Computing Minimal Injective Resolutions of Sheaves on Finite Posets

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Abstract

In this paper we introduce two new methods for constructing injective resolutions of sheaves of finite-dimensional vector spaces on finite posets. Our main result is the existence and uniqueness of a minimal injective resolution of a given sheaf and an algorithm for its construction. For the constant sheaf on a simplicial complex, we give a topological interpretation of the multiplicities of indecomposable injective sheaves in the minimal injective resolution, and give asymptotically tight bounds on the complexity of computing the minimal injective resolution with our algorithm.

1 Introduction

A common strategy for analyzing a complicated mathematical structure is to approximate or represent the given structure with a collection of simpler, or at least more familiar, objects; the goal is to reframe questions concerning the complex structure as questions about the building blocks which represent it. Illustrations of this strategy permeate mathematics. The focus of this paper is a particular instance of this phenomena: injective resolutions of sheaves.

Sheaves use algebra to model relationships between local and global properties of a topological space. When the topological space is a poset (with the Alexandrov topology), a sheaf, $F$, is defined by associating a finite-dimensional vector space, $F(\sigma)$, to each element, $\sigma$, and a linear map, $F(\sigma \leq \tau) : F(\sigma) \rightarrow F(\tau)$, to each relation $\sigma \leq \tau$ (subject to commutativity requirements, see Definition 1). The utility of this definition is also its foil: the high level of generality encompasses many pathologies. For example, each persistence module (including the multi-parameter ones) can be viewed as a sheaf on a poset, and all of the difficulties in analyzing multi-parameter persistence modules arise when studying sheaves. An injective resolution represents a given sheaf (much like a barcode or persistence diagram represents a 1-dimensional persistence module) with an exact sequence of injective sheaves, which admit many desirable properties. Efficient algorithms for computing injective resolutions are a first step toward applying well-established and powerful theoretical results from derived sheaf theory to persistent homology. In this paper we aim to present this theory in an explicit and computationally feasible framework.

Injective Resolutions. An injective sheaf (Definition 2), $I$, is a sheaf such that each morphism of sheaves $G \rightarrow I$ (Definition 3) can be extended to a morphism $F \rightarrow I$, whenever $G \subset F$. Injective sheaves admit many beneficial features which general sheaves lack (see, for example, Lemma 9, Proposition 10 and Lemma 11). From the perspective of homological algebra, injective sheaves are the ‘basic’ objects with which we aim to represent a general sheaf. However, standard operations in linear algebra are insufficient for such a representation. For example, if a sheaf is not already injective, then it does not decompose into a direct sum of injective sheaves. Instead, we will represent a given sheaf $F$ with an injective resolution (Definition 14): an exact sequence, $0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \cdots$, such that each $I^j$ is an injective sheaf.
Injective resolutions, a fundamental ingredient for homological algebra, are used to study sheaves from the ‘derived’ perspective, i.e. as objects in a derived category. These derived categories unify and generalize several forms of cohomology, such as simplicial cohomology, de Rham cohomology, intersection cohomology, etc. For example, simplicial cohomology (and level-set persistent cohomology) can easily be computed from an injective resolution of the constant sheaf (see Example 2 and Section 5), illustrating that even an injective resolution of the constant sheaf contains subtle topological information. Several recent works point to the potential benefits of applying derived sheaf theory to the study of persistent homology [BG21, BGO19, BP21, Cur14, KS18, KS21]. We approach this subject from a computational perspective in order to help bridge gaps between applied topology and derived sheaf theory. With this goal in mind, we aim to limit the mathematical prerequisites of our approach whenever possible (a choice which often comes at the cost of brevity).

**Main Results.** In this paper we develop methods for computing injective resolutions of sheaves of finite-dimensional vector spaces on finite posets. Our main contributions are:

1. We establish the existence and uniqueness of a minimal injective resolution of a given sheaf (Theorem 17 and Corollary 18).
2. For the constant sheaf on a simplicial complex, we give a topological interpretation of the multiplicity of an indecomposable injective sheaf in the minimal injective resolution, in terms of compactly supported cohomology (Theorem 22).
3. We introduce a non-inductive definition of a (non-minimal) injective resolution of a given sheaf (Section 4.1).
4. We introduce an inductive algorithm for computing the minimal injective resolution (Algorithm 2), and prove correctness of the algorithm (Section 4.2).
5. For the constant sheaf on a simplicial complex, we give an asymptotically tight bound on the complexity of Algorithm 2 (Proposition 32 and Corollary 33).

As an application of our results, in Section 5 we explicitly describe the right derived push-forwards, $R^\bullet f_*$ and $R^\bullet f_!$, and show that traditional (level-set) multi-parameter persistence modules can be recovered from $R^\bullet f_* k_\Sigma$.

**Comparison to Prior Work.** Derived sheaf theory is a rich subject which has been thoroughly developed over several decades. There are multiple textbooks on sheaf theory [Bre97, Ive86, KS94], and many publications which study sheaves on finite topological spaces. In [She85], Shepard relates sheaves on finite cell complexes (viewed as posets) to the classical setting of constructible sheaves on stratified topological spaces. In [Lad08], Ladkani studies the homological properties of finite posets and introduces combinatorial criteria guaranteeing derived equivalences between categories of sheaves. In [Cur14], Curry establishes the connection between sheaf theory and persistent homology. More recently, several publications expand on the work initiated by Curry on applications of derived sheaf theory to persistent homology [BG21, BGO19, BP21, KS18, KS21].

Motivated by the above work, and by the potential to develop new techniques for computational topology, we aim to establish results on computational aspects of derived sheaf theory for finite topological spaces. The contributions of this paper are the first of our knowledge to describe and analyze efficient algorithms for computing injective resolutions of sheaves on finite posets. While this is a necessary first step toward utilizing the machinery of derived categories in computational topology, there is still more work to be done. In Section 6 we describe several future directions of research which stem from this paper.

**Remark.** We should comment on a matter of perspective and terminology. Over finite posets, sheaves are closely related (and sometimes equivalent, as in the case of modules over
the incidence algebra of a poset), to several other mathematical objects studied by various 
research communities. Specifically, a great deal of work has been done in commutative 
algebra on minimal projective and free resolutions of modules over various kinds of algebras. 
However, in the present paper we choose to focus on the perspective and terminology which 
most closely aligns with classical sheaf theory, in order to preserve intuition from that 
discipline.

2 Background and Preliminary Results

In this paper we study finite-dimensional vector space valued sheaves on finite posets. We 
begin by recalling preliminary definitions and results, drawing heavily from [Cur14, She85, 
Lad08]. Throughout the paper we fix a field \( k \).

For \( \pi, \tau \) in a poset \( \Pi \), we write \( \pi \overset{1}{<} \tau \) if \( \pi \not< \tau \), and there is no other element between \( \pi \) 
and \( \tau \). We use some of the standard terminology from simplicial complexes for general posets: 
the \textit{star} of an element \( \sigma \) is \( \text{St}_\sigma = \{ \tau \in \Pi \mid \sigma \leq \tau \} \), the \textit{boundary} is \( \text{bnd}(\sigma) = \{ \tau \in \Pi \mid \tau \overset{1}{<} \sigma \} \), 
and the \textit{coboundary} is \( \text{cobnd}(\sigma) = \{ \tau \in \Pi \mid \sigma \overset{1}{<} \tau \} \).

\textbf{Definition 1.} A sheaf \( F \) on a finite poset \( \Pi \) is an assignment of a finite-dimensional 
\( k \)-vector space \( F(\pi) \) to each element \( \pi \in \Pi \), and an assignment of a linear map 
\( F(\tau \leq \gamma) : F(\tau) \to F(\gamma) \), to each face relation \( (\tau \leq \gamma) \in \Pi \), such that

1. \( F(\tau \leq \tau) = \text{id}_{F(\tau)} \)
2. \( F(\tau \leq \gamma) \circ F(\sigma \leq \tau) = F(\sigma \leq \gamma) \)

for each triple \( \sigma \leq \tau \leq \gamma \in \Pi \).

\textbf{Example 2.} The \textit{constant sheaf}, denoted \( k_\Pi \), on a poset \( \Pi \), assigns to each element \( \pi \) 
of \( k \)-dimensional vector space, \( k \), and to each relation, \( (\pi \leq \tau) \in \Pi \), the identity map \( \text{id}_k \).

\textbf{Definition 3.} A natural transformation, \( \eta : F \to G \), between two sheaves on \( \Pi \), is a 
collection of linear maps \( \eta(\pi) : F(\pi) \to G(\pi) \) for each \( \pi \in \Pi \), such that

\[ G(\tau \leq \gamma) \circ \eta(\tau) = \eta(\gamma) \circ F(\tau \leq \gamma), \]

for each \( \tau \leq \gamma \in \Pi \). For a natural transformation \( \eta : F \to G \), the kernel, cokernel, image, 
and coimage are taken point-wise, defining sheaves on \( \Pi \):

\[ (\ker \eta)(\pi) := \ker(\eta(\pi)), \quad (\ker \eta)(\pi \leq \tau) := F(\pi \leq \tau) |_{\ker \eta(\pi)}. \]

Moreover, if \( \ker \eta(\pi) = 0 \) for each \( \pi \in \Pi \), we say that \( \eta \) is \textit{injective}. We write \( G/F := \text{coker} \eta \) 
if \( \eta \) is an injection clear from the context.

\textbf{Definition 4.} A sheaf \( I \) is called injective if for each injective natural transformation \( A \hookrightarrow B \), 
any given natural transformation \( A \to I \) can be extended to \( B \to I \):

\[
\begin{array}{ccc}
0 & \longrightarrow & A & \longrightarrow & B \\
& & & \downarrow \triangle & \\
& & & 1_3 & \mapsto I \\
& & & 1_4 & \\
& & & \downarrow & \\
& & & I & 
\end{array}
\]

This condition is always satisfied for sheaves over a single point space (i.e., the assignment 
of a point to a single vector space): we can extend any linear map on a subspace to the 
whole space by mapping a complement space to 0. This property does not hold in general 
for sheaves. The following two examples show sheaves that are not injective.
Example 5. We define a sheaf $F$ which does not satisfy the condition of Definition 4. Let us fix a vector space $W$, and define two sheaves, $F$ and $G$, on a poset with two elements $\sigma \leq \tau$. Let $F_\sigma = 0$ and $F_\tau = W$, and $G_\sigma = W = G_\tau$ with $G(\sigma \leq \tau) = \text{id}$. Then $F$ embeds into $G$, and we claim that $F \xrightarrow{\text{id}} F$ cannot be extended to $G \to F$.

