METHODS IN THE LOCAL THEORY OF PACKING AND COVERING LATTICES

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ABSTRACT. In this paper we are concerned with three lattice problems: the lattice packing problem, the lattice covering problem and the lattice packing-covering problem. One way to find optimal lattices for these problems is to enumerate all finitely many, locally optimal lattices. For the lattice packing problem there are two classical algorithms going back to Minkowski and Voronoi. For the covering and for the packing-covering problem we propose new algorithms.

Here we give a brief survey about these approaches. We report on some recent computer based computations where we were able to reproduce and partially extend the known classification of locally optimal lattices. Furthermore we found new record breaking covering and packing-covering lattices. We describe several methods with examples to show that a lattice is a locally optimal solution to one of the three problems.

1. INTRODUCTION

Classical problems in the geometry of numbers are the determination of most economical lattice sphere packings and coverings of the Euclidean $d$-space $\mathbb{R}^d$. A lattice $L$ is a full rank, discrete subgroup of $\mathbb{R}^d$. Thus there exist matrices $A \in \text{GL}_d(\mathbb{R})$ with $L = AZ^d$ which we call bases of $L$.

If $B^d$ denotes the Euclidean unit ball, then the Minkowski sum $L + \alpha B^d = \{v + \alpha x : v \in L, x \in B^d\}$, $\alpha \in \mathbb{R}_{>0}$, is a lattice packing if the translates of $\alpha B^d$ have mutually disjoint interiors and a lattice covering if $\mathbb{R}^d = L + \alpha B^d$. The packing radius $\lambda(L)$ of a lattice $L$ is given by

$$\lambda(L) = \max\{\lambda : L + \lambda B^d \text{ is a lattice packing}\},$$

and the covering radius $\mu(L)$ by

$$\mu(L) = \min\{\mu : L + \mu B^d \text{ is a lattice covering}\}.$$

For a lattice $L$ we define its determinant $\det(L) = |\det(A)|$, which is independent of the chosen basis. We consider the following three “quality measures” of $L$:

1. the packing density $\delta(L) = \frac{\lambda(L)^d}{\det(L)} \cdot \kappa_d$,

2. the covering density $\Theta(L) = \frac{\mu(L)^d}{\det(L)} \cdot \kappa_d$,

3. the packing-covering constant $\gamma(L) = \frac{\mu(L)}{\lambda(L)}$.

Here $\kappa_d = \pi^{d/2} / \Gamma(d/2 + 1)$ denotes the volume of the unit ball $B^d$. So the packing density $\delta(L)$ for instance gives the ratio of space covered by spheres in the lattice packing $L + \lambda(L)B^d$. Note that all three quantities are invariant with respect to a scaling $\alpha L$ of $L$ with $\alpha \neq 0$. For each of the quantities we consider the problem of finding extremal lattices attaining a maximum or minimum respectively.

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Problem 1.1 (Lattice Packing Problem). For \( d \geq 2 \), determine \( \delta_d = \max_L \delta(L) \) and lattices \( L \) attaining it.

Problem 1.2 (Lattice Covering Problem). For \( d \geq 2 \), determine \( \Theta_d = \min_L \Theta(L) \) and lattices \( L \) attaining it.

Problem 1.3 (Lattice Packing-Covering Problem). For \( d \geq 2 \), determine \( \gamma_d = \min_L \gamma(L) \) and lattices \( L \) attaining it.

Note that all three optima are attained. All three problems have in common that there exist only finitely many local optima for every \( d \) (see Section 3 for definitions). In this article we want to review some of the major tools available to find such local extrema and to verify their local optimality. In Section 2 we briefly summarize known results of the three problems and in Section 3 we give a short introduction to the connection of lattices and positive definite quadratic forms, which gives the framework in that the problems are usually dealt with. In Section 4 we describe two classical approaches by Minkowski and Voronoi to enumerate all local optima of the lattice packing problem. In Section 5 we are concerned with local optima of the lattice covering and the lattice packing-covering problem, which can be treated in parallel. Most of these techniques are described in greater detail in [SV04a] and [SV04b].

2. KNOWN RESULTS

2.1. The Lattice Packing Problem. The lattice packing problem, arising from the study of positive definite quadratic forms, is the oldest and most popular of the three problems and has been considered by many authors in the past. As shown in Table 1 the solution to the problem was known for dimension \( d \leq 8 \) since 1934. For a description of the extremal root lattices \( A_d, D_d \) and \( E_d \) and the history of the problem we refer the interested reader to the book [CS88b]. Recently, Cohn and Kumar [CK04] showed that the Leech lattice \( \Lambda \) gives the unique densest lattice packing in \( \mathbb{R}^{24} \). Furthermore they showed: The density of any sphere packing (without restriction to lattices) in \( \mathbb{R}^{24} \) cannot exceed the one given by the Leech lattice by a factor of more than \( 1 + 1.65 \cdot 10^{-30} \).

| \( d \) | lattice | density \( \delta_d \) | author(s) |
|-------|---------|-----------------------|-----------|
| 2     | \( A_2 \) | 0.9069...             | Lagrange [Lag73], 1773 |
| 3     | \( A_3 = D_3 \) | 0.7404...             | Gauss [Gau40], 1840 |
| 4     | \( D_4 \) | 0.6168...             | Korkine, Zolotareff [KZ73], 1873 |
| 5     | \( D_5 \) | 0.4652...             | Korkine, Zolotareff [KZ77], 1877 |
| 6     | \( E_6 \) | 0.3729...             | Blichfeldt [Bli34], 1934 |
| 7     | \( E_7 \) | 0.2953...             | Blichfeldt [Bli34], 1934 |
| 8     | \( E_8 \) | 0.2536...             | Blichfeldt [Bli34], 1934 |
| 24    | \( \Lambda \) | 0.0019...             | Cohn, Kumar [CK04], 2004 |

Table 1. Optimal lattice packings.

2.2. The Lattice Covering Problem. The lattice covering problem has only been solved up to dimension 5. Recently, we were able to verify the list of known results computationally. Even more, we found the complete list of 222 local covering optima in dimension 5. Table 2 invites to a question formulated by Ryshkov [Rys67], who asked for the lowest dimension in which \( A_4 \) gives not the thinnest lattice covering. In Section 5 we describe the method used to find a lattice \( L_6^c \subset \mathbb{R}^6 \) with \( \Theta(L_6^c) = 2.4648... < 2.5511... = \Theta(A_6^c) \). Thus the answer to Ryshkov’s question is 6. It remains an open problem to prove that the lattice \( L_6^c \) gives the best lattice covering in dimension 6. We do not even know the exact
coordinates of the lattice yet. Currently, a complete solution in dimension \( d \geq 6 \) seems out of reach without completely new methods.

| \( d \) | lattice | density \( \Theta_d \) | author(s) |
|---|---|---|---|
| 2 | \( A_2 \) | 1.2091 \ldots | Kershner [Ker39], 1939 |
| 3 | \( A_3 \) | 1.4635 \ldots | Bambah [Bam54], 1954 |
| 4 | \( A_4 \) | 1.7655 \ldots | Delone, Ryshkov [DR63], 1963 |
| 5 | \( A_5 \) | 2.1242 \ldots | Ryshkov, Baranovskii [RB75], 1975 |

Table 2. Optimal lattice coverings.

