}0}} & \ldots & 0 & 0 \\
\ldots & & & & & \\
-\lambda \frac{\partial f_{10}^{11}}{\partial \mu_{\bar{m}0}} & \lambda & 0 & \ldots & 0 & 0
\end{pmatrix},
\]

\[
G = \begin{pmatrix}
a \frac{\partial f_{10}^{11}}{\partial \mu_{\bar{m}0}} & 0 & 0 & \ldots & 0 & b^2 \lambda^{-1} \frac{\partial f_{10}^{11}}{\partial \mu_{\bar{m}0}} & b \lambda^{-1} \frac{\partial f_{10}^{11}}{\partial \mu_{\bar{m}0}} \\
b \frac{\partial f_{10}^{11}}{\partial \mu_{\bar{m}0}} & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & \frac{\partial f_{10}^{11}}{\partial \mu_{\bar{m}0}} & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & \frac{\partial f_{10}^{11}}{\partial \mu_{\bar{m}0}} & 1
\end{pmatrix}
\]

First we reduce the Lax pair (3.12) to the appropriate form (3.10) and (3.2) (or to the form (3.11) and (3.3)) by means of the transformation

\[ \Phi = HY \quad (\text{or} \quad \Phi = HY), \]

where the factor \( H \) is lower (respectively \( H \) is upper) block-diagonal matrix

\[ H = \begin{pmatrix}
H_{11} & 0 \\
H_{21} & H_{22}
\end{pmatrix}, \quad \left( \text{or} \quad H = \begin{pmatrix}
H_{11} & H_{12} \\
0 & H_{22}
\end{pmatrix} \right). \]

Here \( H_{11}, H_{22}, H_{11}, H_{22} \) are diagonal matrices, some entries of which might depend on the spectral parameter \( \xi = \lambda^{2N} \) (or \( \zeta = \lambda^{-2N} \)). Examples show that these factors are effectively found. Below, for simplicity, we illustrate all the arguments and calculations using the example of \( N = 3 \).

**Example 1.** For \( N = 3 \) the quad system (2.1) reads as

\[
\begin{align*}
\begin{cases}
\alpha_1 f_{11}^{01} - f_{10}^{11} f_{01}^{11} = b_1 f_{11}^{01} f_{11}^{01}, \\
\alpha_2 f_{11}^{01} - f_{10}^{11} f_{01}^{11} = b_2 f_{11}^{01} f_{11}^{01}, \\
\alpha_3 f_{11}^{01} - f_{10}^{11} f_{01}^{11} = b_3 f_{11}^{01} f_{11}^{01},
\end{cases}
\end{align*}
\]

(13.3)
System (3.13) corresponds to the algebra $A_2^{(1)}$ and relates with the Lax pair
\[ \Phi_{1,0} = F \Phi, \quad \Phi_{0,1} = G \Phi, \] (3.14)
where
\[ F = \begin{pmatrix} \frac{\partial \omega_0}{\partial \mu_0} & -1 & 0 \\ \frac{\partial \omega_0}{\partial \nu_0} & \frac{\partial \nu_0}{\partial \mu_0} & -1 \\ \frac{\partial \omega_0}{\partial \nu_0} & \lambda & \frac{\partial \nu_0}{\partial \nu_0} \end{pmatrix}, \quad G = \begin{pmatrix} a_{\delta_{12}} \frac{\partial \omega_1}{\partial \mu_1} & b_{\delta_{12}} \frac{\partial \omega_1}{\partial \nu_1} & b_{\delta_{12}} \frac{\partial \nu_1}{\partial \mu_1} \\ \frac{\partial \nu_1}{\partial \mu_1} & a_{\delta_{12}} \frac{\partial \omega_1}{\partial \nu_1} & b_{\delta_{12}} \frac{\partial \nu_1}{\partial \nu_1} \\ -
\end{pmatrix}. \]

Let us perform the formal diagonalization (really block-diagonalization) procedure to the system (3.14) around the singular values $\lambda = 0$ and $\lambda = \infty$ of the spectral parameter. We begin with the case $\lambda = \infty$. By changing the variables
\[ \Phi = H Y, \]
where
\[ H = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial \omega_0}{\partial \mu_0} & \xi^{-1} & 0 \\ \frac{\partial \omega_0}{\partial \nu_0} & 0 & 1 \end{pmatrix}, \quad \xi = \sqrt{\lambda}, \]
we reduce (3.14) to the form
\[ Y_{1,0} = f Y, \quad Y_{0,1} = g Y. \] (3.15)
Here $f = f(-1) \xi + f(0) + f(1) \xi^{-1}, g = g(0) + g(1) \xi^{-1} + g(2) \xi^{-2} + g(3) \xi^{-3},$

\[ f(-1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad f(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \omega_0}{\partial \mu_0} + \frac{\partial \nu_0}{\partial \nu_0} \\ 0 & 0 & \frac{\partial \nu_0}{\partial \nu_0} \end{pmatrix}, \]

\[ f(1) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g(0) = \begin{pmatrix} a_{\delta_{12}} \frac{\partial \omega_1}{\partial \mu_1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \]

\[ g(1) = \begin{pmatrix} 0 & 0 & 0 \\ -a_{\delta_{12}} \frac{\partial \omega_1}{\partial \mu_1} & 0 & 0 \\ 0 & 0 & \frac{\partial \nu_1}{\partial \nu_0} \end{pmatrix}, \]

\[ g(2) = \begin{pmatrix} \frac{\partial \omega_0}{\partial \mu_0} \frac{\partial \omega_1}{\partial \mu_1} & 0 & 0 \\ 0 & \frac{\partial \nu_1}{\partial \mu_0} & 0 \\ 0 & 0 & \frac{\partial \nu_1}{\partial \nu_0} \end{pmatrix}, \quad g(3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]
Obviously the leading term $f(-1)$ of the potential $f$ at $\xi \to \infty$ is of the necessary block-diagonal form. According to the above scheme, there is a formal series

$$T = E + T^{(1)} \xi^{-1} + T^{(2)} \xi^{-2} + \ldots$$

with the coefficients having the following block structure

$$T^{(i)} = \begin{pmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad i \geq 1,$$

such that replacement $Y = T\varphi$ brings the system (3.15) to a block-diagonal form

$$\varphi_{1,0} = h\varphi, \quad \varphi_{0,1} = S\varphi,$$

(3.16)

where the potentials $h$ and $S$ are also formal series:

$$h = h^{(-1)} \xi + h^{(0)} + h^{(1)} \xi^{-1} + h^{(2)} \xi^{-2} + \ldots,$$

$$S = S^{(0)} + S^{(1)} \xi^{-1} + S^{(2)} \xi^{-2} + \ldots.$$

Here we assume that the potential $h$ has a block-diagonal form

$$h^{(i)} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad i \geq -1.$$

Then we can check that $S^{(j)}$ for all $j \geq 0$ has the same block-diagonal matrix structure

$$S^{(j)} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Omitting the calculations we give several members of these series

