Abstract. In this article we study a class of stochastic functional differential equations driven by Lévy processes (in particular, \( \alpha \)-stable processes), and obtain the existence and uniqueness of Markov solutions in small time interval. This corresponds to the local solvability to a class of quasi-linear partial integro-differential equations. Moreover, in the constant diffusion coefficient case, without any assumptions on the Lévy generator, we also show the existence of a unique maximal weak solution for a class of semi-linear partial integro-differential equation systems under bounded and Lipschitz assumptions on the coefficients. Meanwhile, in the non-degenerate case (corresponding to \( \Delta^{\alpha/2} \) with \( \alpha \in (1,2) \)), basing upon some gradient estimates, the existence of global solutions is established too. In particular, this provides a probabilistic treatment for non-linear partial integro-differential equations such as the multi-dimensional fractal Burgers and the fractal scalar conservation law equations.

1. Introduction

Consider the following multi-dimensional fractal Burgers equation in \( \mathbb{R}^d \):

\[
\partial_t u + \nu \Delta^{\alpha/2} u - (u \cdot \nabla u), \quad t \geq 0, \quad u_0 = \varphi, \tag{1}
\]

where \( u = (u^1, \cdots, u^d) \) and \( \nu > 0 \) is the viscosity constant, \( \Delta^{\alpha/2} \) with \( \alpha \in (0,2) \) is the usual fractional Laplacian defined by

\[
\Delta^{\alpha/2} u(x) := \lim_{\varepsilon \downarrow 0} \int_{|z| \geq \varepsilon} \frac{u(x+z) - u(x)}{|z|^{d+\alpha}} \, dz.
\]

This is a typical non-linear partial integro-differential equation and regarded as a simplified model for the classical Navier-Stokes equation when \( \alpha = 2 \). Recently, there are great interests for studying the multi-dimensional Burgers turbulence (cf. \([2,15]\)), the fractal Burgers equation (cf. \([3,9,5]\)) and the fractal conservation law equation (cf. \([6]\)), etc. All these works are based on the analytic approaches, especially energy method, Duhamel’s formulation and maximum principle.

The purpose of the present paper is to give a probabilistic treatment for a large class of quasi-linear partial integro-differential equations. Let us first introduce the main idea. By reversing the time variable, one can write Burger’s equation (1) as the following equivalent backward form:

\[
\partial_t u + \nu \Delta^{\alpha/2} u - (u \cdot \nabla u) = 0, \quad t \leq 0, \quad u_0 = \varphi. \tag{2}
\]

Now, consider the case of \( \alpha = 2 \), and for a given smooth solution \( u_t(x) \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d) \) to the above equation, let \( X_{t,s}(x) \) solve the following stochastic differential equation (abbreviated as

* This work is supported by NSFs of China (Nos. 10971076; 10871215).
SDE):
\[ dX_{t,s}(x) = -u_s(X_{t,s}(x))ds + \sqrt{2}vdW_s, \quad s \in [t, 0], \quad X_{t,t}(x) = x, \quad (3) \]

where \((W_s)_{s \in [0]}\) is a \(d\)-dimensional standard Brownian motion on \(\mathbb{R}_- := (-\infty, 0]\). By Itô’s formula and the Markov property of solutions, it is well-known that
\[ u_t(x) = \mathbb{E}\varphi(X_{t,0}(x)). \quad (4) \]

Conversely, assume that \((u, X)\) solves the implicit system (3) and (4), then \(u\) also solves the backward Burgers equation (2). This type of implicit stochastic differential equation has been systematically studied by Freidlin [7, Chapter 5] (see also [4, 14]).

Let us now substitute (4) into (3), then
\[ dX_{t,s}(x) = -[\mathbb{E}\varphi(X_{t,0}(y))]_{y=X_{t,s}(x)}ds + \sqrt{2}vdW_s, \quad s \in [t, 0], \quad X_{t,t}(x) = x. \quad (5) \]

Using the Markov property of solutions, one can write the above equation as a closed form:
\[ dX_{t,s}(x) = -\mathbb{E}^{\mathcal{F}_{t,s}}\varphi(X_{t,0}(x))ds + \sqrt{2}vdW_s, \quad s \in [t, 0], \quad X_{t,t}(x) = x, \quad (6) \]

where \(\mathcal{F}_{t,s} = \sigma\{W_r - W_t : r \in [t, s]\}\), and \(\mathbb{E}^{\mathcal{F}_{t,s}}\) denotes the conditional expectation with respect to \(\mathcal{F}_{t,s}\). The question is coming up: Suppose that stochastic equation (6) admits a unique solution family \(\{X_{t,s}(x) : t \leq s \leq 0, x \in \mathbb{R}^d\}\). Does \(u_t(x)\) defined by (4) solve Burgers equation (2)? For answering this question, the key point is to establish the following Markov property: for all \(t_1 \leq t_2 \leq t_3 \leq 0\) and \(x \in \mathbb{R}^d\),
\[ \mathbb{E}^{\mathcal{F}_{t_1,t_3}}(\varphi(X_{t_1,t_3}(x))) = \mathbb{E}(\varphi(X_{t_2,t_3}(y)))|_{y=X_{t_1,t_3}(x)} \text{ a.s.} \quad (7) \]

so that equation (6) can be written back to (5). This seems not obvious. On the other hand, if we change the Brownian motion in (6) by an \(\alpha\)-stable process as done in [16], then it is naturally expected to give a probabilistic explanation for fractal Burgers equation (2).

Basing on this simple observation, in this paper we are mainly concerned about the following general stochastic functional differential equation (abbreviated as SFDE) driven by a Lévy process \((L_t)_{t \in [0]}\):
\[ dX_{t,s}(x) = G_s(X_{t,s}(x), \mathbb{E}^{\mathcal{F}_{r,s}}(\phi_s(X_{r,s}(x))))dL_s, \quad s \in [t, 0], \quad X_{t,t}(x) = x, \quad (8) \]

where \(\mathcal{F}_s := \sigma\{L_{s'} - L_{s''} : s'' < s' \leq s\}\), \(G\) and \(\phi\) are some Lipschitz functionals (see below). In Section 2, we are devoted to proving the existence and uniqueness of a short time solution as well as Markov property (7) for equation (8) under Lipschitz assumptions on \(G\) and \(\phi\). Moreover, the local maximal solution is also achieved. Since the Lévy process usually has poor integrability, we have to carefully treat the big jump part of the Lévy process. Compared with the classical argument in Freidlin [8], it seems that SFDE (8) is easier to be handled since it is a closed equation.

Next, in Section 3 we apply our result to a class of quasi-linear partial integro-differential equation (abbreviated as PIDE) and obtain the existence of short time solutions. Here, we discuss two cases: \(G\) and \(\phi\) admits to be linear growth, but Lévy process has finite moments of arbitrary orders; \(G\) and \(\phi\) are bounded, but equation (8) has a constant coefficient in big jump part. This is natural since only big jump is related to the moment of Lévy process.

In Section 4, we turn to the investigation of the following system of semi-linear PIDE (non-linear transport equation):
\[
\begin{cases}
\partial_t u_t + L_0 u_t + (G_t(x, u_t) \cdot \nabla) u_t + F_t(x, u_t) = 0, \\
(t, x) \in \mathbb{R}_- \times \mathbb{R}^d, \quad u_0(x) = \varphi(x) \in \mathbb{R}^m,
\end{cases}
\quad (9)
\]
where $L_0$ is the generator of the Lévy process given by (15) below. It is observed that the following scalar conservation law equation can be written as the above form:

$$
\begin{cases}
\partial_t u_t + L_0 u_t + \text{div}(g(x, u_t)) + f_t(x, u_t) = 0, \\
(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad u_0(x) = \varphi(x) \in \mathbb{R}.
\end{cases}
$$

(10)

In particular, one-dimensional fractal Burgers equation (2) takes the above form. In equation (9), since there are no any analytic properties to be imposed on $L_0$, one can not appeal to Duhamel’s formula or energy method to give an analytic treatment. In this situation, probabilistic approach seems to be quite suitable. In fact, by using purely probabilistic argument, we shall prove in Theorem 4.2 below that PIDE (9) admits a unique maximal weak solution in the class of bounded and Lipschitz functions. In the case of non-degenerate (corresponding to subcritical case for $L_0 = \Delta^{\alpha/2}$ with $\alpha \in (1, 2]$), the existence of global solutions is also obtained by applying some gradient estimates. We mention that for one-dimensional Burgers equation (1), it has been proved in [9] that global analyticity solution does exist for time blow up solution also exists for existence of global solutions for general equation (9) is left open.

We conclude this introduction by introducing the following conventions: The letter $L$ with or without subscripts will denote a positive constant, whose value may change in different places. If we write $T = T(K_1, K_2, \cdots)$, this means that $T$ depends only on these indicated arguments.

2. A stochastic functional differential equation: Short time existence

2.1. General facts about Lévy processes. Let $(L_t)_{t \in \mathbb{R}}$ be a $\mathbb{R}^m$-valued Lévy process on the real line and defined on some complete probability space $(\Omega, \mathcal{F}, P)$, which means that

- $(L_t)_{t \in \mathbb{R}}$ has independent and stationary increments, i.e., for all $-\infty < t_1 < t_2 \cdots < t_n < +\infty$, the random variables $(L_{t_2} - L_{t_1}, \cdots, L_{t_n} - L_{t_{n-1}})$ are independent and have the same laws as $(L_{t_2-t_1} - L_0, \cdots, L_{t_n-t_{n-1}} - L_0)$.
- For $P$-almost all $\omega \in \Omega$, the mapping $t \mapsto L_t(\omega)$ is right-continuous and has left-limit (also called càdlàg in French).

Let $\mathcal{N}$ be the total of all $\mathcal{P}$-null sets. For $-\infty \leq t < s < +\infty$, define

$$
\mathcal{F}_{t<s} := \sigma\{L_r - L_{r'}; r, r' \in (t, s]\} \vee \mathcal{N}.
$$

By the independence of increments of Lévy process, it is easy to see that for $-\infty \leq t_1 < t_2 < t_3 < +\infty$, $\mathcal{F}_{t_1,t_2}$ and $\mathcal{F}_{t_2,t_3}$ are independent. For simplicity of notation, we write

$$
\mathcal{F}_t = \mathcal{F}_{-\infty,t}, \quad \mathcal{F}_{s-} := \vee_{t<s} \mathcal{F}_t.
$$

It is clear that $\mathcal{F}_t \subset \mathcal{F}_s$ if $t < s$, and $s \mapsto \mathcal{F}_{s-}$ is left-continuous. Throughout this paper, we shall work on the negative time axes $\mathbb{R}_- := (-\infty, 0]$.

Remark 2.1. For any measurable process $\eta_s \in L^1(\Omega, \mathcal{F}_0, P)$, $s \leq 0$, by the predictable projection theorem (cf. [12], p.173, Theorem 5.3), there always exists a predictable version of $s \mapsto \mathbb{E}(\eta_s|\mathcal{F}_{s-})$, which will be denoted by $\mathbb{E}^{\mathcal{F}_{s-}}(\eta_s)$. Moreover, for any $\xi \in L^1(\Omega, \mathcal{F}_0, P)$, by the regularization theorem of martingales (cf. [12], p.64, Proposition 2.7 and p. 65, Theorem 2.9)), we have

$$
\lim_{s \downarrow t} \mathbb{E}^{\mathcal{F}_{s-}}(\xi) = \mathbb{E}^{\mathcal{F}_{t-}}(\xi) = \mathbb{E}^{\mathcal{F}_t}(\xi), \quad a.s.,
$$

where the second equality is due to $P[L_s = L_{s-}] = 1$.

By Lévy-Khintchine’s formula (cf. [11], p.109, Corollary 2.4.20)), the characteristic function of $L_t$ is given by

$$
\mathbb{E}(e^{i\xi L_t}) = \exp \left\{ t \left[ i |\xi| \varphi(\xi) + \int_{\mathbb{R}^n} \left[ 1 - e^{i \xi \cdot z} + i \xi \cdot z 1_{|z| \leq 1} \right] \nu(dz) \right] \right\} =: e^{|\xi|^2 A_t},
$$

(11)
where $\Psi(\xi)$ is a complex-valued function called the symbol of $(L_t)_{t \leq 0}$, and $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times m}$ is a positive definite and symmetric matrix, $\nu$ is the Lévy measure on $\mathbb{R}^m$, i.e., $\nu(0) = 0$ and
\[
\int_{\mathbb{R}^m} 1 \wedge |z|^2 \nu(dz) < +\infty.
\]
We call
\[
\mathcal{A} := (b, A, \nu)
\] (13)
the characteristic triple. If $b = 0, A = 0, \nu(dz) = \frac{dz}{|z|^\alpha}$, where $\alpha \in (0, 2)$, then $L_t$ is a standard $\alpha$-stable process and its generator is the fractional Laplacian $\Delta^{\alpha/2}$ by multiplying a constant $c_{m, \alpha}$.

By Lévy-Itô’s decomposition (cf. [1] p.108, Theorem 2.4.16), $L_t$ can be written as
\[
L_t = bt + W^A_t + \int_{|z|<1} z\tilde{N}(t, dz) + \int_{|z|>1} zN(t, dz),
\]
(14)
where $W^A_t$ is a Brownian motion with covariance matrix $A = (a_{ij})$, $N(t, dz)$ is the Poisson random point measure associated with $(L_t)_{t \leq 0}$ given by
\[
N(t, \Gamma) := \sum_{i<j=0} 1_{\Gamma}(L_s - L_{s-}), \Gamma \in \mathcal{B}(\mathbb{R}^m),
\]
and $\tilde{N}(t, dz) := N(t, dz) - t\nu(dz)$ is the compensated random martingale measure. Here, $(W^A_t)_{t \leq 0}$ and $(N(t, dz))_{t \leq 0}$ are independent. The generator of $L_t$ is given by
\[
L_0 u(x) = \frac{1}{2}a_{ij}\partial_i\partial_j u + b_i \partial_i u + \int_{\mathbb{R}^m} [u(x + z) - u(x) - \sum_{i=1}^{n} \partial_i u(x)z_i] \nu(dz).
\]
(15)
Here and after, we use the usual convention for summation: the same index in a product will be summed automatically.

