ON SEMIGROUPS GENERATED BY SUMS OF EVEN POWERS OF DUNKL OPERATORS

JACEK DZIUBAŃSKI AND AGNIESZKA HEJNA

ABSTRACT. On the Euclidean space \( \mathbb{R}^N \) equipped with a normalized root system \( R \), a multiplicity function \( k \geq 0 \), and the associated measure \( dw(x) = \prod_{\alpha \in R} |\langle x, \alpha \rangle|^{k(\alpha)} dx \) we consider the differential-difference operator

\[
L = (-1)^{\ell+1} \sum_{j=1}^{m} T^2\ell_{\zeta_j},
\]

where \( \zeta_1, ..., \zeta_m \) are nonzero vectors in \( \mathbb{R}^N \), which span \( \mathbb{R}^N \), and \( T_{\zeta} \) are the Dunkl operators. The operator \( L \) is essentially self-adjoint on \( L^2(dw) \) and its closure generates a semigroup \( \{S_t\}_{t \geq 0} \) of linear self-adjoint contractions, which has the form

\[
S_t f(x) = f \ast q_t(x),
\]

where \( q_t(x) = t^{-N/(2\ell)} q(x/t^{1/(2\ell)}) \), and \( q(x) \) is the Dunkl transform of the function \( \exp(-\sum_{j=1}^{m} \langle \zeta_j, \xi \rangle^{2\ell}) \). We prove that \( q(x) \) satisfies the following exponential decay:

\[
|q(x)| \lesssim \exp(-c \|x\|^{2\ell/(2\ell-1)})
\]

for a certain constant \( c > 0 \). Moreover, if \( q(x, y) = \tau_x q(-y) \), then \( |q(x, y)| \lesssim w(B(x, 1))^{-1} \exp(-cd(x, y)^{2\ell/(2\ell-1)}) \), where \( d(x, y) = \min_{\sigma \in G} \|x - \sigma(y)\| \), \( G \) is the reflection group for \( R \), and \( \tau_x \) denotes the Dunkl translation.

1. Introduction

Let \( \zeta_1, ..., \zeta_m \in \mathbb{R}^N \) be non-zero vectors which span \( \mathbb{R}^N \). For \( \ell \in \mathbb{N} \) (which will be fixed throughout the paper) we consider the symmetric differential-difference operator

\[
L = (-1)^{\ell+1} \sum_{j=1}^{m} T^2\ell_{\zeta_j},
\]

where \( T_{\zeta} \) are Dunkl operators associated with a normalized system of roots \( R \) and a multiplicity function \( k \geq 0 \) (see Section 2 for details). Let \( dw \) denote the related measure (see (2.2)). The operator \( L \) is essentially self-adjoint on \( L^2(dw) \) and its closure generates a semigroup of self-adjoint linear contractions \( \{S_t\}_{t \geq 0} \) on \( L^2(dw) \). The semigroup has the form

\[
S_t f(x) = f \ast q_t(x),
\]

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where \( q_t(x) = \mathcal{F}^{-1}(\exp(-t \sum_{j=1}^{m}(\zeta_j, \cdot)^{2\ell}) (x)) \). Here and subsequently, \(*\) denotes the Dunkl convolution, while \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) stand for the Dunkl transform and its inverse respectively (see (2.10)). Clearly, \( q_t \in \mathcal{S}(\mathbb{R}^N) \), and if we set \( q(x) = q_1(x) \), then, by homogeneity,

\[
q_t(x) = t^{-N/(2\ell)} q\left(\frac{x}{t^{1/(2\ell)}}\right). \tag{1.2}
\]

Our first result is to prove that the decay of \( q(x) \) is exponential. This is stated in the following theorem.

