Towards Optimal Energy Efficiency in Cell-Free Massive MIMO Systems

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Abstract—Motivated by the ever-growing demand for green wireless communications and the advantages of cell-free (CF) massive multiple-input multiple-output (MIMO) systems, we focus on the design of their downlink for optimal energy efficiency (EE). To address this fundamental topic, we assume that each access point (AP) is deployed with multiple antennas and serves multiple users on the same time-frequency resource while the APs are Poisson point process (PPP) distributed, which approaches realistically their opportunistic spatial randomness. Relied on tools from stochastic geometry, we derive a lower bound on the downlink average achievable spectral efficiency (SE). Next, we consider a realistic power consumption model for CF massive MIMO systems. These steps enable the formulation of a tractable optimization problem concerning the downlink EE per unit area, which results in the analytical determination of the optimal pilot reuse factor, the AP density, and the number of AP antennas and users that maximize the EE. Notably, the EE per unit area and not just the EE is the necessary metric to describe CF systems, where we meet multi-point transmission. Hence, we provide useful design guidelines for CF massive MIMO systems relating to fundamental system variables towards optimal EE. Among the results, we observe that a lower pilot reuse factor enables a decrease of the interference, and subsequently, higher EE up to a specific value. Overall, it is shown that the CF massive MIMO technology is a promising candidate for next-generation networks achieving simultaneously high SE and EE per unit area.

Index Terms—Cell-free massive MIMO systems, energy efficiency, stochastic geometry, small cell networks, beyond 5G MIMO.

I. INTRODUCTION

The rapid development of wireless communication systems, by means of the fifth generation (5G) networks and beyond, aim at higher data rates with adequate quality of service (QoS) but with the reduction of energy consumption being of primary concern [1]. In fact, the power consumption of the developing information and communication technology (ICT) sector is emerging as a major societal, economic, and environmental concern [2]. Obviously, achieving higher data rates with less power consumption might seem like contradictory goals [3], but that is not necessarily the case. A promising solution to provide higher data rates is achieved by means of the so-called network densification, which, unfortunately, stumbles at the major bottleneck of increasing interference resulting in higher power consumption [4]. Hence, the fundamental arising question is how to increase the network data rate while achieving optimal energy efficiency (EE) at the same time. Although both academia and industry already have focused on the EE of cellular networks in the past years [5], new innovative architectures should be proposed to address the crucial green specifications and considerations in next-generation networks.

In the direction of network densification, a key 5G technology (in terms of the number of antennas per area unit), known as massive multiple-input multiple-output (MIMO) systems, has emerged by providing $10^3$ higher data rate with comparison to conventional cellular systems [6]–[9]. Its implementation relies on the use of a massive number of base station (BS) antennas serving simultaneously a number of users while exploiting spatial multiplexing and the array gains on the same time-frequency resource. Massive MIMO systems achieve higher data rates by providing asymptotically negligible fast fading and interference [6], [9].

Although massive MIMO systems can effectively deal with interference, the achievable energy efficiency is limited by the large propagation losses that are typical in cellular networks. An interesting alternative is to distribute a large number of antennas over the coverage area and operate these antennas in a network MIMO manner [10], [11]. A practical embodiment of network MIMO is the cell-free (CF) massive MIMO concept described in [12]. Specifically, the main characteristic of CF massive MIMO is the deployment of a large number of access points (APs) that are distributed over the coverage area to coherently serve a large number of users on the same time-frequency resource. According to [12], as the number of APs increases, we manage to take advantage of the favorable propagation and channel hardening properties, and finally, achieve very large spectral efficiency with simplified signal processing needing less overhead. However, herein, it is crucial to mention that the attractive properties of channel hardening and favorable propagation do not hold under all conditions. In particular, despite [12] that accounted for these properties for single-antenna APs, in [13], it was proved the opposite. Fortunately, it was shown that channel hardening and favorable propagation appear in the case of multiple-antennas APs (at least $5 \times 10$ antennas) or low path-loss. As a result, CF massive MIMO can combine the benefits of coordination and low overhead. Moreover, CF massive MIMO is a promising architecture because by increasing the number

1 Although network MIMO has attracted a lot of interest in the last decade [10], [11], its implementation is not feasible for practical systems due to its substantial backhaul overhead.
of APs the path-losses are improved and the macro-diversity is enhanced \cite{12}, which means that the transmit powers can be reduced. Unfortunately, these gains from CF massive MIMO are achieved by deploying more hardware, which in turn, may increase the power consumption. Notably, even though CF massive MIMO systems come with plausible potentials, the study of this technology is limited as literature reveals \cite{12}, \cite{13}, \cite{14}, \cite{16}–\cite{26}. For example, the authors in \cite{17} achieved better data rates by suggesting a user-centric approach of CF massive MIMO systems, where the APs serve a group of users instead of all of them. Another interesting study concerns \cite{26}, where the locations of the APs are Poisson point process (PPP) distributed, and the coverage probability was derived.

Along the line concerning energy consumption, there are many works that have studied the EE of network MIMO \cite{27}–\cite{29}, however, only a few prior works have examined the EE in CF massive MIMO systems which is of particular interest since they are more beneficial but are deployed with more hardware than network MIMO \cite{14}–\cite{16}, \cite{21}. In parallel, the optimal uplink EE of cellular networks was obtained analytically and examined thoroughly in \cite{28} by using tools from stochastic geometry where the BSs are PPP distributed, and in \cite{30}, the same methodology was applied for a multislope path-loss model. In particular, in the case of CF massive MIMO systems, the EE was investigated in \cite{14}–\cite{16}, \cite{27}, \cite{21}, the EE was investigated under a user-centric approach at millimeter-wave frequencies, but these works did not obtain analytical expressions for the EE.

A. Motivation

Most existing works on 5G networks focus on the spectral efficiency (SE) while they neglect the importance of EE which is decreased when interference increases. Network MIMO, mitigating interference by means of coordination, is practically unattainable due to excessively high complexity in terms of hardware and information overhead. Hence, the study of EE of CF massive MIMO systems is of pivotal interest. Luckily, CF massive MIMO systems emerge as a promising feasible solution regarding coordination with low overhead exploiting the favorable propagation and channel hardening properties as the number of each AP antennas increases. Despite some existing works on the numerical optimization of the EE of CF massive MIMO systems \cite{14}, \cite{15}, there is no previous work deriving the optimal system parameters in closed form. Most importantly, existing works, except \cite{19}, \cite{25}, focus on simplified network topologies such as grid-based models, and they do not account for the realistic spatial randomness of the APs. Especially, as the number of APs increases, i.e., their geographical architecture becomes denser, which also agrees with the concept of CF massive MIMO systems, the network becomes increasingly irregular. Above this, the APs are in general deployed opportunistically which means high irregularity. Although previous works mentioned that the APs are randomly located, they consider a fixed number of APs while their randomness is not utilized in the analysis, but only in the simulations. These observations suggest that the analytical derivation of the optimal realistic EE of CF massive MIMO systems, where the APs are distributed according to a PPP, is of paramount importance. In order to extract trustable results, a realistic power consumption model is needed to take both the transmit power and other system parameters into account.

B. Contribution

The main contributions are summarized as follows.

- Contrary to existing works \cite{14}, \cite{15}, which did not account for the spatial randomness of the APs, and thus, are quite idealized, we apply tools from stochastic geometry and assume that the APs are PPP located. In addition, contrary to \cite{26}, our analysis relies on a finite number of APs, and the aim of this work is the study of the EE. Also, we differentiate from \cite{19} that assumed a BPP for the APs which is again idealistic.
- We derive a lower bound of the downlink average achievable SE for a finite number of APs being PPP distributed and having multiple antennas. Furthermore, we present a realistic power consumption model, specialized in CF massive MIMO systems.
- Contrary to the common definition of EE in cellular networks, we provide a novel definition describing the EE per unit area, which is necessary to model the EE in CF massive MIMO systems, and in general, in architectures with coordinated multi-point joint transmission (CoMP JT).
- We obtain the optimal EE per unit area of CF massive MIMO systems with PPP distributed multiple-antenna APs by means of an analytical expression enabling to derive the optimal values for fundamental system parameters such as the network size in terms of AP antennas and serving users.
- We shed light on the impact of the main system parameters on the optimal EE. The results are of high practical interest since the analysis accounts for finite and realistic systems dimensions. Specifically, we obtain the optimal reuse factor, the optimal AP density, and the optimal number of AP antennas and users. For the sake of comparison, we also present results for a corresponding "cellular" massive MIMO system and a small-cells (SCs) network.

C. Paper Outline

The remainder of this paper is organized as follows. Section \( \text{II} \) presents the system model of a CF massive MIMO system with multiple antennas APs being PPP distributed.

