Dual Abrikosov vortices in U(1) and SU(2) lattice gauge theories

Richard W. Haymaker

Department of Physics and Astronomy,
Louisiana State University, Baton Rouge, Louisiana 70803-4001, USA

The spatial distribution of fields and currents in confining theories can give direct evidence of dual superconductivity. We review the behavior of vortices in the lattice Higgs effective theory. We discuss the techniques for finding these properties and calculating the superconductivity parameters in lattice simulations. We have seen dual Abrikosov vortices directly in pure U(1) and SU(2) and others have also seen them in SU(3). We review the duality transformation for U(1) in order to connect the U(1) results to a dual Higgs theory. In the non-Abelian cases the system appears to be near the borderline between type I and II. We also discuss the response of the persistent currents to external fields.

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I. INTRODUCTION

The physics of superconductivity has provided the inspiration for many of the ideas behind the color confinement mechanism in QCD. The most notable property is the behavior of a magnetic field imposed normal to the surface of such a material. For a type II superconductor, the field drills holes in the superconductor forming a lattice of filaments of normal material defining the cores of Abrikosov vortices. The longitudinal magnetic field is a maximum at the center and falls off exponentially in the direction transverse to the core over a distance denoted as the London penetration depth, \( \lambda \). One quantum of magnetic flux is carried by each vortex. Recent photographs using a scanning-tunneling-microscope and electron-holograph-interferometry show this flux confinement very dramatically.

Dual superconductivity is the analogous phenomenon in which persistent currents of magnetic monopoles form dual Abrikosov vortices which confine electric flux. Lattice gauge theory offers the prospect of exploring dual superconductivity in depth as a confining mechanism. In four dimensional U(1) lattice gauge theory there is considerable indirect evidence from “bulk properties” of the vacuum such as the monopole density, monopole susceptibility, and static quark potential that Dirac magnetic monopoles are associated with the phenomenon of confinement.

The existence of a dual Abrikosov vortex between a static quark-antiquark pair leads to more direct evidence for confinement. By studying the relationship between the monopole currents and fields in the neighborhood of static sources we identified the flux tube with a dual vortex in U(1), and in SU(2). Suzuki and collaborators reproduced the SU(2) results with better statistics and verified the effect in SU(3). We have further studied the effect at finite temperature and have seen the vortex disappear above the deconfining temperature. Cea and Cosmai have measured the electric field profiles in a flux tube in U(1) and SU(2) identifying the shape with a dual vortex.

In ordinary superconductivity, the primary issue is the spontaneous breaking of the electromagnetic U(1) gauge symmetry (SSB) signaled by the non-vanishing of the vacuum...
expectation value of a charged field. An immediate consequence is that the curl of the vector potential is proportional to the curl of the electric current known as the London relation [13]. The London relation is violated only near the boundaries of superconducting material within a distance defined as the Ginzburg-Landau [16] coherence length, $\xi$. Combining the London relation with Maxwell’s equations gives mass to the electromagnetic field. The Meissner [17] effect and infinite conductivity follow.

These relationships suggest numerous ways to search for signals of dual superconductivity. Numerical methods may be successful in establishing some of these connections and not others. The focus of this work is to pursue the dual London equation which is central to the phenomenon. There are other promising approaches which we will also describe.

The theory of superconductivity entails (i) the identification of what symmetry is broken and what are the relevant dynamical coordinates, (ii) the mechanism that leads to the instability and hence SSB, e.g. BCS [18] theory, and then (iii) an effective theory of the currents and fields in the broken phase, e.g. Ginzburg-Landau [16] [GL] theory, or equivalently the four dimensional generalization, the Higgs model [19] treated as an effective theory, which allows the calculation of the spatial consequences of the broken symmetry [2].

Our goals are similar to these. We hope to identify relevant dynamical coordinates and a corresponding effective theory. Experience has shown that an analytic approach is very difficult. The duality transformation has not been achieved for the Wilson form of the action which we are using. But this transformation has been implemented in closed form in the closely related Villain [20] form of the action by Polyakov [21] and Banks, Myerson and Kogut [22] in the early development of this subject.

Frölich and Marchetti [23] have developed these techniques further. They have identified the monopole field operator for the Villain action. The non-vanishing vacuum expectation value is identified as the order parameter for dual superconductivity. This has been studied numerically by Polley and Wiese, and Polikarpov [24,25]. Ref [26,27] are other closely related papers.

Similar analytic methods have not been successful for the Wilson form for the U(1) action.
However Del Debbio, Di Giacomo, Paffuti and Pieri [28] have identified the monopole field operator applicable for a more general U(1) action and found the vacuum expectation value to be a very strong signal for dual superconductivity.

While the origin of confinement in U(1) lattice gauge theory is fairly clear, understanding confinement in non-Abelian SU(N) theories has been more difficult [29]. One promising approach is to fix the non-Abelian degrees of freedom in the maximal Abelian gauge as advocated by Schierholz and Suzuki and others [30,31], leaving a residual $U(1)^{N-1}$ gauge freedom, with $(N-1)$ species of $U(1)$ Dirac monopoles. Suzuki and collaborators [32,33] have laid much of the foundation for this lattice approach. The monopoles have been observed to be abundant in the confined phase and dilute in the (finite temperature) unconfined phase [30,31,32,34,35,36,37,38,39]. A number of contributions by Yee [40,41,42] are described in a conference talk [40].

Del Debbio, Di Giacomo, Paffuti and Pieri [28] have applied their U(1) construction to the Abelian projected SU(2) and again saw a clear signal of dual superconductivity. They have considered a number of forms of the Abelian projection but have not yet implemented this method for the maximal Abelian gauge.

Dual vortices have been seen for SU(2) as reported in Refs. [11,12,13]. All these papers report the similar conclusion that the dual GL coherence length $\xi_d$ and the dual London penetration depth $\lambda_d$ are roughly equal. For a type II dual superconductor $\kappa_d \equiv \lambda_d/\xi_d > 1/\sqrt{2}$, and a type I otherwise [2]. Maeda, Matsubara and Suzuki [42] came to a similar conclusion for SU(3) from an entirely independent argument. For superconducting materials the difference between type I and type II is dramatic when a magnetic field is imposed. The penetration of magnetic flux in a type I material leads to very complex patterns of normal and superconducting domains called an ’intermediate state structure’. Numerous photographs of this phenomenon are shown in a monograph by R. P. Huebener [43].

Since U(1) lattice gauge theory does not have a non-trivial continuum limit, one must use the results with caution in describing confinement in QCD. However it is more than just an independent model of confinement. Yee [44] has found using the demon method
that for small $\beta$ the Abelian projected action for SU(2) is essentially the U(1) action with $\beta_{U(1)} = \beta_{SU(2)}/2$. Further this must and indeed Yee finds it does break down in an interesting way as $\beta$ approaches the scaling region. See also Suzuki [32] for a different approach.

