INNER COACTIONS, FELL BUNDLES, AND ABSTRACT UNIQUENESS THEOREMS

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Abstract. We prove gauge-invariant uniqueness theorems with respect to maximal and normal coactions for $C^*$-algebras associated to product systems of $C^*$-correspondences. Our techniques of proof are developed in the abstract context of Fell bundles. We employ inner coactions to prove an essential-inner uniqueness theorem for Fell bundles. As application, we characterise injectivity of homomorphisms on Nica’s Toeplitz algebra $T(G,P)$ of a quasi-lattice ordered group $(G,P)$ in the presence of a finite non-trivial set of lower bounds for all non-trivial elements in $P$.

1. Introduction

Starting with the early constructions of $C^*$-algebras associated to generating families of operators on Hilbert space such as isometries or partial isometries, possibly subject to certain relations, a question of interest arose as to whether the $C^*$-algebra was unique. Coburn’s theorem asserts that the $C^*$-algebra generated by a non-unitary isometry on Hilbert space is unique up to isomorphism, [Cob67]. In [Cun77], Cuntz constructed large classes of $C^*$-algebras, both simple and non-simple, generated by families of isometries satisfying certain relations, and proved that two tuples of isometries on Hilbert space fulfilling the same relation generate isomorphic $C^*$-algebras.

In a remarkable generalization, Nica introduced the notion of a quasi-lattice ordered group $(G,P)$ and constructed a Toeplitz $C^*$-algebra $T(G,P)$ and a universal $C^*$-algebra $C^*(G,P)$, [Nic92]. He obtained both analogues of Coburn’s theorem, and results relating to Cuntz’s uniqueness theorems in the particular case of the quasi-lattice ordered group $(\mathbb{F}_n, \mathbb{F}_n^+)$ consisting of the free group and the free semigroup on $n$ generators. Laca and Raeburn [LR96] discovered a semigroup crossed...
product structure of $C^*(G, P)$, and used it to prove faithfulness results for representations of this algebra in the presence of an amenability hypothesis.

Our starting point is two-fold. For one thing, we noticed that the analysis of the gauge-invariant uniqueness property from [CLSV] involved two crucial ingredients of nonabelian duality, namely maximal and normal coactions. The second motivating fact was that the quasi-lattice ordered group $(\mathbb{F}_n, \mathbb{F}_n^+)$ belongs to the class of those $(G, P)$ for which, as Nica showed, $\mathcal{T}(G, P)$ contains $\mathcal{K}(l^2(P))$. We could see that for such pairs $(G, P)$, the ideal $\mathcal{K}(l^2(P))$ of $\mathcal{T}(G, P)$ contains a family of projections that determines an inner coaction.

Our thrust in this paper is to show how the general theory of coactions gives uniqueness theorems for $C^*$-algebras of Fell bundles in a systematic manner. We apply these results in familiar contexts with sharpened or new characterizations of uniqueness as outcomes.

The first gauge-invariant uniqueness type result was proved by an Huef and Raeburn in [aHR97]. Here we obtain a gauge-invariant uniqueness result in the context of Fell bundles. Since the proofs of the abstract gauge-invariant uniqueness results for $C^*$-algebras of Fell bundles are painless, albeit non-trivial, we chose to place this material in an appendix. The other type of abstract uniqueness results we prove emerges from inner coactions.

The first main application is to establish a gauge-invariant uniqueness property for the Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$ of Sims and Yeend from [SY10] by highlighting the feature observed in [CLSV] that it carries a maximal coaction. If $(G, P)$ is a quasi-lattice ordered group and $X$ is a compactly aligned product system over $P$ of $C^*$-correspondences over a $C^*$-algebra $A$, then Sims and Yeend’s $C^*$-algebra $\mathcal{NO}_X$ is universal for Cuntz-Nica-Pimsner covariant representations of $X$. When $X$ is $\tilde{\phi}$-injective, $\mathcal{NO}_X$ has the desired property of admitting an injective universal Cuntz-Nica-Pimsner covariant representation. For product systems, $\mathcal{NO}_X$ is the appropriate candidate for the Cuntz-Pimsner algebra $\mathcal{O}_Y$ associated in [Kat04] to a single $C^*$-correspondence $Y$, in a generalization of Pimsner’s work from [Pim97].

The gauge-invariant uniqueness property for $\mathcal{NO}_X$ proved in [CLSV] (see Corollaries 4.11 and 4.12) is equivalent to asking for the canonical maximal coaction on $\mathcal{NO}_X$ to be normal. In our treatment here we look at the gauge-invariant uniqueness property in two separate classes, that of $C^*$-algebras with maximal coactions, and of $C^*$-algebras with normal coactions. Thereby we are in the context of coactions and can streamline the proofs by using specific techniques. We obtain gauge-invariant
uniqueness theorems for $\mathcal{N}O_X$, seen in the category of maximal coactions, and for the co-universal algebra $\mathcal{N}O_X^\vee$ identified in [CLSV] and viewed in the category of normal coactions.

As a bonus for sorting out abstract gauge-invariant uniqueness results for Fell bundles, we also obtain a gauge-invariant uniqueness theorem for the Toeplitz-like extension of $\mathcal{N}O_X$. This is the universal $C^*$-algebra for Nica covariant Toeplitz representations of the compactly aligned product system $X$; this algebra was denoted $\mathcal{T}_{\text{cov}}(X)$ in [Fow02], but we shall follow [BaHLR], see their Remark 5.3, and use the notation $\mathcal{NT}(X)$.

Faithfulness of representations of $\mathcal{T}(G,P)$ was characterized by Laca and Raeburn for all amenable quasi-lattice ordered groups $(G,P)$, [LR96]. In coaction terminology, $(G,P)$ amenable means that the canonical maximal coaction on $C^*(G,P)$ is also normal. Here we exploit the fact that $\mathcal{T}(G,P)$ has a natural normal coaction. For a quasi-lattice ordered group $(G,P)$ with the property that there is a finite set $F$ of elements in $P \setminus \{e\}$ such that every non-trivial element in $P$ has a lower bound in $F$, we characterize directly injectivity of homomorphisms from $\mathcal{T}(G,P)$ to a $C^*$-algebra $B$. The crucial observation is that existence of $F$ not only characterizes the fact that $\mathcal{K}(l^2(P))$ is included in $\mathcal{T}(G,P)$, as proved by Nica in [Nic92, Proposition 6.3], but that it also characterizes existence of an inner coaction on the ideal $\mathcal{K}(l^2(P))$. With this card at hand, we can apply our abstract essential-inner uniqueness result, i.e. Corollary 4.3. For the pair $(\mathbb{F}_n, \mathbb{F}_n^+)$, which clearly admits a finite set of lower bounds for elements in $\mathbb{F}_n^+$, our Theorem 6.3 thus provides a characterization of faithful representations of $\mathcal{T}(\mathbb{F}_n, \mathbb{F}_n^+)$ without reference to the amenability of the pair, a property that is by no means trivial to verify.

The organization of the paper is as follows: after a preliminary section in which we recall terminology and facts about coactions, quasi-lattice ordered groups and $C^*$-algebras of product systems, in section 3 we present gauge-invariant uniqueness theorems for the Nica-Toeplitz algebra, the Cuntz-Nica-Pimsner algebra and the co-universal $C^*$-algebra of a class of compactly aligned product systems $X$. In section 4 we prove the abstract inner-uniqueness and essential-inner uniqueness results. In section 5 we place the representation of $C^*(G,P)$ arising from the Toeplitz representation of $P$ in the framework of non-abelian duality. Section 6 contains the essential-inner uniqueness theorem for $\mathcal{T}(G,P)$, namely Theorem 6.3, and a converse to it, Theorem 6.10. The appendix collects the promised gauge-invariant uniqueness results for Fell bundles.
2. Preliminaries

Throughout, $G$ will be a discrete group. If $A$ is a $C^*$-algebra and $\delta : A \rightarrow A \otimes C^*(G)$ is a coaction, we will just say “$(A, \delta)$ is a coaction”. For the theory of coactions we refer to [EKQR06, Appendix A], and for discrete coactions in particular we refer to [EKQ04, Qui96]. For maximalizations and normalizations of coactions we refer to [KQ09, KQ10].

If $(A, \delta)$ is a (full\footnote{and all our coactions will be full}) coaction of $G$, we will let $\mathcal{A}$ denote the associated Fell bundle, and similarly for other capital letters. If $\pi : (A, \delta) \rightarrow (B, \varepsilon)$ is a morphism of coactions, we write $\bar{\pi} : \mathcal{A} \rightarrow \mathcal{B}$ for the corresponding homomorphism of Fell bundles. Note that $\pi$ is surjective if and only if $\{\pi(a_s) : s \in G\}$ generates $B$. Also note that if $(A, \delta)$ and $(B, \varepsilon)$ are coactions, then a homomorphism $\pi : A \rightarrow B$ is $\delta - \varepsilon$ equivariant if and only if $\pi(a_s) \subset B_s$ for all $s \in G$ (because equivariance can be checked on the generators $a_s \in A_s$ for $s \in G$).

