I. INTRODUCTION

Perturbation theory is a well established framework for approximative analytical calculations in quantum field theory. In numerous examples perturbative calculations have been shown to be successful in the description of processes and systems characterized by a weak coupling. For larger values of the coupling strength, however, extrapolated perturbative results do not yield appropriate approximations. So, e. g., the perturbative series of the deconfined QCD thermodynamical potential $\Omega$, which is known up to the order $O(g^4)$, shows even for moderate values of the coupling, say $g \lesssim \frac{1}{2}$, a strongly fluctuating behavior increasing with the order of the expansion, whereas lattice gauge simulations exhibit a rather smooth dependency $\Omega(g)$.

This feature can be understood qualitatively by the asymptotic expansions $Z_n(g^2) = \sum_m z_m(g^2)^m$ of the function $Z(g^2) = \int_{-\infty}^{\infty} dx \exp \left\{ -\frac{1}{4} x^2 - g^2 x^4 \right\}$. This simple example mimics the perturbative and full representation, respectively, of the partition function of an interacting field theory in zero dimension. (In fact, the coefficients $z_m$ count the number of diagrams contributing to the thermodynamical potential of the scalar $\phi^4$ theory considered below.) Although $Z(g^2)$ defined for complex $g^2$ possesses a cut along the negative axis (thus the series expansion is divergent), $Z_n$ still contains useful information as to be expected for perturbative results. The error of the approximation can easily be estimated to be $|Z(g^2) - Z_n(g^2)| \sim (ng^2)^n$, hence the optimal order to truncate the series expansion is $n^* \sim 1/g^2$.

These considerations generalized suggest that for the (at least qualitative) description of strongly coupled systems suitable lower-order perturbative approximations might be more appropriate than elaborated high-order calculations. For the QCD thermodynamical potential, this leading-order conjecture is also supported by the argument of minimal renormalization scale dependence of the truncated perturbative expansion. With this assumption for the regime of larger coupling strength, in Section II the thermodynamical potential of the scalar $\phi^4$ theory is calculated in a selfconsistent nonperturbative approximation. The emerging picture motivates a quasiparticle description of the thermodynamics of gauge systems as discussed in Section III. The model is compared to available lattice data, and a prediction of the equation of state of the quark-gluon plasma (QGP) including strange flavor is given.

II. TADPOLE RESUMMED $\phi^4$ THEORY

It is well known that the (naive) perturbative calculation of the thermodynamical potential $\Omega$ of the massless scalar theory with the bare Lagrangian

$$\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi \right) \left( \partial^\mu \phi \right) - \frac{g^2}{4!} \phi^4$$

is plagued with infrared divergencies. These thermal divergencies can formally be treated by suitably rearranging the Lagrangian, e. g. by adding and subtracting a mass term to $\mathcal{L}$, yielding an equivalent theory with free massive propagators and an additional 2-point interaction. With such a reorganization of the perturbation theory, the expansion of $\Omega$ in the coupling strength, then well-defined up to 5th order, has been calculated. However, this series expansion exhibits a similar bad convergence as that of the QCD thermodynamical potential when being extrapolated to larger values of the coupling. As an alternative way to the perturbative expansion around the (modified) interaction-free limit, the Luttinger-Ward approach, originally derived for nonrelativistic fermion systems but readily generalized, is proposed here. Formulated in terms of full propagators, this formalism appears to be a more adequate description than perturbative calculations using free Green’s functions. In the tadpole approximation considered below it leads to propagators with a mass-like selfenergy providing thus a less formal motivation of the perturbative reorganization technique.
The Luttinger-Ward formulation of the thermodynamical potential of the scalar theory reads
\[
\Omega = \frac{1}{2} TV \sum \left[ \ln(-\Delta^{-1}) + \Delta \Pi \right] + \Omega',
\]
\[
\Omega' = - \sum_\mu \frac{1}{4m} TV \sum \Delta \Pi'_m
\tag{2}
\]
where \( \Delta = (\Delta_0^{-1} - \Pi)^{-1} \) denotes the exact propagator and \( \Pi \) is the exact selfenergy which can be decomposed into skeleton (i.e. 2-particle irreducible) contributions \( \Pi'_m \) of order \( m \). The sum-integral notation is specified below. The expression (2) obeys the fundamental condition
\[
\frac{\delta \Omega}{\delta \Pi} = 0
\tag{3}
\]
which functionally relates microscopic properties (\( \Pi \)) to macroscopic quantities (\( \Omega \)) ensuring thermodynamical selfconsistency. Moreover, the representation (3) is an expedient starting point for a systematic (also referred to as symmetry conserving) approximation scheme. Approximating \( \Omega' \) by the order-\( n \) truncated series \( \Omega_n' \) and the selfenergy by \( \Pi_n = \sum_m \Pi'_m \) yields an approximative expression \( \Omega_n \) for the potential which strictly fulfills (3).

