Abstract

Marginal Structural Models (MSM) (Robins, 2000) are the most popular models for causal inference from time-series observational data. However, they have two main drawbacks: (a) they do not capture subject heterogeneity, and (b) they only consider fixed time intervals and do not scale gracefully with longer intervals. In this work, we propose a new family of MSMs to address these two concerns. We model the potential outcomes as a three-dimensional tensor of low rank, where the three dimensions correspond to the agents, time periods and the set of possible histories. Unlike the traditional MSM, we allow the dimensions of the tensor to increase with the number of agents and time periods. We set up a weighted tensor completion problem as our estimation procedure, and show that the solution to this problem converges to the true model in an appropriate sense. Then we show how to solve the estimation problem, providing conditions under which we can approximately and efficiently solve the estimation problem. Finally we propose an algorithm based on projected gradient descent, which is easy to implement, and evaluate its performance on a simulated dataset.

1. Introduction

One of the basic challenges in causal inference is to estimate a causal quantity from observational data. Often such datasets involve individuals who are subject to treatments over multiple time periods. The main goal is to estimate the effect of a treatment policy on the outcome. Consider a ride-sharing company, for example, which records several variables such as the number of trips, and trip origins and destinations, for each rider, and based on this information decides whether or not to provide monthly discounts. After running this experiment for several months, the company is interested to know whether providing discounts increases the number of trips taken. If the answer is yes, the company might also want to find a policy that would further increase the number of trips taken.

A second example comes from Acemoglu et al. (2014), who consider a fundamental problem in political science: does democracy cause economic development, in relation to autocracy? The authors collect data from 184 countries over more than half a century. The outcome is GDP per capita and is measured every year. The variables recorded include whether the country was under democracy or autocracy, the population across different age brackets, and net financial inflow. The goal is to find out whether democracy increases GDP of the countries over the periods when the country was under democracy.

There are two fundamental differences between these examples: (1) In the first example, not all users who are provided a discount end up using it, whereas in the second example the political status of a country is determined by policy. (2) The ridesharing company is perfectly aware of all the factors that go into the treatment policy, since it uses an algorithm to determine whether or not to assign a pass based on the past history of a rider. On the other hand, factors that affect the decision about the political status of a country may not be present in any data (and those decisions are far from algorithmic). Despite these differences, the same kind of question is of interest: what is the effect of a treatment policy over the subjects who are assigned the treatment? This quantity is known as the average treatment effect over the treated (ATET). There are also additional challenges. The subjects in both cases, the riders or countries, are heterogeneous, i.e., the effect of the same treatment policy can be expected to differ across subjects. Second, the number of time periods over which the policy is applied can be quite large, with treatments varying across time-steps.

Marginal Structural Models (MSMs) (Robins, 2000) are widely used to estimate the causal quantity of interest when subjects receive treatments over multiple periods of time. However, MSMs have two main limitations: (a) they do not capture subject heterogeneity, and (b) they only consider fixed time intervals and do not scale gracefully with longer intervals. This latter limitation comes about because the number of parameters scales linearly with the length of the
time interval, and with a fixed number of agents there is not enough data to estimate the parameters of the model.

In this work, we propose a new form of MSM to address these drawbacks. We assume that potential outcomes are generated from a three-dimensional tensor of low rank, where the dimensions correspond to the agents, time intervals, and set of possible histories. Intuitively, the rank of the tensor can be interpreted as a measure of the heterogeneity of the agents or the time periods. For example, if the rank is \( r \), then each agent can be described as some combination of \( r \) underlying groups. We assume the rank of the tensor is low, but we allow the dimensions of the tensor to increase with the number of agents and time periods.

**Contributions:** In order to estimate the outcome model, we set up a weighted tensor completion problem, and show that the solution converges to the true model. Compared to Robins (2000), we prove convergence for two cases – when the number of agents \( N \) is fixed and the length of the time interval \( T \) increases and when \( T \) is fixed and \( N \) increases. In particular, if the outcome at every time period depends only on the history of length \( k \), then as long as \( k \) is bounded by logarithm of the increasing variable (be it \( N \) or \( T \)), our method guarantees convergence. We solve the weighted tensor completion in two steps. First, we convert it to a weighted tensor approximation problem with an additive loss, where the loss goes to zero as either \( N \) or \( T \) increases. Then we turn to solving this weighted low-rank approximation problem, and provide conditions under which we can approximately solve the estimation problem in polynomial time. To the best of our knowledge, ours is the first additive approximation algorithm for the noisy weighted tensor completion that runs in polynomial time under reasonable conditions. Finally, we propose an algorithm based on projected gradient descent, which is easy to implement, and show that on a simulated dataset, it performs better than MSM and matrix completion in estimating ATET.

### 1.1. Related Work

The fundamental problem of causal inference is that for each unit we observe only one of two possible outcomes—the outcome corresponding to the treatment but not the control. A standard approach is to use the Rubin-Neyman potential outcomes framework (Rubin, 1974). For each unit and each intervention (0 or 1), there are two potential outcomes \( Y_0 \) and \( Y_1 \), and we only observe one of these two outcomes. The traditional focus has been on estimating the average treatment effect (ATE), which measures the difference in average outcomes under treatment than without treatment. When the treatment policy is completely randomized, this quantity can be estimated by taking the average of the outcomes between the treatment and the control group. For observational data, this quantity can estimated through propensity score matching (Rosenbaum & Rubin, 1983), which cleverly accounts for the covariates that predict treatment.

Traditionally, datasets have been too small to discover any heterogeneity in treatment effects. However, with ever-increasing data and improvements in machine learning algorithms, several recent papers have devised algorithms to discover heterogeneous treatment effects. They often involve machine learning techniques such as Bayesian non-parametrics (Hill, 2011), random forests (Wager & Athey, 2018; Athey & Imbens, 2016), and deep learning (Shalit et al., 2016; Johansson et al., 2016; Yoon et al., 2018). Although we will be working with the potential outcomes framework, there has also been significant effort in using graphical models as a framework for causality (Pearl & Mackenzie, 2018), including attention to heterogeneous effects (Shpitser & Pearl, 2012; Pearl, 2017). However, we are not aware of work on combining these methods with the kinds of temporal settings studied here.

Epidemiologists and biostatisticians have also considered the problem of estimating the causal effect of a policy that applies treatments over multiple time periods. Robins (1986) proposed the marginal structural model (MSM), as a way to measure the causal effect of a time-varying treatment in the presence of time-varying confounders. Suppose, for example, that a policy applies a binary treatment over \( T \) time periods. MSM models each of the \( 2^T \) potential outcomes through a parametric model with parameter \( \beta \). Robins (1986) further showed that the solution to a maximum weighted likelihood correctly estimates the quantity \( \beta \). MSM has been adopted in various domains to estimate the causal effect in a longitudinal study, ranging from effect of different drugs on the mortality of HIV patients (Robins et al., 2000) to the effect of loneliness on depression (VanderWeele et al., 2011).

There have been very few attempts to generalize these models to capture important aspects such as heterogeneous effects, large numbers of time-periods, or to the case when the outcome depends on a short history instead of the full history of length \( T \). This is because MSM was developed in the context of clinical trials, and most of these datasets are relatively small. Neugebauer et al. (2007) define a history-adjusted MSM, which considers potential outcomes dependent on a short history instead of the full history of length \( T \). In particular, they propose a parametric model of the potential outcome conditioned on a history of treatments and covariates. Similar to Robins (1986), they propose an estimator based on maximum weighted likelihood, but that fails to capture heterogeneous effects over the population.

The most closely related prior work is that of Athey et al. (2018), who use matrix completion methods to estimate average treatment effects and other related causal quantities for the time-varying treatment setting. They model the
potential outcomes using a matrix of low rank and provide an estimator. The rank of the underlying matrix captures different types of heterogeneous effects in the population. However, they do not consider the effect of past treatments on the outcomes. Rather, the potential outcome at each time step depends only on the current treatment. Boruvka et al. (2018) do consider time-varying treatments, but model treatment effect conditioned on a given history and under the same underlying policy; i.e., what would happen if treatment were switched form 1 to 0 at time \( t \) and then the policy is otherwise unchanged. Since they prefer not to directly model the environment, their method cannot be used to estimate the average treatment effect or other related quantities under a different policy.

In recent years, there have been several applications of tensor methods; e.g., for learning mixture models (Hsu & Kakade, 2013) and learning topic models (Anandkumar et al., 2012), and so forth. Our main optimization problem is the problem of tensor completion (Barak & Moitra, 2016; Yuan & Zhang, 2016; Montanari & Sun, 2018), the probem of weighted tensor completion problem, which tries to estimate the missing entries of a tensor from the observed entries. Although several algorithms have been proposed for the problem of tensor completion (Barak & Moitra, 2016; Yuan & Zhang, 2016; Montanari & Sun, 2018), the problem of weighted tensor completion is relatively unexplored. We convert the weighted tensor completion problem into a weighted tensor approximation problem. A special case of this problem, weighted matrix completion, is intractable in general. Srebro & Jaakkola (2003) developed an alternating minimization algorithm for this problem. Razenshteyn et al. (2016) developed a provably efficient algorithm for this problem using sketching techniques. Subsequently, Song et al. (2017) generalized their methods for the weighted tensor approximation problem.

2. Model

For \( t = 1, \ldots, T \), \( A_{i,t} \) denotes the treatment assigned to subject \( i \) at time \( t \), and \( X_{i,t} \) denotes the observed co-variate at time \( t \). For \( t = 1, \ldots, T \), \( Y_{i,t} \) denotes the observed outcome for unit \( i \) at time \( t \) and depends on the history of the treatments assigned to agent \( i \) at time \( t \). We use the following notation for a sequence of treatments. \( A_{i,t' : t''} \) denotes the sequence of treatments from \( t' \) to \( t'' \) i.e. \( A_{i,t'}, A_{i,t'+1}, \ldots, A_{i,t''} \). A sequence of covariates, \( X_{i,t' : t''} \) is defined analogously. We will use lowercase variables to denote particular realizations of the random variables, e.g., \( a_{i,t} \) denotes a realization of \( A_{i,t} \), the random variable denoting treatment of agent \( i \) at time \( t \). The same notation applies to co-variates, and outcomes.

The directed acyclic graph (Figure 1) represents the relationship among different variables. For each \( i \) and \( t \), a policy determines \( A_{i,t} \), i.e., the treatment to be assigned. In general, such a policy can be randomized and dynamic, such that the action \( A_{i,t} \) depends on the history up to time \( t \). In such a case, we will write \( \Pr \left[ A_{i,t} = a_{i,t} | A_{i,1 : t-1}, X_{i,1 : t-1}, Y_{i,1 : t-1} \right] \) for the probability assigned to the treatment \( a_{i,t} \) given past treatment sequence of length \( t-1 \), \( a_{i,1 : t-1} \), the realization of the past co-variate sequence of length \( t-1 \), \( x_{i,1 : t-1} \), and the past outcome sequence of length \( t-1 \), \( y_{i,1 : t-1} \).

A covariate can also be dynamic, such that \( X_{i,t} \) can depend on the entire history up to time \( t \). To give a concrete example, Robins et al. (2000) consider a clinical trial setting with HIV patients where the outcomes are the health status and the decision to give a particular drug at a time depends on the patient’s CD4 count at that time (the covariate). In full generality, the outcome at any time might also depend on the entire treatment history, but we make the following assumption about the outcome for any agent, say \( i \).

Assumption 1. The outcome at time \( t \), \( Y_{i,t} \) depends only on the past treatment history of length \( k \), \( A_{i,t-k+1 : t} \).

2.1. Outcome Model

Since the outcome at time \( t \) for agent \( i \), \( Y_{i,t} \), may depend on the past treatment history of length \( k \), there are \( 2^k \) potential outcomes for each agent \( i \) and each time \( t \). This implies that there are \( N \times T \times 2^k \) potential outcomes out of which we observe only \( N \times T \) potential outcomes.\(^2\) We now define

\(^2\)We assume the policy is known i.e. the conditional probabilities of the treatment assignments are known. We leave the problem of estimating these probabilities from the data as future work. In particular, it will be interesting to develop a doubly robust estimator which is robust to misspecification in either the treatment model or the outcome model.

\(^3\)In some scenarios, potential outcomes can exhibit structure, and the number of distinct potential outcomes we need to estimate can be smaller. As an example, suppose that a subject’s response at time \( t \) depends only on how many times she was given the treatment in the last \( k \) rounds. This implies, for each \( i \) and \( t \), there are only \( k+1 \) distinct potential outcomes. Our algorithm need not be aware of such a structure, and the results are stated without this requirement. Introducing this assumption would only lead to...
the outcome model. There is a tensor $T$ of dimension of $N \times T \times 2^k$, such that the outcome for subject $i$ at time $t$ is

$$Y_{i,t} = T[i, t, A_{i,t-1:t}] + \xi_{i,t},$$

(1)

where $\xi_{i,t}$ are iid Gaussian random variables with zero mean and unit variance. Equation (1) says that the potential outcomes are indexed by the subject $i$, time period $t$, and the treatment history of length $k$, $A_{i,t-k+1:t}$. The variable $k$ controls the dependence of the outcome on past sequence of treatments. In general, $k$ can be arbitrarily long. However, we need to assume that the $k$ is bounded from above by the larger of $N$ and $T$ in order to estimate the potential outcomes. Otherwise, the number of missing outcomes grows at a rate larger than the number of observed outcomes, and we are unable to estimate all the missing outcomes.\(^3\)

2.2. Sequentially Randomized Experiment

In this paper, we restrict our attention to the case when there is no unobserved confounders. These are variables that both affect the treatment and the outcome, but are not recorded in the covariates. This assumption is true for the ride-sharing example, where the platform determines whether to give a rider a coupon or not based on the rider’s history. However, not all variables that affect a country’s GDP and political situation can be recorded, and this example has unobserved confounders.

