Ultra-analytic effect of Cauchy problem for a class of kinetic equations

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The smoothing effect of the Cauchy problem for a class of kinetic equations is studied. We firstly consider the spatially homogeneous nonlinear Landau equation with Maxwellian molecules and inhomogeneous linear Fokker–Planck equation to show the ultra-analytic effects of the Cauchy problem. Those smoothing effect results are optimal and similar to heat equation. In the second part, we study a model of spatially inhomogeneous linear Landau equation with Maxwellian molecules, and show the analytic effect of the Cauchy problem.

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1. Introduction

It is well known that the Cauchy problem of heat equation possesses the ultra-analytic effect phenomenon, namely, if $u(t,x)$ is the solution of the following Cauchy problem:

$$
\begin{align*}
\partial_t u - \Delta_x u &= 0, \quad x \in \mathbb{R}^d, \quad t > 0, \\
|u|_{t=0} &= u_0 \in L^2(\mathbb{R}^d).
\end{align*}
$$

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then under the uniqueness hypothesis, the solution \( u(t, \cdot) = e^{t\Delta_s}u_0 \) is an ultra-analytic function for any \( t > 0 \). We give now the definition of function spaces \( \mathcal{A}^s(\Omega) \) where \( \Omega \) is an open subset of \( \mathbb{R}^d \).

**Definition 1.1.** For \( 0 < s < +\infty \), we say that \( f \in \mathcal{A}^s(\Omega) \), if \( f \in C^\infty(\Omega) \), and there exist \( C > 0, N_0 > 0 \) such that

\[
\|\partial^\alpha f\|_{L^2(\Omega)} \leq C|\alpha|+1(\alpha t)^s, \quad \forall \alpha \in \mathbb{N}^d, \ |\alpha| \geq N_0.
\]

If the boundary of \( \Omega \) is smooth, by using Sobolev embedding theorem, we have the same type estimate with \( L^2 \) norm replaced by any \( L^p \) norm for \( 2 < p \leq +\infty \). On the whole space \( \Omega = \mathbb{R}^d \), it is also equivalent to

\[
\left(\varrho\beta_0 f\right) \in L^2(\mathbb{R}^d)
\]

for some \( c_0 > 0 \) and \( \beta_0 \in \mathbb{N}^d \), where \( \varrho c_0(\cdot) \) is the Fourier multiplier defined by

\[
e^{-t(\cdot)}\left(\varrho c_0(\xi)\right).
\]

If \( s = 1 \), it is usual analytic function. If \( s > 1 \), it is Gevrey class function. For \( 0 < s < 1 \), it is called ultra-analytic function. Notice that all polynomial functions are ultra-analytic for any \( s > 0 \).

It is obvious that if \( u_0 \in L^2(\mathbb{R}^d) \) then, for any \( t > 0 \) and any \( k \in \mathbb{N} \), we have \( u(t, \cdot) = e^{t(-\Delta_s)^k}u_0 \in \mathcal{A}^\frac{s}{k}(\mathbb{R}^d) \), namely, there exists \( C > 0 \) such that for any \( m \in \mathbb{N} \),

\[
\| \left( t^m \varrho 2^{km} \right) u(t, \cdot) \|_{L^2(\mathbb{R}^d)} \leq C^km \| \left( t(-\Delta_s)^k \right)^m u(t, \cdot) \|_{L^2(\mathbb{R}^d)}
\]

\[
\leq \| u_0 \|_{L^2(\mathbb{R}^d)} C^km! \leq C^{2km+1}((2km)!)^{\frac{1}{2}},
\]

where \( \varrho 2^{km} = \sum_{|\alpha|=2km, \alpha \in \mathbb{N}^d} \varrho_\alpha^k \). We say that the diffusion operators \( (-\Delta)^k \) possess the ultra-analytic effect property if \( k > 1/2 \), the analytic effect property if \( k = 1/2 \) and the Gevrey effect property if \( 0 < k < 1/2 \).

We study the Cauchy problem for spatially homogeneous Landau equation

\[
\begin{aligned}
f_t &= Q(f, f, f) = \nabla_v (\bar{a}(f) \cdot \nabla_v f - \bar{b}(f) f), \quad v \in \mathbb{R}^d, \quad t > 0, \\
f|_{t=0} &= f_0,
\end{aligned}
\]

(11)

where \( \bar{a}(f) = (\bar{a}_{ij}(f)) \) and \( \bar{b}(f) = (\bar{b}_1(f), \ldots, \bar{b}_d(f)) \) are defined as follows (convolution is w.r.t. the variable \( v \in \mathbb{R}^d \))

\[
\bar{a}_{ij}(f) = a_{ij} \ast f, \quad \bar{b}_j(f) = \sum_{i=1}^{d} (\partial_v a_{ij}) \ast f, \quad i, j = 1, \ldots, d,
\]

with

\[
a_{ij}(v) = \left( \delta_{ij} - \frac{v_i v_j}{|v|^2} \right) |v|^\gamma + 2, \quad \gamma \in [-3, 1].
\]

We consider hereafter only the Maxwellian molecule case which corresponds to \( \gamma = 0 \). We introduce also the notation, for \( l \in \mathbb{R}, L^l(\mathbb{R}^d) = \{ f; (1 + |v|^2)^l f \in L^p(\mathbb{R}^d) \} \) is the weighted function space.

We prove the following ultra-analytic effect results for the nonlinear Cauchy problem (11).
**Theorem 1.1.** Let \( f_0 \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \) and \( 0 < T \leq +\infty \). If \( f(t, x) > 0 \) and \( f \in L^\infty([0, T]; L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)) \) is a weak solution of the Cauchy problem (1.1), then for any \( 0 < t < T \), we have

\[
f(t, \cdot) \in A^{1/2}(\mathbb{R}^d),
\]

and moreover, for any \( 0 < T_0 < T \), there exists \( c_0 > 0 \) such that for any \( 0 < t \leq T_0 \)

\[
\left\| e^{-\alpha t\Delta} f(t, \cdot) \right\|_{L^2(\mathbb{R}^d)} \leq e^\frac{4t}{\alpha} \| f_0 \|_{L^2(\mathbb{R}^d)}. \tag{1.2}
\]

In [17], they proved the Gevrey regularity effect of the Cauchy problem for linear spatially homogeneous non-cut-off Boltzmann equation. By a careful revision for the proof of Theorem 1.2 of [17], one can also prove that the solution of the Cauchy problem (1.10) in [17] belongs to \( A^{\frac{3}{5}}(\mathbb{R}^d) \) for any \( t > 0 \), where \( 0 < \alpha < 1 \) is the order of singularity of collision kernel of Boltzmann operator. Hence, if \( \alpha \geq 1/2 \), there is also the ultra-analytic effect phenomenon. Now the above Theorem 1.1 shows that, for Landau equation, the ultra-analytic effect phenomenon holds in nonlinear case, which is an optimal regularity result.

The ultra-analytic effect property is also true for the Cauchy problem of the following generalized Kolmogorov operators

\[
\begin{aligned}
\begin{cases}
\partial_t u + v \cdot \nabla_x u + (-\Delta v)^{\alpha} u = 0, \\
u |_{t=0} = u_0 \in L^2(\mathbb{R}^{2d}),
\end{cases}
\end{aligned}
\]

where \( 0 < \alpha < \infty \), and the classical Kolmogorov operators is corresponding to \( \alpha = 1 \). By Fourier transformation, the explicit solution of the above Cauchy problem is given by

\[
\hat{u}(t, \eta, \xi) = e^{-\int_0^t \frac{\xi}{|s+\eta|^{2\alpha}} ds} \hat{u}_0(\eta, \xi + t\eta).
\]

Since there exists \( c_\alpha > 0 \) (see Lemma 3.1 below) such that

\[
c_\alpha \left( t |\xi|^{2\alpha} + t^{2\alpha+1} |\eta|^{2\alpha} \right) \leq \int_0^t |\xi + s\eta|^{2\alpha} ds, \tag{1.3}
\]

we have

\[
\left\| e^{c_\alpha (t(-\Delta v)^{\alpha} + t^{2\alpha+1}(-\Delta v)^{\alpha})} u(t, \cdot, \cdot) \right\|_{L^2(\mathbb{R}^{2d})},
\]

i.e. \( u(t, \cdot, \cdot) \in A^{1/(2\alpha)}(\mathbb{R}^{2d}) \) for any \( t > 0 \).

