On the Dual Canonical Monoids

Mahir Bilen Can

1mahirbilencan@gmail.com

May 22, 2019

Abstract

We investigate the conjugacy decomposition, nilpotent variety, the Putcha monoid, as well as the two-sided weak order on the dual canonical monoids.

Keywords: Nilpotent variety, Gauss-Jordan elements, asymptotic semigroup

MSC: 20M32, 14C15, 14M27

1 Introduction

Let $M$ be a reductive monoid with unit group $G$. We fix a Borel subgroup $B$ and a maximal torus $T$ such that $T \subseteq B$. The Renner monoid of $M$, denoted by $R$, is a finite semigroup which parametrizes the $B \times B$-orbits in $M$, [18]. The nilpotent variety of $M$, denoted by $M_{\text{nil}}$, is the subvariety consisting of all nilpotent elements of $M$. It is studied by Putcha in a series of papers, [12, 14, 16, 17]. Unlike $M$, the nilpotent variety does not decompose into $B \times B$-orbits. Nevertheless, Putcha showed in [14, Theorem 3.1] that, under the conjugation action, $M$, hence $M_{\text{nil}}$, posses closely related decompositions. More precisely, we have $M = \bigsqcup_{\sigma \in \mathcal{C}} X(\sigma)$ and $M_{\text{nil}} = \bigsqcup_{\sigma \in \mathcal{C}_{\text{nil}}} X(\sigma)$, where $X(\sigma) = \bigcup_{g \in G} Bg \sigma Bg^{-1}$. The indexing object, that is $\mathcal{C}$, is called the Putcha poset; it is defined as a quotient $GJ/\sim$, where $GJ$ is a finite submonoid in $R$, and $\sim$ is the conjugation equivalence relation defined as follows: $\sigma \sim \sigma'$ if there is an element $w \in W$ such that $\sigma = w \sigma' w^{-1}$. Then $\mathcal{C}_{\text{nil}} = \{[\sigma] \in \mathcal{C} : \sigma^k = 0 \text{ for some } k \in \mathbb{N}\}$. The varieties $X(ev)$ ($ev \in GJ$) give a stratification of $M$ in the sense that, for any $ev \in GJ$, we have $X(ev) = \bigcup_{ew \leq ev} X(ew)$. The purpose of our article is to study various partial orders arising from such decompositions for some very specific monoids. Our main focus is on the “dual canonical monoids.” Rather than introducing these objects by their technical definition, let us describe an important member of their family.

The asymptotic semigroup of a semisimple group $G_0$, denoted by $\text{As}(G_0)$, is the algebraic semigroup whose coordinate ring is given by $\text{gr} \ k[G_0]$, where $k[G_0]$ is the coordinate ring of $G_0$. The grading on $k[G_0]$ is the one that comes from a well-known decomposition of
$k[G_0]$ as a $G_0 \times G_0$-module. More precisely, we have $k[G_0] = \bigoplus_{\chi \in \bar{O}^+} V(\chi) \otimes V^*(\chi)$, where $\bar{O}^+$ is the semigroup of dominant weights, and $V(\chi)$ is the finite dimensional irreducible representation of $G_0$ corresponding to the highest weight $\chi \in \bar{O}^+$, and $V^*(\chi)$ is its dual. This remarkable algebraic semigroup is introduced by Vinberg in [28, 27], and studied by Rittatore [21, 22] from the point of view of spherical varieties. By [27, Theorem 2], we know that the union $\text{As}(G_0) \cup G$, where $G \cong k^* \cdot G_0$, has the structure of a normal irreducible algebraic semigroup. In fact, as $1 \in G$, this union is a semisimple monoid. An alternative construction of this monoid, by using one-parameter monoids, is outlined in [19, Section 6.2]. We will call $\text{As}(G_0) \cup G$ the asymptotic monoid of $G_0$. As we alluded before, $\text{As}(G_0)$ is a dual canonical monoid whose precise definition will be given in the sequel.

In this paper, among other things, we will discuss the Putcha posets $C$ and $C_{nil}$ associated with the dual canonical monoids. The main structural properties of these posets are described by Putcha in his papers that are mentioned before. Additional progress, in the cases of canonical and dual canonical monoids, is made by Therkelsen in [25, 26]. Now we will state two of our main results.

**Theorem 1.1.** Let $M$ be a dual canonical monoid, and let $M_{nil}$ denote its nilpotent variety. Then $M_{nil}$ is an equidimensional variety of dimension $\dim G_0 - |S|$, where $G_0$ is the derived subgroup of the unit group of $M$, and $S$ is the set of simple reflections for $G_0$.

Let $e$ be an idempotent from the cross-section lattice contained in a Renner monoid $R$. Let $C$ denote the Putcha poset. By $C(e)$, we will denote the subposet of $C$ whose elements come from the double coset $WeW$. The rook monoid $R_n$ is the finite inverse semigroup, whose elements are the $n \times n$ 0/1 matrices with at most one 1 in each row and each column. It is the Renner monoid for the monoid of $n \times n$ matrices, see [18]. The Bruhat-Chevalley-Renner order on any Renner monoid will be denoted by $\leq$; it is defined by the inclusion relations among the Zariski closures of the $B \times B$-orbits.

**Theorem 1.2.** Let $C$ denote the Putcha poset of the dual canonical monoid with unit group $GL_n$, and let $W$ denote the symmetric group $W = S_n$. Let $k$ be a number such that $\lfloor n/2 \rfloor \leq k \leq n - 1$. If $I$ is the subset $\{s_1, \ldots, s_k\}$ in $S = \{s_1, \ldots, s_{n-1}\}$, then the opposite of the poset $W_I \backslash W/W_I$, or equivalently, the Putcha subposet $C(e_I)$ is isomorphic to $(R_{\lfloor n/2 \rfloor - k}, \leq)$.

Another goal of our paper is to initiate the study of the (two-sided) weak order, denoted by $\leq_{LR}$, on reductive monoids. We define it by using the double Richardson-Springer monoid action on the Renner monoid $R$. This action respects the decomposition $R = \bigsqcup_{\Lambda \in \Lambda} WeW$, where $\Lambda$ is the cross-section lattice, which parametrizes the $G \times G$-orbits in $M$. Here, $G$ is the unit group of the reductive monoid $M$, and $W$ is the unit group of $R$, which is equal to the Weyl group of $G$. As its notation suggests, when restricted to $W$, the weak order agrees with the two-sided weak order on the Coxeter group $W$. It is easy to see from a simple example that the two-sided weak order on a Coxeter group is not a lattice. However, as we will show in the sequel, for dual canonical monoids, if $e$ is from $\Lambda \setminus \{1\}$, then $(WeW, \leq_{LR})$ is a lattice. Furthermore, it is a distribute lattice if and only if $(WeW, \leq_{LR}) \cong (WeW, \leq)$. 

2
A crucial notion that is related to the geometry of the weak order is the “degree” of a covering relation. It essentially measures the generic degree of a morphism that is canonically attached to a covering relation in the weak order. This number can be 0, 1, or a power of 2.

A related result that we prove here is the following.

**Theorem 1.3.** Let $M$ be a dual canonical monoid, and let $W$ and $\Lambda$ denote, as before, the Weyl group and the cross-section lattice of $M$, respectively. If $e$ is an idempotent from $\Lambda \setminus \{1\}$, then all covering relations in $(WeW, \leq_{LR})$ have degree 1.

The two-sided weak order on the symmetric group $S_{n+1}$ is interesting by itself. It turns out that there are many degree 2 covering relations in this case.

**Theorem 1.4.** Let $W$ denote the symmetric group $S_{n+1}$. Then we have
1. the total number of covering relations in $(W, \leq_{LR})$ is $n^2 n!$;
2. the number of covering relations of degree 2 in $(W, \leq_{LR})$ is $nn!$.

We are now ready to describe the individual sections of our paper. In the next preliminaries section we collect some well-known facts about the reductive monoids, Bruhat-Chevalley-Renner order, Putcha posets, and about the nilpotent variety. The purpose of Section 3 is to streamline some important structural results regarding the type map and the $G \times G$-orbits for a dual canonical monoid. In Section 4 we prove one of our main results that the rook monoid appears as an interval in the Putcha poset of the dual canonical monoid with unit group $\text{GL}_n$. In Section 5 we show that the nilpotent variety of the dual canonical monoid is equidimensional. In particular, we give precise descriptions of the some of the intervals in $C_{nil}$. The purpose of Section 6 is to define and study the weak order on the sets $WeW$. We finish our paper by Section 7 where we mention a theorem about the order complex of the poset $(W, \leq_{LR})$ which we plan to report in another paper.

## 2 Preliminaries

Let $G$ be a connected reductive group, let $T$ be a maximal torus, and let $B$ be a Borel subgroup of $G$ such that $T \subset B$. We denote by $W$ the Weyl group $N_G(T)/T$. The Bruhat-Chevalley order on $W$ is defined by $v \leq w \iff B\dot{v}B \subseteq B\dot{w}B$, where $\dot{v}$ and $\dot{w}$, respectively, are two elements from $N_G(T)$ representing the cosets $v$ and $w$. The bar on $B\dot{w}B$ indicates the Zariski closure in $G$. In the sequel, if a confusion is unlikely, then we will omit writing the dots on the representatives of the cosets.

