Strong completeness and semi-flows for stochastic
differential equations with monotone drift

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February 21, 2018

Abstract

It is well-known that a stochastic differential equation (sde) on a Euclidean space driven
by a (possibly infinite-dimensional) Brownian motion with Lipschitz coefficients generates
a stochastic flow of homeomorphisms. If the Lipschitz condition is replaced by an
appropriate one-sided Lipschitz condition (sometimes called monotonicity condition) and
the number of driving Brownian motions is finite, then existence and uniqueness of global
solutions for each fixed initial condition is also well-known. In this paper we show that un-
der a slightly stronger one-sided Lipschitz condition the solutions still generate a stochastic
semiflow which is jointly continuous in all variables (but which is generally neither one-
to-one nor onto). We also address the question of strong $\Delta$-completeness which means
that there exists a modification of the solution which if restricted to any set $A \subset \mathbb{R}^d$ of
dimension $\Delta$ is almost surely continuous with respect to the initial condition.

\textit{2010 Mathematics Subject Classification} Primary 60H10 Secondary 37C10, 35B27

\textbf{Keywords.} Stochastic flow; stochastic semi-flow; stochastic differential equation; monotonic-
ity; strong completeness; strong $\Delta$-completeness.

1 Introduction

In this paper, we study properties of the stochastic differential equation (sde)
\begin{equation}
\mathrm{d}X_t = b(X_t) \, \mathrm{d}t + M(\mathrm{d}t, X_t),
\end{equation}
where $M$ is a continuous martingale field on $\mathbb{R}^d$. Under an appropriate Lipschitz condition, this
sde has a unique solution $X_t$, $t \geq s$ for each initial condition $X_s = x \in \mathbb{R}^d$ and, moreover,
the sde generates a stochastic flow of homeomorphisms ([5 Theorem 4.5.1]). The aim of this paper is to show that a weaker semiflow property still holds in case the Lipschitz condition is replaced by a (local) one-sided Lipschitz condition (also known as monotonicity condition) and a coercivity condition. Under these weaker conditions, we cannot expect to obtain a stochastic flow of homeomorphisms anymore: both the one-to-one property and the onto property may fail – even in the deterministic case. The best we can hope for is a modification of the flow of homeomorphisms. The aim of this paper is to show that a weaker semiflow property still holds in case the Lipschitz condition is replaced by a (local) one-sided Lipschitz condition (also known as monotonicity condition) and coercivity condition. Under these weaker conditions, we cannot expect to obtain a stochastic flow of homeomorphisms anymore: both the one-to-one property and the onto property may fail – even in the deterministic case. The best we can hope for is a modification of the flow of homeomorphisms. The aim of this paper is to show that a weaker semiflow property still holds.

We will denote the standard inner product on \( R^d \) by \( \langle ., . \rangle \), the Euclidean norm on \( R^d \) by \( |.| \), the induced norm on \( R^{d \times d} \) by \( ||.|| \) (which is equal to the largest eigenvalue in case the matrix is positive semi-definite) and the (joint) quadratic variation of continuous semimartingales by \( [.,.] \). We denote the trace of a matrix \( A \in R^{d \times d} \) by \( \text{tr}(A) \). Throughout the paper, we will impose the following assumptions:

- \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) is a filtered probability space satisfying the usual conditions.
- \( b : R^d \to R^d \) is continuous.
- For each \( x \in R^d, t \mapsto M(t, x) \) is a continuous \( R^d \)-valued martingale s.t. \( M(0, x) = 0 \).
- The matrix \( a(x, y) := \frac{d}{dt}[M(., x), M(., y)]_t \) is non-random and independent of \( t \). Further, the map \( (x, y) \mapsto a(x, y) \) is continuous.
- Define \( \mathcal{A}(x, y) := a(x, x) - a(x, y) - a(y, x) + a(y, y) \). For each \( R > 0 \) there exists some \( K_R \geq 0 \) such that \( 2\langle b(x) - b(y), x - y \rangle + \text{tr}(\mathcal{A}(x, y)) \leq K_R|x - y|^2 \) for all \( |x|, |y| \leq R \).

Note that our assumptions imply that the field \( (t, x) \mapsto M(t, x) \) is a centered Gaussian process and for each \( x \in R^d, t \mapsto M(t, x) \) has the same law as \( t \mapsto GW_t \), where the \( d \times d \) matrix \( G \) satisfies \( GG^T = a(x, x) \) and \( W \) is \( d \)-dimensional Brownian motion. We point out that the equation

\[
\text{d}X_t = b(X_t) \, \text{d}t + \sum_{k=1}^m \sigma_k(X_t) \, \text{d}W^k_t
\]

with \( W^1, ..., W^m \) independent standard Brownian motions is a special case of (1.1) if we define \( M(t, x) := \sum_{k=1}^m \sigma_k(x)W^k_t \). Then \( a_{i,j}(x, y) = \sum_{k=1}^m \sigma_k(x)\sigma_k(y) \) and \( \mathcal{A}_{i,j}(x, y) = \sum_{k=1}^m (\sigma_k(x) - \sigma_k(y))(\sigma_k(x) - \sigma_k(y)) \). Note that we have (in general) \( \mathcal{A}(x, y) = \frac{d}{dt}[M(., x) - M(., y)]_t \).

**Definition 1.1.** We say that (1.1) has a (strong) local solution if for each \( x \in R^d \) and \( s \geq 0 \), there exists a stopping time \( \tau := \tau_{s,x} > s \) and an \( R^d \)-valued adapted process \( X_t, t \in [s, \tau) \) with continuous paths such that \( X_t = x + \int_s^t b(X_u) \, \text{d}u + \int_s^t M(\text{d}u, X_u) \) a.s. whenever \( 0 < t < \tau \) and \( \lim_{t \to \tau} |X_t| = \infty \) a.s. on the set \( \{ \tau < \infty \} \). We say that the
local solution is unique if whenever \( \tilde{X}_t, t \geq s \) is another process with these properties with associated stopping time \( \tilde{\tau} \), then \( \tau = \tilde{\tau} \) and \( X = \tilde{X} \) on \([s, \tau)\) almost surely. We will denote such a solution by \( \phi_{s,t}(x) \). For \( t \geq \tau \), we define \( \phi_{s,t}(x) := \infty \).

- We say that (1.1) has a unique global solution or is weakly complete if it has a unique local solution and if for each \( x \in \mathbb{R}^d \) and \( s \geq 0 \), \( \tau = \infty \) almost surely. We will denote such a solution by \( \phi \).

- We say that (1.1) admits a local semiflow, if it has a unique local solution \( \phi_{s,t}(x), t \in [s, \tau(s, x)) \) which admits a modification \( (\varphi_{s,t}(x), \Theta(s, x)) \) which is a local semiflow, i.e.: \( \varphi \) and \( \Theta : [0, \infty) \times \mathbb{R}^d \times \Omega \rightarrow (0, \infty] \) are measurable and for each \( \omega \in \Omega \),
  
  \begin{itemize}
    \item i) \( (s, x) \mapsto \Theta(s, x) \) is lower semicontinuous
    \item ii) \( (s, t, x) \mapsto \varphi_{s,t}(x) \) is continuous on \( \{(s, t, x) : 0 \leq s \leq t < \Theta(s, x)\} \)
    \item iii) For all \( 0 \leq s \leq t \leq u \), and \( x \in \mathbb{R}^d \), we have \( u < \Theta(s, x) \) iff both \( t < \Theta(s, x) \) and \( u < \Theta(t, \varphi_{s,t}(x)) \) and in this case the following identity holds: \( \varphi_{s,u}(x) = \varphi_{t,u}(\varphi_{s,t}(x)) \)
    \item iv) \( \varphi_{s,s} = \text{id}_{\mathbb{R}^d} \) for all \( s \geq 0 \).
    \item v) \( \lim_{t \rightarrow \Theta(s, x)} |\varphi_{s,t}(x)| = \infty \) whenever \( s \geq 0, x \in \mathbb{R}^d \), and \( \Theta(s, x) < \infty \).
  \end{itemize}

- We say that (1.1) admits a global semiflow, if it admits a local semiflow with \( \Theta(s, x) = \infty \) for all \( s \geq 0, x \in \mathbb{R}^d, \omega \in \Omega \).

