Optimally Guarding Perimeters and Regions with Mobile Range Sensors

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Abstract—We investigate the problem of using mobile robots equipped with 2D range sensors to optimally guard perimeters or regions, i.e., 1D or 2D sets. Given such a set of arbitrary shape to be guarded, and $k$ mobile sensors where the $i$-th sensor can guard a circular region with a variable radius $r_i$, we seek the optimal strategy to deploy the $k$ sensors to fully cover the set such that $\max_i r_i$ is minimized. On the side of computational complexity, we show that computing a 1.152-optimal solution for guarding a perimeter or a region is NP-hard, i.e., the problem is hard to approximate. The hardness result on perimeter guarding holds when each sensor may guard at most two disjoint perimeter segments. On the side of computational methods, for the guarding perimeters, we develop a fully polynomial time approximation scheme (FPTAS) for the special setting where each sensor may only guard a single continuous perimeter segment, suggesting that the aforementioned hard-to-approximate result on the two-disjoint-segment sensing model is tight. For the general problem, we first describe a polynomial-time $(2 + \varepsilon)$-approximation algorithm as an upper bound, applicable to both perimeter guarding and region guarding. This is followed by a high-performance integer linear programming (ILP) based method that computes near-optimal solutions. Thorough computational benchmarks as well as evaluation on potential application scenarios demonstrate the effectiveness of these algorithmic solutions.

I. INTRODUCTION

In this paper, we consider the problem of using mobile robots equipped with range sensors to guard (1D) perimeters or (2D) regions. Given an arbitrary bounded one- or two-dimensional set to be secured, and $k$ mobile robots where robot $i$’s sensor covers a circular region of radius $r_i$, we seek a deployment of the robots so that $\max_i r_i$ is minimized. That is, we would like to minimize the maximum single-sensor coverage across all sensors. We denote this multi-sensor coverage problem under the umbrella term optimal set guarding with 2D sensors, or OSG$_{2D}$. The specific problem for guarding perimeters (resp., regions) is denoted as optimal perimeter (resp., region) guarding with 2D sensors, abbreviated as OPG$_{2D}$ (resp., ORG$_{2D}$). Beside direct relevance to sensing, surveillance, and monitoring applications using mobile sensors [1]–[3], OSG$_{2D}$ applies to other robotics related problem domains, e.g., the deployment of ad-hoc mobile wireless networks [4], [5], in which case an optimal solution to OSG$_{2D}$ provides a lower bound on the guaranteed network strength over the targeted 2D region.

As a summary of the study, on the side of computational complexity, we establish that OPG$_{2D}$ is hard to approximate within a factor of 1.152 even when the perimeter is a simple polygonal chain and the distance is bounded by the input size, through a reduction from vertex cover on planar 3-regular graphs. A unique property of our reduction is that it shows the inapproximability gap remains when each sensor can cover at most two disjoint perimeter segments. The proof also shows that ORG$_{2D}$ is at least as hard to approximate. Therefore, no polynomial time algorithm may exist that solves OSG$_{2D}$ to better than the 1.152-optimal lower bound, unless P=NP. On the algorithmic side, we begin by providing an efficient $(1 + \varepsilon)$ approximation algorithm for a specific class of OPG$_{2D}$ problems in which each mobile sensor must cover a continuous perimeter segment. This implies that the aforementioned inapproximability result on OPG$_{2D}$ under the two-disjoint-segment sensing model is tight. For the general OSG$_{2D}$ problem, we first describe a polynomial time $(2 + \varepsilon)$ approximation algorithm as a reasonable approximability upper bound. Then, an integer linear programming (ILP) model is devised that allows the fast computation of highly optimal solutions for fairly large problem instances. Results described in this paragraph, together with the introduction of OSG$_{2D}$ as a practical multi-robot deployment problem focusing on global optimality,
constitute the main contributions of this work.

As an intermediate result toward showing the hardness of the simple polygon coverage problem, we also supply a hardness proof of vertex cover on planar bridgeless\(^2\) 3-regular graphs, which may be of independent interest.

**Related work.** Our work on optimal perimeter and region guarding draws inspiration from a long line of multi-robot coverage planning and control research, e.g., [3], [6]–[10]. In an influential body of work on coverage control [3], [6], a gradient based iterative method is shown to drive one or multiple mobile sensors to a locally optimal configuration with convergence guarantees. Whereas [3], [6] assume that the distribution of sensory information is available \emph{a priori}, it is shown that such information can be effectively learned [9]. Subsequently, the control method is further extended to allow the coverage of non-convex and disjoint 2D domains [7] and to work for mobile robots with varying sensing or actuation capabilities [10]. In contrast to these control-based approaches, which produce iterative locally optimal solutions, OSG\(_{2D}\) emphasizes the direct computation of globally optimal deployment solutions and supports arbitrarily shaped bounded (1D) perimeters and (2D) regions.

Recently, the problems of globally optimally covering perimeters using one-dimensional sensors have been studied in much detail [1], [11]. It is shown that when the sensors are homogeneous, the optimal deployment of sensors can be computed very efficiently, even for highly complex perimeters [1]. On the other hand, the problem becomes immediately intractable, sometimes strongly NP-hard, when sensors are heterogeneous [11]. Our research is distinct from [1], [11] in that we employ a (two-dimensional) range sensing model and work on the coverage of both perimeters and regions, which has much broader applicability.

