Online Simple Knapsack with Reservation Costs

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Abstract

In the \textit{online simple knapsack problem} we are given a knapsack of unit size 1. Items of size smaller or equal to 1 are presented in an iterative fashion and an algorithm has to decide for each item whether to reject or permanently include it into the knapsack without any knowledge about the rest of the instance. The goal is to pack the knapsack as full as possible. In this work, we introduce a third option additional to those of packing and rejecting an item, namely that of reserving an item for the cost of a fixed fraction $\alpha$ of its size. An algorithm may pay this fraction in order to postpone its decision on whether to include or reject the item until after the last item of the instance was presented.

While the classical \textit{online simple knapsack problem} does not admit any constantly bounded competitive ratio in the deterministic setting, we find that adding the possibility of reservation makes the problem constantly competitive, with varying competitive ratios depending on the value of the fraction $\alpha$. We give tight bounds for the whole range of reservation costs, which is split up into four connected areas, depending on the value of $\alpha$. Up to a value of 0.25, the competitive ratio is 2. This is followed by a competitive ratio of $(1 + \sqrt{5 - 4\alpha})/(2(1 - \alpha))$ for values up to $\sqrt{2} - 1$. Afterwards, the competitive ratio is $2 + \alpha$ for values up to $\phi - 1$, where $\phi$ is the golden ratio. Finally, the competitive ratio diverges with a value of $1/(1 - \alpha)$ for values between $\phi - 1$ and 1.

With our analysis, we find a counterintuitive characteristic of the problem: Intuitively, one would expect that the possibility of rejecting items becomes more and more helpful for an online algorithm with growing reservation costs. However, for reservation costs above $\sqrt{2} - 1$, an algorithm that is unable to reject any items tightly matches the lower bound and is thus the best possible. On the other hand, for any positive reservation cost smaller than $\sqrt{2} - 1$, any algorithm that is unable to reject any items performs considerably worse than one that is able to reject.

\textit{Keywords:} Online problem, Simple knapsack, Reservation costs

1. Introduction

Online algorithms can be characterized by receiving their input in an iterative fashion and having to act in an irrevocable way on each piece of the input, e.g., by deciding whether to include an element into the
solution set or not. This has to be done with no additional knowledge about the contents or even the length of the rest of the instance that is still to be revealed. The goal, as with offline algorithms, is to optimize some objective function. In order to measure the performance of an online algorithm, it is compared to that of an optimal offline algorithm on the same instance. The worst-case ratio between the performances of these algorithms over all instances is then called the strict competitive ratio of an online algorithm, as introduced by Sleator and Tarjan [22].

One of the arguably most basic and famous problems of online computation is called the ski rental problem [19]. In this problem, someone is going on a skiing holiday of not yet fixed duration without owning the necessary equipment. On each day, she decides, based solely on that day’s short-term weather report, whether skiing is possible on that day or not. On each day with suitable weather, she can either buy the equipment or rent it for that day for a fixed percentage of the cost of buying. Arguably, the only interesting instances are those in which a selected number of days are suitable for skiing, followed by the rest of the days at which she is unable to go skiing anymore. This is simply due to the fact that, as long as she has not decided to buy a pair of ski yet, a day at which no skiing is possible requires no buy-or-rent decision. Thus, the problem can be simplified as follows: The input is a string of some markers that represent days suitable for skiing, and as soon as the instance ends, skiing is no longer possible.

This notion of delaying a decision for a fixed percentage of the cost is the model that we want to study in this work. Note that, while the ski-rental problem in its above-mentioned form only models a single buy-or-rent decision (buying or renting a single commodity), iterated versions of it have been discussed in the literature, with important applications, e.g., to energy-efficient computing [15, 1, 4]. In this paper, we investigate the power of delaying decisions for another more involved problem, namely the online knapsack problem.

The knapsack problem is defined as follows. Given a set of items $I$, a size function $w: I \to \mathbb{R}$ and a gain function $g: I \to \mathbb{R}$, find a subset $S \subseteq I$ such that the sum of sizes of $S$ is less or equal to the so-called knapsack capacity (which we assume to be normalized to 1 in this paper) and the sum of the gains is maximized. The online variant of this problem reveals the items of $I$ piece by piece, with an algorithm having to immediately decide whether to pack a revealed item or to discard it.

The knapsack problem is a classical hard optimization problem, with the decision variant being shown to be NP-complete as one of Karp’s 21 NP-complete problems [18]. The offline variant of this problem was studied extensively in the past, showing for example that it admits a fully polynomial time approximation scheme [14].

A variant of this problem in which the gains of all items coincide with their respective sizes is called the simple knapsack problem; it is also NP-complete. Both variants do not admit a constantly bounded competitive ratio [20]. In this paper, we focus on the online version of the latter variant, which we simply refer to as the online knapsack problem, for short ONLINEKP, if not explicitly stated otherwise.

We propose a rather natural generalization of the online knapsack problem. Classically, whenever an item of the instance is presented, an online algorithm has to irrevocably decide whether to include it into its solution (i.e., pack it into the knapsack) or to discard it. In our model, the algorithm is given a third option, which is to reserve an item for the cost of a fixed percentage $0 \leq \alpha \leq 1$ of its value. The algorithm may reserve an arbitrary number of items of the instance and decide to pack or discard them at any later point. Philosophically speaking, the algorithm may pay a prize in order to (partially) “go offline.” It is easy to see that, for $\alpha = 0$, the complete instance can be learned before making a decision without any disadvantages, essentially making the problem an offline problem, while, for $\alpha = 1$, reserving an item is not better than discarding it because packing a reserved item does not add anything to the gain.

This extension to the knapsack problem is arguably quite natural: Consider somebody trying to book a flight with several intermediate stops. Since flight prices are subject to quick price changes, it might be necessary to invest some reservation costs even for some flights not ending up in the final journey. The knapsack problem with reservations might also be seen as a simple model for investing in a volatile stock market, where a specific type of derivatives is available (at some cost) that allows the investor to fix the price of some stock for a limited period of time.

One of the key properties of our reservation model applied to the simple knapsack problem is its un-
Table 1: Competitive ratios proven in this work.

| $\alpha$          | Competitive ratio | Lower bound | Upper bound |
|-------------------|-------------------|-------------|-------------|
| $0 < \alpha \leq 0.25$ | $\frac{2}{\sqrt{2} - 1}$ | Thm. 12     | Thm. 1     |
| $0.25 < \alpha \leq \sqrt{2} - 1$ | $\frac{1+\sqrt{2} - 1}{2(1-\alpha)}$ | Thm. 13     | Thm. 1     |
| $\sqrt{2} - 1 < \alpha < \phi - 1$ | $2 + \alpha$ | Thm. 14     | Thm. 12, 13 |
| $\phi - 1 \leq \alpha < 1$ | $\frac{1}{1-\alpha}$ | Thm. 13     | Thm. 11   |

intuitive behavior with respect to rejecting items. It shows sharp thresholds for the competitive ratio at seemingly arbitrary points: For some interval of low reservation charges, the competitive ratio is not affected by the charge at all, on the next interval, it follows a nonlinear function, penultimately it depends linearly on the charge, and on the last interval, the competitive ratio grows even faster. This behavior is further discussed in the following subsection.

The OnlineKP has been extensively studied under many other variations, including buffers of constant size in which items may be stored before deciding whether to pack them, studied by Han et al. [12]. Here, the authors allow for a buffer of size at least the knapsack capacity into which the items that are presented may or may not be packed and from which a subset is then packed into the actual knapsack in the last step. They extensively study the case in which items may be irrevocably removed from the buffer. Our model can be understood as having an optional buffer of infinite size, with the caveat that each buffered item induces a cost. A variant without an additional buffer, but with the option to remove items from the knapsack at a later point was studied by Iwama et al. [16]. The same model with costs for each removal that are proportional to a fraction $f$ of each item from the knapsack was researched by Han et al. [11], which is closer to our model. Their model allows an algorithm to remove items that were already packed into the knapsack, for a proportion of the value of these items. However, we do not know of any simple reduction from one model to the other, which is supported by the considerably different behavior of the competitive ratio relative to the reservation cost. For OnlineKP, Han et al. show that the problem is 2-competitive for a removal cost factor $f \leq 0.5$ and becomes $(1 + f + \sqrt{f^2 + 2f + 5})/2$-competitive for $f > 0.5$.

Other models of the online (simple) knapsack problem have been studied as well, such as the behavior of the resource-augmentation model, where the knapsack of the online algorithm is slightly larger than the one of the offline algorithm on the same instances, studied by Iwama et al. [17]. When randomization is allowed, the OnlineKP becomes 2-competitive using only a single random bit and does not increase its competitive ratio with any additional bits of randomization [6]. A relatively young measure of the informational complexity of a problem is that of advice complexity, introduced by Dobrev, Královič, and Pardubská [6], revised by Hromkovič et al. [13] and refined by Böckenhauer et al. [5]. In the standard model, an online algorithm is given an advice tape of bits that an oracle may use to convey some bits of information to an online algorithm about the instance in order to improve its performance. Not surprisingly, a single advice bit also improves the competitive ratio to 2, but interestingly, any advice in the size of $\Theta(\log n)$ advice bits does not improve this ratio [6].

