Abelianizing vertex algebras

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Abstract

To every vertex algebra \( V \) we associate a canonical decreasing sequence of subspaces and prove that the associated graded vector space \( \text{gr}(V) \) is naturally a vertex Poisson algebra, in particular a commutative vertex algebra. We establish a relation between this decreasing sequence and the sequence \( C_n \) introduced by Zhu. By using the (classical) algebra \( \text{gr}(V) \), we prove that for any vertex algebra \( V \), \( C_2 \)-cofiniteness implies \( C_n \)-cofiniteness for all \( n \geq 2 \). We further use \( \text{gr}(V) \) to study generating subspaces of certain types for lower truncated \( \mathbb{Z} \)-graded vertex algebras.

1 Introduction

Just as with classical (associative or Lie) algebras, abelian or commutative vertex algebras (should be) are the simplest objects in the category of vertex algebras. It was known (see [B]) that commutative vertex algebras exactly amount to differential algebras, namely unital commutative associative algebras equipped with a derivation. Related to the notion of commutative vertex algebra, is the notion of vertex Poisson algebra (see [FB]), where a vertex Poisson algebra structure combines a commutative vertex algebra structure, or equivalently, a differential algebra structure, with a vertex Lie algebra structure (see [K], [P]). As it was showed in [FB], vertex Poisson algebras can be considered as classical limits of vertex algebras.

In the classical theory, a well known method to abelianize an associative algebra is to use a good increasing filtration and then consider the associated graded vector space. A typical example is about the universal enveloping algebra \( U(\mathfrak{g}) \) of a Lie algebra \( \mathfrak{g} \) with the filtration \( \{U_n\} \), where for \( n \geq 0 \), \( U_n \) is linearly spanned by the vectors \( a_1 \cdots a_m \) for \( m \leq n \), \( a_1, \ldots, a_m \in \mathfrak{g} \). In this case, the associated graded Poisson algebra \( \text{gr}U(\mathfrak{g}) \) is naturally a Poisson algebra and the well known Poincaré-Birkhoff-Witt theorem says that the associated graded Poisson algebra \( \text{gr}U(\mathfrak{g}) \) is canonically isomorphic to the symmetric algebra \( S(\mathfrak{g}) \) which is also a Poisson algebra. This result and the canonical isomorphism have played a very important role in Lie theory.

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Motivated by this classical result, in [Li2] we introduced and studied a notion of what we called good increasing filtration for a vertex algebra $V$ and we proved that the associated graded vector space $gr V$ of $V$ with respect to a good increasing filtration is naturally a vertex Poisson algebra. Furthermore, for any $\mathbb{N}$-graded vertex algebra $V = \bigsqcup_{n \in \mathbb{N}} V(n)$ with $V(0) = \mathbb{C}1$, we constructed a canonical good increasing filtration of $V$. This increasing filtration was essentially used in [KL], [NG], [Bu1,2], [ABD] and [NT] in the study on generating subspaces of $V$ with a certain property analogous to the well known Poincaré-Birkhoff-Witt spanning property.

In this paper, we introduce and study “good” decreasing filtrations for vertex algebras. To any vertex algebra $V$ we associate a canonical decreasing sequence $E$ of subspaces $E_n$ for $n \geq 0$ and we prove that the associated graded vector space $gr E(V)$ is naturally an $\mathbb{N}$-graded vertex Poisson algebra, where for $n \in \mathbb{Z}$, $E_n$ is linearly spanned by the vectors

$$u_{-1-k_1}^{(1)} \cdots u_{-1-k_r}^{(r)} v$$

for $r \geq 1$, $u^{(i)}, v \in V$, $k_i \geq 0$ with $k_1 + \cdots + k_r \geq n$. Notice that unlike the increasing filtration which uses the weight grading, this decreasing sequence uses only the vertex algebra structure.

For any vertex algebra $V$, there has been a fairly well known decreasing sequence $C = \{C_n\}_{n \geq 2}$ introduced by Zhu [Z1,2], where for $n \geq 2$, $C_n$ is linearly spanned by the vectors $u_{-n}v$ for $u, v \in V$. The notion of $C_2$ was introduced and used in the fundamental study of Zhu on modular invariance, where the finiteness of $\dim V/C_2$ played a crucial role. It was showed in [Z2] that $V/C_2$ has a natural Poisson algebra structure.

In this paper, we relate our decreasing sequence $E$ with Zhu’s sequence $C$. In particular, we show that $C_2 = E_1$ and $C_3 = E_2$. We then show that the degree zero subspace $E_0/E_1$ of $gr E(V)$, which is naturally a Poisson algebra, is exactly the Zhu’s Poisson algebra $V/C_2$. We further show that $gr E(V)$ as a differential algebra is generated by the degree zero subalgebra $V/C_2$. As an application, we show that for any vertex algebra $V$, if $V$ is $C_2$-cofinite, then $V$ is $E_n$-cofinite and $C_{n+2}$-cofinite for all $n \geq 0$. Similarly we show that if $V$ is a $C_2$-cofinite vertex algebra and if $W$ is a $C_2$-cofinite $V$-module, then $W$ is $C_{n}$-cofinite for all $n \geq 2$.

Under the assumption that $V$ is an $\mathbb{N}$-graded vertex algebra with $\dim V(0) = 1$, it has been proved before by [GN] (see also [NT], [Bu1,2]) that $C_2$-cofiniteness implies $C_{n+2}$-cofiniteness for all $n \geq 0$. On the other hand, the original method of [GN] and [KL] used this assumption in an essential way.

As we show in this paper, for certain vertex algebras, both sequences $E$ and $C$ are trivial in the sense that $E_n = C_{n+2} = V$ for all $n \geq 0$. On the other hand, by using the connection between the two decreasing sequences we prove that if $V = \bigsqcup_{n \geq t} V(n)$
is a lower truncated $\mathbb{Z}$-graded vertex algebra such as a vertex operator algebra in the sense of [FLM] and [FHL], then for any $k$, $C_n, E_n \subset \bigsqcup_{m \geq k} V_m$ for $n$ sufficiently large. Consequently, $\bigcap_{n \geq 0} E_n = \bigcap_{n \geq 0} C_{n+2} = 0$. (In this case, both sequences are filtrations.) Furthermore, using this result and $\text{gr}_E(V)$ we show that if a graded subspace $U$ of $V$ gives rise to a generating subspace of $V/C_2$ as an algebra, then $U$ generates $V$ with a certain spanning property. Similar results have been obtained before in [KL], [NG], [Bu1,2] and [NT] under a stronger condition.

This paper is organized as follows: In Section 2, we define the sequence $\mathcal{E}$ and show that the associated graded vector space is an $\mathbb{N}$-graded vertex Poisson algebra. In Section 3, we relate the sequences $\mathcal{E}$ and $\mathcal{C}$. In Section 4, we study generating subspaces of certain types for lower truncated $\mathbb{Z}$-graded vertex algebras.

## 2 Decreasing sequence $\mathcal{E}$ and the vertex Poisson algebra $\text{gr}_E(V)$

In this section we first recall the definition of a vertex Poisson algebra from [FB] and we then construct a canonical decreasing sequence $\mathcal{E}$ for each vertex algebra $V$ and show that the associated graded vector space $\text{gr}_E(V)$ is naturally a vertex Poisson algebra. We also show that if $V$ is an $\mathbb{N}$-graded vertex algebra, then the sequence $\mathcal{E}$ is indeed a filtration of $V$.

Let $V$ be a vertex algebra. We have Borcherds’ commutator formula and iterate formula:

\[ [u_m, v_n] = \sum_{i \geq 0} \binom{m}{i} (u_i v)_{m+n-i}, \]

\[ (u_m v)_n w = \sum_{i \geq 0} (-1)^i \binom{m}{i} (u_{m-i} v_{n+i} w - (-1)^m v_{m+n-i} u_i w) \]

for $u, v, w \in V, m, n \in \mathbb{Z}$. Define a (canonical) linear operator $D$ on $V$ by

\[ D(v) = v_{-2}1 \quad \text{for } v \in V. \]

Then

\[ Y(v, x)1 = e^{xD}v \quad \text{for } v \in V. \]

Furthermore,

\[ [D, v_n] = (Dv)_n = -nv_{n-1} \]

for $v \in V, n \in \mathbb{Z}$. (See for example [LL] for an exposition of such facts.)
A vertex algebra $V$ is called a **commutative vertex algebra** if
\[ [u_m, v_n] = 0 \text{ for } u, v \in V, \ m, n \in \mathbb{Z}. \] (2.6)

It is well known (see [B], [FHL]) that (2.6) is equivalent to that
\[ u_n = 0 \text{ for } u \in V, \ n \geq 0. \] (2.7)

**Remark 2.1.** Let $A$ be any unital commutative associative algebra with a derivation $d$. Then one has a commutative vertex algebra structure on $A$ with
\[ Y(a, x)b = (e^{x d} a)b \text{ for } a, b \in A \] and with the identity 1 as the vacuum vector (see [B]). On the other hand, let $V$ be any commutative vertex algebra. Then $V$ is naturally a commutative associative algebra with $u \cdot v = u_{-1} v$ for $u, v \in V$ and with 1 as the identity and with $D$ as a derivation. Furthermore, $Y(u, x)v = (e^{x D} u)v$ for $u, v \in V$. Therefore, a commutative vertex algebra exactly amounts to a unital commutative associative algebra equipped with a derivation, which is often called a **differential algebra**.