2.3. The Lattice Packing-Covering Problem. As in the case of the lattice covering problem the lattice packing-covering problem has been solved only for dimensions \( d \leq 5 \).

And, as in the covering case, we recently were able to verify these results computationally. Moreover, we found a new best known lattice \( L_6^{pc} \) in dimension 6, having a slightly lower packing-covering constant \( \gamma(L_6^{pc}) = 1.4110 \ldots \) than the previously best known one \( \gamma(E_6) = \sqrt{2} \).

One reason for studying the lattice packing-covering problem is the open question whether there exists a dimension \( d \) with \( \gamma_d \geq 2 \). If so, then any \( d \)-dimensional lattice packing with spheres would leave space large enough for spheres of the same radius. This would in particular prove that densest sphere packings in dimension \( d \) are non-lattice packings. This phenomenon is likely to be true for large dimensions, but has not been verified for any \( d \) so far.

| \( d \) | lattice | \( \gamma_d \) | author(s) |
|---|---|---|---|
| 2 | \( A_2 \) | 1.1547 \ldots | Ryshkov [Rys74], 1974 |
| 3 | \( A_3 \) | 1.2909 \ldots | Ryshkov [Rys74], 1974 |
| 4 | \( H_4 \) | 1.3625 \ldots | Horváth [Hor82], 1980 |
| 5 | \( H_5 \) | 1.4494 \ldots | Horváth [Hor86], 1986 |

Table 3. Optimal lattice packing-coverings.

3. LATTICES AND POSITIVE QUADRATIC FORMS

It is sometimes convenient to switch from the language of lattices to the language of positive definite quadratic forms (PQFs from now on). In this section we give a dictionary. For further reading we refer to [CS88b] and [SV04a].

Given a \( d \)-dimensional lattice \( L = A\mathbb{Z}^d \) with basis \( A \) we associate a \( d \)-dimensional PQF \( Q[x] = x'A'Ax = x'Gx \), where the Gram matrix \( G = A'A \) is symmetric and positive definite. We will carelessly identify quadratic forms with symmetric matrices by saying \( Q = G \) and \( Q[x] = x'^{\prime}Qx \). The set of quadratic forms is a \( (d+1) \)-dimensional real vector space \( S^d \), in which the set of PQFs forms an open, convex cone \( T^d \). The PQF \( Q \) depends on the chosen basis \( A \) of \( L \). For two arbitrary bases \( A \) and \( B \) of \( L \) there exists a \( U \in \text{GL}_d(\mathbb{Z}) \) with \( A = BU \). Thus, \( \text{GL}_d(\mathbb{Z}) \) acts on \( S^d_+ \) by \( Q \mapsto U'AQU' \). A PQF \( Q \) can be associated to different lattices \( L = A\mathbb{Z}^d \) and \( L' = A'\mathbb{Z}^d \). In this case there exists an orthogonal transformation \( O \) with \( A = OA' \). Note that the packing and covering density, as well as the packing-covering constant, are invariant with respect to orthogonal transformations.

The determinant (or discriminant) of a PQF \( Q \) is defined by \( \det(Q) \). The homogeneous minimum \( \lambda(Q) \) and the inhomogeneous minimum \( \mu(Q) \) are given by

\[
\lambda(Q) = \min_{v \in \mathbb{Z}^d \setminus \{0\}} Q[v], \quad \mu(Q) = \max_{x \in \mathbb{R}^d} \min_{v \in \mathbb{Z}^d} Q[x - v].
\]
If $Q$ is associated to $L$, then $\det(L) = \sqrt{\det(Q)}$, $\mu(L) = \sqrt{\mu(Q)}$, $\lambda(L) = \sqrt{\lambda(Q)/2}$. Using this dictionary we define

$$\delta(Q) = \sqrt{\frac{\lambda(Q)^d}{\det Q}} \frac{\kappa_d}{2^d}, \quad \Theta(Q) = \sqrt{\frac{\mu(Q)^d}{\det Q}} \frac{\kappa_d}{\lambda(Q)} \quad \text{and} \quad \gamma(Q) = 2 \sqrt{\frac{\mu(Q)}{\lambda(Q)}}.$$ 

We say that a lattice $L$ with associated PQF $Q$ gives a locally optimal packing, locally optimal lattice covering or locally optimal lattice packing-covering, if there is a neighborhood of $Q$ in $S^d_{\geq 0}$, so that we have $\delta(Q) \geq \delta(Q')$, $\Theta(Q) \leq \Theta(Q')$ or $\gamma(Q) \leq \gamma(Q')$ respectively, for all $Q'$ in this neighborhood.

### 4. On Packing Lattices

A PQF $Q$ attaining a local maximum of $\delta(Q)$ is called extreme. A PQF attaining $\delta_d = \max_{Q \in S^d_{\geq 0}} \delta(Q)$ is called absolutely extreme or critical. One can characterize an extreme PQF using the geometry of its minimal vectors

$$\Min(Q) = \{v \in \mathbb{Z}^d : Q[v] = \lambda(Q)\}.$$ 

Before we state the characterization in Theorem 4.1, we give some more definitions. A PQF $Q'$ is called perfect if it is uniquely determined by its minimal vectors, i.e. $Q'$ is the unique solution of the linear equations $Q[v] = \lambda(Q')$, $v \in \Min(Q')$. A PQF $Q$ is called eutactic if

$$Q^{-1} \in \relint \cone\{vv^t : v \in \Min(Q)\} = \left\{ \sum_{v \in \Min(Q)} \lambda_v vv^t : \lambda_v > 0, v \in \Min(Q) \right\}.$$ 

The eutaxy and the polyhedral cone $\cone\{vv^t : v \in \Min(Q)\}$ also plays an important role in the lattice packing-covering problem. As a general reference on basic facts about polyhedral cones, which are used throughout this article, we refer to the book of Ziegler [Zie97].

**Theorem 4.1** (Voronoi [Vor07]). A PQF is extreme if and only if it is perfect and eutactic.

This provides an easy way for proving that a given PQF is extreme: after finding the minimal vectors, one has to solve a system of linear equations to show its perfectness. Then, one has to solve a linear programming problem to verify its eutaxy. By scaling we can normalize an extreme PQF $Q$ so that $\lambda(Q)$ is rational. Then, since $Q$ is perfect, the matrix entries of $Q$ are rational as well.

