Lambert $W$-Function Branch Identities

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January 19, 2021

Abstract

After defining in detail the Lambert $W$-function branches, we give a large number of exact identities involving (infinite) symmetric functions of these branches, as well as geometrically convergent series for all the branches. In doing so, we introduce a family of polynomials which may be of independent interest.

1 Introduction: the Lambert $W$-Function Branches

We begin by recalling the definition of the complex branches of the Lambert $W$-function. Originally, the Lambert $W$-function was defined as the solution of the implicit equation $we^w = z$ with $z \in \mathbb{R}$. It is immediate to see that for $z \geq 0$ there is a single solution, denoted $W_0(z)$, with a power series expansion given by Lagrange inversion

$$W_0(z) = \sum_{n \geq 1} \frac{(-1)^n-1}{n!} \frac{n^{n-1}}{n} z^n,$$

This power series converges for $|z| \leq 1/e$ (here and in the sequel $e = \exp(1)$), and provides an analytic continuation of $W_0(z)$ to the whole interval $[-1/e, \infty]$. When $-1/e \leq z < 0$ there exists a second solution $W_{-1}(z)$ characterized by $W_{-1}(z) \leq -1$. Note that $W_0(-1/e) = W_{-1}(-1/e) = -1$ and that $W_{-1}(z) \to -\infty$ as $z \to 0^-$. Finally, if $z < -1/e$ there is no real solution.

For completeness and comparison with later formulas, note that La-
grange inversion allows us to prove additional expansions such as

\[
W_0(z)^k = \sum_{n \geq k} (-1)^{n-k}k^{n-k-1}\frac{n^{n-k-1}}{(n-k)!}z^n
\]

for all \( k \in \mathbb{Z} \setminus \{0\} \), and

\[
\frac{1}{W_0(z) + 1} = \sum_{n \geq 0} (-1)^{n^n}n!z^n.
\]

More generally, if \( F \) is any power series, we have

\[
[z^n]F(W_0(z)) = [t^n]F(t)(1 + t)e^{-nt},
\]

where \([z^n]\) extracts the coefficient of \( z^n \), see [2].

It is natural to extend these definitions to all complex numbers. Taking logarithms in the initial implicit equation, we have

\[
w_k + \log(w_k) = \log(z) + 2k\pi i,
\]

where \( \log \) denotes the principal determination of the logarithm. Note that this is not the same as \( w_k + \log(w_k/x) = 2k\pi i \).

Because of the branching properties of the logarithm we will see that this equation may have 0, 1, or 2 solutions, so we must be more careful. Although the definitions are in the literature, for instance in [2], we prefer doing everything from scratch. The precise result is as follows:

**Proposition 1.1** Let \( z \in \mathbb{C} \setminus \{0\}, k \in \mathbb{Z}, \) and denote by \((E_k)\) the equation \( w + \log(w) = \log(z) + 2k\pi i \).

1. If \( z \notin ]-\infty,0[ \), equation \((E_k)\) has a unique complex solution denoted \( W_k(z) \), and we have \( W_{-k}(z) = W_k(\bar{z}) \).
2. If \( z \in ]-\infty,0[ \) and \( k \neq 0, -1, \) or \( z \in ]-\infty,-1/e[ \) and \( k = 0 \) or \( k = -1 \), equation \((E_k)\) has a unique complex solution denoted \( W_k(z) \), and we have \( W_{-k}(z) = W_{k-1}(z) \).
3. If \( z \in [-1/e,0[ \) and \( k = -1 \), equation \((E_k)\) has no solution (this is the only case in which there is no solution).
4. If \( z \in [-1/e,0[ \) and \( k = 0 \), equation \((E_k)\) has two real solutions denoted \( W_{-1}(z) \) and \( W_0(z) \) characterized by \( W_{-1}(z) \leq -1 \leq W_0(z) \), which coincide if \( z = -1/e \).

\( ^1 \)We evidently use the much more sensible French notation \([a,b]\) and similar for intervals, instead of the dreadful \((a,b)\) which has so many other different meanings.
We give the detailed elementary proof, since it is sometimes obscured in the literature. It can also be proved using the principle of the argument, but the proof is essentially the same, see [4].

Proof. As usual denoting by \log(y) the principal branch of the logarithm, we can write \log(z) = a + \lambda \pi i with \( a \in \mathbb{R} \) and \(-1 < \lambda \leq 1\). Equation \((E_k)\) is \( w + \log(w) = a + (2k + \lambda) \pi i \), so writing \( w = \rho e^{i\theta} \) with \(-\pi < \theta \leq \pi\) gives the system \( S_{2k+\lambda}(a) \), where \( S_{\mu}(a) \) is

\[
\rho \sin(\theta) + \theta = \mu \pi \quad \text{and} \quad \rho \cos(\theta) + \log(\rho) = a .
\]

Let us first find the solutions with \( \theta \in ]0, \pi[ \). The first equation has the unique solution \( \rho = (\mu \pi - \theta) / \sin(\theta) \), and since we have \( \rho > 0 \), we must have \( \theta < \mu \pi \). Thus if \( \mu \leq 0 \) there are no solutions with \( \theta \in ]0, \pi[ \), so assume \( \mu > 0 \). Replacing in the second equation gives

\[
F_{\mu}(\theta) := (\mu \pi - \theta) \cotan(\theta) + \log(\mu \pi - \theta) - \log(\sin(\theta)) = a .
\]

We compute that

\[
F'_{\mu}(\theta) = -\frac{\mu \pi - \theta - \sin(2\theta)}{\sin^2(\theta)} - \frac{1}{\mu \pi - \theta} ,
\]

and a tedious computation shows that \( F'_{\mu}(\theta) < 0 \) for all \( \theta \in ]0, \pi[ \) such that \( \theta < \mu \pi \). It follows that \( F_{\mu} \) is a strictly decreasing function of \( \theta \).

When \( \theta \to 0^+ \) we have \( F_{\mu}(\theta) \to +\infty \). If \( \mu > 1 \), when \( \theta \to \pi^- \) we have \( F_{\mu}(\theta) \to -\infty \), so \( F_{\mu} \) has a single root, and it follows that \( S_{\mu}(a) \) has a single solution. If \( \mu < 1 \), when \( \theta \to \mu \pi^- \) we again have \( F_{\mu}(\theta) \to -\infty \), so again \( S_{\mu}(a) \) has a single solution. Finally, if \( \mu = 1 \), when \( \theta \to \pi^- \) we compute that \( F_{1}(\theta) \to -1 \). It follows that if \( a > -1 \), \( S_{1}(a) \) has a single solution, while if \( a \leq -1 \) it has no solution.

To summarize, \( S_{\mu}(a) \) has a solution with \( \theta \in ]0, \pi[ \) if and only if \( \mu > 0 \) and either \( \mu \neq 1 \) or \( \mu = 1 \) and \( a > -1 \), and that solution is unique.

Since changing \( \theta \) into \(-\theta\) in the system simply changes \( \mu \) into \(-\mu\), we deduce that \( S_{\mu}(a) \) has a solution with \( \theta \in ]-\pi, 0[ \) if and only if \( \mu < 0 \) and either \( \mu \neq -1 \) or \( \mu = -1 \) and \( a > -1 \), and that solution is unique.

A solution of \( S_{\mu}(a) \) with \( \theta = 0 \) is possible if and only if \( \mu = 0 \) and \( \rho + \log(\rho) = a \), and since this is an increasing function of \( \rho \) and tends to \(-\infty \) when \( \rho \to 0^+ \) and to \(+\infty \) when \( \rho \to \infty \), we deduce that there always exists a solution with \( \theta = 0 \) when \( \mu = 0 \), and it is unique (and evidently equal to \( W_0(e^a) \)).

