Transient Convective Spin-up Dynamics

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We study the formation, longevity and breakdown of convective rings during impulsive spin-up in square and cylindrical containers using direct numerical simulations. The rings, which are axisymmetric alternating regions of up- and downwelling flow that can last for $O(100)$ rotation times, were first demonstrated experimentally and arise due to a balance between Coriolis and viscous effects. We study the formation of these rings in the context of the Greenspan-Howard spin-up process, the disruption of which modifies ring formation and evolution. We show that, unless imprinted by boundary geometry, convective rings can only form when the surface providing buoyancy forcing is a free-slip surface, thereby explaining an apparent disagreement between experimental results in the literature. For Prandtl numbers from 1–5 we find that the longest-lived rings occur for intermediate Prandtl numbers, with a Rossby number dependence. Finally, we find that the constant evaporative heat-flux conditions imposed in the experiments are essential in sustaining the rings and in maintaining the vortices that form in consequence of the ring breakdown.

I. INTRODUCTION

The dynamical processes by which a fluid within a spinning container attains the same angular velocity as the vessel is referred to as the “spin-up” (or “spin-down”) problem, and was unified in the theoretical treatment of [1] (hereafter GH). Suppose that the vessel is a right solid of horizontal dimension $L$ containing an isothermal fluid of viscosity $\nu$. At $t=0$ the container is rotated about its vertical axis with a constant angular velocity $\Omega$. The fluid takes a finite amount of time to “spin up” to the angular velocity of the solid container. Clearly, this required transfer of angular momentum is controlled solely by viscosity the spin up time would scale as $\tau_s \propto L^2/\nu$. However, GH showed that $\tau_s = \Omega^{-1} Re^{1/2}$, where the Reynolds number is $Re = L^2\Omega/\nu$, and hence $\tau_s/\tau_s \propto Re^{1/2}$. Therefore, given that $Re$ is typically large, the time required for fluid spin-up is much smaller than if the process were controlled by viscosity alone.

When the surfaces of the container are heated, the interplay between buoyancy and rotational forces complicates the dynamics considerably. For example, when the container is heated from below, the long-term ($\tau \gg \tau_s$) state is characterized by columnar vortices aligned in the direction of gravity, along which fluid is transported. Here, we study the spin up of a convectively unstable impulsively rotated container of fluid to its final vortical state. In particular, we are interested in a transient ring pattern that occurs during convective spin up. This ringed state consists of alternating axisymmetric rings of up- and downwelling flow, which have been reported in experimentally by [2], [3], and [4].

The experiments of [2] were performed in square and circular cross-sectioned containers of water with open upper surfaces cooled by evaporation. They measured the temperature of the free surface and estimated the rate of evaporation, and hence the cooling rate, to be nearly steady. When the upper surface was one of free slip, they observed the transient ringed state for a wide range of rotation and cooling rates (varied by changing the mean temperature of the water). However, for both square and circular cross-sections, when the top surface was covered by a lid, and the bottom surface is heated, they found no ringed state.

In contrast to [2], [3] held the bottom surface at constant temperature and found the ringed state (albeit with fewer rings) in a cylindrical container with a no-slip upper surface. [4] combined particle image velocimetry with infrared thermometry in a square cross-section container of depth $H$ with an evaporating free slip upper surface. They quantified the ringed state as a transient balance between rotational and viscous forces that exists for approximately one Ekman time, $\tau_E = \sqrt{H^2/\Omega \nu}$.

Here, we study the formation and breakdown of these transient convective rings using numerical simulations in a variety of geometries. We find that the ringed state is a universal feature of convective spin-up, and that the Prandtl number plays an important role in the formation and stability of the rings. Additionally, we find that the thermal boundary conditions used–Dirichlet as in [3] and Neumann as in [2] and [4]–influence the ring stability and the dynamics of their breakdown. Our results reconcile the seemingly contradictory observations of [2] and [3].

In [11], we describe the setup of the numerical simulations and the key differences from the experiments. We then discuss the numerical methods used and the resolution requirements for the simulations. The formation, longevity and breakdown of the ringed state into the final vortical state is summarized in [11], wherein we also examine some special cases of ring formation in non standard geometries, and connect these to what is observed experimentally. Conclusions are drawn in [11].

