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A p-ADIC SHIMURA-MAASS OPERATOR ON MUMFORD CURVES

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Abstract. We study a p-adic Shimura-Maass operator in the context of Mumford curves defined by C. Franc in [Fra11]. We prove that this operator arises from a splitting of the Hodge filtration, thus answering a question in [Fra11]. We also study the relation of this operator with generalized Heegner cycles, in the spirit of [BDP13], [HB15], [Kri18] and [AI19].

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1. INTRODUCTION

The main purpose of this paper is to study in the context of Mumford curves a p-adic variant of the Shimura-Maass operator, and relate it to generalized Heegner cycles.

The real analytic Shimura-Maass operator is defined by the formula

$$\delta_k(f(z)) = \frac{1}{2\pi i} \left( \frac{\partial}{\partial z} + \frac{k}{z - \bar{z}} \right) f(z)$$

where $z$ is a variable in the complex upper half plane $\mathcal{H}$, $f(z)$ is a real analytic modular form of weight $k$, and $z \mapsto \bar{z}$ denotes the complex conjugation; here $\delta_k(f(z))$ is a real analytic modular form of weight $k + 2$. The relevance of this operator arises in studying algebraicity properties of Eisenstein series and L-functions: see Shimura [Shi75], Hida [Hid93, Chapter 10]. One of the main results in [Shi75] is the following. Let

$$\delta_k = \delta_{k+2(r-1)} \circ \delta_{k+2(r-2)} \circ \cdots \circ \delta_k$$

for any $r \geq 1$, and let $K$ be an imaginary quadratic field. Then there exists $\Omega_K \in \mathbb{C}^\times$ such that for every CM point $z \in K \cap \mathcal{H}$, every congruence subgroup $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$, every integer $k \geq 0$, $r \geq 1$, and every modular form of weight $k$ and level $\Gamma$ with algebraic Fourier coefficients we have

$$\frac{\delta_k^r(f(z))}{\Omega_{K}^{k+2r}} \in \bar{\mathbb{Q}}.$$ 

Katz described in [Kat68] the Shimura-Maass operator in more abstract terms by means of the Gauss-Manin connection (see also [KO68]). More precisely, let $N \geq 1$ be an integer,
X_1(N) the modular form of level Γ_1(N) over Q, and let π : E → X_1(N) be the universal elliptic curve. Consider the relative de Rham cohomology sheaf

\[ \mathcal{L}_1 = \mathbb{R}^1\pi_* \left( 0 \to \mathcal{O}_E \to \Omega^1_{E/X_1(N)} \right) \]

on X_1(N), and define \( \mathcal{L}_r = \text{Sym}^r(\mathcal{L}_1) \). Let \( \omega = \pi_* \left( \Omega^1_{E/X_1(N)} \right) \). The sheaf \( \omega \) is invertible and we have the Hodge filtration

\[ 0 \to \omega \to \mathcal{L}_1 \to \omega^{-1} \to 0. \]

Once we are given a splitting \( \Psi \) of the Hodge filtration \( \mathfrak{h} \), one may define by means of the Gauss-Manin connection and the Kodaira-Spencer map an operator \( \Theta^{(r)}_\Psi : \omega^r \to \omega^{r+2} \), for any even integer \( r \geq 2 \). In particular, considering the associated real analytic sheaves, which we denote by a superscript \( \text{ra} \), the Hodge exact sequence admits a splitting

\[ \Psi_\infty : \mathcal{L}_{1}^{\text{ra}} \cong \omega_{\text{ra}}^{\text{pr}} \oplus \omega_{\text{ra}}, \]

where \( \omega_{\text{ra}} \) is obtained from \( \omega \) by applying the complex conjugation. The Shimura-Maass operator can then be described as the map \( \Theta^{(r)}_\Psi \) on real analytic differentials by appealing to the general procedure alluded to above applied to the real analytic splitting \( \mathfrak{h} \). For details on this construction, the reader is referred to [Kat78 §1.8] and [BDP13 §1.2]; for the case of Siegel modular forms, see [Har81 §4] while for the case of Shimura curves see [HB15 §3], [Mor11 §2].

As hinted from the above discussion, Katz description of the Shimura-Maass operator rests on the fact that the real analytic Hodge sequences \( \mathfrak{h} \) splits. In [Kat78 §1.11], Katz introduces a \( p \)-adic analogue of this splitting. Suppose that \( p \nmid N \) is a prime number, and let \( X^{\text{ord}}_1(N) \) denote the ordinary locus of the modular curve, viewed as a rigid analytic scheme over \( \mathbb{Q}_p \).

Let \( \mathcal{F}^{\text{rig}} \) be the rigid analytic sheaf associated with a sheaf \( \mathcal{F} \) on \( X_1(N) \). Then \( \mathcal{L}^{\text{rig}}_1 \) splits over \( X^{\text{ord}}_1(N) \) as the direct sum

\[ \Psi_p : \mathcal{L}^{\text{rig}}_1 \cong \omega^{\text{rig}} \oplus \mathcal{L}^{\text{Frob}}_1 \]

(where \( \mathcal{L}^{\text{Frob}}_1 \) has the property that the Frobenius endomorphism acts on this sheaf invertibly).

This allows to define a differential operator \( \Theta^{(r)}_\Psi \), which can be seen as a \( p \)-adic analogue of the Shimura-Maass operator; this operator can be also described in terms of Atkin-Serre derivative. At CM points the splittings \( \Psi_\infty \) and \( \Psi_p \) coincide, and therefore one deduces by comparison rationality results for the values of \( \Theta^{(r)}_\Psi \) at CM points from intro rationality. For details, see [BDP13 Proposition 1.12]. The \( p \)-adic Shimura-Maass operator is then used in [Kat78] and [BDP13] to construct \( p \)-adic L-functions and study their properties.

We now fix an integer \( N \), a prime \( p \nmid N \), and a quadratic imaginary field in which \( p \) is inert. In this context, Kritz introduced in [Kri18] for modular forms of level \( N \) a new \( p \)-adic Shimura-Maass operator by using perfectoid techniques, and define \( p \)-adic L-functions by means of this operator, thus removing the crucial assumption that \( p \) is split in \( K \), but keeping the assumption that \( p \) is a prime of good reduction for the modular curve. Moreover, Andreatta-Iovita [AI19] introduced still an other \( p \)-adic Shimura-Maass operator in [AI19], and obtained results analogue to [BDP13], thus extending their work to the non-split case.

On the other hand, Franc in his thesis [Fra11] proposed still an other \( p \)-adic Shimura-Maass operator for primes \( p \) which are inert in \( K \), in the following context. Let \( N \geq 1 \) be an integer, \( K/\mathbb{Q} \) a quadratic imaginary field, \( p \nmid N \) a prime number which is inert in \( K \), and let \( Np = N^+ \cdot N^- p \) be a factorization of \( Np \) into coprime integers such that \( N^+ \) is divisible only by primes which are split in \( K \), and \( N^- p \) is a square-free product of an even number of primes factors which are inert in \( K \). Let \( \mathcal{B} \) be the indefinite quaternion of discriminant \( N^- \), \( \mathcal{R} \) an Eichler order of \( X \) of level \( N^+ \), and \( \mathcal{C} \) the Shimura curve attached to \( (\mathcal{B}, \mathcal{R}) \). The rigid analytic curve \( X^{\text{rig}} \) over \( \mathbb{Q}_p \) is then a Mumford curve, namely \( X^{\text{ord}}(\mathbb{C}_p) \) is isomorphic to the
rigid analytic quotient of the p-adic upper half plane \( \mathcal{H}_p(\mathbb{Q}_p) = \mathbb{C}_p - \mathbb{Q}_p \) by an arithmetic subgroup \( \Gamma \subseteq \text{SL}_2(\mathbb{Q}_p) \). Franc defines in this context a p-adic Shimura-Maass operator \( \delta_{p,k} \) by mimicking the definition (1) and formally replacing the variable \( z \in \mathcal{H} \) with the p-adic variable \( z \in \mathcal{H}_p(\hat{\mathbb{Q}}_p^\text{unr}) = \hat{\mathbb{Q}}_p^\text{unr} - \mathbb{Q}_p \), and replacing the complex conjugation with the Frobenius map (here \( \hat{\mathbb{Q}}_p^\text{unr} \) is the completion of the maximal unramified extension of \( \mathbb{Q}_p \)). Following the arguments of [Shi75], Franc proves statement analogue to (2) (see [Fra11, Theorem 5.1.5]).

In [Fra11] [§6.1.3], Franc asks for a construction of his p-adic Shimura-Maass operator by means of (non-rigid analytic) splitting \( \Psi_p \) of the Hodge filtration, similar to what happens over \( X_1(N) \) (in the real analytic case [Shi75]) and \( X_1^{\text{ord}}(N) \) (in the p-adic rigid analytic case [Kat78]). The first result of this paper is to provide such a splitting \( \Psi_p \), and define the associated p-adic Shimura-Maass operator. In particular, we show that our splitting \( \Psi_p \) coincides at CM points with the Hodge splitting \( \Psi_\infty \), and therefore, as in [Kat78], we reprove the main results of [Fra11] by the comparison of the two Shimura-Maass operators. We also derive a relation between our p-adic Shimura-Maass operator and generalized Heegner cycles in the context of Mumford curves, which can be vied as an analogue of [BDP13, Proposition 3.24]. In the remaining part of the introduction we describe more precisely the results of this paper.

Instead of the curve \( X \) attached to the Eichler order \( \mathcal{R} \), we follow [HB15] and consider a covering \( C \to X \) where \( C \) is a geometrically connected curve defined over \( \mathbb{Q} \) corresponding to a \( \Gamma_1(N^{\tau}) \)-level structure subgroup of \( \mathcal{R}^\times \), where \( \mathcal{R} = \mathcal{R} \otimes \mathbb{Z} \hat{\mathbb{Z}} \) is the profinite completion of \( \mathcal{R} \). The advantage of using \( C \) is that \( C \) is the solution of a moduli problem, and we have a universal false elliptic curve \( \pi : \mathcal{A} \to C \) (see §2.2). Following [Has95], [Mor11], [HB15], we define a quaternionic projector \( e \), acting on the relative de Rham cohomology of \( \pi : \mathcal{A} \to C \), and define the sheaf

\[
\mathcal{L}_1 = e \cdot \mathcal{H}_p^1(\mathcal{A}/\mathcal{C})
\]

and the line bundle

\[
\omega = e \cdot \pi^*(\Omega^1_{\mathcal{A}/\mathcal{C}}).
\]

We have a corresponding Hodge filtration

\[
0 \to \omega \to \mathcal{L}_1 \to \omega^{-1} \to 0.
\]

The rigid analytic curve \( C^{\text{rig}} \) associated with \( C \) admits a p-adic uniformization

\[
C^{\text{rig}}(\mathbb{C}_p) \simeq \Gamma \backslash \mathcal{H}_p(\mathbb{C}_p)
\]

for a suitable subgroup \( \Gamma \subseteq \text{SL}_2(\mathbb{Q}_p) \). Modular forms on \( C^{\text{rig}} \) are then \( \Gamma \)-invariant sections of \( \mathcal{H}_p \), and therefore, to define a p-adic Shimura-Maass operator on \( C \) one is naturally led to consider the analogue problem for \( \mathcal{H}_p \).

Let \( C^0 \) denote the \( \mathbb{C}_p \)-vector space of continuous (for the standard p-adic topology on both spaces) \( \mathbb{C}_p \)-valued functions on \( \mathcal{H}_p(\hat{\mathbb{Q}}_p^\text{unr}) \), and let \( \mathcal{A} \) denote the \( \hat{\mathbb{Q}}_p^\text{unr} \)-vector space of rigid analytic global sections of \( \mathcal{H}_p(\hat{\mathbb{Q}}_p^\text{unr}) \). We have a map of \( \hat{\mathbb{Q}}_p^\text{unr} \)-vector spaces \( r : \mathcal{A} \to C^0 \) and, following [Fra11], we denote \( \mathcal{A}^r \) the image of the morphism of \( \mathcal{A} \)-algebras \( \mathcal{A}[X,Y] \to C^0 \) defined by sending \( X \) to the function \( z \mapsto 1/(z - \sigma(z)) \) and \( Y \) to the function \( z \mapsto \sigma(z) \), where \( \sigma : \hat{\mathbb{Q}}_p^\text{unr} \to \hat{\mathbb{Q}}_p^\text{unr} \) is the Frobenius automorphism (note that the function \( z \mapsto z - \sigma(z) \) is invertible on \( \mathcal{H}_p(\hat{\mathbb{Q}}_p^\text{unr}) \)). Denote \( \mathcal{H}_p \) the formal \( \mathbb{Z}_p \)-scheme whose generic fiber is \( \mathcal{H}_p \), let \( \mathcal{H}_p^\text{unr} \) be its base change to \( \hat{\mathbb{Q}}_p^\text{unr} \) and let \( \mathcal{G} \to \mathcal{H}_p^\text{unr} \) be the universal SFD-module. Denote \( \omega_{\mathcal{G}} = e^\vee (\Omega^1_{\mathcal{G}/\mathcal{H}_p^\text{unr}}) \), where \( e^\vee : \mathcal{H}_p^\text{unr} \to \mathcal{G} \) is the zero-section, and let \( \mathcal{L}ie_{\mathcal{G}^\vee} \) be the Lie algebra of the Cartier dual \( \mathcal{G}^\vee \) of \( \mathcal{G} \). Then \( \omega_{\mathcal{G}} \) and \( \mathcal{L}ie_{\mathcal{G}^\vee} \) are locally free \( \mathcal{O}_{\mathcal{H}_p^\text{unr}} \)-modules, dual to each
other and we have the Hodge-Tate exact sequence of $\mathcal{O}_{\tilde{H}^\text{unr}_p}$-modules

$$0 \rightarrow \omega_G \rightarrow \mathcal{H}^1_{\text{dR}}(G/\tilde{H}^\text{unr}_p) \rightarrow \mathcal{L}e_{G^\vee} \rightarrow 0.$$ 

Set $\mathcal{L}^0_G = e \cdot \mathcal{H}^1_{\text{dR}}(G/\tilde{H}^\text{unr}_p)$ and $\omega^0_G = e \cdot \omega_G$. Define $\Lambda^*_G = H^0(\mathcal{H}^\text{unr}_p, \mathcal{L}_G^0)$, $\Lambda^*_G = \Lambda^*_G \otimes_A A^*$, $w_G = H^0(\mathcal{H}^\text{unr}_p, \omega^0_G)$, $w_G^* = w_G \otimes_A A^*$. We have then an injective map of $A^*$-algebras

\begin{equation}
(5) \quad w_G^* \hookrightarrow \Lambda^*_G.
\end{equation}

**Theorem 1.1.** The injection (5) of $A^*$-algebras admits a canonical splitting $\Psi_p^* : \Lambda^*_G \rightarrow w_G^*$.

This is the main result of this paper, Theorem 4.7. We may then attach to $\Psi_p^*$ a $p$-adic Shimura-Maass operator $\Theta^{(r)}_p$. We have the following two corollaries.

**Corollary 1.2.** The $p$-adic Shimura-Maass operator $\delta_{p,k}$ defined by Franc in [Fra11] coincides with the $p$-adic Shimura-Maass operator $\Theta^{(r)}_p$ defined by means of the splitting in Theorem 1.1.

The main tool which is used to prove Theorem 1.1 and Corollary 1.2 is Drinfel’d interpretation of $\mathcal{H}^\text{unr}_p$ as moduli space of special formal modules with quaternionic multiplication; following [Tei89], we call these objects SFD-modules. We study the relative de Rham cohomology of the universal SFD-module $G \rightarrow \tilde{H}^\text{unr}_p$ by means of techniques from [Tei89], [Fal97] and [IS03]. The upshot of our analysis is an explicit description of the Gauss-Manin connection and the Kodaira-Spencer isomorphism for $G \rightarrow \tilde{H}^\text{unr}_p$, once we apply to the relevant sheaves the projector $e$. This detailed study is contained in Section 4 which we believe is of independent interest and is the technical heart of the paper.

For the next corollary, define as in the real analytic case

$$\delta_{p,k}^* = \delta_{p,k+2(r-1)} \circ \delta_{p,k+2(r-2)} \circ \cdots \circ \delta_{k}.$$ 

Moreover, we fix an embedding $\bar{Q} \hookrightarrow \bar{Q}_p$, and we say that $\xi \in \bar{Q}_p$ belongs to $\bar{Q}$ if $\xi$ belongs to the image of this embedding.

**Corollary 1.3.** Let $f$ be a modular form on $C$. Then there exists $t_p \in \mathbb{C}_p^\times$, independent of $f$, such that for every CM point $z \in K \cap \mathcal{H}_p(\tilde{H}^\text{unr}_p)$ we have $\frac{\delta_{p,k}^*(f)(z)}{t_p} \in \bar{Q}$.

As remarked above, this is the main result of Franc thesis [Fra11], which he proves via an explicit approach following Shimura. Instead, we derive this result in Theorem 5.3 from a comparison between the values at CM points of our $p$-adic Shimura-Maass operator $\Theta^{(r)}_p$ and the real analytic Shimura-Maass operator $\Theta^{(r)}$.

We explain now the connection with generalized Heegner cycles. These cycles were introduced in [BDP13] with the aim of studying certain anticyclotomic $p$-adic $L$-functions. Generalized Heegner cycles have been also studied in the context of Shimura curves with good reduction at $p$ by [HB15], and in the context of Mumford curves in [Mas12], [LP19]. In this paper we introduce still another variant of Generalized Heegner cycles. Fix a false elliptic curve $A_0$ with CM by $\mathcal{O}_K$. To any isogeny $\varphi : A_0 \rightarrow A$, where $A$ is a false elliptic curve, we construct a cycle $\Upsilon_{\varphi}$ in the Chow group $\text{CH}^m(A \times A_0)$ of the Chow motive $A \times A_0$, where $m = n/2$ with $n = k - 2$. The work of Brooks [HB15] gives us a projector $\epsilon$ in the ring of correspondences of $X_m = A^m \times A_0^n$, which defines the motive $\mathcal{D} = (X^m, \epsilon)$. The generalised Heegner cycle $\Delta_{\varphi}$ in the image of $\Upsilon_{\varphi}$ in $\text{CH}^m(\mathcal{D})$ via this projector. Let $M_k(\Gamma)$ be the $\mathbb{C}_p$-vector space of rigid analytic quaternionic modular forms of weight $k$ and level $\Gamma$; elements of $M_k(\Gamma)$ are functions from $\mathcal{H}_p(\mathbb{C}_p) = \mathbb{C}_p - \mathbb{Q}_p$ to $\mathbb{C}_p$ which transform under the action of $\Gamma$ by the automorphic factor of weight $k$. We construct a $p$-adic Abel-Jacobi map

$$\text{AJ}_p : \text{CH}^m(\mathcal{D}) \rightarrow (M_k(\Gamma) \otimes \text{Sym}^m eH^1_{\text{dR}}(A_0))^\vee$$
where $^\vee$ denotes $\mathbb{C}_p$-linear dual. If follows from our work that $L^0_\varphi = eH^1_{dR}(G/H^0_p)$ is equipped with two canonical sections $\omega_{\text{can}}$ and $\eta_{\text{can}}$, such that $\omega_{\text{can}}$ is a generator of the invertible sheaf $\omega^0_\varphi$. Let $\omega_f \in H^0_\varphi$ be the $\Gamma$-invariant differential form associated with $f \in M_k(\Gamma)$, and let $F_f$ its Coleman primitive satisfying $\nabla(F_f) = \omega_f$, where $\nabla$ is the Gauss-Manin connection. Denote $\langle . , . \rangle$ is the Poincaré pairing on $\text{Sym}^n eH^1_{dR}(A_z)$, where $A_z$ is the fiber of $A$ at $z$. Define the function
\[ H(z) = \langle F_f(z), \omega^\ell_{\text{can}}(z) \rangle. \]

**Theorem 1.4.** Let $\varphi : A_0 \to A$ be an isogeny and $z_A$ the fiber of $A \to C$. Then for each integer $j = n/2, \ldots, n$ we have
\[ \delta^{n-j}_{p,k}(H_n)(z_A) = AJ_p(\Delta_\varphi)(\omega_f \otimes \omega^j_{\text{can}} \eta^n_{\text{can}}). \]

Theorem [1.3] relates the Shimura-Maass operator with generalised Heegner cycles, and corresponds to Corollary [3.3.

We finally make a remark on $p$-adic $L$-functions. It would be interesting to use our $p$-adic Shimura-Maass operator to construct $p$-adic $L$-functions interpolating special values of the complex $L$-function of $f$ twisted by Hecke characters as in [BDP13], [HB15], [Kri18], [AI19]. We would like to come back to this problem in a future work.

2. **Algebraic de Rham cohomology of Shimura curves**

Throughout this section, let $k \geq 2$ be an even integer and $N \geq 1$ an integer. Fix an imaginary quadratic field $K/\mathbb{Q}$ of discriminant $D_K$ prime to $N$ and factor $N = N^+ \cdot N^-$ by requiring that all primes dividing $N^+$ (respectively $N^-$) split in $K$ (respectively, are inert in $K$). Assume that $N^-$ is a square-free of an odd number of primes, and let $p \nmid N$ be a prime number which is inert in $K$ (thus $N^-p$ is a square-free of an even number of primes). Let $f \in S_k(\Gamma_0(Np))$ be a weight $k$ newform of level $\Gamma_0(Np)$. Fix also embeddings $\overline{Q} \hookrightarrow \mathbb{C}$ and $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ for each prime number $p$.

### 2.1. Quaternion algebras

Fix an indefinite quaternion algebra $B/\mathbb{Q}$ of discriminant $N^-p$, a maximal order $\mathcal{R}_{\text{max}} \subseteq B$ and an Eichler order $\mathcal{R} \subseteq \mathcal{R}_{\text{max}}$ of level $N^+$. Fix isomorphisms $\iota_\ell : B_\ell = B \otimes \mathbb{Q}_\ell \simeq M_2(\mathbb{Z}_\ell)$ for each prime $\ell \nmid N^-p$ such that $\iota_\ell(\mathcal{R}_{\text{max}} \otimes \mathbb{Z}_\ell)$ is the subgroup $M_2(\mathbb{Z}_\ell)$ and moreover for each prime $\ell \mid N^+$ we require that $\iota_\ell(\mathcal{R} \otimes \mathbb{Z}_\ell)$ is the subgroup of $\mathcal{R}^\ell_{\text{max}}$ to consisting of elements $x$ such that $\iota_\ell(x) \equiv \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \pmod{N^+}$ (as a general notation, for any $\mathbb{Z}$-algebra $A$, let $\tilde{A} = A \otimes \mathbb{Z}$, where $\mathbb{Z}$ is the pro finite completion of $\mathbb{Z}$).

We need to fix a convenient basis for the $\mathbb{Q}$-algebra $B$, called Hashimoto model. Denote $M = \mathbb{Q}(\sqrt{p_0})$ the splitting field of the quadratic polynomial $X^2 - p_0$, where $p_0$ if an auxiliary prime number fixed as in [Mor11] §1.1 and [HB15] §2.1, such that:

1. for all primes $\ell$ we have $(p_0, pN^-)\ell = -1$ if and only if $\ell \mid pN^-$, where $(a, b)_\ell$ denotes the Hilbert symbol,
2. all primes $\ell \mid N^+$ are split in the real quadratic field $M = \mathbb{Q}(\sqrt{p_0})$, where $\sqrt{p_0}$ is a square root of $p_0$ in $\mathbb{Q}$.

The choice of $p_0$ fixes a $\mathbb{Q}$-basis of $B$ as in [Has95] §2 given by $\{1, i, j, k\}$ with $i^2 = -pN^-$, $j^2 = p_0$, $k = ij = -ji$, and $1$ the unit of $B$; of course, if $x \in \mathbb{Q}$ we will often just write $x$ for $x \cdot 1$.

### 2.2. Moduli problem

A false elliptic curve $A$ over a scheme $S$ is an abelian scheme $A \to S$ of relative dimension 2 equipped with an embedding $\iota_A : \mathcal{R}_{\text{max}} \to \text{End}_S(A)$. An isogeny of false elliptic curves is an isogeny which commutes with the action of $\mathcal{R}_{\text{max}}$. A full level $N^+$-structure on $A$ is an isomorphism of group schemes $\alpha_A : A[N^+] \simeq (\mathcal{R}_{\text{max}} \otimes \mathbb{Z}(\mathbb{Z}/N^+\mathbb{Z}))S$, where for any group $G$ we denote $G_S$ the constant group scheme $G$ over $S$. Note that
The Gauss-Manin connection is then defined as the differential $d$ from the Gauss-Manin connection as follows. Let in the long exact sequence of derived functors obtained from (6). It can also be reconstructed

$$E = \pi \text{ for each integer } i,$$

$$\text{relative differential forms for the morphism } \phi,$$}

We first recall some general notation. For any morphism of schemes $\phi : X \to S$, denote $(\Omega^*_{X/S}, d_X^*)$, or simply $\Omega^*_{X/S}$ understanding the differentials $d^*_X$, the complex of sheaves of relative differential forms for the morphism $\phi$. For a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules over a scheme $X$, we denote $\mathcal{F}^i$ its $\mathcal{O}_X$-linear dual and, for a positive integer $k$, we put $\mathcal{F}^{i\otimes k} = \mathcal{F} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{F}$ ($k$ factors). If $\mathcal{F}$ is invertible, we denote $\mathcal{F}^{-1}$ its inverse, and in this case for $k$ a negative integer, $\mathcal{F}^{i\otimes k}$ denotes $(\mathcal{F}^{-1})^{\otimes k}$ as usual.

