LONG-TIME EXISTENCE OF YAMABE FLOW ON SINGULAR SPACES
WITH POSITIVE YAMABE CONSTANT

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Abstract. In this work we establish long-time existence of the normalized Yamabe flow with positive Yamabe constant on a class of manifolds that includes spaces with incomplete cone-edge singularities. We formulate our results axiomatically, so that our results extend to general stratified spaces as well, provided certain parabolic Schauder estimates hold. The central analytic tool is a parabolic Moser iteration, which yields uniform upper and lower bounds on both the solution and the scalar curvature.

CONTENTS

1. Introduction and statement of the main results ...................................................... 1
2. The evolution of the scalar curvature and lower bounds ....................................... 9
3. Uniform bounds on the solution along the flow .................................................... 14
4. Upper bound on the scalar curvature along the flow .......................................... 18
5. Long-time existence of the normalized Yamabe flow ......................................... 32
6. Future research directions and open problems ................................................... 34
References ................................................................................................................. 34

1. Introduction and statement of the main results

The Yamabe conjecture states that for any compact, smooth Riemannian manifold \((M, g_0)\) there exists a constant scalar curvature metric, conformal to \(g_0\). The first proof of this conjecture was initiated by Yamabe [Yam60] and continued by Trudinger [Tru68] Aubin [Aub76], and Schoen [Sch84]. The proof is based on the calculus of variations and elliptic partial differential equations. An alternative tool for proving the conjecture is due to Hamilton [Ham89]: the normalized Yamabe flow of a Riemannian manifold \((M, g_0)\), which is a family \(g \equiv g(t), t \in [0, T]\) of Riemannian metrics on \(M\) such that the following evolution equation holds

\[
\partial_t g = -(S - \rho)g, \quad \rho := \text{Vol}_g(M)^{-1} \int_M S \, d\text{Vol}_g.
\]

(1.1)

Here \(S\) is the scalar curvature of \(g\), \(\text{Vol}_g(M)\) the total volume of \(M\) with respect to \(g\) and \(\rho\) is the average scalar curvature of \(g\). The normalization by \(\rho\) ensures that the total volume does not change along the flow. Hamilton [Ham89] introduced the Yamabe flow and also showed its long time existence. It preserves the
conformal class of \( g_0 \) and ideally should converge to a constant scalar curvature metric, thereby establishing the Yamabe conjecture by parabolic methods.

Establishing convergence of the normalized Yamabe flow is intricate already in the setting of smooth, compact manifolds. In case of scalar negative, scalar flat and locally conformally flat scalar positive cases, convergence is due to Ye [Ye94]. The case of a non-conformally flat \( g_0 \) with positive scalar curvature is delicate and has been studied by Brendle [Bre05, Bre07]. More specifically, [Bre05, p. 270], [Bre07, p. 544] invoke the positive mass theorem, which is where the dimensional restriction in [Bre05] and the spin assumption in [Bre07, Theorem 4] come from. Assuming [ScYa17] to be correct, [Bre05] and [Bre07] cover all closed manifolds which are not conformally equivalent to spheres.

In the non-compact setting, our understanding is limited. On complete manifolds, long-time existence has been discussed in various settings by Ma [Ma16], Ma and An [MaAn99], and the recent contribution by Schulz [Sch18]. On incomplete surfaces, where Ricci and Yamabe flows coincide, Giesen and Topping [GiTo10, GiTo11] constructed a flow that becomes instantaneously complete.

In this work, we study the Yamabe flow on a general class of spaces that includes incomplete spaces with cone-edge (wedge) singularities or, more generally, stratified spaces with iterated cone-edge singularities. This continues a program initiated in [BaVe14, BaVe19], where existence and convergence of the Yamabe flow has been established in case of negative Yamabe invariant. Here, we study the positive case and, utilizing methods of Akutagawa, Carron, Mazzeo [ACM14], we establish long time existence of the flow under certain mild geometric assumptions. We don’t attempt to prove convergence here, in view of [Bre05, Bre07] and the fact that we don’t have a substitute for the positive mass theorem in the singular setting. Our main result is as follows.

**Theorem 1.1.** Let \((M, g_0)\) be a Riemannian manifold of dimension \( n = \dim M \geq 3 \), such that the following four assumptions (to be made precise below) hold:

1. The Yamabe constant \( \mathcal{Y}(M, g_0) \) is positive;
2. \((M, g_0)\) is admissible and in particular satisfies a Sobolev inequality;
3. Certain parabolic Schauder estimates hold on \((M, g_0)\);
4. The initial scalar curvature \( S_0 \) admits certain Hölder regularity.

Under these assumptions, the normalized Yamabe flow of \( g_0 \) exists within the space of admissible spaces, with infinite existence time.

Examples, where the assumptions of the theorem are satisfied, include spaces with incomplete wedge singularities. More general stratified spaces with iterated cone-edge metrics are also covered, provided parabolic Schauder estimates continue to hold in that setting.

We now proceed with explaining the assumptions in detail.

### 1.1. Normalized Yamabe flow and Yamabe constant.

Consider a Riemannian manifold \((M, g_0)\), with \( g_0 \) normalized such that the total volume \( \text{Vol}_{g_0}(M) = 1 \). The Yamabe flow (1.1) preserves the conformal class of the initial metric \( g_0 \) and,
assuming \( \dim M = n \geq 3 \), we can write \( g = u^{\frac{4}{n-2}}g_0 \) for some function \( u > 0 \) on \( M_T = M \times [0, T] \) for some upper time limit \( T > 0 \). Then the normalized Yamabe flow equation can be equivalently written as an equation for \( u \)

\[
\partial_t \left( u^{\frac{n+2}{n-2}} \right) = \frac{n+2}{4} \left( \rho u^{\frac{n+2}{n-2}} - L_0(u) \right), \quad L_0 := S_0 - 4\frac{(n-1)}{n-2} \Delta_0,
\]

where \( L_0 \) is the conformal Laplacian of \( g_0 \), defined in terms of the scalar curvature \( S_0 \) and the Laplace Beltrami operator \( \Delta_0 \) associated to the initial metric \( g_0 \). The scalar curvature \( S \) of the evolving metric \( g \) can be written \( S = u^{-\frac{n-2}{2}} L_0(u) \), and the volume form of \( g = u^{\frac{2}{n-2}} g_0 \) is given by \( \text{dVol}_g = u^{\frac{2n}{n-2}} \text{d}\mu \), where we write \( \text{d}\mu := \text{dVol}_{g_0} \) for the time-independent initial volume form. One computes

\[
\partial_t \text{dVol}_g = -\frac{n}{2}(S - \rho) \text{dVol}_g.
\]

Hence, the total volume of \((M, g)\) is constant and thus equal to 1 along the flow. The average scalar curvature then takes the form

\[
\rho = \int_M S \text{dVol}_g = \int_M L(u) u^{-\frac{n+2}{n-2}} u^{\frac{2n}{n-2}} \text{d}\mu = \int_M \frac{4(n-1)}{n-2} |\nabla u|^2 + S_0 u^2 \text{d}\mu.
\]

Explicit computations lead to the following evolution equation for the average scalar curvature

\[
\partial_t \rho = -\frac{n-2}{2} \int_M (S - \rho)^2 u^{\frac{2n}{n-2}} \text{d}\mu.
\]

The latter evolution equation in particular implies that \( \rho = \rho(t) \) is non-increasing along the flow. We conclude the exposition with defining the Yamabe constant of \( g_0 \), which incidentally provides a lower bound for \( \rho \). Let \( u \) be a solution of (1.2). We define the \( L^q(M) \) spaces with respect to the integration measure \( \text{d}\mu \).

We define the first Sobolev space \( H^1(M) \) as the space of all \( v : M \to \mathbb{R} \) such that first Sobolev norm, defined with respect to \( \text{d}\mu \) and the pointwise norm associated to \( g_0 \)

\[
\|v\|_{H^1(M)}^2 := \int_M v^2 \text{d}\mu + \int_M |\nabla v|^2 \text{d}\mu < \infty.
\]

Similarly, we define \( H^1(M, g) \) by using \( \text{dVol}_g \) instead of \( \text{d}\mu \), and the pointwise norm associated to \( g \). If \( u \) and \( u^{-1} \) are both bounded, one easily checks \( H^1(M) = H^1(M, g) \).

We define the Yamabe invariant of \( g_0 \) as follows

\[
Y(M, g_0) := \inf_{v \in H^1(M) \setminus \{0\}} \frac{\int_M \frac{4(n-1)}{n-2} |\nabla v|^2 + S_0 v^2 \text{d}\mu}{\|v\|_{L^{\frac{2n}{n-2}}(M)}^2}
\]

\[
\leq \int_M \frac{4(n-1)}{n-2} |\nabla u|^2 + S_0 u^2 \text{d}\mu \overset{\text{(1.4)}}{=} \rho,
\]

where in the inequality we have used that for any solution \( u \) of (1.2), \( \|u\|_{L^{\frac{2n}{n-2}}(M)} = \text{dVol}_g(M) \equiv 1 \). How one proceeds will depend heavily on the
sign of the Yamabe constant. In this paper we will assume $Y(M, g_0) > 0$. In particular, the average curvature $\rho$ is then positive and uniformly bounded away from zero along the normalized Yamabe flow.

**Assumption 1.** The Yamabe constant $Y(M, g_0)$ is positive.

### 1.2. A Sobolev inequality and other admissibility assumptions.

The Moser iteration arguments in this paper are strongly motivated by the related work of Akutagawa, Carron and Mazzeo [ACM14] on the Yamabe problem on stratified spaces. Thus, similar to [ACM14], we impose certain admissibility assumptions, which are naturally satisfied by certain compact stratified spaces with iterated cone-edge metrics.

**Definition 1.2.** Let $(M, g_0)$ be a smooth Riemannian manifold of dimension $n$. We call $(M, g_0)$ admissible, if it satisfies the following conditions.

- $(M, g_0)$ with volume form $d\mu = d\text{Vol}_{g_0}$ has finite volume $\text{Vol}_{g_0}(M) < \infty$.
- For any $\varepsilon > 0$, there exist finitely many open balls $B_{2R_i}(x_i) \subset M$ such that
\[
\text{Vol}_{g_0} \left( M \setminus \bigcup_i B_{R_i}(x_i) \right) \leq \varepsilon. \tag{1.7}
\]
- Smooth, compactly supported functions $C_c^\infty(M)$ are dense in $H^1(M)$.
- $(M, g_0)$ admits a Sobolev inequality of the following kind. Defining $L^q(M)$ spaces with respect to $d\mu$, there exist $A_0, B_0 > 0$ such that for all $f \in H^1(M)$
\[
\|f\|_{L^2(M)}^2 \leq A_0 \|\nabla f\|_{L^2(M)}^2 + B_0 \|f\|_{L^2(M)}^2. \tag{1.8}
\]

The main examples we have in mind are closed manifolds and regular parts of smoothly stratified spaces, endowed with iterated cone-edge metrics. See [ACM14, Section 2.1] for a definition of the latter. That the Sobolev inequality holds in this case is shown in [ACM14, Proposition 2.2]. Note that the list of admissibility assumptions does not contain compactness. Nor do we specify explicitly how the metric $g_0$ looks near the singular strata of $M$, in case of stratified spaces. Restrictions on the local behaviour of the metric will instead be coded in $L^q$-data, like requiring the initial scalar curvature $S_0$ to be in $L^q(M)$ for suitable $q > 0$. These requirements are stated in the theorems below, and will vary from statement to statement.

**Assumption 2.** $(M, g_0)$ is an admissible Riemannian manifold.

In what follows we want to relate the assumption of the Sobolev inequality (1.8) in Definition 1.2 to positivity of the Yamabe constant $Y(M, g_0)$.

**Proposition 1.3.** Assume $S_0 \in L^\infty(M)$ and $Y(M, g_0) > 0$. Then (1.8) holds.

---

1This can be phrased as $H^1_0(M) = H^1(M)$. Note that this rules out $M$ being the interior of a manifold with a codimension 1 boundary.

2This includes finite volume, complete manifolds, since (see [Heb96, Lemma 3.2, pp. 18-19], [Heb96, Remark 2), pp. 56-57]) any finite volume, complete manifold satisfying the Sobolev inequality is compact.
Proof. Indeed, it follows directly from the definition of $Y(M,g_0)$ in (1.6) that
\[
\|f\|_{L^{2n/(n-2)}(M)}^2 \leq \frac{1}{Y(M,g_0)} \left( \frac{4(n-1)}{n-2} \left\| \nabla f \right\|_{L^2(M)}^2 + \left\| S_0 \right\|_{L^\infty(M)} \|f\|_{L^2(M)}^2 \right)
\]
holds for all $f \in H^1(M)$. This is indeed the Sobolev inequality (1.8). \qed

1.3. **Parabolic Schauder estimates and short-time existence.** Our proof requires intricate arguments involving the heat operator and its mapping properties, as is seen in the previous work by the second author jointly with Bahuaud [BAVe14, BAVe19] in the setting of spaces with incomplete wedge singularities. Here, we axiomatize these arguments into a definition of certain parabolic Schauder estimates, having in mind further generalizations to stratified spaces.

**Definition 1.4.** $(M,g_0)$ satisfies parabolic Schauder estimates, if there is a sequence of Banach spaces $\{C^{k,\alpha} \equiv C^{k,\alpha}(M \times [0,T])\}_{k \in \mathbb{N}_0}$ of continuous functions on $M \times [0,T]$, for some $\alpha \in (0,1)$ and any $T > 0$, with the following properties.

(1) **Algebraic properties of the Banach spaces:**

(a) For any $k \in \mathbb{N}_0$, the constant function $1 \in C^{k,\alpha}(M \times [0,T])$.

(b) For any $k \in \mathbb{N}_0$ and any $u \in C^{k,\alpha}(M \times [0,T])$ uniformly bounded away from zero, we have for the inverse $u^{-1} \in C^{k,\alpha}(M \times [0,T])$.

(c) For any $k \in \mathbb{N}_0$ we have $C^{k+1,\alpha}(M \times [0,T]) \subset C^{k,\alpha}(M \times [0,T])$.

(d) For any $k \geq 2$ and $\ell \leq k$ we have $C^{k,\alpha} \cdot C^{\ell,\alpha} \subset C^{\ell,\alpha}$. Writing $\| \cdot \|_{\ell,\alpha}$ for the norm on $C^{\ell,\alpha}$ we have a uniform constant $C_{\ell,\alpha}$ such that for any $u \in C^{k,\alpha}$ and $v \in C^{\ell,\alpha}$
\[
\| u \cdot v \|_{\ell,\alpha} \leq C_{\ell,\alpha} \| u \|_{k,\alpha} \| v \|_{\ell,\alpha}, \tag{1.9}
\]

(2) **Regularity properties of the Banach spaces:**

(a) We have the following inclusions\(^3\)
\[
C^{0,\alpha}(M \times [0,T]) \subset C^0([0,T],L^2(M)),
C^{1,\alpha}(M \times [0,T]) \subset C^0([0,T],H^1(M)) \cap C^1([0,T],H^1(M)),
C^{2,\alpha}(M \times [0,T]) \subset C^1([0,T],H^1(M)) \cap L^\infty(M \times [0,T]).
\tag{1.10}
\]

Moreover, for any $u \in C^{0,\alpha}(M \times [0,T])$ and any fixed $p \in M$, the evaluation $u(p,\cdot)$ still lies in $C^{0,\alpha}$. The map $M \ni p \mapsto \|u(p,\cdot)\|_{0,\alpha}$ is again $L^2(M)$.