A morphism $\pi : (B, \varepsilon) \rightarrow (A, \delta)$ of coactions is a maximization of $(A, \delta)$ if $(B, \varepsilon)$ is maximal and $\pi \times G : B \times \varepsilon G \rightarrow A \times \delta G$ is an isomorphism. Sometimes we call $(B, \varepsilon)$ itself a maximization of $(A, \delta)$. Maximalizations of $(A, \delta)$ always exist, and all are uniquely isomorphic. Choosing one for every coaction, we get a maximization functor that sends $(A, \delta)$ to the maximization

$$q^m_A : (A^m, \delta^m) \rightarrow (A, \delta),$$

and sends a morphism $\pi : (A, \delta) \rightarrow (B, \varepsilon)$ to the unique morphism $\pi^m$, called the maximization of $\pi$, making the diagram

$$\begin{array}{ccc}
(A^m, \delta^m) & \xrightarrow{\pi^m} & (B^m, \varepsilon^m) \\
q^m_A \downarrow & & \downarrow q^m_B \\
(A, \delta) & \xrightarrow{\pi} & (B, \varepsilon)
\end{array}$$

commute. A parallel theory exists for normalizations: $\pi : (A, \delta) \rightarrow (B, \varepsilon)$ is a normalization of $(A, \delta)$ if $(B, \varepsilon)$ is normal and $\pi \times G : A \times \delta G \rightarrow B \times \varepsilon G$ is an isomorphism. We sometimes call $(B, \varepsilon)$ itself a normalization of $(A, \delta)$. Normalizations of $(A, \delta)$ always exist, and all are uniquely isomorphic. Choosing one for every coaction, we get a normalization functor that sends $(A, \delta)$ to the normalization

$$q^n_A : (A, \delta) \rightarrow (A^n, \delta^n),$$
and sends a morphism $\pi : (A, \delta) \to (B, \varepsilon)$ to the unique morphism $\pi^n$, called the normalization of $\pi$, making the diagram

\[
\begin{array}{ccc}
(A, \delta) & \xrightarrow{\pi} & (B, \varepsilon) \\
q^n_A & \downarrow & q^n_B \\
(A^n, \delta^n) & \xrightarrow{i} & (B^n, \varepsilon^n)
\end{array}
\]

commute. Maximalizations and normalizations are automatically surjective. Moreover, if $\pi : (A, \delta) \to (B, \varepsilon)$ is either a maximalization or a normalization, then $\pi$ maps each spectral subspace $A_s := \{ a \in A : \delta(a) = a \otimes s \}$ isometrically onto the corresponding subspace $B_s$, and in particular maps the fixed-point algebra $A^\delta := A_e$ isomorphically onto $B^e$. If $(A, \delta)$ is normal, then the maximalization $q^m_A : (A^m, \delta^m) \to (A, \delta)$ is also a normalization of $(A^m, \delta^m)$, and similarly if $(A, \delta)$ is maximal then the normalization $q^n_A : (A, \delta) \to (A^n, \delta^n)$ is also a maximalization of $(A^n, \delta^n)$.

For every Fell bundle $p : A \to G$, the (full) cross-sectional algebra $C^*(A)$ carries a maximal coaction $\delta_A$, determined on $A$ by $\delta_A(a) = a \otimes p(a)$, the reduced cross-sectional algebra $C^*_r(A)$ carries a normal coaction $\delta^n_A$ determined by the same formula, and the regular representation $\Lambda_A : (C^*(A), \delta_A) \to (C^*_r(A), \delta^n_A)$ is both a maximalization and a normalization.

For $s \in G$, we write $\chi_s$ for the characteristic function of $\{s\}$, viewed as an element of $B(G) = C^*(G)^*$. If $(A, \delta)$ is a coaction, we write

\[
(2.1) \quad \delta_s = (\text{id} \otimes \chi_s) \circ \delta,
\]

which is a projection of norm one from $A$ onto the spectral subspace $A_s$.

If $A$ is a $C^*$-algebra and $P$ is a discrete semigroup with identity $e$, a product system over $P$ of $C^*$-correspondences over $A$ consists of a semigroup $X$ equipped with a semigroup homomorphism $d : X \to P$ such that: (1) $X_p := d^{-1}(p)$ is a $C^*$-correspondence over $A$ for each $p \in P$; (2) $X_e = AA_e$; (3) the multiplication on $X$ implements isomorphisms $X_p \otimes_A X_q \cong X_{pq}$ for $p, q \in P \setminus \{e\}$; and (4) multiplication implements the right and left actions of $X_e = A$ on each $X_p$. For $p \in P$ we let $\phi_p : A \to \mathcal{L}(X_p)$ be the homomorphism that implements the left action. Given $p, q \in P$ with $p \neq e$ there is a homomorphism $\psi^p_q : \mathcal{L}(X_p) \to \mathcal{L}(X_q)$ such that $\psi^p_q(S)(xy) = (Sx)y$ for all $x \in X_p$, $y \in X_q$, and $S \in \mathcal{L}(X_p)$. Upon identifying $K(X_p) = A$, we let $\psi^p_q : K(X_e) \to \mathcal{L}(X_q)$ be given by $\psi^p_q = \phi_q$, see [SY10] §2.2.
Recall that for a $C^*$-correspondence $Y$ over $A$, a map $\psi : Y \to B$ and a homomorphism $\pi : A \to B$ into a $C^*$-algebra form a Toeplitz representation if $\psi(x \cdot a) = \psi(x)\pi(a)$ and $\pi((x, y)) = \psi(x)^*\psi(y)$ for all $a \in A, x, y \in Y$. A map $\psi$ of a product system $X$ into a $C^*$-algebra $B$ is a Toeplitz representation if $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in X$ and $(\psi|_{X_p}, \psi|_{X_e})$ is a Toeplitz representation of the $C^*$-correspondence $X_p$, for all $p \in P$.

We recall from [Nic92] that a quasi-lattice ordered group $(G, P)$ consists of a subsemigroup $P$ of a (discrete) group $G$ such that $P \cap P^{-1} = \{e\}$ and every finite subset of $G$ with a common upper bound in $P$ admits a least common upper bound in $P$, all taken with respect to the left-invariant partial order on $G$ given by $x \leq y$ if $x^{-1}y \in P$. We write $x \lor y < \infty$ to indicate that $x, y$ have a common upper bound in $P$, and then $x \land y$ denotes their least common upper bound in $P$. If no common upper bound of $x, y$ exists in $P$ we write $x \land y = \infty$. The semigroup $P$ is directed if $x \lor y < \infty$ for all $x, y \in P$.

Given a quasi-lattice ordered group $(G, P)$, a product system $X$ over $P$ is called compactly aligned if $\psi_p^{\lor q}(S)\psi_q^{\lor q}(T) \in \mathcal{K}(X_p \otimes q)$ whenever $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$, and $p \lor q < \infty$ cf. [CLSV] or [Fow02, Definition 5.7] in case each $X_p$ is essential. If $\psi : X \to B$ is a Toeplitz representation, there are homomorphisms $\psi^{(p)} : \mathcal{K}(X_p) \to B$ such that $\psi^{(p)}(\theta_{x,y}) = \psi_p(x)\psi_p(y)^*$ for all $p \in P$ and $x, y \in X$, [Pim97]. When $X$ is compactly aligned, $\psi$ is said to be Nica covariant if $\psi^{(p)}(S)\psi^{(q)}(T)$ is $\psi^{(p \lor q)}(\psi_p^{\lor q}(S)\psi_q^{\lor q}(T))$ in case $p \lor q < \infty$ and is zero otherwise, see [Fow02] in each fibre $X_p$.

Fowler introduced a $C^*$-algebra $\mathcal{T}_{\text{cov}}(X)$ and showed it is universal for Nica covariant Toeplitz representations of $X$, [Fow02]. Here we shall use the notation $\mathcal{N}\mathcal{T}(X)$ instead of $\mathcal{T}_{\text{cov}}(X)$ because, as advocated for in [BaHLR, Remark 5.3], the choice of $\mathcal{T}_{\text{cov}}(X)$ for a $C^*$-algebra generated by a universal representation was unfortunate. Fowler introduced also a Cuntz-Pimsner algebra of $X$ by imposing usual Cuntz-Pimsner covariance in the sense of [Pim97] in each fibre $X_p$.

Given a quasi-lattice ordered group $(G, P)$ and a compactly aligned product system $X$ over $P$ of $C^*$-correspondences over $A$, Sims and Yeend [SY10] introduced a new notion of Cuntz-Pimsner covariance for a Toeplitz representation of $X$. The definition is quite complicated and we will not give it here. It was proved in [SY10, Theorem 4.1] that the universal Cuntz-Nica-Pimsner covariant representation $j_X$ of $X$ is injective, meaning that $j_X|_{X_e}$ is injective, if $X$ is $\phi$-injective (see [CLSV, §2.4] for the definition of this concept). The universal $C^*$-algebra for
Cuntz-Nica-Pimsner covariant representations of $X$, denoted $\mathcal{NO}_X$, is then nontrivial.

3. **Gauge-invariant uniqueness for $\mathcal{NT}(X)$ and $\mathcal{NO}_X$**

Fix a quasi-lattice ordered group $(G, P)$ and a compactly aligned product system $X$ over $P$ of $C^*$-correspondences over $A$. There is a canonical coaction $(\mathcal{NT}(X), \delta)$ of $G$, and we let $B$ be the associated Fell bundle. If $X$ is $\tilde{\phi}$-injective, there is also a canonical coaction $(\mathcal{NO}_X, \nu)$ of $G$, whose associated Fell bundle is denoted by $\mathcal{N}$. It was shown in [CLSV, Remark 4.5] that $C^*(B) \cong \mathcal{NT}(X)$ and $C^*(\mathcal{N}) \cong \mathcal{NO}_X$. Equivalently, both coactions $\delta$ on $\mathcal{NT}(X)$ and $\nu$ on $\mathcal{NO}_X$ are maximal in the sense of [EKQ04].

The following terminology was introduced in [CLSV, Definition 4.10]: $\mathcal{NO}_X$ has the gauge-invariant uniqueness property provided that a surjective homomorphism $\varphi : \mathcal{NO}_X \to B$ is injective if and only if:

1. (GI1) there is a coaction $\beta$ of $G$ on $B$ such that $\varphi$ is $\nu - \beta$ equivariant,
2. (GI2) the homomorphism $\varphi|_{\mathcal{J}_X(A)}$ is injective.

The **gauge-invariant uniqueness theorem** for $\mathcal{NO}_X$ is [CLSV, Corollary 4.11] and gives a number of necessary and sufficient conditions for $\mathcal{NO}_X$ to have the gauge-invariant uniqueness property. For instance, $\mathcal{NO}_X$ has the gauge-invariant uniqueness property precisely when the gauge-coaction $\nu$ is normal. Thus the gauge-invariant uniqueness theorem holds for $\mathcal{NO}_X$ provided that $\nu$ is both maximal and normal.

In the next result we recast the gauge-invariant uniqueness property for $\mathcal{NO}_X$ by asking for a maximal coaction on the target algebra. The apparently short proof follows from the general uniqueness theorems worked out in the context of Fell bundles in the appendix, and illustrates the power of coaction techniques.