Fix a field $F$ of characteristic zero. The relative de Rham cohomology bundle for the morphism $A_F \to C_F$ is defined by

$$H^q_{dR}(A_F/C_F) = \mathbb{R}^q \pi^* \left( \Omega^*_{A_F/C_F} \right).$$

We first recall the construction of the Gauss-Manin connection. We have a canonical short exact sequence of locally free sheaves

$$(6) \quad 0 \to \pi^*_F \left( \Omega^1_{\mathcal{O}_F/F} \right) \to \Omega^1_{A_F/C_F} \to \Omega^1_{A_F/F} \to 0$$

(the exactness is because $\pi^*_F$ is smooth). This exact sequence induces maps

$$\Omega^*_{A_F/F} \otimes_{\mathcal{O}_{A_F}} \pi^*_F \left( \Omega^i_{C_F/F} \right) \to \Omega^*_{A_F/F}$$

for each integer $i$, defining a filtration $F^i \Omega^*_{A_F/F} = \text{Im}(\psi^i_{A_F/F})$ on $\Omega^*_{A_F/F}$ with associated graded objects

$$\text{gr}^i \left( \Omega^*_{A_F/F} \right) = \Omega^*_{A_F/F} \otimes_{\mathcal{O}_A} \pi^*_F \left( \Omega^i_{C_F/F} \right).$$

Let $E^{i,q}_p$ denote the spectral sequence associated with this filtration. The $E^{i,q}_p$ terms are then given by $E^{i,q}_1 = \mathbb{R}^{p+q} \pi^*_{F*} \left( \text{gr}^p \left( \Omega^*_{A_F/F} \right) \right)$. Since $\Omega^p_{C_F/F}$ is locally free, and the differentials in the complex $\pi^*_F \left( \Omega^p_{C_F/F} \otimes_{\mathcal{O}_A} \Omega^*_{A_F/C_F} \right)$ are $\pi^{-1}(\mathcal{O}_{C_F})$-linear, one can show that

$$E^{i,q}_1 \simeq \Omega^p_{C_F/F} \otimes_{\mathcal{O}_{C_F}} H^q_{dR}(A_F/C_F)$$

([KO68 (7)]). The Gauss-Manin connection

$$\nabla : H^i_{dR}(A_F/C_F) \to \Omega^1_{C_F/F} \otimes_{\mathcal{O}_{C_F}} H^i_{dR}(A_F/C_F)$$

is then defined as the differential $d^{i,1}_{\nabla_1} : E^{i,1}_0 \to E^{i,1}_1$ in this spectral sequence.

We now recall various descriptions of the Kodaira-Spencer map. It is defined to be the boundary map

$$\text{KS}_{A_F/C_F} : \pi^*_F \left( \Omega^1_{A_F/C_F} \right) \to \mathbb{R}^1 \pi^*_F \left( \pi^*_F \left( \Omega^1_{C_F/F} \right) \right)$$

in the long exact sequence of derived functors obtained from [KO68]. It can also be reconstructed from the Gauss-Manin connection as follows. Let $\pi^*_F : A^1_F \to C_F$ denote the dual abelian
Kodaira-Spencer map can also be seen as a map of $O$ in which the first and the last map come from the Hodge exact sequence (7). Therefore the same symbol, $\text{(2.4).}$ Idempotents and line bundles.

Manin connection as the composition $(R → 0)$ then $\bar{x}$ with the involution $x → x^t$ of $R_{\text{max}}$, defined by $x^t = i^{-1} x$ (as usual, if $x = a + bi + cj + dk$, then $\bar{x} = a - bi - cj - dk$). Using the principal polarization and the isomorphism between $R^1 \pi_{F*}(\Omega^1_{A_F/F})$ and the tangent bundle of $A^\vee_F$, the Hodge exact sequence can be written as

$$0 \to \pi_{F*}(\Omega^1_{A_F/F}) \to \mathcal{H}^1_{\text{dr}}(A_F/C_F) \to (\pi_{F*}(\Omega^1_{A_F/F}))^\vee \to 0 \tag{7}$$

(cf. [Mor11 (2.2)], [HB15 §2.6]). The Kodaira-Spencer map can be defined using the Gauss-Manin connection

$$\text{KS}_{A_F/C_F} : \pi_{F*}(\Omega^1_{A_F/F}) \to \mathcal{H}^1_{\text{dr}}(A_F/C_F) \to \mathcal{H}^1_{\text{dr}}(A_F/C_F) \otimes \mathcal{O}_{C_F} \Omega^1_{C_F/F} \to (\pi_{F*}(\Omega^1_{A_F/F}))^\vee \otimes \mathcal{O}_{C_F} \Omega^1_{C_F/F}$$

in which the first and the last map come from the Hodge exact sequence (7). Therefore the Kodaira-Spencer map can also be seen as a map of $\mathcal{O}_{C_F}$-modules, denoted again with the same symbol,

$$\text{KS}_{A_F/C_F} : \pi_{F*}(\Omega^1_{A_F/F}) \otimes^2 \to \Omega^1_{C_F/F}. \tag{2.4}$$

2.4. Idempotents and line bundles. Let $e = \frac{1}{2} \left( 1 \otimes 1 + \frac{1}{p_0} j \otimes \sqrt{p_0} \right) \in R_M = R_{\text{max}} \otimes \mathbb{Z} \mathcal{O}_M[1/(2p_0)]$ be the idempotent in [Mor11 (1.10)], [HB15 §2.1], where $\mathcal{O}_M$ is the ring of integers of $M$. We have an isomorphism $\iota_M : B \otimes \mathbb{Q} M \simeq M_2(M)$.

Suppose we have an embedding $M → F$, allowing us to identify $M$ with a subfield of $F$; in the cases we are interested in, either $F \subseteq \mathbb{Q}$ (and then we require that $F$ contains $M$), or $F = \mathbb{C}$ (and then we view $M \hookrightarrow \mathbb{C}$ via the fixed embedding $\mathbb{Q} \hookrightarrow \mathbb{C}$) or $F \subseteq \mathbb{Q}_p$ (and then we require that $F$ contains the image of $M$ via the fixed embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$).

Since $M$ is contained in $F$, we have an action of $R_M$ on the sheaves $\pi_{F*}(\Omega^1_{A_F/F})$ and $\mathcal{H}^1_{\text{dr}}(A_F/C_F)$, and we may therefore define the invertible sheaf of $\mathcal{O}_{C_F}$-modules

$$\omega_F = e \cdot \pi_{F*}(\Omega^1_{A_F/F}) \tag{8}$$

and the sheaf of $\mathcal{O}_{C_F}$-modules

$$\mathcal{L}_F = e \cdot \mathcal{H}^1_{\text{dr}}(A_F/C_F). \tag{9}$$

Using that $e$ is fixed by the Rosati involution, the Hodge exact sequence (7) becomes

$$0 \to \omega_F \to \mathcal{L}_F \to \omega_F^{-1} \to 0 \tag{10}$$

(see [HB15 §2.6] for details). For any integer $n \geq 1$, define

$$\mathcal{L}_{F,n} = \text{Sym}^n(\mathcal{L}_F).$$

The Gauss-Manin connection is compatible with the quaternionic action ([Mor11 Proposition 2.2]). Therefore, restricting to $\mathcal{L}_{F,1}$ and using the Leibniz rule (see for example [HB15 §3.2]), the Gauss-Manin connection defines a connection

$$\nabla_n : \mathcal{L}_{F,n} → \mathcal{L}_{F,n} \otimes \Omega^1_{C_F/F}.$$
By [Mor11] Theorem 2.5, restricting the Kodaira-Spencer map to $\omega_{F}^{\otimes 2}$ gives an isomorphism

$$KS_F : \omega_{F}^{\otimes 2} \cong \Omega_{C_F/F}^1.$$ 

We may then define a map $\tilde{\nabla}_n : \mathcal{L}_{F,n} \to \mathcal{L}_{F,n+2}$ by the composition

$$(11)$$

$$\tilde{\nabla}_n : \mathcal{L}_{F,n} \xrightarrow{\nabla_n} \mathcal{L}_{F,n} \otimes \mathcal{O}_{\mathcal{C}_F/F} \xrightarrow{id \circ KS_F^{-1}} \mathcal{L}_{F,n} \otimes \mathcal{O}_{\mathcal{C}_F/F} \omega_{F}^{\otimes 2} \xrightarrow{\mathcal{C}_F/F \otimes \mathcal{O}_{\mathcal{C}_F/F}} \mathcal{L}_{F,n} \otimes \mathcal{O}_{\mathcal{C}_F/F} \mathcal{L}_{F,2} \xrightarrow{\mathcal{L}_{F,n} \otimes \mathcal{O}_{\mathcal{C}_F/F} \mathcal{L}_{F,n+2}}$$

where the last map is the product map in the symmetric algebras.

2.5. Algebraic modular forms. For any $F$-algebra $R$, we define the $R$-algebra

$$S_{k}^{\text{alg}}(V_1(N^+), R) = H^0(C_R, \omega_{R}^{\otimes k})$$

of algebraic modular forms of weight $k$ and level $V_1(N^+)$ over $R$. One can show ([HB15] §3.1) that the $R$-algebra $S_{k}^{\text{alg}}(V_1(N^+), R)$ can be alternatively described in modular terms. Let $R'$ be an $R$ algebra. A test triple over $R'$ is a triplet $(A', t', \omega')$ consisting of a false elliptic curve $A'/R'$, a $V_1(N^+)$-level structure $t'$ and a global section $\omega'$ of $\omega_{A'/R'}$. An isomorphism of test triples $(A', t', \omega')$ and $(A'', t'', \omega'')$ is an isomorphism of false elliptic curves $\phi : A' \to A''$ such that $\phi(t') = t''$ and $\phi^*(\omega'') = \omega'$. A test pair over $R'$ is a pair $(A', t')$ obtained from a test triple by forgetting the datum of the global section. Then one can identify global sections of $\omega_{R}^{\otimes k}$ with:

1. A rule $F$ which assigns, to each $R$-algebra $R'$ and each isomorphism class of test triplets $(A', t', \omega')$ over $R'$, an element $F(A', t', \omega') \in R'$, subject to the base change axiom (for all maps of $R$-algebras $\phi : R' \to R''$, we have $F(A', t', \phi^*(\omega')) = F(A'', \phi(t'), \omega'')$, where $A'$ is the base change of $A''$ via $\phi$) and the weight $k$ condition ($F(A', t', \lambda \omega') = \lambda^{-k} F(A', t', \omega')$ for any $\lambda \in (R')^\times$) ([HB15] Definition 3.2)).

2. A rule $F$ which assigns to each $R$-algebra $R'$ and each isomorphism class of test pairs $(A', t')$ over $R'$, a translation invariant section $F(A', t') \in \omega_{A'/R'}^{\otimes k}$ subject to the base change axiom (for all maps of $R$-algebras $\phi : R' \to R''$, we have $F(A', t') = \phi^*(F(A'', \phi(t'))$, where $A'$ is the base change of $A''$ via $\phi$) ([HB15] Definition 3.3)).

Let us make the relations between these definition more explicit ([HB15] page 4193]). Given a global section $f \in H^0(C_R, \omega_{R}^{\otimes k})$, we get a function as in (2) above associating to each test pair $(A', t')$ over $R'$ the point $x_{(A', t')} \in C_R(R')$, and taking the value of $f$ at $x_{(A', t')}$. If $F$ is as in (2), we get a function $G$ on test triples $(A', t', \omega')$ over $R'$ as in (1) by the formula $F(A', t') = G(A', t', \omega') \omega^{\otimes k}$ where $\omega \in \omega_{A'/R'}$ is the choice of any translation invariant global section.

3. Special values of $L$-series

In this section we review the work of Brooks [HB15] expressing special values of certain $L$-functions of modular forms in terms of CM-values of the Shimura-Maass operator applied to the modular form in question.

3.1. The real analytic Shimura-Maass operator. We denote $(X, O_X) \sim (X^{an}, O_X^{an})$ the analytification functor which takes a scheme of finite type over $\mathbb{C}$ to its associated complex analytic space ([Ser56] §2). For each sheaf $F$ of $O_X$-modules on $X$, we also denote $F^{an}$ the analytification of $F$, and for each morphism $\varphi : F \to G$ of $O_X$-modules, we let $\varphi^{an} : F^{an} \to G^{an}$ the corresponding morphism of analytic sheaves ([Ser56] §3). If $(X, O_X)$ is an analytic space, we denote $O_X^{an}$ the ring of real analytic functions on $X$; this is a sheaf of $O_X$-modules, and for any sheaf $F$ of $O_X$-modules, we let $F^{an} = F \otimes O_X^{an}$; when $F = F^{an}$, we simplify the notation by writing $F^{an}$ instead of $(F^{an})^{an}$.
Since \( C_C \) is proper and smooth over \( \mathbb{C} \), the analytification functor \( F \sim F^{an} \) induces an equivalence of categories between the category of coherent sheaves \( C_C \) and the category of analytic coherent sheaves of \( O^{an}_{C_C} \)-modules ([Ser56 Th. 2, 3]). Also, the analytic sheaf obtained from the sheaf of algebraic de Rham cohomology \( H^1_{dR}(A_C/C_C) \) coincides with the derived functor \( \mathbb{R}^1\pi_{C,*}(\Omega^1_{A_C/C_C}) \) in the category of analytic sheaves over \( C_C^{an} \) ([Ser56 Theorem 1]).

Hodge theory gives a splitting
\[
H^1_{dR}(A_C/C_C)^{r-an} \rightarrow \left( \pi_{C,*}(\Omega^1_{A_C/C_C}) \right)^{r-an}
\]
of the corresponding Hodge exact sequence of real analytic sheaves obtained from [7]. Since this splitting is the identity on the image of \( \left( \pi_{C,*}(\Omega^1_{A_C/C_C}) \right)^{r-an} \) in \( H^1_{dR}(A_C/C_C)^{r-an} \), it gives rise to a map \( \Psi : \mathcal{L}_{C,1}^{r-an} \rightarrow \mathcal{L}_n^{r-an} \) (cf. [Mor11 Proposition 2.8]). We may then consider the induced maps \( \Psi_{\infty,n} : \mathcal{L}_{\infty,n}^{r-an} \rightarrow \mathcal{L}_{C,n+2}^{r-an} \) for any integer \( n \geq 1 \). Further, the map \( \nabla_n \) gives rise to a map \( \nabla_n^{\infty,n} : \mathcal{L}_{C,n}^{r-an} \rightarrow \mathcal{L}_{C,n+2}^{r-an} \) of real analytic sheaves. The composition
\[
\Theta_{\infty,n} : \left( \mathcal{L}_{C,n}^{r-an} \right)^{r-an} \xrightarrow{\nabla_n^{\infty,n}} \mathcal{L}_{C,n}^{r-an} \xrightarrow{\Psi_{\infty,n}} \left( \mathcal{L}_{C,n+2}^{r-an} \right)^{r-an}
\]
is the real-analytic Shimura-Maas operator.

The effect of \( \Theta_{\infty,n} \) on modular forms is described in [HB15 Proposition 3.4] and [Mor11 Proposition 2.9]. Denote \( \Gamma = \Gamma_1(N^+) \) the subgroup of \( B^\infty \cap V_1(N^+) \) consisting of elements of norm equal to 1. Fix an isomorphism \( B \otimes \mathbb{Q} \mathbb{R} \simeq M_2(\mathbb{R}) \) and denote \( \Gamma_\infty \) the image of \( \Gamma \) in \( \text{GL}_2(\mathbb{R}) \). Let \( S_k(\Gamma_\infty) \) denote the \( \mathbb{C} \)-vector space of \textit{holomorphic modular forms} of weight \( k \) and level \( \Gamma_\infty \) consisting of those holomorphic functions on \( \mathcal{H}_\infty \), the complex upper half plane, such that \( f(\gamma(z)) = j(\gamma, z)^k f(z) \) for all \( \gamma \in \Gamma_\infty \); here \( \Gamma_\infty \) acts on \( \mathcal{H}_\infty \) by fractional linear transformations via the map \( B \mapsto B \otimes \mathbb{Q} \mathbb{R} \simeq M_2(\mathbb{R}) \). We have (cf. [HB15 §2.7])
\[
S_k(\Gamma_\infty) \simeq H^0\left( C_C^{an}, (\mathcal{L}_C^{r-an})^{r-an} \right).
\]

Define the space \( S_{k}^{r-an}(\Gamma_\infty) \) of real analytic modular forms of level \( \Gamma_\infty \) and weight \( k \) to be the \( \mathbb{C} \)-vector space of real analytic functions \( f : \mathcal{H}_\infty \rightarrow \mathbb{C} \) such that \( f(\gamma(z)) = j(\gamma, z)^k f(z) \) for all \( \gamma \in \Gamma_\infty \). One then has
\[
S_{k}^{r-an}(\Gamma_\infty) \simeq H^0\left( C_C^{an}, (\mathcal{L}_C^{r-an})^{r-an} \right).
\]
The operator \( \Theta_{\infty,k} \) gives then rise to a map \( \delta_{\infty,k} : S_k(\Gamma_\infty) \rightarrow S_{k+2}(\Gamma_\infty) \) and we have
\[
\delta_{\infty,k}(f(z)) = \frac{1}{2\pi i} \left( \frac{d}{dz} + \frac{k}{z + \bar{z}} \right) f(z).
\]

### 3.2. CM points and triples

Fix an embedding \( \varphi : K \hookrightarrow \mathbb{B} \) there exists a unique \( \tau \in \mathcal{H} \) such that \( \iota_\infty(\varphi(K^\infty))(\tau) = \tau \). The additive map \( K \rightarrow \mathbb{C} \) defined by \( \alpha \mapsto j(\iota_\infty(\varphi(\alpha)), \tau) \) gives an embedding \( K \rightarrow \mathbb{C} \); we say that \( \varphi \) is \textit{normalized} if \( \alpha \mapsto j(\iota_\infty(\varphi(\alpha)), \tau) \) is the identity (with respect to our fixed embedding \( \mathbb{Q} \hookrightarrow \mathbb{C} \)).

We say that \( \tau \in \mathcal{H} \) is a \textit{CM point} if there exists an embedding \( \varphi : K \hookrightarrow \mathbb{B} \) which has \( \tau \) as fixed point as above, and that a CM point \( \tau \) is \textit{normalized} if \( \varphi \) is normalized. Finally, we say that a CM point \( \tau \in \mathcal{H} \) is a \textit{Heegner point} if \( \varphi(\mathcal{O}_K) \subseteq \mathcal{R} \) ([HB15 §2.4 and page 4188]).

Fix a CM point \( \tau \) corresponding to an embedding \( \varphi : K \hookrightarrow \mathbb{B} \). Let \( \alpha \) be an integral ideal of \( \mathcal{O}_K \), and define the \textit{Ramakrishnan-Ideal} \( \mathcal{R}_\alpha = \mathcal{R}_{\max} \cap \varphi(\alpha) \). This ideal is principal, generated by an element \( \alpha = \alpha_0 \in \mathcal{B} \). Right multiplication by \( \alpha \) gives an isogeny \( A_\tau \rightarrow A_0 \), whose kernel is \( A_\tau[a] \). Let \( \Gamma_{\max} \) be the subgroup of \( \tau^{\infty} \) consisting of elements of norm equal to 1. The image of \( \alpha \tau \) by the canonical projection map \( \rho_{\max} : \mathcal{H} \rightarrow \Gamma_{\max} \mathcal{H} \) does not depend on the choice of the representative \( \alpha \), and therefore one may write \( A_{\alpha \tau} \) for the corresponding abelian
surface. Shimura’s reciprocity law states that $\rho_{\text{max}}(\tau)$ is defined over the Hilbert class field $H$ of $K$, and that $\rho_{\text{max}}(\tau)^{a^{-1}, H/K} = \rho_{\text{max}}(a \ast \tau)$, where $(a^{-1}, H/K)$ denotes the Artin symbol.

Fix a primitive $N^+$-root of unity $\zeta$. Fix a normalized Heegner point $\tau$, and fix a point $P_{\tau} \in A_v[N^+]$ of exact order $N^+$ such that $e \cdot P = P$. Let $(A_{\tau}, P_{\tau})$ denote the point on $\mathcal{C}(F)$ corresponding to the level structure $\mu_{N^+} \times \mu_{N^+} \simeq \mathbb{Z}/N^+\mathbb{Z} \times \mathbb{Z}/N^+\mathbb{Z} \to A_v[N^+]$ which takes $(1, 0) \in \mathbb{Z}/N^+\mathbb{Z} \times \mathbb{Z}/N^+\mathbb{Z}$ to $P_{\tau}$. A CM triple is an isomorphism class of triples $(A_{\tau}, P_{\tau}, \omega_{\tau})$ with $(A_{\tau}, P_{\tau})$ as above and $\omega_{\tau} \in e \cdot \Omega_{A_{\tau}/F}$ non vanishing.

There is an action of $\text{Cl}(\mathcal{O}_K)$ on the set of CM triples, given by

$$a * (A_{\tau}, P_{\tau}, \pi^s(\omega)) = (A_{\tau}/A_{\tau}[a], \pi(P_{\tau}), \omega)$$

where $\pi : A_{\tau} \to A_{\tau}/A_{\tau}[a]$ is the canonical projection.

3.3. Special value formulas. Fix a CM triple $(A, P, \omega) = (A_{\tau}, P_{\tau}, \omega_{\tau})$ with $\omega$ defined over $H$, the Hilbert class field of $K$; recall that $A$ is also defined over $H$, while in general $P$ is only defined over a field $L$ as in \[22\]

The complex structure $J_{\tau}$ on $M_2(\mathbb{R})$ defines a differential form $\omega_C = J_{\tau}^*(2\pi idz_1)$, and let $\Omega_{\infty} \in \mathbb{C}$ be define by $\omega = \Omega_{\infty} \cdot \omega_C$; clearly, different choices of $\omega$ correspond to changing $\Omega$ by a multiple in $H$.

We now let $f$ be a modular form of weight $k$, level $\Gamma_1(N^+) \cap \Gamma_0(N^-)$, and character $\varepsilon_f$, and let $f^{\text{JL}}$ be the modular form on the Shimura curve $\mathcal{C}_C$ associated with $f$ by the Jacquet-Langlands correspondence. We can normalise the choice of $f^{\text{JL}}$ so that the ration $\langle f, f \rangle / \langle f^{\text{JL}}, f^{\text{JL}} \rangle$ belongs to $K$ (\[[H\text{B}15, \S 2.7 \text{ and page 4232}]]).

Let $\Sigma^{(2)}$ be the set of Hecke characters $\chi$ of $K$ of infinite type $(\ell_1, \ell_2)$ with $\ell_1 \geq k$ and $\ell_2 \leq 0$. We say that $\chi \in \Sigma^{(2)}$ is central critical if $\ell_1 + \ell_2 = k$, so that the infinite type of $\chi$ is $(k + j, -j)$ for some integer $j \geq 0$. Denote $\Sigma^{(\text{cc})}$ the subset of $\Sigma^{(2)}$ consisting of central critical characters.

For each positive integer $j$, let $\delta_{\infty, j} : S_{k, \text{an}}(\Gamma_{\infty}) \to D_{k+2j}^{\text{an}}(\Gamma_{\infty})$ denote the $j$-th iterate of the Shimura-Mass operator defined by

$$\delta_{\infty, k} = \delta_{\infty, k+2(j-1)} \circ \cdots \circ \delta_{\infty, k+2} \circ \delta_{\infty, k}.$$

For any Hecke character, one may consider the $L$-function $L(f, \chi^{-1}, s)$, and for $\chi \in \Sigma^{(2)}$ central critical define the algebraic part $L_{\text{alg}}(f, \chi^{-1})$ of its special value at $s = 0$ as in \[[H\text{B}15, \text{Proposition 8.7}]]]. By \[[H\text{B}15, \text{Proposition 8.7}]]], if $\chi \in \Sigma^{(2)}_{\text{cc}}$ then $L_{\text{alg}}(f, \chi^{-1}) \in \mathbb{Q}$, and we have

$$L_{\text{alg}}(f, \chi^{-1}) = \left( \sum_{a \in \text{Cl}(\mathcal{O}_K)} \chi_j^{-1}(a) \cdot \delta_{\infty, j}^{(2)}(f^{\text{JL}})(a \ast (A, t, \omega)) \right)^2$$

where $\chi_j = \chi \cdot \text{nr}^{-j}$ and $\text{nr}$ is the norm map on ideals of $\mathcal{O}_K$. In this formula we view the real analytic modular $\delta_{\infty, k}(f^{\text{JL}})$ as a function on test triplets, as in \[[H\text{B}15, \text{Proposition 8.5}]] via \[[12]] (see also the discussion in \[[B\text{D}P\text{T}13, \text{page 1094}]] in the GL$_2$ case).

For each ideal class $a$ in $\text{Cl}(\mathcal{O}_K)$, let $a_0$ be the corresponding element in $\mathcal{B}$, as in \[32\]. Then using the dictionary between real analytic forms as functions on $\mathcal{H}$ or functions on test triplets, and recalling that $A = A_{\tau}$ for a normalized Heegner point $\tau$, we have

$$L_{\text{alg}}(f, \chi^{-1}) = \left( \Omega_{\infty}^{k+2j} \cdot \sum_{a \in \text{Cl}(\mathcal{O}_K)} \chi_j^{-1}(a) \cdot \delta_{\infty, j}^{(2)}(f^{\text{JL}})(a_0 \cdot \tau) \right)^2.$$
4. The Shimura-Maass operator on the $p$-adic upper half plane

In this Section we define a $p$-adic Shimura-Maass operator in the context of Drinfel’d upper half plane. These results will be used in the next section to define a $p$-adic Shimura-Maass operator on Shimura curves, whose values at CM points will be compared with their complex analogue. As in the complex case, we will see that this operator plays a special role in defining $p$-adic $L$-functions.

