Calibration of radii and masses of open clusters with a simulation.

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ABSTRACT

Context. A recent new approach to apply a simple dynamical mass estimate of tidally limited star clusters is based on the identification of the tidal radius in a King profile with the dynamical Jacobi radius. The application to an unbiased open cluster catalogue yields significantly higher cluster masses compared to the classical methods.

Aims. We quantify the bias in the mass determination as function of projection direction and cluster age by analysing a simulated star cluster.

Methods. We use direct \textit{N}-body simulations of a star cluster including stellar evolution in an analytic Milky Way potential and apply a best fit to the projected number density of cluster stars.

Results. We obtain significantly overestimated star cluster masses which depend strongly on the viewing direction. The overestimation is typically in the range of 10-50 percent and reaches a factor of 3.5 for young clusters. Mass segregation reduces the derived limiting radii systematically.

Key words. Galaxy: open clusters and associations: general – methods: \textit{N}-body simulations – stellar dynamics

1. Introduction

In a series of papers (Piskunov et al. 2007, 2008a,b) a new approach to determine the masses of open star clusters (OCs) was developed and used to determine the initial and present day mass function of OCs in the solar neighbourhood. The new method is based on the determination of the tidal radius \(r_t\) from the cumulative number of cluster members as function of projected distance to the cluster center. For each cluster the tidal radius \(r_t\) is determined from projected number density profiles by fitting a King 1962 profile (King 1962). The identification of the King cutoff radius \(r_t\) with the “Jacobi” radius \(r_J\) (i.e. the dynamical tidal radius, which is the distance from the cluster center to the Lagrange points \(L_1\) and \(L_2\)) yields the OC mass from the standard formula (Equation \ref{eq1} below solved for \(M_J\)). The application of this dynamical mass estimate of tidally limited clusters to an unbiased OC catalogue yields an independent mass determination compared to the classical methods. A detailed comparison with other methods of cluster mass determinations is also given. In a second step the method is extended to all OCs of an unbiased cluster catalogue by establishing a transformation of the observed semi-major axis and central surface density to \(r_t\). These results were then used to derive the cluster present day mass function (CPDMF) and the initial mass function of OCs (CIMF) in the extended solar neighbourhood. Adopting a constant cluster formation rate over the last 10 Gyr yields a surface density of 18 \(\text{M}_\odot\text{pc}^{-2}\) of stars born in OCs. This corresponds to a fraction of 37\% of disc stars which were born in OCs (Röser et al. 2010). This is large compared to the classical values of the order of 10\% or less (e.g. Wielen 1971, Miller & Scalo 1978).

Some crucial assumptions enter the dynamical mass determination based on fitting a King profile: a) The OC fills its Roche lobe in the tidal field of the Milky Way. For compact (e.g. Roche-lobe underfilling) clusters \(r_J\) and as a consequence the mass can be underestimated by a large amount. b) The effect of mass segregation can be neglected, i.e. star counts of the upper main sequence, which dominate the observed cluster members, are representative for the mass distribution. c) The elliptic shape of the clusters and the contamination through tidal tail stars do not result in a systematic bias. Shape parameters were measured by Kharchenko et al. (2009) and the distribution of tidal tail stars were investigated in detail (e.g. Just et al. 2009). d) The tidal radius \(r_t\) determined by fitting the cumulative projected mass profile represents the Jacobi radius \(r_J\) to derive the cluster mass. Since the cluster mass depends on the third power of \(r_J\), the method is very sensitive to systematic errors in the derivation of \(r_J\).

In the present paper we quantify the possible bias introduced by the identification of the tidal radius from a King profile fitting \(r_t\) with the Jacobi radius \(r_J\) used for the mass determination by Piskunov et al. (2007). We apply the King profile fitting procedure to a direct \textit{N}-body simulation of a dissolving star cluster at different evolutionary states. We have simulated a star cluster on a circular orbit at \(R_C = 8.5\text{ kpc}\) which evolved in the tidal field of the Milky Way including stellar evolution. We took snapshots of the evolved model with all stellar masses and positions...
Table 1. The list of galaxy component parameters. The first column gives the component, the second the mass, and the third and fourth the Plummer-Kuzmin parameters (equation 1).

| Component | M [M⊙] | a [kpc] | b [kpc] |
|-----------|--------|---------|---------|
| Bulge     | 1.4 × 10^{10} | 0.0 | 0.3 |
| Disk      | 9.0 × 10^{10} | 3.3 | 0.3 |
| Halo      | 7.0 × 10^{11} | 0.0 | 25.0 |

At four different times and projected the snapshots from the perspective of an observer on Earth (at R₀ = 8 kpc) onto the sky, at different positions along its orbit. After all, we determined the model’s limiting radius rₗ by fitting the projected cumulative mass profile with Equation (3) and compared rₗ to the actual Jacobi radius rᵢ.