II. PROBLEM SETUP AND NUMERICAL METHOD

A schematic of the system under study is shown in Figure 1. We consider a container of width $L$ and height $H$ filled with a Boussinesq fluid of density $\rho$, coefficient of
The boundary conditions (BCs) for velocity and temperature determine the nature of the buoyancy forcing and the details of the spin-up process. In the cuboidal geometry, the six bounding surfaces (the top and bottom surfaces, and the four lateral boundaries) are impenetrable and thus have zero normal velocity. Each boundary can have no-slip or free-slip velocity BCs and Dirichlet (if both Dirichlet or both Neumann). Thus there are eight combinations of BCs. Of these, the majority of our results are from combinations listed (in their nondimensionalized form) in Table I in §IIA below. Other combinations are mentioned where relevant.

For simplicity we call all boundaries ‘surfaces’, so that, for instance, a ‘free-slip surface’ is a boundary where the normal velocity and the tangential stress are both zero.

B. Nondimensionalization

We scale time in the problem using the rotation rate, \(\Omega^{-1}\), and the length using the width of the container \(L\) (see Figure 1). The choice of \(\Omega\) instead of \(H\) for the length scale is based on numerical considerations. These together define the velocity scale \(U = \Omega L\). Assuming a temperature scale \(\Delta T\) (to be defined in the case of constant heat-flux), the governing equations (Eqs. 3-2) become

\[
\frac{D\mathbf{u}}{Dt} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} - 2\Omega e_z \times \mathbf{u} + \frac{g\alpha(T - T_0)}{Fr^2} \mathbf{e}_z + \alpha T \mathbf{e}_z, \tag{5}
\]

\[
\frac{DT}{Dt} = \kappa \nabla^2 T \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0. \tag{6}
\]

where \(Pr = \nu/\kappa\) is the Prandtl number, \(Re = \Omega L^2/\nu\) is the Reynolds number, and \(Fr^{-2} = g\alpha \Delta T/\Omega^2 L^2\) is the Froude number, which is a measure of the strength of the buoyancy relative to other forces. The initial velocity is \(\mathbf{u}(t = 0) = -e_z \times r\) and the initial temperature is \(\theta(t = 0) = 0\) everywhere in the container. The BCs are defined in §IIA.

A constant heat flux \(\dot{q}\) implies a constant buoyancy flux \(\dot{B}\), given in terms of \(\dot{q}\) as

\[
\dot{B} = \frac{g\dot{q}}{\rho C_p}, \tag{9}
\]

where \(C_p\) is the heat capacity per unit mass of the fluid at constant pressure. The flux Rossby number, which is a measure of the buoyancy flux, is

\[
Ro_f = \sqrt{\frac{\dot{B}}{\Omega^2 L^2}}, \tag{10}
\]

the flux Rayleigh number \(Ra_f\) is

\[
Ra_f = \frac{\dot{B} H^4}{\nu \kappa^2}, \tag{11}
\]

and the Nusselt number is

\[
Nu = \frac{\theta' H}{\langle \theta \rangle_{z=0} - \langle \theta \rangle_{z=H}}, \tag{12}
\]

where \(\theta'\) is the constant slope imposed at \(z = H\), and \(\langle \cdot \rangle\) denotes the time-average of a quantity at a given plane. The standard Rayleigh number follows from the above

\[
\text{Rayleigh number} = \frac{g\alpha \Delta T}{\nu \kappa}. \tag{5.0}
\]

\[
\Rightarrow \frac{\dot{B}}{\rho C_p} L^2 = \frac{\Delta T}{\nu \kappa}. \tag{5.1}
\]

\[
\Rightarrow \frac{\dot{B} H^4}{\nu \kappa^2} = \frac{\Delta T}{\nu \kappa}. \tag{5.2}
\]

\[
\Rightarrow \frac{\theta' H}{\langle \theta \rangle_{z=0} - \langle \theta \rangle_{z=H}} = \frac{\Delta T}{\nu \kappa}. \tag{5.3}
\]
definitions and is
\[ Ra = \frac{Ra_f}{Nu}. \tag{13} \]

The temperature scale \( \Delta T \) is defined as
\[ \Delta T = \frac{qL}{\rho C_p \kappa}. \tag{14} \]

and hence the Froude number can also be written as
\[ Fr^{-2} = Ro_f^2 Re Pr. \tag{15} \]

For very large Taylor numbers, \( Ta = 4 Re^2/A^4 \), the container of fluid rotates like a solid body, and for small Taylor numbers, the dynamics resemble non-rotating Rayleigh-Bénard convection \[2\]. The boundary between these is defined by the critical Rayleigh number
\[ Ra_c \propto Ta^{2/3} \propto Re^{4/3}, \tag{16} \]

where the constant of proportionality depends on whether the top- and bottom surfaces obey free-slip or no-slip BCs \[2\].

