Torsion pairs and quasi-abelian categories

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Abstract

We define torsion pairs for quasi-abelian categories and give several characterisations. We show that many of the torsion theoretic concepts translate from abelian categories to quasi-abelian categories. As an application, we generalise the recently defined algebraic Harder-Narasimhan filtrations to quasi-abelian categories.

1 Introduction

Torsion classes were introduced for abelian categories by Dickson [Dic66] to generalise the notion of torsion and torsionfree groups. Since then they have been widely studied in various contexts including (τ-)tilting theory [AIR14], [HRS96], lattice theory [DIR17] and, more recently, stability conditions [BST18], [Tre18].

Quasi-abelian categories are a particular class of exact categories (in the sense of Quillen [Qui73]) whose maximal exact structure ([Rum11], [SW11]) coincides with the class of all kernel-cokernel pairs of the category (see Definition 2.1). As the name suggests, they are a weaker structure than abelian categories. Quasi-abelian categories appear naturally in cluster theory [Sha19] and in the context of Bridgeland’s stability conditions [Bri07]. Of particular interest to us is their appearance in torsion theory: Each torsion(free) class in an abelian category is quasi-abelian and every quasi-abelian category appears as the torsion(free) class of an associated abelian category [Rum01b].

In this paper we seek to exploit this relationship to define and study torsion classes in quasi-abelian categories by describing torsion classes of quasi-abelian categories in terms of the torsion classes in the associated abelian category. We note that torsion pairs in pre-abelian and semi-abelian categories, which are weaker structures still than quasi-abelian categories, have been studied in [JT07]. In this more general context, torsion pairs no longer have the well-known characterisations that they have in the abelian set up. In [BG06] torsion theory in non-abelian, so-called homological categories has also been considered.

Based on a characterisation of torsion pairs in an abelian categories [Dic66], we define a torsion pair for a quasi-abelian category as follows.

Definition. (Definition 2.3) Let \( \mathcal{Q} \) be a quasi-abelian category. A torsion pair in \( \mathcal{Q} \) is an ordered pair \( (\mathcal{T}, \mathcal{F}) \) of full subcategories of \( \mathcal{Q} \) satisfying the following.

(a) \( \text{Hom}_\mathcal{Q}(\mathcal{T}, \mathcal{F}) = 0 \).
For all $M$ in $Q$ there exists a short exact sequence

$$0 \to \tau M \to M \to M_f \to 0$$

with $\tau M \in \mathcal{T}$ and $M_f \in \mathcal{F}$.

In this case we call $\mathcal{T}$ a torsion class and $\mathcal{F}$ a torsionfree class.

We establish a correspondence between certain torsion pairs in an abelian category and torsion pairs in a related quasi-abelian category.

**Theorem A.** (Proposition 3.2 and Theorem 4.4.) Let $A$ be an abelian category and let $(\mathcal{C}, \mathcal{D})$, $(\mathcal{C}', \mathcal{D}')$ be twin torsion pairs in $A$ such that $\mathcal{C} \subseteq \mathcal{C}'$. Then the intersection, $\mathcal{C}' \cap \mathcal{D}$, is quasi-abelian and there is an inclusion preserving bijection:

$$\{(X, Y) \text{ torsion pair in } A \mid \mathcal{C} \subseteq X \subseteq \mathcal{C}'\} \leftrightarrow \{(\mathcal{T}, \mathcal{F}) \text{ torsion pair in } \mathcal{C}' \cap \mathcal{D}\}$$

$$(X, Y) \mapsto (X \cap \mathcal{D}, Y \cap \mathcal{C}')$$

$$(\mathcal{C} \ast \mathcal{T}, \mathcal{F} \ast \mathcal{D}') \leftrightarrow (\mathcal{T}, \mathcal{F})$$

We remark that this result generalises the bijection in [Jas14] where functorially finite torsion classes in abelian categories are considered. Furthermore, independently, in the situation of $\mathcal{C}' \cap \mathcal{D}$ being wide [AP19] have also shown the above bijection and that it induces an isomorphism of lattices. As a consequence of Theorem A, we obtain this isomorphism of lattices in our more general setting (see Corollary 4.5). We note that our work differs from [AP19] in that our aim is to understand torsion pairs for quasi-abelian categories, therefore we also consider the cases when $\mathcal{C}' \cap \mathcal{D}$ is not wide.

With this machinery in hand, our strategy for studying properties of torsion classes in the quasi-abelian setting is to translate the problem to abelian categories using the above bijection, utilise the properties of torsion in abelian categories, then translate back to quasi-abelian categories. We see that in general, torsion theoretic concepts of abelian categories carry well to quasi-abelian categories. In particular, we show that the following well-known properties of torsion pairs in the abelian setting still hold in the quasi-abelian case.

**Theorem B.** (Propositions 5.4, 5.7 and 5.9.) Let $Q$ be a quasi-abelian category. Then $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $Q$ if and only if $\mathcal{T} \perp \mathcal{F} = \mathcal{F} \perp \mathcal{T}$. Moreover, the following hold

(a) $\mathcal{T}$ and $\mathcal{F}$ are both quasi-abelian categories.

(b) $\mathcal{T}$ is functorially finite if and only if $\mathcal{F}$ is functorially finite.

As an application of our results, in Section 6 we show that quasi-abelian categories admit algebraic Harder-Narasimhan filtrations as recently studied in the abelian context in [Tre18]. Such filtrations were extensively studied in [Rei03] and named after Harder and Narasimhan for their work [HN75]. Furthermore, Rudakov [Rud97] showed that every stability function on an abelian
category induces a Harder-Narasimhan filtration of each object. In \cite{BKT14} and \cite{BST18} it was observed, for abelian categories, that each stability function induces a chain of torsion classes; and in \cite{Tref18} the above is generalised to show that every chain of torsion classes in an abelian length category induces Harder-Narasimhan filtrations. We show that the same is true for chains of torsion classes in quasi-abelian categories that have an associated abelian category (see Section 5) of finite length. Namely, we show the following.

**Theorem C.** (Corollary 6.8.) Every chain of torsion classes in a quasi-abelian category with an associated abelian length category induces a Harder-Narasimhan filtration of each object that is unique up to isomorphism.

This article is organised as follows. In Section two, we translate the characterisations of torsion pairs in the abelian setting to the quasi-abelian case and show that, in this case, not all characterisations remain equivalent. This leads naturally leads to our choice of definition. In the third Section, we prove that the heart of twin torsion pairs is quasi-abelian. This provides us with a way to generate examples of quasi-abelian categories that are not naturally arising as torsion(free) classes. The fourth Section is devoted to proving the bijection of Theorem A. We furthermore show that, under mild assumptions, this bijection preserves the functorial finiteness of the torsion(free) classes. In the fifth Section we use the results of the previous sections to completely characterise torsion pairs for quasi-abelian categories. As an application of the newly developed theory, in the final section we show the existence of Harder-Narasimhan filtrations for chains of torsion classes in a quasi-abelian category. Furthermore, we also explore topological properties of the set of chains of torsion classes in a quasi-abelian category.

**Acknowledgments.** This work was undertaken as part of the author’s PhD studies supported by the EPSRC. The author thanks Sibylle Schroll and Hipolito Treffinger for many helpful discussions and Gustavo Jasso for communicating the proof of Proposition 4.11(b).

## 2 Defining torsion pairs

In this paragraph, we define a torsion pair in a preabelian category and compare this definition with other candidate formulations coming from the abelian case. We begin by recalling the definitions of the categories that will form the backdrop for our work. Recall that an additive category is a pointed category enriched in abelian groups that admits all binary products and coproducts. We also remark that in an additive category biproducts (direct sums) and coproducts coincide.

**Definition 2.1.** Let $\mathcal{A}$ be an additive category.

(a) $\mathcal{A}$ is **preabelian** if every morphism in $\mathcal{A}$ admits a kernel and a cokernel.

(b) A pair of composable morphisms $(f, g)$ in $\mathcal{A}$ is a **kernel-cokernel pair** if $g = \text{Ker}f$ and $f = \text{Coker}g$. 

(c) $\mathcal{A}$ is quasi-abelian (or almost abelian) if it is preabelian and if the class of all kernel-cokernel pairs in $\mathcal{A}$ forms a Quillen exact structure on $\mathcal{A}$.

**Remark 2.2.** We make some observations.

(a) The data of a kernel-cokernel pair coincides with that of a short exact sequence. In the sequel, we use these terms interchangeably.

(b) Equivalently, a preabelian category $\mathcal{C}$ is quasi-abelian if cokernels (resp. kernels) in $\mathcal{C}$ are stable under pullback (resp. pushout).

(c) Any preabelian category has split idempotents. Indeed, every idempotent morphism admits a kernel.

**Definition 2.3.** Let $\mathcal{A}$ be a preabelian category. A torsion pair in $\mathcal{A}$ is an ordered pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ satisfying the following.

(T1) $\text{Hom}_\mathcal{A}(\mathcal{T}, \mathcal{F}) = 0$.

(T2) For all $M$ in $\mathcal{A}$ there exists a short exact sequence

$$
0 \longrightarrow \tau M \xrightarrow{i_M} M \xrightarrow{p_M} M/F \longrightarrow 0
$$

with $\tau M \in \mathcal{T}$ and $M/F \in \mathcal{F}$.