**Theorem 1.1.** There are constants \( C, c > 0 \) such that for all \( x \in \mathbb{R}^N \) we have

\[
|q(x)| \leq C \exp(-c\|x\|^{2\ell/(2\ell-1)}).
\]

Let \( \tau_x \) denote the Dunkl translation (see (2.15)). Then \( q_t(x,y) = \tau_x q_t(-y) \) are the integral kernels of the operators \( S_t \) with respect to the measure \( dw \), that is,

\[
S_t f(x) = \int_{\mathbb{R}^N} q_t(x,y) f(y) dw(y).
\]

Let

\[
d(x,y) = \min_{\sigma \in G} \|\sigma(x) - y\|
\]

be the distance of the orbit of \( x \) to the orbit of \( y \), where \( G \) denotes the Weyl group associated with \( R \) (see Section 2). We denote by \( B(x,r) \) the (closed) Euclidean ball centered at \( x \in \mathbb{R}^N \) and radius \( r \). Our second result expresses the decay of \( q_t(x,y) \) by means of the distance \( d(x,y) \).

**Theorem 1.2.** There are constants \( C, c > 0 \) such that for all \( x, y \in \mathbb{R}^N \) we have

\[
|q_t(x,y)| \leq C \exp(-cd(x,y)^{2\ell/(2\ell-1)}).
\]

**Remark 1.3.** By a scaling argument applied to (1.3) (see (1.2) and (2.3)) we obtain that there are \( C, c > 0 \) such that for all \( x, y \in \mathbb{R}^N \) and \( t > 0 \) we have

\[
|q_t(x,y)| \leq C \exp\left(-c\frac{d(x,y)^{2\ell/(2\ell-1)}}{t^{1/(2\ell-1)}}\right).
\]

To prove the first theorem we borrow ideas of [9] and [10]. We first introduce a family of weighted \( L^2 \)-spaces with weights of exponential growth and prove that (1.1) defines strongly continuous semigroups of linear operators on these spaces. This is done by proving Garding inequalities for associated weighted linear forms and applying a theorem of J.-L. Lions (see Theorem 5.1). We expect that if a convolution operator preserves weighted \( L^2 \)-spaces with weights of exponential growth and has some smoothness properties, then its convolution kernel should have some fast decay, and in fact it has.

Let us note that the function \( q(x) \) is not radial. Therefore in the proof of Theorem 1.2 we cannot apply the formula of Rößler (see (2.16)) for translations of radial functions. In order to prove Theorem 1.2 we use methods developed in [11] based on the description of the support the Dunkl translations of compactly supported functions combined with the observation that any sufficiently regular
fast decaying function can be written as a convolution of two functions such that one of them is radial (see \cite{11}). Let us emphasis difficulties we have to face when we apply the method of exponential weights. The first one is that the Dunkl operators do not satisfy the Leibniz rule. The second one concerns the lack of knowledge about boundedness of the Dunkl translations on $L^p(dw)$ spaces and the fact that the translations do not form a group of operators as it is in the case of Lie groups.

2. Preliminaries and notation

The Dunkl theory is a generalization of the Euclidean Fourier analysis. It started with the seminal article \cite{6} and developed extensively afterwards (see e.g. \cite{4}, \cite{5}, \cite{7}, \cite{8}, \cite{12}, \cite{15}, \cite{16}, \cite{17}, \cite{20}, \cite{21}). In this section we present basic facts concerning the theory of the Dunkl operators. For details we refer the reader to \cite{6}, \cite{18}, and \cite{19}.

We consider the Euclidean space $\mathbb{R}^N$ with the scalar product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{N} x_j y_j$, $\mathbf{x} = (x_1, ..., x_N)$, $\mathbf{y} = (y_1, ..., y_N)$, and the norm $\| \mathbf{x} \|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$. For a nonzero vector $\alpha \in \mathbb{R}^N$, the reflection $\sigma_\alpha$ with respect to the hyperplane $\alpha \perp$ orthogonal to $\alpha$ is given by

$$\sigma_\alpha(\mathbf{x}) = \mathbf{x} - 2 \frac{\langle \mathbf{x}, \alpha \rangle}{\| \alpha \|^2} \alpha. \tag{2.1}$$

