On the Direction of Casimir Forces

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The Casimir force due to a massless scalar field satisfying Dirichlet boundary conditions may attract or repel a piston in the neck of a flask-like container. Using the world-line formalism this behavior is related to the competing contribution to the interaction energy of two types of Brownian bridges. It qualitatively is also expected from attractive long-range two-body forces between constituents of the boundary. A geometric subtraction scheme is presented that avoids divergent contributions to the interaction energy and classifies the Brownian bridges that contribute to the force. These are all of finite length and the Casimir force can be analyzed and in principle accurately computed without resorting to regularization or analytic continuation. The world-line analysis is robust with respect to variations in the shape of the piston and the flask and the analogy with long-range forces suggests that neutral atoms and particles are also drawn into open-ended pipes (or nano-tubes) by Casimir forces of electromagnetic origin.

I. INTRODUCTION

Contrary to intuition derived from the attractive Casimir force between two conducting plates\[1], Boyer\[2] found that the zero-point energy apparently tends to expand a perfectly conducting spherical shell. Until recently\[3], there was no qualitative explanation for the sign of the Casimir energy. However, the finite negative surface tension of a metallic spherical shell cannot be measured by itself: changing the radius of a real cavity necessarily involves its material properties. The negative Casimir tension of a spherical shell in this sense is a result without direct physical implications. It was later found that Casimir self-energies of many closed cavities are plagued by divergences that cannot be removed without appealing to material properties of their walls\[4].

The Casimir force between disjoint solid bodies on the other hand in principle is observable and ought to be finite. For some simple shapes the force between uncharged conductors has now been measured quite accurately\[5]. Experimentally as well as theoretically the force between conductors is attractive in all cases studied. A theorem by Kenneth and Klich\[6] and its generalization by Bachas\[7] states that reflection positivity implies that the interaction between a mirror-pair of disjoint (charge-conjugate) bodies is attractive. This theorem in particular implies that, contrary to previous suggestions\[8, 9], the Casimir force between two half-spheres is attractive\[6]. The attractive Casimir-Polder\[10] force between polarizable atoms furthermore suggests that the force could be attractive for any shape of the conductors. Such considerations, as well as many failed attempts\[8, 9, 11] to find shapes exhibiting repulsion might give the impression that repulsive Casimir forces between distinct bodies occur for suitable (mixed) boundary conditions\[12, 13, 14] only.

But neither the long list of examples nor the restrictive theorems by Kenneth, Klich and Bachas\[6, 7] apparently imply that the Casimir force is attractive between any conductors. Intuition based on the Casimir-Polder\[10] force between atoms could be misleading\[15]. Polarizable atoms attract just as any distant conducting spheres would and as such do not even qualitatively reproduce the Casimir energy of some geometries. [If vacuum forces are entirely due to attractive two-body forces, a metallic spherical shell apparently would have to have positive surface tension.]

Semiclassical\[16] and numerical\[17] arguments suggest that the Casimir force on a piston depends qualitatively on the shape of the casing. We will see below that a piston in the neck of a flask with a spherical bulb in fact may be attracted or repelled from the bulb and can have a stable equilibrium position. This will be shown for the Casimir force on the piston due to a massless scalar field satisfying Dirichlet boundary conditions on the flask and piston surfaces. Although the world-line approach we use is for a scalar field, the results can be qualitatively understood as due to an attractive long-range interaction between constituents of the boundaries. The outward Casimir pressure on an ideal metallic sphere\[2, 18] and inward pressure on an ideal metallic cylinder\[19] also suggest that a similar competition of vacuum forces occurs in the electromagnetic case with metallic boundaries. Net Casimir forces that change direction or vanish perhaps can be observed in micro-mechanical devices. The existence of stable equilibrium positions with a vanishing Casimir force for a large class of shapes could also be of practical interest in high precision studies of long-range forces.

