Random walks on Convergence Groups

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Abstract

We extend some properties of random walks on Hyperbolic groups to random walks on Convergence groups. In particular, we prove that if a convergence group $G$ acts on a compact metrizable space $M$ with the convergence property, then, using a previous construction by Tukia [Tuk98], we can provide $G \cup M$ with a compact topology, which works as an extension of Gromov’s topology, such that random walks on $G$ converge almost surely to points in $M$. Furthermore, we use Maher and Tiozzo’s result [MT18], to prove that if the random walk is finitely generated, then $M$, with the corresponding hitting measure, can be seen as a model for the Poisson Boundary of $G$.

1 Introduction

Consider a countable group $G$, and a probability measure $\mu$ supported on that group. We can define a random walk on $G$ by fixing a starting point, and successively multiplying it by independent elements of $G$ according to the probability $\mu$, that is, defining

$$w_n := g_0 g_1 \cdots g_n,$$

where $g_0$ is our started fixed point, and $g_i$ for $i \geq 1$ are independent and identically distributed (with distribution $\mu$) random variables with values on $G$. In this thesis we will study the asymptotic behaviour of such processes for a class of hyperbolic-like groups.

In the case where $G$ is a $\delta$-hyperbolic group, we can consider the embedding of $G$ into $G \cup \partial G$, where $\partial G$ is the Gromov boundary of $G$. Kaimonovich showed in [Kai97] that, under some assumptions on the measure $\mu$, the sample paths $(w_n)$ converge almost surely to points in the Gromov boundary. Furthermore, he showed that $\partial G$, together with the corresponding hitting measure $\nu$, form a model for the Poisson boundary; that is, $(\partial G, \nu)$ seen as a measure space, encodes all the asymptotically relevant information of the sample paths (for a formal definition of the Poisson boundary, see [Kai96]). Similar results have been proven for many hyperbolic-like groups (see, for example, [Tio15] and [Kai00]).

We extend these results to groups that act on a space $M$ in the same way that hyperbolic groups act on their Gromov boundaries. In concrete,
Definition. Let $G$ be a discrete countable group acting on a compact metrizable space $M$. $G$ is called a convergence group if for every infinite sequence $(g_n) \subset G$ of distinct elements, there exists a subsequence $(g_{n_k})$ and points $a, b \in M$ such that $g_{n_k}|_{M \setminus a}$ converges to $b$ locally uniformly, that is, for every compact set $K \subset M \setminus a$, and every open neighbourhood $U$ of $b$, there is $N$ such that $g_{n_k}(K) \subset U$ whenever $n_k > N$.

It is fairly easy to see that hyperbolic groups act as convergence groups on their Gromov boundaries (see, for example, [Bow99]), so in this definition $M$ plays the role of the Gromov boundary of the group. Hence, one would hope that the previous results about random walks on Gromov hyperbolic groups also worked in convergence group, replacing $\partial G$ by $M$. Indeed, we prove the following.

Theorem 5.2. Let $G$ be a countable, discrete group acting as a convergence group, non elementary and minimally on a metrizable compact space $M$. Then, there exists a compact topology on $G \cup M$ such that the inclusions $G \hookrightarrow G \cup M$, $M \hookrightarrow G \cup M$ are topological embeddings and, for any generating measure $\mu$ on $G$, the sample paths of the associated random walk on $G$ converges almost surely to points in $M$.

By non elemental action we mean an action such that there is no invariant subset consisting of 2 or 1 point. To prove this result, we use a construction done by Tukia in [], which consists in observing that, just as $M$ is the equivalent of the Gromov boundary, the space of distinct triples $T = \{(a, b, c) \in M^3 | a \neq b \neq c \neq a\}$ is the equivalent of the hyperbolic space upon which $G$ acts. Generalizing from the case of Kleinian groups, he gives a compact Topology to $T \cup M$, and from here we get, in section 5.1 a compact topology on $G \cup M$. To see that the random walk converges to the boundary we use fairly standard methods, and apply directly the definition of convergence groups.

To see whether $M$ works as a model for the Poisson boundary we use a quasimetric $\rho$ for $T$, introduced by Bin Sun in [Sun16], which makes $(T, \rho)$ quasiisometric to a hyperbolic space $(S, d)$, upon which $G$ acts by isometries and with a weakly properly discontinuous (WPD) element, that is, there exists an element $h \in G$ such that, for every $s \in S$ and $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$|\{f \in G | d(s, fs) < \epsilon, d(h^K s, fh^K s) < \epsilon\}| < \infty.$$  

To use this metric, we use a theorem proven by Maher and Tiozzo in [MT18]. Applied to our situation, Maher and Tiozzo’s theorem gives us that, under some restrictions on the measure $\mu$, the Gromov boundary of $S$, together with it’s hitting measure, form a model for the Poisson boundary. In our
case the conditions for the measure are hard to check, as they involve the metric, which is not straightforward. However, the conditions are automatically satisfied whenever the measure has finite support, that is, when the random walk is finitely generated. Using the quasiisometry between $S$ and $T$ we can identify the Gromov boundary of $T$ as the Poisson Boundary. However, the Gromov boundary of $T$ may be a complicated object, and a priori we have no direct to relate it with $M$. We build a $G$-equivariant homeomorphism between a subset of $M$ and a subset of the Gromov boundary of $T$. We also see that the subset of the Gromov boundary of $T$ chosen for the homeomorphism has full measure under the hitting measure, so we get the following theorem.

**Theorem 4.8.** Let $G$ be a finitely generated group acting as a convergence group, minimally and non-elementary on a compact space $M$, and $\mu$ a probability measure generating $G$ with finite support. Then, $(M, \nu)$ is the Poisson boundary of $(G, \mu)$, where $\nu$ is the $\mu$-stationary Borel probability measure on $M$.

As a corollary of the proof, (in particular, of proposition 4.6) we get an alternative proof to the fact that the set of conical limit points of $M$ introduced by Tukia in [Tuk98] has full measure under the stationary measure. A more quantitative statement of this fact has been proven by Gekhtman, Gerasimov, Potyagailo and Yang in [GGPY17] (Theorem 9.14 and 9.15).

Many results about a group apparently not related with random walks can be obtained by studying their asymptotic behaviour. For example, we say that a function $f : G \to \mathbb{R}$ is $\mu$-harmonic if $f(g) = \sum_{h \in G} \mu(h)f(gh)$, that is, if the value at a point is the average of the values at the neighbouring points, using $\mu$ for ponderation. If $(M, \nu)$ is the Poisson boundary of $(G, \mu)$, there exists an explicit isomorphism from $L^\infty(M, \nu)$ to the bounded $\mu$-harmonic functions on $G$. Also, using the convergence of the random walks to $M$ one can show that the action of $G$ on $(M, \nu)$ is strongly almost transitive, that is, given $\epsilon > 0$ and $A \subset M$ with $\nu(A) > 0$, there exists $g \in G$ such that $\mu(gA) > 1 - \epsilon$. Having a non trivial strongly almost transitive action has interesting implications, and we refer to [GW16] for a compilation of some.

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## 2 Preliminaries

### 2.1 Hyperbolicity and quasi-metric spaces

Let $(X, d)$ be a geodesic metric space, i.e., a metric space such that for any two points $a, b \in X$ there exists a curve $[a, b]$, with length equal to
the distance between $a$ and $b$. That curve may not be unique, and by $[a,b]$ we mean an arbitrary choice of that family. Given a set $A \subset X$ and $r > 0$, we will denote by $N(A,r)$ the $r$ neighbourhood of $A$, that is, 

$\{x \in X \mid d(x,A) \leq r\}$.

Given $\delta > 0$, we say $X$ is $\delta$-hyperbolic, or Gromov hyperbolic, if it satisfies the $\delta$-slim triangles condition, that is, given any three points $a,b,c \in X$, the geodesic $[a,b]$ is contained in $N([b,c] \cup [a,c], \delta)$.

In this way, hyperbolic spaces may be seen as generalizations of trees, where the $\delta$-slim triangles condition is satisfied by $\delta = 0$. One could say that the triangles on hyperbolic spaces behave, "from far away", as the triangles on trees. In fact, many theorems which are true for trees turn out to be true as well for hyperbolic spaces, up to some constant depending on $\delta$. The $\delta$-slim condition can be seen to be equivalent to some others, such as the $\delta$-center triangles condition or the 4-point condition.

In this paper we will deal with a relaxation of the notion of metric, where we soften the triangle inequality by an additive constant, and allow two different points to have 0 distance. At large scale, this relaxation is indistinguishable from a metric, so many results about hyperbolicity go through. Here is a precise definition:

**Definition 2.1.** Given $r \geq 0$, an $r$-quasimetric $\rho$ on a set $Q$, is a function $\rho : Q^2 \to [0, +\infty)$, satisfying $\rho(x,x) = 0$, $\rho(x,y) = \rho(y,x)$ and $\rho(x,y) \leq \rho(x,z) + \rho(z,y) + r$ for all $x,y,z \in Q$.

A quasimetric is an $r$-quasimetric for some $r > 0$. Given $s \geq 0$ and a quasimetric space $(Q, \rho)$, an $s$-geodesic segment is a finite sequence of points $x_0, x_1, \ldots, x_n$ such that $|i-j| - s \leq \rho(x_i, x_j) \leq |i-j| + s$ for all $0 \leq i,j \leq n$. We will also denote by $[a,b]$ an arbitrary choice of $s$-geodesic segment between $a$ and $b$, that is, such that $a = x_0$ and $b = x_n$. A quasimetric is a path quasimetric if there exists $s \geq 0$ such that every pair of points can be connected by an $s$-geodesic segment. A path quasimetric is called hyperbolic path quasimetric if the $\delta$-slim triangles condition is satisfied with the $s$-geodesics instead of geodesics. To ease the notation, we will always assume that $r = s = \delta$.

As explained, many results about hyperbolicity translate to hyperbolic path quasimetric spaces. We now cite exactly which results we are going to use, and why they translate.

Given a point $p \in X$, the Gromov product is defined by

$$(x \cdot y)_p = \frac{1}{2}(d(p,x) + d(p,y) - d(x,y)).$$

A useful feature of this product is that for $\delta$-hyperbolic spaces $(x \cdot y)_p$ is equal to the distance between $p$ and any geodesic between $x$ and $y$, up to additive error. That is,

$$(x \cdot y)_p = d(p, [x,y]) + O(\delta), \quad (1)$$
Consider the last point \( a, p \) such that \( \mathcal{A} \) is an \( r \)-geodesic, they are at most at distance \( 2r+1 \), and by hyperbolicity, \( a_{k+1} \) is contained in the \( r \)-neighbourhood of \( [p, y] \). Consider \( w \in [y, p] \) such that \( \rho(a_k, w) \leq 4r+1 \). Then,

\[
\rho(p, y) \geq \rho(p, w) + \rho(w, y) - 4r - 1 \geq \rho(p, w) + \rho(w, a_k) + \rho(a_k, w) + \rho(w, y) - 4r - 1 \\
\geq \rho(p, a_k) + \rho(a_k, y) - 6r - 1.
\]

Doing the same reasoning for \( \rho(p, x) \) and adding both inequalities we get

\[
2 \rho(p, a_k) \leq \rho(p, x) + \rho(p, y) - \rho(x, a_k) - \rho(a_k, y) + 12r + 2 \\
\rho(p, x) + \rho(p, y) - \rho(x, y) + 13r + 2 \leq 2(x \cdot y)_p + 13r + 2.
\]

And we get our result (since \( \rho(p, [x, y]) \leq \rho(p, a_k) \)).

Another important property we will use about the Gromov product is the **inverse triangle inequality**, 

\[
(x \cdot y)_p \geq \min \{ (x \cdot z)_p, (y \cdot z)_p \} + O(r),
\]

which is satisfied as a direct application of the last lemma and the \( r \)-slim triangles conditions, that is, since \( [x, y] \subset N([x, z] \cup [z, y], r) \),

\[
(x \cdot y)_p = \rho(p, [x, y]) + O(r) \geq \rho(p, [x, z] \cup [z, y]) - O(r) = \min \{ \rho(p, [x, z]), \rho(p, [z, y]) \} - O(r) = \min \{ (x \cdot z)_p, (y \cdot z)_p \} - O(r).
\]

Just as we relaxed the notions of metric and trees to something that, on the large scale, looks the same, we can do the same with isometries. Precisely:
Definition 2.3. Let \((Q, \rho)\) and \((Q', \rho')\) be two quasimetric spaces. A map \(f : Q \to Q'\) is called an \((L, C)\)-quasi-isometric embedding of \(Q\) into \(Q'\) if there exists \(L, C > 0\) such that \(\rho(x, y)/L - C < \rho'(f(x), f(y)) < L\rho(x, y) - C\) for all \(x, y \in Q\). If, in addition, there exists \(D > 0\) such that every point of \(Q'\) is within distance \(D\) from the image of \(f\), then we say \(f\) is a quasi-isometry between \(Q\) and \(Q'\).

Note that we do not require \(f\) to be continuous, injective nor exhaustive. Just as in the case of metric spaces, one can show that given an \((L, C)\) quasi-isometry between \(Q\) and \(Q'\), there exists a quasi-inverse \(g : Q' \to Q\), such that \(\rho(gf(x), x) \leq C + O(r)\) and \(\rho'(fg(x'), x') \leq C + O(r)\).

The same idea can be applied to geodesics in the following way. Let \(I\) be a connected subset of \(\mathbb{R}\). An \((L, C)\)-quasigeodesic \(\gamma\) is an \((L, C)\)-quasi-isometric embedding of \(I\) into \(Q\), that is, such that for all \(s\) and \(t\) in \(I\),

\[
\frac{1}{L}|t - s| - C \leq \rho(\gamma(s), \gamma(t)) \leq L|t - s| + C.
\]

If \(I = \mathbb{R}\) we will call the quasigeodesic \(\gamma\) a bi-infinite quasigeodesic. It is easy to see that the image of an \((L, C)\)-quasigeodesic by an \((L', C')\)-quasi-isometric embedding is a \((L'', C'')\)-quasigeodesic, where \(L''\) and \(C''\) depend only on the other constants. If we are on a \(\delta\)-hyperbolic metric space, then it is well known that quasigeodesics have the following stability property, which is often referred to as the Morse Lemma.

Proposition 2.4. Let \((Q, \rho)\) be a hyperbolic path quasimetric space. Given numbers \(L\) and \(C\), there is a number \(D\) such that for any two points \(x\) and \(y\) in \(Q\), any two \((L, C)\)-quasigeodesics connecting \(x\) and \(y\) are contained in \(D\)-neighbourhoods of each other.