Let $\mathcal{H}_p$ denote Drinfel’d $p$-adic upper half plane; this is a $\mathbb{Z}_p$-formal scheme, and we denote $\mathcal{H}_p$ its generic fiber, which is a $\mathbb{Q}_p$-rigid space ([BC91 Chapitre I]).

4.1. Drinfel’d Theorem. Denote $D$ the unique division quaternion algebra over $\mathbb{Q}_p$, and let $\mathcal{O}_D$ be its maximal order. The field $\mathbb{Q}_p$ can be embedded in $D$, and in the following we will see it as a maximal commutative subfield of $D$ without explicitly mentioning it. Let $\sigma$ denote the absolute Frobenius automorphism of $\text{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p)$. If $\mathcal{O}_D$ is a $\mathbb{Z}_p$-module over $\mathbb{Q}_p^{ur}/\mathbb{Q}_p$, then $D = \mathbb{Q}_p^{ur}/\mathbb{Q}_p$. We will denote $x \mapsto \bar{x}$ the restriction of $\sigma$ to $\text{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p)$.

For any $\mathbb{Z}_p$-algebra $B$, a formal $\mathcal{O}_D$-module over $B$ is a commutative 2-dimensional formal group $G$ over $B$ equipped with an embedding $\iota_G: \mathcal{O}_D \rightarrow \text{End}(G)$. A formal $\mathcal{O}_D$-module is said to be special if for each geometric point $P$ of $\text{Spec}(B/pB)$, the representation of $\mathcal{O}_D/\mathcal{O}_D$ over the tangent space $\text{Lie}(G_P)$ of $G_P = G \times k_P$ is one of the two distinct characters of $\mathcal{O}_D/\mathcal{O}_D$, where $k_P$ is the residue field of $P$; see [Tei89, Definition 1] for more details on this definition. By an SFD-module over $B$, we mean a special formal $\mathcal{O}_D$-module over $B$. If $G$ is a SFD-module over $B$, we denote $\mathcal{M}(G)$ the (covariant) Cartier-Dieudonné module of $G$ ([BC91 Chapitre II, §1]); we also denote $\mathcal{F}_G$ and $\mathcal{V}_G$ (or simply $\mathcal{F}$ and $\mathcal{V}$ when there is no confusion) the Frobenius and Verschiebung endomorphisms of $\mathcal{M}(G)$. If $B$ is a $\mathbb{Z}_p$-algebra, and $G$ a formal $\mathcal{O}_D$-module, then we may define

$$\text{Lie}^0(G) = \{m \in \text{Lie}(G): \iota_G(a) = am, a \in \mathbb{Z}_p\},$$

and, since $G$ is special, both $\text{Lie}^0(G)$ and $\text{Lie}^1(G)$ are free $B$-modules of rank 1, (recall that $\bar{x} = \sigma(x)$, so $x \mapsto \bar{x}$ is the non-trivial automorphism of $\text{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p)$). Moreover, $\mathcal{M}(G)$ is also equipped with a graduation $\mathcal{M}(G) = M^0(G) \oplus M^1(G)$ where

$$M^0(G) = \{m \in \mathcal{M}(G): \iota_G(a) = am, a \in \mathbb{Z}_p\},$$

$$M^1(G) = \{m \in \mathcal{M}(G): \iota_G(a) = \bar{a}m, a \in \mathbb{Z}_p\}.$$

Fix a SFD-module $\Phi = G \times G$ over $\mathbb{F}_p$, where $G$ is the reduction modulo $p$ of a Lubin-Tate formal group $\tilde{E}$ of height 2 over $\mathbb{Z}_p^{ur}$, the completion of the valuation ring of the maximal unramified extension $\mathbb{Z}_p^{ur}$ of $\mathbb{Z}_p$; so $\tilde{E}$ is the formal group of a supersingular elliptic curve $E$ over $\mathbb{Z}_p^{ur}$ (see [Tei89] Definition 9 and Remark 27]). The Dieudonné module $\mathcal{M}(\Phi)$ of $\Phi$ is the $\mathbb{Z}_p^{ur}[\mathcal{F}, \mathcal{V}]$-module with $\mathcal{V}$-basis $g^0$ and $g^1$, satisfying the relations $\mathcal{F}(g^0) = \mathcal{V}(g^0)$ and $\mathcal{F}(g^1) = \mathcal{V}(g^1)$. The quaternionic order $\mathcal{O}_D$ acts via the rules $\mathcal{F}(g^0) = \mathcal{V}(g^1)$, $\mathcal{F}(g^1) = \mathcal{V}(g^0)$ and $a(g^0) = ag^0$, $a(g^1) = \bar{a}g^1$ for $a \in \mathbb{Z}_p$, (viewed inside $\mathcal{O}_D$ by the fixed embedding $\mathbb{Q}_p^{ur} \hookrightarrow D$), where $a \mapsto \bar{a}$ is the non-trivial automorphism of $\text{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p)$. By [Tei89 Corollary 30], $\eta^0(\Phi)$ is generated over $\mathbb{Z}_p$ by $[g^0, 0]$ and $[\mathcal{V}(g^1), 0]$, and $\eta^1(\Phi)$ is generated over $\mathbb{Z}_p$ by $[g^1, 0]$ and $[\mathcal{V}(g^0), 0]$.

Let $\text{Nilp}$ denote the category of $\mathbb{Z}_p$-algebras in which $p$ is nilpotent. Denote SFD the functor on $\text{Nilp}$ which associates to each $B \in \text{Nilp}$ the set SFD($B$) of isomorphism classes of triples $(\psi, G, \rho)$ where

1. $\psi: \mathbb{F}_p \rightarrow B/pB$ is an homomorphism,
2. $G$ is a SFD-module over $B$ of height 4,
(3) \( \rho : \psi_* \Phi \to G_{B/pB} = G \otimes_B B/pB \) is a quasi-isogeny of height 0, called rigidification. See \cite{Tei89} page 663 or \cite{BC91} Chapitre II (8.3) for more details on the definition of the functor SFD.

Drinfel’d shows in \cite{Dri76} that the functor SFD is represented by the \( \mathbb{Z}_p \)-formal scheme

\[
\hat{H}_p^{\text{unr}} = \hat{H}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\text{unr}}
\]

(see \cite{Tei89} Theorem 28, \cite{BC91} Chapitre II (8.4)). Note that \( \hat{H}_p^{\text{unr}} \), considered as \( \mathbb{Z}_p^{\text{unr}} \)-formal scheme, represents the restriction \( \text{SFD} \) of SFD to the category \( \text{Nilp} \) of \( \mathbb{Z}_p^{\text{unr}} \)-algebras in which \( p \) is nilpotent (cf. \cite{BC91} Chapitre II, §8). Unless otherwise stated, we will see \( \hat{H}_p^{\text{unr}} \) as a \( \mathbb{Z}_p^{\text{unr}} \)-formal scheme.

For later use, we review some of the steps involved in the proof of Drinfel’d Theorem. The crucial step is the interpretation of the \( \mathbb{Z}_p \)-formal scheme \( \hat{H}_p \) as the solution of a moduli problem. For \( B \in \text{Nilp} \), a compatible data on \( S = \text{Spf}(B) \) consists of a quadruplet \((\eta, T, u, \rho)\) where

1. \( \eta = \eta^0 \oplus \eta^1 \) is a sheaf of flat \( \mathbb{Z}/2\mathbb{Z} \)-graded \( \mathbb{Z}_p[\Pi] \)-modules on \( S \),
2. \( T = T^0 \oplus T^1 \) is a \( \mathbb{Z}/2\mathbb{Z} \)-graded sheaf of \( \mathcal{O}_S[\Pi] \)-modules with \( T^1 \) invertible,
3. \( u : \eta \to T \) is a homogeneous degree zero map such that \( u \otimes 1 : \eta \otimes_{\mathbb{Z}_p} \mathcal{O}_S \to T \) is surjective,
4. \( \rho : (\mathbb{Q}_p^2)_S \to \eta_0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) is a \( \mathbb{Q}_p \)-linear isomorphism, which satisfy natural compatibilities, denoted \((C1), (C2), (C3)\) in \cite{Tei89} page 652, to which we refer for details. The first step in Drinfel’d work is to show that the \( \mathbb{Z}_p \)-formal scheme \( \hat{H}_p \) represents the functor which associates to each \( B \in \text{Nilp} \) the set of admissible quadruplets over \( B \). To each compatible data \( D = (\eta, T, u, \rho) \) on \( S \) one associates a \( S \)-valued point \( \Psi : S \to \hat{H}_p \) of \( \hat{H}_p \), as explained in \cite{Tei89} pages 652-655. The second step to prove the representability of \( \text{SFD} \) is to associate with any \( B \in \text{Nilp} \) and \( X = (\psi, G, \rho) \in \text{SFD}(B) \) a quadruplet \((\eta_X, T_X, u_X, \rho_X)\) which corresponds to an \( S \in \text{Spf}(B) \)-valued point on \( \hat{H}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\text{unr}} \).

We finally discuss rigid analytic parameters (\cite{Tei89}). With an abuse of notation, let SFD be the functor from the category pro-Nilp of projective limits of objects in Nilp associated with SFD. In \cite{Tei89} Def. 10, Teitelbaum introduces a function

\[
(13) \quad z_0 : \text{SFD}(\mathbb{Z}_p^{\text{unr}}) \to \hat{H}_p(\mathbb{Z}_p^{\text{unr}})
\]

such that the map \( X = (\psi, G, \rho) \mapsto (z_0(X), \psi) \) gives a bijection between \( \text{SFD}(\mathbb{Z}_p^{\text{unr}}) \) and \( (\hat{H}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\text{unr}})(\mathbb{Z}_p^{\text{unr}}) \), which we identify with the set \( \hat{H}_p(\mathbb{Z}_p^{\text{unr}}) \times \text{Hom}(\mathbb{Z}_p^{\text{unr}}, \mathbb{Z}_p^{\text{unr}}) \). We call
the map $X \mapsto z_0(X)$ a rigid analytic parameter on SFD. If we let \(\text{pro-Nilp}\) the category of projective limits of objects in \(\text{Nilp}\), and we still denote \(\text{SFD}\) the restriction of \(\text{SFD}\) to \(\text{pro-Nilp}\), this implies that the map $X = (\psi, G, \rho) \mapsto z_0(X)$ gives a bijection between \(\text{SFD}(\mathbb{Z}^{unr}_p)\) and \(\mathcal{H}_p(\mathbb{Z}^{unr}_p)\). By [Tei89] Thm. 45, for each $z \in \mathcal{H}_p(\mathbb{Z}^{unr}_p)$, there exists triple $X = (\psi, G, \rho)$ in \(\text{SFD}(\mathbb{Z}^{unr}_p)\) such that $z_0(X) = z$.

4.2. Filtered $\phi$-modules. Let $F$ be an unramified field extension of $\mathbb{Q}_p$. For an integer $a$, a $\sigma^a$-isocrystal $E$ over $F$ is a pair $E = (V, \phi)$ consisting of a finite dimensional $F$-vector space $V$ with a $\sigma^a$-linear isomorphism $\phi$ (i.e. the isomorphism $\phi : V \to V$ satisfies the relation $\phi(xv) = \sigma(x)^a \cdot v$ for $x \in F$ and $v \in V$; see [Zin84] Chapter VI, §1]. If $a = 1$, $\sigma$-isocrystals are also called $F$-isocrystals (here $F$ stands for Frobenius, do not confuse with our fixed $p$-adic field $F$) of $\phi$-modules, in which case the $\sigma$-linear isomorphism $\phi$ is called Frobenius (in the following we will use both terminologies of $\phi$-modules and $F$-isocrystals).

A filtered $F$-isocrystal, or a filtered $\phi$-module is a $\phi$-module $(V, \phi)$ equipped with an exhaustive and separate filtration $F^*V$.

If $G$ is a $p$-divisible formal group over $\mathbb{F}_p$, one can define its first crystalline cohomology cohomology group as in [Gro73], [HOT87], [BBMS2] Définition 2.5.7, in terms of the crystalline Dieudonné functor (among many other references, see for example [Ill76], [CL98], [dJ98] for self-contained expositions). In the following we will denote $H^1_{\text{cris}}(G)$ the global sections of the crystalline Dieudonné functor (defined as in [BBMS2] Théorème 4.2.8.1) tensored over $\mathbb{Z}^{unr}_p$ with $\mathbb{Q}^{unr}_p$. By construction, $H^1_{\text{cris}}(G)$ is then an $F$-isocrystal. Moreover, the canonical isomorphism between $H^1_{\text{cris}}(G)$ and the first de Rham cohomology group $H^1_{\text{dR}}(G)$ of $G$ equips $H^1_{\text{cris}}(G)$ with a canonical filtration (arising from the Hodge filtration in the de Rham cohomology), making $H^1_{\text{dR}}(G)$ a filtered $F$-isocrystal; see [Oda69].

Let $G$ be a SFD-module over $\mathbb{F}_p$. Then the $F$-isocrystal $H^1_{\text{cris}}(G)$ is a four-dimensional $\mathbb{Q}^{unr}_p$-vector space, equipped with its $\sigma$-linear Frobenius $\phi_{\text{cris}}(G)$. It is also equipped with a $D$-module structure $j_G : D \hookrightarrow \text{End}_{\mathbb{Q}^{unr}_p}(H^1_{\text{cris}}(G))$ which commutes with $\phi_{\text{cris}}(G)$, and a $\mathbb{Q}_p$-algebra embedding $i_G : \mathbb{M}_2(\mathbb{Q}_p) \hookrightarrow \text{End}_{\mathbb{Q}^{unr}_p}(H^1_{\text{cris}}(G))$ induced by the isomorphism $\text{End}_{\mathbb{Q}_p}(G) \simeq \mathbb{M}_2(\mathbb{Q}_p)$, which commutes with the $D$-action. Define $\phi'_{\text{cris}}(G) = j_G(\Pi)^{-1} \phi_{\text{cris}}(G)$ and put

$$V_{\text{cris}}(G) = H^1_{\text{cris}}(G)^{\phi'_{\text{cris}}(G) = 1}.$$ 

Denote $\phi_{\text{cris}}(G) = j_G(\Pi)|_{V_{\text{cris}}(G)}$ the restriction of $j_G(\Pi)$ to $V_{\text{cris}}(G)$. Moreover, denote

$$(\eta'(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)\vee = \text{Hom}_{\mathbb{Q}_p}(\eta'(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \mathbb{Q}_p)$$

the $\mathbb{Q}_p$-linear dual of $\eta'(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

The following lemma is crucial in what follows, and identifies $V_{\text{cris}}(G)$ with $(\eta'(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)\vee$, from which one deduces a complete description of the filtered $F$-isocrystal $H^1_{\text{cris}}(G)$. It appears in a slightly different version in the proof of [IS03] Lemma 5.10]. Since we did not find an reference for this fact in the text we need it, we add a complete proof.

**Lemma 4.1.** There is a canonical isomorphism $V_{\text{cris}}(G) \simeq (\eta'(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)\vee$ of $\mathbb{Q}_p$-vector spaces. Moreover, $H^1_{\text{cris}}(G) = V_{\text{cris}}(G) \otimes_{\mathbb{Q}_p} \mathbb{Q}^{unr}_p$, where the right hand side is equipped with the structure of $\mathbb{Q}^{unr}_p$-vector space given by $x \cdot (v \otimes \alpha) = v \otimes (\sigma(x) \alpha)$ for $v \in V_{\text{cris}}(G), x, \alpha \in \mathbb{Q}^{unr}_p$. Finally, under this isomorphism the Frobenius $\phi_{\text{cris}}(G)$ corresponds to $\phi_{\text{cris}}(G) \otimes \sigma$.

**Proof.** The $F$-isocrystal $H^1_{\text{cris}}(G)$ is canonically isomorphic to the contravariant Dieudonné module of $G$ with $p$ inverted, and with $\mathbb{Q}^{unr}_p$-action twisted by the Frobenius automorphism $\sigma$ of $\mathbb{Q}^{unr}_p$, equipped with the canonical Frobenius of the contravariant Dieudonné module (see [BBMS2] 4.2.14). More precisely, denote $D(G) = \text{Hom}_{\mathbb{Q}^{unr}_p}((M(G)[1/p], \mathbb{Q}^{unr}_p)$ the
\( \hat{Q}_p^{\text{unr}} \)-linear dual of the covariant Dieudonné module \( M(G) \) of \( G \) with \( p \) inverted, and let \( D(G) = D(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}} \), where the tensor product is taken with respect to the Frobenius endomorphism \( \sigma \) of \( \hat{Q}_p^{\text{unr}} \). Then as \( \hat{Q}_p^{\text{unr}} \)-vector spaces, we have \( H^1_{\text{cris}}(G) \cong D(G)/\sigma \). Under this isomorphism the Frobenius \( \phi_{\text{cris}}(G) \) is given by the map \( \varphi \mapsto \sigma \circ \varphi \circ \eta(G) \) for \( \varphi \in D(G) \).

Now, by [BC91 Lemme (5.12)], we have an isomorphism of \( \sigma^{-1} \)-isocrystals
\[
(M^i(G)[1/p], V_G \Pi^{-1}) \cong \left( \eta^i(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}}, \sigma^{-1} \right)
\]
for each index \( i = 0, 1 \) (where the action of \( \sigma^{-1} \) on \( \eta^i(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}} \) is on the second factor only). We may therefore compute \( V_{\text{cris}}(G) \) in terms of the isocrystal \( \left( \eta^i(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}}, \sigma^{-1} \right) \).

As above, define \( D^i(G) = \text{Hom}_{\hat{Q}_p^{\text{unr}}} \left( M^i(G)[1/p], \hat{Q}_p^{\text{unr}} \right) \) (\( \hat{Q}_p^{\text{unr}} \)-linear dual) and let \( D^0(G) \) denote the base change \( D^0(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}} \) via \( \sigma \). Since \( M(G) = M^0(G) \oplus M^1(G) \), we have \( D^0(G) = D^0(G)^\sigma \oplus D^1(G)^\sigma \), and we may write any element \( \varphi \in D^0(G)^\sigma \) as a pair \((\varphi_0, \varphi_1)\) with \( \varphi_i \in D^i(G)^\sigma \), \( i = 0, 1 \). By definition, an element \( \varphi = (\varphi_0, \varphi_1) \in D(G)^\sigma \) belongs to \( V_{\text{cris}}(G) \) if and only if \( \varphi_i(V_G \Pi^{-1}(m_i)) = \sigma^{-1}(\varphi_i(m_i)) \) for all \( m_i \in M^i(G)[1/p] \), and for all \( i = 0, 1 \).

Using [13], identify \( \varphi_i \) with a \( \hat{Q}_p^{\text{unr}} \)-linear homomorphism \( \varphi_i : \eta^i(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}} \to \hat{Q}_p^{\text{unr}} \) denoted with a slight abuse of notation with the same symbol; then the above equation describing \( V_{\text{cris}}(G) \) becomes \( \varphi_i(n \otimes \sigma^{-1}(x)) = \sigma^{-1}(\varphi_i(n \otimes x)) \) for all \( n \in \eta^i(G) \) and all \( x \in \hat{Q}_p^{\text{unr}} \), or equivalently, since \( \varphi_i \) is \( \hat{Q}_p^{\text{unr}} \)-linear, \( \varphi_i(n \otimes 1) = \sigma^{-1}(\varphi_i(n \otimes 1)) \) for all \( n \in \eta^i(G) \), and we conclude that \( \varphi_i(n \otimes 1) \in \hat{Q}_p^{\text{unr}} \) for all \( n \in \eta^i(G) \). So \( \varphi_i \) is the \( \hat{Q}_p^{\text{unr}} \)-linear extension of a \( \hat{Q}_p^{\text{unr}} \)-linear homomorphism \( \eta^i(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}} \to \hat{Q}_p^{\text{unr}} \). Since \( \eta(G) = \eta^0(G) \oplus \eta^1(G) \), we then conclude that \( V_{\text{cris}}(G) \cong (\eta(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}})^\vee \) as \( \hat{Q}_p^{\text{unr}} \)-vector spaces. If \( n_1, \ldots, n_4 \) is a \( \hat{Q}_p^{\text{unr}} \)-basis of \( \eta(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}} \), then \( dn_1, \ldots, dn_4 \) defined by \( dn_i(n_j) = \delta_{i,j} \) (as usual, \( \delta_{i,j} = 1 \) if \( i = j \) and \( 0 \) otherwise) is a basis of \( (\eta(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}})^\vee \) and, by \( \hat{Q}_p^{\text{unr}} \)-linear extension, also of \( (\eta(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}})^\vee \). If we now base change the \( \hat{Q}_p^{\text{unr}} \)-vector space \( (\eta(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}})^\vee \) via \( \sigma \), we see that \( dn_1, \ldots, dn_4 \) is still a \( \hat{Q}_p^{\text{unr}} \)-basis, and we have \( (x \cdot dn_i)(n_j) = \sigma(x) \delta_{i,j} \) for all \( x \in \hat{Q}_p^{\text{unr}} \). Using the above description of \( H^1_{\text{cris}}(G) \) in terms of \( D(G)^\sigma \), the description of \( V_{\text{cris}}(G) \) in terms of \( \eta(G) \), we have an isomorphism of \( \hat{Q}_p^{\text{unr}} \)-vector spaces,
\[
H^1_{\text{cris}}(G) \cong \left( V_{\text{cris}}(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}} \right)^\sigma,
\]
where the upper index \( \sigma \) on the right hand side means that the structure of \( \hat{Q}_p^{\text{unr}} \)-vector space is twisted by \( \sigma \) as explained above. Moreover, the \( \sigma^{-1} \)-linear isomorphism \( \Pi \otimes \sigma \) of \( M(G)[1/p] \) corresponds to the \( \sigma^{-1} \)-linear isomorphism \( \sigma^{-1} \) of \( \eta(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}} \) (acting on the second component only), and therefore the isomorphism \( \varphi \mapsto \sigma \circ \varphi \circ \eta(G) \) of \( M(G)[1/p] \) \( \hat{Q}_p^{\text{unr}} \)-linear dual) corresponds to the isomorphism \( \Pi \otimes \sigma \) of \( (\eta(G) \otimes_{\hat{Q}_p^{\text{unr}}} \hat{Q}_p^{\text{unr}})^\vee \) \( \hat{Q}_p^{\text{unr}} \)-linear dual) given by \( \sigma \cdot \cdot \cdot \to (dn_i \otimes \Pi) \otimes \sigma(x) \) where \( (dn_i \otimes \Pi)(n) = \eta_i(\Pi n) \), which corresponds to \( \phi_{\text{cris}}(G) \) \( \hat{Q}_p^{\text{unr}} \)-vector spaces.

4.3. Filtered convergent \( F \)-isocrystals on \( \hat{Q}_p^{\text{unr}} \). To describe the relative de Rham cohomology of the \( p \)-adic upper half plane, we first need some preliminaries on the notion of filtered convergent \( F \)-isocrystals introduced in [13].

We first recall some preliminaries. Let \( F \subseteq \hat{Q}_p^{\text{unr}} \) be an unramified extension of \( \hat{Q}_p^{\text{unr}} \), with valuation ring \( \mathcal{O}_F \). If \( (X, \mathcal{O}_X) \) is a \( p \)-adic \( \mathcal{O}_F \)-formal scheme, we denote \( (X^{\text{rig}}, \mathcal{O}_X^{\text{rig}}) \) the associated \( F \)-rigid analytic space (or its generic fiber), and if \( F \) is a sheaf of \( \mathcal{O}_X \)-modules, we denote \( F^{\text{rig}} \) its associated sheaf of \( \mathcal{O}_X^{\text{rig}} \)-modules ([13] §7.4, [13] §1]). We say that \( X \)
is a \( p \)-adic \( \mathcal{O}_F \)-formal scheme if \( X \) is a \( \mathcal{O}_F \)-formal scheme which is locally of finite type. We will always assume in the following that \( X \) is analytically smooth, so that \( X ^{\text{rig}} \) is smooth.

An enlargement of \( X \) is a pair \((T, z_T)\) consisting of a flat \( p \)-adic \( \mathcal{O}_F \)-formal scheme \( T \) and a morphism of \( \mathcal{O}_F \)-formal schemes \( z_T : T_0 \to X \) where for each \( \mathcal{O}_F \)-formal scheme \( T \) we denote \( T_0 \) the reduced closed subscheme of the closed subscheme \( T_1 \) of \( T \) defined by the PD ideal \( p \mathcal{O}_T \).

A convergent isocrystal on \( X \) (cf. \cite{ISO3} Definition 3.1) is a rule \( \mathcal{E} \) which assigns to each enlargement \((T, z_T)\) of \( X \) a coherent \( \mathcal{O}_T \otimes \mathcal{O}_F \)-module \( \mathcal{E}_T \) such that for any morpshism \( g : T' \to T \) of \( \mathcal{O}_F \)-formal schemes with \( g_0 : T'_0 \to T_0 \) satisfying \( z_{T'} = z_T \circ g_0 \) (where \( g_0 \) is induced from \( g \)), there is an isomorphism of \( \mathcal{O}_T \otimes \mathcal{O}_F \)-modules \( \theta_g : g^*(\mathcal{E}_T) \simeq \mathcal{E}_{T'} \), satisfying the cocycle condition. The \( \mathcal{O}_T \otimes \mathcal{O}_F \)-module \( \mathcal{E}_T \) also seen as rigid analytic \( \mathcal{O}_T^\text{rig} \)-module on the \( F \)-rigid analytic space \( T ^{\text{rig}} \) (\cite{Ogu84} Remark (1.5)); we distinguish the notation and write \( \mathcal{E}_{T}^{\text{rig}} \) to emphasise this viewpoint. If \( \mathcal{E} \) is a convergent isocrystal over \( X \), for each enlargement \((T, z_T)\) which is analytically smooth over \( \mathcal{O}_F \) we have an integrable connection

\[
\nabla_{T}^{\text{rig}} : \mathcal{E}_{T}^{\text{rig}} \longrightarrow \mathcal{E}_{T}^{\text{rig}} \otimes_{\mathcal{O}_T^{\text{rig}}} \Omega_1^{1} X^{\text{rig}}.
\]

A convergent \( F \)-isocrystal on \( X \) (cf. \cite{ISO3} Definition 3.2)) is a convergent isocrystal \( \mathcal{E} \) on \( X \) equipped with an isomorphism of convergent isoscystals \( \phi_\mathcal{E} : \mathcal{F}^* \mathcal{E} \simeq \mathcal{E} \), where \( \mathcal{F} \) is the absolute Frobenius of \( X_0 \).

