Privacy-Utility Trade-Off

Hao Zhong and Kaifeng Bu

Abstract—In this paper, we investigate the privacy-utility trade-off (PUT) problem, which considers the minimal privacy loss at a fixed expense of utility. Several different kinds of privacy in the PUT problem are studied, including differential privacy, approximate differential privacy, maximal information, maximal leakage, Rényi differential privacy, Sibson mutual information and mutual information. The average Hamming distance is used to measure the distortion caused by the privacy mechanism. We consider two scenarios: global privacy and local privacy. In the framework of global privacy framework, the privacy-distortion function is upper-bounded by the privacy loss of a special mechanism, and lower-bounded by the optimal privacy loss with any possible prior input distribution. In the framework of local privacy, we generalize a coloring method for the PUT problem.

Index Terms—Privacy-utility trade-off, differential privacy, maximal leakage, Rényi differential privacy, Sibson mutual information

I. INTRODUCTION

Advance in digital technologies of data-producing and data-using provides enormous reliable and convenient services in the era of Big Data. Along with the prosperity, security issues and challenges result in increasing privacy concerns. Nowadays, examples include patient data breach, facial recognition data abuse, and surveillance measures without consent during the COVID-19 pandemic.

Traditional semantic security for cryptosystems is not always achievable due to the auxiliary information available to adversary. Hence, differential privacy has been introduced by Dwork et al. [5], [6] to deal with this problem. The original differential privacy is a typical divergence-based privacy since it is exactly the max-divergence between the probability distributions on neighboring databases.

This new measure requires an upper bound on the worst-case change, which may take high expense of utility. As a result, several relaxations of differential privacy have been studied recently. We categorize these existing measures into two main classes, divergence-based privacy and mutual information-based privacy. Besides, δ-approximation can be used for particular privacy notion to relax the strict relative shift.

The most generalized divergence-based privacy for now was given by [20], Rényi differential privacy. This α-Rényi divergence-based measure keeps track of the privacy cost and allows tighter analysis of mechanism composition [13], [21], [22]. All the divergence-based measures consider the worst-case guarantee of privacy, and thus they protect the information of every single individual.

The mutual information-based privacy concerns the expected information leakage from the input (plain) database to the output (synthetic or sanitized) database. A straightforward measure is based on Shannon’s mutual information which makes use of related results in information theory [4], [30]. Another measure called max-information was also given by Dwork et al. [9], which aims at bounding the change in the conditional probability of events relative to their a priori probability. Two most common mutual information-based privacy measures are based on Sibson’s [26] and Arimoto’s α-mutual information [1], respectively. Plenty of works [10], [12], [15], [19], [29], [31] based on the α-mutual information were published, concerning data processing and mechanism composition. Notably, the case α = ∞ leads to a newly defined privacy definition, maximal leakage, which was first proposed by Issa et al. [16]. As compared to most of mutual-information metrics, it is easier to compute and analyze.

In Ref. [7], Dwork et al. defined approximate (ε, δ)-differential privacy. It is a differential privacy notion with a δ shift tolerance, which provides more flexibility while designing mechanisms. In another paper by Dwork [9], she unified this idea with max information. Let x and x′ be two adjacent database, i.e., differing in only one individual. It is worth noting that for (ε, δ)-differential privacy, the mechanism Q is approximate private if and only if there exists a distribution which is statistically δ-close to Q(x) and max-divergently ε-close to Q(x′) [8].

The main goal is to find a privacy-preserving mechanism not destroying statistical utility, or to solve the privacy-utility trade-off (PUT) problem. A rate-distortion theory flavored PUT problem is to determine the minimal compression during a valid communication [18]. As an important topic in information theory, rate distortion has been studied and applied in many fields including complexity and learning theory. For example, the information bottleneck (IB) problem which was initiated by Tishby [27], has attracted lots of attention as it provides an explanation of the success of machine learning [14], [24], [25], [28]. As for PUT problem, global differential privacy-distortion with average Hamming distance was studied in [30]; local differential privacy-distortion with average Hamming distance was studied in [18]; local maximal leakage-distortion was studied in [23]; Arimoto mutual information-distortion with hard Hamming distance was studied in [19].

In this paper, we study the the minimal privacy loss with given Hamming distortion for seven different privacy notions including differential privacy, approximate differential privacy, maximal information, maximal leakage, Rényi differential privacy, Sibson mutual information and mutual information. Our main contributions in this work are listed as follows:

1) We obtain the lower- and upper-bounds of privacy-distortions for the privacy notions mentioned above.

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Moreover, for the prior input distribution is unknown, we obtain the closed-form expression for differential privacy- and rate-distortion.

2) We generalize the coloring method to solve the PUT problems for local approximate differential privacy- and maximal leakage-distortion. The term "local" means the data server is untrusted and the individual information is calibrated before uploading.

3) If the input space consisting of samples independently and identically distributed chosen, we show the analytical closed form results for differential privacy-, maximal leakage- and rate-distortion, together with the lower- and upper-bounds for the rest of privacy-distortions.

The remainder of this paper is organized as follows. In Section II, we introduce the basics notions and concepts. In Section III, we investigate the global privacy distortion, i.e., the privacy-utility trade-off when there is a curator with access to the plain data. In Section IV, we study the local privacy distortion and demonstrate our results by two images. Moreover, we show the privacy-distortion trade-off for parallel composition. Finally, we conclude in Section V.

II. PRELIMINARIES

Let \( X \) be the input data space, which is a finite set. A randomized mechanism \( Q \) is a conditional distribution \( Q_{Y|X} \) mapping the discrete random variable \( Y \in Y \) to an output data. In this paper, we focus on the accuracy-persevering mechanisms that do not distort the data above some threshold \( D \).

**Definition 2.** A mechanism \( Q \) is \((P,D)\)-valid if the expected distortion \( E_{P,Q}[d(x,y)] \leq D \), where \( E_{P,Q} \) denotes the average over the randomness of input and mechanism. Moreover, \( Q \) is \((P,D)\)-valid if \( Q \) is \((P,D)\)-valid for any \( P \in \mathcal{P} \). The set of all \((P,D)\)-valid \((\mathcal{P},D)\)-valid, respectively) mechanisms is denoted by \( Q(P,D) \) \((Q(\mathcal{P},D), \) respectively).

Due to the convexity, it is directly to see that \( Q(\mathcal{P},D) = Q(Conv(\mathcal{P}),D) \). Hence, without loss of generality, we always assume that \( \mathcal{P} \) is convex.

B. DIFFERENT NOTIONS OF PRIVACY

In this paper, we investigate different notions of privacy, which we list as follows.

**Definition 3.** Let \( P_X \) be a distribution with full support on \( X \) and let \( \epsilon \) represent a non-negative real number.

1) \((\epsilon)-\)differential privacy, \([5],[6]\) A mechanism \( Q \) is called \(\epsilon\)-differential private if for any \( y \in Y \), neighboring elements \( x \) and \( x' \in X \),

\[
Q_{Y|X}(y|x) \leq e^{\epsilon}Q_{Y|X}(y|x')
\]

In this case, we denote the minimal \( \epsilon \) as \( \epsilon_{DP}(Q) \) for the mechanism \( Q \).

2) \((\epsilon,\delta)-\)differential privacy, \([8]\) Given a positive number \( \delta < 1 \), a mechanism \( Q \) is called \((\epsilon,\delta)-\)differential private if for any \( y \in Y \), neighboring elements \( x \) and \( x' \in X \),

\[
Q_{Y|X}(y|x) \leq e^{\epsilon}Q_{Y|X}(y|x') + \delta.
\]

In this case, we denote the minimal \( \epsilon \) as \( \epsilon_{DP}(Q) \) for the mechanism \( Q \).

3) [Maximal information, \([9]\)] A mechanism \( Q \) has maximal information \( \epsilon \) if for any \( y \in Y \), \( x \in X \),

\[
Q_{Y|X}(y|x) \leq e^{\epsilon} \mathbb{E}_{x \sim P_X} [Q_{Y|X}(y|x)].
\]

In this case, we denote the minimal \( \epsilon \) as \( \epsilon_{MI}(Q) \) for the mechanism \( Q \).

4) [Maximal leakage, \([16]\)] A mechanism \( Q \) has maximal leakage \( \epsilon \) if

\[
\sum_{y \in Y} \max_{x \in X} Q_{Y|X}(y|x) \leq e^{\epsilon}.
\]

In this case, we denote the minimal \( \epsilon \) as \( \epsilon_{ML}(Q) \) for the mechanism \( Q \).
(5) [Rényi differential privacy, [20]] Given a positive number \( \alpha > 1 \), a mechanism \( Q \) is \( \epsilon \)-Rényi differential private if for any neighboring elements \( x \) and \( x' \in X \), the Rényi divergence

\[
D_\alpha(Q_Y|X(x)|Q_Y|X(x')) \leq \epsilon.
\]

In this case, we denote the minimal \( \epsilon \) as \( \epsilon_{\alpha,D}(Q) \) for the mechanism \( Q \).

(6) \( \alpha \)-mutual information privacy, [10], [26], [29] Given a positive number \( \alpha > 1 \), a mechanism \( Q \) has maximal \( \alpha \)-mutual information \( \epsilon \) if the the Sibson mutual information

\[
I_\alpha(X; Q(X)) \leq \epsilon.
\]

In this case, we denote the minimal \( \epsilon \) as \( \epsilon_{\alpha,M}(Q) \) for the mechanism \( Q \).

(7) [Mutual information privacy, [4], [30]] A mechanism \( Q \) is called \( \epsilon \)-mutual information private if the mutual information

\[
I(X; Q(X)) \leq \epsilon.
\]

In this case, we denote the minimal \( \epsilon \) as \( \epsilon_M(Q) \) for the mechanism \( Q \).

The minimum \( \epsilon^*_\alpha(Q) \) above is called the privacy loss of mechanism \( Q \). We now introduce the privacy-distortion function, which describes the minimal achievable privacy loss under restricted distortion. It is defined as follows.

**Definition 4.** Let \( \mathcal{P} \) be a set with full-support distributions and \( D \) a positive number.

\[
\epsilon^*_\alpha(\mathcal{P}, D) := \min_{Q \in \mathcal{Q}(\mathcal{P}, D)} \epsilon_\alpha(Q)
\]

(2) is called the corresponding privacy-distortion function for \( \star \) representing the privacy notions aforementioned. The set of mechanisms achieving the minimum is denoted by \( \mathcal{Q}^*_{\alpha}(\mathcal{P}, D) \). In particular, if \( \mathcal{P} \) consists of only a single element \( D \), then we write them as \( \epsilon^*_\alpha(\mathcal{P}, D) \) and \( \mathcal{Q}^*_{\alpha}(\mathcal{P}, D) \). Conversely, given restricted privacy cost \( \epsilon \geq 0 \), we denote the smallest expected distortion achievable by

\[
D^*_\epsilon(\mathcal{P}, \epsilon) := \min_{\epsilon_\alpha(Q) \leq \epsilon} \max_{P \in \mathcal{P}} \sum_{(x, y) \in \mathcal{P}} P(d(x, y)).
\]

**III. GLOBAL PRIVACY FRAMEWORK**

A global privacy framework relies on a data server which collects all the data directly from individuals and publish them in a privacy-protected synthetic database (see Fig. 1). The \( n \)-individual database \( X \) is drawn from \( X = \{1, 2, \ldots, m\}^n \) with respect to the probability distribution \( P \) in \( \mathcal{P} \) where \( m \) is the total number of types the individuals belong to. For fixed \( x \), all database in \( X \) can be divided into \( n+1 \) categories based on their Hamming distance to \( x \). Let \( N_l(x) \) be the set consisting of all elements with \( l \) attributes different from \( x \) and \( 0 \leq l \leq n \). Then the size of the set \( N_l(x) \) is \( N_l := (\binom{m}{l}-1)^l \). To avoid notation representation confusion, \( \epsilon^* \) is used to denote the privacy-distortion function of global case. Here, we have the following result which holds for every privacy notions in Definition 3.

**Lemma 5.** Given a source set \( \mathcal{P} \), let \( D \) be a positive integer not greater than \( n \), we have the following properties of privacy-distortion function:

(1) For any \( P \in \mathcal{P} \), \( \epsilon^*(\mathcal{P}, D) \geq \epsilon^*(P, D) \).

(2) Let \( Q_U \) be the uniform mechanism, i.e., \( Q_U(y|x) = \frac{1}{m^n} \) for any \( x, y \in X \). Then for \( \frac{n(m-1)}{m} \leq D \leq 1 \), \( \epsilon^*(Q_U) = \epsilon^*(Q_U) = 0 \).

(3) Let \( Q_D \) be the mechanism given by Wang et al. [30], defined as

\[
Q_D(y|x) = \left(1 - \frac{D}{n}\right)^n \left(\frac{(m-1)(n-D)}{D}\right)^{-d(x, y)},
\]

then \( \epsilon^*(\mathcal{P}, D) \leq \epsilon(Q_D) \).