Notice that this ultra-analytic (if \( \alpha > 1/2 \)) effect phenomenon is similar to heat equations of \( (x, v) \) variables. That is, this means \( v \cdot \nabla_x + (-\Delta v)^{\alpha} \) is equivalent to \( (-\Delta v)^{\alpha} + (-\Delta v)^{\alpha} \) by time evolution in “some sense”, though the equation is only transport for \( x \) variable.

We consider now a more complicate equation, the Cauchy problem for linear Fokker–Planck equation:

\[
\begin{aligned}
\begin{cases}
f_t + v \cdot \nabla_x f = \nabla_v \cdot (\nabla_v f + vf), \\
f |_{t=0} = f_0.
\end{cases}
\end{aligned} \tag{1.4}
\]

This equation is a natural generalization of classical Kolmogorov equation, and a simplified model of inhomogeneous Landau equation (see [20,21]). The local property of this equation is the same as classical Kolmogorov equation since the add terms \( \nabla_v \cdot (vf) \) is a first order term, but for the studies of
The problem is now much more complicated since the solution

difficulty for analysis of this equation.

Remark 1.1. and Villani [10], between the nonlinear Landau equation (with Maxwellian molecules) and the linear
kinetic equations (see [11,14]), and also the relationship, established by Villani [19] and Desvillettes
properties of solutions. From these results, we obtain a good understanding for the hypoellipticity of
linear Landau collision operators

We consider it here only in the linearized framework around the normalized Maxwellian distribution

The existence of weak solution is similar to full Landau equation (see [1,13]). We get also the following
ultra-analytic effect result.

Theorem 1.2. Let \( f_0 \in L^2(\mathbb{R}_x^d) \cap L_1(\mathbb{R}_x^d) \), \( 0 < T \leq +\infty \). Assume that \( f \in L^\infty(\mathbb{R}_x^d) \cap L^2(\mathbb{R}_x^d) \cap L_1(\mathbb{R}_x^d) \) is a weak solution of the Cauchy problem (1.4). Then, for any \( 0 < t < T \), we have

\[
0 < T_0 < T \quad \text{there exists } c_0 > 0 \quad \text{such that for any } 0 < t < T_0, \quad \text{we have}
\]

\[
\| e^{c_0(t_0 + t^3)} f(t, \cdot, \cdot) \|_{L^2(\mathbb{R}_x^d)} \leq e^{\frac{t}{2}} \| f_0 \|_{L^2(\mathbb{R}_x^d)}.
\]

(1.5)

Remark 1.1. The ultra-analyticity results of the above two theorems are optimal for the smoothness
properties of solutions. From these results, we obtain a good understanding for the hypoellipticity of
kinetic equations (see [11,14]), and also the relationship, established by Villani [19] and Desvillettes
and Villani [10], between the nonlinear Landau equation (with Maxwellian molecules) and the linear
Fokker–Planck equation.

We consider now the spatially inhomogeneous Landau equation

\[
\begin{cases}
 ft + v \cdot \nabla_x f = Q(f, f), & (x, v) \in \mathbb{R}_x^d, \quad t > 0, \\
 f|_{t=0} = f_0(x, v).
\end{cases}
\]

(1.6)

The problem is now much more complicated since the solution \( f \) is the function of \((t, x, v)\) variables. We consider it here only in the linearized framework around the normalized Maxwellian distribution

\[
\mu(v) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|v|^2}{2}},
\]

which is the equilibrium state because \( Q(\mu, \mu) = 0 \). Setting \( f = \mu + g \), we consider the diffusion part
of linear Landau collision operators

\[
Q(\mu, g) = \nabla_v (\bar{a}(\mu) \cdot \nabla_v g - \bar{b}(\mu) g),
\]

where

\[
\bar{a}_{ij}(\mu) = a_{ij} \ast \mu = \delta_{ij}(|v|^2 + 1) - v_i v_j,
\]

\[
\bar{b}_j(\mu) = \sum_{i=1}^d (\partial_{v_i} a_{ij}) \ast \mu = -v_j, \quad i, j = 1, \ldots, d.
\]

In particular, it follows that

\[
\sum_{ij=1}^d \bar{a}_{ij}(\mu) \xi_i \xi_j \geq |\xi|^2, \quad \text{for all } (v, \xi) \in \mathbb{R}_x^d.
\]

(1.7)
We then consider the following Cauchy problem

$$
\begin{align*}
g_t + v \cdot \nabla_x g &= \nabla_v (\tilde{a}(\mu) \cdot \nabla_v g - \tilde{b}(\mu) g), \quad (x, v) \in \mathbb{R}^{2d}, \ t > 0, \\
g|_{t=0} &= g_0.
\end{align*}
$$

We can also look this equation as a linear model of spatially inhomogeneous Landau equation, which is much more complicate than linear Fokker–Planck equation (1.4), since the coefficients of diffusion part are now variables. The existence and $C^\infty$ regularity of weak solution for the Cauchy problem have been considered in [1]. We prove now the following:

**Theorem 1.3.** Let $g_0 \in L^2(\mathbb{R}^{2d}) \cap L^1(\mathbb{R}^{2d}), \ 0 < T < +\infty$. Assume that $g \in L^\infty([0, T]; L^2(\mathbb{R}^{2d}) \cap L^1(\mathbb{R}^{2d}))$ is a weak solution of the Cauchy problem (1.8). Then, for any $0 < t < T$, we have

$$
g(t, \cdot, \cdot) \in A^1(\mathbb{R}^{2d}).
$$

Furthermore, for any $0 < T_0 < T$ there exist $C, c > 0$ such that for any $0 < t \leq T_0$, we have

$$
\|e^{ct(-\Delta_x)^{1/2} + t^2(-\Delta_x)^{1/2}} g(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{2d})} \leq e^{ct} \|g_0\|_{L^2(\mathbb{R}^{2d})}.
$$

In this theorem, we only consider the analytic effect result for the Cauchy problem (1.8), neglecting the symmetric term $Q(g, \mu)$ in the linearized operators of Landau collision operator (cf. (1.15) of [1]) because of the technical difficulty, see the remark in the end of Section 4.

There have been many results about the regularity of solutions for Boltzmann equation without angular cut-off and Landau equation, see [1–3,6,7,9,12,15,16] for the $C^\infty$ smoothness results, and [4, 5,8,17,18] for Gevrey regularity results for Boltzmann equation and Landau equation in both cases: the spatially homogeneous and inhomogeneous cases. As for the analytic and Gevrey regularities, we remark that the propagation of Gevrey regularities of solutions is investigated in [5] for full nonlinear spatially homogeneous Landau equations, including non-Maxwellian molecule case, and the local Gevrey regularity for all variables $t, x, v$ is considered in [4] for some semi-linear Fokker–Planck equations. Comparing those results, the ultra-analyticity for $x, v$ variables showed in Theorem 1.1 is strong although the Maxwellian molecule case is only treated. As a related result for spatially homogeneous Boltzmann equation in the Maxwellian molecule case, we refer [8], where the propagation of Gevrey and ultra-analytic regularity is studied uniformly in time variable $t$. Throughout the present paper, we focus the smoothing effect of the Cauchy problem, and the uniform smoothness estimate near to $t = 0$. Concerning further details of the analytic and Gevrey regularities of solutions for Landau equations and Boltzmann equation without angular cut-off, we refer the introduction of [5] and references therein.