Note that for existence and uniqueness of local or global solutions it suffices to check the definition for \( s = 0 \) due to time homogeneity of \( a \) and \( b \). We will also use the following definition.

**Definition 1.2.** We call \( \phi : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d \cup \{\infty\} \) a continuous local map if for each \( x \in \mathbb{R}^d \), there exists some \( \tau(x) \in (0, \infty] \) such that the following hold:

i) \( \phi_0 = \text{id}_{\mathbb{R}^d} \).

ii) \( \phi_t(x) = \infty \) whenever \( t \geq \tau(x) \).

iii) \( \phi \) is (jointly) continuous with respect to the one-point compactification \( \mathbb{R}^d \cup \{\infty\} \).

We call \( \phi : [0, \infty) \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d \cup \{\infty\} \) a random continuous local map in case \( \phi \) is measurable and \( \phi(., \omega) \) is a continuous local map for every \( \omega \in \Omega \). Further, we call \( \phi \) a continuous global map resp. random continuous global map if \( \phi \) is a continuous local map resp. random continuous local map such that \( \tau \equiv \infty \).

**Definition 1.3.** We say that the sde (1.1) is strongly complete in case it has a unique local solution which for initial time \( s = 0 \) admits a modification which is a random continuous global map.
Note that if the SDE (1.1) admits a global semiflow, then it is strongly complete.

In our main results, we will sometimes need the following assumptions. In the following hypotheses, \( \mu, K \geq 0 \) and \( \rho : [0, \infty) \to (0, \infty) \) is a non-decreasing function such that \( \int_0^{\infty} 1/\rho(u) \, du = \infty \).

**Assumption (A_{\mu,K})** For all \( x,y \in \mathbb{R}^d \), we have
\[
2\langle b(x) - b(y), x - y \rangle + \text{tr}(A(x,y)) + \mu \|A(x,y)\| \leq K|x - y|^2
\]

We will show in the Appendix (Proposition 8.5) that the previous assumption holds if it holds locally, i.e., for each \( z \in \mathbb{R}^d \) there is a neighborhood of \( z \) such that the assumption holds for all \( x,y \) in that neighborhood.

**Assumption (A_{\mu,loc})** For each \( R > 0 \) there exists \( K_R \geq 0 \) such that
\[
2\langle b(x) - b(y), x - y \rangle + \text{tr}(A(x,y)) + \mu \|A(x,y)\| \leq K_R|x - y|^2
\]
for all \( |x|, |y| \leq R \).

**Assumption (G_\rho)** For all \( x \in \mathbb{R}^d \), we have
\[
2\langle b(x), x \rangle + \text{tr}(a(x,x)) \leq \rho(\|x\|^2).
\]

**Assumption (G)** There exists a function \( \rho \) as above, for which Assumption (G_\rho) holds.

**Assumption (H_{f,\mu})** \( f : [0, \infty) \to (0, \infty) \) is continuous and nondecreasing, \( \mu \geq 0 \), and
\[
2\langle b(x) - b(y), x - y \rangle + \text{tr}(A(x,y)) + \mu \|A(x,y)\| \leq f(|x| \vee |y|)|x - y|^2
\]
for all \( x,y \in \mathbb{R}^d \).

We will prove in the Appendix (Lemma 8.4) that (A_{0,K}) implies (G_\rho) when \( \rho \) is a suitable multiple of \( x \mapsto x \vee 1 \) and hence (A_{0,K}) implies (G).

The paper is organized as follows. We will state sufficient conditions for existence and uniqueness of local and global solutions in Proposition 2.1. We will state sufficient conditions for strong completeness in Theorem 2.2. Proposition 2.3 contains various explicit sufficient conditions for strong completeness. Theorems 2.4 and 2.5 provide sufficient conditions for the existence of a local, respectively global semiflow. In Section 6 we define what we mean by strong \( \Delta \)-completeness and provide a sufficient condition for this property to hold (applying results of Ledoux and Talagrand on the existence of a continuous modification). In Section 7 we consider the special case of additive noise in which case we can obtain better results (than the ones before if applied in that particular case).

Let us relate our results to prior work. Existence and uniqueness of solutions of the SDE (1.1) have been shown in [10] (based on earlier work by Krylov in [4]) in case the SDE is driven by a finite number of Brownian motions (they allow however random and time dependent coefficients). Our proof follows the one in [10] initially but we apply a stochastic Gronwall lemma which simplifies the proof. The first major result about strong completeness of SDES is [7]. Our results are more general in some sense (we do not require differentiability properties of the coefficients but just continuity and monotonicity) but less general in other respects (we only work...
on $\mathbb{R}^d$ instead of more general manifolds). We believe that our approach has the advantage of being straightforward and short (once existence and uniqueness of solutions and the stochastic Gronwall lemma are available). We point out that [3] contains strong completeness results under more restrictive conditions than ours. A recent paper dealing with strong completeness is [2]. They only consider finite dimensional driving noise but they are more general in other respects (e.g. they consider Lyapunov functions while we only work with functions of the radial part of the solution).

It has been observed before that in order to prove strong completeness, one needs to control both the growth of the driving vector fields and of the local Lipschitz constants which determine the local dispersion of the semiflow. Relaxing one of the conditions will typically require the other one to be strengthened (see Theorem 2.2 and Proposition 2.3). Even if the vector fields are bounded, a local Lipschitz condition is insufficient for strong completeness as was shown in [8]. The additive noise case is somewhat special. The correlation of the driving noise is such that the usual conditions on the drift (linear growth and local Lipschitz condition) suffice to show strong completeness (see Section 7).

2 Main results

Proposition 2.1. a) Equation (1.1) has a unique local solution. Solutions enjoy the following coalescence property: for each pair $x, y \in \mathbb{R}^d$, and $s, s' \geq 0$, the following holds true almost surely: if there exists $t \geq s \vee s'$ such that $\phi_{s,t}(x) = \phi_{s',t}(y)$, then $\phi_{s,u}(x) = \phi_{s',u}(y)$ for all $u \geq t$.

b) If, moreover, Assumption (G) holds, then equation (1.1) has a unique global solution.

We will prove Proposition 2.1 in the following section.

Theorem 2.2. If $(H_{f,\mu})$ holds for some $\mu > d - 2$ (and $\mu \geq 0$) and (1.1) admits a global solution $\phi$ such that there exist $\gamma > 0$ and $t_0 > 0$ such that for any $R > 0$ we have

$$\sup_{|x| \leq R} \sup_{s \in [0, t_0]} \mathbb{E} e^{\gamma f(|\phi_{0,s}(x)|)} < \infty$$

(2.2)

then the sde is strongly complete. Moreover, if $(t, x) \mapsto \varphi_t(x)$ is a continuous modification of $(t, x) \mapsto \phi_{0,t}(x)$, then, for each $T > 0$, the map $x \mapsto \varphi_t(x)$ from $\mathbb{R}^d$ to $C([0, T], \mathbb{R}^d)$ is almost surely Hölder continuous with parameter $1 - \frac{d}{q}$ for every $q \in (d, \mu + 2)$.

