Needles and straw in a haystack: robust confidence for possibly sparse sequences

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Abstract

In the general signal+noise model we construct an empirical Bayes posterior (linked to the penalization method) which we then use for uncertainty quantification for the unknown, possibly sparse, signal. We introduce a novel excessive bias restriction (EBR) condition, which gives rise to a new slicing of the entire space that is suitable for uncertainty quantification. Under EBR and some mild conditions on the noise, we establish the local (oracle) optimality of the proposed confidence ball. In passing, we also get the local optimal (oracle) results for estimation and posterior contraction problems. Adaptive minimax results (also for the estimation and posterior contraction problems) over various sparsity classes follow from our local results.

1 Introduction

The model and the main problem. Suppose we observe $X = X^{(\sigma,n)} = (X_1, \ldots, X_n)$:

$$X_i = \theta_i + \sigma \xi_i, \quad i \in \mathbb{N}_n = \{1, \ldots, n\},$$

where $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ is an unknown high-dimensional parameter of interest, the $\xi_i$'s are random errors with $E\xi_i = 0$, $\text{Var}(\xi_i) \leq C\xi$, $\sigma > 0$ is the known noise intensity. The goal is to make inference about the parameter $\theta$ based on the data $X$: recovery of $\theta$ and uncertainty quantification by constructing an optimal confidence set. We pursue the robust inference in the sense that the distribution of the error vector $\xi = (\xi_1, \ldots, \xi_n)$ is unknown, but assumed to satisfy only certain mild exchangeable exponential moment condition; see Condition (A1) in Section 2. For inference on $\theta$, we exploit the empirical Bayes approach and make connection with the penalization method. We derive the non-asymptotic results, which imply asymptotic assertions if needed. Possible asymptotic regimes are decreasing noise level $\sigma \to 0$, high-dimensional setup $n \to \infty$ (the leading case in the literature for high dimensional models), or their combination, e.g., $\sigma = n^{-1/2}$ and $n \to \infty$.

Useful inference is not possible without some structure on the parameter $\theta$. Popular structural assumptions are smoothness and sparsity, and in this paper we are concerned with the latter. The best studied problem in the sparsity context is that of estimating $\theta$ in

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In this paper, for inference on \( \theta \) we use a penalization method which originates from an empirical Bayes approach. Since any Bayesian approach always delivers a posterior \( \pi(\vartheta|X) \) (in the posteriors for \( \theta \), we will use the variable \( \vartheta \) to distinguish it from the “true” \( \theta \)), an accompanying problem of interest is the contraction of the resulting (empirical Bayes) posterior to the “true” \( \theta \) from the frequentist perspective of the “true” measure \( P_{\theta} \), which is the distribution of \( X \) from model (1). Clearly, the quality of posterior is characterized by the the posterior contraction rate. In this paper we allow this to be a local quantity, i.e., depending on the true \( \theta \), whereas usually in the literature on Bayesian nonparametrics it is related to the minimax estimation rates over certain classes.

A common Bayesian way to model sparsity structure is by the so called two-groups priors. Such a prior puts positive mass on vectors \( \theta \) with some exact zero coordinates (zero group) and the remaining coordinates (signal group) are drawn from a chosen distribution. So the marginal prior for each coordinate is a mixture of a continuous distribution and a point-mass at zero. In [11] it is shown that for a suitably chosen two-groups prior, the posterior concentrates around the true \( \theta \) at the minimax rate (as \( n \to \infty \)) for two sparsity classes, nearly black vectors \( \ell[p_n] \) with \( p_n \) nonzero coordinates and weak \( \ell_s \)-balls \( m_s[p_n] \). As pointed out by [11] (also by [15]), the distributions of non-zero coordinates should not have too light tails, otherwise one gets sub-optimal rates. The important Gaussian case is for example excluded. This has to do with the so called over-shrinkage effect of the normal prior with a fixed mean for nonzero coordinates, which pushes the posterior too much towards the prior mean, missing the true parameter that in general differs from the prior mean. That is why [15] and [11] discard normal priors on non-zero coordinates and use heavy tailed priors. A way to construct such a prior is to put a next level heavy-tailed prior, like half-Cauchy, on the variance in the normal prior, resulting in the so called (one-component) horseshoe prior on \( \theta \) (cf. [9] and [22]). In the present paper we show that normal priors are still usable (cf. [17]) and even lead to strong local results (and even for non-normal models) if combined with empirical Bayes approach.

**Uncertainty quantification problem.** The main aim in this paper is to construct confidence sets with optimal properties. The size of a confidence set is measured by the
smallest radius of a ball containing this set, hence it suffices to consider confidence balls.

For the usual norm $\| \cdot \|$ in $\mathbb{R}^n$, a random ball in $\mathbb{R}^n$ is $B(\hat{\theta}, \hat{r}) = \{ \theta \in \mathbb{R}^n : \| \theta - \theta \| \leq \hat{r} \}$, where the center $\hat{\theta} = \hat{\theta}(X) : \mathbb{R}^n \mapsto \mathbb{R}^n$ and radius $\hat{r} = \hat{r}(X) : \mathbb{R}^n \mapsto \mathbb{R}_+ = [0, +\infty]$ are measurable functions of the data $X$. Let us introduce the optimality framework for uncertainty quantification. The goal is to construct such a confidence ball $B(\hat{\theta}, C\hat{r})$ that for any $\alpha_1, \alpha_2 \in (0, 1]$ and some functional $r(\theta) = r_{\sigma,n}(\theta), r : \mathbb{R}^n \rightarrow \mathbb{R}_+$, there exist $C, c > 0$ such that

$$\sup_{\theta \in \Theta_0} P_{\theta}(\theta \notin B(\hat{\theta}, C\hat{r})) \leq \alpha_1, \quad \sup_{\theta \in \Theta_1} P_{\theta}(\hat{r} \geq cr(\theta)) \leq \alpha_2,$$  \hspace{1cm} (2)

for some $\Theta_0, \Theta_1 \subseteq \mathbb{R}^n$. The function $r(\theta)$, called the radial rate, is a benchmark for the effective radius of the confidence ball $B(\hat{\theta}, C\hat{r})$. The first expression in (2) is called coverage relation and the second size relation. Notice that our approach is local (and hence genuinely adaptive) as the radial rate $r(\theta)$ is a function of the “true” parameter $\theta$. Recall the common (global) minimax adaptive version of (2): given a family of sets $\Theta_\beta$ with corresponding minimax estimation rates $r(\Theta_\beta)$ indexed by a structural parameter $\beta \in \mathcal{B}$ (e.g., smoothness or sparsity), the minimax adaptive version of (2) would be obtained by taking $\Theta_0 = \Theta_1 = \Theta_\beta$ and the radial rate $r(\theta) = r(\Theta_\beta)$ for all $\theta \in \Theta_\beta$ and all $\beta \in \mathcal{B}$.

Coming back to our local framework (2), it is desirable to find the smallest $r(\theta)$ and the biggest $\Theta_0, \Theta_1$, for which (2) holds. These are contrary requirements, so we have to trade them off against each other. There are different ways of doing this, leading to different optimality frameworks. For example, if we insist on $\Theta_0 = \Theta_1 = \mathbb{R}^n$, then the results in [16] and [8] (a more refined version is by [3]) say basically that the radial rate $r$ cannot be of a faster order than $\sigma n^{1/4}$ for every $\theta$ and is at least of order $\sigma n^{1/2}$ for some $\theta$. This means that any confidence ball that is optimal with respect to the optimality framework (2) with $\Theta_0 = \mathbb{R}^n$ will necessarily have a big size, even if the true $\theta$ happens to lie in a very “good”, smooth or sparse, class $\Theta_1$. Many good confidence sets cannot be optimal in this sense (called “honest” in some papers) and effectively excluded from the consideration. For minimax adaptive versions of (2) this means that as soon as we require $\Theta_0 = \Theta_\beta, \beta \in \mathcal{B}$ in the coverage relation, the minimax rate $r(\Theta_\beta)$ in the size relation is unattainable even for $\beta \in \mathcal{B} = \{ \beta_1, \beta_2 \};$ cf. [18] for two nearly black classes. Essentially, the overall uniform coverage and optimal size properties can not hold together, it is necessary to sacrifice at least one of these, preferably as little as possible. We argue that it is unreasonable to pursue an optimality framework with the entire space $\Theta_0 = \mathbb{R}^n \ (\Theta_0 = \Theta_\beta)$, because this leads to discarding many good procedures and optimality of uninteresting ones. Instead, it makes sense to sacrifice in the set $\Theta_0 = \mathbb{R}^n \setminus \Theta'$, by removing a preferably small portion of “deceptive parameters” $\Theta'$ from $\mathbb{R}^n$ so that that the optimal radial rates become attainable in the size relation with interesting (preferably “massive”) sets $\Theta_1$.

This “deceptiveness” phenomenon is well understood for some smoothness structures (e.g., Sobolev scale), especially in global minimax settings; see [20], [7], [1] and [21]. If we now insist on the optimal size property in (2) for all $\Theta_\beta, \beta \in \mathcal{B}$, the coverage relation in (2) will not hold for all $\Theta_0 = \Theta_\beta$, but only for $\Theta_0 = \Theta_\beta \setminus \Theta'$, with some set of “deceptive parameters” $\Theta'$ removed from $\Theta_\beta$. In [21] such parameters are called “inconvenient truths” and an implicit construction of a $\theta' \in \Theta'$ is given. Examples of non-deceptive parameters
are the set of self-similar parameters $\Theta_0 = \Theta_{ss}$ introduced by [19] and studied by [6], [7], [21], and the set of polished tail parameters $\Theta_0 = \Theta_{pt}$ considered by [21]. In all the above mentioned papers global minimax radial rates (i.e., $r(\theta) = r(\Theta_\beta)$ for all $\theta \in \Theta_\beta$) for specific smoothness structures are studied. A local approach, delivering also the adaptive minimax results for many smoothness structures simultaneously, is considered by [2] for posterior contraction rates and by [4] for constructing optimal confidence balls. In [4], yet a more general (than $\Theta_{ss}$ and $\Theta_{pt}$) set of non-deceptive parameters was introduced, $\Theta_0 = \Theta_{eb}$, parameters satisfying the so called excessive bias restriction. More on this can be found in Subsection 4.2.

To the best of our knowledge, there are only two papers about global adaptive minimax results on uncertainty quantification [2]: the case of two nearly black classes is treated by [18]; and a restricted scale of nearly black classes is treated in [23]. This paper attempts to further contribute to this area by covering also the misspecified models (as we allow the $\xi_i$’s to be not necessarily independent normals) and by developing the novel local approach, namely, the radial rate $r(\theta)$ in [2] is allowed to be a function of $\theta$, which, in a way, measures the amount of sparsity for each $\theta \in \mathbb{R}^n$: the smaller $r(\theta)$, the more sparse $\theta$.

The local radial rate $r(\theta)$ is constructed as the best (smallest) rate over a certain family of local rates, therefore called oracle rate. We demonstrate that the local approach is more powerful than global in that we do not need to impose any specific sparsity structure, because the proposed local approach automatically exploits the “effective” sparsity of each underlying $\theta$, and our local results imply a whole panorama of the global minimax results for many scales at once. More on this is in Subsection 3.4.

The scope of this paper. In this paper, we introduce a family of normal mixture priors and propose an empirical Bayes procedure (in fact, two procedures). We use the normal likelihood, whereas the true model (1) does not have to be normal (independence of $\xi_i$’s is not required either), but only satisfying some mild conditions. We derive the local posterior contraction result for the resulting empirical Bayes posterior $\hat{\pi}(\theta|X)$ in the refined non-asymptotic formulation: $\sup_{\theta \in \mathbb{R}^n} \mathbb{E}_\theta\hat{\pi}(\|	heta - \bar{\theta}\|^2 \geq M_0 r^2(\theta) + M\sigma^2|X) \leq H_0 e^{-m_0 M}$ for some fixed $M_0, H_0, m_0 > 0$ and arbitrary $M \geq 0$, as an exponential non-asymptotic concentration bound in terms of $M$, uniformly in $\theta \in \mathbb{R}^n$. This formulation provides a rather refined characterization of the quality of the posterior (finer, than, e.g., asymptotically in terms of the dimension $n$), allowing subtle analysis for various asymptotic regimes. This result is of interest and importance on its own as it actually establishes the contraction of the empirical Bayes posterior with the local rate $r(\theta)$. Besides, we obtain the oracle estimation result (also in similar refined formulation, finer than traditional oracle inequalities) by constructing an estimator, the empirical Bayes posterior mean, which converges to $\theta$ with the local rate $r(\theta)$. This local result, besides being an ingredient for the uncertainty quantification problem [2], is also of interest and importance on its own as it implies the oracle results of [1] and the global (minimax) estimation results of [15] and [11].

Next, we construct a confidence ball by using the empirical Bayes posterior quantities. Since we want the size of our confidence sets to be of an oracle rate order, this comes with
the price that the coverage property can hold uniformly only over some set of parameters satisfying the so called excessive bias restriction (EBR) \( \Theta_0 = \Theta_{eb} \subseteq \mathbb{R}^n \). The main result consists in establishing the optimality (2) of the constructed confidence ball for the optimality framework \( \Theta_0 = \Theta_{eb}, \Theta_1 = \mathbb{R}^n \) and the local radial rate \( r(\theta) \). The important consequence of our local approach is that a whole panorama of adaptive (global) minimax results (for all the three problems: estimation, posterior contraction rate and confidence sets) over all sparsity scales covered by \( r(\theta) \) (see Subsection 3.4) follow from our local results. In particular, our local results imply the same type of adaptive minimax estimation results over sparsity scales as in [15], and the same type of global minimax results on contraction posterior rates as in [11].

Although the original motivation of the EBR condition was to remove the deceptive parameters, it turned out to be a very useful notion in the context of uncertainty quantification. In effect, the EBR condition gives rise to a new sparsity EBR-scale which gives the slicing of the entire space that is very suitable for uncertainty quantification. This provides a new perspective at the above mentioned “deceptiveness” issue: basically, each parameter is deceptive (or non deceptive) to some extent. It is the structural parameter of the new EBR-scale that measures the deceptiveness amount, and the (mild and controllable) price for handling deceptive parameters is the effective amount of inflating of the confidence ball that matches the amount of deceptiveness needed to provide a high coverage. An elaborate discussion on the EBR condition can be found in Subsection 4.2.

The paper is organized as follows. In Section 2 we introduce the notation, the prior, describe the empirical Bayes procedure in detail, make a link with the penalization method, and provide some conditions. Section 3, where we also introduce the EBR, contains the main results of the paper. In Section 4 we discuss some variations of our results, present concluding remarks and discuss the EBR. The theoretical results are illustrated in Section 5 by a small simulation study. The proofs of the lemmas and theorems are given in Sections 6 and 7 respectively.

2 Preliminaries

First we introduce some notation and multivariate normal prior. Next, by applying the empirical Bayes approach to the the normal likelihood, we derive an empirical Bayes posterior which we will use in the construction of the estimator and the confidence ball. The empirical Bayes procedure is linked to the penalization method. We complete this section with some conditions on the errors \( \xi_i \)'s and the prior.

2.1 Notation

Denote the probability measure of \( X \) from the model (1) by \( P_\theta = P_\theta^{(\sigma, n)} \), and by \( E_\theta \) the corresponding expectation. For the notational simplicity, we often skip the dependence on \( \sigma \) and \( n \) of these quantities and many others. Denote by \( 1_E = 1\{E\} \) the indicator function of the event \( E \), by \( |S| \) the cardinality of the set \( S \), the difference of sets \( S \setminus S_0 = \{ s \in S : s \notin S_0 \} \), \( N_k = \{1, \ldots, k\} \) for \( k \in \mathbb{N} = \{1, 2, \ldots\} \). For \( I \subseteq \mathbb{N} \), define \( I^c = \mathbb{N} \setminus I \). If random quantities appear in a relation, this relation should be understood in \( P_\theta \)-almost
sure sense. Throughout \( \phi(x, \mu, \sigma^2) \) will be the density of \( \mu + \sigma Z \sim N(\mu, \sigma^2) \) at point \( x \), where \( Z \sim N(0, 1) \). By convention, \( N(\mu, 0) = \delta_\mu \) denotes a Dirac measure at point \( \mu \). The symbol \( \hat{=} \) will refer to equality by definition.