The method used here to determine the direction of the Casimir force does not require an atomistic interpretation of its origin, but uses a geometrical subtraction scheme to express finite Casimir energies as a sum of finite contributions of definite sign due to classes of Brownian bridges of finite length. The need to compute differences of potentially arbitrary large quantities to obtain Casimir forces is thereby avoided.

II. WORLD-LINE APPROACH

Consider the heat kernel operator $K_{D}(\beta) = e^{\beta \Delta/2}$ for the Laplacian $\Delta$ with Dirichlet boundary conditions on
a bounded domain $\mathcal{D} \subset \mathbb{R}^3$. The eigenvalues $\{\lambda_n \geq \lambda_{n-1} > 0; n \in \mathbb{N}\}$ of the negative Laplace operator in this case are discrete, real and positive and the corresponding spectral function (or trace of the heat-kernel),

$$\phi_{\mathcal{D}}(\beta) = \text{Tr} \mathbb{R}_{\mathcal{D}}(\beta) = \sum_{n \in \mathbb{N}} e^{-\beta \lambda_n/2},$$

is finite and well defined for $\beta > 0$. In principle, the spectral function includes all the information required to compute zero point energies of bounded domains.

$\phi_{\mathcal{D}}(\beta \sim 0)$ has the well-known\textsuperscript{23, 24} asymptotic (short time or high-temperature) expansion,

$$\phi_{\mathcal{D}}(\beta \sim 0) \sim \frac{1}{(2\pi\beta)^{3/2}} \sum_{n=0}^{\infty} (2\pi\beta)^{n/2} a_n(\mathcal{D}) + \mathcal{O}(e^{-\beta^2/\beta}).$$

For smoothly bounded domains, the Hadamard-Minakshisundaram-DeWitt-Seeley coefficients $a_n(\mathcal{D})$ in this series are integrals over powers of the local curvature and reflect average geometric properties of the domain and its boundary\textsuperscript{23, 24}. For a bounded threedimensional flat Euclidean domain $\mathcal{D}$, $a_0(\mathcal{D})$ gives its volume $V_\mathcal{D}$ and $a_1(\mathcal{D}) = -S_\mathcal{D}/4$ gives the surface area $S_\mathcal{D}$ of its boundary\textsuperscript{23, 24}. $a_2(\mathcal{D})$ is proportional to the integrated curvature [sharp edges of the boundary also contribute\textsuperscript{23, 24}] and $a_3(\mathcal{D})$ is a dimensionless coefficient reflecting topological characteristics of the domain [such as the connectivity of its boundary and the number and opening angles of its corners\textsuperscript{23, 26}]. The coefficient $a_4$ is the most crucial for Casimir effects, since $a_4(\mathcal{D}) \neq 0$ implies a logarithmic divergent vacuum energy that prevents one from uniquely defining the Casimir energy. The geometric origin of this coefficient\textsuperscript{24} is, however, not simple to describe. Non-analytic and (for $\beta \sim 0$) exponentially suppressed contributions to the asymptotic expansion of $\phi_{\mathcal{D}}(\beta)$ are associated with classical periodic- and diffractive-orbits\textsuperscript{32} of a minimal length $l$.

The world-line approach to Casimir energies\textsuperscript{27} is based on the observation\textsuperscript{23, 28} that the spectral function for a bounded flat Euclidean domain $\mathcal{D}$ can be expressed in terms of its support of standard Brownian bridges. In three dimensions,

$$\phi_{\mathcal{D}}(\beta) = \int_{\mathcal{D}} \frac{dx}{(2\pi\beta)^{3/2}} P[\ell_\beta(x) \subset \mathcal{D}],$$

where $\ell_\beta(x) = \{B_\tau(x, \beta), 0 \leq \tau \leq \beta; B_0(x, \beta) = B_\beta(x, \beta) = x\}$ is a standard Brownian bridge from $x$ to $x$ in "proper time" $\beta$ and $P[\ell_\beta(x) \subset \mathcal{D}]$ denotes the probability for the bridge to be entirely within the bounded domain $\mathcal{D}$.