Indeed, the only way to make the right square commute is for both maps to be 0, but then the top triangle does not commute.

We can use the same reasoning for a sheaf on any poset with a non-zero vector space one step above a zero vector space. Below we demonstrate one other obstruction to injectivity.

Example 6. We consider a three-element “V” shaped poset, a vector space $W$, and two different endomorphisms $f, g : W \to W$. We define sheaves $A, B$ and $F$ as follows:

We claim that $F$ does not satisfy the condition in Definition 4. The sheaf $A$ embeds into $B$, and we choose $\alpha : A \to F$ to be the analogous embedding. To define an extension $\beta : B \to F$, we only have a choice for the bottom map $\beta_0 : W \to W$. However, commutativity requires $f = \beta_0 = g$, which is impossible to satisfy, since $f \neq g$.

Avoiding the obstructions above, we define the simplest injective sheaves as follows.

Definition 7 (cf. [Cur14, Definition 7.1.3]). For each $\pi \in \Pi$, we define an indecomposable injective sheaf $[\pi]$ as

$$[\pi](\gamma) := \begin{cases} k & \text{if } \gamma \leq \pi, \\ 0 & \text{otherwise,} \end{cases} \quad \text{with} \quad [\pi](\gamma \leq \tau) := \begin{cases} \text{id} & \text{if } \gamma \leq \tau \leq \pi, \\ 0 & \text{otherwise}. \end{cases}$$

For $n \in \mathbb{Z}_{\geq 0}$, we denote by $[\pi]^n$, the direct sum $\bigoplus_{i=1}^n [\pi]$. For a vector space $V$, we denote by $[\pi]^V$, the sheaf $[\pi]^\dim V$, with an implicitly fixed isomorphism between $k^\dim V$ and $V$.

The following results can be found in [Cur14] and [She85] for sheaves on cell complexes. We give a straightforward generalization of the results to sheaves on any finite poset.

Lemma 8 (cf. [Cur14, Lemma 7.1.5]). Indecomposable injective sheaves are injective.

Proof. We show that $I = [\pi]$ for a fixed poset $\Pi$ and $\pi \in \Pi$ satisfies Definition 4. Given an inclusion $A \xrightarrow{\alpha} B$ and a natural transformation $\alpha : A \to I$, we need to find an extension $\beta : B \to I$. For the linear map $A(\pi) \xrightarrow{f(\pi)} B(\pi)$, there is a projection $A(\pi) \xrightarrow{\pi} B(\pi)$ such that $gf(\pi) = \text{id}_{A(\pi)}$. We define

$$\beta(\gamma) := \begin{cases} \alpha(\gamma) \circ g \circ B(\gamma \leq \pi) & \text{if } \gamma \leq \pi, \\ 0 & \text{otherwise}. \end{cases}$$
For every $\gamma$, this satisfies $\beta(\gamma)f(\gamma) = \alpha(\gamma)$, because if $\gamma \leq \pi$, then

$$\beta(\gamma)f(\gamma) = \alpha(\pi)gB(\gamma \leq \pi)f(\gamma) = \alpha(\pi)g\alpha(\gamma)\alpha(\gamma) = \alpha(\gamma),$$

and otherwise both sides are 0. For the commutativity conditions, consider $\gamma \leq \tau \leq \pi$. Then

$$\beta(\tau)B(\gamma \leq \tau) = \alpha(\pi)g\alpha(\gamma)B(\gamma \leq \tau) = \alpha(\pi)g\alpha(\tau)B(\gamma \leq \pi) = \beta(\tau) = I(\gamma \leq \tau)\beta(\gamma).$$

If $\gamma \leq \tau \leq \pi$, then both sides are 0. \hfill \square

**Lemma 9** (cf. [She85, Lemma 1.3.1]). A direct sum of injective sheaves is injective. Additionally, if $I \rightarrow J$ is an injective natural transformation with $I, J$ injective sheaves, then $J \cong I \oplus \text{coker} \alpha$, and $\text{coker} \alpha$ is an injective sheaf.

**Proof.** This proof is standard for any abelian category, we include a sketch for completeness. Suppose $I = A \oplus B$, with $A, B$ injective sheaves. Suppose $F \rightarrow G$ and $F \rightarrow I$. Then composition with projection gives maps $F \rightarrow A$ and $F \rightarrow B$. By injectivity of $A$ and $B$, each map extends to $G \rightarrow A$ and $G \rightarrow B$, respectively. The sum of these maps defines an extension $G \rightarrow I$, proving that $I$ is injective. The second claim follows by extending the identity map $I \rightarrow I$ to $J \rightarrow I$ by $\alpha$ and the injectivity of $J$. Then, the sum of the extension and the quotient map define an isomorphism $J \rightarrow I \oplus \text{coker} \alpha$. The final claim follows by composing a given map $F \rightarrow \text{coker} \alpha$ with the extension by zero map, $\text{coker} \alpha \rightarrow J$, to get $F \rightarrow J$. Then, for $F \rightarrow G$, we define (by the injectivity of $J$) an extension $G \rightarrow J$. By post-composing with the projection map, we get the desired extension $G \rightarrow \text{coker} \alpha$. \hfill \square

**Proposition 10** (cf. [Cur14, Lemma 7.1.6], [She85, Theorem 1.3.2]). Every injective sheaf is isomorphic to a direct sum of indecomposable injective sheaves.

**Proof.** We adapt the proof of [She85, Theorem 1.3.2] to the setting of finite posets on $n$ elements (rather than cell complexes). We fix some linear extension of the partial order, $(\pi_1, \ldots, \pi_n)$, and let $\Pi_i = \{\pi_j | j \leq i\}$. We will proceed with the proof by working inductively through this filtration of $\Pi$. We define support of a sheaf $I$ as

$$\text{supp } I := \{\pi \in \Pi | I(\pi) \neq 0\}.$$

Assume that the result holds for injective sheaves supported on $\Pi_{i-1}$. Suppose $I$ is an injective sheaf with support contained in $\Pi_i$. If $\text{supp } I \subseteq \Pi_{i-1}$, then the inductive assumption implies the result. Therefore, we are left to prove the result for $I$ such that $I(\pi_i) \neq 0$. Set $F_{\pi_i}$ to be the functor which assigns $I(\pi_i)$ to $\pi_i$ and the zero vector space to each other poset element (and the zero linear map to each poset relation). Then the identity map induces injective natural transformations

$$F_{\pi_i} \xrightarrow{\alpha} I \quad \text{and} \quad F_{\pi_i} \xleftarrow{\oplus_{v \in B} [\pi_i]},$$

where $B$ is some basis of $I(\pi_i)$. Because $I$ is injective, we can extend $\alpha$ to a natural transformation $\beta : \bigoplus_{v \in B} [\pi_i] \rightarrow I$. It is injective, because for every $\sigma \leq \pi_i$, the linear map $I(\sigma \leq \pi_i)\beta(\sigma) = \beta(\pi_i) = \alpha(\pi_i)$ is injective. By Lemma 9 this implies that

$$I \cong \text{coker } \beta \oplus \bigoplus_{v \in B} [\pi_i],$$

and that $\text{coker } \beta$ is injective. Because $\text{supp } \text{coker } \beta \subseteq \Pi_{i-1}$, the inductive hypothesis completes the proof. \hfill \square
2.1 Natural transformations between injective sheaves

Before we introduce injective resolutions, we take a brief detour to discuss natural transformations between injective sheaves. To describe a map between two sheaves on a poset, in general, we need to give a linear map for each \( \tau \in \Pi \). For two injective sheaves the situation is simpler.

Given a natural transformation \( \varphi : I \rightarrow J \) between two injective sheaves, and a decomposition into indecomposable injective sheaves as in Lemma 10

\[
I = \bigoplus_{i=1}^{m} [\pi_i]^{p_i} \quad \text{and} \quad J = \bigoplus_{j=1}^{n} [\sigma_j]^{s_j},
\]

\( \varphi \) can be uniquely described by a collection of linear maps \( f_{ij} : k^{p_i} \rightarrow k^{s_j} \), for each pair \( i, j \) such that \( \sigma_j \leq \pi_i \). Moreover, each collection of linear maps defines a natural transformation.

**Lemma 11.** Suppose \( I = \bigoplus_{i=1}^{m} [\pi_i]^{p_i} \) and \( J = \bigoplus_{j=1}^{n} [\sigma_j]^{s_j} \). Then

\[
\text{Hom}(I, J) \cong \bigoplus_{\sigma_j \leq \pi_i} \text{Hom}(k^{p_i}, k^{s_j}) \cong \bigoplus_{\sigma_j \leq \pi_i} k^{p_i \cdot s_j},
\]

where \( \text{Hom}(I, J) \) denotes the set of natural transformations from \( I \) to \( J \) and \( \text{Hom}(k^{p_i}, k^{s_j}) \) denotes the set of linear transformations from \( k^{p_i} \) to \( k^{s_j} \).

