Analysis and comparison of the main characteristics of linear and nonlinear oscillations

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Abstract. In this paper, the problem of determining the positions of stable and unstable equilibrium positions of a mechanical system of the type of a planetary mechanism with one degree of freedom is considered. Analysis of possible movements of the mechanism is carried out using the phase plane. We investigate the dependence of the number of equilibrium positions on the change in the values of the parameters of the system. It is shown that with changes in the values of the parameters, the nature of the motion of the mechanism can change drastically. Some characteristics of nonlinear oscillations of the mechanism near the positions of stable equilibrium are determined. A comparative analysis of the obtained characteristics with similar characteristics for small (linear) oscillations is carried out.

1. Introduction

It is known that to ensure the long-term operation of any mechanical system in the conditions of vibrations it is necessary to conduct her dynamic calculation, that, in particular, must include determination of characteristics of her free oscillations accomplished near positions of stable equilibrium, yet on the stage of planning. Usually at the study of vibrations of the mechanical systems are limited to the consideration of those her deviations from the positions of stable equilibrium, that are described by linear differential equations with constant coefficients. The methods of decision of such equations are well worked out and the use of this method (method of linear oscillations) in many cases gives results, which mainly satisfactorily describe the most personal touches of dynamic processes in the mechanical system [1], [2]. But, as a rule, the vibrations of the mechanical systems are described by nonlinear differential equations, and their simplified description by linear equations with constant coefficients does not allow to reveal all peculiar features of the real process of oscillations. The aim of this work is to study the free nonlinear oscillations of element of the real mechanism [3], determination of some characteristics of such oscillations and comparison of them with analogical characteristics of linear oscillations.

2. Formulation of the Problem

A mechanism located in the vertical plane (Figure 1) containing a satellite 1 of radius \( r \) and of mass \( M_1 \) is considered. The satellite can roll without slipping along a fixed cylindrical surface of radius \( R \). The axis \( A \) of the satellite is fastened with a rod 2 of length \( L \) and of mass \( M_2 \), rotating about the horizontal fixed axis \( O \).
A weightless bar 5 with a counterweight 3 is rigidly attached to the gear 1. The counterweight of mass \( M_3 \) can be fixed on the bar 5 at a distance of \( AB=l_3 \). The load 4 of mass \( M_4 \) is on the rod 2 at a distance \( OD=l_4 \). In the middle C of the rod 2 is located its center of mass.

A mechanism has one degree of freedom. As the generalized coordinate was selected the angle \( \phi \) (radian) of deflection of the rod 2 from the vertical. The Figure 1 shows the mechanism in current moment of time. The superscript index * indicates the position of the points N, A, B when \( \phi=0 \). The position of a counterweight 3 in relation to a gear 1 determined by the angle \( \alpha=const \) between the rays \( OA* \) и \( A*B* \).

In research of vibrations of mechanism we use the expressions of his potential energy \( E_P(\phi) \) and kinetic energy \( E_K \), which have the form [4], [5]:

\[
E_P(\phi) = g \left[ (M_1 + M_2 + M_3) \cdot (R-r) - M_2 \cdot L/2 - M_4 \right] \cdot (1 - \cos \phi) + M_3 g l_3 \left[ \cos \left( \frac{R-r}{r} \varphi - \alpha \right) - \cos \alpha \right],
\]

\[
E_K = 0.5 \cdot M_1 (R-r)^2 + J_{IA} \left( \frac{R-r}{r} \right)^2 + M_2 \left[ \frac{l_1^2}{12} + \left( R-r - \frac{L}{2} \right)^2 \right] +

+ M_3 (R-r)^2 \cdot \left[ 1 + 2 \cdot \frac{l_3}{r} \cos \left( \frac{R}{r} \varphi - \alpha \right) + \frac{l_3^2}{r^2} \right] + M_4 l_4^2 \cdot \dot{\phi}^2 = \frac{1}{2} A(\phi) \cdot \dot{\phi}^2
\]

Where \( J_{IA} \) - a moment of inertia of the gear 1 relative to the horizontal axis A.

