THE $L_\infty$-DEFORMATIONS OF ASSOCIATIVE ROTA-BAXTER ALGEBRAS AND HOMOTOPY ROTA-BAXTER OPERATORS

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Abstract. A relative Rota-Baxter algebra is a triple $(A, M, T)$ consisting of an algebra $A$, an $A$-bimodule $M$, and a relative Rota-Baxter operator $T$. Using Voronov’s derived bracket and a recent work of Lazarev et al., we construct an $L_\infty[1]$-algebra whose Maurer-Cartan elements are precisely relative Rota-Baxter algebras. By a standard twisting, we define a new $L_\infty[1]$-algebra that controls Maurer-Cartan deformations of a relative Rota-Baxter algebra $(A, M, T)$. This $L_\infty[1]$-algebra is an extension of the graded Lie algebra controlling deformations of the AssBimod pair $(A, M)$ by the graded Lie algebra controlling deformations of the relative Rota-Baxter operator $T$. We introduce the cohomology of a relative Rota-Baxter algebra $(A, M, T)$ and study infinitesimal deformations in terms of this cohomology (in low dimensions). Finally, we define homotopy relative Rota-Baxter operators and find their relationship with homotopy dendriform and homotopy pre-Lie algebras.

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1. Introduction

Rota-Baxter operators are algebraic abstraction of integral operators. They first appeared in the fluctuation theory in probability [4]. Subsequently, Rota [25] and Cartier [5] studied these operators from combinatorial aspects. More precisely, a Rota-Baxter operator on an associative algebra $A$ is a linear map $R : A \to A$ satisfying

$$R(a)R(b) = R(R(a)b + aR(b)),$$

for $a, b \in A$.

In the last twenty years, several important applications of Rota-Baxter operators were found in the algebraic approach of the renormalization in quantum field theory [6], splitting of algebras [2], Yang-Baxter solutions [1, 2], quasi-symmetric functions [33], and double Lie algebras [15]. See [16] for more details about Rota-Baxter operators.
operators. As a variation of a noncommutative Poisson structure, Uchino [29] defined a generalization of a Rota-Baxter operator in the presence of a bimodule over the algebra. Such operators are called generalized Rota-Baxter operators or relative Rota-Baxter operators (also called \(O\)-operators). In this paper, we stick with the term relative Rota-Baxter operator.

Rota-Baxter operators and its relative version were also defined on Lie algebras by Kupershmidt [18] as an operator analog of classical \(r\)-matrices. In [21], the authors construct an \(L_\infty\)-algebra which controls the deformation of a relative Rota-Baxter algebra. Subsequently, they find the connection between this \(L_\infty\)-algebra and the graded Lie algebra obtained in [28] that governs the deformation of the relative Rota-Baxter operator.

Our aim in this paper is to follow the approach of [21] to study deformations of relative Rota-Baxter (associative) algebras via \(L_\infty\)-algebras. During the course, we also study deformations of an \textbf{AssBimod}-pair \((A, M)\) consisting of an associative algebra \(A\) and an \(A\)-bimodule \(M\). We construct a graded Lie algebra whose Maurer-Cartan elements are given by \textbf{AssBimod}-pairs. This characterization allows us to construct a differential graded Lie algebra controlling deformations of an \textbf{AssBimod}-pair.

Next, using the higher derived brackets of Voronov [31,32], we construct an \(L_\infty\)-algebra whose Maurer-Cartan elements are given by relative Rota-Baxter algebras. With this characterization and the well-known construction [14] of an \(L_\infty\)-algebra twisted by a Maurer-Cartan element, we construct a new \(L_\infty\)-algebra that controls Maurer-Cartan deformations of a relative Rota-Baxter algebra. This new \(L_\infty\)-algebra also allows us to define the cohomology of a relative Rota-Baxter algebra. Finally, we obtain a long exact sequence connecting the cohomology of the relative Rota-Baxter algebra \((A, M, T)\), cohomology of the underlying \textbf{AssBimod}-pair \((A, M)\) and the cohomology of the relative Rota-Baxter operator \(T\) introduced in [9].

The formal deformation theory of algebraic structures was first developed for associative algebras in the classical work of Gerstenhaber [13] and subsequently extended to Lie algebras by Nijenhuis and Richardson [24]. In [3], Balavoine extended the formal deformation theory to algebras over binary quadratic operads. The more general notion of \(R\)-deformations over a local Artinian ring (or a complete local ring) \(R\) has been described in [10,11,17]. We apply this more general deformation theory to relative Rota-Baxter algebras. We give special attention to infinitesimal deformations of a relative Rota-Baxter algebra and show that there is a one-to-one correspondence between equivalent infinitesimal deformations and elements in the second cohomology space of a relative Rota-Baxter algebra.

In the sequel, we define homotopy relative Rota-Baxter operators on bimodules over \(A_\infty[1]\)-algebras. More precisely, given an \(A_\infty[1]\)-algebra and a bimodule over it, we construct an \(L_{\infty}[1]\)-algebra using Voronov’s construction. This \(L_{\infty}[1]\)-algebra is a graded version of a graded Lie algebra constructed in [9]. We define a homotopy relative Rota-Baxter operator \(T\) as a Maurer-Cartan element in the above \(L_{\infty}[1]\)-algebra. We also explicitly describe the identities satisfied by the components of \(T\). This notion generalizes the strict homotopy relative Rota-Baxter operator defined in [8]. It is known that a strict homotopy relative Rota-Baxter operator induces a \textit{Dend}_{\infty}[1]-algebra structure. In [21], the authors define homotopy relative Rota-Baxter operators on modules over \(L_\infty[1]\)-algebras. We show that our definition is suitable compatible with that of [21].

Finally, we construct a \textit{pre-Lie}_{\infty}[1]-algebra associated to a \textit{Dend}_{\infty}[1]-algebra, which generalizes the non-homotopic case [2]. We also extend various relationships among dendriform algebras, \textit{pre-Lie} algebras, associative algebras, Lie algebras, and relative Rota-Baxter algebras to the homotopy context.

This paper is organized as follows. In Section 2, we recall some basics on relative Rota-Baxter algebras and Voronov’ derived bracket construction of \(L_{\infty}\)-algebras. Maurer-Cartan characterization of \textbf{AssBimod}-pairs and their cohomology are studied in Section 3. In the next Section (Section 4), we construct the \(L_{\infty}\)-algebra that controls Maurer-Cartan deformations of a relative Rota-Baxter algebra. Using this, we also define the cohomology of a relative Rota-Baxter algebra. Deformations of a relative Rota-Baxter algebra are considered in Section 5. Finally, in Section 6, we introduce homotopy relative Rota-Baxter operators and discuss their relationship with homotopy dendriform and homotopy \textit{pre-Lie} algebras.
All (graded) vector spaces, linear maps and tensor products are over a field \(\mathbb{K}\) of characteristic 0. We denote set of all permutations on \(k\) elements by \(S_k\). A permutation \(\sigma \in S_k\) is called an \((i, k-i)\)-shuffle if \(\sigma(1) < \cdots < \sigma(i)\) and \(\sigma(i+1) < \cdots < \sigma(k)\). The set of all \((i, k-i)\)-shuffles in \(S_k\) is denoted by \(S_{(i,k-i)}\). For any permutation \(\sigma \in S_k\) and homogeneous elements \(v_1, \ldots, v_k\) in a graded vector space \(V\), the Koszul sign \(\epsilon(\sigma) = \epsilon(\sigma; v_1, \ldots, v_k)\) is given by

\[
\epsilon(\sigma) = \epsilon(\sigma)\ v_1 \circ \cdots \circ v_k,
\]

where \(\circ\) denotes the product in the reduced symmetric algebra \(\overline{S}(V)\) of \(V\).

2. Preliminaries

In this section, we first recall the graded Lie algebra from \([9]\), whose Maurer-Cartan elements are relative Rota Baxter operators. Then, we recall the notion of \(V\)-data and induced \(L_\infty\)-algebras via higher derived brackets \([31]\).

2.1. Relative Rota-Baxter operators. Let \(A = (A, \mu)\) be an associative algebra. We denote the multiplication \(\mu: A \otimes A \to A\) simply by \(\mu(a, b) = ab\) for \(a, b \in A\). An \(A\)-bimodule is a vector space \(M\) together with two bilinear maps \(l: A \otimes M \to M, (a, m) \mapsto am\) and \(r: M \otimes A \to M, (m, a) \mapsto ma\) (called left and right \(A\)-actions, respectively) satisfying the following identities

\[
(ab)m = a(bm), \quad (ma)b = m(ab), \quad \text{and} \quad (am)b = a(mb), \quad \text{for all} \quad a, b \in A, \quad m \in M.
\]

The associative algebra \(A\) itself is an \(A\)-bimodule, where algebra multiplication in \(A\) gives the left and right actions on itself. This \(A\)-bimodule structure on \(A\) is called the adjoint \(A\)-bimodule.

For any vector space \(A\), consider the graded vector space \(\bigoplus_{n \geq 0} \text{Hom}(A^{\otimes n+1}, A)\) of multilinear maps. There is a graded Lie bracket, called the Gerstenhaber bracket \([13]\) on \(\bigoplus_{n \geq 0} \text{Hom}(A^{\otimes n+1}, A)\) given by

\[
[f, g] := f \circ g - (-1)^{mn} g \circ f, \quad \text{where}
\]

\[
(f \circ g)(a_1, \ldots, a_{m+n+1}) := \sum_{i=1}^{m+1} (-1)^{(i-1)n} f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+n}), a_{i+n+1}, \ldots, a_{m+n+1}),
\]

for \(f \in \text{Hom}(A^{\otimes m+1}, A)\) and \(g \in \text{Hom}(A^{\otimes n+1}, A)\). Note that \((A, \mu)\) is an associative algebra if and only if \(\mu \in \text{Hom}(A^{\otimes 2}, A)\) is a Maurer-Cartan element in the above graded Lie algebra. The coboundary operator

\[
\delta_{\text{Hoch}}(f) = (-1)^n f, \quad \text{for all} \quad f \in \text{Hom}(A^{\otimes n}, A).
\]

induced by the Maurer-Cartan element \(\mu\) is called the Hochschild coboundary operator. The cohomology of the Hochschild cochain complex \((C_{\text{Hoch}}^\bullet(A) := \bigoplus_{n \geq 1} \text{Hom}(A^{\otimes n}, A), \delta_{\text{Hoch}})\) is called the Hochschild cohomology and it is denoted by \(H_{\text{Hoch}}^\bullet(A)\).

2.1. Definition \([29]\). Let \(A\) be an associative algebra and \(M\) be an \(A\)-bimodule. A linear map \(T: M \to A\) is said to be a relative Rota-Baxter operator on \(M\) over the algebra \(A\) if \(T\) satisfies

\[
T(m)T(n) = T(T(m)n + nT(n)), \quad \text{for all} \quad m, n \in M.
\]

It follows from the Definition 2.1 that a Rota-Baxter operator on \(A\) is a relative Rota-Baxter operator on the adjoint \(A\)-bimodule. A relative Rota-Baxter associative algebra is a triple \((A, M, T)\) consisting of an associative algebra \(A\), an \(A\)-bimodule \(M\), and a relative Rota-Baxter operator \(T: M \to A\). A Rota-Baxter algebra is a relative Rota-Baxter algebra with respect to the adjoint \(A\)-bimodule.

Let \((A, M, T)\) be a relative Rota-Baxter algebra, where the associative product in \(A\) is given by \(\mu\), and the \(A\)-bimodule actions on \(M\) are given by \(l\) and \(r\). If we need to emphasize the underlying structure maps, we alternatively use the notation \(((A, \mu), (M, l, r), T)\) to denote the relative Rota-Baxter algebra \((A, M, T)\) with structure maps.

