MOMENTS AND DISTRIBUTION OF THE LOCAL TIMES
OF A TRANSIENT RANDOM WALK ON $\mathbb{Z}^d$

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7 May, 2008

Abstract. Consider an arbitrary transient random walk on $\mathbb{Z}^d$ with $d \in \mathbb{N}$. Pick $\alpha \in [0, \infty)$ and let $L_n(\alpha)$ be the spatial sum of the $\alpha$-th power of the $n$-step local times of the walk. Hence, $L_n(0)$ is the range, $L_n(1) = n + 1$, and for integers $\alpha$, $L_n(\alpha)$ is the number of the $\alpha$-fold self-intersections of the walk. We prove a strong law of large numbers for $L_n(\alpha)$ as $n \to \infty$. Furthermore, we identify the asymptotic law of the local time in a random site uniformly distributed over the range. These results complement and contrast analogous results for recurrent walks in two dimensions recently derived by Černý [Ce07]. Although these assertions are certainly known to experts, we could find no proof in the literature in this generality.

1. Introduction and main results

Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of independent, identically distributed, $\mathbb{Z}^d$-valued random variables. Let $(S_n)_{n \in \mathbb{N}_0}$ be the corresponding random walk:

\[ S_0 := 0 \quad \text{and} \quad S_n := \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N}. \] (1.1)

The main object of the present paper are the so-called local times

\[ \ell(n, x) := \sum_{i=0}^{n} \mathbb{1}_{\{S_i = x\}}, \quad n \in \mathbb{N}, x \in \mathbb{Z}^d, \] (1.2)

the number of visits to $x$ by time $n$. More specifically, we are interested in the large-$n$ asymptotics of the following functional of the local times:

\[ L_n(\alpha) := \sum_{x \in \mathbb{Z}^d} \ell(n, x)^\alpha, \quad \alpha \geq 0. \] (1.3)

This is a rather natural object in the study of random walks on $\mathbb{Z}^d$. Much attention has been focused on $L_n(0) = |\{S_0, \ldots, S_n\}|$, the range of the random walk, which is the number of distinct lattice points visited up to time $n$. The case $\alpha = 1$ is trivial, as $L_n(1) = n + 1$. Furthermore, $L_n(2)$ is the self-intersection local time, the number of self-intersections, which has been much studied from physical motives. More generally, for $\alpha$ an integer, $L_n(\alpha)$ is the number of $\alpha$-fold self-intersections of

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AMS 2000 Subject Classification. 60G50, 60J55, 60F15
Keywords. Random walk on $\mathbb{Z}^d$, local time, self-intersection number
the random walk up to time $n$. The quantity $L_n(2)$ also arises as the variance of the random walk in random scenery [KS79].

In this paper, we first prove a strong law of large numbers for $L_n(\alpha)$ for any $\alpha \geq 0$, and we second show that $\ell(n,Y_n)$, where $Y_n$ is uniformly distributed on the set $\{S_0,\ldots,S_n\}$, has asymptotically a geometric distribution. We restrict to transient random walks in all dimensions. Our precise assumptions in this paper are the following.

We assume that the random walk is genuinely $d$-dimensional in the sense of [Sp64 D7.1]. That is, the set $R^+ - R^+$ is $d$-dimensional, where $R^+ = \bigcup_{n \in \mathbb{N}_0} \{x \in \mathbb{Z}^d : \mathbb{P}(S_n = x) > 0\}$ is the support of the random walk. Furthermore, we assume that the escape probability of the random walk,

$$\gamma = \mathbb{P}(S_n \neq 0 \text{ for any } n \in \mathbb{N}),$$

satisfies $0 < \gamma < 1$. (The condition that $\gamma < 1$ rules out trivial cases). Finally, in the case $d \in \{1,2\}$, we assume that either the second moment of the steps is finite or that there is some $\eta > 0$ such that $\sum_{k=n}^{\infty} \mathbb{P}(S_k = 0) \leq O(n^{-\eta})$ as $n \to \infty$.

We have not been able to decide if any of our two conditions in the case $d \in \{1,2\}$ implies the other or not. Now we formulate the strong law of large numbers.