Let \( \mathcal{I} = \mathcal{I}_n = \mathcal{P}(\mathbb{N}_n) = \{ I : I \subseteq \mathbb{N}_n \} \) be the power set of \( \mathbb{N}_n \), the family of all subsets of \( \mathbb{N}_n \) including the empty set. If the summation range in \( \sum_I \) is not specified (for brevity), this means \( \sum_{I \in \mathcal{I}} \). Throughout we assume the conventions that \( \sum_{I \in \varnothing} a_I = 0 \) for any \( a_I \in \mathbb{R} \) and \( 0 \log(a/0) = 0 \) (hence \( (a/0)^0 = 1 \)) for any \( a > 0 \). Finally, denote \( X(I) = (X_i, i \in I), i \in \mathbb{N}_n \) for \( I \in \mathcal{I} \).

### 2.2 Multivariate normal prior

When deriving all the posterior quantities in the Bayesian analysis below, we will use the normal likelihood \( \ell(\theta, X) = (2\pi\sigma^2)^{-n/2} \exp\{-||X - \theta||^2/2\sigma^2\} \), which is equivalent to imposing the classical high-dimensional normal model \( X = (X_i, i \in \mathbb{N}_n) \sim \mathcal{N}(\theta, \sigma^2) \). Recall however that the “true” model \( X \sim P_\theta \) is not assumed to be normal (but satisfying Condition (A1)).

To model possible sparsity in the parameter \( \theta \), the coordinates of \( \theta \) can be split into two distinct groups of coordinates of \( \theta \): for some \( I \in \mathcal{I} \), \( \theta_I = (\theta_i, i \in I) \) and \( \theta_{I^c} = (\theta_i, i \in I^c) \), so that \( \theta = (\theta_I, \theta_{I^c}) \). The group of coordinates \( \theta_{I^c} = (\theta_i, i \notin I) \) consists of (almost) zeros (or any other fixed value, but we assume it to be zero without loss of generality) and \( \theta_I = (\theta_i, i \in I) \) is the group of non-zeros coordinates. For any \( \theta \in \mathbb{R}^n \) (even “not sparse” one) there is the best (oracle) splitting into two groups, we will come back to this in Section 3. To model sparsity, we propose a prior on \( \theta \) given \( I \) as follows:

\[
\pi_I = \bigotimes_{i=1}^n N(\mu_i(I), \tau_i^2(I)), \quad \mu_i(I) = \mu_1[i \in I], \quad \tau_i^2(I) = \sigma^2 K_n(I) 1\{i \in I\},
\]

and \( K_n(I) = \frac{c_n}{|I|} - 1 \). The indicators in prior (3) ensure the sparsity of the group \( I^c \). The rather specific choice of \( K_n(I) \) is made for the sake of concise expressions in later calculations, many other choices are actually possible. By using normal likelihood \( \ell(\theta, X) = (2\pi\sigma^2)^{-n/2} \exp\{-||X - \theta||^2/2\sigma^2\} \), the corresponding posterior distribution for \( \theta \) is readily obtained:

\[
\pi_I(\theta|X) = \bigotimes_{i=1}^n N\left(\frac{\tau_i^2(I) X_i + \sigma^2 \mu_i(I)}{\tau_i^2(I) + \sigma^2}, \frac{\tau_i^2(I) \sigma^2}{\tau_i^2(I) + \sigma^2}\right).
\]

Next, introduce the prior \( \lambda \) on \( \mathcal{I} \), discussed in Subsection 4.1 below. For \( \kappa > 1 \), draw a random set from \( \mathcal{I} \) with probabilities

\[
\lambda_I = c_{\kappa,n} \exp\{-\kappa |I| \log\left(\frac{\ell_n}{|I|}\right)\} = c_{\kappa,n} \left(\frac{\ell_n}{|I|}\right)^{-\kappa |I|}, \quad I \in \mathcal{I},
\]

where \( c_{\kappa,n} \) is the normalizing constant. Since \( \binom{n}{k} \leq \left(\frac{n}{k}\right)^k \) and \( \binom{n}{0} = 1 \),

\[
1 = \sum_{I \in \mathcal{I}} \lambda_I = c_{\kappa,n} \sum_{k=0}^n \binom{n}{k} \left(\frac{\ell_n}{k}\right)^{-\kappa k} \leq c_{\kappa,n} \sum_{k=0}^n \binom{n}{k} \left(\frac{\ell_n}{k}\right)^{-(\kappa-1)k} \leq c_{\kappa,n} \sum_{k=0}^n e^{-(\kappa-1)k},
\]
so that \( c_{\kappa,n} \geq 1 - e^{1-\kappa} > 0, \ n \in \mathbb{N} \). Combining (3) and (5) gives the mixture prior on \( \theta \):

\[
\pi = \sum_{I \in \mathcal{I}} \lambda_I \pi_I.
\]

This leads to the marginal distribution of \( X \): \( P_X = \sum_{I \in \mathcal{I}} \lambda_I P_{X,I} \), with \( P_{X,I} = \bigotimes_{i=1}^n \mathcal{N}(\mu_i(I), \sigma^2 + \tau_i^2(I)) \), and the posterior of \( \theta \) is

\[
\pi(\theta | X) = \pi(\theta | X) = \sum_{I \in \mathcal{I}} \pi_I(\theta | X) \pi(I | X),
\]

where \( \pi_I(\theta | X) \) is defined by (4) and the posterior \( \pi(I | X) \) for \( I \) is

\[
\pi(I | X) = \frac{\lambda_I P_{X,I}}{P_X} = \frac{\lambda_I \prod_{i=1}^n \phi(X_i, \mu_i(I), \sigma^2 + \tau_i^2(I))}{\sum_{J \in \mathcal{I}} \lambda_J \prod_{i=1}^n \phi(X_i, \mu_i(J), \sigma^2 + \tau_i^2(J))}.
\]

### 2.3 Empirical Bayes posterior

The parameters \( \mu_{1,i} \) are yet to be chosen in the prior. We choose \( \mu_i \) by using empirical Bayes approach. The marginal likelihood \( P_X \) is readily maximized with respect to \( \mu_i \): \( \hat{\mu}_i = X_i \), which we then substitute instead of \( \mu_i \) in the expression (7) for \( \pi(\theta | X) \), obtaining the empirical Bayes posterior

\[
\tilde{\pi}(\theta | X) = \tilde{\pi}(\theta | X) = \sum_{I \in \mathcal{I}} \tilde{\pi}_I(\theta | X) \tilde{\pi}(I | X),
\]

where the empirical Bayes conditional posterior (recall that \( \mathcal{N}(0,0) = \delta_0 \))

\[
\tilde{\pi}_I(\theta | X) = \bigotimes_{i=1}^n \mathcal{N}(X_i, 1 \{ i \in I \}, \frac{K_n(I)\sigma^2_1(i \in I)}{K_n(I) + 1})
\]

is obtained from (3) and (4) with \( \mu_{1,i} = X_i \), and

\[
\tilde{\pi}(I | X) = \frac{\lambda_I \prod_{i=1}^n \phi(X_i, X_i, 1 \{ i \in I \}, \sigma^2 + \tau_i^2(I))}{\sum_{J \in \mathcal{I}} \lambda_J \prod_{i=1}^n \phi(X_i, X_i, 1 \{ i \in J \}, \sigma^2 + \tau_i^2(J))}
\]

is the empirical Bayes posterior for \( I \in \mathcal{I} \), obtained from (3) and (5) with \( \mu_i = X_i \). Let \( \tilde{E} \) and \( \tilde{E}_I \) be the expectations with respect to the measures \( \tilde{\pi}(\theta | X) \) and \( \tilde{\pi}_I(\theta | X) \) respectively. Then \( \tilde{E}_I(\theta | X) = X(I) = (X_i, 1 \{ i \in I \}, i \in \mathbb{N}_n) \). Introduce the **empirical Bayes posterior mean estimator**

\[
\tilde{\theta} = \tilde{E}(_{\theta} | X) = \sum_{I \in \mathcal{I}} \tilde{E}_I(_{\theta} | X) \tilde{\pi}(I | X) = \sum_{I \in \mathcal{I}} X(I) \tilde{\pi}(I | X).
\]

Consider an alternative empirical Bayes posterior. First derive an empirical Bayes variable selector \( I \) by maximizing \( \tilde{\pi}(I | X) \) over \( I \in \mathcal{I} \) (any maximizer will do) as follows:

\[
\hat{i} = \arg\max_{I \in \mathcal{I}} \tilde{\pi}(I | X) = \arg\max_{I \in \mathcal{I}} \left\{ -\sum_{i \in I} \frac{X_i^2}{2\sigma^2} - \frac{|I|}{2} \log(K_n(I) + 1) + \log \lambda_I \right\}
\]

\[
\hat{i} = \arg\min_{I \in \mathcal{I}} \left\{ -\sum_{i \in I} X_i^2 + (2\kappa + 1)\sigma^2 |I| \log \left( \frac{\sigma^2}{\sigma^2 + \tau_i^2} \right) \right\},
\]

7
which is actually the penalization procedure from [5] (cf. also [1]): more detail on connection with the penalization method is in Section 4.1. Now plugging in \( \hat{I} \) into \( \hat{\pi}(\vartheta | X) \) defined by (10) yields yet another empirical (now “double empirical”: with respect to \( \mu_i \)'s and with respect to \( I \)) Bayes posterior and the corresponding empirical Bayes mean estimator for \( \theta \):

\[
\hat{\pi}(\vartheta | X) = \hat{\pi}_f(\vartheta | X), \quad \hat{\theta} = \hat{\mathbb{E}}(\vartheta | X) = X(\hat{I}),
\]

where \( \hat{\mathbb{E}} \) denotes the expectation with respect to the measure \( \hat{\pi}(\vartheta | X) \). Notice that, like (9), \( \hat{\pi}(\vartheta | X) \) defined by (14) can also be seen as mixture

\[
\hat{\pi}(\vartheta | X) = \sum_{I \in \mathcal{I}} \tilde{\pi}_I(\vartheta | X) \hat{\pi}(I | X), \quad \hat{\pi}(I | X) = \delta_{\hat{I}},
\]

where the mixing distribution \( \hat{\pi}(I | X) = \delta_{\hat{I}} \), the (double) empirical Bayes posterior for \( I \), is degenerate at \( \hat{I} \).

2.4 Conditions

We finish this section with some technical conditions and definitions. The following condition (called exchangeable exponential moment condition) on the error vector \( \xi = (\xi_1, \ldots, \xi_n) \) will be assumed throughout.

**Condition (A1).** The random variables \( \xi_i \)'s from (11) satisfy: \( \mathbb{E}\xi_i = 0, \text{Var}(\xi_i) \leq C_{\xi}, \quad i \in \mathbb{N}_n; \) and for some \( \beta, B > 0 \) (without loss of generality assume \( C_{\xi} = 1 \) and \( \beta \in (0, 1] \)),

\[
\mathbb{E}\exp\left\{ \beta \sum_{i \in I} \xi_i^2 \right\} \leq e^{B|I|} \quad \text{for all } I \in \mathcal{I}.
\]

(A1)

The constants \( \beta \in (0, 1] \) and \( B > 0 \) will be fixed throughout and we omit the dependence on these constants in all further notation. A short discussion about this condition can be found in Subsection 4.1. There is no need to assume \( \text{Var}(\xi_i) \leq C_{\xi} \) as this follows from condition (A1), but we provide this just for reader’s convenience. Note also that the \( \xi_i \)'s do not have to be independent. Of course, Condition (A1) is satisfied for independent normals \( \xi_i \overset{\text{ind}}{\sim} N(0, 1) \) and for bounded (arbitrarily dependent) \( \xi_i \)'s. In case of independent normal errors, some bounds in the proofs can be sharpened; we will mention possible refinements in Subsection 4.1.

In the proof of Theorem 1 below, we will need a bound for \( \left[ \mathbb{E}(\sum_{i \in I} \xi_i^2)^2 \right]^{1/2}, I \in \mathcal{I} \). Actually, Condition (A1) ensures such a bound. Indeed, since \( x^\beta \leq e^{x\beta} \) for all \( x \geq 0 \), by using the Hölder inequality and (A1), we derive that for any \( t \in (0, \beta] \)

\[
\mathbb{E}_\theta\left( \sum_{i \in I} \xi_i^2 \right)^2 = \frac{4}{t^2} \mathbb{E}_\theta\left( \sum_{i \in I} \xi_i^2 \right)^2 \leq \frac{4}{t^2} \mathbb{E}_\theta e^{t \sum_{i \in I} \xi_i^2} \leq \frac{4}{t^2} \left[ \mathbb{E}_\theta e^{\beta \sum_{i \in I} \xi_i^2} \right]^{t/\beta} \leq \frac{4}{t^2} e^{B\beta^{-1}|I|}.
\]

To summarize, Condition (A1) implies that for any \( \rho \in (0, B] \) and any \( I \in \mathcal{I} \),

\[
\left[ \mathbb{E}(\sum_{i \in I} \xi_i^2)^2 \right]^{1/2} \leq \frac{B}{\beta \rho} \exp\{\rho |I| \}.
\]

(16)
In some proofs we need a technical condition on the parameter \( \kappa \) appearing in (1). CONDITION (A2). The parameter \( \kappa \) of the prior \( \lambda \) defined by (5) satisfies
\[
\kappa > \bar{\kappa} \triangleq (12 - \beta + 4B)/(4\beta),
\] (A2)
where \( \beta, B \) are from Condition (A1).

Finally, we give one more technical definition which we will need in the claims. For constants \( \beta, B \) from Condition (A1), define the function
\[
\bar{\tau}(\rho) \triangleq \frac{6(\kappa\beta + B)(1 + \rho) + 3\beta}{2\beta(1 - \rho)}, \quad \rho \in [0, 1).
\] (17)
Notice that \( \bar{\tau}(\rho) > 1 \) for all \( \rho \in [0, 1) \).

3 Main results

In this section we give the main results of the paper. From now on, by \( \hat{\pi}(\vartheta|X) \) we denote either \( \hat{\pi}(\vartheta|X) \) defined by (9) or \( \hat{\pi}(\vartheta|X) \) defined by (13), and \( \hat{\vartheta} \) will stand either for \( \hat{\vartheta} \) defined by (12) or for \( \hat{\vartheta} \) defined by (13).

3.1 Oracle rate

The empirical Bayes posterior \( \hat{\pi}(\vartheta|X) \) is a random mixture over \( \hat{\pi}_I(\vartheta|X), I \in \mathcal{I} \). From the \( P_\vartheta \)-perspective, each posterior \( \hat{\pi}_I(\vartheta|X) \) (and the corresponding estimator \( \hat{E}_I(\vartheta|X) = X(I) \)) contracts to the true parameter \( \vartheta \) with the local rate \( R^2(I, \vartheta) = R^2(I, \vartheta, \sigma, n) = \sum_{i \in I} \vartheta_i^2 + \sigma^2 |I| \). Indeed, since \( \hat{E}_I(\vartheta|X) = X(I) = (X_i, i \in I, i \in \mathbb{N}_n) \), (10) and the Markov inequality yield
\[
E_{\vartheta} \hat{\pi}_I(\vartheta - \theta)^2 \geq M^2 R^2(I, \vartheta) |X| \leq \frac{E_{\vartheta} \|X(I) - \theta\|^2 + K_{\vartheta}(I)\sigma^2 |I|}{M^2 R^2(I, \vartheta)} \leq \frac{2}{M^2}.
\]
For each \( \vartheta \in \mathbb{R}^n \), among \( I \in \mathcal{I} \) there exists the best choice \( I_o = I_o(\vartheta) = I_o(\vartheta, \sigma) \) (called the \( R \)-oracle) corresponding to the fastest local rate \( R^2(\vartheta) = R^2(\vartheta, \mathcal{I}) = \min_{I \in \mathcal{I}} R^2(I, \vartheta) = \sum_{i \in I_o} \vartheta_i^2 + \sigma^2 |I_o| \). Ideally, we would like to mimic the \( R \)-oracle, i.e., to construct an empirical Bayesian procedure (e.g., \( \hat{\pi}(\vartheta|X) \)) which performs as good as the oracle empirical Bayes posterior \( \hat{\pi}_{I_o}(\vartheta|X) \) without knowing \( I_o \), uniformly in \( \vartheta \in \mathbb{R}^n \). However, (13) and later (5) showed in the estimation problem (hence, in posterior contraction problem) that the rate \( R(\vartheta) \) is unachievable uniformly even over the scale of sparsity classes, let alone over \( \mathbb{R}^n \). As is shown in (5), the smallest “mimicable” oracle rate is the modified version of \( R \) (in case \( I_o = \emptyset \), plus penalty \( \sigma^2 \)), where the variance term \( \sigma^2 |I_o| \) is inflated by the factor \( \log(en/|I_o|) \) (thought of as payment for not knowing \( I_o \)).

