On operators with bounded approximation property

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ABSTRACT. It is known that any separable Banach space with BAP is a complemented subspace of a Banach space with a basis. We show that every operator with bounded approximation property, acting from a separable Banach space, can be factored through a Banach space with a basis.

§1. Lemmas

Definition 1.1. Let \( T \in L(X, W) \), \( C \geq 1 \). We say that \( T \) has the \( C \)-BAP (\( C \)-bounded approximation property) if for every compact subset \( K \) of \( X \), for any \( \varepsilon > 0 \) there exists a finite rank operator \( R : X \to W \) such that \( ||R|| \leq C ||T|| \) and \( \sup_{x \in K} ||Rx - Tx|| \leq \varepsilon \). The operator \( T \) has the BAP if it has the C-BAP for some \( C \in [1, \infty) \).

Lemma 1.1. \( T \) has C-BAP iff for any finite family \( (x_k)_{k=1}^N \subset X \), for any \( \varepsilon > 0 \) there exists a finite rank operator \( R : X \to W \) such that \( ||R|| \leq C ||T|| \) and \( \sup_{1 \leq k \leq N} ||Rx_k - Tx_k|| \leq \varepsilon \).

Proof. We may (and do) assume that \( ||T|| = 1 \). Fix a compact subset \( K \subset X \) and \( \varepsilon > 0 \). Let \( \varepsilon_0 := \varepsilon/(2 + C) \), \( (x_k)_{k=1}^M \) be an \( \varepsilon_0 \)-net for \( K \) in \( X \), \( R \in X^* \otimes W \), \( ||R|| \leq C \) and \( \sup_{1 \leq k \leq M} ||Rx_k - Tx_k|| \leq \varepsilon_0 \). Take an \( x \in K \), and let \( x_k \) be such that \( ||x - x_k|| \leq \varepsilon_0 \). Then \( ||T x - R x|| \leq ||x - x_k|| + ||T x_k - Rx_k|| + ||x_k - Rx|| \leq \varepsilon_0 + \varepsilon_0 + C \varepsilon_0 = \varepsilon \).

Lemma 1.2. Let \( X, W \) be Banach spaces, \( X \) being separable, and \( T \in L(X, W) \). \( T \) has C-BAP iff there exists a sequence \( (Q_l)_{l=1}^\infty \) of finite rank operators from \( X \) to \( W \) such that

1) for every \( x \in X \) the series \( \sum_{l=1}^\infty Q_l x \) converges and

\[
Tx = \sum_{l=1}^\infty Q_l x, \quad x \in X;
\]

2) \( \sup_N ||\sum_{l=1}^N Q_l|| \leq C ||T|| \).

Proof. Since \( X \) is separable, there exists a sequence \( (x_k)_1^\infty \) which is dense in the closed unit ball \( \bar{B}_1(0) \) of \( X \). Suppose as above that \( ||T|| = 1 \) and \( T \) has the C-BAP, that is for any finite set \( F \subset X \), for every \( \varepsilon > 0 \) there is a finite rank operator \( R : X \to W \) such that \( ||R|| \leq C \) and \( \sup_{f \in F} ||Rf - Tf|| \leq \varepsilon \). Put, for \( N = 1, 2, \ldots \),

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For each $N$, let $R_N$ be a finite rank operator from $X$ to $W$ with the properties that

(i) $\|R_N\| \leq C$

(ii) $\sup_{1 \leq n \leq N} \|R_N x_n - T x_n\| \leq 1/2^{N+1}$.

If $n \in \mathbb{N}$ then for every $N \geq n$ one has

(iii) $\|R_N x_n - T x_n\| \leq \frac{1}{2^{N+1}}$

and, therefore, for a fixed $x_n$

$$R_N x_n \to T x_n$$

as $N$ tends to $\infty$.

Now, fix $\varepsilon > 0$ and let $\delta > 0$ be such that $C\delta + \delta < \varepsilon$. For $x \in B_1(0)$, take an $x_n$ with $\|x_n - x\| < \delta$. Then there is an $N_0$ so that for $N \geq N_0$

$$\|R_N x - T x\| \leq \|R_N\| \|x_n - x\| + \|R_N x_n - T x_n\| + \|T x_n - T x\| \leq C\delta + \|T\|\delta < \varepsilon.$$

Thus, if $x \in X$ then $R_N x \to T x$ as $N \to +\infty$.

To finish the proof of the "only if" part, we apply

**Lemma 1.3.** Let $X, W$ be any Banach spaces, $C \geq 1$ and $T \in L(X, W)$. Suppose that

(*) there exists a sequence $(S_N)_{N=1}^{\infty}$ of finite rank operators from $X$ to $W$ such that if $x \in X$ then $S_N x \to T x$ as $N \to +\infty$ and $\|S_N\| \leq C \|T\|$ for every $N$.