In this case we call $\mathcal{T}$ a torsion class, $\mathcal{F}$ a torsionfree class and the short exact sequence in (T2) is called the $(\mathcal{T}, \mathcal{F})$-canonical short exact sequence of $M$.

**Proposition 2.4.** Let $\mathcal{A}$ be a preabelian category. Then a full subcategory $\mathcal{T} \subseteq \mathcal{A}$ is a torsion class in $\mathcal{A}$ if and only if there exists an admissible subfunctor of the identity $t: \mathcal{A} \rightarrow \mathcal{T}$ that is idempotent, radical, and such that $\mathcal{T} = \{ M \in \mathcal{A} | tM \cong M \}$. Moreover, in this situation such a functor is a right adjoint to the canonical inclusion $\mathcal{T} \hookrightarrow \mathcal{A}$.

Recall that a functor $F: \mathcal{A} \rightarrow \mathcal{A}$ is

(a) an admissible subfunctor of the identity if $FM \hookrightarrow M$ is a kernel and part of a kernel-cokernel pair

$$
FM \hookrightarrow M \longrightarrow M/FM
$$

for all $M \in \mathcal{A}$ and furthermore that for all $f: M \rightarrow N$ in $\mathcal{A}$ the diagram

$$
\begin{array}{ccc}
FM & \xrightarrow{f} & M \\
\downarrow{f} & & \downarrow{f} \\
FN & \leftarrow & N
\end{array}
$$

commutes.

(b) idempotent if $F(FM) \cong FM$ for all $M \in \mathcal{A}$.  

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(c) radical if $F(M/FM) \cong 0$ for all $M \in \mathcal{A}$.

Proof of Proposition 2.4. \((\Rightarrow)\) Let $\mathcal{F}$ be the torsionfree class associated to $\mathcal{T}$. We verify that the assignment $M \mapsto \tau M$ satisfies the conditions above. Firstly, let $f : M \rightarrow N$ in $\mathcal{A}$ then, by (T1), $p_{N}fi_{M} : \tau M \rightarrow NF$ is zero, hence by the universal property of kernels, there exists a unique $\tau f : \tau M \rightarrow \tau N$ such that $i_{N}\tau f = fi_{M}$. It is clear that this defines a functor $\mathcal{A} \rightarrow \mathcal{T}$ which is, by construction, a subfunctor of the identity. To see that it is idempotent, consider the $(\mathcal{T},\mathcal{F})$-canonical short exact sequence of $\tau M$ for any $M \in \mathcal{A}$.

\[
0 \rightarrow \tau(\tau M) \xrightarrow{i_{\tau M}} \tau M \xrightarrow{p_{\tau M}} (\tau M)_{F} \rightarrow 0
\]

and observe that $p_{\tau M} = 0$ by (T1) therefore $i_{\tau M}$ is an isomorphism and also $(\tau M)_{F} \cong 0$. The fact that $\tau(\cdot)$ is radical follows from applying a dual argument to the $(\mathcal{T},\mathcal{F})$-canonical short exact sequence of $MF$. It remains to check that $\mathcal{T} = \{M \in \mathcal{A} \mid \tau M \cong M\}$, but this follows from the fact that, for all $M \in \mathcal{T}$, $p_{M} = 0$ by (T1).

\((\Leftarrow)\) Let $t : \mathcal{A} \rightarrow \mathcal{T}$ be a functor as in the statement and set $\mathcal{F} = \{M \in \mathcal{A} \mid tM \cong 0\}$. Then as $t$ is radical, $M/tM \in \mathcal{F}$, for all $M \in \mathcal{A}$. Thus (T2) is satisfied. To verify (T1), let $M \in \mathcal{T}$, $N \in \mathcal{F}$ and $f : M \rightarrow N$ be a morphism in $\mathcal{A}$, then there is a commutative diagram

\[
\begin{array}{ccc}
tM & \xrightarrow{\cong} & M \\
\downarrow f & & \downarrow f \\
fN & \rightarrow & N
\end{array}
\]

from which we conclude $f = 0$ and (T1) is satisfied.

The fact that such a $t$ is a right adjoint follows from the fact that every morphism $T \rightarrow M$ with $T \in \mathcal{T}$ and $M \in \mathcal{A}$ factors through $tM$ by the universal property of the kernel.

As a direct consequence, we justify some of our terminology.

Corollary 2.5. Let $\mathcal{A}$ be a preabelian category and $(\mathcal{T},\mathcal{F})$ be a torsion pair on $\mathcal{A}$. Then for all $M \in \mathcal{A}$ the $(\mathcal{T},\mathcal{F})$-canonical short exact sequence is unique up to isomorphism.

Proposition 2.6. Let $\mathcal{A}$ be a preabelian category and $(\mathcal{T},\mathcal{F})$ be a torsion pair in $\mathcal{A}$. Then

(a) For all $M \in \mathcal{A}$, if $\text{Hom}_{\mathcal{A}}(M,\mathcal{F}) = 0$ then $M \in \mathcal{T}$.

(b) For all $N \in \mathcal{A}$, if $\text{Hom}_{\mathcal{A}}(\mathcal{T},N) = 0$ then $N \in \mathcal{F}$.

Proof. Let $M \in \mathcal{A}$ be such that $\text{Hom}_{\mathcal{A}}(M,\mathcal{F}) = 0$, then $p_{M} = 0$ and $M \cong \tau M \in \mathcal{T}$. The second statement follows by a dual argument.

We show that the converse of the above statement is true for quasi-abelian categories in Proposition 5.7.
Proposition 2.7. Let $A$ be a preabelian category and $T$ be a torsion class in $A$, then $T$ is closed under extensions and quotients.

Proof. Let $A$ be a pre-abelian category. We begin by showing that a torsion class, $T$, is closed under quotients. Let $e : M \rightarrow N$ be an epimorphism in $A$ with $M \in T$. As $\text{Hom}_A(T,F) = 0$, the composition $p_N e : M \rightarrow N_F$ is zero. Thus, as $e$ is an epimorphism, $p_N$ is zero and $N \cong \tau N \in T$.

We now show that $T$ is closed under extensions. To this end, let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence in $A$ with $M', M'' \in T$. Since $\text{Hom}_A(T,F) = 0$ and $M' \in T$, by the universal property of the kernel and cokernel, there exists a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & M' & \rightarrow & M & \rightarrow & M'' & \rightarrow & 0 \\
& \downarrow & & \downarrow^1 & & \downarrow^f & \\
0 & \rightarrow & \tau M & \rightarrow & M & \rightarrow & M_F & \rightarrow & 0.
\end{array}
$$

As $M'' \in T$, the morphism $f$ is zero and hence so is $p_M$. Thus $M \cong \tau M \in T$.

Remark 2.8. In general, the converse to the above statement is not true: Let $Q$ be the quiver

$$
1 \rightarrow 2 \rightarrow 3
$$

and consider the abelian category $A = \text{mod}KQ$ whose Auslander-Reiten quiver is given by

and consider the subcategory

$$
C = \text{add}\{3 \oplus 2 \oplus \frac{2}{3} \oplus \frac{1}{2} \oplus \frac{1}{3}\}
$$

which is quasi-abelian as it is a torsionfree class of $A$. Now the subcategory $T = \text{add}\{2 \oplus \frac{2}{3}\}$ of $C$ is closed under extensions and quotients in $C$ but it is not a torsion class in $C$. Indeed, $T^\perp = \text{add}\{3\}$ (in $C$) but $\perp(T^\perp) = \text{add}\{2 \oplus \frac{2}{3} \oplus \frac{1}{2} \oplus \frac{1}{3}\} \neq T$ which contradicts Proposition 2.6, thus $T$ is not a torsion class in $C$.

3 The heart of twin torsion pairs

We begin by recalling a result of Rump:
Lemma 3.1. ([Rum01b], §4, Corollary] Every torsion class and torsionfree class of an abelian category has the structure of an quasi-abelian category.

In this section we generalise the above result. Namely, we consider the intersection \( \mathcal{C}' \cap \mathcal{D} \) where \((\mathcal{C}, \mathcal{D}), (\mathcal{C}', \mathcal{D}')\) are torsion pairs in \( \mathcal{A} \) such that \( \mathcal{C} \subseteq \mathcal{C}' \) or, equivalently, \( \mathcal{D}' \subseteq \mathcal{D} \). We shall refer to such couples of torsion pairs as twin torsion pairs and the intersection \( \mathcal{C}' \cap \mathcal{D} \) as their heart. We denote twin torsion pairs by \([([\mathcal{C}, \mathcal{D}], (\mathcal{C}', \mathcal{D}'))\]. We will show that such hearts are quasi-abelian.

Proposition 3.2. Let \( \mathcal{A} \) be an abelian category and let \([([\mathcal{C}, \mathcal{D}], (\mathcal{C}', \mathcal{D}')), (\mathcal{C}', \mathcal{D}'), (\mathcal{C}', \mathcal{D}'))\] be twin torsion pairs on \( \mathcal{A} \). Then the heart, \( \mathcal{C}' \cap \mathcal{D} \), is quasi-abelian.