As pointed out in [3], [9], distributed sensor coverage, as well as OSG\(_{2D}\), has roots in the study of the facility location optimization problem [12], [13], which examines the selection of facility (e.g., warehouses) locations that minimize the cost of delivery of supplies to spatially distributed customers. In theoretical computer science and operations research, these are known as the \(k\)-center, \(k\)-means, and \(k\)-median clustering problems [14], the differences among which are induced by the cost structure. Our investigation of OSG\(_{2D}\) benefits from the vast literature on the study of \(k\)-center clustering and related problems, e.g., [15]–[17]. These clustering problems are in turn related to packing [18], tiling [19], and the well-studied art gallery problems [20], [21].

**Organization.** The rest of the paper is organized as follows. In Section \(\text{II}\) we introduce the OSG\(_{2D}\) formulation. Section \(\text{III}\) is devoted to establishing that OSG\(_{2D}\) is hard to approximate to better than 1.152-optimal, providing a theoretical lower bound. In Section \(\text{IV}\) focusing on the upper bound, we describe algorithms that for OSG\(_{2D}\) and the special OPG\(_{2D}\) variant where a sensor is allowed to cover a continuous perimeter segment. In Section \(\text{V}\) we benchmark the algorithms and illustrate two potential applications. We discuss and conclude the work in Section \(\text{VI}\).

**II. Preliminaries**

Let \(W \subset \mathbb{R}^2\) be a compact (i.e., closed and bounded) workspace, which may contain one or multiple connected components. A \emph{critical subset} of \(W\) needs to be guarded by \(k\) indistinguishable point robots with range sensing capabilities. For example, the workspace may be a forest reserve and the critical subset may be its boundary. Or, the workspace may be a high-security facility, e.g., a prison, and the critical subset the prison yard. The \(i\)th robot, \(1 \leq i \leq k\), located at \(c_i \in \mathbb{R}^2\), can monitor a circular area of radius \(r_i\) centered at \(c_i\) with \(r_i\) being a variable. For example, the robot may be a quadcopter equipped with a vision sensor that can detect intruders. As the quadcopter’s altitude increases, its sensing range also increases; but its monitoring quality will decrease at the same time due to resolution loss. In this study, we seek to compute the optimal strategy to deploy these \(k\) robots so that the required sensing range, \(\max_i r_i\), could be minimized.

More formally, we model a connected component of \(W\) as some 2D region with a simple polygonal boundary containing zero or more simple polygonal obstacles. For a set \(D \subset W\), we define

\[
\text{size}(k, D) = \min_{c_1, \ldots, c_k} \max_{p \in D} \min_{1 \leq i \leq k} \|c_i - p\|_2
\]

and use \(B(c, r)\) to denote the disc of radius \(r\) centered at a point \(c \in \mathbb{R}^2\) (the definition \(\text{size}(k, D)\) is used extensively in later sections). The critical subset of interest in this work is either part of the boundary of \(W\), denoted \(\partial W\), or \(W\) itself. The main problem studied in this work is:

**Problem II.1 (Optimal Set Guarding with 2D Sensors).**

\textit{Given a compact workspace }\(W \subset \mathbb{R}^2\), \textit{let }\(D \subset W\) \textit{be a critical subset to be guarded by }\(k\) \textit{robots each with a variable coverage radius of }\(r\). \textit{Find the smallest }\(r\) \textit{and corresponding robot locations }\(c_1, \ldots, c_k\) \textit{such that }\(D \subset \cup_i B(c_i, r)\).

For making accurate statements about computational complexity, we make the assumption that the length of \(\partial W\) is linear with respect to the complexity of \(W\). Furthermore, we assume the possible sensing radius of the robots is lower and upper bounded by some constants, as is the case in practice.

For convenience, we give specific names to these optimal guarding problems based critical subset types. If the critical subset belongs to \(\partial W\), we denote the problem as \emph{optimal perimeter guarding with 2D sensors} or OPG\(_{2D}\). If the critical subset is \(W\), we denote the problem as \emph{optimal region guarding with 2D sensors} or ORG\(_{2D}\). When there is no need to distinguish, the problem is denoted as \emph{optimal set guarding with 2D sensors} (OSG\(_{2D}\)).

As an example, to guard the boundary of a plus-shaped polygon with 5 robots, an optimal solution could be Fig. 2, where the inner circle covers 4 disconnected boundary segments, such pattern in the optimal solution also renders OPG\(_{2D}\) much more difficult than the simplified 1D sensing model studied in [1] (indeed, OPG\(_{2D}\) becomes hard to

\(^2\)That is, the deletion of any edge does not disconnect the graph.
approximate, as will be shown shortly). The solution is also optimal under the ORG_{2D} formulation.

Fig. 2: An example showing an optimal solution of using five discs to cover the plus-shaped polygon. The solution is optimal for both OPG_{2D} and ORG_{2D} formulations.

III. INTRACTABILITY OF APPROXIMATE OPTIMAL GUARDING OF SIMPLE POLYGON

In this section, we prove that OSG_{2D} with the set being a simple polygon is strongly NP-hard to approximate within a factor of \( \alpha = 1.152 \), through a sequence of auxiliary NP-hardness results. First, in Section III-A, we prove an intermediate result that the vertex cover problem is NP-complete on planar bridgeless 3-regular graphs. Next, in Section III-B starting from a planar bridgeless 3-regular graph, we construct a structure which we call 3-net and prove the the problem of finding the minimum coverage radius of the 3-net is NP-hard to approximate within \( \alpha \). Then, in Section III-C we apply a straightforward reduction to transform the 3-net into a simple polygon to complete the hard-to-approximate proof for OSG_{2D} for simple polygons.