2.2. Definition. Let \((A, M, T)\) and \((A', M', T')\) be two relative Rota-Baxter algebras. A homomorphism of relative Rota-Baxter algebras from \((A, M, T)\) to \((A', M', T')\) is a pair \((\varphi, \psi)\) of an algebra homomorphism...
\(\varphi : A \to A'\) and a linear map \(\psi : M \to M'\) satisfying

\[
T' \circ \psi = \varphi \circ T, \quad (\psi(\imath m)) = (\varphi(a))\psi(m), \quad \text{and} \quad \psi(\imath ma) = \psi(m)\varphi(a), \quad \text{for all } a \in A, \ m \in M.
\]

The morphism \((\varphi, \psi)\) is called an isomorphism if \(\varphi\) and \(\psi\) are linear isomorphisms.

Let \(A\) be an associative algebra and \(M\) be an \(A\)-bimodule. Let \(\mu : A \otimes A \to A\) be the associative product in \(A\), and the linear maps \(l : A \otimes M \to M, \ r : M \otimes A \to M\) be left and right \(A\)-actions on \(M\). Then, the sum \(\mu + l + r \in \text{Hom}(A \oplus M)^{\otimes 2}, A \otimes M)\).

Let us consider a graded vector space \(\bigoplus_{n \geq 1} \text{Hom}(M^{\otimes n}, A)\) with the bracket

\[
[P, Q] = (-1)^{m-1}[[\mu + l + r], P]Q, \quad \text{for } P \in \text{Hom}(M^{\otimes m}, A) \text{ and } Q \in \text{Hom}(M^{\otimes n}, A).
\]

Here, the bracket \([ \ , \ ]\) on the right hand side is the Gerstenhaber bracket on the graded vector space \(\bigoplus_{n \geq 0} \text{Hom}((A \otimes M)^{\otimes n+1}, A \otimes M)\) (see [9] for more details). With the above notations, the main result of [9] can be stated as follows.

2.3. Theorem. Let \(A\) be an associative algebra and \(M\) be an \(A\)-bimodule. Then,

\begin{enumerate}[(i)]
  \item The graded vector space \(\bigoplus_{n \geq 1} \text{Hom}(M^{\otimes n}, A)\) with the bracket \([ \ , \ ]\) is a graded Lie algebra. A linear map \(T : M \to A\) is a relative Rota-Baxter operator on \(M\) over the algebra \(A\) if and only if \(T \in \text{Hom}(M, A)\) is a Maurer-Cartan element in the graded Lie algebra \(\bigoplus_{n \geq 1} \text{Hom}(M^{\otimes n}, A), [ \ , \ ]\). Consequently, a relative Rota-Baxter operator \(T\) induces a differential \(d_T := [T, \ ]\), which makes the graded Lie algebra \((\bigoplus_{n \geq 1} \text{Hom}(M^{\otimes n}, A), [ \ , \ ]\)) into a differential graded Lie algebra with the differential \(d_T\).
  \item For a relative Rota-Baxter operator \(T : M \to A\) and a linear map \(T' : M \to A\), the sum \(T + T'\) is also a relative Rota-Baxter operator if and only if \(T'\) is a Maurer-Cartan element in the differential graded Lie algebra \((\bigoplus_{n \geq 1} \text{Hom}(M^{\otimes n}, A), d_T, [ \ , \ ]\)). \hfill \square
\end{enumerate}

More precisely, an explicit description of the differential \(d_T = [T, \ ]\) is given by

\[
d_T P(m_1, m_2, \ldots, m_{n+1}) = T(P(m_1, m_2, \ldots, m_n)m_{n+1}) - (-1)^n T(m_1 P(m_2, \ldots, m_{n+1}))
\]

\[
- (-1)^n \left\{ \sum_{i=1}^{n} P(m_1, \ldots, m_{i-1}, T(m_i)m_{i+1}, m_{i+2}, \ldots, m_{n+1}) - \sum_{i=1}^{n} P(m_1, \ldots, m_{i-1}, m_i T(m_{i+1}), m_{i+2}, \ldots, m_{n+1}) \right\}
\]

\[
+ (-1)^{n+1} T(m_1)P(m_2, \ldots, m_{n+1}) - P(m_1, \ldots, m_n)T(m_{n+1}),
\]

for all \(P \in \text{Hom}(M^{\otimes n}, A)\) and \(m_1, m_2, \ldots, m_{n+1} \in M\).

Let us define a cochain complex \((C^\bullet(T), d_T)\) with the graded vector space \(C^\bullet(T) = \bigoplus_{n \geq 0} C^n(T)\), where \(C^0(T) = C^1(T) := 0\) and \(C^n(T) := \text{Hom}(M^{\otimes n-1}, A)\), for \(n \geq 2\). The associated cohomology is called the cohomology of \(T\) and denoted by \(H^\bullet(T)\).

2.2. \(L_\infty\)-algebras and Voronov’s construction.\ In this subsection, we give necessary background on \(L_\infty\)-algebras and their construction form a \(V\)-data [31].

The notion of \(L_\infty\)-algebras was introduced by Lada and Stasheff [20] as a homotopy analogy of (graded) Lie algebras. Throughout the paper, we use a ‘shifted’ version of \(L_\infty\)-algebras (called \(L_\infty[1]\)-algebras) in which all multilinear maps are graded symmetric and have degree 1.

2.4. Definition. An \(L_\infty[1]\)-algebra is a graded vector space \(W = \bigoplus_{i \in \mathbb{Z}} W_i\) together with a collection of degree 1 multilinear maps \(\{l_k : W^{\otimes k} \to W\}_{k \geq 1}\) satisfying

\begin{enumerate}[(i)]
  \item graded symmetry: for \(k \geq 1\),
    \[
l_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \epsilon(\sigma) l_k(x_1, \ldots, x_k), \quad \text{for any } \sigma \in S_k,
    \]
  \item shifted higher Jacobi identities: for each \(n \geq 1\),
    \[
    \sum_{i+j+k=n+1} \sum_{\sigma \in S_{(i,n-i)}} \epsilon(\sigma) l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0.
    \]
\end{enumerate}
Here, \( S_{k(n-i)} \) denotes the set of \((i, n - i)\)-shuffles in the permutation group \( S_n \). For any permutation \( \sigma \in S_k \) and \( x_1, x_2, \ldots, x_k \in V \), \( \epsilon(\sigma) := \epsilon(\sigma, x_1, x_2, \ldots, x_k) \) denotes the Koszul sign.

2.5. **Remark.** The equivalence between \( L_\infty \)-algebras and \( L_\infty[1] \)-algebras is given by a degree shift. More precisely, an \( L_\infty \)-algebra structure (in the sense of [26]) on a graded vector space \( V = \sum_{i \in \mathbb{Z}} V_i \) is equivalent to an \( L_\infty[1] \)-algebra structure on the graded vector space \( W := V[1] \), where \( V_i = (V[1])_{i-1} = V_{i+1} \). The correspondence between multi-brackets of these two structures is given by the décalage isomorphisms

\[
V^{\otimes n}[n] \cong (V[1])^{\otimes n}, \quad v_1 \ldots v_n \mapsto (-1)^{(n-1)|v_1 + \cdots + v_{n-1} + |v_n - 1)}v_1 \ldots v_n,
\]

where \( |v_i| \) denotes the degree of homogeneous element \( v_i \) in the graded vector space \( V \).

The notion of filtered \( L_\infty[1] \)-algebras was introduced by Getzler [14], which ensures the convergence of certain infinite sums. For the results in this paper, we only need the following weaker notion [21].

2.6. **Definition** ([21]). A weakly filtered \( L_\infty[1] \)-algebra is a triple \((W, \{ l_k \}_{k \geq 1}, F_* W)\), where \((W, \{ l_k \}_{k \geq 1}) \) is an \( L_\infty[1] \)-algebra and \( F_* W \) is a descending filtration of \( W \) such that \( W = F_1 W \supset \cdots \supset F_n W \supset \cdots \) and the following conditions are satisfied:

\( \text{(i)} \) there exists \( n \geq 1 \) such that \( l_k(W, \ldots, W) \subset F_n W \), for all \( k \geq n \) and

\( \text{(ii)} \) \( W \) is complete with respect to this filtration. Thus, there is an isomorphism of graded vector spaces \( W \cong \varinjlim W/F_n W \).

2.7. **Definition.** Let \((W, \{ l_k \}_{k \geq 1}, F_* W)\) be a weakly filtered \( L_\infty[1] \)-algebra. An element \( \alpha \in W^0 \) is called a Maurer-Cartan element if \( \alpha \) satisfies

\[
\sum_{k=1}^{\infty} \frac{1}{k!} l_k(\alpha, \alpha, \ldots, \alpha) = 0.
\]

The set of all Maurer-Cartan elements in \( W \) is denoted by \( \text{MC}(W) \).

It is known that a filtered \( L_\infty[1] \)-algebra can be twisted by a Maurer-Cartan element [14]. Similarly, for weakly filtered \( L_\infty[1] \)-algebra, we have the following result from [21].

2.8. **Theorem** ([21]). Let \((W, \{ l_k \}_{k \geq 1}, F_* W)\) be a weakly filtered \( L_\infty[1] \)-algebra and \( \alpha \in W^0 \) be a Maurer-Cartan element. Then, \((W, \{ l_k^\alpha \}_{k \geq 1}, F_* W)\) is a weakly filtered \( L_\infty[1] \)-algebra, where

\[
l^\alpha_k(x_1, x_2, \ldots, x_k) = \sum_{n=0}^{\infty} \frac{1}{n!} l_{k+n}^\alpha(\alpha, \ldots, \alpha, x_1, x_2, \ldots, x_k).
\]

Next, let us recall the construction of an \( L_\infty[1] \)-algebra from a \( V \)-data, which is given by Voronov’s higher derived brackets [31].

2.9. **Definition.** A \( V \)-data is a quadruple \((L, a, P, \Delta)\), where \( L \) is a graded Lie algebra (with graded Lie bracket \([ , , ]\)), \( a \subset L \) is an abelian graded Lie subalgebra, \( P : L \to a \) is a projection map whose kernel is a graded Lie subalgebra of \( L \), and \( \Delta \in \text{Ker}(P)_1 \) satisfying \([\Delta, \Delta] = 0\).

For a \( V \)-data \((L, a, P, \Delta)\), the following theorem yields an \( L_\infty[1] \)-algebra structure on \( a \).

2.10. **Theorem** ([31]). Let \((L, a, P, \Delta)\) be a \( V \)-data. Then, the graded vector space \( a \subset L \) together with the operations

\[
l_k(a_1, \ldots, a_k) := P[\cdots[[\Delta, a_1], a_2], \ldots, a_k], \quad \text{for } k \geq 1
\]

is an \( L_\infty[1] \)-algebra.

In fact, there is a larger \( L_\infty[1] \)-algebra in which the \( L_\infty[1] \)-algebra \( a \), defined by Theorem 2.10 is an \( L_\infty[1] \)-subalgebra.
2.11. Theorem. Let \((L, a, P, \Delta)\) be a \(V\)-data. Then, the graded vector space \(L[1] \oplus a\) is an \(L_\infty[1]\)-algebra with multilinear operations

\[
l_1(x[1], a) = (\Delta, x[1]), \quad P(x + [\Delta, a]),
\]

\[
l_2(x[1], y[1]) = (\Delta, x, y)[1],
\]

\[
l_k(x[1], a_1, \ldots, a_{k-1}) = P\left(\cdots [[[x, a_1], a_2], \ldots, a_k]\right), \quad \text{for } k \geq 2,
\]

\[
l_k(a_1, \ldots, a_k) = P\left(\cdots [[[\Delta, a_1], a_2], \ldots, a_k]\right), \quad \text{for } k \geq 2.
\]

Here, \(x, y\) are homogeneous elements in \(L\) and \(a, a_1, \ldots, a_k\) are homogeneous elements in \(a\). Up to the permutations of the above entries, all other multilinear operations vanish.