**Theorem 1.1.** For all $\alpha \in [0,\infty)$ it holds $\mathbb{P}$-almost surely:

$$\lim_{n \to \infty} \frac{L_n(\alpha)}{n} = \sum_{j \in \mathbb{N}} j^\alpha \gamma^2 (1 - \gamma)^{j-1}. \quad (1.5)$$

Note that the assertion for $\alpha = 0$ is already well-known; a proof is contained, e.g., in [Sp64 T4.1].

**Theorem 1.2.** Given the steps $X_1,\ldots,X_n$, let $Y_n$ be a uniformly distributed random variable on the set of visited points, $\{S_0,\ldots,S_n\}$. Then, $\mathbb{P}$-almost surely,

$$\lim_{n \to \infty} \mathbb{P}(\ell(n,Y_n) = u | X_1,\ldots,X_n) = \gamma (1 - \gamma)^{u-1}, \quad u \in \mathbb{N}. \quad (1.6)$$

We prove Theorems 1.1 and 1.2 in Section 4. In Section 2 we analyse the expected value of $L_n(\alpha)$ and in Section 3 its variance.

Theorem 1.2 can be understood as follows. For large $n$, the event $\{\ell(n,Y_n) = u\}$ is realised by returning $u - 1$ times to $Y_n$, which has probability approximately $1 - \gamma$ each, and not returning afterwards, which has probability $\approx \gamma$. This makes plausible that $\ell(n,Y_n)$ converges in distribution towards a geometric random variable $Z$ with parameter $\gamma$. Observe that the right-hand side of (1.5) is equal to $\gamma \mathbb{E}(Z^\alpha)$. Hence also Theorem 1.1 can easily be understood, taking into consideration the well-known fact that $|\{S_0,\ldots,S_n\}| \sim \gamma n$ (see [Sp64 T4.1]), which means that the sum on $x$ in the definition of $L_n(\alpha)$ has effectively only $\approx \gamma n$ summands.

It is remarkable that Theorems 1.1 and 1.2 hold for an arbitrary, genuinely $d$-dimensional random walk in $d \geq 3$, without any integrability, centering or periodicity condition. This is in sharp contrast to the recurrent, two-dimensional case studied in [Ce07], where it was assumed that the steps are centred and have finite second moments. Also observe that our results contain the low-dimensional case with non-zero drift, which leads to the same asymptotics as in the higher-dimensional case.

Analogous results for the two-dimensional case have recently been derived in [Ce07]. To the best of our knowledge Theorem 1.2 has not yet appeared in the literature, although related assertions already appeared in [ET60]. Various special cases of Theorem 1.1 are spread over the literature. [DE51] considered the asymptotics of $\mathbb{E}(L_n(0))$ for simple random walk. An extension to a somewhat more
general random walk can be found in [We94]. Some estimates of the variance of \( L_n(2) \) for simple random walk have been derived in [BS95]. The novelty and main value of the present paper lies in its generality in three aspects: all \( \alpha \geq 0 \), very general transient random walks, and the strong law of large numbers.

2. The expected value \( \mathbb{E}(L_n(\alpha)) \)

In this section we prove the asymptotics in (1.5) for the expected value of \( L_n(\alpha) \) in place of the variable \( L_n(\alpha) \) itself. We use the approach of [ET60] (which considers simple random walk only) and use the opportunity to correct an error in [ET60, Theorem 12].

**Proposition 2.1.** For any \( \alpha \in [0, \infty) \),

\[
\lim_{n \to \infty} \mathbb{E}(L_n(\alpha)) = \sum_{j \in \mathbb{N}} j^\alpha \gamma 2(1-\gamma)_{j-1}.
\]

**Proof.** Introduce the probability that the point visited by random walk in the \( n \)-th step has not been visited before:

\[
\gamma(n) = \mathbb{P}(S_0 \neq S_n, \ldots, S_{n-1} \neq S_n),
\]

in particular \( \gamma(0) = 1 \). Note that \( 1 - \gamma(1) = \mathbb{P}(S_1 = 0) \) may be zero. An easy computation yields that \( \gamma(n) \) is also equal to the probability that the random walk does not return to the origin within the first \( n \) steps:

\[
\gamma(n) = \mathbb{P}(S_1 \neq S_0, \ldots, S_n \neq S_0).
\]

Let \( \tau = \inf\{n \in \mathbb{N} : S_0 = S_n\} \) denote the return time to the origin, then we have \( \mathbb{P}(\tau = n) = \gamma(n-1) - \gamma(n) \) for any \( n \in \mathbb{N} \). By the monotone convergence theorem, \( \gamma = \lim_{n \to \infty} \gamma(n) \).

