Geometry of Brill–Noether Loci on Prym Varieties

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1. Introduction

Given a smooth curve $X$, it is well known that the Brill–Noether loci $W^r_dX$ contain much interesting information about the curve $X$ and its polarized Jacobian $(JX, \Theta_X)$. Given a smooth curve $C$ and an étale double cover $\pi: \tilde{C} \to C$, one can analogously define Brill–Noether loci $V^r$ for the Prym variety $(P, \Theta)$ (see Section 2). Several fundamental results on these loci have been known for some time: the expected dimension is $g(C) - 1 - \left(\binom{r+1}{2}\right)$, the loci are nonempty if the expected dimension is nonnegative [Ber, Thm. 1.4], and they are connected if the expected dimension is positive [D3, Exm. 6.2]. If $C$ is general in the moduli space of curves, then all the Brill–Noether loci are smooth and have the expected dimension [W2, Thm. 1.11]. Whereas the Brill–Noether locus $V^1 \subset P^+$ is the canonically defined theta-divisor and has received the attention of many authors, the study of higher Brill–Noether loci (and the information they contain about the étale cover $\pi: \tilde{C} \to C$) is a more recent development. Casalaina-Martin, Lahoz, and Viviani [CaLV] show that $V^2$ is set-theoretically the theta-dual (cf. Definition 2.1) of the Abel–Prym curve. Lahoz and Naranjo [LN] refine this statement and prove a Torelli theorem: the Brill–Noether locus $V^2$ determines the covering $\tilde{C} \to C$. That finding motivates a more detailed study of the geometry of $V^2$. Our first result is as follows.

1.1. Theorem. Let $C$ be a smooth curve of genus $g(C) \geq 6$, and let $\pi: \tilde{C} \to C$ be an étale double cover such that the Prym variety $(P, \Theta)$ is an irreducible principally polarized abelian variety.

(a) Suppose that $C$ is hyperelliptic. Then $V^2$ is irreducible of dimension $g(C) - 3$.

(b) Suppose that $C$ is not hyperelliptic. Then $V^2$ is a reduced Cohen–Macaulay scheme of dimension $g(C) - 4$. If the singular locus $V^2_{\text{sing}}$ has an irreducible component of dimension at least $g(C) - 5$, then $C$ is a plane quintic, trigonal, or bielliptic.

The condition on the irreducibility is always satisfied unless $C$ is hyperelliptic and $\tilde{C}$ is not. In that case, $(P, \Theta)$ is isomorphic to a product of Jacobians [M2].
In the hyperelliptic case (cf. Proposition 4.2), the statement is a straightforward extension of [CaLV]. In the non-hyperelliptic case, it is based on the following observation: if the singular locus of $V^2$ is large, then the singularities are exceptional in the sense of [B3]. This provides a link with certain Brill–Noether loci on $JC$.

An immediate consequence of the theorem is that $V^2$ is irreducible unless $C$ is a plane quintic, trigonal, or bielliptic (Corollary 3.6). The case of trigonal curves is very simple: $(P, \Theta)$ is isomorphic to a Jacobian $JX$ and $V^2$ splits into two copies of $W^0_{g(C)-4}X$. For a plane quintic, $V^2$ is reducible if and only if $(P, \Theta)$ is isomorphic to the intermediate Jacobian of a cubic threefold; in this case, $V^2$ splits into two copies of the Fano surface $F$. Note that the Fano surface $F$ and the Brill–Noether loci $W^0_{g}X$ are expected to be the only subvarieties of principally polarized abelian varieties having the minimal cohomology class $[\delta_{\Theta}^k]$. By [dCPr], the cohomology class of $V^2$ is $[2\frac{\delta_{\Theta}^{g(C)-4}}{g(C)-4}]$; therefore, a reducible $V^2$ provides an important test for this conjecture.

Our second result is the following theorem.

1.2. Theorem. Let $C$ be a smooth non-hyperelliptic curve of genus $g(C) \geq 6$, and let $\pi: \tilde{C} \rightarrow C$ be an étale double cover. Denote by $(P, \Theta)$ the polarized Prym variety. The Brill–Noether locus $V^2$ is reducible if and only if at least one of the following statements holds:

(a) $C$ is trigonal;
(b) $C$ is a plane quintic and $(P, \Theta)$ an intermediate Jacobian of a cubic threefold;
(c) $C$ is bielliptic and the covering $\pi: \tilde{C} \rightarrow C$ belongs to the family $R_{g(C_1),g(C_1)}$ with $g(C_1) \geq 2$ (cf. Remark 5.11). Then $V^2$ has two or three irreducible components, but none of them has minimal cohomology class.

If $C$ is bielliptic of genus $g(C) \geq 8$, then the Prym variety is not a Jacobian of a curve [S]. Moreover, these Prym varieties form $[\frac{g(C)-1}{2}]$ distinct subvarieties of $A_{g(C)-1}$ [D1]. For exactly one of these families, the general member has the property that the cohomology class of any subvariety is a multiple of the minimal class $[\frac{\delta_{\Theta}}{4}]$. The proof of Theorem 1.2 shows that the Brill–Noether locus $V^2$ is irreducible if and only if the Prym variety belongs to this family! This is the first evidence for Debarre’s conjecture that is not derived from low-dimensional cases or considerations on Jacobians and intermediate Jacobians (cf. [D2, H62, R]).

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2. Notation

Most of our arguments are valid for an arbitrary algebraically closed field of characteristic $\neq 2$. However, we work over $\mathbb{C}$ so that we can apply [ACGH] and [D3],
which are crucial for Theorem 1.1 and its consequences. For standard definitions in algebraic geometry we refer to [Ha] and for Brill–Noether theory to [ACGH].

Given a smooth curve $C$, we denote by $\text{Pic } C$ its Picard scheme and by

$$\text{Pic } C = \bigcup_{d \in \mathbb{Z}} \text{Pic}^d C$$

the decomposition into its irreducible components. We will identify the Jacobian $JC$ and the degree-0 component $\text{Pic}^0 C$ of the Picard scheme. In order to simplify the notation we denote by $L \in \text{Pic } C$ the point corresponding to a given line bundle $L$ on $C$. We will abuse terminology somewhat and say that a line bundle is effective if it has a global section.

For $\varphi: X \to Y$ a finite cover between smooth curves and $D$ a divisor on $X$, we denote the norm by $\text{Nm}_\varphi(D)$. In the same way, $\text{Nm}_\varphi: \text{Pic } X \to \text{Pic } Y$ denotes the norm map. If $F$ is a coherent sheaf on $X$ (in general, $F$ will be the locally free sheaf corresponding to some divisor), then we denote by $\varphi_* F$ the push-forward as a sheaf.

Let $C$ be a smooth curve of genus $g(C)$ and let $\pi: \tilde{C} \to C$ be an étale double cover. We have $(\text{Nm } \pi)^{-1}(K_C) = P^+ \cup P^-$, where $P^- \simeq P^+ \simeq P$ are defined by

$$P^- := \{ L \in (\text{Nm } \pi)^{-1}(K_C) | \dim|L| \equiv 0 \text{ mod } 2 \},$$

$$P^+ := \{ L \in (\text{Nm } \pi)^{-1}(K_C) | \dim|L| \equiv 1 \text{ mod } 2 \}.$$  

For $r \geq 0$ we set

$$W^{r}_{2g(C)-2}\tilde{C} := \{ L \in \text{Pic}^{2g(C)-2}\tilde{C} | \dim|L| \geq r \}.$$  

The Brill–Noether loci of the Prym variety $[W2]$ are defined as the scheme-theoretical intersections

$$V^r := \begin{cases} W^{r}_{2g(C)-2}\tilde{C} \cap P^- & \text{if } r \text{ is even,} \\ W^{r}_{2g(C)-2}\tilde{C} \cap P^+ & \text{if } r \text{ is odd.} \end{cases}$$

The notion of theta-dual was introduced by Pareschi and Popa in their work on Fourier–Mukai transforms (see [PPo2] for a survey).

2.1. Definition. Let $(A, \Theta)$ be a principally polarized abelian variety, and let $X \subset A$ be any closed subset. Then the theta-dual $T(X)$ of $X$ is the maximal subset $Z \subset A$ such that $A - Z \subset \Theta$.

Note that $T(X)$ has a natural scheme structure [PPo2].

