CLUSTER POINTS AND ASYMPTOTIC VALUES OF PLANAR HARMONIC FUNCTIONS

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Abstract. A sufficient condition for a cluster point of a planar harmonic function to be an asymptotic value is given, based on a partitioning into regions of constant valence. A sufficient condition for the cluster set of a planar harmonic function to have non-empty interior is given. An example is given of a planar harmonic function where the image of the critical set is not closed and such that the cluster set has non-empty interior and is a proper subset of the image.

1. Introduction

The behavior of complex-valued harmonic functions in the plane was studied in [Neu 05]. Many of these results also hold for $C^1$ functions $f(z) = (u(z), v(z))$ in an open set $R \subseteq \mathbb{R}^2$, identifying $z = (x, y) \in \mathbb{R}^2$ with $z = x + iy \in \mathbb{C}$ and $f$ with $u + iv$. Recall that critical set of $f$ is the set of points where the Jacobian of $f$ vanishes. The cluster set of $f$ consists of those finite values $w$ such that there exists a sequence $\{z_n\} \subset R$ such that $z_n \to \partial R \cup \{\infty\}$ and $f(z_n) \to w$. It was shown that the image of the critical set combined with the cluster set partitions $\mathbb{R}^2$ into regions where each point has the same number of distinct preimages. This result and some of its consequences are summarized in Theorem 4 below. These results are nicest when the partitioning set has empty interior and that occurs when the cluster set has empty interior.

It was shown that if the cluster set has non-empty interior, then points with infinite valence are dense in the interior of the cluster set. Harmonic functions will be assumed to be complex-valued unless stated otherwise. If $f$ is also assumed to be harmonic in $\mathbb{C}$, it was shown that if the cluster set of $f = h + g$ (where $h$ and $g$ are entire) has empty interior, then both $g$ and $h$ are either polynomials or entire transcendental functions. Theorem 5 below gives a sufficient condition for the critical set to have non-empty interior.

Many examples were given in [Neu 05]. In these examples, the image of the critical set was always closed. Further, when a cluster set had non-empty interior, it filled the entire plane. Is the image of the critical set always closed? Can a cluster
set with non-empty interior be a proper subset of the image? Example 1 below shows that the image of the critical set is not always closed and that a cluster set with non-empty interior can be a proper subset of the image.

In the examples of entire harmonic functions in Section 4 of [Neu 05], where the cluster set has empty interior, the sequence given for \( \{z_n\} \) where \( z_n \to \infty \) and where \( f(z_n) \) approaches a finite limit, can be extended to a curve approaching \( \infty \) whose image approaches the cluster value. In these examples, the cluster set has empty interior and each finite cluster point is an asymptotic value. Does this hold in general for entire harmonic functions? Note that entire harmonic functions can exclude open regions of the plane.

Consider the following version of Iversen’s theorem:

**Theorem 1** ([Car 56] p.25). Suppose that \( f(z) \) is meromorphic for \( |z| > \ell \) and has an essential singularity at infinity. Then if \( f(z) \) has a Picard exceptional value \( a \) (e.g., \( f(z) - a \) has a finite number of zeros), it is an asymptotic value of \( f(z) \). In particular, infinity is an asymptotic value of \( f \).

Recall that a sense-preserving harmonic function \( f(z) \) is said to be K-quasiregular in a domain \( U \subseteq \mathbb{R}^2 \) if there exists a constant \( K \geq 1 \) such that \( (|f_z|^2 + |f_\bar{z}|)^2 \leq K J_f \) holds for all \( z \in U \) (see [CGH 00]). Note that the term K-quasiregular is defined for functions defined in subsets of \( \mathbb{R}^n \) for some \( n \geq 2 \) and does not require that \( f \) be harmonic; see [Ric 93]. K-quasiregular functions can be thought of as analogues to analytic functions and K-quasimeromorphic functions as analogues to meromorphic functions. O. Martio, S. Rickman, and J. Väisälä showed the following analogue of Iversen’s theorem for quasimeromorphic functions:

**Theorem 2** ([Ric 93], page 170). Let \( f: G \to \mathbb{R}^n \) be a K-quasimeromorphic mapping and let \( b \in \partial G \) be an isolated essential singularity of \( f \). Then every point in \( \mathbb{R}^n \setminus \overline{f(G)} \) is an asymptotic value of \( f \).

D. A. Brannan ([Br 86]) considered the case \( f: \mathbb{C} \to \mathbb{C} \) where \( f \) is continuous:

**Theorem 3** (Brannan [Br 86]). Let \( f: \mathbb{C} \to \mathbb{C} \) be continuous. Then at least one of the following occurs: (i) \( f \) has \( \infty \) as an asymptotic value, (ii) \( f \) is bounded on a path going to \( \infty \), or (iii) \( f \) is uniformly bounded on a sequence of closed curves which surround the origin and whose distance from the origin approaches \( \infty \).

Is there an analog to Iversen’s theorem for planar harmonic functions? It seems reasonable to expect that the behavior of \( f \) harmonic in \( \mathbb{C} \) to be nicer than when \( f \) is only continuous, but perhaps not as nice as in the K-quasimeromorphic and meromorphic cases. Clearly, if \( \lim_{z \to \infty} f(z) = \infty \), then \( \infty \) is an asymptotic value of \( f \). Based on the examples in [Neu 05], one might expect

**Conjecture.** Let \( f: \mathbb{C} \to \mathbb{C} \) be harmonic. If the critical set of \( f \) is nowhere dense and if the cluster set of \( f \) has empty interior, then each cluster point of \( f \) is an asymptotic value. In particular, if each value in \( f(\mathbb{C}) \) is taken at most a finite number of times, then every cluster point of \( f \) is an asymptotic value.
Theorem 4 gives a sufficient condition for a cluster value of an entire $C^1$ function on $\mathbb{R}^2$ to be an asymptotic value. Given that this is a $C^1$ result, it is not surprising that it does not settle this conjecture. The proof uses a path-lifting argument. It starts with a sequence $\{z_n\}$ such that $z_n \to \infty$ and $w_n = f(z_n)$ approaches the cluster point $w_0$. It builds a simple asymptotic polygonal path that contains a subsequence of the $\{w_n\}$ and approaches $w_0$. Theorem 4 is then used to lift this path and it is shown that one of the lifted paths approaches $\infty$ asymptotically.

