MATHEMATICAL ANALYSIS OF A PDE MODEL 
DESCRIBING CHEMOTACTIC E. COLI COLONIES 

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ABSTRACT. We consider an initial-boundary value problem describing the formation 
of colony patterns of bacteria Escherichia coli. This model consists of reaction-diffusion 
equations coupled with the Keller-Segel system from the chemotaxis theory in a bounded 
domain, supplemented with zero-flux boundary conditions and with non-negative initial 
data. We answer questions on the global in time existence of solutions as well as on their 
large time behaviour. Moreover, we show that solutions of a related model may blow up 
in a finite time. 

1. INTRODUCTION 

Budrene and Berg [4, 5] performed experiments showing that chemotactic strains of 
bacterias E. coli, inoculated in semi-solid agar, form stable and remarkably complex but 
geometrically regulated spatial patterns such as swarm rings, radial spots, interdigitated 
arrays of spots and rather complex chevron-like patterns as shown in Fig. 1. They sug-
gested that such colonial patterns depend on an initial concentration of nutrient (sub-
strate) which determines how long multicellular aggregate structures remain active. They 
expected that four elements such as substrate consumption, cell proliferation, excretion of 
attractant, and chemotactic motility, when they are suitably combined, can generate com-
plex spatial structures in a self-organized way and that a specialized and more complex 
morphogenetic program is not required. However, this hypothesis does not necessarily 
imply that such complex patterns occur as a consequence of self-organization. 

It is a challenging problem in the field of mathematical biology to understand self- 
organization and, in particular, the influence of chemotaxis on the occurrence of colonial 
patterns. Mimura and Tsujikawa [15] first proposed the following macro model based on 
the chemotaxis and growth of bacteria: 

\begin{align} 
\dot{u} = d_u \Delta u - \nabla \cdot (u \nabla \chi(c)) + f(u) \\
\dot{c} = d_c \Delta c + \alpha u - \beta c, 
\end{align} 

(1.1) 

where \( u = u(x, t) \) denotes the density of cells and \( c = c(x, t) \) is a concentration of chemoat-
tractant. The letters \( d_u, d_c, \alpha, \) and \( \beta \) denote positive constants, \( \chi \) is the sensitivity 
function of chemotaxis and \( f(u) \) is a growth function with an Allee effect. Note that, in 
the absence of the function \( f(u) \), this system reduces to the famous Keller–Segel equa-
tions [14]. The authors of [15] studied the influence of the form of \( \chi(c) \) on the occurrence

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An apparent difference from experiments in [4, 5] consists in the handling of the nutrient: in experiments, the nutrient is only supplied initially. In order to deal with this phenomenon, a new unknown variable \( n(t, x) \) describing the concentration of nutrient is included in the modeling which leads to the system of three equations

\[
\begin{align*}
    u_t &= d_u \Delta u - \nabla \cdot (u \nabla \chi(c)) + g(u)nu, \\
    c_t &= d_c \Delta c + \alpha u - \beta c, \\
    n_t &= d_n \Delta n - \gamma g(u)nu.
\end{align*}
\]

In fact, this approach appears in other models which are basically similar to the one in (1.2), see e.g. Tsimring et al. [23], Tyson et al. [24], Shigesada and Kawasaki [21], and Polezhaev et al. [19]. The authors of these works suggest that the chemotactic effect generates spotty patterns which are a consequence of a chemotaxis-induced instability. However, they have not yet shown that such models generate geometrically regulated patterns which are observed in experiments when an initial nutrient is changed.

For this reason, Mimura and his collaborators [2] proposed a new system of differential equations in which two internal states of bacteria are introduced: active and less-active ones, where the density of active bacteria is denoted by \( u(x, t) \), the density of inactive bacteria by \( w(x, t) \), the density of nutrient by \( n(x, t) \), and the concentration of chemoattractant by \( c(x, t) \), with \( x \in \Omega \) and \( t \in [0, \infty) \); the new diffusion-chemotaxis-growth system has the form

\[
\begin{align*}
    u_t &= \Delta u - \nabla \cdot (u \nabla \chi(c)) + g(u)nu - b(n)u, \\
    c_t &= d_c \Delta c + \alpha u - \beta c, \\
    n_t &= d_n \Delta n - \gamma g(u)nu, \\
    w_t &= b(n)u.
\end{align*}
\]

In the beginning of the next section, we formulate assumptions which are imposed on given parameters and functions in equations (1.3). Here, we only remark that the first three equations are closed for \( u, c \) and \( n \) and that if these are solved, then \( w \) can be
obtained from $u$ and $n$ by the formula

$$w(x, t) = \int_0^t b(n(x, \tau))u(x, \tau) d\tau,$$

where the initial condition $w(x, 0) = 0$ is required from the experiments. The resulting colonial pattern is represented by the total density of $u(x, t) + w(x, t)$.

Numerical simulations presented on Fig. 2 show that, for certain specific functions $g$, $b$ and $\chi$, system (1.3) describes a formation of chevron-like patterns which are closely related to those observed in biological experiments. Moreover, it is especially remarkable that such systems generate geometrically different patterns depending only on the initial nutrient concentration, as it was done in real-life experiments performed by Budrene and Berg [4, 5].

The aim of this paper is to discuss the fundamental property on system (1.3) from mathematical viewpoint. First, we recall the article [12] with proofs that solutions of the one-dimensional initial-boundary value problem for system (1.3) exist for all $t > 0$ and are bounded uniformly in $t > 0$. Moreover, for some specific functions $g$, $b$ and $\chi$, the authors of [12] found an asymptotic profile of such solutions when $t \to \infty$.

In this paper, in Theorem 2.3, we first generalize one-dimensional results from [12] and then, we show an analogous theorem on the global-in-time existence and on the large time behaviour of solutions in the space dimensions two and three, under suitable smallness assumptions on the initial conditions, see Theorem 2.4 below. In fact, such a smallness condition is not needed for a large class of such equations as it is stated in Theorem 2.7.

We also show that sufficiently well-concentrated solutions of a suitable modification of system (1.3) may blow up in finite time, see Theorem 2.10 below for more details.

**Notation.** In the sequel, the usual norm of the Lebesgue space $L^p(\Omega)$ with respect to the spatial variable is denoted by $\| \cdot \|_p$ for all $p \in [1, \infty]$ and $W^{k,p}(\Omega)$ is the corresponding Sobolev space with its usual norm defined by

$$\|u\|_{W^{k,p}} = \left( \sum_{|\alpha| \leq k} \| D^\alpha u \|_p \right)^{\frac 1 p}$$

if $p < +\infty$ and

$$\|u\|_{W^{k,\infty}} = \max_{|\alpha| \leq k} \| D^\alpha u \|_\infty.$$  

We say that a function, say $f = f(t)$, decays exponentially, if there exist constants $\mu > 0$ and $C > 0$ such that $|f(t)| \leq Ce^{-\mu t}$ for all $t > 0$. The letter $C$ corresponds to a generic constant (always independent of $x$ and $t$) which may vary from line to line. Sometimes, we write, $C = C(\alpha, \beta, \gamma, ...)$ when we want to emphasise the dependence of $C$ on parameters $\alpha, \beta, \gamma, ...$. Moreover, for simplicity, we sometimes avoid to explicitly show the dependence of the solution either on $x$ or on $t$. 
Figure 2. Numerical simulation of the functions $u$ and $u+w$ which satisfy an initial-boundary value problem for system (1.3) with $d_c = 10, d_n = 2, \alpha = \beta = \gamma = 1, g(u) = \frac{1}{2}(1 + \tanh(100(u - 0.05))), \chi(c) = \frac{\chi_0 c^2}{c^2 + 0.0025}$ with $\chi_0 = 0.053, b(n) = 0.05$, and supplemented with suitable initial conditions.
2. Results and comments

In this work, we prove results on the existence and the large time behaviour of solutions to the system
\[
\begin{align*}
    u_t &= \Delta u - \nabla \cdot (u \nabla \chi(c)) + g(u)nu - b(n)u, \\
    c_t &= d_c \Delta c + \alpha u - \beta c, \\
    n_t &= d_n \Delta n - \gamma g(u)nu, \\
    w_t &= b(n)u,
\end{align*}
\]
considered in a bounded domain \( \Omega \subset \mathbb{R}^d \) with a smooth boundary \( \partial \Omega \). We supplement these equations with the Neumann boundary conditions
\[
\frac{\partial u}{\partial \nu} = \frac{\partial c}{\partial \nu} = \frac{\partial n}{\partial \nu} = 0 \quad \text{for} \quad x \in \partial \Omega \quad \text{and} \quad t > 0
\]
as well as with non-negative initial data
\[
u(x,0) = u_0(x), \quad c(x,0) = c_0(x), \quad n(x,0) = n_0(x), \quad w(x,0) = w_0(x) \quad \text{for} \quad x \in \Omega.
\]
Here, we impose the following assumptions on coefficients and functions which appear in equations (2.1)–(2.6).

Assumptions 2.1. The diffusion coefficients \( d_c > 0 \) and \( d_n > 0 \) as well as the coefficients \( \alpha > 0, \beta > 0, \gamma > 0 \) in equations (2.2)–(2.3) denote given constants. Moreover, for the functions \( g, b \in C^1([0, \infty)) \) and \( \chi \in C^2([0, \infty)) \), we assume
\begin{enumerate}
    \item \( g(0) = 0 \) and \( g = g(s) \) is increasing for \( s > 0 \) and bounded with \( G_0 = \sup_{s \geq 0} g(s) \);
    \item \( b(0) = B_0 > 0 \) and \( b = b(s) > 0 \) is decreasing for \( s > 0 \);
    \item \( \chi', \chi'' \in L^\infty([0, \infty)) \).
\end{enumerate}

We begin our analysis by showing that problem (2.1)–(2.6) has a unique local-in-time solution for all sufficiently regular initial conditions. Moreover, this solution is non-negative if initial conditions (2.6) are non-negative. This is more-or-less standard reasoning which we recall in Section 3. In this work, we focus mainly on the behaviour of non-negative solutions to problem (2.1)–(2.6) for large values of time. First, we consider spatially homogeneous non-negative solutions. Notice that if initial conditions (2.6) are independent of \( x \), namely, if
\[
u(x) = \bar{u}_0, \quad c(x) = \bar{c}_0, \quad n(x) = \bar{n}_0, \quad w(x) = \bar{w}_0,
\]
for certain constants \( \bar{u}_0, \bar{c}_0, \bar{n}_0, \bar{w}_0 \in [0, \infty) \), then the corresponding solution of problem (2.1)–(2.6) is also independent of \( x \), which is an immediate consequence of the uniqueness of solution. The following theorem describes the large time behaviour of such non-negative, space homogeneous solutions.

Theorem 2.2. Let Assumptions 2.1 be satisfied. For every non-negative, constant initial condition (2.7), the corresponding solution \( (\bar{u}(t), \bar{c}(t), \bar{n}(t), \bar{w}(t)) \) to problem (2.1)–(2.6) is \( x \)-independent and satisfies a system of ordinary differential equations (see equations (4.1)–(4.4), below). Each such solutions is non-negative, global-in-time, and converges exponentially towards the constant vector \((0,0,\bar{n}_\infty,\bar{w}_\infty)\) for some \( \bar{n}_\infty \geq 0 \) and \( \bar{w}_\infty \geq 0 \) depending on initial conditions.
This theorem is proved in Section 4 by analyzing the phase portrait of the corresponding system of ordinary differential equations, see (4.1)–(4.4) below. In Section 4, we also study the large time behaviour of “mass” of space inhomogeneous solutions and we show in Theorem 4.1 below that the vector
\[
\left( \int_\Omega u(x,t) \, dx, \int_\Omega c(x,t) \, dx, \int_\Omega n(x,t) \, dx, \int_\Omega w(x,t) \, dx \right)
\]
behaves for large values of time analogously as space homogeneous solutions.

Next, we consider problem (2.1)–(2.6) in the one dimensional case and we show that all solutions corresponding to sufficiently regular, non-negative initial conditions are global-in-time and converge uniformly towards certain steady states. This result has been already proved in [12] for problem (2.1)–(2.6) with particular functions \(g, b\) and \(\chi\). Here, however, we propose a different approach which allows us to consider more general nonlinearities.

**Theorem 2.3.** Assume that \(d = 1\) and \(\Omega \subset \mathbb{R}\) is an open and bounded interval. Let the constants \(\alpha, \beta, \gamma\) and the functions \(g, b\) and \(\chi\) satisfy Assumptions 2.1. For every non-negative initial condition \(u_0, n_0, w_0, c_0\) \(L^\infty(\Omega)\) and \(c_0 \in W^{1,\infty}(\Omega)\), the corresponding solution \((u,c,n,w)\) to problem (2.1)–(2.6) exists for all \(t > 0\) and is non-negative. Moreover, there exists a constant \(n_\infty \geq 0\) and a non-negative function \(w_\infty \in L^\infty(\Omega)\) such that
\[
(u(x,t), c(x,t), n(x,t), w(x,t)) \xrightarrow{t \to \infty} (0, 0, n_\infty, w_\infty(x))
\]
exponentially in \(L^\infty(\Omega)\).

Analogous results hold true in higher dimensions under a smallness assumption on initial conditions.

**Theorem 2.4.** Let \(d \in \{2, 3\}\) and \((u,c,n,w)\) be a non-negative local-in-time solution to problem (2.1)–(2.6) with the parameters satisfying Assumptions 2.1. Let \(p_0 \in \left(\frac{d}{2}, \frac{d}{d-2}\right)\). There exists \(\varepsilon(p_0) > 0\) such that if
\[
\max(\|u_0\|_{p_0}, \|n_0\|_1, \|\nabla c_0\|_{2p_0}) < \varepsilon(p_0),
\]
then the solution \((u,c,n,w)\) exists for all \(t > 0\) and satisfies
\[
\sup_{t > 0} \|u(t)\|_\infty < \infty.
\]
Moreover, there exist a constant \(n_\infty \geq 0\) and a non-negative function \(w_\infty \in L^\infty(\Omega)\) such that
\[
(u(x,t), c(x,t), n(x,t), w(x,t)) \xrightarrow{t \to \infty} (0, 0, n_\infty, w_\infty(x))
\]
exponentially in \(L^\infty(\Omega)\).

**Remark 2.5.** Because of methods used in the proof of Theorem 2.4, we have to limit ourselves to the dimension \(d \in \{2, 3\}\) (note that the interval \(\left(\frac{d}{2}, \frac{d}{d-2}\right)\) is nonempty only in this case). Obviously, this is not a real constraint from a point of view of applications. In Remark 6.1 below, we explain how to show an analogous result for \(d > 3\).

**Remark 2.6.** Note also that the lower bound \(\frac{d}{2}\) in the interval \(\left(\frac{d}{2}, \frac{d}{d-2}\right)\) corresponds with the exponent of the critical space \(L^\frac{d}{2}\) for the reduced Keller-Segel system (see [6], [7]). This fact suggests that our result cannot be improved.
There is an immediate question if the smallness assumption in Theorem 2.4 is indeed necessary to show both results: the global-in-time existence of non-negative solutions to problem (2.1)–(2.6) and their exponential convergence toward steady states as in (2.8). In fact, these assumptions can be relaxed in the case of chemotactic sensitivities $\chi$ with a suitable behaviour for large values of $c$. The following result is a consequence of our reasoning from the proof of Theorem 2.4 combined with estimates from [26].

**Theorem 2.7.** Let $d \in \{2, 3\}$ and $(u, c, n, w)$ be a non-negative local-in-time solution to problem (2.1)–(2.6) with the parameters satisfying Assumptions 2.1. Assume, moreover, that chemotactic sensitivity function satisfies

\[ 0 \leq \frac{d\chi(s)}{ds} \leq \frac{\chi_0}{(1 + ps)^2} \quad \text{for all } s \geq 0 \]  

with some constants $\chi_0 > 0$, and $p > 0$. There exists a constant $K > 0$ independent of initial conditions such that if $\chi_0/p < K$, then the solution $(u, c, n, w)$ exists globally-in-time. Moreover, there exist a constant $n_\infty \geq 0$ and a non-negative function $w_\infty \in L^\infty(\Omega)$ such that the exponential convergence in the norm of $L^\infty(\Omega)$ stated in expression (2.8) holds true.