For \(\delta\)-hyperbolic spaces, a proof can be found, for example, in [BH99] [Theorem III.1.7]. The proof can be done as well for quasimetric spaces, resulting in a bigger constant \(D'\), but it is not a short proof, so we shall not repeat it here. Another way of seeing it can be achieved using that any hyperbolic path quasimetric space \(Q\) is quasi-isometric to a metric hyperbolic space \(S\) [Bow98], so if we take two \((L, C)\)-quasigeodesics on \(Q\) connecting \(x\) and \(y\), then their images by the \(L', C'\)-quasi-isometry \(f : Q \to S\) will be \((L'', C'')\)-quasigeodesics, so they will be contained on some \(D\) neighbourhood of each other, and hence their preimages will also be contained in some \(D'\) neighbourhood of each other, where \(D'\) depends only on \(L, C\) and our space \(Q\).

By the last proposition, if \(f\) is a quasi-isometry between \((Q, \rho)\) and \((Q', \rho')\), then \([f(x), f(y)]\) and \(f([x, y])\) are contained in \(D\) neighbourhoods of each other. Therefore, using (1),

\[
(x \cdot y)_{\rho} = \rho(p, [x, y]) + O(r) \leq L\rho(f(p), f([x, y])) + C \leq L\rho(f(p), [f(x), f(y)]) + C + DL = L(f(x) \cdot f(y))_{f(p)} + K.
\]
Repeating the process to get a bound from below we get
\[ \frac{(f(x) \cdot f(y))_{f(p)}}{L - K} \leq (x \cdot y)_p \leq L(f(x) \cdot f(y))_{f(p)} + K, \]
that is, the Gromov product stays, in big terms, untouched by quasi-isometries.

The Gromov boundary of a space \( Q \), which we will denote \( \partial Q \), can be defined by means of its Gromov sequences, up to a relation, which by this last observation will turn out to be invariant under quasi-isometries. We say that a sequence \((x_n)_{n \in \mathbb{N}} \subseteq Q\) is a Gromov sequence if \((x_m \cdot x_n)_p \) tends to infinity as \( m, n \) tend to infinity (that is, as \( \min\{m, n\} \) tend to infinity). We say that two Gromov sequences \((x_n)_n \) and \((y_n)_n \) are equivalent if \((x_n \cdot y_n)_p \) converges to infinity as \( n \) tends to infinity. If we don’t know whether \((y_n)_n \) is a Gromov sequence, \( \lim_{n \to \infty} (x_n \cdot y_n)_p = \infty \) implies that it is, since by the inverse triangle inequality we have \((y_n \cdot y_m)_p \geq \min\{(y_n \cdot x_n)_p, (x_n \cdot x_m)_p, (y_m \cdot x_m)_p\} + O(r)\), which goes to infinity. The Gromov boundary is defined as the set of equivalence classes of Gromov sequences. By (1) we have that the choice of \( p \) is irrelevant, and by (2) we have that quasi-isometries can be extended to bijections between the corresponding Gromov boundaries.

The Gromov product can be extended to the boundary by
\[ (x \cdot y)_p = \sup \liminf_{m,n \to \infty} (x_m \cdot y_n)_p, \]
where the supremum is taken over all sequences \((x_m)_m \) of the class of \( x \) and \((y_n)_n \) of the class of \( y \). With this definition, the inverse triangle inequality still holds, but with a larger additive constant. The sets
\[ V(x, R) = \{ y \in Q \cup \partial Q \mid (x \cdot y)_p > R \}, \]
together with all the open sets of \( Q \) (if \( Q \) is a quasimetric, we don’t have an induced topology, so instead we may take the weakest topology which respects the convergences to the boundary), form a basis for a topology on \( Q \cup \partial Q \). When \( Q \) is a proper metric space, that is, the closed metric balls are compact, \( Q \cup \partial Q \) with this topology is compact. The convergence to the points of the boundary is independent of the choice of \( p \), and we can consider the induced topology on \( \partial Q \), which is also well defined. By (2), if \( Q \) and \( Q' \) are two hyperbolic path quasimetric spaces, and \( f \) a quasi-isometry between them, we will have \((x_n) \subseteq Q \cup \partial Q \) converging to \( \lambda \in \partial Q \) if and only if \((f(x_n)) \) converges to \( f(\lambda) \).

On a \( \delta \)-hyperbolic space, the nearest point projection into a geodesic \( \gamma \) is coarsely well defined, i.e. if \( \gamma \) is a geodesic between \( a, b \), there is a constant \( K(\delta) \) such that if \( p \) and \( q \) are nearest points of \( \gamma \) to \( c \), \( d(p, q) \leq K \). Furthermore, \( p \) and \( q \) are also \( O(\delta) \) close to any projection of \( a \) into \([b, c]\). A proof of this fact for \( \delta \)-hyperbolic spaces, as well as the following 2 propositions, can be found in [Mah10] [Section 3]. These proofs can be redone for hyperbolic path spaces, with respect to the \( r \)-geodesics, in the
same way we redid the proof for lemma 2.2 so we shall not redo them here.
Using the coarsely defined projections one can prove the reverse triangle inequality.

**Proposition 2.5.** Let $\gamma$ be a $r$-geodesic in a hyperbolic path quasimetric space $Q$, $x \in Q$ a point, and $p$ a nearest point projection of $x$ to $\gamma$. Then for any $y \in \gamma$ we have

$$\rho(x, y) = \rho(x, p) + \rho(y, p) + O(r)$$

Using twice this proposition one can show the following.

**Proposition 2.6.** Let $\gamma$ be a $r$-geodesic in a hyperbolic path quasimetric space $Q$, and let $x$ and $y$ be two points in $Q$ with nearest points $p_x$ and $p_y$ respectively on $\gamma$. If $\rho(p_x, p_y) \geq K(r)$, then

$$\rho(x, y) = \rho(x, p_x) + \rho(p_x, p_y) + \rho(p_y, y) + O(r).$$

If $G$ is a group acting by isometries on $Q$, we say $g \in G$ is a loxodromic element if the map $\mathbb{Z} \to Q$, $n \to g^n x$ is an $(L(x), C(x))$-quasi-isometric embedding for some (equivalently, any) $x \in Q$, that is, $t \to g^{[t]}x$ is a quasi-geodesic. Of interest to us will be the following property of these elements, well known when $Q$ is a proper hyperbolic metric space.

**Proposition 2.7.** Let $G$ be a group acting by isometries on a hyperbolic path quasimetric space $(Q, \rho)$, and let $g$ be a loxodromic element. Then there exists $N$ big enough and $M > 0$ such that $\inf_{x \in Q} \rho(x, g^{nN}x) \geq nM$.

**Proof.** Fix $s \in Q$. Then, $g^n s$ is an $(L, C)$-quasigeodesic for some $L, C$. Given $x \in Q$, consider one of the closest point projections of $x$ into $\{g^n s\}$, and denote it $g^m s$, that is, a point such that $\rho(x, g^n s) \leq \rho(x, g^m s)$ for all $m \in \mathbb{Z}$. Since $g$ is an isometry, and the set $\{g^n s\}$ is $g$-invariant, the closest point projection of $g^k x$ can be chosen to be $g^{k+n_x} s$. Consider now the $r$-geodesic $\gamma$ between $g^n s$ and $g^{k+n_x} s$, and the projections $p_x$ and $p_{g^k x}$ of $x$ and $g^k x$ on $\gamma$. By Morse Lemma, there is a constant $K$ such that the geodesic $\gamma$ is at $K$ distance from the points $\{g^n s, n_x \leq n \leq k + n_x\}$, so by the triangle inequality,

$$\rho(x, p_x) \geq \rho(x, \{g^n s, n_x \leq n \leq k + n_x\}) - \rho(\{g^n s, n_x \leq n \leq k + n_x\}, p_x) - r \geq \rho(x, g^{n_x} s) - r - K.$$ 

Adding the reverse triangle inequality,

$$\rho(p_x, g^{n_x} s) = \rho(x, g^{n_x} s) - \rho(x, p_x) + O(r) \leq K + r + O(r),$$

The same result can be obtained in the same way for the distance between $g^{k+n_x} s$ and $p_{g^k x}$. Hence,

$$\rho(p_x, p_{g^k x}) \geq \rho(g^{n_x} s, g^{k+n_x} s) - 2K - 2r + O(r) \geq \frac{k}{L} - C - 2K - 2r + O(r),$$

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Since $L$, $C$ and $K$ depend only on $s$, we can take $k$ such that $\frac{k}{L} - C - 2K - 2r + O(r)$ is big enough so we can apply proposition 2.6. Applying it we get,

$$d(x, g^k x) = d(x, p_x) + d(p_x, p_{g^k x}) + d(p_{g^k x}, g^k x) + O(r) \geq \frac{k}{L} - C - 2K - 2r + O(r),$$

so the lemma is satisfied for $0 < M < \frac{1}{L}$ and $N > L(M + C + 2K + 2r)$ (and big enough such that 2.6 applies on the proof).

Finally, we recall that a group is called hyperbolic if it is finitely generated, and the Cayley graphs obtained by taking a finite set of generators and endowing it with the path metric are hyperbolic. Since two Cayley graphs generated by different finite sets of generators are quasi-isometric, and the hyperbolicity property and Gromov boundary are invariant by quasi-isometries, the notion of hyperbolic group is well defined and one can talk about the Gromov boundary of the group. By Švarc–Milnor lemma, any group acting by isometries, properly discontinuously and cocompactly on a proper hyperbolic space is hyperbolic.

### 2.2 Random Walks

Let $G$ be a discrete group and $\mu$ a probability measure on $G$. The **step space** $\Omega := G^\mathbb{N}$ is the space of infinite sequences of group elements, which we consider as a probability space with the product measure $P := \mu^\mathbb{N}$. We will denote random walk on $G$ starting at $g_0$ the stochastic process (indexed by $\mathbb{N} \cup \{0\}$) obtained by associating to each $n$, the $G$-valued random variable $w_n : \Omega \to G$

$$(g_1, g_2, \ldots) \mapsto w_n := g_0 g_1 \ldots g_n.$$ 

In other words, a random walk on $G$ is a time homogeneous, space homogeneous Markov chain with transition probabilities given by $p(g, h) = \mu(g^{-1}h)$. Unless explicitly stated, our random walks will always start at the neutral element, that is, $g_0 = e$.

On all of our cases our group $G$ will act by isometries on some metric space $(X, d)$, and we will be interested on the process we get by applying the random walk to some starting point $x \in X$, i.e., in the process $(w_n x)_{n \in \mathbb{N}}$. This may make the choice of creating the random walk on the group by right multiplication look weird, since this may result in the new process not being a Markov chain. However, by doing it this way we can interpret the steps as going from $w_n x$ to $(w_n g_{n+1} w_n^{-1}) w_n x$, i.e., every step consists on drafting independently an isometry $g_{n+1} \in G$ with probability $\mu$, translating it to the point $w_n x$ (i.e., considering the isometry $w_n g_{n+1} w_n^{-1}$) and applying this new isometry. Since there might be more than one way of translating our isometry, and the way we chose depends on the path $w_n$, we might end up with something that is no longer a Markov chain. However, by making this
choice we get something similar to time and space homogeneity, since the
distribution of every step will be a random translation of the distribution of
the first step, i.e.,
\[ P[w_n = y | w_{n-1} = z] = P[w_{n-1}g_n = y | w_{n-1} = z] = P[g_n = w_{n-1}^{-1}y | x = w_{n-1}^{-1}z]. \]

We will refer to this new process as random walk on \( X \) (generated by \( (G, \mu) \)).
This can also be seen as the projection of the random walk on \( G \) to \( X \).

To interest to us will be the asymptotic behaviour of the random walks.
In particular, whether they converge to some ”infinity”, or rather, to some
boundary, and in which way they converge. Assume \( G \) can be embedded
into a \( G \)-space of the form \( G \cup B \) (that is, a topological space upon which
\( G \) acts by homeomorphisms), and that the sample paths \( (w_n(\omega)) \) converge
almost surely to some point \( w_\infty \in B \). Then, Furstenberg shows in \cite{Fur71}
that this implies that the resulting hitting measure \( \nu \) in \( B \) is \( \mu \) stationary,
that is, \( \nu(A) = \sum_{g \in G} \mu(g) g\nu(A) \) (where \( g\nu(A) := \nu(g^{-1}A) \)) for every Borel
set in \( B \), and the measure \( w_\infty \nu \) converges in the weak* topology to a point
measure. With this in mind, Furstenberg defines the following.

**Definition 2.8.** Let \( G \) be a group acting on a measurable \( G \)-space \( (B, \nu) \)
and \( \mu \) a measure on \( G \). Then \( (B, \nu) \) is a \( \mu \)-boundary (or Furstenberg bound-
ary) of \( (G, \mu) \) if

1. \( \nu \) is a \( \mu \) stationary probability measure
2. For almost every sample path \( (w_n) \), the sequence of measures \( (w_n \nu) \)
converges weakly to a \( \delta \)-measure.

Furstenberg also shows that whenever \((B, \nu)\) is a \( \mu \)-boundary, we can
endow \( G \cup B \) with a topology such that the sample paths of the random walks
converge almost surely to points in the boundary. However, the inclusion
\( G \hookrightarrow G \cup B \) might not be an embedding.

If we consider the \( \mu \) boundaries of a group as measure spaces, we can
establish an order between them by considering \((B_1, \mu_1) \geq (B_2, \mu_2)\) if there
exists a \( G \)-equivariant map \( f : B_1 \to B_2 \) such that \((B_2, \mu_2) = (f(B_1), f(\mu_1))\).
Furstenberg also shows that there exists a unique (up to the equivalence
given by the order relation) maximal \( \mu \)-boundary, which gives us a definition
for the Poisson Boundary.

Whenever \( M \) is a metrizable compact space, the space of probability
measures equipped with the weak* measure, \( \mathcal{M}_1(M) \), is compact (see, for
example, \cite{VO16} [Chapter 2]), which guarantees the existence of \( \mu \)-stationary
measures.