\footnotetext[2]{In \cite{19}, the spatial randomness of the APs was considered. However, the distribution of the APs was again idealized and neglected their irregularity since it was assumed uniform, i.e., a binomial point process (BPP) was applied. Moreover, certain approximations were made that result in a not strict analysis with not reliable expressions. For example, it was made the assumption of the nearest AP and it was considered the mean contribution from the rest of the APs. Regarding our recent work in \cite{29}, it was relied on the deterministic equivalent (DE) analysis to obtain the DE signal-to-interference-plus-noise ratio (SINR) for a large number of APs. Also, it focused on the derivation of the coverage probability and achievable rate for a large APs number.}

\footnotetext[3]{The authors in \cite{13}, which accounted for the PPP distribution, only explored the validity of the channel hardening and favorable propagation properties.
Sections III and IV provide the uplink training and downlink transmission phases, respectively. Section V provides the analysis regarding the EE while Section VI presents the optimization of the EE and obtains the optimal system parameters in closed form. The numerical results are placed in Section VII, and Section VIII concludes the paper.

D. Notation

Vectors and matrices are denoted by boldface lower and upper case symbols, respectively. The symbols \((\cdot)^T\), \((\cdot)^\text{H}\), and \(\text{tr}(\cdot)\) express the transpose, Hermitian transpose, and trace operators, respectively. The expectation operator is denoted by \(\mathbb{E}[\cdot]\). Also, \(b \sim \mathcal{CN}(0, \Sigma)\) represents a circularly symmetric complex Gaussian vector with zero mean and covariance matrix \(\Sigma\). Finally, the superscript \(^*\) is used to represent optimal values.

II. SYSTEM MODEL

We consider a CF massive MIMO system with multiple antennas at the APs and we model the practical spatial randomness of APs by means of stochastic geometry. Specifically, we assume that the APs, each having \(N \geq 1\) antennas, are distributed in the two dimensional Euclidean plane with their locations following a homogeneous PPP \(\Phi_{\text{AP}}\) with intensity \(\lambda_{\text{AP}}\, [\text{AP/km}^2]\). In a specific realization of the PPP \(\Phi_{\text{AP}}\), the number of APs in any region of size \(S(\mathcal{A})\) in \([\text{km}^2]\), denoted by \(M\), is a Poisson random variable with mean value

\[
\mathbb{E}[M] = \lambda_{\text{AP}} S(\mathcal{A}).
\]  

(1)

Following the network MIMO principle, all the APs serve simultaneously all the single-antenna users on the same time-frequency resource.\(^4\) Interestingly, the total number of antennas in \(\mathcal{A}\), in a realization of the spatial process, denoted by \(W = MN\) is a Poisson random variable with mean \(\mathbb{E}[W] = N \lambda_{\text{AP}} S(\mathcal{A})\). We let \(K\) denote the number of users in a given network realization. Their number is fixed and the users are selected at random from a large set based on some scheduling algorithm. Notable, the number of users is an optimization variable while their locations are uniformly distributed\(^5\). To consider a CF massive MIMO scenario, the densities are chosen in order to fulfill the condition \(W \gg K\) in most realizations\(^6\).

All APs are connected via a perfect fronthaul network to a central processing unit for coding and decoding of the data signals\(^7\). Taking advantage of Slepian’s theorem, we focus on a typical user, selected at random among the users and indexed by \(k\), in order to analyze the network performance\(^{33}\). In particular, we assume that the typical user is located at the origin for ease of exposition.

A. Channel Model

In a realization of the PPP \(\Phi_{\text{AP}}\), i.e., given \(M\), let the \(N \times 1\) channel vector \(h_{mk}\) between the \(m\)th AP and the typical user be given by

\[
h_{mk} = l_{mk}^{1/2} g_{mk}, \quad m = 1, \ldots, M \text{ and } k = 1, \ldots, K
\]

(2)

where \(l_{mk} = \min(1, r_{mk}^{-\alpha})\) and \(g_{mk}\) represent independent path-loss and small-scale fading between the \(m\)th AP and the typical user. In particular, the path-loss is described by means of a non-singular bounded model with \(\alpha > 0\) being the path-loss exponent and \(r_{mk}\) being the distance between the \(m\)th AP and the \(k\)th user\(^{34}\). Note that this bounded path-loss model is practical also at short distances\(^{13}\). Given that this work accounts for the spatial randomness of the APs, the following analysis is dependent on the selection of the path-loss model. Although the majority of CF massive MIMO works such as\(^{12},^{14}\) have considered another path-loss model, herein, for the sake of clarity and simplicity, we have considered a famous bounded path-loss model that will result in tractable expressions. Note that the three-slope path-loss model would be too complicated for the analysis. Also, both models provide similar insights regarding the parameters of the system under study in this work. The same reasons have contributed to the wide acceptance of the bounded model in many scenarios modeled in terms of stochastic geometry\(^{34}\). In addition, both types of distances, i.e., the distance between the \(m\)th AP located at \(x_m\) in \(\mathbb{R}^2\) and the typical user as well as the distances between the \(m\)th AP and the other users in \(\mathcal{A}\setminus\{x_m \in \mathcal{A}\}\) follow the uniform distribution and are independent. Also, similar to other works on CF massive MIMO systems, e.g.,\(^{12},^{14}–^{16}\), we assume independent Rayleigh fading where the elements of \(g_{mk}\) are independent and identically distributed (i.i.d.) \(\mathcal{CN}(0,1)\) random variables. Note that this assumption of un-correlated channels is reasonable, since the service antennas (APs) in CF massive MIMO systems are distributed over a large area. Hence, the set of scatterers is likely to be different for each AP and each user.

We consider a time-varying narrowband channel that is divided into coherence blocks, which are blocks of duration \(T_c\) in \(s\) and bandwidth \(B_c\) in \(\text{Hz}\) while the channels are fixed and frequency-flat. Each coherence block consists of \(\tau_{\text{c}} = B_c T_c\) samples (channel uses) and we follow the standard block fading model where independent channel realizations appear in every block\(^8\). We employ the time-division-duplex (TDD) protocol with an uplink training phase of \(\tau_{\text{tr}}\) samples and two data transmission phases of \(\tau_d\) (downlink) and \(\tau_{\text{up}}\) (uplink) samples, respectively. Hence, we have \(\tau_c = \tau_{\text{tr}} + \tau_{\text{up}} + \tau_d\) while the communication strategy is illustrated in Fig. 1. In this work, we focus on the uplink training and downlink data transmission phases. The duration of the latter can be expressed by \(\tau_d = \xi (\tau_c - \tau_{\text{tr}})\) with \(\xi \leq 1\), where \(\xi\) expresses the downlink payload fraction transmission\(^{50}\).
To summarize, we have introduced the reuse factor $k$. The choice of conjugate beamforming relies on its parameterization of the network, where the APs have multiple antennas. In the training phase of one realization of the network, the $k$th user transmits a normalized pilot sequence $\tilde{\psi}_k \in \mathbb{C}^{\tau_{tr} \times 1}$ with $\|\tilde{\psi}_k\|^2 = 1$, and the received $N \times \tau_{tr}$ channel vector by the $m$th AP is given by

$$\hat{y}_{tr}^m = \sum_{i=1}^{K} \sqrt{\rho_{tr} \tau_{tr}} h_{mi}^t \psi_i^t + n_{m}^t, \quad (3)$$

where $\rho_{tr}$ is the average transmit power while $n_{m}^t$ is the $N \times \tau_{tr}$ additive noise vector at the $m$th AP consisted of i.i.d. $\mathcal{CN}(0,1)$ random variables. In other words, $\rho_{tr}$ is actually the normalized signal-to-noise ratio (SNR). By projecting $\hat{y}_{tr}^m$ onto $1/\sqrt{\tau_{tr} \rho_{tr}} \psi_k$, we obtain

$$\tilde{y}_{mk} = g_{mk}^t k_{mk}^2 + \sum_{i \neq k} |\psi_{mk}^t|^2 g_{mi}^t \psi_{mi}^t + \frac{1}{\tau_{tr} \rho_{tr}^t} n_{m}^t |\psi_k|^2. \quad (4)$$

With the assumption that the channel and distances statistics are known a priori and that $\psi_i^t |\psi_k \in \{0,1\}$ for all $i, k$, the $m$th AP obtains the linear minimum mean-squared error (MMSE) estimate according to (3), i.e., $\hat{h}_{mk} = (\hat{y}_{mk} \psi_{mk} |\psi_k)^{-1} \hat{y}_{mk} \psi_{mk} |\psi_k$. Thus, we have

$$\hat{h}_{mk} = \sum_{i=1}^{K} |\psi_{mi}|^2 h_{mi}^t + \frac{1}{\tau_{tr} \rho_{tr}^t} \hat{y}_{mk} \psi_{mk} |\psi_k. \quad (5)$$

The estimation error vector $\tilde{h}_{mk} = h_{mk} - \hat{h}_{mk}$ is independent of $h_{mk}$, and the channel estimation error vector $\tilde{h}_{mk} = (\hat{h}_{mk} \psi_{mk} |\psi_k)^{-1} \hat{y}_{mk} \psi_{mk} |\psi_k$. The matrices $\hat{I}_{mk}$ and $\hat{e}_{mk}$ are given by

$$\hat{I}_{mk} = (\hat{y}_{mk} \psi_{mk} |\psi_k)^{-1} \hat{y}_{mk} \psi_{mk} |\psi_k. \quad (6)$$

$$\hat{e}_{mk} = (\hat{y}_{mk} \psi_{mk} |\psi_k)^{-1} \hat{y}_{mk} \psi_{mk} |\psi_k. \quad (7)$$

To summarize, we have $h_{mk} \in \mathbb{C}^{N \times 1} \sim \mathcal{CN}(0, I_N)$, $\hat{h}_{mk} \in \mathbb{C}^{N \times 1} \sim \mathcal{CN}(0, \sigma_{mk}^2 I_N)$, and $\sigma_{mk}^2 = \frac{1}{\tau_{tr} \rho_{tr}^t}$. The noise is distributed according to $\mathcal{CN}(0, I_N)$, and the noise vector $n_{mk} \in \mathbb{C}^{N \times 1} \sim \mathcal{CN}(0, \sigma_{mk}^2 I_N)$. We define $C_k = \Phi_k^{-1}$ with $C_k |_{w_{w}} = c_{mk} I_N$, where $c_{mk} = \sigma_{mk}^2$.

