THE HODGE SPECTRUM OF ANALYTIC GERMS ON ISOLATED SURFACE SINGULARITIES

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Abstract. We use topological methods to prove a semicontinuity property of the Hodge spectra for analytic germs defined on an isolated surface singularity. For this we introduce an analogue of the Seifert matrix (the fractured Seifert matrix), and of the Levine–Tristram signatures associated with it, defined for null-homologous links in arbitrary three dimensional manifolds. Moreover, we establish Murasugi type inequalities in the presence of cobordisms of links.

It turns out that the fractured Seifert matrix determines the Hodge spectrum and the Murasugi type inequalities can be read as spectrum semicontinuity inequalities.

1. Introduction

In a series of articles [BN, BN2] (see also [BNR, BNR2]) the authors developed a topological method to prove the semicontinuity of the Hodge spectrum in low dimensions, which originally was obtained by purely Hodge theoretical methods (that is, by algebraic or analytic machinery). This topological method worked successfully for local plane curve singularities, or for two-variable complex polynomials (for the mixed Hodge structure at infinity). It was even possible to compare the spectrum at infinity with local spectra of singular points of a fixed fiber of a polynomial map. The approach used only topological, not analytic, arguments; the idea was that upon using the polarization properties of the mixed Hodge structure, the spectrum was characterized by the Levine–Tristram signatures of the Seifert form. Next, in the presence of a deformation, using the corresponding topological cobordism one proved a Murasugi type inequality valid for the Levine–Tristram signatures. This inequality was reinterpreted as a spectrum semicontinuity property.

Having these results, it was natural to ask if these method can be generalized to germs \( g : (X,0) \to (\mathbb{C},0) \) defined on an arbitrary isolated surface singularities \( (X,0) \); in fact, this question was asked explicitly by J. Steenbrink at the meeting in Lille in 2012 during a presentation of the first author.

The point is that a possible generalization was obstructed seriously already at its starting point: if the link \( M \) of \( (X,0) \) (which is an oriented 3–manifold) is not a rational homology sphere, then one cannot associate with the link of \( g, L_g \subset M \), a Seifert form, and all the linking theory of cycles in \( M \), intensively used in the previous cases, was missing.

There is also a second warning. Although in the literature there are a few different proofs for the semicontinuity property of hypersurface singularities (see [St3, Var, Var2]), in this general context the semicontinuity was not even formulated, and it is not so clear

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what would be a possible Hodge theoretical proof for it. In this general case even the computation of the spectrum in concrete examples might create problems, hence we lack even key examples.

The goal of the present note is to surmount all these difficulties, and to propose and prove a possible semicontinuity inequality. Since in the classical case of hypersurfaces, the semicontinuity of the spectrum had several strong applications (mostly as obstructions for deformations), we expect that the present results will also find their applications regarding deformations of these more general objects.

The main novelty is the definition of the fractured Seifert form, defined on the subspace \( \ker(H_1(\Sigma) \to H_1(M)) \) of \( H_1(\Sigma) \) (\( \Sigma \) being the Milnor fiber of \( g \)). Moreover, we establish for this new object all the important properties of the classical Seifert form, and its relation with monodromy and intersection forms. In this presentation we use intensively the language of hermitian variation structures of \([\text{Nem2}]\).

For this fractured Seifert form we can consider the analog of the Levine–Tristram signatures, and we prove Murasugi type inequality in the presence of a cobordism. Furthermore, one of the main results shows that the fractured Seifert form determines the Hodge spectrum, hence, as in the old case, the Murasugi type inequalities for the signatures provide semicontinuity properties for the spectrum.

Unless specified otherwise, the homologies usually mean homologies with rational coefficients. For a set \( A \), the symbol \(|A|\) denotes the cardinality. All the manifolds are assumed to be oriented.

2. Hermitian variation structures – Generalities

Hermitian Variation Structures (in short HVS) were introduced in \([\text{Nem2}]\) as a way to encode the ‘homological Milnor package’ of the Milnor fibration. It turns out that they can be used to connect knot theory with Hodge theory via the Seifert form of the link. This approach was exploited in \([\text{BN}]\): in this language, the Levine–Tristram signatures for links correspond to the spectrum of a HVS.

2.1. Review of hermitian variation structures. First recall the definition of a HVS. In contrast with e.g. \([\text{BN}]\) or \([\text{BN2}]\) we will deal with non-simple variation structures as well.

**Definition 2.1.1.** \([\text{Nem2}]\) For a fixed sign \( \epsilon = \pm 1 \), an \( \epsilon \)-hermitian variation structure consist of a quadruple \((U; b, h, V)\), where

- \( U \) is a complex linear space;
- \( b: U \to U^* \) is an \( \epsilon \)-hermitian endomorphism;
- \( h: U \to U \) is an automorphism preserving \( b \);
- \( V: U^* \to U \) is an endomorphism.

These objects should satisfy the following compatibility relations:

\[
V \circ b = h - I \quad \text{and} \quad V^* = -\epsilon V \circ h^* \quad \text{('Picard–Lefschetz' formulae)}.
\]

Here \( \overline{\cdot} \) denotes the complex conjugate and \( \cdot^* \) the duality.

In our applications in the next sections we shall always choose the sign \( \epsilon = -1 \). Sometimes we refer to the dimension of a HVS as the dimension of the underlying linear space \( U \). The prototype of a HVS is provided by a Milnor fibration of an isolated hypersurface singularity \((\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)\): \( U \) is the middle homology \( H_m(F) \) of the fiber, \( b \) the intersection form on it, \( h \) the monodromy, and \( V \) is the variation operator, see Section \([\text{5.2}]\). In this case \( \epsilon = (-1)^m \).
If $V$ is an isomorphism, then we say that the HVS is simple. In such a case $V$ determines $b$ and $h$ completely by the formulae $h = -\epsilon V(\bar{V})^{-1}$ and $b = -V^{-1} - \epsilon \bar{V}^{-1}$. If $V$ fails to be an isomorphism, then necessarily 1 must be an eigenvalue of $h$.

If $b$ is an isomorphism, we say that the HVS is non-degenerate. Then $V = (h - I)b^{-1}$, hence the HVS is completely determined by the underlying isometric structure $(U; b, h)$, see [Nem2, Remark 2.6a] and [BF] for the definition of the isometric structure.

2.2. **Examples of HVS.** Here we shall follow closely [Nem2, BN] (some sign conventions differ from [Nem2]). For $k \geq 1$, $J_k$ denotes the $(k \times k)$–Jordan block with eigenvalue 1.

**Example 2.2.1.** For $\lambda \in \mathbb{C}^* \setminus S^1$ and $k \geq 1$, the quadruple

$$\mathcal{V}_k^2(\lambda) = \left( \mathbb{C}^{2k}; \frac{0}{\epsilon I}, \lambda J_k, 0 \right), \text{ where } (1/\lambda) \cdot J_k^{*-1} = \left( \frac{0}{\epsilon(\lambda J_k - I)(b^\pm_1 - \lambda)} \right)$$

defines a simple and non-degenerate HVS. Moreover, $\mathcal{V}_k^2$ and $\mathcal{V}_k^2(1/\lambda)$ are isomorphic.

**Example 2.2.2.** For any $k \geq 1$ there are precisely two non-degenerate $\epsilon$–hermitian forms (up to a real positive scaling), denoted by $b^\pm_1$, such that

$$\tilde{b}^* = \epsilon b \text{ and } J_k^* b J_k = b.$$ 

By convention, the signs are fixed by $(b^\pm_1)_{1,k} = \pm i^{-m^2-k+1}$, where $\epsilon = (-1)^m$. The entries of $b$ satisfy: $b_{i,j} = 0$ for $i + j \leq k$ and $b_{i,k+1-i} = (-1)^{i+1}b_{1,k}$. According to this, for $\lambda \in S^1$, there are up to an isomorphism two non–degenerate HVS with $h = \lambda J_k$. These are

$$\mathcal{V}_k^2(\pm 1) = \left( \mathbb{C}^k; b^\pm_1, \lambda J_k, (\lambda J_k - I)(b^\pm_1 - \lambda) \right).$$

The structures are simple for $\lambda \neq 1$, otherwise not.

**Example 2.2.3.** For $k \geq 1$ there are two degenerate simple HVS with $h = J_k$. They are

$$\mathcal{V}_k^1(\pm 1) = \left( \mathbb{C}^k; b^\pm_1, J_k, \bar{V}_k^\pm \right), \text{ where } \bar{b}^\pm_1 = \left( \frac{0}{0} b^\pm_{k-1} \right).$$

The entries of $V^{-1}$ satisfy: $(V^{-1})_{i,j} = 0$ for $i + j \geq k + 2$, $(V^{-1})_{i,k+1-i} = \pm (-1)^{i+1}i^{-m^2-k}$. For $k = 1$ this is $\mathcal{V}_1^1(\pm 1) = (\mathbb{C}, 0, I, \pm e^{2\pi i}^m)$.

We use the following uniform notation for the above simple structures:

\begin{equation}
\mathcal{W}_k^k(\pm 1) = \begin{cases} 
\mathcal{V}_k^k(\pm 1) & \text{if } \lambda \in S^1 \setminus \{1\} \\
\mathcal{V}_k^1(\pm 1) & \text{if } \lambda = 1.
\end{cases}
\end{equation}

2.3. **Classification of simple HVS.** In [Nem2] the second author proved that each simple variation structure is a direct sum of indecomposable ones.

**Proposition 2.3.1.** A simple HVS is uniquely expressible as a sum of indecomposable ones up to ordering of summands and up to an isomorphism. The indecomposable pieces are $\mathcal{W}_k^k(\pm 1)$ (for $k \geq 1$, $\lambda \in S^1$) and $\mathcal{V}_k^2$ (for $k \geq 1$, $0 < |\lambda| < 1$).

Hence, for each simple HVS, say $\mathcal{V}$, there exists a collection of numbers $p^{k}(u)$ (with $k \geq 1$, $\lambda \in S^1$, $u = \pm 1$) and $q^{k}(u)$ (with $k \geq 1$ and $0 < |\lambda| < 1$) such that

\begin{equation}
\mathcal{V} = \bigoplus_{0 < |\lambda| < 1} q^{k}(\lambda) \cdot \mathcal{V}^{2k} \oplus \bigoplus_{|\lambda| = 1} \mathcal{V}^{k}(u) \cdot \mathcal{W}^{k}(u),
\end{equation}

where the symbol $m \cdot \mathcal{V}$ denotes a sum of $m$ copies of the structure $\mathcal{V}$.

If a HVS is not simple, then a direct sum decomposition of the monodromy $h$ (e.g. its Jordan block decomposition) does not imply the existence of splitting of the whole structure.
Example 2.3.3 (see also [Nem2, Example 2.7.9]). Consider
\[
b = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
These matrices define a \((-1)\)-HVS, which is indecomposable.

Nevertheless, the next splitting holds (to see it, write \(V\) as a block matrix and use the assumptions).

Lemma 2.3.4. Let \(V = (U; b, h, V)\) be a HVS. Assume that \(U = U_1 \oplus U_2\) such that both \(b\) and \(h\) have block decomposition \(b = b_1 \oplus b_1\) and \(h = h_1 \oplus h_2\) with \(b_1\) non-degenerate. Then \(V = V_1 \oplus V_2\) as well, hence \(V\) decomposes as a direct sum \(V_1 \oplus V_2\) of HVS’s.

Remark 2.3.5. Regarding our identities we use the following matrix notations.

Assume that in the vector spaces \(V\) and \(W\) we fixed the bases \(\{\xi_i\}_1\) respectively \(\{\zeta_j\}_j\). Then a morphism \(A : V \to W\) is represented by the matrix \(A = \{A_{ij}\}_{ij}\), where \(A(\xi_i) = \sum_j A_{ji}\zeta_j\). (This means that \((A_{11} \cdots A_{nn})\) is the first line.) (In other words, if \(v\) is the column vector representing \(v \in V\) i.e. it has entries \(v_i\)’s, where \(v = \sum_i v_i \xi_i\), then \(A(v)\) is represented by the column vector \(A \cdot v\)’s.) Any base \(\{\xi_i\}_i\) in \(V\) determines a dual base \(\{\xi^*_j\}_j\) in \(V^* = \text{Hom}_R(V, \mathbb{R})\) via \(\xi^*_j(\xi_i) = \delta_{ji}\), where \(\delta_{ji}\) is the Kronecker symbol. If \(A : V \to V\) is a morphism, then its dual \(A^* : V^* \to V^*\) has matrix representation \(A^* = A^t\).

