AN IMPROVED BILINEAR ESTIMATE FOR BENJAMIN-ONO TYPE EQUATIONS

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ABSTRACT. A bilinear estimate in Fourier restriction norm spaces with applications to the Cauchy problem

\[ u_t - |D|^{\alpha} u_x + uu_x = 0 \quad \text{in} \ (-T, T) \times \mathbb{R} \]
\[ u(0) = u_0 \]

is proved, for \( 1 < \alpha < 2 \). As a consequence, local well-posedness in \( H^s(\mathbb{R}) \cap \dot{H}^{-\omega}(\mathbb{R}) \) follows for

\[ s > -\frac{3}{4}(\alpha - 1) \quad \text{and} \quad \omega = 1/\alpha - 1/2 \]

This extends to global well-posedness for all \( s \geq 0 \).

1. INTRODUCTION

We consider the Cauchy problem

\[ u_t - |D|^{\alpha} u_x + uu_x = 0 \quad \text{in} \ (-T, T) \times \mathbb{R} \]
\[ u(0) = u_0 \] \hspace{1cm} (1)

for \( 1 < \alpha < 2 \) and we are interested in well-posedness results in low regularity Sobolev spaces.

Our aim is to give an improvement to our previous results [7], where we proved that the Cauchy problem is locally well-posed in \( H^s(\mathbb{R}) \cap \dot{H}^{-\omega}(\mathbb{R}) \) for \( s \geq 1 - \alpha/2 \) and \( \omega = \frac{1}{\alpha} - \frac{1}{2} \). Due to the conserved Hamiltonian this also implied global well-posedness for \( s \geq \frac{\alpha}{2} \). Moreover, using the counterexamples found by Molinet, Saut and Tzvetkov in [13] it was shown that the condition on the low frequencies is sharp in the sense that for \( \omega < \frac{1}{\alpha} - \frac{1}{2} \) the flow map fails to be \( C^2 \). For further references and results we refer the reader to the works of Colliander, Kenig and Staffiliani [3] and Kenig and Koenig [9] and the introduction of [7]. After [7] was completed, we learned that these results were also an improvement to a similar approach by Molinet and Ribaud [12].

Here, by a refined bilinear estimate we observe that the same local well-posedness result holds for all \( s > -\frac{3}{4}(\alpha - 1) \), which immediately implies global well-posedness for all \( s \geq 0 \).

Our analysis includes the range \( 1 < \alpha < 2 \), without the endpoints \( \alpha = 1, 2 \). Very recently, Kenig and Ionescu [8] studied global well-posedness of the Benjamin-Ono equation \( (\alpha = 1) \) for real valued data in \( L^2 \) (see also [2, 14]). We observe that in the limit for \( \alpha \rightarrow 2 \) our lower bound on \( s \) tends to \( -\frac{3}{4} \) which coincides with the results of Kenig, Ponce and Vega [11] for the Korteweg-de Vries equation and the low frequency condition disappears. In the limit for \( \alpha \rightarrow 1 \) the lower bound for \( s \) tends to \( 0 \) and \( \omega \rightarrow \frac{1}{2} \). We believe that the lower bound for \( s \) is optimal, but this is work in progress.

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2. Notation and Definition of the Spaces

Let \( S(\mathbb{R}^n) \) be the space of Schwartz functions on \( \mathbb{R}^n \) and define the Fourier transform by

\[
\mathcal{F} f(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx.
\]

The partial Fourier transform w.r.t. \( t \in \mathbb{R} \) will be denoted by \( \mathcal{F}_t \) \( (\mathcal{F}_x) \). \(|D|^s\) denotes the Fourier multiplier operator with \( \mathcal{F}[|D|^s v]\) \( = |\xi|^s \mathcal{F} v(\xi) \), and \( J^s \) is the operator with symbol \( \langle \xi \rangle^s \).

We write

\[
W_\alpha(t) : H^{(s, \omega)} \rightarrow H^{(s, \omega)}, \mathcal{F}_x W_\alpha(t) u_0(\xi) = e^{it\xi|\eta|^s} \mathcal{F}_x u_0(\xi)
\]

for the solution operator of the linear homogeneous problem, which defines a unitary group on \( H^{(s, \omega)} \).

Throughout this work let \( \psi \in C_0^\infty([-2, 2]) \) be a nonnegative, symmetric function with \( \psi[-1, 1] \equiv 1 \) and let \( \psi_T(t) := \psi(t/T) \).

We use the same spaces as in [7].

**Definition 2.1.** For \( s \geq 0 \) and \( 0 \leq \omega < \frac{1}{2} \) we define the Sobolev space \( H^{(s, \omega)} \) as the completion of \( S(\mathbb{R}) \) with respect to the norm

\[
\|u\|_{H^{(s, \omega)}}^2 := \int_{\mathbb{R}} \langle \xi \rangle^{2s+2\omega} |\xi|^{-2\omega} |\hat{u}(\xi)|^2 \, d\xi. \tag{2}
\]

Sometimes it is convenient to identify \( H^{(s, \omega)} \) and \( H^s(\mathbb{R}) \cap \dot{H}^{-\omega}(\mathbb{R}) \). Our resolution space, a variant of the Bourgain spaces introduced in [1], will be

**Definition 2.2.** For \( 0 \leq \omega < \frac{1}{2} \) and \( s, b \in \mathbb{R} \) we define the space \( X_{s, \omega, b} \) as the completion of \( S(\mathbb{R}^2) \) with respect to the norm

\[
\|u\|_{X_{s, \omega, b}}^2 := \int_{\mathbb{R}^2} |\xi|^{-2\omega} (\langle \xi \rangle^{2s+2\omega} |\tau| + |\xi|^{1+\alpha} 2^\omega |\tau - \xi| \langle \xi \rangle^\omega |\mathcal{F} u(\tau, \xi)|^2) \, d\tau d\xi. \tag{3}
\]

For \( T > 0 \) we define the restriction norm space

\[
X_{s, \omega, b}^T := \{ u[-T,T] \mid u \in X_{s, \omega, b} \}
\]

with norm

\[
\|u\|_{X_{s, \omega, b}^T} = \inf \{ \|\bar{u}\|_{X_{s, \omega, b}} \mid u = \bar{u}[-T,T], \bar{u} \in X_{s, \omega, b} \}.
\]

3. Main Results

Our aim is to prove the following bilinear estimate.

**Theorem 3.1.** Let \( 1 < \alpha < 2 \), \( s \geq s_0 > -\frac{1}{2}(\alpha - 1) \) and \( \omega = \frac{1}{\alpha} - \frac{1}{2} \). There exists \( b' > -\frac{1}{2} \) and \( b \in \left( \frac{1}{2}, b' + 1 \right) \) such that

\[
\|\partial_x (u_1 u_2)\|_{X_{s, \omega, b'}} \leq c \|u_1\|_{X_{s, \omega, b}} \|u_2\|_{X_{s_0, \omega, b}} + \|u_1\|_{X_{s_0, \omega, b}} \|u_2\|_{X_{s, \omega, b}} \tag{4}
\]

for all \( u_1, u_2 \in S(\mathbb{R}^2) \).