For the poset $(W, \leq)$, the data of $(G, B, T)$ determines a Coxeter generating system $S$ and a length function $\ell : W \to \mathbb{Z}$, where, for $w \in W$, $\ell(w)$ is equal to the minimal number of simple reflections $s_{i_1}, \ldots, s_{i_r}$ from $S$ with $w = s_{i_1} \cdots s_{i_r}$. A subgroup that is generated by a subset $I \subset S$ will be denoted by $W_I$ and it will be called a parabolic subgroup of $W$. For $I \subseteq S$, we will denote by $D_I$ the following set:

$$D_I := \{ x \in W : \ell(xw) = \ell(x) + \ell(w) \text{ for all } w \in W_I \}. \quad (2.1)$$
Let \( M \) be a reductive algebraic group. This means that the unit group of \( M \), denoted by \( G \), is a connected reductive algebraic group. Let \( T \) be a maximal torus in \( G \), and let \( B \) be a Borel subgroup such that \( T \subset B \). The following decompositions are well-known:

1. \( M = \bigsqcup_{r \in R} BrB \) (the Renner decomposition of \( M \));
2. \( M = \bigsqcup_{e \in \Lambda} GeG \) (the Putcha decomposition of \( M \)).

In the first item, the parametrizing object \( R \) is called the Renner monoid of \( M \), and it is defined as \( R := \overline{N_G(T)} / T \), where \( N_G(T) \) is the normalizer of \( T \) in \( G \), and the bar over \( N_G(T) \) denotes the Zariski closure in \( M \). Then \( R \) is a finite inverse semigroup with the unit group \( W := \overline{N_G(T)} / T \), the Weyl group of \( G \). In the second item, the parametrizing object \( \Lambda \) is called the cross-section lattice (or, the Putcha lattice) of \( M \); if \( M \) has a zero, then \( \Lambda \) can be defined as
\[
\Lambda := \{ e \in E(T) : Be = eBe \},
\]
where \( E(T) \) denotes the semigroup of idempotents of \( T \). In fact, \( \Lambda \) and \( B \) determine each other, see [13, Theorem 9.10]. This means also that the cross section lattice determines (and determined by) the set of Coxeter generators for \( W \).

The set that is described in the next lemma is first used by Renner in [18], where, among other things, the Gauss-Jordan elimination method is generalized to arbitrary reductive monoids.

**Lemma 2.2.** If \( GJ = GJ(R, B) \) denotes the set \( GJ := \{ x \in R : Bx \subseteq xB \} \), then \( GJ \) is a submonoid of \( R \).

**Proof.** Clearly, the neutral element of \( R \) is contained in \( GJ \). If \( x \) and \( y \) are two elements from \( GJ \), then \( Bxy \subseteq xBy \) and \( xBy \subseteq xyB \). It follows that \( xy \in GJ \).

We will call \( GJ \) the Gauss-Jordan monoid of \( M \) although, strictly speaking, it is determined by \( (R, B) \). Note that the unit group \( W \) acts on \( R \) by left multiplication, and \( W \times W \) acts on \( R \) by \( (a, b) \cdot x = axb^{-1} \), where \( a, b \in W \) and \( x \in R \). Then the \( W \)-orbits (resp. the \( W \times W \)-orbits) are parametrized by \( GJ \) (resp. by \( \Lambda \)). Indeed, it is easy to see from [19, Proposition 8.9] that
\[
|Wx \cap GJ| = 1 \quad \text{for every } x \in R. \tag{2.3}
\]

The cross section lattice \( \Lambda \) has a natural, semigroup theoretic partial order:
\[
e \leq f \iff e = fe = ef \quad \text{for } e, f \in \Lambda. \tag{2.4}
\]

If we view \( \Lambda \) in \( R \), then (2.4) agrees with the Bruhat-Chevalley-Renner order on \( R \), which is defined by
\[
x \leq y \iff Bx \subseteq ByB \quad \text{for } x, y \in R. \tag{2.5}
\]

For an element \( e \) from \( \Lambda \), we define the following subgroups in \( W \):

\[1\]
1. \( W(e) := \{a \in W : ae = ea\} \),
2. \( W^*(e) := \cap_{f \geq e} W(f) \),
3. \( W_*(e) := \cap_{f \leq e} W(f) = \{a \in W : ae = ea = e\} \).

Then we know from [13, Chapter 10] that \( W(e), W^*(e), \) and \( W_*(e) \) are parabolic subgroups of \( W \), and furthermore, we know that \( W(e) \cong W^*(e) \times W_*(e) \). If \( W(e) = W_I \) and \( W_*(e) = W_K \) for some subsets \( I, K \subset S \), then we define \( D(e) := D_I \) and \( D_*(e) := D_K \).

Let \( B(S) \) denote the Boolean lattice of all subsets of \( S \). The type map of the cross-section lattice of \( M \) is an order preserving map \( \lambda : \Lambda \rightarrow B(S) \) that plays the role of Coxeter-Dynkin diagram for \( M \). It is defined as follows. Let \( e \in \Lambda \). Then \( \lambda(e) := \{s \in S : es = se\} \). Associated with \( \lambda(e) \) are the following sets:

\[ \lambda_*(e) := \cap_{f \leq e} \lambda(f) \quad \text{and} \quad \lambda^*(e) := \cap_{f \geq e} \lambda(f). \]

Then we have

\[ W(e) = W_{\lambda(e)}, \quad W_*(e) = W_{\lambda_*(e)}, \quad W^*(e) = W_{\lambda^*(e)}. \]

**Theorem/Definition (Pennell-Putcha-Renner):** For every \( x \in WeW \) there exist elements \( a \in D_*(e), b \in D(e), \) which are uniquely determined by \( x \), such that

\[ x = aeb^{-1}. \tag{2.6} \]

The decomposition of \( x \) in (2.6) will be called the standard form of \( x \). Let \( e, f \) be two elements from \( \Lambda \). It is proven in [10] that if \( x = aeb^{-1} \) and \( y = cfd^{-1} \) are two elements in standard form in \( R \), then

\[ x \leq y \iff e \leq f, \quad a \leq cw, \quad w^{-1}d^{-1} \leq b^{-1} \quad \text{for some} \quad w \in W(f)W(e). \tag{2.7} \]

Let us write \( D(e)^{-1} \) to denote the set \( \{b^{-1} : b \in D(e)\} \). In this notation, the Gauss-Jordan monoid of \( R \) has the following decomposition:

\[ GJ = \bigsqcup_{e \in \Lambda} eD(e)^{-1}. \tag{2.8} \]

For \( e, f \in \Lambda \), let \( x \) be an element from \( D(e)^{-1} \), and let \( y \) be an element from \( D(f)^{-1} \). Then (2.7) translates to the following statement:

\[ ex \leq fy \iff y \leq wx \quad \text{for some} \quad w \in W(e). \tag{2.9} \]

Another useful method for studying Bruhat-Chevalley-Renner order is introduced by Putcha in [15]. Let \( e \) and \( f \) be two elements from \( \Lambda \) such that \( e \leq f \). Then Putcha defines the associated “upward projection map” \( p_{e,f} : WeW \rightarrow WfW \), and he shows that

\[ \sigma \leq \sigma' \iff p_{e,f}(\sigma) \leq \sigma' \quad \text{for} \quad \sigma \in WeW \quad \text{and} \quad \sigma' \in WfW. \]

In the sequel, we will use the adaptation of these maps to the Putcha posets of dual canonical monoids. This adaptation is already used by Therkelsen in [26].

The main properties of the projection maps are summarized in the next theorem.
Theorem 2.10. \cite[Theorem 2.1]{15} Let $e, f \in \Lambda$ be such that $e \leq f$. Then

1. $p_{e,f} : WeW \to WfW$ is order preserving and $\sigma \leq p_{e,f}(\sigma)$ for all $\sigma \in WeW$.

2. If $\sigma \in WeW, \theta \in WfW$, then $\sigma \leq \theta \iff p_{e,f}(\sigma) \leq \theta$.

3. If $h \in \Lambda$ with $e \leq h \leq f$, then $p_{e,f} = p_{h,f} \circ p_{e,h}$.

4. $p_{e,f}$ is onto if and only if $\lambda_\ast(e) \subseteq \lambda_\ast(f)$.

5. $p_{e,f}$ is 1-1 if and only if $\lambda(f) \subseteq \lambda(e)$.

2.1 The conjugacy decomposition.

The results that we mention in this subsection are obtained by Putcha in a series of papers, \cite{12,14,16,17}.

Let $M$ be a reductive monoid with zero. Let $GJ$ denote its Gauss-Jordan monoid relative to some Borel subgroup $B$. The following equivalence relation on $GJ$ is introduced by Putcha:

$$ey \sim e'y' \iff weyw^{-1} = e'y' \text{ for some } w \in W.$$\hfill (2.11)

Note that, if $ey \sim e'y'$, then we have $e = e'$.

**Definition 2.12.** The set of equivalence classes of $\sim$ will be denoted by $C$, and it will be called the Putcha poset of $M$. For $e \in \Lambda$, we will denote by $C(e)$ the subposet $C(e) := \{ev : ev \in C\}$. We will denote by $C_{nil}$ the subposet consisting of nilpotent elements,

$$C_{nil} := \{[ey] \in C : (ey)^k = 0 \text{ for some } k \in \mathbb{N}\},$$

and we will denote by $C_{nil}(e)$ the subposet $C_{nil} \cap C(e)$. By abusing the terminology, we will call $C_{nil}$ a Putcha poset as well.