A vertex algebra $V$ equipped with a $\mathbb{Z}$-grading $V = \bigsqcup_{n \in \mathbb{Z}} V(n)$ is called a **$\mathbb{Z}$-graded vertex algebra** if $1 \in V(0)$ and if for $u \in V(k)$ with $k \in \mathbb{Z}$ and for $m, n \in \mathbb{Z}$,
\[ u_m V(n) \subset V(n + k - m - 1). \] (2.8)

We say that a $\mathbb{Z}$-graded vertex algebra $V = \bigsqcup_{n \in \mathbb{Z}} V(n)$ is **lower truncated** if $V(n) = 0$ for $n$ sufficiently small. In particular, every vertex operator algebra in the sense of [FLM] and [FHL] is a lower truncated $\mathbb{Z}$-graded vertex algebra. An $\mathbb{N}$-graded vertex algebra is defined in the obvious way. We say that a vertex algebra $V$ is **$\mathbb{Z}$-gradable** ($\mathbb{N}$-gradable) if there exists a $\mathbb{Z}$-grading ($\mathbb{N}$-grading) such that $V$ becomes $\mathbb{Z}$-graded ($\mathbb{N}$-graded) vertex algebra. We see that a commutative $\mathbb{Z}$-graded vertex algebra is naturally a $\mathbb{Z}$-graded differential algebra.

The following definition of the notion of vertex Lie algebra is due to [K] and [P]:

**Definition 2.2.** A **vertex Lie algebra** is a vector space $V$ equipped with a linear operator $D$ and a linear map
\[ Y_- : V \to \text{Hom} \left( V, x^{-1} V[x^{-1}] \right), \]
\[ v \mapsto Y_-(v, x) = \sum_{n \geq 0} v_n x^{-n-1} \] (2.9)

such that for $u, v \in V, \ m, n \in \mathbb{N}$,
\[ (Dv)_n = -nv_{n-1}, \] (2.10)
\[ u_m v = \sum_{i=0}^{m} (-1)^{m+i+1} \frac{1}{i!} D^i v_{m+i} u, \] (2.11)
\[ [u_m, v_n] = \sum_{i=0}^{m} \binom{m}{i} (u_i v)_{m+n-i}. \] (2.12)
A module (see [K]) for a vertex Lie algebra \( V \) is a vector space \( W \) equipped with a linear map
\[
Y_W : V \to \text{Hom}(W, x^{-1}W[x^{-1}]),
\]
\[
v \mapsto Y_W(v, x) = \sum_{n \geq 0} v_n x^{-n-1}
\]
(2.13)
such that (2.10) and (2.12) hold.

Recall the following notion of vertex Poisson algebra from [FB] (cf. [DLM]):

**Definition 2.3.** A vertex Poisson algebra is a commutative vertex algebra \( A \), or equivalently, a (unital) commutative associative algebra equipped with a derivation \( \partial \), equipped with a vertex Lie algebra structure \((Y_-, \partial)\) such that
\[
Y_-(a, x) \in x^{-1}(\text{Der } A)[[x^{-1}]] \quad \text{for } a \in A.
\]
(2.14)

A module for a vertex Poisson algebra \( A \) is a vector space \( W \) equipped with a module structure for \( A \) as an associative algebra and a module structure for \( A \) as a vertex Lie algebra such that
\[
Y_W(u, x)(vw) = (Y_W(u, x)v)w + vY_W(u, x)w
\]
(2.15)
for \( u, v \in V, w \in W \).

The following result obtained in [Li2] gives a construction of vertex Poisson algebras from vertex algebras through certain increasing filtrations:

**Proposition 2.4.** Let \( V \) be a vertex algebra and let \( \mathcal{E} = \{ E_n \}_{n \in \mathbb{Z}} \) be a good increasing filtration of \( V \) in the sense that \( 1 \in E_0 \),
\[
u_n E_s \subset E_{r+s}
\]
(2.16)
for \( u \in E_r, r, s, n \in \mathbb{Z} \) and
\[
u_n E_s \subset E_{r+s-1} \quad \text{for } n \geq 0.
\]
(2.17)
Then the associated graded vector space \( \text{gr}_\mathcal{E} V = \bigsqcup_{n \in \mathbb{Z}} E_{n+1}/E_n \) is naturally a vertex Poisson algebra with
\[
(u + E_{m-1})(v + E_{n-1}) = u_{-1}v + E_{m+n-1},
\]
(2.18)
\[
\partial(u + E_{m-1}) = \partial u + E_{m-1},
\]
(2.19)
\[
Y_-(u + E_{m-1})(v + E_{n-1}) = \sum_{r \geq 0} (u_r v + E_{m+n-2}) x^{-r-1}
\]
(2.20)
for \( u \in E_m, v \in E_n \) with \( m, n \in \mathbb{Z} \).
Furthermore, the following construction of good increasing filtrations was also given in [12]:

**Theorem 2.5.** Let $V = \bigoplus_{n \in \mathbb{N}} V(n)$ be an $\mathbb{N}$-graded vertex algebra such that $V(0) = \mathbb{C}1$. Let $U$ be a graded subspace of $V_+ = \bigoplus_{n \geq 1} V(n)$ such that

$$V = \text{span}\{u^{(1)}_{-k_1} \cdots u^{(r)}_{-k_r} 1 \mid r \geq 0, u^{(i)} \in U, k_i \geq 1\}.$$ 

In particular, we can take $U = V_+$. For any $n \geq 0$, denote by $E^U_n$ the subspace of $V$ linearly spanned by the vectors

$$u^{(1)}_{-k_1} \cdots u^{(r)}_{-k_r} 1$$

for $r \geq 0$, for homogeneous vectors $u^{(1)}, \ldots, u^{(r)} \in U$ and for $k_1, \ldots, k_r \geq 1$ with wt $w^{(1)} + \cdots + wt u^{(r)} \leq n$. Then the sequence $\mathcal{E}_U = \{E^U_n\}$ is a good increasing filtration of $V$. Furthermore, $\mathcal{E}_U$ does not depend on $U$.

Next, we give a construction of vertex Poisson algebras from vertex algebras using decreasing filtrations. First, we formulate the following general result, which is similar to Proposition 2.4 and which is classical in nature:

**Proposition 2.6.** Let $V$ be any vertex algebra and let $\mathcal{F} = \{F_n\}_{n \geq 0}$ be a decreasing sequence of subspaces of $V$ such that $1 \in F_0$ and

$$u_n v \in F_{r+s-n} \quad \text{for} \quad u \in F_r, \ v \in F_s, \ r, s \in \mathbb{N}, \ n \in \mathbb{Z}, \quad (2.21)$$

where by convention $F_m = V$ for $m < 0$. Then the associated graded vector space

$$\text{gr}_\mathcal{F} V = \bigoplus_{n \geq 0} F_n/F_{n+1}$$

is naturally an $\mathbb{N}$-graded vertex algebra with

$$(u + F_{r+1})_n(v + F_{s+1}) = u_n v + F_{r+s-n} \quad (2.22)$$

for $u \in F_r$, $v \in F_s$, $r, s \in \mathbb{N}$, $n \in \mathbb{Z}$ and with $1 + F_1 \in F_0/F_1$ as the vacuum vector. Furthermore, $\text{gr}_\mathcal{F} V$ is commutative if and only if

$$u_n v \in F_{r+s-n} \quad \text{for} \quad u \in F_r, \ v \in F_s, \ r, s, n \in \mathbb{N}. \quad (2.23)$$

Assume $\text{(2.21)}$ and $\text{(2.23)}$. Then the commutative vertex algebra $\text{gr}_\mathcal{F} V$ is a vertex Poisson algebra where

$$\partial(u + F_{r+1}) = D u + F_{r+2}, \quad (2.24)$$

$$Y_-(u + F_{r+1}, x)(v + F_{s+1}) = \sum_{n \geq 0} (u_n v + F_{r+s-n+1}) x^{-n-1} \quad (2.25)$$

for $u \in F_r$, $v \in F_s$ with $r, s \in \mathbb{N}$. 

