Global Existence of Strong and Weak Solutions to 2D Compressible Navier–Stokes System in Bounded Domains with Large Data and Vacuum

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Abstract

The barotropic compressible Navier–Stokes system subject to the Navier-slip boundary conditions in a general two-dimensional bounded simply connected domain is considered. For initial density allowed to vanish, the global existence of strong and weak solutions is established when the shear viscosity is a positive constant and the bulk one proportional to a power of the density with the power bigger than one and a third. It should be mentioned that this result is obtained without any restrictions on the size of initial value. To get over the difficulties brought by boundary, on the one hand, Riemann mapping theorem and the pull-back Green’s function method are applied to get a pointwise representation of the effective viscous flux. On the other hand, since the orthogonality is preserved under conformal mapping due to its preservation on the angle, the slip boundary conditions are used to reduce the integral representation to the desired commutator form whose singularities can be cancelled out by using the estimates on the spatial gradient of the velocity.

1. Introduction and Main Results

We study the barotropic compressible Navier–Stokes system in a two-dimensional (2D) domain $\Omega$:

$$
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P = \mu \Delta u + \nabla((\mu + \lambda)\text{div} u).
\end{cases}
$$

(1.1)

Here $t \geq 0$ is time, $x \in \Omega \subset \mathbb{R}^2$ is the spatial coordinate, $\rho = \rho(x, t)$ and $u = (u_1(x, t), u_2(x, t))$ represent the unknown density and velocity respectively, and the pressure $P$ is given by

$$
P = a\rho^\gamma,
$$

(1.2)
with constants $a > 0$, $\gamma > 1$. We also have the following hypothesis on the shear viscosity coefficient $\mu$ and the bulk one $\lambda$:

$$0 < \mu = \text{constant}, \quad \lambda(\rho) = b \rho^\beta,$$

with positive constants $b$ and $\beta$. We set $a = b = 1$ without loss of generality.

In this paper, we assume that $\Omega$ is a simply connected bounded $C^{2,1}$-domain in $\mathbb{R}^2$. In addition, the system is subject to the given initial data

$$\rho(x, 0) = \rho_0(x), \quad \rho u(x, 0) = m_0(x), \quad x \in \Omega,$$

and Navier slip boundary conditions

$$u \cdot n = 0 \quad \text{and} \quad \text{curl } u = -A u \cdot n^\perp \text{ on } \partial \Omega,$$

where $A$ is a non-negative smooth function on the boundary, $n = (n_1, n_2)$ denotes the unit outer normal vector of the boundary $\partial \Omega$, and $n^\perp$ is the unit tangential vector on the boundary $\partial \Omega$ denoted by

$$n^\perp \triangleq (n_2, -n_1).$$

We may extend $n$, $n^\perp$ and $A$ to the whole region $\overline{\Omega}$ smoothly. The approach of extension is not unique, and we fix one through out the paper.

There is a large amount of literature about the strong solvability for multidimensional compressible Navier–Stokes system with constant viscosity coefficients. The history of the area may trace back to Nash [1] and Serrin [2], who established the local existence and uniqueness of classical solutions respectively for the density away from vacuum. The first result of global classical solutions was due to Matsumura-Nishida [3] for initial data close to a non-vacuum equilibrium in $H^3$. Hoff [4,5] then studied the problem with discontinuous initial data, and introduced a new type of a priori estimates on the material derivative $\dot{u}$. The major breakthrough in the frame of weak solutions was due to Lions [6], where, under the assumption that the energy is finite initially, he obtained the global existence of weak solutions provided the exponent $\gamma \geq \frac{9}{5}$ which was further released to $\gamma > \frac{3}{2}$ by Feireisl et al. [7] for three-dimensional space. Recently, Huang-Li-Xin [8] and Li-Xin [9] established the global existence and uniqueness of classical solutions, merely assumed the initial energy is small enough, where large oscillations were available. The initial density is allowed to vanish and even has compact support. More recently, for the Navier-slip boundary conditions in general bounded domains, Cai-Li [10] obtain the global existence and exponential growth of classical solutions with vacuum provided that the initial energy is suitably small.

In contrast, positive results without limitation on the size of initial value are rather fewer. Vaigant-Kazhikov [11] pioneered in large initial value theory. They obtained a unique global strong solution under the restriction $\beta > 3$ in rectangle domains. Recently, Huang-Li [12,13] applied some new ideas based on commutator theory and blow up criterion, and improved the conclusion in periodic case, and even for the Cauchy problem in the whole space (see [14] also), demanding only $\beta > \frac{4}{3}$, which, as indicated by [15], seems to be the optimal result one may expect up to now even for the periodic case. However, for general domains, the theory of large
initial data is still blank, and boundary terms do bring some essential difficulties. Therefore the aim of the paper is to study the global existence of strong and weak solutions with large initial data in general simply connected domains.

Before stating the main results, we explain the notations and conventions used throughout this paper. For a positive integer $k$ and $1 \leq p < \infty$, the standard $L^p$ spaces and Sobolev ones are denoted as follows:

\[
\begin{align*}
L^p &= L^p(\Omega), \ W^{k,p} = W^{k,p}(\Omega), \ H^k = W^{k,2}(\Omega), \\
\|f\|_{L^p} &= \|f\|_{L^p(\Omega)}, \ \|f\|_{W^{k,p}} = \|f\|_{W^{k,p}(\Omega)}, \ \|f\|_{H^k} = \|f\|_{W^{k,2}(\Omega)}, \\
\tilde{H}^1 &= \{u \in H^1(\Omega) | u \cdot n = 0, \text{curl}u = -Au \cdot n^\perp \text{on} \partial\Omega\}.
\end{align*}
\]

The material derivative and the transpose gradient are given by

\[
\frac{D}{Dt} f = \dot{f} \triangleq \frac{\partial}{\partial t} f + u \cdot \nabla f, \quad \nabla^\perp \triangleq (\partial_2, -\partial_1). \tag{1.7}
\]

First we define weak and strong solution as follows:

**Definition 1.1.** Let $T > 0$ be a finite constant. A solution $(\rho, u)$ to (1.1) is called a weak solution if they satisfy (1.1) in the sense of distribution. Moreover, when all the derivatives involved in (1.1) are regular distributions, and (1.1) holds almost everywhere in $\Omega \times (0, T)$, we call the solution a strong one.

We state the main result concerning the global existence of strong solutions as follows:

**Theorem 1.1.** Let $\Omega$ be a simply connected bounded domain in $\mathbb{R}^2$ with $C^{2,1}$ boundary $\partial\Omega$. Assume that

\[
\beta > \frac{4}{3}, \quad \gamma > 1, \tag{1.8}
\]

and that the initial data $(\rho_0 \geq 0, m_0)$ satisfy, for some $q > 2$,

\[
\rho_0 \in W^{1,q}, \quad u_0 \in \tilde{H}^1, \quad m_0 = \rho_0 u_0. \tag{1.9}
\]

Then the problem (1.1)–(1.5) has a unique strong solution $(\rho, u)$ in $\Omega \times (0, \infty)$ satisfying for any $0 < T < \infty$,

\[
\begin{align*}
\rho &\in C([0, T]; W^{1,q}), \ \rho_t \in L^\infty(0, T; L^2), \\
u &\in L^\infty(0, T; H^1) \cap L^{1+1/q}(0, T; W^{2,q}), \\
\sqrt{t}u &\in L^\infty(0, T; H^2) \cap L^2(0, T; W^{2,q}), \ \sqrt{t}u_t \in L^2(0, T; H^1), \\
\rho u &\in C([0, T]; L^2), \ \sqrt{\rho}u_t \in L^2((0, T) \times \Omega). \tag{1.10}
\end{align*}
\]

The second result concerns the global existence of weak solutions.
Theorem 1.2. Under the conditions of Theorem 1.1 except for $\rho_0 \in W^{1,q}$ in (1.9) being replaced by $\rho_0 \in L^\infty$. Then, there exists at least one weak solution $(\rho, u)$ of the problem (1.1)–(1.5) in $\Omega \times (0, \infty)$ satisfying, for any $0 < \tau \leq T < \infty$ and $p \geq 1$, that

$$
\begin{align*}
\rho &\in L^\infty(0, T; L^\infty) \cap C([0, T]; L^p), \\
u &\in L^\infty(0, T; H^1), \ u_t &\in L^2(\tau, T; L^2), \ \nabla u &\in L^\infty(\tau, T; L^p), \\
\text{curl} u, (2\mu + \lambda)\text{div} u - P &\in L^2(0, T; H^1) \cap L^{3/2}(0, T; L^\infty),
\end{align*}
(1.11)

Moreover, there exists a positive constant $C$ depending on $\Omega$, $T$, $\mu$, $\beta$, $\gamma$, $\|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that

$$
\inf_{(x, t) \in \Omega \times (0, T)} \rho(x, t) \geq C^{-1} \inf_{x \in \Omega} \rho_0(x).
(1.12)
$$

A few more remarks are in order.

Remark 1.1. For bounded domains, the usual Navier-type slip condition can be stated as

$$
u \cdot n = 0, \quad (2D(u) n + \vartheta u)_{\text{tan}} = 0 \quad \text{on} \quad \partial \Omega,
(1.13)$$

where $D(u) = (\nabla u + (\nabla u)^{\text{tr}})/2$ is the shear stress, $\vartheta$ is a scalar friction function which measures the tendency of the fluid to slip on the boundary, and the symbol $v_{\text{tan}}$ represents the projection of tangent plane of the vector $v$ on $\partial \Omega$. As indicated by [10, Remark 1.1], the Navier-type slip condition (1.13) is in fact a particular case of the slip boundary one (1.5).

Remark 1.2. Compared with the previous known results on the global existence of strong solutions with large data [11–14,16], our Theorem 1.1 seems to be the first concerning the global existence of strong solutions to the compressible Navier–Stokes system in general two-dimensional bounded domains with large data.

Remark 1.3. It should be mentioned here that (1.12) implies that the weak solutions obtained by Theorem 1.2 will not exhibit vacuum in any finite time provided that no vacuum is present initially. However, for Lions-Feireisl’s weak solutions [6,7], whether this phenomenon still holds or not remains open.

Remark 1.4. As indicated by [15], $\beta > \frac{4}{3}$ seems to be the optimal result one may expect up to now even for the periodic case. However, it seems that $\beta > 1$ is the extremal case for the system (1.1)–(1.3) (see [11] or Lemma 3.2 below). Therefore, it would be interesting to study the case of $1 < \beta \leq 4/3$ which is left for the future.

Remark 1.5. It is worth noting that our method strongly uses the simple connectedness of the domain which lets the question of more general domains remain open. This will be left for the future.
We now make some comments about the analysis of the whole paper. Similar to [12–14], the key issue of the existence of global solutions is to derive the upper bound of density \( \rho \). One possible way is to rewrite the conservation of mass formally as
\[
\frac{D}{Dt} (\theta(\rho) + \Delta^{-1} \text{div}(\rho u)) + P = u \cdot \nabla \Delta^{-1} \text{div}(\rho u) - \Delta^{-1} \text{div} \text{div}(\rho u \otimes u),
\]
(1.14)
where
\[
\theta(\rho) \triangleq 2\mu \log \rho + \beta^{-1}\rho^\beta,
\]
(1.15)
and \( \Delta^{-1} \) is taken in the proper sense. A remarkable fact is that the combination of the two terms on the righthand side of (1.14) forms a Calderon-type commutator in some cases, which indeed improves the integrability. For example, the theory applies perfectly in the case of whole space \( \mathbb{R}^2 \) ([13]) and that of periodic boundary \( \mathbb{T}^2 \) ([12]) due to the fact that one can interchange the \( \Delta^{-1} \) and \( \nabla \) at both cases, but for general bounded domains, the interchangeability fails, and hence, the classical commutator theory is no longer available, which is indeed a major difficulty in our situation.

To overcome it, we first take a look at the mechanism of commutator for the case \( \mathbb{R}^2 \). Rewriting commutator in singular integral form (see also Remark 3.2) as
\[
[u_k, R_i R_j](\rho u_l)(x) = \text{p.v.} \int_{\mathbb{R}^2} K_{ij}(x - y) (u_k(x) - u_k(y)) \rho u_l(y) \, dy,
\]
for some kernel \( K_{ij}(x)(i, j = 1, 2) \) which has a singularity of the second order at 0 and satisfies
\[
|K_{ij}(x)| \leq C|x|^{-2}, \quad x \in \mathbb{R}^2 \setminus \{0\},
\]
we observe that the key fact about the representation is that the singularity can be actually canceled out once \( u \) lies in any Hölder space, which can be guaranteed by any control upon \( \|\nabla u\|_{L^p} \) with \( p > 2 \). Accordingly, a term of commutator type which can be written in the above integral form plays an important role in carrying out further computations.

Thus, motivated by the above analysis, we find an alternative approach to estimate the upper bound of the density. Note that, equivalently, we have, for \( \theta(\rho) \) as in (1.15),
\[
\frac{D}{Dt} \theta(\rho) + P = -F,
\]
where \( F = (2\mu + \lambda)\text{div} u - P \) is the so-called effective viscous flux. This implies that in order to estimate \( \rho \), we need to establish a proper bound on \( F \) which indeed solves a Neumann problem as follows:
\[
\begin{aligned}
\Delta F &= \text{div}(\rho \dot{u}) \quad &\text{in } \Omega, \\
\frac{\partial F}{\partial n} &= \rho \dot{u} \cdot n - \mu n^\perp \cdot \nabla (Au \cdot n^\perp) \quad &\text{on } \partial \Omega.
\end{aligned}
\]
(1.16)
Generally, we can get a pointwise representation of $F$ via applying Green’s function. We first consider the simplest case that $\Omega_1$ is the unit disc $D$, where Green’s function takes the form (see [17])

$$N(x, y) = -\frac{1}{2\pi} \log |x - y| + \log \left| \frac{x}{|x|} \right|.$$

(1.17)

After applying Green’s identity, we derive the integral representation of $F$, but the singularity is still out of control especially for the integral near the boundary. Fortunately, we observe that for each $x \in \partial D$, $x$ is just the outer normal of $\partial D$, and slip boundary yields $u(x) \cdot x = 0$ on $\partial D$ (see also (2.12)). Such cancellation condition helps us reduce the integral representation to the desired commutator form, thus we may apply the previous idea.

For general domains, unfortunately, the precise formula of Green’s function is unknown. So we need further arguments. The cancelation of singularity on the boundary is indeed a type of geometry restriction, and we do not know whether it remains true in other circumstance or not. However, we observe that the orthogonality which is crucial in our computation is preserved under conformal mapping, since it preserves the angle. Moreover, Riemann mapping theorem ([18, Chapter 9]) makes sure every simply connected domain is conformally equivalent to the unit disc. Consequently, it is reasonable to expect that we can reduce the general case to that of unit disc via making use of conformal mapping. The goal is achieved by pull-back Green’s function (see (3.33)). Finally, after some careful calculations, we obtain the desired upper bound upon $\rho$.

