REPRESENTATION OF NUMERICAL SEMIGROUPS BY DYCK PATHS

MARIA BRAS-AMORÓS AND ANNA DE MIER

ABSTRACT. We introduce square diagrams that represent numerical semigroups and we obtain an injection from the set of numerical semigroups into the set of Dyck paths.

INTRODUCTION

A numerical semigroup is a subset of \( \mathbb{N}_0 \) closed under addition and with finite complement in \( \mathbb{N}_0 \). The genus of a numerical semigroup \( \Lambda \) is the number of elements in \( \mathbb{N}_0 \setminus \Lambda \), which are called gaps. A Dyck path of order \( n \) is a lattice path from \((0, 0)\) to \((n, n)\) consisting of up-steps \( \uparrow = (0, 1) \) and right-steps \( \rightarrow = (1, 0) \) and never going below the diagonal \( x = y \).

We introduce the notion of the square diagram of a numerical semigroup and analyze some properties of numerical semigroups such as their weight or symmetry by means of the square diagram.

We prove that any numerical semigroup is represented by a unique Dyck path of order given by its genus.

1. SQUARE DIAGRAM

Given a numerical semigroup \( \Lambda \) define \( \tau(\Lambda) \) as the path with origin \((0, 0)\) and steps \( e(i) \) given by

\[
e(i) = \begin{cases} \rightarrow & \text{if } i \in \Lambda, \\ \uparrow & \text{if } i \notin \Lambda, \end{cases} \quad \text{for } 1 \leq i \leq 2g.
\]

We denote it as the square diagram of \( \Lambda \). For instance, the set

\( \{0, 4, 8, 12, 16, 17, 19, 20, 21, 23, 24, 25, 27, 28, 29, 31\} \cup \{i \in \mathbb{N}_0 : i \geq 31\} \)

is a numerical semigroup and its square diagram is the following one:

![Square Diagram](image)

The conductor of a numerical semigroup is the unique element \( c \) in \( \Lambda \) such that \( c - 1 \notin \Lambda \) and \( c + \mathbb{N}_0 \subseteq \Lambda \). The enumeration of \( \Lambda \) is the unique bijective increasing map \( \lambda : \mathbb{N}_0 \rightarrow \Lambda \). We use \( \lambda_i \) to denote \( \lambda(i) \). The \( i \)th partial genus may be defined as

\[ g(i) = \lambda_i - i = \# \text{ gaps smaller than } \lambda_i. \]
Note that the following statements are satisfied:

1. \( g(0) = 0 \),
2. \( g(i) \leq g(i+1) \),
3. \( g(i) = g \) for all \( i \geq \lambda^{-1}(c) \),
4. \( g(i) = g \) for all \( i \geq g \) (consequence of (iii)).

The points with integer coordinates in the square diagram of \( \Lambda \) are all points in

\[ \{(i, g(i)) : 0 \leq i \leq g\} \cup \{(i-1, g(i)) : 1 \leq i \leq g\} \]

together with the points contained in the vertical lines from \((i-1, g(i-1))\) to \((i-1, g(i))\) whenever \(g(i-1) < g(i)\). In particular, \(\tau(\Lambda)\) goes from \((0, 0)\) to \((g, g)\). So it is included in the square grid from \((0, 0)\) to \((g, g)\). This is why we call this diagram the square diagram of \(\Lambda\).

2. SQUARE DIAGRAM OF SYMMETRIC SEMIGROUPS

The conductor \(c\) of any numerical semigroup satisfies \(c \leq 2g\), where \(g\) is the genus of the semigroup. When \(c = 2g\) the numerical semigroup is said to be symmetric \([3, 2]\). It is well known that all semigroups generated by two integers are symmetric. As a consequence of the definition of symmetric semigroups we have the following proposition:

**Proposition 2.1.** A numerical semigroup \(\Lambda\) is symmetric if and only if its square diagram satisfies \(e(2g-1) = \uparrow\).

The next proposition is a well known result on symmetric semigroups:

**Proposition 2.2.** A numerical semigroup \(\Lambda\) with conductor \(c\) is symmetric if and only if for any non-negative integer \(i\), if \(i\) is a gap, then \(c - 1 - i\) is a non-gap.

The proof can be found in \([3, \text{Remark 4.2}]\) and \([2, \text{Proposition 5.7}]\). It follows by counting the number of gaps and non-gaps smaller than the conductor and the fact that if \(i\) is a non-gap then \(c - 1 - i\) must be a gap because otherwise \(c - 1\) would also be a non-gap. As a consequence we have the following property on the square diagram of symmetric semigroups:

**Corollary 2.3.** A numerical semigroup with genus \(g\) is symmetric if and only if the intersection of its square diagram with the square determined by the points \((0, 0)\) and \((g-1, g-1)\) is symmetric with respect to the diagonal from \((0, g-1)\) to \((g-1, 0)\).