We prove Theorem 2.7 at the end of Section 6.

**Remark 2.8 (Pattern formation).** We have proved in theorems stated above that there exists an asymptotic inactive bacteria configuration $w_\infty \in L^\infty(\Omega)$ such that

\[ \lim_{t \to \infty} (u(x, t) + w_\infty(x, t)) = w_\infty(x) \]

which can be regarded as a formation of a colonial pattern. Unfortunately, our results do not give any information about the shape of the spatial profile $w_\infty(x)$, as e.g. presented on Fig. 2.

Before we formulate the last results of this work, let us conjecture that a certain decay assumption on the chemotactic sensitivity, such as the one in (2.9), seems to be necessary to show the global-in-time existence of solutions to model (2.1)–(2.6). We base this conjecture on an observation that an analogous phenomenon appears in the Keller-Segel model of chemotaxis, where some solutions may blow up in a finite time, see e.g. [13, 17]. In the following, we use an idea, which is well-known in the study of the Keller-Segel model and we show that solutions to a modified system (2.1)–(2.6) in two dimensions and where the parabolic equation for $c = c(x, t)$ is replaced by its elliptic counterpart may blow up in finite time. Thus, we focus on the following system

\[ u_t = \Delta u - \nabla \cdot (u\nabla \chi(c)) + g(u)nu - b(n)u, \]

\[ 0 = \Delta c + \alpha u - \beta c, \]

\[ n_t = d_n \Delta n - \gamma g(u)nu, \]

in a bounded domain $\Omega \subset \mathbb{R}^d$, supplemented with the Neumann boundary conditions

\[ \frac{\partial u}{\partial \nu} = \frac{\partial c}{\partial \nu} = \frac{\partial n}{\partial \nu} = 0 \quad \text{for } x \in \partial \Omega \quad \text{and } t > 0, \]

and with non-negative initial data

\[ u(x, 0) = u_0(x), \quad n(x, 0) = n_0(x). \]
We construct smooth solutions for initial data with Hölder regularity. This is also a more or less standard procedure (see for example [11] and especially [22]) we recall in section 3.

Then, we quickly get back some of the previously obtained results for this system, namely:

**Theorem 2.9.** Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain with smooth boundary. Let the constants $\alpha$, $\beta$, $\gamma$ and the functions $g,b$ and $\chi$ satisfy Assumptions 2.1. Assume furthermore that $\chi$ is $C^3$ and $\chi'' \in L^\infty((0, +\infty))$. For every non-negative initial data $(u_0, n_0) \in C^{2, \eta}$, $\eta \in (0,1)$ the corresponding solution $(u,c,n)$ of (2.10)–(2.14) constructed in theorem 3.5 satisfies the following properties.

- If $d = 1$ and $\Omega$ is convex, the solution is global-in-time and there exists a constant $n_\infty \geq 0$ such that
  \[ (u(x,t), c(x,t), n(x,t)) \xrightarrow{t\to \infty} (0,0,n_\infty) \]
  exponentially in $L^\infty(\Omega)$.
- If $d \in \{2,3\}$, for all $p_0 \in (\frac{d}{d-2}, \frac{d}{d-1})$, there exists $\varepsilon(p_0) > 0$ such that if
  \[ \max(\|u_0\|_{p_0} , \|n_0\|_1) < \varepsilon(p_0), \]
  then the solution $(u,c,n)$ exists for all $t > 0$ and satisfies
  \[ \sup_{t > 0} \|u(t)\|_\infty < \infty. \]

Moreover, there exists a constant $n_\infty \geq 0$ such that
\[ (u(x,t), c(x,t), n(x,t)) \xrightarrow{t \to \infty} (0,0,n_\infty) \]
exponentially in $L^\infty(\Omega)$.

In the following theorem, we show that some of those solutions cannot be extended for all $t > 0$.

**Theorem 2.10.** Let $d = 2$ and $\chi(c) = \chi_0 c$ with $\chi_0 > 0$. Let the functions $g = g(u)$, $b = b(n)$ and the constant $\gamma$ satisfy Assumptions 2.1. Assume that $\chi_0 > 0$ is constant. For every non-negative initial data $(u_0, n_0) \in C^{2, \eta}$, $\eta \in (0,1)$ satisfying the Neumann boundary condition, consider the corresponding solution $(u, c, n)$ of (2.10)–(2.14) constructed in theorem 3.5 and assume that
\[ M_0 = \int_\Omega u_0(x) \, dx > \frac{8\pi}{\alpha \chi_0}. \]
For every $q \in \Omega$ there exists $\varepsilon(q) > 0$ such that if
\[ \int_\Omega u_0(x) |x-q|^2 \, dx < \varepsilon(q) \]
then the solution $(u,c,n)$ cannot be extended to a global one and satisfies
\[ \limsup_{t \to T_{\text{max}}} \|u(t)\|_\infty = +\infty. \]

This theorem is proved in Section 7. Here, let us only emphasize that comparing with problem (2.1)–(2.6), we now consider a linear function $\chi(c) = \chi_0 c$ with a constant $\chi_0 > 0$. Moreover, we set $d_c = d_n = 1$ and we skip the equation for the function $w(x,t)$, because these constants and the function $w(x,t)$ do not play any role in our reasoning.
3. Local existence of non-negative solutions

3.1. Parabolic-parabolic-parabolic problem.

Let us first show that the initial-boundary value problem (2.1)–(2.6) has a local-in-time, unique, non-negative and regular solution. This is more-or-less standard reasoning and we sketch it only. As a standard practice, local-in-time solutions may be obtained via the Banach fixed point argument applied to the Duhamel formulation of problem (2.1)–(2.6) as the following integral equations

\[ u(t) = e^{\Delta t} u_0 + \int_0^t \nabla \cdot e^{\Delta (t-s)} u(s) \nabla \chi(c(s)) \, ds \]

(3.1)

\[ c(t) = e^{(d,\Delta - \beta)t} c_0 + \alpha \int_0^t e^{(d,\Delta - \beta)(t-s)} u(s) \, ds, \]

(3.2)

\[ n(t) = e^{d_2 \Delta t} n_0 - \gamma \int_0^t e^{d_2 \Delta (t-s)} g(u(s)) n(s) u(s) \, ds. \]

(3.3)

Here, the symbol \( \{ e^{\Delta t} \}_{t \geq 0} \) denotes the semigroup of linear operators on \( L^p(\Omega) \) generated by Laplacian with the Neumann boundary conditions. Below, in Lemma A.1 from Appendix, we recall the usual \( L^p - L^q \) estimates of this semigroup.

The following result can be generalized to several systems of semilinear parabolic equations as it was done e.g. in the classical monographs by Henry [10], Rothe [20] and, more recently, by Yagi [25]. We use the ideas developed by Horstmann and Winkler in [11] for a similar system.

**Theorem 3.1** (Local-in-time solutions). Let Assumptions 2.1 hold. Let \( p \in (d, +\infty) \) For every non-negative initial data satisfying \((u_0, c_0, n_0, w_0) \in C(\Omega) \times W^{1,p}(\Omega) \times C(\Omega) \times C(\overline{\Omega})\) there exists a unique maximal time of existence \( T_{\text{max}} \in (0, +\infty] \) such that for all \( T \in (0, T_{\text{max}}) \), problem (2.1)–(2.6) has a unique solution \((u, c, n, w)\) such that

\[ u, c, n, w \in C(0 \leq t < 0, T_{\text{max}})) \times C^{2,1}(0 \times (0, T_{\text{max}})). \]

Moreover, if \( T_{\text{max}} < +\infty \), then

\[ \limsup_{t \to T_{\text{max}}} (\|u(t)\|_\infty + \|c(t)\|_{W^{1,p}}) = +\infty. \]

**Proof.** We deal with the first three equations in problem (2.1)–(2.6) which are formally written as the system of integral equations (3.1)–(3.3). For fixed \( R > 0 \), we define the ball

\[ B_{T,R}(0) = \{(u, c, n) \in C(0 \leq t \leq T, C(\Omega)) \times L^\infty([0, T), W^{1,p}(\Omega)) \times C([0, T], C(\overline{\Omega})) : \quad \sup_{0 \leq t \leq T} \|u(t)\|_\infty \leq R, \quad \sup_{0 \leq t \leq T} \|c(t)\|_{W^{1,p}} \leq R, \quad \sup_{0 \leq t \leq T} \|n(t)\|_\infty \leq R\}. \]

endowed with the distance

\[ d((u, c, n), (\tilde{u}, \tilde{c}, \tilde{n})) = \max(\sup_{0 \leq t \leq T} \|u(t) - \tilde{u}(t)\|_\infty, \sup_{0 \leq t \leq T} \|c(t) - \tilde{c}(t)\|_{W^{1,p}}, \sup_{0 \leq t \leq T} \|n(t) - \tilde{n}(t)\|_\infty), \]

making it a complete metric space.
Given \((u_0, c_0, n_0) \in C(\bar{\Omega}) \times W^{1,p}(\Omega) \times C(\bar{\Omega})\), we define the nonlinear operator \(\mathcal{F} = \mathcal{F}(u, c, n) : B_{T,R}(0) \rightarrow B_{T,R}(0)\) by the right-hand side of equations (3.1)–(3.3).

Let \((u, c, n), (\tilde{u}, \tilde{c}, \tilde{n}) \in B_{T,R}(0)\), we have

\[
d(\mathcal{F}(u, c, n), \mathcal{F}(\tilde{u}, \tilde{c}, \tilde{n})) = \max(\sup_{0 \leq t \leq T} A_1(t), \sup_{0 \leq t \leq T} A_2(t), \sup_{0 \leq t \leq T} A_3(t))
\]

with

\[
A_1(t) = \left\| \int_0^t \nabla \cdot e^{\Delta(t-s)} \left( u(s) \nabla \chi(c(s)) - \tilde{u}(s) \nabla \chi(\tilde{c}(s)) \right) ds \right\|_\infty,
\]

\[
A_2(t) = \left\| \alpha \int_0^t e^{(d+\beta)(t-s)} \left( u(s) - \tilde{u}(s) \right) ds \right\|_{W^{1,p}};
\]

\[
A_3(t) = \left\| \gamma \int_0^t e^{d\Delta(t-s)} \left( g(u(s)) n(s) u(s) - g(\tilde{u}(s)) \tilde{n}(s) \tilde{u}(s) \right) ds \right\|_\infty.
\]

Then, we have the usual estimates

\[
\left\| u \nabla \chi(c) - \tilde{u} \nabla \chi(\tilde{c}) \right\|_p \leq \left\| u - \tilde{u} \right\|_\infty \left\| \nabla c \right\|_p \sup_{|c| \leq R} |\chi'(c)|
\]

\[
+ \left\| c - \tilde{c} \right\|_\infty \left\| \nabla c \right\|_p \left\| \tilde{u} \right\|_\infty \sup_{|c| \leq R} |\chi''(c)|
\]

\[
+ \left\| \nabla c - \nabla \tilde{c} \right\|_p \left\| \tilde{u} \right\|_\infty \sup_{|c| \leq R} |\chi'(c)|,
\]

(3.5)

\[
\left\| b(n) u - b(\tilde{n}) \tilde{u} \right\|_\infty \leq \left\| u - \tilde{u} \right\|_\infty \sup_{|n| \leq R} |b(n)| + \left\| \tilde{u} \right\|_\infty \sup_{|n| \leq R} |n - \tilde{n}| \sup_{|n| \leq R} |b'(n)|
\]

(3.6)

\[
\left\| g(u) u - g(\tilde{u}) \tilde{u} \right\|_\infty \leq \left\| u - \tilde{u} \right\|_\infty \sup_{|u| \leq R} |g(u)| + \left\| u - \tilde{u} \right\|_\infty \left\| \tilde{n} \right\|_\infty \sup_{|u| \leq R} |g'(u)|.
\]

(3.7)

Since \(p > d\), we can use the Sobolev embedding of \(W^{1,p}(\Omega)\) into \(L^\infty(\Omega)\) in order to get from (3.5) the estimate

\[
\left\| u \nabla \chi(c) - \tilde{u} \nabla \chi(\tilde{c}) \right\|_p \leq \left\| u - \tilde{u} \right\|_\infty \left\| \nabla c \right\|_p \left\| \chi' \right\|_\infty
\]

\[
+ \left\| c - \tilde{c} \right\|_{W^{1,p}} \left\| \nabla c \right\|_p \left\| \tilde{u} \right\|_\infty \left\| \chi'' \right\|_\infty
\]

\[
+ \left\| \nabla c - \nabla \tilde{c} \right\|_p \left\| \tilde{u} \right\|_\infty \left\| \chi' \right\|_\infty.
\]

(3.8)

Using the following estimates on the heat semigroup

\[
\|e^{t\Delta}v\|_\infty \leq \|v\|_\infty \quad \text{and} \quad \|\nabla e^{t\Delta}v\|_\infty \leq Ct^{-\frac{1}{2}}(1+\frac{d}{p})\|v\|_p
\]
which are valid for all \( v \in L^\infty(\Omega) \), \( t > 0 \), and where \( C > 0 \) is independent of \( v \) and of \( t \) (see Lemma \([A.3]\) below), we deduce

\[
\int_0^t \left\| \nabla \cdot e^{\Delta(t-s)} \left( (u(s)\nabla \chi(c(s)) - \tilde{u}(s)\nabla \tilde{\chi}(\tilde{c}(s)) \right) \right\|_\infty \, ds \\
\leq \sup_{0 \leq t \leq T} \left\| u\nabla \chi(c) - \tilde{u}\nabla \tilde{\chi}(\tilde{c}) \right\|_p \int_0^t C s^{-\frac{1}{2}(1+\frac{2}{p})} \, ds \\
\leq \frac{C}{\frac{1}{2}(1+\frac{2}{p})} T^{1-\frac{1}{2}(1+\frac{2}{p})} \sup_{0 \leq t \leq T} \left\| u\nabla \chi(c) - \tilde{u}\nabla \tilde{\chi}(\tilde{c}) \right\|_p
\]

We combine this result with the inequalities \([3.6]\), \([3.7]\) and \([3.8]\) and we bound \( A_2 \) with the method of Horstmann and Winkler in \([11]\), proof of theorem 3.1, which leads to

\[
\sup_{0 \leq t \leq T} A_1(t) \leq \left( \frac{C}{1-\frac{1}{2}(1+\frac{2}{p})} T^{1-\frac{1}{2}(1+\frac{2}{p})} (2R\|\chi\|_\infty + R^2 \|\chi''\|_\infty) + T (G_0 R + R \sup_{0 \leq y \leq R} |g'(y)y + g(y)| + B_0 + R \sup_{0 \leq y \leq R} |b'(y)|) \right) d((u, c, n), (\tilde{u}, \tilde{c}, \tilde{n}))
\]

\[
\sup_{0 \leq t \leq T} A_2(t) \leq \delta_1 T^{1-\delta_2} d((u, c, n), (\tilde{u}, \tilde{c}, \tilde{n}))
\]

\[
\sup_{0 \leq t \leq T} A_3(t) \leq T \gamma \left( G_0 R + R \sup_{0 \leq y \leq R} |g'(y)y + g(y)| \right) d((u, c, n), (\tilde{u}, \tilde{c}, \tilde{n}))
\]

where \( \delta_1, \delta_2 > 0 \) are positive constants such that \( 1 - \delta_2 > 0 \). Therefore \( F \) is a contraction provided that \( T > 0 \) is sufficiently small with respect to \( R \) and the parameters. Hence, the mapping \( F \) has a unique fixed point by the Banach fixed point theorem.

The smoothness properties of the unique mild solution come from the heat Kernel regularising properties and a bootstrap argument. Since \( p > d \), by continuous embedding of \( W^{1,p}(\Omega) \) into \( C(\Omega) \), we have \( C([0, T], C(\Omega)) \). Since the other initial data lies in \( C(\Omega) \), we get the expected regularity by applying the bootstrap argument to the right-hand sides of the Duhamel’s formulae \((3.1)\), \((3.2)\) and \((3.3)\).