**Theorem 2.9.** Let \( G \) be a countable group acting by homeomorphisms on
a compact metric space \( M \), and let \( \mu \) a probability distribution on \( G \). Then
there exists a \( \mu \)-stationary Borel probability measure \( \nu \) on \( M \).
Proof. Consider any Borel probability measure \( \nu_0 \in \mathcal{M}_1(M) \), and consider the Cesàro averages \( \nu_n = \frac{1}{n} (\mu \ast \nu_0 + \ldots + \mu^n \ast \nu_0) \), where \( \mu^n = \mu \ast \mu \ast \ldots \ast \mu \) (n times). Since \( \mathcal{M}_1(M) \) is compact, this sequence has a converging subsequence \( \nu_{n_k} \), to some \( \nu \in \mathcal{M}_1(M) \). Then, for any bounded continuous function \( f : M \rightarrow \mathbb{R} \), writing \( \nu(f) := \int_B f \nu \),

\[
|\nu(f - \mu \ast \nu(f))| \leq |\nu(f) - \nu_{n_k}(f)| + |\nu_{n_k}(f) - \mu \ast \nu_{n_k}(f)| + |\mu \ast \nu_{n_k}(f) - \mu \ast \nu(f)|.
\]

The first term on the RHS goes to 0 by the definition of the topology. For the second term we have

\[
|\nu_{n_k}(f) - \mu \ast \nu_{n_k}(f)| = \frac{1}{n} (\mu \ast \nu_{n_k}(f) - \mu \ast \nu_{n_k+1}(f)) \leq \frac{2}{n} \max_{x \in B} |f(x)|,
\]

so it also goes to 0. For the last term we have

\[
|\mu \ast \nu_{n_k}(f) - \mu \ast \nu(f)| \leq \sum_{g \in G} \mu(g) |g \nu_{n_k}(f) - g \nu(f)|.
\]

g \nu(f) = \int f(x) \nu(g^{-1}(x)) = \int f(gx) \nu(x) = \nu(f(g \cdot)) \), so by definition of the topology every term of this last sum goes to 0 (since \( f(g \cdot) \) is a continuous bounded function). Applying dominated convergence (dominated by \( 2 \max_{x \in B} f(x) \)), we have that this last sum converges to 0, so \( |\nu(f) - \mu \ast \nu(f)| \rightarrow 0 \), meaning, \( \nu(f) = \mu \ast \nu(f) \) for every continuous bounded function, which implies \( \nu = \mu \ast \nu \).

With this, the first condition of the definition of \( \mu \) boundary is automatically satisfied whenever we have \( G \) acting on a compact metric space. For the second condition we have the following result, which goes back to Furstenberg (see also [Mar91][Chapter 6]).

**Theorem 2.10.** Let \( M \) be a compact metric space on which the countable group \( G \) acts continuously, and \( \nu \) a \( \mu \)-stationary Borel probability measure on \( M \). Then for almost every sample path \( w \in \Omega \), the sequence \( w_n \nu \) converges in the weak* topology to some measure on \( M \).

**Proof.** Since \( M \) is compact, \( C(M) \) is separable, so we have a dense countable subset \( f^1 \). For each of these functions we can define the functions on \( G \), \( h^1(g) = \langle f, g \nu \rangle \), which, since \( \nu \) is \( \mu \)-stationary, are \( \mu \)-harmonic. Thus, \( h^1(w_n) \) is a bounded martingale, and hence it converges almost surely (on a set \( \Omega_n \)) to some random variable \( f^i_w \). Since the set of dense functions is countable, we can consider the set of probability one \( \bigcap_{n \in \mathbb{N}} \Omega_n \), where \( f^i_w \) is defined for all \( i \). Hence, we have the positive linear functional defined by \( \lambda_w(f^i) := f^i_w \), which can be extended by continuity to a positive linear functional on the space \( C(M) \), which is thus associated to a Borel measure \( \lambda_w \), and by construction \( w_n \nu \rightarrow \lambda_w \).
So, in many cases, identifying a $\mu$-boundary consists in looking for a compact metrizable space upon which $G$ acts by homeomorphisms, and checking if the limit of the last theorem is indeed a point measure. An example where we find ourselves in this situation is that of a hyperbolic group acting on its Gromov boundary, and was studied by Kaimonovich in [Kai97], where he proves not only that the Gromov boundary is a $\mu$-boundary, but that it is maximal as a measure space (that is, the Gromov boundary with the corresponding stationary measure is a model for the Poisson boundary).

Once determined a $\mu$-boundary, there are some methods for proving that the boundary is maximal. In particular, we remark a method developed by Kaimonovich in [Kai00] [Theorem 6.4], where he uses methods based on the entropies of the conditional random walks. The entropy of a measure $\mu$ is defined by $H(\mu) = \sum_{g \in G} \mu(g) \log \mu(g)$, and $\mu$ is said to have finite entropy if $H(\mu) < \infty$.

**Theorem 2.11** (Strip Criterion). Let $G$ be a group acting by isometries on some metric space $X$, let $\mu$ be a probability measure with finite entropy on $G$, with associated inverse measure $\tilde{\mu}(g) := \mu(g^{-1})$, and let $(B_+, \nu)$, $(B_-, \tilde{\nu})$ be $\mu$- and $\tilde{\mu}$-boundaries, respectively. If there exists a measurable $G$-equivariant map $S$ assigning to $\nu \times \tilde{\nu}$-almost every pair of points $(\alpha, \beta) \in (B_+, B_-)$ a non-empty "strip" $S(\alpha, \beta) \subset G$, such that, for $x_0 \in X$,

\[
\frac{1}{r} \log |S(\alpha, \beta)x_0 \cap B(x_0, r)| \to 0 \text{ a.s. for } r \to \infty,
\]

then $(B_+, \nu)$ and $(B_-, \tilde{\nu})$ are the Poisson boundaries of the random walks $(G, \mu)$ and $(G, \tilde{\mu})$ respectively.

Using this theorem, and assuming that the measure $\mu$ has finite logarithmic moment (that is, $\sum_{g \in G} \mu(g) |\log(d(x, gx))| < \infty$), Maher and Tiozzo prove in [MT18] the following theorem, where they determine the Poisson boundary for wide variety of groups.

**Theorem 2.12.** Let $G$ be a countable group which acts by isometries on a hyperbolic metric space $(X, d)$, and let $\mu$ be a non-elementary probability measure on $G$ with finite logarithmic moment and finite entropy. Suppose that there exists at least one WPD element $h$ in the semigroup generated by the support of $\mu$. Then the Gromov boundary of $X$ with the hitting measure is a model for the Poisson boundary of the random walk $(G, \mu)$.

Just as it happens with hyperbolic spaces, the Gromov boundary, together with a stationary measure, is a model for the Poisson boundary. The proof, however, is more complicated, as the Gromov boundary of a non-proper space is not always compact. This result is an improvement of their previous result, proven in [MT16], where they required the action to be acylindrical.
3 Convergence Groups

The notion of convergence group was originally introduced by Gehring and Martin in [GM87], where they axiomatize the dynamical properties of Kleinian groups acting on the Gromov Boundary of $\mathbb{H}^n$. In particular, they give the following definition:

**Definition 3.1.** Let $G$ be a discrete countable group acting on a compact metrizable space $M$. $G$ is called a convergence group if for every infinite sequence $(g_n) \subset G$ of distinct elements, there exists a subsequence $(g_{n_k})$ and points $a, b \in M$ such that $g_{n_k}|_{M \setminus a}$ converges to $b$ locally uniformly, that is, for every compact set $K \subset M \setminus a$, and every neighbourhood $U$ of $b$, there is $N$ such that $g_{n_k}(K) \subset U$ whenever $n_k > N$.

The points $a$ and $b$ are respectively called the repelling and attracting points of the subsequence $(g_{n_k})$. This property appears naturally when dealing with groups acting on hyperbolic spaces. Indeed, Bowditch proves in [Bow99] the following result.

**Proposition 3.2.** Let $G$ be a group acting by isometries and properly discontinuously on a proper hyperbolic space $X$. Then, $G$ acts as a convergence group on the Gromov boundary $\partial X$. In particular, all hyperbolic groups are convergence groups.

**Proof.** We provide here a more direct approach to the proof. Consider the sequence $(g_i)_{i \in \mathbb{N}} \subset G$. Since $G$ acts properly discontinuously, given $x \in X$, the sequence $g_i x$ is unbounded. Furthermore, since $X$ is proper, $X \cup \partial X$ is compact, so there exists a subsequence $g_{i_k} x$ converging to $b$, which belongs on $\partial X$ since the sequence is unbounded. Doing the same reasoning, we can take a further subsequence $g_{i_{k_j}}$, which we relabel $g_i$, such that $g_i^{-1}x$ converges to some $a \in \partial X$.

Consider now a compact $K \subset \partial X \setminus a$. Since $K$ is compact and does not contain $a$, there exists an $R > 0$ such that for all $\lambda \in K$, $(a \cdot \lambda)_x \leq R$. Since $g_i^{-1}x$ converges to $a$, $(a \cdot g_i^{-1}x)_x >> R$ for all $i \geq i_0$. We have, by the inverse triangle inequality,

$$R \geq (a \cdot \lambda)_x \geq \min((a \cdot g_i^{-1}x)_x, (g_i^{-1}x \cdot \lambda)_x) + O(\delta) = (g_i^{-1}x \cdot \lambda)_x + O(\delta).$$

Hence, the projection of $x$ in the geodesic $[g_i^{-1}x, \lambda]$ is closer than $R + O(\delta)$ to $x$, and hence so is the projection of $g_i^{-1}x$ into $[x, \lambda]$. Since $G$ acts by isometries, we have that the projection of $x$ in the geodesic $[g_i x, g_i \lambda]$ is $R + O(\delta)$ close to $g_i x$, so $(g_i x \cdot g_i \lambda)_x \geq d(x, g_i x) - R + O(\delta)$, which goes to infinity, since $g_i x$ is unbounded. Furthermore, $(b \cdot g_i \lambda)_x \geq \min((b \cdot g_i x)_x, (g_i x \cdot g_i \lambda)_x) + O(\delta)$, so since the two possible values go to infinity, $(b \cdot g_i \lambda)_x$ goes to infinity. Since this lower bound on $(b \cdot g_i \lambda)_x$ is independent of $\lambda \in K$, we get for every neighbourhood $U$ around $b$, an $i_0$ such that $g_i K \subset U$ for $i \geq i_0$. \hfill \Box
Throughout this section, $G$ will represent a fixed group acting on a fixed metrizable space $M$ as a convergence group. Adapting the definition for hyperbolic spaces, we say $G$ is non-elementary if there is no invariant subset of $M$ consisting of at most 2 points. Assume that $N \subset M$ is a proper invariant closed set; then it is immediate to see that $G$ also acts on $N$ as a convergence action. Furthermore, if we have a decreasing chain of proper invariant closed sets $M \supset N_0 \supset N_1 \supset \ldots$, then $G$ also acts as a convergence group on $N := \bigcap_{i \in \mathbb{N}} N_i$. $N$ has at least 3 points (and hence infinitely many), since the action is non-elementary, and the action of $G$ on $N$ is minimal, meaning that $N$ has no proper closed invariant set. Therefore, we will also assume that the action is minimal, since we can always take a subset of $M$ where that is satisfied.

Given a probability measure $\mu$ on $G$ we have, by theorem 2.9, that there exists a $\mu$-stationary measure on $M$. Besides some standard methods, the definition of convergence group implies almost directly the following result.

**Proposition 3.3.** Let $G$ be a group acting as a convergence group, minimally and non-elementary on a compact metric space $M$, then, given a measure $\mu$ on $G$ such that $\text{supp}(\mu)$ generates $G$, and a $\mu$-stationary measure $\nu$ on $M$, $(M, \nu)$ is a $\mu$-boundary of $G$.

**Proof.** We are in the situation of theorem 2.10, so the only thing we have to check is that for almost every $w \in \Omega$ the limit $\lambda_w = \lim_{n \to \infty} w_n \nu$ is a point measure. To do so, we first prove that $\nu$ is non-atomic (which in this case is equivalent to showing that there is no point $p \in M$ with $\nu(p) > 0$). Assume $\nu$ has atoms, then there is an atom of maximal weight, as an infinite sequence of atoms $(b_n)_{n}$ of increasing weights has total measure greater than one. Let $m$ be the maximal weight of any atom, and let $A_m$ be the collection of atoms of weight $m$, which is a finite set. As $\nu$ is $\mu$-stationary, if $b \in A_m$, then

$$
\nu(b) = \sum_{g \in G} \mu(g) \nu(g^{-1}b).
$$

As no atom has weight greater than $m$ and $\text{supp } \mu$ generates $G$, all elements of the orbit of $b$ under $G$ must have the same weight $m$, so $A_m$ is a finite $G$-invariant set, which contradicts the minimality hypothesis.

The next thing we want to do is to apply the convergence property to the sequence $(w_n)$, but to do that we first have to check that the sequence has infinitely many different elements. To see this consider $k$ such that the distribution of the $k$-th step has a support of at least $m$ elements (that is, $|\text{supp}(\mu^k)| \geq m$). Then, the probability that a path touches less than $m$ elements is smaller than the probability of the subpath $w_{nk}$ touching less than $m$ elements, but if the path has less than $m$ elements, then the probability of touching a new one on the next step is positive and bounded.
from below, that is,
\[ P \left[ w_{(n+1)k} \neq w_{ik} \text{ for all } i \leq n \right] \leq m \geq \inf_{A \subset G, |A| < m} P \left[ w_{ik}^{-1}w_{(n+1)k} \notin A \right] = \inf_{A \subset G, |A| < m} \sum_{g \notin A} \mu^k(g) = p > 0, \]
where on the last step we use that all of the support of \( \mu^k \) can not be covered with a set with less than \( m \) elements. Using repeatedly the formula of conditional probability,
\[ P \left[ \{w_{ik}, i \leq n\} \right] = \left[ P \left[ \{w_{ik}, i \leq n\} \right] \leq m \right] \leq (1 - p)^n. \]
So, the probability that the path touches less than \( m \) states before time \( nk \) goes to 0 as \( n \) goes to infinity, and hence \((w_n)\) touches infinitely many points almost surely.

Therefore, we can apply the definition of convergence action, that is, the sequence \( w_n \) has almost surely a convergent subsequence \( w_{n_k} \), with attracting point \( b \) and repelling point \( a \). We will have \( w_n \nu \to \delta_b \) if for every continuous function \( f \) on \( M \),
\[ \int_{x \in M} f(x)w_n \nu \to \int_{x \in M} f(x) \delta_b = f(p). \]
Consider a neighbourhood \( U \) of \( b \) such that \( |f(x) - f(b)| < \epsilon \) for all \( x \in U \), and a neighbourhood \( V \) around \( a \) such that \( \nu(V) \leq \epsilon \) (we can take such an open set since \( \nu \) is non-atomic and, since it is Borel and \( M \) is metrizable, \( \nu \) is also regular). By the convergence property, there exists \( k_0 \) such that for \( k \geq k_0 \),
\[ w_{n_k}M \setminus V \subset U, \]
so
\[ \left| \int_{x \in M} f(x)w_n \nu - f(b) \right| \leq \int_{x \in M} |f(x) - f(b)|w_n \nu = \int_{x \in M} |f(w_n x) - f(b)| \nu = \int_{x \in M \setminus V} |f(w_n x) - f(b)| \nu + \int_{x \in V} |f(w_n x) - f(b)| \nu \leq \nu(M \setminus V) \sup_{x \in U} |f(x) - f(b)| + \nu(V) \sup_{x \in M} |f(x) - f(b)| \leq \epsilon + \epsilon K. \]
Hence, \( w_{n_k} \nu \) converges to \( \delta_b \), so since the whole sequence \( w_n \nu \) converges to a measure, it must be the delta measure \( \delta_b \), and \((M, \nu)\) is a \( \mu \) boundary.