In the following, we denote by $\mathbb{D}$ the space of all cadlag functions from $\mathbb{R}_-$ to $\mathbb{R}^d$, which is endowed with the locally uniform metric $\rho$. Notice that this metric is complete but not separable. For given $t < 0$ and a cadlag function $f : [t, 0] \to \mathbb{R}^d$, we extend $f$ to $\mathbb{R}_-$ in a natural manner by putting $f(s) = f(t)$ for $s < t$ so that $f \in \mathbb{D}$.

2.2. A general case. In this subsection, we consider the following general SFDE in $\mathbb{R}^d$ driven by Lévy process $(L_t)_{t \leq 0}$:
\[
X_{t,s} = \xi + \int_{[t,s]} G_t(X_{t,r}, \mathcal{F}_r - (\phi_t(X_{t,r}))) dL_r, \ t \leq s \leq 0,
\]
where $\xi \in \mathcal{F}_t$, $G : \mathbb{R}_- \times \mathbb{R}^d \times \mathcal{F}_s \to \mathbb{R}^d \times \mathbb{R}^m$ is a measurable function, and $\phi : \mathbb{R}_- \times \mathbb{D} \to \mathbb{R}^k$ is a uniformly Lipschitz continuous functional in the sense that
\[
\|\phi\|_{Lip} := \sup_{x \in \mathbb{R}_-} \sup_{\omega \neq \omega' \in \mathbb{D}} \frac{|\phi_t(\omega) - \phi_t(\omega')|}{\rho(\omega, \omega')} < +\infty,
\]
where $\rho(\omega, \omega') := \sum_{n} 2^{-n} \left(1 \wedge \sup_{s \in [0, n]} |\omega(s) - \omega'(s)|\right)$ is the locally uniform metric on $\mathbb{D}$.

The definition about the solutions to equation (16) is given as follows:

**Definition 2.2.** For fixed $t < 0$ and $\xi \in \mathcal{F}_t$, an $(\mathcal{F}_s)$-adapted cadlag stochastic process $X_s := X_{s,s}(\xi)$ is called a solution of equation (16) if for all $s \in [t, 0]$,
\[
X_s = \xi + \int_{[t,s]} G_t(X_{s,r}, \mathcal{F}_r - (\phi_t(X_{r})) dL_r, \ a.s.
\]
For $T < 0$, we call that equation (16) is (uniquely) solvable on $(T, 0]$ (or $[T, 0]$) if for all $t \in (T, 0]$ (or $t \in [T, 0]$) and $\xi \in \mathcal{F}_t$, equation (16) has a (unique) solution starting from $\xi$ at time $t$. 


Remark 2.3. In this definition, it has been assumed that $\phi_t(X_t) \in L^1(\Omega, \mathcal{F}_t, P)$ so that $\mathbb{E}^{\mathcal{F}_t} (\phi_t(X_t))$ makes sense by Remark 2.1, and further the stochastic integral with respect to the Lévy process in the definition makes sense.

Below, we make the following assumptions on the coefficients and the Lévy measure:

**(H)G** For some $K_0, K_1 > 0$ and all $s \leq t$, $x, x' \in \mathbb{R}^d$, $u, u' \in \mathbb{R}^k$,

$$|G_s(0, 0)| \leq K_0, \quad |G_s(x, u) - G_s(x', u')| \leq K_1(|x - x'| + |u - u'|).$$

**(H)β** For some $\beta > 0$,

$$\int_{|z| \geq 1} |z|^\beta \nu(dz) < +\infty.$$

Remark 2.4. Condition (H)β, which is a restriction on the big jump of the Lévy process, is equivalent to say that the $\beta$-order moment of Lévy process is finite (cf. [13 Theorem 25.3]). It should be noticed that for $\alpha$-stable process, condition (H)β is satisfied only for any $\beta < \alpha$.

Now we prove the following result about the existence and uniqueness of solutions for equation (16) in a short time.

**Theorem 2.5.** Assume that (H)G and (H)β hold for some $\beta > 1$, and $\phi$ is a Lipschitz continuous functional on $\mathbb{D}$ (see [17]). Then there exists a time $T = T(K_1, 2, \beta, \|\phi\|_{Lip}) < 0$ such that equation (16) is uniquely solvable on $[T, 0]$ for any $L^\beta$-integrable initial value, and for some $C = C(T, K_0)$ and any $t \in [T, 0]$, $\xi \in \mathcal{F}_t$,

$$\mathbb{E} \left( \sup_{s \in [t, 0]} |X_{t,s}(\xi)|^\beta \right) \leq C\mathbb{E} |\xi|^\beta. \quad (18)$$

Moreover, if $\xi = x \in \mathbb{R}^d$ is non-random, then for any $t \in [T, 0)$, the unique solution $X_{t,s}$ is $\mathcal{F}_{t,s}$-measurable for all $s \in [t, 0]$.

**Proof.** We prove the theorem for $\beta \in (1, 2)$. For $\beta \geq 2$, the proof is similar and simpler. Fix $t > 0$, which will be determined below. For $\xi \in L^\beta(\Omega, \mathcal{F}_t, P)$, set $X_{t,s}^{(0)} \equiv \xi$ and let $X_{t,s}^{(n)}$ be the Picard iteration sequence defined by the following SDE with random coefficients:

$$X_{t,s}^{(n)} = \xi + \int_{(t,s]} G_s(X_{t,s}^{(n-1)}), \mathbb{E}^{\mathcal{F}_t} (\phi_t(X_{t,s}^{(n-1)})))dL_r,$$

which is uniquely solvable by the classical result (cf. [11 Page 249, Theorem 6]).

Set

$$Z_{t,s}^{(n)} := X_{t,s}^{(n+1)} - X_{t,s}^{(n)},$$

Using Lévy-Itô’s decomposition (14), one can write

$$Z_{t,s}^{(n)} = \int_{(t,s]} G_r^{(n)} \cdot zN(dr, dz) + \int_{(t,s]} G_r^{(n)} \cdot zN(dr, dz)$$

$$+ \int_{(t,s]} G_r^{(n)} \cdot bdr + \int_{(t,s]} G_r^{(n)} dW^A_r$$

$$=: I_1^{(n)}(s) + I_2^{(n)}(s) + I_3^{(n)}(s) + I_4^{(n)}(s).$$

where

$$G_r^{(n)} := G_r(X_{t,r}^{(n-1)}, \mathbb{E}^{\mathcal{F}_r} (\phi_r(X_t^{(n-1)}))) - G_r(X_{t,r}^{(n-1)}, \mathbb{E}^{\mathcal{F}_r} (\phi_r(X_t^{(n-1)}))).$$

By Burkholder’s inequality (cf. [10 Theorem 23.12] and Young’s inequality, thanks to $\beta \in (1, 2)$, we have that for any $\epsilon \in (0, 1)$,

$$\mathbb{E} \left( \sup_{r \in [t, 0]} |I_1^{(n)}(r)|^\beta \right) \leq C \mathbb{E} \left( \int_{(t,0]} \int_{|z| < 1} |G_r^{(n)} \cdot z|^2 N(dr, dz) \right)^{\beta/2}.$$
\[
\begin{align*}
&\leq CE \left( \sup_{r \in [t, 0]} |G^{(n)}_r|^{2-\beta} \int_{|z|<1} |G^{(n)}_r|^{\beta} \cdot |z|^2 N(dr, dz) \right)^{\beta/2} \\
&\leq eE \left( \sup_{r \in [t, 0]} |G^{(n)}_r|^{\beta} \right) + CE \left( \int_{|z|<1} |G^{(n)}_r|^{\beta} \cdot |z|^2 N(dr, dz) \right) \\
&= eE \left( \sup_{r \in [t, 0]} |G^{(n)}_r|^{\beta} \right) + CE \left( \int_{|z|<1} |G^{(n)}_r|^{\beta} \cdot |z|^2 \nu(dz)dr \right) \\
&\leq \left( \epsilon + CE|t| \right) \left( \sup_{r \in [t, 0]} |G^{(n)}_r|^{\beta} \right). 
\end{align*}
\]

Here and below, the constant C or \( C_\epsilon \) is independent of \( t \) and \( n \). For \( I_2^{(n)}(s) \), by Itô’s formula, we have
\[
\mathbb{E} \left( \sup_{r \in [t, 0]} |I_2^{(n)}(r)|^\beta \right) \leq \mathbb{E} \left( \int_{|z|<1} \left| I_2^{(n)}(r) - G^{(n)}_r |z|^{\beta} - I_2^{(n)}(r) \right| |N(dr, dz) \right) \\
= \mathbb{E} \left( \int_{|z|<1} \left| I_2^{(n)}(r) - G^{(n)}_r |z|^{\beta} - I_2^{(n)}(r) \right| \nu(dz)dr \right) \\
\leq CE \left( \int_{|z|<1} |I_2^{(n)}(r)|^{\beta} dr \right) + C \left( \int_{|z|<1} |z|^{\beta} \nu(dz)dr \right) \mathbb{E} \left( \sup_{r \in [t, 0]} |G^{(n)}_r|^{\beta} \right),
\]
which then implies that by \( (H_\beta) \) and Gronwall’s inequality,
\[
\mathbb{E} \left( \sup_{r \in [t, 0]} |I_2^{(n)}(r)|^\beta \right) \leq C|t| \mathbb{E} \left( \sup_{r \in [t, 0]} |G^{(n)}_r|^{\beta} \right).
\]

Similarly, we have
\[
\mathbb{E} \left( \sup_{r \in [t, 0]} |I_3^{(n)}(r)|^\beta \right) \leq (|t| \cdot |b|)^{\beta} \mathbb{E} \left( \sup_{r \in [t, 0]} |G^{(n)}_r|^{\beta} \right),
\]
and for any \( \epsilon \in (0, 1) \),
\[
\mathbb{E} \left( \sup_{r \in [t, 0]} |I_4^{(n)}(r)|^\beta \right) \leq (\epsilon + C_\epsilon|t|) \mathbb{E} \left( \sup_{r \in [t, 0]} |G^{(n)}_r|^{\beta} \right).
\]
Combining the above calculations, we obtain that for any \( \epsilon \in (0, 1) \),
\[
\mathbb{E} \left( \sup_{r \in [t, 0]} |Z^{(n)}_{r, t}|^\beta \right) \leq (\epsilon + C_\epsilon|t|) \mathbb{E} \left( \sup_{r \in [t, 0]} |G^{(n)}_r|^{\beta} \right). \tag{20}
\]

Noticing that by \( (H_G) \),
\[
|G^{(n)}_r| \leq K_1 \left( |Z^{(n)}_{t, s}| + ||\phi||_{L_i} \mathbb{E} \left[ G^{(n)}_r \left( \sup_{s \in [t, 0]} |Z^{(n-1)}_{s, t}| \right) \right] \right),
\]
and in view of \( \beta > 1 \), we further have by Doob’s maximal inequality,
\[
\mathbb{E} \left( \sup_{r \in [t, 0]} |Z^{(n-1)}_{r, t}|^\beta \right) \leq (\epsilon + C_\epsilon|t|)C_0 \left[ \mathbb{E} \left( \sup_{r \in [t, 0]} |Z^{(n)}_{r, t}|^\beta \right) + \mathbb{E} \left( \sup_{r \in [t, 0]} |Z^{(n-1)}_{r, t}|^\beta \right) \right].
\]
Now, let us choose
\[
\epsilon = \frac{1}{8C_0} \text{ and } T := \frac{1}{8C_\epsilon C_0},
\]
then for all \( t \in [T, 0] \),
\[
\mathbb{E} \left( \sup_{r \in [t, 0]} |Z^{(n)}_{r, t}|^\beta \right) \leq \frac{1}{3} \mathbb{E} \left( \sup_{r \in [t, 0]} |Z^{(n-1)}_{r, t}|^\beta \right) \leq \cdots \leq \frac{1}{3^n} \mathbb{E} \left( \sup_{s \in [t, 0]} |Z^{(0)}_{s, t}|^\beta \right). \tag{21}
\]
On the other hand, notice that

\[ Z_{t,s}^{(0)} = X_{t,s}^{(1)} - \xi = \int_{(t,s]} G_r(X_{t,r-}, \mathbb{E}^\mathcal{F}_r(\phi_r(\xi))) dL_r. \]

As above and using Gronwall’s inequality, it is easy to derive that

\[ \mathbb{E} \left( \sup_{s \in [t,0]} |Z_{t,s}^{(0)}|^\beta \right) \leq C\mathbb{E}|\xi|^\beta. \]  

(22)

Hence, there exists an \((\mathcal{F}_s)\)-adapted and càdlàg stochastic process \(X_{t,s}\) such that

\[ \lim_{n \to \infty} \mathbb{E} \left( \sup_{s \in [t,0]} |X_{t,s}^{(n)} - X_{t,s}|^\beta \right) = 0. \]  

(23)

By taking limits for equation (19), it is easy to see that \(X_{t,s}\) solves SFDE (16). Moreover, estimate (18) follows from (21), (22) and (23). The uniqueness is clear from the above proof.