A solution of \( S_{\mu}(a) \) with \( \theta = \pi \) is possible if and only if \( \mu = 1 \) and \(-\rho + \log(\rho) = a \). This is a concave function of \( \rho \) tending to \(-\infty \) when
The solutions to Lemma 1.2 double solution if \( a \rightarrow 0^+ \) and \( \rho \rightarrow \infty \), with a maximum at \( \rho = 1 \) equal to \(-1\), so there exists a solution with \( \theta = \pi \) if and only if \( \mu = 1 \) and \( a \leq -1 \), and there are two solutions if \( a < -1 \) (evidently equal to \( W_0(-e^a) \) and \( W_{-1}(-e^a) \)), and a double solution if \( a = -1 \).

Summarizing, we have proved the following:

**Lemma 1.2** The solutions to \( S_\mu(a) \) are as follows:

1. If \( \mu > 1 \), or \( 0 < \mu < 1 \), a unique solution with \( \theta \in ]0, \pi[ \), if \( \mu < -1 \) or \(-1 < \mu < 0 \) a unique solution with \( \theta \in ]-\pi, 0[ \), and in both cases no solution with \( \theta = 0 \) or \( \theta = \pi \).
2. If \( \mu = -1 \) and \( a > -1 \), a unique solution with \( \theta \in ]-\pi, 0[ \), and if \( \mu = -1 \) and \( a \leq -1 \) no solution.
3. If \( \mu = 1 \) and \( a > -1 \), a unique solution with \( \theta \in ]0, \pi[ \), and if \( \mu = 1 \) and \( a \leq -1 \) two solutions with \( \theta = \pi \) which coincide for \( a = -1 \).
4. If \( \mu = 0 \), a unique solution with \( \theta = 0 \).

Since \( \mu = 2k + \lambda \) with \(-1 < \lambda \leq 1 \) and \( k \in \mathbb{Z} \), \( \mu > 1 \) is equivalent to \( k \geq 1 \), \( \mu < -1 \) is equivalent to \( k \leq -1 \), \(-1 < \mu < 1 \) and \( \mu \neq 0 \) is equivalent to \( k = 0 \) and \( \lambda \neq 0 \), \( \mu = 0 \) is equivalent to \( k = \lambda = 0 \), \( \mu = 1 \) is equivalent to \( k = 0 \) and \( \lambda = 1 \), and \( \mu = -1 \) is equivalent to \( k = -1 \) and \( \lambda = 1 \), so it is clear that this lemma is equivalent to the proposition.

**Corollary 1.3** Keep the same notation.

1. If \( z \not\in ]-\infty, 0[ \), or \( z \in ]-\infty, 0[ \) and \( k \neq 0, -1 \), or \( z \in ]-\infty, -1/e[ \) and \( k = 0 \) or \( k = -1 \), \( W_k(z) \) is the unique solution of \( (E_k) \), and \( W_{-k}(z) = W_k(z) \) if \( z \not\in ]-\infty, 0[ \), and \( W_{-k}(z) = W_{k-1}(z) \) otherwise.
2. If \( z \in ]-1/e, 0[ \), \( W_{-1}(z) \) is the unique real solution of \( E_0 \) (not of \( E_{-1} \)) \less than or equal to \(-1 \), and \( W_0(z) \) is the unique real solution of \( E_0 \) \greater or equal to \(-1 \).

To give a precise idea of what these results mean, let us give the example of \( W_k(x) \) for \(-2 \leq k \leq 1 \), and \( x \) in each of the three real intervals \([-\infty, -1/e[ \), \([-1/e, 0[ \), and \( ]0, \infty[ \); for instance \( x = -1 \), \( x = -0.1 \), and \( x = 1 \):

\[
\begin{align*}
W_{-2}(-1) &= -2.06227... - 7.58863... \\
W_{-2}(-0.1) &= -4.44909... - 7.30706... \\
W_{-1}(-1) &= -0.31813... - 1.33723... \\
W_{-1}(-0.1) &= -3.57715... \\
W_0(-1) &= -0.31813... + 1.33723... \\
W_0(-0.1) &= -0.11183... \\
W_1(-1) &= -2.06227... + 7.58863... \\
W_1(-0.1) &= -4.44909... + 7.30706...
\end{align*}
\]
In both of these cases, $W_{-k}(z) = \overline{W_{k+1}(z)}$, except in the second case for $k = 0$ and 1.

$$W_{-2}(1) = -2.40158... - 10.77629...$$
$$W_{-1}(1) = -1.53391... - 4.37518...$$
$$W_0(1) = 0.56714...$$
$$W_1(1) = -1.53391... + 4.37518...$$

In this case, $W_{-k}(z) = \overline{W_k(z)}$.

Remark. It is possible to define $W_k(z)$ for any $k \in \mathbb{C}$ (not only for $k \in \mathbb{Z}$) by the defining equation $W_k(z) + \log(W_k(z)) = \log(z) + 2k\pi i$. Of course, if $k \notin \mathbb{Z}$, this will not anymore correspond to a solution of $y e^y = z$.

Furthermore, it is clear that we can find $\theta$ such that $-1/2 < \theta \leq 1/2$ and such that $W_k(z) + \log(W_k(z)) = \log(z e^{2\theta \pi i}) + 2m\pi i$, where $m \in \mathbb{Z}$ is such that $|m - k| < 1$, so if for instance $z \notin [-\infty, 0]$, the solution to this equation will simply be $W_m(z e^{2\theta \pi i})$. A similar remark applies to the equation $w + \log(w/x) = 2k\pi i$ whose solution is $W_{k+\varepsilon}(x)$ for some $\varepsilon = -1, 0, 1$ depending on $k$ and $x$.

2 The Fundamental Identity

We begin by the following lemma:

**Lemma 2.1** Let $x \in \mathbb{C} \setminus \{0\}$ be fixed. We have

$$\lim_{t \to \pm \infty} \sum_{k \in \mathbb{Z}} \frac{1}{W_k(x) - t} = \mp \frac{1}{2},$$

where here and below a sum or product for $k \in \mathbb{Z}$ is understood as the limit as $K \to \infty$ of the sum or product for $-K \leq k \leq K$.

**Proof.** We will see below the trivial fact that $W_k(x) = 2k\pi i + O(\log(k))$ when $|k| \to \infty$, so the series indeed converges if we sum symmetrically. Denoting it by $f(t)$, we thus have $f(t) = f_1(t) + f_2(t)$ with

$$f_1(t) = \sum_{k \in \mathbb{Z}} \frac{1}{2k\pi i - t} \quad \text{and} \quad f_2(t) = \sum_{k \in \mathbb{Z}} \frac{2k\pi i - W_k(x)}{(W_k(x) - t)(2k\pi i + t)}.$$

Since $2k\pi i - W_k(x) = O(\log(k))$ the series for $f_2(t)$ converges uniformly in $t$, hence $\lim_{t \to \pm \infty} f_2(t) = 0$. On the other hand it is classical that $f_1(t) = -\coth(t/2)/2$, so the result follows. \qed
Theorem 2.2 For $x \in \mathbb{C} \setminus \{0\}$ we have

$$
\prod_{k \in \mathbb{Z}} \left(1 - \frac{t}{W_k(x)}\right) = e^{-t/2} - \frac{t}{x} e^{t/2} .
$$

Proof. The variable $x$ being fixed, consider the function $F(t) = e^{-t/2} - (t/x)e^{t/2}$. It has order 1, has no pole, and its zeros are the roots of $te^t = x$, so by definition they are the $W_k(x)$. By the Hadamard factorization theorem it thus has a Hadamard product which we can write in the form

$$
F(t) = Ae^{Bt} \prod_{k \in \mathbb{Z}} (1 - t/W_k(x)) .
$$

Note that the standard way of writing the product would be $\prod_{k \in \mathbb{Z}}(1 - t/W_k(x))e^{t/W_k(x)}$, but since we know that the product written as above converges if we interpret it as the limit of the product for $|k| \leq K$, it is preferable to write it as we have done.