We then further show the inapproximability of the special OPG_{2D} setup when each robot can only guard at most two disjoint perimeter segments (Section III-D), contrasting the FPTAS for the special OPG_{2D} setup when each robot can only guard a continuous perimeter segment in Section V-A.

A. Vertex Cover on Planar Bridgeless 3-Regular Graph

Our reduction uses the hardness result on the vertex cover problem for planar graphs with maximum degree 3 [22]. Such a vertex cover problem can be fully specified with a 2-tuple \((G, k)\) where \( G = (V, E) \) is a planar graph with max degree 3 and \( k \) is an integer specifying the allowed number of vertices in a vertex cover. We note that the result has been suggested implicitly in [23]; we provide an explicit account with a simple proof.

**Lemma III.1.** Vertex cover on planar bridgeless 3-regular graph is NP-complete.

**Proof.** For a given planar graph \( G \) with max degree 3 and an integer \( k \), we construct a planar bridgeless 3-regular graph \( G'' \) and provide an integer \( k'' \) such that \( G \) has a vertex cover of size \( k \) if and only if \( G'' \) has a vertex cover of size \( k'' \).

The reduction first makes \( G \) 3-regular by attaching (one or two of) the gadget shown in Fig. 3 to \( v \in G \) that are not degree 3. This results in a 3-regular graph \( G' \). For each attached gadget, \( k \) is bumped up by 3, i.e., we let \( k' \) for \( G' \) be \( k' = k + 3(|V(G)|-2|E(G)|) \). It is straightforward to see that \( G \) has a vertex cover size of \( k \) if and only if \( G' \) has a vertex cover size of \( k' \).

Fig. 3: A gadget that can be attached to a degree one or two vertices (at the point \( A \)) in a max degree 3 graph to make all vertices have degree 3. With each addition of the gadget, we increase the vertex cover by a size of 3, regardless of whether \( A \) is part of a vertex cover.

In the second and last step, we remove bridges in \( G' \). As in Fig. 4 for a bridge \( PQ \) that divides \( G' \) into \( G'_1 \) (containing \( P \)) and \( G'_2 \) (containing \( Q \)), we split the bridge edge \( PQ \) using the illustrated transformation, which yields a new graph \( G'' \) that is planar, bridgeless, and 3-regular, after all bridges are removed this way. For each such augmentation, the size of the vertex cover is bumped up by six. Let \( br(G') \) be the number of bridges in \( G' \). \( G'' \) has a vertex cover of size \( k'' \) if and only if \( G'' \) has a vertex cover of size \( k'' = k' + 6br(G') \). This completes the proof.

B. Hardness on Optimally Guarding a 3-Net

Starting from a planar cubic graph \( G \), we construct a structure that we call 3-net, \( T_G \), as follows. First, similar to [24], to embed \( G \) into the plane, an edge \( uw \in E(G) \) is converted to an odd length path \( uv_1v_2...v_{2m}w \) where \( m > 3 \) is an integer. We note that \( m \) is different in general for different edges of \( G \). Denote such a path as \( u ... w \); each edge along \( u ... w \) is straight and has unit edge length. We also require that each path is nearly straight locally. For a vertex of \( G \) with degree 3, e.g., a vertex \( u \in V(G) \) neighboring \( w, x, y \in V(G) \), we choose proper configurations and lengths for paths, \( u ... w, u ... x, u ... y \) such that these paths meet at \( u \) forming pairwise angles of \( 2\pi/3 \). We denote the resulting graph as \( G' \), which becomes the backbone of the 3-net \( T_G \).

From here, a second modification is made which completes the construction of \( T_G \). In each previously constructed path \( u ... w = uv_1...v_{2m}w \), for each \( v_i, v_{i+1} \), \( 1 \leq i \leq 2m-1 \), we add a line segment of length \( \sqrt{3} \) that is perpendicular
to \(v_i v_{i+1}\) such that \(v_i v_{i+1}\) and the line segment divide each other in the middle. A graphical illustration is given in Fig. 5. \(G'\) and the bars form the 3-net, which we denote as \(T_G\). An example of transforming \(K_4\) into a 3-net is given in Fig. 6.

![Fig. 5: Structure within the odd length path and attached perpendicular “bars” with length \(\sqrt{3}\). Regarding the representation of such non-integral coordinates in the problem input, we may scale the coordinates to some certain extent and round them to integers so that the relative distance between each other is precise enough for the proof.](image)

Let \(L\) be the number of (unit length) edges of \(G'\) (i.e., \(L = \sum_{uv \in E(G')} \text{len}(u \cdot w)\)).

**Lemma III.2.** A planar bridgeless 3-regular graph \(G\) has a vertex cover of size \(k\) if and only if its transformed 3-net \(T_G\) can be covered by \(K = k + (L - |E(G)|)/2\) circles of radius approximately \(\alpha = 1.152\).

**Proof.** If \(G\) has a vertex cover of size \(k\), then we put \(k\) circles of radius 1 at the centers of the corresponding vertices in \(T_G\). For each odd length path \(u \cdot w\), since either \(u\) or \(w\) is already selected as the circle center, applying one coverage pattern shown in Fig. 7 with \((\text{len}(u \cdot w) - 1)/2\) circles will cover the rest of \(u \cdot w\) and all bars on it. So, the total number of circles used is \(K\) to cover all of \(T_G\).