The above discussion motivates the following definition. For \( \tau > 0 \) and \( I \in \mathcal{I} \), introduce the family of the so called local \( \tau \)-rates:
\[
\tau^2(I, \vartheta) = \sum_{i \in I} \vartheta_i^2 + \tau \sigma^2 |I| \log \left( \frac{en}{|I|} \right), \tag{18}
\]
where the inflating logarithmic factor \( \log(\frac{\sigma}{|\tau_o|}) \) appears in the variance part of the local rate. There exists the best choice \( I_o^\tau = I_o^\tau(\theta) = I_o^\tau(\theta, \sigma, n) \in \mathcal{I} \) such that

\[
r^2_\tau(\theta) = r^2_\tau(\theta, \mathcal{I}) = \min_{I \in \mathcal{I}} r^2_\tau(I, \theta) = r^2_\tau(I_o^\tau, \theta) = \sum_{i \in (I_o^\tau)^c} \theta_i^2 + \tau \sigma^2 |\log(\frac{\sigma}{|\tau_o|})|, \tag{19}
\]

where \( r^2_\tau(I, \theta) \) is defined by \((15)\). The oracle \( I_o^\tau \) may not be unique as some coordinates of \( \theta \) can coincide. If \( \min_{I \in \mathcal{I}} r^2_\tau(I, \theta) = r^2_\tau(J^k_o, \theta) \) for different \( J^k_o \in \mathcal{I}, k = 1, \ldots, K, \) we take \( I_o^\tau \) to be such that \( \min_k \sum_{i \in J^k_o} i = \sum_{i \in I_o^\tau} i \). We call the quantity \( r^2_\tau(\theta) \) \( \tau \)-oracle rate and \( I_o^\tau \) is called \( \tau \)-oracle. For \( \tau = 1 \), we denote \( r^2(\theta) = r^2_1(\theta) \) called the oracle rate, and \( I_o = I_o^1 \) called the oracle.

Let us elucidate what the \( \tau \)-oracle means. Let \( \theta_{(1)}^2 \leq \theta_{(2)}^2 \leq \ldots \leq \theta_{(n)}^2 \) be the ordered values of \( \theta_1^2, \ldots, \theta_n^2 \) and denote \( j = o_0(i) \) if \( \theta_i^2 = \theta_{(j)}^2 \), with the convention that if \( \theta_{i_1}^2 = \ldots = \theta_{i_k}^2 \) for \( i_1 < \ldots < i_k \), we set \( o_0(i_{l+1}) = o_0(i_l) + 1, l = 1, \ldots, k - 1 \). Denote for brevity \( i_o = |J^r_o| \). Then the \( \tau \)-oracle \((19)\) classifies the coordinates \((\theta, i \in (I_o^\tau)^c) = (\theta_{(i)}, i \leq n - i_o)\) as \( \tau \)-insignificant, and the coordinates \((\theta, i \in I_o^\tau) = (\theta_{(i)}, i > n - i_o)\) as \( \tau \)-significant. The bias related term \( \sum_{i \in (I_o^\tau)^c} \theta_i^2 \) of the \( \tau \)-oracle rate is called the excessive \( \tau \)-bias. This is the error the \( \tau \)-oracle makes when setting \( \tau \)-insignificant coordinates of \( \theta \) to zero. The variance related term \( \sigma^2 |I_o^\tau| \log(\frac{\sigma}{|\tau_o|}) \) is the error the \( \tau \)-oracle makes when recovering the \( \tau \)-significant coordinates (the log factor being the payment for not knowing the locations).

The idea of introducing the parameter \( \tau \geq 0 \) is to re-weight the variance and the bias parts in the \( \tau \)-oracle rate. Notice that \( I_o^{\tau_1} \subseteq I_o^{\tau_2} \) if \( \tau_1 \geq \tau_2 \) because the function \( x \log(n e/x) \) is increasing for \( x \in (0, n] \). For \( \tau \downarrow 0 \), \( r^2_\tau(\theta) \downarrow 0 \) and the “limiting” \( \tau \)-oracle recovers the active index set (or the sparsity of \( \theta \)) \( I^* = I^*(\theta) = \{i \in \mathbb{N}_n : \theta_i \neq 0\} \) in the sense that \( I_o^\tau \uparrow I^* \) as \( \tau \downarrow 0 \). Clearly, we would like to mimic the \( \tau \)-oracle rate with the smallest possible \( \tau \). However, all \( \tau \)-oracle rates for \( \tau > 0 \) are related to the oracle rate (i.e., with \( \tau = 1 \)) by the trivial relations

\[ r^2(\theta) \leq r^2_\tau(\theta) \leq \tau r^2(\theta) \quad \text{for } \tau \geq 1, \quad r^2_\tau(\theta) \leq r^2(\theta) \leq \tau^{-1} r^2_\tau(\theta) \quad \text{for } 0 < \tau < 1. \]

So, in principle we can obtain the result for any \( \tau \)-oracle rate \( r^2_\tau(\theta) \) via the result for the oracle rate \( r^2(\theta) \) (and vice versa), but at the price of the large factor \( \tau^{-1} \) for small \( \tau \in (0, 1) \). Expectedly, only \( \tau \)-oracle rates with \( \tau > \bar{\tau} \geq 0 \) for some \( \bar{\tau} \) should be mimicable without price, and this condition indeed appears in Theorem 3.

### 3.2 Contraction results with oracle rate

The following theorem establishes that the empirical Bayes posterior \( \hat{\pi}(\theta | X) \) (which is either \( \hat{\pi}(\theta | X) \) defined by \((9)\) or \( \hat{\pi}(\theta | X) \) defined by \((14)\)) contracts to \( \theta \) with the oracle rate \( r(\theta) \) from the frequentist \( P_\theta \)-perspective, and the empirical Bayes posterior mean \( \hat{\theta} \) which is either \( \hat{\theta} \) defined by \((12)\) or \( \hat{\theta} \) defined by \((14)\) converges to \( \theta \) with the oracle rate \( r(\theta) \), uniformly over the entire parameter space.
**Theorem 1.** Let Conditions (A1) and (A2) be fulfilled. Then there exist constants $M_0, M_1, H_0, H_1, m_0, m_1 > 0$ (depending only on $\varphi$) such that for each $\theta \in \mathbb{R}^n$

\[
E_{\theta} \hat{\pi} (\| \theta - \theta \|^2 \geq M_0 r^2(\theta) + M \sigma^2 | X) \leq H_0 e^{-m_0 M} \quad \text{for any} \quad M \geq 0, \quad (i)
\]
\[
P_{\theta} (\| \theta - \theta \|^2 \geq M_1 r^2(\theta) + M \sigma^2) \leq H_1 e^{-m_1 M} \quad \text{for any} \quad M \geq 1. \quad (ii)
\]

**Remark 1.** Notice that already claim (i) of the theorem contains an oracle bound for the estimator $\hat{\pi}$. Indeed, by Jensen’s inequality, and the fact that $\sigma^2 \leq r^2(\theta)$, we derive the oracle inequality

\[
E_{\theta} \| \hat{\theta} - \theta \|^2 \leq E_{\theta} \bar{\hat{\pi}} (\| \theta - \theta \|^2 | X) \leq M_0 r^2(\theta) + H_0 \int_0^{+\infty} e^{-m_0 u/\sigma^2} du = M_0 r^2(\theta) + \frac{H_0 \sigma^2}{m_0}.
\]

However, comparing (20) with (11), we see that claim (ii) is actually stronger than (20) and therefore requires a separate proof.

**Remark 2.** A few more remarks on the theorem are in order.

(i) The above local result implies the minimax optimality over various sparsity scales, see Section 3.4 for more detail on this.

(ii) The constants in the theorem depend only on $\beta, B$ and some also on $\varphi$, the exact expressions can be found in the proof.

(iii) Notice that for a local rate to be mimicable it should include the penalty term $\sigma^2$ which is not included in the oracle rate $r^2(\theta)$ if $I_o = \emptyset$ (e.g., for $\theta = 0$). But of course, the quantity $M_0 r^2(\theta) + M \sigma^2$ from Theorem 1 does include this term.

(iv) The non-asymptotic exponential bounds in terms of the constant $M$ from the expression $M' r^2(\theta) + M \sigma^2$ (with fixed $M'$) in claims (i) and (ii) of the theorem provide a very refined characterization of the quality of the posterior $\hat{\pi}(\cdot | X)$ and estimator $\hat{\theta}$, finer than, e.g., the traditional oracle inequalities like (20). This refined formulation allows for subtle analysis in various asymptotic regimes ($n \to \infty$, $\sigma \to 0$, or their combination) as we can let $M$ depend in any way on $n, \sigma$, or both.

When proving the above theorem, we also obtain in passing a result about the frequentist behavior of the selector $\hat{I}$ and the empirical Bayes posterior for $I$, saying basically that $\hat{I}$ and $\hat{\pi}(I | X)$ “live” on a set that is, in a sense, almost as good as the oracle $I_o = I_o(\theta)$.

**Theorem 2.** Let Conditions (A1) and (A2) be fulfilled, $c_1, c_2, c_3$ be the constants defined in Lemma 2, $C_0 = (1 - e^{-c_1})^{-1}$, and let $\hat{\pi}(I | X)$ be either $\hat{\pi}(I | X)$ defined by (11) or $\hat{\pi}(I | X) = \delta_{\hat{I}}$ defined by (15). Then for any $\theta \in \mathbb{R}^n$ and $M \geq 0$,

\[
E_{\theta} \hat{\pi} (I \in \mathcal{I} : r(I, \theta) \geq c_3 r(\theta) + M \sigma^2 | X) \leq C_0 e^{-c_2 M}.
\]

**Remark 3.** In case $\hat{\pi}(I | X) = \hat{\pi}(I | X) = \delta_{\hat{I}}$, for any $\mathcal{G} \subseteq \mathcal{I}$ we have $\hat{\pi}(I \in \mathcal{G} | X) = \hat{\pi}(I \in \mathcal{G} | X) = 1 \{ \hat{I} \in \mathcal{G} \}$ so that $E_{\theta} \hat{\pi}(I \in \mathcal{G} | X) = P_{\theta}(\hat{I} \in \mathcal{G})$. In particular, the claim of Theorem 2 in this case reads as $P_{\theta}(r(\hat{I}, \theta) \geq c_3 r(\theta) + M \sigma^2) \leq e^{-c_2 M}$. 

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3.3 Confidence ball under excessive bias restriction

Theorem 4 establishes the strong local optimal properties of the empirical Bayes posterior \( \hat{\pi}(\theta|X) \) and the empirical Bayes posterior mean \( \hat{\theta} \), but these do not solve the uncertainty quantification problem yet.

Let us construct a confidence ball by using the empirical Bayes posterior mean \( \hat{\theta} = \hat{\theta}(X) \) and the empirical Bayes variable selector \( \hat{I} = \hat{I}(X) \) defined by (13). Recall that \( \hat{\pi}(\theta|X) \) is either \( \pi(\theta|X) \) defined by (9) or \( \hat{\pi}(\theta|X) \) defined by (14), and the empirical Bayes posterior mean \( \hat{\theta} \) is either \( \hat{\theta} \) defined by (12) or \( \hat{\theta} \) defined by (14). Introduce the data dependent (quadratic) radius

\[
\hat{r}^2 = \hat{r}^2(X) = \sigma^2 + \sigma^2 \log(en/|\hat{I}|).
\]  

(21)

The additive term \( \sigma^2 \) in (21) is needed to handle the case \( \hat{I} = \emptyset \). Introduce the quantity

\[
b_r(\theta) = b_r(\theta, I) = \frac{\sum_{i \in \{I_i \cap \beta \}} (\theta_i - \hat{\theta}_i)^2}{\sigma^2 + \sigma^2 \log(en/|I_0|)}, \quad \theta \in \mathbb{R}^n,
\]

called excessive \( \tau \)-bias ratio, which is the ratio of the excessive \( \tau \)-bias to the sum of \( \sigma^2 \) and the variance part of the \( \tau \)-oracle rate \( r^2(\theta) \) defined by (19).

Now we can use the center \( \hat{\theta} \) and the radius \( \hat{r} \) in constructing a confidence ball for \( \theta \).

The following theorem, which is the main result in the paper, describes the coverage and size properties of the confidence ball \( B(\bar{\theta}, (M_2(b_r(\theta) + \tau))^{1/2} + M \sigma^2)^{1/2} \).

**Theorem 3.** Let Conditions (A1) and (A2) be fulfilled, \( \tau > \check{\tau}(e^{-1}) \) (with \( \check{\tau}(\cdot) \) defined by (17)), and \( b_r(\theta) \) be defined by (22). Then there exist constants \( M_2, H_2, m_2, M_3, H_3, m_3 > 0 \) (depending only on \( \alpha, \tau, \) and the variance part of the \( \tau \)-oracle rate) such that for all \( \theta \in \mathbb{R}^n \) and all \( M \geq 2 \)

\[
P_\theta \hat{\theta} \notin B(\bar{\theta}, (M_2(b_r(\theta) + \tau))^{1/2} + M \sigma^2)^{1/2}) \leq H_0 e^{-m_0M/2} + H_2 \left( \frac{en}{|I_0|} \right)^{-\alpha/|I_0|} e^{-m_2[b_r(\theta) + \tau]^{-1}M},
\]

\[
P_\theta (\hat{r}^2 \geq M_3 \sigma^2(\theta) + M \sigma^2) \leq H_3 e^{-m_3M},
\]

where the constants \( H_0, m_0 \) are defined in Theorem 1 and \( \alpha = \alpha(\tau, e^{-1}) \) is defined in Lemma 3.

Notice that the size relation in the theorem holds uniformly in \( \theta \in \mathbb{R}^n \). Although the coverage relation is also uniform in \( \theta \in \mathbb{R}^n \), the main problem is the dependence of the coverage relation on \( b_r(\theta) \). This motivates introducing a condition which provides control over the quantity \( b_r(\theta) \).

**CONDITION EBR.** We say that parameter \( \theta \) satisfies the excessive bias restriction (EBR) condition with structural parameters \( (t, \tau) \) if \( b_r(\theta) \leq t \), where \( b_r(\theta) \) is defined by (22). We denote the corresponding set (called the EBR class) of parameters by

\[
\Theta_{eb}(t, \tau) = \{ \theta \in \mathbb{R}^n : b_r(\theta) \leq t \}.
\]

(23)

The excessive \( \tau \)-bias is the error that the \( \tau \)-oracle makes when setting the \( \tau \)-insignificant coordinates to zero (whereas they may not be zero). Large ratio \( b_r(\theta) \)
means that this error is relatively large as compared to the variance part of the oracle rate. In a way, such \( \theta \)'s “trick” the \( \tau \)-oracle and can therefore be regarded as deceptive. For each \( \theta \in \mathbb{R}^p \), \( b_r(\theta) \) measures the amount of deceptiveness of \( \theta \): the bigger \( b_r(\theta) \), the more deceptive \( \theta \). The EBR condition says that the deceptiveness has to be restricted: \( \sum_{i \in (I^r_{\theta} \setminus I^e_{\theta})} \theta_i^2 \leq t\alpha^2|I^e_{\theta}| \log\left(\frac{\alpha}{|I^e_{\theta}|}\right) \). An explicit example of EBR parameters is the set of self-similar parameters introduced in [23] which is in our terms \( \Theta_{ss}(p, \tau, c) = \{ \theta \in \ell_0[p] : |I^e_{\theta}(\theta)| = cp \} \) for \( p \in \mathbb{N}_n \), \( c \in (0, 1] \), \( \tau > 0 \). If \( \theta \in \Theta_{ss}(p, \tau, c) \), then \( \theta_i^2 \leq \tau\sigma^2 \log\left(\frac{\alpha}{|I^e_{\theta}|}\right) \) for each \( i \in (I^e_{\theta})^c \) and there are at most \( |I^e_{\theta}| \) nonzero coordinates in \( (I^e_{\theta})^c \), so that \( \sum_{i \in (I^e_{\theta})^c} \theta_i^2 \leq (p - |I^e_{\theta}|)\tau\sigma^2 \log\left(\frac{\alpha}{|I^e_{\theta}|}\right) \leq (c^{-1} - 1)\tau\sigma^2|I^e_{\theta}| \log\left(\frac{\alpha}{|I^e_{\theta}|}\right) \).