A filtered convergent \( F \)-isocrystal on \( X \) (cf. \cite{ISO3} Definition 3.3)) is a \( F \)-isocrystal \( (\mathcal{E}, \phi_\mathcal{E}) \) such that \( \mathcal{E}_{X}^{\text{rig}} \) is equipped with an exhaustive and separated decreasing filtration \( \mathcal{F}^{\text{rig}} \mathcal{E}_{X}^{\text{rig}} \) of coherent \( \mathcal{O}_{X}^{\text{rig}} \)-submodules such that \( \nabla_{X}^{\text{rig}}(\mathcal{F}^{i} \mathcal{E}_{X}^{\text{rig}}) \) is contained in \( \mathcal{F}^{i-1}\mathcal{E}_{X}^{\text{rig}} \otimes_{\mathcal{O}_F^{\text{rig}}} \Omega_1^{1} X^{\text{rig}} \) for all \( i \).

We present two explicit examples of filtered convergent \( F \)-isocrystals. A third example will be discussed in \S 4.4.

**Example 4.2.** The first example (cf. \cite{ISO3} Example 3.4(a))) is the identity object of the additive tensor category of filtered isocrystals on \( X \). This is the convergent isocrystal \( \mathcal{E}(\mathcal{O}_X) \) on \( X \) given by the rule \((T, z_T) \mapsto \mathcal{O}_T \otimes \mathcal{O}_F \mathcal{F} \) equipped with the canonical Frobenius and the filtration given by \( \mathcal{F}^i \mathcal{O}_X^{\text{rig}} = \mathcal{O}_X^{\text{rig}} \) for \( i \leq 0 \) and \( \mathcal{F}^i \mathcal{O}_X^{\text{rig}} = 0 \) for \( i > 0 \) (in loc. cit. this filtered convergent \( F \)-isocrystal is simply denoted \( \mathcal{O}_X \)).

**Example 4.3.** Our second example (cf. \cite{ISO3} pages 345-346) is the filtered convergent \( F \)-isocrystal \( \mathcal{E}(V) \) attached to a representation \( \rho : GL_2 \times GL_2 \to GL(V) \), where \( V \) is a finite dimensional \( \mathbb{Q}_p \)-rational representation, and \( GL_2 \) is the algebraic group of invertible matrix over \( \mathbb{Q}_p \). First, for a given such representation \( \rho : GL_2 \times GL_2 \to GL(V) \), let \( \rho_1 \) and \( \rho_2 \) denote the restrictions of \( \rho \) to the first and second \( GL_2 \)-factor, respectively. As convergent isocrystal, \( \mathcal{E}(V) = V \otimes \mathbb{Q}_p \mathcal{E}(\mathcal{O}_{H_p^{\text{rig}}}) \). The Frobenius, making it a convergent \( F \)-isocrystal, is defined by \( \phi_V \otimes \phi_{\mathcal{E}(\mathcal{O}_{H_p^{\text{rig}}})} \), where \( \phi_V = \rho_2 \left( \left( \begin{array}{cc} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{array} \right) \right) \) (so, the Frobenius depends on \( \rho_2 \) only). To define the filtration, making it a filtered convergent \( F \)-isocrystal, we first recall some preliminaries. First, the filtration only depends on \( \rho_1 : GL_2 \to GL(V) \), and therefore it is enough to define the filtration attached to a given representation \( \rho : GL_2 \to GL(V) \). For this, let \( P_n \) be the \( \mathbb{Q}_p \)-vector space of polynomials in one variable \( X \) of degree at most \( n \), equipped with a right action of \( GL_2 \) by \( P(X) \cdot A = (cX + d)^n P \left( \frac{x + A_{11}}{cX + d} \right) \) for \( A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) and \( P(X) \in P_n \). Then put \( V_n = P_n^* \) (\( \mathbb{Q}_p \)-linear dual), equipped with the left action of \( GL_2 \) by

\[
(A \cdot \varphi)(P(x)) = \varphi(P(X) \cdot A).
\]

Recall that any representation \( \rho : GL_2 \to GL(V) \) can be written as a direct sum of a sum of representations of the form \( V_1^\otimes m \otimes (V_1^\vee)^\otimes n \), where \( m, n \) are non-negative integers. To define the filtration on \( \mathcal{E}(V) \) it is then enough to defined it for \( V = V_1 \). We have a map

\[
\]
The de Rham cohomology sheaf $\mathcal{H}^{unr}$ on the de Rham cohomology $H^\cdot$.

Moreover, the connection $\nabla$ as follows. Denote $\omega$ be the Lie algebra of the Cartier dual $G$ vectorial extension of $\rho : GL_2 \times GL_2 \rightarrow GL(V)$ we obtain a filtered convergent $F$-isocrystal $\mathcal{E}(V)$. We put $F^0\mathcal{E}(V_1) = \mathcal{E}(V_1)$. $F^1\mathcal{E}(V_1) = ker(ev_{X-z})$ and $F^2\mathcal{E}(V_1) = 0$. This defines the filtered convergent $F$-isocrystal $\mathcal{E}(V_1)$ attached to $V_1$, and therefore for any representation $\rho : GL_2 \times GL_2 \rightarrow GL(V)$ we obtain a filtered convergent $F$-isocrystal $\mathcal{E}(V)$.

4.4. The filtered convergent $F$-isocrystal of the universal SFD-module. The third example of filtered convergent $F$-isocrystal arises from relative de Rham cohomology of the universal SFD-module. Since it is more articulated that the previous ones, we prefer to keep it in a separate subsection. We follow [Fal97], [LS03].

Let $(\lambda_G, \mathcal{G}, \rho_G)$ be the universal triple, arising from the representability of the functor SFD by $\mathcal{H}^{unr}_p$; denote $\lambda : G \rightarrow \mathcal{H}^{unr}_p$ be universal map. Let $G^\vee$ be the Cartier dual of $G$ ([Pon77 Chapitre III, §5]), which is equipped with a canonical map $\lambda^\vee : G^\vee \rightarrow \mathcal{H}^{unr}_p$. In this setting one may define a convergent $F$-isocrystal

$$\mathcal{E}(\mathcal{G}) = R^1\lambda_*(O_{\mathcal{G}/\mathcal{Q}^{unr}_p})$$

interpolating crystalline cohomology sheaves (Ogu84, Theorems (3.1), (3.7)): for each enlargement $(T, z_T)$, the value $E(G)_T$ of $E(G)$ at $T$ is defined to be the crystalline cohomology sheaf of coherent $O_T \otimes_{O_F} F$-modules $R^qf_{T,z}O_{\mathcal{G}_{T/zT}} \otimes_{O_F} F$. The notation adopted here is standard, following Ogu84 §3: $f_{T,z} : G \times \mathcal{H}^{unr} \rightarrow T$ is the canonical projection where we use $z_T : T \rightarrow \mathcal{H}^{unr}_{p}$ to form the fiber product $G \times_Z T_1$, and $R^qf_{T,z}O_{\mathcal{G}_{T/zT}}$ is the crystalline cohomology sheaf on the formal scheme $T$ (note that $T_1 \rightarrow T$ is defined by the PD ideal $pO_T$ and $f_{T,z}$ is smooth and proper); see BOS3, BOS5. Since $T$ is noetherian, these are coherent sheaves of $O_T \otimes_{O_F} F$-modules, and therefore $E(G)^{rig}_T$ are coherent $O_T^{rig}$-modules.

The coherent $O^{rig}_{\mathcal{H}^{unr}} = O^{rig}_{\mathcal{H}^{unr}_p}$-module $E(G)^{rig}_{\mathcal{H}^{unr}}$ is equipped as in [15] with a connection $\nabla^{rig}_{\mathcal{H}^{unr}}$. The coherent $O^{rig}_{\mathcal{H}^{unr},\mathcal{G}}$-module $E(G)^{rig}_{\mathcal{H}^{unr}}$ is canonically isomorphic to the relative rigid de Rham cohomology sheaf

$$\mathcal{H}^{1,rig}_{dr}(\mathcal{G}) = \mathcal{H}^{1}_{dr}(\mathcal{G}^{rig}/\mathcal{H}^{unr}_p) = R^1\lambda^{rig}_*(\Omega^r_{\mathcal{G}^{rig}/\mathcal{H}^{unr}_p})$$

Moreover, the connection $\nabla^{rig}_{\mathcal{H}^{unr}}$ corresponds to the Gauss-Manin connection

$$\nabla^{rig}_{\mathcal{G}} : \mathcal{H}^{1,rig}_{dr}(\mathcal{G}) \rightarrow \Omega^1_{\mathcal{H}^{unr}/\mathcal{Q}^{unr}_p} \otimes O^{rig}_{\mathcal{H}^{unr}} \mathcal{H}^{1,rig}_{dr}(\mathcal{G})$$

whose construction in this context follows [KO68], and is the analogue of the construction we outlined in [23, see LS03 Example 3.4(c)], Ogu84 Theorem (3.10)]. The Hodge filtration on the de Rham cohohology $\mathcal{H}^{1,rig}_{dr}(\mathcal{G})$ makes then $E(\mathcal{G})$ a filtered convergent F-isocrystal.

The filtration on $E(\mathcal{G})$ arising from the Hodge filtration on the de Rham cohomology can be described more explicitly. Denote $\mathcal{H}^{1}_{dr}(\mathcal{G}/\mathcal{H}^{unr}_p)$ the dual of the Lie algebra of the universal vectorial extension of $\mathcal{G}$, equipped with its structure of convergent $F$-isocrystal ([Mes72 Chapter IV, §2], [AM74, §§1,9,11])). By [BBM82 §3.3], we have an isomorphism of convergent $F$-isocrystals

$$E(\mathcal{G}) \simeq \mathcal{H}^{1}_{dr}(\mathcal{G}/\mathcal{H}^{unr}_p).$$

The Hodge-Tate filtration on $\mathcal{H}^{1}_{dr}(\mathcal{G}) \simeq \mathcal{H}^{1}_{dr}(\mathcal{G}/\mathcal{H}^{unr}_p)$ can be described in explicit terms as follows. Denote $\omega_G = c^*_{\mathcal{G}}(\Omega^1_{\mathcal{G}/\mathcal{H}^{unr}_p})$, where $c_{\mathcal{G}} : \mathcal{H}^{unr}_p \rightarrow \mathcal{G}$ is the zero-section, and let $L_{\mathcal{G}^\vee}$ be the Lie algebra of the Cartier dual $\mathcal{G}^\vee$ of $\mathcal{G}$. Then $\omega_G$ and $L_{\mathcal{G}^\vee}$ are locally free $O_{\mathcal{H}^{unr}_p}$-modules, dual to each other ([BBM82 §3.3]). We have the Hodge-Tate exact sequence of
where the tensor product is again over $\omega$. In which the first and the last map come from the Hodge exact sequence (16). Recalling the isomorphism of filtered convergent $F$-isocrystals (17) can be reformulated as follows (see also the discussion in [IS03, Chap. III, Lemma 4.4]), we obtain isomorphism (18) can be rewritten in a more compact way as

$${\mathcal{H}}^1_{\text{dR}}(G/\hat{H}_p^\text{unr}) \simeq V_{\text{cris}}(\Phi) \otimes_{\mathbb{Q}_p} \mathcal{E}(O_{\hat{H}_p^\text{unr}}).$$

The isomorphism of filtered convergent $F$-isocrystals (17) can be rewritten as follows (see also the discussion in [IS03, Lemmas 5.10]). Let $\rho : GL_2 \times GL_2 \rightarrow GL(M_2)$ be the representation defined by $\rho_1(A)(B) = AB$ and $\rho_2(A)B = B\tilde{A}$ where if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\tilde{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Note that $\mathcal{E}(M_2)$ is pure of weight 1. The isomorphism (17) can be written in a more compact way as

$${\mathcal{H}}^1_{\text{dR}}(G/\hat{H}_p^\text{unr}) \simeq V_{\text{cris}}(\Phi) \otimes_{\mathbb{Q}_p} \mathcal{E}(O_{\hat{H}_p^\text{unr}}).$$
satisfying $\langle dx, y \rangle^\rig_{G} = \langle x, dy \rangle^\rig_{G}$ for all $x, y$ sections in $\mathcal{H}_{\dR}^{1,\rig}(G)$ and all $d \in D$ (because $\iota_{G}(dy) = d^{1}i_{G}(y)$) which we call rigid polarization pairing. We may therefore construct a map

$$\rho : \mathcal{H}_{\dR}^{1,\rig}(G) \rightarrow \left( \mathcal{H}_{\dR}^{1,\rig}(G) \right)^{\vee} \rightarrow (\omega_{G}^{\rig})^{\vee}$$

where the first map takes a section $s$ to the map defined for a section $t$ by $t \mapsto \langle s, t \rangle^{\rig}$ and the second map is induced by duality from the inclusion $\omega_{G}^{\rig} \hookrightarrow \mathcal{H}_{\dR}^{1,\rig}(G)$. Fix now a section $s \in H^{0}(U, (\Omega_{H_{\overline{\text{univ}}/Q_{p}}^{\rig}/Q_{p})^{\vee})$ over some affinoid $U$. Then we may compose the maps to get

$$\rho_{s} : H^{0}(U, \omega_{G}^{\rig}) \rightarrow H^{0}(U, \mathcal{H}_{\dR}^{1,\rig}(G)) \stackrel{\gamma^{\rig}}{\rightarrow} H^{0}(U, \mathcal{H}_{\dR}^{1,\rig}(G) \otimes \Omega_{H_{\overline{\text{univ}}/Q_{p}}^{\rig}/Q_{p}}^{1}) \xrightarrow{1 \otimes s} \rightarrow H^{0}(U, \mathcal{H}_{\dR}^{1,\rig}(G)) \xrightarrow{\rho} H^{0}(U, (\omega_{G}^{\rig})^{\vee}).$$

The association $s \mapsto \rho_{s}$ defines then a map of sheaves

$$\left( K\Sigma_{G}^{\rig}/\right)^{\vee} : (\Omega_{H_{\overline{\text{univ}}/Q_{p}}^{\rig}/Q_{p}}^{\rig}/Q_{p})^{\vee} \rightarrow \text{Hom}_{\mathcal{O}_{H_{\overline{\text{univ}}}}} \left( \omega_{G}^{\rig}, (\omega_{G}^{\rig})^{\vee} \right).$$

By construction, the dual of this map is the Kodaira-Spencer map, under the canonical identification between $\text{Hom}(\omega_{G}^{\rig}, (\omega_{G}^{\rig})^{\vee})$ and $(\omega_{G}^{\rig})^{\otimes 2}$.

4.5. Universal rigid data. The aim of this subsection is to use the results of [Tei89] to better describe the Hodge filtration $\mathcal{H}_{\text{univ}}$. For this, we need to recall the universal rigid data introduced in [Tei89].

Let $V_{0}$ and $V_{1}$ be constant sheaves of one-dimensional $\mathbb{Q}_{p}$-vector spaces on the $\mathbb{Q}_{p}$-rigid analytic space $\mathcal{H}_{p}$ with basis $t_{0}$ and $t_{1}$ respectively. Define two invertible sheaves $T_{0}^{\text{univ}}$ and $T_{1}^{\text{univ}}$ on $\mathcal{H}_{p}$ by $T_{i}^{\text{univ}} = \mathcal{O}_{H_{p}} \otimes V_{i}$ for $i = 0, 1$, where $\mathcal{O}_{H_{p}}$ is the structural sheaf of rigid analytic functions on $\mathcal{H}_{p}$. Define $T^{\text{univ}} = T_{0}^{\text{univ}} \oplus T_{1}^{\text{univ}}$. For $i = 0, 1$, let $\eta_{i}^{\text{univ}}$ be the constant sheaf of two-dimensional $\mathbb{Q}_{p}$-vector spaces on $\mathcal{H}_{p}$ with basis $e_{i,0}$ and $e_{i,1}$. One fixes

$$\eta_{i}^{\text{univ}} = \eta_{i}^{\text{univ}}(\Phi) \otimes \mathbb{Z}_{p} \otimes \mathbb{Q}_{p}$$

as in [Tei89] page 664. Define $u_{0}^{\text{univ}} : \eta_{0}^{\text{univ}} \rightarrow T_{0}^{\text{univ}}$ by $u_{0}^{\text{univ}}(e_{0,0}) = zt_{0}$ and $u_{0}^{\text{univ}}(e_{1,0}) = t_{0}$, and $u_{1}^{\text{univ}} : \eta_{1}^{\text{univ}} \rightarrow T_{1}^{\text{univ}}$ by $u_{1}^{\text{univ}}(e_{0,1}) = (p/z)t_{1}$ and $u_{1}^{\text{univ}}(e_{1,1}) = t_{1}$, where $z$ denotes the standard coordinate function on $\mathcal{H}_{p}$. Define $\eta_{i}^{\text{univ}} = \eta_{0}^{\text{univ}} \oplus \eta_{1}^{\text{univ}}$ and similarly define $u_{i}^{\text{univ}} = u_{0}^{\text{univ}} \oplus u_{1}^{\text{univ}}$. We write $\rho^{\text{univ}} : (\mathbb{Q}_{p}/\mathbb{Z}_{p} \otimes \mathbb{Q}_{p}) \simeq \eta_{i}^{\text{univ}}$ for the isomorphism determined by the choice of the basis $\{e_{0,0}, e_{0,1}\}$. For $\gamma = (a/b, c/d) \in \mathbb{M}_{2}(\mathbb{Q}_{p})$ and $i = 0, 1$, define endomorphisms $\phi_{i}(\gamma)$ in $\text{End}_{\mathcal{O}_{H_{p}}}(T_{i}^{\text{univ}})$ by

$$\phi_{0}^{\gamma}(f(z) \otimes t_{0}) = (cz + d)f(\gamma(z)) \otimes t_{0},$$

and $\phi_{1}^{\gamma}(f(z) \otimes t_{1}) = (a/b + c/d)f(\gamma(z)) \otimes t_{1}$

for any $f \in \mathcal{O}_{H_{p}}(U)$, and any affinoid $U \subseteq \mathcal{H}_{p}$. Define an action of $\text{SL}_{2}(\mathbb{Q}_{p})$ on $\eta_{i}^{\text{univ}}$ for $i = 0, 1$ in such a way that $u_{i}^{\text{univ}}$ is equivariant with respect to these actions, namely, for $\gamma = (a/b, c/d) \in \text{GL}_{2}(\mathbb{Q}_{p})$, put $\gamma^{\times}(x_{0,0}) = (a/b, c/d)(x_{0,0})$ and $\gamma^{\times}(x_{1,0}) = (a/b, c/d)(x_{1,0})$. Let $\mathbb{Z}_{p}[\Pi]$ act on $T^{\text{univ}}$ by $\Pi t_{0} = (p/z)t_{1}$ and $\Pi t_{1} = zt_{0}$. We let $\mathbb{Z}_{p}[\Pi]$ act on $\eta_{i}^{\text{univ}}$ in such a way that $u_{i}^{\text{univ}}$ commutes with this action. We call the quadruplet

$$\mathcal{D}^{\text{univ}} = (\eta^{\text{univ}}, T^{\text{univ}}, u^{\text{univ}}, \rho^{\text{univ}})$$

the universal rigid data.

Passing to the associated normed sheaves ([Tei89] Definition 6), we obtain from $\mathcal{D}^{\text{univ}}$ a quadruplet $\hat{\mathcal{D}}^{\text{univ}} = (\hat{\eta}^{\text{univ}}, \hat{T}^{\text{univ}}, \hat{u}^{\text{univ}}, \hat{\rho}^{\text{univ}})$ on $\mathcal{H}_{p}$, corresponding to a $\mathcal{H}_{p}$-valued point,
which is universal in the following sense: for each $B \in \text{Nilp}$ and each $\Psi : S = \text{Spec}(B) \to \hat{H}_p$ corresponding to a quadruplet $(\eta, T, u, \rho)$, we have

\[(\eta, T, u, \rho) = (\Psi^{-1} \eta^\text{univ}, \Psi^* \tilde{T}^\text{univ}, \Psi^{-1} u^\text{univ}, \Psi^{-1} \rho^\text{univ}).\]

See [Tei89] Cor. 18 and Thm. 19 for more precise and complete statements. We call $\mathcal{D}^\text{univ}$ the \textit{universal formal data}, and we denote the quadruplet on the RHS of (22) by $\hat{\mathcal{D}}^\text{univ}$ to simplify the notation.

The universal SFD-module $G$ over $\hat{H}_p^\text{unr}$ can be recovered from a universal rigid data $\mathcal{D}^\text{univ}$. Pulling back via the projection $\pi_{\hat{H}_p} : \hat{H}_p^\text{unr} \to \hat{H}_p$, we obtain a quadruplet

\[\mathcal{D}^\text{unr} = (\eta^\text{unr}, \tilde{T}^\text{unr}, u^\text{unr}, \rho^\text{unr}) = (\pi_{\hat{H}_p}^{-1} \eta^\text{univ}, \pi_{\hat{H}_p}^* \tilde{T}^\text{univ}, \pi_{\hat{H}_p}^{-1} u^\text{univ}, \pi_{\hat{H}_p}^{-1} \rho^\text{univ})\]

on $\hat{H}_p^\text{unr}$. Comparing (22) with the universal property satisfied by $G$, we see that the quadruplet $(\eta_G, T_G, u_G, \rho_G)$ associated to $G$ coincides with the quadruplet $\mathcal{D}^\text{unr}$. In particular, the associated quadruplet $(\eta^\text{rig}, T^\text{rig}, u^\text{rig}, \rho^\text{rig})$ on the rigid $\mathbb{Q}_p^\text{unr}$-rigid analytic space $G^\text{rig}$ is the quadruplet

\[\mathcal{D}^\text{rig} = (\eta^\text{rig}, T^\text{rig}, u^\text{rig}, \rho^\text{rig}) = (\pi_{\hat{H}_p}^{-1} \eta^\text{univ}, \pi_{\hat{H}_p}^* T^\text{univ}, \pi_{\hat{H}_p}^{-1} u^\text{univ}, \pi_{\hat{H}_p}^{-1} \rho^\text{univ})\]

obtained from the quadruplet $\mathcal{D}^\text{univ}$, where $\pi_{\hat{H}_p} : \hat{H}_p^\text{unr} \to \hat{H}_p$ is the canonical projection.

Let $(T^\text{unr})^\vee$ denote the $\mathcal{O}^\text{unr}$-dual of $T^\text{unr}$, and, as above, denote $(\eta^\text{unr} \otimes \mathbb{Z}_p \mathbb{Q}_p)^\text{rig}$ the $\mathbb{Q}_p$-linear dual of $\eta^\text{unr} \otimes \mathbb{Z}_p \mathbb{Q}_p$. From the surjective map $u^\text{unr} : \eta^\text{unr} \otimes \mathbb{Z}_p \mathcal{O}^\text{unr} \to T^\text{unr}$ induced by $u^\text{univ}$ we obtain an injective map

\[\tau : (T^\text{unr})^\vee \hookrightarrow (\eta^\text{unr} \otimes \mathbb{Z}_p \mathbb{Q}_p)^\text{rig} \otimes \mathbb{Q}_p \mathcal{O}^\text{unr}\]

**Proposition 4.4.** We have canonical isomorphisms

\[(T^\text{unr})^\vee \simeq \omega^\text{rigG}\]

\[(\eta^\text{unr} \otimes \mathbb{Z}_p \mathbb{Q}_p)^\vee \otimes \mathbb{Q}_p \mathcal{O}^\text{unr} \simeq \mathcal{H}^1_{\mathbb{Q}_p}^\text{rig}(G),\]

under which the map $\tau$ corresponds to the canonical map in (15).

**Proof.** The first statement follows from the canonical isomorphism between $T_G = \text{Lie}_G$ and $T^\text{unr}$, while the second follows from Proposition 4.4 combined with (17). For the statement about $\tau$, note that for each SFD-module $G$ over $\overline{\mathbb{F}}_p$, the map $u_G$ corresponds under the identification between $\eta(G) \otimes \mathbb{Z}_p \mathbb{Q}_p$ and $\mathcal{M}(G) \otimes \mathbb{Z}_p^\text{unr} \mathbb{Q}_p$ to the canonical projection $\mathcal{M}(G) / \mathbb{V}_G \mathcal{M}(G) \to T_G$, where $T_G$ is the tangent space of $G$ at the origin. \[\square\]

### 4.6. The action of the idempotent $e$

Fix an isomorphism $\mathbb{Q}_p(\sqrt{a}) \simeq \mathbb{Q}_p^2$. By means of this isomorphism, and the fixed embedding $\mathbb{Q}_p^2 \hookrightarrow D$, we may identify elements $a + b \sqrt{a}$ in $\mathbb{Q}_p(\sqrt{a})$ (where $a, b \in \mathbb{Q}_p$) with elements of $D$ in what follows without explicitly mentioning it.

**Lemma 4.5.** $e \cdot (\eta(\Phi) \otimes \mathbb{Z}_p^2 \mathbb{Q}_p^2) = \eta^0(\Phi) \otimes \mathbb{Z}_p \mathbb{Q}_p^2$ and $e \cdot (T(\Phi) \otimes \mathbb{Z}_p \mathbb{Q}_p^2) = T^0(\Phi) \otimes \mathbb{Z}_p \mathbb{Q}_p^2$.