![Fig. 6: Illustration of a 3-net obtained from \(K_4\), the complete graph on 4 vertices.](image)

Fig. 7: Two coverage patterns on an odd length path with robots of range sensing radius 1 which is less than \(\alpha\).

The “if” part requires more analysis. Consider \(T_G\) that can be covered by \(K\) circles of radius \(r \approx \alpha\). For a path \(u \cdot w\) on \(T_G\) whose length is \(2m + 1\), there are \(2m - 1\) vertical bars associated with it. Consider the \(4m - 2\) endpoints of these vertical bars. We note that (as shown in Fig. 8), it requires a radius of \(2\sqrt{3}/3 \approx 1.155\) for a circle to cover 4 bar endpoints in an asymmetrical manner (three endpoints on one side of the path, one on the other). Since we set the radius of coverage circle to be about \(\alpha = 1.152\) (actually, between 1.152 and 1.153), a circle may only cover up to 4 bar endpoints. When a circle does cover 4 bar endpoints, it must use a symmetrical coverage pattern, i.e. 4 endpoints on two bars, resulting in fully covering two bars. For the rest of the proof, we use “circle” to mean circles with a radius of \(\alpha\), unless otherwise stated explicitly.

Since there are \(4m - 2\) bar endpoints, it requires \(m\) circles to cover all bar endpoints when \(m \geq 3\). Moreover, at least one circle must cover 4 bar endpoints by the pigeonhole principle. Fixate on such a circle \(S\), which must have symmetric coverage, we examine the bars on one side of it, say the left side, assuming the path \(u \cdot w\) is horizontal. If there are more than two bars to the left, then it is always beneficial to cover the two bars immediately to the left of \(S\) using another circle. To see that this is the case, look at the two bars \((DE\) and \(FH)\) and the two associated unit length edges \((AB\) and \(BC)\) to the left of \(S\) in Fig. 9. It can be computed that the circle \(S\) to the right can cover a maximum length of 0.412 of \(AB\) to \(A'\). Circles to the left of \(S\) then must cover \(A'B\). Let the circle covering \(A'\) be \(S'\). We may assume that \(S'\) covers at least one of \(D\) and \(E\) (otherwise, at least one more circle \(S''\) must be added that fall between \(S'\) and \(S\), in which case \(S''\) must also cover \(A'\)).

![Fig. 8: Asymmetrical coverage of 4 endpoints requiring a circle of radius at least \(2\sqrt{3}/3 \approx 1.155\).](image)

![Fig. 9: When there are enough bars left, it is always better to cover two bars at a time with a circle. Two extremal cases of \(S'\) covering \(A'\) and \(D\) but not \(E\) are shown (as dashed circles), which amounts to rotating the circle \(S'\) around \(D\).](image)

If \(S'\) covers \(A'\) and only one of \(D\) or \(E\), say \(D\), some other circle \(S''\) must cover \(E\). In this case, the coverage region of \(S'\) and \(S''\) are bounded by circles of radius approximately 2.304 with center at \(D\) and \(E\), respectively. It is readily observed that \(S'\) and \(S''\) can reach at most one more bar to the left of \(FH\) (we note that \(S'\) and \(S''\) will not be able reach structures on the 3-net beyond \(u \cdot w\) path). In this case, we can instead move \(S'\) cover \(A'C\), \(DE\), and \(FH\), and move \(S''\) cover the bar to the left of \(FH\) and potentially one more bar. Therefore, we may assume that \(S'\) covers bar endpoints \(D, E, F,\) and \(H\), symmetrically along...
the \(u \cdots w\) path. Following the reasoning, we may assume that all bars on paths are covered, two at a time by a circle in a symmetrical manner, until there are one or two bars left before a path reaches a junction where it meets other paths.

Because there are odd number of bars on a path, the symmetric coverage pattern extends until one side of a path has two bars remaining while the other side has one bar. \(m - 2\) circles have been used so far, which means that at least two more circles are needed to cover the remaining three bars. Without loss of generality, assume two bars out of these three are adjacent and are on the left end of \(u \cdots w\) and one is on the right. Denote these bars as \(b_1, b_2,\) and \(b_3,\) from left to right. We call the end of a path with two bars the \textit{even end} (e.g., the side ending with two bars \(b_1\) and \(b_2\)) and the end of the path with one bar the \textit{odd end} (e.g., the side ending with one bar \(b_3\)).

We now examine the coverage of \(b_2\) (see Fig. 10 where \(b_2\) corresponds to \(CD\) and \(b_2\) corresponds to \(AB\)). Again, if the two endpoints \(A\) and \(B\) of \(b_2\) are covered by more than one circle, then one of these two circles can be replaced with one that fully covers \(b_1\) and \(b_2\) (the solid circle in Fig. 10), since a circle covering only one endpoint of \(b_2\) (e.g., \(B\)) will not be able to reach structures outside \(u \cdots w\). By now, \(m - 1\) circles have been used and to cover \(u \cdots w,\) at least two more circles are needed at the two ends (i.e., \(m + 1\) circles are required to cover \(u \cdots w\)).