2.12. Remark. Let \((L, a, P, \Delta)\) be a \(V\)-data. If \(L'\) is a graded Lie subalgebra of \(L\) such that \([\Delta, L'] \subset L'\), then \(L'[1] \oplus a\) is an \(L_\infty[1]\)-subalgebra of the \(L_\infty[1]\)-algebra \(L[1] \oplus a\) (see [12] for more details).

3. Cohomology of an \AssBimod-pair

Let \(A\) be an associative algebra and \(M\) be an \(A\)-bimodule. Then, we call the pair \((A, M)\) an \AssBimod-pair. The aim of this section is to find a graded Lie algebra associated to the vector spaces \(M\) such that the Maurer-Cartan elements in the graded Lie algebra are in bijective correspondence with the \AssBimod-pair structures on the pair \((A, M)\). Subsequently, we define cohomology and deformations of an \AssBimod-pair.

First, we recall the following notations from [30]. Let \(A_1\) and \(A_2\) be two vector spaces. For any linear map \(f : A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_n} \rightarrow A_j\) with \(i_1, \ldots, i_n, j \in \{1, 2\}\), we define a map \(\hat{f} \in \text{Hom}((A_1 \oplus A_2)^{\otimes n}, A_1 \oplus A_2)\) by

\[
\hat{f} = \begin{cases} 
    f & \text{on } A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_n}, \\
    0 & \text{otherwise}.
\end{cases}
\]

The map \(\hat{f}\) is called a horizontal lift of \(f\) or simply a lift of \(f\). In particular, the lifts of the linear maps \(\mu : A_1 \otimes A_1 \rightarrow A_1\), \(l : A_1 \otimes A_2 \rightarrow A_2\), and \(r : A_2 \otimes A_1 \rightarrow A_2\) are respectively given by

\[
\hat{\mu}((a, u), (b, v)) = (\mu(a, b), 0),
\]

\[
\hat{l}((a, u), (b, v)) = (0, l(a, v)),
\]

\[
\hat{r}((a, u), (b, v)) = (0, r(a, b)), \quad \text{for all } a, b \in A_1, u, v \in A_2.
\]

Let \(A^{k,l}\) be the direct sum of all possible \((k + l)\)-tensor powers of \(A_1\) and \(A_2\) in which \(A_1\) (respectively, \(A_2\)) appears \(k\) times (respectively, \(l\) times). For instance,

\[
A^{2,0} := A_1 \otimes A_1, \quad A^{1,1} := (A_1 \otimes A_2) \oplus (A_2 \otimes A_1), \quad A^{0,2} := A_2 \otimes A_2,
\]

and

\[
A^{1,2} := (A_1 \otimes A_2 \otimes A_2) \oplus (A_2 \otimes A_1 \otimes A_2) \oplus (A_2 \otimes A_2 \otimes A_1).
\]

Note that there is a vector space isomorphism \((A_1 \oplus A_2)^{\otimes n} \cong \oplus_{k+l=n} A^{k,l}\). Therefore, we have the isomorphism (by the horizontal lift)

\[
\text{Hom}((A_1 \oplus A_2)^{\otimes n+1}, A_1 \oplus A_2) \cong \sum_{k+l=n+1} \text{Hom}(A^{k,l}, A_1) \oplus \sum_{k+l=n+1} \text{Hom}(A^{k,l}, A_2)
\]

We say that a linear map \(f \in \text{Hom}((A_1 \oplus A_2)^{\otimes n+1}, A_1 \oplus A_2)\) has bidegree \(k|l\) with \(k + l = n\) if

\[
f(A^{k+l,0}) \subset A_1, \quad f(A^{k+l,0}) \subset A_2, \quad \text{and } f \text{ is zero otherwise}.
\]

Let us denote the bidegree of \(f\) by \(||f|| = k|l|\). In general, multilinear maps may not have a bidegree. A multilinear map \(f\) is called homogeneous if \(f\) has a bidegree. The set of all homogeneous multilinear maps of bidegree \(k|l\) is denoted by \(C^{k,l}(A_1 \oplus A_2, A_1 \oplus A_2)\). Let us observe that

\[
C^{k,0}(A_1 \oplus A_2, A_1 \oplus A_2) \cong \text{Hom}(A_1^{[k+1]}, A_1) \oplus \text{Hom}(A^{1,0}, A_2),
\]

\[
C^{0,1}(A_1 \oplus A_2, A_1 \oplus A_2) \cong \text{Hom}(A_2^{[k+1]}, A_1) \oplus \text{Hom}(A^{0,1}, A_2),
\]

The following result has been proved in [30, Proposition 2.6].
3.1. Proposition. If $f \in \text{Hom}((A_1 \oplus A_2)^{\otimes m+1}, A_1 \oplus A_2)$ has bidegree $k_f$ and $g \in \text{Hom}((A_1 \oplus A_2)^{\otimes n+1}, A_1 \oplus A_2)$ has bidegree $k_g$, then the Gerstenhaber bracket $[f, g]$ is homogeneous and has the bidegree $(k_f + k_g)(l_f + l_g)$.

As a consequence of the above proposition, the following result holds.

3.2. Proposition. The graded subspace $\bigoplus_{l \geq 1} C^{-1+l}(A_1 \oplus A_2, A_1 \oplus A_2)$ is an abelian subalgebra of the graded Lie algebra $C^{\ast+1}(A_1 \oplus A_2, A_1 \oplus A_2)$.

Proof. If $||f|| = -1|l_f|$ and $||g|| = -1|l_g|$, then by Proposition 3.1, we have $[f, g] = 0$. Hence, the result follows.

Also, by a direct application of Proposition 3.1, it follows that

3.3. Proposition. The graded vector space $\bigoplus_{k \geq 1} C^k(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \mathfrak{A}_1 \oplus \mathfrak{A}_2)$ together with the Gerstenhaber bracket is a graded Lie algebra.

Let $A$ and $M$ be two vector spaces. Suppose there are linear maps $\mu : A \otimes A \to A$, $l : A \otimes M \to M$, and $r : M \otimes A \to M$. Then, all these maps $\mu, l, r \in C^1(A \otimes M, A \otimes M)$, hence the sum

$$\pi := \mu + l + r \in C^1(A \otimes M, A \otimes M).$$

The next result shows that AssBimod-pairs are in bijective correspondence with Maurer-Cartan elements in the graded Lie algebra $\bigoplus_{k \geq 1} C^k(A \otimes M, A \otimes M)$.

3.4. Proposition. With the above notations, $\mu$ defines an associative product on $A$ and $l, r$ defines an $A$-bimodule structure on $M$ if and only if $\pi = \mu + l + r \in C^1(A \otimes M, A \otimes M)$ is a Maurer-Cartan element in the graded Lie algebra $\bigoplus_{k \geq 1} C^k(A \otimes M, A \otimes M)$.

Proof. The proof of the proposition is similar to [9, Proposition 2.11].

It follows that a AssBimod-pair $(A, M)$ induces a differential $d_{\mu+l+r} := [\mu + l + r, ]$ on the graded vector space $\bigoplus_{k \geq 1} C^k(A \otimes M, A \otimes M)$, which makes $(\bigoplus_{k \geq 1} C^k(A \otimes M, A \otimes M), d_{\mu+l+r}, [ , ])$ into a differential graded Lie algebra. Moreover, using the standard Maurer-Cartan theory, we can deduce the following theorem.

3.5. Theorem. Let $(A, M)$ be an AssBimod-pair and $\mu + l + r \in C^1(A \otimes M, A \otimes M)$ be the corresponding Maurer-Cartan element. Then, for linear maps $\mu' : A \otimes A \to A$, $l' : A \otimes M \to M$, and $r' : M \otimes A \to M$, the sum $(\mu + l + r) + (l' + r')$ corresponds to an AssBimod-pair structure on $(A, M)$ if and only if $\mu' + l' + r'$ is a Maurer-Cartan element in the differential graded Lie algebra $(\bigoplus_{k \geq 1} C^k(A \otimes M, A \otimes M), d_{\mu+l+r}, [ , ])$. □

We now define the cohomology of an AssBimod-pair $(A, M)$. Let us define the 0-th cochain group $\mathcal{C}^0(A, M)$ to be 0 and for $n \geq 1$, the $n$-th cochain group

$$\mathcal{C}^n(A, M) = C^{(n-1)1}(A \otimes M, A \otimes M) = \text{Hom}(A^{\otimes n}, A) \oplus \text{Hom}(A^{n-1, 1}, M).$$

Here, $A^{n-1, 1}$ is the direct sum of all possible $n$-tensor powers of $A$ and $M$ in which $A$ appears $n-1$ times (and thus, $M$ appears only once). Finally, we define a differential $d : \mathcal{C}^n(A, M) \to \mathcal{C}^{n+1}(A, M)$, which is induced from the Maurer-Cartan element, i.e.

$$(4) \quad df = (-1)^{n-1}[\mu + l + r, f], \quad \text{for } f \in \mathcal{C}^n(A, M).$$

The cohomology of the cochain complex $(\mathcal{C}^\ast(A, M), d)$ is called the cohomology of the AssBimod-pair $(A, M)$.

4. L∞-algebras associated to relative Rota-Baxter algebras

Let $A$ and $M$ be two vector spaces. Then, $L = (\bigoplus_{n \geq 0} \text{Hom}((A \otimes M)^{\otimes n+1}, A \otimes M), [ , ])$ is a graded Lie algebra where the graded Lie bracket is given by the Gerstenhaber bracket. It has been observed in Proposition 3.2 that the graded subspace

$$\mathfrak{a} = \bigoplus_{n \geq 0} C^{-1|n+1}(A \otimes M, A \otimes M) = \bigoplus_{n \geq 0} \text{Hom}(M^{\otimes n+1}, A).$$
is an abelian subalgebra of $L$. Let $P : L \to a$ be the projection onto the subspace $a$ and take $\triangle = 0 \in \ker(P)$. It is also easy to see that the kernel of $P$ is a graded Lie subalgebra of $L$. Hence, with all these notations, the quadruple $(L, a, P, \triangle)$ is a $V$-data. Thus, Theorem 2.11 yields the following result.

4.1. Theorem. There is an $L_\infty[1]$-algebra on the graded vector space $L[1] \oplus a$ with structure maps

$$l_1(Q[1], \theta) = P(Q),$$

$$l_2(Q[1], Q'[1]) = (-1)^{|Q|}[Q, Q'][1],$$

$$l_k(Q[1], \theta_1, \ldots, \theta_{k-1}) = \frac{1}{k!}P([\cdots [[Q, \theta_1], \theta_2], \ldots, \theta_{k-1}],$$

for homogeneous elements $Q, Q' \in L$ and $\theta, \theta_1, \ldots, \theta_{k-1} \in a$.