**Proof.** The action of $\mathcal{O}_D$ on $\eta(\Phi)$ is induced by duality from the action on $\mathcal{M}(\Phi)$, so any element $a \in \mathbb{Z}_p \hookrightarrow \mathcal{O}_D$ acts on $\eta^0(\Phi)$ by multiplication by $a$ and on $\eta^1(\Phi)$ by multiplication by $\bar{a}$. On the other hand, the action of $1 \otimes a$ on $\eta(\Phi) \otimes \mathbb{Q}_p^2$ is given by multiplication by $a$. An immediate calculation shows then that the action of $e$ is just the projection $\eta(\Phi) \to \eta^0(\Phi)$. The argument for $T(\Phi)$ is similar. \[\square\]

Write $\eta^0 = \pi_{\hat{H}_p}^{-1} \eta_0^\text{univ}$, $T_0^\text{unr} = \pi_{\hat{H}_p}^* T_0^\text{univ}$, $u_0^\text{unr} = \pi_{\hat{H}_p}^{-1} u_0^\text{univ}$.

**Proposition 4.6.** $e \cdot \eta^\text{unr} = \eta_0^\text{unr}$ and $e \cdot T^\text{unr} = T_0^\text{unr}$. 

Proof. This is clear from Lemma 4.5 and (20).

For \( i = 0, 1 \), the sheaf \( T_0^{\text{unr}} \) is a free \( \mathcal{O}_{\hat{H}_p^{\text{unr}}} \)-module of rank 1, so it is invertible; denote \( (T_0^{\text{unr}})^\vee \) its \( \mathcal{O}_{\hat{H}_p^{\text{unr}}} \)-dual. Taking duals we get a map \( du_0 : (T_0^{\text{unr}})^\vee \to (\eta_0 \otimes_{\hat{Z}_p} \mathcal{Q}_p)^\vee \otimes \mathcal{Q}_p \mathcal{O}_{\hat{H}_p^{\text{unr}}} \) (where the RHS denotes \( \mathcal{Q}_p \)-duals as above). We set up the following notation:

- \( \omega_0^0_G = e \cdot \omega_G^0 \)
- \( \mathcal{L}_G^0 = e \cdot \mathcal{H}^1_{\text{dR}}(G) \)

Applying the idempotent \( e \) and using Propositions 4.4 and 4.6 we then obtain a diagram with exact rows in which the vertical arrows are isomorphisms:

\[
\begin{array}{ccc}
0 & \to & (T_0^{\text{unr}})^\vee \\
\downarrow{\cong} & & \downarrow{\cong} \\
0 & \to & \omega_0^0_G
\end{array}
\]

4.7. Differential calculus on the p-adic upper half plane. We now set up the following notation. Recall that the map \( u_0 \) takes \( x_0 \cdot e_0 + x_0,1 \cdot e_{0,1} \) to \( (zx_0,0+1) \cdot t_0 \); dualizing, \( du_0 \) can be described in coordinates by the map which takes the canonical generator \( t_0 \) of the \( \mathcal{O}_{\hat{H}_p^{\text{unr}}} \)-module \( (T_0^{\text{unr}})^\vee \) (satisfying the relation \( dt_0(t_0) = 1 \)) to the map \( x_0 \cdot e_0 + x_0,1 \cdot e_{0,1} \mapsto zx_0,0+1 \). If we denote \( de_{0,i} \) the dual basis of \( e_{0,i} \) (satisfying the condition \( de_{0,i}(e_{0,j}) = \delta_{i,j} \)), we may write this map as \( zde_{0,0} + de_{0,1} \). To simplify the notation, we put from now on \( \tau = t_0, d\tau = dt_0, x = e_0,0, y = e_0,1, dx = de_{0,0} \) and \( dy = de_{0,1} \), so that the above map reads simply as

\[ d\tau = zdx + dy. \]

Let \( C = C^0(\mathcal{H}_p(\hat{Q}_p^{\text{unr}}), \mathcal{C}_p) \) denote the \( \mathcal{C}_p \)-vector space of continuous (for the standard \( p \)-adic topology on both spaces) \( \mathcal{C}_p \)-valued functions on \( \mathcal{H}_p(\hat{Q}_p^{\text{unr}}) \). Denote \( \mathcal{A} = H^0(\mathcal{H}_p^{\text{unr}}, \mathcal{O}_{\hat{H}_p^{\text{unr}}}) \) the \( \mathcal{O}_p^{\text{unr}} \)-vector space of global sections of \( \mathcal{O}_{\hat{H}_p^{\text{unr}}} \). Each \( f \in \mathcal{A} \) is, in particular, continuous on \( \mathcal{H}_p^{\text{unr}} \) for the standard \( p \)-adic topology of \( \hat{Q}_p^{\text{unr}} \), and therefore restriction induces a map of \( \mathcal{Q}_p^{\text{unr}} \)-vector spaces \( r : \mathcal{A} \to C \). Denote \( \mathcal{A}^* \) the image of the morphism of \( \mathcal{A} \)-algebras \( \mathcal{A}[X,Y] \to C \) defined by sending \( X \) to the function \( z \mapsto 1/(z - \sigma(z)) \) and \( Y \) to the function \( z \mapsto \sigma(z) \) (note that the function \( z \mapsto z - \sigma(z) \) is invertible on \( \mathcal{H}_p^{\text{unr}}(\hat{Q}_p^{\text{unr}}) \)). To simplify the notation, we put from now on

\[ z^* = \sigma(z). \]

Set up the following notation (here \( n \geq 1 \) is an integer)

- \( \Lambda_G = H^0(\mathcal{H}_p^{\text{unr}}, \mathcal{L}_G^0) \) and \( \Lambda_{G,n} = \Lambda_G^0 \)
- \( \Lambda_G^* = \Lambda_G \otimes_{\mathcal{A}^*} \mathcal{A}^* \) and \( \Lambda_{G,n}^* = (\Lambda_{G,n}^0)^0 \)
- \( w_G = H^0(\mathcal{H}_p^{\text{unr}}, \omega_0^0_G) \) and \( w_{G,n} = w_G^0 \)
- \( w_G^* = \Lambda_G^* \otimes_{\mathcal{A}^*} \mathcal{A}^* \) and \( w_{G,n}^* = (w_{G,n}^0)^0 \)

The \( \hat{Q}_p^{\text{unr}} \)-algebra \( \mathcal{A} \) is equipped with the standard derivation \( \frac{d}{dz} \) on power series. The \( \mathcal{A} \)-module \( \Omega_{\mathcal{A}}^1 = H^0(\mathcal{H}_p^{\text{unr}}, \Omega_{\hat{H}_p^{\text{unr}}}^1) \) is then one dimensional and generated by \( dz \) satisfying \( dz \left( \frac{d}{dz} \right) = 1 \). We extend differential operator \( \frac{d}{dz} \) to a differential operator \( \frac{d}{dz} : \mathcal{A}^* \to \mathcal{A}^* \) by \( \hat{Q}_p^{\text{unr}} \)-linearity using the product formula and setting \( \frac{d}{dz}(z^*) = 0 \) and \( \frac{d}{dz}(\frac{1}{z-z^*}) = -\frac{1}{(z-z^*)^2} \).

Similarly, we define a differential operator \( \frac{d}{dz^*} : \mathcal{A}^* \to \mathcal{A}^* \) setting \( \frac{d}{dz^*}(z) = 0 \) and \( \frac{d}{dz^*}(z^*) = 1 \) and \( \frac{d}{dz^*}(\frac{1}{z-z^*}) = -\frac{1}{(z-z^*)^2} \). Define \( \Omega_{\mathcal{A}}^* \), to be the \( \mathcal{A}^* \)-subalgebra of the algebra of derivations generated by \( dz \) and \( dz^* \) satisfying the usual rules \( dz \left( \frac{d}{dz} \right) = 1, dz \left( \frac{d}{dz^*} \right) = 0, dz^* \left( \frac{d}{dz^*} \right) = 0, dz^* \left( \frac{d}{dz} \right) = 1 \).
4.8. **Splitting of the rigid analytic Hodge filtration.** Recall the notation fixed before for the differential form $d\tau = zdx + dy$. Define

$$d\tau^* = z^* dx + dy.$$ 

Then $d\tau^*$ belongs to $w^*_G$. Taking global sections, restricting to $\hat{\mathbb{Q}}^{unr}_p$, and extending linearly with $\mathcal{A}^*$ we obtain a short exact sequence of $\mathcal{A}^*$-algebras

$$0 \rightarrow w^*_G \rightarrow \Lambda^*_G.$$ 

**Theorem 4.7.** The exact sequence (24) admits a canonical splitting $\Psi : \Lambda^*_G \rightarrow w^*_G$.

**Proof.** We have

$$dx = \frac{d\tau - d\tau^*}{z - z^*}, \quad dy = \frac{z d\tau^* - z^* d\tau}{z - z^*}.$$ 

We may therefore write any differential form $\omega = f(z) dx + g(z) dy$ with $f, g \in \mathcal{A}^*$ as

$$\omega = \left( \frac{f(z) - g(z) z^*}{z - z^*} \right) d\tau + dz \left( \frac{z g(z) - f(z)}{z - z^*} \right).$$

One then defines the sough-for splitting sending $\omega \mapsto \left( \frac{f(z) - g(z) z^*}{z - z^*} \right) d\tau$. \hfill $\square$

4.9. **The $p$-adic Shimura-Maass operator.** Taking global section, the Gauss-Manin connection gives rise to a map $\nabla^\text{rig}_G : \Lambda_G \rightarrow \Lambda_G \otimes \Omega^1_{\mathcal{A}}$. We extend $\nabla^\text{rig}_G$ to a map $\nabla^\text{rig}_G : \Lambda^*_G \rightarrow \Lambda^*_G \otimes \Omega^1_{\mathcal{A}}$, as follows. First define $\nabla^1_0^\text{rig}_G : \Lambda^*_G \rightarrow \Lambda^*_G \otimes \Omega^1_{\mathcal{A}}$ to be the derivation satisfying the rules

$$\nabla^1_0^\text{rig}_G(d\tau) = dx \otimes dz, \quad \nabla^1_0^\text{rig}_G(d\tau^*) = 0, \quad \nabla^1_0^\text{rig}_G(z^*) = 0.$$ 

Define similarly the derivation $\nabla^0_1^\text{rig}_G : \Lambda^*_G \rightarrow \Lambda^*_G \otimes \Omega^1_{\mathcal{A}}$, by the rules $\nabla^0_1^\text{rig}_G(d\tau) = 0, \quad \nabla^0_1^\text{rig}_G(d\tau^*) = dx \otimes dz^*, \quad \nabla^0_1^\text{rig}_G(z^*) = 0$. We finally define

$$\nabla^*_G = \nabla^1_0^\text{rig}_G + \nabla^0_1^\text{rig}_G : \Lambda^*_G \rightarrow \Lambda^*_G \otimes \Omega^1_{\mathcal{A}}.$$ 

Taking global sections, the Kodaira-Spencer map gives rise to a map $\text{KS}_G : w^*_G \otimes^2 \rightarrow \Omega^1_{\mathcal{A}}$, which we extend $\mathcal{A}^*$-linearly to a map

$$\text{KS}^*_G : (w^*_G)^{\otimes 2} \rightarrow \Omega^1_{\mathcal{A}}.$$ 

Note that

$$\nabla^\text{rig}_G(d\tau) = \nabla^*_G(d\tau) = \frac{d\tau - d\tau^*}{z - z^*} \otimes dz.$$ 

and, since $\nabla^*_G(z^*) = 0$, we have

$$\nabla^*_G(d\tau^*) = 0.$$ 

In particular, if $f(z) \otimes d\tau \in w^*_G$ we have

$$\nabla^\text{rig}_G(f(z) \otimes d\tau) = \left( \frac{\partial}{\partial z} f(z) \otimes d\tau + f(z) \otimes \frac{d\tau - d\tau^*}{z - z^*} \right) \otimes dz.$$ 

Taking global sections, we can form the pairing $\langle \cdot, \cdot \rangle^\text{rig}_G : \Lambda_G \otimes_{\mathcal{A}} \Lambda_G \rightarrow \mathcal{A}$. Extending linearly by $\mathcal{A}^*$, we obtain a new pairing

$$\langle \cdot, \cdot \rangle^*_G : \Lambda^*_G \otimes_{\mathcal{A}^*} \Lambda^*_G \rightarrow \mathcal{A}^*.$$ 

Using the description of the Kodaira-Spencer map in the end of §[4.4](#), we see that

$$\langle d\tau, \nabla^\text{rig}_G(d\tau) \rangle^*_G = \langle d\tau, \nabla^*_G(d\tau) \rangle^*_G = \frac{-\langle d\tau, d\tau^* \rangle^*_G}{z - z^*} dz = -(dx, dy)^*_G dz.$$
where for the second equality we use \(29\), while the last equality easily from the equality \(\langle zdx + dy, z^* dx + dy \rangle_{\hat G} = (z - z^*)(dx, dy)_{\hat G}\). Therefore

\[
KS^\text{rig}_{\hat G}(d\tau \otimes d\tau) = -\langle dx, dy \rangle_{\hat G}^\text{rig} dz.
\]

So, to compute \(KS^\text{rig}_{\hat G}(d\tau \otimes d\tau) = KS^\text{rig}_{\hat G}(d\tau \otimes d\tau)\) we are reduced to compute \(\langle dx, dy \rangle_{\hat G}^\text{rig}\). For this, we switch to de Rham homology and follow the computations in \([Mor11\), \([HB15\).

To begin with, let \(W\) denote the order \(O_D\) viewed as free left \(O_D\)-module of rank 1; then \(W \simeq \mathcal{R}_\text{max} \otimes_\mathbb{Z} \mathbb{Z}_p\). By \([BC91\), Ch. III, Lemma 1.9], the collection of bilinear skew-symmetric maps \(\psi : W \times W \to \mathbb{Z}_p\) which satisfy \(\psi(dx, y) = \psi(x, dy)\) (for all \(x, y \in W\) and \(d \in O_D\)) is a free \(\mathbb{Z}_p\)-module of rank 1, and every generator \(\psi_0\) of this \(\mathbb{Z}_p\)-module is a perfect duality on \(W\); the pairing

\[
\psi_0(x, y) = \frac{\text{tr}(iy^\dagger x)}{p}
\]

is such a generator, which we fix once and for all (recall the notation introduced in \(2.1\) and \(2.3\) for \(i\) and \(d^I\).

Recall that \(H^1_{\text{cris}}(\Phi)\) is a free \(D \otimes_{\mathbb{Q}_p} \hat{\mathbb{Q}}^\text{unr}_p\)-module of rank 1 (cf. \([IS03\) page 354]); the structure of \(D \otimes_{\mathbb{Q}_p} \hat{\mathbb{Q}}^\text{unr}_p\)-module is induced from the \(D\)-module structure of \((\eta(\Phi) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)\langle\Phi\rangle\) via the isomorphisms \((\eta(\Phi) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)\langle\Phi\rangle \simeq V_{\text{cris}}(\Phi)\) and \(H^1_{\text{cris}}(\Phi) \simeq V_{\text{cris}}(\Phi) \otimes_{\mathbb{Q}_p} \hat{\mathbb{Q}}^\text{unr}_p\) in Lemma \(4.1\). We have then from Lemma \(4.1\) then canonical isomorphisms of convergent \(F\)-isocrystals:

\[
H^1_{\text{dR}}(\mathcal{G}/\hat{\mathcal{H}}_p) \simeq H^1_{\text{cris}}(\Phi) \otimes_{\hat{\mathbb{Q}}^\text{unr}_p} \mathcal{E}(\mathcal{O}_{\hat{\mathcal{H}}_p})
\]

\[
\simeq (D \otimes_{\mathbb{Q}_p} \hat{\mathbb{Q}}^\text{unr}_p) \otimes_{\hat{\mathbb{Q}}^\text{unr}_p} \mathcal{E}(\mathcal{O}_{\hat{\mathcal{H}}_p})
\]

\[
\simeq (D \otimes_{\mathbb{Q}_p} \hat{\mathbb{Q}}^\text{unr}_p) \otimes_{\mathbb{Q}_p} \mathcal{E}(\mathcal{O}_{\hat{\mathcal{H}}_p})
\]

\[
\simeq D \otimes_{\mathbb{Q}_p} \hat{\mathbb{Q}}^\text{unr}_p \otimes_{\hat{\mathbb{Q}}^\text{unr}_p} \mathcal{E}(\mathcal{O}_{\hat{\mathcal{H}}_p})
\]

\[
\simeq M_2(\mathbb{Q}_p^2) \otimes_{\mathbb{Q}_p^2} \mathcal{E}(\mathcal{O}_{\hat{\mathcal{H}}_p}).
\]

Let \(\psi_0\) denote the \(\hat{\mathbb{Q}}^\text{unr}_p\)-linear extension of \(\psi_0\); under the isomorphism \(28\), \(\psi_0\) defines a pairing \(H^1_{\text{cris}}(\Phi) \times H^1_{\text{cris}}(\Phi) \to \hat{\mathbb{Q}}^\text{unr}_p\) still denoted by \(\psi_0\). If we still denote \(\langle \cdot, \cdot \rangle_{\hat{\mathcal{G}}}\) the restriction of \(\langle \cdot, \cdot \rangle_{\hat{\mathcal{H}}_p}\) to \(H^1_{\text{cris}}(\Phi)\), it follows from the nicety of \(\psi_0\) up to constant that there exists an element \(t_p \in \mathbb{C}_p^\times\) such that

\[
\langle \cdot, \cdot \rangle_{\hat{\mathcal{G}}, W} = \frac{1}{t_p} \cdot \langle \cdot, \cdot \rangle_{\hat{\mathcal{H}}, W} = \psi_0,
\]

Moreover, under the isomorphism \(28\), the element \(d\tau = zdx + dy\) of \(H^1_{\text{dR}}(\mathcal{G})\) corresponds to the element \(e_1 \otimes z + e_2 \otimes 1\), where \(e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\), \(e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), \(e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\), \(e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\) is the standard basis of \(M_2(\mathbb{Q}_p^2)\). We therefore obtain the sought-for recipe to compute the Kodaira-Spencer image of \(d\tau \otimes d\tau\) in terms of \(\psi_0\):

\[
KS^\text{rig}_{\hat{\mathcal{G}}}(d\tau \otimes d\tau) = \frac{1}{t_p} \cdot \psi_0(e_1 \otimes z, e_2 \otimes 1).
\]

**Remark 4.8.** The number \(t_p\) may be viewed as the \(p\)-adic analogue of the complex period \(2\pi i\), relating de Rham cohomology with homology \(([Mor11 (2.7)\), \([HB15\) p. 4197]). This explains why we prefer to keep \(t_p\) at the denominator in \(29\).

We now make more explicit the equations \(29\) and \(30\) using Hashimoto basis. For this part, we follow closely the nice calculations in \([Mor11\ Prop. 2.3\], to which the reader is referred to for details. Recall the Hashimoto basis \(\{1, i, j, k\}\) in \(\text{§2.1}\). As in \([Has95\ (2)\], define \(\epsilon_1 = 1, \epsilon_2 = (1 + j)/2, \epsilon_3 = (i + ij)/2, \epsilon_4 = (apN^2 j + ij)/p_0\) and use these elements to define
a symplectic basis of $W$ with respect to the pairing $\psi_0$ as in [Has95 (5)] by $\eta_1 = e_3 - \frac{m-1}{2} e_4$, $\eta_2 = -a De_1 - e_4$, $\eta_3 = e_1$, $\eta_4 = e_2$ (note that $\psi_0$ we consider above is equal to the pairing $(x,y) \mapsto \text{tr}(x iy^\dagger)$ in [Has95 (3)]). Denote $\eta_1^\vee, \eta_2^\vee, \eta_3^\vee, \eta_4^\vee$ the dual basis of $W^\vee$, and let $\eta_1^\vee$ be the column vector with entries $\eta_1^\vee, \eta_2^\vee, \eta_3^\vee, \eta_4^\vee$. The elements $\eta_i^\vee$ give rise to elements of $\mathcal{H}^1_{dR}(G)$, denoted with the same symbol, which are horizontal with respect to $\nabla^\text{rig}_G$, namely $\nabla^\text{rig}_G(\eta_i^\vee) = 0$. Write $d\tau = \Pi(z) \cdot \eta^\vee$. A simple calculation shows that

$$\Pi(z) = \left( \frac{\alpha}{2\sqrt{p_0}}(\alpha+a\Delta z + 1), \frac{-1}{\sqrt{p_0}}(\alpha+a\Delta z + 1), \frac{1}{2\alpha^+}z \right).$$

Since $\eta_i^\vee$ are horizontal sections of $\nabla^\text{rig}_G$, using (31) to calculate $d\Pi(z)/dz$ shows that (24) becomes

$$\nabla^\text{rig}_G(d\tau) = \left( \frac{\alpha-a\Delta}{2\sqrt{p_0}}, \frac{-a\Delta}{\sqrt{p_0}}, 1, \frac{1}{2\alpha^+} \right) \cdot \eta^\vee \otimes dz.$$

The recipe (19) to compute the Kodaira-Spencer map combined with (20) and (32) gives then

$$\text{KS}^\text{rig}_G(d\tau \otimes d\tau) = \frac{1}{t_p} d\Pi(z) \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \Pi(z)^T dz = \frac{1}{t_p} d\tau.$$

In particular, (33) shows that $\text{KS}^\text{rig}_G$ is an isomorphism, and therefore we may define the $p$-adic Shimura-Maass operator. We also need to consider iterates of this operator. Define

$$\tilde{\nabla}^*_n, \tilde{\nabla}^*_n : \Lambda^\ast_{G,n} \longrightarrow \Lambda^\ast_{G,n+2} \longrightarrow \Lambda^\ast_{G,n+2} \longrightarrow \Lambda^\ast_{G,n+2}$$

where $\tilde{\nabla}^*_n$ is obtained from $\nabla^*_n$ using the Leibniz rule (as before, see for example [HD15 §3.2]). The splitting in Theorem 4.7 induces a morphism of $A^\ast$-modules $\Psi^*_{p,n} : \Lambda^\ast_{G,n} \rightarrow \Lambda^\ast_{G,n}$. The composition

$$\Theta^*_{p,n} : w^\ast_{G,n} \longrightarrow \tilde{\nabla}^*_n \longrightarrow \tilde{\nabla}^*_n \longrightarrow \tilde{\nabla}^*_n$$

is the $p$-adic Shimura-Maass operator. We also need to consider iterates of this operator. Define

$$\tilde{\nabla}^*_{n+2j} = \tilde{\nabla}^*_{n+2j} \circ \cdots \circ \tilde{\nabla}^*_{n+2} \circ \tilde{\nabla}^*_{n}.$$

Define then

$$\Theta^*_{p,n} : w^\ast_{G,n} \longrightarrow \Lambda^\ast_{G,n} \longrightarrow \Lambda^\ast_{G,n+2j} \longrightarrow \Lambda^\ast_{G,n+2j}$$

where the morphism of $A^\ast$-modules $\Psi^*_{p,n+2j} : \Lambda^\ast_{G,n+2j} \rightarrow \Lambda^\ast_{G,n+2j}$ is induced as above by the splitting in Theorem 4.7. We call $\Theta^*_{p,n}$ the $j$-th iterate of the $p$-adic Shimura-Maass operator.

The work accomplished so far allows us to explicitly describe $\Theta^*_{p,k}$. We first introduce some differential operators, similar in shape to the Shimura-Maass operator in the real analytic setting. For each integer $k \geq 0$, we may then define the $\tilde{\nabla}^*_p$-linear function

$$\delta_{p,k} = \frac{\partial}{\partial z} + \frac{k}{z - z^*} : A^\ast \rightarrow A^\ast.$$

For each integer $j \geq 0$ we get a map $\delta^j_{p,k} : A^\ast \rightarrow A^\ast$ defined by

$$\delta^j_{p,k} = \delta_{p,k+2(j-1)} \circ \cdots \circ \delta_{p,k+2} \circ \delta_{p,k}.$$

We call $\delta_{p,k}$ the Shimura-Maass operator, and $\delta^j_{p,k}$ its $j$-th iteration. Applying (33) to compute the inverse of the Kodaira-Spencer map to (24), we obtain

$$\tilde{\nabla}^*_k \left( f(z) \otimes d\tau^{\otimes k} \right) = \frac{1}{t_p} \cdot \left( \frac{\partial}{\partial z} f(z) \otimes d\tau^{\otimes k+2} + kf(z) \otimes \frac{d\tau - d\sigma(\tau)}{z - \sigma(z)} \otimes d\tau^{\otimes k+1} \right).$$
Applying the splitting $\Psi_{p,k}^*$ of the Hodge filtration which annihilate $dt^*$, we finally obtain
\begin{equation}
\Theta_{p,k}^* \left( f(z) \otimes dt^k \right) = \frac{1}{t_p} \cdot \left( \frac{\partial}{\partial z} \frac{f(z)}{z - \sigma(z)} \right) \otimes dt^{k+2} = \frac{1}{t_p} \cdot \delta_{p,k}(f) \otimes dt^{k+2}.
\end{equation}

Iterating (36) we obtain
\begin{equation}
\Theta_{p,k}^{i,j} \left( f(z) \otimes dt^k \right) = \left( \frac{1}{t_p} \right)^j \delta_{p,k}^j(f(z)) \otimes dt^{k+2j}.
\end{equation}

5. The $p$-adic Shimura-Maass operator on Shimura curves

5.1. $p$-adic uniformization of Shimura curves. In this subsection we review the Cerednik-Drinfel’d Theorem. Let $B/\mathbb{Q}$ be the quaternion algebra obtained from $B$ by interchanging the invariants at $\infty$ and $p$; so $B$ is the definite quaternion algebra over $\mathbb{Q}$ of discriminant $N^\pm$. For a subgroup $U \subseteq B^\times$, let $U(p)$ the elements outside of the place $p$. Fix isomorphisms $B_\ell \simeq B_\ell$ for all primes $\ell \neq p$, so that we can view $V_1(N^+(p))$ as a subgroup of $(B^\times(p))$. Define $\tilde{\Gamma}_p = B^\times \cap V_1(N^+(p))$. We still denote $\tilde{\Gamma}_p$ the image of $\tilde{\Gamma}_p$ in $\text{GL}_2(\mathbb{Q}_p)$ via a fixed isomorphism $i_p : B \otimes \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p)$, and we let $\Gamma_p$ denote the subgroup of $\tilde{\Gamma}_p$ consisting of elements whose determinant has even $p$-power order.