The **conjugacy decomposition** of $M$ is given by

$$M = \bigsqcup_{[ey] \in C} X(ey), \quad \text{where} \quad X(ey) := \bigsqcup_{g \in G} gBeyBg^{-1}.$$\hfill (2.12)

If $M$ has a zero, then the **nilpotent variety** of $M$, denoted by $M_{nil}$, is defined by

$$M_{nil} := \{a \in M : a^k = 0 \text{ for some } k \in \mathbb{N}\}.$$\hfill (2.13)

Clearly, the set of nilpotent elements in a semigroup with zero is closed under the conjugacy action of its units. For $M_{nil}$, the conjugacy decomposition of $M$ yields the following decomposition:

$$M_{nil} = \bigsqcup_{[ev] \in C_{nil}} X(ev).$$\hfill (2.14)
Let $R$ denote the Renner monoid of $M$. In relation with the conjugacy decomposition, for $\sigma \in R$, we will call the associated locally closed subvariety $X(\sigma)$ a Pucha sheet in $M$. For $\tau, \sigma \in R$, it is easy to see that

$$\tau \leq \sigma \iff X(\tau) \subseteq X(\sigma).$$

In fact, Putcha shows that, for $[ey], [e'y'] \in C$,

$$[ey] \leq [e'y'] \iff X(ey) \subseteq X(e'y'). \quad (2.13)$$

In particular, for $[ey] \in C$, we know that

$$\overline{X(e'y')} = \bigcup_{[ey] \leq [e'y']} X(ey).$$

It turns out that the order (2.13) is equivalent to the following partial order:

$$[ey] \leq [e'y'] \iff \text{weyw}^{-1} \leq e'y' \quad \text{for some } w \in W. \quad (2.14)$$

**Theorem 2.15.** [17, Theorem 4.2] Let $M$ be a reductive monoid with zero. Let $0 \neq e \in \Lambda$, $y \in D(e)^{-1}$. Then $[ey] \in C_{\text{nil}}$ if and only if $\text{supp}(y) \subseteq \lambda(f)$ for all $f \in \Lambda_{\text{min}}$ with $f \leq e$.

A reductive monoid $M$ is called $J$-coirreducible if $\Lambda \setminus \{1\}$ has a unique maximal element, $e_{\text{max}}$. In this case, the type of $M$ is defined as the subset $I := \lambda(e_{\text{max}})$ in $S$. A reductive monoid $M$ with a zero is called $J$-irreducible if $\Lambda \setminus \{0\}$ has a unique minimal element, $e_{\text{min}}$. In this case, the type of $M$ is defined as the subset $I := \lambda(e_{\text{min}})$ in $S$.

**Theorem 2.16.** [17, Theorem 6.1] Let $M$ be a $J$-coirreducible monoid of type $I$. Then

1. $M$ is semisimple;
2. $e, e' \in \Lambda \setminus \{1\}$, then $e \leq e'$ if and only if $\lambda_*(e') \subseteq \lambda_*(e)$;
3. $e' \in \Lambda \setminus \{1\}$, then $\lambda^*(e) = \{s \in I : ss' = s's \text{ for every } s' \in \lambda_*(e)\}$;
4. If $K \subseteq S$, then $K = \lambda_*(e)$ for some $e \in \Lambda \setminus \{1\}$ if and only if no connected component of $K$ is contained in $I$;
5. If $e \in \Lambda \setminus \{1\}$, then $|\lambda_*(e)| = \text{crk}(e) - 1 = |S| - \text{rk}(e)$. In particular, if $e \in \Lambda_{\text{min}}$, then $\lambda(e) = \lambda_*(e) = S \setminus \{s\}$ for some $s \in S$.

**Definition 2.17.** Let $S := \{s_1, \ldots, s_n\}$ be the generating set of simple reflections for the Coxeter group $W$. An element $v \in W$ is called linear if it is of the form $v := s_{i_1} \cdots s_{i_p}$, where $s_{i_1}, \ldots, s_{i_p}$ are all different from each other. A linear element is called a Coxeter element if $p = |S|$.

**Theorem 2.18.** [17, Theorem 6.2] Let $M$ be a $J$-coirreducible monoid of type $I$. Then the distinct irreducible components of $M_{\text{nil}}$ are $X(e_0x)$ where $x$ is a Coxeter element of $W$ in $D_I^{-1}$.
Finally, we come to the definitions of canonical monoids.

**Definition 2.19.** Let $M$ be a $J$-coirreducible monoid of type $I$. Then $M$ is called a dual canonical monoid if $I = \emptyset$. This means that $\lambda(e_{\text{max}}) = \emptyset$. In this case, we will denote $e_{\text{max}}$ by $e_{\emptyset}$. A canonical monoid is defined similarly; let $M$ be a $J$-irreducible monoid of type $I$. If $I = \emptyset$, then $M$ is called a dual canonical monoid.

**Remark 2.20.** Let $\Lambda$ be the cross-section lattice of a dual canonical monoid $M$ with Renner monoid $R$, and let $e$ be an element of $\Lambda \setminus \{1\}$. Then for every pair $(a, b) \in D(e) \times D(e)$, there exist precisely one element $x = x(a, b, e) \in R$ such that $x = aeb^{-1}$. In particular, this decomposition of $x$ is its standard form.

### 2.2 Double cosets

Let $(W, S)$ be a Coxeter system, let $I$ and $J$ be two subsets from $S$. For $w \in W$, we denote by $[w]$ the double coset $W_I w W_J$. Let

$$\pi : W \to W_I \backslash W / W_J$$

denote the canonical projection onto the set of $(W_I, W_J)$-double cosets. It turns out that the preimage in $W$ of every double coset in $W_I \backslash W / W_J$ is an interval with respect to Bruhat-Chevalley order, hence it has a unique maximal and a unique minimal element, see [6]. Moreover, if $[w], [w'] \in W_I \backslash W / W_J$ are two double cosets, $w_1$ and $w_2$ are the maximal elements of $[w]$ and $[w']$, respectively, then $w \leq w'$ if and only if $w_1 \leq w_2$, see [7]. Therefore, $W_I \backslash W / W_J$ has a natural combinatorial partial ordering defined by

$$[w] \leq [w'] \iff w \leq w' \iff w_1 \leq w_2$$

where $[w], [w'] \in W_I \backslash W / W_J$ and $w_1$ and $w_2$ are the maximal elements, $w_1 \in [w]$ and $w_2 \in [w']$.

Now let $[w]$ be a double coset from $W_I \backslash W / W_J$ represented by an element $w \in W$ such that $\ell(w) \leq \ell(v)$ for every $v \in [w]$. It turns out that the set of all such minimal length double coset representatives is given by $D_I^{-1} \cap D_J$, the intersection of the set of minimal length left coset representatives of $W_I$ in $W$ and the set of minimal length right coset representatives of $W_J$ in $W$. We will denote this intersection by $X_{I, J}$. Set $H := I \cap W J w^{-1}$. Then $uw \in D_J$ for $u \in W_I$ if and only if $u$ is a minimal length coset representative for $W_I / W_H$. In particular, every element of $W_I w W_J$ has a unique expression of the form $uvw$ with $u \in W_I$ is a minimal length coset representative of $W_I / W_H$, $v \in W_J$ and $\ell(uvw) = \ell(u) + \ell(w) + \ell(v)$.

Another characterization of the sets $X_{I, J}$ is as follows. For $w \in W$, the **right ascent set** is defined as

$$\text{Asc}_R(w) = \{ s \in S : \ell(ws) > \ell(w) \}.$$

The **right descent set**, $\text{Des}_R(w)$ is the complement $S \setminus \text{Asc}_R(w)$. Similarly, the **left ascent set** of $w$ is

$$\text{Asc}_L(w) = \{ s \in S : \ell(sw) > \ell(w) \} \quad (= \text{Asc}_R(w^{-1})).$$
Then
\[ X_{\mathcal{I},\mathcal{J}}^- = \{ w \in W : I \subseteq \text{Asc}_L(w) \text{ and } J \subseteq \text{Asc}_R(w) \} \] (2.21)
\[ = \{ w \in W : I^c \supseteq \text{Des}_R(w^{-1}) \text{ and } J^c \supseteq \text{Des}_R(w) \} \] (2.22)

Let us point out that, in general, the Bruhat-Chevalley order on \( X_{\mathcal{I},\mathcal{J}}^- \) is a nongraded poset. For some special choices of \( I \) and \( J \), in type A, we determined the corresponding posets explicitly, see [2, 3].

3 The Type Map of a Dual Canonical Monoid

Most of the results in this section are well-known to the experts. In fact, as observed by Therkelsen in [25], the proofs of many of these results follow by duality from the corresponding facts that hold true in the canonical monoid case. However, since they are important for our purposes, we provide direct proofs for completeness.

The Boolean lattice \( B_n \) is the poset of all subsets of an \( n \)-element set which is ordered with respect to the inclusions of subsets. The opposite-Boolean lattice is the opposite of the poset \( (B_n, \subseteq) \). We will denote it by \( B_{op}^n \). For \( A, B \in B_{op}^n \), we have \( A \leq B \iff A \supseteq B \). For simplifying our notation, we will denote the set \( \{1, \ldots, n\} \) by \( [n] \).

**Lemma 3.1.** Let \( P \) be a graded sublattice of \( B_{op}^n \) with \( \emptyset \in P \) and \( [n] \in P \). If for every element \( I \) in \( B_{op}^n \) there is a collection of elements \( A_1, \ldots, A_r \) in \( P \) such that \( \cap_{i=1}^r A_i = I \), then \( P = B_{op}^n \).