3. The Singular Locus of $V^2$

Throughout this section we denote by $C$ a smooth non-hyperelliptic curve of genus $g(C)$ and by $\pi: \tilde{C} \to C$ an étale double cover. The following lemma will be used repeatedly.

3.1. Lemma. Let $L \in V^r$ be a line bundle such that $\dim|L| = r$. If the Zariski tangent space $T_L V^r$ satisfies
then there exist
(a) a line bundle $M$ on $C$ such that $\dim |M| \geq 1$ and
(b) an effective line bundle $F$ on $\tilde{C}$ such that
$$L \simeq \pi^* M \otimes F.$$
has exactly two irreducible components, \( \Lambda_0 \) and \( \Lambda_1 \), and that both are normal varieties of dimension \( g(C) - 1 \). Let

\[
i : \tilde{C}^{2g(C)-2} \to \text{Pic}^{2g(C)-2} \tilde{C}, \quad D \mapsto O_{\tilde{C}}(D)
\]

be the Abel–Jacobi map; then, up to renumbering,

\[
\varphi(\Lambda_0) = P^+ \quad \text{and} \quad \varphi(\Lambda_1) = \emptyset \subset P^+.
\]

Recall that for all \( L \in \text{Pic} \tilde{C} \) we have the set-theoretic equality \( i^{-1}(L) = |L| \). In particular, we see that

\[
\dim i^{-1}(V^r) \geq \dim V^r + r \tag{2}
\]

for every \( r \geq 0 \).

Suppose now that \( r \) is even (the odd case is analogous and is left to the reader). For a general point \( L \in P^- \) one has \( \dim |L| = 0 \). Thus, for \( r \geq 2 \),

\[
i^{-1}(V^r) \subsetneq \Lambda_1;
\]

hence \( i^{-1}(V^r) \) has dimension at most \( g(C) - 2 \). We conclude by using (2). \( \square \)

3.4. Remark. In the proof we used non-hyperelliptic \( C \) only to show that \( \Lambda_0 \) and \( \Lambda_1 \) are irreducible. Since inequality (2) is valid without this property, we obtain

\[
\dim V^r \leq g(C) - 1 - r \quad \forall r \geq 2.
\]

We will see in Section 4.A that this estimate is optimal.

We can now use Marten’s theorem to give an estimate of the dimension of the singular locus \( V^2_{\text{sing}} \).

3.5. Proposition. Suppose that \( g(C) \geq 6 \) and \( V^2_{\text{sing}} \) has an irreducible component \( S \) of dimension at least \( g(C) - 5 \). Then there exist

(a) a \( d \in \{3, 4\} \) such that

\[
\dim W_d^1 C = d - 3
\]

and

(b) an irreducible component \( W \subset W_d^1 C \) of maximal dimension such that, for every \( M \in W \),

\[
\dim |K_C \otimes M^{\otimes -2}| = g - d - 2.
\]

For every \( L \) in \( S \) we have

\[
L \simeq \pi^* M \otimes F
\]

for some \( M \in W \) and some effective line bundle \( F \) on \( \tilde{C} \). In particular, \( S \) is of dimension \( g(C) - 5 \).

Proof. Let \( L \in S \) be a generic point; then, by Lemma 3.3, \( \dim |L| = 2 \). Since \( V^2 \) is singular in \( L \), it follows that

\[
\dim T_L V^2 > g(C) - 4.
\]

Hence by Lemma 3.1 there exist a line bundle \( M \in W_d^1 C \) for some \( d \leq g(C) - 1 \) and an effective line bundle \( F \) on \( \tilde{C} \) such that
\[ L \simeq \pi^* M \otimes F. \]

The family of such pairs \((M, F)\) is a finite cover of the set of pairs \((M, B)\) for which \(M \in W_d^1 C\) for some \(d \leq g(C) - 1\) and \(B\) is an effective divisor of degree \(2g(C) - 2 - 2d \geq 0\) on \(C\) such that \(B \in |K_C \otimes M^{\otimes -2}|\).

By hypothesis, the parameter space \(T\) of the pairs \((M, B)\) has dimension at least \(g(C) - 5\).

Note that if \(\deg M = g(C) - 1\) then \( \mathcal{O}_C \). Thus \(M\) is a theta-characteristic and the space of pairs \((M, B)\) is finite—a contradiction to \(g(C) - 5 > 0\).

Because \(C\) is not hyperelliptic, \(3 \leq \deg M < g(C) - 1\).

Moreover, by Clifford’s theorem we have
\[ \dim |H^0(C, K_C \otimes M^{\otimes -2})| \leq g(C) - 1 - d - 1. \]

Thus the variety \(W\) parameterizing the line bundles \(M\) has dimension at least \(d - 3\). By construction we have \(W \subset W^1_d\); by Marten’s theorem [ACGH, IV, Thm. 5.1],
\[ \dim W \leq \dim W^1_d \leq d - 3. \]

Therefore, \(T\) and \(S\) each have dimension at least \(g(C) - 5\). Since (by hypothesis) \(S\) has dimension at least \(g(C) - 5\), it follows that (3) and (4) are equalities—at least for \(M \in W\) generic. By upper semicontinuity and Clifford’s theorem, we obtain equality for every \(M \in W\).

The last remaining point is to show that this situation can occur only for \(d \in \{3, 4\}\). We have already established the existence of a finite map
\[ W \to W^{g-d-2}_{2g(C)-2-2dC}, \quad M \mapsto K_C \otimes M^{\otimes -2}. \]

If \(2g(C) - 2 - 2d \leq g(C) - 1\), then, by Marten’s theorem, \(\dim W^{g(C)-d-2}_{2g(C)-2-2dC} \leq 1\). Because \(\dim W = d - 3\), we see that \(d \leq 4\). Now if \(2g(C) - 2 - 2d \geq g(C)\), we use the isomorphism
\[ W^{g(C)-d-2}_{2g(C)-2-2dC} \to W^{d-1}_{2d} C, \quad K_C \otimes M^{\otimes -2} \mapsto M^{\otimes 2} \]

together with Marten’s theorem to show that \(\dim W^{g(C)-d-2}_{2g(C)-2-2dC} \leq 1\); hence, again we obtain \(d \leq 4\).

**Proof of Theorem 1.1.** The hyperelliptic case is settled in Proposition 4.2, so we suppose that \(C\) is not hyperelliptic.

By [D3, Exm. 6.2.1], the Brill–Noether-locus \(V^2\) is a determinantal variety. Since for non-hyperelliptic \(C\) it has the expected dimension, \(V^2\) is Cohen–Macaulay. Since \(\dim V^2_{\text{sing}} \leq g(C) - 5\) by Proposition 3.5, it follows that all the irreducible components of \(V^2\) are generically reduced. Recall that a generically reduced Cohen–Macaulay scheme is itself reduced. If \(\dim V^2_{\text{sing}} \geq g(C) - 5\) then, by Proposition 3.5, \(\dim W^1_d C = d - 3\) for \(d = 3\) or 4. Thus the second statement follows from Mumford’s refinement of Marten’s theorem [ACGH, IV, Thm. 5.2].

**Remark.** Lahoz and Naranjo [LN] use completely different methods to show that \(V^2\) is reduced and Cohen–Macaulay.
3.6. Corollary. Let $C$ be a smooth non-hyperelliptic curve of genus $g(C) \geq 6$, and let $\pi: \tilde{C} \rightarrow C$ be an étale double cover. If $V^2$ is reducible, then $C$ is a plane quintic, trigonal, or bielliptic.

Remark. Teixidor i Bigas [T] uses the Martens–Mumford theorem to determine when the singular locus of a Jacobian of a curve is reducible.

Proof of Corollary 3.6. By a theorem of Debarre [D3, Exm. 6.2.1], the locus $V^2$ is $(g(C) - 5)$-connected. In other words, if $V^2$ is not irreducible then there exist two irreducible components $Z_1, Z_2 \subset V^2$ such that $Z_1 \cap Z_2$ has dimension at least $g(C) - 5$ in one point [D3, p. 287]. So if $V^2$ is reducible, its singular locus has dimension at least $g(C) - 5$. Now conclude using Theorem 1.1.

4. Examples

4.A. Hyperelliptic Curves

Let $C$ be a smooth hyperelliptic curve of genus $g(C)$. Let $\pi: \tilde{C} \rightarrow C$ be an étale double cover such that the Prym variety $(P, \delta \Theta E)$ is an irreducible principally polarized abelian variety (i.e., $\tilde{C}$ is also a hyperelliptic curve). Let $\sigma: \tilde{C} \rightarrow \tilde{C}$ be the involution induced by $\pi$.