The paper is organized as follows: In Section 2 we discuss the method of N-body simulations in the external potential of the Milky Way, Section 3 contains the theory of the cluster geometry in a tidal field. In Section 4, we show a simple iterative method to determine the Jacobi radius of an N-body model of a star cluster in a tidal field. In Section 5 the projection and fitting methods are described. Finally, Section 6 contains the results and section 7 the conclusions.

2. Numerical simulation

We analyse in detail a numerical simulation of a star cluster with initial mass M₀ = 10^6M⊙ on a circular orbit in an analytic Milky Way potential. It is the fiducial cluster simulation (run 10) discussed in Just et al. (2009). We have chosen this cluster, because it is a typical representative for the high-mass end of the observed OCs. Since the total mass of the cluster system is dominated by the high-mass end, the correction of biases in the mass determination are most important in that parameter regime. The cluster is set up as a W₀ = 6 King model with a half-mass radius of 8 pc. The extension of the cluster initially exceeds the Roche lobe initially leading to an enhanced mass loss in the first 0.5 Gyr. We used a Salpeter IMF and included mass loss by stellar evolution. The total lifetime of the cluster at a circular orbit with Rₖ = 8.5 kpc is 6.3 Gyr. For details of the evolution see Just et al. (2009).

For the high-resolution simulation of a dissolving star cluster with N = 40404 particles in the tidal field of the Milky Way the direct N-body code φGRAPE^4^ (Harfst et al. 2007) has been used in combination with the micro-GRAPE special-purpose hardware at the Astronomisches Rechen-Institut (ARI) in Heidelberg. φGRAPE is an acronym for Parallel Hermite Integration with GRAPE. The code is written in ANSI-C and uses a fourth-order Hermite scheme (Makino & Aarseth 1992) for the orbit integration. It is parallelised and uses the MPI library for communication between the processors. The force computations are executed on the fast special-purpose hardware GRAPE. The special-purpose micro-GRAPE6 hardware cards are especially designed to calculate gravitational forces in N-body simulations very fast using parallelization with pipelining (see Harfst et al. 2007 and references therein).

The code φGRAPE does not use regularization as the codes NBODY4 or NBODY6++ (Aarseth 1999, 2003; Spurzem 1999) but a standard Plummer type N-body gravitational softening. The softening length in the model used for the current work was ε = 10⁻³ pc. We tested with different softening lengths ε = 10⁻³, 10⁻⁴ and 10⁻⁵ pc that there are no significant differences regarding shape evolution and star cluster mass loss.

For the simulation of a star cluster in the tidal field of the Galaxy the N-body problem is solved in an analytic background potential. We use an axi-symmetric 3-component model, where bulge, disc and halo are described by Plummer-Kuzmin models (Miyamoto & Nagai 1975) with the potential

\[
\Phi(R, z) = -\frac{GM}{\sqrt{R^2 + (a + \sqrt{R^2 + z^2})^2}}, \tag{1}
\]

The parameters a, b and M of the Milky Way model are given in Table II for the three components.

The top panel of Figure 1 shows the rotation curve of the 3-component Plummer-Kuzmin model of the Milky Way. The parameters of the 3-component model are chosen such that the rotation curve matches that of the Milky Way (Dauphole & Colin 1995). At the solar radius R₀ = 8.0 kpc, which was assumed in this study, the value of the circular velocity is V₀ = 234.2 km/s. The values of Oort’s constants A and B are consistent with the observed values (A, B) = (14.5 ± 0.8, −13.0 ± 1.1)
where \( \kappa, \nu \) and \( \Omega \) are the epicyclic, vertical and circular frequencies of a near-circular orbits and \( \rho \) is the local galactic density (see Oort 1965 for the derivation of \( \delta^2 \)). The bottom panel of Figure 1 shows the course of the epicyclic and vertical frequency parameters \( \beta \) and \( \delta \).