After substituting (5) into (7), the received signal by the typical user is given by

$$y^d_k = \sqrt{\rho_{d}} \sum_{m=1}^{M} \sum_{i=1}^{K} g_{mi}^t h_{mk}^t \psi_{mi}^t + z^d_k \quad (9)$$

$$= \sqrt{\rho_{d}} \left( \mathbb{E} \left[ \sum_{m=1}^{M} g_{mi}^t h_{mk}^t \tilde{h}_{mk} \psi_{mi}^t \right] q_k + \sum_{m=1}^{M} r_{mk}^t h_{mk}^t \tilde{h}_{mk} \psi_{mi}^t \right) \quad (10)$$

where we have written (9) as (10), similar to [36], in order to derive the SINR based on the fact that the users do not have any indication for distributed architectures due to no need for CSI exchange among the APs and the central unit [12]. Given that all APs serve jointly all users, the received signal by the typical user is given by

$$y^d_k = \sqrt{\rho_{d}} \sum_{i=1}^{K} h_{mi}^t s_i + z^d_k \quad (6)$$

$$= \sqrt{\rho_{d}} \sum_{m=1}^{M} h_{mk}^t s_m + z^d_k. \quad (7)$$

In (6), the vector $\tilde{h}_{mk}$ describes the channel between the $i$th AP located at $x_i \in \mathbb{R}^2$ and the typical user including small-scale fading and path-loss, $\rho_{d} > 0$ denotes the corresponding transmit power, while $s_i$ is the transmitted signal from the $i$th AP and $z^d_k \sim \mathcal{CN}(0,1)$ is the additive white Gaussian noise at the $k$th user. Since a realization of the system includes $M$ APs, the signal model described by (6) can be written as in (7). Notably, the number $M$ is a random variable changing in every spatial realization of the APs. In (7), the vector $\tilde{h}_{mk}$ expresses the channel between the $m$th AP and the typical user while $s_m$ is the transmit signal from the $m$th AP, which is written as

$$s_m = \sum_{k=1}^{K} \sqrt{\eta_{mk} f_{mk} q_k}, \quad (8)$$

where $q_k \in \mathbb{C}$ is the normalized transmit data symbol for user $k$ satisfying $\mathbb{E}[|q_k|^2] = 1$. The vector $f_{mk} = \hat{h}_{mk} \in \mathbb{C}^N$ expresses the linear precoder. This selection regarding $\eta_{mk}$ aims at easing the following algebraic manipulations. Actually, it corresponds to a statistical channel inversion power-control policy [28]. It allows each AP to allocate more power to the most distant users and less power to the closest ones. Note that the scaling does not result in any loss in the performance since the parameter $\mu$ is changed accordingly. We denote $\eta_{mk} = \mu \sigma_{mk}^{-4}$, where the parameter $\mu$ is obtained by means of the constraint of the transmit power $\mathbb{E}[\frac{1}{\eta_{mk}} |\psi_k|^2] = \rho_{d}$. Henceforth, for the sake of algebraic manipulations, we denote $h_{mk} = [h_{mk}^T \cdots h_{MK}^T] \sim \mathcal{CN}(0, L_k)$, $\hat{h}_{mk} = [\hat{h}_{mk}^T \cdots \hat{h}_{MK}^T] \sim \mathcal{CN}(0, \Phi_k)$, and $\hat{e}_{mk} = \mathcal{CN}(0, L_k)$. The matrices $I_k \in \mathbb{C}^{N \times N}$, $\Phi_k = \mathcal{L}_k \mathcal{D}^{-1} \in \mathbb{C}^{N \times N}$, and $\mathcal{D} \in \mathbb{C}^{N \times N}$ are block diagonal matrices with elements given by the matrices $[L_{kw} | \mathbb{I}_N]$ and $[D_{kw} | d_m I_N]$, respectively, for $w = 1, \ldots, W$ and $\mathcal{W} = MN$. We also define $C_k = \Phi_k^{-1}$ with $C_k |_{w_{w}} = c_{mk} I_N$, where $c_{mk} = \sigma_{mk}^{-2}$.
knowledge of the instantaneous CSI given by \( \mathbf{h}_{mk} \), but they are aware of its statistics \( \mathbb{E} \left[ \sum_{m=1}^{M} \mathbf{h}_{mk}^H \mathbf{h}_{mk} \right] \). Note that the second term in \( (10) \) expresses the desired signal while the fourth term describes the multi-user interference. By applying the well-established bounding technique in \([36]\), we consider that \( (10) \) represents a single-input single-output (SISO) system, where the APs treat the unknown terms as uncorrelated additive noise. Thus, we obtain the effective SINR of the downlink transmission from all the multi-antenna APs to the typical user, conditioned on the number of APs and their distances from the users, as

\[
\gamma_k = \frac{\mathbb{E} \left[ \mathbf{h}_k^H \mathbf{C}_k \mathbf{h}_k \right]^2}{\sum_{i=1}^{K} \mathbb{E} \left[ \left| \mathbf{h}_k^H \mathbf{C}_k \mathbf{h}_m \right|^2 \right] - \mathbb{E} \left[ \left| \mathbf{h}_k^H \mathbf{C}_k \hat{\mathbf{h}}_k \right|^2 \right] + \frac{1}{\mu_p \rho_d}}. \tag{11}
\]

Notably, the matrices in \( (11) \) are random because they include the number of APs and the distances between the APs and the users that both are cases of random variables changing in each realization.

**Proposition 1:** Given a realization of the network with \( M \) APs and \( K \) users, the effective SINR of the downlink transmission from the PPP distributed \( N \) antennas APs to the typical user in a CF massive MIMO system, accounting for pilot contamination and conjugate beamforming, is given by \( (12) \) at the top of the next page.

**Proof:** See Appendix A.

**Remark 1:** The scaling in the numerator with \( N \) corresponds to the array gain resulting from the coherent transmission of the \( N \) antennas per AP. Moreover, the summations in the denominator take place over the number of users \( K \) because as their number increases, the interference increases.

V. EE ANALYSIS

In this section, we provide the definition of the EE per unit area of CF massive MIMO systems where the APs locations follow a PPP distribution. Note that this definition is novel and also necessary to model CF massive MIMO systems, and in general, architectures with CoMP. Next, we focus on the analytical derivation of the downlink EE per unit area by first obtaining a lower bound on the average SE, and then, presenting a realistic power consumption model. The power consumption is expected to increase rapidly with the number of APs, i.e., their density. Hence, it is of paramount importance to quantify the relevant efficiency of a CF massive MIMO system.

**Definition 1:** The EE per unit area expresses the amount of reliably transmitted information per unit of energy and area, which is defined mathematically as

\[
\text{EE} = \frac{\text{Throughput}}{\text{Area power consumption}} = \frac{B_w \cdot \text{TSE}}{\text{APC}}, \tag{13}
\]

where \( B_w \), TSE, and APC describe the transmission bandwidth, the total SE (TSE), and the area power consumption (APC), respectively.

**Remark 2:** Notably, \( (13) \) defines the EE per unit area, and not simply the EE, since it is obtained by means of the fraction between the total network SE (the sum SE of \( K \) users) and the area power consumption. In other words, contrary to the common definition for the EE in cellular systems with no cooperation \([28]\), \([30]\), the CF massive MIMO architecture necessitates to define the EE per unit area. Specifically, in CF massive MIMO systems each user receives joint transmission from multiple sources (APs), and the received SINR at the user is obtained from the sum of received signals from all these serving APs. Therefore, this received SINR is not the same as the received SINR computed in a single BS association network. Consequently, the definition of area spectral efficiency (ASE), where the received user rate (i.e., per transmission link rate) is multiplied with the AP density does not hold in this scenario. Hence, we focus on the EE per unit area.

We continue with the derivations of TSE and APC.