3.1 Kleinian groups and the space of distinct triples

A Kleinian group is a discrete subgroup of Möbius transformations of the \( n \)-sphere \( S^n \). The action can be extended to act on the \((n + 1)\)-ball \( B^{n+1} \), and the ball can be equipped with an hyperbolic metric \( d_H \) such that the extension of the Möbius transformations act by isometries. Hence, Kleinian groups are discrete groups acting by isometries on the hyperbolic space.
Given any \( y \in (\text{preimages by } p \in \mathcal{P}) \) the preimage of a point is compact, and that given two points in \( B \) their preimages by \( p \) are homeomorphic. Therefore, in the case of Kleinian groups, \( T \) can be seen as a bigger version of \( B \). As Tukia points out in [Tuk98], \( T \) works as a rough equivalent to the hyperbolic space for convergence groups. For example, Bowditch shows in [Bow99] [Lemma 1.1] that the action of \( G \) on \( M \) is a convergence action if and only if the induced action on \( T \) is properly discontinuous, bearing some similarity to proposition 3.2. To add \( M \) into the analogy in the same way that Tukia does, we prove the following lemma.

**Lemma 3.4.** Let \( (x_n) \subset B^{n+1} \) be a sequence with \( x_n \to \lambda \in \partial B^{n+1} = S^n \). Then, given a neighbourhood \( U \) of \( \lambda \) on \( S^n \), there exists \( n_0 \) such that for all \( n \geq n_0 \), every member of \( p^{-1}(x_n) \) has at least two components inside \( U \).

**Proof.** Fix \( x \in B^{n+1} \) and consider \( R > 0 \) such that the neighbourhood \( \lambda \) on \( B^{n+1} \), \( U(\lambda, r) := \{ y \in B^{n+1} | (\lambda, y)_x > R \} \) has \( U(\lambda, R) \cap S^n \subset U \). Since \( x_n \to \lambda \), given \( C > 0 \), there exists \( n_0 \) such that \( x_n \subset U(\lambda, R + C) \) for all \( n \geq n_0 \). Fix then \( n \geq n_0 \) and \( (a, b, c) \in p^{-1}(x_n) \). Consider the Gromov products \( (\lambda \cdot a)_x \), \( (\lambda \cdot b)_x \) and \( (\lambda \cdot c)_x \). If none of them is smaller than \( R \), we are done. Assume \( (\lambda \cdot x)_x \) to be the smallest, and that it is smaller than \( R \). We have \( (\lambda \cdot x)_x \geq \min((c \cdot x_n)_x, (x_n \cdot \lambda)_x) \), so

\[
d(x, [c, x_n]) = (c \cdot x_n)_x + O(\delta) \leq (\lambda \cdot x)_x + O(\delta) \leq R + O(\delta).
\]

Given any \( y \in [a, b] \), the projection of \( y \) in the geodesic \([c, x_n]\) is \( x_n \). Since \( (x_n \cdot \lambda)_x \geq R + C \), \( d(x, x_n) \geq R + C \), so if \( p_x \) is the projection of \( x \) in \([c, x_n]\), \( d(p_x, x_n) \geq d(x, x_n) - d(x, p_x) \geq C \). Taking \( C \) big enough so proposition 3.2 applies, we get \( d(x, y) \geq R + C + O(\delta) \). Hence, \( (a \cdot x_n)_x \), \( (x_n \cdot b)_x \) are at a bounded distance from \( p(a, b, c), \) and take \( C \) a little bigger.

This lemma shows how the notion of convergence to the boundary on \( B^{n+1} \) can be translated to \( T \cup M \) via neighbourhoods defined in the following
way: Given $U \subset M$ an open set, we define the associated set on $T \cup M$ by

$$\tilde{U} = \{x \in T | x \text{ has at least two components in } U \} \cup U.$$ 

Adding to these sets the open sets of $T$ we get a family of sets $B$ such that $\bigcup_{U \in B} U = T \cup M$ (that is, the elements of $B$ cover $T \cup M$) and, for $U_1, U_2 \in B$, and $x \in U_1 \cap U_2$ we get an element $U_3 \in B$ such that $x \in U_3 \subset U_1 \cap U_2$. To see this last claim, if $x \in T$, we can consider $U_1 \cap U_2 \cap T$, which is open in the topology of $T$; if $x \in M$, we can consider $V = U_1 \cap U_2 \cap M$, which is open in the topology of $M$, and the extension $\tilde{V}$, which is in $B$, and contained in $U_1 \cap U_2$. Therefore, we have the base of a unique topology on $T \cup M$. From lemma 3.3 if $x_n \to \lambda \in S^n$ on $B^{n+1}$, then any sequence of preimages $\tilde{x}_n \in p^{-1}(x_n)$ will also converge to the same $\lambda \in M = S^n$ on $T \cup M$, and if $(\tilde{y}_n)$ converges to $\lambda$ on $T \cup S^n$, then $(p(\tilde{y}_n))$ will be $O(\delta)$ close to a geodesic with two endpoints that converge to $\lambda$, so it also converges to $\lambda$ on $B^{n+1}$. Therefore, just as $T$ can be regarded as a rough equivalent of the hyperbolic space, $M$ can be seen as a rough equivalent of its Gromov boundary, and this way of pasting them together works as an equivalent of Gromov’s topology.

We can then consider the random walk on the space $T$, and we get the following.

**Proposition 3.5.** Let $G$ be a group actioning non elementary, minimally and as a convergence group on a compact metrizable space $M$, and let $\mu$ be a probability measure on $G$ such that its support generates $G$. Then, given $x \in T$, the sample paths $(w_n x)$ of the associated random walk converge almost surely to $M$.

**Proof.** By proposition 3.3 if $\nu$ is a $\mu$-stationary Borel probability measure on $M$, $w_n \nu$ converges to $\delta_{p(w)}$ for some $p(w) \in M$ almost surely. Assume we have $w \in \Omega$ such that $w_n \nu$ converges to $\delta_p$ but $w_n x$ does not converge to $p$. Then, there exists a neighbourhood $U$ of $p$ in $M$ such that $w_n x \notin \tilde{U}$ for infinitely many $n_k$. We have $w_n \nu \to \delta_p$, so if $(w_{n_k})$ has finitely many elements, there exists some $s$ such that $w_{n_k} \nu = \delta_p$, so $\nu = \delta_{w_{n_k} x}$, which can not be, as $\nu$ is non atomic (as seen in the proof of 3.3). Hence, $(w_{n_k})$ has infinitely many elements, and we may take a convergent subsequence, relabeled $(w_i)$. The attracting point $p'$ of $w_i$ can not be $p$, since then, taking a neighbourhood $V$ around the repelling point small enough so it does not contain at least two components of $x$ (that is, such that $x \notin \tilde{V}$), we have by the definition of convergence action, that there exists $i_0$ such that, for $i > i_0$, $w_i(M \setminus V) \subset U$, and hence $w_i x \in \tilde{U}$. Therefore, as done in the proof of 3.3, we can show that $w_i \nu$ converges to $\delta_{p'}(\neq \delta_p)$, and we have a contradiction.

From this, we can deduce that any Borel $\mu$-stationary probability measure $\nu$ is actually the hitting measure, and hence, that $\nu$ is unique.
Proposition 3.6. Let $G$ be a group acting non elementary, minimally and as a convergence group on a compact metrizable space $M$, and $\mu$ a probability measure on $G$ such that its support generates $G$. Then there is a unique $\mu$-stationary Borel probability measure $\nu$ on $M$ (which is the hitting measure of the associated random walk on $T \cup M$).

Proof. Let $\nu$ be a Borel $\mu$-stationary probability measure on $M$, and let $A \subset M$. Then, $g\nu(A)$ is a $\mu$-harmonic bounded function on $G$, since $g\nu(A) = \sum_{h \in G} \mu(h) g h \nu(A)$. Furthermore, denoting $\pi(w) := \lim_{n \to \infty} w_n x$, since $w_n \nu$ converges to $\delta_{\pi(w)}$ in the weak* topology we have that, if $A$ is open, then $\liminf w_n \nu(A) \geq \delta_{\pi(w)}(A) = 1_A(\pi(w))$, and if $A$ is closed, $\limsup w_n \nu(A) \leq 1_A(\pi(w))$ (see [VO16] Chapter 2). Hence, since $w_n \nu(A)$ is a martingale, we have, for $U \subset M$ open,

$$
\nu(U) = \mathbb{E}[w_n \nu(U)] = \lim_{n \to \infty} \mathbb{E}[w_n \nu(U)] = \mathbb{E}[\lim_{n \to \infty} w_n \nu(U)] \geq \mathbb{E}[1_U(\pi(w))] = \mathbb{P}[\pi(w) \in U],
$$

where we used dominated convergence (as $w_n \nu(A)$ is bounded) to put the limit inside. Similarly, for $K \subset M$ closed,

$$
\nu(K) \leq \mathbb{P}[\pi(w) \in K].
$$

Furthermore, since $\nu$ is Borel and $M$ metrizable, $\nu$ is regular, and since $M$ is compact, $K$ is compact. Therefore, $\nu(K) = \inf \{ \nu(U); K \subset U, U \text{ open} \}$. Hence,

$$
\mathbb{P}[\pi(w) \in K] \geq \nu(K) = \inf \{ \nu(U); K \subset U, U \text{ open} \} \geq \inf \{ \mathbb{P}[w_n \to U; K \subset U] \} \geq \mathbb{P}[\pi(w) \in K],
$$

So for every closed set we have $\nu(K) = \mathbb{P}[w_n \to K]$. That is, the measure of the closed sets is equal for all the Borel $\mu$-stationary probability measures on $M$. As $\nu$ is regular, the measures of the closed sets defines the measures of the other sets (as $\nu(A) = \sup \{ \nu(K); K \subset A, K \text{ closed} \}$), so $\nu$ is unique. Furthermore, the stable measure is the hitting measure, which, as we conclude, does not depend on the basepoint $x \in T$ of the random walk. \hfill \Box

3.2 The metric of the space of triples

As Bin Sun shows in [Sun16], the analogy from the last section can be taken a step further, by actually endowing $T$ with an hyperbolic path quasimetric $\rho$, in such a way that $G$ acts by isometries in $(T, \rho)$. To define the quasimetric we first have to introduce some concepts:

Definition 3.7. An annulus, $A$, is an ordered pair, $(A^-, A^+)$, of disjoint closed subsets of $M$ such that $M \setminus (A^- \cup A^+) \neq \emptyset$. A set of annuli $\mathcal{A}$ is an annulus system, and it is symmetric if $A \in \mathcal{A}$ implies $-A := (A^+, A^-) \in \mathcal{A}$.
For an annulus $A$ and $g \in G$, we write $gA$ for the annulus $(gA^-, gA^+)$. Given a subset $K \subset M$, we define the relations $K < A$ if $K \subset \text{int} A^-$, and $A < K$ if $K \subset \text{int} A^+$. If $B$ is another annulus, we write $A < B$ if $\text{int} A^+ \cup \text{int} B^- = M$. Since $B^+ \subset (B^-)^c$, this implies $A^+ \supset B^+$ and, in the same way, $A^- \subset B^-$. For an annulus system, $\mathcal{A}$ on $M$, and $K,L \subset M$, we define $(K|L) = n \in \{0, 1, \ldots, \infty\}$, where $n$ is the maximal number of annuli $A_i$ in $\mathcal{A}$ such that we can build the chain $K < A_1 < A_2 < \ldots < A_n < L$. This gives us two sequences of inclusions, $K \subset A^-_1 \subset A^-_2 \ldots \subset A^-_n \subset L^c$ and $K^c \supset A^+_1 \supset \ldots \supset A^+_n \supset L$. For finite sets we drop the braces and write $(a,b|c,d)$ to mean $(\{a,b\}|\{c,d\})$. With all of this, we can define the function which will give us the quasimetric:

**Definition 3.8.** Given an annulus system $\mathcal{A}$ on $M$. Define the function $\rho : T^2 \to [0, \infty]$ by

$$\rho((x^1, x^2, x^3), (y^1, y^2, y^3)) := \max((x^i, x^j)|y^k, y^l) : i \neq j, k \neq l).$$

In [Bow98] it is shown that, if the annulus system is $G$-invariant and symmetric, and such that $\mathcal{A}/G$ is finite, then the previous function takes values in $[0, \infty)$ and is a $G$-invariant hyperbolic path quasimetric. By considering the geometric realization of the graph obtained by considering the points of $T$ as vertices, and joining them by edges whenever their $\rho$ distances are smaller than $s + 1$, Bin Sun obtains an hyperbolic metric space $(S, \rho')$, upon which the induced action by $G$ is an isometric action, and the inclusion $T \hookrightarrow S$ is a $G$-equivariant quasi-isometry.

The remaining step is to choose a convenient annulus system. The following result, found in [Tuk94], will play an important role on that.

**Theorem 3.9.** If $G$ is a non-elementary convergence group, then there is an element $g \in G$ such that $g$ fixes two distinct points $a, b$ and that $g^n|_{M\setminus a}$ converges to $b$ locally uniformly as $n \to \infty$.

Consider such $g$, and two closed sets $A^-, A^+$ such that $A^- \cap A^+ = \emptyset$ and $a \in \text{int} A^-$, $b \in \text{int} A^+$. Consider the annulus $A := \{A^-, A^+\}$, and the annulus system generated by $A$,

$$\mathcal{A} := \{g(\pm A) | g \in G\}.$$ 

Then, $\mathcal{A}$ is symmetric, and $\mathcal{A}/G$ is finite. Bin Sun proves that there exists some $N$ such that $h := g^N$ is a loxodromic element. He also sees that $h$ is a WPD element. In [Tuk94] it is also shown that non elementary convergence groups can not be virtually cyclic, so, applying the following theorem from Osin [Osi16], Sun proves that $G$ is acylindrically hyperbolic.

**Theorem 3.10.** A group $G$ acting isometrically on a Gromov hyperbolic space $S$ is acylindrically hyperbolic if and only if $G$ is not virtually cyclic.
and there is at least one element of $G$ which is loxodromic and satisfies the weak proper discontinuity condition.

We do not have from this theorem, however, that the action of $G$ on $S$ is acylindrical, as the proof of the theorem is based on building another hyperbolic space, which may be not quasi-isometric to the original space, upon which the action is acylindrical (and non elementary). We have, however, that the action of $G$ on $S$ has a WPD element, so we can apply theorem 2.12.

3.3 The Gromov boundary of $(T, \rho)$

We have now two possible boundaries for $T$, the one given by the Gromov boundary for the quasi-metric $\rho$, $T \cup \partial T$, and the one given by Tukia’s topology, $T \cup M$. In this section we will determine what relation do these two boundaries have.

We relate the two boundaries by using sequences of $T$ converging to points in the boundaries. In particular, we first see that given a sequence $(x_\lambda)$ in $T$ such that $x_\lambda \to \lambda \in \partial T$, then there exists a $p \in M$, which only depends on $\lambda$, such that $x_\lambda \to p$. This gives us an application $\phi : \partial T \to M$, which actually is $G$-equivariant and continuous. It is not possible to repeat the process in the other direction, as some sequences converging to some points in $M$ might be bounded in $(T, \rho)$ (and hence not converging to any point in $\partial T$). However, we are able to create an inverse restricted to the points of $M$ where such sequences do not exist, which is also continuous, and hence we get an homeomorphism between such subset and the corresponding subset of $\partial T$.

