Effective Dilaton Potential in Linearized Gravity

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Considering the linearized gravity with matter fields, the effective potential of the “conformal dilaton” in the string frame is generated semiclassically by one-loop contribution of heavy matter fields. This in turn generates a non-trivial potential for the physical dilaton in the Einstein frame with the trace of the graviton in the Einstein frame gauged away. The remaining manifest local spacetime symmetry is only the volume preserving diffeomorphism symmetry. The consistency of this procedure is examined and the possibility of spontaneous diffeomorphism symmetry breaking is suggested.

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One of the most elusive parts of particle physics is the mystery of scalar particles. Introduction of spontaneous symmetry breaking induced by scalar particles in the $SU(2) \times U(1)$ gauge theory has led us to the triumphant electroweak theory, but we are still bothered by the missing existence of such a scalar particle. In much higher energy scale, we have encountered another elusive scalar particle called the dilaton, which often appears in the context of string motivated supergravity models. The actual role of the dilaton is yet to be fully understood, but we normally anticipate that it would determine coupling constants near the Planck scale\cite{1} and clarify the relation between the supersymmetric structure of string theory and that of 4-d supergravity below the compactification scale, and perhaps play a role in 4-d supersymmetry breaking itself\cite{2}\cite{3}.

In fact, the appearance of the dilaton is quite generic in any theory inherited from a scale-invariant or conformally invariant gravitational theory. We however do not know what actually controls the dynamics of the dilaton because its potential is unknown. As alluded in \cite{4}, we suspect the difficulty might partly lie on the existence of conformal diffeomorphisms. From this point of view, here we shall investigate the dilaton in the context of the linearized gravity with heavy matter fields.

The linearized theory we deal with is not renormalizable as soon as higher order terms are included. Thus we should accommodate the theory more or less in spirit of effective field theory\cite{5}\cite{6}\cite{7}\cite{8}. This approach is fairly reasonable as long as we remain in the scale where all the higher order contributions are sufficiently suppressed. We also assume there is a well-defined quantum gravity at the Planck scale, e.g. superstring theory, whose effective linearized gravity limit looks like the one we consider here. This allows us to avoid any anomalous situation that may arise. In principle, linearization of gravity must exist for any quantum gravity if we wish to regard the graviton as a particle because particles only make sense in a local Lorentz frame. Under this circumstance we can also treat the local spacetime symmetry in an equal footing as other local internal symmetries. Then we claim that it is possible to show that the conformal dilaton and the graviton behave differently\cite{1}. In a naive sense, this amounts either an anomaly or spontaneous diffeomorphism symmetry breaking. But the effective potential itself is Diff (i.e. diffeomorphism) invariant, hence we could interpret it as spontaneous Diff symmetry breaking and the conformal dilaton gets a vacuum expectation value. Furthermore, if we accept such a symmetry breaking, we can argue that the physical dilaton incorporating the string dilaton

\footnote{To serve our purpose the best, we shall, from here on, call the trace of the graviton the conformal dilaton and the rest of the graviton just the graviton.}
becomes massive, while the graviton remains to be massless. The remaining manifest symmetry is the SDiff (i.e. volume-preserving diffeomorphism) symmetry. Note that this is different from the approach in [1], where the mass of the graviton is allowed in equal footing.

Before we integrate out all the matter degrees of freedom in the linearized gravity, the renormalizable part of the Lagrangian is only approximately Diff-invariant, that is, up to higher orders of $\kappa$, where $1/\kappa \equiv m_{pl} = 1/\sqrt{8\pi G} \sim 10^{18}$GeV. The matter part of the Lagrangian however is exactly SDiff-invariant. Usually, there is a difficulty of computing the semi-classical effective potential involving external gauge fields in a gauge invariant way. However, in our case we shall only consider up to the tadpole contribution, which turns out to be Diff-invariant. The higher order terms are only SDiff-invariant. Nonetheless, it is sufficient for our purpose.