A. Total Spectral Efficiency

Taking advantage of the property of the typical user, stating that it is statistically equivalent with any other user in the network, the TSE is provided by

\[
\text{TSE} = K R \quad \text{[bit/s/Hz]}, \tag{14}
\]

where \( R \) is the average SE per user. Given that our analysis relies on user \( k \), being statistical equivalent with any other user in the network, we have \( R = R_k \), where \( R_k \), provided below, corresponds to the average downlink SE of the typical user \( k \) over the channel realizations and APs locations. Notably, the multiplicative factor \( K R \) corresponds to the sum SE of all users.

Since the downlink capacity for this network including imperfect CSI in not known, we follow the common approach, especially in the area of massive MIMO \([6]\), \([37]\), focusing on the derivation of achievable lower bounds on the ergodic capacity. In particular, the following lemma provides a tractable lower bound on the ergodic capacity for any given realization of \( \Phi_{AP} \).

**Lemma 1 (\([38]\)):** A lower bound on the downlink ergodic channel capacity of the typical user \( k \) in a CF massive MIMO system with conjugate beamforming and PPP distributed APs for any given realization of \( \Phi_{AP} \) is provided by

\[
R_k = \left( 1 - \frac{K}{\zeta c} \right) \log_2 \left( 1 + \gamma_k \right) \quad \text{b/s/Hz}, \tag{15}
\]

where \( K \) is the number of users, \( \zeta \) is the pilot reuse factor, and \( \tau_c \) is the channel coherence interval in number of samples while \( \gamma_k \) is given by \( (12) \).
The average SE per user is obtained by applying the expectation at \([15]\) over the APs locations. We resort to Jensen’s inequality to derive a closed-form lower bound for the downlink achievable \(R_k\) and avoid intractable lengthy numerical integral evaluations with respect to the APs distances.

**Theorem 1:** A lower bound on the downlink average SE per user with conjugate beamforming precoding in a CF massive MIMO system with multi-antenna APs is obtained by

\[
\bar{R}_k = \left(1 - \frac{K}{\zeta \tau_c}\right) \log_2 \left(1 + \gamma_k\right) \text{ b/s/Hz,}
\]

where \(\gamma_k = 1/\bar{\gamma}_k\) with \(\bar{\gamma}_k\) given by

\[
\bar{\gamma}_k = \sum_{j=1}^{K} \left| \psi_j \right|^2 \left( \frac{\alpha - 2}{\alpha \pi N_p d^2} + K - 1 \right) + \frac{\zeta}{\alpha \pi K P_{tx}} \left( (K-1)(\alpha-2) + \frac{(\alpha-1)}{N_p d^2} \right) + \lambda_{AP}(K-1).
\]

**Proof:** See Appendix B.

Notably, if we shed further light into \((16)\), we observe that the TSE is a strictly quasi-concave function of the number of users \(K\) while the optimal number of antennas per AP depends on the AP density and the quality of CSI in terms of \(N\) and \(\zeta\), respectively. These observations are in line with \([26]\), accounting also for the spatial AP randomness.

Although we have applied the law of large numbers regarding the number of APs during the derivation of this proof, it is known that this law is applicable and valid in the case of a finite number of APs obeying to \(M > 8\) \([39]\). Obviously, this range is of practical interest in CF massive MIMO systems. The agreement of the analytical results with Monte Carlo simulations in Section VII for finite system dimensions confirms this assertion. Thus, Theorem 1 and the following results describe realistic systems of finite dimensions.

### B. Area Power Consumption

The sources of the area power consumption of a CF massive MIMO system are the power usage during the transmission \(P_{TX}\) and the circuitry of the system \(P_{CPC}\). Following a similar approach to \([30], [40]\) but specialized to CF massive systems, we have

\[
APC = \lambda_{AP} \left( \frac{1}{\alpha_{eff}} P_{TX} + P_{CPC} \right).
\]

where \(\alpha_{eff} \in [0, 1]\) is the power amplifier efficiency. Note that \(P_{TX}\) concerns both the average powers for the uplink pilot and downlink payload transmissions. Regarding \(P_{CPC}\), it describes the circuitry dissipation in terms of cooling, power supply, backhaul signaling, digital signal processing, etc. Although the majority of works assume that \(P_{CPC}\) is a fixed constant, this is not a realistic assumption, and obviously, not a good design methodology. In practice, each antenna is accompanied by dedicated circuits that contribute to the system power consumption. Above this, if APC was independent of \(N\), the TSE, increasing with \(N\), would result in an unbounded EE as \(N\) increases, which is irrational \([40]\). Hence, it is of dire necessity to incorporate in our EE analysis an accurate model for the power consumption.

**Proposition 2:** A generic realistic model for the downlink APC of CF massive MIMO systems is given by

\[
APC(\theta) = \lambda_{AP} \left( C_0 + C_1 K + C_2 K^2 + D_0 N + D_1 N K - D_2 N K^2 + A B_w \text{TSE} \right),
\]

where \(C_0 = P_{UP} + P_{LO}, C_1 = \frac{B_w}{\tau_c}, C_2 = \frac{1}{\alpha_{eff} K P_{tx}}, D_0 = P_{AP}, D_1 = \frac{3B_w}{\tau_c}, D_2 = \frac{3B_w}{\tau_c},\) and \(A = (F_{COD} + P_{DEC} + P_{GET})\).

**Proof:** See Appendix C.

It is worthwhile to mention that \([19]\) is written in a polynomial structure that will facilitate the optimization taking place in the following section.

### VI. EE Maximization

This section elaborates on the main objective of this work, which is the maximization of the constrained EE with respect to the parameters defining the size of the network (e.g., the AP density and the number of users) under generic hardware and transmission characteristics. In other words, we scrutinize the tuple of system parameters \(\theta = (\zeta, \lambda_{AP}, K, N)\) that obey to the problem

\[
\theta^* = \arg \max_{\theta \in \Theta} \text{EE}(\theta) = \frac{B_w \text{TSE}(\theta)}{\text{APC}(\theta)}
\]

subject to \(\gamma_k(\theta) = \gamma_0\),

where \(\text{TSE}(\theta) = K R\) with \(R = \bar{R}_k\), where \(\bar{R}_k\) is given by Theorem 1, APC(\(\theta\)) is provided by Proposition 2, \(\gamma_k\) is obtained by Theorem 1 while \(\gamma_0 > 0\) is a design parameter. The set \(\Theta\), including the feasible parameters values, is defined as \(\Theta = \{\theta : \lambda_{AP} \geq 0, \zeta \geq 1, K/\zeta \leq \tau_c, (K, N) \in \mathbb{Z}_+\}\). The constraint in \((20)\) prevents from an optimal tuple of parameters with a low unacceptable achievable rate while it demands a specific QoS \([28], [30]\).

We aim at solving \((20)\) for either \(\lambda_{AP}, N, K\) when the remaining parameters are fixed. The advantage of this approach is to obtain closed-form expressions for the optimal EE and to shed light into the interplay among these parameters.

### A. Feasibility

The optimization problem in \((20)\) is feasible for a certain range of values of \(\gamma_0\) because of the multiuser interference.
Lemma 2: The feasibility range of values of $\gamma_0$, obtained from the maximization problem for CF massive MIMO systems, can be described by

$$\gamma_0 < \frac{1}{\lambda_{AP}}.$$  

Proof: In order to obtain the range of values of $\gamma_0$, we simplify the expression of the SINR, being the inverse of (17) by noticing that it is a monotonically increasing function of $N$. Hence, deriving its upper limit as $N \to \infty$, we obtain (22) at the top of the next page. Since the upper limit is a decreasing function of the optimizable variable $\zeta$, we exploit the constraint $\zeta = K/\tau_c$ by taking its minimal value when $K = 1$, and we obtain the feasible $\gamma_0$.

This lemma reveals that the upper limit of the SINR depends only on the AP density $\lambda_{AP}$ as $N \to \infty$. In the case of CF massive MIMO, the typical value concerning the number of APs is $100 - 200$ [12] which is equivalent to a density $\lambda_{AP} \approx 10^{-4}$ m$^{-2}$. In such case, e.g., $\lambda_{AP} = 10^{-4}$ m$^{-2}$, the average SE per user is $\log_2(1 + 100) \approx 13.29$ b/s/Hz. This value, showing the feasibility of the optimization problem described by (20), is larger than the SE of currently applied systems [41]. Hence, the optimization problem under study is quite meaningful for practical systems.

B. Optimal Pilot Reuse Factor

Herein, we derive the optimal pilot reuse factor $\zeta^*$ while the rest of the parameters are fixed.

