Matrix Models

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Matrix models and their connections to String Theory and noncommutative
gometry are discussed. Various types of matrix models are reviewed. Most of
interest are IKKT and BFSS models. They are introduced as 0+0 and 1+0
dimensional reduction of Yang–Mills model respectively. They are obtained
via the deformations of string/membrane worldsheet/worldvolume. Classical
solutions leading to noncommutative gauge models are considered.

1 Introduction

At the beginning let us define the topic of the present lectures. As follows
from the title, “Matrix Models” are theories in which the fundamental vari-
able is a matrix. The matrix variable can be a just a constant or a function
of time or even be defined as a function over some space-time manifold. With
this definition almost any model existing in modern physics e.g. Yang–Mills
theory, theories of Gravity etc., will be a “matrix theory”. Therefore, when
speaking on the matrix theory usually a simple structure is assumed, e.g.
when fundamental variables are constant or at most time dependent. In the
first case, the models of random matrices, one has no time therefore no dy-
namics. This is a statistical theory describing random matrix distributions.
These models are popular in many areas e.g. in the context of description of
integrable systems in QCD, or nuclear systems as well as in the study of the
lattice Dirac operators (for a review see e.g. [28, 48, 10, 19, 31, 49, 37] and
references therein). The special case of interest for us are the Yang–Mills type
matrix models arising in String Theory such like the [21].

Another case of of interest are so called matrix mechanics, i.e. theories of
time-evolutive matrices. These models along with the random matrix models
are of special interest in String Theory. Thus, the Yang–Mills type matrix
models appear to nonperturbatively describe collective degrees of freedom in string theory called branes. Branes are extended objects on which the “normal” fundamental strings can end. It was conjectured that including the brane degrees of freedom in the “conventional” superstring theories leads to their unification into the M-theory, a model giving in its different perturbative regimes all known superstring models. The M-theory is believed to be related to the twelve-dimensional membrane. In the light-cone frame it was conjectured to be described by an Yang–Mills type matrix mechanics (BFSS matrix model) [4]. As we will see in the next section, this as well as the IKKT matrix model can be obtained by quantization/deformation of, respectively, the worldvolume of the membrane and the worldsheet of the string.

As it is by now clear, in this notes we are considering mainly these two models, which sometimes are called “matrix theories” to underline their fundamental role in string theory.

The plan of these notes is as follows. In the next section we give the string motivation and introduce the matrix models as dimensional reductions of supersymmetric Yang–Mills model. Next, we consider Nambu–Goto description of the string and membrane and show that the noncommutative deformation of the respectively, worldsheet or worldvolume leads to IKKT or BFSS matrix models. In the following section we analyze the classical solutions to these matrix models and interpret them as noncommutative gauge models. The fact that these models have a common description in terms of the original matrix model allows one to establish the equivalence relations among them.

2 Matrix models of String Theory

2.1 Branes and Matrices

A breakthrough in the development of string theory, “the second string revolution” happened when it was observed that in the dynamics of fundamental string on has additional degrees of freedom corresponding to the dynamics at the string ends [33] (see [2] for a review).

In the open string mode expansion the dynamics at the edge is described by an Abelian gauge field (particle) (for a modern introduction to string theory see e.g. [32, 23]). The corresponding charge of the end of the string is called Chan–Paton factor. Allowing a superposition of several, say $N$ such factors, which correspond to an “$N$-valent” string end, gives rise to a nonabelian $U(N)$ super Yang–Mills gauge field in the effective lagrangian of the open string. This is the so called nine-brane. As it was shown in [33] string theory allows brane configurations of different other dimensions $p$, $0 \leq p + 1 \leq 10$. Depending on the type of the string model they preserve parts of supersymmetry.

So, descending down to the lower dimensional $p$-branes one gets the $p + 1$-dimensional reductions of the ten dimensional Super Yang–Mills model.
It appears that out of all possibilities only two cases are fundamental, namely, this of \( p = 0 \) and \( p = -1 \). All other cases can be obtained from either \( p = -1 \) or \( p = 0 \) by condensation of \(-1\)- or 0-branes into higher dimensional objects.

### 2.2 The IKKT matrix model family

As it follows from the space-time picture, the \(-1\) branes are non-dynamical and, therefore, should be described by a random matrix model which is the reduction of the 10d SYM down to zero dimensions:

\[
S_{-1} = -\frac{1}{4g^2} \text{tr}[X_\mu, X_\nu]^2 - \text{tr} \bar{\psi} \gamma^\mu [X_\mu, \psi],
\]  

where \( g \) is some coupling constant depending on SYM coupling \( g_{\text{YM}} \) and the volume of compactification. Matrices \( X_\mu, \mu = 1, \ldots, 10 \) are Hermitian \( N \times N \) matrices, \( \psi \) is a 10d spinor which has \( N \times N \) matrix index, \( \gamma^\mu \) are 10d Dirac \( \gamma \)-matrices.

From the 10d SYM the matrix model \( S_{-1} \) inherits the following symmetries:

- **Shifts:**
  \[
  X_\mu \rightarrow X_\mu + a_\mu \cdot \mathbb{1},
  \]  
  where \( a_\mu \) is a c-number.

- **SO(10) rotation symmetry**
  \[
  X_\mu \rightarrow \Lambda_\mu \nu X_\nu,
  \]  
  where \( \Lambda \in \text{SO}(10) \). This is the consequence of the (euclideanized) Lorenz invariance of the ten dimensional SYM model.

- **SU(N) gauge symmetry**
  \[
  X_\mu \rightarrow U^{-1} X_\mu U,
  \]  
  where \( U \in \text{SU}(N) \), and this is the remnant of the SYM gauge symmetry invariance.

- **Also one has left from the SYM model the supersymmetry invariance:**

  \[
  \delta_1 X_\mu = \bar{\epsilon} \gamma_\mu \psi,
  \]

  \[
  \delta_1 \psi = [X_\mu, X_\nu] \gamma^{\mu\nu} \epsilon,
  \]

  as well as the second one which is simply the shift of the fermion,

  \[
  \delta_2 X_\mu = 0,
  \]

  \[
  \delta_2 \psi = \eta,
  \]

where \( \epsilon \) and \( \eta \) are the supersymmetry transformation parameters.
Exercise 1. Find the relation between the coupling $g$ in (1) on one side and SYM coupling $g_{YM}$ and the size/geometry of compactification on the other side.

Hint: Use an appropriate gauge fixing.

Exercise 2. Show that (3–5) are indeed the symmetries of the action (1).

The purely bosonic version of IKKT matrix model can be interpreted as the algebraic version of much older Eguchi–Kawai model [18]. The last is formulated in terms of SU(N) group valued fields $U_{\mu}$ (in contrast to the algebra valued $X_{\mu}$). The action for the Eguchi–Kawai model reads as,

$$S_{EK} = -\frac{1}{4g_{EK}^2} \sum_{\mu,\nu} \text{tr}(U_{\mu}U_{\nu}U_{\mu}^{-1}U_{\nu}^{-1} - I).$$  (9)

By the substitution, $U_{\mu} = \exp \alpha X_{\mu}$, $g_{EK}^2 = g^2 a^{4-d}$ and taking the limit $a \to 0$ one formally comes to the bosonic part of the IKKT action (1).

Note: From the string interpretation we will discuss in the next section it is worth to add an extra term to the IKKT action (1) and, namely, the chemical potential term,

$$\Delta S_{chem} = -\beta \text{tr} I, \quad (10)$$

which “controls” the statistical behavior of $N$. In the string/brane picture $\beta$ plays the role of the chemical potential for the number of branes. This produces the relative weights for the distributions with different $N$, which can not be catched from the arguments we used to write down the action (1).

2.3 The BFSS model family

Let us consider another important model which describes the dynamics of zero branes [4]. Basic ingredients of this model are roughly the same as for the previous one, the IKKT model, except that now the matrices depend on time. The action for this model is the dimensional reduction of the ten dimensional SYM model down to the only time dimension:

$$S_{BFSS} = \frac{1}{g_{BFSS}} \int dt \text{tr} \left\{ \frac{1}{2} (\nabla_0 X_i)^2 + \bar{\psi} \nabla_0 \psi - \frac{1}{4} [X_i, X_j]^2 - \bar{\psi} \gamma_i [X_i, \psi] \right\}, \quad (11)$$

where, now, the index $i$ runs from one to nine.

The action (11) describes the dynamics of zero branes in IIA string theory, but it was also proposed as the action for the M-theory membrane in the light-cone approach. As we are going to see in the next section, this model along with the IKKT model can be obtained by worldvolume quantization of the membrane action.
Another known modification of this action for the pp-wave background was proposed by Berenstein–Maldacena–Nastase (BMN) [7, 6, 8]. It differs from the BFSS model additional terms which are introduced in order to respect the pp-wave supersymmetry. The action of the BMN matrix model reads:

\[ S_{\text{BMN}} = \int dt \, \text{tr} \left[ \frac{1}{2(2R)} (\nabla_0 X)^2 + \bar{\psi} \nabla_0 \psi \right. \\
\left. + \frac{(2R)}{4} [X_i, X_j]^2 - i(2R) \bar{\psi} \gamma^i [X_i, \psi] \right] + S_{\text{mass}}, \quad (12) \]

where \( S_{\text{mass}} \) is given by

\[ S_{\text{mass}} = \int dt \, \text{tr} \left[ \frac{1}{2(2R)} \left( -\left( \frac{\mu}{3} \right) \sum_{i=1,2,3} X_i^2 - \left( \frac{\mu}{6} \right) \sum_{i=4,\ldots,9} X_i^2 \right) \\
- \frac{\mu}{4} \bar{\psi} \gamma^{123} \psi - \frac{\mu}{3} \sum_{ijk=1,\ldots,3} \epsilon_{ijk} X_i X_j X_k \right] \quad (13) \]

The essential difference of this model from the standard BFSS one is that due to the mass and the Chern-Simons terms this matrix model allows stable vacuum solutions which can be interpreted as spherical branes (see e.g. [46, 45]). Such vacuum configurations can not exist in the original BFSS model.

3 Matrix models from the noncommutativity

In this section we show that the Matrix models which we introduced in the previous section arise when one allows the worldsheets of the string/worldvolume of the membrane to possess noncommutativity. It is interesting to note from the beginning that the “quantization” of the string worldsheet leads to the IKKT matrix model, while the space noncommutative membrane is described by the BFSS model. Let us remind that the above matrix models were introduced to describe, respectively, the \(-1\) - and \(0\)-branes, while the string and the membrane are respectively \(1\)- and \(2\)-brane objects. In the shed of the next section this can be interpreted as deconstruction of the \(1\)- and \(2\)-branes into their basic components, namely \(-1\)- and \(0\)-brane objects.

In this section we consider only the bosonic parts. The extension to the fermionic part is not difficult, so this is left to the reader as an exercise.

3.1 Noncommutative string and the IKKT matrix model

In trying to make the fundamental string noncommutative one immediately meets the following problem: The noncommutativity parameter is a dimensional parameter and, therefore, hardly compatible with the worldsheet conformal symmetry which plays a fundamental role in the string theory. Beyond this
there is no theoretical reason to think that the worldsheet of the fundamental string should be noncommutative. On the other hand, the are other string-like objects in the nonperturbative string theory: D1-branes or D-strings. As it was realized, in the presence of the constant nonzero Neveu-Schwarz B-field the brane can be described by a noncommutative gauge models \[12, 13, 14, 35\]. Then, in contrast to the fundamental string, it is natural to make the D-string noncommutative.

Let us start with the Euclidean Nambu–Goto action for the string,

\[
S_{NG} = T \int d^2 \sigma \sqrt{\det \partial_a X^\mu \partial_b X_\mu},
\]

where \(T\) is the D-string tension and \(X^\mu = X^\mu(\sigma)\) are the embedding coordinates. The expression under the square root of the r.h.s. of (14) can equivalently rewritten as follows,

\[
\det \partial X \cdot \partial X = \frac{1}{2} \Sigma^2,
\]

where,

\[
\Sigma^{\mu\nu} = \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu,
\]

which is the induced the worldsheet volume form of the embedding \(X^\mu(\sigma)\).