We will prove Theorem 2.2 and the following proposition in Section 4. Next, we provide some examples for functions $f$ satisfying the assumptions of the previous corollary under suitable conditions on the coefficients $b$ and $a$ of the sde.

Proposition 2.3. For each of the following combinations of $b, a$, and $f$, the assumptions of Theorem 2.2 are satisfied and hence the sde is strongly complete. In each of the cases $\beta, c > 0$ are arbitrary.

a) $\langle b(x), x \rangle, \text{tr}(a(x, x)) \leq c(1 + |x|^2)$, \hspace{1cm} $f(u) = c(u^2 + 1)$

b) $2 \langle b(x), x \rangle + \text{tr}(a(x, x)) \leq c(1 + |x|^2)$, \hspace{1cm} $f(u) = c(u^2 + 1)$

c) $b, a$ bounded, \hspace{1cm} $f(u) = c(u^2 + 1)$.  


Theorem 2.4. Assume that \((A_{\mu,\text{loc}})\) holds for some \(\mu > d + 2\). Then equation (1.1) admits a local semiflow.

Theorem 2.5. Assume that unique global solutions of (1.1) exist. If \((H_{f,\mu})\) holds for some \(\mu > d + 2\) and there exist \(\gamma > 0\) and \(t_0 > 0\) such that for any \(R > 0\) we have

\[
\sup_{\|x\| \leq R} \sup_{s \in [0, t_0]} Ee^{\gamma f(\|\phi_{0,s}(x)\|)} < \infty,
\]

then the sde admits a global semiflow.

We will prove Theorems 2.4 and 2.5 in Section 5. We do not know if Theorem 2.5 remains true under the slightly weaker assumptions of Theorem 2.2.

Remark 2.6. The case \(d = 1\) is special and in that case better results can be achieved due to the fact that \(R\) is totally ordered. In this case weak completeness plus existence of a local semiflow are (more than) enough to guarantee strong completeness (and even existence of a global semiflow).

3 Existence and uniqueness of a local and global solution

Proof of Proposition 2.1. We first show existence of a local solution. The first part of the proof is an adaptation of arguments in Chapter 3 of [10] (based on previous work of Krylov [4]), while the final part is similar to a corresponding proof for stochastic functional differential equations (with finite dimensional noise) in [11] (using a stochastic Gronwall lemma). Due to time-homogeneity of the coefficients, we can and will assume that \(s = 0\). Fix \(x \in \mathbb{R}^d\). Increasing the number \(K_R\) if necessary, we can and will assume that

\[
\sup_{\|x\| \leq R} \sup_{s \in [0, t_0]} \|b(\phi_{0,s}(x))\| \leq K_R
\]

for each \(R > 0\). To prove existence of a solution, we employ an Euler scheme. For \(n \in \mathbb{N}\) we define the process \((\phi_{t}^{(n)})_{t \in [0, \infty)}\) by \(\phi_{0}^{(n)} := x \in \mathbb{R}^d\) and – for \(k \in \mathbb{N}_0\) and \(t \in \left(\frac{k}{n}, \frac{k+1}{n}\right]\) – by

\[
\phi_{t}^{(n)} := \phi_{\frac{k}{n}}^{(n)} + \int_{\frac{k}{n}}^{t} b(\phi_{\frac{s}{n}}^{(n)}) \, ds + \int_{\frac{k}{n}}^{t} M(ds, \phi_{\frac{s}{n}}^{(n)}),
\]

which is equivalent to

\[
\phi_{t}^{(n)} = x + \int_{0}^{t} b(\phi_{\frac{s}{n}}^{(n)}) \, ds + \int_{0}^{t} M(ds, \phi_{\frac{s}{n}}^{(n)})
\]

for \(t \in [0, \infty)\), where \(\phi_{\frac{n}{n+1}}^{(n)} := \phi_{\frac{k}{n+1}}^{(n)}\). Defining \(p_{t}^{(n)} := \phi_{t}^{(n)} - \phi_{t}^{(n)}\), we obtain

\[
\phi_{t}^{(n)} = x + \int_{0}^{t} b(\phi_{\frac{s}{n}}^{(n)} + p_{s}^{(n)}) \, ds + \int_{0}^{t} M(ds, \phi_{s}^{(n)} + p_{s}^{(n)})
\]
for \( t \in [0, \infty) \). Observe that \( t \mapsto \phi_t^{(n)} \) is adapted and continuous. Using Itô’s formula, we obtain for \( t \geq 0 \)

\[
\left| \phi_t^{(n)} - \phi_t^{(m)} \right|^2 = \int_0^t 2 \left\langle \phi_s^{(n)} - \phi_s^{(m)} , b(\phi_s^{(n)}) - b(\phi_s^{(m)}) \right\rangle \, ds \\
+ \int_0^t 2 \left\langle \phi_s^{(n)} - \phi_s^{(m)} , M(ds, \phi_s^{(n)}) - M(ds, \phi_s^{(m)}) \right\rangle + \int_0^t \text{tr}(A(\phi_s^{(n)}, \phi_s^{(m)})) \, ds \\
+ \int_0^t 2 \left\langle \phi_s^{(n)} - \phi_s^{(m)} , b(\phi_s^{(n)}) - b(\phi_s^{(m)}) \right\rangle \, ds + \int_0^t \text{tr}(A(\phi_s^{(n)}, \phi_s^{(m)})) \, ds \\
+ M_t^{(n,m)} + \int_0^t 2 \left\langle p_s^{(m)} - p_s^{(n)} , b(\phi_s^{(n)}) - b(\phi_s^{(m)}) \right\rangle \, ds,
\]

where

\[
M_t^{(n,m)} := \int_0^t 2 \left\langle \phi_s^{(n)} - \phi_s^{(m)} , M(ds, \phi_s^{(n)}) - M(ds, \phi_s^{(m)}) \right\rangle
\]

is a continuous local martingale starting at 0. Let \( R > 3|x| \) and define the stopping times

\[
\tau^{(n)}(R) := \inf \left\{ t \geq 0 : \left| \phi_t^{(n)} \right| \geq \frac{R}{3} \right\}.
\]

Then

\[
|p_t^{(n)}| \leq \frac{2R}{3} \quad \text{and} \quad \left| \phi_t^{(n)} \right| \leq \frac{R}{3} \quad \text{for} \quad t \in [0, \tau^{(n)}(R)] \cap [0, \infty).
\]

(3.7)

For \( 0 \leq s \leq \tau^{(n)}(R) \wedge \tau^{(m)}(R) =: \gamma^{(n,m)}(R) \), we have

\[
\left\langle p_s^{(m)} - p_s^{(n)} , b(\phi_s^{(n)}) - b(\phi_s^{(m)}) \right\rangle \leq 2K_R \left| p_s^{(m)} - p_s^{(n)} \right| \leq 2K_R \left( |p_s^{(m)}| + |p_s^{(n)}| \right).
\]

Therefore, for \( t \leq \gamma^{(n,m)}(R) \), we get

\[
\left| \phi_t^{(n)} - \phi_t^{(m)} \right|^2 \leq \int_0^t \left( K_R \left| \phi_s^{(m)} - \phi_s^{(m)} \right|^2 + 2 \left\langle p_s^{(m)} - p_s^{(n)} , b(\phi_s^{(n)}) - b(\phi_s^{(m)}) \right\rangle \right) \, ds + M_t^{(n,m)} \\
\leq \int_0^t 2K_R \left| \phi_s^{(m)} - \phi_s^{(m)} \right|^2 + \left| p_s^{(m)} - p_s^{(n)} \right|^2 + 4K_R \left( |p_s^{(m)}| + |p_s^{(m)}| \right) \, ds + M_t^{(n,m)} \\
\leq \int_0^t 2K_R \left| \phi_s^{(m)} - \phi_s^{(m)} \right|^2 \, ds + 4 \int_0^t K_R \left( |p_s^{(m)}| + |p_s^{(m)}| + |p_s^{(m)}|^2 + |p_s^{(m)}|^2 \right) ds + M_t^{(n,m)}.
\]