Theorem 2: Let any set of $\{\lambda_{AP}, K, N\}$ resulting in the feasibility of the maximization of EE given by (20). The optimal pilot reuse factor, satisfying the SINR constraint, is obtained by

$$\zeta^* = \frac{\alpha \pi KN\rho_{tr}\rho_4 - \gamma_0 Q_1}{\gamma_0 Q_2}.$$  

Proof: The reuse factor $\zeta^*$ is obtained by means of simple algebraic manipulations. Specifically, we focus on the constraint and we collect the terms including $\zeta$ in the SINR given by $\gamma_k = 1/\zeta_k$ as

$$\gamma_0 = \frac{\alpha \pi N\rho_{tr}\rho_4}{Q_1 - \zeta Q_2},$$  

where we set

$$Q_1 = K\rho_{tr}\left[(\alpha - 2) \sum_{j=1}^{K} |\psi_j^\rho|^2 + \alpha \pi N\rho_4 (K - 1) \left( \sum_{j=1}^{K} |\psi_j^\rho|^2 + \lambda_{AP} \right) \right],$$

$$Q_2 = (\alpha - 1 + N\rho_4 (\alpha - 2)(K - 1))/K,$$

and we solve (24) with respect to $\zeta$.

Theorem 2 provides the dependence of $\zeta^*$ on the rest of the system parameters. According to its physical interpretation, a smaller pilot reuse factor, meaning a larger training phase, results in both more precise channel estimation and less pilot contamination. Intuitively, a better channel estimation increases the SE, or equivalently, a better SINR constraint $\gamma_0$ is allowed, which comes to agreement with (23). It is shown that $\zeta^*$ is a decreasing function of $Q_1$ and $Q_2$, which both are increasing functions of $K$. However, a larger $K$ means higher interference, requiring a better channel estimation, i.e., a lower $\zeta$ which admits to the dependence shown by (23).

C. Optimal APs Density

After plugging (23) into the optimization problem, (20) is written as

$$\text{EE}(\zeta^*, K, N) = \frac{B_w \text{TSE}(\zeta^*, K, N)}{\text{APC}(\zeta^*, K, N)},$$

subject to $1 \leq \frac{\alpha \pi N\rho_{tr}\rho_4 - \gamma_0 Q_1}{\gamma_0 Q_2} \leq \frac{K}{\tau_c}$.

Theorem 3: Let any set of $\{K, N\}$ keeping the optimization problem (26) feasible. For fixed $K$ and $N$, the EE per unit area is maximized by

$$\lambda_{AP}^* = \min \{\max(\lambda_{AP_0}, \lambda_{AP_1}), \lambda_{AP_2}\},$$

where

$$\lambda_{AP_0} = \frac{(a_1 + a_3)G + \sqrt{a_2 a_3 a_4 (a_1 + a_3)G}}{a_2 a_3 G},$$

$$G = a_2 (a_4 + a_5 + a_6 K \log (1 + \gamma_0))$$

with

while $\lambda_{AP_1} = a_3 - a_1$, $\lambda_{AP_2} = \tau_c/(K a_3 + a_1 a_2)$, and the parameters $\{a_i\}$ are provided in Table I.

Proof: Both the TSE and APC include the term $\zeta^*\tau_c/K$. Hence, we proceed with its computation which gives $\zeta^*\tau_c/K = a_3 \lambda_{AP}^* - a_1$. Then, after substituting this term into the objective function of (26), the EE becomes

$$\text{EE}(\zeta^*) = \frac{B_w \text{TSE}(\zeta^*, K, N)}{a_4 + a_5 a_2 + a_6 K \xi (1 - \bar{a}) \log (1 + \gamma)}$$

where $\bar{a} = a_3/a_2\lambda_{AP}^* - a_1$. Following the approach in [40] Lem. 3, it can be shown that (30) is a quasi-concave function of $\lambda_{AP}$.

Thus, (28) is obtained by taking the first derivative of (30) and equating to zero. Given that the constraint in (26) depends on $\lambda_{AP}$, we obtain $\lambda_{AP_1}$ and $\lambda_{AP_2}$.
\[
\lim_{N \to \infty} \gamma_k = \frac{\alpha \pi \rho_{11} K}{\alpha \pi K \left( \sum_{j=1}^{K} |\psi_j \psi_k^*|^2 (K-1) + K \lambda_{AP} \right) \rho_{11} + (\alpha - 2) (K-1) \zeta}.
\]

(22)

**TABLE II**

| Parameter | Value |
|-----------|-------|
| \(b_1\)   | \(K (a - 1)\) |
| \(b_2\)   | \(\rho_{11} K (1 - (a - 2)\) |
| \(b_3\)   | \(\alpha \pi \rho_{11} \tau_0 (1 - \frac{\gamma_{N+1}}{\gamma_{N+2}}) (K-1)\) |
| \(b_4\)   | \(\rho_{11} \tau_0 (a - 2) \sum_{j=1}^{K} |\psi_j \psi_k^*|^2\) |
| \(b_5\)   | \(\rho_{11} \tau_0 (C_0 + C_1 K) / \lambda_{AP}\) |
| \(b_6\)   | \((D_0 + D_1 K - D_2 K^2) / \lambda_{AP}\) |
| \(b_7\)   | \((\varphi_{22} - d_{22} - c_{12}) K^2 / \lambda_{AP}\) |
| \(b_8\)   | \((P_{COD} + P_{DEC} + P_{BT}) / \lambda_{AP}\) |

**TABLE III**

| Parameter | Value |
|-----------|-------|
| \(e_1\)   | \(\gamma N \rho_{11} (a - 2)\) |
| \(e_2\)   | \(\gamma N \rho_{11} (1 - 3 (a - 2))\) |
| \(e_3\)   | \(2 \gamma (1 - (a - 2))\) |
| \(e_4\)   | \(-\alpha \pi N \rho_{11} \tau_0 \rho_{11} (\gamma_{N+1} + \lambda_{AP} - 1)\) |
| \(e_5\)   | \(\rho_{11} \tau_0 (a - 2) (\gamma_{N+1} + \lambda_{AP} - 1)\) |
| \(e_6\)   | \((\rho_{11} \tau_0 (a - 2) (\gamma_{N+1} + \lambda_{AP} - 1)) / \lambda_{AP}\) |
| \(e_7\)   | \((C_0 + D_0 N) / \lambda_{AP}\) |
| \(e_8\)   | \((P_{COD} + P_{DEC} + P_{BT}) / \lambda_{AP}\) |
| \(e_9\)   | \((C_2 - D_2 N) / \lambda_{AP}\) |
| \(e_{10}\) | \((P_{COD} + P_{DEC} + P_{BT}) / \lambda_{AP}\) |

**D. Optimal Number of AP Antennas and Users**

The optimal values of \(N\) and \(K\) are found by means of the maximization problem (20) in the case of optimal \(\zeta^*\). Initially, we consider the integer-relaxed problem where \(K\) and \(N\) can be any positive scalars, but then, we select the corresponding integer values.

**Theorem 4**: Let the maximized problem (20) with \(\lambda_{AP}, K,\) and \(N\) real variables. For any fixed \(\lambda_{AP}, K > 0,\) the optimal number of AP antennas \(N^*\) is given by

\[
N^* = \min \left( \max \{N_0, N_1, N_2\} \right)
\]

(31)

with \(N_0 = \frac{a_1 - \sqrt{a_1}}{a_2 - a_1}\) while \(N_1 = \frac{b_1 + b_2}{b_3 - b_2}\) and \(N_2 = \frac{c_1}{c_1 - c_2}\), where \(a_1 = \frac{b_1^2 + d_1^2}{b_1 - b_2}\), \(b_1 = 1 + b_2 D_1 + b_1 K (b_1 d_{21} K^2 + b_2 c_{11} + b_1 d_{11} + d_{22} K),\)

\(q_2 = 2 \left(b_3 + b_1 K (b_4 - b_2 K) (b_2 d_{21} K^2 - b_4 d_{11})\right),\)

\(c_3 = q_2^2 + 4 q_4 (b_2 d_{21} K^2 - b_4 d_{11})\)

\(c_{11} = c_{12} (c_{11} + c_{22} K),\)

\(c_{22} = \frac{c_4}{\rho_{11} \lambda_{AP}}, d_{22} = \frac{3}{\rho_{11} \lambda_{AP}}\)

The parameters \(c_{12}\) are provided in Table II.