It turns out that there exist only finitely many pairwise non-equivalent perfect PQFs in $S^d_{\geq 0}$. We want to describe two classical algorithms to attain all perfect forms of a given dimension $d$. The first one goes back to Minkowski, the second one is due to Voronoi. Here we only state definitions and main results. Additionally, we briefly sketch the computations which we were able to perform in low dimensions. We compare them with corresponding results in the literature.

For history and further remarks we refer to [GL87], [RB79], [vdW56], [Mar03] and to references therein. Chapter 3 of Gruber and Lekkerkerker’s book [GL87] gives a comprehensive survey about history, results and literature of the reduction theory of PQFs. The article [RB79] introduces to methods for studying the geometry of PQFs and contains many proofs. Van der Waerden’s paper [vdW56] is a classic resource for Minkowski’s approach. The recent book [Mar03] of Martinet gives a contemporary view on Voronoi’s approach and on possible generalizations.

#### 4.1. Minkowski’s Approach.

**Definition 4.2.** A PQF $Q = (q_{ij}) \in S^d_{\geq 0}$ is called Minkowski reduced if

- (i) $Q[x] \geq q_{ii}$ whenever $\gcd(x_1, \ldots, x_d) = 1$, $i = 1, \ldots, d$,
- (ii) $q_{i,i+1} \geq 0$, $i = 1, \ldots, d - 1$. 

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**References:**

- [GL87], [RB79], [vdW56], [Mar03]
- [Zie97]

**Key Concepts:**

- Locally optimal lattice packing
- Locally optimal lattice covering
- Locally optimal lattice packing-covering
- Extreme PQF
- Absolutely extreme PQF
- Critical PQF
- Perfect PQF
- Eutactic PQF
- Relint cone
- Polyhedral cone
- Linear equations
- System of linear equations
- Relative interior
- Minimal vectors
- Perfect form
- Eutaxy
- Reduction theory of PQFs
- Minkowski’s approach
- Voronoi’s approach
- Martinet’s book
- Contemporary view on Voronoi’s approach
- Possible generalizations
Every PQF is equivalent to a Minkowski reduced PQF. The following procedure, which is nothing but an algorithmic interpretation of the definition, finds a Minkowski reduced PQF equivalent to a given PQF \( Q \). Choose a minimal vector \( v_1 \) of \( Q \). Then, choose among all vectors in \( \mathbb{Z}^d \), which can complement \( v_1 \) to a lattice basis of \( \mathbb{Z}^d \), a vector \( v_2 \) for which the value \( Q[v_2] \) is minimal. Using this greedy strategy we get a basis \( V = (v_1, \ldots, v_d) \in \text{GL}_d(\mathbb{Z}) \). Now we choose signs for \( v_i \) so that \( v^t_i Q v_i \geq 0 \). Hence, \( V^t Q V \) is Minkowski reduced.

The set of Minkowski reduced PQFs forms an unbounded cone in \( S_{\geq 0}^d \) which is defined by the linear inequalities (i) and (ii). By \( M \) we denote the cone defined by the linear inequalities (i) and by \( M^+ \) the one which is defined by the linear inequalities (i) and (ii). Minkowski [Min05] showed that \( M \), and hence \( M^+ \), is a polyhedral cone, i.e. that finitely many inequalities (i) imply all others. He showed that every extreme PQF is equivalent to a PQF lying on a ray (a one-dimensional face) of \( M^+ \). Ryshkov [Rys70] proved that every perfect PQF is equivalent to a PQF lying on a ray of \( M^+ \). On the other hand Cohn, Lomakina and Ryshkov [CLR82] found a ray of \( M^+ \subset S_{\geq 0}^5 \) which contains non-perfect PQFs.

Minkowski [Min87] gave a list of conditions implying all others in (i) up to dimension 6. Tammela [Tam81] enlarged this list to dimension 7. Besides the \( d - 1 \) inequalities \( q_{11} \leq \ldots \leq q_{dd} \), the linear conditions for \( M \), \( d = 2, \ldots, 7 \), are attained by plugging the values \( x = (x_1, \ldots, x_d) \in \mathbb{Z}^d \) from Table 4 into (i), where the indices \( i_1, \ldots, i_d \) run through all permutations of \( \{1, \ldots, d\} \). If \( d < 7 \) one has to omit the columns \( d+1, \ldots, 7 \) and the rows with more than \( d \) non-zero entries.

| \( x_{i1} \) | \( \pm x_{i2} \) | \( \pm x_{i3} \) | \( \pm x_{i4} \) | \( \pm x_{i5} \) | \( \pm x_{i6} \) | \( \pm x_{i7} \) |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| 2 | 1 | 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 1 | 1 | 1 | 1 | 1 |
| 3 | 2 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 1 | 1 | 1 | 1 | 1 |
| 3 | 2 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 1 | 1 | 1 | 1 |
| 3 | 2 | 2 | 1 | 1 | 1 | 1 |
| 3 | 2 | 2 | 2 | 1 | 1 | 1 |
| 4 | 2 | 2 | 2 | 2 | 1 | 1 |
| 3 | 3 | 2 | 1 | 1 | 1 | 1 |
| 4 | 3 | 2 | 1 | 1 | 1 | 1 |
| 3 | 2 | 2 | 2 | 1 | 1 | 1 |
| 4 | 3 | 2 | 2 | 2 | 1 | 1 |

**Table 4.** Tammela’s list of linear inequalities defining \( M \)

We checked these conditions for redundancy using the software *lrs* of Avis [Avi04]. Our computations show that in row \(^1\) for \( d = 6 \), the entries with \( x_1 \neq 3 \) are redundant (as already mentioned by Tammela in [Tam73]). For \( d = 7 \) redundant entries are those with \( x_1 \neq 0, 3 \) and \( x_1 = 0, x_2 \neq 3 \). In row \(^2\) the entries with \( x_1 \neq 3 \) or \( x_2 \neq 3 \), and in row \(^3\) the entries with \( x_1 \neq 4 \) or \( x_2 \neq 3 \) are redundant. The remaining conditions
are all non-redundant and define a facet of $\mathcal{M}$. In Table 5 we list the number of facets and rays as far as we were able to compute them with cdd \cite{Fuk03}. We hereby confirm earlier results by $^a$ Minkowski \cite{Min7} and $^b$ Barnes and Cohn \cite{BC7}. We made the data available from the arXiv.org e-print archive. To access it, download the source files for the paper arXiv:math.MG/0412320. The files mink2.ine, ..., mink6.ine (due to its size the file mink7.ine is only available from the authors) contain the facets of $\mathcal{M}$, the files mink2.ext, ..., mink6.ext contain the rays of $\mathcal{M}$, the files minkp2.ine, ..., minkp6.ine contain the facets of $\mathcal{M}^+$, and the files minkp2.ext, ..., minkp5.ext contain the rays of $\mathcal{M}^+$. For the data format we chose the common convention (Polyhedra $H$-format for the *.ine-files and Polyhedra $V$-format for the *.ext-files) of the software packages cdd and lrs.