Now, we apply Lemma 8.1 to the process \( Z_t := \left| \phi_{t \wedge \gamma^{(n,m)}(R)}^{(n)} - \phi_{t \wedge \gamma^{(n,m)}(R)}^{(m)} \right|^2 \). Note that the assumptions are satisfied with \( \psi := 2K_R \), \( M_t := M_{t \wedge \gamma^{(n,m)}(R)}^{(n,m)} \), and

\[
H_t := H_t^{(n,m)} := 4 \int_0^{t \wedge \gamma^{(n,m)}(R)} K_R \left( |p_s^{(m)}|^2 + |p_s^{(m)}| + |p_s^{(m)}|^2 + |p_s^{(m)}|^2 \right) ds.
\]
Therefore, for $p \in (0, 1)$ and all $T > 0$, we have

$$\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \phi_{t,t_0}^{(n)}(x) - \phi_{t,t_0}^{(m)}(x) \right|^2 \right] \leq c_p e^{2pK^2T} \mathbb{E} \left[ \sup_{t \in [0,T]} \left( H_{t,m}^{n,m} \right)^p \right].$$

(3.8)

It is easy to check that $\sup_{s \geq 0} \mathbb{E} |p_{s\wedge \tau}^{(n)}(R)|$ converges to 0 as $n \to \infty$ since the coefficients of the sde are bounded on bounded subsets of $\mathbb{R}^d$. Thanks to (3.7), the right hand side of (3.8) therefore converges to 0 as $n, m \to \infty$. Now, by standard arguments, we can find a stopping time $\tau_R > 0$, a process $\phi_t$, $t \in [0, \tau(R)] \cap [0, \infty)$, such that $\tau_R = \inf\{t \geq 0 : |\phi_t| = R\}$ such that $\phi^n \to \phi_t$ on $[0, \tau_R] \cap [0, t]$ uniformly in probability for each $t > 0$. Further, $\phi$ solves the sde (1.1) with initial condition $x$ on the interval $[0, \tau_R] \cap [0, \infty)$ (see [11] for details – in the more general case of a stochastic delay differential equation).

Assume that there is another solution $\tilde{\phi}_t$, $t \in [0, \tilde{\tau}_R] \cap [0, \infty)$ with associated stopping time $\tilde{\tau} = \inf\{t \geq 0 : |\tilde{\phi}_t| = R\}$ which solves (1.1) with initial condition $x$ on the interval $[0, \tau_R] \cap [0, \infty)$. Then we can apply Itô’s formula to the square of the norm of the difference of the two processes up to the minimum of the two stopping times and use Lemma 8.2 to see that both solutions (and the associated stopping times) agree almost surely.

Now we let $R \to \infty$. Define $\tau := \lim_{R \to \infty} \tau_R$. Then the construction above shows that there exists a unique solution $\phi$ of (1.1) with initial condition $x$ on the interval $[0, \tau]$ and that $\tau$ has all properties stated in the first part of Definition 1.1.

Now we show the coalescence property. Let $x, y \in \mathbb{R}^d$ and $0 \leq s \leq s'$. Define the stopping time $T := \inf\{t \geq s' : \phi_{s,t}(x) = \phi_{s',t}(y)\}$. On the set where $\phi_{s,T}(x) = \phi_{s',T}(y) = \infty$ and on the set $\{T = \infty\}$ there is nothing to prove. Therefore, we define

$$Z_t := |\phi_{s,T+t}(x) - \phi_{s',T+t}(y)|^2 1_{\{T < \tau_{s,x} \wedge \tau_{s',y}\}}, \quad t \geq 0.$$

Applying Itô’s formula and Lemma 8.2 (as above), we see that $Z \equiv 0$, so the proof of part a) is complete.

Next, we prove part b) of the proposition. W.l.o.g. we assume that $s = 0$. Let $(X, \sigma)$ be a maximal strong solution of the sde (1.1) starting at $x$. We want to show that $\sigma = \infty$ almost surely. Note that $\lim \sup_{t \to \infty} |X_t| = \infty$ almost surely on the set $\{\sigma < \infty\}$. For a stopping time $0 \leq \tau < \sigma$, Itô’s formula implies that

$$X^2_\tau - X^2_0 = \int_0^\tau 2\langle b(X_u), X_u \rangle + \text{tr}(a(X_u, X_u)) \, du + 2 \int_0^\tau \langle X_u, M(du, X_u) \rangle$$

$$\leq \int_0^\tau \rho(|X_u|^2) \, du + \tilde{M}_\tau,$$

where $\tilde{M}$ is a continuous local martingale. Applying Lemma 8.1 to $Z_t := |X_t|^2$ finishes the proof.

The following proposition is a straightforward consequence of the uniqueness of local solutions.
Proposition 3.1. Under the assumptions of part a) of the previous proposition, the following holds for any modification of the (unique) local solution \((\phi, \tau)\): for each \(0 \leq s \leq t \leq u\) and \(x \in \mathbb{R}^d\), there exists a set of full measure \(\Omega_0\), such that on the set \(\Omega_0\) the following holds:

- \(u < \tau(s, x)\) if both \(t < \tau(s, x)\) and \(u < \tau(t, \phi_{s,t}(x))\)
- \(\phi_{s,u}(x, \omega) = \phi_{t,u}(\phi_{s,t}(x, \omega), \omega)\) whenever \(u < \tau(s, x)\).

4 Strong completeness

Our next aim is to establish sufficient conditions for strong completeness of an SDE. We will show the existence of a (Hölder) continuous modification with the help of Kolmogorov’s continuity theorem. Therefore, we will start by providing suitable \(L^p\)-estimates for the difference of solutions with different initial conditions.

Lemma 4.1. Let \(p \geq 2\) and assume that \((H_{f,p-2})\) holds and that global solutions exist (for which \((G)\) is sufficient). Further, let \(0 < q < p\) and \(P, Q > 1\) be such that \(\frac{1}{P} + \frac{1}{Q} = 1\) and \(qQ/p < 1\). Then, for each \(0 \leq s \leq t\),

\[
\mathbb{E} \sup_{s \leq t \leq T} |\phi_{s,t}(x) - \phi_{s,t}(y)|^q \leq |x - y|^q \tilde{c}^{1/Q}_{qQ/p} \left( \mathbb{E} \exp \left\{ \frac{Pq}{2} \int_s^T f(\langle \phi_{s,u}(x) \rangle \vee |\phi_{s,u}(y)|) \, du \right\} \right)^{1/P},
\]

where the constant \(\tilde{c}_r\) is defined before Lemma 8.2.