Remark 3.3. We remark that, in general, distinct twin torsion pairs can have the same heart. Indeed, consider the quiver \( A_3 \) as in Remark 2.8 and the twin torsion pairs

\[
\begin{align*}
&\left( \text{add}\{1\}, \text{add}\{2 \oplus 3 \oplus \frac{1}{2} \oplus \frac{2}{3} \oplus \frac{1}{3}\} \right), \quad \left( 0, \text{add}\{1 \oplus 2 \oplus 3 \oplus \frac{1}{3} \oplus \frac{2}{3} \}\right) \\
&\left( \text{add}\{3\}, \text{add}\{1 \oplus 2 \oplus \frac{1}{2}\} \right), \quad \left( \text{add}\{1 \oplus 3\}, \text{add}\{1 \oplus 2 \oplus \frac{1}{2}\} \right)
\end{align*}
\]

which both have heart \( \text{add}\{1\} \).

The first step of the proof follows the argument in [Rum01b, Theorem 2] and does not require the assumption that the torsion pairs are twin.

Lemma 3.4. Let \( \mathcal{A} \) be an abelian category and let \((\mathcal{C}, \mathcal{D}), (\mathcal{C}', \mathcal{D}')\) be torsion pairs in \( \mathcal{A} \). Then \( C' \cap D \) is preabelian.

Proof. We check the existence of kernels in \( C' \cap D \), whence existence of cokernels will follow by duality. Let \( f : X \to Y \) be a morphism in \( C' \cap D \), let \( g : \text{Ker}f \to X \) be a kernel of \( f \) in \( \mathcal{A} \) and let \( g : C'(\text{Ker}f) \to \text{Ker}f \) be the monic \( C' \)-approximation of \( \text{Ker}f \). Set \( K := C'(\text{Ker}f) \). We claim that \( hg : K \to X \) is a kernel of \( f \) in \( C' \cap D \). Firstly, note that \( K \in D \). Indeed, \( D \) is closed under subobjects, and \( K \) is a subobject of \( \text{Ker}f \) which in turn is a subobject of \( X \).

Now let \( u : Z \to X \) be a morphism in \( C' \cap D \) such that \( fu = 0 \). Then by the universal property of kernels, there exists a unique morphism \( v : Z \to \text{Ker}f \) such that \( vg = u \). Since \( h \) is a right \( C' \)-approximation of \( \text{Ker}f \) and \( Z \in C' \) there exists a morphism \( w : Z \to K \) such that \( wh = v \). Together, we have that \( u = vg = whg \), thus \( u \) factors through \( hg \).

\[
\begin{array}{c}
Z \\
\leftarrow \exists w \\
K \xrightarrow{h} \text{Ker}f \xrightarrow{g} X \xrightarrow{f} Y \\
\end{array}
\]

It remains to show that this factorisation is unique. Let \( w' : Z \to K \) be such that \( u = w'(hg) \). Observe that \( h \) and \( g \) are both monomorphisms and hence so is \( hg \). Then \( w(hg) = u = w'(hg) \) and we conclude that \( w = w' \).

\[\square\]
The previous result shows that kernels (resp. cokernels) in $C' \cap D$ are given by kernels in $C'$ (resp. cokernels in $D$).

**Notation 3.5.** When they exist, we denote the kernel of a morphism $f$ in a subcategory $\mathcal{C}$ of an ambient category $\mathcal{A}$ by $\text{Ker}_\mathcal{C} f$.

The proof of Proposition 3.2 follows from the following:

**Proposition 3.6.** Let $\mathcal{A}$ be an abelian category and let $[(C, D), (C', D')]$ be twin torsion pairs in $\mathcal{A}$. Then a pair of composable morphisms $(f, g)$ in $C' \cap D$ is a kernel-cokernel pair in $C' \cap D$ if and only if it is a kernel-cokernel pair in $\mathcal{A}$.

In other words, the exact structure on $C' \cap D$ inherited from $\mathcal{A}$ and the exact structure arising from kernel-cokernel pairs in $C' \cap D$ coincide.

**Proof of 3.6.** Let $(f: X \to Y, g: Y \to Z)$ be a kernel-cokernel pair in $C' \cap D$. Then it follows from Lemma 3.4 that $X = \text{Ker}_{C'} g = C' (\text{Ker} g)$ and $Z = \text{Coker}_D f = (\text{Coker} f)_D$. Consider the commutative diagram with rows that are exact in $\mathcal{A}$

$$
\begin{array}{ccccccc}
0 & & & & & & 0 \\
& & & \downarrow & & & \\
0 & & \downarrow & & \downarrow & & \\
& & \downarrow & & \downarrow & & \\
X & \overset{f}{\longrightarrow} & Y & \longrightarrow & \text{Coker} f & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Ker} g & \longrightarrow & Y & \overset{g}{\longrightarrow} & Z \\
& & \downarrow & & \downarrow & & \\
& & \downarrow & & \downarrow & & \\
& & (\text{Ker} g)_{D'} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 \\
\end{array}
$$

By the Snake lemma we see that $C(\text{Coker} f) \cong (\text{Ker} g)_{D'} \in C \cap D'$. But as $C \subseteq C'$ we have that $C \cap D' = 0$. Thus $X \cong \text{Ker} g$, $Z \cong \text{Coker} f$ proving the assertion.

The reverse implication is trivial.

**Remark 3.7.** Not every quasi-abelian subcategory of an abelian category arises this way. For example, consider the linearly oriented quiver $Q$ of type $A_3$ as in Remark 2.8. Then the subcategory $\mathcal{X} = \text{add}\{\frac{3}{2} \oplus \frac{1}{2}\}$ of $\text{mod}KQ$ is quasi-abelian. Indeed, the kernel and cokernel of the morphism $\frac{3}{2} \to \frac{1}{2}$ are both the zero morphism and there are no non-trivial short exact sequences. Suppose that $\mathcal{X} = C' \cap D$ for some twin torsion pairs $[(C, D), (C', D')]$. Then $\text{add}\{2\} \subset \text{Fac}\mathcal{X} \subseteq C'$ and $\text{add}\{2\} \subset \text{Sub}\mathcal{X} \subseteq D$, but $\text{add}\{2\} \not\subseteq \mathcal{X}$.
4 A bijection of torsion pairs

In this section, we develop a bijection between the torsion pairs of the heart of two twin torsion pairs and a class of torsion pairs of the ambient category. We begin with a technical lemma.

**Lemma 4.1.** Let $\mathcal{A}$ be an abelian category and let $[(\mathcal{C}, \mathcal{D}), (\mathcal{C}', \mathcal{D}')]$ be twin torsion pairs in $\mathcal{A}$. Then for all $M \in \mathcal{A}$, we have

(a) $(cM)_\mathcal{D} \cong c'(M_D) =: c'M_D$.

(b) $c(M_{\mathcal{D}'}) \cong (cM)_{\mathcal{D}'} \cong 0$.

(c) $c(c'M) \cong c'M \cong c(cM)$.

(d) $(M_D)_{\mathcal{D}'} \cong M_{\mathcal{D}'} \cong (M_{\mathcal{D}'})_\mathcal{D}$.

**Proof.** Let $M \in \mathcal{A}$, using the $(\mathcal{C}, \mathcal{D})$-canonical short exact sequence of $M$ and the $(\mathcal{C}', \mathcal{D}')$-canonical short exact sequence of $X_{\mathcal{D}}$, we build the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & cM & \to & E & \to & c'(M_D) & \to & 0 \\
\downarrow &   & \downarrow &   & \downarrow f &   & \downarrow &   & \downarrow &   \\
0 & \to & cM & \to & M & \to & M_{\mathcal{D}} & \to & 0 \\
\downarrow &   & \downarrow &   & \downarrow &   & \downarrow &   & \downarrow &   \\
&   & (M_D)_{\mathcal{D}'} &   &   &   &   &   &   \\
&   & 0. &   &   &   &   &   &   \\
\end{array}
\]

We make some observations. First note that as $\mathcal{C} \subseteq \mathcal{C}'$ and $\mathcal{C}'$ is closed under extensions, the upper short exact sequence shows that $E \in \mathcal{C}'$. Secondly, by using the Snake Lemma we see that $f$ is a monomorphism and we have a short exact sequence

\[
0 \to E \xrightarrow{f} M \to (M_D)_{\mathcal{D}'} \to 0
\]

with first term in $\mathcal{C}'$ and last term in $\mathcal{D}'$. Hence, by uniqueness of torsion canonical short exact sequences, we have that $E \cong c'M$. Now the top row can be written as

\[
0 \to cM \to c'M \to c'(M_D) \to 0
\]

which has first term in $\mathcal{C}$ and, as $\mathcal{D}$ is closed under submodules, last term in $\mathcal{D}$. Thus we conclude that $(c'M)_\mathcal{D} \cong c'(M_D)$ and $c(c'M) \cong c'M$. 

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The fact that \( C(M_D) \cong 0 \) and \( (M_D)_{D'} \cong M_{D'} \) follows from the commutative
diagram with exact rows
\[
\begin{array}{cccccc}
0 & \longrightarrow & C(M_D) & \longrightarrow & M_{D'} & \longrightarrow & (M_D)_{D'} & \longrightarrow & 0 \\
\downarrow & & \downarrow 1 & & \downarrow 1 & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & M_{D'} & \longrightarrow & M_D & \longrightarrow & 0
\end{array}
\]
and the uniqueness of torsion short exact sequences. The remaining isomorphisms are proved similarly.