We formalize this requirement of no unobserved confounders by time-varying generalizations of standard properties in the literature on causal inference, namely consistency (the observed outcome is the same as the potential outcome corresponding to the treatment applied), and ignorability (the treatment is independent of the potential outcomes conditioned on the covariate). Let $A^{obs}_{i,t}$ denote the observed outcome, and $A_i(t)$ denote the corresponding random variable dependent on the covariate. The same notation holds for the outcomes and the covariates. We define the following properties:

1. **Consistency**: The observed data $(Y_{i,1}, A_{i,1}, X_{i,1}, Y_{i,2}, A_{i,2}, X_{i,2}, \ldots)$ is equal to the potential outcomes as follows. For every history $h_{i,t} = (a_{i,1:t}, x_{i,1:t}, y_{i,1:t-1})$, we have $Y^{obs}_{i,t} = Y_{i,t}(h_{i,t}) = Y_{i,t}(a_{i,t-k+1:t}), X^{obs}_{i,t+1} = X_{i,t+1}(h_{i,t}),$ and $A^{obs}_{i,t+1} = A_{i,t+1}(h_{i,t})$.

2. **Sequential Ignorability**: For each $t$, the potential outcomes are independent of the treatment conditioned on the

\(^3\)This seems reasonable in settings with time-varying treatments, e.g., the number of trips taken by a rider will depend on his coupons for the past couple of months, but not on whether she received coupons several years back.

history at time $t$, i.e.,

$$Y_{i,t} \perp A_{i,t} \mid A_{i,1:t} = a_{i,1:t}, X_{i,1:t} = x_{i,1:t}, Y_{i,1:t-1} = y_{i,1:t-1}$$

(2)

3. **Positivity**: There exists a $\delta > 0$ such that for each $a_{i,1:t-1}, x_{i,1:t-1}, y_{i,1:t-1}$, we have

$$\delta < \Pr[a_{i,t}|a_{i,1:t-1}, x_{i,1:t-1}, y_{i,1:t-1}] < 1 - \delta$$

Consistency maps the observed outcomes to the potential outcomes. In particular, the outcome observed at time $t$, $Y^{obs}_{i,t}$ is completely determined by the past treatment history of length $k$, i.e., there are no additional factors such as the subject’s motivation that affects both the actual treatment assignment and the outcome. If the treatment at time $t$, $A_{i,t}$ is chosen based on the history up to time $t$, then sequential ignorability automatically holds (Boruvka et al., 2018). On the other hand, in an observational study, we must assume there are no unmeasured confounders for sequential ignorability to hold. If the policy systematically violates positivity, then it might be that some units do not get a particular treatment at all, and it would be impossible to estimate the outcome model.

2.3. Quantities to Estimate

The literature on causal inference has proposed various quantities to estimate in a setting with time-varying treatments. In the introduction, we talked about the average treatment effect over the units that actually received the treatment at all, and it would be impossible to estimate the

$$\text{ATET} = \frac{1}{\{([i,t] : a_{i,t} = 1\} \cap \{t, a_{i,t} = 1\}} \sum_{(i,t): a_{i,t} = 1} E[Y_{i,t}(a_{i,t-k+1:t})] - E[Y_{i,t}(a_{i,t-k+1:t}, 0)]$$

According to the outcome model specified in (1), this becomes

$$\text{ATET} = \frac{1}{\{([i,t] : a_{i,t} = 1\} \cap \{t, a_{i,t} = 1\}} \sum_{(i,t): a_{i,t} = 1} T[i,t, a_{i,t-k+1:t}] - T[i,t, a_{i,t-k+1:t}, 0],$$

(3)

and can be computed easily once we have an estimate of the tensor $T$. We can also generalize ATET by considering the effect of switching from one history $h_1$ to another history $h_2$ of length at most $k$:

$$\text{ATET}(h_1, h_2) = \frac{1}{\{([i,t] : a_{i,t-|h_1|-1:t} = h_1\}} \sum_{(i,t): a_{i,t-|h_1|-1:t} = h_1} E[Y_{i,t}(a_{i,t-k+1:t})] - E[Y_{i,t}(a_{i,t-k+1:t-|h_2|:h_2})]$$
and it is straightforward to write $\text{ATET}(h_1, h_2)$ using the outcome model specified in (1).

### 2.4. Marginal Structural Models

Our work builds on the marginal structural models, proposed by Robins et al. (2000). At each time $t$, for every possible sequence of treatments $a_{i,t}$, MSMs define the following model of the corresponding potential outcome.

$$E[Y_{it}(a_{i,1:t})] = g(a_{i,1:t}, \beta)$$  \hspace{1cm} (4)

Here $g$ is the link function, usually chosen to be either a linear function or a logistic function.

The standard maximum likelihood based estimator of $\beta$ will be biased. Robins (2000) showed that the parameter can be estimated in an unbiased way through an inverse probability of treatment weighting (IPTW) approach. Suppose the observed data is given as $\{a_{i,t}, x_{i,t}, y_{i,t}\}_{i,t}$. Then consider the following weight for each agent $i$ and each time period $t$:

$$w_{it} = \frac{P_r[a_{i,s}|a_{i,1:s-1}]}{P_r[a_{i,s}|a_{i,1:s-1}, z_{i,1:s-1}, y_{i,1:s-1}]}$$

The denominator of each term is the probability of the corresponding treatment given the history up to that point. The numerator of each term is the marginal probability of the corresponding treatment conditioned only on the past sequence of treatments and is used to stabilize the weights. Now if we compute a maximum likelihood estimator where the observation of subject $i$ at time $t$ is weighted by $w_{it}$, then $\beta$ can be identified. If we know the policy, we can directly compute the marginal probabilities and get the weights. When the policy is unknown the probabilities are estimated from the data and substituted to compute the weights.

### 3. Estimation

The goal is to design an unbiased and consistent estimator $\hat{T}$ of the $N \times T \times 2^k$ tensor $T$. We will assume that the tensor $T$ has low rank $r$. $T$ has rank $r$ if there exist vectors $\{u_i\}_{i=1}^r$, $\{v_i\}_{i=1}^r$ and $\{w_i\}_{i=1}^r$ such that $T = \sum_{i=1}^r u_i \otimes v_i \otimes w_i$ and $r$ is the smallest integer such that $T$ can be written in this form. Here $u_i \otimes v_i \otimes w_i$ denotes the outer-product of the three vectors $u_i$, $v_i$, and $w_i$ with entries $u_i \otimes v_i \otimes w_i(a, b, c) = u_i(a) \cdot v_i(b) \cdot w_i(c)$.

Without loss of generality, we can assume that the tensor $T$ is written in the following form, where each of the vectors $u_i$, $v_i$, and $w_i$ are normalized.

$$T = \sum_{i=1}^r \lambda_i u_i \otimes v_i \otimes w_i$$  \hspace{1cm} (5)

We use $\lambda_i(T)$ to denote the $i$-th singular value of $T$. For $p = 1, \ldots, B$, let $O_p$ be the set of observations that lead to the realization of history corresponding to the $p$-th slice. Formally, $O_p = \{(i, t) : A_{i,t-k+1:t} = p\}$. Then we propose to solve the following optimization problem:

$$\min_{T \in \mathbb{R}^{N \times T \times B}, \operatorname{rank}(T) \leq r} \frac{1}{NT} \sum_{p=1}^B \sum_{(i,t) \in O_p} w_{i,t} (Y_{i,t} - T(i,t,p))^2.$$  \hspace{1cm} (6)

The weights $w_{i,t}$ are defined as:

$$w_{i,t} = \prod_{s=t-k+1}^{t} \frac{P_r[a_{i,s}|a_{i,t-2k+1:s-1}]}{P_r[a_{i,s}|a_{i,t-2k+1:s-1}, y_{i,t-2k+1:s-1}]}.$$  \hspace{1cm} (7)

For each term, the denominator denotes the probability of the treatment given the history from time $t - 2k + 1$ to that time. The numerator can be any marginal probability not involving the outcome variables. But, why are we interested in the optimization problem eq. (13)? The objective function is the weighted log-likelihood given tensor $T$, and we prove next that if we could solve this problem exactly, the corresponding estimator will be consistent. We make some additional assumptions:

1. **Bounded Singular Value**: For each $N$ and $T$, each of the $r$ singular values of $T_{N,T}$, $\lambda_i(T)$, are bounded, i.e. $\|T_{N,T}\|_\infty = \max_i |\lambda_i(T)| \leq L$ for some $L$.

2. **Decaying Covariance**: There exists a constant $\gamma \geq 1$, such that for all $t' > t + 3k$, we have

$$\frac{\text{cov}(f(A_{i,t'}), g(A_{i,t',t+k}))}{\text{E}[f(A_{i,t'})]} \leq O((t' - t)^{-\gamma}).$$

for functions $f$ and $g$ of the following form:

$$h(A_{i,t'}) = \prod_{j=1}^r h_j(A_{i,j})$$

where each $h_j$ is either $A_j$ or $1 - A_j$.

The first assumption implies that each entry of the tensor is bounded between $-L$ and $L$. The second assumption implies that the treatments chosen at two time periods that are far apart are, almost independent.

#### 3.1. Consistency

For any $N$ and $T$, we assume that the data is generated from an underlying tensor $T_{N,T}$. We will also write $\hat{T}_{N,T}$ to denote the solution to eq. (13). Consider the weighted log-likelihood function:

$$L_{N,T}(T_{N,T}) = \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T w_{i,t} \log P_r[Y_{i,t}|T_{N,T}].$$  \hspace{1cm} (9)

The estimate $\hat{T}_{N,T}$ maximizes $L_{N,T}(T_{N,T})$ over all possible choices of $T_{N,T}$. Our goal is to show that with high probability, $\|\hat{T}_{N,T} - T_{N,T}\|_2^2 / \sqrt{NT}$ converges to zero as $N$ and $T$ increases. We normalize the difference in norm by...
both \( N \) and \( T \). This is necessary, as with increasing \( N \) and \( T \) the number of parameters we are estimating also grows.

**Theorem 3.1.** Suppose \( T^*_N,T \) exists for all \( N \) and \( T \). Then

- If \( k \leq O \left( \log_{2(1-\delta)/\delta} N \right) \), then for any \( \varepsilon > 0 \),
  \[
  \Pr \left[ \frac{\|N \rightarrow T^*_N,T\|}{\sqrt{NT}} > \varepsilon \right] \to 0 \text{ as } N \to \infty.
  \]
- If Assumption 15 holds, and \( k \leq \max \{ O \left( \log_4(T/T \log T) \right), O \left( \log_{2(1-\delta)/\delta} T \right) \} \),
  then \( \Pr \left[ \frac{\|N \rightarrow T^*_N,T\|}{\sqrt{NT}} > \varepsilon \right] \to 0 \text{ as } T \to \infty. \)

The full proof is given in the appendix. Here we sketch the main challenges. The proof follows the ideas presented in Newey & McFadden (1994), but there are some subtle differences. Unlike the traditional maximum likelihood estimation, we are not estimating a fixed parameter. As either \( N \) or \( T \) increases, we are estimating a sequence of tensors increasing in either \( N \) or \( T \). This is why we prove that the normalized distance between \( T^*_N,T \) and \( T_N,T \) goes to zero, instead of the actual L2 distance. There are two more challenges in the proof. First the parameter space \( \Theta_{N,T} = \{ T \in \mathbb{R}^N \times \mathbb{T}^2 : \text{rank}(T) \leq r \} \) need not be a closed set, as we can have a sequence of rank \( r \) tensors converging to a rank 1 tensor. However, the concavity of the log-likelihood function in \( T \) helps us to circumvent this problem. Second, the standard way to prove the consistency of the maximum likelihood estimation is to consider a neighborhood around the true parameter, say \( \mathcal{B} \). Then there will be a gap of \( \varepsilon \) between the maximum over \( \mathcal{B} \) and the maximum outside of \( \mathcal{B} \), and for large number of samples the gap between the objective value of the true parameter and the estimate will be less than \( \varepsilon \), and the estimate will be inside the neighborhood \( \mathcal{B} \). However, in our case, the gap \( \varepsilon \) is also changing with \( N \) and \( T \) as the entire parameter space is changing, and it might be possible that this gap goes to zero with increasing \( N \) and \( T \). However, we can provide a lower bound on the gap in terms of the radius of the neighborhood and the parameters \( N \) and \( T \), and this helps to complete the proof.