2. Spatially homogeneous Landau equations

We consider the Cauchy problem (1.1) and prove Theorem 1.1 in this section. We refer to the works of C. Villani [19,20] for the essential properties of homogeneous Landau equations. We suppose the existence of weak solution $f(t, v) > 0$ in $L^\infty([0, T]; L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))$. The conservation of mass, momentum and energy reads

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(t, v) \left( \frac{1}{|v|} v \right) dv \equiv 0.$$

Without loss of generality, we can suppose that
\[ \int_{\mathbb{R}^d} f(t, v) \, dv = 1, \quad \text{unit mass}, \]
\[ \int_{\mathbb{R}^d} f(t, v)v_j \, dv = 0, \quad j = 1, \ldots, d, \quad \text{zero mean velocity}, \]
\[ \int_{\mathbb{R}^d} f(t, v)|v|^2 \, dv = T_0, \quad \text{unit temperature}, \]
\[ \int_{\mathbb{R}^d} f(t, v)v_j v_k \, dv = T_j \delta_{jk}, \quad \sum_j T_j = T_0, \]
\[ T_j = \int_{\mathbb{R}^d} f(t, v)v_j^2 \, dv > 0, \quad j = 1, \ldots, d, \quad \text{directional temperatures}. \]

Then we have
\[ \tilde{a}_{jk}(f) = \delta_{jk}(|v|^2 + T_0 - T_j) - v_j v_k, \quad (2.1) \]
\[ \tilde{b}_j(f) = -v_j, \quad (2.2) \]
\[ \sum_{j,k} \tilde{a}_{jk}(f)\xi_j \xi_k \geq C_1 |\xi|^2, \quad \forall (\nu, \xi) \in \mathbb{R}^{2d}, \quad (2.3) \]

where \( C_1 = \min_{1 \leq j \leq d} (T_0 - T_j) > 0. \)

Now for \( N > \frac{d}{4} + 1 \) and \( 0 < \delta < 1/N, \) \( c_0 > 0, \) \( t > 0, \) set
\[ G_\delta(t, |\xi|) = \frac{e^{c_0 |\xi|^2}}{(1 + \delta e^{c_0 |\xi|^2})(1 + \delta c_0 |\xi|^2)^N}. \]

Since \( G_\delta(t, \cdot) \in L^\infty(\mathbb{R}^d) \), we can use it as Fourier multiplier, denoted by
\[ G_\delta(t, Dv) f(t, v) = \mathcal{F}^{-1}(G_\delta(t, |\xi|) \hat{f}(t, \xi)). \]

Then, for any \( t > 0, \)
\[ G_\delta(t) = G_\delta(t, Dv) : L^2(\mathbb{R}^d) \rightarrow H^{2N}(\mathbb{R}^d) \subset C^2(\mathbb{R}^d). \]

The object of this section is to prove the uniform bound (with respect to \( \delta > 0 \)) of
\[ \| G_\delta(t, Dv) f(t, \cdot) \|_{L^2(\mathbb{R}^d)}. \]

Since \( f(t, \cdot) \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \) is a weak solution, we can take
\[ G_\delta(t)^2 f(t, \cdot) = G_\delta(t, Dv)^2 f(t, \cdot) \in H^{2N}(\mathbb{R}^d), \]
as test function in the equation of (1.1), whence we have
\[
\frac{1}{2} \frac{d}{dt} \|G_\delta(t) f(t, \cdot)\|_{L^2(\mathbb{R}^d)}^2 + \sum_{j, k = 1}^d \int \tilde{a}_{jk}(f) (\partial_{v_j} G_\delta(t) f(t, v)) (\partial_{v_k} G_\delta(t) f(t, v)) dv \\
= \frac{1}{2} \| (\partial_t G_\delta(t)) f, G_\delta(t) f \|_{L^2(\mathbb{R}^d)}^2 + \sum_{j=1}^d \int (\partial_{v_j} (v_j f(t, v))) \overline{G_\delta(t)^2 f(t, v)} dv \\
+ \sum_{j, k = 1}^d \int (\tilde{a}_{jk}(f) (G_\delta(t) \partial_{v_j} f(t, v)) - G_\delta(t) (\tilde{a}_{jk}(f) \partial_{v_j} f(t, v))) (\partial_{v_k} G_\delta(t) f(t, v)) dv.
\]

To estimate the terms in the above equality, we prove the following two propositions.

**Proposition 2.1.** We have

\[
C_1 \| \nabla_v G_\delta(t) f(t) \|_{L^2(\mathbb{R}^d)}^2 \leq \sum_{j, k = 1}^d \int \tilde{a}_{jk}(f) (\partial_{v_j} G_\delta(t, D_v) f(t, v)) (\partial_{v_k} G_\delta(t, D_v) f(t, v)) dv, \quad (2.4)
\]

\[
|((\partial_t G_\delta(t)) f, G_\delta(t) f)|_{L^2} \leq c_0 \| \nabla_v G_\delta(t) f(t) \|_{L^2}^2, \quad (2.5)
\]

\[
\text{Re} \sum_{j=1}^d \int (\partial_{v_j} (v_j f(t, v))) \overline{G_\delta(t)^2 f(t, v)} dv \leq \frac{d}{2} \| G_\delta(t) f(t) \|_{L^2}^2 + 2c_0 t \| \nabla_v G_\delta(t) f(t) \|_{L^2}^2. \quad (2.6)
\]

**Proof.** The estimate (2.4) is exactly the elliptic condition (2.3). By using the Fourier transformation, (2.5) is deduced from the following calculus

\[
(\partial_t G_\delta(t, |\xi|)) = c_0 |\xi|^2 G_\delta(t, |\xi|) \left( \frac{1}{1 + \delta e^{\xi t|^\xi^2|}} - \frac{N\delta}{1 + \delta c_0 t |\xi|^2} \right) = c_0 |\xi|^2 G_\delta(t, |\xi|) J_{N, \delta},
\]

where

\[
|J_{N, \delta}| = \left| \frac{1}{1 + \delta e^{\xi t|^\xi^2|}} - \frac{N\delta}{1 + \delta c_0 t |\xi|^2} \right| \leq 1.
\]

To treat (2.6), we use

\[
(\partial_{\xi_j} G_\delta(t, |\xi|)) = 2c_0 t \xi_j G_\delta(t, |\xi|) J_{N, \delta}. \quad (2.7)
\]

Then, we have

\[
\text{Re} \sum_{j=1}^d \int (\partial_{v_j} (v_j f(t, v))) \overline{G_\delta(t, D_v)^2 f(t, v)} dv = -\text{Re} \sum_{j=1}^d \int v_j G_\delta(t, D_v) f(t, v) (\overline{\partial_{v_j} G_\delta(t, D_v) f(t, v)}) dv \\
- \text{Re} \sum_{j=1}^d \int \left[ (G_\delta(t, D_v), v_j] f(t, v) \right) (\overline{\partial_{v_j} G_\delta(t, D_v) f(t, v)}) dv.
\]
By using Fourier transformation, we have
\[ \frac{d}{2} \left\| G_{\delta}(t) f(t, \cdot) \right\|_{L^2(\mathbb{R}^d)}^2 - \text{Re} \sum_{j=1}^{d} \int \left( [G_{\delta}(t, D_{v_j}) f(t, v_j)] \partial_{v_j} G_{\delta}(t, D_{v_j}) f(t, v_j) \right) dv. \]

Using Fourier transformation and (2.7), we have that for \( t > 0 \),
\begin{align*}
& \sum_{j=1}^{d} \int \left( [G_{\delta}(t, D_{v_j}) f(t, v_j)] \partial_{v_j} G_{\delta}(t, D_{v_j}) f(t, v_j) \right) dv \\
& = - \sum_{j=1}^{d} \int \left( G_{\delta}(t, D_{v_j}) v_j f(t, v) - v_j G_{\delta}(t, D_{v_j}) f(t, v) \right) \overline{\partial_{v_j} G_{\delta}(t, D_{v_j}) f(t, v)} dv \\
& = \sum_{j=1}^{d} \int \left( i \partial_{\xi_j} \left( G_{\delta}(t, |\xi|) \hat{f}(t, \xi) \right) - G_{\delta}(t, |\xi|) \left( i \partial_{\xi_j} \hat{f}(t, \xi) \right) \right) \overline{G_{\delta}(t, |\xi|) t \xi_j \hat{f}(t, \xi)} d\xi \\
& = - \sum_{j=1}^{d} \int \left( \partial_{\xi_j} G_{\delta}(t, |\xi|) \right) \hat{f}(t, \xi) \xi_j G_{\delta}(t, |\xi|) \overline{\hat{f}(t, \xi)} d\xi \\
& = 2c_0 t \int_{\mathbb{R}^d} |\xi|^2 |G_{\delta}(t, |\xi|) \hat{f}(t, \xi)|^2 J_{N,\delta} d\xi 
\end{align*}

which give (2.6). The proof of Proposition 2.1 is now complete. \( \square \)

For the commutator term, the special structure of the operator implies

**Proposition 2.2.**
\[ \sum_{j,k=1}^{d} \int \left( \bar{a}_{jk}(f) \left( G_{\delta}(t, D_{v_j}) \partial_{v_j} f(t, v) \right) - G_{\delta}(t, D_{v_j}) (\bar{a}_{jk}(f) \partial_{v_j} f(t, v)) \right) \overline{\left( \partial_{v_k} G_{\delta}(t, D_{v_j}) f(t, v) \right)} dv = 0. \]