1.1. Our Contributions

We study the behavior of the knapsack problem with reservation costs for a reservation factor $0 < \alpha < 1$. We analyze four subintervals for $\alpha$ separately with borders at $1/4$, $\sqrt{2} - 1$ and $\phi - 1 \approx 0.618$, where $\phi$ is the golden ratio. The bounds that we are providing, which are illustrated in Figure 1, can be found in Table 1. We prove matching upper and lower bounds for each interval.

We also take a look at a subclass of algorithms for this problem that never discard presented items up to a point where they stop processing the rest of the input, with arguably paradoxical results: One would expect that, for very small reservation costs, reserving items instead of rejecting them right away is more helpful than doing the same with large reservation costs and that rejecting becomes more and more helpful with increasing reservation costs. But we prove that, while, for small values of $\alpha$, every algorithm that is unable to reject items is strictly dominated by algorithms that are able to reject items, for larger values of...
α this is not the case, with our algorithms used for \( \alpha \geq \sqrt{2} - 1 \) being nonrejecting and matching their lower bounds.

We cannot give a definitive answer on why this behavior can be observed. Our intuition is that, for higher reservation costs, the behavior is more similar to the model without reservations. In this classical model, any errors that occur by ultimately not packing an item are punished severely, so while reserving everything is costly, it is not worse than rejecting any items.

The remainder of this paper is structured as follows: In Section 2 we present the upper bounds for all intervals. We divide this section by first presenting a more general algorithm that provides us with the wanted upper bounds for values of \( \alpha \) up to \( \sqrt{2} - 1 \) and continue with much simpler algorithms for values of \( \alpha \) larger than \( \sqrt{2} - 1 \). In Section 3 we look at matching lower bounds, with a specialized lower bound for values of \( \alpha \) smaller than 0.25 and a uniform approach for all higher values. Section 4 is devoted to a discussion of algorithms that are unable to reject items. We conclude the paper in Section 5.

1.2. Preliminaries

Our model can be defined as follows. Consider a knapsack of size 1 and a reservation factor \( \alpha \), with \( 0 \leq \alpha \leq 1 \). A request sequence \( I \) is a list of item sizes \( x_1, x_2, \ldots, x_n \) with \( x_i \leq 1 \) for \( 1 \leq i \leq n \), which arrive sequentially. At time step \( i \), the knapsack is filled up to some size \( t_i \leq 1 \) and the reserved items add up to size \( r_i \). When an item with size \( x_i \) arrives, an online algorithm may pack this item into the knapsack if \( x_i + t_i \leq 1 \), it may also reject the item, or reserve the item at cost \( \alpha \cdot x_i \).

At step \( n+1 \), no new items arrive and the knapsack contains all items that were taken with total size \( t_n \) and the reserved items have size \( r_n = R \). An algorithm can additionally pack any of the reserved items which still fit into the knapsack, up to some size \( t \leq 1 \). The gain of an algorithm \( A \) solving \textsc{ReserveKP} with a reservation factor \( \alpha \) on an instance \( I \) is \( \text{gain}_A(I) = t - \alpha \cdot R \).

The strict competitive ratio of an algorithm \( A \) on an instance \( I \) is, given a solution with optimal gain \( \text{gain}_{\text{opt}}(I) \) on this instance, \( \rho_A(I) = \text{gain}_{\text{opt}}(I)/\text{gain}_A(I) \). The general strict competitive ratio of an algorithm \( A \) is taken as the worst case over all possible instances, \( \rho_A = \max_I \{ \rho_A(I) \} \).

The strict competitive ratio as defined above is a special case of the well-known competitive ratio which relaxes the above definition by allowing for a constant additive term in the definition. Note that this generalized definition only makes sense for online problems in which the optimal solution has unbounded gain. Since the gain of an optimal solution for \textsc{ReserveKP} is bounded by the knapsack capacity, we only work with strict competitive ratios in this paper and will simply call it competitive ratio from now on. For a thorough introduction to competitive analysis of online algorithms, see the textbooks by Borodin and El-Yaniv [7] and by Komm [19].
Algorithm 1  Competitive ratio of 2 for $\alpha \leq 0.25$ and of $\frac{1+\sqrt{\alpha}-\frac{3\alpha}{2(1-\alpha)}}{2}$ for $\alpha \leq \sqrt{2} - 1$

1: $R := 0$;
2: $R_s := \emptyset$;
3: for $k = 1, \ldots, n$ do
4: if $x_k + R(1 - \alpha) < 1/\rho$ then $R := R + x_k$, $R_s := R_s \cup \{x\}$
5: else if $x_k + R \leq 1$ then pack optimally and stop
6: else if $\forall j \in R_s : x_j \leq 1 - \mu$ then pack optimally and stop
7: else if $POPT(x_k \cup R_s) - R\alpha \geq 1/\rho$ then pack optimally and stop
8: else reject $x_k$
9: pack the reserved items optimally

2. Upper Bounds

We start by presenting an algorithm that gives us bounds for smaller values of $\alpha$. While the algorithm that is presented in the following subsection does not diverge from the lower bounds for values of $\alpha$ bigger than $\sqrt{2} - 1$, we handle the two function segments above this threshold separately, using much simpler algorithms. This is done to show a curious observation that is further discussed in section 3. This observation states that an algorithm that tightly matches the lower bound for these segments no longer has to reject any items in order to be optimal.

2.1. Upper Bound for $0 < \alpha < \sqrt{2} - 1$

We present Algorithm 1 and first look at the competitive ratio after each of the stop statements is reached. The interesting part of the analysis is then done in Theorem 1 which deals with the case that none of the stop statements are reached during the run of the algorithm.

We define $\mu = 1/(\rho(1 - \alpha))$, where we will choose $\rho$ as our claimed competitive ratio, depending on the area of values for $\alpha$ that we want to analyze. We keep track of the reserved items in the set $R_s$. We denote the largest packing of a set $I$ of items that is of size at most 1 by $POPT$.

The algorithm is split up into several parts, while a reservation is only made in line 4 and if one criterion is met: If the gain from the current item plus the gain of everything reserved so far is still below our wanted gain, we reserve the item. This especially means that the current item and the gain from our reserve itself are both smaller than $1/\rho$ if this condition triggers.

If we do not reserve, we know that the new item together with the gain of our reserve exceeds our wanted gain. It could however be the case that this sum also exceeds 1 and thus does not constitute a valid packing. Thus, we pack in line 9 if the sum is smaller than 1 and are done or we continue otherwise.

We also know that $R(1 - \alpha) < 1/\rho$, and thus $R < 1/(\rho(1 - \alpha)) = \mu$ always holds, as we do not reserve items that would cause a violation of this condition. This, together with the knowledge that $x_k$ is too big to pack it together with $R$ lets us derive that $x_k > 1 - \mu$. Thus, if all other items from $R_s$ are smaller than $1 - \mu$, we can simply remove them iteratively from our knapsack until $x_k + R \leq 1$ holds. This leads to our wanted result in line 3 as we know that the reservation costs up to this point are at most $\mu \alpha$ (since $R < \mu$), our final packing is of size at least $\mu$ and thus our gain is $\mu - \mu \alpha = 1/\rho$.

This leaves us with a corner case in which the previous conditions hold, except that there is at least one other item in $R_s$ which is bigger than $1 - \mu$. We then simply calculate in line 7 if we are able to achieve our wanted competitive ratio using the new item $x_k$ anyway and reject the item in line 8 otherwise.

Thus, we are only left to analyze the competitive ratio of the algorithm in the case that none of the stop statements triggered at the end of the instance, which we do in the following theorem.

Theorem 1. There is an algorithm for RESERVEKP with a competitive ratio of $\max\{\frac{1+\sqrt{\alpha}-\frac{3\alpha}{2(1-\alpha)}}{2}, 2\}$ if $0 < \alpha < \sqrt{2} - 1$.

Proof. In order to show this property, we analyze Algorithm 1 and show that it provides us with the wanted upper bound. We already saw in the description of the algorithm that in all of the stop statements, the
claim of the theorem holds if we set $\rho$ to our wanted competitive ratios. We are thus left with analyzing the performance of the algorithm in the case that none of the stop statements are reached during its run.

We first argue that, whenever a new item $x_k$ is rejected, the reserve contains exactly one item $x_i$ of size at least $1 - \mu$. We already know that for $x_k$ to be rejected, the stop condition in line 6 did not trigger, which tells us that, in line 8 there is at least one item bigger than $1 - \mu$ in our reserve already. We now look at the time where the first item of size at least $1 - \mu$ has already been reserved and assume that $x_k$ is of size at least $1 - \mu$ as well. Then we do not reserve this item since

$$x_k + R(1 - \alpha) \geq (1 - \mu) + (1 - \mu)(1 - \alpha)$$

$$= \frac{1}{\rho(1 - \alpha)}(2 - \alpha)$$

$$\geq 1/\rho$$

where the last inequality holds for $\rho = 2$ and all $\alpha \leq 1 - 1/\sqrt{2} \approx 0.293$, in particular for $\alpha \leq 0.25$. The inequality remains true for $\rho = 1 + \sqrt{\frac{4 - \alpha}{2(1 - \alpha)}}$ and any $0 < \alpha < 1$, in particular for the interval considered in the theorem statement.