The Nambu–Goto action then becomes:

\[
S_{NG} = T \int d^2 \sigma \sqrt{\frac{1}{2} \Sigma^2}.
\]

This action is nonlinear and still quite complicate. A much simple form can be obtained using the Polyakov trick. To illustrate the idea of te trick which is widely used in the string theory consider first the example of a particle.

**Polyakov’s trick**

The relativistic particle is described by the following reparametrization invariant action,

\[
S_p = m \int d\tau \sqrt{x^2},
\]

where \(m\) is the mass and \(x\) is the particle coordinate. The dynamics of the particle (15) is equivalent, at least classically to one described by the following action,

\[
S_{pp} = \int d\tau \left( \frac{1}{2} e^{-1} \dot{x}^2 + m^2 e \right).
\]

In this form one has a new variable \(e\) which plays the role of the line einbein function, or better to say of the one-dimensional volume form.

To see the classical equivalence between (19) and (15) one should write down the equations of motion arising from the variation of \(e\),
and use it to substitute $e$ in the action (19) which should give exactly (18).

**Exercise 3.** Show this!

As one can see, both actions (18) and (19) are reparametrization invariant, the difference being that the Polyakov action (19) is quadratic in the particle velocity $\dot{x}$. This trick is widely used in the analysis of nonlinear systems with gauge symmetry. In what follows we will apply it too.

Let us turn back to our string and the action (17). Applying the Polyakov trick, one can rewrite the action (17) in the following (classically) equivalent form,

$$S_{NGP} = \int d^2 \sigma \left( \frac{1}{4} \eta^{-1} \{ X_\mu, X_\nu \}^2 + \eta T^2 \right),$$

(21)

where $\eta$ is the string “area” density and we introduced the Poisson bracket notation,

$$\{ X, Y \} = \epsilon^{ab} \partial_a X \partial_b Y.$$  \hspace{1cm} (22)

It is not very hard to see that the bracket defined by (22) satisfies to all properties a Poisson bracket is supposed to satisfy.

**Exercise 4.** Do it!

Let us note, that the Poisson bracket (22) is not an worldsheet reparametrization invariant quantity. Under the reparametrizations $\sigma \mapsto \sigma'(\sigma)$ it transforms like density rather than scalar the same way as $\eta$ is:

$$\{ X, Y \} \mapsto \det \left( \frac{\partial \sigma'}{\partial \sigma} \right) \{ X, Y \}',$$

(23a)

$$\eta \mapsto \det \left( \frac{\partial \sigma'}{\partial \sigma} \right) \eta(\sigma').$$  \hspace{1cm} (23b)

Having two densities one can master a scalar,

$$\{ X, Y \}_s = \eta^{-1} \{ X, Y \},$$

(24)

which is invariant. Actually, these two definitions coincide in the gauge $\eta = 1$, which in some cases may be possible only locally. In terms of the scalar Poisson bracket the action is rewritten in the form as follows

$$S_{NGP} = \int d^2 \sigma \eta \left( \frac{1}{4} \{ X_\mu, X_\nu \}^2 + T^2 \right),$$

(25)

where $d^2 \sigma \eta$ is the invariant worldsheet area form.
“Quantization”

Consider the naive quantization procedure we know from the quantum mechanics. The classical mechanics is described by the canonical classical Poisson bracket,

\[ \{ p, q \} = 1, \]  

and the quantization procedure consists, roughly speaking, in the replacement of the canonical variables \((p, q)\) by the operators \(\hat{p}, \hat{q}\). At the same time the 

\[ -i\hbar \times \text{(Poisson bracket)} \]

is replaced by the commutator of the corresponding operators. In particular,

\[ \{ p, q \} \mapsto [\hat{p}, \hat{q}] = -i\hbar. \]

Afterwards, main task consists in finding the irreducible representation(s) of the obtained algebra\(^4\). From the undergraduate course of quantum mechanics we know that there are many unitary equivalent ways to do this, e.g. the oscillator basis representation is a good choice.

Under the quantization procedure functions on the phase space are replaced by operators acting on the irreducible representation space of the algebra \(27\). For these functions and operators one have the correspondence between the tracing and the integration over the phase space with the Liouville measure

\[ \int \frac{dp dq}{2\pi\hbar} \ldots \mapsto \text{tr} \ldots \]

Let us turn to our string model. As in the case of quantum mechanics, under the quantization we mean the replacing the fundamental worldsheet variables \(\sigma^1\) and \(\sigma^2\) by corresponding operators: \(\hat{\sigma}^1\) and \(\hat{\sigma}^2\), such that the invariant Poisson bracket is replaced by the commutator according to the rule:

\[ \{ \cdot, \cdot \}_\text{PB} = i\theta [\cdot, \cdot], \]

where \(\theta\) is the deformation parameter (noncommutativity). The worldsheat functions are replaced by the operators on the Hilbert space on which \(\hat{\sigma}^a\) act irreducibly. As we have two forms of the Poisson bracket the question is wether one should use the density form of the Poisson bracket \(22\) or the invariant form \(24\)? The correct choice is the invariant form \(24\). This is imposed by the fact that the operator commutator is invariant with respect of the choice of basic operator set (in our case it is given by operators \(\hat{\sigma}^a\)).

Let us note that with the choice of invariant Poisson bracket in \(29\) the operators \(\hat{\sigma}^a\), generally, do not have standard Heisenberg commutation relations. Rather than that, they commute to a nontrivial operator,

\[ [\hat{\sigma}^1, \hat{\sigma}^2] = i\theta \eta^{-1}, \]

\(^4\) In fact, the enveloping algebra rather the Lie algebra itself.
where the operator $\eta^{-1}$ corresponds to the inverse density of the string worldsheet area (i.e., its classical limit gives this density). At the same time the trace in the quantum case corresponds to the worldsheet integration with the invariant measure

$$\int d^2\sigma \eta \mapsto 2\pi \theta \text{tr}[\cdot]. \tag{31}$$

Having the “quantization rules” (29) and (31) one is able to write down the noncommutative analog of the Nambu–Goto–Polyakov string action (21). It looks as follows,

$$S = \alpha \text{tr} \frac{1}{4} [X_\mu, X_\nu]^2 + \beta \text{tr} \mathbb{I}, \tag{32}$$

where $\alpha$ and $\beta$ are the couplings of the matrix model. In terms of the string and the deformation parameters they read,

$$\alpha = \frac{2\pi}{\theta}, \tag{33}$$

$$\beta = \frac{2\pi T^2}{\theta}. \tag{34}$$

After the identification of couplings the model (32) is identical with the IKKT model (11). As a bonus we have obtained the chemical potential (10). As we see from the construction, the dimensionality of matrices depend on the irreducibility representation of the noncommutative algebra. As one can expect from what is familiar in quantum mechanics, the compact worldsheets should lead to finite-dimensional representations and thus are described, respectively, by matrices of finite dimensions. There is no exact equivalence between the worldsheet geometry and the matrix description. However, the consistency requires that one should recover the worldsheet geometry in the semi-classical limit ($\theta \to 0$).

Another interesting remark is that in this picture the Heisenberg operator basis correspond to the worldsheet parametrization for which $\eta$ is constant. As it is well known such parametrization can exist globally only for the topologically trivial worldsheets. On the other hand, in the algebra of operators acting on a separable infinite dimensional Hilbert space one can always find a Heisenberg operator basis.

**Example I: Torus**

To illustrate the above consider the example of quantization of toric worldsheet. The torus can be described by one complex modulus (or two real moduli). We are not interested here in the possible form of the toric metric, so we can choose the parametrization of the torus for which $\eta = 1$ and the flat worldsheet coordinates span the range

$$0 \leq \sigma^1 < l_1, \quad 0 \leq \sigma^2 < l_2. \tag{35}$$
The first problem arises when one tries to quantize variables with the range \( 35 \). In spite of the fact that the (invariant) Poisson bracket is canonical the operators \( \hat{\sigma}^1, \hat{\sigma}^2 \) can not satisfy the Heisenberg algebra,

\[
\left[ \hat{\sigma}^1, \hat{\sigma}^2 \right] = i\theta,
\]

(36)

and have bounded values like in \( 35 \) at the same time.

**Exercise 5.** Prove this!

To conciliate the compactness and noncommutativity one should use the compact coordinates \( U_a \) instead,

\[
U_a = \exp \frac{2\pi i \hat{\sigma}_a}{l_a}, \quad a = 1, 2.
\]

(37)

The compact coordinates \( U_a \) satisfy the following (Weyl) commutation relations

\[
U_1 U_2 = q U_2 U_1,
\]

(38)

where \( q \) is the toric deformation parameter,

\[
q = e^{\frac{2\pi^2 i}{l_1 l_2}}.
\]

(39)

If \( q^N = 1 \) for some \( N \in \mathbb{Z}_+ \), then \( U_a \) generate an irreducible representation of dimension \( N \). In this case an arbitrary \( N \times N \) matrix \( M \) can be expanded in powers of \( U_a \), e.g.

\[
M = \sum_{m,n=0}^{N-1} M_{mn} U_1^m U_2^n.
\]

(40)

Expansion (40) is in terms of monomials in \( U_1 \) and \( U_2 \) ordered in such a way that all \( U_1 \)'s are to the left of all \( U_2 \) one can alternatively use the Weyl functions \( W_{mn} \) defined as

\[
W_{mn} = \exp \left( 2\pi i m \hat{\sigma}_1/l_1 + 2\pi i n \hat{\sigma}_2/l_2 \right),
\]

(41)

which differs from the product \( U_1^m U_2^n \) by a polynomial of lower degree, but is symmetrized in \( \hat{\sigma}_1 \) and \( \hat{\sigma}_2 \). Using this expansion in terms of the Weyl functions leads one to the description of matrices in terms of the Weyl symbols — ordinary functions subject to the star product algebra. Weyl symbols as well as the star product algebras we are going to consider in the next sections.

As a result we have that quantization of the torus surface leads to the description in terms of \( N \times N \) matrices where the dimensionality \( N \) of the matrices depends on the torus moduli.
Example II: Fuzzy sphere

Another case of interest is the deformation of the spherical string worldsheet. On the sphere there is no global flat parametrization with $\eta = 1$. It is convenient to represent the two-sphere worldsheet parameters embedded into the three-dimensional Euclidean space:

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 1,$$

with the induced metric and volume form $\eta$. The (invariant) Poisson bracket is given by the following expression:\footnote{We drop out the subscript of the invariant Poisson bracket since it creates no confusion while it is the only used from now on.}

$$\{\sigma_i, \sigma_j\} = (1/r)\epsilon_{ijk}\sigma_k.$$  \hspace{1cm} (43)

Quantization of the Poisson algebra (43) leads to the su(2) Lie algebra commutator,

$$[\hat{\sigma}_i, \hat{\sigma}_j] = i(\theta/r)\epsilon_{ijk}\hat{\sigma}_k,$$  \hspace{1cm} (44)

whose unitary irreducible representations are the well known representations of the su(2) algebra. They are parameterized by the spin of the representation $J$. The dimensionality of such representation is $N = 2J + 1$. The two dimensional parameters: the radius of the sphere and the noncommutativity parameter are not independent. They satisfy instead,

$$r^4 = \theta^2 J(J+1).$$  \hspace{1cm} (45)

Again, arbitrary $(2J+1) \times (2J+1)$ matrix can be expanded in terms of symmetrized monomials in $\sigma_i$ — noncommutative spherical harmonics, which are the spherical analogues of the Weyl functions.

Turning back to the action one get exactly the same model as in the previous example with $N = 2J + 1$. As a result we get that independently from which geometry one starts one gets basically the same deformed description. The only meaningful parameter is the dimensionality of the matrix and it depends only on the worldsheet area. This is a manifestation of the universality of the matrix description which we plan to explore in the next sections.

