BAYESIAN ANALYSIS OF TRAFFIC FLOW ON INTERSTATE I-55: THE LWR MODEL

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Transportation departments take actions to manage traffic flow and reduce travel times based on estimated current and projected traffic conditions. Travel time estimates and forecasts require information on traffic density which are then combined with a model to project travel flow, typically the Lighthill-Whitham-Richards (LWR) traffic flow model. In this paper, we develop a particle filtering and learning method to estimate the current traffic state (density) and key parameters of the LWR model. These are critical inputs to central relationship of the LWR model, called the fundamental diagram, which describes the relationship between traffic flow and density. Our particle filtering algorithm improves on existing methods by allowing on-line updating of the posterior distribution of the critical density and capacity parameters via on-line learning. We apply the methodology to traffic flow data measured on interstate highway I-55 in Chicago and show that particle filtering algorithm allows us to accurately estimate uncertainty of the traffic state even at shock waves, where the uncertainty follows a mixture distribution. Furthermore, we show that Bayesian parameter learning can correct the estimation bias that is present when parameters are fixed.

1. Introduction. Effectively managing traffic flow to reduce congestion can improve communities by reducing travel times, reducing pollution, and improving economic efficiency. Transportation departments use information on current and projected travel times to adjust ramp metering and traffic lights; travelers use projected travel time to make travel plans and to adjust departure time, transportation mode, and route. Estimated travel times are developed using sophisticated models of traffic flow that begin with observations on speed and density and use those to develop estimates of road capacity based on estimates of current density and flow.

In their seminal paper Lighthill and Whitham (1955) describe the theory of kinematic wave motion which they then apply to model highway traffic flow. Richards (1956) independently proposed a similar application. The key assumption of these models is that there is a relation between traffic flow and traffic density. The model is calibrated using the characteristics of road segments, such as the number of lanes, free flow speed, and road type. However, these characteristics themselves do not explain all of the variations in model parameters. Thus, parameter estimates need to be corrected using observations on current speed, density, and lane configurations at sparse points throughout the network; much of the recent improvement in travel time estimation and forecasting has come from improving the estimates of network characteristics used in the model.

The underlying data on traffic speed and density are typically sparse and noisy observations. They may be observed at specific points in the traffic network using fixed loop-sensors or they may be observed at random points via GPS-equipped probe vehicles. Estimated road capacity varies continuously [Brilon, Geistefeldt and Regler (2005)] as drivers change speed in response to...
congestion, weather conditions, and the behavior of other drivers and as the number of available lanes change due to weather conditions, traffic issues, and other events. Accurately estimating road capacity from these sparse and noisy observations of traffic speed and density at points in the traffic network is a significant challenge and improving on these estimates has yielded better travel time forecasts.

In this paper we develop a particle filtering approach to estimating road capacity from the underlying sparse and noisy observations. The particle filtering approach improves on existing estimation methods because it allows for flexible and frequent updating of the estimates. Our particle filtering approach also incorporates parameter learning so that we update the model in real-time using information on the accuracy of recent forecasts. We apply our methodology to data from Chicago’s interstate I-55 highway and demonstrate how the parameter learning feature of our methodology effectively handles estimation in a dynamic environment that includes shock waves. We show that the Bayesian parameter learning embedded in our approach can correct the estimation bias that results from estimation with fixed parameters.

Our methodology builds on existing MCMC methods for estimating these inputs in a number of ways. The lower computational complexity of particle filtering in this setting makes frequent updating feasible. This is because the computational complexity of the particle filter does not grow with the length of the observation sequence, while MCMC sampler’s computational cost grows linearly. Our particle filtering methodology also allows for real time assessment of the current density state and sequential posterior parameter distributions. For MCMC application in transportation, see work on inferring network route flows in Tebaldi and West (1998). Westgate et al. (2013) use MCMC to estimate travel time reliability for ambulances using noisy GPS for both path travel time and an individual road segment travel time distributions. Anacleto, Queen and Albers (2013) develop a dynamic Bayesian network to model external intervention techniques to accommodate situations with suddenly changing traffic variables. Chiou, Lan and Tseng (2013) provide a non-parametric mixture prediction model for traffic flow trajectories. Chiou (2012) proposes using a functional mixture prediction approach.