In general, any matter-gravity couplings break the diffeomorphism symmetry $h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ in the linearized gravity, hence we tend to think they are not allowed. Note that this linearized diffeomorphism transformation is reduced from the metric transformation $g_{\mu\nu} \rightarrow g_{\mu\nu} + \kappa (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu)$ such that

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \kappa (\xi^\alpha \partial_\alpha h_{\mu\nu} + h_{\mu\alpha} \partial_\nu \xi^\alpha + h_{\alpha\nu} \partial_\mu \xi^\alpha) \; ,$$  

(1)

where $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ and

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\alpha} h^{\nu}_\alpha + \cdots$$  

(2)

Then the above linear diffeomorphism transformation is obtained by taking $\kappa \rightarrow 0$ limit. Similarly, a scalar $S$ transforms

$$\delta S = \kappa \xi^\alpha \partial_\alpha S$$  

(3)

and $\delta S = 0$ is used in the linearized gravity. Thus one can easily see that any matter-gravity couplings are not allowed. This seems to be inevitable if we want the linearized theory to be related to the Einstein gravity. Nevertheless, such a missing matter-gravity coupling never conflicts with the physics in the Newtonian limit because no individual particle can have a significant matter-gravity coupling in this limit. This is why, in the Newtonian limit, matters are usually dealt in a bulk. The missing matter-gravity coupling however becomes an issue if we need to deal with a heavy particle.

This difficulty however can be overcome if we require the Diff symmetry in a more realistic way. We are interested in the system in which there is another mass parameter not extremely small compared to $m_{pl}$, yet small enough to treat the gravity classically. So we could require the
theory approximately Diff-invariant, i.e. invariant up to higher orders of \( \kappa \), under eqs.\((\ref{eq:1})(\ref{eq:3})\). Then \( S + \kappa hS \) type of matter-gravity coupling can be introduced.

This in turn creates another problem. Naive truncation in the \( 1/m_{\text{pl}} \) expansion leaves a nonrenormalizable derivative-coupling matter-gravity interaction. In our case however more careful truncation drops such a term because the \( \kappa hS \) coupling can be relatively enhanced by another big parameter as follows.

The system we investigate contains two massive fields: a scalar field \( \varphi \) and a fermion \( \psi \). If the mass \( \mu \) of \( \varphi \) and the fermion mass \( m \) are heavy enough, then \( \mu/m_{\text{pl}} \) and \( m/m_{\text{pl}} \) are not negligible compared to the contribution of the linear graviton. It turns out that \( \mu \) and \( m \) need to be of similar order to be reasonable. In this case the renormalizable scalar field couplings include only conformal dilaton couplings. The conformal dilaton also couples to the fermion with a Yukawa coupling constant \( \bar{\lambda} \) induced from the fermion mass as \( \bar{\lambda} \equiv m/2m_{\text{pl}} \). The rest of matter-gravity couplings are all suppressed by inverse powers of the modified Planck mass \( m_{\text{pl}} \).

Thus we write the action:

\[
S_{1.g.} = \int d^4x L_{1.g.} \equiv \int d^4x (L_c + L_0 + L_h + L_\varphi + L_\psi + \cdots), \tag{4}
\]

where

\[
L_c = \Lambda_0 \left( 1 + \frac{1}{2} \kappa h + \frac{1}{8} \kappa^2 h^2 - \frac{1}{4} \kappa^2 h^{\alpha\beta} h_{\alpha\beta} + \mathcal{O}(\kappa^3) \right), \tag{5}
\]

\[
L_0 + L_h = \frac{1}{8} \partial^\mu h^{\alpha\beta} \partial_\mu h_{\alpha\beta} - \frac{1}{4} \partial_\mu h^{\alpha\mu} \partial^\nu h_{\nu\alpha} - \frac{1}{8} \partial^\mu h \partial_\mu h + \frac{1}{2} \partial^\mu h \partial^\nu h_{\mu\nu}, \tag{6}
\]

\[
L_\varphi = -\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi + \frac{1}{2} \mu^2 \varphi^2 + \frac{1}{2} \lambda_1 h \varphi^2 + \frac{1}{2} \lambda_2 h^2 \varphi^2 - \frac{1}{2} \lambda_3 h^{\alpha\beta} h_{\alpha\beta} \varphi^2 - \frac{\lambda_4}{4} \varphi^4, \tag{7}
\]

