FIG. 1. Dependence of $T_c$ on the carrier density $N_0/2N_1$ in the case of weak (1-3) and strong (curves 4, 5) hybridization for $N_2/N_1 = 1$ (curves 1 and 4); 0.75 (2); 0.5 (3, 5).
FIG. 2. Temperature dependence of the chemical potential ($\eta = -\mu$) for various values of the carrier density $N_0$. Curve 1 corresponds to $N_0/2N_1 = 0.01$; 2 – 0.015; 3 – 0.02; 4 – 0.03.
FIG. 3. Superconductivity transition temperature $T_c$ dependenc on chemical potential $\eta = -\mu$: (1) $\lambda_{11} = 0.1$, $\lambda_{22} = 0.19$, $\lambda_{12}\lambda_{21} = 0.0024$; (2) $\lambda_{11} = -0.37$, $\lambda_{22} = -0.03$, $\lambda_{12}\lambda_{21} = 0.14$. 
SUPERCONDUCTIVITY IN TWO-BAND SYSTEM WITH
LOW CARRIER DENSITY

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Received 25 October 2003

In the article a review over the theory of superconductors with energy bands that overlap on Fermi surface at arbitrary densities of charge carriers (including reduced and very low) is done. All pairings of electrons that result in formation Cooper pairs of electrons from different energy bands as well as in every energy band are considered.

The system of equations for four order parameters $\Delta_{11}$, $\Delta_{22}$, $\Delta_{12}$, $\Delta_{21}$ and chemical potential $\mu$ is derived. Self-consistent approach is strictly necessary at $\mu \sim \Delta_{nm}$. Transition to the effective four-band model leading to the temperature of superconducting transition is performed. Analytic and numeric solutions have been performed for two mechanisms of superconducting pairing – non-phonon and phonon one. The account of the additional in regards to Moscalenco model pairings of electrons from different bands or their hybridization gives an option to obtain the high values of $T_C$ even at reduced density of charge carriers. This hybridization influences greatly the form of the dependence of the quantity $T_C$ on the density of charge carriers as well as the jump of the electron heat capacity at the point $T = T_C$.

High and low values of the relative jump of electronic heat capacity at the point $T = T_C \left( \frac{C_S - C_N}{C_N} > 1.43 \right.$ and $\left. \frac{C_S - C_N}{C_N} < 1.43 \right)$ are obtained with varying the density of charge carriers.

These four effective energy bands in the considered system favors essentially in
The experimental observation of the kink of the chemical potential $\mu(T)$ at the point $T = T_C$.

The mechanism of superconductivity (non-phonon and phonon) is shown to affect differently with varying the density of charge carriers as well as the dependence of $T_C$ and on this density.

Possible peculiarities in behavior of thermodynamic quantities in $MgB_2$ compound with reducing the density of charge carriers are predicted.

Influence of the overlapping energy bands on Fermi surface at $T = 0$ and very low densities of charge carriers ($\mu < 0$) on the superconductivity in the system that is in the state of Bose condensation of localized pairs is studied.

An application of the path integral method to the two-band model is developed and on this basis, the process of transition from the Fermi to the Bose pattern of elementary excitations at $T \neq 0$ in the presence of a two-particle bound state in the system is demonstrated. The expression for the temperature of Bose condensation $T_k$ is obtained and the contribution of the residual boson interactions is estimated for systems with different dimensions.

The overlapping of the energy bands on the fermi surface is favorable for superconductivity and intensify some peculiarities of thermodynamic properties at low carrier density. This helps to their experimental confirmation.

*Keywords:* Tow-band superconductor, hybridization, lowered and low carrier density, thermodinamical properties, magnesium borid, BCS-Bose crossover.

1. INTRODUCTION

A rich body of experimental and theoretical research material on oxide ceramics has been accumulated since the discovery of high-temperature superconductivity. However, the description of the physical properties of these materials remains one of the most difficult problems in low-temperature physics today. This is because of the complexity of
the objects of study: they have a complex crystal structure, strong anisotropy, anomalies in the electronic energy spectrum, a variable concentration of charge carriers, strong electronic correlations, and so on. Models for the analysis of such systems apparently should be based on the Hubbard model, which takes into account the strong electronic correlations due to the Coulomb interaction of the electrons, and should take into account the strong electron-phonon interaction. A review of the different approaches to this problem and the approximations used in them is given in Ref.1, for example. This theory contains dielectric and magnetic phase transitions and the possible onset of superconductivity. However, because of the great mathematical difficulties it is hard to obtain any meaningful physical results without making some substantial simplifications. Moreover, at a certain carrier density a metallic states arises in the system, in which the electronic states are modified but not destroyed by correlations. Consequently, there can be a transition to the superconducting state, with the formation of Cooper pairs (the BCS scenario) or local pairs (the Schafroth scenario). In this connection it is unquestionably of interest to apply Fermi-liquid concepts to the study of the superconducting properties of high-$T_c$ superconductors with allowance for their peculiar features, such as overlapping of the energy bands at the Fermi surface and the presence of various kinds of van Hove-Lifshitz singularities, strong anisotropy, variable carrier density (including small values) etc., when treating both phononic and nonphononic mechanisms of superconductivity.

Many theoretical papers have by now been published on various aspects of the two-band model [2], [3]. The great interest in this model and in its generalizations is due, firstly, to band calculations [4], [5] showing that in metal-oxide ceramics the energy bands on the Fermi surface overlap (a similar situation obtains, apparently, also in systems with heavy fermions [6]), and secondly to the possibility of using the aforementioned model to describe the properties of systems with two groups of electrons (e.g., layered compounds).

The main development stages of the superconducting theory of the systems with overlapping energy bands are presented in comment [7].

The consideration of the overlapping of the energy bands leads not only to the quanti-
tative difference of results from the case of one-band superconductor, but in some cases to the qualitatively new results. For example:

1) In two-band superconductors high temperature superconductivity is possible not only in the case of attractive interaction between the electrons, but even if the interaction between the electrons has repulsive character ($\lambda_{nm} < 0, n, m = 1-2$), but relation $\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} < 0$ is fulfilled.

2) In impurity two-band superconductors, for example, Anderson theorem is violated at $\Delta_1 \neq \Delta_2$, and appears the dependence of thermodynamical quantities on concentrations of non-magnetic impurity due to the interband scattering of electrons on impurity atoms.

3) In two-band superconductors appears collective oscillations of exciton-type Leggett mode, caused by the fluctuations of phase of order parameters for different bands. In the three-band systems, and also in the two-band with lowered density of charge carrier, which reduces to the three-band model, such oscillatory modes can be two.

4) On the basis of the theory of superconductivity with overlapping energetical bands one can explain a great number of experimental results in High-$T_c$ materials. In particular, by using the two-band model and assuming moderate values of the coupling constants one can obtain high $T_c$, two energy gaps $2\Delta_1/T_c > 3, 5$ and $2\Delta_2/T_c < 3, 5$, large values of negative $dlnT_c/dlnV$ ($V$ is the volume), a positiv curvature of the upper critical field near the transition temperature, and others [8] - [13]. Furthermore, in the two-band model it is possible to describe the decrease of $T_c$ with increase of the oxygen disorder, as well as when copper atoms are replaced by a nonmagnetic dopant (Al, Zn, etc) [14] - [16].

5). An important role in the determination of the thermodynamic and magnetic properties of a two-band superconductor is played also by the location of the Fermi level, which is changed by doping or by introduction of oxygen. Having assumed a non-phonon pairing mechanism of superconductivity as well as phonon mechanism in the multi-band systems with lowered densities of charge carriers the account of the peculiarities mentioned above is very crucial. Particular interest attaches to the possibility of obtaining a bell-shaped dependence of $T_c$ and of the heat-capacity discontinuity ($C_S - C_N$) at the point $T = T_c$ on
the carrier density [17]-[19]. In the three-band model with a nonphonon superconductivity mechanism it is possible to obtain the "step" which is observed for $Y_1Ba_2Cu_3O_{7-\delta}$ in the dependence of $T_c$ on the carrier density [20], [21]. An investigation of the properties of high-temperature ceramics, based on allowance for energy-band overlap of the energy bands and for electronic topological transitions, was reviewed in Ref [22]. This review contains the classical achievements of the problem. One can find there references to experimental research results that can be described by allowance for the singularities in the electron energy spectrum of complex systems.

An increase of the number of energy bands on the Fermi surface increases the overall electron-state density and leads to the onset of an additional interband electron-electron interaction that contributes to the onset of superconductivity. This interaction violates the universal BCS relations and leads to a substantial dependence of a number of physical characteristics on the properties of an anisotropic system [22]-[27].

The discovery [28] of superconductivity in $MgB_2$ with a high transition temperature $T_c \sim 39K$ has attracted considerable attention. The theoretical investigations [29]-[32] as well experimental observations [33], [34] have led to the conclusion that $MgB_2$ is multi-band superconductor with phonon-mediated BCS superconductivity mechanism.

Therefore, the known classical results for the system with overlapping energy bands can be used as a basis to describe the superconducting properties of $MgB_2$.

Nowadays, many authors rediscover unfortunately well-known already results, obtained early on the basis of Moskalenko [2] two-band model. At the same time the method, taking into account the overlapping energy bands as well as the anisotropy of energy gap, is developed in a number of studies (see, for example [35]-[37]). This method gives a good option to incorporate the theoretical results to the experimental data about the specific heat in $MgB_2$.

As mentioned above, an interesting feature of the two-band model is independence of the superconducting-transition temperature of the sign of the inter-and intraband interaction constants. This model can therefore be used for the usual electron-phonon mechanism of
superconductivity as well as for a mechanism based on the repulsion between carriers. In all the references cited above, the two-band model can be used to describe the properties of superconductors for which the relation $\mu \gg T_c$ is satisfied ($\mu$ is the chemical potential). This relation is satisfied in a number of cases also in high-temperature compound. The existence of this relation between $\mu$ and $T_c$ makes it possible to use in the calculations an approximation diagonal in the band indices, [38], [39] which leads to neglect of off-diagonal parameters such as $\Delta_{12}$ and $\Delta_{21}$.

In systems with low carrier density, however, the relation $\mu \gg T_c$ does not hold. It becomes therefore necessary to develop a superconductivity theory for two-band systems, without constraints on the Fermi energy. This is the task of the present paper. We consider simultaneously two possible superconductivity mechanisms - phonon and electron. A characteristic feature of systems with low carrier density is a substantial dependence of the chemical potential $\mu$ on the order parameter in the superconducting phase. This circumstances has been noted in many papers, and the feasibility of experimentally observing anomalies in the temperature dependence of the chemical potential was first suggest in Ref.40. It was shown there, with the BCS model as the example, the $\mu(T)$ curve has an experimentally observable bend at the point $T = T_c$. We shall show below that in the two-band case this effect is enhanced by the presence of four order parameters ($\Delta_{nm}; m = 1, 2$) and is manifested at $\mu$ values easier to observe in experiment.

In the articles referred to above, investigations were conducted on the basis of the idea of Cooper pairing. In systems with small carrier concentrations, such as semiconductors or metallic oxides, bound states may arise following a decrease in the carrier concentration, and a transition to a Bose condensate of local pairs with a final binding energy may occur (the Schaffroth scenario [41]). The issue of such transitions in one-band systems was discussed in a number of articles [42] - [48].

To realize the Schaffroth scenario, a bound two-particle state [45] must exist in the system. The appearance of this state in the presence of attractive interaction depends on the dimension of the system [45] and is realized in the region of small carrier concentrations.
As is shown in [43], [47], the change of sign in the chemical potential with a decrease in the carrier concentration corresponds to a transition from the BCS to the Schaffroth scenario. Condensation of local pairs occurs at concentrations of carriers for which $\mu \leq 0$.

This review article is devoted to investigations of superconducting ordering in systems with two characteristic features - a small concentration of charge carriers and overlapping energy bands on the Fermi surface. As mentioned above, both of these features are characteristic of superconducting compounds.

Our review article is organized as follows. In Sec. 2 we present a theory of the superconductivity of two-band system that is valid for any carrier density and takes into account all possible pairing of electrons due to intraband and interband interactions on the basis of the idea of Cooper pairing. The critical temperature $T_c$, the chemical potential $\mu$, the heat capacity ($C_S - C_N$) at the point $T = T_c$ as function of carrier density are illustrated. Section 3 is devoted to a self-consistent discussion in the mean field approximation of a system of equations in order parameters $\Delta_n$ and $\mu$ at $T = 0$ and to revealing the influence of energy band overlapping on these quantities and the carrier concentration at which the system experiences a transition from Cooper pairing ($\mu > 0$) to the Schaffroth ($\mu < 0$) scenario. The equation for the binding energy $\varepsilon_b$ of a two-particle state is also obtained and the relationship between $\varepsilon_b$ and $\mu$ is established. In Sec. 4 the path integral method as applied to the two-band model is developed and, on this basis, the procedure for a transition from the Fermi to Bose elementary excitations is given. The condensation temperature of the Bose system $T_k$ is also determined. In the last section, the results of the investigation are summarized.