Our approach also builds on existing work focused on parameter learning in transportation problems. For example, Dervisoglu et al. (2009) develop a quantile regression methodology that re-estimates parameters every 5 minutes based on flow and density measurements. Our approach incorporates continuous updating via continuous filtering. Wang and Papageorgiou (2005) propose a filtering method using an extended Kalman filter with boundary condition estimation. The advantage of particle filter over this Kalman filter approach is that it does not require linearizion of the system dynamics equation and does not assume normal distribution over the state vector.

Real-time estimation and short-run prediction of traffic conditions plays a key role in Intelligence Transportation Systems (ITS). Current Vehicle Navigation Systems and Traffic Management Systems use forecasts of traffic flow variables, such as traffic volume, travel speed, or travel times ranging from 5-30 minutes ahead. There are a number of real-world applications:

**Advanced Traveler Information Services (ATIS)** Multiple studies show the positive impacts of providing information on traffic flow conditions to the public [Chorus, Molin and Van Wee (2006)], as it can potentially lead to congestion relief [Arnott, De Palma and Lindsey (1991)]. Travel information is provided in multiple ways; for example, by transportation system managers such as local departments of transportation via variable message signs or radio, automakers through in-dash navigation, technology companies through phone apps or web, fleet managers and transit operators.

**Transportation Planning** The benefits of Intelligent Transportation Systems are studied by local governments based on the previous implementations. System performance data is studied
before and after ITS is deployed. An accurate comparison of the benefits to travel times requires efficient estimation of the network states.

**Control of Transportation Operations** For traffic control applications, it is necessary to efficiently estimate the formation of traffic congestion. Accurate knowledge of current state allows transportation system managers to provide a reasonable forecast of road conditions and to improve traffic flows using such techniques as ramp metering and speed harmonization.

The rest of the paper is outlined as follows. Section 2 treats the LWR model as a nonlinear state space model. Data measurements from the Illinois Department of Transportation are used to estimate traffic flows on Chicago’s interstate highway I-55. We discuss two version of the fundamental diagram (or flux function) that links traffic flow and density. It is central to specifying the LWR model. Section 3 provides a particle filtering algorithm for inference and prediction that provides on-line real-time inference. Section 4 illustrates our methodology with a simulation study of rush hour traffic on Chicago’s I-55. Finally, Section 5 concludes with directions for future research.

2. LWR Traffic Flow Model.

2.1. *Data Description.* The data was provided by the Lake Michigan Interstate Gateway Alliance ([http://www.travelmidwest.com/](http://www.travelmidwest.com/)) formally Gary-Chicago-Milwaukee Corridor (GCM). The data is available from the loop-detector sensors installed on interstate highways. A loop detector is a very simple presence sensor that senses when a vehicle is on top of it and generates an on/off signal. There are slightly more then 900 loop-detector sensors that cover a large portion of the Chicago metropolitan area. Figure 1 illustrates the locations of the detectors in the region.

![Figure 1. Locations of the Loop Detectors in Chicago](image)

Since 2008, Argonne National Laboratory archives traffic flow data every five minutes from the grid of sensors. Data contains averaged *speed, flow, and occupancy.* Occupancy is defined as percent of time a point on the road is occupied by a vehicle and count is number of off-on switches. Illinois uses a single loop detector setting and speed is estimated based on the assumption of an average vehicle length.

2.2. *The LWR Model: A State Space Formulation.* The LWR model is a macroscopic traffic flow model. It a combination of a conservation law defined via a partial differential equation and a flow-density relation, which is called the fundamental diagram. The non-linear first-order partial
differential equation describes the aggregate behavior of drivers. The density \( \rho(x,t) \) and flow \( q(x,t) \) satisfy the equation

\[
\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0.
\]

This can be solved numerically by discretizing time and space. In its simplest form, imagine a homogeneous road segment cut into \( M \) cells. Let \( \rho_i \) be the density in cell \( i \) (in veh/m) and \( q_i \) the exit flow of cell \( i \) (in veh/s). For a road segment, with given boundary conditions, the LWR computes the conditions inside the domain, as shown in Figure 2. Boundary conditions can be either measured by fixed sensors such as loop detectors or estimated from GPS probe data based on Claudel and Bayen (2010). The statistical inference problem is to fill in the missing states, learn the parameters of interest, and predict forward using the dynamics of the LWR model.

\[
\begin{array}{cccc}
\text{Boundary} & \text{Conditions} & \text{Boundary} & \text{Conditions} \\
\rho_0 & \rho_1 & \cdots & \rho_M & \rho_{M+1} \\
0 & & & L \\
\end{array}
\]