**Definition 4.2.** Let \( A \) be a pre-abelian category. For two subcategories \( X, Y \) of \( A \), by \( X \ast Y \) we denote the subcategory of \( A \) consisting of objects \( M \in A \) for which there exists an exact sequence
\[
0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0
\]
with \( M' \in X \) and \( M'' \in Y \).

**Remark 4.3.** It follows immediately from the definition that \( X, Y \subseteq X \ast Y \).

The main result of this section is the following.

**Theorem 4.4.** Let \( A \) be an abelian category and let \( [(C, D), (C', D')] \) be twin torsion pairs in \( A \). Then there is an inclusion preserving bijection:

\[
\{ (X, Y) \text{ torsion pair in } A \mid C \subseteq X \subseteq C' \} \longleftrightarrow \{ (T, F) \text{ torsion pair in } C' \cap D \}
\]

\[
(X, Y) \mapsto (X \cap D, Y \cap C') \quad (C \ast T, F \ast D') \mapsto (T, F).
\]

**Proof.** We begin by showing the maps are well-defined. First, let \( (X, Y) \ast Y \) be a torsion pair on \( A \) and suppose that \( C \subseteq X \subseteq C' \). Observe that \( X \cap D \) and \( Y \cap C' \) are subcategories of \( C' \cap D \) and we have that \( \text{Hom}_{C' \cap D}(X \cap D, Y \cap C') = 0 \) thus (T1) is satisfied. To verify (T2), let \( M \in C' \cap D \) and consider the \( (X, Y) \)-canonical short exact sequence of \( M \)
\[
0 \longrightarrow X_M \longrightarrow M \longrightarrow M_Y \longrightarrow 0.
\]

Now as \( D \) is closed under subobjects, \( X_M \in D \) and thus \( X_M \in X \cap D \). Similarly, as \( C' \) is closed under quotients we have that \( M_Y \in Y \cap C' \). Thus, \( (X \cap D, Y \cap C') \) is a torsion pair in \( C' \cap D \).

Conversely, let \( (T, F) \ast Y \) be a torsion pair in \( C' \cap D \). By definition, we have that \( C \subseteq C \ast T \) and as \( C' \) is closed under extensions and \( T \subseteq C' \) we have that \( C \ast T \subseteq C' \). Now to show that \( (C \ast T, F \ast D') \) satisfies (T1), let \( f : M \to N \) be an arbitrary morphism with \( M \in C \ast T \) and \( N \in F \ast D' \). Consider the diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
\downarrow \circ f' & & \downarrow f & & \downarrow f'' & & \downarrow & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0
\end{array}
\]
where the top (respectively bottom) row shows that $M$ (respectively $N$) is an element of $\mathcal{C} \ast \mathcal{T}$ (respectively $\mathcal{F} \ast \mathcal{D}'$). That is, $M' \in \mathcal{C}$, $M'' \in \mathcal{T}$, $N' \in \mathcal{F}$ and $N'' \in \mathcal{D}'$. Observe that as $\text{Hom}_A(\mathcal{C}, \mathcal{D}') = 0$, by the universal property of kernels (respectively, cokernels) there exists $f' : M' \to N'$ (resp. $f'' : M'' \to N''$) rendering the diagram commutative. But since $F \subseteq \mathcal{D}$, we have that $\text{Hom}_A(\mathcal{C}, \mathcal{F}) = 0$, so $f' = 0$. Similarly, as $T \subseteq \mathcal{C}'$, $f'' = 0$. By the Snake Lemma there is an exact sequence

$$0 \to M' \to \text{Ker}f \to M'' \xrightarrow{\delta} N' \to \text{Coker}f \to N'' \to 0.$$ 

Then $\delta = 0$ as $\text{Hom}_A(T, F) = \text{Hom}_{\mathcal{C}' \cap \mathcal{D}}(T, F) = 0$. We conclude that $\text{Ker}f \cong M$, $\text{Coker}f \cong N$ and $f = 0$.

To show (T2) let $M \in A$, we begin by using the $(\mathcal{C}', \mathcal{D}')$-canonical short exact sequence of $M$ and the $(\mathcal{C}, \mathcal{D})$-canonical short exact sequence of $\mathcal{C}' \cdot M$ to form the pushout of short exact sequences

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{C} \cdot M & \to & M' & \to & M'' & \delta & N' & \to \text{Coker}f & \to & N'' & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & \to & \mathcal{C} \cdot M & \to & M & \to & M_{D'} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & \to & \mathcal{C} \cdot M_{D} & \to & P & \to & M_{D'} & \to & 0 \\
& & & & \downarrow & & & & \downarrow & & & & 0.
\end{array}
\]

Note that, by the Snake Lemma we have a short exact sequence

$$0 \to \mathcal{C} \cdot M \to M \to P \to 0.$$ \hspace{1cm} (2)

Now, we use the lower short exact sequence of the above diagram and the $(\mathcal{T}, \mathcal{F})$-canonical short exact sequence of $\mathcal{C} \cdot M_{D}$ to form the pushout of short
exact sequences

\[
\begin{array}{c}
0 \\
\downarrow \\
\tau(C'M_D) \\
\downarrow \\
0 \rightarrow C'M_D \rightarrow P \rightarrow M_{D'} \rightarrow 0 \\
\downarrow \\
0 \rightarrow (C'M_D)_F \rightarrow Q \rightarrow M_{D'} \rightarrow 0 \\
\downarrow \\
0.
\end{array}
\]

Then the lower short exact sequence shows that \( Q \in \mathcal{F} \ast \mathcal{D}' \) and by the Snake Lemma we have a short exact sequence

\[
0 \rightarrow \tau(C'M_D) \rightarrow P \rightarrow Q \rightarrow 0.
\]

Finally we use this short exact sequence and Sequence (2) to form the pullback of short exact sequences

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \rightarrow C'M \rightarrow R \rightarrow \tau(C'M_D) \rightarrow 0 \\
\downarrow^{1} \downarrow^{f} \downarrow \\
0 \rightarrow C'M \rightarrow M \rightarrow P \rightarrow 0 \\
\downarrow \\
Q \\
\downarrow \\
0.
\end{array}
\]

We observe that the upper short exact sequence shows that \( R \in \mathcal{C} \ast \mathcal{T} \). Now by the Snake lemma, there is a short exact sequence

\[
0 \rightarrow R \rightarrow M \rightarrow Q \rightarrow 0
\]

which shows that (T2) is satisfied.

We show that the mappings are mutually inverse. Let \((\mathcal{X}, \mathcal{Y})\) be a torsion pair in \( \mathcal{A} \) such that \( \mathcal{C} \subseteq \mathcal{X} \subseteq \mathcal{C}' \). We claim that \( \mathcal{X} = \mathcal{C} \ast (\mathcal{X} \cap \mathcal{D}) \). Let \( M \in \mathcal{X} \). Observe that, as \( \mathcal{C}' \) is closed under quotients and \( \mathcal{X} \subseteq \mathcal{C}' \), we have \( M_D \in \mathcal{C}' \cap \mathcal{D} \).
Therefore we may build the pullback of short exact sequences using the \((\mathcal{C}, \mathcal{D})\)-canonical short exact sequence of \(M\) in \(\mathcal{A}\) and the \((\mathcal{X} \cap \mathcal{D}, \mathcal{Y} \cap \mathcal{C}')\)-canonical short exact sequence of \(M_D\) in \(\mathcal{C}' \cap \mathcal{D}\):

\[
\begin{array}{c}
0 \\
\downarrow  \quad \downarrow f
\end{array}
\]

\[
\begin{array}{ccccccccc}
0 & \to & C & \to & E & \to & \mathcal{X} \cap \mathcal{D}(M_D) & \to & 0 \\
\downarrow & & \downarrow f & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & C & \to & M & \to & M_D & \to & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
& & & & (M_D)_{\mathcal{Y} \cap \mathcal{C}'} & & & & \\
& & & & \downarrow & & \downarrow & & \downarrow \\
& & & & 0 & & 0 & & \\
\end{array}
\]

Observe that the upper short exact sequence shows \(E\) is an element of \(\mathcal{C} \ast (\mathcal{X} \cap \mathcal{D})\). By the Snake Lemma, we see that \(\text{Coker} f \cong (M_D)_{\mathcal{Y} \cap \mathcal{C}'}\). As \(M \in \mathcal{X}\), \(\text{Hom}_\mathcal{A}(M, \mathcal{Y}) = 0\) and so \(\text{coker} f = 0\). Thus \(M \cong E \in \mathcal{C} \ast (\mathcal{X} \cap \mathcal{D})\). The reverse inclusion is clear since both \(\mathcal{C}\) and \(\mathcal{X} \cap \mathcal{D}\) are contained in \(\mathcal{X}\) and \(\mathcal{X}\) is closed under extensions. The fact that \(\mathcal{Y} = (\mathcal{Y} \cap \mathcal{C}') \ast \mathcal{D}'\) follows by a dual argument.