At the radius \( R_C = 8.5 \) kpc of the circular orbit considered in this study we obtain \( (\beta_C, \delta_C) = (1.37, 2.86) \) and the circular velocity \( V_C = 233.3 \) km/s. The orbital time scale at \( R_C = 8.5 \) kpc is \( T_{\text{orb}} \approx 224 \) Myr.

### 3. Cluster geometry

According to King (1962), the projected density profile \( \Sigma(r) \) of a star cluster can be approximated by

\[
\Sigma(r) = k \left\{ X^{-1/2} - C^{-1/2} \right\}^2 \quad \text{for} \quad r \leq r_t \tag{4}
\]

with normalisation constant \( k \) and

\[
X(r, r_c) = 1 + (r/r_c)^2 \quad \text{and} \quad C(r_c, r_t) = 1 + (r_t/r_c)^2, \tag{5}
\]

where \( r_c \) is the core radius and \( r_t \) is the (tidal) cutoff radius where the projected density of the model drops to zero. Integration yields the cumulative form of the King 1962 profile,

\[
M_p(r) = 2\pi \int_0^r \Sigma(r')r'dr' \quad \text{for} \quad r \leq r_t
= \pi r_c^2 k \left\{ \ln(X) - 4 X^{1/2} - 1 + X - 1 \right\}, \tag{6}
\]

where \( M_p(r) \) is the projected cumulative mass of the model, i.e. the mass in projection on the sky within a circle of radius \( r \). For \( r > r_t \) we force the integrated profile (Equation 4) to the finite value

\[
M_p(r_t) = \pi r_c^2 k \left\{ \ln(C) - 3 + \frac{4}{C^{1/2}} - \frac{1}{C} \right\}, \tag{7}
\]

We will not use the identification of \( M_p(r_t) \) with the cluster mass, since in practice the equations are applied to star counts and the effective mass-of-light ratio enters the normalisation constant \( k \).

Already King (1961) remarks that the tidal forces from the galaxy distort only the outer regions of a star cluster. We quantify these deviations from spherical symmetry due to the tidal field in this Section. We employ a coordinate system \((x, y, z)\) of “principal axes of the star cluster”. Its origin is the cluster centre. The \( x \)-axis points away from the galactic centre, the \( y \)-axis points in the direction of the galactic rotation and the \( z \)-axis is directed towards the galactic north pole. Figure 2 shows a “principal axis plane cut” through the equipotential surfaces of the effective potential around a star cluster on a circular orbit in the tidal field of the Milky Way. To second order, the effective potential is given by

\[
\Phi_{\text{eff}} = \Phi_{\text{eff},0} - \frac{GM_{\text{cl}}}{\sqrt{x^2 + y^2 + z^2}} + \frac{1}{2} \left( \beta^2 - 4 \right) \Omega^2 x^2 + \frac{1}{2} \delta^2 \Omega^2 z^2 \tag{8}
\]

For the cluster we assumed a Kepler potential, which is a very good approximation in the outer parts (Just et al. 2009). The unit in Figure 2 is the Jacobi radius \( r_J \). The Jacobi radius is defined as the distance from the cluster centre to the Lagrange points \( L_1 \) and \( L_2 \). It is given by

\[
r_J = \left( \frac{GM_{\text{cl}}}{(4 - \beta^2)\Omega^2} \right)^{1/3} \tag{9}
\]

(see King 1962). The value of the effective potential on the critical equipotential surface which connects \( L_1 \) and \( L_2 \) can be easily calculated from (8) and (9). It is given by

\[
\Phi_{\text{eff},\text{crit}} = \Phi_{\text{eff},0} - \frac{3 GM_{\text{cl}}}{2 r_J}. \tag{10}
\]

Assuming that \( x_{\text{max}} = r_J \), the radius in \( y \)-direction of the last closed (critical) equipotential surface follows from (8) and (10),

\[
y_{\text{max}} = \frac{2}{3} r_J. \tag{11}
\]