4.2. Remark. From Proposition 3.3, $L' = \bigoplus_{n \geq 0} C^{n,0}(A \oplus M, A \oplus M), [ , ]$ is a graded Lie subalgebra of the graded Lie algebra $L$. Now, Remark 2.12 implies that there is an $L_\infty[1]$-algebra on the graded vector space $L'[1] \oplus a$, whose structure maps are given by

$$l_2(Q[1], Q'[1]) = (-1)^{|Q|}[Q, Q'][1],$$

$$l_k(Q[1], \theta_1, \ldots, \theta_{k-1}) = \frac{1}{k!}P([\cdots [[Q, \theta_1], \theta_2], \ldots, \theta_{k-1}],$$

for homogeneous elements $Q, Q' \in L'$ and $\theta, \theta_1, \ldots, \theta_{k-1} \in a$. Furthermore, this $L_\infty$-algebra is weakly filtered with $n = 3$ and the filtration is given by

$$\mathcal{F}_1 = L'[1] \oplus a, \mathcal{F}_2 = P[L'[1] \oplus a, a] \text{ and } \mathcal{F}_k = P[\cdots [L'[1] \oplus a, a], \ldots, a] \text{ for } k \geq 3.$$

Note that, the degree 0 component of the graded vector space $L'[1] \oplus a$ is given by

$$(L'[1] \oplus a)_0 = L'_1 \oplus a_0 = \frac{\text{Hom}(A^{\otimes 2}, A) \oplus \text{Hom}(A \otimes M, M) \oplus \text{Hom}(M \otimes A, M) \oplus \text{Hom}(M, A)}{L'_1 \oplus a_0}.$$

Let $A$ and $M$ be two vector spaces. Suppose there are maps

$$(5) \quad \mu \in \text{Hom}(A^{\otimes 2}, A), \quad l \in \text{Hom}(A \otimes M, M), \quad r \in \text{Hom}(M \otimes A, M) \quad \text{and} \quad T \in \text{Hom}(M, A).$$

Consider the element $\pi = \mu + l + r \in C^{1,0}(A \oplus M, A \oplus M) = L'_1$. Then we have $(\pi[1], T) \in (L'[1] \oplus a)_0$.

4.3. Theorem. With the above notations, the triple $(\pi, \mu, (M, l, r), T)$ is a relative Rota-Baxter algebra if and only if $(\pi[1], T)$ is a Maurer-Cartan element in the $L_\infty[1]$-algebra $(L'[1] \oplus a, \{l_k\}_{k \geq 1})$.

Proof. The triple $((A, \mu), (M, l, r), T)$ is a relative Rota-Baxter algebra if and only if $(A, M)$ is a $\text{AssBimod}$-pair and $T : M \to A$ is a relative Rota-Baxter operator on $M$ over the algebra $A$. Equivalently, by Theorem 2.3 and Proposition 3.4, $((A, \mu), (M, l, r), T)$ is a relative Rota-Baxter algebra if and only if

$$(6) \quad [\mu + l + r, \mu + l + r] = 0 \quad \text{and} \quad [T, \mu] = [[\mu + l + r, T], T] = 0.$$

If $\pi := \mu + l + r$, then Remark 4.2 implies that

$$l_2(\pi[1], T), (\pi[1], T)) = -[\pi, \pi][1],$$

$$l_3((\pi[1], T), (\pi[1], T), (\pi[1], T)) = [[\pi, T], T] = [T, T].$$

Let us consider the sum

$$\sum_{k=1}^{\infty} \frac{1}{k!}l_k((\pi[1], T), (\pi[1], T), \ldots, (\pi[1], T)) = \frac{1}{2}l_2((\pi[1], T), (\pi[1], T)) + \frac{1}{6}l_3((\pi[1], T), (\pi[1], T), (\pi[1], T))$$

$$= \left(- \frac{1}{2}[\pi, \pi][1], \frac{1}{6}[[\pi, T], T]\right) = \left(- \frac{1}{2}[\pi, \pi][1], \frac{1}{6}[T, T]\right).$$

From equation (6) and equation (7), it is clear that $((A, \mu), (M, l, r), T)$ is a relative Rota-Baxter algebra if and only if

$$\sum_{k=1}^{\infty} \frac{1}{k!}l_k((\pi[1], T), (\pi[1], T), \ldots, (\pi[1], T)) = 0.$$

Hence, the theorem holds true. \qedsymbol
The above theorem implies that relative Rota-Baxter algebras can be interpreted as Maurer-Cartan elements in a $L_\infty[1]$-algebra. This allows us to construct a new $L_\infty[1]$-algebra twisted by the Maurer-Cartan element associated with a given relative Rota-Baxter algebra. More precisely, let $((A, \mu), (M, l, r), T)$ be a relative Rota-Baxter algebra. Consider the Maurer-Cartan element $\alpha = (\pi[1], T) \in (L'[1] \oplus a)_0$ in the $L_\infty[1]$-algebra $(L'[1] \oplus a)_0$. By applying Theorem 2.8 to this case, we get the twisted $L_\infty[1]$-algebra which controls deformations of the relative Rota-Baxter algebra.

4.4. Theorem. Let $((A, \mu), (M, l, r), T)$ be a relative Rota-Baxter algebra. Then there is a twisted $L_\infty[1]$-algebra $(L'[1] \oplus a, \{l_k(\pi[1], T)\}_{k \geq 1})$. Moreover, for any linear maps $\mu', l', r', T'$ as of (5), the triple

$$((A, \mu + \mu'), (M, l + l', r + r'), T + T')$$

is a relative Rota-Baxter algebra if and only if $(\pi'[1], T')$ is a Maurer-Cartan element in the $L_\infty[1]$-algebra $(L'[1] \oplus a, \{l_k(\pi'[1], T')\}_{k \geq 1})$, where $\pi' = \mu' + l' + r' \in L'_1$.

Let $A$ be an associative algebra and $M$ be an $A$-bimodule. Suppose $T : M \to A$ is a relative Rota-Baxter operator which makes the triple $(A, M, T)$ a relative Rota-Baxter algebra. We have already seen various graded Lie algebras (hence $L_\infty[1]$-algebras by degree shift) and $L_\infty[1]$-algebras.

- In Proposition 3.3, we find a graded Lie algebra structure on $\bigoplus_{n \geq 0} C^n(A \oplus M, A \oplus M)$, hence an $L_\infty[1]$-algebra on $\bigoplus_{n \geq 0} (\bigoplus_{\geq 0} C^n(A \oplus M, A \oplus M))[1]$.
- In Theorem 2.3, we have a graded Lie algebra structure on $\bigoplus_{n \geq 1} \Hom(M \otimes^n, A)$, hence get an $L_\infty[1]$-algebra.
- Finally, in Theorem 4.4, we obtain an $L_\infty$-algebra $(L'[1] \oplus a, \{l_k(\pi[1], T)\}_{k \geq 1})$.

Similar to the Lie algebra case [21, Theorem 3.16], one can prove the following relationship between these $L_\infty[1]$-algebras.

4.5. Theorem. There is a short exact sequence of $L_\infty[1]$-algebras

$$0 \to (\bigoplus_{n \geq 1} \Hom(M \otimes^n, A))[1] \xrightarrow{i} (L'[1] \oplus a)[1] \xrightarrow{p} \bigoplus_{n \geq 0} C^n(A \oplus M, A \oplus M)[1] \to 0$$

by strict morphism of $L_\infty[1]$-algebras, where $i(\theta) = (0, \theta)$ and $p(f[1], \theta) = f[1]$.

4.1. Cohomology of relative Rota-Baxter algebras. Using the twisted $L_\infty[1]$-algebra constructed in Theorem 4.4, we define the cohomology of a relative Rota-Baxter algebra $(A, M, T)$. This cohomology is related to the cohomology of the relative Rota-Baxter operator $T$ and the cohomology of the AssBimod pair $(A, M)$ by a long exact sequence.

Let $((A, \mu), (M, l, r), T)$ be a relative Rota-Baxter algebra. Consider the twisted $L_\infty[1]$-algebra

$$(L'[1] \oplus a, \{l_k(\pi[1], T)\}_{k \geq 1})$$

as given in Theorem 4.4. Then it follows from the definition of an $L_\infty[1]$-algebra that $l_1^1(\pi[1], T) \circ l_1^1(\pi[1], T) = 0$. In other words, $l_1^1(\pi[1], T)$ is a differential. In the following, we will use this differential to define the cohomology of the relative Rota-Baxter algebra $(A, M, T)$.

Define the space of 0-cochains $C^0(A, M, T)$ to be 0, and the space of 1-cochains $C^1(A, M, T)$ to be $\Hom(A, A) \oplus \Hom(M, M)$. For $n \geq 2$, the space of $n$-cochains $C^n(A, M, T)$ to be defined by

$$C^n(A, M, T) = C^n(A, M) \oplus C^n(T)$$

$$= (\Hom(A^\otimes n, A) \oplus \Hom(A^{n-1}, M)) \oplus \Hom(M^\otimes n, A).$$

To define the coboundary, we observe that if $(f, P) \in C^n(A, M, T)$, then $(f[1], P) \in (L'[1] \oplus a)_{n-2}$. We define the coboundary $D : C^n(A, M, T) \to C^{n+1}(A, M, T)$ by

$$D(f, P) = (-1)^{n-2} l_1^1(\pi[1], T)(f[1], P).$$
Then, it follows that $D^2 = 0$. The explicit description of $D$ is given by

$$D(f, P) = (−1)^{n−2} \left(−[π, f]. [[π, T], P] + \frac{1}{n!} [\cdots [[f, T], T], \ldots, T] \right).$$

It follows that the coboundary map $D : C^n(A, M, T) → C^{n+1}(A, M, T)$ can also be written as

$$D(f, P) = (\partial f, (−1)^{n}d_T P + Ω f),$$

where $\partial f = [μ + l + r, f]$ (given by equation (4)), $d_T P = [T, P]$ (given in Theorem 2.3), and $Ω$ is defined by

$$(Ω f)(a_1, \ldots, a_n) = (−1)^n (f(Ta_1, \ldots, Ta_n) − \sum_{i=1}^{n} T f(Ta_1, \ldots, a_i, \ldots, Ta_n)).$$

Therefore, $(C^∗(A, M, T), D)$ is a cochain complex. The homology of this complex is called the homology of the relative Rota-Baxter algebra $(A, M, T)$ and denoted by $H^∗(A, M, T)$.

### 4.6. Theorem

Let $(A, M, T)$ be a relative Rota-Baxter algebra. Then there is a long exact sequence of cohomology groups

$$\cdots → H^n(T) → H^n(A, M, T) → H^n(A, M) → \cdots,$$

where the connecting homomorphism $δ^n$ is given by $δ^n([a]) = [h_T(a)]$, for $[a] ∈ H^n(A, M)$. □

### 4.2. Cohomology of Rota-Baxter algebras

In this subsection, we focus on Rota-Baxter algebras, i.e. relative Rota-Baxter algebras with respect to the adjoint representation. We deduce cohomology of Rota-Baxter algebras following the cohomology of relative Rota-Baxter algebras modulo certain modifications.

