The dynamically asymmetric SQUID: Münchhausen effect

A.U. Thomann, V.B. Geshkenbein, and G. Blatter

Theoretische Physik, ETH Zurich, CH-8093 Zurich, Switzerland
L.D. Landau Institute for Theoretical Physics, 117940 Moscow, Russia

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We report on a complex zero-temperature decay channel of a classical object in a metastable state coupled to a quantum degree of freedom. This setting can be realized in a dc-SQUID where both Josephson-junctions have identical critical currents but strongly asymmetric dynamical parameters; more precisely, selecting both parameters $C$ and $1/R$ adequately large for one and small for the other junction makes the first junction behave essentially classical but lets quantum effects be present for the second one. The decay process is initiated by the tunneling of the quantum junction, which distorts the trapping potential of the classical junction; the metastable state of the latter then becomes unstable if the distortion is large enough. We present the dynamical phase diagram of this system providing the dependence of this decay channel on the external bias current $I$ and on the coupling strength between the two junctions, determined by the loop inductance $L$.

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I. INTRODUCTION

It is a basic feature of the classical world that a massive object residing in a metastable potential well cannot decay and, at zero temperature, is determined to reside in the well. There are, however, situations, where this doctrine is no longer adequate, namely if the system consists of two parts, a classical and a smaller quantum one. In that case, a decay process is possible under certain circumstances: The decay sequence is initiated by the tunneling of the quantum degree of freedom; through its coupling to the classical part the trapping potential is distorted, eventually turning the metastable state into an unstable one, provided that the distortion is large enough. We name this decay process the ‘Münchhausen effect’ after the famous baron telling the story of pulling himself out of a swamp by his own hair.

The above situation can be implemented experimentally in a dynamically asymmetric dc-SQUID; it is this specific realization we will study in detail in this paper, see Fig. 1. The dynamical degrees of freedom in the dc-SQUID are the gauge-invariant phase differences $\varphi_j$, $j = 1, 2$, across the two Josephson junctions. The potential energy (of a single Josephson junction) is given by $V_j = E_j(1-\cos \varphi_j)$, $j = 1, 2$, involving the Josephson energy $E_j = \Phi_0 I_c/2\pi e$ (with the flux unit $\Phi_0 = \hbar c/2e$ and the critical current $I_c$ of the junction). The kinetic energy reads $T_j = \hbar/2e^2 C_j \dot{\varphi}_j^2/2$, where the capacitances $C_j$ assume the role of effective masses. Hence, a SQUID featuring two Josephson junctions with equal critical current $I_c$ but adequately chosen and strongly asymmetric capacitances, one big and one small ($C_1 \gg C_2$), effectively provides us with a classical and a quantum degree of freedom. The fabrication of large, classical junctions is easily achieved today; however, this is not the case for small quantum junctions whose realization is more difficult. Nevertheless, small junctions exhibiting quantum tunneling and quantum coherence can be routinely fabricated today.

The decay process of the biased dynamically asymmetric SQUID proceeds in the following manner: The bias $I$, leading to a term $-I(\varphi_1 + \varphi_2)$ in the potential, turns a stable state of the washboard potential into a metastable one. As junction 1 features a large capacitance, we assume its dynamics to be strictly classical. If the bias $I$ is large enough, a (imaginary time) decay process involving only the quantum junction (at constant $\varphi_1$) is enabled. This phase slip leads to the entry of magnetic flux into the ring. Given the inductive coupling $1/L$ (inductance $L$), the current through the classical junction is enhanced and it may eventually become overcritical, thus decaying via a classical real time trajectory.