We trivially have $A = F(0) = 1$. To determine $B$, we compute the logarithmic derivative:

$$
\frac{F'(t)}{F(t)} = B - \sum_{k \in \mathbb{Z}} \frac{1}{W_k(x) - t} .
$$

We immediately compute that $\lim_{t \to \pm\infty} F'(t)/F(t) = \pm1/2$, so making $t \to \infty$ or $t \to -\infty$ and using the lemma gives $B = 0$, proving the theorem. \hfill \Box

Remarks.

1. In the proofs of the lemma and the theorem, we have implicitly used the evident fact that for $t$ real with $|t|$ sufficiently large, for all $k \in \mathbb{Z}$ we have $t \neq W_k(x)$ and $t \neq 2k\pi i$.

2. Since $W_k(x)$ is close to $2k\pi i$, the above theorem should be compared with the identity (equivalent to the one we used above for $\coth(t/2)$):

$$
-t \prod_{k \in \mathbb{Z}\setminus\{0\}} \left(1 - \frac{t}{2k\pi i}\right) = e^{-t/2} - e^{t/2} .
$$

The following corollary gives the usual form of the Hadamard product:

Corollary 2.3 We have

$$
\prod_{k \in \mathbb{Z}} \left(1 - \frac{t}{W_k(x)}\right) e^{t/W_k(x)} = e^{t/x} \left(1 - \frac{t}{x} e^t\right) .
$$
Proof. Immediate from the theorem since identification of the coefficients of \( t^1 \) gives \( \sum_{k \in \mathbb{Z}} 1/W_k(x) = 1/2 + 1/x \).

Using the theorem, differentiating with respect to \( t \) or \( x \), and/or specializing, we easily find a large number of identities which we regroup in the following corollary:

**Corollary 2.4** We have the following identities:

\[
\sum_{k \in \mathbb{Z}} \frac{1}{W_k(x) - t} = \frac{1}{2} + \frac{1 + t}{xe^{-t} - t}, \quad \sum_{k \in \mathbb{Z}} \frac{1}{W_k(x) + 1} = \frac{1}{2}, \\
\sum_{k \in \mathbb{Z}} \frac{1}{W_k(x)} = \frac{1}{2} + \frac{1}{x}, \quad \sum_{k \in \mathbb{Z}} \frac{1}{W_k(x)^2} = \frac{2x + 1}{x^2}, \quad \sum_{k \in \mathbb{Z}} \frac{1}{W_k(x)^3} = \frac{3x^2 + 6x + 2}{2x^3}. \\
\prod_{k \in \mathbb{Z}} \left(1 - \frac{t}{W_k(x)}\right) = e^{-t/2} - \frac{xe^{t/2}}{x}, \quad \sum_{k \in \mathbb{Z}} \frac{1}{(W_k(x) + 1)(W_k(x) - t)} = \frac{1}{xe^{-t} - t}. \\
\sum_{k \in \mathbb{Z}} \frac{1}{(W_k(x) - t)^2} = \frac{xe^{-t}(2 + t) + 1}{(xe^{-t} - t)^2}, \quad \sum_{k \in \mathbb{Z}} \frac{1}{(W_k(x) + 1)^2} = \frac{1}{xe + 1}. \\
\sum_{k \in \mathbb{Z}} \frac{1}{(W_k(x) + 1)(W_k(x) - t)^2} = \frac{xe^{-t} + 1}{(xe^{-t} - t)^2}, \quad \sum_{k \in \mathbb{Z}} \frac{1}{(W_k(x) + 1)^3} = \frac{1}{xe + 1}. \\
\sum_{k \in \mathbb{Z}} \frac{W_k'(x)}{W_k(x) - t} = \frac{1}{x} \left(\frac{1}{2} + \frac{t}{xe^{-t} - t}\right), \quad \sum_{k \in \mathbb{Z}} \frac{W_k'(x)}{W_k(x)} = \frac{1}{2x}. \\
\sum_{k \in \mathbb{Z}} \frac{1}{W_k(x)^2 + t^2} = \frac{x(cos(t) + sin(t)/t) + 1}{x^2 + 2xt sin(t) + t^2} \quad \text{and} \quad \sum_{k \in \mathbb{Z}} \frac{W_k(x)}{W_k(x)^2 + t^2} = \frac{1}{2} + \frac{x(cos(t) - t sin(t)) - t^2}{x^2 + 2xt sin(t) + t^2}.
\]

**Acknowledgment:** I am grateful to I. Pinelis for having given the identity \( \sum_{k \in \mathbb{Z}} 1/(W_k(x) + 1) = 1/2 \) in the mathoverflow forum, which led to the present section of this paper, but his proof is completely different, using an integral representation of this sum coming from the residue theorem.
3 Asymptotics and Sum and Product Formulas

In this section we are going to show that even nonconvergent symmetric sums or products of the $W_k$ can be evaluated explicitly after removing the asymptotic main term. First note the following easy proposition:

**Proposition 3.1** The variable $x$ being fixed, as $|k| \to \infty$ we have

$$W_k(x) = 2k\pi i - \log(2k\pi i) + \log(x) + \frac{\log(2k\pi i) - \log(x)}{2k\pi i} + O\left(\frac{\log(|k|^2)}{k^2}\right).$$

**Proof.** For notational simplicity set $K = 2k\pi i$, $L = \log(2k\pi i) = \log(K)$, $X = \log(x)$, and $M = L - X = \log(2k\pi i) - \log(x)$. By successive approximation, we find that if we set $w_k = K - M + M/K + \varepsilon/K$ we have $\log(w_k) = L - M/K + O(L^2/K^2)$, hence $w_k + \log(w_k) = K + X + \varepsilon + O(L^2/K^2)$, and using that $w_k + \log(w_k) = \log(x) + 2k\pi i$ we see that $\varepsilon = O(L^2/K^2)$. \qed

Note that we will give below the complete asymptotic expansion.

**Corollary 3.2** Let the variable $x$ be fixed.

1. For $k > 0$ sufficiently large we have $(2k - 1)\pi < |W_k(x)| < (2k + 1)\pi$.

2. If $x$ is positive real, then for $K$ sufficiently large, the solutions to the equation $w^x = x$ with $|w| < (2K + 1)\pi$ are the $W_k(x)$ for $-K \leq k \leq K$.

**Proof.** Immediate from the proposition and left to the reader. \qed

Note that if $x$ is not a positive real, there may be an additional solution for $k = \pm(K + 1)$.

**Theorem 3.3** We have

$$\lim_{K \to \infty} \left( \sum_{-K \leq k \leq K} W_k(x) + \log(\pi^{2K}(2K)!) - (2K + 1/2)\log(x) \right) = \frac{\log(2)}{2}.$$  

**Proof.** We will prove this theorem in two stages. First we will prove that the left-hand side is a constant independent of $x$. Second, the appendix due to Letong Hong and Shengtong Zhang proves that this constant is equal to $\log(2)/2$, which I conjectured in the first version of this paper. Note that
their proof also shows that the left-hand side is independent of \( x \), but we have chosen to give the present proof first since it handles more easily the case where \( x \) is not a positive real.