2. Notation and Main Results

Let $R \subseteq \mathbb{R}^2$ be open and let $f : R \to \mathbb{R}^2$ be $C^1$. Let

$$B(z_0, r) = \{z : |z - z_0| < r\}$$
$$J_f = |f_x|^2 - |f_y|^2$$
$$S = \{z \in R : J_f(z) = 0\}$$
$$C(f) = \{w \in \mathbb{R}^2 : \exists \{z_n\} \subseteq R \text{ where } z_n \to z_0 \in \partial R \cup \{\infty\} \text{ and } f(z_n) \to w\}$$
$$Val(f, w) = \# \{z \in R : f(z) = w\}$$
$$\lambda_f = |f_x| + |f_y|$$
$$\lambda_f = ||f_x| - |f_y||$$
$$\mu_f = f_x/|f_x|$$

$S$ is called the critical set of $f$ and $C(f)$ the cluster set. It is well known that $C(f)$ is closed. We also note that $f(S) \cup C(f)$ is closed (see [Neu 05]). If $R = \mathbb{R}^2$, we denote the cluster set by $C(f, \infty)$. $Val(f, w)$ is the valence of $f$ at $w$ and counts the number of distinct preimages of $w$ in $R$; it does not count multiplicity. $\mu_f$ is referred to as the complex dilatation of $f$.

We summarize some results from [Neu 05] as follows:

**Theorem 4.** Let $R \subseteq \mathbb{R}^2$ be open and let $f : R \to \mathbb{R}^2$ be $C^1$. Then $f(S) \cup C(f)$ partitions $\mathbb{R}^2$ into regions of constant, finite valence. Moreover, each component $R_0 \subseteq \mathbb{R}^2 \setminus f^{-1}(f(S) \cup C(f))$ is a covering space for $f(R_0)$ with $f|_{R_0}$ as the covering map. Note that $f(R_0)$ is a component of $\mathbb{R}^2 \setminus (f(S) \cup C(f))$.

We will refer to $f(S) \cup C(f)$ as the partitioning set of the image and to $f^{-1}(f(S) \cup C(f))$ as the partitioning set of the preimage. Unless explicitly stated otherwise, a component of the image is a component of $\mathbb{R}^2 \setminus (f(S) \cup C(f))$ and a component of the preimage is a component of $\mathbb{R}^2 \setminus f^{-1}(f(S) \cup C(f))$.

Let $w_0 \in C(f)$. If there exists an asymptotic path $\gamma : 0 \to 0 \in \partial R \cup \{\infty\}$ such that $f(\gamma) \to w_0$, then $w_0$ is said to be an asymptotic value of $f$. Recall that an asymptotic path is a curve which is the continuous image of $[0, \infty)$ such that $\gamma(t) \in R$ for all $0 \leq t < \infty$ and $\gamma \to z_0 \in \partial R \cup \{\infty\}$. If $\gamma \to z_0 \in \partial R$ where $z_0$ is finite, we also refer to $\gamma$ as an end-cut.

Recall that a set $U$ is uniformly locally connected if for each $\epsilon > 0$, there exists $\delta > 0$ such that if $z_1, z_2 \in U$ with $|z_1 - z_2| < \delta$, then there exists a connected subset of $U$ joining $z_1$ and $z_2$ of diameter less than $\epsilon$ ([New 64], page 160). If instead of
requiring a connected subset joining the two points, we require an arc, the set is set to be uniformly locally arcwise connected (ulac) ([HY 61], page 129).

Our two main results are:

**Theorem 5.** Let $f : \mathbb{C} \to \mathbb{C}$ be harmonic. Suppose that $\{z_n\}$ is such that $z_n \to \infty$, $w_n = f(z_n) \to w_0$, and $J_f(z_n) > 0$ for all $n$. Suppose that the following hold:

1. $d(\{z_n\}, S) = \delta > 0$,  
2. $\inf_n J_f(z_n) = \eta^2 > 0$, and  
3. There exists $\rho$ in $(0, \delta)$ such that $\sup_{z \in B(z_n, \rho)\setminus z_n} J_f(z)/J_f(z_n) < \infty$.

If $\sup_n \{|\mu_f(z) : z \in B(z_n, \rho)\setminus z_n|\} < 1$, then $\text{int } C(f, \infty) \neq \emptyset$ and $\text{Val}(f, w) = \infty$ in a neighborhood of $w_0$.

**Remark 1.** The condition that $J_f(z_n) > 0$ is not needed in Theorem 5, given that the sequence stays off the critical set. Given $\{z_n\}$ such that $z_n \to \infty$ and $d(\{z_n\}, S) > 0$, we can find a subsequence such that $J_f(z_n)$ does not change signs; this follows from noting that our sequence is off the critical set, applying the pigeonhole principle, and passing to a subsequence. If $J_f(z_n) < 0$ for a subsequence, we can work with $\overline{f}$ instead.

**Theorem 6.** Let $f$ be a $C^1$ function on $\mathbb{R}^2$ and suppose that $S$ is nowhere dense. Let $w_0 \in C(f, \infty)$ be such that either (i) $\text{Val}(f, w_0) < \infty$ or (ii) $\text{Val}(f, w_0) = \infty$ and $w_0 \notin f(S)$. Suppose that there exists $\epsilon > 0$ such that:

1. $C(f, \infty)$ is nowhere dense in $B(w_0, \epsilon)$.
2. $B(w_0, \epsilon) \setminus (f(S) \cup C(f, \infty))$ consists of finitely many components, each of which is uniformly locally arcwise connected.

Then $w_0$ is an asymptotic value of $f$.

### 3. Proof of Theorem 5

S. Bochner [Boc 46] proved a version of Bloch’s theorem for sense-preserving, $K$-quasiregular harmonic mappings in $\mathbb{R}^n$ for $n \geq 2$. H. Chen, P. M. Gauthier, and W. Hengartner [CGH 00] studied the value of the Bloch constant for $n = 2$. For example,

**Theorem 7** (Chen, Gauthier, and Hengartner [CGH 00]). Let $f$ be a $K$-quasiregular harmonic mapping of the unit disk $\mathbb{D}$ such that $J_f(0) = 1$. Then $f(\mathbb{D})$ contains a schlicht disk of radius at least $r_1 = \frac{\pi \sqrt{2K}}{1+2K}$.