Hence, \( \Theta_{ss}(p, \tau, c) \subseteq \Theta_{eb}(c^{-1} - 1, \tau, \tau) \). In particular, if \( \theta \in \Theta_{ss}(p, \tau, 1) = \{ \theta \in \ell_0[p] : \theta_i^2 > \tau\sigma^2 \log\left(\frac{\alpha}{|I^e_{\theta}|}\right), i \in I^e(\theta) \} \), then \( I^e_{\theta}(\theta) = I^e(\theta) = \{ i \in \mathbb{N}_n : \theta_i \neq 0 \} \) and \( |I^e_{\theta}(\theta)| = p \), implying \( \Theta_{ss}(p, \tau, 1) \subseteq \Theta_{eb}(0, \tau) \). This class consists of the “nicest”, least deceptive parameters with zero excessive bias (hence \( b_r(\theta) = 0 \)). This implies that the \( \tau \)-insignificant coordinates of such \( \theta \)'s are the true zeros and the \( \tau \)-significant coordinates are sufficiently distinct from zero. The uncertainty quantification result is the strongest for this class because the inflating factor is the smallest and the coverage is the highest, as \( t = 0 \). More about the EBR condition is in Section 1.2.

The first claim from Theorem 3 immediately implies the uniformity of the coverage relation over the EBR class which is summarized by the following corollary.

**Corollary 1.** Under the conditions of Theorem 3, the following relation holds for any \( t \geq 0 \):

\[
\sup_{\theta \in \Theta_{eb}(t, \tau)} P_\theta(\theta \notin B(\hat{\theta}, (M^2_{\hat{\theta}}\alpha^2 + M\alpha^2)^{1/2})) \leq H^2_2 e^{-m_{2}^2M},
\]

where \( M_{2} = M_{2}(t + \tau) \), \( H^2_2 = H_0 + H_2 \), \( m_{2} = \min\{m_{0}, m_{2}/(t + \tau)\}/2 \).

**EBR for a restricted oracle.** Instead of the complete family \( \mathcal{I} = \mathcal{P}(\mathbb{N}_n) \) of candidate index sets for modeling significant coordinates of the parameter \( \theta \), we can consider a subfamily \( \mathcal{I}_r \subset \mathcal{I} \), when, say, we have some extra knowledge about the locations of significant coordinates. Similar to (19), the corresponding restricted \( \tau \)-oracle \( I^r_{\theta} \in \mathcal{I}_r \) is defined by \( r^2(\theta, I_r) = \min_{I \in \mathcal{I}_r} r^2(\theta, I) = r^2(\theta, I_r) \). By restricting the prior \( \lambda_I \) on \( \mathcal{I}_r \) (instead of \( \mathcal{I} \)), we can obtain exactly the same results as before with the restricted \( \tau \)-oracle rate \( r^2(\theta, I_r) \) instead of the \( \tau \)-oracle rate \( r^2(\theta, \mathcal{I}) \). Clearly, as \( r^2(\theta, \mathcal{I}) \leq r^2(\theta, I_r) \) for all \( \theta \in \mathbb{R}^n \), there is no gain in the size relation. But the corresponding EBR for the restricted oracle \( \Theta_{eb}(t, I_r) \) can be weaker than \( \Theta_{eb}(t, \tau) \).

For example, consider the subfamily \( \mathcal{I}_r(k) = \{ I \in \mathcal{I} : |I| \geq k \}, k \in \mathbb{N}_n \) (note \( \mathcal{I}_r(0) = \mathcal{I} \)). For a small \( k \), mimicking the oracle over \( \mathcal{I}_r(k) \) is almost as good as over the complete family \( \mathcal{I} \) since \( r^2(\theta, \mathcal{I}_r(k)) \leq r^2(\theta, I_r(k)) \leq r^2(\theta, \mathcal{I}) + \tau^2\alpha^2 k \log(en/k) \). Besides, the cardinality of the restricted oracle \( I^r_{\theta}(k) = \arg\min_{I \in \mathcal{I}_r(k)} r^2(\theta, I) \) is at least \( k \geq 1 \), so that the second term in the right hand side of the coverage relation from Theorem 3 (formulated for the restricted family \( \mathcal{I}_r(k) \)) is bounded by \( H^2_2 \left(\frac{en}{I^r_{\theta}(k)}\right)^\alpha \right) \leq H^2_2 n^{-\alpha} \) and hence vanishes to zero as \( n \to \infty \). Other terms will converge to zero as well by choosing \( M = \log n \), so that the resulting radius of the ball is at most \( (M^2_{2} + 1)^{1/2} \hat{r} \).
The largest value of $t$ for the EBR $\Theta_{eb}(t, \tau, \mathcal{I}_r(k))$ to cover the whole space $\mathbb{R}^n$ is
\[
\sup_{\theta \in \mathbb{R}^n} b_r(\theta, \mathcal{I}_r(k)) \leq \sup_{\theta \in \mathbb{R}^n} \frac{\tau \sigma^2 |(I_T^r)|^c}{\sigma^2 + \sigma^2 |I_T^r| \log(\frac{en}{|I_T^r|})} \leq \frac{\tau (n-k)}{1 + k \log(\frac{en}{|I_T^r|})} \triangleq t_n(\tau, \mathcal{I}_r(k)).
\]
Then $\mathbb{R}^n \subseteq \Theta_{eb}(t_n(\tau, \mathcal{I}_r(k)), \tau)$, but $t_n(\tau, \mathcal{I}_r(k))$ is large for large $n$, meaning that there are “very deceptive” parameters in $\mathbb{R}^n$. Clearly, $t_n(\tau, \mathcal{I}_r(k))$ is smaller for bigger $k$. To summarize this discussion, considering a restricted family $\mathcal{I}_r$ leads to a bigger EBR set $\Theta_{eb}(t, \tau, \mathcal{I}_r)$ in the coverage relation at the price of a worse resulting restricted oracle rate $r_3^2(\theta, \mathcal{I}_r)$ in the size relation.

The largest value of $b_r(\theta)$ for the parameters from the traditional sparsity class $\ell_0[p]$ is
\[
\sup_{\theta \in \ell_0[p]} b_r(\theta, \mathcal{I}_r(k)) \leq \sup_{\theta \in \mathbb{R}^n} \frac{\tau \sigma^2 |(I_T^r)|^c}{\sigma^2 + \sigma^2 |I_T^r| \log(\frac{en}{|I_T^r|})} \leq \frac{\tau \min\{p, n-k\}}{1 + k \log(\frac{en}{|I_T^r|})}. \tag{24}
\]
Typically $p = p_n \to \infty$ as $n \to \infty$ and $k$ is a small fixed number (say, $k = 0$ or $k = 1$ in order to have a small restricted oracle rate), so that the upper bound in the above display is still large for large $n$. This means that the sparsity classes $\ell_0[p_n]$ (even with $p_n = o(n)$) still contain “very deceptive” parameters.

### 3.4 Implications: the minimax results over sparsity classes

In this subsection we elucidate the potential strength of the local approach. In particular, we demonstrate how the global adaptive minimax results over certain scales can be derived from the local results. Note that the oracle rate $r(\theta)$ is a local quantity in that it quantifies the level of accuracy of inference about specific $\theta$ and originally it is not linked to any particular scale of classes. However, it is always possible to relate the oracle rate to various scales. Precisely, if we want to establish global adaptive minimax results over certain scale, say, $\{\Theta_\beta, \beta \in \mathcal{B}\}$, with corresponding minimax rates $\{r(\Theta_\beta), \beta \in \mathcal{B}\}$ (the minimax rate over $\Theta_\beta$ is $r^2(\Theta_\beta) \triangleq \inf_\beta \sup_\theta \|\hat{\theta} - \theta\|^2$, where the infimum is taken over all estimators), the only thing we need to show is
\[
\sup_{\theta \in \Theta_\beta} r^2(\theta) \leq cr^2(\Theta_\beta), \quad \text{for all } \beta \in \mathcal{B}.
\]
If the above property holds, we say the oracle rate $r(\theta)$ covers the scale $\{\Theta_\beta, \beta \in \mathcal{B}\}$. In this case, the local results (with the oracle rate $r(\theta)$) on the estimation, the posterior contraction and the size relation of the confidence ball will immediately imply the corresponding global adaptive minimax results over the covered scale, (actually, simultaneously for all scales that are covered by the oracle rate $r(\theta)$). As to the coverage property, according to Corollary 11 it holds uniformly only over the EBR class $\Theta_{eb}(t, \tau)$, whichever scale we consider. Thus, specializing the coverage property to a particular scale boils down to intersecting this scale with the EBR class $\Theta_{eb}(t, \tau)$ in the coverage property.

Next we consider two sparsity scales $\{\Theta_\beta, \beta \in \mathcal{B}\}$ for which the adaptive minimax results (on the estimation problem, the contraction rate of the empirical Bayes posterior, and the size property of the confidence ball $B(\hat{\theta}, (M_2r^2 + M_1^{1/2})$) follow from our local results Theorems 11 and 3. The results for other (covered) scales can also be readily derived.
Nearly black vectors. For \( p_n \in \mathbb{N} \) such that \( p_n = o(n) \) as \( n \to \infty \) (we use the usual \( o \), \( O \) notation to describe the asymptotic behavior of certain quantities as \( n \to \infty \)), introduce the class
\[
\ell_0[p_n] = \{ \theta \in \mathbb{R}^n : s(\theta) = |I^*(\theta)| \leq p_n \}, \quad \text{where} \quad I^*(\theta) = \{ i \in \mathbb{N} : \theta_i \neq 0 \}.
\]
By \( I^*(\theta) \) and \( s(\theta) \) we denote the active index set and the sparsity of \( \theta \in \mathbb{R}^n \).

The minimax estimation rate over the class of nearly black vectors \( \ell_0[p_n] \) with the sparsity parameter \( p_n \) is known to be \( r^2(\ell_0[p_n]) = O(\sigma^2 p_n \log(\frac{n}{p_n})) \) as \( n \to \infty \); see [12].

By the definition (19) of the oracle rate \( r \), the class \( I \)

THEOREM 4. Under the conditions of Theorem 2, there exist \( I \)

Corollary 2. Under the conditions of Theorem 3,\( I \)

Remark 4. Clearly, the restricted \( \tau \)-oracle rate \( r_\tau(I, \mathcal{L}_\tau(1)) = \min_{\ell_0[p_n]} r^2(I, \theta) \) (where \( \mathcal{L}_\tau(1) = \{ I \in \mathcal{I} : |I| \geq 1 \} = \mathcal{I} \setminus \emptyset \) also covers the scale \( \ell_0[p_n] \). Moreover, from [21] it follows that if \( p_n \leq C \log(en) \), then \( \ell_0[p_n] \subseteq \Theta_{eb}(\tau C, \tau, \mathcal{L}_\tau(1)) \), which means that the EBR class, with sufficiently large but finite \( t \), contains the nearly black vectors \( \ell_0[p_n] \) with sparsity \( p_n \leq C \log n \) (the case of high degree of sparsity). This implies that the coverage relation (formulated for the restricted family \( \mathcal{L}_\tau(1) \) instead of \( \mathcal{I} \)) holds for all classes \( \ell_0[p_n] \) with \( p_n \leq C \log n \), if we take \( t = \tau C \) in the constants \( M_2 \) and \( m'_2 \).

The next assertion describes some “over-dimensionality” (or “undersmoothing”) control of the empirical Bayes posterior \( \hat{\pi}(I|X) \) from the \( \mathbb{P}_\theta \)-perspective.

Theorem 4. Under the conditions of Theorem 2 there exist \( M_4, \theta \in \mathbb{R}^n \)

In particular, there exist constants \( M'_4, m'_4 > 0 \) such that
\[
\mathbb{E}_\theta \hat{\pi}(I : |I| > M_4 s(\theta)|X) \leq C_0 \exp \left\{ -m'_4 s(\theta) \left[ (M - M_4) \log(\frac{en}{\sigma^2}) - M \log M \right] \right\}.
\]

The above assertion is a local type result, but can readily be specialized to the sparsity class \( \theta \in \ell_0[s_n] \) in the minimax sense. If \( s(\theta) \geq 1 \), the probability bound goes to 0 as \( n \to \infty \).
For $s \in (0, 2)$, the weak $\ell_s$-ball with the sparsity parameter $p_n$ is defined by

$$m_s[p_n] = \{ \theta \in \mathbb{R}^n : \theta_{(i)}^2 \leq (p_n/n)^2(n/i)^{2/s}, \text{ } i \in \mathbb{N}_n \}, \quad p_n = o(\sigma n) \text{ as } n \to \infty,$$

where $\theta_{(i)}^2 \geq \ldots \geq \theta_{(n)}^2$ are the ordered $\theta_1, \ldots, \theta_n$. This scheme can be thought of as Sobolev hyper-rectangle for ordered (with unknown locations) coordinates: $m_s[p_n] = \mathcal{H}(\beta, \delta_n) = \{ \theta \in \mathbb{R}^n : |\theta_{(i)}| \leq \delta_n i^{-\beta} \}$, with $\delta_n = n^{1/s} R/n$ and $\beta = 1/s > 1/2$.

Denote $j = O_0(i)$ if $\theta_{(i)}^2 = \theta_{(j)}^2$, with the convention that in the case $\theta_{(i)}^2 = \ldots = \theta_{(k)}^2$ for $i_1 < \ldots < k$ we let $O_0(i, j+1) = O_0(i, 1) + 1, l = 1, \ldots, k - 1$. The minimax estimation rate over the class is $r^2(m_s[p_n]) = n(\log(n/p_n))^{1-s/2}$ when $n^{2/s}(\log(n/p_n))^2 \geq \sigma^2 \log n$, and $r^2(m_s[p_n]) = n^{2/s}(\log(n/p_n))^2 + \sigma^2$ when $n^{2/s}(\log(n/p_n))^2 < \sigma^2 \log n$, as $n \to \infty$; see [14] and [3]. Then take $I^*(\theta) = \{ i \in \mathbb{N}_n : O_0(i) \leq p_n^* \}$, with $p_n^* = eN(\log(n/p_n)^s)^{-s/2}$ in the case $n^{2/s}(\log(n/p_n))^2 \geq \sigma^2 \log n$, to derive

$$\sup_{\theta \in m_s[p_n]} r^2(I^*(\theta)) \leq \sup_{\theta \in m_s[p_n]} r^2(I^*(\theta), \theta) \leq \sigma^2 p_n^* \log(e n/p_n) + n^{2/s}(p_n^*)^2 \sum_{i > p_n^*} i^{-2/s},$$

$$\leq K_1 \sigma^2 p_n^* \log(e n/p_n) + K_2 n^{2/s}(p_n^*)^2(p_n^*)^{1-2/s} \leq Kn(\log(n/p_n))^s \sigma^2 \log((\log(n/p_n))^{1-s/2} = O(r^2(m_s[p_n])),$$

for some $K = K(s)$. The case $n^{2/s}(\log(n/p_n))^2 < \sigma^2 \log n$ is treated similarly by taking $p_n^* = 0$.

Theorems [1] and [3] imply the minimax adaptive results for the scale $m_s[p_n]$.

**Corollary 3.** Under the conditions of Theorem 3.

$$\sup_{\theta \in m_s[p_n]} \mathbb{E}_\theta(\|\theta - \theta\|^2) \geq M_0 K r^2(m_s[p_n]) + M \sigma^2 |X| \leq H_0 e^{-m_0 M},$$

$$\sup_{\theta \in m_s[p_n]} \mathbb{P}_\theta(\|\theta - \theta\|^2 \geq M_1 K r^2(m_s[p_n]) + M \sigma^2) \leq H_1 e^{-m_1 M},$$

$$\sup_{\theta \in m_s[p_n]} \mathbb{P}_\theta(\hat{r}^2 \geq M_3 K r^2(m_s[p_n]) + M \sigma^2) \leq H_3 e^{-m_3 M}.$$
4 Concluding remarks and EBR

4.1 Concluding remarks

Normal case. In case \( \xi_i \sim N(0, 1) \), we can sharpen up some bounds in the proofs. In the proof of Lemma 1 we can compute exactly the right hand side of (28) by using the elementary identity: for \( Y \sim N(\mu_y, \sigma_y^2) \),

\[
E \exp \left\{ \frac{aY^2}{2} \right\} = \exp \left\{ \frac{a\mu_y^2}{2(1-a\sigma_y^2)} - \frac{1}{2} \log(1-a\sigma_y^2) \right\}, \text{ for any } a < \sigma_y^{-2}.
\]  

(26)

By some tedious but straightforward calculations, we obtain the claim of Lemma 1 for any \( h \in [0, 1) \) with the constants \( A_h = \frac{h}{2(1+h)} \), \( B_h = \frac{h}{2(1-h)} \), \( C_h = \frac{h}{2} \) and \( D_h = \frac{h}{2} + \frac{1}{2} \log(1-h) \).