The analysis of possible motions of the gear 1 will be conducted by means of the phase plane – the Cartesian plane, along the abscissa axis of which the angular coordinate \( \phi(t) \) is plotted, and along the ordinates axis – the corresponding angular velocity \( \dot{\phi}(t) \). The trajectory of point with coordinates \( \phi(t), \dot{\phi}(t) \) (visual point) on a phase plane is called the phase trajectory [2]. Her kind depends on initial values \( \phi_0 \) and \( \dot{\phi}_0 \), and family of phase trajectories illustrates the possible processes in the mechanical system under consideration.
3. Research of the Movements

To obtain the equation of the phase trajectory, we use the fact that in a conservative system the mechanical energy remains constant:

\[ E_k + E_H = \text{const} = h, \quad (3) \]

where \( h \) is initial reserve of mechanical energy.

From (3) with an allowance (1) and (2), the equation of a phase trajectory has the form:

\[ \dot{\phi} = \pm \sqrt{\frac{2(h - E_H(\phi))}{A(\phi)}}, \quad (4) \]

As can be seen, there are no phase trajectories in the areas, where \( h < E_H(\phi) \).

The following parameter values are accepted: \( M_1=0.2 \) kg; \( M_2=M_4=0.5 \) kg; \( M_3=0.4 \) kg; \( R=0.55 \) m; \( r=0.1 \) m; \( l_4=0.2 \) m; \( L=0.8 \) m; \( \alpha=\pi \). Note that then if \( \phi=0 \) a counterweight 3 is in the lowest position.

For a start, leaving the listed parameters unchanged, find out the influence of the distance \( l_3 \) a counterweight from the center \( C \) of gear on the amount of positions of equilibrium and types of motion of the mechanism.

3.1 The case \( l_3=0 \) m.

The Figure 2 shows for this variant the potential energy curve (Figure 2a), constructed in accordance with the equation (1), and family of phase trajectories (Figure 2b), constructed in accordance with the equation (4). As follows from a theory [2] and Figure 2a, the mechanism is in the equilibrium position at \( \phi_1=0, \phi_2,3=\pm\pi \) rad. Moreover the angle \( \phi_1=0 \) (a minimum of potential energy) is the stable equilibrium position, and the angles \( \phi_2,3=\pm\pi \) (a maximum of potential energy) are unstable equilibrium positions.

The Figure 2b shows the trajectories of the visual point on a phase portrait, constructed for different values of initial reserve of mechanical energy \( h \) (Joule), which can be realized, in particular, only due to the initial angle of deflection of the rod \( \phi_0 \):

\[ h_1=0.08; \quad h_2=0.4; \quad h_3=1.0; \quad h_4=1.9; \quad h_5=2.7; \quad h_6=3.6; \quad h_7=4.3162; \quad h_8=4.9; \quad h_9=5.8. \]

Phase trajectories 1, 2, 3, …, 6 near stable equilibrium position \( \phi_1=0 \) are the disjoint closed curves similar to ellipses. It means that the motions of the rod of mechanism, described by these trajectories, will be almost harmonic motions. The unstable equilibrium positions \( (\phi_2,3=\pm\pi) \) are described by two
intersecting trajectories 7 (separatrices) along which the visual point moves with the initial reserve of mechanical energy \( h = 4.3162 \). Such movement is practically unrealized. Firstly, because even a small change of initial reserve of mechanical energy will lead to the exit of the visual point from this trajectory, and, secondly, even with accurate observance of the necessary energy reserve, the time of motion along the separatrices tends to infinity [1].

Increasing the initial mechanical energy higher than value \( h_1 \), we pass to the new type of the motion (trajectories 8, 9) – rotation of the mechanism in one direction with a variable angular velocity, which is maximal near stable equilibrium position and is minimum near unstable equilibrium position.