This leads to local well-posedness by an application of the contraction mapping principle in a straightforward way. For the general outline of the proof we refer the reader to e.g. [1][4][11]. The minor modifications of these arguments in the \( X_{s, \omega, b} \) spaces are carried out in detail in our previous work [7].

**Theorem 3.2.** Let \( 1 < \alpha < 2 \) and \( \omega = \frac{1}{\alpha} - \frac{1}{2} \). Then, for \( s \geq s_0 > -\frac{1}{2}(\alpha - 1) \), there exists \( b > \frac{1}{2} \) and a non-increasing function \( T' : (0, \infty) \rightarrow (0, \infty) \), such that for any \( u_0 \in H^{(s, \omega)} \) and \( \bar{T} = T'(\|u_0\|_{H^{(s, \omega)}}) \), there exists a solution

\[
u \in X_{s, \omega, b}^T \subset C([-T', T'], H^{(s, \omega)})
\]
of the Cauchy problem
\[ u_t - |D|^{\alpha}u_x + uu_x = 0 \quad \text{in} \ (-T, T) \times \mathbb{R} \]
\[ u(0) = u_0 \]
which is unique in the class of \( X^T_{s_0, \omega, b} \) solutions. Moreover, for any \( r > 0 \) there exists \( T = T(r) \), such that for \( B = \{ v_0 \in H^{(s, \omega)} \mid \| v_0 \|_{H^{(s, \omega)}} \leq r \} \) the flow map
\[ F : H^{(s, \omega)} \supset B \to C([\tau, \xi], H^{(s, \omega)}) \cap X^T_{s, \omega, b} \ , \ u_0 \mapsto u \]
is analytic.

Remark 1. Here, solution always means fixed point of (an extension of) the operator
\[ \Phi_T(u)(t) = \psi(t) W_{\alpha}(t) u_0 - \frac{1}{2} \psi''(t) \int_0^t W_{\alpha}(t - t') \partial_x(u^2)(t') \, dt' \]
in \( X_{s, \omega, b} \). These solutions are solutions in the sense of distributions at least\(^1\) for \( s \geq 0 \).

Together with the a priori bound from Lemma 6.1, this also shows the following

**Theorem 3.3.** Let \( 1 < \alpha < 2 \) and \( \omega = \frac{1}{2} - \frac{1}{2\alpha} \) as well as \( s_0 > -\frac{3}{2}(\alpha - 1) \). Then, for \( s \geq 0 \) there exists \( b > \frac{1}{2} \) such that for every \( T > 0 \) and real valued \( u_0 \in H^{(s, \omega)} \) there exists a real valued solution
\[ u \in X^T_{s, \omega, b} \subset C([\tau, \xi], H^{(s, \omega)}) \]
of the Cauchy problem
\[ u_t - |D|^{\alpha}u_x + uu_x = 0 \quad \text{in} \ (-T, T) \times \mathbb{R} \]
\[ u(0) = u_0 \]
which is unique in \( X^T_{s_0, \omega, b} \). Moreover, the flow map
\[ F : H^{(s, \omega)} \to C([\tau, \xi], H^{(s, \omega)}) \cap X^T_{s, \omega, b} \ , \ u_0 \mapsto u \]
is real analytic.

**4. Preparatory Lemmata**

In this section we will summarize our main tools for the proof of the bilinear estimate. First, we recall the \( L^4_{t}L^\infty_{x} \) Strichartz estimate.

**Lemma 4.1.** For \( b > \frac{1}{2} \) we have
\[ \| J^{\frac{\alpha-1}{2}} u \|_{L^4_t L^\infty_x} \leq c \| u \|_{X_{0,0,b}} \]  \( 5 \)

**Proof.** From [10] Theorem 2.1, we know that
\[ \| |D|^{\frac{\alpha-1}{2}} W_{\alpha}(t) u_0 \|_{L^4_t L^\infty_x} \leq c \| u_0 \|_{L^2} \]
By the general properties of Bourgain spaces, see e.g. [21] Lemma 3.3, the estimate
\[ \| |D|^{\frac{\alpha-1}{2}} u \|_{L^4_t L^\infty_x} \leq c \| u \|_{X_{0,0,b}} \]  \( 6 \)
follows. By smooth cutoffs in frequency, we split \( u \) into a low frequency part \( u^{\text{low}} \) with
\[ F u^{\text{low}}(\tau, \xi) = \psi(\xi) F u(\tau, \xi) \]
and a high frequency part \( u^{\text{high}} := u - u^{\text{low}} \). Then,
\[ \| J^{\frac{\alpha-1}{2}} u \|_{L^4_t L^\infty_x} \leq \| J^{\frac{\alpha-1}{2}} u^{\text{low}} \|_{L^4_t L^\infty_x} + \| J^{\frac{\alpha-1}{2}} u^{\text{high}} \|_{L^4_t L^\infty_x} \]
By an application of the Sobolev inequality, the first part is bounded by
\[ c \| J^{\frac{\alpha-1}{2} + \varepsilon} u^{\text{low}} \|_{L^4_t L^2_x} \leq c \| u \|_{L^4_t L^2_x} \leq c \| u \|_{X_{0,0,b}} \]
\[^1\]Even for \( s < 0 \) one can still use some smoothing properties to verify this
Lemma 4.2. We define the bilinear operator $I^*_F$ via
\[ \mathcal{F} I^*_F(u_1, u_2)(\tau, \xi) = \int_{t = t_1 + t_2}^{t = t_1 + t_2} ||\xi_1|^{2s} - |\xi_2|^{2s}|^\frac{1}{2} \mathcal{F}u_1(\tau_1, \xi_1)\mathcal{F}u_2(\tau_2, \xi_2) d\tau_1 d\xi_1 \]
For $b > \frac{1}{2}$
\[ ||I^*_F(u_1, u_2)||_{L^2}\leq c ||u_1||_{X_0,0,b} ||u_2||_{X_0,0,b} \quad u_1, u_2 \in X_0,0,b \tag{7} \]
Moreover, we define $K^*_F$ as
\[ \mathcal{F} K^*_F(u_1, u_2)(\tau, \xi) = \int_{t = t_1 + t_2}^{t = t_1 + t_2} |||\xi|^\alpha - |\xi_1|^\alpha||^\frac{1}{2} \mathcal{F}u_1(\tau_1, \xi_1)\mathcal{F}u_2(\tau_2, \xi_2) d\tau_1 d\xi_1 \]
$K^*_F$ is the formal adjoint of $u_2 \mapsto I^*_F(u_1, u_2)$ with respect to $L^2_{xt}$ and for $b > \frac{1}{2}$
\[ ||K^*_F(u_1, u_2)||_{X_0,0,-b} \leq c ||u_1||_{X_0,0,b} ||u_2||_{L^2_{xt}} \quad u_1 \in X_0,0,b , u_2 \in L^2_{xt} \tag{8} \]
Finally, we note the elementary resonance relation, which is crucial to exploit the weights in our resolution space.