### C. Numerical Method

The numerical simulations are performed with the finite volume code Megha-5, which uses uniform grids and second order central differences in space and second order Adams Bashforth timestepping. The momentum equation is solved using the projection operator \[3\] and the resulting Poisson equation for the pressure is solved using cosine transforms with the PFFT Library of \[6\]. The scalar equation is solved using a local upwind scheme \[7\] that avoids Gibbs oscillations while retaining overall second order accuracy. Alternatively, the second order scheme of \[8\] can also be used. Megha-5 is based on an extensively validated earlier version \[9\] and has been used in studies of jets and plumes \[10\], and mammatus clouds \[11\].

The thickness of the thermal boundary layer adjacent to a surface is defined as the distance at which the mean temperature of the volume would be reached starting at the surface temperature with the slope from the first two gridpoints from the surface, following the convention of Belmonte et al. \[12\] and Verzicco and Sreenivasan \[13\]. We ensure that the thermal boundary layers at the top surface are resolved with at least 6 gridpoints for Reynolds numbers \( Re \leq 7.5 \times 10^3 \) (grid size of \( 256^2 \times 128 \), with a time step of \( 2.5 \times 10^{-3} \)), and up to 12 gridpoints (grid size of \( 512^2 \times 256 \), with a time step of \( 1.25 \times 10^{-3} \)) for \( Re \geq 10^4 \), as required in turbulent Rayleigh-Bénard convection [see \[14\] and refs. therein]. The results were found to be grid independent and we report those from the lower resolution grid here. We have also verified that the choice of local-upwinding or Kurganov-Tadmor discretization does not affect the results (the former is used unless otherwise mentioned).

Simulations in the cylindrical geometry and other geometries mentioned in \[\S\text{III}E\] are performed using the volume penalization method \[15, 16\], with insulating BCs for the simulations in the cylindrical geometry applied following \[17\]. Our results are verified to be independent of the penalization parameters used.

### III. RESULTS AND DISCUSSION

We begin by summarising the spin-up process in the absence of buoyancy forcing, following GH. Consider the case where the top surface is free-slip and the bottom surface and the four lateral boundaries are no-slip surfaces, as shown schematically in Figure 2(a). The flow at the bottom surface is that due to a plate impulsively rotated about an axis perpendicular to its plane [see e.g., Chapter 5.2.4, p. 119 of \[18\]]. Fluid is centrifuged outwards from the axis of rotation along the surface. Continuity drives fluid downward towards the bottom surface. As the centrifuged fluid reaches the periphery of the container, it ascends up the lateral surfaces, driven by a vorticity gradient that exists as a result of the boundary layers on the lateral surfaces. Once this fluid reaches the upper free surface, it is driven towards the axis, eventually becoming part of the downward flow. In this manner, fluid is driven from larger to smaller radii. Conservation of angular momentum (excepting for small viscous losses) insures that the fluid near the axis is replaced with fluid that is rotating more rapidly. GH show that this process takes a time \( O(\Omega^{-1} Re^{1/2}) = O(\sqrt{L^2/\nu \Omega}) \).
A. Type I BCs

We first discuss results from simulations with Type I BCs (see Table I); the sides and the bottom are all no-slip, thermally insulating surfaces and the upper free-slip surface is driven with a constant heat flux. The dynamical time scale for the circulation shown schematically in Figure 1(a) is fast relative to the build up of negatively buoyant fluid at the upper surface. As cold plumes emerge, they are sequentially forced towards the axis of rotation as buoyancy and rotational forces balance, the oldest and more central of which are deeper. A given plume evolves into an axisymmetric ring as this quasi-steady balance is attained, thereby leading to a sequence of upwelling and downwelling ring pairs. Up to three pairs are seen for such $Re \leq 10^4$. The rings eventually reach the bottom of the box, where they interact with the boundary layer and are influenced by the shape of the container if a sufficiently long time passes. As the system approaches solid-body rotation, the system must become unstable and break up into cyclonic vortices, in which fluid sinks surrounded by regions of slower upwelling flow. While this generic process remains similar across a wide parameter range, the ring and vortex numbers are a function of the Reynolds, flux Rossby and Prandtl numbers. A sequence of images showing this evolution is presented in Figure 3.