![Figure 10](image-url)

Fig. 10: When there are two bars at the end of a path, it is preferred to cover them with a single circle (in red). The dotted circle shows that a circle of radius \(2 + \alpha\) covering \(B\) will not be able to reach structures outside \(u \cdots w.\)

Next, instead of examining \(b_3,\) we examine the possible configurations at junctions where paths meet. There are four possible cases that contains 0-3 odd ends. For the case where only even ends meet, one additional circle is needed to cover the rest of the junction (Fig. 11(a)). When there is one odd end and two even ends (Fig. 11(b)), it requires one more circle to cover the junction. This constraint is how the radius \(\alpha = 1.152\) is obtained (more precisely, with circles with radius \(1.153\), no additional circles are needed at the junction). When there are two odd ends and one even end (Fig. 11(c)), at least one more circle is needed to cover the junction. For the last case (Fig. 11(d)), no additional circles are needed. The cases where additional cycles are needed correspond to the junction vertex being selected as a vertex cover. It is straightforward to observe that the constructed vertex cover is a valid one. The cover has size of \(K - (L - |E(G)|)/2.\)

\[\text{Fig. 11: The four possible patterns at the junction using circles of radius less than } \alpha = 1.152.\text{ When we increase the radius to } r = 1.152259,\text{ circles shown in (b) and (c) can successfully cover the junction and the odd ends.}\]

It is clear that Lemma III.2 holds for discs with radius in \([1, \alpha)\). Thus, approximating \(\text{size}(T_G, K)\) to less than a factor of \(\alpha\) will decide whether \(G\) has a vertex cover of \(k,\) yielding the hard-to-approximate result. Also, it can be observed that all lengths are polynomial with respect to the problem input size, which implies strongly NP-hardness.

\[\text{Theorem III.1. Finding the minimum radius for cover a 3-net using } k \text{ circular discs is strongly NP-hard to approximate within a factor of } \alpha = 1.152.\]

\[\text{C. From 3-Nets to Simple Polygons}\]

We proceed to show that OSG\(_{2D}\) is hard to approximate for a simply polygon by converting a 3-net into one. Along the backbone \(G',\) of a 3-net \(T_G,\) we first expand the line segments by \(\delta\) to get a 2D region (see Fig. 12(a)). We may describe the interior of the resulting polygon as

\[P = \{ p \in \mathbb{R}^2 \mid \min_{q \in T_G} (||p − q||_1) \leq \delta/2\}\]

For small enough \(\delta,\) it’s clear that \(P\) is a polygon with holes. Let \(K = ((L − |E|)/2 + k),\) it holds that

\[\text{size}(K, T_G) \leq \text{size}(K, P) \leq \text{size}(K, T_G) + \delta,\]

\[\text{size}(K, T_G) \leq \text{size}(K, \partial P) \leq \text{size}(K, T_G) + \delta.\]

To convert the structure into a simple polygon, we can open “doors” of width \(\delta\) on the structure to get rid of the holes (see Fig. 12(b)). Each opening removes one hole from \(P.\) This is straightforward to check; we omit the details.

\[\text{Fig. 12: (a) A 3-net } T_G \text{ maybe readily converted into a simple polygon } P \text{ with holes by expanding along its backbone. (b) Creating a “door” of width } \delta \text{ will remove one hole from } P.\]

Denoting the resulting simple polygon as \(P',\) we have

\[\text{size}(K, P) − \delta \leq \text{size}(K, P') \leq \text{size}(K, P),\]

\[\text{size}(K, \partial P) − \delta \leq \text{size}(K, \partial P') \leq \text{size}(K, \partial P).\]

Therefore, both \(\text{size}(k, P')\) and \(\text{size}(k, \partial P')\) are between \(\text{size}(k, T_G) − \delta\) and \(\text{size}(k, T_G) + \delta.\) Suppose the OSG\(_{2D}\) for \(\partial P'\) or \(P'\) has a polynomial approximation algorithm with approximation ratio \(1.152 − \varepsilon\) where \(\varepsilon > 0,\) let \(\delta =\)
\( \varepsilon/2 \), then the optimal guarding problem for the \( T_G \) can be
approximated within 1.152 disobeying the inapproximability gap by Theorem III.1. Therefore,

**Theorem III.2.** OSG\( _{2D} \) is NP-hard and does not admit a polynomial time approximation within a factor of \( \alpha \) with
\( \alpha = 1.152 \), unless \( P=NP \).

**D. OSG\( _{2D} \) with Sensor Guarding Restrictions**

The inapproximability gap from Theorem III.2 prompts us to further consider restrictions on the setup with the
hope that meaningful yet more tractable problems may arise. On natural restriction is to limit the number of continuous
segments a mobile sensor may cover. As will be shown in Section IV-A if a mobile sensor may only guard a single
continuous perimeter segment, a \((1+\varepsilon)\)-optimal solution can be computed efficiently. On the other hand, it turns out that if
a sensor can guard up to two continuous perimeter segments, OSG\( _{2D} \) remains hard to approximate.

**Theorem III.3.** OSG\( _{2D} \) of a simple polygon cannot be approximated within \( \alpha \approx 1.152 \) even when each robot can
guard no more than two continuous boundary segments, unless \( P=NP \).

**Proof.** Due to [25], every bridgeless 3-regular graph \( G \) has
a perfect matching. We can obtain such a perfect matching of the 3-regular graph using Edmonds' Blossom algorithm
in polynomial time [26]. Doubling the edges in the perfect
matching, we can then obtain a 4-regular graph \( G' \).

![Fig. 13: (a) Part of the augmented Eulerian path for non-doubled paths. (b) Part of the augmented Eulerian path for
doubled paths.](image)

The Eulerian tour on \( T_G \) may have self-intersections, which will prevent the tour from being a simple polygon. To address this, we may use one of two possible solutions outlined in Fig. 14 to eliminate the self-intersections.