Let \((T, \mathcal{F})\) be a torsion pair in \(\mathcal{C}' \cap \mathcal{D}\). We claim that \(T = (\mathcal{C} \ast T) \cap \mathcal{D}\). Let \(M' \in (\mathcal{C} \ast T) \cap \mathcal{D}\). As \(M \in \mathcal{C} \ast T\) there is a short exact sequence

\[
\begin{array}{c}
0 \\
\downarrow
\end{array}
\]

\[
\begin{array}{ccccccccc}
0 & \to & C & \to & M & \to & T & \to & 0 \\
\downarrow & & \downarrow f & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & C & \to & M & \to & T & \to & 0 \\
\end{array}
\]

with \(C \in \mathcal{C}\) and \(T \in \mathcal{T}\). Now, as \(M \in \mathcal{D}\), \(\text{Hom}_\mathcal{A}(C, M) = 0\) and, in particular, \(f = 0\). Thus \(M \cong T \in \mathcal{T}\). The reverse inclusion is clear since \(\mathcal{T} \subseteq \mathcal{D}\) by assumption and \(\mathcal{T} \subseteq \mathcal{C} \ast \mathcal{T}\) trivially. The fact that \(\mathcal{F} = (\mathcal{F} \ast \mathcal{D}') \cap \mathcal{C}'\) follows by a dual argument.

The following Corollary is a direct consequence of the inclusion preserving property of the bijection in Theorem 4.4. We note that this generalises Theorem 4.2 in \[A\text{P}19\], where the same result is shown to hold in the case that \(\mathcal{C}' \cap \mathcal{D}\) is wide.

**Corollary 4.5.** \[A\text{P}19, \text{4.2} \]. Let \(\mathcal{A}\) be an abelian category and let \([(\mathcal{C}, \mathcal{D}), (\mathcal{C}', \mathcal{D}')]\) be twin torsion pairs in \(\mathcal{A}\). Then the set of all torsion classes in \(\mathcal{C}' \cap \mathcal{D}\) is a complete lattice isomorphic to the lattice interval \([\mathcal{C}, \mathcal{C}']\) of the complete lattice of torsion classes in \(\mathcal{A}\).

The proof of the following Lemma, which is needed in Section 6, is straightforward and is left to the reader.

**Lemma 4.6.** Let \(\mathcal{A}\) be an abelian category, \([(\mathcal{C}, \mathcal{D}), (\mathcal{C}', \mathcal{D}')]\) be twin torsion pairs in \(\mathcal{A}\) and let \(T\) be a torsion class in \(\mathcal{C}' \cap \mathcal{D}\). Then for \(X \in \mathcal{A}\), we have
that $X \in C \ast T$ if and only if $X_D \in T$. In particular, any short exact sequence showing $X$ as an element of $C \ast T$ is isomorphic to the $(C, D)$-canonical short exact sequence of $X$.

The hearts of twin torsion pairs are preserved under the bijection of Theorem 4.4.

**Lemma 4.7.** Let $A$ be an abelian category, $[(C, D), (C', D')]$ twin torsion pairs in $A$ and let $[(T, F), (T', F')]$ be twin torsion pairs in $C' \cap D$. Then

$$T' \cap F = (C \ast T') \cap (F \ast D').$$

**Proof.** Let $X \in (C \ast T') \cap (F \ast D')$ and consider the commutative diagram

$$0 \rightarrow cX \rightarrow X \rightarrow X_D \rightarrow 0$$

with top (resp. bottom) row being the $(C, D)$-canonical (resp. $(C', D')$-canonical) short exact sequences of $X$. The existence of the vertical maps $f$ and $g$ follows from the fact that $C \subseteq C'$. Moreover, it follows from Lemma 4.6 and its dual that $cX, X_D \in T$; in particular, $cX, X_D \in C' \cap D$. Thus, as $\text{Hom}_A(C, D) = 0$, $f = 0$ and we deduce that $X_c \cong 0$ and $X \cong X_D \in T'$. Similarly, $X \cong cX \in F$ and we have $X \in T' \cap F$. The reverse inclusion is trivial. □

### 4.1 Functorial finiteness

In this section, we investigate how the bijection in Theorem 4.4 reflects the functorially finite property of torsion(free) classes. We begin by recalling the relevant definitions.

**Definition 4.8.** Let $\mathcal{A}$ be a preabelian category, $\mathcal{X} \subseteq \mathcal{A}$ be a full subcategory and let $M \in \mathcal{A}$. A right $\mathcal{X}$-approximation of $M$ is a morphism $\alpha : X \rightarrow M$ with $X \in \mathcal{X}$ such that all morphisms $X' \rightarrow M$ with $X' \in \mathcal{X}$ factor through $\alpha$:

$$X \xrightarrow{\alpha} M$$

Dually, we define a left $\mathcal{X}$-approximation of $M$. The subcategory $\mathcal{X}$ is called contravariantly finite (resp. covariantly finite) in $\mathcal{A}$ if every $M \in \mathcal{A}$ admits a right (resp. left) $\mathcal{X}$-approximation. $\mathcal{X}$ is called functorially finite if it is both contravariantly and covariantly finite in $\mathcal{A}$.

**Remark 4.9.** It follows immediately from the definitions that for any torsion pair $(T, F)$ on a pre-abelian category $\mathcal{A}$, $T$ is a contravariantly finite subcategory with right $T$-approximations given by the functor $\tau(-)$. Dually, $F$ is a covariantly finite subcategory of $\mathcal{A}$. 

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In the case of torsion pairs in abelian categories, there is a well-known symmetry:

**Proposition 4.10.** ([Sma84]) Let $\mathcal{A}$ be an abelian category and $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. Then $\mathcal{T}$ is functorially finite in $\mathcal{A}$ if and only if $\mathcal{F}$ is functorially finite in $\mathcal{A}$.

In the situation of the above proposition, we call the torsion pair $(\mathcal{T}, \mathcal{F})$ funtorially finite. In the following result, we see how this symmetry extends to the bijection of Theorem 4.4. The proof of part (b) was privately communicated by Gustavo Jasso who used a similar argument in his work on $\tau$-tilting reduction ([Jas14, Theorem 3.13]).

**Proposition 4.11.** Let $\mathcal{A}$ be an abelian category and let $[(\mathcal{C}, \mathcal{D})$, $(\mathcal{C}', \mathcal{D}')]$ be twin torsion pairs in $\mathcal{A}$.

(a) If $(\mathcal{X}, \mathcal{Y})$ is a functorially finite torsion pair in $\mathcal{A}$ such that $\mathcal{C} \subseteq \mathcal{X} \subseteq \mathcal{C}'$, then $(\mathcal{X} \cap \mathcal{D}, \mathcal{Y} \cap \mathcal{C}')$ is a functorially finite torsion pair in $\mathcal{C}' \cap \mathcal{D}$.

(b) Suppose that $\mathcal{A}$ has enough projectives and injectives and that $(\mathcal{C}, \mathcal{D})$ and $(\mathcal{C}', \mathcal{D}')$ are functorially finite as torsion pairs on $\mathcal{A}$. Suppose $(\mathcal{T}, \mathcal{F})$ is a functorially finite torsion pair in $\mathcal{C}' \cap \mathcal{D}$, then $(\mathcal{C}^* \mathcal{T}, \mathcal{F}^* \mathcal{D}')$ is a functorially finite torsion pair in $\mathcal{A}$.

For the proof we will need the following result.

**Lemma 4.12.** ([IO13, Proposition 5.33]) Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ and $\mathcal{Y}$ be full subcategories of $\mathcal{T}$. If $\mathcal{X}$ and $\mathcal{Y}$ are contravariantly finite in $\mathcal{T}$, then so is $\mathcal{X} \ast \mathcal{Y}$.

**Proof of Proposition 4.11.** (a) Let $(\mathcal{X}, \mathcal{Y})$ be a functorially finite torsion pair in $\mathcal{A}$. In light of Remark 4.9 we only need to check that $\mathcal{X} \cap \mathcal{D}$ is covariantly finite in $\mathcal{C}' \cap \mathcal{D}$ and that $\mathcal{Y} \cap \mathcal{C}'$ is contravariantly finite in $\mathcal{C}' \cap \mathcal{D}$. We will show the first property, the second will follow by a dual argument. Let $M \in \mathcal{C}' \cap \mathcal{D}$ and let $\beta : M \to \mathcal{X}$ be a left $\mathcal{X}$-approximation of $M$, which exists as $M \in \mathcal{A}$ and $\mathcal{X}$ is closed under factor objects, $\mathcal{X} \mathcal{D} \in \mathcal{X}$ and therefore $\mathcal{X} \mathcal{D} \in \mathcal{X} \cap \mathcal{D}$. We claim that $g \beta : M \to \mathcal{X} \mathcal{D}$ is a left $\mathcal{X} \cap \mathcal{D}$-approximation of $M$ in $\mathcal{C}' \cap \mathcal{D}$. Indeed, let $r : M \to \mathcal{X}'$ be a morphism with $\mathcal{X}' \in \mathcal{X} \cap \mathcal{D}$ then, as $\mathcal{X}' \in \mathcal{X}$ and $g$ is a left $\mathcal{X}$-approximation of $M$, there exists a morphism $\gamma : X \to \mathcal{X}'$ such that $\gamma \beta = r$. Now, as $\mathcal{X}' \in \mathcal{D}$, $(\gamma f : cX \to \mathcal{X}') = 0$ and as $g = \text{Cok} f$, there exists a morphism $\delta : \mathcal{X} \mathcal{D} \to \mathcal{X}'$ such that $\delta g = \gamma$. Together we have $r = \gamma \beta = \delta (g \beta)$ and thus $r$ factors through $g \beta$ as required.