2. THERMODYNAMICAL PROPERTIES ON BASIS OF THE COOPER PAIRS OF THE CHARGE CARRIERS.

2. 1. System Hamiltonian and basic equations.
The considered two-band system is described by the Hamiltonian [49]

\[
H = \sum_{n\vec{k}\sigma} \left[ \varepsilon_n(\vec{k}) - \mu \right] a_{n\vec{k}\sigma}^\dagger a_{n\vec{k}\sigma} - \frac{1}{V} \sum_{m_1...m_4,\vec{k}\vec{k}'} V_{m_1m_2}^{m_3m_4} (\vec{k} - \vec{k}'\); \; \vec{k}k') ^{m_1} \hat{a}_{m_1\vec{k_1}^\uparrow} a_{m_2\vec{k_2}^\downarrow} a_{m_3\vec{k_3}^\downarrow} a_{m_4\vec{k_4}^\uparrow},
\]

(1)

were \( a_{n\vec{k}\sigma}^\dagger \) and \( a_{n\vec{k}\sigma} \) are creation and annihilation operators for a electron band \( n \) with spin \( \sigma \) and quasiwave vector \( \vec{k} \), \( \mu \) is the chemical potential, and \( V_{m_1m_2}^{m_3m_4} \) are the intra- and interband interaction constants. Expression (1) is a generalization of the BCS - Bogolyubov model Hamiltonian to include the two-band case. Account is taken here of all possible methods of electron pairing within each band and of electron pairing from different bands. If \( m_1 = m_2 \) and \( m_3 = m_4 \), the Hamiltonian (1) is equal to that of the Moskalenko model [2], which considers only intraband pairing and transitions of Cooper pair as a whole from one band to another are considered; this model is widely used to describe the properties of high-temperature superconductors [8] - [27] and magnesium diboride [30] - [32]; [35] - [37]. Examination of the more general Hamiltonian (1) uncovers additional possible onsets of superconductivity (on account of single-particle hybridization and of all interband-interaction constants) and makes possible a description of the properties of a system with a low density of states (\( \mu \sim T_c \)). Applying the Green’s-function method [50] to the Hamiltonian (1) we obtain the set of equations for the order parameters \( \Delta_{np} \) [49].

\[
\Delta_{np} = \frac{1}{4V} \sum_{\vec{k}l,r\neq l} V_{np}^{ll} \Delta_{ll} \left[ \left( \frac{\xi_l^2 - \xi_r^2 + 2\Delta_{12}\Delta_{21}(1 + 1/z(12))}{d} + 1 \right) \right]
\]

\[
\times \frac{\beta E_1}{2E_1} \left( \frac{\xi_l^2 - \xi_r^2 + 2\Delta_{12}\Delta_{21}(1 + 1/z(12))}{d} - 1 \right) \frac{\beta E_2}{2E_2} + \right]
\]

\[
\frac{1}{4V} \sum_{\vec{k}} \left( V_{np}^{12} \Delta_{12} \frac{z(13)}{z(14)} + V_{np}^{21} \Delta_{21} \frac{z(14)}{z(13)} \right) \Gamma,
\]

(2)

where

\[
E_{1,2} = \sqrt{\frac{a \pm d}{2}}, \; a = \xi_1^2 + \xi_2^2 + 2\Delta_{12}\Delta_{21},
\]

(3)

\[
d^2 = (\xi_1^2 - \xi_2^2) + 4\Delta_{12}\Delta_{21} \left[ (\bar{\xi}_1 - \bar{\xi}_2)^2 + (\Delta_{11} + \Delta_{22})^2 \right] ,
\]

8
\[
\Gamma = \left[ \frac{(\varepsilon_1 - \varepsilon_2)^2 + (\Delta_{11} + \Delta_{22})^2}{d} + 1 \right] \frac{\sinh \frac{\beta E_1}{2}}{E_1} - \\
- \left[ \frac{(\varepsilon_1 - \varepsilon_2)^2 + (\Delta_{11} + \Delta_{22})^2}{d} - 1 \right] \frac{\sinh \frac{\beta E_2}{2}}{E_2},
\]

(4)

\[
\xi_n^2 = \xi_n^2 \vec{n} = \varepsilon_n + \Delta_{nn}, \quad \bar{\varepsilon}_n = \varepsilon_n - \mu_n, \quad (n; p; l; r = 1, 2),
\]

\[
z(12) = \Delta_{11}/\Delta_{22}, \quad z(13) = \Delta_{11}/\Delta_{12}, \quad z_{14} = \Delta_{11}/\Delta_{21}.
\]

(5)

The system (2) determines the order parameters \( \Delta_{11} \) and \( \Delta_{22} \) of an ordinary two-band superconductor (and can be simplified in this case by putting \( \Delta_{12} = \Delta_{21} = 0 \)) as well as of a superconductor with low carrier density \( \mu \sim T_c \). We supplement the system (2) with the expression [49].

\[
N_0 = \sum_{knm, n \neq m} \left[ 1 - \left( \frac{\varepsilon_n + \varepsilon_m}{2} + \frac{\varepsilon_n - \varepsilon_m}{2} \right) \frac{\sinh \frac{\beta E_n}{2}}{E_n} \right].
\]

(6)

The self consistent set of Eqs. (2) and (6) determines the order parameters \( \Delta_{np} \) and the chemical potential \( \mu \) for a specified temperature \( T \) and a carrier density \( N_0 \). A characteristic feature of the ground state in a system with low carrier density is a substantial change of the position of the Fermi level following the onset of the superconducting gap. The order parameters \( \Delta_{np} \) become of the same order as the chemical potential \( \mu (\mu \sim \Delta_{np}) \). This leads to an anomalous behavior of the chemical potential as a function of temperature. In particular, in the single-band BCS model [40] and in the Hubbard model [51] the chemical potential has in rarefied systems a kink at the point \( T = T_c \). As first noted by Van der Marel [40], this kink is observable in experiment and consequently its observation can help explain the superconductivity mechanism. In Ref. [40] has demonstrated the possibility of a kink at \( \mu \leq 2 \text{meV} \), which is the lower limit of present-day accuracy [52]. Since the overlap of the energy bands plays a major role in the explanation of the properties of high-temperature superconductors, it is of interest to investigate the anomalous behavior of the chemical potential in a two-band model with low carrier density, and the possible onset of a kink at values of \( \mu \) more conducive to experimental observation. We represent the order parameter \( \Delta_{nm} \) near the superconducting transition temperature \( T \sim T_c \) in the form
\[ \Delta_{nm} = c_{nm}(\beta - \beta_c)^{1/2} + c_{nm}^{1}(\beta - \beta_c)^{3/2} + \ldots . \]  

(7)

The expression for the chemical potential \( \mu \) near \( T_c \) is now

\[ \mu(T) = \mu_0(T) - R_0\Delta_{11}^2, \]

(8)

were \( \mu_0(T) \) is the chemical potential of the normal phase, and \( R_0 \) is determined from the constancy of the carrier density in the superconducting and normal phases. Substituting (7) in (8) we readily obtain a jumplike change of \( d\mu/dT \). Contributions to this change are made by all the order parameters \( \Delta_{nm} \), so that the results can differ from those for the case of one band [40].

2. 2. Self-consistent system of equations for \( T = T_c \)

To investigate the properties of a two-band system near the superconducting - transition temperature we expand in Eqs.(2) and (6) in terms of the small parameters \( \Delta_{nm} \Delta_{nn} \) with account taken of expansions (7) and (8).

We introduce the dispersion law of the \( n \)th band

\[ \varepsilon_n = -\zeta_n - \frac{k_x^2 + k_y^2}{2m_n}, \]

(9)

and change in these equations from summation over \( \vec{k} \) to integration over the energy in accordance with the dispersion law (9)\( (\varepsilon_{n0} = \varepsilon_n - \mu_{0n}, \mu_{0n} = \mu_0 - A_n) \):

\[ \frac{1}{V} \sum_k \Phi(\varepsilon_n - \mu_{0n}) = 2N_n \int_{-D_{cn}}^{D_{cn}} dx \Phi(-x), \]

(10)

where \( N_n = m_n k_{zn}^0 / (2\pi)^2 \) is the density of the electron states in the \( n \)th band, \( A_n \) is a quantity that renormalizes the chemical potential in the self-consistent-field approximation [49]. The integration limits are chosen to be able to consider simultaneously two possible superconductivity mechanisms: the values of \( D_n = \eta_{0n} - \zeta_n \leq \omega_{D_n} \) and \( D_{cn} = \omega_{D_n} \) (\( \omega_{D_n} \) is the phonon cutoff frequency in the \( n \)th band) correspond to the phonon mechanism of superconductivity, and the quantities \( D_n = \eta_{0n} - \zeta_n \) and \( D_{cn} = \zeta_{cn} - \eta_{0n} \) (\( \zeta_{cn} \) is a cutoff energy of the order of that of the electron; \( \eta_{0n} = -\mu_{0n} \)) corresponds to the hole mechanism.
Integrating next over the energy and equating the coefficients of equal powers in the difference \((\beta - \beta_c)\), we obtain for \(c_{nm}\) and \(c_{nm}^{(1)}\) the set of equations [49]. In the "pseudoband" representation these equations take formally the form of the four - band model [2], [23].

\[
c_{(n)} = - \sum_{(m)=1}^{4} N_{(m)} J_{m} U_{(n)(m)} c_{(m)}, \quad (11)
\]

\[
c_{(n)}^{1} = - \sum_{(m)=1}^{4} N_{m} J_{m} U_{(n)(m)} c_{(m)}^{(1)} - \sum_{(m)=1}^{4} \Phi_{(m)} U_{(n)(m)}, \quad (12)
\]

where

\[
\Phi_{(m)} = \left(-\frac{\Theta_{(m)}}{\beta} + (\beta c_{(m)})^2 F_{m} \right) N_{(m)}. \quad (13)
\]

At \(m = 1, 2\) we have

\[
J_{m} = J_{0}(-\beta D_{cm}) - J_{0}(\beta D_{m}), \quad J_{0}(x) = \int_{0}^{x} \frac{th y/2}{y} dy
\]

\[
\Theta_{m} = th \frac{\beta D_{cm}}{2} - th \frac{\beta D_{m}}{2}, \quad n = 1 - 4. \quad (14)
\]

The functions \(N_{(m)} F_{(m)}\) at \(m = 1 - 4\) and \(N_{(m)} J_{m}, \quad N_{(m)} \Theta_{m}\) at \(m = 3, 4\) are complicated functions, contained the summations on zon indexis [49]. We introduce quasiband indices in accord with the rule

\[
11 \rightarrow (1), \quad 22 \rightarrow (2), \quad 12 \rightarrow (3), \quad 21 \rightarrow (4), \quad (15)
\]

and also the symbols:

\[
V^{pr}_{nm} \rightarrow V_{nm \rightarrow (n')(m')}; \quad n', m' = 1 - 4.
\]

We omit hereafter the parentheses of the pseudoband subscripts. It is convenient to rewrite Eqs.(11) and (12) in the matrix form

\[
D_{0} c = 0 \quad (16)
\]

\[
D_{0} c^{(1)} = -\Phi, \quad (17)
\]
were $c$, $c^{(1)}$, and $\Phi$ are single-column matrices in the indices 1 to 4, and

$$D_0 = \begin{pmatrix}
1 + N_1 J_1 U_{11} & N_2 J_2 U_{12} & N_3 J_3 U_{13} & N_4 J_4 U_{14} \\
N_1 J_1 U_{12} & 1 + N_2 J_2 U_{22} & N_3 J_3 U_{23} & N_4 J_4 U_{24} \\
N_1 J_1 U_{31} & N_2 J_2 U_{32} & 1 + N_3 J_3 U_{33} & N_4 J_4 U_{34} \\
N_1 J_1 U_{41} & N_2 J_2 U_{42} & N_3 J_3 U_{43} & 1 + N_4 J_4 U_{44}
\end{pmatrix}. \quad (18)$$

From the condition that the system (16) have a solution, we obtain an equation for the critical temperature $T_c$:

$$||D_0|| = 0 \quad (19)$$

where $||...||$ designates the determinant of a matrix. It follows from the system (17) that

$$c_1^{(1)} = ||D_1||/||D_0||. \quad (20)$$

Since $||D_0|| = 0$, we get

$$||D_1|| = 0, \quad (21)$$

where $D_1$ is a $4 \times 4$ matrix that differs from $D_0$ in that the first column is replaced by the elements of the matrix $\Phi$. We have on the basis of (20) the expression for $c_1^2$

$$c_1^2 = \frac{1}{\beta c_1^2} \sum_{n=1}^{N} N_n \theta_n f_n/z_{1n}^2 = \frac{c_1^2}{\beta c_1^2}. \quad (22)$$

Assuming the particle number $N_0$ to be fixed and using the expansion (7) for the chemical potential near $T_c$ we obtain

$$\mu(T) = \mu_0(T) - \frac{R_0}{\beta} c_1^2 (T_c - T), \quad (23)$$

and $\mu_0$ is determined from the equation

$$N_0 = 2 \sum_l \left[ D_{cl} + D_l - \frac{2}{\beta} \ln \frac{ch \theta_{D_l}}{ch \theta_{D^{D_l}}} \right]. \quad (24)$$

The determinations of the functions $f_n/z_{1n}$ and $R_0$ see in [49].
It is possible to calculate the dependence of the superconducting temperature $T_c$ on the carrier density $N_0$ for all values of the interaction constants $\lambda_{nm} = N_m U_{nm} \ (n, m = 1 - 4)$, and also the temperature dependence of the chemical potential $\mu(T)$ on basis of (18) - (24).