**Theorem 3.1** (The gauge-invariant uniqueness theorem for $\mathcal{NO}_X$ and maximal coactions). Let $(G, P)$ be a quasi-lattice ordered group and $X$ a $\tilde{\phi}$-injective compactly aligned product system over $P$ of $C^*$-correspondences over $A$. A surjective homomorphism $\pi : \mathcal{NO}_X \to B$ is injective if and only if $\pi$ is injective on $\mathcal{NO}_X^e$ and there is a maximal coaction $\beta$ on $B$ such that $\pi$ is $\nu - \beta$ equivariant.

**Proof.** Apply Corollary A.2 to $(\mathcal{NO}_X, \nu)$ and $\pi$. \qed

Compared to [CLSV, Corollary 4.12], Theorem 3.1 does not require amenability of $G$, and can be applied to arbitrary $\tilde{\phi}$-injective compactly
aligned product systems over \( P \) (for which \( j_X \) is an injective representation). The drawback is that \( \pi \) needs to be injective on the entire fixed-point algebra for \( \nu \), and not just on the coefficient algebra.

In practice, the injectivity of \( \pi \) on \( \mathcal{NO}_X^\nu \) is likely to be difficult to establish. However, when the compactly aligned product system satisfies one of the two conditions: the left actions on the fibres of \( X \) are all injective, or \( P \) is directed and \( X \) is \( \tilde{\phi} \)-injective, then \cite{CLSV} Theorem 3.8 says that \( \pi \) is injective on \( \mathcal{NO}_X^\nu \) precisely when it is injective as a Toeplitz representation, i.e. its restriction to \( j_X(A) \) is an injective homomorphism.

Example 3.2. Suppose that \( G \) is a nonabelian finite-type Artin group. Then \( G \) and its positive cone \( P \) form a quasi-lattice ordered group. By \cite{CL02}, \( P \) is directed and \( G \) is not amenable. Then, if \( X \) is the product system over \( P \) with fibers \( C \), the algebra \( \mathcal{NO}_X \) is isomorphic to \( C^*(G) \) and does not have the gauge-invariant uniqueness property (see \cite{CLSV} Remark 5.4] for details). However, since \( \mathcal{NO}_X^\nu = C \), Theorem 3.1 implies that a surjective homomorphism \( \pi: C^*(G) \to B \) is injective if and only if \( B \) carries a compatible maximal coaction.

Since also \( (\mathcal{NT}(X), \delta) \) is a maximal coaction, we have a version of Theorem 3.1 for \( \mathcal{NT}(X) \).

**Theorem 3.3** (The gauge-invariant uniqueness theorem for \( \mathcal{NT}(X) \) and maximal coactions). Let \( (G, P) \) be a quasi-lattice ordered group and \( X \) a compactly aligned product system over \( P \) of \( C^* \)-correspondences over \( A \). A surjective homomorphism \( \pi: \mathcal{NT}(X) \to B \) is injective if and only if \( \pi \) is injective on \( \mathcal{NT}(X)^\delta \) and there is a maximal coaction \( \beta \) on \( B \) such that \( \pi \) is \( \delta - \beta \)-equivariant.

**Proof.** Apply Corollary A.2 to \( (\mathcal{NT}(X), \delta) \) and \( \pi \). □

**Corollary 3.4.** Let \( (G, P) \) be a quasi-lattice ordered group and \( X \) a compactly aligned product system over \( P \) of \( C^* \)-correspondences over \( A \). The coaction \( (\mathcal{NT}(X), \delta) \) is normal precisely when the following is satisfied: a surjective homomorphism \( \pi: \mathcal{NT}(X) \to B \) is injective if and only if \( \pi \) is injective on \( \mathcal{NT}(X)^\delta \) and there is a coaction \( \beta \) on \( B \) such that \( \pi \) is \( \delta - \beta \)-equivariant.

**Proof.** Apply Corollary A.4. □

**Corollary 3.5.** Let \( (G, P) \) be a quasi-lattice ordered group and \( X \) a \( \tilde{\phi} \)-injective compactly aligned product system over \( P \) of \( C^* \)-correspondences over \( A \). The coaction \( (\mathcal{NO}_X, \nu) \) is normal precisely when the following is satisfied: a surjective homomorphism
\[ \pi : \mathcal{NO}_X \to B \] is injective if and only if \( \pi \) is injective on \( \mathcal{NO}_X^\nu \) and there is a coaction \( \beta \) on \( B \) such that \( \pi \) is \( \nu - \beta \) equivariant.

**Proof.** Apply Corollary A.4 \( \Box \)

Next we recall from [CLSV] that given a quasi-lattice ordered group \( (G, P) \) and a compactly aligned product system \( X \) over \( P \) satisfying one of the following two conditions: the left actions on the fibres of \( X \) are all injective, or \( P \) is directed and \( X \) is \( \phi \)-injective, then the \( C^* \)-algebra \( \mathcal{NO}_X^\nu := C^*(\mathcal{N}) \) and the normalization \( \nu^n \) of \( \nu \) have the co-universal property of [CLSV] Theorem 4.1. This co-universal property was used to identify various reduced crossed product type \( C^* \)-algebras in the form \( \mathcal{NO}_X^\nu \) for appropriate \( X \), and also to investigate the gauge-invariant uniqueness property in several contexts.

Our abstract uniqueness results for Fell bundles allow us to give a characterization of injectivity of homomorphisms \( \pi : B \to \mathcal{NO}_X^\nu \) that is an alternative to [CLSV, Corollary 4.9].

**Theorem 3.6** (The gauge-invariant uniqueness theorem for \( \mathcal{NO}_X^\nu \) and normal coactions). Let \( (G, P) \) be a quasi-lattice ordered group and \( X \) a \( \tilde{\phi} \)-injective compactly aligned product system over \( P \) of \( C^* \)-correspondences over \( A \). A homomorphism \( \pi : B \to \mathcal{NO}_X^\nu \) is injective if and only if there is a normal coaction \( \beta \) of \( G \) on \( B \) such that \( \pi \) is \( \beta - \nu^n \) equivariant and \( \pi|_{B_e} \) is injective.

**Proof.** Apply Corollary A.3 \( \Box \)

To see how this relates to [CLSV], suppose \( X \) is a compactly aligned product system over \( P \) such that the left actions on the fibres of \( X \) are all injective, or \( P \) is directed and \( X \) is \( \phi \)-injective. Suppose also that \( \pi \) arises from the co-universal property of \( \mathcal{NO}_X^\nu \) applied to an injective Nica covariant Toeplitz representation \( \psi : X \to B \), where there is a coaction \( \beta \) of \( G \) on \( B \) making \( \pi \) a \( \beta - \nu^n \) equivariant homomorphism. It is proved in [CLSV] Corollary 4.9 that \( \pi \) is injective if and only if \( \beta \) is normal and \( \psi \) is Cuntz-Nica-Pimsner covariant. In Theorem 3.6 the last condition is replaced by \( \pi|_{B_e} \) being injective.

**Corollary 3.7.** Let \( (G, P) \) be a quasi-lattice ordered group and \( X \) a \( \tilde{\phi} \)-injective compactly aligned product system over \( P \) of \( C^* \)-correspondences over \( A \). The coaction \( (\mathcal{NO}_X^\nu, \nu^n) \) is maximal precisely when the following is satisfied: a homomorphism \( \pi : B \to \mathcal{NO}_X^\nu \) is injective if and only if there is a coaction \( \beta \) of \( G \) on \( B \) such that \( \pi \) is \( \beta - \nu^n \) equivariant and \( \pi|_{B_e} \) is injective.

**Proof.** Apply Corollary A.5 \( \Box \)
4. Inner coactions

In this section we study inner coactions in relation to faithfulness of representations. First we recall some notation. The multiplier algebra \( M(C_0(G) \otimes C^*(G)) \) is identified with the algebra of continuous bounded functions on \( G \) with values in \( M(C^*(G)) \) equipped with the strict topology. Let \( w_G \) be the unitary element of \( M(C_0(G) \otimes C^*(G)) \) given by the canonical embedding of \( G \) in \( M(C^*(G)) \). Given a coaction \( (A,\delta) \) and a \( C^* \)-algebra \( D \), nondegenerate homomorphisms \( \mu : C_0(G) \rightarrow M(D) \) and \( \pi : A \rightarrow M(D) \) form a covariant pair for \( (A,\delta) \) provided that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\delta} & A \otimes C^*(G) \\
\downarrow{\pi} & & \downarrow{\pi \otimes \operatorname{id}} \\
M(D) & \xrightarrow{\operatorname{Ad} \mu \otimes \operatorname{id}(w_G) \otimes (\operatorname{id} \otimes 1)} & M(D \otimes C^*(G))
\end{array}
\]

commutes, or, equivalently, since \( G \) is discrete, provided that

\[
(4.1) \quad \pi(a_x)\mu(\chi_y) = \mu(\chi_{xy})\pi(a_x)
\]

for all \( a_x \in A_x \) and all \( x,y \in G \) (see e.g., [EQ99, Section 2]).

By [Qui94, Lemma 1.11], any nondegenerate homomorphism \( \mu : C_0(G) \rightarrow M(A) \) implements an inner coaction \( \delta^\mu \) on \( A \) via

\[
\delta^\mu(a) = \operatorname{Ad} \mu \otimes \operatorname{id}(w_G)(a \otimes 1).
\]

Note that \((\operatorname{id}_A,\mu)\) forms a covariant pair for every inner coaction \((A,\delta^\mu)\).

Every inner coaction is normal, by [Qui94, Proposition 2.3] (see also [BKQ11, Lemma A.2]). If \((A,G,\delta^\mu)\) is an inner coaction, then \( a \in A_e \) if and only if \( a \) commutes with \( \{\mu(\chi_x) : x \in G\} \). Indeed, if \( a \in A_e \) then \( a \) commutes with every \( \mu(\chi_x) \) by \((4.1)\). Conversely, if \( a \) commutes with every \( \mu(\chi_x) \) then \( a \) commutes with \( \mu(C_0(G)) \), hence \( a \otimes 1 \) commutes with \( \mu(C_0(G)) \otimes C^*(G) \), and therefore with \( \mu \otimes \operatorname{id}(w_G) \), so \( a \in A_e \).