\textbf{Fig 2. Underlying state space for a road segment}

Let \( y_{t+1} \) be observed traffic flow, the observation matrix \( H_{t+1} \) picks out cells with measurements available. Let \( \theta_t \) be a hidden state vector of traffic densities, where the boundary conditions \( \rho_{0t} \) and \( \rho_{(M+1)t} \) are given. We denote

\[
\theta_t = (\rho_{1t}, \ldots, \rho_{Mt})
\]

The next state \( \theta_{t+1} = f_\phi(\theta_t) \) is given by the solution of the LWR model. A discretized Godunov’s scheme computes \( f_\phi(\theta_t) \) given the parameters, \( \phi \), of the fundamental diagram.

\[
\begin{align*}
y_{t+1} &= H_{t+1} \theta_{t+1} + v_{t+1} \quad \text{where} \quad v_{t+1} \sim \mathcal{N}(0, V_{t+1}) \\
\theta_{t+1} &= f_\phi(\theta_t) + w_{t+1} \quad \text{where} \quad w_{t+1} \sim \mathcal{N}(0, W_{t+1}),
\end{align*}
\]

with given error covariances \( V_t \) and \( W_t \). Section 3 provides a full Bayesian analysis of this nonlinear state space model.

\subsection*{2.3. The Fundamental Diagram}

An important feature of the LWR model is the emergence of a shock waves of traffic due to the density-dependent local propagation velocities. The triangular fundamental diagram has two velocities of density variations; one for free-flow traffic and one for congested traffic. This specification allows for an efficient Godunov’s scheme to solve the nonlinear evolution dynamics. However, we also need to provide the model with an accurate assessment of the current density state vector and the parameters of the fundamental diagram. The fundamental diagram is a key input into the specification of the LWR model, which expresses the relationship between traffic density and flow. To motivate the choice of a triangular flux function Figure 3 shows the empirical volume and occupancy for a highway segment in Chicago metropolitan area together
with the speed at cell. In this paper we assume that occupancy and traffic density are linearly related and treat the interchangeably.

![Fundamental Diagram Based on Measured data on I-55 North Bound](image)

We assume a homogeneous road segment with a flux function that does not depend on time and space. Two common flux functions are Greenshields and Newell-Daganzo (or triangular) where

\[
\text{Greenshields: } q(\rho) = \frac{v_f}{\rho_{jam}} \rho (\rho_{jam} - \rho)
\]

(2.2) \[
\text{Newell-Daganzo: } q(\rho) = \begin{cases} 
\frac{q_c}{\rho_{c}} \rho & \rho < \rho_{c} \\
\frac{q_c}{\rho_{jam} - \rho_{c}} \rho_{jam} - \rho & \rho \geq \rho_{c}
\end{cases}
\]

![Fundamental diagram](image)

The parameters of the model are \( \phi = (q_c, \rho_c, \rho_{jam}) \), representing the critical flow (capacity), critical density and jam density, respectively.
Under uninterrupted flow, the flow speed, $v(x,t)$, is related to density and flux via

$$v(x,t) = \frac{q(x,t)}{\rho(x,t)}$$

The Rankine-Hugoniot relation [LeVeque (2002)] determines the shock wave velocity, as the velocity of the shock $w$ times the jump in density which equals the jump in flow in the two regions separated by the shock where

$$w = \frac{q(\rho_l) - q(\rho_r)}{\rho_l - \rho_r} \quad \text{and} \quad v_f = \frac{q_c}{\rho_c}.$$

The direction of the shock wave propagation depends on the sign of $q(\rho_l) - q(\rho_r)$. Here $v_f$ is a free flow speed on a link and $q_c = \max_{\rho} q(\rho)$ is the critical flow or capacity of the link. Correspondingly, $\rho_c = \arg \max_{\rho} q(\rho)$ is called the critical density. The pair $(q_c, \rho_c)$ are the so called breakdown point of a road segment.

Figure 5 demonstrates speed-density relation based on Greenshields and triangular fundamental diagrams.

Calibrating the model parameters can be done in a number of ways. The standard approach uses values from the Highway Capacity Manual [Transportation Research Board (2010)]. In practice, we wish to learn parameters in real time and will develop a particle filter for this task.