Next, we argue that any item that is rejected is of size larger than $1/2$. This can be easily verified using the fact that the stop statement in line 6 did not trigger, and thus $x_k > 1 - R$ holds. Since we additionally know that $R$ contains at least one item of size at least $1 - \mu$, it has to hold that $x_k > \mu$, which is bigger than $1/2$ for $0 < \alpha < \sqrt{2} - 1$.

This also means that any optimal solution can contain at most one of the items that the algorithm discards. Thus, we can ignore any items of an instance that both our algorithm and an optimal algorithm discard and fix one item of size larger than $1/2$ (there can only be one in any valid solution) as the unique item that is part of the optimal solution, but discarded by our algorithm.

Let now $x_i$ be the unique item that the algorithm rejects, but that is part of the optimal solution and let $x_j$ be the largest reserved item of the algorithm at the time that $x_i$ is rejected. As we discard $x_i$, we know by the conditionals that we passed and that were not triggered that both $x_i \leq \text{POPT}(x_k \cup R) \leq 1/\rho + R\alpha$ and $x_i + x_j > 1$ have to hold. We also know that the only rejected item from the optimal solution is $x_i$ and that every other item of this solution was reserved by our algorithm.

Let $b$ be the total size of all items reserved by our algorithm before $x_j$ arrived. On the other hand, let $a$ be the total size of all items reserved by our algorithm after item $x_i$ is rejected. At the moment that $x_i$ is rejected, we know by line 4 that $x_i \leq \text{POPT}(x_k \cup R) \leq 1/\rho + (x_j + b)\alpha$. We use this term between steps 4 and 2 of the following calculation. Together with $x_i + x_j > 1$, we can derive $1 - x_j < x_i \leq 1/\rho + (x_j + b)\alpha$ which solved for $x_j$ yields $\frac{1 - x_j}{x_j + 1/\rho} \leq x_j$. This is used between steps 3 and 4 of the following inequation chain. We can now derive the following. To summarize, the optimal algorithm has a gain of $x_i + b + a$. Algorithm 1 also has a gain of $a + b$ but packs $x_j$ instead of $x_i$ and has to pay reservation costs of $\alpha$ for $a$, $b$ and $x_i$. Hence, we have

$$\frac{\text{gain}_{\text{opt}}}{\text{gain}_{k}} \leq \frac{x_i + b + a}{(1 - \alpha)(x_j + b + a)}$$

Use: $x_i \leq 1/\rho + (x_j + b)\alpha$ (1)

$$\leq \frac{1}{1 - \alpha} \cdot \frac{1/\rho + (x_j + b)\alpha + b + a}{x_j + b + a}$$

Add $a \cdot \alpha$ to numerator (2)

$$\leq \frac{1}{1 - \alpha} \cdot \frac{1/\rho + x_j\alpha + (b + a)(1 + \alpha)}{x_j + b + a}$$

Take $1 + \alpha$ as common factor (3)

$$= \frac{1 + \alpha}{1 - \alpha} \cdot \frac{x_j + b + a}{x_j + b + a}$$

(4)

If the fraction to the right of $\frac{1 + \alpha}{1 - \alpha}$ is at most 1, we are done: It holds that $\frac{1 + \alpha}{1 - \alpha} \leq 2$ for $\alpha \leq 1/3$ and also $\frac{1 + \alpha}{1 - \alpha} \leq 1 + \sqrt{\frac{4 - \alpha}{2(1 - \alpha)}}$ for $\alpha \leq \sqrt{2} - 1$. Otherwise, we use that removing the term $a + b$ from both the numerator and the denominator can only increase the value of the right fraction and continue as follows.
Algorithm 2: Competitive ratio of $2 + \alpha$ for $0 < \alpha \leq \phi - 1$

$R := 0$;

for $k = 1, \ldots, n$ do
  if $x_k + (1 - \alpha)R \geq 1/(2 + \alpha)$ then
    pack $x_1, \ldots, x_k$ optimally;
    stop
  else
    reserve $x_k$;
    $R := R + x_k$
  pack $x_1, \ldots, x_n$ optimally

This gives us our wanted competitive ratio of at most $\rho = 2$ for $\alpha \leq 0.25$ and likewise of $\rho = \frac{1 + \sqrt{5} - 2}{2(1 - \alpha)}$ for $0.25 \leq \alpha \leq \sqrt{2} - 1$.

2.2. Upper Bound for $\frac{1}{6} < \alpha < \phi - 1$

We now prove a general upper bound for $0 < \alpha < \phi - 1$. We split the proof, into two pieces: Theorem 2 handles the case for values of $\alpha$ up to $\frac{1}{6}$. Theorem 3 contains an induction over the number of large elements in an instance, which proves the upper bound for the rest of the interval up to $\phi - 1$.

Theorem 2. There exists an algorithm for RESERVEKP with a competitive ratio of at most $2 + \alpha$ if $0 < \alpha \leq \frac{1}{6}$.

Theorem 3. There exists an algorithm for RESERVEKP with a competitive ratio of at most $2 + \alpha$ if $\frac{1}{6} < \alpha < \phi - 1$.

We consider Algorithm 2, which, unlike Algorithm 1, does not reject any offered item until it reaches the desired gain and stops processing the rest of the input. In Section 4 we further discuss this class of algorithms and when they are optimal. We need the following technical lemmas.

Lemma 4. The size of $R$ in Algorithm 2 is never larger than $1/(2 + \alpha)(1 - \alpha)$.

Proof. Let us consider a request sequence $x_1, \ldots, x_n$ for Algorithm 2 such that $x_k$ triggers the packing, i.e., $x_k + (1 - \alpha)R \geq 1/(2 + \alpha)$.

We know that $x_{k-1} + (1 - \alpha)R' < \frac{1}{2(1 - \alpha)}$, where $R$ is defined as $R = x_{k-1} + R'$. But $x_{k-1} \geq (1 - \alpha)x_{k-1}$ which means that $(1 - \alpha)R = (1 - \alpha)x_{k-1} + (1 - \alpha)R' \leq x_{k-1} + (1 - \alpha)R' < \frac{1}{2 - \alpha}$. Thus, $R < \frac{1}{2(1 - \alpha)}$.

Observe that this is, by construction, the largest possible value that $R$ can take before the algorithm triggers the packing, thus, we have proved the claim.

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Note that, for every considered value of $\alpha$, the upper bound on $R$ is positive, that is, $(2 + \alpha)(1 - \alpha) > 0$ if $\alpha$ is smaller than 1. With Lemma 4 we can prove the following claim.

**Lemma 5.** If Algorithm 2 packs at least $1/(2 + \alpha)(1 - \alpha)$ into the knapsack, then its competitive ratio is at most $2 + \alpha$.

*Proof.* We know that, if the algorithm packs at least $\frac{1}{(2 + \alpha)(1 - \alpha)}$, the gain of the algorithm, $\text{gain}_k$, is at least $\frac{1}{(2 + \alpha)(1 - \alpha)} - \alpha R$. But, by Lemma 4 $R < 1/(2 + \alpha)(1 - \alpha)$, which means that $\text{gain}_k \geq \frac{1}{(2 + \alpha)(1 - \alpha)} - \alpha R < 1$, which guarantees that $R < 1$. With Lemma 4, we can prove the following claim.

The next lemma allows us to restrict our attention to ordered sequences of items. It proves that, if an instance violates the upper bound for Algorithm 2 we can rearrange the first $k - 1$ items in decreasing order and get another hard instance.

**Lemma 6.** If $x_1, \ldots, x_n$ is an instance for Algorithm 2 whose packing is triggered by $x_k$, with a competitive ratio larger than $2 + \alpha$, then there is another instance $x_1, x_2, x_3, \ldots, x_{k-1}, x_k, \ldots, x_n$ where $(i_1, i_2, \ldots, i_{k-1})$ is a permutation of $(1, \ldots, k - 1)$ and $x_{i_1} \geq x_{i_2} \geq \ldots \geq x_{i_{k-1}}$, that also has a competitive ratio larger than $2 + \alpha$.

*Proof.* We show how to transform the original sequence such that the number of inversions is reduced by one. Repeating this transformation yields the desired result.