For the radius in \( z \)-direction we have to solve a cubic equation

\[
-\frac{K}{z_{\text{max}}} + \frac{1}{2} L z_{\text{max}}^2 = \frac{3 K}{2 M} \tag{12}
\]

with the constants

\[
K = GM_{\text{cl}}, \quad L = \delta\Omega, \quad M = r_J. \tag{13}
\]

The only real solution is given by
cluster mass within $r_G$ stant model iteratively from Equation 9. The gravitational con-
Our iteration is given by

$$z_{\text{max}} = \frac{K [K + 2L (LM^3 + \sqrt{KM^3 + L^2M^6})]^{1/3} - K^{2/3}}{L (LM^3 + \sqrt{KM^3 + L^2M^6})^{1/3}}$$

(14)

For $(\beta, \delta) = (\beta_C, \delta_C) = (1.37, 2.86)$ we obtain

$$z_{\text{max}} \approx 0.503 \, r_J$$

(15)

(e.g. Wielen 1974). More generally, the ratios of the principal axes $y_{\text{max}} : x_{\text{max}}$ and $z_{\text{max}} : x_{\text{max}}$ of all closed equipotential surfaces are shown in Figure 3 as a function of the parameter $\gamma = x/r_J$. At the centre of the cluster we have $\gamma = 0$ while $\gamma = 1$ corresponds to the critical equipotential surface through $x = r_J$.

Figure 3 shows the logarithmically colour-coded surface density $\Sigma$ of projections of the simulated N-body model of the star cluster onto its principal axis planes at time $T_K = 1.31$ Gyr. The surface density $\Sigma$ has been calculated from the N-body snapshot file with the method of Casertano & Hut (1985) (their Equation II.6 with $\gamma = 20$). The contours correspond to $\Delta \log \Sigma \approx 2$ dex. The extent of the last closed (critical) equipotential surface is marked with dashed lines. The contours of constant surface density roughly follow the equipotential surfaces from Figure 3.

4. Jacobi radius of the model

We determined the Jacobi radius of our simulated N-body model iteratively from Equation 9. The gravitational constant $G$ and the quantities $\beta$ and $\Omega$ are known but $M_{cl}$, the cluster mass within $r = r_J$ is unknown since $r_J$ is unknown. Our iteration is given by

$$r_{n+1} = \left(\frac{GM_{\text{enc}}(r_n)}{(4 - \beta^2)R^2} \right)^{1/3}$$

(16)

where $M_{\text{enc}}(r)$ is the enclosed mass of cluster stars within radius $r$ around the cluster centre. Starting with $r_0 = \infty$ the iteration theoretically converges towards $r_\infty = r_J$. In practice, a few iterations are sufficient to determine $r_J$ accurately. It is interesting to note that this simple method does not rely on numerical fits of the effective potential to the envelope of the spatial distribution of Jacobi energies of the cluster stars. It can easily be applied to an N-body snapshot file which contains masses and positions of the individual particles at a certain time.

5. Projection and fitting

We calculate the projection at the sky in Galactic coordinates $(l,b)$ and ignore here for simplicity the perspective effects. Strictly speaking the projection of the cluster model is consistent only in the midplane $b = 0^\circ$ corresponding to the circular orbit in the Galactic plane. The rotation angle $\alpha$ is related to the galactic longitude $l$ by the law of sines,

$$\sin \alpha = -\sin l \frac{R_0}{R_C} \quad R_C > R_0$$

(17)

with $R_0 = 8$ kpc for the solar circle and $R_C = 8.5$ kpc for the cluster orbit. At the Galactic coordinates $(1, b) = (0, 0)$ and $(1, b) = (180^\circ, 0)$ the “system $(x,y,z)$ of principal axes of the cluster” with origin at the cluster centre is not rotated. Note that at $l = 90^\circ$ and $l = 270^\circ$ we have the maximum rotation angle $\alpha = -70.25^\circ$ and $\alpha = 70.25^\circ$, respectively.

A sketch of the projection of the simulated N-body model of the star cluster onto the plane of the sky is illustrated in Figure 3. Four orbital positions of the dissolving star cluster with its tidal tails are marked in the sketch.