Base changing from $\mathbb{Z}_p$ to the valuation ring $\mathbb{Z}_{p^2}$ of $\mathbb{Q}_{p^2}$ gives a $\mathbb{Z}_{p^2}$-formal scheme $\mathcal{H}_{p^2}$, whose generic fiber $\mathcal{H}_{p^2}$ is the base change of the $\mathbb{Q}_p$-rigid analytic space $\mathcal{H}_p$ to $\mathbb{Q}_{p^2}$. The group $\text{GL}_2(\mathbb{Q}_p)$ acts on the $\mathbb{Z}_{p^2}$-formal scheme $\mathcal{H}_p$ ($\text{BC91}$ Chapitre I, §6) and acts on $\text{Spf}(\mathbb{Z}_{p^2})$ via the inverse of the arithmetic Frobenius raised to the determinant map ($\text{BC91}$ Chapitre II, §9)). Therefore, the group $\text{GL}_2(\mathbb{Q}_p)$ also acts on the $\mathbb{Z}_{p^2}$-formal scheme $\mathcal{H}_{p^2}$ and the $\mathbb{Z}_{p^2}$-rigorous scheme $\mathcal{H}_{p^2}$, and the associated rigid analytic spaces. We may then form the quotient $\Gamma_p \backslash \mathcal{H}_{p^2}$, in the category of $\mathbb{Z}_{p^2}$-formal schemes, and the quotient $\Gamma_p \backslash \mathcal{H}_{p^2}$, in the category of $\mathbb{Z}_{p^2}$-formal schemes, and similarly for the associated rigid analytic spaces. The formal completion $\hat{\mathcal{A}}_{p_2}$ of the universal abelian variety $\mathcal{A}_{p_2}$ over $\mathbb{Z}_{p^2}$ along its special fiber is a SFD-module over the formal completion $\hat{\mathcal{C}}_{p_2}$ of $\mathcal{C}_{p_2}$, along its special fiber. We may base change $\hat{\mathcal{A}}_{p_2}$, and $\hat{\mathcal{C}}_{p_2}$ to $\mathbb{Z}_{p^2}$ obtaining a SFD-module $\hat{\mathcal{A}}_{p_2}$ over the formal scheme $\mathcal{C}_{p_2}$; of course, $\hat{\mathcal{A}}_{p_2}$ is the completion of $\mathcal{C}_{p_2}$ along its special fiber. The Cerednik-Drinfel’d Theorem ($\text{DT76}$, $\text{BC91}$ Théorème 5.3)) states the existence of an isomorphism of $\mathbb{Z}_{p^2}$-formal schemes $\Gamma_p \backslash \mathcal{H}_{p^2} \simeq \mathcal{A}_{p_2}$ which induced an isomorphism of $\mathbb{Z}_{p^2}$-formal schemes $\Gamma_p \backslash \mathcal{G} \simeq \mathcal{A}_{p_2}$ on the universal objects.

Under our assumptions, there is an isomorphism of $\mathbb{Z}_{p^2}$-formal schemes $\Gamma_p \backslash \mathcal{H}_{p^2} \simeq \Gamma_p \backslash \mathcal{H}_{p^2}$ ($\text{BC91}$ §3.5.3, $\text{JL85}$ Theorem 4.3’) from which we deduce an isomorphism of $\mathbb{Z}_{p^2}$-formal schemes $\Gamma_p \backslash \mathcal{H}_{p^2} \simeq \mathcal{C}_{p_2}$ which induces an isomorphism of $\mathbb{Z}_{p^2}$-formal schemes, equivalent for the quaternionic actions on both sides, $\Gamma_p \backslash \mathcal{G} \simeq \mathcal{A}_{p_2}$.

Denote $(X, \mathcal{O}_X) \sim (X_{rig}, \mathcal{O}_{X_{rig}})$ the rigidification functor which takes a proper scheme over a complete extension $F$ of $\mathbb{Q}_p$ to its associated rigid analytic space over $F$ ($\text{Bos14}$ §5.4). For each coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on $X$, we also denote $\mathcal{F}_{rig}$ the rigidification of $\mathcal{F}$, and for each morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules, we let $\varphi_{rig} : \mathcal{F}_{rig} \to \mathcal{G}_{rig}$ the corresponding morphism of rigid analytic sheaves ($\text{Bos14}$ §6). We have a rigid version of GAGA stating that $\mathcal{F} \sim \mathcal{F}_{rig}$ is an equivalence of categories between coherent $\mathcal{O}_X$-modules and coherent $\mathcal{O}_{X_{rig}}$-modules; we refer to $\text{Bos14}$ §6.3, Theorems 11, 12, 13), or [?] for details. Moreover, if $X$ is a proper $\mathcal{O}_F$-scheme, where $\mathcal{O}_F$ is the valuation ring of $F$, the generic fiber of the formal
completion $\hat{X}$ of $X$ along its special fiber coincides with $X^\text{rig}_F$, where $X_F = X \otimes_{\mathcal{O}_p} F$. Passing to the generic fiber, the Cerednik-Drinfel’d theorem then implies that there are isomorphisms of $\mathbb{Q}_p$-rigid analytic spaces

\[(38) \quad \Gamma_p \backslash \mathcal{H}_{p^2} \simeq \mathcal{C}_{\mathbb{Q}_p}^{\text{rig}}\]

and an isomorphism of $\mathbb{Q}_p$-rigid analytic spaces which is equivariant with respect to the quaternionic actions on both sides:

\[(39) \quad \Gamma_p \backslash \mathcal{G}^{\text{rig}} \simeq \mathcal{A}_{\mathbb{Q}_p}^{\text{rig}}.\]

5.2. Rigid analytic modular forms. A rigid analytic function $f : \mathcal{H}_p(\mathbb{C}_p) \to \mathbb{C}_p$ is said to be a rigid analytic modular form of weight $k$ and level $\Gamma_p$ if

\[f(\gamma z) = (cz + d)^k f(z)\]

for all $z \in \mathcal{H}_p(\mathbb{C}_p)$ and $\gamma \in \Gamma_p$, where $\gamma(z) = (az + b)/(cz + d)$. Denote $S_{k, \text{rig}}(\Gamma_p)$ the $\mathbb{C}_p$-vector space of rigid analytic modular forms of weight $k$ and level $\Gamma_p$. See [Dar04, §5.2] for details.

Given a $\mathbb{Z}[\Gamma_p]$-module $M$, we denote $M^{\Gamma_p}$ the submodule consisting of $\Gamma_p$-invariant elements of $M$. With notation as in [138], define $\hat{Q}^\text{unr}$-submodule $w_\mathcal{G}^{\Gamma_p}$ of $w_\mathcal{G}$ consisting of global sections which are invariant for the $\Gamma_p$-action. In particular, $w_\mathcal{G}^{\Gamma_p}$ is a $A^{\Gamma_p}$-module. Given $f \in S_2^{\text{rig}}(\Gamma_p)$, define $\omega_f = f(z) \otimes d\tau^{\otimes k}$ in $w_\mathcal{G}[\mathbb{C}_p] = w_\mathcal{G} \otimes_{\hat{Q}^\text{unr}} \mathbb{C}_p$.

**Lemma 5.1.** The correspondence $f \mapsto \omega_f = f(z) \otimes d\tau^{\otimes k}$ sets up a $\mathbb{C}_p$-linear isomorphism between $S_k^{\text{rig}}(\Gamma_p)$ and $w_\mathcal{G}^{\Gamma_p}[\mathbb{C}_p]$.

**Proof.** That $\omega_f$ belongs to $w_\mathcal{G}^{\Gamma_p}$ is because of [21]. The map $f \mapsto \omega_f$ has clearly an inverse because $\omega_f^0$ is an invertible $\mathcal{O}_{\mathcal{H}_p^\text{unr}}$-module, and the result follows. \qed

For any sheaf $\mathcal{F}$ on $\mathcal{H}_p^\text{unr}$, denote $\mathcal{F}^{\Gamma_p}$ the sheaf on $\Gamma_p \backslash \mathcal{H}_p^\text{unr}$ defined by taking $\Gamma_p$-invariant sections. Also, recall the sheaves $\omega_{\hat{Q}^\text{unr}}$ and $\mathcal{L}_{\hat{Q}^\text{unr}}$ introduced in [38] and [39].

**Lemma 5.2.** The isomorphisms [38] and [39] induce isomorphisms $(\omega_{\hat{Q}^\text{unr}})^{\Gamma_p} \simeq \omega_{\hat{Q}^\text{unr}}^{\text{rig}}$ and $(\mathcal{L}_{\hat{Q}^\text{unr}})^{\Gamma_p} \simeq \mathcal{L}_{\hat{Q}^\text{unr}}^{\text{rig}}$ of sheaves.

**Proof.** Recall that a basis of affinoid subsets of $\Gamma_p \backslash \mathcal{H}_p^\text{unr}$ is given by $\text{Sp}(A^{\Gamma_p})$ where $\text{Sp}(A)$ ranges over the affinoid subsets of $\mathcal{H}_p^\text{unr}$ such that $\Gamma_p$ acts on $A$ by a finite group ([?, §6]). It follows that the structural sheaf $\mathcal{O}_{\mathcal{H}_p^\text{unr}}^{\text{rig}} = \mathcal{C}_{\hat{Q}^\text{unr}}^{\text{rig}}$ of $\mathcal{C}_{\hat{Q}^\text{unr}}^{\text{rig}}$ is identified with the sheaf of $\Gamma_p$-invariant sections of $\mathcal{H}_p^\text{unr}$. The result follows from this, in light of [38], [39] and the construction of differentials and de Rham cohomology. \qed

**Proposition 5.3.** There are canonical isomorphisms of $\mathbb{C}_p$-vector spaces:

\[S_2^{\text{rig}}(\Gamma_p) \simeq w_\mathcal{G}^{\Gamma_p}[\mathbb{C}_p] = H^0(\mathcal{H}_p^\text{unr}, \omega_f^{\Gamma_p}) \mathcal{C}_p \simeq H^0(\mathcal{C}_{\hat{Q}^\text{unr}}^{\text{rig}}, \omega_{\hat{Q}^\text{unr}}^{\text{rig}}) \mathcal{C}_p.\]

**Proof.** Put together Lemmas [5.1] and [5.2]. \qed

5.3. The $p$-adic Shimura-Maass operator. Taking $\Gamma_p$-invariants defines a map, for integers $k \geq 0$ and $j \geq 0$ and understanding that $\mathcal{O}_p^{\text{rig}} = \mathcal{O}_p$, $\Theta_{p,k}^{\text{rig}} : (w_\mathcal{G}^{\Gamma_p})^p \to (w_\mathcal{G}^{\Gamma_p})^p$ where recall that $\Theta_{p,k}^{\text{rig}}$ was introduced in [34].
An alternative way to introduce $\Theta_{p,n}^{j,*}$ is the following. Recall the operator $\nabla_n$ in (11) and, for any integer $j \geq 0$, define $\nabla_n^j : \mathcal{L}_{Q_p^{unr},n}^{\rho} \rightarrow \mathcal{L}_{Q_p^{unr},n+2j}^{\rho}$ by the formula

$$
\nabla_n^j = \nabla_{n+2j} \circ \cdots \circ \nabla_{n+2} \circ \nabla_n.
$$

Considering the associated rigid analytic sheaves, and taking global sections, we obtain a map of $\mathcal{A}$-modules $\nabla_n^{j,\text{rig}} : \Gamma_{p,n}^{\rho} \rightarrow \Gamma_{p,n+2j}^{\rho}$. One may define the operator

$$
\Theta_{p,n}^{j} : w_{G,n}^{\rho} \xrightarrow{\text{ig.} \, \sigma} \Lambda_{G,n}^{\rho} \xrightarrow{\nabla_n^{j,\text{rig}}} \Lambda_{G,n+2j}^{\rho} \xrightarrow{\left(\Lambda_{G,n+2j}^{\rho} \ast \Psi_{\rho,n+2j} \ast \left(w_{G,n+2j}^{\rho}\right)\right)} \Gamma_{p,n+2j}^{\rho}.
$$

By (20), $d\tau^*$ is horizontal for $\nabla_n^\rho$, and therefore $\Theta_{p,n}$ coincides with the restriction of $\Theta_{p,k}^{j,*}$ to $w_{G,n}^{\rho}$.

5.4. Comparison of Shimura-Maass operators at CM points. Identify the set of $Q_p^i$-points in the rigid space $H_p$ with the set of $Q_p^i$-algebra homomorphisms $\text{Hom}(Q_p^i, M_2(Q_p^i))$ as follows: any $\Psi \in \text{Hom}(Q_p^i, M_2(Q_p^i))$ defines an action of $Q_p^i$ on $H_p(Q_p^i) = Q_p^i - Q_p^i$ by fractional linear transformations, and the point $z \in H_p(Q_p^i)$ associated with $\Psi$ is characterised by the property $\Psi(a)(\xi) = a(\xi)$, for all $a \in Q_p^i$.

Given a representation $\rho = (\rho_1, \rho_2) : GL_2 \times GL_2 \rightarrow GL(V)$, the stalk $\mathcal{E}(V)_{\Psi}$ of $\mathcal{E}(V)$ at a point $\Psi \in \text{Hom}(Q_p^i, M_2(Q_p^i))$ can be described explicitly. One first observes that the structure of filtered convergent $F$-isocrystal of $\mathcal{E}(V)$ induces a structure of filtered Frobenius module (IS03, §2) on the fiber $\mathcal{E}(V)_{\Psi}$. On the other hand, one attaches to such a pair $(V, \Psi)$ a filtered Frobenius module $V_{\Psi}$ in a natural way as follows. The underlying vector space $V_{\Psi}$ is $V_{Q_p^{unr}} = V \otimes_{Q_p^i} Q_p^{unr}$. The Frobenius is given by $\phi_V \otimes \sigma$, where $\phi_V = \rho_2 \left(\begin{smallmatrix} 0 & \rho_1 \\ \rho_1 & 0 \end{smallmatrix}\right)$ as before (thus, only depending on $\rho_2$). The filtration, only depending on $\rho_1$, has a more involved definition. Recall that any representation $V$ can be split into the direct sum of sub-representations $(V^{\rho_1}, \rho^{(\rho_1)})$ which are pure of weight $n$, and therefore it is enough to define the filtration for a representation $\rho : GL_2 \rightarrow GL(V)$ which is pure of weight $n$, since in the general case, the filtration $F^iV_{Q_p^{unr}}$ is by definition the direct sum of the filtrations $F^{i,n}V_{Q_p^{unr}}$ for all $n \in \mathbb{Z}$. If $V$ is pure of weight $n$, define $V_j$ to be the subspace of $V_{Q_p^{unr}}$ consisting of elements $v \in V$ satisfying the property $\rho(\Psi(a))(v) = a^j \sigma(a)^{n-j} v$ for all $a \in Q_p^i$. Define the filtration $F^iV_{Q_p^{unr}}$ of $V_{Q_p^{unr}}$ as the direct sum of $V_j$ for $j \geq i$. This equips $V_{Q_p^{unr}}$ with a structure of filtered Frobenius module, denoted $V_{\Psi}$. By [IS03] Lemma 4.2, $V_{\Psi} \simeq \mathcal{E}(V)_{\Psi}$ as filtered Frobenius modules.

To stress the dependence on $\Psi$, we denote $F^*V_{\Psi}$ the filtered of the Frobenius module $V_{\Psi}$; this is then a filtration on $V_{Q_p^{unr}}$ which depends on $\Psi$. Let $\text{gr}^i(F^*V_{\Psi}) = F^iV_{\Psi}/F^{i-1}V_{\Psi}$ be the graded pieces of the filtration. If $V$ is pure of weight $n$, we have a canonical isomorphism $\text{gr}^i(F^*V_{\Psi}) \simeq V_i$ as well as a decomposition $V_{Q_p^{unr}} = \bigoplus_{i \in \mathbb{Z}} \text{gr}^i(F^*V_{\Psi})$.

For $\Psi \in \text{Hom}(Q_p^i, M_2(Q_p^i))$, denote $\bar{\Psi}$ the morphism of $Q_p^i$-algebras obtained by composition $\Psi$ with the main involution of $M_2(Q_p^i)$; therefore, if $\Psi(x) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then $\bar{\Psi}(x) = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. If $V$ is pure of weight $n$, then the graduate pieces $\text{gr}^i(F^*V_{\Psi})$ and $\text{gr}^{n-i}(F^*V_{\Psi})$ are equal, for all $i \in \mathbb{Z}$. In particular, for $V = M_2$ we have

$$
\text{gr}^1(F^*(M_2)) \simeq \text{gr}^2(F^*(M_2)).
$$

and therefore there is an exact sequence:

$$
0 \rightarrow \text{gr}^1(F^*(M_2)) \rightarrow (M_2)_{\Psi} \rightarrow \text{gr}^1(F^*(M_2)) \rightarrow 0
$$

and a canonical decomposition

$$
(M_2)_{\Psi} \simeq \text{gr}^1(F^*(M_2)) \bigoplus \text{gr}^1(F^*(M_2)).
$$
One can choose generators $\omega_1, \omega_2$ of the $\hat{Q}_p^{\text{unr}}$-vector space $\text{gr}^1(\mathbb{F}^*\mathbb{M}_2)_{\psi}$ so that $\omega_1$ and $\omega_2$ are defined over $\mathbb{Q}_p$. Then $\overline{\omega}_1$ and $\overline{\omega}_2$ are generators of the $\hat{Q}_p^{\text{unr}}$-vector space $\text{gr}^1(\mathbb{F}^*\mathbb{M}_2)_{\psi}$, where $\omega_i \mapsto \overline{\omega}_i$ for $i = 1, 2$ denotes the action of $\text{Gal}(\hat{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $\omega_i$. If therefore follows that the Hodge splitting coincides on quadratic points with the projection $(\mathbb{M}_2)_{\psi} \to \text{gr}^1(\mathbb{F}^*\mathbb{M}_2)_{\psi}$ to the first factor in the decomposition \( (\hat{\mathbb{Q}}, \mathbb{Q}) \).

We now apply the above results to the situation of the previous sections. Recall that $K$ is a imaginary quadratic field and $f \in H^0(\mathbb{C}_\mathbb{Q}, \mathcal{O}_\mathbb{Q}^k)$ is an algebraic modular form of weight $k$ and level $N^+N^-$ with $p \nmid N = N^+N^-$ and $(N^+, N^-) = 1$. We write $f_\infty : \mathcal{H}_p \to \mathbb{C}$ and $f_p : \mathcal{H}_p \to \mathbb{C}_p$ for the holomorphic and the rigid analytic modular forms corresponding to $f$, respectively. Assume $N^p$ is a product of an even number of distinct primes, each of them inert in $K$, and that all primes dividing $N^+$ are split in $K$. Let $P \in \mathcal{C}_\mathbb{Q}(K)$ be a Heegner point, and assume that $P \in \mathcal{C}_\mathbb{C}(\mathbb{C})$ represented by the point $\tau_\infty \in \mathcal{H}_\infty$ modulo $\Gamma_\infty$, and $P \in \mathcal{C}_\mathbb{C}_p(\mathbb{C}_p)$ is represented by the point $\tau_p \in \mathcal{H}_p$ modulo $\Gamma_p$. Fix embeddings $\mathbb{Q} \hookrightarrow \hat{\mathbb{Q}}_p$ and $\hat{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, which allows us to view algebraic numbers as complex and $p$-adic numbers.

**Theorem 5.4.** For any positive integer $j$ we have the equality

$$\Theta^j_{\infty,k}(f_\infty)(\tau_\infty) = \Theta^j_{p,k}(f_p)(\tau_p).$$

**Proof.** We mimic a well known argument of Katz when $p$ is split in $K$ ([Kat78, Theorems 2.4.5, 2.4.7]; see also [BDP13, Proposition 1.12], [HLM, Theorem 3.5], [Mor11, Proposition 2.12]). Let $A_P$ be the false elliptic curve corresponding to the Heegner point $P$. The algebraic CM splitting of $A_P$ coincides both with the Hodge splitting and the $p$-adic setting, and therefore the values of $\Psi_{\infty,n}$ and $\Psi_{p,n}$ at CM points are the same. Since the construction of the Shimura-Maass operators is algebraic, we see that $\nabla^\text{an}_n(f_\infty)$ coincides with $\nabla^\text{rig}_n(f_p)$, and the same still holds for the iterates of the Shimura-Maass operator, which also admit an algebraic construction. The result follows. \( \square \)

### 5.5. Nearly rigid analytic modular forms.

In this subsection we make explicit the relation between the results of this paper and those of Franc’s thesis [Fra11]; it is independent from the rest of the paper.

We first introduce a $\mathbb{C}_p$-subspace of $\mathcal{C}$, which plays a role analogue to that of nearly holomorphic functions in the real analytic setting. For this part, we closely follow [Fra11]. The assignment $X \mapsto 1/(z - z^*)$ defines an injective homomorphism $\mathcal{A}[X] \hookrightarrow \mathbb{C}$ ([Fra11, Proposition 4.3.3]). Define the $\mathcal{A}$-algebra $\mathcal{N}$ of nearly rigid analytic functions on to be the image of this map (cf. [Fra11, Definition 4.3.5]). By definition, $\mathcal{N}$ is a sub-$\mathcal{A}$-algebra of $\mathcal{A}^*$. The $\mathcal{A}$-algebra $\mathcal{N}$ is equipped with a canonical graduation $\mathcal{N} = \bigoplus_{j \geq 0} \mathcal{N}^{(j)}$ where for each integer $j \geq 0$, we denote $\mathcal{N}^{(j)}$ the sub-$\mathcal{A}$-algebra of $\mathcal{N}$ consisting of functions $f$ which can be written in the form

$$f(z) = \sum_{i=0}^{j} f_i(z) \frac{1}{(z - \sigma(z))^i}$$

with $f_i \in \mathcal{A}$. The Shimura-Maass operator $\delta_{p,k}$ restricts to an operator (denoted with the same symbol) $\delta_{p,k} : \mathcal{N} \to \mathcal{N}$ which takes $\mathcal{N}^{(j)}$ to $\mathcal{N}^{(j+2)}$.

Define now $\mathcal{N}_k(\Gamma_p) = \mathcal{N}^{(j)}_{k,\text{rig}}$ to be the $\mathbb{C}_p$-subalgebra of $\mathcal{N}$ consisting of functions which are invariant under the weight $k$ action of $\Gamma_p$ on $\mathcal{N}$, namely, those functions satisfying the transformation property $f(\gamma z) = (cz + d)^k f(z)$ for all $z \in \hat{\mathbb{Q}}_p^{\text{unr}} - \mathbb{Q}_p$ and $\gamma \in \Gamma_p$. Note that $S_{k,\text{rig}}^{\text{rig}}(\Gamma_p) \subseteq \mathcal{N}_k(\Gamma_p)$. We call $\mathcal{N}_k(\Gamma_p)$ the $\mathbb{C}_p$-vector space of nearly rigid analytic modular forms of weight $k$ and level $\Gamma_p$. Define also $\mathcal{N}_k^{(j)}(\Gamma_p) = \mathcal{N}_k(\Gamma_p) \cap \mathcal{N}^{(j)}$. The operator $\delta_{p,k}$ introduced in 4.7 restricts to a map $\delta_{p,k} : \mathcal{N}_k(\Gamma_p) \to \mathcal{N}_{k+2}(\Gamma_p)$ ([Fra11, Lemma 4.3.8]). By [Fra11]...
Theorem 4.3.11], for each integer \( r \geq 0 \) we have an isomorphism of \( \mathbb{C}_p \)-vector spaces
\[
\bigoplus_{j=0}^{r} S_{k+2(r-j)}^{\text{rig}}(\Gamma_p) \simeq \Lambda_{p,k+2r}^{(r)}(\Gamma_p)
\]
which maps \((h_j)_{j=0}^{k+2(r-j)}\) to \(\sum_{j=0}^{k+2(r-j)} \delta_{p,k}^j(h_j)\).

**Corollary 5.5** (Franc). Let \( \tau_p \in \mathcal{H}_p \) corresponds to a Heegner point. The values \( \Theta_{\infty,k}^j(f)(\tau_p) \)
to \(\mathcal{H}_p\) are algebraic for each integer \( j \geq 0 \).

**Proof.** The result is clear from Theorem 5.4 since this is known for \( \Theta_{\infty,k}^j(f)(\tau_\infty) \).

**Remark 5.6.** Equation (37) answers affirmatively one of the questions left in [Fra11, §6.1] whether if it was possible to describe the \( p \)-adic Shimura-Maass operator \( \delta_{p,k}^j \), introduced in [Fra11] in a more conceptual way, similar to that in the complex case. Corollary 5.5 is the main result of [Fra11], which was obtained via a completely different method, following more closely the complex analytic approach of Shimura.