The computational bottlenecks of Minkowski’s approach are apparent. It is not easy to find a sufficiently small system of linear inequalities defining $\mathcal{M}$ (or of $\mathcal{M}^+$). Even if one has a minimal system of linear inequalities, then computing its rays is a very difficult computational problem in higher dimensions.

| $d$ | $(d+1)_2$ | $\# \text{Facets } \mathcal{M}$ | $\# \text{Rays } \mathcal{M}$ | $\# \text{Facets } \mathcal{M}^+$ | $\# \text{Rays } \mathcal{M}^+$ |
|-----|----------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 2   | 3        | $3^a$                         | $3^a$                         | $3^a$                         | $3^a$                         |
| 3   | 6        | $12^a$                        | $19^b$                        | $9^a$                         | $11^b$                        |
| 4   | 10       | $39^a$                        | $323^b$                       | $26^b$                        | $109^b$                       |
| 5   | 15       | 200                           | 15971                         | 117                           | 4105                          |
| 6   | 21       | 1675                          | ?                             | 1086                          | ?                             |
| 7   | 28       | 65684                         | ?                             | ?                             | ?                             |

Table 5. Known number of facets and rays of $\mathcal{M}$ and $\mathcal{M}^+$.

4.2. Voronoi’s Approach. Now we describe Voronoi’s algorithm \cite{Vor07} for finding all perfect forms of a given dimension. Let $m$ be a positive number. In the remaining of this section we assume that every perfect form $Q$ is scaled so that $\lambda(Q) = m$. The set

$$\mathcal{P}_m = \{ Q \in S^d_{>0} : Q[v] \geq m \text{ for all } v \in \mathbb{Z}^d \setminus \{0\} \}$$

is a convex, locally finite polyhedral cone. Its boundary consists of the PQFs with homogeneous minimum $m$. A PQF $Q$ is perfect if and only if it is a vertex of $\mathcal{P}_m$. The set of perfect PQFs of a given dimension $d$ naturally carries a graph structure which we denote as the Voronoi graph in dimension $d$: Two perfect PQFs $Q, Q'$ are connected by an edge if the line segment $\text{conv}\{Q, Q'\}$ (convex hull of $Q$ and $Q'$) is an edge of $\mathcal{P}_m$. In this case we say that $Q$ and $Q'$ are Voronoi neighbors. The group $\text{GL}_d(\mathbb{Z})$ acts on $\mathcal{P}_m$, on its vertices and on its edges by $Q \mapsto U^t QU$. Therefore, one can enumerate perfect PQFs by a graph traversal algorithm which we shall now describe.

Voronoi’s first perfect form $Q[x] = \sum_{i=1}^d x_i^2 + \sum_{i<j} x_i x_j$, which is associated to the root lattice $A_2$, can serve as a starting point in any dimension. For implementing a graph traversal algorithm one has to find the Voronoi neighbors of a given perfect form $Q$. Consider the unbounded polyhedral cone

$$\mathcal{P}(Q) = \{ Q' \in S^d : Q'[v] \geq m, v \in \text{Min}(Q) \}.$$ We compute the rays $Q + \mathbb{R}_{\geq 0} R_i$, $i = 1, \ldots, n$ of $\mathcal{P}(Q)$. The $R_i$ turn out to be indefinite quadratic forms. So there are $v \in \mathbb{Z}^d$ with $R_i[v] < 0$. Then, the Voronoi neighbors of $Q$ are $Q + \rho_i R_i$ where $\rho_i$ is the smallest positive number so that $\lambda(Q + \rho_i R_i) = m$ and $\text{Min}(Q + \rho_i R_i) \not\subseteq \text{Min}(Q)$. It is possible to determine $\rho_i$, for example with the following procedure:
(α, β) ← (0, 1)
while Q + βRᵢ ∉ Sₙₒ or λ(Q + βRᵢ) = m do
  if Q + βRᵢ ∉ Sₙₒ then β ← α + β
  else (α, β) ← (β, 2β)
end if
end while
while Min(Q + αRᵢ) ⊆ Min Q do
  γ ← α + β
  if λ(Q + γRᵢ) < m then β ← γ
  else α ← γ
end if
end while
ρᵢ ← α

Perfect forms were classified up to dimension 7. A list of all these forms is given in the paper [CS88a] of Conway and Sloane. One can find an electronic version in the Catalogue of Lattices [1] by Nebe and Sloane. On his homepage [2], Martinet reports that up to now, 10916 pairwise inequivalent perfect forms are known in dimension 8 and lists them.

We verified the results for dimensions d ≤ 6 using the programs lrs by Avis [Avi04], isom by Plesken and Souvignier [PS97] and shvec by Vallentin [Val99]. In Table 6 we give the known classifications of perfect forms, extreme forms and absolute extreme forms together with the references where the classifications were established.

| d  | (d+1)/2 | # perfect | # extreme | # critical |
|----|---------|-----------|-----------|------------|
| 2  | 3       | 1ᵇ        | 1ᵃ        | 1ᵃ         |
| 3  | 6       | 1ᵇ        | 1ᵇ        | 1ᵇ         |
| 4  | 10      | 2ᶜ        | 2ᶜ        | 1ᶜ         |
| 5  | 15      | 3ᵈ        | 3ᵈ        | 1ᵈ         |
| 6  | 21      | 7ᶠ        | 6ᶜ        | 1ᶠ         |
| 7  | 28      | 33ᵇ       | 30ᵇ       | 1ᵍ         |
| 8  | 36      | ⩾10916ᵇ   | ?         | 1ᵍ         |
| 9  | 45      | > 500 000ᵇ | ?        | ?          |
| 24 | 300     | ?         | ?         | 1ʲ         |

| a | Lagrange [Lag73] |
|---|-----------------|
| b | Gauß [Gau40]   |
| c | Korkine, Zolotareff [KZ73] |
| d | Korkine, Zolotareff [KZ77] |
| e | Hofreiter [Hol33] |
| f | Barnes [Bar57] |
| g | Vetchinkin [Vet82] |
| h | Jaquet-Chiffelle [JC93] |
| i | Laihem, Baril, Napias, Batut, Martinet (see [http://www.math.u-bordeaux.fr/~martinet]) |
| j | Cohn, Kumar [CK04] |

**Table 6.** Classifications of perfect, extreme and absolute extreme forms

The computational bottleneck of Voronoi’s approach is mainly the enumeration of all rays of the polyhedral cone P(Q) in case of a large set Min(Q) of minimal vectors. Martinet writes in [Mar03], Ch. 7.11: “The existence of E₈ [...] makes hopeless any attempt to
construct the Voronoi graph in dimension 8". Another problem is the combinatorial explosion, when \( d \geq 9 \). We found more than 500000 inequivalent perfect forms in dimension 9 and we strongly believe there exist millions of them.