(b) If $C^{k,\alpha}([0,T]) \subset C^{k,\alpha}(M \times [0,T])$ consists of functions that are constant on $M$, then the spaces $C^{2k,\alpha}([0,T])$ are characterized as follows
\[
C^{2k,\alpha}([0,T]) = \{ u \in C^{0,\alpha}([0,T]) \mid \partial_t^k u \in C^{0,\alpha}([0,T]) \}.
\tag{1.11}
\]

\(^3\)The space $C^{1,\alpha}$ introduced in [BAVe14] does not specify regularity under time differentiation. However, one can extend its definition such that $\partial_t u \equiv \Delta u + u \in C^{0,\alpha}$ for any $u \in C^{1,\alpha}$. This does not affect the other arguments in [BAVe14] and [BAVe19].
(c) For any $k \in \mathbb{N}_0$ the following maps are bounded
\[\partial_t, \Delta_0 : C^{k+2,\alpha}(M \times [0, T]) \to C^{k,\alpha}(M \times [0, T]),\]
\[\nabla : C^{k+1,\alpha}(M \times [0, T]) \to C^{k,\alpha}(M \times [0, T]).\]  

(3) Weak maximum principle for elements of the Banach spaces:
(a) Any $u \in C^{2,\alpha}(M \times [0, T])$ satisfies a weak maximum principle, that is for any Cauchy sequence $\{q_l\}_{l \in \mathbb{N}} \subset M$ we have the following
\[\inf_M u = \lim_{l \to \infty} u(q_l) \Rightarrow \lim_{l \to \infty} (\Delta_0 u)(q_l) \geq 0.\]  

(4) Mapping properties of the heat operator:
(a) The heat operator $e^{t\Delta_0}$ admits following mapping properties
\[e^{t\Delta_0} : C^{k,\alpha}(M \times [0, T]) \to C^{k+2,\alpha}(M \times [0, T]),\]
\[e^{t\Delta_0} : C^{k,\alpha}(M \times [0, T]) \to L^\alpha C^{k+1,\alpha}(M \times [0, T]),\]
\[e^{t\Delta_0} : L^\infty(M \times [0, T]) \to C^{1,\alpha}(M \times [0, T]).\]  

If $e^{t\Delta_0}$ acts without convolution in time, then we have a bounded map
\[e^{t\Delta_0} : C^{k,\alpha}(M) \to C^{k,\alpha}(M \times [0, T]).\]  

(5) Mapping properties of other solution operators:
(a) For any positive $\alpha \in C^{1,\alpha}(M \times [0, T])$, uniformly bounded away from zero, there is a solution operator $Q$ for $(\partial_t - \alpha \cdot \Delta_0)u = f, u(0) = 0$, such that
\[Q : C^{0,\alpha}(M \times [0, T]) \to C^{2,\alpha}(M \times [0, T]).\]  

If $\alpha \in C^{2,\alpha}$, then additionally, $Q : C^{1,\alpha} \to C^{3,\alpha}$ is bounded.

(b) For any positive $\alpha \in C^{1,\alpha}(M \times [0, T])$, uniformly bounded away from zero, there is a solution operator $R$ for $(\partial_t - \alpha \cdot \Delta_0)u = 0, u(0) = f$, such that
\[R : C^{2,\alpha}(M) \to C^{2,\alpha}(M \times [0, T]),\]  

where $C^{k,\alpha}(M)$ denotes the subspace of $C^{k,\alpha}(M \times [0, T])$ consisting of time-independent functions. If $\alpha \in C^{2,\alpha}$, then additionally, $R : C^{3,\alpha}(M) \to C^{3,\alpha}(M \times [0, T])$ is bounded.

Clearly, parabolic Schauder estimates hold on smooth compact Riemannian manifolds. By [BAVe14, BAVe19] a manifold with a wedge singularity satisfies the parabolic Schauder estimates\(^4\), assuming that the wedge metric is feasible in the sense of [BAVe19, Definition 2.2]. The proof is based on the microlocal heat kernel description in [MAVe12]. Note that the choice of Banach spaces is not canonical, and one can e.g. use the scale of weighted Hölder spaces as in [VER20] instead. In view of the recent work by Albin and Gell-Redman [AlGe17],

\(^4\)In fact in the mapping properties of solution operators $Q$ and $R$ we require here less than in [BAVe19]: in case $\alpha \in C^{2,\alpha}$ we only ask for $Q : C^{1,\alpha} \to C^{3,\alpha}$ and $R : C^{3,\alpha}(M) \to C^{3,\alpha}$, while in [BAVe19] these additional mapping properties are proved for one order higher.
we expect same parabolic Schauder estimates to hold on general stratified spaces with iterated cone-wedge metrics.

**Assumption 3.** \((M, g_0)\) satisfies parabolic Schauder estimates.

Using parabolic Schauder estimates, we can prove short time existence and regularity of the renormalized Yamabe flow, exactly as in [BaVe14, Theorem 1.7 and 4.1] and by a slight adaptation of [BaVe19, Proposition 4.8].

**Theorem 1.5.** Let \((M, g_0)\) satisfy parabolic Schauder estimates. Assume moreover that the scalar curvature \(S_0\) of \(g_0\) lies in \(C^{1,\alpha}(M)\). Then the following holds.

1. The Yamabe flow (1.2) admits for some \(T > 0\) sufficiently small, a solution \(u \in C^{2,\alpha}(M \times [0, T]) \subseteq C^1([0, T), H^1(M)) \cap L^\infty(M \times [0, T])\),

   that is positive and uniformly bounded away from zero\(^5\).

2. If a solution \(u \in C^{2,\alpha}(M \times [0, T])\) to the Yamabe flow (1.2) exists for a given \(T > 0\) and is uniformly bounded away from zero, then in fact \(u \in C^{3,\alpha}(M \times [0, T])\). In particular we obtain

\[
S \in C^{1,\alpha}(M \times [0, T]) \subseteq C^0([0, T], H^1(M)) \cap C^1((0, T), H^1(M)).
\]

**Proof.** We shall only provide a brief proof outline. The first statement is proved by setting up a fixed point argument in the Banach space \(C^{2,\alpha}(M \times [0, T])\). If \(u = 1 + v \in C^{2,\alpha}(M \times [0, T])\) is a solution to (1.2), then \(v\) satisfies the equation

\[
\partial_t v - (n-1)\Delta_0 v = -\frac{n-2}{4}S_0 + \Phi(v), \tag{1.18}
\]

where \(\Phi : C^{2,\alpha}(M \times [0, T]) \to C^{0,\alpha}(M \times [0, T])\) is a bounded map, in view of the algebraic and regularity properties (1.12) in Definition 1.4. Moreover, \(\Phi\) is quadratic in its argument, i.e. writing \(\| \cdot \|_{k,\alpha}\) for the norm on \(C^{k,\alpha}\) for any \(k \in \mathbb{N}\), there exists a uniform \(C > 0\), such that by (1.9) (cf. [BaVe14, Lemma 5.1])

\[
\forall w, w' \in C^{2,\alpha}: \quad \| \Phi(w) \|_{0,\alpha} \leq C \| w \|_{2,\alpha}^2,
\]

\[
\| \Phi(w) - \Phi(w') \|_{0,\alpha} \leq C (\| w \|_{2,\alpha} + \| w' \|_{2,\alpha}) \| w - w' \|_{2,\alpha}. \tag{1.19}
\]

Now a solution \(v\) of (1.18) (and hence also a solution \(u = 1 + v\) of (1.2)) is obtained as a fixed point of the map

\[
C^{2,\alpha}(M \times [0, T]) \ni v \mapsto e^{t(n-1)\Delta_0} \left( -\frac{n-2}{4}S_0 + \Phi(v) \right) \in C^{2,\alpha}(M \times [0, T]), \tag{1.20}
\]

which is a contraction mapping on a subset of \(C^{2,\alpha}(M \times [0, T])\) for \(T > 0\) sufficiently small\(^6\), by (1.14) in Definition 1.4. One argues exactly as in [BaVe14, Theorem 4.1]. Note that the regularity of the scalar curvature \(S\) along the flow is then \(S \in C^{0,\alpha}(M \times [0, T])\).

Note also, that the fixed point argument is performed in a small ball around zero in \(C^{2,\alpha}(M \times [0, T])\), and thus for \(T > 0\) sufficiently small, the norm of \(v\) is small. Hence \(u = 1 + v\) is positive and bounded away from zero.

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\(^5\)Later on, we will prove uniform lower bounds on \(u\) for any finite \(T > 0\).

\(^6\)We need to assume that \(T > 0\) is sufficiently small in order to control \(e^{t(n-1)\Delta_0}(S_0)\).
The second statement improves regularity of $S$. By the regularity properties (1.10) in Definition 1.4, we conclude that $\rho, \partial_t \rho \in C^{0,\alpha}([0,T])$. By (1.11), this implies that $\rho \in C^{2,\alpha}([0,T])$. We can now apply the mapping properties (1.16) and (1.17) in Definition 1.4 to obtain a solution $u' \in C^{3,\alpha}(M \times [0,T])$ to

$$
\partial_t u' - (n - 1)u^{-\frac{4}{n-2}} \Delta_g u' = \frac{n-2}{4} \left( \rho u - S_0 u^{\frac{n+4}{n-2}} \right), \quad u'(0) = 1. \quad (1.21)
$$

The given solution $u \in C^{2,\alpha}$ satisfies the same equation, and we can prove by the weak maximum property (1.13) of elements in $C^{2,\alpha}$, that $u \equiv u'$. Thus, indeed $u \in C^{3,\alpha}$ and hence $S \in C^{1,\alpha}$. This is basically the argument also used in [BAVE19, Proposition 4.8].

**Remark 1.6.** If we assume $Q : C^{2,\alpha} \to C^{4,\alpha}$ and $R : C^{4,\alpha}(M) \to C^{4,\alpha}$ in Definition 1.4, as has been proved in [BAVE19], then the condition $S_0 \in C^{2,\alpha}(M)$ implies by similar arguments as in Theorem 1.5, that any solution $u \in C^{2,\alpha}$ is actually in $C^{4,\alpha}$. This would lead to $S \in C^{2,\alpha}$, in particular the scalar curvature would stay bounded along the flow. Here, we decided to require less in Definition 1.4, assume less regularity for $S_0$, and conclude boundedness of $S$ by Moser iteration methods instead.

### 1.4. Regularity of the initial scalar curvature.

In view of Theorem 1.5 we arrive at our final assumption on a regularity of the initial scalar curvature $S_0$ with respect to the scale of Banach spaces in Definition 1.4.

**Assumption 4.** Assuming that $(M, g_0)$ satisfies parabolic Schauder estimates, we also ask the initial scalar curvature $S_0 \in C^{1,\alpha}(M)$. This implies in view of Theorem 1.5

$$S \in C^0([0,T], H^1(M)) \cap C^1((0,T), H^1(M)).$$

In particular, since the flow $u : C^{2,\alpha}(M \times [0,T])$ is bounded from above and below for $T > 0$ sufficiently small, norms on the Sobolev space $H^1(M)$ with respect to $g_0$ and on the Sobolev space $H^1(M, g)$ with respect to $g = u^{-\frac{4}{n-2}} g_0$, are equivalent. Thus $S$ lies in the Sobolev space $H^1(M, g)$

$$S \in C^0([0,T], H^1(M, g)) \cap C^1((0,T), H^1(M, g)).$$

Our arguments below will use regularity of $S$ to show that given $S_0 \in L^q(M)$ for $q = \frac{n^2}{2(n-2)} = \frac{n}{2} + \frac{n}{n-2} > n/2$, we may conclude by Moser iteration that $S \in L^\infty(M)$ for positive times. We close this subsection with an observation, that on stratified spaces, $S_0 \in L^q(M)$ for $q > n/2$ and $S_0 \in L^\infty(M)$ basically carry the same geometric restriction. Indeed, consider a cone $(0,1) \times N$ over a Riemannian manifold $(N, g_0)$, with metric $g_0 = dx^2 + x^2 g_N + h$, where $h$ is smooth in $x \in [0,1]$ and $|h|_\infty = O(x)$ as $x \to 0$, where we write $\mathfrak{g} := dx^2 + x^2 g_N$. Then

$$S_0 \sim \frac{\text{scal}(g_N) - \text{dim} N (\text{dim} N - 1)}{x^2} + O(x^{-1}), \quad \text{as } x \to 0, \quad (1.22)$$

where the higher order term $O(x^{-1})$ comes from the perturbation $h$. Both assumptions $S_0 \in L^\infty(M)$ and $S_0 \in L^q(M)$ for $q > n/2$ imply that the leading term of the metric $g_0$ is scalar-flat, i.e. scal($g_N$) = dim $N$ (dim $N - 1$).

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7Here we use the assumption that $u$ is uniformly bounded away from zero, and that $1 \in C^{3,\alpha}$ by the algebraic properties of the Banach spaces.
1.5. The overarching strategy. Studies of the Yamabe flow usually follow the following very rough pattern. One first argues that (1.2) has a short-time solution. This is the step we have been concerned with in this section. This step doesn’t invoke the sign of the Yamabe constant.

The next step is to show that the flow can be extended to all times. The way one does this is to assume the flow is defined for \( t \in (0, T) \) for some maximal time \( T < \infty \) and then derive a priori bounds on the solution \( u \) and the scalar curvature \( S \), showing that neither of them develop singularities as \( t \to T \). One can thus keep flowing past \( T \), establishing long-time existence. This is the step we are concerned with for the rest of the paper.

2. The evolution of the scalar curvature and lower bounds

In this section we derive a lower bound on the scalar curvature \( S \) along the normalized Yamabe flow. We present an argument that does not require the maximum principle, but rather the following assumptions

\[
S \in C^0([0, T], H^1(M, g)) \cap C^1((0, T), H^1(M, g)), \\
C^\infty_c(M) \text{ is dense in } H^1(M), \\
H^1(M) = H^1(M, g), \\
Y(M, g_0) > 0.
\]

These properties follow from Assumptions 1, 2, 3 and 4.

**Lemma 2.1.** Let \( g = u^{\frac{4}{n-2}} g_0 \) be a family of metrics evolving according to the normalized Yamabe flow (1.2) satisfying\(^8\) (2.1). Then \( S \) evolves according to

\[
\partial_t S - (n - 1) \Delta S = S(S - \rho).
\]

where \( \Delta \) denotes the Laplacian with respect to the time-evolving metric \( g \). We write \( S_+ := \max\{S, 0\} \) and \( S_- := -\min\{S, 0\} \). Then \( S_\pm \in C^1((0, T), H^1(M, g)) \) and satisfy

\[
\partial_t S_+ - (n - 1) \Delta S_+ \leq S_+ (S_+ - \rho), \\
\partial_t S_- - (n - 1) \Delta S_- \leq -S_- (S_- + \rho).
\]

**Remark 2.2.** The equation (2.2) is to be understood in the weak sense: for any compactly supported smooth test function \( \phi \in C^\infty_c(M) \) we have

\[
\int_M \partial_t S \cdot \phi \, dVol_g + (n - 1) \int_M (\nabla S, \nabla \phi)_g dVol_g = \int_M S(S - \rho) \cdot \phi \, dVol_g.
\]

Similarly for the partial differential inequalities (2.3) and (2.4) and \( \phi \geq 0 
\]

\[
\int_M \partial_t S_\pm \cdot \phi \, dVol_g + (n - 1) \int_M (\nabla S_\pm, \nabla \phi)_g dVol_g \leq \pm \int_M S_\pm (S_\pm - \rho) \cdot \phi \, dVol_g.
\]

By (2.1), \( C^\infty_c(M) \) is dense in \( H^1(M) = H^1(M, g) \). Hence we can as well assume \( \phi \in H^1(M, g) \) in the weak formulation above.