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ a_{i,0}^2 \xi^{(2)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xi^{-1}$$

$$+ \begin{pmatrix} 0 & 0 & a_{1,1} \xi^{(1)} \\ 0 & 0 & 0 \\ a_{i,0}^2 (t_{1,0}^2 t_{1,0}^2 + t_{1,0} t_{1,0}^2 t_{1,0} t_{1,0}) & 0 & 0 \end{pmatrix} \xi^{-2}$$

$$+ \begin{pmatrix} 0 & 0 & a_{1,1}^2 \xi^{(2)} \\ a_{i,0}^2 (t_{1,0}^2 t_{1,0}^2 + t_{1,0} t_{1,0}^2 t_{1,0} t_{1,0}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xi^{-3} + \ldots.$$
\[ h = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right) \xi + \left(\begin{array}{ccc} 0 & 0 & 0 \\ \frac{a_{1}t_{1}^{2}}{r_{1}^{2}t_{1}^{2}} + \frac{a_{1}t_{1}}{r_{1}t_{1}^{2}} & 0 & 0 \\ 0 & 0 & \frac{\alpha_{1}t_{1}^{2}}{r_{1}t_{1}^{2}} \end{array}\right) + \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r_{1}t_{1}^{2}} \end{array}\right) \xi^{-1} \\
+ \left(\begin{array}{ccc} \frac{a_{1}t_{1}^{2}r_{1}^{2}}{r_{1}^{2}t_{1}^{2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{a_{1}t_{1}^{2}r_{1}^{2}}{r_{1}^{2}t_{1}^{2}} \end{array}\right) \xi^{-2} + \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{a_{1}t_{1}^{2}}{r_{1}t_{1}^{2}} \end{array}\right) \xi^{-3} \\
+ \left(\begin{array}{ccc} -\frac{a_{1}t_{1}}{r_{1}^{2}} & \frac{a_{1}t_{1}^{2}r_{1}^{2}}{r_{1}^{2}t_{1}^{2}} & 0 \\ \frac{a_{1}t_{1}^{2}r_{1}^{2}}{r_{1}^{2}t_{1}^{2}} & 0 & 0 \\ 0 & 0 & \frac{a_{1}t_{1}^{2}}{r_{1}t_{1}^{2}} \end{array}\right) \xi^{-4} + \ldots, \]

where \( h_{33}^{4} = \frac{a_{12}r_{1}^{2}r_{1}^{1}}{t_{1}^{2}t_{1}^{1}} + \frac{a_{12}r_{1}^{2}r_{1}^{1}}{t_{1}^{2}t_{1}^{1}} + \frac{a_{12}r_{1}^{2}r_{1}^{1}}{t_{1}^{2}t_{1}^{1}} \),

\[ S = \left(\begin{array}{ccc} a_{12}r_{1}^{2}r_{1}^{1} & 0 & 0 \\ 0 & 0 & -\frac{a_{12}r_{1}^{2}r_{1}^{1}}{t_{1}^{2}t_{1}^{1}} \\ 0 & 0 & 0 \end{array}\right) \xi^{-1} + \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -\frac{a_{12}r_{1}^{2}r_{1}^{1}}{t_{1}^{2}t_{1}^{1}} \\ 0 & 0 & 0 \end{array}\right) \xi^{-3} + \left(\begin{array}{ccc} -\frac{a_{12}r_{1}^{2}r_{1}^{1}}{t_{1}^{2}t_{1}^{1}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{a_{12}r_{1}^{2}r_{1}^{1}}{t_{1}^{2}t_{1}^{1}} \end{array}\right) \xi^{-4} + \ldots. \]

The consistency condition of the system (3.16) can be written as \( D_{n}(S)h = D_{n}(h)S \). Due to the representation

\[ h = \left(\begin{array}{cc} h_{11} & 0 \\ 0 & h_{22} \end{array}\right), \quad S = \left(\begin{array}{cc} S_{11} & 0 \\ 0 & S_{22} \end{array}\right) \]

the latter implies

\[ (D_{n} - 1) \log \det(S_{ii}) = (D_{m} - 1) \log \det(h_{ii}), \quad i = 1, 2. \] (3.17)

We can derive an infinite set of the local conservation laws from the equation (3.17). Let us give some of them:

1. \( (D_{n} - 1) \left( -\frac{\beta_{1}r_{1}^{1}}{t_{1}^{2}t_{1}^{1}} \right) = (D_{m} - 1) \left( \frac{t_{1}^{2}}{t_{1}^{2}t_{1}^{1}} + \frac{t_{1}^{2}t_{1}^{1}}{t_{1}^{2}t_{1}^{1}} + \frac{t_{1}^{2}t_{1}^{1}}{t_{1}^{2}t_{1}^{1}} \right) \)

2. \( (D_{n} - 1) \left( -\frac{\beta_{1}r_{1}^{1}}{t_{1}^{2}t_{1}^{1}} - \frac{\beta_{1}r_{1}^{1}}{t_{1}^{2}t_{1}^{1}} - \frac{1}{2} \frac{\beta_{1}r_{1}^{1}}{t_{1}^{2}t_{1}^{1}} \right) \)

\( = (D_{m} - 1) \left( \frac{t_{1}^{2}t_{1}^{1}}{t_{1}^{2}t_{1}^{1}} + \frac{t_{1}^{2}t_{1}^{1}}{t_{1}^{2}t_{1}^{1}} + \frac{t_{1}^{2}t_{1}^{1}}{t_{1}^{2}t_{1}^{1}} + \frac{1}{2} \left( \frac{t_{1}^{2}t_{1}^{1}}{t_{1}^{2}t_{1}^{1}} + \frac{t_{1}^{2}t_{1}^{1}}{t_{1}^{2}t_{1}^{1}} + \frac{t_{1}^{2}t_{1}^{1}}{t_{1}^{2}t_{1}^{1}} \right) \right). \)
3. \((D_n - 1) \left( \frac{\mu_1}{\tau_1 t_0} - \frac{a_1 h_1}{\mu_1 t_1} + \frac{b_1 d_1 h_1}{\tau_1 t_0} \right) = (D_m - 1) \left( \frac{a_2 h_1}{\tau_1 t_0} + \frac{b_2 d_1 h_1}{\tau_1 t_0} \right) + \frac{1}{2} \left( \frac{b_1 d_1 h_1}{\tau_1 t_0} \right)^2 \right)

where

\[
\Phi = \bar{g} + \bar{f} \chi = \bar{g} + \bar{f} \chi^{-1} + \bar{f} \chi^{-2} + \bar{f} \chi^{-3},
\]

and arrive at the system

\[
Y_{1,0} = \bar{f} Y, \quad Y_{0,1} = \bar{g} Y
\]

with potentials

\[
\bar{g} = \bar{g}(-1) \chi + \bar{g}(0) \chi^{-1} + \bar{g}(1) \chi^{-2} + \bar{g}(2) \chi^{-3},
\]

where

\[
\bar{g}(-1) = \begin{pmatrix} 0 & \frac{\rho_1 \rho_2}{\tau_1 t_1} & 0 \\ \frac{\rho_2 \rho_1}{\tau_1 t_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{g}^{(0)} = \begin{pmatrix} 0 & 1 + \frac{\rho_1 \rho_2}{\tau_1 t_1} & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

and

\[
\bar{f}^{(1)} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{f}^{(2)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},
\]