In a long series of papers and a review article, Baker, Ball, Zachariasen [15] and co-workers have postulated a dual form of QCD in the continuum. Their formulation describes a non-Abelian dual superconductor and hence it confines color. They have calculated flux tubes, static quark potentials, temperature dependent effects and many other quantities in the tree approximation. Although their formulation does not focus on monopoles, there is a large potential overlap between their work and the lattice description developed here.

In Section II we review the Abrikosov vortex in the Higgs theory [19] which is the basis of the many interrelated signals of superconductivity. We will emphasize the lattice form of the Higgs model. The singularities giving vortices arise in an interesting way naturally on the lattice.

In Section III we report the results of numerical studies of U(1) for correlations between a Wilson loop and a plaquette giving a very clear signal for the London penetration depth. These numerical results suggest that a dual form of the model should exhibit this phenomenon in a simple way. In Section IV we show that the simpler Villain [20] form of the action does allow an analytic treatment of the question and we review the results of Frölich and Marchetti [23], Polley and Wiese [24], and Polikarpov [25].

Results for SU(2) and SU(3) are given in Section V. The analysis involves an approximate treatment of the GL theory in order account for the behavior of the fields in the dual vortex.

Fields and currents exist only near the boundaries of dual superconductors over a distance scale set by the London penetration depth $\lambda_d$. It is these spatially transient effects that provide spatial structures where the local properties of dual superconductivity can be studied and that is the focus of this work. One way to study these phenomena is to impose an external field. In Section VI we introduce an external field and discuss some of the prospects and problems.

The lattice approach provides a very concrete way to visualize physical quantities as living
on links, plaquettes, dual plaquettes etc. The use of the abstract language of differential forms is gaining use in the lattice literature. This formulation focuses on $k$ dimensional volumes and boundaries of the volumes in a systematic way. The technology in some ways removes one from the nitty gritty of the lattice but with some familiarity it actually enhances one's ability to see general relationships. Polley Weise [24] give a brief introduction to the basics of this formulation. In the appendix we go one step further and give a concrete translation of differential forms into lattice difference operators.

II. ABRIKOSOV VORTICES

The Ginzburg-Landau effective theory of superconductivity is a time independent description based on a complex order parameter, $\psi(\vec{x})$, where $|\psi(\vec{x})|^2$ is the number density of superconducting charge carriers. In normal material $|\psi(\vec{x})| = 0$. We will focus on the special circumstance in which $|\psi(\vec{x})|$ is constant in the superconducting phase and discontinuous at the surface. This is the extreme type II limit, also called the London limit.

The Higgs [19] model is the four dimensional generalization of GL theory with the Higgs field $\phi(x)$ corresponding to the GL order parameter $\psi(\vec{x})$. The London limit is obtained by constraining the Higgs field:

$$V_{\text{Higgs}}(|\phi(x)|^2) \Rightarrow |\phi(x)|^2 = v^2.$$  

In other words the London theory is equivalent to the effective non-linear Higgs theory.

A. London theory

A concise statement of the London [15] theory is contained in the relation

$$\vec{A} + \lambda^2 \vec{J} = 0; \quad (\nabla \cdot \vec{A} = 0).$$
If the charge density is zero, then in this gauge the electric field is given by $-\dot{A}$ and therefore $\vec{E} = \lambda^2 \vec{J}$. This describes a perfect conductor and is just Newton’s law for free carriers, $e\vec{E} = m\vec{v}$. By taking the curl of Eqn. (3) we obtain the condition for a perfect diamagnet.

$$\vec{\nabla} \times \vec{J} = -\frac{1}{\lambda^2} \vec{B}. \quad (3)$$

This relation together with Ampere’s law $\vec{\nabla} \times \vec{B} = \vec{J}$ gives $\nabla^2 \vec{B} = \vec{B}/\lambda^2$ which implies that the magnetic field falls off in the interior of the superconductor with a skin depth $\lambda$.

Finally the fluxoid is given by the integral

$$\int_S (\vec{B} + \lambda^2 \vec{\nabla} \times \vec{J}) \cdot \hat{n} da = \int_C (\vec{A} + \lambda^2 \vec{J}) \cdot d\vec{l} = \frac{N 2\pi}{e} = N e_m = \Phi_m. \quad (4)$$

If the curve $C$ is in a simply connected region of a superconductor, then $N = 0$. However if the curve encircles a hole in the material then $N$ need not be zero but must be an integer due to the quantization of flux. In an extreme type II Abrikosov vortex, a very small core is comprised of normal material. A single unit of magnetic flux of radius $\sim \lambda$ passes through the vortex. The fluxoid density $\vec{B} + \lambda^2 \vec{\nabla} \times \vec{J}$ is zero everywhere except in the region of the normal material. In the limit in which the core is a delta function we obtain:

$$\vec{B} + \lambda^2 \vec{\nabla} \times \vec{J} = \Phi_m \delta^2(x_\perp) \hat{n}_z. \quad (5)$$

Further if we use Ampere’s law we can get an analytic expression for the $B_z$ profile of a vortex.

$$B_z(r_\perp) - \lambda^2 \nabla^2_\perp B_z(r_\perp) = \Phi_m \delta^2(x_\perp); \quad B_z = \frac{\Phi_m}{2\pi\lambda^2} K_0(r_\perp/\lambda). \quad (6)$$

**B. Lattice Higgs in the London limit**

I would like to review the salient features of the lattice Higgs theory in the tree approximation. It is interesting to see how the the London relation arises as a direct consequence of the spontaneous breaking of the U(1) gauge theory. Also we compare the simulations of pure gauge theories with this model interpreted as a dual theory.
Consider the lattice action:

\[
S = \beta \sum_{x,\mu > \nu} (1 - \cos \theta_{\mu\nu}(x)) - \kappa \sum_{x,\mu} (\phi^* (x) e^{i\varphi_\mu(x)} \phi(x + e^{\mu}) + H.c.) + \sum_{x} V_{\text{Higgs}}(|\phi(x)|^2). \tag{7}
\]

We need a background field in order to form an Abrikosov vortex. To keep as close as possible to our simulations we generate the field from a monopole loop, i.e. the dual analog of the Wilson loop. Hence we will digress to define the monopoles on the lattice using the DeGrand-Toussaint \cite{6} construction.