Figure 1 shows the dependence of the superconducting temperature $T_c$ on the carrier density $N_0/2N_1$ for different degrees of hybridization in the case of electron mechanism of superconductivity: a) weak: $\lambda_{11} = \lambda_{22} = 0.2, \lambda_{12} = \lambda_{21} = \lambda_{33} = \lambda_{44} = 0.01, \lambda_{34} = \lambda_{43} = 0.105$, the remaining ones: $\lambda_{nm} = 0.001 \ (n, m = 1 - 4)$; b) strong $\lambda_{11} = \lambda_{22} = 0.2, \lambda_{12} = \lambda_{21} = \lambda_{33} = \lambda_{44} = 0.1, \lambda_{34} = \lambda_{43} = 0.15$, the remaining ones $\lambda_{nm} = 0.01 \ (n, m = 1 - 4)$.

Calculations are performed for the following values of the parameter

$$\zeta_1 = 0 \ eV, \ \zeta_2 = 0.03 \ eV, \ \zeta_{c1} = 0.05 \ eV, \ \zeta_{c2} = 0.08 \ eV.$$

It follows from Fig.1 that one can obtain a bell-shaped dependence of $T_c$ on $N_0$ (curves 1-3) as well as the weak dependence $T_c(N_0)$ curves(1 - 3). Inclusion of strong hybridization raise $T_c$. The character of the dependence of the transition temperature $T_c$ on the carrier density $N_0$ is strongly influenced by the relation between the electronic - state densities of the different bands. Lowering the ratio $N_2/N_1$ slows down the growth of $T_c$ with increase of $N_0$ (for case b) and accelerates the decrease for both cases (a, b)

In the case of weak hybridization (curves 1 – 3) the $T_c(N_0)$ plot acquires two maxima. The degree to which then become pronounced, given the parameters $\lambda_{nm}$, is determined by the electron state-density ratio $N_2/N_1$. The presence of weakly pronounced maxima (curve 1) corresponds to the case $N_1 = N_2$, and more strongly pronounced maxima (curves 2 and 3) appear at $N_1 \neq N_2$ and are determined by the anisotropy of the system (by the difference between the bands). Each of these maxima is connected with the occupation of the corresponding band. In the absence of interband interaction (energy - band overlap) the plot would consist of two nonoverlapping curves. The onset of interband interactions produces simultaneous superconductivity in both bands, with a single superconducting temperature determined by all the interaction constants $\lambda_{nm}$. With increase of the interband-interaction constants ($\lambda_{34}, \lambda_{43} \sim \lambda_{11}, \lambda_{22}$) the contribution due to overlap of the two bands begins to
predominate over the individual contribution of each band, so that the $T_c(N_0)$ plot is a single bell-shaped curve (4,5).

Figure 2 shows the temperature dependence of $\eta = -\mu$ for a non-phonon superconductivity mechanism at various carrier densities $N_0$ and with weak hybridization. We see that this plot has at $T = T_c$ (curves 1 - 3) a kink that becomes less peaked with increase of the carrier density and vanishes at $\eta \sim 8$ meV (curve 4). The behavior of $\eta(T)$ under strong hybridization is similar. The anomaly of the temperature dependence of the chemical potential $\eta = -\mu$ at the point $T = T_c$ is due to the appearance, on the Fermi surface, of a superconducting gap that does not differ excessively from the chemical potential. This gap influences substantially the chemical potential $\mu$ at $T < T_c$, since the values of $\eta$ and $\Delta_{nm}$ are self-consistently determined from the set of equations (2) and (6).

Allowance for the energy-band overlap leads to a kink on the temperature dependence of the chemical potential at $T = T_c$ for the sufficiently high values $\eta \leq 8$ meV ($\eta \leq 2$ meV in the single band case [40], which undoubtedly facilitates experimental verification of this effect).

2. 3. Critical temperature $T_c$ and ratio $2\Delta_n/T_c$ in the limiting case $\Delta_{12} = \Delta_{21} = 0$.

In the limiting case $\Delta_{12} = \Delta_{21}$ ($N_3 = N_4 = 0$) the problem is simplify essentialy. Such model corresponds to [2], [23] the equation for $T_c$ has the following form [17]:

$$1 + \lambda_{11}J_1 + \lambda_{22}J_2 + aJ_1J_2 = 0,$$

(25)

$$\lambda_{nm} = V_{nm}N_m, \quad a = \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}.$$  

(26)

Besides, the $\eta$ in the case of nonphonon mechanism of superconductivity is defined from

$$N_0 = 2 \sum_n N_n \left[ \zeta_{cn} - \zeta_n - |\zeta_{cn} - \eta| + |\zeta_n - \eta| - 2T_c \ln \frac{1 + e^{\beta_c|\zeta_{cn} - \eta|}}{1 + e^{\beta_c|\zeta_n - \eta|}} \right].$$  

(27)

From the relation (22) we have
where you $J_m$ and $\Theta_m$ are determined by (14), and

$$F_m = -\frac{1}{4} \int_{-\beta_cD_m}^{\beta_cD_m} dx \frac{shx - x}{x^3ch^2x/2}, z = \Delta_1/\Delta_2. \tag{29}$$

For the sake of definiteness we choose $\zeta_1 = 0 < \zeta_2 < \zeta_{c1} < \zeta_{c2}$ and consider the equation for the superconducting transition temperature (25). On the basis of this equation one can obtain the analytic expressions for the transition temperature $T_c$ if the following conditions are satisfied: $|\eta - \zeta_n|/T_c \gg 1$ and $|\zeta_{cn} - \eta|/T_c \gg 1$. So, for definite values of $\eta$ we obtain the following expressions:

1. For $0 < \eta < \zeta_2$:

$$T_c = \frac{2\gamma}{\pi} \sqrt{\eta(\zeta_c1 - \eta)} \exp \left[ -\frac{1}{4} \int_{-\beta_cD_m}^{\beta_cD_m} dx \frac{shx - x}{x^3ch^2x/2}, z = \Delta_1/\Delta_2. \right] \tag{30}$$

$$N_0 = 4N_1\eta.$$  

2. For $\eta = \zeta_2$:

$$T_c = \frac{2\gamma}{\pi} \eta^{1/4}(\zeta_c1 - \eta)^{1/4} \sqrt{\zeta_c2 - \eta} \exp \left[ -\frac{2\lambda_{11} + \lambda_{22}}{4a} \pm \frac{1}{4a} \left( a \ln \eta(\zeta_c1 - \eta) \right)^2 + \frac{\lambda_{11} - \lambda_{22}}{2} + 8\lambda_{12}\lambda_{21} \right], \tag{31}$$

$$N_0 = 4N_1\eta.$$  

3. For $\zeta_2 < \eta < \zeta_{c1}$:

$$T_c = \frac{2\gamma}{\pi} \sqrt{\eta(\zeta_c1 - \eta)(\zeta_c2 - \eta)(\eta - \zeta_2)}^{1/4} \exp \left[ -\frac{\lambda_{11} + \lambda_{22}}{4a} \pm \frac{1}{4a} \left( a \ln \eta(\zeta_c1 - \eta) \right) \frac{(\zeta_c2 - \eta)(\eta - \zeta_2)}{(\zeta_c2 - \eta)(\eta - \zeta_2)} + (\lambda_{11} - \lambda_{22})^2 + 4\lambda_{12}\lambda_{21} \right], \tag{32}$$

$$N_0 = 4N_1\eta_1 + 4N_2(\eta - \zeta_2).$$

4. For $\eta = \zeta_{c1}$

$$T_c = \frac{2\gamma}{\pi} \sqrt{\eta(\zeta_c2 - \eta)}^{1/4}(\eta - \zeta_2)^{1/4}.$$
\[
\exp \left[ \frac{\lambda_{11} + 2\lambda_{22}}{4a} \pm \frac{1}{4a} \left( a \ln \frac{\eta^2}{(\zeta_2 - \eta)(\eta - \zeta_2)} + \lambda_{11} - 2\lambda_{22} \right) + 8\lambda_{12}\lambda_{21} \right],
\]

(33)

\[N_0 = 2N_1(\eta + \zeta_1) + 4N_2(\eta - \zeta_2).\]

(5) For \(\zeta_1 < \eta < \zeta_2\)

\[T_c = \frac{2\gamma}{\pi} \sqrt{(\zeta_2 - \eta)(\eta - \zeta_2)} \exp \left[ -\frac{1}{2} \frac{1 + \lambda_{11} \ln((\eta - \zeta_1)/\eta)}{\lambda_{22} + a \ln((\eta - \zeta_1)/\eta)} \right],
\]

(34)

\[N_0 = 4N_1 \zeta_1 + 4N_2(\eta - \zeta_2).\]

The expressions (31) \(\sim (33)\) contain two solutions which correspond to the sign “±” in the brackets. A negative value of the quantity in the exponent is a necessary condition for the solution’s selection. In the case where both solutions satisfy the above necessary condition, the solution giving the greater \(T_c\) must be selected. Note that we will obtain the results of Ref. [2] provided \(V_{11} = V_{22} = 0\) and \(\zeta_1 = \zeta_2\).

The analysis of the above \(T_c\) expressions permits us to make the conclusion that high \(T_c\) values can be achieved both for \(\lambda_{nm} > 0\) (carrier attraction) and \(\lambda_{nm} < 0\) (repulsion). In the latter case the condition \(a = \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} < 0\), imposing restrictions on the constants \(\lambda_{nm}\), must be satisfied. The dependence of \(T_c\) on \(\eta\) is demonstrated in fig. 3 at definite values of the parameters \(\lambda_{nm}\). Here and hereafter in computations we choose the following values of the parameters: \(\zeta_1 = 0\ eV, \ zeta_2 = 0.11\ eV, \ zeta_{c_1} = 0.2\ eV\) and \(\zeta_{c_2} = 0.3\ eV\). Curves 1 and 2 correspond to the case \(\lambda_{nm} > 0\) and to the case \(\lambda_{nm} < 0\), respectively. As follows from this figure, high values of \(T_c\) can be achieved as the chemical potential changes. The \(T_c\) dependence on \(\tilde{N}_0 = N_0/4N_1\) can be presented easily. In this case the ratio \(N_2/N_1\) must be given. Then the \(T_c\) dependence will be defined by this ratio. The \(\eta\) dependence on \(\tilde{N}_0\) at different values of \(N_2/N_1\) is shown in fig.4. As follows from this figure, the rate of growth of the quantity \(\eta\) is defined by the ratio \(N_2/N_1\) as \(N_0\) increases. The critical temperature \(T_c\) as well as the order paramiters \(\Delta_n\) as function of \(\eta\) in this limit case is studied detaly in [17].

This investigations permit us to define the ratios \(2\Delta_1/T_c\) and \(2\Delta_2/T_c\) as function of \(\eta\) (see fig.5). The curves 1 and 2 correspond to the ratios \(2|\Delta_1(0)|/T_c\) and \(2|\Delta_2(0)|/T_c\) respectively for \(\lambda_{nm} > 0\), and the curves 1′, 2′ correspond to the ones for \(\lambda_{nm} < 0\). As is seen from this
figure the behavior of the quantities $2|\Delta_1(0)|/T_c$ and $2|\Delta_2(0)|/T_c$ as functions of $\eta$ depends essentially on the values of the parameters $\lambda_{nm}$. When $\lambda_{nm} > 0$ we have a step decrease of $2|\Delta_1(0)|/T_c$ and a smooth growth of $2|\Delta_2(0)|/T_c$ as $\eta$ increases. When $\lambda_{nm} < 0$ the ratio $2|\Delta_1(0)|/T_c$ essentially increases and can achieve values near 7.5 and the ratio $2|\Delta_2(0)|/T_c$ slowly decreases, achieving the value 3.5 as $\eta$ increases. So the values of $2|\Delta_1(0)|/T_c$ and $2|\Delta_2(0)|/T_c$ depend on $\eta$ and can be essentially different from the BCS theory ones where $2|\Delta(0)|/T_c = 3.5$ and from the two-band theory with phonon superconductivity mechanism [2], [23], where these ratios are independent of the chemical potential (carrier concentration). Note that for $T_c$, as for $|\Delta_n(0)|$, the ratio $2|\Delta_n(0)|/T_c$ can be shown in dependence on the carrier concentration, by using relations between $\eta$ and $\bar{N}_0$ (see fig.4).