Finally, we want to remind of Coxeter’s \( A_d \)-hypothesis: Although finding perfect forms with maximal packing density is a very difficult problem, finding perfect forms with minimal packing density might be very easy. In [Cox51] Coxeter formulates the following conjecture:

**Conjecture 4.3.** (Coxeter’s \( A_d \)-hypothesis) Voronoi’s first perfect form gives the minimal packing density among all perfect forms of a given dimension.

Our computations support Coxeter’s conjecture. But on the contrary, Conway and Sloane [CS88a] conjecture that it is false for sufficiently large \( d \).

5. Covering and Packing-Covering Lattices

The lattice covering problem and the lattice packing-covering problem can be treated in parallel. We describe below that both problems have only finitely many local optima which can be found by solving finitely many convex optimization problems. This is mainly due to Voronoi’s theory of Delone subdivisions, which we briefly review. For a detailed account we refer to [SV04a].

With an implementation of the proposed algorithms we found all local optima in dimension \( d \leq 5 \) and some new best known lattices in dimension \( d \geq 6 \).

For both problems, recognition of local optima is not as easy as for the lattice packing problem. Due to the involved convexity we can give sufficient conditions for local optima, allowing to compute a certificate for the local optimality of a lattice. This is in particular applicable, if the Delone subdivision is a triangulation which is the generic case (for definitions see below). Exemplarily we give a proof of the local packing-covering optimality of the lattices \( A_d^* \).

In some cases it is possible to attain good or even tight “local lower bounds” for the lattice covering density and the packing-covering constant. This is demonstrated for the local packing-covering optimality of the Leech lattice. A similar proof of the local covering optimality of the Leech lattice is given in [SV04b].

5.1. Voronoi’s Theory of Delone Subdivisions.** Let \( Q \) be a positive semidefinite quadratic form. A polyhedron \( P = \text{conv}\{v_1, v_2, \ldots\} \) with \( v_1, v_2, \ldots \in \mathbb{Z}^d \), is called a Delone polyhedron of \( Q \) if there exists a \( c \in \mathbb{R}^d \) and a real number \( r \in \mathbb{R} \) with \( Q[v_i - c] = r^2 \) for all \( i = 1, 2, \ldots \), and \( Q[v - c] > r^2 \) for all other \( v \in \mathbb{Z}^d \setminus \{v_1, v_2, \ldots\} \). The set \( \text{Del}(Q) \) of all Delone polyhedra is called the Delone subdivision of \( Q \). It is a periodic face-to-face tiling of \( \mathbb{R}^d \). Therefore \( \text{Del}(Q) \) is completely determined by all Delone polytopes having a vertex at the origin 0. We call two Delone polyhedra \( L, L' \) equivalent if there exists a \( v \in \mathbb{Z}^d \) so that \( L = v \pm L' \). Note moreover that the inhomogeneous minimum \( \mu(Q) \) is at the same time the maximum squared circumradius of its Delone polyhedra. We say that the Delone subdivision of a positive semidefinite quadratic form \( Q' \) is a refinement of the Delone subdivision of \( Q \), if every Delone polytope of \( Q' \) is contained in a Delone polytope of \( Q \).

By a theory of Voronoi [Vor08], the set of positive semidefinite quadratic forms with a fixed Delone subdivision \( D \) is an open (with respect to its affine hull) polyhedral cone in \( S_{d \geq 0}^d \). We refer to this set as the secondary cone \( \Delta(D) \) of the subdivision. In the literature the secondary cone is sometimes called \( L \)-type domain of the subdivision. The topological closure \( \overline{\Delta(D)} \) of a secondary cone is a closed polyhedral cone. The relative interior of each face in \( S_{d \geq 0}^d \) is the secondary cone of another Delone subdivision. If a face is contained in the boundary of a second face, then the corresponding Delone subdivision of the first is a true refinement of the second one.

The interior of faces of maximal dimension \( \binom{d+1}{2} \) contain PQFs whose Delone subdivision is a triangulation, that is, it consists of simplices only. We refer to such a subdivision
as a simplicial Delone subdivision or Delone triangulation. As mentioned in Section 3, the group \( \text{GL}_d(\mathbb{Z}) \) acts on \( S^d_{\geq 0} \). One of the key observations of Voronoi is that under this group action there exist only finitely many inequivalent Delone subdivisions, respectively secondary cones.

**Theorem 5.1** (Voronoi [Vor08]). The topological closures of secondary cones of Delone triangulations give a face-to-face tiling of \( S^d_{\geq 0} \). The group \( \text{GL}_d(\mathbb{Z}) \) acts on the tiling, and under this group action there are only finitely many non-equivalent secondary cones.

Given a Delone triangulation \( \mathcal{D} \), the Delone triangulations \( \mathcal{D}' \) with \( \Delta(\mathcal{D}') \) sharing a facet with \( \Delta(\mathcal{D}) \) are attained by bistellar operations (flips). These change a triangulation only in certain repartitioning polytopes associated to the facet. By this operation it becomes possible to enumerate all Delone triangulations, and hence all Delone subdivisions in a given dimension. For details we refer to [SV04a].

### 5.2. Obtaining Local Optima via Convex Optimization

For a fixed triangulation \( \mathcal{D} \), we can formulate the lattice covering, as well as the lattice packing-covering problem in the framework of Determinant Maximization Problems. Following Vandenberghe, Boyd, and Wu [VBW98] their general form is

\[
\begin{align*}
\text{minimize} & \quad c^T x - \log \det G(x) \\
\text{subject to} & \quad G(x) \succ 0, F(x) \succeq 0.
\end{align*}
\]

(1)

Here, the optimization vector is \( x \in \mathbb{R}^D \). The objective function contains a linear part given by \( c \in \mathbb{R}^D \) and \( G : \mathbb{R}^D \rightarrow \mathbb{R}^{m \times m}, F : \mathbb{R}^D \rightarrow \mathbb{R}^{n \times n} \) are both affine maps

\[
G(x) = G_0 + x_1 G_1 + \cdots + x_D G_D,
\]

\[
F(x) = F_0 + x_1 F_1 + \cdots + x_D F_D,
\]

where \( G_i \in \mathbb{R}^{m \times m}, F_i \in \mathbb{R}^{n \times n}, i = 0, \ldots, D \), are symmetric matrices. The notation \( G(x) \succ 0 \) and \( F(x) \succeq 0 \) gives the constraints “\( G(x) \) is positive definite” and “\( F(x) \) is positive semidefinite”. Note that we are dealing with a so-called semidefinite programming problem, if \( G(x) \) is the identity matrix for all \( x \in \mathbb{R}^D \).