3.2 Noncommutative membrane and the BFSS matrix model

Let us consider slightly more complicate example, namely that of the membrane. For the membrane one can write a Nambu–Goto action too,

$$S_{NG} = T_m \int_{\Sigma_3} d^3\sigma \sqrt{\det \partial_a X_\mu \partial_b X_\mu},$$  \hspace{1cm} (46)

where $T_m$ is the membrane tension and $X$ are the membrane embedding functions.
In the case when the topology of the worldvolume $\Sigma^3$ is of the type $\Sigma^3 = I \times \mathcal{M}_2$, where $\mathbb{R}^1$ is the time interval $I = [0, t_0]$ and $\mathcal{M}_2$ is a two dimensional manifold, one has the freedom to choose the worldsheet parameters $\sigma^i$, $i = 1, 2, 3$ in such a way that the time like tangential will be always orthogonal to the space-like tangential,

$$\partial_0 X^\mu \partial_a X_\mu = 0.$$  \hfill (47)

In this case the Nambu–Goto action takes the following form

$$S_{NG} = T_m \int d\tau d^2 \sigma \sqrt{\frac{1}{2} \dot{X}^2 \Sigma_{\mu\nu}},$$ \hfill (48)

where,

$$\Sigma_{\mu\nu} = \epsilon_{ab} \partial_a X_\mu \partial_b X_\nu.$$ \hfill (49)

In the complete analogy to the case of the string let us rewrite the Nambu–Goto action in the Polyakov form,

$$S_{NGP} = \int d^3 \sigma \eta \left[ \frac{T^2}{4} \dot{X}^2 + \frac{1}{4} \{X_\mu, X_\nu\}^2 \right],$$ \hfill (50)

where the (invariant) Poisson bracket is defined as

$$\{X, Y\} = \eta^{-1} \epsilon_{ab} \partial_a X \partial_b Y.$$ \hfill (51)

Since we partially fixed the reparametrization gauge invariance by choosing the time direction we have the constraint (47). This leads to the following constraint,

$$\{\dot{X}^\mu, X_\mu\} = 0.$$ \hfill (52)

Now, straightforwardly repeating the arguments of the previous subsection one can write down the matrix model action. In the present case the action takes the following form:

$$S_m = \int dt \left( \beta \text{tr} \frac{1}{2} \dot{X}^2 + \alpha \text{tr} \frac{1}{4} \{X_\mu, X_\nu\}^2 \right),$$ \hfill (53)

where $\beta = 2\pi T^2/\theta$ and $\alpha = 2\pi/\theta$, respectively. The action (53) should be supplemented with the following constraint:

$$[\dot{X}_\mu, X_\mu] = 0.$$ \hfill (54)

The constraint (54) can be added to the action (53) with the Lagrange multiplier $A_0$. In this case the action acquires the following form:

$$S_{gi} = \int dt \left( \beta \text{tr} \frac{1}{2} (\nabla_0 X_\mu)^2 + \alpha \text{tr} \frac{1}{4} \{X_\mu, X_\nu\}^2 \right),$$ \hfill (55)

which is identic (upto definition of parameters $\alpha$ and $\beta$) to the bosonic part of the BFSS action (11). By the redefinition of the matrix fields and rescaling
of the time one can eliminate the constants $\alpha$ and $\beta$, so in what follows we can put both to unity.

So far we have considered only the bosonic parts of the membrane. Including the fermions (when they exist) introduces no conceptual changes. Therefore, derivation of the fermionic parts of the IKKT and BFSS matrix model description of the string and membrane is entirely left to the reader.

**Exercise 6.** Derive the fermionic part of both matrix models starting from the superstring/supermembrane.

### 4 Equations of motion. Classical solutions

In this section we consider two types of theories, namely the string and the membrane in the Nambu–Goto–Polyakov form and the corresponding matrix models. One can write down equations of motion and try to find out some simple classical solutions in order to compare these cases among each other.

The static equations of motion in the membrane case coincide with the string equations of motion. Therefore, it is enough to consider only the last case: Any solution in the IKKT model has also the interpretation as a classical vacuum of the BFSS theory.

#### 4.1 Equations of motion before deformation: Nambu–Goto–Polyakov string

Consider first the equations of motion corresponding to the Nambu–Goto–Polyakov string in the form one gets just before the deformation procedure.

Variation of $X_\nu$ produces the following equations,

$$\{X_\mu, \eta^{-1} \{X_\mu, X_\nu\}\} = 0,$$

while the variation of $\eta$ produces the constraint

$$\eta^2 = \frac{1}{4} \frac{\{X_\mu, X_\nu\}^2}{T^2}.$$

(As in the Polyakov particle case the last equation can be used to eliminate $\eta$ from the action in order to get the original Nambu–Goto action.)

The equations of motion possess a large symmetry related to the reparametrization invariance. In order to find some solutions it is useful (but not necessary!) to fix this gauge invariance. As the use of the model is to describe branes, one may be interested in solutions corresponding to infinitely extended branes, which have the topology of $\mathbb{R}^2$. In this, simplest case one can impose the gauge $\eta = 1/4T^2$. Then, the equations of motion are reduced to
\[ \{ X_\mu, \{ X_\mu, X_\nu \} \} = 0, \quad \{ X_\mu, X_\nu \}^2 = 1. \] (57)

In the case of two dimensions \((\mu, \nu = 1, 2)\), one can find even the generic solution. It is given by an arbitrary canonical transformation \(X_{1,2} = X_{1,2}(\sigma_1, \sigma_2)\). This is easy to see if to observe that the second equation in (57) requires that the \(XX\) Poisson bracket must be a canonical one. The first equation is then satisfied automatically. One can also see that all the arbitrariness in the solution is due to the remnant of the reparametrization invariance which is given by the \textit{area preserving diffeomorphisms}. This situation is similar to one met in the case of two dimensional gauge theories where there are no physical degrees of freedom left to the gauge fields beyond the gauge arbitrariness. As we will see later, this similarity is not accidental, in some sense the above matrix model is indeed a two-dimensional gauge theory.

The situation is different in more than two dimensions. In this case we are not able to write down the generic solution, but one can find a significant particular one. The simplest solutions of (57) can be obtained by just lifting up the two-dimensional ones to higher dimensions. In particular, one has the following solution

\[ X_1 = \sigma_1, \quad X_2 = \sigma_2, \quad X_i = 0, \quad i = 3, \ldots, 10. \] (58)

It is not difficult to check that the solution (58) satisfy to both equations (57). The physical meaning of this solution is an infinite Euclidean brane extended in the plane \((1,2)\).

One can see, that by the nature of the model in which fields \(X_\mu\) are functions of a two dimensional parameter the solutions to the equations of motion are forced always to describe two dimensional surfaces i.e. single brane configurations. One can go slightly beyond this limitation allowing \(X\)'s to be multivalent functions of \(\sigma\)'s. In this case one is able to describe a certain set of multibrane systems, each sheet of \(X\) corresponding to an individual brane. This situation in application to spherical branes was analyzed in more details in [45].

Another question one may ask is whether one can find solutions describing a compact worldsheet. We are not going to give any proof of the fact that such type of solutions do not exist. Rather we consider a simple example of a cylindrical configuration and show that the equations of motion are not satisfied by it. An infinite cylinder as an extremal case of the torus can be given by the following parametric description:

\[ X_1 = \sin \sigma_1, \quad X_2 = \cos \sigma_1, \quad X_3 = \sigma_2. \] (59)

The eq. (59) describes a cylinder obtained from moving the circle in the plane \((1,2)\) along the axe 3. The parametrization (59) satisfy the constraint (47), therefore to see weather such surface is a classically stable it is enough to check the first equation of (57). The explicit evaluation of the equations of motion gives
\{X_\mu, \{X_\mu, X_1\}\} = -X_1 \neq 0, \quad (60)
\{X_\mu, \{X_\mu, X_2\}\} = -X_2 \neq 0, \quad (61)
\{X_\mu, \{X_\mu, X_0\}\} = 0. \quad (62)

As one see, only the equation of motion for the third noncompact direction is satisfied. Other equations can be satisfied if one modifies the action of the model by adding mass terms for e.g. \(X_1\) and \(X_2\):

\[ S \rightarrow S + m^2(X_1^2 + X_2^2). \quad (63) \]

**Exercise 7.** Modify the classical action in a way to allow the spherical brane solutions. Worldsheet quantize this model and compare it to the BMN matrix model.

Another interesting type of solutions is given by singular configurations with trivial Poisson bracket,

\[ \{X_\mu, X_\nu\} = 0. \quad (64) \]

Obviously, these configurations satisfy the equations of motion. This solution corresponds to an arbitrary open or closed smooth one-dimensional line embedded in \(\mathbb{R}^D\). The problem appears when one tries to make this type of solution to satisfy the constraint [177] arising from the gauge fixing \(\eta^2 = 1/4T^2\). This configuration, however is still an acceptable solution before the gauge fixing. The degeneracy of the two dimensional surface into the line results into the degeneracy of the two-dimensional surface reparametrization symmetry into the subgroup of the line reparametrizations. This means in particular that \(\eta^2 = 1/4T^2\) is not an acceptable gauge condition in this point, one must impose \(\eta = 0\) instead.

Let us now turn to the noncommutative case and see how the situation is changed there.

### 4.2 Equations of motion after deformation: IKKT/BFSS matrix models

After quantization of the worldsheet/worldvolume we are left with no Polyakov auxiliary field \(\eta\). The role of this field in the noncommutative theory is played by the choice of the representation. As most cases we can not smoothly variate the representation, we have no equations of motion corresponding to this parameter. So, we are left with only equations of motion corresponding to the variation of \(X\)’s. For the IKKT model these equations read

\[ [X_\mu, [X_\mu, X_\nu]] = 0, \quad (65) \]

while for the BFSS model the variation of \(X\) leads to the following dynamical equations,
\[ \ddot{X}_\mu + [X_\mu, [X_\mu, X_\nu]] = 0, \]  

(66)

where we also put the brane tension to unity: \( T = 1 \). If one is interested in only the static solutions (\( \dot{X} = 0 \)) to the BFSS equations of motion, then the equation (66) is reduced down to the IKKT equation of motion. Therefore, in what follows we consider only the last one.

By the first look at the equation (65) it is clear that one can generalize the string solution (58) from the commutative case. Namely, one can check that the configuration

\[ X_1 = \hat{\sigma}_1, \quad X_2 = \hat{\sigma}_2, \quad X_i = 0, \quad i = 3, \ldots, D, \]  

(67)

satisfy the equations of motion (65). By the analogy with the commutative case we can say that this configuration describes either Euclidean D-string (IKKT) or a static membrane (BFSS). The solution (67) corresponds to the Heisenberg algebra

\[ [X_1, X_2] = 1, \]  

(68)

which allows only the infinite-dimensional representation. The value of \( X \) are not bounded, therefore this solution corresponds to a noncompact brane.

What is the role of the \( \eta \)-constraint here? The algebra (68) does not completely specify the solution unless the nature of its representation is also given. In particular, the algebra of \( \hat{\sigma} \)'s can be irreducibly represented on the whole Hilbert space. In the semiclassical limit this can be seen to correspond to the constraint of the previous subsection.

As we discussed in the case of commutative string, any solution to the equations of motion describes a two dimensional surface and, therefore, has the Poisson bracket of the rank (in indices \( \mu \) and \( \nu \)) two or zero. In contrast to this, in the noncommutative case one may have solutions with an arbitrary even rank between zero and \( D \). Indeed, consider a configuration,

\[ X_a = p_a, \quad a = 1, \ldots, p + 1, \quad X_i = 0, \quad i = p + 2, \ldots, D, \]  

(69)

such that

\[ [p_a, p_b] = iB_{ab}, \quad \text{det } B \neq 0, \]  

(70)

where \( B \) is the matrix with c-number entries \( B_{ab} \). Such set of operators always exists if the Hilbert space is infinite dimensional separable. The set of operators \( p_a \) generate a Heisenberg algebra. Interesting cases are when the Heisenberg algebra (70) is represented irreducibly on the Hilbert space of the model, or when this irreducible representation is \( n \)-tuple degenerate. Analysis of these cases we will do in the next sections.