2.4. Heat-capacity jump at the point $T = T_c$.

In the section 2.2 we changed over to the pseudo-band representation, which allowed us to write down Eqs. (11) and (12), as it were, for a four-band model.

It can also be shown that the difference between the free energies in the superconducting and normal phases in the pseudoband representation generalizes the corresponding expression of the Moskalenkov two-band model [2], [23]. We obtain

$$\frac{\Psi_S - \Psi_N}{V} = \sum_{nmp} \int_0^{\Delta_p} \Delta_n \Delta_m (\delta U^{-1})_{nm} \delta \Delta_p, \quad (35)$$

where $n, m$ and $p$ are the pseudoband numbers ($n, m, p = 1 - 4$), and $U^{-1}$ is the inverse of the interaction matrix $U$ in (11). We expand, in the pseudoband formalism, the set of Eqs. (2) for the order parameters $\Delta_n$ in powers of the small quantity $(\beta \Delta_n)^2$ in the vicinity of the critical temperature $T_c$:

$$\Delta_m = - \sum_n N_n U_{nm} (J_n + (\beta \Delta_n)^2 \mathcal{F}_n + ...), \quad (36)$$

where $J_n$ and $\mathcal{F}_n$ are defined in [49]. Using the calculation method of Refs. [23] and [17] for Eqs.(35) and (36), we obtain for the heat-capacity jump at the point $T = T_c$

$$\frac{C_S - C_N}{V} = -T \frac{\partial^2}{\partial T^2} \frac{\Psi_S - \Psi_N}{V} = \beta_c^5 \sum_{n=1}^{4} N_n \mathcal{F}_n c_n^4 =$$
\[ T_c \left[ \frac{\Sigma_{n=1}^{4} N_n \theta_n f_n / z_{1n}^2}{\Sigma_{n=1}^{4} N_n \bar{F}_n f_n / z_{1n}^4} \right]^2 \sum_{n=1}^{4} \frac{N_n \bar{F}_n}{z_{1n}^4} \] \quad (37)

The equation for the electronic heat capacity in the normal phase is [17]

\[ C_N = 4T \sum_{n=1}^{2} N_n \phi_n(\eta), \] \quad (38)

where

\[ \phi_n(\eta) = \int_{-\beta(\eta-\zeta_n)}^{\infty} \frac{x^2 dx}{(1 + e^x)(1 + e^{-x})}. \] \quad (39)

In accordance with Eqs. (37) and (38) we have for the relative heat - capacity jump

\[ \frac{C_S - C_N}{C_N} = \left[ \frac{\Sigma_{n=1}^{4} N_n \theta_n f_n / z_{1n}^2}{\Sigma_{n=1}^{4} N_n \bar{F}_n f_n / z_{1n}^4} \right]^2 \frac{\Sigma_{n=1}^{4} N_n \bar{F}_n}{4 \Sigma_{n=1}^{2} N_n \phi_n(\eta)}. \] \quad (40)

Figures 6 shows the dependences of the absolute and relative jump of the electron heat capacity at the point \( T = T_c \) on the carrier density \( N_0 \), obtained by numerical methods using the equations given above. The numbers of the curves in these figures are the same as in Fig. 1.

As seen from Fig. 6a, the dependence of \( (C_S - C_N)_{T=T_c} \) on \( N_0 \) has a maximum. At the same time, this dependence does not duplicate the behavior of \( T_c(N_0) \). This circumstance indicates that a substantial contribution to the dependence of \( (C_S - C_N)_{T=T_c} \) on \( N_0 \) is made not only by \( T_c \) but also by the complicated function in the right-hand side of (37). Analysis of the curves of Fig.6 shows that the character of the plot of \( (C_S - C_N)_{T=T_c} \) versus \( N_0 \) is determined by the type of hybridization (strong or weak) and by the ratio \( N_2/N_1 \) of the electron - state densities. It is possible for \( T_c \) and \( (C_S - C_N)_{T=T_c} \) to have maxima at one and the same value of \( N_0 \) (curves 4 and 5). This situation is observed in experiment, for example in \( La_{2-x}Sr_xCu_2O_4 \) [53].

The possibility of obtaining small \( (C_S-C_N)/C_N < 1.43 \), as well as large \( (C_S-C_N)/C_N > 1.43 \) values of the relative jump of the electron specific heat is demonstrated by Fig. 6b. This picture is observed in high - temperature ceramics [54] - [57]. The complicated dependence of the relative electron-heat-capacity jump, shown in Fig. 6a, is determined by the competition
between the behavior of the difference \( C_S - C_N \) shown in Fig.6a as a function of \( N_0 \) and the quantity \( C_N \) which increases as \( N_0 \) increases.

We have considered in the present study quasi-two-dimensional systems with a simple dispersion law (9). This approach is dictated, in particular, by the lower dimensionality of a number of high-temperature ceramics. Since, however, the electron-state densities \( N_n(\varepsilon), n = 1, 2 \) have no singularities for the dispersion law (9), we obtain the very same equations also for a three-dimensional system. Only the values of \( N_n \) will differ. Just as in the case of single-band superconductors, [40] in our case the ratio \( C_S - C_N > 1.43 \) is governed to a considerable degree by the small \( C_N \) in the considered range of \( N_0 \) compared with the case of ordinary metals (or by the faster increase of \( C_S - C_n \) with increase of \( N_0 \) compared with the increase of \( C_S - C_N \) with increase of \( N_0 \) compared with the increase of \( C_N \)).

In the case \( N_3 = N_4 = 0 \) (this limit corresponds to usual two-band model [2]), we presents the specific heat jump in the form Ref. (17):

\[
\frac{C_S - C_N}{V} = T_c \left[ \frac{N_1 \theta_1(\eta) + N_2 \theta_2(\eta)}{N_1 F_1(\eta) + N_2 F_2(\eta)} \right]^{2},
\]

\[
\frac{C_S - C_N}{C_N} = T_c \left[ \frac{N_1 \theta_1(\eta) + N_2 \theta_2(\eta)}{N_1 F_1(\eta) + N_2 F_2(\eta)} \right]^{2} \left[ \frac{N_1 \varphi_1(\eta) + N_2 \varphi_2(\eta)}{N_1 \phi_1(\eta) + N_2 \phi_2(\eta)} \right].
\]

The presence of the functions \( \theta_n(\eta) \) and \( F_n(\eta) \) distinguishes this expression from the two-band theory with phonon superconductivity. In the above approximation \( \eta >>> T_c \), values of these functions at the different values of \( \eta \) are given in table 1 Ref. [17]. So the specific heat jump depends on \( \eta \) (on \( N_0 \) through the \( T_c \) dependence on this quantity and by the supplementary function in the right hand part of eq. (41). By using table 1 [17] and the above values of \( T_c \) the specific heat jump dependence \( (C_S - C_N) \) on \( N_0 \) can be easily presented on the basis of eq. (41). This dependence is shown in fig.7 at different values of the ratio \( N_2/N_1 \). The curves with non-primed numbers correspond to \( \lambda_{nm} > 0 \) and the ones with primed numbers correspond to \( \lambda_{nm} < 0 \). We obtain the result that the quantity \( (C_S-C_N)/VN_1 \) increases as the carrier concentration increases. The maximum of this quantity
is achieved at the values of \( N_0 \) which correspond to the superconducting transition temperature. Then this quantity decreases as \( N_0 \) increases for all values of \( N_0/N_1 \). It is interesting to note also that the same dependence of the absolute specific heat jump at the point \( T = T_c \) on \( S_r \) concentration has been observed in \( La_{2-x}Sr_xCu_2O_4 \) \[53\]. By using table 1 of Ref. [17] we present expressions for the relative jump of electron specific heat in some points.

1) For \( 0 < \eta < \zeta_2 \):
\[
\frac{C_S - C_N}{C_N} = \frac{12}{7\zeta(3)} = 1.43.
\]

(43)

2) For \( \eta = \zeta_2 \):
\[
\frac{C_S - C_N}{C_N} = \frac{12}{7\zeta(3)} \frac{(N_1 + N_2/z^2)^2}{(N_1 + N_2/z^2)(N_1 + N_2/2)}.
\]

(44)

3) For \( \zeta_2 < \eta < \zeta_{c1} \):
\[
\frac{C_S - C_N}{C_N} = \frac{12}{7\zeta(3)} \frac{(N_1 + N_2/z^2)^2}{(N_1 + N_2/z^4)(N_1/2 + N_2)}.
\]

(45)

4) For \( \eta = \zeta_{c1} \):
\[
\frac{C_S - C_N}{C_N} = \frac{12}{7\zeta(3)} \frac{(N_1 + N_2/z^2)^2}{(N_1/2 + N_2/z^4)(N_1/2 + N_2)}.
\]

(46)

5) For \( \zeta_{c1} < \eta < \zeta_{c2} \):
\[
\frac{C_S - C_N}{C_N} = \frac{12}{7\zeta(3)}.
\]

(47)

The formulae (43)-(46) implicitly demonstrate the consequence of inclusion and turning-off of the overlapping bands as the parameter \( \eta \) increases. The expression (45) corresponds to the existence of both bands and to the maximum value of the superconducting transition temperature. This expression coincides in form with the one obtained in the model with phonon superconductivity [2], [23].

By using the above formulae the relative jump of the electron specific heat one has \((C_S - C_N)/C_N < 1.43\) at any values of the parameter \( \eta \) (or \( N_0 \)) and the former can achieve sufficiently small values. Note that the small values of the quantity \((C_S - C_N)/C_N = 0.43 - 1.14\) have been considered in the thallium ceramics [54].
The specific heat jump dependence on $N_0$ corresponding to $\lambda_{nm} > 0$ and to $\lambda_{nm} < 0$ is presented in fig. 8(a) and (b), respectively. In both cases the ratio $\frac{(C_S - C_N)}{C_N}$ is equal to 1.43 in the range where only one energy band exists ($\eta < \zeta_2$ and $\eta > \zeta_{c1}$). This value corresponds to the BCS theory. This ratio decreases after inclusion of both bands. In the case $\lambda_{nm} > 0$ we obtain a smooth dependence with a slightly spread minimum. Nevertheless, in the case $\lambda_{nm} < 0$ we observe a quick decrease of this ratio as $N_0$ increases followed by a slower rate of growth in the range with overlapping energy bands.

The mentioned above formulas for the jump of heat capacity (42), and (43)-(47) as well, correspond to the isotropic two-band superconductor at $\Delta_{12} = \Delta_{21} = 0$. In particular, for $\Theta_n = F_n = 1$ formula (42) gives the case of electron-phonon mechanism of superconductivity [2], [23] in two-band isotropic system. In case of anisotropic electron-phonon interaction the result is presented in Ref. [35]- [37] and have succeeded to describe the behavior of the heat capacity at $T = T_c$ in $MgB_2$ compound.

Note that our investigations were made in the meanfield approximation, and it is in this approximation that the proposed superconductivity theory, with two overlapping energy bands, describes the properties of the system for an arbitrary ratio of $T_c$ to $N_0$. The meanfield approximation itself, however, may turn out to be insufficient when very low carrier densities are considered. It becomes necessary here to take into account the fluctuations of the order parameters near the superconducting transition temperature. Our numerical calculations were made mainly for values $T_c/\varepsilon_F \sim 10^{-1} - 10^{-2}$ (cf. the data of Figs. 7 and 9), and in this respect we can be assumed to have a physical picture that is qualitatively close to the real one. To be sure, we are still faced here with the question of the nature of the superconductivity, namely, will it be based on the Cooper-pair production mechanism or will it be determined by the Bose condensate. Good results in the ground state are obtained in the BCS theory for $\Delta \ll |\mu|$, but a condensate of noninteractiong bosons is produced in the opposite limit. Analysis of the intermediate region at finite temperature is still an unsolved problem, and our results are an interpolation of the BCS mechanism to this intermediate region.
2.5 Electron-phonon mechanism of superconductivity

In the previous sections we have derived the equations and the analytical expressions in particular cases for the quantities $T_C$, $\mu(T)$, and for the absolute and relative jump of heat capacity at the point $T = T_C$. The equations are valid both for electron and phonon mechanism of superconducting pairing. Formally speaking, this feature appears in a way of cut-off while having performed the integration over energy in the main equations (see section 2.2). The graph plots of the presented here dependences of various quantities (fig. 1-9) correspond the non-phonon superconductivity.