In the following, we specify the setup (Sec. II). In Sec. III we describe the decay process in more detail and present the resulting dynamical phase diagram. We finish with a few concluding remarks in Sec. IV.
II. SETUP

We start from the capacitively shunted junction model (CSJ), where the dc-SQUID, biased with a current \( I \), is described by the Lagrangian

\[
\mathcal{L} = \sum_{j=1}^{2} \left[ \left( \frac{\Phi_0}{2\pi c} \right)^2 \frac{C_j}{2} \dot{\varphi}_j^2 - E_J(1 - \cos \varphi_j) \right] + \frac{\Phi_0 I}{2\pi c} (\varphi_1 + \varphi_2) - \left( \frac{\Phi_0}{2\pi c} \right)^2 \frac{(\varphi_1 - \varphi_2)^2}{2L};
\]

\( j \), we have assumed that the inductance \( L \) of the SQUID is symmetrically distributed. The Lagrangian \( \mathcal{L} \) generates the equations of motion

\[
m_j \ddot{\varphi}_j + \eta_j \dot{\varphi}_j = -\partial_\varphi v(\varphi_1, \varphi_2),
\]

with the ‘masses’ \( m_j = \Phi_0 C_j / 2\pi c I_L \) and where we have added the dissipative terms \( \eta_j \dot{\varphi}_j \) with the damping parameters \( \eta_j = \Phi_0 / 2\pi c I_L R_j \propto 1 / R_j \), the normal resistances of the junctions; the potential (illustrated in Fig. 2) is given by

\[
v(\varphi_1, \varphi_2) = 2 - \cos \varphi_1 - \cos \varphi_2 - i(\varphi_1 + \varphi_2) + \frac{k}{2} (\varphi_1 - \varphi_2)^2,
\]

with the dimensionless current \( i = I / 2I_L \), the coupling constant \( k = \Phi_0 / 2\pi c I_L L \) and where energy is measured in units of \( E_J \).

Residing in a symmetric \( (\varphi_1 = \varphi_2 = \arcsin i) \) metastable state of the potential \( v(\varphi_1, \varphi_2) \) at finite bias current \( i \), the classical version of the system described through Eq. (2) cannot decay at zero temperature. Here, we are investigating the case where junction one, featuring a large capacitance \( C_1 \), is assumed to behave strictly classical, whereas the dynamics of junction 2 is characterized by large quantum fluctuations.\(^1\) This can be achieved through a suitable choice of parameters, i.e., small capacitance \( C_2 \), while keeping \( E_J \gtrsim E_{C2} = e^2 / 2C_2 \) such that we remain in a quasi-classical regime.

In the system under consideration, different scenarios can arise depending on the strength of the dissipation as quantified by the dimensionless damping parameter

\[
\alpha_j = (2R_j C_j \omega_{pj})^{-1}, \quad j = 1, 2,
\]

where, at \( i = 0 \), the plasma frequency \( \hbar \omega_{pj} = (8E_J E_{Cj})^{1/2} \). The simplest case is the overdamped situation \( \alpha_1, \alpha_2 > 1 \), where the dynamics of the classical junction is viscous and both relaxation and tunneling of the quantum junction are incoherent.\(^5\). We will analyze this situation in detail in the next section; the obtained results are also relevant for other choices of parameters, c.f. below.

III. DECAY PROCESS AND PHASE DIAGRAM

In the following, we determine for which currents \( i \) and coupling constants \( k \) a zero temperature decay of a symmetric metastable state \( (\varphi_1 = \varphi_2 = \arcsin i, \text{ up to an arbitrary multiple of } 2\pi) \) is allowed in the interferometer potential \( v(\varphi_1, \varphi_2) \). The result is displayed in a dynamical phase diagram in the \( i-k \)-plane, see Fig. 3, where the critical line \( i_c(k) \) separates regions where this decay is prohibited (localized state) from regions where it is allowed (delocalized).

Assuming junction 1 to behave strictly classical, (i.e. considering the limit of very large \( C_1 \)), a quantum decay of the SQUID in a metastable state can only occur at fixed \( \varphi_1 \), i.e. through an imaginary-time trajectory of \( \varphi_2 \) in the effective potential \( v_{\text{eff}}(\varphi_2) = v(\varphi_1 = \text{const.}, \varphi_2) \). We will adopt this approximation for all tunneling processes throughout the discussion.