Now the singular point is \(\zeta = \infty\) and the coefficient \(\bar{g}^{(-1)}\) at \(\zeta\) is of the necessary block-diagonal form. Therefore we can find the formal series \(T\) and \(h\) from the equation

\[
T_0 h = (\bar{g}^{(-1)} \chi + \bar{g}^{(0)} + \bar{g}^{(1)} \chi^{-1}) T.
\]
Then knowing $T$ we find the series $S$ from

$$\bar{T}_{1,0}S = f\bar{T}.$$  

As a result we obtain

$$\bar{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\frac{\partial_{l_1}^{(1)}(\zeta_{0})^2}{\partial l_2^{(1)} \partial l_0^{(1)}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \zeta^{-1}$$

$$+ \begin{pmatrix} \frac{\partial_{l_1}^{(1)}(\zeta_{0})^2}{\partial l_2^{(1)} \partial l_0^{(1)}} - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \zeta^{-2}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial_{l_1}^{(1)}(\zeta_{0})^2}{\partial l_2^{(1)} \partial l_0^{(1)}} - \frac{\partial_{l_1}^{(1)}(\zeta_{0})^2}{\partial l_2^{(1)} \partial l_0^{(1)}} \zeta^{-3} + \ldots \end{pmatrix}$$

$$\bar{h} = \begin{pmatrix} 0 & \frac{\partial_{l_2}^{(1)}(\zeta_{0})^2}{\partial l_0^{(1)}} & 0 \\ 0 & \frac{\partial_{l_2}^{(1)}(\zeta_{0})^2}{\partial l_0^{(1)}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \zeta + \begin{pmatrix} \frac{\partial_{l_1}^{(1)}(\zeta_{0})^2}{\partial l_2^{(1)} \partial l_0^{(1)}} & 0 & 0 \\ 0 & 1 + \frac{\partial_{l_1}^{(1)}(\zeta_{0})^2}{\partial l_2^{(1)} \partial l_0^{(1)}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \zeta^{-1}$$

$$+ \begin{pmatrix} 0 & \frac{\partial_{l_1}^{(1)}(\zeta_{0})^2}{\partial l_2^{(1)} \partial l_0^{(1)}} & 0 \\ 0 & \frac{\partial_{l_1}^{(1)}(\zeta_{0})^2}{\partial l_2^{(1)} \partial l_0^{(1)}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \zeta^{-2}$$

$$+ \begin{pmatrix} 0 & \frac{\partial_{l_1}^{(1)}(\zeta_{0})^2}{\partial l_2^{(1)} \partial l_0^{(1)}} & 0 \\ 0 & \frac{\partial_{l_1}^{(1)}(\zeta_{0})^2}{\partial l_2^{(1)} \partial l_0^{(1)}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \zeta^{-3} + \ldots$$

$$\bar{S} = \begin{pmatrix} \frac{\partial_{l_2}^{(1)}(\zeta_{0})^2}{\partial l_0^{(1)}} & 0 & 0 \\ 0 & \frac{\partial_{l_2}^{(1)}(\zeta_{0})^2}{\partial l_0^{(1)}} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \zeta^{-1}$$

$$+ \begin{pmatrix} \frac{\partial_{l_1}^{(1)}(\zeta_{0})^2}{\partial l_2^{(1)} \partial l_0^{(1)}} & 0 & 0 \\ 0 & \frac{\partial_{l_1}^{(1)}(\zeta_{0})^2}{\partial l_2^{(1)} \partial l_0^{(1)}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \zeta^{-2}$$

$$+ \begin{pmatrix} \frac{\partial_{l_1}^{(1)}(\zeta_{0})^2}{\partial l_2^{(1)} \partial l_0^{(1)}} & 0 & 0 \\ 0 & \frac{\partial_{l_1}^{(1)}(\zeta_{0})^2}{\partial l_2^{(1)} \partial l_0^{(1)}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \zeta^{-3} + \ldots$$
Passing to the blocks
\[ \overline{h} = \begin{pmatrix} h_{11} & 0 \\ 0 & h_{22} \end{pmatrix}, \quad \overline{S} = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix} \]

we can write the relations
\[ (D_m - 1) \log \det(S_i) = (D_m - 1) \log \det(h_i), \quad i = 1, 2, \]
from which we can derive the local conservation laws:

1. \( (D_m - 1) \left( \frac{r_{i0}^2 r_{i1}^2}{m_{i1}^2} + \frac{r_{i0}^2 r_{i1}^2}{h_{i1}^2} + \frac{r_{i0}^2 r_{i1}^2}{m_{i1}^2} \right) = (D_m - 1) \left( \frac{r_{i0}^2 r_{i1}^2}{m_{i1}^2} \right) \)

2. \( (D_m - 1) \left( \frac{(t_{i1} r_{i1})^2}{m_{i1}^2} - \frac{r_{i0}^2 r_{i1}^2}{h_{i1}^2} + \frac{r_{i0}^2 r_{i1}^2}{m_{i1}^2} \right) + \frac{1}{2} \left( \frac{(t_{i1} r_{i1})^2}{m_{i1}^2} \right) \)

3. \( (D_m - 1) \left( \frac{a_{i0}^2 + r_{i0}^2 - b_{i0}^2}{h_{i0}^2} \right) \left( \frac{a_{i1}^2 + r_{i1}^2 - b_{i1}^2}{h_{i1}^2} \right) + \frac{1}{2} \left( \frac{(t_{i1} r_{i1})^2}{m_{i1}^2} \right) \)

4. Higher symmetries

Quad system (2.1) possess higher symmetries. However, presented in the variables \( t_{i0}^2 \), the symmetries have non-localities. They become local in new variables introduced by potentiation or, more precisely, by setting \( r_{i1}^j = \frac{t_{i1}^j}{t_{i0}^j} \). Actually in terms of these variables quad system (2.1) turns into

\[ ar_{i1}^1 = r_{i0}^1 + r_{i0}^2 r_{i1}^1 \left( \frac{m_{i0}}{h_{i0}} - \frac{1}{h_{i0}} \right), \]
\[ ar_{i1}^1 = r_{i0}^1 + r_{i0}^2 r_{i1}^1 \left( \frac{m_{i0}}{h_{i0}} - \frac{1}{h_{i0}} \right), \quad 2 \leq j \leq N - 1, \]
\[ ar_{i1}^N = r_{i0}^N + r_{i0}^2 r_{i1}^N \left( \frac{m_{i0}}{h_{i0}} - \frac{1}{h_{i0}} \right). \]

More precisely in what follows we present higher symmetries to this quad system. We note that obtained system (4.1) does not already contain the parameter \( b \). Everywhere below we write \( r^j \) instead of \( r_{i0}^j \).