Suppose now that \(\xi = x\) is non-random. From Picard’s iteration (19), one sees that for each \(n \in \mathbb{N}\) and \(s \in (t,0]\), \(X_{t,s}^{(n)}\) is \(\mathcal{F}_{t,s}\)-measurable. Indeed, suppose that \(X_{t,s}^{(n-1)}\) is \(\mathcal{F}_{t,s}\)-measurable for each \(s \in (t,0]\), then it is clear that \(\phi_r(X_{t,s}^{(n-1)})\) is independent of \(\mathcal{F}_r\). Noticing that for \(r > t\), \(\mathcal{F}_r = \mathcal{F}_{t,r} \vee \mathcal{F}_t\) and \(\mathcal{F}_{t,r}\) is independent of \(\mathcal{F}_t\), we have

\[ \mathbb{E}^{\mathcal{F}_r}(\phi_r(X_{t,s}^{(n-1)})) = \mathbb{E}^{\mathcal{F}_{t,r}}(\phi_r(X_{t,s}^{(n-1)})). \]

By induction method, starting from equation (19) with \(\xi = x\), one finds that \(X_{t,s}^{(n)}\) is also \(\mathcal{F}_{t,s}\)-measurable for each \(s \in (t,0]\). So, the limit \(X_{t,s}\) is also \(\mathcal{F}_{t,s}\)-measurable. \(\square\)

**Remark 2.6.** In this theorem, if \(G\) does not depend on \(u\), then the short time solution can be extended to any large time by the usual time shift technique.

2.3. A special case. In Theorem 2.5 since we require \(\beta > 1\), the result rules out the \(\alpha\)-stable process with \(\alpha \in (0,1)\). In this subsection, we drop assumption \((H_2')\) in Theorem 2.5 and consider the following special form:

\[ X_{t,s} = \xi + \int_{(t,s]} \int_{|z| < 1} G_r(X_{t,r-}, \mathbb{E}^{\mathcal{F}_r}(\phi_r(X_{t,r}))) \cdot zN(dr, dz) + \int_{(t,s]} \int_{|z| > 1} zN(dr, dz) \]

\[ + \int_{(t,s]} G_r(X_{t,r-}, \mathbb{E}^{\mathcal{F}_r}(\phi_r(X_{t,r}))) \cdot bdr + \int_{(t,s]} G_r(X_{t,r-}, \mathbb{E}^{\mathcal{F}_r}(\phi_r(X_{t,r}))) dW^A_r, \]  

(24)

where \(\xi \in \mathcal{F}_r\). In this equation, the big jump part has a constant coefficient. In order to make sense for the integrals, we need to assume that \(G\) and \(\phi\) are bounded. We have:

**Theorem 2.7.** In addition to \((H_2')\), we assume that \(G\) is bounded, and \(\phi\) is a bounded Lipschitz continuous functional on \(\mathcal{D}\). Then there exists a time \(T = T(K_1, \mathcal{A}, \|\phi\|_{\text{Lip}}) < 0\) such that SFDE (24) is uniquely solvable on \([T,0]\). Moreover, if \(\xi = x \in \mathbb{R}^d\) is non-random, then for any \(t \in [T,0]\), the unique solution \(X_{t,s}\) is \(\mathcal{F}_{t,s}\)-measurable for all \(s \in [t,0]\).

**Proof.** Since we do not assume any integrability on the Lévy process, the proof of Theorem 2.5 has to be carefully rewritten.

For \(t < 0\) and \(\xi \in \mathcal{F}_r\), set \(X_{t,s}^{(0)} \equiv \xi\) and let \(X_{t,s}^{(n)}\) be the Picard iteration sequence defined by the following SDE with random coefficients:

\[ X_{t,s}^{(n)} = \xi + \int_{(t,s]} \int_{|z| < 1} zN(dr, dz) + \int_{(t,s]} \int_{|z| > 1} G_r(X_{t,r-}, \mathbb{E}^{\mathcal{F}_r}(\phi_r(X_{t,r-}))) \cdot zN(dr, dz) \]

\[ + \int_{(t,s]} G_r(X_{t,r-}, \mathbb{E}^{\mathcal{F}_r}(\phi_r(X_{t,r-}))) \cdot bdr + \int_{(t,s]} G_r(X_{t,r-}, \mathbb{E}^{\mathcal{F}_r}(\phi_r(X_{t,r-}))) dW^A_r, \]  

(25)
which is uniquely solvable by the classical result (cf. [11, Page 249, Theorem 6]).

Set

\[ Z_{t,s}^{(n)} := X_{t,s}^{(n+1)} - X_{t,s}^{(n)} \,.
\]

Then,

\[
Z_{t,s}^{(n)} = \int_{(t,s)} \mathcal{G}_r^{(n)} \cdot z\tilde{N}(dr, dz) + \int_{(t,s)} \mathcal{G}_r^{(n)} \cdot bdr + \int_{(t,s)} \mathcal{G}_r^{(n)} dW_r
\]

\[ =: I_1^{(n)}(s) + I_2^{(n)}(s) + I_3^{(n)}(s),
\]

where

\[
\mathcal{G}_r^{(n)} := G_r(X_{t,r}^{(n+1)} - E \mathbb{E}_{r-r} - (\phi_r(X_{t,r}^{(n)}))).
\]

Since it is prior not known whether \( Z_{t,s}^{(n)} \) is integrable, we have to use the stopping time technique. For \( R > 0 \), define

\[
\tau_R^n := \inf \{s \in [t,0] : |Z_{t,s}^{(n)}| \geq R\}.
\]

By Burkholder’s inequality, (Hc) and (12), we have

\[
\mathbb{E} \left( \sup_{s \in [t,\tau_R^n]} |I_1^{(n)}(s)|^2 \right) \leq C \mathbb{E} \left( \int_{(t,\tau_R^n]} \int_{[0,1]} |\mathcal{G}_r^{(n)} \cdot z|^2 N(dr, dz) \right)
\]

\[ \leq C \mathbb{E} \left( \int_{(t,\tau_R^n]} \int_{[0,1]} (|Z_{t,r}^{(n)}|^2 + \Phi_r^{(n)}) \cdot |z|^2 \nu(dz)dr \right)
\]

\[ \leq C |t| \mathbb{E} \left( \sup_{s \in [t,\tau_R^n]} |Z_{t,r}^{(n)}|^2 \right) + C |t| \mathbb{E} \left( \sup_{s \in [t,0]} \Phi_r^{(n)} \right),
\]

where

\[
\Phi_r^{(n)} := \left( \mathbb{E}_{r-r} \left[ \phi_r(X_{t,r}^{(n)}) - \phi_r(X_{t,r}^{(n-1)}) \right] \right)^2.
\]

Here and below, the constant \( C \) is independent of \( t, R \) and \( n \). Similarly, we have

\[
\mathbb{E} \left( \sup_{s \in [t,\tau_R^n]} (|I_2^{(n)}(s)|^2 + |I_3^{(n)}(s)|^2) \right) \leq C |t| \mathbb{E} \left( \sup_{s \in [t,\tau_R^n]} |Z_{t,r}^{(n)}|^2 \right) + C |t| \mathbb{E} \left( \sup_{s \in [t,0]} \Phi_r^{(n)} \right).
\]

Combining the above calculations, we obtain

\[
\mathbb{E} \left( \sup_{s \in [t,\tau_R^n]} |Z_{t,s}^{(n)}|^2 \right) \leq C_1 |t| \mathbb{E} \left( \sup_{s \in [t,\tau_R^n]} |Z_{t,r}^{(n)}|^2 \right) + C_2 |t| \mathbb{E} \left( \sup_{s \in [t,0]} \Phi_r^{(n)} \right).
\]

Now, let us choose

\[
T := - \frac{1}{4(C_1 + 4C_2 \|\phi\|_{Lip}^2)},
\]

then for all \( t \in [T,0] \),

\[
\mathbb{E} \left( \sup_{s \in [t,\tau_R^n]} |Z_{t,r}^{(n)}|^2 \right) \leq \frac{C_2}{3(C_1 + 4C_2 \|\phi\|_{Lip}^2)} \mathbb{E} \left( \sup_{s \in [t,0]} \Phi_r^{(n)} \right).
\]

Since the right hand side is finite and independent of \( R \), by letting \( R \to \infty \), we obtain

\[
\lim_{R \to \infty} \tau_R^n = 0, \text{ a.s.,}
\]

and so, by Fatou’s lemma and Doob’s maximal inequality,

\[
\mathbb{E} \left( \sup_{s \in [t,0]} |Z_{t,r}^{(n)}|^2 \right) \leq \frac{1}{3} \mathbb{E} \left( \sup_{s \in [t,0]} |Z_{t,r}^{(n-1)}|^2 \right) \leq \cdots \leq \frac{1}{3^n} \mathbb{E} \left( \sup_{s \in [t,0]} |Z_{t,r}^{(0)}|^2 \right) \leq \frac{C}{3^n}.
\]
Hence, there exists an ($\mathcal{F}_t$)-adapted and cadlag stochastic process $X_{t,s}$ such that

$$\lim_{n \to \infty} \mathbb{E} \left( \sup_{t \leq r \leq s} |X_{n,t}^{s} - X_{t,s}|^2 \right) = 0.$$ 

By taking limits for equation (25), it is easy to see that $X_{t,s}$ solves SFDE (24). The rest proof is the same as in Theorem 2.5. \hfill \Box

2.4. Markov property. In this subsection, we prove the Markov property for the solutions of equations (16) and (24), which is crucial for the development of the next section.

We first show the continuous dependence of the solutions with respect to the initial values.

**Proposition 2.8.** In the situation of Theorem 2.5 for $t \in [T, 0]$, let $\xi^{(n)}, \xi \in L^\beta(\Omega, \mathcal{F}_t, \mathbb{P})$. If $\xi^{(n)}$ converges to $\xi$ in probability as $n \to \infty$, then $X_{t,s}^{(n)}$ converges to $X_{t,s}$ uniformly with respect to $s \in [t, 0]$ in probability as $n \to \infty$, where $\{X_{t,s}^{(n)}, t \leq s \leq 0\}$ and $\{X_{t,s}, t \leq s \leq 0\}$ are the solutions of SFDE (16) corresponding to the initial values $\xi^{(n)}$ and $\xi$.

**Proof.** Define

$$A_n := \{ |\xi^{(n)} - \xi| \leq 1 \} \in \mathcal{F}_t.$$ 

Then we can write

$$1_{A_n}(X_{t,s}^{(n)} - X_{t,s}) = 1_{A_n}(\xi^{(n)} - \xi) + \int_{(t,s]} \int_{|z| < 1} 1_{A_n} G^{(n)}_t \cdot \tilde{Z}(dr, dz)$$

$$+ \int_{(t,s]} \int_{|z| \geq 1} 1_{A_n} G^{(n)}_t \cdot \tilde{Z}(dr, dz)$$

$$+ \int_{(t,s]} 1_{A_n} G^{(n)}_t \cdot \mathbb{B} + \int_{(t,s]} 1_{A_n} G^{(n)}_t \mathbb{D}W^A_t,$$

where

$$G^{(n)}_t := G_t(X_{t,s}^{(n)}, \mathbb{E}^{\mathcal{F}_t -} (\phi_s(X_{t,s}^{(n)}))) - G_t(X_{t,s}, \mathbb{E}^{\mathcal{F}_t -} (\phi_s(X_{t,s}))).$$

As in estimating (21), we can prove that for all $t \in [T, 0]$,

$$\mathbb{E} \left( 1_{A_n} \cdot \sup_{s \in [t, 0]} |X_{t,s}^{(n)} - X_{t,s}|^\beta \right) \leq C \mathbb{E}(1_{A_n} \cdot |\xi^{(n)} - \xi|^\beta),$$

(26)

where $C$ is independent of $n$.

Now, for any $\varepsilon > 0$, we have

$$P \left( \sup_{s \in [t, 0]} |X_{t,s}^{(n)} - X_{t,s}| \geq \varepsilon \right) \leq P \left( 1_{A_n} \cdot \sup_{s \in [t, 0]} |X_{t,s}^{(n)} - X_{t,s}| \geq \varepsilon \right) + P(A_n^c)$$

$$\leq \frac{1}{\varepsilon^\beta} \mathbb{E} \left( 1_{A_n} \cdot \sup_{s \in [t, 0]} |X_{t,s}^{(n)} - X_{t,s}|^\beta \right) + P(A_n^c)$$

$$\leq C \frac{\varepsilon^\beta}{\varepsilon^\beta} \mathbb{E}(1_{A_n} \cdot |\xi^{(n)} - \xi|^\beta) + P(A_n^c).$$

The proof is then complete by letting $n \to \infty$. \hfill \Box

**Remark 2.9.** In the situation of Theorem 2.7, the conclusion of this proposition still holds, which can be proven by the same procedure.