Although the spectral function of a bounded domain of finite volume thus is evidently finite, divergences arise in the corresponding zero-point energy. The formal zero-point energy of a massless scalar satisfying Dirichlet boundary conditions on $\mathcal{D}$,

$$\mathcal{E}_{\text{vac}}(\mathcal{D}) \sim \frac{1}{2} \sum_{n=0}^{\infty} \sqrt{\lambda_n} \sim -\pi \int_0^{\infty} \frac{d\beta}{(2\pi\beta)^{3/2}} \phi_{\mathcal{D}}(\beta),$$

diverges due to the behavior of the integrand for $\beta \sim 0$ implied by Eq.(3). To calculate finite Casimir energies one customarily regulates the integral in Eq.(1) for instance by analytic continuation or with a finite lower limit in the integration over the proper time in Eq.(3). One then must show that the physical effect of interest remains finite after analytic continuation or when the cutoff is removed. The same result can also be achieved using a numerically more suitable and physically more transparent cutoff-independent procedure by noting that only the first few terms (first five in three spatial dimensions) of the asymptotic high-temperature expansion in Eq.(2) lead to divergent contributions to the vacuum energy. The behavior of the integrand for $\beta \rightarrow 0$ therefore can be improved by considering a (finite) linear combination of spectral functions for domains $D_k$, $k = 0, 1, \ldots$,

$$\tilde{\phi}(\beta) = \sum_k c_k \phi_{D_k}(\beta).$$

When the coefficients $c_k$ are chosen so that

$$\sum_k c_k a_n(D_k) = 0 \quad \text{for} \quad n = 0, \ldots, 4,$$

the "interaction" vacuum energy,

$$\mathcal{E}_{\text{int}} = -\pi \int_0^{\infty} \frac{d\beta}{(2\pi\beta)^{3/2}} \tilde{\phi}(\beta) = \sum_k c_k \mathcal{E}_{\text{vac}}(D_k),$$

is finite because the integrand of Eq.(7) is $O(\beta^{-1/2})$. Due to the geometrical nature of the asymptotic heat kernel expansion, the linear combination $\mathcal{E}_{\text{int}}$ of zero-point energies $\mathcal{E}_{\text{vac}}(D_k)$ may be interpreted as the difference in vacuum energy for domains with the same total volume, total surface area, average curvature, topology, etc... It of course is here understood that the subtractions at most affect the physical quantity of interest in a calculable way. The most convenient subtractions thus could depend on the physical problem at hand.

In the context of Casimir effects, such a "geometric" scheme was first used by Power\textsuperscript{29} to derive the original Casimir force\textsuperscript{1} between parallel metallic plates without intermediate regularization. Power compared the vacuum energy of a metallic box of fixed dimensions with a moveable plate for different positions of the plate. Swaiter\textsuperscript{30} recognized and succinctly emphasized the physical nature of this scheme. In\textsuperscript{1} finite contribution to the interaction vacuum energy due to periodic orbits were computed in leading semiclassical approximation. However, diffractive orbits and non-vanishing higher terms in the asymptotic power series $\phi(\beta)$ may sometimes contribute significantly to $\mathcal{E}_{\text{int}}$\textsuperscript{17}.

III. THE CASIMIR FORCE ON A PISTON IN THE NECK OF A FLASK

The world-line formalism of the previous section can be used to obtain the Casimir force on the piston in the
only allows one to compute the difference in the force on the piston within the neck and within a cylinder of length $2L$ of the same radius $r$. For finite $L > R > r > 0$ the domains $D_0 \ldots D_5$ are all bounded, but we will be especially interested in the limit of large $L/R$ where the force on the piston near the center of the cylinder is negligible. Eqs. (8), (9) and (10) give the second expression of $\mathcal{E}_{\text{int}}$ in Eq. (8). The volumes $D_4$ and $D_5$ above the piston in the flask and in the cylinder are identical and give no net contribution to the alternating sum in Eq. (8). By conditioning on whether a loop pierces certain surfaces one can show that the so defined interaction vacuum energy is finite for any height $0 < a < L$ of the piston. Only bridges that pierce (or touch) the piston contribute to the alternating sum in Eq. (8). If they are entirely within $D_0$ or $D_4$ (the parts of the flask below and above the piston) then they also are entirely within $D_3$ (the whole flask) and if they are entirely within the cylinder below or above the piston ($D_1$ or $D_5$) then they also are within the whole cylinder ($D_2$). A loop that only pierces the piston, or in addition pieces both cylinder and flask also gives no contribution. Only two types of Brownian bridges (shown schematically in Fig.1) therefore contribute to $\mathcal{E}_{\text{int}}$. They either