The Lax pair of the system (4.1) is written as

\[ \Phi_{0,1} = F \Phi, \quad \Phi_{1,0} = G \Phi, \]

where \( F \) and \( G \) are matrices:
Example 2. Consider the system (4.1) for $N = 2$. For the sake of simplicity here we use notations $u^t := r^1, v^t := r^2$

\[au_{1,1} = u_{1,0} + v_{0,1} \left( \frac{2}{a} - \frac{1}{m_{1,1}} \right), \]
\[av_{1,1} = v_{1,0} + u_{2,0} \left( \frac{2}{a} - \frac{1}{m_{2,0}} \right). \]  

(4.4)

The simplest higher symmetry of (4.4) in the direction of $n$ is given by

\[u^t = v + \frac{a}{v_{1,0}} u_{1,0}, \]
\[v^t = au_{1,0} + \frac{v^2}{v_{1,0}}. \]  

(4.5)

The symmetry admits Lax pair $\Phi_{1,0} = F\Phi, \Phi_{1,t} = A\Phi$ where

\[F = \begin{pmatrix} \frac{2}{a} & -1 & 0 & \ldots & 0 & 0 \\ \frac{2}{a} & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\lambda \frac{2}{a} & \lambda & 0 & \ldots & 0 & \frac{ar_{1,0}}{r^2} \end{pmatrix}, \]
\[A = \begin{pmatrix} \frac{2}{a} & \lambda & \frac{2}{a} & \ldots & 0 & 0 \\ \frac{2}{a} & \lambda & \frac{2}{a} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2}{a} & \lambda & \frac{2}{a} & \ldots & 0 & 0 \end{pmatrix}. \]

For the particular case $a = 1$, quad system (4.4) and also its symmetry (4.5) have already been found in [1]. We also indicate the symmetry in the direction of $m$ [33]:

\[u^t = \frac{u^2}{m_{1,0} - 1}, \]
\[v^t = \frac{u_{1,0}}{m_{1,0} - 1}. \]  

(4.6)

with the Lax pair $\Phi_{0,1} = G\Phi, \Phi_{t} = B\Phi$ where

\[G = \begin{pmatrix} \frac{1}{a} & \frac{(m_{1,1} - u)(m_{1,1} - v)}{m_{1,1} - u} & \frac{m_{1,1} - u}{\lambda_m} \\ \frac{m_{1,1} - v}{m_{1,1} - u} & \frac{m_{1,1} - v}{\lambda_{m_{1,1} - v}} & \frac{m_{1,1} - v}{\lambda} \end{pmatrix}, \]
\[B = \begin{pmatrix} \frac{a}{m_{1,1} - v} & \frac{a}{m_{1,1} - v} & \frac{a}{\lambda(m_{1,1} - v)} \\ \frac{a}{m_{1,1} - v} & \frac{a}{m_{1,1} - v} & \frac{a}{\lambda} \end{pmatrix}. \]

For the special choice $a = 1$ of the parameter the quad system (4.4) and its symmetries take the most simple form. Namely the coupled lattice (4.5) in that case is converted to a scalar lattice.
\[ R_{n,m,t} = R_{n+1,m} + \frac{R^2_{n,m}}{R_{n-1,m}}. \]

where \( R_{2n,m} = u_{n,m} \) and \( R_{2n+1,m} = v_{n,m} \). Hence the desire to transform the discrete system itself to a scalar equation. However, the symmetry (4.6) in the other direction is reduced to a scalar form

\[ p_{n,m,\tau} = \frac{1}{P_{n,m-1} - P_{n,m+1}} \]

under the different transformation \( P_{n,2m} = \frac{1}{r_m} \). \( P_{n,2m-1} = v_{n,m} \). Therefore the system of equation (4.4) is not reduced to a scalar autonomous equation [34]. If \( a \neq 1 \) then none of the symmetries are reduced to an autonomous scalar lattice.

**Example 3.** Now we represent the higher symmetries to the system (4.1) for \( N = 3 \). Here we use notations \( u^1 = r^1 \), \( v^1 := r^2 \) and \( w^1 := r^3 \)

\[ \begin{align*}
    au_{1,0} &= u_{1,0} + v_{0,1}w_{0,1} \left( \frac{a}{u} - \frac{1}{w_{1,0}} \right), \\
    av_{1,0} &= v_{1,0} + u_{0,1}w_{0,1} \left( \frac{a}{v} - \frac{1}{w_{1,0}} \right), \\
    aw_{1,0} &= w_{1,0} + u_{0,1}v_{0,1} \left( \frac{a}{w} - \frac{1}{w_{1,0}} \right). \\
\end{align*} \]

(4.7)

The Lax pair to the system takes the form

\[ \Phi_{1,0} = F \Phi, \quad \Phi_{0,1} = G \Phi, \]

(4.8)

where

\[ \begin{pmatrix}
    \frac{a}{u} - \frac{1}{w_{1,0}} & 0 & 0 \\
    0 & \frac{a}{v} - \frac{1}{w_{1,0}} & 0 \\
    -\frac{\lambda}{u} & \frac{\lambda}{w_{1,0}} & \frac{1}{w_{1,0}} \\
\end{pmatrix}, \quad \begin{pmatrix}
    \frac{a}{u} - \frac{1}{w_{1,0}} & \frac{a}{v} - w_{0,1} & \frac{a}{w} - w_{0,1} \\
    \frac{a}{u} - w_{0,1} & 1 & 0 \\
    \frac{a}{v} - w_{0,1} & 0 & 1 \\
\end{pmatrix}. \]

The simplest higher symmetry of the system (4.7) is given by

\[ \begin{align*}
    u_t &= w + \frac{aw}{w_{-1,0}} + \frac{a^2}{w_{-1,0}}, \\
    v_t &= au_{1,0} + \frac{aw}{v} + \frac{a^2}{w_{-1,0}}, \\
    w_t &= av_{1,0} + \frac{aw}{w} + \frac{a^2}{w}. \\
\end{align*} \]

(4.9)

The linear system of the form

\[ \Phi_{1,0} = F \Phi, \quad \Phi_{0,1} = A \Phi \]

is a Lax pair for (4.9), where \( F \) is given in (4.8) and \( A \) as

\[ \begin{pmatrix}
    \frac{a}{u} - \frac{1}{w_{-1,0}} & \frac{a}{v} - w_{-1,0} & 1 \\
    0 & \frac{a}{w} - w_{-1,0} & \frac{a}{w} \\
    \frac{a}{u} - w_{-1,0} & 0 & \frac{a}{w} \\
\end{pmatrix}. \]

The quad system (4.7) admits also the classical symmetries

\[ u_t = u, \quad v_t = v, \quad w_t = w \]

and
\[ u_t = (-1)^n u, \quad v_t = (-1)^n v, \quad w_t = (-1)^n w. \]

In the direction of \( n \) the system has a symmetry of the form
\[ 
\begin{align*}
\nu_t & = (\frac{\partial}{\partial n}) (\frac{\partial}{\partial n}) v, \\
\nu & = (\frac{\partial}{\partial n}) (\frac{\partial}{\partial n}) v, \\
w_t & = (\frac{\partial}{\partial n}) (\frac{\partial}{\partial n}) w.
\end{align*}
\]

The symmetry is related to the Lax pair \( \Phi_{0,1} = G\Phi, \Phi_{0,1} = B\Phi \) with \( G \) and \( B \) defined by (4.8) and, respectively, by
\[ B = 
\begin{pmatrix}
\frac{uv}{(\partial n_1 - \nu)(\partial n_1 - w)} & 0 & \frac{uv}{(\partial n_1 - \nu)(\partial n_1 - w)} \\
0 & \frac{uv}{(\partial n_1 - \nu)(\partial n_1 - w)} & \frac{uv}{(\partial n_1 - \nu)(\partial n_1 - w)} \\
-\frac{uv}{(\partial n_2 - \nu)(\partial n_2 - w)} & \frac{uv}{(\partial n_2 - \nu)(\partial n_2 - w)} & \frac{uv}{(\partial n_2 - \nu)(\partial n_2 - w)}
\end{pmatrix}.
\]