However, they also note that in harmonic analogues of Bloch’s theorem, the schlicht disk contained in the image of the unit disk is not necessarily centered at $f(0)$. Because we are interested in finding disks in the image that overlap in a neighborhood of our cluster point, we need a somewhat different result. Instead of adapting Bochner’s proof, we will use another result in [CGH 00].
Lemma 2. Let \( w \) be a schlicht disk \( B(0, r_0) = \{ \rho \lambda_f(z_n) : z_n \in \overline{B(z_n, \rho)} \} \). Then, \( f \) is univalent on \( B(0, \rho_0) \) with \( \rho_0 = \frac{\pi}{4(1+\Lambda)} \) and \( f(B(0, \rho_0)) \) contains a schlicht disk \( B(0, r_0) \) with \( r_0 = \frac{1}{2}\rho_0 = \frac{\pi}{8(1+\Lambda)} \).

Proof of Theorem 5. Let \( M = \sup_n |\mu_f(z)| : z \in \overline{B(z_n, \rho)} \). By assumption, \( M < 1 \). Let \( K = (1 + M)/(1 - M) \). Note that on each \( B(z_n, \rho) \), we have \( J_f(z_n, \rho) < (1 + |\mu_f|)/(1 - |\mu_f|) \leq (1 + M)/(1 - M) = K \), so \( f \) is \( K \)-quasiregular on each \( B(z_n, \rho) \). Thus, on each \( B(z_n, \rho) \), \( \lambda_f^2 \geq J_f/K \). In particular, for each \( n \), \( (\lambda_f(z_n))^2 \geq J_f(z_n)/K \).

Following Bochner's proof, we define the complex-valued harmonic function

\[
F_n(z) = (f(z_n + \rho z) - f(z_n))/\rho \lambda_f(z_n)
\]

for \( z \in \mathbb{D} \). We now check that each \( F_n \) satisfies the conditions of Theorem 8 on \( \mathbb{D} \). Note that \( F_n(0) = 0 \) and \( J_{F_n}(z) = J_f(z_n + \rho z)/\lambda_f(z_n)^2 \). Also, \( (\lambda_{F_n}^2(z))^2 = (\lambda_f(z_n + \rho z))^2/\lambda_f(z_n)^2 \leq K J_f(z_n + \rho z)/\lambda_f(z_n)^2 \). Thus, \( \lambda_{F_n}^2(z) \leq K J_f(z_n + \rho z) \). In particular, for each \( n \), \( \lambda_{F_n}^2(z_n) \geq J_f(z_n)/K \).

4. Proof of Theorem 5

Suppose that we have a sequence \( \{z_n\} \) such that \( z_n \to \infty \) and \( w_n = f(z_n) \to w_0 \). We set things up so that we may choose \( w_n \) off of the partitioning set of the image. We show that \( w_0 \) is an asymptotic value of \( f \) by constructing a path \( \gamma \) containing a subsequence of the \( \{w_n\} \) so that \( \gamma \to w_0 \) and then lifting \( \gamma \) to a component of the preimage. We then show that a lift of \( \gamma \) asymptotically approaches \( \infty \). To simplify some of the arguments involving the lifts of \( \gamma \), we will require that \( \gamma \) be a simple path; this construction is carried out by inductively constructing a sequence of simple polygonal paths.

We begin with some elementary results concerning polygonal paths. The proofs of Lemmas 1 and 2 are sufficiently simple that they will be omitted.

**Lemma 1.** Let \( \Gamma \) be a polygonal path consisting of a finite number of line segments where \( \Gamma(0) = a \) and \( \Gamma(1) = b \). Then there exists a simple polygonal path (consisting of a finite number of line segments) \( \gamma \subseteq \Gamma \) such that \( \gamma(0) = a \) and \( \gamma(1) = b \).

**Lemma 2.** Let \( \gamma \) be a simple path in an open set \( U \subseteq \mathbb{R}^2 \), where \( \gamma(0) = a \) and \( \gamma(1) = b \). Choose an integer \( n > 0 \). Suppose that the diameter of \( \gamma \) is less than \( \epsilon \).
Then there exists a simple polygonal path in $U$ of diameter less than $(1 + \frac{1}{2^n}) \epsilon$ which starts at $a$ and ends at $b$.

Remark 2. Note that we are not attempting to approximate $\gamma$ arbitrarily closely by a simple polygonal path. We are merely trying to find a polygonal path in $U$ connecting $a$ and $b$ given an arbitrary path in $U$ connecting these two points. This way, we may use polygonal paths instead of arbitrary paths. In particular, if a uniformly locally arcwise connected open set contains a path of diameter less than $\epsilon$ connecting $z_1$ and $z_2$ if $|z_1 - z_2| < \epsilon$, then it also contains a simple polygonal path of diameter less than $2\epsilon$ connecting these points. Hence, without loss of generality, we may use simple polygonal paths in place of paths in arguments involving a uniformly locally arcwise connected open set.

Lemma 3. Let $U$ be an open subset of $\mathbb{R}^2$ and let $\gamma \subset U$ be a simple polygonal path with $\gamma(0) = a$ and $\gamma(1) = b$ of diameter less than $\epsilon$. Choose $\delta_0 > 0$. Let $c = \gamma(t_c)$ for some $t_c \in (0, 1)$ and suppose that $\zeta \in B(c, \delta) \setminus \gamma$ where $0 < \delta < \delta_0$ is chosen so that $B(c, \delta) \subset U$. Then there exists a simple polygonal path $\tilde{\gamma} \subset U$ of diameter less than $\epsilon + 2\delta_0$ such that $\tilde{\gamma}(0) = \zeta$, $\tilde{\gamma}(1) = b$, and $\tilde{\gamma} \cap \gamma = \{b\}$.

Proof. While it is possible to give an explicit construction of the desired path, we will spare the reader the details and instead follow a suggestion of D. Sarason.

Cover each $z \in \gamma$ with a ball of radius $r_z \leq \delta$ such that $B(z, r_z) \subset U$. Include $B(c, \delta)$ to cover $\{\zeta\}$. Because $\gamma \cup \{\zeta\}$ is compact, we may choose a finite subcover, which consists of balls centered at points $z_1, z_2, \ldots, z_N$ of $\gamma$. Let $G = \bigcup_{j=1}^{N} B(z_j, r_{z_j})$. By construction, the diameter of $G$ is at most $\epsilon + 2\delta$. Let $B = G \setminus \gamma$. Clearly, $B$ is open. It also follows that $B$ is connected. (Let $F_1 = \mathbb{R}^2 \setminus G$ and $F_2 = \gamma$. Then $F_1 \cap F_2 = \emptyset$. Since $G$ is open, $F_1$ is closed. By construction, $\mathbb{R}^2 \setminus F_1 = G$ is connected. Clearly, $F_2 = \gamma$ is closed. Since $\gamma$ is a simple arc, it does not separate the plane and $\mathbb{R}^2 \setminus F_2$ is connected. It follows from Corollary 1, page 112 of [New 64] that $B = \mathbb{R}^2 \setminus (F_1 \cup F_2)$ is connected.)