For \( k > 0 \) we have

\[
W_k(x) + W_{-k}(x) = -2 \log(2k\pi) + 2 \log(x) + 1/(2k) + O(\log(k)^2/k^2) .
\]

It follows that

\[
\sum_{-K \leq k \leq K} W_k(x) = -2 \log(K!) + 2K \log(x/(2\pi)) + \log(K)/2 + F(x) + o(1)
\]

for some function \( F \). By Stirling’s formula, we have

\[
-2 \log(K!) - 2K \log(2\pi) + \log(K)/2 = -(2K + 1/2) \log(2K) + 2K - 2K \log(\pi) - \log(4\pi) + o(1)
\]

so

\[
\sum_{-K \leq k \leq K} W_k(x) = -\log((\pi/x)^{2K}(2K)!) + A(x) + o(1)
\]

for some other function \( A(x) \). To find \( A(x) \), we note that

\[
A'(x) + \frac{2K}{x} + o(1) = \sum_{-K \leq k \leq K} W'_k(x) .
\]

Now since \( W'_k(x) = (1/x)W_k(x)/(W_k(x) + 1) = (1/x)(1 - 1/(W_k(x) + 1)) \), we have by Pinelis’s formula

\[
\sum_{-K \leq k \leq K} W'_k(x) = \frac{2K + 1/2}{x} + o(1),
\]

so \( A'(x) = 1/(2x) \), hence \( A(x) = \log(x)/2 + A \) for some constant \( A \). The fact that \( A \) is real follows by choosing \( x > 0 \) and using that \( W_{-k}(x) = \overline{W_k(x)} \), proving that the left-hand side is indeed a real constant. We refer to the appendix for the proof that \( A = \log(2)/2 \) and a second proof of the fact that the left-hand side is constant. \( \square \)

The proof shows that the implicit \( o(1) \) term is a \( O(\log(K)^2/K) \).

\[\text{[Footnote]}\]

This is not quite rigorous, but one can prove that the derivative of the \( o(1) \) is still a \( o(1) \).
Corollary 3.4 We have
\[ \lim_{K \to \infty} \frac{\prod_{-K \leq k \leq K} W_k(x)}{\pi^{2K}(2K)!} = \sqrt{x/2}, \]
and more generally
\[ \lim_{K \to \infty} \frac{\prod_{-K \leq k \leq K} (W_k(x) - t)}{\pi^{2K}(2K)!} = \sqrt{x/2(e^{-t/2} - (t/x)e^{t/2})}, \]

Proof. By definition, we have \( \log(W_k(x)) = \log(x) + 2k\pi i - W_k(x) \). It follows from the theorem that
\[ \sum_{-K \leq k \leq K} \log(W_k(x)) = (2K + 1)\log(x) - \sum_{-K \leq k \leq K} W_k(x) \]
\[ = \log(\pi^{2K}(2K)!)) + \log(x)/2 - \log(2)/2 + o(1), \]
so the result follows for \( t = 0 \), and the more general result from the formula for \( \prod_k (1 - t/W_k(x)) \) given above. \( \square \)

4 A Family of Polynomials

Before going further, we are now going to study in great detail a family of polynomials important for our work. These polynomials date back at least to Comtet [1].

Definition 4.1 Let \( A(X) \) be a given polynomial such that \( A(0) = 0 \).

1. We define the Lambert family of polynomials \( P_n(X) = P_{n,A}(X) \) associated to \( A \) by \( P_0(X) = A(X) \) and for \( n \geq 0 \) by the recursion
\[ P_{n+1}(X) = -P_n(X) + n \int_0^X P_n(t) \, dt. \]

2. We denote by \( F_A(T,X) = \sum_{n \geq 0} P_{n,A}(X)T^n \) the generating function of the \( P_n(X) \).

Lemma 4.2 \( P_n \) is the Lambert family associated to \( A \) if and only if
\[ (1 + T) \frac{\partial F_A}{\partial X} = A'(X) + T^2 \frac{\partial F_A}{\partial T} \]
and \( P_n(0) = 0 \) for all \( n \).
Proof. Since $\partial F_A/\partial X = \sum_{n \geq 0} P_n'(X)T^n$ we have

$$(1 + T)\partial F_A/\partial X = dP_0/dX + \sum_{n \geq 1} (P_n'(X) + P_{n-1}'(X))T^n$$

$$= A'(X) + \sum_{n \geq 1} (n - 1)P_{n-1}'(X)T^n = A'(X) + T^2\partial F_A/\partial T .$$

Conversely, if this is satisfied we have $P_0'(X) = A'(X)$ and $P_n'(X) + P_{n-1}'(X) = (n - 1)P_{n-1}'(X)$, and since we assume in addition that $P_n(0) = 0$ we have $P_0(X) = A(X)$ and $P_n(X) + P_{n-1}(X) = (n - 1) \int_0^X P_{n-1}(t) \, dt$. □

**Lemma 4.3** If $P_n$ is the Lambert family associated to $A(X)$ then $\int_0^X P_n(t) \, dt$ is the Lambert family associated to $\int_0^X A(t) \, dt$.

Proof. If $Q_n(X) = \int_0^X P_n(t) \, dt$ it is clear that $Q_{n+1}'(X) + Q_n'(X) = nQ_n(X)$, so the result follows since $Q_n(0) = 0$. □

**Theorem 4.4** Let $L_n$ be the Lambert family associated to $A(X) = -X$, $F(T,X) = \sum_{n \geq 0} L_n(X)T^n$ its generating function, and $k \geq 1$. By abuse of notation, write $F^k(T,X) = \sum_{n \geq 0} L_{n,k}(X)T^n$ (including for $k \leq -1$), so that $L_{n,1}(X) = L_n(X)$.

1. For $n \geq 1$ we have

$$L_{n,2}(X) = -2(n + 1) \int_0^X L_n(t) \, dt = -\frac{2(n + 1)}{n} (L_{n+1}(X) + L_n(X)) .$$

Equivalently, $L_{n,2}(X)/(n + 1)$ is the Lambert family associated to $A(X) = X^2$.

2. The function $F$ satisfies the differential equation in $T$

$$TFF' + (1 + T)F' + F = 0 .$$

3. The function $F$ satisfies the implicit equation

$$\log(1 + TF) + F = -X .$$

Proof. (1). By Lemma 4.2 applied to $A(X) = -X$ we have

$$(1 + T)\partial (F^k)/\partial X = kF^{k-1}(1 + T)\partial F/\partial X$$

$$= kF^{k-1}(A'(X) + T^2\partial F/\partial T) = -kF^{k-1} + T^2\partial (F^k)/\partial T .$$

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Identifying coefficients of $T^n$, it follows that

$$L'_{n,k}(X) + L'_{n-1,k}(X) = (n-1)L_{n-1,k}(X) - kL_{n,k-1}(X).$$

We prove (1) by induction. Since $L_{0,2}(X) = (-X)^2 = (-2) \int_0^X (-t) \, dt$, it is true for $n = 0$. Assume that it is true for $n-1$: we thus have

$$L'_{n,2}(X) = 2nL_{n-1,1}(X) - 2(n-1)n \int_0^X L_{n-1,1}(t) \, dt - 2L_{n,1}(X)$$

and since $(n-1) \int_0^X L_{n-1,1}(t) \, dt = L_{n,1}(X) + L_{n-1,1}(X)$ we obtain $L'_{n,2}(X) = -2(n+1)L_{n,1}(X)$, proving our induction hypothesis since $L_{n,k}(0) = 0$ for all $n, k$. It follows from Lemma 4.3 that $L_{n,2}(X)/(n+1)$ is the Lambert family associated to $X^2$.

(2). For $n \geq 1$ the coefficient of $T^n$ in $T F F' + (1 + T)F' + F = (T/2)(F^2)' + F' + T F F' + F$ is equal to

$$(n/2)L_{n,2}(X) + (n+1)L_{n+1,1} + nL_{n,1} + L_{n,1}$$

$$= -n(n+1) \int_0^X L_{n,1}(t) \, dt + (n+1)n \int_0^X L_{n,1}(t) \, dt = 0,$$

and the coefficient of $T^0$ is $L_{0,1}(X) + L_{1,1}(X) = -X + X = 0$, proving the differential equation.