\(^8\)In fact here we do not require \( S \in C^0([0, T], H^1(M, g)) \).
Furthermore, we compute for the derivatives in case $x > 0$. For $v \in H^1(M, g)$ we may rewrite this as

$$\frac{\partial_t}{\partial_t} S = \frac{n+2}{4}(S-\rho)S - \frac{n-2}{4}(S-\rho)\Delta S - \frac{n-2}{4}(S-\rho)\Delta v.$$

This proves the formula (2.2). In order to derive the differential inequality for $S_+$, consider any $\varepsilon > 0$ and define

$$\psi_{\varepsilon}(x) := \begin{cases} \sqrt{x^2 + \varepsilon^2} - \varepsilon, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

For $v \in H^1(M, g)$ it is readily checked that $\psi_{\varepsilon}(v) \in H^1(M, g)$ and $\lim_{\varepsilon \to 0} \psi_{\varepsilon}(v) = v_+$. Furthermore, we compute for the derivatives in case $x > 0$

$$\psi'_{\varepsilon}(x) = \frac{x}{\sqrt{x^2 + \varepsilon^2}}, \quad \psi''_{\varepsilon}(x) = \frac{\varepsilon^2}{(x^2 + \varepsilon^2)^{3/2}}.$$

These are both bounded for a fixed $\varepsilon > 0$, so the chain rule applies. Next up, we claim for any $v \in H^1(M, g)$ in the weak sense

$$\Delta \psi_{\varepsilon}(v) \geq \frac{v}{\sqrt{v^2 + \varepsilon^2}} \Delta v = \psi'_{\varepsilon}(v) \Delta v.$$

This is seen as follows. Let $0 \leq \xi \in C_c^\infty(M)$ be arbitrary and compute

$$\int_M \xi \Delta \psi_{\varepsilon}(v) \, d\text{Vol}_g := -\int_M \nabla \xi \cdot \nabla \psi_{\varepsilon}(v) \, d\text{Vol}_g = -\int_M \frac{v}{\sqrt{v^2 + \varepsilon^2}} \Delta \psi_{\varepsilon}(v) \, d\text{Vol}_g$$

$$= -\int_M \left( \nabla \psi_{\varepsilon} \right)_g \nabla \left( \frac{v}{\sqrt{v^2 + \varepsilon^2}} \xi \right) \, d\text{Vol}_g + \int_M \frac{\varepsilon^2 |v|}{\sqrt{v^2 + \varepsilon^2}} \, d\text{Vol}_g$$

$$\geq -\int_M \left( \nabla \psi_{\varepsilon} \right)_g \nabla \left( \frac{v}{\sqrt{v^2 + \varepsilon^2}} \xi \right) \, d\text{Vol}_g =: \int_M \frac{\xi v}{\sqrt{v^2 + \varepsilon^2}} \Delta v \, d\text{Vol}_g.$$
This proves (2.5), which allows us to deduce
\[ \partial_t \psi_\varepsilon(S) - (n - 1)\Delta \psi_\varepsilon(S) \leq \begin{cases} \psi_\varepsilon'(S)(\partial_t S - (n - 1)\Delta S), & S \geq 0 \\ 0, & S < 0 \end{cases} \]
\[ = \begin{cases} \psi_\varepsilon'(S)(S - \rho), & S \geq 0 \\ 0, & S < 0 \end{cases} \]
\[ = \frac{S}{\sqrt{S^2 + \varepsilon^2}} S_+(S - \rho). \]

Letting \( \varepsilon \to 0 \) results in (2.3). To prove (2.4), observe that \( S_- = S_+ - S \). Hence
\[ \partial_t S_- - (n - 1)\Delta S_- = (\partial_t S_+ - (n - 1)\Delta S_+) - (\partial_t S - (n - 1)\Delta S) \]
\[ \leq (S_+(S - \rho) - S(S - \rho)) - S_-(S - \rho), \]
where we used (2.2) and (2.3) in the inequality step. The only thing which remains to be observed is that \( S_- \cdot S = S_-(S_+ - S_-) = -S_-^2 \).

We can now derive lower bounds for \( S \) by studying the evolution (in-) equalities above. This is usually done by invoking the weak maximum principle for \( S \), which is not available under the current assumptions (2.1). Thus, we provide an alternative novel argument, which does not use a maximum principle and which we could not find elsewhere in the literature.

**Proposition 2.3.** Let \( g = u^{n/2} g_0 \) be a family of metrics evolving according to the normalized Yamabe flow (1.2) satisfying (2.1). Then
\[ \|S_-\|_{L^p(M,g)}(t) \leq e^{\int_0^t \frac{\rho}{2p} \|S_-(S - \rho)\|_{L^p(M)} dt} \]
holds for all \( 2 \leq p \leq \infty \). In particular, if \( (S_0)_- \in L^\infty(M) \), then \( S_- \in L^\infty \) on \( [0, T] \) with uniform bounds depending only on \( T \) and \( S_0 \). Moreover, if \( S_0 \geq 0 \), then \( S \geq 0 \) along the normalized Yamabe flow for all time.

**Proof.** The weak formulation of (2.4) is that for any \( 0 \leq \xi \in H^1(M, g) \)
\[ \int_M \xi \partial_t S_- \, dV_g + (n - 1) \int_M (\nabla S_-, \nabla \xi)_g \, dV_g \leq -\int_M \xi S_-(S_+ + \rho) \, dV_g \] (2.6)
holds. A problem when manipulating this is of course that the chain rule fails to hold in general, so we use the same workaround as [ACM14, pp. 10-13] (who in turn are following [Gur93, pp. 349-352]). Let \( L > 0, \beta \geq 1 \) and define
\[ \Phi_{\beta,L}(x) := \begin{cases} x^\beta, & x \leq L, \\ \beta L^{\beta - 1}(x - L) + L^\beta, & x > L. \end{cases} \]
(2.7)
\[ G_{\beta,L}(x) := \int_0^x \Phi_{\beta,L}(y)^2 \, dy = \begin{cases} \frac{\beta^2}{2\beta - 1} x^{2\beta - 1}, & x \leq L, \\ \beta^2 L^{2(\beta - 1)} x - \frac{2\beta^2 L^{2\beta - 1}(\beta - 1)}{2\beta - 1}, & x > L. \end{cases} \]
(2.8)
Finally, we define $H_{\beta,L}(x) := \int_0^x G_{\beta,L}(y) \, dy$ and conclude

$$H_{\beta,L}(x) = \begin{cases} \frac{\beta x^{2\beta}}{2(2\beta - 1)}, & x \leq L, \\ \frac{\beta^2 L^2(\beta - 1)}{2} (x^2 - L^2) - \frac{2\beta^2 L^{2\beta-1}(\beta - 1)}{2\beta - 1} (x - L) + \frac{\beta L^{2\beta}}{2(2\beta - 1)}, & x > L. \end{cases}$$

The crucial features of these definitions are as follows

$$\phi_{\beta,L}(x) \xrightarrow{L \to \infty} x^\beta, \quad G_{\beta,L}(x) \xrightarrow{L \to \infty} \frac{\beta^2}{2\beta - 1} x^{2\beta - 1}, \quad H_{\beta,L}(x) \xrightarrow{L \to \infty} \frac{\beta}{2(2\beta - 1)} x^{2\beta}.$$ 

These functions are also dominated by simpler expressions. For instance, $H_{\beta,L}(x) = \beta^2 x^{2\beta}$ holds for all $L > 0$ and $\beta \geq 1$, as one sees as follows. For $x \leq L$ there is nothing to show. For $x > L$, we first observe that

$$H_{\beta,L}(x) = \frac{\beta^2}{2} L^{2(\beta - 1)} x^2 - \frac{2\beta^2 (\beta - 1)}{2\beta - 1} L^{2\beta - 1} x + \frac{\beta (\beta - 1)}{2} L^{2\beta}.$$ 

Dropping the non-positive middle term and estimating by $x \geq L$ we find

$$H_{\beta,L}(x) \leq \frac{\beta^2}{2} x^{2\beta} + \frac{\beta (\beta - 1)}{2} x^{2\beta} < \beta^2 x^{2\beta}.$$ 

Another important property is $\phi_{\beta,L} \in C^1(\mathbb{R}_+)$, with $\phi'_{\beta,L} \in L^\infty(\mathbb{R}_+)$ for all $L > 0$, and so we may apply the chain rule to $\phi_{\beta,L}(S_-)$. Finally, since we are assuming a $C^1$ time-dependence, we have $\partial_t H_{\beta,L}(S_-) = (\partial_t S_-) G_{\beta,L}(S_-)$. We will use $\xi := G_{\beta,L}(S_-)$ as a test function in (2.6). Note that by definition, $G_{\beta,L}(x)$ is linear for $x > L$ and hence $G_{\beta,L}(f) \in H^1(M, g)$ whenever $f \in H^1(M, g)$ (here we are also using that $\text{Vol}(M) < \infty$). Then (2.6) implies

$$\int_M \partial_t H_{\beta,L}(S_-) \, d\text{Vol}_g \leq -(n - 1) \int_M |\nabla \phi_{\beta,L}(S_-)|_g^2 \, d\text{Vol}_g - \int_M G_{\beta,L}(S_-) S_-(S_- + \rho) \, d\text{Vol}_g. \quad (2.9)$$

We then use (1.3) to conclude

$$\int_M \partial_t H_{\beta,L}(S_-) \, d\text{Vol}_g = \partial_t \int_M H_{\beta,L}(S_-) \, d\text{Vol}_g + \frac{n}{2} \int_M H_{\beta,L}(S_-)(S_- - \rho) \, d\text{Vol}_g$$

$$= \partial_t \int_M H_{\beta,L}(S_-) \, d\text{Vol}_g - \frac{n}{2} \int_M H_{\beta,L}(S_-)(S_- + \rho) \, d\text{Vol}_g,$$

$$\quad (2.10)$$

where the last step uses $SH_{\beta,L}(S_-) \equiv (S_+ - S_-) H_{\beta,L}(S_-) = -S_- H_{\beta,L}(S_-)$. Finally, we need a Sobolev inequality given to us by the positivity of the Yamabe constant, namely for any $f \in H^1(M, g)$ we have by the definition of $Y(M, g_0)$ (note that $Y(M, g_0) = Y(M, g)$ by conformal invariance)

$$Y(M, g_0) \|f\|_{L^{2\alpha} (M, g)}^2 \leq 4 \frac{n - 1}{n - 2} \|\nabla f\|_{L^2(M, g)}^2 + \int_M S \rho^2 \, d\text{Vol}_g. \quad (2.11)$$
We set $f = \phi_{\beta,L}(S_-)$. Observe $\phi_{\beta,L}(S_-)^2 S = -\phi_{\beta,L}(S_-)^2 S_-$. Then (2.11) implies
\[
(n - 1) \| \nabla \phi_{\beta,L}(S_-) \|^2_{L^2(M,g)} \geq \frac{n - 2}{4} Y(M, g_0) \| \phi_{\beta,L}(S_-) \|^2_{L^{2\beta}(M,g)} + \frac{n - 2}{4} \int_M \phi_{\beta,L}(S_-)^2 S_- \, d\text{Vol}_g \\
\geq \frac{n - 2}{4} \int_M \phi_{\beta,L}(S_-)^2 S_- \, d\text{Vol}_g.
\]
Combining (2.9), (2.10) and (2.12) yields
\[
\partial_t \int_M H_{\beta,L}(S_-) \, d\text{Vol}_g \leq \int_M \left( \frac{n}{2} H_{\beta,L}(S_-) - G_{\beta,L}(S_-)S_- - \frac{n - 2}{4} \phi_{\beta,L}(S_-)^2 \right) S_- \, d\text{Vol}_g \\
+ \int_M \rho \left( \frac{n}{2} H_{\beta,L}(S_-) - G_{\beta,L}(S_-)S_- \right) \, d\text{Vol}_g.
\]
The claim is that the first group of terms on the right hand side is non-positive, which follows by a direct computation
\[
\frac{n}{2} H_{\beta,L}(x) - xG_{\beta,L}(x) - \frac{n - 2}{4} \phi_{\beta,L}(x)^2
\]
\[
\begin{align*}
\left( \begin{array}{c} \frac{4(2\beta - 1)}{4(2\beta + n)(\beta - 1) + 2} x^{2\beta}, & x \leq L, \\
\frac{L^{2\beta}}{4} \left( -2\beta^2 \left( \frac{x}{L} \right)^2 - \frac{2(n - 2)\beta(\beta - 1)}{2\beta - 1} \left( \frac{x}{L} \right) + (\beta - 1)(n + 2(\beta - 1)) \right), & x > L.
\end{array} \right)
\end{align*}
\]
In both cases one checks that the expressions are non-positive\(^9\) for $\beta \geq 1$. Hence using that $G_{\beta,L}(S_-) \geq 0$ and $\rho$ is non-increasing by (1.5), we conclude
\[
\partial_t \int_M H_{\beta,L}(S_-) \, d\text{Vol}_g \leq \int_M \frac{n\rho}{2} H_{\beta,L}(S_-) \, d\text{Vol}_g \leq \frac{n\rho(0)}{2} \int_M H_{\beta,L}(S_-) \, d\text{Vol}_g.
\]
Integrating this shows
\[
\int_M H_{\beta,L}(S_-) \, d\text{Vol}_g(t) \leq e^{t \frac{n\rho(0)}{2}} \int_M H_{\beta,L}(S_-) \, d\text{Vol}_g(t = 0).
\]
The conclusion will follow when we take the limit $L \to \infty$, which we can do for the following reason\(^10\). On the left hand side we appeal to Fatou’s lemma and the pointwise convergence of $H_{\beta,L}$;
\[
\liminf_{L \to \infty} \int_M H_{\beta,L}(S_-) \, d\text{Vol}_g \geq \int_M \liminf_{L \to \infty} H_{\beta,L}(S_-) \, d\text{Vol}_g = \frac{\beta}{2(2\beta - 1)} \int_M S_-^{2\beta} \, d\text{Vol}_g.
\]
The right hand side we deal with by the dominated convergence theorem. We showed above that $H_{\beta,L}(x) \leq \beta^2 x^{2\beta}$ holds for all $L > 0$ and $\beta \geq 1$. Since we are assuming $(S_0)_{-} \in L^\infty(M)$, can use $\beta^2 ((S_0)_{-})^{2\beta}$ as a dominating integrable function to deduce
\[
\liminf_{L \to \infty} \int_M H_{\beta,L}((S_0)_{-}) \, d\mu = \lim_{L \to \infty} \int_M H_{\beta,L}((S_0)_{-}) \, d\mu = \frac{\beta}{2(2\beta - 1)} \int_M ((S_0)_{-})^{2\beta} \, d\mu
\]
\(^9\)For the $x \geq L$ case observe that the polynomial is negative for $x = L$, and the expression for $x > L$ clearly has a negative derivative. So the expression remains negative for $x > L$.
\(^10\)This argument is applied several times, without writing out the details in the latter instances.
Combined we conclude for $\beta \geq 1$
\[
\int_M S^{2\beta} \, d\operatorname{Vol}_g \leq e^{\mu(0)} \int_M (S_0)^{2\beta} \, d\mu.
\]
This gives the conclusion when writing $2\beta = p$. \hfill $\square$

Remark 2.4. Let us again emphasize the novelty of this argument: it circumvents the maximum principle, and one only needs to know that $S \in C^1((0, T); H^1(M, g)) \cap C^0([0, T]; H^1(M, g))$, as assumed in (2.1).

For completeness, let us also provide the classical widely known argument, cf. [BRE05], using the weak maximum principle: we assume that $S$ satisfies (1.13), which is the case if $S \in C^{2,\alpha}(M \times [0, T])$. See Remark 1.6 for conditions which ensure this regularity of $S$ along the flow.

