A Coxeter type classification of Dynkin type $\mathbb{A}_n$ non-negative posets

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Abstract. We continue the Coxeter spectral analysis of finite connected posets $I$ that are non-negative in the sense that their symmetric Gram matrix $G_I := \frac{1}{2}(C_I + C_I^T) \in \mathbb{M}_n(\mathbb{Q})$ is positive semi-definite of rank $n \geq 0$, where $C_I \in \mathbb{M}_m(\mathbb{Z})$ is the incidence matrix of $I$ encoding the relation $\preceq_I$. We extend the results of [Fundam. Inform., 139.4(2015), 347–367] and give a complete Coxeter spectral classification of finite connected posets $I$ of Dynkin type $\mathbb{A}_n$.

We show that such posets $I$, with $|I| > 1$, yield exactly $\left\lfloor \frac{m^2}{2} \right\rfloor$ Coxeter types, one of which describes the positive (i.e., with $n = m$) ones. We give an exact description and calculate the number of posets of every type. Moreover, we prove that, given a pair of such posets $I$ and $J$, the incidence matrices $C_I$ and $C_J$ are $\mathbb{Z}$-congruent if and only if $\text{spec}e_I = \text{spec}e_J$, and present deterministic algorithms that calculate a $\mathbb{Z}$-invertible matrix defining such a $\mathbb{Z}$-congruence in a polynomial time.

Keywords: non-negative poset, unit quadratic form, Coxeter-Dynkin type, Coxeter spectrum

1. Introduction

Coxeter spectral study of finite posets and, more generally, edge-bipartite graphs, is inspired by their applications in the representation theory of posets, finite groups, classical orders, finite-dimensional algebras over a field $K$, and cluster $K$-algebras, see [1–4], and [5–10]. The algebraic framework of this study is given in [6, 11].

We note that the Coxeter spectral classification of finite posets, up to the Gram $\mathbb{Z}$-congruences defined in Section 2, grew up from many different branches of mathematics and computer science.
and is successfully applied in Lie theory, Diophantine geometry, algebraic combinatorics, representation theory, matrix analysis, graph theory, combinatorial and graph algorithms, singularity theory, and related areas. The reader is referred to [5, 6, 12] and [13, Section 6.1.2] for a more detailed discussion.

In the present paper, we study posets that are non-negative of Dynkin type $A_m$, as defined in Section 2. In [14] we give a characterization of such posets in terms of their Hasse digraphs (see Fact 2.18). One of the main results of the present paper is Theorem 1.1, which gives a detailed description of such posets up to the strong Gram $Z$-congruence (see Definition 2.9). This yields a complete Coxeter spectral classification of this class of posets and generalizes the results of [9, 15, 16]. In particular, we show that the complex Coxeter spectrum $\text{spec}_{C} \subseteq C (2.8)$ determines a connected non-negative poset $I$ of Dynkin type $\text{Dyn}_{I} = A_m$ uniquely, up to the strong Gram $Z$-congruence.

**Theorem 1.1.** Let $I = (I, \leq)$ be a finite connected non-negative poset of corank $\text{crk}_I \geq 0$ and Dynkin type $\text{Dyn}_{I} = A_{|I| - \text{crk}_I}$.

(a) Poset $I$ is either positive or principal, that is, $\text{crk}_I \in \{0, 1\}$.

(b) If $I$ is positive, then

(b1) $I$ is strongly Gram $Z$-congruent with a one peak poset $0A_{n-1}^*$ (4.6),

(b2) $\text{cox}_{I}(t) = t^{n+1} + \ldots + t + 1 \in \mathbb{Z}[t],$

(b3) $c_I = n + 1.$

(c) If $I$ is principal, then

(c1) $I$ is strongly Gram $Z$-congruent with a canonical two peak poset $pA_n$ (4.2), where $p := c(I)$ is the cycle index of $I$ (3.11),

(c2) $\text{cox}_{I}(t) = t^n - t^p - t^{n-p} + 1 \in \mathbb{Z}[t],$

(c3) $c_I = \infty$ and $\hat{c}_I = \text{lcm}(p, n-p).$

In other words, there exist precisely $\lfloor \frac{n}{2} \rfloor$ Coxeter types of connected Dynkin type $A_n$ non-negative posets, and exactly one of them describes positive ones. In Theorem 1.2 we give formulae for the exact number of all, up to the isomorphism, posets of a given Coxeter polynomial.

**Theorem 1.2.** Assume that $n$ is a natural number. Up to poset isomorphism, there exist exactly:

(a) $N(P_n)$ connected non-negative posets $I$ with a Coxeter polynomial $\text{cox}_{I}(t) = t^{n+1} + \ldots + t + 1,$ where

$$N(P_n) = \begin{cases} 2^{n-2}, & \text{if } n \geq 2 \text{ is even}, \\ 2^{\frac{n-3}{2}} + 2^{n-2}, & \text{if } n \geq 1 \text{ is odd}, \end{cases}$$

and every such a poset $I$ is positive, i.e., $\text{crk}_I = 0;$
(b) \( N(C_n,p) - 1 \) connected non-negative posets \( I \) with the Coxeter polynomial \( \text{cox}_I(t) = t^n - t^p - t^{n-p} + 1 \), where \( 2 \leq \frac{n}{2} \leq p \leq n - 2 \),

\[
N(C_n,p) = \begin{cases} \\
\frac{1}{n} \sum \gcd(n,p) \varphi(d) \frac{n/d}{p/d}, & \text{if } p \in \left\{ \left\lceil \frac{n}{2} \right\rceil, \ldots, n - 2 \right\} \text{ and } p \neq \frac{n}{2}, \\
\frac{1}{2n} \sum \gcd(n,\frac{n}{2}) \varphi(d) \frac{n/d}{\lfloor n/2d \rfloor} + 2^{n/2-2}, & \text{if } p = \frac{n}{2}.
\end{cases}
\]

where \( \varphi \) is Euler’s totient function. Moreover, every such a poset \( I \) is principal, i.e., \( \text{crk}_I = 1 \).

As a direct result of Theorem 1.1 we obtain a useful algebraic tool for checking whether two posets are strongly Gram \( \mathbb{Z} \)-congruent. Namely, every two connected non-negative posets \( I \) and \( J \) of Dynkin type \( \text{Dyn}_I = \text{Dyn}_J = \mathbb{A}_m \) are strongly Gram \( \mathbb{Z} \)-congruent if and only if \( \text{cox}_I = \text{cox}_J \).

**Corollary 1.3.** If \( I \) and \( J \) are non-negative connected posets of Dynkin type \( \mathbb{A}_m \), then \( I \approx_{\mathbb{Z}} J \) if and only if \( \text{cox}_I(t) = \text{cox}_J(t) \) or, equivalently, \( \text{specc}_I = \text{specc}_J \).

This is a solution to the following variant of the Coxeter spectral analysis problem formulated by Simson in [6, 11] and studied in [8–10, 12–19], for a wide class of connected non-negative posets \( I \) of Dynkin type \( \mathbb{A}_m \).

**Problem 1.4.** When the Coxeter polynomial \( \text{cox}_I(t) \in \mathbb{Z}[t] \) (equivalently: the Coxeter spectrum \( \text{specc}_I \subseteq \mathbb{C} \)) of a finite poset \( I \) determines the incidence matrix \( C_I \in \mathbb{M}_n(\mathbb{Z}) \) uniquely, up to the strong \( \mathbb{Z} \)-congruence?

There is also a natural question of not only determining whether posets \( I \) and \( J \) are strong Gram \( \mathbb{Z} \)-congruent, but computing a matrix that defines this congruence, see [6, 11].

**Problem 1.5.** Construct an algorithm that computes such a \( \mathbb{Z} \)-invertible matrix \( B \in \text{Gl}(n; \mathbb{Z}) \), that \( B^{tr} \cdot C_I \cdot B = C_J \), for any pair of strong Gram \( \mathbb{Z} \)-congruent connected non-negative posets \( I \) and \( J \).

We present a solution to Problem 1.5 for a wide class of connected non-negative posets, i.e., posets of Dynkin type \( \mathbb{A}_m \), in the form of efficient polynomial-time deterministic algorithms: Algorithm 5.2 (for \( \text{crk}_I = 0 \)) and Algorithm 5.7 (for \( \text{crk}_I \neq 0 \)), see Section 5 for thorough description and complexity analysis.

## 2. Preliminaries and notation

Throughout the paper, by \( \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \) we denote the set of non-negative integers, the ring of integers, the rational, the real, and the complex number fields, respectively. We view \( \mathbb{Z}^n \), with \( n \geq 1 \), as a free abelian group, and we denote by \( e_1, \ldots, e_n \) the standard \( \mathbb{Z} \)-basis of \( \mathbb{Z}^n \). We use a row notation for vectors \( v = [v_1, \ldots, v_n] \) and we write \( v^{tr} \) to denote a column vector. Given \( n \geq 1 \), by \( \mathbb{M}_n(\mathbb{Z}) \) we denote the \( \mathbb{Z} \)-algebra of all square \( n \) by \( n \) matrices, by \( E \in \mathbb{M}_n(\mathbb{Z}) \) the identity matrix, and by \( \text{Gl}(n,\mathbb{Z}) := \{ A \in \mathbb{M}_n(\mathbb{Z}), \det A \in \{-1, 1\} \} \subseteq \mathbb{M}_n(\mathbb{Z}) \) the general integral \( \mathbb{Z} \)-linear group.
We say that two square integral matrices $X \in \mathbb{M}_n(\mathbb{Z})$ and $Y \in \mathbb{M}_n(\mathbb{Z})$ are $\mathbb{Z}$-congruent if there exists such a matrix $B \in \text{Gl}(n; \mathbb{Z})$, that $B^\text{tr} \cdot X \cdot B = Y$. We denote this relation by $X \sim_\mathbb{Z} Y$ and write $X \sim_B^Y$ to denote the matrix $B \in \mathbb{M}_n(\mathbb{Z})$ defining the congruence.

By a finite signed graph we mean a triplet $G = (V_G, E_G, \text{sgn}_G)$ consisting of a finite set of vertices $V_G$, a finite set of edges $E_G$ and a sign function $\text{sgn}_G : E_G \to \{-1, 1\}$. By an edge, we mean a pair of (not necessarily distinct) vertices and, for simplicity of presentation, throughout the paper we assume that $V_G = \{1, \ldots, n\}$. By a directed [signed] graph (digraph or quiver) we mean a graph $G$, whose edges $\alpha \in E_G$ have designated source $s(\alpha) \in V_G$ and target $t(\alpha) \in V_G$. Following [6, 11], by a finite edge-bipartite graph (bigraph) we mean a signed graph $\Delta = (\Delta_0, \Delta_1, \text{sgn} : \Delta_1 \mapsto \{+1, -1\})$, with the sign map constant on the multiset $\Delta_1(u, v) = \Delta_1(v, u) \subseteq \Delta_1$ of edges adjacent with the vertices $u, v \in \Delta_0$. Graphically, we represent bigraphs as graphs with multiple edges, where:

- **positive** edges $\Delta^+_1 := \{e \in \Delta_1; \text{sgn}(e) = +1\}$ are denoted by dotted lines $u \rightsquigarrow v$ and
- **negative** edges $\Delta^-_1 := \{e \in \Delta_1; \text{sgn}(e) = -1\}$ are denoted by full lines $u \rightarrow v$.