We can now apply the maximum principle to the third equation describing the evolution of \( n \). For non-negative initial data, \((u, c, n)\) remains non-negative (see Theorem \([3.3]\) below) and hence the term \( g(u)n c \) remains non-negative. The maximum principle applied to the linear equation for \( n \) yields (see also remark \([3.4]\) below for another point of view)

\[
\forall t \in (0, T_{\max}), \quad \|n(t)\|_\infty \leq \|n_0\|_\infty.
\]

Finally, we argue by contradiction in order to prove \((3.4)\). Assume that the maximal time of existence \( T_{\max} \) for the unique solution which we have constructed is finite and that

\[
\sup_{t \in [0, T_{\max})} \left( \|u(t)\|_\infty + \|c(t)\|_{W^{1,p}} + \|n(t)\|_\infty \right) < R < +\infty.
\]

In the construction of local-in-time solutions done above, we can choose a constant \( R \) which is valid for every \((u(t), c(t), n(t))\) taken as an initial condition. As stated above, the local time of existence \( T(R) \) from any initial datum \( u(t), t < T_{\max} \) only depends on \( R \) and the other constants. Hence, we can choose \( t \) close enough to \( T_{\max} \) such that \( t + T(R) > T_{\max} \). It is a contradiction with \( T_{\max} \) being the maximal time of existence.
Therefore, in view of (3.9) we have,
\[
\sup_{t \in (0, T_{\text{max}})} \left( \|u(t)\|_{\infty} + \|c(t)\|_{W^{1,p}} \right) = +\infty.
\]

Finally, the function \( w = w(x,t) \) is obtained by integrating equation (2.4) over the time interval \((0, t)\); thus we deduce the unique local-in-time solution of the initial-boundary value problem \((2.1)-(2.6)\).

\[\square\]

**Remark 3.2.** The local-in-time solution constructed in Theorem 3.1 may be continued to global-in-time provided we find suitable \(a \text{ priori}\) estimates on its norms, see [25, Ch. 4, Sec. 1.4].

**Theorem 3.3 (Nonnegativity).** If \(u_0 \geq 0, c_0 \geq 0, n_0 \geq 0, w_0 \geq 0\) almost everywhere then the local-in-time solution of problem \((2.1)-(2.6)\) is non-negative for all \(0 < t < T\).

**Proof.** We employ a standard truncation method. Let \(H = H(u)\) be a cutoff function such that
\[
H(u) := \begin{cases} 
1 & \text{for } -\infty < u < 0, \\
\frac{1}{2}u^2 & \text{for } 0 \leq u < \infty.
\end{cases}
\]

Then, the function \(\psi(t) = \int_{\Omega} H(u(t)) \, dx\) is continuously differentiable and \(\psi(0) = 0\) for non-negative \(u_0\). Following the reasoning in [25, Ch. 12, Sec. 1.3] one may show that \(\psi'(t) \leq C\psi(t)\) for all \(t \in [0, T]\). Then, \(\psi(0) = 0\) implies \(\psi(t) \equiv 0\) and consequently \(u(t) \geq 0\) for all \(t \in [0, T]\). An analogous reasoning should be applied to the functions \(c(t)\) and \(n(t)\). Now, it is clear that by equation (2.4), we have \(w(t) \geq 0\) for all \(t \in [0, T]\) if \(u\) is non-negative. \(\square\)

**Remark 3.4.** Notice, that by an analogous reasoning as in the proof of Theorem 3.3 we may show that
\[
0 \leq n(x, t) \leq \|n_0\|_{\infty} \quad \text{for all } x \in \Omega, \ t \in [0, T]
\]
for \(\gamma g(u)u \geq 0\) and \(n_0 \geq 0\).

### 3.2. Parabolic-elliptic-parabolic problem.

With the same notations, we write a Duhamel’s formula for problem \((2.10)-(2.14)\).

\[
u(t) = e^{\Delta t}u_0 + \int_0^t \nabla \cdot e^{\Delta(t-s)}u(s)\nabla \chi(c(s)) \, ds \\
+ \int_0^t e^{\Delta(t-s)}u(s)(g(u)n - b(n))(s) \, ds,
\]

(3.10)

\[
n(t) = e^{d_n \Delta t}n_0 - \gamma \int_0^t e^{d_n \Delta(t-s)}g(u(s))n(s)u(s) \, ds.
\]

(3.11)

where \(c(t)\) is the unique solution of

\[
-\Delta c(t) + \beta c(t) = \alpha u(t) \quad \text{for } x \in \Omega \quad \text{and } \quad t > 0,
\]

\[
\frac{\partial c}{\partial \nu} = 0 \quad \text{for } x \in \partial \Omega \quad \text{and } \quad t > 0.
\]

(3.12)
Theorem 3.5 (Local-in-time solutions). Let Assumptions 2.1 hold. Assume furthermore that \( \chi \) is \( C^3 \) and \( \chi'''' \in L^\infty((0, +\infty)) \). For every non-negative initial data \( u_0, n_0 \in C^{2,\eta}(\Omega) \) satisfying the Neumann boundary condition, there exists a unique maximal time \( T_{\text{max}} \in (0, +\infty) \) such that problem (2.10)–(2.14) has a unique solution

\[
(u, c, n) \in C^{2+\eta, 1+\eta/2}(\Omega \times [0, T_{\text{max}}]).
\]

Moreover, if \( T_{\text{max}} < +\infty \), then

\[
\limsup_{t \to T_{\text{max}}} ||u(t)||_\infty = +\infty.
\]

Proof. First, by Lemma B.2 if \( u \in C^{\eta, \eta}(\Omega \times [0, T]) \), then the solution \( c \) of (3.12) satisfies \( c \in C^{2+\eta, \eta}(\Omega \times [0, T]) \) and for all \( t > 0 \)

\[
\|\nabla c(t)\|_\infty \leq K\|u(t)\|_\infty,
\]

for some constant \( K > 0 \).

Then, we fix \( R > 0 \) such that

\[
\max(||u_0||_\infty, ||n_0||_\infty) \leq \frac{R}{2},
\]

and we define the space

\[
V_{T,R} = \{(u, n) \in C^{\eta, \eta}(\Omega \times [0, T])^2 \sup_{0 \leq t \leq T} \|u(t)\|_\infty \leq R, \sup_{0 \leq t \leq T} \|n(t)\|_\infty \leq R \}.
\]

Let \( \mathcal{F} = \mathcal{F}(u, n) : V_{T,R} \to V_{T,R} \) be the mapping defined by the right-hand side of equations (3.10)–(3.12).

Let \((u, n) \in V_{T,R}\), we denote

\[
F(u, n) = (\bar{u}, \bar{n}).
\]

Hence,

\[
\|\bar{u}(t)\|_\infty \leq \|e^{t\Delta} u_0\|_\infty + \left\| \int_0^t \nabla \cdot e^{\Delta(t-s)} \left( u(s) \nabla \chi(c(s)) \right) \, ds \right\|,
\]

and

\[
\|\bar{n}(t)\|_\infty \leq \|e^{t\Delta} n_0\|_\infty + \gamma \left\| \int_0^t e^{d\Delta(t-s)} \left( g(u(s))n(s) - b(n)(s) \right) \, ds \right\|_\infty.
\]

As in the fully parabolic case, we write,

\[
\|u \nabla \chi(c)\|_\infty \leq \|u(t)\|_\infty \|\chi'\|_\infty \|\nabla c\|_\infty,
\]

\[
\|(g(u)n - b(n))u\|_\infty \leq (G_0\|n\|_\infty + B_0)\|u\|_\infty,
\]

we combine these inequalities with (3.14) and the estimates of the heat semigroup

\[
\|e^{t\Delta} v\|_\infty \leq \|v\|_\infty \quad \text{and} \quad \|\nabla e^{t\Delta} v\|_\infty \leq Ct^{-\frac{1}{2}}\|v\|_\infty
\]

\[
\mathcal{F}(u, n) = (\bar{u}, \bar{n}).
\]
(see Lemma A.1 below) to deduce

\begin{align}
\sup_{0 \leq t \leq T} \|\tilde{u}(t)\|_\infty & \leq \frac{R}{2} + 2\sqrt{T} CK \|\chi'\|_\infty R^2, \\
\sup_{0 \leq t \leq T} \|\tilde{n}(t)\|_\infty & \leq \frac{R}{2} + T\gamma G_0 R^2
\end{align}

Furthermore, as we said above, by Lemma B.2 if \( u \in C^{\eta, \frac{n}{2}}(\bar{\Omega} \times [0, T]) \), then the solution \( c \) of (3.12) satisfies \( c \in C^{2+\eta, \frac{n}{2}}(\bar{\Omega} \times [0, T]) \). Hence, \( \nabla \chi(c) \in C^{1+\eta, \frac{n}{2}}(\bar{\Omega} \times [0, T]) \) and thus \( u\nabla \chi(c) \in C^{\eta, \frac{n}{2}}(\bar{\Omega} \times [0, T]) \). Then, by definition of \( \{e^{\Delta t}\}_{t \geq 0} \), the semigroup of linear operators on \( L^p(\Omega) \) generated by Laplacian with the Neumann boundary conditions, and by property of the integral, we have for all \( s, t > 0 \)

\[ e^{\Delta(t-s)}u(s)\nabla \chi(c(s)) \in C^{2, \eta}(\bar{\Omega}), \]

and thus

\[ t \mapsto \int_0^t \nabla \cdot e^{\Delta(t-s)}(u(s)\nabla \chi(c(s))) \, ds \in C^{2+\eta, \frac{n}{2}}(\bar{\Omega} \times [0, T]). \]

Applying these arguments to each term in the right-hand sides of the Duhamel’s formulae (3.10) and (3.12), we are able to conclude, thanks to the hypothesis \( u_0, n_0 \in C^{2, \eta} \),

\[ \tilde{u}, \tilde{n} \in C^{2+\eta, \frac{n}{2}}(\bar{\Omega} \times [0, T]) \subset C^{\eta, \frac{n}{2}}(\bar{\Omega} \times [0, T]). \]

We chose \( T(R) \) small enough so that the right-hand sides of inequalities (3.15) and (3.16) are bounded from above by \( R \). Then, we have \( \mathcal{F}(V_{T,R}) \subset V_{T,R} \).

Since the embedding \( C^{2+\eta, \frac{n}{2}}(\bar{\Omega} \times [0, T]) \hookrightarrow C^{\eta, \frac{n}{2}}(\bar{\Omega} \times [0, T]) \) is compact, \( \mathcal{F}(V_{T,R}) \) is a relatively compact subset of \( V_{T,R} \). Hence, the continuous mapping \( \mathcal{F} \) has a fixed point by the Schauder fixed point theorem.

We now prove uniqueness: assume there exist two solutions \( (u_1, c_1, n_1) \) and \( (u_2, c_2, n_2) \) on \([0, T]\). Let \( U, C, N \) be defined by \( U = u_1 - u_2, C = c_1 - c_2 \) and \( N = n_1 - n_2 \). We choose \( R \) large enough such that

\[ \sup_{t \in [0, T]} \|u_i(t)\|_\infty \leq R, \quad \sup_{t \in [0, T]} \|c_i(t)\|_\infty \leq R, \quad \sup_{t \in [0, T]} \|n_i(t)\|_\infty \leq R \]

for \( i = 1, 2 \).

Since, using the equation for \( c_i, \ i = 1, 2 \),

\[ \Delta \chi(c_i) = \chi''(c_i)|\nabla c_i|^2 + \chi'(c_i)(\beta c_i - \alpha u_i), \]

we have, for \( i = 1, 2 \),

\begin{align}
\frac{\partial u_i}{\partial t} - \Delta u_i = -\chi'(c_i)\nabla u_i \cdot \nabla c_i \\
- u_i \left( \chi''(c_i)|\nabla c_i|^2 + \chi'(c_i)(\beta c_i - \alpha u_i) \right) + (g(u_i)n_i - b(n_i))u_i.
\end{align}
Hence,

\[
(3.19) \quad \frac{\partial U}{\partial t} - \Delta U = -\chi'(c_1) \nabla U \cdot \nabla c_1 - \nabla u_2 \cdot (\chi'(c_1) \nabla c_1 - \chi'(c_2) \nabla c_2) \\
+ \left( g(u_1)n_1 - b(n_1) + \chi''(c_1) |\nabla c_1|^2 + \chi'(c_1)(\beta c_1 - \alpha u_1) \right) u_1 \\
- \left( g(u_2)n_2 - b(n_2) + \chi''(c_2) |\nabla c_2|^2 + \chi'(c_2)(\beta c_2 - \alpha u_2) \right) u_2.
\]

We have

\[
|b(n_1)u_1 - b(n_2)u_2| \leq |U| \sup_{|n| \leq R} |b(n)| + |N| \sup_{s \in [0,T]} \|u_2(s)\|_\infty \sup_{|n| \leq R} |b'(n)|,
\]

\[
|g(u_1)u_1n_1 - g(u_2)u_2n_2| \leq |N| \sup_{|u| \leq R} |g(u)| |u| + |U| \sup_{s \in [0,T]} \|n_2(s)\|_\infty \sup_{|u| \leq R} |g'(u)u + g(u)|,
\]

\[
|u_1\chi''(c_1) |\nabla c_1|^2 - u_2\chi''(c_2) |\nabla c_2|^2| \leq \|U\|\chi''_{\infty} \sup_{s \in [0,T]} \|\nabla c_1(s)\|_{\infty}^2 \\
+ |C| \|\chi''_{\infty}\| \sup_{s \in [0,T]} \|u_2(s)\|_\infty \sup_{s \in [0,T]} \|\nabla c_1(s)\|_{\infty}^2 \\
+ |\nabla C| \|\chi''_{\infty}\| \sup_{s \in [0,T]} \|u_2(s)\|_\infty \sup_{s \in [0,T]} \|\nabla c_1(s) + \nabla c_2(s)\|_\infty.
\]

We will need below the following estimates

\[
|\chi'(c_1) \nabla U \cdot \nabla c_1 U| \leq \frac{1}{2} |\nabla U|^2 + \frac{1}{2} (\|\chi'\|_{\infty} \sup_{s \in [0,T]} \|\nabla c_1(s)\|_{\infty}^2) U^2
\]

\[
|\nabla u_2 \cdot (\chi'(c_1) \nabla c_1 - \chi'(c_2) \nabla c_2 U| \leq \frac{1}{2} (\sup_{s \in [0,T]} \|\nabla u_2(s)\|_\infty) \|\chi'\|_{\infty}^2 |\nabla C|^2 + (\sup_{s \in [0,T]} \|\nabla c_1(s)\|_{\infty}^2) \|\chi''\|_{\infty}^2 |C|^2.
\]

Hence, multiplying equation (3.19) by \( U \) and integrating, we obtain

\[
(3.20) \quad \frac{d}{dt} \int_{\Omega} U^2 + \int_{\Omega} |\nabla U|^2 \leq \kappa_1 \left( \int_{\Omega} U^2 + \int_{\Omega} C^2 + \int_{\Omega} |\nabla C|^2 + \int_{\Omega} N^2 \right),
\]

for some large enough constant \( \kappa_1 > 0 \).

Then, we have

\[
-\Delta C + \beta C = \alpha U.
\]

We multiply by \( C \) and we integrate in order to obtain,

\[
\int_{\Omega} |\nabla C|^2 + \beta \int_{\Omega} C^2 = \alpha \int_{\Omega} CU \leq \alpha \left( \frac{\beta^2}{2\alpha} \int_{\Omega} C^2 + \frac{\alpha}{2\beta} \int_{\Omega} U^2 \right),
\]

and as a consequence

\[
\int_{\Omega} |\nabla C|^2 + \beta \frac{\alpha}{\beta} \int_{\Omega} C^2 \leq \alpha \frac{\alpha^2}{2\beta} \int_{\Omega} U^2.
\]

Last, we have

\[
\frac{\partial N}{\partial t} = d_n \Delta N + \gamma (g(u_1)n_1u_1 - g(u_2)n_2u_2).
\]
We multiply by $N$ and we integrate. With the same reasoning as above we find that, for some large enough constant $\kappa_2 > 0$,

$$
\frac{d}{dt} \int_{\Omega} N^2 + 2d_n \int_{\Omega} |\nabla N|^2 \leq \kappa_2 \left( \int_{\Omega} N^2 + \int_{\Omega} U^2 \right)
$$

We deduce from (3.20), (3.21) and (3.22) that, for some large enough constant $\lambda > 0$,

$$
\frac{d}{dt} \int_{\Omega} (U^2 + N^2) \leq \lambda \int_{\Omega} (U^2 + N^2),
$$

and – since $U(0)^2 + N(0)^2 = 0$ – applying Gronwall’s lemma gives $U(t) = 0$ and $N(t) = 0$ for all $t \in [0, T]$. As a consequence, $C(t) = 0$ for all $t \in [0, T]$. This complete the proof of the uniqueness of the solution.