Let $(A, T)$ be a Rota-Baxter algebra. We define the space of $0$-cochains $C^0_{RB}(A, T)$ to be $0$, and the space of $1$-cochains $C^1_{RB}(A, T)$ to be $\text{Hom}(A, A)$. The space of $n$-cochains $C^n_{RB}(A, T)$, for $n ≥ 2$, is defined by

$$C^n_{RB}(A, T) := \text{Hom}(A^⊙n, A) ⊕ \text{Hom}(A^{⊙n−1}, A).$$

We define the coboundary map using the following inclusion map

$$i : C^n_{RB}(A, T) → C^n(A, A, T), \quad (f, P) → (f, f, P).$$

Here, $C^n(A, A, T)$ is the $n$-th cochain group of the relative Rota-Baxter algebra $(A, A, T)$. Let us write an element $f ∈ \text{Hom}(A^⊙n, A) ⊕ \text{Hom}(A^{⊙n−1}, M)$ as a pair $(f_A, f_M)$ with the components $f_A ∈ \text{Hom}(A^⊙n, A)$ and $f_M ∈ \text{Hom}(A^{⊙n−1}, M)$. Then, we can write the differential $\partial : \text{Hom}(A^⊙n, A) ⊕ \text{Hom}(A^{⊙n−1}, M) → \text{Hom}(A^{⊙n+1}, A) ⊕ \text{Hom}(A^{⊙n−1}, M)$ as follows

$$\partial(f) = ((\partial f)_A, (\partial f)_M).$$

In the case, when $M = A$ is the adjoint $A$-bimodule, it is clear that

$$\partial(f)_A = \partial(f)_M = (−1)^{n−1}[μ, f] = δ_Hoch f.$$

Thus, we have the following proposition.

### 4.7. Proposition

The complex $(C^∗_{RB}(A, T), D)$ is a subcomplex of the cochain complex $(C^∗(A, A, T), D)$ associated to the relative Rota-Baxter algebra $(A, A, T)$. □

The induced coboundary map $D : C^n_{RB}(A, T) → C^{n+1}_{RB}(A, T)$ is explicitly given by

$$D(f, P) = (δ_{Hoch} f, (−1)^{n}d_T P + Ω f),$$

where $δ_{Hoch}$ is the Hochschild coboundary (given by equation (1)), $d_T P = [T, P]$ (given in Theorem 2.3), and $Ω$ is given by

$$(Ω f)(a_1, \ldots, a_n) = (−1)^n (f(Ta_1, \ldots, Ta_n) − \sum_{i=1}^{n} T f(Ta_1, \ldots, a_i, \ldots, Ta_n)).$$

The cohomology of the cochain complex $(C^∗_{RB}(A, T), δ)$ is called the cohomology of the Rota-Baxter algebra $(A, T)$. We denote the cohomology groups by $H^∗_{RB}(A, T)$. 
4.8. Remark. As a consequence of Theorem 4.6, there is a short exact sequence of cochain complexes

\[ 0 \longrightarrow (C^\bullet(T), \delta_T) \overset{i}{\longrightarrow} (C^\bullet_{\text{RB}}(A, T), D) \overset{p}{\longrightarrow} (C^\bullet_{Hoch}(A), \delta_{Hoch}) \longrightarrow 0, \]

where \( i(P) = (0, P) \) and \( p(f, P) = f \). Consequently, there is a long exact sequence in cohomology groups

\[ \cdots \longrightarrow H^n(T) \overset{H^n(i)}{\longrightarrow} H^n_{\text{RB}}(A, T) \overset{H^n(p)}{\longrightarrow} H^n_{Hoch}(A) \overset{\delta^n}{\longrightarrow} H^{n+1}(T) \longrightarrow \cdots, \]

where the connecting homomorphism \( \delta^n \) is given by \( \delta^n[a] = [\Omega a] \).

5. Deformations of relative Rota-Baxter algebras

In this section, we study \( R \)-deformations of a relative Rota-Baxter algebra, where \( R \) is a pro-Artinian \( K \)-algebra. In particular, we consider the case \( R = K[[t]]/(t^2) \), that is the infinitesimal deformations of a relative Rota-Baxter algebra. We show that equivalence classes of infinitesimal deformations of a relative Rota-Baxter algebra \((A, M, T)\) are in bijective correspondence with the cohomology classes in \( H^2(A, M, T) \).

Let \( R \) be a pro-Artinian ring. One may define \( R \)-relative Rota-Baxter algebras and morphisms between them similar to Definition 2.1 and Definition 2.2 by replacing the vector spaces and the linear maps by \( R \)-modules and \( R \)-linear maps. Since \( R \) is pro-Artinian \( K \)-algebra, there is an augmentation \( \epsilon : R \rightarrow K \). Thus, any relative Rota-Baxter algebra \((A, M, T)\) can be realised as an \( R \)-relative Rota-Baxter algebra, where the \( R \)-module structures on \( A \) and \( M \) are respectively given by

\[ ra = \epsilon(r)a, \quad ru = \epsilon(r)u, \quad \text{for } r \in R, a \in A, \text{ and } u \in M. \]

5.1. Definition. An \( R \)-deformation of a relative Rota-Baxter algebra \(((A, \mu), (M, l, r), T)\) consists of an \( R \)-algebra structure \( \mu_R \) on \( R \otimes_K A \), and two \( R \)-linear maps \( l_R : (R \otimes_K A) \otimes (R \otimes_K M) \rightarrow (R \otimes_K M) \), \( r_R : (R \otimes_K M) \otimes (R \otimes_K A) \rightarrow (R \otimes_K M) \), and an \( R \)-linear map \( T_R : R \otimes_K M \rightarrow R \otimes_K A \) which makes the triple \(((R \otimes_K A, \mu_R), (R \otimes_K M, l_R, r_R), T_R)\) into an \( R \)-relative Rota-Baxter algebra such that

\[ (\epsilon \otimes_K id_A, \epsilon \otimes_K id_M) : ((R \otimes_K A, \mu_R), (R \otimes_K M, l_R, r_R), T_R) \rightarrow ((A, \mu), (M, l, r), T) \]

is a morphism of \( R \)-relative Rota-Baxter algebras.

5.2. Definition. Let \(((R \otimes_K A, \mu_R), (R \otimes_K M, l_R, r_R), T_R)\) and \(((R \otimes_K A, \mu_R)', (R \otimes_K M, l'_R, r'_R), T'_R)\) be two \( R \)-deformations of a relative Rota-Baxter algebra \(((A, \mu), (M, l, r), T)\). Then, they are said to be equivalent if there exists an \( R \)-relative Rota-Baxter algebra isomorphism

\[ (\Phi, \Psi) : ((R \otimes_K A, \mu_R), (R \otimes_K M, l_R, r_R), T_R) \rightarrow ((R \otimes_K A, \mu_R)', (R \otimes_K M, l'_R, r'_R), T'_R), \]

satisfying \( (\epsilon \otimes_K id_A) \circ \Phi = (\epsilon \otimes_K id_A) \) and \( (\epsilon \otimes_K id_M) \circ \Psi = (\epsilon \otimes_K id_M) \).

Next, we explicitly describe \( R \)-deformations of a relative Rota-Baxter algebra for \( R = K[[t]]/(t^2) \). Such deformations are called infinitesimal deformations. Note that the augmentation \( \epsilon : K[[t]]/(t^2) \rightarrow K \) is given by \( \epsilon(f) = f(0) \), the evaluation of \( f \) at 0. Thus, an infinitesimal deformation of a relative Rota-Baxter algebra \(((A, \mu), (M, l, r), T)\) consists of maps \( \mu_0, \mu_1 \in \text{Hom}(A \otimes A, M) \), \( l_0, l_1 \in \text{Hom}(A \otimes M, M) \), \( r_0, r_1 \in \text{Hom}(M \otimes A, M) \), and \( T_0, T_1 \in \text{Hom}(M, A) \) such that the maps

\[ \mu_R := \mu_0 + t\mu_1, \quad l_R := l_0 + tl_1, \quad r_R := r_0 + tr_1, \quad T_R : T_0 + tT_1 \]

makes \(((R \otimes_K A, \mu_R), (R \otimes_K M, l_R, r_R), T_R)\) into an \( K[[t]]/(t^2) \)-relative Rota-Baxter algebra and

\[ (\epsilon \otimes_K id_A, \epsilon \otimes_K id_M) : ((R \otimes_K A, \mu_R), (R \otimes_K M, l_R, r_R), T_R) \rightarrow ((A, \mu), (M, l, r), T) \]

is a morphism of \( K[[t]]/(t^2) \)-relative Rota-Baxter algebras. The morphism condition implies that \( \mu_0 = \mu \), \( l_0 = l \), \( r_0 = r \), and \( T_0 = T \). Therefore, an infinitesimal deformation is determined by the quadruple \((\mu_1, l_1, r_1, T_1)\).

5.3. Theorem. Let an infinitesimal deformation of a relative Rota-Baxter algebra \(((A, \mu), (M, l, r), T)\) be determined by the quadruple \((\mu_1, l_1, r_1, T_1)\). Then, the quadruple is a 2-cocycle in the cohomology of the relative Rota-Baxter algebra \(((A, \mu), (M, l, r), T)\). Conversely, any infinitesimal deformation is obtained by a 2-cocycle in this way.
Proof. Let $R = \mathbb{K}[t]/(t^2)$ and $(\mathbb{R} \otimes_{\mathbb{K}} A, \mu_R), (\mathbb{R} \otimes_{\mathbb{K}} M, l_R, r_R), T_{R})$ be an infinitesimal deformation of relative Rota-Baxter algebra $((A, \mu), (M, l, r), T)$. Then, equivalently we have the following conditions.

(i) $(\mathbb{R} \otimes_{\mathbb{K}} A, \mu_R = \mu + t\mu_1)$ is an $R$-associative algebra, i.e.

$$\mu_R(a, \mu_R(b, c)) = \mu_R(\mu_R(a, b), c), \quad \text{for all } a, b, c \in A.$$  

On comparing the coefficients of $t$ from both sides of the above identity, we have

$$\mu_1(a, bc) + a\mu_1(b, c) = \mu_1(ab, c) + \mu_1(a, b)c, \quad \text{for all } a, b, c \in A,$$

(ii) the triple $(\mathbb{R} \otimes_{\mathbb{K}} M, l_R, r_R)$ is an $(\mathbb{R} \otimes_{\mathbb{K}} A, \mu_R = \mu + t\mu_1)$-bimodule, i.e.

$$l_R(\mu_R(a, b), m) = l_R(a, l_R(b, m)), \quad r_R(r_R(m, a), b) = r_R(r_R(m, \mu_R(a, b)), \quad \text{and } r_R(l_R(a, m), b) = l_R(r_R(m, a), b),$$

for all $a, b \in A$ and $m \in M$. On comparing the coefficients of $t$ from both sides of the above identities, we have

$$l_1(ab, m) + \mu_1(a, b)m = l_1(a, bm) + a l_1(b, m)$$

$$r_1(ma, b) + r_1(m, ab) = r_1(m, ab) + m\mu_1(a, b)$$

$$l_1(a, m)b + r_1(am, b) = l_1(a, mb) + ar_1(m, b), \quad \text{for all } a, b \in A, m \in M,$$

(iii) the map $T_R : \mathbb{R} \otimes_{\mathbb{K}} M \to \mathbb{R} \otimes_{\mathbb{K}} A$ is a relative Rota-Baxter algebra, i.e.

$$\mu_R(T_R(m), T_R(n)) = T_R(l_R(T_R(m), n) + r_R(m, T_R(n)))$$

On comparing the coefficients of $t$ from both sides of the above identity, we have

$$\mu_1(T(m), T(n)) - T(l_1(T(m), n) + r_1(m, T(n))$$

$$= T_1(T(m)n + mT(n)) + T(T_1(m)n + mT_1(n)) - T_1(m)T(n) - T(m)T_1(n), \quad \text{for all } m, n \in M.$$  

By definition of the map $\delta$ is clear that equation (8) and equation (9) are equivalent to $\delta(\mu_1 + l_1 + r_1) = 0$. Moreover (10) is equivalent to $\Omega(\mu_1 + l_1 + r_1 - \delta(T_1) = 0$. Thus, $(\mathbb{R} \otimes_{\mathbb{K}} A, \mu_R), (\mathbb{R} \otimes_{\mathbb{K}} M, l_R, r_R), T_{R})$ is an infinitesimal deformation of relative Rota-Baxter algebra $((A, \mu), (M, l, r), T)$ if and only if

$$\mathcal{D}(\mu_1 + l_1 + r_1, T_1) = 0.$$

Hence, the statement of the theorem follows. \hfill \Box

5.4. Theorem. Let $((A, \mu), (M, l, r), T)$ be a relative Rota-Baxter algebra. There is a bijective correspondence between the equivalence classes of infinitesimal deformations of $((A, \mu), (M, l, r), T)$ and the cohomology classes in the second cohomology space $H^2(A, M, T)$.