We note that graphs $G = (V_G, E_G)$ can be viewed as bigraphs $\Delta = (V_G, E_G, \text{sgn})$ with a constant sign function $\text{sgn}(e) := -1$ for every $e \in E_G$.

Two (di)graphs $G = (V_G, E_G)$ and $G' = (V'_{G'}, E'_{G'})$ are called isomorphic $G \simeq G'$ if there exists a bijection $f : V_G \to V_{G'}$ that preserves (directed) edges. By underlying graph $\overline{D}$, we mean a graph obtained from signed digraph $D$ by forgetting the orientation and signs of its edges. We call graph $G$ a path graph if $V_G$ is an empty set or $G \simeq P_n(u, v) := u \rightsquigarrow \ldots \rightsquigarrow v$ and $u \neq v$ (if $u = v$, we call $G$ a cycle). We say that a digraph $D$ is an oriented path if $D \simeq P_n(u, v)$ and $a \neq b$ (if $a = b$ we call $D$ an oriented cycle). A digraph $D$ is called acyclic if it contains no oriented cycle $\overline{P}(a, a) := a \rightsquigarrow \ldots \rightsquigarrow a$, i.e., induced subdigraph isomorphic to $\overline{P}(a, a)$. A graph $G = (V_G, E_G)$ is connected if $P(u, v) \subseteq G$ for every $u \neq v \in V_G$. A digraph $D$ (bigraph $\Delta$) is connected if the graph $\overline{D}$ ($\overline{\Delta}$) is connected. A connected (di)graph is called a tree if it does not contain any cycle. We call a vertex $v$ of a digraph $D = (V, A)$ a source (minimum) if it is not a target of any edge $\alpha \in A$. Analogously, we call $v \in D$ a sink (maximum) if it is not a source of any edge. By degree $\text{deg}(v)$ of a vertex $v$ in (di)graph $G$ we mean a number of edges incident with $v$. A (di)graph $G$ is called 2-regular if $\text{deg}(v) = 2$ for every $v \in G$.

Every (bi)graph $\Delta$ is uniquely determined by the non-symmetric Gram matrix $\hat{G}_\Delta \in \mathbb{M}_n(\mathbb{Z})$

$$\hat{G}_\Delta := \begin{bmatrix} 1 + d_{11}^\Delta & d_{12}^\Delta & \cdots & d_{1n}^\Delta \\ 0 & 1 + d_{22}^\Delta & \cdots & d_{2n}^\Delta \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 + d_{nn}^\Delta \end{bmatrix}, \quad \text{where } d_{ij}^\Delta := |\Delta^+_1(i, j)| - |\Delta^-_1(i, j)| \tag{2.1}$$

and the symmetric Gram matrix $G_\Delta := \frac{1}{2}(\hat{G}_\Delta + \hat{G}_\Delta^\text{tr}) \in \mathbb{M}_n(\mathbb{Q})$, see [6].

By a finite poset $I = \{1, \ldots, n\}$, $\preceq$) we mean a partially ordered set $I$, with respect to a partial order relation $\preceq$. Every finite poset $I$ is uniquely encoded in the form of its incidence matrix

$$C_I = [c_{ij}] \in \mathbb{M}_n(\mathbb{Z})$$

where $c_{ij} = 1$ if $i \preceq j$ and $c_{ij} = 0$ otherwise, \quad \tag{2.2}
see [7]. We often use the Hasse digraph $\mathcal{H}(I)$ representation of $I$, where $\mathcal{H}(I) = (V, A)$ is an acyclic digraph with vertices $V = \{1, \ldots, n\}$ and edges defined as follows: there is an oriented edge (arrow) $i \to j \in A$ if and only if $i \leq j$ and there is no such a $k \in \{1, \ldots, n\}$ such that $i \leq k \leq j$, see [3, Section 14.1]. We note that $\mathcal{H}(I)$ encodes $I$ uniquely. We associate with $I$ its (incidence) bigraph $\Delta_I = (V, A)$, where $V = \{1, \ldots, n\}$ and $i \longrightarrow j \in A$ iff $i \leq j$. We call $I$ connected if the bigraph $\Delta_I$ or, equivalently, the digraph $\mathcal{H}(I)$ is connected. Following [7] we associate with $I = (\{1, \ldots, n\}, \leq)$:

- the symmetric Gram matrix $G_I := \frac{1}{2} (C_I + C_I^{tr}) \in M_n(\mathbb{Q})$,
- the incidence bilinear form $b_I: \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$, $b_I(x, y) := \sum_{i \leq j} x_i y_j = x \cdot C_I \cdot y^{tr}$,  
- the incidence quadratic form $q_I: \mathbb{Z}^n \to \mathbb{Z}$, $q_I(x) := b_I(x, x) = \sum_{i \leq j} x_i x_j = \sum_i x_i^2 + \sum_{i < j} x_i x_j = x \cdot C_I \cdot x^{tr} = x \cdot G_I \cdot x^{tr}$,  
- the Coxeter matrix $\text{Cox}_I := -C_I \cdot C_I^{-tr} \in M_n(\mathbb{Z})$, where $C_I^{-tr} := (C_I^{tr})^{-1} = (C_I^{-1})^{tr}$,  
- the Coxeter polynomial $\text{cox}_I(t) := \det(tE - \text{Cox}_I) \in \mathbb{Z}[t]$,  
- the Coxeter spectrum $\text{spec}_I := \{ \lambda \in \mathbb{C}; \text{cox}_I(\lambda) = 0 \} \subseteq \mathbb{C}$,

that is, the multiset of all eigenvalues of the Coxeter matrix $\text{Cox}_I \in M_n(\mathbb{Z})$ (with multiplicities).

We note that $C_I = \hat{G}_{\Delta_I}$ if and only if vertices of the digraph $\mathcal{H}(I)$ are topologically ordered.

A poset $I$ is called non-negative [of corank $\text{crk}_I \geq 0$] if its symmetric Gram matrix (2.3) is positive semi-definite of rank $n - \text{crk}_I \geq 0$, see [7]. By a positive [principal] poset, we mean a non-negative $I$ of corank $\text{crk}_I = 0 [\text{crk}_I = 1]$.

**Definition 2.9.** Two posets $I$ and $J$ are said to be: strongly [weakly] Gram $\mathbb{Z}$-congruent and denoted by $I \approx_{\mathbb{Z}} J [I \sim_{\mathbb{Z}} J]$ if their incidence matrices (2.2) [symmetric Gram matrices (2.3)] are $\mathbb{Z}$-congruent, i.e., $C_I \sim_{\mathbb{Z}} C_J [G_I \sim_{\mathbb{Z}} G_J]$. A poset $I$ is strongly [weakly] Gram $\mathbb{Z}$-congruent with a bigraph $\Delta'$, i.e., $I \approx_{\mathbb{Z}} \Delta' [I \sim_{\mathbb{Z}} \Delta']$ if $C_I \sim_{\mathbb{Z}} \hat{G}_{\Delta'} [G_I \sim_{\mathbb{Z}} G_{\Delta'}]$. It is straightforward to check that

$$I \simeq J \Rightarrow I \approx_{\mathbb{Z}} J$$

(2.10)

for any two posets $I$ and $J$, that is, isomorphic posets are strongly Gram $\mathbb{Z}$-congruent. In this case, the $\mathbb{Z}$-congruence is defined by the permutation matrix $B_\sigma \in \text{Gl}(n; \mathbb{Z})$, where $\sigma: I \to J$ is a bijection defining $I \simeq J$ isomorphism.

Given a non-negative poset $I$, following [7] and [6, 11, 20], we use the following definitions.

- The Coxeter number $c_I \in \mathbb{N}$ is the order of $\text{Cox}_I$ (2.6) in the group $\text{Gl}(n; \mathbb{Z})$, that is, such a minimal integer $c_I \geq 1$ that $\text{Cox}_I^c = E$. If such a number does not exist, we set $c_I := \infty$. (2.11)
- The reduced Coxeter number $\hat{c}_I \in \mathbb{N}$, that is, such a minimal $\hat{c}_I \geq 1$, that for every $1 \leq i \leq n$

$$e_i \cdot (\text{Cox}_I^{\hat{c}_I} - E) \in \text{Ker } q_I,$$

where $\text{Ker } q_I := \{ v \in \mathbb{Z}^n ; q_I(v) = 0 \}$, (2.12)
The basic properties of a strong Gram $\mathbb{Z}$-congruence are summarized in the following fact. In particular, we note that Coxeter spectrum (2.8), Coxeter polynomial (2.7), Coxeter number (2.12) and reduced Coxeter number (2.12) are invariant under strong Gram $\mathbb{Z}$-congruence.

**Fact 2.13.** Assume that $I$ and $J$ are finite partially ordered sets, and $\Delta$ is a bigraph.

(a) $I \approx_{\mathbb{Z}} J \Rightarrow \text{spec}_c I = \text{spec}_c J, \ cox_I(t) = cox_J(t), \ c_I = c_J$ and $\check{c}_I = \check{c}_J$

(b) $I \approx_{\mathbb{Z}} J \Rightarrow I \sim_{\mathbb{Z}} J$

(c) If poset $I$ [bigraph $\Delta$] is non-negative of corank $r$ and $I \sim_{\mathbb{Z}} J$ [$I \sim_{\mathbb{Z}} \Delta$], then poset $J$ [bigraph $\Delta$] is non-negative of corank $r$.

**Proof:**
Apply arguments of [6, Lemma 2.1] and [7, Lemma 3].

In the case of non-negative posets $I$, the kernel $\text{Ker} q_I$ is a group that admits a $(k_1, \ldots, k_r)$-special $\mathbb{Z}$-basis in the following sense.

**Fact 2.14.** Assume that $I = \{1, \ldots, n\}$ is a connected non-negative poset of corank $r \geq 1$.

(a) There exist integers $1 \leq j_1 < \ldots < j_r \leq n$ such that free abelian group $\text{Ker} q_I \subseteq \mathbb{Z}^n$ of rank $r \geq 1$ admits a $(k_1, \ldots, k_r)$-special $\mathbb{Z}$-basis $h^{(k_i)}$, $\ldots$, $h^{(k_r)} \in \text{Ker} q_I$, that is, $h^{(k_i)}_{k_j} = 0$ for $1 \leq i, j \leq r$ and $i \neq j$.

(b) $I^{(k_i)} := I \setminus \{k_i\}$ is a connected non-negative poset of corank $r - 1 \geq 0$.