Before starting to build $\phi$, we prove a lemma that we use in many occasions:

**Lemma 3.11.** Let $G$ be a convergence group acting on $M$. Also, let $A^-, A^+$ be two non intersecting closed sets, and $B_1, C_1, B_2, C_2 \subset M$, such that $B_1, C_1$ have separating neighbourhoods (i.e., there exists open sets such that $B_1 \subset V_i, C_1 \subset U_i, \text{cl} V_i \cap \text{cl} U_i = \emptyset$), then

$$|\{g \in G | gA^- \cap B_1, gA^- \cap C_1, gA^+ \cap B_2, gA^+ \cap C_2 \neq \emptyset\}| < \infty.$$ 

**Proof.** Assume we have $(g_n)_{n \in \mathbb{N}}$ (with $g_i \neq g_j$ for $i \neq j$) on that set. Then, by the convergence property, we can take a convergent subsequence $(g_{n_k})$.

Since $g_{n_k} A^-$ intersects $B_1$ and $C_1$, the repelling point has to be in $A^-$, since if that weren’t true, we could choose an open $W$ around the repelling point, not intersecting $A^-$, and $k_0$ big enough such that $g_{n_{k_0}} (M - W) \subset O$, where $O$ is an arbitrary open set around the attracting point. Therefore $g_{n_{k_0}} A^- \subset O$, so taking an open set small enough such that $O \cap V_1 = \emptyset$ or $O \cap U_1 = \emptyset$ (which we can do, since the closures don’t intersect) we get a
contradiction to the definition of the set. Doing the same reasoning for \(i = 2\) we get that the repelling point also has to be in \(A^+\), which is not possible since \(A^- \cap A^+ = \emptyset\). \(\square\)

The main application we will have of the previous lemma goes as follows: if \(\mathcal{A} = \{g(\pm A) | g \in G\}\) is the annulus system used to generate the quasimetric \(\rho\), where \(A = (A^-, A^+)\) is the generating annulus, we have, keeping the notation from the previous lemma,

\[
|\{A_j \in \mathcal{A} | A_j^- \cap B_1, A_j^- \cap C_1, A_j^+ \cap B_2, A_j^+ \cap C_2 \neq \emptyset\}| \leq \|g \in G | gA^+ \cap B_1, gA^- \cap C_1, gA^+ \cap B_2, gA^- \cap C_2 \neq \emptyset\| + |\{g \in G | gA^+ \cap B_1, gA^- \cap C_1, gA^- \cap B_2, gA^- \cap C_2 \neq \emptyset\}| < \infty.
\]

3.3.1 One direction, from \(\partial T\) to \(M\)

Given \(\lambda \in \partial T\), take any sequence \((a_n) = ((a_n^1, a_n^2, a_n^3))\) such that \((a_n) \sim \lambda\). Since each component of the sequence is in the compact space \(M\), we can take a subsequence, relabeled \(a_{n_k}\), such that \(a_{n_k}^1 \rightarrow a^1, a_{n_k}^2 \rightarrow a^2\) and \(a_{n_k}^3 \rightarrow a^3\). If \((a^1, a^2, a^3) \in T\), given \(x \in T\), \(\rho(x, a_n)\) is bounded. To see this, denote \(x = (x^1, x^2, x^3)\). Consider neighbourhoods \(V_1, V_2\) of \(a^1\) and \(a^2\) respectively, with disjoint closures. Then, by 3.11

\[
|\{A_j \in \mathcal{A} | \{x^1, x^2\} \subset A_j^-, A_j^+ \cap V_1, A_j^+ \cap V_2 \neq \emptyset\}| \leq K < \infty.
\]

Hence, taking \(n_0\) such that, for \(n \geq n_0\), \(a_{n_k}^1 \in V_1\) and \(a_{n_k}^2 \in V_2\), we have \((x^1, x^2 | a_{n_k}^1, a_{n_k}^2) \leq K\). Repeating the reasoning for all possible combinations, we get that, for \(n \geq n_0\), \(\rho(x, a_n) \leq K\).

Therefore, at least two components will converge to the same point \(p\). We will define the function as \(\phi(\lambda) := p\), so we have to see that \(p\) does not depend on the sequence \((a_n)\) (or the convergent subsequence).

**Lemma 3.12.** Given \(\lambda \in \partial T\) and a representing sequence \((a_n)_{n \in \mathbb{N}} = ((a_n^1, a_n^2, a_n^3))_{n \in \mathbb{N}} \subset T\) such that two elements converge to \(p\), then every other \((b_n) \subset T\) converging to \(\lambda\) with Gromov topology, converges to \(p\) with Tukia’s topology. In particular, \(\phi(\lambda)\) is well defined.

**Proof.** Assume the lemma is false, i.e., that there exists \((b_n) = (b_n^1, b_n^2, b_n^3)\) with \((a_n \cdot b_n)_x \rightarrow \infty\) for some \(x \in T\), but \((b_n)\) does not converge to \(p\). By definition of the topology, given an open set \(V\) around \(p\) there exists a subsequence of \((b_n)\) (which we relabel as \((b_n)\)) such that there are always two components outside that open set. Hence, by compacity, we can take again a converging subsequence of \((b_n)\) (relabel again \((b_n)\)) such that two components converge to points outside \(V\), and since \(\rho(x, b_n) \rightarrow \infty\), we can
assume they converge to the same point \( p' \neq p \). Fixing \( x = (p, t, p') \), let’s compute the Gromov product

\[
2(a_n \cdot b_n)_x = \rho(a_n, x) + \rho(x, b_n) - \rho(a_n, b_n).
\]

Assume, reordering the components if necessary, that the first two components of \( a_n \) and \( b_n \) are the ones that converge to \( p \) and \( p' \) respectively. If the third component of \( (a_n) \) converges to \( \alpha \neq p \), then by lemma \( \text{3.11} \) taking as sets separated open set around \( p \) and \( \alpha \), and any possible combinations of points of \( x \), we get that \( \max((a_n^1, a_n^2) \cap (v, w) : v, w \in \{p, t, p'\}, i \in \{1, 2\}) < C \).

Since \( \rho(a_n, x) \to \infty \), we have \( \rho(a_n, x) = \max((a_n^1, a_n^2) \cap (v, w) : v, w \in \{p, t, p'\}) \) for \( n \) greater than some \( n_0 \). If \( a_n^3 \to p \), we can reorder the components of each \( a_n \) such that the distance to \( x \) is always \( \max((a_n^1, a_n^2) \cap (v, w) : v, w \in \{p, t, p'\}) \).

Consider now \( V_o, V_i \) neighbourhoods of \( p \) not containing \( p' \) nor \( t \) in their closures, and such that \( \text{cl} V_i \subset V_o \). By lemma \( \text{3.11} \)

\[
\{|A_j \in \mathcal{A}\{p, t\} \subset A_j^+ \cap V_i = \emptyset, A_j^+ \cap V_i^c \neq \emptyset\}| \leq K_1. \tag{3}
\]

Assume \( (p, t)|_{a_n^1, a_n^2} = r \geq K_1 + 2 \). Then we have sequence

\[
\{p, t\} < A_1 < \ldots < A_r < \{a_n^1, a_n^2\}.
\]

We will show now that \( A_{K_1+1} \) is contained in \( V_o \). To see this, we recall that the definition of the relation between annulus implies \( A_i^+ \supset A_{i+1}^+ \), so if \( A_i^+ \) is contained on \( V_0 \) for \( i \leq K_1 \) are done. Assume \( A_i^+ \) is not contained in \( V_o \) for all \( i \leq K_1 \), that is, \( A_i^+ \cap V_i^c \neq \emptyset \) for all \( i \leq K_1 \). We also have not just for \( i \leq K_1 \) but for all \( i \leq r \), that \( A_i^+ \) intersects \( V_i \), since \( a_n^1 \in V_i \) and \( A_i^+ \) contains \( p \) and \( t \). Hence, by \( \text{3} \), \( A_{K_1+1}^+ \) cannot intersect \( V_i^c \) (since there are no more annulus intersecting it, while having \( \{p, t\} < A_k < \{a_n^1\} \)). By definition of the relation, \( A_{K_1+2}^- \) contains \( V_o^c \), so we will have the chain

\[
\{p', t\} < A_{K_1+2} < \ldots < A_r < \{a_n^1, a_n^2\},
\]

and therefore, \( (a_n^1, a_n^2)|_{p, t} \leq (a_n^1, a_n^2)|_{t, p'} + K_1 + 2 \). With the same reasoning, \( (a_n^1, a_n^2)|_{p, t} \leq (a_n^1, a_n^2)|_{t, p'} + K_1 + 2 \), and in total we have, for \( n \) big enough,

\[
\rho(a_n, x) \leq (a_n^1, a_n^2)|_{t, p'} + K_1 + 2. \tag{2}
\]

Doing the same reasoning for \( b_n \) we get, for \( n \) big enough,

\[
2(a_n, b_n)_x \leq (a_n^1, a_n^2)|_{t, p'} + (p, t)|_{b_n} - \rho(a_n, b_n) + K_1 + K_2 + 4.
\]

To bound from below the remaining term we just have to build a chain separating the two pairs of elements. Consider \( (a_n^1, a_n^2)|_{b_n} \), and consider \( V_i', V_o' \) neighbourhoods of \( p' \), such that \( \text{cl} V_i' \subset V_o' \), \( \text{cl} V_o \cap \text{cl} V_o' = \emptyset \), and \( t \) is not contained in the closure of any. If \( (a_n^1, a_n^2) = r \) we have the sequence \( \{a_n^1, a_n^2\} < A^+_r < \ldots < A^+_r < \{t, p'\} \), and hence, doing the same reasoning as before

\[
\{a_n^1, a_n^2\} < A^+_r < \ldots < A^+_r < V_o^c.
\]
If \((p, t|b_n^1, b_n^2) = s\), for the sequence \(\{p, t\} < A_1^b < \ldots < A_s^b < \{b_n^1, b_n^2\}\), we will have

\((V_o')^c < A_{K_4+2}^b < \ldots < A_s^b < \{b_n^1, b_n^2\}\).

Therefore, since \(V_o \subset (V_o')^c\) and \(V_o^c \supset V_o'\), we can concatenate both sequences and we get

\(\{a_n^1, a_n^2\} < A_1^a < \ldots < A_{K_3-2}^a < A_{K_4+2}^b < \ldots < A_s^b < \{b_n^1, b_n^2\}\).

Hence, taking the sequences associated to \((a_n^1, a_n^2|t, p')\) and \((p, t|b_n^1, b_n^2)\), we get \((a_n^1, a_n^2|b_n^1, b_n^2) \geq (a_n^1, a_n^2|t, p') + (p, t|b_n^1, b_n^2) - K_3 - K_4 - 4\), so, finally,

\[2(a_n \cdot b_n)_x \leq K_1 + K_2 + K_3 + K_4 + 8.\]

So, since the bound does not depend on \(n\), we have, by the inverse triangle inequality, \((a_n \cdot b_n)_x \geq \min((a_n \cdot \lambda)_x, (\lambda \cdot b_n)_x) + O(r)\), so since \((a_n \cdot \lambda)_x\) goes to infinity, \((\lambda \cdot b_n)_x \leq (a_n \cdot b_n)_x + O(r) \leq K + O(r)\), and hence \(b_n\) does not converge to \(\lambda\).

Since \(\phi\) has been defined by using the convergence of the sequence we can get the following.

**Lemma 3.13.** The map \(\phi\) is \(G\)-equivariant and continuous.

**Proof.** Since \(G\) respects the convergences to the boundaries on \(T \cup \partial T\) and \(T \cup M\), we get that \(\phi\) is \(G\)-equivariant. That is, if \(x_n \to \lambda \in \partial T\), then \(gx_n \to g\lambda \in \partial T\), and if \(x_n \to p \in M, gx_n \to gp \in M\), so \(\phi(g\lambda) = g\phi(\lambda)\).

To see the continuity, take \(\lambda_n \to \lambda\), and assume \(\phi(\lambda_n)\) does not converge to \(p := \phi(\lambda)\). Since \(\phi(\lambda_n))\) is contained on a compact, we can take a convergent subsequence (relabeled \((\phi(\lambda_n))\)), converging to some \(p'\neq p\).

Consider sequences \((a_n^m)_m\) associated to each \(\lambda_n\), and a sequence \((b_n^m)_m\) associated to \(\lambda\). The same reasoning as in the last part of the proof of Lemma 3.12 can be repeated for each pair of sequences \((a_n^m)_m\) and \((b_n^m)_m\). Since \(\phi(\lambda_n)\) converges to \(p'\neq \phi(\lambda)\), we can take \(V_i(n), V_o(n), V'_i(n)\) and \(V'_o(n)\) to be fixed neighbourhoods of the fixed points \(p\) and \(p'\) respectively, such that \(\text{cl} V_i \subset V_o, \text{cl} V'_i \subset V'_o\) and \(\text{cl} V_o \cap \text{cl} V'_i = \emptyset\). In this way, the proof can be repeated, but the bound we get is fixed, as it only depends on the obtained constants, which only depend on our choices of open sets. Hence, given \(n\) big enough (i.e., such that \(\phi(\lambda_n) \in V_i\)) there exists \(m(n)\) big enough such that for all \(m \geq m(n)\), \((a_m^n \cdot b_m)_x \leq K\), so \((a_m^n \cdot \lambda)_x \leq K + O(\delta)\) which by the definition of the topology of the Gromov boundary, contradicts the hypothesis \(\lambda_n \to \lambda\).

### 3.3.2 Finite boundary points

As we explained, we would like to be able to do the same for going from \(M\) to \(\partial T\). That is, given \(p \in M\), take any sequence \((x_n) \subset T\) with \(x_n \to p\),
and see that in $T \cup \partial T$, $x_n \to \lambda \in \partial T$. However, there may be problematic points for which $\rho(x_0, x_n)$ can be bounded. In concrete, we give the following definition.

**Definition 3.14.** Let $p \in M$, and $\rho$ a quasimetric on $T$ created in the way Sun explains. We say that $p$ is a finite boundary point if there exists a sequence $(x_n) \subset T$ and a number $R < \infty$ such that $x_n \to p$ with Tukia’s topology, and $\rho(x_0, x_n) \leq R$.

It is immediate to check, using the triangle inequality, that the notion of finite boundary point does not depend on the basepoint $x_0$. It might, however, depend on the choice of quasimetric, that is, on the annulus we chose to define it. The set of all finite boundary points will be called finite boundary and will be denoted $M_F$. If a point is not a finite boundary point, we will call it infinite boundary point, and the set of all infinite boundary points will be denoted $M_\infty$ ($= M_F$).