**Example 4.** The next example concerns the algebra \( A_3^{(1)} \). In this case the quad system (4.1) reads as
\[ 
\begin{align*}
ar_1^{1,1} & = r_1^{1,0} + r_0^{1,0} r_{0,1}^{1,0} \left( \frac{a}{n} - \frac{1}{r_{0,1}^{1,0}} \right), \\
ar_1^{2,1} & = r_1^{2,0} + r_{0,1}^{2,0} r_1^{2,0} \left( \frac{a}{n} - \frac{1}{r_{0,1}^{2,0}} \right), \\
ar_1^{3,1} & = r_1^{3,0} + r_{0,1}^{3,0} r_1^{3,0} \left( \frac{a}{n} - \frac{1}{r_{0,1}^{3,0}} \right), \\
ar_1^{4,1} & = r_1^{4,0} + r_{0,1}^{4,0} r_1^{4,0} \left( \frac{a}{n} - \frac{1}{r_{0,1}^{4,0}} \right).
\end{align*}
\]

The Lax pair for (4.10) is given by a system of the form
\[ \Phi_{1,0} = F\Phi, \quad \Phi_{0,1} = G\Phi, \]
where the potentials are
\[ F = 
\begin{pmatrix}
\frac{a}{n} & -1 & 0 & 0 \\
0 & \frac{a}{n} & -1 & 0 \\
0 & 0 & \frac{a}{n} & -1 \\
-\lambda^2 & \lambda & 0 & \frac{a r_{0,1}^{2,0}}{n}
\end{pmatrix},
\]
\[ G = 
\begin{pmatrix}
\frac{a r_{0,1}^{2,0}}{n} & 0 & \frac{(a r_{0,1}^{2,0} - r)}{r^2} & \frac{a r_{0,1}^{2,0} - r}{r^2} & \frac{a r_{0,1}^{2,0} - r}{r^2} & \frac{a r_{0,1}^{2,0} - r}{r^2} \\
0 & \frac{a r_{0,1}^{2,0} - r}{r^2} & 1 & 0 & 0 \\
0 & 0 & \frac{a r_{0,1}^{2,0} - r}{r^2} & 1 & 0 \\
0 & 0 & 0 & \frac{a r_{0,1}^{2,0} - r}{r^2} & 1 \\
\end{pmatrix}.
\]

The following multield lattices are symmetries for the quad system (4.10)
where

\[ \Phi_{1,0} = F \Phi, \Phi_t = A \Phi \text{ and } \Phi_{0,1} = G \Phi, \Phi_\tau = B \Phi \] are given by

\[ A = \begin{pmatrix} \alpha_{1}^{(1)} \tau_{1,0} + \lambda & \alpha_{1}^{(1)} \tau_{1,0} & \alpha_{1}^{(1)} \tau_{1,0} & 1 \\ \frac{\tau_{1,0}}{\tau} \lambda & \frac{\tau_{1,0}}{\tau} \lambda & \frac{\tau_{1,0}}{\tau} \lambda & 0 \\ \frac{\tau_{1,0}}{\tau} \lambda & 0 & \frac{\tau_{1,0}}{\tau} \lambda & 0 \\ \frac{\tau_{1,0}}{\tau} \lambda & 0 & 0 & \frac{\tau_{1,0}}{\tau} \lambda \end{pmatrix}, \]

\[ B = \begin{pmatrix} \frac{-r^{2} r^{2} r^{2}}{P^{(3)}(r^{2}, r^{2}, r^{2})} & 0 & 0 & \frac{-r^{2} r^{2} r^{2} \lambda^{-1}}{P^{(3)}(r^{2}, r^{2}, r^{2})} \\ \frac{-r^{1} r^{1} r^{1}}{P^{(1)}(r^{1}, r^{1}, r^{1})} & 0 & 0 & \frac{-r^{1} r^{1} r^{1} \lambda^{-1}}{P^{(1)}(r^{1}, r^{1}, r^{1})} \\ \frac{-r^{2} r^{2} r^{2}}{P^{(2)}(r^{2}, r^{2}, r^{2})} & 0 & 0 & \frac{-r^{2} r^{2} r^{2} \lambda^{-1}}{P^{(2)}(r^{2}, r^{2}, r^{2})} \\ \frac{-r^{1} r^{1} r^{1}}{P^{(1)}(r^{1}, r^{1}, r^{1})} & 0 & 0 & \frac{-r^{1} r^{1} r^{1} \lambda^{-1}}{P^{(1)}(r^{1}, r^{1}, r^{1})} \end{pmatrix}, \]

where \( P^{(3)}(h, k, l) = (a h_{0,1} - h)(a k_{0,1} - k)(a l_{0,1} - l), P^{(2)}(h, k) = (a h_{0,1} - h)(a k_{0,1} - k). \)

**Example 5.** Finally, we concentrate on the general case of the system (4.1). Symmetries of the system are given as

\[ r_{1}^{1} = r^{N} + \frac{a(r^{1})^{2}}{r_{1,0}^{2}} + \frac{a(r^{1})^{2}}{r_{1,0}^{2}} + \frac{a r^{1} r^{1}}{r_{1,0}^{2}} + \cdots + \frac{a r^{1} r^{1}}{r_{1,0}^{2}}, \]

\[ r_{2}^{2} = a r_{1,0}^{2} + \frac{a(r^{2})^{2}}{r_{2,0}^{2}} + \frac{a(r^{2})^{2}}{r_{2,0}^{2}} + \frac{a r^{2} r^{2}}{r_{2,0}^{2}} + \cdots + \frac{a r^{2} r^{2}}{r_{2,0}^{2}} + 2 r^{2}, \]

\[ r_{3}^{3} = a r_{1,0}^{2} + \frac{a(r^{3})^{2}}{r_{3,0}^{2}} + \frac{a(r^{3})^{2}}{r_{3,0}^{2}} + \frac{a r^{3} r^{3}}{r_{3,0}^{2}} + \cdots + \frac{a r^{3} r^{3}}{r_{3,0}^{2}} + 2 r^{3}, \]

\[ \cdots, \]

\[ r_{N}^{N} = a r_{1,0}^{2} + \frac{a(r^{N})^{2}}{r_{N,0}^{2}} + \frac{a(r^{N})^{2}}{r_{N,0}^{2}} + \frac{a r^{N} r^{N}}{r_{N,0}^{2}} + \cdots + \frac{a r^{N} r^{N}}{r_{N,0}^{2}} + 2 r^{N}, \]
In this section we discuss the recursion operators for the quad systems associated with $A^{(1)}_{N-1}$.

5. Evaluation of the recursion operators

Proposition 3. The systems (4.1) and (4.11) are not related by a point transformation.
operators, based on the Lax representation [35, 36], Hamiltonian operators [37, 38] and generalized invariant manifolds [39]. In our opinion the most convenient of them is one using the Lax pair, moreover in some cases it is reasonable to derive the bi-Hamiltonian structure from the recursion operator.

Since the quad system admits two hierarchies of symmetries, it admits two recursion operators as well, corresponding to the variables $n$ and $m$. Below we concentrate on that related to $n$. To study the problem we use the ideas of [35, 36].

It can be shown that the procedure of the formal diagonalization allows us to construct a formal series (see [28])

$$A = A^{(0)} + A^{(1)}\lambda^{-1} + A^{(2)}\lambda^{-2} + \ldots$$

commuting with the operator $L = D_n^{-1}F$ such that

$$[L, A] := LA - AL = 0,$$

where $F = F^{(0)} + F^{(1)}\lambda$ is the potential of the Lax equation (4.2) and the expression $D_n^{-1}F$ means a composition of the backward shift operator $D_n^{-1}$ and the operation of multiplication by $F$.

