An Electromagnetic Perpetuum Mobile?

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A charge moving freely in orbit around the Earth radiates according to Larmor’s formula. If the path is closed, it would constitute a perpetuum mobile. The solution to this energy paradox is found in an article by C. M. De Witt and B. De Witt from 1964. The main point is that the equation of motion of a radiating charge is modified in curved spacetime. In the present article we explain the physics behind this modification, and use the generalized equation to solve the perpetuum mobile paradox.

I. INTRODUCTION

A surprising possibility of a perpetuum mobile seems to exist according to the usual equations of classical electromagnetism. A situation leading to this possibility was described recently by Chiao \cite{1}. He considered two objects, one neutral and one charged, orbiting in free fall around the Earth, and wrote:

"The charged object will gradually spiral in towards the Earth, since it is undergoing constant centripetal acceleration in uniform circular motion, and will thereby in principle lose energy due to the emission of electromagnetic radiation at a rate determined by Larmor’s radiation-power formula [using here SI-units]:"

$$P_{EM} = \alpha_q a^2 \quad \alpha_q = \frac{q^2}{6\pi\varepsilon_0 c^2}$$

where $P_{EM}$ is the total amount of power emitted in electromagnetic radiation by the charged object with charge $q$ undergoing centripetal acceleration $a$. The energy escaping to infinity in the form of electromagnetic radiation emitted by the orbiting charged object must come from its gravitational potential energy (which is related by the virial theorem to its kinetic energy), and therefore this object will gradually spiral inwards towards the surface of the Earth.

Although everything Chiao says is not only perfectly reasonable, but must also be correct due to the principle of conservation of energy, the Lorentz-Abraham-Dirac equation of motion (the LAD-equation) has a strange consequence as applied to the situation he describes. The covariant form of this equation is \cite{2}:

$$f^\mu_{\text{ext}} + \Gamma^\mu = m\dot{u}^\mu$$

where the dot denotes covariant differentiation with respect to the proper time of the charge, and

$$\Gamma^\mu = \alpha_q \left( \dot{a}^\mu - \frac{1}{c^2} a^\beta a_\beta u^\mu \right)$$

is the Abraham four-force, also called the field reaction force. Here $a^\mu$ is the four-acceleration of the charged body.

The four-acceleration vanishes for geodesic motion. Hence there is no field reaction force in this case. This means that according to the LAD-equation the neutral and the charged particle motions are identical, although the charged particle radiates, and the neutral particle does not.

Since the particles return to the same point of the path after one period, they can continue this motion for an unlimited amount of time. And the charged particle would continue radiating all the time. Hence, this is a perfect perpetuum mobile. One gets radiation energy for nothing from a system working cyclically.

Obviously this in conflict with the principle of conservation of energy, so something must be wrong. There are two equations involved in this energy paradox, the Larmor equation and the LAD-equation.

One solution of this energy paradox would be that a charge falling freely does not radiate. One could argue for this by noting that a freely moving particle defines a local inertial reference frame, i.e. a frame in which Newton’s 1st law is valid. Let the neutral object mentioned by Chiao be an observer. As measured by this observer the charge is at rest in an inertial frame, and hence it does not radiate. However, a thorough analysis proves that this is not the solution to the paradox \cite{3, 4, 5, 6}. It has been shown that, as measured by an observer that is not falling freely, a freely falling charge radiates with a power given by Larmor’s formula, and it extension to a non-inertial reference frame. No generalization to curved spacetime has been made, but since the above result shows that it does not matter for the radiated effect whether a charge is accelerated by gravitational or normal forces in flat spacetime, we will assume that this also holds, at least in the Newtonian limit of weak field and low relative velocities, in curved spacetime.

The other equation is the equation of motion of the charged, radiating particle, the LAD-equation. Maybe there is something wrong with this. Could there be a non-vanishing field reaction force on a freely falling charge? An argument against this possibility is the following. According to the principle of equivalence no local measurement should be able to distinguish between being at rest
in flat space and being freely falling in curved space. As observed by the comoving neutral object there is no force acting upon the charged particle. Hence it should remain at rest relative to the neutral object, and this is a reference independent result.

This argument has, however, a weak point. The principle of equivalence has a local character. The mentioned equivalence is only valid as far as the measurements do not reveal a possible curvature of space. So, if there is a non-local interaction between the charge and the curvature of spacetime, due to the non-local character of its electromagnetic field, this may modify the equation of motion of the charge. The necessity and nature of such a modification will be explained in the next sections.