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Proof. Notice that the condition (2.21) guarantees that the operations given in (2.22) are well defined. Just as with any classical algebras, it is straightforward to check that \(\text{gr}_F V\) is an \(N\)-graded vertex algebra and it is also clear that \(\text{gr}_F V\) is commutative if and only if (2.23) holds. Assuming (2.21) and (2.23) we have a commutative associative \(N\)-graded algebra \(\text{gr}_F(V)\) with derivation \(\partial\) defined by

\[
\partial(u + F_{n+1}) = (u + F_{n+1})_{-2}(1 + F_1) = u_{-2}1 + F_{n+2} = Du + F_{n+2},
\]

noticing that by (2.21) we have \(Du = u_{-2}1 \in F_{n+1}\). The condition (2.23) guarantees that the linear map \(Y_\cdot\) in (2.24) is well defined. It is straightforward to check that \(\text{gr}_F(V)\) equipped with \(Y_\cdot\) and \(\partial\) is a vertex Lie algebra.

Now, we check the compatibility condition (2.14). Let \(u \in F_r, v \in F_s, w \in F_k\) with \(r, s, k \in \mathbb{N}\). For \(m \geq 0\), using the Borcherds’ commutator formula for \(V\) we have

\[
u_m(v_{-1}w) = v_{-1}(u_mw) + \sum_{i=0}^{m} \binom{m}{i} (u_{i}v)_{m-i}w = v_{-1}(u_mw) + (u_mv)_{-1}w + \sum_{i=0}^{m-1} \binom{m}{i} (u_{i}v)_{m-i}w. \tag{2.26}\]

For \(0 \leq i \leq m-1\), using (2.23) (twice) we have

\[
(u_{i}v)_{m-i}w \in F_{r+s+k-m+1}.
\]

Thus

\[
u_m(v_{-1}w) + F_{r+s+k-m+1} = v_{-1}(u_mw) + (u_mv)_{-1}w + F_{r+s+k-m+1}. \tag{2.27}\]

This proves \(Y_\cdot(u,x) \in x^{-1}(\text{Der}(\text{gr}_E(V)))[x^{-1}]\). Therefore, \(\text{gr}_F(V)\) is a vertex Poisson algebra.

In the following, for each vertex algebra we construct a canonical decreasing sequence \(\mathcal{E} = \{E_n\}_{n \geq 0}\) which satisfies all the conditions assumed in Proposition 2.6.

**Definition 2.7.** Let \(V\) be a vertex algebra and let \(W\) be a \(V\)-module. Define a sequence \(\mathcal{E}_W = \{E_n(W)\}_{n \in \mathbb{Z}}\) of subspaces of \(W\), where for \(n \in \mathbb{Z}\), \(E_n(W)\) is linearly spanned by the vectors

\[
u_{-1-k_1-\cdots-k_r}w \tag{2.28}\]

for \(r \geq 1, u^{(1)}, \ldots, u^{(r)} \in V, w \in W, k_1, \ldots, k_r \geq 0\) with \(k_1 + \cdots + k_r \geq n\).

Our main task is to establish the properties (2.21) and (2.23) for the sequence \(\mathcal{E}\). The following are some immediate consequences:
Lemma 2.8. For any $V$-module $W$ we have

\[ E_n(W) \supset E_{n+1}(W) \quad \text{for any } n \in \mathbb{Z}, \]  
\[ E_n(W) = W \quad \text{for any } n \leq 0, \]  
\[ u_{-1-k}E_n(W) \subset E_{n+k}(W) \quad \text{for } u \in V, \ k \geq 0, \ n \in \mathbb{Z}. \]

The following gives a stronger spanning property for $E_n(W)$:

Lemma 2.9. Let $W$ be a $V$-module. For any $n \geq 1$, we have

\[ E_n(W) = \text{span}\{u_{-1-i}w \mid u \in V, \ i \geq 1, \ w \in E_{n-i}(W)\}. \]  
Furthermore, for $n \geq 1$, $E_n(W)$ is linearly spanned by the vectors

\[ u_{-k_1-1}^{(1)} \cdots u_{-k_r-1}^{(r)}w \]

for $r \geq 1$, $u^{(1)}, \ldots, u^{(r)} \in V$, $w \in W$, $k_1, \ldots, k_r \geq 1$ with $k_1 + \cdots + k_r \geq n$.

Proof. Notice that (2.33) follows from (2.32) and induction. Denote by $E'_n(W)$ the space on the right-hand side of (2.32). To prove (2.32), we need to prove that each spanning vector of $E_n(W)$ in (2.28) lies in $E'_n(W)$. Now we use induction on $r$. If $r = 1$, we have $k_1 \geq n \geq 1$ and $w \in W = E_{n-k_1}(W)$, so that $u_{-1-k_1}^{(1)}w \in E'_n(W)$. Assume $r \geq 2$. If $k_1 \geq 1$, we have

\[ u_{-1-k_1}^{(1)}u_{-1-k_2}^{(2)} \cdots u_{-1-k_r}^{(r)}w \in E'_n(W) \]

because $u_{-1-k_2}^{(2)} \cdots u_{-1-k_r}^{(r)}w \in E_{n-k_1}(W)$ with $k_2 + \cdots + k_r \geq n - k_1$. If $k_1 = 0$, we have $k_2 + \cdots + k_r \geq n$, so that $u_{-1-k_2}^{(2)} \cdots u_{-1-k_r}^{(r)}w \in E_n(W)$. By the inductive hypothesis, we have $u_{-1-k_2}^{(2)} \cdots u_{-1-k_r}^{(r)}w \in E'_n(W)$. Furthermore, for any $b \in V$, $k \geq 1$, $w' \in E_{n-k}(W)$, we have

\[ u_{-1}^{(1)}b_{-1-k}w' = b_{-1-k}u_{-1}^{(1)}w' + \sum_{i \geq 0} \binom{-1}{i}(u_{-1}^{(1)}b)_{-2-k-i}w'. \]

From definition we have $u_{-1}^{(1)}w' \in E_{n-k}(W)$, so that $b_{-1-k}u_{-1}^{(1)}w' \in E'_n(W)$. On the other hand, for $i \geq 0$, we have $w' \in E_{n-k}(W) \subset E_{n-k-i-1}(W)$, so that

\[ (u_{-1}^{(1)}b)_{-2-k-i}w' \in E'_n(W). \]

Therefore, $u_{-1}^{(1)}b_{-1-k}w' \in E'_n(W)$. This proves that $u_{-1-k_1}^{(1)}u_{-1-k_2}^{(2)} \cdots u_{-1-k_r}^{(r)}w \in E'_n(W)$, completing the induction. \( \square \)

We have the following special case of (2.21) and (2.28) for $\mathcal{E}$:
Lemma 2.10. Let $W$ be any $V$-module. For $a \in V$, $m, n \in \mathbb{Z}$, we have

$$a_mE_n(W) \subset E_{n-m-1}(W).$$  \hfill (2.34)

Furthermore,

$$a_mE_n(W) \subset E_{n-m}(W) \quad \text{for } m \geq 0. \quad \hfill (2.35)$$

Proof. By (2.31), (2.34) holds for $m \leq -1$. Assume $m \geq 0$. Since $E_{n-m}(W) \subset E_{n-m-1}(W)$ it suffices to prove (2.35). We now prove the assertion by induction on $n$. If $n \leq 0$, we have $E_{n-m}(W) = W$ (because $n - m \leq 0$), so that $a_mE_n(W) \subset W = E_{n-m}(W)$. Assume $n \geq 1$. From (2.32), $E_n(W)$ is spanned by the vectors $u_{-1-k}w$ for $u \in V$, $k \geq 1$, $w \in E_{n-k}(W)$. Let $u \in V$, $k \geq 1$, $w \in E_{n-k}(W)$. In view of Borcherds’ commutator formula we have

$$a_mu_{-1-k}w = u_{-1-k}a_mw + \sum_{i \geq 0} \binom{m}{i}(a_iu)_{m-k-i-1}w.$$  

Since $w \in E_{n-k}(W)$ with $n - k < n$, from inductive hypothesis we have

$$a_mw \in E_{n-k-m}(W),$$

$$(a_iu)_{m-k-i-1}w \in E_{n-m+i}(W) \subset E_{n-m}(W) \quad \text{for } i \geq 0.$$  

Furthermore, using inductive hypothesis and Lemma 2.8 we have

$$u_{-1-k}a_mw \in u_{-1-k}E_{n-k-m}(W) \subset E_{n-m}(W).$$

Therefore, $a_mu_{-1-k}w \in E_{n-m}(W)$. This proves $a_mE_n(W) \subset E_{n-m}(W)$, completing the induction and the whole proof.