We introduce now the number of points that have been visited exactly \( j \) times up to time \( n \):

\[
Q_j(n) = |\{x \in \mathbb{Z}^d : \ell(n, x) = j\}|.
\]

Its expectation can be calculated as follows.

\[
\mathbb{E}(Q_j(n)) = \sum_{x \in \mathbb{Z}^d} \mathbb{P}(\ell(n, x) = j) = \sum_{0 \leq k_1 < k_2 < \ldots < k_j \leq n} \mathbb{P}(S_{k_1} = \ldots = S_{k_j} = x, S_k \neq x \text{ for } k \in \{0, \ldots, n\} \setminus \{k_1, \ldots, k_j\})
\]

\[
= \sum_{0 \leq k_1 < k_2 < \ldots < k_j \leq n} \gamma(k_1) \left[ \prod_{i=1}^{j-1} \mathbb{P}(\tau = k_{i+1} - k_i) \right] \gamma(n-k_j),
\]

where we used the Markov property at the times \( k_1, \ldots, k_j \) and the definition of \( \tau \). We use (2.2) to identify the generating function \( q_j(s) = \sum_{n \in \mathbb{N}_0} \mathbb{E}(Q_j(n)) s^n \) of \( (Q_j(n))_{n \in \mathbb{N}} \) as

\[
q_j(s) = \left( \sum_{n=0}^{\infty} s^n \gamma(n) \right)^2 - \left( \sum_{n=1}^{\infty} s^n \mathbb{P}(\tau = n) \right)^{j-1}.
\]

Since \( \lim_{n \to \infty} \gamma(n) = \gamma \) and by the monotone convergence theorem, its behaviour as \( s \nearrow 1 \) is identified as follows

\[
q_j(s) \sim \left( \frac{\gamma}{1-s} \right)^2 \left( \sum_{n=1}^{\infty} s^n \mathbb{P}(\tau = n) \right)^{j-1} \sim (1-s)^{-2} \gamma^2 (1-\gamma)^{j-1},
\]
where we note that\( \sum_{n=1}^{\infty} P(\tau = n) = P(\tau < \infty) = 1 - \gamma \).
By the Tauberian theorem (see \cite[Theorem XIII.5]{Fe71}) we obtain
\[
\lim_{n \to \infty} \mathbb{E}\left( \frac{Q_j(n)}{n} \right) = \gamma^2 (1 - \gamma)^{j-1}.
\] (2.5)
(At this point we see that in \cite[Theorem 12]{ET60} a factor \( \gamma \) is missing. Let us remark that (2.5) was also derived in \cite{P74} using entirely different methods.)

Now we return to the expected value of \( L_n(\alpha) \). It is easy to see that
\[
\mathbb{E}(L_n(\alpha)) = \sum_{j \in \mathbb{N}} j^\alpha \mathbb{E}(Q_j(n)).
\]

We will now give an upper and a lower asymptotic bound for the generating function of \( (\mathbb{E}(L_n(\alpha)))_{n \in \mathbb{N}} \) which will turn out to be identical. With the help of (2.3), similarly to (2.4), we deduce, for \( s \in [0,1) \),
\[
\sum_{n \in \mathbb{N}} s^n \mathbb{E}(L_n(\alpha)) = \sum_{j \in \mathbb{N}} j^\alpha \left( \sum_{n=0}^{\infty} s^n \gamma(n) \right)^2 (1 - \gamma)^{j-1}
\sim \left( \frac{\gamma}{1 - s} \right)^2 \sum_{j \in \mathbb{N}} j^\alpha (1 - \gamma)^{j-1}, \quad \text{as } s \nearrow 1.
\]

For the lower bound we use Fatou’s lemma and obtain:
\[
\lim \inf_{s \uparrow 1} (1 - s)^2 \sum_{n \in \mathbb{N}} s^n \mathbb{E}(L_n(\alpha)) = \lim \inf_{s \uparrow 1} \sum_{j \in \mathbb{N}} j^\alpha (1 - s)^2 q_j(s)
\geq \sum_{j \in \mathbb{N}} j^\alpha \lim \inf_{s \uparrow 1} (1 - s)^2 q_j(s) = \sum_{j \in \mathbb{N}} j^\alpha \gamma^2 (1 - \gamma)^{j-1}.
\]