**Proof.** Using projection and inclusion maps of the direct sum, proj, incl, a map \( \varphi : I \rightarrow J \) can be decomposed as a sum of maps \( \varphi_{ij} = \text{proj}_{I,\pi_i} \circ \varphi \circ \text{incl}_{J,\sigma_j} \) between the powers of indecomposable injective sheaves. Consider \( \tau \in \Pi \). If \( \tau \leq \sigma_j \), then \( \varphi_{ij}(\tau) = 0 \). Otherwise, \( \varphi_{ij}(\tau) = \varphi_{ij}(\tau) \circ [\sigma_j]^{s_j} \circ (\pi_\tau) = \varphi_{ij}(\sigma_j) \circ [\pi_i]^{p_i} \circ (\tau \leq \sigma_j) = \varphi_{ij}(\sigma_j) \). This shows that \( \varphi_{ij} \) is determined by the linear map \( f_{ij} : = \varphi_{ij}(\sigma_j) \). Moreover, this map is necessarily 0 whenever \( \sigma_j \leq \pi_i \), and it can be any linear map otherwise. \( \square \)

In other words, we can represent a natural transformation \( \varphi : I \rightarrow J \) as just one \((\sum_{i=1}^{n} s_j) \times (\sum_{i=1}^{m} p_i)\) matrix with rows and columns labeled by the indecomposable injective sheaves in the decomposition of \( J \) and \( I \), respectively. The value \( \varphi[[\sigma], [\pi]] \) at position labeled by \((\sigma), (\pi)\) gives the linear map between \([\pi](\sigma)\) and \([\sigma](\sigma)\). It is always 0 if \( \sigma \leq \pi \) (see Figure 3).

3 Injective Resolutions

In this section we give definitions of injective hull and resolution, and present our main theoretical results about the minimal injective resolutions.

**Definition 12.** An injective hull of a sheaf \( F \) is an injective sheaf \( I \) together with an injective natural transformation \( F \rightarrow I \).

**Definition 13.** A (bounded) complex of sheaves, denoted \( A^\bullet \), is a sequence of sheaves \( A^i \) and natural transformations \( \mu^i \)

\[
\cdots \rightarrow A^i \xrightarrow{\mu_{i+1}} A^{i+1} \xrightarrow{\mu_{i+2}} A^{i+2} \xrightarrow{\mu_{i+3}} \cdots
\]

such that \( \mu_{i+1} \circ \mu_i = 0 \) for each \( i \), and \( A^i = 0 \) for \( |i| \) sufficiently large. A complex is exact if \( \text{im} \mu_i = \ker \mu_{i+1} \) for each \( i \). A morphism \( \alpha^\bullet : A^\bullet \rightarrow B^\bullet \) between complexes of sheaves is a collection of natural transformations \( \alpha^i : A^i \rightarrow B^i \) such that the diagrams commute.
Definition 14. An injective resolution of a sheaf $F$ is an exact sequence
$$0 \to F \xrightarrow{\alpha} I^0 \xrightarrow{\eta^0} I^1 \xrightarrow{\eta^1} I^2 \xrightarrow{\eta^2} \cdots$$
where $I^j$ is an injective sheaf for each $j$. We denote by $I^\bullet$ the complex
$$\cdots \to 0 \to I^0 \xrightarrow{\eta^0} I^1 \xrightarrow{\eta^1} \cdots$$

A classical result of sheaf theory is that each sheaf admits an injective resolution (though it need not be unique) [Ive86]. In the remainder of the paper, we will introduce and study explicit algorithms for computing injective resolutions of a given sheaf $F$.

3.1 Minimal Injective Resolutions

We will now define minimal injective resolutions, and show that they are unique up to isomorphism of complexes. We fix a sheaf $F$ on a finite poset $\Pi$.

Definition 15. A vector $s \in F(\pi)$ is maximal if $F(\pi \leq \tau)(s) = 0$ for each $\tau \geq \pi$. Let $M_F(\pi)$ be the subspace of maximal vectors in $F(\pi)$, i.e.
$$M_F(\pi) := \bigcap_{\pi < \sigma} \ker F(\pi \leq \sigma).$$

Note that it is sufficient to take only the intersection of $\ker F(\pi \leq \sigma)$ for each $\pi <_1 \sigma$.

Definition 16. An injective resolution $I^\bullet$ of $F$ is minimal if, for each $i$, the number of indecomposable injective summands of $I^i$ is minimal among all injective resolutions of $F$.

Theorem 17. Let $I^\bullet$ be an injective resolution of a sheaf $F$. The following are equivalent:

1. $I^\bullet$ is minimal.

2. For any injective resolution $J^\bullet$ of $F$, there exists a morphism of complexes $\delta^\bullet : I^\bullet \to J^\bullet$ such that $\delta^i$ is injective for each $i$.

3. For each $i > 0$, each $\pi \in \Pi$, and each maximal vector $s \in I^i(\pi)$, $s \in \im \eta^i(\pi)$. For each $\pi \in \Pi$ and each maximal vector $s \in I^0(\pi)$, $s \in \im \alpha(\pi)$.

4. For each $i$, each $\pi \in \Pi$, and each maximal vector $s \in I^i(\pi)$, $\eta^i(\pi)(s) = 0$.

Proof. $1 \Rightarrow 4$: Assume there exists a maximal vector $s \in I^i(\pi)$ such that $\eta^i(\pi)(s) \neq 0$. Let $[\pi]_s$ be an indecomposable injective subsheaf supported on the down-set of $\pi$ with $s \in [\pi]_s(\pi)$, and $\hat{I}^i := I^i/[\pi]_s$ the quotient, with quotient map $q$. Similarly, let $\hat{I}^{i+1} := I^{i+1}/[\pi]'_{q(\pi)(s)}$. By Lemma $\mathbb{[9]}$ $I^i \cong \hat{I}^i \oplus [\pi]_s$, $I^{i+1} \cong \hat{I}^{i+1} \oplus [\pi]'_{q(\pi)(s)}$, and $\hat{I}^i, \hat{I}^{i+1}$ are injective sheaves. Then
with columns and the top two rows exact. We will show that the bottom row is also exact. Suppose \( x \in \ker \hat{\eta}_i^{-1}(\sigma) \) for some \( \sigma \in \Pi \). Then \( \eta_i^{-1}(\sigma)(x) = [\pi]_{\eta_i}(\pi)(s) \).

Because \( I^\bullet \) is a chain complex, \( \eta_i(\sigma) \circ \eta_i^{-1}(\sigma)(x) = 0 \). By assumption the restriction of \( \eta_i(\sigma) \) to \([\pi]_{\eta_i}(\pi)\) is an injective linear map. Therefore, \( \eta_i^{-1}(\sigma)(x) = 0 \). By similar diagram chasing arguments, one can show that

\[
\ker \hat{\eta}_i^{-1} = \ker \eta_i^{-1} = \text{im} \eta_i^{-1} = \text{im} \hat{\eta}_i^{-1}.
\]

Therefore, 

\[
\begin{array}{cccccccc}
0 & \rightarrow & F & \rightarrow & I^0 & \rightarrow & \cdots & \\
& & \downarrow \alpha & & \downarrow \eta^0 & & \downarrow \eta^1 & \\
& & I^0 & \rightarrow & I^1 & \rightarrow & \cdots & \\
\end{array}
\]

is an exact sequence, and an injective resolution of \( F \), with fewer indecomposable injective summands than \( I^\bullet \), which shows that \( I^\bullet \) is not minimal.

4 \Leftrightarrow 3: Follows from the exactness of the injective resolution \( I^\bullet \).

3 \Rightarrow 2: Let

\[
\begin{array}{cccccccc}
0 & \rightarrow & F & \rightarrow & J^0 & \rightarrow & \cdots & \\
& & \downarrow \beta & & \downarrow \lambda^0 & & \downarrow \lambda^1 & \\
& & J^0 & \rightarrow & J^1 & \rightarrow & \cdots & \\
\end{array}
\]

be an injective resolution of \( F \) which satisfies criteria 3, and

\[
\begin{array}{cccccccc}
0 & \rightarrow & F & \rightarrow & J^0 & \rightarrow & \cdots & \\
& & \downarrow \beta & & \downarrow \lambda^0 & & \downarrow \lambda^1 & \\
& & J^0 & \rightarrow & J^1 & \rightarrow & \cdots & \\
\end{array}
\]

be any injective resolution of \( F \). We will inductively construct a chain complex morphism \( \delta^\bullet : I^\bullet \rightarrow J^\bullet \) such that \( \delta^i : I^i \rightarrow J^i \) is an injective natural transformation for each \( i \). We begin by extending the natural transformation \( \beta : F \rightarrow J^0 \) through the injection \( 0 \rightarrow F \rightarrow I^0 \), by the injectivity of \( J^0 \), resulting in the commutative diagram

\[
\begin{array}{cccccccc}
0 & \rightarrow & F & \rightarrow & I^0 & \rightarrow & \cdots & \\
& & \downarrow \text{id} & & \downarrow \beta^0 & & \downarrow \lambda^0 & \\
0 & \rightarrow & F & \rightarrow & J^0 & \rightarrow & \cdots & \\
\end{array}
\]

Similarly, we can extend the map \( \alpha : F \rightarrow I^0 \) to a map \( \gamma^0 : J^0 \rightarrow I^0 \), resulting in a commutative diagram

\[
\begin{array}{cccccccc}
0 & \rightarrow & F & \rightarrow & I^0 & \rightarrow & \cdots & \\
& & \downarrow \text{id} & & \downarrow \gamma^0 & & \downarrow \gamma^0 & \\
0 & \rightarrow & F & \rightarrow & J^0 & \rightarrow & \cdots & \\
\end{array}
\]
We claim that \( \ker \gamma^0 \circ \delta^0 = 0 \). By assumption on \( I^* \), for each maximal vector \( s \in I^0(\sigma) \), there exists \( x \in F(\sigma) \) such that \( \alpha(\sigma)(x) = s \). Because the above diagrams commute, \( \beta(\sigma)(x) = \delta^0(\sigma)(s) \). Moreover, again by commutativity,

\[
\gamma^0(\sigma) \circ \delta^0(\sigma)(s) = \gamma^0(\sigma) \circ \beta(\sigma)(x) = \alpha(\sigma)(x) = s,
\]

which proves that \( \ker \gamma^0 \circ \delta^0 = 0 \), because every non-zero vector maps to some non-zero multiple of a maximal vector via the sheaf maps. In particular, \( \delta^0 \) is injective.