Proof. Let us fix $R = \mathbb{K}[t]/(t^2)$. Let $(\mathbb{R} \otimes_{\mathbb{K}} A, \mu_R), (\mathbb{R} \otimes_{\mathbb{K}} M, l_R, r_R), T_{R})$ and $(\mathbb{R} \otimes_{\mathbb{K}} A, \mu_R', (\mathbb{R} \otimes_{\mathbb{K}} M, l_R', r_R'), T_{R}')$ be two infinitesimal deformations of a relative Rota-Baxter algebra $((A, \mu), (M, l, r), T)$. Then, Theorem 5.3 implies that these infinitesimal deformations are determined by 2-cocycles (say) $(\mu_1, l_1, r_1, T_1)$ and $(\mu_1', l_1', r_1', T_1')$, respectively. Both of the infinitesimal deformations are equivalent if there exists an $R$-relative Rota-Baxter algebra isomorphism

$$(\Phi, \Psi) : ((\mathbb{R} \otimes_{\mathbb{K}} A, \mu_R), (\mathbb{R} \otimes_{\mathbb{K}} M, l_R, r_R), T_{R}) \to ((\mathbb{R} \otimes_{\mathbb{K}} A, \mu_R'), (\mathbb{R} \otimes_{\mathbb{K}} M, l_R', r_R'), T_{R}'),$$

satisfying

$$(\epsilon \otimes_{\mathbb{K}} \text{id}_A) \circ \Phi = (\epsilon \otimes_{\mathbb{K}} \text{id}_A) \quad \text{and} \quad (\epsilon \otimes_{\mathbb{K}} \text{id}_M) \circ \Psi = (\epsilon \otimes_{\mathbb{K}} \text{id}_M).$$

From equation (11), it follows that

$$\Phi = \text{id}_A + t\phi_1, \quad \Psi = \text{id}_V + t\psi_1, \quad \text{for some } \phi_1 \in \text{Hom}(A, A) \text{ and } \psi_1 \in \text{Hom}(V, V).$$

If $(\Phi, \Psi) : ((\mathbb{R} \otimes_{\mathbb{K}} A, \mu_R), (\mathbb{R} \otimes_{\mathbb{K}} M, l_R, r_R), T_{R}) \to ((\mathbb{R} \otimes_{\mathbb{K}} A, \mu_R'), (\mathbb{R} \otimes_{\mathbb{K}} M, l_R', r_R'), T_{R}')$ is an $R$-relative Rota-Baxter algebra isomorphism, then one can observe the following.
Moreover, comparing the coefficients of \( t \),

\[
\Phi(\mu_R(a, b)) = \mu'_R(\Phi(a), \Phi(b)), \quad \text{for all } a, b \in A.
\]

On comparing the coefficients of \( t \),

\[
\mu_1 - \mu'_1 = \delta_{\text{Hoch}} \phi_1.
\]

(ii) The map \((\Phi, \Psi)\) is an isomorphism of \( R \)-relative Rota-Baxter operators from \( T_R \) to \( T'_R \). Then,

\begin{enumerate}
\item \( \Psi(ln(a, m)) = l'_R(\Phi(a), \Psi(m)), \quad \Psi(rn(m, a)) = r'_R(\Psi(m), \Phi(a)), \quad \text{for all } m \in M, \ a \in A \)
\item \( \Phi \circ T_R = T'_R \circ \Psi. \)
\end{enumerate}

Thus, comparing the coefficients of \( t \) in the above identities given in the condition (1), we have

\[
\psi_1(am) + l_1(a, m) = l'_1(\Phi(a), \Psi(m)) + \phi_1(a)m + a\psi_1(m)
\]

\[
\psi_1(ma) + r_1(m, a) = r'_1(m, a) + \psi_1(m)a + m\phi_1(a).
\]

Moreover, comparing the coefficients of \( t \) from both sides of the identity in the condition (2), we get

\[
T_1 + \phi_1 \circ T = T'_1 + T \circ \psi_1.
\]

Therefore, from equations (12)-(14), the pair \((\Phi, \Psi)\) is an \( R \)-relative Rota-Baxter algebra isomorphism if and only if

\[
(\mu_1, l_1, r_1, T_1) - (\mu'_1, l'_1, r'_1, T_1) = \mathcal{D}(\phi_1, \psi_1).
\]

Hence, there is a bijective correspondence between the equivalence classes of infinitesimal deformations of \(((A, \mu), (M, l, r), T)\) and the second cohomology space \( H^2(A, M, T) \).

\]

6. Homotopy relative Rota-Baxter operators

In this section, we introduce homotopy relative Rota-Baxter operators on bimodules over strongly homotopy associative algebras. Our notion will generalize (strict) Rota-Baxter operators on \( A_{\infty} \)-algebras introduced in [8]. We compare homotopy relative Rota-Baxter algebras and homotopy relative Rota-Baxter Lie algebras introduced in [21]. Finally, we construct a pre-Lie\(_{\infty}\)[1]-algebra from any \( \text{Dend}_{\infty}[1] \)-algebra generalizing the similar result from non-homotopic case [2]. This construction suitably fits with the various relations among dendriform algebras, pre-Lie algebras, associative algebras, Lie algebras and relative Rota-Baxter algebras in the homotopy context.

6.1. Definition. An \( A_{\infty}[1] \)-algebra is a graded vector space \( A = \bigoplus_{i \in \mathbb{Z}} A_i \), together with a collection of degree 1 multilinear maps \( \{\mu_k : A^\otimes k \to A\}_{k \geq 1} \) satisfying the following identities

\[
\sum_{i+j=n+1} \sum_{\lambda=1}^{j} (-1)^{|a_1|+\cdots+|a_{\lambda-1}|} \mu_j(a_1, \ldots, a_{\lambda-1}, \mu_1(a_{\lambda}, \ldots, a_{\lambda+i-1}), a_{\lambda+i}, \ldots, a_n) = 0, \quad \text{for } n \geq 1.
\]

Note that \( A_{\infty}[1] \)-algebra structure on a graded vector space \( A \) can be described by Maurer-Cartan element in a suitable graded Lie algebra [27]. Let \( A \) be a graded vector space. Consider the free reduced tensor algebra \( T(A) = \bigoplus_{n \geq 1} A^\otimes n \) and for each \( n \in \mathbb{Z} \), define \( C^n(A) = \text{Hom}_n(T(A), A) \) the space of degree \( n \) linear maps. An element \( \mu \in \text{Hom}_n(T(A), A) \) is the sum of degree \( n \) multilinear maps \( \mu_k : A^\otimes k \to A \), for \( k \geq 1 \).

With these notations \( \bigoplus_{n \in \mathbb{Z}} C^n(A) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(T(A), A) \) is a graded Lie algebra with the bracket given by \( [\mu_k, \gamma_l] = \mu_k \circ \gamma_l - (-1)^{mn} \gamma_l \circ \mu_k \), where

\[
(\mu_k \circ \gamma_l)(a_1, \ldots, a_{k+l-1}) = \sum_{i=1}^{k+l} (-1)^{|a_1|+\cdots+|a_{i-1}|} \mu_k(a_1, \ldots, a_{i-1}, \gamma_l(a_i, \ldots, a_{i+l-1}), a_{i+l}, \ldots, a_{k+l-1}),
\]

for \( \mu = \sum_{k \geq 1} \mu_k \in C^n(A) \) and \( \gamma = \sum_{l \geq 1} \gamma_l \in C^n(A) \). An element \( \mu \in C^1(A) = \sum_{k \geq 1} \mu_k \) is a Maurer-Cartan element in the above graded Lie algebra if and only if \( (A, \{\mu_k\}_{k \geq 1}) \) is an \( A_{\infty}[1] \)-algebra.

6.2. Definition. Let \( (A, \{\mu_k\}_{k \geq 1}) \) be an \( A_{\infty}[1] \)-algebra. A bimodule over it consists of a graded vector space \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) together with a collection of degree 1 maps \( \{\eta_k : A^{k-1,1} \to M\}_{k \geq 1} \) satisfying the identities
(15) with exactly one of $a_1, \ldots, a_n$ is from $M$ and the corresponding multilinear operation $\mu_i$ or $\mu_j$ replaced by $\eta_i$ or $\eta_j$.

It follows from the definition that any $A_\infty[1]$-algebra is a bimodule over itself with $\eta_k = \mu_k$, for $k \geq 1$. This is called the adjoint bimodule.

6.1. **Homotopy relative Rota-Baxter operators.** Here, we use a homotopy version of Theorem 2.3 to describe relative Rota-Baxter operators in the homotopy context.

Let $(A, \{\mu_k\}_{k \geq 1})$ be an $A_\infty[1]$-algebra and $(M, \{\eta_k\}_{k \geq 1})$ be a bimodule over it. We consider the graded Lie algebra

$$L = (\bigoplus_{n \in \mathbb{Z}} C^n(A \oplus M) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(T(A \oplus M), A \oplus M), [ \cdot , \cdot ]).$$

It is easy to see that $a = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(T(M), A)$ is an abelian subalgebra of $L$. Let $P : L \to a$ be the projection onto the subspace $a$. Then $\Delta = \sum_{k \geq 1} (\mu_k + \eta_k) \in \ker(P)_1$ and satisfies $[\Delta, \Delta] = 0$. Therefore, we get that $(L, a, P, \Delta)$ is a V-data. Hence by Theorem 2.10, we get an $L_\infty[1]$-algebra generalizing the graded Lie algebra of Theorem 2.3 to the homotopy context.

6.3. **Theorem.** Let $(A, \{\mu_k\}_{k \geq 1})$ be an $A_\infty[1]$-algebra and $(M, \{\gamma_k\}_{k \geq 1})$ be a bimodule over it. Then there is an $L_\infty[1]$-algebra structure on the graded vector space $a = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(T(M), A)$ with structure maps given by (2). Moreover, this $L_\infty[1]$-algebra is weakly filtered. \hfill \Box

Note that the filtration here is given by $\mathcal{F}_n(a) = \bigoplus_{i \geq n} \text{Hom}(M^{\otimes i}, A)$, for $n \geq 1$. Inspired by the characterization of relative Rota-Baxter operators in the usual case, we define the following definition.

6.4. **Definition.** Let $(A, \{\mu_k\}_{k \geq 1})$ be an $A_\infty[1]$-algebra and $(M, \{\gamma_k\}_{k \geq 1})$ be a bimodule. A Maurer-Cartan element $T = \sum_{k \geq 1} T_k \in \text{Hom}_0(T(M), A)$ in the above $L_\infty[1]$-algebra is called a homotopy relative Rota-Baxter operator on $(M, \{\gamma_k\}_{k \geq 1})$ over the $A_\infty[1]$-algebra $(A, \{\mu_k\}_{k \geq 1})$.

In the next, we give a characterization of a homotopy relative Rota-Baxter operator $T = \sum_{k \geq 1} T_k$ in terms of its components.