Now apply once more the Tauberian theorem to complete the proof of (2.1). \( \square \)

### 3. The variance \( \mathbb{V}(L_n(\alpha)) \)

In this section we prove an upper bound on the variance of \( L_n(\alpha) \) that is sufficient for the application of the second-moment method in Section 4. For this, it suffices to show that \( \mathbb{V}(L_n(\alpha)) = o(\mathbb{E}(L_n(\alpha))^{2-\kappa}) = o(n^{2-\kappa}) \) for some \( \kappa > 0 \) (see Proposition 2.1). We now give a bound that seems optimal in \( d \geq 3 \). Our method is a combination of ideas from \cite{Ce07} (which are based on ideas from \cite{Bo89}) and from the proof of \cite[Prop. 3.1]{BS95}, where the normalized self-intersection number of simple random walk is estimated. We are also able to apply a result from \cite{JP71}.

**Proposition 3.1.** Fix \( \alpha \in \mathbb{N} \).

(i) In the case \( d \in \{1,2\} \), if the walker’s steps have a finite second moment, then there exists a constant \( C > 0 \) such that, for any \( n \in \mathbb{N} \),
\[
\mathbb{V}(L_n(\alpha)) \leq C \times \begin{cases} n^{3/2} \log n & \text{in } d = 1, \\ n \log^2 n & \text{in } d = 2. \end{cases}
\] (3.1)

On the other hand, if \( \sum_{k=0}^{\infty} \mathbb{P}(S_k = 0) = O(n^{-\eta}) \) for some \( \eta > 0 \), then there exists a constant \( C > 0 \) such that,
\[
\mathbb{V}(L_n(\alpha)) \leq C n^{2-\eta}, \quad n \in \mathbb{N}.
\] (3.2)
(ii) In the case $d \geq 3$, there exists a constant $C > 0$ such that, for any $n \in \mathbb{N}$,

$$\mathbb{V}(L_n(\alpha)) \leq C \begin{cases} n^{3/2} & \text{in } d = 3, \\ n \log n & \text{in } d = 4, \\ n & \text{in } d \geq 5. \end{cases} \quad (3.3)$$

**Proof.** Since $\alpha$ is an integer we may rewrite the variance as follows.

$$\mathbb{V}(L_n(\alpha)) = \mathbb{V} \left[ \sum_{x \in \mathbb{Z}^d} \sum_{k_1, \ldots, k_\alpha = 0}^n \mathbb{I}(S_{k_1} = \ldots = S_{k_\alpha} = x) \right]$$

$$= \sum_{k_1, \ldots, k_\alpha = 0}^n \left[ \mathbb{P}(S_{k_1} = \ldots = S_{k_\alpha}, S_{l_1} = \ldots = S_{l_\alpha}) - \mathbb{P}(S_{k_1} = \ldots = S_{k_\alpha}) \mathbb{P}(S_{l_1} = \ldots = S_{l_\alpha}) \right]$$

$$= \sum_{\alpha} C(\alpha, \beta, \gamma)a_{\beta, \gamma}(n),$$

where

$$a_{\beta, \gamma}(n) = \sum_{0 \leq k_1 < \ldots < k_\beta \leq n} \mathbb{P}(S_{k_1} = \ldots = S_{k_\beta}, S_{l_1} = \ldots = S_{l_\gamma}) - \mathbb{P}(S_{k_1} = \ldots = S_{k_\beta}) \mathbb{P}(S_{l_1} = \ldots = S_{l_\gamma}) \quad (3.4)$$

and

$$C(\alpha, \beta, \gamma) = \left| \{(\bar{k}_1, \ldots, \bar{k}_\alpha) \in \{0, \ldots, n\}^\alpha : (k_1, \ldots, k_\beta) = (\bar{k}_1, \ldots, \bar{k}_\alpha) \} \right|$$

$$\times \left| \{(\bar{l}_1, \ldots, \bar{l}_\alpha) \in \{0, \ldots, n\}^\alpha : (l_1, \ldots, l_\gamma) = (\bar{l}_1, \ldots, \bar{l}_\alpha) \} \right|.$$