Recall from [BiLa, Chap. 12, Sec. 5] that in this case, for a fixed $p_0 \in C$, the Abel–Prym map

$$\alpha: \tilde{C} \rightarrow P, \quad p \mapsto \sigma(p) - p + \sigma(p_0) - p_0$$

is two-to-one onto its image $C'$ (which is a smooth curve) and the Prym variety $(P, \Theta)$ is isomorphic to $(J(C'), \Theta_{C'})$.

In [CaLV, Lemma 2.1] the authors show that, for $C$ not hyperelliptic, $V^2$ is a translate of the theta-dual of the Abel–Prym embedded curve $\tilde{C} \subset P$. In fact, their argument works also for $C$ hyperelliptic if one replaces $\tilde{C} \subset P$ by $\alpha(\tilde{C}) = C' \subset P$. Thus we have the following statement.

4.1. Lemma. The Brill–Noether locus $V^2$ is a translate of the theta-dual $T(C')$.

Since the Prym variety $(P, \Theta)$ is isomorphic to $(J(C'), \Theta_{C'})$, it follows that the theta-dual of $C'$ is a translate of $W^0_{g(C)-3}C'$. In particular, $V^2$ is irreducible of dimension $g(C) - 3$.

4.2. Proposition. Let $C$ be a smooth hyperelliptic curve of genus $g(C) \geq 6$, and let $\pi: \tilde{C} \rightarrow C$ be an étale double cover such that the Prym variety $(P, \Theta)$ is an irreducible principally polarized abelian variety. Then $V^2$ is irreducible of dimension $g(C) - 3$ and, set-theoretically, it is a translate $W^0_{g(C)-3}C'$.

For any point $L \in V^2$ we have

$$L \simeq \pi^*H \otimes F,$$

where $H$ is the unique $g^1_2$ on $C$ and $F$ is an effective line bundle on $\tilde{C}$. 

Proof. By Remark 3.4 we have a proper inclusion $V^4 \subsetneq V^2$, so a general $L \in V^2$ satisfies $\dim |L| = 2$. By Lemma 3.1 there exists a line bundle $M \in W_1^dC$ for some $d \leq g(C) - 1$ and an effective line bundle $F$ on $\tilde{C}$ such that

$$L \cong \pi^* M \otimes F.$$ 

We can now argue as in the proof of Proposition 3.5 to obtain the statement. We need only observe that the inequality

$$\dim |H^0(C, K_C \otimes M^{\otimes 2})| \leq g(C) - 1 - d - 1$$

is also valid on a hyperelliptic curve unless $M$ is a multiple of the $g_2^1$.

4.B. Plane Quintics

Let $C \subset \mathbb{P}^2$ be a smooth plane quintic and let $\pi : \tilde{C} \to C$ be an étale double cover. We denote by $H$ the restriction of the hyperplane divisor to $C$ and by $\eta \in \text{Pic}^0 C$ the 2-torsion line bundle inducing $\pi$. Let $\sigma : \tilde{C} \to \tilde{C}$ be the involution induced by $\pi$.

4.3. Example. Suppose that $h^0(C, O_C(H) \otimes \eta)$ is odd—that is, suppose the Prym variety $P^-$ is isomorphic to the intermediate Jacobian $J(X)$ of a cubic threefold $X$ [CIG]. Let us fix such an isomorphism of principally polarized abelian varieties $J(X) \cong P^-$. The Fano variety $F$ parameterizing lines on the threefold $X$ is a smooth surface that has a natural embedding in the intermediate Jacobian $J(X)$. By [CIG], the surface $F \subset P$ has minimal cohomology class $[\Theta/\pi^*]$. Moreover, it follows from [Hö1] and [PPo1] that the theta-dual satisfies $T(F) = -F$. It is well known that $\tilde{C} \subset F$ (up to translation), so

$$-F = T(F) \subset V^2 = T(\tilde{C}).$$

Since the condition $\dim |L| \geq 2$ is invariant under isomorphism, the Brill–Noether locus $V^2$ is stable under the map $x \mapsto -x$. Thus $-F \subset V^2$ implies that $F \subset V^2$. Since the cohomology class of $V^2$ is $[2\Theta/\pi^*]$, we see that (up to translation) $V^2$ is a union of $F$ and $-F$. In particular, $V^2$ is reducible and its singular locus is the intersection of the two irreducible components. Since $V^2$ is Cohen–Macaulay, the singular locus has pure dimension 1.

We will now prove the converse of this example.

4.4. Proposition. The Brill–Noether locus $V^2$ is reducible if and only if $h^0(C, O_C(H) \otimes \eta)$ is odd—in other words, if the Prym variety is isomorphic to the intermediate Jacobian of a cubic threefold. In this case, the singular locus $V^2_{\text{sing}}$ is a translate of $\tilde{C}$.

Proof. Suppose that $V^2_{\text{sing}}$ has a component $S$ of dimension 1. Since $C$ is not trigonal, we know from Proposition 3.5 that $S$ corresponds to a 1-dimensional component $W \subset W_1^dC$ such that, for every $[M] \in W$,
By adjunction we have $K_C \simeq O_C(2H)$, and from [B2, Sec. 2, (iii)] it follows that $M \simeq O_C(H - p)$, where $p \in C$ is a point. Hence $K_C \otimes M^{\otimes -2} \simeq O_C(2p)$ and a general point $L \in S$ is of the form

$$L \simeq K_C \otimes M \otimes -2 \otimes O_C(q_1 + q_2),$$

where $q_1, q_2$ are points in $\tilde{C}$. Since $N_{\pi}(L) \simeq O_C(2H)$ and $C$ is not hyperelliptic, we obtain that $q_i \in \pi^{-1}(p)$. Then we can write

$$L \simeq \pi^*O_C(H) \otimes \tilde{O}_{\tilde{C}}(q - \sigma(q))$$

for some $q \in \tilde{C}$.

Because $L$ varies in a 1-dimensional family, we can exclude the first case. By Mumford’s description of a Prym variety whose theta-divisor has a singular locus of dimension $g(C) - 5$, we know that $h^0(C, O_C(H) \otimes \eta)$ is even if and only if $h^0(\tilde{C}, \pi^*O_C(H) \otimes O_C(q - \sigma(q)))$ is even [M2, p. 347]. Since $V^2 \subset P^+$, this shows the statement.

The description of the general points $L \in S$ shows that $V^2_{\text{sing}}$ has a unique 1-dimensional component and that $V^2_{\text{sing}}$ is the translate by $\pi^*O_C(H)$ of the Abel–Prym embedded $\tilde{C} \subset P$. □

4.C. Trigonal Curves

Let $C$ be a trigonal curve of genus $g(C) \geq 6$. Let $\pi : \tilde{C} \to C$ be an étale double cover and $(P, \Theta)$ the corresponding Prym variety. By a theorem of Recillas [Re], the Prym variety is isomorphic as a principally polarized abelian variety to the polarized Jacobian $(JX, \Theta_X)$ of a tetragonal curve $X$ of genus $g(C) - 1$. By Recillas’s construction [BiLa, Chap. 12.7] we also know how to recover the double cover $\pi : \tilde{C} \to C$ from the curve $X$. Namely, let $s : X(2) \times X(2) \to X(4)$ be the sum map; then

$$\tilde{C} \simeq p_1(s^{-1}(\mathbb{P}^1)),$$

where $\mathbb{P}^1 \subset X(4)$ is the linear system giving the tetragonal structure and $p_1$ is the projection onto the first factor. In particular, we see that

$$\tilde{C} \subset X(2) \simeq W^0_2 X.$$

Therefore, up to choosing an isomorphism $(P, \Theta) \simeq (JX, \Theta_X)$ (and appropriate translates),

$$T(W^0_2 X) \subset T(\tilde{C}) \simeq V^2.$$

By [PPo1, Exm. 4.5], the theta-dual of $W^0_2 X$ is $-W^0_{g(C)-4} X$. As in the case of the intermediate Jacobian described in Example 4.3, we see that (up to translation)

$$V^2 = -W^0_{g(C)-4} X \cup W^0_{g(C)-4} X;$$

moreover, the singular locus of $V^2$ is the union of $\pm(W^0_{g(C)-4} X)_{\text{sing}}$, which has dimension at most $g(C) - 6$, and the intersection of the two irreducible components, which has dimension $g(C) - 5$. 