The following lemma is a direct consequence of the uniqueness of solutions.
Lemma 2.10. Suppose that SFDE (16) is uniquely solvable on the time interval \((T, 0]\). Then for all \(T < t_1 < t_2 < t_3 \leq 0\) and \(\xi \in F_{t_1}\), we have
\[
X_{t_2,t_3}(X_{t_1,t_2}(\xi)) = X_{t_1,t_3}(\xi), \quad a.s.
\]
(27)
Moreover, for any \(T < t < s \leq 0\), \(x_i \in \mathbb{R}^d, i = 1, \cdots, n\) and disjoint \(\Lambda_i \in F_{t}, i = 1, \cdots, n\) with \(\cup_i \Lambda_i = \Omega\),
\[
X_{t,s}(\sum_i 1_{\Lambda_i} \xi_i) = \sum_i 1_{\Lambda_i} X_{t,s}(\xi_i), \quad a.s.
\]
(28)
Proof. For \(T < t_1 < t_2 < s \leq 0\), we can write
\[
X_{t_1,s} = X_{t_1,t_2}(\xi) + \int_{[t_2,s]} G_r(X_{t_1,r}, \mathbb{E}^{F_r}(\phi(X_{t_1}))) dL_r, \quad a.s.
\]
On the other hand, if we set
\[
Y_s := X_{t_2,s}(X_{t_1,t_2}(\xi)), \quad \forall s \in [t_2, 0],
\]
then \(Y_s\) satisfies
\[
Y_s = X_{t_1,t_2}(\xi) + \int_{[t_2,s]} G_r(Y_{r-}, \mathbb{E}^{F_r}(\phi(Y))) dL_r, \quad a.s.
\]
Equation (27) follows by the uniqueness.
As for (28), noticing that for all \(r \in (t, 0]\),
\[
\sum_i 1_{\Lambda_i} G_r(X_{t_1,r}(x_i), \mathbb{E}^{F_r}(\phi_r(X_{t_1}(x_i))))
= \sum_i G_r(1_{\Lambda_i} X_{t_1,r}(x_i), 1_{\Lambda_i} \mathbb{E}^{F_r}(\phi_r(X_{t_1}(x_i))))
= G_r\left(\sum_i 1_{\Lambda_i} X_{t_1,r}(x_i), \mathbb{E}^{F_r}(\phi_r(\sum_i 1_{\Lambda_i} X_{t_1}(x_i)))\right),
\]
it follows by the uniqueness as above. \(\Box\)

Now we can prove the following Markov property.

Proposition 2.11. In the situation of Theorem 2.5 or Theorem 2.7 let \(\{X_{t,s}(x); T \leq t < s \leq 0\}\) be the solution family of SFDE (16) or (24). Then for any \(T \leq t_1 < t_2 < t_3 \leq 0\), \(x \in \mathbb{R}^d\), and bounded continuous function \(\varphi\), we have
\[
\mathbb{E}^{\mathcal{F}_{t_3}}(\varphi(X_{t_1,t_3}(x))) = \mathbb{E}(\varphi(X_{t_2,t_3}(y)))|_{y=X_{t_1,t_3}(x)} \quad a.s.
\]
(29)
Proof. We only prove (29) in the case of Theorem 2.5. By Proposition 2.8, the mapping \(y \mapsto \mathbb{E}(\varphi(X_{t_2,t_3}(y))) := \Phi(y)\) is continuous. So, \(\Phi(X_{t_2,t_3}(x))\) is \(\mathcal{F}_{t_2}\)-measurable. Thus, for proving (29), it only needs to prove that for any \(\Lambda \in \mathcal{F}_{t_2}\),
\[
\mathbb{E}(1_{\Lambda} \varphi(X_{t_1,t_3}(x))) = \mathbb{E}(1_{\Lambda} \Phi(X_{t_1,t_3}(x))).
\]
Let \(\xi^{(n)} = \sum_i x_i 1_{\Lambda_i}\) be a sequence of simple functions, where \(x_i \in \mathbb{R}^d, \Lambda_i \in \mathcal{F}_{t_2}\) disjoint and \(\cup_i \Lambda_i = \Omega\), and such that
\[
\xi^{(n)} \to X_{t_1,t_2}(x) \quad \text{in } L^\beta \text{ as } n \to \infty.
\]
By Proposition 2.8 again, we have
\[
\mathbb{E}(1_{\Lambda} \varphi(X_{t_1,t_3}(x))) \overset{(27)}{=} \lim_{n \to \infty} \mathbb{E}(1_{\Lambda} \varphi(X_{t_2,t_3}(\xi^{(n)})))
= \lim_{n \to \infty} \mathbb{E}(1_{\Lambda} \varphi(X_{t_2,t_3}(\xi^{(n)})))
\]
Theorem 2.12. In this case, we have the following existence result of a unique maximal solution.

\[
\lim_{n \to \infty} \sum_{i} \mathbb{E}\left(1_{A} \cdot 1_{A, \varphi}(X_{t_{2}, t_{3}}(x_{i}))\right).
\]

Since \( X_{t_{2}, t_{3}}(x_{i}) \) is \( \mathcal{F}_{t_{2}, t_{3}} \)-measurable and independent of \( \mathcal{F}_{t_{2}} \), we further have

\[
\mathbb{E}\left(1_{A} \varphi(X_{t_{1}, t_{3}}(x))\right) = \lim_{n \to \infty} \sum_{i} \mathbb{E}\left(1_{A} \cdot 1_{A, \varphi}(x_{i})\right)
= \lim_{n \to \infty} \mathbb{E}\left(1_{A} \Phi(\xi^{(n)})\right) = \mathbb{E}\left(1_{A} \Phi(X_{t_{1}, t_{2}}(x))\right).
\]

The proof is complete.

2.5. Local maximal solutions. Now, suppose that \( \phi \) takes the following form:

\[
\phi_{s}(\omega) = \varphi(\omega(0)) + \int_{s}^{0} f_{r}(\omega(r))dr, \quad \omega \in \mathbb{D},
\]

(30)

where \( \varphi : \mathbb{R}^{d} \to \mathbb{R}^{k} \) and \( f : \mathbb{R}_{-} \times \mathbb{R}^{d} \to \mathbb{R}^{d} \) satisfy that for some \( K_{2} > 0 \) and all \( s \in \mathbb{R}_{-} \) and \( x, x' \in \mathbb{R}^{d} \),

\[
|\varphi(x) - \varphi(x')| + |f_{s}(x) - f_{s}(x')| \leq K_{2}|x - x'|.
\]

(31)

In this case, we have the following existence result of a unique maximal solution.

Theorem 2.12. Assume that (27), (H_{G}) and (H_{\beta}) hold for some \( \beta > 1 \). Then there exists a time \( T = T(K_{1}, K_{2}, \varphi', \beta) < \infty \) such that SFDE (16) is solvable on \( [T, 0] \) for any initial value \( x \in \mathbb{R}^{d} \), and

\[
\lim_{t \uparrow T} \|u_{t}\|_{Lip} := \sup_{x \neq x', x' \in \mathbb{R}^{d}} \frac{|u_{t}(x) - u_{t}(x')|}{|x - x'|} = +\infty,
\]

(32)

where

\[
u_{t}(x) := \mathbb{E}\left(\varphi(X_{t_{1}, 0}(x)) + \int_{t}^{0} f_{s}(X_{t_{1}, s}(x))ds\right).
\]

(33)

Moreover, the family of solutions \{\( X_{t_{1}, s}(x), T < t < s \leq 0, x \in \mathbb{R}^{d} \)\} is unique in the class that for all \( T < t_{1} < t_{2} < t_{3} \leq 0 \) and \( x \in \mathbb{R}^{d} \),

\[
X_{t_{1}, t_{3}}(x) \in L^{\beta}(\Omega, \mathcal{F}_{t_{1}, t_{3}}, P), \quad X_{t_{1}, t_{3}}(x) = X_{t_{2}, t_{3}}(X_{t_{1}, t_{2}}(x)) \ a.s.
\]

We also have the following uniform estimate: for any \( T' \in (T, 0) \) and \( x \in \mathbb{R}^{d} \),

\[
\sup_{t \in [T', 0]} \mathbb{E}\left(\sup_{s \in [t, 0]} |X_{t, s}(x)|^{\beta}\right) \leq C_{x}.
\]

(34)

Proof. First of all, let \( T_{1} \) be the existence time in Theorem 2.5. By (26), there exists a constant \( C = C(K_{1}, K_{2}, \varphi', \beta) > 0 \) such that for all \( x, x' \in \mathbb{R}^{d} \) and \( t \in [T_{1}, 0] \),

\[
\mathbb{E}\left(\sup_{s \in [t, 0]} |X_{t, s}(x) - X_{t, s}(x')|^{\beta}\right) \leq C|x - x'|^{\beta}.
\]

Using this estimate and (31), it is easy to check that

\[
\|u_{T_{1}}\|_{Lip} < +\infty.
\]

Next, we consider the following SFDE on \([T, T_{1}]\),

\[
X_{t, s}(x) = x + \int_{(t, s)} G_{r}(X_{t, r}(x), \mathbb{E}^{\mathcal{F}_{r}}(u_{T_{1}}(X_{t, T_{1}}(x)) + \int_{r}^{T_{1}} f_{r}(X_{t, r}(x))dr'))dL_{r}.
\]
Repeating the proof of Theorem 2.5, one can find another $T_2 < T_1$ so that this SFDE is uniquely solvable on $[T_2, T_1]$. Meanwhile, one can patch up the solution by setting

$$X_{t,s}(x) := X_{T_1,s}(X_{t,T_1}(x)), \quad \forall s \in [T_1, 0], t \in [T_2, T_1].$$

It is easy to verify that $\{X_{t,s}(x), T_2 \leq t < s \leq 0, x \in \mathbb{R}^d\}$ solves SFDE (16) on $[T_2, 0]$. Proceeding this construction, we obtain a sequence of times

$$0 > T_1 > T_2 > \cdots > T_n \downarrow T,$$

and a family of solutions

$$\{X_{t,s}(x), T < t < s \leq 0, x \in \mathbb{R}^d\}.$$ 

From the construction of $T$, one knows that (32) holds. As for the uniqueness, it can be proved piecewisely on each $[T_n, T_{n+1}]$. Estimate (34) follows from (18) and induction. \hfill \Box

**Remark 2.13.** By this theorem, for obtaining the global solution, it suffices to give an a priori estimate for $\|u_T\|_{L^p} = \|\nabla u_T\|_{\infty}$.

The following result can be proved similarly. We omit the details.

**Theorem 2.14.** In addition to (37) and (H_G), we assume that $G, \varphi$ and $f$ are uniformly bounded. Then there exists a time $T = T(K_1, K_2, \varphi') < 0$ such that SFDE (24) is solvable on $(T, 0)$ and estimate (32) holds. Moreover, the family of solutions $\{X_{t,s}(x), T < t < s \leq 0, x \in \mathbb{R}^d\}$ is unique in the class that for all $T < t_1 < t_2 < t_3 \leq 0$ and $x \in \mathbb{R}^d$,

$$X_{t_1,t_2}(x) \in \mathcal{F}_{t_1,t_2}, \quad X_{t_1,t_3}(x) = X_{t_2,t_3}(X_{t_1,t_2}(x)) \text{ a.s.}$$

### 3. Application to quasi-linear partial integro-differential equations

In this section, we establish the connection between stochastic functional differential equation and a class of quasi-linear partial integro-differential equations. For this aim, we consider $\varphi$ taking the form (30) and assume that for some $k \in \mathbb{N}$,

- (H_k) $G, f$ and $\varphi$ are continuous functions, and for any $j = 1, \cdots, k$, $\partial_j G_s(x, u), \partial_j f_s(x), \partial_j \varphi(x)$ are uniformly bounded continuous functions with respect to $s \in \mathbb{R}_-$, where $\partial_j$ denotes the $j$-th order gradient with respect to $x, u$. We also denote

$$\mathcal{K} := \sup_{s \in \mathbb{R}_-} \left( \|\nabla G_s\|_{\infty} + \|\nabla f_s\|_{\infty} \right) + \|\nabla \varphi\|_{\infty}. \quad (35)$$

Under this assumption, it is clear that (31) and (H_f) hold. Let $u_t(x)$ be defined by (33). By Theorem 2.12 the mapping $x \mapsto u_t(x)$ is Lipschitz continuous. However, it is in general not $C^2$-differentiable since we have poor integrabilities for $\nabla X_{t,s}(x)$. This difficulty is caused by the non-constancy of the big jump. We shall divide two cases to discuss this problem.

#### 3.1. Unbounded data and $\nu$ has finite moments of arbitrary orders

In this subsection, we consider equation (16), and assume that (H_k) holds for some $k \geq 3$, and (H'_\beta) holds for all $\beta \geq 2$.

In this case, we can write

$$L_t = \dot{b}t + W_t^A + \int_{\mathbb{R}^m} z\tilde{N}(t, dz),$$

where $\dot{b} = b + \int_{|z| > 1} z\nu(dz) \in \mathbb{R}^m$.

Let $T < 0$ be the maximal time given in Theorem 2.12 and $\{X_{t,s}(x), T < t < s \leq 0, x \in \mathbb{R}^d\}$ the solution family of equation (16). For simplicity of notation, below we shall write

$$G_{t,r} := G_{t,r}(x) := G_t\left(X_{t,r}(x), \mathbb{E}^{\mathcal{F}_r}\left(\varphi(X_{r,0}(x)) - \int_r^0 f_s'(X_{r,s}'(x))dr'\right)\right). \quad (36)$$
Let $g : \mathbb{R}^d \to \mathbb{R}^k$ be a $C^2$-function with bounded first and second order partial derivatives. By Itô’s formula (cf. [11] p.226, Theorem 4.4.7), we have

$$g(X_{t,s}) = g(x) + \int_{(t,s)} \int_{\mathbb{R}^m} \left[ g(X_{t,r} + \mathcal{G}_{t,r} \cdot z) - g(X_{t,r}) - \partial_i g(X_{t,r}) \mathcal{G}^{ij}_{t,r} z \right] \nu(dz)dr$$

$$+ \int_{(t,s)} \partial_i g(X_{t,r}) \mathcal{G}^{ij}_{t,r} \hat{b}_j dr + \frac{1}{2} \int_{(t,s)} \partial_i \partial_j g(X_{t,r})(\mathcal{G}^i_{t,r} A \mathcal{G}^j_{t,r}) d\mathcal{M}^2_{t,s} ,$$

where

$$\mathcal{M}^2_{t,s} := \int_{(t,s)} \int_{\mathbb{R}^m} \left[ g(X_{t,r} + \mathcal{G}_{t,r} \cdot z) - g(X_{t,r}) \right] \tilde{N}(dr, dz) + \int_{(t,s)} \partial_i g(X_{t,r}) \mathcal{G}^{ij}_{t,r} d(W^A)^j$$

is a square integrable $(\mathcal{F}_t)$-martingale by (34). Here and below, the superscript “t” denotes the transpose of a matrix.