Proof. It suffices to prove the lemma in case \(s = 0\). Fix \(x, y \in \mathbb{R}^d, x \neq y\). Define

\[
D_t := \phi_t(x) - \phi_t(y), \quad Z_t := |D_t|^p.
\]

Then, by Itô’s formula,

\[
dZ_t = p|D_t|^{p-2}\langle b(\phi_t(x)) - b(\phi_t(y)), D_t \rangle \, dt + p|D_t|^{p-2}\langle D_t, M(\langle D_t, \phi_t(x) \rangle) - M(\langle D_t, \phi_t(y) \rangle) \rangle \\
+ \frac{1}{2}p|D_t|^{p-2}\langle A(\phi_t(x), \phi_t(y)), D_t \rangle \, dt + \frac{1}{2}p(p-2)|D_t|^{p-4}\langle D_t, A(\phi_t(x), \phi_t(y))D_t \rangle \, dt,
\]

where the last term should be interpreted as zero when \(D_t = 0\) even if \(p < 4\). Therefore, using \((H_{f,p-2})\), we get

\[
Z_t \leq |x - y|^p + \frac{p}{2} \int_0^t Z_u f(|\phi_u(x)| \vee |\phi_u(y)|) \, du + N_t,
\]

where \(N\) is a continuous local martingale starting at 0. Lemma 8.2 implies

\[
\mathbb{E} \sup_{0 \leq s \leq t} Z^r_s \leq |x - y|^{pr} \tilde{c}^{1/Q}_{Qr} \left( \mathbb{E} \exp \left\{ \mathbb{E}^{Pf, \psi_u du} \right\} \right)^{1/P},
\]

where \(\psi_u := \mathbb{E} f(|\phi_u(x)| \vee |\phi_u(y)|), r := q/p, P, Q > 1, rQ < 1,\) and \(\frac{1}{P} + \frac{1}{Q} = 1\), so the assertion of the lemma follows. \(\square\)
Remark 4.2. Clearly, Lemma 4.1 remains true if the expectations are replaced by the conditional expectations given \( F_s \) since \( \phi \) has independent increments.

Proof of Theorem 2.2. Fix \( p, q, P, Q \) as in Lemma 4.1. Using Jensen’s inequality, we get for \( t > 0 \)

\[
\sup_{|x|,|y| \leq R} E \exp \left\{ \frac{Pq}{2} \int_0^t f(|\phi_u(x)| \lor |\phi_u(y)|) \, du \right\} \\
\leq \sup_{|x|,|y| \leq R} \sup_{u \in [0,t]} E \exp \left\{ \frac{Pq}{2} tf(|\phi_u(x)| \lor |\phi_u(y)|) \right\} \\
\leq 2 \sup_{|x| \leq R} \sup_{u \in [0,t]} E \exp \left\{ \frac{Pq}{2} tf(|\phi_u(x)|) \right\}.
\]

Choosing \( t > 0 \) sufficiently small, the right hand side is finite for any choice of \( R \). Using Lemma 4.1 it follows from Kolmogorov’s continuity theorem with \( q \in (\mu, 2) \) that there exists a modification \( \varphi \) of \( \phi \) which is continuous on \( R^d \times [0, t_1] \) for some \( t_1 > 0 \) and that this modification has the stated Hölder continuity property. Iterating, we obtain a Hölder continuous modification on \( R^d \times [0, \infty) \). □

Proof of Proposition 2.3. To show b), we use the first equality in (3.9) and then apply Lemma 8.4 to get

\[
\sup_{|x| \leq R} \sup_{0 \leq s \leq t} |\phi_{0,s}(x)|^p < \infty \text{ for every } R, t > 0 \text{ and } p \in (0, 2). \]

Therefore b) follows.

To show a), apply Itô’s formula to \( Y_t := \log (|X_t|^2 + 1) \) and use the first equality in (3.9) to see that \( Y \) has Gaussian tails uniformly on compact subsets of \( R^d \times [0, \infty) \). Therefore, a) follows.

To show c), apply Itô’s formula to \( Y_t := (|X_t|^2 + 1)^{1/2} \) and use the first equality in (3.9) to see that \( Y \) has Gaussian tails uniformly on compact subsets of \( R^d \times [0, \infty) \). Therefore, c) follows. □

5 Local and global semiflows

Lemma 5.1. Assume that \((A_{\mu,K})\) holds for some \( \mu, K \geq 0 \) and that \( a \) and \( b \) are globally bounded. For each \( T > 0 \) and \( q \in (\mu + 2) \), there exists a constant \( c \geq 0 \) such that for all \( x, y \in R^d \) and all \( 0 \leq s \leq t \leq T, 0 \leq s' \leq t' \leq T \), we have

\[
E(|\phi_{s,t}(x) - \phi_{s',t'}(y)|^q) \leq c (|x - y|^q + |t - t'|^{q/2} + |s' - s|^{q/2}).
\]

Proof. Fix \( T > 0 \). We will assume without loss of generality that \( 0 \leq s \leq t \leq t' \leq T \) and \( 0 \leq s' \leq t' \leq T \). Further, \( c \) denotes a constant (possibly depending on \( a, b, \mu, K, q \) and \( T \)) whose value may change from line to line. First note that Lemma 8.4 implies weak
completeness.
\[
E\left(\left|\phi_{s,t}(x) - \phi_{s',t'}(y)\right|^q\right) \leq c \left( E\left(\left|\phi_{s,t}(x) - \phi_{s,t}(y)\right|^q\right) + E\left(\left|\phi_{s,t}(y) - \phi_{s',t'}(y)\right|^q\right) + E\left(\left|\phi_{s,t}(y) - \phi_{s',t'}(y)\right|^q\right) \right) \quad (5.10)
\]

We estimate the three terms separately. Concerning the first term, Lemma 4.1 (with \(f\) equal to the constant \(K\)) implies
\[
E\left(\left|\phi_{s,t}(x) - \phi_{s,t}(y)\right|^q\right) \leq c|x - y|^q.
\]
Applying Burkholder’s inequality, the second term in (5.10) can be estimated as follows:
\[
E\left(\left|\phi_{s,t}(y) - \phi_{s',t'}(y)\right|^q\right) = E\left(\int_t^{t'} b(\phi_{s,u}(y)) \, dy + \int_t^{t'} M(\phi_{s,u}(y), du)\right)^q \leq c|t' - t|^{q/2}. \quad (5.11)
\]
Finally, we estimate the third term in (5.10). Assuming w.l.o.g. that \(s \leq s'\), we get
\[
E\left(\left|\phi_{s,t}(y) - \phi_{s',t'}(y)\right|^q\right) = E\left(E\left(\left|\phi_{s',t'}(\phi_{s,u}(y)) - \phi_{s',t'}(y)\right|^q\right)\right) \leq c E\left|\phi_{s,u}(y) - y\right|^q,
\]
where we used Lemma 4.1 and Remark 4.2 in the last step. The last term can be estimated by \(c|s' - s|^{q/2}\) just like in (5.11). Therefore, the assertion of the lemma follows.

We continue by proving a version of Theorem 2.5 under stronger assumptions.

**Proposition 5.2.** Let the assumptions of the previous lemma hold for some \(\mu > d + 2\). Then the sde (1.1) admits a global semiflow.

**Proof.** Lemma 5.1 and Kolmogorov’s continuity theorem (as formulated e.g. in [5], Theorem 1.4.1) show that there exists a modification \(\varphi\) of the solutions \(\phi\) which is jointly continuous in all three variables for all \(\omega \in \Omega\). Since \(\varphi_{s,s}(x) = \phi_{s,s}(x) = x\) almost surely for each fixed \(s\) and \(x\) and since \((s, x) \mapsto \varphi_{s,s}(x)\) is continuous, there exists a set \(\mathbb{N}\) of measure 0 such that \(\varphi_{s,s}(x) = x\) for all \(x, s\) and all \(\omega \notin \mathbb{N}\). Redefining \(\varphi_{s,t}(x) := x\) on \(\mathbb{N}\), we obtain \(\varphi_{s,s}(x) = x\) for all \(s, x, \omega\).