In words: $C(\alpha, \beta, \gamma)$ is the number of pairs of unordered tuples $(\bar{k}_1, \ldots, \bar{k}_\alpha)$ and $(\bar{l}_1, \ldots, \bar{l}_\alpha)$ (with possible repetitions) that give the same sequence $0 \leq k_1 < \ldots < k_\beta \leq n$ respectively $0 \leq l_1 < \ldots < l_\gamma \leq n$. Note that $C(\alpha, \beta, \gamma)$ only depends on $\alpha, \beta$ and $\gamma$. Hence, we only have to show that $a_{\beta, \gamma}(n)$ satisfies the bound in (3.3) for any $\beta$ and $\gamma$.

Let $\beta, \gamma \in \{1, \ldots, \alpha\}$ be fixed. In order to prepare for the application of the Markov property, we encode the tuples of the numbers $0 \leq k_1 < \ldots < k_\beta \leq n$ and $0 \leq l_1 < \ldots < l_\gamma \leq n$ in terms of a new set of variables $(j_i, \kappa_i) \in \{0, \ldots, n\} \times \{0, 1\}$ for $i \in \{1, \ldots, \beta + \gamma\}$ satisfying

$$\{j_i : \kappa_i = 0\} = \{k_1, \ldots, k_\beta\} \quad \text{and} \quad \{j_i : \kappa_i = 1\} = \{l_1, \ldots, l_\gamma\},$$

and such that $(j_i, \kappa_i)_{i=1}^{\beta+\gamma}$ is alphabetically ordered, i.e., $j_i \leq j_{i+1}$ for any $i$, and if $j_i = j_{i+1}$ then $\kappa_i < \kappa_{i+1}$. Then we introduce $m_0 = j_1$ and $m_{\beta+\gamma} = n - j_{\beta+\gamma}$ and

$$m_i = j_{i+1} - j_i \quad \text{and} \quad \epsilon_i = \kappa_{i+1} - \kappa_i \quad \text{for } i = 1, \ldots, \beta + \gamma - 1.$$

In this way, we have mapped tuples of numbers $0 \leq k_1 < \ldots < k_\beta \leq n$ and $0 \leq l_1 < \ldots < l_\gamma \leq n$ one-to-one onto sequences $(m_i, \epsilon_i)_{i=0}^{\beta+\gamma}$ in $\mathbb{N}_0 \times \{-1, 0, 1\}$ satisfying $\sum_{i=0}^{\beta+\gamma} m_i = n$. In words, $m_i$ is the difference between the $(i+1)$-th largest and the $i$-th largest of the numbers $k_1, \ldots, k_\beta, l_1, \ldots, l_\gamma$, and $\epsilon_i$ is 1 if the $m$-sequence switches from an $k$-value to an $l$-value, and $\epsilon_i = -1$ if it switches in the reversed way. Since $\beta \geq 1$ and $\gamma \geq 1$, one easily sees that $\# \{i : \epsilon_i \neq 0\} \geq 1$.

Let us look at the case that $\# \{i : \epsilon_i \neq 0\}$ is equal to one, i.e., either the $k_j$’s all all smaller than the $l_j$’s or the other way around. By use of the Markov property, we easily see that in this case all the summands on the right-hand side of the definition (3.4) of $a_{\beta, \gamma}(n)$ vanish. Thus we may restrict to the
case $\# \{i: \varepsilon_i \neq 0\} \geq 2$. In this case, it is clear that $\varepsilon_u = 1 = -\varepsilon_v$, for some $u, v \in \{1, \ldots, \beta + \gamma - 1\}$. For our purposes, it will turn out to suffice to estimate the negative terms on the right-hand side of (3.4) against zero. Hence, we obtain

$$a_{\beta, \gamma}(n) \leq \sum_{0 \leq k_1 < \ldots < k_\beta \leq n} \sum_{0 \leq l_1 < \ldots < l_\gamma \leq n} \mathbb{1}_{\{\exists u, v: \varepsilon_u = 1 = -\varepsilon_v\}} \mathbb{P}(S_{k_1} = \ldots = S_{k_\beta}, S_{l_1} = \ldots = S_{l_\gamma})$$