Fix $t \in (T, 0]$ and $h > 0$ so that $t - h \in (T, 0]$. By taking expectations for both sides of (37), we have

$$\frac{1}{h} [\mathbb{E} g(X_{t-h,t}) - g(x)] = I^1_1(h) + I^2_2(h) + I^3_3(h),$$

where

$$I^1_1(h) := \frac{1}{h} \mathbb{E} \left( \int_{j-h}^t \int_{\mathbb{R}^m} \left[ g(X_{t-h,r} + \mathcal{G}_{t-h,r} \cdot z) - g(X_{t-h,r}) - \partial_i g(X_{t-h,r}) \mathcal{G}^{ij}_{t-h,r} z \right] \nu(dz)dr \right),$$

$$I^2_2(h) := \frac{1}{h} \mathbb{E} \left( \int_{j-h}^t \partial_i g(X_{t,r}) \mathcal{G}^{ij}_{t-r} \hat{b}_j dr \right), \quad I^3_3(h) := \frac{1}{2h} \mathbb{E} \left( \int_{j-h}^t \partial_i \partial_j g(X_{t,r})(\mathcal{G}^i_{t,r} A \mathcal{G}^j_{t,r}) d\mathcal{M}^2_{t,s} \right).$$

We have

**Lemma 3.1.** As $h \downarrow 0$, it holds that

$$I^1_1(h) \to \int_{\mathbb{R}^m} \left[ g(x + \mathcal{G}_i(x) \cdot z) - g(x) - \partial_i g(x) \mathcal{G}^{ij}_i(x) \cdot z \right] \nu(dz)dr,$$

$$I^2_2(h) \to \partial_i g(x) \mathcal{G}^{ij}_i(x) \hat{b}_j, \quad I^3_3(h) \to \frac{1}{2} \partial_i \partial_j g(x)(\mathcal{G}^i_{t} A \mathcal{G}^j_{t}),(37),$$

where

$$\mathcal{G}_i(x) := \left( x, \mathbb{E}(\varphi(X_{0,t}(x)) + \int_0^t f_i(X_{t,s}(x))ds) \right).$$

**Proof.** We only prove the first limit, the others are analogous. By the change of variables, we can write

$$I^1_1(h) = \mathbb{E} \left( \int_{j-h}^t \int_{\mathbb{R}^m} \left[ g(X_{t-h,t-h} + \mathcal{G}_{t-h,t-h} \cdot z) - g(X_{t-h,t-h}) - \partial_i g(X_{t-h,t-h}) \mathcal{G}^{ij}_{t-h,t-h} z \right] \nu(dz)ds \right).$$

Notice that

$$X_{t-h,t-h}(x) - x = \int_{(t-h,t-h)} \int_{\mathbb{R}^m} \mathcal{G}_{t-h,r}(x) \cdot z \tilde{N}(dr, dz) + \int_{(t-h,t-h)} \mathcal{G}_{t-h,r}(x) \cdot \hat{b} dr$$

$$+ \int_{(t-h,t-h)} \mathcal{G}_{t-h,r}(x) dW^A_r =: J_1(h) + J_2(h) + J_3(h).$$

By the isometric property of stochastic integrals, we have

$$\mathbb{E} |J_1(h)|^2 = \mathbb{E} \left( \int_{t-h}^{t-h} \int_{\mathbb{R}^m} |\mathcal{G}_{t-h,r}(x) \cdot z|^2 \nu(dz)dr \right)$$
\[ |h| \mathbb{E} \left( \sup_{r \in I_{t-h,t-hs}} |G_{t-h,r}(x)|^2 \int_{\mathbb{R}^m} |z|^2 \nu(dz) \right) \]
\[ \leq C|h| \mathbb{E} \left( \sup_{r \in [t-h,0]} |X_{t-r}(x)|^2 \right) \rightarrow 0 \text{ as } h \downarrow 0. \]

Similarly,
\[ \mathbb{E}|J_2(h)|^2 + \mathbb{E}|J_3(h)|^2 \rightarrow 0 \text{ as } h \downarrow 0. \]

Hence, for fixed \( t, s, x \),
\[ \lim_{h \downarrow 0} \mathbb{E}|X_{t-h,t-hs}(x) - x|^2 = 0. \tag{38} \]

Notice that
\[ \mathbb{E}\left| g(X_{t-h,t-hs} + G_{t-h,t-hs} \cdot z) - g(X_{t-h,t-hs}) - \partial_i g(X_{t-h,t-hs}) G_{t-h,t-hs}^{ij} z_j \right| \]
\[ = \mathbb{E}\left| \left( \int_0^1 [\partial_i g(X_{t-h,t-hs} + \theta G_{t-h,t-hs} \cdot z) - \partial_i g(X_{t-h,t-hs})] d\theta \right) G_{t-h,t-hs}^{ij} z_j \right| \]
\[ \leq C\mathbb{E}|G_{t-h,t-hs}^{ij} z_j|^2 \leq C\mathbb{E} \left( \sup_{r \in [t-h,0]} |X_{t-r}(x)|^2 \right) |z|^2 \leq C|z|^2. \]

Thus, for proving the first limit, by the dominated convergence theorem, it suffices to prove that for fixed \( s \in [0, 1] \) and \( z \in \mathbb{R}^m \),
\[ \mathbb{E}\left( \int_0^1 [\partial_i g(x + \theta G_{t-h} \cdot z) - \partial_i g(x)] d\theta \right) G_{t-h}^{ij} z_j \text{ as } h \downarrow 0. \]

By (38) and Remark 2.1, this limit is easily obtained. \( \square \)

We also need the following differentiability of the solution \( X_{t,s}(x) \) with respect to \( x \) in the \( L^p \)-sense.

**Lemma 3.2.** For any \( p \geq 2 \), there exists a time \( T_* = T_*(p, k, \mathcal{A}, \mathcal{X}) \in (T, 0) \), where \( \mathcal{A} \) is defined by (13), and \( \mathcal{X} \) is defined by (35), such that for any \( T_* \leq t \leq s \leq 0 \), the mapping \( x \mapsto X_{t,s}(x) \) is \( C^{k-1} \)-differentiable in the \( L^p \)-sense and for any \( j = 1, \ldots, k - 1 \),
\[ \sup_{x \in \mathbb{R}^d} \sup_{x \in [t,0]} \mathbb{E} |\nabla_j X_{t,s}(x)|^p < +\infty. \]

**Proof.** Since the proof is standard, we sketch it. Let \( \{e_i, i = 1, \ldots, d\} \) be the canonical basis of \( \mathbb{R}^d \). For \( \delta > 0 \) and \( i = 1, \ldots, d \), define
\[ X_{t,s}^{\delta,i} := X_{t,s}^{\delta,i}(x) = \frac{X_{t,s}(x + \delta e_i) - X_{t,s}(x)}{\delta} \]
and
\[ G_{t,s}^{\delta,i} := G_{t,s}^{\delta,i}(x) = \frac{G_{t,s}(x + \delta e_i) - G_{t,s}(x)}{\delta}, \]
where \( G_{t,s}(x) \) is defined by (36). Then,
\[ X_{t,s}^{\delta,i} = e_i + \int_{(t,s]} \int_{\mathbb{R}^n} G_{t,r}^{\delta,i} \cdot \tilde{N}(dr, dz) + \int_{(t,s]} G_{t,r}^{\delta,i} \cdot b dr + \int_{(t,s]} G_{t,r}^{\delta,i} dW_r. \tag{39} \]

As in estimating (20), by Burkholder’s inequality, we have that for any \( p \geq 2 \),
\[ \mathbb{E} \left( \sup_{r \in [t,0]} |X_{t,r}^{\delta,i}|^p \right) \leq C_{p,\mathcal{A}} |t| \mathbb{E} \left( \sup_{r \in [t,0]} |G_{t,r}^{\delta,i}|^p \right). \tag{40} \]
Moreover, by \((H_k)\), we easily derive that
\[
\mathbb{E}
\left(
\sup_{t \in \mathbb{R}^d} \left| \mathcal{G}_{t, t}^{\delta, i} \right|^p
\right)
\leq C_{p, \delta, \mathcal{X}} \mathbb{E}
\left(
\sup_{t \in \mathbb{R}^d} \left| X_{t, t}^{\delta, i} \right|^p
\right).
\]
Substituting this into (40), we find that for some \(C_{p, \mathcal{A}, \mathcal{X}} > 0\) independent of \(x, t\) and \(\delta\),
\[
\mathbb{E}
\left(
\sup_{t \in \mathbb{R}^d} \left| X_{t, t}^{\delta, i} \right|^p
\right) \leq C_{p, \mathcal{A}, \mathcal{X}} \mathbb{E}
\left(
\sup_{t \in \mathbb{R}^d} \left| X_{t, t}^{\delta, i} \right|^p
\right).
\]
From this, we deduce that there exists a time \(T_* = T_\epsilon(p, \mathcal{A}, \mathcal{X}) \in (T, 0)\) such that for all \(t \in [T_*, 0]\),
\[
\sup_{\delta \in (0, 1)} \mathbb{E}
\left(
\sup_{t \in \mathbb{R}^d} \left| X_{t, t}^{\delta, i} \right|^p
\right) < +\infty. \tag{41}
\]
On the other hand, let \(Y_{t, t}^{i, j} = Y_{t, t}^j(x)\) satisfy the following SFDE:
\[
Y_{t, t}^{i, j} = e_i + \int_t^\infty \nabla_i G_r \left(X_{t, r}(x), \mathbb{E}^\mathcal{P}_t \left( \Phi(X_{t, 0}(x)) - \int_r^0 f_r(X_{t, r}^j(x))dr \right) \right) Y_{t, r}^{j, i} \ dL_r
\]
\[
+ \int_t^\infty \nabla_i G_r \left(X_{t, r}, \mathbb{E}^\mathcal{P}_t \left( \Phi_r(X_{t, r}) \right) \right) \cdot \nabla \Phi(X_{t, 0}) Y_{t, r}^{j, i} \ + \int_r^\infty \nabla \Phi_r(X_{t, r}) Y_{t, r}^{j, i} \ dL_r \tag{42}
\]
which can be solved on \([T_*, 0]\) as in Theorem 2.5. Using the uniform estimate (41), it is standard to deduce that
\[
\lim_{\delta \to 0} \mathbb{E}
\left(
\sup_{t \in \mathbb{R}^d} \left| X_{t, t}^{\delta, i} \right|^p - Y_{t, t}^{i, j}(x) \right) = 0.
\]
In particular,
\[
\sup_{x \in \mathbb{R}^d} \mathbb{E}
\left(
\sup_{t \in \mathbb{R}^d} \left| Y_{t, t}^{i, j}(x) \right|^p
\right) < +\infty.
\]
The higher derivatives can be estimated similarly from (42). \(\square\)

Now we can prove the following result, which is originally due to [4, 14, 7].

**Theorem 3.3.** Assume that \((H_k)\) holds for some \(k \geq 3\), and \((H^\beta)\) holds for all \(\beta \geq 2\). Let \(\{X_{t, s}(x), T < t \leq s \leq 0, x \in \mathbb{R}^d\}\) be the maximal solution of SFDE (24) in Theorem 2.12 and \(u_t(x)\) be defined by
\[
\begin{align*}
&u_t(x) := \mathbb{E} \Phi(X_{t, 0}(x)) + \mathbb{E} \left( \int_t^0 f_r(X_{t, s}(x))ds \right). \tag{43}
\end{align*}
\]
Then there exists a time \(T_* = T_\epsilon(k, \mathcal{A}, \mathcal{X}) \in (T, 0)\) such that for each \(t \in [T_*, 0]\), \(x \mapsto u_t(x)\) has bounded derivatives up to \((k - 1)\)-order, and solves the following quasi-linear partial integro-differential equation:
\[
\begin{align*}
u_t(x) & = \varphi(x) + \int_t^\infty \left[ \mathcal{L}^c u_t(x) + \mathcal{L}^d u_t(x) + f_t(x) \right]ds, \forall (t, x) \in [T_*, 0] \times \mathbb{R}^d, \\
 \text{where} \\
\mathcal{L}^c u_t(x) & := \partial_i u_t(x) G^{ij}_t(x, u_t(x)) \hat{b}_j + \frac{1}{2} \partial_i \partial_j u_t(x) (G^{T}_t(x, u_t(x))AG_t(x, u_t(x)))^{ij} \\
\text{and} \\
\mathcal{L}^d u_t(x) & := \int_{\mathbb{R}^d} \left[ u_t(x + G_t(x, u_t(x)) \cdot z) - u_t(x) - \partial_i u_t(x) G_t^{ij}(x, u_t(x)) \cdot z \right] \nu(\text{d}z) \text{d}r.
\end{align*}
\]
Proof. We follow the argument of Friedman [8]. By Proposition 2.11, for \( T < t - h < t \leq 0 \), we have
\[
    u_{t-h}(x) = \mathbb{E} \left[ \left. \mathbb{E} \varphi(X_{t,0}(y)) \right|_{y=X_{t-h}(x)} \right] + \mathbb{E} \left[ \left. \mathbb{E} \left( \int_t^0 f_s(X_{t,s}(y)) ds \right) \right|_{y=X_{t-h}(x)} \right]
\]
\[
    + \mathbb{E} \left( \int_t^{t-h} f_s(X_{t-h,s}(y)) ds \right)
\]
\[
    = \mathbb{E} u_t(X_{t,h,t}(x)) + \mathbb{E} \left( \int_{t-h}^t f_s(X_{t-h,s}(y)) ds \right).
\]
By Lemma 3.2, it is easy to see that there exists a time \( T_\ast = T_\ast(k, \mathcal{A}, \mathcal{X}) < 0 \) such that for each \( t \in [T_\ast, 0] \), \( u_t(x) \) has bounded first and second order continuous derivatives. Thus, we can invoke Lemma 3.1 to derive that
\[
\frac{1}{h}(u_{t-h}(x) - u_t(x)) = \frac{1}{h} \mathbb{E} u_t(X_{t-h,t}(x)) - u_t(x) + \frac{1}{h} \mathbb{E} \left( \int_{t-h}^t f_s(X_{t-h,s}(y)) ds \right)
\]
\[
\to \mathcal{L}^c u_t(x) + \mathcal{L}^d u_t(x) + f_t(x) \text{ as } h \to 0.
\]
On the other hand, from the above proof, it is also easy to see that for fixed \( x \in \mathbb{R}^d \), \( t \mapsto u_t(x) \) is Lipschitz continuous. Hence,
\[
    u_t(x) - \varphi(x) = - \int_t^0 \partial_i u_s(x) ds = \int_t^0 \left[ \mathcal{L}^c u_s(x) + \mathcal{L}^d u_s(x) + f_s(x) \right] ds.
\]
The proof is thus complete. \( \square \)