To show the heat transport by the rings, we plot the cross sectional area-averaged dimensionless buoyancy flux, defined as

$$\langle B \rangle (z, t) = \frac{1}{Fr^2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} dx dy \langle w\theta \rangle,$$  \hspace{1cm} (17)$$

at a horizontal section at $z = 0.455$. The first two peaks of buoyancy flux seen in Figure 4 correspond to the formation of the first ring and the maximally ringed state respectively.

In the limit of very small $Ro_f$ (Eq. 10), convection is strongly suppressed. For large $Ro_f$, the heat flux dominates the effects of rotation. The dynamics of ring formation are most prominent at some intermediate value of $Ro_f$, as found by [2] and [4]. As $Ro_f$ increases, the time for the first ring to form decreases and its radius increases.

The Prandtl number strongly influences the dynamics—particularly the stability of the ringed state. For a given $Re$, the widths of plumes that emerge from the top boundary layer are inversely proportional to $Pr$. For low
Figure 4: The buoyancy flux at $z = 0.455$ as a function of time, for $Re = 7500$ and $Pr = 5$. The peaks in buoyancy flux, labelled (i) and (ii) in the figure, correspond to the time when the first ring forms and the time at which the ringed state is maximal respectively. The first ring forms sooner when the flux Rossby number is larger.

Figure 5: The longevity of the ringed state for three different Reynolds numbers and various flux Rossby numbers, showing the variation of $Pr^*(Re, Ro_f)$ for (a): $Re = 5000$, (b): $Re = 7500$, (c): $Re = 10000$. The legends show the values of $Ro_f$ for which the lifetimes are plotted. The legends for $Re = 7500$ and $Re = 10000$ are shared.

$Pr$, coherent rings dissipate rapidly and for high $Pr$, the increased inter-ring shear drives instability. Thus, the stability of the rings peaks at an intermediate Prandtl number $Pr^*$. The parameter $\Phi$ measures deviations from axisymmetry of a flow-variable $\phi$ as

$$\Phi = \int_0^{r_{max}} dr [\phi(x, y, z_0, t) - \phi(r, z_0, t)]^2$$

where $r_{max} = 0.45$ and $\phi(r, t)$ is the average value at radius $r$ at time $t$, and $z_0 = 0.47$. When $\Phi \leq \Phi_b(t = t_b)$, we can define the longevity of the ringed state as $t_b$. Figure 6 shows the variation of the lifetime of the ringed state with the system parameters. (Clearly $\Phi$ is also zero if $\phi = 0$ everywhere. Thus, a threshold for $\Phi$ is used.) It can be seen that $Pr^*$ is a decreasing function of $Re$ and $Ro_f$, as shown in Figure 6.

The ringed state breaks down into columnar vortices at a time $t_{breakdown}(Re, Ro_f, Pr)$ that follows the second buoyancy flux peak as seen in Figures 4 and 5. The Okubo-Weiss parameter is used to identify vortices, and

Figure 6: The Prandtl number $Pr^*(Re, Ro_f)$ at which the rings are the longest-lived. See also Fig. 5.

Figure 7: (a) Nusselt number as a function of the flux Rayleigh number for the three different values of the Reynolds number. The dotted lines are $Nu \sim Ra_f^{3/4}$. (b) The curves from (a) collapse onto each other when scaled appropriately. The dotted line is $Nu \sim Ra_f^{3/4}Re^{-1}$.

is defined as

$$W = \Omega_{ij}^2 - S_{ij}^2,$$

where $\Omega_{ij}$ and $S_{ij}$ are the symmetric and anti-symmetric parts of the gradients of the horizontal components of the velocity. The number of vortices typically increases with the Reynolds and Rossby numbers.

The steady state Nusselt number is calculated from Eq. (12) as a function of the other parameters in the system. In geostrophic convection [19, 20], because $Nu \propto Ra^3Re^{-4} = Ra_f^3Nu^3Re^{-4}$, then we have

$$Nu \propto Ra_f^{3/4}Re^{-1}.$$  \hspace{1cm} (20)

The Nusselt numbers obtained for different $Re$ are plotted as a function of the flux Rayleigh number $Ra_f$ in Figure 7(a). These curves collapse onto a single curve, as shown in Figure 7(b), when plotted against $Ra_f^{3/4}Re^{-1}$.