3.2 The case \( l_3 = 0.1 \) m.

In the Figure 3 the results of these calculations are presented. Now the potential energy (Figure 3a) has instead of one three local minimums (\( \varphi_1 = 0 \); \( \varphi_{2,3} = \pm 2.6583 \) rad) and has instead of two four local maximums (\( \varphi_{4,5} = \pm 2.3347 \); \( \varphi_{6,7} = \pm 3.4157 \) rad). Now there are three special points of type "center" and four special points of type "saddle" [1] on a phase portrait (Figure 3b). Phase trajectories are built for the next values of \( h \) (Joule): \( h_1 = 0.1848 \); \( h_2 = 1.3848 \); \( h_3 = 1.7848 \); \( h_4 = 2.9848 \); \( h_5 = 4.1433 \); \( h_6 = 4.1433 \); \( h_7 = 4.2278 \); \( h_8 = 4.7048 \); \( h_9 = 4.9978 \); \( h_{10} = 5.5848 \).

The periodic oscillations of the rod are determined by the phase trajectories, covered by separatrices 7 and 9. Inside the links of the separatrices 7 there are trajectories 1, 2, ..., 6. Trajectories 1, 2, ..., 5 describe periodic oscillations of the rod near the basic position of stable equilibrium, determined by the angle \( \varphi = 0 \), and trajectories 6 – near the additional positions of stable equilibrium, determined by the angles \( \varphi_{2,3} = \pm 2.6583 \) rad. Vibrations near the basic position of stable equilibrium (\( \varphi = 0 \)) practically harmonic, if the initial reserve of mechanical energy \( h = 0.1848 \). The amplitude of the rotation angle of rod is somewhere about 0.2 rad (Figure 3b, trajectory 1). With increasing of the initial reserve of mechanical energy \( h \) the vibrations near this position of equilibrium become nonharmonic with varying "jerks" of angular velocity. The amplitude of the angle \( \varphi \) of rotation of the rod can thus reach the value somewhere about 2.18 rad (Figure 3b, trajectory 5) at an initial reserve of mechanical energy \( h = 4.1433 \). From the phase portrait it follows that with fluctuations of mechanism near additional positions of stable equilibrium, for example, near \( \varphi = \pm 2.6583 \) rad, the angle of rotation of the rod \( 2.3347 < \varphi < 2.8400 \), that is the scope of the angle (\( \Delta \varphi = 0.5053 \)) much less than, with fluctuations near the basic position of stable equilibrium.

If the mechanism is given energy \( 4.2278 < h < 4.9978 \) at the initial moment of time, then the rod will also perform periodic motions near the basic stable equilibrium position (\( \varphi = 0 \)), but at the same time covering the positions of stable equilibrium \( \varphi_{2,3} = \pm 2.6583 \) rad. The amplitude of the angle of deflection of the rod \( 2.8400 < \varphi < 3.4157 \). If the mechanism received energy \( h > 4.9978 \) Joules (figure 3b, trajectory 10), the rod will rotate all the time in one direction with a sharply changing angular velocity: for one turn she five times takes on values of local maximum and four times takes on values of local minimum.