5. Prym Varieties of Bielliptic Curves, I

5.A. Special Subvarieties

We recall some well-known facts about special subvarieties that we will use in the next section.

Let \( \varphi : X \rightarrow Y \) be a double cover (which may be étale or ramified) of smooth curves. We suppose that \( g(Y) \) is at least 1 and denote by \( \text{Nm} : \text{Pic} X \rightarrow \text{Pic} Y \) the norm morphism. Let \( M \) be a globally generated line bundle of degree \( d \geq 2 \) on \( Y \). Denote by \( \mathbb{P}^r \subset Y(d) \), where \( r := \dim |M| \), the set of effective divisors in the linear system \( |M| \). If \( \text{Nm} : X(d) \rightarrow Y(d) \) is the norm map, then \( \Lambda := \text{Nm} \varphi^{-1}(\mathbb{P}^r) \) is a reduced Cohen–Macaulay scheme of pure dimension \( r \) and the map \( \Lambda \rightarrow |M| \) is étale of degree \( 2^d \) over the locus of smooth divisors in \( |M| \) that do not meet the branch locus of \( \varphi \).

If \( \varphi \) is étale then \( \Lambda \) has exactly two connected components, \( \Lambda_0 \) and \( \Lambda_1 \) \([W1]\). If \( \varphi \) is ramified, the scheme \( \Lambda \) is connected \([N, \text{Prop. 14.1}]\). Let \( i_Y : Y(d) \rightarrow \text{Pic}^d Y, \ D \mapsto \mathcal{O}_Y(D) \) and \( i_X : X(d) \rightarrow \text{Pic}^d X, \ D \mapsto \mathcal{O}_X(D) \) be the Abel–Jacobi maps; then we have the commutative diagram

\[
\begin{array}{ccc}
\Lambda & \rightarrow & X(d) \\
\downarrow & & \downarrow i_X \\
\mathbb{P}^r & \rightarrow & Y(d) \\
\downarrow \text{Nm} \varphi & & \downarrow \text{Nm} \varphi \\
& & \text{Pic}^d Y.
\end{array}
\]

The fibre of \( i_X(X(d)) \rightarrow i_Y(Y(d)) \) over the point \( M \)—and thus the intersection of \( i_X(X(d)) \) with \( \text{Nm} \varphi^{-1}(M) \)—is equal (at least set-theoretically) to \( i_X(\Lambda) \).

Fix now a connected component \( S \subset \Lambda \). Then we call \( V := i_X(S) \) a special subvariety associated to \( M \). (In general it is not true that \( S \) is irreducible; in particular, the special subvariety may not be a variety. Note also that in general it should be obvious which covering we consider, and otherwise we say that \( V \) is a \( \varphi \)-special subvariety associated to \( M \).) It is clear that

\[
\dim V = r - \dim|\mathcal{O}_X(D)|, \tag{5}
\]

where \( D \in S \) is a general point.

The following technical definition will be crucial in the next section.

5.1. Definition. Let \( \varphi : X \rightarrow Y \) be a double cover of smooth curves. An effective divisor \( D \subset X \) is not simple if there exists a point \( y \in Y \) such that \( \varphi^*y \subset D \), and it is simple if this is not the case.

Note that if an effective divisor \( D \subset X \) is not simple then \( \text{Nm} \varphi(D) \) is not reduced. Hence, if \( Y \) is an elliptic curve and \( M \) a line bundle of degree \( d \geq 2 \) on \( Y \), then a
general divisor \( D \in X(d) \) such that \( \text{Nm} \varphi(D) \in |M| \) is simple: the linear system \(|M|\) is base point free, so a general element is reduced.

5.2. Lemma. Let \( \varphi : X \to Y \) be a ramified double cover of smooth curves such that \( Y \) is an elliptic curve. Denote by \( \delta_\varphi \) the line bundle of degree \( g(X) - 1 \) defining the cyclic cover \( \varphi \). Let \( M \not\cong \delta_\varphi \) be a line bundle of degree \( 2 \leq d \leq g(X) - 1 \) on \( Y \). Then the following statements hold:

(a) \( \Lambda \) is smooth and irreducible;
(b) a general divisor \( D \in \Lambda \) is simple and satisfies \( \dim |O_X(D)| = 0 \).

In particular, there exists a unique special subvariety associated to \( M \) and it is irreducible of dimension \( d - 1 \).

Proof. We start by showing part (b). By the foregoing, \( D \) is simple and so, according to [M1, p. 338], we have an exact sequence

\[ 0 \to O_Y \to \varphi_* O_X(D) \to O_Y(\text{Nm} \varphi(D)) \otimes \delta_\varphi^* \to 0. \]

Since \( \deg D \leq \deg \delta_\varphi \) and \( O_Y(\text{Nm} \varphi(D)) \not\cong M \not\cong \delta_\varphi \), we have

\[ h^0(Y, O_Y(\text{Nm} \varphi(D)) \otimes \delta_\varphi^*) = 0. \]

Therefore, \( 1 = h^0(Y, O_Y) = h^0(Y, \varphi_* O_X(D)) \).

For the proof of part (a) we note first that, since \( \Lambda \) is connected, it is sufficient to show the smoothness. Let \( D \in \Lambda \) be any divisor. Then we have a unique decomposition

\[ D = \varphi^* A + R + B, \]

where \( A \) is an effective divisor on \( Y \); the divisor \( R \) is effective, with support contained in the ramification locus of \( \varphi \); and \( B \) is effective, simple, and has support disjoint from the ramification locus of \( \varphi \). Since \( Y \) is an elliptic curve, we have

\[ h^0(Y, M \otimes O_Y(-A - \text{Nm} \varphi(R))) = h^0(Y, M) - \deg(A + \text{Nm} \varphi(R)) \]

unless \( \deg M = \deg(A + \varphi_* R) \) and \( M \otimes O_Y(-A - \text{Nm} \varphi(R)) \) is not trivial. Because \( \deg M = \deg D \), this last case could occur only when \( A = 0 \) and \( B = 0 \); hence \( D = R \). Yet by construction we have \( M \cong O_Y(\text{Nm} \varphi(D)) = O_Y(\text{Nm} \varphi(R)) \), so \( M \otimes O_Y(-A - \text{Nm} \varphi(R)) \) is trivial. By [N, Prop. 14.3] this shows the smoothness of \( \Lambda \). The statement on the dimension follows by part (b) and equation (5). \( \square \)

5.B. The Irreducible Components of \( V^2 \)

In this section \( C \) will be a smooth curve of genus \( g(C) \geq 6 \) that is bielliptic; in other words, we have a double cover \( p : C \to E \) onto an elliptic curve \( E \). As usual, \( \pi : \tilde{C} \to C \) will be an étale double cover. In this section we suppose that the covering \( p \circ \pi : \tilde{C} \to E \) is Galois. Then one sees easily that the Galois group is \( \mathbb{Z}_2 \times \mathbb{Z}_2 \).
Using the Galois action on $\tilde{C}$ yields the commutative diagram

$$
\begin{array}{ccc}
\pi & \downarrow \pi_1 & \pi_2 \\
C & \rightarrow & \tilde{C} \\
P & \downarrow P_1 & \downarrow P_2 \\
C_1 & \rightarrow & C_2.
\end{array}
$$

(The presentation here follows [D1, Chap. 5], to which we refer for details.) It is straightforward to see that $g(C_1) + g(C_2) = g(C) + 1$, and we will assume without loss of generality that $1 \leq g(C_1) \leq g(C_2) \leq g(C)$.

Denote by $\delta$ the branch locus of $p$ and by $\delta$ the line bundle inducing the cyclic cover $p$. Then $2\delta \simeq \Delta$ and, by the Hurwitz formula, $\deg K_C = \deg \delta_\Delta E$; hence $\deg \delta = g(C) - 1$.

The cyclic covers $p_1$ and $p_2$ are analogously given by line bundles $\delta_1$ and $\delta_2$ such that $\deg \delta_1 = g(C_1) - 1$ and $\deg \delta_2 = g(C_2) - 1$.