Finally, we argue by contradiction in order to prove (3.13). Assume that the maximal time of existence $T_{\text{max}}$ for the unique solution is finite. The same maximum principle argument as in the parabolic-parabolic-parabolic case yields for all $t \in [0, T_{\text{max}})$

$$
\|n(t)\|_\infty \leq \|n_0\|_\infty.
$$

Assume that

$$
\sup_{t \in [0, T_{\text{max}})} \|u(t)\|_\infty < \frac{\tilde{R}}{2} < +\infty.
$$

Then, in the construction of local-in-time solutions done above, we can choose a constant $\tilde{R}$ that works for every $(u(t), n(t))$ taken as an initial condition. As stated above, the local time of existence $T(\tilde{R})$ from any initial datum $u(t), t < T_{\text{max}}$ depends only on $\tilde{R}$ and other constants. Hence, we can choose $t$ close enough to $T_{\text{max}}$ so as to achieve $t + T(\tilde{R}) > T_{\text{max}}$. It’s a contradiction with $T_{\text{max}}$ being the maximal time of existence. Therefore,

$$
\sup_{t \in [0, T_{\text{max}})} \|u(t)\|_\infty = +\infty.
$$

Remark 3.6. The results obtained for the full parabolic problem still hold with the same proofs combined with the elliptic maximum principle : solutions are non-negative for non-negative initial data and $\|n(t)\|_\infty \leq \|n_0\|_\infty$.

4. Space homogeneous solutions and large time behaviour of mass

Proof of Theorem 2.2. Obviously, the chemotactic term $-\nabla \cdot (u \nabla \chi(c))$ as well as the terms in equations (2.1)–(2.4) containing Laplacian disappear in the case of $x$-independent solutions. Hence, we shall focus on the following system of corresponding ordinary differential
equations

\begin{align}
(4.1) & \quad \frac{d}{dt} \bar{u} = g(\bar{u}) \bar{n} \bar{u} - b(\bar{n}) \bar{u} \\
(4.2) & \quad \frac{d}{dt} \bar{c} = \alpha \bar{u} - \beta \bar{c} \\
(4.3) & \quad \frac{d}{dt} \bar{n} = -\gamma g(\bar{u}) \bar{n} \bar{u} \\
(4.4) & \quad \frac{d}{dt} \bar{w} = b(\bar{n}) \bar{u}. 
\end{align}

where the vector \((0, \bar{n}_\infty)\) is a steady state for each non-negative constant \(\bar{n}_\infty\). Here, we notice that, by a standard reasoning, every solution \((\bar{u}(t), \bar{n}(t))\) of \((4.5)-(4.6)\) which starts in the first quadrant \((\bar{u} > 0, \bar{n} > 0)\) at \(t = 0\) has to remain in this quadrant of the \((\bar{u}, \bar{n})\)-plane for all future times (in fact, this is also proved in Theorem 3.3, below). Observe also that, by equation \((4.6)\) and by Assumptions 2.1, the derivative \(\frac{d}{dt} \bar{n}\) is always nonpositive in the first quadrant.

Now, we split the first quadrant into two regions, where \(\frac{d}{dt} \bar{u}\) has a fixed sign, in the following way. Since \(g\) is a bounded and increasing function, it has a limit \(G_0 = \lim_{u \to \infty} g(s)\). Thus, the bijection \(g^{-1} : [0, G_0) \to [0, \infty)\) is an increasing function and the equation \(g(\bar{u}) \bar{n} \bar{u} - b(\bar{n}) \bar{u} = 0\) for \(\bar{u} > 0\), can be written as \(\bar{u} = f(\bar{n}) = g^{-1}(b(\bar{n})/\bar{u})\). By properties of the functions \(g\) and \(b\), we immediately obtain that \(f\) is a decreasing function such that

\[ f(\bar{n}) \to 0 \quad \text{as} \quad n \to \infty \quad \text{and} \quad f(\bar{n}) \to +\infty \quad \text{as} \quad n \to N_0, \]

where the constant \(N_0 > 0\) is defined by the equation \(g(\bar{n}_o)/\bar{n} = G_0\). The curve \(\bar{u} = f(\bar{n})\) divides the first quadrant of the \((\bar{u}, \bar{n})\)-plane into two regions, where \(\frac{d}{dt} \bar{u} < 0\) in region I and \(\frac{d}{dt} \bar{u} > 0\) in region II (see the dashed curve on Fig. 3).

For every \((\bar{u}(0), \bar{n}(0))\) in region I, the functions \(\bar{n}(t)\) and \(\bar{u}(t)\) are non increasing, hence, \(\lim_{t \to \infty}(\bar{u}(t), \bar{n}(t)) = (0, \bar{n}_\infty)\) for some \(\bar{n}_\infty \geq 0\). In fact, this is an exponential convergence because, for a solution \((\bar{u}(t), \bar{n}(t))\) in region I we have \(\frac{d}{dt} \bar{u} \leq (g(\bar{u}(0)) \bar{n}(0) - b(\bar{n}(0))) \bar{u}\), where \(g(\bar{u}(0)) \bar{n}(0) - b(\bar{n}(0)) < 0\) by properties of the functions \(g\) and \(b\). Moreover, by equation \((4.6)\), we have that \(\bar{n}_\infty = \bar{n}_0 - \gamma \int_0^\infty g(\bar{u}(s)) \bar{n}(s) \bar{u}(s) ds\). Hence

\[ |\bar{n}(t) - \bar{n}_\infty| = \gamma \int_t^\infty g(\bar{u}(s)) \bar{n}(s) \bar{u}(s) ds \leq C \int_t^\infty \bar{u}(s) ds \to 0 \]

exponentially as \(t \to \infty\), due to exponential decay of \(\bar{u}(t)\).

Now, let us prove that if a trajectory starts in region II, then this has to enter region I. Indeed, if we suppose that it is not the case, and we have \(\bar{u}(t) \geq \bar{u}(0)\) for all \(t > 0\). Then, from equation \((4.6)\) we have that \(\bar{n}(t) \leq -\gamma g(\bar{u}(0)) \bar{u}(0) \bar{n}\) and so \(\bar{n}(t) \leq \bar{n}_0 e^{-t(\gamma g(\bar{u}(0)))} \to 0\) as \(t \to \infty\) which leads to a contradiction. Thus, we have proved that \((\bar{u}(t), \bar{n}(t)) \to (0, \bar{n}_\infty)\) exponentially as \(t \to \infty\).
Finally, we describe the large time behaviour of $\bar{c}(t)$ and $\bar{w}(t)$. Solving equation (4.2) with respect to $\bar{c} = \bar{c}(t)$, we may easily show that $\lim_{t \to \infty} \bar{c}(t) = 0$ exponentially, using exponential decay of $\bar{u}(t)$. Moreover, we have

$$\lim_{t \to \infty} \bar{w}(t) = \bar{w}_0 + \int_0^\infty b(\bar{n}(s))\bar{u}(s) \, ds \equiv \bar{w}_\infty,$$

where the quantity on the right-hand side is finite and positive, because $b(\bar{n}(t))$ is bounded for $t > 0$ and $\bar{u}(t)$ decays exponentially. \hfill $\square$

Now, we consider solutions of problem (2.1)–(2.6) with nonconstant initial conditions and we prove a similar result to the one in Theorem 2.2 on the large time behaviour of the integrals $\int_\Omega u(x,t) \, dx$, $\int_\Omega c(x,t) \, dx$, $\int_\Omega n(x,t) \, dx$ and $\int_\Omega w(x,t) \, dx$, which correspond to masses with the densities $u$, $c$, $n$ and $w$, respectively.

**Theorem 4.1.** Assume that a non-negative solution $(u,c,n,w)$ of problem (2.1)–(2.6) exists for all $t > 0$. Let Assumptions 2.1 hold true. Then

$$\int_\Omega u(t) \, dx \to 0 \quad \text{and} \quad \int_\Omega c(t) \, dx \to 0 \quad \text{as} \quad t \to \infty,$$

and there are constants $\tilde{n}_\infty \geq 0$ and $\tilde{w}_\infty > 0$ such that

$$\int_\Omega n(t) \, dx \to \tilde{n}_\infty \quad \text{and} \quad \int_\Omega w(t) \, dx \to \tilde{w}_\infty \quad \text{as} \quad t \to \infty.$$
Moreover, since we conclude that there exists a constant \( u \) is also nonincreasing, hence, it has a limit as \( t \rightarrow \infty \). Consequently, we have

\[
(4.17)
\
\frac{d}{dt} \int_{\Omega} u \, dx = \int_{\Omega} g(u)nu \, dx - \int_{\Omega} b(n)u \, dx
\]

(4.8)

\[
\frac{d}{dt} \int_{\Omega} c \, dx = \alpha \int_{\Omega} u \, dx - \beta \int_{\Omega} c \, dx
\]

(4.9)

\[
\frac{d}{dt} \int_{\Omega} n \, dx = -\gamma \int_{\Omega} g(u)nu \, dx
\]

(4.10)

\[
\frac{d}{dt} \int_{\Omega} w \, dx = \int_{\Omega} b(n)u \, dx.
\]

(4.11)

Since, \( \frac{d}{dt} \left( \int_{\Omega} u(t) \, dx + \frac{1}{\gamma} \int_{\Omega} n(t) \, dx + \int_{\Omega} w(t) \, dx \right) = 0 \), we get the conservation of mass in the following sense

\[
(4.12)
\int_{\Omega} u(t) \, dx + \frac{1}{\gamma} \int_{\Omega} n(t) \, dx + \int_{\Omega} w(t) \, dx = \int_{\Omega} u_0 \, dx + \frac{1}{\gamma} \int_{\Omega} n_0 \, dx + \int_{\Omega} w_0 \, dx,
\]

for all \( t > 0 \). In particular, since all functions are non-negative, we have

\[
(4.13)
\int_{\Omega} u(x,t) \, dx \leq \int_{\Omega} u_0 \, dx + \frac{1}{\gamma} \int_{\Omega} n_0 \, dx + \int_{\Omega} w_0 \, dx \quad \text{for all} \quad t > 0.
\]

Now, we improve this estimate by adding equation (4.8) to equation (4.10) multiplied by \( \gamma^{-1} \) and integrating resulting equation over \([0,t]\) to obtain the relation

\[
(4.14)
\int_{\Omega} u(t) + \gamma^{-1}n(t) \, dx = \int_{\Omega} u_0 + \gamma^{-1}n_0 \, dx - \int_{0}^{t} \int_{\Omega} b(n(s))u(s) \, dx \, ds,
\]

which by positivity of \( b \) and \( u \) implies that

\[
(4.15)
\int_{\Omega} u(t) \, dx \leq \int_{\Omega} u_0 \, dx + \frac{1}{\gamma} \int_{\Omega} n_0 \, dx.
\]

Next, we observe that, since \( g(u)nu \geq 0 \), it follows from equation (4.10) that \( \int_{\Omega} n(t) \, dx \) is nonincreasing and since it is also non-negative, the following finite limit exists

\[
(4.16)
\lim_{t \rightarrow \infty} \int_{\Omega} n(t) \, dx = \tilde{n}_\infty \geq 0.
\]

Now, since \( b(n)u \geq 0 \), equation (4.14) implies that the mapping \( t \mapsto \int_{\Omega} u(t) + \gamma^{-1}n(t) \, dx \) is also nonincreasing, hence, it has a limit as \( t \rightarrow \infty \). Consequently, using relations (4.16), we conclude that there exists a constant \( u_\infty \geq 0 \) such that

\[
\lim_{t \rightarrow \infty} \int_{\Omega} u(t) \, dx = u_\infty.
\]

Moreover, since \( \int_{\Omega} u(t) + \gamma^{-1}n(t) \, dx \) is bounded for \( t \geq 0 \), identity (4.14) implies that \( b(n)u \in L^1((0, \infty); L^1(\Omega)) \). However, since \( b(n) \geq b(\|n_0\|_\infty) > 0 \), it follows that

\[
(4.17)
u \in L^1((0, \infty); L^1(\Omega))
\]

Consequently, we have \( u_\infty = 0 \).

Since \( b(n)u \in L^1((0, \infty), L^1(\Omega)) \) we obtain from equation (4.11)

\[
\lim_{t \rightarrow \infty} \int_{\Omega} w(t) \, dx \equiv \int_{\Omega} w_0 \, dx + \int_{0}^{\infty} \int_{\Omega} b(n)u \, dx \, ds.
\]
Finally, $\lim_{t \to \infty} \int_\Omega c(t) \, dx = 0$ due to equation (4.9) because $\lim_{t \to \infty} \|u(t)\|_1 = 0$. This completes the proof of Theorem 4.1.

**Remark 4.2.** Under the following extra assumption $G_0\|n_0\|_\infty - b(\|n_0\|_\infty) < 0$, where $b(\|n_0\|_\infty) = \inf_n b(n)$ and $G_0 = \sup_n g(u)$, we obtain the exponential decay of $\int_\Omega u(x,t) \, dx$. This is an immediate consequence of equation (4.8) and the estimate

$$g(u)nu - b(n)u \leq (G_0\|n_0\|_\infty - b(\|n_0\|_\infty))u < 0,$$

since $n(x,t) \leq \|n_0\|_\infty$ by equation (2.3) (cf. Remark 3.4).

**Remark 4.3.** Let us point out that the method from the proof of Theorem 4.1 can be also used to show Theorem 2.2.

### 5. Problem in One Space Dimension

The proof of Theorem 2.3 requires the following two auxiliary lemmas. First, we find an estimate of $c_x(t)$ which is uniform in time.

**Lemma 5.1.** Let the assumptions of Theorem 2.3 hold true and denote by $(u,c,n,w)$ the corresponding non-negative local-in-time solution to problem (2.1)–(2.6) on $[0,T]$ constructed in Theorem 3.1. For each $p \in [1,\infty)$ there exists a constant $C = C(p) > 0$ independent of $T$ such that $\|c_x(t)\|_p \leq C$ for all $t \in (0,T]$. Moreover, if the solution is global-in-time, then $\lim_{t \to \infty} \|c_x(t)\|_p = 0$ for each $p \in [1,\infty)$.

**Proof.** Using the Duhamel principle (3.2) and the estimate of the heat semigroup (A.3) we obtain

$$\|c_x(t)\|_p \leq \|\partial_x e^{\Delta - \beta t}c_0\|_p + \alpha \int_0^t \|\partial_x e^{(\Delta - \beta)(t-s)}u(s)\|_p \, ds$$

$$\leq Ce^{-\beta t}\|c_0\|_p + C\int_0^t (t-s)^{-\frac{1}{2}(1-\frac{1}{p})\frac{1}{2}} e^{(\beta + \lambda_1)(t-s)}\|u(s)\|_1 \, ds$$

for all $t \in (0,T]$ and a constant $C > 0$ independent of $t > 0$. The right-hand side of this inequality is bounded uniformly in $t > 0$ and independent of $T > 0$ because of estimate (4.15). Moreover, if solution is global-in-time, this converges to zero (cf. Lemma A.2 below) since $\lim_{t \to \infty} \|u(t)\|_1 = 0$ by Theorem 4.1.

Next, we show the boundedness of the $L^2$-norm of $u$ using usual energy estimates. This result was already obtained in [12] and we recall it for the completeness of the exposition.

**Lemma 5.2.** Let the assumptions of Theorem 2.3 hold true. Moreover, let $(u,c,n,w)$ be the non-negative local-in-time solution to problem (2.1)–(2.6) constructed in Theorem 3.1. Then, there exists $C$ independent of $T$ such that $\|u(t)\|_2 \leq C$ for all $t \in [0,T]$.