**Proof.** Clearly our claim is true for \( n = 1 \) as well as for \( n = 2 \). We will prove the general case by induction, so we assume that our lemma holds true for the opposite-Boolean poset \( B_{op}^{n-1} \).

Now, let \( P \) be a graded sublattice of \( B_{op}^n \) which satisfies the hypothesis of our lemma. Clearly, for every \( i \in [n] \), the set \( A_i := [n] \setminus \{i\} \) is an element of \( P \). These are precisely the atoms in \( P \). Note that if \( K \) is a subset in \( [n] \), then \( K = \cap_{i \in K} A_i \).

Let \( B(i) \) denote the opposite-Boolean sublattice in \( B_{op}^n \) which consists of all subsets containing the element \( i \). Then \( A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n \) are elements of \( B(i) \), and furthermore, any other element in \( B(i) \) can be written as their intersections. Therefore, by our induction hypothesis the sublattice generated by \( A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n \) is equal to \( B(i) \). This arguments is true for all \( i \in [n] \). Finally, we note that \( \{\emptyset\} \cup \bigcup_{i \in [n]} B(i) = B_{op}^n \). This finishes the proof.

The opposite-Boolean lattice of subsets of \( S \) will be denoted by \( B_{op}(S) \). Let \( \Lambda \) be the cross-section lattice of a dual canonical monoid \( M \). If \( I \in B_{op}(S) \) is such that \( \lambda(e) = I \), then sometimes we will write \( e_I \) to specify \( e \).

**Proposition 3.2.** Let \( M \) be a dual canonical monoid. Then \( \Lambda \setminus \{1\} \) is isomorphic to the opposite-Boolean lattice \( B_{op}(S) \).
Proof. The cross section lattice of $M$ contains 0 as an element. It corresponds to $e_S$. Indeed, by part 4 of Theorem 2.16, for $f \in \Lambda_{\text{min}}$, we have $\lambda(f) = S \setminus \{s\}$ for some $s \in S$. This implies that $\lambda(0) = S$.

Since $M$ is of type 0, by part 3 of Theorem 2.16, for any $K \subseteq S$ we have an idempotent $e \in \Lambda \setminus \{1\}$ such that $\lambda_*(e) = K$. We know that the type map $\lambda : \Lambda \to B^\text{op}(S)$ is 1-1 in our case, therefore, $\Lambda \setminus \{1\}$ isomorphic to its image under $\lambda$. Since for every $e \in \Lambda \setminus \{1\}$, we have $\lambda_*(e) = \cap_{f \leq e} \lambda(f)$, we see that $\Lambda \setminus \{1\}$ satisfies the hypothesis of Lemma 3.1. This finishes the proof. □

Corollary 3.3. Let $M$ be a dual canonical monoid. Then $\lambda_*(e) = \lambda(e)$ for all $e \in \Lambda \setminus \{1\}$.

Proof. Let $e$ be an idempotent in $\Lambda \setminus \{1\}$. It follows from Proposition 3.2 that if $f \in \Lambda \setminus \{1\}$ is such that $f \leq e$, then $\lambda(f) \supseteq \lambda(e)$. Therefore, $\lambda_*(e) = \cap_{f \leq e} \lambda(f) = \lambda(e)$.

For an idempotent $e$ in $\Lambda$, let us denote by $P(e)$ and $P(e)^-$ the subgroups

$$P(e) = \{g \in G : ge = ege\} \quad \text{and} \quad P(e)^- = \{g \in G : eg = ege\}.$$ 

Then $P(e)$ and $P(e)^-$ are opposite parabolic subgroups in $G$. The centralizer of $e$ in $G$ will be denoted by $C_G(e)$. In other words, we have $C_G(e) := \{g \in G : ge = eg\} = P(e) \cap P(e)^-$.

Theorem 3.4. Let $M$ be a dual canonical monoid, and let $e$ be an idempotent from the cross section lattice $\Lambda = \Lambda(B)$. Then the $G \times G$-orbit $GeG$ is a fiber bundle over $G/P(e) \times G/P(e)^-$ with fiber $eBe$ at the identity double coset $idP(e) \times idP(e)^-$.

Proof. The following fibre bundle structure on $GeG$ is observed in [4, Lemma 3.5 and 3.6]:

$$eC_G(e) \to GeG \to G/P(e) \times G/P(e)^-. \tag{3.5}$$

By Corollary 3.3, we know that $W(e) = W_*(e) = \{w \in W : we = ew = e\}$. We know from [13, Proposition 10.9 (i)] that the Weyl group of $C_G(e)$ is given by $W(e)$. Let $B_1$ denote the Borel subgroup of $C_G(e)$ such that $C_G(e) = B_1W(e)B_1$ (the Bruhat-Chevalley decomposition for $C_G(e)$). Then we see that

$$eC_G(e) = B_1eW(e)B_1 = B_1eW_*(e)B_1 = B_1eB_1 = eB_1.$$

But $eB_1 = eC_B(e) = eBe$ by [13, Corollary 7.2]. This finishes the proof. □

Corollary 3.6. If $e$ is the idempotent $e = e_0$ in $\Lambda$, then $GeG$ is a torus fiber bundle over $G/B \times G/B^-$. More precisely, we have

$$T_0 \to Ge_0G \to G/B \times G/B^-,$$

where $T_0$ is the maximal torus of the derived subgroup of the unit group $G$.

Proof. This follows from the fact that if $e = e_0$, then $P(e) = B$, $P(e)^- = B^-$, and $C_G(e) = T$. Finally, we note that $e_0T \cong T_0$ since $e_0$ is the maximal element of $\Lambda \setminus \{1\}$, and the height of $\Lambda \setminus \{1\}$ is equal to $\dim T_0$. □
4 The Rook Monoid As an Interval

As Putcha showed in [17, Theorem 4.4], if \( M \) is a semisimple monoid, then \( J \cap M_{nil} \neq \emptyset \) for every \( J \)-classes \( J \neq G \) of \( M \). The following result is recorded by Therkelsen in his PhD thesis [25, Theorem 5.2.2].

**Lemma 4.1.** Let \( M \) be a dual canonical monoid with \( e \in \Lambda \setminus \{1\} \). Then \( C(e) \) is isomorphic to the dual of \( W(e) \setminus W/W(e) \). That is,

\[
[e_y] \leq [e_x] \iff W(e)xW(e) \leq W(e)yW(e) \iff x \leq y,
\]

for \( x, y \in D^*(e) = D(e) \cap D(e)^{-1} \).

It is a natural (and important) question to ask for which idempotents \( e \in \Lambda \setminus \{1\} \) the double coset \( W(e) \setminus W/W(e) \) is graded. For \( e = e_\emptyset \) this is the case. In type A, our results in [2] shows that if \( e = e_{S \setminus \{s\}} \), then \( W(e) \setminus W/W(e) \) is a graded lattice. We anticipate this result will hold true in other types as well.

The *rook monoid* on the set \( \{1, \ldots, n\} \), denoted by \( R_n \), is the full inverse semigroup of injective partial transformations \( \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \). It is the Renner monoid of the reductive monoid of \( n \times n \) matrices. The unit group of \( R_n \) is the symmetric group \( S_n \). Let \( w \) be a permutation from \( S_n \). The *one-line notation* for \( w \) is a string of numbers \( w_1 \ldots w_n \), where \( w_i = w(i) \) for \( i \in \{1, \ldots, n\} \). In a similar manner, the *one-line notation* for \( \sigma \in R_n \) is a string of numbers \( \sigma_1 \ldots \sigma_n \), where, for \( i \in \{1, \ldots, n\} \), \( \sigma_i = \sigma(i) \) if \( \sigma(i) \) is defined; otherwise \( \sigma_i = 0 \).

For example, \( \sigma = 02501 \) is the injective partial transformation \( \sigma : \{2, 3, 5\} \rightarrow \{1, 2, 3, 4, 5\} \) with \( \sigma(2) = 2, \sigma(3) = 5, \) and \( \sigma(5) = 1 \).

Let \( \sigma = \sigma_1 \ldots \sigma_n \) and \( \tau = \tau_1 \ldots \tau_n \) be two elements from \( R_n \). We will write \( \tilde{\sigma}_i \) for the non-increasing rearrangement of the string \( \sigma_1 \sigma_2 \ldots \sigma_i \). For example, if \( \sigma = 02501 \), then \( \tilde{\sigma}_4 = 5200 \). If \( a := a_1 \ldots a_m \) and \( b := b_1 \ldots b_m \) are two strings of integers of the same length, then we will write \( a \leq_c b \) if \( a_i \leq b_i \) for all \( i \in \{1, \ldots, m\} \). The following characterization of the Bruhat-Chevalley-Renner order is proven in [5]:

\[
\tau \leq \sigma \iff \tilde{\tau}_i \leq_c \tilde{\sigma}_i \quad \text{for all } i \in \{1, \ldots, n\}.
\] (4.2)

Our next result describes a surprising connection between \( R_n \) and the Putcha monoid of the dual canonical monoid with unit group \( GL_n \).

**Theorem 4.3.** Let \( W \) denote the symmetric group \( W = S_{2m} \). If \( I \) denotes the subset \( \{s_1, \ldots, s_{m-1}\} \) in \( S = \{s_1, \ldots, s_{2m-1}\} \), then the opposite of the poset \( W_I \setminus W/W_I \), or equivalently, the Putcha subposet \( C(e_I) \) is isomorphic to the poset \( (R_m, \leq) \).