(c) Poset $I^{(k_1, \ldots, k_r)} := I \setminus \{k_1, \ldots, k_r\}$ is of corank 0 (i.e., positive) and connected. (2.15)

**Proof:**
Since, without loss of generality, one may assume that $C_I = \mathcal{G}_{\Delta_I}$ (i.e., vertices of the digraph $\mathcal{H}(I)$ are topologically ordered), apply [21, Proposition 5.1] and [19, Theorem 2.1].

| $A_n$: | $1 \quad 2 \quad \ldots \quad n-1 \quad n$ (n\geq 1); |
|-------|-----------------------------------------------|
| $D_n$: | $1 \quad 2 \quad \ldots \quad n-1 \quad n$ (n\geq 1); |
| $E_6$: | $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$ |
| $E_7$: | $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$ |
| $E_8$: | $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8$ |

**Table 2.16:** Simply-laced Dynkin diagrams

Following [7, 13, 20] with any connected non-negative poset $I$ we associate its **Dynkin type** $\text{Dyn}_I \in \{A_n, D_n, E_6, E_7, E_8\}$. 


**Definition 2.17.** Assume that $I$ is a connected non-negative poset of corank $r \geq 0$. The Dynkin type $	ext{Dyn}_I$ is defined to be the unique simply-laced Dynkin diagram of Table 2.16 viewed as a bigraph

$$\text{Dyn}_I \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$$

such that $\check{\Delta}_I$ is weakly Gram $\mathbb{Z}$-congruent with $\text{Dyn}_I$, where

- $\check{\Delta}_I := \Delta_I$ if $r = 0$ (i.e., $I$ is positive),
- $\check{\Delta}_I := \Delta_I^{(k_1, \ldots, k_r)} = \Delta_I \setminus \{k_1, \ldots, k_r\} \leq \Delta_I$ (2.15) if $r > 0$.

The bigraph $\text{Dyn}_I$ can be obtained by means of the inflation algorithm [12, Algorithm 4.6], in a polynomial time [12, Corollary 4.1].

In [14], we described the non-negative posets $I$ of Dynkin type $\text{Dyn}_I = \mathbb{A}_m$ in terms of their Hasse digraphs as follows.

**Fact 2.18.** Assume that $I = (\{1, \ldots, n\}, \preceq)$ is a finite connected poset.

(a) $I$ is non-negative of Dynkin type $\text{Dyn}_I = \mathbb{A}_n$ if and only if $H(I)$ is an oriented path.

(b) $I$ is non-negative of Dynkin type $\text{Dyn}_I = \mathbb{A}_{n-1}$ if and only if $H(I)$ is 2-regular and $I$ has at least two maximal elements.

(c) If $I$ is non-negative of Dynkin type $\text{Dyn}_I = \mathbb{A}_{n-\text{crk}_I}$, then $\text{crk}_I \in \{0, 1\}$.

Summing up, every non-negative poset $I$ of Dynkin type $\mathbb{A}_m$ has its Hasse quiver isomorphic with either an oriented path (and is positive) or an oriented cycle (and is principal). While every oriented path yields a positive poset, this is not the case for oriented cycles: only a cycle with at least two sinks (sources) yields a Hasse digraph of a principal poset of Dynkin type $\mathbb{A}_m$, see [14] for a detailed analysis.

### 3. Combinatorial tools

Inspired by the applications of $(\min, \max)$-equivalence, introduced by Bondarenko [22] in the study of posets with respect to the non-negativity of Tits quadratic form [23, 24], we introduce the notion of a $(\min, \max)$-reflection. By the *neighbourhood* $N_I(a) \subseteq I$ of $a \in I = (I, \preceq)$ in $I$ we mean the set

$$N_I(a) := \{b \in I; a \preceq b \land \neg\exists c \neq b a \preceq c \preceq b\} \cup \{d \in I; d \preceq a \land \neg\exists c \neq d a \preceq c \preceq a\}.$$

In other words $N_I(a) \subseteq I$ is a set of elements that either *covers* $a$ or are *covered* by $a$.

**Definition 3.1.** Let $I = (I, \preceq)$ be a finite poset. For a minimal (resp. maximal) element $a \in I$ the $(\min, \max)$-reflection of $I$ at $a$ is the poset $S_a I := (I, \preceq_\circ)$, where $\preceq_\circ \in I \times I$ is a transitive closure of the $\preceq_\circ$ relation, defined as follows:

- for every $b \in N_I(a)$, we have $b \preceq_\circ a$ if and only if $a \preceq b$ (resp. $a \preceq_\circ b$ if and only if $b \preceq a$),
• for every \( c, d \in I \setminus \{a\} \), we have \( c \preceq d \) if and only if \( c \leq d \).

The following example illustrates the \((\min, \max)\)-reflection operation.

**Example 3.2.** Consider 5 element poset \( J = (\{1, 2, 3, 4, 5\}, \{2 \preceq 1, 3 \preceq 1, 4 \preceq 1, 3 \preceq 2, 4 \preceq 2\}, 3 \preceq 5, 4 \preceq 5\}) \). For the maximal element element \( 5 \in J \), with \( N_J(5) = \{3, 4\} \), we have

\[
S_5 J = (\{1, 2, 3, 4, 5\}, \{2 \preceq 1, 3 \preceq 1, 4 \preceq 1, 3 \preceq 2, 4 \preceq 2\}, 5 \preceq 3, 3 \preceq 5, 4 \preceq 5\}),
\]

where the relation \( \preceq \) is a transitive closure of the relation \( \preceq \) defined as

\[
\{2 \preceq 1, 3 \preceq 1, 4 \preceq 1, 3 \preceq 2, 4 \preceq 2, 5 \preceq 3, 3 \preceq 5, 4 \preceq 5\} \subseteq J \times J.
\]

The \((\min, \max)\)-reflection \( J \mapsto S_5 J \) has the following interpretation at the Hasse quiver level.

\[
\mathcal{H}(J): \quad \begin{array}{c}
1 \\
\Rightarrow \\
2 \\
\Rightarrow \\
3 \\
\Rightarrow \\
4 \\
\Rightarrow \\
5 \\
\end{array} \quad \xrightarrow{S_5} \quad \mathcal{H}(S_5 J): \quad \begin{array}{c}
1 \\
\Rightarrow \\
2 \\
\Rightarrow \\
3 \\
\Rightarrow \\
4 \\
\Rightarrow \\
5 \\
\end{array}
\]

As Example 3.2 illustrates, \((\min, \max)\)-reflection \( S_a I \) at \( a \in I \) can be viewed as a reflection \( s_a \) of Hasse quiver \( \mathcal{H}(I) \) [1, Chapter VII.4]. Contrary to the quiver operation and \((\min, \max)\)-equivalence, the \( I \mapsto S_a I \) operation does not preserve equivalence of quadratic forms, in general.

**Example 3.3.** Consider 5 element poset \( I = (\{1, 2, 3, 4, 5\}, \preceq) \) of Example 3.2. The quadratic form \( q_I : \mathbb{Z}^5 \to \mathbb{Z} (2.5) \) is indefinite, as \( q_I([-4,-4,5,7,-6]) = -10 \), while the quadratic form \( q_{S_5 I} : \mathbb{Z}^5 \to \mathbb{Z} \) is positive, since \( S_5 I \) is isomorphic with the positive one-peak poset \( 1 \mathbb{D}^*_4 \circ \mathbb{A}_0 \), see [15].

**Definition 3.4.** Let \( I = (I, \preceq) \) be a finite poset. We call a minimal [maximal] element \( a \in I \) junction free if and only if for all \( b, c \in N_I(a) \) and \( d \in I \setminus \{a\} \) we have

\[
(a \preceq b \preceq d) \land (a \preceq c \preceq d) \Rightarrow b = c \quad [d \preceq b \preceq a] \land (d \preceq c \preceq a) \Rightarrow b = c.
\]

In the poset \( S_5 J \) of Example 3.2, the element 1 is junction free and 5 is not, since \( 2 \preceq 3 \preceq 5 \) and \( 2 \preceq 4 \preceq 5 \).

**Proposition 3.5.** Let \( I = (\{1, \ldots, n\}, \preceq) \) be a finite poset and \( a \in I \) be a minimal or maximal element that is junction free in both \( I \) and \( S_a I \).

(a) The diagram

\[
\begin{array}{ccc}
\mathbb{Z}^n \times \mathbb{Z}^n & \xrightarrow{b_I} & \mathbb{Z} \\
\downarrow{s_a \times s_a} & & \downarrow{b_{S_a I}} \\
\mathbb{Z}^n \times \mathbb{Z}^n & \cong & \mathbb{Z}^n
\end{array}
\]

is commutative, where the group isomorphism \( \mathbb{Z}^n \ni x \xmapsto{s_a} y \in \mathbb{Z}^n \), is defined as follows:

\[
y_a := -x_a, \quad y_i := x_i + x_a \quad \text{for} \quad i \in N_I(a), \quad \text{and} \quad y_i := x_i \quad \text{otherwise}.
\]
(b) Poset $I$ is strongly Gram $\mathbb{Z}$-congruent with poset $S_a I$.

(c) $I \mapsto S_a I$ operation is involution, that is $S_a(S_a I) = I$.

\textbf{Proof:}

(a) Without loss of generality, we may assume that $a = 1$ is a minimal element of $I = (\{1, \ldots, n\}, \preceq)$ and $N_I(a) = \{2, \ldots, k\}$. By assumptions, $\mathbb{Z}$-bilinear form $b_I : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$ (2.4) has a form

$$b_I(x, y) = x_1 y_1 + \sum_{i=2}^{k} y_i \left( x_1 + x_i + \sum_{j=k+1}^{n} c_{j,i} x_j \right) + \sum_{i=k+1}^{n} y_i \left( x_i + \sum_{j\neq i}^{n} c_{j,i} x_j \right)$$

$$= x_1 \left( \sum_{i=1}^{k} y_i + \sum_{i=k+1}^{n} c_{1,i} y_i \right) + b_I(x, y),$$

where $c_{i,j} \in \{0, 1\}$ and $b_I(x, y) := b_I(x|x_1=0, y|y_1 = 0)$ is the $\mathbb{Z}$-bilinear form associated with the poset $I \setminus \{1\}$. By direct calculations, one easily checks that

$$b_I(s_1(x), s_1(y)) = x_1 \sum_{i=k+1}^{n} y_i \left( \sum_{j=2}^{k} c_{j,i} - c_{1,i} \right) + y_1 \left( \sum_{i=1}^{k} x_i + \sum_{i=k+1}^{n} x_i \sum_{j=2}^{k} c_{j,i} \right) + b_I(x, y).$$

We claim that $\mathbb{Z}$-bilinear form $b_I(s_1(x), s_1(y))$ describes poset $S_1 I$, i.e., $b_{S_1 I} = b_I(s_1(x), s_1(y))$. Indeed, for any $i > k$ and $j \in \{2, \ldots, k\}$ we have the following.

(i) At most one $c_{j,i} \neq 0$, since $1 \preceq j$ and $1$ is \textit{junction free} in $I$. If that is the case, $c_{1,i} = 1$ by transitivity of the relation $\preceq$ and, consequently, we have $x_1 \sum_{i=k+1}^{n} y_i \left( \sum_{j=2}^{k} c_{j,i} - c_{1,i} \right) = 0$.

(ii) At most one $c_{i,j} \neq 0$, otherwise 1 would not be \textit{junction free} in $S_1 I$.

Summing up, $\mathbb{Z}$-bilinear form $b_I(s_1(x), s_1(y))$ describes such a poset $J = (I, \preceq_\ast)$, that $J \setminus \{1\}$ coincides with $I \setminus \{1\}$, $1$ is a maximal element in $J$ and $b \preceq_\ast 1$ iff $c \preceq b$ for some $c \in N_I(1)$. Hence $J = S_1 I$ by Definition 3.1.