Consider the unit 3-volume on the lattice at fixed \(x_4\) as shown in Fig.1. The link angles are compact, \(-\pi < \varphi_\mu \leq \pi\). The plaquette angle is also compact, \(-4\pi < \theta_{\mu\nu} \leq 4\pi\) and defined

\[
e a^2 F_{\mu\nu}(x) = \theta_{\mu\nu}(x) = \Delta^+_{\mu} \varphi_\nu(x) - \Delta^+_{\nu} \varphi_\mu(x), \tag{8}
\]

where \(a\) is the lattice spacing and the difference operators are defined in the appendix. This measures the electromagnetic flux through the face. Consider a configuration in which all link angles on this cube are small compared to \(\pi\). Gauss’ theorem applied to this cube then clearly gives zero total flux. Because of the 2\(\pi\) periodicity of the action we decompose the plaquette angle into two parts

\[
\theta_{\mu\nu}(x) = \bar{\theta}_{\mu\nu}(x) + 2\pi n_{\mu\nu}(x) \tag{9}
\]

where \(-\pi < \bar{\theta}_{\mu\nu} \leq \pi\). If the four angles shown in Fig.1 are adjusted so that e.g. \(\theta_{\mu\nu} > \pi\) then there is a discontinuous change in \(\bar{\theta}_{\mu\nu}\) by \(-2\pi\) and a compensating change in \(n_{\mu\nu}\). We can clearly choose the configuration that leaves the plaquette angles on all the other faces safely away from a discontinuity. We then define a Dirac string \(n_{\mu\nu}\) passing through this face (or better a Dirac sheet since the lattice is 4D). \(\bar{\theta}_{\mu\nu}\) measures the electromagnetic flux through the face. The important points are the following:

- The volume shown contains a magnetic monopole as indicated by a net flux of \(2\pi/e = e_m\).
Note that if one does a gauge transformation e.g. at the point $x + \epsilon^{(\mu) + \epsilon^{(\nu)}$ then if one of the affected links picks up a discontinuity of $2\pi$, the two contiguous plaquettes will be affected but the flux out of the volume is unchanged. The monopole is a gauge invariant construction.

This example created a monopole-antimonopole pair in the two space cubes separated by a lattice spacing in the $x_3$ direction. The same holds in the $x_4$ direction (not shown in fig.1).

In the dual description, cubes in the original lattice are represented by links on the dual lattice orthogonal to the 3-volume.

Hence for $n_{\mu\nu} = 1$, (a Dirac sheet threading the plaquette) a closed $1 \times 1$ monopole loop has been created on the dual lattice in the $x_3, x_4$ plane forming the boundary of a Dirac sheet.

The monopole loops are closed, hence the current is conserved.

This construction gives the following definition of the magnetic monopole current.

$$\frac{a^3}{e_m} J^m_\mu (x) = \epsilon_{\mu \nu \sigma \tau} \Delta^+_{\nu} \tilde{\theta}_{\sigma \tau} (x)$$ (10)

Which lives on the dual lattice. It satisfies the conservation law $\Delta^+_{\mu} J^m_\mu (x) = 0$.

We obtain a classical solution by minimizing the action. Our initial configuration contains a single closed magnetic monopole loop in the $x_3, x_4$ plane analogous to the Wilson loop projector in pure gauge simulations. This can be easily accomplished by choosing a configuration with a Dirac sheet threading all 1-2 plaquettes in a the $3-4$ plane that spans the desired monopole loop. Our algorithm then rejects all updated links that change the initial monopole configurations.

We accept configurations with Dirac sheets with no accompanying monopoles. A lump of Dirac sheets is rather easily created in a single local update. The update of a single link can produce six Dirac sheets in the plaquettes contiguous to this link. Each of these
produces a monopole loop on the dual lattice. One can see as shown in Fig. 2 that the six dual monopole loops form the faces of a cube on the dual lattice in such a way that all the monopole currents on the edges of the cube cancel.

We use the method of simulated annealing, slowly increasing \( \beta \), holding \( \lambda^2/a^2 = \beta/\kappa = 1/e^2\kappa \) constant, where \( \lambda/a \) is the London penetration depth in lattice units.

The electric current is given by

\[
\frac{a^3}{e\kappa} J^e_\mu(x) = Im(\phi^*(x)e^{i\phi_\mu(x)}\phi(x + \epsilon(\mu))).
\]  

Writing the Higgs field \( \phi(x) = \rho(x)e^{i\varphi(x)} \), the curl of the current is hence given by

\[
\frac{a^4}{e\kappa} \left( \Delta^+_\mu J^e_\mu(x) - \Delta^+_\nu J^e_\nu(x) \right) = \rho(x) \rho(x + \epsilon(\mu)) \sin[-\omega(x) + \varphi_\mu(x) + \omega(x + \epsilon(\mu))] + \rho(x + \epsilon(\mu)) \rho(x + \epsilon(\mu) + \epsilon(\nu)) \sin[-\omega(x + \epsilon(\mu)) + \varphi_\nu(x + \epsilon(\mu)) + \omega(x + \epsilon(\mu) + \epsilon(\nu))] - \rho(x + \epsilon(\nu)) \rho(x + \epsilon(\nu) + \epsilon(\mu)) \sin[-\omega(x + \epsilon(\nu)) + \varphi_\mu(x + \epsilon(\nu)) + \omega(x + \epsilon(\mu) + \epsilon(\nu))] - \rho(x) \rho(x + \epsilon(\nu)) \sin[-\omega(x) + \varphi_\nu(x) + \omega(x + \epsilon(\nu))].
\]  

(12)

Compare this with the electromagnetic field tensor

\[
e a^2 F_{\mu\nu} = \sin[\varphi_\mu(x) + \varphi_\nu(x + \epsilon(\mu)) - \varphi_\nu(x + \epsilon(\nu)) - \varphi_\mu(x)].
\]  

(13)

If the U(1) gauge symmetry is spontaneously broken, these two quantities are equal which is the London relation. To be more precise: if (i) \( \rho \) is nonvanishing and independent of position (absorb the normalization into \( \kappa \)) and (ii)

\[
\sin[\theta + 2N\pi] \approx \theta,
\]  

(14)

then

\[
F_{\mu\nu} \equiv F_{\mu\nu} - \frac{a^2}{e^2\kappa} \left( \Delta^+_\mu J^e_\mu(x) - \Delta^+_\nu J^e_\nu(x) \right) = \frac{2\pi N}{e} \frac{1}{a^2} = Ne_m \frac{1}{a^2}.
\]  

(15)

1 An apparent sign difference with Eqn. (2) is a notational one, see the appendix A2.
\mathcal{F}_{\mu \nu} a^2 \text{ summed over the transverse plane is called the fluxoid and hence } \mathcal{F}_{\mu \nu} \text{ is the fluxoid density. The magnetic charge } e_m \text{ is the Dirac monopole charge. Condition (i) is satisfied by freezing out the modulus of the Higgs field with the constraint Eqn. (1), and (ii) is satisfied if the variation of the field modulo } 2\pi N \text{ is much smaller than a lattice spacing. (Note that this singularity is not an isolated Dirac sheet. That would act differently, adding } 2\pi n \text{ to both terms in Eqn. (15)} \text{ and giving a zero contribution on the right.)}

Eqn. (15), together with Maxwell’s equations, gives the lattice version of the Meissner effect: \( \vec{B} - \lambda^2 \Delta \vec{B} = 0 \), where \( \lambda^2 = m^2 - 2 \gamma \); infinite conductivity \( \vec{E} = \lambda^2 \Delta^+ \vec{J} \) (assuming \( \rho = 0 \)); and an Abrikosov vortex

\[
B_z - \lambda^2 (\vec{\Delta}^+ \times \vec{J})_z = N \frac{e_m}{a^2} \delta^2 \varepsilon_{x,z,0}. \tag{16}
\]

Fig. 3 shows the profile of the R.H.S. of eqn. (12) and eqn. (13) in the directions perpendicular to the 5 x 5 magnetic monopole loop on a 12 x 1 lattice, with \( \beta/\kappa = 1 \). We used the constrained form of the Higgs potential, eqn. (1), which corresponds to an extreme type II superconductor. The graphs show the expected behavior: i.e. the equality of the two quantities everywhere except at \( r = 0 \) where they should differ by \( 2\pi \). There are significant violations only at \( r = 0 \) and smaller violations at \( r = 1 \) due to the breakdown of Eqn. (14). We anticipate that the violations will decrease with increasing loop size. This solution is a possible way to treat the complications that are ignored in the continuum London theory described in Sec. II A.