\begin{equation}
(+ \text{) pierce piston and cylinder, but not the flask, or}
\end{equation}

\begin{equation}
(- \text{) pierce piston and flask, but not the cylinder.}
\end{equation}

The shortest bridge that contributes to $\mathcal{E}_{\text{int}}$ is of type (+) and has extent $a$: it just touches the piston and pierces the cylinder (but not the flask) near the base of the neck. The length of all loops that contribute to $\mathcal{E}_{\text{int}}(a > 0)$ thus is bounded below by $2a$. For $\beta \to 0$ the probability of a loop of finite extent is exponentially small. The asymptotic expansion of $\phi(\beta)$ for $\beta \sim 0$ therefore vanishes to all orders and $\mathcal{E}_{\text{int}}$ is finite.

A slight elaboration on the previous argument gives the direction of the force on the piston for some extreme configurations. Bridges of type (+) are within the whole flask (domain $D_3$) but not entirely within any of the other five domains. They therefore give a positive contribution to $\mathcal{E}_{\text{int}}$. Bridges of type (–) on the other hand are within domain $D_2$ (the whole cylinder) only and give a negative contribution to $\mathcal{E}_{\text{int}}$ in Eq. (8). The sign of $\mathcal{E}_{\text{int}}$ thus depends on which of the two types of bridges occurs more frequently and can be readily established for the following cases.

$L > R = r$

Since any loop that pierces the cylinder also pierces this flask with a hemispherical bottom, there is no contribution from (+)-loops and $\mathcal{E}_{\text{int}}$ is negative (11) for any height $a > 0$.

$a \ll r \ll R \ll L/2$:

For this flask with a very long, thin neck, the probability of a bridge over time $\beta$ of type (–) is much less than one of type (+) when the piston is near the base of the neck. (–)-loops in this case have a length greater than $2R + a$ but a transverse extent of less than $2r$ and are very elongated. Their contribution, $\mathcal{E}_{\text{int}}(–)$, to $\mathcal{E}_{\text{int}}$ may be estimated by noting that longitudinal and transverse components of a bridge are statistically independent. The probability that a bridge starting and ending at $x = (x^z, z)$ lies entirely within a cylinder, $C(r, l)$, of radius $r$ and length $l$ therefore is the product of the probabilities for the transverse bridge to remain within the disk $D(r)$ of radius $r$, and

\begin{equation}
\mathcal{E}_{\text{int}} = \sum_{k=0}^{3} (-1)^k \mathcal{E}_{\text{vac}}(D_k)
\end{equation}

\begin{equation}
= -\frac{1}{8\pi^2} \int_{0}^{\infty} \rho \frac{d\beta}{\beta^3} \int d\mathbf{x} \sum_{k=0}^{3} (-1)^k \mathcal{P}[\ell_{\beta}(x) \subset D_k],
\end{equation}

$D_0 = \text{flask below piston, } D_1 = \text{cylinder below piston,}$

$D_2 = \text{whole cylinder, } D_3 = \text{whole flask,}$

$D_4 = D_5 = \text{cylinder above piston}$
for the one-dimensional longitudinal bridge to lie within the interval \([0, l]\),
\[
\mathcal{P}[\ell_\beta(x) \in C(r, l)] = \mathcal{P}[\ell^{\pm}_\beta(x^\perp) \in D(r)] \mathcal{P}[\ell^{\parallel}_\beta(z) \in [0, l]] .
\]
(9)