To empirically illustrate the stochastic nature of the parameters, we estimate capacity and critical density from the measurements for 242 days in 2009, on a segment of interstate highway I-55 in Chicago. Holidays, weekends as well as days with unreliable measurements were excluded. Figure 6 plots $\rho_c$ and $q_c$ across the days. Clearly, there is a nonlinear relation between $q_c$ and $\rho_c$. Capacity has a lot of variation from day to day and the distribution has a heavy left tail. On the other hand, critical density $\rho_c$ has relatively tight distribution around 0.023 [veh/m] for our dataset.
2.4. Godunov’s scheme for solving the LWR Model. The LWR model shown in Equation 2.1 describes the evolution of traffic flow on a road segment with uniform topology (see also Appendix A). The change in road segment characteristics (crossing, number of lanes, speed limit, curvature, etc.) can be modeled using a junction. The treatment of junctions requires specific efforts for physical consistency and mathematical compatibility with the link model. For uniqueness of the solution of the junction problem, different conditions have been used: for instance, maximizing the incoming flow through the junction was suggested by Daganzo (1995) and Coclite, Garavello and Piccoli (2005). Holden and Risebro (1995) consider maximizing a concave function of the incoming flow. A formulation using internal dynamics for the junction [Lebacque (2005)] has been shown to be equivalent to the vertex models for the merge and diverge junction, see Garavello and Piccoli (2006) for more details.

Given an initial condition \( \rho_0(x), \ x \in [0, L] \), propagating the LWR model requires solving the associated Cauchy problem. If the initial condition is piecewise constant (which is the case for many numerical approximations) and self-similar, the Cauchy problem reduces to a Riemann problem. The Godunov’s scheme solves a Riemann problem between each cell. This is a Cauchy problem (initial value problem) with initial conditions having a single discontinuity

\[
\rho_0(x) = \begin{cases} 
\rho_l, & x < 0 \\
\rho_r, & x > 0.
\end{cases}
\]
For the Riemann problem, the speed of the shock wave propagation is given by the Rankine-Hugoniot relation

\[ w = \frac{q(\rho_l) - q(\rho_r)}{\rho_l - \rho_r}. \]

Heuristically, imagine at initial time \( t = 0 \), that there are two regions in the domain with different values of thermodynamic parameters (flow, density and speed in our case). Two regions are divided by a thin membrane and at the initial time the membrane is being removed. The computational problem is to find the values of thermodynamic parameters at all future times.

Standard finite difference schemes are too inaccurate for solving LWR model, Godunov (1959) showed that a first order finite difference scheme is inaccurate for calculating with a small time step. Moreover, none of the second order difference scheme preserve monotonicity of the \( \rho_0 \) and thus are not applicable.

According to Godunov’s scheme, we calculate iterates

\[ \rho_i^{n+1} = \rho_i^n + \frac{\tau}{h} \left( q_G(\rho_{i-1}^n, \rho_i^n) - q_G(\rho_i^n, \rho_{i+1}^n) \right) \]

where \( \rho_i^n \) is the density value at point with coordinates \( x = ih, t = n\tau \), with \( h \) a space discretization step and \( \tau \) is time discretization step. The function \( q_G(\rho_l, \rho_r) \) is defined by

\[ q_G(\rho_l, \rho_r) = \begin{cases} 
q(\rho_l), & \rho_r < \rho_l \leq \rho_c \\
q(\rho_c), & \rho_r \leq \rho_c \leq \rho_l \\
q(\rho_r), & \rho_c \leq \rho_r < \rho_l \\
\min(q(\rho_l), q(\rho_r)), & \rho_l < \rho_r.
\end{cases} \]

Typically a virtual cell is introduced on both sides of the domain to include boundary conditions (in and our flow) given by left boundary

\[ \rho_0^{n+1} = \rho_0^n + \frac{\tau}{h} \left( q_G(\rho_{-1}^n, \rho_0^n) - q_G(\rho_0^n, \rho_1^n) \right), \text{ with } \rho_{-1}^n = \frac{1}{\tau} \int_{(n-1)/2\tau}^{(n+1)/2\tau} \rho(0, t) dt 
\]

and right boundary

\[ \rho_M^{n+1} = \rho_M^n + \frac{\tau}{h} \left( q_G(\rho_{M-1}^n, \rho_M^n) - q_G(\rho_M^n, \rho_{M+1}^n) \right), \text{ with } \rho_{M+1}^n = \frac{1}{\tau} \int_{(n-1)/2\tau}^{(n+1)/2\tau} \rho(L, t) dt. \]

Numerical stability in space and time is ensured by the Courant-Friedrichs-Lewy type condition: \( \tau \leq h/|v_{\text{max}}| \), where \( v_{\text{max}} \) is the maximum wave velocity present in the meshed domain at any given point in time, see, for example, Courant, Friedrichs and Lewy (1928).