Another major technical difficulty arises from lower order estimates, especially, the trace of $\nabla u$ on the boundary $\partial \Omega$, which seems ought to be controlled by the $L^p$-norm of $\nabla^2 u$ in general case. Unfortunately, we have no a priori estimate on $\nabla^2 u$ at this stage. To overcome it, we mainly adapt the ideas due to Cai-Li [10]. On the one hand, observe that the slip boundary condition $u \cdot n|_{\partial \Omega} = 0$ admits

$$u = (u \cdot n^\perp) n^\perp, \quad (u \cdot \nabla) u \cdot n = -(u \cdot \nabla) n \cdot u,$$

(1.18)

with $n^\perp$ as in (1.6). On the other hand, as observed by Cai-Li [10], it holds that for smooth function $f$

$$\left| \int_{\partial \Omega} u \cdot \nabla f \, ds \right| = \left| \int_{\partial \Omega} (u \cdot n^\perp) n^\perp \cdot \nabla f \, ds \right|$$

$$= \left| \int_{\Omega} \nabla f \cdot \nabla^\perp (u \cdot n^\perp) \, dx \right| \leq C \| f \|_{H^1} \| u \|_{H^1},$$

where the operator $\nabla^\perp = (\partial_2, -\partial_1)$ is as in (1.7). With these key observations at hand, after some delicate calculations, all the boundary term including $\nabla u$ caused by integration by parts can be handled. See Lemma 4.2 for details. Finally, after all obstacles are eliminated, the conclusion follows in a rather routine way.

**Remark 1.6.** In order to derive the proper representation on $F$, we spend much effort on studying the property (see Lemma 3.6) of the pull-back Green’s function.
\(\tilde{N}\) (3.33) and computing corresponding representation (3.46), instead of using respectively the common Green's function \(G\) (see [19, 20]) whose precise formula is still unknown for general bounded smooth domains. We comment that such procedure is necessary because the usual Green's function can not meet our requirements. Indeed, for the usual Green's function \(G\), the known estimates we can use are the order of singularity (see [19, 20]):
\[
|\partial^\alpha x \partial^\beta y G(x, y)| \leq C|x - y|^{-\alpha - \beta},
\]
with \(\alpha, \beta\) nonnegative integers such that \(\alpha + \beta > 0\). This information on singularity only guarantees (3.43) dominated by
\[
\int_{\Omega} \frac{|u|}{|x - y|^2} \rho |u|(y) \, dy,
\]
whose singularity is indeed out of control in dimension 2. But after using \(\tilde{N}\), we change the key integral representation of the effective viscous flux \(F\) into (3.46) which together with the slip boundary (2.12) further expresses the form of above term (1.19) explicitly as
\[
\int_{\Omega} \frac{|u(x) - u(y)|}{|x - y|^2} \rho |u|(y) \, dy.
\]
Such a term can improve the integrability just like the common Calderon-type commutator (see Proposition 3.3) and meets our request.

The rest of paper is organized as follows: in Section 2, we state some elementary preliminaries which will be used later. Sections 3 and 4 are devoted to the lower order and higher one a priori estimates which will be carried out in details. Finally, with all necessary estimates in hand, we will prove the main results, Theorems 1.1 and 1.2, in Section 5 shortly in a standard way.

2. Preliminaries

First of all, we quote the well-known local existence theory in [21, 22], where the initial density is strictly away from vacuum. Actually it is the cornerstone of our proof on global situation.

**Lemma 2.1.** Assume that \((\rho_0, m_0)\) satisfies
\[
\rho_0 \in H^2, \quad \inf_{x \in \Omega} \rho_0(x) > 0, \quad u_0 \in H^2 \cap \tilde{H}^1, \quad m_0 = \rho_0 u_0.
\]  

Then there is a small time \(T > 0\) and a constant \(C_0 > 0\) both depending only on \(\Omega, \mu, \beta, \gamma, \|\rho_0\|_{H^2}, \|u_0\|_{H^2}\), and \(\inf_{x \in \Omega} \rho_0(x)\) such that there exists a unique strong solution \((\rho, u)\) to the problem (1.1)–(1.5) in \(\Omega \times (0, T)\) satisfying

\[
\begin{aligned}
\rho &\in C([0, T]; H^2), \quad \rho_t \in C([0, T]; H^1), \\
u &\in L^2(0, T; H^3), \quad u_t \in L^2(0, T; H^2) \cap H^1(0, T; L^2),
\end{aligned}
\]  

\[\text{(2.2)}\]
and
\[ \inf_{(x,t)\in\Omega\times(0,T)} \rho(x,t) \geq C_0 > 0. \] (2.3)

Next, to deduce the proper estimates upon \( \nabla u \), we need the following div-curl control which can be found in [23–25]:

**Lemma 2.2.** Let \( 1 < p < \infty \) and \( \Omega \) be a simply connected bounded domain in \( \mathbb{R}^2 \) with Lipschitz boundary \( \partial \Omega \). Then there exists a positive constant \( C \) depending only on \( \Omega \) and \( p \) such that every \( v \in W^{1,p}(\Omega) \) with \( v \cdot n = 0 \) on \( \partial \Omega \) satisfies
\[ \| \nabla v \|_{L^p} \leq C(\| \text{div} \: v \|_{L^p} + \| \text{curl} \: v \|_{L^p}). \] (2.4)

Moreover, for technical reasons, the modified version of classical Poincaré–Sobolev inequality will be used frequently.

**Lemma 2.3.** ([26]) There exists a positive constant \( C \) depending only on \( \Omega \) such that every function \( u \in H^1(\Omega) \) satisfies for \( 2 < p < \infty \),
\[ \| u \|_{L^p} \leq C p^{1/2} \| u \|_{L^2}^{2/p} \| u \|_{H^1}^{1-2/p}, \] (2.5)
where \( \| u \|_{H^1} \) can be replaced by \( \| \nabla u \|_{L^2} \) provided that
\[ u \cdot n|_{\partial \Omega} = 0 \text{ or } \int_{\Omega} u \, dx = 0. \]

Next, to obtain the estimate on the \( L^\infty(0, T; L^p) \)-norm of \( \nabla \rho \), we need the following Beale-Kato-Majda type inequality which was first proved in [27] when \( \text{div} \: u = 0 \).

**Lemma 2.4.** ([10, 27]) For \( 2 < q < \infty \), there is a constant \( C \) depending only on \( \Omega \) and \( q \) such that every \( u \in \{ W^{2,q}(\Omega) \} \) satisfies
\[ \| \nabla u \|_{L^\infty} \leq C(\| \text{div} \: u \|_{L^\infty} + \| \text{curl} \: u \|_{L^\infty}) \log(e + \| \nabla^2 u \|_{L^q}) + C \| \nabla u \|_{L^2} + C. \] (2.6)

Next, let \( \Omega \) be as in Theorem 1.1 and \( \mathbb{D} \) the unit disc. Then, by Riemann mapping theorem ([18, Chapter 9]), there exists a conformal mapping \( \varphi = (\varphi_1, \varphi_2) : \tilde{\Omega} \to \mathbb{D} \) with \( \varphi_1 \) and \( \varphi_2 \) satisfying the following Cauchy–Riemann equations:
\[ \begin{cases} \partial_1 \varphi_1 = \partial_2 \varphi_2, \\ \partial_2 \varphi_1 = -\partial_1 \varphi_2. \end{cases} \] (2.7)

Moreover, the conformal mapping \( \varphi(x) \) shares the following important properties:

**Lemma 2.5.** ([18, 28, 29]) The conformal mapping \( \varphi(x) : \tilde{\Omega} \to \mathbb{D} \) is smooth and satisfies
(i) \( \varphi(x) \) is a one to one holomorphic mapping from \( \Omega \) to \( \mathbb{D} \) and maps the boundary \( \partial \Omega \) onto the boundary \( \partial \mathbb{D} \).
(ii) For any integer \( k = 0, 1, 2 \), there exists some constant \( C \) depending only on \( \Omega \) and \( k \) such that
\[
|\nabla^k \varphi(x)| \leq C, \quad \forall x \in \tilde{\Omega}.
\]  
(2.8)

Consequently,
\[
|\nabla^k \varphi(x) - \nabla^k \varphi(y)| \leq C|x - y|, \quad \forall x, y \in \tilde{\Omega}.
\]  
(2.9)

(iii) There exist two positive constants \( c_1, c_2 \) such that
\[
c_1|x - y| \leq |\varphi(x) - \varphi(y)| \leq c_2|x - y|.
\]  
(2.10)

(iv) Angle is preserved by \( \varphi(x) \), that is, for any two smooth curves \( \gamma_1(t), \gamma_2(t) : (0, 1) \to \tilde{\Omega} \),
\[
\left\langle \frac{d}{dt} \gamma_1(t), \frac{d}{dt} \gamma_2(t) \right\rangle \bigg|_{t=0} = \langle n, v \rangle = 0.
\]

(v) For any harmonic function \( h(y) \) in \( \mathbb{D} \), \( h(\varphi(x)) \) is still harmonic in \( \Omega \), that is,
\[
\Delta h(\varphi(x)) = 0 \text{ in } \Omega,
\]
provided
\[
\Delta h(y) = 0 \text{ in } \mathbb{D}.
\]

Finally, since conformal mapping preserves angles, one immediately has the following conclusion:

**Lemma 2.6.** Let \( n \) be the unit outer normal vector of \( \partial \Omega \) at \( x_0 \). Then
\[
n \cdot \nabla \varphi(x_0) = |\nabla \varphi_1(x_0)| \tilde{n},
\]  
(2.11)

where \( \tilde{n} \) is the unit outer normal vector of \( \partial \mathbb{D} \) at \( \varphi(x_0) \).

**Proof.** Let \( \gamma(s) : [-1, 0] \to \tilde{\Omega} \) be a smooth curve satisfying
\[
\gamma(s) \in \Omega \text{ for any } s \in [-1, 0], \quad \text{and } \gamma(0) = x_0, \quad \frac{d}{ds} \gamma(s)|_{s=0} = n.
\]

Suppose that \( \gamma_0(s) : (-1, 1) \to \partial \Omega \) is a curve which lies entirely on the boundary \( \partial \Omega \) satisfying
\[
\gamma_0(0) = x_0, \quad \frac{d}{ds} \gamma_0(s)|_{s=0} = v.
\]

Then \( v \) is the tangent vector of \( \partial \Omega \) at \( x_0 \), and
\[
\left\langle \frac{d}{ds} \gamma(s), \frac{d}{ds} \gamma_0(s) \right\rangle|_{s=0} = \langle n, v \rangle = 0.
\]
By Lemma 2.5, we have
\[
\left| \frac{d}{ds} \varphi(y(s)) \right|_{s=0} = \left| \frac{d}{ds} \varphi_0(s) \right|_{s=0} = 0,
\]
which implies that \( \frac{d}{ds} \varphi(y(s)) \) is the normal vector of \( \partial \mathbb{D} \) at \( \varphi(x_0) \) since \( \frac{d}{ds} \varphi(y_0(s)) \) is just the tangent vector of \( \partial \mathbb{D} \) at \( \varphi(x_0) \).

We check that
\[
\frac{d}{ds} \varphi(y(s))|_{s=0} = \left( \frac{d}{ds} \varphi(s)|_{s=0} \cdot \nabla \right) \varphi(x_0) = n \cdot \nabla \varphi(x_0),
\]
and by the Cauchy–Riemann equations (2.7),
\[
|n \cdot \nabla \varphi| = (n \cdot \nabla \varphi_1)^2 + (n \cdot \nabla \varphi_2)^2 = (n_1 \cdot \partial_1 \varphi_1)^2 + 2n_1n_2 \cdot \partial_1 \varphi_1 \partial_2 \varphi_1 + (n_2 \cdot \partial_2 \varphi_1)^2 + (n_1 \cdot \partial_1 \varphi_2)^2 + 2n_1n_2 \cdot \partial_1 \varphi_2 \partial_2 \varphi_2 + (n_2 \cdot \partial_2 \varphi_2)^2 = |\partial_1 \varphi_1|^2 + |\partial_2 \varphi_1|^2 = |\nabla \varphi_1|^2 = |\nabla \varphi_2|^2.
\]
As a result,
\[
n \cdot \nabla \varphi(x_0) = |\nabla \varphi_1(x_0)| \tilde{n},
\]
where \( \tilde{n} \) is the unit outer normal of \( \mathbb{D} \) at \( \varphi(x_0) \).

**Remark 2.1.** With the boundary condition \( u \cdot n = 0 \) on \( \partial \Omega \), we can check that for any \( y_0 \in \partial \Omega, \varphi(y_0) \) and \( (u \cdot \nabla)\varphi(y_0) \) are the outer normal and tangent vector at \( \varphi(y_0) \) on \( \partial \mathbb{D} \) respectively. Consequently,
\[
u_i(y_0) \cdot \partial_i \varphi_j(y_0) \cdot \varphi_j(y_0) = \langle (u \cdot \nabla)\varphi(y_0), \varphi(y_0) \rangle = 0,
\]
which is an important observation for further work.

### 3. A Priori Estimates (I): Upper Bound of the Density

In this section and the next, we always assume that \( (\rho, u) \) is the strong solution to (1.1)–(1.5) on \( \Omega \times (0, T) \) whose existence is guaranteed by Lemma 2.1.