Ignoring (for \(R \gg r\) small) corrections due to the curvature of the flask bottom, the contribution \(\mathcal{E}^{(-)}_{\text{int}}\) of bridges of type (−) in this regime becomes,
\[
\mathcal{E}^{(-)}_{\text{int}}(a; L > R) \sim -\int_0^\infty d\beta \phi_D(\beta) \times \int_{2R+a}^\infty ds \left(\frac{s+2R-a}{4\pi^2} \right) \mathcal{P}[\ell^{\parallel}_\beta] < s] \frac{ds}{2}\sqrt{2\pi s/3} \int_{2R+a}^\infty ds \left(\frac{s-2R-a}{2\sqrt{2\pi s/3}}\right)
\]
(10)

Here \(\ell_\beta\) denotes the maximal extent of a standard (one-dimensional) Brownian bridge over time \(\beta\) [the overall extension of a bridge evidently does not depend on its starting point]. The factor \((s-2R-a)\) in Eq. (10) is the translational volume available to a one-dimensional loop of extent \(s\) that pierces the piston (at height \(a\)) as well as the bulb at the bottom of the flask (at height \(-2R\)). \(\phi_D(\beta) = \int d\mathbf{x} \mathcal{P}[\ell^{\parallel}_\beta(x) \subset D(r)] / (2\pi\beta)\) is the spectral function for the disk \(D(r)\). The last line of Eq. (10) uses that the spectral function of the interval \([0, s]\) is \(\phi_{[0, s]}(\beta) = \int_0^s dx \mathcal{P}[\ell^{\parallel}_\beta] < x] / \sqrt{2\pi s}\). The spectral functions of an interval and of a disk are known \[32\] but for our estimate it suffices that the analog of Eq. (3) for two dimensions \[23\] implies that \(\phi_D(\beta) < r^2 / (2\beta)\) and that
\[
\phi_{[0, s]}(\beta) = \frac{8}{\sqrt{2\pi \beta}} \left(1 + 2 \sum_{n=1}^\infty e^{-2a^2n^2/\beta}\right).
\]
(11)

\(\mathcal{E}^{(-)}_{\text{int}}\) for \(R \gg r\) thus is bounded by,
\[
0 > \mathcal{E}^{(-)}_{\text{int}}(a; L > R) > -\sum_{n=1}^\infty \frac{d^2}{\beta^2} (2R + a) e^{-\frac{4}{\beta}(a^2(2R + a)^2)} = -\frac{\pi^3 r^2}{1440(2R + a)^3}.
\]
(12)

The lower bound in Eq. (12) is just the Casimir energy due to a massless scalar associated with two parallel flat plates of area \(\pi r^2\) that are separated by a distance \(d = 2R + a\). [This lower bound of \(\mathcal{E}^{(-)}_{\text{int}}\) remarkably holds for \(d \gg 0\)!]

The sign of \(\mathcal{E}_{\text{int}}\) for finite \(a/r\) in the limit \(r/R \to 0\) follows from the fact that \(\mathcal{E}^{(-)}_{\text{int}}\) vanishes in this limit whereas the corresponding positive contribution, \(\mathcal{E}^{(+)}_{\text{int}}\), of (−)-loops does not (it even slightly increases for \(R \to \infty\)). For any piston height, \(a\), and neck radius, \(r\), the interaction energy \(\mathcal{E}_{\text{int}}\) therefore is positive when \(L \) and \(R\) are sufficiently large,
\[
\mathcal{E}_{\text{int}}(a; r) = \lim_{R \to \infty} \lim_{L \to \infty} \left(\mathcal{E}^{(+)}_{\text{int}} + \mathcal{E}^{(-)}_{\text{int}}\right)
\]
\[-\mathcal{E}^{(-)}_{\text{int}}(a; r, L > R) \to \mathcal{E}^{(+)}_{\text{int}}(a, r; L > R) > 0 .
\]
(13)

\(L > a \gg r\):