(b) Suppose that $(\mathcal{C}, \mathcal{D})$ and $(\mathcal{C}', \mathcal{D}')$ are functorially finite torsion pairs in $\mathcal{A}$ and let $(\mathcal{T}, \mathcal{F})$ be a functorially finite torsion pair in $\mathcal{C}' \cap \mathcal{D}$. We claim that $(\mathcal{C}^* \mathcal{T}, \mathcal{F}^* \mathcal{D}')$ is a functorially finite torsion pair in $\mathcal{A}$. By Remark 4.9 we only
need to show that $\mathcal{C} \ast T$ (resp. $\mathcal{F} \ast D'$) is covariantly (resp. contravariantly) finite in $A$. Both facts follow from Lemma 4.12 and its dual by using the equivalences $D^b(A) \cong K^- (\text{proj} A)$ and $D^b(A) \cong K^+ (\text{inj} A)$ respectively which hold as $A$ has enough projectives (resp. injectives) together with the observation that, in this case, $A$ is a functorially finite subcategory of $D^b(A)$.

\section{5 Torsion pairs in quasi-abelian categories}

The aim of this section is to characterise torsion pairs in quasi-abelian categories. For a torsion class $\mathcal{T}$ of an abelian category $\mathcal{A}$ (which are quasi-abelian by [Rum01b, §4, Corollary]) we have already done this in the previous sections: By taking the twin torsion pairs $[(0, \mathcal{A}), (\mathcal{T}, \mathcal{F})]$. Theorem 4.4 tells us that torsion classes in $\mathcal{T}$ are precisely torsion classes of $\mathcal{A}$ that lie in $\mathcal{T}$ with corresponding torsionfree classes obtained by intersecting with $\mathcal{T}$. Using a result of Rump, we may do this for all quasi-abelian categories.

**Lemma 5.1.** [Rum01b, Theorem 2] Let $\mathcal{Q}$ be a quasi-abelian category. Then $\mathcal{Q}$ is a torsion class in an abelian category $\mathcal{R} = \mathcal{R}_\mathcal{Q}$.

Following [Rum01a], we give a construction of $\mathcal{R}_\mathcal{Q}$ which is sometimes referred to as the (right) associated abelian category of $\mathcal{Q}$. Recall the homotopy category of $\mathcal{Q}$, $K(\mathcal{Q})$, whose objects are chain complexes of objects of $\mathcal{Q}$ and morphisms are chain complex morphisms modulo homotopy, see [Sch99, 1.2.1] for details. Let $\mathcal{X}$ be the subcategory of $K(\mathcal{Q})$ consisting of complexes concentrated in degrees 0 and 1 with the non-trivial differential being an epimorphism. That is, complexes of the form

$$\cdots \longrightarrow 0 \longrightarrow X^0 \overset{f}{\longrightarrow} X^1 \longrightarrow 0 \longrightarrow \cdots$$

that are exact in $X^1$. In practice, we identify the above complex with the epimorphism $f$.

**Remark 5.2.** We make some observations.

(a) A morphism, $(\alpha, \beta) : f \rightarrow f'$ in $\mathcal{X}$ is just a commutative square

$$\begin{array}{ccc}
X & \overset{f}{\longrightarrow} & Y \\
\downarrow \alpha & & \downarrow \beta \\
X' & \overset{f'}{\longrightarrow} & Y'
\end{array} \quad (3)
$$

and is null homotopic if there exists $h : Y \rightarrow X'$ in $\mathcal{Q}$ such that $hf = \alpha$ and $f'h = \beta$. Observe that, as $f$ is epic, if $hf = \alpha$ then $f'h = \beta$ is automatically satisfied.

(b) [Rum01a, Proposition 6] We may describe kernels and cokernels of a morphism as in (3) explicitly in $\mathcal{X}$. Consider the commutative diagrams in
Then it is easily verified that the morphisms in $X'$ give a kernel and cokernel of (3) in $X$ respectively. It follows that a morphism as in (3) is epic if and only if it is a pushout and it is regular (that is, both monic and epic) if and only if it is an exact square in $Q$. Furthermore, we can naturally decompose any morphism as in (3):

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{r} & & \downarrow{1} \\
X' & \xrightarrow{q} & Q \\
\end{array}
\quad
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{q} & & \downarrow{1} \\
Q & \xrightarrow{s} & Y' \\
\end{array}
$$

This shows that $X'$ is semi-abelian.

By [Rum01a] Proposition 1, Proposition 3], $X$ is integral. Thus we may formally invert all regular morphisms to obtain the category $R$ which is abelian by [Fre66 3.2].

**Remark 5.3.** There is a canonical inclusion

$$
\begin{array}{ccc}
Q & \hookrightarrow & R_Q \\
x & \mapsto & (x \to 0)
\end{array}
$$

which is full, faithful and additive. We implicitly identify $Q$ with its image in $R_Q$. Moreover, it follows from [Fre66 1.5] that $\text{inj} R_Q \cong \text{add} Q$.

**Proposition 5.4.** Every torsion and torsionfree class in a quasi-abelian category has the structure of a quasi-abelian category.
Proof. By Theorem 4.4 every torsion class, \( T \), of a quasi-abelian category \( Q \) is a torsion class of \( R_Q \) that happens to lie in \( Q \) and is therefore quasi-abelian. Theorem 4.4 also tells us that the associated torsionfree class is \( A \cap F \) where \( F = T^\perp \) in \( R_Q \). But \( Q \cap F \) is the intersection of a torsion and torsionfree class, and as \( T \subseteq Q \), by Proposition 3.2, it is quasi-abelian. \( \square \)

Remark 5.5. Dually, for a quasi-abelian category \( Q \), one may construct an abelian category \( L = L_Q \) such that \( Q \) is torsionfree in \( L \). This gives another proof of Proposition 3.2. Moreover, the categories \( R \) and \( L \) are derived equivalent and are related by tilting induced by \( Q \), see [Fio16] and [Sch99] for more details.

There is an immediate consequence of Lemma 4.7.

Corollary 5.6. The heart of twin torsion pairs in a quasi-abelian category is quasi-abelian.

We now prove that the converse of Proposition 2.6 holds in quasi-abelian categories giving a familiar characterisation of torsion classes.

Proposition 5.7. Let \( Q \) be a quasi-abelian category. Then a pair of full subcategories \((T, F)\) is a torsion pair on \( Q \) if and only if the following hold

(a) For all \( M \in Q \), if \( \text{Hom}_Q(M, F) = 0 \) then \( M \in T \).

(b) For all \( N \in Q \), if \( \text{Hom}_Q(T, N) = 0 \) then \( N \in F \).

Proof. The fact that the conditions are sufficient was proved in Proposition 2.6. We now prove that they are necessary. Let \( T, F \) be full subcategories of \( Q \) such that

\[
T = \{ M \in Q \mid \text{Hom}_Q(M, F) = 0 \}
\]

\[
F = \{ N \in Q \mid \text{Hom}_Q(T, N) = 0 \}.
\]

Observe that if \((T, F \ast Q^\perp)\) is a torsion pair on \( R = R_Q \), then \((T, (F \ast Q^\perp) \cap Q) = (T, F)\) is a torsion pair on \( Q \) which proves the statement. It remains to show that \((T, F \ast Q^\perp)\) is a torsion pair on \( R \). Since \( R \) is abelian, it suffices to show that

\[
T = \{ M \in R \mid \text{Hom}_R(M, F \ast Q^\perp) = 0 \}
\]

\[
F \ast Q^\perp = \{ N \in R \mid \text{Hom}_R(T, N) = 0 \}.
\]

Let \( M \in R \) be such that \( \text{Hom}_R(M, F \ast Q^\perp) = 0 \). In particular, we have that \( \text{Hom}_R(M, Q^\perp) = 0 \) thus \( M \in Q \) and as \( 0 = \text{Hom}_R(M, F) = \text{Hom}_Q(M, F) \), \( M \in T \). Now let \( N \in A \) be such that \( \text{Hom}_R(T, N) = 0 \) and consider the \((Q, Q^\perp)\)-canonical short exact sequence of \( N \)

\[
0 \longrightarrow QN \longrightarrow N \longrightarrow N_{Q^\perp} \longrightarrow 0.
\]

Observe that \( \text{Hom}_R(T, QN) = \text{Hom}_Q(T, QN) = 0 \), else by composing with the monomorphism \( i \) we would obtain a morphism \( T \rightarrow N \). Thus \( QN \in F \) and the sequence shows \( N \) is an element of \( F \ast Q^\perp \). \( \square \)
Lemma 5.8. Let $Q$ be a quasi-abelian category. Then $Q$ is functorially finite in $\mathcal{R} = \mathcal{R}_Q$. Moreover, $\mathcal{R}$ has enough injectives.

Proof. As $Q$ is a torsion class on $\mathcal{R}$, it is contravariantly finite. It remains to show that it is covariantly finite. Let $(f : X \to Y) \in \mathcal{R}$, we claim that the morphism

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow 1 & & \downarrow 1 \\
X & \to & 0
\end{array}
\quad (4)
\]

in $\mathcal{R}$ is a left $Q$-approximation of $f$. Indeed, if $(\alpha, \beta) : f \to (Z \to 0)$ is some morphism in $\mathcal{R}$ (note that necessarily $\beta = 0$), then $(\alpha, 0) : (X \to 0) \to (Z \to 0)$ gives the required factorisation.