Lemma 4.3. Let $1 < \alpha < 2$. Define
\[ h(\xi_1, \xi_2, \xi) = \xi|\xi|^\alpha - \xi_1|\xi|^\alpha - \xi_2|\xi|^\alpha \]
Then, for $\xi = \xi_1 + \xi_2$ it holds that
\[ |h(\xi_1, \xi_2, \xi)| \geq c|\xi_{\min}||\xi_{\max}|^\alpha, \tag{9} \]
with $|\xi_{\min}| := \min\{|\xi_1|, |\xi_2|, |\xi|\}$ and $|\xi_{\max}| := \max\{|\xi_1|, |\xi_2|, |\xi|\}$.

5. PROOF OF THE BILINEAR ESTIMATE

Let us fix notation. We define $\sigma = |\tau| + |\xi|^{1+\alpha}$ and $\sigma_i = |\tau_i| + |\xi_i|^{1+\alpha}$ as well as $\lambda = |\tau - \xi|^\alpha$ and $\lambda_i = |\tau_i - \xi_i|^\alpha$. Moreover, we set
\[ f_i(\tau_i, \xi_i) = |\xi_i|^{-\omega}(\xi_i)^{s-\omega}(\lambda_i)^b(\sigma_i)^\omega \mathcal{F}u_i(\tau_i, \xi_i) \]
and
\[ \mathcal{F}u_i(\tau_i, \xi_i) := f_i(\tau_i, \xi_i)(\lambda_i)^{-b}. \]

We use the notation
\[ \int_{t = t_1 + t_2}^{t = t_1 + t_2} g(\tau_1, \xi_1)h(\tau_2, \xi_2) d\tau_1 d\xi_1 \]
We first consider the case $s = s_0 = -\frac{1}{2}(\alpha - 1) + \varepsilon$ for small $\varepsilon > 0$. Our goal is to bound
\[ \left\| \partial_x(u_1u_2) \right\|_{L^2_{xt}} \leq \left\| \xi|^{1-\omega}(\xi)^{s-\omega}(\lambda)^b(\sigma)^\omega \int_{t_1}^{t_2} \left| \int_{t_1}^{t_2} f_i(\tau_i, \xi_i) \right| \right\|_{L^2_{xt}} \]
by the product of the $L^2$ norms of the $f_i$, where we may assume that $0 \leq f_i \in \mathcal{S}(\mathbb{R}^2)$.

Due to the symmetry in $\xi_1, \xi_2$ it suffices to consider the subregion of the domain of integration where $|\xi_1| \leq |\xi_2|$. By the convolution constraint $\xi = \xi_1 + \xi_2$ we then have $|\xi| \leq 2|\xi_2|$. This region is split again into
1. Region $D_1$: $4|\xi_1| \leq |\xi_2|$. Therefore, $|\xi_1| \leq \frac{1}{4}|\xi_2| \leq \frac{1}{3}|\xi| \leq \frac{2}{3}|\xi_2|$. 

2. Region $D_2$: $|\xi_1| \leq |\xi_2| \leq 4|\xi_1|$. Therefore, $|\xi| \leq 2|\xi_2|$, $|\xi| \leq 5|\xi_1|$. 

Let $A, A_1, A_2$ be subregions of the domain of integration, such that in $A$ we have $\langle \lambda \rangle \geq \langle \lambda_1 \rangle, \langle \lambda_2 \rangle$, in $A_1$ we have $\langle \lambda_1 \rangle \geq \langle \lambda \rangle, \langle \lambda_2 \rangle$ and in $A_2$ the inequalities $\langle \lambda_2 \rangle \geq \langle \lambda \rangle, \langle \lambda_1 \rangle$ hold.

We first consider the region $D_1$ and subdivide it into two parts $D_1 = D_{11} \cup D_{12}$, where in $D_{11}$ we have $|\xi_1| \leq 2$ and in $D_{12}$ we have $|\xi_1| \geq 2$. In $D_1$ we see by Lemma 4.3

$$|\lambda - \lambda_1 - \lambda_2| = |h(\xi_1, \xi_2, \xi)| \geq c|\xi_1||\xi|$$

because $|\xi_1| = |\xi_{\text{min}}|$ and $|\xi| \leq 2|\xi_{\text{max}}|$.

Now we start the analysis in the subregion $D_{11}$ where the arguments remain close to those in $\mathcal{U}$. We exploit

$$|\xi|^{\frac{1}{2}} - \frac{\theta}{2} = |\xi|^\alpha \leq c|\xi_1|^\alpha(\chi_A(\lambda)^\omega + \chi_{A_1}(\lambda_1)^\omega + \chi_{A_2}(\lambda_2)^\omega).$$

Therefore in $D_{11}$ the bilinear estimate follows from

$$\sum_{k=0}^2 \| J_{11,k} \|_{L^2} \leq c \prod_{i=1}^2 \| f_i \|_{L^2},$$

where

$$J_{11,0} = \int_s \chi_{D_{11} \cap A}|\xi|^{\frac{1}{2} - \omega} \langle \xi \rangle^{s - \alpha \omega} \langle \lambda \rangle^{b + \omega} \langle \sigma \rangle^{\omega} \|\xi_2\|^2 \prod_{i=1}^2 \frac{f_i(\tau_i, \xi_i)\langle \xi_i \rangle^{\alpha \omega - s}}{\langle \lambda_i \rangle^b \langle \sigma_i \rangle^\omega}$$

and for $k = 1, 2$

$$J_{11,k} = \int_s \chi_{D_{11} \cap A_k}|\xi|^{\frac{1}{2} - \omega} \langle \xi \rangle^{s - \alpha \omega} \langle \lambda \rangle^{b + \omega} \langle \sigma \rangle^{\omega} \|\xi_2\|^2 \prod_{i=1}^2 \frac{f_i(\tau_i, \xi_i)\langle \xi_i \rangle^{\alpha \omega - s}}{\langle \lambda_i \rangle^b \langle \sigma_i \rangle^\omega}$$