Proposition 2.5. Assume $S \in C^0(M \times [0, T])$ satisfies the weak maximum principle (1.13) and that $Y(M, g_0) > 0$. Then $S$ admits a uniform lower bound

$$S \geq \min \{0, \inf_M S_0\}.$$  

Proof. By the weak maximum principle, we have for $S_{\text{min}} := \inf_M S$

$$\partial_t S_{\text{min}} \geq S_{\text{min}}(S_{\text{min}} - \rho).$$

If $S_{\text{min}}$ is negative for all times, then the right hand side becomes positive, and we we get $S_{\text{min}} \geq \inf_M S_0$. If $S_{\text{min}}$ is positive for all times, we can further estimate the right hand side using $\rho \leq \rho(0)$, cf. (1.5). Dividing, we then get

$$\frac{\partial_t S_{\text{min}}}{S_{\text{min}}(\rho(0) - S_{\text{min}})} \geq -1.$$

Integrating this differential inequality we find (writing $S_{\text{min}}^0 := \inf_M S_0$)

$$S_{\text{min}}(t) \geq \frac{\rho(0)(S_0)_{\min}}{e^{\rho(0)t}(\rho(0) - (S_0)_{\min}) + (S_0)_{\min}} \geq 0.$$

If $S_{\text{min}}$ changes the sign along the flow, the statement follows by a combination of both estimates. \hfill $\square$

3. Uniform bounds on the solution along the flow

The arguments of this section employ the following assumptions

\begin{align*}
(M, g_0) & \text{ is an admissible manifold,} \\
u & \in C^1((0, T); H^1(M)) \cap C^0([0, T]; H^1(M)), \\
S & \in C^1((0, T); H^1(M, g)) \cap C^0([0, T]; H^1(M, g)), \\
H^1(M) & = H^1(M, g), \ Y(M, g_0) > 0.
\end{align*}

These properties follow from Assumptions 1, 2, 3 and 4.

We begin with the upper bound on $u$, which follows easily from the lower bound on the scalar curvature $S$, obtained in Proposition 2.3.
Proposition 3.1. Let \( g = u^{\frac{4}{n-2}} g_0 \) be a family of metrics, \( u > 0 \), such that (3.1) holds and the normalized Yamabe flow equation (1.2) holds weakly with \( u(0) = 1 \). Assume furthermore that \( (S_0)_- \in L^\infty(M) \), where \( S_0 \) is the scalar curvature of \( g_0 \). Then there exists some uniform constant \( 0 < C(T) < \infty \), depending only on \( T > 0 \) and \( S_0 \), such that \( u \leq C(T) \) for all \( t \in [0, T] \), \( T < \infty \).

Proof. We have by (1.1) and (1.5) that
\[
\partial_t u = -\frac{n-2}{4} (S - \rho) u \leq \frac{n-2}{4} (S_+ + \rho) u \leq \frac{n-2}{4} (S_+ + \rho(0)) u.
\]

By Proposition 2.3 we have \( \|S_\|_{L^\infty(M)} \leq \|(S_0)_-\|_{L^\infty(M)} \), and hence setting \( C := \frac{n-2}{4} \left( \|(S_0)_-\|_{L^\infty(M)} + \rho(0) \right) \), we conclude
\[
\partial_t u \leq Cu \implies u \leq e^{CT} u_0 = e^{CT}.
\]

\[\Box\]

The lower bound is more intricate and in many ways more interesting. The argument will rely on the upper bound on \( u \) and the lower bound on \( S \). The proof will be a mixture and modification of the methods of [ACM14, pp. 20-21] and [BRE05, pp. 221-222].

Theorem 3.2. Let \( g = u^{\frac{4}{n-2}} g_0 \) be a family of metrics, \( u > 0 \), such that (3.1) holds and the normalized Yamabe flow equation (1.2) holds weakly with \( u(0) = 1 \). Assume furthermore that \( (S_0)_- \in L^\infty(M) \) and that \( S_0 \in L^q(M) \) for some \( q > \frac{n}{2} \). Then there exists some uniform constant \( c(T) > 0 \), depending only on \( T > 0 \) and \( S_0 \), such that \( c(T) \leq u \) for all \( t \in [0, T] \).

Proof. By combining (1.2) and (1.1) we may solve away the term \( \partial_t u \) and get
\[
-4 \frac{n-1}{n-2} \Delta_0 u = \left( u^{\frac{4}{n-2}} S - S_0 \right) u.
\]

Using \( (S_0)_- \in L^\infty(M) \) and \( u \in L^\infty(M \times [0, T]) \) by Proposition 3.1, we may define
\[
P := \frac{n-2}{4(n-1)} \left( S_0 + \|u\|_{L^\infty(M \times [0, T])} \| (S_0)_-\|_{L^\infty(M)} \right) \in L^q(M),
\]

Note that \( P \) only depends on \( S_0 \) and \( T \). Furthermore, Proposition 2.3 yields
\[
(-\Delta_0 + P) u \geq 0. \tag{3.2}
\]

Let us explain the proof idea. Assume we can show that there is some \( \delta > 0 \) such that \( u^{-\delta} \in H^1(M) \) uniformly in \( t \in [0, T] \). Then (3.2) implies
\[
(-\Delta_0 - \delta P) u^{-\delta} = \delta u^{-1-\delta} \Delta_0 u - \delta(1+\delta) u^{-2-\delta} |\nabla u|^2 - \delta P u^{-\delta} = -\delta u^{-1-\delta} (-\Delta_0 + P) u - \delta(1+\delta) u^{-2-\delta} |\nabla u|^2 \leq 0. \tag{3.3}
\]

This is precisely the setting of [ACM14, Proposition 1.8], which then concludes by Moser Iteration and Sobolev inequality (1.8)
\[
\|u^{-\delta}\|_{L^\infty(M)} \leq C \|u^{-\delta}\|_{H^1(M)},
\]

Proof. We have by (1.1) and (1.5) that
where the constant $C > 0$ depends on $\delta P$, hence only on $T$ and $S_0$, but not on $t$. Under our temporary assumption (3.2), we thus get a uniform bound on $u^{-\delta}$, which gives a uniform lower bound on $u$.

Hence we only need to show that $u^{-\delta} \in H^1(M)$ uniformly. Let $\epsilon, \delta > 0$ and (following [ACM14, pp. 20-21]) define the functions $\psi_\epsilon(u) := (u + \epsilon)^{-\delta}$ and $\phi_\epsilon(u) := (u + \epsilon)^{-1-2\delta}$. These are both in $H^1(M)$ since $u$ is. Using $\phi_\epsilon$ as a test function in the weak formulation of (3.2) we deduce

$$\frac{1}{2} \int_M \Delta \psi_\epsilon(u) \, d\mu = 0$$

and, using that $u \phi_\epsilon(u) \leq \psi_\epsilon(u)^2$ along with the Hölder inequality, we find

$$\|\nabla \psi_\epsilon(u)\|^2_{L^2(M)} \leq \left(\frac{\delta^2}{1 + 2\delta}\right) \|P\|_{L^q(M)} \|\psi_\epsilon(u)\|^2_{L^{q/(q-1)}(M)}. \tag{3.4}$$

Since $q > \frac{n}{2}$ we have $\frac{\delta^2}{1 + 2\delta} < \frac{n}{q-1}$ and thus $\|\psi_\epsilon(u)\|^2_{L^{q/(q-1)}(M)} \leq \|\psi_\epsilon(u)\|^2_{L^{q/(q-2)}(M)}$. By the Sobolev inequality (1.8) we know

$$\|\psi_\epsilon(u)\|^2_{L^{q/(q-2)}(M)} \leq A_0 \|\nabla \psi_\epsilon(u)\|^2_{L^2(M)} + B_0 \|\psi_\epsilon(u)\|^2_{L^2(M)}. \tag{3.5}$$

Next we need a Poincaré inequality. Let $B \subset M$ be a ball. Then, exactly as in [ACM14, Lemma 1.14], there exists a constant $C_B > 0$ such that

$$\|\nabla \psi_\epsilon(u)\|^2_{L^2(M)} \leq C_B \left(\|\nabla \psi_\epsilon(u)\|^2_{L^2(M)} + \|f\|^2_{L^2(B)}\right), \tag{3.6}$$

holds for all $f \in H^1(M)$. Plugging (3.5) and (3.6) in (3.4) results in

$$\|\nabla \psi_\epsilon(u)\|^2_{L^2(M)} \leq \left(1 - \frac{\delta^2}{1 + 2\delta}\right) \|P\|_{L^q(M)} \left(A_0 + B_0 C_B\right) \|\nabla \psi_\epsilon(u)\|^2_{L^2(M)} + B_0 C_B \|\psi_\epsilon(u)\|^2_{L^2(B)},$$

which is equivalent to the following inequality

$$\left(1 - \frac{\delta^2}{1 + 2\delta}\right) \|P\|_{L^q(M)} \left(A_0 + B_0 C_B\right) \|\nabla \psi_\epsilon(u)\|^2_{L^2(M)} \leq \left(1 - \frac{\delta^2}{1 + 2\delta}\right) \|P\|_{L^q(M)} B_0 C_B \|\psi_\epsilon(u)\|^2_{L^2(B)}.$$

Choosing $\delta > 0$ small enough so that the left hand side becomes positive, we get a uniform (meaning now both $t$- and $\epsilon$-independent) bound on $\|\nabla \psi_\epsilon(u)\|_{L^2(M)}$ if we can get a uniform bound on $\|\psi_\epsilon(u)\|_{L^2(B)}$. The uniform bound on $\|\psi_\epsilon(u)\|_{L^2(B)}$ will come from the local theory for elliptic supersolutions. Observe that since $u$ satisfies (3.2), $u^{\frac{2n}{n-2}}$ satisfies (by the same computation as in (3.3))

$$-\Delta_0 u^{\frac{2n}{n-2}} + \frac{2n}{n-2} P u^{\frac{2n}{n-2}} \geq 0.$$  

Let $R > 0$ be such that $B_{2R}(x) \subset M$ for some $x \in M$. Then, according to [GiTr01, Theorem 8.18, p. 194] the following weak Harnack inequality holds on $B_{2R}(x)$, namely there is a constant $C > 0$ independent of $u$ but depending on $g_0$, $R$ and $n$ such that

$$\text{Vol}_g(B_{2R}(x)) \equiv \left\| u^{\frac{2n}{n-2}} \right\|_{L^1(B_{2R}(x))} \leq C \inf_{B_R(x)} u^{\frac{2n}{n-2}}, \tag{3.7}$$
where in the first identification we recalled \( d\text{Vol}_g = u^{\frac{2n}{n-2}} d\mu \). By admissibility of \((M, g_0)\), the assumption (1.7) holds and we may take a collection of balls \( B_{4R_i}(x_i) \subset M \), indexed by \( i = 1, \cdots, N < \infty \), with the property that
\[
\left( 1 - \text{Vol}_{g_0} \left( \bigcup_{i=1}^N B_{2R_i}(x_i) \right) \right) \|u\|_{L^\infty(M, \gamma)}^{\frac{2n}{n-2}} < 1. \tag{3.8}
\]
Let \( C_i \) be the constant in (3.7) for the ball \( B_{2R_i}(x_i) \). By summing all the individual inequalities (3.7) for each \( i = 1, \cdots, N \), we have
\[
\sum_{i=1}^N \text{Vol}_g(B_{2R_i}(x_i)) \leq \sum_{i=1}^N C_i \inf_{B_{R_i}(x_i)} u^{\frac{2n}{2n-2}} \leq NC \max_i \left( \inf_{B_{R_i}(x_i)} u^{\frac{2n}{2n-2}} \right)
\]
with \( C := \max_i C_i \). The left hand side we can bounded from below by
\[
\sum_{i=1}^N \text{Vol}_g(B_{2R_i}(x_i)) \geq \text{Vol}_g \left( \bigcup_{i=1}^N B_{2R_i}(x_i) \right) = 1 - \text{Vol}_g \left( M \setminus \bigcup_{i=1}^N B_{2R_i}(x_i) \right)
\]
\[
\geq 1 - \text{Vol}_{g_0} \left( M \setminus \bigcup_{i=1}^N B_{2R_i}(x_i) \right) \|u\|_{L^\infty(M, \gamma)}^{\frac{2n}{n-2}} =: c
\]
which is positive by choice of the balls subject to (3.8). Thus
\[
0 < c \leq NC \max_i \left( \inf_{B_{R_i}(x_i)} u^{\frac{2n}{2n-2}} \right).
\]
This shows that there has to be a ball \( B_{R_i}(x_i) \) with \( u \) uniformly bounded from below by \( c(T) > 0 \) for \( t \in [0, T) \). On this ball we thus get a uniform bound \( \psi_\varepsilon(u) \geq c(T)^{-\delta} \), which gives our desired \( t \)- and \( \varepsilon \)-independent bound on \( \|\psi_\varepsilon(u)\|_{L^1(B)} \), and thereby that \( u^{-\delta} \in H^1(M) \) uniformly. \( \square \)

**Corollary 3.3.** Under the conditions of Theorem 3.2, one can find uniform constants \( 0 < A(T), B(T) < \infty \), depending only on \( T > 0 \) and initial scalar curvature \( S_0 \) (but not dependent on \( t \)), such that for all \( f \in H^1(M, g) \)
\[
\|f\|_{L^{\frac{2n}{n-2}}(M, g)}^{\frac{2n}{n-2}} \leq A(T) \|\nabla f\|_{L^2(M, g)}^2 + B(T) \|f\|_{L^2(M, g)}^2, \tag{3.9}
\]
\( i.e. \) (1.8) holds for the time-dependent metric but with time-independent constants.

**Proof.** Due to (1.8) we have for all \( f \in H^1(M) = H^1(M, g) \)
\[
\|f\|_{L^{\frac{2n}{n-2}}(M, g_0)}^{\frac{2n}{n-2}} \leq A_0 \|\nabla f\|_{L^2(M, g_0)}^2 + B_0 \|f\|_{L^2(M, g_0)}^2.
\]
Using \( g = u^{\frac{1}{n-2}} g_0 \) we conclude a similar estimate with respect to \( g \)
\[
\|f\|_{L^{\frac{2n}{n-2}}(M, g)}^{\frac{2n}{n-2}} \leq A(T) \|\nabla f\|_{L^2(M, g)}^2 + B(T) \|f\|_{L^2(M, g)}^2, \tag{3.10}
\]
where \( A(T) := A_0 \left( \sup_{M_T} u \right)^2 \left( \inf_{M_T} u \right)^2 \), \( B(T) := B_0 \left( \frac{\left( \sup_{M_T} u \right)^2}{\left( \inf_{M_T} u \right)^2} \right) \).
Now the statement follows, since \( u, u^{-1} \in L^\infty(M \times [0, T]) \) by Proposition 3.1 and Theorem 3.2.

We shall need this Sobolev inequality (3.9) when we tackle the upper bound on the scalar curvature \( S \) in section 4.

4. Upper bound on the scalar curvature along the flow

The arguments of this section employ the following assumptions

\[
(M, g_0) \text{ is an admissible manifold,}
S \in C^1((0, T); H^1(M, g)) \cap C^0([0, T]; H^1(M, g)),
\]

The Sobolev inequality (3.9) holds,

\[
H^1(M) = H^1(M, g), \ Y(M, g_0) > 0,
C_c^\infty(M) \text{ is dense in } H^1(M).
\]

These properties follow from Assumptions 1, 2, 3, 4, as in the previous section. The Sobolev inequality (3.9) holds under the same assumptions in view of Corollary 3.3. In this section we use (4.1) to show a uniform upper bound on the scalar curvature. More precisely, we will show the following result.

**Theorem 4.1.** Let \( S \) evolve according to (2.2) with initial curvature \( S_0 \in L^{\frac{n^2}{n-2}}(M) \), and its negative part \( (S_0)_- \in L^\infty(M) \). Then, assuming (4.1) holds, there exists a uniform constant \( 0 < C(T) < \infty \), depending only on \( T > 0 \) and \( S_0 \), such that

\[
\|S\|_{L^\infty(M \times [\frac{T}{2}, T])} \leq C(T).
\]

The proof proceeds in two steps. First step is to prove an \( L^{\frac{n^2}{n-2}}(M, g) \)-norm bound on \( S \), uniform in \( t \in [0, T] \). That uniform bound rests on a chain of arguments of [Bre05, Lemma 2.2, Lemma 2.3, Lemma 2.5] (also to be found in [ScSt03, Lemma 2.3]) that apply in our setting as well. In the second step we perform a Moser iteration argument by following [MCZ12]. Our proofs are close to those in [Bre05] with some additional arguments due to lower regularity.