In light of Remark 5.3, to show that $\mathcal{R}$ has enough injectives, it is enough to show that the morphism (4) is monic. By computation as in 5.2(b), the kernel is given by

\[
\begin{array}{ccc}
X & \xrightarrow{1} & X \\
\downarrow 1 & & \downarrow f \\
X & \xrightarrow{f} & Y
\end{array}
\]

which is a null homotopic by the identity morphism $X \to X$. Thus the kernel of (4) is zero and therefore it is monic.

Proposition 5.9. Let $Q$ be a quasi-abelian category and $(T, F)$ be a torsion pair in $Q$. Then $T$ is functorially finite if and only if $F$ is functorially finite.

Proof. Let $(T, F)$ be a torsion pair in a quasi-abelian category $Q$ and suppose that $T$ is functorially finite in $Q$. We begin by showing $T$ is functorially finite on $\mathcal{R} = \mathcal{R}_Q$ (this was not done in Proposition 4.11). As $T$ is a torsion class in $\mathcal{R}$, it is contravariantly finite on $\mathcal{R}$. We now show that every $X \in \mathcal{R}$ admits a left $T$-approximation. Let $X \to Q$ be a left $Q$-approximation of $X$, which exists by Lemma 5.8, and let $Q \to T$ be a left $T$-approximation of $Q$, which exists by assumption. Then it is easily verified that the composition $X \to T$ is a left $T$-approximation of $X$. Thus $T$ is a functorially finite torsion class in $\mathcal{A}$ and therefore so its associated torsion free class $F \ast Q^\perp$ in $\mathcal{R}$. It now follows from Proposition 4.11(a) that $F$ is functorially finite in $Q$.

For the converse, we reverse the argument but use Proposition 4.11(b) to see that $F \ast Q^\perp$ is functorially finite in $\mathcal{R}$ which we may do since $\mathcal{R}$ has enough injectives by Proposition 5.8.

Lemma 5.10. The intersection of torsion classes in a quasi-abelian category is again quasi-abelian.

Proof. This property is immediately inherited from the abelian setting. 

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6 Harder-Narasimhan filtrations

In this section we apply the results of the previous sections to show the existence of Harder-Narasimhan filtrations for quasi-abelian categories.

For a pre-abelian category, $\mathcal{C}$, let $\mathcal{T}(\mathcal{C})$ be the set of order reversing functions of posets, $\eta$, from the real interval $[0, 1]$ to the set of all torsion classes of $\mathcal{C}$ such that $\eta(0) = \mathcal{C}$ and $\eta(1) = 0$. Equivalently, the data of such a map is a chain of torsion classes in $\mathcal{C}$

$$\eta : 0 = T_1 \subseteq \cdots \subseteq T_r \subseteq \cdots \subseteq T_0 = \mathcal{C}$$

with $r \in [0, 1]$ satisfying $T_r \subseteq T_{r'}$ if and only if $r \geq r'$.

We begin with a generalised version of [Tre18, 2.3], where only the abelian case was considered.

**Lemma 6.1.** Let $\mathcal{Q}$ be a quasi-abelian category and $\eta = (T_i)_{i \in [0, 1]} \in \mathcal{T}(\mathcal{Q})$. Then for every $r \in [0, 1]$, the subcategories

$$\bigcap_{s < r} T_s$$

and

$$\bigcup_{s > r} T_s$$

are torsion classes in $\mathcal{Q}$. Moreover, for all $\mathcal{X} \subset \text{tors} \mathcal{A}$, if $\mathcal{X} \subset T_s$ for all $s < r$ then $\mathcal{X} \subset \bigcap_{s < r} T_s$. Similarly, if $T_s \subset \mathcal{X}$ for all $s > r$ then $\bigcup_{s > r} T_s \subset \mathcal{X}$.

**Proof.** This follows from [Tre18, 2.3] by translating to $R\mathcal{Q}$. $\square$

Based on [Tre18, 2.8] we define the following.

**Definition 6.2.** Let $\mathcal{C}$ be a pre-abelian category and $\eta = (T_i)_{i \in [0, 1]} \in \mathcal{T}(\mathcal{C})$. Define the subcategories $\mathcal{P}_\eta^r$ as follows

$$\mathcal{P}_\eta^r = \begin{cases} 
\left( \bigcup_{s > r} T_s \right) & \text{if } r = 0 \\
\left( \bigcap_{s < r} T_s \right) \cap \left( \bigcup_{s > r} T_s \right) & \text{if } r \in (0, 1) \\
\bigcap_{s < 1} T_s & \text{if } r = 1
\end{cases}$$

**Remark 6.3.** In a quasi-abelian category $\mathcal{A}$, for every $\eta = (T_i)_{i \in [0, 1]} \in \mathcal{T}(\mathcal{A})$, each $\mathcal{P}_\eta^r$ is quasi-abelian. For $r = 0, 1$ this is obvious. For $r \in (0, 1)$ observe that

$$\bigcup_{s > r} T_s \subseteq T_r \subseteq \bigcap_{s < r} T_s$$

thus $\bigcup_{s > r} T_s$ and $\bigcap_{s < r} T_s$ define twin torsion pairs with heart $\mathcal{P}_\eta^r$. We also note that every heart of twin torsion pairs occurs as a $\mathcal{P}_\eta^r$ for some chain of torsion classes $\eta$ and some $r \in [0, 1]$.

**Proposition 6.4.** [Tre18, 2.9]. Let $\mathcal{A}$ be an abelian length category and $\eta = (T_i)_{i \in [0, 1]} \in \mathcal{T}(\mathcal{A})$. Then for all $M \in \mathcal{A}$ there exists a unique (up to isomorphism) Harder-Narasimhan filtration of $M$ with respect to $\eta$ in $\mathcal{A}$. That is, a filtration

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

such that
(hn1) $M_k/M_{k-1} \in \mathcal{P}_{rk}$ for all $1 \leq k \leq n$.

(hn2) $r_k > r_{k'}$ if and only if $k < k'$.

### 6.1 Harder-Narasimhan filtrations in quasi-abelian categories

We want to show that quasi-abelian categories admit Harder-Narasimhan filtrations in the above sense.

**Set-up 6.5.** Let $\mathcal{A}$ be an abelian length category and fix twin torsion pairs $[(\mathcal{C}, \mathcal{D}), (\mathcal{C}', \mathcal{D}')]$ in $\mathcal{A}$ and set $\mathcal{Q} = \mathcal{C}' \cap \mathcal{D}$. By Theorem 4.4, we may identify $\mathcal{T}(\mathcal{Q})$ bijectively with a subset of $\mathcal{T}(\mathcal{A})$ along the map

$$\phi_C = \phi : \mathcal{T}(\mathcal{Q}) \hookrightarrow \mathcal{T}(\mathcal{A})$$

where

$$\mathcal{X}_i = \begin{cases} 
\mathcal{A} & \text{if } i = 0 \\
\mathcal{C} \ast \mathcal{T}_i & \text{if } i \in (0, 1) \\
0 & \text{if } i = 1.
\end{cases}$$

We denote the image of $\phi_C$ by $\mathcal{T}_C(\mathcal{Q})$. Thus $\mathcal{T}_C(\mathcal{Q})$ consists of all $\eta = (\mathcal{T}_i)_{i \in [0, 1]} \in \mathcal{T}(\mathcal{A})$ such that $\mathcal{C} \subseteq \mathcal{T}_i \subseteq \mathcal{C}'$ for all $i \in (0, 1)$. We remark that, in light of Remark 3.3, this map does indeed depend on $\mathcal{C}$ (since then $\mathcal{Q}$ determines $\mathcal{C}'$ by Theorem 4.4).

We investigate the subcategories $\mathcal{P}_{\phi(r)}^{\phi(\eta)}$.

**Lemma 6.6.** Let $\eta = (\mathcal{T}_i)_{i \in [0, 1]} \in \mathcal{T}(\mathcal{Q})$. Then

$$\mathcal{P}_{\phi(r)}^{\phi(\eta)} = \begin{cases} 
\mathcal{P}_0^\eta \ast \mathcal{D}' & \text{if } r = 0 \\
\mathcal{P}_r^\eta & \text{if } r \in (0, 1) \\
\mathcal{C} \ast \mathcal{P}_1^\eta & \text{if } r = 1.
\end{cases}$$

**Proof.** Let $r \in (0, 1)$, and observe

$$\mathcal{P}_{\phi(r)}^{\phi(\eta)} = \left( \bigcap_{s \in (0, r)} (\mathcal{C} \ast \mathcal{T}_s) \right) \cap \left( \bigcup_{s \in (r, 1)} (\mathcal{C} \ast \mathcal{T}_s) \right)^\perp$$

$$= \left( \mathcal{C} \ast \bigcap_{s \in (0, r)} \mathcal{T}_s \right) \cap \left( \mathcal{C} \ast \bigcup_{s \in (r, 1)} \mathcal{T}_s \right)^\perp$$

$$= \left( \mathcal{C} \ast \bigcap_{s \in (0, r)} \mathcal{T}_s \right) \cap \left( \left( \bigcup_{s \in (r, 1)} \mathcal{T}_s \right)^\perp \ast \mathcal{D}' \right)$$

$$= \left( \bigcap_{s \in (0, r)} \mathcal{T}_s \right) \cap \left( \bigcup_{s \in (r, 1)} \mathcal{T}_s \right)^\perp = \mathcal{P}_{\phi(r)}^{\phi(\eta)}.$$
where the first and last equalities follow from the definitions knowing that $T_1 = 0$ and $T_0 = Q$. The second equality is straightforward set theory, the third equality follows from Theorem 4.4 and the fourth equality holds by Lemma 4.7. The cases $r = 0, 1$ follow by similar arguments.