For any $a \in \mathbb{Z}$ we define closed subsets $Z_a \subset \text{Pic } C_1 \times \text{Pic } C_2$ by

$$
\left\{ (L_1, L_2) \mid L_1 \in W_{g(C_1)-1+a}^0 C_1, \right. \\
\left. L_2 \in W_{g(C_2)-1-a}^0 C_2, \text{Nm } p_1(L_1) \otimes \text{Nm } p_2(L_2) \simeq \delta \right\}.
$$

We note that the sets $Z_a$ are empty unless $1 - g(C_1) \leq a \leq g(C_2) - 1$. Pulling back to $\tilde{C}$ we obtain natural maps

$$(\pi_1^*, \pi_2^*): Z_a \rightarrow \text{Pic } \tilde{C}, \quad (L_1, L_2) \mapsto \pi_1^* L_1 \otimes \pi_2^* L_2,$$

and by [D1, p. 230] the image $\langle \pi_1^*, \pi_2^* \rangle(Z_a)$ is in $P^-$ if and only if $a$ is odd. Moreover, we can argue as in [D1, Prop. 5.2.1] to show that

$$V^2 \subset \langle \pi_1^*, \pi_2^* \rangle \left( \bigcup_{a \text{ odd}} Z_a \right).$$

5.3. Lemma. For $a$ odd, the sets $Z_a$ are empty or

$$\dim Z_a = g(C) - 1 - a.$$  \hfill (7)$$

Furthermore, $Z_a$ is irreducible unless $g(C_1) = 1$ and $a \geq g(C_2) - 2$.

Proof. We divide the proof into two cases as follows.

Case 1: $g(C_1) > 1$. We prove the statement for positive $a$ (the argument is analogous for negative $a$). The projection onto the second factor gives a surjective
map $Z_a \to W^0_{g(C_2) - 1 - a} C_2$, and the fibres of this map are parameterized by effective line bundles $L_1$ with fixed norm. Because $a \geq 1$, the line bundles $L_1$ are of degree at least $g(C_1)$ and so are automatically effective. Thus the fibres identify to \textit{irreducible}. Since all the irreducible components of $Z_a$ map into the component $P$, the surjective map induced by the projection onto the first factor $Z_a \to \text{Pic} C_1$. Since the double covering $p_1$ is ramified, it follows that the $(Nm p_1)$-fibres are irreducible of dimension $g(C_1) - 1$; hence $Z_a$ is irreducible of the expected dimension.

\textit{Case 2:} $g(C_1) = 1$. The sets $Z_a$ are empty for $a$ negative, so suppose that $a$ is positive. Arguing as in the first case, we obtain the statement on the dimension. In order to see that $Z_a$ is irreducible for $a \leq g(C_2) - 3$, we consider the surjective map induced by the projection onto the first factor $Z_a \to \text{Pic} g(C_1) - 1 + a C_1$. The fibre over a line bundle $L_1$ is the union of the $p_2$-special subvarieties associated to $\delta \otimes Nm p_1 L_1^*$. Since $2 \leq \deg \delta \otimes Nm p_1 (L_1^*) \leq g(C_2) - 2$, it follows from Lemma 5.2 that the unique special subvariety is irreducible and so the fibres are irreducible.

Since all the irreducible components of $V^2$ have dimension $g(C) - 4$, by (6) and (7) we have

$$V^2 \subseteq (\pi_1^*, \pi_2^*) \left( \bigcup_{a \text{ odd, } |a| \leq 3} Z_a \right). \quad (8)$$

If $(L_1, L_2) \in Z_{\pm 3}$ then, by the Riemann–Roch theorem, it follows that $\dim |L_1| \geq 2$ and $\dim |L_2| \geq 2$; therefore,

$$(\pi_1^*, \pi_2^*)(Z_{\pm 3}) \subset V^2.$$ 

For the sets $Z_{\pm 1}$ this cannot be true, since equation (7) shows that they have dimension $g(C) - 2$. We introduce the following smaller loci:

$$W_1 := \{(L_1, L_2) \in Z_1 \mid L_1 \in W^1_{g(C_1)} C_1\};$$
$$W_{-1} := \{(L_1, L_2) \in Z_{-1} \mid L_2 \in W^1_{g(C_2)} C_2\}.$$ 

Note that if $g(C_1) = 1$ then $W_1 = \emptyset$: there is no $g_1^1$ on a nonrational curve. Because $\dim W^1_{g(C_1)} C_1 = g(C_1) - 2$ (resp., $\dim W^1_{g(C_2)} C_1 = g(C_2) - 2$), one may easily deduce (from the proof of Lemma 5.2) that the sets $W_{\pm 1}$ are either empty or irreducible of dimension $g(C) - 4$.

By the same lemma we see that, for fixed $L_1$ (resp. $L_2$) and general $L_2$ (resp. $L_1$) such that $(L_1, L_2) \in W_1$ (resp. $(L_1, L_2) \in W_{-1}$), the linear system $|L_1|$ (resp. $|L_2|$) contains a unique effective divisor and this divisor is simple.

Observe that if $(L_1, L_2) \in W_{\pm 1}$ then $\dim |(\pi_1^*, \pi_2^*)(L_1, L_2)| \geq 1$. Since these sets map into the component $P^-$, we obtain

$$(\pi_1^*, \pi_2^*)(W_{\pm 1}) \subset V^2.$$ 

5.4. Proposition. We have

$$V^2 = (\pi_1^*, \pi_2^*)(Z_{-3} \cup W_{-1} \cup W_1 \cup Z_3).$$

The proof requires some technical preparation.
5.5. Definition. Let $\varphi : X \to Y$ be a double cover of smooth curves, and let $L$ be a line bundle on $X$ such that $\dim |L| \geq 1$. Then the line bundle $L$ is simple if every divisor in $D \in |L|$ is simple in the sense of Definition 5.1.

5.6. Lemma [D1, Cor. 5.2.8]. In our situation, let $L_1 \in \text{Pic} \ C_1$ and $L_2 \in \text{Pic} \ C_2$ be effective line bundles such that $L \simeq \pi_1^*L_1 \otimes \pi_2^*L_2$. If $L_1$ is $p_1$-simple, then

$$h^0(\tilde{C}, L) \leq 2h^0(C_2, L_2) + g(C_2) - 1 - \deg L_2.$$ 

Analogously, if $L_2$ is $p_2$-simple then

$$h^0(\tilde{C}, L) \leq 2h^0(C_1, L_1) + g(C_1) - 1 - \deg L_1.$$

Proof of Proposition 5.4. Let $L \in V^2$ be an arbitrary line bundle. By the inclusion (8) we need only show that if $L \in (\pi_1^*, \pi_2^*)(Z_{\pm 1})$ and $L \notin (\pi_1^*, \pi_2^*)(Z_{\pm 3})$ then $L \in (\pi_1^*, \pi_2^*)(W_{\pm 3})$. We will suppose that $L \in (\pi_1^*, \pi_2^*)(Z_3)$; the other case is analogous and is left to the reader. Because $L \in (\pi_1^*, \pi_2^*)(Z_1)$, we can write

$$L \simeq \pi_1^*L_1 \otimes \pi_2^*L_2$$

with $L_1$ effective of degree $g(C_1)$ and $L_2$ effective of degree $g(C_1) - 2$. If $L_2$ is not simple then $L$ is in $(\pi_1^*, \pi_2^*)(Z_3)$, which we have already excluded. Hence $L_2$ is simple and so, by Lemma 5.6,

$$3 \leq h^0(\tilde{C}, L) \leq 2h^0(C_1, L_1) + g(C_1) - 1 - g(C_1).$$

Therefore, $\dim |L_1| \geq 1$ and $L \in (\pi_1^*, \pi_2^*)(W_1)$. \hfill $\Box$

5.7. Corollary. If $g(C_1) = 1$ then

$$V^2 = (\pi_1^*, \pi_2^*)(Z_3).$$

In particular, $V^2$ is irreducible.

Proof. Since the sets $Z_{\pm 1}$, $W_{\pm 1}$, and $W_1$ are empty for $g(C_1) = 1$, the first statement is immediate from Proposition 5.4. Since $g(C_1) = 1$ implies that $g(C_2) = g(C)$ and $g(C) \geq 6$ by hypothesis, it follows from Lemma 5.3 that $Z_3$ is irreducible. \hfill $\Box$

We now focus on the case $g(C_1) \geq 2$. Proposition 5.4 reduces the study of $V^2$ to understanding the sets $W_{\pm 1}, Z_{\pm 3}$ and their images in $P^-$. We start with the following observation.