**Lemma 4.2.** Under the conditions of Theorem 4.1, there exists for any finite \( T > 0 \) a uniform constant \( 0 < C(T) < \infty \), depending only on \( T \) and \( S_0 \), such that for all \( t \in [0, T] \) we have the following estimate \(^{11}\)

\[
\int_0^T \left( \int_M \frac{n^2}{n-2} \, dV_{g(t)} \right)^{\frac{n-2}{n}} \, dt \leq C(T), \quad \|S\|_{L^\frac{n^2}{n-2}(M, g)} \leq C,
\]

where the second constant \( C \) only depends on \( S_0 \), not on \( T \).

**Proof.** It suffices to prove the statement for \( S_+ \) and \( S_- \) individually. By Proposition 2.3, the statement holds for the negative part \( S_- \). Thus we only need to prove the claim for \( S_+ \). We may therefore assume without loss of generality that \( S \geq 0 \), so that \( S \equiv S_+ \) and use (2.3) as the evolution equation.

\(^{11}\) Below, we will denote all uniform positive constants, depending only on \( T \) and \( S_0 \) either by \( C(T) \) or \( C_T \), unless stated otherwise.
The claim will follow from the evolution equation (2.2), but we have to argue a bit differently depending on whether $3 \leq n \leq 4$ or $n > 4$. The idea is the same in all dimensions $n \geq 3$ however. Let us start with $3 \leq n \leq 4$. Fix any $\sigma > 0$, and set $\beta = \frac{n}{4}$. Since $\beta \leq 1$, the function $x \mapsto (x + \sigma)^\beta$ is in $C^1([0, \infty)$ with bounded derivative. Thus, we may apply the chain rule to $(S + \sigma)^\beta$ and conclude that $(S + \sigma)^\beta \in C^1([0, \infty); H^1(M, g))$. We use $\frac{\beta}{2\beta - 1}(S + \sigma)^{2\beta - 1}$ as a test function with $\beta = \frac{n}{4}$ in the weak formulation of (2.3), which yields the following inequality
\[
\frac{\beta^2}{2(2\beta - 1)} \int_M (S + \sigma)^{2\beta - 1} \partial_t (S + \sigma) \, dVol_g + (n - 1) \int_M |\nabla (S + \sigma)\beta|^2 dVol_g \leq \frac{\beta^2}{2\beta - 1} \int_M (S - \rho)(S + \sigma)^{2\beta - 1} \, dVol_g.
\]
Using (1.3) results in
\[
\frac{\beta}{2(2\beta - 1)} \int_M (S + \sigma)^{2\beta} \, dVol_g + (n - 1) \int_M |\nabla (S + \sigma)\beta|^2 dVol_g \leq \frac{\beta^2}{2\beta - 1} \int_M (S - \rho)(S + \sigma)^{2\beta - 1} \, dVol_g
\]
\[
= -\frac{\beta^2\sigma}{2\beta - 1} \int_M (S - \rho)(S + \sigma)^{2\beta - 1} \, dVol_g
\]
\[
= -\frac{\beta^2\sigma}{2\beta - 1} \int_M (S + \rho)(S + \sigma)^{2\beta - 1} \, dVol_g + \frac{\sigma^2\beta^2}{2\beta - 1} \int_M (S + \sigma)^{2\beta - 1} \, dVol_g
\]
\[
\leq \frac{\sigma\beta^2(\rho + \sigma(0))}{2\beta - 1} \int_M (S + \sigma)^{2\beta - 1} \, dVol_g \leq \frac{\sigma\beta^2(\rho(0))}{2\beta - 1} \int_M (S + \sigma)^{2\beta} \, dVol_g,
\]
where the first equality is due to $\beta = \frac{n}{4}$, the penultimate inequality uses $\rho(0) \geq \rho(t)$, and the final inequality is due to Hölder with $p = \frac{2\beta - 1}{\beta}$ and $q = \beta$. We want to integrate this inequality in time. Note that any inequality of the form $\partial_t w(t) + a(t) \leq bw(t)$ with $a(t) \geq 0$ yields $\partial_t w \leq bw$ and hence $w(t) \leq e^{bt}w(0)$. Plugging this estimate into the original differential inequality leads to $\partial_t w + a \leq be^{bt}w(0)$. Integrating the latter inequality in time yields $w(t) + \int_0^1 a(s) ds \leq e^{bt}w(0)$. We therefore conclude that
\[
\int_M (S + \sigma)^{2\beta} \, dVol_g(T) + \frac{4(n-2)(n-1)}{n} \int_0^T \int_M |\nabla (S + \sigma)\beta|^2 dVol_g \leq \frac{e^{\sigma(\rho(0))inT}}{2\beta - 1} \int_M (S_0 + \sigma)^{2\beta} \, d\mu.
\]
(4.3)
This is for any $\sigma > 0$. Sending $\sigma \to 0$, using Fatou’s lemma on the left hand side and the monotone convergence theorem on the right hand side yields (on dropping the non-negative term with $\nabla S$)
\[
\int_M S^{2\beta} \, dVol_g(T) \leq \int_M S_0^{\frac{n}{2}} \, d\mu.
\]
This yields our uniform $L^{\frac{n}{2}}(M, g)$-bound on $S$ in (4.2). Returning to (4.3), we appeal to the Sobolev inequality (3.9) to deduce
\[
\int_0^T \| (S + \sigma)^{2\beta} \|^2_{L^{\frac{n}{2}}(M, g)} \, dt \leq \left( \frac{A(T)n}{4(n-2)(n-1)} + TB(T) \right) e^{\frac{\sigma(\rho(0))inT}{2}} \int_M (S_0 + \sigma)^{\frac{n}{2}} \, d\mu,
\]
hence also

$$\int_0^T \left( \int_M |S|^{\frac{n^2}{2n-2r}} \, d\text{Vol}_g \right)^{\frac{n-2}{n}} \, dt \leq C(T).$$

This proves the claim for $3 \leq n \leq 4$.

For $n > 4$ the claim will follow similarly, but the above test function does not have bounded derivative for $n > 4$, and we neither know that it is in $H^1$, nor do we know that the chain rule applies. We therefore argue similarly to the proof of Proposition 2.3, where we introduced the functions $\phi_{\beta,L}$, $G_{\beta,L}$ and $H_{\beta,L}$. We set here again $\beta = n/4$. Using $G_{\beta,L}(S)$ as a test function in (2.2) we find

$$\int_M G_{\beta,L}(S)(\partial_t S) \, d\text{Vol}_g + (n-1) \int_M |\nabla \phi_{\beta,L}(S)|^2 \, d\text{Vol}_g \leq \int_M S(S-\rho) G_{\beta,L}(S) \, d\text{Vol}_g.$$

Using the evolution equation (1.3) for the volume form, we have

$$\partial_t \int_M H_{\beta,L}(S) \, d\text{Vol}_g + (n-1) \int_M |\nabla \phi_{\beta,L}(S)|^2 \, d\text{Vol}_g \leq \int_M (S-\rho) \left( SG_{\beta,L}(S) - \frac{n}{2} H_{\beta,L}(S) \right) \, d\text{Vol}_g. \quad (4.4)$$

One readily checks from the definition of $G_{\beta,L}$ and $H_{\beta,L}$ in Proposition 2.3 that

$$x G_{\beta,L}(x) - \frac{n}{2} H_{\beta,L}(x)$$

$$= \begin{cases} \frac{\beta}{2\beta-1} x^{2\beta} \left( \beta - \frac{n}{4} \right), & x \leq L, \\ \beta^2 L^{2\beta} \left( 1 - \frac{n}{4} \right) \left( \frac{x}{L} \right)^2 + \frac{2(\beta-1)}{2\beta-1} \left( \frac{n}{2} - 1 \right) \frac{x}{L} - \frac{n(\beta-1)}{4\beta}, & x > L, \end{cases} \quad (4.5)$$

and from this one sees that $x G_{\beta,L}(x) - \frac{n}{2} H_{\beta,L}(x) \leq 0$ for $\beta = \frac{n}{4}$ and $n \geq 4$ as follows. For $x \leq L$ there is nothing to show. For $x > L$, notice that

$$\beta^2 L^{2\beta} \left( 1 - \frac{n}{4} \right) \left( \frac{x}{L} \right)^2 + \frac{2(\beta-1)}{2\beta-1} \left( \frac{n}{2} - 1 \right) \frac{x}{L} - \frac{n(\beta-1)}{4\beta}$$

$$= -\beta^2 (\beta-1) L^{2\beta} \left( \frac{x}{L} - 1 \right)^2 \leq 0$$

where we have substituted $n = 4\beta$ and recognized a square.\(^{12}\) Hence, using again that $\rho$ is non-increasing along the flow, we conclude that

$$\partial_t \int_M H_{\beta,L}(S) \, d\text{Vol}_g + (n-1) \int_M |\nabla \phi_{\beta,L}(S)|^2 \, d\text{Vol}_g \leq 0$$

\(^{12}\)This is the point where we need $n \neq 3$, since in this case $\beta-1 < 0$ and the above expression fails to be negative for $x > L$. 
LONG-TIME EXISTENCE OF THE SINGULAR YAMABE FLOW

holds for any $L \geq \rho(0)$. This is a differential inequality of the same kind as in the above $3 \leq n \leq 4$ case. Integrating it we deduce for any $t \in [0, T]$

$$
\int_M H_{\beta,L}(S) \, d\text{Vol}_g(T) + (n-1) \int_0^T |\nabla \phi_{\beta,L}(S)|^2 \, d\text{Vol}_g \, dt
\leq \int_M H_{\beta,L}(S_0) \, d\mu,
$$

(4.6)

Using $\beta = \frac{n}{4}$ and letting $L \to \infty$ this yields, using Fatou’s lemma and dominated convergence exactly as in the final step of the proof of Proposition 2.3 (neglecting the positive second summand on the left hand side of (4.6)) the following inequality

$$
\|S\|_{L^n(M,g)} = \left( \int_M S^n \, d\text{Vol}_g \right)^{\frac{1}{n}} \leq \left( \int_M |S_0|^n \, d\mu \right)^{\frac{1}{n}} \equiv C,
$$

(4.7)

where the constant $C(T) > 0$ depends only on $T$ and $S_0$. This yields the second estimate in (4.2), in the case of $n > 4$. For the first estimate in (4.2) note that $\phi_{\beta,L}(S) \in H^1(M, g)^{13}$ and thus by (3.9) and (4.6) we deduce

$$
\int_0^T \left( \int_M |\phi_{\beta,L}(S)|^{\frac{2n}{n-2}} \, d\text{Vol}_g \right)^{\frac{n-2}{n}} \, dt
\leq A(T) \int_M |\nabla \phi_{\beta,L}(S)|^2 \, d\text{Vol}_g \, dt + B(T) \int_0^T \int_M |\phi_{\beta,L}(S)|^2 \, d\text{Vol}_g \, dt
\leq A(T) \left( \int_M H_{\beta,L}(S_0) \, d\mu - \int_M H_{\beta,L}(S) \, d\text{Vol}_g(T) \right)
+ B(T) \int_0^T \int_M |\phi_{\beta,L}(S)|^2 \, d\text{Vol}_g \, dt.
$$

Thus, letting $L \to \infty$ we conclude using Fatou’s lemma and dominated convergence as before, with $\beta = n/4$ and (4.7)

$$
\int_0^T \left( \int_M S^{\frac{2n}{n-2}} \, d\text{Vol}_g \right)^{\frac{n-2}{n}} \, dt
\leq \left( \frac{nA(T)}{4(n-1)(n-2)} + B(T)T \right) \int_M |S_0|^n \, d\mu \equiv C(T),
$$

(4.8)

where the uniform constant $C(T) > 0$ depending only on $T$ and $S_0$. This proves the first estimate in (4.2) for $n > 4$. \qed

13 Note that a priori we do not know if $S^{n/4} \in H^1(M; g)$ and thus cannot directly apply the Sobolev inequality (3.9) to $S^{n/4}$. However we do know that $\phi_{\beta,L}(S) \in H^1(M, g)$, since $\phi_{\beta,L}(x)$ is linear for $x > L$ and $S \in H^1(M, g)$ for each fixed time argument.
Lemma 4.3. Under the conditions of Theorem 4.1, there exists for any finite \( T > 0 \) a uniform constant \( 0 < C(T) < \infty \), depending only on \( T \) and \( S_0 \), such that for all \( t \in [0, T] \) we have the following estimate
\[
\int_M |S|^{\frac{n^2}{n-2}} \, dVol_g \leq C(T).
\]