We present also the results for the electron-phonon superconductivity, having considered as earlier the weak and strong hybridization, and chosen the same values of parameters $\lambda_{nm}$ as in section 2.2.

We choose also the following values $\varsigma_1 = 0$ eV, $\varsigma_2 = 0.01$ eV, $\varsigma_{c1} = 0.02$ eV, $\varsigma_{c2} = 0.03$ eV. The results of numerical calculations are presented on the fig.1 - 10. The shape of dependences of the quantity $T_C$ on $N_0$ presented on the fig.1 and 10 is easily observed not to be the same. This difference appears clearer especially in the case of weak hybridization. In the case of strong hybridization there is a bell-shape dependence of the quantity $T_C$ on the density of charge carriers.

The kink of chemical potential $\mu(T)$ at the point $T = T_C$ (fig. 11) is well distinguished and disappears at the values $\mu \sim 6$ meV what is as in the case of electron pairing mechanism (fig. 2) achievable to the experimental observation.

The dependence of the jump of electron heat capacity $(C_S - C_N)|_{T=T_C}$ on the density of charge carriers differs distinctly at the weak hybridization as well (compare fig.6a and 12). In the case of phonon pairing mechanism of superconductivity and weak hybridization the maximum of the quantity $(C_S - C_N) / V N_1$ is shifted towards low densities of charge carriers in comparison with the non-phonon pairing mechanism. The curves (fig.6a and 12) that correspond to the strong hybridization differ as well. The dependence of the relative jump of electron heat capacity on the density of charge carriers in both cases of hybridization...
has complex shape (see fig.6b and 13) and depends essentially upon theory parameters. Regarding the value of $N_0$ the quantity $(C_S - C_N) / C_N$ at $T = T_C$ can be as greater as lower than 1.43 (the value that corresponds to the isotropy BCS model with phonon pairing mechanism of superconductivity). Detailed analysis of these results and their comparison with experimental data might shed light upon the mechanism of superconductivity in two-band system.

As was noticed above the discovery of high-$T_C$ superconductivity in magnesium boride $MgB_2$ $T_C \sim 39K$ has lead to rapid development in experimental and theoretical studies. Confirmation of two-band nature of this system (presence of two energy gaps) [34] and electron-phonon mediated pairing mechanism of superconductivity [29] - [32] is very important discovery. The singularities of two-band system in temperature dependence of heat capacity, penetration depth of magnetic field, upper critical magnetic field and other physical quantities are observed. The background of all these studies is the model of Moscalenco that was suggested by him in 1959 [2]. On the basis of this model Moscalenco and his colleagues have made a great number of studies long before the discovery the compound $MgB_2$. Many modern authors to our pity represent their own results as originally new ones not making any references on the pioneer works of Moscalenco and his colleagues (see [7], [22]).

In order to describe the physical properties of $MgB_2$ it is necessary to determine the parameters of theory on the basis of present experimental data, to take into account anisotropy of matrix element of electron-phonon interaction [35] - [37], important role of Van Hove singularities [55], it is apt to take into consideration strong electron-phonon interaction and other singularities as well as multi-band nature of the system.

Because parameters of theory in the two-band model presented in this article are unknown, our studies performed here permit us to make qualitative conclusions about superconducting properties of $MgB_2$.

1. The quantity $T_C$ as a function of the density of charge carriers has bell-shape dependence. At small hybridization there is wide field of densities of charge carriers $N_0$ at which
$T_C$ depends slightly upon $N_0$. At great hybridization the maximum in the dependence of $T_C$ on $N_0$ is notably distinguished. The height and position of this maximum is determined by relationship $N_1/N_2$ as well as by values of coupling constants $\lambda_{nm}$.

2. The chemical potential at the point $T = T_C$ has kink at low densities of charge carriers. This kink disappears with increasing $N_0$. The overlapping of energy bands favors the experimental confirmation of this kink.

3. The positions of maximums of $T_C$ and difference $(C_S - C_N)$ at $T = T_C$ as a functions of density of charge carriers do not coincide.

4. The relative jump of electron heat capacity $(C_S - C_N)/C_N$ at the point $T = T_C$ depends on the density of charge carriers and can get as great (greater than 1.43) as small (less than 1.43) values. The more detailed study of the two-band model considered in this article with reduced density of charge carriers and account of all possible pairings of electrons at phonon mechanism of superconductivity can be found in the second article of the Ref. [49].

Note that in the most of works about $MgB_2$ theoretical and experimental studies of superconducting properties are made at given value of density of charge carriers. But there are studies (see for example, [56], [57]) where the dependence of the quantity $T_C$ on the density of charge carriers is investigated on the basis of simplified Moscalenco model assuming $V_{11} = V_{22} = 0, V_{12} \neq 0$. In order to achieve high values of $T_C$ it was assumed $V_{12} > 1$.

The change of density of charge carriers can be done by doping as it was suggested in works [58], [59] where compounds $MgB_{2-x}C_x, Mg_{1-y}D_yB_2 (D = Li, Al$ and others) have been studied. Having introduced impurity in such compounds, the interband scattering of electrons on a impurity, that leads to the decrease of the quantity $T_C$ with increase of impurity concentration [60], occurs as well as the change of chemical potential. So, in order to confirm the properties 1 - 4 in the experimental perspective we have to exclude the interband impurity scattering, to neutralize it somehow. It can be done, for example, by introducing the impurity outside the plane that is responsible for superconductivity. Therefore, in this plane the arrangement of electrons is preserved and the scattering on the impurity potential
is absent [60].

2.6 Conclusion

We have developed a superconductivity theory for a system with two overlapping energy bands on the Fermi surface. This theory is valid in the weak-field approximation for any carrier density, including a low one ($\mu \sim T_c$).

The main results are the following.

1. We have introduced a system Hamiltonian (1) which can account for superconducting pairing of electrons both within each band and from different bands. A system of self-consistent equations was derived for the order parameter $\Delta_{nm}$ (2) and for the chemical potential $\mu$ (6).

2. We have used a sub-band representation, in which the basic equations at temperatures close to critical can be expressed as the set (16)-(19) for the four-band model. This set can be used to consider both the phonon and non-phonon superconductivity mechanisms and can be used to describe superconductivity in a system with low carrier density ($\mu \sim T_c$). The low carrier density notwithstanding, this system can lead to rather high $T_c$ in view of the inclusion of more interband interactions than in Ref. [2], which are connected with formation of superconducting pairs of electrons from different bands.

3. Our model offers more possibilities of describing various two-band systems, since a major role is played in the theory by the ratio $N_2/N_1$ of state densities of electrons from different bands, as well as by interaction constants $\lambda_{nm}(n; m = 1 - 4)$. The foregoing is in fact clearly demonstrated in Figs.1 and 6a which show respectively the dependences of the critical temperature $T_c$ and of the electronic heat capacity $(C_S - C_N)_{T=T_c}$ on the carrier density for a nonphonon superconductivity mechanism and strong (curves 4 - 5) hybridization.

4. The plot of the chemical potential of a superconductor with low carrier density versus temperature has at the point $T = T_c$ a sharp kink that becomes less peaked and vanishes when the carrier density is increased. It vanishes at $\mu \approx 2$ meV in the BCS model [40] and at $\mu \approx 8$ meV in the two-band case (Fig. 2). Overlap of the energy band produces
thus favorable conditions for revealing anomalies in the \( \mu(T) \) dependence and hence for elucidating the superconductivity mechanisms.

5. Analytic expressions were obtained for the absolute \((C_S - C_N)\) (37) and relative \((C_S - C_N)/C_N\) (40) jumps of the electron heart capacity at \( T = T_c \). The behavior of \((C_S - C_N)\) as a function of the carrier density \( N_0 \) does not, generally speaking, duplicate the behavior of \( T_c(N_0) \) (see Figs. 1 and 6a) and is determined by the parameters of the theory. Situations are possible in which \( T_c \) and \((C_S - C_N)_{T=T_c}\) have maxima at one and the same density \( N_0 \) (e.g., curves 3 - 5). The character of the dependence of the ratio \((C_S - C_N)/C_N\) at the point \( T = T_c \) on \( N_0 \) is also determined by the parameters of the theory and depend substantially on the carrier density. This quantity can be either large or small: \((C_S - C_N)/C_N > 1.43\) or \((C_S - C_N)/C_N < 1.43\).

6. At large carrier densities \( (\mu \gg T_c) \) we have \( \Delta_{12}, \Delta_{21} \ll \Delta_{11}, \Delta_{22} \), so that simpler equations for \( T_c, \Delta_{nm}, (C_S - C_N)_{T=T_c} \) and \((C_S - C_N)/C_N\) can be obtained by putting \( \Delta_{12} = \Delta_{21} = 0 \) in (2) and \( N_3 = N_4 = 0 \) in (19), (37) and (40). The ensuing results agree with the corresponding equations for phonon [2], [23] and non-phonon [17] superconductivity mechanisms.

We have considered two cases: (a) all constants both for the interband and for the intraband interaction are positive: \( (\lambda_{nm} > 0, \text{ this corresponds to effective attraction between carriers}) \); (b)\( \lambda_{nm} < 0, \text{ (this corresponds to repulsion)} \). We suppose that case (2) corresponds to the strong Coulomb interaction when any attraction between carriers is suppressed. The interband interaction coupling constants which are involved in the \( T_c \) square definition are here the unique reason for generating a superconductivity mechanism, with the intraband interactions preventing one. Nevertheless, high values of \( T_c \) can be achieved provided \( \lambda_{11} \lambda_{22} < \lambda_{12} \lambda_{21} \). In the case of \( \lambda_{nm} > 0 \) the high values of \( T_c \) are easily acquired without imposing any restrictions on the values of the parameters \( \lambda_{nm} \). The results concerning the dependences of \( 2|\Delta_n(0)|/T_c \) on the carrier concentration are of great interest to us. These dependences can be interpreted as the dependences on the oxygen content in the yttrium ceramics or on the doping content in the lanthanum ones. The supplemental
dependence on the carrier concentration results in our model in a difference from that with a two-band phonon superconductivity mechanism [2], [23].

The theory, represented in this work, gives the essential dependence of the relative specific heat jump $(C_s - C_N)/C_N$ on the carrier concentration. In particular, sufficiently small values of this ratio are possible (see fig. 6a and 6b). As has been noted before, small values of this quantity have been observed, for example, in the thallium ceramics [54]. So the specific heat jump dependence on the carrier concentration which has a maximum (see fig. 7) is of great interest to us. The value of $N_0$ which corresponds to the above maximum coincides with the one which defines the maximum of the superconductivity transition temperature. The specific heat dependence on the doping concentration has been observed, for instance, in $La_{2-x}Sr_xCu_2O_4$ [53].

In the case of electron-phonon mechanism of superconductivity, the results obtained in this work can be used to describe the thermodynamic properties of superconducting MgB$_2$ compound in all scope of densities of charge carriers (see section 2.5). In this way we have to determine the parameters of theory for this compound and find suitable experimental technique to lower the density of charge carriers preferably up to values where $\mu \sim T_C$. As it was shown in many works, the superconductivity in MgB$_2$ compound with its inherent density of charge carriers is described by Moscalenko two-band model [2].

3. SUPERCONDUCTIVITY IN TWO-BAND SYSTEM PASSED TO THE STATE OF A BASE CONDENSATE OF LOCALIZED PAIRS AT $T = 0$.

3.1. Analytical solutions for the basic equations in the meanfield approximation at $T = 0$.

The two-band superconducting system with an arbitrary carrier concentration is described by the Hamiltonian (1). Using this Hamiltonian and the Green functions method [50], we have obtain the self consistent system of equations for the order parameters $\Delta_{nm}(n;m = 1,2)$ and the chemical potential $\mu$ (2), (6). These equations we will study at $T = 0$. 

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Since this system of equations cannot be solved in the general form, our goal is to find these solutions for the region of low carrier concentrations when the system passes to the state of Bose condensate of localized pairs \[45\], \[43\] \((\mu_n < 0)\) at \((\Delta_{mn}/\mu_{n,m})^2 \ll 1\). For these conditions, in expressions (2)-(6), expansion in terms of \(\Delta_{nm}^2\) can be performed.

For the sake of simplicity, redefine the indices 11 → 1; 22 → 2; 12 → 3. As a result, the system of equations (2) and (6) can be rewritten as a system of equations for the order parameters of the three-band model \[64\]. Limiting ourselves to terms of order \(\Delta_n^2\), we obtain the system of equations for finding the values of the chemical potential \(\mu\) and order parameters \(\Delta_n(n = 1, 2, 3)\).