How about the compact branes? As we have already discussed in the previous section, the compact worldsheet solution corresponds to finite dimensional matrices \( X_\mu \). As it appears for such matrices the only solution to the equation of motion which exists is one with the trivial commutator,

\[ [X_\mu, X_\nu] = 0. \]  

(71)
To prove this fact, suppose we find such a solution with $B_{\mu\nu} = [X^{(0)}_\mu, X^{(0)}_\nu] \neq 0$ and satisfying the equations of motion (65). The IKKT action (BFSS energy) computed on such a solution is

$$S(X) = -\frac{1}{2} B_{\mu\nu}^2 \mathrm{tr} \mathbb{I} \neq 0.$$  

(72)

Since this is a solution to the equations of motion the variation of the action should vanish on the solution,

$$\delta S = \mathrm{tr} \frac{\delta S}{\delta X_\mu} (X^{(0)}) \delta X_\mu = 0, \quad \text{for } \forall \delta X_i,$$

(73)

which is not the case: Take $\delta X_\mu = \epsilon X^{(0)}_\mu$ to find out that $\delta S|_{X^{(0)}} \neq 0$. So there are no solutions with nontrivial commutator for the finite dimensional matrix space.

Consider now the extremal case of singular solutions with vanishing commutators,

$$[X_\mu, X_\nu] = 0.$$  

(74)

Obviously, from the equation (74) automatically follows that the equations are satisfied too. This solution exists in both finite as well as infinite-dimensional cases. Since the commutativity of $X_\mu$’s allows their simultaneous diagonalization

$$X_\mu = \begin{pmatrix} x^\mu_1 \\ x^\mu_2 \\ \ddots \end{pmatrix},$$

(75)

this means that the branes which are described by the matrix models are localized $x^\mu_k$ being the coordinates of the $k$-th brane.

The symmetry of the solutions

The various types of solutions have different symmetry properties. Thus, the solution of the type (69) with the algebra of $p_\alpha$’s irreducibly represented over the Hilbert space of the model has no internal symmetries. Indeed, by the Schur’s lemma any operator commuting with all $p_\alpha$ is proportional to the identity. In the case when the representation is $n$-tuple degenerate one has an $U(n)$ symmetry mixing the representations. The degenerate case (74), when $B_{\mu\nu} = 0$ give rise to some symmetries too. Indeed, an arbitrary diagonal matrix commute with all $X_\mu$ given by (75). If no two branes are in the same place: $x^\mu_m \neq x^\mu_n$ for any $m \neq n$, then the configuration breaks the $U(N)$ symmetry group (in the finite-dimensional case) down to the the Abelian subgroup $U(1)^N$. 
5 From the Matrix Theory to Noncommutative Yang–Mills

This and the following section is mainly based on the papers [40, 41, 43, 42, 39, 24], the reader is also referred to the lecture notes [44] and references therein.

The main idea is to use the solutions from the previous section both as classical vacua, such that arbitrary matrix configuration is regarded as a perturbation of this vacuum configuration, and as a basic set of operators in terms of which the above perturbations are expanded. Now follow the details.

5.1 Zero commutator case: gauge group of diffeomorphisms

Consider first the case of the solution with the vanishing commutator (74). We are interested in configurations in which the branes form a \( p \)-dimensional lattice. Using the rotational symmetry of the model, one can choose this lattice to be extended in the dimensions 1, ..., \( p \):

\[
X_a \equiv p_a, \quad a + 1, \ldots, p; \quad X_I = 0, \quad I = p + 1, \ldots, D. \tag{76}
\]

Then an arbitrary configuration can be represented as

\[
X_a = p_a + A_a, \quad X_I = \Phi_I. \tag{77}
\]

Let us take the limit \( N \to \infty \) and take such a distribution of the branes in which they form an infinite regular \( p \)-dimensional lattice:

\[
p_a \to \lambda n_a, \quad n_a \in \mathbb{Z}, \tag{78}
\]

such that the Hilbert space can be split in the product of \( p \) infinite-dimensional subspaces \( \mathcal{H}_a \)

\[
\mathcal{H} = \otimes_{a=1}^p \mathcal{H}_a, \tag{79}
\]

such that each eigenvalue \( \lambda n_a \) is non-degenerate in \( \mathcal{H}_a \). In this case the operators \( p_a \) can be regarded as \((-i\) times\) partial derivatives on a \( p \)-dimensional torus of the size \( 1/\lambda \),

\[
p_a = -i \partial_a. \tag{80}
\]

Now let us turn to the perturbation of the vacuum configuration and try to write it in terms of operators \( p_a \). Since the algebra of \( p_a \)'s is commutative, they alone fail to generate an irreducible representation in terms of which one can expand an arbitrary operator acting on the Hilbert space \( \mathcal{H} \). One must instead supplement this set with \( p \) other operators \( x^a \), which together with \( p_a \) form a Heisenberg algebra irreducibly represented on \( \mathcal{H} \),

\[
[x^a, x^b] = 0, \quad [p_a, x^b] = -i \delta^a_b. \tag{81}
\]
From the algebra (81) follows that the operators $x^a$ have a continuous spectrum which is bounded: $-\pi/\lambda \leq x^a < \pi/\lambda$. This precisely means that $x^a$ are operators of coordinates on the $p$-dimensional torus. Then, an arbitrary matrix $X$ can be represented as an operator function of the operators $p_a$ and $x^a$,

$$X = \hat{X}(\hat{p}, \hat{x}).$$

In the “$x$-picture” this will be a differential operator $X(-i\partial, x)$. There are many ways to represent a particular operator $X$ as a operator function of $p_a$ and $x^a$ which is related to the ordering. The Weyl ordering we will consider in the next subsection, here let us use a different one in which all operators $p_a$ are on the right to all $x^a$. In such an ordering prescription one can write down a Fourier expansion of the operator in the following form

$$X = \frac{1}{(2\pi)^p} \int d^p z \tilde{X}(z,x)e^{i\hat{p} \cdot z}. \quad (82)$$

In this parametrization the product of two operators is given by an involution product of the symbols:

$$\tilde{XY}(z,x) = \tilde{X} \ast \tilde{Y}(z,x) = \frac{1}{(2\pi)^p} \int d^p y \tilde{X}(y,x)\tilde{Y}(z-y,x+y). \quad (83)$$

The trace of an operator can be computed in a standard way, namely

$$\text{tr} X = \int d^px \langle x | X | x \rangle = \int d^px \tilde{X}(0,x) = \int d^px d^pl X(l,x), \quad (84)$$

where in the last part $X(l,x)$ is the normal symbol of which is obtained by the replacement of operator $\hat{p}_a$ by an ordinary variable $l_a$ in the definition (82),

$$X(l,x) = \frac{1}{(2\pi)^p} \int d^p z \tilde{X}(z,x)e^{il \cdot z}, \quad (85)$$

$$\tilde{X}(z,x) = \text{tr} e^{-i\hat{p} \cdot z} X. \quad (86)$$

Now we are ready to write down the whole matrix action (82) in terms of the normal symbols. It looks as follows,

$$S = \int d^pl d^px \left( -\frac{1}{4} \mathcal{F}_{ab}^2 + \frac{1}{2} (\nabla_a \Phi_I)^2 - \frac{1}{4} [\Phi_I, \Phi_J]_s^2 \right), \quad (87)$$

where

$$\mathcal{F}_{ab}(l,x) = \partial_a A_b(l,x) - \partial_b A_a(l,x) - [A_a, A_b]_s(l,x), \quad (88)$$

$$\nabla_a \Phi = \partial_a \Phi + [A_a, \Phi]_s(l,x), \quad (89)$$

$$[A, B]_s(l,x) = A \ast B(l,x) - B \ast A(l,x) \quad (90)$$
and the star product is defined as in (83).

The model defined by the action (87) has the meaning of Yang–Mills theory with the infinite dimensional gauge group of diffeomorphism transformations generated by the operators

\[ T_f = i f^a(x) \partial_a. \]  

(91)

Because of the noncommutative nature of the products involved in the action (87) the local gauge group is not commutative. However, if one tries to write down the group of global gauge symmetry, one finds out that this group is, in fact nothing else that U(1). Changing only slightly the character of the solution one can also get a non-Abelian global group. Indeed, consider the solution as in (76) with the exception that the Hilbert space is not just (79), but is given by the product of parts \( \mathcal{H}_a \) at some (positive integer) power \( n \):

\[ \mathcal{H} = (\otimes_{a=1}^p \mathcal{H}_a)^\otimes n. \]  

(92)

Repeating with this solution the same manipulations which lead us to (87) with the only exception that in this case an arbitrary matrix is represented by a \((n \times n)\)-matrix valued function instead of just “ordinary” one, we arrive to the action similar to (87) with the exception that the fields take their value in the \( u(n) \) algebra and the global gauge group is, respectively, U(\( n \)). We hope, that the things will clarify a lot when the reader will pass the next subsection.

**Ordinary gauge model?**

A question one may ask oneself is if the fluctuations of the matrix models can be restricted in such a way to get a “normal” Yang–Mills theory with a compact Lie group. In the present case one may restrict the fluctuations around the background (76) to depend on \( \hat{x}^a \) operators only. This aim can be achieved by imposing the following constraints on the matrices \( X_\mu \):

\[ [x_a, X_b] = i \delta_{ab}, \quad [x_a, X_I] = 0. \]  

(93)

Let us note that \( X_a \) and \( x_a \) do not form the Heisenberg algebra because the commutator between \( X_a \) do not necessarily vanish:

\[ [X_a, X_b] \equiv F_{ab} \neq 0. \]  

(94)

Dynamically, the constraint (93) can be implemented through the modification of the matrix action by the addition of the constraint (93) with the Lagrange multiplier. The modified matrix model action reads:

\[ S_c = \text{tr} \left( \frac{1}{4} [X_\mu, X_\nu]^2 + \rho_{\mu\nu} ([x_\mu, X_\nu] - \Delta_{\mu\nu}) + T^2 \right), \]  

(95)

where \( \rho_{\mu\nu} \) are the Lagrange multipliers, \( x_\mu = (x_a, 0) \) and \( \Delta_{\mu\nu} \) is equal to \( \delta_{ab} \) when \( (\mu\nu) = (ab) \) and zero otherwise. The limit \( N \to \infty \) of the matrix model specified by the action (95) produces the Abelian gauge model. Under similar setup one can obtain also nonabelian gauge models.
5.2 Nonzero commutator: Noncommutative Yang–Mills model

In this subsection we consider the matrix action as a perturbation of the background configuration given by (69) and (70). Here we plan to give a more detailed approach also partly justifying the result of the previous subsection. The operators \( p_a \) generate a \((p+1)/2\)-dimensional Heisenberg algebra. If this algebra is represented irreducibly on the Hilbert space of the model (which in fact is our choice), then an arbitrary operator acting on this space can be represented as an operator function of \( p_a \). Let us consider this situation in more details.

Irreducibility of the representation in particular means that any operator commuting with all \( p_a \) is a \( c \)-number constant. From this follows that the operators \( P_a = [p_a, \cdot] \),

\[
P_a = [p_a, \cdot], \quad (96)
\]

which are Hermitian on the space of square trace operators equipped with the scalar product \((A,B) = \text{tr} A^\ast B\), are diagonalizable and have non-degenerate eigenvalues.