Then there exists a sequence $(Q_l)_{l=1}^{\infty}$ of finite rank operators from $X$ to $W$ such that

1) for every $x \in X$ the series $\sum_{l=1}^{\infty} Q_l x$ converges and

$$Tx = \sum_{l=1}^{\infty} Q_l x, \quad x \in X;$$

2) $\sup_N \|\sum_{l=1}^{N} Q_l\| \leq C \|T\|$. 

**Proof.** We assume again that $\|T\| = 1$. Put $Q_1 := S_1, Q_l := S_l - S_{l-1}$ for $l > 1$, so that

$$S_N = S_1 + (S_2 - S_1) + \cdots + (S_{N-1} - S_{N-2}) + (S_N - S_{N-1}) = Q_1 + Q_2 + \cdots + Q_N.$$ 

It follows that

$$\sum_{l=1}^{\infty} Q_l x \forall x \in X$$

and

$$\sum_{l=1}^{N} Q_l x = \sup_N \|\sum_{l=1}^{N} Q_l\| = \sup_N \|S_N\| \leq C.$$

The "if" part of the proof of Lemma 1.2 follows from

**Lemma 1.4.** Let $X, W$ be any Banach spaces, $C \geq 1$ and $T \in L(X, W)$. If there exists a sequence $(R_N)_{N=1}^{\infty}$ of finite rank operators from $X$ into $W$ which converges pointwise to $T$ and such that $\|R_N\| \leq C \|T\|$ for all $N$ then $T$ has the C-BAP.
and a compact subset $K \subset X$. Put $\varepsilon_0 := \varepsilon(\|T\| + 1 + C\|T\|)^{-1}$. Take a finite $\varepsilon_0$-net $F \subset X$ for $K$ and consider $R_{N_0}$ such that $\sup_{f \in F} \|R_{N_0} f - T f\| \leq \varepsilon_0$. Then, for any $x \in K$ there is an $f_0 \in F$ with $\|f_0 - x\| \leq \varepsilon_0$, and one has:

$$
\|T x - R_{N_0} x\| \leq \|T\| \varepsilon_0 + \varepsilon_0 + \|R_{N_0}\| \varepsilon_0 \leq \varepsilon_0(\|T\| + 1 + C\|T\|) = \varepsilon.
$$

**Corollary 1.1.** If $X$ is separable and $T \in L(X,W)$, then $T$ has the C-BAP iff there exists a sequence $(R_N)_1^\infty$ of finite rank operators from $X$ into $W$ which converges pointwise to $T$ and such that $\|R_N\| \leq C\|T\|$ for all $N$.

**Corollary 1.2.** If $X$ is separable and $T \in L(X,W)$, then $T$ has the BAP iff there exists a sequence of finite rank operators from $X$ to $W$ convergent to $T$ pointwise.

**§2. Theorem.**

Now, we redenote some objects from §1. Let $X, W$ be any Banach spaces, $T \in L(X,W)$ and $T$ possesses the property (*) from Lemma 1.3. Consider the sequence $(Q_i)_{i=1}^\infty$, given by assertion of Lemma 1.3, and put $A_p := Q_p$ ($p = 1, 2, \ldots$) and $K := C(\geq 1)$, assuming that $\|T\| = 1$. We are now in notations (partially) of the paper [1].

**Theorem 2.1.** If $T : X \to W$ has the property (*), then there exist a Banach space $Y$ with a Schauder basis and two operators $\tilde{A} : X \to Y$ and $j : Y \to W$ so that $T = j \tilde{A}$.

**Proof.** In the above notation (assuming that $\|T\| = 1$), we have:

$$
Tx = \sum_{p=1}^\infty A_p x, \ \forall x \in X; \ \ A_p \in X^* \otimes W; \ \sup_{n \in \mathbb{N}} \| \sum_{p=1}^n A_p \| \leq K
$$

(note that for every $n$ $\|A_n\| \leq \| \sum_{p=1}^n A_p - \sum_{p=1}^{n-1} A_p \| \leq 2K$). Let $E_p = A_p(X) \subset W$, $m_p := \dim E_p$ for $p \geq 1$ and $m_0 = 0$. We will proceed as in [1].

By Auerbach, there exist one-dimensional operators $B_j^{(p)} : E_p \to E_p$ with $\|B_j^{(p)}\| = 1$ for $j = 1, 2, \ldots, m_p$, and so that

$$
\sum_{j=1}^{m_p} B_j^{(p)}(e) = e, \ e \in E_p.
$$

Set $C_i^{(p)} := \frac{1}{m_p} B_j^{(p)}$ for $i = rm_p + j$ (where $r = 0, 1, \ldots, m_p - 1; j = 1, 2, \ldots, m_p$). Then, for $e \in E_p$,

$$
\sum_{i=1}^{m_p^2} C_i^{(p)}(e) = m_p \sum_{j=1}^{m_p} \frac{1}{m_p} B_j^{(p)} e = \sum_{j=1}^{m_p} B_j^{(p)} e = e.
$$
Also, for any \( q \geq 1, q \leq m_p^2 \) and some \( l < m_p \) and \( k \leq m_p \) we have:

\[
\| \sum_{i=1}^{q} C_i^{(p)} \| = \| \sum_{i=1}^{lm_p} C_i^{(p)} + \sum_{lm_p+1}^{lm_p+k} C_i^{(p)} \| \leq l \cdot \frac{1}{m_p} \| \sum_{j=1}^{m_p} B_j^{(p)} \| + \frac{1}{m_p} \| \sum_{j=1}^{k} B_j^{(p)} \| \leq 1 + 1 = 2.
\]