B. Type II and Type IV BCs

The results of [11, 12] are qualitatively similar to experiments of [2] and [4] because Type I BCs are similar.
to the experimental BCs, which have free upper surfaces that are cooled by the evaporation of water. [2] comment that they observe no rings if the cooled top surface is one of no-slip. Since [2] report experiments in both square-cross-sectioned and cylindrical containers, it was presumed that they meant this for both geometries. [3] perform experiments in cylindrical containers and their rings eventually break up into vortices as in the cylindrical geometry experiments of [2], but they form much further away from the axis of rotation. The first ring forms close to the outer lateral surface.

We implement the cylindrical geometry using volume penalization, as discussed in §II. A sequence of images showing the evolution for a particular case is shown in Figure 8, which may be compared with that in Figure 3.

The role that the lateral boundaries play in the dynamics can be seen by comparing simulations with Type II and Type IV BCs. The latter involve a square cross-sectioned container with six no-slip boundaries. The evolution is similar to spin-up in a closed container, with radially outwards flow at the upper and lower surfaces (compare Figure 9 with Figure 2b), with radially outwards flow at the upper and lower surfaces. Because these boundary layers eventually reach the lateral surfaces, the container geometry creates alternating sheets of up- and down-welling convection that take the form of square annuli. The foregoing argument implies that ring formation with the no-slip top surface in the [3] experiments is strongly influenced by the cylindrical shape of the container.

We examine the processes necessary for ring-formation in terms of the nature of the upper surface boundary conditions. Namely, if the upper surface is one of free-slip, but lacks buoyancy forcing. For example, when the no-slip bottom surface provides the buoyancy forcing rings do not form, as can be seen in Figure 10. This follows from the mechanism described above; the warm fluid at the bottom surface is centrifuged outwards and collects at the upper boundary at the periphery of the container, where it remains, taking the shape of the container.

Thus, for containers that are not axisymmetric, the necessary and sufficient condition for convective ring formation during impulsive spin-up is that the surface providing the buoyancy forcing be stress-free. This criterion explains the apparent disagreement between the experiments of [2] and [3].

**D. Type III BCs: the influence of Dirichlet vs Neumann thermal BCs**

The thermal BCs play an important role in the dynamics of convective ring formation. Whereas rings form for both Dirichlet and Neumann thermal BCs, their formation times, locations and lifetimes are markedly different. In addition, the columnar vortical state is less well defined with Dirichlet than with Neumann BCs.

Since the temperature difference between the horizontal boundaries is prescribed instead of the buoyancy flux, we use $\Delta T$ to nondimensionalize Equation 2. Hence, the nondimensionalization of §II is modified, with the Rossby number defined as $Ro = g\alpha \Delta T / (\Omega^2 L)$ (note that
Figure 10: Evolution of the square sheets of up- and down-welling in simulations with Type V BCs. The horizontal sections are drawn at $z \approx 0.025$ at $t = 20, 40, 60, 80$ as in Figure 9.

$Ro = Fr^{-2}$; see Eq. 15). The definitions of the Reynolds and Prandtl numbers remain unchanged. The Rayleigh number is

$$Ra = \frac{g \alpha \Delta T H^3}{\nu \kappa} = \frac{Re^2 Ro Pr}{A^3},$$

along with the Nusselt number, which may be defined as

$$Nu = \frac{\langle \partial \theta / \partial z \rangle_{z=0}}{A},$$

with $\langle \rangle$ denoting averages over space and time.

Figure 11 shows the ring formation for the case $Re = 7500$, $Pr = 5$, and $Ro = 0.03125$, where the evolution can be compared to that in Figure 3 for Type I BCs ($Re = 7500$, $Pr = 5$, $Ro_I = 0.00442$). However, the first ring forms earlier and at a larger radius for Type III BCs, the difference being associated with the thermal boundary layers. Namely, for Type III BCs, the thickness of the thermal boundary layer changes significantly with time; fluid from the bottom surface ($\theta = 0$) is forced towards the top surface ($\theta = 1$) where the boundary layer grows, eventually becoming thicker than the corresponding case with Type I BCs. In Figure 12 we see that the overall ring structure has a larger radius with Type III BCs and in Figure 13 the surfaces of constant temperature show that the first ring forms at a larger radius and is thinner for Type III BCs. Moreover, Figure 14 shows that the ratio of the maximum buoyancy flux to the long-time average is much larger for Type III BCs than for Type I BCs (Figure 4). However, after the rings have broken up into vortices, the thicker thermal boundary layers for Type III BCs leads to vortices that gather buoyancy from a broader spatial extent and hence are more diffuse relative to those for Type I BCs (compare Figures 11 and 3).