**Proof**: Similar to the proof of Theorem 3, we have \(\zeta^* \tau_0 / K = \frac{\alpha \pi N \rho_{11} e}{\lambda_{AP} N - e},\) where the parameters \(b_{12}\) are provided in Table II. Then, after substituting this term into the objective function of (26), the EE becomes

\[
EE(\zeta^*) = \frac{K \xi (1 - \tilde{b}) \log_2 (1 + \gamma)}{b_2 + b_0 N + b_1 + b_2 N + b_2 K \xi (1 - \tilde{b}) \log_2 (1 + \gamma)},
\]

(32)

which represents a quasi-concave function of \(N\). Note that \(\tilde{b} = \frac{b_1 + b_2 N}{b_2 N - b_3}\). The optimal value of \(N\) is obtained by computing its first derivative with respect to \(N\) and equating it to zero. The resultant value, satisfying the unconstraint problem, is given by (31). Taking into account the constraint in (26), this can be written as \(\frac{N}{K} \leq \frac{b_1 + b_2 N}{b_2 N - b_3} \leq 1,\) which results in \(N_1 < N^* \leq N_2.\)

**Theorem 5**: Let the maximization problem (20) with \(\lambda_{AP}, K\) and \(N\) real variables. For any fixed \(\lambda_{AP}, N > 0,\) the optimal number of users \(K^*\) is given by

\[
K^* = \max \left( K_2, \max \left( K_{11}, \min \{K_0, K_{12}\} \right) \right),
\]

(33)

where \(K_0\) is one of the real roots of a quintic equation, i.e., a polynomial of degree five given by \(\sum_{i=0}^{5} \rho_i x_i = 0\) with 

\[p_0 = e_1 - e_2, p_1 = (e_2 - e_3) + 2 e_4, p_2 = 3 (e_3 - 2 e_1), p_3 = (e_2 + e_1) - e_1 e_3, p_4 = 2 e_3 (e_1 - e_2), p_5 = (e_2 - e_1) - e_1.\]

Also, we have \(K_2 = \frac{e_5 e_7 - e_2}{e_1 - e_3 - e_4}\) and

\[K_{11} = -\frac{(e_1 - e_2) - \sqrt{(e_1 - e_2)^2 - 4 e_4 e_5}}{2 e_4},\]

(34)

\[K_{12} = -\frac{(e_1 - e_2) + \sqrt{(e_1 - e_2)^2 - 4 e_4 e_5}}{2 e_4}.
\]

(35)

**Proof**: We notice that the term \(A = \sum_{k=1}^{K} |\psi_k^* \psi_k|^2,\) appearing in \(\zeta^* \tau_0 / K,\) depends on \(K\) by means of its superscript. In fact, \(\zeta^* \tau_0 / K\) is an increasing function regarding \(A.\) Hence, we apply the bound on \(A\) by using the Welch inequality [42], and we obtain

\[A \geq \frac{\tau_0 (K - 3) + K - 1}{\tau_0 (K - 2)}\]

(36)

since the summation becomes \(\sum_{k=1}^{K} |\psi_k^* \psi_k|^2 = K - 1 - \tau_0 (K - 2)\) by using the inequality. Substituting (36) into (32) and rearranging with respect to \(K,\) the objective function can be written as

\[EE(\zeta^*) = \frac{K \xi (1 - \bar{e}) \log_2 (1 + \gamma)}{e_7 + e_8 K + e_9 K^2 + e_9 \xi (1 - \bar{e}) \log_2 (1 + \gamma)},\]

(37)

where \(\bar{e} = \frac{e_5 K^2 + e_5 K + e_5}{e_5 K^2 + e_5 K + e_5} + e_5 \xi (1 - \bar{e}) \log_2 (1 + \gamma),\)

while the parameters \(e_i\) are provided in Table II. Taking the first derivative of (37) with respect to \(K\) and equating it to zero, we obtain a polynomial fifth degree with roots provided by an exhaustive search over the domain set while using a bisection method and the help of Mathematica [43]. We obtain three real roots and one pair of complex roots. Note that the constraint results in \(K_2.\)

**VII. NUMERICAL RESULTS**

This section presents illustrations of the analytical results provided by means of Theorems 2 concerning the optimal EE. Notably, the tightness of the derived bounds, denoting their values as good approximations, is demonstrated in Fig. 4 by Monte Carlo simulations. For the sake of comparison, we have considered a conventional "cellular" massive MIMO
We assume that the locations of the APs are simulated as realizations of the PPP with density \( \lambda_{\text{AP}} = 100 \text{ APs/km}^2 \). Based on a wraparound topology to keep the translation invariance, we consider a sufficiently large squared area of 1 km\(^2\), where the locations of the APs are simulated as realizations of the PPP with density \( \lambda_{\text{AP}} = 100 \text{ APs/km}^2 \). Based on a wraparound topology to keep the translation invariance, we simulate the EE per unit area for the downlink. Notably, the outperformance of the CF massive MIMO setting is depicted. In particular, in CF massive MIMO systems, the EE per unit area is higher and the required AP density is much lower.

In addition, in the case of SCs, studied in Fig. 3(b), we have considered the system model in [44], where independent users are associated with their nearest multi-antenna AP, while the remaining APs act as interferers. In particular, we have set \( N = 4 \) antennas per AP serving a single user, i.e., \( K = 1 \). Also, the imperfect CSI model in that scenario is replaced by the current one while no hardware impairments and channel aging have been assumed. Especially, we have denoted \( \rho_{\text{tr}}^c = \rho_{\text{tr}}^d = \frac{N}{K} \rho_{\text{tr}} \) and \( \rho_{\text{tr}}^d = \rho_{\text{tr}}^d = \frac{N}{K} \rho_{\text{tr}} \), where \( \rho_{\text{tr}}^c \) and \( \rho_{\text{tr}}^d \) are the normalized uplink training and downlink transmit powers in the case of SCs, in order to guarantee that the total radiated power is equal in both architectures [12]. Clearly, the EE per unit area is maximized after a large AP density, being \( \lambda_{\text{AP}} = 70 \text{ APs/km}^2 \), while in the case of CF systems we need only \( \lambda_{\text{AP}} = 25 \text{ APs/km}^2 \).

We consider a sufficiently large squared area of 1 km\(^2\), where the locations of the APs are simulated as realizations of the PPP with density \( \lambda_{\text{AP}} = 100 \text{ APs/km}^2 \). Based on a wraparound topology to keep the translation invariance, we assume that the system bandwidth is \( B_w = 20 \text{ MHz} \) and that each coherence block consists of \( T_c = 200 \) samples corresponding to a coherence bandwidth of 200 KHz and a coherence time of 1 ms [12]. Moreover, we assume that \( N = 20 \) antennas per AP and \( K = 10 \) users in total while \( \zeta = 4 \). Also, we assume that \( \rho_{\text{tr}} = 100 \text{ mW} \), \( \rho_{\text{d}} = 200 \text{ mW} \), \( \alpha = 4 \), and \( \xi = 1/3 \). Moreover, the normalized uplink training transmit power per pilot symbol \( \rho_{\text{tr}} \) and downlink transmit power \( \rho_{\text{d}} \) result by dividing \( \rho_{\text{tr}} \) and \( \rho_{\text{d}} \) with the noise power \( N_f \) given in W by \( N_f = k_B T_0 T_c N_f \). For the sake of reference, the descriptions and values of the various system parameters are found in Table IV unless otherwise stated. Note that the circuit power parameters have been taken from [8].

![Fig. 2. Energy efficiency per unit area (Mbit/J/km\(^2\)) of CF massive MIMO systems versus the AP density \( \lambda_{\text{AP}} \) and pilot reuse factor \( \zeta \). The optimal EE per unit area is star-marked and the corresponding parameters are provided.](image)

In Fig. 4, we examine the impact of the SINR constraint \( \gamma_0 \) on the EE per unit area. Moreover, we shed light on the tightness of the lower bound on the average SE given by Theorem 1 and an upper bound provided by averaging the instantaneous SE presented by Lemma 1. In particular, we assume that \( \gamma_0 \in \{1, 3, 7\} \) to result in an average SE \( \log_2 (1 + \gamma_0) \) equal to 1, 2, and 3, respectively. It is illustrated that the EE per unit area decreases with \( \gamma_0 \). This observation is a pseudo-concave function with respect to \( \zeta \) with a unique global maximum at \( \zeta^* = 3 \) while the corresponding optimal EE per unit area is EE\(^*\) = 5.92 Mbit/Joule. Regarding the AP density, the EE per unit area is a quasi-concave function with respect to \( \lambda_{\text{AP}} \) as was stated by Theorem 2. Nevertheless, this figure shows the optimal AP density to achieve maximum EE. Hence, we observe that when \( \lambda_{\text{AP}} = 25 \text{ APs/km}^2 \), the EE per unit area takes its maximum value. It is worthwhile to mention that the optimal \( \lambda_{\text{AP}} \) depends on fundamental system parameters such as the transmit power and the number of antennas per AP as the corresponding theorem shows.
Fig. 3. Energy efficiency per unit area (Mbit/J/km²) of versus the AP density \(\lambda_{\text{AP}}\) and pilot reuse factor \(\zeta\) in the cases of a) “cellular” massive MIMO systems and b) SCs systems, respectively. The optimal EE per unit area is star-marked and the corresponding parameters are provided.

Fig. 4. Energy efficiency per unit area (Mbit/J/km²) of CF massive MIMO systems versus the AP density \(\lambda_{\text{AP}}\) for different SINR constraints. “Solid-bullet” and “dashed” lines correspond to the lower bound due to the Theorem 1 and upper bound due to Monte Carlo simulation of the average SE.

Fig. 5. Energy efficiency per unit area (Mbit/J/km²) of CF massive MIMO systems versus the TSE for different SINR constraints. Notably, these values are confirmed analytically by means Theorems \(4\) and \(5\).