Now, let

\[
\tilde{A}_s := C_i^{(p)} A_p
\]

for \( p \in \mathbb{N}, i = 1, 2, \ldots, m_p^2 \) and \( s = m_0^2 + m_1^2 + \cdots + m_{p-1}^2 + i \). 1-dimensional operator \( \tilde{A}_s \) maps \( X \) into \( E_p \subset W \) in the following way:

\[
\tilde{A}_s : X \rightarrow E_p = A_p(X) \rightarrow E_p(\subset W).
\]

Since, for any \( n \in \mathbb{N} \), for some \( k \) and \( r \leq m_k^2 \)

\[
\sum_{s=1}^{n} \tilde{A}_s = \sum_{p=1}^{k-1} \sum_{i=1}^{m_p^2} C_i^{(p)} A_p + \sum_{i=1}^{r} C_i^{(k)} A_k,
\]

we get that

\[
\| \sum_{s=1}^{n} \tilde{A}_s \| \leq \| \sum_{p=1}^{k-1} A_p \| + \| \sum_{i=1}^{r} C_i^{(k)} A_k \| \leq K + 2\|A_k\| \leq 5K.
\]

(To get an estimation "4K" as in [1], it it enough to consider simultaniously, in the center, the given sum and the sum like \( \| \sum_{p=1}^{k} A_p \| + \| \sum_{i=1}^{m_k^2} C_i^{(k)} A_k \| \).

Since, for every \( x \in X \), \( A_k x \to 0 \) as \( k \to \infty \), we have:

\[
\lim_{n \to \infty} \sum_{s=1}^{n} \tilde{A}_s x = \lim_{N \to \infty} \sum_{p=1}^{N} A_p x = Tx.
\]

Now, consider the space

\[
Y := \{(y(s))_{s=1}^{\infty} : y(s) \in \tilde{A}_s(X), \sum_{s=1}^{\infty} y(s) \text{ converges in } W\}.
\]

Set \( ||(y(s))_{s=1}^{\infty}|| := \sup_{n} \| \sum_{s=1}^{n} y(s) \|_W \). Note that \( (\tilde{A}_s(x))_{s=1}^{\infty} \in Y \) for every \( x \in X \), \( \sum_{s=1}^{\infty} \tilde{A}_s(x) = Tx \) and \( ||(\tilde{A}_s(x))_{s=1}^{\infty}||_Y \leq 5K ||x||_X \). Therefore, the map \( \tilde{A} : X \to Y \), defined by \( \tilde{A}(x) = (\tilde{A}_s(s))_{s=1}^{\infty} \), is linear and continuous (and \( ||\tilde{A}|| \leq 5K \)). Let \( j : Y \to W \) be the natural map which takes \( (y(s))_{s=1}^{\infty} \) to \( \sum_{s=1}^{\infty} s y(s) \).

Since

\[
|| \sum_{s=1}^{\infty} y(s) ||_W = \lim_{N} \| \sum_{s=1}^{N} y(s) \|_W \leq \sup_{n} \| \sum_{s=1}^{n} y(s) \|_W,
\]

then \( ||j||_{L(Y,W)} \leq 1 \). Therefore, \( Tx = j\tilde{A} : X \to Y \to W \). It remains now to consider the space \( Y \).

For each \( s \), let \( \tilde{y}_s \in \tilde{A}_s \) be of norm 1. If \( (y_s)_{s=1}^{\infty} \in Y \), then \( y_s = c_s \tilde{y}_s \). Define \( \bar{y}_s \in Y \) by \( \bar{y}_s(t) = 0 \) for \( t \neq s \) and \( \bar{y}_s(s) = \tilde{y}_s \) \((s = 1, 2, \ldots)\). Then, every \( y = (y_s) \in Y \) is of
type $\sum_{s=1}^{\infty} c_s \bar{y}_s$, if we consider $(\bar{y}_s)_{s=1}^{\infty}$ as a basis in $Y$. And this basis is monotone: for all scalars $(c_s)$ we have that

$$||| \sum_{s=1}^{m} c_s \bar{y}_s ||| \leq ||| \sum_{s=1}^{m+1} c_s \bar{y}_s |||$$

(by definition of the norm in $Y$). Finally, the space $Y$ is Banach (cf. [2,p. 18, Prop. 3.1]).

Remark. The theorem just obtained is a spade-theorem for some further investigations in a next paper.

**Corollary 2.1.** If $X$ is separable and $T \in L(X,W)$ then $T$ has the bounded approximation property if and only if $T$ can be factored through a Banach space with a basis.

**References**

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