**Exercise 8.** Prove this!

By a direct check one can verify that the operator \( e^{ik_a \hat{x}^a} \), where \( \hat{x}^a = \theta^{ab} \hat{p}_b \), \( \theta \equiv B^{-1} \) is an eigenvector for \( P_a \) with the eigenvalue \( k_a \):

\[
P_a e^{ik_a \hat{x}^a} = [p_a, e^{ik_a \hat{x}^a}] = k_a e^{ik_a \hat{x}^a}. \quad (97)
\]

This set of eigenvectors form an orthogonal basis (\( P_a \)'s are Hermitian). One can normalize the eigenvectors to delta function trace,

\[
E_k = c_k e^{ik_a \hat{x}^a}, \quad \text{tr} E_k^* E_k = \delta(k' - k). \quad (98)
\]

The normalizing coefficients \( c_k \) can be found from evaluating explicitly the trace of \( e^{i(k-k')\hat{x}} \) in (95) and equating it to the Dirac delta. Let us compute this trace and find the respective quotients. To do this, consider the basis where the set of operators \( x^\mu \) splits in pairs \( p_i, q_i \) satisfying the standard commutation relations: \([p_i, q_j] = -i\theta \delta_{ij}\).

As we know from courses of Quantum Mechanics the trace of the operator

\[
e^{-ik'\hat{x}} e^{ik\hat{x}} = e^{i(k-k')\hat{x}} e^{\hat{x} \cdot k}, \quad (99)
\]

can be computed in \( q \)-representation as,

\[
\text{tr} e^{i(k-k')\hat{x}} e^{\hat{x} \cdot k} = \int dq \langle q | e^{-i(l_i' - l_i)q^i + (z_i' - z_i)p_i} | q \rangle = 1/|c_k|^2 \delta(k' - k), \quad (100)
\]

where \( |q \rangle \) is the basis of eigenvectors of \( q^i \),

\[
q^i | q \rangle = q^i | q \rangle, \quad \langle q' | q \rangle = \delta(q' - q). \quad (101)
\]
and \( l_i, z^i \) \((l_i, z^i)\) are components of \( k_\mu \) \((k'_\mu)\) in the parameterizations: \( x^\mu \to p_i, q^i \). Explicit computation gives,

\[
\frac{1}{|c_k|^2} = \frac{(2\pi)^2}{\sqrt{\text{det} \theta}}.
\]  

(102)

Now, we have the basis of eigenvectors \( E_k \) and can write any operator \( F \) in terms of this basis,

\[
\hat{F} = \int dk \tilde{F}(k) e^{ik\hat{x}},
\]

(103)

where the “coordinate” \( \tilde{F}(k) \) is given by,

\[
\tilde{F}(k) = \frac{\sqrt{\text{det} \theta}}{(2\pi)^{\frac{p}{2}}} \text{tr}(e^{-ik\hat{x}} \hat{F}).
\]

(104)

Function \( \tilde{F}(k) \) can be interpreted as the Fourier transform of a \( L^2 \) function \( F(x) \),

\[
F(x) = \int dk \tilde{F}(k) e^{ikx} = \sqrt{\text{det} \theta} \int \frac{dk}{(2\pi)^{p/2}} e^{ikx} \text{tr} e^{-ik\hat{x}} \hat{F}.
\]

(105)

And vice versa, to any \( L^2 \) function \( F(x) \) from one can put into correspondence an \( L^2 \) operator \( \hat{F} \) by inverse formula,

\[
\hat{F} = \int \frac{dx}{(2\pi)^{p/2}} \int \frac{dk}{(2\pi)^{p/2}} F(x) e^{ik(x-x)}. 
\]

(106)

Equations (105) and (106) providing a one-to-one correspondence between \( L^2 \) functions and operators with finite trace,

\[
\text{tr } \hat{F} \cdot F < \infty,
\]

(107)
give in fact formula for the Weyl symbols. By introducing distributions over this space of operators one can extend the above map to operators with unbounded trace.

**Exercise 9.** Check that (105) and (106) lead in terms of distributions to the correct Weyl ordering prescription for polynomial functions of \( p_\mu \).

Let us note, that the map (105) and (106) can be rewritten in the following form,

\[
F(x) = (2\pi)^{p/2} \sqrt{\text{det} \theta} \text{tr} \delta(x-x) \hat{F}, \quad \hat{F} = \int d^p x \delta(x-x) F(x),
\]

(108)

where we introduced the operator,

\[
\delta(x-x) = \int \frac{d^p k}{(2\pi)^p} e^{ik(x-x)}.
\]

(109)
This operator satisfy the following properties,

\[ \int d^p x \hat{\delta}(\hat{x} - x) = \mathbb{I}, \]  
(110a)

\[ (2\pi)^{p/2} \sqrt{\det \theta} \text{tr} \hat{\delta}(\hat{x} - x) = 1, \]  
(110b)

\[ (2\pi)^{p/2} \sqrt{\det \theta} \text{tr} \hat{\delta}(\hat{x} - x) \hat{\delta}(\hat{x} - y) = \delta(x - y), \]  
(110c)

where in the r.h.s. of last equation is the ordinary delta function. Also, operators \( \hat{\delta}(\hat{x} - x) \) for all \( x \) form a complete set of operators,

\[ [\hat{\delta}(\hat{x} - x), F] \equiv 0 \Rightarrow F \propto i. \]  
(110d)

The commutation relations of \( \hat{x}^\mu \) also imply that \( \hat{\delta}(\hat{x} - x) \) should satisfy,

\[ [\hat{x}^\mu, \hat{\delta}(\hat{x} - x)] = i\theta^{\mu\nu} \partial_\nu \hat{\delta}(\hat{x} - x). \]  
(110e)

In fact one can define alternatively the noncommutative plane starting from operator \( \hat{\delta}(\hat{x} - x) \) satisfying (110), with \( \hat{x}^\mu \) defined by,

\[ \hat{x}^\mu = \int d^p x x^\mu \hat{\delta}(\hat{x} - x). \]  
(111)

In this case (110e) provides that \( \hat{x}^\mu \) satisfy the Heisenberg algebra (69), while the property (110d) provides that they form a complete set of operators. Relaxing these properties allows one to introduce a more general noncommutative spaces.

Let us the operator \( \hat{\delta}(x) \) in the simplest case of two-dimensional noncommutative plane. The most convenient is to find its matrix elements \( D_{mn}(x) \) in the oscillator basis given by,

\[ |n\rangle = (\hat{a}^\dagger)^n |0\rangle, \quad \hat{a} |0\rangle = 0, \]  
(112)

where the oscillator operators \( \hat{a} \) and \( \hat{a}^\dagger \) are the noncommutative analogues of the complex coordinates,

\[ \hat{a} = \sqrt{\frac{1}{2\theta}} (\hat{x}^1 + i\hat{x}^2), \quad \hat{a}^\dagger = \sqrt{\frac{1}{2\theta}} (\hat{x}^1 - i\hat{x}^2); \quad [\hat{a}, \hat{a}^\dagger] = 1. \]  
(113)

Then the matrix elements read

\[ D_{mn}(x) = \langle m | \hat{\delta}^{(2)}(\hat{a} - z) |n\rangle = \text{tr} \hat{\delta}^{(2)}(\hat{a} - z) P_{nm}, \]  
(114)

where \( P_{nm} = |n\rangle \langle m| \).

As one can see, up to a Hermitian transposition the matrix elements of \( \hat{\delta}(\hat{x} - x) \) correspond to the Weyl symbols of operators like \( |m\rangle \langle n| \), or so called Wigner functions. The computation of (110) gives,\(^6\)

\(^6\) For the details of computation see e.g. [20].
\[ D_{mn}^\theta(z, \bar{z}) = (-1)^n \left( \frac{2}{\sqrt{\theta}} \right)^{m-n+1} \sqrt{\frac{n!}{m!}} e^{-\frac{z\bar{z}}{\theta}} \left( \frac{z^m}{\bar{z}^n} \right) L_n^{m-n}(2z\bar{z}/\theta), \]  

where \( L_n^{m-n}(x) \) are Laguerre polynomials,

\[ L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \left( \frac{d}{dx} \right)^n (e^{-x} x^{\alpha+n}). \]

It is worthwhile to note that in spite of its singular origin the symbol of the delta operator is a smooth function which is rapidly vanishing at infinity. The smoothness comes from the fact that the operator elements are written in an \( L^2 \) basis. In a non-\( L^2 \) basis, e.g. in the basis of \( x_1 \) eigenfunctions \( D^\theta \) would have more singular form.

The above computations can be generalized to \( p \)-dimensions. Written in the complex coordinates \( z_i, \bar{z}_i \) corresponding to oscillator operators (113), which diagonalize the noncommutativity matrix this looks as follows,

\[ D_{mn} = D_{m_1 n_1}^{\theta(1)}(z_1, \bar{z}_1)D_{m_2 n_2}^{\theta(2)}(z_2, \bar{z}_2) \ldots D_{m_p n_p}^{\theta(p/2)}(z_{p/2}, \bar{z}_{p/2}), \]

where,

\[ [z_i, \bar{z}_j]_* = \delta_{ij}, \quad i = 1, \ldots, p/2. \]

Having the above map one can establish the following relations between operators and their Weyl symbols.

1. It is not difficult to derive that,

\[ (2\pi)^{p/2} \sqrt{\det \theta} \text{tr} F = \int dx \, F(x). \]

2. The (noncommutative) product of operators is mapped into the star or Moyal product of functions,

\[ F \cdot G \rightarrow F \ast G(x), \]

where \( F \ast G(x) \) is defined as,

\[ F \ast G(x) = e^{-\frac{i}{2} \theta_{\mu\nu} \partial_\mu \partial_\nu} F(x) G(x') \bigg|_{x' = x}. \]

In terms of operator \( \hat{\delta}(\hat{x} - x) \), this product can be written as follows,

\[ F \ast G(x) = \int d^p y d^p z K(x; y, z) F(y) G(z), \]

where,

\[ K(x; y, z) = (2\pi)^{p/2} \sqrt{\det \theta} \text{tr} \hat{\delta}(\hat{x} - x) \hat{\delta}(\hat{x} - y) \hat{\delta}(\hat{x} - z) = e^{\frac{i}{2} \theta_{\mu\nu} \partial_\mu \partial_\nu} \delta(y - x) \delta(z - x), \]
\[ \partial_y^\mu \text{ and } \partial_z^\mu \text{ are, respectively, } \partial/\partial y^\mu \text{ and } \partial/\partial z^\mu, \text{ and in the last line one has ordinary delta functions.} \]

On the other hand the ordinary product of functions was not found to have any reasonable meaning in this context.

3. One property of the star product is that in the integrand one can drop it once because of,

\[ \int d^p x F \star G(x) = \int d^p x F(x)G(x), \tag{124} \]

were in the r.h.s the ordinary product is assumed.

4. Interesting feature of this representation is that partial derivatives of Weyl symbols correspond to commutators of respective operators with \[ i \mathbf{p}_\mu, \]

\[ [i \mathbf{p}_\mu, \mathbf{F}] \rightarrow i(p_\mu * F - F * p_\mu)(x) = \frac{\partial F(x)}{\partial x^\mu}, \tag{125} \]

where \[ p_\mu \] is linear function of \[ x^\mu \]: \[ p_\mu = -\theta^{-1} \mu^\nu x^\nu. \]

This is an important feature of the star algebra of functions distinguishing it from the ordinary product algebra. In the last one can not represent the derivative as an \textit{internal automorphism} while in the star algebra it is possible due to its nonlocal character. This property is of great importance in the field theory since, as it will appear later, it is the source of duality relations in noncommutative gauge models which we turn to in the next section.

\textbf{Exercise 10.} Derive equations (119)–(125).

Let us turn back to the matrix model action (32) and represent an arbitrary matrix configuration as a perturbation of the background (69):

\[ X_a = p_a + A_a, \quad X_I = \Phi_I, \quad a = 1, \ldots, p + 1, \quad I = p + 2, \ldots, D. \tag{126} \]

Passing from operators \[ A_a \text{ and } \Phi \] to their Weyl symbols using (108), (120) and (125) one gets following representation for the matrix action (32):

\[ S = \int d^p x \left( -\frac{1}{4}(F_{ab} - B_{ab})^2 + \frac{1}{2}(\nabla_a \Phi_I)^2 - \frac{1}{4}[\Phi_I, \Phi_J]^2 \right), \tag{127} \]

where,

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]. \tag{128} \]

In the case of the irreducible representation of the algebra (70) this describes the U(1) gauge model.