**Proof.** After multiplying equation (2.1) by $u$ and integrating over $\Omega$ we obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 \, dx + \int_\Omega (u_x)^2 \, dx + \int_\Omega b(n)u^2 \, dx = \int_\Omega g(u)nu^2 \, dx + \int_\Omega uc_x\chi'(c)u_x \, dx.$$
Thus, by the Cauchy inequality and Assumptions 2.1 on the functions $g$, $b$ and $\chi$, we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dx + \frac{1}{2} \int_{\Omega} (u_x)^2 \, dx + b(\|n_0\|_\infty) \int_{\Omega} u^2 \, dx \\
\leq G_0 \|n_0\|_\infty \int_{\Omega} u^2 \, dx + \frac{\|\chi'\|_{L^\infty(\mathbb{R})}^2}{2} \int_{\Omega} u^2(c_x)^2 \, dx,
\]
where constants $b(\|n_0\|_\infty) = \inf_n b(n) > 0$ and $G_0 = \sup_u g(u) > 0$ are finite by Assumptions 2.1.

To deal with the last term on the right-hand side of (5.2), we use estimate (4.15) and Lemma 5.1 combined with the Hölder, Sobolev (cf. e.g. [3, Eq. (42), pp. 233]) and the $\varepsilon$-Cauchy inequalities in the following way
\[
\int_{\Omega} u^2(c_x)^2 \, dx \leq \|u\|_2^4 \|c_x\|_4^2 \leq C\|u\|_{W^{1,2}}^2 \|u\|_1^2 \|c_x\|_4^2 \leq \varepsilon \|u\|_{W^{1,2}}^2 + C(\varepsilon),
\]
where $C(\varepsilon) = C(\varepsilon, \|u\|_1, \|c_x\|_4)$ is uniformly bounded in $t$. Moreover, by the Sobolev inequality and the Young inequality, we have
\[
\int_{\Omega} u^2 \, dx \leq C\|u\|_{W^{1,2}}^2 \|u\|_1^4/3 \leq \varepsilon \|u\|_{W^{1,2}}^2 + C_\varepsilon \|u\|_1^2.
\]
Therefore, for every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dx + \frac{1}{2} \int_{\Omega} (u_x)^2 \, dx + b(\|n_0\|_\infty) \int_{\Omega} u^2 \, dx \leq \varepsilon \|u\|_{W^{1,2}}^2 + C(\varepsilon).
\]

The term on the right-hand side of (5.4) containing small $\varepsilon > 0$ can be absorbed by the corresponding two terms on the left-hand side. Thus, we obtain the following differential inequality
\[
\frac{d}{dt} \int_{\Omega} u^2 \, dx + C\|u\|_{W^{1,2}}^2 \leq C,
\]
with a constant $C > 0$, which, in particular, implies that $\|u(t)\|_2$ has to be bounded uniformly in $t$. \qed

The reminder of this section is devoted to the proof of Theorem 2.3 on the large time behaviour of solutions to problem (2.1)–(2.6) in a one dimensional domain.

**Proof of Theorem 2.3** The local-in-time solutions constructed in Theorem 3.1 can be extended to all $t > 0$ due to a priori estimates, which we will obtain below in the study of the large time behaviour of solution. We skip this standard reasoning and we proceed directly to estimates of solutions for large values of $t > 0$.

**Step 1:** $\lim_{t \to \infty} \|u(t)\|_\infty = 0$. We apply the Duhamel principle (3.1) in the following way
\[
\|u(t) - e^{\Delta t}u_0 - \int_0^t e^{\Delta(t-s)}u(s)(g(u)n - b(n))(s) \, ds\|_\infty
\leq \|\int_0^t \partial_x e^{\Delta(t-s)}u(s)c_x(s)\chi'(c) \, ds\|_\infty.
\]
Using the property of the heat semigroup \( A.3 \), the Hölder inequality, and Assumptions 2.1 on the function \( \chi \) we estimate the right-hand side of equation (5.5) as follows

\[
\| \int_0^t \partial_x e^{A(t-s)} u(s)c_x(s)\chi'(c) \, ds \|_\infty \\
\leq C\| \chi' \|_\infty \int_0^t (t-s)^{-\frac{3}{2}} e^{-\lambda_1(t-s)} \| u(s) c_x(s) \|_{3/2} \, ds \\
\leq C\| \chi' \|_\infty \int_0^t (t-s)^{-\frac{3}{2}} e^{-\lambda_1(t-s)} \| u(s) \|_2 \| c_x(s) \|_6 \, ds.
\]

(5.6)

Thus, by Lemma A.2 below, the integral on the right-hand side of inequality (5.6) tends to zero because \( \| u(t) \|_2 \) is bounded by Lemma 5.2 and because \( \| c_x(t) \|_6 \) tends to zero, which is proved in Lemma 5.1. Hence, coming back to identity (5.5), we see that

\[
\| u(t) - v(t) \|_\infty \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty,
\]

where \( v(x,t) \) is a solution to the problem

\[
\begin{align*}
vt &= v_{xx} + g(u)nu - b(n)u, \\
v(x,0) &= u_0(x),
\end{align*}
\]

(5.8)

(5.9)

supplemented with the Neumann boundary conditions. We denote the nonlinear term on the right-hand side of (5.8) by \( f \equiv g(u)nu - b(n)u \) and since \( g, b \) and \( n \) are bounded, there exist a constant \( C > 0 \) such that

\[
\| f(\cdot,t) \|_1 \leq C\| u(t) \|_1 \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty,
\]

by Theorem 4.1. Hence, by Lemma A.3 we obtain

\[
\| v(t) - \frac{1}{|\Omega|} \int_{\Omega} v(t) \, dx \|_\infty \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.
\]

(5.10)

However, integrating equation (5.8) with respect to \( x \) and comparing the resulting formula with equation (4.8), it is easy to see that \( \int_{\Omega} u(t) \, dx = \int_{\Omega} v(t) \, dx \) for all \( t > 0 \). Therefore, using (5.7) and (5.10) we obtain the convergence

\[
\| u(t) - \frac{1}{|\Omega|} \int_{\Omega} u(t) \, dx \|_\infty \leq \| u(t) - v(t) \|_\infty + \| v(t) - \frac{1}{|\Omega|} \int_{\Omega} v(t) \, dx \|_\infty \rightarrow 0
\]

as \( t \rightarrow \infty \) which, in virtue of Theorem 4.1 completes the proof that \( \lim_{t \rightarrow \infty} \| u(t) \|_\infty = 0 \).

**Step 2: Exponential decay of \( \int_{\Omega} u(t) \, dx \).** Recall that the function \( b(n(x,t)) \) is bounded from below by \( b(\| n_0 \|_\infty) > 0 \) because \( b \) is nonincreasing, cf. Assumptions 2.1. Hence, since \( \| u(t) \|_\infty \rightarrow 0 \) as \( t \rightarrow \infty \) and since \( g(0) = 0 \), there exist constants \( T > 0 \) and \( \mu > 0 \) such that for all \( t \geq T \) and all \( x \in \Omega \) we have

\[
(g(u)n - b(n))(x,t) \leq -\mu.
\]

Thus, using equation (4.8) we get the following differential inequality

\[
\frac{d}{dt} \int_{\Omega} u(t) \, dx \leq -\mu \int_{\Omega} u(t) \, dx;
\]

which implies the exponential decay

\[
\| u(t) \|_1 \leq \| u_0 \|_1 e^{-\mu t} \quad \text{for all} \quad t > 0.
\]

(5.11)

Now, we use this estimate to improve Lemma 5.1.
Step 3: Exponential decay of \( \|c_x(t)\|_p \) for each \( p \in [1, \infty) \). Using the exponential decay of \( \|u(t)\|_1 \) from inequality (5.11) in estimate (5.1), we obtain
\[
\|c_x\|_p \leq Ce^{-\beta t}\|c_0,x\|_p + C \int_0^t (t - s)^{-\frac{1}{2}}(1 + s)^{-\frac{1}{2}}e^{-(\beta + \lambda_1)(t-s)}e^{-\mu s} ds,
\]
where the integral on the right-hand side decays exponentially by Lemma A.2.

Step 4: Exponential decay of \( \|c(t)\|_\infty \). Applying the Duhamel principle (3.2), computing the \( L^\infty \)-norm, and using the heat semigroup estimate (A.2), we have
\[
\|c(t)\|_\infty \leq Ce^{-\beta t}\|c_0\|_\infty + C \int_0^t \left( 1 + (t - s)^{-\frac{1}{2}} \right) e^{-\beta(t-s)} \|u(s)\|_1 ds
\]
for all \( t > 0 \) and a constant \( C > 0 \) independent of \( t > 0 \). Since \( \|u(t)\|_1 \) decays exponentially, see (5.11), we complete the proof of this step by Lemma A.2, again.

Step 5: Exponential decay of \( \|u(t)\|_\infty \). Here, it suffices to repeat all the estimates from Step 1 using the exponential decay estimates of \( \|c_x(t)\|_0 \) established in Step 3 and the decay of \( \|u(t)\|_1 \) from Step 2.

Step 6: Exponential convergence \( \lim_{t \to \infty} \|n(t) - n_\infty\|_\infty = 0 \). By Theorem 4.1, the limit
\[
\lim_{t \to \infty} \int \Omega n(t) dx \equiv \tilde{n}_\infty = \int \Omega n_0 dx - \int_0^\infty \int \Omega \gamma g(u)n u dx ds
\]
exists and is non-negative. This is, in fact, exponential convergence, because by equation (4.10) and by Step 2 we have
\[
\left| \int \Omega n(t) dx - \tilde{n}_\infty \right| \leq \gamma \int_0^\infty \int \Omega |g(u)n u| dx ds \leq \gamma G_0 \|n_0\|_\infty \int_0^\infty \|u(s)\|_1 ds \leq Ce^{-\mu t}.
\]
Now, applying Lemma A.3 with \( f(x,t) = -\gamma g(u)n u \) to equation (2.3), since \( \|f(\cdot,t)\|_1 \to 0 \) exponentially as \( t \to \infty \), we obtain
\[
\left\| n(t) - \frac{1}{|\Omega|} \int \Omega n(t) dx \right\|_\infty \to 0 \quad \text{exponentially as} \quad t \to \infty.
\]
Combining these two convergence results we complete the proof of Step 6 with \( n_\infty = |\Omega|^{-1}\tilde{n}_\infty \).

Step 7: \( \|w(t) - w_\infty\|_\infty \to 0 \) exponentially as \( t \to \infty \). Here, we define
\[
(5.12) \quad w_\infty(x) = w_0(x) + \int_0^\infty b(n(x,t))u(x,t) dt.
\]
Notice, that since \( b \) is bounded and \( \|u(t)\|_\infty \) decays exponentially, the right-hand side of (5.12) belongs to \( L^\infty(\Omega) \). Moreover, it is easy to see that
\[
\left\| w(t) - w_\infty(x) \right\|_\infty = \left\| \int_t^\infty b(n(x,t)) \|u(s)\|_\infty ds \right\|_\infty \leq C \int_t^\infty \|u(s)\|_\infty ds \leq C \int_t^\infty e^{-\mu s} ds \to 0
\]
exponentially as \( t \to \infty \). This completes the proof of Step 7 and of Theorem 2.3. □
Proof of Theorem 2.4. As in the one dimensional case, we consider a unique non-negative local-in-time solution to problem (2.1)–(2.6) which is constructed in Theorem 3.1. This solution can be continued to the global one due to estimates proved below.

Our first goal is to obtain an estimate for an $L^p$-norm of $u(t)$ for a certain fixed $p$, which is uniform in time. Here, we use the Duhamel formula (3.1) in the following way

$$\|u(t)\|_p \leq \left\| e^{\Delta t} u_0 + \int_0^t e^{\Delta (t-s)} u(s) (g(u)n - b(n))(s) \, ds \right\|_p$$

(6.1)

$$+ \left\| \int_0^t \nabla e^{\Delta (t-s)} u(s) \nabla \chi(c(s)) \, ds \right\|_p.$$

**Step 1: Estimate of $\|u(t)\|_p$ for fixed $p = p_0 \in \left( \frac{d}{2}, \frac{d}{d-2} \right)$**. As in Step 1 of the proof of Theorem 2.3, the first term on the right-hand side of (6.1) will be denoted by $\|v(t)\|_{p_0}$, where $v(x,t)$ is a solution to the auxiliary problem (A.5)–(A.7) with $f = u(g(u)n - b(n))$ and $v_0 = u_0$. Recall, that

$$\|f(t)\|_1 \leq \|u(t)(g(u)n - b(n))(t)\|_1 \leq C\|u(t)\|_1 \text{ for all } t > 0.$$ 

Hence, using Lemma A.3 (note that $p_0 < \frac{d}{d-2}$), inequality (4.15), and the elementary estimate $\|u_0\|_1 \leq C(\Omega)\|u_0\|_{p_0}$, we obtain

$$\left\| e^{\Delta t} u_0 + \int_0^t e^{\Delta (t-s)} u(s) (g(u)n - b(n))(s) \, ds \right\|_{p_0} \leq C(\|u_0\|_p + \|n_0\|_1),$$

(6.2)

for some constant $C > 0$ independent of $t > 0$.

Now, we deal with the second term on the right-hand side in (6.1). First, using the heat semigroup estimate (A.3), the assumption $\chi' \in L^\infty([0,\infty))$ and the Hölder inequality with $\frac{1}{q} = \frac{1}{p_0} + \frac{1}{2p_0}$ we obtain

$$\left\| \int_0^t \nabla e^{\Delta (t-s)} u(s) \nabla \chi(c(s)) \, ds \right\|_{p_0} \leq C \int_0^t (t-s)^{-\frac{d}{4} + \frac{1}{2q} - \frac{1}{2}} e^{-\lambda_1(t-s)} \|u\|_{p_0} \|\nabla \chi(c(s))\|_q \, ds$$

(6.3)

$$\leq C \|\chi'\|_\infty \int_0^t (t-s)^{-\frac{d}{4p_0} - \frac{1}{2}} e^{-\lambda_1(t-s)} \|u(s)\|_{p_0} \|\nabla c(s)\|_{2p_0} \, ds.$$

Notice that since $p_0 > d/2$, the function $(t-s)^{-\frac{d}{4p_0} - \frac{1}{2}}$ is integrable at $s = t$.

We proceed in a similar way using equation (3.2) and inequality (A.2) to estimate

$$\|\nabla c(t)\|_{2p_0} \leq \|e^{(\Delta - \beta)t} \nabla c_0\|_{2p_0} + \int_0^t \|\nabla e^{(\Delta - \beta)(t-s)} u(s)\|_{2p_0} \, ds$$

(6.4)

$$\leq e^{-\beta} \|\nabla c_0\|_{2p_0} + C \int_0^t (t-s)^{-\frac{d}{4p_0} - \frac{1}{2}} e^{-\lambda_1(t-s)} \|u(s)\|_{p_0} \, ds.$$

Now, we define function

$$f(t) \equiv \sup_{0 \leq s \leq t} \|u(s)\|_{p_0}.$$
Then, by inequality \((6.4)\) we have that
\[
\| \nabla c(t) \|_{2p_0} \leq \| \nabla c_0 \|_{2p_0} + Cf(t).
\]

Finally, applying estimates \((6.2), (6.3)\) and \((6.5)\) into \((6.1)\) we obtain
\[
\| u(t) \|_{p_0} \leq C(\| u_0 \|_{p_0} + \| n_0 \|_1) + Cf(t)(\| \nabla c_0 \|_{2p_0} + Cf(t)),
\]
which leads the following inequality
\[
f(t) \leq C_1(\| u_0 \|_{p_0} + \| n_0 \|_1) + C_2\| \nabla c_0 \|_{2p_0}f(t) + C_3f^2(t)
\]
for positive constants \(C_1, C_2\) and \(C_3\) independent of \(t > 0\) and of the solution. Now, we prove that, for a sufficiently small initial datum, inequality \((6.6)\) implies that \(f(t)\) has to be bounded function.