If \( I \setminus I_0 = \emptyset \), the bound holds also for \( h = 1 \) with \( A_1 = \frac{1}{4}, B_1 = 0, C_1 = D_1 = \frac{1}{2} \).

Next, since Lemma 1 now holds for any \( h \in [0, 1) \), we can try to optimize the choice of \( h \) in Lemma 2. For example, take \( h_0 = \arg\min_{h \in [0,1]} \lambda_{h}^{2} \exp(-D_h |I| \log(\frac{\sigma}{\sigma_y})) = \arg\max_{h \in (0,1)} \left( (1+2\kappa)h + \log(1-h) \right) = \frac{\sigma_y}{2 \tau+1} \) (also, this is such \( h_0 \) that \( \kappa h_0 + C h_0 = B h_0 \)).

The claim of Lemma 2 follows with the constants \( c_1 = \kappa h_0 + D h_0 - A h_0 = \kappa - \frac{1}{2} \log(2\kappa + 1) - \frac{\kappa}{2 \tau + 1}, c_2 = \frac{\kappa}{2 \tau + 1} \) and \( c_3 = 4 \kappa + 1 \). We need to ensure that \( c_1 > 2 \), which is provided in this case by taking \( \kappa > \hat{\kappa} = 3.24 \).

The constants in the proof of Lemma 3 can also be improved in the normal case. Indeed, we apply Lemma 1 with \( h = 1 \) (instead of \( h = \beta \)) to obtain the main claim with \( a(\tau, \phi) = \frac{\kappa}{\tau} (1 - \phi) - \kappa (1 + \phi) - \frac{1}{2} > 0, C \phi = (1 - e^{1-\kappa})^{-1} \) and \( m_{\phi} = r + \kappa \) under the conditions \( \kappa > 1 \) and \( \tau > \hat{\tau}(\phi) \triangleq \frac{4 \kappa (1+\phi) + 2}{1-\phi} \). Finally, Lemma 1 can be specialized to the normal case \( \xi_i \sim N(0, 1) \) and we can use the bound

\[
E(\sum_{i \in I} \xi_i^2)^2 = |I|^2 + 2|I| \leq 3|I|^2
\]

instead of (16) in the proof of Theorem 1.

Product prior. If, instead of the prior \( \pi \), we take a prior \( \bar{\pi} = \bar{\pi}_{K, \kappa} = \sum_{I \in \mathcal{I}} \lambda_I \pi_I \) with \( \tau^{(2)}_{\pi}(I) = K \sigma^2 |I| \) for any fixed \( K > 0 \) (we can even allow \( K = K_n \to \infty \), but \( K_n = O(n) \), as \( n \to \infty \)) in (3) and \( \lambda_I = c_{\kappa, \phi} \exp(-\kappa |I| \log n) \) (with \( \kappa = \kappa_0 \) for some \( \kappa_0 > 0 \) in (6)), then all the results will hold with \( \log n \) instead of \( \log(\frac{\pi}{\pi_n}) \) in the oracle rate (19). This case was studied in the first version of the arXiv-preprint of this paper. Thus, the results for the prior \( \bar{\pi} \) are weaker than the results obtained in this paper. For example, the minimax rates for the sparsity classes (Corollaries 2, 3) follow from these weaker results only if the sparsity parameter \( p_n = O(n^{\gamma}) \) for \( \gamma \in [0, 1) \) as \( n \to \infty \), otherwise we obtain only the near-minimax rates, with the factor \( \log n \) instead of \( \log(\frac{\pi}{\pi_n}) \).

However, there is an advantageous feature of the prior \( \bar{\pi} \) as compared with \( \pi \). Namely, it is of the product structure: for \( \lambda_I = c_{\lambda} \prod_{i \in I} \lambda_i \) with \( c_{\lambda} = \prod_{i=1}^{n} (1 + \lambda_i)^{-1} \), we compute

\[
\bar{\pi} = \sum_{I \in \mathcal{I}} \lambda_I \pi_I = \bigotimes_{i=1}^{n} \left[ \omega_i N(\mu_{1,i}, K \sigma^2) + (1 - \omega_i) \delta_0 \right], \omega_i = \frac{\lambda_{\pi_i}}{\lambda_{\bar{\pi}_i}} \text{ (} \omega_i = \lambda(i \in I) \text{ is the prior probability that the (random) set } I \text{ contains } i \text{).}
\]

This leads to the product structure of the empirical Bayes posterior, so that the computation of the corresponding empirical Bayes estimator can easily be done in the coordinatewise fashion. Indeed, in our case
\( \lambda_i = \lambda = n^{-\kappa} \) and some computations give the following empirical Bayes posterior

\[
\pi(\theta | X) = \prod_{i=1}^{n} \left[ p_i N(X_i, K \sigma_i^2) + (1 - p_i) \delta_0 \right], \quad p_i = 1/\left[ 1 + h \exp\{-\frac{X_i^2}{2\sigma^2}\} \right],
\]

where \( p_i = \pi(\theta_i \neq 0 | X) \) and \( h = h_{\kappa,K} = \sqrt{\frac{K+1}{\lambda}} = n^{\kappa}(K+1)^{1/2} \). The mean with respect to \( \pi(\theta | X) \) is readily obtained: \( \hat{\theta} = E_\pi(\theta | X) = (p_i X_i, i \in \mathbb{N}_n) \), a shrinkage estimator with easily computable shrinkage factors \( p_i \). Coordinatewise empirical Bayes medians can also be easily computed.

**Cardinality dependent prior \( \lambda \).** Notice that the prior \( \lambda = (\lambda_I, I \in \mathcal{I}) \) defined by \([5]\) depends on the set \( I \in \mathcal{I} \) only via its cardinality \( |I| \), i.e., \( \lambda_I = g(|I|) \) for some nonnegative function \( g(k), k = 0, 1, \ldots, n \). It is easy to see that in this case \( \pi_n(k) = g(k)(\binom{n}{k}) \), \( k = 0, 1, \ldots, n \), determines the prior on the cardinality of \( I \). Hence, the prior \( \lambda_I \) can always be modeled in two steps: first draw the random cardinality \( K \) according to the prior \( \pi_n(k) \), and then given \( K = k \), draw a random set \( I \) uniformly from the family of all subsets of \( \mathcal{I} \) of cardinality \( k \). Such priors \( \lambda \) are used in \([11]\), where the cardinality prior \( \pi_n(k) \) can be taken to be a so called “complexity prior” \( \pi_n(k) = \exp\{-ak \log(bn/k)\} \) for some \( a, b > 0 \). Since \( e^{k \log(n/k)} \leq \binom{n}{k} \leq e^{k \log(nk/k)} \), the resulting prior mass \( \lambda_I \) on \( I \) is bounded below and above by expressions of the type \( \exp\{-a_I |I| \log(bn/|I|)\} \), resembling the prior \([5]\).

The condition on the complexity prior from \([11]\) essentially corresponds to our condition \( \kappa > \bar{\kappa} \) for some \( \bar{\kappa} > 0 \) (Condition \((A2)\)).

**Connection of double empirical Bayes with penalization method.** Notice that the estimator \( \hat{\theta} = X(\hat{I}) \) defined by \([11]\) is the penalized estimator as introduced in \([5]\) (cf. also \([1]\)), where the selector (“estimator of the oracle”) \( \hat{I} \) is determined by the penalization criterion \([13]\) with the penalty \( P(I) = (2\kappa + 1)\sigma^2 |I| \log \left( \frac{n}{|I|} \right) \). This penalty is from the family of penalties corresponding to the complete variable selection case in \([5]\) with the penalty constant \( 2\kappa + 1 \). Recall our rather specific choice of parameter \( K_n(I) \) in \([3]\) resulting in this penalty. As we mentioned, other choices of \( K_n(I) \) are also possible, which would lead to other penalties. But the main term \( \log \left( \frac{n}{|I|} \right) \) would always be present in the penalty because of the choice of prior \( \lambda_I \). This reiterates the conclusion in \([1]\) that essentially only this kind of penalties lead to adaptive penalized estimators.

Interestingly, recall that we require \( \kappa > \bar{\kappa} \) for some \( \bar{\kappa} \) (see Condition \((A2)\)). For each specific situation, one can try to relax this bound. For example, in the normal case \( \kappa > 3.24 \). By more subtle evaluations (and at the price of bigger constants elsewhere), this bound can be relaxed to \( \kappa > \kappa_0 \) for some \( 0 < \kappa_0 < 3.24 \), but \( \kappa_0 \) cannot be arbitrarily close to zero (in order for prior \( \lambda \) to sufficiently penalize big values of cardinality \( |I| \)). In a way, the condition \( \kappa > \bar{\kappa} > 0 \) (Condition \((A2)\)) corresponds to the requirement in \([5]\) (argued in \([5]\) from a different perspective) that the penalty constant for penalized estimators should be bounded away from 1. It is argued in \([5]\) that large penalty constants \( \kappa \) should also be avoided. We get the same conclusion by observing that the constants in claims (i)-(ii) of Theorem \([1]\) become worse as \( \kappa \to \infty \).
Computing the estimators. Note that the estimator (12) is a shrinkage estimator, and the penalized estimator (14) is a hard thresholding estimator. Indeed, the estimator (12) is \( \hat{\theta}_i = p_i X_i \) where \( p_i = \sum_{I: i \in I} \pi(I|X) \), and the estimator (14) is \( \tilde{\theta}_i = X_i 1_{|X_i| \geq \tilde{t}} \), where \( \tilde{t} = |X[k]|, |X[1]| \geq \ldots \geq |X[n]| \), and \( k \) is the minimizer of \( \text{crit}(k, X) = \sum_{i=k+1}^n X_i^2 + (2\kappa + 1)\sigma^2 k \log(en/k) \).

The thresholding procedure is easy to implement, whereas the values \( p_i \) in the shrinkage procedure are more difficult to compute. It is demonstrated in [11] how one can use the partial product structure (in the model and in \( \pi(I) \), but not in \( \lambda(I) \)) to facilitate the computation of \( p_i \)'s. Other estimators can be considered, for example, the coordinatewise median with respect to \( \tilde{\pi} \), which is going to be something in between shrinkage and thresholding.

Condition (A1). It is interesting to relate Condition (A1) to the so called subgaussianity condition on the error vector \( \xi = (\xi_i, i \in \mathbb{N}_n) \). Let \( \langle x, y \rangle = \sum_i x_i y_i \) denote the usual scalar product between \( x, y \in \mathbb{R}^n \). The random vector \( \xi \) is called sub-gaussian with parameter \( \rho > 0 \) if

\[
P(\langle v, \xi \rangle > t) \leq e^{-\rho t^2} \text{ for all } t \geq 0 \text{ and all } v \in \mathbb{R}^n \text{ such that } ||v|| = 1.
\]

The sub-gaussianity condition and Condition (A1) are close, but in general incomparable. If the \( \xi_i \)'s are independent, then the sub-gaussianity condition implies Condition (A1). Indeed,

\[
\mathbb{E}e^{\beta\xi_i^2} = \int_{1}^{\infty} P(\xi_i^2 > \beta^{-1} \log t) dt \leq \int_{1}^{\infty} t^{-\beta/\rho} dt = C(\rho, \beta),
\]

for \( \beta < \rho \), so that \( \mathbb{E}\exp\{\beta \sum_{i \in I} \xi_i^2\} \leq e^{\beta |I| \log C(\rho, \beta)} \) for any \( \beta < \rho \). In a way, the case of independent normal \( \xi_i \)'s is the worst for inference on \( \theta \) in the model (1).

All the results still hold, if, instead of Condition (A1), we assume the weaker condition: \( \mathbb{E}\exp\{\beta \sum_{i \in I} \xi_i^2\} \leq C_{\beta} e^{B|I| \log(en/|I|)} \) for all \( I \in \mathcal{I} \) and some \( \beta \in (0, 1], B, C_{\beta} > 0 \). However, we leave Condition (A1) in its present form, to provide a cleaner mathematical exposition.

4.2 The EBR condition

The original motivation of EBR. As mentioned in the introduction, it is impossible to construct optimal (fully) adaptive confidence set in the minimax sense over traditional smoothness and sparsity scales with a prescribed high coverage probability. Namely, there exist “deceptive” parameters \( \theta \in \Theta'_0 = \mathbb{R}^n \setminus \Theta_0 \) for which the coverage property in (2) may not hold for arbitrarily small \( \alpha_1 \). Removing deceptive parameters \( \Theta'_0 \) and restricting to the remaining set \( \Theta_0 \) of non-deceptive parameters resolves this issue. This was the original motivation of introducing the EBR condition.

Such sets of non-deceptive parameters have been well investigated only for certain smoothness structures. Examples of sets of non-deceptive parameters are: self-similar parameters introduced by [19] for a related problem and and later adopted by a number of authors; a more general class of polished tail parameters introduced in [21]; and yet
As demonstrated in [21], self-similarity or polished tail conditions can be considered mild from topological, minimax and measure theoretic points of view, the same certainly applies to the EBR. Moreover, the topological and measure theoretic justifications employed for self-similarity or polished tail conditions are trivially satisfied for EBR, because the EBR condition, unlike self-similarity or polished tail conditions, leads to slicing of the entire parameter space $\mathbb{R}^n$. This opens up a new perspective on the EBR and its role in the deceptiveness issue, which we explain next.

**A new perspective on EBR – the EBR scale.** Note that the EBR condition $\theta \in \Theta_{eb}(t, \tau)$ is actually a family of imbedded conditions parametrized by $t \geq 0$: $\Theta_{eb}(t_1, \tau) \subseteq \Theta_{eb}(t_2, \tau)$ for $t_1 \leq t_2$. An important observation is that this family of conditions effectively introduces a new scale $\cup_{t \geq 0} \Theta_{eb}(t, \tau)$ (for any fixed $\tau > 0$), to be called the *EBR scale*, with the structural parameter $t \geq 0$ measuring the allowed amount of deceptiveness for parameters $\theta \in \Theta_{eb}(t, \tau)$. Indeed, this scale “slices” $\mathbb{R}^n$ in the sense that $\mathbb{R}^n = \cup_{t \geq 0} \Theta_{eb}(t, \tau) = \cup_{0 \leq t \leq \tau \in R} \Theta_{eb}(t, \tau)$. The main benefit of introducing the EBR scale is that it gives the slicing of the entire space that is very suitable for uncertainty quantification. Indeed, the dictum “removing deceptive parameters” becomes a very natural notion in terms of the scale $\cup_{t \geq 0} \Theta_{eb}(t, \tau)$ as it is nothing else but restricting the amount of deceptiveness $t$. This provides a new perspective at the above mentioned “deceptiveness” issue: basically, each parameter $\theta \in \mathbb{R}^n$ has a certain amount of “deceptiveness” that is measured by the excessive $\tau$-bias ratio $b_{\tau}(\theta)$, or the smallest $t$ for which $\theta \in \Theta_{eb}(t, \tau)$. The more $t$, the more deceptive parameters are allowed in $\Theta_{eb}(t, \tau)$. A mild and controllable price for the uniformity over $\Theta_{eb}(t, \tau)$ in the coverage relation is the amount of inflating of the confidence ball needed to provide a guaranteed high coverage for the parameters of deceptiveness at most $t$.

Interestingly, slicing is also possible by the parameter $\tau > 0$: $\mathbb{R}^n = \cup_{\tau \geq 0} \Theta_{eb}(t, \tau)$ (for any $t > 0$), the embedding goes in the opposite direction: the smaller the $\tau$, the weaker the EBR. Namely, $\Theta_{eb}(t, \tau_2) \subseteq \Theta_{eb}(t, \tau_1)$ for any $0 \leq \tau_1 \leq \tau_2$, $t > 0$, and the “limiting” EBR set $\lim_{\tau \downarrow 0} \Theta_{eb}(t, \tau)$ expands to the entire space: $\Theta_{eb}(t, 0) = \mathbb{R}^p$. Besides, notice that the inflating factor in the confidence ball from Corollary 1 will not not increase as $\tau \downarrow 0$ (in fact, it will decrease). A paradox seems to have emerged: by considering very small $\tau$’s, we can have less deceptiveness without any price in the coverage relation. However, this paradox is resolved by reminding that the coverage relation from Theorem 3 does not hold for arbitrarily small $\tau$ because of the condition $\tau > \bar{\tau}$ for some $\bar{\tau} > 0$, showing that “there is no free lunch”.