Remark 4.1. We note that a necessary and sufficient condition for a coaction \( \delta \) on \( A \) to be inner is that there is a family \( \{p_x : x \in G\} \) of orthogonal projections in \( M(A) \) that sum strictly to 1 in \( M(A) \) and satisfy

\[
(4.2) \quad a_x p_y = p_{xy} a_x
\]

for all \( a_x \in A_x \) and \( x, y \in G \). Indeed, if \((A,\delta)\) is an inner coaction, there is a nondegenerate homomorphism \( \mu : C_0(G) \rightarrow M(A) \) such that \( \operatorname{id}_A \) and \( \mu \) satisfy \((4.1)\), which turns into \((4.2)\) by letting \( p_y = \mu(\chi_y) \).
Conversely, given a coaction \((A, \delta)\) and a family of projections satisfying (1.2), let \(\mu : C_0(G) \to M(A)\) be the unique homomorphism satisfying \(\mu(x_y) = p_y\) for \(y \in G\). Then \(\mu\) is nondegenerate because \(\sum_{y \in G} p_y = 1\) strictly in \(M(A)\), and \((\text{id}_A, \mu)\) forms a covariant pair by (1.2). Unravelling the definitions, we have \(\delta = \delta \mu\).

**Theorem 4.2** (Abstract uniqueness theorem). Let \((A, \delta)\) be an inner coaction. A surjective homomorphism \(\varphi : A \to B\) onto a C*-algebra \(B\) is injective if and only if \(\varphi|_{A_e}\) is injective.

**Proof.** Since \(\delta\) is inner, there is a nondegenerate homomorphism \(\mu : C_0(G) \to M(A)\) such that \(\delta = \delta \mu\). Define a nondegenerate homomorphism \(\nu : C_0(G) \to M(B)\) by \(\nu = \varphi \circ \mu\). Then \(\delta \nu\) is an inner coaction on \(B\), and the computation

\[
\varphi(a_x)\nu(x_y) = \varphi(a_x)\overline{\varphi}(\mu(x_y)) = \varphi(a_x)\mu(x_y)
\]

\[
= \varphi(\mu(x_y)a_x) \quad \text{by (1.1)}
\]

\[
= \nu(x_{xy})\varphi(a_x)
\]

for \(a_x \in A_x\) and \(x, y \in G\) shows that \(\varphi\) is \(\delta - \delta \nu\) equivariant. The theorem therefore follows from Proposition [A.1] (2). \(\square\)

Suppose that \((A, \delta)\) is a coaction of \(G\). An ideal \(I\) in \(A\) is \(\delta\)-invariant if the restriction of \(\delta\) to \(I\) gives rise to a coaction of \(G\) on \(I\). If this is the case, we let \(\delta|_I\) be the restricted coaction on \(I\).

**Corollary 4.3** (Essential-inner uniqueness theorem). Let \((A, \delta)\) be a coaction of \(G\) and \(I\) a \(\delta\)-invariant ideal in \(A\) such that the coaction \(\delta|_I\) of \(G\) on \(I\) is inner. If \(I\) is an essential ideal in \(A\), then a homomorphism \(\varphi : A \to B\) is injective if and only if \(\varphi|_{A^\delta}\) is injective.

**Proof.** For the non-trivial direction, suppose that \(\varphi|_{A^\delta}\) is injective. Then \(\varphi\) is injective on \(I^{\delta|_I} = A^\delta \cap I\). Since \(\delta|_I\) is inner, Theorem 4.2 implies that \(\varphi|_I\) is injective. But \(I\) is an essential ideal, and so \(\varphi\) is injective. \(\square\)

5. C*-algebras of quasi-lattice ordered groups

In this section we recall Nica’s constructions of C*-algebras associated to isometric representations of quasi-lattice ordered groups, we give a quick review of subsequent constructions, and we make connections with coaction theory.

Let \((G, P)\) be a quasi-lattice ordered group. A semigroup homomorphism \(V\) of \(P\) into the isometries on a Hilbert space \(H\) such that \(V_e = I\) and \(V_sV_t = V_{st}\) for all \(s, t \in P\) is called an (isometric) representation of \(P\). Let \(\{e_t\}_{t \in P}\) be the canonical orthonormal basis of \(l^2(P)\).
The Toeplitz or Wiener-Hopf representation of $P$ on $l^2(P)$ is given by $T_{st} = \varepsilon_{st}$, for $s, t \in P$. The Toeplitz algebra (or Wiener-Hopf algebra) $\mathcal{T}(G, P)$ is the $C^*$-subalgebra of $B(l^2(P))$ generated by the image of $T$. Nica noticed that $T_s T_s^* T_t^* = T_{st} T_{st}^*$ when $s \land t < \infty$ and is zero otherwise. Such representations of $P$ are now called Nica coactions, and $C^*(G, P)$ is the universal $C^*$-algebra generated by a Nica covariant representation $v$ of $P$ (see [Nic92, LR96]).

By [Nic92, Proposition 3.2], the family $\{T_t^* : s, t \in P\}$ spans a dense subalgebra of $\mathcal{T}(G, P)$. The diagonal subalgebra of $\mathcal{T}(G, P)$ is $D = \text{span} \{T_s T_s^* : s \in P\}$.

We next recall some facts from [LR96]. Let $(G, P)$ be a quasi-lattice ordered group, and for each $s \in P$ write $1_s$ for the characteristic function of the set $\{t \in P : s \leq t\}$. Then $B_P = \text{span} \{1_s : s \in P\}$ is a commutative $C^*$-subalgebra of $l^\infty(P)$, and $C^*(G, P)$ is the semigroup crossed product $B_P \rtimes P$ arising from translation $t \mapsto (1_s \to 1_{st})$ on $B_P$, see [LR96, Corollary 2.4]. By [LR96, §6.1], there is a coaction $\delta$ of $G$ on $C^*(G, P)$ such that $\delta(v_s) = v_s \otimes s$ for all $s \in P$, and $B_P$ is the fixed-point algebra $C^*(G, P)^\delta$. Moreover, [LR96, Proposition 2.3] shows that every representation of $C^*(G, P)$ is determined by a Nica covariant representation of $P$. We let $\lambda_T$ denote the representation of $C^*(G, P)$ determined by $T$, and note that it carries $1_s$ to $T_s T_s^*$ for all $s \in P$.

It follows from [SY10, Proposition 5.6] that if $X = \mathbb{C} \times P$ is the trivial product system over $P$ with fibers $X_p = c\mathbb{C}_C$ for all $p \in P$, then $\mathcal{N}\mathcal{T}(X) \cong C^*(G, P)$. Since $\delta$ is maximal by [CLSV, Remark 4.5], we shall view it as a coaction on $C^*(G, P) = C^*(\mathcal{B})$ (recall that we let $\mathcal{B}$ denote the associated Fell bundle over $G$) with fixed point algebra equal to $B_P$. Recall from [EQ99] that $C^*_r(\mathcal{B})$ is identified with the normalization $(C^*_r(\mathcal{B}))^\mathrm{nc}$.

**Proposition 5.1.** The representation $\lambda_T$ is both a maximalization and a normalization from $(C^*(G, P), \delta)$ onto $(\mathcal{T}(G, P), \delta^\mathrm{nc})$. In particular, $\mathcal{T}(G, P) \cong C^*_r(\mathcal{B})$.

**Proof.** Using reduced coactions, it was shown in [QR97, Proposition 6.5] that there is a normal coaction $\eta$ on $\mathcal{T}(G, P)$ such that $\eta(T_s T_t^*) = T_s T_t^* \otimes s t^{-1}$ for $s, t \in P$. Then $\lambda_T : (C^*(G, P), \delta) \to (\mathcal{T}(G, P), \eta)$ is equivariant.

We noted in the preliminaries that the regular representation $\Lambda_B : (C^*(G, P), \delta) \to (C^*_r(\mathcal{B}), \delta^\mathrm{nc})$ is both a maximalization and a normalization. Since $\lambda_T$ is injective on $B_P$ by [LR96, Corollary 2.4(1)], Proposition [A.1, parts (3) and (4)], imply that $\lambda_T$ is also both a maximalization and a normalization. Since all maximalizations are isomorphic, and
similarly for normalizations, we therefore have $C^*(G, P) \cong (\mathcal{T}(G, P))^n$ and $\mathcal{T}(G, P) \cong (C^*(G, P))^n$. □

Nica [Nic92, Definition 4.2] defined $(G, P)$ to be amenable if the representation $\lambda_T$ is an isomorphism. His definition motivated Exel’s definition of amenable Fell bundles in [Exe97]. Our Proposition 5.1 shows that the Fell bundle $\mathcal{B}$ is amenable when $(G, P)$ is amenable in Nica’s sense.

6. Finite exhaustive sets of strictly positive elements

Throughout this section let $(G, P)$ be a quasi-lattice ordered group.

**Definition 6.1.** A FESSPE of $(G, P)$ is a finite subset $F \subset P \setminus \{e\}$ such that $FP = P \setminus \{e\}$.

“FESSPE” stands for “finite exhaustive set of strictly positive elements”, and the existence of such an $F$ is easily seen to be equivalent to the existence, for each $x \in G$, of a finite set of strict upper bounds $S$ of $x$ (i.e., $x \preceq y$ for all $y \in S$) that is exhaustive in the sense that every strict upper bound of $x$ has a lower bound in $S$ — namely, take $S = xF$. This condition was introduced in [Nic92], and was shown in [Nic92, Proposition 6.3] to be equivalent, among others, to the fact that $\mathcal{T}(G, P)$ contains the compact operators $\mathcal{K}(l^2(P))$.

As remarked in [Nic92], all pairs $(G, P)$ with $P$ finitely generated have a FESSPE. In particular, $(\mathbb{F}_n, \mathbb{F}_n^+)$ has a FESSPE for all $n \geq 1$. The pair $(\mathbb{F}_\infty, \mathbb{F}_\infty^+)$ does not have a FESSPE since in this case the Toeplitz algebra is isomorphic to $\mathcal{O}_\infty$ and is therefore simple. Another example of a quasi-lattice ordered group not having a FESSPE is $(\mathbb{Q}_+^*, \mathbb{N}^*)$, endowed with the order given by $r \leq s \iff r$ divides $s$. No finite set of non-zero positive integers different from 1 can contain a lower bound for every element in $\mathbb{N}^* \setminus \{1\}$.