We continue inductively. Suppose we have defined \( \delta^0 \) through \( \delta^k \) and \( \gamma^0 \) through \( \gamma^k \) such that

\[
\begin{array}{ccccccc}
0 & \longrightarrow & F & \longrightarrow^\alpha & I^0 & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow^\beta & J^0 & \longrightarrow & \cdots \\
\end{array}
\]

commutes for each square and \( \ker \gamma^i \circ \delta^i = 0 \) for each \( i \).

Then \( \delta^k \) and \( \gamma^k \) induce natural transformations \( \delta^k : \text{coker} \eta^{k-1} \rightarrow \text{coker} \lambda^{k-1} \) and \( \gamma^k : \text{coker} \lambda^{k-1} \rightarrow \text{coker} \eta^{k-1} \), respectively. Using the injectivity of \( I^{k+1} \) and \( J^{k+1} \), we extend the maps from \( \text{coker} \eta^{k-1} \rightarrow J^{k+1} \) and \( \text{coker} \lambda^{k-1} \rightarrow J^{k+1} \), respectively:

\[
\begin{array}{ccc}
I^k & \xrightarrow{\gamma^k} & I^{k+1} \\
\downarrow & & \downarrow \\
\text{coker} \lambda^{k-1} & \xrightarrow{\eta^{k-1}} & \text{coker} \eta^{k-1} \\
\downarrow & & \downarrow \\
\text{coker} \eta^{k-1} & \xrightarrow{\delta^k} & \text{coker} \eta^{k-1} \\
\downarrow & & \downarrow \\
J^k & \xrightarrow{\lambda^{k-1}} & J^{k+1} \\
\end{array}
\]

By diagram chasing, \( \gamma^k \circ \delta^k (\text{im} \eta^{k-1}) \subseteq \text{im} \eta^{k-1} \). By the inductive assumption, \( \ker \gamma^k \circ \delta^k = 0 \). Therefore, as a map on \( \text{coker} \eta^{k-1} \), \( \gamma^k \circ \delta^k \) is injective. By an argument analogous to the above proof that \( \ker \gamma^0 \circ \delta^0 = 0 \) is injective, we have that \( \ker \gamma^{k+1} \circ \delta^{k+1} = 0 \). This implies that \( \delta^{k+1} \) is injective.

By [2] \( \Rightarrow \text{1} \). By condition 2, for each injective resolution \( J^* \), there are injective maps \( \delta^i : I^i \rightarrow J^i \) for each \( i \). The injectivity of \( \delta^i \) implies that the number of indecomposable injective summands of \( J^i \) is greater than that of \( I^i \), which proves that \( I^* \) is minimal.

The proof of the theorem yields several immediate corollaries.

**Corollary 18.** For each sheaf \( F \) on a finite poset \( \Pi \), there exists a unique (up to isomorphism of chain complexes) minimal injective resolution.

**Proof.** Notice that the proof of \( 1 \Rightarrow 4 \) in Theorem 17 shows that from any injective resolution \( J^* \) of \( F \), and any maximal vector \( s \in J^i(\pi) \) such that \( \eta^i(\pi)(s) \neq 0 \), we can construct an injective resolution \( I^* \) of \( F \) by taking a quotient of \( J^i \) and \( J^{i+1} \) by \( [\pi]_s \) and \( [\pi][\eta^i(\pi)(s)] \), respectively. Therefore, by applying this procedure inductively, we can construct from any injective resolution \( J^* \), a minimal injective resolution \( I^* \). Existence of a minimal injective resolution then follows from the existence of injective resolutions. By Theorem 17 property 2, any two minimal injective resolutions must be isomorphic as chain complexes. □

We also explicitly illustrate the existence of the minimal injective resolution in Section 4 where we provide an algorithm to construct it.
Corollary 19. The minimal injective resolution of a sheaf $F$ on a finite poset $\Pi$ of height $d$ consists of at most $d+1$ non-zero injective sheaves.

Proof. The length of the longest chain of non-zero vector spaces in $I^j$ is strictly decreasing in $j$ in the minimal injective resolution. This is implied by properties 3 and 4: If $I^j(\pi) = 0$, then property 3 implies that there are no maximal vectors in $I^{j+1}(\pi)$. Therefore, if $I^j(\pi) = 0$ for all $\tau \geq \pi$, then also $I^{j+1}(\pi) = 0$ for all $\tau \geq \pi$. Moreover, if $I^j(\pi) \neq 0$ and $I^j(\pi) = 0$ for all $\tau > \pi$, then all vectors in $I^j(\pi)$ are maximal, and by property 4 and the argument above, $I^k(\pi) = 0$ for all $k > j$. \qed

Corollary 20. If $F \xrightarrow{\alpha} I$ is an injective hull such that for all $\pi \in \Pi$, all maximal vectors of $I(\pi)$ are in $\text{im} \alpha(\pi)$, then it is the minimal injective hull.

Proof. The inductive construction in the proof of 3 $\Rightarrow$ 2 in Theorem 17 only depends on the initial segments of the resolution. Therefore, if the property 3 is satisfied in an initial segment, then this initial segment injects in any injective resolution. In particular, this shows that if $F \xrightarrow{\alpha} I$ is an injective hull such that for all $\pi \in \Pi$ the maximal vectors in $I(\pi)$ are in $\text{im} \alpha(\pi)$, then it is the minimal injective hull of $F$. \qed

3.2 Indecomposable multiplicities of the minimal injective resolution

Because the minimal injective resolution of a sheaf $F$ is unique, the multiplicity of an indecomposable injective sheaf in the minimal injective resolution is a well-defined invariant of $F$. It is natural to ask what topological information is captured with these multiplicities. Below we answer this question for the constant sheaf on a finite simplicial complex.

Definition 21. Let $I^\bullet$ be the minimal injective resolution of $F$. By $m^j_F(\sigma)$ we denote the multiplicity of $[\sigma]$ in $I^j$:

$$I^j \cong \bigoplus_{\sigma \in \Pi} [\sigma] m^j_F(\sigma).$$

Equivalently, we can define $m^j_F(\sigma) := \dim M_{I^j}(\sigma)$, where $M_{I^j}(\sigma)$ is as in Definition 15.

Theorem 22. Let $\Sigma$ be a finite simplicial complex, $k_\Sigma$ the constant sheaf on $\Sigma$ (viewed as a poset with the face relation), and $H^\bullet(\text{St} \sigma, k)$ be the singular cohomology with compact support of the geometric realization of $\text{St} \sigma$. Then

$$m^j_{k_\Sigma}(\sigma) = \dim H^{j + \dim \sigma}_c(\text{St} \sigma, k).$$

Proof. For $\sigma \in \Sigma$, there exists an abstract simplicial complex $K = \sigma^0 \cup \{\tau \setminus \{\sigma\} | \tau \in \text{St} \sigma \setminus \{\sigma\}\}$ with $\sigma^0 \in K$ a zero simplex, such that $\text{St} \sigma^0 = \text{St} \sigma$ as posets, and the geometric realization $|\text{St} \sigma|$ is homeomorphic to $\mathbb{R}^{\dim \sigma} \times |\text{St} \sigma^0|$. Let $I^\bullet$ be the minimal injective resolution of $k_\Sigma$. Then $I^\bullet|_{\text{St} \sigma}$ is a minimal injective resolution of $k_{\text{St} \sigma}$, because $\text{St} \sigma = \text{St} \sigma^0$ as posets. We will briefly abuse notation and think of $I^\bullet|_{\text{St} \sigma}$ as a complex of injective sheaves on $\text{St} \sigma^0$. Let $f : \text{St} \sigma^0 \to \text{pt}$. Because $\dim \sigma^0 = 0$, the map $f$ is a fibred cellular map in the sense of [She85 §3.3]. Therefore, by [She85 §1.6] (cf. [Cur14 Definition 5.1.15]), $f_!(J) = \{s \in J(\sigma^0) : s \text{ is a maximal vector}\}$, for each injective sheaf $J$ on $\text{St} \sigma^0$. By standard results of sheaf theory (for example, [She85 Theorem 3.4.14] and [Cur14 Section 13.2]), we have

$$R^j f_! I^\bullet|_{\text{St} \sigma} \cong H^j_c(\text{St} \sigma^0, k)$$

(where $R^j f_! I^\bullet|_{\text{St} \sigma}$ are the cohomology groups of the complex $f_! I^\bullet|_{\text{St} \sigma}$; see [She85] [Bor97] [Cur14] for an introduction to the right derived functors $R^j f_!$). It follows from Theorem 14 that $f_! I^\bullet|_{\text{St} \sigma}$ is the complex

$$0 \to [\sigma^0] m^0_{E^\bullet}(\sigma) \to [\sigma^0] m^1_{E^\bullet}(\sigma) \to \cdots,$$
with all chain maps equal to zero. Therefore,
\[
\dim R^j f_! I^*_0 |_{\text{St } \sigma} = m^j_{k+2} (\sigma).
\]
The result then follows from the isomorphisms
\[
H^j_c (| \text{St } \sigma^0|, k) \cong H^{j+\dim \sigma}_c (\mathbb{R}^{\dim \sigma} \times | \text{St } \sigma^0|, k) \cong H^{j+\dim \sigma}_c (| \text{St } \sigma|, k).
\]

\[\square\]

4 Algorithms for Computing Injective Resolutions

We describe two methods for constructing an injective resolution of a sheaf \( F \) on a poset \( \Pi \).

4.1 Injective Resolutions via the order complex

We begin with a non-inductive construction of a (not necessarily minimal) injective resolution of a given sheaf \( F \). This section generalizes, from the constant sheaf to general sheaves, Lemma 1.3.17 of \([\text{Lad}08]\). On a practical level, this allows one to compute \( k \)-th right derived functors without first computing the full injective resolution (see Section 5).