Choose $r$ such that $B(b, r) \subset B(z_N, r_N)$ and such that $B(b, r)$ intersects exactly one line segment of $\gamma$ (this is possible because $b$ is the last point of $\gamma$). Choose $\xi \in B(b, r)$. It is well known (see, for example, page 15 of [Con 78]) that an open set $U \subset \mathbb{R}^2$ is connected if and only if every pair of points in $U$ can be connected by a polygonal path contained in $U$. In particular, we can find a polygonal path, say $\Gamma$, in $B$ connecting $\zeta$ and $\xi$. By construction, $\gamma \cap \Gamma = \emptyset$. By our choice of $r$, we may find a line segment $\ell$ in $B(b, r) \subset B(z_N, r_N)$ connecting $\xi$ to $b$, such that $\gamma \cap \ell = \{b\}$. Apply Lemma 1 to find a simple polygonal path $\tilde{\gamma} \subset \Gamma \cup \ell$ connecting $\zeta$ and $b$. By construction, the diameter of $\tilde{\gamma}$ is at most $\epsilon + 2\delta < \epsilon + 2\delta_0$. By construction, $\tilde{\gamma}$ is a simple polygonal path, starting at $\zeta$ and ending at $b$, such that $\tilde{\gamma} \cap \gamma = \{b\}$. □

We now apply these tools to build an asymptotic path.

Lemma 4. Suppose that $U$ is a connected, open subset of $\mathbb{R}^2$ and that $\{\zeta_n\} \subset U$ such that $\zeta_n \rightarrow \zeta_0 \in \partial U$, where $\zeta_0$ is finite. Also suppose that there exists $\epsilon > 0$ such that $B(\zeta_0, \epsilon) \cap U$ is uniformly locally arcwise connected. Then there exists an asymptotic path $\Gamma \subset U$ containing a subsequence of $\{\zeta_n\}$ such that $\Gamma \rightarrow \zeta_0$. 
Proof. Without loss of generality, we will let \( U = U \cap B(\zeta_0, \epsilon) \). Choose \( \epsilon_0 > 0 \) such that \( \epsilon_0 < \epsilon \). Since \( U \) is ulac and by Remark 2 there exists \( \delta_0 > 0 \) such that whenever \( z_1, z_2 \in U \) with \( |z_1 - z_2| < \delta_0 \), then there exists a simple polygonal path in \( U \) connecting \( z_1 \) and \( z_2 \) of diameter less than \( \epsilon_0/2 \).

By passing to a subsequence, we may assume that \( |\zeta_1 - \zeta_0| < \delta_0/2 \) and that \( |\zeta_n - \zeta_0| < \frac{1}{2^n} |\zeta_n - \zeta_0| \). Thus, \( |\zeta_j - \zeta_0| < \delta_0/2^j \) and \( |\zeta_j - \zeta_0| \leq |\zeta_j - \zeta_0| + |\zeta_k - \zeta_0| < \delta_0/2^{j-1} \) for \( k > j \). We will construct an asymptotic path which consists of a countable number of simple polygonal paths in \( U \) and which contains \( \{w_m\}_{m=1}^{\infty} \), a subsequence of \( \{\zeta_n\} \).

Let \( w_0 = \zeta_0 \) and \( w_1 = \zeta_1 \). Let \( \rho_1 = \epsilon_0 \) and let \( d_2 = \rho_1/4 \). Choose \( j > 1 \) so that (i) there exists a path in \( U \) of diameter less that \( d_2 \) connecting \( z \) and \( \xi \) whenever \( |z - \xi| < \delta_0/2^{j-1} \) (by ulac of \( U \)) and (ii) \( d(\zeta_j, w_0) < \frac{1}{4} d(w_1, w_0) \) (since \( d(\zeta_n, w_0) \) is strictly decreasing). Let \( w_2 = \zeta_j \). Since \( |w_2 - w_1| = |\zeta_j - \zeta_1| < \delta_0 \), there is a simple polygonal path, say \( \gamma_{1,2} \), in \( U \) of diameter less than \( \epsilon_0/2 \) connecting \( w_1 \) and \( w_2 \). Let \( \Gamma_1 = \emptyset \). Let \( \Gamma_2 = \gamma_{1,2} \) and let \( \rho_2 = d(\gamma_{1,2}, w_0) = d(\Gamma_2, w_0) \).

We will build the other segments inductively for \( n > 1 \). Let \( \gamma_{n,n+1} \) be the simple polygonal path constructed in this procedure connecting \( w_n \) and \( w_{n+1} \). Let \( \Gamma_{n+1} = \Gamma_n \cup \gamma_{n,n+1} \). Let \( \rho_{n+1} = d(\Gamma_{n+1}, w_0) \). The path \( \gamma_{n,n+1} \) is chosen so that \( \gamma_{n,n+1} \cap \Gamma_n = \{w_n\} \) and \( \gamma_{n,n+1} \cap \Gamma_{n-1} = \emptyset \). We also require that \( \gamma_{n,n+1} \subset U \cap B(w_0, \frac{\epsilon}{n+1} \rho_{n-1}) \); this guarantees that \( \rho_{n} \to 0 \). Let \( \Gamma = \lim_{n \to \infty} \Gamma_n \). If each \( \gamma_{n,n+1} \) for \( n > 1 \) satisfies these assumptions, then \( \Gamma \) is a simple asymptotic path which approaches \( \zeta_0 \) and contains a subsequence of \( \{\zeta_j\} \).

It remains to construct \( \gamma_{n,n+1} \) for \( n > 1 \). It is enough to choose \( w_{n+1} \) so that (i) \( \gamma_{n,n+1} \cap \Gamma_n = \{w_n\} \), (ii) \( \gamma_{n,n+1} \cap \Gamma_{n-1} = \emptyset \) and (iii) \( \gamma_{n,n+1} \subset U \cap B(w_0, \frac{\epsilon}{n+1} \rho_{n-1}) \).

We start by choosing \( j \) so that (a) \( |\zeta_j - w_0| < \rho_n/4 \) and (b) for all \( z \in U \) such that \( |z - \zeta_j| < \delta_0/2^{j-1} \), there exists a simple polygonal path in \( U \) from \( z \) to \( \zeta_j \) of diameter less than \( d_{n+1} = \min(\epsilon_0/2^{n+1}, \rho_{n}/4) \). Condition (a) forces \( \zeta_j \) to be closer to \( w_0 \) than \( \Gamma_n \) is. Condition (b) lets us control the diameter of simple polygonal paths from from \( \zeta_j \) to \( \zeta_k \) for all \( k > j \). It is clear that (a) is possible and that \( j \) will be greater than the corresponding \( k \) in \( w_j = \zeta_k \) for \( w_1, w_2, ..., w_n \). Remark 2 allows us to satisfy (b). Let \( w_{n+1} = \zeta_j \).