![Fig. 14: In order to eliminate possible self-intersections in
(a), we may transform it into one of the solutions given in
(b) and (c) to make the Eulerian tour remain connected (one
of the two solutions will satisfy this).](image)

At this point, we readily observe that Theorem III.2 applies. Furthermore, an optimal solution always allows
each mobile sensor to cover only two continuous perimeter
segments. This is clear in the middle of any paths of \( T_G \):
at junctions, the polygon boundary will be either one of two
possibilities shown in Fig. 15 where a sensor again covers at
most two continuous segments of the simple polygon.  

![Fig. 15: The figure shows two possible types of boundaries
near a vertex with degree of 4. A robot near the vertex will
only be able to cover two disjoint but individually continuous
boundary segments with sensing radius less than \( \alpha \) if the
solution is to be optimal.](image)

**IV. EFFECTIVE ALGORITHMIC SOLUTIONS FOR OSG\( _{2D} \)**

In this section, we present several algorithmic solutions for
OSG\( _{2D} \). First, an efficient \((1+\varepsilon)\)-approximation algorithm
(i.e., an FPTAS) is presented that solve OSG\( _{2D} \) with the
additional requirement that each sensor is responsible for a
continuous perimeter segment. This contrasts Theorem III.3. Then, we show that there exist polynomial time algorithms
that readily guarantee a \((2+\varepsilon)\)-approximation for OSG\( _{2D} \).
This is followed by an integer linear programming (ILP)
method that delivers high-quality solutions (as compared
with the \((2+\varepsilon)\)-approximate one) and has good scalability.

In preparation for introducing the result, we first describe
a method that used for discretizing the problem. For a simple
discretely represented its boundary \( \partial P \) as a set of balls with radius \( \varepsilon \) along \( \partial P \), by splitting \( \partial P \) into
\( N = \lceil \text{len}(\partial P)/(2\varepsilon) \rceil \) continuous pieces of length at
most \( 2\varepsilon \) and putting the balls’ centers at their midpoints.

**A. OSG\( _{2D} \) with Single Segment Guarding Restriction**

By Theorem III.3 if a mobile sensor can guard up to two
continuous perimeter segments, OSG\( _{2D} \) is hard to approx-
imate within 1.152-optimal. Translating this into guarding
elements of \( S_O \), this means that a sensor can guard two
chains of elements from \( S_O \), where each chain contains
some \( m \) elements \( o_1, \ldots, o_m \) that are neighbors along \( \partial P \).
Interestingly, if each sensor may only guard a single chain
of elements from \( S_O \), we may compute an optimal cover
for \( S_O \) using \( O(N^2 \log(N)) \) time. This readily turns into a
fully polynomial time approximation scheme (FPTAS) for OPG$_{2D}$. The algorithm operates by checking multiple times whether a given radius $r$ is sufficient for $k$ discs of the given radius to cover elements of $S_O$ where each disc covers only a single chain of elements.

A single feasibility check is outlined in Algorithm 1. In the pseudo code, it is assumed that the indices are modulo $N$, i.e. $M[N+1] = M[1]$ (corresponding to $o_{N+1} = o_1$). Algorithm 1 is based on an efficient implementation of a subroutine MIN_ENCLOSURE_DISC (from e.g., [27], [28]) that computes the disc with minimum radius to enclose a given set of points in expected linear time. With this, a sliding window can be applied to find the rightmost end for each $1 \leq i \leq N$ such that $o_1, \ldots, o_{end}$ can be enclosed in a circle of radius $r$. The length of this sequence is stored in $M[i]$.

As $o_{end}$ cannot come around and meet $o_i$, the total call to MIN_ENCLOSURE_DISC is no more than $2N$. After this, the algorithm simply tries to put discs from each $o_i$ to cover as many centers as possible to see whether $S_O$ can be enclosed with $k$ discs. An optimization can be made by only examining starting point as $o_1, \ldots, o_{M[1]+1}$, since there is no circle of radius $r$ that can cover them together by the definition of $M$. The apparent complexity of Algorithm 1 is $O(N^2)$. Since there are a total of $N$ points and $k$ robots, in a majority of cases a circle would enclose about $N/k$ points. This effectively lowers the complexity to $O(N^2/k)$.

**Algorithm 1: OPG$_{2D}$-CONT_FEASIBLE**

Data: $S_O = \{o_1, \ldots, o_N\}$, sample points in circular order $k$, the number of robots

$r$, the candidate sensing radius

Result: true or false, indicating whether $S_O$ can be covered with $k$ discs with radius $r$

1 if MIN_ENCLOSURE_DISC(o$_1, \ldots, o_N$) \(\leq r\) then
2 return true
3 end
4 $M$ \(\leftarrow\) an array of length $N$; end \(\leftarrow\) 1;
5 for $i = 1$ to $N$ do
6 while MIN_ENCLOSURE_DISC(o$_1, \ldots, o_{end+1}$) \(\leq r\) do
7 \hspace{1em} end \(\leftarrow\) end + 1;
8 \hspace{1em} end
9 $M[i]$ \(\leftarrow\) end - $i$ + 1;
10 end
11 \%Phase 2: try to tile from each $o_i$.
12 for $i = 1$ to $N$ do
13 \hspace{1em} $j$ \(\leftarrow\) $i$, cnt \(\leftarrow\) $k$;
14 \hspace{1em} while cnt \(>\) 0 do
15 \hspace{2em} $j$ \(\leftarrow\) $j + M[j]$;
16 \hspace{2em} if $j - i \geq N$ then
17 \hspace{3em} return true
18 \hspace{3em} end
19 \hspace{2em} cnt \(\leftarrow\) cnt - 1
20 end
21 return false