### 2.5. State Uncertainty is a Mixture.

When uncertainty about state gets propagated from one time step to another using Godunov’s scheme, the unimodal distribution might become a mixture distribution. This happens, at the location of a shock wave, when cell on the right is in a free flow regime and cell on the left is in congested regime. This can be demonstrated by a Monte Carlo experiment. Consider two consecutive cells, with densities \( \rho_l \) and \( \rho_r \) correspondingly. Assume, \( \rho_l \sim TN(0.02, 0.01, 0.0, 0.2) \) and \( \rho_r \sim TN(0.03, 0.01, 0.0, 0.2) \). Using triangular fundamental diagram with \( q_c = 1600 \text{ veh/h}, \rho_c = 0.025 \text{ veh/m}, \) and \( \rho_{\text{jam}} = 0.2 \text{ veh/m} \), we can calculate the speed of the shock wave propagation given by the Equation 2.4. We simulate the distribution over \( \sigma \) using Monte Carlo experiment with \( N = 1000 \) samples. Figure 7 shows the results of the experiment. We can see that the uncertainty over speed propagation is a mixture distribution. Thus, the uncertainty about the density in the in the future times is also a mixture. We further demonstrate this concept in Section 4 by plotting the histogram of a flow density estimated by particle filter.
3. Bayesian Analysis and Particle Filtering of the LWR Model.

3.1. LWR as a Nonlinear State Space Model. Our model has a state-space formulation of observation and evolution system given by

\begin{align}
\text{Observation: } & y_{t+1} = H_{t+1}\theta_{t+1} + v_{t+1}; \quad v_{t+1} \sim N(0, V_{t+1}) \tag{3.1} \\
\text{Evolution: } & \theta_{t+1} = f_\phi(\theta_t) + w_{t+1}; \quad w_{t+1} \sim N(0, W_{t+1}) \tag{3.2}
\end{align}

The state $\theta_t = (\rho_{1t}, \ldots, \rho_{Mt})$, corresponds to a graphical model with conditional independence structure given in Figure 8. Particle filtering will provide samples from the filtered posteriors $p(\theta_t|y^t)$ and $p(\phi|y^t)$.

![Parameter learning graphical model]

We are modeling $\theta \in \mathbb{R}^M$ as a vector of densities for each volume cell of the road segment. The non-linear operator $f_\phi$ is defined by equations (2.5) - (2.6). The operator $H : \mathbb{R}^M \rightarrow \mathbb{R}^k$ in the measurement model depends on the sensor, and in our setting we make it linear. If speed is measured, $H$ can be non-linear, depending on the functional form of the fundamental diagram. For
example, in Greenshields diagram, the speed-density relation is linear. For a triangular diagram the relationship is non-linear, but is accurately approximated with a piece-wise linear function. Conversely, when the traffic flow \( q \) is measured, the flow-density relation is piece-wise linear for triangular diagram and non-linear for Greenshields diagram.

3.2. A Fully Adapted Particle Filter. Let \( y_t \) denote observed data and \( y^T = (y_1, \ldots, y_t) \), \( \theta_t \) be a state variable and \( \phi \) denote unknown parameters. Particle filtering methods are designed to provide sequential state inference from the set of filtered posteriors \( p(\theta_t|y^T) \); see, for example, Gordon, Salmond and Smith (1993), Carpenter, Clifford and Fearnhead (1999), Pitt and Shephard (1999), Liu and West (2001), Storvik (2002), Carvalho et al. (2010). We will develop an algorithm based on a fully adapted version of the Liu and West filter.