We observe that in $D_{11}$

$$\langle \xi_2 \rangle^{\alpha \omega - s} \langle \xi \rangle^{s - \alpha \omega} \leq c \text{ and } \langle \xi_1 \rangle^{\alpha \omega - s} \leq c$$

and $b + \omega \leq 0$ and $\|\xi_2\|^\omega \leq c|\xi|^\omega$ to show that

$$\| J_{11,0} \|_{L^2} \leq c \left( \int_s \chi_{D_{11} \cap A} \|\xi\|^{\frac{1}{2} - \omega} \langle \xi \rangle^{s - \alpha \omega} \langle \lambda \rangle^{b + \omega} \langle \sigma \rangle^{\omega} \|\xi_2\|^2 \prod_{i=1}^2 \frac{f_i(\tau_i, \xi_i)}{\langle \lambda_i \rangle^b \langle \sigma_i \rangle^\omega} \right)^{\frac{1}{2}}$$

Because of the convolution constraint $(\tau, \xi) = (\tau_1, \xi_1) + (\tau_2, \xi_2)$ we also have

$$\frac{\langle \sigma \rangle}{\langle \sigma_1 \rangle \langle \sigma_2 \rangle} \leq c \frac{1}{\min_{1, 2} \|\sigma_i\|} \leq c $$

which implies

$$\| J_{11,0} \|_{L^2} \leq c \int_s \chi_{D_{11} \cap A} \|\xi\|^{\frac{1}{2} - \omega} \langle \xi \rangle^{s - \alpha \omega} \langle \lambda \rangle^{b + \omega} \langle \sigma \rangle^{\omega} \|\xi_2\|^2 \prod_{i=1}^2 \frac{f_i(\tau_i, \xi_i)}{\langle \lambda_i \rangle^b \langle \sigma_i \rangle^\omega} \left| \right|_{L^2}$$

We observe that in $D_{11}$

$$|\xi|^{\frac{1}{2}} \leq c|\xi_2|^\alpha - |\xi_1|^\alpha$$

such that with (7)

$$\| J_{11,0} \|_{L^2} \leq c \left( \int_s \|\xi_2\|^\alpha - |\xi_1|^\alpha \|\xi\|^{\frac{1}{2} - \omega} \langle \xi \rangle^{s - \alpha \omega} \langle \lambda \rangle^{b + \omega} \langle \sigma \rangle^{\omega} \|\xi_2\|^2 \prod_{i=1}^2 \frac{f_i(\tau_i, \xi_i)}{\langle \lambda_i \rangle^b \langle \sigma_i \rangle^\omega} \right)^{\frac{1}{2}}$$

$$\leq c \left( \int_s \|\xi_2\|^\alpha - |\xi_1|^\alpha \|\xi\|^{\frac{1}{2} - \omega} \langle \xi \rangle^{s - \alpha \omega} \langle \lambda \rangle^{b + \omega} \langle \sigma \rangle^{\omega} \|\xi_2\|^2 \prod_{i=1}^2 \frac{f_i(\tau_i, \xi_i)}{\langle \lambda_i \rangle^b \langle \sigma_i \rangle^\omega} \right)^{\frac{1}{2}}$$

$$\leq c \| \mathcal{I}_s^\frac{1}{2} (\nu_1, \nu_2) \|_{L^2} \leq c \prod_{i=1}^2 \|\nu_i\|_{X_{0, 0, b}} = c \prod_{i=1}^2 \| f_i \|_{L^2}$$
since $b > 1/2$. For $J_{11,1}$ we use (11) and (12) again and get

$$
\|J_{11,1}\|_{L^2} \leq c \left\| \int \chi_{D_{11} \cap A_1} |\xi|^{\frac{\sigma}{2}} (\lambda)^{b'} f_1(\tau_1, \xi_1)(\lambda_1)^{-b} f_2(\tau_2, \xi_2)(\lambda_2)^{-b} \right\|_{L^2}
$$

We may assume that $|\lambda_1| \geq 2|\lambda|$, because otherwise the same argument as for $J_{11,0}$ applies. If $\langle \sigma_1 \rangle \leq \langle \sigma_2 \rangle$ we have $\langle \lambda_1 \rangle^\omega \leq \min_{i=1,2} (\sigma_i)^\omega$. If we suppose that $\langle \sigma_2 \rangle \leq \langle \sigma_1 \rangle$ we see

$$
|\lambda_1| = |\tau_1 - \xi_1| \leq |\tau_2 - \xi_2| + |\xi_1| \leq |\lambda| + 16|\sigma_2|
$$

since we are in region $D_{11}$. This implies $\langle \lambda_1 \rangle \leq c\langle \sigma_2 \rangle$ and we also have

$$
\langle \lambda_1 \rangle^\omega \leq c \min_{i=1,2} (\sigma_i)^\omega.
$$

Therefore,

$$
\|J_{11,1}\|_{L^2} \leq c \left\| \int \chi_{D_{11} \cap A_1} |\xi|^{\frac{\sigma}{2}} (\lambda)^{b'} f_1(\tau_1, \xi_1)(\lambda_1)^{-b} f_2(\tau_2, \xi_2)(\lambda_2)^{-b} \right\|_{L^2}
$$

In $D_{11}$ we have $|\xi|^{\frac{\sigma}{2}} \leq c |\xi_2|^{\alpha} - |\xi_1|^{\alpha} \frac{\sigma}{2}$ and by assumption $b' \leq 0$, such that we may proceed as above with $J_0$ and use the estimate (7) to conclude

$$
\|J_{11,1}\|_{L^2} \leq c \left\| \int |\xi_2|^{\alpha} - |\xi_1|^{\alpha} \frac{\sigma}{2} \prod_{i=1}^2 f_i(\tau_i, \xi_i) \right\|_{L^2} \leq c \prod_{i=1}^2 \|f_i\|_{L^2}
$$

For $J_{11,2}$, we have by (11) and (12)

$$
\|J_{11,2}\|_{L^2} \leq c \left\| \int \chi_{D_{11} \cap A_2} |\xi|^{\frac{\sigma}{2}} (\lambda)^{b'} f_1(\tau_1, \xi_1)(\lambda_1)^{-b} f_2(\tau_2, \xi_2)(\lambda_2)^{-b} \right\|_{L^2}
$$

In $D_{11} \cap A_2$ we have

$$
|\xi|^{\frac{\sigma}{2}} \leq c |\xi|^{\alpha} - |\xi_1|^{\alpha} \frac{\sigma}{2} \text{ and } \langle \lambda_2 \rangle^{-b} \leq \langle \lambda \rangle^{-b}
$$

such that, because of $b' + \omega \leq 0$,

$$
\|J_{11,2}\|_{L^2} \leq c \left\| K^{\sigma} (\tau_1, F^{-1} f_2) \right\|_{X_{0,0,-b}} \leq c \|v_1\|_{X_{0,0,b}} \| F^{-1} f_2 \|_{L^2} = c \prod_{i=1}^2 \|f_i\|_{L^2}
$$

for $b > 1/2$ by the estimate (5).