Note that for the optimal coverage radius $r^*$, it holds that $r_{min} = 0 < r^* \leq len(\partial P)/(2k) = r_{max}$. Hence, after at most $\log(len(\partial P)/(2k)) = O(\log N/k) = O(\log N)$ times of binary search on the optimal radius $r^*$ by calling OPG$_{2D}$-CONT_FEASIBLE, the search range of $r^*$ or the gap between $r_{max}$ and $r_{min}$ will be reduced to within $\varepsilon$. So, it takes expected $O(N^2\log(N))$ time in total to get an approximate solution with radius at most $\varepsilon$ more than size($k, \partial S_O$) or size($k, \partial P$).

**Theorem IV.1.** Under the restriction of continuous coverage, OPG$_{2D}$ for a simple polygon can be approximated to $(1+\varepsilon)$-optimal in expected $O(N^2\log N)$ time, and $O(N^2\log N/k)$ in most cases, where $N = \lceil\text{len}(\partial P)/(2\varepsilon)\rceil$.

**Remark.** In the running time complexity analysis, we implicitly used the assumption that $\text{len}(\partial P)$ is proportional to problem input size (see Section III). Also, the algorithm as given computes an $OPT + \varepsilon$ optimal solution. However, due to the assumption that sensing radius is lower bounded, $(OPT + \varepsilon)$ directly translates into a $(1 + \varepsilon)$-optimal solution. Lastly, using techniques similar to those from [1], we mention that results in this subsection readily extends to multiple simple polygons. These arguments continue to apply throughout the rest of this section. Regarding the choice in implementation, the minimum enclosing disc problem (1-center problem) also has deterministic linear solution [29], but the randomized algorithm is considered to be more efficient [27] and easier to implement.

**B. $(2 + \varepsilon)$ Approximation**

In dealing with Euclidean $k$-clustering problems, two seminal methods are often brought out, both of which compute 2-approximation solutions for $k$-center problem in polynomial time. This is fairly close to the inapproximability gap of 1.822 [24]. The first [15], [30] transforms the clustering problem to a dominating set problem and then applies parametric search on the cluster size (radius), resulting in a 2-approximation in time $O(n^2 \log(n))$. A second method [16] takes a simpler farthest clustering approach by iteratively choosing the furthest point from the current centers as the new center. The method runs in $O(nk)$ but is subsequently improved to $O(n \log(k))$ in [24]. So, by applying either of them on $S_O$, we have

**Proposition IV.1.** OSG$_{2D}$ can be approximated to $(2 + \varepsilon)$-optimal in polynomial time with $N = O(\text{len}(\partial P)/\varepsilon)$ samples for perimeter guarding and $N = O((\text{len}(\partial P)/\varepsilon)^2)$ samples for region guarding.

For evaluation, we implemented the farthest clustering approach [16].

**C. Grid and Integer Programming-based Algorithm**

Approximation using grids [14] often exhibits good optimality guarantees and bounded time complexity. Seeing that and knowing that OSG$_{2D}$ is hard in general, we attempted grid-based integer programming (ILP) methods for solving
Consider bounding the polygon $P$ of interest by an $m \times n$ square grid where each cell is $\varepsilon \times \varepsilon$, denote $g_{ij}$ as the center of the cell at row $i$ and column $j$. If we limit the possible locations of each robot to the center of some grid cell, the optimal radius with this restriction will only be at most $\sqrt{2}/2\varepsilon$ away from size($k$, $S_O$).

So, given a candidate radius $r$, to check the feasibility of whether $\partial P$ can be covered by $k$ circles of radius $r$, we adopt an approach for solving the $k$-center problem [17] with integer linear programming. Specifically, we create $m \times n$ boolean variables $y_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$, indicating whether there is a robot at $g_{ij}$, then start to check the feasibility of following integer programming model.

$$\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n} y_{ij} \leq k \quad (1)$$

$$y_{ij} \geq 1 \quad \text{for each } 1 \leq \ell \leq N \quad (2)$$

$$y_{ij} \in \{0, 1\} \quad 1 \leq i \leq m, \ 1 \leq j \leq n \quad (3)$$

The first constraint says the number of locations is no more than $k$, and the second ensures each $g_{ij}$ can be covered by at least one circle with radius $r$ illustrated in Fig. 16.

![Fig. 16: This perimeter guarding example illustrates constraint (2) for $o_{67}$ with $r = 7$. The black dots are the sampled $S_O = \{o_1, \ldots, o_{100}\}$. In order to cover $o_{67}$, at least one among the red color grid cell centers need to be selected as robot location.](image)

When the ILP model has a feasible solution, $r^* = \text{size}(k, S_O) \leq r$ and $r \leq r^* = \text{size}(k, S_O)$ otherwise. This means that we can do a binary search on $r^*$, from an initial range of $r^* = \text{size}(k, S_O)$: $r^*_{\text{min}} = 0 < r^* \leq \text{len}(\partial P)/(2k) = r^*_{\text{max}}$, until finally $r^*_{\text{max}} - r^*_{\text{min}}$ is reduced to the selected granularity of $\varepsilon$.