II. THE FIELD REACTION FORCE IN CURVED SPACE

Before we can understand the solution to the energy paradox introduced above, we need to know the correct form of the equation of motion of a charged particle in curved space. It was, in fact, deduced more than forty years ago [8, 9], but is not well known outside a rather small group of physicists having performed research within this field. The reason it not that this knowledge is not important - the equation of motion of a charged particle in curved space is one of the fundamental equations in classical electrodynamics - but that the equation is mathematically rather complicated. Nevertheless, the physics behind the equation can be understood, as we shall explain in this article.

A. An introduction to the electromagnetic self force problem in curved spacetime

Let us consider the motion of a classical charged point particle due to the electromagnetic force. The path of the particle is \( x'(\tau) \), and we shall also need a label for a general point in space-time, which will be \( x \). To make the notation compact, it is usual to associate the prime also with components of vectors taken at a given point. Thus, \( j^\mu \) is understood to be \( j^\mu(x) \), while \( j^\mu = j^\mu(x') \).

In general, the force on a body due to the electromagnetic interaction is given by the Lorentz equation

\[
\frac{dF^\mu}{dV} = F^\mu_{\nu} j^\nu
\]  

(4)

where \( F^\mu_{\nu} \) is the force per unit volume on the body, the tensors \( F^\mu_{\nu} = A^\mu_{\nu,\mu} - A^\mu_{\mu,\nu} \) and \( j^\mu \) are the electromagnetic field and current density, respectively, and \( A^\mu \) is the electromagnetic potential. The force on the whole body, then, is

\[
f^\mu = \int_V F^\mu_{\nu} j^\nu dV
\]  

(5)

In the case of a point particle with charge \( q \) four-velocity \( u \) and current density \( j^\mu(x) = q \int u^\nu \delta(x, x'(\tau)) d\tau \), where \( \tau \) is the proper time of the particle, the Lorentz equation reduces to

\[
f^\mu = q F^\mu_{\nu} u^\nu
\]  

(6)

The \( \delta(x, x') \equiv g^{\mu \nu} \delta(x - x') \) used in the expression for the four-current is the invariant delta-function, which is defined such that any volume integral over \( x \) is 1 if it contains \( x' \), and 0 otherwise. Furthermore, \( q \) is the absolute value of the determinant of the metric.

Since the particle is charged, it not only reacts to \( F_{\mu
u} \), but also acts as a source of it, according to the electromagnetic wave function

\[
A^\mu_{\nu, \alpha} - R^\mu_{\alpha \nu} A^\alpha = -\mu_0 j^\mu
\]  

(7)

with \( R_{\mu\nu} \) being the Ricci curvature tensor. The solution of this equation depends on the motion of the particle, encoded in \( j^\mu \), which is again determined by the Lorentz equation, so these two equations are coupled, the coupling being the particle’s reaction to its own field.

B. Green functions

Since these equations are linear, it is possible to split \( j^\mu \) into points, solve the equation for each point, and then take the sum of the results. Furthermore, we do not need to know the amplitude of \( j^\mu \) at each of these points when we solve the equation, as the linearity makes it possible to multiply by this afterwards. This way, much of the work in solving the equation can be done without involving \( j^\mu \) itself. This is called the Green function approach to solving differential equations.

In general, given some linear differential equation

\[
L(\phi(x)) = f(x)
\]  

(8)

where \( L \) is a general linear differential operator (for example \( L = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y} \)), a Green function defined by

\[
L(G(x, x')) = \delta(x - x')
\]  

(9)

where \( \delta(x - x') \) is the Dirac delta function, solves the original differential equation through

\[
\phi(x) = \int G(x, x') f(x') dx' + S
\]  

(10)

where \( S \) is a surface term that can be taken to be zero when the integral goes over all of spacetime.

In many cases, it is easier to find the Green function than the unknown function, and so it is mainly used for convenience, but it also serves to make the causal connections inherent in the differential equation explicit. From the definition, we see that the \( G(x, x') \) is the effect on the spacetime point \( x \) of the point \( x' \), and thus can be seen
as the degree to which the two points are causally connected (as far as the differential equation is concerned). For example, in flat, 3+1-dimensional spacetime, electromagnetic radiation moves only on the light cone, and the Green function for electromagnetic radiation is therefore zero everywhere outside of the light cone, so in this case \( G \propto \delta(\sigma) \). The \( \sigma \) appearing here is commonly used in the theory of geodesics, and is defined as \( \sigma \equiv \frac{1}{2} s^2 \), with \( s \) being the geodetic interval between two points \( (x) \) and \( (x') \) in this case. It is negative for a time-like interval, positive for a space-like interval, and zero for a light-like interval.