**Definition 23.** The order complex, \( K(\Pi) \), of a finite poset \( \Pi \), is the poset of strictly increasing chains \( \pi_* = \pi_0 < \pi_1 < \cdots < \pi_k \) in \( \Pi \). The order complex has the structure of an abstract simplicial complex. Let \( K^i(\Pi) \) denote the \( i \)-simplices of \( K(\Pi) \), i.e. the set of chains \( \pi_0 < \pi_1 < \cdots < \pi_i \) of length \( i + 1 \).

**Definition 24.** A signed incidence relation on \( K(\Pi) \) is an assignment to each pair of simplices \( \sigma_*, \gamma_* \in K(\Pi) \) a number \([\sigma_* : \gamma_*] \in \{-1, 0, 1\}, \) such that

1. if \([\sigma_* : \gamma_*] \neq 0\), then \( \sigma_* \leq \gamma_* \), and
2. for each pair of simplices \( (\sigma_*, \gamma_*) \),
\[
\sum_{\tau_* \in K(\Pi)} [\sigma_* : \tau_*][\tau_* : \gamma_*] = 0.
\]

**The construction.** Given a sheaf \( F \) on \( \Pi \), we define (recalling the notation of Definition 7)
\[
I^k := \bigoplus_{\pi_* \in K^k(\Pi)} [\pi_0]^{F(\pi_k)}.
\]
Suppose \( \pi_* \in K^k(\Pi) \) and \( \pi_0 < \tau_* \) (i.e. the chain \( \pi_* \) is obtained from the chain \( \tau_* \) by removing one element). Then \( \pi_0 \geq \tau_0 \) and \( \pi_k \leq \tau_{k+1} \). Therefore,
\[
F(\pi_k \leq \tau_{k+1}) \in \text{Hom}(F(\pi_k), F(\tau_{k+1})) \cong \text{Hom}([\pi_0]^{F(\pi_k)}, [\tau_0]^{F(\tau_{k+1})}).
\]
Using this identification, we define the natural transformation \( \eta^k : I^k \to I^{k+1} \) so that on the \( \pi_* \)-summand \([\pi_0]^{F(\pi_k)}\) of \( I^k \),
\[
\eta^k|_{[\pi_0]^{F(\pi_k)}} = \sum_{\pi_* < \tau_*} [\pi_* : \tau_*] F(\pi_k \leq \tau_{k+1}),
\]
where \( F(\pi_k \leq \tau_{k+1}) \in \text{Hom}([\pi_0]^{F(\pi_k)}, [\tau_0]^{F(\tau_{k+1})}) \) is understood to have its codomain as the \( \tau_* \)-summand of \( I^{k+1} \). Let \( \alpha : F \to I^0 \) be the natural transformation given by the maps
\[
\alpha(\sigma) := \sum_{\sigma \leq \gamma} F(\sigma \leq \gamma) : F(\sigma) \leftrightarrow \bigoplus_{\sigma \leq \gamma} F(\gamma) =: I^0(\sigma).
\]

\[\square\]
Theorem 25. The complex $0 \rightarrow F \xrightarrow{\alpha} I^0 \xrightarrow{\eta^0} I^1 \xrightarrow{\eta^1} \cdots$ defined above is an injective resolution of $F$.

Proof. By construction, each sheaf $I^j$ is injective, and each map $\eta^j$ (as well as $\alpha$) is a natural transformation. It remains to show that the sequence is an exact chain complex. It is enough to show that for each $\pi \in \Pi$, the sequence $0 \rightarrow F(\pi) \xrightarrow{\alpha(\pi)} I^0(\pi) \xrightarrow{\eta^0(\pi)} I^1(\pi) \xrightarrow{\eta^1(\pi)} \cdots$ is exact.

We first describe an explicit construction of the minimal injective hull of a sheaf $\tau$. We claim that $\tau$ to each chain $K \xrightarrow{\eta} \tau$ is an exact functor.

To each chain $K$, we choose the extension $\tau_k \xrightarrow{\eta_k} \tau(k) \xrightarrow{\eta_k} \tau(\pi)$ for any natural transformation $\eta : F \rightarrow G$, it clear that $T$ is an injective hull of $\tau$.

Notice that $0 \rightarrow F(\pi) \xrightarrow{\alpha(\pi)} I^0(\pi) \xrightarrow{\eta^0(\pi)} I^1(\pi) \xrightarrow{\eta^1(\pi)} \cdots$ is identical to the compactly supported cochain complex of the sheaf $T(F|_{St\pi})$ on the simplicial complex $K(St\pi)$ [Cur14, Definition 6.2.1 and Definition 6.2.3]. Therefore, exactness in $I^0(\pi)$ follows from

$$\ker \eta_0(\pi) \cong \Gamma(T(F|_{St\pi})) \cong F(\pi) \cong \text{im } \alpha(\pi),$$

and it remains to prove a vanishing property for the cohomology of $T(F|_{St\pi})$, namely that $H^j(K(St\pi);T(F|_{St\pi})) = 0$ for $j > 0$.

Let $J^*$ be an injective resolution of the sheaf $F|_{St\pi}$ on the poset $St\pi$. Because $T$ is an exact functor (and maps injective sheaves to injective sheaves), $T(J^*)$ is an injective resolution of $T(F|_{St\pi})$. Let $f : K(St\pi) \rightarrow pt$. Then $R^j f_*(T(J^*)) \cong H^j(K(St\pi);T(F|_{St\pi}))$ (see Section 5). Moreover, $R^j f_*(T(J^*))$ is isomorphic to the $j$-th cohomology group of the complex of vector spaces $J^*(\pi)$, which, by the exactness of $J^*$, is zero for $j > 0$. \qed

4.2 Minimal injective resolutions via inductive algorithm

We first describe an explicit construction of the minimal injective hull of a sheaf $F$ on a poset $\Pi$, and then give an algorithm to inductively compute the minimal injective resolution with the minimal injective hull as the input. Lastly, we focus on constant sheaves, give an example, and analyze complexity of the algorithm.

**Minimal injective hull.** To construct the minimal injective hull of $F$, we first find the space of maximal vectors $M_F(\pi)$ for each $\pi \in \Pi$ (see Definition 15). Then $M_F$, with zero linear maps, is a subsheaf of $F$. Recalling the notation described below Definition 7 we define $I^0 = \bigoplus_{\pi} [\pi] \cong \bigoplus_{\pi} [\pi]^\dim M_F(\pi)$, where $[\pi] M_F(\pi)$ is the injective sheaf with $[\pi] M_F(\pi) = M_F(\pi)$ if $\sigma \leq \pi$, and 0 otherwise. We can naturally include $M_F \xrightarrow{\sim} I^0$, and extend this inclusion to $F \xrightarrow{\alpha} I^0$, using the injectivity of $I^0$. We choose the extension $\alpha = \sum \alpha_\pi$, where $\alpha_\pi$ is the extension of $\text{proj}_{[\pi] M_F(\pi)} \circ \gamma$ to $F$ that we describe in the proof of Lemma 8. That is,

$$\alpha_\pi(\sigma) = \begin{cases} \text{proj}_{M_F(\pi)} \circ F(\sigma \leq \pi) & \text{if } \sigma \leq \pi, \\ 0 & \text{otherwise}. \end{cases}$$

**Proposition 26.** This construction yields the minimal injective hull $F \xrightarrow{\alpha} I^0$.

Proof. We claim that $\alpha$ is injective. Let $u \in \ker \alpha(\sigma)$. Then

$$0 = \alpha(\sigma)(u) = \sum_{\pi} \alpha_\pi(\sigma)(u) = \sum_{\sigma \leq \pi} \text{proj}_{M_F(\pi)} F(\sigma \leq \pi)(u),$$

12
which is equivalent to $\text{proj}_{M_F}(\pi) F(\sigma \leq \pi)(u) = 0$ for every $\pi \geq \sigma$, since the images of different $\alpha$, only intersect in 0. But this means that $u = 0$, because every non-zero vector is either maximal or maps onto some non-zero maximal vector via the sheaf maps.

By Corollary 20, the minimality of the injective hull is equivalent to the condition that every maximal vector of $I^0$ is in $\text{im} \alpha$. This is satisfied, as $M_F(\pi)$ are exactly the maximal vectors in $I^0(\pi)$.

We give an explicit formulation of an algorithm computing $\alpha(\pi)$ as Algorithm 1. We first fix bases in $F$. For each $\pi \in \Pi$, we fix a basis $B(\pi) = (v_1, \ldots, v_l, w_{l+1}, \ldots, w_{l+k})$, with $l, k$ dependent on $\pi$, such that $(w_{l+1}, \ldots, w_{l+k})$ is a basis of $M_F(\pi)$, which we also use for $I^0(\pi)$. We assume that all maps $F(\pi \leq \sigma)$ are expressed with respect to those bases.