To implement our particle filter, we factorize the joint conditional distribution as

\[
p(y_{t+1} | \theta_t, \phi) = \int p(y_{t+1} | \theta_{t+1}, \phi) p(\theta_{t+1} | \theta_t, \phi) d\theta_{t+1}.
\]

Propagation of states requires the conditional posterior for the next state given by \( p(\theta_{t+1} | \theta_t, \phi, y_{t+1}) \). These densities can be computed using the model assumptions via

\[
p(y_{t+1} | \theta_t, \phi) \sim N(H_{t+1} \theta_{t+1}, V_t)
p(\theta_{t+1} | \theta_t, \phi) \sim N(f_\phi(\theta_t), W_t).
\]

Therefore, we have distributions

\[
p(y_{t+1} | \theta_t, \phi) \sim N(H_{t+1} f_\phi(\theta_t), H_{t+1}^T W_t H_{t+1} + V_t)
p(y_{t+1} | \theta_t, \phi) \sim N(H_{t+1} \theta_{t+1}, V_{t+1}).
\]

For propagation of \( \theta_{t+1} \), we use Bayes’ rule

\[
p(\theta_{t+1} | \theta_t, \phi, y_{t+1}) \sim N(\mu_{t+1}, C_{t+1})
\]

where the mean and variance follow the Kalman recursion

**Forecast:** \( \mu_f = f_\phi(\theta_t), C_f = W_{t+1} \)

**Kalman Gain:** \( K = C_f H_{t+1}^T (H_{t+1} C_f H_{t+1}^T + V_{t+1})^{-1} \)

**Measurement Assimilation:** \( \mu_{t+1} = \mu_f + K(y_{t+1} - H_{t+1} \mu_f), C_{t+1} = (I - KH_{t+1}) C_f \)

To develop our particle filter, we factorize the joint conditional distribution as

\[
p(y_{t+1}, \theta_{t+1} | \theta_t, \phi) = p(y_{t+1} | \theta_{t+1}, \phi) p(\theta_{t+1} | \theta_t, \phi)
\]

\[
= p(y_{t+1} | \theta_t, \phi) p(\theta_{t+1} | \theta_t, \phi, y_{t+1})
\]

The goal is to obtain the new filtering distribution \( p(\theta_{t+1} | y^{t+1}) \) from the current \( p(\theta_t | y^T) \) and to provide a particle approximation to the parameter posterior, \( p(\phi | y^T) \). We start with a particle (a.k.a random histogram of draws) filtering approximation for the joint distribution

\[
p^N(\theta_t, \phi | y^T) = \frac{1}{N} \sum_{i=1}^{N} \delta_{(\theta_t, \phi)^{(i)}}
\]
where $\delta$ is a Dirac measure. As the number of particles increases $N \to \infty$ the law of large numbers guarantees that this distribution converges to the true filtered distribution $p(\theta_t, \phi | y^t)$. This is in contrast to MCMC methods which rely on ergodic results to guarantee convergence.

For the next posterior distribution, we can use Bayes rule to write

$$p^N(\theta_{t+1} | y^{t+1}) = \sum_{i=1}^{N} w_t^{(i)} p \left( \theta_{t+1} | (\theta_t, \phi)^{(i)}, y_{t+1} \right)$$

where the particle weights are given by

$$w_t^{(i)} = \frac{p \left( y_{t+1} | (\theta_t, \phi)^{(i)} \right)}{\sum_{i=1}^{N} p \left( y_{t+1} | (\theta_t, \phi)^{(i)} \right)}.$$

The algorithm consists of three steps:

Step 1. (Resample) Draw an index $k(i) \sim Mult_N \left( w_t^{(1)}, ..., w_t^{(N)} \right)$ for $i = 1, ..., N$

Step 2. (Propagate) Draw $\theta_t^{(i)} \sim p \left( \theta_{t+1} | (\theta_t, \phi)^{k(i)}, y_{t+1} \right)$ for $i = 1, ..., N$.

Step 3. (Replenish) Draw $\phi_t^{(i)} \sim \frac{1}{N} \sum_{i=1}^{N} \delta_{[-\epsilon, \epsilon]}(\phi^{k(i)})$

where $\delta_{[-\epsilon, \epsilon]}(\cdot)$ denotes the Dirac measure in an interval $[-\epsilon, \epsilon]$. The re-sampling creates a new set of particles $(\theta_t, \phi)^{k(i)}$.

A practical advantage of our approach is that it does not suffer from particle degeneracy which plagues standard sample-importance re-sample filter.