**Proof of Lemma 5.** By Definition 4, for any mechanism \( Q \) in \( \mathcal{Q}^*(\mathcal{P}, D) \), \( Q \) is \( (P, D) \)-valid. Thus, \( Q \) is \( (P, D) \)-valid for any \( P \in \mathcal{P} \). Hence, \( \epsilon^*(Q) = \epsilon^*(\mathcal{P}, D) \geq \epsilon^*(P, D) \).

Conversely, for any \( (P, D) \)-valid mechanism \( Q \), the definition of privacy distortion function implies \( \epsilon^*(\mathcal{P}, D) \leq \epsilon^*(Q) \).

Besides, it is easy to verify that \( Q_U \) is \( (P, D) \)-valid for \( \frac{n(m-1)}{m} \leq D \leq 1 \) and \( Q_D \) is \( (P, D) \)-valid for \( 0 < D \leq 1 \). This completes the proof.

**Theorem 6.** Let \( \mathcal{P} \) be a source set such that each probability distribution in \( \mathcal{P} \) has full support and let \( \theta^* := m^n \cdot \max_{P \in \mathcal{P}} \min_{x \in X} P(x) \). Then we have the following results on the privacy-distortion function with respect to the privacy notions in Definition 3.

(1) (Differential privacy)

\[
\max\left\{0, \log(m-1) - \frac{\theta^* n - D}{D}\right\} \leq \epsilon_{DP}^*(\mathcal{P}, D) \leq \max\left\{0, \log\left(\frac{n(m-1) - D}{D}\right)\right\}.
\]

(2) (Approximate differential privacy)

\[
\max\left\{0, \log(m-1) - \frac{\theta^* n(1 - \delta m^{n-1}) - D}{D}\right\} \leq \epsilon_{\delta}^*(\mathcal{P}, D) \leq \max\left\{0, \log(m-1) - \frac{n - D}{D} \left(1 - \delta\left(1 - \frac{D}{n}\right)^n\right)\right\}.
\]

(3) (Maximal information)

\[
\max\left\{0, \log m(1 - \frac{D}{\theta^* n})\right\}.
\]
\[ \leq e^*_M(\mathcal{P},D) \]
\[ \leq \max \left\{ 0, -\log \min_{P_x \in \mathcal{P}, y \in Y} \sum_{i=0}^{m} \left( m - 1 \frac{n-D}{D} \right)^{i-1} P_X(N_i(y)) \right\}. \]

(4) (Maximal leakage)
\[ \max \left\{ 0, \log m \left( 1 - \frac{D}{\theta^a n} \right) \right\} \]
\[ \leq e^*_M(\mathcal{P},D) \]
\[ \leq \max \left\{ 0, n \log m \left( 1 - \frac{D}{n} \right) \right\}. \]

(5) (Rényi differential privacy)
\[ \max \left\{ 0, \frac{\log n}{\theta^a (m+1)} \right\} \]
\[ \leq \varepsilon^*_{\alpha, DP}(\mathcal{P},D) \]
\[ \leq \max \left\{ 0, \frac{n}{\alpha-1} \log \frac{D}{n(m-1)} \left( m - 2 + \frac{(n-D)(m-1)}{D} \right) \right\}. \]

(6) (Sibson mutual information)
\[ \max \left\{ 0, \frac{\alpha - n}{\alpha - 1} \log m + \frac{\alpha}{\alpha - 1} \log \frac{1 - D/n}{\theta^a} \right\} \]
\[ \leq \varepsilon^*_{\alpha, M}(\mathcal{P},D) \]
\[ \leq \max \left\{ 0, n \log m \left( D^a (m-1)^{1-a} \right)^{1/(\alpha-1)} \right\}. \]

(7) (Mutual information)
\[ \max \left\{ 0, \theta^a \log \frac{1 - D/m}{n^a \theta^a (m-D)} \right\} \]
\[ \leq e^*_M(\mathcal{P},D) \]
\[ \leq \max \left\{ 0, n \log m \left( 1 - \frac{D}{n} \right) + \log \frac{D}{(m-1)(m-D)} \right\}. \]

where \( \eta = \max_{x \in X} P'(x) \) and \( P' = \arg\max_{P \in \mathcal{P}} \min_{x \in X} P(x) \).

In particular, \( e^*(\mathcal{P},D) = 0 \) if \( n(1 - 1/m) \leq D \leq 1 \) for all notions aforementioned.

Proof of Theorem 6. Based on Lemma 5, we have already known \( \mathcal{Q}_U \) is \((P,D)-\)valid for \( \frac{(n-m)}{m} \leq D \leq 1 \). Thus \( e^*(\mathcal{P},D) = 0 \) if \( \frac{m-n}{m} \leq D \leq 1 \) for all notions aforementioned. Otherwise, for \( 0 < D < \frac{m-n}{m} \) we consider the upper- and lower-bounds, respectively.

(1) First, let us consider the upper bound on the privacy-distortion functions. By Lemma 5, \( e^*(\mathcal{Q}_D) \) is upper-bounded by \( e^*(\mathcal{Q}_D) \). Hence, it suffices to calculate or upper-bound \( e^*(\mathcal{Q}_D) \). For simplicity, we denote \( \frac{(m-n)}{m} \) and \( \frac{(1-D)^a}{n} \) by \( A \) and \( B \), respectively. Note that if \( 0 < D < n(m-1)/m \) then \( m^{-n} < B < 1 < A \). Let us start with \( \alpha \)-type notions, i.e., \( e^*_{\alpha, DP}(\mathcal{P},D) \) and \( e^*_{\alpha, M}(\mathcal{P},D) \).

1.1) For \( e^*_{\alpha, DP}(\mathcal{P},D) \), we have
\[ \varepsilon^*_{\alpha, DP}(\mathcal{Q}_D) = \max_{\alpha} \frac{1}{\alpha - 1} \log \sum_y \mathcal{Q}_B(y|x) \mathcal{Q}_D^{1-\alpha}(y|x') \]
\[ = \max_{\alpha} \frac{1}{\alpha - 1} \log \left( \sum_{d(x,x')=1} A^{-d(x,y)} B \right) \]
\[ + \sum_{d(x,y)=d(x',y)+1} A^{-d(x,y)} A^{\alpha-1} \]
\[ + \sum_{d(x,y)=d(x',y)-1} A^{-d(x,y)} A^{\alpha-1} \]
\[ = \max_{\alpha} \frac{1}{(m-1)\alpha - 1} \log \left( \sum_{l=1}^{n} \frac{(m-1)}{(l-1)} (m-1)^{l-1} A^{-l} B \right) \]
\[ + \sum_{l=0}^{n-1} (m-1)^{l-1} A^{-l} A^{\alpha-1} \]
\[ + \sum_{l=0}^{n-1} A^{-l} A^{\alpha-1} \]
\[ = \max_{\alpha} \frac{1}{(m-1)\alpha - 1} \log \left( \sum_{l=1}^{n-1} (m-1)^{l-1} (m-2) A^{-l} B \right) \]
\[ + A^{-\alpha} B + A^{\alpha-1} \]
\[ = \frac{1}{\alpha} \log \frac{B}{A} \left( (m-2) + A^{-\alpha} + A^{\alpha} \right) \left( 1 + (m-1) A^{-1} \right)^{-1} \]
\[ = \frac{1}{\alpha-1} \log \frac{D}{n(m-1)} \left( m - 2 + \left( \frac{n-D}{m} \right)^a \right) \]
\[ + \left( \frac{n-D}{m} \right)^{a-1} \].

1.2) For \( e^*_{\alpha, M}(\mathcal{P},D) \), by the definition of Sibson \( \alpha \)-mutual information, we have
\[ \varepsilon^*_{\alpha, M}(\mathcal{Q}_D) = \max_{\alpha} \frac{\alpha}{\alpha-1} \log \sum_y \sum_x P(x) \mathcal{Q}_D^a(y|x)^{1/a} \]
\[ = \max_{\alpha} \frac{\alpha}{\alpha-1} \log B + \frac{\alpha}{\alpha-1} \log \max_{P \in \mathcal{P}} \sum_y \left( \sum_x P(x) A^{-\alpha(d(x,y))} \right)^{1/a} \]
\[ (4) \]

We now show that if \( \mathcal{P} \) contains the uniform distribution \( \mathcal{P}_U \), then \( e^*_{\alpha, M}(\mathcal{Q}_D) \) obtains the maximal value at \( P = \mathcal{P}_U \). Otherwise, \( e^*_{\alpha, M}(\mathcal{Q}_D) \) is upper-bounded by
\[ \frac{\alpha}{\alpha-1} \log B + \frac{\alpha}{\alpha-1} \log \sum_y \left( \sum_x P(x) A^{-\alpha(d(x,y))} \right)^{1/a} \]
\[ \cdot \]

To prove this claim, let us consider the following optimization problem.
\[ \min_{P} - \sum_x \left( \sum_y P(x) A^{-\alpha(d(x,y))} \right)^{1/a} \]
\[ \text{subject to} \]
\[ (c1) P(x) \geq 0, \]
\[ (c2) \sum_x P(x) = 1. \]

It is obvious that the feasible region of (5) consists of all the probability distributions. By the convexity of Sibson mutual information (see Lemma 21 in Appendix A), this
optimization problem is convex. To find the optimal value, we utilize the Karush-Kuhn-Tucker (KKT) conditions [2]. The KKT conditions for convex problem (5) are as follows.

a) For \( x \in \mathcal{X} \),
\[
\nabla_{P(x)} L(P(x), \lambda, \alpha_x) = \lambda - \sum_y \frac{1}{\alpha} \left( \sum_x P(x) A^{-\alpha d(x, y)} \right) + \alpha_x = 0
\]
where \( L(P(x), \lambda, \alpha_x) \) is the Lagrangian defined as follows.
\[
L(P(x), \lambda, \alpha_x) = -\sum_y \left( \sum_x P(x) A^{-\alpha d(x, y)} \right) + \lambda (\sum_x P(x) - 1) - \sum_x \alpha_x P(x).
\]
b) \( \sum_x P(x) = 1 \)
c) \( \alpha P(x) = 0, \alpha_x \geq 0, P(x) \geq 0. \)
It is easy to verify that
\[
(P = P_U, \lambda = \frac{1}{m} m(n-1) (1 + (m-1) A^{-\alpha})^n, \alpha_x = 0)
\]
satisfies the above KKT conditions. Note that feasible solution of KKT conditions is also optimal for convex problem [2]. Thus
\[
\tilde{\epsilon}_{\alpha, M}(Q_D) = \lim_{\alpha \to 0} \tilde{\epsilon}_{\alpha, M}(Q_D)
\]
1.3) For \( \tilde{\epsilon}_{DP} \), let \( c_1 = \frac{D}{m-1} \) and \( c_2 = n-D, \) then \( 0 < c_1 < n/m < c_2 < n \) and we have
\[
\tilde{\epsilon}_{DP}(Q_D) = \lim_{\alpha \to \infty} \tilde{\epsilon}_{\alpha, DP}(Q_D)
\]
1.4) For \( \tilde{\epsilon}_{ML}(Q_D) \), by the relation between Sibson mutual information and maximal leakage (see Lemma 20 in Appendix A), we have
\[
\tilde{\epsilon}_{ML}(Q_D) = \lim_{\alpha \to \infty} \tilde{\epsilon}_{\alpha, M}(Q_D)
\]
1.5) For \( \tilde{\epsilon}_{M}(Q_D) \), by the relation between Sibson mutual information and mutual information (see Lemma 20 in Appendix A), we have
\[
\tilde{\epsilon}_{M}(Q_D) = \lim_{\alpha \to 0} \tilde{\epsilon}_{\alpha, M}(Q_D)
\]

1.6) For \( \tilde{\epsilon}_{\delta}(Q_D) \), by Definition 3,
\[
\epsilon_{\delta}(Q_D) = \max_{y \in \mathcal{Y}, x \in \mathcal{X}} \log \left( \frac{Q_D(y|x) - \delta}{Q_D(y|x')} \right)
\]
Note that \( |d(x, y) - d(x, y')| \leq d(x, x') \).