(3). If we set $G = \log(1 + T F) + F$, we have

$$G' = (T F' + F)/(1 + T F) + F'$$

$$= ((1 + T)F' + F + T F F)/ (1 + T F) = 0$$

by (2), so $G(T) = G(0) = F(0) = -X$. \qed

Remarks.

1. The proof of (3) shows that the general solution to the differential equation in (2) is a solution to the implicit equation $\log(1 + T F) + F = A(X)$, where $A(X) = F(0)$.

2. It is clear that the theorem can be proved by using Lagrange inversion, but I have preferred to give the above direct proof.

It is natural to call the $L_n(X)$ the Lambert polynomials.
Proposition 4.5  In addition to the partial differential equation \( (1+T)\partial F/\partial X = -1 + T^2 \partial F/\partial T \) of Lemma 4.2, the function \( F \) also satisfies the following ones:

\[
\frac{\partial F}{\partial X} = -\frac{TF + 1}{TF + T + 1} \quad \text{and} \quad F \frac{\partial F}{\partial X} = (TF + 1) \frac{\partial F}{\partial T} = -T \frac{\partial F}{\partial T} - F.
\]

Proof. By Lemma 4.2 and Theorem 4.4 (2) we have

\[
\frac{\partial F}{\partial X} = \frac{1}{1+T} \left( -1 + T^2 \frac{\partial F}{\partial T} \right)
= -\frac{1}{1+T} \left( 1 + T^2 \frac{F}{TF + T + 1} \right) = -\frac{TF + 1}{TF + T + 1}.
\]

Thus, by Theorem 4.4 once again we have \( TF + T + 1 = -F/F' \), hence \( \partial F/\partial X = (TF + 1)(\partial F/\partial T)/F \), and the last formula again follows from the theorem. \( \square \)

Corollary 4.6  We have the identity

\[
\sum_{n \geq 1} \frac{L_n(X)}{n} T^n = -\log \left( \frac{F(T,X)}{-X} \right).
\]

Proof. Indeed, if we differentiate with respect to \( T \), the derivative of the left hand side is \( \sum_{n \geq 1} L_n'(X)T^{n-1} = (\partial F/\partial X + 1)/T \), while that of the right-hand side is \(- (\partial F/\partial T)/F \), and equality follows from the last formula given by the proposition. The identity follows since both sides vanish for \( T = 0 \), \( \square \)

Corollary 4.7  1. Generalizing (1) of the theorem, for \( k \geq 2 \) we have

\[
L_{n,k}(X) = -\frac{k}{k-1} \frac{(n+k-1)}{n} \int_0^X L_{n,k-1}(t) \, dt
\]

and \( L_{n,k}(X)/(n+k-1) \) is the Lambert family associated to \( A(X) = (-X)^k \).

2. For all \( k \geq 2 \) we have

\[
L_{n,k}(X) = (-1)^{k-1} k \frac{(n+k-1)}{n} \int_0^X (X-t)^k L_n(t) \, dt
= (-1)^{k-1} k \frac{(n+k-1)}{n} \int_0^X (X-t)^k L_n'(t) \, dt.
\]
3. For any polynomial \( A(X) \) such that \( A(0) = 0 \), the Lambert family associated to \( A \) is given by

\[
P_{n,A}(X) = -\int_0^X A'(X - t)L'_n(t) \, dt = -A'(0)L_n(X) - \int_0^X A''(X - t)L_n(t) \, dt.
\]

Proof. (1). Multiplying by \( F^{k-2} \) the second identity of the proposition we obtain \( F^{k-1}\partial F/\partial X = -TF^{k-2}\partial F/\partial T - F^{k-1} \), and identifying the coefficients of \( T^n \) gives \( \L_{n,k}/k = -((n + k - 1)/(k - 1))L_{n,k-1} \), proving the first formula since \( L_{n,k}(0) = 0 \).

It is clear that \( (n!/(n + k - 1)!L_{n,k}(X) \) is a Lambert family for \( k = 1 \). Assume that this is the case for \( k - 1 \) with \( k \geq 2 \). Using the formula just proved and using the induction hypothesis, we have

\[
\frac{n!}{(n + k - 1)!}L'_{n,k}(X) + \frac{(n - 1)!}{(n + k - 2)!}L'_{n-1,k}(X)
= -\frac{k}{k-1} \left( \frac{n!}{(n + k - 2)!}L_{n,k-1}(X) + \frac{(n - 1)!}{(n + k - 3)!}L_{n-1,k-1}(X) \right)
= -\frac{k}{k-1}(n - 1)\frac{(n - 1)!}{(n + k - 3)!} \int_0^X L_{n-1,k-1}(t) \, dt
= (n - 1)\frac{(n - 1)!}{(n + k - 2)!}L_{n-1,k}(X),
\]

proving that \( (n!/(n+k-1)!L_{n,k}(X) \) is a Lambert family, and it is associated to \( L_{0,k}(X)/(k - 1)! = (-X)^k/(k - 1)! \). Using this and replacing in the first formula proves the second, proving (1).

(2). This follows immediately from the formula giving iterated integrals.

(3). By linearity, it is clear that

\[
P_{n,A}(X) = -A'(0)L_n(X) - \int_0^X A''(X - t)L_n(t) \, dt,
\]

so the result follows by integrating by parts. \( \square \)

The polynomials \( L_{n,k} \) can be given explicitly:

**Proposition 4.8** We have

\[
L_{n,k}(X) = (-1)^{n+k}k! \binom{n + k - 1}{n} \sum_{j=1}^n s(n, n + 1 - j) \frac{X^{k-1+j}}{(k-1+j)!},
\]

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where the $s(n,m)$ are the Stirling numbers of the first kind. In particular,

$$L_n(X) = (-1)^{n+1} \sum_{j=1}^{n} s(n, n+1 - j) \frac{X^j}{j!}, \quad \text{or equivalently,}$$

$$\int_0^\infty e^{-xt} L_n(t) \, dt = X^{-(n+1)} \prod_{j=1}^{n-1} (j - X).$$

**Proof.** The result for $k = 1$ follows immediately from the recursion for Stirling numbers $s(n+1, k) = s(n, k - 1) - ns(n, k)$, the general result by successive integration, and the integral formula from the definition of Stirling numbers. \hfill \Box

**Remark.** In view of the above results, it would seem that the most natural polynomials are the $L_n'(X)$ and not the $L_n(X)$: they are monic, have constant term equal to $(-1)^{n-1}$, satisfy the same Lambert-type recursion (but they are not a Lambert family since $L_n'(0) \neq 0$), but it is immediate to see that the equations involving them are more complicated, so we have preferred to take the $L_n(X)$ themselves as fundamental building blocks. We will see in the next section that the higher derivatives of $L_n$ enter in the expansion of $F^k$ for $k \leq -1$.

The first few $L_n(X)$ are the following:

$L_0(X) = -X$, $L_1(X) = X$, $L_2(X) = X^2/2 - X$,
$L_3(X) = X^3/3 - (3/2)X^2 + X$, $L_4(X) = X^4/4 - (11/6)X^3 + 3X^2 - X$.

## 5 Further Results on Lambert Polynomials

The preceding section dealt essentially with the explicit computation of the polynomials $L_{n,k}$ when $k \geq 1$. The present section deals with the case $k \leq -1$.