We define the effective viscous flux \( F \) and the vorticity \( \omega \) in the usual manner:
\[
F \triangleq (2\mu + \lambda(\rho)) \text{div} \, u - P, \quad \omega \triangleq \nabla \times u = \partial_2 u_1 - \partial_1 u_2.
\]

Then, we set
\[
A_1^2(t) \triangleq 1 + \int_{\Omega} \left( \omega^2(t) + \frac{F^2(t)}{2\mu + \lambda(\rho(t))} \right) \, dx + \int_{\partial \Omega} A|u|^2 \, ds,
\]
\[
A_2^2(t) \triangleq \int_{\Omega} \rho(t)|\dot{u}(t)|^2 \, dx,
\]
and
\[
R_T \triangleq 1 + \sup_{0 \leq t \leq T} \| \rho(t) \|_{L^\infty}.
\]

We now state the standard energy estimate.
Lemma 3.1. There exists a positive constant $C$ depending only on $\gamma$, $\|\rho_0\|_{L^\gamma}$, and $\|\sqrt{\rho_0}u_0\|_{L^2}$ such that

$$\sup_{0 \leq t \leq T} \int_{\Omega} (\rho |u|^2 + \rho^\gamma) \, dx + \int_0^T \int_{\Omega} \left( (2\mu + \lambda(\rho))(\text{div} \, u)^2 + \mu \omega^2 \right) \, dx \, dt \leq C.$$  

(3.5)

Proof. It is easy to check that

$$P_t + \text{div}(Pu) + (\gamma - 1) P \text{div} u = 0,$$  

(3.6)

due to (1.1)$_1$. Then, integrating (3.6) over $\Omega$ and using slip boundary condition (1.5) yields

$$\frac{d}{dt} \int_{\Omega} \frac{\rho^\gamma}{\gamma - 1} \, dx + \int_{\Omega} P \text{div} u \, dx = 0.$$  

Since $\Delta u = \nabla \text{div} u + \nabla \|_\perp \omega$, we rewrite the equation of conservation of momentum (1.1)$_2$ as

$$\rho \ddot{u} + \nabla P = \nabla((2\mu + \lambda(\rho))\text{div} \, u) + \mu \nabla \|_\perp \omega.$$  

(3.7)

Multiplying (3.7) by $u$ and integrating over $\Omega$, one gets by the boundary condition (1.5)

$$\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) \, dx + \int_{\Omega} \left( (2\mu + \lambda(\rho))(\text{div} \, u)^2 + \mu \omega^2 \right) \, dx$$

$$+ \mu \int_{\partial \Omega} A |u|^2 \, ds = 0.$$  

Integrating this over $(0, T)$ yields (3.5). \hfill \Box

Next, we state a known result concerning the estimate on the $L^\infty(0, T; L^p(\Omega))$-norm of the density whose proof is similar to that of [11, (36)].

Lemma 3.2. ([11]) Let $\beta > 1$. Then, for any $1 \leq p < \infty$, there is a constant $C$ depending on $\Omega$, $T$, $\mu$, $\beta$, $\gamma$, $\|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that

$$\|\rho\|_{L^p} \leq C(T)p^{\frac{\beta}{p-1}},$$  

(3.8)

where and in what follows, we use the convention that $C$ denotes a generic positive constant depending on $\Omega$, $T$, $\mu$, $\beta$, $\gamma$, $\|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$, and we write $C(\alpha)$ to emphasize that $C$ depends on $\alpha$.

We also need extra integrability up on the momentum $\rho u$ where we modify the proof of [12, Lemma 3.7] slightly due to the boundary effect.
Lemma 3.3. There exists a constant $C$ depending on $\Omega$, $T$, $\mu$, $\beta$, $\gamma$, $\|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that
\[
\sup_{0 \leq t \leq T} \int_\Omega \rho |u|^{2+v} \, dx \leq C,
\] (3.9)
where
\[
v \triangleq R_T^{-\frac{\beta}{2}} v_0,
\] (3.10)
for some suitably small generic constant $v_0 \in (0, 1)$ depending only on $\mu$ and $\Omega$.

Proof. First, combining (2.4), (3.8), and Poincaré’s inequality gives
\[
\|u\|^2_{H^1} \leq C \|\nabla u\|^2_{L^2} \leq CA_1^2(t) \leq CR_T^\beta \|\nabla u\|^2_{L^2} + C,
\] (3.11)
which together with (3.5) and (3.8) implies
\[
\int_0^T \left( \|u\|^2_{H^1} + A_1^2 \right) \, dt \leq C.
\] (3.12)

Then, following the proof of [12, Lemma 3.7], multiplying (3.7) by $|u|^v u$ and integrating the resulting equality over $\Omega$ by parts, we arrive at
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |u|^{2+v} \, dx + \int_\Omega |u|^v ((2\mu + \lambda)(\nabla u)^2 + \mu \omega^2) \, dx
\] 
\[+ \mu \int_{\partial \Omega} A |u|^{2+v} \, ds \leq v \int (2\mu + \lambda) |\nabla u| |u|^v |\nabla u| \, dx + v\mu \int |u|^v |\nabla u|^2 \, dx
\] 
\[+ C \int \rho^v |\nabla u| \, dx \leq \frac{1}{2} \int |u|^v (\nabla u)^2 \, dx + \frac{v_0^2 (\mu + 1)}{2} \int |u|^v |\nabla u|^2 \, dx
\] 
\[+ \int |\nabla u|^2 \, dx + C \int \rho |u|^{2+v} \, dx + C \int \rho \frac{4\mu + 2v}{2-v} \frac{2v}{2-v} \, dx,
\]
which, together with (3.12), Lemma 3.2, and the next Lemma 3.4 yields (3.9) and finishes the proof of Lemma 3.3.

Lemma 3.4. There exist positive constants $\tilde{v}$ and $C$ both depending only on $\Omega$ such that for any $v \in (0, \tilde{v})$,
\[
\int |u|^v |\nabla u|^2 \, dx \leq C \int |u|^v (|\nabla u|^2 + \omega^2) \, dx.
\] (3.13)

Proof. First, direct computation shows
\[
\frac{1}{2} |u|^v |\nabla u|^2 \leq |\nabla (\rho u)|^2 + v^2 |u|^v |\nabla u|^2,
\]
which implies that
\[
|u|^v |\nabla u|^2 \leq 4 |\nabla (\rho u)|^2,
\] (3.14)
provided \( \nu < 1/2 \).

Next, let us calculate that
\[
(\text{div}(|u|^{\nu} u))^2 = (|u|^{\nu}\text{div} u - \nabla |u|^{\nu} \cdot u)^2 
\leq C |u|^\nu (\text{div} u)^2 + C \nu^2 |u|^{\nu} |\nabla u|^2.
\] (3.15)

Similarly, we have
\[
(\text{curl}(|u|^{\nu} u))^2 \leq C |u|^\nu (\text{curl} u)^2 + C \nu^2 |u|^{\nu} |\nabla u|^2.
\] (3.16)

Finally, since \( |u|^{\nu} u \cdot n \bigg|_{\partial \Omega} = 0 \), we apply (2.4) to deduce that
\[
\int_{\Omega} \left| \nabla (|u|^{\nu} u) \right|^2 \, dx \leq C(\Omega) \int_{\Omega} \left( \left| \text{div}(|u|^{\nu} u) \right|^2 + \left| \text{curl}(|u|^{\nu} u) \right|^2 \right) \, dx,
\]
which, together with (3.14)–(3.16), proves Lemma 3.4.

Next, the following typical estimate on the \( L^p \)-norm of \( \nabla u \) will be used frequently:

**Lemma 3.5.** For \( p > 2 \) and \( \varepsilon > 0 \), there exists some positive constant \( C(p, \varepsilon) \) depending only on \( p, \varepsilon, \) and \( \Omega \) such that
\[
\|\nabla u\|_{L^p} \leq C(p, \varepsilon) R_\Omega^{1/2} T A_2 \left( \frac{A_2^2}{A_1^2} \right)^{1/p} + C(p, \varepsilon) R_\Omega^{\varepsilon} A_1.
\] (3.17)

**Proof.** First, we rewrite the momentum equations as
\[
\rho \dot{u} = \nabla F + \mu \nabla \perp \omega,
\] (3.18)
which together with the boundary condition (1.5) yields that \( F \) solves the Neumann problem (1.16) and that \( \omega \) solves the related Dirichlet problem:
\[
\begin{cases}
\mu \Delta \omega = \nabla \perp \cdot (\rho \dot{u}) & \text{in } \Omega, \\
\omega = -Au \cdot n & \text{on } \partial \Omega.
\end{cases}
\] (3.19)

Then, standard \( L^p \) estimate of elliptic equations (see [30], [31, Lemma 4.27]) implies that for \( k \geq 0 \) and \( p \in (1, \infty) \),
\[
\|\nabla F\|_{W^{k,p}} + \|\nabla w\|_{W^{k,p}} \leq C(p, k, \Omega)(\|\rho \dot{u}\|_{W^{k,p}} + \|\nabla u\|_{W^{k,p}}).
\] (3.20)

In particular, we have
\[
\|\nabla F\|_{L^2} + \|\nabla w\|_{L^2} \leq C(\rho \|\dot{u}\|_{L^2} + \|\nabla u\|_{L^2}) \leq C R_\Omega^{1/2} A_2,
\]
which together with the Poincaré inequality and (3.8) yields
\[
\|F\|_{H^1} + \|\omega\|_{H^1} \leq C(\|\nabla F\|_{L^2} + \|\nabla w\|_{L^2}) + \frac{C}{|\Omega|} \int_F dx 
\leq C R_\Omega^{1/2} A_2 + C A_1.
\] (3.21)
Finally, according to (2.4) and Lemma 3.2, we get
\[
\|\nabla u\|_{L^p} \leq C(p) \left( \|\text{div } u\|_{L^p} + \|\omega\|_{L^p} \right)
\]
\[
\leq C(p) \left\| \frac{F}{2\mu + \lambda} \right\|_{L^p} + C(p) \|\omega\|_{L^p} + C(p)
\]
\[
\leq C(p) \left\| \frac{F}{2\mu + \lambda} \right\|_{L^p}^{\frac{2}{p} - \varepsilon} \|F\|_{L^2}^{\frac{2}{p} + 1 + \varepsilon} + C(p) \|\omega\|_{L^p} + C(p) (322)
\]
\[
\leq C(p, \varepsilon) A_1^{\frac{2}{p} - \varepsilon} \|F\|^\varepsilon_{L^2} \|F\|^\frac{1 - 2}{H^{1\varepsilon}} + C(p) A_1^{\frac{2}{p}} \|\omega\|^\frac{1 - 2}{H^{1\varepsilon}} + C(p)
\]
\[
\leq C(p, \varepsilon) R_T^{\frac{2}{p} - \varepsilon} A_1^{\frac{2}{p}} (\|F\|^\frac{1}{H^1} + \|\omega\|_{H^1})^{1 - \frac{2}{p}} + C(p),
\]
which together with (3.21) gives (3.17) and finishes the proof of Lemma 3.5.

At present stage, we are in a position to prove the following estimate on the upper bound of \(\log(1 + \|\nabla u\|_{L^2})\) in terms of \(R_T\) which will play an important role in obtaining the upper bound of the density:

**Proposition 3.1.** For any \(\alpha \in (0, 1)\), there is a constant \(C(\alpha)\) depending only on \(\alpha, \Omega, T, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}\), and \(\|u_0\|_{H^1}\) such that
\[
\sup_{0 \leq t \leq T} \log A_1^2(t) + \int_0^T \frac{A_2^2(t)}{A_1^2(t)} \, dt \leq C(\alpha) R_T^{1 + \alpha}. \tag{3.23}
\]

**Proof.** We modify the ideas of [12] to overcome the difficulties arising from the boundary. First, direct calculations show that
\[
\nabla \cdot \hat{u} = \frac{D}{Dt} \omega - (\partial_1 u \cdot \nabla)u_2 + (\partial_2 u \cdot \nabla)u_1 = \frac{D}{Dt} \omega + \omega \text{div } u, \tag{3.24}
\]
and that
\[
\text{div } \hat{u} = \frac{D}{Dt} \text{div } u + (\partial_1 u \cdot \nabla)u_1 + (\partial_2 u \cdot \nabla)u_2
\]
\[
= \frac{D}{Dt} \left( \frac{F}{2\mu + \lambda} \right) + \frac{D}{Dt} \left( \frac{P}{2\mu + \lambda} \right) - 2 \nabla u_1 \cdot \nabla u_2 + (\text{div } u)^2. \tag{3.25}
\]
Then, multiplying (3.18) by \(2 \hat{u}\) and integrating the resulting equality over \(\Omega\), we obtain after using (3.24), (3.25), and the boundary condition (1.5) that
\[
\frac{d}{dt} A_1^2 + 2A_2^2 = -\mu \int_\Omega \omega^2 \text{div } u \, dx + 4 \int_\Omega F \nabla u_1 \cdot \nabla u_2 \, dx
\]
\[
- 2 \int_\Omega F(\text{div } u)^2 \, dx - \int_\Omega \frac{(\beta - 1)\lambda - 2\mu}{(2\mu + \lambda)^2} F^2 \text{div } u \, dx
\]
\[
+ 2\beta \int_\Omega \frac{\lambda P}{(2\mu + \lambda)^2} F \text{div } u \, dx - 2\gamma \int_\Omega \frac{P}{2\mu + \lambda} F \text{div } u \, dx
\]
\[
+ \int_{\partial \Omega} Fu \cdot \nabla u \cdot n \, ds + \mu \int_{\partial \Omega} \omega (\hat{u} \cdot n^\perp) \, ds \triangleq \sum_{i=1}^8 I_i. \tag{3.26}
\]
Now we estimate each $I_i$ as follows:
First, combining (2.5), (3.11) and the Hölder inequality leads to

$$I_1 \leq C \|\omega\|_{L^4}^2 \|\text{div} u\|_{L^2} \leq C \|\omega\|_{L^2} \|\nabla\omega\|_{L^2} A_1 \leq C A_1^2 \|\nabla\omega\|_{L^2}.$$  \hspace{1cm} (3.27)

Next, $I_2$ is the most difficult term which requires careful calculations. For $\alpha \in (0, 1)$, letting $p \geq 8\beta/\alpha$, we use (3.22), Hölder’s and Sobolev’s inequalities to get

$$|I_2| \leq \int F|\nabla u|^2 \, dx \leq \|F\|_{L^p} \|\nabla u\|_{L^2}^{2(\frac{p-1}{p-2})} \|\nabla u\|_{L^p}^{\frac{2}{p-2}} \leq C(p, \varepsilon) \|F\|_{L^p} \left( R_T^{-\frac{\beta}{2}} A_1^{-\frac{\beta}{2}} \left( \|F\|_{H^1} + \|\omega\|_{H^1} \right)^{1-\frac{2}{p}} + 1 \right)^{\frac{2}{p-2}} \leq C(\alpha) R_T^{\frac{\beta}{2}} A_1 \|F\|_{H^1} + C(\alpha) A_1^2,$$  \hspace{1cm} (3.28)

where in the last line, we have used $\|F\|_{L^p} \leq C(p) \|F\|_{L^2}^{\frac{2}{p}} \|F\|_{H^1}^{1-\frac{2}{p}}$ and $\|F\|_{L^2} \leq C R_T^{\frac{\beta}{2}} A_1$.

Next, the Hölder inequality yields that for $0 < \alpha < 1$,

$$\sum_{i=3}^{6} I_i \leq C \int_{\Omega} \frac{F^2 |\text{div} u|}{2\mu + \lambda} \, dx + C \int_{\Omega} \frac{P}{2\mu + \lambda} |F||\text{div} u| \, dx \leq C A_1 \left\|\frac{F}{2\mu + \lambda}\right\|_{L^2} + C A_1 \|F\|_{L^{2+4\beta/\beta}} \leq C A_1 \left\|\frac{F}{(2\mu + \lambda)^{1/2}}\right\|_{L^2} \|F\|_{H^1}^{1-\alpha} + C A_1 \|F\|_{H^1} \leq C(\alpha) A_1^{2-\alpha} \|F\|_{L^2}^{\alpha} \|F\|_{H^1} + C(\alpha) A_1\|F\|_{H^1} \leq C(\alpha) R_T^{\frac{\beta}{2}} A_1^{2-\alpha} \|F\|_{H^1},$$  \hspace{1cm} (3.29)

where in the last inequality, we have used $\|F\|_{L^2} \leq C R_T^{\frac{\beta}{2}} A_1$.