1.7) For \( \tilde{\epsilon}_{M1}(Q_D) \), by Definition 3,
\[
\epsilon_{M1}(Q) = \max_{P \in \mathcal{P}, x \in \mathcal{X}} \log \left( \frac{Q_D(y|x)}{\sum_{x \in \mathcal{X}} P_X(x) Q_D(y|x)} \right)
\]
(2) Second, let us consider the lower bounds on the privacy-distortion functions. For any mechanism \( Q \) in \( Q(\mathcal{P}, D) \), it is \( (P, D) \)-valid for any \( P \in \mathcal{P} \). Thus, for
\[
P' = \arg \max P(x) \text{ and } \theta' = m^n \min_P P(x),
\]
we have
\[
\sum_{x, y} m^{-n} \theta' Q(y|x) d(x, y)
\]
\[
\leq \sum_{x, y} P'(x) Q(y|x) d(x, y)
\]
\[
= \mathbb{E}_{P', Q}[d(X, Y)] \leq D.
\]
Therefore, we have
\[
\sum_{x, y} Q(y|x) d(x, y) \leq D m^n \theta'^{-1}.
\]
\[
\begin{align*}
\text{Contradiction! Thus } & \quad \text{we have} \\
& \quad n(m - 1) \sum_{x \in N_{l-1}(y)} Q(y|x) \\
& \quad < e^\epsilon \sum_{l=1}^n \left( \sum_{x \in N_{l-1}(y)} Q(y|x) \right) + lN_l \delta \\
\Rightarrow & \quad n(m - 1) \sum_{x \in N_{l-1}(y)} Q(y|x) \\
& \quad < e^\epsilon \sum_{x \in N_{l-1}(y)} Q(y|x) \\
& \quad + (m - 1)(l - 1) \sum_{x \in N_{l-1}(y)} Q(y|x) + n(m - 1)m^{n-1} \delta \\
\Rightarrow & \quad n(m - 1) \sum_{x \in N_{l-1}(y)} Q(y|x) \\
& \quad < e^\epsilon \sum_{l=1}^n \left( \sum_{x \in N_{l-1}(y)} Q(y|x) \right) + lN_l \delta \\
& \quad + (m - 1)(l - 1) \sum_{x \in N_{l-1}(y)} Q(y|x) + n(m - 1)m^{n-1} \delta \\
\Rightarrow & \quad n(m - 1) \sum_{x \in N_{l-1}(y)} Q(y|x) \\
& \quad < e^\epsilon \sum_{x \in N_{l-1}(y)} Q(y|x) \\
& \quad + (m - 1)(l - 1) \sum_{x \in N_{l-1}(y)} Q(y|x) + n(m - 1)m^{n-1} \delta \\
\Rightarrow & \quad \sum_{x,y} Q(y|x)d(x,y) > n(m - 1)m^{n-1}(1 - m^{n-1}) \delta \\
& \quad = \frac{Dm^n}{\theta^*}.
\end{align*}
\]

Contradiction! Thus \( \varepsilon_{sDP, \Delta}(P,D) \geq \log(m(1 - \frac{D\Delta}{\theta^*})). \)

2.4 For \( \varepsilon_{ML} \), assume there exists a \( Q \in Q(\mathcal{P},D) \) such that \( \varepsilon_{ML}(Q) < \epsilon := \log(m(1 - \frac{D\Delta}{\theta^*})), \) then
\[
\sum_{y} \max_{x} Q(y|x) < e^\epsilon \\
\Rightarrow \sum_{y} \max_{x \in N(y)} Q(y|x) < e^\epsilon \text{ for } 0 \leq l \leq n \\
\Rightarrow \sum_{x,y} Q(y|x)d(x,y) > nm^n(1 - \frac{e^\epsilon}{m}) = \frac{Dm^n}{\theta^*}.
\]

Contradiction! Thus \( \varepsilon_{ML}(P,D) \geq \log(m(1 - \frac{D\Delta}{\theta^*})). \)

2.5 For \( \varepsilon_{\Delta,DP} \), assume there exists a \( Q \in Q(\mathcal{P},D) \) such that \( \varepsilon_{\Delta,DP}(Q) < \epsilon := \log \left( \frac{\log(m(1 - \frac{D\Delta}{\theta^*}))}{\theta^*} \right) - \frac{1}{\alpha-1} \log mn^{n-1}. \)

For any vector \( y \in X \), we can always find \( x \in N_{l-1}(y) \) and \( x' \in N_{l}(y) \) for any positive integer \( l \) no larger than \( n \). Since \( x \) and \( x' \) are two neighbors, then by the probability preservation property of Rényi divergence (see Lemma 22 in Appendix A), we have
\[
Q(y|x) < (e^\epsilon Q(y|x')) \frac{a_{l-1}}{a_l} \\
\Rightarrow \sum_{x} \max_{y \in N_{l-1}(y)} Q(y|x) < e^\epsilon \\
\Rightarrow \sum_{x} \max_{y \in N_{l}(y)} Q(y|x) < e^\epsilon \text{ for } 0 \leq l \leq n \\
\Rightarrow \sum_{x,y} Q(y|x)d(x,y) > nm^n(1 - \frac{e^\epsilon}{m}) = \frac{Dm^n}{\theta^*}.
\]

Contradiction! Thus \( \varepsilon_{ML}(P,D) \geq \log(m(1 - \frac{D\Delta}{\theta^*})). \)
By Hölder inequality,
\[ n(m-1)m^n < \left( e^\epsilon \sum_{x,y} Q(y|x)d(x,y) \right)^{\frac{\alpha-1}{\alpha}} \left( \sum_{x,y} d(x,y) \right)^{\frac{1}{\alpha}} + (m-1) \sum_{x,y} Q(y|x)d(x,y) \]
\[ = (n(m-1)m^{2n-1})^{\frac{1}{\alpha}} \left( e^\epsilon \sum_{x,y} Q(y|x)d(x,y) \right)^{\frac{\alpha-1}{\alpha}} + (m-1) \sum_{x,y} Q(y|x)d(x,y). \]
However, the inequality (6) implies that
\[ \left( n(m-1)m^{2n-1} \right)^{\frac{1}{\alpha}} \left( e^\epsilon \sum_{x,y} Q(y|x)d(x,y) \right)^{\frac{\alpha-1}{\alpha}} + (m-1) \frac{Dm^n}{\theta^*} = n(m-1)m^n. \]

Contradiction! Thus
\[ \epsilon_{\alpha,D_P}(P,D) \geq \log \frac{(m-1)(n-D)^{\frac{n}{\alpha-1}}}{D} - \frac{1}{\alpha-1} \log mn^{n-1}. \]

2.7) We restrict the input distribution to
\[ P' = \arg \max_{P \in \mathcal{P}} \min_{x \in X} P(x). \]
By Lemma 5, \( \epsilon'(P,D) = \epsilon'(P',D). \) To evaluate \( \epsilon'(P',D), \) let us consider the following convex optimization problem.
\[ \min_{Q(y|x)} \sum_{x,y} P'(x)Q(y|x) \log \frac{Q(y|x)}{\sum_x P_U(x)Q(y|x)} \]
subject to
\[ (c1) \sum_{x,y} P'(x)Q(y|x)d(x,y) \leq D, \]
\[ (c2) \sum_y Q(y|x) = 1, \]
\[ (c3) Q(y|x) \geq 0. \]
All the conditions in (7) imply feasible \( Q \) is \( (P',D) \)-valid and thus the optimal value is \( \epsilon'(P',D). \) It is difficult to solve (7) directly. Instead, let us consider the following relaxed optimization problem.
\[ \min_{Q(y|x)} \sum_{x,y} P'(x)Q(y|x) \log \frac{Q(y|x)}{\sum_x \eta Q(y|x)} \]
subject to
\[ (c1) \sum_{x,y} Q(y|x)d(x,y) \leq \frac{Dm^n}{\theta^*}, \]
\[ (c2) \sum_y Q(y|x) = 1, \]
\[ (c3) Q(y|x) \geq 0 \]
where \( \eta = \max_{x \in X} P'(x). \) Note that (c1) in (7) implies (c1) in (8). Thus, the optimal value of (8) is not greater than \( \epsilon'(P',D). \) By the convexity of mutual information (see Lemma 21 in Appendix A), optimization problem (8) is convex. To find the optimal value, we utilize the KKT conditions again. The KKT conditions for convex problem (8) are as follows.

a) For any \( x, y \in X, \]
\[ \nabla Q L(Q, \lambda, \alpha_x, \beta_x) = 0 \]
where \( L(Q, \lambda, \alpha_x, \beta_x) \) is the Lagrangian of (8) defined as
\[ L(Q, \lambda, \alpha_x, \beta_x) := m^{-n}\theta^* \sum_{x,y} Q(y|x) \log \frac{Q(y|x)}{\sum_x \eta Q(y|x)} + \lambda \left( \sum_{x,y} Q(y|x)d(x,y) - \frac{Dm^n}{\theta^*} \right) \]
\[ + \sum_x \alpha_x \left( \sum_{x,y} Q(y|x) - 1 \right) - \sum_{x,y} \beta_x Q(y|x). \]

b) \( \lambda \left( \sum_{x,y} Q(y|x)d(x,y) - \frac{Dm^n}{\theta^*} \right) = 0, \lambda \geq 0 \)
\[ \sum_{x,y} Q(y|x)d(x,y) \leq \frac{Dm^n}{\theta^*}. \]
c) For \( x \in X, \sum_y Q(y|x) = 1. \)
d) For any \( x, y \in X, \beta_x Q(y|x) = 0, \beta_x \geq 0, \quad Q(y|x) \geq 0. \)
It is easy to verify that \( \left( Q = Q_{D/\theta^*}, \lambda = \frac{\theta^*}{m^n} \log \frac{(m-1)(n-D)}{D}, \alpha_x = \frac{n\theta^*}{m^n} \log \frac{n!}{m^{(n-1)}(n-1)!}, \beta_x = 0 \right) \) satisfies the KKT conditions. Thus
\[ \epsilon_{\alpha,D_P}(P,D) \geq \epsilon_{\alpha,M}(P',D) \]
Theorem 7. Let $\mathcal{P}$ be a source set over $X = \{1, \ldots, m\}$ in Class I, then we have the following results on the privacy-distortion function with respect to the privacy notions in Definition 3.

1. (Differential privacy)
$$
\tilde{\varepsilon}_{DP}(\mathcal{P}, D) = \max \left\{ 0, \log \frac{(m-1)(n-D)}{D} \right\}.
$$

2. (Approximate differential privacy)
$$
\max \left\{ 0, \log (m-1) \frac{n-\delta m^{\alpha-1}}{D} \right\} \leq \tilde{\varepsilon}_D^\alpha(\mathcal{P}, D) \leq \max \left\{ 0, \log (m-1) \frac{n-D}{D} (1-\delta (1-\frac{D}{n})^{-\alpha}) \right\}.
$$

3. (Maximal information)
$$
\max \left\{ 0, \log \frac{m}{n} \frac{m-D}{n} \right\} \leq \tilde{\varepsilon}_M^\alpha(\mathcal{P}, D) \leq \max \left\{ 0, \tilde{\varepsilon}_M(\mathcal{Q}(D)) \right\},
$$
where
$$
\tilde{\varepsilon}_M(\mathcal{Q}(D)) = -\log \min_{P_X \in \mathcal{P}, y \in \mathcal{Y}} \left\{ \sum_{i=1}^{m} \left( \frac{m-1}{n-D} \right)^{-1} P_X(N_i(y)) \right\}.
$$

4. (Maximal leakage)
$$
\max \left\{ 0, \log \frac{m}{n} \frac{m-D}{n} \right\} \leq \tilde{\varepsilon}_{ML}^\alpha(\mathcal{P}, D) \leq \max \left\{ 0, \log \frac{m}{n} \frac{m-D}{n} \right\}.
$$

5. (Rényi differential privacy)
$$
\max \left\{ 0, \log (m-1) (1-n^{-\alpha}(\alpha-1)) \right\} \leq \tilde{\varepsilon}_{\alpha,DP}(\mathcal{P}, D) \leq \max \left\{ 0, \frac{1}{\alpha-1} \log mn^{\alpha-1} \right\}.
$$

6. (Sibson mutual information)
$$
\max \left\{ 0, \tilde{\varepsilon}_M(\mathcal{P}, D) \left. \right| \frac{1}{\alpha-1} \log m^{\alpha-n} (1-D)^{\alpha} \right\}
$$

It is seen in Theorem 6 that the difference between lower- and upper-bounds is decreasing in $\theta^\alpha$. In fact, the difference vanishes for some special cases. Assume that the source set $\mathcal{P}$ in Theorem 6 contains the uniform distribution $P_U$. Then $\arg \max_{P \in \mathcal{P}} \min_{x \in X} P(x) = P_U$, which implies $\theta^\alpha = 1$ and $\eta = m^{-n}$. Hence, we have the following results on privacy-distortion functions.

In particular, $\tilde{\varepsilon}^\alpha(\mathcal{P}, D) = 0$ if $\frac{n(m-1)}{m} \leq D \leq 1$ for all notions aforementioned.