3.3 The case \( l_3 = 0.2 \) m.

The calculations results are presented on Figure 4. The graph shows that the potential energy (Figure 4a) has now five local minimums and six local maximums, accordingly a mechanism has five stable and six unstable positions of the equilibrium. Because phase trajectories are symmetrical about the ordinate axis (for \( \varphi = 0 \)), then on the Figure 4b for comfort of analysis they are given only for \( \varphi > 0 \). For such values we will consider the types of motions of mechanism. Three special points of type "center" (\( \varphi_1 = 0 \); \( \varphi_2 = 1.2583 \); \( \varphi_3 = 2.7388 \) rad)) and three special points of type "saddle" (\( \varphi_4 = 0.7989 \); \( \varphi_5 = 2.2084 \); \( \varphi_6 = 3.4493 \) rad)) are visible on the abscise axis of the phase portrait. Coordinates \( \varphi_1, \varphi_2, \varphi_3 \) define the first, second and third positions of the stable equilibrium. The parameter \( h \) (Joule) has following values: \( h_1 = 0.3696 \); \( h_2 = 0.9696 \); \( h_3 = 1.6696 \); \( h_4 = 1.8696 \); \( h_5 = 2.1431 \); \( h_6 = 1.8696 \); \( h_7 = 1.6696 \); \( h_8 = 2.9696 \); \( h_9 = 1.1896 \); \( h_{10} = 4.4696 \); \( h_{11} = 4.9115 \); \( h_{12} = 4.4696 \); \( h_{13} = 4.1896 \); \( h_{14} = 5.3696 \); \( h_{15} = 5.7711 \); \( h_{16} = 7.1696 \); \( h_{17} = 9.5696 \). As follows from a phase portrait, the rod performs periodic oscillations around the angle \( \varphi_3 = 2.7388 \) rad(third position of stable equilibrium), if a mechanism in this position possesses a reserve.
of energy $4.1666 < h < 4.9115$. The angle $\varphi$ of the deflection of the rod 2 from the vertical line belongs to the interval $2.2084 < \varphi < 3.0882$ (for example, trajectories 12 and 13 on the phase portrait), that is the maximal angle of turn is equal $0.8798$ rad. As seen, the maximal angle of deflection of the rod from this position of stable equilibrium depends on direction of rotation: when turning in the direction of increasing the angle $\varphi$ it is equal to $0.3494$ rad, and in the opposite direction $-0.5304$ rad.

Near angle $\varphi = 0$ rad (first position of stable equilibrium) the rod, depending on the initial reserve of mechanical energy, can perform three types of periodic movements:

- if $0 < h < 2.1431$, then amplitude of the angle of deflection of the rod from the vertical line $\phi_0 < 0.7988$ rad (trajectories 1, 2, 3, 4 on the phase portrait);
- if $2.1431 < h < 4.9115$, then rod vibrations cover the first and second stable equilibrium positions, the amplitude of the angle of turn belongs to the interval $1.5140 < \phi_0 < 2.2084$ (rad) (trajectories 8, 9, 10 on the phase portrait);
- if $4.9115 < h < 5.7711$ (Joules), then rod vibrations cover all three positions of stable equilibrium, the amplitude of the angle of turn rod belongs to the interval $3.0882 < \phi_0 < 3.4493$ (rad) (trajectory 14 on the phase portrait).

If to increase the energy of mechanism higher than value $h = 5.7711$ Joules, then the rod passes to a non-oscillatory motion – one-way rotation (trajectories 16 and 17 on the phase portrait).

Further, supposing $l_3 = 0.2$ m, will define some characteristics of the free nonlinear oscillations of the mechanism and will compare them to analogical characteristics of linear oscillations. In this case the potential energy (1) and the kinetic energy (2) of mechanism are determined by next expressions:

$$E_p(\varphi) = 2.943 - 2.158 \cos \varphi - 0.785 \cos (4.5 \varphi), \quad E_k = 0.5 \cdot \left( 0.5049 - 0.324 \cos (5.5 \varphi) \right) \cdot \dot{\varphi}^2 = 0.5 A(\varphi) \cdot \dot{\varphi}^2.$$

Will analyze the motion of mechanism near the position of the stable equilibrium, determined by the angle $\varphi = 0$.