**Proof.** We introduce now polar coordinates on \( \mathbb{R}^d_\xi \) by setting \( r = |\xi| \) and \( \omega = \xi / |\xi| \in S^{d-1} \). Note that \( \partial / \partial \xi_j = \omega_j \partial / \partial r + r^{-1} \Omega_j \) where \( \Omega_j \) is a vector field on \( S^{d-1} \), and (see [14, Proposition 14.7.1])
\[ \sum_{j=1}^{d} \omega_j \Omega_j = 0, \quad \sum_{j=1}^{d} \Omega_j \omega_j = d - 1. \tag{2.8} \]

By using Fourier transformation, we have
\begin{align*}
- \sum_{j,k=1}^{d} \int & \left( \bar{a}_{jk}(f) \left( G_{\delta}(t, D_{v_j}) \partial_{v_j} f(t, v) \right) - G_{\delta}(t, D_{v_j}) (\bar{a}_{jk}(f) \partial_{v_j} f(t, v)) \right) \overline{\left( \partial_{v_k} G_{\delta}(t, D_{v_j}) f(t, v) \right)} dv \\
& = \int_{\mathbb{R}^d} \left\{ \sum_{j,k=1}^{d} \xi_k \left[ (\delta_{jk} \Delta_{\xi} - \bar{a}_{\xi_k} \partial_{\xi_j}), G_{\delta}(t, |\xi|) \right] \hat{f}(t, \xi) \right\} \times G_{\delta}(t, |\xi|) \overline{\hat{f}(t, \xi)} d\xi.
\end{align*}
Noting, in polar coordinates on \( \mathbb{R}^d_\xi \),

\[
\Delta_\xi = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \sum_{j=1}^d \Omega_j^2,
\]

we have, denoting by \( \hat{\mathcal{C}}(r^2) = G_\delta(t, r) \),

\[
\sum_{j,k=1}^d \omega_k \left[ \left( \Omega_{jk} \left\{ \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r}, \hat{\mathcal{C}}(r^2) \right\} - \left\{ \left( \sum_{k=1}^d (\omega_k \partial/\partial r + r^{-1} \Omega_k) \right) \sum_{j=1}^d (\omega_j \partial/\partial r + r^{-1} \Omega_j \omega_j) \right\} \right) \right] \hat{\mathcal{C}}(r^2) \omega_j
\]

\[
= \left[ \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r}, \hat{\mathcal{C}}(r^2) \right] - \left[ \left( \sum_{k=1}^d (\omega_k \partial/\partial r + r^{-1} \Omega_k) \right) \sum_{j=1}^d (\omega_j \partial/\partial r + r^{-1} \Omega_j \omega_j) \right] \hat{\mathcal{C}}(r^2) = 0.
\]

where we have used (2.8). Then we finish the proof of Proposition 2.2. \( \square \)

**Remark 2.1.** In the above proof of Proposition 2.2, we have used the polar coordinates in the dual variable of \( v \), which is essentially related to a form of the Landau operator with Maxwellian molecules. We notice that the same relation (in \( v \) variable) was described by Villani [19] and Desvillettes and Villani [10].

**End of proof of Theorem 1.1.** From Propositions 2.1 and 2.2, we get

\[
\frac{1}{2} \frac{d}{dt} \left\| G_\delta(t) f(t, \cdot) \right\|_{L^2(\mathbb{R}^d)}^2 + \left( C_1 - \frac{1}{2} c_0 - 2c_0 t \right) \left\| \nabla_v G_\delta(t) f(t, \cdot) \right\|_{L^2(\mathbb{R}^d)}^2 
\]

\[
\leq \frac{d}{2} \left\| G_\delta(t) f(t, \cdot) \right\|_{L^2(\mathbb{R}^d)}^2.
\]

For any \( 0 < T_0 < T \), choose \( c_0 \) small enough such that \( C_1 - \frac{1}{2} c_0 - 2c_0 T_0 \geq 0 \). Then we get

\[
\frac{d}{dt} \left\| G_\delta(t) f(t, \cdot) \right\|_{L^2(\mathbb{R}^d)} \leq \frac{d}{2} \left\| G_\delta(t) f(t, \cdot) \right\|_{L^2(\mathbb{R}^d)}.
\]  

(2.9)

Integrating the inequality (2.9) on \( [0, t] \), we obtain

\[
\left\| G_\delta(t) f(t, \cdot) \right\|_{L^2(\mathbb{R}^d)} \leq e^{\frac{d}{2} t} \left\| f_0 \right\|_{L^2(\mathbb{R}^d)}.
\]  

(2.10)

Take limit \( \delta \to 0 \) in (2.10). Then we get

\[
\left\| e^{-c_0 t \Delta} f(t, \cdot) \right\|_{L^2(\mathbb{R}^d)} \leq e^{\frac{d}{2} t} \left\| f_0 \right\|_{L^2(\mathbb{R}^d)}
\]

(2.11)

for any \( 0 < t \leq T_0 \). We have now proved \( f(t, \cdot) \in \mathcal{A}^{1/2}(\mathbb{R}^d) \) and Theorem 1.1. \( \square \)
3. Linear Fokker–Planck equations

In the paper [19], there is an exact solution for spatially homogeneous linear Fokker–Planck equation. In the inhomogeneous case we can also obtain an exact solution of the Cauchy problem (1.4). Denote by

$$\hat{f}(t, \eta, \xi) = \mathcal{F}_{x,v}(f(t, x, v))$$

the partial Fourier transformation of $f$ with respect to $(x, v)$ variable. Then, by Fourier transformation for $(x, v)$ variables, the linear Fokker–Planck equation (1.4) becomes

$$\begin{cases}
\frac{\partial}{\partial t} \hat{f}(t, \eta, \xi) - \eta \cdot \nabla_\xi \hat{f}(t, \eta, \xi) + \xi \cdot \nabla_\xi \hat{f}(t, \eta, \xi) = -|\xi|^2 \hat{f}(t, \eta, \xi), \\
\hat{f}|_{t=0} = \mathcal{F}(f_0)(\eta, \xi).
\end{cases}$$

Therefore we obtain the exact solution

$$\hat{f}(t, \xi, \eta) = \hat{f}(0, \xi e^{-t} + \eta(1 - e^{-t}), \eta) \exp \left(-\int_0^t |\xi e^{r-t} + \eta(1 - e^{r-t})|^2 dr \right).$$

Note that

$$\int_0^t |\xi e^{r-t} + \eta(1 - e^{-r})|^2 dr$$

$$= \frac{1 - e^{-2t}}{2} |\xi|^2 + (1 - e^{-t})^2 \xi \cdot \eta + \left(t - \frac{3 + e^{-2t}}{2} + 2e^{-t}\right) |\eta|^2$$

$$= \left(X - \frac{X^2}{2}\right) |\xi|^2 + X^2 \xi \cdot \eta + \left(-\log(1 - X) - X - \frac{X^2}{2}\right) |\eta|^2,$$

where $X = 1 - e^{-t} \sim t$. We have for $0 < K < 2/3$

$$\int_0^t |\xi e^{r-t} + \eta(1 - e^{-r})|^2 dr \geq X(1 - 1/(2K) - X/2)|\xi|^2 + (1/3 - K/2)X^3 |\eta|^2.$$

Hence for $t \sim X < 2 - 1/K$, we get

$$f(t, \cdot, \cdot) \in A^{1/2}(\mathbb{R}^{2d}),$$

so that the ultra-analytic effect holds for any $t > 0$ by means of the semi-group property. But we cannot get the uniform estimate (1.5).

We present now the proof of (1.5) which implies the ultra-analytic effect, by commutator estimates similarly as for homogeneous Landau equation. Set

$$w(t, \eta, \xi) = \hat{f}(t, \eta, \xi - t\eta).$$
Then the Cauchy problem (1.4) is equivalent to

\[
\begin{align*}
\frac{dw}{dt}(t, \eta, \xi) &= -|\xi - t\eta|^2 w(t, \eta, \xi) - (\xi - t\eta) \cdot \nabla_\xi w(t, \eta, \xi), \\
w|t=0 &= \mathcal{F}(f_0)(\eta, \xi).
\end{align*}
\]  

(3.1)

Since we need to study the function \(\int_0^t |\xi - s\eta|^2 \, ds\), we prove the following estimate.