5.8. Lemma. For $g(C_1) \geq 2$,

$$(\pi_1^*, \pi_2^*)(W_1) = (\pi_1^*, \pi_2^*)(W_{-1}).$$

Proof. We claim that the following holds: If $L_1 \in W_{g(C_1)}^1 C_1$ is a general point then

(a) $L_1$ is not simple and
(b) there exists a point $x \in E$ such that

$$L_1 \simeq p_1^*\mathcal{O}_E(x) \otimes \mathcal{O}_{C_1}(D_1),$$

with $D_1$ an effective divisor such that $\mathcal{O}_E(Nm p_1(D_1) + x) \simeq \delta_1$. 


Assuming this for the time being, let us show how to conclude. If $L \in (\pi_1^*, \pi_2^*)(W_1)$ is a general point, then $L \simeq \pi_1^*L_1 \otimes \pi_2^*L_2$ with $L_1 \in W_{g-1}(C_1)$ a general point and $L_2$ a $p_2$-simple line bundle. Thus, by the claim we can write

$$L \simeq \pi_1^*O_{C_1}(D_1) \otimes \pi_2^*(L_2 \otimes p_2^*O_E(x)).$$

Since $O_E(Nm p_1(D_1) + x) \simeq \delta_1$ and $\delta \simeq \delta_1 \otimes \delta_2$, a short computation shows that $Nm p_2(L_2) \otimes O_E(x) \simeq \delta_2$. Moreover $L_2$ is $p_2$-simple and so, by [D1, Prop. 5.2.7], $\dim L_2 \otimes p_2^*O_E(x) \geq 1$. Hence $L$ is in $(\pi_1^*, \pi_2^*)(W_{-1})$. This shows one inclusion; the proof of the other is analogous.

**Proof of the claim.** Set

$$S := \{ (x, D_1) \in E \times C_1^{(g(C_1)-2)} \mid x + Nm p_1(D_1) \in |\delta_1| \}.$$

(For $g(C_1) = 2$, the symmetric product $C_1^{(g(C_1)-2)}$ is a point; it corresponds to the zero divisor on $C_1$.) Observe that the projection $p_2: S \to C_1^{(g(C_1)-2)}$ on the second factor is an isomorphism, so $S$ is not uniruled. For $(x, D_1) \in S$ general, the divisor $D_1$ is $p_1$-simple by Lemma 5.2 and so, by [M1, p. 338], we have the exact sequence

$$0 \to O_E(x) \to (p_1)_*O_{C_1}(p_1^*x + D_1) \to O_E(x + Nm p_1(D_1)) \otimes \delta_1^* \to 0.$$

By construction we have $O_E(x + Nm p_1(D_1)) \otimes \delta_1^* \simeq O_E$. Thus $H^1(E, O_E(x)) = 0$ implies that $h^0(C_1, O_{C_1}(p_1^*x + D_1)) = 2$. Hence the image of

$$\tau: S \to \Pic C_1, \quad (x, D_1) \mapsto O_{C_1}(p_1^*x + D_1)$$

is contained in $W_{g-1}(C_1)$. Because $S$ is not uniruled, the general fibre of $S \to \tau(S)$ has dimension 0. By Riemann–Roch, the residual map $W_{g-1}(C_1) \to W_{g-1}(C_1)$ is an isomorphism and so $W_{g-1}(C_1)$ is irreducible of dimension $g(C_1) - 2$. Hence $\tau$ is surjective on $W_{g-1}(C_1)$.

Suppose that $g(C_1) \geq 2$. Let $(JC_1, \Theta_{C_1})$ and $(JC_2, \Theta_{C_2})$ be the Jacobians of the curves $C_1$ and $C_2$ with their natural principal polarizations. Since $p_1$ and $p_2$ are ramified, the pull-backs $\pi_1^*: JE \to JC_1$ and $\pi_2^*: JE \to JC_2$ are injective and the restricted polarizations $B_1 := \Theta_{C_1}|_{JE}$ and $B_2 := \Theta_{C_2}|_{JE}$ are of type (2) [M1, Chap. 3]. We define

$$P_1 := \ker(Nm p_1: JC_1 \to JE), \quad P_2 := \ker(Nm p_2: JC_2 \to JE).$$

We set $A_1 := \Theta_{C_1}|_{P_1}$ and $A_2 := \Theta_{C_2}|_{P_2}$; then the polarizations $A_1$ and $A_2$ are of type $(1, \ldots, 1, 2)$ [BiLb, Cor. 12.1.5].

If $p_j^* \times i_P: JE \times P_j \to JC_j$ denotes the natural isogeny, then $(p_j^* \times i_P)^*\Theta_{C_j} \equiv B_j \boxtimes A_j$. Thus if $a_j: JC_j \to JE \times \widehat{P}_j$ is the dual map then

$$\Theta_{C_j}^{\otimes 2} \equiv a_j^*(\widehat{B}_j \boxtimes \widehat{A}_j) \quad (9)$$

[BiLb, Prop. 14.4.4], where $\widehat{B}_j$ and $\widehat{A}_j$ are the dual polarizations. We remark that $\widehat{A}_j$ has type $(1, 2, \ldots, 2)$. 
By [D1, Prop. 5.5.1], the pull-back maps \( P_1 \) and \( P_2 \) into the Prym variety \( P \) and we obtain an isogeny \( (\pi_1^*, \pi_2^*)|_{P_1 \times P_2} : P_1 \times P_2 \to P \) such that

\[
(\pi_1^*, \pi_2^*)|_{P_1 \times P_2} \circ \Theta \equiv A_1 \boxtimes A_2.
\]

In particular, if \( g : P \to P_1 \times P_2 \) denotes the dual map then

\[
\Theta^{\otimes 2} \equiv g^*(A_1 \boxtimes A_2).
\]

5.9. Proposition. If \( g(C_1) \geq 3 \) then the cohomology classes of \((\pi_1^*, \pi_3^*) (Z_{-3})\), \((\pi_1^*, \pi_2^*) (W_1)\), and \((\pi_1^*, \pi_3^*) (Z_3)\) are not minimal. Moreover, their cohomology classes are distinct and so they are distinct irreducible components of \( V^2 \).

If \( g(C_1) = 2 \) then the same holds for \((\pi_1^*, \pi_3^*) (W_1)\) and \((\pi_1^*, \pi_3^*) (Z_3)\).

Proof. In order to simplify the notation, we denote the pull-back of the polarizations \( \widehat{A}_1 \) and \( \widehat{A}_2 \) to \( P_1 \times P_2 \) by the same letter.

We start by observing that is sufficient to show that \([ (\pi_1^*, \pi_3^*) (Z_{-3}) ] \) (resp. \([ (\pi_1^*, \pi_3^*) (Z_3) ] \)) is a nonnegative multiple of \( g^* \widehat{A}_1^3 \) (\( g^* \widehat{A}_2^3 \)). Indeed, once we have shown this property, we can use that

\[
[(\pi_1^*, \pi_3^*) (Z_{-3})] + [(\pi_1^*, \pi_3^*) (Z_3)] + [(\pi_1^*, \pi_3^*) (W_1)] = \Theta^3 = \frac{23}{3!}
\]

and the identity (10) to compute that

\[
[(\pi_1^*, \pi_3^*) (W_1)] = \frac{1}{3!} [1 - a_1] g^* \widehat{A}_1^3 + 3 g^* \widehat{A}_1^2 \widehat{A}_2 + 3 g^* \widehat{A}_1 \widehat{A}_2^2 + (1 - a_2) g^* \widehat{A}_2^3,
\]

where \( a_1, a_2 \geq 0 \) correspond to the cohomology class of \( Z_{\pm 3} \). It is clear that none of these classes is a multiple of a minimal cohomology class. If \( g(C_1) \geq 4 \) then all the classes are nonzero and distinct, in which case the images of \( Z_{\pm 3} \) and \( W_1 \) are distinct irreducible components of \( V^2 \). If \( 2 \leq g(C_1) \leq 3 \) then the set \( Z_{-3} \) is empty (and the corresponding class zero), so we obtain only two irreducible components.