IV. LOCAL PRIVACY FRAMEWORK

In this section, we assume $|\mathcal{X}| = m > 0$ and consider the local privacy framework, i.e., the privacy mechanism is applied before data uploading (see Fig. 2). The distance metric $d(x, y)$ is actually the discrete distance, i.e.,
$$
d(x, y) = \begin{cases} 
0, & x = y \\
1, & x \neq y.
\end{cases}
$$

Hence, for any $x \neq x'$, $x$ and $x'$ are neighborhood. Given $0 < D \leq 1$, $Q$ belongs to $Q(\mathcal{P}, D)$ if and only if $\sum_{i=1}^{m} P_Q(i|j) \geq 1-D$. We claim that a good choice of output space is the synthetic one i.e., $\mathcal{Y} = \hat{\mathcal{X}}$ (See Lemma 24 in Appendix C).

A. Source Set of Class I

Let us start with the source set of Class I, for which we take $n = 1$ into Theorem 7 and get the following results directly.

Proposition 8. Let $\mathcal{P}$ be a source set over $X = \{1, \ldots, m\}$ in Class I, then we have the following results on the privacy-distortion function with respect to the privacy notions in Definition 3.

1. (Differential privacy)
$$
\varepsilon_{DP}(\mathcal{P}, D) = \begin{cases} 
\log \frac{(m-1)(1-D)}{D}, & 0 < D < \frac{m-1}{m} \\
0, & \frac{m-1}{m} \leq D \leq 1;
\end{cases}
$$

2. (Approximate differential privacy)
$$
\varepsilon_{\sigma}(\mathcal{P}, D) = \begin{cases} 
\log \frac{(m-1)(1-D)}{D} - \frac{D}{(m-1)} \left( 1 - \delta \frac{D}{m} \right)^{1-\alpha}, & 0 < D < (1-\delta) \frac{m-1}{m} \\
0, & (1-\delta) \frac{m-1}{m} \leq D \leq 1;
\end{cases}
$$

3. (Maximal information)
$$
\max \left\{ 0, \varepsilon_M(\mathcal{P}, D) \left. \right| \frac{1}{\alpha-1} \log m^{\alpha-n} (1-D)^{\alpha} \right\}
$$

Fig. 2. Local Privacy Framework: the individuals do not trust the data server, so the privacy mechanism is applied before data uploading.

$$
\leq \varepsilon^\alpha_M(\mathcal{P}, D) \left. \right| \frac{1}{\alpha-1} \log m^{\alpha-n} (1-D)^{\alpha} \right\}
$$

$$
\leq \max \left\{ 0, \tilde{\varepsilon}_M(\mathcal{Q}(D)) \right\}
$$

$$
\text{where } \varepsilon^\alpha_M(\mathcal{Q}(D)) = n \log m((n-D)^{\alpha} + D^{\alpha}(m-1)^{-\alpha} - \frac{n}{\alpha-1}) n \log n.
$$

$$
\tilde{\varepsilon}_M(\mathcal{P}, D) = \max \left\{ 0, n \log m(1 - D) + D \log \left( \frac{D}{(m-1)(n-D)} \right) \right\}.
$$

In particular, $\tilde{\varepsilon}^\alpha(\mathcal{P}, D) = 0$ if $\frac{n(m-1)}{m} \leq D \leq 1$ for all notions aforementioned.
where \( \epsilon_{\mathcal{M}}(Q_D) = \log \frac{1-D}{p_\pi(1-D - \frac{m}{m+1})} \) and \( P^*_p = \min_{1 \leq i \leq m} \{P_i | P \in \mathcal{P}\}; 

(4) (Maximal leakage)

\[
\epsilon_{\mathcal{M}}^*(\mathcal{P}, D) = \begin{cases} 
\log m(1-D), & 0 < D < \frac{m-1}{m}, \\
0, & \frac{m-1}{m} \leq D \leq 1;
\end{cases}
\]

(5) (Rényi differential privacy)

\[
\max \{0, \log \left( \frac{(m-1)(1-D)^{a/(\alpha-1)}}{D} \right) \} \leq \epsilon_{\alpha, DP}^*(\mathcal{P}, D) \leq \max \{0, \epsilon_{\alpha, DP}(\mathcal{Q}(D)) \},
\]

where

\[
\epsilon_{\alpha, DP}(\mathcal{Q}(D)) = \frac{1}{\alpha-1} \log \frac{D}{m-1} (m-2 + \left( \frac{(1-D)(m-1)}{D} \right)^{\alpha} + \left( \frac{(1-D)(m-1)}{D} \right)^{1-\alpha});
\]

(6) (Sibson mutual information)

\[
\max \left\{ \epsilon_{\mathcal{M}}^*(\mathcal{P}, D), \log m(1-D)^{\frac{a}{\alpha-1}} \right\} \leq \epsilon_{\alpha, M}(\mathcal{P}, D) \leq \max \{0, \epsilon_{\alpha, M}(\mathcal{Q}(D)) \},
\]

where \( \epsilon_{\alpha, M}(\mathcal{Q}(D)) = \log m(1-D)^{\alpha} + D^{\alpha}(m-1)^{1-\alpha}; \)

(7) (Mutual information)

\[
\epsilon_{\mathcal{M}}(\mathcal{P}, D) = \left\{ m(1-D) \left( \frac{D}{(m-1)(1-D)} \right)^D, \frac{m-1}{m} \leq D \leq 1. \right\}
\]

B. Source Set of Class II

By taking \( n = 1 \) into Theorem 6, one can obtain the properties of privacy-distortion functions for any source set \( \mathcal{P} \). However, if \( \mathcal{P} \) does not contain the uniform distribution \( P_U \), then Theorem 6 fails to give the lower bounds sometimes. Thus, in this subsection, we introduce a different method to address the PUT problem for the source set of Class II, and we mainly study the differential privacy, approximate differential privacy, and maximal leakage-distortion trade-off. First, we consider the scenario in which the distribution on \( X \) is given.

**Theorem 9.** Let \( P \) be a distribution with full support on \( X \) such that \( 0 \leq P_m \leq P_{m-1} \leq \cdots \leq P_1 \leq 1 \). Let \( D^{(k)} = \sum_{j=m-k+1}^{m} P_j \) for \( 0 \leq k \leq m \). Then

\[
\epsilon_{\alpha}(P, D) = \max \left\{ 0, \min_{D > (1-\delta)D^{(k-1)}} \log \left( \frac{(m-k)(1-D-\delta)}{D - (1-D)D^{(k-1)}} \right) \right\}.
\]

Before proving Theorem 9, we need introduce the following useful lemmas.

**Lemma 10.** For \( 0 < \delta < 1 \), \( \epsilon_{\delta}(P, D) = 0 \) if and only if \( D \geq (1-\delta)D^{(m-1)} \).

**Proof of Lemma 10.** On one hand, if \( \epsilon_{\delta}(P, D) = 0 \), then there exists a mechanism \( Q \) such that \( \epsilon_{\delta}(Q) = 0 \) and \( \mathbb{E}_{P_i Q}[d(x,y)] \leq D \). Thus, \( |Q(j|i) - Q(j|k)| \leq \delta \) for any \( i, j, k \). Hence,

\[
\mathbb{E}_{P_i Q}[d(x,y)] = \sum_{j=1}^{m} P_j (1 - Q(j|j))
\]

On the other hand, if \( D \geq (1-\delta)D^{(m-1)} \), we define mechanism \( Q_\delta \) as follows.

\[
Q_\delta = \begin{pmatrix}
Q_\delta(1|1) & Q_\delta(1|2) & \cdots & Q_\delta(1|m) \\
Q_\delta(2|1) & Q_\delta(2|2) & \cdots & Q_\delta(2|m) \\
\cdots & \cdots & \cdots & \cdots \\
Q_\delta(m|1) & Q_\delta(m|2) & \cdots & Q_\delta(m|m)
\end{pmatrix}
\]

Note that \( \epsilon_{\delta}(Q_\delta) = 0 \) and \( \mathbb{E}_{P_i Q_\delta}[d(x,y)] = (1-\delta)D^{(m-1)} \leq D \). Thus, \( \epsilon_{\delta}(P, D) = 0 \).

**Lemma 11.** For \( 0 < \delta < 1 \), if \( \epsilon_{\delta}(P, D) > 0 \), then there exists a mechanism \( Q \in \mathcal{Q}^*_\delta(\mathcal{P}, D) \) such that

(1) \( \delta \leq \mathbb{E}Q(m|m) \leq \mathbb{E}Q(m-1|m-1) \leq \cdots \leq \mathbb{E}Q(1|1) \leq 1 \);

(2) \( Q(j|j) = e^{\epsilon_{\delta}(P, D)} Q^*(j|j) + \delta \) for \( 2 \leq j \leq m \).

To prove Lemma 11, we improve the coloring method introduced by Kalantari et al. [18]. The details of the proof are provided in Appendix B, C and D.

**Proof of Theorem 9.** For \( D \geq (1-\delta)D^{(m-1)} \), Lemma 10 proves the statement.

For \( 0 < D < (1-\delta)D^{(m-1)} \), let \( Q^* \) be a mechanism satisfying the conditions in Lemma 11. Then \( Q^*(j|j) = e^{\epsilon_{\delta}(P, D)} (Q^*(j|j) - \delta) \) for \( 2 \leq j \leq m \), which implies

\[
\epsilon_{\delta}(P, D) = \log \frac{\sum_{j=2}^{m} (Q^*(j|j) - \delta)}{1 - Q^*(1|1)}.
\]

Let us consider the following optimization problem,

\[
d_1 := \min \left\{ \frac{\sum_{j=2}^{m} (P_j \alpha_j - \delta)}{1 - \alpha_1} \right\}
\]

subject to

(1) \( \sum_{j=1}^{m} P_j \alpha_j \geq 1 - D, \)

(2) \( \sum_{j=1}^{m} (\alpha_j - \delta) \geq 1 - \delta, \)

(3) \( \alpha_j \leq \alpha_{j-1} \) for \( 1 \leq j \leq m + 1 \).
where \(\alpha_0 = 1\) and \(\alpha_{m+1} = \delta\). Note that \(\{Q'(j|i)\}_{1 \leq j \leq m}\) is a feasible solution of (9), and hence \(e^*_j(P,D) \geq d_1\). Conversely, let \(\{\alpha^*_1, \alpha^*_2, \ldots, \alpha^*_m\}\) be the optimal solution of (9). We define the following mechanism.

\[
Q(j|i) = \begin{cases} 
\frac{a_i^*}{\sum_j (a_j^* - \delta)} & \text{if } i = j, \\
\frac{a_i^*}{\sum_j (a_j^* - \delta)} & \text{if } i \neq j.
\end{cases}
\]

For any \(j \in [1, m]\), \(\sum_{i=1}^m Q(j|i) = 1\), that is, \(Q\) is a conditional probability. The first condition in (9) implies that \(Q\) is within \(Q(P,D)\). The other two imply that

\[
e^*_j(Q) = \max_{1 \leq i \leq m} \frac{\sum_{j\neq i} (a_j^* - \delta)}{1 - a_i^*} = \frac{\sum_{j\neq i} (a_j^* - \delta)}{1 - a_i^*} = d_1.
\]

Thus, we have \(d_1 = e^*_j(P,D)\). Moreover, condition (c1) in (9) combined with the assumption \(P_j\) is non-increasing in \(j\) implies that

\[
a_1 = \sum_{j=1}^m P_j a_1 \geq \sum_{j=1}^m P_j a_j \geq 1 - D.
\]

Thus,

\[
d_1 = \min_{1 \leq i \leq m} \min_{a_1, a_2, \ldots, a_m} \log \frac{\sum_{j=1}^m (a_j - \delta)}{1 - a_1},
\]

where \(d_1 := 0\) for \(a_1 = 1\). For fixed \(a_1 \in [1 - D, 1]\), consider the following optimization problem,

\[
d_2(a_1) := \min_{a_2, a_3, \ldots, a_m} \sum_{j=2}^m (a_j - \delta)
\]

subject to

1. \(\sum_{j=2}^m P_j a_j \geq 1 - D - P_1 a_1\),
2. \(\sum_{j=2}^m (a_j - \delta) \geq 1 - a_1\),
3. \(a_j \leq a_{j-1}\) for \(2 \leq j \leq m + 1\).

The optimal values of optimization problem (9) and (10) satisfy \(d_1 = \min_{1 \leq D \leq a_1 \leq 1} \log \frac{d_1(a_1)}{1 - a_1}\). Denote \(\beta_j := a_j - a_{j+1}\) for \(1 \leq j \leq m - 1\). Then problem (10) can be rewritten as the following optimization problem.

\[
d_3(a_1) := \max_{\beta_1, \beta_2, \ldots, \beta_{m-1}} \sum_{j=1}^{m-1} (m - j) \beta_j
\]

subject to

1. \(\sum_{j=1}^{m-1} D^{(m-j)} \beta_j \leq a_1 - 1 + D\),
2. \(\sum_{j=1}^{m-1} (m - j) \beta_j \leq m(a_1 - \delta) - 1 + \delta\),
3. \(\sum_{j=1}^{m-1} \beta_j \leq a_1 - \delta\),
4. \(\beta_j \geq 0\) for \(1 \leq j \leq m - 1\).