3.2. Bounded data and constant big jump. In this subsection we assume that \((H_k)\) holds for some \( k \geq 3 \), and \( G, \varphi \) and \( f \) are uniformly bounded and continuous functions. Consider the following SFDE:
\[
    X_{t,s}(x) = x + \int_{(t,s]} \int_{|z|<1} G_{t,r}(x) \cdot z \bar{N}(dr, dz) + \int_{(t,s]} \int_{|z| \geq 1} z N(dr, dz)
\]
\[
+ \int_{(t,s]} G_{t,r}(x) \cdot b dr + \int_{(t,s]} G_{t,r}(x) dW^A_r,
\]
where \( G_{t,r}(x) \) is defined by (36). In this case, Lemmas 3.1 and 3.2 still hold. We just want to mention that (38) should be replaced by
\[
    X_{t-h, t-h, s}(x) \to x \text{ in probability as } h \to 0,
\]
and (39) becomes
\[
    X_{t,s}^{d,i} = e_i + \int_{(t,s]} \int_{|z|<1} G_{t,r}^{d,i} \cdot z \bar{N}(dr, dz) + \int_{(t,s]} G_{t,r}^{d,i} \cdot b dr + \int_{(t,s]} G_{t,r}^{d,i} dW^A_r.
\]
Thus, the following result can be proved along the same lines as in Theorem 3.3. We omit the details.

Theorem 3.4. Assume that \((H_k)\) holds for some \( k \geq 3 \), and \( G, \varphi \) and \( f \) are uniformly bounded and continuous functions. Let \( \{X_{t,s}(x), T < t < s \leq 0, x \in \mathbb{R}^d\} \) be the short time solution of SFDE (2.4) in Theorem 2.14 and \( u_t(x) \) be defined by (2.2). Then there exists a time \( T_\ast = T_\ast(k, \mathcal{A}, \mathcal{X}) \in (T, 0) \) such that for each \( t \in [T_\ast, 0] \), \( x \mapsto u_t(x) \) has bounded derivatives up to \((k-1)\)-order, and solves the following quasi-linear partial integro-differential equation:
\[
    u_t(x) = \varphi(x) + \int_t^0 \left[ \mathcal{L}^c u_s(x) + \mathcal{L}^d u_s(x) + f_s(x) \right] ds, \forall (t, x) \in [T_\ast, 0] \times \mathbb{R}^d,
\]
where
\[ L^0 u_t(x) := \partial_t u_t(x)G_t^{ij}(x, u_t(x))b_j + \frac{1}{2} \partial_i \partial_j u_t(x)(G_t^i(x, u_t(x)))AG_t(x, u_t(x))^{ij} \]
and
\[ L^d u_t(x) := \int_{|z|1} \left[ u_t(x + G_t(x, u_t(x)) \cdot z) - u_t(x) - \partial_i u_t(x)G_t^{ij}(x, u_t(x)) \cdot z_j \right] \nu(dz) + \int_{|z|>1} \left[ u_t(x + z) - u_t(x) \right] \nu(dz). \]

4. **Semi-linear partial integro-differential equation: Existence and uniqueness of weak solutions**

In this section we consider the following semi-linear partial integro-differential equation:
\[ \partial_t u_t + L_0 u_t + G_i^j(x, u_t)\partial_j u_t + F_i(x, u_t) = 0, \quad u_0 = \varphi, \quad t \leq t_0, \tag{44} \]
where \( L_0 \) is the generator of Lévy process \( L \), given by (15), and
\[ G \in \mathcal{B}(\mathbb{R}^d; \mathbb{W}^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^d)), \ F \in \mathcal{B}(\mathbb{R}^d; \mathbb{W}^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^k)), \ \varphi \in \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}). \tag{45} \]

Here and below, \( \mathbb{W}^{1,\infty} \) denotes the space of bounded and Lipschitz continuous functions, \( \mathcal{B} \) or \( \mathcal{B}_{\text{loc}} \) denotes the space of uniformly or locally bounded measurable functions.

Let us first give the following definition about the maximal weak solution for equation (44).

**Definition 4.1.** For \( T < 0 \), we call \( u \in \mathcal{B}_{\text{loc}}((T, 0]; \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k)) \) a maximal weak solution of equation (44) if
\[ \lim_{t \uparrow T} \| \nabla u_t(x) \|_\infty = +\infty, \]
and for all \( \psi \in C^0_b(\mathbb{R}^d; \mathbb{R}^k) \) and \( t \in (T, 0] \),
\[ \langle u_t, \psi \rangle = \langle \varphi, \psi \rangle + \int_0^t \langle u_s, L_0^* \psi \rangle \, ds + \int_0^t \left\langle G_i^j(u_s)\partial_j u_s + F_i(u_s), \psi \right\rangle \, ds, \tag{46} \]
where \( \langle \varphi, \psi \rangle := \int_{\mathbb{R}^d} \varphi(x) \cdot \psi(x) \, dx \), and \( L_0^* \) is the adjoint operator of \( L_0 \) and given by
\[ L_0^* \psi(x) := \frac{1}{2} a_{ij} \partial_i \partial_j \psi - b_j \partial_j \psi + \int_{\mathbb{R}^d} \left[ \psi(x - z) - \psi(x) + 1_{|z|<1} \partial_i \psi(x) z_i \right] \nu(dz). \]

The main aim of this section is to prove the following existence and uniqueness of a maximal weak solution as well as global solution for equation (44).

**Theorem 4.2.** (i) **(Local maximal weak solution)** Under (45), there exists a unique maximal weak solution \( u_t(x) \) for equation (44) in the sense of Definition 4.1. Moreover, let \( T \) be the maximal existence time, then for any \( t \in (T, 0] \),
\[ \| u_t \|_\infty \leq \| \varphi \|_\infty + |t| \sup_{x \in [T, 0]} \| F_i \|_\infty. \tag{47} \]

(ii) **(Non-negative solution)** If for some \( j = 1, \ldots, k \), the components \( \varphi^j \) and \( F^j \) are non-negative, then the corresponding component \( u^j \) of weak solution in (i) are also non-negative.

(iii) **(Global solution)** Let \( \Psi(\xi) \) be the Lévy symbol defined in (11) with \( b = A = 0 \). If for some \( \alpha \in (1, 2) \),
\[ \text{Re}(\Psi(\xi)) \asymp |\xi|^\alpha \text{ as } |\xi| \to \infty, \tag{48} \]
where \( a \asymp b \) means that for some \( c_1, c_2 > 0 \), \( c_1 b \leq a \leq c_2 b \), then the maximal existence time \( T \) in (i) equals to \(-\infty\). In the case that \( b = v = 0 \) and \( A \) is strictly positive, then \( T \) also equals to \(-\infty\).
Remark 4.3. Since we have estimate (47), it is easy to see that the assumption on \(G\) in (45) can be replaced by

\[G \in B(\mathbb{R}_-; \mathcal{W}^{1,\infty}(\mathbb{R}^d \times \mathbb{B}_R; \mathbb{R}^d)), \forall R > 0,\]

where \(\mathbb{B}_R := \{x \in \mathbb{R}^k : |x| \leq R\} \).

For proving this theorem, let us begin with studying

4.1. **Linear partial integro-differential equation.** In this subsection, we firstly study the existence and uniqueness of weak solutions for the following linear PIDE:

\[
\partial_t u_t + L_0 u_t + G_i'(x)\partial_i u_t + H_i(x)u_t + f_i(x) = 0, \quad u_0 = \varphi, \quad t \leq 0, \quad (49)
\]

where \(G : \mathbb{R}_- \times \mathbb{R}^d \to \mathbb{R}^d, H : \mathbb{R}_- \times \mathbb{R}^d \to \mathbb{R}^k \times \mathbb{R}^k, f : \mathbb{R}_- \times \mathbb{R}^d \to \mathbb{R}^k\) and \(\varphi : \mathbb{R}^d \to \mathbb{R}^k\) are bounded measurable functions.

Let us start with the following case of smooth coefficients (cf. [16]), which is the classical Feynman-Kac formula. Here, the main point is to prove the uniqueness.

**Theorem 4.4.** (Feynman-Kac formula). Assume that \(G \in B(\mathbb{R}_-; C^\infty_b(\mathbb{R}^d; \mathbb{R}^d)), H \in B(\mathbb{R}_-; C^\infty_b(\mathbb{R}^d; \mathbb{R}^k \times \mathbb{R}^k)), f \in B(\mathbb{R}_-; C^\infty_b(\mathbb{R}^d; \mathbb{R}^k)), \varphi \in C^\infty_b(\mathbb{R}^d; \mathbb{R}^k)\),

where \(C^\infty_b\) denotes the space of bounded smooth function with bounded derivatives of all orders. Let \(\{X_{t,s}(x), t \leq s \leq 0, x \in \mathbb{R}^d\}\) solve the following SDE:

\[X_{t,s}(x) = x + \int_t^s G_r(X_{t,r}(x))dr + \int_t^s dL_r, \]

and \(\{Z_{t,s}(x), t \leq s \leq 0, x \in \mathbb{R}^d\}\) solve the following ODE:

\[Z_{t,s}(x) = \mathbb{I}_{m \times m} + \int_t^s H_r(X_{t,r}(x)) : Z_{t,r}(x)dx. \]

Define

\[ u_t(x) := \mathbb{E}[Z_{t,0}(x)\varphi(X_{t,0}(x))] + \mathbb{E} \left[ \int_t^0 Z_{t,s}(x)f_t(X_{t,r}(x))ds \right]. \]

Then \(u \in C(\mathbb{R}_-; C^\infty_b(\mathbb{R}^d; \mathbb{R}^d))\) uniquely solves the following linear PIDE:

\[u_t(x) = \varphi(x) + \int_t^0 [L_0 u_t(x) + G_i'(x)\partial_i u_t(x) + H_i(x)u_t(x) + f_i(x)]ds, \quad \forall (t, x) \in \mathbb{R}_- \times \mathbb{R}^d. \]

**Proof.** By smoothing the time variable and then taking limits, as in Section 3, by careful calculations, one can find that \(u\) defined by (50) belongs to \(C(\mathbb{R}_-; C^\infty_b(\mathbb{R}^d; \mathbb{R}^d))\) and satisfies (51).

We now prove the uniqueness by the duality argument. Let \(\hat{X}_{t,s}(x)\) solve the following SDE:

\[\hat{X}_{t,s}(x) = x - \int_t^s G_r(\hat{X}_{t,r}(x))dr - \int_t^s dL_r, \]

and \(\hat{Z}_{t,s}(x)\) solve the following ODE:

\[\hat{Z}_{t,s}(x) = \mathbb{I}_{m \times m} + \int_t^s [H_r(\hat{X}_{t,r}(x))^t + \text{div}G_r(\hat{X}_{t,r}(x))\mathbb{I}_{m \times m}] : \hat{Z}_{t,r}(x)dx. \]

Fix \(T < 0\) and \(\psi \in C^\infty_0(\mathbb{R}^d; \mathbb{R}^k)\), define

\[\hat{u}_t(x) := \mathbb{E}[\hat{Z}_{T,0}(x)\psi(\hat{X}_{T,0}(x))]. \]
As above, one can check that
\[ \hat{u}_t(x) = \psi(x) + \int_0^t \left[ \mathcal{L}_0^s \hat{u}_s(x) - G^i_s(x) \partial^i_j \hat{u}_s(x) + H^j_s(x) \hat{u}_s(x) + \text{div} G^i_s(x) \hat{u}_s(x) \right] ds. \]
Let \( u \in C(\mathbb{R}_-; C^0_0(\mathbb{R}^d; \mathbb{R}^k)) \) satisfy equation (51) with \( \varphi = f = 0 \). Then by the integration by parts formula, we have for almost all \( t \in [T, 0) \),
\[ \partial_t \langle u_t, \hat{u}_t \rangle = - \left\langle \mathcal{L}_0 u_t + G^i_t \partial^i_j u_t + H^j_t u_t, \hat{u}_t \right\rangle + \left\langle u_t, \mathcal{L}_0^s \hat{u}_s - G^j_s \partial^j_i \hat{u}_s + H^j_s \hat{u}_s + \text{div} G^i_s \hat{u}_s \right\rangle = 0. \]
From this, we get
\[ \langle u_T, \psi \rangle = \langle u_T, \hat{u}_T \rangle = \langle u_0, \hat{u}_0 \rangle = 0, \]
which leads to \( u_T(x) = 0 \) by the arbitrariness of \( \psi \).