**Proposition 5.1** We have the additional partial differential equation

$$\frac{\partial^2 F}{\partial X^2} = -T^2 \frac{\partial^2 (1/F)}{\partial T^2}.$$  

**Proof.** By Proposition 4.5 we have $\partial F/\partial X = -(TF + 1)/(TF + T + 1)$. Differentiating once more we deduce that

$$\frac{\partial^2 F}{\partial^2 X} = -\frac{T^2}{(TF + T + 1)^2} \frac{\partial F}{\partial X} = T^2 \frac{TF + 1}{(TF + T + 1)^3}.$$
On the other hand, writing for simplicity \( \frac{1}{F}' \) instead of \( \frac{\partial}{\partial T} \), by Theorem 4.4 (2) we have \((1/F)' = -F'/F^2 = 1/(F(TF + T + 1))\). Now,

\[
(TF^2 + TF + F)' = F^2 + 2TFF' + F + TF' + F' = F^2 + TFF'
\]

\[
= F^2 - TF^2/(TF + T + 1) = F^2(TF + 1)/(TF + T + 1)
\]

hence \((1/F)'' = -(TF + 1)/(TF + T + 1)^3\), proving the proposition. \(\Box\)

**Corollary 5.2** Recall that we have defined \(L_{n,k}\) for all \(k \in \mathbb{Z}\) by \(F_k(T, X) = \sum_{n \geq 0} L_{n,k}(X) T^n\).

1. We have \(L_{0,-1}(X) = L_{1,-1}(X) = -1/X\), and for \(n \geq 2\)

\[
L_{n,-1}(X) = -\frac{L_n''(X)}{n(n-1)}, \quad \text{in other words}
\]

\[
\frac{1}{F(T, X)} = -1/X - (1/X)T - \sum_{n \geq 2} \frac{L_n''(X)}{n(n-1)} T^n.
\]

2. More generally, let \(k \leq -2\). We have

\[
(n + k)L_{n,k}(X) = -\frac{k}{(k+1)} L_{n,k+1}'(X),
\]

and for \(n \geq 1 - k\) we have

\[
L_{n,k}(X) = (-1)^{k-1} k \frac{(n + k - 1)!}{n!} L_n^{(1-k)}(X).
\]

**Proof.** (1). The formulas for \(L_{0,-1}(X)\) and \(L_{1,-1}(X)\) are obtained directly, and the formula for \(L_{n,-1}(X)\) is equivalent to the PDE of the proposition. It can also be easily obtained by differentiating the formula of Corollary 4.6.

(2). Multiplying by \(F^{k-1}\) the last PDE of Proposition 4.5, we obtain \(F^k \partial F/\partial X = -TF^{k-1}\partial F/\partial T - F^k\), and identifying the coefficients of \(T^n\) gives \(L_{n,k+1}'/(k + 1) = -((n + k)/k) L_{n,k}\), proving the first formula, and the second follows by induction. \(\Box\)

The above corollary does not give \(L_{n,k}\) for \(n \leq -k\). For this, we need to introduce another family of polynomials, closely linked to \(L_n\) as follows:

**Definition 5.3** For \(n \geq 1\) we define a family \(M_n\) of functions by \(M_1(X) = -1/X\) and \(M_{n+1}(X) = M_n'(X)/n - M_n(X)\).
Note that the $M_n$ are polynomials in $1/X$ (for instance $M_2(X) = (X + 1)/X^2$ and $M_3(X) = -(2X^2 + 3X + 2)/(2X^3)$), but the derivative $M'_n(X)$ is of course taken with respect to $X$, not with respect to $1/X$.

**Proposition 5.4** Let $k \leq -1$.

1. We have

$$L_{n,k} = \begin{cases} \frac{k M_n^{(-k-n)}(X)}{n \ (-k-n)!} & \text{for } 1 \leq n \leq -k, \\ (-X)^k & \text{for } n = 0. \end{cases}$$

2. In other words, for $k \leq -1$ we have

$$F(T, X)^k = (-X)^k - k \sum_{1 \leq n \leq -k} \frac{M_n^{(-k-n)}(X)}{n(-k-n)!} T^n \right.$$

$$\left. + (-1)^{k-1} k \sum_{n \geq 1-k} \frac{(n+k-1)!}{n!} L_n^{(1-k)}(X) T^n. \right.$$

**Proof.** In the proof of Theorem 4.4 (1) we have seen the recursion

$$L'_{n,k}(X) + L'_{n-1,k}(X) = (n-1)L_{n-1,k}(X) - kL_{n,k-1}(X),$$

which is valid for all $k$ including negative ones. Thus, for $n \geq 1$ we have

$$L'_{n+1,\ -n}(X) + L'_{n,-n}(X) = nL_{n,-n}(X) + nL_{n+1,-n-1}(X).$$

On the other hand, the recursion given in (2) of the above corollary implies that $L'_{n+1,-n}(X) = 0$ for $n \geq 1$. We thus have the recursion

$$L'_{n+1,-n}(X) = L'_{n,-n}(X)/n - L_{n,-n}(X),$$

which is the recursion for $M_n$, proving the result for $n = -k$ since $L_{1,-1}(X) = -1/X = M_1(X)$.

The same recursion for $1 \leq n < -k$ gives

$$L_{n,k}(X) = \frac{-k}{(k+(k+1)(k+n))} L_{n,k+1}(X),$$

so by induction for $1 \leq j \leq -k - n$ we have, since $n \geq 1$ hence $j \leq -k - 1$:

$$L_{n,k}(X) = \frac{k}{k+j} \frac{(-k-j-n)!}{(-k-n)!} L_{n,j+k}^{(j)},$$

and choosing $j = -k - n$ and using what we just proved gives the result for $1 \leq n \leq -k$. The case $n = 0$ is trivial directly since $L_{0,k}$ is the coefficient of $T^0$ in $F^k$, hence equal to $(-X)^k$. \hfill \Box

The functions $M_n$ can also be given explicitly in terms of Stirling numbers:
Proposition 5.5

\[ M_n(X) = \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^{j-1} \frac{s(n, j+1)j!}{X^{j+1}} = \frac{(-1)^n}{(n-1)!} \int_0^\infty e^{-Xt} \prod_{j=1}^{n-1} (t+j) \, dt. \]

\[ \sum_{j=0}^{n} (-1)^{j-1} \frac{s(n, j+1)j!}{X^{j+1}} = \left( \frac{-1}{(n-1)!} \int_0^\infty e^{-Xt} \prod_{j=1}^{n-1} (t+j) \, dt \right). \]

Proof. Left to the reader. \qed

The coefficients of \( M_n(X) \) are of course closely related to those of \( L_n(X) \), but I do not see any natural way of expressing \( M_n(X) \) in terms of the \( L_n(X) \), except by the following formula coming from the definition of the beta function:

Proposition 5.6

\[ M_n(X) = \frac{n(n+1)}{X^{n+1}} \int_0^\infty \frac{L_n(-Xt)}{(t+1)^{n+2}} \, dt. \]

Proof. Simply use the expansions of \( L_n(X) \) and \( M_n(X) \) in terms of Stirling numbers. In addition, one can integrate once or twice by parts and obtain similar formulas involving \( L'_n \) and \( L''_n \). \qed

6 Convergent Series for the Lambert Branches

The result of Proposition 3.1 is asymptotic, and can be pushed further if desired. The result is that not only does it give an asymptotic expansion to any desired number of terms, but in fact a convergent series for \( W_k(x) \) even for small \( k \). We can even put these series in a slightly more general setting as follows.

Proposition 6.1 Fix \( x \in \mathbb{C} \setminus \{0\} \), let \( K \in \mathbb{C} \setminus \{0\} \), and define

\[ M = K + \log(K) - \log(x) - 2k\pi i. \]

1. As \( |K| \to \infty \) we have the asymptotic expansions

\[ W_k(x) = K + \sum_{n \geq 0} \frac{L_n(M)}{K^n} \quad \text{and} \quad \log(W_k(x)) = \log(K) - \sum_{n \geq 1} \frac{L_n(M)}{K^n}, \]

where the \( L_n(z) \) are as usual the Lambert polynomials.