### 3.2. Solving Tensor Completion

In this section, we focus on solving the weighted tensor completion to estimate the underlying tensor \( T^*_N,T \). We proceed in two steps. First, we convert the weighted tensor completion problem to a weighted tensor approximation problem with an additive error that goes zero as either the number of units \( N \) or the number of time intervals \( T \) increases to infinity. Then we provide a \((1+\varepsilon)\)-approximation to the weighted tensor approximation problem under reasonable assumptions on the policy generating the treatment assignment. A combination of these two steps gives us an approximate solution to the original objective function defined in eq. (13).

Recall that \( O_p \) refers to all observations for which we observe the counterfactual outcome corresponding to the \( p \)-th slice, i.e., \( O_p = \{ (i,t) : A_{i,t-k+1:t} = p \} \). Consider the objective function defined in (13):

\[
\frac{1}{NT} \sum_{p=1}^{2^k} \sum_{(i,t) \in O_p} w_{i,t} \left( Y_{i,t} - T(i,t,p) \right)^2
\]

\[
= \frac{1}{NT} \sum_{p=1}^{2^k} \sum_{(i,t) \in O_p} w_{i,t} \left( Y_{i,t}^2 - 2Y_{i,t}T(i,t,p) + T(i,t,p)^2 \right).
\]

Since we are optimizing over the tensor \( T \), we can drop the first term above, and consider the following objective:

\[
\frac{1}{NT} \sum_{p=1}^{2^k} \sum_{(i,t) \in O_p} -2w_{i,t}Y_{i,t}T(i,t,p)
\]

\[
+ \frac{1}{NT} \sum_{p=1}^{2^k} \sum_{(i,t) \in O_p} w_{i,t}T(i,t,p)^2.
\]

The main idea to convert this objective into a tensor approximation problem is to replace the second term by its population variant and define a weight tensor so that the first sum is defined over all the entries in \( T \). Let \( \Pr \left[ (i,t) \in O_p \right] \) be the marginal probability that the underlying policy selects slice \( p \) for agent \( i \) at time \( t \). The supplementary material proves that the expected value of the second term is \( \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{p=1}^{2^k} \Pr \left[ (i,t) \in O_p \right] T^2(i,t,p) \), a weighted norm of \( T \). So we replace the second term above by its corresponding population variant,

\[
\frac{1}{NT} \sum_{p=1}^{2^k} \sum_{(i,t) \in O_p} -2w_{i,t}Y_{i,t}T(i,t,p)
\]

\[
+ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{p=1}^{2^k} \Pr \left[ (i,t) \in O_p \right] T^2(i,t,p) \tag{10}
\]

Let us define the following tensor:

\[
Y_w(i,t,p) = \begin{cases} 
\frac{w_{i,t}Y_{i,t}}{\Pr \left[ (i,t) \in O_p \right]} & \text{if } (i,t) \in O_p \\
0 & \text{otherwise} \end{cases}
\]

and the “weight” tensor, \( W(i,t,p) = \sqrt{\Pr \left[ (i,t) \in O_p \right]} \). This leads to the following form of objective (10):

\[
- \frac{2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{p=1}^{2^k} \left( W(i,t,p) \right)^2 Y_w(i,t,p)T(i,t,p)
\]

\[
+ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{p=1}^{2^k} \Pr \left[ (i,t) \in O_p \right] T^2(i,t,p) \tag{11}
\]
Finally, we add additional terms involving the tensor \( Y_w \) to make the objective function (11) a square:

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{p=1}^{2^k} (W(i, t, p))^2 \{ Y_w(i, t, p) - T(i, t, p) \}^2
= \frac{1}{NT} \| Y_w - T \|_W^2. \tag{5}
\]

This leads us to the following tensor approximation problem, instead of the tensor completion problem in (13):

\[
\min_{\text{rank}(T) \leq \| T \|_W^2} \frac{1}{NT} \| Y_w - T \|_W^2. \tag{12}
\]

Objective (14) computes a weighted low rank approximation of \( Y_w \). Let \( \tilde{T}_{N,T} \) be the solution to (14). We first show that this estimator approximately optimizes the original objective (13). Let OPT be the optimal value of (13).

**Lemma 3.2.** If \( k \leq O \left( \frac{\log(1-\delta)/\delta}{N} \right) \), then

\[
\frac{1}{NT} \sum_{p=1}^{2^k} \sum_{i \in O_p} W_{i,t} \left( T_{N,T}(i, t, p) - \tilde{T}_{N,T}(i, t, p) \right)^2 \leq \text{OPT} + O \left( \frac{L^2}{N^\alpha} \right) \text{ w.p. at least } 1 - \exp(-N^{1/4}).
\]

If \( k \leq O \left( \frac{\log(1-\delta)}{\delta} T \right) \) and assumption 15 holds, then

\[
\frac{1}{NT} \sum_{p=1}^{2^k} \sum_{i \in O_p} W_{i,t} \left( T_{N,T}(i, t, p) - \tilde{T}_{N,T}(i, t, p) \right)^2 \leq \text{OPT} + O \left( \frac{L^2}{T^{3/4}} \right) \text{ w.p. at least } 1 - O \left( \frac{L^2}{\sqrt{T}} \right).
\]

When \( N \) is fixed and \( T \) is increasing, the additive error holds with probability \( 1 - O \left( \frac{L^2}{\sqrt{T}} \right) \) instead of probability \( 1 - \exp(-T) \). This is because we do not have independence in the treatments for different values of \( t \) for a given subject \( i \). However, Assumption 15 helps us to bound the variance of the estimated norm of the tensor and thereby bound the probability of failure by \( O \left( \frac{L^2}{\sqrt{T}} \right) \).

### 3.3. Solving Low-Rank Tensor Approximation

Now we focus on solving problem (14). Although weighted low-rank approximation of a tensor is in general intractable, we provide two methods to solve this problem. The first method provides a \((1 + \varepsilon)\)-multiplicativc approximation using techniques derived by (Song et al., 2017). However, the algorithm can be hard to implement and can be slow depending on the choice of different parameters. We develop a second method based on projected gradient descent, which is easy to implement and use this method in simulation.

#### 3.3.1. A \((1 + \varepsilon)\)-MULTIPLICATIVC APPROXIMATION

Song et al. (2017) show that there is an algorithm that takes as input a tensor \( A \in \mathbb{R}^{n \times n \times n} \), a weight tensor \( W \in \mathbb{R}^{n \times n \times n} \), and outputs a tensor \( A' \) of rank \( r \) such that \( \| A - A' \|_W^2 \leq (1 + \varepsilon) \min_{\| B \| \leq r} \| A - B \|_W^2 \). The authors consider the case when the weight tensor \( W \) has \( s \) distinct faces in two dimensions (e.g. \( s \) distinct rows, and columns). Then their algorithm runs in time \( nzz(A) + nzz(W) + nzz(A') \) time, where \( nzz(A) \) is the number of nonzero entries in \( A \). The algorithm works by choosing a sketching matrix for each of the three unfoldings of the tensor. The sketching matrices project the rows, columns, and tubes of the tensor to a low-dimensional space. This allows to convert the tensor approximation problem to a polynomial system verification problem in a low-dimensional space.

We want to find a rank \( r \) approximation of tensor \( Y_w \in \mathbb{R}^{N \times T \times 2^k} \). There are two challenges. First, the algorithm proposed in Song et al. (2017) works with tensors whose dimensions across the three axes are the same. However, the algorithm can be easily generalized so that it works for tensors of arbitrary dimensions by choosing the sketching matrices to be of appropriate dimensions. Second, we want to enforce an additional constraint that the singular values are bounded between \( -L \) and \( L \). This can be handled by introducing \( r \) additional constraints in the polynomial system verifier of the algorithm in Song et al. (2017). The full algorithm and an analysis of its running time is given in the supplementary material.

Recall that we want to compute a low-rank approximation of the tensor \( Y_w \in \mathbb{R}^{N \times T \times 2^k} \). Although \( nzz(Y_w) = NT \), positivity implies that the number of nonzero entries in \( W \) is \( nzz(W) = NT 2^k \). Therefore, the resulting algorithm runs in time \( O \left( NT 2^k + \max\{N, T, 2^k\} \hat{O}(s^2 r^2 / \varepsilon) \right) \) and outputs a tensor \( \tilde{T}_{N,T} \) such that \( \| Y_w - \tilde{T}_{N,T} \|_W^2 \leq (1 + \varepsilon) \min_{\text{rank}(T) \leq r} \| Y_w - T \|_W^2 \) with probability at least 9/10. The next lemma shows that \( \tilde{T}_{N,T} \) approximately optimizes our original objective.

**Lemma 3.3.** If \( k \leq O \left( \frac{\log(1-\delta)}{\delta} N \right) \), then

\[
\frac{1}{NT} \sum_{p=1}^{2^k} \sum_{i \in O_p} W_{i,t} \left( T_{N,T}(i, t, p) - \tilde{T}_{N,T}(i, t, p) \right)^2 \leq (1 + \varepsilon) \text{OPT} + O \left( \frac{L^2}{N^\alpha} \right) \text{ w.p. at least } 4/5.
\]

If \( k \leq O \left( \frac{\log(1-\delta)}{\delta} T \right) \) and assumption 15 holds, then

\[
\frac{1}{NT} \sum_{p=1}^{2^k} \sum_{i \in O_p} W_{i,t} \left( T_{N,T}(i, t, p) - \tilde{T}_{N,T}(i, t, p) \right)^2 \leq (1 + \varepsilon) \text{OPT} + O \left( \frac{L^2}{T^{3/4}} \right) \text{ w.p. at least } 4/5.
\]
These two assumptions together imply that $W$ has $s$ distinct faces along the two dimensions.

1. There are $s$ groups of subjects such that the policy treats all the subjects in a group identically.

2. There are $s$ groups of time periods such that for any two time $t$ and $t'$ belonging to the same group we have the same marginal probabilities across all the subjects ($\Pr [(i, t) \in O_p] = \Pr [(i, t') \in O_p] \forall i, p$).

These two assumptions together imply that $W$ has $s$ distinct faces in two dimensions, and allows an efficient $(1 + \varepsilon)$-multiplicative approximation of problem 14.

3.3.3. Projected Gradient Descent

We now provide a simple algorithm for the weighted tensor approximation problem (14) based on projected gradient descent. Algorithm 1 repeatedly applies two steps. Line 5 computes a gradient step to compute the new tensor $T_u$. However, the tensor $T_u$ might not be of rank $r$, so line 6 computes a projection of tensor $T_u$ into the space of tensors of rank $r$. As the projection step is a standard rank $r$ approximation of a tensor, we use the parafac method from the TensorLy package (Kossaifi et al., 2018) for this step.

![Algorithm 1](image)

**Algorithm 1** Weighted Tensor Approximation

1: **Input:** Tensor $S \in \mathbb{R}^{N \times T \times 2^k}$, weight tensor $W \in \mathbb{R}^{N \times T \times 2^k}$, rank $r$, and $R$.
2: **Initialize** $T$.
3: **for** $j = 1$ to $R$ **do**
4: $T_u \leftarrow T + \lambda 2W^2(S - T)$
5: $T \leftarrow \text{Project}(T_u, r)$
6: **if** Relative Change in Loss $\leq \varepsilon$ **then**
7: **return** $T$
8: **end if**
9: **end for**
10: **return** $T$

3.3.4. Simulation

We now evaluate the effectiveness of Algorithm 1 through a simulation. We pick $N = 50$ agents, $T = 50$ time periods and $k = 5$. We first fix a tensor $T \in \mathbb{R}^{50 \times 50 \times 2^5}$ of rank 6 by two steps. First, we choose the vectors $\{u_i\}_{i=1}^r$ and $\{v_i\}_{i=1}^r$ and $\{w_i\}_{i=1}^r$ by selecting each entry uniformly at random from the interval [0, 1] and then normalizing the vectors. Second, we select the singular values $\{\lambda_i\}_{i=1}^r$ uniformly at random from the interval [100, 200]. Having fixed this tensor, we generate the data i.e. the treatment assignment $\{A_{i,t}\}_{i,t}$ and the outcome $\{Y_{i,t}\}_{i,t}$ according to two policies.

1. **Simple:** The treatment at every period is either 0 or 1 with equal probability.

2. **Dynamic:** The policy counts the number of ones in the previous three rounds. If the count is zero then $A_{i,t}$ is 1 with probability 0.75. If it is one or two then
4. Conclusion

In this work, we introduced a new form of marginal structural models. We used tensors to model the potential outcomes for time-varying treatments and showed how to efficiently estimate the parameters of the model. There are several directions for future work including handling of time-varying unobserved confounders, and developing a doubly robust estimator which works when either the outcome model or the treatment model is correctly specified.

References

Acemoglu, D., Naidu, S., Restrepo, P., and Robinson, J. A. Democracy does cause growth. Technical report, National Bureau of Economic Research, 2014.

Anandkumar, A., Foster, D. P., Hsu, D. J., Kakade, S. M., and Liu, Y.-K. A spectral algorithm for latent dirichlet allocation. In Advances in Neural Information Processing Systems, pp. 917–925, 2012.