**Remark.** With minor modifications, the ILP model applies to 2D region guarding, where the number of constraint (2) will then be $O(mn)$ with one for each grid that intersects with the polygon in an $m \times n$ grid. The initial upper bound set as $r^* = \text{len}(\partial P)$ and lower bound set as $\sqrt{\text{area}(P)}/[k\pi]$. It is also possible to apply the $(2+\varepsilon)$-approximation algorithm and set the result as the initial upper bound with the half of it as the initial lower bound.

### V. Evaluation and Application Scenarios

For the three algorithms described in Section IV, we developed implementations in C++ and evaluated them on an Intel Core i7 PC with a boost clock of 4.2GHz and 16GB RAM. For solving ILP models, Gurobi solver [31] is used. To evaluate the algorithms, we first generate a set of performance benchmarks obtained by subjecting these algorithms through a large set of benchmark cases. Following the synthetic benchmarks, we applied the algorithms on two potential application scenarios: guarding the outer perimeter of the Warwick Castle and monitoring a building for potential fire eruption points.

#### A. Performance Benchmarks

For creating synthetic benchmarks, to generate the test set $W$, we created simple polygons with the number of vertices ranging between 10 and 200. For each instance of the tested polygon, vertices are picked uniform at random from $[0,1] \times [0,1]$ and the TSP tour among these vertices are used for generating a simple polygon of a reasonable shape. An example is given in Fig. 16.

We first evaluate the computational performance of the special OPG$_{2D}$ algorithm where each sensor may cover a single continuous perimeter segment; denote this algorithm as A$_{OPG\_2D\_CONT}$. Table I lists the running time in seconds for various $N$ (number of discretized samples) and $k$ (number of robots). Each data point is an average of 100 examples. Since the variances are fairly small, they are not reported here. As we can observe, the method has very good scalability. It also demonstrates the behavior that running time is inverse proportional to the number of robots. This is due to the larger range of sensing radius that must be searched. In practice, sensing radius is often lower and upper bounded, which means that the algorithm will generally perform better. The normalized average standard deviation is about 0.06, which is fairly small.

| $N$ | $k$ | 5  | 10 | 20 | 30 | 50 | 100 |
|-----|-----|----|----|----|----|----|-----|
| 500 | 0.097 | 0.044 | 0.019 | 0.013 | 0.007 | 0.004 |
| 800 | 0.257 | 0.118 | 0.054 | 0.036 | 0.019 | 0.011 |
| 1000 | 0.385 | 0.183 | 0.082 | 0.055 | 0.029 | 0.016 |
| 1500 | 0.912 | 0.436 | 0.203 | 0.120 | 0.073 | 0.039 |
| 2000 | 1.597 | 0.743 | 0.345 | 0.225 | 0.123 | 0.062 |

**TABLE I: Running time (seconds) for A$_{OPG\_2D\_CONT}$.**

Since the $(2+\varepsilon)$-optimal algorithm is extremely efficient, we do not report its running time. For the ILP methods, Table II and Table III provide the running times for solving OPG$_{2D}$ and ORG$_{2D}$, respectively (for convenience, denote these two methods as A$_{ILP\_2D}$ and A$_{ILP\_2D\_ILP}$). Each data point is an average over 10 cases. $GS$ denotes the discrete grid size. We observe that the ILP method is highly effective for solving OPG$_{2D}$ and fairly good for solving ORG$_{2D}$. The normalized average standard deviation is about 0.125 for A$_{ILP\_2D\_ILP}$ (which is
ILP since the perimeter is suitable for continuous guarding while the ILP method is slightly limited by the chosen resolution.

In a second application, we took the footprint of the Brazil National Museum and use 40 mobile robots to monitor it. The solution, shown in Fig. 18 is computed using \(A_{\text{ORG}}^{2D,\text{ILP}}\). This could be useful when a building is on fire and drones equipped with heat sensors can monitoring “hot spots” on top of the building to prioritize fire extinguishing effort. There are also many other similar application scenarios.

![Fig. 17: Solutions for deploying 15 mobile sensors to guard the perimeter of the Warwick Castle. Methods: (a) \((2 + \varepsilon)\)-optimal. (b) \(A_{\text{OPG}}^{2D,\text{CONT}}\). (c) \(A_{\text{OPG}}^{2D,\text{ILP}}\).](image1)

![Fig. 18: A near-optimal solution for deploying 40 mobile robots for monitoring the Brazil National Museum, which caught fire in 2019.](image2)

### VI. Conclusion and Discussions

In this study, we examine \(\text{OSG}_{2D}\), the problem of directly computing a deployment strategy for covering 1D or 2D critical sets using many mobile sensors while minimizing the maximum sensing radius. After showing that \(\text{OSG}_{2D}\) is computationally intractable to even approximate within 1.152, we describe several algorithmic solutions with optimality and/or computation time guarantees. Subsequent thorough evaluation demonstrates the effectiveness of these algorithmic solutions. Finally, we demonstrate the utility of our algorithmic solutions with two application scenarios. Due to space limit, guarding perimeters with gaps (see, e.g., [1]) is
not discussed in this work. However, because our algorithms work with a grid-based discretization, the results directly apply to arbitrary bounded 1D and 2D sets.

Many intriguing questions follow; we mention two here concerning the sensing capabilities. First, OSG2D works with circular regions which is perhaps the simplest one due to symmetry. What if the sensor region is not circular? Whereas such cases appear to be hard [33], effective scalable solutions may still be possible. Secondly, currently we assume that all parts of the critical set to be guarded have equal importance. What if certain subsets are more important?

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