**Lemma 3.1.** For any \(\alpha > 0\), there exists a constant \(c_\alpha > 0\) such that

\[
\int_0^t |\xi - s\eta|^\alpha \, ds \geq c_\alpha (t|\xi|^\alpha + t^{\alpha+1}|\eta|^\alpha).
\]  

(3.2)

**Remark 3.1.** If \(\alpha = 2\), we can get the above estimate by direct calculation. The following simple proof is due to Seiji Ukai.

**Proof of Lemma 3.1.** Setting \(s = t\tau\) and \(\tilde{\eta} = t\eta\), we see that the estimate is equivalent to

\[
\int_0^1 |\xi - \tau \tilde{\eta}|^\alpha \, d\tau \geq c_\alpha (|\xi|^\alpha + |\tilde{\eta}|^\alpha).
\]

Since this is trivial when \(\tilde{\eta} = 0\), we may assume \(\tilde{\eta} \neq 0\). If \(|\xi| < |\tilde{\eta}|\) then

\[
\begin{align*}
\int_0^1 |\xi - \tau \tilde{\eta}|^\alpha \, d\tau &\geq |\tilde{\eta}|^\alpha \int_0^1 \tau - \left| \frac{\xi}{|\tilde{\eta}|} \right|^\alpha \, d\tau \\
&= |\tilde{\eta}|^\alpha \left\{ \int_0^{\left| \frac{\xi}{|\tilde{\eta}|} \right|} \left( \frac{|\xi|}{|\tilde{\eta}|} - \tau \right)^\alpha \, d\tau + \int_{\left| \frac{\xi}{|\tilde{\eta}|} \right|}^1 \left( \tau - \frac{|\xi|}{|\tilde{\eta}|} \right)^\alpha \, d\tau \right\} \\
&\geq |\tilde{\eta}|^\alpha \min_{0 \leq \theta \leq 1} (\theta^{\alpha+1} + (1 - \theta)^{\alpha+1}) = \frac{|\tilde{\eta}|^\alpha}{2^\alpha (\alpha + 1)} \\
&\geq \frac{1}{2^{\alpha+1} (\alpha + 1)} (|\xi|^\alpha + |\tilde{\eta}|^\alpha).
\end{align*}
\]

If \(|\xi| \geq |\tilde{\eta}|\) then

\[
\begin{align*}
\int_0^1 |\xi - \tau \tilde{\eta}|^\alpha \, d\tau &\geq |\xi|^\alpha \int_0^1 \left( 1 - \frac{|\tilde{\eta}|}{|\xi|} \right)^\alpha \, d\tau \geq |\xi|^\alpha \int_0^1 (1 - \tau)^\alpha \, d\tau \\
&= \frac{|\xi|^\alpha}{\alpha + 1} \geq \frac{1}{2(\alpha + 1)} (|\xi|^\alpha + |\tilde{\eta}|^\alpha).
\end{align*}
\]

Hence we obtain (3.2). \(\square\)
Set now
\[ \phi(t, \eta, \xi) = c_0 \left( \int_0^t |\xi - s\eta|^2 ds - \frac{c_2}{2} t^3 |\eta|^2 \right), \]

where \( c_0 > 0 \) is a small constant to choose later, and \( c_2 \) is the constant in (3.2) with \( \alpha = 2 \). Then (3.2) implies
\[ \phi(t, \eta, \xi) \geq c_0 \frac{c_2}{2} \left( t^2 |\xi|^2 + t^3 |\eta|^2 \right). \quad (3.3) \]

Let \( N = (2d + 1)/4 \). For \( 0 < \delta < 1/4N^2 \) and \( t > 0 \), set
\[ G_{\delta} = G_{\delta}(t, \eta, \xi) = \frac{e^{\phi(t, \eta, \xi)}}{(1 + \delta e^{\phi(t, \eta, \xi)})(1 + \delta (|\eta|^2 + |\xi|^2))^N}. \quad (3.4) \]

Since \( G_{\delta}(t, \cdot, \cdot) \in L^\infty(\mathbb{R}^{2d}) \), we can use it as Fourier multiplier, denoted by
\[ (G_{\delta}(t, D_x, D_v)u)(t, x, v) = \mathcal{F}^{-1}_{t, \xi}(G_{\delta}(t, \eta, \xi) \hat{u}(t, \eta, \xi)). \]

**Lemma 3.2.** Assume that \( f(t, \cdot) \in L^2(\mathbb{R}^{2d}_{x,v}) \cap L^1(\mathbb{R}^{2d}_{x,v}) \) for any \( t \in ]0, T[ \). Then \( \nabla_\xi w(t, \eta, \xi) \in L^\infty(\mathbb{R}^{2d}_{\eta, \xi}) \), and
\[ |\xi - t\eta|G_{\delta}(t, \eta, \xi)^2 \hat{\omega}(t, \eta, \xi), \quad |\eta|G_{\delta}(t, \eta, \xi)^2 \hat{\omega}(t, \eta, \xi), \quad \nabla_\xi(G_{\delta}(t, \eta, \xi)^2 \hat{\omega}(t, \eta, \xi)) \quad (3.5) \]
belong to \( L^2(\mathbb{R}^{2d}_{\eta, \xi}) \) for any \( t \in ]0, T[ \).

**Proof.** Since \( \partial_{\xi_j} w = -i \mathcal{F}(\nu_j f) \), it follows from \( f \in L^1_{x,v}(\mathbb{R}^{2d}_{x,v}) \) that \( \nabla_\xi w(t, \eta, \xi) \in L^\infty(\mathbb{R}^{2d}_{\eta, \xi}) \). Noting
\[ |\xi - t\eta|G_{\delta}(t, \eta, \xi)^2, \quad |\eta|G_{\delta}(t, \eta, \xi)^2 \in L^\infty(\mathbb{R}^{2d}_{\eta, \xi}), \]
we see that the first two terms of (3.5) are obvious. To check the last term in (3.5), note
\[ \partial_{\xi_j} G_{\delta}(t, \eta, \xi) = 2c_0 t \left( \xi_j - \frac{t}{2} \eta_j \right) G_{\delta}(t, \eta, \xi) \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} \]
\[ - \frac{2N\delta \xi_j}{(1 + \delta (|\eta|^2 + |\xi|^2)))} G_{\delta}(t, \eta, \xi). \quad (3.6) \]

Then, we have
\[ \nabla_\xi(G_{\delta}(t, \eta, \xi)^2 \hat{\omega}(t, \eta, \xi)) = G_{\delta}(t, \eta, \xi)^2 \nabla_\xi \hat{\omega}(t, \eta, \xi) + \nabla_\xi \left( G_{\delta}(t, \eta, \xi)^2 \hat{\omega}(t, \eta, \xi) \right) \]
\[ = G_{\delta}(t, \eta, \xi)^2 \nabla_\xi \hat{\omega}(t, \eta, \xi) \]
\[ + 4c_0 t \left( \xi - \frac{t}{2} \eta \right) \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} G_{\delta}(t, \eta, \xi)^2 \hat{\omega}(t, \eta, \xi) \]
\[ - \frac{4N\delta \xi}{(1 + \delta (|\eta|^2 + |\xi|^2)))} G_{\delta}(t, \eta, \xi)^2 \hat{\omega}(t, \eta, \xi). \]
Since $G_\delta(t, \eta, \xi)^2 \in L^2(\mathbb{R}^{2d})$ we have

$$G_\delta(t, \eta, \xi)^2 \nabla_\xi \tilde{w}(t, \eta, \xi) \in L^2(\mathbb{R}^{2d}).$$

Using

$$\left| \frac{1}{1 + \delta e^{\phi(t, \eta, \xi)}} \right| \leq 1, \quad \left| \frac{2N\delta \xi}{1 + \delta (|\eta|^2 + |\xi|^2)} \right| \leq 1,$$

and

$$\left| \left( \frac{\xi - t}{2} \eta \right) G_\delta(t, \eta, \xi)^2 \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} \tilde{w}(t, \eta, \xi) \right| \leq \left| \xi - t \eta \right| G_\delta(t, \eta, \xi)^2 |\tilde{w}(t, \eta, \xi)| \leq \left| \xi - t \eta \right| G_\delta(t, \eta, \xi)^2 |\tilde{w}(t, \eta, \xi)| \in L^2(\mathbb{R}^{2d}).$$