**Proof.** First, we will determine the elements of \( D^*(e_I) \). Let \( w = w_1 \ldots w_{2m} \) be an element from \( D^*(e_I) \). Notice that the set \( I \) indicates the positions of the descents in \( w \); if \( s_i \in I \), then \( w_i > w_{i+1} \). Since \( w^{-1} \) is also in \( D(e_I) \), we see that if \( w_i = 2m, w_{i+1} = 2m-1, \ldots, w_{im} = m+1 \), then \( i_1 < \cdots < i_m \). At the same time, \( w \) is of minimal possible length. These requirements imply that the intersection

\[
\{1, \ldots, m\} \cap \{i_1, \ldots, i_m\} = \{i_1, \ldots, i_k\}
\]
uniquely determines \( w \); we place \( 2m, \ldots, 2m - k + 1 \) at the positions \( i_1, \ldots, i_k \), and we place \( 2m - k + 2, \ldots, m + 1 \) at the positions \( m + 1, m + 2, \ldots, 2m - k \). The numbers \( i_1, \ldots, i_k \) are placed, in a decreasing order, at the positions \( 2m - k + 1, \ldots, 2m \). The remaining entries are filled in the increasing order with what remains of \( 1, 2, 3, \ldots, 2m \). But now such a permutation, \( w \in S_{2m} \) defines a unique partial permutation with its first \( m \) entries; we define \( \sigma = \sigma(w) \) by \( \sigma_i := w_i - i \) for \( i \in \{1, \ldots, m\} \). It is not difficult to show conversely that any \( \sigma \in R_m \) gives a permutation \( w = w(\sigma) \in D^*(e_1) \subset S_{2m} \). Furthermore, it is now clear from (4.2) that, for two elements \( \tau \) and \( \sigma \) from \( R_m \), \( \tau \leq \sigma \) if and only if \( w(\tau) \leq w(\sigma) \). This finishes the proof.

The proofs of the next two corollaries follow from the proof of Theorem 4.3.

**Corollary 4.4.** Let \( W \) denote the symmetric group \( W = S_{2m+1} \). If \( I \) denotes the subset \( \{s_1, \ldots, s_m\} \) in \( S = \{s_1, \ldots, s_{2m}\} \), then the opposite of the poset \( W_I \setminus W/W_I \), or equivalently, the Putcha subposet \( C(e_I) \) is isomorphic to the poset \( (R_m, \leq) \).

**Corollary 4.5.** Let \( W \) denote the symmetric group \( W = S_n \). Let \( k \) be a number such that \( \lfloor n/2 \rfloor \leq k \leq n - 1 \). If \( I \) is the subset \( \{s_1, \ldots, s_k\} \) in \( S = \{s_1, \ldots, s_{n-1}\} \), then the opposite of the poset \( W_I \setminus W/W_I \), or equivalently, the Putcha subposet \( C(e_I) \) is isomorphic to \( (R_{\lfloor n/2 \rfloor - k}, \leq) \).

![Figure 4.1: The Putcha subposet C(e_{s1,s2}) is isomorphic to the rook monoid (R_2, \leq).](image)

5 The Nilpotent Variety of a Dual Canonical Monoid

Let \( M \) be a dual canonical monoid, and let \( C \) denote the corresponding Putcha monoid. Let \( [ev] \) \( (v \in D(e)^{-1}) \) be an element from \( C \). By Theorem 2.15 we know that \( [ev] \in C_{nil} \) if and
only if $\text{supp}(v) \not\subseteq \lambda(f)$ for all $f \in \Lambda_{\text{min}}$ with $f \leq e$. Also, we know from the previous section that for such $f$, $\lambda(f) = S \setminus \{s\}$ for some $s \in S$, and $f \leq e$ if and only if $\lambda(e) \subseteq \lambda(f)$. Therefore, $\text{supp}(v)$ contains every $s$ that lies in the complement of the set $\lambda(e)$. In other words, we have

$$\text{supp}(v) \supseteq S \setminus \lambda(e). \quad (5.1)$$

As a consequence of this observation, we identify the maximal elements of the subposet $C_{\text{nil}}(e) \subseteq C(e)$ for $e \in \Lambda \setminus \{1\}$.

**Proposition 5.2.** Let $K$ be a subset of $S$. Then the set of maximal elements of the poset $C_{\text{nil}}(e_K)$ consists of linear elements of the form $s_{i_1}s_{i_2}\cdots s_{i_k}$, where $\{s_{i_1}, \ldots, s_{i_k}\} = S \setminus K$. In particular, $C_{\text{nil}}(e_K)$ has a unique maximal element if and only if $s_is_j = s_js_i$ for all $s_i, s_j$ in $S \setminus K$.

*Proof.* Let $[ex]$ and $[ey]$ be two elements from $C_{\text{nil}}(e_K)$. By Lemma 4.1, $[ex] \leq [ey]$ if and only if $y \leq x$. Therefore, by (5.1), the maximal elements of $C_{\text{nil}}(e_K)$ are of the form $[ey] = [e(s_{i_1}s_{i_2}\cdots s_{i_k})]$, where $\{i_1, \ldots, i_k\} = S \setminus \lambda(e_K)$. The second claim is obvious. $\Box$

**Corollary 5.3.** Let $e_K$ be a minimal nonzero idempotent from $\Lambda \setminus \{1\}$. Then $C_{\text{nil}}(e_K)$ has a unique maximal and a unique minimal element.

*Proof.* If $e_K$ is a minimal nonzero element in $\Lambda \setminus \{1\}$, then by Proposition 5.2 we know that $K = S \setminus \{s\}$ for some $s \in S$. Therefore, $S \setminus K = \{s\}$. In other words, $C_{\text{nil}}(e_K)$ has a unique maximal and a unique minimal element. $\Box$

**Remark 5.4.** In type $A$, for $K = S \setminus \{s\}$, the poset $C(e_K)$, hence $C_{\text{nil}}(e_K)$, is a chain. In fact, this fact holds true in some other types as well, see [9, Proposition 3.2] and [24, Theorem 2.3].

Let $e_I$ and $e_J$ be two different elements from $\Lambda \setminus \{1\}$. Comparisons between the elements belonging to $C(e_I)$ and $C(e_J)$ are described by another result of Therkelsen.

**Proposition 5.5.** The interval between $[e_0w_0]$ and $[e_S]$ ($e_S = 0$) in $C_{\text{nil}}$ is isomorphic to $B^K(S)$.

*Proof.* Let $I$ be a subset of $S$, and let $[e_Iy]$ be the minimal element of interval $C(e_I)$. Then $[e_Iy] \in C_{\text{nil}}$. Let $J$ be another subset of $S$. If $[e_Jz]$ is the minimal element of $C(e_J)$, then we will prove that

$$J \subseteq I \iff [e_Iy] \leq [e_Jz].$$

Note that $(\Leftarrow)$ direction is clearly true. To prove the other direction, we will prove the stronger statement that $e_Iy \leq e_Jz$ in the Bruhat-Chevalley-Renner order. By [16, Lemma 2.1 (i)] this will show that $[e_Iy] \leq [e_Jz]$ in $C$.

To prove the latter statement, first, we will show that

$$p_{e_Ie_0}(e_Iy) = e_0w_0. \quad (5.6)$$

13
By the last part of Theorem 2.16 and Corollary 3.3, we know that the upward projection maps are 1-1. Thus we will conclude that $[e_I y] \leq [e_I z]$ in $C$. Now we proceed to prove (5.6). But this can be seen directly from the description of the Bruhat-Chevalley-Renner order (2.7); we write $w_0$ in the form $w^{-1} y^{-1} = w_0$ for some $w^{-1} \in W(e_I)$. Then (2.7) shows that $[e_I y] \leq [e_\emptyset w_0]$, hence, it shows that (5.6). This finishes the proof.

Theorem 5.7. Let $M$ be a dual canonical monoid, and let $G$ denote its unit group. The nilpotent variety $M_{nil}$ of $M$ is an equidimensional variety. If $v \in W$ is a Coxeter element, then the dimension of the corresponding irreducible component is given by

$$\dim \overline{X(e_\emptyset v)} = \dim G_0 - |S|,$$

where $G_0$ is the derived subgroup of $G$.

Proof. The proof of the first claim follows immediately from the proof of the second claim, so we will prove the second one.

By [14, Theorem 2.2], for every subset $K \subset S$, we have a corresponding decomposition of the $J$-class $Ge_K G$ in the form

$$Ge_K G = \bigsqcup_{y \in D^*(e_K)} X(e y).$$

If $K = \emptyset$, then $D^*(e_K) = W$, and the Putcha order on $C(e_\emptyset)$ agrees with the opposite of the Bruhat-Chevalley order on $W$. In particular, the inclusion relations between the varieties $\overline{X(e_\emptyset y)}$ with $e y \in D^*(e_\emptyset)$ correspond to the inclusion relations between the $B \times B$-orbit closures $\overline{Be_\emptyset y B}$ that they contain. It follows from this fact that the dimension of $\overline{X(e_\emptyset y)}$ is given by the difference

$$\dim \overline{X(e_\emptyset y)} = \dim Ge_\emptyset G - \text{corank}_{C(e_\emptyset)}(e_\emptyset y).$$

If $v$ is a Coxeter element, its corank in $C(e_\emptyset) \cong W^{op}$ is $|S|$. Thus, the proof will be finished once we compute $\dim Ge_\emptyset G$. But since $e_\emptyset$ is the unique corank 1 element in $\Lambda$, we know that

$$M = G \sqcup Ge_\emptyset G.$$

The $G \times G$-orbit of 1 is $G \cong C^*G_0$, and the $G \times G$-orbit of $e_\emptyset$ is the dense orbit in $Ge_\emptyset G$. This is the unique $G \times G$-stable divisor in $M$. (This can be taken as the definition of a $J$-coirreducible monoid.) Therefore, $\dim Ge_\emptyset G = \dim M - 1 = \dim G_0$. This finishes the proof.