\[
\lambda_1 \equiv \frac{1}{2} \mu^2, \quad \lambda_2 \equiv \frac{1}{8} \mu^2, \quad \lambda_3 \equiv \frac{1}{4} \mu^2, \quad \lambda_4 \equiv \frac{m^2}{2m_{\text{pl}}^2},
\]

\[
L_\psi = \overline{\psi} (i\gamma^\mu \partial_\mu - m) \psi - \bar{\lambda} h \overline{\psi} \psi, \quad \bar{\lambda} \equiv \frac{m}{2m_{\text{pl}}}, \tag{8}
\]

where \( h \equiv h_{\alpha\beta} \) and the indices are raised and lowered by the Minkowski metric \( \eta_{\mu\nu} \). The ellipsis contains nonrenormalizable terms suppressed by the order \( \mathcal{O}(\frac{1}{m_{\text{pl}}}) \). \( L_c \) is included to provide the necessary counter terms. This linearized action has the approximate local gauge symmetry under eqs.\((\ref{eq:1})(\ref{eq:3})\) as well as the local Lorentz symmetry. In particular, the action is not invariant term by term.

Once we decide to count the order \( \kappa \) terms in eqs.\((\ref{eq:1})(\ref{eq:3})\), the separation of \( h \) and \( \gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{4} \eta_{\mu\nu} h \) is no longer practical because \( \delta h = 0 \) is not the complete volume-preserving
condition. So we introduce a new parametrization for the conformal dilaton $\hat{h}$ and the graviton $\hat{\gamma}_{\mu\nu}$, incorporating order $\kappa$ terms, to serve our purpose the best:

$$\hat{h} \equiv h - \frac{1}{2}\kappa h^{\alpha\beta}h_{\alpha\beta},$$  \hspace{1cm} (9)

$$\hat{\gamma}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu}h + \kappa \left( \frac{1}{32}\eta_{\mu\nu}h^2 + \frac{1}{8}\eta_{\mu\nu}\eta_{\alpha\beta}h^{\alpha\beta}h_{\alpha\beta} - \frac{1}{4}h_{\mu\nu}h \right).$$  \hspace{1cm} (10)

The fields now transform under Diff as

$$\delta\hat{\gamma}_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\xi^\alpha + \kappa \left( \xi^\alpha\partial_\alpha\hat{\gamma}_{\mu\nu} + \hat{\gamma}_{\alpha\nu}\partial_\mu\xi^\alpha + \hat{\gamma}_{\mu\alpha}\partial_\nu\xi^\alpha - \frac{1}{2}\hat{\gamma}_{\mu\nu}\partial_\alpha\xi^\alpha \right),$$  \hspace{1cm} (11)

$$\delta\hat{h} = 2\partial_\alpha\xi^\alpha + \kappa \xi^\alpha\partial_\alpha\hat{h},$$  \hspace{1cm} (12)

such that $(\eta^{\mu\nu} - \kappa h^{\mu\nu})\delta\hat{\gamma}_{\mu\nu} = 0$. This gives the linearized version of the metric h-decomposition used in [4]. Under SDiff,

$$\partial_\alpha\xi^\alpha + \frac{1}{2}\kappa\xi^\alpha\partial_\alpha\hat{h} = 0$$  \hspace{1cm} (13)

so that

$$\delta\hat{\gamma}_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \kappa \left( \hat{\gamma}_{\alpha\nu}\partial_\mu\xi^\alpha + \hat{\gamma}_{\mu\alpha}\partial_\nu\xi^\alpha + \xi^\alpha\partial_\alpha\hat{\gamma}_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}\xi^\alpha\partial_\alpha\hat{h} \right),$$  \hspace{1cm} (14)

$$\delta\hat{h} = 0.$$  \hspace{1cm} (15)

Under Weyl transformations, simply

$$\delta\hat{\gamma}_{\mu\nu} = 0, \quad \delta\hat{h} = \rho.$$  \hspace{1cm} (16)

In this decomposition the conformal dilaton and the graviton never mix under SDiff or Weyl.