Now we have the following general case:

Proposition 2.11. Let $W$ be a $V$-module and let $u \in E_r(V)$, $w \in E_s(W)$ with $r, s \in \mathbb{Z}$. Then

$$u_nw \in E_{r+s-n-1}(W) \quad \text{for } n \in \mathbb{Z}. \quad \hfill (2.36)$$

Furthermore, we have

$$u_nw \in E_{r+s-n}(W) \quad \text{for } n \geq 0. \quad \hfill (2.37)$$

Proof. We are going to use induction on $r$. By Lemma 2.10 we have $u_nw \in E_{s-n-1}(W)$. If $r \leq 0$, we have $r + s - n - 1 \leq s - n - 1$, so that $u_nw \in E_{s-n-1}(W) \subset E_{r+s-n-1}(W)$. Assume $r \geq 0$ and $u \in E_{r+1}(V)$. In view of (2.32) it suffices to
consider \( u = a_{-i}b \) for some \( a \in V, \ 0 \leq i \leq r, \ b \in E_{r-i}(V) \). By the iterate formula \((2.2)\) we have

\[
(a_{-i}b)_n w = \sum_{j \geq 0} (-1)^j \binom{-i}{j} (a_{-i-j}b_{n+j}w - (-1)^j b_{n-2-i-j}a_j w).
\]\\(2.38)\\

If \( n \geq 0 \), using the inductive hypothesis (with \( b \in E_{r-i}(V) \)) and Lemma \((2.10)\) we have

\[
a_{-i-j}b_{n+j}w \in a_{-i-j}E_{r-i+s-n-j}(W) \subset E_{r+1+s-n}(W),
b_{n-2-i-j}a_j w \in b_{n-2-i-j}E_{s-j}(W) \subset E_{r+1+s-n}(W),
\]

from which we have that \((a_{-i}b)_n w \in E_{r+1+s-n}(W)\). If \( n \leq -1 \), we have

\[
a_{-i-j}b_{n+j}w \in a_{-i-j}E_{r-i+s-n-j-1}(W) \subset E_{r+s-n}(W),
b_{n-2-i-j}a_j w \in b_{n-2-i-j}E_{s-j}(W) \subset E_{r+1+s-n}(W) \subset E_{r+s-n}(W),
\]

so that \((a_{-i}b)_n w \in E_{r+s-n}(W) = E_{(r+1+s)-n-1}(W)\). This concludes the proof. \(\square\)

Combining Propositions \((2.6)\) and \((2.11)\) we immediately have:

**Theorem 2.12.** Let \( V \) be any vertex algebra and let \( \mathcal{E} = \{E_n(V)\} \) be the decreasing sequence defined in Definition \((2.7)\) for \( V \). Set

\[
gr_{\mathcal{E}}(V) = \prod_{n \geq 0} E_n/E_{n+1}.
\]\\(2.39)\\

Then \( gr_{\mathcal{E}}(V) \) equipped with the multiplication defined by

\[
(a + E_{r+1})(b + E_{s+1}) = a_{-1}b + E_{r+s+1}
\]\\(2.40)\\
is a commutative and associative \( \mathbb{N} \)-graded algebra with \( 1 + E_1 \in E_0/E_1 \) as identity and with a derivation \( \partial \) defined by

\[
\partial(u + E_{n+1}) = D(u) + E_{n+2} \quad \text{for } u \in E_n, \ n \in \mathbb{N}.
\]\\(2.41)\\

Furthermore, \( gr_{\mathcal{E}}(V) \) is a vertex Poisson algebra where

\[
Y_-(a + E_{r+1}, x)(b + E_{s+1}) = \sum_{n \geq 0} (u_n v + E_{r+s-n+1})x^{-n-1}
\]\\(2.42)\\

for \( a \in E_r, \ b \in E_s \) with \( r, s \in \mathbb{N} \).
Proposition 2.13. Let $W$ be any $V$-module and $\mathcal{E}_W$ the decreasing sequence defined in Definition 2.7 for $W$. Then the associated graded vector space $\text{gr}_{\mathcal{E}}(W) = \bigsqcup_{n \geq 0} E_n(W) / E_{n+1}(W)$ is naturally a module for the vertex Poisson algebra $\text{gr}_{\mathcal{E}}(V)$ with

$$(v + E_{r+1}(V)) \cdot (w + E_{s+1}(W)) = v_{-1}w + E_{r+s+1}(W),$$

(2.43)

$$Y_-(v + E_{r+1})(w + E_{s+1}(W)) = \sum_{n \geq 0} (v_n w + E_{r+s-n+1}) x^{-n-1}$$

(2.44)

for $v \in E_r(V)$, $w \in E_s(W)$.

Proof. With the properties (2.36) and (2.37) the actions given by (2.43) and (2.44) are well defined. Clearly, $1 + E_1$ acts on $\text{gr}_{\mathcal{E}}(W)$ as identity and we have

$$(u + E_{r+1}(V)) \cdot ((v + E_{s+1}(V)) \cdot (w + E_{k+1}(W))) = u_{-1}v_{-1}w + E_{r+s+k+1}(W).$$

By the iterate formula (2.2) we have

$$(u_{-1}v)_{-1}w = \sum_{i \geq 0} (u_{-1-i}v_{-1+i}w + v_{-2-i}u_i w),$$

where for $i \geq 1$, using (2.36) and (2.37) we have

$$u_{-1-i}v_{-1+i}w \in u_{-1-i}E_{s+k+1-i}(W) \subset E_{r+s+k+1}(W)$$

and for $i \geq 0$, similarly we have

$$v_{-2-i}u_i w \in v_{-2-i}E_{s+k-i}(W) \subset E_{r+s+k+1}(W).$$

Thus

$$(u_{-1}v)_{-1}w \in u_{-1}v_{-1}w + E_{r+s+k+1}(W).$$

This proves that $\text{gr}_{\mathcal{E}}(W)$ is a module for $\text{gr}_{\mathcal{E}}(V)$ as an associative algebra. It is straightforward to check that it is a module for the vertex Lie algebra. Other properties are clear from the proof of Proposition 2.6. \hfill \Box

Notice that so far we have not excluded the possibility that the associated sequence $\mathcal{E}_V$ is trivial in the sense that $E_n(V) = V$ for all $n \geq 0$. Indeed, as we shall see in the next section, for some vertex algebras the associated sequence $\mathcal{E}$ is trivial.

Nevertheless, we have:

Lemma 2.14. Let $V = \bigsqcup_{n \geq 0} V(n)$ be an $\mathbb{N}$-graded vertex algebra and $\mathcal{E} = \{E_n\}$ be the decreasing sequence defined in Definition 2.7 for $V$. Then

$$E_n(V) \subset \bigsqcup_{m \geq n} V(m) \quad \text{for } n \geq 0.$$

(2.45)

Furthermore, the associated decreasing sequence $\mathcal{E} = \{E_n\}$ for $V$ is a filtration, i.e.,

$$\cap_{n \geq 0} E_n(V) = 0.$$

(2.46)
Proof. By definition we have $E_0 = V = \bigcup_{n \geq 0} V(n)$. For $n \geq 1$, recall that $E_n$ is linearly spanned by the vectors

$$u_{1-k_1}^{(1)} \cdots u_{1-k_r}^{(r)} v$$

for $r \geq 1$, $u^{(1)}, \ldots, u^{(r)}, v \in V$, $k_1, \ldots, k_r \geq 1$ with $k_1 + \cdots + k_r \geq n$. If the vectors $u^{(1)}, \ldots, u^{(r)}, v$ are homogeneous, we have

$$\text{wt}\left(u_{1-k_1}^{(1)} \cdots u_{1-k_r}^{(r)} v\right) = \text{wt}\left(u^{(1)} + k_1 + \cdots + \text{wt}\left(u^{(r)} + k_r + \text{wt}\left(v \geq k_1 + \cdots + k_r \geq n.\right.\right.\right.\right.\right.$$

This proves (2.45) for $n \geq 1$. Clearly, each subspace $E_n$ of $V$ is graded. From (2.45) we immediately have (2.46).

In the next section we shall generalize Lemma 2.14 from an $\mathbb{N}$-graded vertex algebra to a lower truncated $\mathbb{Z}$-graded vertex algebra by using a relation between the decreasing sequence $E$ and a sequence introduced by Zhu.

### 3 The relation between the sequences $E$ and $C$

In this section we first recall the sequence $C$ introduced by Zhu and we then give a relation between the two decreasing sequences $E$ and $C$. We show that if $V$ is a lower truncated $\mathbb{Z}$-graded vertex algebra, then both sequences are decreasing filtrations of $V$.

