Fundamentals of the theory of self-injection locking of multi-frequency laser diode to high-Q optical microresonator

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Abstract. In this work we present a theoretical model describing a phenomenon of self-injection locking of multi-frequency laser to high-Q optical microresonator. Such process allows high-power single-frequency emission with sub-kHz linewidth from the compact multi-frequency diode laser. Elaborated theory takes into account fast optical feedback due to the Rayleigh scattering, mode competition effect and Bogatov asymmetric mode interaction.

1. Introduction

The effect of the self-injection locking of laser to high-Q whispering gallery mode (WGM) microresonators is well-known and is widely used for laser stabilization and narrowing the output spectrum of lasers [1, 2, 3]. Self-injection locking to a WGM microresonator uses resonant Rayleigh scattering [4] on internal and surface inhomogeneities when a fraction of incoming radiation in resonance with the frequency of the selected WGM reflects back to the laser. This effect provides fast optical feedback and may result in a significant reduction of the laser linewidth. This effect was demonstrated for various lasers, including quantum cascade [5], fiber-loop [6] and DFB lasers [7] achieving the instantaneous linewidth down to sub-Hz levels [8]. Recently, an obvious theoretical description of this effect was demonstrated [3]. Mostly, self-injection locking effect was studied and applied for stabilization of single-frequency lasers. However, latterly it was shown the possibility of the self-injection locking of multi-frequency laser resulting in a spectrum collapse to a single line with a sub-kHz [9] and even to the generation of dissipative Kerr solitons [10]. It was also shown that spectrum of locked multifrequency laser had the characteristic asymmetric form first observed and explained by Bogatov [11]. Similar results were also observed with on-chip microresonators [12]. To support recent experimental work and obtain accurate analytical estimations for the parameters critical for the considered effect in this work we developed an original theoretical model of multi-frequency self-injection locking taking into account fast optical feedback, mode competition [13] and Bogatov asymmetric mode interaction [11] and performed numerical modeling. All numerical results obtained from the developed model are in a good agreement with our experimental data.
2. Model

We assume for simplicity a reduced model of a laser cavity consisting of two mirrors (FP-cavity) which was successfully used for the theory of single-frequency laser self-injection locking [3]. Distributed mirror losses and distributed optical feedback could be derived from the boundary condition. In this model the slowly-varying laser cavity field \( E_l(t) e^{i \phi_l(t)} \) of a mode \( l \) with the amplitude \( E_l(t) \) and phase \( \phi_l(t) \) reflected from the front laser mirror can be found as a sum of the field \( B(t) \) that was transmitted back to the laser cavity due to backscattering in the WGM microresonator and retarded field reflected from both mirrors. The boundary equation for the front end of the laser is written as:

\[
E_l(t + \tau_d) e^{i \phi_l(t+\tau_d)} + i \omega \tau_d = i T_o B_l(t) + R_e R_o E_l(t) e^{2(\gamma_l - \gamma_l)L + i \phi_l(t)},
\]

(1)

where \( \gamma_l \) is the distributed gain in the mode \( l \); \( \gamma_i \) is the laser internal loss coefficient; \( \tau_d = \frac{2L}{v_g} \) is round-trip time of the laser diode \( (v_g \sim \text{group velocity}) \); \( \omega_l \) is frequency of a mode \( l \) \( (\omega_l \tau_d = 2\pi k, \ k \in \mathbb{N}) \), \( R_o \) and \( R_e \) is the front and the end facet reflectivity of the laser, respectively. The expression for the optical field amplitude \( B(t) \) reflected from the WGM microresonator is given by [3]:

\[
B_l(t) = \frac{i T_o \Gamma(\omega_l)}{R_o} E_l(t - \tau_d) e^{i \omega_l \tau_d + i \phi_l(-\tau_d)},
\]

(2)

where \( \tau \) is the round trip time from laser to the reflector, \( \Gamma(\omega_l) \) is frequency-dependent complex reflection coefficient of the WGM microresonator due to Rayleigh backscattering [4].

In order to determine the condition for the state of equilibrium lasing (equalization of the gain and laser losses), we rewrite the boundary equation in the form

\[
E_l(t + \tau_d) e^{i \phi_l(t+\tau_d)} + i \omega \tau_d = \zeta E_l(t) e^{i \phi_l(t)},
\]

\[
\zeta = e^2 \left( \frac{\gamma_l - \gamma_i}{\frac{\pi L}{v_g}} \right) - \frac{1}{2} \ln \left( 1 - \frac{i T_o B_l(t)}{E_l(t) e^{i \phi_l(t+\tau_d)}} \right).
\]

(3)

(4)