Note that there is a sign change in the \( \text{curl} J^e \) profile. Eqn. (1) shows that there must be a sign change since it must integrate to zero over the plane. The vortices have an exponential profile. All the current circulates about the core in the same sense. However since the profile falls faster than \( 1/r \), the line integral around a small patch at the origin will have the opposite sign than the line integral around a small patch elsewhere.

It is interesting to note that the interior surface spanned by the monopole loop is in the normal phase since the London relation is violated there. All other regions are superconducting. This translates in the following sections in which the Wilson loop projects out.
the normal phase in the plane spanned by the loop and the remaining regions are a dual superconductor.

We chose the Higgs field $|\phi(x)| = 1$ for this discussion. If instead the field was subjected to a SSB potential

$$V_{Higgs}(|\phi(x)|^2) = \lambda(|\phi(x)|^2 - v^2)^2,$$

there are important differences in the solution. In the region near a superconducting normal boundary, one expects a soliton like deformation of the Higgs field, constrained to vanish in the normal region and favoring a non-vanishing expectation value deep inside the superconducting side. This transition region defines the Ginzburg-Landau coherence length $\xi$, which is the distance over which the order parameter rises to its asymptotic value. For distances into the superconducting material deeper than the coherence length, one expects the London relation to be satisfied.

### III. DUAL SUPERCONDUCTIVITY IN PURE U(1) GAUGE THEORY

We now turn to the dynamical simulations in pure U(1) gauge theory given by

$$S = \beta \sum_{x,\mu > \nu} (1 - \cos \theta_{\mu\nu}(x)).$$

We have given evidence ref. [9] for the dual fluxoid density relation signaling a dual Abrikosov vortex:

$$^* F_{\mu\nu} \equiv F_{\mu\nu} - \lambda^2 \frac{(\Delta^-_\mu J^m_\nu(x) - \Delta^-_\nu J^m_\mu(x))}{a} = \frac{e}{a^2} \delta_{x\mu,0} \delta_{x\nu,0}.$$

(19)

where $^*F_{\mu\nu} = -\frac{1}{2} \epsilon_{\mu\nu\sigma\tau} F_{\sigma\tau}$. The magnetic monopole current is defined in Eqn.(10).

The correlation of $\vec{\Delta} \times \vec{J}^m$ with a Wilson loop gives this signal for the solenoidal behavior of the currents surrounding the electric flux between oppositely charged particles. The correlators for the electric field $\vec{E}$ and $\vec{\Delta}^- \times \vec{J}^m$ are

$$ea^2 E_z = -\frac{e^{i\theta_W} e^{i\theta_P}}{e^{i\theta_W}} = \frac{\sin \theta_W \sin \theta_P}{\cos \theta_W}$$

$$\frac{a^4}{e_m} (\vec{\Delta}^- \times \vec{J}^m)_z = -\frac{e^{i\theta_W} i (\vec{\Delta}^- \times \vec{J}^m_{lat})_z}{e^{i\theta_W}} = \frac{\sin \theta_W (\vec{\Delta}^- \times \vec{J}^m_{lat})_z}{\cos \theta_W}$$

(20)
The notation \((\cdots)_{\text{lat}}\) denotes an integer valued lattice quantity.

The identification of \(\sin \theta_P\) with the electric field is arbitrary, any quantity with the correct naive continuum limit will do. However Zach, Faber, Kainz and Skala made the interesting observation that this choice in conjunction with the Wilson action satisfies Gauss’ law [10].

For \(x_\mu\) out of the plane of the Wilson loop, the RHS of Eqn.\((19)\) vanishes and the London relation is satisfied. Hence we interpret the Wilson loop as a projector onto the normal phase in the plane bounded by the loop and a dual superconductor elsewhere.

Fig. 4 shows the lattice operators for the electric field and the curl of the magnetic monopole current. The longitudinal electric field, \((a)\), is given by a \(z,t\) plaquette which is depicted by a bold line for fixed time. The curl of the magnetic monopole current, \((b)\), is built from four 3-volumes which appear as squares since the time dimension is not shown. Passing through the center of each square is the link dual to the 3-volume. One takes the monopole number \(n\) in each 3-volume and associates the value \(ne_m\) with the corresponding dual link. The ‘line integral’ around the dual plaquette completes the picture. Notice from this construction that \(\vec{E}\) and \(\Delta^- \times \vec{J}_m\) take values at the same location within the unit cell of the lattice, both are indicated by the bold face line in the \(z\) direction. Both operators live on the same \(z,t\) plaquette. However \(\Delta^- \times \vec{J}_m\) is a hefty operator involving many links and as a consequence gives lattice artifacts for correlations at modest separations.

Fig. 5 shows the profile for the quantities making up the London relation on the same plot as for the effective lattice Higgs case, Fig. 3. We did 800 measurements on a \(12^4\) lattice, \(\beta = 0.95\), and a \(3 \times 3\) Wilson loop was used to project onto the \(q, \bar{q}\) sector using methods described in Ref. [9].

Fig. 6 shows the behavior of the dual fluxoid density. We did a \(\chi^2\) fit of the London relation, eqn.\((19)\), using the \(R \neq 0\) points and determined \(\lambda_d = 0.49(3)\). The fit strongly favors the small \(R\) points. That determined the \(R = 0\) value of the dual fluxoid in eqn.\((19)\), giving a prediction of \(e = 1.07(7)\), compared to the expected value \(1/\sqrt{\beta} = 1.03\). All these features correspond directly to the classical solution of the lattice Higgs model reinterpreted.
as a dual theory.

Our goal here is to learn how to measure parameters of the dual superconducting medium. The Wilson loop projects out a normal region giving a normal-dual superconducting boundary and a chance to discover a London relation in the spatially transient region on the dual superconducting side. The U(1) model has a rather small correlation length for $\beta \approx 1$ and it is difficult to probe large distances. One small improvement is to replace the Wilson loop projector by a plaquette and study the tails of the correlators, Eqn(20). We present some preliminary observations but this small scale simulation is inadequate to tie down a consistent determination of $\lambda_d$.