Let $x_1, \ldots, x_n$ be an instance for Algorithm 2 such that packing is triggered by $x_k$ with a competitive ratio larger than $2 + \alpha$. Assume that $x_i < x_{i+1}$ for some $i < k - 1$. We prove that we will get again an instance with a competitive ratio larger than $2 + \alpha$ if we swap the positions of $x_i$ and $x_{i+1}$. It is easy to see that this transformation preserves the sum of the first $k - 1$ items. To show that it is a sequence that does not trigger the packing after the $i$th or $(i + 1)$st item, we have to verify that

$$x_{i+1} < \frac{1}{2 + \alpha} - (1 - \alpha)(x_1 + \cdots + x_{i-1})$$

and

$$x_i < \frac{1}{2 + \alpha} - (1 - \alpha)(x_1 + \cdots + x_{i-1} + x_{i+1}).$$

Here, (10) is straightforward, since $x_k$ for $k > i + 1$ triggering the packing in the unmodified instance implies $x_{i+1} < \frac{1}{2 + \alpha} - (1 - \alpha)(x_1 + \cdots + x_i)$. We prove (11) by means of a contradiction. Assume, that

$$x_i \geq \frac{1}{2 + \alpha} - (1 - \alpha)(x_1 + \cdots + x_{i-1} + x_{i+1}).$$

We also know that, for $1 \leq j \leq i + 1$,

$$x_j \leq \frac{1}{2 + \alpha} - (1 - \alpha)(x_1 - \cdots - x_{j-1}),$$

and by construction

$$x_i < x_{i+1}.$$  

If we define $z = \frac{1}{2 + \alpha} - (1 - \alpha)(x_1 + \cdots + x_{i-1})$, then we can rewrite (12), (13) for $j = i + 1$, and (14) as

$$x_i \geq z - (1 - \alpha)x_{i+1}$$

$$x_{i+1} \leq z - (1 - \alpha)x_i$$

$$x_i < x_{i+1}.$$  

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If we modify the first two equations to isolate the term \((1-\alpha)^{-1}z\), we obtain

\[
\frac{x_i}{(1-\alpha)} + x_{i+1} \geq \frac{z}{(1-\alpha)}
\]

\[
\frac{x_{i+1}}{(1-\alpha)} + x_i \leq \frac{z}{(1-\alpha)}
\]

which means that

\[
\frac{x_{i+1}}{(1-\alpha)} + x_i \leq \frac{x_i}{(1-\alpha)} + x_{i+1},
\]

that is, these two equations are satisfiable if and only if \(x_{i+1} \leq x_i\).

It remains to show that the new sequence has a competitive ratio larger than \(2+\alpha\). Although the order of \(x_1, \ldots, x_{k-1}\) has been changed, the requests are still the same and all of them are reserved and trigger a packing in \(x_k\) with the same reservation size, just as before the transformation. Hence, if there was no way to pack the items well enough before the transformation, it also cannot be done afterwards.

From Lemma 5, we know that, if Algorithm 2 packs at least \(1/(2+\alpha)(1-\alpha)\), this guarantees a competitive ratio of at most \(2+\alpha\). Thus, if we have enough elements that are smaller than the gap of size \(1-1/(2+\alpha)(1-\alpha)\), those elements can be packed greedily and always achieve the desired competitive ratio. Let us call small items those of size smaller than \(\phi\) and large items those of size larger than \(\phi\) for \(\alpha\) such that the desired competitive ratio is achieved.

This definition is only valid when \((15)\) is positive, that is, when \(1-\alpha-\alpha^2 \geq 0\), which is the case if \(0 < \alpha \leq \phi - 1\), including our desired range. We call large items those of larger size. Let \(\alpha_4\) be the unique positive real root of the polynomial \(1-2\alpha-\alpha^2+\alpha^3\), i.e.,

\[
\alpha_4 = \frac{1}{3} + \frac{2\sqrt{7}}{3} \cos \left(\frac{1}{3} \arccos \left(\frac{1}{2\sqrt{7}}\right) - \frac{2\pi}{3}\right) \approx 0.445.
\]

**Lemma 7.** Given any request sequence \(x_1 \geq x_2 \geq \ldots \geq x_{k-1}, x_k, \ldots, x_n\), where \(x_k\) triggers the packing in Algorithm 3, there is at most one large item if \(0 < \alpha \leq \alpha_4\) and there are at most two large items if \(\alpha_4 < \alpha \leq 0.5\).

**Proof.** Assume by contradiction that there exists a request sequence where \(x_1 \geq x_2 \geq \ldots \geq x_i \geq \frac{1-\alpha-\alpha^2}{(2+\alpha)(1-\alpha)}\), with \(k > i\). Any item \(x_j\) with \(j \leq i\) satisfies

\[
x_j \leq \frac{1}{2+\alpha} - (1-\alpha)(x_1 + x_2 + \ldots + x_{j-1}),
\]

in particular,

\[
x_i \leq \frac{1}{2+\alpha} - (1-\alpha)(x_1 + x_2 + \ldots + x_{i-1}).
\]

All of the contributions of previous requests are negative, so in order to obtain a maximal value for \(x_i\), we need that \(x_1, \ldots, x_{i-1}\) are minimal, but by construction, still greater or equal than \(x_i\). Thus, the maximal value is obtained when \(x_1 = x_2 = \ldots = x_i\). In this case, we obtain from \((16)\) the upper bound

\[
x_i \leq \frac{1}{2+\alpha} - (1-\alpha)(i-1)x_i \Rightarrow x_i \leq \frac{(2+\alpha)(1-\alpha)}{(2+\alpha)(1-\alpha) + \alpha}.
\]

We also know that \(x_i\) is a large item, thus we can also state \(x_i \geq \frac{1-\alpha-\alpha^2}{(2+\alpha)(1-\alpha)}\) as a lower bound for \(x_i\).

Thus we get \(\frac{1}{2+\alpha} - (1-\alpha)(i-1)x_i \geq \frac{1-\alpha-\alpha^2}{(2+\alpha)(1-\alpha) + \alpha}\) which we solve for \(i\) to obtain \(i \leq 1 + \frac{2\alpha^2}{(2+\alpha)(1-\alpha) + \alpha}\). In particular, for \(i = 2\) we get \(2(1-\alpha)(1-\alpha - \alpha^2) \leq (1-\alpha)(1-\alpha - \alpha^2) + \alpha^2\) which is equivalent to \((1-\alpha)(1-\alpha - \alpha^2) \leq \alpha^2\) and thus to \(1 - 2\alpha - \alpha^2 + \alpha^3 \leq 0\), which means that the number of large items is strictly smaller than \(2\) for \(\alpha \leq 0.4\).

For \(i = 3\), we get \(3(1-\alpha)(1-\alpha - \alpha^2) \leq (1-\alpha)(1-\alpha - \alpha^2) + \alpha^2\) or equivalently \(2(1-2\alpha + \alpha^3) \leq \alpha^2\). Hence, \(2 - 4\alpha - \alpha^2 + 2\alpha^3 \leq 0\), which means that the number of large items is strictly smaller than \(3\) for \(\alpha \leq 0.5\), since \(0.5\) is the unique positive real root of the left-hand-side polynomial.
Now we are ready to prove the claimed competitive ratio of Algorithm 2.

**Proof of Theorem 2**. Let us consider first the case where $\alpha \leq \alpha_3$, which means that only one large element can appear in the request sequence without triggering the packing. We know that, if $x_1 \geq 1/(2+\alpha)$, then the algorithm will take it and our competitive ratio will be at most $2 + \alpha$, and if a request sequence ends without triggering the packing, then the optimum would be $x_1 + \ldots + x_n = s < 1$ and Algorithm 2 would have a gain of $(1 - \alpha)s$, so the competitive ratio will be at most $1/(1 - \alpha) < 2 + \alpha$.

Otherwise, assume that there is a shortest sequence containing at least two elements, which at some point, namely with request $x_k$, triggers Algorithm 2 to pack. We now consider all possible cases.

If $x_1 + \ldots + x_k \leq 1$, then we know that the gain of the algorithm is $x_k + (1 - \alpha)(x_1 + \ldots + x_{k-1})$, which is larger than $1/(2 + \alpha)$ by construction.

We are now in the case where $x_1 + \ldots + x_k > 1$. If all of the elements of the request sequence are small, Algorithm 2 packs $x_k$ and then it can greedily pack elements from $x_1, \ldots, x_{k-1}$. Because these elements all have size smaller than $1 - \frac{1}{(2 + \alpha)(1 - \alpha)}$, we know that Algorithm 2 will be able to pack at least $\frac{1}{(2 + \alpha)(1 - \alpha)}$ into the knapsack, obtaining a competitive ratio of at most $2 + \alpha$ by Lemma 5.

If not all elements are small, we know that at most $x_1$ is a large element, and we do a further case distinction. If $x_1 + x_k < 1$, then we pack them, and we greedily pack the rest of the items, obtaining a competitive ratio of at most $2 + \alpha$ with the same argument as in the previous case. If $x_1 + x_k > 1$, then we consider two cases. If $x_2 + \ldots + x_k > \frac{1}{(2 + \alpha)(1 - \alpha)}$, we can again greedily pack the small items until we obtain the desired competitive ratio. Otherwise, we know that $x_k > 1 - x_1$ and $x_2 + \ldots + x_k < \frac{1}{(2 + \alpha)(1 - \alpha)}$ and thus $x_2, \ldots, x_k$ can all be packed into the knapsack. In this case, we can calculate the gain of Algorithm 2 as

$$x_2 + \ldots + x_k - \alpha(x_1 + \ldots + x_{k-1})$$

$$= x_k + (1 - \alpha)(x_2 + \ldots + x_{k-1}) - \alpha x_1$$

$$> 1 - x_1 - \alpha x_1 + (1 - \alpha)(x_2 + \ldots + x_{k-1})$$

$$= 1 - (1 + \alpha)x_1 + (1 - \alpha)(x_2 + \ldots + x_{k-1})$$

$$> 1 - \frac{1}{(2 + \alpha)} + (1 - \alpha)(x_2 + \ldots + x_{k-1})$$

$$= \frac{1}{(2 + \alpha)} + (1 - \alpha)(x_2 + \ldots + x_{k-1}),$$

where we used the fact that $x_2 > 1 - x_1$, and that $x_1 < 1/(2 + \alpha)$ since it did not trigger the packing. This gain is larger than $1/(2 + \alpha)$ and we obtain the desired bound for the competitive ratio.