We have proved Lemma 3.2. □

We take now $G_\delta(t, \eta, \xi)^2 \tilde{w}(t, \eta, \xi)$ as test function in the equation of (3.1). Then we have

$$\frac{d}{dt} \left\| G_\delta(t, \cdot, \cdot) w(t, \cdot, \cdot) \right\|_{L^2(\mathbb{R}^{2d})}^2 + 2 \int_{\mathbb{R}^{2d}} |(\xi - t \eta) G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 d\eta d\xi$$

$$= 2 \sum_{j=1}^{d} \int_{\mathbb{R}^{2d}} w(t, \eta, \xi) \left( \partial_{\xi_j} (\xi_j - t \eta_j) G_\delta(t, \eta, \xi)^2 w(t, \eta, \xi) \right) d\eta d\xi$$

$$+ \left( \left( \partial_t G_\delta(t, \cdot, \cdot) w(t, \cdot, \cdot), G_\delta(t, \cdot, \cdot) w(t, \cdot, \cdot) \right)_{L^2(\mathbb{R}^{2d})} \right). \tag{3.7}$$

We prove now the following:

**Proposition 3.1.** We have

$$\left( \left( \partial_t G_\delta(t, \cdot, \cdot) w, G_\delta(t, \cdot, \cdot) w \right)_{L^2(\mathbb{R}^{2d})} \right)$$

$$= c_0 \int_{\mathbb{R}^{2d}} |(\xi - t \eta) G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 d\eta d\xi$$

$$- \frac{3}{2} c_0 c_2 t^2 \int_{\mathbb{R}^{2d}} |\eta|^2 |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} d\eta d\xi. \tag{3.8}$$
\[
\text{Proof.} \quad \text{The estimate (3.8) is deduced from}
\]
\[
\partial_t G_\delta(t, \eta, \xi) = c_0 \left( |\xi - t\eta|^2 - \frac{3}{2} c_2 t^2 |\eta|^2 \right) G_\delta(t, \eta, \xi) \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})}.
\]

Since it follows from (3.6) that

\[
I = \text{Re} \sum_{j=1}^{d} \int_{\mathbb{R}^{2d}} w(t, \eta, \xi) \partial_{\xi_j} ((\xi_j - t\eta_j) G_\delta(t, \eta, \xi)^2 w(t, \eta, \xi)) d\eta d\xi
\]

\[
= \text{Re} 2c_0 t \sum_{j=1}^{d} \int \left( \xi_j - t\eta_j \right) \left( \xi_j - \frac{t}{2} \eta_j \right) |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} d\eta d\xi
\]

\[
- \text{Re} \sum_{j=1}^{d} \int \frac{2N \delta \xi_j (\xi_j - t\eta_j)}{(1 + \delta (|\eta|^2 + |\xi|^2))} |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 d\eta d\xi
\]

\[
- \text{Re} \sum_{j=1}^{d} \int (\xi_j - t\eta_j) (\partial_{\xi_j} G_\delta(t, \eta, \xi) w(t, \eta, \xi)) \overline{G_\delta(t, \eta, \xi) w(t, \eta, \xi)} d\eta d\xi,
\]

we get

\[
I = 2c_0 t \sum_{j=1}^{d} \int (\xi_j - t\eta_j) \left( \xi_j - \frac{t}{2} \eta_j \right) |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} d\eta d\xi
\]

\[
- \sum_{j=1}^{d} \int \frac{2N \delta \xi_j (\xi_j - t\eta_j)}{(1 + \delta (|\eta|^2 + |\xi|^2))} |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 d\eta d\xi + \frac{d}{2} \|G_\delta(t, \eta, \xi) w(t, \eta, \xi)\|_{L^2(\mathbb{R}^{2d})}^2
\]

\[
= 2c_0 t \int |\xi - t\eta|^2 |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} d\eta d\xi
\]

\[
+ c_0^2 \int (\xi - t\eta) \cdot \eta |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} d\eta d\xi
\]
\[- \sum_{j=1}^{d} \int \frac{2N\delta \xi_j (\xi_j - t \eta_j)}{(1 + \delta (|\eta|^2 + |\xi|^2))} |G_\delta(t, \eta, \xi)w(t, \eta, \xi)|^2 \, d\eta \, d\xi \]
\[+ \frac{d}{2} \|G_\delta(t, \cdot, \cdot)w(t, \cdot, \cdot)\|^2_{L^2(\mathbb{R}^d)} . \]

For the last term, noting
\[\sum_{j=1}^{d} \frac{2N\delta \xi_j (\xi_j - t \eta_j)}{(1 + \delta (|\eta|^2 + |\xi|^2))} \leq \frac{(N^2/c_0)\delta^2 |\xi|^2 + c_0 |\xi| - t \eta|^2}{(1 + \delta (|\eta|^2 + |\xi|^2))} \leq N^2 \delta/c_0 + c_0 |\xi| - t \eta|^2 , \]
we finally obtain
\[I \leq \left( 2c_0 t + \frac{c_0 t^2}{3c_2} + c_0 \right) \int \frac{|(\xi - t \eta)G_\delta(t, \eta, \xi)w(t, \eta, \xi)|^2}{\mathbb{R}^d} \, d\eta \, d\xi \]
\[+ \frac{d + 2N^2 \delta/c_0}{2} \|G_\delta(t, \cdot, \cdot)w(t, \cdot, \cdot)\|^2_{L^2(\mathbb{R}^d)} \]
\[+ \frac{3}{4} c_0 c_2 t^2 \int \left| |\eta|^2 |G_\delta(t, \eta, \xi)w(t, \eta, \xi)| \right|^2 \left( \frac{1}{(1 + \delta e^{|t| \eta, |\xi|})} \right) \, d\eta \, d\xi . \]

Thus we have proved Proposition 3.1. \(\square\)

**End of proof of Theorem 1.2.** Now Eq. (3.7), the estimate (3.8) and (3.9) deduce
\[\frac{d}{dt} \|G_\delta(t, \cdot, \cdot)w(t, \cdot, \cdot)\|^2_{L^2(\mathbb{R}^d)} + \left( 2 - 3c_0 - 4c_0 t - \frac{2c_0 t^2}{3c_2} \right) \int \frac{|(\xi - t \eta)G_\delta(t, \eta, \xi)w(t, \eta, \xi)|^2}{\mathbb{R}^d} \, d\eta \, d\xi \]
\[\leq \left( d + 2N^2 \delta/c_0 \right) \|G_\delta(t, \cdot, \cdot)w(t, \cdot, \cdot)\|^2_{L^2(\mathbb{R}^d)} . \]

Then for any \(0 < T_0 < T\) choose \(c_0 > 0\) (depends on \(T_0\)) small enough such that
\[2 - 3c_0 - 4c_0 t_0 - \frac{2c_0 T_0^2}{3c_2} \geq 0 , \]
then for any \(0 < t \leq T_0\),
\[\frac{d}{dt} \|G_\delta(t, \cdot, \cdot)w(t, \cdot, \cdot)\|^2_{L^2(\mathbb{R}^d)} \leq \frac{d + 2N^2 \delta/c_0}{2} \|G_\delta(t, \cdot, \cdot)w(t, \cdot, \cdot)\|^2_{L^2(\mathbb{R}^d)} , \]
which gives
\[\|G_\delta(t, \cdot, \cdot)w(t, \cdot, \cdot)\|^2_{L^2(\mathbb{R}^d)} \leq e^{\frac{d + 2N^2 \delta/c_0}{2} t} \|f_0\|^2_{L^2(\mathbb{R}^d)} . \]

Take \(\delta \to 0\), we have
\[
\int_{\mathbb{R}^2} e^{c_0 \int_0^t |\xi - s\eta|^2 ds - c_1 t^3 |\eta|^2} \left| \mathbf{\hat{f}}(t, \eta, \xi - t\eta) \right|^2 d\eta d\xi
\]

\[
= \int_{\mathbb{R}^2} e^{c_0 \int_0^t |\xi + (t-s)\eta|^2 ds - c_1 t^3 |\eta|^2} \left| \mathbf{\hat{f}}(t, \eta, \xi) \right|^2 d\eta d\xi \leq e^{dt} \| f_0 \|_{L^2(\mathbb{R}^2d)}^2.
\]