Athey, S. and Imbens, G. Recursive partitioning for heterogeneous causal effects. 113(27):7353–7360, 2016. doi: 10.1073/pnas.1510489113.

Athey, S., Bayati, M., Doudchenko, N., Imbens, G., and Khosravi, K. Matrix completion methods for causal panel data models. Technical report, National Bureau of Economic Research, 2018.

Barak, B. and Moitra, A. Noisy tensor completion via the sum-of-squares hierarchy. In Conference on Learning Theory, pp. 417–445, 2016.

Bini, D. Border rank of $m \times n \times (mn - q)$ tensors. Linear Algebra and Its Applications, 79:45–51, 1986.

Boruwka, A., Almirall, D., Witkiewitz, K., and Murphy, S. A. Assessing time-varying causal effect moderation in mobile health. Journal of the American Statistical Association, 113(523):1112–1121, 2018.

Hill, J. L. Bayesian nonparametric modeling for causal inference. Journal of Computational and Graphical Statistics, 20(1):217–240, 2011. doi: 10.1198/jcgs.2010.08162.

Hsu, D. and Kakade, S. M. Learning mixtures of spherical gaussians: moment methods and spectral decompositions. In Proceedings of the 4th conference on Innovations in Theoretical Computer Science, pp. 11–20. ACM, 2013.

Johansson, F., Shalit, U., and Sontag, D. Learning representations for counterfactual inference. In International Conference on Machine Learning, pp. 3020–3029, 2016.

Kossaifi, J., Panagakis, Y., Anandkumar, A., and Pantic, M. Tensorly: Tensor learning in python. CoRR, abs/1610.09555, 2018.

Montanari, A. and Sun, N. Spectral algorithms for tensor completion. Communications on Pure and Applied Mathematics, 71(11):2381–2425, 2018.

Neugebauer, R., van der Laan, M. J., Joffe, M. M., and Tager, I. B. Causal inference in longitudinal studies with history-restricted marginal structural models. Electronic journal of statistics, 1:119, 2007.

Newey, W. K. and McFadden, D. Large sample estimation and hypothesis testing. Handbook of econometrics, 4: 2111–2245, 1994.

Pearl, J. Detecting latent heterogeneity. Sociological Methods & Research, 46(3):370–389, 2017.

Pearl, J. and Mackenzie, D. The Book of Why: The New Science of Cause and Effect. Basic Books, Inc., New York, NY, USA, 1st edition, 2018. ISBN 046509760X, 9780465097609.

Razenshtein, I., Song, Z., and Woodruff, D. P. Weighted low rank approximations with provable guarantees. In Proceedings of the forty-eighth annual ACM symposium on Theory of Computing, pp. 250–263. ACM, 2016.

Robins, J. A new approach to causal inference in mortality studies with a sustained exposure periodapplication to control of the healthy worker survivor effect. Mathematical modelling, 7(9-12):1393–1512, 1986.

Robins, J. M. Marginal structural models versus structural nested models as tools for causal inference. In Statistical models in epidemiology, the environment, and clinical trials, pp. 95–133. Springer, 2000.
Robins, J. M., Hernan, M. A., and Brumback, B. Marginal structural models and causal inference in epidemiology, 2000.

Rosenbaum, P. R. and Rubin, D. B. The central role of the propensity score in observational studies for causal effects. *Biometrika*, 70(1):41–55, 1983.

Rubin, D. B. Estimating causal effects of treatments in randomized and nonrandomized studies. *Journal of educational Psychology*, 66(5):688, 1974.

Shalit, U., Johansson, F. D., and Sontag, D. Estimating individual treatment effect: generalization bounds and algorithms. *arXiv preprint arXiv:1606.03976*, 2016.

Shpitser, I. and Pearl, J. Identification of conditional interventional distributions. *arXiv preprint arXiv:1206.6876*, 2012.

Song, Z., Woodruff, D. P., and Zhong, P. Relative error tensor low rank approximation. *arXiv preprint arXiv:1704.08246*, 2017.

Srebro, N. and Jaakkola, T. Weighted low-rank approximations. In *Proceedings of the 20th International Conference on Machine Learning (ICML-03)*, pp. 720–727, 2003.

VanderWeele, T. J., Hawkley, L. C., Thisted, R. A., and Cacioppo, J. T. A marginal structural model analysis for loneliness: implications for intervention trials and clinical practice. *Journal of consulting and clinical psychology*, 79(2):225, 2011.

Wager, S. and Athey, S. Estimation and inference of heterogeneous treatment effects using random forests. *Journal of the American Statistical Association*, 113(523):1228–1242, 2018. doi: 10.1080/01621459.2017.1319839.

Yoon, J., Jordon, J., and van der Schaar, M. GAN-ITE: Estimation of individualized treatment effects using generative adversarial nets. In *International Conference on Learning Representations*, 2018. URL https://openreview.net/forum?id=ByKWuEWA-.

Yuan, M. and Zhang, C.-H. On tensor completion via nuclear norm minimization. *Foundations of Computational Mathematics*, 16(4):1031–1068, 2016.
A. Recap

Recall that our main estimation problem is the following:

\[ \frac{1}{NT} \min_{T \in \mathbb{R}^{N \times T \times 2^k}, \text{rank}(T) \leq r} \sum_{p=1}^{2^k} \sum_{(i,t) \in O_p} w_{i,t} (Y_{i,t} - T(i,t,p))^2. \quad (13) \]

But, we converted the tensor completion problem into the following low-rank tensor approximation problem.

\[ \min_{T \in \mathbb{R}^{N \times T \times 2^k}, \text{rank}(T) \leq r, \|T\|_F \leq L} \frac{1}{NT} \|Y - T\|^2_W. \quad (14) \]

We will often use \( B = 2^k \) to denote the third dimension of the tensor.

We will use four tensors throughout the proofs – (a) \( T_{N,T}^* \) denotes the true underlying tensor, (b) \( \hat{T}_{N,T} \) denotes the solution of objective 13, (c) \( \hat{T}_{N,T} \) denotes the solution of objective 14, and (d) \( \hat{T}_{N,T} \) denotes the tensor which provides \((1 + \epsilon)\)-multiplicative approximation to the problem 14.

Finally, we recall two assumptions from the main paper:

1. **Bounded Singular Value**: The maximum entry of the underlying tensor is bounded, i.e., for each \( N \) and \( T \), \( \|T_{N,T}^*\|_\ast = \max_i |\lambda_i(T_{N,T}^*)| \leq L \) for some \( L \).

2. **Decaying Covariance**: There exists a constant \( \gamma \geq 1 \), such that for all \( t' > t + 3k \)

\[
\frac{|\text{cov}(f(A_{i,t}), g(A_{i,t',t+k}))|}{E[f(A_{i,t})]} \leq O((t' - t)^{-\gamma}). \quad (15)
\]

for any function \( f \) and \( g \) of the following form: \( h(A_{i,t}) = \prod_{j=1}^s h_j(A_{i,j}) \) where \( h_j(A_{i,j}) \) is either \( A_{i,j} \) or \( 1 - A_{i,j} \).

B. \((1 + \epsilon)\)-approximation algorithm

In this section, we provide the details of the \((1 + \epsilon)\)-approximation algorithm for weighted tensor approximation. We will write \( B \) to denote \( 2^k \). As input, we are given a tensor \( T \in \mathbb{R}^{N \times T \times B} \), a weight tensor \( W \in \mathbb{R}^{N \times T \times B} \) and our goal is to provide an approximately solve

\[ \min_{B: \text{rank}(B) \leq r} \frac{1}{NT} \|T - B\|^2_W. \]

We are guaranteed that \( W \) has \( s \) distinct rows and \( s \) distinct columns. This also guarantees that the number of distinct tubes of \( s \) is at most \( S = 2^{O(s \log s)} \).

Algorithm 2 closely follows algorithm G.4 in (Song et al., 2017) with modifications to handle asymmetric tensors and additional constraint on the bound for the largest singular value. It chooses three sketching matrices of appropriate dimension to solve the original low-rank approximation problem in a low-dimensional space. The main idea is that the entries of \( \hat{U}_1 \) can be represented with as polynomials of the variables for \( i = 1 \) to \( s \) (line 10). This is possible because the weight matrix has \( s \) distinct rows and columns, which implies that it’s flattening along the rows has \( s \) distinct faces. The same thing holds for \( \hat{U}_2 \). However, this need not be true for \( \hat{U}_3 \), so they are represented through \( S \) distinct denominators (line 16). With this setup (Song et al., 2017) shows that the number of variables in the polynomial system verifier is \( O \left( r^2 s / \epsilon \right) \) and the number of constraints is \( 2s + S \). In line 18, we add additional \( r \) constraints. So the total number of constraints is \( 2s + r + 2^{O(s \log s)} \) and the total number of variables is \( O \left( r^2 s / \epsilon \right) \). Moreover, the degree of the new constraints in line 18 is at most \( \text{poly}(r, s, S) \). A polynomial system can be verified in time \( \# \text{max degree of any polynomial} \times S \) number of variables. In our case, this takes time

\[
\left( \text{poly}(r, s) \text{poly} \left( 2^{O(s \log s)} \right) \right)^{O(r^2 s / \epsilon)} = \left( \text{poly}(r, s) 2^{O(s \log s)} \right)^{O(r^2 s / \epsilon)} = 2^{O(r^2 s^2 / \epsilon)}.
\]
Algorithm 2: Weighted Low Rank Tensor Approximation

1: **Input:** Tensor $T \in \mathbb{R}^{N \times T \times B}$, weight tensor $W \in \mathbb{R}^{N \times T \times B}$, rank $r$, rank of weight tensor $s$, and $\varepsilon$.
2: **Output:** Tensor $T'$ of rank $k$ such that $\|T - T'\|_W \leq (1 + \varepsilon) \min_{B: \text{rank}(B) \leq k} \|T - B\|_W$.
3: for $j = 1$ to 3 do
4: $s_j \leftarrow O(r/\varepsilon)$
5: end for
6: Choose three sketching matrices $S_1 \in \mathbb{R}^{TB \times s_1}$, $S_2 \in \mathbb{R}^{NB \times s_2}$, and $S_3 \in \mathbb{R}^{NT \times s_3}$
7: for $j = 1$ to 2 do
8: for $i = 1$ to $s$ do
9: Create $r \times s_j$ variables for matrix $P_{i,j}$
10: Set $(\hat{U}_i)^j = T_i^j W_i^j S_j^T (P_{i,j} P_{i,j}^T)^{-1}$
11: end for
12: end for
13: for $i = 1$ to $S$ do
14: Set $(\hat{U}_3)^i = T_3^i W_3^i S_3^T (P_3^T P_3)^{-1}$
15: end for
16: Form $\|W \cdot (\hat{U}_1 \otimes \hat{U}_2 \otimes \hat{U}_3 - T)\|_F$
17: for $i = 1$ to $r$ do
18: Add constraint $\|U_1^i\|_2^3 \|\hat{U}_2^i\|_2^3 \|\hat{U}_3^i\|_2^3 \leq L$
19: end for
20: Run Polynomial System Verifier to get $U_1, U_2$, and $U_3$
21: return $U_1 \otimes U_2 \otimes U_3$

C. Missing Proofs

**Theorem C.1.** Suppose $T_{N,T}^*$ exists for all $N$ and $T$. Then

- $k \leq O\left(\log_2(1 - \delta)/N\right)$, then for any $\varepsilon > 0$, $\Pr\left[\|T_{N,T} - T_{N,T}^*\|_2/\sqrt{NT} > \varepsilon\right] \to 0$ as $N \to \infty$.
- If assumption 15 holds and $k \leq \max\{O(\log_4(T/\log T)), O\left(\log_2(1 - \delta)/T\right)\}$, $\Pr\left[\|T_{N,T} - T_{N,T}^*\|_2/\sqrt{NT} > \varepsilon\right] \to 0$ as $T \to \infty$.

**Proof.** The weighted log-likelihood function with respect to a tensor $T_{N,T}$ is given as:

$$L_{N,T}(T_{N,T}) = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} w_{i,t} \log \Pr[Y_{i,t}|T_{N,T}]$$

First we compute the expected value of the weighted log-likelihood with respect to the policy $\mathcal{P}$ (i.e. the random variables $\{Y_{i,t}\}_{i=1}^{N}, \{A_{i,1,t}\}_{i=1}^{N}, \{X_{i,1,t}\}_{i=1}^{N}$ and the true underlying tensor $T_{N,T}^*$). We write $\ell^*_{N,T}(T_{N,T})$ to denote this quantity as it only depends on the tensor $T_{N,T}$.

$$\ell^*_{N,T}(T_{N,T}) = E_{P,T_{N,T}^*} [L_{N,T}(T_{N,T})]$$

$$= \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} E_{A_{i,1,t}, Y_{i,1,t}, X_{i,1,t}} \left[w_{i,t} \log \Pr[Y_{i,t}|T_{N,T}]\right]$$

$$= \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} 2^k \sum_{a_{i,t-k+1:1}} \Pr[a_{i,t-k+1:1}] \int \log \Pr[Y_{i,t} = Y_{i,t}(a_{i,t-k+1:1})|T_{N,T}] \Pr[Y_{i,t} = Y_{i,t}(a_{i,t-k+1:1})|T_{N,T}] dY_{i,t}$$

(16)

The last line follows from lemma C.2. We want to show that $\|\hat{T}_{N,T} - T_{N,T}^*\|$ becomes small as either $N$ or $T$ increases. Our proof is based on the proof of the consistency of the maximum likelihood given in (Newey & McFadden, 1994). We
write $\Theta_{N,T}$ to denote the parameter space \( \{ T \in \mathbb{R}^{T \times N \times B} : \text{rank}(T) \leq r, \|T\|_{\infty} \leq L \} \). $\Theta_{N,T}$ is bounded but need not be closed because of issues with border tensor. It is known that there might exist a sequence of rank $r$ tensors whose limit is a rank $r + 1$ tensor (Bini, 1986). However, we can exploit the concavity of the log-likelihood function to overcome this problem.