**Example 6.2.** It is possible for $(G, P)$ to have a FESSPE but not be finitely generated. For example, consider $G = (\mathbb{R}, +)$ and $P = 0 \cup [1, \infty)$. Then $(G, P)$ is quasi-lattice ordered and has a FESSPE (and $P - P = G$), but is not finitely generated.

The following result is the essential-inner uniqueness theorem for $\mathcal{T}(G, P)$ when $(G, P)$ has a FESSPE.

**Theorem 6.3.** Let $(G, P)$ be a quasi-lattice ordered group and $\delta^n$ the canonical normal coaction on $\mathcal{T}(G, P)$. Assume $(G, P)$ has a FESSPE. Then the following assertions hold.

(a) $\mathcal{K}(l^2(P))$ is a $\delta^n$-invariant ideal in $\mathcal{T}(G, P)$ and $\delta^n|_{\mathcal{K}(l^2(P))}$ is an inner coaction.
Let $\varphi$ be a homomorphism of $\mathcal{T}(G,P)$ into a $C^*$-algebra $B$. Then the following are equivalent:

1. The homomorphism $\varphi$ is injective.
2. The homomorphism $\varphi$ is injective on $\mathcal{D}$.
3. We have $\varphi(p_e) \neq 0$, where $p_e$ is the rank-one projection onto $\varepsilon_e$.

To prove this theorem we shall need some preparation. The equivalence of (1) and (4) in the next result is implicit in [Nic92, Proposition 6.3]. We first recall a couple of facts about the Nica spectrum of $(G,P)$.

The spectrum of the commutative algebra $\mathcal{D}$ is the space $\Omega$ of all non-empty, hereditary, directed subsets $A$ of $P$, see [Nic92, §6] for definitions and details. Assigning the set $A_\gamma = \{s \in P : \gamma(T_sT_s^*) = 1\}$ to a character $\gamma$ of $\mathcal{D}$ gives a homeomorphism of the character space of $\mathcal{D}$ onto $\Omega$. Let $\iota : P \to \Omega$ be the map $t \mapsto [e,t]$ from [Nic92, §6.3, Remark 1], where $[e,t] := \{s \in P : s \leq t\}$. Since $\lambda_T$ is an isomorphism of $B_P$ onto $\mathcal{D}$, there is a homeomorphism $\hat{B}_P \to \Omega$ given by $\gamma \to A_\gamma$ for $A_\gamma = \{t \in P : \gamma(1_t) = 1\}$, see [Lac99]. Under this homeomorphism, $[e,t]$ corresponds to the character $\gamma$ of $B_P$ given by $\gamma(1_x) = 1_x(t)$ for all $x \in P$.

**Lemma 6.4.** Let $(G,P)$ be a quasi-lattice ordered group. The following statements are equivalent:

1. $(G,P)$ has a FESSPE.
2. $c_0(P)$ is contained in $B_P$.
3. $c_0(P)$ is an essential ideal in $B_P$.
4. $c_0(\iota(P))$ is an essential ideal in $\mathcal{D}$.

**Proof.** Let $F$ be a FESSPE for $(G,P)$, and for $x \in P$ define $1_{\{x\}} \in l^\infty(P)$ by

$$1_{\{x\}}(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

Then $c_0(P)$ is generated by the projections $1_{\{x\}}$ for $x \in P$. To establish $(\textcircled{1}) \Rightarrow (\textcircled{2})$ it suffices to prove that

$$(6.1) \prod_{a \in F} (1_x - 1_{xa}) = 1_{\{x\}}$$

for all $x \in P$. Take $y \in P$, and note that if $x = y$ then $1_{xa}(x) = 0$ for all $a \in F$, so the left hand side of (6.1) evaluated at $y$ is equal to $1_x(x) = 1$.

If $x^{-1}y \in P \setminus \{e\}$, there is $a \in F$ such that $a \leq x^{-1}y$. Hence $(1_x - 1_{xa})(y) = 1_x(y) - 1_{xa}(y) = 0$, and so the product on the left hand side of (6.1) evaluated at $y$ is zero.
The remaining possibility for \( y \) is \( x^{-1}y \notin P \), in which case the left side (6.1) is obviously zero; this establishes \( 1 \Rightarrow 2 \).

The implication \( 2 \Rightarrow 3 \) is clear, and \( 3 \Rightarrow 4 \) follows because the isomorphism \( B_P \rightarrow D \) carries the ideal \( c_0(P) \) onto \( c_0(\iota(P)) \).

It remains to prove \( 4 \Rightarrow 1 \). Since \( c_0(\iota(P)) \) is essential, \( \iota(P) \) is an open and dense subset of \( \Omega \). Thus the relative topology on \( \iota(P) \) is the original topology on \( P \), and so each interval \([e, t] \) is open in \( \Omega \). Then the implication \( 2 \Rightarrow 4 \) from [Nic92, Proposition 6.3] shows that \((G, P)\) has a FESSPE.

The next result is a sharpening of [Nic92, remark 2.2.3].

**Lemma 6.5.** Let \((G, P)\) be a quasi-lattice ordered group. Assume \( a, b, z \in G \) are such that at least one of \( a, b \) is in \( P \) and at least one of \( za, zb \) is in \( P \). Then \( a \lor b < \infty \) precisely when \( z(a \lor b) < \infty \), in which case \( z(a \lor b) = za \lor zb \) as elements of \( P \).

**Proof.** Suppose \( w := za \lor zb \in P \). Then \( a \leq z^{-1}w \) and \( b \leq z^{-1}w \). Since at least one of \( a, b \) is in \( P \) we necessarily have \( z^{-1}w \in P \). Thus \( a \lor b < \infty \). Since the order is left-invariant, \( w \leq z(a \lor b) \). Then \( z^{-1}w \leq a \lor b \), so necessarily \( z^{-1}w = a \lor b \).

Now suppose \( a \lor b < \infty \). Then by left invariance \( za \leq z(a \lor b) \) and \( zb \leq z(a \lor b) \). It follows that \( z(a \lor b) \in P \). Therefore \( za \lor zb \leq z(a \lor b) \), from which equality follows as in the previous paragraph. \( \square \)

**Lemma 6.6.** Let \((G, P)\) be a quasi-lattice ordered group having a FESSPE. The assignment

\[
e_y = \begin{cases} 1_{\{y\}} & \text{if } y \in P \\ 0 & \text{if } y \in G \setminus P \end{cases}
\]

for \( y \in G \) defines a family of mutually orthogonal projections in \( C^*(G, P) \) such that

\[
c_x e_y = e_{xy} c_x
\]

for all \( c_x \in C^*(G, P)_x \), and \( x, y \in G \).

Note that the family \( \{e_y\} \) gives rise to a nondegenerate homomorphism \( \mu : c_0(G) \rightarrow c_0(P) \).

**Proof.** Since \( C^*(G, P) \) is the closed span of monomials \( v_p v_q^* \), we have

\[
C^*(G, P)_x = \begin{cases} \overline{\text{span}} \{v_p v_q^* : x = pq^{-1}\} & \text{if } x \in PP^{-1} \\ 0 & \text{otherwise.} \end{cases}
\]

It therefore suffices to prove (6.3) when \( c_x \) is of form \( v_p v_q^* \) with \( x = pq^{-1} \in PP^{-1} \).
Case 1: \( y, xy \in P \). We must show that \( v_pv_q^*e_y = e_xyv_pv_q^* \). By \((6.1)\) we have \( e_y = \prod_{a \in F}(1_y - 1_{ya}) \) and \( e_{xy} = \prod_{a \in F}(1_{xy} - 1_{xya}) \).

If \( y \vee q = \infty \) then Lemma \((6.5)\) implies \( xy \vee p = \infty \). Hence Nica covariance of \( v \) implies that \( v_q^*1_y = v_q^*v_yv_q^* = 0 \) and \( 1_{xy}v_p = v_{xy}v_q^*v_{xy} = 0 \). Since also \( ya \vee q = \infty \) for all \( a \in F \) (because \( y \leq ya \) for all \( a \in F \)), we likewise have \( xya \vee p = \infty \), and therefore \( c_x1_{ya} = 1_{xya}c_x \). In all, \((6.3)\) is satisfied.

If \( y \vee q < \infty \) then \( xy \vee p = x(q \vee y) < \infty \) by Lemma \((6.5)\). We then have

\[
\begin{align*}
c_x1_y &= v_pv_q^*1_y = v_pv_q^*v_yv_q^* \\
&= v_pv_q^{-1}(q \vee y)v_q^*v_yv_q^* \\
&= v_{x(q \vee y)}v_q^* \\
&= v_{xy \vee p}v_q^* \\
&= v_{xy}v_{(xy)}^{-1}(xy \vee p)v_{xy}v_{(xy)}^{-1}(x \vee p)v_q^* \\
&= 1_{xy}v_pv_q^* = 1_{xy}c_x.
\end{align*}
\]

Let \( a \in F \). Two sub-cases arise: \( ya \vee q = \infty \), in which case also \( xya \vee p = \infty \), and \( c_x1_{ya} = 1_{xa}c_x = 0 \) follows as in the previous paragraph. The second sub-case has \( ya \vee q < \infty \), which entails \( xya \vee p < \infty \), and replacing \( y \) with \( ya \) in the computations leading to \((6.6)\) shows that \( c_x1_{ya} = 1_{xa}c_x \). The equality \((6.3)\) is thus satisfied in case 1.

Case 2: \( y \in P, xy \notin P \). We must show \( v_pv_q^*e_y = 0 \). Equivalently, we must show

\[
\prod_{a \in F} v_pv_q^*(1_y - 1_{ya}) = 0.
\]

Again, two sub-cases arise. If \( q \vee y = \infty \), then also \( q \vee ya = \infty \) for all \( a \in F \), and by Nica covariance we see that \( v_q^*v_y = 0 = v_q^*v_{ya} \) for all \( a \in F \). Hence \((6.7)\) follows. In case \( q \vee y \in P \), we have \( v_pv_q^*1_y = v_{xy}v_pv_q^*v_{qy} \) by \((6.5)\) (where the use of Lemma \((6.5)\) is legitimate because \( p, q, y \in P \)).