6.5. **Proposition.** An element $T = \sum_{k \geq 1} T_k \in \text{Hom}_0(T(M), A)$ is a homotopy relative Rota-Baxter operator if and only if its components satisfy the following identities: for each $p \geq 1$,

$$\sum_{k_1 + \cdots + k_n = p} \frac{1}{n!} \mu_n(T_{k_1}(u_1, \ldots, u_{k_1}), T_{k_2}(u_{k_1+1}, \ldots, u_{k_1+k_2}), \ldots, T_{k_n}(u_{k_1+\cdots+k_{n-1}+1}, \ldots, u_p))$$

$$= \sum_{i=1}^n \sum_{\substack{j+k_1+\cdots+k_{i-1}+1 \, \text{and} \, k_{i+1}+\cdots+k_n+t=p \, \text{if} \, t=0 \, \text{or} \, n-1}} \left(\frac{(-1)^{\left|u_1+\cdots+u_p\right|}}{(n-1)!} \right) T_{j+1+t}\left(u_1, \ldots, u_j, \eta(T_{k_1}(u_{j+1}, \ldots, u_{j+k_1}), \ldots, T_{k_{i-1}}(\cdots, u_{j+k_1+\cdots+k_{i-1}}), \ldots, T_{k_n}(\cdots, u_{p-t})), u_{p-t+1}, \ldots, u_p\right).$$

Proof. We will use the interpretation of the elements of $C^\bullet(A \oplus M, A \oplus M)$ as coderivations on the free tensor coalgebra $\overline{T} (A \oplus M)$ and the Gerstenhaber bracket on $C^\bullet(A \oplus M, A \oplus M)$ as the commutator bracket of coderivations on $\overline{T} (A \oplus M)$. Let $\overline{\mu}, \overline{\eta}$ and $\overline{T}$ denote the coderivations on $\overline{T} (A \oplus M)$ corresponding to the maps $\sum_{k \geq 1} \mu_k$, $\sum_{k \geq 1} \eta_k$ and $\sum_{k \geq 1} T_k$ in $C^\bullet(A \oplus M, A \oplus M)$. Hence $T$ is a Maurer-Cartan element if and only if

$$\sum_{n=1}^\infty \frac{1}{n!} \overline{T}[[\overline{\mu} + [\overline{\eta}, \overline{T}], \overline{T}], \ldots, \overline{T}] = 0,$$
where \( \overline{T} \) denotes the projection onto the abelian subalgebra \( \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(\overline{T}(M), A) \). First observe that

\[
\begin{align*}
\overline{T}^n : \cdots &\to [\overline{T} + \overline{\eta}, \overline{T}], \overline{T}_{n-1}, \ldots, \overline{T}(u_1, \ldots, u_p) \\
&\overset{\text{pr}_A}{=} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \overline{T} \circ \cdots \circ \overline{T} \circ [\overline{T} + \overline{\eta}] \circ \overline{T} \circ \cdots \circ \overline{T}(u_1, \ldots, u_p) \\
&= (\overline{T} \circ \cdots \circ \overline{T})(u_1, \ldots, u_p) - n(\overline{T} \circ \cdots \circ \overline{T})(u_1, \ldots, u_p) \\
\text{(the other terms are zero).}
\end{align*}
\]

By a straightforward calculation, we obtain

\[
\begin{align*}
\overline{T}^n &\overset{\text{pr}_A}{=} \sum_{k_1 + \cdots + k_n = p} \mu_n(T_{k_1}(u_1, \ldots, u_k), T_{k_2}(u_{k+1}, \ldots, u_{k+k_2}), \ldots, T_{k_n}(u_{k+\cdots+k_{n-1}+1}, \ldots, u_p)), \\
n\overline{T}^n &\overset{\text{pr}_A}{=} \sum_{k_1 + \cdots + k_n = p} \mu_n(T_{k_1}(u_1, \ldots, u_k), T_{k_2}(u_{k+1}, \ldots, u_{k+k_2}), \ldots, T_{k_n}(u_{k+\cdots+k_{n-1}+1}, \ldots, u_p)) - n(\overline{T} \circ \cdots \circ \overline{T})(u_1, \ldots, u_p).
\end{align*}
\]

Therefore, it follows from equations (17) and (18) that \( T \) is a homotopy relative Rota-Baxter operator if and only if the identity (16) holds. \( \square \)

6.6. **Remark.** The identities given by equation (16) can be alternatively used as a definition of a homotopy relative Rota-Baxter operator. A homotopy Rota-Baxter operator on an \( A_{\infty}[1] \)-algebra \( (A, \{\mu_k\}_{k \geq 1}) \) is a homotopy relative Rota-Baxter operator on the adjoint bimodule.

6.7. **Definition.** (i) A homotopy relative Rota-Baxter algebra is a triple \( (A, \{\mu_k\}_{k \geq 1}, (M, \{\gamma_k\}_{k \geq 1}, \{T_k\}_{k \geq 1}) \) consisting of an \( A_{\infty}[1] \)-algebra, a bimodule and a homotopy relative Rota-Baxter operator.

(ii) A homotopy Rota-Baxter algebra is a pair \( (A, \{\mu_k\}_{k \geq 1}, \{T_k\}_{k \geq 1}) \) of an \( A_{\infty}[1] \)-algebra and a homotopy Rota-Baxter operator on it.

In [21], the authors introduce homotopy relative Rota-Baxter operators on modules over \( L_{\infty}[1] \)-algebra. Here we first recall their definition and then we compare with homotopy relative Rota-Baxter operator on bimodules over \( A_{\infty}[1] \)-algebras. Let \( (W, \{l_k\}_{k \geq 1}) \) be an \( L_{\infty}[1] \)-algebra and \( (V, \{\rho_k\}_{k \geq 1}) \) be a representation of it. Then

\[
(L = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(\overline{S}(W \oplus V), W \oplus V), \ a = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(\overline{S}(V), W), P, \triangle = \sum_{k \geq 1} (l_k + \rho_k))
\]

is a \( V \)-data, where the graded vector space \( L \) is equipped with the standard Nijenhuis-Richardson bracket [21,24]. Hence by Theorem 2.10, the graded space \( a = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(\overline{S}(V), W) \) carries an \( L_{\infty}[1] \)-algebra structure which turns out to be weakly filtered. A Maurer-Cartan element \( T = \sum_{k \geq 1} T_k \in \text{Hom}(S(V), W) \) on this \( L_{\infty}[1] \)-algebra is said to be a homotopy relative Rota-Baxter operator on the module \( (V, \{\rho_k\}_{k \geq 1}) \) over the \( L_{\infty}[1] \)-algebra \( (W, \{l_k\}_{k \geq 1}) \). A homotopy relative Rota-Baxter operator \( T = \sum_{k \geq 1} T_k \) is said to be strict if \( T_k = 0 \) for \( k \geq 2 \). It has been shown in [21] that a strict homotopy relative Rota-Baxter operator \( T = T_1 \) induces a pre-Lie \( A_{\infty}[1] \)-algebra structure on \( V \) given by

\[
\theta_k(v_1, \ldots, v_k) := \rho_k(Tv_1, \ldots, Tv_{k-1}, v_k), \quad k \geq 1.
\]

Let \( (A, \{\mu_k\}_{k \geq 1}) \) be an \( A_{\infty}[1] \)-algebra and \( (M, \{\eta_k\}_{k \geq 1}) \) be a bimodule over it. It has been proved in [19] that the standard symmetrization process yields an \( L_{\infty}[1] \)-algebra \( (A, \{l_k\}_{k \geq 1}) \) and a representation
\((M, \{ \rho_k \}_{k \geq 1})\) of it, where

\begin{align}
(19) \quad l_k(a_1, \ldots, a_k) := \sum_{\sigma \in S_k} \epsilon(\sigma) \mu_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}), \quad \text{for } k \geq 1 \text{ and } a_1, \ldots, a_k \in A, \\
(20) \quad \rho_k(a_1, \ldots, a_k) := \sum_{\sigma \in S_k} \epsilon(\sigma) \eta_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}), \quad \text{for } k \geq 1 \text{ and } a_1, \ldots, a_{k-1} \in A, a_k \in M.
\end{align}

Moreover, the above symmetrization process yields a morphism of graded Lie algebras

\[ \Psi : \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(T(A \oplus M), A \oplus M) \longrightarrow \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(S(A \oplus M), A \oplus M). \]

As a consequence, we get the following.

6.8. Proposition. With the above notations, if \(T\) is a homotopy relative Rota-Baxter operator on \((M, \{ \rho_k \}_{k \geq 1})\) over the \(A_\infty[1]\)-algebra \((A, \{ \mu_k \}_{k \geq 1})\), then \(\Psi(T)\) is a homotopy relative Rota-Baxter operator on the module \((M, \{ \rho_k \}_{k \geq 1})\) over the \(L_\infty[1]\)-algebra \((A, \{ \eta_k \}_{k \geq 1})\).

6.2. Homotopy dendriform and homotopy pre-Lie algebras. Dendriform algebras were introduced by Loday [22] as a Koszul dual of associative dialgebras. The notion of homotopy dendriform algebras \((Dend_\infty, \{ \mu_k \}_{k \geq 1})\) was defined in [23] and explicitly described in [8]. Here, we recall the definition of a \(Dend_\infty[1]\)-algebra from [8] in a slightly different way using Maurer-Cartan elements in a graded Lie algebra.

Let \(C_k\) be the set of first \(n\) natural numbers. For convenience, we denote the elements of \(C_k\) by

\[ C_k = \{[1], [2], \ldots, [k]\}. \]

There are some distinguished maps \(R_0(k; 1, \ldots, \underbrace{1, \ldots, 1}_{i-\text{th}}) : C_{k+1} \rightarrow C_k\) and \(R_i(k; 1, \ldots, \underbrace{1, \ldots, 1}_{i-\text{th}}) : C_{k+1} \rightarrow \mathbb{K}[C_k]\) given by

\[
\begin{array}{c|c|c|c|c}
| & 1 \leq r \leq i-1 & 1 \leq r \leq i + l - 1 & i + l \leq r \leq k + l - 1 \\
---&---&---&---
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
R_0(k; 1, \ldots, \underbrace{1, \ldots, 1}_{i-\text{th}}) & [r] & [r] & [r + 1] & [r + 1] + \cdots + [l] \\
R_i(k; 1, \ldots, \underbrace{1, \ldots, 1}_{i-\text{th}}) & [1] + \cdots + [i] & [i] & [r - i + 1] & [1] + \cdots + [l] \\
\end{array}
\]

Let \(D = \bigoplus_{i \in \mathbb{Z}} D_i\) be a graded vector space. Suppose \(\mu_k : D \otimes D \otimes D \otimes D \rightarrow D\) is a map which is nonzero only on \(\bigoplus_{1 \leq i+j, k, m \geq 1, i+j \geq 1} D^{\otimes i} \otimes D \otimes D^{\otimes j} \otimes D^{\otimes k} \otimes D^{\otimes l} \rightarrow D\). Then \(\mu_k\) induces \(k\) many maps \(\mu_{k, [1]}, \ldots, \mu_{k, [k]} : D^{\otimes k} \rightarrow D\) by

\[
\mu_{k, [1]}(a_1, \ldots, a_k) = \mu_k(1 \otimes a_1 \otimes \cdots \otimes a_k), \quad \mu_{k, [2]}(a_1, \ldots, a_k) = \mu_k(a_1 \otimes a_2 \otimes \cdots \otimes a_k), \ldots
\]

\[
\mu_{k, [k]}(a_1, \ldots, a_k) = \mu_k(a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k \otimes 1).
\]