Computation of the cohomology class of \((\pi_1^*, \pi_3^*) (Z_{\pm 3})\). We will prove the claim for \( Z_3 \); the proof for \( Z_{-3} \) is analogous. We have the commutative diagram

\[
P \xrightarrow{\iota_p} J\widehat{C} \xrightarrow{\sim} \widehat{J\widehat{C}} \xrightarrow{\iota_{\widehat{p}}} \widehat{P} \simeq P
\]

\[
\xymatrix{ J\mathcal{C}_1 \times J\mathcal{C}_2 \ar[r]^-{\sim} \ar[d]_-g & \widehat{J\mathcal{C}_1 \times J\mathcal{C}_2} \ar[d]_-g \ar[r]^-q & \widehat{P}_1 \times \widehat{P}_2 },
\]

therefore, if \( X \subset J\mathcal{C}_1 \times J\mathcal{C}_2 \) is a subvariety such that \((\pi_1^*, \pi_3^*) (X) \subset P \), then its cohomology class is determined (up to a multiple) by the class of \( g(X) \) in \( P_1 \times P_2 \). We choose a translate of \( Z_3 \) that is in \( J\mathcal{C}_1 \times J\mathcal{C}_2 \) and denote it by the same letter. We want to understand the geometry of \( q(Z_3) \). Since the norm maps \( N \rho \)}
are dual to the pull-backs $p_j^* [M1, Chap. 1]$, the map $q$ fits into an exact sequence of abelian varieties

$$0 \rightarrow JE \times JE \xrightarrow{P_1^* \times P_2^*} JC_1 \times JC_2 \xrightarrow{q} \hat{P}_1 \times \hat{P}_2 \rightarrow 0. \quad (11)$$

Recall from the proof of Lemma 5.3 that $Z_3$ is a fibre space over $W^0_{g(C_1)-4}$ such that, for given $L_2 \in W^0_{g(C_2)-4}$, the fibre identifies to the fibre of $Nm p_1: \operatorname{Pic}^g(C_1)+2 C_1 \rightarrow \operatorname{Pic}^g(C_1)+2 E$ over $\delta \otimes \operatorname{Nm} p_2(L_2^*)$. Thus $Z_3$ identifies to a fibre product

$$\operatorname{Pic}^g(C)+2 C_1 \times_{JE} W^0_{g(C_2)-4}.$$ Together with the exact sequence (11) this shows that

$$q(Z_3) = \hat{P}_1 \times q_2(W^0_{g(C_2)-4}),$$

where $q_2: JC_2 \rightarrow P_2$ is the restriction of $q$ to $JC_2$.

Thus we are left to compute the cohomology class of $q_2(W^0_{g(C_2)-4})$. Note first that $q_2$ is the composition of the isogeny $\alpha_2: JC_2 \rightarrow JE \times \hat{P}_2$ with the projection on $\hat{P}_2$. Since the polarization $\hat{B}_2$ is numerically equivalent to a multiple of $e \times \hat{P}_2 \subset JE \times \hat{P}_2$ and since the cohomology class of $W^0_{g(C_2)-4}$ is $\frac{\Theta_{C_2}^2}{4}$, it follows from the identity (9) that the cohomology class of $q_2(W^0_{g(C_2)-4})$ is a multiple of $\hat{A}_2^3$.

5.10. Remark. With some additional effort one can prove the following statement. If $g(C_1) \geq 2$, then the following equalities in $H^6(P, \mathbb{Z})$ hold:

$$[(\pi_1^*, \pi_2^*)(Z_{-3})] = \frac{1}{4^g} g^* p_1^* \hat{A}_1^3; \quad (12)$$

$$[(\pi_1^*, \pi_2^*)(Z_3)] = \frac{1}{4^g} g^* p_2^* \hat{A}_2^3; \quad (13)$$

$$[(\pi_1^*, \pi_2^*)(W_1)] = \frac{1}{8^g} (p_1^* \hat{A}_1^3 p_2^* \hat{A}_2^3 + p_2^* \hat{A}_2^3 p_1^* \hat{A}_1^3). \quad (14)$$

The polarization $\hat{A}_j$ is of type $(1, 2, \ldots, 2)$ and so, by [BiLa, Thm. 4.10.4], $\frac{1}{8^g} \hat{A}_j^3$ is a “minimal” cohomology class for $(P_j, \hat{A}_j)$; in other words, it is in $H^6(P_j, \mathbb{Z})$ and is not divisible.

5.11. Remark. Let $\mathcal{R}_{g(C)}$ be the moduli space of pairs $(C, \pi)$, where $C$ is a smooth projective curve of genus $g(C)$ and $\pi: \bar{C} \rightarrow C$ is an étale double cover. We denote by

$$\operatorname{Pr}: \mathcal{R}_{g(C)} \rightarrow \mathcal{A}_{g(C)-1}$$

the Prym map associating to $(C, \pi)$ the principally polarized Prym variety $(P, \Theta)$.

Let $\mathcal{B}_{g(C)}$ be the moduli space of bielliptic curves of genus $g(C) \geq 6$, and let $\mathcal{R}_{g(C)} \subset \mathcal{R}_{g(C)}$ be the moduli space of étale double covers over them. Let $\mathcal{R}_{g(C), g(C)}$ be those étale double covers such that $\bar{C} \rightarrow C \rightarrow E$ has Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the curve $C_1$ has genus $g(C_1)$. 


By [D1, Thm. 4.1(i)], the closure of \( \text{Pr}(R_{\mathcal{B}g(C),1}) \) in \( A_{g(C)-1} \) contains the locus of Jacobians of hyperelliptic curves of genus \( g(C) - 1 \). A general hyperelliptic Jacobian has the property that the cohomology class of every subvariety is an integral multiple of the minimal class [Bis]. Hence the same property holds for a general element in \( \text{Pr}(R_{\mathcal{B}g(C),1}) \). So if \( V^2 \) were reducible then the irreducible components would have minimal cohomology class.

6. Prym Varieties of Bielliptic Curves, II

6.A. Tetragonal Construction and \( V^2 \)

Denote by \( C \) an irreducible nodal curve of arithmetic genus \( p_a(C) \geq 6 \) and by \( \pi: \tilde{C} \to C \) a Beauville admissible cover. By [B1], the corresponding Prym variety \( (P, \Theta) \) is a principally polarized abelian variety. Suppose that \( C \) is a tetragonal curve—that is, suppose there exists a finite morphism \( f: C \to \mathbb{P}^1 \) of degree 4. We set \( H := f^*\mathcal{O}_{\mathbb{P}^1}(1) \). By Donagi’s tetragonal construction ([Do]; see also [BLa, Chap. 12.8]), the corresponding special subvarieties give Beauville admissible covers \( \tilde{C}' \to C' \) and \( \tilde{C}'' \to C'' \) such that \( C' \) and \( C'' \) are tetragonal and the Prym varieties are isomorphic to \( (P, \delta \Theta E) \).

Consider now the residual line bundle \( K_C \otimes H^* \). By Riemann–Roch, the linear series \( |K_C \otimes H^*| \) is a \( 2^{p_a(C)-6} \) to which we can apply the construction of special subvarieties (cf. Section 5.A). If \( S \subset \Lambda \) is a connected component then, by [B3, Thm. 1 and Rem. 4], the cohomology class of \( V := \iota_C(S) \) is \( [2^{3/2}] \). Denote by \( (P^+, \Theta^+) \) the canonically polarized Prym variety; thus,

\[ \Theta^+ = \{ L \in (\text{Nm } \pi)^{-1}(K_C) \mid |L| \neq \emptyset, \dim|L| \equiv 0 \mod 2 \}. \]

Up to exchanging \( \tilde{C}' \) and \( \tilde{C}'' \), we can suppose that the image of the natural map

\[ V \times \tilde{C}' \to J\tilde{C} \]

is contained in \( P^+ \). By construction, the image is then contained in \( \Theta^+ \): hence a translate of \( -V \) is contained in the theta-dual \( T(\tilde{C}') \). Since \( T(\tilde{C}') \) equals the Brill–Noether locus \( (V^2)' \) of the covering \( \tilde{C}' \to C' \), it has cohomology class \( [2^{3/2}] \) and the inclusion is a (set-theoretical) equality.

The preceding argument shows that the special subvariety \( V \) is isomorphic to the Brill–Noether locus \( (V^2)' \) of a tetragonally related covering. The following technical lemma shows that this special subvariety is irreducible unless we are in a very special situation.