Similarly, if \(B : V \times V \to \mathbb{R}\) is a bilinear form, then it is represented by the matrix \(B = \{B_{ij}\}_{ij}\), where \(B_{ij} := B(\xi_i, \xi_j)\). (This means that \(B(v, w)\) in matrix calculus is given by \(v^t \cdot B \cdot w\).) Furthermore, if \(b : V \to V^*\) is defined by \(b(v) = B(v, \cdot)\), then the identities \(B_{ik} = B(\xi_i, \xi_k) = b(\xi_i)(\zeta_k) = \sum_j b_{ji}\zeta_j(\zeta_k) = b_{ik}\) show that \(B = b^t\).

2.4. Spectrum and signature of a real simple HVS. Let \(V\) be a simple HVS defined over the real numbers. For simplicity we will also assume that the coefficients \(q^k_{\lambda}\) in the decomposition \((\text{2.3.2})\) are all zero. In the sequel we define the spectrum \(\text{Sp}\) and the signature \(\text{signature}\) of \(V\). For more details regarding this subsection, see [BN].

Definition 2.4.1. The spectrum \(\text{Sp}\) is a finite set of real numbers from the interval \((0, 2]\) with integral multiplicities such that any real number \(\alpha\) occurs in \(\text{Sp}\) precisely \(s(\alpha)\) times, where
\[
s(\alpha) = \sum_{n=1}^{\infty} \sum_{u=\pm 1} \left( \frac{2n - 1 - u(-1)^{|\alpha|}}{2} \right) p_\alpha^u p_{\lambda}^u + np_{\lambda}^u(u), \quad (e^{2\pi i} = \lambda).
\]

Definition 2.4.2. The signature \(\text{signature}\) of \(\zeta\) is the map \(\sigma_\zeta : S^1 \setminus \{1\} \to \mathbb{Z}\) given by
\[
z \mapsto \text{signature of } (\{(1 - z)V + (1 - \overline{z})V^t\}.
\]

The spectrum and the signature are related, cf. [BN, Corollary 4.15].

Lemma 2.4.3. Let \(x \in [0, 1]\) be such that \(\{x, 1 + x\} \cap \text{Sp} = \emptyset\). Let \(z = e^{2\pi i x}\). Then
\[
|\text{Sp} \cap (x, x + 1)| = \frac{1}{2}(\dim U - \sigma(z))
\]
\[
|\text{Sp} \setminus (x, x + 1)| = \frac{1}{2}(\dim U + \sigma(z)).
\]

This lemma will allow us to use topological methods developed in Section 8 (namely, cobordism) to study semicontinuity of the spectrum.

\[\text{1This is Corollary 4.4.9 in the arxiv version of [BN].}\]
3. Links in 3-Manifolds

In this section we study links in oriented closed 3-manifolds. Our approach depends on several choices, for example, choices of the Seifert surface. However, in the applications in singularity theory, there will always be a natural choice, dictated by singularity theory.

3.1. Fractured linking number. Let $M$ be a closed connected oriented 3-manifold. Let $\alpha, \beta \subset M$ be two disjoint one-dimensional cycles. If $M \cong S^3$, the linking number $\text{lk}(\alpha, \beta) \in \mathbb{Z}$ is a well-defined number. We refer to [Rol, Chapter V.D] for various equivalent definitions. In this section we extend the definition for arbitrary 3-manifold $M$, but for special 1-cycles.

Assume that $[\alpha] = [\beta] = 0$ in $H_1(M; \mathbb{Q})$. Then there exists a 2-chain $A$ such that $\partial A = \alpha$. We denote the algebraic intersection number of $A$ and $\beta$ in $M$ by $A \cdot \beta$ (which counts intersection points with signs provided that $A$ and $\beta$ are in general position). If $A$ and $A'$ are two different 2-chains such that $\partial A = \partial A' = \alpha$, then $A \cdot \beta = A' \cdot \beta$. Indeed, the cycle $A \cup -A'$ defines an element in $H_2(M; \mathbb{Q})$. Then $(A \cup -A') \cdot \beta$ as an intersection product in $M$, is zero, since $[\beta] = 0 \in H_1(M)$.

Definition 3.1.1. We define the fractured linking number $\text{flk}(\alpha, \beta)$ as $A \cdot \beta \in \mathbb{Q}$, where $\partial A = \alpha$. (By the above discussion it is independent of the choice of $A$.)

Remark 3.1.2. If $M$ is a 3-manifold, one has a linking form on the torsion part of its first homology with values in $\mathbb{Q}/\mathbb{Z}$. Our construction is different: it assigns an element from $\mathbb{Q}$ to any two disjoint rationally null-homologous cycles. By choosing its name ‘fractured linking number’ (instead of ‘linking number’) we emphasize the difference and avoid confusions.

As the classical linking number, the fractured linking number is symmetric too.

Lemma 3.1.3. For any two disjoint null-homologous cycles $\alpha, \beta$ on $M$ we have $\text{flk}(\alpha, \beta) = \text{flk}(\beta, \alpha)$.

Proof. It is enough to prove the statement for $\alpha$ and $\beta$ integral cycles. Let $A, B$ be surfaces such that $\partial A = \alpha$, $\partial B = \beta$, and $A$ and $B$ intersect transversely. Then $A \cap B$ is an oriented cobordism between $A \cap \partial B$ and $\partial A \cap B$. This proof extends to the level of chains as well. □

In the classical case, one has another definition of the linking pairing. Namely, given two disjoint 1-cycles $\alpha, \beta \subset S^3 = \partial B^4$, one takes two 2-chains $A, B$ in the ball $B^4$ such that $\partial A = \alpha$ and $\partial B = \beta$. Then $\text{lk}(\alpha, \beta) = A \cdot B$. We extend this result in a way that we allow an arbitrary four manifold instead of $B^4$, but we need to impose additional conditions on the chains $A$ and $B$.

Lemma 3.1.4. Assume that $W$ is a four manifold such that $\partial W = M$. Let $\alpha, \beta$ be two disjoint null-homologous 1-cycles in $M$. Then for any 2-chains $A, B \subset W$ such that $\partial A = \alpha$, $\partial B = \beta$ and $[A] = [B] = 0 \in H_2(W, M; \mathbb{Q})$ we have

$$\text{flk}(\alpha, \beta) = A \cdot B.$$ 

Proof. First we show that $A \cdot B$ does not depend on the specific choice of $A$ and $B$. To this end, assume that we have a 2-chain $A'$ such that $\partial A' = \alpha$ and $[A'] = 0 \in H_2(W, M; \mathbb{Q})$. Then $A \cup -A'$ is an absolute cycle in $W$ so it represents a class $[A - A'] \in H_2(W)$ and $(A \cup -A') \cdot B = [A - A'] \cdot [B]$, where the last product is the intersection pairing $H_2(W) \times H_2(W, M) \to \mathbb{Q}$. But this is zero since $[B] = 0$. Thus $A \cdot B$ is well-defined.

By picking concrete $A$ and $B$ we will show that $A \cdot B = \text{flk}(\alpha, \beta)$. To this end, choose a collar $M \times [0, 1] \subset W$, such that $M \times \{0\}$ is identified with $M = \partial W$. Let $C$ be a 2-chain in $M$ such that $\partial C = \beta$. We define $B$ as $C$ with its interior pushed slightly inside $M \times [0, 1/2]$. Clearly $[B] = 0 \in H_2(W, M; \mathbb{Q})$. Let $A$ be arbitrary 2-chain satisfying the hypothesis of the lemma. We isotope $A$ so that $A \cap (M \times [0, 1]) = \alpha \times [0, 1]$. Then, by construction, all the intersection points of $A \cap B$ correspond to the intersections of $\alpha$ with $C$. □
Example 3.1.5. The condition that $A$ and $B$ represent $0 \in H_2(W, M)$ is essential, even if $M$ is the 3-sphere. For example, consider $\mathbb{C}^2$ with coordinates $x, y$ and let $M = \{|x|^2 + |y|^2 = 1\}$, $W_0 = \{|x|^2 + |y|^2 < 1\}$. Let $A_0 = \{x = 0\}$, $B_0 = \{y = 0\}$ and put $\alpha = A_0 \cap M$, $\beta = B_0 \cap M$. Then $\text{lk}(\alpha, \beta) = 1$, as is the algebraic intersection number of $A_0$ with $B_0$.

But now we can define $W$ as $W_0$ blown up in the origin and $A, B$ as the strict transforms of $A_0$ and $B_0$ respectively. Then $A \cap B = \emptyset$, but still we have $\partial A = \alpha$, $\partial B = \beta$. Of course, $A$ and $B$ do not represent $0$ in $H_2(W, M)$.

We end up this subsection with an alternative construction of the fractured linking number. Let $\alpha$ and $\beta$ be disjoint cycles in $M$, which represent $0 \in H_1(M; \mathbb{Q})$. Assume that $\alpha$ can be represented by a simple closed loop. Then $\beta$ defines an element in $H_1(M \setminus \alpha; \mathbb{Q})$, which is mapped to $0$ by the map $H_1(M \setminus \alpha) \to H_1(M)$. We have

$$U := \text{ker}(H_1(M \setminus \alpha; \mathbb{Q}) \to H_1(M; \mathbb{Q})) \cong \mathbb{Q}$$

and there is a canonical choice of the isomorphism, such that the oriented meridian of $\alpha$ goes to $1$. Then we can define $\text{flk}(\alpha, \beta)$ to be the class $[\beta] \in \mathbb{Q}$. We leave it as an exercise to check that the two definitions are in fact equivalent.

3.2. Links and fractured Seifert matrices.

Definition 3.2.1. A link $L$ in an oriented connected 3-manifold $M$ is a disjoint union of embedded oriented circles $K_1 \sqcup \cdots \sqcup K_n$ in $M$. A Seifert surface of a link $L \subset M$ is a connected oriented surface $\Sigma \subset M$ such that $\partial \Sigma = L$, and the interior of $\Sigma$ is disjoint from $L$.

If such surface exists, we know that $[L] = 0 \in H_1(M; \mathbb{Z})$. Conversely, if $[L] = 0 \in H_1(M; \mathbb{Z})$, the arguments of [Er] or [BNR2] guarantee the existence of $\Sigma$. However, in the present paper, all the links that we shall consider will have a Seifert surfaces.

Let $\Sigma$ be a Seifert surface for a link $L$ and $j : \Sigma \hookrightarrow M$ be the inclusion map. We set

$$(3.2.2) \quad U^\Sigma = \text{ker}(j_+: H_1(\Sigma; \mathbb{Q}) \to H_1(M; \mathbb{Q})).$$

For any $\beta \in U^\Sigma$ let $\beta^+$ be the cycle $\beta$ pushed slightly off $\Sigma$ in the positive normal direction. Obviously, $[\beta^+] = 0 \in H_1(M; \mathbb{Q})$.

Definition 3.2.3. The fractured Seifert pairing $S : U^\Sigma \times U^\Sigma \to \mathbb{Q}$ for $\Sigma$ is defined by $(\alpha, \beta) \mapsto \text{flk}(\alpha, \beta^+)$. A fractured Seifert matrix is a rational square matrix of size $\dim U^\Sigma$ such that in some basis of $U^\Sigma$ the fractured Seifert pairing is represented by $S$.

Usually we shall not make a distinction between a Seifert pairing and a Seifert matrix, see Remark 2.3.5

In general, $S$ is not $\pm$-symmetric, nevertheless Lemma 3.1.3 implies the following.