To see that $M_F$ is not empty in some cases, we recall a classical definition of the context of Kleinian groups, which can be easily generalized for convergence groups.

**Definition 3.15.** Let $p \in M$. We say that $p$ is a conical limit point if there exists $a, b \in M$ different, and a sequence $(g_n) \subset G$ such that $g_n p$ converges to $a$ but $g_n x$ converges to $b$ for all $x \neq p$.

On the context uniformly convergence group (or, in particular, Kleinian groups) Tukia shows that parabolic fixed points (that is, points fixed by some parabolic element) are exactly the non conical limit points. We will show that non conical limit points are finite boundary points, and hence that $M_F$ might not be empty.

First we introduce a sufficient condition for being a finite boundary point (which is actually an equivalent condition, as we see in 3.3).

**Lemma 3.16.** Let $p \in M$. If there are two points $a, b \in M \setminus p$ and $R > 0$ such that for all $t$, $(a, b|t, p) \leq R$ (equivalently, $(a, b|p) \leq R$), then $p \in M_F$.

Proof. To prove this we fix $x = (a, b, p)$, we need to find $(x_n) \subset T$ such that $x_n \to p$, and $\rho(x, x_n) \leq K$. We take as candidate $x_n = (a, t_n, p)$, where $t_n \to p$. We have $\rho(x, x_n) = \max \{(a, b|t_n, p), (b, p|a, t_n)\}$, and by hypothesis, $(a, b|t_n, p)$ is bounded, so the only possibility of $p$ being in the infinite boundary is $(b, p|a, t_n) \to \infty$. Assume it happens, and take $t_{k_l}$ such that $(b, p|a, t_{k_l}) \geq l$. We have the chain

$$\{b, p\} < A_{1, l} \prec \ldots \prec A_{l, l} \prec \{a, t_{k_l}\}.$$  

Consider $V$ around $p$, separated from $a$, and such that $t_{k_l} \subset V$ for $l \geq l_0$. By lemma 3.11 there exists $K < \infty$ such that

$$|\{A_i \in A|\{b, p\} \subset A_i^-, a \subset A_i^+, V \cap gA_i^+ \neq \emptyset\}| \leq K.$$  

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So taking $l \geq \max(l_0, K + 1)$ we get a contradiction.

If $(a, b|p) \leq R$, then $(a, b|t, p) \leq R$, since every chain of annulus between \{a, b\} and \{t, p\} is also a chain between \{a, b\} and \{p\}. If $(a, b|t, p) \leq R$, then $(a, b|p) \leq R$, since if we had $(a, b|p) \geq R + 1$, we would have the chain
\[
\{a, b\} < A_1 < \ldots < A_{R+1} < p,
\]
so we could take $t_{R+1} \in \text{int} A_{R+1}$, and we would have, using the same chain, $(a, b|t_{R+1}, p) \geq R + 1$. \hfill \Box

**Proposition 3.17.** Let $p \not\in M_F$. Then, there exists a sequence $(g_k) \subset G$ such that $g_k p \to c \in A^\pm$, and $g_k x \to b \in A^+$ for all $x \neq p$. In particular, $p$ is a conical limit point.

**Proof.** Since $p \not\in M_F$, for any $a, b \in M \setminus p$, $(a, b|p)$ is unbounded, that is, we can build arbitrary long chains of the form
\[
\{a, b\} < A_1 < \ldots < A_{2R} < \{p\},
\]
and then we can take subchains such that
\[
\{a, b\} < g_1(-1)^{e(R)} A < \ldots < g_R(-1)^{e(R)} A < \{p\},\]
that is, such that $e(R)$ is the same for all the chain. Assume that as $R \to \infty$, $e(R) = 2$ infinitely many times. Then, we have infinitely many $h \in G$ such that \{a, b\} $\subset h A^-$ and $p \in h A^+$. Taking a convergent subsequence $(h_i)$, by the reasoning in 3.11 the repelling point is in $A^-$. Also, $p \in h_i A^+$, so $h_i^{-1} p \in A^+$. Since $M$ is compact, we can take a subsequence such that $h_i^{-1} p$ converges, and since $A^+$ is closed, converges to a point in $A^+$. Also, since the repelling point of $h_i$ is in $A^-$, the proposition is satisfied with $(g_k) = (h_i^{-1})$ (since that will have as attracting point the previous repelling point (so it will be in $A^-$), but $g_k p$ converges to a point in $A^+$, and $A^+ \cap A^- = \emptyset$). If instead we had $e(R) = 1$ infinitely many times, we would have gotten $g_k p \to c \in A^-$, and $g_k x \to b \in A^+$ for all $x \neq p$. \hfill \Box

So, in particular, parabolic fixed points are finite boundary points, and so $M_F$ is not empty in some cases.

### 3.3.3 Inverse, from $M_\infty$ to $\phi^{-1}(M_\infty)$

We have a continuous $G$-invariant map from $\partial T$ to $M$, which has been built by observing that, given $\lambda \in \partial T$, any sequence converging to $\lambda$ with Gromov’s topology, converges to a fixed point $\phi(\lambda) \in M$ with Tukia’s topology. As we have seen, the same reasoning can not be used to build the inverse, as some sequences converging to finite boundary points may be bounded (and hence, not converge to $\partial T$ with Gromov’s topology). Furthermore, the Gromov boundary of a non proper space may not be compact, so it might
Lemma 3.18. The application \( \psi : M_\infty \to \partial T \) is well defined.

Proof. Fix \( x = (\alpha, t_0, p) \), and choose \( (t_n) \subset M \) with \( t_n \to p \). Then denoting \( x_n = (\alpha, t_n, p) \), since \( p \in M_\infty \) and \( x_n \to p \), \( \rho(x, x_n) \to \infty \). We have

\[
\rho(x, x_n) = \max((p, t_0|t_n, \alpha), (\alpha, t_0|t_n, p)).
\]

Since \( t_n \to p \neq \alpha \), we can apply lemma 3.11 to conclude \((p, t_0|t_n, \alpha) < K\). Hence \( \rho(x_0, x_n) = (\alpha, t_0|t_n, p) \) for \( n \) big enough. Therefore, the Gromov product of the sequence \( (x_n) \) results

in, for \( n, m \) big enough,

\[
2(x_n \cdot x_m)_x = \rho(x, x_n) + \rho(x, x_m) - \rho(x_n, x_m) = (\alpha, t_0|t_n, p) + (\alpha, t_0|t_m, p) - \max((\alpha, t_n|t_m, p), (p, t_n|t_m, \alpha)).
\]

To bound from above the last term we just have to use the same reasoning as in the proof of lemma 3.12. Take \( V_o, V_i \) neighbourhoods of \( p \), with \( \text{cl} V_i \subset V_o \) and \( t_0, \alpha \) not in \( \text{cl} V_o \). Then,

\[
\{|A_\alpha \in A| \alpha \in A^-_i, A^+_i \cap V_i \neq \emptyset, A^+_i \cap V_i \neq \emptyset, A^+_i \not\subseteq V_o\} < K.
\]

With this we get that, given \( n, m \) such that \( t_n \) and \( t_m \) are inside \( V_i \), if the distance \( \rho(x_n, x_m) \) is longer than \( K + 2 \), then all the chain except the
first \( K + 2 \) elements will serve as a chain between \( \alpha, t_0 \) and \( t_n, p \). Hence, 
\[
\rho(x_n, x_m) \leq \rho(x_n, x) + K + 2,
\]
and we get
\[
2(x_n \cdot x_m)_x \geq \rho(x, x_m) - K - 2.
\]
which goes to infinity as \( n, m \) go to infinity. The same reasoning can be applied to see that any other sequence with \( t'_n \to p \) will satisfy \( (x_n \cdot (\alpha, t'_m, p))_{x_0} \to \infty \).

The only remaining thing to check to see that the function is well defined is seeing that it does not depend on \( \alpha \). We just have to see that a different \( \alpha \) displaces the tail of the sequence by a finite amount (which may depend on \( \alpha \) and \( \beta \)). I.e., that for \( n \) big enough,
\[
\rho((\alpha, t_n, p), (\beta, t_n, p)) = \max((\alpha, t_n|\beta, p), (\alpha, p|\beta, t_n)) < C
\]
which follows easily from applying lemma 3.11 in the same way as before, choosing the neighbourhood of \( p, V_0 \) such that \( \text{cl} V_0 \) does not contain \( \alpha \) nor \( \beta \).

The next thing we want to see is that this is actually the inverse of the previous application, that is, all Gromov sequences converging to \( p \in M_\infty \), are actually similar (and hence related to the same \( \lambda \in \partial T \)).

**Lemma 3.19.** The restriction of the function \( \phi : \partial T \to M \) to \( \phi^{-1}(M_\infty) \to M_\infty \) has an inverse, given by the function \( \psi \) described above.

**Proof.** It is easy to see from the definitions that \( \phi(\psi(p)) = p \), since \( \psi(p) = ((\alpha, t_n, p))_n \) with \( t_n \to p \), so for any open set \( V \) around \( p \) there will be \( n_0 \) such that, for \( n \geq n_0, t_n \in V \).

We have to check \( \psi(\phi(\lambda)) = \lambda \). I.e., that given \( \lambda \in \phi^{-1}(p) \), where \( p \) is not a finite boundary point, and a sequence \( (a_n) = ((x_n, y_n, z_n))_n \sim \lambda \), we have \( (x_n, y_n, z_n)_n \sim ((\alpha, t_n, p))_n \).

The first step will be seeing \( ((x_n, y_n, z_n))_n \sim ((\alpha, y_n, z_n))_n \). We begin fixing \( x = (\alpha, t, p) \) and taking a subsequence of \( (a_n) \) such that the three elements converge to some point. By lemma 3.11 at least two of these have to converge to \( p \), which we assume are the two last components. If the first component converges to \( \beta \neq p \), doing a similar reasoning as before we have that the tails of the sequences \( (x_n, y_n, z_n) \) and \( (\alpha, y_n, z_n) \) are separated by a finite distance, so we have the similarity between the sequences. If \( x_n \to p \), shuffling the components if necessary, we assume that the distance from \( x \) is always achieved with the last two components (i.e., \( \rho(x, a_n) = \max((v, w|y_n, z_n) : v, w \in \{\alpha, t, p}\))).

Evaluating the Gromov product,
\[
2(a_n \cdot (\alpha, y_n, z_n))_x = \rho(x, a_n) + \rho(x, (\alpha, y_n, z_n)) - \rho((x_n, y_n, z_n), (\alpha, y_n, z_n)),
\]
the last term is equal to \( \max((x_n, y_n|\alpha, z_n), (x_n, z_n|\alpha, y_n)) \), so applying lemma 3.11 as before, \( \rho((x_n, y_n, z_n), (a_n, y_n, z_n)) \leq \rho(x, a_n) + K, \)
and
\[
2(a_n \cdot (\alpha, y_n, z_n))_x \geq \rho(x, (\alpha, y_n, z_n)) - K.
\]
Since we assumed the distance between $x$ and $a_n$ is achieved with the last two components of $a_n$, $\rho(x, (\alpha, y_n, z_n)) \geq \rho(x, a_n)$, the Gromov product goes to infinity, and both sequences are similar.

The next step will be checking $((\alpha, y_n, z_n))_n \sim ((\alpha, y_n, p))_n$. The Gromov product is

$$2((\alpha, y_n, z_n) \cdot (\alpha, y_n, p))_x = \rho(x, (\alpha, y_n, z_n)) + \rho(x, (\alpha, y_n, p)) - \rho((\alpha, y_n, z_n), (\alpha, y_n, p)).$$

The last term is

$$\rho((\alpha, y_n, z_n), (\alpha, y_n, p)) = \max((\alpha, y_n, p), (y_n, z_n | \alpha, p))$$

The second term is

$$\rho((\alpha, t, p), (\alpha, y_n, p)) = \max((\alpha, t | y_n, p), (t, p | \alpha, y_n)).$$

By the same reasoning as before, $(t, p | \alpha, y_n) \leq K$, and since $p$ is in the infinite boundary, $(\alpha, t | y_n, p) \to \infty$ as $y_n \to p$. Using the usual reasoning, for $n$ big enough

$$\rho((\alpha, t, p), (\alpha, y_n, p)) = (\alpha, t | y_n, p) \geq (\alpha, z_n | y_n, p) - C.$$

For the first term we have, by definition of the distance,

$$\rho(x, (\alpha, y_n, z_n)) \geq (y_n, z_n | \alpha, p).$$

Adding the three inequalities we get

$$\rho((\alpha, y_n, z_n), (\alpha, y_n, p)) \leq \max(\rho(x, (\alpha, y_n, p)), \rho(x, (\alpha, y_n, z_n))) + C,$$

so we have

$$2((\alpha, y_n, z_n) \cdot (\alpha, y_n, p))_x \geq \min(\rho(x, (\alpha, y_n, p)), \rho(x, (\alpha, y_n, z_n))) - C,$$

which goes to infinity, since the $p$ is an infinite boundary point (and the first possible value goes to infinity) and we had chosen $y_n$ and $z_n$ such that $\rho(x, (\alpha, y_n, z_n)) \geq \rho(x, a_n) \to \infty$.

Finally, by the proof of 3.12 for $p$ on the infinite boundary all sequences of the form $(\alpha, t_n, p)$ with $t_n \to p$ are equivalent, so we have $(a_n) \sim ((\alpha, t_n, p))$, and hence $\psi(\phi(\lambda)) = \lambda$.  

The only thing we have to check now, to get an homeomorphism, is the continuity of $\psi$, which follows from a reasoning similar to the one done in the last part of the previous lemma.

**Lemma 3.20.** The function $\psi : M_\infty \to \partial T$ is continuous.

**Proof.** Consider $(p_m) \subset M_\infty$ converging to $p \in M_\infty$. We want to see if $\lambda_m = \psi(p_m)$ converges, and if it converges to $\lambda = \psi(p)$. Fix, as before, $x = (\alpha, t, p)$, consider $t_n \to p$ and the maximal sequences $(\alpha, t) < A_{1,n} < \ldots < A_{k(n),n} < \{t_n, p\}$, associated to $(\alpha, t | t_n, p)$. For the last term of the
sequence we have \( p \in \text{int} \, A_{k(n),n}^+ \), so we can take \( m(n) \) such that \( p_m \) is inside \( \text{int} \, A_{k(n),n}^+ \) for all \( m \geq m(n) \). Then, for all \( y \in \text{int} \, A_{k(n),n}^+ \) we have \( \{\alpha, t\} < A_{1,n} < \ldots A_{k(n),n} < \{y, p_m\} \), so, \( \rho(x, (\alpha, y, p_m)), \rho(x, (\alpha, y, p)) \geq (\alpha, t|\mathcal{t}_n, p) \).