For the lattice covering, as well as the lattice packing-covering problem, we can express \( \mu(Q) \leq 1 \) as a linear matrix inequality \( F(q_{ij}) \succeq 0 \) with optimization vector \( Q = (q_{ij}) \). To see this, it is crucial to observe that an inner product \( (\cdot, \cdot) \) defined by \( (x, y) = x^T Q y \) gives a linear expression in the parameters \( (q_{ij}) \) for any fixed choice \( x, y \in \mathbb{Z}^d \) (or \( \mathbb{R}^d \)). Delone, Dolbilin, Rychkov and Stogrin [DDRS70] showed

**Proposition 5.2.** Let \( L = \text{conv} \{0, v_1, \ldots, v_d\} \subseteq \mathbb{R}^d \) be a \( d \)-dimensional simplex. Then \( L \)'s circumsphere is at most 1 with respect to \( (\cdot, \cdot) \) if and only if

\[
\text{BR}_L(Q) = \begin{pmatrix}
4 & (v_1, v_1) & (v_2, v_2) & \cdots & (v_d, v_d)
(v_1, v_1) & (v_1, v_1) & (v_1, v_2) & \cdots & (v_1, v_d)
(v_2, v_2) & (v_2, v_1) & (v_2, v_2) & \cdots & (v_2, v_d)
\vdots & \vdots & \vdots & \ddots & \vdots
(v_d, v_d) & (v_d, v_1) & (v_d, v_2) & \cdots & (v_d, v_d)
\end{pmatrix} \succeq 0.
\]

Since a block matrix is semidefinite if and only if the blocks are semidefinite, we conclude

**Proposition 5.3.** Let \( Q = (q_{ij}) \in S^d_{\geq 0} \) be a PQF. Let \( \mathcal{D} \) be a Delone triangulation refining \( \text{Del}(Q) \), and let \( L_1, \ldots, L_n \) be a representative system of all non-equivalent \( d \)-dimensional Delone polytopes in \( \mathcal{D} \). Then

\[
\mu(Q) \leq 1 \iff \begin{pmatrix}
\text{BR}_{L_1}(Q) & 0 & 0 & \cdots & 0 \\
0 & \text{BR}_{L_2}(Q) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \text{BR}_{L_n}(Q)
\end{pmatrix} \succeq 0.
\]
Thus $\mu(Q) \leq 1$ can be brought into one linear matrix inequality of type $F(q_{ij}) \succeq 0$. We can moreover add linear constraints on the parameters $q_{ij}$ by extending $\mathcal{F}$ by a $1 \times 1$ block matrix for each linear inequality. In this way we can get one linear matrix inequality for the two constraints $\mu(Q) \leq 1$ and $Q \in \Delta(D)$.

For a fixed Delone triangulation $\mathcal{D}$, we can therefore determine the optimal solutions of the lattice covering problem of all PQFs $Q$ for which $\mathcal{D}$ is a refinement of $\text{Del}(Q)$. Recall that the covering density of a PQF $Q$ in $d$ variables is $\Theta(Q) = \sqrt{\frac{\mu(Q)}{\det(Q)} \cdot \kappa_d}$. Scaling of $Q$ by a positive real number $\alpha$ leaves $\Theta$ invariant. Thus we may maximize $\det(Q)$ while $\mu(Q) \leq 1$. For all $Q \in \Delta(D)$ this can be achieved by solving

\[
\begin{align*}
\text{minimize} & \quad -\log \det(Q) \\
\text{subject to} & \quad Q > 0, \\
& \quad Q \in \Delta(D), \mu(Q) \leq 1.
\end{align*}
\]

With an analogue specialization of problem (1), we are able to attain optimal solutions of the lattice packing-covering problem among all PQFs $Q$ for which $\mathcal{D}$ is a refinement of $\text{Del}(Q)$. Because $\gamma(Q) = 2 \cdot \sqrt{\mu(Q)/\lambda(Q)}$, we have to maximize $\lambda(Q)$ while $\mu(Q) \leq 1$. Maximizing $\lambda(Q)$ is not as straightforward as maximizing the determinant, since we do not know which vector is the shortest. By a theorem of Voronoi [Vor08] we know though that among the (at most $2(2^d - 1)$) edges $[0, v] = \text{conv}(0, v) \in \mathcal{D}$, there exist some with $v \in \text{Min}(Q)$. Consequently, if we require $Q[v] \leq Q[w]$ for all $w$ with $[0, w] \in \mathcal{D}$, we know $v \in \text{Min}(Q)$, respectively $\lambda(Q) = Q[v]$. So we can solve for each $[0, v] \in \mathcal{D}$ the semidefinite programming problem

\[
\begin{align*}
\text{minimize} & \quad -Q[v] \\
\text{subject to} & \quad Q[v] \leq Q[w], \text{ for all } w \text{ with } [0, w] \in \mathcal{D}, \\
& \quad Q \in \Delta(D), \mu(Q) \leq 1.
\end{align*}
\]

Note that in many cases the constraints have no feasible solutions, since in general not all of the $v$ with $[0, v] \in \mathcal{D}$ are elements of $\text{Min}(Q)$.

So both, the lattice covering as well as the lattice packing-covering problem, are reduced to convex programming problems if restricted to the closure of a secondary cone. Consequently, there is at most one local minimum of $\Theta$, respectively $\gamma$, for each of the cones $\Delta(D)$. This was first observed by Barnes and Dickson [BD67] in the covering case and by Ryshkov [Rys74] for the packing-covering problem.

**Proposition 5.4.** Let $\mathcal{D}$ be a Delone triangulation. Then there exists a unique minimum of

1. $\{\Theta(Q) : Q \in \Delta(D)\}$ and the set of PQFs attaining it is equal to all positive multiples of a single PQF.
2. $\{\gamma(Q) : Q \in \Delta(D)\}$ and the set of PQFs attaining it is convex.

In case of a triangulation $\mathcal{D}$, it follows that $Q \in \Delta(D)$ is a locally optimal solution with respect to $\Theta$ or $\gamma$ if and only if it is an optimal solution within $\Delta(D)$. In general we have the trivial

**Proposition 5.5.** A PQF $Q$ is a locally optimal solution with respect to $\Theta$ or $\gamma$, if and only if it is an optimal solution for all Delone triangulations $\mathcal{D}$ with $Q \in \Delta(D)$.