4. Simulation Study. The goal of our study is to model traffic flow on Chicago’s I-55 highway. Figure 9 below shows traffic patterns on February 6th, 2009 (Friday) and all five work days of the following week (week of February 9th). We pick a specific day for that purpose, which is very similar from work day to work day. From our previous discussions, the empirical measurements of the flow-density relation imply a triangular fundamental diagram.
Several conclusions can be drawn from the traffic patterns:

(i) The break down start time is different from day to day, even on the same day of week (Fridays) of different weeks

(ii) the duration of the flow at the lowest speed is different, Wednesday being the worst and Thursday the best

(iii) the breakdown period is shorter then the recovery period

The last point follows from the asymmetric shape of the triangular fundamental diagram, where the free flow speed \( v_f \) (speed at which drivers arrive to the end of the congestion queue) is higher than backward wave propagation speed \( w \) (speed at which drivers depart from the front of the congestion queue).

We simulate a road segment of length 1.5 kilometers and time horizon of 1600 seconds. Figure 10 shows the graphical representation of our road segment model and its discretization scheme.
Our discretization grid cell is of length $h = 300$ meters and time interval of $\tau = 5$ seconds. The initial conditions are set to be uniform traffic density of 0.01 $veh/m$ and boundary conditions are shown in Figure 11.

As a typical morning traffic pattern, our simulated data set begins with free-flow traffic regime followed by breakdown and then recovery. The breakdown starts 3 minutes into the simulation and the recovery starts at 10 minute mark. All three traffic flow characteristics (flow, density and speed) are simulated for each cell of the road segment.

To illustrate the dynamics of parameter learning, we change the LWR parameters $\rho_c$ and $q_c$ several times throughout the simulation. Figure 12 shows the times at which the parameters were changed and the corresponding values. These shifts are typical of the actual traffic flows.
We choose a speed profile that simulates a typical traffic pattern that occurs during a morning peak hour traffic flow. The speed profile is depicted in Figure 13.

Figure 13(b) shows the morning period speed profile measured on one of the I-55 north bound on the specific date of February 6, 2009.

Figure 14 compares the true density simulated for cell 3 with the particle filter estimate that relies on measurements at the ends of the domain. We can see the sensitivity to parameter learning. Without learning, the density profile is shifted in a meaningful way. Clearly, full Bayesian parameter learning corrects this bias.
Figure 14 shows the estimated density at cell 3.

Figure 15 shows the expected value and 95th percentile of the filtered posterior distribution of the model parameters. We can see, as expected, there is a certain delay between the underlying parameter changed and the filtering algorithm captures the change. Change in capacity is picked up faster than change in critical density.

Figure 16 shows the particle histogram plots of the capacity and critical density posteriors at a specific time, $t = 1315$ seconds.
In Subsection 2.5 we showed that the distribution over traffic density is a mixture distribution at the locations when the density in two consecutive cells lies on two sides of critical density. To further demonstrate this fact, Figure 17 shows the distribution over density at cell 3 before, and after the shock wave travels through the cell.

Figure 13(a) shows that a shock wave travels through the cell 3 between $t = 600$ and $t = 700$. We can see that the distribution over state is a unimodal at those time steps. However, in between, it is a mixture.

5. Discussion. This paper analyses the LWR traffic flow model applied to Chicago’s interstate I-55 highway. We show how particle filtering method can be used to provide real-time estimate of the
density states. We sequentially learn the parameters of the fundamental diagram which is the central input for the LWR dynamics of traffic flow. Our results have a number of important implications for transportation system management applications. In particular, real-time assessment of model states and parameters can correct biases in estimating the current density of states used for forecasting.

We apply our method on simulated experiments based on data from Chicago’s interstate I-55 highway. When the measurements are sparse, and the parameters are fixed, the filtering algorithm mis-estimates the current state. This can be corrected by having the same measurements and including parameters estimated simultaneously. This leads to an accurate estimation of traffic density.

There are a number of possible extensions to our approach. First, the LWR model is only valid when the relationship between flow and density is time independent. Second, the model does not describe traffic behavior within a queue or when particular instabilities such as stop and go traffic exist. Third, the model is not realistic for free-flowing traffic, as the vehicle bypassing that happens frequently in this regime is not captured. Although, from a system management perspective free flow traffic is not an issue, extending our approach to higher order traffic flow models will lead to improvements in estimation.