6.1. Lemma. Let \( C \) be an irreducible nodal curve of arithmetic genus \( p_a(C) \geq 6 \), and let \( \pi: \tilde{C} \to C \) be a Beauville admissible cover. Suppose that \( C \) is a tetragonal curve but is not hyperelliptic, trigonal, or a plane quintic. Assume that the normalization \( v: T \to C \) is a hyperelliptic curve, and denote by \( h: T \to \mathbb{P}^1 \) the hyperelliptic covering.

Denote by \( f: C \to \mathbb{P}^1 \) the morphism of degree 4 and set \( H := f^*\mathcal{O}_{\mathbb{P}^1}(1) \). Suppose that the base locus of the linear system \( |K_C \otimes H^*| \) does not contain any points of \( C_{\text{sing}} \), and suppose that the special subvariety \( S \) corresponding to \( |K_C \otimes H^*| \) is reducible. Then the following claims hold.
(a) There exist points $p, q$ in the smooth locus $C_{sm}$ such that we have a two-to-one cover

$$\varphi_{|H \otimes \mathcal{O}_C(p+q)|}: C \rightarrow \tilde{C} \subset \mathbb{P}^2$$

onto a singular plane cubic $\tilde{C}$. This morphism factors through the hyperelliptic covering; that is, we have a commutative diagram

$$\begin{array}{ccc}
T & \xrightarrow{\nu} & C \\
\downarrow & & \downarrow \varphi_{|H \otimes \mathcal{O}_C(p+q)|} \\
\mathbb{P}^1 & \xrightarrow{i} & \tilde{C}.
\end{array}$$

(b) If $x_1, x_2 \in T$ such that $\nu(x_1) = \nu(x_2)$, then $h(x_1) = h(x_2)$ unless $\tilde{C}$ is nodal and $h(x_1)$ and $h(x_2)$ are mapped onto the unique node.

**Proof.** Since $C$ is not a plane quintic, the linear system $|K_C \otimes H^*|$ is base point free. By [B3, Sec. 2, Cor.] applied to the pull-back of the linear system to $T$, we know that $\bar{S}$ is irreducible if the linear system $|K_C \otimes H^*|$ induces a map $f: C \rightarrow \mathbb{P}_{H^0(C)}$ that is birational onto its image $f(C)$. Suppose now that this is not the case; then, for every generic point $p \in C$, there exists another generic point $q \in C$ such that

$$h^0(C, K_C \otimes H^* \otimes \mathcal{O}_C(-p-q)) = h^0(C, K_C \otimes H^* \otimes \mathcal{O}_C(-p)).$$

By Riemann–Roch, this equality implies that the linear system $|H \otimes \mathcal{O}_C(p+q)|$ is a base point free $g_2^2$. Because $C$ is not hyperelliptic, we obtain in this way a $1$-dimensional subset $W \subset W^2_6 C$. Consider now the morphism $\varphi_{|H \otimes \mathcal{O}_C(p+q)|}: C \rightarrow \tilde{C} \subset \mathbb{P}^2$. Because $C$ is irreducible and not trigonal, the curve $\tilde{C}$ is an irreducible cubic or sextic curve.

Since $H$ and $H \otimes \mathcal{O}_C(p+q)$ are base point free, it is easy to see that $|\nu^*(H \otimes \mathcal{O}_C(p+q))|$ is a $g_2^2$. Thus we have $\nu^*(H \otimes \mathcal{O}_C(p+q)) \simeq h^*\mathcal{O}_{p1}(3)$, and a factorization $\tilde{\nu}: \mathbb{P}^1 \rightarrow \tilde{C}$ such that $\tilde{\nu} \circ h = \varphi_{|H \otimes \mathcal{O}_C(p+q)|} \circ \nu$.

In particular, $\varphi_{|H \otimes \mathcal{O}_C(p+q)|}$ is not birational onto its image and $\tilde{C}$ is a singular cubic. A look at the lemma’s commutative diagram shows that if $x_1, x_2 \in T$ such that $\nu(x_1) = \nu(x_2)$, then $h(x_1) = h(x_2)$ unless $\tilde{C}$ is nodal and $h(x_1)$ and $h(x_2)$ are mapped onto the unique node. \hfill \Box

**Remark.** For the sake of completeness we also consider the case where, in Lemma 6.1, the normalization $T$ is not hyperelliptic. In this case the pull-backs $\nu^*(H \otimes \mathcal{O}_C(p+q))$ define a $1$-dimensional subset $\bar{W} \subset W^2_6 T$. It follows from [ACGH, p. 198] that $T$ is bielliptic, and if $h: T \rightarrow E$ is a two-to-one map onto an elliptic curve $E$ then $\nu^*(H \otimes \mathcal{O}_C(p+q)) \simeq h^*L$, where $L \in \text{Pic}^3 E$. As before, we have a factorization $\tilde{\nu}: E \rightarrow C'$, which is easily seen to be an isomorphism. In particular, $C$ is obtained from $T$ by identifying points that are in a $h$-fibre.

### 6.B. The Irreducible Components of $V^2$

Let $C'$ be a smooth curve of genus $g(C) \geq 6$ that is bielliptic; in other words, we have a double cover $\nu': C' \rightarrow E$ onto an elliptic curve $E$. As usual, $\pi': \tilde{C}' \rightarrow C'$
will be an étale double cover. We suppose that the morphism $p' \circ \pi': \tilde{C}' \to E$ is not Galois (in the terminology of $[D1; N]$, the covering belongs to the family $R_{g(C)} \subset R_2(C)$; cf. Remark 5.11).

If we apply the tetragonal construction to a general $g^4_4$ on $C$, the result is a Beauville admissible cover $\pi: \tilde{C} \to C$ such that the normalization $\nu: T \to C$ is a smooth hyperelliptic curve $T$ of genus $g(C) - 2$. Denote by $h: T \to \mathbb{P}^1$ the hyperelliptic structure. Then $\nu$ identifies two pairs of points, $x_1, x_2$ and $y_1, y_2$, such that $h(x_1), h(x_2), h(y_1), h(y_2)$ are four distinct points in $\mathbb{P}^1$ (this follows from the “figure locale” in $[D1, 7.2.4]$).

By $[N, Chap. 15]$, a tetragonal structure on $C$ can be constructed as follows. There exists a unique double cover $j: \mathbb{P}^1 \to \mathbb{P}^1$ sending each pair $h(x_1), h(x_2)$ and $h(y_1), h(y_2)$ onto a single point. The four-to-one covering $j \circ h: T \to \mathbb{P}^1$ factors through the normalization $\nu$, so we have a four-to-one cover $f: C \to \mathbb{P}^1$. After applying the tetragonal construction to $H := f^*\mathcal{O}_{\mathbb{P}^1}(1)$, we recover the original étale double cover $\pi': \tilde{C}' \to C'$. We have already seen in Section 6.A that the Brill–Noether locus $V_2$ associated to $\pi'$ is isomorphic to a special subvariety associated to $|K_C \otimes H^*|$. Now, by considering the exact sequence

$$0 \to \nu_*(K_T \otimes \nu^*H^*) \to K_C \otimes H^* \to \mathbb{C}_{\nu(x_1)} \oplus \mathbb{C}_{\nu(y_1)} \to 0,$$

one sees easily that the linear system $|K_C \otimes H^*|$ is base point free yet does not separate the singular points $\nu(x_1)$ and $\nu(y_1)$. Since the points $h(x_1), h(x_2), h(y_1), h(y_2)$ are distinct, it follows from Lemma 6.1 that the special subvarieties are irreducible. Our final proposition summarizes these considerations.

6.2. Proposition. Let $C'$ be a smooth curve of genus $g(C') \geq 6$ that is bielliptic (i.e., we have a double cover $p': C' \to E$ onto an elliptic curve $E$). Let $\pi': \tilde{C}' \to C'$ be an étale double cover such that the cover $\tilde{C}' \to E$ is not Galois. Then $V_2$ is irreducible.

7. Proof of Theorem 1.2

If $V_2$ is reducible then, by Corollary 3.6, $C$ is trigonal, a plane quintic, or bielliptic. The first two cases are settled in Sections 4.B and 4.C, respectively. If $C$ is bielliptic, we distinguish two cases depending on whether or not the four-to-one cover $\tilde{C} \to C \to E$ is Galois. In the Galois case we conclude by Corollary 5.7 and Proposition 5.9; otherwise, we use Proposition 6.2.

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