Finally, thanks to (1.18), we deal with $I_7$ via:

$$|I_7| = \left| \int_{\partial\Omega} F(u \cdot \nabla n \cdot u) \, ds \right| \leq C \|F\|_{H^1} \|\nabla u\|_{L^2}^2 \leq C \|F\|_{H^1} A_1^2.$$  \hspace{1cm} (3.30)
And for $I_8$:

$$I_8 = \mu \int_{\partial \Omega} \omega (\dot{u} \cdot n^\perp) \, ds$$

\[= -\mu \int_{\partial \Omega} A(u \cdot n^\perp) \cdot \frac{d}{dt} (u \cdot n^\perp) \, ds - \mu \int_{\partial \Omega} A(u \cdot n^\perp)(u \cdot \nabla)u \cdot n^\perp \, ds\]

\[= -\frac{\mu}{2} \cdot \frac{d}{dt} \int_{\partial \Omega} A(u \cdot n^\perp)^2 \, ds - \mu \int_{\partial \Omega} A(u \cdot n^\perp)^2 (n^\perp \cdot \nabla)u \cdot n^\perp \, ds\]

\[= -\frac{\mu}{2} \cdot \frac{d}{dt} \int_{\partial \Omega} A|u|^2 \, ds - \frac{\mu}{3} \int_{\partial \Omega} A(n^\perp \cdot \nabla)(u \cdot n^\perp)^3 \, ds\]

\[\leq -\frac{\mu}{2} \cdot \frac{d}{dt} \int_{\partial \Omega} A|u|^2 \, ds - \frac{\mu}{3} \int_{\partial \Omega} \text{div} (\nabla^\perp (u \cdot n^\perp)^3 A) \, ds + C \|\nabla u\|_{L^2}^3\]

\[\leq -\frac{\mu}{2} \cdot \frac{d}{dt} \int_{\partial \Omega} A|u|^2 \, ds + C \int_{\Omega} |\nabla u| \cdot |u|^2 \, dx + C \|\nabla u\|_{L^2}^3\]

\[\leq -\frac{\mu}{2} \cdot \frac{d}{dt} \int_{\partial \Omega} A|u|^2 \, ds + CA^3.\]

Putting all the estimates (3.27)–(3.31) into (3.26) yields that for any $\alpha \in (0, 1)$,

$$\frac{d}{dt} A_1^2(t) + A_2^2(t) \leq C(\alpha) R_T^{\frac{\alpha}{2}} (\|F\|_{H^1} + \|\omega\|_{H^1}) A_1^2 + C(\alpha) A_1^2$$

\[\leq C(\alpha) R_T^{\frac{\alpha}{2}} A_2 A_1^2 + C(\alpha) A_1^3\]

\[\leq C(\alpha) R_T^{1+\alpha\beta} A_1^2 + \frac{1}{2} A_2^2,\]

where in the second inequality we have used (3.21). Combining this with (3.12) proves (3.23) and finishes the proof of Proposition 3.1.

With all preparation done, we turn to the crux of our problem which is to estimate the effective viscous flux $F$ defined as in (3.1). Such term can be estimated via solving the Neumann boundary problem (1.16) with the help of classical Green’s function. However, except for some special domains, it is difficult to write down Green’s function for arbitrary domains $\Omega$ in an explicit form, which is a crucial obstacle in further calculations.

To get over this, we start with the simplest case, that is, $\Omega$ is the unit disc $\mathbb{D}$. We note that Green’s function of the unit disc (see [17]) takes the form (1.17). It meets our requirements perfectly and provides a prototype of our proof. Moreover, thanks to Riemann mapping theorem [18, Chapter 9], every simply connected domain is conformally equivalent to the unit disc. Thus we may pull back Green’s function of the unit disc via conformal mapping, and reduce the general case to that of unit disc in some sense.

Move precisely, let $\varphi = (\varphi_1, \varphi_2) : \tilde{\Omega} \rightarrow \mathbb{D}$ be the conformal mapping which satisfies (2.7) and Lemma 2.5. We then define the pull-back Green’s function $\tilde{N}$ of $\Omega$ as: 

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\[ \tilde{N}(x, y) \triangleq N(\varphi(x), \varphi(y)) \]
\[ = -\frac{1}{2\pi} \left[ \log|\varphi(x) - \varphi(y)| + \log \left| \varphi(x)|\varphi(y) - \frac{\varphi(x)}{|\varphi(x)|} \right| \right]. \quad (3.33) \]

Since the outer normal derivative of \( \tilde{N} \) on the boundary \( \partial \Omega \) is no longer constant which is illustrated by the next Lemma 3.6, \( \tilde{N} \) is not the “real” Green’s function of \( \Omega \) in the classical sense but still sufficient for our further calculations.

**Lemma 3.6.** Let \( n \) be the unit outer normal at \( y_0 \in \partial \Omega \) and \( \tilde{N} = \tilde{N}(x, y) \) be given by (3.33), then \[ \frac{\partial \tilde{N}}{\partial n}(x, y_0) = -\frac{1}{2\pi} |\nabla \varphi_1(y_0)|. \]

**Proof.** By Lemma 2.5, it is clear that \( \varphi(y) \in \partial D \). A direct calculation shows that
\[ \frac{\partial}{\partial n} \tilde{N}(x, y_0) = n \cdot \nabla_y \tilde{N}(x, y)|_{y=y_0} \]
\[ = n \cdot \nabla_y N(\varphi(x), \varphi(y))|_{y=y_0} \]
\[ = n_i \cdot \partial_i \varphi_j \partial_j N(\varphi(x), \varphi(y))|_{y=y_0} \]
\[ = (n \cdot \nabla) \varphi_j \partial_j N(\varphi(x), \varphi(y))|_{y=y_0} \]
\[ = |\nabla \varphi_1| \hat{n} \cdot \nabla N(\varphi(x), \varphi(y))|_{y=y_0} \]
\[ = -\frac{1}{2\pi} |\nabla \varphi_1|, \]
where in the fifth equality we have used (2.11). \( \square \)

Now we use the pull-back Green’s function \( \tilde{N} \) defined by (3.33) to obtain a pointwise representation of \( F \) in \( \Omega \) via Green’s identity as follows:

**Lemma 3.7.** Let \( F \in C^1(\tilde{\Omega}) \cap C^2(\Omega) \) solve the problem (1.16). Then there holds for \( x \in \Omega \)
\[ F(x) = -\int_{\Omega} \nabla_y \tilde{N}(x, y) \cdot \rho \hat{u}(y) \, dy + \int_{\partial \Omega} \frac{\partial \tilde{N}}{\partial n}(x, y) F(y) \, dS_y \]
\[ + \mu \int_{\partial \Omega} \tilde{N}(x, y)n^\perp \cdot \nabla (Au \cdot n^\perp) \, dS_y. \quad (3.34) \]

**Proof.** Denote \( \tilde{N}(x, y) = N_1(x, y) + N_2(x, y) \), where
\[ N_1(x, y) = -\frac{1}{2\pi} \log|\varphi(x) - \varphi(y)|, \quad N_2(x, y) = -\frac{1}{2\pi} \log \left| \varphi(x)|\varphi(y) - \frac{\varphi(x)}{|\varphi(x)|} \right|. \]

By Lemma 2.5, conformal mapping preserves harmonicity which implies that \( N_2(x, \cdot) \) is harmonic in \( \Omega \). Moreover, since \( N_2(x, \cdot) \) has no singular point in \( \Omega \), we apply Green’s second identity to \( F \) and \( N_2(x, y) \) in \( \Omega \):
\[ -\int_{\partial \Omega} \Delta F \, dy = \int_{\partial \Omega} \left( \frac{\partial N_2}{\partial n} F - N_2 \frac{\partial F}{\partial n} \right) \, dS_y. \quad (3.35) \]
Up on applying Green’s second identity to $F$ and $N_1(x, \cdot)$ in $\Omega \setminus \tilde{B}_r(x)$, where $\tilde{B}_r(x) = \phi^{-1}(B_r(\phi(x)))$ is just the inverse image under $\phi$ of the ball centered at $\phi(x)$ of radius $r$, we deduce that for $r$ small enough

$$
\int_{\Omega \setminus \tilde{B}_r(x)} (\Delta N_1 F - N_1 \Delta F) \, dy
= \int_{\partial \Omega} \left( \frac{\partial N_1}{\partial n} F - N_1 \frac{\partial F}{\partial n} \right) \, dS_y - \int_{\partial \tilde{B}_r(x)} \left( \frac{\partial N_1}{\partial n} F - N_1 \frac{\partial F}{\partial n} \right) \, dS_y.
$$

Letting $r \to 0$ and using the fact that $\Delta N_1 = 0$ in $\Omega \setminus \tilde{B}_r(x)$, we have

$$
- \int_{\Omega} N_1 \Delta F \, dy = \int_{\partial \Omega} \left( \frac{\partial N_1}{\partial n} F - N_1 \frac{\partial F}{\partial n} \right) \, dS_y
- \lim_{r \to 0} \int_{\partial \tilde{B}_r(x)} \left( \frac{\partial N_1}{\partial n} F - N_1 \frac{\partial F}{\partial n} \right) \, dS_y.
$$

(3.36)

For $r$ small enough,

$$
\left| \int_{\partial \tilde{B}_r(x)} N_1 \frac{\partial F}{\partial n} \, dS_y \right|
\leq \int_{\partial \tilde{B}_r(x)} |N_1(x, y)| \, dS_y \cdot \sup_{\partial \tilde{B}_r(x)} |\nabla F|
\leq \int_{\partial \tilde{B}_r(\phi(x))} |\log|\phi(x) - \phi(y)|| \frac{1}{|\nabla \phi_1(y)|} \, dS_{\phi(y)} \cdot \sup_{\partial \tilde{B}_r(x)} |\nabla F|
\leq C r |\log r| \cdot \sup_{\partial \tilde{B}_r(x)} |\nabla \phi_1|^{-1} \sup_{\tilde{B}_r(x)} |\nabla F| \to 0 \text{ as } r \to 0,
$$

(3.37)

and

$$
\int_{\partial \tilde{B}_r(x)} F \frac{\partial N_1}{\partial n} \, dS_y
= \int_{\partial \tilde{B}_r(x)} n_y \cdot \nabla_y N_1(x, y) F(y) \, dS_y
= -\frac{1}{2\pi} \int_{\partial \tilde{B}_r(x)} \frac{\nabla \phi(y)[\phi(y) - \phi(x)]}{|\nabla \phi_1(y)||\phi(x) - \phi(y)|} \cdot \frac{\nabla \phi(y)[\phi(x) - \phi(y)]}{|\phi(x) - \phi(y)|^2} F(y) \, dS_y
= -\frac{1}{2\pi r} \int_{\partial \tilde{B}_r(x)} |\nabla \phi_1(y)| F(y) \, dS_y
= -\frac{1}{2\pi r} \int_{\partial B_r(\phi(x))} F(\phi^{-1}(\tilde{y})) \, dS_{\tilde{y}} \to F(x) \text{ as } r \to 0,
$$

(3.38)
where we use (2.7) and the same method as in previous Lemma 3.6. Adding (3.35) and (3.36) together, we have by using (3.37) and (3.38)

\[
F(x) = \int_{\Omega} \tilde{N}(x, y) \text{div}(\rho \dot{u})(y) \, dy + \int_{\partial \Omega} \frac{\partial \tilde{N}}{\partial n}(x, y) F(y) \, dS_y \\
- \int_{\partial \Omega} \tilde{N}(x, y)(\rho \dot{u} \cdot n(y) - \mu n^\perp \cdot \nabla (Au \cdot n^\perp)) \, dS_y \\
= - \int_{\Omega} \nabla_y \tilde{N}(x, y) \cdot \rho \dot{u}(y) \, dy + \int_{\partial \Omega} \frac{\partial \tilde{N}}{\partial n}(x, y) F(y) \, dS_y \\
+ \mu \int_{\partial \Omega} \tilde{N}(x, y)n^\perp \cdot \nabla (Au \cdot n^\perp) dS_y,
\]

which gives (3.34) and finishes the proof of Lemma 3.7. \(\square\)

Now, the central point is to estimate \(F\) defined by (3.34). For the last two terms on the righthand side of (3.34), by Lemma 3.6, one has

\[
\max_{x \in \Omega} \left| \int_{\partial \Omega} \frac{\partial \tilde{N}}{\partial n}(x, y) F(y) dS_y \right| \leq C \|F\|_{H^1}, \quad (3.39)
\]

and

\[
\left| \int_{\partial \Omega} \tilde{N}(x, y)n^\perp \cdot \nabla (Au \cdot n^\perp) dS_y \right| \\
= \left| \int_{\Omega} \text{div}(\nabla^\perp (Au \cdot n^\perp) \tilde{N}(x, y))dy \right| \\
\leq C \int_{\Omega} \|\nabla \tilde{N}(x, y)\| \cdot (|\nabla u| + |u|) \, dx \\
\leq C \|\nabla \tilde{N}\|_{L^3} \cdot \|\nabla u\|_{L^4} \\
\leq C R_T^{1+\varepsilon} A_1^{1} A_2^{1} + C R_T^{\varepsilon} A_1,
\]

where we have applied (3.22) in last line. It thus follows from (3.21), (3.23), (3.39), and (3.40) that

\[
\int_0^T \max_{x \in \Omega} \int_{\partial \Omega} \frac{\partial \tilde{N}}{\partial n}(x, y) F(y) dS_y \, dt \\
+ \int_0^T \max_{x \in \Omega} \mu \int_{\partial \Omega} \tilde{N}(x, y)n^\perp \cdot \nabla (Au \cdot n^\perp) dS_y \, dt \\
\leq C(\varepsilon) \int_0^T \left( \|F\|_{H^1} + R_T^{1+\varepsilon} A_1^{1} A_2^{1} + R_T^{\varepsilon} A_1 \right) \, dt \\
\leq C(\varepsilon) \int_0^T \left( R_T^{\frac{1}{2}} A_2 + R_T^{1+\varepsilon} A_1^{\frac{1}{2}} A_2^{\frac{1}{2}} + R_T^{\varepsilon} A_1 \right) \, dt \\
\leq C(\varepsilon) R_T^{\frac{1}{2}} \left( \int_0^T A_2^2 \, dt \right)^{\frac{1}{2}} \left( \int_0^T A_1^2 \, dt \right)^{\frac{1}{2}}
\]
\[
+C(\varepsilon) R_T^{1+\varepsilon} \left( \int_0^T \frac{A^2_2}{A^2_1} \, dt \right)^\frac{1}{2} \left( \int_0^T \frac{A^2_1}{A^2_2} \, dt \right)^\frac{1}{2} + C(\varepsilon) \\
\leq C(\varepsilon) R_T^{1+\varepsilon}.
\] (3.41)

Then, we use the mass equation and the boundary condition \( u \cdot n|_{\partial\Omega} = 0 \) to rewrite the first term on the righthand side of (3.34) as follows:

\[
- \int_{\Omega} \nabla_y \tilde{N}(x, y) \cdot \rho \hat{u}(y) \, dy \\
= - \int_{\Omega} \nabla_y \tilde{N}(x, y) \cdot \left( \frac{\partial (\rho u)}{\partial t} + \text{div}(\rho u \otimes u) \right) \, dy \\
= - \frac{\partial}{\partial t} \int_{\Omega} \partial_{y_j} \tilde{N}(x, y) \rho u_j(y) \, dy + \int_{\Omega} \partial_{y_i} \partial_{y_j} \tilde{N}(x, y) \rho u_i u_j(y) \, dy \\
= - (\partial_t + u \cdot \nabla) \int_{\Omega} \partial_{y_j} \tilde{N}(x, y) \rho u_j(y) \, dy + J,
\] (3.42)

with

\[
J \triangleq \int_{\Omega} \left[ \partial_{x_i} \partial_{y_j} \tilde{N}(x, y) u_i(x) + \partial_{y_i} \partial_{y_j} \tilde{N}(x, y) u_i(y) \right] \rho u_j(y) \, dy.
\] (3.43)

Next, we have the following crucial point-wise estimate on \( J \).