For \( \ell \in \mathbb{N} \), we introduce a family of mollifiers in \( \mathbb{R}^\ell \). Let \( \rho : \mathbb{R}^\ell \to [0, 1] \) be a smooth function satisfying that
\[ \rho(x) = 0, \forall |x| > 1, \int_{\mathbb{R}^\ell} \rho(x) dx = 1. \]
We shall call \( \rho_\varepsilon(x) := \varepsilon^{-\ell} \rho(x/\varepsilon), \varepsilon \in (0, 1) \) a family of mollifiers in \( \mathbb{R}^\ell \).

Next, we relax the regularity assumption on \( G, H, f \) and \( \varphi \), and prove that

**Theorem 4.5.** Assume that
\[ G \in \mathcal{B}(\mathbb{R}_-; \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)), \quad H \in \mathcal{B}(\mathbb{R}_-; \mathbb{W}^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^k)), \]
\[ f \in \mathcal{B}(\mathbb{R}_-; \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k)), \varphi \in \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k). \]
Let \( u_\varepsilon(x) \) be defined as in (50). Then \( u_\varepsilon(x) \in \mathcal{B}_{\text{loc}}(\mathbb{R}_-; \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k)) \) is a unique weak solution of equation (49) in the sense of Definition 4.1.

**Proof.** We only prove the uniqueness. As for the existence, it follows by smoothing the coefficients and then taking limits as done in Theorem 4.3 below.

Suppose that \( u \in \mathcal{B}_{\text{loc}}(\mathbb{R}_-; \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k)) \) is a weak solution of equation (49) with \( \varphi = f = 0 \) in the sense of Definition 4.1. We want to prove that \( u \equiv 0 \). Let \( \rho_\varepsilon \) be a family of mollifiers in \( \mathbb{R}^d \). Define
\[ u_\varepsilon^\ell(x) := u_\varepsilon \ast \rho_\varepsilon(x), \quad G_\varepsilon^i_t(x) := G_t \ast \rho_\varepsilon(x), \quad H_\varepsilon^j_t(x) := H_t \ast \rho_\varepsilon(x). \]
Taking \( \psi(x) = \rho_\varepsilon(x - \cdot) \) in (46), one finds that \( u_\varepsilon^\ell(x) \) satisfies
\[ u_\varepsilon^\ell(x) = \int_0^T \left[ \mathcal{L}_0 u_\varepsilon^\ell_s(x) + G_\varepsilon^{i,j}_s(x) \partial^i_j u_\varepsilon^\ell_s(x) + H_\varepsilon^j_s(x) u_\varepsilon^\ell_s(x) + f_\varepsilon^s(x) \right] ds, \]
where
\[ f_\varepsilon^s(x) = (G_\varepsilon^{i,j} \partial^i_j u_\varepsilon^\ell_s(x) - G_\varepsilon^{i,j} \partial^j_i u_\varepsilon^\ell_s(x)) + (H_s u_\varepsilon^\ell_s(x) - H_\varepsilon^j_s(x) u_\varepsilon^\ell_s(x)). \]
By the property of convolutions, we have
\[ \|f_\varepsilon^s\|_\infty \leq 2 \|G_s\|_\infty \|\nabla u\|_\infty + 2 \|H_s\|_\infty \|u\|_\infty \]
and for fixed \( s \) and Lebesgue almost all \( x \in \mathbb{R}^d \),
\[ f_\varepsilon^s(x) \to 0, \quad \varepsilon \to 0. \]
Let \( X_{t,s}^\varepsilon(x) \) solve the following SDE:
\[ X_{t,s}^\varepsilon(x) = x + \int_t^s G_r^e(X_{t,r}^\varepsilon(x)) dr + \int_t^r dL_r, \]
and \( Z_{t,s}^\varepsilon(x) \) solve the following ODE:
\[ Z_{t,s}^\varepsilon(x) = \mathbb{I}_{m \times m} + \int_t^s H_r^e(X_{t,r}^\varepsilon(x)) \cdot Z_{t,r}^\varepsilon(x) dx. \]
By Theorem 4.4, $u_t^\varepsilon(x)$ can be uniquely represented by

$$u_t^\varepsilon(x) := \mathbb{E} \left[ \int_{t}^{0} Z_{l,s}^\varepsilon(x)f_s^\varepsilon(X_{l,s}(x))ds \right].$$

For completing the proof, it suffices to prove that for each $(t, x) \in \mathbb{R} \times \mathbb{R}^d$,

$$u_t^\varepsilon(x) \to 0, \ \varepsilon \to 0.$$  

Since by (55), $Z_{l,s}^\varepsilon(x)$ is uniformly bounded with respect to $x, \varepsilon$ and $s \in [t, 0]$, we need only to show that for any nonnegative $\psi \in C_0^\alpha(\mathbb{R}^d)$,

$$I_t^\varepsilon := \mathbb{E} \left[ \int_{t}^{0} \int_{\mathbb{R}^d} |f_s^\varepsilon(X_{l,s}(x))\psi(x)dx| ds \right] \to 0, \ \varepsilon \to 0.$$  

For any $R > 0$, by (52), we have

$$I_t^\varepsilon \leq \mathbb{E} \left[ \int_{t}^{0} \int_{|X_{l,s}(x)| \leq R} |f_s^\varepsilon(X_{l,s}(x))\psi(x)dx| ds \right] + C_t \mathbb{E} \left[ \int_{t}^{0} \int_{|X_{l,s}(x)| > R} \psi(x)dxds \right]$$

$$\leq \mathbb{E} \left[ \int_{t}^{0} \int_{B_R} |f_s^\varepsilon(x)| \psi(x)dxds \right] \det(\nabla X_{l,s}^{\varepsilon,-1}(x)) + C_t \int_{t}^{0} \int_{|x| \leq R} P(|X_{l,s}^\varepsilon(x)| > R)\psi(x)dxds. \quad (56)$$

From equation (54), there exists at most one weak solution for equation (44).

Thus, for fixed $R > 0$, the first term in (56) is less than

$$C_0 \|\psi\|_\infty \int_{t}^{0} \int_{|x| \leq R} |f_s^\varepsilon(x)| dxds \to 0, \ \varepsilon \to 0.$$  

Moreover, by equation (54), we also have

$$\lim_{R \to \infty} \sup_{\varepsilon} P(|X_{l,s}^\varepsilon(x)| > R) \leq \lim_{R \to \infty} P \left\{ |x| + \int_{t}^{0} \|G_r\|_\infty dr + |L_s - L_t| > R \right\} = 0.$$  

The proof is complete by first letting $\varepsilon \to 0$ and then $R \to \infty$ in (56). \hfill \Box

As an easy corollary of this theorem, we firstly establish the uniqueness for equation (44).

**Theorem 4.6.** Under (45), there exists at most one weak solution for equation (44).

**Proof.** Let $u^{(i)} \in \mathcal{B}_{loc}((T, 0]; \mathbb{W}_0^{1, \infty}(\mathbb{R}^d; \mathbb{R}^k)), i = 1, 2$ be two weak solutions of equation (44) in the sense of Definition 4.1. Define $u_t(x) := u_t^{(1)}(x) - u_t^{(2)}(x)$. Then $u_t(x)$ satisfies that for all $\psi \in C_0^\alpha(\mathbb{R}^d; \mathbb{R}^k),$

$$\langle u_t, \psi \rangle = \int_{t}^{0} \langle u_s, \mathcal{L}_0^\varepsilon \psi \rangle dr + \int_{t}^{0} \left\langle G_s(u_s^{(1)})\partial_t u_s, \psi \right\rangle ds + \int_{t}^{0} \langle H_s u_t, \psi \rangle ds,$$

where

$$H_s(x) := \left( \int_{0}^{1} \nabla_s G_s(x, u_s^{(1)}(x) + \theta u_t(x))d\theta \right)\partial_t u_t^{(2)}(x) + \int_{0}^{1} \nabla_s F_s(x, u_s^{(1)}(x) + \theta u_t(x))d\theta.$$

By (45), it is easy to verify that

$$u_t^{(1)}(x), u_t^{(2)}(x) \in \mathcal{B}_{loc}((T, 0]; \mathbb{W}_0^{1, \infty}(\mathbb{R}^d; \mathbb{R}^d)).$$
functions, and for some $K$

**Proposition 4.7.** Suppose that $(G^{(i)}, f^{(i)}, \varphi^{(i)}), i = 1, 2$ are two groups of bounded measurable functions, and for some $K > 0$ and all $t \in \mathbb{R}_-$, $x, x' \in \mathbb{R}^d$ and $u, u' \in \mathbb{R}^k$,

$$|G_t^{(i)}(x, u) - G_t^{(i)}(x', u')| + |f_t^{(i)}(x) - f_t^{(i)}(x')| + |\varphi_t^{(i)}(x) - \varphi_t^{(i)}(x')| \leq K(|x - x'| + |u - u'|).$$

Then there exists a time $T < 0$ depending only on $K$ such that for all $t \in [T, 0]$ and $x, y \in \mathbb{R}^d$,

$$\sup_{s \in [t, 0]} \mathbb{E}|X_{t,s}^{(1)}(x) - X_{t,s}^{(2)}(y)| \leq 2|x - y| + 2 \int_t^0 \|G_t^{(1)} - G_t^{(2)}\|_\infty dr + 2K|T| \left(\|\varphi_t^{(1)} - \varphi_t^{(2)}\|_\infty + \int_t^0 \|f_t^{(1)} - f_t^{(2)}\|_\infty dr\right),$$

where $X_{t,s}(x)$ is the solution of (57) corresponding to $(G^{(i)}, f^{(i)}, \varphi^{(i)})$.

**Proof.** Set

$$Z_{t,s} := X_{t,s}^{(1)}(x) - X_{t,s}^{(2)}(y).$$

By (57) and the assumption, we have

$$\mathbb{E}|Z_{t,s}| \leq |x - y| + \int_t^\infty \|G_r^{(1)} - G_r^{(2)}\|_\infty dr + K \int_t^\infty \left(\mathbb{E}|Z_{t,r}| + \|\varphi_t^{(1)} - \varphi_t^{(2)}\|_\infty + K\mathbb{E}|Z_{t,0}|\right) dr$$

$$+ \int_t^0 \|f_r^{(1)} - f_r^{(2)}\|_\infty dr + K \int_t^0 \mathbb{E}|Z_{t,r}| dr$$

$$\leq |x - y| + \int_t^\infty \|G_r^{(1)} - G_r^{(2)}\|_\infty dr + K|t| \left(\|\varphi_t^{(1)} - \varphi_t^{(2)}\|_\infty + \int_t^0 \|f_r^{(1)} - f_r^{(2)}\|_\infty dr\right)$$

$$+ K \int_t^\infty \mathbb{E}|Z_{t,r}| dr + K^2|t| \left(\mathbb{E}|Z_{t,0}| + \int_t^0 \mathbb{E}|Z_{t,r}| dr\right)$$

$$\leq |x - y| + \int_t^0 \|G_r^{(1)} - G_r^{(2)}\|_\infty dr + K|t| \left(\|\varphi_t^{(1)} - \varphi_t^{(2)}\|_\infty + \int_t^0 \|f_r^{(1)} - f_r^{(2)}\|_\infty dr\right)$$

$$+ \left(K|t| + K^2|t|^2\right) \sup_{r \in [t, 0]} \mathbb{E}|Z_{t,r}|.$$

From this, we immediately conclude the proof. \qed

**Theorem 4.8.** Assume that $(G, f, \varphi)$ are bounded measurable functions and satisfies for some $K > 0$ and all $t \in \mathbb{R}_-$, $x, x' \in \mathbb{R}^d$ and $u, u' \in \mathbb{R}^k$,

$$|G_t(x, u) - G_t(x', u')| + |f_t(x) - f_t(x')| + |\varphi(x) - \varphi(x')| \leq K(|x - x'| + |u - u'|).$$
Then there exists a time $T = T(K) < 0$ such that
\[
    u_t(x) := \mathbb{E}\varphi(X_{t,0}(x)) + \mathbb{E}\left(\int_t^0 f_t(X_{t,r}(x))\,dr\right)
\]  
(59)
is a unique weak solution of equation \((44)\) on $[T, 0]$.