Now, let us consider the case where $\alpha \leq 0.5$, which means that the request sequence either has at most two large elements or it is shorter than 3 elements. If we get a request sequence of 2 elements, the first one has to be $x_1 < 1/(2 + \alpha)$, otherwise we already have the desired competitive ratio by taking that element. Now, if $x_2 \leq 1 - 1/(2 + \alpha)$, then both elements fit into the knapsack and we get again a competitive ratio of at most $1/(1 - \alpha) < 2 + \alpha$. Otherwise, $x_2 > 1 - 1/(2 + \alpha)$, so the algorithm can pack $x_2$ and obtain a gain of at least $1 - 1/(2 + \alpha) - \alpha/(2 + \alpha) = 1/(2 + \alpha)$.

If the request sequence has at least length 3, we do a further case distinction. As in the previous case, if a request sequence ends without triggering the packing, then the optimal packing is $x_1 + \ldots + x_n = s < 1$ and Algorithm 2 has a gain of $(1 - \alpha)s$, so the competitive ratio is at most $1/(1 - \alpha) < 2 + \alpha$. Otherwise, assume that there is a shortest contradictory sequence which at some point $x_2$ triggers Algorithm 2 to pack. If $x_1 + \ldots + x_k \leq 1$, then we know that the gain of the algorithm is $x_k + (1 - \alpha)(x_1 + \ldots + x_{k-1})$, which is larger than $1/(2 + \alpha)$ by construction.

Otherwise, $x_1 + \ldots + x_k > 1$. In this case, if all of the elements of the request sequence are small or there is at most one large element, we are in the same case as before and we obtain the desired competitive ratio. Let us now assume that we have two large elements in our request sequence. We know that $x_1 < \frac{1}{(2 + \alpha)}$ and $x_2 < 1/(2 + \alpha) - (1 - \alpha)x_1$. By Lemma 6 we can also assume that $x_1 \geq x_2$ and with this we can deduce that $x_2 < \frac{1}{(2 + \alpha)} - (1 - \alpha)x_2$ and thus $(2 - \alpha)x_2 < \frac{1}{(2 + \alpha)}$ which allows us to estimate $x_2$ as $x_2 < \frac{1}{(2 + \alpha)(1 - \alpha)}$.

Now, we can fall in one of the following cases:
1. If $x_1 + x_2 + x_k \leq 1$, filling up greedily from the small items, we obtain the desired competitive ratio.

2. If $x_1 + x_2 + x_k > 1$, but $x_1 + x_k \leq 1$, we pack $x_1$ and $x_k$ and pack the small items greedily. If the small items fill the knapsack until at least $\frac{1}{2 + \alpha} (1 - \alpha)$, we know that we obtain the desired competitive ratio. Otherwise, the gain obtained by the algorithm is

$$x_1 + x_3 + \ldots + x_k - \alpha(x_1 + \ldots + x_{k-1})$$

$$= x_k + (1 - \alpha)(x_1 + x_3 + \ldots + x_{k-1}) - \alpha x_2$$

$$> 1 - x_1 - x_2 + (1 - \alpha)x_1 - \alpha x_2 + (1 - \alpha)(x_3 + \ldots + x_{k-1})$$

$$= 1 - \alpha x_1 - (1 + \alpha)x_2 + (1 - \alpha)(x_3 + \ldots + x_{k-1})$$

$$> 1 - \frac{\alpha}{2 + \alpha} - \frac{1 + \alpha}{(2 + \alpha)(2 - \alpha)} + (1 - \alpha)(x_3 + \ldots + x_{k-1})$$

$$= \frac{3(1 - \alpha)}{(2 + \alpha)(2 - \alpha)} + (1 - \alpha)(x_3 + \ldots + x_{k-1})$$

$$\geq \frac{1}{2 + \alpha} + (1 - \alpha)(x_3 + \ldots + x_{k-1}).$$

where the last step is true for any $\alpha \leq 1/2$, and we used $x_1 > 1 - x_1 - x_2$ and the upper bounds $x_1 < \frac{1}{2 + \alpha}$ and $x_2 < \frac{1}{(2 + \alpha)(2 - \alpha)}$ for $x_1$ and $x_2$.

3. If $x_1 + x_2 > 1$, but $x_2 + x_k \leq 1$, we pack $x_2$ and $x_k$ into the knapsack and the small items greedily. Here, again, if the small items fill the knapsack until at least $\frac{1}{2 + \alpha} (1 - \alpha)$, we know that we obtain the desired competitive ratio, otherwise, the gain obtained by the algorithm is $x_2 + \ldots + x_k - \alpha(x_1 + \ldots + x_{k-1})$ which is exactly the same as in the case for $\alpha \leq \alpha_4$, and we can perform the same operations and obtain the desired bound for the competitive ratio.

4. If $x_2 + x_k > 1$, we pack $x_k$ into the knapsack and pack the small items greedily. If these fill the knapsack with at least $\frac{1}{2 + \alpha} (1 - \alpha)$, we know that we obtain the desired competitive ratio, otherwise, the gain obtained by the algorithm is

$$x_3 + \ldots + x_k - \alpha(x_1 + \ldots + x_{k-1})$$

$$= x_k + (1 - \alpha)(x_3 + \ldots + x_{k-1}) - \alpha x_1$$

$$\geq 1 - x_2 - \alpha x_1 - \alpha x_2 + (1 - \alpha)(x_3 + \ldots + x_{k-1})$$

$$= 1 - \alpha x_1 - (1 + \alpha)x_2 + (1 - \alpha)(x_3 + \ldots + x_{k-1}),$$

which is exactly the same as in the case where $x_1 + x_2 + x_k > 1$ but $x_1 + x_k \leq 1$.

\[ \square \]

We continue with proving an upper bound for the rest of the interval, using induction over the number of large elements. Before we start our proof, we provide a lemma that allows us to ignore possible small elements during the proof of Theorem 3.

**Lemma 8.** Given a request sequence without small elements for which Algorithm 4 does not achieve a competitive ratio of $2 + \alpha$, adding small elements to it will only improve its competitive ratio.

**Proof.** Let us consider a request sequence containing only large elements $x_1 \geq \ldots \geq x_{k-1}, x_k, \ldots, x_n$, where $x_k$ is the element triggering the packing for Algorithm 2 and for which the achieved competitive ratio is larger than $2 + \alpha$. This means, by Lemma 5, that the total size of the items packed into the knapsack is smaller than $1/(2 + \alpha)(1 - \alpha)$. By definition, if we add enough small elements to the request sequence before
$x_k$, the small elements can be packed greedily until the knapsack is filled up to $1/(2 + \alpha)(1 - \alpha)$, achieving the desired competitive ratio. If not enough of them are added before $x_k$, the small elements requested before $x_k$ will be reserved but will still be able to be packed, so they will never contribute negatively to the total packing gain.

This lemma brings us to the point where it is possible to prove Theorem 3.

**Proof of Theorem 3.** We prove by induction that, for any finite number of large elements before the packing, Algorithm 2 achieves the desired competitive ratio for $\alpha < \phi - 1$. The base case for zero large elements is trivial, as we can greedily reserve and later pack all small elements to get the desired competitive ratio. Let us assume that Algorithm 2 achieves the desired competitive ratio for any request sequence with less than $k - 1$ large elements before the algorithm packs and stops.

If a request sequence has $k - 1$ large elements, we can assume by Lemma 6 that the smallest of those is $x_{k-1}$, and we can also assume by Lemma 8 that $x_k$ triggers the packing. Let us assume, using Lemma 10, that we have an instance $x_1 \geq x_2 \geq \ldots \geq x_{k-1}, x_k, \ldots x_n$. Moreover, note that

$$\frac{\alpha}{1-\alpha} < 1 - \alpha \text{ for } \alpha < \phi - 1.$$ (17)

We distinguish two cases.