This system of equations and theirs solutions have study in paper \[65\]. As follows from this paper the solutions for \(\mu\) and \(\Delta_n^2\) are rather cumbersome and contain a complicated dependence on the carrier concentration and the intra- and interband interaction constants. To clarify the role or the overlapping energy bands, some particular cases will be considered below. Let us consider two simpler cases here.

A. The Moskalenko model \[2\] assumes the existence of intraband \(V_{11}\) and \(V_{22}\) and interband \(V_{12}\) electron-electron interactions corresponding to electron pairing inside one band and transitions of pairs as a whole from one band to the other. In this case, \(\Delta_3 = 0\) and in the region of a small concentrations \((\mu < 0, \Delta_n^2/\mu^2 \ll 1)\) we reduce the equations for \(\Delta_1, \Delta_2\) and \(\mu\) to the following form:

\[
\mu(1 - \exp \alpha_1) + \frac{\Delta_2^2}{4} f_1 = \tilde{\zeta}_{c_1}, \quad \mu(1 - \exp \alpha_2) + \frac{\Delta_2^2}{4} f_2 = \tilde{\zeta}_{c_2}, \quad n_1 F_1 \frac{\Delta_1^2}{4} + n_2 F_2 \frac{\Delta_2^2}{4} = \tilde{n},
\]

where

\[
\alpha_1 = \frac{V_{22} - V_{12} \Delta_2/\Delta_1}{N'_{1}(V_{11}V_{22} - V_{12}V_{21})}, \quad \alpha_2 = \frac{V_{11} - V_{21} \Delta_1/\Delta_2}{N'_{1}(V_{11}V_{22} - V_{12}V_{21})},
\]

\[
N'_n = \frac{N_n}{2}, \quad \tilde{n} = \frac{N}{2(N_1 + N_2)}, \quad n_n = \frac{N_n}{N_1 + N_2},
\]

\[
f_n = \frac{(1 - \exp \alpha_n)(1 - \exp 2\alpha_n)}{\zeta_{cn} \exp \alpha_n}, \quad F_n = \frac{(1 - \exp \alpha_n)^2}{\zeta_{cn} \exp \alpha_n},
\]

\[
\tilde{\zeta}_{cn} = \zeta_{cn} + \zeta_n (1 - \exp \alpha_n),
\]

\(28\)
and where $N_n$ is the density of electron states in the $n$th sheet of the Fermi surface and $\zeta_{cn}$ is the cutoff energy for the integrals. The solutions of Eqs.(48) have the form

$$\mu = \left( n_1 \tanh \frac{\alpha_1}{2} + n_2 \tanh \frac{\alpha_2}{2} \right)^{-1} \left\{ -\frac{1}{2} \left[ n_1 \tilde{\zeta}_{c_1} \left( 1 - \tanh \frac{\alpha_1}{2} \right) + n_2 \tilde{\zeta}_{c_2} \left( 1 - \tanh \frac{\alpha_2}{2} \right) \right] + \tilde{n} \right\}. \quad (50)$$

$$\Delta_1^2 = \frac{\zeta_{c_1}}{\sinh \frac{\alpha_1}{2} \cosh \frac{\alpha_1}{2}} \left( n_1 \tanh \frac{\alpha_1}{2} + n_2 \tanh \frac{\alpha_2}{2} \right)^{-1} \times$$

$$\times \left\{ \tilde{n} + n_2 \left[ \zeta_{c_1} \tanh \frac{\alpha_2}{2} \left( \coth \frac{\alpha_1}{2} - 1 \right) + \zeta_{c_2} \left( \tanh \frac{\alpha_2}{2} - 1 \right) \right] \right\}. \quad (51)$$

The expression for $\Delta_2$ can be obtained from Eq.(51) by replacing indices $1 \leftrightarrow 2$. On the basis of Eq.(51) we obtain the equation for the ratio $\Delta_1/\Delta_2$,

$$\left( \frac{\Delta_1}{\Delta_2} \right)^2 = \frac{\zeta_{c_1} \tilde{n} + n_2 \zeta_{c_1} \left[ \zeta_{c_1} \tanh \frac{\alpha_2}{2} \left( \coth \frac{\alpha_1}{2} - 1 \right) + \zeta_{c_2} \left( \tanh \frac{\alpha_2}{2} - 1 \right) \right] \sinh \alpha_2}{\zeta_{c_2} \tilde{n} + n_1 \zeta_{c_2} \left[ \zeta_{c_2} \tanh \frac{\alpha_1}{2} \left( \coth \frac{\alpha_1}{2} - 1 \right) + \zeta_{c_1} \left( \tanh \frac{\alpha_1}{2} - 1 \right) \right] \sinh \alpha_1}. \quad (52)$$

Let us pass to the limit of symmetrical bands. In this case, $\alpha_1 = \alpha_2$, $\zeta_{c_1} = \zeta_{c_2}$, $n_1 + n_2 = 1$ and formulas (51) transform into

$$\mu = -\tilde{\zeta}_{c_1} \left[ \coth \frac{\alpha_1}{2} - 1 \right] + \tilde{n} \coth \frac{\alpha_1}{2}, \quad \Delta_1^2 = \Delta_2^2 = \frac{\tilde{n} \zeta_{c_1}}{\sinh^2 \alpha_1/2}. \quad (53)$$

Formulas (53) coincide in form with analogous expressions for one-band superconductors [47], and the difference is in the determination of $\alpha_1$ (49), including interband interactions in addition to intraband interactions. This fact leads to a decrease in the denominator in the second formula in (53), and consequently, to an increase in $\Delta_1$, in comparison with the one-band case. At the same time, a more essential shift of $\mu$ towards the negative region occurs under the condition of weak changes in the value of $\zeta_{c_1}$. As follows from (51), in the case of an anisotropic two-band system, the formulas are modified due to the difference in the parameters of the considered systems. So, for example, in the determination of $\Delta_1$ for the carrier concentration $\tilde{n}$, besides renormalization of the coefficient, an additional term arises due to the presence of the second band ($n_2 \neq 0$).

As follows from formulas (50), (51), the dependence of $\mu$ and $\Delta_n$ on the carrier concentration has a complex character due to the ratio $\Delta_1/\Delta_2$ entering the expressions determining
The $\Delta_1/\Delta_2$ ratio itself is determined by Eq. (52). From this formula it is easy to see that for a weakly anisotropic system (the bands are nearly similar), the dependence of $\Delta_1/\Delta_2$ on the carrier concentration can be neglected. As a result, $\mu$ and $\Delta_n$ are determined by the explicit dependence on $\tilde{n}$ (51), as in the case of a one-energy band. In the case of a strongly anisotropic system, the dependences of $\mu$ and $\Delta_n$ must be determined from formulas (50), (51) and equation (52). The result can be essentially different from the case of one-band superconductors.

In the behavior of a superconducting system with a small carrier concentration, the existence of a two-particle bound state plays an important role [45], [46]. The equation for determining the binding energy $\varepsilon_b$ in a two-band system is obtained by considering the two-particle "density-density" Green’s function at $\varepsilon_F = 0$ and the corresponding equation for the vertices [64], [66],

$$\Gamma_{nm} = -V_{nm} + \sum_{n_1, m_1} V_{nm_1} \xi_{n_1}(|\varepsilon_b|) \Gamma_{n_1 m}.$$  

$$\xi_n(|\varepsilon_b|) = \sum_{\vec{k}} \left[ 2 \varepsilon_n(\vec{k}) + |\varepsilon_b| \right]^{-1}.$$  

(54)

This system is written for the case of $\Delta_n = 0$ and $\varepsilon_F = 0$ and corresponds to the two-particle interaction without accounting for multiparticle effects. The bound state of two particles arises when the vertex diverges. Consequently, on the basis of Eq.(54), the following equation for the bound-state energy $\varepsilon_b$ is obtained:

$$1 - V_{11} \xi_1(|\varepsilon_b|) - V_{22} \xi_2(|\varepsilon_b|) + (V_{11} V_{22} - V_{12}^2) \xi_1(|\varepsilon_b|) \xi_2(|\varepsilon_b|) = 0.$$  

(55)

From the condition $N_0 \rightarrow 0$, we obtain [67]

$$1 - V_{11} \xi_1(2\eta) - V_{22} \xi_2(2\eta) + (V_{11} V_{22} - V_{12}^2) \xi_1(2\eta) \xi_2(2\eta) = 0.$$  

(56)

Comparing the two last equations, we can state that as $N_0 \rightarrow 0$,

$$|\varepsilon_b| = 2\eta, \quad (\eta = -\mu).$$  

(57)

B. All the interaction constants are equal to zero ($V^{ke}_{nm} = 0$), except $V_{12}^{V_{21}}$. In this case $\Delta_1 = \Delta_2 = 0$ and we have the following expression for $\Delta_3$ and $\mu$:  

30
\[ \Delta_3^2 = \Delta_{12}^2 = \frac{\tilde{n}(1 + m_1/m_2)(\zeta_{c1} + \zeta_{c2})}{8n_1 \sinh^2 \beta_1/2} \]

(58)

\[ \mu = - (\zeta_{c1} + \zeta_{c2}) \frac{1}{4} \left( \coth \frac{\beta_1}{2} - 1 \right) + \frac{\tilde{n}(1 + m_1/m_2)}{4n_1} \coth \frac{\beta_1}{2}, \]

(59)

Where \(m_n\) is the effective mass of the electron in \(n\)-band.

\[ \zeta_{c2} = \frac{m_1}{m_2} \zeta_{c1}, \quad \beta_1 = \frac{1 + m_1/m_2}{4N'_1V_{12}^2}. \]

(60)

If we assume that the bands are symmetric, i.e., \(\zeta_{c1} = \zeta_{c2}, m_1 = m_2, n_1 = 1/2\), the above formulas are modified into the expressions

\[ \mu = - \zeta_{c1} \frac{1}{2} \left( \coth \frac{\beta_1}{2} - 1 \right) + \tilde{n} \coth \beta_1/2, \quad \Delta_3^2 = \frac{\zeta_{c1} \tilde{n}}{\sinh^2 \beta_1/2}, \]

(61)

whose form resembles (53).

The above considered case of the presence of one order parameter (\(\Delta_{12}\)) in the system leads to a concentration dependence analogous to the case of symmetrical bands.

For a system with Hamiltonian (1), the expression for the difference in thermodynamic potentials for the states with \(\Delta_{nm} \neq 0\) and \(\Delta_{nn} = 0\) can be written as (35). Using the calculation technique reported in [17] and the system of equations for \(\Delta_n\), we can reduce the difference (35) to the form

\[ \frac{\Psi(\Delta) - \Psi(0)}{V} = \frac{1}{8N'_1} \left[ f_1 \Delta_1^4 + f_2 \Delta_2^4 \frac{N_2}{N_1} + 2f_3 \Delta_3^4 \right], \]

(62)

where \(f_n < 0\) [65]. As a consequence, the state with \(\Delta_n \neq 0\) is more advantageous than the trivial solution \(\Delta_n = 0\). The advantage increases as the number of order parameters increases in connection with accounting for all kinds of inter- and intraband interactions.

In the case of symmetrical bands, expression (62) can be rewritten in the form

\[ \frac{\Psi(\Delta) - \Psi(0)}{V} = -2N' \coth \frac{\alpha_1}{2} \tilde{n}^2 \]

(63)

This formula coincides in form with an analogous expression for the one-band superconductor [47].
3.2 Application of the path integral method.

In this section the path integral method as applied to the two-band model is developed and, on this basis, the procedure for transition from the Fermi to Bose elementary excitations is given.

We start from following Hamiltonian describing the two-band systems:

\[
H = \int dr \sum_{n\sigma} \Psi_{n\sigma}^{+}(\tau) H_{0n} \Psi_{n\sigma}(\tau) - \frac{1}{2} \sum_{nm} V_{nm} \int dr \sum_{\sigma \sigma'} \Psi_{n\sigma}^{+}(\tau) \Psi_{n\sigma'}^{+}(\tau) \Psi_{n\sigma}(\tau) \Psi_{n\sigma'}(\tau)(1 - \delta_{\sigma \sigma'}),
\]

where the first term corresponds to the kinetic energy, the second is responsible for the superconductivity, the band indices \(n; m = 1, 2\), \(V_{nm}\) are the constants of intra- and interband interaction, \(\Psi_{n\sigma}^{+}(\tau)\) is the particle creation operator in the band \(n\) with spin \(\sigma\), and \(H_{0n} = \frac{\nabla^2}{2m_n} - \mu_n\). Hamiltonian (1) correspond to expression of the Moskalenko model [2] considering only interband pairing and the transitions of Cooper pairs as a whole from one band to another.