**Proposition 3.2.** For \( J \) as in (3.43), there exists a generic positive constant \( C(\Omega) \) such that for any \( x \in \Omega \) with \( \phi(x) \neq 0 \),

\[
|J(x)| \leq C(\Omega) \int_{\Omega} \frac{\rho |u|^2(y)}{|x - y|} \, dy + C(\Omega) \int_{\Omega} \frac{|u(x) - u(y)|}{|x - y|^2} \rho |u|(y) \, dy \\
+ C(\Omega) \int_{\Omega} \frac{|u(x') - u(y)|}{|x' - y|^2} \rho |u|(y) \, dy,
\] (3.44)

where

\[
x' \triangleq \phi^{-1} \left( \frac{\phi(x)}{|\phi(x)|} \right) \in \partial\Omega.
\] (3.45)

**Proof.** Using (3.33), we rewrite \( J \) as

\[
J(x) = \int_{\Omega} \partial_{x_i} \partial_{y_j} \tilde{N}(x, y)[u_i(x) - u_i(y)] \rho u_j(y) \, dy \\
- \frac{1}{2\pi} \int_{\Omega} \Lambda_{i,j}(\phi(y), \phi(x)) \rho u_i u_j(y) \, dy \\
- \frac{1}{2\pi} \int_{\Omega} \Lambda_{i,j}(\phi(y), w(x)) \rho u_i u_j(y) \, dy \triangleq \sum_{l=1}^{3} J_l(x),
\] (3.46)

with
\[ \Lambda_{i,j}(\varphi(y), v(x)) \triangleq (\partial_x \partial_{y_j} + \partial_y \partial_{x_j}) \log |\varphi(y) - v(x)|, \quad w(x) \triangleq \frac{\varphi(x)}{|\varphi(x)|^2}. \]

Thus, for \( J_1(x) \), direct computation yields that
\[ |J_1(x)| \leq C(\Omega) \int_{\Omega} \frac{|u(x) - u(y)|}{|x - y|^2} \rho|u|(y) \, dy. \]  

Then, to estimate \( J_2(x) \) and \( J_3(x) \), we check that for \( v(x) \in \{\varphi(x), w(x)\} \),
\[ \Lambda_{i,j}(\varphi(y), v(x)) = \partial_{y_j} \left( (v_k(x) - \varphi_k(y))(\partial_t v_k(x) - \partial_t \varphi_k(y)) \right) \]
\[ = \partial_{y_j} \left[ \frac{(v_k(x) - \varphi_k(y))\partial_t v_k(x) - \partial_t \varphi_k(y)}{|v(x) - \varphi(y)|^2} \right] \]
\[ = \frac{(\varphi_k(y) - v_k(x))\partial_j \varphi_k(y)}{|v(x) - \varphi(y)|^2} + \frac{\partial_j \varphi_k(y)(\partial_t \varphi_k(y) - \partial_t v_k(x))}{|v(x) - \varphi(y)|^2} + 2(\varphi_k(y) - v_k(x))(\partial_t v_k(x) - \partial_t \varphi_k(y))(\varphi_s(y) - v_s(x))\partial_j \varphi_s(y) \]
\[ \frac{|v(x) - \varphi(y)|^4}{|v(x) - \varphi(y)|^2}. \]  

Thus, for \( J_2(x) \), it follows from (2.8)–(2.10) and (3.49) that
\[ |\Lambda_{i,j}(\varphi(y), \varphi(x))| \leq C(|\Omega|)|x - y|^{-1}, \]
which in particular implies
\[ |J_2(x)| \leq C(\Omega) \int_{\Omega} \frac{\rho|u|^2(y)}{|x - y|} \, dy. \]  

Finally, it remains to estimate the more difficult term \( J_3 \). According to (2.8), we have
\[ |\Lambda_{i,j}(\varphi(y), w(x))u_i(y)| \]
\[ \leq \frac{C|u|}{|\varphi(y) - w(x)|} + C \sum_{k=1}^{2} \frac{|(\partial_t w_k(x) - \partial_t \varphi_k(y))u_i(y)|}{|\varphi(y) - w(x)|^2}. \]  

Now, the central issue is to estimate \((\partial_t w_k(x) - \partial_t \varphi_k(y))u_i(y)\).
First, for \( \varphi(x), \varphi(y) \in \mathbb{D} \) with \( \varphi(x) \neq 0 \), we have
\[ \left| \frac{\varphi(y) - \varphi(x)}{|\varphi(x)|} \right| \leq |\varphi(y) - w(x)|, \quad |\varphi(y) - \varphi(x)| \leq |\varphi(y) - w(x)|. \]  

A direct consequence of (3.52) shows that for \( x, y \in \Omega \) with \( \varphi(x) \neq 0 \),
\[ |\varphi(y) - w(x)| \geq 1 - |\varphi(x)|, \]  

which implies that
\[ |\varphi(y) - w(x)|^{-1} \leq (1 - |\varphi(x)|)^{-1} \leq 4, \]
provided \(|\phi(x)| \leq 3/4\). Thus, from now on, we always assume that

\[|\phi(x)| > 3/4.\]  \hfill (3.54)

Next, direct computation shows that for \(w\) as in (3.47)

\[
\partial_{x_i} w_k(x) - \partial_{y_i} \phi_k(y) = \frac{\partial_{x_i} \phi_k(x)}{|\phi(x)|^2} - \frac{2 \phi_k(x) \phi_l(x) \partial_{x_i} \phi_l(x)}{|\phi(x)|^4} - \partial_{y_i} \phi_k(y) \tag{3.55}
\]

where \(x'\) is as in (3.45) and

\[
\tilde{K}_{i,k} \triangleq \frac{\partial_{x_i} \phi_k(x)}{|\phi(x)|^2} - \partial_{y_i} \phi_k(y) - 2 \phi_k(x') \phi_l(x') \left( \frac{\partial_{x_i} \phi_l(x)}{|\phi(x)|^2} - \partial_{y_i} \phi_l(y) \right). \tag{3.56}
\]

Now it follows from (2.8) and (3.52)–(3.54) that

\[
\left| \frac{\partial_{x_i} \phi_k(x)}{|\phi(x)|^2} - \partial_{y_i} \phi_k(y) \right| = \left| \partial_{x_i} \phi_k(x) - \partial_{y_i} \phi_k(y) + \partial_{x_i} \phi_k(x) \frac{1 - |\phi(x)|^2}{|\phi(x)|^2} \right| \\
\leq C|x - y| + C(1 - |\phi(x)|) \\
\leq C|\phi(y) - w(x)|,
\]

which together with (3.56) gives

\[|\tilde{K}_{i,k}| \leq C|\phi(y) - w(x)|. \]

Combining this with (3.55) leads to

\[
|\partial_{x} w_k(x) - \partial_{y} \phi_k(y) u_{i}(y)| \\
\leq C|\phi(y) - w(x)||u| + C|\phi_l(x') \partial_{y_i} \phi_l(y) u_{i}(y)|. \tag{3.57}
\]

Then, since \(x' \in \partial \Omega\) and \(u \cdot n = 0\) on \(\partial \Omega\), we have by (2.12)

\[\phi_l(x') \partial_{y_i} \phi_l(x') u_{i}(x') = 0,
\]

which yields that

\[
|\phi_l(x') \partial_{y_i} \phi_l(y) u_{i}(y)| \\
= |\phi_l(x') (\partial_{y_i} \phi_l(y) - \partial_{y_i} \phi_l(x')) u_{i}(y) + \phi_l(x') \partial_{y_i} \phi_l(x') (u_{i}(y) - u_{i}(x'))| \\
\leq C|y - x'||u| + C|u(y) - u(x')| \\
\leq C|\phi(y) - w(x)||u| + C|u(y) - u(x')| \tag{3.58}
\]

where in the last inequality we have used

\[|y - x'| \leq C|\phi(y) - \phi(x')| \leq C|\phi(y) - w(x)|, \tag{3.59}\]

which comes from (2.10) and (3.52).

Consequently, combining (3.57) with (3.58) gives

\[|(\partial_{x_i} w_k(x) - \partial_{y_i} \phi_k(y)) u_{i}(y)| \leq C|\phi(y) - w(x)||u| + C|u(y) - u(x')|,
\]
which together with (3.51) shows

\[|\Lambda_{i,j}(\varphi(y), w(x))u_i(y)| \leq \frac{C|u|}{|\varphi(y) - w(x)|} + \frac{C|u(y) - u(x')|}{|\varphi(y) - w(x)|^2}\]

\[\leq \frac{C|u|}{|y - x|} + \frac{C|u(y) - u(x')|}{|y - x'|^2},\]

where in the last inequality we have used (3.59). As a direct consequence, we arrive at

\[|J_3(x)| \leq C \int_{\Omega} \frac{\rho|u|^2}{|y - x|} \, dy + C \int_{\Omega} \frac{|u(y) - u(x')|}{|y - x'|^2} \rho|u| \, dy,
\]

which together with (3.50) and (3.48) gives (3.44) and finishes the proof of Proposition 3.2.

\[\quad \square\]

**Remark 3.1.** In particular, when \(\Omega\) is the unit disc \(\mathbb{D}\) itself, the computation above can be greatly simplified, provided we set \(\varphi\) to be the identity. Actually such reduced case is exactly the starting point of our proof.

**Remark 3.2.** We remark that the first term on the right hand side of (3.44) having singularity of order 1, and the last two terms are of commutator type.

Indeed, for the Riesz transformation \(R_i\) given by (see Chapter 3 of [32])

\[R_i f(x) \triangleq \text{p.v.} \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y_i}{|y|^3} f(x - y) \, dy,
\]

the usual Caldron-type commutator of \(\mathbb{R}^2\) takes the form of \([u_k, R_i R_j](\rho u_i)\) and can be written in singular integral form (see also [33, 34]) as

\[(\rho u_i)(x) = -\text{p.v.} \frac{1}{2\pi} \int_{\mathbb{R}^2} \delta_{ij} \frac{u_k(x) - u_k(y)}{|x - y|^2} \rho u_i(y) \, dy
\]

\[+ \text{p.v.} \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{2(x_i - y_i)(x_j - y_j)}{|x - y|^4} (u_k(x) - u_k(y)) \rho u_i(y) \, dy.
\]

For simplicity, we can find some kernel \(K_{ij}(x)(i, j = 1, 2)\) which has a singularity of the second order at 0 and satisfies

\[|K_{ij}(x)| \leq C|x|^{-2}, \ x \in \mathbb{R}^2 \setminus \{0\},\]

and deduce that

\([u_k, R_i R_j](\rho u_i)(x) = \text{p.v.} \int_{\mathbb{R}^2} K_{ij}(x - y) (u_k(x) - u_k(y)) \rho u_i(y) \, dy.
\]

Such representation coincides with the terms of commutator type mentioned above except for the integral domains.
Remark 3.3. It should be mentioned here that the situation is much more different from [12] in which the domain is assumed to be periodic. The setting on periodic case is equivalent to consider the compressible Navier–Stokes system (1.1) on the torus of dimension 2. Such manifold is compact orientable and out of boundary. Consequently, the usual commutator theory can be applied directly in [12] to get a proper control of \( F \) (see [12, (3.34)] for example). But in this paper, we consider the domain with boundary which is essentially different from the periodic case. Such difference is reflected in the technical difficulties that classical commutator theory is no longer available. The main purpose of Proposition 3.2 is to get over it. In other words, we are trying to establish a suitable commutator theory for bounded domains, and the approach we adapted is the combination of Green’s functions in a disc with conformal mapping.

Now, for the three terms on the righthand side of (3.44), as mentioned in Remark 3.2, we note that the second one

\[
\int_{\Omega} \left| \frac{u(x) - u(y)}{|x - y|^2} \right| \rho |u|(y) \, dy
\]

corresponds to the usual commutator just like [12]. Thus, because of the effect of the boundary and the Navier-slip conditions, the rest two terms appear, represent the difference between the periodic domains and general bounded ones, and can be regarded as the “error” terms we must face during treating the general bounded domain case. In particular, the last term coincides with the usual commutator except that \( x' \) is on the boundary and can be regarded as the commutator on the boundary.

Now, to derive the precise control of \( F \), we focus on commutator type terms which are of vital importance. Although such terms are not “real” commutator, and the classical \( L^p \) theory seems difficult to be applied directly, we still have alternative method to obtain proper estimates.