1. Assume that, when the packing is triggered, $x_{k-1}$ is not part of an optimal packing.

In this case,

$$\left( \sum_{i \text{ is packed}} x_i \right) + x_{k-1} > 1.$$ (18)

We also have

$$x_{k-1} + (1 - \alpha) \sum_{j<k-1} x_j < \frac{1}{2 + \alpha},$$ (19)

since $x_{k-1}$ would trigger the packing otherwise, and thus

$$x_{k-1} < \frac{1}{2 + \alpha}.$$ (20)

Thus, the gain that Algorithm 2 achieves is

$$\left( \sum_{j \text{ is packed}} x_j \right) - \alpha R$$

$$> 1 - x_{k-1} - \alpha \sum_{j \leq k-1} x_j$$

$$= 1 - (1 + \alpha)x_{k-1} - \alpha \sum_{j<k-1} x_j$$

$$= 1 - 2\alpha x_{k-1} - (1 - \alpha) \left( x_{k-1} + \frac{\alpha}{1-\alpha} \sum_{j<k-1} x_j \right)$$

$$\geq 1 - 2\alpha x_{k-1} - (1 - \alpha) \left( x_{k-1} + (1 - \alpha) \sum_{j<k-1} x_j \right)$$

$$\geq 1 - 2\alpha x_{k-1} - (1 - \alpha) \frac{1}{2 + \alpha}$$

$$> 1 - \frac{2\alpha}{2 + \alpha} - \frac{1 - \alpha}{2 + \alpha}$$

$$= \frac{1}{2 + \alpha},$$

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as we wanted.

2. Assume that when the packing is triggered, \( x_{k-1} \) is part of an optimal packing. We consider two subcases.

(a) Taking \( x_{k-1} \) out of the request sequence still triggers the packing.

This means that \( x_k + (1 - \alpha)(R - x_{k-1}) \geq \frac{1}{2 + \alpha} \) holds. We can thus consider the sequence \( x_1, \ldots, x_{k-2}, x_k \). This sequence has \( k - 2 \) large elements, and \( x_{k-1} \) cannot be part of its optimal solution, thus its competitive ratio is at most \( 2 + \alpha \) by the induction hypothesis, and the competitive ratio after adding \( x_{k-1} \) can only get better.

(b) Taking \( x_{k-1} \) out of the request sequence does not trigger the packing.

This means that

\[
x_k + (1 - \alpha)R \geq \frac{1}{2 + \alpha}, \tag{21}
\]

and, if we let \( x_t \) be the smallest element that does not get packed, \( x_t \geq x_{k-1} \) holds. Also, because of the optimality of the packing \( x_k \geq x_t \), (otherwise one can take all of the reserved elements as the packing and obtain a better bound) and

\[
\sum_{j \text{ is packed}} x_j - x_{k-1} + x_t > 1 \tag{23}
\]

holds. With these bounds, the gain incurred by the algorithm is at least

\[
\left( \sum_{j \text{ is packed}} x_j \right) - \alpha R
\]

\[
> 1 - x_t + x_{k-1} - \alpha R \quad \text{using (23)}
\]

\[
= 1 - x_t + \frac{x_k}{1 - \alpha} + R - \frac{1}{(1 - \alpha)(2 + \alpha)} - \alpha R \quad \text{using (21)}
\]

\[
= 1 - x_t + \frac{x_k}{1 - \alpha} + (1 - \alpha)R - \frac{1}{(1 - \alpha)(2 + \alpha)}
\]

\[
\geq 1 - x_t + \frac{\alpha x_k}{1 - \alpha} + \frac{1}{2 + \alpha} - \frac{1}{(1 - \alpha)(2 + \alpha)} \quad \text{using (22)}
\]

\[
= \frac{1}{2 + \alpha} + \frac{1 - \alpha - \alpha^2}{(1 - \alpha)(2 + \alpha)} - x_t + \frac{\alpha x_k}{1 - \alpha}
\]

\[
\geq \frac{1}{2 + \alpha} + \frac{1 - \alpha - \alpha^2}{(1 - \alpha)(2 + \alpha)} + \frac{a - (1 - \alpha)x_k}{1 - \alpha} \quad \text{using } x_k \geq x_t
\]

\[
= \frac{1}{2 + \alpha} + \frac{1 - \alpha - \alpha^2}{(1 - \alpha)(2 + \alpha)} + \frac{2a - 1}{1 - \alpha x_k}
\]

\[
\geq \frac{1}{2 + \alpha},
\]

where the last step is trivially true for any \( \alpha \geq 1/2 \). Thus we get the desired competitive ratio in the considered range for \( \alpha \).

This proves the induction step, and thus the desired upper bound on the competitive ratio. \( \square \)
Algorithm 3 Algorithm for $\phi - 1 \leq \alpha < 1$

\[ R := 0; \]
\hspace{1em}for \( k = 1, \ldots, n \) do
\hspace{2em}if \( x_k + (1 - \alpha)R \geq 1 - \alpha \) then
\hspace{3em}pack \( x_1, \ldots, x_k \) optimally;
\hspace{3em}stop
\hspace{2em}else
\hspace{3em}reserve \( x_k \);
\hspace{3em}\( R := R + x_k \)
\hspace{2em}pack \( x_1, \ldots, x_n \) optimally

2.3. Upper Bound for $\phi - 1 \leq \alpha < 1$

For proving an upper bound, we consider Algorithm 3 and first bound its reservation costs.

Lemma 9. For Algorithm 3, the reservation cost $R$ is never larger than 1.

Proof. For any $j$, let $R_j$ denote the reservation cost of the algorithm after the items $x_1, \ldots, x_j$ have been presented. If, for any $k$,

\[ x_k + (1 - \alpha)R_{k-1} < 1 - \alpha , \]

then the algorithm reserves the item $x_k$ and has a new reservation cost of $R_k = x_k + R_{k-1}$. Since obviously $x_k \geq (1 - \alpha)x_k$, we have

\[ (1 - \alpha)R_k = (1 - \alpha)x_k + (1 - \alpha)R_{k-1} \leq x_k + (1 - \alpha)R_{k-1} < 1 - \alpha . \]

Thus, $R_k < 1$. Because we can apply this reasoning to any $x_k$ that does not trigger the if-condition in the algorithm, we have proven the claim. \hfill \qedsymbol

And we also need a lemma, analogous to Lemma 6, allowing us to reorder the items.

Lemma 10. If $x_1, \ldots, x_n$ is an instance for Algorithm 3, whose packing is triggered by $x_k$, with a competitive ratio larger than $\frac{1}{1-\alpha}$, then there is another instance $x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_{k-1}}, x_k, \ldots, x_n$ where $(i_1, i_2, \ldots, i_{k-1})$ is a permutation of $(1, \ldots, k-1)$ and $x_{i_1} \geq x_{i_2} \geq \ldots \geq x_{i_{k-1}}$, that also has a competitive ratio larger than $\frac{1}{1-\alpha}$.

Proof. We show how to transform the original sequence such that the number of inversions is reduced by one. Repeating this transformation yields the desired result.

Let $x_1, \ldots, x_n$ be an instance for Algorithm 3 such that packing is triggered by $x_k$ with a competitive ratio larger than $\frac{1}{1-\alpha}$. Assume that $x_i < x_{i+1}$ for some $i < k - 1$. We prove that we will get again an instance with a competitive ratio larger than $\frac{1}{1-\alpha}$ if we swap the positions of $x_i$ and $x_{i+1}$. It is easy to see that this transformation preserves the sum of the first $k - 1$ items. To show that it is a sequence that does not trigger the packing after the $i$th or $(i+1)$st item, we have to verify that

\[ x_{i+1} < (1 - \alpha) - (1 - \alpha)(x_1 + \cdots + x_{i-1}) \]  \hspace{1em} (24)

and

\[ x_i < (1 - \alpha) - (1 - \alpha)(x_1 + \cdots + x_{i-1} + x_{i+1}) . \]  \hspace{1em} (25)

Here, (24) is straightforward, since $x_k$ for $k > i + 1$ triggering the packing in the unmodified instance implies $x_{i+1} < (1 - \alpha) - (1 - \alpha)(x_1 + \cdots + x_i)$. We prove (25) by means of a contradiction. Assume, that

\[ x_i \geq (1 - \alpha) - (1 - \alpha)(x_1 + \cdots + x_{i-1} + x_{i+1}) . \]  \hspace{1em} (26)

We also know that, for $1 \leq j \leq i + 1$,

\[ x_j \leq (1 - \alpha) - (1 - \alpha)(x_1 - \ldots - x_{j-1}) , \]  \hspace{1em} (27)
and by construction

\[ x_i < x_{i+1} . \]  

(28)

If we define \( z = (1 - \alpha) - (1 - \alpha)(x_1 + \cdots + x_{i-1}) \), then we can rewrite (26), (27) for \( j = i + 1 \), and (28) as

\[
\begin{align*}
x_i &\geq z - (1 - \alpha)x_{i+1}, \\
x_{i+1} &\leq z - (1 - \alpha)x_i, \\
x_i &< x_{i+1}.
\end{align*}
\]

As in the proof of Lemma 6, we can solve this system of equations and conclude that the first two conditions are only satisfiable if \( x_{i+1} \leq x_i \), which gives us the desired contradiction.

Thus, the new sequence has a competitive ratio larger than \( \frac{1}{1-\alpha} \). Again, although the order of \( x_1, \ldots, x_{k-1} \) has been changed, the same reserved requests trigger a packing with \( x_k \) with the same reservation size, just as before the transformation. Hence, if there was no way to pack the items well enough before the transformation, it also cannot be done afterwards.