Relying in the following on the conjecture stated in Section II, \( \Omega' \) is approximated in leading-loop order by \( \Omega'_1 = -3 \sum m \Pi'_m \) where double lines represent the full propagator \( \Delta_1 = (\Delta_0^{-1} - \Pi_1)^{-1} \) within the approximation considered. The corresponding selfenergy is given by
\[
\Pi_1 = 12 \sum \Delta_1 = 12 \left( \frac{g_0^2}{4!} \right) T \sum \Delta_1 .
\tag{4}
\]
This implicit equation for \( \Pi_1 \) amounts to the summation of all super daisy selfenergy diagrams of the (naive) perturbation theory. Due to the tadpole interaction process, \( \Pi_1 \) is real and momentum independent. Thus, \( \Delta_1 \) takes the form \( \Delta_{m^2}(P) = (P^2 - m^2)^{-1} \) of a free propagator with a mass term so the sum-integration in (4) is elementary, though divergent. It can be regularized in 4 – 2\( \epsilon \) dimensions using in the imaginary time formalism
\[
T \sum = \left( \frac{e^\gamma \bar{\mu}^2}{4\pi} \right)^\epsilon T \sum \int \frac{d^{3-2\epsilon} p}{(2\pi)^{3-2\epsilon}} , \quad p_0 = i 2\pi T n , \quad \gamma \text{ being Euler’s number and } \bar{\mu} \text{ the renormalization scale},
\]
\[
T \sum \Delta_{m^2} = \frac{m^2}{(4\pi)^2} \left[ \frac{1}{\epsilon} \ln \left( \frac{\bar{\mu}^2}{m^2} + 1 \right) + \frac{1}{2} \int_0^\infty dp \frac{p^2}{\omega_p} n_B(\omega_p/T) + \mathcal{O}(\epsilon) \right].
\tag{5}
\]
Here the notation \( \omega_p = (m^2 + p^2)^{1/2} \) and the Bose function \( n_B(x) = \exp(x) - 1 \) are used. The first expression of the right hand side of equation (5) corresponds to the vacuum selfenergy term of a massive scalar particle. It contains a divergent contribution which in the present case cannot be compensated by a counter term since it is temperature dependent. However, both the divergent and the scale dependent term are absorbed when expressing \( g_0 \) by the physical coupling \( g \) in equation (3). To be consistent with the tadpole topology class for the selfenergy, \( g \) is defined as the truncated vacuum scattering amplitude.

At the momentum scale \( s \):
\[
g^2 = g_0^2 - \frac{1}{2} g^2 g_0^2 L(s) \text{ with } L(s) = (4\pi)^{-2} \left( \epsilon - 1 - \ln(s/\bar{\mu}^2) + 2 + \mathcal{O}(\epsilon) \right).
\]
With the identification \( \Pi_1 = m^2 \) and choosing \( \sqrt{s} = \epsilon T \), the renormalized equation (3) reads at \( \epsilon \to 0 \)
\[
m^2 = \frac{g^2}{2} \left( \frac{m^2}{(4\pi)^2} \left[ \ln \frac{m^2}{T^2} - 1 \right] + \frac{1}{2\pi^2} \int_0^\infty dp \frac{p^2}{\omega_p^2} n_B(\omega_p/T) \right).
\tag{6}
\]

The solution \( m^2(g) \) of this gap equation, when expanded in \( g \), coincides with the perturbative selfenergy up to \( \mathcal{O}(g^4) \). For larger values of \( g \), in contrast to the perturbative result in the various known orders, it has a smooth behavior.