Evaluating the Gromov product,

\[
2((\alpha, y, p_m) \cdot (\alpha, y, p)_x) = \rho(x, (\alpha, y, p_m)) + \rho(x, (\alpha, y, p)) - \rho((\alpha, y, p_m), (\alpha, y, p)).
\]

As before, the last term is \( \rho((\alpha, y, p_m), (\alpha, y, p)) = \max((\alpha, p_m|y, p), (y, p_m|\alpha, p)). \)

The second term is \( \rho((\alpha, t, p), (\alpha, y, p)) = \max((\alpha, t|y, p), (t, p|\alpha, y)). \) As before, \( (t, p|\alpha, y) < K \), so we can take \( n \) big enough (modifying \( p_m \) accordingly) such that \( \rho((\alpha, t, p), (\alpha, y, p)) = (\alpha, t|y, p) \), which is greater than \( (\alpha, p_m|y, p) - C \). Again, by definition of the distance, the first term is \( \rho(x, (\alpha, y, p_m)) \geq (y, p_m|\alpha, p). \) Coupling all together,

\[
2((\alpha, y, p_m) \cdot (\alpha, y, p)_x) \geq \min\{\rho(x, (\alpha, y, p_m)), \rho(x, (\alpha, y, p))\} - C \\
\geq \rho(x, (\alpha, t_n, p))) - C.
\]

Given \( L > 0 \), consider a neighbourhood \( V \) of \( p \) such that \( ((\alpha, t_i, p) \cdot (\alpha, t_j, p))_x \geq L \) for all \( t_i, t_j \in V \) (if there is no such neighbourhood we get a contradiction with the proof of lemma 3.12, since we could make a sequence \( (t_n) \) converging to \( p \) where \( (\alpha, t_n, p) \) doesn’t define a point on the Gromov boundary). Take any \( t_j \in V \), and consider \( U := \text{int} \, A_{k(j),j}^+ \), and \( m(j) \) as defined at the beginning of the proof. Take \( m \geq m(j) \) big enough so \( p_m \in V \cap U \).

For all points on \( t^m, t^p \in U \cap V \) we have

\[
((\alpha, t^m, p_m) \cdot (\alpha, t^p, p))_x \geq L - C
\]

and

\[
((\alpha, t^m, p) \cdot (\alpha, t^p, p))_x \geq L.
\]

So, by the triangle inequality for the Gromov product,

\[
((\alpha, t^m, p_m) \cdot (\alpha, t^p, p))_x \geq L - C - O(r).
\]

Therefore, for any \( L \) there exists \( m_0 \) such that for \( m \geq m_0 \), if \( t^m_n \rightarrow p_m \) and \( t^p_n \rightarrow p \), the Gromov products between elements of the tails of the associated Gromov sequences \( (\alpha, t^m_n, p_m) \) and \( (\alpha, t^p_n, p) \) is bigger than \( K - C - O(r) \).

Hence, given \( K \) there exists \( m \) such that for \( m \geq m_0 \), \( (\lambda_m \cdot \lambda)_x \geq K \), so \( \lambda_m \rightarrow \lambda \).

\[\Box\]

In total, we have the following.
**Proposition 3.21.** Let $G$ be a group acting as a convergence group, minimally and non-elementary on a metric space $M$. Let $(T, \rho)$ be the set of distinct triples equipped with the quasimetric described by Bin Sun. Then, there exists a $G$-equivariant continuous map $\phi : \partial T \to M$ such that the restriction $\phi : \phi^{-1}(M_\infty) \to M_\infty$ becomes an homeomorphism.

**4 Zero sets of $M$ under the stationary measure**

In this section we will show how, under the stationary measure, the finite boundary has 0 mass, which implies that random walks converge to the infinite boundary and, in particular, to conical limit points. The main ingredient we will use is the following lemma, found in [MT16][Lemma 4.5].

**Lemma 4.1.** Let $G$ a countable group acting by homeomorphisms on a metric space $M$, $\mu$ a probability distribution on $G$ whose support generates $G$, and $\nu$ a $\mu$-stationary probability measure on $M$. Moreover, let us suppose that $Y \subset M$ has the property that there is a sequence of positive numbers $(\epsilon_n)$ such that for any translate $fY$ of $Y$ there is a sequence $(g_n)$ of group elements (which may depend on $f$), such that the translates $fY$, $g_1^{-1}fY$, $g_2^{-1}fY$, ... are all disjoint, and for each $g_n$, there is an $m \in N$, such that $\mu_m(g_n) > \epsilon_n$. Then $\nu(Y) = 0$.

Given $x \in T$ and $R > 0$, we will consider the sets of finite boundary points which are at "distance" smaller or equal to $R$ from $x$, that is, the points $p \in M$ such that there exists a sequence $(x_n) \subset T$ with $\rho(x, x_n) \leq R$ such that $x_n \to p$. In other words, we define

$$D_M(x, R) := \overline{B(x, R)} \cap M,$$

where the closure is done with respect to Tukia’s topology. A critical observation is that, by the definition of $M_F$, if $(R_i)_{i \in \mathbb{N}}$ goes to infinity we get $D_M(x, R_i) \subset D_M(x, R_{i+1})$, and $\bigcup_{i \in \mathbb{N}} D_M(x, R_i) = M_F$, so if we prove that each of $D_M(x, R)$ has zero measure, then the ascending limit $M_F$ also has zero measure.

The first step we need to take to apply the lemma is proving that these balls behave well under the action by $G$, that is,

**Lemma 4.2.** $gD_M(x, R) = D_M(gx, R)$, or, equivalently, $p \in D_M(x, R) \iff gp \in D_M(gx, R)$.

**Proof.** $p$ belongs in $D_M(x_0, R)$ if and only if there exists a sequence $(x_n) \subset B(x, R)$ with $x_n \to p$. Since $G$ acts by isometries on $T$ and by homeomorphisms on $M$, the previous is equivalent to $(gx_n) \subset gB(x, R) = B(gx, R)$ and $gx_n \to gp$, which is equivalent to $gp$ belonging in $D_M(gx_0, R)$. $\square$
Next we need to see that if the centers of the balls are separated enough with respect to the radius, then the balls are separated. To prove that first we need a small lemma, which will come up later.

**Lemma 4.3.** If \( x = (x^1, x^2, x^3) \) and \( p \in D_M(x, R) \), then \( (x^i, x^j|p) \leq R \) for all \( i \neq j, 1 \leq i, j \leq 3 \).

**Proof.** Assume \( p \in D_M(x, R) \), i.e., that there exists \( x_n \to p \) with \( \rho(x, x_n) \leq R \), and that the conclusion is false, that is, there exists \( i, j \) such that \( (x^i, x^j|p) \geq R + 1 \). Then, there exists an annulus sequence of length \( R + 1 \) such that \( \{x^i, x^j\} \subset A_1 \subset A_2 < \ldots < A_{R+1} \subset \{p\} \).

By definition of the relation, \( p \in \text{int} A_{R+1}^+ \), so \( \text{int} A_{R+1}^+ \) is a neighbourhood of \( p \). By definition of the convergence to the boundary, there exists \( n_0 \) big enough such that for all \( n \geq n_0 \) at least two components of \( x_n \) are in \( \text{int}(A_{R+1}^-) \). Therefore, we will also have the chain

\[
\{x^i, x^j\} \subset A_1 \subset A_2 < \ldots < A_{R+1} \subset \{x^k_n, x^l_n\},
\]

and hence \( \rho(x, x_n) \geq R + 1 \).

As a side note, coupling this last result with lemma 3.16 we get an equivalent definition of finite boundary point

**Corollary 4.4.** Let \( p \in M \). Then \( p \) is a finite boundary point if and only if there are two points \( a, b \in M \setminus p \) and \( R > 0 \) such that \( (a, b|p) \leq R \).

Next we prove that if the centers of the balls are separated enough, then the balls are also separated. The idea is to assume that they intersect, and use the annulus sequence between the two centers to build two sequences between each of the centers and the intersecting point, such that the sum of the lengths of these sequences is almost the length of the total sequence, which will be a contradiction by the previous lemma (since then, at least one of them has to bee too large).

**Lemma 4.5.** If \( \rho(x, y) \geq R + 2 \), then \( D_M(x, R) \cap D_M(y, R) = \emptyset \)

**Proof.** Denote \( x = (x^1, x^2, x^3), y = (y^1, y^2, y^3) \) and assume \( p \in D_M(x, R) \cap D_M(y, R) \). Assume that the distance between \( x \) and \( y \) is realized by \( (x^1, x^2|y^1, y^2) \). Then, \( (x^1, x^2|y^1, y^2) \geq 2R + 2 \), and hence we have the chain

\[
\{x^1, x^2\} \subset A_1 \subset A_2 < \ldots < A_{2R+2} \subset \{y^1, y^2\}.
\]

By definition of the relation, \( \text{int} A_{i+1}^- \cup \text{int} A_i^+ = M \), so \( p \) belongs to \( \text{int} A_{i+1}^- \) or \( \text{int} A_i^+ \) for all \( i \). Let \( i_0 \) be the biggest \( i \) such that \( p \in \text{int} A_i^+ \). We have the chain

\[
\{x^1, x^2\} \subset A_1 \subset A_2 < \ldots < A_{i_0} < \{p\},
\]

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so if \( i_0 \geq R + 1 \) we get a contradiction with lemma \(^{1.3} \) since \((x^1, x^2)p \geq R + 1\). If \( i_0 \leq R \) we have that \( p \not\in \text{int } A_{i_0+1}^+ \), so, by definition of the relation, \( p \in \text{int } A_{i_0+2}^+ \). We get the chain
\[
\{p\} < A_{i_0+2} < A_{i_0+3} < \ldots < A_{2R+2} < \{y^1, y^2\},
\]
and we have again a contradiction with lemma \(^{4.3} \).

With this we can prove the result we anticipated at the beginning of the section

**Proposition 4.6.** Let \( G \) be a group acting minimally, non elementary and as a convergence group on \( M \), and \( \mu \) a probability measure on \( G \) such that its support generates \( G \). If \( \nu \) is the \( \mu \)-stationary Borel probability measure, then \( \nu(M_F) = 0 \).

**Proof.** We want to apply lemma \(^{1.1} \) with \( Y = D_M(x, R) \), where \( x \in T \) fixed. By lemma \(^{1.2} \), all translations of \( Y \) will be of the form \( D_M(y, R) \). Let \( g \in G \) be the loxodromic element determined by Sun. By proposition \(^{2.7} \), we can take \( N > 0 \) such that \( \inf(\rho(x, g^{nN}x)) \geq n(2R + 2) \). By lemma \(^{4.5} \), for any \( f \in G \), the sets \( fY, g^{-N}fY, g^{-2N}fY, \ldots \) are disjoint, since the distance between any of the centers of the balls is greater or equal to \( 2R+2 \). Since the support of \( \mu \) generates \( G \), there is \( m(n) \) such that \( \mu_{m(n)}(g^{nN}) > 0 \), so labeling \( \epsilon_n := \mu_{m(n)}(g^{nN}) \) we can apply the lemma and we get \( \nu(D_M(x, R)) = 0 \). We finish by recalling that, as \( R \to \infty \), \( D_M(x, R) \to M_F \).

The set where the measure has all of its mass can be restricted a little further. To do this, we observe that \( M_\infty \), which has full mass, may depend on the metric \( \rho \), which in turn only depends on the chosen annulus system. Since we always deal with annulus systems generated by one annulus \( A := \{A^-, A^+\} \), the infinite boundary depends only on the annulus \( A \), or more specifically, on the sets \( A^-, A^+ \), so we can write \( M_\infty(A^-, A^+) \). Therefore, if we choose a countable family of annuli \( \{A_i^-, A_i^+\}_{i \in \mathbb{N}} \) such that Sun’s construction works, we will get a countable family of sets, \( M_i := M_\infty(A_i^-, A_i^+) \), where \( \nu(M_i) = 1 \), and intersecting them we will have \( \nu(\bigcap_{i \in \mathbb{N}} M_i) = 1 \).

Looking at Sun’s construction we see that, for the construction to work, the conditions on \( A^- \) and \( A^+ \) are the following:

- \( A^- \) and \( A^+ \) are closed, and \( A^- \cap A^+ = \emptyset \)
- There exists an element \( g \in G \) behaving like the one described in \(^{3.9} \) such that if \( a^-, a^+ \in M \) are its fixed points, \( a^- \in \text{int } A^- \) and \( a^+ \in \text{int } A^+ \).

Choosing a particular family of acceptable generating annulus we get the following.

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Proposition 4.7. Let $\nu$ be the Borel $\mu$-stationary measure on $M$, and let $g \in G$ such that $g$ fixes two distinct points $a^-, a^+$ and that $g^n|_{M \setminus a^-}$ converges to $a^+$ locally uniformly as $n \to \infty$, then, denoting $\mathcal{M}^g_\infty$ the set of points $p \in M$ such that there exists a sequence $g_n$ (depending on $p$) with $g_np \to a^+$ and $g_nx \to a^+$ for all $x \neq p$, we have $\nu(\mathcal{M}^g_\infty) = 1$.

Proof. Equip $M$ with a metric $d_M$, and define the sets

$$A^-_i = \overline{B(a^-, d_M(a^-, a^+) / i)}, A^+_i = \overline{B(a^+, d(a^-, a^+) / i)}.$$

For $i \geq 3$ these sets define acceptable annulus, we get a family of annulus systems as described above, and an associated family of infinite boundary points $M^\infty_i$. As explained, $\mathcal{M}^g_\infty := \bigcap_{i \geq 3} M^\infty_i$ has full mass, so let’s see how $p \in \mathcal{M}^g_\infty$ behaves. By 3.17 for each $i$ we will have $(g^i_n)_n$ with either $g^i_n p \to A^-_i$ and $g^i_n x \to A^+_i$ for all $x \neq p$ or $g^i_n p \to A^+_i$ and $g^i_n x \to A^-_i$ for all $x \neq p$. We assume now that the first one happens infinitely many times for $(i_k)$, and we take a convergent subsequence of each $(g^{i_k}_n)$ (which we relabel as $(g^{i_k}_n)$). Taking $V$ open set around $p$, for each $i_k$ there exists $n_k$ big enough so $g^i_{n_k} p \subset A^-_{i_k}$ and $g^i_{n_k} (M - V) \subset A^+_{i_k - 1}$. By definition of the sets, as $k \to \infty$ $g^i_{n_k} p \to a^-$, and $g^i_{n_k} (M - V) \to a^+$, so taking a convergent subsequence $(h_j) \subset (g^{i_k}_n)_{k}$, $p$ will satisfy the first possible condition of the lemma (that is, $h_j p \to a^-$ and $h_j x \to a^+$ for all $x \neq p$). Hence, $\mathcal{M}^g_\infty \subset M^\infty_\infty$, so $\nu(\mathcal{M}^g_\infty) = 1$.