The Green function defined above is a scalar quantity. However, the equation we are considering, the electromagnetic wave equation, is a vector equation. A straightforward generalization of equation (10) for this case is

\[
L_{\mu \nu}(G_{\mu \nu}(x,x')) = \delta_{\mu \nu} \delta(x-x')
\]

(11)

\[
A_{\mu} = \int G_{\mu \nu} \phi^\nu \sqrt{g} dx'
\]

(12)

This equation is correct\(^{[12]}\) as it stands, but to make it an invariant expression, we need to replace the delta function with its invariant version \( \delta(x,x') \), and the integral needs a volume element of \( \sqrt{g} \). Rectifying this, we arrive at

\[
L_{\mu \nu}(G_{\mu \nu}(x,x')) = \delta_{\mu \nu} \delta(x-x')
\]

(13)

\[
\phi^\mu = \int G_{\mu \nu} \phi^\nu \sqrt{g} dx'
\]

(14)

Applying this to the electromagnetic wave equation, we get

\[
\Box G_{\mu \nu} - R_{\mu \nu} G_{\mu \nu} = -\mu_0 \delta_{\mu \nu} \delta(x,x')
\]

(15)

\[
A_{\mu} = \int G_{\mu \nu} \phi^\nu \sqrt{g} dx'
\]

(16)

where the constant factor \( -\mu_0 \) has been absorbed into \( G \) due to convention.

C. Generalized LAD equation for curved space

Equation (15) differs from equation (7) in one important aspect: It is independent of \( j^\mu \), and thus not coupled to the motion of the particle. This is the main reason for introducing Green functions. It may still be difficult to solve, but it can be solved once and for all, and then applied to specific cases with equation (10).

Let us assume that we already know the Green function for the wave equation; we can then use it to find the electromagnetic potential:

\[
A_\mu(x) = \int G_{\mu \nu}(x,x') j^\nu(x') \sqrt{g'} d^4 x'
\]

\[
= q \int G_{\mu \alpha}(x,x'(\tau)) u^\alpha(\tau') d\tau
\]

(17)

Since this is an explicit expression for \( A_\mu \), we can insert it into the Lorentz equation, resulting in

\[
f_\mu(\tau) = q \int d\tau' (G_{\rho \mu;\rho}(x'(\tau),x'(\tau')) u^\rho(\tau')) u^\mu(\tau)
\]

(18)

Here, the general point \( x \) has been replaced by the position of the particle, \( x' \), as that is the point where we use the Lorentz equation. The physical interpretation of this equation is that each point \( x'(\tau') \) on the world line of the particle gives a contribution to the force acting on some chosen point \( x'(\tau) \), the effect of \( x'(\tau') \) on the potential, and therefore on the force, being given by the Green function \( G_{\mu \nu}(x'(\tau),x'(\tau')) \).

The case \( \tau = \tau' \), where emission of and reaction to the field happens at the same time, stands out as both obvious due to its locality, and daunting due to the fact that the infinite charge density of a point charge sets up an infinitely strong field at its location.

The contribution from this case was computed by DeWitt and Brehme\(^{[3]}\) in 1960 and Hobbs\(^{[3]}\) in 1968 as

\[
\Gamma^\mu = -\frac{1}{2} \alpha q \left( R^\mu_{\nu \rho \sigma} u^\nu f^\rho u^\sigma + R_{\alpha \beta} u^\alpha f^\beta u^\nu \right)
\]

(19)

where \( \Gamma^\mu \) is the Abraham four-force. The part of the integral \( [13] \) where the “absorption” time \( \tau \) is earlier than the emission time \( \tau' \), can be eliminated on grounds of causality, and we are left with

\[
f_\mu(\tau) = \Gamma^\mu = -\frac{1}{2} \alpha q \left( R^\mu_{\nu \rho \sigma} u^\nu f^\rho u^\sigma + R_{\alpha \beta} u^\alpha f^\beta u^\nu \right)
\]

\[
+ q \int_{-\infty}^{\tau} d\tau' (G_{\rho \mu;\rho} - G_{\mu \alpha;\rho}) u^\alpha(\tau') u^\rho(\tau)
\]

(20)

This is a generalization of the LAD equation we used earlier, which stops after the first term on the right hand side, and is therefore a local equation. It is on this generalized equation that DeWitt and DeWitt\(^{[10]}\) base their calculations.

Though we have seen how the integral term, usually called the tail term, originates analytically, it is still not obvious that it can be anything but zero. We will therefore take another detour before returning to the DeWitts’ results, in order to show how this can be.