$$\leq \sum_{m_0, m_1, \ldots, m_{\beta+\gamma} \in \mathbb{N}_0} \sum_{m_1 + \ldots + m_{\beta+\gamma} = n} \mathbb{1}_{\{\exists u, v: \varepsilon_u = 1 = -\varepsilon_v\}} \sum_{x, y \in \mathbb{Z}^d} \mathbb{P}(S_{m_0} = x) \prod_{i=1}^{\beta - \gamma - 1} \mathbb{P}(S_{m_i} = \varepsilon_i y),$$

where $x$ plays the role of $S_{k_1 \wedge l_1}$ and $y$ the role of $S_{l_1} - S_{k_1}$ respectively $S_{k_1} - S_{l_1}$.

In the following, we use $C$ to denote a generic positive constant that depends on $\beta$, $\gamma$, $d$ and the step distribution only and may change its value from appearance to appearance. The summation over $m_{\beta+\gamma}$ is redundant, as $m_{\beta+\gamma}$ can be computed from the other $m_i$ and $n$. Furthermore, we execute the sum of the terms $\mathbb{P}(S_{m_0} = x)$ over $x \in \mathbb{Z}^d$, and consequently the summation over $m_0$ delivers an additional factor $n$ as an upper bound. Now we also execute the sum over all $m_i$ with $i \neq u, v$ and use that $\sup_{y \in \mathbb{Z}^d} \sum_{m_i = 0}^{\infty} \mathbb{P}(S_{m_i} = \varepsilon_i y) \leq C$, by transience. Hence, we obtain

$$a_{\beta, \gamma}(n) \leq C n \sum_{u, v = 1}^{\beta + \gamma - 1} \sum_{y \in \mathbb{Z}^d} \sum_{m_u, m_v = 0}^{n} \mathbb{P}(S_{m_u} = y) \mathbb{P}(S_{m_v} = -y)$$

$$\leq C n \sum_{y \in \mathbb{Z}^d} \sum_{m, \bar{m} = 0}^{n} \mathbb{P}(S_m = y) \mathbb{P}(S_{\bar{m}} = -y).$$

(3.5)

Now, in dimensions $d \geq 3$, our assertion in (3.3) directly follows from [JP71, Lemma 3], which implies that, for every $n \in \mathbb{N}$,

$$\sum_{y \in \mathbb{Z}^d} G_n(0, y)G_n(0, -y) \leq C \times \begin{cases} n^{1/2} & \text{for } d = 3, \\ \log n & \text{for } d = 4, \\ 1 & \text{for } d \geq 5, \end{cases}$$

(3.6)

where $G_n(0, y) = \sum_{m=1}^{n} \mathbb{P}(S_m = y)$ denotes the $n$-step Green’s function. The main tool in the proof of (3.6) is the estimate

$$\sup_{x \in \mathbb{Z}^d} \mathbb{P}(S_m = x) \leq C m^{-\frac{d}{4}}, \quad m \in \mathbb{N},$$

(3.7)

which is proved in any dimension $d$ as [JP71, Lemma 1], based on [Sp64, P7.6]. Hence, the proof of (ii) is finished.

Now we consider the case $d \in \{1, 2\}$. First we work under the assumption of the existence of the second moment of the steps. Consider their expected value, $v = \mathbb{E}(X_1) \in \mathbb{R}^d \setminus \{0\}$. Consider the closed half-plane, respectively half-axis, $A = \{x \in \mathbb{R}^d: x \cdot v \leq 0\}$, where $x \cdot v$ is the standard inner product. Then $-A^c \subset A$. Splitting the sum on the right hand side of (3.5) into $y \in A$ and $y \in A^c$.
and using the symmetry of the summands in $y$, we obtain:

$$a_{\beta, \gamma}(n) \leq Cn \sum_{y \in A} \sum_{m, \tilde{m}=1}^{n} \mathbb{P}(S_m = y) \mathbb{P}(S_{\tilde{m}} = -y)$$