Lemma 3.2.4. If $\alpha, \beta \in U^\Sigma$ and $\alpha$ is disjoint from $\beta$, then $S(\alpha, \beta) = \text{flk}(\alpha, \beta)$. In particular, if $\alpha_1, \ldots, \alpha_k \in U^\Sigma$ are pairwise disjoint, then $S$ restricted to the subspace spanned by $\alpha_1, \ldots, \alpha_k$ is symmetric.

In this paper we assume that all the links satisfy the following additional assumptions.

Definition 3.2.5. (a) A link will be called 0–link if all component are (rational) null-homologous: $[K_j] = 0 \in H_1(M; \mathbb{Q})$ for any $j = 1, \ldots, n$, and if $L$ admits a Seifert surface.

(b) A 0–link is called special if $\text{flk}(K_i, K_j) > 0$ for all $i \neq j$.

Consider the fractured linking matrix $L(K_i, K_j)$ associated with a special link $L$. Here, $L(K_i, K_j) := \text{flk}(K_i, K_j)$ for $i \neq j$, while $L(K_i, K_i)$ is determined by the imposed identities $L(\sum_i K_i, K_j) = 0$ for any $j$. 
Lemma 3.2.6. If $L$ is special, then $\mathcal{L}$ is negative semi-definite with 1-dimensional null space generated by $\sum_i K_i$.

Proof. (Cf. [Neu1, Sec. 3]) If $R = \sum_i r_i K_i$, then $\mathcal{L}(R, R) = -\sum_{i<j} (r_i - r_j)^2 \text{flk}(K_i, K_j)$.

3.3. Fibred links and monodromy. Our goal in this section is to study the fractured Seifert matrix associated to a fibred link. Understanding a decomposition of a fractured Seifert matrix with respect to generalized eigenspaces of the monodromy operator will lead us to a decomposition of HVS defined for a fibred link.

We begin with the following definition.

Definition 3.3.1. We shall refer to an open book decomposition $(M, L, p)$ with binding $L$ and projection $p: M \setminus L \rightarrow S^1$ simply as a fibred link. We define its (canonical) Seifert surface $\Sigma$ as the page $p^{-1}(1)$. The monodromy diffeomorphism (well defined up to an isotopy) will be denoted by $h^\Sigma: \Sigma \rightarrow \Sigma$. (Notice that since we consider $L$ to be an oriented link, we require that $p$ restricted to the oriented meridians $\mu_1, \ldots, \mu_n$ of components of $L$ is an orientation preserving diffeomorphism.)

For any $t \in [0, 1]$, set $\Sigma_t = p^{-1}(e^{2\pi it})$ with $\Sigma = \Sigma_0$. Since $p$ is locally trivial, there exist a smooth family of diffeomorphisms $h^\Sigma_t: \Sigma \rightarrow \Sigma_t$ induced by trivialization over $[0, t]$, such that $h^\Sigma_0$ is the identity and $h^\Sigma_1 = h^\Sigma$ is the monodromy. These diffeomorphisms are also well defined only up to isotopy. Let $h: H_\ast(\Sigma; R) \rightarrow H_\ast(\Sigma; R)$ be the homological monodromy induced by $h^\Sigma$ for any coefficient ring $R$.

Remark 3.3.2. In the usual definition of the Seifert matrix, to any cycle $\beta \in H_1(\Sigma)$ we associate $\beta^+ = \text{the push off of } \beta \text{ in the positive direction}$. In the case of the above fibred situation, we set $\beta^+ = h^\Sigma_1/2 \beta$. This is common in singularity theory too, see e.g. [AGV][Zol].

The Wang sequence of the fibration $p: M \setminus L \rightarrow S^1$ is

$\cdots \rightarrow H_1(\Sigma) \rightarrow H_1(M \setminus L) \xrightarrow{h-I} H_1(M \setminus L) \xrightarrow{q} H_0(\Sigma) \rightarrow \cdots$

The map $q$ is the following: a cycle $\alpha \in H_1(M \setminus L)$ in general position with respect to $\Sigma$ is mapped to $(\alpha \cdot \Sigma)$ times the generator of $H_0(\Sigma) \simeq \mathbb{Z}$. Since $q(\mu_i) = 1$, $q$ is onto.

Let $L$ be a special fibred link. Let $j: \Sigma \hookrightarrow M$ be the inclusion. We define

$$U_\partial := \ker (\overline{j}_*: H_1(\Sigma)/\text{im}(h-I) \rightarrow H_1(M)),$$

where $\overline{j}_*$ is induced by $j_*$. Later on we shall define a lift of $U_\partial$ to a subspace of $H_1(\Sigma)$.

Let $\mu_i$ be the oriented meridian of $K_i$ in $M$. We have the following commutative diagram,
Corollary 3.3.9. In the sequel we will not make distinction between about $\odromy$ invariant and orthogonal with respect to the corresponding to eigenvalue subspace of (3.3.8) $H$.

Proof. (a) Use $H_2(M, M \setminus L) = \mathbb{Q}\langle \mu_i \rangle_{i=1}^n$ and the fact that $H_2(M) \to H_2(M, M \setminus L)$ is trivial due to the fact that each $[K_i] = 0$.

(b) is clear, the map $\overline{m}$ sends a meridian to $M \setminus L$ and $q$ sends it further to 1, for each meridian intersects $\Sigma$ precisely once.

(c) surjectivity of $\overline{j}_*$ follows from (b) and diagram chasing. The rest is clear.

The intersection form $\alpha \cdot \Sigma \beta$ on $H_1(\Sigma)$ has the following compatibility properties. (Part (b) is the analogue of a Picard–Lefschetz formula from singularity theory.)

Lemma 3.3.6. Assume that $L$ is a special fibred link and $\alpha = (h - I)\gamma$ for some $\gamma \in H_1(\Sigma; \mathbb{Q})$. Then $\alpha \in U^\Sigma$. Moreover, the following hold.

(a) If $\beta \in \ker(h - I)$ then $\alpha \cdot \Sigma \beta = 0$.

(b) If $\beta \in U^\Sigma$ then $\flk(\alpha, \beta^+) = \gamma \cdot \Sigma \beta$.

(c) Denote the homology classes in $H_1(\Sigma)$ determined by the boundary components by $\{K_i\}_{i=1}^n$ (subject to the single relation $\sum_i K_i = 0$). Let $K$ be the subspace of $H_1(\Sigma)$ generated by these components. Then $K \cap \im(h - I) = 0$, hence $K$ injects to $H_1(\Sigma)/\im(h - I)$ with image exactly $U_\beta$.

Proof. The first statement follows from Wang exact sequence, which shows that the class of $\alpha$ is zero already in $H_1(M \setminus L)$.

(a) $\gamma \cdot \Sigma \beta = h_\gamma \cdot \Sigma h_\beta = (\gamma + \alpha) \cdot \Sigma \beta = \gamma \cdot \Sigma \beta + \alpha \cdot \Sigma \beta$, since $\cdot \Sigma$ is $h$-invariant.

(b) Consider $A = \bigcup_{\gamma \in [0,1]} h_\gamma^* \gamma$. The boundary of $A$ is $h_\gamma - \gamma$, which is homologous to $\alpha$. Hence $\flk(\alpha, \beta^+) = A \cdot \beta^+$. We can assume that $A$ is in general position with respect to $\beta^+ = h_{1/2}^{\Sigma} \beta$.

$$A \cdot \beta^+ = (A \cap \Sigma_{1/2}) \cdot \Sigma_{1/2} \beta^+ = h_{1/2}^{\Sigma} \cdot \Sigma_{1/2} h_{1/2}^{\Sigma} \beta = \gamma \cdot \Sigma \beta.$$  

(c) Set $R = \sum_i r_i K_i = (h - I)\gamma$. Then by part (b) $\flk(R, K_j) = \gamma \cdot \Sigma K_j = 0$. Hence, by Lemma 3.2.6 $R = 0$ in $H_1(\Sigma)$. Since each $K_j$ is zero-homologous, $K$ injects in $U_\beta$. Since these spaces have the same dimension, they are isomorphic.

Convention 3.3.7. In the sequel we will not make distinction between $K$ and $U_\beta$, we think about $U_\beta$ as the kernel of the intersection pairing on $H_1(\Sigma)$, that is $U_\beta$ is regarded as a subspace of $U^\Sigma$.

Consider the generalized eigenspace decomposition of $h$

$$H_1(\Sigma; \mathbb{Q}) = U_{\neq 1} \oplus U_{= 1},$$

corresponding to eigenvalue $\neq 1$, respectively $= 1$. Both subspaces $U_{\neq 1}$ and $U_{= 1}$ are monodromy invariant and orthogonal with respect to $\cdot \Sigma$.

Corollary 3.3.9. (a) The subspace $U_{\neq 1}$ belongs to $U^\Sigma$. Furthermore $U_{= 1} \cap U^\Sigma = U_{\im} \oplus U_\beta$, where $U_{\im} = U_{= 1} \cap \im(h - I)$. 

(b) The monodromy $h$ preserves the direct sum decomposition
\begin{equation}
U^\Sigma = U_{\neq 1} \oplus \text{im} \oplus U_{\partial}.
\end{equation}

(c) The components in (3.3.10) are pairwise orthogonal with respect to the intersection form $\cdot_\Sigma$.

(d) The restriction of the intersection form on $U_{\neq 1}$ is non-degenerate.

\textbf{Proof.} For (a) use Wang exact sequence, Lemma 3.3.5 and part (c) of Lemma 3.3.6. Part (b) is clear. Next, we prove (c). The kernel of the intersection form, $U_{\partial}$ is orthogonal to everything. Next, $\text{im} \perp U_{\neq 1}$, since $U_{=1} \perp U_{\neq 1}$. For (d), take $\alpha$ from the kernel of $\cdot_\Sigma|U_{\neq 1}$. Then $\alpha \in \ker(\cdot_\Sigma)$, hence $\alpha \in U_{\partial}$. But $U_{\partial} \cap U_{\neq 1} = \emptyset$. □

\textbf{Proposition 3.3.11.} For the fractured Seifert pairing $S$ the following facts hold.

(a) $S$ is monodromy invariant, i.e. $S(h\alpha, h\beta) = S(\alpha, \beta)$ for any $\alpha, \beta \in U^\Sigma$,

(b) $S$ satisfies $S(\alpha, \beta) = S(h\beta, \alpha)$,

(c) $S$ satisfies $S(\alpha, \beta) = S(\beta, \alpha) = 0$ for $\alpha \in U_{\neq 1}$ and $\beta \in \text{im} \oplus U_{\partial}$, and

(d) $S$ has block structure with respect to the decomposition (3.3.10).

\textbf{Proof.} (a) The monodromy $h^\Sigma$ extends to an automorphism of $M$, still denoted by the same $h^\Sigma$. We clearly have $\text{flk}(\alpha, \beta^+) = \text{flk}(h^\Sigma \alpha, (h^\Sigma \beta)^+)$. Hence $S(\alpha, \beta) = S(h\alpha, h\beta)$.

(b) We have $S(\alpha, \beta) = \text{flk}(\alpha, \beta^+) = \text{flk}(\alpha^+, h^\Sigma \beta) = \text{flk}(h^\Sigma \beta, \alpha^+) = S(h\beta, \alpha)$.

(c) Take $\gamma \in U_{\neq 1}$ such that $(h - I)\gamma = \alpha$. Then $S(\alpha, \beta) = \gamma \cdot_\Sigma \beta = 0$ by Corollary 3.3.9. Similarly, $S(\alpha, \beta) = S(h\alpha, h\beta) = S(\alpha, h^{-1}\beta) = h^{-1}_\Sigma h^{-1}\beta = 0$.

(d) By part (c) it is enough to ensure that $S$ has block structure on $\text{im} \oplus U_{\partial}$. Let $\alpha = (h - I)\gamma \in \text{im}$ and $\beta \in U_{\partial}$. Then $S(\alpha, \beta) = \gamma \cdot_\Sigma \beta = 0$. □

Corresponding to part (d) of Proposition 3.3.11 we write $S_{\neq 1}$, $S_{\text{im}}$ and $S_{\partial}$ for the fractured Seifert pairing restricted to $U_{\neq 1}$, $U_{\text{im}}$ and $U_{\partial}$ respectively.