As we remember from the derivation of the term, it represents the force from a field that is generated at the point \( x'(\tau') \), and later reaches the point \( x'(\tau) \). The behavior of the electromagnetic field is encoded in \( G \), but before calculating, we already have some expectations about its form from our experience with electromagnetism:

1. Electromagnetic fields in vacuum should move at the speed of light, \( c \).
2. It should be impossible for a particle to catch up with a light-like field it has emitted.
From these two points, we would conclude that the only possible way a particle can interact with itself is when emission and absorption of the field happens at the same point, so there should be nonlocal contributions, and therefore no tail term!

Our intuition turns out to be wrong on both these points, and both of them can give contributions to the tail term.

III. THE PHYSICS BEHIND THE NON-LOCAL FIELD REACTION IN CURVED SPACE-TIME

A. Overtaking light

The latter premise for the assertion that there should be no tail term, namely that it should be impossible for a particle to catch up with a light-like field it has emitted, is the easiest to prove wrong, as gravitational lensing provides a simple counterexample. As an extreme case of gravitational lensing, consider a particle suspended at a constant Schwarzschild coordinate of $r = \frac{3}{2} R_s$, i.e. at the innermost stable orbit. It then emits a photon perpendicularly to the direction towards the black hole. The photon will then reencounter the particle after one orbit. The particle has, effectively, caught up with the photon by taking another path through space-time.

Less extreme examples of lensing also make the same effect possible, though a relative velocity between the mass and the particle will be necessary in those cases.

B. Slow light

In the above case, it was possible to catch up to light because light took another path through spacetime, though it still moved at the speed of light. It is more difficult to see how light can end up moving more slowly than $c$ through empty space. However, this turns out to be the most important contribution to the tail term.

To see how the wave equation can lead to parts of the field moving more slowly than the speed of light, and thus causing dispersion, so that an initially sharp pulse develops a trailing tail, let us first consider the spherically symmetric case in flat spacetime, and vary the number of spatial dimensions. These cases provide one of the conceptually simplest examples of dispersion, and will turn out to be mathematically related to cases in curved, 3+1-dimensional spacetime.

In flat spacetime, it is possible to decompose the wave equation, which is a vector equation, into one scalar wave equation for each dimension of space-time, with all the scalar equations being independent. These scalar equations have the form

$$\phi^{\mu \nu} = -\rho$$

and solving this equation is equivalent to solving the vector equation. In curved spacetime, this is no longer the case, but the solutions to the vector and scalar equations generally display similar behavior.

In polar coordinates with spherical symmetry, equation (21) takes the form

$$\left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{d - 1}{r} \frac{\partial}{\partial r} \right) \phi = -\rho,$$

with $\rho$ being the angular integral of $\phi$, and $d$ being the number of spatial dimensions. The equation for the Green function corresponding to this is

$$\left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{d - 1}{r} \frac{\partial}{\partial r} \right) G = -2c\delta(t)\delta(r) \tag{22}$$

Note that the only thing separating the cases with different number of dimensions, is the factor in front of the $\frac{\partial}{\partial r}$ term, but this factor is critical for its behavior. It possesses simple solutions for the cases $d = 1$ and $d = 3$, but is complicated to solve for other cases, especially the even ones. Brown \cite{11} discusses this thoroughly in an article on the Huygens’ Principle. He only considers the homogeneous version of the equation,

$$\left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{d - 1}{r} \frac{\partial}{\partial r} \right) \phi = 0 \tag{23}$$

but since the right hand side of equation (22) is given by delta functions, they only differ at the initial moment, and so the solution to equation (22) can be found by solving equation (23) with an appropriate initial condition.

To see that the case for 1 spatial dimension and our familiar 3 spatial dimensions are exceptionally simple, Brown suggests that one makes a change of variable by introducing $\psi = r \frac{\partial}{\partial r} \phi$ (our notation switches the role of $\phi$ and $\psi$ compared to his), which leads to the differential equation

$$\left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + \frac{(d - 1)(d - 3)}{4r^2} \frac{\partial}{\partial r} \right) \psi = 0 \tag{24}$$

The first order term vanishes for $d = 1$ and $d = 3$, giving the simple equation $\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi = \frac{\partial^2}{\partial r^2} \psi$, which can be rewritten as $\left( \frac{\partial^2}{\partial r^2} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi = 0$, and so obviously has solutions

$$\psi = f(r - ct) + g(r + ct) \tag{25}$$

for any function $f$ and $g$ \cite{14} and then It is easy to see that this solution consists of signals moving at a single, well-defined speed: $c$. This exceptional case ($d = 3$) is what our intuition for the wave equation, and thus behavior of light, is based on, and the behavior of electromagnetic fields in other numbers of dimensions therefore end up being counterintuitive. For a general $d$, we cannot expect the first order term to cancel, and therefore do not end up with an an equation that leads to this kind of simple propagating solution.