First consider a neighborhood $\mathcal{B}$ of radius $d$ centered at $T^*_{N,T}$ and contained within the interior of $\Theta_{N,T}$.

\[ \mathcal{B} = \{ T \in \mathbb{R}^{N \times T \times B} : \| T - T^*_{N,T} \|_2 / \sqrt{NT} \leq d \} \]

Lemma C.8 proves that $L_{N,T}(\cdot)$ is concave over $\Theta_{N,T}$. Since a concave function is continuous over the interior of its domain, $L_{N,T}(\cdot)$ is continuous over $\mathcal{B}$. Moreover, unlike $\Theta_{N,T}$, set $\mathcal{B}$ is a compact set. This implies that there exists a maximizer for $L_{N,T}(\cdot)$ over $\mathcal{B}$. Suppose $T^*_N$ be the maximizer of $L_{N,T}(\cdot)$ over $\mathcal{B}$. Consider any $T \in \Theta_{N,T} \setminus \mathcal{B}$. Then there exists $\lambda < 1$ such that $T' = \lambda T^*_N + (1 - \lambda) T$ and $T' \in \mathcal{B}$. This gives us the following:

\[ L_{N,T}(T) \geq L_{N,T}(T') = L_{N,T}(\lambda T^*_N + (1 - \lambda) T) \geq \lambda L_{N,T}(T^*_N) + (1 - \lambda) L_{N,T}(T) \]

This first line uses the concavity of $L_{N,T}(\cdot)$ (lemma C.8). This proves that $T^*_N$ is actually the maximizer of $L_{N,T}(\cdot)$ over the entire parameter space $\Theta_{N,T}$. Moreover, any other maximizer $T_{N,T}$ of $L_{N,T}(\cdot)$ must be inside $\mathcal{B}$. Otherwise, suppose $T_{N,T}$ maximizes $L_{N,T}(\cdot)$ and $T_{N,T} \in \Theta_{N,T} \setminus \mathcal{B}$. Then for $\varepsilon = 2k^2d^2\delta$ we have with probability at least $1 - O(1/\varepsilon^2 N^p)$, $\varepsilon/3 > L_{N,T}(T) + \varepsilon > L_{N,T}(T) + 2\varepsilon/3$

The first and the third inequality uses lemma C.4 and the second inequality uses lemma C.13. Therefore, with probability at least $1 - O(1/4k^2d^2\delta N^p)$ all the maximizers of $L_{N,T}(\cdot)$ must be inside the ball $\mathcal{B}$. This proves that for any $d$ we can choose $N$ large enough such that with high probability $L_{N,T}(\cdot)$ lies within a $d$ neighborhood of $T^*_N$. This proves the consistency of the estimate when $N$ increases to infinity. The proof of consistency when the number of time periods $T$ increases to infinity is similar. \( \square \)

**Lemma C.2.**

\[ \text{E}_{y_{i,t}, y_{i,t}} \left[ w_{i,t} \log \Pr \left[ y_{i,t} | T \right] \right] = 2^k \sum_{a_{i,t-k+1:t}} \Pr \left[ a_{i,t-k+1:t} \right] \int \log \Pr \left[ y_{i,t} = y_{i,t}(a_{i,t-k+1:t}) | T \right] \Pr \left[ y_{i,t} = y_{i,t}(a_{i,t-k+1:t}) | T^*_N \right] dY_{i,t} \]

**Proof.**

\[ \text{E}_{y_{i,t}, y_{i,t}} \left[ w_{i,t} \log \Pr \left[ y_{i,t} | T \right] \right] = \text{E}_{y_{i,t}} \left[ \text{E}_{y_{i,t}} \left[ w_{i,t} \log \Pr \left[ y_{i,t} | T \right] \right] \right] \]

\[ = \text{E}_{y_{i,t}} \left[ w_{i,t} \int \log \Pr \left[ y_{i,t} | T \right] \Pr \left[ y_{i,t} | T^*_N \right] dY_{i,t} \right] \]

\[ = \int \sum_{y_{i,t-1}} \sum_{a_{i,t-1}} \Pr \left[ a_{i,t-1}, x_{i,t-1:t}, d_{i,t} = 1 \right] w_{i,t}(a_{i,t-1}, y_{i,t-1}) \times \]

\[ \int \log \Pr \left[ y_{i,t} = y_{i,t}(a_{i,t-k+1:t}) | T \right] \Pr \left[ y_{i,t} = y_{i,t}(a_{i,t-k+1:t}) | T^*_N \right] dY_{i,t} dY_{i,t-1} \]

\[ = \int \sum_{y_{i,t-2k+1:t}} \sum_{a_{i,t-2k+1:t}} \Pr \left[ a_{i,t-2k+1:t}, y_{i,t-2k+1:t-1} \right] w_{i,t}(a_{i,t-2k+1:t}, y_{i,t-2k+1:t-1}) \times \]

\[ \int \log \Pr \left[ y_{i,t} = y_{i,t}(a_{i,t-k+1:t}) | T \right] \Pr \left[ y_{i,t} = y_{i,t}(a_{i,t-k+1:t}) | T^*_N \right] dY_{i,t} dY_{i,t-2k+1:t-1} \times \]
The last step marginalizes out the history from time 1 to time \( t - 2k \) and the covariates from time \( t - 2k + 1 \) to \( t \).

\[
\int \sum_{Y_{i,t-2k+1:t-1}} \Pr \left[ a_{i,t-2k+1:t}, Y_{i,t-2k+1:t-1} \right] \frac{\prod_{s=k+1}^{t} \Pr \left[ a_{i,s} \mid a_{i,t-2k+1:s-1} \right]}{\Pr \left[ a_{i,t-2k+1:t} \mid a_{i,t-2k+1:k}, Y_{i,t-2k+1:t-1} \right]} \times 
\int \log \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T} \right] \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T} \right] dY_{i,t} dY_{i,t-2k+1:t-1} \\
= \int \sum_{Y_{i,t-2k+1:t-1}} \Pr \left[ a_{i,t-2k+1:t}, Y_{i,t-2k+1:t-1} \right] \Pr \left[ a_{i,t-2k+1:t} \mid a_{i,t-2k+1:k} \right] \times 
\int \log \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T} \right] \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T} \right] dY_{i,t} dY_{i,t-2k+1:t-1} \\
= \sum_{a_{i,t-k+1:t}} \Pr \left[ a_{i,t-k+1:t} \right] \int \log \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T} \right] \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T} \right] dY_{i,t} \\
= 2^{k} \sum_{a_{i,t-k+1:t}} \Pr \left[ a_{i,t-k+1:t} \right] \int \log \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T} \right] \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T} \right] dY_{i,t}
\]

Lemma C3. \( \ell_{N,T}^{*}(\mathbf{T}_{N,T}) \leq \ell_{N,T}^{*}(\mathbf{T}_{N,T}^{*}) \) for any tensor \( \mathbf{T}_{N,T} \).

Proof. Fix \( i, t \) and \( a_{i,t-k+1:t} \). Then conditioned on \( \mathbf{T}_{N,T} \), \( Y_{i,t} \) is distributed according to a normal distribution with mean \( \mathbf{T}_{N,T}(i, t, a_{i,t-k+1:t}) \) (= \( \mu^{*} \) say) and variance 1. And conditioned on \( \mathbf{T}_{N,T} \), \( Y_{i,t} \) is distributed according to a normal distribution with mean \( \mathbf{T}_{N,T}(i, t, a_{i,t-k+1:t}) \) (= \( \mu \) say) and variance 1. Now consider the following term:

\[
\int \log \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T} \right] \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T} \right] dY_{i,t} \\
- \int \log \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T}^{*} \right] \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T} \right] dY_{i,t} \\
= \int \log \left( \frac{\Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T} \right]}{\Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T}^{*} \right]} \right) \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T}^{*} \right] dY_{i,t} \\
\leq \int \left( \frac{\Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T} \right]}{\Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T}^{*} \right]} - 1 \right) \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T} \right] dY_{i,t} \\
= \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T} \right] dY_{i,t} - \int \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T}^{*} \right] dY_{i,t} = 1 - 1 = 0
\]

This proves that

\[
\int \log \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T} \right] \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T} \right] dY_{i,t} \\
\leq \int \log \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T}^{*} \right] \Pr \left[ Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) \mid \mathbf{T}_{N,T}^{*} \right] dY_{i,t}
\]

Since the above holds for any \( i, t \) and \( a_{i,t-k+1:t} \), the claim follows from the expression of \( \ell_{N,T}^{*}(\mathbf{T}_{N,T}) \) given in eq. (16). \( \square \)

Lemma C4. Suppose \( \| \mathbf{T}_{N,T} \|_{\infty} \leq L \). Then,

1. If \( k \leq O \left( \log_{2}(1-\delta)/\delta \right) N \), \( \Pr \left[ \left\| \mathbf{L}_{N,T}(\mathbf{T}_{N,T}) - \ell^{*}(\mathbf{T}_{N,T}) \right\| > \varepsilon \right] \to 0 \) as \( N \to \infty \).

2. If assumption 15 holds and \( k \leq \max \{ O \left( \log_{4} T - \log_{4} \log T \right), O \left( \log_{2}(1-\delta)/\delta \right) T \} \), \( \Pr \left[ \left\| \mathbf{L}_{N,T}(\mathbf{T}_{N,T}) - \ell^{*}(\mathbf{T}_{N,T}) \right\| > \varepsilon \right] \to 0 \) as \( T \to \infty \).
We now consider the remaining terms. There are two cases. Let \( \delta \) be the parameter. This gives the following bound.

\[
\Pr[[L_{N,T}(T_{N,T}) - \epsilon^*(T_{N,T})] > \epsilon] \leq \frac{\text{Var} \left( L_{N,T}(T_{N,T}) \right)}{\epsilon^2} \leq \frac{1}{\epsilon^2 N^2 T^2} \sum_{i,t} \text{Var} \left( w_{i,t} \log \Pr \left[ Y_{i,t} | T_{N,T} \right] \right)
\]

Now we bound each term in the summation. Fix any \( a_{t-1:t+1} = p \). This fixes the distribution of \( Y_{i,t} \). Then we have,

\[
\mathbb{E}_{Y_{i,t}} \left[ \log^2 \Pr \left[ Y_{i,t} | T_{N,T} \right] \right] = \int_{0}^{1} \log^2 \Pr \left[ Y_{i,t} | T_{N,T} \right] \Pr \left[ Y_{i,t} | T_{N,T}^* \right] dY_{i,t}
\]

Since both \( T_{N,T}^*(i,t,p) \) and \( T_{N,T}(i,t,p) \) are bounded by \( L \), there exists a constant \( L_1 \) such that \( \mathbb{E}_{Y_{i,t}} \left[ \log^2 \Pr \left[ Y_{i,t} | T_{N,T} \right] \right] \leq L_1 \). This gives the following bound.

\[
\mathbb{E}_{Y_{i,t}} \left[ w_{i,t} \log \Pr \left[ Y_{i,t} | T_{N,T} \right] \right] \leq L_1 \left( \frac{2(1-\delta)}{\delta} \right)^k \]

We now consider the remaining terms. There are two cases.

**Case 1:** Suppose \( t < t' \leq t + 2k \).

\[
\mathbb{E} \left[ w_{i,t} w_{i,t'} \log \Pr \left[ Y_{i,t} | T_{N,T} \right] \log \Pr \left[ Y_{i,t'} | T_{N,T} \right] \right] \leq \mathbb{E} \left[ w_{i,t} w_{i,t'} \log \Pr \left[ Y_{i,t} | T_{N,T} \right] \right] \log \Pr \left[ Y_{i,t'} | T_{N,T} \right]
\]

By an argument same as before, for any realization of \( a_{t'-k+1:t'} \) we can bound \( \mathbb{E}_{Y_{i,t'}} \left[ \log \Pr \left[ Y_{i,t'} \right] \right] \) by a constant, say \( L_2 \). This gives us the following bound.

\[
L_2 \left( \frac{2(1-\delta)}{\delta} \right)^k \mathbb{E}_{Y_{i,t}} \left[ \log \Pr \left[ Y_{i,t} | T_{N,T} \right] \right]
\]
Case 2: Suppose $t' > t + 2k$. We can bound the corresponding covariance term by $L_2^2 \left( \frac{2(1-\delta)}{\delta} \right)^k$ by an argument same as before. This gives us the following bound:

$$\Pr \left[ |L_{N,T}(T_{N,T}) - \ell^*(T_{N,T})| > \varepsilon \right] \leq \frac{1}{\varepsilon^2 N^2 T^2} \left[ NTL_1 \left( \frac{2(1-\delta)}{\delta} \right)^k + 2NTL_2 \left( \frac{2(1-\delta)}{\delta} \right)^k \right] = O \left( \frac{1}{\varepsilon^2 N^p} \right)$$

for some $0 < p < 1$ as long as $k = O \left( \log_{2(1-\delta)/\delta} N \right)$. This gives us the first result.