3.4. Non-degeneracy of $S$. Simple fibred links. In this subsection we give sufficient conditions for the fractured Seifert matrix of a fibred link to be non-degenerate.

\textbf{Proposition 3.4.1.}

(a) $S_{\neq 1}$ and $S_{\partial}$ are non-degenerate.

(b) If $U_{\text{im}} \subset \ker(h - I)$, then $S_{\text{im}}$ is non-degenerate as well.

\textbf{Proof.} We begin with (a). Since $h - I$ is invertible on $U_{\neq 1}$ and the intersection pairing on $U_{\neq 1}$ is non-degenerate, the statement follows from Lemma 3.3.6(b). The non-degeneracy of $S_{\partial}$ is built in our assumptions: Lemma 3.2.4 together with Definition 3.2.5 and Lemma 3.2.6 show that $S_{\partial}$ is actually a negative definite symmetric matrix. We note, that in the statement of Lemma 3.2.6, we use the word ‘negative semi-definite’, but on $U_{\partial}$ this 1-dimensional null–space of the fractured linking form is killed by Lemma 3.3.3(c).

We continue with (b). Recall that by Lemma 3.3.4 we have $U_{\partial} \cap U_{\text{im}} = \emptyset$. Set $U_{=1} := U_{=1}/U_{\partial}$. Then $U_{\text{im}}$ projects isomorphically to a subspace of $U_{=1}$. Let $\alpha_1, \ldots, \alpha_k$ be linearly independent elements in $U_{\text{im}}$, such that their representatives form a basis in $U_{\text{im}}/U_{\partial}$. Let us choose $\beta_1, \ldots, \beta_k$ in $U_{=1}$ such that $(h - I)\beta_j = \alpha_j$. Let $U_B$ be the space spanned by $\beta_1, \ldots, \beta_k$. Clearly $U_B \to U_B/U_{\text{im}}$ is an isomorphism. Finally, take a subspace $U_{\text{fix}}$ of $U_{=1}$ such that
\begin{equation}
U_{=1} = U_{\text{fix}} \oplus U_{\text{im}} \oplus U_{\partial} \oplus U_B,
\end{equation}
and such that $U_{\text{fix}}$ is $h$–invariant. This is possible because the assumption guarantees that there are no Jordan block of size 3 or larger and $U_B \oplus U_{\text{im}}$ corresponds to Jordan blocks of size 2.

We have $U_{\text{im}} \oplus U_{\text{fix}} \subset \ker(h - I)$, so by Lemma 3.3.6(a) $U_{\text{im}} \perp (U_{\text{im}} \oplus U_{\text{fix}})$. On the other hand, the induced intersection form on $U_{\text{im}} \oplus U_{\text{fix}} \oplus U_B$ is non-degenerate; this follows from the fact that the kernel of the intersection form on $U_{\text{im}}$ is exactly $U_B$, see Corollary 3.3.9(c). We conclude that the block matrix $\{\alpha_i \cdot \beta_j\}_{i,j}$ should be non-degenerate. But by Lemma 3.3.6(b) this is the matrix of $S_{\text{im}}$ in the basis $\alpha_1, \ldots, \alpha_k$. □

**Remark 3.4.3.** The subspaces $U_B$ and $U_{\text{fix}}$ of $U_{\text{im}}$ that were defined in the above proof, will be important in Section 3.5. Their definition depends on various choices, in Section 3.5 we will put additional conditions on $U_B$ and $U_{\text{fix}}$.

The assumption that $U_{\text{im}} \subset \ker(h - I)$ is equivalent to the absence of Jordan blocks of size 3 or larger of eigenvalue 1 in the Jordan block decomposition of $h$. This always holds if $L$ is a graph link in a graph manifold (see e.g. [EN]). More generally, if in the Nielsen–Thurston decomposition of $h$ there are no pseudo–Anosov pieces, then $h$ cannot have Jordan blocks of size larger than 2 (with whatever eigenvalue) by [FM] Corollary 13.3.

**Definition 3.4.4.** A special fibred link $(M, L, p)$ is called simple if the monodromy has no Jordan block of size more than 2 with eigenvalue 1.

**Corollary 3.4.5.** The fractured Seifert matrix of a simple fibred link is non-degenerate.

3.5. **Complementary space to $U^\Sigma$.** Assume that $L$ is a simple fibred link. Let us consider the space $U_{\text{fix}}$ in the proof of Proposition 3.4.1. Since the intersection form on $U_{\text{im}} \times U_B$ is non-degenerate we can choose $U_{\text{fix}}$ (by adding vectors from $U_{\text{im}}$ to base elements of the original $U_{\text{fix}}$) such that $U_B \perp U_{\text{fix}}$ too and $U_{\text{fix}}$ still remains $h$–invariant. Decomposition 3.4.2 has the property that the space $U_{\text{fix}}$ is orthogonal to $U_{\text{im}} \oplus U_B \oplus U_B$, $U_{\text{im}} \perp U_{\text{im}}$, and $\ker(h - I) = U_{\text{fix}} \oplus U_{\text{im}} \oplus U_B$. On $U_{\text{fix}} \oplus U_{\text{im}} \oplus U_B$ the intersection form is non-degenerate. Then automatically one also has the following result.

**Corollary 3.5.1.** The intersection form of $H_1(\Sigma)$ restricted to $U_{\text{fix}} \times U_{\text{fix}}$ is non-degenerate, and $U_{\text{fix}}$ is even dimensional.

Decomposition 3.4.2 together with Corollary 3.3.9 gives also

$$H_1(\Sigma) = U^\Sigma \oplus U_B \oplus U_{\text{fix}}.$$  (3.5.2)

To conclude this section let us write a corollary to Lemma 3.3.5(c).

**Corollary 3.5.3.** For any special fibred link (not necessary simple) we have

$$\dim U^\Sigma = \dim H_1(\Sigma) - \dim H_1(M).$$

**Proof.** Lemma 3.3.5(c) tells that $j_*$ is onto, hence $H_1(\Sigma) \to H_1(M)$ is onto, and $U^\Sigma$ is defined as the kernel of this map. □

4. **Cobordisms and signatures**

4.1. **Cobordisms of links.** We begin with the following definition.

**Definition 4.1.1.** Fix two links $L_0 \subset M_0$ and $L_1 \subset M_1$ as in 3.2.1. A cobordism of links connecting $(M_0, L_0)$ and $(M_1, L_1)$ is a pair $(W, Y)$, where $W$ is a 4–manifold and $Y \subset W$ is a surface, such that $\partial W = -M_0 \cup M_1$, $\partial Y = -L_0 \cup L_1$.

A cobordism will be called standard if $W \cong M_0 \times [0, 1]$.

Compatibility with Seifert surfaces of links is formulated as follows.
**Definition 4.1.2.** Let \((M_0, L_0)\) and \((M_1, L_1)\) be two links and \(\Sigma_0, \Sigma_1\) Seifert surfaces respectively for \((M_0, L_0)\) and \((M_1, L_1)\). A pair \((W, \Omega)\) is a *Seifert cobordism* between links \((M_0, L_0)\) and \((M_1, L_1)\) and their Seifert surfaces \(\Sigma_0\) and \(\Sigma_1\) if

- \(W\) is a 4-manifold with boundary \(-M_0 \sqcup M_1\);
- \(\Omega\) is a 3-manifold with corners;
- \(\Sigma_0 := \Omega \cap M_0\) and \(\Sigma_1 := \Omega \cap M_1\) are the Seifert surfaces for \(L_0 = \partial \Sigma_0\) and \(L_1 = \partial \Sigma_1\);
- \(\partial \Omega = \Sigma_0 \cup Y \cup \Sigma_1\), where \(Y \subset W\) and \(\partial Y = -L_0 \cup L_1\).

**Remark 4.1.3.** The condition that \(W\) is a manifold might be slightly relaxed. More precisely, we can allow \(W\) to be singular away from \(\partial W \cup \Omega\). For example, we can assume that there exists a finite number of points \(w_1, \ldots, w_s \in W \setminus (\partial W \cup \Omega)\) such that \(W \setminus \{w_1, \ldots, w_s\}\) is a smooth manifold. In fact, in the applications we do not use the smoothness of \(W\). On the other hand, the smoothness of \(\Omega\) is exploited by its Poincaré duality.

We wish to study how the fractured Seifert matrices does change under the Seifert cobordism. The situation is standard in the classical case. To simplify the notation, we shall first restrict ourselves to the case when \(M_1, L_1\) and \(\Sigma_1\) are empty. The manifolds \(M_0, L_0\) and \(\Sigma_0\) will be denoted by \(M, L, \Sigma\). The inclusions \(\Sigma \hookrightarrow \Omega, \Sigma \hookrightarrow M\) and \((\Omega, \Sigma) \hookrightarrow (W, M)\) will be denoted by \(i, j, \text{ resp. } k\).

Let \(U_{\text{null}} \subset U^\Sigma\) be the space of those elements \(\alpha \in \ker i_*\) for which there exists \(A \in H_2(\Omega, \Sigma; \mathbb{Q})\) such that \(\partial A = \alpha\) and \(k_*A = 0\) (see diagram (4.1.5)).

**Proposition 4.1.4.** Let \(\alpha, \beta \in U_{\text{null}}\). Then, \(S(\alpha, \beta) = 0\).

**Proof.** Let \(B^+\) be the cycle \(B\) pushed off \(\Omega\) in the positive normal direction. Obviously \(\partial B^+ = \beta^+\) and \(B^+\) is a zero element in \(H_2(W, M; \mathbb{Q})\). By Lemma [3.1.4] we have \(\text{flk}(\alpha, \beta^+) = A \cdot B^+\). But \(A \subset \Omega\) and \(B^+\) is disjoint from \(\Omega\). \(\Box\)

Next we search for a bound for \(\dim U_{\text{null}}\). We will need the following diagram

\[
\begin{array}{ccc}
H_2(\Omega, \Sigma) & \xrightarrow{\partial} & H_1(\Sigma) \\
\downarrow{k_*} & & \downarrow{j_*} \\
0 & \xrightarrow{i_*} & H_1(\Omega)
\end{array}
\]

\[(\text{4.1.5})\]

(\text{where the rows are long exact sequences of pairs}) and the next terminology as well.

**Definition 4.1.6.** The *irregularity* of the Seifert cobordism \((W, \Omega)\) is

\[
\text{Irr}_2 := \dim \ker (H_2(M; \mathbb{Q}) \to H_2(W; \mathbb{Q})).
\]

(\text{The subscript 2 suggests that the irregularity is related to the map on the second homologies. Later, in a more specific case, we shall also introduce } \text{Irr}_1.\)

We have the following estimates.

**Lemma 4.1.8.**

(a) \(\dim U_{\text{null}} \geq \dim (\ker i_* \cap \ker j_*) - \text{Irr}_2\)

(b) \(\dim (\ker i_* \cap \ker j_*) \geq \dim \ker i_* - \dim \ker a_*\).

**Proof.** (a) Since \(U_{\text{null}} = \partial(k_*^{-1}(0))\) and \(\ker i_* \cap \ker j_* = \partial(k_*^{-1}(\ker \partial'))\), one has

\[
\dim (\ker i_* \cap \ker j_*)/U_{\text{null}} \leq \dim(k_*^{-1}(\ker \partial'))/k_*^{-1}(0) = \dim \ker \partial'.
\]

(b) \(0 \to \ker i_* \cap \ker j_* \to \ker i_* \xrightarrow{j_*} \ker a_*\) is exact. \(\Box\)
We remark that \( \dim \ker a_* \) does not depend on the cobordism of the link itself (that is, on \( L, \Sigma, \Omega \)), but only on \( M \) and \( W \). Finally, we estimate \( \dim \ker i_* \).