Consider the graded space \(\bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(TD \otimes D \otimes TD, D)\) of such maps. It carries a graded Lie bracket defined by \([\mu_k, \eta] = \mu_k \circ \eta - (-1)^{\mu \eta} \eta \circ \mu_k\), where

\[
(\mu_k \circ \eta)(a_1 \cdots a_r \otimes a_r \otimes a_{r+1} \cdots a_{k+l-1}) = \sum_{i=1}^{k} (-1)^{|a_1| + \cdots + |a_{i-1}|} \mu_k, R_0(k; 1, \ldots, \underbrace{1, \ldots, 1}_{i-\text{th}}) \mu_{i, R_0(k; 1, \ldots, \underbrace{1, \ldots, 1}_{i-\text{th}})}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{k+l-1}),
\]

for \(\mu = \sum_{k \geq 1} \mu_k \in \text{Hom}_{n}(TD \otimes D \otimes TD, D)\) and \(\eta = \sum_{l \geq 1} \eta_l \in \text{Hom}_{n}(TD \otimes D \otimes TD, D)\). A \(Dend_\infty[1]\)-algebra is a graded vector space \(D = \bigoplus_{i \in \mathbb{Z}} D_i\), together with a Maurer-Cartan element \(\mu = \sum_{k \geq 1} \mu_k\) of the graded Lie algebra \(\bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(TD \otimes D \otimes TD, D)\). Note that each \(\mu_k\) determines \(k\) many degree 1 maps \(D^{\otimes k} \rightarrow D\). Therefore, for now onward, we will denote a \(Dend_\infty\)-algebra by \((D, \{ \mu_{k, [r]} \}_{k \geq 1, [r] \in C_k})\). If \((D, \{ \mu_{k, [r]} \}_{k \geq 1, [r] \in C_k})\) is a \(Dend_\infty[1]\)-algebra, then it has been proved in [8, 23, 21] that \((D, \{ \mu_k \}_{k \geq 1})\) is an \(A_\infty[1]\)-algebra, where

\[
(21) \quad \mu_k = \mu_{k, [1]} + \cdots + \mu_{k, [k]}, \quad \text{for } k \geq 1.
\]

The notion of \(pre-Lie_\infty[1]\)-algebras are defined by Chapoton and Livernet [7]. Here we will not recall the definition rather mention that there is a graded Lie bracket \([\cdot, \cdot]_{MN}\), known as Matsushita-Nijenhuis bracket on \(\bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(S(W) \otimes W, W)\), for any graded vector space \(W\). See [7, 21] for details. A \(pre-Lie_\infty[1]\)-algebra
structure on a graded vector space $W$ is by definition a Maurer-Cartan element in the graded Lie algebra $\bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(S(W) \otimes W, [\; , \;]_{\text{MN}})$.

A pre-Lie$_{\infty}$-algebra induces an $L_{\infty}$-algebra structure on $W$ with structure maps given by

$$l_k(x_1, \ldots, x_k) = \sum_{i=1}^{k} (-1)^{|x_i|(|x_{i+1}| + \cdots + |x_k|)} \theta_k(x_1, \ldots, \hat{x_i}, \ldots, x_k, x_i), \quad \text{for } k \geq 1.$$  

This is called the subadjacent $L_{\infty}[1]$-algebra of the pre-Lie$_{\infty}$-[1]-algebra.

It is known that a dendriform algebra gives rise to a pre-Lie$_{\infty}$-algebra. Here we prove a homotopy version of this result.

**6.9. Theorem.** Let $(D, \{\mu_k, [\; , \;]_{\infty}\}_{k \geq 1, [\; , \;]_{\infty} \in \mathcal{C}_k})$ be a Dend$_{\infty}[1]$-algebra. Then $(D, \{\theta_k\}_{k \geq 1})$ is a pre-Lie$_{\infty}[1]$-algebra, where

$$\theta_k(a_1, \ldots, a_k) = \sum_{\sigma \in S_k} \epsilon(\sigma) \mu_k,_{\sigma-1(1)}(a_{\sigma(1)}, \ldots, a_{\sigma(k)}), \quad \text{for } k \geq 1.$$  

**Proof.** We define a map

$$\Psi: \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(TD \otimes D \otimes TD, D) \rightarrow \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(S(D) \otimes D, D)$$

by

$$\Psi(\mu_k)(a_1, \ldots, a_k) = \sum_{\sigma \in S_k} \epsilon(\sigma) \mu_k,_{\sigma-1(1)}(a_{\sigma(1)}, \ldots, a_{\sigma(k)}).$$

It is a straightforward but tedious calculation to show that $\Psi$ preserves the graded Lie algebra structures. Hence the result follows from the definition of pre-Lie$_{\infty}[1]$-algebra structure by Maurer-Cartan element. \qed

The above construction is functorial with respect to strict morphisms in the categories of Dend$_{\infty}[1]$-algebras and pre-Lie$_{\infty}[1]$-algebras.

**6.10. Proposition.** With the above construction, the following diagram commutes

$$\begin{array}{ccc}
\text{Dend}_{\infty}[1] & \xrightarrow{\Psi} & \text{pre-Lie}_{\infty}[1] \\
\downarrow & & \downarrow \\
\text{A}_{\infty}[1] & \xrightarrow{\text{Maurer-Cartan element}} & \text{L}_{\infty}[1].
\end{array}$$

**Proof.** Let $(D, \{\mu_k, [\; , \;]_{\infty}\}_{k \geq 1, [\; , \;]_{\infty} \in \mathcal{C}_k})$ be a Dend$_{\infty}$-algebra with the corresponding A$_{\infty}[1]$-algebra $(D, \{\mu_k\}_{k \geq 1})$ where $\mu_k$’s are given by equation (21). Therefore, the corresponding $L_{\infty}[1]$-algebra structure on $D$ is given by

$$l_k(a_1, \ldots, a_k) = \sum_{\sigma \in S_k} \epsilon(\sigma) \mu_k,_{\sigma-1(1)}(a_{\sigma(1)}, \ldots, a_{\sigma(k)})$$

$$= \sum_{\sigma \in S_k} \epsilon(\sigma) \sum_{i=1}^{k} \mu_k,_{i}(a_{\sigma(1)}, \ldots, a_{\sigma(k)}).$$  

On the other hand, consider the pre-Lie$_{\infty}[1]$-algebra $(D, \{\theta_k\}_{k \geq 1})$ where $\theta_k$’s are given in equation (22). Hence, the subadjacent $L_{\infty}[1]$-algebra on $D$ has structure maps

$$l_k'(a_1, \ldots, a_k) = \sum_{i=1}^{k} (-1)^{|a_i|(|a_{i+1}| + \cdots + |a_k|)} \theta_k(a_1, \ldots, \hat{a_i}, \ldots, a_k, a_i)$$

$$= \sum_{i=1}^{k} (-1)^{|a_i|(|a_{i+1}| + \cdots + |a_k|)} \sum_{\sigma \in S_k} \epsilon(\sigma \tau) \mu_k,_{\tau-1(\tau-1(1))}(a_{\sigma(1)}, \ldots, a_{\sigma(k)})$$

$$= \sum_{i=1}^{k} \sum_{\sigma \in S_k} \epsilon(\sigma) \mu_k,_{\tau-1(\tau-1(1))}(a_{\sigma(1)}, \ldots, a_{\sigma(k)}).$$

In the second equality, given a fixed $i$, we use the permutation $\tau \in S_k$ given by

$$\tau(j) = j \text{ for } j \leq i - 1, \quad \tau(j) = j + 1 \text{ for } i \leq j \leq k - 1 \quad \text{and} \quad \tau(k) = i.$$
Finally, the third equality follows as $e(\tau) = (-1)^{|a_i|(|a_i + 1| + \cdots + |a_k|)}$. From equations (24) and (25), it follows that $l_k = l'_k$ for $k \geq 1$. Hence the result follows.

6.3. **Strict homotopy relative Rota-Baxter operators.** A homotopy relative Rota-Baxter operator $T = \sum_{k \geq 1} T_k$ on a bimodule $(M, \{\eta_k\}_{k \geq 1})$ over an $A_\infty[1]$-algebra is said to be strict if $T_k = 0$ for $k \geq 2$. It follows from equation (16) that a degree 0 linear map $T : M \rightarrow A$ is a strict homotopy relative Rota-Baxter operator if $T$ satisfies

$$\mu_k(Tu_1, \ldots, Tu_k) = \sum_{r=1}^{k} T(\eta_k(Tu_1, \ldots, u_r, \ldots, Tu_k)), \text{ for } k \geq 1.$$ 

Strict homotopy relative Rota-Baxter operators are considered in [8] and the following result is proved in the same reference.

6.11. **Proposition.** Let $(A, \{\mu_k\}_{k \geq 1})$ be an $A_\infty[1]$-algebra and $(M, \{\eta_k\}_{k \geq 1})$ be a bimodule. If $T$ is a strict homotopy relative Rota-Baxter operator, then $M$ carries a $Dend_\infty[1]$-algebra structure, where

$$\mu_{k, [r]}(u_1, \ldots, u_k) = \eta_k(Tu_1, \ldots, u_r, \ldots, Tu_k), \text{ for } k \geq 1 \text{ and } [r] \in C_k.$$ 

The $Dend_\infty[1]$-algebra constructed in the above proposition is called the subadjacent $Dend_\infty[1]$-algebra associated to the strict homotopy relative Rota-Baxter operator $T$.

6.12. **Proposition.** The following diagram commutes

$$\begin{align*}
\text{strict homotopy relative Rota – Baxter algebra} & \quad \quad \rightarrow Dend_\infty[1] \\
\text{strict homotopy relative Rota – Baxter Lie algebra} & \quad \quad \rightarrow \text{pre-Lie}_\infty[1].
\end{align*}$$

**Proof.** Let $T : M \rightarrow A$ be a strict homotopy relative Rota-Baxter operator on the bimodule $(M, \{\eta_k\}_{k \geq 1})$ over the $A_\infty[1]$-algebra $(A, \{\mu_k\}_{k \geq 1})$. Let $(M, \{\mu_{k, [r]}\}_{k \geq 1, [r] \in C_k})$ denote the corresponding $Dend_\infty[1]$-algebra, where $\mu_{k, [r]}$’s are given by equation (26). Hence by Theorem 6.9, the corresponding $\text{pre-Lie}_\infty[1]$-algebra structure on $M$ are given by

$$\theta_k(u_1, \ldots, u_k) = \sum_{\sigma \in S_k} \epsilon(\sigma) \mu_{k, [\sigma^{-1}(k)]}(u_{\sigma(1)}, \ldots, u_{\sigma(k)})$$

$$= \sum_{\sigma \in S_k} \epsilon(\sigma) \eta_k(Tu_{\sigma(1)}, \ldots, u_{\sigma(j)}, \ldots, u_{\sigma(k)})|_{\sigma(j)=k}.$$ 

On the other hand, by considering $T$ as a strict homotopy relative Rota-Baxter operator on the module $(M, \{\rho_k\}_{k \geq 1})$ over the $L_\infty[1]$-algebra $(A, \{l_k\}_{k \geq 1})$ given in equations (19) and (20), the $\text{pre-Lie}_\infty[1]$-algebra on $M$ is given by

$$\rho_k(Tu_1, \ldots, Tu_{k-1}, u_k) = \sum_{\sigma \in S_k} \epsilon(\sigma) \eta_k(Tu_{\sigma(1)}, \ldots, u_{\sigma(j)}, \ldots, Tu_{\sigma(k)})|_{\sigma(j)=k}.$$ 

Thus we have $\theta_k = \rho_k$, for $k \geq 1$ which proves the result.

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