Developing methods to incorporate model monitoring is an important area of future research. For example, alternative models might correspond to different assumptions about the shape of the fundamental diagram. We can statistically discriminate two models using a sequential likelihood ratio (Bayes factor), $B_t$, given by

$$B_t = \frac{p(y_1, ..., y_t | M_1)}{p(y_1, ..., y_t | M_0)},$$

where $p(y_1, ..., y_t | M_i) = \prod_{j=1}^{t} p(y_j | y_{1:j-1}, M_i)$. This is simply a product of marginal predictive densities, which the particle filter approximates by

$$p^N(y_j | y_{1:j-1}, M) = \frac{1}{N} \sum_{i=1}^{N} p(y_j | \theta^i_{j-1}, M).$$

We plan to extend our particle algorithm to a transportation network with simultaneous tracking of multiple segments. This will make out methodology applicable to real life transportation networks of a large metropolitan area. Within our framework, it is feasible to filter over the boundary conditions. This also applies in the case of GPS probes where inferring the boundary conditions is a hard task as location and time of the measurement are random and one rarely observes the boundary conditions.

**APPENDIX A: DERIVATION OF FLOW MODEL**

Let $q(x, t)$, $\rho(x, t)$ and $v(x, t)$ denote traffic flow, density and speed at position $x$ at time $t$. Kinematic wave theory establishes a relationship between density $\rho$ and flow $q$, which is known as the fundamental diagram given by the functional equation $q(x, t) = q(\rho(x, t), x)$ where $q$ is called flux.

The conservation law implies that with no inflow or outflow

$$\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial q(x, t)}{\partial x} = 0$$

(A.1)

Combined with flux function, we obtain the equation for $\rho(x, t)$

$$\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial q(\rho(x, t), x)}{\partial x} = 0$$

$$\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial q(\rho(x, t), x)}{\partial x} = 0$$
or
\[
\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial q(\rho, x)}{\partial \rho} \frac{\partial \rho(x,t)}{\partial x} = 0
\]

The term \( w = \frac{\partial q(\rho, x)}{\partial \rho} \) is called the wave velocity. To get a more intuitive understanding of the problem it is convenient to use the cumulative flow \( N(x,t) \), number of vehicles that pass location \( x \) by time \( t \). Then the conservation law can be derived by evaluating
\[
\frac{\partial N}{\partial t} = q(x,t), \quad \frac{\partial N}{\partial x} = -\rho(x,t)
\]
Assuming that \( N \) is smooth
\[
\frac{\partial^2 N}{\partial x \partial t} = \frac{\partial^2 N}{\partial t \partial x}
\]
we get the conservation law (A.1). In practice the function \( N \) has discontinuity of the first kind (first derivative), however, the conservation law holds in case of discontinuities as long as \( N(x,t) \) is continuous along the shock path. Method of characteristics can be used to solve the equation (A.1). Specifically, from (A.1) \( \rho(x,t) \) is constant \( (d\rho/ds = 0) \) along a characteristic curve (wave) described by
\[
\frac{dt}{ds} = q'(\rho)
\]
Eliminating \( s \), gives
\[
\rho(x,t) = \rho(x - q'(\rho_0)t)
\]
Thus density is constant along the straight line with slope \( dq/d\rho \) (characteristic line) and the slope is nothing but a shock propagation speed. For a free flow speed the shock moves forward and for jammed traffic it moves backward. In Newell’s case the forward shock propagation speed is \( v_f \) and the backward shock propagations speed is given by \( w \).

**APPENDIX B: DERIVATION OF KALMAN RECURSION**

Measurements are taken at the first and last cell of the road segment and noise independently distributed, with covariance structure \( V_t = V = vI_2 \) and \( W_t = W = wI_5 \). The operator \( H_t \) and Kalman gain matrix \( K_t \) are of the following form
\[
H_t = H = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
K_t = K = \begin{pmatrix}
w \\
v + w
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
This leads us to the following Kalman updates
\[
C_{t+1} = \begin{pmatrix}
w \left(1 - \frac{w}{v+w}\right) & 0 & 0 & 0 & 0 \\
0 & w & 0 & 0 & 0 \\
0 & 0 & w & 0 & 0 \\
0 & 0 & 0 & w & 0 \\
0 & 0 & 0 & 0 & w \left(1 - \frac{w}{v+w}\right)
\end{pmatrix},
\]
\[
\mu_{t+1} = \left(\mu_1^f + \frac{w \left(y_1 - \mu_1^f\right)}{v + w}, \mu_2^f, \mu_3^f, \mu_4^f, \mu_5^f + \frac{w \left(y_2 - \mu_5^f\right)}{v + w}\right)^T.
\]
The variance of the predictive likelihood distribution is given by

\[ HWH^T + V = \begin{pmatrix} v + w & 0 \\ 0 & v + w \end{pmatrix}. \]
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