Brown goes on to consider the case $d = 2$, and by assuming separability, $\phi(r,t) = f(r)g(t)$, rewrites equation
Figure 1: Time series for the solution of the equation 
\[
-\frac{1}{c^2} \partial^2_t + \partial^2_r + \frac{d-1}{r} \partial_r \frac{d}{dr} G = -2c\delta(t)\delta(r)
\]
with \(d = 2\) and Gaussians instead of delta functions, and \(c = 1\). \(r\) runs along the horizontal axis, and \(G\) along the vertical one. This is a set of superimposed curves, one for each time step. The general behavior is that of a pulse propagating to the right, but the value behind the pulse does not reach 0, making the final result a lopsided Gaussian. Since we have spherical symmetry, the time series in this figure corresponds to an initial central peak evolving into a spreading circular “hill”, with a nonzero “elevation” inside, which is the “tail”.

The equation for \(f\) is the zeroth order Bessel equation, and is solved by the corresponding zeroth order Bessel functions \(J_0(r)\) and \(Y_0(r)\). By using the integral representation for \(J_0, J_0(r) = \frac{2}{\pi} \int_0^\infty \sin(\cosh(\theta)r) \, d\theta\), and a solution \(g(t) = \sin(ct)\) of the equation for \(g\), Brown finds a solution for \(\phi\) to be

\[
\phi = \frac{1}{\pi} \int_0^\infty \left[ \cos(\cosh(\theta)r - ct) - \cos(\cosh(\theta)r + ct) \right] \, d\theta
\]

This, he points out, is a superposition of waves traveling with speeds of \(v = \frac{c}{\cosh(\theta)}\), which takes all values from 0 to \(c\). So in this case, light moves at the speed of light, as well as all lower velocities!

To see how this applies to the actual Green equation, which differs from the equation considered by Brown by being inhomogeneous, we present numerical solutions for the cases of 2, 3 and 4 spatial dimensions, and with delta functions approximated by Gaussians. The result can be seen in the figures 1-3.

From these solutions, we see that the factor in front of the \(\frac{d}{dr}\) term of equation 22 controls how fast the Green function falls after the wave front has passed. For \(d = 2\), which gives a factor of \(\frac{1}{r}\), the fall after the wave front is less than the rise that preceded it, so the function develops a tail. For \(d = 4\), with a factor of \(\frac{1}{r^2}\), the fall is too great, and the solution overshoots zero, again leaving a tail. The case \(d = 3\) has the exact balancing between fall and rise needed to avoid a tail. We again see that not having a tail is not a general feature of the wave equation, but a special case.

To see how this applies to curved 3+1-dimensional spacetime, we now consider the form of the spherically
symmetric scalar wave equation in the Schwarzschild spacetime:

$$
\left( -\frac{\beta}{c^2} \frac{\partial^2}{\partial\tau^2} + \frac{1}{\beta} \frac{\partial^2}{\partial \rho^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) G = -2c\delta(t)\delta(r)
$$

(28)

with the shorthand notation $\beta = 1 - \frac{R_S}{r}$. This equation is expressed in terms of Schwarzschild radial coordinates, which behave as flat-space polar coordinates only in the limit $r \to \infty$. This prevents us from directly comparing it to equation (22). To remedy this, one can introduce orthonormal coordinates $\tau = \sqrt{1 - \frac{R_S}{r}}t$, $\rho = \frac{r}{\sqrt{1 - \frac{R_S}{r}}}$ around $r = r_0$. With these, equation (28) takes the form

$$
\left( -\frac{1}{c^2} \frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial \rho^2} + \frac{2 - 3\frac{R_S}{r_0} + \frac{R_S^2}{r_0^2}}{\rho} \frac{\partial}{\partial \rho} \right) G = -2c\delta(\tau)\delta(\rho)
$$

(29)

This equation is of the same form as equation (22), but this time the factor in front of the first order radial derivative varies between 0 (in the limit $r_0 \to \infty$) and 2 (in the limit $r_0 \to 0$). Except for in the latter limit, these factors are too small to make the Green function reach zero behind the wave front. Hence, there will be dispersion.

The right hand side of this equation is problematic, as the source term is in the middle of the black hole. This does not matter much, though, since we are interested in the fate of an initially sharp wave front, and we only need the homogeneous version of the equation to find the time evolution of such a wave. One could, for example, start with an initial wave concentrated in a thin shell centered on the black hole (a shell because we assume spherical symmetry here). The homogeneous equation then tells us that this will evolve as in a $3(1 - \frac{R_S}{r_0}) + \frac{R_S^2}{r_0^2}$-dimensional spacetime, and thus disperse accordingly.