By using (3.3), we get finally
\[
\| e^{-\tilde{c}_0 (t\Lambda + t^3 \Lambda')} f(t, \cdot, \cdot) \|_{L^2(\mathbb{R}^2d)} \leq e^{\frac{4t}{\tilde{c}_0}} \| f_0 \|_{L^2(\mathbb{R}^2d)}
\]

for any \( 0 < t \leq T_0 \), where \( \tilde{c}_0 = \frac{c_0 c_2}{2} > 0 \). This is the desired estimate (1.5), which implies
\[
f(t, \cdot, \cdot) \in \mathcal{A}^{1/2}(\mathbb{R}^2d).
\]

We have thus proved Theorem 1.2. \( \square \)

4. Linear model of inhomogeneous Landau equations

We prove now Theorem 1.3 in this section. By the change of variables \((t, x, v) \to (t, x + vt, v)\), the Cauchy problem (1.8) is reduced to
\[
\begin{cases}
  \mathbf{f}_t = (\nabla_v - t \nabla_x) (\bar{a}(\mu) \cdot (\nabla_v - t \nabla_x) f - \bar{b}(\mu) f), \\
  f|_{t=0} = g_0(x, v),
\end{cases}
\] (4.1)

where \( f(t, x, v) = g(t, x + vt, v) \). Recall that
\[
\bar{a}_{ij}(\mu) = a_{ij} \star \mu = \delta_{ij} (|v|^2 + 1) - v_i v_j,
\]
\[
\bar{b}_j(\mu) = \sum_{i=1}^{d} (\partial_{vi} a_{ij}) \star \mu = -v_j, \quad i, j = 1, \ldots, d,
\]

and
\[
\sum_{ij=1}^{d} \bar{a}_{ij}(\mu) \xi_i \xi_j \geq |\xi|^2, \quad \text{for all } (v, \xi) \in \mathbb{R}^{2d}.
\]

In view of this Cauchy problem, we set
\[
\Psi(t, \eta, \xi) = c_0 \int_0^t |\xi - s\eta| ds,
\]

for a sufficiently small \( c_0 > 0 \) which will be chosen later on. Then we can use (3.2) with \( \alpha = 1 \) to estimate \( \Psi \). Set
\[
F_{\delta}(t, \eta, \xi) = \frac{e^\psi}{(1 + \delta e^\psi)(1 + \delta \Psi)^N}
\]
for \( N = d + 1, 0 < \delta \leq \frac{1}{N} \). If \( A \) is a first order differential operator of \((t, \eta, \xi)\) variables then we have

\[
AF_\delta = \left(\frac{1}{1 + \delta e^\Psi} - \frac{N\delta}{1 + \delta \Psi}\right)(A\Psi)F_\delta.
\]  

(4.2)

and

\[
\left|\frac{1}{1 + \delta e^\Psi} - \frac{N\delta}{1 + \delta \Psi}\right| \leq 1.
\]

Taking

\[
F_\delta(t, D_x, D_\eta)^2 f = F_\delta(t)^2 f \in H^{2N}(\mathbb{R}^{2d})
\]
as a test function in the weak solution formula of (4.1), we have

\[
\frac{1}{2} \frac{d}{dt} \left\| F_\delta(t) f \right\|_{L^2(\mathbb{R}^{2d})}^2 + (\bar{a}(\mu))((\nabla_x - t \nabla_x)F_\delta(t) f), ((\nabla_x - t \nabla_x)F_\delta(t) f)\right\|_{L^2(\mathbb{R}^{2d})}
\]

\[
= - \sum_{j=1}^d \int v_j f ((\partial_{\xi_j} - t \partial_{\xi_j})F_\delta(t)^2 f) dx dv + \frac{1}{2} ((\partial_t F_\delta) f, F_\delta(t) f)_{L^2(\mathbb{R}^{2d})}
\]

\[
+ \sum_{j,k=1}^d \int \{\bar{a}_{jk}(\mu)(F_\delta(t)(\partial_{\xi_j} - t \partial_{\xi_j})) f - F_\delta(t)\{\bar{a}_{jk}(\mu)(\partial_{\xi_j} - t \partial_{\xi_j}) f\}\}((\partial_{\xi_k} - t \partial_{\xi_k})F_\delta(t)f) dx dv.
\]

We prove now the following results.

**Proposition 4.1.** We have

\[
\left\| (\nabla_x - t \nabla_x)F_\delta(t) f \right\|_{L^2(\mathbb{R}^{2d})}^2 \leq (\bar{a}(\mu))((\nabla_x - t \nabla_x)F_\delta(t) f), ((\nabla_x - t \nabla_x)F_\delta(t) f)\right\|_{L^2(\mathbb{R}^{2d})},
\]

(4.3)

\[
\left| (\partial_t F_\delta(t) f, F_\delta(t) f)_{L^2} \right| \leq c_0 \left\| (\nabla_x - t \nabla_x)F_\delta(t) f \right\|_{L^2} \left\| F_\delta(t) f \right\|_{L^2}.
\]

(4.4)

\[
- \text{Re} \sum_{j=1}^d \int v_j f ((\partial_{\xi_j} - t \partial_{\xi_j})F_\delta(t)^2 f) \leq \frac{d}{2} \left\| F_\delta(t) f \right\|_{L^2}^2 + c_0 t \left\| (\nabla_x - t \nabla_x)F_\delta(t) f \right\|_{L^2} \left\| F_\delta(t) f \right\|_{L^2}.
\]

(4.5)

**Proof.** The estimate (4.3) is a direct consequence of the elliptic condition (1.7). Using the Fourier transformation and noting (4.2), we see that (4.4) is derived from

\[
\partial_t F_\delta(t, \eta, \xi) = \left(\frac{1}{1 + \delta e^\Psi} - \frac{N\delta}{1 + \delta \Psi}\right)(\partial_t \Psi)F_\delta, \quad \partial_t \Psi = c_0 |\xi - t\eta|.
\]

For (4.5), we have firstly

\[
- \text{Re} \sum_{j=1}^d \int v_j F_\delta(t) f ((\partial_{\xi_j} - t \partial_{\xi_j})F_\delta(t) f) = \frac{d}{2} \left\| F_\delta(t) f \right\|_{L^2}^2.
\]

For the commutators \([v_j, F_\delta(t)]\), using Fourier transformation, we have that for \( t > 0 \) and \( \hat{f} = \hat{f}(t, \eta, \xi) \).
Proof. In order to prove (4.6), we introduce the polar coordinates of $\xi$ centered at $t\eta$, that is,

$$
r = |\xi - t\eta| \quad \text{and} \quad \omega = \frac{\xi - t\eta}{|\xi - t\eta|} \in \mathbb{S}^{d-1}.
$$

Note again that $\partial/\partial \xi_j = \omega_j \partial/\partial r + r^{-1} \Omega_j$ where $\Omega_j$ is a vector field on $\mathbb{S}^{d-1}$. We have again

$$
\sum_{j=1}^d \omega_j \Omega_j = 0, \quad \sum_{j=1}^d \Omega_j \omega_j = d - 1.
$$

By means of Plancherel formula, we have

$$
\sum_{j,k=1}^d \int \left[ \left( \partial_{\xi_j} F_\delta(t)(\partial_{\xi_j} - t\partial_{\eta_j}) \right) F_\delta(t) \right] \left( \partial_{\xi_k} F_\delta(t) \right) \, d\xi \, d\eta.
$$

Thus (4.5) has been proved. $\Box$

For the commutator terms, we have

$$
\left| \sum_{j=1}^d (\partial_{\xi_j} \Psi)(t, \eta, \xi) \times (\xi_j - t\eta_j) \right| \leq c_0 \int_0^1 \left| \sum_{j=1}^3 \frac{\xi_j - s\eta_j}{|\xi - s\eta|} (\xi_j - t\eta_j) \right| ds \leq c_0 t |\xi - t\eta|.
$$

where, in view of (4.2), we have used the fact that

$$
\sum_{j=1}^d (\partial_{\xi_j} \Psi)(t, \eta, \xi) \times (\xi_j - t\eta_j) \leq c_0 t |\xi - t\eta|.
$$