2. These series converge to \( W_k(x) \) and \( \log(W_k(x)) \) when \( 2(1+|M|) < |K| \) (we will see below the exact domain of convergence in special cases).
3. More generally, we have the following formulas:

\[
(W_k(x) - K)^j = \sum_{n \geq 0} \frac{L_{n,j}(M)}{K^n} \quad \text{for all } j \in \mathbb{Z} \setminus \{0\},
\]

\[
-\frac{M}{W_k(x) - K} = 1 + \frac{1}{K} + \sum_{n \geq 2} \frac{M L''_n(M)}{n(n-1)K^n};
\]

\[
\frac{1}{1 + W_k(x)} = \sum_{n \geq 1} \frac{L'_n(M)}{K^n}, \quad \text{and } \log \left( \frac{W_k(x) - K}{-M} \right) = -\sum_{n \geq 1} \frac{L'_n(M)}{nK^n}
\]

with the same convergence properties.

Proof. (1). Denote by \( R \) the right-hand side of the expansion for \( W_k(x) \), and recall that \( F(T, X) = \sum_{n \geq 0} L_n(X)T^n \). By Theorem 4.4, we have

\[ \log(R) = \log(K) + \log(1 + (1/K)F(1/K, M)) = \log(K) - M - F(1/K, M) . \]

It follows that

\[ R + \log(R) = K + \log(K) - M = \log(x) + 2k\pi i , \]

which is exactly equation \((E_k)\) defining the Lambert branches, and since \( W_k(x) \) is the solution of \((E_k)\) when \( k \neq 0, 1 \), we deduce that it gives its asymptotic expansion. The expansion for \( \log(W_k(x)) \) follows from \( W_k(x) + \log(W_k(x)) = \log(x) + K \).

(2). Let us study the convergence properties. By the recursion formula for Stirling numbers, it is immediate to show that \(|s(n, k)| \leq 2^n n! / k!\), so replacing in the explicit formula for \( L_n(X) \) we deduce that \(|L_n(M)| \leq 2^n (1 + |M|)^n\), proving the convergence result. Using the stronger bound \(|s(n, n-k)| \leq (n/2)^k (n-1)^{k-1}\) conjectured by René Gy and proved by Mike Earnest in the stackexchange forum, we can replace \( 2^n \) by \( e^{n/2} \).

(3). These formulas follow from (1), the definition, and using \( W'_k(x) = W_k(x) / (x(1 + W_k(x))) \) and \( L'_0(X) = -1 \).

Corollary 6.2 1. We have the asymptotic expansions

\[ W_k(x) = K + \sum_{n \geq 0} \frac{L_n(M)}{K^n} \]

(as well as all the others given in the proposition) for instance for \( K = 2k\pi i \) and \( M = \log(K) - \log(x) \) (which gives an asymptotic expansion
when $|k| \to \infty$), or for $K = 2k\pi i + \log(x)$ and $M = \log(K)$ (which
gives another asymptotic expansion when $|k| \to \infty$, and also when
$x \to \infty$ or $x \to 0$).

2. If $x \in [-1/e, 0]$, we have the same expansions also for $k = -1$ by
choosing $K = \log(-x)$ and $M = \log(-K)$.

3. All these expansions converge if and only if

$$\log(|K|) + 1 - \Re(M) + \min_{m \in \{-1, 0\}} \Re(W_m(-e^{M-1})) > 0$$

(note that this condition is automatically satisfied in case (2)).

Proof. (1) is clear, (2) is proved in exactly the same way as the propo-
sition, and (3) is proved in [2] and [3].

Remarks.

1. The series with $K = 2k\pi i + \log(x)$ is the one usually given, for instance
in [1] and [2]. However, when $k \neq 0$ the series with $K = 2k\pi i$ often
converges faster.

2. We could choose $K = W_k(x)$, in which case $M = 0$ by definition, so
the identity is a triviality since $L_n(0) = 0$ for all $n \geq 0$.

3. Even for $k = 1$ (which is certainly not in the “asymptotic” regime)
and $x = 1$, we can compute $W_1(1)$ to 38 decimals using slightly more
than 100 terms of the series (of course the usual iterative methods
to compute $W_k(x)$ are much more efficient since they are based on
Newton or Halley iterations).

4. We have remarked above that it is possible, although not very useful,
to define $W_k(x)$ even when $k \notin \mathbb{Z}$. It is clear that by construction
the above series are still valid in this more general context.

In [3], some alternate expansions are given for the principal branch $W_0$, but
they are trivially generalizable to all branches. For instance, instead of
choosing $K$ as main variable, we can choose $K_1 = K + 1$. Thus, we write

$$W_k(x) = K + F(1/K, M) = K_1 - 1 + F(1/(K_1 - 1), M)$$

$$= K_1 - 1 + \sum_{n \geq 0} L_n(M)/(K_1 - 1)^n$$

$$= K_1 - 1 + \sum_{n \geq 0} L_n(M)/K_1 \sum_{j \geq 0} \left( \frac{n + j - 1}{n - 1} \right) / K_1^j$$

$$= K_1 - 1 + \sum_{N \geq 0} \left( \frac{1}{K_1^N} \right) \sum_{n = 0}^N \left( \frac{N - 1}{n - 1} \right) L_n(M),$$

so

$$W_k(x) = K_1 - 1 + \sum_{N \geq 0} \frac{P_n(M)}{K_1^N} \quad \text{with} \quad P_n(X) = \sum_{n = 0}^N \left( \frac{N - 1}{n - 1} \right) L_n(X),$$

and these new polynomials $P_n(X)$ can be expressed using another type of Stirling numbers, and also have recursion properties coming from those of $L_n(X)$, such as

$$P_{n+1}(X) = \int_0^X \left( nP_n(t) - (n - 1)P_n(t) \right) dt ,$$

whose proof is left to the reader. The first few $P_n(X)$ are

$$P_0(X) = -X, \quad P_1(X) = X, \quad P_2(X) = X^2/2, \quad P_3(X) = X^3/3 - X^2/2, \quad P_4(X) = X^4/4 - (5/6)X^3, \quad P_5(X) = X^5/5 - (13/12)X^4 + X^3/2.$$

In particular, note that $P_N(X)$ is divisible by $X^{\lfloor N/2 \rfloor + 1}$.

For the branch $k = 0$, [3] mentions that the domain of convergence is much larger than that of the original series, but for the other branches the difference is not that large since $K + 1$ is not so different from $K$.

## 7 Additional Properties and Conjectures

**Conjecture 7.1**  1. The roots of $L_n(X)$ are all nonnegative real, and less than or equal to $n + \log(n) + \gamma + O(1/n)$, where $\gamma$ is Euler’s constant.

2. More generally, the $j$th root of $L_n(X)$ in decreasing order is asymptotic to $n/j + \sum_{j \leq m \leq n} 1/m$. 

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3. The roots of $L_n(X)$ and $L_{n+1}(X)$ interlace.

4. More generally (and more precisely), for $k \geq 1$ the roots of $L_{n,k}(X)$ are all nonnegative real, and their maximum has an asymptotic expansion

$$n + \log(n) + \gamma + k - 1 + C_0(k)/n + C_1(k)/n^2 + \ldots,$$

where $C_m(k)$ is a polynomial of degree $m$ in $k$, in particular $C_0(k)$ is independent of $k$. This is still true for $k \leq 0$ if we interpret $L_{n,k}(X)$ as $L_n^{(1-k)}(X)$.

Note that the fact that the real roots of $L_{n,k}(X)$ are nonnegative follows immediately from Proposition 4.8 and the fact that Stirling numbers alternate in sign, and the fact that the roots of $L_{n+1}(X)$ and $L_n(X)$ interlace follows from the recursion once proved that all the roots of $L_n(X)$ are real.