5.3. **Computational Results.** By the foregoing propositions we know that the number of local optima is bounded from above by the number of pairwise inequivalent Delone triangulations in $\mathbb{R}^d$. Voronoi [Vor08] classified these triangulation in dimension 2, 3 (only one each) and 4 (three). By the work of Baranovskii and Ryshkov [BR73], [RB78] Engel [Eng98], and Engel and Grishukhin [EG02] we know of exactly 222 Delone triangulations in dimension $d = 5$. Using lrs [Avi04] and an implementation (in C++) of Voronoi’s algorithm for enumerating Delone triangulations, we were able to confirm these results [SV04a]. For dimension $d \geq 6$ we experience a combinatorial explosion, e.g. Engel...
ENG04 reports on more than 2,129,120 pairwise inequivalent Delone triangulations for $d = 6$.

| $d$ | $(d+1)/2$ | # covering optima | # packing-covering optima |
|-----|-----------|-----------------|-----------------------------|
| 2   | 3         | $1^a$           | $1^c$                       |
| 3   | 6         | $1^c$           | $1^b$                       |
| 4   | 10        | $3^d$           | $3^c$                       |
| 5   | 15        | 222$f$          | 47$f$                       |

$^a$Kershner [Ker39]  
$^b$Ryshkov [Rys74]  
$^c$Bambah [Bam54]  
$^d$Baranovski, Dickson [Dic67]  
$^e$Horváth [Hor82]  
$^f$Schürmann, Vallentin [SV04a]

Table 7. Classifications of locally optimal covering and packing-covering lattices.

For each of the triangulations in dimension $d \leq 5$ we determined the local optima with respect to $\Theta$ and $\gamma$ using the software package MAXDET of Wu, Vandenberghe, and Boyd as a subroutine. By this we confirmed the known results for dimensions $d = 2, 3, 4$ and extended them to $d = 5$ (see Table 7). Note that for the lattice covering problem there exists a local optimum for each triangulation, while this is not the case in higher dimensions (see [SV04b]) and for the lattice packing-covering problem.

Using our implementation we also found two lattices which currently give the best known covering and packing-covering in dimension 6:

**Theorem 5.6 ([SV04a]).** In dimension 6, there exits a lattice $L_6^c$ with $\Theta(L_6^c) = 2.4648\ldots$ and a lattice $L_6^{pc}$ with $\gamma(L_6^{pc}) = 1.4110\ldots$.

In [SV04b] we show that the root lattice $E_8$ does not give a locally optimal lattice covering, by constructing a refining triangulation $D$ of Del$(Q_{E_8})$ in which $\Theta$’s local optimum is not attained by the PQF $Q_{E_8}$. The PQF found in this way even beats the formerly best known value $\Theta(A_8^c)$ by more than 12%. By looking at a bistellar neighbor of the triangulation $D$, we found the currently best known covering lattice in dimension 8.

**Theorem 5.7 ([SV04b]).** In dimension 8, there exists a lattice $L_8^c$ with $\Theta(L_8^c) = 3.1423\ldots$.

Looking at the results in dimension $d = 6, 8$ it is interesting to observe that we found the new covering lattices by looking at triangulations refining the Delone subdivisions of the lattices $E_d^a$. By looking at a corresponding refinement of $E_d^a$, we also found a new covering record in dimension 7. It remains to see if these results have a common explanation. . .

5.4. **Sufficient Conditions for Local Optima.** A disadvantage of finding local optima via convex programming is that solutions can only be approximated. But this is an inherent problem: In contrast to the lattice packing problem, local optima to the other two problems can in general not be represented by rational numbers. One has to use algebraic numbers instead. In some cases it might be possible, e.g. with additional information on the automorphism group, to attain exact coordinates from a first approximation (see [SV04a] for an example). In other cases we might have a conjectured optimal form $Q'$ and want to compute a “certificate” verifying its local optimality. The following two propositions give such a criterion in terms of the gradient $g_L(Q') = \text{grad} |BR_L|(Q')$ of the regular surfaces $|BR_L(Q)| = 0$ at $Q'$. Both are a consequence of the geometric fact that we

http://www.stanford.edu/~boyd/MAXDET.html
have a local optimum at $Q' \in \Delta(\mathcal{D})$ with $\mu(Q') = 1$ if and only if there exists a hyperplane through $Q'$, separating the convex sets $\{Q \in S_{\mathcal{D}}^d : L \in \mathcal{D}, |BR_L(Q)| \geq 0\}$ and $\{Q \in S_{\mathcal{D}}^d : \det(Q) \geq \det(Q')\}$, respectively $\{Q \in S_{\mathcal{D}}^d : \lambda(Q) \geq \lambda(Q')\}$.

**Proposition 5.8** (Barnes and Dickson [BD67]). Let $\mathcal{D}$ be a Delone triangulation. Then $Q \in \Delta(\mathcal{D})$ with $\mu(Q) = 1$ is a locally optimal solution to the lattice packing-covering problem if and only if

$$Q^{-1} \in -\text{cone}\{g_L(Q) : L \in \mathcal{D} \text{ with } |\text{BR}_L(Q)| = 0\}.$$

The corresponding result for the lattice packing-covering problem is

**Proposition 5.9** ([SV04a]). Let $\mathcal{D}$ be a Delone triangulation. Then $Q \in \Delta(\mathcal{D})$ with $\mu(Q) = 1$ is a locally optimal solution to the lattice packing-covering problem if and only if

$$\text{cone}\{vv^t : v \in \text{Min}(Q)\} \cap -\text{cone}\{g_L(Q) : L \in \mathcal{D} \text{ with } |\text{BR}_L(Q)| = 0\} \neq \emptyset.$$

Combining these two propositions we get

**Corollary 5.10.** Let $\mathcal{D}$ be a Delone triangulation. Then $Q \in \Delta(\mathcal{D})$ is a unique locally optimal solution to the lattice packing-covering problem, if $Q$ is eutactic and a locally optimal solution to the lattice covering problem.

**Example 5.11.** We can use Corollary 5.10 to show that $\mathcal{A}_d^*$, $d \geq 2$, with $\gamma(\mathcal{A}_d^*) = \sqrt{\frac{d+2}{2}}$ is locally optimal for the lattice packing-covering problem. Ryshkov [Rys74] gave another proof of this fact. The lattice $\mathcal{A}_d^*$ is known to give a locally optimal lattice covering (see [Gam62], [Gam63], [Ble62]). A quadratic form $Q_{\mathcal{A}_d^*}$ associated with $\mathcal{A}_d^*$ is

$$Q_{\mathcal{A}_d^*} = \begin{pmatrix} d & -1 & \cdots & -1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & d \end{pmatrix}$$

with $\mu(Q_{\mathcal{A}_d^*}) = \frac{d(d+2)}{2}$ and $\lambda(Q) = d$. The set $\text{Min}(Q_{\mathcal{A}_d^*})$ contains exactly $2(d+1)$ elements, namely the standard basis vectors $e_1, \ldots, e_d$, their negatives and $\pm \sum_{i=1}^d e_i$ (see [CS88b]). Thus in particular

$$(d+1) \cdot Q_{\mathcal{A}_d^*}^{-1} = Q_{\mathcal{A}_d^*} = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{pmatrix} = \sum_{vv^t : v \in \text{Min}(Q_{\mathcal{A}_d^*})} vv^t$$

Hence, $Q_{\mathcal{A}_d^*}$ is eutactic because $Q_{\mathcal{A}_d^*}^{-1} \in \text{relint} \{vv^t : v \in \text{Min}(Q_{\mathcal{A}_d^*})\}$ and therefore the assertion follows.