Moreover, \(d_2(a_1) = (m - 1)(a_1 - \delta) - d_3(a_1)\). The dual of linear program (11) is the minimization linear program as below.

\[
\tilde{d}_3(a_1) := \min_{\gamma_1, \gamma_2} (a_1 - 1 + D) \gamma_1 + (m(a_1 - \delta) - 1 + \delta) \gamma_2 + (a_1 - \delta) \gamma_3
\]

subject to

1. \(D^{(i)} \gamma_1 + j \gamma_2 + \gamma_3 \geq j\) for \(1 \leq j \leq m - 1\)
2. \(\gamma_1, \gamma_2, \gamma_3 \geq 0\).

By strong duality theorem, we have \(d_3(a_1) = \tilde{d}_3(a_1)\). Fix \(\gamma_2 \geq 0\), we define the following linear program.

\[
d_4(a_1, \gamma_2) := \min_{\gamma_1 \gamma_3} (a_1 - 1 + D) \gamma_1 + (a_1 - \delta) \gamma_3
\]

subject to

1. \(D^{(i)} \gamma_1 + \gamma_3 \geq j(1 - \gamma_2)\) for \(1 \leq j \leq m - 1\),
2. \(\gamma_1, \gamma_3 \geq 0\).

Hence

\[
d_3(a_1) = \min_{\gamma_2 \geq 0} \{d_4(a_1, \gamma_2) + (m(a_1 - \delta) - 1 + \delta) \gamma_2\}.
\]

Case I. \(\gamma_2 \geq 1\). Let us consider the optimization problem (13), where \(d_4(a_1, \gamma_2) = \min_{\gamma_1 \gamma_3} (a_1 - 1 + D) \gamma_1 + (a_1 - \delta) \gamma_3\) with \(\gamma_1, \gamma_3 \geq 0\). Thus, \(d_4(a_1, \gamma_2) = 0\).

Case II. \(0 \leq \gamma_2 < 1\). Let \(l_j\) denote the line \(\gamma_3 = -D^{(j)} \gamma_1 + (j - 1) \gamma_2\) for \(1 \leq j \leq m - 1\). Assume that \(l_j\) intersects with \(l_{j-1}\) at \(Z_{ij}(u_{ij}, v_{ij})\) for \(i \neq j\). It is easy to calculate the coordinate of the intersection point, i.e.,

\[
\begin{align*}
u_{ij} &= \frac{(1 - \gamma_2) (j - i)}{D^{(i)} - D^{(j)}} \gamma_3, \\
\gamma_3 &= \frac{(1 - \gamma_2) (j - i) (D^{(i)} - D^{(j)})}{D^{(i)} - D^{(j)}}.
\end{align*}
\]

We claim that for any \(j \in [1, m - 1]\),

\[
\begin{align*}
u_{1j} &> u_{2j} > \cdots > u_{j-1,j} > u_{j+1,j} > \cdots > u_{m-1,j} > 0, \\
0 < v_{1j} < v_{2j} < \cdots < v_{j-1,j} < v_{j+1,j} < \cdots < v_{m-1,j}.
\end{align*}
\]

Therefore, the feasible region of (13) is unbounded, and corner points are located from top left to bottom right as follows.

\[
(0, (m - 1)(1 - \gamma_2)) \rightarrow Z_{m-1,m} \rightarrow \cdots Z_{2,3} \rightarrow Z_{1,2} \rightarrow \left(\frac{1 - \gamma_2}{P_m}, 0\right).
\]

We now prove (14). By the definition of \(D^{(k)}\), \(D^{(k)}\) is increasing in \(k\), and hence \(u_{ij}\) is positive. If \(j > i\) then \(f(D^{(i)} - iD^{(j)}) = (j - i)D^{(j)} - iD^{(i)} = \sum_{a=m-i}^{m} \sum_{b=m-j}^{m} (P_a - P_b) < 0\). Thus \(v_{ij}\) is positive.

For \(k \neq j\) and \(1 \leq i < k \leq m\),

\[
u_{kj} - u_{ij} = \frac{k - j}{D^{(k)} - D^{(j)}} - \frac{i - j}{D^{(i)} - D^{(j)}} = \frac{(k - i)D^{(j)} + (i - k)D^{(i)} + (i - j)(D^{(i)} - D^{(j)})}{(D^{(k)} - D^{(j)})(D^{(i)} - D^{(j)})} = \frac{(k - i)(D^{(i)} - D^{(j)}) - (i - j)(D^{(k)} - D^{(j)})}{(D^{(k)} - D^{(j)})(D^{(i)} - D^{(j)})} = \begin{cases} \frac{\sum_{a=m-i}^{m} \sum_{b=m-j}^{m} (P_a - P_b)}{(D^{(k)} - D^{(j)})(D^{(i)} - D^{(j)})} & \text{if } k > i > j, \\
\frac{\sum_{a=m-j}^{m} \sum_{b=m-k}^{m} (P_a - P_b)}{(D^{(k)} - D^{(j)})(D^{(i)} - D^{(j)})} & \text{if } j > k > i, \\
\frac{\sum_{a=m-k}^{m} \sum_{b=m-j}^{m} (P_a - P_b)}{(D^{(k)} - D^{(j)})(D^{(i)} - D^{(j)})} & \text{if } k > j > i.
\end{cases}
\]
Thus, \( a_{ij} \) is decreasing in \( i \). On the other hand, line \( l_j \) with negative slope \( -D^{(j)} \) passes point \( Z_{ij} \) for any \( 1 \leq i \leq m \), which implies \( v_{ij} \) is increasing in \( i \).

The slope of the objective function in (13) determines which corner point will be reached last. Hence, we divide \([1-D, \infty]\) into \( m \) closed intervals. For \( \frac{1-D-D^{(k-1)}}{1-D^{(k-1)}} \leq \alpha_1 \leq \frac{1-D-D^{(k)}}{1-D^{(k)}} \), \( k = 1, 2, \cdots, m \), we have \(-D^{(k)} \leq \frac{1-D-D^{(k-1)}}{1-D^{(k-1)}} \leq \frac{1-D-D^{(k)}}{1-D^{(k)}} \). Thus, \( Z_{k-1,k} \) is optimal solution of (13) and

\[
d_4(\alpha_1, \gamma_2) = \frac{1}{P_{m-k+1}} \left( (\alpha_1 - 1 + D) + (\alpha_1 - \delta)((k-1)P_{m-k+1} - D^{(k-1)}) \right),
\]

for \( 0 \leq \gamma_2 < 1 \). Let

\[
c_k(\alpha_1) = \frac{1}{P_{m-k+1}} \left( (\alpha_1 - 1 + D) + (\alpha_1 - \delta)((k-1)P_{m-k+1} - D^{(k-1)}) \right)
\]

Then for \( \frac{1-D-D^{(k-1)}}{1-D^{(k-1)}} \leq \alpha_1 \leq \frac{1-D-D^{(k)}}{1-D^{(k)}} \) with \( k \) satisfying \( D > (1-\delta)D^{(k-1)} \), we have

\[
d_3(\alpha_1) = \min_{\gamma_2 \geq 0} \{ d_4(\alpha_1, \gamma_2) + (m(\alpha_1 - \delta) - 1 + \delta)\gamma_2 \}
\]

\[
= \min_{0 \leq \gamma_2 \leq 1} \{ c_k(\alpha_1)(1-\gamma_2) + (m(\alpha_1 - \delta) - 1 + \delta)\gamma_2 \}
\]

\[
= \min \{ c_k(\alpha_1), m(\alpha_1 - \delta) - 1 + \delta \}.
\]

Thus,

\[
d_3(\alpha_1) = d_3(\alpha_1) = \min \{ c_k(\alpha_1), m(\alpha_1 - \delta) - 1 + \delta \}
\]

\[
\Rightarrow d_2(\alpha_1) = \max \{ 1 - \alpha_1, (m-1)(\alpha_1 - \delta) - c_k(\alpha_1) \}
\]

\[
\Rightarrow d_1 = \min \{ 1, \frac{\log \max \left\{ 1, \frac{(m-1)(\alpha_1 - \delta) - c_k(\alpha_1)}{1-\alpha_1} \right\}}{\alpha_1} \}
\]

\[
\Rightarrow \epsilon^*_\delta(P,D) = \min \left\{ 0, \log \left( \frac{m-k}{D - (1-\delta)D^{(k-1)}} \right) \right\}.
\]

In a summary, we have that if \( 0 < D < (1-\delta)D^{(1)} \), then

\[
\epsilon^*_\delta(P,D) = \log \left( \frac{m-k}{D - (1-\delta)D^{(k-1)}} \right);
\]

if \( (1-\delta)D^{(k-1)} \leq D < (1-\delta)D^{(k)} \) for \( 2 \leq k \leq m-1 \), then

\[
\epsilon^*_\delta(P,D) = \min \left\{ 0, \log \left( \frac{m-j}{D - (1-\delta)D^{(j-1)}} \right) \right\};
\]

if \( (1-\delta)D^{(m-1)} \leq D \leq 1 \), then

\[
\epsilon^*_\delta(P,D) = 0.
\]

Next, let us consider the case when the input distribution belongs to some set \( P \) of Class II.

**Proposition 12.** The approximate differential privacy-distortion with \( P \) of Class II is

\[
\epsilon^*_\delta(P,D) = \min_{1-D < \alpha_1 \leq 1} \frac{(m-1)(\alpha_1 - \delta) - \min_{\gamma_2 \geq 0} \{ d(\alpha_1, \gamma_2) \}}{1-\alpha_1}
\]

where \( d(\alpha_1, \gamma_2) \) is the optimal value of PV.

**Proof.** Recall the optimization problem (13), we can also obtain the approximate differential privacy function as the solution to the following optimization problem:

\[
\min_{\gamma_2 \geq 0} \frac{(m-1)(\alpha_1 - \delta) - \min_{\gamma_2 \geq 0} \{ d(\alpha_1, \gamma_2) \}}{1-\alpha_1}
\]

subject to

\[
\sum_{P \in \mathcal{P}} \gamma_P \frac{m-k}{D - (1-\delta)D^{(k-1)}} \geq j(1-\gamma_2), 1 \leq j \leq m-1
\]

where \( D^{(k)} = \sum_{j=m-k+1}^m P_j \) for the corresponding \( P \) in \( \mathcal{P} \). Thus, we have the result.

\[\square\]

By taking \( \delta = 0 \), we get the following result of differential privacy-distortion function for given input distribution, which is stronger than that in [18], where it only provides the optimization problem instead of the solution.

**Theorem 13.** Given distribution \( P \) with \( P_1 \geq P_2 \geq \cdots \geq P_m \), we have

\[
\epsilon^*_{DP}(P,D) = \max \left\{ 0, \min \left( \frac{m-k}{D - (1-\delta)D^{(k-1)}} \right) \right\}.
\]

A recent paper [23] studied the privacy-utility trade-off by taking maximal leakage as the privacy measure. We regain the results by coloring argument which we put in Appendix E.

**Theorem 14.** Let \( D^{(k)} = \sum_{j=m-k+1}^m P_j \) for \( 0 \leq k \leq m \). Then

\[
\epsilon^*_{ML}(P,D) = \max \left\{ 0, \log \left( \frac{m-k}{D - (1-\delta)D^{(k-1)}} \right) \right\}
\]

if \( D^{(k-1)} < D \leq D^{(k)} \) and \( 1 \leq k \leq m \).

An equal expression of Theorem 14 showing smallest distortion with fixed leakage was given by Saedian et al. [23]. Assume that the prior belongs to some set \( P \) of Class II. Recall the definition of \( D^*(\mathcal{P}, \epsilon) \) in Definition 4, we have a straightforward result as follows.

**Corollary 15.** [23]. Given \( \epsilon \geq 0 \),

\[
D^*_{ML}(\mathcal{P}, \epsilon) = \max_{P \in \mathcal{P}} \epsilon^*_{ML}(P,D)
\]

where \( D^*_{ML}(\mathcal{P}, \epsilon) = D^{(m-k)} + P_{k+1}(k-e^\epsilon) \) if \( \log k \leq \epsilon < \log(k+1) \) and \( 1 \leq k \leq m \).