We denote \( \gamma_{mr} = \frac{1}{L} \ln \left( \frac{1}{R_e R_o} \right) \) and \( \gamma_{\kappa} = \frac{1}{L} \ln \left( 1 - \frac{i T_o B_l(t)}{E_l(t+\tau_d) e^{i \phi_l(t+\tau_d)}} \right) \) to be distributed mirror losses and distributed optical feedback, respectively. Therefore, when the laser radiation is in equilibrium, the following relationship can be obtained:

\[
\gamma_l - (\gamma_i + \gamma_{mr} + \gamma_{\kappa}) = \Delta \gamma \rightarrow 0.
\]

(5)

Further, it is proposed to consider a laser in the regime close to the equilibrium radiation, so we can expand \( E_l(t + \tau_d) \) and \( \phi_l(t + \tau_d) \) with respect to the small parameters \( \tau_d \) and \( \Delta \gamma \).

\[
\dot{E_l}(t) + i \dot{\phi_l}(t) E_l(t) = \frac{1}{\tau_d} \left[ 2 \left( \gamma_l - \gamma_i - \gamma_{mr} \right) L - \ln \left( 1 - \frac{i T_o B_l(t)}{E_l(t+\tau_d) e^{i \phi_l(t+\tau_d)}} \right) \right] E_l(t).
\]

(6)

In the work we consider the weak feedback mode \( \frac{|i T_o B_l(t)|}{E_l(t+\tau_d) e^{i \phi_l(t+\tau_d)}} \ll 1 \), therefore one gets

\[
\dot{E_l}(t) + i \dot{\phi_l}(t) E_l(t) = \frac{2L}{\tau_d} \left( \gamma_l - \gamma_i - \gamma_{mr} \right) \dot{E_l}(t) + \frac{1}{\tau_d} \frac{i T_o B_l(t)}{e^{i \phi_l(t)}}.
\]

(7)

In a dispersive medium, the ratio of energy flux to energy density is the group velocity \( v_g = \frac{2L}{\tau_d} \) \[14, 15\]. Hence, the rate of stimulated emission minus the rate of loss is given by:

\[
v_d [\gamma_l - (\gamma_i + \gamma_{mr} + \gamma_{\kappa})] = G_t - G_{th} + \kappa_{ad},
\]

(8)
where $G_l$ is the gain in the mode $l$, $G_{\text{th}}$ is the threshold gain, $\kappa$ is the optical feedback.

The amplitude $E_l$ of the field of the mode $l$ determines the number of contained photons $S_l$ according to the relationship

$$S_l = 2\epsilon_0 n_d^2 \hbar \omega_l |E_l|^2,$$

(9)

where $n_d$ is the refractive index of the active region of a laser.

From the equation (7) one can derive an equation for number of photons $S_l$ in the laser mode of the electromagnetic field $E_l$. Our physical model describing the self-injection locking of a multi-frequency laser to modes of a high-Q WGM microresonator [16, 17] is based on two equations, one of them is for number of photons $S_l$ in the laser mode of the electromagnetic field $E_l$, another is for the electron density $N$, which includes the effects of diffusion, spontaneous and stimulated recombination:

$$\dot{N} = \frac{I}{e} - \frac{N}{\tau_s} - \sum_l G_l^{(1)} S_l,$$

(10)

$$\dot{S}_l = (G_l - G_{\text{th}}) S_l + N F_l + \delta S_{\text{feedback}},$$

(11)

where $I$ is the diode current, $e$ is the electron charge, $N$ is the number of excited electrons, $\tau_s$ is the lifetime of the excited electron, $S_l$ is the number of photons; $F_l$ is the spontaneous emission coefficient

$$F_l = \frac{\tilde{\beta}}{[2(\lambda_l - \lambda_{\text{peak}})/\Delta \lambda]^2 + 1},$$

(12)

where $\tilde{\beta}$ is the spontaneous emission factor normalized over the time of radiative recombination of excited electrons, $\Delta \lambda$ is the spontaneous emission width, $\lambda_l$ is the wavelength of the mode $l$, $\lambda_{\text{peak}}$ is the central wavelength of the laser. The $\delta S_{\text{feedback}}$ is the optical feedback, which is derived from Eq. (7) and Eq. (2):

$$\delta S_{\text{feedback}} = 2\kappa_{\text{od}} \sqrt{S_l(t - \tau)} S_l \cos(\psi_l + \phi_l(t) - \phi_l(t - \tau)),$$

(13)

where $\kappa_{\text{od}} \approx \frac{1 - R_o^2}{R_o \tau_d} \Gamma(\omega_l)$ is feedback rate, $\phi_l(t)$ is the phase of the mode $l$, $\psi_l = \omega_l \tau + \arg(\Gamma(\omega_l))$, $\tau$ is the round trip time from laser to the reflector and back and $\tau_d$ is the round trip time within the laser.