In terms of $\hat{h}$ and $\hat{\gamma}_{\mu\nu}$ each term in the Lagrangian now reads

$$L_c = \Lambda_0 \left( 1 + \frac{1}{2}\kappa\hat{h} + \frac{1}{8}\kappa^2\hat{h}^2 + O(\kappa^3) \right),$$  \hspace{1cm} (17)

$$L_0 = \frac{1}{8}\partial^\mu\hat{\gamma}^{\alpha\beta}\partial_\mu\hat{\gamma}_{\alpha\beta} - \frac{1}{4}\partial_\mu\hat{\gamma}^{\alpha\mu}\partial^\nu\hat{\gamma}_{\nu\alpha},$$  \hspace{1cm} (18)

$$L_{\hat{h}} = -\frac{3}{64}\partial^\mu\hat{h}\partial_\mu\hat{h} + \frac{1}{8}\partial^\mu\hat{h}\partial^\nu\hat{\gamma}_{\mu\nu},$$  \hspace{1cm} (19)

$$L_\varphi = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi + \frac{1}{2}\mu^2\varphi^2 + \frac{1}{2}\lambda_1\hat{h}\varphi^2 + \frac{1}{2}\lambda_2\hat{h}^2\varphi^2 - \frac{\lambda}{4}\varphi^4,$$  \hspace{1cm} (20)

$$L_\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \lambda\psi^\dagger\psi.$$  \hspace{1cm} (21)

Note that there is no renormalizable matter-graviton couplings in this parametrization. The graviton appears only in the kinetic energy term.

\[2\]From here on, the equality always stands for up to the leading order of $\kappa$. 

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Now we would like to call the reader’s attention to the fact that each term in the matter part of the action is in fact exactly SDiff-invariant. This could tempt us to start with the SDiff-invariance as a fundamental symmetry rather than Diff-invariance. Since the difference between Diff and SDiff are conformal diffeomorphisms, if the theory has conformal fixed points where the beta function of the Newton’s constant vanishes, the SDiff symmetry will be enhanced to Diff symmetry at these points. And we can interpret the points away from the fixed points are those with broken Diff symmetry by a condensate or a nonperturbative effect. But this requires to include the quantum effects of the gravity, which is beyond the scope of this paper.

We have chosen the fermion coupled to the conformal dilaton heavy enough to be nonnegligible. Yet we want it to be still light enough compared to the Planck scale so that we could still treat the gravity classically. So we assume there is such an intermediate mass scale fermion in nature. Perhaps we could use any grand-unification scale fermion. In this sense it is quite natural to include the scalar-gravity coupling because scalar fields are to be present for the gauge symmetry breaking. We however would not bother any gauge field contributions partly because their gravitational couplings are not renormalizable and suppressed. Now the matter fields are heavy enough, thus the conformal dilaton-matter coupling contribution is no longer negligible. We are interested in the effect of such heavy fields in this linearized gravity.

In this intermediate regime we can safely use the semiclassical method to integrate out the scalar and fermionic contribution to obtain the effective potential \( V_{\text{eff}}(\hat{h}) \), incorporating all the necessary tree level contributions. This effective potential only makes sense if \( h/m_{\text{pl}} \ll 1 \) so that only a few leading terms are meaningful. To compute the leading terms, let us set \( \varphi_c^2 = \mu^2/\lambda \) which can be taken from the gauge symmetry breaking parameter. Then we obtain

\[
V_{\text{eff}}(\hat{h}) = \Lambda + a_1 \kappa \hat{h} + a_2 \kappa^2 \hat{h}^2 + \cdots ,
\]

\[
\Lambda = -\Lambda_0 - \frac{\mu^4}{4\lambda} + \frac{\mu^4}{16\pi^2} \left( 2 \log \frac{2\mu^2}{M^2} - \frac{3}{2} \right) - \frac{m^4}{16\pi^2} \left( \log \frac{2\mu^2}{M^2} - \frac{3}{2} \right) ,
\]

\[
a_1 = -\frac{1}{2} \Lambda_0 - \frac{\lambda_1 \mu^2}{2\kappa \lambda} - \frac{\mu^4}{32\pi^2} \left( \log \frac{2\mu^2}{M^2} - 1 \right) - \frac{m^4}{8\pi^2} \left( \log \frac{2\mu^2}{M^2} - 1 \right) ,
\]