**Lemma 4.1.9.** One has the following estimate

\[
\dim \ker i_* \geq b_1(\Sigma) - \frac{1}{2} b_1(\Sigma \cup Y).
\]

**Proof.** Decompose \( i_* \) as \( H_1(\Sigma) \xrightarrow{b_*} H_1(\Sigma \cup Y) \xrightarrow{c_*} H_1(\Omega) \). Then

\[
\dim \ker i_* - \dim \ker b_* = \dim(\im b_* \cap \ker c_*) \geq \dim \im b_* + \dim \ker c_* - b_1(\Sigma \cup Y).
\]

Clearly, \( \dim \ker b_* + \dim(\im b_*) = b_1(\Sigma) \). As \( \Sigma \cup Y = \partial \Omega \) and \( \Omega \) has dimension three, by the Poincaré duality arguments one gets \( \dim \ker c_* = b_1(\Sigma \cup Y)/2 \). \( \square \)

Now we combine our last three statements.

**Corollary 4.1.10.** If \((W, \Omega)\) is a Seifert cobordism such that \((M_1, L_1, \Sigma_1) = \emptyset\), then \(U^\Sigma\) contains the subspace \(U_{\text{null}}\) of dimension at least

\[
b_1(\Sigma) - \frac{1}{2} b_1(Y \cup \Sigma) - \text{Irr}_2 - \dim \ker a_*
\]

on which the Seifert form identically vanishes.

We will use Corollary 4.1.10 to control the fractured signatures, which we now define.

### 4.2. Fractured signatures.

**Definition 4.2.1.** Let \((M, L)\) be a special link, \(\Sigma\) its Seifert surface, and \(S\) the fractured Seifert pairing on \(U^\Sigma\). The **fractured signature** is a function \(\sigma: S^1 \setminus \{1\} \to \mathbb{Z}\) defined as

\[
z \mapsto \text{signature}((1 - z)S + (1 - \overline{z})S^t).
\]

The **fractured nullity** is a function \(n: S^1 \setminus \{1\} \to \mathbb{Z}\) defined as

\[
z \mapsto \dim \ker((1 - z)S + (1 - \overline{z})S^t).
\]

Here \(\cdot^t\) denotes the transpose (in the matrix notation), or simply \(S^t(\alpha, \beta) = S(\beta, \alpha)\).

**Remark 4.2.2.** If \(S\) is non-degenerate then \(n(z) = 0\) for almost all values of \(z\). Furthermore, if \((M, L)\) is fibred and \(\Sigma\) is a fiber, then \(\sigma\) is a piecewise constant function with possible jumps only at roots of the characteristic polynomial of the monodromy \(\Delta(z) := \det(h^{-1} - I \cdot z)\). Indeed, by Proposition 3.3.11(b) one has the identity \((1 - z)S + (1 - \overline{z})S^t = (1 - \overline{z})S(h^{-1} - I \cdot z)\).

Fractured signatures are motivated by cobordisms: the next Theorem 4.2.4 is at the core of the present article. To prove it, we start with the following standard fact from linear algebra.

**Lemma 4.2.3.** If \(U_{\text{null}} \subset U^\Sigma\) is a subspace such that for \(\alpha, \beta \in U_{\text{null}}\) we have \(S(\alpha, \beta) = 0\), then for all \(z \in S^1 \setminus \{1\}\):

\[
|\sigma(z)| \leq \dim U^\Sigma - 2 \dim U_{\text{null}} + n(z).
\]

The next main result is the starting point in proving our semicontinuity results.

**Theorem 4.2.4.** Let \((W, \Omega)\) be a Seifert cobordism of links \((M_0, L_0)\) and \((M_1, L_1)\). Then for all \(z \in S^1 \setminus \{1\}\) we have

\[
|\sigma_0(z) - \sigma_1(z)| \leq \dim U^\Sigma_0 + \dim U^\Sigma_1 - 2b_1(\Sigma_0) - 2b_1(\Sigma_1) + b_1(Y \cup \Sigma_0 \cup \Sigma_1) + 2 \text{Irr}_2 + 2 \dim \ker(H_1(M_0 \cup M_1) \to H_1(W)) + n_0(z) + n_1(z),
\]

where \(U^\Sigma_j = \ker(H_1(\Sigma_j) \to H_1(M_j))\) and \(\Sigma_j = \Omega \cap M_j\) is the Seifert surface of \(L_j\) \((j = 0, 1)\).
Proof. We define \((M,L) = (M_1 \cup -M_0, L_1 \cup -L_0)\). Let also \(\Sigma = \Sigma_1 \cup -\Sigma_0\). Then the fractured signatures of \(L\) are \(\sigma(z) = \sigma_0(z) - \sigma_1(z)\) and the nullities satisfy \(n(z) = n_0(z) + n_1(z)\). Indeed, if \(S_0\) and \(S_1\) are fractured Seifert matrices for \((M_0,L_0)\) and \((M_1,L_1)\), respectively, then \(S_1 \oplus -S_0\) is a fractured Seifert matrix for \(S\). The theorem is now a direct consequence of Corollary 4.1.10 and Lemma 4.2.3.

5. HERMITIAN VARIATION STRUCTURES ASSOCIATED WITH A SIMPLE FIBRED LINK

In this section we will associate with a simple fibred link two HVSs. One of them is given by the fractured Seifert matrix, the other by the classical variation map. We prove that they determine each other.

5.1. The ‘fractured’ and ‘mended’ HVS for simple fibred links. Let \((M,L,p)\) be a simple fibred link. Let \(\Sigma\) be its Seifert surface, \(h: H_1(\Sigma) \to H_1(\Sigma)\) the monodromy, \(b\) the hermitian intersection form on \(H_1(\Sigma)\), and \(S: U^\Sigma \times U^\Sigma \to \mathbb{C}\) the fractured Seifert pairing.

Definition 5.1.1. The fractured hermitian variation structure associated to the simple fibred link \(L\) is the structure \(V_{\text{frct}} = (U^\Sigma; b|_{U^\Sigma}, h|_{U^\Sigma}, (S^t)^{-1})\) (defined already over \(\mathbb{Q}\)).

By Lemma 3.3.6(b) and Proposition 3.3.11(b) (and using the last line of Remark 2.3.5 too) one checks that the above system forms a HVS. Since \((S^t)^{-1}\) is invertible, cf. Proposition 3.4.1 \(V_{\text{frct}}\) is simple.

According to the direct sum \(U^\Sigma := U_{\neq 1} \oplus U_0 \oplus U_{\text{im}}\) (cf. Section 3.3) the fractured HVS decomposes into a direct sum \(V_{\text{frct}} := V_{\neq 1} \oplus V_0 \oplus V_{\text{im}}\) of HVS as well. This follows from the Splitting Lemma 2.3.4 since the pair \((b|_{U^\Sigma}, h|_{U^\Sigma})\) admits a direct sum decomposition, and \(b|_{U^\Sigma}\) is non-degenerate on \(V_{\neq 1} \oplus V_{\text{im}}\), cf. Corollary 3.3.9. The components are the following.

- The quadruple \(V_{\neq 1} = (U_{\neq 1}; b_{\neq 1}, h|_{U_{\neq 1}}, (S_{\neq 1}^t)^{-1})\) (the natural restrictions on \(U_{\neq 1}\));
- On \(U_0\) the \(h|_{U_0}\) is trivial and \(b|_{U_0} = 0\). Hence \(V_0 = (U_0; 0, I, (S_0^t)^{-1})\). By Lemma 3.2.6 \((S_0^t)^{-1}\) is negative definite, thus \(V_0 = (n - 1) \cdot W_1^1(1)\).
- On \(U_{\text{im}}\) the form \(b|_{U_{\text{im}}}\) is 0 (by Lemma 3.3.6(a)) and \(h|_{U_{\text{im}}} = 0\) the identity (since \(L\) is simple). Thus \(V_{\text{im}} = (U_{\text{im}}; 0, I, (S_{\text{im}}^t)^{-1})\). In particular, \(U_{\text{im}}\) is a union of copies of \(W_1^1(1)\) and \(W_1^1(-1)\).

Although the operators \(b\) and \(h\) of \(V_{\text{frct}}\) can be extended to (the intersection form and monodromy on) \(H_1(\Sigma,\mathbb{C})\), the extension of \((S^t)^{-1}\) is not immediate.

Nevertheless, we wish to define such an extension, however the extension will fail to be simple. First, we introduce a HVS on \(U_{\text{fix}}\) (see Section 3.5).

Definition 5.1.2. The structure \(V_{\text{fix}}\) on \(U_{\text{fix}}\) is the HVS structure determined by non-degenerate isometric structure \((U_{\text{fix}}; b|_{U_{\text{fix}}}, h|_{U_{\text{fix}}})\) (cf. Corollary 3.5.1).

Note that \(h\) is the identity, \(V = 0\), and \(b|_{U_{\text{fix}}}\) is fully antisymmetric. Hence \(V_{\text{fix}}\) is a direct sum of \(\frac{1}{2} \dim U_{\text{fix}}\) copies of \(V_1^1(+1) \oplus V_1^1(-1)\).

Definition 5.1.3. The mended HVS associated with a simple fibred link, denoted by \(V_{\text{mend}}\), is a direct sum \(V_{\text{mend}} := V_0 \oplus V_{\neq 1} \oplus V' \oplus V_{\text{fix}}\), where \(V'\) is constructed from \(V_{\text{im}}\) by replacing each summand \(V_1^1(\pm 1)\) of \(V_{\text{im}}\) by a copy of \(V_1^1(\pm 1)\).

It is straightforward that the operator \(h\) of \(V_{\text{mend}}\) is the monodromy \(H_1(\Sigma) \to H_1(\Sigma)\), the form \(b\) of \(V_{\text{mend}}\) is the intersection form of \(\Sigma\), and the restriction of \(V_{\text{mend}}\) on \(U^\Sigma\) is \(V_{\text{frct}}\). The operator \(V\) of \(V_{\text{mend}}\) is zero on \(U_{\text{fix}} \oplus U_B\), which is exactly the kernel of \(V\). This space is isomorphic to \(H_1(M;\mathbb{C})\), since \(j_* : H_1(\Sigma) \to H_1(M)\) is onto, cf. Lemma 3.3.5(c).

The form of the above extensions is motivated by Theorem 5.2.1.
5.2. The ‘classical’ homological variation structure on $H_1(\Sigma)$. Fix a fibred link $(M, L, p)$ and consider the fibration $p: M \setminus L \to S^1$ with fiber $\Sigma = p^{-1}(1)$. Let $h^\Sigma : \Sigma \to \Sigma$ be the geometric monodromy as in Definition 3.3.1. We can assume that $h^\Sigma|_{\partial \Sigma}$ is the identity (this is how usually we recover $M$ from $h^\Sigma$).

The variation map $\text{Var} : H_1(\Sigma, \partial \Sigma) \to H_1(\Sigma)$ is given by $[x] \mapsto [h^\Sigma(x) - x]$ for any relative cycle $x$ of $(\Sigma, \partial \Sigma)$, see e.g. [Lo]. After identifying $H_1(\Sigma, \partial \Sigma)$ with the dual of $H_1(\Sigma)$ we obtain the homological (real) VHS associated with the fibration $p$, $\mathcal{V}_{\text{fib}} := (H_1(\Sigma); b, h, \text{Var})$, where $b$ and $h$ are the intersection form and the algebraic monodromy. ($\mathcal{V}_{\text{fib}}$ is defined already over $\mathbb{Z}$.)

It is known that when $M$ is a $\mathbb{Q}$– or $\mathbb{Z}$–homology sphere, then $\text{Var} = (S^1)^{-1}$ (over $\mathbb{Q}$ or $\mathbb{Z}$ respectively). This fact is generalized in the next statement.

Theorem 5.2.1. $\mathcal{V}_{\text{fib}} = \mathcal{V}_{\text{mend}}$. Moreover, $\mathcal{V}_{\text{fib}}$ determines the fractured Seifert form $S$, while $S$ and the integer $n$ (the number of components of $L$) determine $\mathcal{V}_{\text{fib}}$ as well.