**The EBR does not affect the minimaxity over the sparsity scale $\ell_0[p]$.** The EBR condition is mild from the minimax point of view in the following sense: if we take the traditional sparsity class $\ell_0[p] = \{ \theta \in \mathbb{R}^n : |I^*(\theta)| = ||\theta||_0 \leq p \}$ for $p \in \mathbb{N}$ and remove non-EBR parameters, then the minimax rate over the remaining part will not change (up to a constant). We outline the argument below. The minimax estimation rate was established by Birgé and Massart [5] (Theorem 4 from [5], formulated in our notation):
for some universal constant \( c > 0 \),
\[
  r^2(\ell_0[p]) \triangleq \inf_{\hat{\theta}} \sup_{\theta \in \ell_0[p]} \mathbb{E}_\theta \| \hat{\theta} - \theta \|^2 \geq c\sigma^2 p \log(en/p).
\]

The proof is based on considering the following subset \( \mathcal{B}_1(p) = \{ \theta \in \mathbb{R}^n : |I^*(\theta)| \leq p, |\theta_i| \leq \sigma^2 \log(en/p) \} \subset \ell_0[p] \) and establishing the required lower bound for the minimax risk \( R(\mathcal{B}_1(p)) \) over the set \( \mathcal{B}_1(p) \), thus obtaining \( r^2(\ell_0[p]) \geq r^2(\mathcal{B}_1(p)) \geq c\sigma^2 p \log(en/p) \).

Inspecting all the steps in the proof, we see that essentially the same lower bound (with a different constant \( c \)) holds for another subset of \( \ell_0[p] \): \( \mathcal{B}_2(p, \tau) = \{ \theta \in \mathbb{R}^n : |I^*(\theta)| = p, 2\tau \sigma^2 \log(en/p) \leq |\theta_i| \leq (2\tau + 1)\sigma^2 \log(en/p) \) for all \( i \in I^*(\theta) \), for \( \tau > 0 \). For each \( \theta \in \mathcal{B}_2(p, \tau) \), we have \( I^*_c(\theta) = I^*(\theta) \) so that \( |I^*_c(\theta)| = \varnothing, |I^*_0(\theta)| = p \), and the EBR condition is trivially satisfied for any \( t \geq 0 \):
\[
b_t(\theta) = \frac{\sum_{i \notin I^*_c(\theta)} \theta_i^2}{\sigma^2 + \sigma^2 p \log(en/p)} = 0 \leq t.
\]

This means that \( r^2(\ell_0[p] \cap \Theta_{eb}(t, \tau)) \geq r^2(\mathcal{B}_2(p, \tau)) \geq c\sigma^2 p \log(en/p) \).

## 5 Simulations

Here we present a small simulation study according to the model (1) with \( \xi \overset{\text{ind}}{\sim} \mathcal{N}(0, 1) \), \( \sigma = 1 \) and \( n = 500 \). We used signals \( \theta = (\theta_1, \ldots, \theta_n) \) of the form \( \theta = (0, \ldots, 0, A, \ldots, A) \), where \( p = \#(\theta_0, i \neq 0) \) last coordinates of \( \theta \) are equal to a fixed number \( A \). Different sparsity levels \( p_n \in \{25, 50, 100\} \) and “signal strengths” \( A \in \{3, 4, 5\} \) are considered. We construct a confidence ball by using the empirical Bayes posterior \( \hat{\pi}(\theta|X) \) defined by (14) with parameter \( \varpi = 0.7 \). Namely, consider the ball \( B(\hat{\theta}, \sqrt{M\hat{r}}) \) around \( \hat{\theta} \) defined by (14) with the radius \( \hat{r} = \left( [I] \log(en/|I|) \right)^{1/2} \) given by (21). The multiplicative factor \( \sqrt{M} \) is intended to trade-off the size of the ball against its coverage probability. For each sparsity level \( p \in \{25, 50, 100\} \) and signal strength \( A \in \{3, 4, 5\} \), we simulated 100 data vectors \( X \) of dimension \( n = 500 \) from the model (1) and computed the average squared radius by \( \frac{M\hat{r}^2}{r_0^2} \). Table II shows the ratio of \( M\hat{r}^2 \) to the oracle radial rate \( r_0^2(\theta) = |I_0| \log(en/|I_0|) = p \log(en/p) \), where \( I_0 \) is defined by (13), and the frequency \( \alpha \) of the event that confidence ball \( B(\hat{\theta}, \sqrt{M\hat{r}}) \) contains the signal \( \theta \), respectively. The higher the signal strength, the closer the ratio \( \frac{M\hat{r}^2}{r_0^2(\theta)} \) to 1.

## 6 Technical lemmas

First we provide a couple of technical lemmas used in the proofs of the main results.

**Lemma 1.** Let Condition (A1) be fulfilled. Then for any \( \theta \in \mathbb{R}^n \), any \( I, I_0 \in \mathcal{I} \), the following bound holds:
\[
  \mathbb{E}_{\theta} \hat{\pi}(I|X) \leq \left[ \frac{\lambda}{N_{I_0}} \right]^h \exp \left\{ B_h \sum_{i \in I \setminus I_0} \frac{\theta_i^2}{\sigma^2} - A_h \sum_{i \in I_0 \setminus I} \frac{\theta_i^2}{\sigma^2} + C_h |I_0| \log \left( \frac{en}{|I_0|} \right) - D_h |I| \log \left( \frac{en}{|I|} \right) \right\},
\]
where \( h = \frac{2\beta}{3} \) and the constants \( A_h = \frac{\beta}{6}, B_h = \frac{2\beta}{3}, C_h = \frac{\beta + B}{3} \) and \( D_h = \frac{\beta - 2B}{3} \). If \( I \setminus I_0 = \emptyset \), the bound holds also for \( h = \beta \) with \( A_\beta = \frac{\beta}{2}, C_\beta = \frac{\beta}{2} + B, D_\beta = \frac{\beta}{2}, B_\beta = 0 \).

**Proof of Lemma [4]** In case \( \hat{\pi}(I | X) = \tilde{\pi}(I | X) \), we get by (11) that, for any \( I, I_0 \in \mathcal{I} \) and any \( h \in [0, 1] \),

\[
E_\theta \tilde{\pi}(I | X) = E_\theta \tilde{\pi}(I | X) = E_\theta \left[ \frac{\lambda_I \prod_{i=1}^n \phi(X_i, X_i, 1 \{i \in I\}) \sigma^2 + \tau^2(I) - \tau^2(J)}{\sum_{J \in \mathcal{I}} \lambda_I \prod_{i=1}^n \phi(X_i, X_i, 1 \{i \in J\}) \sigma^2 + \tau^2(J)} \right]^{h} \leq E_\theta \left[ \frac{\lambda_I \prod_{i=1}^n \phi(X_i, X_i, 1 \{i \in \hat{I}\}) \sigma^2 + \tau^2(I) - \tau^2(J)}{\sum_{J \in \mathcal{I}} \lambda_I \prod_{i=1}^n \phi(X_i, X_i, 1 \{i \in \hat{I}\}) \sigma^2 + \tau^2(J)} \right]^{h} \leq \left( \frac{\lambda}{\lambda_0} \right)^h \sum_{I \in \mathcal{I}} X^2_{\sigma^2} - \sum_{i \in I_0 \setminus I} X^2_{\sigma^2} = |I_0| \left( \frac{\log \frac{\lambda}{\lambda_0}}{\log \frac{\lambda}{\lambda_0}} \right)^h.
\]
If \( |I|_0 = \emptyset \), we take \( h = \beta \) in (27) and (28) and combine this with \( E_\theta \exp \left\{ -\frac{\beta}{2} \sum_{i \in I_0 \setminus I} \frac{X_i^2}{\sigma^2} \right\} \leq \exp \left\{ -\frac{\beta}{2} \sum_{i \in I_0 \setminus I} \frac{\theta_i^2}{\sigma^2} + B|I_0 \setminus I| \right\} \). The last relation holds in view of Condition (A1) and \(-X_i^2 \leq -\frac{\theta_i^2}{3\sigma^2} + 2\zeta_i^2\), as \((a + b)^2 \geq 2a^2/3 - 2b^2\).

\[ \square \]

**Remark 6.** Notice that we proved the above lemma for the both cases \( \hat{\pi}(I|X) = \hat{\pi}(I|X) \) and \( \hat{\pi}(I|X) = \hat{\pi}(I|X) \). From this point on, the proof of the properties of \( \hat{\pi}(\theta|X) \) and \( \hat{\theta} \) proceeds exactly in the same way as for \( \hat{\pi}(\theta|X) \) and \( \hat{\theta} \), with the only difference that everywhere (in the claims and in the proofs) \( \hat{\pi}(I \in \mathcal{G}|X) \) should be read as \( \hat{\pi}(I \in \mathcal{G}|X) \) in case \( \hat{\pi} = \hat{\pi} \); and as \( 1\{I \in \mathcal{G}\} \) in case \( \hat{\pi} = \hat{\pi} \), for all \( \mathcal{G} \subseteq \mathbb{I} \) that appear in the proof. Hence, \( E_\theta \hat{\pi}(I \in \mathcal{G}|X) = E_\theta \hat{\pi}(I \in \mathcal{G}|X) \) in the former case, and \( E_\theta \hat{\pi}(I \in \mathcal{G}|X) = P_\theta(I \in \mathcal{G}|X) \) in the latter case.

Note that above lemma holds for any set \( I_0 \in \mathbb{I} \). By taking \( I_0 = I_o \) defined by (19), we obtain the following lemma.

**Lemma 2.** Let Conditions (A1) and (A2) be fulfilled. Then there exist positive constants \( c_1 = c_1(\kappa) > 2 \), \( c_2 \) and \( c_3 = c_3(\kappa) \) such that for any \( \theta \in \mathbb{R}^n \)

\[
E_\theta \hat{\pi}(I|X) \leq \left( \frac{m}{|\mathcal{F}|} \right)^{-c_1|I|} \exp \left\{ -c_2\kappa^2 \left[ \kappa^2(I, \theta) - c_3\kappa^2(\theta) \right] \right\}.
\]

**Proof of Lemma 2.** With constants \( h, A_h, B_h, C_h, D_h \) given in Lemma 1 define the constant \( c_1 = c_1(\kappa) = \kappa h + D_h - A_h = \frac{2\kappa h}{3} + \frac{\beta - 2B}{3} - \frac{\beta}{6} > 2 \) as \( \kappa > \kappa \) by condition (A2).

Since \( \kappa h + D_h = c_1 + A_{h} \), the definition (5) of \( \lambda_l \) entails that

\[
\left( \frac{\lambda_l}{\lambda_0} \right)^h \exp \left\{ C_h|I_0| \log(\frac{m}{|\mathcal{F}|}) - D_h|I| \log(\frac{m}{|\mathcal{F}|}) \right\} = \left( \frac{m}{|\mathcal{F}|} \right)^{-c_1|I|} \exp \left\{ (\kappa h + C_h)|I_0| \log(\frac{m}{|\mathcal{F}|}) - A_h|I| \log(\frac{m}{|\mathcal{F}|}) \right\}.
\]

Using the last relation and Lemma 1 with \( I_0 = I_o \), we bound \( E_\theta \hat{\pi}(I|X) \) by

\[
\left[ \frac{\lambda_l}{\lambda_0} \right]^h \exp \left\{ B_h \sum_{i \in I \setminus I_0} \frac{\theta_i^2}{2} - A_h \sum_{i \in I \setminus I_0} \frac{\theta_i^2}{2} + C_h|I_0| \log(\frac{m}{|\mathcal{F}|}) - D_h|I| \log(\frac{m}{|\mathcal{F}|}) \right\} = \left( \frac{m}{|\mathcal{F}|} \right)^{-c_1|I|} \exp \left\{ -A_h \sum_{i \in I \setminus I_0} \frac{\theta_i^2}{2} - A_h|I| \log(\frac{m}{|\mathcal{F}|}) + B_h \sum_{i \in I \setminus I_0} \theta_i^2 + (\kappa h + C_h)|I_0| \log(\frac{m}{|\mathcal{F}|}) \right\}.
\]

The claim of the lemma follows with the constants \( c_1 = (4\beta\kappa + \beta - 4B)/6 > 2 \), \( c_2 = A_h = \beta/6 \) and \( c_3 = c_3(\kappa) = \max\{B_h, \kappa h + C_h\}/A_h = (\kappa h + C_h)/A_h = 4\kappa + 2(\beta + B)/\beta \).

\[ \square \]

**Lemma 3.** Let Condition (A1) be fulfilled, \( \kappa > \beta^{-1} \) (Condition (A2) implies this), the \( \tau \)-oracle \( I^*_\tau = I^*_\tau(\theta) \) be defined by (19). Then for any \( \varrho \in (0, 1) \) and \( \tau > \bar{\tau}(\varrho) \) with \( \bar{\tau}(\cdot) \) defined by (17), there exists an \( m_1^\tau > 0 \) such that, for any \( \theta \in \mathbb{R}^n \) and any \( M \geq 0 \),

\[
E_{\theta} \hat{\pi}(I : |I| \log(\frac{m}{|\mathcal{F}|}) \leq \varrho|I^*_\tau| \log(\frac{m}{|\mathcal{F}|}) - M|X| \leq C_\kappa \left( \frac{m}{|\mathcal{F}|} \right)^{-\alpha|I^*_\tau|} e^{-m_1^\tau M},
\]

where \( C_\kappa = (1 - e^{1-\kappa\beta})^{-1} \) and \( \alpha = \alpha(\tau, \varrho) = \frac{\varrho}{\kappa}(1 - \varrho) - (\kappa\beta + B)(1 + \varrho) - \frac{\beta}{2} > 0 \).
Proof of Lemma 3. For each \( \theta \in \mathbb{R}^n \) and \( I \in \mathcal{I} \) such that \(|I| \leq \varrho |I_0^c|\), define \( I_0 = I_0(I, \theta) = I \cup I_0^c\), where \( I_0^c = I_0^c(\theta) \) is given by (19). Then

\[
|I| \leq \varrho |I_0^c| \leq |I_0^c| \leq |I_0| \leq |I| + |I_0^c|.
\]

From the definition of \( I_0 \) and the definition (19) of the \( \tau \)-oracle \( I_0^\tau \), it follows that, for each \( I \in \mathcal{G} = \{ I : |I| \log \left( \frac{cn}{|I|} \right) \leq \varrho |I_0^c| \log \left( \frac{cn}{|I_0^c|} \right) - M \}, \)

\[
\sum_{i \in I \setminus I_0} \frac{\theta_i^2}{\sigma_i^2} \geq \frac{1}{\sigma^2} \left( \sum_{i \in I} \theta_i^2 - \sum_{i \in (I_0)^c} \theta_i^2 \right) \geq \tau\left( |I_0^c| \log \left( \frac{cn}{|I_0^c|} \right) - |I| \log \left( \frac{cn}{|I|} \right) \right)
\]

\[
\geq \tau(1 - \varrho)|I_0^c| \log \left( \frac{cn}{|I_0^c|} \right) + \tau M.
\]

Besides, we have

\[
|I_0| \log \left( \frac{cn}{|I_0|} \right) = |I \cup I_0^c| \log \left( \frac{cn}{|I_0^c|} \right) \leq |I| \log \left( \frac{cn}{|I|} \right) + |I_0^c| \log \left( \frac{cn}{|I_0^c|} \right).
\]