The expression for the statistical sum of a two-band superconductor after introducing additional scalar fields \(\Delta_n\), Hubbard-Stratonovich transformations, integrations under Fermi fields, and transition into \(nk\Omega\) representation has the following form [68] (the detailed version see [67]):

\[
Z = C \int \left( \prod_{i=1,2} d\Delta_{i}^{*} d\Delta_{i} \right) \exp \left( - S_{eff}^{(2)} \right),
\]

where

\[
S_{eff}^{(2)} = \frac{1}{\beta} \sum_{nm} \sum_{\vec{q}\omega} \Delta_{n}^{*}(\vec{q}\omega) \left\{ V_{nm}^{-1} + \Pi_n(\vec{q}\omega) \delta_{nm} \right\} \Delta_{m}(\vec{q}\omega),
\]

\[
\Pi_n(\vec{q},\omega) = \frac{1}{\beta} \sum_{\vec{k}\Omega} G_{-n}(\vec{k}\Omega) G_{+n}(\vec{k} - \vec{q},\Omega - \omega) |\chi_n(\vec{k},\vec{k} - \vec{q})|^2,
\]

\[
G_{\pm n}(\vec{k}\Omega) = -i \Omega \pm \frac{k^2}{2m_n} \mp \mu_n.
\]
Here $V_{nm}^{-1}$ is the matrix element of inverse to the interaction matrix

$$
\hat{V} = \begin{pmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{pmatrix}.
$$

and $\chi_n(\vec{k}, \vec{k} - \vec{q})$ are the integrals over the elementary cell under the Bloch functions.

Further calculations are connected as in the case of a single band [46] with the existence of a nonzero solution of Eq. (55), determining the energy of the bound state $\varepsilon_b$, and we consider the case $T \ll |\varepsilon_b|$.

In the mean-field approximation $\mu = \varepsilon_b/2$ (57). Let us suppose that $\mu$ is near $\varepsilon_b/2$. After integrating under $\Omega$ in formula (67) we make the expansion under $\omega, q$ and the difference $\mu^* - \varepsilon_b$. After that we acquire

$$
\Pi_n(q, \omega) = -\xi_n(|\varepsilon_b|) + \gamma_n \left[ -i\omega + \frac{q^2}{2m^*_n} - \mu^* \right],
$$

were

$$
\gamma_n = -\frac{\partial \Pi_n}{\partial (2\mu)} \quad \mu^* = 2\mu - \varepsilon_b, \quad \frac{1}{m^*_n} = \gamma_n^{-1} \frac{\partial^2 \Pi_n}{\partial \overline{q}^2}.
$$

Let us substitute (66) into (65) and perform the functional integration under additional Bose fields, leading to the canonical form by orthogonal transformation. We obtain

$$
\ln Z = - \sum_{\overline{q}, \omega} \ln \left\{ 1 + V_{11} \Pi_1(\overline{q}, \omega) + V_{22} \Pi_2(\overline{q}, \omega) + \left( V_{11} V_{22} - V_{12}^2 \right) \Pi_1(\overline{q}, \omega) \Pi_2(\overline{q}, \omega) \right\}. \quad (71)
$$

We substitute in this formula expressions for the $\Pi_n(q, \omega)$ [see Eq. (69)] and take into account Eq. (55), determining the energy of the bound state $\varepsilon_b$, and also relationships

$$
V_{11} - \left( V_{11} V_{22} - V_{12}^2 \right) \xi_2 = V_{21} \Delta_1/\Delta_2, \quad V_{22} - \left( V_{11} V_{22} - V_{12}^2 \right) \xi_1 = V_{12} \Delta_2/\Delta_1,
$$

resulting from the system of equations for the order parameters $\Delta_n$. Thus we obtain

$$
\ln Z = - \sum_{\overline{q}, \omega} \ln \left\{ \gamma_1 V_{21} \frac{\Delta_1}{\Delta_2} \left( -i\omega + \frac{q^2}{2m^*_1} - \mu^* \right) + \gamma_2 V_{12} \frac{\Delta_2}{\Delta_1} \left( -i\omega + \frac{q^2}{2m^*_2} - \mu^* \right) \right\}. \quad (73)
$$

For the charge carrier density we have
\[ n = \frac{1}{\beta} \frac{\partial}{\partial \mu^*} \ln Z = \frac{1}{\beta} \sum_{\vec{q}} \left( -i\omega + \frac{q^2}{2M^*} - \mu^* \right)^{-1} = \sum_{\vec{q}} \left( \exp \left[ \beta \left( \frac{q^2}{2M^*} - \mu^* \right) \right] - 1 \right)^{-1}, \quad (74) \]

were

\[ (M^*)^{-1} = \frac{1}{m_1^*} \left( \gamma_1 \frac{\Delta_1}{\Delta_2} + \gamma_2 \frac{\Delta_2}{\Delta_1} \frac{m_1^*}{m_2^*} \right) \left( \gamma_1 \frac{\Delta_1}{\Delta_2} + \gamma_2 \frac{\Delta_2}{\Delta_1} \right)^{-1}. \quad (75) \]

Expression (75) has been obtained in the approximation in which only the quadratic part of complete action is taken into account. In this approximation we obtain the ideal Bose gas with effective mass \( M^* \) and chemical potential \( \mu^* = 2\mu - \varepsilon_b \). In distinction from the case of a single band [46], here \( \varepsilon_b \) and \( \mu \) are determined on the two-band basis (55), (50) and \( M^* \) is defined by (75). In the case where the effective masses from different bands are equivalent \( (m_1^* = m_2^*), \) the relationship \( M^* = m_1^* = m_2^* \) results from the last formula. If the energy band with effective mass \( m_1^* \) overlaps with the wider band \( (m_1^* > m_2^*), \) in the field with low charge carrier density a lighter Bose gas arises \( (M^* < m_1^* ) \) in comparison with the case of a single-band superconductor effective electron mass of which is equal to \( m_1^* \).

The system undergoes transition to the state of Bose condensation at the point \( \mu^* = 0 \) and this relationship is the condition for determining the condensation temperature \( T_k \). After summation under \( \vec{q} \) in (74) at \( \mu^* = 0 \), we obtain

\[ T_k = a_D \frac{n^{2/D}}{M^*}, \quad (76) \]

where \( D \) is the dimension of system and \( a_D \) is a constant. We see that \( T_k \) depends on the charge carrier density according to the law \( n^{2/D} \) as in the case of a single-band superconductor but only for systems with equivalent bands \( (M^* = m_1^* = m_2^*). \) If the system is strongly anisotropic \( (m_1^* \neq m_2^*), \) the quantity \( M^* \) has the additional inexplicit dependence on the charge carrier density through the same dependence of the relation \( \Delta_1/\Delta_2 \). From the formulas (75) and (76) we see that the existence of a second wider energy band can lead to an increase in the condensation temperature \( T_k \) in comparison with the case of a single energy band. The expression for the effective action (66) can be considered as an improved mean-field approximation, which can be also applied at \( T \neq 0 \).
Taking into account the residual interaction between bosons, the effective action has the following form:

\[ S_{\text{eff}} = S_{\text{eff}}^{(2)} + S_{\text{eff}}^{(4)} + \ldots, \]  

(77)

where

\[ S_{\text{eff}}^{(4)} = \frac{1}{\beta} \sum_{\vec{q}\omega} \left\{ |\Delta_1|^4 \pi_1 + |\Delta_2|^4 \pi_2 \right\}, \]  

(78)

\[ \pi_n = \frac{1}{2\beta} \sum_{k\Omega} \left[ G_{+n}(k\Omega) \right]^2 \left[ G_{-n}(k\Omega) \right]^2 = \sum_{\vec{k}} \left[ 2\varepsilon_n(\vec{k}) + \varepsilon_b \right]^{-3}. \]  

(79)

After renormalization of the Bose fields

\[ \Delta_n = \left( \frac{\partial \Pi_n}{\partial (2|\mu|)} \right)^{-1/2} \varphi_n \]  

(80)

we have

\[ S_{\text{eff}}^{(4)} = \frac{1}{\beta} \sum_n \sum_{\vec{q}\omega} |\varphi_n(\vec{q}\omega)|^2 \pi'_n(\vec{q}, \omega), \]  

(81)

\[ \pi'_n = \left( \frac{\partial \Pi_n}{\partial (2|\mu|)} \right)^{-2} \pi_n. \]  

(82)

For the quantity \( \pi'_n \) determining the potential of two-particle interaction in the \( n \)th band we obtain

\[ \pi'_n \sim m_n^{-D/2} |\varepsilon_b|^{1-D/2}. \]  

(83)

This quantity depends essentially on the system dimension and is negligible in the case of three-dimensional. This conclusion consides with the case of a single band [46]. Thus, omiting the residual interaction, which is equivalent to the mean-field approximation, is valid only for three-dimensional systems.

3.3 Conclusions and discussion
In the section 3.1 the theory of superconductivity for a two-band system with a low carrier density at $T = 0$ was constructed. All types of electron-electron interband and intraband interactions were accounted for. As a result, three different order parameters, $\Delta_{11}, \Delta_{22},$ and $\Delta_{12},$ [65] arise in the system. The system of equations for the order parameters and the chemical potential $\mu$ was obtained. The state of the system in the region of low carrier concentrations, where $\mu_n < 0$ ($n = 1, 2$) (corresponding to a Bose condensate of localized pairs), was considered. Under the assumption that $\Delta_{nm}^2/\mu_{n,m}^2 \ll 1$, the expansion in terms of $\Delta_{nm}^2$ was conducted. This allowed the analytical expressions for $\mu$ and $\Delta_n$ to be obtained. The expression for the difference in thermodynamic potentials for $\Delta_{nm} \neq 0$ and $\Delta_{nn} = 0$ was also obtained.

On the basis of these investigations the following conclusions are made:

1. The overlapping of energy bands on the Fermi surface in a system with low carrier density gives rise to three order parameters. In the limit of the symmetrical bands ($\Delta_{11} = \Delta_{22} = \Delta_{12}$), the values of these parameters are higher than in the case of the one-energy band, i.e., the overlapping of energy bands is favorable to setting up the superconducting properties. At the same time, the value of $\mu$ shifts towards the negative value region, accelerating the transition of the system from the BCS pattern to the mode of a Bose condensate of localized pairs.

2. The order parameters $\Delta_{nm}$ and the chemical potential $\mu$ are complex functions of inter- and intraband interaction constants, as well as of carrier concentrations. In the weak anisotropy case (the bands are nearly symmetrical) the dependences of $\Delta_{nm}$ and $\mu$ on the carrier concentrations are analogous to those for the case of one-band superconductors. For strongly anisotropic systems (the bands are different), the concentration dependences of $\Delta_{nm}$ and $\mu$ can differ from the case of one-band superconductors. This is because, in addition to the explicit dependence on the carrier concentration, a nonexplicit dependence exists through the ratios $\Delta_{11}/\Delta_{22}, \Delta_{12}/\Delta_{22},$ etc.

3. The difference in the thermodynamic potentials (62) shows that the considered condensed state is advantageous at $\Delta_{nm} \neq 0$. The advantage of this state increases as the
number of order parameters and, consequently, the number of intraelectron constants, increases.

To make the results more illustrative, two simpler cases - the limiting case of the Moskalenko model [2], where $\Delta_{11}, \Delta_{22} \neq 0, \Delta_{12} = 0$, as well as the case where $\Delta_{11} = \Delta_{22} = 0, \Delta_{12} \neq 0$ - were considered. We have shown that overlapping of the energy bands facilitate superconductivity and increase the critical concentration of the carriers at which a transition from the BCS scenario to the local-pair condensate scenario occurs. In the case of different bands (especially, for strong anisotropy), the dependence of $\Delta_{nm}$ and $\mu$ on the carrier concentration may be different from the one-band superconductor case because of their additional implicit dependence on $\bar{n}$ through the ratio $\Delta_1/\Delta_2$. We also obtained an equation for the bound-state energy $\varepsilon_b$ on a two-band basis and established the relation $\varepsilon_b = 2\mu$.

In section 3.2 is developed the path integral method for two-band superconductors and, on this basis, improve the mean field approximation to make it applicable for $T \neq 0$.

To do this, we introduced action, which takes into account the existence of intra- and interband interactions and additional boson fields. In the equation for the statistical sum, we performed the Hubbard-Stratanovich transformation generalized to the two-band case. Having performed the integration over Fermi fields, and expansion in terms of $\hat{\Delta}_n$ with an accuracy to quadratic terms, we came to effective action (66), (67). We connected the calculation of $\Pi_n(\vec{q}, \omega)$ with the existence in the system of a bound state with the bound-state energy $|\varepsilon_b| \gg T$ and performed the expansion of $\Pi_n(\vec{q}, \omega)$ in powers of $\omega, q^2$, and $(\varepsilon_b - 2\mu)$. Along the way, we came to statistical sum for the ideal Bose gas with a renormalized chemical potential $\mu^* = 2\mu - \varepsilon_b$ and effective mass $M^*$ (75).