Now suppose assumption 15 holds. Then, lemma C.12 gives us a bound of $O \left( 4^k (t' - t)^{-\gamma} \right)$ for some constant $\gamma > 0$. This gives us the following bound:

$$\Pr \left[ |L_{N,T}(T_{N,T}) - \ell^*(T_{N,T})| > \varepsilon \right] \leq \frac{1}{\varepsilon^2 N^2 T^2} \left[ NTL_1 \left( \frac{2(1-\delta)}{\delta} \right)^k + 4NTkL_2 \left( \frac{2(1-\delta)}{\delta} \right)^k + 2N4^k T \log T \right]$$

$$= O \left( \frac{1}{\varepsilon^2 T^p} \right)$$

for some $0 < p < 1$ as long as $k = O \left( \log_{2(1-\delta)/\delta} T \right)$ and $4^k \log T \leq T$.

**Lemma C.5.** $w_{i,t} \leq \left( \frac{1-\delta}{\delta} \right)^k$

**Proof.**

$$w_{i,t} = \prod_{s=t-k+1}^{t} \frac{\Pr \left[ A_{i,s} | A_{i,t-2k+1:s-1} \right]}{\Pr \left[ A_{i,s} | A_{i,t-2k+1:s-1}, Y_{i,t-2k+1:s-1} \right]}$$

Recall that the given policy satisfies positivity with constant $\delta$ i.e. for each $a_{i,1:t-1}, x_{i,1:t-1}, y_{i,1:t-1}$, we have

$$\delta < \Pr \left[ a_{i,t} | a_{i,1:t-1}, x_{i,1:t-1}, y_{i,1:t-1} \right] < 1 - \delta$$

Positivity implies that $\Pr \left[ A_{i,s} | A_{i,t-2k+1:s-1} = A_{i,s} \right] = \Pr \left[ A_{i,s} | A_{i,1:s-1}, X_{i,1:s-1}, Y_{i,1:s-1} \right] \Pr \left[ A_{i,1:s-1}, X_{i,1:s-1}, Y_{i,1:s-1} \right] \leq 1 - \delta$. Now consider the term in the denominator.

$$\Pr \left[ A_{i,s} | A_{i,t-2k+1:s-1}, Y_{i,t-2k+1:s-1} = A_{i,s} \right] = \sum_{x_{i,t-2k+1:s-1}} \Pr \left[ A_{i,s} | A_{i,t-2k+1:s-1}, x_{i,t-2k+1:s-1}, Y_{i,t-2k+1:s-1} \right] \times \Pr \left[ x_{i,t-2k+1:s-1} | A_{i,t-2k+1:s-1}, Y_{i,t-2k+1:s-1} \right]$$

$$\geq \delta \sum_{x_{i,t-2k+1:s-1}} \Pr \left[ x_{i,t-2k+1:s-1} | A_{i,t-2k+1:s-1}, Y_{i,t-2k+1:s-1} \right] = \delta$$

These two results imply that each term in the product of $w_{i,t}$ is bounded by $(1 - \delta)/\delta$ and we get the desired bound on $w_{i,t}$.

**Lemma C.6.** Let $S(i, t)$ be the random variable denoting the slice chosen by the policy for user $i$ at time $t$. If assumption 15 holds, then $\text{cov} \left( w_{i,t}T^2(i, t, S(i, t)), w_{i,t}T^2(i, t', S(i, t')) \right) \leq O \left( L_2^2 \left( \frac{1-\delta}{\delta} \right)^{2k} (t' - t)^{-\gamma} \right)$.

**Proof.** The proof is analogous to the proof of lemma C.8 and is omitted.

**Lemma C.7.** $\ell^*_{N,T}(T)$ is concave in $T$. 

Proof. From equation 16 we get

\[ \ell_{N,T}^a(T) = \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T 2^k \sum_{a_{i,t-k+1:t}} \Pr [a_{i,t-k+1:t}] \int \log \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) | T] \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) | T_{N,T}^a] dY_{i,t} \]

Now fix any value of \( a_{i,t-k+1:t} \). Then \( \log \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) | T] \) is concave in \( T \). Then

\[ \int \log \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) | \lambda T_1 + (1 - \lambda) T_2] \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) | T_{N,T}^a] dY_{i,t} \]

\[ \geq \frac{\lambda}{\int \log \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) | T_1] \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) | T_{N,T}^a] dY_{i,t}} \]

\[ + (1 - \lambda) \frac{\log \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) | T_2] \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) | T_{N,T}^a] dY_{i,t}}{\int \log \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) | T_2] \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t}) | T_{N,T}^a] dY_{i,t}} \]

This proves that the function inside the integral is concave. Since \( \ell_{N,T}^a(T) \) is just a non-negative weighted sum of such functions, it is also concave. \( \square \)

**Lemma C.8.** \( L_{N,T}(T_{N,T}) \) is concave in \( T_{N,T} \).

Proof.

\[ L_{N,T}(T_{N,T}) = \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T w_{i,t} \log \Pr [Y_{i,t} | T_{N,T}] \]

\[ = - \frac{1}{2NT} \sum_{p=1}^B \sum_{(i,t) \in O_p} w_{i,t} (Y_{i,t} - T_{N,T}(i,t,p))^2 + \text{const} \]

Each term inside the summation i.e. \(- (Y_{i,t} - T_{N,T}(i,t,p))^2 \) is a concave function. The likelihood function is a non-negative weighted sum of concave functions and is also concave. \( \square \)

**Lemma C.9.**

\[ E \left[ \sum_{p=1}^B \sum_{(i,t) \in O_p} w_{i,t} T^2(i,t,p) \right] = \sum_{i=1}^N \sum_{t=1}^T \sum_{p=1}^B \Pr [(i,t) \in O_p] T^2(i,t,p) \]

Proof. Let \( S(i,t) \) be the random variable denoting the slice selected by the policy for subject \( i \) at time \( t \). From the linearity of expectation, we have

\[ E \left[ \sum_{p=1}^B \sum_{(i,t) \in O_p} w_{i,t} T^2(i,t,p) \right] = \sum_{i=1}^N \sum_{t=1}^T E \left[ w_{i,t} T^2(i,t,S(i,t)) \right] \]
We now consider each term inside the summation.

\[
E \left[ w_{i,t} T^2(i, t, S(i, t)) \right] = \int_{y_{i,t-1}} \sum_{x_{i,t-1}} \sum_{a_{i,t-1}} \Pr[a_{i,t-1}, x_{i,t-1}, y_{i,t-1}] w_{i,t}(a_{i,t-1}, x_{i,t-1}, y_{i,t-1}) T^2(i, t, s(i, t))
\]

\[
= \int_{y_{i,t-1}} \sum_{x_{i,t-1}} \sum_{a_{i,t-1}} \Pr[a_{i,t-2k+1:t}, x_{i,t-2k+1:t}, y_{i,t-2k+1:t}] w_{i,t}(a_{i,t-2k+1:t}, x_{i,t-2k+1:t}, y_{i,t-2k+1:t}) T^2(i, t, s(i, t))
\]

\[
= \int_{y_{i,t-1}} \sum_{x_{i,t-1}} \sum_{a_{i,t-1}} \Pr[a_{i,t-2k+1:t}] \Pr[a_{i,t-2k+1:t-1}] T^2(i, t, s(i, t))
\]

\[
= \sum_{a_{i,t-2k+1:t}} \Pr[a_{i,t-2k+1:t}] \Pr[a_{i,t-2k+1:t-1}] T^2(i, t, s(i, t)) = \sum_{a_{i,t-2k+1:t}} \Pr[a_{i,t-2k+1:t}] T^2(i, t, s(i, t)) = \sum_{p=1}^B \Pr[S(i, t) = p] T^2(i, t, p)
\]

Lemma C.10.

\[
\Pr \left[ \frac{1}{NT} \sum_{p=1}^B \sum_{(i,t) \in O_p} w_{i,t} T^2(i, t, p) - \frac{1}{NT} \|T\|_{W}^2 \geq \varepsilon \right] \leq 2 \exp \left( -\frac{2N \varepsilon^2}{L^4 \left( \frac{1}{\delta} \right)^{2k}} \right)
\]

**Proof.** Suppose \(S(i, t)\) be the slice selected by the policy for agent \(i\) at time \(t\). Then we have \(1/NT \sum_{p=1}^B \sum_{(i,t) \in O_p} w_{i,t} T^2(i, t, p) = 1/NT \sum_{i=1}^N \sum_{t=1}^T w_{i,t} T^2(i, t, S(i, t)).\) Observe that for each \(i\),

\[
\frac{1}{T} \sum_{t=1}^T w_{i,t} T^2(i, t, S(i, t)) \in \left[ 0, L^2 \left( \frac{1 - \delta}{\delta} \right)^k \right]
\]

Now we apply the Hoeffding inequality considering the random variables \(\left\{ \frac{1}{T} \sum_{t=1}^T w_{i,t} T^2(i, t, S(i, t)) \right\}_{i=1}^N\) as independent random variables and get the following bound.

\[
\Pr \left[ \frac{1}{NT} \sum_{p=1}^B \sum_{(i,t) \in O_p} w_{i,t} T^2(i, t, p) - \frac{1}{NT} \|T\|_{W}^2 \geq \varepsilon \right] \leq 2 \exp \left( -\frac{2N \varepsilon^2}{L^4 \left( \frac{1}{\delta} \right)^{2k}} \right)
\]

Lemma C.11.

\[
\Pr \left[ \frac{1}{NT} \sum_{p=1}^B \sum_{(i,t) \in O_p} w_{i,t} T^2(i, t, p) - \frac{1}{NT} \|T\|_{W}^2 \geq \varepsilon \right] \leq O \left( \frac{L^4 \left( \frac{1 - \delta}{\delta} \right)^{2k}}{\varepsilon^2 T/\log T} \right)
\]

**Proof.** Suppose \(S(i, t)\) be the slice selected by the policy for agent \(i\) at time \(t\). Then we have \(1/NT \sum_{p=1}^B \sum_{(i,t) \in O_p} w_{i,t} T^2(i, t, p) = 1/NT \sum_{i=1}^N \sum_{t=1}^T w_{i,t} T^2(i, t, S(i, t)).\) Observe that for each \(t\),

\[
\frac{1}{N} \sum_{i=1}^N w_{i,t} T^2(i, t, S(i, t)) \in \left[ 0, L^2 \left( \frac{1 - \delta}{\delta} \right)^k \right]
\]
Since logarithm of the probabilities are negative, the second equality changes them to their absolute values. The next equality

\[
\Pr \left[ \frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t) \in O_p} w_{i,t} T_2^2(i,t,p) - \frac{1}{NT} \| T_2^2 \|_W \geq \varepsilon \right] \leq \frac{\Var \left( \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} w_{i,t} T_2^2(i,t,S(i,t)) \right)}{\varepsilon^2}
\]

\[
= \frac{1}{\varepsilon^2 T^2} \left( \sum_{t=1}^{T} \Var \left( \frac{1}{N} \sum_{i=1}^{N} w_{i,t} T_2^2(i,t,S(i,t)) \right) + 2 \sum_{t'=t+2k} \cov \left( \frac{1}{N} \sum_{i=1}^{N} w_{i,t} T_2^2(i,t,S(i,t)), \frac{1}{N} \sum_{i=1}^{N} w_{i,t'} T_2^2(i,t',S(i,t')) \right) \right)
\]

\[
\leq \frac{1}{\varepsilon^2 T^2} \left( TL^4 \left( \frac{1-\delta}{\delta} \right)^{2k} + 2 TL^4 \left( \frac{1-\delta}{\delta} \right)^{2k} \right)
\]

Lemma C.12. Suppose \( t' > t + 2k \) and assumption 15 holds. Then the following is true.

\[
|\cov ( w_{i,t} \log \Pr [ Y_{i,t} | T_{N,T} ], w_{i,t'} \log \Pr [ Y_{i,t'} | T_{N,T} ]) | \leq O \left( 4^{k} (t'-t)^{-\gamma} \right)
\]

Proof. Let us write \( \mathcal{H}_{i,1:t} = (a_{i,1:t}, x_{i,1:t}, y_{i,1:t-1}) \) to denote the history upto time \( t \) excluding the outcome at time \( t \).