In order to demonstrate the effect of different projections also in galactic latitude, we add the cases of a certain height of the cluster orbit above and below the plane of the solar circle (dotted lines in Figure 5). The effect of the corresponding vertical oscillation of the cluster orbit on the intrinsic structure of the cluster and of a variation of $\delta$ in Equation 9 are neglected in this study.

For all positions of the cluster on its orbit, we rotate the cluster around the $z$ axis and then around the $y'$ axis by the ordered pair of angles $(\alpha, -b)$ in order to simulate the perspective of the cluster for an observer on earth. After the projection, we determine the polar symmetric profile of the projected cumulative mass $M_{p}(r)$ from our N-body data file by summations over radius and polar angle and
Fig. 4. Surface density of projections onto the principal axis planes of the cluster at $T_4 = 1.31$ Gyr. The dashed lines show the theoretical values of $x_{\text{max}}/r_J$, $y_{\text{max}}/r_J$, and $z_{\text{max}}/r_J$. The contours correspond to $\Delta \log \Sigma \approx 2$ dex.

Fig. 5. Sketch of the galactic coordinate system in which the projection is done. The galactocentric radius of the sun is assumed to be $R_0 = 8.0$ kpc while the cluster orbits at $R_C = 8.5$ kpc on a circular orbit. The abbreviations denote the Galactic North (NGP) and South (SGP) Pole and the Galactic Centre (GC).

apply a fit with Equation 6. For the fitting, we used the MPFIT package in IDL (Markwardt 2009; Moré 1978 for the Levenberg-Marquardt algorithm).

6. Results

We first discuss projections according to Equation 17 in detail. Figure 6 shows examples of fits (upper panels) with the corresponding projections (lower panels). The resulting parameter ratio $r_t/r_J$ is given in the upper panels and the $(l, b)$ coordinates in the corresponding lower panels. In the upper panels, the solid (black) line represents the data and dotted (blue) line the fit. The dashed (red) lines mark $r_c$ (left dashed line) and $r_t$ (right dashed line) from the fit with Equation 6. In the lower panels, the dashed (red) line marks $r_J$.

Fig. 8. Histogram of the fraction on the sky per bin in $M_t/M_D$ corresponding to the $T_4 = 1.31$ Gyr parameter surface from Figure 4. $M_t$ and $M_D$ are the “tidal masses” calculated with Equation 9 from $r_t$ and $r_J$, respectively.

Fig. 9. Mean mass $\langle M_t/M_D \rangle$ as a function of the galactic latitude $b$. $M_t$ and $M_D$ are the “tidal masses” calculated with Equation 9 from $r_t$ and $r_J$, respectively.
Fig. 6. Examples of fits (upper panels) with the corresponding projections (lower panels) at time $T_3 = 1.31$ Gyr. The resulting parameter ratio $r_t/r_J$ is given in the upper panels and the ($l$, $b$) coordinates in the corresponding lower panels. In the upper panels, the solid (black) line represents the data and dotted (blue) line the fit. The dashed (red) lines mark $r_c$ (left dashed line) and $r_t$ (right dashed line) from the fit with Equation 6. In the lower panels, the dashed (red) line marks $r_J$. 
We derived the ratio $r_t/r_J$ for all projection directions in $(l, b)$ at four different evolution times of the cluster. The projections are done in steps of 4 degrees from $b = -90^\circ$ to $b = +90^\circ$ and $l = 0$ to $l = 360^\circ$. The full N-body snapshot file with $N = 40404$ particles has been used for the projection and fitting procedure as described in Section 5. Figure 7 shows parameter surfaces of $r_t/r_J$ as a function of Galactic coordinates of the cluster centre at times $T_1 = 0.62$ Gyr (top left), $T_2 = 0.84$ Gyr (top right), $T_3 = 1.06$ Gyr (bottom left) and $T_4 = 1.31$ Gyr (bottom right). A squeezed Hammer-Aitoff projection of Galactic coordinates has been used (see Appendix A). The total number of fits for each plot in Figure 7 is $N_{\text{fits}} = 4186$. Note that in the plot for $T_1$, a peak around $(l, b) \approx (270^\circ, 0)$ (see also Figure A.1 in Appendix A) is not resolved (white colored area). Here $r_t/r_J$ reaches a factor of 1.52.