**Proposition 3.3.** For any \( \varepsilon > 0 \), we have

\[
\int_0^T \max_{x \in \Omega} \int_{\Omega} \left| \frac{u(x) - u(y)}{|x - y|^2} \right| \rho |u|(y) \, dy \, dt \leq C(\varepsilon) R_T^{1 + \frac{\beta}{4} + 3\varepsilon}. \tag{3.60}
\]

**Proof.** First, for \( 2 < p < 4 \) which will be determined later, by Sobolev’s embedding theorem (Theorem 5 of [35, Chapter 5]), we have for any \( x, y \in \Omega \),

\[
|u(x) - u(y)| \leq C(p) \| \nabla u \|_{L^p} |x - y|^{1 - \frac{2}{p}},
\]

which implies

\[
\int_{\Omega} \left| \frac{u(x) - u(y)}{|x - y|^2} \right| \rho |u|(y) \, dy \leq C(p) \| \nabla u \|_{L^p} \int_{\Omega} |x - y|^{-\left(1 + \frac{2}{p}\right)} \rho |u|(y) \, dy.
\] \tag{3.61}
Then, for $\delta > 0$ and $\varepsilon_0 \in (0, (p - 2)/8)$ which will be determined later, on the one hand, we use (2.5) to get

$$
\int_{|x - y| < 2\delta} |x - y|^{-(1 + \frac{2}{p})} \rho |u|(y) \, dy \\
\leq C(p) R_T \left( \int_{|x - y| < 2\delta} |x - y|^{-(1 + \frac{2}{p})(1 + \varepsilon_0)} \, dy \right)^{\frac{1}{1 + \varepsilon_0}} \|u\|_{L^{1+\varepsilon_0}}^{1+\varepsilon_0} \tag{3.62}
$$

$$
\leq C(p) R_T \delta^{1 - \frac{2}{p} - \frac{2\varepsilon_0}{1 + \varepsilon_0}} \left( \frac{1 + \varepsilon_0}{\varepsilon_0} \right)^{\frac{1}{2}} \|u\|_{H^1} \\
\leq C(p) R_T \varepsilon_0^{\frac{1}{2}} A_1 \delta^{1 - \frac{2}{p} - \frac{2\varepsilon_0}{1 + \varepsilon_0}}.
$$

On the other hand, for $\nu = \frac{\beta}{p} R_T^{-\frac{2}{p-2}}$ as in (3.10), we use Lemma 3.3 to derive

$$
\int_{|x - y| > \delta} |x - y|^{-(1 + \frac{2}{p})} \rho |u|(y) \, dy \\
\leq C(p) \left( \int_{|x - y| > \delta} |x - y|^{-(1 + \frac{2}{p})(\frac{2\nu}{\lambda^2 + \nu})} \, dy \right)^{\frac{1 + \nu}{\frac{2\nu}{\lambda^2 + \nu}}} \left( \int_{\Omega} \rho^{2+\nu} |u|^{2+\nu} \, dx \right)^{\frac{1}{\frac{2\nu}{\lambda^2 + \nu}}} \\
\leq C(p) R_T^{\frac{1 + \nu}{\lambda^2 + \nu}} \delta^{-\frac{2}{p} + \frac{\nu}{\lambda^2 + \nu}}. \tag{3.63}
$$

Now, we choose $\delta > 0$ such that

$$
\delta^{\frac{2}{p} + \frac{\nu}{\lambda^2 + \nu}} = A_1^{\frac{2}{\nu}}, \tag{3.64}
$$

which, in particular, implies that

$$
A_1 \delta^{1 - \frac{2}{p} - \frac{2\varepsilon_0}{1 + \varepsilon_0}} = A_1^{\frac{2}{\nu}}, \tag{3.65}
$$

provided we set

$$
\varepsilon_0 = \frac{(p - 2)\nu}{8 + (6 - p)\nu} \in \left(0, \frac{p - 2}{8}\right). \tag{3.66}
$$

Then, it follows from (3.62)–(3.65) that

$$
\int_{\Omega} |x - y|^{-(1 + \frac{2}{p})} \rho |u|(y) \, dy \\
\leq \left( \int_{|x - y| < 2\delta} + \int_{|x - y| > \delta} \right) |x - y|^{-(1 + \frac{2}{p})} \rho |u|(y) \, dy \\
\leq C(p) R_T^{-\frac{1}{2}} A_1^{\frac{2}{p}} + C(p) R_T^{\frac{1 + \nu}{\lambda^2 + \nu}} A_1^{\frac{2}{p}} \\
\leq C(p) R_T^{1 + \beta/4} A_1^{2/p},
$$
where in the last line we have used $\varepsilon_0^{-1/2} \leq C(p) v^{-1/2} \leq C(p) R_T^{\beta/4}$ due to (3.66).

Combining this, (3.61), and (3.17) shows that for any $\varepsilon > 0$,

$$
\int \frac{|u(x) - u(y)|}{|x - y|^2} \rho|u|(y) \, dy 
\leq C(p, \varepsilon) R_T^{3 - \frac{1}{p} + \frac{\beta}{4} + \frac{\beta}{2}} A_1^{1 + \frac{2}{p}} \left( \frac{A_2^2}{A_1^2} \right)^{\frac{1}{2} - \frac{1}{p}} + C(p, \varepsilon) R_T^{1 + \frac{\beta}{4}} A_1^{1 + \frac{2}{p}}.
$$

Finally, integrating this with respect to $t$ and using the Hölder inequality, we arrive at

$$
\int_0^T \max_{x \in \Omega} \int \frac{|u(x(t)) - u(y)|}{|x - y|^2} \rho|u|(y) \, dy \, dt 
\leq C(p, \varepsilon) R_T^{2 - \frac{2}{p} + \frac{\beta}{4} + 2\varepsilon},
$$

where in the last line we have used (3.12) and (3.23). This, after choosing $p = 2/(1 - \varepsilon)$, in particular yields (3.60) and finishes the proof of Proposition 3.3. \qed

Now we are in a position to obtain the upper bound of the density which plays an essential role in the whole procedure.

**Proposition 3.4.** Assume that (1.8) holds. Then there exists some positive constant $C$ depending only on $\Omega$, $T$, $\mu$, $\beta$, $\gamma$, $\|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that

$$
\sup_{0 \leq t \leq T} (\|\rho\|_{L^\infty} + \|u\|_{H^1}) + \int_0^T \left( \|\omega\|_{H^1}^2 + \|F\|_{H^1}^2 + \|\sqrt{\rho_0} u\|_{L^2}^2 \right) \, dt \leq C.
$$

(3.67)

**Proof.** First, for $\theta(\rho)$ as in (1.15), we have by (1.1), (3.1), and (1.7)

$$
\frac{D}{Dt} \theta(\rho) = -(2\mu + \lambda) \text{div} u \leq -F
$$
which together with (3.34), (3.42), (3.39), and (3.44) gives

\[
\frac{D}{Dt} \theta(\rho) \leq \frac{D}{Dt} \int_{\Omega} \partial_y j \tilde{N}(x, y) \rho u_j(y) \, dy + \max_{x \in \Omega} \left| \int_{\partial \Omega} \frac{\partial \tilde{N}}{\partial n}(x, y) F(y) \, dS_y \right|
\]

\[
+ \max_{x \in \Omega} \int_{\partial \Omega} \left| \mu \tilde{N}(x, y)(n^+ \cdot \nabla (A u \cdot n^+)) \right| dS_y + |J|
\]

\[
\leq \frac{D}{Dt} \int_{\Omega} \partial_y j \tilde{N}(x, y) \rho u_j(y) \, dy + \max_{x \in \Omega} \left| \int_{\partial \Omega} \frac{\partial \tilde{N}}{\partial n}(x, y) F(y) \, dS_y \right|
\]

\[
+ \max_{x \in \Omega} \int_{\partial \Omega} \left| \mu \tilde{N}(x, y)(n^+ \cdot \nabla (A u \cdot n^+)) \right| dS_y
\]

\[
+ \max_{x \in \Omega} \int_{\Omega} \frac{\rho|u|^2}{|x-y|} \, dy + \max_{x \in \Omega} \int_{\Omega} \frac{|u(x) - u(y)|}{|x-y|^2} \rho |u(y)| \, dy.
\]

Then, on the one hand, direct computation yields that for \( \nu = R_T^{-\frac{\beta}{2}} \) as in Lemma 3.3,

\[
\left| \int_{\Omega} \partial_y j \tilde{N}(x, y) \rho u_j(y) \, dy \right|
\]

\[
\leq C \int_{\Omega} |x-y|^{-1} \rho(y)|u(y)| \, dy
\]

\[
\leq C \left( \int_{\Omega} |x-y|^{-\frac{2+\nu}{2+\nu}} \, dy \right)^{\frac{1}{2+\nu}} \left( \int_{\Omega} \rho^{2+v}|u|^{2+v} \, dy \right)^{\frac{1}{2+\nu}}
\]

\[
\leq C \nu^{\frac{1}{2+\nu}} R_T^{\frac{1+\nu}{2+\nu}} \left( \int_{\Omega} \rho |u|^{2+v} \, dy \right)^{\frac{1}{2+\nu}}
\]

\[
\leq C R_T^{\frac{1+\nu}{2+\nu}}
\]

\[
\leq C R_T^{\frac{2+\nu}{2+\nu}},
\]

where in the fourth inequality we have used (3.9).

On the other hand, Sobolev’s inequality leads to

\[
\int_{\Omega} \frac{\rho|u|^2(y)}{|x-y|} \, dy \leq R_T \left( \int_{\Omega} |x-y|^{-\frac{3}{2}} \, dy \right)^{\frac{2}{3}} \left( \int_{\Omega} |u|^6 \, dy \right)^{\frac{1}{3}}
\]

\[
\leq C R_T \|u\|_{H^l}^2.
\]

which together with (3.11) and (3.5) shows

\[
\int_0^T \max_{x \in \Omega} \int_{\Omega} \frac{\rho|u|^2(y)}{|x-y|} \, dy \, dr \leq C R_T.
\]

Finally, integrating (3.68) with respect to \( t \), we obtain after using (3.41), (3.60), (3.69), and (3.70) that

\[
R_T^\beta \leq C(\varepsilon) R_T^{\max \left\{ 1+\beta, 2+\beta, \frac{2+\beta}{3} \right\}}.
\]
Since $\beta > 4/3$, this in particular implies
\begin{equation}
\sup_{0 \leq t \leq T} \| \rho \|_{L^\infty} \leq C,
\end{equation}
which together with (3.32), (3.12), (3.21), and Gronwall’s inequality gives (3.67) and finishes the proof of Proposition 3.4.

\section*{4. A Priori Estimates (II): Lower and Higher Order Ones}

Once obtaining the upper bound of the density, we will proceed to study the lower and higher order estimates which are indeed quite similar to those of [12], except that the boundary terms do bring some trouble that requires further considerations. We mainly focus on the boundary terms occurring along the way to our final goal, which is different from standard process. We mainly borrow some ideas from [10,12].

First, following [10], we give a Poincaré type estimate to treat the effect of boundary conditions where the construction depends on slip condition $u \cdot n|_{\partial \Omega} = 0$.

\begin{lemma}
For $p \geq 1$, there exist positive constants $C_1(p, \Omega)$ and $C_2(\Omega)$ such that
\begin{align}
\| \dot{u} \|_{L^p} &\leq C_1(\| \nabla \dot{u} \|_{L^2} + \| \nabla u \|_{L^2}^2), \\
\| \nabla \dot{u} \|_{L^2} &\leq C_2(\| \text{div} \dot{u} \|_{L^2} + \| \text{curl} \dot{u} \|_{L^2} + \| \nabla u \|_{L^4}^2).
\end{align}
\end{lemma}

\begin{proof}
First, for $n^\perp \triangleq (n_2, -n_1)$ as in (1.6), we have by $u \cdot n|_{\partial \Omega} = 0$,
\begin{equation}
u = (u \cdot n^\perp)n^\perp \text{ on } \partial \Omega,
\end{equation}
and
\begin{align*}
\dot{u} \cdot n &= (u \cdot \nabla)u \cdot n = -(u \cdot \nabla)n \cdot u \\
&= -(u \cdot n^\perp)(u \cdot \nabla)n \cdot n^\perp = (u \cdot n^\perp)(u \cdot \nabla)n^\perp \cdot n \text{ on } \partial \Omega,
\end{align*}
which gives
\begin{equation}
(\dot{u} - (u \cdot n^\perp)(u \cdot \nabla)n^\perp) \cdot n = 0.
\end{equation}

Next, it follows from (4.4) and Poincaré’s inequality that
\begin{equation}
\| \dot{u} - (u \cdot n^\perp)(u \cdot \nabla)n^\perp \|_{L^{3/2}} \leq C \| \nabla (\dot{u} - (u \cdot n^\perp)(u \cdot \nabla)n^\perp) \|_{L^{3/2}},
\end{equation}
which leads to
\begin{equation}
\| \dot{u} \|_{L^{3/2}} \leq C(\| \nabla \dot{u} \|_{L^{3/2}} + \| \nabla u \|_{L^2}^2).
\end{equation}
Combining this and the Sobolev embedding theorem implies that for $p \geq 1$,

$$\|\dot{u}\|_{L^p} \leq C(p)(\|\dot{u}\|_{L^2} + \|\nabla \dot{u}\|_{L^2})$$

$$\leq C(p)(\|\dot{u}\|_{L^{3/2}} + \|\nabla \dot{u}\|_{L^{3/2}}) + C(p)\|\nabla \dot{u}\|_{L^2}$$

$$\leq C(p)(\|\nabla \dot{u}\|_{L^2} + \|u\|_{L^2}^2),$$

which proves (4.1).

Finally, (4.2) is a direct consequence of (4.4) and (2.4).

Now we are ready to derive the lower order a priori estimates step by step.