The following definition is (essentially) due to Zhu ([Z1,2]):

**Definition 3.1.** Let $V$ be a vertex algebra and $W$ a $V$-module. For any $n \geq 2$ we define $C_n(W)$ to be the subspace of $W$, linearly spanned by the vectors $v_{-n} w$ for $v \in V$, $w \in W$. A $V$-module $W$ is said to be $C_n$-cofinite if $W/C_n(W)$ is finite-dimensional. In particular, if $V/C_n(V)$ is finite-dimensional, we say that the vertex algebra $V$ is $C_n$-cofinite.

The following are easy consequences:

**Lemma 3.2.** Let $V$ be any vertex algebra, let $W$ be a $V$-module and let $n \geq 2$. Then

$$C_m(W) \subset C_n(W) \quad \text{for } m \geq n,$$

$$u_{-k} C_n(W) \subset C_n(W) \quad \text{for } u \in V, \quad k \geq 0,$$

$$u_{-n} v_{-k} w \equiv v_{-k} u_{-n} w \mod C_{n+k}(W) \quad \text{for } u, v \in V, \quad w \in W, \quad k \geq 0. \quad (3.3)$$

**Proof.** For $v \in V$, $r \geq 2$ we have $v_{-r-1} = \frac{1}{r}(Dv)_{-r}$. From this we immediately have $C_{r+1}(W) \subset C_r(W)$ for $r \geq 2$, which implies (3.1). Let $u, v \in V$, $w \in W$, $k \geq 0$. Using the commutator formula (2.1) and (3.1) we have

$$u_{-k} v_{-n} w = v_{-n} u_{-k} w + \sum_{i \geq 0} \binom{k}{i} (u_i v)_{-k-n-i} w \in C_n(W),$$
proving that \( u_{-k}C_n(W) \subset C_n(W) \). We also have
\[
u_{-n}v_{-k}w - v_{-k}u_{-n}w = \sum_{i \geq 0} \binom{-n}{i} (u_i v)_{-n-k-i} w \in C_{n+k}(W),
\]
proving \( (3.3) \).

We also have the following more technical results:

**Lemma 3.3.** Let \( V \) be any vertex algebra, let \( W \) be a \( V \)-module and let \( k \geq 2 \). Then
\[
u_{-k}C_k(W) \subset C_{k+1}(W) \quad \text{for } u \in V.
\]

**Proof.** For \( u, v \in V, w \in W \), in view of the iterate formula \( (2.2) \) we have
\[
(u_{-1}v)_{-2k+1}w = \sum_{i \geq 0} (u_{-1-i}v_{-2k+1+i}w + v_{-2k-i}u_i w).
\]

Now we examine each term in \( (3.5) \). Notice that \( (u_{-1}v)_{-2k+1}w \in C_{k+1}(W) \) as \(-2k + 1 \leq -k - 1\) and that \( v_{-2k-i}u_i w \in C_{k+1}(W) \) for \( i \geq 0 \) as \(-2k - i \leq -k - 1\). If \( i \geq k \), we have \(-1 - i \leq -k - 1\), so that \( u_{-1-i}v_{-2k+1+i}w \in C_{k+1}(W) \). For \( 0 \leq i \leq k-2 \), we have \(-2k + 1 + i \leq -k - 1\), so that \( v_{-2k+1+i}w \in C_{k+1}(W) \). Then by Lemma \( 3.2 \) we have \( u_{-1-i}v_{-2k+1+i}w \in C_{k+1}(W) \) for \( 0 \leq i \leq k-2 \). Therefore, the only remaining term \( u_{-k}v_{-k}w \) in \( (3.5) \) must also lie in \( C_{k+1}(W) \). This proves \( u_{-k}C_k(W) \subset C_{k+1}(W) \).

**Proposition 3.4.** Let \( V \) be any vertex algebra, let \( W \) be a \( V \)-module and let \( n \) be any nonnegative integer. Then
\[
u_{-k_1}^{(1)} \cdots u_{-k_r}^{(r)} w \in C_{n+2}(W)
\]
for \( r \geq 2^n \), \( u^{(1)}, \ldots, u^{(r)} \in V, w \in W, k_1, \ldots, k_r \geq 2 \).

**Proof.** Since \( u_{-r}C_{n+2}(W) \subset C_{n+2}(W) \) for \( u \in V, i \geq 0 \) (by Lemma \( 3.2 \)), it suffices to prove the assertion for \( r = 2^n \). Also, since \( u_{-k} = \frac{1}{(k-1)!} (D^{k-2}u)_{-2} \) for \( u \in V, k \geq 2 \), it suffices to prove the assertion for \( k_1 = \cdots = k_r = 2 \). We are going to use induction on \( n \). If \( n = 0 \), by definition we have \( v_{-2}w \in C_2(W) \) for \( v \in V, w \in W \). Assume the assertion holds for \( n = p \), some nonnegative integer. Assume that \( r = 2^{p+1} \) and set \( s = 2^p \). Let \( u^{(1)}, \ldots, u^{(r)} \in V, w \in W \). By inductive hypothesis we have
\[
u_{-2}^{(s+1)} \cdots u_{-2}^{(r)} w \in C_{p+2}(W),
\]
so that
\[
u_{-2}^{(1)} \cdots u_{-2}^{(r)} w \in u_{-2}^{(1)} \cdots u_{-2}^{(s)} C_{p+2}(W).
\]
Consider a typical spanning vector $a_{-p-2}w'$ of $C_{p+2}(W)$ for $a \in V$, $w' \in W$. Using (3.12) and (3.2) we have

$$u^{(1)}_{-2} \cdots u^{(s)}_{-2} a_{-p-2}w' \equiv a_{-p-2} u^{(1)}_{-2} \cdots u^{(s)}_{-2} w' \mod C_{p+4}(W). \tag{3.8}$$

Furthermore, by inductive hypothesis, we have

$$u^{(1)}_{-2} \cdots u^{(s)}_{-2} w' \in C_{p+2}(W),$$

which together with Lemma 3.3 gives

$$a_{-p-2} u^{(1)}_{-2} \cdots u^{(s)}_{-2} w' \in a_{-p-2} C_{p+2}(W) \subset C_{p+3}(W). \tag{3.9}$$

Thus by (3.8) we have

$$u^{(1)}_{-2} \cdots u^{(s)}_{-2} a_{-p-2} w' \in C_{p+3}(W),$$

proving that

$$u^{(1)}_{-2} \cdots u^{(s)}_{-2} C_{p+2}(W) \subset C_{p+3}(W). \tag{3.10}$$

Therefore, by (3.7) we have

$$u^{(1)}_{-2} \cdots u^{(r)}_{-2} w \in C_{p+3}(W).$$

This finishes the induction steps and completes the proof. \qed

The relation between the two decreasing sequences \{E_n(W)\} and \{C_n(W)\} is described as follows:

**Theorem 3.5.** Let $W$ be any module for vertex algebra $V$ and let $\mathcal{E}_W = \{E_n(W)\}$ be the associated decreasing sequence. Then for any $n \geq 2$,

$$C_n(W) \subset E_{n-1}(W), \tag{3.11}$$

$$E_m(W) \subset C_n(W) \quad \text{whenever } m \geq (n-2)2^{n-2}. \tag{3.12}$$

Furthermore,

$$\cap_{n \geq 0} E_n(W) = \cap_{n \geq 0} C_{n+2}(W). \tag{3.13}$$

**Proof.** From the definitions of $C_n(W)$ and $E_{n-1}(W)$ we immediately have $C_n(W) \subset E_{n-1}(W)$. Consider a generic spanning element of $E_m(W)$:

$$X = u^{(1)}_{-1-k_1} \cdots u^{(r)}_{-1-k_r} w$$

where $r \geq 1$, $u^{(1)}, \ldots, u^{(r)} \in V$, $w \in W$, $k_1, \ldots, k_r \geq 1$ with $k_1 + \cdots + k_r \geq m$. If $k_i \geq n-1$ for some $i$, by (3.1) we have $u^{(i)}_{-1-k_i} w \subset C_{-1-k_i}(W) \subset C_n(W)$ and then by (3.2) we have $X \in C_n(W)$. If $r \geq 2^{n-2}$, by Proposition 3.3 $X \in C_n(W)$. Since $k_1 + \cdots + k_r \geq m \geq (n-2)2^{n-2}$, we have either $k_i \geq n-1$ for some $i$ or $r \geq 2^{n-2}$. Therefore, $X \in C_n(W)$ whenever $m \geq (n-2)2^{n-2}$. This proves (3.12). Combining (3.12) and (3.11) we have (3.13). \qed
Corollary 3.6. For any vertex algebra $V$ and any $V$-module $W$, we have

$$E_1(W) = C_2(W), \quad E_2(W) = C_3(W).$$  \hspace{1cm} (3.14)

Proof. By (3.11) we have $C_2(W) \subset E_1(W)$ and $C_3(W) \subset E_2(W)$. On the other hand, by (3.12) with $m = 1, n = 2$ we have $E_1(W) \subset C_2(W)$ and by (3.12) with $m = 2, n = 3$ we have $E_2(W) \subset C_3(W)$.