Numerically,

$$C_0(k) = -4.3469909700078207218721533281574751800\ldots$$

The above conjecture is certainly intimately linked to the formula

$$\int_0^\infty e^{-Xt} L_n(t) \, dt = X^{-(n+1)} \prod_{j=1}^{n-1} (j - X)$$

given by Proposition 4.8, but I do not yet see a proof.

8 Integral Representation of the Branches

The following is part of a result given in [4] and is given for completeness:

**Proposition 8.1** 1. Set $P = \pi/2 + 1$, $K = 2k\pi i + \log(x)$, and $f(t) = t - \log(t) + K$. Assume that either $k \neq 0, -1$, or that $k = 0$ or $k = -1$ and $x \notin [-1/e, 0]$. We have $W_k(x) = N/D$, with

$$N = \frac{K}{K^2 + P^2} + \int_0^\infty \frac{tdt}{(t^2 + 1)(f(t)^2 + \pi^2)}$$

$$D = \frac{P}{K^2 + P^2} - \int_0^\infty \frac{dt}{(t^2 + 1)(f(t)^2 + \pi^2)}.$$

2. Set $P = \pi/2 - 1$, $K = (2k - \text{sign}(k))\pi i + \log(x)$, and $f(t) = t + \log(t) - K$. Assume that $k \neq -1, 0, 1$. Then $W_k(x) = -N/D$ with the same formulas as in (1).
 Remarks

1. I refer to [4] for an explicit but much more complicated formula for \( k = 0 \) or \( k = -1 \) and \( x \in [-1/e, 0] \).

2. The conditions given by the author for the validity of these integral formulas are not quite the same as those given here. I believe at least that the above are correct.

3. As the author mentions, the change of variable \( t = \sinh(u) \) transforms the integrals into exponentially decaying ones, which can thus be easily computed. But in fact the doubly-exponential integration method does this automatically.
Appendix: Proof of Theorem 3.3

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Without loss of generality assume that $x$ is a fixed positive real number\(^3\), and set $f(w) = we^w - x$. By Corollary 3.2, for $K$ sufficiently large the set of zeros $z$ of $f$ with $|z| \leq (2K + 1)\pi$ is precisely $\{w_k : -K \leq k \leq K\}$, where we set $w_k = W_k(x)$. By Jensen’s formula applied to $f$ we have

$$\log x = \sum_{-K \leq k \leq K} \log |w_k| - (2K + 1) \log((2K + 1)\pi) + \frac{I((2K + 1)\pi)}{2\pi},$$

with

$$I(\rho) = \int_0^{2\pi} \log |f(\rho e^{i\theta})| \, d\theta.$$  

Since $w_k e^{w_k} = x$, we have $\log |w_k| = \log x - \Re w_k$, so summing over $k \in [-K, K]$ we obtain

$$\sum_{-K \leq k \leq K} \Re w_k = 2K \log x - (2K + 1) \log((2K + 1)\pi) + \frac{I((2K + 1)\pi)}{2\pi}. \quad (1)$$

We now study the integral $I((2K + 1)\pi)$ when $K$ is large, so we set $\rho(K) = (2K + 1)\pi$. The function $f(w) = we^w - x$ behaves like $we^w$ when $\Re w \geq 0$ and is near $-x$ when $\Re w \leq -2\log \rho(K)$. We now handle the intermediate case.

**Lemma 8.2** For $K$ sufficiently large and any $w$ with $|w| = \rho(K)$, we have

$$|f(w)| \gg 1.$$  

**Proof.** If $|we^w| \notin [x/2, 2x]$ the result follows by the triangle inequality. Otherwise, the real part of $w$ must be $O(\log x + \log(\rho(K)))$, therefore the imaginary part of $w$ must be $\pm \rho(K) + o(1)$. Since $\rho(K)$ is an odd multiple of $\pi$, it follows that the argument of $we^w$ must be $\pm \frac{\pi}{2} + o(1)$. Since $x$ is positive real, this implies that $|we^w - x|$ is also bounded from below. \qed

We can now give a precise estimate for $I((2K + 1)\pi)$:

**Lemma 8.3** For $K$ sufficiently large, we have

$$\int_0^{2\pi} \log |f(\rho(K)e^{i\theta})| \, d\theta = \pi \log \rho(K) + 2\rho(K) + \pi \log x + o(1). \quad (2)$$

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\(^3\)This is purely for simplicity. The argument works for any $x \in \mathbb{C}\setminus\{0\}$ with minor modifications.
Proof. For simplicity, write $\rho$ instead of $\rho(K)$, set $w = \rho e^{i\theta}$, and let $\epsilon = \frac{3 \log \rho}{\rho}$. We split the integral into three terms corresponding to $\theta \in [0, \pi] \cup \left[\frac{3\pi}{2}, 2\pi\right]$, $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2} + \epsilon\right] \cup \left[\frac{3\pi}{2} - \epsilon, \frac{3\pi}{2}\right]$, and $\theta \in \left[\frac{\pi}{2} + \epsilon, \frac{3\pi}{2} - \epsilon\right]$. In the first term, we have $\Re w \geq 0$, so $|we^w| = \rho e^{\Re w} \geq \rho$. It follows that $\log |f(w)| = \log |we^w| + O(\rho^{-1})$, hence

$$\int_{\theta \in [0, \pi] \cup \left[\frac{3\pi}{2}, 2\pi\right]} \log |f(\rho e^{i\theta})| = \int_{\theta \in [0, \pi] \cup \left[\frac{3\pi}{2}, 2\pi\right]} \log |we^w| + O(\rho^{-1})$$

$$= \int_{\theta \in [0, \pi] \cup \left[\frac{3\pi}{2}, 2\pi\right]} (\log \rho + \Re w) + O(\rho^{-1})$$

$$= \pi \log \rho + 2 \rho + O(\rho^{-1}).$$

In the second term, since $w$ has negative real part, we have the upper bound $|f(w)| \leq x + |we^w| \leq 2\rho$, and by the previous lemma, it is bounded from below by a constant. It follows that $\log |f(w)| \ll \log \rho$, so the second term tends to 0.

Finally, in the third term we have $\Re w \leq -2 \log \rho$, so that $|we^w| = \rho e^{\Re w} \ll \rho^{-1}$, therefore we have $\log |f(w)| = \log x + O(\rho^{-1})$. It follows that

$$\int_{\theta \in [\frac{\pi}{2} + \epsilon, \frac{3\pi}{2} - \epsilon]} \log |f(\rho e^{i\theta})| = \pi \log x + o(1).$$

Summing the three terms proves the lemma. \hfill \Box

Plugging this estimate into (1), and using that for $x > 0$ we have $\Im(w_{-k}) = -\Im(w_k)$, we deduce that as $K \to \infty$:

$$\sum_{-K \leq k \leq K} W_k(x) = (2K + \frac{1}{2}) \log x - (2K + \frac{1}{2}) \log((2K + 1 \pi) + (2K + 1) + o(1)).$$

Using Stirling’s formula, it is immediate to deduce that

$$\sum_{-K \leq k \leq K} W_k(x) = (2K + \frac{1}{2}) \log x - \log(\pi^{2K}(2K)!) + \frac{\log(2)}{2} + o(1),$$

proving Theorem 3.3.

References

[1] L. Comtet, *Advanced Combinatorics*, Springer (1974).
[2] R. Corless, G. Gonnet, D. Hare, D. Jeffrey, and D. Knuth, *On the Lambert W Function*, Advances in Comp. Math. 5 (1996), pp. 329–359.

[3] G. Kalugin and D. Jeffrey, *Convergence in $\mathbb{C}$ of series for the Lambert W Function*, arXiv:1208.0754.

[4] A. Kheyfits, *Explicit solutions of transcendental equations and the Lambert W function*, Fractional Calculus and Applied Analysis, Jan. 2004, 12p.