Fig. 7 shows such a plot. This represents 2000 measurements for $\beta = 1.0$ and for a plaquette projector. An approximately exponential fall off is expected. The curve is only to guide the eye, it is an eyeball fit to the tail of the electric field:

$$E(R) = E_0 e^{-R/R_0} \frac{R^3}{2}$$

where $E_0 = 0.2$ and $R_0 = 0.7$. We use the 4-dimensional behavior for the tail for these 'point like' operators.

One can read off the point by point determination of the dual London penetration depth from Fig. 7:

$$\lambda = \sqrt{\frac{E}{2\pi [\text{curl} J^m]}}$$

On axis, at $R = 0$ we have instead

$$\lambda = \sqrt{\frac{E}{2\pi [\text{curl} J^m E/(E - \Phi_m)]}}$$

and the quantity in square brackets is plotted instead of $[\text{curl} J^m(\Box)]$ in order that the $R = 0$ determination of $\lambda_d$ can be read from the graph the same as the $R \neq 0$ cases.

If we look along a lattice axis transverse to the projector plaquette, i.e. integer R, then $E/curl J^m \approx 1.0$ giving $\lambda_d \approx 0.4$. compared to the eyeball fit of 0.7. Off the lattice axis, $R$ noninteger, $E/curl J^m \approx 2.0$ giving $\lambda_d \approx 0.56$. We expect the point by point determinations
of $\lambda_d$ should be consistent with each other and with the exponential tail determination only at large distances. This small scale simulation is inadequate to make a thorough check of this.

A run of 2420 sweeps at $\beta = 0.95$ gave similar results. An eyeball fit to the $E$ data gave $E_0 = 0.4$ and $R_0 = 0.5$ which is consistent with the Wilson loop determination above. This agrees with the expected result that $\lambda_d$ decreases with decreasing $\beta$.

IV. DUALITY TRANSFORMATIONS IN U(1)

We arrived at the London relation in the U(1) theory purely numerically in Section III. This begs the question of whether the dual formulation of U(1) lattice gauge theory is equivalent to a lattice Higgs theory with a spontaneously broken gauge symmetry. Fr"olich and Marchetti [23] have found this to be the case which we review briefly in this section.

The duality transformation has not been performed for the Wilson form of the action which we are using. But this transformation has been implemented in closed form in the closely related Villain [20] form of the action by Fr"olich and Marchetti [23] and developed further by Polley and Wiese, and Polikarpov and others [24,25,26,27]. We draw on these later papers to gain more insight into the dual London relations.

Recent literature on this subject have found it very convenient to use the differential form notation adapted to the lattice. Polley and Wiese [24, Sec. 2.1] have given a brief introduction to the subject. The advantage of this formulation is that it forces one to focus on the k-volume cell on the lattice for objects with k indices. Further it dispenses with the indices and reduces the typical manipulations to some very simple general operator relations. As a supplement to Polley and Wiese’s introduction, we give the reader a concrete realization of the differential form algebra in the appendix A1 in order to make the transition to this notation a little easier.

Consider the Wilson action
\[ Z = \prod \int_{-\pi}^{\pi} d\varphi \exp\{ -\beta \sum \frac{1 - \cos d\varphi}{d\varphi} \}, \tag{24} \]

where the link angle \( \varphi \) is a 1-form and the plaquette \( d\varphi \) is a 2-form constructed with the exterior differential, \( d \). We can expand each plaquette in a Fourier series

\[ \exp\{ -\beta (1 - \cos z) \} = \sum_k e^{ikz} e^{-\beta I_k(\beta)}, \tag{25} \]

\[ \approx \sum_k e^{ikz} e^{-k^2/(2\beta)}, \tag{26} \]

where we have used the asymptotic form of the Bessel function \( I_k(\beta) \) for large \( \beta \) and further for large \( k \). This gives the Villain \textsuperscript{[20]} form of the action

\[ Z = \prod \int_{-\pi}^{\pi} d\varphi \prod \sum_k \exp\{ i(k, d\varphi) - \frac{1}{2\beta} \| k \|^2 \}, \tag{27} \]

where the inner product, \( (\Phi, \Psi) \), is defined in the appendix\textsuperscript{[A]} and \( \| \Phi \|^2 \equiv (\Phi, \Phi) \). We now use the property of the inner product, \( (k, d\varphi) = (\delta k, \varphi) \). We can now integrate over the links \( \varphi \) and we obtain

\[ Z = \prod \sum_k \exp\{ -\frac{1}{2\beta} \| k \|^2 \} \Big|_{\delta k = 0}. \tag{28} \]

The variable \( k \) is an integer valued 2-form analogous to the field tensor \( F_{\mu \nu} \) and the constraint \( \delta k = 0 \) is analogous to \( \partial_\mu F_{\mu \nu} = J_\nu^e = 0 \). Next, using the general operator relation \( \delta^2 = 0 \) we solve the constraint by introducing

\[ k = \delta p, \tag{29} \]

\[ Z = \prod \sum_p \exp\{ -\frac{1}{2\beta} \| \delta p \|^2 \} = \prod \sum_{*p} \exp\{ -\frac{1}{2\beta} \| d^*p \|^2 \}. \tag{30} \]

The variable \( p \) is a 3-form analogous to a dual potential written as an antisymmetric three index object, \( B_{\alpha \beta \gamma} \) and Eqn.\textsuperscript{(29)} is analogous to \( F_{\mu \nu} = \partial_\alpha B_{\alpha \mu \nu} \). If we switch to the dual objects this last result looks much more familiar: \( *p \) is a 1-form, \( *k = d^*p \iff *F_{\mu \nu} = \partial_\mu^*B_\nu - \partial_\nu^*B_\mu \).

In order to verify this interpretation we should introduce external sources and observe how they couple to the fields. Equation \textsuperscript{(30)} naively looks like a free field theory, except
that the variables are integer valued and this indeed makes it an interacting theory. One can trace the integer value to the fact that the link angles $\varphi$ are compact and hence their Fourier transform is a discrete variable. But this compactness is also responsible for the existence of monopoles.

Our immediate concern in this paper is to show that this interacting theory given by Eqn.(30) is in fact a dual lattice Higgs model [23]. It is interesting how this comes about and gives us some insight into the significance of an integer valued dual gauge field.

Following references [23,24] we introduce a non-compact 1-form gauge field $*A$ on the dual links. We choose the Higgs field to be compact 0-form $*\Psi = \exp\{i*\chi\}$ on the dual sites constrained on the unit circle. Finally we introduce an integer valued 0-form $*\ell$ on the dual sites

$$Z = \prod \int_{-\infty}^{\infty} d^*A \prod \int_{-\pi}^{\pi} d^*\chi \prod \sum_{*\ell} \exp\left\{ -\frac{1}{2e_m^2} \| d^*A \|^2 - \frac{\kappa_d}{2} \| d^*\chi + 2\pi*\ell - *A \|^2 \right\}. \quad (31)$$

This differs from the lattice Higgs action, Eqn.(44), in two respects: (i) The gauge field is non-compact, and (ii) the Villain [20] form is employed. The variables $*\ell$ are introduced in order to make the action periodic in $*\chi$.