(b) Follows directly from (a). In particular, $I \cong_{\mathbb{Z}} S_a I$, where $B_a = [b_{i,j}^a] \in \mathbb{M}_n(\mathbb{Z})$

$$b_{i,j}^a = \begin{cases} -1 & \text{if } i = j = a, \\ 1 & \text{if } i = j \text{ and } i \neq a \text{ or } i \in N_I(a) \text{ and } j = a, \\ 0 & \text{otherwise} \end{cases} \quad (3.7)$$

is the matrix defining the group isomorphism $s_a : x \mapsto x \cdot B_a^t$. Since $s_a$ is an involution (equivalently: $B_a$ is an involutory matrix), (c) follows and the proof is finished. \hfill $\square$

Proposition 3.5 describes conditions under which $(\min, \max)$-reflection defines strong Gram $\mathbb{Z}$-
congruence. In the following example, we illustrate its applications.
Example 3.8. Consider poset $J$ of Example 3.2, described by the Hasse quiver $\mathcal{H}(J)$ (3.9). We note that every element $a \in \{1, 3, 4, 5\}$ is junction free in $J$, but $a$ is junction free in $S_a J$ only in the case of $a = 1$. Hence, in view of Proposition 3.5, $J$ is strong Gram $\mathbb{Z}$-congruent with $S_1 J$.

$$\mathcal{H}(J): \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \xrightarrow{S_1} \mathcal{H}(S_1 J): \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \quad \text{(3.9)}$$

In particular, we have $\mathbb{Z}^5 \ni [x_1, x_2, x_3, x_4, x_5] \xmapsto{S_1} [-x_1, x_1 + x_2, x_3, x_4, x_5] \in \mathbb{Z}^5$,

$$b_J(x, y) = y_1 (x_1 + x_2 + x_3 + x_4) + y_2 (x_2 + x_3 + x_4) + y_3 x_3 + y_4 x_4 + y_5 (x_3 + x_4 + x_5),$$

$$b_J(s_1(x), s_1(y)) = -y_1 (x_2 + x_3 + x_4) + (y_1 + y_2)(x_1 + x_2 + x_3 + x_4) + y_3 x_3 + y_4 x_4 + y_5 (x_3 + x_4 + x_5)$$

$$= y_1 x_1 + y_2 (x_1 + x_2 + x_3 + x_4) + y_3 x_3 + y_4 x_4 + y_5 (x_3 + x_4 + x_5)$$

$$= b_{S_1 J}(x, y)$$

and

$$B_1^{tr} \cdot C_J \cdot B_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = C_{S_1 J}.$$

Assume that $I = (I, \preceq), \vert I \vert = n$, is a finite connected principal poset of Dynkin type $\mathbb{A}_{n-1}$. By Fact 2.18(b), elements $a_i$ of $I$ can be enumerated in such an order $a_1, \ldots, a_n$, that

- $a_{r_0}, \ldots, a_{r_k-1}$ denote all elements that are either minimal or maximal in $I$,

- for every $j \in \{0, \ldots, k-1\}$ and $j' = j + 1 \mod k$ either
  - $a_{r_s} \preceq a_{r_t}$ for every $a_{r_s}, a_{r_t} \in \{a_{r_j}, \ldots, a_{r_{j'}}\}$ where $r_s < r_t$, or
  - $a_{r_s} \succeq a_{r_t}$ for every $a_{r_s}, a_{r_t} \in \{a_{r_j}, \ldots, a_{r_{j'}}\}$ where $r_s < r_t$.

Since $I$ has at least two maximal elements and $\mathcal{H}(I)$ is a 2-regular directed graph, it easily follows that $2 \leq k = 2s$ is an even number.

Definition 3.10. The cycle index $c(I) \in \{\frac{n}{2}, \ldots, n\}$ of the Dynkin type $\mathbb{A}_{\vert I \vert - 1}$ principal connected poset $I = (\{1, \ldots, n\}, \preceq)$ that has exactly $k = 2s$ elements that are either minimal or maximal is

$$c(I) := \max \left( \sum_{j=0}^{s-1} \left| \{a_{r_2j}, \ldots, a_{r_2j+1}\} \right|, \sum_{j=1}^{s-1} \left| \{a_{r_2j-1}, \ldots, a_{r_2j}\} \right| + \left| \{a_{r_{k-1}}, \ldots, a_n, a_1\} \right| \right) - s. \quad \text{(3.11)}$$

The cycle index $c(I)$ has the following combinatorial interpretation at the Hasse quiver $\mathcal{H}(I)$ level.
Remark 3.12. Let \( I = (\{1, \ldots, n\}, \preceq) \) be a finite connected principal poset of Dynkin type \( \mathbb{A}_{n-1} \). We may assume that the (planar) digraph \( \mathcal{H}(I) \) is visualized graphically in the circle layout, that is

\[
\mathcal{H}(I): \quad a_{r_0} = a_1 \rightarrow \cdots \rightarrow a_1 \rightarrow a_{r_1} \rightarrow \cdots \rightarrow a_{r_1},
\]

where every arrow \((a_i, a_{(i+1) \mod n})\) in (3.13) is oriented either clockwise or counterclockwise and every subquiver \(\{a_{r_j}, \ldots, a_{r_j'}\} \subseteq \mathcal{H}(I)\), where \(j' = j + 1 \mod k\), is an oriented chain. The cycle index \(c(I)\) equals \(\max(l, r)\), where \(r\) denotes the number of arrows oriented clockwise, and \(l\) denotes the number of arrows oriented counterclockwise in (3.13).

We show in Theorem 1.1 that \(c(I)\) is an invariant of strong Gram \(\mathbb{Z}\)-congruence. Moreover, it is uniquely determined by the Coxeter polynomial \(\text{cox}_I(t) \in \mathbb{Z}[t]\).

4. Proof of the main theorem

The general idea of the proof of the Theorem 1.1 is to reduce an arbitrary connected non-negative poset \(I\) of Dynkin type \(\mathbb{A}_m\) to a canonical one.

Definition 4.1. Let \(n \geq 4\) be a natural number. By a canonical two peak poset \(p \tilde{\mathbb{A}}_n\) we mean any of the finite connected posets defined by the following Hasse quiver

\[
\mathcal{H}(p \tilde{\mathbb{A}}_n): \quad 1 \rightarrow 3 \rightarrow 4 \rightarrow \cdots \rightarrow p \rightarrow n-1 \rightarrow \cdots \rightarrow n, \quad (4.2)
\]

where \(\frac{n}{2} \leq p \leq n-2\).

In the following lemma, we sum up some of the properties of canonical posets \(p \tilde{\mathbb{A}}_n\) that are important from the Coxeter spectral analysis point of view.

Lemma 4.3. If \(I := p \tilde{\mathbb{A}}_n = (\{1, \ldots, n\}, \preceq)\) is a canonical two peak poset (4.2), then:

(a) \(I\) is principal, \(\text{Dyn}_I = \mathbb{A}_{n-1}\) and \(\text{Ker} I = \mathbb{Z} \cdot h_I \subseteq \mathbb{Z}^n\), where \(h_I = [-1, -1, 0, \ldots, 0, 1, 1]\),

(b) \(\text{cox}_I(t) = t^n - t^p - t^{n-p} + 1 = (t-1)^2 \nu_p \nu_{n-p} \in \mathbb{Z}[t]\), where \(\nu_p := 1 + t + t^2 + \cdots + t^{n-1}\),

(c) \(c_I = \infty\) and \(\check{c}_I = \text{lcm}(p, n-p)\),

(d) \(c(p \tilde{\mathbb{A}}_n) = p\).
Proof:
(a) The incidence quadratic form \( q_I : \mathbb{Z}^n \to \mathbb{Z} \) (2.5) is given by the formula:

\[
q_I(x) = \sum_i x_i^2 + \sum_{i<j, i,j \in I_1} x_i x_j + \sum_{i<j, i,j \in I_2} x_i x_j + x_{n-1} (x_1 + x_2)
\]

\[
= \frac{1}{2} \sum_{i=3}^{n-2} x_i^2 + \frac{1}{2} \left( \sum_{i \in I_1} x_i \right)^2 + \frac{1}{2} \left( \sum_{i \in I_2} x_i \right)^2 + \frac{1}{2} (x_1 + x_{n-1})^2 + \frac{1}{2} (x_2 + x_{n-2})^2,
\]

where \( I_1 := \{1, 3, \ldots, p, n\} \) and \( I_2 := \{2, p+1, \ldots, n-2, n\} \). It follows that \( q_I(v) \geq 0 \) for every \( v \in \mathbb{Z}^n \) and \( h_I \in \text{Ker } I \), that is, \( I \) is non-negative of corank \( \text{crk}_I > 0 \). Since \( h_{n-1}^I = 1 \) and, by [15, Theorem 5.2], the subposet \( I^{(n-1)} := I \setminus \{n-1\} = p\mathbb{A}_{n-1}^+ \) is positive of Dynkin type \( \text{Dyn}_I^{(n-1)} = \mathbb{A}_{n-1} \), we conclude that \( \text{crk}_I = 1 \) (\( I \) is principal), \( \text{Dyn}_I = \mathbb{A}_{n-1} \) and \( \text{Ker}_I = h^I \mathbb{Z} \subseteq \mathbb{Z}^n \), see Definition 2.17.

To prove (b) and (c) we follow the line graph [25] technique developed by Jiménez González [10, 17, 18] in the context of Coxeter spectral classification of Dynkin type \( \mathbb{A}_n \) non-negative edge-bipartite signed graphs (bigraphs). First, we note that elements of the poset \( p\mathbb{A}_n = (\{1, \ldots, n\}, \leq) \) are topologically sorted, i.e., \( i \leq j \) implies that \( i \leq j \), and the incidence matrix \( C_I \in M_n(\mathbb{Z}) \) is upper triangular. Hence \( \hat{G}_I = C_I \) and we conclude that \( \text{cok}_I(t) = \text{cox}_I(t), c_I = c_{\Delta I} \) and \( \hat{c}_I = \hat{c}_{\Delta I} \).

We recall from [18] (see also [10]) that the line graph (known also as root graph or incidence graph) associated with a finite quiver (digraph) \( Q = (Q_0, Q_1, s, t) \) is the signed graph \( L(Q) = (Q_1, E) \) whose set of edges \( E \) is defined as follows: two vertices \( e_1, e_2 \in Q_1 \) are connected:

- by a positive edge \( e_1 \rightarrow e_2 \in E \) iff \( s(e_1) = s(e_2) \) or \( t(e_1) = t(e_2) \),

- by a negative edge \( e_1 \leftarrow e_2 \in E \) iff \( s(e_1) = t(e_2) \) or \( t(e_1) = s(e_2) \).