Proof. As in the previous lemma we have to split the argument into cases based on the dimension. We first show the statement for \( n \geq 4 \). We will again use the inequality (4.4). However while in Lemma 4.2 we have set \( \beta = n/4 \), here we will use the inequality (4.4) for \( \beta = \frac{n}{4(n-2)} \). For this choice of \( \beta \) the expression \( xG_{\beta,L}(x) - \frac{2}{\beta}H_{\beta,L}(x) \) is not necessarily non-positive any longer and we estimate it against a new approximation function
\[
f_{\beta,L}(x) := \begin{cases} 
\beta x^{2\beta}, & x \leq L, \\
n\beta^2 L^{2\beta-1} x, & x > L.
\end{cases}
\]
By (4.5) one sees that \( xG_{\beta,L}(x) - \frac{2}{\beta}H_{\beta,L}(x) \leq f_{\beta,L}(x) \) holds for all \( \beta \geq 1 \) and \( L > 0 \), in case \( n \geq 4 \). One important aspect to notice is that \( f_{\beta,L}(x) \) is linear in \( x \) for \( x > L \), as opposed to quadratic for \( H_{\beta,L}(x) \) and \( xG_{\beta,L}(x) \). This will become important below. Returning to (4.4), and applying (3.9) to the term \( \|\nabla \phi_{\beta,L}(S)\|_{L^2(M,g)}^2 \) we find after some reshuffling
\[
\partial_t \|H_{\beta,L}(S)\|_{L^1(M,g)}^2 
\leq \frac{(n-1)B(T)}{A(T)} \|\phi_{\beta,L}(S)\|_{L^1(M,g)}^2 - \frac{(n-1)}{A(T)} \|\phi_{\beta,L}(S)\|_{L^{\frac{n^2}{n-2}}(M,g)}^2 \\
\quad + \rho(0) \left\|SG_{\beta,L}(S) - \frac{n}{2}H_{\beta,L}(S)\right\|_{L^1(M,g)} + \|Sf_{\beta,L}(S)\|_{L^1(M,g)}.
\]
A straightforward computation shows for all \( \beta \geq 1 \) and \( L > 0 \)
\[
12\beta H_{\beta,L}(x) \geq \phi_{\beta,L}(x)^2, \quad 4\beta H_{\beta,L}(x) \geq xG_{\beta,L},
\]
holds, and here is a way of seeing this. For \( x \leq L \) these are both obvious from the definitions, so we look at \( x > L \). One first notices that
\[
\phi_{\beta,L}(x)^2 = \beta^2 L^{2(\beta-1)} x^2 - 2\beta(\beta-1) L^{2\beta-1} x + (\beta-1)^2 L^{2\beta} \leq \beta^2 L^{2(\beta-1)}(x^2 + L^2) \leq 3\beta^2 L^{2(\beta-1)} x^2,
\]
where the first inequality comes from dropping the non-positive linear term and estimating \( 1 \leq \beta \), and the final inequality is simply \( L^2 < x^2 \). We similarly estimate \( H_{\beta,L}(x) \) from below for \( x > L \) and find
\[
H_{\beta,L}(x) \geq \left( \frac{\beta^2}{2} \left( \frac{x}{L} \right)^2 - \frac{2\beta^2(\beta-1)}{2\beta-1} \left( \frac{x}{L} \right) + \frac{\beta(\beta-1)}{2} \right) L^{2\beta} \geq \left( \frac{\beta^2}{2(2\beta-1)} \left( \frac{x}{L} \right)^2 + \frac{\beta(\beta-1)}{2} \right) L^{2\beta} \geq \frac{\beta^2}{2(2\beta-1)} x^2 L^{2(\beta-1)},
\]
where the first inequality uses \( -\frac{1}{x^2} \geq -\frac{x^2}{2} \) and the second inequality comes from dropping the non-negative constant term. Using these two estimates one readily
sees that
\[ 12\beta H_{\beta,L}(x) \geq \frac{2\beta}{2\beta-1} 3\beta^2 x^2 L^{2(\beta-1)} \geq 3\beta^2 x^2 L^{2(\beta-1)} \geq \phi_{\beta,L}(x)^2, \]
showing half of the claim in (4.11). To see the other half, first observe that (for \( x > L \)), we have 
\[ xG_{\beta,L}(x) \leq \beta^2 x^2 L^{2(\beta-1)} \]
by dropping the non-positive term in (2.8). Using (4.12) again we deduce
\[ 4\beta H_{\beta,L}(x) \geq \frac{2\beta}{2\beta-1} \beta^2 x^2 L^{2(\beta-1)} \geq \beta^2 x^2 L^{2(\beta-1)} \geq xG_{\beta,L}(x). \]
This finishes the proof of (4.11), so we arrive by overestimating the right hand side of (4.10) at the following inequality
\[ \partial_t \|H_{\beta,L}(S)\|_{L^1(M,g)} \leq C_T \|H_{\beta,L}(S)\|_{L^1(M,g)} + \|Sf_{\beta,L}(S)\|_{L^1(M,g)} - \frac{(n-1)}{A(T)} \|\phi_{\beta,L}(S)\|_{L^{\frac{2n}{n-2}}(M,g)}^2, \quad (4.13) \]
where the uniform constant \( C_T > 0 \) is explicitly given by
\[ C_T := 12(n-1)\beta \frac{A(T)}{B(T)} + \rho(0) \left( \frac{n}{2} + 4\beta \right). \]
Introduce the non-negative, real function \( F_{\beta,L} \) via
\[ F_{\beta,L}(x) := (xf_{\beta,L}(x))^{\frac{1}{2\beta+1}}. \]
Assume \( \beta > \frac{n}{2} \), which holds e.g. for \( \beta = \frac{n^2}{4(n-2)} \). Set \( \alpha := \frac{n}{4\beta} < 1 \) and choose any \( \delta > 0 \). Observe that by the Hölder inequality in the first estimate and the Young inequality in the second, we obtain
\[ \left\| F_{\beta,L}(S)^{2\beta+1} \right\|_{L^1(M,g)} \leq \left\| F_{\beta,L}(S) \right\|_{L^{2\beta}(M,g)}^{2\alpha\beta} \|F_{\beta,L}(S)\|_{L^{2\beta}(M,g)}^{1+2(1-\alpha)\beta} \leq \delta\alpha \left\| F_{\beta,L}(S) \right\|_{L^{\frac{2n}{n-2}}(M,g)}^{\frac{2\alpha}{n-2}} + \delta^{-\frac{\alpha}{1-\alpha}} (1-\alpha) \left\| F_{\beta,L}(S) \right\|_{L^{\frac{2n}{n-2}}(M,g)}^{\frac{1-\alpha}{1-\alpha} + 2\beta}. \quad (4.14) \]
These norms are finite for finite \( L > 0 \), as one can see as follows: the claim is clear for \( S \leq L \) and the delicate point is the behaviour of the function for \( S \) large. For \( S > L \), \( F_{\beta,L}(S) \sim S^{\frac{2\beta}{2\beta+1}} \), and (since \( \frac{2\beta}{2\beta+1} \leq 1 \)) the terms \( \left\| F_{\beta,L}(S)^{\beta} \right\|_{L^{\frac{2n}{n-2}}(M,g)}^{\frac{2\alpha}{n-2}} \) and \( \left\| F_{\beta,L}(S) \right\|_{L^{\frac{2n}{n-2}}(M,g)}^{\frac{2\alpha}{n-2}} \) can be controlled via \( S \left\| \phi_{\beta,L}(S) \right\|_{L^{\frac{2n}{n-2}}(M,g)} \) and \( S \left\| \phi_{\beta,L}(S) \right\|_{L^{\frac{2n}{n-2}}(M,g)} \) respectively. These latter norms are bounded\(^{14}\) because of \( S \in C^0([0,T];H^1(M,g)) \) and (3.9).

\(^{14}\)This is the point where the estimate \( xG_{\beta,L} - \frac{1}{4}H_{\beta,L} \leq f_{\beta,L} \) was necessary. Otherwise, defining \( F_{\beta,L} \) in terms of \( xG_{\beta,L} - \frac{1}{4}H_{\beta,L} \) would have caused \( F_{\beta,L}(S) \) to go as \( S^{\frac{2\beta}{2\beta+1}} \) for large \( L \) and we would not be able to guarantee that \( \left\| F_{\beta,L}(S)^{\beta} \right\|_{L^{\frac{2n}{n-2}}(M,g)}^{\frac{2\alpha}{n-2}} \) is finite.
We can compare $\|F_{β,L}(S)^β\|_L^{2nβL}(M,g)$ and $\|φ_{β,L}(S)^β\|_L^{2nβL}(M,g)$ since we have the following pointwise estimates. Directly from the definition we have

$$F_{β,L}(x)^β = \begin{cases} β \frac{2β}{T^β} x^β ≤ β x^β, & x ≤ L, \\ (nβ)^2 \frac{2β}{T^β} L^β \left( \frac{x}{L} \right)^{\frac{2β}{T^β}} ≤ nβ L^{β-1} x, & x > L. \end{cases}$$

Similarly, we may estimate $φ_{β,L}$ from below

$$φ_{β,L}(x) = \begin{cases} x^β = x^β, & x ≤ L, \\ βL^{β-1} x - (β - 1)L^β ≥ xL^{β-1}, & x ≤ L. \end{cases}$$

so combined we find $nβφ_{β,L}(x) ≥ F_{β,L}(x)^β$. By sufficiently shrinking $δ > 0$ (choosing $δ ≤ \frac{4(n-1)}{nβA(T)}$ to be precise), we can thus ensure for all $L > 0$

$$δα \|F_{β,L}(S)^β\|_L^{2nβL}(M,g) ≤ \frac{(n-1)}{A(T)} \|φ_{β,L}(S)^β\|_L^{2nβL}(M,g),$$

and therefore deduce from (4.13) and (4.14)

$$δ_l \|H_{β,L}(S)^β\|_{L^1(M,g)} ≤ C_T \|H_{β,L}(S)^β\|_{L^1(M,g)} + C'_T \|F_{β,L}(S)^β\|_{L^1(M,g)},$$

for uniform constants $C_T, C'_T > 0$ where $C_T$ is given above and

$$C'_T := δ^\frac{n}{n-α} \left( \frac{4β - n}{4β} \right)$$

and $δ ≤ \frac{4(n-1)}{nβA(T)}$. The point is that both depend only on $T > 0$ and $S_0$.

We then compare $F_{β,L}(x)^{2β}$ to $H_{β,L}(x)$ as follows. From the definition of $F_{β,L}(x)$ again we find

$$F_{β,L}(x)^{2β} = \begin{cases} β \frac{2β}{T^β} x^{2β} ≤ β x^{2β}, & x ≤ L, \\ (nβ)^2 \frac{2β}{T^β} L^{β} \left( \frac{x}{L} \right)^{\frac{2β}{T^β}} ≤ nβ^2 L^{2(β-1)} x^2, & x > L. \end{cases}$$

Consulting (4.12) we find

$$4nβH_{β,L}(x) ≥ \frac{2β}{2β-1} \begin{cases} nβ x^{2β} & x ≤ L, \\ nβ^2 L^{2(β-1)} x^2 & x > L. \end{cases}$$

We therefore conclude $4nβH_{β,L}(x)^{2β} ≥ F_{β,L}(x)^{2β}$. Defining

$$C'' := \max \left\{ (4nβ)^{1+\frac{2β}{T^β}} C'_T, C_T \right\},$$

we deduce from (4.15) that

$$δ_l \|H_{β,L}(S)^β\|_{L^1(M,g)} ≤ C'' \left( 1 + \|H_{β,L}(S)^β\|_{L^1(M,g)} \right) \|H_{β,L}(S)^β\|_{L^1(M,g)}.$$

Setting $β = \frac{n^2}{4(n-2)}$, we can rewrite this differential inequality as

$$δ_l \log \left( \|H_{β,L}(S)^β\|_{L^1(M,g)} \right) ≤ C'' \left( 1 + \|H_{β,L}(S)^β\|_{L^1(M,g)} \right).$$
Integrating this differential inequality in time, we conclude
\[
\log \left( \|H_{\beta,L}(S(T))\|_{L^1(M,g)} \right) \\
\leq \log \left( \|H_{\beta,L}(S_0)\|_{L^1(M,g_0)} \right) + C''T + C'' \int_0^T \|H_{\beta,L}(S)\|_{L^1(M,g)}^\alpha \, dt.
\]
Taking the limit \( L \to \infty \) (using Fatou’s lemma and dominated convergence as before in the final step of the proof of Proposition 2.3) and using Lemma 4.2, we deduce
\[
\log \left\| S(T) \right\|_{L^1(M,g)}^{\frac{n}{n-2}} \leq \log \left\| S_0 \right\|_{L^1(M,g)}^{\frac{n}{n-2}} + C''T + C'' C(T),
\]
which proves the statement for \( n \geq 4 \).

The above proof would almost work for \( n = 3 \). The problem is that \( xG_{\beta,L} - \frac{n}{2}H_{\beta,L} \leq f_{\beta,L} \) no longer holds true, and one would have a problem showing that the norms in (4.14) are finite. One solution is to redefine the approximation functions \( \phi_{\beta,L}, G_{\beta,L} \), and \( H_{\beta,L} \) to ensure \( xG_{\beta,L}(x) - \frac{n}{2}H_{\beta,L}(x) \) is dominated by a function \( f_{\beta,L} \) going like at most \( x \) for large \( x \), rather than \( x^2 \). This is a non-trivial task, because it is also important for the above argument that one can find constants (depending on \( n, \beta \), but not \( L \)) such that \( C_1 H_{\beta,L}(x) \geq \phi_{\beta,L}(x)^2, C_2 H_{\beta,L}(x) \geq xG_{\beta,L}(x), C_3 F_{\beta,L}(x)^\beta \leq \phi_{\beta,L}(x) \) and \( C_4 H_{\beta,L}(x) \geq F_{\beta,L}(x)^{2\beta} \), where \( F_{\beta,L}(x) = (xf_{\beta,L}(x))^{\frac{1}{n-1}} \).

Consider the following family of approximation functions, \( \nu \leq 1 \) and \( \nu \not\in \{0, \frac{1}{2}\} \).

\[
\tilde{\phi}_{\beta,L}(x) := \begin{cases} 
\chi^\beta, & x \leq L \\
\frac{\beta}{\nu} L^{\beta-\nu} x^{\nu} + L^\beta \left( 1 - \frac{\beta}{\nu} \right), & x > L.
\end{cases}
\]

(4.16)

\[
\tilde{G}_{\beta,L}(x) := \int_0^x \tilde{\phi}_{\beta,L}'(y)^2 \, dy = \begin{cases} 
\frac{\beta^2}{2\beta - 1} x^{2\beta - 1}, & x \geq L,
\\
\frac{\beta^2 L^{2(\beta-\nu)}}{2\nu - 1} x^{2\nu - 1} - \frac{2\beta^2 L^{2\beta-1}(\beta - \nu)}{(2\nu - 1)(2\beta - 1)}, & x > L.
\end{cases}
\]

(4.17)

\[
\tilde{H}_{\beta,L}(x) := \int_0^x \tilde{G}_{\beta,L}'(y) \, dy = \begin{cases} 
\frac{\beta}{2(2\beta - 1)} x^{2\beta}, & x \leq L,
\\
\frac{\beta^2 L^{2(\beta-\nu)}}{2\nu(2\nu - 1)} x^{2\nu} - \frac{2\beta^2 L^{2\beta-1}(\beta - \nu)}{(2\nu - 1)(2\beta - 1)} x - C_{\beta,\nu} L^{2\beta}, & x > L,
\end{cases}
\]

and

where

\[
C_{\beta,\nu} := \frac{\beta (2\beta - 1) + 4\nu\beta (\nu - \beta) + \nu (1 - 2\nu)}{2\nu(2\beta - 1)(2\nu - 1)}.
\]

In the \( n \geq 4 \) case we considered these functions with \( \nu = 1 \). These functions have the same qualitative properties of as before, namely that \( \tilde{\phi}_{\beta,L} \xrightarrow{L\to\infty} x^\beta \) and \( \tilde{\phi}_{\beta,L} \in C^1(\mathbb{R}^+) \) with \( \tilde{\phi}_{\beta,L} \in L^\infty(\mathbb{R}^+) \), and so on for \( \tilde{G}_{\beta,L} \) and \( \tilde{H}_{\beta,L} \). We can therefore
use $\tilde{G}_{\beta,L}(S)$ as test function in (2.2) and deduce the analogue of (4.4), namely
\[
\partial_t \int_M H_{\beta,L}(S) \, d\text{Vol}_g + (n-1) \int_M |\nabla \tilde{\Phi}_{\beta,L}(S)|^2 \, d\text{Vol}_g \\
\leq \int_M (S - \rho) \left( \tilde{S}G_{\beta,L}(S) - \frac{n}{2} \tilde{H}_{\beta,L}(S) \right) \, d\text{Vol}_g.
\] (4.18)

Consider the expression $x\tilde{G}_{\beta,L}(x) - \frac{n}{2} \tilde{H}_{\beta,L}(x)$ for $x > L$,
\[
x\tilde{G}_{\beta,L}(x) - \frac{n}{2} \tilde{H}_{\beta,L}(x) = \frac{\beta 2^{2(\beta - \nu)}}{2\nu(2\nu - 1)} \left( 2\nu - \frac{n}{2} \right) x^{2\nu} + \frac{2\beta^2 (\beta - \nu) L^{2\beta - 1}}{(2\beta - 1)(2\nu - 1)} \left( \frac{n}{2} - 1 \right) x + \frac{n}{2} C_{\beta,\nu} L^{2\beta}.
\]
From this one sees that when $0 < \nu \leq \frac{n}{4}$ and $\beta \geq \frac{n}{2}$, the first two terms become negative. So assume from now on that $0 < \nu \leq \frac{n}{4}$ and later we will make a choice of $\beta \geq \frac{n}{4}$. Introduce the function
\[
\tilde{f}_{\beta,L}(x) := \begin{cases} \beta x^{2\beta} & x \leq L \\ \frac{n}{2} C_{\beta,\nu} L^{2\beta} & x > L \end{cases}
\]
which has the property that $x\tilde{G} - \frac{n}{2} \tilde{H}_{\beta,L} \leq \tilde{f}_{\beta,L}(x)$ for all $x \geq 0$ and $L > 0$, as long as $\beta \geq \frac{n}{4} \geq \nu$. Proceeding exactly as in the $n = 4$ case we deduce
\begin{align*}
\partial_t \|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)} &
\leq (n-1) \frac{B(T)}{A(T)} \|\tilde{\Phi}_{\beta,L}(S)^2\|_{L^1(M,g)} - \frac{(n-1)}{A(T)} \|\tilde{\Phi}_{\beta,L}(S)\|_{L^2_{\text{comp}}(M,g)}^2 \\
&
\quad + \rho(0) \|\tilde{S}G_{\beta,L}(S) - \frac{n}{2} \tilde{H}_{\beta,L}(S)\|_{L^1(M,g)} + \|\tilde{S}\tilde{f}_{\beta,L}(S)\|_{L^1(M,g)}.
\end{align*} (4.19)