**C. Illustration of Local Results**

In this subsection, we demonstrate the results in local privacy framework by figures. Our results are first applied to binary datasets. In particular, each attribute associated with individuals is the answer of some binary classification problem, that is a yes/no question or a setting with 0-1 outcome. In general, \( X = \{0,1\} \) and \( m = 2 \). For local privacy mechanism, we compare the privacy-distortion functions with input distribution unknown by taking \( m = 2 \) into Theorem 8. It is shown in Fig. 3 that high accuracy requirement (i.e., \( D < 0.5 \)) leads to the phenomena-the privacy costs for divergence-based measures (i.e., \( \epsilon_{DP}, \epsilon_\delta \) and \( \epsilon_{\alpha,DP} \)) are numerically higher than the ones for mutual information (i.e.,
10 is the special mechanism $Q_D$ differential privacy and approximate differential privacy jump among these three popular measures. Notably, the curves for guarantees, the data server usually collects the i.i.d. samples and mutual information ($\varepsilon_P$ Rényi differential privacy ($\alpha$) and prior.

We now turn to the privacy-distortion functions with the knowledge of prior distribution. To demonstrate the power of Theorem 9, we set a certain example with $m = 4$ and prior distribution $P = (P_1, P_2, P_3, P_4) = (0.4, 0.3, 0.2, 0.1)$. As shown in Fig. 4, maximal leakage is the most relaxed privacy notion among these three popular measures. Notably, the curves for differential privacy and approximate differential privacy jump to zero when $D = 0.6$ and 0.54, respectively. This is because the special mechanism $Q_D$ mentioned in the proof of Lemma 10 is $(P, D)$-valid if and only if $D \geq (1 - \delta)(1 - P_1)$.

D. Parallel Composition

To create a database consisting of data with local privacy guarantees, the data server usually collects the i.i.d. samples from all possible individual uploads (see Fig. 5). In other words, the input space $X^n$ contains the elements which are independently and identically distributed samples from $X = \{1, 2, \cdots, m\}$ with full-support distribution $P_X$, i.e.,

$$P_X(x) = P_X(x_1, \cdots, x_n) = \prod_{j=1}^n P_{x_j}.$$  

Then the global privacy mechanisms on $X^n$ by $Q$ on $X$ is given as follows.

$$Q_{Y|X}(y|x) = Q_{Y|X}(y_1, \cdots, y_n|x_1, \cdots, x_n) = \prod_{j=1}^n Q_{Y|X}(y_j|x_j).$$

Due to the independence of sampling, we have

$$\mathbb{E}_{P, Q}[d(x, y)] = n\mathbb{E}_{P, Q}[d(x, y)].$$

Lemma 16. Let $\hat{\varepsilon}^*(P, D)$ be the privacy-distortion function over $X^n$. Then

1) $\hat{\varepsilon}^*(P, D) = \varepsilon^*(P, D/n)$ for differential privacy and Rényi differential privacy.

2) $\hat{\varepsilon}^*(P, D) = n\varepsilon^*(P, D/n)$ for maximal information, maximal leakage, Sibson mutual information and mutual information.

Proof of Lemma 16. By the relations among these privacy notions (See Lemma 20 in Appendix A), it suffices to prove the cases for maximal information and the $\alpha$-type notions. Let $Q$ be a mechanism from $X^n$ to $Y^n$.

1) (Maximal information)

$$\hat{\varepsilon}_{M1}(Q) = \max_{x, y} \log \sum_{x, y} P(x)Q(y|x)$$

$$= \max_{x, y} \log \frac{\sum_{x, y} P(x)Q(y|x)}{\prod_{j=1}^n P(x_j)Q(y_j|x_j)}$$

$$= \max_{y_1, \cdots, y_n} \log \frac{\prod_{j=1}^n P(x_j)Q(y_j|x_j)}{\prod_{j=1}^n \sum_{x_j} P(x_j)Q(y_j|x_j)}$$

$$= \sum_{j=1}^n \log \frac{Q(y_j|x_j)}{P(x_j)Q(y_j|x_j)}$$

$$= n\varepsilon_{M1}(Q).$$
2) (Rényi differential privacy)
\[
\exp((\alpha-1)\hat{\epsilon}_{\alpha,DP}(Q)) \\
= \max_{d(x,x')} \sum_{y} Q^\alpha(y|x)Q^{1-\alpha}(y|x') \\
= \max_{x \neq x', y_1, \cdots, y_n} \sum_{j=1}^{n} Q(y_j|x)Q^{1-\alpha}(y_j|x') \\
= \exp((\alpha-1)\epsilon_{\alpha,DP}(Q)).
\]

3) (Sibson mutual information)
\[
\hat{\epsilon}_{\alpha,M}(Q) = \frac{\alpha}{\alpha-1} \log \left( \sum_{x} P(x)Q^\alpha(y|x) \right)^{1/\alpha} \\
= \frac{\alpha}{\alpha-1} \log \sum_{y_j} \left( \sum_{x} \sum_{j=1}^{n} P(x_j)Q^\alpha(y_j|x_j) \right)^{1/\alpha} \\
= \frac{\alpha}{\alpha-1} \log \sum_{j=1}^{n} \left( \sum_{y_j} P(x_j)Q^\alpha(y_j|x_j) \right)^{1/\alpha} \\
= n\epsilon_{\alpha,M}(Q).
\]

Since \( \mathbb{E}_P[Q(d(x,y))] = n\mathbb{E}_P[Q(d(x,y))] \), the results above complete the proof. \(\square\)

Based on the additivity of composition and Theorem 8, we get the following results directly.

**Theorem 17.** For any \( P \) of Class I, we have

1) (Differential privacy)
\[
\hat{\epsilon}_{DP}^*(P,D) = \left\{ \begin{array}{ll}
\log \left( \frac{(m-1)(n-D)}{D} \right), & 0 < D < \frac{n(m-1)}{m} \\
0, & n \leq D \leq 1;
\end{array} \right.
\]

2) (Approximate differential privacy)
\[
\max \left\{ 0, \log \left( \frac{(m-1)(n-D)}{D} \right) \right\} \\
\leq \hat{\epsilon}_{DP}^*(P,D) \\
\leq \max \left\{ 0, \log \left( \frac{n-D}{D} \left( 1 - \frac{1}{n} \right)^\delta \right) \right\};
\]

3) (Maximal leakage)
\[
\left\{ \begin{array}{ll}
n \log \left( \frac{(m-1)(n-D)}{D} \right) \leq \hat{\epsilon}_{ML}^*(P,D) \leq \hat{\epsilon}_{ML}(Q,D), & 0 < D < \frac{n(m-1)}{m} \\
\hat{\epsilon}_{ML}(P,D) = 0, & n \leq D \leq 1;
\end{array} \right.
\]

where \( \hat{\epsilon}_{ML}(Q,D) = n \log \frac{n-D}{D} \) and \( \hat{\epsilon}_{ML}^*(P,D) = \min_{1 \leq i \leq m} \{ \epsilon_i \mid P_i \in \mathcal{P} \}; \)

4) (Rényi differential privacy)
\[
\hat{\epsilon}_{DP}^*(P,D) = \left\{ \begin{array}{ll}
n \log \left( \frac{(m-1)(n-D)}{D} \right), & 0 < D < \frac{n(m-1)}{m} \\
0, & n \leq D \leq 1;
\end{array} \right.
\]

5) (Maximal leakage)
\[
\hat{\epsilon}_{ML}^*(P,D) = \left\{ \begin{array}{ll}
n \log \left( \frac{(m-1)(n-D)}{D} \right), & 0 < D < \frac{n(m-1)}{m} \\
0, & n \leq D \leq 1;
\end{array} \right.
\]

\( \leq \hat{\epsilon}_{DP}^*(P,D) \)
\( \leq \max \{ 0, \hat{\epsilon}_{DP}^*(Q,D) \} \)

where
\[
\hat{\epsilon}_{DP}(Q,D) = \frac{1}{\alpha-1} \log \frac{D}{n(m-1)} \left( m - 2 + \left( \frac{n-D}{D} \right)^\alpha \right);
\]

6) (Sibson mutual information)
\[
\hat{\epsilon}_{M}^*(P,D), n \log \frac{D}{n(m-1)} \right\};
\]

7) (Mutual information)
\[
\hat{\epsilon}_{ML}^*(P,D) = \max \left\{ 0, \log \left( \frac{D}{n(m-1)} \right) \right\};
\]

Proof of (2) in Theorem 17. For given \( \delta \in (0,1) \),
\[
\hat{\epsilon}_{\delta}^*(Q) = \max_{y,d(x,y) = 1} \log \frac{Q(y|x)}{Q(y|x')},
\]

Thus,
\[
\epsilon_{\hat{\delta}}^* = \max_{y,x \neq x'} (Q_j(Q_j^\delta)) = \epsilon_{\hat{\delta}}^*(Q),
\]

where \( Q_j = \max_{y} Q_j(Q_j) \).

**Theorem 18.** Let \( P \) be a given prior distribution such that \( P_1 \geq P_2 \geq \cdots \geq P_m \). Let \( D^{(k)} = \sum_{j=m-k+1}^{m} P_j \) for \( 0 \leq k \leq m \). Then

1) (Approximate differential privacy)
\[
\epsilon_{\hat{\delta}}^*(P,nD) = \max \left\{ 0, \min_{(m-k)(1-D) < D} \right\}
\]

2) (Maximal leakage)
\[
\epsilon_{ML}^*(P,nD) = \max \left\{ 0, \log \left( \frac{m-k}{D-(1-D)} \right) \right\};
\]

if \( D^{(k)} < D \leq D^{(k+1)} \) and \( 1 \leq k \leq m \).

V. CONCLUSION

This paper investigates the privacy-utility trade-off problem for seven popular privacy measures in two different scenarios: global and local privacy. For both global and local privacy, our results provide some upper and lower bounds on privacy-distortion function, which reveals the relationships between privacy and distortion in a quantitative way. In particular, with known prior distributions, we obtain the analytical closed form of privacy-distortion function for approximate differential privacy. To the best of our knowledge, this is the first result
on the privacy-utility tradeoff for approximate differential privacy.

In addition to the privacy measures we used here, it is still possible to consider other privacy measures, such as the privacy measure defined by Arimoto $\alpha$-mutual information and $f$-mutual information. Moreover, the application privacy-utility trade-offs in machine learning is an important problem, which we leave it for further study.

VI. ACKNOWLEDGMENTS

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A. Basic Properties of Privacy

To clearly show the properties of different notions for any given mechanism \( Q \), we turn to the following definition equivalent to Definition 3.

**Definition 19.** Let \( P_X \) be a fixed distribution with full support on \( X \).

1. (Maximal divergence) The differential privacy loss of \( Q \) is defined as
   \[
   \epsilon_{DP}(Q) := \max_{y \in \mathcal{Y}, d(x,x')=1} \frac{Q_Y|X(y|x) - Q_Y|X(y|x')}{Q_Y|X(y|x')}. \]

2. (Approximate maximal divergence) Given a positive real number \( \delta < 1 \), the approximate differential privacy loss of \( Q \) is defined as
   \[
   \epsilon_\delta(Q) := \max_{y \in \mathcal{Y}, d(x,x')=1} \frac{Q_Y|X(y|x) - \delta}{Q_Y|X(y|x')}. \]

3. (Maximal information) The maximal information between \( X \) and \( Q(X) \) is denoted by
   \[
   \epsilon_M(Q) := I_\infty(X; Q(X)) = \max_{x \in X, y \in \mathcal{Y}} \frac{Q_Y|X(y|x)}{\sum_{x' \in X} P_X(x) Q_Y|X(y|x')}. \]

4. (Maximal leakage) The maximal leakage from \( X \) to \( Q(X) \) is denoted by
   \[
   \epsilon_{ML}(Q) := \mathcal{L}(X \rightarrow Q(X)) = \sum_{x \in X} \max_{y \in \mathcal{Y}} Q_Y|X(y|x). \]

5. (Rényi divergence) Given a positive real number \( \alpha > 1 \), the Rényi differential privacy loss of \( Q \) is defined as
   \[
   \epsilon_{\alpha,DP}(Q) := \max_{d(x,x')=1} D_\alpha \left( Q_Y|X(.|x)|| Q_Y|X(.|x') \right) = \max_{d(x,x')=1} \frac{1}{\alpha - 1} \log \sum_{y \in \mathcal{Y}} Q_Y^\alpha|X(y|x) Q_Y^{1-\alpha}|X(y|x'). \]

6. (Sibson’s mutual information) Given a positive real number \( \alpha > 1 \), the \( \alpha \)-mutual informations of \( Q \) is denoted as
   \[
   \epsilon_{\alpha,M}(Q) := I_\alpha(X; Q(X)) = \frac{\alpha}{\alpha - 1} \log \left( \sum_{x \in X} P_X(x) Q_Y^\alpha|X(y|x) \right)^{1/\alpha}. \]

7. (Mutual information) The mutual information of \( Q \) is denoted as
   \[
   \epsilon_M(Q) := I(X; Q(X)) = \sum_{x \in X} P_X(x) Q_Y|X(y|x) \log \frac{Q_Y|X(y|x)}{\sum_{x} P_X(x) Q_Y|X(y|x)}. \]

Then we have the following well-known results.