\[
a_2 = -\frac{1}{8} \Lambda_0 - \frac{\lambda_2 \mu^2}{2\kappa^2 \lambda} - \frac{\mu^4}{128\pi^2} \left( \frac{1}{2} \log \frac{2\mu^2}{M^2} - 1 \right) - \frac{m^4}{32\pi^2} \left( 3 \log \frac{2\mu^2}{M^2} - 1 \right) ,
\]

where \( M \) is the renormalization scale. If we truncate at the first order term, i.e. counting only the tadpole contribution, \( V_{\text{eff}} \) is Diff-invariant. However, if we include the quadratic term, it is no longer Diff-invariant because \( a_2 \neq a_1/4 \). As we pointed out before, this could be due to
our inability to compute the effective action in a Diff-invariant way. Perhaps, the conformal dilaton should not be really treated as a slowly varying external field in certain energy region above the mass scale of the matter fields. Therefore, we should really consider only the first two Diff-invariant terms.

One may wonder if there is any missing contribution from nonrenormalizable terms. Indeed there is and it changes numerically, but does not modify our conclusion. In fact, the dimension-five derivative matter-gravity couplings contribute to $a_i$, removing the logarithmic terms, if we use the prescription $\int d^4p1 \propto (\text{mass})^4$.

There is another issue to take care of. We want to require any renormalized cosmological constant to vanish. This can be done by properly choosing $\Lambda_0$ without a serious fine tuning as follows: First of all, we take the scale $M$ to be small for weak gravity. We need $\mu$ and $m$ to be large enough as we want $V_{\text{eff}}$ is larger than all the rest suppressed nonrenormalizable terms. Then we can show that there is $\Lambda_0$ to satisfy this requirement as long as $\mu$ and $m$ are of the same order. This determines the numerical relation between $m$ and $\mu$ in terms of $\lambda$, $M$ and $\Lambda_0$. Actual computation requires to locate the true vacuum first, but unfortunately the true vacuum cannot be located in this linearized gravity. It only indicates the true vacuum, which should exist because one can always make the potential positive asymptotically, is probably located in the strong gravity regime.

Thus, at least in semi-classical analysis, it certainly indicates that the original vacuum is no longer stable simply because $\hat{h}$ and $\hat{\gamma}_{\mu\nu}$ behave differently now. More precisely, if $a_1 \neq \frac{1}{2}\Lambda$, it indicates the vacuum instability in the linearized gravity. In other words, from the curved spacetime point of view, $\eta_{\mu\nu} + \kappa h_{\mu\nu}$ does not behave like a metric. Thus, from the generation of this effective potential, we can anticipate that the symmetry, eqs.(1)(3), is probably spontaneously broken. The remaining symmetry is nothing but the SDiff symmetry, which can be seen easily from eq.(14). Once we accept that the symmetry breaking occurs, we can use eq.(22) to compute the mass of the dilaton consistently because the effective potential is still SDiff-invariant.

$\mathcal{L}_{\hat{h}}$ contains a mixed term of $\hat{h}$ and $\hat{\gamma}_{\mu\nu}$. To obtain a system in which $\hat{h}$ and $\hat{\gamma}_{\mu\nu}$ are completely independent, we need to get rid of such a term. In the usual linearized gravity this term is removed by a gauge fixing in the Newtonian limit. Here we can remove this mixing term by incorporating the string dilaton. In fact we need to introduce the string dilaton if the linearized gravity is to be inherited from a well-defined quantum gravity like string theory. The string
dilaton is defined in the string frame such that it couples to the world-sheet curvature scalar \(\Phi\). Therefore the dilaton in the Einstein frame appears as a combination of the conformal dilaton and the string dilaton. Such manipulation for the linearized gravity in the string context can be found, for example, in [\textcolor{red}{11}][\textcolor{red}{12}][\textcolor{red}{13}]. This is due to the fact that the natural string frame and the Einstein frame are different.