In the tadpole approximation, the thermodynamic potential (2) is given by
\[
\Omega_1 = \frac{1}{2} TV \sum \left[ \ln(-\Delta_1^{-1}) + \frac{1}{2} m^2 \Delta_1 \right] - \frac{TV}{2\pi^2} \int_0^\infty dp \frac{p^2}{2} \ln \left( 1 - \exp(-\omega_p/T) \right) - \int_0^\infty dp \frac{p^2}{\omega_p} n_B(\omega_p/T) + \frac{m^4 V}{128\pi^2} ,
\tag{7}
\]
where (3) and \( \partial_m \ln(-\Delta_1^{-1}) = -\Delta_1 \) were used for evaluating the sum-integral over \( \ln(-\Delta_1^{-1}) \) up to an irrelevant integration constant. As to be expected for a consistent approximation, the divergent and renormalization scale dependent terms from the sum-integrals exactly cancel each other in \( \Omega_1 \). In analogy to the approximation \( \Pi_1 \) of the selfenergy, \( \Omega_1 \) sums up all perturbative super daisy diagrams of the thermodynamical potential.

It is emphasized that in equation (3) the first contribution is the thermodynamical potential of an ideal gas of quasiparticles with mass \( m(T) \) as determined by the gap equation (3). The remaining interaction contribution, when calculating the entropy \( S_1 = -\partial \Omega_1 / \partial T \), compensates the temperature derivative terms of the quasiparticle mass. Thus, according to (3), \( S_1 \) is given by the entropy of the ideal massive gas.

Figure (3) shows the pressure \( p_1 = -\Omega_1/V \) as a function of the coupling strength. In the range of the coupling considered, the positive interaction pressure turns out to be a correction to the ideal gas contribution \( p_{\text{id}}(T, m) = T/(2\pi^2) \int dp p^2 \ln(1 - \exp(-\omega_p/T)) \) at the 10% level.
FIG. 1. The pressure $p_1$ and the ideal gas contribution $p_{id}$ as functions of the coupling strength $g$ in units of $\rho_{SB} = \pi^2/90 T^4$. For comparison, the leading and next-to-leading order perturbative results are shown as well.

For large values of $g$ the pressure $p_1$ does not deviate as much from the free limit as the naive extrapolation of perturbative results might indicate. This general feature is confirmed by other nonperturbative approaches. So the approximation $\Omega_1$ agrees with the result [12] derived by considering the $O(N)$ scalar theory in the limit $N \rightarrow \infty$. Without an expansion in $g$, the reorganized (screened) perturbation theory [13], where a gap equation has to be taken as an ‘external information’, leads to similar results.

Compared to these approaches, the Luttinger-Ward ansatz is favored here because, by construction, it yields symmetry conserving approximations. Furthermore, expressing the thermodynamical potential in terms of full propagators, the formulation (2) presumably guarantees a well-organized asymptotic expansion [14]. Finally, it is at least conceptually straightforward to extend the selfconsistent resummation scheme to the next-to leading order where the functional $\Omega'$ is approximated by

$$\Omega'_2 = \Omega'_1 - 12$$

Due to the rising sun contribution the selfenergy then becomes momentum dependent and complex thus the quasiparticles acquire a finite spectral width, a fact which requires more sophisticated techniques as those utilized here.

III. A QUASIPARTICLE MODEL FOR THE QGP

In QCD the difficulties mentioned at the end of the last section arise already at the 1-loop level. Therefore, in the following a more phenomenological approach, though closely related to the Luttinger-Ward formalism, is applied to describe the strongly coupled QGP. Temperatures closely above the confinement temperature $T_c \sim 200$ MeV, as they are possibly attainable in heavy-ion collisions, thereby require the appropriate treatment of the strange flavor. With a mass $m_{0s} \sim 150$ MeV the strange quarks represent a relevant non-light degree of freedom while the heavier flavors are suppressed.

The plasma is characterized by broken Lorentz symmetry which leads to additional partonic excitations not existing in the vacuum. The effect of the symmetry breaking is yet marginal for excitations with momentum $p \gtrsim T$. Therefore, the hard ‘medium-bound’ (longitudinal gluonic and helicity-flipped fermionic) modes are essentially not populated while the persisting excitations bear resemblance to the physical vacuum states. These thermodynamically relevant excitation are approximately described by a dispersion law $\omega^2(p) = m^2 + p^2$ with the effective gluon and quark masses given by

$$m_g = \sqrt{\frac{1}{6} \left( N_c + \frac{1}{2} n_f \right) g^2 T^2},$$

$$m_q = \frac{1}{2} \left( m_{0q} + \sqrt{m_{0q}^2 + \frac{N_c^2 - 1}{2 N_c} g^2 T^2} \right).$$

These gauge invariant masses are generated dynamically and thus not in conflict with two propagating gluon excitations and with chiral symmetry for vanishing current quark masses $m_{0q}$. Within the mass parametrization [6] of interaction, sub-leading effects in $g$ like damping are neglected so the approach outlined below amounts to a partial resummation of relevant contributions similar to the Luttinger-Ward formalism. The expansion in $m/T \sim g$ yields the leading order perturbative QCD results [17].