Let \( \eta \) be a simple polygonal path in \( U \) connecting \( w_n \) and \( w_{n+1} \) of diameter less than \( d_n \) (this is possible by the previous stage of the construction). Recall that \( d_n \leq \rho_{n-1}/4 \) and that \( \rho_n \leq d(w_n, w_0) < \rho_{n-1}/4 \). Recall also that \( \Gamma_1 = \emptyset \), so we only need to check (ii) for \( n > 2 \). When \( n > 2, \eta \subset B(w_0, r) \) where \( r \leq |w_n - w_0| + d_n < \rho_{n-1}/4 + \rho_{n-1}/4 = \rho_{n-1}/2 \), so \( \eta \) will not intersect \( \Gamma_{n-1} \).

The only possible way for \( \eta \cup \Gamma_n \) to fail being a simple polygonal path is if \( \eta \) intersects \( \gamma_{n-1,n} \) at a point other than \( w_n \). Let \( c \) be the first point of intersection of \( \eta \) with \( \gamma_{n-1,n} \) as we travel from \( w_{n+1} \) to \( w_n \). Choose \( \delta_0 \) (in Lemma 2) less than \( d_{n+1} \). Choose \( \zeta \) in Lemma 3 so that \( \zeta \) is in the portion of \( \eta \) starting from \( w_{n+1} \) before \( c \). Apply Lemma 4 to construct \( \tilde{\eta} \) connecting \( \zeta \) to \( w_n \) and let \( \eta' \) be the portion of \( \eta \) from \( w_{n+1} \) to \( \zeta \). Let \( \gamma_{n,n+1} = \eta' \cup \tilde{\eta} \) and apply Lemma 1 to find \( \gamma_{n,n+1} \).
This new path satisfies (i) above. We note that if (iii) holds, so will (ii). We now check (iii): By Lemma 3, \( \gamma_{n+1} \subset U \cap B(w_0, r_n) \) where \( r_n < \rho_{n-1}/2 + (a_n + 2\delta) \), \( \rho_{n-1}/2 + \rho_{n-1}/4 + 2\rho_n/4 < 3\rho_{n-1}/4 + \rho_{n-1}/8 = \frac{5}{8}\rho_{n-1} \). This completes the construction. \( \square \)

**Remark 3.** It is known that a boundary point of a uniformly locally connected open set is accessible. Note that a ulac open set is also uniformly locally connected.

**Lemma 5.** Suppose that \( U \) is an open subset of \( \mathbb{R}^2 \) and that \( \{\zeta_n\} \subset U \) such that \( \zeta_n \to \zeta_0 \in \partial U \), where \( \zeta_0 \) is finite. Also suppose that there exists \( \epsilon > 0 \) such that \( B(\zeta_0, \epsilon) \cap U \) consists of finitely many components and that each component is uniformly locally arcwise connected. Then there exists a simple continuous curve \( \Gamma \subset U \) containing a subsequence of \( \{\zeta_n\} \) such that \( \Gamma \to \zeta_0 \).

**Proof.** Without loss of generality, we assume that our sequence consists of distinct points, that \( \{\zeta_n\} \subset B(w_0, \epsilon) \), and that \( |\zeta_{n+1} - \zeta_n| < \frac{1}{2}|\zeta_n - \zeta_{n-1}| \). By the pigeonhole principle, we may find a subsequence of \( \{\zeta_n\} \) that lies in one of the components of \( B(\zeta_0, \epsilon) \cap U \), say \( \Omega_0 \). Our result follows by applying Lemma 4 to \( \Omega_0 \).

We now apply Theorem 4 to lift our asymptotic path.

**Lemma 6.** Let \( R \subset \mathbb{R}^2 \) be open and let \( f : R \to \mathbb{R}^2 \) be \( C^1 \). Let \( w_0 \in C(f) \) and \( \{z_n\} \subset R \) be such that \( f(z_n) \to w_0 \) and \( z_n \to z_0 \in \partial R \cup \{\infty\} \). Suppose that \( \{z_n\} \subset R_0 \) where \( R_0 \) is a component of \( R \setminus f^{-1}(f(S) \cup C(f)) \). Let \( w_n = f(z_n) \) and let \( \Omega_0 = f(R_0) \). Suppose also that there exists \( \epsilon > 0 \) such that \( B(w_0, \epsilon) \cap \Omega_0 \) can be partitioned into a finite number of components, each of which is uniformly locally arcwise connected. Then there exists a simple, half-open path \( \gamma \subset R_0 \) which contains a subsequence of the \( \{z_n\} \).

**Proof.** By Theorem 4, \( \Omega_0 \) is a component of \( \mathbb{R}^2 \setminus (f(S) \cup C(f, \infty)) \) and \( f |_{\Omega_0} \) is a covering map. Let \( N = \text{Val}(f |_{\Omega_0}) \). By the pigeonhole principle, a subsequence of \( \{w_n\} \) is contained in one of the components of \( B(w_0) \cap \Omega_0 \). By Lemma 5 there exists a simple asymptotic path \( \Gamma \subset \Omega_0 \) that contains a pairwise distinct subsequence of \( \{w_n\} \) such that \( \Gamma \to w_0 \). We will denote this subsequence by \( \{w_n\} \) and the corresponding subsequence of \( \{z_n\} \) by \( \{z_n\} \). Because \( \Gamma \) is the continuous image of \([0, \infty)\) (whose fundamental group is trivial) and \( f |_{\Omega_0} \) is a covering map, we can lift \( \Gamma \) from \( \Omega_0 \) to \( R_0 \). Since \( f \) is locally 1-1 in \( R_0 \) and \( \Gamma \) is simple, \( \Gamma \) lifts to \( N \) distinct, simple half-open paths in \( R_0 \). By the pigeonhole principle, at least one of these paths, say \( \gamma \), contains an infinite subsequence of the \( \{z_n\} \).