4.1 The Poisson boundary of convergence groups

By Sun’s construction [Sim16], the action of $G$ on $(T, \rho)$ (or rather, on $(S, \rho')$) has a WPD element and is non elementary, so, using Maher and Tiozzo’s theorem [MT18][Theorem 1.4], if $\mu$ is a measure generating $G$ with finite support (and hence it automatically has finite entropy and finite logarithmic moment), then the Gromov boundary of $S$, coupled with the hitting measure $\nu$, is a model for the Poisson boundary of $(G, \mu)$. Using the application between $\partial T$ and $M$ built in 3.21 and proposition 3.6 we are able to prove our main result.

Theorem 4.8. Let $G$ be a finitely generated group acting as a convergence group, minimally and non-elementary on a compact space $M$, and $\mu$ a probability measure generating $G$ with finite support. Then, $(M, \nu)$ is the Poisson boundary of $(G, \mu)$, where $\nu$ is the $\mu$-stationary Borel probability measure on $M$.

Proof. By theorem 2.12 $(\partial S, \tilde{\nu})$ is the Poisson boundary of the random walk $(G, \mu)$, where $S$ is the hyperbolic space quasi-isometric to $T$ by a $G$-equivariant quasi-isometry $f$, and given $x \in S$, $\tilde{\nu}$ is the $\mu$-stationary hitting measure of the random walk $w_n x$ on $S$. $\partial S$ and $\partial T$ are homeomorphic by
the induced action of $f$, which is $G$-equivariant, so $\tilde{\nu} := f(\tilde{\nu})$ is $\mu$-stationary, and $(\partial T, \tilde{\nu})$ is equivalent to $(\partial S, \tilde{\nu})$ as measure space (and hence it is the Poisson boundary).

Let $\phi$ be the $G$-equivariant application build in section 3.3, and $\psi$ the inverse on $M_\infty$. By $G$-equivariance, the probability measure on $M$, $\nu := \phi(\tilde{\nu})$, is also $\mu$-stationary, and by continuity it is Borel, so $\nu(M_\infty) = 1$. Therefore, $\psi(\nu) = \tilde{\nu}$, so the two spaces, as measure spaces, are equivalent via a $G$-equivariant map, and hence $(M, \nu)$ is the Poisson boundary.

5 Applications

5.1 Compactification of $G$ and the Dirichlet problem

Using the topology we used to paste $M$ to $T$ we can try to paste $M$ to $G$ in a similar way. That is, fixing $x \in T$, given $U \subset M$ open, we can consider the subset of $G \cup M$, 

$$\tilde{U}_G := \{g \in G | gx \text{ has two components in } U\} \cup U.$$ 

Considering the the family $B$ of sets of this form, together with the sets $P(G)$ (that is, all open sets of $G$, as it has the discrete topology), we get a basis for a topology in $G \cup M$. Indeed, we have a cover, and for $U_1, U_2 \in B$ and $x \in U_1 \cap U_2$, if $x \in G$, we can simply take $\{x\} \in B$, and we have $x \in \{x\} \subset U_1 \cap U_2$; if $x \in M$, we can take $V = U_1 \cap U_2 \cap M$, which is open, so $V_G \in B$, for which we will have $x \in V_G \subset U_1 \cap U_2$. Therefore, we may consider the unique generated topology (which by the following proposition does not depend on $x$).

**Proposition 5.1.** The topology we defined on $G \cup M$ does not depend on the basepoint $x$.

**Proof.** Given a point in $M$, we can take a countable neighbourhood basis in $M$, and we get a corresponding countable neighbourhood basis in $G \cup M$. Since $G$ has the discrete topology, a point in $G$ itself is a neighbourhood, so we have a countable neighbourhood basis. Therefore, $G \cup M$ is first countable, and the topology is defined by the convergence along sequences.

Consider a sequence $(g_n) \subset G$ with $g_n \to p$, which by definition is equivalent to $g_n x \to p$ with Tukia’s topology. Take $y \in T$, and assume that $g_n y$ does not converge to $p$. Then, there exists an open neighbourhood of $p$ on $T \cup M$ of the form $\tilde{U}$, and a subsequence of $(g_{n_k}) \subset (g_n)$, such that $g_{n_k} y$ does not enter $\tilde{U}$. However, by the convergence property, we can take a convergent subsequence of $g_{n_k}$ (in the sense of convergence groups), which since $g_n x \to p$, will have $p$ as attracting point, and hence $g_{n_k} y$ will enter $\tilde{U}$ eventually (and hence, $g_{n_k}$ will enter $\tilde{U}_G$). That is, $g_n$ converges to $p$ with the topology generated by taking $y$ as a basepoint, so doing the same
reasoning the other way, \( g_n \) converges to \( p \) with the topology generated by taking \( y \) as basepoint if and only if it also converges to \( p \) with the topology generated by taking \( x \) as basepoint.

For \( h \in G \),

\[
h\hat{U}_G = \{ hg \in G | gx \in \hat{U} \} \cup hU = \{ g \in G | gx \in h\hat{U} \} \cup hU = \hat{hU}_G,
\]

so \( G \) acts by homeomorphisms on \( G \cup M \).

This construction is indeed a compactification. To see this we observe that, if \((g_n)\) is a converging sequence in the sense of convergence group with attracting point \( a \in M \), then \((g_n,x)\) converges to \( a \) for any \( x \in T \), so we will have \((g_n)\) converging to \( a \). Therefore, by definition of convergence group, any sequence \((h_n) \subset G\) will have a converging sequence (in the sense of convergence groups) which will converge (in the topology of \( G \cup M \)). Adding that \( M \) is also compact, and that it is topologically embedded into \( G \cup M \), we get that any sequence \( g_n \in G \cup M \) has a converging subsequence (we can restrict ourselves to either a subsequence of points in \( G \), or to a subsequence of points in \( M \)).

All in all, since for sequence \((g_n) \subset G\) we have convergence to a point \( p \in M \) if and only if \((g_n,x) \subset T\) converges to the same point \( p \in M \) we have, by 3.5, the following result.

**Corollary 5.2.** Let \( G \) be a discrete, countable group acting as a convergence group, non elementary and minimally on a metrizable compact space \( M \). Then, there exists a compact topology on \( G \cup M \) such that the inclusions \( G \hookrightarrow G \cup M, M \hookrightarrow G \cup M \) are topological embeddings and, for any generating measure \( \mu \) on \( G \), the sample paths of the associated random walk on \( G \) converges almost surely to points in \( M \).

Whenever \( G \) is an hyperbolic group, we can consider a finite set of generators \( S \) and the Cayley graph \( \Gamma(G,S) \). Then, we can add the Gromov boundary to \( \Gamma(G,S) \), getting \( \Gamma(G,S) \cup \partial G \), and then we can take the induced topology on \( G \cup \partial G \). As changing the generating set \( S \) induces a quasi-isometry, this topology on \( G \cup \partial G \) does not depend on \( S \), so it is well defined. The topology we have explained for a convergence group can be seen as a generalization of Gromov’s one. Indeed, we have the following.

**Proposition 5.3.** Let \( G \) be an hyperbolic group, and assume its Gromov boundary \( M \) has more than two points. Then, the topology we obtain on \( G \cup M \) by considering \( G \) acting as a convergence group on \( M \) following the procedure explained in this section coincides with Gromov’s topology.

**Proof.** Both restrictions to \( M \) and to \( G \) have the same topology in both cases, so we only have to check if the sequences of \( G \) converging to points in \( M \) have the same limit in both topologies (as there are no sequences of elements of \( M \) converging to elements of \( G \)).
Consider \((g_n)_{n \in \mathbb{N}} \subset G\) converging to \(\lambda \in M\) with the topology of convergence groups, and assume the sequence does not converge to \(\lambda\) in Gromov’s topology. Then, given a finite set of generators \(S\), there exists a subsequence \((g_{n_k}) \subset (g_n)\) such that \((g_{n_k} \cdot \lambda)_{n_k} \leq K\) (where the Gromov product is done with respect to the path metric on \(\Gamma(G, S)\)). Taking a convergent subsequence (in the sense of convergence groups) we get a subsequence \((h_i) \subset (g_{n_k})\), which, since it converges to \(\lambda\) in the convergence group topology, has \(\lambda\) as attracting point. Take \(\alpha \in M\) different from the repelling point of \((h_i)\). Then, \(h_i \alpha\) converges to \(\lambda\), so \((h_i \alpha \cdot \lambda)_{n_k}\) goes to infinity. Hence, by the inverse triangle inequality,

\[
K \geq (h_i \cdot \lambda)_e \geq \min((h_i \alpha \cdot h_i)_e, (h_i \alpha \cdot \lambda)_e) + O(\delta) = (h_i \alpha \cdot h_i)_e + O(\delta).
\]

Hence, \(K \geq d(e, [h_i \alpha, h_i]) + O(\delta) = d(h_i^{-1}, [\alpha, e]) + O(\delta)\). Therefore, \((h_i^{-1})\) converges to \(\alpha\) with Gromov’s topology. Choosing \(\alpha'\) different than \(\alpha\) and the repelling point of \((h_i)\) we get a contradiction, so \((g_n)\) converges to \(\lambda\) with Gromov’s topology.

Assume now \((g_n)\) converges to \(\lambda \in M\) with Gromov’s topology, but it does not converge to \(\lambda\) with the convergence group topology. Then, there exists a subsequence \(g_{n_k}\) which converges to \(\lambda' \neq \lambda\) with the convergence group topology, and by the previous paragraph, \(g_{n_k}\) converges to \(\lambda'\) with Gromov’s topology, giving us a contradiction. 

We have that \((M, \nu)\) is a \(\mu\) boundary of \((G, \mu)\), with \(\nu\) being the hitting measure of the random walk \((w_n x) \subset T\), that is, we have \(w_n x \to p \in A \in M\) with probability \(\nu(A)\). Hence, for the random walk \((w_n) \subset G\) we have \(w_n \to p \in A\) with probability \(\nu(A)\), that is, the random walk \(w_n\) converges pointwise to a random variable \(w_\infty\) on \(M\), with distribution \(\nu\). We are, then, in the situation introduced when introducing the Poisson boundary, so again we can define the hitting measures of the random walk starting at any \(g \in G\),

\[
\nu_g(A) := \mathbb{P}[gw_\infty \in A|w_0 = e] = \mathbb{P}[w_\infty \in g^{-1}A|w_0 = e] = g \nu_e(A).
\]

Given this setting, a frequent question is whether the Dirichlet problem at infinity is solvable, that is, whether every continuous function \(f : M \to \mathbb{R}\) admits a continuous extension to \(G \cup M\), such that it is harmonic on \(G\) with respect to the transition probability. For this we will use the following theorem, proved, for example, in [Woe00] [Theorem 20.3]

**Theorem 5.4.** The Dirichlet problem with respect a measure \(\mu\) and a compactification of \(G, G \cup B\), is solvable if and only if

1. The random walks \((w_n)\) converge almost surely to the boundary \(B\),

2. for the corresponding harmonic measures,

\[
\lim_{g \to p} \nu_g = \delta_p \text{ weakly for every } p \in B
\]
We have already seen that the first requisite is satisfied. For the second we just have to observe that every sequence with \( g_n \to p \) has a converging subsequence (in the sense of converging groups), for which \( g_n \nu \) will converge to \( \delta_p \). Therefore, \( g_n \nu \to \delta_p \) (since every subsequence will have a converging subsequence, and hence we can not take a fully non converging subsequence).

Hence, we get the following

**Proposition 5.5.** Assume \( G \) is a group acting as a convergence group, non elementary and minimally on a metrizable space \( M \). Then, the Dirichlet problem on \( G \cup M \) solvable with respect to the topology we explain above.

If \( f : M \to \mathbb{R} \) is the continuous function on the boundary, the extension to \( G \) will be given by the Poisson formula

\[
h(g) = \int_M f(p) g \nu.
\]

### 5.2 Strongly almost transitive actions

Let \( G \) be a second countable group acting measurably on a standard Lebesgue space \((X, \mathcal{B}, \nu)\), in such a way that the action preserves the measure class of \( \mu \) (that is, for all \( g \in G \) and \( A \in \mathcal{B} \) we have \( \nu(A) = 0 \iff \nu(gA) = 0 \)). We say that the action is **strongly almost transitive** if, given a set \( A \subset X \) such that \( \nu(A) > 0 \), and \( \epsilon > 0 \), there exists \( g \in G \) such that \( \nu(gA) > 1 - \epsilon \).

That is, the action is strongly almost transitive if every set of positive measure can be blown up to almost full measure. These actions were introduced by Jaworski in [Jaw94], where he proves the following theorem.

**Theorem 5.6.** Let \((M, \nu)\) be a \( \mu \)-boundary of \( G \). Then, the action of \( G \) on \((M, \nu)\) is strongly almost transitive.

**Corollary 5.7.** Let \( G \) be group acting non elementary, minimally and as a convergence group on a compact space \( M \), and \( \mu \) a measure on \( G \) such that its support generates \( G \). Then, there exists a measure \( \nu \) such that the action on the probability space \((M, \nu)\) is strongly almost transitive.

We refer to the paper by Glasner and Weiss, [GW16], for a reccompilation of some implications of having a non trivial strongly almost transitive action. We write here one of the consequences explained in that paper, which we find particularly interesting.

**Corollary 5.8** (of [GW16][Proposition 4.3]). Let \( G \) be group acting non elementary, minimally and as a convergence group on a compact space \( M \). Then, there is no non-constant Borel measurable equivariant map \( \phi : M \to Z \), where \((Z, d)\) is a separable metric space on which \( G \) acts by isometries (that is, \( M \) is ergodic with isometric coefficients).
5.3 $F_{\mu}$-proximality

Given a measure $\mu$ on a discrete countable group $G$, we can define the Cesàro averages $\mu_n := \frac{1}{n}(\mu + \mu^2 + \ldots + \mu^n)$. We say that a compact metric $G$-space $X$ is $F_{\mu}$-proximal if for each $x, y \in X$, $\mu_n\{g : d(gx, gy) > \epsilon\} \to 0$ as $n \to \infty$ for any $\epsilon > 0$. Furstenberg introduced this notion in [Fur73], where he also shows (among other equivalences, see Theorem 14.1 of that same article; see also [GW16] Theorems 8.4 and 8.5) that $X$ is $F_{\mu}$-proximal if and only if for any $\mu$-stationary Borel probability measure $\nu$ on $X$, $(X, \nu)$ is a $\mu$-boundary of $G$. As we have seen in proposition 3.3, this is indeed the case for convergence groups, so we get the following result.

**Corollary 5.9.** Let $G$ be group acting non elementary, minimally and as a convergence group on a compact space $M$, and $\mu$ a measure on $G$ such that its support generates $G$. Then, $M$ is $F_{\mu}$-proximal.

In [GW16] Theorem 8.5, Glasner and Weiss show that being $F_{\mu}$-proximal implies having a unique $\mu$-stationary Borel probability measure. Hence, by the corollary above, we get another way of proving that the Borel $\mu$-stationary probability measure is unique.

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