Proof. Let us write $\tilde{U} := U_{\text{im}} \oplus U_B$. Note that the monodromy $h$ and the intersection form $b$ on $U = H_1(\Sigma)$ have block decomposition according to the direct sum decomposition $\tilde{U} = U_{\neq 1} \oplus U_{\text{fix}} \oplus \tilde{U} \oplus U_B$. Since the kernel of $b$ is exactly $U_B$, by Lemma 2.3.1 the whole $\mathcal{V}_{\text{fib}}$ splits into $\mathcal{V}_{\text{fix}} \oplus \mathcal{V}_{\text{fix}} \oplus \tilde{V} \oplus \mathcal{V}_B$.

Consider now the homological exact sequence of the pair $(M, \Sigma)$. In the presence of the open book decomposition $p$, the boundary operator $\partial : H_2(M, \Sigma) \to H_1(\Sigma)$ can be identified with $\text{Var} : H_1(\Sigma, \partial \Sigma) \to H_1(\Sigma)$. Hence, one has the exact sequence (see also [St2, (2.6)(b)])

$$0 \to H_2(M) \to H_2(\Sigma, \partial \Sigma) \xrightarrow{\text{Var}} H_1(\Sigma) \to H_1(M) \to 0.$$  

(5.2.2)

This has several consequences. First, $\text{im} \text{Var} = U^\Sigma$. Hence, if $\alpha \in H_1(\Sigma, \partial \Sigma)$ and $\beta \in U^\Sigma$, and $\alpha(\beta)$ denotes the pairing (duality) $H_1(\Sigma, \partial \Sigma) \otimes H_1(\Sigma) = U^* \otimes U \to \mathbb{R}$, then $S(\text{Var} \alpha, \beta)$ is well-defined, and, in fact, it equals $\alpha(\beta)$ (whose proof is analogous to the proof of 3.3.6(b)). In matrix notations, $\alpha(\beta) = S^t(\text{Var} \alpha)(\beta)$, hence $S^t \cdot \text{Var}$ is the identity whenever $S$ is well-defined. In particular, $\text{Var}$ extends $(S^t)^{-1}$ from $U^\Sigma$ to $U$.

The extension is special: by (5.2.2) the rank of the kernel of $\text{Var}$ is $\dim H_1(M)$ which equals $\dim(\mathcal{V}_{\text{fix}} \oplus U_B)$ (cf. the end of 5.1), the complementary space of $U^\Sigma$ in $U$.

Since $b|_{U_{\text{fix}}}$ is non-degenerate on $U_{\text{fix}}$, it determines the HVS, hence this component agrees with the extension 5.1.2 of $\mathcal{V}_{\text{mend}}$. Finally, the extension from $U_{\text{im}}$ to $U$ with the imposed kernel property mentioned above, imposes the modification $\mathcal{W}^1_1(\pm 1) \mapsto \mathcal{V}^2_1(\pm 1)$ from 5.1.3.

This ends the proof of the identity $\mathcal{V}_{\text{fib}} = \mathcal{V}_{\text{mend}}$.

Let us recall the type of the blocks of $(\mathcal{V}_{\text{mend}})_{=1}$. $\mathcal{V}_{\text{fix}}$ is a direct sum of $\frac{1}{2} \dim U_{\text{fix}}$ copies of $\mathcal{V}^1_1(+1) \oplus \mathcal{V}^1_1(-1)$, $\mathcal{V}_\partial = (n - 1) \cdot \mathcal{W}^1_1(1)$, while $\tilde{V}$ has (say) $c_{\pm}$ copies of $\mathcal{V}^2_1(\pm 1)$.

Since all these types are different, it is easy to delete the extended part: $\mathcal{V}_{\text{fix}}$ is deleted, $\mathcal{V}_{\partial}$ is preserved, while $\oplus_{\pm} c_{\pm} \cdot \mathcal{V}^2_1(\pm 1)$ is modified into $\oplus_{\pm} c_{\pm} \cdot \mathcal{W}^1_1(1)$ (the restriction to $U_{\text{im}} = \text{im}(h|_{U_{\text{im}}}) - 1) \subset U$). Hence, $\mathcal{V}_{\text{fib}}$ determines $S$.

Conversely, the matrix $S$ itself almost determines $\mathcal{V}_{\text{fib}}$. The only missing data is to know, how to separate the sum $n - 1 + c_+$ (which is determined from $S^t$) into two pieces $n - 1$ and $c_+$.

6. ISOLATED COMPLEX ANALYTIC SURFACE SINGULARITIES

6.1. Links in isolated surfaces singularities. Let $(X, 0)$ be a complex analytic isolated surface singularity (germ). We fix an embedding of $(X, 0)$ into some $\mathbb{C}^N$. The link of $(X, 0)$ is the oriented 3-manifold $M$ obtained as the intersection $X \cap S^2_{\mathbb{C}^N - 1}$, where $S^2_{\mathbb{C}^N - 1}$ is a sphere of sufficiently small radius $\varepsilon$ and centered at 0. The diffeomorphism type of $M$ does not depend on the choice of the embedding and on the radius of the sphere [Le3, Le, Mi2].
Assume that \( g: (X, 0) \to (\mathbb{C}, 0) \) is the germ of an analytic function, which determines an isolated singularity \( \{ g = 0 \} \subset (X, 0) \). If \( \varepsilon \) is sufficiently small, then the intersection \( L_g := M \cap \{ g = 0 \} \) is transverse.

**Definition 6.1.1.** The pair \( L_g \subset M \) is called the link of the germ \( g \) at 0.

For a germ \( g \) as above one defines two fibrations. The first one is the Milnor fibration (see [Mi2] when \( X \) is smooth and [Ham] in the general case).

**Proposition 6.1.2.** The map \( \arg g: M \setminus L_g \to S^1 \) defines an open book decomposition of \( (M, L_g) \).

In parallel, let \( \eta > 0 \) be sufficiently small, \( D_\eta \subset \mathbb{C} \) be the disk of radius \( \varepsilon \) centre 0, and \( B_\varepsilon \subset \mathbb{C}^N \) be the \( \varepsilon \)-ball around 0. Then one has the tube filtration (see [Lé2]):

**Proposition 6.1.3.** The map \( g: (g^{-1}(D_\eta \setminus 0) \cap B_\varepsilon) \to D_\eta \setminus 0 \) is a locally trivial fibration.

**Proposition 6.1.4.** [Mi2, Lé3, CSS] The fibrations \( g \) of 6.1.2 and the restriction of the fibration of 6.1.3 to \( S^1 = \partial D_\eta \) are equivalent. In particular, their fibres are diffeomorphic and the monodromy maps coincide.

Take \( \Sigma := (\arg g)^{-1}(1) \subset M \) to be the Seifert surface of \( L_g \) and denote the components of \( L_g \) by \( K_1, \ldots, K_n \). For the pair \( \Sigma \subset M \) we will use all the notation of sections 2 and 3.

The following result proves that \( L_g \subset M \) is a 0–link in the sense of Definition 3.2.5.

**Lemma 6.1.5.** Each component \( K_i \) of \( L_g \) represents \( 0 \in H_1(M; \mathbb{Q}) \).

**Proof.** In general it is not true that there exist analytic germs \( g_i: (X, 0) \to (\mathbb{C}, 0) \) (1 \( \leq i \leq n \)), such that the link of \( g_i \) is \( K_i \subset M \). However, if we allow to modify the analytic structure supported on the topological type of \( (X, 0) \) (that is, if we keep the pair \( (M, L_g) \) up to an isotopy, but we change the analytic structure into some \( (X_i, 0) \)), then such a germ \( g_i: (X_i, 0) \to (\mathbb{C}, 0) \) exists; see [NP], or page 3 of [NPP]. Then the Milnor fiber \( \Sigma_i \subset M \) of \( g_i \) satisfies \( \partial \Sigma_i = K_i \).

**Lemma 6.1.6.** \( L_g \) is special fibred in the sense of Definition 3.2.5. In particular, the form \( \text{flk} \) on \( U_0 \) is negative definite (cf. Lemma 3.2.6).

**Proof.** We need to show that \( \text{flk}(K_i, K_j) > 0 \) for any \( i < j \). By resolution of singularities, the pair \( g^{-1}(0) \subset X \) has an embedded resolution, hence the pair \( L_g \subset M \) has a plumbing representation, where each \( K_i \) is represented by an arrowhead. Let \( \Sigma_i \) be the Seifert surface of \( K_i \) provided by the Milnor fiber of \( g_i: (X_i, 0) \to (\mathbb{C}, 0) \) (identified topologically, cf. the proof of Lemma 6.1.5). If the arrowhead associated with \( K_j \) is supported by the vertex \( v_j \), then \( \text{flk}(K_i, K_j) = \Sigma_i \cdot K_j \) is exactly the multiplicity of the germ \( g_i \) along the exceptional divisor marked by \( v_j \). This is a positive integer (which can be identified combinatorially from the plumbing graph of \( (M, L_g) \)).

Finally, we verify that \( L_g \) is simple in the sense of Definition 3.4.4.

**Proposition 6.1.7** (The Monodromy Theorem [EN, Lé1]). The eigenvalues of monodromy \( h \) are roots of unity, and \( h \) does not have any Jordan blocks of size \( \geq 3 \).

**Corollary 6.1.8.** The fibred link \( (M, L_g, \arg g) \) is a simple fibred link.
Proposition 6.1.9. For any eigenvalue \( \lambda \) the block \( V_\lambda^2(-1) \) does not appear in \( V_{\text{fib}} \).

Proof. Consider first the case \( \lambda = 1 \). Using the terminology of [EN, Sec. 13], the twist of the monodromy is nonpositive. This follows similarly as in [EN], since it involves only local computation regarding local analytic germs of type \( f(x, y) = x^ay^b \) with \( a, b \) positive integers and \( (x, y) \in (\mathbb{C}^2, 0) \) corresponding to the edges of an embedded resolution graph. This means, that if \( \alpha \in H_1(\Sigma, \partial \Sigma) \) then \( \alpha(\text{Var} \alpha) \) should be nonpositive (here we use for \( \alpha(\beta) \) the notation of the proof of Theorem 5.2.1 regarding the pairing \( (\alpha, \beta) \in H_1(\Sigma, \partial \Sigma) \otimes H_1(\Sigma) \)). Since \( \alpha(\beta) = S(\text{Var} \alpha, \beta) \) (cf. the proof of 5.2.1) we get that \( \alpha(\text{Var} \alpha) = S(\text{Var} \alpha, \text{Var} \alpha) \). Let us consider \( V_1^2(\pm 1) \). It is the structure

\[
\left( \mathbb{C}^2; \begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mp 1 & 0 \\ 0 & 0 \end{pmatrix} \right).
\]

Suppose \( V_1^2(\pm 1) \) is a summand of \( V_{\text{fib}} \), let \( \alpha \) be in \( \mathbb{C}^2 \) corresponding to \( V_1^2(\pm 1) \), then \( \text{Var} \alpha \) is a vector of type \((a, 0)\) for some \( a \in \mathbb{C} \) and then \( \alpha(\text{Var} \alpha) = \mp a^2 \). This is nonpositive for any \( \alpha \) if and only if from \( \mp \) we take the minus sign.

In the literature, there is another test for the sign of the twist which works uniformly for any \( \lambda \). In [Neu1] the notations are the following: Neumann’s \( L \) is our \( S^\ell \), while his \( S \) is our \( b^\ell \). (This can be verified by identifying his identities with ours from Definition 2.1.1.) Then, for any \( \lambda \), an \( N \)-root of unity, the test from pages 228–229 from [Neu1] requires that \( S(\alpha, (h^N - 1)\alpha) \) is nonpositive. This, in our language, means that \( b^\ell_{\pm}(J_2 - 1) \) must have on the diagonal nonpositive entries (where \( b^\ell_\pm \) is the \( b \)-operator of \( V_\lambda^2(\pm 1) \), or of \( V_1^2(\pm 1) \) given above). By a computation, \( b^\ell_{\pm}(J_2 - 1) = \begin{pmatrix} 0 & 0 \\ 0 & \mp 1 \end{pmatrix} \), hence in \( \mp 1 \) only the – sign is allowed. \( \square \)

Remark 6.1.10. As a corollary of Proposition 6.1.9 the components \( V_{\text{fib}} \oplus V_\theta \oplus \tilde{V} \) of \( (V_{\text{fib}})_{\lambda = 1} \) (generalized eigenspace for \( \lambda = 1 \)) are the following: \( V_{\text{fib}} \) is a direct sum of \( \frac{1}{2} \dim U_{\text{fix}} \) copies of \( V_1^1(1) \oplus V_1^1(-1) \), \( V_\theta = (n - 1) \cdot W_1^1(1) \), and \( \tilde{V} \) is a direct sum of \( \dim U_{\text{im}} \) copies of \( V_1^2(\pm 1) \). All these ranks can be read from the dual embedded resolution graph of \((M, L_\theta)\).