\[
|\cov ( w_{i,t} \log \Pr [ Y_{i,t} | T_{N,T} ], w_{i,t'} \log \Pr [ Y_{i,t'} | T_{N,T} ] ) |
\]

\[
= \left| \E [ w_{i,t} \log \Pr [ Y_{i,t} | T_{N,T} ], w_{i,t'} \log \Pr [ Y_{i,t'} | T_{N,T} ] ] \right|
\]

\[
= \left| \E [ w_{i,t} \log \Pr [ Y_{i,t} | T_{N,T} ] ] \log \Pr [ Y_{i,t'} | T_{N,T} ] \right| - \left| \E [ w_{i,t'} \log \Pr [ Y_{i,t'} | T_{N,T} ] ] \log \Pr [ Y_{i,t} | T_{N,T} ] \right|
\]

\[
= \left| \sum_{a_{i,1:t}, x_{i,1:t} \uparrow y_{i,1:t}} \Pr [ \mathcal{H}_{i,1:t}, w_{i,t} | \mathcal{H}_{i,1:t} ] \log \Pr [ Y_{i,t} | Y_{i,t} = Y_{i,t} (a_{i,t-k+1:t}) | T_{N,T} ] \Pr [ Y_{i,t'} | Y_{i,t'} = Y_{i,t'} (a_{i,t'-k+1:t'}) | T_{N,T} ] dY_{i,t} \right|
\]

\[
\times \sum_{a_{i,1:t}, x_{i,1:t} \uparrow y_{i,1:t}} \Pr [ \mathcal{H}_{i,1:t}, w_{i,t'} | \mathcal{H}_{i,1:t} ] \log \Pr [ Y_{i,t'} | Y_{i,t'} = Y_{i,t'} (a_{i,t'-k+1:t'}) | T_{N,T} ] dY_{i,t'}
\]

Since logarithm of the probabilities are negative, the second equality changes them to their absolute values. The next equality
just expands the individual terms. Integrating out the covariates we get,

\[
\sum_{a_{i,t}} \int_{y_{i,t}} \Pr [a_{i,t}, y_{i,t-1}] w_{i,t}(a_{i,t}, y_{i,t-1}) \log \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t})|T_{N,T}] \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t})|T_{N,T}^*] dY_{i,t} \\
\times \sum_{a_{i,t}} \int_{y_{i,t}} \Pr [\tilde{a}_{i,t+1:t'}, \tilde{y}_{i,t+1:t'}-1] w_{i,t'}(\tilde{a}_{i,t+1:t'}, \tilde{y}_{i,t+1:t'}-1) \log \Pr [\tilde{Y}_{i,t'} = Y_{i,t'}(\tilde{a}_{i,t-k+1:t'})|T_{N,T}] \\
\Pr [\tilde{Y}_{i,t'} = Y_{i,t'}(\tilde{a}_{i,t-k+1:t'})|T_{N,T}^*] dY_{i,t+1:t'} \\
- \sum_{a_{i,t}} \int_{y_{i,t}} \Pr [a_{i,t}, y_{i,t-1}] w_{i,t}(a_{i,t}, y_{i,t-1}) \log \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t})|T_{N,T}] \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t})|T_{N,T}^*] dY_{i,t} \\
\times \sum_{a_{i,t}} \int_{y_{i,t}} \Pr [\tilde{a}_{i,t+1:t'}, \tilde{y}_{i,t+1:t'}-1] w_{i,t'}(\tilde{a}_{i,t+1:t'}, \tilde{y}_{i,t+1:t'}-1) \log \Pr [\tilde{Y}_{i,t'} = Y_{i,t'}(\tilde{a}_{i,t-k+1:t'})|T_{N,T}] \\
\Pr [\tilde{Y}_{i,t'} = Y_{i,t'}(\tilde{a}_{i,t-k+1:t'})|T_{N,T}^*] dY_{i,t+1:t'} \\
\geq \sum_{a_{i,t}} \int_{y_{i,t}} \Pr [a_{i,t}, y_{i,t-1}] w_{i,t}(a_{i,t}, y_{i,t-1}) \log \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t})|T_{N,T}] \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t})|T_{N,T}^*] dY_{i,t} \\
\times \sum_{a_{i,t}} \int_{y_{i,t}} \Pr [\tilde{a}_{i,t+1:t'}, \tilde{y}_{i,t+1:t'}-1] w_{i,t'}(\tilde{a}_{i,t+1:t'}, \tilde{y}_{i,t+1:t'}-1) \log \Pr [\tilde{Y}_{i,t'} = Y_{i,t'}(\tilde{a}_{i,t-k+1:t'})|T_{N,T}] \\
\Pr [\tilde{Y}_{i,t'} = Y_{i,t'}(\tilde{a}_{i,t-k+1:t'})|T_{N,T}^*] dY_{i,t+1:t'} \\
\times \log \Pr [\tilde{Y}_{i,t'} = Y_{i,t'}(\tilde{a}_{i,t-k+1:t'})|T_{N,T}] \Pr [\tilde{Y}_{i,t'} = Y_{i,t'}(\tilde{a}_{i,t-k+1:t'})|T_{N,T}^*] dY_{i,t'}. \\
(18)
\]

Now we marginalize the history from time \( t + 1 \) to \( t' - 2k \) from the last summation and integration. This gives us the following bound on the covariance:

\[
\sum_{a_{i,t}} \int_{y_{i,t}} \Pr [a_{i,t}, y_{i,t-1}] w_{i,t}(a_{i,t}, y_{i,t-1}) \log \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t})|T_{N,T}] \Pr [Y_{i,t} = Y_{i,t}(a_{i,t-k+1:t})|T_{N,T}^*] dY_{i,t} \\
\times \sum_{a_{i,t}} \int_{y_{i,t}} \Pr [\tilde{a}_{i,t+1:t'}, \tilde{y}_{i,t+1:t'}-1] w_{i,t'}(\tilde{a}_{i,t+1:t'}, \tilde{y}_{i,t+1:t'}-1) \log \Pr [\tilde{Y}_{i,t'} = Y_{i,t'}(\tilde{a}_{i,t-k+1:t'})|T_{N,T}] \\
\Pr [\tilde{Y}_{i,t'} = Y_{i,t'}(\tilde{a}_{i,t-k+1:t'})|T_{N,T}^*] dY_{i,t'}. \\
(19)
\]
We simplify two terms.

\[
\Pr \left[ \tilde{a}_{i,t'-2k+1:t'}; \tilde{y}_{i,t'-2k+1:t'} \right] \times \frac{\Pr \left[ \tilde{a}_{i,t'-k+1:t'}; \tilde{a}_{i,t'-2k+1:t'} \right]}{\Pr \left[ \tilde{a}_{i,t'-k+1:t'}; \tilde{a}_{i,t'-2k+1:t'} \right]} = \Pr \left[ \tilde{a}_{i,t'-k+1:t'}; \tilde{y}_{i,t'-2k+1:t'} \right] \times \frac{\Pr \left[ \tilde{a}_{i,t'-2k+1:t'} \right]}{\Pr \left[ \tilde{a}_{i,t'-2k+1:t'} \right]} \]

And,

\[
\Pr \left[ \tilde{a}_{i,t'-2k+1:t'}, \tilde{y}_{i,t'-2k+1:t'} \mid \tilde{a}_{i,1:t}, \tilde{y}_{i,1:t} \right] \times \frac{\Pr \left[ \tilde{a}_{i,t'-k+1:t'}; \tilde{a}_{i,t'-2k+1:t'} \right]}{\Pr \left[ \tilde{a}_{i,t'-k+1:t'}; \tilde{a}_{i,t'-2k+1:t'} \right]}
\]

Substituting the two results above in eq. 19 and integrating out \( \tilde{y}_{i,t'-2k+1:t'} \) we get,

\[
\sum_{\tilde{a}_{i,t'-2k+1:t'}} \int \left( \Pr \left[ \tilde{a}_{i,t'-2k+1:t'} \mid \tilde{a}_{i,1:t}, \tilde{y}_{i,1:t} \right] \Pr \left[ \tilde{y}_{i,t'-k+1:t'} \mid \tilde{a}_{i,t'-2k+1:t'} \right] \right) d\tilde{y}_{i,t'-k+1:t'}
\]

Now integrating out \( \tilde{y}_{i,t'-k+1:t'} \) we get

\[
\sum_{\tilde{a}_{i,t'-2k+1:t'}} \int \left( \Pr \left[ \tilde{a}_{i,t'-2k+1:t'} \mid \tilde{a}_{i,1:t}, \tilde{y}_{i,1:t} \right] - \Pr \left[ \tilde{a}_{i,t'-2k+1:t'} \mid \tilde{a}_{i,1:t}, \tilde{y}_{i,1:t} \right] \right) d\tilde{y}_{i,t'}
\]

Now observe that \( \Pr \left[ \tilde{a}_{i,t'-2k+1:t'} \mid \tilde{a}_{i,1:t}, \tilde{y}_{i,1:t} \right] = \frac{\Pr \left[ \tilde{a}_{i,t'-2k+1:t'} \mid \tilde{a}_{i,1:t}, \tilde{y}_{i,1:t} \right] \Pr \left[ \tilde{y}_{i,t'} \mid \tilde{a}_{i,1:t}, \tilde{y}_{i,1:t} \right]}{\Pr \left[ \tilde{a}_{i,1:t}, \tilde{y}_{i,1:t} \right]} = \frac{\Pr \left[ \tilde{y}_{i,t'} \mid \tilde{a}_{i,1:t}, \tilde{y}_{i,1:t} \right]}{\Pr \left[ \tilde{a}_{i,1:t}, \tilde{y}_{i,1:t} \right]} \Pr \left[ \tilde{a}_{i,t'-2k+1:t'} \mid \tilde{a}_{i,1:t}, \tilde{y}_{i,1:t} \right] \)

Substituting this result we get

\[
\sum_{\tilde{a}_{i,t'-2k+1:t'}} \int \left( \Pr \left[ \tilde{a}_{i,t'-2k+1:t'} \mid \tilde{a}_{i,1:t}, \tilde{y}_{i,1:t} \right] - \Pr \left[ \tilde{a}_{i,t'-2k+1:t'} \mid \tilde{a}_{i,1:t}, \tilde{y}_{i,1:t} \right] \right) d\tilde{y}_{i,t'}
\]
Using assumption 15 we can bound the difference \((\Pr[\tilde{a}_{i,t'-k+1:t'} | a_{i,1:t}] - \Pr[\tilde{a}_{i,t'-k+1:t'}])\) by \(c(t' - t)^{-\gamma}\) for some constant \(c > 0\). This follows from the following observations:

- \(f(A_{i,1:t}) = \prod_j f_j(A_{i,j})\) where \(f_j(A_{i,j}) = A_{i,j}\) if \(a_{i,j} = 1\) and \(f_j(A_{i,j}) = 1 - A_{i,j}\) otherwise. We choose \(g(A_{i,t'-k+1:t'})\) analogously.

- With these choices of \(f\) and \(g\) we have \(\text{cov}(f(A_{i,1:t}), A_{i,t'-k+1:t'}) = \Pr[a_{i,1:t}, a_{i,t'-k+1:t'}] - \Pr[a_{i,1:t}] \Pr[a_{i,t'-k+1:t'}]\) and \(E[f(A_{i,1:t})] = \Pr[a_{i,1:t}]\).