Corollary 5.8. Let $w_0$ denote the longest element in $W$. Then the dimension of $X(e_\emptyset w_0)$ is given by $\dim G_0 - \dim G/B = \dim U$.

Proof. It follows from the arguments of the proof of the previous theorem that the corank of $e_\emptyset w_0$ in $C(e_\emptyset)$ is equal to $\dim G/B$. Since $\dim Ge_\emptyset G = \dim G_0$, we see that $\dim X(e_\emptyset w_0) = \dim G_0 - \dim G/B$. 

14
6 A Richardson-Springer Monoid Action

Let $M$ be a dual canonical monoid, and let $M_{\text{nil}}$ denote its nilpotent variety. The irreducible components of $M_{\text{nil}}$ are indexed by the Coxeter elements of the Weyl group of the unit group of $M$. It is well-known that all Coxeter elements are conjugate to each other. However, they (Coxeter elements) do not necessarily form a single conjugacy class in a Weyl group. Therefore, the conjugation action of $W$ on the set of Coxeter elements does not give an additional structure on the Chow group of $M_{\text{nil}}$. The structure that we are looking for is given by a finite monoid that is canonically associated with $W$, which is first used by Richardson and Springer in [20] for studying the weak order on symmetric varieties.

Definition 6.1. Let $(W, S)$ be a Coxeter group. The Richardson-Springer monoid $O(W)$ of $W$ is the quotient of the free monoid generated by $S$ modulo the relations $s^2 = s$ for $s \in S$ and

\[ stst \cdots = tsts \cdots \]  \hspace{1cm} (6.2)

for $s, t \in S$, where both sides of (6.2) are the product of exactly order of $st$ many elements.

$O(W)$ is a finite monoid, and its elements are in canonical bijection with the elements of $W$. We write $m(w)$ for the element of $O(W)$ corresponding to $W$. If $w = s_1s_2\cdots s_l$ is any reduced expression of $w \in W$, then $m(w) = m(s_1)m(s_2)\cdots m(s_l)$. Furthermore, for $s \in S$ and $w \in W$, we have

\[ m(s)m(w) = \begin{cases} m(sw) & \text{if } \ell(sw) > \ell(w); \\ m(w) & \text{if } \ell(sw) < \ell(w). \end{cases} \]  \hspace{1cm} (6.3)

From now on, we write $w$ for $m(w)$ when discussing an element $w \in O(W)$.

There is a useful geometric interpretation of (6.3). Let $X$ be a $G$-variety, and let $B$ be a Borel subgroup in $G$. The set of all nonempty, irreducible, $B$-stable subvarieties of $X$ will be denoted by $B(B : X)$. For $w \in W$, let $X_w$ denote the Zariski closure of $BwB$ in $G$. Clearly, every closed irreducible $B \times B$-subvariety of $G$ is of this type. For $w, w' \in W$, we set $X_{w*w'} := X_wX_{w'}'$. It is not difficult to check that if $s \in S$, $w \in W$, then

\[ X_{sw} = X_{m(s)m(w)}, \]

and that $Xs \neq X_w$ if and only if $\ell(sw) = \ell(w) + 1$.

Next, we will introduce the Richardson-Springer monoid action on $B(B : X)$. For $Y \in B(B : X)$, we have a morphism defined by the action, $\pi : G \times Y \rightarrow X (g, z) \mapsto gz$. Let $w$ be an element from $O(W)$. The restriction of $\pi$ to $X_w \times Y$ is equivariant with respect to $B$-action that is given by $b \cdot (a, z) := (ab^{-1}, bz)$ for $b \in B$ and $(a, z) \in X_w \times Y$. Passing to the quotient, we get a new morphism

\[ \pi_{Y,w} : X_w \times^B Y \rightarrow \overline{X_wY}. \]

Following [8], let us denote $\overline{X_wY}$ by $w*Y$. Next definitions are due to Brion [1, Section 1]. Since $1 \in X_w$, we always have $Y \subseteq w*Y$. Note that it may happen that $Y = w*Y$ although
If the morphism $\pi_{Y,w}$ is generically finite, then we will denote the degree of $\pi_{Y,w}$ by $\deg(Y,w)$; if it is not generically finite, then we set $\deg(Y,w) := 0$. Finally, we define the W-set of $Y$, denoted $W(Y)$, as the set of $w$ from $O(W)$ such that $\pi_{Y,w}$ is generically finite and $BwY$ is $G$-invariant. The following facts are proven in [1, Lemma 1.1]

**Lemma 6.4.** Let $Y$ be a variety from $\mathcal{B}(B : X)$.

1. For any $\tau, w \in W$ such that $\ell(w\tau) = \ell(w) + \ell(\tau)$, we have
   \[ d(Y, \tau w) = d(Y, \tau)d(BwY, \tau). \]

2. For any $w \in W$ such that $BwY$ contains only finitely many $B$-orbits the integer $d(Y, w)$ is either 0 or a power of 2.

3. For any $w \in W$ such that $d(Y, w) \neq 0$, we have
   \[ W(BwY) = \{ \tau \in W : \ell(\tau w) = \ell(\tau) + \ell(w) \text{ and } \tau w \in W(Y) \}. \]

4. The set $W(Y)$ is nonempty.

5. Assume that $X = G/P$, where $P$ is a parabolic subgroup with $B \subset P$, and with a Levi subgroup $L$ such that $T \subset L$. If $Y = BwP/P$ with $\tau$ is a minimal length coset representative for $W_L$ in $W$, then $W(Y) = \{w_0w_{0,L}w^{-1} \}$, where $w_{0,L}$ denotes the longest element of $W_L$. Moreover, we have
   \[ d(Y, w_0w_{0,L}w^{-1}) = 1. \]

**Definition 6.5.** Let $Y_1$ and $Y_2$ be two elements from $\mathcal{B}(B : X)$. We will write

\[ Y_1 \leq Y_2 \text{ if } Y_2 = w \ast Y_1 \text{ for some } w \in O(W). \]

From now on, we will refer to the partial order that is defined by the transitive closure of the relations in (6.6) the weak order on $X$. If $Y_2 = B\mathbf{s}Y_1$ for some $s \in S$ and $Y_2 \neq Y_1$, then we will call the cardinality $|W(Y_2)|$, the degree of the covering relation $Y_1 < Y_2$. In this case, we will write $\deg(Y_1, Y_2)$ for $|W(Y_2)|$.

**Example 6.7.** Let $I$ be a subset of $S$, and let $P = BW_IB$ denote the corresponding parabolic subgroup in $G$. We set $X := G/P$, and let $Y$ be a Schubert variety in $X$ such that $Y = BwP/P$, where $w \in D_I$. For $s \in S$, either $\dim s \ast Y = \dim Y$ or $\dim s \ast Y = \dim Y + 1$. In the latter case, $\ell(sw) = \ell(w) + 1$, and we get a covering relation for the left weak order on $D_I$. In other words, the weak order on $X$ as defined in Definition 6.5 agrees with the well-known left weak order on $D_I$. Furthermore, Brion’s lemma shows that all covering relations in this case have degree 1.
Now we will apply this development in the setting of reductive monoids. By Bruhat-Chevalley-Renner order, we know that the set $\mathcal{B}(B \times B : M)$ is parametrized by the Renner monoid of $M$. Therefore, if we view $M$ as a $G \times G$-variety, then we have the “doubled” Richardson-Springer monoid action, $*: O(W \times W) \times \mathcal{B}(B \times B : M) \to \mathcal{B}(B \times B : M)$, which is defined as follows: Let $s \in S$ and $\sigma \in R$. Then

$$(s, 1) \ast \sigma = \begin{cases} 
  s\sigma & \text{if } \ell(s\sigma) > \ell(\sigma), \\
  \sigma & \text{if } \ell(s\sigma) \leq \ell(\sigma),
\end{cases} \quad (6.8)$$

and

$$(1, s) \ast \sigma = \begin{cases} 
  \sigma s & \text{if } \ell(\sigma s) > \ell(\sigma), \\
  \sigma & \text{if } \ell(\sigma s) \leq \ell(\sigma).
\end{cases} \quad (6.9)$$

The operation in (6.8) corresponds to $Y \rightsquigarrow BsBY$, where $Y = B\sigma B$, and the operation in (6.9) corresponds to $Y \rightsquigarrow YBsB$. We will denote the weak order on $M$ by $(R, \leq_{LR})$. (The notation will be explained in the sequel.)

Let $X$ be a $G$-variety, and let $Z$ be an element from $\mathcal{B}(B : X)$. If $Z \subseteq Y$, where $Y$ is a $G$-orbit closure in $X$, then $w * Z \subseteq Y$ for all $w \in O(W)$. Consequently, we see that the weak order on $X$ is a disjoint union of various weak order posets, one for each $G$-orbit. It is easy to see from our definitions that

$$(R, \leq_{LR}) = \bigsqcup_{e \in \Lambda} (WeW, \leq_{LR}).$$

Note that if $e$ is the neutral element of $G$, then we have $(WeW, \leq_{LR}) \cong (W, \leq_{LR})$. On the latter poset, the subscript $LR$ in the partial order stands for the two-sided weak order on the Coxeter group, so, our choice of notation is consistent with the notation in the literature.