To establish \((6.7)\) we claim that there exists \( a' \in F \) such that \( v_pv_q^*(1_y - 1_{ya'}) = 0 \). The assumption \( xy \notin P \) implies \( q^{-1}y \notin P \), and so \( y^{-1}(q \vee y) \in P \setminus \{e\} \). Thus by assumption there is \( a' \in F \) with \( a' \leq y^{-1}(q \vee y) \). This gives \( ya' \leq q \leq q \vee y \), and since the reverse inequality is satisfied because \( F \subset P \) we get \( q \vee ya' = q \vee y \in P \). Applying Lemma \((6.5)\) yields \( xya' \vee p = x(q \vee ya') = x(q \vee y) = xy \vee p \), and invoking equation \((6.5)\) where \( y \) is replaced by \( ya' \) gives \( v_pv_q^*1_{ya'} = v_{xya'}v_pv_q^*v_{q', ya'}. \)

The claim is therefore proved, and case 2 is finished.

Case 3: \( y \notin P, xy \in P \). We must show that \( e_xyv_pv_q^* = 0 \). Either \( xy \vee p = \infty \), in which case \( xya \vee p = \infty \) for all \( a \in F \), and \((6.7)\) follows.
by Nica covariance, or \( xy \lor p \in P \). If this last alternative happens, the choice of \( y \) implies that \( p \not\in xy \), so \( (xy)^{-1}(xy \lor p) \in P \setminus \{e\} \). The FESSPE \( F \) supplies \( a' \in F \) with \( a' \leq (xy)^{-1}(xy \lor p) \), and similarly to case 2 we get \( 1_{xy}v_pv_q^* = 1_{xya'}v_pv_q^* \), from which (6.7) again follows.

**Case 4:** \( y \notin P, xy \notin P \). Then both sides of (6.3) are zero. \( \square \)

**Proof of Theorem 6.3.** Since \((G, P)\) has a FESSPE, [Nic92, Proposition 6.3] gives an inclusion \( \mathcal{K}(l^2(P)) \subset \mathcal{T}(G, P) \). Clearly \( \mathcal{K}(l^2(P)) \) is an ideal in \( \mathcal{T}(G, P) \).

To prove part (a) we will show that \( \mathcal{K}(l^2(P)) \) is \( \delta^n \)-invariant. This will give a coaction \( \delta^n_{\mathcal{K}(l^2(P))} \) on \( \mathcal{K}(l^2(P)) \) obtained as restriction of \( \delta^n \).

We shall then construct mutually orthogonal projections \( \{ p_x : x \in G \} \) in \( B(l^2(P)) \) such that \( \sum_{x \in G} p_x = I \) in weak-operator topology on \( B(l^2(P)) \) and \( a_x p_y = p_y a_x \) for all \( a_x \in \mathcal{K}(l^2(G, P))_x \) and \( x, y \in G \).

**Remark 4.1** therefore provides an inner coaction \( \delta^\mu \) on \( \mathcal{K}(l^2(P)) \) with \( \delta^n_{\mathcal{K}(l^2(P))} \) is the maximalization by Proposition 5.1, it carries \( C^*(G, P)_x \) isometrically onto \( \mathcal{T}(G, P)_x \) for every \( x \in G \). Lemma 6.6 implies that

\[ p_x = \prod_{a \in F} (T_x T_x^* - T_x a T_x a^*). \]

With \( \xi \otimes \eta \) denoting the rank-one operator \( \langle \xi \otimes \eta \rangle(\zeta) = \eta \langle \xi, \zeta \rangle \) in \( B(l^2(P)) \) we see that \( p_e = \varepsilon_e \otimes \varepsilon_e \). So \( p_e \in \mathcal{K}(l^2(P)) \). Since also \( p_e \in \mathcal{D} \), it follows that \( \delta^n(p_e) = p_e \otimes 1 \). For \( x, y \in P \), the product \( T_x p_e T_y^* \) is the rank-one operator \( e_y \otimes e_x \), see also the proof of [Nic92, Proposition 6.3]. Thus \( \mathcal{K}(l^2(P)) \) is the closed span of monomials \( T_x p_e T_y^* \) for \( x, y \in P \). But

\[ \delta^n(T_x p_e T_y^*) = \delta^n(T_x) \delta^n(p_e) \delta^n(T_y^*) = (T_x p_e T_y^*) \otimes xy^{-1}, \]

showing that \( \delta^n(\mathcal{K}(l^2(P))) \subset \mathcal{K}(l^2(P)) \otimes C^*(G) \). Since

\[ (T_x p_e T_y^*) \otimes z = (T_x p_e T_y^* \otimes xy^{-1})(1 \otimes yx^{-1}z), \]

we have

\[ \overline{\text{span}} \delta^n(\mathcal{K}(l^2(P)))(1 \otimes C^*(G)) = \mathcal{K}(l^2(P)) \otimes C^*(G). \]

Thus \( \mathcal{K}(l^2(P)) \) is \( \delta^n \)-invariant, and by restriction \( \delta^n \) gives a coaction \( \delta^n_{\mathcal{K}(l^2(P))} \) on \( \mathcal{K}(l^2(P)) \).

Since \( \lambda_T \) is a maximalization by Proposition 5.1, it carries \( C^*(G, P)_x \) isometrically onto \( \mathcal{T}(G, P)_x \) for every \( x \in G \). Lemma 6.6 implies that
\(a_x p_y = p_{x y} a_x\) for every \(x \in PP^{-1}, a_x \in T(G, P)_x\) and \(y \in G\). In particular, we may take \(a_x \in \mathcal{K}(l^2(P)) \cap T(G, P)_x\), which shows that
\[
\delta^n|_{\mathcal{K}(l^2(P))_x} = \delta^n|_{\mathcal{K}(l^2(P)) \cap T(G, P)_x}.
\]

Hence \(\delta^n|_{\mathcal{K}(l^2(P))}\) coincides with \(\delta^n\), and thus is an inner coaction, as claimed in (a).

For (b), obviously it suffices to show \(2 \implies 1\) and \(3 \implies 2\). The implication \(2 \implies 1\) follows from (a) and Corollary 4.3 because \(\mathcal{K}(l^2(P))\) is an essential ideal in \(B(l^2(P))\), hence in \(T(G, P)\).

For the implication \(3 \implies 2\), suppose \(\varphi(p_e) \neq 0\). For every \(x \in P\) we have \(T_x p_e = p_x T_x\), and \(T_x\) is an isometry, so \(\varphi(T_x) \varphi(p_e) \neq 0\). Then \(\varphi(p_x) \varphi(T_x) \neq 0\), and hence \(\varphi(p_x) \neq 0\). By linearity and density it follows that \(\varphi(M_f) \neq 0\) for all \(f \in c_0(\iota(P)) \subset \mathcal{D}\). Since \(c_0(\iota(P))\) is an essential ideal in \(\mathcal{D}\), it follows that \(\varphi\) is injective on \(\mathcal{D}\).

\(\square\)

**Remark 6.7.** In the above proof of Theorem 6.3, we appealed to Corollary 4.3 for the implication \(2 \implies 1\), and then for \(3 \implies 2\) we employed an elementary argument. In fact, however, in this particular case we can prove \(3 \implies 1\) directly, as follows. We have mentioned that FESSPE guarantees \(\mathcal{K}(l^2(P)) \subset T(G, P)\), and then \(3\) implies that \(\varphi\) is nonzero on the simple, essential ideal \(\mathcal{K}(l^2(P))\), and hence is faithful. Nevertheless, we wanted to show how the method involving Corollary 4.3 can be applied, because we feel that it will be useful more generally.

A result similar to Theorem 6.3 (a) can be proved for \(C^*(G, P)\), as follows.

**Corollary 6.8.** In the notation of Lemma 6.6, the set
\[
(6.8) \quad \mathcal{J} := \text{span} \{ v_x e_v v_y^* : x, y \in P \}
\]
is a \(\delta\)-invariant ideal of \(C^*(G, P)\), and \(\mathcal{J}\) is an inner coaction.

**Proof.** Since \(C^*(G, P) = \text{span} \{ v_s v_t^* : s, t \in P \}\), it suffices to show that \(\mathcal{J}\) is a subalgebra of \(C^*(G, P)\) and that \(v_s v_t^* v_x e_v v_y^*\) and \(v_x e_v v_y^* v_s v_t^*\) are in \(\mathcal{J}\) for all \(s, t, x, y \in P\). By (6.3), \(v_t^* v_x e_v = v_t^* e_v v_x = 0\) unless \(t^{-1} x \in \mathcal{P}\), in which case \(v_s v_t^* v_x e_v v_y^* = v_s v_t^{-1} e_v v_y^* \in \mathcal{J}\). If \(y \vee s = \infty\), Nica covariance of \(v\) implies that \(v_y^* v_s = 0\). Otherwise \(v_y^* v_s = v_{y^{-1}(y \vee s)} v_{s^{-1}(y \vee s)}\), and (6.3) implies that \(e_v v_{y^{-1}(y \vee s)} = 0\) unless \(y^{-1}(y \vee s) = e\). If \(y \vee s = y\) it follows that \(v_x e_v v_y^* v_s v_t^* = v_x e_v v_t^* v_{(s^{-1}y)} \in \mathcal{J}\). Now clearly \(\mathcal{J}\) is closed under taking adjoints, and by the previous computations it follows that \(v_x e_v v_y^* v_s e_v v_t^*\) is zero unless \(s = y\), in which case it equals \(v_x e_v v_t^*\), so it lies in \(\mathcal{J}\).
That $\mathcal{J}$ is $\delta$-invariant follows as in the proof of part (a) of Theorem 6.3 because $\delta(v_x e_x v_y^*) = v_x e_x v_y^* \otimes xy^{-1}$ and $(v_x e_x v_y^* \otimes z = (v_x e_x v_y^* \otimes xy^{-1})(1 \otimes yx^{-1}z)$. Hence Lemma 6.6 implies that $\delta_{|\mathcal{J}}$ is $\delta^\mu$, and therefore is an inner coaction. □

We obtain an essential-inner uniqueness theorem for $C^*(G,P)$ if the ideal $\mathcal{J}$ of (6.8) is essential.