Recall the following result of Zhu [Z1,2]:

Proposition 3.7. Let $V$ be any vertex algebra. Then $V/C_2(V)$ is a Poisson algebra with

$$\bar{u} \cdot \bar{v} = u_{-1}v, \quad [\bar{u}, \bar{v}] = u_0v \quad \text{for } u, v \in V,$$

where $\bar{u} = u + C_2(V)$, and with $1 + C_2(V)$ as the identity element.

It is clear that the degree zero subspace $E_0/E_1$ of $\text{gr}_e(V)$ is a Poisson algebra where

$$(u + E_1)(v + E_1) = u_{-1}v + E_1, \quad [u + E_1, v + E_1] = u_0v + E_1$$

for $u, v \in V$. With $E_0(V) = V$ and $E_1(V) = C_2(V)$, we see that this Poisson algebra is nothing but the Zhu’s Poisson algebra $V/C_2(V)$.

Thus we have:

Proposition 3.8. Let $V$ be any vertex algebra. The degree zero subspace $\text{gr}_e(V)(0) = E_0(V)/E_1(V)$ of the $\mathbb{N}$-graded vertex Poisson algebra $\text{gr}_e(V)$ is naturally a Poisson algebra which coincides with the Zhu’s Poisson algebra $V/C_2(V) = E_0/E_1$.

The following result generalizes the result of Lemma 2.14:

Proposition 3.9. Let $V = \bigoplus_{n \geq 1} V(n)$ be a lower truncated $\mathbb{Z}$-graded vertex algebra. Then

$$C_n(V) \subset \bigoplus_{k \geq 2t+n-1} V(k)$$

for $n \geq 2$. Furthermore,

$$E_m(V) \subset \bigoplus_{k \geq 2t+n-1} V(k) \quad \text{whenever } m \geq (n - 2)2^{n-2},$$

$$\cap_{n \geq 0} E_n(V) = \cap_{n \geq 2} C_n(V) = 0.$$
Proof. For homogeneous vectors \(u, v \in V\) and for any \(n \geq 2\) we have
\[
\text{wt } (u_n v) = \text{wt } u + \text{wt } v + n - 1 \geq 2t + n - 1.
\]

In view of this we have
\[
C_n(V) \subset \bigcap_{k \geq 2t+n-1} V(k)
\]
for \(n \geq 2\). This proves (3.16), from which we immediately have that \(\cap_{n \geq 0} C_{n+2}(V) = 0\). Using Theorem 3.5 we obtain (3.17) and (3.18).

For the rest of this section, we consider vertex algebras whose associated decreasing sequence \(E\) is trivial.

First we have:

**Lemma 3.10.** Let \(V\) be a vertex algebra and let \(W\) be a \(V\)-module. If \(W = C_2(W)\), then
\[
E_n(W) = C_{n+2}(W) = W \quad \text{for all } n \geq 0.
\]

**Proof.** Since \(W = C_2(W)\), we have \(E_1(W) = C_2(W) = W\). Assume that \(E_k(W) = W\) for some \(k \geq 1\). Then
\[
v_{-2}W = v_{-2}E_k(W) \subset E_{k+1}(W) \quad \text{for } v \in V.
\]

From this we have \(W = C_2(W) \subset E_{k+1}(W)\), proving \(E_{k+1}(W) = W\). By induction, we have \(E_n(W) = W\) for all \(n \geq 0\). In view of Theorem 3.5 we have \(C_n(W) = W\) for all \(n \geq 2\).

Suppose that \(V\) is a vertex algebra such that \(C_2(V) = V\). By Lemma 3.10 we have \(C_{n+2}(V) = V\) for \(n \geq 0\), so that \(V = \cap_{n \geq 0} C_{n+2}(V)\). Furthermore, if there exists a lower truncated \(\mathbb{Z}\)-grading \(V = \bigcup_{n \in \mathbb{Z}} V(n)\) with which \(V\) becomes an \(\mathbb{Z}\)-graded vertex algebra, by (3.18) (Proposition 3.9) we have \(\cap_{n \geq 0} C_{n+2}(V) = 0\), so that \(V = \cap_{n \geq 0} C_{n+2}(V) = 0\). Therefore we have proved:

**Proposition 3.11.** Let \(V\) be a nonzero vertex algebra such that \(C_2(V) = V\). Then there does not exist a lower truncated \(\mathbb{Z}\)-grading \(V = \bigcup_{n \in \mathbb{Z}} V(n)\) with which \(V\) becomes an \(\mathbb{Z}\)-graded vertex algebra.

From [B] and [FLM], associated to any nondegenerate even lattice \(L\) of finite rank, we have a vertex algebra \(V_L\). Furthermore, \(V_L\) is a vertex operator algebra if and only if \(L\) is positive-definite in the sense that \(\langle \alpha, \alpha \rangle > 0\) for \(0 \neq \alpha \in L\). In this case, \(V_L\) is \(\mathbb{N}\)-graded by \(L(0)\)-weight (with 1-dimensional weight-zero subspace), so that Lemma 2.14 (and Proposition 3.9) applies to \(V_L\). On the other hand, we have:
Proposition 3.12. Let $L$ be a finite rank nondegenerate even lattice that is not positive-definite and let $V_L$ be the associated vertex algebra. Then $C_{n+2}(V_L) = E_n(V_L) = V_L$ for $n \geq 0$. Furthermore, there does not exist a lower truncated $\mathbb{Z}$-grading on $V_L$ with which $V_L$ becomes a lower truncated $\mathbb{Z}$-graded vertex algebra.

Proof. First we show that there exists $\alpha \in L$ such that $\langle \alpha, \alpha \rangle < 0$. Since $L$ is not positive-definite, there exists $0 \neq \beta \in L$ such that $\langle \beta, \beta \rangle \leq 0$. If $\langle \beta, \beta \rangle \neq 0$, that is, $\langle \beta, \beta \rangle < 0$, then we can simply take $\alpha = \beta$. Suppose $\langle \beta, \beta \rangle = 0$. Since $L$ is nondegenerate, there exists $\gamma \in L$ such that $\langle \gamma, \beta \rangle \neq 0$. For $m \in \mathbb{Z}$, we have

$$\langle \gamma + m\beta, \gamma + m\beta \rangle = \langle \gamma, \gamma \rangle + 2m\langle \gamma, \beta \rangle.$$  

We see that $\langle \gamma + m\beta, \gamma + m\beta \rangle < 0$ for some $m$. Then we can take $\alpha = \gamma + m\beta$ with the desired property.

Let $\alpha \in L$ be such that $\langle \alpha, \alpha \rangle < 0$ and set $\langle \alpha, \alpha \rangle = -2k$ with $k \geq 1$. Using the explicit expression of the vertex operators in [FLM], we have $(e^{\alpha})_{-2k-1}e^{-\alpha} = 1$, so that $1 \in C_{2k+1}(V_L) \subset C_2(V_L)$. Then $v = v_{-1}1 \in C_2(V_L)$ for $v \in V_L$. Thus $C_2(V_L) = V_L$. By Lemma 3.10 we have $C_{n+2}(V_L) = E_n(V_L) = V_L$ for $n \geq 0$. The last assertion follows immediately from Proposition 3.11. \hfill \Box

4 Generating subspaces of vertex algebras

In this section we shall use the differential algebra structure on $\text{gr}_E(V)$ to study certain kinds of generating subspaces of lower truncated $\mathbb{Z}$-graded vertex algebras.

First we prove the following results for classical algebras:

Lemma 4.1. Let $(A, \partial)$ be an $\mathbb{N}$-graded (unital) differential algebra such that $(\partial A)A = A_+$, where

$$A_+ = \prod_{n \geq 1} A_{(n)}. \quad (4.1)$$

Let $S$ be a generating subspace of $A_{(0)}$ as an algebra. Then $A$ is linearly spanned by the vectors

$$\partial^{n_1}(a_1) \cdots \partial^{n_r}(a_r) \quad (4.2)$$

for $r \geq 0$, $n_1 \geq n_2 \geq \cdots \geq n_r \geq 0$, $a_1, \ldots, a_r \in S$, or equivalently, $S$ generates $A$ as a differential algebra. In particular, $A_{(0)}$ generates $A$ as a differential algebra. Furthermore, $A$ is linearly spanned by the vectors

$$\partial^{n_1}(a_1) \cdots \partial^{n_r}(a_r) \quad (4.3)$$

for $r \geq 1$, $n_1 > n_2 > \cdots > n_r \geq 0$, $a_1, \ldots, a_r \in A_{(0)}$. 