Let us now consider the region $D_{12}$. We define the contributions

$$
J_{12,0} = \int \chi_{D_{12} \cap A} |\xi|^{1-\omega} (\xi)^{s-\alpha\omega} (\lambda)^{b'} (\sigma)^{-\omega} \prod_{i=1}^2 \frac{|\xi_i|^{\alpha\omega-s} f_i(\tau_i, \xi_i)}{(\lambda_i)^b (\sigma_i)^\omega}
$$

and, for $k = 1, 2$,

$$
J_{12,k} = \int \chi_{D_{12} \cap A_k} |\xi|^{1-\omega} (\xi)^{s-\alpha\omega} (\lambda)^{b'} (\sigma)^{-\omega} \prod_{i=1}^2 \frac{|\xi_i|^{\alpha\omega-s} f_i(\tau_i, \xi_i)}{(\lambda_i)^b (\sigma_i)^\omega}
$$

In the subregion $D_{12} \cap A$ we use

$$
|\xi|^{-b'} (\xi_1)^{-b'} \leq c(\lambda)^{-b'}
$$

and

$$
\|J_{12,0}\|_{L^2} \leq c \left\| \int \chi_{D_{12} \cap A} |\xi|^{1-\alpha\omega} (\xi)^{s-\alpha\omega} (\lambda)^{b'} (\xi_1)^{-\omega-s} (\xi_2)^{\alpha\omega-s} (\lambda_1)^b (\sigma_1)^\omega \prod_{i=1}^2 \frac{f_i(\tau_i, \xi_i)}{(\lambda_i)^b (\sigma_i)^\omega} \right\|_{L^2}
$$
Using \(\langle \xi \rangle^{\alpha - s} \langle \xi \rangle^s - \alpha \omega \leq c \) and (12), this is bounded by
\[
\left\| \int \chi_{D_{12} \cap A} \langle \xi \rangle^{1 + \alpha b'} \langle \xi \rangle^{\alpha \omega + s + \omega} \left( \prod_{i=1}^{2} f_i(\tau_i, \xi_i) \right) \right\|_{L^2}
\]
Now, for \( b' + \omega \leq 0 \) we estimate
\[
\langle \xi \rangle^{1 + \alpha b'} \leq c \langle \xi \rangle^{\frac{1}{2} + \alpha b'}
\]
since \( 1 - \frac{\omega}{2} + \alpha b' \leq 0 \). Moreover,
\[
1 - \frac{\alpha}{2} + \alpha b' + b' + \alpha \omega - s + \omega - (1 + \alpha)\omega = 1 - \frac{\alpha}{2} + \alpha b' + b' - s
\]
which is negative for
\[
s \geq \alpha \left( -\frac{1}{2} + b' \right) + 1 + b'
\]
Therefore, choosing \( b' \leq \min\{ -\omega, -\frac{1}{2} \} \), we continue for \( s \geq -\frac{\alpha}{4}(\alpha - 1) \) with
\[
\left\| \int \chi_{D_{12} \cap A} \langle \xi \rangle^{\frac{1}{2}} \left( \prod_{i=1}^{2} f_i(\tau_i, \xi_i) \right) \right\|_{L^2} \leq c \left\| J_{12}^2(v_1, v_2) \right\|_{L^2} \leq c \prod_{i=1}^{2} \left\| f_i \right\|_{L^2}
\]
Next, we study the contribution of \( J_{12,1} \). We may assume that \( \langle \lambda_1 \rangle \geq 2 \langle \lambda \rangle \), because otherwise we use the same argument as in \( D_{12} \cap A \). In \( D_{12} \cap A_1 \) we exploit
\[
\langle \xi \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^{\frac{1}{2}} \leq c \langle \lambda_1 \rangle^{\frac{1}{2}}
\]
We observe that
\[
|\lambda_1| = |\tau_1 - \xi_1| |\xi_1|^{\alpha} \leq |\lambda| + c \langle \sigma_2 \rangle \Rightarrow \langle \lambda_1 \rangle \leq c \langle \sigma_2 \rangle
\]
and therefore
\[
\langle \lambda_1 \rangle^{\omega} \leq c \min_{i=1,2} \langle \sigma_i \rangle^{\omega}
\]
This shows
\[
\| J_{12,1} \|_{L^2} \leq c \left\| \int \chi_{D_{12} \cap A_1} \langle \lambda \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^{\frac{1}{2}} \langle \xi \rangle^{\alpha b' + \alpha \omega - s} \langle \lambda_1 \rangle^{\frac{1}{2} - b} \langle \lambda_2 \rangle^{\frac{1}{2} - b} \left( \prod_{i=1}^{2} f_i(\tau_i, \xi_i) \right) \right\|_{L^2}
\]
We choose \( b > \frac{1}{2} \) and in \( D_{12} \) we have \( |\xi_1| \leq |\xi_2| \). Since we only consider \( s \leq \frac{1}{2} - \frac{\alpha}{2} \) (which means \( \varepsilon \leq \frac{\alpha - 1}{2} \)), we have
\[
\| J_{12,1} \|_{L^2} \leq c \left\| \int \langle \lambda \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2} - \frac{\alpha}{2} - s} \langle \lambda_2 \rangle^{\frac{1}{2} - b} \left( \prod_{i=1}^{2} f_i(\tau_i, \xi_i) \right) \right\|_{L^2}
\]
With \( b' \leq -\frac{1}{4} \) and Sobolev in time we see
\[
\| J_{12,1} \|_{L^2} \leq c \left\| F^{-1} f_1. J^{\frac{1}{2} - \frac{\alpha}{2} - s} v_2 \right\|_{L^{1/3} L^2_x}
\leq c \| f_1 \|_{L^1 L^2_x} \| J^{\frac{1}{2} - \frac{\alpha}{2} - s} v_2 \|_{L^1 L^2_x}
\]
Finally, by (5)
\[
\| J^{\frac{1}{2} - \frac{\alpha}{2} - s} v_2 \|_{L^1 L^2_x} \leq c \| v_2 \|_{X_{0,0,b}} = \| F_2 \|_{L^2}
\]
if \( \frac{1}{2} - \frac{\alpha}{2} - s \leq \frac{\alpha - 1}{2} \), which is equivalent to \( s \geq -\frac{\alpha}{4}(\alpha - 1) \).
Now we turn to the contribution of \( D_{12} \cap A_2 \), where we use
\[
\langle \xi \rangle^{-\alpha b'} \langle \xi_1 \rangle^{-b'} \leq c \langle \lambda_2 \rangle^{-b'}
\]
and it follows