**Lemma 20.** Relations and implications.

1. The privacy losses \( \epsilon_{DP}(Q), \epsilon_\delta(Q), \epsilon_{ML}(Q) \) and \( \epsilon_{\alpha,DP}(Q) \) do not depend on \( P \).
2. The privacy losses \( \epsilon_{DP}(Q), \epsilon_\delta(Q) \) and \( \epsilon_{\alpha,DP}(Q) \) provide worst-case privacy guarantees. (3) The privacy losses \( \epsilon_{M1}(Q), \epsilon_{ML}(Q), \epsilon_{\alpha,M}(Q) \) and \( \epsilon_M(Q) \) provide global privacy guarantees. (4) For \( 0 \leq \delta_1 \leq \delta_2 \), \( \epsilon_{\delta_1}(Q) \leq \epsilon_{\delta_2}(Q) \). In particular, \( \epsilon_{\delta_1}(Q) = \epsilon_{DP}(Q) \).

**Lemma 21** (Convexity of privacy). For fixed \( P_X \) in \( \mathcal{Y} \).

1. \( \epsilon_{DP}(Q) \) and \( \epsilon_{ML}(Q) \) are quasi-convex;
2. \( \epsilon_{ML}(Q) \) is convex;
3. \( \epsilon_{ML}(Q) \) is convex;
4. \( \epsilon_{M}(Q) \) is convex.

Proof of Lemma 21. The results of (2)-(5) are well-known and one may refer to the publications mentioned above. We now prove (1). Let \( Q_1 \) and \( Q_2 \) be \( (\epsilon_1, \delta_1) \) and \( (\epsilon_2, \delta_2) \)-differential private mechanisms respectively for some \( \delta \in [0, 1) \). Then for any \( y \in \mathcal{Y} \) and neighbors \( x, x' \in X \), \( Q_1(y|x) \leq e^{\epsilon_1} Q_1(y|x') + \delta \), \( i = 1, 2 \). For \( 0 \leq \alpha \leq 1 \), we have
   \[
   Q(y|x) := \lambda Q_1(y|x) + (1 - \lambda) Q_2(y|x) \leq \lambda (e^{\epsilon_1} Q_1(y|x') + \delta) + (1 - \lambda) (e^{\epsilon_2} Q_2(y|x') + \delta) \leq e^{\max(\epsilon_1, \epsilon_2)} (\lambda Q_1(y|x') + (1 - \lambda) Q_2(y|x')) + \delta = e^{\max(\epsilon_1, \epsilon_2)} Q(y|x') + \delta. \]

Thus, \( \epsilon_\delta(Q) \leq \max(\epsilon_1, \epsilon_2) \). Next, let \( \epsilon_M(Q_i) \leq \epsilon_i \) for \( i = 1 \) and \( 2 \). Then for any \( y \in \mathcal{Y} \) and \( x \in X \), we have
   \[
   Q(y|x) := \lambda Q_1(y|x) + (1 - \lambda) Q_2(y|x) \leq \lambda e^{\epsilon_1} B_\alpha(Q_1(y|x)) + (1 - \lambda) e^{\epsilon_2} B_\alpha(Q_2(y|x)) \leq e^{\max(\epsilon_1, \epsilon_2)} (\lambda B_\alpha(Q_1(y|x)) + (1 - \lambda) B_\alpha(Q_2(y|x))) = e^{\max(\epsilon_1, \epsilon_2)} Q(y|x). \]

Thus, \( \epsilon_M(Q) \leq \max(\epsilon_M(Q_1), \epsilon_M(Q_2)) \). This completes the proof.

The following two lemmas reveal the probability preservation properties of the \( \alpha \)-type privacy mechanisms.

**Lemma 22** ([20]). For \( \alpha > 1 \), let \( D_\alpha(P||Q) \) denote the Rényi divergence between \( P \) and \( Q \) over \( \mathcal{R} \). Then we have

1. Monotonicity. \( D_\alpha(P||Q) \) is non-decreasing in \( \alpha \).
2. Probability preservation. For any arbitrary event \( A \subset R \),
   \[
   P(A) \leq \left( e^{D_\alpha(P||Q)} Q(A) \right)^{\frac{\alpha - 1}{\alpha}}. \]

**Lemma 23** ([10]). Let \( (X \times \mathcal{Y}, F, P_X Y) \) and \( (X \times \mathcal{Y}, F, P_X Y) \) be two probability spaces, and assume \( P_X Y \ll\]
Given $E \in \mathcal{E}$, let $E \gamma := \{x : (x, y) \in E\}$. For $\alpha > 1$, let $I_{\alpha}(X; Y)$ denote Sibson mutual information. Then

$$P_{XY}(E) \leq \max_y \left( P_X(E) \exp(I_{\alpha}(X; Y)) \right)^{\frac{1}{\alpha}}.$$  

### B. Coloring Scheme

Let $P$ be a prior distribution such that $P_1 \geq P_2 \geq \ldots \geq P_m$. For any mechanism $Q$, we write it as the matrix form below.

$$Q = \begin{pmatrix}
Q(1|1) & Q(2|1) & \cdots & Q(m|1) \\
Q(1|2) & Q(2|2) & \cdots & Q(m|2) \\
\vdots & \vdots & \ddots & \vdots \\
Q(1|m) & Q(2|m) & \cdots & Q(m|m)
\end{pmatrix}.$$  

To reinforce the understanding of prior distribution, we define the accumulation function $D^{(0)} = 0$ $D^{(k)} := \sum_{j=m-k+1}^{m} P_j$ for $1 \leq k \leq m$.

By Lemma 10 and 33, there exists mechanism costing zero approximate privacy loss as well as mechanism causing no maximal leakage under specific assumption sacrificing accuracy. In this appendix, we assume that privacy loss is inevitable, i.e., the threshold $D$ is at most $(1 - \delta)D^{(m-1)}$ for approximate differential privacy and $D^{(m-1)}$ for maximal leakage.

For any mechanism $Q$ with $\epsilon(\delta)(Q) > 0$, we say two distinct entries $Q(j|i)$ and $Q(j|k)$ of the same column consist of a critical pair if $Q(j|i) = e^{\epsilon(\delta)(Q)}Q(j|k) + \delta$. We color every single entry of $Q$ following the rules below.

- Color the element black if it is the bigger one in the critical pair.
- Color the element red if it is the smaller one in the critical pair.
- Color the element white, otherwise.

Notice that if $Q(j|i) = 0$ then for the $j$th column, every element is not greater than $\delta$ with equality if the element is black.

Similarly, for any mechanism $Q$ with $\epsilon_{\alpha}(Q) > 0$, we color every single entry of $Q$ following the rules below.

- Color the element black if it is the biggest among the entries of the same column.
- Color the element red if it is the smallest one among the entries of the same column.
- Color the element white, otherwise.

### C. Special Transformers

**Lemma 24.** (Size of the output space) There exists a mechanism $Q \in Q_{\epsilon}(\mathcal{P}, D)$ such that $|\hat{Q}(X)| \leq m$ for the notions aforementioned except for the approximate privacy with positive $\delta$.

To prove Lemma 24, we introduce the transformation on mechanism given by Kalantari et al [18].

**Proof of Lemma 24.** For any $Q \in Q(\mathcal{P}, D)$, assume that $|\hat{Q}(X)| = N > m$, that is $Q(j|i) > 0$ for $j \leq N$ and $Q(j|i) = 0$ for $j > N$. We create a mechanism $\hat{Q}$ defined as follows.

$$\hat{Q}(j|i) = \begin{cases} 
Q(N-1|i) + Q(N|i) & \text{if } j = N-1, \\
Q(j|i) & \text{if } j < N-1, \\
0 & \text{if } j > N-1.
\end{cases}$$

Note that $|\hat{Q}(X)| = N - 1$, and for any $P \in \mathcal{P}$, $\mathbb{E}_{P, \hat{Q}}[d(x, y)] = \sum_{i=1}^{m} P_i(1 - \hat{Q}(i|i)) \leq \mathbb{E}_{P, Q}[d(x, y)]$ where the equality holds if and only if $N > m + 1$. It is followed by the fact $\hat{Q}$ belongs to $Q(\mathcal{P}, D)$. Hence, it is sufficient to prove that $\epsilon(\hat{Q}) \leq \epsilon(\hat{Q})$.

1) For $1 \leq j \leq N - 2$ and $1 \leq i \neq k \leq m$,

$$\hat{Q}(j|i) = Q(j|i) \leq e^{\epsilon(\hat{Q})}Q(j|k) = e^{\epsilon(\hat{Q})}\hat{Q}(j|k),$$

$$\hat{Q}(N-1|i) = Q(N-1|i) + Q(N|i) \leq e^{\epsilon(\hat{Q})}Q(N-1|k) + e^{\epsilon(\hat{Q})}Q(N|k) = e^{\epsilon(\hat{Q})}\hat{Q}(N-1|k).$$

Hence, $\epsilon_{\alpha}(\hat{Q}) \leq \epsilon_{\alpha}(\hat{Q})$.

2) For maximal information, one can obtain that $\epsilon_{\alpha,M}(\hat{Q}) \leq \epsilon_{\alpha,M}(Q)$ by changing $Q(j|k)$ to $\mathbb{E}_k[Q(j|k)]$ in 1).

3) The inequality

$$\mathcal{L}(X \rightarrow \hat{Q}(X)) = \sum_{j=1}^{N-1} \max_{1 \leq i \leq m} \hat{Q}(j|i)$$

$$= \sum_{j=1}^{N-2} \max_{1 \leq i \leq m} \{\hat{Q}(N-1|i), \hat{Q}(N|i)\} + \max_{1 \leq i \leq m} \hat{Q}(N|N)$$

$$\leq \sum_{j=1}^{N} \max_{1 \leq i \leq m} (Q(j|i) = \mathcal{L}(X \rightarrow Q(X))$$

yields the result $\epsilon_{\alpha,M}(\hat{Q}) \leq \epsilon_{\alpha,M}(Q)$.

4) Let $f_{\alpha}(x, y) = x^{\alpha}y^{1-\alpha}$ for $0 \leq x, y \leq 1$. Then $f_{\alpha}$ is convex since $\nabla^2 f_{\alpha}$ is positive semi-definite. Therefore, we have for $1 \leq i \neq k \leq m$,

$$\sum_{j=1}^{N-1} \hat{Q}(j|i)\hat{Q}^{1-\alpha}(j|k)$$

$$= \sum_{j=1}^{N-2} \hat{Q}(j|i)\hat{Q}^{1-\alpha}(j|k) +$$

$$Q(N-1|i) + Q(N|i) \left( Q(N-1|k) + Q(N|k) \right)^{1-\alpha}$$

$$\leq \sum_{j=1}^{N-2} \hat{Q}(j|i)\hat{Q}^{1-\alpha}(j|k) +$$

$$Q(N-1|i)\hat{Q}^{1-\alpha}(N-1|k) + Q(N|i)\hat{Q}^{1-\alpha}(N|k)$$

$$= \sum_{j=1}^{N} \hat{Q}(j|i)\hat{Q}^{1-\alpha}(j|k).$$

Hence, $\epsilon_{\alpha,D}(\hat{Q}) \leq \epsilon_{\alpha,D}(Q)$.

5) The function $\frac{1}{\alpha} \exp\left( -\frac{1}{\alpha} \epsilon_{\alpha,M}(Q) \right)$ is convex (refer to Lemma 21 in Appendix A), and thus so is the situation for fixed $y$. Hence, we have

$$\sum_{j=1}^{N-1} \left( \sum_{i=1}^{m} P_i \hat{Q}^{\gamma}(j|i) \right)^{\frac{1}{\gamma}}$$

$$= \sum_{j=1}^{N-2} \left( \sum_{i=1}^{m} P_i \hat{Q}^{\gamma}(j|i) \right)^{\frac{1}{\gamma}}$$

$$+ \left( \sum_{i=1}^{m} P_i \left( \hat{Q}(N-1|i) + Q(N|i) \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{\gamma}}$$
followed by $\epsilon_{a,M}(\tilde{Q}) \leq \epsilon_{a,M}(Q)$.  
6) As $\alpha$ tends to 1 in 5), we have $\epsilon_{M}(\tilde{Q}) \leq \epsilon_{M}(Q)$. }

This explains why we choose synthetic version as the released database in local privacy framework. Next, inspired by the construction of transformation above, we introduce several interesting transformers which are useful in coloring scheme. For any mechanism $Q$ with

$$Q(\sigma(1)|\sigma(1)) \geq Q(\sigma(2)|\sigma(2)) \geq \cdots \geq Q(\sigma(m)|\sigma(m))$$

for some permutation $\sigma$ over $\{1,2,\cdots,m\}$. We define the transformer $T_1$ over $Q$ as follows.

$$T_1 : Q \mapsto T_1Q$$

for $1 \leq i, j \leq m$.