It can be seen that the cluster masses are typically overestimated. The strongest bias to high masses at time $T_1$ (including the peak) shows that the unbound stars stay for a long time close to the cluster and contribute to the outer density profile in the fitting procedure. This effect depends on the initial conditions. At later times $T_3$, $T_4$ the cluster mass distribution becomes stable and the bias is independent of the age of the cluster.

Figure 8 shows a histogram of the fraction on the sky per bin in $M_t/M_{cl} = (r_t/r_J)^3$ for the parameter surface in Figure 7 corresponding to time $T_4$, where the “tidal” masses $M_t$ and $M_{cl}$ are calculated with Equation 9 from $r_t$ and $r_J$, respectively. The solid line shows the distribution for projections in the range between $-20^\circ < b < +20^\circ$. The most frequent overestimation of the mass is $M_t/M_{cl} = 1.18$. The dashed line shows the distribution for projections in the range between $-40^\circ < b < +40^\circ$. The most frequent ratio is the same as for the solid line. However, the ratios extend up to $M_t/M_{cl} \approx 2.1$.

Figure 9 shows the mean mass $\langle M_t/M_{cl} \rangle$ as a function of the galactic latitude $b$, where $M_t$ is the mass of cluster stars within radius $r_t$ which has been obtained by the fitting procedure. Shown are the curves corresponding to times $T_1 - T_4$. All curves show a local minimum which is located roughly at $b = 0^\circ$. For time $T_1$ the mean overestimation of the mass reaches 2.5 in the direction of the Galactic poles.
Figure 10. Parameter surfaces of $r_t/r_J$ and $\Delta r_J/r_t$ as a function of Galactic coordinates for a fit of King (1962) models to projections on the sky of a simulated model at different positions on its theoretical orbit. We used a squeezed Hammer-Aitoff projection. The time is $T_4 = 1.31$ Gyr. The color denotes the value of $r_t/r_J$ (and $\Delta r_J/r_t$, bottom plot) on a linear scale. The upper plots show the parameter surface for the 400 brightest stars in the simulated cluster (data courtesy of J. Beuria). The middle and bottom plots show the result of the bootstrap analysis (for explanations see the text). The middle plot shows the mean value of $r_t/r_J$ averaged over 100 small-number samples. The bottom plot shows the corresponding relative standard deviation $\Delta r_J/r_t$.

Usually only a small fraction of cluster stars are identified as members leading to an increased statistical uncertainty in the fitting procedure. The uppermost plot in Figure 10 shows the parameter surface derived for the 400 most massive (i.e. the brightest) stars in the simulation at time $T_4$ (data courtesy of J. Beuria). The reason for the low values of $r_t/r_J$ is mass segregation. The stronger asymmetries in $\pm b$ compared to the plots in Figure 7 are due to a slightly asymmetric distribution of the 400 mass-segregated stars in position space.

In order to measure the statistical scatter in $r_t$ we applied a bootstrap analysis. We divided the $N$-body snapshot file for time $T_4$ with $N = 40404$ particles into 100 small-number samples of $N_{\text{sample}} = 400$ particles each (the remaining particles were dropped out of the analysis). For each of these small-number samples we applied the procedure described above. The resulting total number of projections and fits was therefore $N_{\text{fits}} = 418600$. The middle and bottom plots in Figure 10 show the result of the bootstrap analysis. The middle plot of Figure 10 shows the mean value of $r_t/r_J$ averaged over the 100 samples. The bottom plot shows the relative standard deviation $\Delta r_J/r_t$ which resulted from the averaging over the 100 samples. The uncertainty on a single determination of $r_t$ lies in the range between 10 and 20 percent. The highest uncertainty is expected for $r_J$ determinations in the vicinity of the peak around $(l, b) \approx (270^\circ, 0)$.

A comparison with the uppermost plot in Figure 10 shows that the derived limiting radii for the sample of the most massive stars are systematically lower due to mass segregation. The differences are typically on a $1 - 2\sigma$ level.

7. Conclusions

The result of the analysis in this paper is the confirmation that the star cluster masses are typically overestimated if the method by Piskunov et al. (2007) is applied on a complete sample. Moreover, we quantified the methodological error in our analysis.