Though we considered the scalar wave equation here, one will in general find a tail for the Green function in curved spacetime also for the electromagnetic wave equation [12].

The existence of a tail of the Green function means that parts of the field spread at speeds slower than the speed of light, and thus move inside the light cone. In general, parts of the field will be found to move at all speeds slower than the speed of light [12]. When a sharp wave front enters an area of curvature from flat spacetime, it will begin developing a trailing tail stretching not only to the point where the curved area starts, but also beyond this, since dispersion also leads to part of the field moving in the opposite direction to that of the initial wavefront. This part can propagate back to the particle and interact with it.

There are, then, 4 ways for the particle to interact with its own field:

- Locally, where the particle reacts to the field it just emitted.
- By taking a different path through spacetime than the field, which lets it encounter even the parts of the field that move at the speed of light.
- By catching up to parts of the field it emitted earlier, because the field is moving slower than $c$.
- By having parts of the field scattered back at it.

IV. THE SOLUTION OF THE ENERGY PARADOX

The question is: How does energy conservation come about in a situation where a charged particle moves in the Schwarzschild spacetime, say in the equatorial plane, under the action of gravity alone? Assuming the particle moves in a circular orbit far away at a distance $r$ from a mass $M$, it will have a motion given by

$$
r = r e_{\hat{r}} \quad v = r \omega e_{\hat{\theta}} \\
\mathbf{a} = -r \omega^2 e_{\hat{r}} \quad \dot{\mathbf{a}} = -r \omega^3 e_{\hat{\theta}}
$$

(30)

with $a \equiv |\mathbf{a}| = GMr^{-2}$, giving a radiated effect of

$$
P = \frac{q^2 G^2 M^2}{6\pi r^4 \varepsilon_0 c^3}
$$

(31)

by the use of equation (11), and the particle needs to lose energy at the same rate for energy to be conserved.

The answer to the question above is contained in the generalized LAD-equation for curved space, equation (20). In the present case the local term containing the Ricci curvature tensor does not contribute since this tensor vanishes in the external Schwarzschild spacetime. It is the non-local so-called tail-term that matters here.

Before we discuss the solution of the paradox, let us note an interesting conceptual difference between the special relativistic version of the equation of motion as deduced by Lorentz and Abraham on the one hand, and the fully covariant version of it deduced by Dirac. In the present case we are not concerned with velocity-dependent effects. Hence we may start by looking at the non-relativistic version of the equation,

$$
\mathbf{F}_{\text{ext}} + \alpha_q \dot{\mathbf{a}} = m \mathbf{a}
$$

(32)

The special relativistic (and Newtonian) acceleration is absolute and arises when the particle is acted upon by an external force (or during pre-acceleration or run-away motion). Hence, during circular motion around the Earth under the action of the gravitational force, there is an external gravitational force $\mathbf{F}_{\text{ext}} = \frac{GMm}{r^2}$ acting in the radial direction. The change of the acceleration is directed opposite to the motion, and so the field reaction force acts like a frictional force, decreasing the velocity of the charge, which thereby starts spiraling inwards. The radiated energy is accounted for by a decrease in mechanical energy.
As explained by DeWitt and DeWitt \[10\], the contents of the covariant equation \[2\] interpreted within the general theory of relativity is different. According to this theory, gravity is not reckoned as a force. In Newton’s theory and in special relativity we say that a freely falling particle is acted upon by a gravitational force. In Einstein’s general theory of relativity we say that a freely falling particle is not acted upon by any force. Such a particle has vanishing four-acceleration. DeWitt and DeWitt then say:

When a charged particle is accelerated by means of nongravitational forces, the electric field lines which emanate from it bend and redistribute themselves in the vicinity of the particle (i.e., within a distance of the order of the classical radius) in such a way as to exert, on the average, a net retarding force, over and above the force of inertial reaction. With purely gravitational forces, however, this is not the way things happen. The field in the immediate vicinity of the particle tends to fall freely with the particle, and although it suffers a local tidal distortion characteristic of an explicit occurrence of the Riemann tensor... the net retarding force due to this distortion is zero when integrated over solid angle\[16\].

Hence, the usual Abraham force plus local contributions to the field reaction force from terms depending upon the curvature vanish for free particles. However, DeWitt and DeWitt continue, saying:

The deviation of the particle motion from geodesic when \( F_{\text{ext}}^\mu = 0 \) is caused not by the local field of the particle but by a field which originates well outside the classical radius and which is manifested by the nonlocal term of equation \[20\].