**Algorithm 1 Minimal injective hull**

**Input:** $F$ with fixed bases as described above, $\pi \in \Pi$

**Output:** $\alpha(\pi)$ as a $(\sum_{\sigma \leq \pi} \dim M_F(\sigma)) \times (\dim F(\pi))$ matrix

1: procedure $\text{INCL}(\sigma, w \in M_F(\sigma))$
2: return inclusion of $w$ into $\bigoplus_{\pi < \tau} M_F(\tau)$ \hspace{1cm} \text{▷ just adding extra zeros}
3: end procedure
4: for $u_i \in \{v_1, \ldots, v_l\}$ do \hspace{1cm} \text{▷ $(v_1, \ldots, v_l, w_{l+1}, \ldots, w_{l+k})$ is the fixed basis of $F(\pi)$}
5: $D \leftarrow$ empty dictionary \hspace{1cm} \text{▷ keys: elements $\pi \in \Pi$, values: vectors in $F(\pi)$}
6: $D[\pi] \leftarrow v_i$
7: $u_i \leftarrow 0$ \hspace{1cm} \text{▷ vector of length $\sum_{\pi \leq \tau} \dim M_F(\tau)$}
8: for each $\sigma \geq \pi$ in some topological ordering do
9: \hspace{1cm} if $\sigma \in \text{Keys}(D)$ and $D(\sigma) \neq 0$ then
10: \hspace{1cm} \hspace{1cm} $w \leftarrow D[\sigma]$ \hspace{1cm} \text{▷ $D[\sigma] = F(\pi \leq \sigma)(v_i)$}
11: \hspace{1cm} for each $\tau > 1 \sigma$ do
12: \hspace{1cm} \hspace{1cm} if $\tau \notin \text{Keys}(D)$ then
13: \hspace{1cm} \hspace{1cm} $D[\tau] \leftarrow F(\pi \leq \tau)(w)$
14: \hspace{1cm} \hspace{1cm} $u_i \leftarrow u_i + \text{INCL}(\sigma, \text{proj}_{M_F(\pi)}(D[\tau]))$
15: \hspace{1cm} end if
16: \hspace{1cm} end for
17: \hspace{1cm} end if
18: clear $D[\sigma]$ \hspace{1cm} \text{▷ optional, just to free up memory}
19: end for
20: return a block matrix $\begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}$, where $U = (u_1, \ldots, u_l)$, and $I$ is the identity matrix of order $k = \dim M_F(\pi)$

To express $\alpha(\pi)$ with respect to the fixed bases, we need to find the image of each $v_1, \ldots, v_l, w_{l+1}, \ldots, w_{l+k}$. The maximal vectors $w_i$ are mapped identically to $M_F(\pi) \subseteq I_0(\pi)$. For the other vectors, $v_i$, we need to find $u_i := \sum_{\tau < \pi} \text{proj}_{M_F(\pi)}(F(\pi \leq \tau)(v_i))$. The algorithm does that while avoiding redundant computations. Each $\sigma$ is added to $D$ at most once. If it is added, then $D[\sigma] = F(\pi \leq \sigma)(v_i)$, and $\text{proj}_{M_F(\pi)}(F(\pi \leq \sigma)(v_i))$ is added to $u_i$. If $\sigma$ is never added to $D$, then $F(\pi \leq \sigma)(v_i) = 0$. In the end, $u_i$ contains the desired sum.

**Minimal injective resolution.** Next, we describe an algorithm that takes $I^{k-1} \overset{\eta^{k-1}}{\rightarrow} I^k$ as the input, and gives $I^k \overset{\eta^k}{\rightarrow} I^{k+1}$ on the output. Essentially the same algorithm can be used to compute $I^0 \overset{\eta^0}{\rightarrow} I^1$ from $F \overset{\Delta}{\rightarrow} I^0$, with the difference that $\alpha$ can not be stored the same way we store $\eta^k$—we discuss this in more detail later.
Representing the sheaves and the maps. We represent each sheaf \( I^k \) as a tuple of poset elements, \( (\pi_1, \ldots, \pi_l) \), with possible repetitions, such that \( I^k = \bigoplus_{l=1}^{\ell}[\pi_l] \). We refer to the elements in that tuple as \textit{generators} of \( I^k \). We describe the natural transformation \( \eta^k : I^k \rightarrow I^{k+1} \) by a matrix with columns labeled by the tuple of generators of \( I^k \), and rows labeled by the tuple of generators of \( I^{k+1} \), as discussed in Section 2.1. In the following, we abuse the notation and denote matrices by the same symbols as the maps they represent. The linear map \( \eta^k(\sigma) \) is described by a submatrix

\[
\eta^k(\sigma) := \eta^k[St \sigma, St \sigma],
\]

where we take all the rows and columns that are labeled by simplices from \( St \). Note that Lemma 11 also implies that \( \eta^k[St \sigma, II \setminus St \sigma] \) is a zero matrix. Hence, \( \eta^k[St \sigma, St \sigma] \) and \( \eta^k[St \sigma, II] \) only differ by zero-columns.

The algorithm going from \( \eta^{k-1} \rightarrow I^k \) to \( \eta^k \rightarrow I^{k+1} \). The construction of \( I^{k+1} \) and \( \eta^k \) is described by Algorithm 2. We start with an empty sheaf and an empty matrix. Then we inductively add generators to \( I^{k+1} \) and corresponding rows to \( \eta^k \). Fix a total order, \( (\sigma_1, \ldots, \sigma_n) \), extending the poset \( II \). We go through the elements in reverse, and for each \( \sigma \), we make sure that \( \ker \eta^k(\sigma) = \im \eta^{k-1}(\sigma) \). We do that by adding new linearly independent rows to \( \eta^k(\sigma) \) from \( \im \eta^{k-1}(\sigma) \). The entries of the new rows in positions outside of \( St \sigma \) are 0. The new rows are labeled by \( \sigma \), and for each added row we put a new element \( \sigma \) in the tuple representing \( I^{k+1} \).

Algorithm 2 Step in the minimal injective resolution

\begin{verbatim}
Input: \( I^k, \eta^{k-1} \)
Output: \( I^{k+1}, \eta^k \)
1: for \( \sigma \in (\sigma_n, \ldots, \sigma_1) \) do \( \triangleright (\sigma_1, \ldots, \sigma_n) \) is a total order extending \( II \)
2: \( B \leftarrow \) basis of \( \im \eta^{k-1}(\sigma) \) \( \triangleright \) see discussion of the algorithm
3: for all \( b \in B \) do
4: if \( b \) is linearly independent from the rows of \( \eta^k(\sigma) \) then
5: append \( \sigma \) to \( I^{k+1} \)
6: add \( b \) as the next row of \( \eta_k \), labeled by \( \sigma \)
\( \triangleright \) with 0 entries for places labeled by \( II \setminus St \sigma \)
7: end if
8: end for
9: end for
\end{verbatim}

Correctness of Algorithm 2. We claim that starting with minimal injective hull of \( F \), iterative application of the algorithm yields the minimal injective resolution of \( F \). Each sheaf is injective by definition. We need to show exactness at each point, and minimality.

Proposition 27. If \( I^k, \eta^{k-1} \) is the input and \( I^{k+1}, \eta^k \) the output of Algorithm 2 then \( I^{k-1} \xrightarrow{\eta^{k-1}} I^k \xrightarrow{\eta^k} I^{k+1} \) is exact in \( I^k \).

Proof. We claim that \( \im \eta^{k-1}(\sigma) = \ker \eta^k(\sigma) \) for every \( \sigma \in II \). Note that once we process an element \( \pi \), the submatrix \( \eta^k(\pi) \) does not change, because all elements \( \tau \in St \pi \) were already processed before \( \pi \).

We analyze step \( \sigma \). For all \( \pi \in St \sigma \), \( \pi \) was already processed, and we assume that \( \im \eta^{k-1}(\pi) = \ker \eta^k(\pi) \). Since any row labeled by \( St \pi \) has only zero entries in columns labeled by simplices not in \( St \pi \), we start with \( \ker \eta^k(\sigma) \supseteq \im \eta^{k-1}(\sigma) \). Adding new rows from \( \im \eta^{k-1}(\sigma) \) preserves this inclusion, and we keep adding new rows until

\[
\text{rank } \eta^k(\sigma) = \dim (\im \eta^{k-1}(\sigma)).
\]
The dimension \( \dim I^k(\sigma) \) is the number of columns of the matrix \( \eta^k(\sigma) \), and also the number of rows of \( \eta^{k-1}(\sigma) \). Rank-nullity theorem then implies

\[
\text{rank } \eta^k(\sigma) + \dim \ker \eta^k(\sigma) = \dim I^k(\sigma) = \dim (\text{im } \eta^{k-1}(\sigma))^\perp + \dim \text{im } \eta^{k-1}(\sigma). \tag{2}
\]

Together, (1) and (2) imply that \( \dim \ker \eta^k(\sigma) = \dim \text{im } \eta^{k-1}(\sigma) \), and therefore \( \ker \eta^k(\sigma) = \text{im } \eta^{k-1}(\sigma) \) at the end of processing \( \sigma \), as we claimed.

To show the minimality, we apply the condition on maximal vectors from Theorem 17.

**Lemma 28.** If \( I^{k+1}, \eta^k \) is the output of Algorithm 2 then for all \( \sigma \in \Pi \) all maximal vectors in \( I^{k+1}(\sigma) \) are in the image of \( \eta^k(\sigma) \).

**Proof.** Maximal vectors over \( \sigma \) are exactly the new vectors added at step \( \sigma \). That is, if rows \( i, \ldots, l \) were added to \( \eta^k(\sigma) \) at step \( \sigma \), the space of maximal vectors is \( \text{span}(e_i, \ldots, e_l) \), with \( e_j \) being the \( j \)-th canonical vector. We show that this space is in the image of \( \eta^k(\sigma) \).

Let \( A_j \) be the matrix consisting of the first \( j \) rows of \( \eta^k(\sigma) \). As we only add new rows when they are linearly independent from the previous, we have \( \ker A_j \subseteq \ker A_{j-1} \) for every \( j \in \{i, \ldots, l\} \). That is, there exists \( u_j \) such that \( A_{j-1} u_j = 0 \), and \( A_j u_j \neq 0 \). This is only possible if \( A_j u_j = \lambda e_j \) for some \( \lambda \neq 0 \), which means that \( \eta^k(\sigma) \cdot u_j = A_0 u_j \) is a vector with zeros at positions \( 1, \ldots, j \), and a non-zero at the \( j \)-th coordinate. Therefore, \( \ker \eta^k(\sigma) = \text{im } \eta^k(\sigma) \).
4.3 Examples

We demonstrate how Algorithm 2 works for constant sheaves with two examples.

Example 29. Consider the 3-skeleton of the 4-simplex, with two extra edges attached to vertex 1. We compute the minimal injective resolution of $\Pi := St(1)$. We describe simplices as lists of vertices, and for brevity omit the vertex 1—e.g., $234 = \{1, 2, 3, 4\}$. See Figure 1.

![Figure 1: The poset considered in Example 29. We omit vertex 1 from the labels.](image)

We construct $I^0, I^1, I^2, I^3$, with inputs $I^0, \eta^{-1}$. Initialize $I^1$ and $\eta^0$ empty, and go through the simplices row-by-row left-to-right as they are in Figure 1. Starting with $234$, the space $I^0(234)$ is 1-dimensional and equal to $\text{im} \eta^{-1}(234)$, so there is nothing to be added, and $\eta^0(234) = 0$. The same happens for all the maximal simplices.