The series $A$ provides an effective tool for constructing the higher symmetries of the quad system (4.1). To explain the method we consider the formal series $B$ obtained from $A$ by multiplying by $\lambda^k$, where $k$ is a positive integer:

$$B := \lambda^k A = B^{(k)}\lambda^k + B^{(k-1)}\lambda^{k-1} + \ldots$$ (5.1)

Then we take a polynomial part of the series (5.1):

$$B_+ = B^{(k)}\lambda^k + B^{(k-1)}\lambda^{k-1} + \ldots + B^{(1)}\lambda + B^{(0)}.$$

Here the last summand is chosen in a nontrivial way. Its upper diagonal part coincides with that of the matrix $B^{(0)}$. The other entries vanish except one located at the left upper corner, denoted by $x$. The crucial point is that the consistency condition of the linear equation

$$\frac{dy}{d\tau} = B_+ y$$ (5.2)

with (4.2) gives exactly $N + 1$ equations which allow us to determine the unknown $x$ and to derive a symmetry of the quad system (4.1) with time $\tau$.

In addition to (5.1) we take one more series $C := \lambda^{k+1}A$, such that

$$C = C^{(k+1)}\lambda^{k+1} + C^{(k)}\lambda^{k} + C^{(k-1)}\lambda^{k-1} + \ldots$$ (5.3)

By applying the algorithm used above to (5.3) we get the polynomial

$$C_+ = C^{(k+1)}\lambda^{k+1} + C^{(k)}\lambda^{k} + \ldots + C^{(1)}\lambda + C^{(0)}_+,$$

which provides the time part of the Lax pair

$$\frac{dy}{d\tau_1} = C_+ y.$$ (5.4)

Our goal now is to find the relation between two symmetries defined by the polynomials $B_+, C_+$. Evidently we have

$$C - \lambda B = 0.$$ (5.5)

We replace in (5.5) the summands $B$ and $C$ with their representations (5.1) and (5.3). As a result we get
\[ C^{(k+1)}\lambda^{k+1} + \cdots + C^{(1)}\lambda + C^{(0)}_+ + C^{(-1)}_-\lambda^{-1} + \cdots = -\lambda \left( B^{(k)}\lambda^k + \cdots + B^{(1)}\lambda + B^{(0)}_+ + B^{(-1)}_-\lambda^{-1} + \cdots \right) = 0, \quad (5.6) \]

where \( B^{(0)} := B^{(0)} - B^{(0)}_+ \), \( C^{(0)} := C^{(0)}_+ \).

Let us rewrite (5.6) as follows:

\[ R_N := C^{(k+1)}\lambda^{k+1} + \cdots + C^{(1)}\lambda + C^{(0)}_+ - \lambda \left( B^{(k)}\lambda^k + \cdots + B^{(1)}\lambda + B^{(0)}_+ \right) = -C^{(0)}_+ - C^{(-1)}_-\lambda^{-1} - \cdots + \lambda \left( B^{(0)}_+ + B^{(-1)}_-\lambda^{-1} + \cdots \right). \]

It is easily checked that \( R_N \) is a linear function of \( \lambda \):

\[ R_N = B^{(0)}_+ \lambda + C^{(0)} := r\lambda + s. \quad (5.7) \]

The consistency conditions of the equations (4.2), (5.2) and (4.2), (5.4) can be written in the form

\[ L_{\tau_1} = [L, C_+] \quad \text{and} \quad L_{\tau} = [L, B_+]. \]

Therefore we obtain

\[ L_{\tau_1} - \lambda L_{\tau} = [L, C_+ - \lambda B_+] = [L, R_N]. \]

Since \( L = (\alpha + \beta\lambda)D^{-1}_n \) where \( \alpha = D^{-1}_n(F^{(0)}) \) and \( \beta = D^{-1}_n(F^{(1)}) \) (recall that \( F = F^{(0)} + F^{(1)}\lambda \)) then due to (5.7) we have an equation

\[ (\alpha_{\tau_1} + (\beta_{\tau_1} - \alpha_{\tau}) \lambda - \beta_{\tau} \lambda^2)D^{-1}_n = [(\alpha + \beta\lambda)D^{-1}_n, r\lambda + s]. \]

which produces a system of the equations

\[ \begin{aligned}
\alpha_{\tau_1} &= \alpha s_{-1} - s\alpha, \\
\beta_{\tau_1} - \alpha_{\tau} &= \alpha r_{-1} + \beta s_{-1} - r\alpha - s\beta, \\
\beta_{\tau} &= r\beta - \beta r_{-1}
\end{aligned} \quad (5.8) \]

for determining unknown coefficients \( r \) and \( s \) and a relation between the symmetries \( \{r^j\}_{j=1}^N \) and \( \{s^j\}_{j=1}^N \), which should allow us to derive the recursion operator.

### 5.1. Examples

Let us construct the recursion operator in the direction of \( n \) for a particular case of the system (4.1) for \( N = 2 \). In an explicit form the system is presented in (4.4). In this case, functions \( \alpha \) and \( \beta \) used above have the form

\[ \begin{pmatrix}
\frac{\mu_{-1}n}{\nu_{-1}n} & -1 \\
0 & \frac{\nu_{-1}n}{\mu_{-1}n}
\end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (5.9) \]

We look for the coefficients \( r \) and \( s \) in the form

\[ r = \begin{pmatrix} r^{(11)} & 0 \\ r^{(21)} & 0 \end{pmatrix}, \quad s = \begin{pmatrix} s^{(11)} & s^{(12)} \\ 0 & s^{(22)} \end{pmatrix}. \quad (5.10) \]
Let us rewrite system (5.8) taking into account (5.9) and (5.10):

(1) \( \left( \frac{u_{−1,0}}{v_{−1,0}} \right) τ_1 - \left( \frac{u_{−1,0}}{v_{−1,0}} \right) \left( s^{(11)}_{−1,0} - s^{(11)} \right) = 0, \)

(2) \( s^{(22)}_{−1,0} - s^{(11)} + \frac{v}{u_{−1,0}} s^{(12)} - \frac{u_{−1,0}}{v_{−1,0}} s^{(12)} = 0, \)

(3) \( \left( \frac{v}{u_{−1,0}} \right) τ_1 - \left( \frac{v}{u_{−1,0}} \right) \left( s^{(22)} - s^{(22)} \right) = 0, \)

(4) \( \left( \frac{u_{−1,0}}{v_{−1,0}} \right) τ_1 - \left( \frac{u_{−1,0}}{v_{−1,0}} \right) \left( s^{(11)}_{−1,0} + \frac{v}{u_{−1,0}} s^{(12)} + \frac{v}{u_{−1,0}} s^{(21)} \right) = 0, \)

(5) \( s^{(12)}_{−1,0} = r^{(11)} = 0, \)

(6) \( \left( \frac{u_{−1,0}}{v_{−1,0}} \right) τ_1 - \left( \frac{u_{−1,0}}{v_{−1,0}} \right) \left( s^{(11)} + s^{(22)} \right) + \frac{v}{u_{−1,0}} r^{(21)} = 0, \)

(7) \( \left( \frac{v}{u_{−1,0}} \right) τ - s^{(22)} + s^{(22)} + r^{(21)} = 0, \)

(8) \( \left( \frac{u_{−1,0}}{v_{−1,0}} \right) τ - r^{(21)}_{−1,0} + \frac{u_{−1,0}}{v_{−1,0}} r^{(11)}_{−1,0} = 0. \)