It is straightforward to check that \( \Delta_I = L(Q_I) \), where

\[
Q_I:
\]

and \( \Delta_I: \)

that is, \( \Delta_I \) is a line graph of \( Q_I \). Moreover, quiver \( Q_I \) contains exactly two \textit{minimally decreasing walks} in the sense of [17]:

- \( v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \cdots \rightarrow v_{n-p-1} \rightarrow v_{n-p} \rightarrow v_1 \)

- \( v_{n-p-1} \rightarrow v_{n-p+2} \rightarrow \cdots \rightarrow v_{n-1} \rightarrow v_n \rightarrow v_{n-p+1} \)
of length $n - p$ and $p$, respectively. Hence, [17, Theorem 6.3] and [17, Corollary 6.4] yield

- $\text{cox}_\Phi(t) = (t - 1)^{\text{er}k_{\Delta_I} - 1}((n-p-1)(t^p - 1) = t^n - t^p - t^{n-p} + 1 \in \mathbb{Z}[t]$ and

- $c_\Phi = \infty$ and $\tilde{c}_\Phi = \text{lcm}(p, n-p)$,

where $\Phi := -G_{\Delta_I}^{-1} \cdot G_{\Delta_I}^{tr}$. Since $\Phi = \text{Cox}_{\Delta_I}^{-1} = \text{Cox}_{\Delta_I}$, we conclude that $c_I = c_\Phi = \infty$, $\tilde{c}_I = \tilde{c}_\Phi = \text{lcm}(p, n-p)$ and, in view of [4, Lemma 2.8], $\text{cox}_I(t) = t^n \text{cox}_\Phi(t) = t^n - t^p - t^{n-p} + 1 \in \mathbb{Z}[t]$.

(d) In order to apply Definition 3.10 we enumerate the elements $\{1, \ldots, n\}$ of $\tilde{\mathcal{A}}_n$ as follows.

We note that:

- $\alpha_{r_0} = a_1, \alpha_{r_1} = a_p, \alpha_{r_2} = a_{n-1}, \alpha_{r_3} = a_n$ denote all $k = 4 = 2s$ elements that are either minimal or maximal in $\tilde{\mathcal{A}}_n$,

- for every $j \in \{0, \ldots, k-1\}$ and $j' = j \mod k + 1$, either:
  - $u \preceq v$ for every $u < v \in \{a_{r_j}, \ldots, a_{r_{j'}}\}$ or
  - $u \succeq v$ for every $u < v \in \{a_{r_{j}}, \ldots, a_{r_{j'}}\}$.

Since $p \geq n - p$, the cycle index (3.11) equals

$$c(\tilde{\mathcal{A}}_n) = \max(|\{a_1, \ldots, a_p\}| + |\{a_{n-1}, a_n\}|, |\{a_p, \ldots, a_{n-1}\}| + |\{a_n, a_1\}|) - s = \max(p + 2, n - p + 2) - 2 = p,$$

and the proof is finished.

**Remark 4.4.** We note that Lemma 4.3(a) follows by the results of Simson [26, Proposition 2.12] and Boldt [27, Proposition 3.3]. In our proof, we use the line graph technique to determine not only Coxeter polynomial (2.7) but also Coxeter number (2.11) and reduced Coxeter number (2.12).

Now, we have all the necessary tools to prove one of the paper’s main results.

**Proof of Theorem 1.1:**

Part (a) of the theorem follows from Fact 2.18(c), see [14] for details.

In view of Fact 2.13(a), [15, Theorem 5.2] and Lemma 4.3, to prove parts (b) and (c) it suffices to show that (b1) and (c1) hold true.

(b1) Assume that $I$ is a positive connected poset of the Dynkin type $\mathbb{A}_n$. It follows from Fact 2.18, that $\overrightarrow{\mathcal{H}}(I) \simeq P_n(1, n) := 1 \bullet \cdots \bullet n$, hence, by (2.10), without loss of generality we may assume that the elements of $I = \{1, \ldots, n\}$ are enumerated in such a way, that $N_I(1) = \{2\}$, $N_I(n) = \{n - 1\}$ and $N_I(i) = \{i - 1, i + 1\}$, for every $i \in \{2, \ldots, n - 1\}$.
We have two possibilities: either (i) $1 \leq 2$ or (ii) $2 \leq 1$. First, we assume that $1 \leq 2$, i.e., $\mathcal{H}(I)$ contains the oriented chain $\vec{P}(1, k)$, where $1 < k \leq n$, as an induced subquiver.

$$\mathcal{H}(I): \bullet_1 \rightarrow \bullet_2 \rightarrow \cdots \rightarrow \bullet_k \rightarrow \cdots \rightarrow \bullet_{n-1} \rightarrow \bullet_n$$

If $k = n$ then $I = 0\tilde{A}_{n-1}^+$ (4.6) and we are done. Otherwise, we set $I_1 := I$ and define poset $I_2$ as the result of composition of $k$ ($\min, \max$)-reflections $S_k, S_{k-1}, \ldots, S_1$

$$I_2 = I[k] := S_1 \cdots S_{k-1} S_k I.$$ (4.5)

We note that $\mathcal{H}(I_2)$ contains the oriented chain $\vec{P}(1, k')$, where $k < k' \leq n$ and $k' \geq k + 1$ is minimal in $I_2$. If $k' \neq n$, we repeat the procedure and define $I_3 := I[k']$ (4.5). It is easy to see that this procedure has to end in at most $r \leq n - k$ steps at $I_r \simeq 0\tilde{A}_{n-1}^+$.

$$\mathcal{H}(0\tilde{A}_{n-1}^+) = \bullet_1 \rightarrow \bullet_2 \rightarrow \cdots \rightarrow \bullet_{n-1} \rightarrow \bullet_n$$ (4.6)

Hence, in view of Proposition 3.5(b), we conclude that $I \simeq_{\mathbb{Z}p} \tilde{A}_n$. To finish this part of the proof, we note that in the case (ii), it suffices to apply the arguments of (i) to the $I_0 := S_1 I$ poset.

(c1) We proceed analogously as in the previous case. Without loss of generality, by (2.10) and Fact 2.18(b), we may assume that the elements of $I = \{1, \ldots, n\}$ are enumerated in such a way that $1$ is a minimal element in $I$, $N_I(1) = \{2, n\}$, $N_I(n) = \{1, n-1\}$ and $N_I(i) = \{i-1, i+1\}$ for $i \in \{2, \ldots, n-1\}$. Given a minimal or maximal element $i \in I$, we define:

$$\text{shift}_I(i) := \begin{cases} \min_{k \in \mathbb{N}} \{k; \ i \leq k \text{ and } i \neq (i + k) \mod n + 1\}, & \text{if } i \text{ is a minimal element in } I, \\ \min_{k \in \mathbb{N}} \{k; \ k \leq i \text{ and } (i + k) \mod n + 1 \neq i\}, & \text{if } i \text{ is a maximal element in } I, \end{cases}$$

and $\text{nxt}_I(i) := (i + \text{shift}_I(i)) \mod n + 1 \in I$. It is straightforward to check that $\text{nxt}_I(i) \in I$ is a minimal (maximal) element in $I$ if $i \in I$ is a maximal (minimal) one. Furthermore, by $S^b_a I$, where $a < b < n$, we denote a composition of $b - a - 1$ ($\min, \max$)-reflections $S^b_a I := S_{a+1} \cdots S_{b-1} S_b I$ and by $S^\text{nxt}_I$ we mean $S^r_I$ with $r := \text{nxt}_I(i)$.

Our aim is to show that $I \simeq_{\mathbb{Z}p} \tilde{A}_n$ (4.2), where $p$ is a non-zero integer. We do it by successively applying ($\min, \max$)-reflections to reduce $I$ to $\tilde{A}_n$. We set $I_1 := I$ and proceed as follows.

Step 1° If $2 \in I_1$ is a maximal element in $I_1$ we set $I_2 := I_1$. Otherwise, we set $I_2 := S^\text{nxt}_I I_1$.

Step 2° If $3 \in I_2$ is a minimal element in $I_2$ we set $I_3 := I_2$; otherwise: $I_3 := S^\text{nxt}_I I_2$. We set $k := 2$.

Step 3° We set $k := k + 1$, $r_k := \text{nxt}_{I_k}(3)$ and $t_k := \text{nxt}_{I_k}(r_k)$.

Step 4° If $t_k = 1$, then $I_k = \tilde{A}_n$ and we are done. Otherwise:

Step 5° we set $I_{k+1} := S^r_{t_k} I_k$ and proceed to Step 3°.
We claim that this procedure finishes in a finite number of steps. Indeed, given $t_k := \text{nxt}_k(r_k) \neq 1$ in Step $3^\circ$, after applying $I_{k+1} := S_{r_{k+1}}^I I_k$ in Step $4^\circ$, we have $r_{k+1} = \text{nxt}_k(3) = t_k > r_k > 3$.

Since $I$ is a finite poset and $r_k$ is a strictly monotonically increasing sequence, we conclude that $t_k := \text{nxt}_k(r_k) = 1$ for some $k \geq 2$.

To finish the proof, we note that every poset $I_k$ has at least two minimal and at least two maximal elements. It follows that every minimal (maximal) element is junction-free in the sense of Definition 3.4. Hence, by Proposition 3.5(b), we obtain that $I_k$ is a finite poset and $r_k$ is a strictly monotonically increasing sequence, we conclude that $t_k := \text{nxt}_k(r_k) = 1$ for some $k \geq 2$.

To finish the proof, we note that every poset $I_k$ has at least two minimal and at least two maximal elements. It follows that every minimal (maximal) element is junction-free in the sense of Definition 3.4. Hence, by Proposition 3.5(b), we obtain that $I_k$ is a finite poset and $r_k$ is a strictly monotonically increasing sequence, we conclude that $t_k := \text{nxt}_k(r_k) = 1$ for some $k \geq 2$.

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In the following corollaries, we show that connected non-negative posets $I$ of Dynkin type $\mathbb{A}_m$ are determined uniquely, up to strong Gram $\mathbb{Z}$-congruence, by the Coxeter spectrum $\text{specc}_I \subseteq \mathbb{C}$. In the case of positive (i.e., $\text{crk}_I = 0$) posets, we have a stronger result: in this case strong Gram $\mathbb{Z}$-congruence coincides with weak Gram $\mathbb{Z}$-congruence (see Definition 2.9).

**Corollary 4.7.** If $I$ and $J$ are finite connected positive posets of Dynkin type $\mathbb{A}_m$, then:

(a) $I \cong \mathbb{Z} J$,

(b) $I \approx \mathbb{Z} J$,

(c) $\text{specc}_I = \text{specc}_J [\text{cox}_I(t) = \text{cox}_J(t)]$.

**Proof:**

In view of Theorem 1.1(b1), for every two finite connected positive posets $I$ and $J$ of Dynkin type $\mathbb{A}_m$ we have $I \cong \mathbb{Z} 0 \mathbb{A}_{m-1} \approx \mathbb{Z} J$, hence $I \approx \mathbb{Z} J$ by transitivity of strong Gram $\mathbb{Z}$-congruence. This, by Fact 2.13, implies that $I \cong \mathbb{Z} J$, $\text{specc}_I = \text{specc}_J$ and $\text{cox}_I(t) = \text{cox}_J(t)$. □

**Corollary 4.8.** If $I$ and $J$ are finite connected principal posets of Dynkin type $\mathbb{A}_m$, the following conditions are equivalent:

(a) $I \approx \mathbb{Z} J$,

(b) $\text{specc}_I = \text{specc}_J [\text{cox}_I(t) = \text{cox}_J(t)]$,

(c) $c(I) = c(J)$.