We are now ready to prove the desired competitive ratio for Algorithm 3.

**Theorem 11.** Algorithm 3 is an online algorithm for RESERVEKP achieving a competitive ratio of at most \( \frac{1}{1-\alpha} \), for all \( \phi - 1 \leq \alpha < 1 \).

**Proof.** Let us first assume that we run Algorithm 3 on an instance \( x_1, \ldots, x_n \), and no element triggers the packing. This means that all elements are reserved. But we know by Lemma 9 that the reservation never exceeds the capacity of the knapsack. This means that the optimal solution packs all offered elements. Thus, the algorithm achieves a competitive ratio of

\[
\frac{\sum_{i=1}^{n} x_{k-1}}{\sum_{i=1}^{n} x_{k-1} - \alpha \sum_{i=1}^{n} x_{k-1}} = \frac{1}{1-\alpha}.
\]

Now, it remains to analyze the case where an instance \( x_1, \ldots, x_n \) triggers the packing, for some \( x_k \). Let us do an induction on the value of \( k \). If \( k = 1 \), the first item offered triggers the packing, thus it holds that \( x_1 \geq 1 - \alpha \) and the gain of the algorithm is at least \( 1 - \alpha \) as we expected. Now, we assume that Algorithm 3 has a gain of at least \( 1 - \alpha \) on any request sequence triggering the algorithm to pack and stop before \( k \) elements are offered.

We proceed similarly to the proof of Theorem 3. Let us assume, using Lemma 10, that we have an instance \( x_1 \geq x_2 \geq \cdots \geq x_{k-1}, x_k, \ldots, x_n \) where the algorithm packs after receiving \( x_k \). We distinguish two cases.

1. When the packing is triggered, \( x_{k-1} \) is not part of an optimal packing.

   In this case,

   \[
   \left( \sum_{j \text{ is packed}} x_j \right) + x_{k-1} > 1 .
   \]  

   (29)

   We consider the following bounds. Since \( x_{k-1} \) did not trigger the packing, we know that

   \[
   x_{k-1} + (1 - \alpha) \sum_{j < k-1} x_j < 1 - \alpha ,
   \]

   (30)

   and thus also \( x_{k-1} < 1 - \alpha \).
Thus, the gain that Algorithm 3 achieves is

\[
\left( \sum_{j \text{ is packed}} x_j \right) - \alpha R \\
\geq 1 - x_{k-1} - \alpha \sum_{j \leq k-1} x_j \quad \text{using (29)}
\]

\[
= 1 - (1 + \alpha)x_{k-1} - \alpha \sum_{j \leq k-1} x_j \\
> 1 - (1 + \alpha)x_{k-1} - \alpha \left( 1 - \frac{x_{k-1}}{1 - \alpha} \right) \quad \text{using (30)}
\]

\[
= 1 - \alpha + \left( \frac{\alpha}{1 - \alpha} - (1 + \alpha) \right)x_{k-1} \geq 1 - \alpha + -1 + \alpha + \alpha^2 \left( 1 - \alpha \right)x_{k-1} \geq 1 - \alpha \quad \text{using } \alpha \geq \phi - 1.
\]

Thus we obtain the expected gain in this case.

2. When the packing is triggered, \( x_{k-1} \) is part of an optimal packing. We can consider two subcases.

(a) Taking \( x_{k-1} \) out of the request sequence still triggers the packing.

This means that

\[
x_k + (1 - \alpha)(R - x_{k-1}) \geq 1 - \alpha
\]

holds. We can thus consider the sequence \( x_1, \ldots, x_{k-2}, x_k \). This sequence has \( k-1 \) items before the packing is triggered, and \( x_{k-1} \) cannot be part of its optimal solution, thus its competitive ratio is at most \( 1/(1-\alpha) \) by the induction hypothesis, and the competitive ratio after adding \( x_{k-1} \) can only get better.

(b) Taking \( x_{k-1} \) out of the request sequence does not trigger the packing.

This means that

\[
x_k + (1 - \alpha)(R - x_{k-1}) < 1 - \alpha , \quad (31)
\]

but also

\[
x_k + (1 - \alpha)R \geq 1 - \alpha , \quad (32)
\]

and, if we let \( x_j \) be the smallest element that does not get packed, \( x_j \geq x_{k-1} \) holds. Also because of the optimality of the packing, \( x_k \geq x_j \) (otherwise one can take all of the reserved elements as the packing and obtain a better bound) and it holds that

\[
\left( \sum_{t \text{ is packed}} x_t \right) - x_{k-1} + x_j > 1 . \quad (33)
\]

Moreover, we can bound \( x_j \leq x_1 \leq 1 - \alpha \). With these bounds, the gain incurred by the algorithm
is at least
\[
\left( \sum_{t \text{ is packed}} x_t \right) - \alpha R \\
\geq 1 - x_j + x_{k-1} - \alpha R \\
\geq 1 - x_j + \frac{x_k}{1 - \alpha} + R - 1 - \alpha R \\
= -x_j + \frac{\alpha}{1 - \alpha} x_k + x_k + (1 - \alpha)R \\
\geq -x_j + \frac{\alpha}{1 - \alpha} x_k + (1 - \alpha) \\
= 1 - \alpha - x_j + \frac{\alpha}{1 - \alpha} x_k \\
\geq 1 - \alpha - x_j + \frac{\alpha}{1 - \alpha} x_j \\
= 1 - \alpha + \frac{2\alpha - 1}{1 - \alpha} x_j \\
\geq 1 - \alpha
\]
as we wanted.

This proves the induction step, and thus the desired competitive ratio upper bound.

We continue by providing matching lower bounds to the upper bounds of this section.

3. Lower Bounds

First we present an adversarial strategy that works for all values of \( \alpha \). Then we proceed to analyze the case where only four objects are presented as a generic adversarial strategy and find improved lower bounds for some values of \( \alpha \).

3.1. Tight Lower Bound for \( 0 \leq \alpha \leq 0.25 \)

**Theorem 12.** For \( \alpha > 0 \) there exists no algorithm for reservation knapsack achieving a competitive ratio better than 2.

**Proof.** Consider the following set of adversarial instances as depicted in Figure 2. Given any \( \varepsilon > 0 \), the adversary presents first an object of size \( \frac{1}{2} + \delta \) with \( 0 < \delta \ll \varepsilon \). If an algorithm takes this object, an object of size 1 will follow, making its competitive ratio \( \frac{1}{(\frac{1}{2} + \delta)} > 2 - \varepsilon \). If an algorithm rejects this object, no more objects will follow and it will not be competitive. If an algorithm reserves this object, then an object of size \( \frac{1}{2} + \delta^2 \) will be presented. Observe, that these two objects do not fit together into the knapsack. If an algorithm takes this object, an object of size 1 will be presented, and again the algorithm will achieve a competitive ratio worse than \( 2 - \varepsilon \). If an algorithm reserves this object, then an object of size \( \frac{1}{2} - \delta^2 \) will be presented. This object does not fit in the knapsack with the first one, thus the algorithm can only pack the first object, obtaining a competitive ratio worse than \( 2 - \varepsilon \). If an algorithm reserves this object, then an object of size \( \frac{1}{2} + \delta^3 \) will be presented. The adversary can follow this procedure on and on, and in each step the competitive ratios for algorithms that accept or reject the offered item only get worse due to the additional reservation costs.

The adversary can stop offering items as soon as the reservation costs are such that filling the knapsack will only result in a competitive ratio worse than 2. This shows that, for every \( \varepsilon > 0 \), the competitive ratio is at least \( 2 - \varepsilon \), so the best competitive ratio is at least 2.

The adversarial strategy depicted in Figure 2 works for every positive value of \( \alpha \), but is of course not tight when \( \alpha > 0.25 \). Observe that, if an algorithm continues to reserve items, the cumulated reservation cost will eventually exceed any remaining gain.
3.2. Tight Lower Bound for $0.25 \leq \alpha \leq 1$

The previous strategy provided us with a lower bound that does not match the upper bound for $\alpha > 0.25$. We improve the lower bound for larger $\alpha$ in a way that it provides a matching bound by designing a generic adversarial strategy with up to four items as shown in Figure 3, which contains the competitive ratios for all possible outcomes. For each item, an algorithm can decide to either reject, pack or reserve it. Depending on an algorithm’s behavior on the previous item, it is decided whether another item is presented and, if so, which item. We assume by construction that $s < t < u < 1$ and $s + t > 1$. The competitive ratio of such a strategy is therefore bounded by

$$\rho_k \geq \min \left\{ \frac{1}{s}, \frac{t}{(1-\alpha)s}, \frac{1}{t-\alpha t}, \frac{u}{t-\alpha(s+t)}, \frac{1}{u-\alpha(s+t)}, \frac{u}{u-\alpha(s+t+u)} \right\}. \quad (34)$$

To prove a lower bound for every $0 < \alpha < 1$, we can choose $s$, $t$ and $u$ in order to make (34) as large as possible. Standard calculus leads to the bounds in the following theorem.