Indeed, if \( \Gamma \) is the (abstract) dual resolution graph of \((X, 0)\), let \( c(\Gamma) \) be the number of independent cycles in \( \Gamma \) (that is, the first Betti number of the topological realization of \( \Gamma \)), and let \( g(\Gamma) \) be the sum of all genus decorations of the vertices. Then, by [NS], \( c(\Gamma) = \dim U_{\text{im}} \) (which equals the number of \( 2 \times 2 \)-Jordan blocks with eigenvalue one), and \( \dim U_{\text{fix}} = 2g(\Gamma) \). In particular, all these numbers are independent of the choice of the germ \( g \) (that is, of the choice of the link \( L_\theta \)). Moreover, \( \dim H_1(M) = 2g(\Gamma) + c(\Gamma) \).

On the other hand, \( n \) obviously is the number of arrowheads of the graph of \((M, L_\theta)\).

Regarding the notations of the proof of Theorem 5.2.1 in this analytic case \( c_- = 0 \), and \( c_+ = c(\Gamma) \). Hence \((V_{\text{fix}})_{\lambda = 1} \) consists of \( n - 1 + c(\Gamma) \) copies of \( V_1^1(1) \). Hence, the last statement of Theorem 5.2.1 can be reformulated also as follows: \( S \) and \( c(\Gamma) \) determine \( V_{\text{fib}} \).

In particular, if \( \Gamma \) is a tree, then \( S \) and \( V_{\text{fib}} \) determine each other.

6.2. Mixed Hodge Structures on the vanishing (co)homology of \( g \). If \( g \) is an isolated hypersurface singularity (in any dimension) then the cohomology of its Milnor fiber carries a mixed Hodge structure by the work of Steenbrink and Varchenko. The structure is compatible with the monodromy action (the semisimple and the nilpotent parts are morphisms of type \((0, 0)\) and \((-1, -1)\) respectively), and has several polarization properties induced by the intersection and variation forms. Steenbrink and Varchenko considered also the associated spectrum, which are rational numbers \( \alpha \), one number for each eigenvalue \( \lambda = e^{2\pi i \alpha} \) of the monodromy, such that the choice of \( \alpha \) reflects the position in the Hodge filtration of the corresponding eigenvector.
The more general case when \( (X,0) \) is a space germ with an isolated singularity, and \( g : (X,0) \to (\mathbb{C},0) \) is an analytic function germ which also defines an isolated singularity, is treated in \[\text{[St2]}\]. In this case, if \( \dim(X,0) = 2 \), then the spectrum \( \text{Sp}_{\text{MHS}} \) is situated in the interval \( (0,2] \) (or shifted to \( (-1,1] \), but here we prefer the first version). For precise definitions and particular cases see the articles of Steenbrink and Varchenko in the present bibliography (e.g. \[\text{[St1]} (5.3)\] or \[\text{[St4]}\]), and also their references.

In fact, the (co)homologies of the link \( M \) itself carry mixed Hodge structure as well (see e.g. \[\text{[St2]}\]). For example, if \( \dim(X,0) = 2 \), then \( \dim \text{Gr}_1^W H_1(M) = 2g(\Gamma) \) and \( \dim \text{Gr}_0^W H_1(M) = c(\Gamma) \) (in \( H^1(M) \) the weights are \( +1 \) and \( 0 \)). The Hodge numbers for \( H_1(M) \) are \( h_{-1,0} = h_{0,-1} = g(\Gamma) \) and \( h_{0,0} = c(\Gamma) \). Moreover, the natural geometric exact sequences (like \( (5.2.2) \)) are compatible with MHS (see e.g. the proof of Theorem \[\text{[6.2.1]}\]).

If \( H_1(M,\mathbb{Q}) = 0 \) then \( \text{Sp}_{\text{MHS}} \) is symmetric with respect to \( 1 \), see \[\text{[SSS]}\]. Hence, in this case \( \text{Sp}_{\text{MHS}} \subset (0,2] \). However, in general, \( \text{Sp}_{\text{MHS}} \cap \mathbb{Z} \) fails to be symmetric, see below.

In our approach, one can consider the fractured HVS \( \mathcal{V}_{\text{frct}} \) and its spectrum \( \text{Sp}_{\text{frct}} \subset (0,2] \) determined from \( \mathcal{V}_{\text{frct}} \) as in Section \[\text{[2.4]}\].

**Theorem 6.2.1.**

(a) \( \text{Sp}_{\text{MHS}} \cap \mathbb{Z} = \text{Sp}_{\text{frct}} \setminus \mathbb{Z} \). In particular, they are both symmetric with respect to 1. Hence \( \text{Sp}_{\text{frct}} \) is also symmetric.

(b) In \( \text{Sp}_{\text{MHS}} \) the spectral number 2 appears with multiplicity \( c(\Gamma) + g(\Gamma) \), while 1 with multiplicity \( c(\Gamma) + g(\Gamma) + n - 1 \).

(c) All integral spectral numbers of \( \mathcal{V}_{\text{frct}} \) are concentrated at 1 with multiplicity \( c(\Gamma) + n - 1 \).

(d) The spectrum \( \text{Sp}_{\text{MHS}} \) coincides with the spectrum of \( \mathcal{V}_{\text{fib}} \) (hence also with the spectrum of \( \mathcal{V}_{\text{mend}} \), by Theorem \[\text{[5.2.1]}\]).

(e) In particular, \( \text{Sp}_{\text{MHS}} = \text{Sp}_{\text{frct}} + g(\Gamma) \cdot \{1,2\} + c(\Gamma) \cdot \{2\} \).

**Proof.** Here all the spaces are considered with complex coefficients. As the monodromy preserves the decomposition \( H_1(M) = U_{\neq 1} \oplus U_{\text{fix}} \oplus U_{\partial} \oplus (U_B \oplus U_{\text{im}}) \), the spectrum of MHS is a union of contributions on \( U_{\neq 1} \), \( U_{\text{fix}} \), \( U_{\partial} \) and \( U_B \oplus U_{\text{im}} \). On \( U_{\neq 1} \oplus U_{\text{fix}} \) the intersection form is non–degenerate, so the polarization property of the MHS (as in \[\text{[Nem2]}\] Section 6) shows that the spectrum of the MHS agrees with the spectrum \( \text{Sp}_{\#1} \cup \text{Sp}_{\text{fix}} \). This shows (a). Notice that \( U_{\text{fix}} \) is the sum of the same amount of copies of blocks with different polarizations (signs), hence \( \text{Sp}_{\text{fix}} \) is a union of the same amount of copies of \( \{1\} \) and \( \{2\} \).

On \( U_{\text{im}} \oplus U_B \), the monodromy is the union of two–dimensional Jordan blocks with eigenvalue 1. Each Jordan block corresponds to either \( \mathcal{V}_2^2(1) \), or \( \mathcal{V}_2^1(-1) \), but the contribution of both structures to the spectrum is the same: each contributes with \( \{1,2\} \). Indeed, the nilpotent monodromy operator shifts the Hodge filtration by \(-1\); in particular \( U_{\text{im}} \) contributes with spectral numbers 1 and \( U_B \) with 2.

The contribution of \( U_{\partial} \) to \( \text{Sp}_{\text{MHS}} \) follows from an extension of the argument in \[\text{[SSS]}\] Theorem \[\text{[6.1.10]}\] the term \((\#A - 1)(0,1)\) in that article corresponds to the element \((n-1) \cdot \{1\}\) in \( \text{Sp}_{\text{MHS}} \). The above discussion (see also Remark \[\text{[6.1.10]}\]) shows that \( \text{Sp}_{\text{MHS}} \) agrees with the spectrum of \( \mathcal{V}_{\text{fib}} \). So (b) and (d) are also proved.

(c) For \( \lambda = 1 \) we have only blocks of type \( \mathcal{W}_1^1(+1) \) (cf. Proposition \[\text{[6.1.9]}\]); then use Definition \[\text{[2.4.1]}\].

Part (e) is a consequence of (a)–(d) and the comparison of \( \mathcal{V}_{\text{frct}} \) with \( \mathcal{V}_{\text{fib}} \) in Remark \[\text{[6.1.10]}\].

An alternating way to check the Hodge types of the blocks \( U_{\partial} \) and \( U_{\text{fix}} \oplus U_B \) is via exact sequences. There are two main exact sequences (usually written in cohomology and compact support cohomology), both of them being sequences of mixed Hodge structures,
Definition 4.2.1) is equal to the signature of \( V \) and use the Hodge types of curve singularities. Hence they have the same type of spectrum contributions, namely 1.

For dimensional reason it supports only one Hodge type, which is the same as for curves sitting on surface singularities with rational homology sphere links, or even as for plane curve singularities. Hence they have the same type of spectrum contributions, namely 1.

To identify the term \( U_{\text{fix}} \oplus U_B \simeq \text{coker(Var)} \) we consider the ‘variation exact sequence’:

\[
0 \to H_2(M) \to H_1(\Sigma, \partial \Sigma) \xrightarrow{\text{Var}} H_1(\Sigma) \to H_1(M) \to 0
\]

and use the Hodge types of \( M \), cf. Section 6.2. \( \square \)

We emphasize again that \( \text{Sp}_{\text{frct}} \) can be connected with the signatures of \( \mathcal{V}_{\text{frct}} \) by Lemma 2.4.3. They agree with the signatures of the Seifert form because of the following lemma.

**Lemma 6.2.2.** The fractured signature of the link \( L_g \) (defines via the Seifert matrix, cf. Definition 4.2.1) is equal to the signature of \( \mathcal{V}_{\text{frct}} \) (cf. Definition 2.4.2).

**Proof.** Note that \( V = (S^1)^{-1} \). But then we have

\[
S \left( (1 - z)V + (1 - \overline{z})V^t \right) S^t = (1 - z)S + (1 - \overline{z})S^t,
\]

hence the two forms \((1 - z)V + (1 - \overline{z})V^t\) and \((1 - z)S + (1 - \overline{z})S^t\) are congruent. Their signatures coincide. \( \square \)

6.3. Deformations of singularities. In this section we establish notation, which will allow us to formulate and prove Theorem 6.4.1.

**Definition 6.3.1.** A deformation of an isolated singularity \( g_0 : (X, 0) \to (\mathbb{C}, 0) \) is a complex 3-dimensional variety \( X \subset \mathbb{C}^N \times D \) (where \( D \) is a small disk in \( \mathbb{C} \) centered at the origin) together with an analytic function \( G : X \to \mathbb{C} \) and a projection \( \pi : X \to D \) such that:

- \( \pi \) is a flat morphism;
- for \( t \in D \), the inverse image \( X_t := \pi^{-1}(t) \) is a surface with isolated singularities;
- the function \( g_t = G|_{X_t} \) has only isolated singularities;
- the central fiber \( X_0 \) has a single singularity \( x_0 \) and \( g_0 \) is regular away from \( x_0 \).

Given such a deformation, let us choose a small ball \( B_0 \subset \mathbb{C}^N \) and put \( S_0 = \partial B_0 \). Suppose that the ball is such that \( X_0 \cap S_0 \) is the link \( M_0 \) of the singularity \( x_0 \in X_0 \). Shrinking \( B_0 \) if necessary, we can assume that \( g_0^{-1}(0) \cap S_0 \) is the link of singularity of \( g_0 \) at \( x_0 \). We shall denote this link by \( L_0 \).