Corollary 6.9. If the ideal $\mathcal{J}$ of (6.8) is essential in $C^*(G,P)$, then $C^*(G,P)$ has the essential-inner uniqueness property of Corollary 4.3. This holds in particular if $(G,P)$ has the approximation property for positive definite functions in the sense of Nica.

Proof. If $\mathcal{J}$ is essential, then the conclusion follows immediately from Corollary 6.8 and Corollary 4.3. For the other part, note that, as remarked in [Lac99], if $(G,P)$ has the approximation property of Nica then for every ideal $\mathcal{I}$ of $C^*(G,P)$ we have $\mathcal{I} = \{X \in C^*(G,P) : \Phi(X^*X) \in \Phi(\mathcal{I})\}$, where $\Phi$ is the conditional expectation from $C^*(G,P)$ into $B_P$, see [LR96, Corollaries 2.4 and 3.3]. Now $\Phi$ is faithful by [Nic92, §4.3 and 4.5]. Therefore, if $\mathcal{I}$ is non-trivial then by faithfulness of $\Phi$ also $\Phi(\mathcal{I})$ is non-trivial as an ideal of $B_P$. By Lemma 6.4 there exists $e_x \in c_0(P)$ such that $e_x \in \Phi(\mathcal{I})$. It follows that $e_x \in \mathcal{I}$, so $\mathcal{I} \cap \mathcal{J}$ is non-trivial, and hence $\mathcal{J}$ is essential. □

The next result is a converse to Theorem 6.3.

Theorem 6.10. Suppose there is a family $\{q_x : x \in G\} \subset \mathcal{T}(G,P)$ of mutually orthogonal projections such that

1. $q_y \in \mathcal{D}'$ and $T_p q_y = q_y T_p$ for all $y \in G$ and $p \in P$, and
2. $\sum_{y \in G} q_y = I$ in the weak operator topology of $B(l^2(P))$.

Then $(G,P)$ has a FESSPE.

To prove this theorem we will need a lemma.

Lemma 6.11. Let $(G,P)$ be a quasi-lattice ordered group and $\mathcal{D}$ the diagonal subalgebra of $\mathcal{T}(G,P)$. Then the commutant $\mathcal{D}'$ is contained in $l^\infty(P)$.

Proof. Let $M \in \mathcal{D}' \subset B(l^2(P))$. We claim that there is $g \in l^\infty(P)$ such that $M = M_g$. For each $p \in P$ define $f_p := M_{\varepsilon_p}$ in $l^2(P)$. Using that $MM_{t_p} = M_{t_p} M$ implies that $f_p = M_{t_p} f_p$, and therefore $f_p$ has support included in $\{t \in P : p \leq t\}$. On the other hand, if $p \leq t$ and $p \neq t$, then the commutation relation $MM_{t_p} = M_{t_p} M$ implies that $M_{t_p} f_p = 0$, showing that $f_p$ has support the single point $\{p\}$. 

Thus there is \( g : P \rightarrow \mathbb{C} \) such that \( M_{\varepsilon_p} = g(p)\varepsilon_p \) for all \( p \in P \). Since \( \|M_{\varepsilon_p}\|_2 \leq \|M\| \) for all \( p \in P \), it follows that \( |g(p)| \leq \|M\| \) for all \( p \in P \). This means \( g \in l^\infty \). The claim, hence the lemma, are proved. \[ \square \]

**Proof of Theorem 6.10** We will show that \( \mathcal{T}(G, P) \supseteq \mathcal{K}(l^2(P)) \), and then apply [Nic92, Proposition 6.3] to conclude that \( (G, P) \) has a FESSPE.

It suffices to show that \( \mathcal{T}(G, P) \) contains all rank one projections \( \varepsilon_y \otimes \varepsilon_x \) on \( l^2(P) \), where \( x, y \in P \). For this it suffices to establish that \( q_e = p_e \), because then we will have \( \varepsilon_y \otimes \varepsilon_x = T_x q_e T_y^* \) as in the proof of Theorem 6.3.

By assumption (1), \( T_p T_p^* q_y = q_y T_p T_p^* \) for all \( p \in P \) and \( y \in G \). Thus \( q_y \in \mathcal{D}' \) for all \( y \in G \), and so \( q_y \in l^\infty(P) \) by Lemma 6.11. Write \( q_y = M_{\chi_{E(y)}} \) where \( \emptyset \neq E(y) \subset P \) for every \( y \in G \). Since \( \sum_{y \in G} q_y = I \), the family \( \{E(y)\}_{y \in G} \) is a mutually disjoint family such that \( P = \bigcup_{y \in P} E(y) \). We claim that

\[
(6.9) \quad E(y) = \{y\} \text{ for all } y \in P.
\]

Towards the claim, we prove first that \( p E(y) = E(py) \) for all \( p, y \in P \). By assumption (1), \( T_p M_{\chi_{E(y)}} = M_{\chi_{E(py)}} T_p \) for \( p \in P \). Applying both sides to \( \varepsilon_u \) gives

\[
\begin{cases}
\varepsilon_{pu} & \text{if } u \in E(y) \\
0 & \text{if } u \notin E(y)
\end{cases} = \begin{cases}
\varepsilon_{pu} & \text{if } pu \in E(py) \\
0 & \text{if } pu \notin E(py)
\end{cases}
\]

when \( p, u, y \in P \). Thus it suffices to prove (6.9) when \( y = e \). Let \( y \in P \) such that \( e \in E(y) \). Then \( e \in E(y) = y E(e) \subset y P \). This forces \( y \in P \cap P^{-1} \), so \( y = e \). Hence \( e \in E(e) \), which also implies \( p \in E(p) \) for all \( p \in P \). If \( p \in E(e) \), then \( p \in E(p) \cap E(e) \). This intersection is non-empty precisely when \( p = e \). In other words, we have established \( E(e) = \{e\} \), from which (6.9) and hence the theorem follow. \[ \square \]

**Remark 6.12.** It was asserted in [Nic92, §6.3, Remark 4] that \( \mathcal{K}(l^2(P)) \) is an induced ideal from \( \mathcal{D} \) when \( (G, P) \) has a FESSPE. However, no proof was given of this claim. Here we show that \( \mathcal{K}(l^2(P)) \) is contained in the ideal of \( \mathcal{T}(G, P) \) induced from \( c_0(P) \). We conjecture that the two are equal, but we have not been able to prove this.

To recall terminology, let \( (G, P) \) be a quasi-lattice ordered group. Let \( \Phi \) be the conditional expectation from \( C^*(G, P) \) onto \( B_P \) constructed in [LR96] and \( \Phi^n \) the conditional expectation from \( \mathcal{T}(G, P) \) to \( \mathcal{D} \) associated to the coaction \( \delta^n \) of Proposition 5.1. The representation \( \lambda_T \) intertwines \( \Phi \) and \( \Phi^n \). Since \( \delta^n \) is normal, \( \Phi^n \) is faithful on positive
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elements. In [Nic92, §6] Nica associates to an invariant ideal \( I \) in \( D \) the induced ideal \( \text{Ind} I = \{ X \in \mathcal{T}(G, P) : \Phi^n(X^*X) \in D \} \) in \( \mathcal{T}(G, P) \).

Suppose \((G, P)\) has a FESSPE. Lemma 6.4 says that \( \mathcal{I} := c_0(\iota(P)) \) is an essential ideal in \( D \). Further, \( \mathcal{I} \) is generated by the projections

\[
p_y = \prod_{a \in F} (T_y T^*_y - T_ya T^*_ya),
\]

for all \( y \in P \). We claim that \( \mathcal{I} \) is invariant in Nica’s sense. To see this, let \( x \in G \) and write it as \( x = \sigma(x)\tau(x)^{-1} \) with \( \sigma(x) \in P \) the least upper bound of \( x \). For \( y \in P \), equation (6.3) implies that

\[
T_{\sigma(x)} T_{\tau(x)}^* p_y (T_{\sigma(x)} T_{\tau(x)}^*) = p_{xy} T_{\sigma(x)} T_{\tau(x)}^* T_{\tau(x)} T_{\tau(x)}^* T_{\sigma(x)},
\]

which is \( p_{xy} T_{\sigma(x)} T_{\sigma(x)}^* \), and lies in \( \mathcal{I} \) because \( \mathcal{I} \) is an ideal in \( D \). Since \( p_y \) span \( \mathcal{I} \), the ideal \( \mathcal{I} \) is indeed invariant.

Now the rank-one operator on \( l^2(P) \) taking \( \varepsilon_y \) to \( \varepsilon_x \) is \( X = T_x p_e T^*_y \) and

\[
\Phi^n(X^*X) = T_y p_e T^*_y = p_y \in \mathcal{I},
\]

so \( \text{Ind} \mathcal{I} \) contains all rank-one operators in \( B(l^2(P)) \). Hence \( \mathcal{K}(l^2(P)) \subset \text{Ind} \mathcal{I} \).

Appendix A. Gauge-invariant uniqueness for Fell bundles

Here we present an abstract “gauge-invariant uniqueness” result for Fell bundles over discrete groups. As applications we obtain gauge-invariant uniqueness results for maximal and for normal coactions.

**Proposition A.1.** If \( \pi : (A, \delta) \to (B, \varepsilon) \) is a surjective morphism of coactions such that \( \pi|_{A_e} \) is injective, then

\[
\pi \times G : A \times_\delta G \to B \times_\varepsilon G
\]

is an isomorphism. Consequently:

1. if \( \varepsilon \) is maximal, then \( \delta \) is maximal and \( \pi \) is an isomorphism;
2. if \( \delta \) is normal, then \( \varepsilon \) is normal and \( \pi \) is an isomorphism;
3. if \( \delta \) is maximal, then \( \pi \) is a maximalization of \( (B, \varepsilon) \), and there is a unique morphism \( \varphi : (B, \varepsilon) \to (A^n, \delta^n) \) such that the diagram

\[
(A, \delta) \xrightarrow{\pi} (B, \varepsilon) \xleftarrow{\varphi} (A^n, \delta^n)
\]
commutes, and moreover \( \varphi \) is a normalization.