One can consider a \( n \)-tuple degenerate representation in this case as well. As in the previous case the index labelling the representations become an internal symmetry index and the global gauge group of the model becomes U(\( n \)). Indeed, the operator basis in which one can expand an arbitrary operator now is given by,
\[ E_k^\alpha = \sigma^\alpha \otimes e^{i k \cdot \hat{x}}, \]  

(129)

where \( \sigma^\alpha, \alpha = 1, \ldots, n^2 \) are the adjoint generators of the \( u(n) \) algebra. The can be normalized to satisfy,

\[ [\sigma^\alpha, \sigma^\beta] = i \epsilon^{\alpha\beta\gamma} \sigma^\gamma, \quad \text{tr}_{su(2)} \sigma^\alpha \sigma^\beta = \delta^{\alpha\beta}, \]

(130)

where \( \epsilon^{\alpha\beta\gamma} \) are the structure constants of the \( u(n) \) algebra:

\[ \epsilon^{\alpha\beta\gamma} = -i \text{tr}_{su(2)} [\sigma^\alpha, \sigma^\beta] \sigma^\gamma, \]

(131)

which follows from (130). Then an operator \( \hat{F} \) is mapped to the following function \( F(x) \):

\[
F^\alpha(x) = \sqrt{\det \theta} \int \frac{d^p k}{(2\pi)^{p/2}} e^{i k \cdot \hat{x}} \text{tr} \left\{ (\sigma^\alpha \otimes e^{i k \cdot \hat{x}} \cdot \hat{F}) \right\} = (2\pi)^{p/2} \sqrt{\det \theta} \text{tr} \left\{ (\sigma^\alpha \otimes \hat{\delta}(\hat{x} - x)) \cdot \hat{F} \right\}. \quad (132)
\]

The equation (132) gives the most generic map from the space of operators to the space of \( p \)-dimensional \( u(n) \)-algebra valued functions.

**Exercise 11.** Prove that \( p \) is always even.

Just for the sake of completeness let us give also the formula for the inverse map,

\[ \hat{F} = \int d^p x (\sigma^\alpha \otimes \hat{\delta}(\hat{x} - x)) F^\alpha(x), \]

(133)

Applying the map (132) and (133) to the IKKT matrix model or to the BFSS one, one gets, respectively, the \( p \) or \( p + 1 \) dimensional noncommutative \( u(n) \) Yang–Mills model.

**Exercise 12.** Derive the \( p \)- and \( (p + 1) \)-dimensional noncommutative supersymmetric gauge model from the matrix actions (32) and (55), using the map (132) and its inverse (133).

Some comments regarding both gauge models described by the actions (87) and (128) are in order. In spite of the fact that both models look very similar to the “ordinary” Yang–Mills models, the perturbation theory of this models are badly defined in the case of noncompact noncommutative spaces. In the first case the non-renormalizable divergence is due to extra integrations over \( l \) in the “internal” space. In the case of noncommutative gauge model the behavior of the perturbative expansion is altered by the IR/UV mixing. The supersymmetry or low dimensionality improves the situation allowing the “bad” terms to cancel (see [34, 9, 38, 11]). On the other hand the compact noncommutative spaces provide both IR and UV cut off and the field theory on such spaces is finite [36]. In the case of zero commutator background the behavior of the perturbative expansion depends on the eigenvalue distribution.
Faster the eigenvalues increase, better the expansion converge. However there is always the problem of the zero modes corresponding to the diagonal matrix excitations (functions of commutative $p_a$’s). There is a hope that integrating over the remaining modes helps to generate a dynamical term for the zero modes too. Indeed, for purely bosonic model one has a repelling potential after the one-loop integration of the non-diagonal modes. The fermions contribute with the attractive potential. In the supersymmetric case the repelling bosonic contribution is cancelled by the attractive fermionic one and diagonal modes remain non-dynamical [26].

Exercise 13. Consider the Eguchi–Kawai model given by the action (9). Write down the equations of motion and find the classical solutions analogous to (69). One can have noncommutative solutions even for finite $N$, explain why? Consider arbitrary matrix configuration as a perturbation of the above classical backgrounds and find the resulting models. What is the space on which this models live? How the same space can be obtained from a non-compact matrix model.

We considered exclusively the bosonic models. When the supersymmetric theories are analyzed one has to deal also with the fermionic part. In the case of compact noncommutative spaces which correspond to finite size matrices one has a discrete system with fermions. In the lattice gauge theories with fermions there is a famous problem related to the fermion doubling [30]. Concerning the theories on the compact noncommutative spaces it was found that in some cases one can indeed have fermion doubling [41] some other cases were reported to be doubling free and giving alternative solutions to the long standing lattice problem [3].

6 Matrix models and dualities of noncommutative gauge models

In the previous section we realized that the matrix model from different “points” of the moduli space of classical solutions looks like different gauge models. These models can have different dimensionality or different global gauge symmetry group, but they all are equivalent to the original IKKT or BFSS matrix model. This equivalence can be used to pass from some noncommutative model back to the matrix model and then to a different noncommutative model and viceversa. Thus, one can find a one-to-one map from one model to an equivalent one.

In reality one can jump the intermediate step by writing a new solution direct in the noncommutative gauge model and passing to Weyl (re)ordered description with respect to the new background. From the point of view of noncommutative geometry this procedure is nothing else that the change of

\footnote{For the case of the unitary Eguchi–Kawai-type model with fermions see [29].}
the noncommutative variable taking into account also the ordering. Let us go to the details. Consider two different background solutions given by \( p_{\mu(i)}^{(i)} \), where \( \mu(i) = 1, \ldots, p(i) \) and the index \( i = 1, 2 \) labels the backgrounds. Denote the orders of degeneracy of the backgrounds by \( n(i) \). The commutator for both backgrounds is given by,

\[
[p_{\mu(i)}^{(i)}, p_{\nu(i)}^{(i)}] = iB_{\mu(i)\nu(i)}^{(i)}. \tag{134}
\]

Applying to a \( p_{(1)} \)-dimensional \( \mathfrak{u}(n_{(1)}) \) algebra valued field \( F^{(1)}(x(1)) \) first the inverse Weyl transformation which maps it in the operator form and then the direct transformation from the operator form to the second background one gets a \( p_{(2)} \)-dimensional \( \mathfrak{u}(n_{(2)}) \) algebra valued field \( F^{(2)}(x(2)) \) defined by

\[
F^{(2)}(x(2)) = \int d^{p(1)} x(1) K^{(1)}_{(2)}(x(2)|x(1)) F^{(1)}(x(1)), \tag{135}
\]

where the kernel \( K^{(1)}_{(2)}(x(2)|x(1)) \) is given by,

\[
K^{(1)}_{(2)}(x(2), x(1)) = (2\pi)^{p(2)/2} \sqrt{\det \theta(2)} \times
\]

\[
\text{tr} \left\{ (\sigma^{(2)}_\alpha \otimes \hat{\delta}(\hat{x}(2) - x(2))) \cdot (\sigma^{(1)}_\alpha \otimes \hat{\delta}(\hat{x}(1) - x(1))) \right\}, \tag{136}
\]

where \( x(i) \) and \( \sigma^{(i)}_\alpha \) are the coordinate and algebra generators corresponding to the background \( p_{\mu(i)}^{(i)} \).

The equation still appeals to the background independent operator form by using the \( \hat{\delta} \)-operators and trace. This can be eliminated in the following way. Consider the functions \( x^{(2)}(x^{(1)}, \sigma^{(1)}_\alpha) = x^{(2)}(x^{(1)}|\sigma^{(1)}_\alpha) \) and \( \sigma^{(2)}(x^{(1)} \cdot \sigma^{(1)}_\alpha) = \sigma^{(2)}(x^{(1)}|\sigma^{(1)}_\alpha) \) which are the symbols of the second background \( x^{(2)} \) which are Weyl-ordered with respect to the first background. Namely, they are the solution to the equation,

\[
\sigma^{(2)}_\alpha * (x^{(2)}\sigma^{(2)}_\beta - x^{(2)}\sigma^{(2)}_\gamma) = \theta^{(2)}_{\mu(2)\nu(2)}, \tag{137}
\]

and for \( \sigma^{(2)} \)

\[
\sigma^{(2)}_\alpha * (\sigma^{(2)}_\beta - \sigma^{(2)}_\gamma) = i e^{\alpha(2)\beta(2)\gamma(2)} \sigma^{(2)}_\gamma \tag{138}
\]

where \( * \) includes both the noncommutative with \( \theta(1) \) and the \( \mathfrak{u}(n_{(1)}) \) matrix products and we did not write explicitly the arguments \( (x^{(1)}, \sigma^{(1)}_\alpha) \) and \( \mathfrak{u}(n_{(1)}) \) matrix indices of \( x^{(2)} \) and \( \sigma^{(2)} \). Then, the kernel can be rewritten in the \( x^{(1)} \) background as follows,

\[
K^{(1)}_{(2)}(x(2), x(1)) = \sqrt{\frac{\det 2\pi\theta(2)}{\det 2\pi\theta(1)}} \sigma^{(1)}_{\gamma(1)} \delta(2) \left( \sigma^{(2)}_\alpha | * (\sigma^{(2)}_\gamma (x^{(2)}(x^{(1)}) - x^{(2)})) \right), \tag{139}
\]
where $\delta^{\gamma}_{\alpha\beta} = \text{tr}(\sigma^\gamma(1) \sigma^\beta(1))$ and

$$\delta^{\gamma}_{\alpha\beta}(x^{(2)}(x^{(1)}) - x^{(2)}) = \int \frac{d^p(2)}{(2\pi)^{p/2}} \text{tr}(\sigma^\gamma(1) e^{i(x^{(2)}(x^{(1)}) - x^{(2)})})$$

(140)

e_{\star}(x^{(2)}) is the star exponent computed with the noncommutative structure corresponding to $\star$.

General expression for the basis transform (135) with the kernel (136) or (139) looks rather complicated almost impossible to deal with. Therefore it is useful to consider some particular examples which we take from (23) which show that in fact the objects are still treatable.

6.1 Example 1: The $U(1) \rightarrow U(n)$ map

Let us present the explicit construction for the map from $U(1)$ to $U(2)$ gauge model in the case of two-dimensional non-commutative space. The map we are going to discuss can be straightforwardly generalised to the case of arbitrary even dimensions as well as to the case of arbitrary $U(n)$ group.

The two-dimensional non-commutative coordinates are,

$$[x^1, x^2] = i\theta.$$ 

(141)

As we already discussed, non-commutative analog of complex coordinates is given by oscillator rising and lowering operators,

$$a = \sqrt{\frac{1}{2\theta}}(x^1 + ix^2), \quad \bar{a} = \sqrt{\frac{1}{2\theta}}(x^1 - ix^2)$$

(142)

$$a \ket{n} = \sqrt{n} \ket{n-1}, \quad \bar{a} \ket{n} = \sqrt{n+1} \ket{n+1},$$

(143)

where $\ket{n}$ is the oscillator basis formed by eigenvectors of $N = \bar{a}a$,

$$N \ket{n} = n \ket{n}.$$ 

(144)

The gauge symmetry in this background is non-commutative $U(1)$.

We will now construct the non-commutative $U(2)$ gauge model. For this, consider the $U(2)$ basis which is given by following vectors,

$$\ket{n'}, a = \ket{n'} \otimes e_a, \quad a = 0, 1$$

(145)

$$e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

(146)

where $\{\ket{n'}\}$ is the oscillator basis and $\{e_a\}$ is the “isotopic” space basis.

The one-to-one correspondence between $U(1)$ and $U(2)$ bases can be established in the following way (29),

$$\ket{n'} \otimes e_a \sim \ket{n} = \ket{2n' + a},$$

(147)
where $|n\rangle$ is a basis element of the U(1)-Hilbert space and $|n\rangle \otimes e_\alpha$ is a basis element of the Hilbert space of U(2)-theory. (Note, that they are two bases of the same Hilbert space.)