As in the literature, we will denote the left (resp. right) weak order by $\leq_L$ (resp. $\leq_R$).

**Proposition 6.10.** Let $\Lambda$ be a cross-section lattice of a reductive monoid, and let $e$ be an element from $\Lambda \setminus \{1\}$. If $\lambda^*(e) = \emptyset$, then we have the following poset isomorphisms.

1. $(WeW, \leq) \cong (D(e), \leq) \times (D(e), \leq)^{op}$.
2. $(WeW, \leq_{LR}) \cong (D(e), \leq_L) \times (D(e), \leq_L)^{op}$.

Furthermore, $(WeW, \leq_{LR})$ is a lattice.

**Proof.** The proofs of the items (1) and (2) are similar, so, we will prove the latter only. If $\lambda^*(e) = \emptyset$, then by using the standard forms of elements in $WeW$, we see that $WeW = D(e)eD(e)^{-1}$. Let $\sigma = xey$ and $\sigma' = x'ey'$ be two elements from $D(e)eD(e)^{-1}$. Then $\sigma$ covers $\sigma'$ in $\leq_{LR}$ if and only if there exists $s \in S$ such that either $(s, 1) \ast \sigma' = \sigma$, or $(1, s) \ast \sigma' = \sigma$. In the former case, $x$ covers $x'$ in $\leq_L$ and $y = y'$; in the latter case $y'$ covers $y$ in $\leq_R$, hence $y'^{-1}$ covers $y^{-1}$ in $\leq_L$, and we have $x = x'$. This shows that the posets $(WeW, \leq_{LR})$ and $(D(e), \leq_L) \times (D(e), \leq_L)^{op}$ are canonically isomorphic. It is well known that the weak order on a quotient is a lattice. Since a product of two lattices is a lattice, the proof is finished. $\square$
Let $W$ be an irreducible Coxeter group, and let $I$ be a subset of the set of simple roots $S$ for $W$. The set $D_I$ ($\cong W/W_I$) is said to be \textit{minuscule} if the parabolic subgroup $W_I$ is the stabilizer of a “minuscule” weight. Here, a weight $\nu$ is said to be \textit{minuscule} if there is a representation of a semisimple linear algebraic group $G$ with Weyl group $W$ whose set of weights is the $W$-orbit of $\nu$.

The following result can be seen as an extension of [23, Theorem 7.1] into our setting.

**Corollary 6.11.** Let $e$ be an idempotent from a cross-section lattice of a reductive monoid $M$. We assume that $e$ is not the neutral element. If $\lambda^*(e) \notin \{\emptyset, S\}$ and $\lambda^*(e) = \emptyset$, then the following are equivalent.

1. $(W e W, \leq)$ is a lattice.
2. $(W e W, \leq)$ is a distributive lattice.
3. $(W e W, \leq_{LR})$ is a distributive lattice.
4. $(W e W, \leq_{LR}) = (W e W, \leq)$.
5. $D(e)$ is minuscule.

**Proof.** Let $A$ and $B$ be two posets. The product poset $A \times B$ is a distributive lattice if and only if both of $A$ and $B$ are distributive lattices. Also, $A$ is a distributive lattice if and only if its opposite $A^{op}$ is a distributive lattice. Now, by Proposition 6.10, $(W e W, \leq_{LR})$ is always a lattice, and $(D(e), \leq)$ is a lattice if and only if $(W e W, \leq)$ is a lattice. The rest of the proof follows from the proof of [23, Theorem 7.1].

Next, we discuss the degrees of the covering relations for $\leq_{LR}$. Clearly, $(s, 1) * 1 = s = (1, s') * 1$, therefore, the degree of the covering relation $1 < s$ in $(W, \leq_{LR})$ is always 2.

**Proposition 6.12.** Let $x, y$ be two elements from $W$. If $x$ is covered by $y$ in $(W, \leq_{LR})$, then the degree of the covering relation is either 1 or 2. In the latter case, there exists $s, s' \in S$ such that $y = (s, 1) * x = (1, s') * x$.

**Proof.** Clearly, if $(s, 1) * x = (s', 1) * x = y$ for some $s, s' \in S$, then $s = s'$. Similarly, if $(1, s) * x = (1, s') * x = y$ for some $s, s' \in S$, then $s = s'$. Therefore, if the degree of $x < y$ is at least 2, then we can only have $(s, 1) * x = (1, s') * x = y$ for some $s, s' \in S$. By the same argument, if they exist, then $s$ and $s'$ are unique. Therefore, the degree of a covering relation in $(W, \leq_{LR})$ is always $\leq 2$.

In Figure 6.1 we depicted $(S_4, \leq_{LR})$ together with its degree 2 covering relations.

**Theorem 6.13.** Let $\Lambda$ be a cross-section lattice of a reductive monoid, and let $e$ be an element from $\Lambda \setminus \{1\}$. Then $\lambda^*(e) \neq \emptyset$ if and only if there is a covering relation $x <_{LR} y$ in $W e W$ such that $\deg(x, y) = 2$. 


Figure 6.1: The two-sided weak order on $S_4$ and its double edges.
Proof. If $\lambda^*(e) \neq \emptyset$, then we know that $W^*(e) \neq \emptyset$, hence, there is a simple reflection $s$ in $W^*(e)$ such that $es = se \neq e$. But this means that $\deg(e, es) = 2$.

Conversely, let $x$ be an element in $WeW$. Let $aeb^{-1}$ be the standard form of $x$, where $a \in D_*(e)$ and $b \in D(e)$. By Proposition 6.12, if a covering relation $x <_{LR} y$ in $WeW$ has degree 2, then $(s, 1)*x = (1, s')*x = y$ for some $s, s' \in S$. By the uniqueness of the standard form for the elements of $R$, the equality $aeb^{-1} = aeb^{-1}s'$ implies that $s$ commutes with $a$ and $se = e$. Similarly, $s'$ commutes with $b^{-1}$ and $es' = e$. Since $R$ is a symmetric inverse semigroup, these equalities imply that $se = e = es$ and $es' = e = s'e$, hence $W^*(e) \neq \emptyset$. In other words, $\lambda^*(e) \neq \emptyset$. \hfill \Box

Corollary 6.14. If $M$ is a dual canonical monoid and $e$ is an idempotent from $\Lambda \setminus \{1\}$, then all covering relations in $(WeW, \leq_{LR}) = (D(e)eD(e)^{-1}, \leq_{LR})$ are of degree 1.

Proof. This follows from Theorem 6.13 and the fact that in a dual canonical monoid we have $\lambda^*(e) = \emptyset$ for all $e \in \Lambda \setminus \{1\}$, see part 3 of Theorem 2.16. \hfill \Box

We will denote the monoid of $n \times n$ matrices by $M_n$. The unit group of $M_n$ is given by $GL_n$. Let $B_n$ denote the Borel subgroup consisting of upper triangular matrices in $GL_n$. Then the corresponding cross-section lattice is the set of diagonal matrices that are given by

$e_i := \text{diag}(1, \ldots, 1, 0, \ldots, 0)$ with $i$ 1’s for $i = 0, \ldots, n$.

Proposition 6.15. Let $e_i$ be an element from the cross-section lattice $\Lambda \setminus \{0\}$ for $M_{n+1}$. Let $W$ denote $S_{n+1}$, the Weyl group of the unit group of $M_{n+1}$. Then $i = 1$ if and only if $\deg(x, y) = 1$ for all covering relations $x <_{LR} y$ in $WeW$. Furthermore, in this case, poset $(We_1W, \leq_{LR})$ is isomorphic to $(We_1W, \leq)$, where the latter partial order is the Bruhat-Chevalley-Renner ordering.

Proof. For the monoid $M_{n+1}$, it is easy to check that $\lambda^*(e_i) \neq \emptyset$ if and only if $i \in \{2, \ldots, n+1\}$. It is also easy to check that $\lambda_*(e_1) = \{s_2, \ldots, s_n\}$. Therefore, our first claim follows from Theorem 6.13, and our second claim follows from Corollary 6.11. \hfill \Box

Let $x = x_1 \ldots x_{n+1}$ be a permutation in one-line notation. A right ascent in $x$ is a string of two consecutive integers $\alpha := i i + 1$ such that $x_{i+1} > x_i$. A small (right) ascent in $x$ is a string of two consecutive integers $\alpha := i i + 1$ such that $x_{i+1} = x_i + 1$. A left ascent in $x$ is a pair of integers $\alpha := i j$ such that $1 \leq i < j \leq n + 1$ and $x_j = x_i + 1$.

Theorem 6.16. Let $W$ denote the symmetric group $S_{n+1}$. Then we have

(1) the total number of covering relations in $(W, \leq_{LR})$ is $n^2 n!$;

(2) the number of covering relations of degree 2 in $(W, \leq_{LR})$ is $nn!$.

Proof. We start with (2). Let $x <_{LR} y$ be a covering relation of degree 2 in $S_{n+1}$. Then there exist $s_i, s_j \in \{(12), (23), \ldots, (n n + 1)\}$ such that $s_i x = x s_j = y$. The left multiplication of $x$ by $s_i$ interchanges the values $x_i$ and $x_{i+1}$ in $x$, and the right multiplication of $x$ by
$s_j$ interchanges the occurrence of $j$ and $j + 1$ in $x$. Therefore, $x_i = j$ and $x_{i+1} = j + 1$.