Indeed, denote \(H(y) = C_3y^2 + (B - 1)y + A\), where \(B = C_2\| \nabla c_0 \|_{2p_0}\) and \(A = C_1(\| u_0 \|_{p_0} + \| n_0 \|_1)\). It is easy to check that for \(4AC_3 < (B - 1)^2\), the equation \(H(y) = 0\) has two roots, say \(y_1\) and \(y_2\). Moreover, for \(H'(0) = B - 1 < 0\), those roots are both positive. Hence, since \(f(t)\) is non-negative and continuous, if we assume that \(f(0) = \| u_0 \|_{p_0} \in (0, y_1)\) then \(f(t) \in [0, y_1]\) for all \(t > 0\). Note here that \(f(0) \leq A\) because we can choose \(C_1 \geq 1\) without loss of generality. Moreover, by a direct calculation, we have \(A < y_1\). Hence, \(f(0) \in (0, y_1)\), and this completes the proof of Step 1.

**Step 2: Estimate of \(\sup_{t \geq 0} \| u(t) \|_{\infty}\).** We come back to inequality \((6.1)\) with \(p = \infty\). Let \(p_0 \in \left(\frac{d}{2}, \frac{d}{d - 2}\right)\). We can apply step 1 to \(p_0\) and get an uniform \(L^{p_0}\) estimate on \(u\). Now we are going to build on this uniform \(L^{p_0}\) estimate in order to obtain an uniform \(L^\infty\) estimate via a bootstrap method.

Applying Lemma \(A.4\) with \(p = +\infty\) and \(q = p_0\) we obtain the following estimate of the first term in the right-hand side of \((6.1)\):
\[
\left\| e^{\Delta t} u_0 + \int_0^t e^{\Delta(t-s)}u(s)(g(u)n - b(n))(s)ds \right\|_{\infty} \leq C(\| u_0 \|_{p_0} + \| n_0 \|_1 + \sup_{t \in \mathbb{R}_+} \| u(t) \|_{p_0}).
\]

Now, we consider the second term in the right-hand side of \((6.1)\). Let \(p \in (1, +\infty)\). Using estimate \((A.3)\) and Hölder inequality, cf. \((6.4)\), we obtain for all \(q \in (1, +\infty)\),
\[
\| \nabla c(t) \|_q \leq e^{-\beta t} \| \nabla c_0 \|_q + C \int_0^t (t - s)^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{1}{2}} e^{-\lambda_1(t-s)} \| u(s) \|_q ds.
\]

The function \(s \mapsto (t - s)^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{1}{2}}\) is integrable near \(t\) for all \(q < \frac{dp_0}{d - p_0}\). Hence, for each \(q \in [1, \frac{dp_0}{d - p_0}]\) there exists a constant \(C > 0\) such that \(\| \nabla c(t) \|_q \leq C\) for all \(t > 0\).

Assume \(q < \frac{dp_0}{d - p_0}\). Then, for all \(p_1 \in (1, +\infty)\), we have
\[
\left\| \int_0^t \nabla e^{\Delta(t-s)}u(s)\nabla \chi(c(s))ds \right\|_{p_1}
\leq C \left( \int_0^t (t - s)^{-\frac{d}{2}(\frac{1}{p_1} - \frac{1}{q}) - \frac{1}{2}} e^{-\lambda_1(t-s)} \| u \|_q \| \nabla \chi(c(s)) \|_{p_1} ds + \int_0^t (t - s)^{-\frac{d}{2}(\frac{1}{p_1} - \frac{1}{q}) - \frac{1}{2}} e^{-\lambda_1(t-s)} \| u(s) \|_{p_0} \| \nabla c(s) \|_q ds.
\]
where \( \frac{1}{r} = \frac{1}{p_0} + \frac{1}{q} \), or equivalently \( r = \frac{p_0 q}{p_0 + q} \). We have
\[
\sup_{t \in \mathbb{R}^+} \| u(t) \|_{p_1} < +\infty,
\] (6.9)
as soon as \( p_1 \) satisfies
\[
p_1 < \frac{dr}{d - r}.
\]
Computing their derivatives, we can see that the functions \( x \mapsto \frac{dx}{d - x} \) and \( x \mapsto \frac{p_0 x}{p_0 + x} \)
are increasing; thus inequality (6.9) holds for all \( r \) such that
\[
r < \frac{p_0 \frac{dp_0}{d - p_0}}{p_0 + \frac{dp_0}{d - p_0}} = \frac{dp_0^2}{2dp_0 - p_0^2},
\]
or equivalently for all \( p_1 \) such that
\[
p_1 < \frac{d - \frac{dp_0^2}{2dp_0 - p_0^2}}{\frac{dp_0^2}{2dp_0 - p_0^2}} = \frac{1}{2} \frac{dp_0}{d - p_0}.
\]
Let’s consider the sequence \( (u_k)_{k \in \mathbb{N}} \) defined by
\[
u_{k+1} = \frac{1}{2} \frac{du_k}{d - u_k}.
\]
If \( \frac{d}{2} < u_k < d \),
\[
u_{k+1} - u_k = \frac{u_k}{d - u_k} \left( u_k - \frac{d}{2} \right) > 0.
\]
Moreover, if \( u_0, \ldots, u_k \in (\frac{d}{2}, d) \), then
\[
u_{k+1} - u_0 = \sum_{i=0}^{k-1} \frac{u_i}{d - u_i} \left( u_i - \frac{d}{2} \right) > k \frac{u_0}{d - u_0} \left( u_0 - \frac{d}{2} \right),
\]
Therefore, if \( u_0 \in (\frac{d}{2}, d) \) there exists a first rank \( k_0 \) such that \( u_{k_0} > d \) and we have for all \( q \in (\frac{d}{2}, d) \),
(6.10)
\[
\sup_{t \in \mathbb{R}^+} \| u(t) \|_q < +\infty.
\]
Then, for all \( p \in [d, +\infty) \) we can choose a value \( u_0 = q \) close enough to \( d \) in the previous procedure which yields
(6.11)
\[
\sup_{t \in \mathbb{R}^+} \| u(t) \|_p < +\infty.
\]
First, we consider the case \( d = 2 \). By the heat semigroup estimate (A.3) and the Hölder inequality, we obtain the inequalities
\[
\left\| \int_0^t \nabla e^{\Delta(t-s)} u(s) \nabla \chi(c(s)) \, ds \right\|_\infty \leq C \int_0^t (t - s)^{-\frac{1}{2}} \frac{1}{2} e^{-\frac{1}{2} \lambda_1(t-s)} \| u \nabla \chi(c(s)) \|_3 \, ds
\]
\[
\leq C \| \chi' \|_\infty \int_0^t (t - s)^{-\frac{1}{2}} \frac{1}{2} e^{-\frac{1}{2} \lambda_1(t-s)} \| u(s) \|_6 \| \nabla c(s) \|_6 \, ds,
\]

where the right-hand side is bounded uniformly in \( t > 0 \).

Next, we consider the case \( d = 3 \). Repeating the previous estimates for \( p = \infty \), we obtain
\[
\left\| \int_0^t \nabla e^{\Delta (t-s)} u(s) \nabla \chi(c(s)) \, ds \right\|_{\infty} \leq C \int_0^t (t-s)^{-\frac{7}{2}} \frac{1}{2} e^{-\lambda_1 (t-s)} \left\| u \nabla \chi(c(s)) \right\|_4 \, ds \\
\leq C \| \chi \|_\infty \int_0^t (t-s)^{-\frac{7}{2}} e^{-\lambda_1 (t-s)} \| u(s) \|_5 \| \nabla c(s) \|_{20} \, ds,
\]
where the right-hand side is uniformly bounded in \( t > 0 \). This completes the proof of Step 2.

\textit{Step 3: Exponential convergence of} \((u(t), c(t), n(t), w(t))\). First, we show that
\[ \lim_{t \to \infty} \| u(t) \|_p = 0 \quad \text{for every} \quad p \in [1, \infty). \] (6.12)
Here, it suffices to combine the standard interpolation inequality of \( L^p \)-norms
\[ \| u(t) \|_p \leq C \| u(t) \|_1^{1/p} \| u(t) \|_\infty^{1-1/p}, \] (6.13)
with the relation \( \lim_{t \to \infty} \| u(t) \|_1 = 0 \) proved in Theorem 4.1 and with the estimate \( \sup_{t > 0} \| u(t) \|_1 \leq \infty \) by Step 2.

Using relation (6.12), we may show immediately that \( \lim_{t \to \infty} \| u(t) \|_\infty = 0 \) following the reasoning from Step 2 again. Next, we prove the exponential decay of \( \| u(t) \|_1 \) in the same way as in Step 2 of the proof of Theorem 2.3. Therefore, using interpolation equation (6.13) again, we get the exponential decay of \( \| u(t) \|_p \) for every \( p \in [1, \infty) \) as well. By this fact, one can follow the reasoning from Step 2 once again, to obtain that \( \| u(t) \|_\infty \to 0 \) exponentially as \( t \to \infty \). Moreover, by equation (3.2) we immediately show the exponential decay of \( \| c(t) \|_\infty \).

Finally, to obtain the exponential convergence of \( n(t) \) and \( w(t) \) towards a number \( n_\infty \) and a bounded function \( w_\infty \), it suffices to repeat arguments from Step 6 and 7 of the proof of Theorem 2.3.

\( \square \)

\textbf{Remark 6.1.} For \( d \geq 3 \) and under suitable smallness assumptions on initial data, we may show an exponential decay of \( \| u(t) \|_p \) for each \( p \in (1, \infty) \) in the following way. First, multiplying equation (2.1) by \( u^{p-1} \) and integrating over \( \Omega \) we obtain
\[
\frac{d}{dt} \int_\Omega u^p(t) \, dx \leq -C \int_\Omega |\nabla u^{\frac{p}{2}}(t)|^2 \, dx + C \int_\Omega u^{\frac{p}{2}} \nabla u^{\frac{p}{2}}(t) \cdot \nabla \chi(c(t)) \, dx \\
+ p \int_\Omega u^p(t)(g(u)n - b(n))(t) \, dx,
\] (6.14)
for some positive constant \( C = C(p) \). Now, let us notice that by Assumptions 2.1 we have \( \inf_{n} b(n) = b(\| n_0 \|_\infty) > 0 \). Hence, choosing \( \| n_0 \|_\infty \) so small to have
\[ g(u)n - b(n) \leq G_0 \| n_0 \|_\infty - b(\| n_0 \|_\infty) = -r < 0, \]
we obtain the estimate
\[
\int_\Omega u^p(g(u)n - b(n)) \, dx \leq -r \int_\Omega u^p \, dx = -r \| u^{\frac{p}{2}}(t) \|_2^2.
\]
Thus, we get from equation (6.14) the following estimate
\begin{equation}
\frac{d}{dt} \int_{\Omega} u^p(t) \, dx \leq -C_1 \|u^{p/2}(t)\|_{W^{1,2}(\Omega)}^2 + C \|\chi\|_\infty \int_{\Omega} |\nabla u^{p/2}(t)| u^{p/2}(t) |\nabla c(t)| \, dx
\end{equation}
with the constant \(C_1 = \min\{r, 4(p - 1)/p\}\). Moreover, we use the Hölder inequality to obtain
\begin{equation}
\frac{d}{dt} \int_{\Omega} u^p(t) \, dx \leq -C_1 \|u^{p/2}(t)\|_{W^{1,2}(\Omega)}^2 + 2(p - 1) \|u^{p/2}(t)\|_2 \|\nabla u^{p/2}(t)\|_2 \|\nabla c(t)\|_d.
\end{equation}

Next, it suffices to estimate \(\|\nabla c(t)\|_d\) using the Duhamel formula (3.2) and arguments analogous to those in the proof of Theorem 2.4 to obtain, for each \(q \in (\frac{d}{2}, d]\), the estimates
\begin{equation}
\|\nabla c(t)\|_d \leq C e^{-2(\beta + \lambda_1)t} \|\nabla c_0\|_d + C \int_0^t (t - s)^{-\frac{d}{2} - 1} \frac{1}{s} \|u(s)\|_q \, ds
\end{equation}
\begin{equation}
\leq C e^{-2(\beta + \lambda_1)t} \|\nabla c_0\|_d + C \sup_{0 \leq s \leq t} \|u(s)\|_q.
\end{equation}
Moreover, we recall the following Sobolev inequality
\begin{equation}
\|u^p\|_{\frac{2d}{d-2}} \leq C \|u^p\|_{W^{1,2}(\Omega)}.
\end{equation}
Hence, using inequalities (6.16) and (6.17) in (6.15) and choosing \(p = q \in (\frac{d}{2}, d]\) we obtain
\begin{equation}
\frac{d}{dt} \int_{\Omega} u^p(t) \, dx \leq \|u^{p/2}(t)\|_{W^{1,2}(\Omega)}^2 \left(C(\|\nabla c_0\|_d + \sup_{0 \leq s \leq t} \|u(s)\|_p) - C_1\right).
\end{equation}

Observe now that, if the norms \(\|\nabla c_0\|_d\) and \(\|u_0\|_p\) are small enough, the right-hand side of inequality (6.18) is negative for \(t \in [0, \epsilon]\) for small \(\epsilon > 0\). Thus, \(\|u(t)\|_p\) decreases on \(0, \epsilon\). We can repeat this argument on \([\epsilon, 2\epsilon]\) and, by induction, for all \(t > 0\). Since \(\|u(t)\|_p\) is decreasing, we have
\begin{equation}
C(\|\nabla c_0\|_d + \sup_{0 \leq s \leq t} \|u(s)\|_p) - C_1 \leq C(\|\nabla c_0\|_d + \|u_0\|_p) - C_1 < 0.
\end{equation}

Hence, using the obvious inequality \(\|u^{p/2}\|_{W^{1,2}(\Omega)}^2 \geq \int_{\Omega} u^p \, dx\), we obtain from (6.18) the exponential decay of \(\|u(t)\|_p\) for each \(p \in (d/2, d]\). To show the exponential decay of this norm for other \(p \in [1, \infty]\), and to show the exponential convergence of \((c, n, w)\) toward \((0, n_\infty, w_\infty(x))\), it suffices to use a reasoning similar to that one in the proof of Theorem 2.4.

Now, we prove a global-in-time existence result for all solutions (not necessarily small) under the extra condition imposed on the chemotactic sensitivity \(\chi(c)\).