### 6. The Coleman primitive

Write \( \nabla = \nabla^*_G, n \), \( \nabla^{1,0} = \nabla^*_{G, 0} \), \( \nabla^{0,1} = \nabla^*_{G, 0} \) and \( \langle , \rangle = \langle , \rangle^*_{G, n} \) to simplify the notation. For any \( n \) and any \( j \), whenever there is not possible confusion, we write \( \Theta_p = \Theta_{\infty,n}^* \) and \( \Theta_{p,n}^j = \Theta_{p,k}^j \) for the \( p \)-adic Shimura-Maass operator, and \( \Psi_p = \Psi_{p,n}^* \) for the splitting of the Hodge filtration.

We set up the notation \( \omega_{\text{can}} = dz \) and \( \eta_{\text{can}} = dx^*_{z^* - z} \). Since \( \langle dx, dy \rangle = -1 \), we have \( \langle \omega_{\text{can}}, \eta_{\text{can}} \rangle = 1 \). We also write \( \omega_{\text{can}}^{n-j} = \omega_{\text{can}}^j \otimes \eta_{\text{can}}^{n-j} \).

The computation of the Gauss-Manin connection gives
\[
\nabla(\omega_{\text{can}}) = \left( \frac{\omega_{\text{can}}}{z^* - z} + \eta_{\text{can}} \right) \otimes dz,
\]
\[
\nabla(\eta_{\text{can}}) = \frac{\omega_{\text{can}} \otimes dz^*}{(z^* - z)^2} + \eta_{\text{can}} \otimes dz.
\]

Let \( f : \mathcal{H}_p \to \mathbb{C}_p \) be a rigid modular form giving rise to a section \( \omega_f = f(z) \otimes d\tau^* \). Put \( n = k - 2 \). Using the Kodaira-Spencer map, we identify this with \( \omega_f = f(z)dz \otimes d\tau^n \). Let \( F_j \) be the Coleman primitive of the differential form \( \omega_f \), satisfying the differential equation
\[
\nabla(F_j) = \omega_f.
\]

Define for \( j = n/2, \ldots, n \) an integer
\[
G_j(z) = \langle F_j(z), \omega_{\text{can}}^{n-j} \rangle \otimes \omega_{\text{can}}^{n-2j}.
\]

**Theorem 6.1.** \( \Theta_{p}^{j+1}(G_j) = j! \omega_f \).

**Proof.** This result, which is proved by means of a simple and explicit computation, is the analogue of [BDP13, Proposition 3.24] (and also of [HB15, Theorem 7.3]), but we provide a complete proof since our formalism is quite different from that in [BDP13], where one can use the Tate curve and the \( q \)-expansion principle. As in loc. cit. we show that \( \Theta_p G_0(z) = \omega_f \) and \( \Theta_p(G_j(z)) = j! G_{j-1}(z) \).

We first compute \( \nabla(G_0(z)) \). We have:
\[
\nabla(G_0(z)) = \nabla(\langle F_j(z), \eta_{\text{can}} \rangle) \otimes \omega_{\text{can}}^n
\]
\[
= \langle \nabla(F_j(z)), \eta_{\text{can}}^{n} \rangle \otimes \omega_{\text{can}}^{n} + \langle F_j(z), \nabla(\eta_{\text{can}}^{n}) \rangle \otimes \omega_{\text{can}}^{n} + \langle F_j(z), \eta_{\text{can}}^{n} \rangle \otimes \nabla(\omega_{\text{can}}^{n})
\]
\[
= \langle f(z)dz \otimes \omega_{\text{can}}^{n}, \eta_{\text{can}}^{n} \rangle \otimes \omega_{\text{can}}^{n} + \langle F_j(z), \nabla(\eta_{\text{can}}^{n}) \rangle \otimes \omega_{\text{can}}^{n} + \langle F_j(z), \eta_{\text{can}}^{n} \rangle \otimes \nabla(\omega_{\text{can}}^{n}).
\]
We now compute the last two pieces:
\[
\langle F_j(z), \nabla (\eta_{\text{can}}^n) \rangle \otimes \omega_{\text{can}}^n = \langle F_j(z), n\eta_{\text{can}}^{n-1} \nabla (\eta_{\text{can}}) \rangle \otimes \omega_{\text{can}}^n \\
= \left( \langle F_j(z), n\eta_{\text{can}}^{n-1} \frac{-\omega_{\text{can}} \otimes dz^*}{(z^* - z)^2} + \eta_{\text{can}} \otimes dz \rangle \right) \otimes \omega_{\text{can}}^n \\
= -\left( \langle F_j(z), n\eta_{\text{can}}^{n-1} \omega_{\text{can}} \otimes dz^* \rangle \otimes \omega_{\text{can}}^n + \langle F_j(z), \frac{n\eta_{\text{can}}^n \otimes dz}{z^* - z} \rangle \otimes \omega_{\text{can}}^n \right) \\
= -\langle F_j(z), \eta_{\text{can}}^{n-1} \omega_{\text{can}} \rangle \otimes \frac{n\omega_{\text{can}}^n \otimes dz^*}{(z^* - z)^2} + \langle F_j(z), \eta_{\text{can}}^n \rangle \otimes \frac{n\omega_{\text{can}}^n \otimes dz}{z^* - z} 
\]
and
\[
\langle F_j(z), \eta_{\text{can}}^n \rangle \otimes \nabla (\omega_{\text{can}}^n) = \langle F_j(z), \eta_{\text{can}}^n \rangle \otimes \nabla (\omega_{\text{can}}^n) \\
= \langle F_j(z), \eta_{\text{can}}^n \rangle \otimes n\omega_{\text{can}}^{n-1} \nabla (\omega_{\text{can}}) \\
= \langle F_j(z), \eta_{\text{can}}^n \rangle \otimes n\omega_{\text{can}}^{n-1} \left( -\frac{\omega_{\text{can}}}{z^* - z} + \eta_{\text{can}} \right) \otimes dz \\
= -\langle F_j(z), \eta_{\text{can}}^n \rangle \otimes \frac{n\omega_{\text{can}}^n \otimes dz}{z^* - z} + \langle F_j(z), \eta_{\text{can}}^n \rangle \otimes n\omega_{\text{can}}^{n-1} \eta_{\text{can}} \otimes dz. 
\]
Therefore the sum of these two pieces gives:
\[
-\langle F_j(z), \eta_{\text{can}}^{n-1} \omega_{\text{can}} \rangle \otimes \frac{n\omega_{\text{can}}^n \otimes dz^*}{(z - z^*)^2} + \langle F_j(z), \eta_{\text{can}}^n \rangle \otimes n\omega_{\text{can}}^{n-1} \eta_{\text{can}} \otimes dz. 
\]
Recall now that \( \Psi(\eta_{\text{can}}) = 0 \) and \( \Psi(dz^*) = 0 \). Therefore, using the Kodaira-Spencer map to replace \( dz \) with \( \omega_{\text{can}}^2 \), and applying \( \Psi \) we have
\[
\Theta_p(G_0(z)) = \omega_f(\omega_{\text{can}}^n, \eta_{\text{can}}^n) = \omega_f. 
\]
We now compute \( \nabla(G_j(z)) \) for \( j \geq 1 \). The Gauss-Manin connection
\[
\nabla(G_j(z)) = \nabla (\langle F_j(z), \omega_{\text{can}}^j \eta_{\text{can}}^{n-j} \rangle \otimes \omega_{\text{can}}^{n-2j}) 
\]
is the sum of three terms
(43)
\[
\langle \nabla(F_j(z)), \omega_{\text{can}}^j \eta_{\text{can}}^{n-j} \rangle \otimes \omega_{\text{can}}^{n-2j} + \langle F_j(z), \nabla(\omega_{\text{can}}^j \eta_{\text{can}}^{n-j}) \rangle \otimes \omega_{\text{can}}^{n-2j} + \langle F_j(z), \omega_{\text{can}}^j \eta_{\text{can}}^{n-j} \rangle \otimes \nabla(\omega_{\text{can}}^{n-2j}) 
\]
which we calculate separately as before. First, since \( j > 0 \), we have
\[
\langle \nabla(F_j(z)), \omega_{\text{can}}^j \eta_{\text{can}}^{n-j} \rangle \otimes \omega_{\text{can}}^{n-2j} = \langle f(z) dz \otimes \omega_{\text{can}}^n, \omega_{\text{can}}^j \eta_{\text{can}}^{n-j} \rangle = 0. 
\]
Next, a simple computation shows that
\[
\nabla(\omega_{\text{can}}^j \eta_{\text{can}}^{n-j}) = (n - 2j) \omega_{\text{can}}^j \eta_{\text{can}}^{n-j} \otimes \frac{dz}{z^* - z} + j \omega_{\text{can}}^j \eta_{\text{can}}^{n-j} \otimes dz - (n - j) \omega_{\text{can}}^{j+1} \eta_{\text{can}}^{n-j-1} \otimes \frac{dz^*}{(z^* - z)^2} 
\]
and therefore the second summand in (43) is
\[
\langle F_j(z), \nabla(\omega_{\text{can}}^j \eta_{\text{can}}^{n-j}) \rangle \otimes \omega_{\text{can}}^{n-2j} = (n - 2j) \langle F_j(z), \omega_{\text{can}}^j \eta_{\text{can}}^{n-j} \rangle \otimes \omega_{\text{can}}^{n-2j} \otimes \frac{dz}{z^* - z} - \\
\quad j \langle F_j(z), \omega_{\text{can}}^j \eta_{\text{can}}^{n-j} \rangle \otimes \omega_{\text{can}}^{n-2j} \otimes dz + \\
\quad - (n - j) \langle F_j(z), \omega_{\text{can}}^{j+1} \eta_{\text{can}}^{n-j-1} \rangle \otimes \omega_{\text{can}}^{n-2j} \otimes \frac{dz^*}{(z^* - z)^2} 
\]
Thus the Hodge filtration of the de Rham cohomology is given by

\( \epsilon \)

We define \( (46) \)

\( \epsilon \)

On the other hand, by \([HB15, Proposition 6.4]\), we can define a projector \( (45) \)

\( \epsilon \)

As shown in \([HB15, Corollary 6.3]\), one can define a projector \( \Theta_p(G_j(z)) = j(F_j(z), \omega_{can}^{-1}(j-1)) \subset \omega_{can}^{-2(j-1)} = jG_{j-1}(z). \)

The result follows.

7. The generalised Kuga-Sato motive

Fix an even integer \( k \geq 2 \) and put \( n = k - 2, m = n/2 \). Let \( A_0 \) be a false elliptic curve with quaternionic multiplication and full level-\( M \) structure, defined over \( H \) (the Hilbert class field of \( K \)) and with complex multiplication by \( \mathcal{O}_K \); the action of \( \mathcal{O}_K \) is required to commute with the quaternionic action, and this implies that \( A_0 \) is isogenous to \( E \times E \) for an elliptic curve \( E \) with CM by \( \mathcal{O}_K \). Fix a field \( F \supset H \) and consider the \( (2n+1) \)-dimensional variety \( X_m \) over \( F \) given by

\[ X_m := A^m \times A_0^m. \]

Here and in the following we simplify the notation and simply write \( A, C \) and \( A_0 \) for \( A_F, C_F \) and \( (A_0)_F \), unless we need to stress the field of definition in which case we keep the full notation. The variety \( X_m \) is equipped with a proper morphism \( \pi: X_m \rightarrow C \) with \( 2n \)-dimensional fibers. The fibers above points of \( C \) are products of the form \( A^m \times A_0^m \).

The de Rham cohomology of \( C \) attached to \( \mathcal{L}_n \), denoted \( H^1_{\text{dR}}(C, \mathcal{L}_n, \nabla) \), is defined to be the 1-st hypercohomology of the complex

\[ 0 \rightarrow \mathcal{L}_n \rightarrow \mathcal{L}_n \otimes \Omega^1_C \rightarrow 0. \]

As shown in \([HB15, Corollary 6.3]\), one can define a projector \( \epsilon_A \) (denoted \( P \) in \textit{loc. cit.}) in the ring of correspondences \( \text{Corr}_C(A^m, A^m) \), such that

\[ \epsilon_A H^*_{\text{dR}}(A_m/F) \subseteq H^{n+1}_{\text{dR}}(A_m/F), \]

\[ \epsilon_A H^*_{\text{dR}}(A_m/F) \cong H^1_{\text{dR}}(C, \mathcal{L}_n, \nabla). \]

On the other hand, by \([HB15, Proposition 6.4]\), we can define a projector \( \epsilon_A \in \text{Corr}(A_0^m, A_0^m) \) (which is defined by means of \( \epsilon_A \)) such that

\[ \epsilon_{A_0} H^*_{\text{dR}}(A_0^m/F) = \text{Sym}^n e H^1_{\text{dR}}(A_0/F). \]

The projectors \( \epsilon_A \) and \( \epsilon_A \) are commuting idempotents when viewed in the ring \( \text{Corr}_C(X_m, X_m) \). We define \( \epsilon = \epsilon_A \epsilon_{A_0} \) and denote \( D \) the motive \( (X_m, \epsilon) \). By \([HB15, Proposition 6.5]\) and \([14] \), \([15] \), \([44] \) we see that

\[ \epsilon H^i_{\text{dR}}(X_m/F) = \begin{cases} H^i_{\text{dR}}(C, \mathcal{L}_n, \nabla) \otimes \text{Sym}^n e H^1_{\text{dR}}(A_0^m/F), & \text{if } i = 2n + 1, \\ 0, & \text{if } i \neq 2n + 1. \end{cases} \]

Thus the Hodge filtration of the de Rham cohomology is given by

\[ F^{n+1} \left( \epsilon H^i_{\text{dR}}(X_m/F) \right) = F^{n+1} \left( H^i_{\text{dR}}(C, \mathcal{L}_n, \nabla) \right) \otimes \text{Sym}^n e H^1_{\text{dR}}(A_0^m/F). \]
Finally we have a map ([HB15 page 4221])

\[ H^0(\mathcal{C}, \omega_{\mathcal{O}}^{\otimes n+2}) \rightarrow F^n (H^1_{\text{dR}}(\mathcal{C}, \mathcal{L}_n, \nabla)). \]

8. The Abel-Jacobi Map

Let \( \Delta \) be the class a null-homologous codimension-(n+1) cycle \( \Delta \) in \( CH^{n+1}(D)(F) \), where \( F \) is as in Section [a field containing the Hilbert class field of \( K \). One may associate to \( [\Delta] \) the isomorphism class of the extension

\[ 0 \rightarrow \epsilon H^{2n+1}_{\text{et}}(\overline{X}_m, \mathbb{Q}_p(n+1)) \rightarrow E \rightarrow \mathbb{Q}_p \rightarrow 0 \]

in

\[ \text{Ext}^1_{G_F}(\mathbb{Q}_p, \epsilon H^{2n+1}_{\text{et}}(\overline{X}_m, \mathbb{Q}_p(n+1))) \]

(where \( \text{Ext}^1_{G_F} \) denotes the first Ext group in the category of \( G_F = \text{Gal}(\overline{F}/F) \)-modules) given by the pull-back of

\[ 0 \rightarrow \epsilon H^{2n+1}_{\text{et}}(\overline{X}_m, \mathbb{Q}_p(n+1)) \rightarrow \epsilon H^{2n+1}_{\text{et}}(\overline{X}_m - [\Delta], \mathbb{Q}_p(n+1)) \rightarrow \text{Ker} \left( \epsilon H^{2n+2}_{\text{et}}_{/\Delta}(\overline{X}_m, \mathbb{Q}_p(n+1)) \rightarrow \epsilon H^{2n+2}_{\text{et}}(\overline{X}_m, \mathbb{Q}_p(n+1)) \right) \rightarrow 0 \]

via the map \( \mathbb{Q}_p \rightarrow \epsilon H^{2n+2}_{\text{et}}(\overline{X}_m, \mathbb{Q}_p(n+1)) \) sending 1 to the cycle class \( c_{\overline{X}_m}([\Delta]) \) of \( \Delta \) in \( H^{2n+2}_{\text{et}}(\overline{X}_m, \mathbb{Q}_p(n+1)) \). This association defines a map, called \( p \)-adic étale Abel-Jacobi map

\[ A\text{J}_p: CH^{n+1}(D)(F) \rightarrow \text{Ext}^1_{G_F}(\mathbb{Q}_p, \epsilon H^{2n+1}_{\text{et}}(\overline{X}_m, \mathbb{Q}_p(n+1))). \]

Let \( v \) be the place of \( F \) above \( p \) induced by the inclusion \( F \subseteq \overline{Q} \hookrightarrow \mathbb{C}_p \), which for simplicity we assume to be unramified over \( p \). We now describe the restriction of \( A\text{J}_p \) to \( CH^{n+1}(D)(F_v) \). Consider the base change of \( X_m \) and \( C \) to \( F_v \) that we still denote by \( X_m \) and \( C \) in this section. Since the motive \( X_m \) has semistable reduction at \( v \), the image of the Abel-Jacobi map is contained in the first Ext group in the category of semistable representations; using [IS03 Lemma 2.1], and following the argument in [IS03 page 362] (see also [LP19 §4.2]) the Abel-Jacobi map gives a map, denoted with the same symbol by a slight abuse of notation,

\[ A\text{J}_p: CH^{n+1}(D)(F_v) \rightarrow \text{Ext}^1_{G_F}(\mathbb{Q}_p, \epsilon H^{2n+1}_{\text{et}}(\overline{X}_m, \mathbb{Q}_p(n+1))). \]

where \( D_{\text{st}, F_v} \) is the Fontaine’s semistable functor from the category of \( G_{F_v} = \text{Gal}((\overline{F}_v/F_v) \)-representations to the category \( M_{F_v}^{\text{st}} \) of filtered Frobenius monodromy modules over \( F_v \), and we denote as usual by \( F^i(D) \) the \( i \)-step filtration of a filtered Frobenius monodromy module \( D \).

By [Tsu98, Fal02] (see also [Tsu99]) we know that \( D_{\text{st}, F_v}(H^{2n+1}_{\text{et}}(X_m, \mathbb{Q}_p)) \) is isomorphic to the de Rham cohomology group \( H^{2n+1}_{\text{dR}}(X_m/F_v) \) as filtered Frobenius monodromy modules. Therefore, applying the idempotent \( \epsilon \), we obtain the isomorphism

\[ \frac{D_{\text{st}, F_v}(\epsilon H^{2n+1}_{\text{et}}(\overline{X}_m, \mathbb{Q}_p(n+1)))}{F^{n+1}(D_{\text{st}, F_v}(\epsilon H^{2n+1}_{\text{et}}(\overline{X}_m, \mathbb{Q}_p(n+1))))} \simeq \frac{\epsilon H^{2n+1}_{\text{dR}}(X_m/F_v)(n+1)}{F^{n+1}(\epsilon H^{2n+1}_{\text{dR}}(X_m/F_v)(n+1))}. \]

By Poincaré duality,

\[ \frac{\epsilon H^{2n+1}_{\text{dR}}(X_m/F_v)(n+1)}{F^{n+1}(\epsilon H^{2n+1}_{\text{dR}}(X_m/F_v)(n+1))} \simeq (F^{n+1}(\epsilon H^{2n+1}_{\text{dR}}(X_m/F_v)(n+1)))^\vee \]

where \( V^\vee \) denotes dual of \( F_v \)-vector spaces. Combining [HS] and dualizing (49) we obtain a map

\[ (F^{n+1}(\epsilon H^{2n+1}_{\text{dR}}(X_m/F_v)(n+1)))^\vee \rightarrow (M_k(C, F_v) \otimes \text{Sym}^n \epsilon H^1_{\text{dR}}(A^m_0/F))^\vee. \]
The $p$-adic Abel-Jacobi map for the nullhomologous $(n + 1)$-th Chow cycles of the motive $D$ can thus be viewed as a map, denoted with the same symbol by an abuse of notation:

$$AJ_p : CH^{n+1}_0(D)(F_v) \to (M_{k_0}(X, F_v) \otimes \text{Sym}^n H^1_{\text{dR}}(A/F_v))^\vee.$$ 

9. Generalized Heegner cycles

9.1. Definition. Let $\varphi : A_0 \to A$ be an isogeny (defined over $K$) of false elliptic curves, of degree prime to $N^+$, i.e. whose kernel intersects the level structures of $A_0$ trivially. Let $P_A$ be the point on $C$ corresponding to $A$ with level structure given by composing $\varphi$ with the level structure of $A_0$. We associate to any pair $(\varphi, A)$ a codimension $n + 1$ cycle $\Upsilon_\varphi$ on $X_m$ by defining

$$\Upsilon_\varphi := (\Gamma_\varphi)^m \subset (A \times A_0)^m$$

where $\Gamma_\varphi = \{(\varphi(x), x) : x \in A_0\} \subset A \times A_0$ is the graph of $\varphi$. We then set

$$\Delta_\varphi := \epsilon \Upsilon_\varphi.$$

The cycle $\Delta_\varphi$ of $D$ is supported on the fiber above $P_A$ and has codimension $n + 1$ in $A^m \times A_0^m$, thus $\Delta_\varphi \in CH^{n+1}_0(D)$. By (47), the cycle $\Delta_\varphi$ is homologous to zero.

We now compute the image of $\Delta_\varphi$ under the Abel-Jacobi map. The de Rham cohomology group $H^1_{\text{dR}}(A/F)$ of a false elliptic curve $A$ defined over a field $F$ is equipped with the Poincaré pairing $\langle ., . \rangle_{H^1_{\text{dR}}(A/F)}$, which we simply denote $\langle ., . \rangle_A$. Fix a nonvanishing differential $\omega_{A_0}$ in $\epsilon \Omega^1_{A_0/F}$. This fixed differential determines a class $\eta_{A_0} \in eH^1(A_0, O_{A_0})$ dual to $\omega_{A_0}$ under the Poincaré duality pairing $\langle ., . \rangle_{A_0}$, normalised so that $\langle \omega_{A_0}, \eta_{A_0} \rangle_{A_0} = 1$. We can view $\{\omega_{A_0}, \eta_{A_0}\}$ as a basis of $eH^1_{\text{dR}}(A_0/F)$ since the Hodge exact sequence

$$0 \to \Omega^1_{A_0/F} \to H^1_{\text{dR}}(A_0/F) \to H^1(A_0, O_{A_0}) \to 0$$

splits, because $A_0$ has CM. This yields a basis for $\text{Sym}^n eH^1_{\text{dR}}(A_0/F)$ given by the elements $\omega_{A_0}^j \otimes \eta_{A_0}^{n-j}$ for $j$ an integer such that $0 \leq j \leq n$.

Let $\omega_f$ be the global section of the sheaf $\omega_n \otimes \Omega^1_G$ associated to the modular form over the Shimura curve $C$ which corresponds to $f$ under the Jacquet-Langlands correspondence. The aim of this section is to compute

$$AJ_p(\Delta_\varphi)(\omega_f \otimes \omega^j \eta^{n-j})$$

for $j = 0, \ldots, n$, following [BDP13], [HB15], and [IS03].

Define

$$L_{n,n} = L_n \otimes \text{Sym}^n eH^1_{\text{dR}}(A_0/F).$$

The Gauss-Manin connection on $L_n$ combined with the trivial connection on $H^1_{\text{dR}}(A_0/F)$, gives rise to the connection

$$\nabla : L_{n,n} \to L_{n,n} \otimes \Omega^1_G.$$ 

The de Rham cohomology groups attached to $(L_{n,n}, \nabla)$ are defined to be the hypercohomology of the complex

$$0 \to L_{n,n} \xrightarrow{\nabla} L_{n,n} \otimes \Omega^1_G \to 0.$$

We have

$$H^1_{\text{dR}}(C, L_{n,n}, \nabla) = H^1_{\text{dR}}(C, L_n, \nabla) \otimes \text{Sym}^n eH^1_{\text{dR}}(A_0/F).$$

Let $z_A$ be the point in $C$ corresponding to $A$, and denote $X_{m,z_A}$ the fiber of $X_m$ over $z_A$. Recall the cycle class map

$$c_{z_A} : CH^n(X_{m,z_A})_Q(F) \to H^2_{\text{et}}(X_{m,z_A}, Q_p)(n).$$

Since $\Delta_\varphi = \epsilon \Upsilon_\varphi$, the image $c_{z_A}(\Delta_\varphi)$ of $\Delta_\varphi$ under the cycle class map $c_{z_A}$ belongs to $\epsilon H^2_{\text{et}}(X_{m,z_A}, Q_p)(n)$. Since $D_{st,F_v}(H^2_{\text{et}}(X_m, Q_p))$ is isomorphic to the de Rham cohomology group $H^2_{\text{dR}}(X_m/F_v)$ as filtered Frobenius monodromy modules, we still denote $c_{z_A}(\Delta_\varphi)$,
with a slight abuse of notation, the image of $cl_{\Delta}(\Delta_{\varphi})$ under the functor $D_{at,F_v}$ and the isomorphism with Rham cohomology group; so we finally end up with the element

$$\cl_{\Delta}(\Delta_{\varphi}) \in \epsilon H^1_{\text{dR}}(X_m/F_v) = H^1_{\text{dR}}(\mathcal{C}, \mathcal{L}_n, \nabla) \otimes \text{Sym}^n \epsilon H^1_{\text{dR}}(A_0^m/F).$$

9.2. Rigid analysis on Mumford curves. In this subsection we work over $\hat{\mathbb{Q}}_p^{unr}$. To simplify the notation, we suppress the symbol $\hat{\mathbb{Q}}_p^{unr}$; thus we write $\mathcal{C} = C_{\hat{\mathbb{Q}}_p^{unr}}$, $\mathcal{C}_{\text{rig}} = C_{\hat{\mathbb{Q}}_p^{unr}}^{\text{rig}}$, $\mathcal{L}_n = L_{n,\hat{\mathbb{Q}}_p^{unr}}$, $\mathcal{L}_n^{\text{rig}} = L_{n,\hat{\mathbb{Q}}_p^{unr}}^{\text{rig}}$, and $V_n = V_n \otimes_{\hat{\mathbb{Q}}_p} \hat{\mathbb{Q}}_p^{unr}$.