Thus (4.5) has been proved. $\Box$

**Proposition 4.2.** There exists a constant $C_1 > 0$ independent of $\delta > 0$ such that

$$
\left| \sum_{j,k=1}^d \int \left[ \partial_{\xi_j} \Psi(t)(\partial_{\xi_j} - t\partial_{\eta_j}) \right] F_\delta(t) \left( \partial_{\xi_k} \Psi(t)(\partial_{\xi_k} - t\partial_{\eta_k}) \right) \right| \leq C_1 \left\| (\nabla \Psi - t\nabla \Psi) F_\delta(t) \right\|_{L^2}^2.
$$

**Proof.** In order to prove (4.6), we introduce the polar coordinates of $\xi$ centered at $t\eta$, that is,

$$
r = |\xi - t\eta| \quad \text{and} \quad \omega = \frac{\xi - t\eta}{|\xi - t\eta|} \in \mathbb{S}^{d-1}.
$$

Note again that $\partial/\partial \xi_j = \omega_j \partial/\partial r + r^{-1} \Omega_j$ where $\Omega_j$ is a vector field on $\mathbb{S}^{d-1}$. We have again

$$
\sum_{j=1}^d \omega_j \Omega_j = 0, \quad \sum_{j=1}^d \Omega_j \omega_j = d - 1.
$$

By means of Plancherel formula, we have

$$
\sum_{j,k=1}^d \int \left[ \left( \partial_{\xi_j} \Psi(t)(\partial_{\xi_j} - t\partial_{\eta_j}) \right) - F_\delta(t) \left( \partial_{\xi_j} \Psi(t)(\partial_{\xi_j} - t\partial_{\eta_j}) \right) \right] \left( \partial_{\xi_k} F_\delta(t) \right) \, d\xi \, d\eta.
$$
Putting

\[ \int \varepsilon \left( \xi - t \eta \right) \left[ \left( \delta_{jk} \Delta \xi - \partial_{kj} \partial_{\xi j} \right), F_{\delta}(t, \eta, \xi) \right] \left( \xi - t \eta \right) \hat{f}(t) \right] F_{\delta}(t, \eta, \xi) \hat{f}(t) \, d\xi \, d\eta
\]

\[ J. \]

Noting again

\[ \Delta \xi = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \sum_{l=1}^{d} \Omega_l^2, \]

we have with \( \tilde{F}_{\delta}(t, \eta, r, \omega) = F_{\delta}(t, \eta, r \cdot \omega + t \eta) = F_{\delta}(t, \eta, \xi) \)

\[ - \sum_{j,k=1}^{d} \omega_k \left[ \left( \delta_{jk} \left( \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \sum_{l=1}^{d} \Omega_l^2 \right) \right) - \left( \left( \omega_k \frac{\partial}{\partial r} + r^{-1} \Omega_k \right) \left( \omega_j \frac{\partial}{\partial r} - r^{-1} \Omega_j \right) \right) \right] \tilde{F}_{\delta} \]

\[ \frac{1}{r^2} \sum_{j=1}^{d} \omega_j \left[ \sum_{l=1}^{d} \Omega_l^2 \right] \tilde{F}_{\delta} \]

\[ \omega_j = A_1 + A_2 + A_3. \]

Note again that

\[ A_1 + A_2 = \left[ \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} \tilde{F}_{\delta} \right] + \left[ \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} - \frac{1}{r} \tilde{F}_{\delta} \right] = 0. \]

On the other hand, we have in view of (4.2)

\[ A_3 = - \frac{1}{r^2} \sum_{j,l=1}^{d} \omega_j \left( 2 \Omega_j [\Omega_j, \tilde{F}_{\delta}] - [\Omega_j, [\Omega_j, \tilde{F}_{\delta}]] \right) \omega_j \]

\[ \frac{1}{r^2} \sum_{j,l=1}^{d} \omega_j \left( 2 \Omega_j \left( \frac{\Omega_j \Psi}{1 + \delta e^\Psi} - \frac{N\delta(\Omega_j \Psi)}{1 + \delta \Psi} \right) \tilde{F}_{\delta} \right) \]

\[ \left[ \left( \frac{\Omega_j \Psi}{1 + \delta e^\Psi} - \frac{N\delta(\Omega_j \Psi)}{1 + \delta \Psi} \right)^2 + \left( \Omega_j \left( \frac{\Omega_j \Psi}{1 + \delta e^\Psi} - \frac{N\delta(\Omega_j \Psi)}{1 + \delta \Psi} \right) \right) \tilde{F}_{\delta} \right] \omega_j. \]

Putting \( w_j = \omega_j \tilde{F}_{\delta} \) with \( w(t, \eta, r, \omega) = \hat{f}(t, \eta, r \cdot \omega + t \eta) \), we have

\[ J = \text{Re} \int \int_{\mathbb{R}^d} \int_{S^{d-1}} r^2 (A_3 w) \tilde{F}_{\delta} w dr \, d\omega \, d\eta \]

\[ = - \sum_{j,l=1}^{d} \text{Re} \int \int_{\mathbb{R}^d} \int_{S^{d-1}} \left( 2 \Omega_j \left( \frac{\Omega_j \Psi}{1 + \delta e^\Psi} - \frac{N\delta(\Omega_j \Psi)}{1 + \delta \Psi} \right) w_j \right) \tilde{F}_{\delta} w dr \, d\omega \, d\eta \]
\[ \frac{1}{2} \frac{d}{dt} \left( (F_\delta f)(t) \right)^2_{L^2(R^{2d})} + \left( 1 - (c_0 t)^2 \right) C_2 \left\| (\nabla_v - t \nabla_x)(F_\delta f)(t) \right\|^2_{L^2(R^{2d})} \leq C_3 \left\| (F_\delta f)(t) \right\|^2_{L^2(R^{2d})}. \]
So that if $\frac{1}{2} - (c_0 t)^2 C_2 \geq 0$, we have
\[
\frac{d}{dt} \left\| (F_\delta f)(t) \right\|_{L^2(\mathbb{R}^d)} \leq C_3 \left\| (F_\delta f)(t) \right\|_{L^2(\mathbb{R}^d)}.
\]
(4.9)

Using the fact $(F_\delta f)(0) = \frac{1}{1+t^2} g_0$, we get
\[
\left\| (F_\delta f)(t) \right\|_{L^2(\mathbb{R}^d)} \leq e^{C_3 t} \left\| g_0 \right\|_{L^2(\mathbb{R}^d)}.
\]

Take the limit $\delta \to 0$. Then we have
\[
\int_{\mathbb{R}^d} e^{2\psi(t,\eta,\xi)} \left| \hat{f}(t,\eta,\xi) \right|^2 d\eta d\xi \leq e^{2C_3 t} \left\| g_0 \right\|_{L^2(\mathbb{R}^d)}^2,
\]
(4.10)

On the other hand, by Lemma 3.1, there exists a $c_1 > 0$ such that
\[
\int_{\mathbb{R}^d} e^{2\psi(t,\eta,\xi)} \left| \hat{f}(t,\eta,\xi) \right|^2 d\eta d\xi = \int_{\mathbb{R}^d} e^{2c_0 \int_0^t |\xi - s\eta| ds} \left| \hat{g}(t,\eta,\xi - t\eta) \right|^2 d\eta d\xi
\]
\[
= \int_{\mathbb{R}^d} e^{2c_0 \int_0^t |\xi + (t-s)\eta| ds} \left| \hat{g}(t,\eta,\xi) \right|^2 d\eta d\xi
\]
\[
\geq \int_{\mathbb{R}^d} e^{2c_0 c_1 (t|\xi| + t^2|\eta|)} \left| \hat{g}(t,\eta,\xi) \right|^2 d\eta d\xi.
\]

Finally, for any $0 < T_0 < T$, choosing $c_0 > 0$ small enough such that $\frac{1}{2} - (c_0 T_0)^2 C_2 \geq 0$, we have proved
\[
\int_{\mathbb{R}^d} \left| e^{c_0 c_1 (t(-\Delta x)^{1/2} + t^2(-\Delta v)^{1/2})} g(t, x, v) \right|^2 dx dv \leq e^{2C_3 t} \left\| g_0 \right\|_{L^2(\mathbb{R}^d)}^2
\]
for any $0 < t \leq T_0$.

which completes the proof of Theorem 1.3 with $C = 2C_3$ depending only on $d$. \qed

Remark 4.1. The formulas (4.7) and (4.8) show that we cannot get the ultra-analytic effect of order $1/2$ as in Theorem 1.2. It is the same reason why we do not consider the symmetric term $Q(g, \mu)$ in Eq. (1.8) as in [1].

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