Let now \( t \in D \setminus \{0\} \) be sufficiently small. Then \( X_t \cap S_0 \cong M_0 \). Furthermore, by choosing \( t \) small enough we can guarantee that \((X_t \cap S_0, g_t^{-1}(0) \cap X_t \cap S_0) \cong (M_0, L_0)\) as pairs. Let \( x_1, \ldots, x_k \) be the critical points of \( g_t \) on \( X_t \). (If \( x \in X_t \) is a singular point of \( X_t \) and \( g_t(x) = 0 \), then \( x \) has to be considered as a critical point of \( g_t \).) Let \( B_1, \ldots, B_k \) be small pairwise disjoint balls near \( x_1, \ldots, x_k \) such that \( B_i \subset B_0 \) and the pair \((M_i, L_i) := (\partial B_i \cap X_t, \partial B_i \cap X_t \cap g_t^{-1}(0))\) is the link of the singularity of \( g_t \) at \( x_i \). Finally let

\[
W = B_0 \cap X_t \setminus (B_1 \cup \cdots \cup B_k).
\]

Then \( W \) is a cobordism between a disjoint union \( M_1 \cup \cdots \cup M_k \) and \( M_0 \). In general, \( W \) can have a finite number of singular points: these are all those singular points of \( X_t \) where \( g_t \) does not vanish. See Figure 1.

Let us consider the map \( \arg g_t : W \to S^1 \). This is a surjection and let us pick a regular value \( \delta \) such that \( \Omega := \arg g_t^{-1}(\delta) \) omits all the singular points of \( W \). We have the following observation.
Lemma 6.3.2. For any $i = 0, \ldots, k$ the intersection $\Omega \cap \partial B_i$ is the Seifert surface $\Sigma_i$ for $L_i$ cut out by its Milnor open book.

Let $Y = \partial \Omega \setminus \bigcup_{i=0}^k \Sigma_i$. Observe that $Y = g_t^{-1}(0) \cap W$. Let also $Z = Y \cup \Sigma_1 \cup \cdots \cup \Sigma_k$. (This has some ‘corners’ along $\partial Y$, but they can be smoothed.)

Lemma 6.3.3. The manifold $Z$ is diffeomorphic to a Seifert surface $\Sigma_0$.

Proof. By Proposition 6.1.4, $\Sigma_0 \cong g_0^{-1}(0) \cap B_0$ if $0 \notin \mathbb{C} \setminus \{0\}$ is sufficiently small. Then, since $t$ is sufficiently close to 0, $\Sigma_0 \cong g_t^{-1}(0) \cap B_0$. Now

$$g_t^{-1}(0) \cap B_0 = (g_t^{-1}(0) \cap W) \cup \bigcup_{i=1}^k (g_t^{-1}(0) \cap B_i).$$

Applying Proposition 6.1.4 again, we have $g_i^{-1}(0) \cap B_i \cong \Sigma_i$. On the other hand, since $\delta$ is very small and $g_t^{-1}(0)$ has no singular points, we have $Y = g_t^{-1}(0) \cap W \cong g_t^{-1}(0) \cap W$. □

6.4. Semicontinuity of $\text{Sp}^i_{\text{frct}}$. Given the notation introduced in Section 6.3 we are ready to formulate and prove the next semicontinuity result regarding the spectrum. $\text{Sp}^i_{\text{frct}}$ denotes the spectrum associated with the corresponding local fractured HVS, $i = 0, \ldots, k$.

Theorem 6.4.1. If $s \in [0,1]$ is such that $z = e^{2\pi is}$ is not an eigenvalue of the monodromy of $L_0$, then

$$|\text{Sp}^0_{\text{frct}} \cap (s, s + 1)| \geq \sum_{i=1}^k |\text{Sp}^i_{\text{frct}} \cap (s, s + 1)| - \text{Irr}_2 - \text{Irr}_1$$

$$|\text{Sp}^0_{\text{frct}} \setminus [s, s + 1]| \geq \sum_{i=1}^k |\text{Sp}^i_{\text{frct}} \setminus [s, s + 1]| - \text{Irr}_2 - \text{Irr}_1,$$

where

$$\text{Irr}_1 = \dim \ker(H_1(M_0 \cup M_1 \cup \cdots \cup M_k) \to H_1(W)) - \sum_{i=1}^k b_1(M_i).$$

Proof. The pair $(W, \Omega)$ is a Seifert cobordism of links $(M_0, L_0)$ and $(M_1, L_1) \sqcup \cdots \sqcup (M_k, L_k)$. For $i = 0, \ldots, k$, let $\sigma_i(z)$ denotes the fractured signature of the link $L_i$. Suppose first that
are neighborhoods of the critical points of $W$. Then

$$\sum_{i=1}^{k} \sigma_i(z) - \sigma_0(z) \leq \sum_{i=1}^{k} (\dim U_i^\Sigma - 2b_1(\Sigma_i)) + b_1(\partial \Omega) + \dim U_0^\Sigma - 2b_1(\Sigma_0) + 2 \dim \ker(H_1(M_0 \cup M_1 \cup \cdots \cup M_k) \to H_1(W)).$$

Here $U_i^\Sigma = \ker(H_1(\Sigma_0) \to H_1(M_i))$ and $\dim U_i^\Sigma$ is the size of the fractured Seifert matrix for $M_i$. Therefore $\dim U_i^\Sigma - 2b_1(\Sigma_i) = - \dim U_i^\Sigma - 2b_1(M_i)$. On the other hand, by Lemma 6.3.3, we have $\partial \Omega \cong \Sigma_0 \cup \Sigma_0$, hence $b_1(\partial \Omega) = 2b_1(\Sigma_0)$. We obtain

$$\sum_{i=1}^{k} \sigma_i(z) - \sigma_0(z) \leq 2 \dim \ker(H_1(M_0 \cup M_1 \cup \cdots \cup M_k) \to H_1(W)) + 2 \sum_{i=1}^{k} b_1(M_i).$$

The proof now follows from Lemma 2.4.3. It remains to deal with the case where $z$ is an eigenvalue of $h_j$ for some $j > 0$. This is done by choosing $z'$ sufficiently close to $z$ and using the result for $z'$. The argument is as in [BNR, Section 4.1], we omit here the details. □

### 6.5. Special cases of Theorem 6.4.1

Theorem 6.4.1 is stated in a rather general form, $X_t$ might have many singular points, $W$ itself is allowed to be singular. Sometimes it is more convenient to have some special cases. We begin with the following lemma

**Lemma 6.5.1.** We have

$$\text{Irr}_1 + \text{Irr}_2 = \dim H_2(W, M) - \sum_{i=1}^{k} b_1(M_i),$$

where $M = \partial W = M_0 \cup M_1 \cup \cdots \cup M_k$.

**Proof.** By the long exact sequence of the pair $(W, M)$ we obtain

$$\dim \text{coker}(H_2(M) \to H_2(W)) + \dim \ker(H_1(M) \to H_1(W)) = \dim H_2(W, M).$$

□

**Proposition 6.5.2.** Suppose that $M_0 \cong M_1$ are rational homology cobordant and $M_2 \cong \cdots \cong M_k \cong S^3$. Suppose additionally that $W$ is built from a rational $H$–cobordism (that is the inclusions $M_0 \to W'$ and $M_1 \to W'$ induce isomorphism on rational homologies) $W'$ between $M_0$ and $M_1$ by removing $k - 1$ balls, then $\text{Irr}_1 + \text{Irr}_2 = 0$.

**Proof.** Clearly $H_2(W', M) \cong H_2(W, M)$. Furthermore $\dim H_2(W, M) = b_1(M_0) = b_1(M_1)$. The statement follows by definition. □

**Corollary 6.5.3.** If $X$ is a trivial deformation, that is $X_t \cong X_0$, then $\text{Irr}_1 + \text{Irr}_2 = 0$.

**Proof.** We can choose $M_1$ to be equal to $M_0$. Then $W$ is obtained from $M_0 \times [0, 1]$ by removing a finite number of 4-balls; these balls are neighbourhoods of the critical points of $g_t$ on $g_t^{-1}(0)$. Thus $W$ satisfies the assumptions of Proposition 6.5.2. □
6.6. **Semicontinuity results for $\text{Sp}_{\text{MHS}}$.** Using Theorem 6.2.1(e) we can now deduce semicontinuity property for $\text{Sp}_{\text{MHS}}$ from Theorem 6.4.1. For $i = 0, 1, \ldots, k$ let $c_i$ and $g_i$ be the quantities $c(\Gamma)$ and $g(\Gamma)$ corresponding to $M_i$, as in Remark 6.1.10. That is $c_i$ is the number of independent cycles in the graph $\Gamma_i$ representing the link $M_i$, while $g_i$ is the sum of all genus decorations of the vertices. Let us set

$$
\Delta_1 = c_0 - \sum_{i=1}^k c_i, \quad \Delta_2 = c_0 + g_0 - \sum_{i=1}^k (c_i + g_i).
$$

Then one has the following result.

**Theorem 6.6.1.** If $s \in [0, 1]$ is such that $z = e^{2\pi i s}$ is not an eigenvalue of the monodromy operator of $L_0$, then

$$
|\text{Sp}_{\text{MHS}}^0 \cap (s, s + 1)| \geq \sum_{i=1}^k |\text{Sp}_{\text{MHS}}^i \cap (s, s + 1)| - \text{Irr}_2 - \text{Irr}_1 + \Delta_1
$$

$$
|\text{Sp}_{\text{MHS}}^0 \setminus [s, s + 1]| \geq \sum_{i=1}^k |\text{Sp}_{\text{MHS}}^i \setminus [s, s + 1]| - \text{Irr}_2 - \text{Irr}_1 + \Delta_2.
$$

**Proof.** Suppose $s \in (0, 1)$. We use the fact that $|(\text{Sp}_{\text{MHS}}^i \setminus \text{Sp}_{\text{frct}}^i) \cap (s, s + 1)| = c_i$ and $|(\text{Sp}_{\text{MHS}} \setminus \text{Sp}_{\text{frct}}^i) \setminus (s, s + 1)| = c_i + g_i$, which follows from Theorem 6.2.1(d).

If $s = 0$ or $s = 1$, then the assumptions imply that 1 is not an eigenvalue of the monodromy operator of $L_0$, in particular, neither 1 nor 2 are in $\text{Sp}_{\text{MHS}}^0$. By Remark 6.1.10 we infer that $c_0 = g_0 = 0$, hence $\Delta_1, \Delta_2 \leq 0$. Clearly, $\text{Sp}_{\text{MHS}}^i \cap (0, 1) = \text{Sp}_{\text{frct}}^i \cap (0, 1)$, so the first inequality holds if $s = 0, 1$. If $s = 0$, we have $\text{Sp}_{\text{MHS}}^i \setminus [0, 1] = \text{Sp}_{\text{frct}}^i \setminus [0, 1] \cap \{g_i + c_i\} \cdot \{2\}$, so the second inequality in that case follows from the case $s \in (0, 1)$. As $\text{Sp}_{\text{MHS}}^i \setminus [1, 2] = \text{Sp}_{\text{frct}}^i \setminus [1, 2]$ and $\Delta_2 \leq 0$, we infer that the second inequality holds for $s = 1$ as well. $\square$

As a corollary, we prove the semicontinuity in the case of Proposition 6.5.2.

**Proposition 6.6.2.** Under the assumptions of Proposition 6.5.2, for instance if $X_i \cong X_0$, we have $\Delta_1 = \Delta_2 = 0$. Thus for any $s \in [0, 1]$ such that $e^{2\pi i s}$ is not an eigenvalue of the monodromy operator of $L_0$ we have

$$
|\text{Sp}_{\text{MHS}}^0 \cap (s, s + 1)| \geq \sum_{i=1}^k |\text{Sp}_{\text{MHS}}^i \cap (s, s + 1)|
$$

$$
|\text{Sp}_{\text{MHS}}^0 \setminus [s, s + 1]| \geq \sum_{i=1}^k |\text{Sp}_{\text{MHS}}^i \setminus [s, s + 1]|.
$$

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