**Example 2.4.** The set

\[ \{0, 4, 8, 12, 16, 17, 18, 20, 21, 22, 24, 25, 26, 28, 29, 30, 32\} \cup \{i \in \mathbb{N}_0 : i \geq 32\} \]

is a numerical semigroup and it is symmetric. Its square diagram is therefore symmetric with respect to the diagonal from \((0, g-1)\) to \((g-1, 0)\).
Remark 2.5. There exist paths from $(0,0)$ to $(g-1, g-1)$ which are symmetric with respect to the diagonal from $(0,g-1)$ to $(g-1,0)$ but which do not correspond to a numerical semigroup. For example,

This diagram does not correspond to a numerical semigroup because otherwise, $\lambda_1$ would be 2, but 4 would not belong to the semigroup.

3. Weight of a Semigroup

The notion of the weight of a numerical semigroup has been widely used in the context of Weierstrass semigroups [5,1].

Definition 3.1. Let $\Lambda$ be a numerical semigroup with genus $g$ and let $l_1, \ldots, l_g$ be its gaps. The weight of $\Lambda$ is the sum

$$\sum_{i=1}^{g} (l_i - i).$$

In a sense, the weight measures how complicated the semigroup is. For example, the simplest semigroup is that with gaps $1, 2, \ldots, g$ and it has weight 0.

Proposition 3.2. The weight of a numerical semigroup is equal to the area over the path in the square diagram of the semigroup.

Proof. Let $\lambda$ be the enumeration of the semigroup. The area below the path is equal to the sum $\sum_{i=1}^{g} g(i)$, while the total area of the square diagram is $g^2$. So, it is enough to prove that

$$\sum_{i=1}^{g} g(i) + \sum_{i=1}^{g} (l_i - i) = g^2.$$

But $\sum_{i=1}^{g} g(i) + \sum_{i=1}^{g} (l_i - i) = \sum_{i=1}^{g} (\lambda_i - i) + \sum_{i=1}^{g} (l_i - i) = \sum_{i=1}^{2g} i - 2 \sum_{i=1}^{g} i = \sum_{i=g+1}^{2g} i - \sum_{i=1}^{g} i = \sum_{i=1}^{g} (g+i) - \sum_{i=1}^{g} i = g^2.$$

□

4. The Square Diagram of a Numerical Semigroup Represents a Dyck Path

Lemma 4.1. Let $\Lambda$ be a numerical semigroup with genus $g$ and enumeration $\lambda$. If $g(i) < i$ then $\lambda_{i+1} = \lambda_i + 1$.

Proof. If $g(i) < i$ then there are more non-gaps than gaps in the interval $[1, \lambda_i]$. So by the Pigeonhole Principle there must be at least one pair $a, b \in \Lambda$ with $a + b = \lambda_i + 1$. Therefore, $\lambda_{i+1} = \lambda_i + 1$. □

Lemma 4.2. Let $\Lambda$ be a numerical semigroup with genus $g$, conductor $c$ and enumeration $\lambda$. If $g(i) < i$ then $\lambda_i \geq c$. 

Proof. If $g(i) < i$ then there are more non-gaps than gaps in the interval $[1, \lambda_i]$. So by the Pigeonhole Principle there must be at least one pair $a, b \in \Lambda$ with $a + b = \lambda_i + 1$. Therefore, $\lambda_{i+1} = \lambda_i + 1$. □
Proof. Let us show by induction that \( \lambda_i + k \in \Lambda \) for all \( k \geq 0 \). It is obvious for \( k = 0 \). If \( \lambda_i + k' \in \Lambda \) for all \( 0 \leq k' \leq k \) then \( g(i+k) = g(i) < i \leq i + k \) and, by Lemma 4.1, \( \lambda_i + (k+1) \in \Lambda \).

Theorem 4.3. The path \( \tau(\Lambda) \) associated to a numerical semigroup \( \Lambda \) is a Dyck path.

Proof. Let \( g \) and \( c \) be the genus and the conductor of \( \Lambda \). It is enough to show that for all \( i \) with \( 0 \leq i \leq g \) we have \( g(i) \geq i \). Indeed, if \( g(i) < i \), by Lemma 4.2, \( \lambda_i \geq c \) and \( g(i) = g \), so \( i > g \), a contradiction.

Corollary 4.4. There are at most \( \frac{1}{g+1}(2g) \) numerical semigroups of genus \( g \).

There are at most \( \binom{g-1}{g-1/2} \) symmetric numerical semigroups of genus \( g \).

Proof. The first assertion is a consequence of the fact that the number of Dyck paths of order \( n \) is given by the Catalan number \( C_n = \frac{1}{n+1}(2n) \) (see [7] Corollary 6.2.3(v)).

For the second, recall that the square diagram of a symmetric numerical semigroup is symmetric in the square determined by \((0,0)\) and \((g-1,g-1)\). Such a symmetric path is determined by the first half of the path. The result follows from the fact (see [4]) that the number of paths with \( m \) steps that start at \((0,0)\) and never go below the line \( x = y \) is \( \binom{m}{[m/2]} \). \( \square \)

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