Let us note that the identification (147) is not unique. For example, one can put an arbitrary unitary matrix in front of $|n\rangle$ in the r.h.s. of (147). This in fact describes all possible identifications and respectively maps from U(1) to U(2) model.

Under this map, the U(2) valued functions can be represented as scalar functions in U(1) theory. For example, constant U(2) matrices are mapped to particular functions in U(1) space. To find these functions, it suffices to find the map of the basis of the $u(2)$ algebra given by Pauli matrices $\sigma_\alpha$, $\alpha = 0, 1, 2, 3$.

In the U(1) basis Pauli matrices look as follows,

$$\sigma_0 = \sum_{n=0}^{\infty} (|2n\rangle \langle 2n| + |2n + 1\rangle \langle 2n + 1|) \equiv I,$$  

$$\sigma_1 = \sum_{n=0}^{\infty} (|2n\rangle \langle 2n + 1| + |2n + 1\rangle \langle 2n|),$$  

$$\sigma_2 = -i \sum_{n=0}^{\infty} (|2n\rangle \langle 2n + 1| - |2n + 1\rangle \langle 2n|),$$  

$$\sigma_3 = \sum_{n=0}^{\infty} (|2n\rangle \langle 2n| - |2n + 1\rangle \langle 2n + 1|),$$

while the “complex” coordinates $a'$ and $\bar{a}'$ of the U(2) invariant space are given by the following,

$$a' = \sum_{n=0}^{\infty} \sqrt{n} (|2n - 2\rangle \langle 2n| + |2n - 1\rangle \langle 2n + 1|),$$  

$$\bar{a}' = \sum_{n=0}^{\infty} \sqrt{n + 1} (|2n + 2\rangle \langle 2n| + |2n + 3\rangle \langle 2n + 1|).$$

One can see that when trying to find the Weyl symbols for operators given by (148), (149), one faces the problem that the integrals defining the Weyl symbols diverge. This happens because the respective functions (operators) do not belong to the non-commutative analog of $L^2$ space (are not square-trace).

Let us give an alternative way to compute the functions corresponding to operators (148) and (149). To do this let us observe that operators

$$H_+ = \sum_{n=0}^{\infty} |2n\rangle \langle 2n|,$$

and
\[ \Pi_+ = \sum_{n=0}^{\infty} |2n + 1\rangle \langle 2n + 1|, \quad (151) \]

can be expressed as\(^8\)
\[ \Pi_+ = \frac{1}{2} \sum_{n=0}^{\infty} \left(1 + \sin \pi \left(n + \frac{1}{2}\right)\right) |n\rangle \langle n| \rightarrow \frac{1}{2} \left(1 + \sin_\ast \pi \left(\bar{z}z + \frac{1}{2}\right)\right), \quad (152) \]

and,
\[ \Pi_- = i - \Pi_+ = \frac{1}{2} \left(1 - \sin \pi \left(\bar{z}z + \frac{1}{2}\right)\right) = \frac{1}{2} \left(1 - \sin \pi |z|^2\right), \quad (153) \]

where \(\sin_\ast\) is the “star” \(\sin\) function defined by the star Taylor series,
\[ \sin_\ast f = f - \frac{1}{3!} f * f * f + \frac{1}{5!} f * f * f * f * f - \cdots, \quad (154) \]

with the star product defined in variables \(z, \bar{z}\) as follows,
\[ f * g(\bar{a}, a) = e^{\bar{a}\partial \bar{a} - \bar{a}\partial a} f(\bar{z}, z)g(z', z')|_{z'=z}, \quad (155) \]

where \(\partial = \partial / \partial z, \ \bar{\partial} = \partial / \partial \bar{z}\) and analogously for primed \(z'\) and \(\bar{z'}\). For convenience we denoted Weyl symbols of \(a\) and \(\bar{a}\) as \(z\) and \(\bar{z}\).

The easiest way to compute \(152\) and \(153\) is to find the Weyl symbol of the operator,
\[ I_k^\pm = \frac{1 \pm \sin (\bar{a}a + \frac{1}{2})}{(\bar{a}a + \gamma)^k}, \quad (156) \]

where \(\gamma\) is some constant, mainly \(\pm 1/2\).

For sufficiently large \(k\), the operator \(I_k^\pm\) becomes square trace for which the formula \(154\) defining the Weyl map is applicable. The result can be analytically continued for smaller values of \(k\), using the following recurrence relation,
\[ I_{k-m}^\pm(\bar{z}, z) = \left(|z|^2 + \gamma - \frac{1}{2}\right) * \cdots * \left(|z|^2 + \gamma - \frac{1}{2}\right) * I_k^\pm(\bar{z}, z). \quad (157) \]

The last equation requires computation of only finite number of derivatives of \(I_k^\pm(\bar{z}, z)\) arising from the star product with polynomials in \(\bar{z}, z\).

**Exercise 14.** Compute the Weyl symbol for the operator \(156\).

\(^8\) Weyl symbols of \(a\) and \(\bar{a}\) are denoted, respectively, as \(z\) and \(\bar{z}\). The same rule applies also to primed variables.
6.2 Example 2: Map between different dimensions

Consider the situation when the dimension is changed. This topic was considered in [43, 39].

Consider the Hilbert space $\mathcal{H}$ corresponding to the representation of the two-dimensional non-commutative algebra (141), and $\mathcal{H} \otimes \mathcal{H}$ (which is in fact isomorphic to $\mathcal{H}$) which corresponds to the four-dimensional non-commutative algebra generated by

$$[x^1, x^2] = i\theta_{(1)}, \quad [x^3, x^4] = i\theta_{(2)}.$$

(158)

In the last case non-commutative complex coordinates correspond to two sets of oscillator operators, $a_1$, $a_2$ and $\bar{a}_1$, $\bar{a}_2$, where,

$$a_1 = \sqrt{\frac{1}{2\theta_{(1)}}}(x^1 + ix^2), \quad \bar{a}_1 = \sqrt{\frac{1}{2\theta_{(1)}}}(x^1 - ix^2) \quad (159a)$$

$$a_1 |n_1\rangle = \sqrt{n_1} |n_1 - 1\rangle, \quad \bar{a}_1 |n_1\rangle = \sqrt{n_1 + 1} |n_1 + 1\rangle, \quad (159b)$$

$$a_2 = \sqrt{\frac{1}{2\theta_{(2)}}}(x^3 + ix^4), \quad \bar{a}_2 = \sqrt{\frac{1}{2\theta_{(2)}}}(x^3 - ix^4) \quad (159c)$$

$$a |n_2\rangle_2 = \sqrt{n_2} |n_2 - 1\rangle, \quad \bar{a}_2 |n_2\rangle = \sqrt{n_2 + 1} |n_2 + 1\rangle, \quad (159d)$$

and the basis elements of the “four-dimensional” Hilbert space $\mathcal{H} \otimes \mathcal{H}$ are $|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle$.

The isomorphic map $\sigma : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$ is given by assigning a unique number $n$ to each element $|n_1, n_2\rangle$ and putting it into correspondence to $|n\rangle \in \mathcal{H}$. So, the problems is reduced to the construction of an isomorphic map from one-dimensional lattice of e.g. nonnegative integers into the two-dimensional quarter-infinite lattice. This can be done by consecutive enumeration of the two-dimensional lattice nodes starting from the angle (00). The details of the construction can be found in Refs. [43, 39].

As we discussed earlier, this map induces an isomorphic map of gauge and scalar fields from two to four dimensional non-commutative spaces.

This can be easily generalized to the case with arbitrary number of factors $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ corresponding to $p/2$ “two-dimensional” non-commutative spaces. In this way, one obtains the isomorphism $\sigma$ which relates two-dimensional non-commutative function algebra with a $p$-dimensional one, for $p$ even.

7 Example 3: Change of $\theta$

So far, we have considered maps which relate algebras of non-commutative functions in different dimensions or at least taking values in different Lie algebras. Due to the fact that they change considerably the geometry, these maps could not be deformed smoothly into the identity map. In this section
we consider a more restricted class of maps which do not change either dimensionality or the gauge group but only the non-commutativity parameter. Obviously, this can be smoothly deformed into identity map, therefore one may consider infinitesimal transformations.

The new non-commutativity parameter is given by the solution to the equations of motion. In this framework, the map is given by the change of the background solution $p_\mu$ by an infinitesimal amount: $p_\mu + \delta p_\mu$. Then, a solution with the constant field strength $F^{(\delta p)}_{\mu\nu}$ will change the non-commutativity parameter as follows,

$$\theta^{\mu\nu} + \delta\theta^{\mu\nu} \equiv (\theta^{-1}_{\mu\nu} + \delta\theta^{-1}_{\mu\nu})^{-1} = (\theta^{-1}_{\mu\nu} + F_{\mu\nu})^{-1}. \quad (160)$$

Note, that the above equation does not require $\delta\theta$ to be infinitesimal.

Since we are considering solutions to the gauge field equations of motion $A_\mu = \delta p_\mu$, one should fix the gauge for it. A convenient choice would be e.g. the Lorentz gauge, $\partial_\mu \delta p_\mu = 0$. Then, the solution with

$$A^{(\delta p)}_\mu \equiv \delta p_\mu = (1/2)\epsilon_{\mu\alpha} \theta^{\alpha\beta} p_\beta \quad (161)$$

with antisymmetric $\epsilon_{\mu\nu}$ has the constant field strength

$$F^{(\delta p)}_{\mu\nu} \equiv \delta\theta^{-1}_{\mu\nu} = \epsilon_{\mu\nu} + (1/4)\epsilon_{\mu\alpha} \theta^{\alpha\beta} \epsilon_{\beta\nu} = \epsilon_{\mu\nu} + O(\epsilon^2). \quad (162)$$

This corresponds to the following variation of the non-commutativity parameter,

$$\delta\theta^{\mu\nu} = -\theta^{\mu\alpha} \epsilon_{\alpha\beta} \theta^{\beta\nu} - \frac{1}{4} \theta^{\mu\alpha} \epsilon_{\alpha\gamma} \theta^{\gamma\rho} \epsilon_{\rho\beta} \theta^{\beta\nu} = -\theta^{\mu\alpha} \delta\theta^{-1}_{\alpha\beta} \theta^{\beta\nu} + O(\epsilon^2). \quad (163)$$

Let us note that such kind of infinitesimal transformations were considered in a slightly different context in [22].

Let us find how non-commutative functions are changed with respect to this transformation. In order to do this, let us consider how the Weyl symbol transforms under the variation of background (161). For an arbitrary operator $\phi$ after short calculation we have,

$$\delta\phi(x) = \frac{1}{4} \delta\theta^{\alpha\beta} (\partial_\alpha \phi * p_\beta(x) + p_\beta * \partial_\alpha \phi(x)). \quad (164)$$

In obtaining this equation we had to take into consideration the variation of $p_\mu$ as well as of the factor $\sqrt{\det \theta}$ in the definition of the Weyl symbol (132).

By the construction, this variation satisfies the “star-Leibnitz rule”,

$$\delta(\phi * \chi)(x) = \delta\phi(x) * \chi(x) + \phi(x) * \delta\chi(x) + \phi(x) \delta\chi(x), \quad (165)$$

where $\delta\phi(x)$ and $\delta\chi(x)$ are defined according to (164) and variation of the star-product is given by,
\[\phi(\delta \ast \chi)(x) = \frac{1}{2} \delta \theta^{\alpha \beta} \partial_\alpha \phi \ast \partial_\beta \chi(x). \quad (166)\]

The property (165) implies that \(\delta\) provides an homomorphism (which is in fact an isomorphism) of star algebras of functions.