They calculated this by considering the whole electromagnetic field (and not just the part of it in the neighborhood of the particle, as one normally would) using the weak field approximation and the assumption of a static gravitational field. The result is that the curved spacetime around a massive object disperses and scatters the electromagnetic field. Thus, it is possible for the field to emanate from the particle, propagate until it reaches an area with non-negligible curvature, be scattered there, and then propagate back to the particle. The DeWitts write

Physically [this] arises from a back-scatter process in which the Coulomb field of the particle, as it sweeps over the “bumps” in spacetime, receives “jolts” which are propagated back to the particle.

This non-local interaction between the particle and its field provides a mechanism for reducing the particle’s energy and balancing the energy budget. But as opposed to the radiation reaction for nongravitational forces, this force is determined not by the particle’s current motion, but by its motion at some earlier time, and so there is no reason to expect these to match, or even resemble each other.

This non-locality also makes the equation of motion a time delay differential equation, a difficult class of differential equations that often lack analytical solutions. However, by going to the non-relativistic limit, DeWitt and DeWitt succeed in evaluating the extended field reaction for a point charge in free fall near some point mass, and near some arbitrary mass distribution, the result being

\[
\begin{align*}
\mathbf{F} &= \mathbf{F}_C + \mathbf{F}_{NC} \\
\mathbf{F}_C &= q^2 GM \frac{r}{r^4} \\
\mathbf{F}_{NC} &= -\frac{2}{3} q^2 GM \left( \frac{1}{r^3} - 3 \frac{r r}{r^5} \right) \cdot \mathbf{r}
\end{align*}
\]

where \( M \) is the mass of the source of the gravitational field and \( q \) is the charge and \( \mathbf{r} \) is the separation 3-vector between the charge and the mass as measured in the rest frame of the mass. This can be written

\[
\begin{align*}
\mathbf{F}_C &= -m \mathbf{\nabla} \psi \\
\mathbf{F}_{NC} &= -\frac{2}{3} q^2 \mathbf{r} \cdot \mathbf{\nabla} \phi = -\frac{2}{3} q^2 \frac{d}{dt} \left( \mathbf{\nabla} \phi \right)
\end{align*}
\]

\[\psi(\mathbf{r}) = \frac{1}{2} \frac{q^2}{m} G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \frac{d^3 \mathbf{r}'}{r^2} \]

\[\phi(\mathbf{r}) = -G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}' \]

where \( \rho \) is the mass density and \( m\phi \) is the normal gravitational field. The mathematical expressions for \( \mathbf{F}_{NC} \) are explained in Appendix A. DeWitt and DeWitt write

\( \mathbf{F}_{NC} \) is the nonconservative force which gives rise to radiation damping. Owing to its dependence on velocity it is small in magnitude compared to the force \( \mathbf{F}_C \) which is conservative. ... It [the conservative part] arises from the fact that the total mass of the particle is not concentrated at a point but is partly distributed as electric field energy in the space around the particle.

So \( \mathbf{F}_{NC} \) is the force that should correspond to the radiated effect in equation \[31\]. However, this effect was calculated under the assumption of a circular orbit, which is not possible under the influence of a nonconservative damping force. But if \( \mathbf{F}_{NC} \) is very small, as it should be, because the field has traversed a distance of \( 2r \) before returning to the particle, and only a small part of the field is scattered back in the first place, then the particle motion will be dominated by the force of gravity, so \( F_{\text{total}} \approx F_G = -m \mathbf{\nabla} \phi \).

By inserting this into the expression \[31\] for \( \mathbf{F}_{NC} \), we see that the nonconservative force, which with its dependence on velocity and the local shape of \( \phi \) is quite...
different from the Lorentz-Abraham force $\mathbf{F}_{LA} = \alpha_q \mathbf{a}$, still reduces to it in the non-relativistic limit. The work this force does on the particle is given by $P = \mathbf{F}_{NC} \cdot \mathbf{v}$, and using equation (30), we find

$$P = \alpha_q \dot{\mathbf{a}} \cdot \mathbf{v} = -\alpha_q \dot{r}^2 \omega^4 = -\alpha_q a^2 = \frac{q^2 G^2 M^2}{6 \pi r^4 \epsilon_0 c^3}$$

(35)

which is of the same size but opposite magnitude as the radiated effect in equation (31), meaning that the particle loses energy at the same rate as it is radiated away, and energy is thus balanced.

This applies not only for the circular orbit considered here; DeWitt and DeWitt showed (still using the assumption that the motion is dominated by the effects of gravity) that the average energy loss rate during any closed orbit in free fall matches that given by the Larmor formula. This follows from the fact that the non-local, nonconservative, backscattering-based force $\mathbf{F}_{NC}$ reduces to normal Lorentz-Abraham force $\mathbf{F}_{LA}$ in the non-relativistic limit. This, then, is what balances the energy budget, and prevents an electron in orbit around the Earth from being a perpetual source of energy.