At triangle $23$, we have $I^0(23) = k^2$, since two generators are above $23$. At the moment, $\eta^0(23)$ is empty, so its kernel is $k^2$. We need $\ker \eta^0(23) = \text{im} \eta^{-1}(23) = \text{span}\{1, 1\}$. The orthogonal complement of $\text{im} \eta^{-1}(23)$ is generated by the vector $(1, -1)$. We add it as a new row in $\eta^0(23)$. Therefore, we add $23$ to $I^1$, and add a first row to $\eta^0$; see Figure 3. Similarly, we add one row for each other triangle.

Now for the edges. We have $I^0(2) = k^3$, and $\eta^0(2)$ a $3 \times 3$ matrix, highlighted as a green solid rectangle in Figure 3. We already have $\ker \eta^0(2) = \text{span}\{1, 1, 1\} = \text{im} \eta^{-1}(2)$, so we do not add any new generators over 2. The same goes for $\eta^0(3), \eta^0(4), \eta^0(5)$, each of which you can see highlighted in Figure 3 with a different color and line style.

Finally, we get to the vertex $\emptyset$, with $\eta^0(\emptyset)$ starting as the part of the matrix in Figure 3 above the horizontal line. Its rank is 3, and its nullity is 3. We need the kernel to be 1-
dimensional, so we need to add two additional rows from \( \text{span}\{ (1, 1, 1, 1, 1) \} \). We also add \( \emptyset \) to \( I^1 \) twice. This completes the construction of \( I^1 \) and \( \eta^0 \).

The resolution goes on for two more steps: \( I^2 \) is generated by \((2, 3, 4, 5)\), \( I^3 \) by \((\emptyset)\). The matrices \( \eta^k \) are in Figure 3 and the whole resolution is schematically shown in Figure 2.

![Figure 3: Matrices \( \eta^0, \eta^1, \eta^2 \) in Example 29, with highlighted submatrices \( \eta^k(2) \) (solid green), \( \eta^k(3) \) (dotted blue), \( \eta^k(4) \) (rounded corners magenta), \( \eta^k(5) \) (dashed red). Recall that \( \eta^k(\sigma) = \eta^k[\text{St } \sigma, \text{St } \sigma] \), and note that if \( \sigma \not\leq \tau \), then \( \eta^k[\sigma, \tau] = 0 \).](image)

Example 30. Let \( \Sigma := \Delta^{(2)}_4 \) be the 2-skeleton of a tetrahedron (whose geometric realization is homeomorphic to the sphere). We give the minimal injective resolution of the constant sheaf \( k_\Sigma \) in Figure 4.

### 4.4 Complexity Analysis

We analyze the complexity of finding the minimal injective resolution of the constant sheaf, \( k_\Pi \), on a poset \( \Pi \), with \( n \) elements and height \( d \), computed by an iterative application of Algorithm 2. That is, we start with \( k_\Pi \xrightarrow{\eta^{-1}} I^0 \) the minimal injective hull of the constant sheaf as described above, and then iteratively apply Algorithm 2 until \( I^k = 0 \).

The body of the outer-most for loop in Algorithm 2 consists of finding a basis of \((\text{im}(\eta^{k-1}(\sigma)))^\perp\), and checking for linear independence of rows of \( \eta^k(\sigma) \). Both of those operations can be computed in time at most \( O(c^3) \) with \( c \) the maximum of the number of rows of \( \eta^{k-1}(\sigma) \) and \( \eta^k(\sigma) \). In our analysis we ignore the complexity of finding \( \text{St } \sigma \) to extract the submatrices from \( \eta^k \) in the first place, since it is less expensive than \( O(c^3) \) when we estimate \( c \) by the size of \( \text{St } \sigma \).

By Corollary 19, the length of the minimal injective resolution is at most \( d + 1 \). Therefore, we find it in time \( O(d \cdot n \cdot c^3) \), where

\[
c = \max_{j, \sigma} \sum_{\pi \in \text{St } \sigma} m^j_{k_\Pi}(\pi)
\]

is the maximal number of generators over any star throughout the resolution. This analysis is output-sensitive. To give complexity bounds dependent only on the input, we compare \( c \) to the maximal size of a star in \( \Pi \). How well we can approximate \( c \) this way depends on the structure of \( \Pi \).

**Definition 31.** For \( \sigma \in \Pi \) we define

\[
m^j(\text{St } \sigma) := \sum_{\pi \in \text{St } \sigma} m^j_{k_\Pi}(\pi)
\]

to be the number of generators over \( \text{St } \sigma \) in the \( j \)-th step of the minimal injective resolution of the constant sheaf on \( \Pi \). Furthermore, we define the \( j \)-th star complexity of \( \sigma \) as

\[
\text{SC}^j(\sigma) := \frac{m^j(\text{St } \sigma)}{\# \text{St } \sigma}
\]
For general posets, $SC^j(\sigma)$ can be arbitrarily large even when lengths of chains are bounded, because sizes of boundaries and coboundaries can be arbitrarily large. For simplicial complexes, we give an upper bound on $SC^j(\sigma)$ depending on the dimension.

**Proposition 32.** Let $\Pi$ be a simplicial complex, $\sigma \in \Pi$ and $k := \dim \Pi - \dim \sigma$. Then

$$SC^j(\sigma) \leq \binom{k}{j}.$$  

This bound is asymptotically tight. If $\Pi = \Delta_n^{(k)}$ is the $k$-skeleton of the $n$-simplex, $v$ is a vertex in $\Delta_n^{(k)}$, and $j$ is fixed, then

$$SC^j(v) \xrightarrow{n \to \infty} \binom{k}{j}.$$  

**Proof.** We prove the upper bound using Theorem 22 and bounding dimensions of homology groups by dimensions of chain groups:

$$m^j(St \sigma) = \sum_{\tau \in St \sigma} m^j(\tau) = \sum_{\tau \in St \sigma} \dim H_c^{\dim \tau}(St \tau) \leq \sum_{\tau \in St \sigma} \dim C_c^{\dim \tau}(St \tau)$$

$$= \sum_{\tau \in St \sigma} \# \{ \pi \mid \tau < \pi \} = \sum_{\tau \in St \sigma} \# \{ \tau \in St \sigma \mid \tau < \pi \} = \sum_{\tau \in St \sigma} \# \{ \pi \setminus \sigma \} \leq \# St \sigma \cdot \binom{k}{j},$$

where $k = \dim St \sigma - \dim \sigma \leq \dim \Sigma - \dim \sigma$.

Now we analyse $SC^j(\sigma)$ in the $k$-skeleton of the $n$-simplex, $\Delta_n^{(k)}$. We use the fact that $St \sigma$ in $\Delta_n^{(k)}$ is combinatorially the same as $\Delta_n^{(k)} \cup \{\emptyset\}$, with $n' = n - \dim \sigma - 1$ and $k' = k - \dim \sigma - 1,$
using the correspondence $\text{St} \sigma \ni \tau \mapsto \tau \setminus \sigma$. This map induces an isomorphism between the cochain complexes

$$C^*_{\sigma}(\text{St} \sigma) \cong \tilde{C}^{\cdot - \dim \sigma - 1}(\Delta^{(k')}_{n'}) ,$$

which, using Theorem 22, implies

$$m^j(\sigma) = \dim \tilde{H}^{j+\dim \sigma}(\text{St} \sigma) = \dim \tilde{H}^{j-1}(\Delta^{(k')}_{n'}).$$

The reduced cohomology $\tilde{H}^i(\Delta^{(k')}_{n'})$ is trivial for all $i \neq k'$, and for $i = k'$, we compute the dimension from the Euler characteristic:

$$\dim \tilde{H}^{k'}(\Delta^{(k')}_{n'}) = (-1)^{k'} \tilde{\chi}(\Delta^{(k')}_{n'}) = (-1)^{k'} \left( 1 + \sum_{i=0}^{k'} \binom{n'+1}{i+1} (-1)^i \right)$$

$$= (-1)^{k'} \left( \sum_{i=0}^{k'+1} \binom{n'+1}{i} (-1)^i \right) = (-1)^{k'-1} \cdot (1 - 1)^{k'+1} = \binom{n'}{k'+1}.$$ 

Therefore,

$$m^j(\sigma) = \begin{cases} \binom{n - \dim \sigma - 1}{k - \dim \sigma} & \text{if } j = k' + 1 = k - \dim \sigma, \\ 0 & \text{otherwise}. \end{cases}$$

Finally, we compute $m^j(\text{St} \, v)$ for a vertex $v$:

$$m^j(\text{St} \, v) = \sum_{\sigma \in \text{St} \, v} m^j(\sigma) = \sum_{\sigma \in \text{St} \, v} \binom{n - k + j - 1}{j} = \binom{n}{k-j} \binom{n-k+j-1}{j}.$$ 

We rearrange this as follows

$$m^j(\text{St} \, v) = \frac{n!}{(n-k+j)! (k-j)!} \cdot \frac{(n-k+j-1)!}{(n-k-1)! j!}$$

$$= \frac{n!}{(n-k)! k!} \cdot \frac{n-k}{(n-k+j)! j!} = \binom{n}{k} \binom{k}{j} \frac{n-k}{n-k-1}.$$ 

Now we can easily compare this with $\# \text{St} \, v = \sum_{i=0}^{k} \binom{n}{i}$. When we fix $k$ and $j$, we get

$$\lim_{n \to \infty} \frac{m^j(\text{St} \, v)}{\# \text{St} \, v} = \binom{k}{j}.$$ 

**Corollary 33.** For a fixed dimension $d$, the Algorithm 2 computes the minimal injective resolution of the constant sheaf on a $d$-dimensional simplicial complex $\Sigma$ in time $O(n \cdot s^3)$, where $n$ is the cardinality of $\Sigma$ (as an abstract simplicial complex), and $s$ is the cardinality of the largest star in $\Sigma$.