For geostrophic convection with Type III BCs, the Nusselt number should scale with the Rayleigh number as

$$Nu \propto Ra^3 Re^{-4},$$

but this scaling is not seen in Figure 15, as opposed to the collapse shown in Figure 7(b).

We conclude this section by noting that the nature of the global heat transport in non-rotating Rayleigh-Bénard convection is associated with nature of the boundary layer-core interaction, modulated by plumes. This is heuristically similar to our findings, wherein the nature of the thermal boundary layers differs for Type I and Type III BCs.

E. Special Cases

As described in IIIA each step of the GH spin-up process plays a role in the formation of convective rings. Thus, altering any of these alters the ring formation process. This is seen in the examples presented in IIIE1–IIIE3 below. Furthermore, a case where the fluid is
Figure 12: A comparison of the flow evolution for Type I and Type III BCs. In each half, the four figures are horizontal sections of the vertical velocity (a) and temperature (b), and vertical sections of the vertical velocity (c) and temperature (d). The horizontal sections are plotted at $z = 0.47$ (same as in Figure 3). The parameters are $Re = 7500$, $Pr = 5$, (both as in Figure 3) and $Ro = 0.00442$ ($t = 60$, Type I) and $Ro = 0.03125$ ($t = 50$, Type III).

Figure 13: Isocontours of the temperature, drawn at a value $\theta^* = 0.5(\theta)_{\text{free-surface}}$ for (a): Type I and $\theta^* = 0.8(\theta)_{\text{free-surface}}$ (b): Type III BCs. The parameters are $Re = 7500$, $Pr = 5$, $Ro_f = 0.00442$ ($t = 60$, Type I) and $Ro = 0.03125$ ($t = 50$, Type III).

Figure 14: The buoyancy flux at $z = 0.455$ as a function of time, for the same parameters ($Re = 7500$, $Pr = 5$) as in Figure 3 but with Type III BCs. The peaks of buoyancy flux, labelled (i) and (ii) in the figure, correspond to the formation of the first ring and the maximal ringed state. As with Type I BCs, the time at which the first ring forms decreases for increasing Rossby number (see Figure 3). However, the second peaks of the buoyancy flux, corresponding to the maximally ringed state, occur much sooner here than in Figure 3.

Figure 15: The Nusselt number vs the Rayleigh number for Type III BCs, analogous to Figure 7 (b). Note that the Nusselt number does not scale like $Ra^3 Re^{-4}$, the abscissa, but instead only as $(Ra^3 Re^{-4})^{1/4}$. This lower Nusselt number is responsible for the vortices being less well defined.

1. Free-slip lateral boundaries

The lateral boundaries play an important role in the spin-up process. GH observe that the diffusion of vorticity from the lateral surfaces to the fluid results in the suction of flow out of the boundary layer on the bottom surface into the boundary layers on the lateral surfaces. It is therefore reasonable to ask; what happens if these are free-slip surfaces that do not support boundary layers when the no-slip bottom surface continues to centrifuge fluid outwards?
The formation of convective “rings” with free-slip lateral surfaces for the same parameters as in Figure 3. The horizontal sections of velocity (a) and temperature (b) are plotted at the same location $z = 0.47$ as in Figure 3, and the vertical sections ((c) and (d) respectively) are plotted on planes passing through the axes. Note that the bottom boundary is a no-slip surface.

To this end, Figure 16 shows that while ring formation does occur, the ‘rings’ are no longer axisymmetric as they were for the Type I BCs. The radially inward flow in the bulk created by the boundary layers on the lateral surfaces is thus also responsible for pushing the rings that form towards the center, which thereby become axisymmetric. When these boundary layers are absent, the rings reflect the shape of the container.