**Proof:**

Assume that $I$ and $J$ are finite connected principal posets of Dynkin type $\mathbb{A}_m$. Implication (a) $\Rightarrow$ (b) follows by Fact 2.13(a) and (b) $\Rightarrow$ (c) is a consequence of Theorem 1.1(c).

(c) $\Rightarrow$ (a) By Theorem 1.1(c1), we know that $I \cong \mathbb{Z} p \mathbb{A}_m \cong \mathbb{Z} J$, where $p = c(I) = c(J)$, and we conclude that $I \cong \mathbb{Z} J$ by transitivity of strong Gram $\mathbb{Z}$-congruence.
We finish this section by noting that Corollary 1.3 follows by Corollary 4.7 and Corollary 4.8.

**Proof of Corollary 1.3:**
Assume that $I$ and $J$ are non-negative connected posets of Dynkin type $\mathbb{A}_m$. Our aim is to show that $I \approx_\mathbb{Z} J$ if and only if $\text{cox}_I(t) = \text{cox}_J(t)$. Since the “$\Rightarrow$” implication follows from Fact 2.13(a), it suffices to show “$\Leftarrow$”.

Assume that $\text{cox}_I(t) = \text{cox}_J(t)$. It follows that $|I| = |J|$ and, consequently, $\text{crk}_I = \text{crk}_J$. By Fact 2.18(c), we know that either $\text{crk}_I = 0$ or $\text{crk}_I = 1$. The implication follows by Corollary 4.7 in the first case and by Corollary 1.3 in the second one. □

5. Algorithms

Since isomorphic posets are strongly Gram $\mathbb{Z}$-congruent (2.10), we can view $\mathbb{Z}$-congruence as a generalization of the notion of isomorphism. Similar to the isomorphism case [28], we have two general strategies for devising algorithms that construct a matrix $B \in \text{Gl}(n; \mathbb{Z})$ that defines strong Gram $\mathbb{Z}$-congruence between two non-negative posets $I$ and $J$ of Dynkin type $\text{Dyn}_I = \text{Dyn}_J = \mathbb{A}_m$:

(i) compute such a $B$ that $I \approx_\mathbb{Z} J$
directly;

(ii) compute a canonical representative $R$ and $\mathbb{Z}$-congruences $I \approx_\mathbb{Z} R$ and $J \approx_\mathbb{Z} R$, then calculate $B$ as $B := B_I \cdot B_J^{-1}$.

In the first strategy, we are given two posets, encoded in, e.g., incidence matrix form (2.2), and our aim is to construct such a matrix $B \in \text{Gl}(n; \mathbb{Z})$, that $B^{tr} \cdot C_I \cdot B = C_J$. The reader is referred to [13, 15, 29] for discussion of some of the algorithms of this type.

The main advantage of the second strategy is its efficiency in a general classification setting defined as follows: divide a (possibly large) list of posets into sublists of $\mathbb{Z}$-congruent ones. Since two posets $I$ and $J$ are $\mathbb{Z}$-congruent if and only if their canonical representatives are equal, it is more efficient to compute representatives first and then use a sorting, hashing, or balanced tree algorithm to efficiently group the input list into desired sublists. This methodology is analogous to canonical labeling in a graph isomorphism setting, see [28]. It is the strategy we follow in the current work.

By Theorem 1.1, statements (b1) and (c1), every connected non-negative poset $I$ of Dynkin type $\text{Dyn}_I = \mathbb{A}_n$ is strongly Gram $\mathbb{Z}$-congruent with either the one-peak poset $0\mathbb{A}_{n-1}$ or a canonical two peak poset $p\tilde{\mathbb{A}}_n$,

$$\mathcal{H}(0\mathbb{A}_{n-1}) = 1 \rightarrow 2 \rightarrow \ldots \rightarrow n-1 \rightarrow n, \quad \mathcal{H}(p\tilde{\mathbb{A}}_n) = 1 \rightarrow 3 \rightarrow 4 \rightarrow \ldots \rightarrow n, \quad 2 \rightarrow p+1 \rightarrow \ldots \rightarrow n-2 \rightarrow n.$$ 

Since $I \approx_\mathbb{Z} 0\mathbb{A}_{n-1}$ if and only if $\text{crk}_I = 0$ and $I \approx_\mathbb{Z} p\tilde{\mathbb{A}}_n$ if and only if $\text{crk}_I = 1$, we prepare two algorithms for each of these cases.
Algorithm 5.2 is the implementation of the procedure described in the proof of Theorem 1.1(b1). Its main idea is to use the \((\min, \max)\)-reflection operation, implemented in the form of \textsc{ReflMatrix()} function (see Listing 5.1), to perform the reduction \(J \mapsto 0 \tilde{A}_n\#-1\) and construct matrix \(B \in \text{GL}(n; \mathbb{Z})\) as the composition of \((\min, \max)\)-reflection matrices. We note that this procedure is guaranteed to finish at the poset isomorphic with \(0 \tilde{A}_n\#-1\) and not necessarily equal to it. In lines 25-28 of the algorithm, we take this into consideration and reorder the resulting matrix to ensure that the algorithm returns a matrix that defines the strong Gram \(\mathbb{Z}\)-congruence with the canonically numbered poset \(0 \tilde{A}_n\#-1\) (4.6).

**Algorithm 5.2** Strong Gram \(\mathbb{Z}\)-congruence of a positive connected poset \(J\) with \(\text{Dyn}_J = A_n\) with a one-peak poset \(0 \tilde{A}_n\#-1\).

**Input** Incidence matrix \(C_J \in M_n(\mathbb{Z})\) of a positive connected poset \(J\) of the Dynkin type \(A_n\).

**Output** Nonsingular matrix \(B \in M_n(\mathbb{Z})\) defining strong Gram \(\mathbb{Z}\)-congruence \(J \approx Z 0 \tilde{A}_n\#-1\).

```python
Listing 5.1 \textsc{ReflMatrix()} function
1: function \textsc{ReflMatrix}(i, a, b, n)
2: \(S \leftarrow E_n \in M_n(\mathbb{Z})\) \quad \triangleright \text{Identity matrix}
3: \(S[i, i] \leftarrow -1; S[a, i] \leftarrow S[b, i] \leftarrow 1\) \quad \triangleright S[x, y] \equiv s_{x,y}
4: return \(S\)
```
Corollary 5.3. Algorithm 5.2 performs at most $2(n - 2) = O(n)$ matrix multiplications, hence it is of $O(n^4)$ time complexity (assuming naïve matrix multiplication algorithm).

Proof:
It is straightforward to see that matrix multiplication is the dominant (i.e., most time-consuming) operation of Algorithm 5.2. The total number of these operations equals at most

- $n - 2$ in line 22 and
- $n - 2$ in line 23.

Hence we have at most $2(n - 2) = O(n)$ matrix multiplications.

The number of matrix multiplications performed by Algorithm 5.2 depends on the shape of the Hasse quiver $\mathcal{H}(J)$. In order to verify that the number $2(n - 2)$ is actually attainable, consider a poset $J$ with $\mathcal{H}(J)$ of the shape

$$\mathcal{H}(J) = \begin{array}{ccc}
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
1 & & n
\end{array} \quad \text{or} \quad \mathcal{H}(J) = \begin{array}{ccc}
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
3 & & 2
\end{array}.$$

(5.4)

It is easy to verify that in this case, Algorithm 5.2 performs exactly $n - 2$ steps (line 17), and each step performs exactly 2 matrix multiplications: one in line 22 and second in line 23.

Remark 5.5. The reduction procedure described in the proof of Theorem 1.1(b1), applied to poset $J$ of the shape (5.4), is as follows:

$$J_1 := J \mapsto J_2 := S_1S_2J_1 \mapsto J_3 := S_1S_2S_3J_2 \mapsto \cdots \mapsto J_{n-1} := S_1S_2\cdots S_{n-1}J_{n-2} \simeq 0\mathbb{A}_{n-1}^*.$$

Hence, we can construct matrix $B$ that defines strong Gram $\mathbb{Z}$-congruence $J \simeq B$ as:

$$B := (B_2B_1)(B_3B_2B_1)\cdots (B_{n-1}\cdots B_2B_1) \in \text{Gl}(n; \mathbb{Z}),$$

where $B_a = [b_{ij}^a] \in \mathbb{M}_n(\mathbb{Z})$ (3.7) is a matrix defining $(\min, \max)$-reflection $S_a$, see Proposition 3.5(b). The straightforward implementation of this procedure yields $\frac{1}{2}(n - 2)(n + 1) = O(n^2)$ matrix multiplications. In Algorithm 5.2, we reuse partial results at each step (see lines 21-23) to significantly reduce the number of multiplications.
Assume now, that \( J = (\{1, \ldots, n\}, \preceq) \) is a principal connected poset of Dynkin type \( A_{n-1} \). By Fact 2.18(b), the Hasse quiver \( \mathcal{H}(J) \) is 2-regular, and \( J \) has at least two maximal elements. It follows that \( \mathcal{H}(J) \) is a cycle graph, and \( J \) has at least two minimal elements.

**Listing 5.6** \textsc{ReflMatrixCirc()}, \textsc{NextOnCircle()} and \textsc{MinMaxMove}() functions

```python
1: function ReflMatrixCirc(u, n, circle)
2: S ← \( E_n \in \mathbb{M}_n(\mathbb{Z}) \)  \(\triangleright\) Identity matrix
3: a ← \( \text{circle}[u - 1 \mod n] \); i ← \( \text{circle}[u \mod n] \); b ← \( \text{circle}[u + 1 \mod n] \)
4: \( S[i, i] \leftarrow -1; S[a, i] \leftarrow S[b, i] \leftarrow 1 \)  \(\triangleright\) \( S[x, y] \equiv s_{x,y} \)
5: return S

6: function NextOnCircle(i, n, is_mm, circle)
7: while \( \neg is\_mm[\text{circle}[i \mod n]] \) do
8: \( i \leftarrow i + 1 \)
9: return \( i \mod n \)

10: function MinMaxMove(k, p, is_mm, circle)
11: B ← \( E_n \in \mathbb{M}_n(\mathbb{Z}) \)
12: for \( i \in \{k, k - 1, \ldots , p + 1\} \) do
13: \( B \leftarrow B \cdot \text{ReflMatrixCirc}(i, n, circ) \)
14: \( is\_min\_\text{mm}[\text{circle}[i - 1]] \leftarrow \neg is\_\text{min}\_\text{mm}[\text{circle}[i - 1]] \)
15: \( is\_\text{min}\_\text{max}[\text{circle}[i + 1]] \leftarrow \neg is\_\text{min}\_\text{max}[\text{circle}[i + 1]] \)
16: return B
```

Algorithm 5.7 is the implementation of the procedure described in the proof of Theorem 1.1(c1). Similarly as in case of Algorithm 5.2, its main idea is to use the \((\text{min}, \text{max})\)-reflection operation (see \textsc{ReflMatrixCirc}() function in Listing 5.6) to perform the reduction \( J \mapsto p\tilde{A}_n \). Finally, in lines 27-35, we ensure that the algorithm returns a matrix that defines the strong Gram \( \mathbb{Z} \)-congruence with the canonically numbered poset \( p\tilde{A}_n \) (4.2).