This leads us to use the following Lagrangian for the string dilaton \(\Phi\):

\[
L_{\Phi} = a \partial_{\mu} \Phi \partial^{\mu} \Phi + \frac{3}{4} \partial_{\mu} \Phi \partial^{\mu} \tilde{h} - \partial_{\mu} \Phi \partial^{\mu} \hat{\gamma}_{\mu\nu},
\]

(23)

where \(a\) is a constant. \(a = 2\) corresponds to the string effective action for conformal backgrounds written in the string frame. Thus \(\Phi\) in this case actually denotes the string dilaton. We shall however leave \(a\) arbitrary for future purposes.

Now let us introduce a field redefinition

\[
h_{\mu\nu} = 2 \eta_{\mu\nu} \Phi + \tilde{h}_{\mu\nu} + \kappa (2 \phi \tilde{h}_{\mu\nu} + 2 \eta_{\mu\nu} \Phi^2).
\]

(24)

This field redefinition is nothing but the linearized version of \(g_{\mu\nu} = \text{e}^{2\kappa \phi} \tilde{g}_{\mu\nu}\) and makes sense, despite that \(\Phi\) transforms like a scalar. It corresponds to mixing of the string dilaton \(\phi\) and the conformal dilaton \(\tilde{h}\). One can also easily check out that \(\tilde{h}_{\mu\nu}\) is in fact in the Einstein frame. This field redefinition only affects the conformal dilaton so that \(\hat{\gamma}_{\mu\nu} = \hat{\gamma}_{\mu\nu}\). We then obtain the identity

\[
L_{\Phi} + L_{\tilde{h}} + L_{\phi} = L_{\Phi} + L_{\tilde{h}} + (3 + a) \partial_{\mu} \phi \partial^{\mu} \phi.
\]

(25)

If we choose \(\hat{h} \equiv \tilde{h} - \frac{1}{2} \kappa \hat{h}_{\alpha\beta} \hat{h}^{\alpha\beta} = 0\), \(L_{\tilde{h}}\) drops out so that we can successfully diagonalize the kinetic energy terms in the Lagrangian. In fact this corresponds to choosing the traceless gauge in the Einstein frame and \(\hat{h} = 8 \phi\). \(\hat{h} = 0\) can be chosen because of the presence of \(\phi\) degrees of freedom. In this context, \(\hat{h}\) takes the role of the would-be-Goldstone boson.

Thus we obtain the SDiff-invariant effective Lagrangian

\[
L_{\text{eff}}(\tilde{\gamma}, \phi) = L_{\Phi}(\tilde{\gamma}) + (a + 3) \partial_{\mu} \phi \partial^{\mu} \phi - V_{\text{eff}}(8\phi)
\]

(26)

The vev of \(\phi\) in principle can be determined by minimizing \(V_{\text{eff}}(8\phi)\).

We often hesitate to abandon the manifest Diff-invariance because we are afraid that it may lead to inconsistency of a theory. However, the lesson we learned from gauge theory is that, as long as a local symmetry is not explicitly broken, we can have a consistent theory without a manifest local symmetry. Here we are in a similar situation. The Diff symmetry is treated the
same way as any local internal symmetry and is (probably) spontaneously broken because the
manifestly symmetric vacuum has been destabilized. The conformal dilaton and the graviton
do not behave in the same way. As a first step to check the consistency, we need to ask if the
weak gravity limit could be in fact governed by such a theory in a low energy scale. The only
criterion we need to satisfy is the existence of the correct Newtonian limit in this framework.

From eq.\((26)\) we can derive the equations of motion in the string frame for \(\gamma_{\mu\nu} = \tilde{\gamma}_{\mu\nu}(\kappa = 0)\)
and \(h\) as
\[
\frac{1}{4} \partial^\mu \partial_{\mu} \gamma_{\alpha\beta} + \left(\frac{a + 3}{32}\right) \eta_{\alpha\beta} \partial^\mu \partial_{\mu} h + \eta_{\alpha\beta} \tilde{V}_\text{eff}'(h) = -\frac{1}{2} T_{\alpha\beta}
\]  
with further gauge fixing \(\partial^\mu \gamma_{\mu\nu} = 0\). Eq.\((27)\) factorizes into the trace part and the rest. The
trace part depends only on the conformal dilaton and reads
\[
\frac{a + 3}{8} \partial^\mu \partial_{\mu} \phi + 4 \tilde{V}_\text{eff}'(h) = -\frac{1}{2} T \equiv -\frac{1}{2} T_\mu.
\]  
The rest takes a familiar form
\[
\partial^\mu \partial_{\mu} \gamma_{\alpha\beta} = -2 \tilde{T}_{\alpha\beta},
\]
where \(\tilde{T}_{\alpha\beta} \equiv T_{\alpha\beta} - \frac{1}{4} \eta_{\alpha\beta} T\). This clearly shows that the conformal dilaton and the graviton
behave independently.