If to examine the linear oscillations of the mechanism, then the simplified expressions for the kinetic and potential energies have the form:

$$E^*_k = \frac{1}{2} A(\varphi)_{\varphi=0} \cdot \dot{\varphi}^2 = \frac{1}{2} a \cdot \dot{\varphi}^2, \quad a = 0.1809 \text{ J (rad}^1)^2;$$
$$E^*_p = \frac{1}{2} \left( \frac{\partial^2 E_{\Pi}}{\partial \varphi^2} \right)_{\varphi=0} \cdot \dot{\varphi}^2 = \frac{1}{2} c \cdot \dot{\varphi}^2, \quad c = 18.0504 \text{ J rad}^2.$$  

Oscillation period of the mechanism

$$T = 2\pi \sqrt{a/c} = 0.629 c$$

In the moment of the maximal deviation $h = E^*_p = \frac{1}{2} c \cdot \phi^2_0$. Hence the amplitude of the linear oscillations

$$\phi_0 = \sqrt{2h/c}$$
If $\phi = 0$, then $E_H = 0$, and $\dot{\phi} = \dot{\phi}_{\text{max}}$. In this position $h = E_k^* = \frac{1}{2} a \dot{\phi}_{\text{max}}^2 = 0$, that is, the high angular velocity of the rod is

$$\dot{\phi}_{\text{max}} = \sqrt{2h/a} \quad (9)$$

Now will consider the nonlinear oscillations of the mechanism near this stable equilibrium position in detail. The Figure 5 shows the phase trajectories constructed for the different values of $h$. As can be seen, the family of disjoint trajectories 1-7 has a common center at the origin. It means that the considered position of equilibrium is stable, and the vibrations of the mechanism near him are periodic oscillations. The trajectories 1 and 2 are almost ellipses. It can be argued, that in these cases the mechanism, having obtained the corresponding energy reserves $h_1 = 0.02$ and $h_2 = 0.18$, performs almost harmonic oscillations with amplitudes $\dot{\phi}_{01} = 0.047$ rad (2.7°) and $\dot{\phi}_{02} = 0.143$ rad (8.2°). Such energy reserves can be realized, in particular, by the initial deflection of the rod. With an increase in the value $h$ ($h_3 = 0.7$; $h_4 = 1.1$; $h_5 = 1.5$; $h_6 = 1.9$; $h_7 = 2.0$ (Joule)), the trajectories become more and more diamond-shaped, and the oscillations of the rod are increasingly different from harmonic. The curve 8 in Figure 5 (separatrix) is a curve with selfintersection. The abscissas of points selfintersection correspond to the local maximums on the potential energy graph $E_H(\phi)$ (Figure 4a), and therefore, correspond to unstable positions of the equilibrium, that will be realized with the initial reserve of mechanical energy $h_8 = 2.143$. If $h > h_8$ (trajectory 9 on the Figure 5) the mechanism makes a different kind of movement, which differs from that considered in this part of the work.

The important characteristic of the mechanism is the period of his free oscillations, which can be defined in the following way. If the abscissas of the points of selfintersections of any closed phase trajectory in the phase portrait (Figure 5) denote by $\phi_1$, $\phi_2$, then a period of nonlinear oscillations will be calculated by the formula

$$T_1 = 2 \int_{\phi_1}^{\phi_2} \frac{A(\phi)}{\sqrt{2[h - E_H(\phi)]}} d\phi \quad (10)$$

The Figure 6 shows the graphic dependencies on $h$ following parameters: 1 – period $T$ of linear oscillations of the mechanism, calculated in accordance with a formula (7); 2 – period $T_1$ of nonlinear oscillations of the mechanism, calculated in accordance with (10); 3 – relation of the periods $T_1 / T$. The period $T_1$ of the nonlinear oscillations increases with increasing the value $h$, in contrast to the permanent period $T$ of linear oscillations. And when approaching the value $h$ to $h_8$, which characterizes the position of unstable equilibrium, $T_1$ tends to infinity. In practice, except the period of oscillations, it is important to know also the amplitude of the angle of deflection of the rod depending on the $h$. 

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**Figure 5.** Phase portrait.