It remains to show that for each 0 \(\leq s \leq t \leq u\), \(x \in \mathbb{R}^d\), and \(\omega \in \Omega\), we have
\[
\varphi_{s,u}(x) = \varphi_{t,u}(\varphi_{s,t}(x)). \quad (5.12)
\]
Observe that (5.12) holds up to a null set depending on \(s, t, u, x\) by Proposition 3.1. Since both sides of (5.12) are continuous functions of \((s, t, u, x)\), we can find a null set \(\mathbb{N}\) in \(\Omega\) such that \(\varphi_{s,u}(x) = \varphi_{t,u}(\varphi_{s,t}(x))\) holds for all \((s, t, u, x)\) outside of \(\mathbb{N}\). Redefining \(\varphi_{s,t}(x) := x\) on \(\mathbb{N}\), we see that \(\varphi\) is a global semiflow associated to (1.1).

Next, we relax the assumptions in the previous section and provide sufficient conditions for a local and a global semiflow to exist.
Proof of Theorem 2.4. Let Assumption $(\mathcal{A}_\mu,\text{loc})$ be satisfied and let $N \in \mathbb{N}$. We can find a function $b^N : \mathbb{R}^d \to \mathbb{R}^d$ and a continuous martingale field $M^N$ on the given filtered probability space such that $b^N(x) = b(x)$ and $M^N(t, x) = M(t, x)$ for all $|x| \leq N$ and all $t \geq 0$ and such that $b^N$ and $a^N$ corresponding to $M^N$ satisfy assumptions $(\mathcal{A}_\mu,K)$ for some $K \geq 0$ (depending on $N$) and are globally bounded. For example, we can take any function $\psi : \mathbb{R} \to [0,1]$ which is $C^\infty$ and non-increasing and which satisfies $\psi(s) = 1$ for $s \leq 1$ and $\psi(s) = 0$ for $s \geq 2$ and define

$$b^N(x) := \left(\psi\left(\frac{|x|}{N}\right)\right)^2 b(x), \quad M^N(t, x) := \psi\left(\frac{|x|}{N}\right) M(t, x), \quad x \in \mathbb{R}^d, \quad t \geq 0.$$ 

By Theorem 2.5 there exists an associated global semiflow $\varphi^N$. Clearly, there exists a set $\Omega_0$ of full measure such that for all $\omega \in \Omega$, and all $N > M$, $M, N \in \mathbb{N}$ the semiflows $\varphi^M$ and $\varphi^N$ agree inside a ball of radius $M$ in the following sense: For all $0 \leq s \leq u$, $x \in \mathbb{R}^d$, we have $\sup_{s \leq t \leq u} |\varphi^M_{s,t}(x)| < M$ iff $\sup_{s \leq t \leq u} |\varphi^N_{s,t}(x)| < M$ and in this case $\varphi^M_{s,u}(x) = \varphi^N_{s,u}(x)$. For $\omega \in \Omega_0$, we define $\Theta(s, x, \omega) := \lim_{N \to \infty} \inf \{t \geq s : |\varphi^N_{s,t}(x)| \geq N\}$ and for $t \in [s, \Theta(s, x, \omega))$, define $\varphi^N_{s,t}(x, \omega) := \varphi^N_{s,t}(x, \omega)$, where $N$ is any positive integer satisfying $\sup_{s \leq u \leq t} |\varphi^N_{s,u}(x)| < N$ (such an $N$ exists and the definition is independent of the choice of $N$). On the complement of $\Omega_0$, we define $\Theta \equiv \infty$ and $\varphi \equiv \text{id}_{\mathbb{R}^d}$. It is straightforward to check that $(\varphi, \Theta)$ is a local semiflow of equation (1.1).

Proof of Theorem 2.5. From the previous section, we know that (1.1) admits a local semiflow $\varphi$ and Theorem 2.2 shows that for every $s \geq 0$, there exists a set $\Omega_s$ of full measure such that $\Theta(s, x, \omega) = \infty$ for all $x \in \mathbb{R}^d$ and all $\omega \in \Omega_s$. Let $\tilde{\Omega}$ be the intersection of the sets $\Omega_s$ for all rationals $s \geq 0$ and redefine $\varphi$ as the identity on the complement of $\tilde{\Omega}$. Then we have $\Theta(s, x, \omega) = \infty$ for all $x \in \mathbb{R}^d$, $s \geq 0$, and all $\omega \in \Omega$, i.e. $\varphi$ is a global semiflow.

6 Strong $\Delta$-completeness

So far, we have discussed weak and strong completeness of an sde which mean that images of single points, respectively, subsets of full dimension survive almost surely under a locally continuous modification of the solution. One can consider intermediate concepts of completeness. The concept of strong $p$-completeness for integer $p$ was introduced in [7] meaning that $p$-dimensional submanifolds survive under the local semiflow. It seems natural to consider a corresponding concept of strong $\Delta$-completeness for arbitrary subsets of dimension $\Delta \in [0, d]$. For a precise definition, we need to agree on a particular notion of dimension like Hausdorff dimension or upper Minkowski dimension. In the following definition we will choose the upper Minkowski dimension (also known as box (counting) dimension) only because we can prove some result for it. We will in fact not even assume that a local semiflow exists.

Definition 6.1. Let $\Delta \in [0, d]$. We say that the sde (1.1) is strongly $\Delta$-complete if for any (deterministic) set $A \subset \mathbb{R}^d$ of upper Minkowski dimension $\Delta$ there exist a set $\Omega_0$ of $\Omega$ of full measure and a modification $\varphi$ of the solution $\phi$ starting at time 0 for which $(t, x) \mapsto \varphi_t(x)$ is continuous on $[0, \infty) \times A$ for all $\omega \in \Omega_0$. 

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Proposition 6.2. Assume that \((A_{\mu,K})\) holds. Then the sde (1.1) is strongly \(\Delta\)-complete for each \(\Delta < \mu + 2\) (satisfying \(\Delta \in [0,d]\)).

Proof. For \(q \in (\Delta, \mu + 2)\) and \(T > 0\), Lemma 8.4 implies weak completeness and Lemma 4.1 implies that
\[
\mathbb{E} \sup_{0 \leq t \leq T} |\phi_t(x) - \phi_t(y)| \leq |x - y|^q c \exp\{qKT/2\},
\]
for some constant \(c\) and all \(x, y \in \mathbb{R}^d\). Now, a combination of Theorems 11.1 and 11.6 in [6] applied to a set \(A\) in \(\mathbb{R}^d\) of upper Minkowski dimension \(\Delta\) implies our claim. \(\□\)

Remark 6.3. The image under \(\varphi_t\) of the set \(A\) in the proof of the previous proposition is even almost surely bounded for each \(t > 0\) (this follows from the same theorems in [6]).

7 Additive noise

Consider the sde
\[
dX_t = b(X_t) \, dt + \sigma dW_t, \tag{7.13}
\]
where \(b : \mathbb{R}^d \to \mathbb{R}^d\), \(\sigma > 0\) and \(W\) is a \(d\)-dimensional Wiener process and \(b\) satisfies our standing assumption. If \(b\) has linear growth, i.e. there exists some \(c \geq 0\) such that \(|b(x)| \leq c(1+|x|)\), then the flow generated by the sde is strongly complete, since for each initial condition \(x\), we have
\[
|X_t(x)| \leq |x| + c \int_0^t (|X_s(x)| + 1) \, ds + \sigma |W_t|
\]
for every \(t > 0\) and an application of Gronwall’s lemma yields
\[
|X_t(x)| \leq \left(|x| + \sigma \sup_{0 \leq s \leq t} |W_s| + ct\right) \exp\{ct\}.
\]
This is of course well-known, see [2] also for a discussion in case \(b\) does not have linear growth.