\[ \| J_{12,2} \|_{L^2} \leq c \left\| \int_{\mathbb{R}^n} \chi_{D_{12} \cap A_2} |\xi|^{1+\alpha b'} \frac{\langle \xi \rangle^{b' + \omega + \alpha \omega - s}}{\min_{i=1,2} \langle \sigma_i \rangle^\omega} \langle \lambda \rangle^{b'} \langle \lambda_2 \rangle^{-b'} \langle \lambda_1 \rangle - b \prod_{i=1}^2 f_i(\tau_i, \xi_1) \right\|_{L^2} \]
We have
\[ \langle \lambda \rangle^{b'} \langle \lambda_2 \rangle^{-b'} \leq \langle \lambda \rangle^{-b} \]
and
\[ \min_{i=1,2} \langle \sigma_i \rangle^\omega \geq \langle \xi_1 \rangle^{(1+\alpha)\omega} \]
and if \( b' \leq -\omega \) we have \( 1 + \alpha b' - \frac{\omega}{2} \leq 0 \) and therefore
\[ |\xi|^{1+\alpha b'} \leq c |\xi|^{\frac{\omega}{2}} \langle \xi_1 \rangle^{1+\alpha b' - \frac{\omega}{2}} \]
If \( b' \leq -\frac{1}{2} \) and \( s \geq -\frac{3}{4}(\alpha - 1) \) we estimate \( b' - s + 1 + \alpha b' - \frac{\omega}{2} \leq 0 \) and
\[ |\xi|^{\frac{\omega}{2}} \leq c |\xi|^{\alpha} - |\xi_1|^{\frac{\omega}{2}} \]
and therefore, by the dual bilinear Strichartz estimate \( (8) \)
\[ \| J_{12,2} \|_{L^2} \leq c \left\| \int_{\mathbb{R}^n} |\xi|^\alpha - |\xi_1|^\alpha \frac{1}{2} \langle \lambda \rangle^{-b} \langle \lambda_1 \rangle - b \prod_{i=1}^2 f_i(\tau_i, \xi_1) \right\|_{L^2} \]
\[ \leq c \prod_{i=1}^2 \| f_i \|_{L^2} \]
This completes the discussion of the subregion \( D_1 \).
Let us now consider the domain \( D_2 \), where \( |\xi_1| \leq |\xi_2| \leq 4|\xi_1|, |\xi| \leq 2|\xi_2| \) and \( |\xi| \leq 5|\xi_1| \). We subdivide \( D_2 = D_{21} \cup D_{22} \), where in
\[ D_{21} : \xi_1 \xi_2 > 0 \text{ or } |\xi| \geq \frac{1}{2}|\xi_1| \text{ or } |\xi_2| \leq 1 \]
and in
\[ D_{22} : \xi_1 \xi_2 < 0 \text{ and } |\xi| \leq \frac{1}{2}|\xi_1| \text{ and } |\xi_2| \geq 1 \]
additionally hold. As above, we define for \( j = 1, 2 \)
\[ J_{2j,0} = \int_{\mathbb{R}^n} \chi_{D_{2j} \cap A} |\xi|^{1-\omega} \langle \xi \rangle^{s-\alpha \omega} \langle \lambda \rangle^{b'} \langle \sigma \rangle^\omega \prod_{i=1}^2 |\xi|^{\omega} \langle \xi_i \rangle^{\alpha \omega - s} f_i(\tau_i, \xi_1) \]
and for \( k = 1, 2 \)
\[ J_{2j,k} = \int_{\mathbb{R}^n} \chi_{D_{2j} \cap A_k} |\xi|^{1-\omega} \langle \xi \rangle^{s-\alpha \omega} \langle \lambda \rangle^{b'} \langle \sigma \rangle^\omega \prod_{i=1}^2 |\xi|^{\omega} \langle \xi_i \rangle^{\alpha \omega - s} f_i(\tau_i, \xi_1) \]
We start with the discussion of \( D_{21} \), where all frequencies are of comparable size or smaller then a constant, which shows that
\[ \frac{|\xi|^{1-\omega} \langle \xi \rangle^{s-\alpha \omega} |\xi_1|^{\omega} |\xi_2|^{\omega} \langle \sigma \rangle^\omega}{\langle \xi_1 \rangle^{s-\alpha \omega} \langle \xi_2 \rangle^{s-\alpha \omega} \langle \sigma_1 \rangle^\omega \langle \sigma_2 \rangle^\omega} \leq c(\xi)^{1-s} \]
Therefore,
\[ \| J_{21,0} \|_{L^2} \leq c \left\| \int_{\mathbb{R}^n} \chi_{D_{21} \cap A} |\xi|^{1-s} \langle \lambda \rangle^{b'} \prod_{i=1}^2 f_i(\tau_i, \xi_1) \right\|_{L^2} \]
In \( A \) we have
\[ \langle \xi \rangle^{-b' (1+\alpha)} \leq c \langle \lambda \rangle^{-b'} \]
and we use the Strichartz estimate \( \text{(5)} \) to conclude
\[
\| J_{21,0} \|_{L^2} \leq c \left\| \int_\tau (\xi^{1+s+b'(1+\alpha)} - \frac{\alpha-1}{2}\chi^{\alpha-1}) F v_1(\tau_1, \xi_1) F v_2(\tau_2, \xi_2) \right\|_{L^2} \\
\leq c \| J^{\frac{\alpha-1}{2}} v_1 \|_{L^2_t L^\infty_x} \| v_2 \|_{L^2_t L^2_x} \leq c \prod_{i=1}^2 \| f_i \|_{L^2}
\]
since \( 1 - s + b'(1+\alpha) - \frac{\alpha-1}{2} \leq 0 \), which is equivalent to \( \frac{s}{2} + b' - \frac{\alpha}{2} + \alpha b' \leq s \). This is fulfilled for \( b' \leq -\frac{1}{2} + \frac{\alpha}{2} \). In \( A_1 \) we have
\[
\langle \xi \rangle \langle (1+\alpha) \rangle \leq c \langle \lambda \rangle^b
\]
and we use Sobolev in time and the Strichartz estimate \( \text{(5)} \) to conclude for \( b' \leq -\frac{1}{4} \)
\[
\| J_{21,1} \|_{L^2} \leq c \left\| \int_\tau (\xi^{1-s-b(1+\alpha)} - \frac{\alpha-1}{2}\chi^{\alpha-1}) F f_1(\tau_1, \xi_1) F f_2(\tau_2, \xi_2) \right\|_{L^2} \\
\leq c \| F^{-1} F^{\frac{\alpha-1}{2}} v_1 \|_{L^2} \| v_2 \|_{L^2} \leq c \| f_1 \|_{L^2} \| J^{\frac{\alpha-1}{2}} v_2 \|_{L^2} \\
\leq c \prod_{i=1}^2 \| f_i \|_{L^2}
\]
The same argument applies to \( J_{21,2} \) by exchanging the roles of \( f_1, f_2 \).