In this paper we fix a normalized root system in $\mathbb{R}^N$, that is, a finite set $R \subset \mathbb{R}^N \setminus \{0\}$ such that $\sigma_\alpha(R) = R$ and $\| \alpha \| = \sqrt{2}$ for every $\alpha \in R$. The finite group $G$ generated by the reflections $\sigma_\alpha \in R$ is called the Weyl group (reflection group) of the root system. A multiplicity function is a $G$-invariant function $k : R \to \mathbb{C}$ which will be fixed and $\geq 0$ throughout this paper.

Let

$$dw(\mathbf{x}) = \prod_{\alpha \in R} |\langle \mathbf{x}, \alpha \rangle|^{k(\alpha)} d\mathbf{x} \tag{2.2}$$

be the associated measure in $\mathbb{R}^N$, where, here and subsequently, $d\mathbf{x}$ stands for the Lebesgue measure in $\mathbb{R}^N$. We denote by $N = N + \sum_{\alpha \in R} k(\alpha)$ the homogeneous dimension of the system. Clearly,

$$w(B(t\mathbf{x}, tr)) = t^N w(B(\mathbf{x}, r)) \text{ for all } \mathbf{x} \in \mathbb{R}^N, \ t, r > 0 \tag{2.3}$$

and

$$\int_{\mathbb{R}^N} f(\mathbf{x}) dw(\mathbf{x}) = \int_{\mathbb{R}^N} t^{-N} f(\mathbf{x}/t) dw(\mathbf{x}) \text{ for } f \in L^1(dw) \text{ and } t > 0. \tag{2.4}$$

Observe that

$$w(B(\mathbf{x}, r)) \sim r^N \prod_{\alpha \in R} (|\langle \mathbf{x}, \alpha \rangle| + r)^{k(\alpha)}, \tag{2.5}$$

\footnote{The symbol $\sim$ between two positive expressions means that their ratio remains between two positive constants.}
so $dw(x)$ is doubling, that is, there is a constant $C > 0$ such that
\[(2.6) \quad w(B(x, 2r)) \leq Cw(B(x, r)) \quad \text{for all} \quad x \in \mathbb{R}^N, \quad r > 0.
\]

For $\xi \in \mathbb{R}^N$, the Dunkl operators $T_\xi$ are the following $k$-deformations of the directional derivatives $\partial_\xi$ by a difference operator:
\[(2.7) \quad T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, \xi \rangle}.
\]

The Dunkl operators $T_\xi$, which were introduced in [6], commute and are skew-symmetric with respect to the $G$-invariant measure $dw$. Suppose that $\xi \neq 0$, $f, g \in C^1(\mathbb{R}^N)$ and $g$ is radial. The following Leibniz rule can be confirmed by a direct calculation:
\[(2.8) \quad T_\xi (fg) = f(T_\xi g) + g(T_\xi f).
\]

For fixed $y \in \mathbb{R}^N$ the Dunkl kernel $E(x, y)$ is the unique analytic solution to the system
\[T_\xi f = \langle \xi, y \rangle f, \quad f(0) = 1.
\]

The function $E(x, y)$, which generalizes the exponential function $e^{\langle x, y \rangle}$, has the unique extension to a holomorphic function on $\mathbb{C}^N \times \mathbb{C}^N$. Let $\{e_j\}_{1 \leq j \leq N}$ denote the canonical orthonormal basis in $\mathbb{R}^N$ and let $T_j = T_{e_j}$. For multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_N) \in \mathbb{N}_0^N$, we set
\[|\beta| = \beta_1 + \beta_2 + \ldots + \beta_N,
\]
\[\partial^\beta = \partial_1^{\beta_1} \circ \partial_2^{\beta_2} \circ \ldots \circ \partial_N^{\beta_N},
\]
\[T^\beta = T_1^{\beta_1} \circ T_2^{\beta_2} \circ \ldots \circ T_N^{\beta_N}.
\]

In our further consideration we shall need the following lemma.