One might be tempted to conjecture that one can replace the linear growth property of \(b\) by the slightly weaker property
\[
\langle b(x), x \rangle \leq c(1 + |x|^2) \tag{7.14}
\]
for some \(c \geq 0\) and all \(x \in \mathbb{R}^d\) but this does not seem to be true – not even in case \(c = 0\). Instead of providing an example we just indicate how an example could look like (without making any claims that these ideas can be turned into rigorous mathematics):

Let \(d = 2\), \(\sigma = 1\) and let \(\rho : [0, \infty) \to \mathbb{R}\) be a smooth function such that \(\rho(0) = 0\) and \(\rho\) has heavy and increasingly quick oscillations. Consider \(b(x_1, x_2) := \rho(|x|)(-x_2)/|x|\) for \(x \neq 0\). Then \(\langle b(x), x \rangle = 0\) for all \(x\). Assume we observe the motion of the unit ball under the flow. If – in a short time interval – the ball is pushed a bit in the positive direction of the first coordinate (say), then the huge tangential drift will ensure that the expansion in the negative first coordinate direction is (at least) almost as large as in the first. So the outer boundary of the image of the ball will remain to look almost like a sphere with center 0 and the radius will on average increase at
least by order \( \delta^{1/2} \) during a time interval of length \( \delta \). Since \( \delta \) can be chosen arbitrarily small, it follows that strong completeness cannot hold.

These considerations suggest that in addition to (7.14) one should impose some control on the growth of the tangential part of \( b \). The following proposition shows that a quadratic (not just linear!) growth of that component guarantees strong completeness in the additive noise case.

**Proposition 7.1.** Let \( b : \mathbb{R}^d \to \mathbb{R}^d \) satisfy our standing assumptions and assume that – in addition – there exists some \( c \geq 0 \) such that for all \( x \in \mathbb{R}^d \) the following hold true:

(i) \( \langle b(x), x \rangle \leq c(1 + |x|^2) \),

(ii) \( |b(x) - \frac{\langle b(x), x \rangle}{|x|^2} x| \leq c(1 + |x|^2) \).

Then, for each \( \sigma \geq 0 \), the sde is (7.13) is strongly complete (and even admits a global flow).

**Proof.** For \( x \in \mathbb{R}^d \) let \( X_t(x), t \geq 0 \) be the solution of (7.13) with initial condition \( x \) and define \( Y_t(x) := X_t(x) - \sigma W_t \). Then

\[
\frac{d}{dt} |Y_t(x)|^2 = 2 \langle Y_t(x), b(Y_t(x) + \sigma W_t) \rangle. \tag{7.15}
\]

Fix \( T > 0 \) and \( \omega \in \Omega \) and let \( u \in \mathbb{R}^d \) be such that \( |u| \leq \sigma \sup_{0 \leq t \leq T} |W_t| \). Once we succeed in showing that there exists some \( C = C(T, \omega, \sigma, c) \) such that

\[
\langle y, b(y + u) \rangle \leq C(1 + |y|^2) \tag{7.16}
\]

for all \( y \in \mathbb{R}^d \) and \( u \) as above, then the claim in the proposition follows by applying Gronwall’s lemma to (7.15). Fix \( y \) and \( u \) as above such that \( |y| \geq \sigma \sup_{0 \leq t \leq T} |W_t| + 1 \) (there is no need to consider smaller \( |y| \)). Then \( b(y + u) \) can be uniquely decomposed as

\[
b(y + u) = \alpha(y + u) + \beta v,
\]

where \( \alpha, \beta \in \mathbb{R} \) and \( v \) is a unit vector which is orthogonal to \( y + u \). Assumption (i) and the assumed lower bound on \( y \) guarantee that

\[
\alpha \leq c \frac{1 + |y + u|^2}{|y + u|^2} \leq 2c
\]

and Assumption (ii) implies

\[
|\beta| = |\langle b(y + u), v \rangle| \leq c(1 + |y + u|^2).
\]

Therefore, using \( \langle y, y + u \rangle \geq |y|^2 - |y||u| \geq 0 \), and \( \langle y + u, v \rangle = 0 \), we get

\[
\langle y, b(y + u) \rangle \leq \alpha \langle y, y + u \rangle + \beta \langle y, v \rangle \leq \alpha \langle y, y + u \rangle + \alpha \langle y, y + u \rangle + \beta \langle y + u, v \rangle - \beta \langle u, v \rangle \\
\quad \leq 2c \langle y, y + u \rangle + c(1 + |y + u|^2)|u||v| \\
\quad \leq |y|^2(2c + |u||c|) + |y|(2c|u| + 2c|u|^2 + c(|u|^3 + |u|) \leq C(1 + |y|^2)
\]

for an appropriate \( C = C(T, \omega, \sigma, c) \) showing (7.16) thus completing the proof of the proposition. \( \Box \)
Remark 7.2. Note that the proof of the previous proposition does not use any properties of the Brownian motion $W$ other than almost sure local boundedness, so it can – for example – also be applied to additive Lévy noise. It is of interest to compare Proposition 7.1 with [2, Section 3.3]. They provide an example for $d = 2$ in which the drift has no radial component and the tangential component grows like the third power of the distance to the origin. That example is strongly complete but there exists a continuous function $g$ starting at 0 such that the solution of (7.13) with $W$ replaced by $g$ blows up for some initial conditions.

8 Appendix

We use the notation $Z^*_T = \sup_{0 \leq t \leq T} Z_t$ for a real-valued process $Z$.

The following lemma is taken from [11].

Lemma 8.1. Let $\sigma > 0$ be a stopping time and let $Z$ be an adapted non-negative stochastic process with continuous paths defined on $[0, \sigma]$ which satisfies the inequality

$$Z_t \leq \int_0^t \rho(Z^*_u) \, du + M_t + C,$$

and $\lim_{t \to \sigma} Z^*_t = \infty$ on $\{ \sigma < \infty \}$ almost surely. Here, $C \geq 0$ and $M$ is a continuous local martingale defined on $[0, \sigma]$, $M_0 = 0$ and $\rho : [0, \infty] \to [0, \infty]$ is non-decreasing, and $\int_0^\infty 1/\rho(u) \, du = \infty$. Then $\sigma = \infty$ almost surely.

The following stochastic Gronwall lemma is taken from [12] (for a more recent paper providing optimal constants see [1]). For $p \in (0, 1)$ define

$$\tilde{c}_p := (4 \wedge \frac{1}{p}) \frac{\pi p}{\sin(\pi p)} + 1.$$

Lemma 8.2. Let $Z$ and $H$ be nonnegative, adapted processes with continuous paths and assume that $\psi$ is nonnegative and progressively measurable. Let $M$ be a continuous local martingale starting at 0. If

$$Z_t \leq \int_0^t \psi_s Z_s \, ds + M_t + H_t$$

holds for all $t \geq 0$, then for $p \in (0, 1)$, and $\mu, \nu > 1$ such that $\frac{1}{\mu} + \frac{1}{\nu} = 1$ and $p \nu < 1$, we have

$$\begin{align*}
\mathbb{E} \sup_{0 \leq s \leq t} Z^p_s &\leq (\tilde{c}_p)^{1/\nu} \left( \mathbb{E} \exp \left\{ p\mu \int_0^t \psi_s \, ds \right\} \right)^{1/\mu} \left( \mathbb{E} (H^*_t)^{p\nu} \right)^{1/\nu}.
\end{align*}$$

If $\psi$ is deterministic, then

$$\begin{align*}
\mathbb{E} \sup_{0 \leq s \leq t} Z^p_s &\leq \tilde{c}_p \exp \left\{ p \int_0^t \psi_s \, ds \right\} \left( \mathbb{E} (H^*_t)^p \right),
\end{align*}$$

and

$$\begin{align*}
\mathbb{E} Z_t &\leq \exp \left\{ \int_0^t \psi_s \, ds \right\} \mathbb{E} H^*_t.
\end{align*}$$
We will need the following lemma at two different places.