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Proof. First, we show that $A$ as a differential algebra is generated by $A(0)$. Let $A'$ be the differential subalgebra of $A$, generated by $A(0)$. We are going to show (by induction) that $\prod_{n=0}^{k} A(n) \subset A'$ for all $k \geq 0$. From definition, we have $A(0) \subset A'$. Assume that $\prod_{n=0}^{k} A(n) \subset A'$ for some $k \geq 0$. Consider the subspace $A(k+1)$ of $A$. From our assumption, we have

$$A(k+1) \subset A_+ = A\partial A,$$

so $A(k+1)$ is linearly spanned by the vectors $a\partial b$ for $a \in A(r)$, $b \in A(s)$ with $r+s+1 = k+1$. For any $a \in A(r)$, $b \in A(s)$ with $r+s+1 = k+1$, since $r, s \leq k$ (with $r, s \geq 0$), by the inductive hypothesis, we have $a, b \in A'$. Consequently, $a\partial b \in A'$. Thus $A(k+1) \subset A'$. This proves that $\prod_{n=0}^{k} A(n) \subset A'$ for all $k \geq 0$. Therefore, we have $A = A'$, proving that $A$ as a differential algebra is generated by $A(0)$. It follows that if $S$ generates $A(0)$ as an algebra, then $S$ generates $A$ as a differential algebra.

For a positive integer $n$, let $A''(n)$ be the subspace of $A(n)$ spanned by the vectors

$$\partial^{k_1}(a_1) \cdots \partial^{k_r}(a_r)b,$$

for $r \geq 1$, $k_1 > k_2 > \cdots > k_r \geq 1$, $a_1, \ldots, a_r, b \in A(0)$ with $k_1 + \cdots + k_r = n$. We must prove $A''(n) = A''(n)$ for all $n \geq 1$.

For a positive integer $n$, denote by $P_n$ the set of partitions of $n$. We now endow $P_n$ with the reverse order of the lexicographic order on $P_n$. Set $P = \cup_{n \geq 1} P_n$. For $\alpha \in P_m$, $\beta \in P_n$, combining $\alpha$ and $\beta$ together we get a partition of $m+n$, which we denote by $\alpha * \beta$. Clearly, this defines an abelian semigroup structure on $P$. Furthermore, for $\alpha, \beta \in P_n$, $\gamma \in P$, if $\alpha > \beta$, then $\alpha * \gamma > \beta * \gamma$. That is, the order is compatible with the multiplication.

For $\alpha \in P_n$, define $A''(n)$ to be the linear span of the vectors

$$\partial^{k_1}(a_1) \cdots \partial^{k_r}(a_r)b$$

for $r \geq 1$, $k_1 \geq k_2 \geq \cdots \geq k_r \geq 1$, $a_1, \ldots, a_r, b \in A(0)$ with $k_1 + \cdots + k_r = n$ and $(k_1, \ldots, k_r) \leq \alpha$. Since $A(0)$ generates $A$ as a differential algebra, $\{A''(n)\}$ is a (finite) increasing filtration of $A(n)$.

For $a, b \in A(0)$ and $k \geq 1$, we have

$$\partial^{2k}(ab) = \sum_{i=0}^{2k} \binom{2k}{i} \partial^{2k-i}(a)\partial^{i}(b),$$

which can be rewritten as

$$\binom{2k}{k} \partial^{k}(a)\partial^{k}(b)$$

$$= \partial^{2k}(ab) - \partial^{2k}(a)b - \partial^{2k}(b)a - \sum_{i=1}^{k-1} \binom{2k}{i} \left(\partial^{2k-i}(a)\partial^{i}(b) + \partial^{i}(a)\partial^{2k-i}(b)\right).$$

(4.5)
We see that \((k, k) > (2k), (2k - i, i)\) for \(1 \leq i \leq k - 1\).

Now consider a typical element of \(A_n^\alpha\)

\[
X = \partial^{k_1}(a_1) \cdots \partial^{k_r}(a_r)b
\]

for \((k_1, \ldots, k_r) \in P_n, a_1, \ldots, a_r, b \in A_{(0)}\) with \((k_1, \ldots, k_r) \leq \alpha\). If all \(k_1 > k_2 > \cdots > k_r\), then \(X \in A_{(n)}^\alpha\). Otherwise, using (4.5) we see that

\[
X \in \sum_{\beta < \alpha} A_{(n)}^\beta.
\]

Now it follows immediately from induction that \(A_{(n)} = A_{(n)}^\alpha\).

**Lemma 4.2.** Let \(V\) be a vertex algebra and let \(A = \text{gr}_\mathcal{E}(V)\) be the vertex Poisson algebra, obtained in Theorem 2.12, which is in particular an \(\mathbb{N}\)-graded (unital) differential algebra. Then \(A_+ = A\partial A\). Furthermore, for any \(V\)-module \(W\), the associated graded vector space \(\text{gr}_\mathcal{E}(W)\) is an \(A\)-module with

\[
(u + E_{m+1}(V)) \cdot (w + E_{n+1}(W)) = u_{-1}w + E_{m+n+1}(W)
\]

for \(u \in E_m(V), w \in E_n(W)\) with \(m, n \in \mathbb{N}\), and \(\text{gr}_\mathcal{E}(W)\) as an \(A\)-module is generated by \(E_0(W)/E_1(W)\), i.e.,

\[
\text{gr}_\mathcal{E}(W) = A(E_0(W)/E_1(W)).
\]

**Proof.** We have \(A = \bigsqcup_{n \in \mathbb{N}} A_{(n)}\), where \(A_{(n)} = E_n/E_{n+1}\) for \(n \in \mathbb{N}\). For \(n \geq 1\), from Lemma 2.8, \(E_n\) is linearly spanned by the vectors \(u_{-2-i}v \in E_n\) where \(u \in V, v \in E_{n-1-i}\) for \(0 \leq i \leq n - 1\), and furthermore, we have

\[
u_{-2-i}v + E_{n+1} = \frac{1}{(i+1)!}(D^{i+1}u)_{-1}v + E_{n+1}
\]

\[
= \frac{1}{(i+1)!}(D^{i+1}u + E_{i+2})(v + E_{n-i})
\]

\[
= \frac{1}{(i+1)!}A^{i+1}(u + E_1)(v + E_{n-i})
\]

\[
\in A\partial A,
\]

noticing that for any \(r \in \mathbb{Z}\), \(DE_r \subset E_{r+1}\) from the definition of \(D\) and Lemma 2.11.

This proves \(E_n/E_{n+1} \subset A\partial A\) for \(n \geq 1\), so that \(A_+ \subset A\partial A\). We also have that \(A\partial A \subset AA_+ \subset A_+\). Therefore, \(A\partial A = A_+\).

For a \(V\)-module \(W\), from Proposition 2.13, \(\text{gr}_\mathcal{E}(W)\) is a module for \(\text{gr}_\mathcal{E}(V)\) as an algebra. We must prove that \(E_n(W)/E_{n+1}(W) \subset A(E_0(W)/E_1(W))\) for \(n \geq 1\).
By Lemma 2.8, $E_n(W)$ is linearly spanned by the subspaces $u_{-2-i}E_{n-1-i}(W)$ for $u \in V, 0 \leq i \leq n - 1$. For $w \in E_{n-1-i}(W)$, we have

$$u_{-2-i}w + E_{n+1}(W) = \frac{1}{(i+1)!}\partial^{i+1}(u + E_1)(w + E_{n-i}(W)).$$

Then it follows immediately from induction.

Combining Lemmas 4.1 and 4.2 we immediately have:

**Corollary 4.3.** Let $V$ be a vertex algebra and let $\text{gr}_E(V)$ be the vertex Poisson algebra obtained in Theorem 2.12. Then $\text{gr}_E(V)$ is linearly spanned by the vectors

$$\partial^{k_1}(v^{(1)} + E_1) \cdots \partial^{k_r}(v^{(r)} + E_1)$$

for $r \geq 1$, $v^{(i)} \in V$, $k_1 > \cdots > k_r \geq 0$. In particular, $\text{gr}_E(V)$ as a differential algebra is generated by the subspace $E_0/E_1 (= V/C_2(V))$.

The following result generalizes a theorem of [GN] (see also [NT]):

**Proposition 4.4.** Let $V$ be any vertex algebra. If $V$ is $C_2$-cofinite, then $V$ is $E_n$-cofinite and $C_{n+2}$-cofinite for any $n \geq 0$.