Using the last three relations, (14) and Lemma 1 with \( h = \beta \), we evaluate for each \( I \in \mathcal{G} \)

\[
E_\theta \tilde{\pi}_I \leq \left[ \frac{\lambda}{\lambda_0} \right]^{\frac{\beta}{3}} \exp \left\{ -\frac{\beta}{3} \sum_{i \in I_0 \setminus I} \frac{\theta_i^2}{\sigma_i^2} + (\kappa \beta + B)|I_0| \log \left( \frac{cn}{|I_0|} \right) - \beta \frac{|I|}{2} |I_0| \log \left( \frac{cn}{|I_0|} \right) \right\}
\]

\[
\leq \left[ \frac{\lambda}{\lambda_{c,r,n}} \right]^{\frac{\beta}{3}} \exp \left\{ -\frac{\beta}{3} \sum_{i \in I_0 \setminus I} \frac{\theta_i^2}{\sigma_i^2} + (\kappa \beta + B)|I_0| \log \left( \frac{cn}{|I_0|} \right) + \beta \frac{|I_0|}{2} |I_0| \log \left( \frac{cn}{|I_0|} \right) \right\}
\]

\[
\leq \left[ \frac{\lambda}{\lambda_{c,r,n}} \right]^{\frac{\beta}{3}} \exp \left\{ -\frac{\beta}{3} \left( 1 - \varrho \right) + (\kappa \beta + B)(1 + \varrho) \right| I_0^c \log \left( \frac{cn}{|I_0^c|} \right) - \frac{\beta}{3} |I_0^c| \log \left( \frac{cn}{|I_0^c|} \right) \right\}
\]

Since \( \kappa > \beta^{-1} \), by the same reasoning as in (49), we have that \( \sum_I \left[ \frac{\lambda}{\lambda_{c,r,n}} \right]^{\frac{\beta}{3}} \leq (1 - e^{1-\kappa \beta})^{-1} \equiv C_\kappa \). This relation and the last display imply the claim of the lemma:

\[
E_\theta \tilde{\pi}(I \in \mathcal{G}|X) = \sum_{I \in \mathcal{G}} E_\theta \tilde{\pi}_I \leq C_\kappa \exp \left\{ -\alpha |I_0^c| \log \left( \frac{cn}{|I_0^c|} \right) - \left( \frac{\tau \beta}{3} + \kappa \beta + B \right) M \right\},
\]

with \( \alpha = \alpha(\tau, \varrho) = \frac{\tau \beta}{3} (1 - \varrho) - (\kappa \beta + B)(1 + \varrho) - \beta > 0 \) and \( m_1 = \frac{\tau \beta}{3} + \kappa \beta + B \). \( \square \)

Lemma 4. Let \( Y_1, \ldots, Y_n \) be nonnegative random variables such that, for any \( I \in \mathcal{I} \), \( E c^t \sum_{i \in I} Y_i \leq A_I(t) \) for some \( t > 0 \) and \( A_k(t) \). Let \( Y_{[1]} \geq Y_{[2]} \geq \ldots \geq Y_{[n]} \). Then, for any \( k \in \mathbb{N}_n \) and \( C, c \geq 0 \),

\[
P \left( \sum_{i=1}^k Y_{[i]} \geq C k \log \left( \frac{cn}{k} \right) + c \right) \leq A_k(t) \exp \left\{ -(Ct - 1) k \log \left( \frac{cn}{k} \right) - ct \right\},
\]

\[
E \sum_{i=1}^k Y_{[i]} \leq t^{-1} \left[ k \log \left( \frac{cn}{k} \right) + \log (A_k(t)) \right].
\]
In particular, if \( \xi_1, \ldots, \xi_n \overset{\text{ind}}{\sim} N(0, 1) \), then for any \( k \in \mathbb{N}_n, C, c \geq 0 \)
\[
\mathbb{P}\left( \sum_{i=1}^{k} \xi_{[i]}^2 \geq Ck \log \left( \frac{en}{k} \right) + c \right) \leq \left( \frac{en}{k} \right)^{-0.4(C-2)k} e^{-0.4c} , \quad \mathbb{E} \left( \sum_{i=1}^{k} \xi_{[i]}^2 \right) \leq 6k \log \left( \frac{en}{k} \right).
\]

**Proof.** By Jensen’s inequality, we derive
\[
\exp \left\{ t \mathbb{E} \sum_{i=1}^{k} Y_{[i]} \right\} \leq \mathbb{E} \exp \left\{ t \sum_{i=1}^{k} Y_{[i]} \right\} \leq \sum_{I : |I| = k} \mathbb{E} \exp \left\{ t \sum_{i \in I} Y_{[i]} \right\} \leq \binom{n}{k} A_k(t).
\]
Then \( \mathbb{E} \exp \left\{ t \sum_{i=1}^{k} Y_{[i]} \right\} \leq \binom{n}{k} A_k(t) \leq e^{k \log \left( \frac{en}{k} \right) + \log(A_k(t))} \), where we used \( \binom{n}{k} \leq \left( \frac{en}{k} \right)^k \).

This and the (exponential) Markov inequality yield the first relation:
\[
\mathbb{P} \left( \sum_{i=1}^{k} Y_{[i]} \geq Ck \log \left( \frac{en}{k} \right) + c \right) \leq A_k(t) \exp \left\{ -(Ct - 1)k \log \left( \frac{en}{k} \right) - ct \right\}.
\]

The first display implies also the second relation: \( \mathbb{E} \sum_{i=1}^{k} Y_{[i]} \leq t^{-1} \log \left( \binom{n}{k} \right) + \log(A_k(t)) \).

As to the normal case, for any \( I \in \mathcal{I} \) and any \( t < \frac{1}{2} \) we have that \( \mathbb{E} \exp \left\{ t \sum_{i \in I} \xi_{[i]}^2 \right\} = (1 - 2t)^{-|I|/2} = A_{|I|}(t) \). Since \( A_k(t) \leq e^k \leq e^{k \log \left( \frac{en}{k} \right)} \) for any \( t \leq (1 - e^{-2})/2 < 0.43 \), the first assertion for the normal case follows by taking \( t = 0.4 \). By taking \( t = \frac{1}{4} \), the second assertion follows since \( \mathbb{E} \sum_{i=1}^{k} \xi_{[i]}^2 \leq 4k \log \left( \frac{en}{k} \right) + 2k \log 2 \leq 6k \log \left( \frac{en}{k} \right) \). \( \square \)

This lemma is useful if \( A_k(t) \leq C_1 \left( \frac{en}{k} \right)^{C_2k} \) for some \( t, C_1, C_2 > 0 \); in particular, for \( Y_i = \xi_i^2 \), where the \( \xi_i \)’s satisfy Condition (A1). Then Lemma 4 applies with \( t = \beta \) and \( A_k(\beta) = e^{Bk} \):
\[
\mathbb{P} \left( \sum_{i=1}^{k} \xi_{[i]}^2 \geq \frac{(1+\beta)}{\beta} k \log \left( \frac{en}{k} \right) + M \right) \leq \exp \left\{ -\beta M \right\}, \quad k \in \mathbb{N}_n, \ M \geq 0.
\]

**7 Proofs of the theorems**

Here we gather the proofs of the theorems. By \( C_0, C_1, C_2 \) etc., denote constants which are different in different proofs. Recall that \( Y_{[1]} \geq Y_{[2]} \geq \ldots \geq Y_{[n]} \) denote the ordered \( Y_1, \ldots, Y_n \).

**Proof of Theorem 1** Recall the constants \( c_1, c_2, c_3 \) defined in the proof of Lemma 2. Let \( M_0 = 2c_3(6 + \frac{1+\beta}{2}) \). Introduce the subfamily of index sets \( \mathcal{S}_M = \mathcal{S}_M(\theta) = \{ I \in \mathcal{I} : r^2(I, \theta) \leq c_3 \sigma^2(\theta) + \frac{\beta}{4(1+\beta)} M \sigma^2 \}, \ m = m_M(\theta) = \max \{|I| : I \in \mathcal{S}_M\} \), and the event \( A_M = A(\theta) = \{ \sum_{i=1}^{m} \xi_{[i]}^2 \leq \frac{(1+\beta)}{\beta} m \log \left( \frac{en}{m} \right) + \frac{M}{4} \}. \)

We have
\[
\hat{\pi}(||\vartheta - \theta||^2 \geq M_0 \sigma^2(\theta) + M \sigma^2 |X|) \leq 1_{A_M} \hat{\pi}(I \in \mathcal{S}'_M |X) + \sum_{I \in \mathcal{S}_M} 1_{A_M} \hat{\pi}(I ||\vartheta - \theta||^2 \geq M_0 r^2(\theta) + M \sigma^2 |X|) \hat{\pi}(I |X) = T_1 + T_2 + T_3.
\]

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Now we bound the quantities $E_\theta T_1$, $E_\theta T_2$ and $E_\theta T_3$.

First, we bound $E_\theta T_1$ by using Lemma 4 (see also (29)):

$$E_\theta T_1 = P_\theta(A_M^*) = P \left( \sum_{i=1}^{m} \xi_i^2 > (1+B) \beta m \log \left( \frac{en}{m} \right) + \frac{M}{8} \right) \leq \exp \left\{ - \beta M/8 \right\} .$$ (30)

Let us bound $E_\theta T_2$. Since $\binom{n}{k} \leq \left( \frac{en}{k} \right)^k$ and $c_1 > 2$, the following relation holds:

$$\sum_{I \in S} \left( \frac{ne}{|I|} \right)^{c_1 |I|} = \sum_{k=0}^{\binom{n}{k}} \left( \frac{en}{k} \right)^{c_1 k} \leq \sum_{k=0}^{\binom{n}{k}} \left( \frac{en}{k} \right)^{-k(c_1 - 1)} \leq (1 - e^{1-c_1})^{-1} \triangleq C_0 .$$ (31)

If $I \in S^*_M$, then $r^2(I, \theta) > c_3 r^2(\theta) + \frac{\beta}{40(1+B)} M \sigma^2$. Using this, Lemma 2 and (31), we bound $E_\theta T_2$:

$$E_\theta T_2 = \sum_{I \in S^*_M} E_\theta \hat{\pi}(I | X) \leq \sum_{I \in S^*_M} \left( \frac{ne}{|I|} \right)^{-c_1 |I|} \exp \left\{ - c_2 r^2(\theta) - c_3 r^2(\theta) \right\} \exp \left\{ - e^{1-c_1} \right\} \leq C_0 \exp \left\{ - c_2 \frac{\beta}{40(1+B)} \right\} .$$ (32)

It remains to bound $E_\theta T_3$. For each $I \in S_M$, $\sigma^2 |I| \log (en/|I|) \leq r^2(I, \theta) \leq c_3 r^2(\theta) + \frac{\beta}{40(1+B)} M \sigma^2$. Thus, for any $I \in S_M$, the event $A_M$ implies that $\sum_{i \in I} \xi_i^2 \leq \sum_{i=1}^{\binom{m}{2}} \xi_i^2 \leq \frac{(1+B)}{\beta} \beta m \log \left( \frac{en}{m} \right) + \frac{M}{8} \leq \frac{(1+B)}{\beta} c_3 \sigma^2 r^2(\theta) + \frac{3M}{20}$. Denote for brevity $\Delta_M(\theta) = M_0 r^2(\theta) + M \sigma^2$ and recall that $\sum_{i \in I} \theta_i^2 \leq r^2(I, \theta) \leq c_3 r^2(\theta) + \frac{\beta}{40(1+B)} M \sigma^2$ for any $I \in S_M$. Then for any $I \in S_M$

$$A_M \subseteq \left\{ \frac{\Delta_M(\theta)}{2} - \sigma^2 \sum_{i \in I} \xi_i^2 \geq \sum_{i \in I} \theta_i^2 \geq \left[ \frac{M_0}{2} - \frac{1+B}{\beta} c_3 \right] r^2(\theta) + \frac{13M}{40} \sigma^2 \right\} .$$ (33)

According to (10), $\hat{\pi}_I(\theta | X) = \prod_{i=1}^{n} N(X_i(I), \nu_i^2(I))$, with $X_i(I) = X_{i1} \{ i \in I \}$ and $\nu_i^2(I) = \frac{K_n(I) \sigma^2}{1} \{ i \in I \}$. Let $P_Z$ be the measure of $Z = (Z_1, \ldots, Z_n)$, with $Z_i \sim N(0,1)$. By using (33), the fact that $\frac{r^2(\theta)}{\sigma^2} \geq c_3^{-1} (m \log (en/m) - \frac{\beta}{40(1+B)} M)$ and Lemma 4 (now applied to the
Gaussian case), we obtain that, for any $I \in \mathcal{S}_M$,

$$
\hat{\pi}_I (\|\theta - \hat{\theta}\|^2 \geq M_0 r^2(\theta) + M\sigma^2 |X|) 1_{A_M}
$$

$$
= P_{Z} \left( \sum_{i=1}^{n} (\nu_i(I) Z_i + X_i(I) - \theta_i)^2 \geq \Delta_M(\theta) \right) 1_{A_M}
$$

$$
\leq P_{Z} \left( \sum_{i=1}^{n} \nu_i^2(I) Z_i^2 \geq \frac{\Delta_M(\theta)}{2} - \frac{n}{\sqrt{2}} (X_i(I) - \theta_i)^2 \right) 1_{A_M}
$$

$$
\leq P_{Z} \left( \sum_{i \in I} \sigma^2 Z_i^2 \geq \frac{\Delta_M(\theta)}{2} - \sigma^2 \xi_i^2 - \sum_{i \in I^c} \theta_i^2 \right) 1_{A_M}
$$

$$
\leq P_{Z} \left( \sum_{i \in I} \frac{\sigma^2 Z_i^2}{\beta} \geq \left( \frac{M_0}{2c_3} - \frac{1 + B}{\beta} - 1 \right) \left[ m \log(\frac{c_4}{m}) - \frac{\beta}{40(1 + B) M} \right] + \frac{13M}{40} \right)
$$

$$
\leq P_{Z} \left( \sum_{i = 1}^{m} Z_i^2 \geq 5 m \log(\frac{c_4}{m}) + \frac{M}{5} \right) \leq \exp\{-2M/25\},
$$

where we also used in the last step that $\frac{M_0}{2c_3} - \frac{1 + B}{\beta} - 1 = 5$. Hence,

$$
E_{\theta} T_3 = E_{\theta} \sum_{I \in \mathcal{S}_M} 1_{A_M} \hat{\pi}_I (\|\theta - \hat{\theta}\|^2 \geq M_0 r^2(\theta) + M\sigma^2 |X|) \hat{\pi}(I|X)
$$

$$
\leq \exp\{-2M/25\} E_{\theta} \sum_{I \in \mathcal{L}} \hat{\pi}(I|X) \leq \exp\{-2M/25\}.
$$

This completes the proof of assertion (i) since, in view of (31), (32) and the last display, we established that $E_{\theta} \hat{\pi}(\|\theta - \hat{\theta}\|^2 \geq M_0 r^2(\theta) + M\sigma^2 |X|) \leq E_{\theta}(T_1 + T_2 + T_3) \leq (2 + C_0) e^{-m_0 M}$, with constants $M_0 = 2c_3 (6 + \frac{1 + B}{\beta})$, $H_0 = 2 + C_0$, $m_0 = \min\{\frac{\beta}{8}, \frac{c_2^2}{40(1 + B)}, \frac{2}{25}\}$ and $C_0$ defined in (31).