Finally, we turn to the contributions from the region \( D_{22} \). Here, we have \( \xi_1 \xi_2 < 0 \). Therefore, we may write \( \xi_1 = \beta \xi_2 \) for \( \beta \in [-1, -\frac{1}{4}] \). By the mean value theorem, this shows
\[
|\langle \xi_1 \rangle^\alpha - \langle \xi_2 \rangle^\alpha|^{\frac{1}{2}} \leq |\beta|^\alpha - 1 |\xi_2|^{\frac{1}{2}} \geq \frac{1}{2} |\beta| - 1 |\xi_2|^{\frac{1}{2}} = \frac{1}{2} |\xi_2|^{\frac{1}{2}} |\xi_2|^{\frac{\alpha-1}{2}} (13)
\]
Let us start with the subregion \( A \). We have
\[
\langle \sigma \rangle^\omega \leq c \langle \lambda \rangle^\omega + c \chi_{|\xi| \geq 1} \langle \xi \rangle^{\omega + \alpha \omega}
\]
which shows
\[
\| J_{22,0} \|_{L^2} \leq c \left\| \int_\tau \chi_{D_{22} \cap A} |\xi|^{1-\omega} \langle \xi \rangle^{s-\alpha \omega} \langle \lambda \rangle^{b'} \chi \prod_{i=1}^2 \frac{|\xi_i|^\omega \langle \xi_i \rangle^{\omega - \omega} f_i(\tau_i, \xi_i)}{\langle \lambda_i \rangle^b}\langle \sigma_i \rangle^\omega \right\|_{L^2} \\
+ c \left\| \int_\tau \chi_{D_{22} \cap A} \chi_{|\xi| \geq 1} |\xi|^{1+s} \langle \lambda \rangle^{b'} \chi \prod_{i=1}^2 \frac{|\xi_i|^\omega \langle \xi_i \rangle^{\omega - \omega} f_i(\tau_i, \xi_i)}{\langle \lambda_i \rangle^b}\langle \sigma_i \rangle^\omega \right\|_{L^2}
\]
Using
\[
|\xi|^{-b'-\omega} \langle \xi_2 \rangle^{-\alpha b'-\alpha \omega} \leq c \langle \lambda \rangle^{-b'-\omega}
\]
and \( \text{(13)} \) we see that the first term is bounded by
\[
\left\| \int_\tau \chi_{D_{22} \cap A} |\xi|^{1+b'} \langle \xi \rangle^{s-\alpha \omega} \langle \xi_2 \rangle^{-2s+\alpha b'+\alpha \omega} \chi \prod_{i=1}^2 \frac{f_i(\tau_i, \xi_i)}{\langle \lambda_i \rangle^b} \right\|_{L^2} \\
\leq c \left\| \int_\tau \langle \xi \rangle^{\frac{\alpha}{2} + b'} \langle \xi_2 \rangle^{-2s+\alpha b'+\alpha \omega} \chi \prod_{i=1}^2 \frac{|\xi_i|^\omega \langle \xi_i \rangle^{\omega - \omega} f_i(\tau_i, \xi_i)}{\langle \lambda_i \rangle^b} \right\|_{L^2} \\
\]
If \( b' \leq -\frac{1}{2} \) and \( \epsilon \leq \frac{1}{4} \), then \( \frac{1}{2} + b' + s - \alpha \omega \leq 0 \). Moreover, for \( b' \leq -\frac{1}{2} + \epsilon \), we have
\[-2s + \alpha b' + \alpha \omega - \frac{\alpha-1}{2} \leq 0 \]. Then, by the bilinear Strichartz estimate \( \text{(7)} \) this is bounded by
\[
\cdots \leq c \left\| \prod_{i=1}^2 \frac{f_i(\tau_i, \xi_i)}{\langle \lambda_i \rangle^b} \right\|_{L^2} \\
\leq c \prod_{i=1}^2 \| f_i \|_{L^2}
\]
For the second term we use
\[ |\xi|^{-b'} \langle \xi_2 \rangle^{-\alpha b'} \leq c(\lambda)^{-b'} \]
and find with (13)
\[
\cdots \leq c \left\| \int_{s} \chi \mathcal{D}_{2} = \chi \chi_{|\xi| \geq 1} \langle \xi \rangle^{s + b'} \langle \xi_2 \rangle^{2s} \sum_{i=1}^{2} f_i(\tau_i, \xi) \right\|_{L^2} \\
\leq c \left\| \int_{s} \langle \xi \rangle^{s + b'} \langle \xi_2 \rangle^{2s - \frac{a}{2}} \left| |\xi_1|^{a} - |\xi_2|^{a} \right| \frac{1}{\langle \lambda_{1} \rangle^{b}} \frac{\sum_{i=1}^{2} f_i(\tau_i, \xi)}{\langle \lambda_{i} \rangle^{b}} \right\|_{L^2} \\
\]
We only consider \( \varepsilon < \frac{3}{4}(\alpha - 1) \). Then, for \( b' \leq -\frac{1}{2} + \frac{3}{4}(\alpha - 1) - \varepsilon \) we observe that \( s + b' \leq 0 \). Moreover \( a b' - 2s - \frac{a}{2} \leq 0 \) for \( b' \leq -\frac{1}{2} + \varepsilon \). Using the bilinear Strichartz estimate (7), we arrive at
\[
\left\| J_{22,0} \right\|_{L^2} \leq c \sum_{i=1}^{2} \left\| f_i \right\|_{L^2} \\
\]
Next, we consider the subregion \( A_1 \). We have
\[
\langle \sigma \rangle^{\omega} \leq c\langle \lambda_1 \rangle^{\omega} + c\chi_{|\xi| \geq 1} \langle \xi \rangle^{\omega + \alpha \omega} \\
\]
which shows
\[
\left\| J_{22,1} \right\|_{L^2} \leq c \left\| \int_{s} \chi \mathcal{D}_{2} \cap A_1 \left\| |\xi|^{1-\omega} \langle \xi \rangle^{s - \alpha \omega} \langle \xi_2 \rangle^{-2s} \langle \lambda \rangle^{b'} \langle \lambda_1 \rangle^{-b + \omega} \langle \lambda_2 \rangle^{-b} \frac{\sum_{i=1}^{2} f_i(\tau_i, \xi)}{\langle \lambda_{i} \rangle^{b}} \right\|_{L^2} \\
+ c \left\| \int_{s} \chi \mathcal{D}_{2} \cap A_1 \chi_{|\xi| \geq 1} \langle \xi \rangle^{1+s} \langle \lambda \rangle^{b'} \langle \lambda_1 \rangle^{-b} \langle \lambda_2 \rangle^{-2s} \frac{\sum_{i=1}^{2} f_i(\tau_i, \xi)}{\langle \lambda_{i} \rangle^{b}} \right\|_{L^2} \\
\]
As above, by
\[
\left| |\xi|^{-b'} - \omega \langle \xi_2 \rangle^{-\alpha b' - \omega} \right| \leq c(\lambda_1)^{-b' - \omega} \\
\]
we see that the first term is bounded by
\[
\left\| \int_{s} \chi \mathcal{D}_{2} \cap A_1 \left\| |\xi|^{1+b'} \langle \xi \rangle^{s - \alpha \omega} \langle \xi_2 \rangle^{-2s + a b' + \alpha \omega} \langle \lambda \rangle^{-b} \langle \lambda_2 \rangle^{-b} \frac{\sum_{i=1}^{2} f_i(\tau_i, \xi)}{\langle \lambda_{i} \rangle^{b}} \right\|_{L^2} \\
\leq c \left\| \int_{s} \langle \xi \rangle^{1+b'} + s - \alpha \omega \langle \xi_2 \rangle^{-2s + a b' + \alpha \omega - \frac{a}{2}} \left| |\xi|^{a} - |\xi_2|^{a} \right| \frac{\sum_{i=1}^{2} f_i(\tau_i, \xi)}{\langle \lambda_{i} \rangle^{b}} \right\|_{L^2} \\
\]
Here, we used that due to \( |\xi| \leq \frac{3}{4}|\xi_2| \) and \( |\xi_2| \geq 1 \) we have
\[
\left| |\xi|^{a} - |\xi_2|^{a} \right| \frac{1}{\langle \lambda_{2} \rangle^{b}} \leq c(\xi_2)^{-\frac{1}{b}} \\
\]
By estimating \( \langle \xi \rangle^{\frac{1}{2}} \leq \langle \xi_2 \rangle^{\frac{1}{2}} \) and with the same restrictions on \( s, b' \) as above we may apply the dual bilinear Strichartz estimate (6) and get
\[
\cdots \leq c \left\| \int_{s} \langle |\xi|^{a} - |\xi_2|^{a} \rangle^{\frac{1}{2}} \langle \lambda \rangle^{-b} \langle \lambda_2 \rangle^{-b} \frac{\sum_{i=1}^{2} f_i(\tau_i, \xi)}{\langle \lambda_{i} \rangle^{b}} \right\|_{L^2} \leq c \left\| f_i \right\|_{L^2} \\
\]
For the second term we use
\[
\left| |\xi|^{-b'} \langle \xi_2 \rangle^{-\alpha b'} \right| \leq c(\lambda_1)^{-b'} \\
\]
and find
\[
\ldots \leq c \left\| \left( \xi^{1+s+b'} \right)^{\frac{2}{3}-2s-\frac{2}{3}} \right\| L^2 \leq c \prod_{i=1}^{2} \| f_i \|_{L^2}
\]
by (8) with the same restrictions on \( s, b' \) as in the region \( A_2 \), since \( \langle \xi \rangle^\frac{2}{3} \leq \langle \xi_2 \rangle^\frac{2}{3} \).