2. Free-slip top- and bottom boundaries

When the top or bottom surfaces obey the no-slip condition, they centrifuge fluid outwards. As we have seen, this radially outward flow plays a crucial role in the process of ring-formation. We further illustrate this by making both the top- and bottom-surfaces free-slip (while the lateral surfaces are no-slip). Rings form in this case, but at larger radii than in the standard case. A representative snapshot is shown in Figure 17.

3. All boundaries free-slip

The examples presented thus far demonstrated the important role of the boundary layers on the process of ring-formation. Therefore, it should not be surprising that if all the boundaries of the container are made free-slip, the convective structures that form only have a qualitative resemblance to rings. A representative snapshot from the evolution of the flow is shown in Figure 18, which should be compared with the evolution in Figures 3 and 16.

4. Convective spin-down

Variations of the mechanism discussed here are also relevant in spin-down: i.e. the case of a rotating container of fluid undergoing a negative step-change in angular speed (at the same moment at which heating/cooling is switched on at one of its boundaries). In this case the ratio of initial to final angular velocity (which is zero in spin-up) is also a parameter. We present here results when the container slows abruptly from $2\Omega$ to $\Omega$. The fluid velocity at $t = 0$, in the frame of reference rotating with the container, is thus exactly the negative of the fluid velocity in the spin-up case. All the other equations remain unchanged. We consider two cases: (a) with a cooled top surface and (b) with a heated bottom surface. In both cases, the bottom boundary is no-slip and the upper boundary is free-slip.
Figure 19: The formation of rings during top-cooled convective spin-down. The other BCs and parameters are the same as in Figs. 9 and 12 and the figures are plotted in a similar way. The horizontal sections are at $z = 0.47$.

Figure 20: The formation of rings during convective spin-down with a heated bottom surface. The other BCs and parameters are the same as in Fig. 19. The horizontal sections are at $z = 0.23$.

During spin-down, the flow at the bottom surface is reversed: fluid moves radially inwards along the surface, and is pushed outwards along the axis away from the surface. Hence, in case (a) warm fluid impinges on the top boundary at the axis and moves radially outwards. This leads to plumes forming at the top surface near the periphery, with subsequent plumes forming closer to the axis, as shown in Figure 19.

In case (b), the heating from the surface adds to the Ekman suction at the bottom surface, with warm fluid forced upwards along the axis. This fluid is now at the temperature of the bottom surface, and is pushing against a background of colder fluid, creating an instability. The interface splits into rings between the top and bottom surfaces, which break down into vortices as usual. A snapshot of this is shown in Figure 20.

IV. CONCLUSION

In summary, we have performed a range of numerical experiments to study the formation, longevity and breakdown of a quasi steady ringed state during the convective spinup of a Boussinesq fluid. We have studied the role of the GH spin-up process on ring-formation, and found that the centrifugal radially outwards flow at the bottom surface, the reversal of the Ekman layer due to vorticity diffusion at the side surfaces, and the radially inward flow at the free-slip surface are all important factors. We show that whereas disrupting any one of these disrupts ring formation, and the rings take on the shape of the container, but disrupting all of these completely suppresses ring formation. The ring formation criteria we provide for convective spin-up explain the apparent disagreement in experiments regarding whether rings can form with a solid upper surfaces.

Because the rings arise due to a transient balance between convective and rotational dynamics, our finding that the Prandtl number, which we varied from 1 to 5, plays a key role in the formation and stability of the rings is intuitive. We found that the ring lifetime is longest for intermediate Prandtl numbers, with a Rossby and Reynolds number dependence. We have also described the role played by the thermal boundary conditions on the stability of the ringed state and the heat flux in the system. In the transient dynamics considered here, Dirichlet boundary conditions lead to thinner boundary layers and large heat fluxes initially, and lower Nusselt numbers in the steady state, than corresponding cases with Neumann boundary conditions.

Finally, given the broad relevance of the basic processes we study here, whereby Ekman-layer suction drives the boundary layer fluid towards the lateral boundaries at which it may achieve the same speed, a wide range of problems may be examined within our general numerical framework through systematic manipulation of the boundary conditions to a far greater extent than we have explored here. Indeed, the generality is extended due to the direct mathematical connection between rotating and stratified fluids [21], which are uniquely combined in transient rotating convection. Classical problems that arise in this context include those in which a homogeneous or stratified column of fluid may spin-up or spin-down due to topographic effects [22] and topographic eddy [23], Rossby wave [24] and edge-wave generation [25].

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