(4) if \( \varepsilon \) is normal, then \( \pi \) is a normalization of \( (A, \delta) \), and there is a unique morphism \( \varphi : (B^m, \varepsilon^m) \to (A, \delta) \) such that the diagram

\[
\begin{array}{ccc}
(B^m, \varepsilon^m) & \xrightarrow{\varphi} & (A, \delta) \\
\downarrow_{q_A^m} & \searrow \varphi & \downarrow \pi \\
(B, \varepsilon) & \searrow \pi & (A, \delta) \\
\end{array}
\]

commutes, and moreover \( \varphi \) is a maximalization.

Proof. We first show that

\[ \pi(A_s) = B_s \quad \text{for all} \quad s \in G. \]

Indeed, it is easy to check on the generators that

\[ \varepsilon_s \circ \pi = \pi \circ \delta_s \quad \text{for all} \quad s \in G. \]

Then we have

\[
B_s = \varepsilon_s(B) \\
= \varepsilon_s(\pi(A)) \\
= \pi(\delta_s(A)) \\
= \pi(A_s).
\]

Since \( \pi|_{A_s} \) is injective, it follows that for each \( s \in G \) the restriction \( \pi|_{A_s} \) maps \( A_s \) isometrically onto \( B_s \), and hence the associated Fell-bundle homomorphism \( \tilde{\pi} : A \to B \) is an isomorphism.

The normalization

\[ \pi^n : (A^n, \delta^n) \to (B^n, \varepsilon^n) \]

of \( \pi \) is an isomorphism of coactions, because \( A^n \cong C^*_r(A) \) and \( B^n \cong C^*_r(B) \). Let \( q^n_A : (A, \delta) \to (A^n, \delta^n) \) and \( q^n_B : (B, \varepsilon) \to (B^n, \varepsilon^n) \) be the normalizing maps.

We have a commuting diagram

\[
\begin{array}{ccc}
(A, \delta) & \xrightarrow{q^n_A} & (A^n, \delta^n) \\
\downarrow \pi & & \downarrow \pi^n \\
(B, \varepsilon) & \xrightarrow{q^n_B} & (B^n, \varepsilon^n) \\
\end{array}
\]
of coaction morphisms, hence a commuting diagram

\[
\begin{array}{ccc}
A \times \delta G & \xrightarrow{q^n \times G} & A^n \times \delta^n G \\
\pi \times G & \downarrow & \pi^n \times G \\
B \times \varepsilon G & \xrightarrow{q^n \times G} & B^n \times \varepsilon^n G
\end{array}
\]

of homomorphisms. Thus \(\pi \times G\) is an isomorphism.

Now (1)–(4) follow from the theory of maximalizations and normalizations: First of all, (1) and (2) follow immediately from [KQ09, Proposition 3.1].

For (3), [BKQ11, Proposition 6.1.11] shows that \(\pi\) is a maximalization. Let \(q^n_B : (B, \varepsilon) \to (B^n, \varepsilon^n)\) be the normalization of \((B, \varepsilon)\). Then \(q^n_B \circ \pi : (A, \delta) \to (B^n, \varepsilon^n)\) also is a normalization, by [BKQ11, Proposition 6.1.7]. Since all normalizations of \((A, \delta)\) are isomorphic, there is an isomorphism \(\theta\) making the diagram

\[
\begin{array}{ccc}
(A, \delta) & \xrightarrow{\pi} & (B, \varepsilon) \\
q^n_A & \downarrow & q^n_B \\
(A^n, \delta^n) & \xrightarrow{=} & (B^n, \varepsilon^n)
\end{array}
\]

commute. Put \(\varphi = \theta \circ q^n_B : (B, \varepsilon) \to (A^n, \delta^n)\). Then \(\varphi\) is a normalization since \(q^n_B\) is and \(\theta\) is an isomorphism, and the diagram

\[
\begin{array}{ccc}
(A, \delta) & \xrightarrow{\pi} & (B, \varepsilon) \\
q^n_A & \downarrow & q^n_B \\
(A^n, \delta^n) & \xrightarrow{=} & (B^n, \varepsilon^n)
\end{array}
\]

commutes.

To see that \(\varphi\) is the unique morphism making the diagram (A.1) commute, suppose that \(\varphi'\) is another. Since \(q^n_A\) is also a maximalization (by [BKQ11, Proposition 6.1.15]) it follows from the theory of maximalization that both \(\varphi\) and \(\varphi'\) have the same maximalization (namely \(\text{id}_A\)), and hence are equal since the maximalization functor is faithful (by [BKQ11, Corollary 6.1.19]).
(4) is proved similarly to (3): [BKQ11, Proposition 6.1.7] shows that \( \pi \) is a normalization, and if \( q^m_A : (A^m, \delta^m) \to (A, \delta) \) is a maximalization then \( \pi \circ q^m_A \) is also a maximalization, by [BKQ11, Proposition 6.1.11], so there is an isomorphism \( \theta \) making the diagram
\[
\begin{array}{ccc}
(B^m, \varepsilon^m) & \xrightarrow{\theta} & (A^m, \delta^m) \\
\downarrow q^m_B & & \downarrow q^m_A \\
(B, \varepsilon) & & (A, \delta)
\end{array}
\]
commute. Then \( \varphi := q^m_A \circ \theta \) is a maximalization of \( (A, \delta) \) making the diagram
\[
\begin{array}{ccc}
(B^m, \varepsilon^m) & \xrightarrow{\theta} & (A^m, \delta^m) \\
\downarrow q^m_B & & \downarrow q^m_A \\
(B, \varepsilon) & & (A, \delta)
\end{array}
\]
commute.

To prove that \( \varphi \) is the unique morphism making the diagram \((A.2)\) commute, if \( \varphi' \) is another then, since \( q^m_B \) is also a normalization (by [BKQ11, Proposition 6.1.14]) both \( \varphi \) and \( \varphi' \) have the same normalization (namely \( \text{id}_B \)), and hence are equal since the normalization functor is faithful (by [BKQ11, Corollary 6.1.19]).

**Corollary A.2** (Abstract GIUT for maximal coactions). Let \( (A, \delta) \) be a maximal coaction and \( \pi : A \to B \) a surjective homomorphism. Then \( \pi \) is injective if and only if \( \pi|_{A_c} \) is injective and there is a maximal coaction \( \varepsilon \) of \( G \) on \( B \) such that \( \pi \) is \( \delta - \varepsilon \) equivariant.

**Proof.** The forward direction is immediate. Assume now that \( \pi|_{A_c} \) is injective and there is a maximal coaction \( \varepsilon \) of \( G \) on \( B \) such that \( \pi \) is \( \delta - \varepsilon \) equivariant. Then \( \pi : (A, \delta) \to (B, \varepsilon) \) is a surjective morphism of coactions. Hence \( \pi \) is an isomorphism by Proposition A.1 part (1). □

The following is parallel to Corollary A.2

**Corollary A.3** (Abstract GIUT for normal coactions). Let \( (B, \varepsilon) \) be a normal coaction and \( \pi : A \to B \) a surjective homomorphism. Then \( \pi \)
is injective if and only if there is a normal coaction \( \delta \) of \( G \) on \( A \) such that \( \pi \) is \( \delta - \varepsilon \) equivariant and \( \pi|_{A_e} \) is injective.

**Proof.** The forward direction is immediate. Assume now that there is a normal coaction \( \delta \) of \( G \) on \( A \) such that \( \pi \) is \( \delta - \varepsilon \) equivariant and \( \pi|_{A_e} \) is injective. Then \( \pi : (A, \delta) \to (B, \varepsilon) \) is a surjective morphism of coactions. Hence \( \pi \) is an isomorphism by Proposition A.1, part (2). \( \square \)

**Corollary A.4.** Let \( (A, \delta) \) be a coaction. The following are equivalent:

1. \( \delta \) is normal;
2. A surjective homomorphism \( \pi : A \to B \) is injective if and only if \( \pi|_{A_e} \) is injective and there is a coaction \( \varepsilon \) on \( B \) such that \( \pi \) is \( \delta - \varepsilon \) equivariant.

**Proof.** Assume (1). Let \( \pi : A \to B \) be an isomorphism. Then trivially \( \pi|_{A_e} \) is injective and \( \pi \) carries \( \delta \) to a (normal) coaction on \( B \). If on the other hand \( \pi : A \to B \) is surjective, \( \pi|_{A_e} \) is injective, and \( B \) carries a coaction \( \varepsilon \) such that \( \pi \) is \( \delta - \varepsilon \) equivariant, then by Proposition A.1, part (2) \( \pi \) is an isomorphism. This proves (1) \( \Rightarrow \) (2).

Now assume (2). Since the normalization map \( q^n_A : (A, \delta) \to (A^n, \delta^n) \) is equivariant and satisfies \( q^n_A|_{A_e} \) is injective, by hypothesis \( q^n_A \) is injective. Hence it is an isomorphism, so \( \delta \) is normal since \( \delta^n \) is.

The following is parallel to Corollary A.4.

**Corollary A.5.** Let \( (B, \varepsilon) \) be a coaction. The following are equivalent:

1. \( \varepsilon \) is maximal;
2. A surjective homomorphism \( \pi : A \to B \) is injective if and only if there is a coaction \( \delta \) on \( A \) such that \( \pi \) is \( \delta - \varepsilon \) equivariant and \( \pi|_{A_e} \) is injective.

**Proof.** Assume (1). Let \( \pi : A \to B \) be an isomorphism. Then trivially \( \pi|_{A_e} \) is injective and \( \pi^{-1} \) carries \( \varepsilon \) to a (maximal) coaction on \( A \) and \( \pi|_{A_e} \) is injective. If on the other hand \( \pi : A \to B \) is surjective, \( A \) carries a coaction \( \delta \) such that \( \pi \) is \( \delta - \varepsilon \) equivariant and \( \pi|_{A_e} \) is injective, then by Proposition A.1, part (1) \( \pi \) is an isomorphism. This proves (1) \( \Rightarrow \) (2).

Now assume (2). Since the maximalization map \( q^m_B : (B^m, \varepsilon^m) \to (B, \varepsilon) \) is equivariant and satisfies \( q^m_B|_{B^m} \) is injective, by hypothesis \( q^m_B \) is injective. Hence it is an isomorphism, so \( \varepsilon \) is maximal since \( \varepsilon^m \) is. \( \square \)

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