VIII. CONCLUSION

Given that network densification is a promising way for high EE, we considered its investigation in a CF massive MIMO architecture by assuming both many APs and many antennas per AP. In order to rely on a realistic scenario, we assumed that the APs are PPP distributed. In parallel, we introduced a realistic power consumption model for this setting. Notably, we achieved to derive a new lower bound on the downlink average SE for CF massive MIMO systems and we provided a novel definition for the EE per unit area which is necessary in the case of CoMP-JT architectures. In this direction, we formulated an EE maximization problem for the downlink that enabled the analytical determination of tractable closed-form
where in (41), we have used that (39) results after substituting (40), (45) and (47) into (41). The proof is concluded by substituting (40), (48) and (47) into (41).

The second part, not depending on the contamination, results due to the independence between the two random vectors. The last equation is obtained by simple algebraic manipulations. On the contrary, if \( i = k \), we have

\[
\mathbb{E} \left[ |\mathbf{h}_k^i \mathbf{C}_i \mathbf{h}_i| \right]^2 = \mathbb{E} \left[ |\mathbf{h}_k^i \mathbf{C}_i \mathbf{h}_k|^2 \right] = \text{tr}^2 \mathbf{I}_{MN} + \text{tr} \mathbf{I}_{MN} = M^2 N^2 + MN,
\]

where in [45], we have applied [45] Lemma 2. In total, we have

\[
\mathbb{E} \left[ |\mathbf{h}_k^i \mathbf{C}_i \mathbf{h}_i| \right]^2 = N^2 \text{tr}^2 \left( \mathbf{L}_i^{-1} \mathbf{L}_k \right) + \frac{N^2 \text{tr} (\mathbf{C}_i \mathbf{L}_k)}{MN}, \quad i \neq k
\]

\[
= N^2 \text{tr}^2 \left( \mathbf{L}_k^{-1} \mathbf{L}_k \right) + \frac{N^2 \text{tr} (\mathbf{C}_i \mathbf{L}_k)}{MN}, \quad i = k.
\]

Also, the normalization parameter can be easily written as

\[
\mu = \frac{K}{\mathbb{E} \sum_{i=1}^K h_i^i C_i^2 h_i^i} = \left( \frac{N}{K} \sum_{i=1}^K \text{tr} C_i \right)^{-1}
\]

\[
(48)
\]

and on the number of points in this area. Next, we apply the law of large numbers. Afterwards, we remove the conditioning regarding the number of points, and we assume that the area

\[
(47)
\]

the proof starts with the application of Jensen's inequality that will allow us to derive a tractable lower bound of the downlink average SE by moving the expectation inside the logarithm and continues with the derivation of the expectation of the inverse SINR over the APs distances. Application of the Jensen inequality to the downlink average SE results in

\[
\mathbb{E} \left[ \log_2 \left( 1 + \frac{1}{\gamma_k} \right) \right] \geq \log_2 \left( 1 + \frac{1}{\gamma_k} \right),
\]

\[
(49)
\]

where the expectation applies directly to the inverse SINR \( \frac{1}{\gamma_k} = \mathbb{E} \left[ \frac{1}{\gamma_k} \right] \).

The inverse SINR provided by (12) can be written as (50) at the top of the next page, where the trace of each matrix is replaced by the sum of its entry-wise elements. For the derivation of the expectation, let a ball of radius \( R \) centered at the origin that contains \( M = \Phi (B(o, R)) \) points with \( S(A) = |B(o, R)| \).

The first step includes conditioning on this area of radius \( R \) and on the number of points in this area. Next, we apply the law of large numbers. Afterwards, we remove the conditioning regarding the number of points, and we assume that the area

\[
\mathbb{E} \left[ (X + Y)^2 \right] = \mathbb{E} \left[ X^2 \right] + \mathbb{E} \left[ Y^2 \right] \quad \text{holding between two independent random variables with } \mathbb{E} \left[ X \right] = 0.
\]

\[
(42)
\]

where in (41), we have used that \( \mathbf{h}_k = \mathbf{h}_k + \tilde{e}_k \) and the identity \( \mathbb{E} \left[ |X + Y|^2 \right] = \mathbb{E} \left[ |X|^2 \right] + \mathbb{E} \left[ |Y|^2 \right] \) holding between two independent random variables with \( \mathbb{E} \left[ X \right] = 0 \). In (42), we have applied the property concerning the estimated channels between pilot contaminated users, i.e., \( \mathbf{h}_k = \mathbf{L}_k \mathbf{L}^{-1} \mathbf{h}_k \) [22]. The first part of the next equality follows by using [45] Lemma 2, and
The computation of the first term of (51) is infinite, i.e., $R \to \infty$. Specifically, we have

$$
\mathbb{E} [\gamma_k^{-1}] = \lim_{R \to \infty} \left[ \sum_{i=1}^{K} \sum_{m=1}^{M} d_m l_{m,i}^{-2} \left( N l_{m,i} + \frac{1}{K p_d} \right) \right] / (M^2 N),
$$

where in (51), we have let the ball of radius $R$ going to infinity.

We continue with the computation of the first term of (51). We have

$$
I_1 = \lim_{R \to \infty} \mathbb{E} \left[ \frac{1}{M^2 N} V \right]
$$

$$
= \lim_{R \to \infty} \mathbb{E}_M \left[ \frac{1}{M^2 N} V | M = \Phi (B (o, R)) \right],
$$

$$
= \sum_{i=1}^{K} \mathbb{E} \left[ \frac{1}{M} d_m l_{m,i}^{-2} \left( N l_{m,i} + \frac{1}{K p_d} \right) \right],
$$

$$
= \sum_{i=1}^{K} \mathbb{E} \left[ \frac{1}{N} d_m l_{m,i}^{-2} \left( N l_{m,i} + \frac{1}{K p_d} \right) \right],
$$

$$
= I_{11} + I_{12},
$$

where $V = \sum_{i=1}^{K} \sum_{m=1}^{M} d_m l_{m,i}^{-2} \left( N l_{m,i} + \frac{1}{K p_d} \right) I_{11} = \sum_{i=1}^{K} \mathbb{E} [d_m l_{m,i}^{-2} l_{m,i}]$ and $I_{12} = \frac{1}{K N p_d} \sum_{i=1}^{K} \mathbb{E} [d_m l_{m,i}^{-2}]$. In (52), we compute the conditional expectation given the number of points inside the ball, while in (53), we apply the law of large numbers given the number of APs. Also, the remaining $M$ in the denominator cancels out with the number of points inside the ball. Next, we derive $I_{11}$. Specifically, we have

$$
I_{11} = \mathbb{E} \left[ \sum_{i=1}^{K} \sum_{j=1}^{K} \psi_j \psi_k \left[ l_{m,j} + \frac{1}{\tau_r \rho_{rt}} \right] l_{m,i}^{-2} I_{m,i} l_{m,i} \right],
$$

$$
= \mathbb{E} \left[ \sum_{i=1}^{K} \sum_{j=1}^{K} \psi_j \psi_k \left[ l_{m,j} l_{m,i} l_{m,i}^{-2} \right] + \frac{1}{\tau_r \rho_{rt}} \mathbb{E} \left[ \sum_{i=1}^{K} l_{m,i}^{-2} l_{m,i} \right] \right],
$$

where the first part of (57) can be written as

$$
\mathbb{E} \left[ \sum_{i=1}^{K} \sum_{j=1}^{K} \psi_j \psi_k \left[ l_{m,j} l_{m,i} l_{m,i}^{-2} \right] \right],
$$

and the second part of (57) is obtained due to the independence between the random variables $l_{m,i}$ and $l_{m,i}$ while (60) has accounted for Jensen’s inequality. Notably, (61) is obtained since the two variables have the same marginal distribution. In the condition that $i = k$, the result is the same. Following the same procedure, the expectation in the second branch gives the same result. The expectation in the last branch becomes

$$
\mathbb{E} \left[ l_{m,i}^{-1} l_{m,i}^{-1} \right] = \mathbb{E} \left[ l_{m,i}^{-1} \right] \mathbb{E} \left[ l_{m,i}^{-1} \right] \text{ if } i = k,
$$

$$
\mathbb{E} \left[ l_{m,i}^{-1} l_{m,i}^{-1} \right] = \mathbb{E} \left[ l_{m,i}^{-1} \right] \mathbb{E} \left[ l_{m,i}^{-1} \right] \text{ if } i \neq k.
$$

Herein, the first branch is identical to (59), and results in the same expression. The second branch in (62) becomes

$$
\mathbb{E} \left[ l_{m,i}^{-1} l_{m,i}^{-1} \right] = \mathbb{E} \left[ l_{m,i}^{-1} \right] \mathbb{E} \left[ l_{m,i}^{-1} \right] \mathbb{E} \left[ l_{m,i}^{-1} \right] \mathbb{E} \left[ l_{m,i}^{-1} \right] \geq 1,
$$

which is obtained after following similar steps with (61). The second part of (57) is written as

$$
\mathbb{E} \left[ \sum_{i=1}^{K} l_{m,i}^{-2} I_{m,i} \right] = \mathbb{E} \left[ l_{m,i}^{-1} \right] \sum_{i=1}^{K} \mathbb{E} \left[ l_{m,i}^{-1} \right] \text{ if } i = k,
$$

$$
\mathbb{E} \left[ l_{m,i}^{-1} l_{m,i}^{-1} \right] = \mathbb{E} \left[ l_{m,i}^{-1} \right] \sum_{i=1}^{K} \mathbb{E} \left[ l_{m,i}^{-1} \right] \text{ if } i \neq k.
$$

In (63), we have taken into consideration the independence among the variables, and then, in (64), we have applied the inequality $\mathbb{E} [x^2] \geq \mathbb{E} [x]^2$. Eq. (65) is obtained after following similar steps with (61). The second part of (57) is written as

$$
\mathbb{E} \left[ l_{m,i}^{-q} \right] \geq \frac{1}{\mathbb{E} \left[ l_{m,i}^{-1} \right]},
$$

where we have applied Jensen’s inequality. Not that

$$
\mathbb{E} \left[ l_{m,i}^{-q} \right] = 2 \pi \left( \int_0^1 y dy + \int_1^\infty y^{-qa+1} dy \right) = \frac{q\alpha \pi}{\alpha - 2},
$$

where $q \rightarrow 1$ and $\alpha = \frac{1}{2}$.