This gives us the following bound on the previous term.

\[
\begin{align*}
\sum a_{i,t'-k+1:t'} \int Y_i, t' | \tilde{Y}_i, t' & \Pr[\tilde{Y}_i, t' = Y_i, t' | T_{N,T}^*] d\tilde{Y}_i, t' \\
& = \sum a_{i,t'-k+1:t'} \mathbb{E}[\log \Pr[Y_i, t'], T_{N,T}^*] || \tilde{Y}_i, t'
\end{align*}
\]

Since both \(T_{N,T}^*(i, t, p)\) and \(T_{N,T}(i, t, p)\) are bounded by \(L\) there exists a constant \(L_3 > 0\) such that \(\mathbb{E}[\log \Pr[Y_i, t, T_{N,T}^*]] \leq L_3\). This gives us a bound of \((c(t' - t)^{-2}) L_3\). Substituting this bound in eq. 18 we get the following bound on covariance.

\[
\begin{align*}
(c(t' - t)^{-2} L_3 & \sum a_{i,1:t} \int y_{i,1:t} \Pr[a_{i,1:t}, y_{i,1:t}] \log \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}] | \\
& \times \Pr[Y_i, t = Y_i, t(a_{i,i+k+1:t}) | T_{N,T}^*] dY_i,1:t \\
& = c(t' - t)^{-2} L_3 \sum a_{i,1:t} \int y_{i,1:t} \Pr[a_{i,1:t}, y_{i,1:t}] \log \Pr[a_{i,t-k+1:t}, a_{i,t-k+1:t-k}] | \\
& \times \Pr[a_{i,t-k+1:t-k} | a_{i,t-k+1:t-k} - 1] \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}] | \\
& \times \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}^*] dY_i,1:t \\
& = c(t' - t)^{-2} L_3 \sum a_{i,1:t} \int y_{i,1:t} \log \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}] | \\
& \times \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}^*] dY_i,1:t \\
& = c(t' - t)^{-2} L_3 \sum a_{i,1:t} \int y_{i,1:t} \log \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}] | \\
& \times \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}^*] dY_i,1:t \\
& = c(t' - t)^{-2} L_3 \sum a_{i,1:t} \log \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}] | \\
& \times \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}^*] dY_i,1:t \\
& = c(t' - t)^{-2} L_3 \sum a_{i,1:t} \log \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}] | \\
& \times \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}^*] dY_i,1:t \\
& = c(t' - t)^{-2} L_3 \sum a_{i,1:t} \log \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}] | \\
& \times \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}^*] dY_i,1:t \\
& = c(t' - t)^{-2} L_3 \sum a_{i,1:t} \log \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}] | \\
& \times \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}^*] dY_i,1:t \\
& = c(t' - t)^{-2} L_3 \sum a_{i,1:t} \log \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}] | \\
& \times \Pr[Y_i, t = Y_i, t(a_{i,t-k+1:t}) | T_{N,T}^*] dY_i,1:t
\end{align*}
\]

As we argued before, we can bound \(\mathbb{E}[\log \Pr[Y_i, t, T_{N,T}^*] | a_{i,t-k+1:t}]\) by \(L_3\) for each choice of \(a_{i,t-k+1:t}\). This gives an overall bound of \((c(t' - t)^{-2}) L_3^2\) on the covariance.

\[
\text{Lemma C.13. Let } N \text{ be a d-neighborhood of } T_{N,T}^* \text{ i.e. } \mathcal{N} = \{T : \|T_{N,T}^* - T\|/\sqrt{NT} \leq d\}. \text{ Then for any } T' \notin N \text{ we have } \ell_{N,T}^*(T_{N,T}^*) > \ell_{N,T}^*(T') + 2^d d^2.
\]
Proof. Fix \( a_{i,t-k+1:t} = p \). Then

\[
\int \log \left( \frac{\Pr [Y_{it} = Y_{it}(p)|T_{N,T}^*]}{\Pr [Y_{it} = Y_{it}(p)|T']} \right) \Pr [Y_{it} = Y_{it}(p)|T_{N,T}^*] \, dY_{it} = - \int \frac{1}{2} \left( (Y_{it} - T_{N,T}^*(i,t,p))^2 - (Y_{it} - T'(i,t,p))^2 \right) \Pr [Y_{it} = Y_{it}(p)|T_{N,T}^*] \, dY_{it}
\]

\[
= \int \left\{ Y_{it} \left( T_{N,T}^*(i,t,p) - T'(i,t,p) \right) - \frac{1}{2} \left( (T_{N,T}^*(i,t,p))^2 - (T'(i,t,p))^2 \right) \right\} \Pr [Y_{it} = Y_{it}(p)|T_{N,T}^*] \, dY_{it}
\]

\[
= \frac{1}{2} (T_{N,T}^*(i,t,p) - T'(i,t,p))^2
\]

This gives us the following bound on the difference in log-likelihood

\[
\ell_{N,T}^* - \ell_{N,T}' = \frac{1}{NT} \sum_{i,t} 2^k \sum_p \Pr [p] \frac{1}{2} (T_{N,T}^*(i,t,p) - T'(i,t,p))^2
\]

\[
\geq \frac{2^k \delta}{NT} \| T_{N,T}^* - T' \|_2^2 > 2^k \delta d^2
\]

\[\square\]

Theorem C.14. Suppose \( \| T_{N,T} \|_\infty \leq L \).

- If \( k \leq O \left( \frac{\log(1-\delta)}{\delta} N \right) \), then \( \frac{1}{NT} \sum_{p=1}^B \sum_{(i,t) \in O_p} w_{i,t} Y_{i,t} - \hat{T}_{N,T}(i,t,p) \) \( \leq \) OPT + O \( \left( \frac{\epsilon^2}{N^{1/4}} \right) \) with probability at least 1 \(-\exp\left(-N^{1/4}\right)\).

- If assumption 15 holds, then \( \frac{1}{NT} \sum_{p=1}^B \sum_{(i,t) \in O_p} w_{i,t} Y_{i,t} - \hat{T}_{N,T}(i,t,p) \) \( \leq \) OPT + O \( \left( \frac{\epsilon^2 T}{T^{1/4}} \right) \) with probability at least 1 \(-O\left(\log \frac{T}{\epsilon^2}\right)\).

Proof. Lemma C.10 proves that \( \Pr \left[ \sum_{p=1}^B \sum_{(i,t) \in O_p} w_{i,t} T(i,t,p)^2 \neq \|T\|_W^2 - \epsilon, \|T\|_W^2 + \epsilon \right] \leq O \left( \exp \left( -\frac{2N^2 \epsilon^2}{L^4 \left( \frac{1}{T^{1/4}} \right)^2} \right) \right) \) Suppose \( T_{N,T}^* \) solves 13 and \( \hat{T}_{N,T} \) solves 14, then we get the following bound with
probability at least \(1 - \exp\left(-\frac{2N^2}{L^2\left(\frac{1}{2^k}\right)^2}\right)\):

\[
\frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t) \in O_p} w_{i,t} \left(Y_{i,t} - \hat{T}_{N,T}(i,t,p)\right)^2
\]

\[
= \frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t) \in O_p} w_{i,t} Y_{i,t}^2 - \frac{2}{NT} \sum_{p=1}^{B} \sum_{(i,t) \in O_p} w_{i,t} Y_{i,t} \hat{T}_{N,T}(i,t,p) + \frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t) \in O_p} w_{i,t} \left(\hat{T}_{N,T}(i,t,p)\right)^2
\]

\[
\leq \frac{1}{NT} \left(\|Y_w\|_W^2 + \varepsilon\right) - 2 \sum_{i,t,p} W(i,t,p)Y_w(i,t,p)\hat{T}_{N,T}(i,t,p) + \frac{1}{NT} \left(\|\hat{T}_{N,T}\|_W^2 + \varepsilon\right)
\]

\[
= \frac{1}{NT} \|Y_w - \hat{T}_{N,T}\|_W^2 + \frac{2\varepsilon}{NT}
\]

\[
\leq \frac{1}{NT} \|Y_w - T^*_{N,T}\|_W^2 + \frac{2\varepsilon}{NT}
\]

\[
= \frac{1}{NT} \left[\|Y_w\|_W^2 - 2 \sum_{i,t,p} W(i,t,p)Y_w(i,t,p)T^*_N(i,t,p) + \|T^*_{N,T}\|_W^2\right] + \frac{2\varepsilon}{NT}
\]

\[
\leq \frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t) \in O_p} w(i,t)\left(Y_{i,t} - T^*_N(i,t,p)\right)^2 + \frac{4\varepsilon}{NT}
\]

The first and the third inequality use lemma C.10 and the second inequality uses the fact that \(\hat{T}_{N,T}\) is the optimal solution to 14. Now if we substitute \(k = 1/8 \log(1-\delta)/\delta \) and \(\varepsilon = O\left(\frac{L^2}{NT}\right)\), we get the first result.

Now suppose \(N\) is fixed. If assumption 15 holds, then using lemma C.11 we get with probability at least \(1 - O\left(\frac{L^4(1-\delta)^2k}{\varepsilon^2T/\log T}\right)\),

\[
\frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t) \in O_p} w_{i,t} \left(Y_{i,t} - \hat{T}_{N,T}(i,t,p)\right)^2 \leq \text{OPT} + \frac{4\varepsilon}{NT}
\]

If we substitute \(k = 1/8 \log(1-\delta)/\delta \) and \(\varepsilon = O\left(\frac{L^2}{NT^2}\right)\) we get the second result. \(\square\)

**Theorem C.15.** Suppose \(\|\hat{T}_{N,T}\|_\infty \leq L\).

- If \(k \leq O\left(\log(1-\delta)/\delta \right)N\), then \(\frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t) \in O_p} w_{i,t} \left(Y_{i,t} - \hat{T}_{N,T}(i,t,p)\right)^2 \leq (1 + \varepsilon)\text{OPT} + O\left(\frac{L^2}{N^{3/4}}\right)\) with probability at least 4/5.

- If assumption 15 holds, then \(\frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t) \in O_p} w_{i,t} \left(Y_{i,t} - \hat{T}_{N,T}(i,t,p)\right)^2 \leq (1 + \varepsilon)\text{OPT} + O\left(\frac{L^2}{T^{3/4}}\right)\) with probability at least 4/5.

**Proof.** Lemma C.10 proves that \(\Pr\left[\sum_{p=1}^{B} \sum_{(i,t) \in O_p} w_{i,t} T(i,t,p)^2 \neq \|\hat{T}\|_W^2 - \varepsilon_1, \|\hat{T}\|_W^2 + \varepsilon_1\right] \leq \frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t) \in O_p} w_{i,t} \left(Y_{i,t} - \hat{T}_{N,T}(i,t,p)\right)^2 + \frac{4\varepsilon}{NT}\).
$O\left( \exp\left(-\frac{2N^2}{L^4(\frac{1}{16})^2}\right) \right)$. Therefore, with probability at least $1 - \exp\left(-\frac{2N^2}{L^4(\frac{1}{16})^2}\right)$ we have,

$$
\frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t)\in O_p} w_{i,t} \left( Y_{i,t} - \hat{T}_{N,T}(i,t,p) \right)^2
$$

$$
= \frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t)\in O_p} w_{i,t} Y_{i,t}^2 - \frac{2}{NT} \sum_{p=1}^{B} \sum_{(i,t)\in O_p} w_{i,t} \hat{T}_{N,T}(i,t,p) + \frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t)\in O_p} w_{i,t} \left( \hat{T}_{N,T}(i,t,p) \right)^2
$$

$$
\leq \frac{1}{NT} \left( \|Y_w\|_W + \varepsilon_1 \right) - 2 \sum_{i,t,p} W(i,t,p)Y_w(i,t,p)\hat{T}_{N,T}(i,t,p) + \frac{1}{NT} \left( \|\hat{T}_{N,T}\|_W^2 + \varepsilon_1 \right)
$$

$$
= \frac{1}{NT} \|Y_w - \hat{T}_{N,T}\|_W^2 + \frac{2\varepsilon_1}{NT}
$$

Now $\hat{T}_{N,T}$ approximately solves objective function 14 and $\hat{T}_{N,T}$ exactly solves the objective function 14. Therefore, with probability at least $9/10$ we have $\|Y_w - \hat{T}_{N,T}\|_W^2 \leq (1 + \varepsilon)\|Y_w - \hat{T}_{N,T}\|_W$. Therefore, with probability at least $9/10 - \exp\left(-\frac{2N^2}{L^4(\frac{1}{16})^2}\right)$ we have,

$$
\frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t)\in O_p} w_{i,t} \left( Y_{i,t} - \hat{T}_{N,T}(i,t,p) \right)^2
$$

$$
\leq \frac{1}{NT} \|Y_w - \hat{T}_{N,T}\|_W^2 + \frac{2\varepsilon_1}{NT}
$$

$$
\leq \frac{1 + \varepsilon}{NT} \|Y_w - \hat{T}_{N,T}\|_W^2 + \frac{2\varepsilon_1}{NT}
$$

$$
\leq \frac{1 + \varepsilon}{NT} \left( \|Y_w\|_W^2 - 2 \sum_{i,t,p} W(i,t,p)Y_w(i,t,p)\hat{T}_{N,T}(i,t,p) + \|\hat{T}_{N,T}\|_W^2 \right) + \frac{2\varepsilon_1}{NT}
$$

$$
\leq \frac{1 + \varepsilon}{NT} \sum_{p=1}^{B} \sum_{(i,t)\in O_p} w_{i,t} \left( Y_{i,t} - \hat{T}_{N,T}(i,t,p) \right)^2 + \frac{4\varepsilon_1}{NT}
$$

The last inequality uses lemma C.10. Therefore with probability at least $9/10 - \exp\left(-\frac{2N^2}{L^4(\frac{1}{16})^2}\right)$ we get

$$
\frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t)\in O_p} w_{i,t} \left( Y_{i,t} - \hat{T}_{N,T}(i,t,p) \right)^2 \leq (1 + \varepsilon)OPT + \frac{4\varepsilon_1}{NT}
$$

Now substituting $k = 1/8 \log(1-\delta)/\delta$ $N$ and $\varepsilon_1 = O\left(\frac{L^2}{N^{1/2}}\right)$ and for $N$ large enough we get the first result.

Now suppose $N$ is fixed. If assumption 15 holds, then using lemma C.11 we get with probability at least $1 - O\left(\frac{L^4(\frac{1}{16})^{2k}}{\varepsilon_1^2/T}\right)$,

$$
\frac{1}{NT} \sum_{p=1}^{B} \sum_{(i,t)\in O_p} w_{i,t} \left( Y_{i,t} - \hat{T}_{N,T}(i,t,p) \right)^2 \leq (1 + \varepsilon)OPT + \frac{4\varepsilon}{NT}
$$

If we substitute $k = 1/8 \log(1-\delta)/\delta$ $T$ and $\varepsilon_1 = O\left(\frac{L^2}{T^{1/2}}\right)$ and for $T$ large enough, we get the second result. □