The above transformation (164) do not apply, however, to the gauge field \(A_\mu(x)\) and gauge field strength \(F_{\mu \nu}(x)\). This is the case because the respective fields do not correspond to invariant operators. Indeed, according to the definition \(A_\mu = X_\mu - p_\mu\), where \(X_\mu\) is corresponds to such an operator. Therefore, the gauge field \(A_\mu(x)\) transforms in a nonhomogeneous way:

\[\delta A_\mu(x) = \frac{1}{4} \delta \theta^{\alpha \beta} (\partial_\alpha A_\mu \ast p_\beta + p_\beta \ast \partial_\alpha A_\mu) + \frac{1}{2} \theta_{\mu \alpha} \delta \theta^{\alpha \beta} p_\beta. \quad (167)\]

The transformation law for \(F_{\mu \nu}(x)\) can be computed using its definition (128) and the "star-Leibnitz rule" (165) as well as the fact that it is the Weyl symbol of the operator,

\[F_{\mu \nu} = i[X_\mu, X_\nu] - \theta_{\mu \nu}. \quad (168)\]

Of course, both approaches give the same result,

\[\delta F_{\mu \nu}(x) = \frac{1}{4} \delta \theta^{\alpha \beta} (\partial_\alpha F_{\mu \nu} \ast p_\beta + p_\beta \ast \partial_\alpha F_{\mu \nu})(x) - \delta \theta^{-1}_{\mu \nu}. \quad (169)\]

The infinitesimal map described above has the following properties:

\(i\). It maps gauge equivalent configurations to gauge equivalent ones, therefore it satisfies the Seiberg–Witten equation,

\[U^{-1} \ast A \ast U + U^{-1} \ast dU \rightarrow U'^{-1} \ast A' \ast U' + U'^{-1} \ast d'U'. \quad (170)\]

\(ii\). It is linear in the fields.

\(iii\). Any background independent functional is invariant under this transformation. In particular, any gauge invariant functional whose dependence on gauge fields enters through the combination \(X_{\mu \nu}(x) = F_{\mu \nu} + \theta_{\mu \nu}^{-1}\) is invariant with respect to (164)–(169). This is also the symmetry of the action provided that the gauge coupling transforms accordingly.

\(iv\). Formally, the transformation (164) can be represented in the form,

\[\delta \phi(x) = \delta x^\alpha \partial_\alpha \phi(x) = \phi(x + \delta x) - \phi(x), \quad (171)\]

where \(\delta x^\alpha = -\theta^{\alpha \beta} \delta p_\beta\) and no star product is assumed. This looks very similar to the coordinate transformations.

The map we just constructed looks very similar to the famous Seiberg–Witten map, which is given by the following variation of the background \(p_\mu\).

\[\text{In fact the same happens in the map between different dimensions.}\]
\[ \delta_{\text{SW}} p_\mu = -\frac{1}{2} \epsilon_{\mu\nu} \theta^{\nu\alpha} A_\alpha. \] (172)

In (161) we have chosen \( \delta p_\mu \) independent of gauge field background. (In fact the gauge field background was switched-on later, after the transformation.) An alternative way would be to have nontrivial field \( A_\mu(x) \) from the very beginning and to chose \( \delta p_\mu \) to be of the Seiberg–Witten form. Then, the transformation laws corresponding to such a transformation of the background coincide exactly with the standard SW map. This appears possible because the function \( p_\mu = -\theta^{-1} x^\nu \) has the same gauge transformation properties as \(-A_\mu(x)\),

\[ p_\mu \rightarrow U^{-1} \ast p_\mu \ast U(x) - U^{-1} \ast \partial_\mu U(x). \] (173)

8 Discussion and outlook

This lecture notes were designed as a very basic and very subjective introduction to the field. Many important things were not reflected and even not mentioned here. Among these, very few was said about the brane dynamics and interpretation which was the main motivation for the development of the matrix models, while the literature on this topic is enormously vast. For this we refer the reader to other reviews and lecture notes mentioned in the introduction (as well as to the references one can find inside these papers).

Recently, the role of the matrix models in the context of AdS/CFT correspondence became more clear. Some new matrix models arise in the description of the anomalous dimensions of composite super-Yang–Mills operators (see e.g [1, 5].

Another recent progress even not mentioned here but which is related to matrix models is their use for the computation of the superpotential of \( \mathcal{N} = 1 \) supersymmetric gauge theories [17, 16, 15].

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References

1. Abhishek Agarwal and Sarada. G. Rajeev. The dilatation operator of \( n = 4 \) sym and classical limits of spin chains and matrix models. *Mod. Phys. Lett.,* A19:2549, 2004, hep-th/0405116.
2. Ofer Aharony, Steven S. Gubser, Juan M. Maldacena, Hirosi Ooguri, and Yaron Oz. Large n field theories, string theory and gravity. *Phys. Rept.*, 323:183–386, 2000, hep-th/9905111.

3. A. P. Balachandran and Giorgio Immirzi. The fuzzy ginsparg-wilson algebra: A solution of the fermion doubling problem. *Phys. Rev.*, D68:065023, 2003, hep-th/0301242.

4. Tom Banks, W. Fischler, S. H. Shenker, and Leonard Susskind. M theory as a matrix model: A conjecture. *Phys. Rev.*, D55:5112–5128, 1997, hep-th/9610043.

5. Stefano Bellucci and Corneliu Sochichiu. On matrix models for anomalous dimensions of super yang-mills theory. 2004, hep-th/0410010.

6. David Berenstein, Edi Gava, Juan M. Maldacena, K. S. Narain, and Horatiu Nastase. Open strings on plane waves and their yang-mills duals. 2002, hep-th/0203249.

7. David Berenstein, Juan M. Maldacena, and Horatiu Nastase. Strings in flat space and pp waves from n = 4 super yang mills. *JHEP*, 04:013, 2002, hep-th/0202021.

8. David Berenstein and Horatiu Nastase. On lightcone string field theory from super yang-mills and holography. 2002, hep-th/0205048.

9. W. Bietenholz, F. Hofheinz, and J. Nishimura. The renormalizability of 2d yang-mills theory on a non-commutative geometry. *JHEP*, 09:009, 2002, hep-th/0203151.

10. T. A. Brody et al. Random matrix physics: Spectrum and strength fluctuations. *Rev. Mod. Phys.*, 53:385–479, 1981.

11. Maja Buric and Voja Radovanovic. On renormalizability of the quantum electrodynamics on noncommutative space. 2003, hep-th/0305236.

12. Yeuk-Kwan E. Cheung and Morten Krogh. Noncommutative geometry from 0-branes in a background b-field. *Nucl. Phys.*, B528:185–196, 1998, hep-th/9803031.

13. Chong-Sun Chu and Pei-Ming Ho. Noncommutative open string and d-brane. *Nucl. Phys.*, B550:151–168, 1999, hep-th/9812219.

14. Chong-Sun Chu and Pei-Ming Ho. Constrained quantization of open string in background b field and noncommutative d-brane. *Nucl. Phys.*, B568:447–456, 2000, hep-th/9906192.

15. Robbert Dijkgraaf and Cumrun Vafa. Matrix models, topological strings, and supersymmetric gauge theories. *Nucl. Phys.*, B644:3–20, 2002, hep-th/0206255.

16. Robbert Dijkgraaf and Cumrun Vafa. On geometry and matrix models. *Nucl. Phys.*, B644:21–39, 2002, hep-th/0207106.

17. Robbert Dijkgraaf and Cumrun Vafa. A perturbative window into nonperturbative physics. 2002, hep-th/0208048.

18. Tohru Eguchi and Hikaru Kawai. Reduction of dynamical degrees of freedom in the large N gauge theory. *Phys. Rev. Lett.*, 48:1063, 1982.

19. Thomas Guhr, Axel Muller-Groeling, and Hans A. Weidenmueller. Random matrix theories in quantum physics: Common concepts. *Phys. Rept.*, 299:189–425, 1998, cond-mat/9707301.

20. Jeffrey A. Harvey. Komaba lectures on noncommutative solitons and d-branes. 2001, hep-th/0102076.

21. N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya. A large-n reduced model as superstring. *Nucl. Phys.*, B498:467–491, 1997, hep-th/9612115.
22. Tomomi Ishikawa, Shin-Ichiro Kuroki, and Akifumi Sako. Noncommutative cohomological field theory and gms soliton. *J. Math. Phys.*, 43:872–896, 2002, hep-th/0107033.

23. Elias Kiritsis. Introduction to superstring theory. 1997, hep-th/9709062.

24. Elias Kiritsis and Corneliu Sochichiu. Duality in non-commutative seiberg-witten theory as a non-perturbative seiberg-witten map. 2002, hep-th/0202065.

25. Naofumi Kitsunezaki and Jun Nishimura. Unitary iib matrix model and the dynamical generation of the space time. *Nucl. Phys.*, B526:351–377, 1998, hep-th/9707162.

26. Youri Makeenko. Privat communication.

27. Shiraz Minwalla, Mark Van Raamsdonk, and Nathan Seiberg. Noncommutative perturbative dynamics. *JHEP*, 02:020, 2000, hep-th/9912072.

28. A. Morozov. Challenges of matrix models. 2005, hep-th/0502010.

29. V. P. Nair and A. P. Polychronakos. On level quantization for the noncommutative chern-simons theory. *Phys. Rev. Lett.*, 87:030403, 2001, hep-th/0102181.

30. Holger Bech Nielsen and M. Ninomiya. No go theorem for regularizing chiral fermions. *Phys. Lett.*, B105:219, 1981.

31. J. C. Osborn, D. Toublan, and J. J. M. Verbaarschot. From chiral random matrix theory to chiral perturbation theory. *Nucl. Phys.*, B540:317–344, 1999, hep-th/9806110.

32. J. Polchinski. *String theory*. Cambridge, 1998.

33. Joseph Polchinski. Dirichlet-branes and ramond-ramond charges. *Phys. Rev. Lett.*, 75:4724–4727, 1995, hep-th/9510017.

34. Swarnendu Sarkar. On the uv renormalizability of noncommutative field theories. *JHEP*, 06:003, 2002, hep-th/0202171.

35. Nathan Seiberg and Edward Witten. String theory and noncommutative geometry. *JHEP*, 09:032, 1999, hep-th/9908142.

36. M. M. Sheikh-Jabbari. Renormalizability of the supersymmetric yang-mills theories on the noncommutative torus. *JHEP*, 06:015, 1999, hep-th/9903107.

37. Edward V. Shuryak and J. J. M. Verbaarschot. Random matrix theory and spectral sum rules for the dirac operator in qcd. *Nucl. Phys.*, A560:306–320, 1993, hep-th/9212088.

38. A. A. Slavnov. Consistent noncommutative quantum gauge theories? *Phys. Lett.*, B565:246–252, 2003, hep-th/0304141.

39. Corneliu Sochichiu. Exercising in k-theory: Brane condensation without tachyon. 2000, hep-th/0012262.

40. Corneliu Sochichiu. M(any) vacua of iib. *JHEP*, 05:026, 2000, hep-th/0004062.

41. Corneliu Sochichiu. Matrix models: Fermion doubling vs. anomaly. *Phys. Lett.*, B485:202–207, 2000, hep-th/0005156.

42. Corneliu Sochichiu. A note on noncommutative and false noncommutative spaces. 2000, hep-th/0010149.

43. Corneliu Sochichiu. On the equivalence of noncommutative models in various dimensions. *JHEP*, 08:048, 2000, hep-th/0007127.

44. Corneliu Sochichiu. Gauge invariance and noncommutativity. 2002, hep-th/0202014.

45. Corneliu Sochichiu. Continuum limit(s) of bmn matrix model: Where is the (nonabelian) gauge group? *Phys. Lett.*, B574:105–110, 2003, hep-th/0206239.

46. Paolo Valtancoli. Stability of the fuzzy sphere solution from matrix model. *Int. J. Mod. Phys.*, A18:967, 2003, hep-th/0206075.
47. Mark Van Raamsdonk and Nathan Seiberg. Comments on noncommutative perturbative dynamics. *JHEP*, 03:035, 2000, hep-th/0002186.
48. J. J. M. Verbaarschot. Qcd, chiral random matrix theory and integrability. 2005, hep-th/0502029.
49. Jacobus J. M. Verbaarschot. The spectrum of the qcd dirac operator and chiral random matrix theory: The threefold way. *Phys. Rev. Lett.*, 72:2531–2533, 1994, hep-th/9401059.