Thus, to find the necessary coefficients \( r^{(11)}, r^{(21)}(s^{(11)}, s^{(12)} \) and \( s^{(22)}, \) we obtained an over-determined system of equations. Using equation (5), we exclude the function \( s^{(12)} \) by taking \( s^{(12)} = r^{(11)}. \) Combining equations (1), (2), (4), (6) and (7) we find the function \( r^{(11)}): \)

\[
 r^{(11)} = \frac{v_r}{v}. 
\]

(5.11)

From equation (4), by virtue of (5.11), we find \( r^{(21)} = \frac{u_r}{v}. \)

We find the remaining two coefficients \( s^{(11)} \) and \( s^{(22)}. \) To do this, use the equations (1), (2) and (6). From which we obtain

\[
 s^{(22)} = \frac{u_r}{v} + \frac{v_r}{u_{−1,0}} + (D_n - 1)^{−1} \left[ \left( \frac{1}{v} - \frac{v_{1,0}}{u^2} \right) u_r + \left( \frac{1}{u_{−1,0}} - \frac{u}{v^2} \right) v_r \right],
\]

(5.12)

\[
 s^{(11)} = \frac{v_r}{u_{−1,0}} + \frac{v(u_{−1,0})_r}{u_{−1,0}} + (D_n - 1)^{−1} \left[ \left( \frac{1}{v} - \frac{v_{1,0}}{u^2} \right) u_r + \left( \frac{1}{u_{−1,0}} - \frac{u}{v^2} \right) v_r \right].
\]

(5.13)

Let us write two more equations connecting \( u_{τ_1}, v_{τ_1} \), \( u_{τ_1} \), \( v_{τ_1} \) and \( v_τ. \) Using equations (1) and (3) we obtain

\[
 u_{τ_1} = -u \left( s^{(11)}_{1,0} + s^{(22)} \right), \quad v_{τ_1} = -v \left( s^{(11)} + s^{(22)} \right).
\]

(5.14)

We substitute (5.12) and (5.13) in (5.14) and get

\[
 \left( \frac{u}{v} \right)_{τ_1} = R \left( \frac{u}{v} \right)_τ,
\]

where

\[
 R = \begin{pmatrix}
 0 & 1 \\
 0 & 0 \\
 \end{pmatrix} D_n + \frac{2}{\tau_2} \begin{pmatrix}
 2 \frac{u}{u_{−1,0}} - \frac{s^{(2)}}{v_r} \\
 1 & \frac{2 \frac{v}{u_{−1,0}} - \frac{s^{(1)}}{u_r}}{v} \\
 \end{pmatrix} + \begin{pmatrix}
 0 & 0 \\
 0 & 1 \\
 \end{pmatrix} D_n^{-1}

- 2 \left( \frac{u}{v} \right) (D_n - 1)^{−1} \left( \frac{v_{1,0}}{u^2} - \frac{1}{v^2} \frac{u}{u_{−1,0}} - \frac{1}{u_{−1,0}} \right).
\]

(5.15)
Applying the operator (5.15) to the classical symmetry \( u_\tau = u, v_\tau = v \) of the system (4.4) we obtain the higher symmetry (4.5). Operator (5.15) has been found in our article [40].

As a next example, we consider the system (4.1) with \( N = 3 \) (see (4.7)). For this system, we construct the recursion operator using the method presented above. Omitting the calculations, we give only the final answer

\[
R = R^{(0)} + S \left[ (D_n - 1)^{-1} R^{(1)} + (D_n + 1)^{-1} R^{(2)} + (D_n + 1)^{-1} A^{(1)} (D_n - 1)^{-1} R^{(3)} + (D_n + 1)^{-1} A^{(2)} (D_n + 1)^{-1} R^{(4)} \right],
\]

(5.16)

where

\[
R^{(0)} = \begin{pmatrix} 0 & 1 & \frac{u}{v} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} D_n + \left( \begin{array}{ccc} 2 \frac{w}{w-1} & -1 & \frac{w}{w-1} \\ -2 \frac{w}{w+w} & 0 & \frac{w}{w+w} \\ 2 \frac{w}{w} & 1 & \frac{w}{w} \end{array} \right)
\]

\[
+ \begin{pmatrix} \frac{w^2}{w-1} & 0 & 0 \\ \frac{w}{w-1} & \frac{w^2}{w-1} & 0 \\ \frac{w}{w-1} & \frac{w^2}{w-1} & 0 \end{pmatrix} D_n^{-1},
\]

\[
S = \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix},
\]

\[
R^{(1)} = -\frac{3}{2} \begin{pmatrix} \frac{v_0}{w} - \frac{1}{w} & 0 & 0 \\ 0 & \frac{v_1}{w} & 0 \\ 0 & 0 & \frac{w}{w-1} \end{pmatrix},
\]

\[
R^{(2)} = \begin{pmatrix} \frac{v_1}{w} & -\frac{1}{w} & \frac{w}{w-1} \\ 0 & \frac{w}{w} & \frac{w}{w} \\ 0 & \frac{w}{w} & \frac{w}{w} \end{pmatrix},
\]

\[
R^{(3)} = \begin{pmatrix} \frac{v_1}{w} & -\frac{1}{w} & \frac{w}{w-1} \\ 0 & \frac{w}{w} & \frac{w}{w} \\ 0 & \frac{w}{w} & \frac{w}{w} \end{pmatrix},
\]

\[
R^{(4)} = \begin{pmatrix} \frac{v_1}{w} & -\frac{1}{w} & \frac{w}{w-1} \\ 0 & \frac{w}{w} & \frac{w}{w} \\ 0 & \frac{w}{w} & \frac{w}{w} \end{pmatrix},
\]

\[
A^{(1)} = -\frac{1}{2} \begin{pmatrix} \frac{v_1}{w} & -\frac{1}{w} & \frac{w}{w-1} \\ 0 & \frac{w}{w} & \frac{w}{w} \\ 0 & \frac{w}{w} & \frac{w}{w} \end{pmatrix},
\]

\[
A^{(2)} = \frac{1}{2} \begin{pmatrix} \frac{v_1}{w} & -\frac{1}{w} & \frac{w}{w-1} \\ 0 & \frac{w}{w} & \frac{w}{w} \\ 0 & \frac{w}{w} & \frac{w}{w} \end{pmatrix},
\]

where \( \alpha = \frac{w}{w-1} + \frac{v_1}{w}, \beta = \frac{w}{w-1} + \frac{v_1}{w}, \gamma = \frac{w}{w-1} + \frac{v_1}{w} \).
Usually, recursion operators for integrable systems are pseudo-differential (or pseudo-difference for the case of lattices) operators with so-called weak nonlocalities [38]. However, among lattices there are several representatives which have recursion operators with a more complex nonlocalities. This case is illustrated, for example, in [37] by the Narita–Itoh–Bogoyavlensky equations. Obviously, the recursion operator (5.16) also belongs to this class, since it contains strongly non-local terms.