**Lemma 4.2.** There is a positive constant $C$ depending only on $\Omega, T, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}, \|u_0\|_{H^1}$ such that

$$\sup_{0 \leq t \leq T} \sigma \int_{\Omega} \rho |\dot{u}|^2 \, dx + \int_0^T \sigma \|\nabla \dot{u}\|_{L^2}^2 \, dt \leq C,$$  

with $\sigma(t) \triangleq \min\{1, t\}$. Moreover, for any $p \in [1, \infty)$, there is a positive constant $C(p)$ depending only on $p, \Omega, T, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}, \|u_0\|_{H^1}$ such that

$$\sup_{0 \leq t \leq T} \sigma \|\nabla u\|_{L^p}^2 \leq C(p).$$

**Proof.** First, operating $\dot{u}_j[\frac{\partial}{\partial t} + \text{div}(u \cdot \cdot)]$ to (1.1)$_j^2$, summing with respect to $j$, integrating the resulting equation over $\Omega$, and using the boundary condition (1.5), one gets after integration by parts that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |\dot{u}|^2 \, dx = \int_{\Omega} (\dot{u} \cdot \nabla F + \dot{u}_j \text{div}(\partial_j F u)) \, dx$$

$$+ \mu \int_{\Omega} (\dot{u} \cdot \nabla \omega + \dot{u}_j \partial_k((\nabla \omega)_{j} u_k)) \, dx$$

$$= \tilde{J}_1 + \tilde{J}_2.$$  

Now, we will estimate $\tilde{J}_1$ and $\tilde{J}_2$ respectively. First, using the boundary condition (1.5), we obtain after integration by parts that

$$\tilde{J}_1 = \int_{\Omega} (\dot{u} \cdot \nabla F + \dot{u}_j \text{div}(\partial_j F u)) \, dx$$

$$= \int_{\Omega} (\dot{u} \cdot \nabla F + \dot{u}_j \partial_j (u \cdot \nabla F) + \dot{u}_j \partial_j F \text{div} u + \dot{u} \cdot \nabla u \cdot \nabla F) \, dx$$

$$= \int_{\partial \Omega} (F_t + u \cdot \nabla F)(\dot{u} \cdot n) \, ds - \int_{\Omega} (F_t + u \cdot \nabla F) \text{div} \dot{u} \, dx$$

$$+ \int_{\Omega} (\dot{u}_j \partial_j F \text{div} u + \dot{u} \cdot \nabla u \cdot \nabla F) \, dx$$

$$= \int_{\partial \Omega} F_t(\dot{u} \cdot n) \, ds + \int_{\partial \Omega} u \cdot \nabla F(\dot{u} \cdot n) \, ds - \int_{\Omega} (2\mu + \lambda)(\text{div} \dot{u})^2 \, dx$$  

(4.8)
\[ + \int_{\Omega} \lambda'(\rho) \rho (\text{div } u)^2 \text{div } \dot{u} \, dx + \int_{\Omega} (2\mu + \lambda) \partial_t u \partial_j u_i \text{div } \dot{u} \, dx + \gamma \int_{\Omega} P \text{div } u \text{div } \dot{u} \, dx + \int_{\Omega} (\dot{u}_j \partial_j F \text{div } u + \dot{u} \cdot \nabla u \cdot \nabla F) \, dx \]

\[ \leq \int_{\partial \Omega} F_t (\dot{u} \cdot n) \, ds + \int_{\partial \Omega} u \cdot \nabla F (\dot{u} \cdot n) \, ds - \mu \int_{\Omega} (\text{div } \dot{u})^2 \, dx \]

\[ + C + C \| \nabla u \|_{L^4}^3 + C \int |\dot{u}| |\nabla F| |\nabla u| \, dx, \]

where in the third equality we have used the following fact

\[ F_t + u \cdot \nabla F = \lambda_t \text{div } u + (2\mu + \lambda) \text{div } u_t + u \cdot \nabla ((2\mu + \lambda) \text{div } u) + P_t + u \cdot \nabla P \]

\[ = (\lambda_t + u \cdot \nabla \lambda) \text{div } u + (2\mu + \lambda) \text{div } \dot{u} - (2\mu + \lambda) \text{div } (u \cdot \nabla u) \]

\[ + (2\mu + \lambda) \nabla \text{div } u + \gamma P \text{div } u \]

\[ = -\rho \lambda' (\rho) (\text{div } u)^2 + (2\mu + \lambda) \text{div } \dot{u} + (2\mu + \lambda) \partial_t u_j \partial_j u_i + \gamma P \text{div } u. \]

For the first term on the righthand side of (4.8), we have

\[ \int_{\partial \Omega} F_t (\dot{u} \cdot n) \, ds \]

\[ = \int_{\partial \Omega} F_t (u \cdot \nabla u \cdot n) \, ds \]

\[ = -\frac{d}{dt} \int_{\partial \Omega} F (u \cdot \nabla n \cdot u) \, ds + \int_{\partial \Omega} F (u \cdot \nabla n \cdot u_t) \, ds \]

\[ = -\frac{d}{dt} \int_{\partial \Omega} F (u \cdot \nabla n \cdot u) \, ds + \int_{\partial \Omega} F (u_t \cdot \nabla n \cdot u) \, ds + \int_{\partial \Omega} F (u \cdot \nabla u_t) \, ds \quad (4.9) \]

\[ = -\frac{d}{dt} \int_{\partial \Omega} F (u \cdot \nabla n \cdot u) \, ds + \left[ \int_{\partial \Omega} F (\dot{u} \cdot \nabla n \cdot u) \, ds + \int_{\partial \Omega} F (u \cdot \nabla u) \, ds \right] \]

\[ - \int_{\partial \Omega} F (u \cdot \nabla u) \cdot \nabla (u \cdot \nabla u) \, ds - \int_{\partial \Omega} F (u \cdot \nabla n \cdot (u \cdot \nabla u)) \, ds \]

\[ = -\frac{d}{dt} \int_{\partial \Omega} F (u \cdot \nabla n \cdot u) \, ds + K_1 + K_2 + K_3. \]

Then, by (3.21), (3.3), (4.1), (3.71) and (3.67), we get

\[ K_1 = \int_{\partial \Omega} F (\dot{u} \cdot \nabla n \cdot u) \, ds + \int_{\partial \Omega} F (u \cdot \nabla n \cdot \dot{u}) \, ds \]

\[ \leq C \| u \|_{H^1} \| \dot{u} \|_{H^1} \| F \|_{H^1} \quad (4.10) \]

\[ \leq C \| \nabla u \|_{L^2} (\| \nabla \dot{u} \|_{L^2} + \| \nabla u \|_{L^2}^2) (\| \sqrt{\rho} \dot{u} \|_{L^2} + \| \nabla u \|_{L^2}) \]

\[ \leq \varepsilon \| \nabla \dot{u} \|_{L^2}^2 + C(\varepsilon) \| \sqrt{\rho} \dot{u} \|_{L^2}^2 + C(\varepsilon). \]

For \( K_2 \), by taking advantage of (4.3), we have

\[ |K_2| = \left| \int_{\partial \Omega} F ((u \cdot \nabla u) \cdot \nabla n \cdot u) \, ds \right| \]
Similarly, for $K_3$, which together with (4.9)–(4.11) leads to

\[
\int_{\partial\Omega} F(\hat{u} \cdot n) \, ds \leq \left| \int_{\partial\Omega} F(u \cdot n) \, ds \right| - \frac{d}{dt} \int_{\partial\Omega} F(u \cdot n \cdot u) \, ds + \bar{\varepsilon} \|\nabla \hat{u}\|^2_{L^2} + C(\varepsilon) \left( \|\sqrt{\rho} \hat{u}\|^2_{L^2} + 1 + \|\nabla u\|^4_{L^4} \right). \tag{4.12}
\]

For the second term on the right-hand side of (4.8),

\[
\int_{\partial\Omega} (u \cdot \nabla F)(\hat{u} \cdot n) \, ds \\
= \int_{\partial\Omega} (u \cdot n^\perp) n^\perp \cdot \nabla F(\hat{u} \cdot n) \, ds \\
= \int_{\Omega} \nabla^\perp \cdot ((u \cdot n^\perp) \nabla F(\hat{u} \cdot n)) \, dx \\
= \int_{\Omega} \nabla F \cdot \nabla^\perp ((u \cdot n^\perp)(\hat{u} \cdot n)) \, dx \tag{4.13}
\]

which implies that the last term of $\tilde{J}_1$ can be also bounded by the right-hand side of (4.13). Putting (4.12) and (4.13) into (4.8) leads to

\[
\tilde{J}_1 \leq -\mu \int_{\Omega} (\text{div} \hat{u})^2 \, dx - \frac{d}{dt} \int_{\partial\Omega} F(u \cdot n \cdot u) \, ds + C \varepsilon \|\nabla \hat{u}\|^2_{L^2} + C(\varepsilon) \left( 1 + \|\sqrt{\rho} \hat{u}\|^2_{L^2} \right) \left( \|\nabla u\|^4_{L^4} + 1 \right). \tag{4.14}
\]
Next, we check that
\[
\begin{align*}
\mathcal{J}_2 &= \mu \int_{\Omega} (\dot{u} \cdot \nabla^\perp \omega_t + \dot{u}_j \partial_k ((\nabla^\perp \omega)_j u_k)) \, dx \\
&= \mu \int_{\Omega} \dot{u} \cdot \nabla^\perp \omega_t \, dx + \mu \int_{\Omega} \dot{u} \cdot \nabla^\perp (\text{div}(u \omega)) \, dx - \mu \int_{\Omega} \dot{u}_j \partial_k ((\nabla^\perp u_k)_j \omega) \, dx \\
&= \mu \int_{\partial \Omega} (\dot{u} \cdot n^\perp) \omega_t \, ds + \mu \int_{\partial \Omega} (\dot{u} \cdot n^\perp) \text{div} u \cdot \omega + (\dot{u} \cdot n^\perp)(u \cdot \nabla \omega) \, ds \\
&\quad - \mu \int_{\Omega} (\text{curl} \, \dot{u})^2 \, dx + \mu \int_{\Omega} \text{curl} \, \dot{u} \text{curl}(u \cdot \nabla u) \, dx \\
&\quad - \mu \int_{\Omega} \text{curl} \, \dot{u} \text{div}(u \omega) \, dx - \mu \int_{\Omega} \dot{u}_j \partial_k ((\nabla^\perp u_k)_j \omega) \, dx.
\end{align*}
\]

For the boundary term, we have
\[
\begin{align*}
\mu \int_{\partial \Omega} (\dot{u} \cdot n^\perp) \omega_t \, ds + \mu \int_{\partial \Omega} (\dot{u} \cdot n^\perp) \text{div} u \cdot \omega + (\dot{u} \cdot n^\perp)(u \cdot \nabla \omega) \, ds \\
&= \mu \int_{\partial \Omega} (\dot{u} \cdot n^\perp)\dot{\omega} \, ds + \frac{\mu}{2\mu + \lambda} \int_{\partial \Omega} (\dot{u} \cdot n^\perp)F(-Au \cdot n^\perp) \, ds \\
&\quad + \frac{\mu}{2\mu + \lambda} \int_{\partial \Omega} (\dot{u} \cdot n^\perp)\rho(-Au \cdot n^\perp) \, ds \\
&\leq -\mu \int_{\partial \Omega} A(\dot{u} \cdot n^\perp)^2 \, ds - \mu \int_{\partial \Omega} (u \cdot \nabla A)(\dot{u} \cdot n^\perp)(u \cdot n^\perp) \, ds \\
&\quad - \mu \int_{\partial \Omega} (\dot{u} \cdot n^\perp)Au \cdot (u \cdot \nabla)n^\perp \, ds + C \int_{\partial \Omega} (|F| + 1) \cdot |\dot{u}| \cdot |u| \, ds \\
&\leq -\mu \int_{\partial \Omega} A(\dot{u} \cdot n^\perp)^2 \, ds + C \int_{\partial \Omega} (|F| + 1)|\dot{u}| \cdot |u| \, ds + C \int_{\partial \Omega} |\dot{u}| \cdot |u|^2 \, ds \\
&\leq -\mu \int_{\partial \Omega} A(\dot{u} \cdot n^\perp)^2 \, ds + C(\|\dot{u}\|_{H^1}^2 + (\|F\|_{H^1} + 1)\|\dot{u}\|_{H^1}^2) \\
&\leq -\mu \int_{\partial \Omega} A(\dot{u} \cdot n^\perp)^2 \, ds + \varepsilon \|\nabla \dot{u}\|_{L^2}^2 + C(\varepsilon)\|\nabla u\|_{L^2}^4 + C(\varepsilon)\|
\]

where we have taken advantage of (4.10) and (4.1) in the last inequality.

The remaining terms are handled via
\[
\begin{align*}
- \mu \int_{\Omega} (\text{curl} \, \dot{u})^2 \, dx + \mu \int_{\Omega} \text{curl} \, \dot{u} \text{curl}(u \cdot \nabla u) \, dx \\
&\quad - \mu \int_{\Omega} \text{curl} \, \dot{u} \text{div}(u \omega) \, dx - \mu \int_{\Omega} \dot{u}_j \partial_k ((\nabla^\perp u_k)_j \omega) \, dx \\
&= -\mu \int_{\Omega} (\text{curl} \, \dot{u})^2 \, dx + \mu \int_{\Omega} \text{curl} \, \dot{u} (\nabla^\perp u)_T : \nabla u \, dx \\
&\quad - \mu \int_{\Omega} \text{curl} \, \dot{u} \, \text{div} u \omega \, dx + \mu \int_{\Omega} \partial_k \dot{u}_j (\nabla^\perp u_k)_j \omega \, dx \\
&\leq -\mu \int_{\Omega} (\text{curl} \, \dot{u})^2 \, dx + \varepsilon \|\nabla \dot{u}\|_{L^2}^2 + C(\varepsilon)\|\nabla u\|_{L^4}^4.
\end{align*}
\]
Putting this and (4.14) into (4.7) gives
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |\dot{u}|^2 \, dx + \mu \int_\Omega \left( (\text{div} \, \dot{u})^2 + (\text{curl} \, \dot{u})^2 \right) \, dx + \mu \int_{\beta \Omega} A(\dot{u} \cdot n^+)^2 \, ds \\
\leq - \frac{d}{dt} \int_{\beta \Omega} F(u \cdot \nabla n \cdot u) \, ds + C \varepsilon \|\nabla \dot{u}\|_{L^2}^2 \\
+ C(\varepsilon) \left( 1 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \right) \left( \|\nabla u\|_{L^4}^4 + 1 \right) \\
\leq - \frac{d}{dt} \int_{\beta \Omega} F(u \cdot \nabla n \cdot u) \, ds + C \varepsilon \|\nabla \dot{u}\|_{L^2}^2 + C(\varepsilon) \left( 1 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \right)^2,
\]
where in the last inequality we have used
\[
\|\nabla u\|_{L^4}^2 \leq C \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C,
\]
due to (3.17) and (3.67). Combining (4.17), (4.2), (4.18), and choosing \(\varepsilon\) suitably small yields
\[
\frac{d}{dt} \int_{\Omega} \rho |\dot{u}|^2 \, dx + C^{-1} \mu \int_{\Omega} |\nabla \dot{u}|^2 \, dx \\
\leq - \frac{d}{dt} \int_{\beta \Omega} 2F(u \cdot \nabla n \cdot u) \, ds + C \left( \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + 1 \right)^2.
\]
Multiplying (4.19) by \(\sigma\), one gets (4.5) after using (3.30), (3.67), and Gronwall’s inequality.