**Algorithm 5.7** Strong Gram \( \mathbb{Z} \)-congruence of a principal connected poset \( J \) with \( \text{Dyn}_J = A_n \) with a canonical two-peak poset \( p\tilde{A}_n \)

\begin{tabular}{|l|}
\hline
\textbf{Input} & Incidence matrix \( C_J \in \mathbb{M}_n(\mathbb{Z}) \) of a principal connected poset \( J \) of the Dynkin type \( A_n \). \\
\textbf{Output} & Nonsingular matrix \( B \in \mathbb{M}_n(\mathbb{Z}) \) defining strong Gram \( \mathbb{Z} \)-congruence \( J \preceq \mathbb{Z} p\tilde{A}_n \). \\
\hline
\end{tabular}

```python
1: function StrongCongPosita(C_J \in \mathbb{M}_n(\mathbb{Z}))
2: A ← \( C_J^{-1} \in \mathbb{M}_n(\mathbb{Z}) \);
3: B ← \( E_n \in \mathbb{M}_n(\mathbb{Z}) \)
4: D ← \( \text{zeros} \in \mathbb{M}_n(\mathbb{Z}) \) \(\triangleright\) Zero (null) matrix
5: neigh ← []; is_mm ← []
6: for \( j \in \{0, \ldots , n - 1\} \) do
7: ins ← \{i; \( a_{i,j} = -1\)\}; outs ← \{i; \( a_{j,i} = -1\)\}
8: neigh.append(ins \cup outs)
9: is_mm.append((ins = \emptyset) \text{ or } (outs = \emptyset))  \(\triangleright\) List of minimal and maximal vertices
```
Algorithm 5.7 performs at most $20 \circ v$ operations. Assume that $(3, n, is_{\text{mm}}, circ)$.

```
if ins = ∅ then v ← j
    circ ← [v, neigh[v].pop()]
while (w ← (neigh[circ[len(circ)] − 1]] \ set(circ[len(circ)] − 2])).pop()) ̸= v do
    circ.append(w)
for i ∈ {1, 2} do
    if ¬is_{\text{mm}}[circ[i]] then
        B ← B · MINMAXMOVE(NEXTONCIRCLE(i, n, is_{\text{mm}}, circ), i, is_{\text{mm}}, circ)
    p = NEXTONCIRCLE(p + 1, n, is_{\text{mm}}, circ)
while (k ← NEXTONCIRCLE(p + 1, n, is_{\text{mm}}, circ)) ̸= 0 do
    B ← B · MINMAXMOVE(k, p, is_{\text{mm}}, circ)
    p = NEXTONCIRCLE(p + 1, n, is_{\text{mm}}, circ)
if p < n − p + 2 then
    SWAP(circ[0], circ[2])
    for i ∈ {3, . . . , ⌊n−3 2 ⌋ + 2} do
        SWAP(circ[i], circ[n − i + 2])
    p ← n − p + 2
    perm ← [0, . . . , 0] ∈ Z^n
    perm[0] ← circ[2]; perm[1] ← circ[0]
    perm[n − 1] ← circ[p]; perm[n − 2] ← circ[1]
    for i ∈ {3, . . . , p − 1} do
        perm[i − 1] ← circ[i]
    for i ∈ {1, . . . , n − p − 1} do
        perm[p + i − 2] ← circ[n − i]
    for i ∈ {1, . . . , n} do
        D.setColumn(i, B.getColumn(perm[i]))
return D
```

**Corollary 5.8.** Algorithm 5.7 performs at most $\lfloor \frac{n−2}{2} \rfloor \cdot \lfloor \frac{n−2}{2} \rfloor = O(n^2)$ matrix multiplications, hence it is of $O(n^5)$ time complexity (assuming naïve matrix multiplication algorithm).

**Proof:**
It is straightforward to see that matrix multiplication is the dominant (i.e., most time-consuming) operation of Algorithm 5.7. Assume that $J$ is a principal connected poset of Dynkin type $A_{n−1}$ and

$$
H(J) = \begin{array}{c}
  \bullet & \bullet & \bullet & \bullet & \bullet \\
  k+1 & k & \ldots & 3 & 2 & 1 \\
  k+1 & \ldots & n & n & n & n−1 \\
\end{array},
$$

where $k = \begin{cases} 
  \frac{n}{2}, & \text{if } n \text{ is even,} \\
  \frac{n−1}{2}, & \text{if } n \text{ is odd.}
\end{cases}$ (5.9)

One checks that:
• the number of matrix multiplication performed by Algorithm 5.7 applied to the poset \( J \) equals:
\[
\left\lfloor \frac{n-2}{2} \right\rfloor \cdot \left\lceil \frac{n-2}{2} \right\rceil = \begin{cases} (k-1)^2, & \text{if } n \text{ is even,} \\ k(k-1), & \text{if } n \text{ is odd;} \end{cases}
\]
• poset \( J \), up to the isomorphism, represents the pessimistic case that requires the most matrix multiplications.

The details are left to the reader.

\[\square\]

6. Enumeration of Dynkin type \( \mathbb{A}_m \) posets

Throughout this section, we assume that \( I \) is a non-negative poset of a Dynkin type \( \mathbb{A}_m \). Our aim is to give formulas for the exact number of all (up to the isomorphism) posets \( I \) with a given Coxeter polynomial (called Coxeter type). The total number of such non-negative posets \( I \) is given in [14] and is as follows.

**Fact 6.1.** Let \( N_{\text{neg}}(\mathbb{A}_n) \) be the number of all non-negative posets \( I \) of size \( n = |I| \) and Dynkin type \( \text{Dyn}_I = \mathbb{A}_{n-\text{crk}_I} \). Then
\[
N_{\text{neg}}(\mathbb{A}_n) = \begin{cases} 1 & \text{if } n \in \{1, 2\}, \\ 2^{n-2} + \frac{1}{2n} \sum_{d|n} \left( 2^{\frac{n}{2}} \phi(d) \right) + 2^{\frac{n-3}{2}} - \left\lfloor \frac{n+1}{2} \right\rfloor, & \text{if } n \geq 3 \text{ is odd,} \\ 2^{n-2} + \frac{1}{2n} \sum_{d|n} \left( 2^{\frac{n}{2}} \phi(d) \right) + 2^{\frac{n-2}{2}} - \left\lfloor \frac{n+1}{2} \right\rfloor, & \text{if } n \geq 4 \text{ is even,} \end{cases}
\]
where \( \phi \) is Euler’s totient function. Moreover, there are exactly
\[
N(P_n) = \begin{cases} 2^{n-2}, & \text{if } n \geq 2 \text{ is even,} \\ 2^{\frac{n-3}{2}} + 2^{n-2}, & \text{if } n \geq 1 \text{ is odd,} \end{cases}
\]
positive posets, and \( N_{\text{neg}}(\mathbb{A}_n) - N(P_n) \) principal ones.

In view of Theorem 1.1, to obtain the number of posets \( I \) of a given Coxeter polynomial \( \text{cox}_I(t) \in \mathbb{Z}[t] \) (2.7), we count oriented paths and (some) oriented cycles. Indeed

(i) \( I \) is positive and \( \text{cox}_I(t) = t^{n+1} + t^n + \ldots + t + 1 \in \mathbb{Z}[t] \) iff \( \mathcal{H}(I) \) is an oriented path,

(ii) \( I \) is principal and \( \text{cox}_I(t) = t^n - t^p - t^{n-p} + 1 \in \mathbb{Z}[t] \) iff \( \mathcal{H}(I) \) is such an oriented cycle, that
• there are at least two sinks,
• there are exactly \( p \) edges oriented clockwise, see Remark 3.12.

Since there are exactly \( N(P_n) \) (6.3) oriented paths (see [14]), it is sufficient to count oriented cycles described in (ii). We start with counting such oriented cycles \( D = (V_D, E_D) \), that have exactly \( n := |V_D| \geq 3 \) vertices and \( p \geq \frac{n}{2} \) edges oriented clockwise.

Assume that \( D = (\{1, \ldots, n\}, E_D) \) is an oriented cycle. Without loss of generality, we may assume that \( D \) is depicted in the circle layout on the Euclidean plane and its edges \( e \in E_D \) are labeled with two colors:
• **black**: if $e$ is clockwise oriented, and
• **white**: if $e$ is counterclockwise oriented.

This way, every cycle $D$ can be viewed as a binary combinatorial necklace $N_2(n)$. We recall that a binary necklace of length $n$ is an equivalence class of binary strings $s \in \{0, 1\}^n$ on length $n$, with all **rotations** considered equivalent. By a **rotation** we mean a **circular shift**, i.e., a cyclic permutation consisting of a single nontrivial cycle. In other words, a binary combinatorial necklace represents a structure of $n$ circularly connected beads, each of which is either black or white.

**Proposition 6.4.** The number $\text{Nck}_2(p, n)$ of binary combinatorial necklaces that have exactly $p$ black and $n - p$ white beads:

(a) is given by the coefficient $a_{p,n-p} \in \mathbb{Z}$ of the monomial $b^p w^{n-p}$ in the generating function

$$G_{N_2(n)}(b, w) = \sum_{p=0}^{n} a_{p,n-p} b^p w^{n-p} = \frac{1}{n} \sum_{d|n} (b^{n/d} + w^{n/d})^d \varphi \left( \frac{n}{d} \right) \in \mathbb{Z}[b, w]; \quad (6.5)$$

(b) equals:

$$\text{Nck}_2(p, n) = \frac{1}{n} \sum_{d|\gcd(n,p)} \varphi(d) \binom{n/d}{p/d}. \quad (6.6)$$

**Proof:**

Part (a) of the proposition follows directly from the Pólya enumeration theorem [30] and to show (b) we use the Burnside’s lemma (also known as the Cauchy–Frobenius lemma) as follows.