The proof of assertion (ii) proceeds along similar lines. Recall the constants $c_1 > 2$, $c_2$, $c_3$ from Lemma 2 and define $M_1 = 4c_3 (1 + B + \beta)/\beta$. Introduce the subfamily of sets

$$
\tilde{S}_M = \tilde{S}_M(\theta) = \{ I \in \mathcal{L} : r^2(I, \theta) \leq 2c_3 r^2(\theta) + \frac{\beta}{8(1 + B)} M\sigma^2 \},
$$

and the event $\tilde{A}_M = \tilde{A}_M(\theta) = \{ \sum_{i=1}^{\tilde{m}} \xi_i^2 \leq \frac{(1 + B)}{\beta} \tilde{m} \log(\frac{c_4}{\tilde{m}}) + \frac{M}{6} \}$, where $\tilde{m} = \tilde{m}_M(\theta) = \max\{|I| : I \in \tilde{S}_M\}$. Introduce the notation $\tilde{\Delta}_M(\theta) = M_1 r^2(\theta) + M\sigma^2$ for brevity. By the definition of $\theta$ and the Cauchy-Schwartz inequality, we have that $\|\theta - \hat{\theta}\|^2 \leq \sum_{I \in \mathcal{L}} |X(I)|$.
\[ \theta^2 \hat{\pi}(I|X), \text{ where } \|X(I) - \theta\|^2 = \sigma^2 \sum_{i \in I} \xi_i^2 + \sum_{i \in I^c} \theta_i^2. \] Using this, we derive
\[ P_\theta(\|\hat{\theta} - \theta\|^2 \geq \bar{\Delta}_M(\theta)) \leq P_\theta \left( \sum_{I \in \mathcal{I}} \|X(I) - \theta\|^2 \hat{\pi}(I|X) \geq \bar{\Delta}_M(\theta) \right) \leq P_\theta(\bar{A}_M) + P_\theta \left( \left\{ \sum_{I \in \mathcal{I}} \left[ \sigma^2 \sum_{i \in I} \xi_i^2 + \sum_{i \in I^c} \theta_i^2 \right] \hat{\pi}(I|X) \geq \bar{\Delta}_M(\theta)/2 \right\} \cap \bar{A}_M \right) + P_\theta \left( \sum_{I \in \mathcal{S}_M} \left[ \sigma^2 \sum_{i \in I} \xi_i^2 + \sum_{i \in I^c} \theta_i^2 \right] \hat{\pi}(I|X) \geq \bar{\Delta}_M(\theta)/2 \right) = \bar{T}_1 + \bar{T}_2 + \bar{T}_3. \]

Similar to (30), we bound the term \( \bar{T}_1 \) by Lemma 4 (see also (29)):
\[ \bar{T}_1 = P_\theta(\bar{A}_M) = P \left( \sum_{i=1}^{\bar{m}} \xi_i^2 > \frac{(1+B)}{\beta} \bar{m} \log \left( \frac{\bar{m}}{m} \right) + \frac{M}{\beta} \right) \leq \exp \left\{ -M \beta/6 \right\}. \]

Now we evaluate the term \( \bar{T}_2 \). Since \( \bar{m} = \max \{ |I| : I \in \mathcal{S}_M \} \), \( \sigma^2 \bar{m} \log \left( \frac{\bar{m}}{m} \right) \leq 2c_3 r^2(\theta) + \frac{\beta}{(1+B)} M \sigma^2 \). Then for any \( I \in \mathcal{S}_M \), the event \( \bar{A}_M \) implies that \( \sum_{i \in I} \xi_i^2 \leq \sum_{i=1}^{\bar{m}} \xi_i^2 \leq \frac{\beta}{(1+B)} \bar{m} \log \left( \frac{\bar{m}}{m} \right) + \frac{M}{\beta} \leq 2c_3(1+B)r^2(\theta)/\sigma^2 + \frac{M}{\beta} \). Also \( \sum_{i \in I^c} \theta_i^2 \leq r^2(\theta) \leq 2c_3 r^2(\theta) + \frac{\beta}{(1+B)} M \sigma^2 \) for any \( I \in \mathcal{S}_M \). Hence, for any \( I \in \mathcal{S}_M \), we obtain the implication
\[ \bar{A}_M \subseteq \left\{ \sigma^2 \sum_{i \in I} \xi_i^2 + \sum_{i \in I^c} \theta_i^2 \leq \frac{2c_3(1+B+\beta)}{\beta} r^2(\theta) + \left( \frac{1}{\beta} + \frac{M}{6(1+B)} \right) M \sigma^2 \right\}. \]

As \( M_1 = 4c_3(1+B+\beta)/\beta \), the last relation entails
\[ \bar{T}_2 \leq P_\theta \left( \left\{ \sum_{I \in \mathcal{S}_M} \left( \sigma^2 \sum_{i \in I} \xi_i^2 + \sum_{i \in I^c} \theta_i^2 \right) \hat{\pi}(I|X) \geq \bar{\Delta}_M(\theta) \right\} \cap \bar{A}_M \right) \leq P_\theta \left( \frac{2c_3(1+B+\beta)}{\beta} r^2(\theta) + \left( \frac{1}{\beta} + \frac{M}{6(1+B)} \right) M \sigma^2 \leq \frac{M_1}{\beta} r^2(\theta) + \frac{M}{\beta} \sigma^2 \right) = 0. \]

It remains to handle the term \( \bar{T}_3 \). Applying first the Markov inequality and then the Cauchy-Schwarz inequality, we obtain
\[ \bar{T}_3 \leq \frac{E_\theta \left( \sum_{I \in \mathcal{S}_M^c} \left( \sigma^2 \sum_{i \in I} \xi_i^2 + \sum_{i \in I^c} \theta_i^2 \right) \hat{\pi}(I|X) \right) \Delta_M(\theta)/2 \leq \frac{\sum_{I \in \mathcal{S}_M^c} \left( \sigma^2 \left[ E_\theta \left( \sum_{i \in I} \xi_i^2 \right) \right]^{1/2} \left[ E_\theta \left( \hat{\pi}(I|X) \right) \right]^{1/2} + r^2(\theta) E_\theta \hat{\pi}(I|X) \right) \Delta_M(\theta)/2}{\Delta_M(\theta)/2} = T_{31} + T_{32}. \]

For any \( I \in \mathcal{S}_M^c \), we have \( c_3 r^2(\theta) \leq \frac{r^2(\theta)}{2} - \frac{\beta}{(1+B)} M \sigma^2 \), yielding the bound
\[ \frac{c_2}{2} \left( r^2(\theta) - c_3 r^2(\theta) \right) \geq C_1 r^2(\theta) - C_2 M \sigma^2 \quad \text{for any } I \in \mathcal{S}_M^c, \quad (34) \]
where \( C_1 = c_2/4 \) and \( C_2 = c_2 \beta/[24(1+B)] \). By (34) and Lemma 2
\[ \left[ E_\theta \hat{\pi}(I|X) \right]^{1/2} \leq \left( \frac{M}{\beta} \right)^{-c_1/2} \exp \left\{ -C_1 \sigma^{-2} r^2(\theta) - C_2 M \right\} \quad \text{for any } I \in \mathcal{S}_M^c. \quad (35) \]
Since $c_1 > 2$, \((31)\) gives $\sum_{I \in \mathcal{I}} \left(\frac{p_e}{\theta} \right)^{-c_1|I|/2} \leq (1 - e^{-c_1/2})^{-1} \triangleq C_3$. According to \((10)\) with $ho = \min\{C_1, B\}$, 
\[
E\left(\frac{c}{\sum_{I \in \mathcal{I}} c_e^2} \right)^{1/2} \leq \frac{B}{\rho} \exp\{|I|\}. 
\] 
Besides, $\sigma^2 / \Delta_M(\theta) \leq M^{-1} \leq 1$ (as $M \geq 1$) and $\sigma^{-2} r^2(I, \theta) \geq |I| \log(e n / |I|) \geq |I|$. Piecing all these relations together with \((35)\), we derive 
\[
T_{31} \leq \frac{2B}{\rho} \sum_{I \in \mathcal{S}_M} \exp\{|I|\} \left(\frac{p_e}{\theta} \right)^{-c_1|I|/2} \exp \left\{-C_1 \sigma^{-2} r^2(I, \theta) - C_2 M \right\} \leq C_4 \exp\{-C_2 M\}, 
\] 
where $C_4 = 2BC_3/(\beta \rho) = 2BC_3/(\beta \min\{C_1, B\})$. Finally, by \((31), (35)\) and the facts that $\max_{x \geq 0}\{xe^{-cx}\} \leq (ce)^{-1}$ (for any $c > 0$) and $\sigma^2 / \Delta_M(\theta) \leq 1$, we bound the term $T_{32}$:
\[
T_{32} = \frac{2}{\Delta_M(\theta)} \sum_{I \in \mathcal{S}_M} r^2(I, \theta) E_\theta \hat{\pi}(I \mid X) 
\leq \frac{2}{\Delta_M(\theta)} \sum_{I \in \mathcal{S}_M} r^2(I, \theta) \left(\frac{p_e}{\theta} \right)^{-c_1|I|} \exp \left\{-C_1 \sigma^{-2} r^2(I, \theta) - 2C_2 M \right\} \leq C_5 \exp\{-2C_2 M\}, 
\] 
where $C_5 = C_0/(C_1 e)$. The assertion \((\text{iii})\) is proved since we showed that $P_\theta(\|\hat{\theta} - \theta\|^2 \geq M_1 r^2(\theta) + M \sigma^2) \leq H_1 e^{-m_1 M}$ with $M_1 = 4c_3(1 + B + \beta) / \beta$, $H_1 = 1 + C_4 + C_5$ and $m_1 = \min\{\beta, C_2\}$. \(\Box\)

**Proof of Theorem 2**. Denote $\mathcal{G}_1 = \mathcal{G}_1(\theta, M) = \{I : r^2(I, \theta) \geq c_3 r^2(\theta) + M \sigma^2\}$, where the constants $c_1 > 2, c_2, c_3$ are defined in Lemma 2. Applying Lemma 2 (and Remark 3) and using the fact \((31)\), we obtain 
\[
E_\theta \hat{\pi}(I \in \mathcal{G}_1 \mid X) = \sum_{I \in \mathcal{G}_1} E_\theta \hat{\pi}(I \mid X) \leq e^{-c_2 M} \sum_{I \in \mathcal{I}} \left(\frac{p_e}{\theta} \right)^{-c_1|I|} \leq C_0 e^{-c_2 M},
\] 
which completes the proof. \(\Box\)

**Proof of Theorem 3**. We first establish the coverage property. According to Theorem 1, we have that, for any $\theta \in \mathbb{R}^n$ and $M \geq 2$,
\[
P_\theta(\|\hat{\theta} - \theta\|^2 \geq M_1 r^2(\theta) + M \sigma^2) \leq H_1 e^{-m_1 M/2}, \tag{36}
\] 
where the constants $M_1, H_1$ and $m_1$ are defined in the proof of Theorem 1.

Take $M_2 = eM_1$. Since $\tau > \tilde{r}(e^{-1}) > 1$, from \((19)\) and \((22)\), it follows that $\tau^2(\theta) \leq r^2(\theta) = (b_r(\theta) + \tau) \sigma^2 |I^*_0| \log(\theta) + b_r(\theta) \sigma^2 \leq (b_r(\theta) + \tau) \sigma^2 |I^*_0| \log(\theta) + 1)$. Combining this with Lemma 3 (and Remark 3), the definition \((21)\) of $\tilde{r}$ and \((30)\), yields the coverage property:
\[
P_\theta(\theta \notin B(\hat{\theta}, (M_2(b_r(\theta) + \tau)r^2 + M \sigma^2)^{1/2})) = P_\theta(\|\hat{\theta} - \theta\|^2 > eM_1 (b_r(\theta) + \tau) r^2 + M \sigma^2) 
\leq P_\theta(\|\hat{\theta} - \theta\|^2 > eM_1 (b_r(\theta) + \tau) r^2 + M \sigma^2, \tilde{r}^2 \geq \frac{\tau^2(\theta) - M \sigma^2 / 2M_1}{e(b_r(\theta) + \tau)}) 
\leq P_\theta(\|\hat{\theta} - \theta\|^2 > M_1 r^2(\theta) + M \sigma^2) + P_\theta(\tilde{r} > e^{-1} |I^*_0| \log(\theta) + 1) 
\leq H_1 e^{-m_1 M/2} + H_2 \left(\frac{m}{|I^*_0|}\right)^{-\alpha |I^*_0|} e^{-m_2 (b_r(\theta) + \tau) |I^*_0|} 
\leq H_1 e^{-m_1 M/2} + H_2 \left(\frac{m}{|I^*_0|}\right)^{-\alpha |I^*_0|} e^{-m_2 (b_r(\theta) + \tau) |I^*_0|}. 
\] 
29
where $H_2 = C_x$, $m_2 = m_1'(1/2eM)$; $\alpha = \alpha(\tau, e^{-1})$, $C_x$ and $m_1'$ are defined in Lemma $3$

Let us show the size property. For $M \geq 1$, introduce the set $G(M) = G(M, \theta) = \{ I \in I : \sigma^2 |I| \log(en/|I|) \geq c_3 r^2(\theta) + \sigma^2 (M - 1) \}$, where $c_3$ is defined in Lemma $2$. Then for all $I \in G(M)$,

$$r^2(I, \theta) - c_3 r^2(\theta) \geq \sigma^2 |I| \log(en/|I|) - c_3 r^2(\theta) \geq (M - 1) \sigma^2.$$ 

In view of Remark $3$ from Lemma $2$ and the last relation, it follows that for all $I \in G(M)$

$$P_\theta(I = I) \leq (e^{n/|I|})^{-c_1 |I|} \exp \left\{ - c_2 \sigma^{-2} (r^2(I, \theta) - c_3 r^2(\theta)) \right\} \leq (e^{n/|I|})^{-c_1 |I|} e^{-c_2 (M - 1)}.$$

Using the last relation and $31$, we establish the size relation with $M_3 = c_3$ and $m_3 = c_2$:

$$P_\theta \left( \hat{\sigma}^2 \geq c_3 r^2(\theta) + M \sigma^2 \right) = \sum_{I \in G(M)} P_\theta(I = I) \leq e^{-c_2 (M - 1)} \sum_{I \in I} (e^{n/|I|})^{-c_1 |I|} \leq H_3 e^{-c_2 M}.$$ 

for any $\theta \in \mathbb{R}^n$, where $H_3 = C_0 e^{c_2}$ and $C_0$ is defined in $31$.

**Proof of Theorem 4** Observe that $r^2(\theta) \leq r^2(I^*(\theta), \theta) \leq \sigma^2 s(\theta) \log(en/s(\theta))$. Since the function $x \mapsto x \log(en/x)$ is increasing over $(0, n)$, $|I| \geq M s(\theta)$ implies that $r^2(I, \theta) \geq r^2(I^*(\theta), \theta) \geq \sigma^2 |I| \log(en/|I|) \geq \sigma^2 |I| \log(en/|I|) \geq \sigma^2 M s(\theta) \log(en/s(\theta))$.

Thus, if $|I| \geq M s(\theta)$, then

$$r^2(I, \theta) \geq \sigma^2 |I| \log(en/|I|) \geq \sigma^2 s(\theta) [\log(en/s(\theta)) - \log M'] \geq \frac{M'}{2} \sigma^2 s(\theta) \log(en/s(\theta)),$$

provided that $s(\theta) \leq \log(M')$. Since $r^2(\theta) \leq r^2(I^*(\theta), \theta) \leq \sigma^2 s(\theta) \log(en/s(\theta))$, the relation above implies that $r^2(I, \theta) \geq M r^2(\theta) + \sigma^2 s(\theta) \log(en/s(\theta))$. Hence by Theorem $2$, the assertion holds for $M' = M'$ whenever $s(\theta) \leq \log(M)$.

If $s(\theta) \geq \log(M)^{1/2}$, the result trivially holds by choosing $M' = (M')^{1/2}$. Hence the choice $M' = \max\{M', (M')^{1/2}\}$ ensures the result with $m_4 = m_4(M'/2 - M - 4)$ for any $\theta \in \mathbb{R}^n$.

**Proof of Theorem 5** Recall (25): $\alpha^2 r^2(\theta) \leq K n \left( \frac{\alpha^s}{\frac{en}{p_n}} \right)^{1-s/2}$ for each $\theta \in m_4[p_n]$ with some $K = K(s)$. On the other hand, if $|I| > M p_n^* = M n \left( \frac{\alpha^s}{\frac{en}{p_n}} \right)^{1-s/2}$, then $r^2(I, \theta) \geq |I| \log(en/|I|) \geq M n \left( \frac{\alpha^s}{\frac{en}{p_n}} \right)^{1-s/2} \geq M n \left( \frac{\alpha^s}{\frac{en}{p_n}} \right)^{1-s/2} - c_3 K e^{-1} p_n^* \log \left( \frac{\alpha^s}{\frac{en}{p_n}} \right)$. Then, for any $\theta \in m_4[p_n]$, $M > c_3 K/s(\alpha)$, we have that, for sufficiently large $n$,

$$\sigma^2 r^2(\theta) - c_3 r^2(\theta) \geq M p_n^* \log \left( \frac{\alpha^s}{\frac{en}{p_n}} \right) - c_3 K e^{-1} p_n^* \log \left( \frac{\alpha^s}{\frac{en}{p_n}} \right) \geq M p_n^* \log \left( \frac{\alpha^s}{\frac{en}{p_n}} \right) - c_3 K e^{-1} p_n^* \log \left( \frac{\alpha^s}{\frac{en}{p_n}} \right).$$

Finally, applying Theorem $2$ we obtain

$$\sup_{\theta \in m_4[p_n]} E_\theta \pi(I : |I| > M p_n^* |X|) \leq C_0 \exp \left\{ - c_2 s (M - c_3 K/s(\alpha)) p_n^* \log \left( \frac{\alpha^s}{\frac{en}{p_n}} \right) \right\},$$

which gives the claim with $m_5 = c_2 s$ and $M_5 = c_3 K/s(\alpha)$.
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