**Lemma 7.** Let \( R = \mathbb{R}^2 \). Let \( f, w_0 \in C(f, \infty) \) and \( \{z_n\} \) be defined as in Lemma 4. Suppose that \( \{z_n\} \subset R_0 \) where \( R_0 \) is a component of \( \mathbb{R}^2 \setminus f^{-1}(f(S) \cup C(f, \infty)) \). Let \( w_n = f(z_n) \) and let \( \Omega_0 = f(R_0) \). Suppose also that there exists \( \epsilon > 0 \) such that \( B(w_0, \epsilon) \cap \Omega_0 \) can be partitioned into a finite number of components, each of which is uniformly locally arcwise connected. If either (i) \( \text{Val}(f, w_0) < \infty \) or (ii) \( \text{Val}(f, w_0) = \infty \) and \( w_0 \notin f(S) \), then \( w_0 \) is an asymptotic value of \( f \).
Proof. By Lemma 7 there exists a simple, half-open path \( \gamma \subset R_0 \) such that \( \gamma \) contains a subsequence of \( \{z_n\} \) and \( f(\gamma) \to w_0 \). It remains to show that \( \gamma \to \infty \).

Suppose not. Then there exists \( \rho > 0 \) such that every unbounded subarc of \( \gamma \) intersects \( B(0, \rho) \). In other words, we can find a subsequence of \( \{z_n\} \) such that \( |z_n| \) is strictly increasing and such that \( \gamma_n \), the lift in \( \gamma \) starting at \( z_n \) of the path joining \( w_n \) and \( w_{n+1} \) in \( f(\gamma) \), intersects \( B(0, \rho) \). In particular, choose an integer \( m > \rho \). Then there exists \( N \) such that \( |z_n| > m \) for all \( n > N \) in our subsequence. Let \( C_m = \{z : |z| = m\} \). Then, for each \( n > N \) in our subsequence, there exists \( \zeta_n \in \gamma \cap C_m \subset \gamma \cap C_m \). By construction, the \( \zeta_n \) will be pairwise distinct. Since \( f(\gamma) \to w_0 \), \( f(\zeta_n) \to w_0 \). Since \( C_m \) is compact, \( \{\zeta_n\} \) has a convergent subsequence. Hence, \( \zeta_m^{(m)} \), the limit of this subsequence, is a preimage of \( w_0 \) in \( C_m \). We can repeat this construction for each integer \( m > \rho \). Since the \( C_m \) are pairwise disjoint, the \( \zeta_m^{(m)} \) are distinct and \( \text{Val}(f, w_0) = \infty \), a contradiction when \( \text{Val}(f, w_0) \) is finite.

On the other hand, if \( \text{Val}(f, w_0) = \infty \), then we may assume that \( w_0 \notin f(S) \). This also leads to a contradiction. Consider the closed annulus \( A = \{z : m \leq |z| \leq m + 1\} \), where \( m > \rho \). By the construction above, each circle \( C_r \) where \( m \leq r \leq m + 1 \) contains a point \( \zeta_r \) such that \( f(\zeta_r) = w_0 \). Thus the \( \zeta_r \) accumulate in \( A \), a compact set. Let \( \zeta \in A \) be one such accumulation point. Then \( f \) cannot be \( 1 \) in any neighborhood of \( \zeta \). By the inverse function theorem, \( J_f(\zeta) = 0 \). Hence \( \zeta \in S \) and \( w_0 = f(\zeta) \in f(S) \), a contradiction. \( \Box 

Remark 4. The construction used in the proof of Lemma 7 uses our additional requirement that our lift is a simple path, which guarantees that each time the path intersects the circle \( C_m \), it does so at different points. Thus \( \{\zeta_n\} \) consists of an infinite number of distinct points.

Remark 5. The construction used in the proof of Lemma 7 can be extended to the case where \( R \) is an open disk, \( w_0 \in C(f) \), and \( R_0 \) is a component of \( R \cap (f(S) \cup C(f)) \) by intersecting \( R \) with a sequence of circles that approach \( \partial R \). The lift of \( \gamma \) approaches \( \partial R \); otherwise, we can argue as above to get the contradiction that \( \text{Val}(f, w_0) = \infty \). However, it is unclear whether the lift approaches a particular point on \( \partial R \), so we cannot claim that \( w_0 \) is an asymptotic value.

Lemma 8. Let \( R \subset \mathbb{R}^2 \) be open and let \( f : R \to \mathbb{R}^2 \) be \( C^1 \). Suppose that \( w_0 \in C(f) \) and that \( S \) is nowhere dense in \( R \). Suppose that there exists \( \epsilon > 0 \) such that \( B(w_0, \epsilon) \cap (f(S) \cup C(f)) \) consists of finitely many components and such that \( C(f) \) is nowhere dense in \( B(w_0, \epsilon) \). Then there exists a sequence \( \{\zeta_n\} \subset R_0 \), where \( R_0 \) is a component of \( R \setminus f^{-1}(f(S) \cup C(f)) \), and such that \( \zeta_n \to \zeta_0 \in \partial R \cup \{\infty\} \).

Proof. By definition, there exists \( \{z_n\} \subset R \) such that \( w_n = f(z_n) \to w_0 \) and \( z_n \to z_0 \in \partial R \cup \{\infty\} \). To construct \( \{\zeta_n\} \), we start by guaranteeing that this sequence does not intersect the partitioning set of the preimage. Without loss of generality, we may choose a subsequence of \( \{w_n\} \) such that \( |w_{n+1} - w_0| < \frac{1}{2}|w_n - w_0| \) for each \( n \). We may choose \( N \) such that for \( n > N \), \( w_n \in B(w_0, \epsilon) \). Let \( \epsilon_n = \frac{1}{2} \min\{\epsilon - |w_n - w_0|, |w_n - w_0|\} \). Choose \( 0 < \delta < 1/n \) such that \( f(B(z_n, \delta)) \subset B(w_0, \epsilon_n) \). Since \( S \) is nowhere dense, \( \exists \delta_n' \in B(z_n, \delta) \setminus S \) and we may choose an open set \( U \subset B(z_n, \delta) \) such that \( z_n' \in U \) and \( f \) is a homeomorphism on \( U \). Recall that \( f(S) \cup C(f) \) is closed. By Sard’s theorem (see, for example, page 69 of [Hir 70]), \( f(S) \) has empty interior. Thus, the restriction of \( f(S) \cup C(f) \) to \( B(w_0, \epsilon_0) \) has empty interior iff the
restriction of $C(f)$ does (see Remark 3.8 of [Neu 05]). In particular, since the closed set $C(f)$ is nowhere dense in $B(w_0, \epsilon_0)$, $f(S) \cup C(f)$ is nowhere dense in $B(w_0, \epsilon_0)$. Therefore there exists $w'_n \in f(U) \subseteq B(w_n, \epsilon_n)$ such that $w'_n \notin f(S) \cup C(f)$. Let $\zeta_n = f^{-1}(w'_n) \cap U$. Then $\zeta_n \to z_0$ and $f(\zeta_n) \to w_0$.