Assume that $X$ is the set of binary necklaces with $n$ beads, $p$ of which is black, and $G$ denotes the rotations group. By Burnside’s lemma, we have that

$$\text{Nck}_2(p, n) = \frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where $X^g$ denotes the set on necklaces fixed by $g \in G$, i.e. $X^g := \{ x \in X; gx = x \}$. Now, we fix an element $g \in G$ and describe necklaces $x \in X^g$. Note that $|G| = n$, since every rotation $g \in G$ can be viewed as a clockwise rotation by $0 \leq k < n$ beads. It follows that every $x \in X^g$ consists of $d := \frac{n}{k}$ identical **linear** (continuous) sections of length $k = \frac{n}{d}$. These sections can be chosen in

$$\binom{n/d}{p/d, (n-p)/d} = \frac{n!}{d! (n-p)!} = \binom{n/d}{p/d}$$

ways and we conclude that $X^g \neq \emptyset$ if and only if $d|\gcd(n,p)$. Moreover, for each divisor $d$ of $\gcd(n,p)$, there are $\varphi(d)$ rotations with the same number of symmetries as rotation by $k$ beads, that is:

$$\text{Nck}_2(p, n) = \frac{1}{|G|} \sum_{g \in G} |X^g| = \frac{1}{n} \sum_{d|\gcd(n,p)} \varphi(d) \binom{n/d}{p/d},$$

where $\varphi(d) := |\{ 1 \leq k < n; \gcd(k,n) = 1 \}|$ is Euler’s totient function. \qed
Example 6.7. Consider the set \( \mathcal{N}_2(10) \) of all binary necklaces consisting of 10 beads. The generating function \( G(b, w) := G_{\mathcal{N}_2(10)}(b, w) \in \mathbb{Z}[b, w] \) is given by the formula

\[
G(b, w) = b^{10} + b^9 w + 5b^8 w^2 + 12b^7 w^3 + 22b^6 w^4 + 26b^5 w^5 + 12b^4 w^6 + 5b^3 w^7 + bw^8 + bw^9 + w^{10}.
\]

Hence, the total number of such necklaces equals \( |\mathcal{N}_2(10)| = G_{\mathcal{N}_2(10)}(1, 1) = 108 \). In particular, there exists exactly \( \mathcal{N} \geq 2 \) necklaces that have 6 black beads. Obviously, this number coincides with the number of necklaces that have exactly 6 white beads and with the coefficient \( a_{6,4} \) of \( b^6 w^4 \) monomial in the \( G(b, w) \in \mathbb{Z}[b, w] \) generating function (6.5).

Assume that \( C_n = (\{1, \ldots, n\}, E_{C_n}) \) is an oriented cycle. Since \( C_n \) is a planar digraph, we may assume that it is depicted on the Euclidean plane in such a way that no two edges intersect, for example, in the circle layout. By a cycle index \( c(C_n) \in \{\lceil \frac{n}{2}\rceil, \ldots, n\} \) we mean \( c(C_n) := \max(r_{C_n}, l_{C_n}) \), where \( r_{C_n}(l_{C_n}) \) denotes the number of clockwise (counterclockwise) oriented edges in \( C_n \). It is straightforward to see that the cycle index is invariant under digraph isomorphism.

Proposition 6.8. Number \( N(C_n, p) \) of all, up to digraph isomorphism, oriented cycles \( D \) with the cycle index \( c(D) \) equal \( p \) is given by the formulae

\[
N(C_n, p) = \begin{cases} 
\frac{1}{n} \sum_{d \mid \gcd(n, p)} \varphi(d) \left( \frac{n/d}{p/d} \right), & \text{if } p \in \{\lfloor \frac{n}{2} \rfloor, \ldots, n\} \text{ and } p \neq \frac{n}{2}, \\
\frac{1}{2n} \sum_{d \mid \gcd(n, \frac{n}{2})} \varphi(d) \left( \frac{n/d}{2n/2d} \right) + 2^{\frac{p}{2}-2}, & \text{if } p = \frac{n}{2}.
\end{cases}
\]

Proof:

We use the combinatorial necklace model. If \( p \neq \frac{n}{2} \), it is straightforward to see that \( N(C_n, p) \) coincides with the number of combinatorial necklaces that have exactly \( p > n - p \) black beads. Hence, the first part of equality (6.9) follows from Proposition 6.4(b). To prove the second part, we assume that \( p = \frac{n}{2} = n - p \), i.e., we consider combinatorial necklaces that have an equal number of white and black beads. We have two possibilities: we can identify black beads with clockwise or counterclockwise edges. Since switching beads color yields isomorphic necklaces in exactly \( 2^{\frac{p}{2}-1} \) cases, we conclude that \( N(C_n, \frac{n}{2}) = 2^{\frac{p}{2}-1} + (Nck_2(\frac{n}{2}, n) - 2^{\frac{p}{2}-1})/2 \) and equality (6.9) follows.

As discussed earlier in this section, every non-negative poset \( I \) of a Dynkin type \( \text{Dyn}_I = \mathbb{A}_m \) can be viewed as a binary combinatorial necklace. Moreover, the cycle index \( c(I) \) (3.11) has a natural interpretation in this model: it is the number of black beads. By combining the results of Theorem 1.1 and Proposition 6.8 we get the proof of Theorem 1.2.

Proof of Theorem 1.2:

Apply the results of Fact 6.1, Theorem 1.1 and Proposition 6.8. In particular, to prove (b), assume that \( n \geq 4 \) and \( \frac{n}{2} \leq p \leq n - 2 \) are fixed and note that there exists only one poset \( I \) (oriented cycle \( C_n \)) that has exactly one maximal element (one sink) and \( c(I) = c(C_n) = p \).
Example 6.10. For \( n = 34 \), there exist exactly 4,547,647,110 non-isomorphic connected posets \( I \) of size \( n = |I| \), that are non-negative of Dynkin type \( \text{Dyn}_I = \tilde{A}_{n-\text{crk}_I} \). Moreover, these posets yield \( \lfloor \frac{n}{2} \rfloor = 17 \) Coxeter types (Coxeter polynomials). By Theorem 1.2, there are exactly 4,294,967,296 such positive (\( \text{crk}_I = 0 \)) posets with \( \text{cox}_I = t^{34} + t^{33} + t^{32} + \cdots + t + 1 \in \mathbb{Z}[t] \) and 252,679,814 principal (\( \text{crk}_I = 1 \)) ones, as described in Table 6.11.

| \( p \) | \( \text{cox}_I(t) \) | \#\( I \) | \( p \) | \( \text{cox}_I(t) \) | \#\( I \) |
|---|---|---|---|---|---|
| 17 | \( t^{34} - 2t^{17} + 1 \) | 34,350,506 | 25 | \( t^{34} - t^{25} - t^9 + 1 \) | 1,542,683 |
| 18 | \( t^{34} - t^{18} - t^{16} + 1 \) | 64,823,109 | 26 | \( t^{34} - t^{26} - t^8 + 1 \) | 534,075 |
| 19 | \( t^{34} - t^{19} - t^{15} + 1 \) | 54,587,279 | 27 | \( t^{34} - t^{27} - t^7 + 1 \) | 158,223 |
| 20 | \( t^{34} - t^{20} - t^{14} + 1 \) | 40,941,031 | 28 | \( t^{34} - t^{28} - t^6 + 1 \) | 39,575 |
| 21 | \( t^{34} - t^{21} - t^{13} + 1 \) | 27,293,639 | 29 | \( t^{34} - t^{29} - t^5 + 1 \) | 8,183 |
| 22 | \( t^{34} - t^{22} - t^{12} + 1 \) | 16,128,423 | 30 | \( t^{34} - t^{30} - t^4 + 1 \) | 1,367 |
| 23 | \( t^{34} - t^{23} - t^{11} + 1 \) | 8,414,639 | 31 | \( t^{34} - t^{31} - t^3 + 1 \) | 175 |
| 24 | \( t^{34} - t^{24} - t^{10} + 1 \) | 3,856,891 | 32 | \( t^{34} - t^{32} - t^2 + 1 \) | 16 |

Table 6.11: Coxeter polynomials of connected principal posets \( I \) with \( |I| = 34 \)

![Log-log plot](image)

Figure 12: Log–log plot of \#\( I \) with \( \text{cox}_I = t^n - t^p - t^{n-p} + 1 = N(C_n, p) - 1 \)
Remark 6.13. We recall from [14] that the number of connected non-negative posets $I$ of Dynkin type $\mathbb{A}_{n-\text{crk}_I}$ grows exponentially when $n \to \infty$. Theorem 1.2(b) gives a description of the number $N(C_n, p) - 1$ of principal ($\text{crk}_I = 0$) posets $I$ with $c(I) = p$. In particular, $N(C_n, p) = \lceil \frac{n}{2} \rceil$ for $p = n - 2$, i.e., in this case the number of posets grows linearly. Otherwise, for $p \neq n - 2$, the growth rate is exponential, see Figure 12 for illustration.

7. Summary and future work

In the present work, we give a complete Coxeter spectral classification of connected non-negative posets $I$ of Dynkin type $\text{Dyn}_I = \mathbb{A}_{|I| - \text{crk}_I}$. A complete description and classification of Dynkin type $\mathbb{D}_n$ posets is unknown (partial results are given in [9, 15, 16]). Therefore, the following problem remains open.

Problem 7.1. Give a structural description and Coxeter spectral classification of $\mathbb{D}_n$, Dynkin type non-negative connected posets.

In the case of $\mathbb{A}_m$ type posets, we show in the present work that there are exactly $\left\lfloor \frac{n}{2} \right\rfloor$ Coxeter types:

- $\text{cos}_I(t) = t^n + t^{n-1} + \ldots + t + 1 \in \mathbb{Z}[t]$ if $I$ is positive ($\text{crk}_I = 0$),
- $\text{cos}_I(t) = t^n - t^p - t^{n-p} + 1 \in \mathbb{Z}[t]$, where $\frac{n}{2} \leq p \leq n - 2$, if $I$ is principal ($\text{crk}_I = 1$).

Moreover, we prove that, given a pair of non-negative posets $I$ and $J$ of Dynkin type $\text{Dyn}_I = \text{Dyn}_J = \mathbb{A}_{|I| - \text{crk}_I}$, the incidence matrices $C_I$ and $C_J$ are $\mathbb{Z}$-congruent if and only if $\text{specc}_I = \text{specc}_J$. It is known that the assumption that Dynkin types coincide cannot be omitted in general (see [9, Example 4.1]). Our experiments show that this is possible for small posets.

Proposition 7.2. If $n \leq 8$ and $I$ is such a poset that $\text{cos}_I(t) = t^n + t^{n-1} + \ldots + t + 1$, then $I$ is connected, positive, and $\text{Dyn}_I = \mathbb{A}_n$.

Proof:

The proof is a computational one. We list all, up to isomorphism, posets $I$ of size $|I| \leq 8$, and verify that posets $J$ that have their Coxeter polynomial $\text{cos}_J(t) \in \mathbb{Z}[t]$ (2.7) equal $\text{cos}_J(t) = t^n + t^{n-1} + \ldots + t + 1$ are exactly non-negative ones, of Dynkin type $\text{Dyn}_J = \mathbb{A}_n$. □

Example 7.3. Let $I = (\{1, \ldots , 9\}, \preceq)$ be a connected poset defined by the following Hasse quiver.

\[ \mathcal{H}(I) = \begin{array}{c}
\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet
\end{array}
\]

$C_I = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

$\text{Cox}_I = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$

It is straightforward to check that:
• $q_I([2, -1, -2, -2, 0, -3, -1, 3, 2]) = -3$, i.e., poset $I$ is indefinite (is not non-negative),

• $\text{cox}_I(t) = t^9 + t^8 + t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + t + 1$.

In other words, the Coxeter spectrum (Coxeter polynomial) does not preserve the positivity of posets. Nevertheless, the following conjecture holds for posets of at most 13 elements.

**Conjecture 7.4.** If $I$ is such a poset, that $\text{cox}_I(t) = t^n + t^{n-1} + \cdots + t + 1$, then $I$ is connected. If, additionally, $I$ is non-negative, then $\text{Dyn}_I = \mathbb{A}_{|I|}$.

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