V. ACKNOWLEDGMENTS

We would like to thank Jon Magne Leinaas for useful discussion concerning the electromagnetic perpetuum mobile paradox discussed here.

Appendix A

Given two vectors $A^\mu$ and $B^\mu$, two obvious products can be formed: the inner product $P = A^\mu B_\mu$, and the outer product $Q_{\mu\nu} = A^\mu B^\nu$. DeWitt and DeWitt indicate the former with the $\cdot$ operation, and the latter by adJoining two vectors without any intervening notation. Thus, the $\mathbf{rr}$ seen in equation (33) is

$$\mathbf{rr} = r_i r_j = \begin{pmatrix} r_1 r_1 & r_1 r_2 & r_1 r_3 \\ r_2 r_1 & r_2 r_2 & r_2 r_3 \\ r_3 r_1 & r_3 r_2 & r_3 r_3 \end{pmatrix}$$

(31)

and $(\mathbf{rr}) \mathbf{r} = r(r \cdot \mathbf{r})$. The matrices in equation (33) simply serve to let one write the $\mathbf{r}$ as a single factor outside the parentheses; otherwise, it could be written without them as

$$\mathbf{F}_{NC} = -\frac{2}{3} q^2 G M \left( \frac{\mathbf{r}}{r^3} - 3 \frac{r \cdot \mathbf{r}}{r^5} \right)$$

(32)

The above also applies to the vector operator $\nabla$, which is $\dot{\mathbf{a}} = \dot{a}_x \mathbf{e}_x + \dot{a}_y \mathbf{e}_y + \dot{a}_z \mathbf{e}_z$ in Cartesian coordinates. The inner product formed from $\nabla$ is $\nabla^2 \equiv \nabla \cdot \nabla = \partial_x^2 + \partial_y^2 + \partial_z^2$, but one can also form the outer product

$$\nabla \nabla = \begin{pmatrix} \partial_x \partial_x & \partial_x \partial_y & \partial_x \partial_z \\ \partial_y \partial_x & \partial_y \partial_y & \partial_y \partial_z \\ \partial_z \partial_x & \partial_z \partial_y & \partial_z \partial_z \end{pmatrix}$$

(33)

While $\nabla^2 \phi = 0$ in areas where the mass density $\rho = 0$, the same does not apply to $\nabla \nabla \phi$; otherwise, $F_{NC}$ would be 0 at the particle’s position, and we would have no extended field reaction.

Note also that the fact that the gravitational field is static has been used in the transition $\mathbf{v} \cdot \nabla \phi = \frac{d}{dt} \nabla \phi$. In general, the total time derivative of some field $\xi$ as experienced by a particle moving with velocity $\mathbf{v}$ is given by

$$\frac{d}{dt} \xi = \frac{\partial}{\partial t} \xi + \mathbf{v} \cdot \nabla \xi$$

(34)

but the partial derivative of $\xi = \nabla \phi$ with respect to time is 0 here.

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point, the meaning of the components is position dependent, and one should therefore be careful when directly comparing them. A generalization of the Kronecker delta $\delta^{\mu}_{\mu'}$ called the parallel propagator $g^{\mu}_{\mu'}$ takes care of this. In principle, then, we should replace the Kronecker delta with the parallel propagator in equation \ref{eq:12}. However, since the delta function constrains the point $x'$ to coincide with $x$, this complication does not occur here, and we can keep the Kronecker delta.

[14] Since $r$ is a radial coordinate we also have a mirror boundary condition or $\frac{\partial \psi}{\partial r} = 0$ at $r = 0$. Due to the linearity of the equation, this is equivalent to extending the domain of $r$ to $(-\infty, \infty)$, solving the equation, and then finding the real solution as $\psi_{r \in (0, \infty)}(t, r) = \psi_{r \in (-\infty, 0)}(t, r) + \psi_{r \in (-\infty, 0)}(t, -r)$. The boundary condition does not change the point that the solution moves at a single, well-defined speed, $c$.

[15] If we are to have any hope of solving the energy paradox, the orbit cannot really be circular - the particle must spiral inwards - but far away from the mass this effect will be small, and only makes for a negligible correction to the circular orbit.

[16] The statement that there is no local contribution to the force on the particle for free fall is not strictly correct. It is based on a result by DeWitt and Brehme \cite{8}, which was later corrected by Hobbs \cite{9}. But this correction only applies in the case of $R_{\mu \nu} \neq 0$ at the position of the particle, and so does not apply to the case of a charged test particle in orbit around a mass, which they go on to consider.