**Theorem 13.** The competitive ratio of ReserveKP is at least

(a) $\frac{1+\sqrt{5-4\alpha}}{2(1-\alpha)}$ for $0.225 \leq \alpha \leq \sqrt{2} - 1$;
(b) $2 + \alpha$, for $\sqrt{2} - 1 \leq \alpha \leq \phi - 1$; and
(c) $\frac{1}{1 - \alpha}$, for $\phi - 1 \leq \alpha \leq 1$.

**Proof.** Let us consider an adversary that presents an item of size $s$ first. The size of this item has to be smaller than $\frac{1}{\alpha}$, otherwise an algorithm can take this item and get a competitive ratio smaller than 2. If $s$ is taken, the adversary presents an item of size 1. If $s$ is rejected, no more items will be offered. If $s$ is reserved, an item of size $t$ is presented, where $s + t > 1$. If $t$ is taken, the adversary presents an item of size 1 and if $t$ is rejected, no more items will be offered, and, if $t$ is reserved, an item of size $u$ is presented with $u > t$, if $u$ is taken, the adversary presents an item of size 1 and otherwise the request sequence ends in this third item.

For this adversarial strategy, we can compute the competitive ratio for all possible algorithms as can be seen in Figure 3.

Thus, the competitive ratio of any algorithm $A$ confronted with this adversary is bounded by Equation 34, which we want to maximize.

Choosing $s = \frac{2}{3 + \sqrt{5} - 4\alpha} + \varepsilon$, $t = \frac{\sqrt{5} - 1 + 2\alpha}{2(1 + \alpha)}$, and $u = \frac{\alpha + \sqrt{4(1 - \alpha) + \alpha^2}}{2}$, we see that, if $0.225 \leq \alpha \leq \sqrt{2} - 1$, then $\rho_k \geq 1 + \frac{\sqrt{5} - 4\alpha}{2(1 - \alpha)}$.

If we compare the given competitive ratio to all of the expressions from (34), and substitute the values of $s$, $t$ and $u$ but omit the $\varepsilon$ in the $s$, we see immediately that $1/s = \frac{3 + \sqrt{5} - 4\alpha}{2} \geq \frac{1 + \sqrt{5} - 4\alpha}{2(1 - \alpha)}$ for $\alpha \leq \sqrt{2} - 1$, and $1/\alpha = \frac{1}{s - \alpha(s + t)} = 1 + \frac{\sqrt{5} - 4\alpha}{2(1 - \alpha)}$.

The last three expressions are $\frac{1}{u - \alpha}$, $\frac{1}{t - \alpha}$ and $\frac{1}{u(1 - \alpha) - s}$. For the given $u$, we see that $\frac{1}{u - \alpha} = \frac{u}{u(1 - \alpha) - s} \geq \frac{1 + \sqrt{5} - 4\alpha}{2(1 - \alpha)}$ for $\alpha \geq \alpha_0$, where $\alpha_0 \approx 0.224$ is the unique positive root of $x^4 + 2x^3 - 2x^2 + 5x - 1$. Finally, we observe that $\frac{u}{u(1 - \alpha) - s} \geq \frac{1 + \sqrt{5} - 4\alpha}{2(1 - \alpha)}$ for $\alpha \geq 0.19$ and in particular through the desired range.

Note that $2 \geq \frac{1 + \sqrt{5} - 4\alpha}{2(1 - \alpha)}$ for $\alpha \leq 0.25$, so while the lower bound is correct for values of $\alpha$ down to approximately 0.224, our previous lower bound dominates it up to $\alpha = 0.25$.

For the range $\sqrt{2} - 1 \leq \alpha \leq \phi - 1$, we choose $s = \frac{1}{2 + \alpha}$, $t = 1 - s + \varepsilon$ and no element $u$. We can rewrite Equation (34) as

$$\rho_k \geq \min \left\{ \frac{1}{s}, \frac{1}{t - \alpha s}, \frac{1}{t - \alpha s}, \frac{t}{t - \alpha(s + t)} \right\}.$$ \hspace{1cm} (35)

Substituting the appropriate values of $s$ and $t$ in Equation (35) yields

$$\rho_k \geq \min \left\{ 2 + \alpha, 1 + \frac{1 + \alpha}{1 - \alpha}, \frac{1 + \alpha}{1 - \alpha^2} \right\}.$$ But $\frac{1 + \alpha}{1 - \alpha} \geq 2 + \alpha$ for $\alpha \geq \sqrt{2} - 1$ and $\frac{1 + \alpha}{1 - \alpha^2} \geq \frac{1 + \alpha}{1 - \alpha}$ as long as the denominator is not negative, that is, as long as $\alpha \leq \phi - 1$.

Finally, for the range $\phi - 1 \leq \alpha \leq 1$, we choose $s = 1 - \alpha$ and no elements $t$ or $u$. We can rewrite Equation (34) as

$$\rho_k \geq \min \left\{ \frac{1}{s}, \frac{s}{s - \alpha s} \right\}.$$ \hspace{1cm} (36)

Substituting the appropriate values of $s$ in Equation (36) yields $\rho_k \geq \frac{1}{1 - \alpha}$ as expected. \hfill \Box

### 4. Nonrejecting Algorithms

A seemingly plausible intuition for RESERVEKP might be the following: If the cost of reservation is very small, rejecting an item should not be necessary, as even when an item cannot be packed, the cost of reserving it is negligible. On the other hand, when the reservation cost is rising, aggressively reserving items...
may seem like a very bad strategy, as the risk of not being able to utilize reserved items may come to mind. Interestingly, both of these intuitions turn out to be wrong, which we will show by first giving a lower bound for nonrejecting algorithms (i.e., algorithms that reject nothing until they pack the knapsack, after which all remaining items are rejected) that exceeds the upper bound of a rejecting algorithm for small $\alpha$ and that tightly matches the upper bound of a nonrejecting algorithm for bigger $\alpha$.

We first provide a lower bound for algorithms that are unable to reject items.

**Theorem 14.** There exists no deterministic online algorithm for $\text{ReserveKP}$ that does not reject any elements with a competitive ratio better than $2 + \alpha$ for any $0 < \alpha \leq 1$.

**Proof.** Let $\varepsilon > 0$. Consider the following set of adversarial instances depicted in Figure 4. First the adversary presents an item of size $\frac{1}{2} - \frac{\alpha}{4 + 2\alpha} = \frac{1}{2 + \alpha}$. If an algorithm takes this item the adversary will present an item of size 1, and the algorithm will have a competitive ratio of $2 + \alpha$ as claimed. If an algorithm reserves this first item, the adversary will present a second item of size $\frac{1}{2} + \frac{\alpha}{4 + 2\alpha} + \varepsilon = \frac{1 + \alpha}{2 + \alpha} + \varepsilon$. As both items do not fit into the knapsack together, an algorithm can decide to either take the larger of the two items or to reserve this second item as well. If it takes the larger item, the adversary will again present an item of size one. An algorithm can thus achieve a gain of

$$\frac{1 + \alpha}{2 + \alpha} + \varepsilon - \alpha \frac{1}{2 + \alpha} = \frac{1}{2 + \alpha} + \varepsilon$$

which results in the claimed competitive ratio. Finally, whenever an algorithm decides to reserve an item, from this point onwards, another item of size $\frac{1 + \alpha}{2 + \alpha} + \varepsilon$ is presented. At any point at which the algorithm decides to take a presented item, an item of size 1 is presented afterwards. As there are no two items in the instance that fit into the knapsack together, the gain of any deterministic nonrejecting algorithm will strictly decrease with every reservation. 

Combined with the upper bound given in Theorem 1, we see that an algorithm that is unable to reject items performs quite a bit worse than one that is able to reject items, such as Algorithm 1. Thus, an algorithm needs to be able to reject items to become 2-competitive for small values of $\alpha$. On the other hand, the lower bound provided in Theorem 14 matches the upper bound of Theorem 2, which is based on the nonrejecting Algorithm 2. We conjecture that there are nonrejecting algorithms for every $\alpha \geq \sqrt{2} - 1$ that are at least as good as any other algorithms that are able to reject items.

5. Further Work

In this work, we provide tight competitive ratios for the $\text{ReserveKP}$. While the model seems natural to analyze, there are possible variations that could be studied in order to see how a variation of the reservation model influences the competitive ratio. One could consider a variant where reservation costs are refunded if the item is used, modelling real-world applications such as a down payment. Another reasonable model could look at the largest sum of items in a reserve at any time and ask for reservation costs relative to this size, such as when companies have to rent storage space.
The concept of reservation may be applied to other online problems such as online call admission problems in networks or problems of embedding guest graphs into a host graph. In the online path packing problem, one packs paths in a edge-disjoint way (sometimes node-disjointly) into a graph, which is a generalization of \textsc{ReserveKP}, thus inheriting all lower bounds. The offline version was studied on many types of graphs, with an incomplete selection being \cite{2, 23, 8}. The challenge when applying the idea of reservation is to find an appropriate cost function that can be measured against the competitive ratio.

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