Finally, it follows from (3.17), (3.67), and (4.5) that for \(p \geq 2\),
\[
\sigma \|\nabla u\|_{L^p}^2 \leq C(p) \sigma \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C(p) \leq C(p),
\]
which shows (4.6) and finishes the proof of Lemma 4.2. \(\square\)

**Lemma 4.3.** Assume that (1.8) holds. Then for any \(p > 2\), there is a positive constant \(C\) depending only on \(\Omega\), \(p\), \(T\), \(\mu\), \(\beta\), \(\gamma\), \(\|\rho_0\|_{L^\infty}\), and \(\|u_0\|_{H^1}\) such that
\[
\int_0^T (\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} + \|\sqrt{\rho} \dot{u}\|_{L^p})^{1/p} \, dt + \int_0^T (\|F\|_{L^\infty}^{3/2} + \|\omega\|_{L^\infty}^{3/2}) \, dt \\
+ \int_0^T t (\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} + \|\dot{u}\|_{H^1}^2) \, dt \leq C,
\]
and that
\[
\inf_{(x,t) \in \Omega \times (0,T)} \rho(x,t) \geq C^{-1} \inf_{x \in \Omega} \rho_0(x).
\]

**Proof.** First, it follows from (4.1) and (3.67) that
\[
\|\sqrt{\rho} \dot{u}\|_{L^2}^2 \leq C \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \|\dot{u}\|_{L^2}^{2(p-1)/(p^2-2)} \|\dot{u}\|_{H^1}^{p(p-2)/(p^2-2)} \]
\[
\leq C \|\rho \dot{u}\|_{L^2}^2 + C \|\rho \dot{u}\|_{L^2}^{2(p-1)/(p^2-2)} \|\nabla u\|_{L^2}^{p(p-2)/(p^2-2)},
\]
which, together with (3.67), (4.1), (4.5), and (4.18), implies that
\[ \int_0^T \left( \| \rho \dot{u} \|_{L_p}^{1+1/p} + t \| \dot{u} \|_{H^1}^2 \right) \, dt \leq C. \] (4.22)

Then, the Gagliardo–Nirenberg inequality, (3.20), and (3.67) yield that
\[ \| F \|_{L^\infty} + \| \omega \|_{L^\infty} \leq C \| F \|_{L^2} + C \| \omega \|_{L^2} + C \| F \|_{L^p}^{p-2} \| \nabla F \|_{L^p}^p + C \| \omega \|_{L^p}^{p-2} \| \nabla \omega \|_{L^p}^p \leq C + C \| \rho \dot{u} \|_{L^{p-1}}^{p-1}, \] (4.23)

which together with (4.22) and (3.20) gives (4.20).

Finally, (4.21) is a direct consequence of (4.20) and (1.1). \[ \square \]

Upon now, we have finished the lower order a priori estimates, and will turn to the higher order ones. Since no other boundary terms needed to be controlled, we just follow [12] to derive our final a priori estimates. For the sake of completeness, we sketch the proof here.

**Proposition 4.1.** Assume that (1.8) holds. Then, for \( q > 2 \) as in Theorem 1.1, there is a constant \( \tilde{C} \) depending only on \( \Omega, T, q, \mu, \gamma, \beta, \| u_0 \|_{H^1}, \) and \( \| \rho_0 \|_{W^{1,q}} \) such that
\[ \sup_{0 \leq t \leq T} \left( \| \rho \|_{W^{1,q}} + t \| u \|_{H^2}^2 \right) \leq \tilde{C}. \] (4.24)

**Proof.** We will follow the proof of [12]. First, denoting by \( \Psi = (\Psi_1, \Psi_2) \) with \( \Psi_i \equiv (2\mu + \lambda(\rho))\partial_i \rho \) \((i = 1, 2)\), one deduces from (1.1) that \( \Psi_i \) satisfies
\[ \partial_t \Psi_i + (u \cdot \nabla) \Psi_i + (2\mu + \lambda(\rho))\nabla \rho \cdot \partial_i u + \rho \partial_i F + \rho \partial_i P + \Psi_i \text{div} u = 0. \] (4.25)

For \( q > 2 \), multiplying (4.25) by \( |\Psi|^{q-2} \Psi_i \) and integrating the resulting equation over \( \Omega \), we obtain after integration by parts and using (1.5) to cancel out boundary term that
\[ \frac{d}{dt} \| \Psi \|_{L^q} \leq \tilde{C} \left( 1 + \| \nabla u \|_{L^\infty} \right) \| \Psi \|_{L^q} + \tilde{C} \| \nabla F \|_{L^q}. \] (4.26)

Next, we deduce from standard \( L^p \)-estimate for elliptic system with boundary condition (1.5), (4.23), and (3.20) that
\[ \| \nabla^2 u \|_{L^q} \leq \tilde{C} \left( \| \text{div} u \|_{L^q} + \| \nabla \omega \|_{L^q} \right) \leq \tilde{C} \left( \| \nabla (2\mu + \lambda) \text{div} u \|_{L^q} + \| \text{div} u \|_{L^\infty} \| \nabla \rho \|_{L^q} + \| \nabla \omega \|_{L^q} \right) \leq \tilde{C} \left( \| \text{div} u \|_{L^\infty} + 1 \right) \| \nabla \rho \|_{L^q} + \tilde{C} \| \nabla F \|_{L^q} + \tilde{C} \| \nabla \omega \|_{L^q} \leq \tilde{C} \left( \| \rho \dot{u} \|_{L^q} + 1 \right) \| \nabla \rho \|_{L^q} + \tilde{C} \| \rho \dot{u} \|_{L^q}. \] (4.27)
Then it follows from Lemma 2.4, (4.23), and (4.27) that
\[
\|\nabla u\|_{L^\infty} \leq \tilde{C} (\|\text{div } u\|_{L^\infty} + \|\omega\|_{L^\infty}) \log(e + \|\nabla^2 u\|_{L^q}) + \tilde{C} \|\nabla u\|_{L^2} + \tilde{C}
\]
(4.28)

Noticing that
\[
2\mu \|\nabla \rho\|_{L^q} \leq \|\Psi\|_{L^q} \leq \tilde{C} \|\nabla \rho\|_{L^q},
\]
(4.29)
substituting (4.28) into (4.26), and using (3.20), one gets
\[
\frac{d}{dt} \log(e + \|\Psi\|_{L^q}) \leq \tilde{C} \left(1 + \|\dot{\rho} u\|_{L^q}\right) \log(e + \|\Psi\|_{L^q}) + \tilde{C} \|\dot{\rho} u\|_{L^q}^{1+1/q},
\]
which together with Gronwall’s inequality, (3.20), (4.20), and (4.29) yields that
\[
\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^q} \leq \tilde{C}.
\]
(4.30)
Combining (4.27), (4.30), and (4.20) gives
\[
\int_0^T \left(\|\nabla^2 u\|_{L^q}^{1+1/q} + t \|\nabla^2 u\|_{L^q}^2\right) dt \leq \tilde{C}.
\]
(4.31)
Finally, it follows from (4.1), (4.2), (4.18), (3.67), (4.5), and (4.31) that
\[
\int_0^T t \|u_t\|_{H^1}^2 dt \leq \tilde{C} \int_0^T t \left(\|\dot{u}\|_{H^1}^2 + \|u \cdot \nabla u\|_{H^1}^2\right) dt
\[
\leq \tilde{C} \int_0^T t \left(1 + \|\nabla u\|_{L^q}^2 + \|\nabla u\|_{L^4}^4 + \|u\|_{H^1}^2 \|\nabla u\|_{L^q}^2\right) dt
\[
\leq \tilde{C}.
\]
(4.32)
We obtain from (3.67), (3.20), and (4.30) that
\[
\|\nabla^2 u\|_{L^2} \leq \tilde{C} (\|\nabla \text{div } u\|_{L^2} + \|\nabla \omega\|_{L^2})
\leq \tilde{C} (\|\nabla (2\mu + \lambda) \text{div } u\|_{L^2} + \|\text{div } u\|_{L^{2q/(q-2)}} \|\nabla \rho\|_{L^q} + \|\nabla \omega\|_{L^2})
\leq \tilde{C} \|\nabla F\|_{L^2} + \tilde{C} \|\nabla \omega\|_{L^2} + \tilde{C} + \tilde{C} \|\nabla u\|_{L^2}^{(q-2)/q} \|\nabla^2 u\|_{L^2}^{2/q}
\leq \frac{1}{2} \|\nabla^2 u\|_{L^2} + \tilde{C} \|\rho \dot{u}\|_{L^2} + \tilde{C}
\]
which, together with (4.5), gives
\[
\sup_{0 \leq t \leq T} t \|\nabla^2 u\|_{L^2}^2 \leq \tilde{C}.
\]
Combining this, (4.30)–(4.32), and (3.67) yields (4.24) and finishes the proof of Proposition 4.1.
5. Proofs of Theorems 1.1 and 1.2

After all preparation done, the proof of Theorem 1.1 is in the same way as that of [12, Theorem 1.1] after some slight modifications.

We first state the global existence of strong solution \((\rho, u)\) provided that (1.8) holds and that \((\rho_0, m_0)\) satisfies (2.1) whose proof is similar to that of [12, Proposition 5.1] after some slight modifications.

**Proposition 5.1.** Assume that (1.8) holds and that \((\rho_0, m_0)\) satisfies (2.1). Then there exists a unique strong solution \((\rho, u)\) to (1.1)–(1.5) in \(\Omega \times (0, \infty)\) satisfying (2.2) for any \(T \in (0, \infty)\). In addition, for any \(q > 2\), \((\rho, u)\) satisfies (4.24) with some positive constant \(C\) depending only on \(\Omega, T, q, \mu, \gamma, \beta, \|u_0\|_{H^1}\), and \(\|\rho_0\|_{W^{1,q}}\).

5.1. Proof of Theorem 1.1

Let \((\rho_0, m_0)\) satisfying (1.9) be the initial data as described in Theorem 1.1. We construct a sequence of \(\mathcal{C}^\infty\) initial value \((\tilde{\rho}_0^\delta, \tilde{u}_0^\delta)\), and \(\tilde{u}_0^\delta\) should satisfy slip boundary condition. Using the standard approximation theory to \((\tilde{\rho}_0^\delta, \tilde{u}_0^\delta)\) (see [35] for example), we can find a sequence of \(\mathcal{C}^\infty\) functions \((\tilde{\rho}_0^\delta, \tilde{u}_0^\delta)\) satisfying

\[
\lim_{\delta \to 0} \|\tilde{\rho}_0^\delta - \rho_0\|_{W^{1,q}} + \|\tilde{u}_0^\delta - u_0\|_{H^1} = 0. \tag{5.1}
\]

However, \(\tilde{u}_0^\delta\) may violate the slip boundary condition, we must go further.

Let \(u_0^\delta\) be the unique smooth solution of elliptic system

\[
\begin{cases}
\Delta u_0^\delta = \Delta \tilde{u}_0^\delta & \text{in } \Omega, \\
u_0^\delta \cdot n = 0, \quad \text{curl } u_0^\delta = -Au_0^\delta \cdot n^\perp & \text{on } \partial\Omega.
\end{cases} \tag{5.2}
\]

We define \(\rho_0^\delta = \tilde{\rho}_0^\delta + \delta\), and set \(m_0^\delta = \rho_0^\delta \cdot u_0^\delta\). Then, it is easy to check that

\[
\lim_{\delta \to 0} (\|\rho_0^\delta - \rho_0\|_{W^{1,q}} + \|u_0^\delta - u_0\|_{H^1}) = 0.
\]

After applying Proposition 5.1, we can construct a unique global strong solution \((\rho^\delta, u^\delta)\) with initial value \((\rho_0^\delta, u_0^\delta)\). Such solution satisfy (4.24) for any \(T > 0\) and for some \(C\) independent with \(\delta\).

Letting \(\delta \to 0\), standard compactness assertions (see [11,36,37]) make sure the problem (1.1)–(1.5) has a global strong solution \((\rho, u)\) satisfying the properties listed in Theorem 1.1. Note that the uniqueness can be obtained via similar method in Germain [38].
5.2. Proof of Theorem 1.2

With all a priori estimations and Theorem 1.1 done, to deduce the existence of weak solution, we may follow the proof (especially the compactness assertions) in [12]. We just sketch the proof here. For more details see [12].

**Proof.** Let \((\rho_0, u_0)\) be the initial data as in Theorem 1.2, we construct an approximation initial value \((\rho_0^\delta, u_0^\delta)\) in the same manner as (5.1) and (5.2), but this time we have for any \(p \geq 1\),
\[
\lim_{\delta \to 0} (\|\rho_0^\delta - \rho_0\|_{L^p} + \|u_0^\delta - u_0\|_{H^1}) = 0.
\]
Moreover,
\[
\rho_0^\delta \to \rho_0 \text{ in } W^* \text{ topology of } L^\infty \text{ as } \delta \to 0.
\]
We apply Proposition 5.1 to deduce the existence of unique global strong solution \((\rho^\delta, u^\delta)\) of problem (1.1)–(1.5) with initial value \((\rho_0^\delta, u_0^\delta)\) which satisfy (3.67), (4.5), (4.6) and (4.20) for any \(T > 0\). The constants involved are all independent of \(\delta\).

From (3.67) and (4.20), we argue that
\[
\sup_{0 \leq t \leq T} \|u^\delta\|_{H^1} + \int_0^T t\|u^\delta_t\|_{L^2}^2 \, dt \leq C.
\]
We apply Aubin–Lions Lemma to get a subsequence that
\[
u^\delta \rightharpoonup u \text{ weakly * in } L^{4/3}(0, T; L^\infty),
\]
\[
u^\delta \rightarrow u \text{ in } C([\tau, T]; L^p),
\]
for any \(\tau \in (0, T)\) and \(p \geq 1\).

Let \(F^\delta = (2\mu + (\rho^\delta)^\beta) \text{div } u^\delta + P(\rho^\delta)\) be the effect viscous flux with respect to approximation solution \((\rho^\delta, u^\delta)\). Up on applying (3.67) and (4.20) we have
\[
\int_0^T \left( \|F^\delta\|_{L^\infty}^{4/3} + \|\omega^\delta\|_{H^1}^2 + \|F^\delta\|_{H^1}^2 + t\|\omega^\delta_t\|_{L^2}^2 + t\|F^\delta_t\|_{L^2}^2 \right) \, dt \leq C.
\]
Once again, Aubin–Lions Lemma yields
\[
F^\delta \rightarrow F \text{ weakly * in } L^{3/2}(0, T; L^\infty),
\]
\[
\omega^\delta \rightarrow \omega, \quad F^\delta \rightarrow F \text{ in } L^2(\tau, T; L^p),
\]
for any \(\tau \in (0, T)\) and \(p \geq 1\).

Now we are in the same position as that of [12, (5.13)], and we may follow the assertions in [12] to deduce the strong convergence of \(\rho^\delta\), say
\[
\rho^\delta \rightarrow \rho \text{ in } C([0, T]; L^p(\Omega)),
\]
for any \(p \geq 1\). Combining this with (5.3), we argue that
\[
F^\delta \rightarrow F \text{ in } L^2((\tau, T) \times \Omega)
\]
for any \(\tau \in (0, T)\). And (1.11) follows directly. We conclude that \((\rho, u)\) is exactly the desired weak solution in Theorem 1.2 which finishes our proof. \(\Box\)
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Declarations

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