Figures 7 and 10 show that at certain Galactic coordinates the King (1962) profile fits are particularly biased. A high bias is predicted for $(l, b) \approx (270^\circ, 0)$ (see Figures 7 and 10). The corresponding rotation angle of the cluster is $\alpha = 70^\circ$, where the projection is parallel to the inner end of the tidal arms (see lower left plots in Figure 6). For $(l, b) = (90^\circ, 0)$ there is no corresponding peak in the parameter surface (best visible for $T_1$) due to the asymmetry between the leading and trailing tidal tails (see the sketch in Figure 5). Also, for Galactic latitudes beyond $b \approx \pm 40^\circ$ (cf. Figure A.1) the bias becomes large but only few OCs are located in that regime.

For the parameter surfaces in Figure 7 the masses are biased within the ranges $[1.3 M_\odot, 3.5 M_\odot]$ (at $T_1$), $[1.1 M_\odot, 2.3 M_\odot]$ (at $T_2$), $[1.0 M_\odot, 2.0 M_\odot]$ (at $T_3$) and $[1.0 M_\odot, 2.1 M_\odot]$ (at $T_4$) depending on the evolutionary state of the star cluster in the tidal field of the Galaxy. The bias depends strongly on the projection angles, which transform differently to Galactic coordinates for different orbital radii of the OC.

Furthermore, a bootstrap analysis showed that the relative error on a single determination of the limiting radius $r_t$ lies in the range between 10 and 20 percent (at time $T_4$) corresponding to an uncertainty in the mass of approximately 50 percent for samples of $N_{\text{sample}} = 400$ particles (which are typical for rich OCs).

Mass segregation of the brightest stars in a cluster can alter the $r_t/r_J$ factor significantly which is important for the data analysis of observations. The mass segregation results in a concentrated core which leads to an underesti-
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Appendix A: Squeezed Hammer-Aitoff projection

The squeezed Hammer-Aitoff projection is given by

\begin{align}
  x &= 2f \frac{\cos(b) \sin(l/2)}{\sqrt{1 + \cos(b) \cos(l/2)}}, \quad (A.1) \\
  y &= \frac{2}{f} \frac{\sin(b)}{\sqrt{1 + \cos(b) \cos(l/2)}}, \quad (A.2)
\end{align}

It is the standard Hammer-Aitoff equal-area projection where we introduced a free squeezing factor $f$. The squeezing leaves the area element $dA = dzdy = \sqrt{\det(g) \, dbdl}$ invariant, where $g$ is the first fundamental form calculated from equations (A.1) and (A.2). The ratio of diameters of the elliptic projection area is given by $d_y / d_x = f^2$. In the standard Hammer-Aitoff projection we have $f = \sqrt{2}$. For a projection onto a circular area one can set $f = 1$. The inverse projection is given by

\begin{align}
  z &= \sqrt{2 - \left( \frac{x}{2f} \right)^2 - \left( \frac{y}{2} \right)^2} \quad (A.3) \\
  b &= \arcsin \left( \frac{f}{2} \frac{y}{z} \right), \quad (A.4) \\
  l &= 2 \arctan \left( \frac{1}{2f} \frac{zx}{z^2 - 1} \right), \quad (A.5)
\end{align}

where we have introduced the auxiliary variable $z$.

We used $f = 1.2$ for Figures 7 and 10. The resulting coordinate system (which is hidden in Figures 7 and 10) can be seen in Figure A.1. We have modified IDL routines by W. B. Landsman to incorporate the free squeezing factor.

Fig. A.1. Squeezed Hammer-Aitoff projection. The Galactic latitude runs from $b = -90^\circ$ to $b = +90^\circ$ and the longitude from $l = 0$ to $l = 360^\circ$.

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\[ \text{rotation of the tidal radius. For a younger cluster age one would expect lesser mass segregation.} \]

For a quantitative correction of the bias in the cluster mass determination by identifying $r_1$ with $r_j$ an extensive parameter study of cluster parameters is necessary. The influence of mass-segregation on selection effects concerning the brightness limit of the observations should be included, because stellar evolution is taken into account. Especially for young clusters the bias factor can be sensitive to the initial conditions. The final goal is to find an agreement of OC mass determinations by the different methods. This would allow an interesting insight in the OC properties like the IMF, mass-to-light ratio and mass segregation.