\textbf{Proof of Theorem 2.7.} In the following calculations, we are inspired by [26, Lemma 3.1]. We are going to prove that \(\sup_{t>0} \|u(t)\|_2 < \infty\). We fix small \(\delta > 0\) which will be chosen later and define
\begin{equation}
\varphi(s) := e^{(1+ps)^{-\delta}}.
\end{equation}
Using equations (2.1)–(2.2) and integration by parts we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \varphi(c) \, dx = \int_{\Omega} \varphi(c) u_t \, dx + \frac{1}{2} \int_{\Omega} u^2 \varphi'(c) \, dx
\]

\[
= \int_{\Omega} \varphi(c) \Delta u \, dx - \int_{\Omega} \varphi(c) \nabla \cdot (u\chi'(c) \nabla c) \, dx + \int_{\Omega} u^2 \varphi(c)(g(u)n - b(n)) \, dx
\]

\[
+ \frac{d_c}{2} \int_{\Omega} u^2 \varphi'(c) \Delta c \, dx - \frac{\beta}{2} \int_{\Omega} u^2 c \varphi'(c) \, dx
\]

\[
= - \int_{\Omega} \varphi(c) |\nabla u|^2 \, dx - \int_{\Omega} u\varphi'(c) \nabla u \nabla c \, dx + \int_{\Omega} u^2 \varphi(c)(g(u)n - b(n)) \, dx
\]

\[
+ \int_{\Omega} \varphi(c) \chi'(c) \nabla u \nabla c \, dx + \int_{\Omega} u^2 \varphi'(c) \chi'(c)|\nabla c|^2 \, dx
\]

\[
- d_c \int_{\Omega} \varphi'(c) \nabla u \nabla c \, dx - \frac{d_c}{2} \int_{\Omega} u^2 \varphi''(c)|\nabla c|^2 \, dx
\]

\[
- \frac{\beta}{2} \int_{\Omega} u^2 c \varphi'(c) \, dx + \frac{\alpha}{2} \int_{\Omega} u^3 \varphi'(c) \, dx
\]

Since \(\chi'(s) \geq 0\), \(\varphi'(s) \leq 0\), and function \((g(u)n - b(n))\) is bounded in \(\Omega \times [0, \infty)\), we get

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \varphi(c) \, dx + \int_{\Omega} \varphi(c) |\nabla u|^2 \, dx + \frac{d_c}{2} \int_{\Omega} u^2 \varphi''(c)|\nabla c|^2 \, dx
\]

\[
\leq -(1 + d_c) \int_{\Omega} \varphi'(c) \nabla u \nabla c \, dx + C \int_{\Omega} u^2 \varphi(c) \, dx
\]

\[
+ \int_{\Omega} \varphi(c) \chi'(c) \nabla u \nabla c \, dx - \frac{\beta}{2} \int_{\Omega} u^2 c \varphi'(c) \, dx
\]  

for some constant \(C > 0\). By the inequality \(2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}\), we have

\[
-(1 + d_c) \int_{\Omega} \varphi'(c) \nabla u \nabla c \, dx \leq \frac{1}{4} \int_{\Omega} \varphi(c) |\nabla u|^2 \, dx
\]

\[
+ (1 + d_c)^2 \int_{\Omega} u^2 \frac{(\varphi'(c))^2}{\varphi(c)} |\nabla c|^2 \, dx
\]

as well as

\[
\int_{\Omega} \varphi(c) \chi'(c) \nabla u \nabla c \, dx \leq \frac{1}{4} \int_{\Omega} \varphi(c) |\nabla u|^2 \, dx + \int_{\Omega} u^2 \varphi(c)(\chi'(c))^2 |\nabla c|^2 \, dx
\]

\[
\leq \frac{1}{4} \int_{\Omega} \varphi(c) |\nabla u|^2 \, dx + \chi_0^2 \int_{\Omega} u^2 (1 + pc)^{-4} \varphi(c) |\nabla c|^2 \, dx.
\]

Moreover, by the definition of function \(\varphi(c)\) we have the relations

\[-s \varphi'(s) = \delta \left( \frac{ps}{1 + ps} \right)^{\delta - 1} e^{(1 + ps)^{-\delta}} \leq \delta e^{(1 + ps)^{-\delta}} = \delta \varphi(s)\]

which leads to the estimate

\[
- \frac{1}{2} \int_{\Omega} u^2 c \varphi'(c) \, dx \leq \frac{\delta}{2} \int_{\Omega} u^2 \varphi(c) \, dx.
\]
Using estimates (6.20)–(6.22) in inequality (6.19) we obtain
\begin{equation}
\frac{1}{2} \frac{d}{dt} \int u^2 \varphi(c) \, dx + \frac{1}{2} \int \varphi(c) |\nabla u|^2 \, dx + \frac{d_2}{2} \int u^2 \varphi''(c) |\nabla c|^2 \, dx \\
\leq (1 + d_c)^2 \int u^2 \frac{(\varphi'(c))^2}{\varphi(c)} |\nabla c|^2 \, dx + \chi_0^2 \int u^2 (1 + pc)^{-4} \varphi(c) |\nabla c|^2 \, dx + C \int u^2 \varphi(c) \, dx
\end{equation}
for some constant \(C > 0\).

Now, we are going to show that the terms on the right-hand side of (6.23) containing \(\varphi(c)\) can be compensated by \(\frac{d_2}{2} \int u^2 \varphi''(c) |\nabla c|^2 \, dx\). In order to do so, let us denote
\begin{align*}
I_1 &= (1 + d_c)^2 \frac{(\varphi'(c))^2}{\varphi(c)} = (1 + d_c)^2 p^2 \delta^2 (1 + ps)^{-2\delta-2} e^{(1+ps)^{-\delta}}, \\
I_2 &= \chi_0^2 (1 + pc)^{-4} \varphi(c) = \chi_0^2 (1 + pc)^{-4} e^{(1+ps)^{-\delta}}, \\
I_3 &= \frac{d_2}{2} \varphi''(c) = \frac{d_c}{2} p^2 \delta (\delta + 1) (1 + ps)^{-\delta-2} e^{(1+ps)^{-\delta}} + \frac{d_c}{2} p^2 \delta^2 (1 + ps)^{-2\delta-2} e^{(1+ps)^{-\delta}}.
\end{align*}

Then, by direct calculations, we obtain the estimates
\begin{equation}
\frac{I_1}{2 I_3} \leq \frac{(1 + d_c)^2 p^2 \delta^2 (1 + ps)^{-2\delta-2} e^{(1+ps)^{-\delta}}}{\frac{d_2}{2} p^2 \delta^{(\delta + 1)} (1 + ps)^{-\delta-2} e^{(1+ps)^{-\delta}}} = \frac{4(d_c + 1)^2 \delta}{d_c (\delta + 1)} (1 + ps)^{-\delta} \leq \frac{4(d_c + 1)^2 \delta}{d_c}
\end{equation}
and
\begin{equation}
\frac{I_2}{2 I_3} \leq \frac{\chi_0^2 (1 + pc)^{-4} e^{(1+ps)^{-\delta}}}{\frac{d_2}{2} p^2 \delta^{(\delta + 1)} (1 + ps)^{-\delta-2} e^{(1+ps)^{-\delta}}} = \frac{4 \chi_0^2}{d_c p^2 \delta^{(\delta + 1)} (1 + ps)^{\delta+2-4}}.
\end{equation}

Now, we chose \(\delta = \frac{d_c}{4(d_c+1)^2}\) and notice that if \(\frac{d_c^2}{\chi_0^2} \leq 1\) which implies
\begin{equation}
(1 + d_c)^2 \int u^2 \frac{(\varphi'(c))^2}{\varphi(c)} |\nabla c|^2 \, dx + \chi_0^2 \int u^2 (1 + pc)^{-4} \varphi(c) |\nabla c|^2 \, dx \leq \frac{d_c}{2} \int u^2 \varphi''(c) |\nabla c|^2 \, dx
\end{equation}
and hence, from (6.23) we have
\begin{equation}
\frac{d}{dt} \int u^2 \varphi(c) \, dx + \int \varphi(c) |\nabla u|^2 \, dx \leq C \int u^2 \varphi(c) \, dx,
\end{equation}
for some constant \(C > 0\). Next, by the definition of function \(\varphi(s)\) and the following Sobolev inequality
\[ \|f\|_2 \leq C \|f\|_{W^{1,2}} \|f\|_1^{1-a} \]
with \(a = \frac{d}{d+2} < 1\), we obtain
\begin{equation}
\int u^2 \varphi(c) \, dx \leq c \|u\|_2^2 \leq C \|u\|_{W^{1,2}}^2 \|u\|_1^{2(1-a)} \leq C(\|\nabla u\|_2 + \|u\|_1)^{2a} \|u\|_1^{2(1-a)},
\end{equation}
since for all \(f \in W^{1,2}(\Omega)\) we have \(\|f\|_{W^{1,2}} \leq C(\|\nabla f\|_2 + \|f\|_1)\). Now, by the fact that \(\|u(t)\|_1\) is bounded (see Theorem 4.1), inequality (6.25) implies
\[ \int u^2 \varphi(c) \, dx \leq C(\|\nabla u\|_2^2 + 1)^a. \]
Hence, we have
\[
\|\nabla u\|_2^2 \geq C \left( \int_{\Omega} u^2 \varphi(c) \, dx \right)^{1/a} - 1.
\]
Thus, since \( \varphi \geq 1 \) we obtain
\[
\int_{\Omega} \varphi(c) |\nabla u|^2 \, dx \geq \|\nabla u\|_2^2 \geq C \left( \int_{\Omega} u^2 \varphi(c) \, dx \right)^{1/a} - 1,
\]
for some constant \( C > 0 \) and \( a = \frac{d}{d+2} < 1 \). Using estimate (6.26) in inequality (6.24) we get
\[
\frac{d}{dt} \int_{\Omega} u^2 \varphi(c) \, dx \leq -C_1 \left( \int_{\Omega} u^2 \varphi(c) \, dx \right)^{1/a} + C_2 \int_{\Omega} u^2 \varphi(c) \, dx + 1,
\]
which means that the function \( y(t) = \int_{\Omega} u^2 \varphi(c) \, dx \) satisfies
\[
y'(t) \leq -C_1 y^{\kappa}(t) + C_2 y(t) + 1,
\]
for some positive constants \( C_1, C_2 \) and \( \kappa > 1 \). This immediately implies the boundedness of the function \( y = y(t) \). However, since \( \varphi(s) \geq 1 \), we obtain that \( \sup_{t>0} \|u(t)\|_2 < \infty \).

From now on, it suffices to repeat the reasoning from Steps 2 and 3 of the proof of Theorem 2.4. □

7. Blow up of solutions in two dimensional case

We first give a sketch of the proof of Theorem 2.9. It consists mainly in replacing parabolic estimates of \( \nabla c \) with elliptic estimates and following the remainder of the proofs without many changes.

Proof of Theorem 2.9.

• We first notice that all results in section 4 are still valid for system (2.10)–(2.14) as the proofs do not involve the equation for \( c \).

• In space dimension \( d = 1 \):

  1. Step 2 of the proof of Theorem 2.3 does not involve the \( c \) equation, so we still have \( \lim_{t \to +\infty} \|u(t)\|_1 = 0 \).

  2. Combining this \( L^1 \) decay with lemma B.1 we get the exponential decay of \( \|c_x\|_\infty \), and as a consequence the exponential decay of \( \|c_x\|_p \) for every \( p \in [1, +\infty) \).

  3. This implies Lemma 5.1 for system (2.10)–(2.14) and we can now redo the proof of Lemma 5.2 as well. Then, the remainder of the proof does not involve the elliptic equation except for an upgrade of Lemma 5.1 which we already have thanks to the exponential decay already achieved. Therefore, the result holds true for \( d = 1 \).

• In space dimension \( d = 2 \) or \( d = 3 \) with smallness assumption on initial data:

  1. First, we prove that \( \sup_{t>0} \|u(t)\|_{p_0} < +\infty \) for all \( p_0 \in \left( \frac{d}{2}, \frac{d}{d-2} \right) \) : We follow step 1 of the proof of Theorem 2.4 until we reach the inequality

\[
\left\| \int_0^t \nabla e^{\Delta(t-s)}u(s) \nabla \chi(c(s)) \, ds \right\|_{p_0} \leq C\|\chi\|_\infty \int_0^t (t-s)^{-\frac{d}{2} - \frac{1}{2}} e^{-\lambda_1(t-s)} \|u(s)\|_{p_0} \|\nabla c(s)\|_{2p_0} \, ds.
\]

(7.1)
Then we use the continuous embedding of $W^{1,p_0}(\Omega)$ into $L^{2,p_0}(\Omega)$ (notice that this embedding holds true when $p_0 > d/2$) for two and three dimensional domains along with Lemma B.2 to get
\[
\|\nabla c(s)\|_{L^{2,p_0}(\Omega)} \leq K_1\|\nabla c(s)\|_{W^{1,p_0}(\Omega)} \leq K_2\|u(s)\|_{p_0},
\]
for constants $K_1, K_2 > 0$ depending only on $\Omega, p_0$ and the parameters.

It allows us to obtain, denoting $f(t) = \sup_{0 \leq s \leq t} \|u(s)\|_{p_0}$,
\[
f(t) \leq C_1(\|u_0\|_{p_0} + \|n_0\|_1) + C_3 f^2(t)
\]
for positive constants $C_1$ and $C_3$ independent of $t > 0$ and of the solution. Then, we can finish step 1 without any other change, with the assumption
\[
\max(\|u_0\|_{p_0}, \|n_0\|_1) < \varepsilon,
\]
for small enough $\varepsilon > 0$.

(2) Then we follow line by line the steps 2 and 3 of the proof of Theorem 2.4 to conclude, using the chain of inequalities (7.2) whenever required. Therefore, Theorem 2.4 holds true for system (2.10)–(2.14).

Now we prove the main theorem about problem (2.10)–(2.14).

**Proof of Theorem 2.10** Here, we are inspired by analogous proofs of a blow up of solutions to the parabolic-elliptic model of chemotaxis and we follow the work of Nagai [17]. For given numbers $r_1$ and $r_2$ satisfying $0 < r_1 < r_2 < \text{dist}(q, \partial \Omega)$, we define the function $\phi \in C^1([0, \infty)) \cap W^{2,\infty}(\mathbb{R}^2)$ by the formula
\[
\phi(r) := \begin{cases} r^2 & \text{for } 0 \leq r \leq r_1, \\ a_1 r^2 + a_2 r + a_3 & \text{for } r_1 \leq r \leq r_2, \\ r_1 r_2 & \text{for } r > r_2, \end{cases}
\]
where
\[
a_1 = -\frac{r_1}{r_2 - r_1}, \quad a_2 = \frac{2 r_1 r_2}{r_2 - r_1}, \quad a_3 = -\frac{r_1^2 r_2}{r_2 - r_1}.\]

Hence, for fixed $q \in \Omega$, the function $\varphi(x) = \phi(|x - q|)$ satisfies $\varphi \in C^1(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$. Moreover, by direct computations, we obtain
\[
\nabla \varphi(x) = \begin{cases} 2x & \text{for } |x| \leq r_1, \\ \frac{2 r_1}{r_2 - r_1} \left( r_2 - |x| \right) \frac{x}{|x|} & \text{for } r_1 \leq |x| \leq r_2, \\ 0 & \text{for } |x| > r_2, \end{cases}
\]
\[
|\nabla \varphi(x)| \leq 2(\varphi(x))^{1/2},
\]
\[
\Delta \varphi(x) = 4 \text{ for } |x| \leq r_1 \quad \text{and} \quad \Delta \varphi(x) \leq 2 \text{ for } |x| > r_1.
\]

Now, we consider a non-negative solution $(u, c, n)$ of problem (2.10)–(2.14) on an interval $[0, T_{\text{max}})$ and define mass and the generalized moment by the formulas
\[
M(t) = \int_{\Omega} u(x, t) \, dx \quad \text{and} \quad I(t) = \int_{\Omega} u(x, t) \varphi(x) \, dx.
\]
Integrating by parts and by relation (7.6) it is clear that
\[
(7.7) \quad \int_{\Omega} u(x,t) \Delta \varphi(x) \, dx \leq 4M(t).
\]
Moreover, since the functions \(b, g\) and \(n\) are bounded and non-negative, we obtain the following estimate
\[
(7.8) \quad \int_{\Omega} (g(u)n - b(u))(x,t)u(x,t)\varphi(x) \, dx \leq C_3 I(t),
\]
where \(C_3 = G_0\|n_0\|_{\infty}^4\).

Next, we recall an estimate which is a straightforward adaptation of the result from [17] to system (2.10)–(2.12).

**Lemma 7.1** ([17] Lemma 3.1). Let \(q \in \Omega, 0 < r_1 < r_2 < \text{dist}(q, \partial\Omega)\) and \(\varphi(x) = \phi(|x - q|)\) be defined as above. Then, for all \(t \in (0, T_{\text{max}})\), we have the following estimate
\[
\int_{\Omega} u(x,t) \nabla \varphi(x) \cdot \nabla c(x,t) \, dx \leq \frac{\alpha}{2\pi} M(t)^2 + C_1 M(t) I(t) + C_2 M(t)^{3/2} I(t)^{1/2}
\]
for some constants \(C_1, C_2\) depending on \(r_1, r_2\) and dist\((q, \partial\Omega)\), only.