The pressure of this system of effective quasiparticles with temperature dependent masses $m_i(T)$ is given reminiscently to (3) by

$$p_{qp}(T) = \sum_i p_{id}(T, m_i(T)) - B(T).$$

The function $B(T)$ resembles the interaction term in (3). It is not an independent quantity. From the thermodynamical relation (4), which in the present framework takes the form $\partial p_{qp}/\partial m_i = 0$ (implying for chiral quarks $\langle \bar{q}q \rangle = 0$, thus the model respects chiral symmetry restoration in the plasma), $B$ is determined by

$$\frac{dB}{dT} = \sum_i \frac{\partial p_{id}}{\partial m_i} \frac{dm_i}{dT}.$$
To account for nonperturbative behavior near $T_c$, $g^2$ in (8) is specified by the effective coupling

$$G^2(T) = \frac{48\pi^2}{(11N_c - 2n_f) \ln[(T + T_s)/(T_s/\lambda)]^2} \quad (11)$$

where the parameter $T_s/T_c$ is a phenomenological regulator. For large temperatures, $G^2$ approaches the 1-loop running coupling with $T_c/\lambda \sim \Lambda_{\text{QCD}}$.

Equations (8) - (11) constitute the effective quasiparticle model which self-consistently ‘continues’ the leading order perturbative QCD thermodynamics. Corroborating the leading-order conjecture, it quantitatively reproduces thermodynamical lattice gauge data by suitably adjusting the parameters $\lambda$ and $T_s/T_c$. Hereby it turns out that fixing the integration constant in (10) at the confinement temperature yields positive values $B_0 = \sum_i p(xT_c,m) - p_{\text{lattice}}(T_c)$. This fact suggests a bag constant interpretation of $B_0$. In both the pure gauge system and the flavored plasma, however, at a temperature $T \sim 2T_c$ the ‘bag’ function $B(T)$ becomes negative due to the asymptotic behavior of the coupling which is opposed to the scalar model.

Motivated by the comparison to available lattice data, the quasiparticle model is now applied to predict the equation of state of the QGP with realistic quark masses which so far cannot be simulated numerically. The confinement temperature and the bag constant are chosen to be $T_c = 170$ MeV and $B_0 = (180\text{MeV})^4$. Equating the pressure of the confined phase (described by a hadron resonance gas) and $p_{\text{QP}}$ at $T_c$ to fulfill Gibbs’ condition relates the remaining two parameters $\lambda$ and $T_s/T_c$. In Figure 2 the predicted QGP pressure, energy density and the function $B(T)$ are shown for the large variation $3 \leq \lambda \leq 10$ of the free parameter.

![Figure 2](image.png)

**FIG. 2.** The equation of state of the QGP with massive strange flavor as predicted by the quasiparticle model, and the function $B(T)$. Decreasing values of the model parameter $3 \leq \lambda \leq 10$ are indicated by the arrow; the dotted short line marks the asymptotically free limit. The dashed lines represent pressure and energy density of the confined QCD phase approximated by a hadron resonance gas.

Due to the stationary condition, the prediction is rather insensitive to the choice of $\lambda$, $T_c$ and $B_0$. Remarkably, for temperatures $T \gtrsim 2T_c$ the energy density shows a saturation-like behavior at some 80-90% of the free limit. It is worthwhile pointing out that the strange flavor yields an important contribution to thermodynamical quantities because the effective strange quark mass is yet smaller than the thermal gluon mass.

In summary, a QGP quasiparticle model motivated by the Luttinger-Ward approach to $\phi^4$ thermodynamics is presented. Being successfully tested on lattice data of various gauge systems, it provides a realistic prediction of the equation of state of the QGP with strangeness. This allows further quantitative estimates of relevant observables in relativistic heavy-ion collisions.

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