To connect this theory to Eqn.(30) we first choose a gauge for which $*\chi = 0$. Then note that for large $\kappa_d$ the integral over $*A$ is peaked at the values $*A = 2\pi*\ell$. In the limit $\kappa_d \to \infty$ this becomes

$$Z = \prod \sum_{*\ell} \exp\left\{ -\frac{(2\pi)^2}{2e_m^2} \| d^*\ell \|^2 \right\}. \quad (32)$$

Identifying $*\ell$ with $*p$ this is the same as Eqn.(44) with $1/\beta \equiv e^2 = (2\pi)^2/e_m^2$.

Following Sec.III the dual London penetration depth in this theory is given by $\lambda_d^2/a^2 = 1/(m_s a)^2 = 1/(e_m^2 \kappa_d)$. Therefore in the weak magnetic coupling limit, $e_m \to 0$, the U(1) gauge theory with the Villain form of the action becomes a dual Higgs theory with an infinite photon mass, i.e. zero London penetration depth, $\lambda_d$. We find in our simulations that $\lambda_d$ decreases for decreasing $\beta$. But we have only measured it for a very limited range of $0.9 < \beta < 1.0$.  

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The dual Higgs field, $*\Psi$ has been connected to the monopole field operator $^{[23,24]}$. Polikarpov, Polley and Wiese $^{[25]}$ have calculated the constraint effective potential for this operator and showed that there is a global symmetry breaking in the dual superconducting phase.

V. GENERALIZATION TO PURE SU(2) AND SU(3) GAUGE THEORIES

In ref. $^{[11]}$ we applied these same techniques and showed that dual Abrikosov vortices also occur in SU(2) pure gauge theory in the maximal Abelian gauge. Matsubara et.al. $^{[12]}$ confirmed these results with better statistics and generalized them to SU(3). The SU(2) link matrices are $U_\mu(x)$ and the action is

$$S = \beta \sum_{x,\mu,\nu} (1 - \frac{1}{2} Tr U_{\mu\nu}(x)).$$

(33)

The maximal Abelian gauge is defined by maximizing the quantity

$$R = \sum_{x,\mu} Tr [\sigma_3 U_\mu(x) \sigma_3 U^\dagger_\mu(x)].$$

(34)

The Abelian link angle is then taken as the phase of $[U_\mu(x)]_{11}$ and the calculation can proceed with little change $^{[30,31]}$.

Recently we have done the calculation at finite temperature in order to check this picture on each side of the deconfining phase transition $^{[13]}$ which we report here. The results are shown in the confining phase, fig.8, $\beta = 2.28$ and the deconfining phase, fig.9, $\beta = 2.40$ on a lattice $4 \times 17^2 \times 19$, with 800 measurements for each case. Gauge fixing required about 600 sweeps for each configuration.

Fig.8 shows an important difference from the U(1) case. Whereas in the U(1) case a linear combination of $E$ and $\text{curl} J^m$ for $r \neq 0$ can vanish giving the London relation, it is clearly not possible here. The behavior of $-\text{curl} J^m$ does not match that of $E$. We interpret this discrepancy as a signal of a non-zero Ginzburg-Landau coherence length, $\xi_d$. Unlike the U(1) case, the extreme type II limit, in which the superconducting order parameter turns
on at the surface, the evidence here is that the order parameter turns on over a distance $\xi_d$. The value of the coherence length is approximately the radius where the London relation is restored.

The analysis in this more general situation precludes a point by point comparison of the $E$ and $\text{curl} J^m$ data. We adapt the analysis from Tinkham [4].

In terms of a Higgs model, the Higgs field is not constrained to a certain vacuum expectation value but rather is subjected to a SSB potential Eqn.(17). The equivalent GL theory provides a way to calculate the deformation of the Higgs field $\phi(x)$ between the normal core of a vortex and the asymptotic value at large distance from the vortex, $\phi_\infty$. We write $\phi(x) = \phi_\infty f(r) \exp i\alpha(x)$, where $f(R)$ rises from 0 → 1 over a distance $\xi_d$. Introducing a dual vector potential, $\vec{A}_d$, the dual London equation is replaced by the dual GL equations for the azimuthal component of the monopole current.

$$\vec{E} = \vec{\nabla} \times \vec{A}_d,$$

$$\vec{J}_m = \frac{f^2(r)}{\lambda_d^2}(\vec{A}_d - \frac{\Phi_e}{2\pi} \vec{\nabla} \alpha), \quad (35)$$

and $f$ satisfies

$$-\xi_d^2 \nabla^2 f + \xi_d^2(\vec{\nabla} \alpha - \frac{2\pi}{\Phi_e} \vec{A}_d)^2 f - f + f^3 = 0. \quad (36)$$

The generalized fluxoid relation becomes

$$\vec{E}(r) - \lambda_d^2 \vec{\nabla} \times \left( \frac{\vec{J}_m(r)}{f(r)^2} \right) = N\Phi_e \delta(x_\perp). \quad (37)$$

An approximate solution of Eqn.(35) is

$$f(r) = \tanh(\nu r / \xi_d), \quad (38)$$

with $\nu \approx 1$.

An interpolation of the data is required to fit Eqn.(37). An example of the fit for $\beta = 2.4$ in Ref. [11] is $\lambda_d/a = 1.05(12)$ and $\xi_d = 1.35(11)$.

The behavior of all the SU(2) and SU(3) examples in the confined phase is similar to that shown in fig.8. The interesting conclusion is that
\[ \kappa_d \equiv \frac{\lambda_d}{\xi_d} \approx 1. \] (39)

(no relation to Higgs \( \kappa \)) For a type II dual superconductor \( \kappa_d > 1/\sqrt{2} \) and a type I otherwise. These simulations indicate that the non-Abelian dual superconductors lie at the borderline between type I and type II.

Matsubara, Ejiri and Suzuki [12] have measured \( \kappa_d \) for SU(2) for \( 3 \times 3 \) and \( 5 \times 5 \) loops in the range \( 2.4 < \beta < 2.6 \); and for SU(3) for \( 3 \times 3 \) loop in the range \( 5.6 < \beta < 5.9 \). Although they are not asymptotic, they find clear signals in these data for \( \kappa_d \) both larger and smaller than \( 1/\sqrt{2} \).

The simulation [13] shown in fig.8 was identical to fig.8 except that \( \beta \) was increased to 2.40 on a lattice \( 17^2 \times 19 \times 4 \) putting us in the deconfining phase. The dramatic decrease of \( \text{curl} J^m \) results in the failure of the dual superconductor interpretation as expected for the deconfining phase. Also \( E \) falls more slowly with radius.