*Proof.* Since $\dim V/C_2 < \infty$, it follows from Corollary 4.3 that for each $n \geq 0$, the degree $n$ subspace $E_n/E_{n+1}$ of $\text{gr}_E(V)$ is finite dimensional. Consequently, $\dim V/E_n = \dim E_0/E_n < \infty$ for all $n \geq 0$. For any $n \geq 2$, by (3.17) we have $E_m \subset C_n$ for $m = (n-2)2^{n-2}$. Then $\dim V/C_n \leq \dim V/E_m < \infty$. \[\square\]

Furthermore we have (cf. [Bu1,2]):

**Proposition 4.5.** Let $V$ be any vertex algebra and $W$ any $V$-module. If $V$ and $W$ are $C_2$-cofinite, then $W$ is $C_n$-cofinite for all $n \geq 2$.

*Proof.* In the proof of Proposition 4.4, we showed that $\text{gr}_E(V)$ is an $\mathbb{N}$-graded differential algebra with finite-dimensional homogeneous subspaces. Since $\dim W/C_2(W) < \infty$, it follows from (4.7) that all the homogeneous subspaces of $\text{gr}_E(W)$ are finite-dimensional. The same argument of Proposition 4.4 shows that $W$ is $C_n$-cofinite for all $n \geq 2$. \[\square\]

**Remark 4.6.** It has been proved in [Bu1] and [NT] that if $V$ is a vertex operator algebra with nonnegative weights and with $V_{(0)} = \mathbb{C}1$ and if $V$ is $C_2$-cofinite, then any irreducible $V$-module $W$ is $C_n$-cofinite for all $n \geq 2$.

The following result generalizes a theorem of [NG] (cf. [Bu1-2], [ABD]):
Theorem 4.7. Let $V = \bigsqcup_{n \geq 1} V(n)$ be any lower truncated $\mathbb{Z}$-graded vertex algebra such as a vertex operator algebra in the sense of [FLM] and [FHL]. Then for any graded subspace $U$ of $V$, $V = U + C_2(V)$ if and only if $V$ is linearly spanned by the vectors

$$u_{-n_1}^{(1)} \cdots u_{-n_r}^{(r)} 1$$

for $r \geq 0$, $n_1 > \cdots > n_r \geq 1$, $u^{(1)}, \ldots, u^{(r)} \in U$.

Proof. Assume that $V = U + C_2(V)$. Denote by $A$ the vertex Poisson algebra $\text{gr}_\mathcal{E}(V)$ obtained in Theorem 2.12. In particular, $A$ is an $\mathbb{N}$-graded (unital) differential algebra. Recall that $A = \bigsqcup_{n \in \mathbb{N}} A(n)$, where $A(n) = E_n/E_{n+1}$ for $n \in \mathbb{N}$.

Let $K$ be the subspace of $V$, spanned by those vectors in (4.9). Clearly, $K$ is a graded subspace. For $m \geq 0$, set $K_m = K \cap E_m$. For any linear operator $F$ on a vector space and for any nonnegative integer $n$, we set $F^{(n)} = F^n/n!$. From Corollary 4.3, for any $m \geq 0$, $E_m/E_{m+1}$ is linearly spanned by the vectors

$$\partial^{(k_1)}(u^{(1)} + E_1) \cdots \partial^{(k_r)}(u^{(r)} + E_1)$$

for $r \geq 1$, $u^{(i)} \in U$, $k_1 > k_2 > \cdots > k_r \geq 0$ with $k_1 + \cdots + k_r = m$. By definition we have

$$\partial^{(k_1)}(u^{(1)} + E_1) \cdots \partial^{(k_r)}(u^{(r)} + E_1) = (D^{(k_1)}u^{(1)} + E_{k_1+1}) \cdots (D^{(k_r)}u^{(r)} + E_{k_r+1}) = u^{(1)}_{-1-k_1} \cdots u^{(r)}_{-1-k_r} 1 + E_{m+1}.$$  

It follows that $E_m = K_m + E_{m+1}$. Then

$$V = E_0 = K_0 + K_1 + \cdots + K_n + E_{n+1} \subset K + E_{n+1}$$

for any $n \geq 0$. Since $K$ and $E_{n+1}$ are graded subspaces and since $E_m \subset \bigsqcup_{k \geq 2t+n-1} V(k)$ for $m \geq (n-2)2^n$ by (4.17), we must have

$$V = K = K_0 + K_1 + K_2 + \cdots,$$

proving the desired spanning property.

Conversely, assume the spanning property. Notice that if $r \geq 2$, we have $n_1 \geq 2$, so that $u_{-n_1}^{(1)} \cdots u_{-n_r}^{(r)} 1 \in C_2(V)$. If $n_r \geq 2$, we also have $u_{-n_1}^{(1)} \cdots u_{-n_r}^{(r)} 1 \in C_2(V)$. Then we get $V \subset U + C_2(V)$, proving $V = U + C_2(V)$. \hfill $\square$

By slightly modifying the proof of Theorem 4.7 we immediately obtain the following result (cf. [KL]):
Theorem 4.8. Let $V = \coprod_{n \geq 0} V_n$ be a lower truncated $\mathbb{Z}$-graded vertex algebra such as a vertex operator algebra in the sense of [FLM] and [FHL] and let $S$ be a graded subspace of $V$ such that \( \{ u + C_2(V) \mid u \in S \} \) generates $V/C_2(V)$ as an algebra. Then $V$ is linearly spanned by the vectors
\[
u_{-n_1}^{(1)} \cdots u_{-n_r}^{(r)} 1
\]
for $r \geq 0$, $u^{(1)}, \ldots, u^{(r)} \in S$, $n_1 \geq \cdots \geq n_r \geq 1$. Furthermore, if $S$ is linearly ordered, $V$ is linearly spanned by the above vectors with $u^{(i)} > u^{(i+1)}$ when $n_i = n_{i+1}$.

Definition 4.9. Let $S$ be a subset of a vertex algebra $V$. We say that $S$ is a type 0 generating subset of $V$ if $V$ is the smallest vertex subalgebra containing $S$, $S$ is a type 1 generating subset of $V$ if $V$ is linearly spanned by the vectors
\[
u_{-k_1}^{(1)} \cdots u_{-k_r}^{(r)} 1\quad (4.10)
\]
for $r \geq 0$, $u^{(i)} \in S$, $k_i \geq 1$. $S$ is called a type 2 generating subset of $V$ if for any linear order on $S$ (if $S$ is a vector space, replace $S$ with a basis), $V$ is linearly spanned by the above vectors with $u^{(i)} > u^{(i+1)}$ when $n_i = n_{i+1}$.

Remark 4.10. A type 0 generating subset is just a generating subset in the usual sense and a type 1 generating subset of $V$ is also called a strong generating subset $V$ in $[K]$.

Theorem 4.11. Let $V$ be a lower truncated $\mathbb{Z}$-graded vertex algebra and let $U$ be a graded subspace. Then the following three statements are equivalent: (a) $U$ is a type 1 generating subspace of $V$. (b) $U$ is a type 2 generating subspace of $V$. (c) $U/C_2(V) = \{ u + C_2(V) \mid u \in U \}$ generates $V/C_2(V)$ as an algebra.

Proof. By definition, (b) implies (a) and by Theorem 4.8 (c) implies both (a) and (b). Now it suffices to prove that (a) implies (c). Assuming (a) we have that $V/C_2(V)$ is linearly spanned by the vectors $u_{-k_1}^{(1)} \cdots u_{-k_r}^{(r)} 1 + C_2(V)$ for $r \geq 0$, $u^{(i)} \in U$, $k_i \geq 1$. If $k_i \geq 2$ for some $i$, we have $u_{-k_1}^{(1)} \cdots u_{-k_r}^{(r)} 1 \in C_2(V)$. Then $V/C_2(V)$ is linearly spanned by the vectors $u_{-1}^{(1)} \cdots u_{-1}^{(r)} 1 + C_2(V)$ for $r \geq 0$, $u^{(i)} \in U$. That is, $U/C_2(V)$ generates $V/C_2(V)$ as an algebra. \(\square\)

With Lemma 4.2 from the proof of Theorem 4.7 we immediately have:

Proposition 4.12. Let $V$ be a lower truncated $\mathbb{Z}$-graded vertex algebra and let $U$ be a graded subspace of $V$ such that $U$ generates $V/C_2(V)$ as an algebra. Let $W$ be a lower truncated $\mathbb{Z}$-graded $V$-module and let $W^0$ be a graded subspace of $W$ such that $W = W^0 + C_2(W)$. Then $W$ is spanned by the vectors
\[
u_{-k_1}^{(1)} \cdots u_{-k_r}^{(r)} w
\]
for $r \geq 1$, $u^{(1)}, \ldots, u^{(r)} \in U$, $w \in W^0$, $k_1 > \cdots > k_r \geq 0$. 

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