9.2.1. Structure of filtered Frobenius monodromy modules. We first derive a description of $H^1_{\text{dR}}(\mathcal{C}_{\text{rig}}, L_{n,n}^{\text{rig}}, \nabla)$ by means of $V_n$-valued differential forms on $\mathcal{H}_p^{unr}$.

**Lemma 9.1.** $\epsilon H^1_{\text{dR}}(\mathcal{G}/\hat{\mathcal{H}}) \simeq \mathcal{E}(V_1)$ as filtered convergent $F$-isocrystals on $\mathcal{H}_p^{unr}$.

**Proof.** The representation $(M_2, \rho_1, \rho_2)$ is isomorphic to $V_1 \otimes V_1 = (V_1 \otimes V_1, \sigma_1, \sigma_2)$, where $\sigma_1(A)(R_1 \otimes R_2) = (A \cdot R_1) \otimes R_2$ and $\sigma_2(A)(R_1 \otimes R_2) = R_1 \otimes (A^t \cdot R_2)$. Recall that the isomorphism $t_p$ satisfies $t_p(e) = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$. The result then follows from [IS03].

Write $V_n = \mathcal{E}(V_n)$ to simplify the notation. It follows from Lemma 9.1 that

$$H^1_{\text{dR}}(\mathcal{C}_{\text{rig}}, L_{n,n}^{\text{rig}}, \nabla) \simeq H^1_{\text{dR}}(\mathcal{C}_{\text{rig}}, V_n)$$

as filtered Frobenius monodromy modules, and the right hand side is isomorphic to the $\hat{\mathbb{Q}}_p^{unr}$-vector space of $V_n$-valued, $\Gamma_p$-invariant differential forms of the second kind on $\mathcal{H}_p^{unr}$ modulo forms $\omega$ such that $\nabla(\omega) = 0$. Define

$$V_{n,n} = V_n \otimes \text{Sym}^n \epsilon H^1_{\text{dR}}(A_0^m).$$

Then we have

$$H^1_{\text{dR}}(\mathcal{C}_{\text{rig}}, L_{n,n}^{\text{rig}}, \nabla) \simeq H^1_{\text{dR}}(\mathcal{C}_{\text{rig}}, V_{n,n})$$

as filtered Frobenius monodromy modules.

We now describe the monodromy operator. Let $T$ denote the Bruhat-Tits tree of $\text{PGL}_2(\mathbb{Q}_p)$, and denote $\mathcal{E}$ and $\mathcal{V}$ the set of oriented edges and vertices of $T$, respectively. If $e = (v_1, v_2) \in \mathcal{E}$, we denote by $\tau$ the oriented edge $(v_2, v_1)$. Let $C^0(V_n)$ be the set of maps $\mathcal{V} \to V_n$ and $C^1(V_n)$ the set of maps $\mathcal{E} \to V_n$ such that $f(\tau) = -f(e)$ for all $e \in \mathcal{E}$. The group $\Gamma_p$ acts on $f \in C^1(V_n)$ by $\gamma(f) = \gamma \circ f \circ \gamma^{-1}$. Let

$$\epsilon : C^1(V_n) \Gamma \to H^1(\Gamma, M)$$

be the connecting homomorphism arising from the short exact sequence

$$0 \to V_n \to C^0(V_n) \xrightarrow{\delta} C^1(M) \to 0,$$

where $\delta$ is the homomorphism defined by $\delta(f)(e) = f(V_n) - f(v_2)$ for $e = (v_n, v_2)$. The map $\epsilon$ induces the following isomorphism that we also denote by $\epsilon$

$$\epsilon : C^1(V_n)^{\Gamma} / C^0(V_n)^{\Gamma} \to H^1(V_n).$$

Let $A_e \subset \mathcal{H}_p^{unr}$ be the oriented annulus in $\mathcal{H}_p$ corresponding to $e$ and $U_e \subset \mathcal{H}_p^{unr}$ be the affinoid corresponding to $v \in \mathcal{V}$, which are obtained as inverse images of the reduction map (see [IS03 page 342]). Let $\omega$ be a $V_n$-valued $\Gamma$-invariant differential of the second kind on $\mathcal{H}_p$. We define $I(\omega)$ to be the map which assigns to an oriented edge $e \in \mathcal{E}$ the value $I(\omega)(e) = \text{Res}_e(\omega)$, where $\text{Res}_e$ denotes the annular residue along $A_e$. If $\omega$ is exact, $I(\omega) = 0$. Thus $I$ gives a well-defined map

$$I : H^1_{\text{dR}}(\mathcal{C}_{\text{rig}}, V_n) \to C^1(V_n)^{\Gamma}.$$
Since the set \( \{ U_v \}_{v \in \mathcal{V}} \) is an admissible covering of \( H_p \), the Mayer-Vietoris sequence yields an embedding

\[
C^1(V_n)^{\Gamma_p} / C^0(V_n)^{\Gamma_p} \hookrightarrow H^1_{dR}(C^{rig}, V_n).
\]

Precomposing with \( \epsilon \), we obtain an embedding

\[
(52) \quad \iota: H^1(\Gamma_p, V_n) \hookrightarrow H^1_{dR}(C^{rig}, V_n)
\]

This map admits a natural left inverse

\[
(53) \quad P: H^1_{dR}(C^{rig}, V_n) \rightarrow H^1(\Gamma_p, V_n),
\]

which takes \( \omega \) to the class of the cocycle \( \gamma \mapsto \gamma(F_\omega) - F_\omega \), where \( F_\omega \) is a Coleman primitive of \( \omega \) satisfying \( \nabla(F_\omega) = \omega \).

Define now the monodromy operator \( N_n \) on \( H^1_{dR}(C^{rig}, V_n) \) as the composite \( \iota \circ (-) \circ I \). The monodromy operator \( N_{S_n} \) on the filtered \((\phi, N)\)-module

\[
S_n = \text{Sym}^n eH^1_{dR}(A^m_0)
\]

is trivial. Therefore, the monodromy operator on \( D_{st, \mathcal{Q}_p}(H^{2n+1}_p(D)) \) is given by

\[
(54) \quad N = \text{id}_n \otimes N_{S_n} + N_n \otimes \text{id}_{S_n}.
\]

We now describe the Frobenius operator on \( D_{st, \mathcal{Q}_p}(H^{2n+1}_p(D)) \). First, \( H^1(\Gamma_p, V_n) \) has a Frobenius endomorphism induced by the map \( p \hat{\tau} \otimes \sigma \) on \( V_n \), where \( \sigma \) denotes the absolute Frobenius automorphism on \( \mathcal{Q}_p \). By [IS03] page 348, there exists a unique operator \( \Phi_n \) on \( H^1_{dR}(C^{rig}, V_n) \) satisfying \( N_n \Phi_n = p\Phi_n N_n \) and which is compatible, with respect to \( \iota \), with the Frobenius on \( H^1(\Gamma_p, V_n) \). On the other hand, the Frobenius on the filtered \((\phi, N)\)-module \( S_n \) is given by \( \Phi_{S_n} = p \hat{\tau} \otimes \sigma \) acting on the underlying vector space \( S_n \). The Frobenius operator on \( D_{st, \mathcal{Q}_p}(H^{2n+1}_p(D)) \) is then given by

\[
\Phi = \Phi_n \otimes \Phi_{S_n}.
\]

Note that \( N \) and \( \Phi \) satisfy the relation \( N\Phi = p\Phi N \).

For any \( D \in MF_{\mathcal{Q}_p} \), write \( D = \oplus_{\lambda \in \mathbb{Q}} D_\lambda \) for its slope decomposition, where \( \lambda \in \mathbb{Q} \) (\[IS03\] (2))). We now note that \( N \) induces an isomorphism

\[
(55) \quad N : H^1_{dR}(C^{rig}, V_{n,n})_{n+1} \simeq H^1_{dR}(C^{rig}, V_{n,n})_{n}.
\]

To see this, note that since the monodromy operator \( N \) and the Frobenius \( \Phi \) on \( H^1_{dR}(C^{rig}, V_{n,n}) \) satisfy the relation \( N\Phi = p\Phi N \), we have \( N(H^1_{dR}(C^{rig}, V_{n,n})_{n+1}) \subseteq H^1_{dR}(C^{rig}, V_{n,n})_{n} \). Since \( S_n \) is isotypical of slope \( n/2 \), we have

\[
H^1_{dR}(C^{rig}, V_{n,n})_{n+1} = H^1_{dR}(C^{rig}, V_{n,n})_{n/2+1} \otimes S_n
\]

doing

\[
H^1_{dR}(C^{rig}, V_{n,n})_{n} = H^1_{dR}(C^{rig}, V_{n,n})_{n/2} \otimes S_n.
\]

By [IS03] Lemma 6.1], the operator \( N_n \) on \( H^1_{dR}(C^{rig}, V_{n,n})_{n+1} \) is an isomorphism, the therefore the same is true in \( S_n \) by the definition of the monodromy operator \( N \) given in \( (54) \).

9.2.2. Calculation of Ext groups by the Gysin sequence. For \( f \in M_k(\Gamma) \) and \( v \in S_n \), \( (55) \) allows us to use the description in [IS03] Lemma 2.1] to calculate some Ext groups.

Define \( U_{z_A} = C^{rig} - \{ z_A \} \), and put

\[
(56) \quad H^1_{dR}(U_{z_A}, V_{n,n}) = H^1_{dR}(U_{z_A}, V_n) \otimes S_n.
\]
Let $\text{Res}_{z_A}: H^1_{\text{dR}}(U_{z_A}, V_{n, \text{rig}}) \to (V_{n, \text{rig}})_{z_A}$ be the residue map at a point $z_A$. We have the Gysin sequence ([IS03] Theorem 5.13) in $\text{MF}_{\hat{\phi}, \text{unr}}$.

$$0 \to H^1_{\text{dR}}(\mathcal{V}^{\text{rig}}, V_{n, n})[-(n+1)] \to H^1_{\text{dR}}(U_{z_A}, V_{n, n})[-(n+1)] \xrightarrow{\text{Res}_{z_A}} (V_n \otimes S_n)[-n] \to 0$$

where we write $V_n$ for the stalk $(V_n)_{z_A}$ of $V_n$ at $z_A$.

We have the cycle class map

$$\text{cl}_A = \text{cl}^{(n)}_{(A^m \times A^m_0, \epsilon_{A \times A_0})} : \text{CH}^n((A^m \times A^m_0, \epsilon_{A \times A_0})) \to \Gamma((V_n \otimes S_n)[-n])$$

where for a filtered Frobenius monodromy module $M$ with filtration $F^*(M)$, Frobenius $\phi$ and monodromy $N$, we put $\Gamma(M) = F^0(M) \cap M^{\phi = \text{id}, N = 0}$. Next, from ([57]) we obtain a connecting homomorphism in the sequence of Ext groups

$$\Gamma((V_n \otimes S_n)[-n]) \xrightarrow{\partial} \text{Ext}_{M^\phi, N}^1(\hat{\mathcal{Q}}_{\text{unr}}^n, H^1_{\text{dR}}(\mathcal{V}^{\text{rig}}, V_{n, n})[-(n+1)]) \cong \text{Ext}_{M^\phi, N}^1(\hat{\mathcal{Q}}_{\text{unr}}^{n+1}, H^1_{\text{dR}}(\mathcal{V}^{\text{rig}}, V_{n, n})) \cong (M_k(\Gamma) \otimes S_n)^{\vee}$$

where the last isomorphism comes from ([IS03] Lemma 2.1). On the other hand, we have a canonical map

$$i : \text{CH}^n((A^m \times A^m_0, \epsilon_{A \times A_0})) \to \text{CH}^{n+1}(\mathcal{D}).$$

The definition of the Abel-Jacobi map shows that the following diagram is commutative:

$$(58) \quad \begin{array}{ccc}
\text{CH}^{n+1}(\mathcal{D}) & \xrightarrow{i} & \Gamma((V_n \otimes S_n)[-n]) \\
\downarrow & & \downarrow \\
\text{AJ}_p(\Delta_{\mathcal{D}}) & \xrightarrow{\partial} & (M_k(\Gamma) \otimes S_n)^{\vee}
\end{array}$$

Then $\text{AJ}_p(\Delta_{\mathcal{D}})$ is the extension class determined by the following diagram (in which the right square is cartesian)

$$(59) \quad \begin{array}{ccc}
0 & \to & H^1_{\text{dR}}(\mathcal{V}^{\text{rig}}, V_{n, n}) \\
\downarrow & & \downarrow \\
0 & \to & H^1_{\text{dR}}(U_{z_A}, V_{n, n})
\end{array} \xrightarrow{\text{Res}_{z_A}} \begin{array}{c}
((V_n \otimes S_n)[1]) \to 0 \\
\uparrow \\
E \to \hat{\mathcal{Q}}_{\text{unr}}^{n+1}[n+1] \to 0
\end{array}$$

where the vertical left map sends $1 \mapsto \text{cl}_A(\Delta_{\mathcal{D}})[n+1]$.

9.2.3. Computation of the Abel-Jacobi map. Choose $\alpha \in H^1_{\text{dR}}(U_{z_A}, V_{n, n})_{n+1}$ such that

$$\text{Res}_{z_A}(\alpha) = \text{cl}_A(\Delta_{\mathcal{D}})$$

and $N(\alpha) = 0$. Choose $\beta$ in $H^1_{\text{dR}}(\mathcal{V}^{\text{rig}}, V_{n, n})$ such that

$$j_*(\beta) \equiv \alpha \mod F^{n+1}(H^1_{\text{dR}}(U_{z_A}, V_{n, n})).$$

Then the image of the extension $\text{cl}_A(\Delta_{\mathcal{D}})$ in

$$H^1_{\text{dR}}(\mathcal{V}^{\text{rig}}, V_{n, n})/F^{n+1}(H^1_{\text{dR}}(\mathcal{V}^{\text{rig}}, V_{n, n})) \simeq (M_k(\Gamma) \otimes S_n)^{\vee}$$

is the class of $\beta$ (which we denote by the same symbol $\beta$) in this quotient (for the isomorphism, see [IS03] Proposition 6.1]).
We have the Poincaré duality pairing \( \langle \cdot, \cdot \rangle_{n,n} : V_{n,n} \otimes \mathcal{O}_{\text{Crig}} \to \mathcal{O}_{\text{Crig}} \) arising from Poincaré duality on the fibers \( X \times A_0 \). We therefore obtain a Poincaré pairing, still denoted \( \langle \cdot, \cdot \rangle_{n,n} \),

\[
\langle \cdot, \cdot \rangle_{n,n} : H^1_{\text{dR}}(C^{\text{rig}}, V_{n,n}) \times H^1_{\text{dR}}(C^{\text{rig}}, V_{n,n}) \overset{\cup}{\longrightarrow} H^2_{\text{dR}}(C^{\text{rig}}, \mathcal{O}_{\text{Crig}}) \simeq \mathbb{Q}_p^{\text{unr}}.
\]

Let \( \omega_f \) be the class in \( F^{n+1} (H^1_{\text{dR}}(C^{\text{rig}}, V_n)) \) corresponding to \( f \in M_k(\Gamma) \). Then by definition

\[
\langle \omega_f \otimes v, \beta \rangle_{n,n} = (\omega_f \otimes v, \beta)_{n,n}.
\]

By [IS03 (39)] and the fact that \( S_n \) is isotypical of slope \( n/2 \), we obtain a decomposition

\[
H^1_{\text{dR}}(C^{\text{rig}}, V_{n,n}) \simeq H^1_{\text{dR}}(C^{\text{rig}}, V_{n,n})_n \oplus F^{n+1} (H^1_{\text{dR}}(C^{\text{rig}}, V_{n,n})).
\]

We may therefore assume that the element \( \beta \) considered above belongs to \( H^1_{\text{dR}}(C^{\text{rig}}, V_{n,n})_n \).

We now compute \( \langle \omega_f \otimes v, \beta \rangle_{n,n} \). By [IS03 Theorem 6.4],

\[
\ker(N_n) = \iota \left( H^1(\Gamma, V_n) \right) = H^1_{\text{dR}}(C^{\text{rig}}, V_{n,n})_{n/2}.
\]

To simplify the notation we put

\[
H^1(\Gamma, V_{n,n}) = H^1(\Gamma, V_n) \otimes S_n.
\]

We now extend \( \iota \) to a map, still denoted by the same symbol,

\[
\iota = \iota \otimes \text{id}_{(V_n)_{n,n}} : H^1(\Gamma, V_{n,n}) \hookrightarrow H^1_{\text{dR}}(C^{\text{rig}}, V_{n,n})
\]

and (62) shows that there exists an isomorphisms \( \ker(N) = \iota \left( H^1(\Gamma, V_{n,n}) \right) \), so we may also assume \( \beta = \iota(c) \) for some \( c \in H^1(\Gamma, V_{n,n}) \). Let \( C_{\text{har}}(V_n)^\Gamma \) denote the \( \mathbb{Q}_p \)-vector space of \( \Gamma \)-invariant \( V_n \)-valued harmonic cocycles and denote

\[
\langle \cdot, \cdot \rangle_\Gamma : C_{\text{har}}(V_n)^\Gamma \otimes H^1(\Gamma, V_n) \longrightarrow \mathbb{Q}_p
\]

the pairing introduced in [IS03 (75)]. To simplify the notation, we set

\[
C_{\text{har}}(V_n)^\Gamma = C_{\text{har}}(V_n) \otimes S_n.
\]

We then define the pairing

\[
\langle \cdot, \cdot \rangle_\Gamma : C_{\text{har}}(V_n)^\Gamma \otimes H^1(\Gamma, V_{n,n}) \longrightarrow \mathbb{Q}_p
\]

by \( \langle \cdot, \beta \rangle_\Gamma = \langle \cdot, \beta \rangle_{n,n} \), where \( \langle \cdot, \beta \rangle_{n,n} \) is the Poincaré pairing on \( A_0 \).

**Lemma 9.2.** \( \langle \omega_f \otimes v, \beta \rangle_{n,n} = -(\iota(\omega_f) \otimes v, c)_{\Gamma} \).

**Proof.** Write \( \beta = \sum_i \beta_i \otimes v_i \), and \( c = \sum_j c_j \otimes w_j \). Recall that \( \iota(c) = \beta \) with \( \iota \) injective, so that \( \iota(\beta_i) = c_i \) and \( v_i = w_i \) for all \( i \). By [IS03 Theorem 10.2] we know that for each \( i \) we have

\[
\langle \omega_f, \beta_i \rangle_{V_n} = -(\iota(\omega_f, c_i)_{\Gamma}).
\]

The definitions of \( \langle \cdot, \cdot \rangle_{n,n} \) and \( \langle \cdot, \cdot \rangle_{\Gamma} \) imply the result. \( \square \)

Write \( \alpha - j_*(\beta) = \sum_i \gamma_i \otimes v_i \). For each \( i \), let \( \chi_i \) be a \( \Gamma \)-invariant \( V_n \)-valued meromorphic differential form on \( H_p \) which is holomorphic outside \( \pi^{-1}(U_{z_A}) \), with a simple pole at \( z_A \), and whose class \( [\chi_i] \in F_\mathbb{Z}^{n+1}(H^1_{\text{dR}}(U_{z_A}, V_n)) \) represents \( \gamma_i \). Then the class of \( \chi = \sum_i \chi_i \otimes v_i \) represents \( \alpha - j_*(\beta) \).

Having identified \( H^1_{\text{dR}}(C^{\text{rig}}, V_n) \) with the \( \mathbb{Q}_p^{\text{unr}} \)-vector space of \( \Gamma \)-invariant \( V_n \)-valued differential forms of the second kind on \( H_p \) modulo horizontal forms for \( \nabla \), denote \( F_{\omega_f} \in H^0_{\text{dR}}(C^{\text{rig}}, V_n) \) the Coleman primitive of \( \omega_f \) ([?], §3.2).

**Lemma 9.3.** \( -(\iota(\omega_f) \otimes v, c)_{\Gamma} = (F_{\omega_f}(z_A) \otimes v, \text{Res}_{z_A}(\chi))_{A \times A_0} \), where \( \langle \cdot, \cdot \rangle_{A \times A_0} \) is the Poincaré pairing on \( A \times A_0 \).
Proof. As in the proof of Lemma 9.2 write \( c = \sum_j c_j \otimes w_j \). By definition,
\[
\langle I(\omega_f) \otimes v, c \rangle = \sum_j \langle I(\omega_f), c_j \rangle \cdot \langle v, w_j \rangle_{A_0}.
\]
By [IS03 Corollary 10.7],
\[
\langle I(\omega_f), c_j \rangle_f = \langle F_{\omega_f}(z_A), \text{Res}_{z_A}(\chi_j) \rangle_A
\]
where in the last pairing is the Poincaré pairing on \( A \); note that we can apply [IS03 Corollary 10.7] because the proof of [IS03 Theorem 10.6] still holds in our setting because the analogues of [IS03 (87), (88)] are true. The result follows now from the definition of the pairing \( \langle \cdot \rangle_{n,n} \).

\[\square\]

Lemma 9.4. \( A\mathcal{J}_p(\Delta_\varphi)(\omega_f \otimes \omega_{A_0}^{n-j}) = \langle F_j(z_A), \omega_{A_0}^{n-j}, \text{cl}_{z_A}(\Delta_\varphi) \rangle_{A \times A_0} \).

Proof. Taking into account (63) and (64), this follows from combining Lemma 9.2 and Lemma 9.3.

\[\square\]

Lemma 9.5. \( \langle F_j(z_A), \omega_{A_0}^{n-j} \rangle = \langle \varphi^*(F_j(z_A)), \omega_{A_0}^{n-j} \rangle_{A_0} \).

Proof. This follows from (9.3) and the functoriality property of the Poincaré pairing, as in [BDP13 Proposition 3.21].

\[\square\]

Let \( \omega_A \in eH^1_{\text{dR}}(A/F) \) be such that \( \omega_A = \varphi^*\omega_A \) and let \( \eta_A \in eH^1(A, \mathcal{O}_A) \) be as before the dual class to \( \omega_A \) under the Poincaré duality pairing \( \langle \cdot, \cdot \rangle_A \), normalised so that \( \langle \omega_A, \eta_A \rangle_A = 1 \).

Proposition 9.6. \( A\mathcal{J}_p(\Delta_\varphi)(\omega_f \otimes \omega_{A_0}^{n-j}) = d_\varphi \langle F_j(z_A), \omega_{A_0}^{n-j} \rangle_{A_0} \), where \( d_\varphi \) is the degree of \( \varphi \).

Proof. Observe that \( \varphi^*\eta_A = d_\varphi\eta_{A_0} \). It follows that
\[
\langle \varphi^*(F_j(z_A)), \omega_{A_0}^{n-j} \rangle_{A_0} = d_\varphi^{-n} \langle \varphi^*(F_j(z_A)), \varphi^*(\omega_{A_0}^{n-j}) \rangle_{A_0}.
\]
The functoriality properties of Poincaré pairing show that \( \langle \varphi^*\omega, \varphi^*\eta \rangle_{A_0} = d_\varphi \langle \omega, \eta \rangle_A \) for \( \omega \) and \( \eta \) in \( H^1_{\text{dR}}(A/F) \). It follows that
\[
\langle \varphi^*(F_j(z_A)), \varphi^*(\omega_{A_0}^{n-j}) \rangle_{A_0} = d_\varphi \langle F_j(z_A), \omega_{A_0}^{n-j} \rangle_{A_0}.
\]
The result follows combining Lemma 9.4 and Lemma 9.5 with equations (63) and (64).

\[\square\]

Recall the canonical differentials \( \omega_{\text{can}}, \eta_{\text{can}} \) introduced in Section 6. Since \( A_0 \) and \( A \) have CM by \( \mathcal{O}_K \), \( \omega_{\text{can}} \) and \( \eta_{\text{can}} \) defined pair of differentials \( \omega_{A_0}, \eta_{A_0} \) and \( \omega_A, \eta_A \). Recall now the definition of the function \( G_j \) in (42), and for integers \( j = n/2, \ldots, n \) define the function
\[
H_j(z) = \langle F_j(z), \omega_{A_0}^{n-j}(z) \eta_{\text{can}}^{n-j}(z) \rangle.
\]

Theorem 9.7. Let \( \omega_{A_0}, \eta_{A_0}, \omega_A \) and \( \eta_A \) be defined by means of \( \omega_{\text{can}} \) and \( \eta_{\text{can}} \). Then for each \( j = n/2, \ldots, n \) we have
\[
H_j(z) = A\mathcal{J}(\Delta_\varphi)(\omega_f \otimes \omega_{A_0}^{n-j}).
\]

Proof. The differentials on \( A_0 \) and \( A \) thus defined satisfy the conditions \( \langle \omega_{A_0}, \eta_{A_0} \rangle_{A_0} = 1 \), \( \langle \omega_A, \eta_A \rangle_A = 1 \) and \( \varphi^*\omega_A = \omega_{A_0} \). Therefore we may apply Proposition 9.6 and the result follows.

\[\square\]

Corollary 9.8. Let \( \omega_{A_0}, \eta_{A_0}, \omega_A \) and \( \eta_A \) be defined by means of \( \omega_{\text{can}} \) and \( \eta_{\text{can}} \). Then for each \( j = n/2, \ldots, n \) we have
\[
\delta_p^{n-j}(H_n)(z_A) = \frac{n!}{j!} A\mathcal{J}(\Delta_\varphi)(\omega_f \otimes \omega_{A_0}^{n-j}).
\]

Proof. From the proof of Theorem 9.1 we see that \( \Theta_p(G_j(z)) = jG_{j-1}(z) \), and therefore we have \( \Theta_p^{n-j}(G_n(z)) = \frac{n!}{j!}G_j(z) \). Therefore, \( \delta_p^{n-j}(H_n(z)) = \frac{n!}{j!}H_j(z) \) The result follows from Theorem 9.7.

\[\square\]
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