Propositions 5.8 and 5.9 assume that $\mathcal{D} = \text{Del}(Q)$ is a Delone triangulation. If this is not the case, the situation becomes more complicated, in particular for the lattice packing-covering problem.

For the covering problem we only have to add a condition on the set

$$\mathcal{V}_\mathcal{D} = \bigcup_{\mathcal{D}' \subset \mathcal{D}} \{Q \in \overline{\Delta(\mathcal{D}')} : |\text{BR}_L(Q)| \geq 0 \text{ for all } L \in \mathcal{D}'\}$$

where $\mathcal{D}' < \mathcal{D}$ denotes that $\mathcal{D}'$ is a Delone triangulations refining $\mathcal{D}$. This set is a subset of $\{Q \in S_{\mathcal{D}}^d : \mu(Q) \leq 1\}$. We require that $\mathcal{V}_\mathcal{D}$ is separable at $Q$, that is, there exists a supporting hyperplane of $\mathcal{V}_\mathcal{D}$ through $Q$. This is in particular the case, if there exists a small $r > 0$ such that $(Q + rB^{(d+1)}) \cap \mathcal{V}_\mathcal{D}$ is convex.

**Proposition 5.12.** Let $\mathcal{D}$ be a Delone subdivision and $Q \in \Delta(\mathcal{D})$ with $\mu(Q) = 1$. Then
(1) $Q$ is a locally optimal solution to the lattice covering problem, if and only if $\mathcal{V}_T$ is separable at $Q$ and
\[ Q^{-1} \in -\text{cone}\{g_L(Q) : L \in \mathcal{D}' < \mathcal{D} \text{ with } |BR_L(Q)| = 0\}. \]

(2) $Q$ is a locally optimal solution to the lattice packing-covering problem, if $\mathcal{V}_T$ is separable at $Q$ and
\[ \text{cone}\{v\nu^t : v \in \text{Min}(Q)\} \cap -\text{cone}\{g_L(Q) : L \in \mathcal{D}' < \mathcal{D} \text{ with } |BR_L(Q)| = 0\} \neq \emptyset. \]

In case of the lattice packing-covering problem the “only if” part is missing, because we cannot exclude the case of a locally optimal solution $Q'$ with $\mathcal{V}_T$ not being separable at $Q'$. This is due to the fact that $\{Q \in S^{d}_{>0} : \lambda(Q) \geq \lambda(Q')\}$ is not smooth in contrast to $\{Q \in S^{d}_{>0} : |Q| \geq |Q'|\}$. This phenomenon seems to happen to PQFs associated to the root lattice $E_8$. Nevertheless computational experiments support the

**Conjecture 5.13.** The root lattice $E_8$ gives a locally optimal solution for the lattice packing-covering problem.

Zong [Zon02] even conjectured that $E_8$ gives the unique globally optimal solution to the lattice packing-covering problem in dimension 8.

### 5.5. Local Optima via Local Lower Bounds

In [SV04a] we describe a way to attain local lower bounds for the covering density and the packing-covering constant due to Ryshkov and Delone. A variant of this method is successfully used in [SV04b] to prove the local covering optimality of the Leech lattice. Here we describe a corresponding local lower bound for the lattice packing-covering problem. As an example we use it to prove the local packing-covering optimality of the Leech lattice directly.

**Proposition 5.14.** Let $L_1, \ldots, L_n$ be a collection of Delone simplices of a PQF $Q$. Then
\[ \gamma(Q) \geq 2\sqrt{\frac{\text{trace}(F \cdot Q_F)}{(d+1)\lambda(Q_F)}} \]
with the PQF $F = \frac{1}{n(d+1)} \sum_{i \neq j} v_{i,k} v_{i,l}^t$ and a PQF $Q_F$ with $F \in \text{cone}\{v\nu^t : v \in \text{Min}(Q_F)\}$.

One can prove this Proposition by doing obvious modifications to the proof of Proposition 10.6 in [SV04a]. As in Proposition 5.9 we use the following fact: A linear function $f(Q) = \text{trace}(F \cdot Q_F)$, with a PQF $F$, has a minimum on the homogeneous minimum $\lambda$ surface $\{Q \in S^{d}_{>0} : \lambda(Q) = \lambda\}$ at $Q_F$ if and only if $F \in \text{cone}\{v\nu^t : v \in \text{Min}(Q_F)\}$. In particular, if $Q$ is eutactic and $F = Q^{-1}$, then Proposition 5.14 is immediately applicable with $Q_F = Q$.

**Example 5.15.** We use Proposition 5.14 to show that the Leech lattice is a locally optimal packing-covering lattice.

Let us briefly review some necessary properties of the Leech lattice $\Lambda$. For further reading we refer to [CS88b]. An associated PQF $Q_\Lambda$ has (up to congruences) 23 different Delone polytopes attaining the maximum squared circumradius $\mu(Q_\Lambda) = 2$. One of them is the Delone simplex $L$ of type $A_24$.

Now we apply Proposition 5.14 to the orbit of $L$ under the automorphism group $\text{Co}_0 = \{T \in \text{GL}_{24}(\mathbb{Z}) : T\Lambda T = \Lambda\}$ of $Q_\Lambda$. We get $F = \frac{1}{24} \sum_{T \in \text{Co}_0} \sum_{e} ee^t$, where $e$ runs through all the edge vectors of $TL$. In [SV04b] it was shown that $F = \frac{s^2}{2^2 \cdot 3} Q_\Lambda^{-1}$. Due
to the fact that $\text{Min} \ Q_\Lambda$ is a spherical 2-design with respect to the inner product given by $Q_\Lambda$, we know (see [SV04b] for details)

$$\sum_{v \in \text{Min}(Q_\Lambda)} vv^t = \frac{|\text{Min}(Q_\Lambda)|}{d} Q_\Lambda^{-1}.$$ 

Thus, $Q_\Lambda$ is eutactic and we may use $Q_F = Q_\Lambda$ in Proposition 5.14. With $\lambda(Q_\Lambda) = 4$ we derive

$$\gamma(Q) \geq 2 \sqrt{\frac{2^2 \cdot 24}{25 \cdot 4}} = \sqrt{2} = \gamma(Q_\Lambda)$$

for all PQFs $Q$ with Delone simplices $TL$, $T \in C_{00}$, which proofs the assertion.

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