In the case \( \alpha_1, \alpha_2 > 0 \) the preparation in the assumed symmetric initial state is straightforward: A SQUID is cooled close to zero temperature at zero external magnetic field before a bias current is ramped to the desired value \( i \). The ramp needs to be fast enough, such that the final current is reached before the decay of the quantum junction; any initial potential energy is dissipated as the phases relax to the bottom of the metastable well.

For undercritical currents \( i < 1 \) the described initial state, \( \varphi_1 = \varphi_2 = \arcsin i \), is stable against a decay involving the classical junction; however it is, for \( k < i / (\pi - \arcsin i) \), unstable with respect to a quantum decay of \( \varphi_2 \) since the minimum of \( v_{\text{eff}}(\varphi_2) \) near \( \varphi_2 \approx 2\pi \) is lowered below the initial one and one or more phase slips of \( \varphi_2 \) are possible. In order to determine whether the SQUID will remain in a stable or enter a finite voltage state at given \( i, k \) we proceed in two steps: First, we have to determine where the successive quantum tunneling of \( \varphi_2 \) comes to a halt, i.e. which side minimum is quantum-stable. This is equivalent to finding the global minimum of the effective potential \( v_{\text{eff}}(\varphi_2) \). Fixing the phase across the classical junction \( \varphi_1 = \arcsin i \), one immediately sees that the quantum-stable minimum is the one near \( \varphi_2 \approx 2\pi n \) if \( n \) is the largest integer such that

\[
\frac{i}{2n - 1} \pi - \arcsin i. \quad (5)
\]

The sequence of phase slips of the quantum junction leads to an accumulation of magnetic flux in the SQUID loop, inducing a screening current. Consequently, the current through junction 2 is reduced whilst that through junction 1 is increased. The magnitude of the induced current depends strongly on the coupling constant \( k \); for

\(^1\) Hence, a description via an equation of motion, Eq. (2), is not suitable for the quantum junction and dissipative effects have to be included through a path integral formalism.
given \( n \), it is only large enough to drive the classical junction overcritical if

\[
k > k_{c,n}(i) \approx \frac{1 - i}{(2n - 1/2)\pi + \arcsin(2i - 1)}.
\]  

(6)

If this is the case, the SQUID enters a finite voltage state where quantum tunneling of junction 2 (an approximate flux unit enters the loop) and classical relaxation of junction 1 (flux leaves the loop) interchange sequentially (see Fig. 2). If \( k < k_{c,n}(i) \), the SQUID resides in a localized state and a further increase in \( i \) is necessary to drive the system unstable. The two conditions Eqs. (5) and (6) generate a web of crossing lines in the \( i,k \)-plane, determining the critical line \( i_c(k) \) marking the dynamic transition from a localized to a delocalized state (Fig. 3). In the limit \( k \to 0 \), where the cosine in the potential Eq. (3) becomes a small correction to the parabola, the critical line \( i_c(k) \) approaches 1/2; this indicates that all current is redirected through junction 1 and delocalization takes place at \( I = I_c \), the critical current of a single junction.

The simple arguments above have to be refined in order to obtain the precise location of the critical line \( i_c(k) \). First, the condition of classical stability is but the standard determination of the critical current of a dc-SQUID’s asymmetric minimum. In our case, where classical stability along the \( \varphi_2 \)-direction is guaranteed, the relevant set of equations is given by

\[
\begin{align*}
\sin(\varphi_1^n) &= i - k_{c,n}(\varphi_1^n - \varphi_2^n), \\
\sin(\varphi_2^n) &= i + k_{c,n}(\varphi_1^n - \varphi_2^n), \\
\cos \varphi_1^n \cos \varphi_2^n &= -k_{c,n}(\cos \varphi_1^n + \cos \varphi_2^n),
\end{align*}
\]