Now, multiplying equation (2.10) by \(\varphi(x)\), integrating over \(\Omega\) and using estimates (7.7)–(7.8) together with Lemma 7.1, we obtain
\[
\frac{d}{dt} I(t) \leq 4M(t) - \frac{\alpha\chi_0}{2\pi} M^2(t) + (C_1\chi_0 M(t) + C_3) I(t) + C_2\chi_0 M(t)^{3/2} I(t)^{1/2}.
\]

Note that for all \(s > 0\) and \(\varepsilon > 0\) we have the inequality \(s^{1/2} \leq \varepsilon + \frac{1}{4\varepsilon} s\). Hence, for fixed \(\varepsilon > 0\), which will be chosen later, we use inequality (4.15) to obtain
\[
(7.9) \quad \frac{d}{dt} I(t) \leq 4M(t) + \varepsilon - \frac{\alpha\chi_0}{2\pi} M^2(t) + C_4 I(t),
\]
where
\[
(7.10) \quad C_4 = C_3 + C_1\chi_0(\|u_0\|_1 + \frac{1}{\gamma}\|n_0\|_1) + \frac{C_2^2\chi_0^2(\|u_0\|_1 + \frac{1}{\gamma}\|n_0\|_1)^3}{4\varepsilon}.
\]
Estimate (7.9) immediately implies that
\[
(7.11) \quad \frac{d}{dt} \left( I(t)e^{-C_4t} \right) \leq \left( 4M(t) + \varepsilon - \frac{\alpha\chi_0}{2\pi} M^2(t) \right) e^{-C_4t}.
\]
Next, integrating equation (2.10) over \(\Omega\) and using the inequalities \(0 \leq g(u)n \leq C_3 = G_0\|n_0\|_{\infty}\) and \(0 < b(t) \leq B_0\), we deduce that
\[
\frac{d}{dt} M(t) \leq C_3 M(t) \quad \text{and} \quad \frac{d}{dt} M(t) \geq -B_0 M(t),
\]
and hence,
\[
(7.12) \quad M(t) \leq M_0 e^{C_4t} \quad \text{and} \quad M(t) \geq M_0 e^{-B_0 t} \quad \text{for all} \quad t > 0.
\]
Substituting estimates (7.12) in (7.11) we obtain the inequality
\[
\frac{d}{dt} \left( I(t)e^{-C_4t} \right) \leq 4M_0 + \varepsilon - \frac{\alpha\chi_0 M_0^2}{2\pi} e^{-(C_4 + 2B_0)t},
\]
which implies
\begin{equation}
I(t)e^{-C_4t} \leq I(0) + (4M_0 + \varepsilon)t - \frac{\alpha\chi_0M_0^2}{2\pi(C_4 + 2B_0)}(1 - e^{-(C_4 + 2B_0)t}).
\end{equation}

To complete the proof of the nonexistence of global-in-time solutions, it suffices to show that right-hand side of inequality (7.13) is negative for some \( t > 0 \). Hence, it suffices to study the function
\[ f(t) = A + Bt - D(1 - e^{-(C_4 + 2B_0)t}). \]
First, note that \( f(t) \) attains its minimum at a certain point if and only if \( B < \frac{D}{2} \), which is the case if the number \( \frac{1}{2\pi}M_0(8\pi - \alpha\chi_0M_0) + \varepsilon \) is negative. Here, one can choose for instance \( \varepsilon = \frac{1}{4\pi}M_0(\alpha\chi_0M_0 - 8\pi) \). Thus, for sufficiently small \( f(0) = I(0) = A \) there exist \( t > 0 \) such that \( f(t) < 0 \).

Hence, under these assumptions, the function \( I(t) \) becomes negative in a finite time, which is impossible due to positivity of \( \int_\Omega u(x, t)\varphi(x)\,dx \). This means that a solution \( u(t) \) with sufficiently small initial generalized moment \( I(0) \) and the initial mass satisfying
\[ M_0 > \frac{8\pi}{\alpha\chi_0} \]
cannot be continued for all \( t > 0 \).

Finally, since the solution cannot be global-in-time (\( T_{\text{max}} < +\infty \)), theorem 3.5 implies
\[ \limsup_{t \to T_{\text{max}}} \|u(t)\|_\infty = +\infty. \]

\( \square \)

**Remark 7.2.** The same method could be used to extend Nagai’s result about finite time blow-up on the boundary of the domain, assuming \( \partial\Omega \) contains a line segment and requiring an initial mass such that
\[ M_0 > \frac{4\pi}{\alpha\chi_0}. \]

**Appendix A. Parabolic estimates**

First, we recall estimates on the heat semigroup \( \{e^{t\Delta}\}_{t \geq 0} \) in a bounded domain \( \Omega \) with the Neumann boundary condition.

**Lemma A.1.** Let \( \lambda_1 > 0 \) denote the first nonzero eigenvalue of \( -\Delta \) in \( \Omega \) under the Neumann boundary conditions. For all \( 1 \leq q \leq p \leq +\infty \), there exist constants \( C = C_1(p, q, \Omega) \) such that
\begin{align}
\|e^{t\Delta}f\|_{L^p(\Omega)} &\leq Ct^{-\frac{d}{2}(\frac{1}{q} - \frac{1}{p})}e^{-\lambda_1t}\|f\|_{L^q(\Omega)} &\text{for all } f \in L^q(\Omega) \text{ satisfying } \int_\Omega f\,dx = 0 \text{ and all } t > 0; \\
\|e^{t\Delta}f\|_{L^p(\Omega)} &\leq C(1 + t^{-\frac{d}{2}(\frac{1}{q} - \frac{1}{p})})\|f\|_{L^q(\Omega)} &\text{for all } f \in L^q(\Omega) \text{ and all } t > 0; \\
\|\nabla e^{t\Delta}f\|_{L^p(\Omega)} &\leq Ct^{-\frac{d}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{1}{2}}e^{-\lambda_1t}\|f\|_{L^q(\Omega)} &\text{for all } f \in L^q(\Omega) \text{ and all } t > 0.
\end{align}
Proof. Inequalities (A.1)–(A.3) are well-known in a much more general case of an analytic semigroup of bounded operators in $L^p(\Omega)$ generated by a elliptic partial differential operator. The proof of inequality (A.2) is given e.g. in [20] Lemma 3 on p. 25. Inequality (A.1) can be obtained immediately from the abstract theory developed by Amann in [1] by applying it to the heat semigroup with the Neumann boundary conditions on the Banach space $L^p_0(\Omega) = \{ v \in L^p(\Omega) : \int_{\Omega} v(x) \, dx = 0 \}$. Finally, to show inequality (A.3), we combine results from [25], Thm. 16.10 on p. 553; the usual properties of an analytic semigroup, and estimate (A.1) (because one may always assume that $\int_{\Omega} f(x) \, dx = 0$ in (A.3)) to obtain
\begin{align}
\| \nabla e^{t\Delta} f \|_{L^p(\Omega)} &\leq C \| (-\Delta)^{1/2} e^{(t/2)\Delta} (e^{(t/2)\Delta} f) \|_{L^p(\Omega)} \\
&\leq Ct^{-1/2} \| e^{(t/2)\Delta} f \|_{L^p(\Omega)} \\
&\leq Ct^{-\frac{\alpha}{2}(\frac{1}{q} - \frac{1}{p})} e^{-\frac{Ct}{2}} \| f \|_{L^p(\Omega)}
\end{align}
(A.4)

for all $t > 0$. $\square$

Next, we recall a technical lemma which we use systematically in this work.

Lemma A.2. Let $f \in L^\infty(0, \infty)$ satisfy $\lim_{t \to \infty} f(t) = 0$. Then, for each $k \in [0, 1)$ and $M > 0$, we have $\lim_{t \to \infty} \int_0^t (t-s)^{-k} e^{-M(t-s)} f(s) \, ds = 0$. Moreover, the speed of decaying is exponential if the function $f(t) \to 0$ exponentially as $t \to \infty$.

Proof. By direct estimates, we obtain
\begin{align}
\int_0^t (t-s)^{-k} e^{-M(t-s)} f(s) \, ds &= \int_0^{t/2} (t-s)^{-k} e^{-M(t-s)} f(s) \, ds + \int_{t/2}^t (t-s)^{-k} e^{-M(t-s)} f(s) \, ds \\
&\leq \| f \|_\infty \int_0^{t/2} \left( \frac{t}{2} \right)^{-k} e^{-\frac{M}{2}t} \, ds + \sup_{s \in [t/2, t]} f(s) \int_{t/2}^t (t-s)^{-k} e^{-M(t-s)} \, ds \\
&= \| f \|_\infty \left( \frac{t}{2} \right)^{-k+1} e^{-\frac{M}{2}t} + \sup_{s \in [t/2, t]} f(s) \int_{t/2}^t (t-s)^{-k} e^{-M(t-s)} \, ds.
\end{align}

The right-hand side tends to zero since $\sup_{t > 0} \int_{t/2}^t (t-s)^{-k} e^{-M(t-s)} \, ds < \infty$ and since the function $f(s)$ tends to zero as $s \to \infty$. $\square$

The following result on the large time behaviour of solutions to the nonhomogeneous heat equation seems to be known. However, for the completeness of the exposition, we present its proof.

Lemma A.3. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let
\[ p \in [1, \infty) \quad \text{if} \quad d = 1, \quad p \in [1, \infty) \quad \text{if} \quad d = 2, \quad p \in \left[1, \frac{d}{d-2}\right) \quad \text{if} \quad d \geq 3. \]
Assume that $v_0 \in L^p(\Omega)$ and $f(x,t) \in L^\infty([0,\infty), L^1(\Omega))$ Then, the solution to the following initial value problem

(A.5)  \[ v_t = \Delta v + f(x,t) \quad \text{for} \quad x \in \Omega, \ t > 0, \]

(A.6)  \[ \frac{\partial v}{\partial \nu} = 0, \quad \text{for} \quad x \in \partial \Omega, \ t > 0, \]

(A.7)  \[ v(x,0) = v_0(x) \quad \text{for} \quad x \in \Omega \]

satisfies

(A.8)  \[ \|v(t)\|_p \leq C(\|v_0\|_p + \|v(t)\|_1 + \sup_{s>0} \|f(s)\|_1) \quad \text{for all} \quad t > 0, \]

where a constant $C$ is independent of $t > 0$. Moreover, if $\|f(\cdot, t)\|_1 \to 0$ as $t \to \infty$ then we have

(A.9)  \[ \left\| v(t) - \frac{1}{|\Omega|} \int_\Omega v(t) \, dx \right\|_p \to 0 \quad \text{as} \quad t \to \infty. \]

In addition, if $\|f(\cdot, t)\|_1 \to 0$ exponentially as $t \to \infty$, then the convergence in (A.9) is exponential, as well.

Proof. The function

(A.10)  \[ w(x,t) = v(x,t) - \frac{1}{|\Omega|} \int_\Omega v(x,t) \, dx \]

is a solution to the following initial value problem

\[ w_t = \Delta w + f(x,t) - \frac{1}{|\Omega|} \int_\Omega f(x,t) \, dx \quad \text{for} \quad x \in \Omega, \ t > 0, \]

\[ w(x,0) = w_0(x) = v_0(x) - \frac{1}{|\Omega|} \int_\Omega v_0(x) \, dx, \]

supplemented with the Neumann boundary condition. We estimate the $L^p$-norm of $w$ using its Duhamel representation

(A.11)  \[ w(t) = e^{\Delta t} w_0 + \int_0^t e^{\Delta (t-s)} \left( f - \frac{1}{|\Omega|} \int_\Omega f \, dx \right) \, ds. \]

Obviously we have the inequality \[ \left\| f(s) - \frac{1}{|\Omega|} \int_\Omega f(x,s) \, dx \right\|_1 \leq 2 \| f(s) \|_1. \] Thus, we may use estimate (A.1) (note that \( \int_\Omega w(x,t) \, dx = 0 \) for all $t \geq 0$) in the following way

(A.12)  \[ \|w(t)\|_p \leq Ce^{-\lambda_1 t} \|w_0\|_p + C \| f \|_1 \int_0^t (t-s)^{-\frac{p}{2}(1-\frac{1}{p})} e^{-\lambda_1 (t-s)} \| f(s) \|_1 \, ds. \]

Now, the inequality \[ -\frac{d}{2}(1-\frac{1}{p}) > -1 \] holds true due to the assumption on $p$. Moreover, notice that by the definition of $w$ in (A.10), we have the following, elementary inequalities

(A.13)  \[ \|v(t)\|_p \leq \|w(t)\|_p + |\Omega| \frac{1}{1-p} \|v(t)\|_1 \]

(A.14)  \[ \|w_0\|_p \leq \|v_0\|_p + |\Omega| \frac{1}{1-p} \|v_0\|_1 \leq C \|v_0\|_p. \]

Thus, applying estimates (A.13), (A.14) in inequality (A.12) we obtain bound (A.8) because sup_{t>0} \( \int_0^t (t-s)^{-\frac{1}{2}(1-\frac{1}{p})} e^{-\lambda_1 (t-s)} \, ds < \infty. \)
To show the convergence in (A.9), we apply Lemma A.2 in inequality (A.12) which completes the proof of Lemma A.3.

Next, we slightly generalise estimates from Lemma A.3.

**Lemma A.4.** Let \( \Omega \subset \mathbb{R}^d \) be bounded and fix \( p \in [1, \infty] \). Assume that \( v_0 \in L^p(\Omega) \) and \( f \in L^q((0, \infty), L^q(\Omega)) \) for some \( \frac{dp}{2p+2} \leq q \leq p \). Then there exist a constant \( C > 0 \) such that the solution of problem (A.5)–(A.7) satisfies

\[
\|v(t)\|_p \leq C\left(\|v_0\|_p + \|v(t)\|_1 + \sup_{t>0} \|f(s)\|_1 + \sup_{t>0} \|f(s)\|_q\right)
\]

for all \( t > 0 \). Moreover, if \( \|f(\cdot, t)\|_1 \to 0 \) and \( \|f(\cdot, t)\|_q \to 0 \) as \( t \to \infty \) then we have

\[
(A.15) \quad \left\|v(t) - \frac{1}{|\Omega|} \int_{\Omega} v(t) \, dx\right\|_p \to 0 \quad \text{as} \quad t \to \infty.
\]

In addition, if \( \|f(\cdot, t)\|_1 \to 0 \) and \( \|f(\cdot, t)\|_q \to 0 \) exponentially as \( t \to \infty \), then the convergence in (A.15) is exponential as well.

**Proof.** We proceed in the same way as in the proof of Lemma A.3. The only difference consists in writing inequality (A.12) in the following way

\[
\|w(t)\|_p \leq C\left(e^{-\lambda_1 t}\|w_0\|_p + \int_0^t (t-s)^{-\frac{d}{2(q-\frac{d}{2})}} e^{-\lambda_1 (t-s)} \left(\|f(s)\|_q + |\Omega|^{1-\frac{d}{q}}\|f(s)\|_1\right) \, ds\right).
\]

This ends the proof of Lemma A.4. \( \square \)

**APPENDIX B. ELLIPTIC ESTIMATES**

The following lemma can be found with its proof in [27], Lemma 2.1.

**Lemma B.1.** Let \( \Omega \) a bounded open set of \( \mathbb{R} \) and \( u \in C^0(\overline{\Omega}) \), the unique solution \( c \in C^2(\Omega) \) of

\[
-c_{xx} + c = u
\]

with Neumann boundary condition satisfies

\[
\inf_{x \in \Omega} u(x) \leq c(y) \leq \sup_{x \in \Omega} u(x), \quad y \in \Omega
\]

and, for all \( p \in [1, +\infty] \),

\[
\|c\|_p \leq \|u\|_p,
\]

as well as

\[
\|c_x\|_{\infty} \leq 2\|u\|_1.
\]

If moreover \( u \) is non-negative, \( c \) is non-negative as well and

\[
\|c_x\|_{\infty} \leq \|u\|_1.
\]

In higher dimension, we use standard elliptic estimates one can found, for example, in [9], page 98, Theorem 6.6 for Schauder estimates, Theorem 8.13 for \( L^p \) estimates or the recent and detailed [18] for the very similar Poisson equation.
Lemma B.2. Let $\eta \in (0, 1)$. Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$ of $C^{2, \eta}$ boundary. Let $u \in C^{0, \eta}(\bar{\Omega})$. There exists a unique solution $c \in C^{2, \eta}(\bar{\Omega})$ of the boundary-value problem

$$\begin{cases}
-\Delta c + \beta c = \alpha u & \text{for } x \in \Omega \\
\frac{\partial c}{\partial \nu} = 0 & \text{for } x \in \partial \Omega
\end{cases}$$

(B.5)

and it satisfies

$$\|c\|_{C^{2, \eta}} \leq K_1 \|u\|_{C^{0, \eta}}$$

for some constant $K_1$ depending on $\Omega$, $\alpha$, $\beta$ and $d$. Moreover, for all $p \in (1, +\infty)$,

$$\|c\|_{W^{2, p}} \leq K_p \|u\|_p \quad \text{and} \quad \|
abla c\|_{\infty} \leq K \|u\|_{\infty}$$

for some constants $K_p, K$ depending on $\Omega$, $\alpha$, $\beta$, $d$ and $p$.

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