We then show that there is a subsequence of $\{\zeta_n\}$ in one component of the preimage. Note that a component of $B(w_0, \epsilon) \setminus (f(S) \cup C(f))$ is a subset of some component of $\mathbb{R}^2 \setminus (f(S) \cup C(f))$. Since $B(w_0, \epsilon) \setminus (f(S) \cup C(f))$ consists of finitely many components, $B(w_0, \epsilon)$ intersects finitely many components of the partition of the image. Since each region in the partition of the image has finite valence, $\{\zeta_n\}$ is contained in finitely many regions of the partition of the preimage. Applying the pigeonhole principle gives us a subsequence of $\{\zeta_n\}$ contained in one component of the partition of the preimage. □

**Proof of Theorem 6.** By Lemma 5 there exists a sequence $\{\zeta_n\} \subset R_0$, where $R_0$ is a component of $\mathbb{R}^2 \setminus f^{-1}(f(S) \cup C(f, \infty))$, such that $\zeta_n \to \infty$ and $f(\zeta_n) \to w_0$. The result then follows from Lemma 7. □

5. Examples and discussion

**Example 1.** A harmonic function with a cluster set with non-empty interior such that $f(\mathbb{C}) \setminus (f(S) \cup C(f, \infty)) \neq \emptyset$ and such that $\overline{f(S) \setminus f(S)} \neq \emptyset$.

$$f(z) = \Re e^{\pi} + \frac{i}{2} \Im z^2 = e^{\pi} \cos y + i xy$$

A calculation shows that

$$S = \{z = x + iy : x = -y \tan y, \ y \neq (2k + 1)\pi/2, k \in \mathbb{Z}\}$$

$$f(S) = \{e^{-y \tan y} \cos y - i y^2 \tan y, \ y \neq (2k + 1)\pi/2, k \in \mathbb{Z}\}$$

$f(S)$ intersects the real axis only at $\Re w = \pm 1$. These are the only points of intersection of $f(S)$ with the vertical lines $\Re w = \pm 1$. This follows by first noting that $e^{-y \tan y} \cos y$ is strictly monotone on the interval $(2k - 1)\pi/2 < y < (2k + 1)\pi/2$ when $k \neq 0$. When $k = 0$, $e^{y \tan y} \geq 1$ and thus will intersect $\cos y$ at most once on our interval. By considering $y = k\pi - b/(k\pi)^2$, we can check that $f(S)$ is arbitrarily close to $(-1)^k + ib$ as $|k| \to \infty$. Thus, $\{w : \Re w = \pm 1, 3 w \neq 0\} \subseteq \overline{f(S) \setminus f(S)}$.

We claim that $C(f, \infty) = \{w : -1 \leq \Re w \leq 1\}$. Let $w = a + ib$. Suppose that $f(z_n) \to w$ as $z \to \infty$. If $a > 1$, we must have $\lim x_n > 0$. If $x_n \to \infty$, we must have $y_n \to (2k + 1)\pi/2$; however, $x_n y_n \to b$ is then impossible. Moreover, if $y_n \to \infty$, we must have $x_n \to 0$. Thus, $\Re w = a > 1$ is omitted from $C(f, \infty)$. Similarly, $\Re w < -1$ is omitted from $C(f, \infty)$. Thus, if $w \in C(f, \infty)$, then $|\Re w| \leq 1$. Now consider the case when $|a| \leq 1$. Choose $b \in \mathbb{R}$. It’s clear that $\cos y$ and $ae^{-b/y}$ have an infinite number of intersections when (i) $|y| > 1$ and, (ii) when $b$ is nonzero, $y$ also must have the same sign as $b$. Let $\{y_n\}$ be chosen from these points of intersection and let $\{z_n = (b/y_n) + iy_n\}$. Thus, $\text{Val}(f, w) = \infty$ for $w \in C(f, \infty)$.
We also note that \( f(C) \setminus (f(S) \cup C(f, \infty)) \neq \emptyset \). For example, \( w = a > 1 \), so \( w \notin f(S) \cup C(f, \infty) \). Then \( f^{-1}(w) = \{ \log a \} \). Note that \( f \) omits the real axis for \( a < -1 \).

**Example 2.** An example for Theorem 6

\[
f(z) = e^z + i \Im z = e^x \cos y + i (e^x \sin y + y)
\]

A calculation shows that \( \{ z_k = \log((4k + 3)\pi/2) + i (4k + 3)\pi/2 : k \in \mathbb{Z}^+ \} \subset f^{-1}(0) \). This sequence lies in the right half plane, with \( \Re z_k > 1 \) and \( z_k \to \infty \). We check the conditions of Theorem 6. We note that no point in the critical set of \( w \) is in the right half plane, because \( \Re \) away from 1 on each \( B \). A calculation shows that condition (3) is satisfied, since \( f \). Thus, \( C \) all asymptotic values.

**Example 3.** The cluster points of \( f(z) = z + \Re e^z \) are all asymptotic values.

\( C(f, \infty) \) consists of the horizontal lines where the imaginary part of \( w \) is an odd multiple of \( \pi/2 \). Note that while the horizontal lines in the cluster set can be thought of as horizontal asymptotes to the image of the critical set as \( z \to \infty \), \( f(S) \cap C(f, \infty) = \emptyset \). Further, \( \text{Val}(f) \) is finite. For details, see Example 4.2 in [Neu 05]. If we choose \( w \in C(f, \infty) \), the conditions in Theorem 6 are satisfied and \( w \) is an asymptotic value.

**Example 4.** The cluster points of \( f(z) = 2(\Re(z^3) + i z) \) are all asymptotic values.

\( C(f, \infty) \) is the real axis and \( \text{Val}(f) \) is finite. For details, see Example 4.3 in [Neu 05]. While \( f(S) \cap C(f, \infty) = \emptyset \), \( f(S) \) is asymptotic to \( C(f, \infty) \) as \( z \to \infty \). If we choose \( w \in C(f, \infty) \), the conditions in Theorem 6 are satisfied and \( w \) is an asymptotic value.

**Example 5.** The finite valence cluster points of \( f(z) = 2[\Re(z^2) + i(\Re(z) - \Im(z))] \) are all asymptotic values.