A further observation is that although this dual superconductivity signal drops rapidly above the deconfining temperature, the monopole density does not change dramatically. The monopole density \( \rho \) is defined

\[ \rho = \frac{\langle \sum_{x,\mu} |J^m_{\mu \text{lat}}| \rangle}{4N_{\text{sites}}} \] (40)

where \( |J^m_{\mu \text{lat}}| \) is the integer valued lattice quantities and \( N_{\text{sites}} \) is the number of lattice sites. In Ref. [13] we find

\[ \begin{align*}
\text{confining} : & \quad \beta = 2.28 \quad \rho = 0.0567(1) \\
\text{deconfining} : & \quad \beta = 2.40 \quad \rho = 0.0213(1)
\end{align*} \]

**VI. EXTERNAL ELECTRIC FIELD**

Many interesting superconducting properties can be elucidated by studying the properties of the material in the presence of a magnetic field. We report a preliminary look at the corresponding problem of dual superconductivity in the presence of an external electric
field. The Wilson loop provides such a field by projecting a $q\bar{q}$ out of vacuum configurations. But it may be interesting to see the spontaneous breaking of translation invariance as the electric flux forms a vortex rather than impose the vortex position at a given location and with quantized flux. Type I and type II dual superconductors respond very differently to background uniform external fields.

For a periodic U(1) lattice the sum of the plaquette angles over any plane is identically zero. Referring to Fig.10 a classical uniform electric field for U(1) on a particular time slice can still be obtained by constraining one plaquette angle in each $z,t$ plane to the value $(1/(N_zN_t) - 1)\theta_c$. Then the remaining $(N_zN_t - 1)$ plaquette angles will take the common value $\theta_c/(N_zN_t) = ea^2F_{34}$ giving a uniform electric field on all but one time slice. (N labels the lattice dimensions.) Choosing $\theta_c = \pi$ gives the largest field. For an $8^4$ lattice and $\beta = 1$ for example, there would be enough electric flux to form about 3 vortices in the $x,y$ plane in the dual superconducting phase. This field configuration can also be obtained by multiplying the plaquette that was singled out in the action by a minus sign and such configurations have been studied in non-Abelian theories [47]. We avoided an alternative method of imposing an external field by introducing a non-zero equilibrium value locally for each plaquette angle since we eventually want to see the field break translation invariance. Turning on interactions for $\beta < 1$ brings up other interesting features [48].

Our goal here is to try to see a signal showing that $\text{curl}J^m$ responds to the external field. The immediate problem is that the sum of $(\text{curl}J^m)_{xy}$ over the any $x,y$ plane is identically zero, and the sum of $E_z$ is not. A sample configuration is given in Fig.11. Yet we expect the London relation to be satisfied. The only possibility is that translation invariance is broken which is expected since vortices segregate the superconducting and normal phases.

We make the following rough ansatz that the local London relation is due to a the alignment of local current loops in the external field. This suggests that we truncate $\text{curl}J^m$ to include only values $\pm 2, \pm 3, \pm 4$ representing the winding of the current around the dual plaquette. We denote this $J^m_{\mu}(2 + 3 + 4)$. In this discussion we assume that the current $J^m_{\mu}$ takes only the values $-1, 0, 1$. The fractional population with values outside this range is
small enough to be neglected.

Fig. 11 is a typical configuration for the confined phase. A value $\pm 4$ indicates that the current makes a complete loop, the value $\pm 3$ a partial loop, etc. The value $\pm 1$ corresponds to a single isolated straight current segment in the plane. One expects the local behavior of $\pm 4, \pm 3, \pm 2$ contributions to align themselves in the local external field. But there is no comparable local behavior for the $\pm 1$ contributions. Globally of course they must all sum to zero.

On a large lattice where size effects are irrelevant, one can still expect $\text{curl} \, J_m$ to average to zero. This is precisely the behavior in an isolated vortex as discussed in Sec. II B.

Now we get a large signal for $\text{curl} \, J^m_\mu (2 + 3 + 4)$ as shown in fig. 12 with a sign that agrees with the sign in fig. 5 for the superconducting region, $r \neq 0$. Further if we recalculate fig. 5 with the truncated current, we find a large suppression at $r = 0$ and a moderate enhancement at the other points as shown in Fig. 13. In other words this choice biases in favor of the dual superconducting phase.

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APPENDIX A: NOTATION CONVENTIONS

1. Lattice differential forms

The language of differential forms on the lattice is particularly useful for the Villain form of the action. Frölich and Marchetti use this language in the mathematical physics literature. Polley and Wiese Sec. 2.1] have given a nice introduction to the subject. This
appendix is meant only as a supplement to that introduction by relating the differential form operators to ordinary lattice finite difference operators. The differential form operators have coordinate independent definitions [24]. The correspondences in this appendix allows one to check the algebra of differential forms on a specific coordinate system naturally defined by the hypercubic lattice.

The lattice forward and backward difference operators are defined

\[ \Delta_+^\mu \Phi(x) \equiv \Phi(x + \epsilon^\mu(x)) - \Phi(x), \quad \text{(A1)} \]
\[ \Delta_-^\mu \Phi(x) \equiv \Phi(x) - \Phi(x - \epsilon^\mu(x)). \quad \text{(A2)} \]

We offer the following examples of the use of the exterior differential d, and the codifferential \( \delta \) acting on objects defined on the original (rather than the dual) lattice.

0-form defined on points \( \Phi(c_0) \leftrightarrow \Phi(x) \):

\[ d\Phi \leftrightarrow \Delta_+^\mu \Phi(x), \quad \text{(A3)} \]
\[ \delta\Phi \leftrightarrow 0. \quad \text{(A4)} \]

1-form defined on links \( \Phi(c_1) \leftrightarrow \Phi_\mu(x) \):

\[ d\Phi \leftrightarrow \Delta_+^\mu \Phi_\nu(x) - \Delta_+^\nu \Phi_\mu(x), \quad \text{(A5)} \]
\[ \delta\Phi \leftrightarrow \Delta_-^\mu \Phi_\mu(x). \quad \text{(A6)} \]

2-form defined on plaquettes \( \Phi(c_2) \leftrightarrow \Phi_{\mu\nu}(x) \) (antisymmetric):

\[ d\Phi \leftrightarrow \Delta_+^{\alpha} \Phi_{\mu\nu}(x) \equiv \Delta_+^\alpha \Phi_{\mu\nu}(x) - \Delta_+^\nu \Phi_{\alpha\mu}(x) - \Delta_+^\mu \Phi_{\alpha\nu}(x), \quad \text{(A7)} \]
\[ \delta\Phi \leftrightarrow \Delta_-^\mu \Phi_{\mu\nu}(x). \quad \text{(A8)} \]

3-form defined on 3-volume \( \Phi(c_3) \leftrightarrow \Phi_{\mu\nu\lambda}(x) \) (completely antisymmetric):

\[ d\Phi \leftrightarrow \Delta_+^{[\alpha} \Phi_{\mu\nu\lambda]}(x) \equiv \Delta_+^\alpha \Phi_{\mu\nu\lambda}(x) - \Delta_+^\nu \Phi_{\alpha\mu\lambda}(x) - \Delta_+^\mu \Phi_{\alpha\nu\lambda}(x) - \Delta_+^\lambda \Phi_{\mu\nu\alpha}(x), \quad \text{(A9)} \]
\[ \delta\Phi \leftrightarrow \Delta_-^\mu \Phi_{\mu\nu\lambda}(x). \quad \text{(A10)} \]
4-form defined on 4-volume $\Phi(c_4) \iff \Phi_{\mu\nu\lambda\tau}(x)$ (completely antisymmetric):

\begin{align*}
  d\Phi & \iff 0, \quad (A11) \\
  \delta\Phi & \iff \Delta^-\Phi_{\mu\nu\lambda\tau}(x). \quad (A12)
\end{align*}