(7)-(9)

and has to be solved (numerically) for \( k_{c,n}(i) \) and \( \varphi_{1,2}^n \), the coordinates of the true minima near \( \varphi_1 = \arcsin i, \varphi_2 = 2\pi n \). Eq. (6) is an approximate solution to Eqs. (7)-(9) in the limit of \( k \ll 1 \). Second, as tunneling of the quantum junction might be enabled only after the relaxation of the classical junction to a minimum, condition Eq. (5) has to be corrected to

\[
k_{c,n}(i) = \frac{i}{(2n - 1)\pi - \varphi_{1,n}^{-1}},
\]

(10)

taking into account the change of the effective potential \( v_{\text{eff}}(\varphi_2) \) upon a change in \( \varphi_1 \). The exact numerical solutions of Eqs. (7)-(9) and (10) are shown in the inset of Fig. 3, where the approximate solution is seen to be rather precise.

FIG. 2: Illustration of the decay sequence of the dynamically asymmetric SQUID \( i = 0.5, k = 0.04 \). The initial metastable well is unstable w.r.t. the macroscopic quantum tunneling of the small junction 2. A continuous sequence of phase slips takes the system to a state which is classically unstable. In the following, the (classical) relaxation of the large junction 1 and the quantum decay of junction 2 alternate and lead to a finite voltage state of the SQUID.

FIG. 3: Dynamical phase diagram of the dynamically asymmetric dc-SQUID. The critical line (solid) separates regions in the \( i,k \)-plane where the SQUID enters a continuous finite voltage state (delocalized) from regions where the phases remain localized. On the lower-bias side of the critical line, the dotted lines mark the entry of a additional flux into the SQUID loop. On the higher-bias side of the critical line, the dashed lines show how many flux units are at least needed to render the SQUID unstable. The inset shows a close up on the critical line and additionally displays the exact numerical solution (dashed).

A few remarks on the above results are in place. First, the setup as described above may not be suitable for experimental investigations as an overdamped quantum junction suffers from strongly suppressed tunneling rates. However, the analogous results can be obtained by using two underdamped junctions and ramping the current sufficiently slowly, such that the junctions have dissipated all energy before crossing any line (solid or dotted) in the phase diagram Fig. 3. The hallmark of the “Münchhausen-effect” in this case is the two different types of decay that finally initiate a running state: If the critical line is crossed on a part with positive slope, the quantum stable minimum is classically unstable immediately and a finite voltage state is initiated by the last quantum tunneling process. On the other hand, if the critical line is crossed on a part with negative slope, the quantum-stable minimum is still classically stable after the tunneling and needs to be turned overcritical by increasing the bias. Then, the finite voltage state is initiated by a purely classical decay. The difference between
the two types of decay should be visible upon analyzing the decay histograms of multiple measurements, being broad for a quantum decay but narrow for a classical process.

Second, one has to take into account that for realistic systems, where the capacitance asymmetry is introduced by a capacitive shunt\footnote{M. Steffen, M. Ansmann, R. McDermott, N. Katz, R.C. Bialczak, E. Lucero, M. Neeley, E.M. Weig, A.N. Cleland, and J.M. Martinis, Phys. Rev. Lett. \textbf{97}, 050502 (2006).}, the maximum asymmetry is limited. Corrections due to the finite mass of the classical junction might appear in the form of two-dimensional quantum tunneling, where junction 1 takes part in the tunneling process\footnote{C. Morais Smith, B. Ivlev, and G. Blatter, Phys. Rev. B \textbf{49}, 4033 (1994).}.

\section*{IV. CONCLUSIONS}

We have shown that a dc-SQUID can decay out of a symmetric metastable state, even if one of the junctions behaves \textit{fully} classical, provided the second junction shows quantum behavior. The “Münchhausen-decay” involves tunneling of the quantum junction, where magnetic flux accumulates in the superconducting ring and eventually redirects enough current through the classical junction as to drive it overcritical. We thank A. Ustinov and A. Wallraff for fruitful discussions.

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