**Lemma 8.3.** Let \( x, y \in \mathbb{R}^d \), \( x \neq y \), \( 0 = \gamma_0 < \gamma_1 < \cdots < \gamma_n = 1 \) and define \( x_i := x + \gamma_i(y - x) \), \( i = 0, \ldots, n \). Then

\[
\frac{\|A(x, y)\|}{|x - y|} \leq \sum_{i=1}^{n} \frac{\|A(x_{i-1}, x_i)\|}{|x_{i-1} - x_i|}, \quad \frac{\text{tr} A(x, y)}{|x - y|} \leq \sum_{i=1}^{n} \frac{\text{tr} A(x_{i-1}, x_i)}{|x_{i-1} - x_i|}.
\]

**Proof.** Let \( v \in \mathbb{R}^d \) and \( \alpha_1, \ldots, \alpha_n > 0 \) such that \( \sum_{i=1}^{n} \alpha_i = 1 \). Define \( A_i(t) := \sum_{k=1}^{n} v_k(M_k(t, x_i) - M_k(t, x_{i-1})) \). Using Jensen’s inequality, we get

\[
\langle A(x, y)v, v \rangle = \frac{d}{dt} \left[ \sum_{i=1}^{n} A_i \right]_t \leq \frac{d}{dt} \sum_{i=1}^{n} \frac{1}{\alpha_i} [A_i]_t = \sum_{i=1}^{n} \frac{1}{\alpha_i} \langle A(x_i, x_{i-1})v, v \rangle.
\]

Let \( \alpha_i := |x_i - x_{i-1}|/|x - y| \). Taking the supremum over all \( v \) with norm 1, the first claim follows. Choosing \( v \) to be the \( j \)-th unit vector and summing over \( j = 1, \cdots, d \), the second claim follows. \( \square \)

**Lemma 8.4.** Let \( a \) and \( b \) satisfy our general assumptions and assume that, in addition, there exists \( K \geq 0 \) such that

\[
2\langle b(x) - b(y), x - y \rangle + \text{tr}(A(x, y)) \leq K|x - y|^2 \quad \text{for all } x, y \in \mathbb{R}^d.
\]

Then

\[
2\langle b(x), x \rangle + \text{tr}(a(x, x)) \leq K|x|^2 + O(|x|).
\]

In particular, \((A_0, K)\) implies \((\mathcal{G}_\rho)\) for a positive multiple \( \tilde{\rho}(x) := x \lor 1 \).

**Proof.** Let \( x \in \mathbb{R}^d \setminus \{0\} \). For \( n \in \mathbb{N} \) and \( 0 = \gamma_0 < \cdots < \gamma_n = 1 \) and \( x_i := \gamma_i x \), we have

\[
2\langle b(x), x \rangle + \text{tr}(a(x, x)) = 2\langle b(0), x \rangle + \text{tr}(a(x, x)) + 2 \sum_{i=1}^{n} \langle b(x_i) - b(x_{i-1}), x_i - x_{i-1} \rangle \frac{|x|}{|x_i - x_{i-1}|} \leq K|x|^2 + 2\langle b(0), x \rangle + |x| \left( \frac{1}{|x|} \text{tr}(a(x, x)) - \sum_{i=1}^{n} \frac{\text{tr}(A(x_i, x_{i-1}))}{|x_i - x_{i-1}|} \right).
\]

Therefore,

\[
2\langle b(x), x \rangle + \text{tr}(a(x, x)) \leq K|x|^2 + 2\langle b(0), x \rangle + |x| \left( \frac{1}{|x|} \text{tr}(a(x, x)) - H(0, x) \right),
\]

where

\[
H(y, z) := \sup \left\{ \sum_{i=1}^{n} \frac{\text{tr}(A(\xi_i, \xi_{i-1}))}{|\xi_i - \xi_{i-1}|} \right\},
\]

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where the supremum is extended over all partitions of the line segment from \(y\) to \(z\). By Lemma 8.3, the sum in the definition of \(H(x, y)\) is non-decreasing when the partition is refined and hence \(H\) has the following additivity property: \(H(y, z) = H(y, \xi) + H(\xi, z)\) whenever \(\xi\) lies on the line segment from \(y\) to \(z\). Further, for \(\alpha, \beta > 0\) such that \((1 - \alpha)(1 + \beta) = 1\) and \(y, z \in \mathbb{R}^d\), we have

\[
0 \leq \left( \sqrt{(1 - \alpha)a_{i,i}(z, z)} - \sqrt{(1 + \beta)a_{i,i}(y, y)} \right)^2 \\
= (1 - \alpha)a_{i,i}(z, z) + (1 + \beta)a_{i,i}(y, y) - 2\sqrt{a_{i,i}(y, y)}\sqrt{a_{i,i}(z, z)} \\
\leq (1 - \alpha)a_{i,i}(z, z) + (1 + \beta)a_{i,i}(y, y) - a_{i,i}(y, z) - a_{i,i}(z, y),
\]

where we used the Kunita-Watanabe inequality in the last step. Therefore,

\[
\text{tr}(A(y, z)) \geq \alpha \text{tr}(a(z, z)) - \beta \text{tr}(a(y, y)).
\]

For \(\gamma > 1\), \(y = x\), and \(z = \gamma x\), we therefore get

\[
\frac{\text{tr}(a(\gamma x, \gamma x))}{\gamma|x|} - \frac{\text{tr}(a(x, x))}{|x|} \leq \frac{\text{tr}(A(x, \gamma x))}{(\gamma - 1)|x|} \leq H(x, \gamma x).
\]

Using the additivity property of \(H\), we see that the function \(\gamma \mapsto \frac{\text{tr}(a(x, \gamma x))}{\gamma|x|} - H(0, \gamma x)\) is non-increasing. Since \(a\) and \(b\) are locally bounded, we obtain

\[
2\langle b(x), x \rangle + \text{tr}(a(x, x)) \leq K|x|^2 + O(|x|),
\]

as required. \(\square\)

Finally we show that for Assumption \((A_{\mu,K})\) to hold it suffices that it holds locally.

**Proposition 8.5.** If

\[
2\langle b(y) - b(x), y - x \rangle + \text{tr}(A(x, y)) + \mu\|A(x, y)\| \leq C|y - x|^2
\]

holds locally, then it holds also globally.

**Proof.** Let \(x, y \in \mathbb{R}^d\). For an equidistant partition \(x_0 = x, \ldots, x_n = y\) of the straight line connecting \(x\) and \(y\), Lemma 8.3 implies

\[
2\langle b(y) - b(x), y - x \rangle + \text{tr}(A(x, y)) + \mu\|A(x, y)\| \\
\leq \sum_{i=0}^{n-1} \left( 2\langle b(x_{i+1}) - b(x_i), y - x \rangle + n\left( \text{tr}(A(x_{i+1}, x_i)) + \mu\|A(x_{i+1}, x_i)\| \right) \right) \\
\leq C|y - x|^2,
\]

provided the partition is fine enough. Therefore the assertion follows. \(\square\)

**Acknowledgement.** We thank Sebastian Riedel for valuable discussions related to strong completeness in the additive noise case.
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