$$\leq Cn \left( \sum_{m=1}^{n} \sum_{y \in A} \mathbb{P}(S_m = y) \right)^{\frac{n}{2}} \sum_{m=1}^{n} \sup_{\tilde{m} \in A} \mathbb{P}(S_{\tilde{m}} = -y) \leq Cn \sum_{m=1}^{n} \mathbb{P}(S_m \in A) \sum_{m=1}^{n} \tilde{m}^{-\frac{d}{2}},$$

where we also used (3.1). We have, for any $m \in \mathbb{N}$, using Chebyshev’s inequality,

$$\mathbb{P}(S_m \in A) \leq \mathbb{P}(|S_m - vn| \geq \|v\|m) = \mathbb{P}(|S_m - \mathbb{E}(S_m)| \geq \|v\|m) \leq \frac{\mathbb{V}(S_m)}{\|v\|^2m^2} \leq \frac{C}{m}.$$ (3.9)

Using this in (3.8) yields the result in (3.1).

Finally, we work in $d \in \{1, 2\}$ under the assumption that $\sum_{k=0}^{\infty} \mathbb{P}(S_k = 0) = O(n^{-\eta})$. We go back to (3.5) and use the Markov property to rewrite

$$a_{\beta, \gamma}(n) \leq Cn \sum_{m, \tilde{m}=0}^{n} \mathbb{P}(S_{m+\tilde{m}} = 0) \leq Cn \sum_{j=0}^{2n} \sum_{k=j}^{\infty} \mathbb{P}(S_k = 0) \leq Cn \sum_{j=0}^{2n} j^{-\eta} \leq Cn^{2-\eta}.$$

This means that we have derived (3.2). The proof of the proposition is complete. \hfill \Box

4. PROOF OF THE THEOREMS

In this section we prove our two main results, Theorems 1.1 and 1.2. Our strategy is the same as in [Ce07]: the proof will be done in three steps. First we prove the assertion in Theorem 1.1 for all integers $\alpha \geq 0$. Second we provide the proof for Theorem 1.2, and the last step is the proof of the general assertion in Theorem 1.1 for $\alpha \in [0, \infty)$.

**Proof of Theorem 1.1** for $\alpha \in \mathbb{N}_0$. For $\alpha = 0$ the assertion is proved in [Sp64, T4.1]. For $\alpha = 1$ the assertion is obvious, so let $\alpha \geq 2$. From Proposition 3.1 we have, for some $\kappa > 0$, that $\mathbb{V}(L_n(\alpha)) \leq Cn^{2-\kappa}$ for all $n \in \mathbb{N}$, with a suitable constant $C > 0$. Consider the subsequence $n_k := [k^{2/\kappa} + 1]$. With the help of Chebyshev’s inequality, Propositions 2.1 and 3.1 we get, for a suitable constant $C$ and any $\varepsilon > 0$:

$$\mathbb{P}(\|L_{n_k}(\alpha) - \mathbb{E}L_{n_k}(\alpha)\| \geq \varepsilon \mathbb{E}(L_{n_k}(\alpha))) \leq \frac{\mathbb{V}(L_{n_k}(\alpha))}{\varepsilon^2 (\mathbb{E}(L_{n_k}(\alpha)))^2} \leq C \varepsilon^{-2} n_k^{-\kappa} \leq \frac{C}{\varepsilon^2 k^2}. $$ (4.1)

Since this is summable over $k \in \mathbb{N}$, the Borel-Cantelli lemma yields that, $\mathbb{P}$-almost surely,

$$\lim_{k \to \infty} \frac{L_{n_k}(\alpha)}{\mathbb{E}(L_{n_k}(\alpha))} = 1. $$ (4.2)

It remains to fill the gaps between the $n_k$. With the help of the monotonicity of $n \mapsto L_n(\alpha)$ and the fact that $\lim_{k \to \infty} \mathbb{E}(L_{n_k+1}(\alpha))/\mathbb{E}(L_{n_k}(\alpha)) = \lim_{k \to \infty} n_{k+1}/n_k = 1$ (see Proposition 2.1), we can easily deduce that, $\mathbb{P}$-almost surely,

$$\lim_{n \to \infty} \frac{L_n(\alpha)}{\mathbb{E}(L_n(\alpha))} = 1.$$  

Now Proposition 2.1 finishes the proof of Theorem 1.1 for all $\alpha \in \mathbb{N}$. \hfill \Box