To take the Newtonian limit we first transform into the Einstein frame. Eq.\((29)\) remains
the same, but the trace part changes. It is more instructive if we rewrite the trace part in terms
of \(\phi\) so that
\[
(a + 3) \partial^\mu \partial_{\mu} \phi + \frac{1}{2} \tilde{V}_\text{eff}'(\phi) = -\frac{1}{2} T,
\]
where \(\tilde{V}_\text{eff}'(\phi) \equiv 8 \tilde{V}_\text{eff}'(h)\). Then, taking \(T_{00} = \rho, T_{0i} = 0 = T_{ij}\), one can easily see that eq.\((29)\)
leads to the correct Newtonian limit in the Einstein frame. We also need to turn off any quantum
effect so that we set \(V_\text{eff} = 0\). Then \(a = -5\) is required to be consistent at the Newtonian limit
so that \(\nabla^2 \phi = -\rho/4\). Thus spontaneous Diff symmetry breaking can be consistent with the
Newtonian limit.

Having \(a = -5\) is rather unpleasant because the string effective action with a conformal
background does not satisfy this condition. But, this does not necessarily mean that such
symmetry breaking does not occur in string theory. Furthermore, we do not really expect the
perturbative string action with a conformal background will describe the physics at the low
energy limit because of the strong coupling nature of the string theory\([14]\). If we wish, we
could always rescale the stress-energy tensor to meet the requirement.
It is also important to point out that $\phi$ is not the Brans-Dicke field\[15\], which provides additional gravitational degrees of freedom. $\phi$ simply replaces the trace part of the graviton with additional dynamics at higher energy scale.

Although we are not able to show the symmetry breaking explicitly because of our inability to precisely locate the true vacuum, we have at least shown the instability of the original Diff-symmetric vacuum. Therefore, the dilaton gets a vev and presumably there is a true vacuum because the potential is asymptotically positive.

Let us recapture the essence of the spontaneous symmetry breaking. At first sight, it looks quite different from gauge symmetry breaking, yet it has certain resemblance. Although no separate symmetry breaking sector is introduced as in the case of dynamical symmetry breaking, but the conformal dilaton takes its role and develops a vev. $\hat{\tilde{h}}$ takes the role of the would-be-Goldstone boson eaten by $\phi$ and, as a result, $\phi$ (presumably) becomes massive. One may think that a gauge field disappears to provide a mass to a scalar field, but it is not. In fact, $\phi$ in the Einstein frame is equivalent to $\hat{h}$ in the string frame so that one can think of $\phi$ disappearing on behalf of $\hat{h}$ in $\hat{\tilde{h}} = 0$ gauge. We have simply renamed fields in terms of a field redefinition. Therefore, we anticipate radiative spontaneous Diff symmetry breaking and the remaining symmetry is the SDiff symmetry.

The true vacuum is likely to be located in the strong gravity region, indicating the breaking of the Diff symmetry down to the SDiff symmetry occurs at much higher energy scale. We need a strong gravity formulation to locate the true vacuum precisely. We hope further investigation along the line of ref.\[1\] in 4-d theory would shed some light on the location of the true vacuum and a rigorous proof of such a symmetry breaking. A rigorous proof needs to show two things: quantization of the linear graviton and gauge invariant computation of higher order terms. We plan to address these issues elsewhere.

This also further supports the conjecture that generation of the nontrivial dilaton potential in string theory might necessarily require spontaneous breaking of Diff-invariance down to SDiff-invariance\[4\].

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