Miscellaneous relations on k-forms:

\begin{align*}
  \delta^2 & = 0, \quad (A13) \\
  d^2 & = 0, \quad (A14) \\
  \delta d + d\delta & = \Delta \iff \Delta^-\Delta^+, \quad (A15) \\
  \Phi & = (\delta\Delta^{-1}d + d\Delta^{-1}\delta)\Phi, \quad (A16) \\
  (\Phi, \Psi) & \iff \sum_{x,\mu>\nu\ldots} \Phi_{\mu\nu\ldots}(x)\Psi_{\mu\nu\ldots}(x), \quad (A17) \\
  (\Phi, d\Psi) & = (\delta\Phi, \Psi). \quad (A18)
\end{align*}

Note that the two arguments of the inner product must of course live on the same k-form.

There is a completely equivalent description of these k-forms, $\Phi(c_k)$, in terms of $(4-k)$-forms on the dual lattice $\Phi(c_k)$.

\begin{align*}
  \Phi(c_k) = \Phi(c_k). \quad (A19)
\end{align*}

This is realized in conventional notation by contracting $\epsilon_{\mu\nu\lambda\tau}$ into the object defined on the original lattice to obtain the object on the dual lattice, properly normalized so that acting twice is the identity operation.

Finally we wish to note a correspondence involving $d$ and $\delta$. Given the equivalence Eqn.(A19) then there is a correspondence between the $(k \pm 1)$-form and dual $(4 - (k \pm 1))$-form

\begin{align*}
  d\Phi & = \delta^*\Phi, \\
  \delta\Phi & = d^*\Phi. \quad (A20)
\end{align*}

On the original lattice the position $x$ refers to the ‘lower left etc.’ corner of a k-form. (Links point out of their coordinate address.) This dictates the choice of $\Delta^+$ and $\Delta^-$ in
these examples. Since the dual lattice is displaced a half spacing in the positive direction, the ‘upper right etc.’ corner is used to label the position $x$ of the dual $k$-forms. (Dual links point in to their coordinate address.) As a consequence, when the above rules are applied to $k$-forms on the dual lattice, $\Delta^+$ and $\Delta^-$ should be interchanged.

2. Euclidean Maxwell equations

We have taken the following sign conventions for the Euclidean field tensors with electric and magnetic sources.

$$\partial_\mu F_{\mu\nu} = J^e_\nu, \quad * F_{\alpha\beta} = -\frac{1}{2} \epsilon_{\alpha\beta\mu\nu} F_{\mu\nu}, \quad \epsilon_{1234} = 1, \quad \partial_\mu * F_{\mu\nu} = J^m_\nu,$$

$$\mathbf{\vec{\nabla}} \cdot \mathbf{E} = J^e_4, \quad \mathbf{\vec{\nabla}} \times \mathbf{B} - \partial_4 \mathbf{E} = \mathbf{J}^e,$$

$$\mathbf{\vec{\nabla}} \cdot \mathbf{B} = J^m_4, \quad \mathbf{\vec{\nabla}} \times \mathbf{E} - \partial_4 \mathbf{B} = \mathbf{J}^m. \quad (A21)$$

For $J^m_\mu = 0$ and with the standard definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we get the unconventional sign $\mathbf{B} = -\mathbf{\vec{\nabla}} \times \mathbf{A}$.
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FIGURES

FIG. 1. DeGrand-Toussaint construction to identify a monopole in a spacial 3-volume.

FIG. 2. (a) Six space-time plaquettes (time not shown) contiguous to a time-like link. (b) Six space-space plaquettes dual to those in (a) forming a cube.

FIG. 3. Profile of $ea^2 F_{12}(x)$, Eqn.(13), and $-(a^3/e\kappa)(\Delta^+ J_2^x(x) - \Delta^+ J_1^x(x))$, Eqn.(12), as a function of distance perpendicular to the plane of the monopole current loop in lattice units.

FIG. 4. Operators for (a) $ea^2 F_{34}(x)$, and (b) $(a^3/e_m)(\Delta^+ J_2^m(x) - \Delta^+ J_1^m(x))$.

FIG. 5. Profile of $ea^2 F_{34}(x)$ and $-(a^3/e_m)(\Delta^+ J_2^m(x) - \Delta^+ J_1^m(x))$ as a function of distance perpendicular to the plane of the Wilson loop in lattice units for U(1).

FIG. 6. Profile of $ea^2 F_{34}(x) - \lambda_3^2(a^3/e_m)(\Delta^+ J_2^m(x) - \Delta^+ J_1^m(x))$ as a function of distance perpendicular to the plane of the Wilson loop projector in lattice units for U(1).

FIG. 7. Profile of $ea^2 F_{34}(x)$ and $(a^3/e_m)(\Delta^+ J_2^m(x) - \Delta^+ J_1^m(x))$ as a function of distance perpendicular to the plane of the plaquette projector in lattice units. The on-axis point for $\text{curl}J^m < 0$ is ‘corrected’, see text.

FIG. 8. Profile of $ea^2 F_{34}(x)$ and $-(a^3/e_m)(\Delta^+ J_2^m(x) - \Delta^+ J_1^m(x))$ as a function of distance perpendicular to the plane of the Polyakov lines separated by $R = 3$ in lattice units for SU(2) in the confined phase.

FIG. 9. Profile of $ea^2 F_{34}(x)$ and $-(a^3/e_m)(\Delta^+ J_2^m(x) - \Delta^+ J_1^m(x))$ as a function of distance perpendicular to the plane of the Polyakov lines separated by $R = 3$ in lattice units for SU(2) in the unconfined phase.

FIG. 10. Plaquette angles for a classical external field configuration.
FIG. 11. Sample values of the $z$ component of $\text{curl}J^m_{\text{lat}}$ on an $x, y$ plane for fixed $z$ and $t$ on a $12^4$ lattice, $\beta = 0.99$.

FIG. 12. Average of $ea^2F_{34}$ and $-(a^3/e_m)(\Delta_1 J^m_2(2 + 3 + 4) - \Delta_2 J^m_1(2 + 3 + 4))$ on each time slice for an external field $= \pi/64$ (the horizontal line). The constrained plaquette is at $t = 1$, the field is classical at $t = 2, 8$ and $\beta = .99$ for $t = 3 - 7$.

FIG. 13. Profile of $-(a^3/e_m)(\Delta_1 J^m_2(x) - \Delta_2 J^m_1(x))(2 + 3 + 4)$ as a function of distance perpendicular to the plane of the Wilson loop in lattice units for U(1) superimposed on Fig.\ref{fig:fig11}.
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