6. Conclusions

We conjecture that the discrete exponential type system on a quad graph (1.3) is an integrable discretization of the Drinfeld–Sokolov hierarchy [10]. Evidently (1.3) goes to (1.1) in the continuum limit for the appropriate choice of the parameters $a$ and $b$. As for the proof of integrability of (1.3), this is a much more complicated problem. So far it was proved in [20] that (1.3) is integrable in the sense of Darboux for the simple Lie algebras $A_N, B_N$. For the system (1.3) corresponding to the affine algebras $D_N^{(2)}$ and $A_1^{(1)}$ the Lax pairs were found in [1]. For the cases $A_1^{(1)}$ and $A_2^{(2)}$ the higher symmetries have been constructed [1]. For the cases $C_2, G_2, D_1$ the Darboux integrability was proved [1].

In the present article we studied the quad system corresponding to $A_1^{(1)}$. We derived for it the Lax representation, allowing us to describe the local conservation laws. We constructed the higher symmetries by using this Lax pair. We derived the recursion operator for $N = 3$, which turned out to be rather complicated.

Acknowledgments

The authors gratefully acknowledge financial support from a Russian Science Foundation Grant (project 15-11-20007).

ORCID iDs

I T Habibullin @ https://orcid.org/0000-0003-4658-9175

References

[1] Garifullin R N, Habibullin I T and Yangubaeva M V 2012 Affine and finite Lie algebras and integrable Toda field equations on discrete space-time SIGMA 8 33
[2] Darboux G 1896 Lecons Sur la Théorie générale Des Surfaces Et Les Applications Géométriques Du Calcul Infinitesimal vol 1–4 (Paris: Gauthier-Villars) p 513, p 579, p 512, p 547
[3] Toda M 1967 Vibration of a chain with nonlinear interaction J. Phys. Soc. Japan 22 431–6
[4] Mikhailov A V 1979 Integrability of a two-dimensional generalization of the Toda chain JETP Lett. 30 414–8
[5] Leznov A N 1980 On the complete integrability of a nonlinear system of partial differential equations in two-dimensional space Theor. Math. Phys. 42 225–9
[6] Mikhailov A V, Olshanetsky M A and Perelomov A M 1981 Two-dimensional generalized Toda lattice Commun. Math. Phys. 79 473–88
[7] Shabat A B and Yamilov R I 1981 Exponential systems of type I and Cartan matrices preprint, OFM BFAN SSSR, Ufa
[8] Leznov A N, Smirnov V G and Shabat A B 1982 The group of internal symmetries and the conditions of integrability of two-dimensional dynamical systems Theor. Math. Phys. 51 322–30
[9] Wilson G 1981 The modified Lax and two-dimensional Toda lattice equations associated with simple Lie algebras Ergod. Theor. Dynam. Syst. 1 361–80
[10] Drinfeld V G and Sokolov V V 1985 Lie algebras and equation of KdV type J. Sov. Math. 30 1975–2036
[11] Frenkel I B and Kac V G 1980 Basic representations of affine lie algebras and dual resonance models Inventiones Math. 62 23–66
[12] Ward R S 1995 Discrete Toda field equations Phys. Lett. A 199 45–8
[13] Suris Yu B 1991 Generalized Toda chains in discrete time Leningr. Math. J. 2 339–52
[14] Hirota R 1981 Discrete analogue of a generalized Toda equation J. Phys. Soc. Japan 50 3785–91
[15] Miwa T 1982 On Hirota’s difference equations Proc. Japan Acad. 58A 9–12
[16] Willows R and Hattori M 2015 Discretisations of constrained KP hierarchies J. Math. Sci. Univ. Tokyo 22 613–61
[17] Fordy A P and Gibbons J 1980 Integrable nonlinear Klein–Gordon equations and Toda lattices Commun. Math. Phys. 77 21–30
[18] Fordy A P and Gibbons J 1983 Nonlinear Klein–Gordon equations and simple Lie algebras Proc. R. Ir. Acad. A 83 33–44
[19] Kuniba A, Nakanishi T and Suzuki J 2011 T-systems and Y-systems in integrable systems J. Phys. A: Math. Theor. 44 103001
[20] Smirnov S V 2015 Darboux integrability of discrete two-dimensional Toda lattices Theor. Math. Phys. 182 189–210
[21] Fordy A P and Xenitidis P 2017 Zn graded discrete Lax pairs and integrable difference equations J. Phys. A: Math. Theor. 50 165205
[22] Fu W 2018 Direct linearisation of the discrete-time two-dimensional Toda lattices J. Phys. A: Math. Theor. 51 334001
[23] Atkinson J, Lobb S B and Nijhoff F W 2012 An integrable multicomponent quad-equation and its Lagrangian formulation Theor. Math. Phys. 173 1644–53
[24] Habibullin I T and Poptsova M N 2015 Asymptotic diagonalization of the discrete Lax pair around singularities and conservation laws for dynamical systems J. Phys. A: Math. Theor. 48 115203
[25] Nijhoff F W, Papageorgiou V G, Capel H W and Quispel G R W 1992 The lattice Gelfand–Dikii hierarchy Inverse Problems 8 597421
[26] Doliwa A 2013 Non-commutative lattice–modified Gelfand–Dikii systems J. Phys. A: Math. Theor. 46 205202
[27] Zaborodn A V 1997 Hirota’s difference equations Theor. Math. Phys. 113 1347–92
[28] Habibullin I T and Yangubaeva M V 2013 Formal diagonalization of a discrete Lax operator and conservation laws and symmetries of dynamical systems Theor. Math. Phys. 177 1655–79
[29] Mikhailov A V 2015 Formal diagonalisation of Lax–Darboux schemes Model. Anal. Inform. Sist. 22 795–817
[30] Novikov S P and Dynnikov I A 1997 Discrete spectral symmetries of low-dimensional differential operators and difference operators on regular lattices and two-dimensional manifolds Russ. Math. Surv. 52 1057–116
[31] Adler V E and Startsev S Ya 1999 Discrete analogues of the Liouville equation Theor. Math. Phys. 121 1484–95
[32] Wasow W 1987 Asymptotic expansions for ordinary differential equations Dover Books on Advanced Mathematics (New York: Dover) p 374
[33] Pavlova E V, Habibullin I T and Khakimova A R 2017 On one integrable discrete system (in Russian) Differential Equations: Mathematical Physics (Itogi Nauki Tekh. Ser. Sovrem. Mat. Prilozh. Temat. Obz. vol 140) (Moscow: VINITI) p 30–42
[34] Garifullin R N, Gubbio G and Yamilov R I 2019 Integrable discrete autonomous qud-equations admitting, as generalized symmetries, known five-point differential–difference equations J. Nonlinear Math. Phys. 26 333–57
[35] Gürses M, Karasu A and Sokolov V V 1999 On construction of recursion operators from Lax representation J. Math. Phys. 40 6473–90
[36] Khanizadeh F, Mikhailov A V and Wang J P 2013 Darboux transformations and recursion operators for differential-difference equations Theor. Math. Phys. 177 1606–54
[37] Zhang H, Tu G Z, Oevel W and Fuchssteiner B 1991 Symmetries, conserved quantities, and hierarchies for some lattice systems with soliton structure J. Math. Phys. 32 1908–18
[38] Maltsev A Y and Novikov S P 2001 On the local Hamiltonian systems in the weakly non-local Poisson brackets Physica D 156 53–80
[39] Habibullin I T and Khakimova A R 2018 On the recursion operators for integrable equations J. Phys. A: Math. Theor. 51 22
[40] Habibullin I T and Khakimova A R 2017 Invariant manifolds and Lax pairs for integrable nonlinear chains Theor. Math. Phys. 191 793–810