We compare $\tilde{\Phi}_{\beta,L}(x)^2$, $x\tilde{G}_{\beta,L}(x)$ and $\tilde{H}_{\beta,L}(x)$ as in as for (4.11), and conclude by similar arguments for all $\beta \geq 1$, $L > 0$, and some $L$-independent constants $C_1$, $C_2$
\[
C_1 \tilde{H}_{\beta,L}(x) \geq \tilde{\Phi}_{\beta,L}(x)^2, \quad C_2 \tilde{H}_{\beta,L}(x) \geq x\tilde{G}_{\beta,L}(x).
\] (4.20)

We now proceed as before, getting
\begin{align*}
\partial_t \|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)} &
\leq C_T \|\tilde{H}_{\beta,L}(S)\|_{L^1(M,g)} \\
&
\quad + \|\tilde{S}\tilde{f}_{\beta,L}(S)\|_{L^1(M,g)} - \frac{(n-1)}{A(T)} \|\tilde{\Phi}_{\beta,L}(S)\|_{L^2_{\text{comp}}(M,g)}^2,
\end{align*} (4.21)
where the uniform constant $C_T > 0$ is explicitly given by
\[
C_T := C_1 (n-1) \frac{A(T)}{B(T)} + \rho(0) \left( \frac{n}{2} + C_2 \right).
\]

Introduce the non-negative, real function $\tilde{f}_{\beta,L}$ via
\[
\tilde{f}_{\beta,L}(x) := (x\tilde{f}_{\beta,L}(x))^{\frac{1}{\nu + 1}}.
\]
Assume $\beta > \frac{n}{4}$, which holds e.g. for $\beta = \frac{n^2}{4(n-2)}$. Set $\alpha := \frac{n^2}{4n} < 1$ and choose any $\delta > 0$. Observe that by the Hölder inequality in the first estimate and the Young
inequality in the second, we obtain
\[ \| \tilde{f}_{\beta,L}(S)^{2\beta+1} \|_{L^1(M,g)} \leq \| \tilde{f}_{\beta,L}(S)^{2\beta} \|_{L^{2\beta}(M,g)} \| \tilde{f}_{\beta,L}(S) \|_{L^{2\beta}(M,g)}^{1+2(1-\alpha)\beta} \]
\[ \leq \delta \alpha \| \tilde{f}_{\beta,L}(S)^{2\beta} \|_{L^{2\beta}(M,g)}^2 + \delta^{\frac{\alpha}{1-\alpha}(1-\alpha)} \| \tilde{f}_{\beta,L}(S) \|_{L^{2\beta}(M,g)}^{1+2\beta}. \]
(4.22)

These integrals are finite for the same reasons as in the \( n = 4 \) case.

We shall from now on set \( \nu = \frac{\beta}{2\beta+1} \) and \( \beta = \frac{n^2}{4(n-2)} \), which translates into \( \nu = \frac{\beta}{2} \) for \( n = 3 \). Notice that this choice satisfies \( \nu \leq \frac{\beta}{4} \), so the manipulations up until now are allowed. The reason for choosing this \( \nu \) is that then
\[ \tilde{f}_{\beta,L}(x)^{\beta} = \begin{cases} \beta^{\nu}x^{\beta}, & x \leq L, \\ \left( \frac{n}{2}|C_{\beta,\nu}| \right)^{\nu} L^{\beta-\nu}x^{\nu}, & x > L. \end{cases} \]

This is easily comparable to \( \phi_{\beta,L}(x) \). Since
\[ \phi_{\beta,L}(x) \geq \frac{\beta}{\nu} L^{\beta-\nu}x^{\nu} \]
for \( x > L \), we see that defining
\[ C^{-1}_3 := \max \left\{ \beta^{\nu}, \frac{\nu}{\beta} \left( \frac{n}{2}|C_{\beta,\nu}| \right)^{\nu} \right\} \]
we achieve\(^{15}\) \( C_3 \tilde{f}_{\beta,L}(x)^{\beta} \leq \phi_{\beta,L}(x) \). So if we choose \( \delta \leq \frac{(n-1)4\beta^2}{\Lambda(T) n} \), then
\[ \delta \alpha \| \tilde{f}_{\beta,L}(S)^{\beta} \|_{L^{2\beta}(M,g)}^2 - \frac{(n-1)2\beta}{\Lambda(T)} \| \phi_{\beta,L}(S) \|_{L^{2\beta}(M,g)}^2 \leq 0 \]
holds for all \( L > 0 \), and we deduce from (4.21) and (4.22) that
\[ \delta_1 \| H_{\beta,L}(S) \|_{L^1(M,g)} \leq C_T \| H_{\beta,L}(S) \|_{L^1(M,g)} + C'_T \| \tilde{f}_{\beta,L}(S) \|_{L^{2\beta}(M,g)}^{1+2\beta}, \]
(4.23)
for uniform constants \( C_T, C'_T > 0 \) where \( C_T \) is given above and
\[ C'_T := \delta^{\frac{n}{1-\alpha}} \left( \frac{4\beta - n}{4\beta} \right) \]
and \( \delta \leq \frac{(n-1)4\beta^2}{\Lambda(T) n} \). The point is that both depend only on \( T > 0 \) and \( S_0 \). The final comparison we need is that \( C_4 H_{\beta,L}(x) \geq \tilde{f}_{\beta,L}(x)^{2\beta} \) holds for some \( C_4 > 0 \) independent of \( L \) and here is a way to see that this is doable. For \( x \leq L \) both functions are proportional, so there is nothing to show. Inserting \( \nu = \frac{\beta}{2\beta+1} \) into the definition of \( H_{\beta,L}(x) \) yields (for \( x > L \))
\[ H_{\beta,L}(x) = L^{2\beta} \left( \frac{4\beta^4}{2\beta - 1} \left( \frac{x}{L} \right) - \frac{\beta(2\beta + 1)^2}{2} \left( \frac{x}{L} \right)^{2\nu} + \beta^2 \right), \]

\(^{15}\)This is a somewhat delicate point. If one chooses \( \nu \) small, it is easy to make \( xG - \frac{\nu}{2} \tilde{f} \) sublinear, but if \( \nu \) is too small, \( \tilde{f} \) will increase faster than \( \phi \), ruining the comparison. On the other hand, if \( \nu \) is bigger than \( \frac{\beta}{4} \) we see above that \( xG - \frac{\nu}{2} \tilde{f} \) becomes too large to guarantee the finiteness of the integrals in (4.22).
which shows that $\tilde{F}_{\beta,L}(x)$ is dominated by a positive linear term for $x > L$, which will dominate the sublinear term $x^{2\nu}$ of $\tilde{F}_{\beta,L}(x)^{2\beta}$. Defining
\[ C_T'' := \max \left\{ C_4 \frac{2^n}{n^2} \frac{2}{n} C'_T, C_T \right\}, \]
we deduce from (4.23) that
\[ \partial_t \| \tilde{F}_{\beta,L}(S) \|_{L^1(M,g)} \leq C''_T \left( 1 + \| \tilde{F}_{\beta,L}(S) \|_{L^{2n/(n-2)}(M,g)} \right) \| \tilde{F}_{\beta,L}(S) \|_{L^1(M,g)}, \]
The rest of the proof then runs exactly as in the $n \geq 4$ case, giving us our required bound also for $n = 3$. □

This completes the first step on the way to Theorem 4.1, proving a uniform $L^{2n/(n-2)}(M,g)$-norm bound on $S$. Before we can go on to prove Theorem 4.1 by a Moser iteration argument, we need the following parabolic Sobolev inequality.

**Lemma 4.4.** Let $A(T)$ and $B(T)$ denote the constants of the (elliptic) Sobolev inequality (3.9). Then for any $f \in C^0([0,T]; H^1(M,g))$ we have (writing $M_T := M \times [0,T]$\footnote{We write $L^p(M_T,g) \equiv L^p(M_T,g \oplus dt^2)$ for any $p \geq 1.$})
\[ \| f \|_{L^1(M_T,g)}^{n+2} \leq \frac{n}{n+2} \left( A(T) \| \nabla f \|_{L^2(M_T,g)}^2 + B(T) \| f \|_{L^2(M_T,g)}^2 \right) \]
\[ + \frac{2}{n+2} \sup_{t \in [0,T]} \| f(t) \|_{L^2(M_T,g)}^2. \]

**Proof.** The statement and the proof are close to [MCZ12, Eqn. 12]. We compute
\[ \int_0^T \int_M f^{2(n+2)/n} \, d\text{Vol}_g \, dt = \int_0^T \int_M f^2 \frac{q}{n+2} \, d\text{Vol}_g \, dt \]
\[ \leq \int_0^T \left( \| f \|_{L^{2n/(n-2)}(M,g)}^{2n/(n-2)} \| f \|_{L^2(M,g)}^{2n/(n-2)} \right) \, dt \]
\[ \leq \int_0^T \left( A(T) \| \nabla f \|_{L^2(M,g)}^2 + B(T) \| f \|_{L^2(M,g)}^2 \right) \left( \| f \|_{L^2(M,g)}^{2n/(n-2)} \right) \, dt \]
\[ \leq \left( A(T) \| \nabla f \|_{L^2(M_T,g)}^2 + B(T) \| f \|_{L^2(M_T,g)}^2 \right) \sup_{t \in [0,T]} \left( \| f \|_{L^2(M_T,g)}^{2n/(n-2)} \right). \]
where in the first estimate we applied the Hölder inequality with $p = n/2$ and $q = n/(n-2)$, and in the second estimate applied (3.9). Raising both sides of the inequality to $\frac{n}{n+2}$ and using Young’s inequality $AB \leq \frac{A^p}{p} + \frac{B^q}{q}$ with $p = \frac{n+2}{2}$ and $q = \frac{n^2}{2}$ we arrive at the estimate as claimed.

We are now ready to tackle Theorem 4.1.

**Proof of Theorem 4.1.** Since we assume that $(S_0)_- \in L^\infty(M)$, we have uniform bounds on $S_-$ by Proposition 2.3. Thus, it suffices to prove the statement for $S_+$. Therefore, we may replace $S$ by $S_+$, replacing the evolution equation (2.2) for $S$ by the inequality (2.3) for $S_+$. Hence we continue under the assumption $S \equiv S_+ \geq 0$, subject to (2.3).
Let $\eta \in C^1([0,T],\mathbb{R}_+)$ be non-decreasing with $\eta(0) = 0$ and $\|\eta\|_\infty \leq 1$. We want to use $\beta^2\eta^2S^{2\beta-1}/(2\beta - 1)$ (with $\beta > 1$) as a test function in the weak formulation of (2.3). The problem is of course that the chain rule fails to hold in general, so we use the same workaround as in Proposition 2.3 and Corollary 4.3. Let $L > 0$ and define $\phi_{\beta,L}, G_{\beta,L},$ and $H_{\beta,L}$ as before. Using $\eta(s)^2G_{\beta,L}(S)$ as test function in (2.3) we get
\[
\int_M (\partial_s S)\eta^2 G_{\beta,L}(S) \, dV_0 + (n-1) \int_M \eta^2 |\nabla \phi_{\beta,L}(S)|^2 \, dV_0 \leq \int_M S G_{\beta,L}(S)|S - \rho| \, dV_0.
\]
On the right hand side we observe (by a direct computation) that $S G_{\beta,L}(S) \leq \frac{\beta^2}{2\beta - 1} \phi_{\beta,L}(S)^2$. We integrate this in time for any $t \in [0,T]$
\[
\int_0^t \int_M (\partial_s S)\eta^2 G_{\beta,L}(S) \, dV_0 \, ds + (n-1) \int_M \eta^2 |\nabla \phi_{\beta,L}(S)|^2 \, dV_0 \, ds 
\leq \frac{\beta^2}{2\beta - 1} \int_0^t \int_M \eta^2 \phi_{\beta,L}(S)^2 |S - \rho| \, dV_0 \, ds.
\]
We rewrite the first term on the left hand side of (4.25) using (1.3)
\[
\int_0^t \int_M \eta^2 (\partial_s S)G_{\beta,L}(S) \, dV_0 \, ds = \int_0^t \int_M \eta^2 \partial_s H_{\beta,L}(S) \, dV_0 \, ds 
= \int_M \eta^2 H_{\beta,L}(S) \, dV_0 (s = t) - 2 \int_0^t \int_M \eta \partial_s H_{\beta,L}(S) \, dV_0 \, ds 
+ \frac{n}{2} \int_0^t \int_M \eta^2 H_{\beta,L}(S)(S - \rho) \, dV_0 \, ds,
\]
where we write $\eta \equiv \partial_s \eta$ and used $\eta(0) = 0$. Plugging this into (4.25), we obtain
\[
\int_M \eta^2 H_{\beta,L}(S) \, dV_0 (s = t) + (n-1) \int_M \eta^2 |\nabla \phi_{\beta,L}(S)|^2 \, dV_0 \, ds 
\leq \int_0^t \int_M \eta^2 \left( \frac{\beta^2}{2\beta - 1} \phi_{\beta,L}(S)^2 + \frac{n}{2} H_{\beta,L}(S) \right) |S - \rho| \, dV_0 \, ds 
+ 2 \int_0^t \int_M \eta \partial_s H_{\beta,L}(S) \, dV_0 \, ds.
\]
We now take the supremum over $t \in [0,T]$ and appeal to the parabolic Sobolev inequality (4.24) with $f = \eta \phi_{\beta,L}(S)$. The result is
\[
\frac{(n-1)}{nA(T)} \left( (n+2) \|\eta^2 \phi_{\beta,L}(S)^2\|_{L^{nA(T)}} - 2 \sup_{t \in [0,T]} \|\eta \phi_{\beta,L}(S)\|^2_{L^2(M,T)} \right) 
-nB(T) \|\eta \phi_{\beta,L}(S)\|^2_{L^2(M,T)} + \sup_{t \in [0,T]} \int_M \eta^2 H_{\beta,L}(S) \, dV_0 
\leq \int_0^T \int_M \eta^2 \left( \frac{\beta^2}{2\beta - 1} \phi_{\beta,L}(S)^2 + \frac{n}{2} H_{\beta,L}(S) \right) |S - \rho| \, dV_0 \, dt 
+ 2 \int_0^T \int_M \eta \partial_s H_{\beta,L}(S) \, dV_0 \, dt.
\]
By increasing $\Lambda(T) > 0$ if needed, we may assume (also noting that $H_{\beta,L}$ and $\phi_{\beta,L}^2$ are comparable by (4.11))

$$
\sup_{t \in [0,T]} \int_M \eta^2 H_{\beta,L}(S) \, d\text{Vol}_g - \frac{2(n - 1)}{n \Lambda(T)} \sup_{t \in [0,T]} \|\eta \phi_{\beta,L}(S)\|_{L^2(M,g)}^2 \geq 0
$$

for all $\beta \geq 1$ and $L > 0$. We may therefore drop these terms from (4.26). Taking the limit $L \to \infty$ (using Fatou’s lemma and the dominated convergence theorem) we get

$$
(n - 1) \frac{n + 2}{n \Lambda(T)} \|\eta^2 S^{2\beta}\|_{L^\infty(M_t)} - \frac{B(T)(n - 1)}{\Lambda(T)} \|\eta S^\beta\|_{L^2(M_t,g)}^2
\leq \left( \frac{\beta^2}{2\beta - 1} + \frac{\beta}{4n(2\beta - 1)} \right) \int_0^T \int_M \eta^2 S^{2\beta} |S - \rho| \, d\text{Vol}_g \, dt
\quad + \frac{\beta}{2\beta - 1} \int_0^T \int_M \eta \eta S^\beta \, d\text{Vol}_g \, dt.
$$

Introducing $C := \frac{n \Lambda(T)}{(n+2)(n-1)}$ we get from this inequality for any $\beta > 1$

$$
\|\eta^2 S^{2\beta}\|_{L^\infty(M_t)} \leq \frac{n B(T)}{n + 2} \|\eta S^\beta\|_{L^2(M_t)}^2 + C \int_0^T \int_M \eta \eta S^\beta \, d\text{Vol}_g \, dt
\quad + 2C\beta \int_0^T \int_M \eta^2 S^{2\beta} |S - \rho| \, d\text{Vol}_g \, dt.
$$

We apply the Hölder inequality with $p = \frac{n^2}{2(n-2)}$ to the last integral on the right hand side of the last inequality. Using Corollary 4.3 to get a bound on the integral of $|S - \rho|^p$, we conclude

$$
\|\eta^2 S^{2\beta}\|_{L^\infty(M_t)} \leq \frac{n B(T)}{n + 2} \|\eta S^\beta\|_{L^2(M_t)}^2 + C \int_0^T \int_M \eta \eta S^\beta \, d\text{Vol}_g \, dt + C(T)\beta \|\eta^2 S^{2\beta}\|_{L^N(M_t)}, \quad (4.27)
$$

with $N := \frac{p}{p-1} = \frac{n^2}{n^2 - 2n + 4} < \frac{n+2}{n}$. This is almost the expression we want to iterate, but the presence of $\eta$ means we have to shrink our time interval in the iteration (as is standard for parabolic Moser iteration). Here are the details (inspired by [MCZ12, pp. 889-890]).