For the gain $G_l$, the following expression can be written [18, 16]:

$$G_l = G_l^{(1)} - G_l^{(3)} S_l - \sum_{k \neq l} (G_l^{(3)} - G_{l(k)}^{\text{Bogatov}}) S_k,$$

(14)

where $G_l^{(1)}$ is the stimulated emission coefficient in the laser mode, $G_l^{(3)}$ is the self-saturation coefficient (spectral hole burning), $G_l^{(3)}$ is the coefficient of the symmetric cross-saturation (spectral hole burning due to the neighboring modes), $G_{l(k)}^{\text{Bogatov}}$ is the asymmetric mode interaction coefficient responsible for the Bogatov effect [11].

The coefficient $G_l^{(1)}$ is determined both by the number of excited electrons and by the dispersion of the linear gain:

$$G_l^{(1)} = \theta(N - N_g - D(\lambda_l - \lambda_{\text{peak}})^2),$$

(15)

where $\theta$ is the differential gain, $N_g$ is the number of excited electrons at which the laser diode becomes optically transparent, $D$ is the linear gain dispersion coefficient.
Figure 1. a: Experimental (blue line) and numerically calculated (red line) emission spectrum of the free-running multi-frequency diode laser (10)–(11) without $\delta S_{\text{feedback}}$; b: Experimental (blue line) and numerically calculated (red line) emission spectrum of the self-injection locked multi-frequency diode laser (10)–(11).

The effect of asymmetric interaction of modes was first described by Bogatov [11], where a model of stimulated scattering of laser light influence on the dynamic electron density inhomogeneities was introduced as the theoretical explanation of this effect. The model proposed by Bogatov describes the change in the permittivity $\delta \varepsilon$, caused by the dynamic inhomogeneity of the electron density due to the stimulated emission of the excited electrons under the influence of mode interference. This change creates a temporal phase lattice. Scattering on this lattice could be considered as four-wave mixing. The expression obtained by Bogatov for the variation of the dielectric constant can be rewritten in terms of the gain of a laser active region [9]:

$$G_{\text{Bogatov}}(k) = \frac{3}{4} \theta^2 (N - N_g) \left[ \frac{1}{\tau_s^2} + \frac{3}{2} \theta S + \alpha \Omega_{l(k)} \right] \left[ \frac{1}{\tau_s^2} + \frac{3}{2} \theta S \right]^2 + \Omega_{l(k)}^2,$$

(16)

where $\Omega_{l(k)} = \omega_l - \omega_k$ are laser modes offsets, $S = \sum S_i$ is the total number of photons, $\alpha$ is the linewidth enhancement factor.

It was found out that if a laser mode $p$ is self-injection locked to a resonator mode, the feedback coefficient of this mode effectively adds extra gain to the mode. The total gain of this mode then exceeds the gain of the other modes. This enhances the power of the locked mode $p$. Further feedback enhancement can lead to the strong feedback - “complete” suppression of other modes. Consequently, this process effectively transfers the energy from other laser modes into the locked mode. It is the strong feedback when a multi-frequency laser becomes effectively a single-frequency one. In case of strong self-injection locking it is observed:

$$S_l \ll S_p$$

(17)

Numerical modeling based on the developed model Eq. (10)-(11) was performed for the parameters reported in [9]. In the experiment multi-frequency (but spatially single-mode) InP laser diodes operating at 1535 nm with the free spectral range 17.6 GHz and total output power 200 mW was locked to a MgF$_2$ WGM resonator. Performing numerical modeling for the experimental parameters presented in Tab.1, we observed a good agreement between the theory and the experiment [see Fig.1].
Table 1. Parameters

| Symbol | Definition | Value       | Unit       | Source             |
|--------|------------|-------------|------------|--------------------|
| $\theta$ | differential gain coefficient | $2 \times 10^4$ | $s^{-1}$    | model/experiment   |
| $G_{th}$ | threshold gain | $10^{11}$ | $s^{-1}$ | experiment         |
| $L_d$ | length of diode | 2500 | $\mu m$ | documentation      |
| $n_d$ | refractive index of active region | 3.513 | - | documentation      |
| $\alpha$ | $\alpha$-factor | 4 | - | article            |
| $\tau_s$ | electron life-time | $10^9$ | $s^{-1}$ | article            |
| $\tau_d$ | round-trip time of laser | $5 \times 10^{-11}$ | $s$ | documentation      |
| $R_o$ | front facet reflectivity | 0.54 | $s$ | documentation      |
| $R_e$ | back facet reflectivity | 0.9 | $s$ | documentation      |

3. Conclusion
We developed a fundamentals of the theory of multi-frequency self-injection locking based on the multimode laser model of of Yamada with the Bogatov asymmetric mode interaction and Lang and Kobayashi model an optical feedback to the model of lasing. This combined model was shown to provide an adequate description of the experimental data on the self-injection locking of multi-frequency lasers.

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