The computation of the first term of (51) is infinite, i.e., $R \to \infty$. Specifically, we have

$$
\mathbb{E} [\gamma_k^{-1}] = \lim_{R \to \infty} \left[ \sum_{i=1}^{K} \sum_{m=1}^{M} d_m l_{m,i}^{-2} \left( N l_{m,i} + \frac{1}{K p_d} \right) \right] / (M^2 N),
$$

where in (51), we have let the ball of radius $R$ going to infinity.
Regarding the second branch in (66), we have
\[ \mathbb{E} \left[ I_{mi}^2 l_{mk} \right] = \mathbb{E} \left[ I_{mi}^2 \right] \mathbb{E} \left[ l_{mk} \right] \geq \mathbb{E} \left[ I_{mi}^2 \right] \mathbb{E} \left[ l_{mk} \right] \geq \frac{1}{\mathbb{E} \left[ l_{mi}^2 \right]} \mathbb{E} \left[ l_{mk} \right] = \frac{1}{\mathbb{E} \left[ l_{mi}^2 \right]} = \frac{\alpha - 2}{\alpha \pi}, \] (74)
where we have applied a property of variance in (71), and the Jensen’s inequality in (72). Next, in (74), we have used (69). With respect to the second part of (55) and by following a similar procedure, we have
\[ I_{12} = \frac{1}{KN_p d} \mathbb{E} \left[ \sum_{i=1}^{K} \sum_{j=1}^{l} |\psi_j \psi_k|^2 I_{mj} + \frac{1}{\tau d \rho d} \right] \frac{l_{mi}^2}{(\alpha-2)^2} \] (75)
\[ = \frac{1}{\alpha \pi N_p d} \sum_{j=1}^{K} |\psi_j \psi_k|^2 (\alpha-2) + \frac{\alpha-1}{\alpha \pi \tau d \rho d}. \] (76)
Substituting the results concerning \( I_{11} \) and \( I_{12} \), we obtain \( I_1 \). The second term in (51) becomes
\[ I_2 = \lim_{R \to \infty} \mathbb{E} \left[ \frac{1}{M^2} \sum_{i \neq k} U_i \right] \]
\[ = \lim_{R \to \infty} \mathbb{E}_M \mathbb{E} \left[ \frac{1}{M^2} \sum_{i \neq k} U_i \right] M = \Phi (B (o, R)), \] (77)
\[ = \lim_{R \to \infty} \mathbb{E}_M \left[ \sum_{i \neq k} \mathbb{E} \left[ l_{mk}^2 \right] \frac{1}{M} = \Phi (B (o, R)) \right] \]
\[ = \lim_{R \to \infty} \frac{1}{B (o, R)} \mathbb{E}_M [M] \sum_{i \neq k} \mathbb{E} \left[ l_{mk}^2 \right] \]
\[ = \lambda_{AP} \sum_{i \neq k} \mathbb{E} \left[ l_{mk}^2 \right], \] (80)
\[ = \lambda_{AP} (K-1), \] (81)
where \( U_i = \sum_{m = \Phi (\lambda_{AP} \cap B (o, R))} I_{mk}^2 \). In (77), we have applied the law of large numbers, and in (80) we have taken into account that \( \mathbb{E}_M [M] = \lambda_{AP} [B (o, R)] \). In (81), we have used similar steps to (61). Similarly, the third term in (51) is obtained as
\[ I_3 = \lim_{R \to \infty} \mathbb{E} \left[ \frac{1}{M^2} \sum_{m \in \Phi (\lambda_{AP} \cap B (o, R))} l_{mk}^{-1} \right] \]
\[ = \mathbb{E} \left[ \left( \sum_{j=1}^{K} |\psi_j \psi_k|^2 I_{mj} + \frac{1}{\tau d \rho d} \right) l_{mk}^{-1} \right] \]
\[ = \frac{K}{\alpha \pi \tau d \rho d} \] (84).}

Regarding the last term in (51), we have
\[ I_4 = \lim_{R \to \infty} \mathbb{E} \left[ \frac{1}{MN} \right] \]
\[ \geq \lim_{R \to \infty} \mathbb{E} \left[ \frac{1}{NE [M]} \right] \]
\[ = \lim_{R \to \infty} \frac{1}{N \lambda_{AP} [B (o, R)]} \]
\[ = 0, \] (88)
where in (88), we have used Jensen’s inequality. Next, we have used that \( \mathbb{E}_M [M] = \lambda_{AP} [B (o, R)] \), and we have computed the limit \( R \to \infty \). Substituting \( I_1, I_2, I_3, \) and \( I_4 \) into (51), we conclude the proof.

**Appendix C**

**Proof of Proposition 2**

The proof, split in two parts, starts with the expression of \( P_{TX} \) by means of a lemma, and continues with the presentation of \( P_{CPC} \).

**Lemma 3:** The total average transmit power consumption due to uplink pilot and downlink data transmissions of an arbitrary AP is
\[ P_{TX} = \frac{K}{\tau_c} \] (89)
where \( \tau_d = \xi (\tau_c - \tau_d) \).

**Proof:** In each coherence block, each user transmits pilot symbols for a fraction of \( \tau_d / \tau_c \) with power \( \rho_d \), while each AP transmits data symbols for a fraction of \( \tau_d / \tau_c \) with power \( \rho_d \).

The second part of (18), concerning the \( P_{CPC} \) of an arbitrary AP, is given by (80)
\[ P_{CPC} = P_{FP} + P_{TC} + P_{C-BC} + P_{CE} + P_{LP}, \] (90)
where these terms correspond to the power consumptions of circuitry parts. Specifically, \( P_{FP} \) expresses the power consumed for site-cooling and control signaling and the traffic-independent mixed power consumption of each backhaul, \( P_{TC} \) for the transceiver chain, \( P_{C-BC} \) for coding and load-dependent backhauling cost, while \( P_{CE} \) and \( P_{LP} \) describe the powers consumed for the processes of channel estimation process and linear processing. Actually, each term depends on the system parameters. Especially, we have that \( P_{TC} = N P_{AP} + P_{LO} + K P_{UE} \), where \( P_{AP} \), \( P_{LO} \), and \( P_{UE} \) are the powers per AP antenna, AP local oscillator, and the power per user antenna. Moreover, we have \( P_{C-BC} = B_{w} TSE (P_{COD} + P_{DEC} + P_{BT}) \), where the terms from left to right denote the bandwidth, the powers for data coding and decoding as well as the total power for the backhaul traffic. Regarding the computation of \( P_{CE} \), we have that the MMSE estimation involves \( N \tau_d + \nu \) operations for the calculations of \( \psi_k \psi_k^T \) and \( \hat{h}_{mk} \) in (4) and (5), respectively. In total, the MMSE estimation requires \( KN \left( \tau_d + 1 \right) \) operations needing 3 flops per operation with AP computational efficiency \( \alpha_{eff} \). Given that this procedure takes \( \frac{\nu}{\alpha} \) coherence blocks per
second and $\tau_{tr} = \frac{K}{\zeta}$, we have
\[
P_{CE} = \frac{3}{L_{AP}} \frac{B_w}{\tau_c} KN(\tau_c - \tau_{tr}) + \frac{K}{\zeta}.
\]
(91)
The linear processing power $P_{LP}$ is a result of the powers consumed by precoding/transmitting the data and computation of the precoder, i.e., $P_{LP}$, and $P_{LP_{tr}}$, respectively. Hence, we have
\[
P_{LP} = P_{LP_{tr}} + P_{LP_{tr}},
\]
(92)
where $P_{LP_{tr}} = \frac{3}{L_{AP}} \frac{B_w}{\tau_c} KN(\tau_c - \tau_{tr})$ with $\tau_{tr} = \frac{K}{\zeta}$, and the power consumed by the conjugate beamformer is given by [8, 30] as $P_{LP_{tr}} = \frac{3}{L_{AP}} \frac{B_w}{\tau_c} KN$. Substituting [89] and the power expressions in (90) into (18), we conclude the proof.

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