The piston is very high in the neck of the flask. The magnitudes of \(\mathcal{E}^{(-)}_{\text{int}}(a, r)\) and \(\mathcal{E}^{(+)}_{\text{int}}(a, r)\) both vanish at least as fast as \((r/a)^3\). Since both types of loops are highly elongated in the limit of large \(a/r\), this can be seen by slightly adapting the previous proof that the \(\mathcal{E}^{(-)}_{\text{int}}\) contribution vanishes for large values of \(r/(2R + a)\). We therefore have that
\[
\mathcal{E}_{\text{int}}(L > a) \sim 0 .
\]
(14)

Eq. (14) together with Eq. (13) implies that \(\mathcal{E}^{(-)}_{\text{int}}(a, r)\) cannot be monotonically increasing with piston height \(a\). For sufficiently large \(L\) and \(R\), the force on the piston at some positions (close to the bulb), is directed away from the (large) bulb. The piston in this this sense is repulsed by the bulb, but it probably is more accurate to say that the piston is being drawn into the thin neck of such a flask. Whichever point of view is adopted, the upward force on the piston of Fig. 1 extends to all \(a\) in the limit \(L > R \to \infty\) (i.e. for a half-space with a hole of radius \(r\)), because the contribution \(\mathcal{E}_{\text{int}}\) is negligible and \(\mathcal{E}^{(+)}_{\text{int}}\) evidently decreases monotonically with the piston height \(a\). This repulsion thus is not directly related to the negative vacuum contribution to the surface tension of a spherical shell. The latter vanishes inversely proportional to \(R^2\) (for metallic- \[2\] and semiclassically \[33\] also for Dirichlet-boundary conditions). It rather is the net vacuum force on the piston due to the flask bulb and the long thin cylindrical neck. The latter is finite even if the force due to the spherical bulb may be neglected.

IV. CONCLUSION

The direction of the Casimir force on a body can depend relatively sensitively on the shape of the surrounding boundary. Such shape dependence was previously conjectured \[11\] on the basis of a change in sign of the Casimir self-energy of a parallelepiped \[24\]. However, it was shown \[35\] and has been proven \[4, 7\] on more general grounds that the Casimir force always attracts the piston to the nearest end-plate in this case. We here examined more asymmetric piston configurations for which reflection positivity \[4, 7\] does not imply the direction of the force. We used the world-line formalism and a geometrical subtraction scheme to determine the Casimir force on the piston due to a massless scalar field satisfying Dirichlet boundary conditions in these more complicated geometries. Although numerical calculations in principle are possible in this scheme, the quantitative results depend strongly on the precise geometry and are of limited practical interest. We therefore restrict ourselves to a qualitative analysis that applies to a large class of geometries. Our considerations in fact do not depend on the details of the shape of the “flask” and of the “piston” and can be extended to pistons that are not disks and/or do not touch the cylindrical neck. The bulb of the flask
also may be replaced by a more general cavity with average dimensions that are larger than the neck radius. The force on a piston in the neck of such a "flask" still depends on competing contributions to the Casimir energy from just two types of Brownian bridges (with properties given in (I)). Since these are of finite extent, their contributions to the interaction energy can be estimated and compared. For the flask of Fig. 1, we find that at any given height \( a \) the direction of the Casimir force on the piston depends on the ratio \( r/R \) of the neck’s radius to that of the bulb (assuming the neck is sufficiently long for its end to be ignored). The piston is always drawn into the neck for very small values of \( r/R \), whereas it is always attracted to the bulb of a flask with hemispheric bottom \( r/R = 1 \). We conclude that for finite values of the ratio \( 0 < r/R < 1 \), the Casimir force on the piston vanishes at some (finite) height \( a \). The existence of such an equilibrium position in some geometries perhaps is of interest to high precision measurements of forces. These results based on the world-line formalism can be qualitatively understood as due to long-range attractive two-body forces between constituents of the boundary. Since Van-der-Waals and Casimir-Polder forces between neutral atoms and molecules are of this nature, we expect them to qualitatively also obtain in realistic systems.

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