We also determined the condensation temperature $T_k$, Eq. (76). In the case of different energy bands, considerable renormalization connected with the existence of two energy bands occurs. At the same time, the dependence of $T_k$ on the carrier concentration is explicitly determined by the dimension of the system and is implicitly determined by the dependence of $M^*$ on the carrier concentration.
The contribution of the residual Bose interaction to the effective action was also calculated. In analogy to the case of a one-band superconductor, this contribution is unessential only in three-dimensional systems [46]. For such systems, the mean field approximation can be applied, while for systems of reduced dimension, the mean field approximation appears to be insufficient.

The main results of the section 3.2 are following:

1. On the basis of the functional integration method, the procedure of the transition from the BCS scenario to the Schafroth one is developed for the two-band superconductor, when changing the charge carrier density. In the mean-field approximation the ideal Bose gas with renormalized chemical potential \( \mu^* = 2\mu - \varepsilon_b \) and effective mass \( M^* \) is obtained. These quantities are determined on the basis of the two-band model. The anisotropy of the energy bands plays the critical role in the renormalization of the effective mass. In particular, at \( m_1 > m_2 \) the easier Bose gas \( (M^* < m_1^*) \) appears in comparison with the case of a single band with effective mass \( m_1 \).

2. In the case of strongly anisotropic two-band systems \( (\Delta_1 \neq \Delta_2) \) the possibility of increasing the temperature of the Bose condensation of localized pairs \( T_k \) in comparison with the case of the single energy band is shown.

3. The contribution of residual interaction between bosons to the effective action depends essentially on the system dimension and has a low value only for a three-dimensional systems. Therefore for systems with reduced dimension it is necessary to depart from the mean-field approximation.

4. Summary.

High-\( T_C \) superconductors as was mentioned in Introduction are very complicated compounds, and it is impossible to take into account all their features simultaneously. So, the theoretical studies are performing on the basis of the models, which take into account some characteristic features of these systems.
The goal of this article is to make the review of the papers, in which two peculiarities are taken into consideration - the overlapping of the energy bands on the Fermi surface and the reduced (or low) density of the charge carriers.

In the two-band model proposed by prof. Moskalenko [2] the Cooper pairs appear in every energy band and pass as a whole from one energy band to another.

In the systems with the low density of the charge carriers there is the necessity to take into consideration all possible additional pairings of electrons from different energy bands [49] and go beyond the approximation, in which are taken into account only the diagonal over the band indices Green functions [23]. The basic equations of the theory of superconductivity in such two-band systems is reduced to the representation of four energy pseudo-bands. In this case there appears three energy gaps $\Delta_{11}$, $\Delta_{22}$ and $\Delta_{12} = \Delta_{21}$ and it is also possible to obtain the values of the temperature of the superconducting transition $T_c$, characteristic to the high-$T_C$ materials, even at the low density of the charge carriers as in the case of the electron attraction ($\lambda_{nm} > 0$), as in the case of their repulsion ($\lambda_{nm} < 0$). The main attention in this paper is devoted to the dependence of the thermodynamic quantities on the density of charge carriers. In particular, the behavior of the quantity $T_c$ and the jump of electron heat capacity $(C_S - C_N)$ at the point $T = T_c$ as a function of the density of the charge carriers is determined by the degree of the overlapping of the energy bands, of the filling of these bands, relation between the constants of the electron-electron interaction $\lambda_{nm}$ ($n; m = 1 - 4$) and the ratio of electron density states $N_1/N_2$.

As, for example, in the case of strong hybridization $T_c$ and $(C_S - C_N)$ as function of the concentration of the charge carriers are represented by the bell-shaped dependence, which is observed in the experiment in the number of the oxide ceramics. The represented theory allows to obtain as small $(C_s - C_N)/C_N < 1.43$, as large $(C_s - C_N)/C_N > 1.43$ values for the relative jump of the electron heat capacity. This picture is observed in the high-temperature materials [54]- [57]. These studies have fulfilled in the representation of the BCS superconducting Cooper pairs. At decreasing the density of charge carriers at the point $\mu = 0$ there takes place the crossover BCS state-Bose condensation of the localized pairs.
In the state of the deep Bose-condensation, when $\mu < 0$ and the relation $\Delta_n^2/\mu^2 \ll 1$ takes place at $T = 0$, we obtain that analytical expressions for the quantities $\Delta_n$ and $\mu$ contain additional inexplicit dependence on the density of charge carriers in comparison with the single-band system, due to the consideration of the overlapping of the energy bands on the Fermi surface.

Given in the article the method of the functional integration applied to the two-band system with low density of the charge carriers demonstrates the transition procedure from the Fermi to the Bose elementary excitations at $T \neq 0$. We have obtain ideal Bose-gas with the effective mass $M^*$ and chemical potential $\mu^* = 2\mu - \varepsilon_b$ ($\varepsilon_b$ is the energy of the bound two-particle state). These quantities are determined on the two-band basis.

The important results of the given work are: the overlapping of the energy bands on the Fermi surface promote to the appearing of the superconductivity, to explain some experimental data in the investigations of the thermodynamical properties of the system, intensity the number of the effects, characteristical to the systems with lowered density of the charge carriers, and assist their experimental confirmation. These effects, for example, are the appearing of the kink in the temperature dependence of the chemical potential $\mu(T)$ in the point $T = T_c$, and also crossover the state BCS - Bose condensation of the local pairs. The theory proposed in the paper can be applied to the oxide ceramics and also to the high-temperature composition $MgB_2$ and other compositions, in which takes place the overlapping of the energy bands on the Fermi surface and there is possibility to change the concentration of the charge carriers.

In addition to the overlapping of the energy band, an important factor of the band structure of high-temperature materials is the existence of high symmetry points in the momentum space that lead to singularities in the electron density of states (Van Hove singularities), as well as to topological electron transitions. The behavior of these singularities dependends, to a large extent, on the dimension of the system.

Numerous theoretical and experimental investigations have confirmed that in quasi two-dimensional systems (the majority of high-temperature materials are just such system), a
logarithmic divergence of the density of states occurs in the vicinity of half filling of the energy band at the point where the Fermi surface intersects the Brillouin zone boundary. This corresponds to the change from electron to hole topology on the Fermi surface.

This singularity gives rise to high superconducting transition temperatures $T_c$ and anomalies in the isotope effect and specific heat (see the review in [69]), as well as providing an explanation for experimental data on the temperature dependence of the resistance and thermal EMF in high-$T_c$ materials [70], etc. Photoemission experiments have shown that logarithmic divergence is a characteristic of the $La_{2-x}Sr_xCuO_4$ compound [71]. At the same time, the densities of electron states may have stronger divergences, such as $E^{-1/4}$ and $E^{-1/2}$, in $YBa_2Cu_3O_{7-δ}$ and $BiSrCaCu_2O_8$ compounds, respectively ($E$ is energy counted from the Van Hove singularity), and correspond to the so-called extended Van Hove singularities [72].

In a number of articles, several attempts have been made to explain high-$T_c$ by accounting only for these singularities of the density of electron states and assuming that the pair interaction is universal for all compounds. In the weak-coupling approximation, accounting for these singularities results in a larger increase in the superconducting transition temperature $T_c$ in comparison with the case of logarithmic divergence. In the case of strong coupling, the picture changes, leading to an essential decrease in the role of the extended Van Hove singularities that are spread apart due to the inelastic scattering of electrons [73].

It should be noted that we investigated the influence of the density of the state singularities of type $E^{-1/2}$ on the thermodynamic and kinetic properties of superconductors long before the discovery of high-$T_c$ superconductivity [74], [75]. In particular, we showed that the scattering of electrons on the impurity potential decreased the role of that singularity in the determination of $T_c$.

In studying the properties of high-temperature materials, it is probably insufficient to consider only the existence of extended Van Hove singularities. It is necessary to use a more complex approach that accounts for energy band overlapping, reduced carrier concentrations, nonadiabacity and the other factors characteristic of high-temperature materials, especially
becous the two-band model is capable of explaining a large amount of experimental a great number [14] - [22], [49], [76], [77], in pure and doped high - \( T_c \) superconductors including \( MgB_2 \) [78].

The discovery of high \( T_c \) superconductivity in \( MgB_2 \) and in some other new materials with the lowered density of the charge curriers persuade us that given review article is well useful for theoretical and experimental investigations.

We note that there are many authors at present which carry out the theoretical investigations of the physical properties of \( MgB_2 \) on the bases of two - band model (see, for example, in book [79], also in [29]- [32], [59], [60] and al. and references in them).

The information about theory investigations of thermodynamic and kinetic properties of many-band superconductors, elaborated long before discovery of superconductivity in \( MgB_2 \), are in arXiv: Cond. Mat. [7], [22], [80] and in this review article. The comparison of this theory with the investigations of last years permits to physisists, working with above-mentioned problems, to do conclusions about situation in given field.

Acknowledgments.

The author gratefully acknowledges Prof. to V.A. Moskalenko, for his interest in this problem and is thankful to Dr. F.G.Kochorbe for long-term collaboration and to Prof. T.Mishonov for his active position in the recognition of the priority accomplishment of Moldavian physicists to the development of the theory of multi-band superconductors.
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Fig. 4. Chemical potential \( \eta \) dependence on \( \bar{x} = N_0/4N_1 \): (1) \( N_2/N_1 = 0.1 \), (2) \( N_2/N_1 = 0.4 \), (3) \( N_2/N_1 = 1.0 \)
Fig. 5. $2|\Delta_1(0)|/T_c$ and $2|\Delta_2(0)|/T_c$ dependences on $\eta$: (1) $2|\Delta_1(0)|/T_c$, (2) $2|\Delta_2(0)|/T_c$ at $\lambda_{11} = 0.1$, $\lambda_{22} = 0.19$, $\lambda_{12}\lambda_{21} = 0.0024$; (1') $2|\Delta_1(0)|/T_c$, (2') $2|\Delta_2(0)|/T_c$ at $\lambda_{11} = -0.37$, $\lambda_{22} = -0.03$, $\lambda_{12}\lambda_{21} = 0.14$
Fig. 6. Dependences of the absolute $(C_S - C_N)/VN_1$ (6a) and relative $(C_S - C_N)/C_N$ (6b) electron-temperature jumps on the carrier density $N_0/2N_1$. The curves labels are the same as in Fig.1.
Fig. 6a.

\[ (C_S - C_N) / VN_1, eV \]

\[ (0, 0.05, 0.10, 0.15, 0.20) \]

\[ (0, 100, 200) \]

\[ N_0/2N_1, meV \]
Fig. 7. $(C_s - C_N)/VN_1$ dependence on carrier concentration $\tilde{x} = N_0/4N_1$: (1, 1') $N_2/N_1 = 0.1$, (2, 2') $N_2/N_1 = 0.4$, (3, 3') $N_2/N_1 = 1.0$. 
Fig. 8. (a) \((C_S - C_N)/C_N\) dependence on \(\tilde{x} = N_0/4N_1\) (\(\lambda_{nm} > 0\)): (1) \(N_2/N_1 = 0.1\), (2) \(N_2/N_1 = 0.4\), (3) \(N_2/N_1 = 1.0\). (b) \((C_S - C_N)/C_N\) dependence on \(\tilde{x} = (\lambda_{nm} < 0)\): (1) \(N_2/N_1 = 0.1\), (2) \(N_2/N_1 = 0.4\), (3) \(N_2/N_1 = 1.0\).
Fig. 8a.
Fig. 9. Dependence of the chemical potential ($\eta = -\mu$) on the carrier density $N_0/2N_1$. The numbering of the plots corresponds to the same $N_0/N_1$ as in Fig. 1.
Fig. 10. Dependence of $T_c$ on the carrier density $N_0/2N_1$. Curves 1 and 2 correspond to weak hybridization and ratio $N_2/N_1 = 1, 0.5$, respectively; curves 3 and 4 correspond to strong hybridization and ratio $N_2/N_1 = 1, 0.5$, respectively.
Fig. 11. Dependence of the chemical potential on the temperature. Curve 1 corresponds to $N_2/N_1 = 1$ and $N_0/2N_1 = 0.005$ eV, curves 2 and 2’ to $N_2/N_1 = 1$, 0.5 and $N_0/2N_1 = 0.01$ eV; curves 3 and 3’ to $N_2/N_1 = 1$, 0.5 and $N_0/2N_1 = 0.015$ eV; curves 4 and 4’ to $N_2/N_1 = 1$, 0.5 and $N_0/2N_1 = 0.02$ eV.
Fig. 12. Dependence of the jump of the specific heat on the carrier density $N_0/2N_1$. 

\[(C_S - C_N)/V N_1, eV\]
Fig. 13. Dependence of the relative jump of the specific heat on the carrier density $N_0/2N_1$. 

$(C_S - C_N)/C_N$