\( C(f, \infty) \) is the real axis. Note that every cluster point other than the origin is omitted and that the origin has infinite valence. Also note that \( f(S) = \{ 0 \} \). For details, see Example 4.4 in [Neu 05]. If we choose \( w \in C(f, \infty) \setminus \{ 0 \} \), the conditions in Theorem 6 are satisfied and \( w \) is an asymptotic value. Note that \( w = 0 \) is also an asymptotic value; choose \( \Re z = 3z \) and let \( z \to \infty \).

**Example 6.** The finite valence cluster points of \( f(z) = \Im(\exp(z^2)) + i(\Im(z^2)) \) are all asymptotic values.
Let $z = x + iy$. Calculations show that

$$S = \{xy = 0\} \cup \{y = k\pi/(2x) : k \in \mathbb{Z}\}
\quad f(S) = \{ik\pi : k \in \mathbb{Z}\}
$$

One can easily check that $\text{Val}(f, w) = \infty$ for $w \in f(S)$ and that each such $w$ is an asymptotic value. Thus $f(S) \subseteq C(f, \infty)$. Moreover, points other than $w = ik\pi$ on the imaginary axis and on the horizontal lines $\Im w = k\pi$ are omitted. The valence is constant within each of the remaining horizontal strips in the right half plane and the left half plane. When the points in a horizontal strip in the right half plane have valence 2, points in that strip in the left half plane are omitted. Similarly, if the points in a horizontal strip in the right half plane are omitted, points in that strip in the left half plane have valence 2. Whether the valence is 2 or 0 depends on whether or not $\Re w$ and $\sin(\Im w)$ have the same sign. We note that $S$ is nowhere dense. We see from Theorem 4 that $C(f, \infty)$ contains the horizontal lines $\Im w = k\pi$, as well as the imaginary axis, because none of these points has a neighborhood where the valence is constant. In fact, this exhausts $C(f, \infty)$. (Otherwise, we would have $\{z_n\} \to \infty$ with $f(z_n) \to w \in C(f, \infty)$, satisfying $\Re f(z_n) \to \Re w \neq 0$ and $2x_ny_n \to 3w \neq k\pi$. Since this implies that $\sin(2x_ny_n) \to \sin(3w) \neq 0$, we see that $\Re f(z_n)$ having a finite limit requires $|y_n| \to \infty$ and thus $\Re w$ must be zero, a contradiction.) If $w \in C(f, \infty) \setminus f(S)$, then the conditions in Theorem 6 are satisfied and $w$ is an asymptotic value. For example, if $w = ib$ for $b \neq k\pi$, let $y = b/(2x)$ and let $x \to 0^+$. As another example, if $w = 1$, let $y = \exp(-x^2)/(2x)$ and let $x \to \infty$.

**Remark 6.** Example 4 above illustrates why we require that the cluster set have empty interior in some neighborhood of the cluster point. Choose $w \in C(f, \infty)$ such that $\Re w = \pm 1$. Although for $\epsilon > 0$, $B(w, \epsilon) \cap C(f, \infty) \neq \emptyset$, $w$ is not an asymptotic value of $f$. In order for $\Re f$ to approach $\Re w$, our asymptotic path would either have $\Re z$ bounded (but then $\Im z \to \pm \infty$, which gives a contradiction because of the oscillation in $\Re f$) or have $\Re z \to +\infty$ (however, $3f \to 3w$ requires that $\Im z \to 0$, which is impossible as $|\Re f|$ would then be unbounded).

**Corollary 1.** Suppose that $f : \mathbb{C} \to \mathbb{C}$ is harmonic and that $\text{Val}(f, w)$ is finite for all $w \in \mathbb{C}$. Suppose that $S$ is nowhere dense. Let $w_0 \in C(f, \infty)$. Then at least one of the following holds for some subsequence of any given sequence $\{z_n\}$ such that $z_n \to \infty$ and $f(z_n) \to w_0$:

1. $d(z_n, S) \to 0$ as $z_n \to \infty$.
2. $J_f(z_n) \to 0$ as $z_n \to \infty$.
3. There is no $\rho > 0$ such that $|J_f(z)/J_f(z_n)| < M$ on $B(z_n, \rho)$ for all $n$ for some finite constant $M$.
4. There is no $\rho > 0$ such that $\sup_n \{|\mu_f(z)| : z \in B(z_n, \rho)\} < 1$.

**Proof.** By Theorem 3.9 in [Neu 05], if $S$ is nowhere dense and all points have finite valence, then the cluster set has empty interior. Then apply Theorem 5. □

**Remark 7.** If $f$ is an entire harmonic function, with finite valence and $S$ nowhere dense, Corollary 4 holds. Let $w_0$ be a cluster point of $f$. If condition (1) holds, $\{z_n\}$ gets arbitrarily close to $S$ (though this does not guarantee that $J_f(z_n) \to 0$).

If condition (2) holds, there is no region containing the $\{z_n\}$ where the function is quasiconformal or quasiregular.
Remark 8. Comparing the Conjecture above with Theorem 6, it remains to show that harmonicity implies condition (2) on $B(w_0, \epsilon)$ in Theorem 6. It would also be interesting to find a necessary and sufficient condition for $C(f, \infty)$ to have empty interior.

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References

Bal 91. M. B. Balk, Polyanalytic Functions, Mathematical research, volume 63, Akademie Verlag GmbH (1991).
Boc 46. S. Bochner, Bloch’s theorem for real variables, Bull. Amer. Math. Soc. 52 (1946), 715-719.
Br 86. D. A. Brannan, On the behaviour of continuous functions near infinity, Complex Variables 5 (1986), 237-244.
Car 56. M. L. Cartwright, Integral Functions, Cambridge University Press (1956).
CGH 00. H. Chen, P. M. Gauthier, and W. Hengartner, Bloch constants for planar harmonic mappings, Proc. Amer. Math. Soc. 128 (2000), 3231-3240.
Con 78. J. B. Conway, Functions of one complex variable 1, second edition, Springer-Verlag (1978).
Hir 76. M. W. Hirsch, Differential Topology, Springer-Verlag (1976).
HY 61. J. G. Hocking and G. S. Young, Topology, Addison-Wesley (1961).
Neu 05. G. Neumann, Valence of complex-valued planar harmonic functions, Trans. Amer. Math. Soc. 357 (2005), 3133-3167.
New 64. M. H. A. Newman, Elements of the topology of plane sets of points, second edition, Cambridge University Press, (1964).
Ric 93. S. Rickman, Quasiregular mappings, Springer-Verlag (1993).

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