### 5 Right Derived Functors

As an application of our main results, we define, in terms of injective resolutions, two examples of right derived functors.
The right derived pushforward, $R^* f_*$. Let $f : \Sigma \to \Lambda$ be a continuous (relative to the Alexandrov topology) map of posets. Let $I^*$ be an injective resolution of a sheaf $F$ on $\Sigma$. Define the integers $n^i_F(\pi)$ so that

$$I^j = \bigoplus_{\pi \in \Sigma} [\pi]^{n^j_F(\pi)}.$$  

We describe each chain map $\eta^j : I^j \to I^{j+1}$, as in Section §2.4 by a matrix with columns and rows indexed by the indecomposable summands of $I^j$, and $I^{j+1}$, respectively. Let $\eta^j(f^{-1}(St \lambda))$ be the submatrix of $\eta^j$ consisting of rows and columns corresponding to the indecomposable summands $[\pi]$ with $\pi \in f^{-1}(St \lambda)$, so that

$$\eta^j(f^{-1}(St \lambda)) : \bigoplus_{\pi \in f^{-1}(St \lambda)} [\pi]^{n^j_F(\pi)} \to \bigoplus_{\pi \in f^{-1}(St \lambda)} [\pi]^{n^{j+1}_F(\pi)}.$$  

Note that if $\kappa \leq \lambda$, then $f^{-1}(St \lambda) \subset f^{-1}(St \kappa)$, and the projection

$$\text{proj} : \bigoplus_{\pi \in f^{-1}(St \kappa)} [\pi]^{n^j_F(\pi)} \to \bigoplus_{\pi \in f^{-1}(St \lambda)} [\pi]^{n^{j+1}_F(\pi)}$$

induces linear maps $\ker \eta^j(f^{-1}(St \kappa)) \to \ker \eta^j(f^{-1}(St \lambda))$ and $\operatorname{im} \eta^{-1}(f^{-1}(St \kappa)) \to \operatorname{im} \eta^{-1}(f^{-1}(St \lambda))$ (because $\operatorname{Hom}([\pi], [\tau]) = 0$ if $\tau \not\leq \pi$).

**Definition 34.** Define a sheaf $R^j f_* F$ on $\Lambda$ by

$$R^j f_* F(\lambda) := \ker \eta^j(f^{-1}(St \lambda)) / \operatorname{im} \eta^{-1}(f^{-1}(St \lambda)),$$

with linear maps $R^j f_* F(\kappa \leq \lambda)$ induced by the projections described above.

The right derived pushforward with compact support, $R^* f_!$. Pushforwards with compact support are a critical structure in the machinery of derived categories of sheaves. We would, therefore, like to explicitly describe how to compute $R^* f_! F$ for a given sheaf $F$ and continuous map $f$. However, the topological notion of ‘compactly supported’ does not adapt to the setting of finite posets in a canonical or straightforward way. Subtle topological constraints must be placed on the maps $f$ in order for the discrete calculation to agree with the classical definitions. See [SheSS §3.3] for one approach to establish such topological criteria. In order to keep our methods as transparent and accessible as possible, we will instead describe $R^* f_!$ for a smaller family of functions, which satisfy more familiar topological constraints. We expect that this family of functions is large enough to handle most interesting applications.

For the remainder of this section, let $\tilde{f} : \Sigma \to \Lambda$ be a simplicial map between finite simplicial complexes. Let $i : U \hookrightarrow \Sigma$ be the inclusion of an open subset $U$ in $\Sigma$. Let $f : U \to \Lambda$ be the restriction of $\tilde{f}$ to $U$.

**Definition 35.** Given a sheaf $F$ on $U$, define the sheaf $i_* F$ on $\Sigma$ by

$$i_* F(\sigma) := \begin{cases} F(\sigma) & \text{if } \sigma \in U \\ 0 & \text{else,} \end{cases}$$

with the linear maps $i_* F(\gamma \leq \sigma) = F(\gamma \leq \sigma)$ when $(\gamma \leq \sigma) \in U$, and 0 else. Finally, define the sheaf $R^j f_* F$ on $\Lambda$ by

$$R^j f_* F := R^j \tilde{f}_*(i_* F).$$

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Derived functors and persistent cohomology. The sheaves $R^j f_* F$ and $R^j f_! F$ may be regarded as level-set multi-parameter persistence modules. With this perspective, we can easily compute, from a single injective resolution $I^\bullet$ of $F$, level-set persistence modules associated to any filtration function $f$. Below, we relate the persistence module $R^j f_* k_U$ to the singular cohomology of level-sets, and $R^j f_! k_U$ to the compactly supported singular cohomology of level-sets.

Proposition 36. As sheaves on $\Lambda$,

$$R^j f_* k_U \cong H^j(|f^{-1}(\text{St} -)|, k)$$

where $H^j(|f^{-1}(\text{St} -)|, k)$ is the sheaf defined by associating the simplex $\lambda$ to the singular cohomology of the geometric realization of $f^{-1}(\text{St} \lambda)$ (with linear maps induced by inclusion). Moreover, $R^j f_* k_U$ captures the compactly supported singular cohomology of the fibers of $f$:

$$R^j f_* k_U(\lambda) \cong H^j_\text{c.s.}(|f^{-1}(\text{St} \lambda)|, k).$$

Proof. Let $I^\bullet$ be the injective resolution described in Section 4.1. Let $I^\bullet_{f^{-1}(\text{St} \lambda)}$ be the chain complex of vector spaces consisting of only the linear combinations of generators for indecomposable sheaves $[\pi] \subset I^\bullet$ such that $\pi \in f^{-1}(\text{St} \lambda)$ and chain maps $\eta^*(f^{-1}(\text{St} \lambda))$. Then $I^\bullet_{f^{-1}(\text{St} \lambda)}$ is identical to the simplicial cochain complex of $K(f^{-1}(\text{St} \lambda))$. The cohomology groups of this chain complex are isomorphic to the singular cohomology of the geometric realization of $f^{-1}(\text{St} \lambda)$:

$$R^j f_* k_U := H^j \left(I^\bullet_{f^{-1}(\text{St} \lambda)}\right) \cong H^j(|f^{-1}(\text{St} \lambda)|, k),$$

and the linear maps $R^j f_*(\kappa \leq \lambda)$ are the usual cohomology maps

$$H^j(|K(f^{-1}(\text{St} \kappa))|, k) \rightarrow H^j(|K(f^{-1}(\text{St} \lambda))|, k)$$

induced by inclusion (cf. [Ive86, Chapter II Proposition 5.11]). A similar argument proves the analogous result for $R^j f_! k_U$. We also note that because $f$ is assumed to be a simplicial map between finite simplicial complexes, $f$ is proper, and the result follows by applying the proper base change theorem for sheaves (see [Ive86, Chapter VII Theorem 1.4] or [KS94, Proposition 2.6.7]).

6 Discussion

An injective resolution represents a given sheaf with an exact sequence of injective sheaves. The homological properties of the given sheaf can then be deduced from its injective resolution (which has many more theoretically and practically desirable properties). The results of this paper address fundamental aspects of computing injective resolutions. First, we prove the existence and uniqueness of a minimal injective resolution, and provide several of its defining characteristics (Theorem 17 and Corollary 18). We give a topological interpretation of the multiplicities of indecomposable injective sheaves in the minimal injective resolution of the constant sheaf over a simplicial complex (Theorem 22). We introduce two new methods for constructing injective resolutions. The first defines the $k$-th term of the resolution without referencing earlier terms (Section 4.1). The second is an inductive algorithm which computes the minimal injective resolution of a given sheaf (Section 4.2). Finally, we give asymptotically tight bounds on the complexity of computing the minimal injective resolution of the constant sheaf on a simplicial complex using Algorithm 2 (Proposition 32 and Corollary 33).

There are many directions in which to extend this work, and several interesting questions which arise from studying derived categories of sheaves from the perspective of computational topology. To make full use of the derived category machinery in computational topology,
it is necessary to develop algorithms for computing each of Grothendieck’s six functors on derived categories: $f_*, f^*, f_!, f^!, \text{Hom}$, and $\otimes$. To this end, it will be useful to extend the results of this paper to injective resolutions of complexes of sheaves: to each complex $F^\bullet$, compute a quasi-isomorphic complex of injective sheaves $I^\bullet$. We plan to pursue this in future work.

Theorem 22 also suggests an interesting connection between the minimal injective resolution, the $\mathfrak{p}$-canonical stratifications of Goresky–MacPherson [GM83], and the cohomological stratification of Nanda [Nan20]. Briefly, the canonical $\mathfrak{p}$-stratification is constructed by inductively identifying subsets of $\Sigma$ for which the dualizing complex $\omega^\bullet_{\Sigma}$ is cohomologically locally constant. Similarly, the cohomological stratification of [Nan20] is constructed by inductively identifying subsets of $\Sigma$ for which the cosheaf $\sigma \mapsto H^j_c(\text{St} \sigma, k)$ is cohomologically locally constant. The (co)homology groups $H^{-j}(\omega^\bullet_{\Sigma}(\sigma)) \cong H^j_c(\Sigma, \Sigma - \text{St} \sigma)$, are closely related, by Theorem 22, to the multiplicities of indecomposable injective sheaves in the minimal injective resolution of the constant sheaf. However, to compute the canonical $\mathfrak{p}$-stratification or cohomological stratification, it is necessary to investigate the linear maps induced by these (co)sheaves. It is not currently clear to us how we can recover the linear maps between cohomology groups: $H^k_c(\text{St} \sigma, k) \to H^k_c(\text{St} \tau, k)$, from the minimal injective resolution of $k_{\Sigma}$, without taking a barycentric subdivision of the simplicial complex. However, we conjecture that the invertibility of such linear maps can be deduced from the minimal injective resolution of $k_{\Sigma}$. If this is true, then it would be possible to efficiently compute canonical $\mathfrak{p}$-stratifications and cohomological stratifications directly from the minimal injective resolution of $k_{\Sigma}$.

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