**Theorem 6.7.** In the situation of Set-up 6.5. Let $\eta = (T_i)_{i \in [0,1]} \in \mathfrak{T}(Q)$. Then for all $M \in Q$ there exists a unique (up to isomorphism) Harder-Narasimhan filtration of $M$ with respect to $\eta$ in $Q$. That is, a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of $M$ in $Q$ such that

(HN1) $M_k/M_{k-1} \in \mathcal{P}^\eta_{r_k}$ for all $1 \leq k \leq n$.

(HN2) $r_k > r_k'$ if and only if $k < k'$.

**Proof.** Let $M \in Q$ and $\eta = (T_i)_{i \in [0,1]} \in \mathfrak{T}(Q)$. We claim that the Harder-Narasimhan filtration of $M$ with respect to $\phi(\eta)$ in $A$:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

also is the Harder-Narasimhan filtration of $M$ with respect to $\eta$ in $Q$. We first show that for all $1 \leq k \leq n$, $M_k/M_{k-1} \in \mathcal{P}^\eta_{r_k}$ where the $r_k$ are defined as in Proposition 6.4(hn1). When $r_k \in (0,1)$, this is trivially true by Lemma 6.6. It remains to check for $r_k = 0, 1$. Observe that by Proposition 6.4(hn2), the only case where $r_k = 0$ (resp. $r_k = 1$) is for $k = n$ (resp. $k = 1$). So suppose that $r_n = 0$, then $M_n/M_{n-1} = M/M_{n-1} \in \mathcal{P}^\phi(n) = \mathcal{P}_0^\eta * \mathcal{D}'$. As $M \in Q \subset C'$, so is the quotient $M \to M/M_{n-1}$. Thus $M \to M/M_{n-1} \in C' \cap (\mathcal{P}_0^\eta * \mathcal{D}') = \mathcal{P}_0^\eta$ by Theorem 4.3. Similarly, we see that $M_k/M_0 = M_k \in \mathcal{D} \cap \mathcal{P}_1^\phi(n) = \mathcal{D} \cap (C * \mathcal{P}_0^\eta) = \mathcal{P}_0^\eta$. By Lemma 6.6 this implies that (HN1) holds. Note that (HN2) holds as it is inherited from $A$ as is the uniqueness of the filtration up to isomorphism.

It remains to show that $M_1 \in Q$ for all $1 \leq k \leq n$. We proceed by induction on $k$. For $k = 1$ we have shown that $M_1 = M_1/M_0 \in \mathcal{P}_0^\eta \subset Q$ for some $r_1 \in [0,1]$. The $k > 1$ case follows by using the short exact sequences

$$0 \longrightarrow M_{k-1} \longrightarrow M_k \longrightarrow M_k/M_{k-1} \longrightarrow 0$$

as $M_k/M_{k-1} \in \mathcal{P}^\eta_{r_k} \subset Q$ and since $Q$ is closed under extensions. \hfill \Box

**Corollary 6.8.** Let $Q$ be a quasi-abelian category such that $\mathcal{R}_Q$ is of finite length and $\eta = (T_i)_{i \in [0,1]} \in \mathfrak{T}(Q)$. Then every $M \in Q$ admits a Harder-Narasimhan filtration with respect to $\eta$ in $Q$.

**Proof.** The result follows from Theorem 6.7 as $Q$ appears as $\mathcal{P}_r$ for some $r \in [0,1]$ in, for example, the chain of torsion classes

$$0 \subset Q \subset \mathcal{R}_Q$$

in $\mathfrak{T}(\mathcal{R}_Q)$.

\hfill \Box
6.2 Topological properties of $\mathfrak{T}(\mathcal{Q})$

We recall that, by [Bri07] 6.1 and [Tre18] 7.1, $\mathfrak{T}(\mathcal{A})$ has a topological structure. Namely, we have the following.

**Proposition 6.9.** Let $\mathcal{A}$ be an abelian length category. Then $\mathfrak{T}(\mathcal{A})$ is a topological space with pseudometric given by

$$d(\eta, \eta') = \inf \{ \varepsilon \in [0, 1] \mid \mathcal{P}_r^{\eta'} \subseteq \mathcal{P}_r^{\eta} \forall r \in [0, 1] \}$$

for $\eta, \eta' \in \mathfrak{T}(\mathcal{A})$. Where

$$\mathcal{P}_r^{\eta} := \text{Filt} \left( \bigcup_{s \in [a,b]} \mathcal{P}_s^{\eta} \right)$$

for $0 \leq a \leq b \leq 1$ and we set $\mathcal{P}_0^{\eta} = 0$ for all $r \notin [0, 1]$.

**Remark 6.10.** Note that, for $\eta, \eta' \in \mathfrak{T}(\mathcal{A})$, we have $d(\eta, \eta') = 0$ if and only if $\mathcal{P}_r^{\eta} = \mathcal{P}_r^{\eta'}$ for all $r \in [0, 1]$.

**Proposition 6.11.** In the situation of Set-up 6.6, $\mathfrak{T}_C(\mathcal{Q})$ is a closed set of the topological space $\mathfrak{T}(\mathcal{A})$.

**Proof.** We show that $\mathfrak{T}_C(\mathcal{Q})$ contains all of its accumulation points. To this end, let $\eta = (T_i)_{i \in [0,1]} \in \mathfrak{T}(\mathcal{A})$ such that there exists $\eta' \in \mathfrak{T}_C(\mathcal{Q})$ with $d(\eta, \eta') = d(\eta', \eta) = 0$. Then by, Proposition 6.9 for all $r \in [0, 1]$, $\mathcal{P}_r^{\eta} = \mathcal{P}_r^{\eta'}$. In particular, $\mathcal{P}_0^{\eta} \subseteq \mathcal{C}$ and $\mathcal{P}_0^{\eta} \subseteq \mathcal{D}'$, thus $\mathcal{C} \subseteq T_i \subseteq \mathcal{C}'$ for all $i \in (0, 1)$ and we conclude that $\eta \in \mathfrak{T}_C(\mathcal{Q})$. \[\square\]

As we remarked earlier, the embedding of $\mathfrak{T}(\mathcal{Q})$ (of Set-up 6.6) in $\mathfrak{T}(\mathcal{A})$ depends on $\mathcal{C}$. So when $\mathcal{Q}$ occurs as the heart of many twin torsion pairs, $\mathfrak{T}(\mathcal{Q})$ can be embedded into $\mathfrak{T}(\mathcal{A})$ in as many ways. To finish, we see that the various embeddings of $\mathcal{Q}$ are ‘far apart’ in $\mathfrak{T}(\mathcal{A})$.

**Proposition 6.12.** Let $\mathcal{A}$ be an abelian length category and for $j \in \{0, 1\}$ let $[(\mathcal{C}_j, \mathcal{D}_j), (\mathcal{C}_j', \mathcal{D}_j')]$ be distinct twin torsion pairs with the same heart $\mathcal{Q} = \mathcal{C}_j \cap \mathcal{D}_j$. Then

$$d(\mathfrak{T}_{\mathcal{C}_0}(\mathcal{Q}), \mathfrak{T}_{\mathcal{C}_1}(\mathcal{Q})) = 1$$

where $\mathfrak{T}_{\mathcal{C}_j}(\mathcal{Q})$ are defined following Set-up 6.6.

**Proof.** We show that for all $\eta_j \in \mathfrak{T}_{\mathcal{C}_j}(\mathcal{Q})$, $d(\eta_0, \eta_1) = 1$. By Proposition 6.9 it is enough to show the existence of some

$$X \in (\mathcal{P}_r^{\eta_0} \setminus \mathcal{P}_r^{\eta_1}) \cap (\mathcal{P}_r^{\eta_1} \setminus \mathcal{P}_r^{\eta_0}). \quad (5)$$

Let $Y \in \mathcal{D}_0 \setminus \mathcal{D}_1$, we claim that $X := c_0 Y$ satisfies (5). Clearly, $X \in \mathcal{C}_1 \subseteq \mathcal{P}_1^{\eta_1}$ and, as $\mathcal{D}_0$ is closed under subobjects, $X \subseteq \mathcal{D}_0' \subseteq \mathcal{P}_0^{\eta_0}$.

We now show that $X \notin \mathcal{P}_0^{\eta_1} = \mathcal{D}_1' * \mathcal{P}_0^{\phi_{\mathcal{C}_1}(\eta_1)}$. Observe that as $X \in \mathcal{C}_1', X_{\mathcal{D}_1'} = 0$, thus, by Lemma 4.6, $X \in \mathcal{P}_0^{\eta_1}$ if and only if $X \in \mathcal{P}_0^{\phi_{\mathcal{C}_1}(\eta_1)} \subseteq \mathcal{Q}$. But as $X \in \mathcal{C}_1$, $X \notin \mathcal{Q}$ and in particular, $X \notin \mathcal{P}_0^{\eta_1}$. Similarly, one verifies that $X \notin \mathcal{P}_1^{\eta_0}$ and we are done. \[\square\]
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