Conversely, for each such consecutive pair $x_i, x_{i+1}$ in $x = x_1 \ldots x_{n+1}$ we obtain a covering relation of degree 2 by interchanging $x_i$ and $x_{i+1}$. Therefore, our count is equal to

$$c_{n+1} := \text{the total number of small ascents occurring in permutations in } S_{n+1}.$$ 

To find this number let us first fix a small ascent $\alpha = i \, i + 1$. Clearly, we choose the integer $i$ in $n$ different ways, and $\alpha$ can appear in any of the $n!$ permutations of the set $\{1, \ldots, i - 1, \alpha, i + 2, \ldots, n + 1\}$. In particular, we see that there are $n \cdot n!$ permutations where $\alpha$ can appear. This completes the proof of (2).

Next, we will prove (1). To this end, we will compute

$$a_{n+1} := \text{the total number of left ascents in } S_{n+1},$$

$$b_{n+1} := \text{the total number of right ascents in } S_{n+1}.$$

Then the total number of covering relations is given by $a_{n+1} + b_{n+1} - c_{n+1}$. To find $a_{n+1}$, first, choose two positions $i$ and $j$ in $x \in S_{n+1}$, and set $x_i := k$ and $x_j := k + 1$ for some $k \in \{1, \ldots, n\}$. Clearly, there are $(n+1)n$ possible choices. Then we choose the remaining entries of $x$ in $(n - 1)!$ ways. Therefore, the total number of left ascents in all permutations in $S_{n+1}$ is given by

$$a_{n+1} = \binom{n + 1}{2} n(n - 1)! = \frac{n}{2}(n + 1)!. $$

By a similar argument we find that

$$b_{n+1} = \frac{n}{2}(n + 1)!. $$

Therefore,

$$a_{n+1} + b_{n+1} - c_{n+1} = n(n + 1)! - nn! = n^2 n!, $$

hence, the proof of (1) is complete. \hfill \Box

7 Final Remarks

A graded poset $P$ with rank function $\rho : P \to \mathbb{N}$ is called Eulerian if the equality

$$|\{ z \in [x, y] : \rho(z) \text{ is even} \}| = |\{ z \in [x, y] : \rho(z) \text{ is odd} \}|$$

holds true for every closed interval $[x, y]$ in $P$. The order complexes of such posets enjoy remarkable duality properties.

Another topological property that we are interested in is the notion of “shellability” on the order complex of $P$. Let us assume that $P$ has a minimum and a maximum elements denoted by $\hat{0}$ and $\hat{1}$, respectively. We denote by $C(P)$ the set of pairs $(x, y)$ from $P \times P$ such that $y$ covers $x$. The poset $P$ is called lexicographically shellable, or EL-shellable, if there exists a map $f : C(P) \to [n]$ such that
(1) in every interval \([x, y] \subseteq P\), there exists a unique maximal chain
\[
A : x = x_0 < x_1 < \cdots < x_{k+1} = y
\]
such that \(f(x_i, x_{i+1}) \leq f(x_{i+1}, x_{i+2})\) for \(i = 0, \ldots, k-1\);
(2) the sequence \(f(A) := (f(x, x_1), \ldots, f(x_k, y))\) of the unique chain \(A\) of (1) is lexicographically first among all sequences of the form \(f(B)\), where \(B\) is a maximal chain in \([x, y]\).

If \(P\) is an EL-shellable poset, then the order complex of \(P\) is a Cohen-Macaulay complex.

It is well known that the left (resp. right) weak order on a Coxeter group is a graded poset. However, in general, left (resp. right) weak order is neither EL-shellable nor Eulerian. For example, consider the weak order on \(S_3\). It has two maximal chains, which we label from bottom to top by the sequences \(\alpha := (\alpha_1, \alpha_2, \alpha_3)\) and \(\beta := (\beta_1, \beta_2, \beta_3)\). If the \(\alpha\)-sequence is increasing, then the \(\beta\)-sequence cannot. But this implies that either \(\beta_1 > \beta_2 < \beta_3\), or \(\beta_1 < \beta_2 > \beta_3\). In any of these two possibilities we get a non EL-shellable subinterval in \((S_3, \leq_L)\).

Nevertheless, we have the following result whose proof will be written somewhere else.

**Theorem 7.1.** Let \(\Lambda\) be a cross-section lattice of a reductive monoid, and let \(e\) be an idempotent from \(\Lambda \setminus \{1\}\). Then \((\text{We} \Lambda, \leq_{LR})\) is an Eulerian, EL-shellable poset. Moreover, if \(W\) is an arbitrary finite Coxeter group, then the same statement holds true for \((W, \leq_{LR})\).

**Remark 7.2.** The order complex of \((W, \leq_{LR})\) is known to be shellable, see [11].

**References**

[1] Michel Brion. The behaviour at infinity of the Bruhat decomposition. *Comment. Math. Helv.*, 73(1):137–174, 1998.
[2] Mahir Bilen Can. The cross-section of a spherical double cone. *Adv. in Appl. Math.*, 101:215–231, 2018.

[3] Mahir Bilen Can and Tien Le. Diagonal orbits in a double flag variety of complexity one, type A. https://arxiv.org/abs/1810.06513, 2018.

[4] Mahir Bilen Can and Lex E. Renner. $H$-polynomials and rook polynomials. *Internat. J. Algebra Comput.*, 18(5):935–949, 2008.

[5] Mahir Bilen Can and Lex E. Renner. Bruhat-Chevalley order on the rook monoid. *Turkish J. Math.*, 36(4):499–519, 2012.

[6] C. W. Curtis. On Lusztig’s isomorphism theorem for Hecke algebras. *J. Algebra*, 92(2):348–365, 1985.

[7] C. Hohlweg and Skandera M. A note on Bruhat order and double coset representatives. https://arxiv.org/abs/math/0511611, 2005.

[8] Friedrich Knop. On the set of orbits for a Borel subgroup. *Comment. Math. Helv.*, 70(2):285–309, 1995.

[9] Peter Littelmann. On spherical double cones. *J. Algebra*, 166(1):142–157, 1994.

[10] E.A. Pennell, M.S. Putcha, and L.E. Renner. Analogue of the Bruhat-Chevalley order for reductive monoids. *J. Algebra*, 196(2):339–368, 1997.

[11] T. Kyle Petersen. A two-sided analogue of the Coxeter complex. *Electron. J. Combin.*, 25(4):Paper 4.64, 28, 2018.

[12] Mohan S. Putcha. Conjugacy classes in algebraic monoids. *Trans. Amer. Math. Soc.*, 303(2):529–540, 1987.

[13] Mohan S. Putcha. *Linear algebraic monoids*, volume 133 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[14] Mohan S. Putcha. Conjugacy classes and nilpotent variety of a reductive monoid. *Canad. J. Math.*, 50(4):829–844, 1998.

[15] Mohan S. Putcha. Bruhat-Chevalley order in reductive monoids. *J. Algebraic Combin.*, 20(1):34–53, 2004.

[16] Mohan S. Putcha. Conjugacy decomposition of reductive monoids. *Math. Z.*, 250(4):841–853, 2005.

[17] Mohan S. Putcha. Nilpotent variety of a reductive monoid. *J. Algebraic Combin.*, 27(3):275–292, 2008.
[18] L.E. Renner. Analogue of the Bruhat decomposition for algebraic monoids. *J. Algebra*, 101(2):303–338, 1986.

[19] L.E. Renner. *Linear algebraic monoids*, volume 134 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005. Invariant Theory and Algebraic Transformation Groups, V.

[20] R.W. Richardson and T.A. Springer. The Bruhat order on symmetric varieties. *Geom. Dedicata*, 35(1-3):389–436, 1990.

[21] Alvaro Rittatore. *Monôides algébriques et plongements des groupes*. PhD thesis, Institut Fourier, 1997.

[22] Alvaro Rittatore. Very flat reductive monoids. *Publ. Mat. Urug.*, 9:93–121 (2002), 2001.

[23] John R. Stembridge. On the fully commutative elements of Coxeter groups. *J. Algebraic Combin.*, 5(4):353–385, 1996.

[24] John R. Stembridge. Tight quotients and double quotients in the Bruhat order. *Electron. J. Combin.*, 11(2):Research Paper 14, 41, 2004/06.

[25] Ryan K. Therkelsen. *The conjugacy poset of a reductive monoid*. ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)–North Carolina State University.

[26] Ryan K. Therkelsen. Conjugacy decomposition of canonical and dual canonical monoids. In *Algebraic monoids, group embeddings, and algebraic combinatorics*, volume 71 of *Fields Inst. Commun.*, pages 189–208. Springer, New York, 2014.

[27] Èrnest B. Vinberg. The asymptotic semigroup of a semisimple Lie group. In *Semigroups in algebra, geometry and analysis (Oberwolfach, 1993)*, volume 20 of *De Gruyter Exp. Math.*, pages 293–310. de Gruyter, Berlin, 1995.

[28] Èrnest Borisovich Vinberg. On reductive algebraic semigroups. In *Lie groups and Lie algebras: E. B. Dynkin’s Seminar*, volume 169 of *Amer. Math. Soc. Transl. Ser. 2*, pages 145–182. Amer. Math. Soc., Providence, RI, 1995.