Consider the sequence $t_k := \left( \frac{1}{k} - \frac{1}{k+1} \right) T$ for integers $k \geq 1$. Let $M_k := M \times [t_k, T]$, $M_1 = M_T$ and $M_\infty = M \times [T, \infty)$. Choose test functions $\eta_k \in C^1([0,T], \mathbb{R}_+)$, non-decreasing with $\|\eta_k\|_\infty \leq 1$, such that

$$
\eta_k(t) = \begin{cases} 
0, & t \leq t_{k-1}, \\
1, & t \geq t_k. 
\end{cases}
$$
The expression $\prod \gamma$ will be iterating. We therefore deduce for some uniform constant $C$, which we henceforth assume. Using these functions in (4.27), we find
\[
\|S^{2\beta}\|_{L^4(M_k)} = \|\eta^2 S^{2\beta}\|_{L^4(M_k)} \leq \|\eta_k^2 S^{2\beta}\|_{L^{4}(M_T)} 
\leq \frac{nB(T)}{n+2} \|\eta_k^2 S^{2\beta}\|_{L^2(M_T)}^2 + C \int_0^T \int_M \eta_k \eta_k S^{2\beta} \, dV \, dt + C(T) \beta \|\eta_k^2 S^{2\beta}\|_{L^N(M_T)} \tag{4.28} 
\leq \tilde{C}(T) \beta 2^{k+1} \|S^{2\beta}\|_{L^N(M_{k-1})},
\]
where the second inequality uses (4.27) and last step uses $\eta \leq 2^{k+1}/T$ together with the Hölder inequality to compare $L^1$ and $L^N$ norms. This is the equation we will be iterating. Introduce $\gamma := 2\beta N$ and $\rho := \frac{n+2}{nN} = \frac{n^2+8}{n^2} > 1$. Then (4.28) reads
\[
\|S\|_{L^\gamma(M_k)} \leq (\tilde{C}(T) \gamma 2^k)^{\frac{N}{T}} \|S\|_{L^\gamma(M_{k-1})}.
\]
Replacing $\gamma$ by $\rho^m \gamma$ for $m \geq 0$ results in
\[
\|S\|_{L^{\rho^{m+1}\gamma}(M_{k+m})} \leq (\tilde{C}(T) \rho^m \gamma 2^{k+m})^\frac{N}{\rho^m} \|S\|_{L^{\rho^m\gamma}(M_{k+m-1})},
\]
which can be iterated down to
\[
\|S\|_{L^{\rho^{m+1}\gamma}(M_{k+m})} \leq \prod_{i=0}^m (\tilde{C}(T) \rho^i \gamma 2^{k+i})^\frac{N}{\rho^i} \|S\|_{L^\gamma(M_{k-1})}.
\]
The expression $\prod_{i=0}^m (\tilde{C}(T) \rho^i \gamma 2^{k+i})^\frac{N}{\rho^i}$ converges as $m \to \infty$, as one checks by computing the logarithm
\[
\lim_{m \to \infty} \log \prod_{i=0}^m (\tilde{C}(T) \rho^i \gamma 2^{k+i})^\frac{N}{\rho^i} = \frac{N}{\gamma} \sum_{i=0}^\infty \left( \log(2^k \tilde{C}(T) \gamma) \frac{1}{\rho^i} + \log(2\rho) \frac{i}{\rho^i} \right) 
= \frac{N}{\gamma} \left( \frac{\rho}{\rho-1} \log(\tilde{C}(T) \gamma 2^k) + \log(2\rho) \frac{\rho}{(\rho-1)^2} \right).
\]
We therefore deduce for some uniform constant $C_T > 0$
\[
\|S\|_{L^\infty(M \times [\frac{T}{2}, T])} \leq \lim_{m \to \infty} \|S\|_{L^{\rho^{m+1}\gamma}(M_{k+m})} \leq C_T \|S\|_{L^\gamma(M_{k-1})} \leq C_T \|S\|_{L^\gamma(M_{T})}.
\]
Choosing $\beta = \frac{n^2-2n+4}{4(n-2)}$ $\iff$ $\gamma = \frac{n^2}{2(n-2)}$, we can estimate the right hand side using Corollary 4.3 and deduce for some uniform constant $C(T) > 0$
\[
\|S\|_{L^\infty(M \times [\frac{T}{2}, T])} \leq C(T).
\]
\[\square\]

**Remark 4.5.** It is worth pointing out that we do not assume $S_0 \in L^\infty(M)$, only that $S_0 \in L^{\frac{n+2}{n}}(M)$. The above proof tells us that $S \in L^\infty(M)$ for positive times, even if the initial curvature is unbounded. This is analogous to the well known behaviour of the heat equation, where the solutions for positive times are often much more regular than the initial data.
5. Long-time existence of the normalized Yamabe flow

We can now establish our main Theorem 1.1, which explicitly reads as follows.

**Theorem 5.1.** Let \((M, g_0)\) be a Riemannian manifold of dimension \(n = \dim M \geq 3\), such that the following four assumptions hold:

1. The Yamabe constant \(Y(M, g_0)\) is positive, i.e. Assumption 1 holds;
2. \((M, g_0)\) is admissible, i.e. Assumption 2 holds;
3. Parabolic Schauder estimates hold on \((M, g_0)\), i.e. Assumption 3 holds;
4. \(S_0 \in C^{1,\alpha}(M)\), i.e. Assumption 4 holds.

Moreover we require \(S_0 \in L^{\frac{n^2}{n-2}}(M)\), and its negative part \((S_0)_- \in L^\infty(M)\).

Under these assumptions, the normalized Yamabe flow \(u_{\frac{4}{n-2}} g_0\) exists with \(u \in C^{3,\alpha}(M \times [0, \infty))\) with infinite existence time and scalar curvature \(S(t) \in L^\infty(M)\) for all \(t > 0\).

**Proof.** Short time existence of the flow with \(u \in C^{3,\alpha}(M \times [0, T'])\) for some small \(T' > 0\) is due to Theorem 1.5. Let \(T > 0\) be the maximal existence time, so that \(u \in C^{3,\alpha}(M \times [0, T))\) without locally uniform control of the Hölder norms in \([0, T)\), but with no uniform control of the norms up to \(t = T\). If \(T = \infty\), there is nothing to prove. Otherwise, we proceed as follows.

Proposition 2.3 yields a uniform (i.e. depending only on \(S_0\) and the finite \(T\)) lower bound on the scalar curvature \(S\). Proposition 3.1 and Theorem 3.2 yield uniform upper and lower bounds on the solution \(u\), so that \(u \in L^\infty(M_T)\). This in turn gives us bounds on the Sobolev constants \(A(T), B(T)\) (Corollary 3.3), so we use Theorem 4.1 to argue that \(S \in L^\infty(M_T)\). By the evolution equation

\[\partial_t u = -\frac{4}{n-2}(S - \rho)u,\]

we deduce \(\partial_t u \in L^\infty(M_T)\). Then, arguing exactly as in [BAVE19, Proposition 2.8], we may then restart the flow and extend the solution past \(T\). For the purpose of self-containment, we provide the argument here.

Let us consider the linearized equation (1.18) with \(u = 1 + v\)

\[\partial_t v - (n-1)\Delta_0 v = -\frac{n-2}{4} S_0 + \Phi(v), \quad v(0) = 0,\]

where \(\Phi(v) \in L^\infty(M_T)\), since \(u, \partial_t u, \rho \in L^\infty(M_T)\). By the third mapping property in (1.14), we conclude that \(v \in C^{1,\alpha}(M \times [0, T])\).7 Rewrite flow equation (1.2) using \(N = \frac{n-2}{n-2}\) as follows

\[\partial_t u - (n-1) u^{1-N} \Delta_0 u = \frac{n-2}{4} (\rho u - S_0 u^{2-N}).\]

Treat the right hand side of this equation as a fixed element of \(C^{0,\alpha}(M \times [0, T])\). Since \(u^{1-N} \in C^{1,\alpha}(M \times [0, T])\) is positive and uniformly bounded away from zero, we may apply (1.16) and (1.17) to obtain a solution \(u' \in C^{2,\alpha}(M \times [0, T])\) with initial condition \(u'(0) = 1\).

---

7 Note that we have uniform control of the \(C^{1,\alpha}\) norm up to \(t = T\) now.
Note that \( w := u - u' \) solves \( \partial_t w - \frac{n-1}{n}u^{1-N} \Delta_0 w = 0 \) with zero initial condition. By the weak maximum principle (1.13), \( \partial_t w_{\max} \leq 0 \) and \( \partial_t w_{\min} \geq 0 \). Due to the initial condition \( w(0) = 0 \), we deduce \( w \equiv 0 \) and hence \( u = u' \in C^{2,\alpha}(M \times [0,T]) \). Thus \( u' \in C^{2,\alpha}(M \times [0,T]) \) extends \( u(t) \) up to \( t = T \), and we conclude

\[ u \in C^{2,\alpha}(M \times [0,T]). \]

By the second statement of Theorem 1.5, we even have \( u \in C^{3,\alpha}(M \times [0,T]) \) and can now restart the flow as follows. Consider \( u_0 = u(T) \in C^{3,\alpha}(M) \) as the initial condition for the normalized Yamabe flow. By (1.15), \( e^{t\Delta_0}u_0 \in C^{3,\alpha}(M \times [0,T]) \), where the heat operator acts without convolution in time.

We write \( u = f + e^{t\Delta_0}u_0 \) and plug this into the Yamabe flow equation (1.2) with rescaled time \( \tau = (t - T) \). This yields an equation for \( f \)

\[
\left[ \partial_t - (n-1)(e^{t\Delta_0}u_0)^{1-N} \Delta_0 \right] f = Q_1(f) + Q_2(f, \partial_tf), \quad u'(0) = 0,
\]

where \( Q_1 \) and \( Q_2 \) denote linear and quadratic combinations of the elements in brackets, respectively, with coefficients given by polynomials in \( e^{t\Delta_0}u_0 \) and \( \partial_t e^{t\Delta_0}u_0, \Delta_0 e^{t\Delta_0}u_0 \). Since these coefficients are of higher Hölder regularity \( C^{1,\alpha}(M) \), we may set up a contraction mapping argument in \( C^{3,\alpha}(M) \) and thus extend \( u \) past the maximal existence time \( T \) ad verbatim to the proof of Theorem 1.5. This proves long-time existence.

\[ \square \]

**Corollary 5.2.** In the setting of the above theorem, we have

\[
\lim_{t \to \infty} \int_M (S - \rho)^2 \, dVol_g = 0
\]

and there exists \( u_\infty \in L^2(M) \) such that

\[
\lim_{t \to \infty} \int_M (u - u_\infty)^2 \, d\mu = 0.
\]

**Proof.** By (1.5) we have

\[
\partial_t \rho = -\frac{n-2}{2} \int_M (S - \rho)^2 \, dVol_g.
\]

This shows that \( \rho(t) \) is monotonously decreasing, and we know it’s bounded from below by \( Y(M, g_0) > 0 \), so \( \lim_{t \to \infty} \rho(t) \) exists. Thus \( \int_0^\infty \partial_t \rho(t) \, dt < \infty \) and thus \( \partial_t \rho(t) \) must converge to zero as \( t \to \infty \). This gives the conclusion on \( \int_M (S - \rho)^2 \, dVol_g \). By (1.1) we also conclude that

\[
\int_M \left( \partial_t u^{\frac{2n}{n-2}} \right) \, d\mu = -\frac{n-2}{2} \int_M (S - \rho)u^{\frac{2n}{n-2}} \, d\mu = 0,
\]

and using \( u \) as a test function in (1.2) leads to

\[
\frac{n+2}{2n} \int_M \partial_t u^{\frac{2n}{n-2}} \, d\mu + (n-1) \int_M |\nabla u|^2 \, d\mu = \frac{n+2}{4} \left( \rho(t) - \int_M u^2 S_0 \, d\mu \right),
\]

so

\[
\int_M |\nabla u|^2 \, d\mu \leq \frac{n+2}{4} \left( \rho(0) + \|S_0\|_{L^\infty(M)} \right),
\]
where we have used $\int_M u^{\frac{2n}{n-2}} \, d\mu = 1$. This shows that $u$ is uniformly bounded in $H^1(M)$, independent of $t$ for all $t \geq 0$. Since the Sobolev embedding $H^1(M) \hookrightarrow L^q(M)$ is compact for $q < \frac{2n}{n-2}$ (see [ACM14, Proposition 1.6]), we in particular get that $u$ has a convergent subsequence in $L^2(M)$ as $t \to \infty$, and we call this limit $u_\infty$. □

Remark 5.3. The above methods would also show that $\partial_t u^{\frac{n+2}{n-2}} \to 0$ in $L^1(M)$, since we may use (1.1) and the Hölder inequality to write

$$\|\partial_t u^{\frac{n+2}{n-2}}\|_{L^1(M)} \leq \frac{n+2}{4} \| (S - \rho) u^{\frac{n}{n-2}}\|_{L^2(M)} \| u^{\frac{2}{n-2}}\|_{L^2(M)}.$$

We then use the first part of the corollary to show that the right hand side tends to 0.

6. Future research directions and open problems

Long time existence alone does not guarantee regularity of the limit solution $u_\infty \in L^2(M)$. Indeed, this has to be obstructed for the following two reasons. In the case of closed manifolds, we know that the Yamabe problem is not uniquely solvable on a round sphere, but so far we have not assumed that $(M, g_0)$ is not a sphere. In the singular setting, the Yamabe problem doesn’t always have a solution, as demonstrated by Viaclovsky [VIA10]. We suspect that demanding

$$Y(M, g_0) < \lim_{R \to 0} Y(B_R(p), g_0),$$

for all $p \in \overline{M}$ is the required condition in our setting. Under this assumption, Akutagawa, Carron, and Mazzeo [ACM14] are able to solve the Yamabe problem for smoothly stratified spaces. For closed manifolds, this condition becomes $Y(M, g_0) < Y(S^n, g_{S^n})$, with the round metric $g_{S^n}$, which is the assumption used by Brendle [BRE05] in his study of the Yamabe flow. Brendle’s proof of convergence of the Yamabe flow relies on the positive mass theorem, which is not available in the singular setting.

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LONG-TIME EXISTENCE OF THE SINGULAR YAMABE FLOW

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