**Figure 6.** The dependence of the periods from $h$. 

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Taking into account that at the maximal deviation of the mechanism $E_k = 0$, then the amplitude of the angle is the root of equation:

$$h = E_{II}(\varphi)$$  \hspace{1cm} (11)

The Figure 7 shows the graphic dependencies on $h$ following parameters: 1 – amplitude $\varphi_0$ (rad) of the linear oscillations of the mechanism, calculated in accordance with a formula (8); 2 – amplitude $\varphi_1$ of nonlinear oscillations of the mechanism, calculated in accordance with (11); 3 – relation $\varphi_1/\varphi_0$. Evidently, that at identical $h$ always $\varphi_1 > \varphi_0$ (except $h=0$), but the relation $\varphi_1/\varphi_0$ although increases with increasing $h$, tends to the value 1.65 at $h \to h_8$. From that at $h \to h_8$ the period of nonlinear oscillations $T_1 \to \infty$, it follows that at the initial reserve of mechanical energy $h_8$ the mechanism practically will not be able to perform oscillations, because it requires infinitely long time.

![Figure 7. The dependence of oscillation amplitudes from $h$.](image1)

![Figure 8. The dependence of the maximum angular velocity from $h$.](image2)

Interest is also the dependence of the top angular velocity of the rod on the initial reserve of mechanical energy $\varphi_{max}(h)$. For linear oscillations she is determined by a formula (9), and for nonlinear – by a transformed formula (4). The maximum value of $\dot{\varphi}$ is reached at $\varphi=0$. In this position $E_{II} = 0$ and, in accordance with (4) and (5), $A(\varphi) = a$. Consequently, the dependences $\dot{\varphi}_{max}(h)$ for nonlinear and linear oscillations are identical and determined by the formula (9).

The Figure 8 shows the graphic dependencies $\dot{\varphi}_{max}(h)$. As expected the maximum angular velocity of the rod increases with increasing $h$. At $h=2.14$, which is close to the value of $h_8$, at which the positions of unstable equilibrium takes place, the angular velocity is $\dot{\varphi} = 4.87 \text{ rad} \cdot \text{c}^{-1}$.

4. Conclusion
For the real planetary mechanism, the dependence of the angular velocity of the rod on its rotation angle was determined; and the positions of equilibrium for $\alpha = \pi$, which define the location of the counterweight in the plane of the gear was determined. Using the method of phase plane the types of the movements of mechanism are found and analyzed depending on the distance $l_3$ of the counterbalance to the gear center. It is established that with increasing the parameter $l_3$ the number of stable equilibrium positions increases and can be equal to one, three or five. Depending on the initial reserve of mechanical energy the mechanism can perform free oscillations not only near separately considered stable equilibrium positions, but also at once near three or five positions of stable equilibrium. When the mechanism is oscillated near a certain stable equilibrium positions (except for the case $\varphi_1=0$), the maximum angle of deflection of the rod depends on the direction of turn: when
turning in the direction of increasing $|\phi|$ it is less than when turning in the direction of decreasing $|\phi|$. The angular velocity of the rod is always maximal near the stable equilibrium positions and is minimal near the unstable equilibrium positions.

For the case $\alpha=\pi$ the graphs of the amplitudes of the angle of rotation of the rod, the periods of free oscillations of the mechanism and the maximum angular velocity of the rod on the parameter $h$ are plotted for linear and nonlinear oscillations of mechanism near the stable equilibrium position determined by the angle $\phi=0$. The carried out analysis confirms the necessity of using methods of nonlinear mechanics to obtain more accurate results in comparison with the results obtained by linear mechanics methods.

The bifurcations observed on the phase portraits are demonstrated also on the mockup of the real mechanism. Therefore it is expedient to study some parameters of trajectories belonging to classes, bounded by separatrices and singular points. The article considers the methodology of studying of these parameters. This is the novelty of researches and of the results obtained.

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