**Proof of Theorem 1.2**. Let $Y_n$ be uniformly distributed on $\{S_0, \ldots, S_n\}$, and let $Z_n = \ell(n, Y_n)$. Denote $R(n) = |\{S_0, \ldots, S_n\}| = L_n(0)$ and recall that $\lim_{n \to \infty} R(n)/n = \gamma$, according to [Sp64, T4.1].
Hence, for $\alpha \in \mathbb{N}_0$, the conditional $\alpha$-th moments of $Z_n$ given $X_1, \ldots, X_n$ can be asymptotically identified as
\[
E(Z_n^\alpha|X_1, \ldots, X_n) = \sum_{x \in \mathbb{Z}^d} \frac{1}{R(n)} \ell(n, x)^\alpha = \frac{\gamma_n \ell(n, x)^\alpha}{R(n) \gamma_n} \sim \sum_{j \in \mathbb{N}} j^\alpha \gamma(1 - \gamma)^{j-1}.
\] (4.3)

In other words, these moments converge to the moments of a geometrically distributed random variable with parameter $\gamma$. With the help of the theorem of Fréchet-Shohat (see [Sc98, V.1]) we get the weak convergence of the conditional distribution of $Z_n$ given $X_1, \ldots, X_n$ towards a geometric distribution with the parameter $\gamma$. This finishes the proof of Theorem 1.2.

Proof of Theorem 1.1 in the general case. Recall that it suffices to consider $\alpha \in [0, \infty) \setminus \mathbb{N}_0$. The continuous mapping theorem and Theorem 1.2 imply the weak convergence of the conditional distribution of $Z_n^\alpha$ given $X_1, \ldots, X_n$ towards $Z^\alpha$ if $Z$ is geometrically distributed with parameter $\gamma$. Since that sequence is uniformly integrable, we also have
\[
\lim_{n \to \infty} E(Z_n^\alpha|X_1, \ldots, X_n) = E(Z^\alpha) = \sum_{j \in \mathbb{N}} j^\alpha \gamma(1 - \gamma)^{j-1}.
\] (4.4)

As in (4.3), we see that
\[
\frac{L_n(\alpha)}{n} = \frac{R(n)}{n} E(Z_n^\alpha|X_1, \ldots, X_n).
\] (4.5)

Now use (4.4) and the fact that $\lim_{n \to \infty} R(n)/n = \gamma$ (recall [Sp64, T4.1]) to finish the proof of Theorem 1.1.

Acknowledgement. We thank two anonymous reviewers whose suggestions lead to improvements of our results.

Bibliography

[BS95] D.C. Brydges and G. Slade, The diffusive phase of a model of self-interacting walks. *Probab. Theory Related Fields* **103**, 285–315 (1995).

[Bo89] E. Bolthausen, A central limit theorem for two-dimensional random walks in random sceneries. *Ann. Prob.* **17**, 108–115 (1989).

[Ce07] J. Černý, Moments and distribution of the local time of a two-dimensional random walk. *Stoch. Proc. Appl.* **117**, 262–270 (2007).

[DE51] A. Dvoretzky and P. Erdős, Some problems on random walk in space. Proc. 2nd Berkeley Symp. Math. Statist. Probab., 353–367 (1950).

[ET60] P. Erdős and S.J. Taylor, Some problems concerning the structure of random walk paths. *Acta Math. Acad. Sci. Hungar* **11**, 137–162 (1960).

[Fe71] W. Feller, *An Introduction to Probability Theory and its Applications*. Wiley, New York (1971).

[JP71] N. C. Jain and W. E. Pruitt, The range of transient random walk. *J. Analyse Math.* **24**, 369–393 (1971).

[KS79] H. Kesten and F. Spitzer, A limit theorem related to a new class of self similar processes. *Z. Wahrsch. Verw. Gebiete* **50**, 5–25 (1979).

[P74] J.H. Pitt, Multiple points of transient random walks. *Proc. Amer. Math. Soc.* **43**, 195-199 (1974).
[Sc98] K. Schürger, *Wahrscheinlichkeitstheorie*. Oldenbourg, München (1998).
[Sp64] F. Spitzer, *Principles of Random Walk*. Van Nostrand, Princeton (1964).
[We94] G. H. Weiss, *Aspects and Applications of the Random Walk*. North-Holland, Amsterdam (1994).