**Proof.** Let $(G^\varepsilon, f^\varepsilon, \varphi^\varepsilon)$ be the smooth approximation of $(G, f, \varphi)$ defined by
\[
    G^\varepsilon_t(x, u) := G \ast \rho_1(t, x, u), \quad f^\varepsilon_t(x) := f \ast \rho_1(t, x), \quad \varphi^\varepsilon(x) = \varphi \ast \rho_1(t, x),
\]
where $\rho_1(t, x, u)$ (resp. $\rho_2(t, x, u)$ and $\rho_3(t, x, u)$) is the mollifiers in $\mathbb{R}^{d+k+1}$ (resp. $\mathbb{R}^{d+1}$ and $\mathbb{R}^d$). It is clear that
\[
    ||\nabla G^\varepsilon_t||_\infty + ||\nabla f^\varepsilon_t||_\infty + ||\nabla \varphi^\varepsilon||_\infty \leq K.
\]
By Theorem 2.7 and Proposition 4.7, there exists a time $T = T(K)$ such that for all $T \leq t \leq s \leq 0$ and $x \in \mathbb{R}^d$,
\[
    \lim_{\varepsilon \downarrow 0} \mathbb{E}|X_{t,s}^\varepsilon(x) - X_{t,s}(x)| = 0,
\]
where $X_{t,s}^\varepsilon$ (resp. $X_{t,s}$) is the solution family of SFDE \((57)\) corresponding to the coefficients $(G^\varepsilon, f^\varepsilon, \varphi^\varepsilon)$ (resp. $(G, f, \varphi)$). Using this limit, by the dominated convergence theorem, it is easy to verify that for each $(t, x) \in [T, 0] \times \mathbb{R}^d$,
\[
    u_t^\varepsilon(x) \rightarrow u_t(x),
\]
(60)
where $u_t^\varepsilon(x)$ is defined through $\varphi^\varepsilon, f^\varepsilon$ and $X_{t,s}^\varepsilon(x)$ as in \((59)\). Moreover, by Proposition 4.7, we also have
\[
    \sup_{\varepsilon \in (0, 1)} \sup_{t \in [T, 0]} ||\nabla u_t^\varepsilon||_\infty + \sup_{t \in [T, 0]} ||\nabla u_t||_\infty \leq C_{T, K} < +\infty.
\]
(61)
On the other hand, thanks to \((61)\), by Theorem 3.4 there exists another time $T' = T'(K) \in [T, 0)$ independent of $\varepsilon$ such that
\[
    u_t^\varepsilon(x) = \varphi^\varepsilon(x) + \int_t^0 \mathcal{L}_0 u_r^\varepsilon(x)\,dr + \int_t^0 G^\varepsilon_t(x, u_r^\varepsilon(x))\partial_i u_r^\varepsilon(x)\,dr + \int_t^0 f^\varepsilon_t(x)\,dr.
\]
In particular, for all $\psi \in C^\infty_0(\mathbb{R}^d)$ and all $t \in [T', 0]$,
\[
    \langle u_t^\varepsilon, \psi \rangle = \langle \varphi^\varepsilon, \psi \rangle + \int_t^0 \langle u_r^\varepsilon, \mathcal{L}_r^\varepsilon \psi \rangle\,dr + \int_t^0 \langle G^\varepsilon_t(u_r^\varepsilon)\partial_i u_r^\varepsilon, \psi \rangle\,dr + \int_t^0 \langle f^\varepsilon_t, \psi \rangle\,dr.
\]
(62)
We want to take limits for both sides of the above identity by \((60)\). The key point is to prove
\[
    \int_t^0 \langle G^\varepsilon_t(u_r^\varepsilon)\partial_i u_r^\varepsilon, \psi \rangle\,dr \rightarrow \int_t^0 \langle G_t(u_r)\partial_i u_r, \psi \rangle\,dr,
\]
which will be obtained by proving the following two limits:
\[
    \int_t^0 \langle (G^\varepsilon_t(u_r^\varepsilon) - G_t(u_r))\partial_i u_r^\varepsilon, \psi \rangle\,dr \rightarrow 0, \quad \varepsilon \rightarrow 0,
\]
\[
    \int_t^0 \langle G_t(u_r)\partial_i (u_r^\varepsilon - u_r), \psi \rangle\,dr \rightarrow 0, \quad \varepsilon \rightarrow 0.
\]
The first limit is clear by \((60), (61)\) and the dominated convergence theorem. The second limit follows by \((60), (61)\) and the integration by parts formula. \(\square\)

Now we are in a position to give
4.3. **Proof of Theorem 4.2.** We divide the proof into three steps. (Step 1). For \( h \in \mathcal{B}(\mathbb{R}^d; \mathcal{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k)) \), define
\[
f^h_t(x) := F_t(x, h_t(x))
\]
and
\[
\mathcal{K} := \sup_{x \in \mathbb{R}^d} \left( \|\nabla G_s\|_\infty + \|\nabla F_s\|_\infty \right) + \|\nabla \varphi\|_\infty.
\]
In this step, we prove the following claim:

**Claim:** For given \( U > 4\|\nabla \varphi\|_\infty \), there exists a time \( T = T(\mathcal{K}, U) < 0 \) such that for any bounded measurable function \( h : \mathbb{R}^d \rightarrow \mathbb{R}^m \) satisfying \( \sup_{t \in [T, 0]} \|\nabla h_t\|_\infty \leq U \), it holds that
\[
\|u^h_t\|_\infty \leq \|\varphi\|_\infty + |t| \sup_{s \in [t, 0]} \|f^h_s\|_\infty,
\]
and
\[
\sup_{t \in [T, 0]} \|\nabla u^h_t\|_\infty \leq U,
\]
where \( u^h_t(x) \) is defined by (59) in terms of \( \varphi, f^h \) and \( X^h_{t,s}(x) \), and \( \{X^h_{t,s}(x), T \leq t \leq s \leq 0, x \in \mathbb{R}^d\} \) is the unique solution family of SFDE (57) corresponding to \( (G, f^h, \varphi) \).

**Proof of the claim.** By Proposition 4.7, there exists a time \( T_1 = T_1(\mathcal{K}, U) < 0 \) such that for all \( x, y \in \mathbb{R}^d \),
\[
\sup_{T_1 \leq t \leq s \leq 0} \mathbb{E}|X_{t,s}(x) - X_{t,s}(y)| \leq 2|x - y|.
\]
Using this and by the definition of \( u^h_t(x) \) (see (59)), we have
\[
|u^h_t(x) - u^h_t(y)| \leq 2\|\nabla \varphi\|_\infty |x - y| + 2 \int_t^0 (\|\nabla F_r\|_\infty + \|\nabla u^h_r\|_\infty U)|x - y|dr.
\]
So,
\[
\sup_{s \in [t, 0]} \|\nabla u^h_s\|_\infty \leq 2\|\nabla \varphi\|_\infty + 2|t| \sup_{s \in [t, 0]} (\|\nabla F_s\|_\infty + \|\nabla u^h_s\|_\infty U) \leq 2\|\nabla \varphi\|_\infty + 2|t|\mathcal{K}(U + 1).
\]
Since \( U > 4\|\nabla \varphi\|_\infty \), choosing \( T = \frac{U - 2\|\nabla \varphi\|_\infty \mathcal{K}(U + 1)}{2\|\nabla \varphi\|_\infty} \wedge T_1 \), we obtain (64). Estimate (63) follows from definition (59).

(Step 2). Set \( u^0_t(x) = \varphi(x) \). We construct the following iteration approximation sequence: for \( n \in \mathbb{N} \),
\[
X^n_{t,s}(x) = X^n_{s,t}(x), \quad u^n_t(x) := u^{n-1}_t(x), \quad f^n_t(x) := f^{n-1}_t(x) = F_t(x, u^{n-1}_t(x)).
\]
By the above claim, there exists a time \( T_1 = T_1(\mathcal{K}, U) < 0 \) such that for all \( n \in \mathbb{N} \),
\[
\|u^n_t\|_\infty \leq \|\varphi\|_\infty + |t| \sup_{s \in [T_1, 0]} \|F_s\|_\infty, \quad \sup_{t \in [T_1, 0]} \|\nabla u^n_t\|_\infty \leq 4\|\nabla \varphi\|_\infty.
\]
Hence,
\[
\|\nabla f^n_t\|_\infty \leq \|\nabla X^n_t\|_\infty + \|\nabla F_t\|_\infty \|\nabla u^{n-1}_t\|_\infty \leq \|\nabla X^n_t\|_\infty + 4\|\nabla F_t\|_\infty \|\nabla \varphi\|_\infty.
\]
Thus, by the definition of \( u^m_t(x) \) (see (59)) and Proposition 4.7 again, there exists another time \( T = T(\mathcal{K}, U) \) such that for all \( n, m \in \mathbb{N} \) and \( t \in (T, 0) \),
\[
\|u^n_t - u^m_t\|_\infty \leq \|\nabla \varphi\|_\infty \sup_{x \in \mathbb{R}^d} |X^n_{t,0}(x) - X^m_{t,0}(x)| + \int_t^0 \|f^n_t - f^m_t\|_\infty dr
\]
\[
+ \int_t^0 \|\nabla f^n_t\|_\infty \sup_{x \in \mathbb{R}^d} |X^n_{t,r}(x) - X^m_{t,r}(x)| dr.
\]
Hence, there exists an \( u_t \in \mathcal{B}([T, 0] \times \mathbb{R}^d; \mathbb{R}^k) \) such that
\[
\lim_{n,m \to \infty} \sup_{t \in [0, T]} \|u^n_t - u^m_t\|_\infty = 0.
\]

On the other hand, by Theorem 4.8, \( u^n_0(x) \) satisfies that for all \( \psi \in C^0_\text{c}(\mathbb{R}^d; \mathbb{R}^k) \),
\[
\langle u^n_0, \psi \rangle = \langle \varphi, \psi \rangle + \int_0^t \langle u^n_s, \mathcal{L}_0^s \psi \rangle \, ds + \int_0^t \langle G_s(u^n_s) \partial_t u^n_s + F_s(u^n_0), \psi \rangle \, ds.
\]

Thus, one can take limits as in Theorem 3.4 to obtain the existence of a short time weak solution for equation (44). Moreover, (ii) follows from (59). The existence of a maximal weak solution can be obtained as in the proof of Theorem 2.12 by shifting the time and induction. Thus, we conclude the proof of (i) and (ii).

(Step 3). Let \( u \in \mathcal{B}_{\text{loc}}((T, 0]; \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k)) \) be a maximal weak solution of equation (44). Define for \( (t, x) \in (T, 0] \times \mathbb{R}^d \),
\[
b_t(x) := G_t(x, u_t(x)), \quad f_t(x) := F_t(x, u_t(x)).
\]

Then it is clear that
\[
b \in \mathcal{B}_{\text{loc}}((T, 0]; \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)), \quad f \in \mathcal{B}_{\text{loc}}((T, 0]; \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k)).
\]

For \( t \in (T, 0] \), let \( \{X_{t,s}(x), t \leq s \leq 0, x \in \mathbb{R}^d \} \) solve the following SDE:
\[
X_{t,s}(x) = x + \int_t^s b_r(X_{t,r}(x)) \, dr + \int_t^s dL_r, \quad s \in [t, 0].
\]

Define
\[
\tilde{u}_t(x) := \mathbb{E}(\varphi(X_{t,0}(x))) + \int_0^t \mathbb{E}(f_s(X_{t,s}(x))) \, ds. \tag{66}
\]

By Theorem 4.5 we have
\[
\tilde{u}_t(x) = u_t(x), \quad \forall (t, x) \in (T, 0] \times \mathbb{R}^d.
\]

Suppose now that \( T > -\infty \). For completing the proof, it is enough to show that
\[
\lim_{t \uparrow T} \|\nabla \tilde{u}_t(x)\|_\infty < +\infty.
\]

It immediately follows from (66) and the following claim proved in [16] Theorem 4.5, which is stated in a slight variant.

Claim: Under (158) or A non-degenerate, for any bounded continuous function \( \varphi \) and \( T < t < s \leq 0 \),
\[
\|\nabla \mathbb{E}\varphi(X_{t,s}(\cdot))\|_\infty \leq C_1 |t - s| \wedge 1)^{-1/\alpha} \|\varphi\|_\infty,
\]
where \( C_1 \) only depends on \( d, \alpha, T \) and the bound of \( b \).
REFERENCES

[1] Applebaum D.: Lévy processes and stochastic calculus. Cambridge Studies in Advance Mathematics 93, Cambridge University PRess, 2004.
[2] Bec J., Khanin K.: Burgers turbulence. Phys. Rep. 447 (2007), no. 1-2, 1–66.
[3] Biler P., Funaki T., Woyczynski W.A.: Fractal Burgers equations. J. Diff. Equa., 148, 9-46(1998).
[4] Blagovescenskii Yu. N.: The Cauchy problem for quasi-linear parabolic equations in the degenerate case. Prob. Theory and Appl., 378-382(1964).
[5] Chan C.H., Czubak M. and Silvestre L.: Eventual regularization of the slightly supercritical fractional Burgers equation. Discrete and Continuous Dynamical Systems. Vol. 27, no. 2, 847-861(2010).
[6] Droniou J. and Imbert C.: Fractal first order partial equations. Archive for Rational Mechanics and Analysis, Volume 182, Issue 2, pp.299-331 (2006).
[7] Freidlin M.: Functional Intergration and Partial Differential Equations. Annals of Math. Studies, Princeton Univ. Press, Princeton, 1985.
[8] Friedman A.: Stochastic Differential Equations and Applications. Volume 1, Academic Press, New York, 1975.
[9] Kiselev A., Nazarov F., Schterenberg R.: Blow up and regularity for fractal Burgers equation. Dynamics of PDE, Vol. 5, No. 3, 211-240, 2008.
[10] Kallenberg, O.: Foundations of Modern Probability. Springer, New York, 1997.
[11] Protter P.: Stochastic integration and differential equations. Springer-Verlag, Berlin (2004).
[12] Revuz, D., Yor, M.: Continuous martingales and Brownian motion. Grund. math. Wiss. 293, Springer-Verlag 1999.
[13] Sato K.: Lévy processes and infinitely divisible distributions. Cambridge University Press, 1999.
[14] Tanaka H.: Local solutions of stochastic differential equations associated with certain quasi-linear parabolic equations. J. of the Faculty of Science, Univ. of Tokyo, 1967. Sec. I, 14, 313-326.
[15] Woyczyński W.: Burgers-KPZ Turbulence. Gottingen lectures. Springer-Verlag, New York, 1998.
[16] Zhang X.: Stochastic Lagrangian particle approach to fractal Navier-Stokes equations. [http://arxiv.org/abs/1103.0131](http://arxiv.org/abs/1103.0131)