**Lemma 25.** For approximate differential privacy and maximal leakage, if mechanism $Q$ satisfying $\epsilon(Q) > 0$, then $T_1Q$ is still a well-defined mechanism such that

1) $T_1Q$ is $(P, \mathbb{E}_{P,Q}[d(x,y)])$-valid for any prior distribution $P$;  
2) $\epsilon(T_1Q) = \epsilon(Q)$;  
3) $T_1Q(1|1) \geq T_1Q(2|2) \geq \cdots \geq T_1Q(m|m)$.

We skip the proof for simplicity. It is worth noting that if we color $T_1Q$ following the rules aforementioned, then $T_1Q(j|i)$ and $Q(\sigma(j)|\sigma(i))$ have the same color for $1 \leq i, j \leq m$.

**Lemma 26.** There exists $Q \in Q^*(P,D)$ for approximate differential privacy or maximal leakage, such that all the off-diagonal elements in the same row have the same color.

**Proof.** We denote the number of black (red, white, respectively) off-diagonal elements in the $k$th row by $n_B^{(k)}$, $n_R^{(k)}$, $n_W^{(k)}$, respectively.

1) If $n_B^{(k)}, n_R^{(k)} > 0$, then we define

$$Q'(j|i) = \begin{cases} 
Q(j|k) - \Delta/n_B^{(k)}, & \text{if } i = k \neq j \text{ and } Q(j|k) \text{ is black}, \\
Q(j|k) + \Delta/n_B^{(k)}, & \text{if } i = k \neq j \text{ and } Q(j|k) \text{ is white}, \\
Q(j|i), & \text{otherwise}
\end{cases}$$

where $\Delta$ is small enough such that $0 \leq Q' \leq 1$ and $\epsilon$ does not increase. Moreover, $Q'$ is also a $(P,D)$-valid mechanism.

2) If $n_B^{(k)} > 0$ and $n_W^{(k)} > 0$, then we define

$$Q'(j|i) = \begin{cases} 
Q(j|k) - \Delta/n_R^{(k)}, & \text{if } i = k \neq j \text{ and } Q(j|k) \text{ is black}, \\
Q(j|k) + \Delta/n_R^{(k)}, & \text{if } i = k \neq j \text{ and } Q(j|k) \text{ is white}, \\
Q(j|i), & \text{otherwise}
\end{cases}$$

where $\Delta$ is small enough such that $0 \leq Q' \leq 1$ and $\epsilon$ does not increase. Moreover, $Q'$ is also a $(P,D)$-valid mechanism.
D. Special Mechanism

In this subsection, we consider the properties of $Q \in \mathcal{T}^{k}_1(\mathcal{Q}(P, D))$. By Lemma 25, 26 and 27, $Q$ satisfies the following properties.

1) $Q(1|1) \geq Q(2|2) \geq \cdots \geq Q(m|m)$.
2) All the off-diagonal elements of a row in $Q$ have the same color.
3) If a diagonal element is not black, then the off-diagonal elements of the same row are red.

There are more traits for the color pattern of $Q$.

**Proposition 28.** The elements of a row in $Q$ are not all black nor red.

**Proof.** Assume that all elements of the $k$th row are black. Then, there exists a row, denoted by $i$th row, such that $Q(j|k) > Q(i|i)$ for $j \neq i$. Hence, we have

$$1 = Q(i|i) + \sum_{j \neq i} Q(j|k) > Q(i|i) + Q(j|i) = 1.$$  

Contradiction. The proof for no row with all red elements is similar.

By Property 28, there exists no black-off-diagonal element. Moreover, if a diagonal element is not black, then it is white.

**Proposition 29.** There is only one possible non-black diagonal element in $Q$.

**Proof.** Assume for $1 \leq i < k \leq m$, $Q(i|i)$ and $Q(k|k)$ are not black. Then by Lemma 27, $Q(i|i)$ is red. Thus, $Q(i|i)$ is black for some $1 \leq i \leq m$. Because $Q(i|i)$ is not black, $l \neq i$. Thus $Q(i|i)$ is a black-off-diagonal element which contradicts Property 28.

**Proposition 30.** For the coloring scheme of approximate differential privacy, if $Q(k|k)$ is the only possible non-black diagonal element, then $Q(k|k) = Q(1|1)$.

**Proof.** By Lemma 27, $Q(j|k)$ is red for $j \neq k$. By Property 29, $Q(i|i)$ is black for $i \neq k$. Then by Property 28, $Q(j|k)$ is not black for $j \neq i$. Therefore, $Q(j|i) > Q(j|k)$ for $j \neq i$ and $k$.

$$\sum_{j=1}^{m} Q(j|i) = \sum_{j=1}^{m} Q(j|k) = 1$$

$$\Rightarrow Q(k|k) + Q(k|i) > Q(i|i) + Q(k|i)$$

$$\Rightarrow Q(k|k) \geq \delta + Q(k|i) \geq \delta.$$  

Because $Q(k|k)$ is white, there exists no line number $j$ such that $Q(j|k) = 0$. Thus, $Q(j|k) > 0$. If $Q(i|i) = 0$ then $Q(k|i) \geq Q(i|i)$. If $Q(i|i) > 0$ then $\frac{Q(i|i) - \delta}{Q(i|i) - Q(k|i)} > \frac{Q(k|k) - \delta}{Q(k|k) - Q(k|i)}$. Assume that $Q(k|k) < Q(i|i)$. Then $0 < Q(i|i) - Q(k|k) \leq Q(i|i) - Q(k|k)$. Thus

$$\frac{Q(i|i) - Q(k|k)}{Q(i|i) - Q(k|k)} \leq 1.$$  

On the contrary,

$$\frac{Q(i|i) - Q(k|k)}{Q(i|i) - Q(k|k)} = \frac{Q(i|i) - \delta - (Q(k|k) - \delta)}{Q(i|i) - Q(k|k)}$$

$$\Rightarrow Q(i|i) - \delta > Q(i|i) - Q(k|k) > 1.$$  

Contradiction! Hence, $Q(k|k) \geq Q(i|i)$ for $1 \leq i \leq m$.  

**Proposition 31.** For the coloring scheme of approximate differential privacy, all the off-diagonal elements in the first row of $Q$ are red.

**Proof.** Assume that all the off-diagonal elements in the $k$th row are red and all the off-diagonal elements in the $i$th row are white. Since the sum of each row is one, we have

$$Q(k|k) + Q(k|i) \geq Q(i|i) + Q(k|i).$$

Similar with the proof of Property 30, this inequality implies $Q(k|k) \geq Q(i|i)$. In other words, all the red rows are arranged above the white ones in $Q$.

**Proposition 32.** For the coloring scheme of maximal leakage, all the diagonal elements are black in $Q$.

**Proof.** We claim that if the only possible white diagonal element is $Q(k|k)$, then $Q(j|k) = 0$ for $j \neq k$. Otherwise, we can transfer some non-zero off-diagonal elements’ value of the $k$th row to $Q(k|k)$ until it turns black, resulting in a optima 1 mechanism with smaller distortion. Contradiction to $Q \in \mathcal{Q}(P, D)$. Thus $Q(j|k) = 0$ for $j \neq k$, leading to the fact $Q(k|k) = 1$. This is not possible since $Q(k|k)$ is white.

Combing all aforementioned properties of $Q$, one can obtain Lemma 11.

E. Proof of Theorem 14

The condition of equivalence for zero maximal leakage is given as follows

**Lemma 33.** For given $P$ such that $P_1 \geq P_2 \geq \cdots \geq P_m$, $\epsilon_{ML}(P, D) = 0$ if and only if $D \geq D^{(m-1)}$.

**Proof of Lemma 33.** Necessity. If $\epsilon_{ML}(P, D) = 0$ then there exists a mechanism $Q$ such that $\epsilon_{ML}(Q) = 0$ and $\mathbb{E}_{P,Q}[d(x,y)] \leq D$. Thus, $\sum_{j=1}^{m} Q(j|i) = 1$. Hence, $\sum_{j=1}^{m} Q(j|i) \leq 1$. The expected distortion

$$\mathbb{E}_{P,Q}[d(x,y)] = 1 - \sum_{j=1}^{m} P_j Q(j|i)$$

$$\geq 1 - P_1 \sum_{j=1}^{m} Q(j|i) \geq D^{(m-1)}.$$  

Thus $D^{(m-1)} \leq \mathbb{E}_{P,Q}[d(x,y)] \leq D$.

Sufficiency. For $D \geq D^{(m-1)}$, we define mechanism $Q$ as follows.

$$Q = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0
\end{bmatrix}$$

One can easily obtain that $\epsilon_{ML}(Q) = 0$ and $\mathbb{E}_{P,Q}[d(x,y)] = D^{(m-1)} \leq D$. Thus, $\epsilon_{ML}(P, D) = 0$.  

$\square$
Combing all properties of $Q$ in Appendix D, one can obtain the following lemma.

**Lemma 34.** If $\epsilon_{ML}^*(P, D) > 0$ then there exists a mechanism $Q \in Q_{ML}(P, D)$ such that

1. $0 \leq Q(m|m) \leq Q(m - 1|m - 1) \leq \cdots \leq Q(1|1) \leq 1$;
2. $\max_i Q(j|i) = Q(j|j)$ for $1 \leq j \leq m$.

Combining these two lemmas leads to the proof of Theorem 14.

**Proof of Theorem 14.** For $D \geq D^{(m-1)}$, Lemma 33 proves the statement.

For $0 < D < D^{(m-1)}$, let $Q^*$ be a mechanism satisfying the conditions in Lemma 34. Then $\epsilon_{ML}^*(P, D) = \log \sum_{j=1}^{m} Q^*(j|j)$. Consider the following optimization problem,

$$d_6 := \min_{\alpha_1, \alpha_2, \ldots, \alpha_m} \log \sum_{j=1}^{m} \alpha_j$$
subject to

1. $\sum_{j=1}^{m} P_j \alpha_j \geq 1 - D$, (15)
2. $\alpha_j \geq 1$,
3. $\alpha_j \leq \alpha_{j-1}$ for $1 \leq j \leq m + 1$

where $\alpha_0 = 1$ and $\alpha_{m+1} = 0$. Notice that $\{Q^*(j|i)\}_{1 \leq j \leq m}$ is a feasible solution of (15), and hence $\epsilon_{ML}^*(P, D) \geq d_6$. Conversely, let $\{\alpha_1^*, \alpha_2^*, \ldots, \alpha_m^*\}$ be the optimal solution of (15). We define the following mechanism.

$$Q(j|i) = \begin{cases} 
\alpha_j^* & \text{if } i = j, \\
\frac{\alpha_j^*}{\sum_{j=1}^{m} \alpha_j^*} & \text{if } i \neq j.
\end{cases}$$

For any $j \in [1, m]$, $\sum_{i=1}^{m} Q(j|i) = 1$ which leads to the fact that $Q$ is a conditional probability. The first restricted condition in the scenario of (15) implies that $Q$ is within $Q(P, D)$. The other two imply that

$$\epsilon_{ML}^*(P, D) \leq \epsilon_{ML}(Q) = \log \sum_{j=1}^{m} \max_{1 \leq i \leq m} Q(j|i) = \log \sum_{j=1}^{m} \alpha_j^* = d_6.$$

Moreover, $d_6 = \log(-d_7)$. The dual of (16) is the minimization linear program as below.

$$\begin{align*}
d_7^* := \min_{\gamma_1, \gamma_2, \gamma_3} & \quad (-1 + D) \gamma_1 - \gamma_2 + \gamma_3 \\
\text{subject to} & \quad (1) \frac{D^{(m-j)} - 1}{P_{m+1-k}} - (m+1-k) \\
& \quad (2) \gamma_1, \gamma_2, \gamma_3 \geq 0.
\end{align*}$$

By strong duality theorem, we have $d_6 = d_7$. Similar with the proof of Theorem 9, we can obtain the following result by fixing $\gamma_2 \geq 0$.

$$d_7 := \min_{\gamma_1, \gamma_2, \gamma_3} \left\{ (-1 + D) \gamma_1 - \gamma_2 + \gamma_3 \right\}$$
subject to

1. $D^{(m-j)} \gamma_1 - \gamma_2 + \gamma_3 \geq -j$ for $1 \leq j \leq m$
2. $\gamma_1, \gamma_2, \gamma_3 \geq 0.$

$$d_7 = \min \left\{ -1, \frac{D^{(m-j)} - 1}{P_{m+1-k}} - (m+1-k) \right\}$$
if $D^{(k-1)} < D \leq D^{(k)}$ and $1 \leq k \leq m$. Thus,

$$\epsilon_{ML}^* = \max \left\{ \log \left( m-k - \frac{D^{(k-1)} - 1}{P_{m+1-k}} \right) \right\}$$
if $D^{(k-1)} < D \leq D^{(k)}$ and $1 \leq k \leq m$. \qed