Finally, we return to the region \( A_2 \). In \( D_{22} \) the frequencies \( \xi_1 \) and \( \xi_2 \) are of comparable size and due to \( |\xi| \leq \frac{3}{2} |\xi_1| \) and \( |\xi_1| \geq \frac{1}{3} |\xi_2| \geq \frac{1}{3} \) we have
\[
|\xi|^{\alpha} - |\xi_1|^{\alpha} \geq c \langle \xi_1 \rangle^{\frac{2}{3}}.
\]
Now we use the same argument as \( A_1 \) with the roles of \( f_1, f_2 \) exchanged.

This finishes the proof of the bilinear estimate for \( s = s_0 = -\frac{3}{4} (\alpha - 1) + \varepsilon \). Let \( \rho = s - s_0 \). Because of
\[
\langle \xi \rangle^\rho \leq c \langle \xi_1 \rangle^\rho + c \langle \xi_2 \rangle^\rho
\]
we see
\[
\| \partial_x (u_1 u_2) \|_{X_{s,w,b'}} \leq c \| \partial_x (J^\rho u_1 u_2) \|_{X_{s_0,w,b'}} + \| \partial_x (u_1 J^\rho u_2) \|_{X_{s_0,w,b'}}
\]
\[
\leq c \| u_1 \|_{X_{s_0,w,b'}} \| u_2 \|_{X_{s_0,w,b'}} + \| u_1 \|_{X_{s_0,w,b'}} \| u_2 \|_{X_{s_0,w,b'}}
\]
This proves that for all \( s, s_0 > -\frac{3}{4} (\alpha - 1) \) we find suitable \( b' \in (-\frac{1}{2}, 0) \) and \( b \in (\frac{1}{2}, b' + 1) \) such that the bilinear estimate holds true. \( \square \)

6. AN A PRIORI BOUND

This section is devoted to the proof of an a priori bound for the \( H^{(0,\omega)} \) norm, which allows an iteration of the local argument to prove global well-posedness for \( s \geq 0 \).

**Lemma 6.1.** Let \( s \geq 0 \). There exists \( C > 0 \), such that for all smooth, real valued solutions \( u \) of (11), we have
\[
\sup_{t \in [-T, T]} \| u(t) \|_{H^{(0,\omega)}} \leq C \| u(0) \|_{H^{(0,\omega)}} + CT \| u(0) \|_{H^{(0,\omega)}}^2
\]
\[
(14)
\]

**Proof.** We easily verify the conservation law
\[
\| u(t) \|_{L^2}^2 = \| u(0) \|_{L^2}^2, \quad t \in (-T, T)
\]
Therefore it suffices to prove an a priori estimate for the low frequency part in \( \dot{H}^{-\omega} \). Let \( \psi \in C_0^\infty([-2, 2]) \) be nonnegative with \( \psi_{[-1,1]} = 1 \). We define
\[
\mathcal{F}_x v(t)(\xi) = \psi(\xi) \xi^{-\omega} \mathcal{F}_x u(t)(\xi)
\]
The function \( v \) solves the equation
\[
v_t - |D|^\alpha v_x = f \quad \text{in} \quad (-T, T) \times \mathbb{R}
\]
\[
v(0) = v_0
\]
where \( \mathcal{F}_x v_0(\xi) = \psi(\xi) \xi^{-\omega} \mathcal{F}_x u(0)(\xi) \) and
\[
\mathcal{F}_x f(t)(\xi) = -\frac{i}{2} \psi(\xi) \xi^{-\omega} \mathcal{F}_x u^2(t)(\xi)
\]
For fixed $t$ we estimate
\[ \|f(t)\|_{L^2} \leq c \|\psi(\xi)F_x u^2(t)(\xi)\|_{L^2} \leq c \|F_x u^2(t)\|_{L^\infty} \]
\[ \leq c \|u^2(t)\|_{L^1} \leq c \|u(t)|^2 \|_{L^2} \]
This shows
\[ \|v\|_{L^\infty T L^2} \leq c \|v_0\|_{L^2} + c \|f\|_{L^1 T L^2} \leq c \|u(0)\|_{H^{(0,\omega)}} + cT \|u\|^2_{L^\infty T L^2} \]
\[ \leq c \|u(0)\|_{H^{(0,\omega)}} + cT \|u(0)\|^2_{H^{(0,\omega)}} \]
\[ \square \]

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