**Lemma 2.1.** For all $x \in \mathbb{R}^N$, $z \in \mathbb{C}^N$ and $\nu \in \mathbb{N}_0^N$ we have
\[|\partial^\nu z E(x, z)| \leq \|x\|^{\nu_1} \exp(\|x\| \Re z).
\]

In particular,
\[(2.9) \quad |E(i\xi, x)| \leq 1 \quad \text{for all} \quad \xi, x \in \mathbb{R}^N.
\]

**Proof.** See [16, Corollary 5.3].

The Dunkl transform
\[(2.10) \quad \mathcal{F}f(\xi) = c_k^{-1} \int_{\mathbb{R}^N} E(-i\xi, x) f(x) \, dw(x),
\]
where
\[c_k = \int_{\mathbb{R}^N} e^{-\frac{\|x\|^2}{4}} \, dw(x) > 0,
\]
originally defined for $f \in L^1(dw)$, is an isometry on $L^2(dw)$, i.e.,
\[(2.11) \quad \|f\|_{L^2(dw)} = \|\mathcal{F}f\|_{L^2(dw)} \quad \text{for all} \quad f \in L^2(dw),
\]
and preserves the Schwartz class of functions $S(\mathbb{R}^N)$ (see [3]). Its inverse $\mathcal{F}^{-1}$ has the form

$$\mathcal{F}^{-1}g(x) = c_k^{-1} \int_{\mathbb{R}^N} E(i\xi, x)g(\xi) \, dw(\xi).$$

Obviously, for all $f \in S(\mathbb{R}^N)$, we have

$$\mathcal{F}(T_\zeta f)(\xi) = -i\langle \zeta, \xi \rangle \mathcal{F}f(\xi) \text{ for all } \zeta, \xi \in \mathbb{R}^N,$$

and, consequently,

$$\mathcal{F}(Lf)(\xi) = -\left( \sum_{j=1}^{m} \langle \zeta_j, \xi \rangle^{2j} \right) \mathcal{F}f(\xi) \text{ for all } \xi \in \mathbb{R}^N.$$

The Dunkl transform $\mathcal{F}$ is an analogue of the classical Fourier transform.

The Dunkl translation $\tau_x f$ of a function $f \in S(\mathbb{R}^N)$ by $x \in \mathbb{R}^N$ is defined by

$$\tau_x f(y) = c_k^{-1} \int_{\mathbb{R}^N} E(i\xi, x) E(i\xi, y) \mathcal{F}f(\xi) \, dw(\xi).$$

It is a contraction on $L^2(dw)$, however it is an open problem if the Dunkl translations are bounded operators on $L^p(dw)$ for $p \neq 2$.

The following specific formula was obtained by Rösler [17] for the Dunkl translations of (reasonable) radial functions $f(x) = \tilde{f}(\|x\|)$:

$$\tau_x f(-y) = \int_{\mathbb{R}^N} (\tilde{f} \circ A)(x, y, \eta) \, d\mu_x(\eta) \text{ for all } x, y \in \mathbb{R}^N.$$

Here

$$A(x, y, \eta) = \sqrt{\|x\|^2 + \|y\|^2 - 2\langle y, \eta \rangle} = \sqrt{\|x\|^2 - \|\eta\|^2 + \|y - \eta\|^2}$$

and $\mu_x$ is a probability measure, which is supported in the set $\operatorname{conv} O(x)$, where $O(x) = \{\sigma(x) : \sigma \in G\}$ is the orbit of $x$. Formula (2.16) implies that for all radial $f \in L^1(dw)$ and $x \in \mathbb{R}^N$ we have

$$\|\tau_x f(y)\|_{L^1(dw(y))} \leq \|f(y)\|_{L^1(dw(y))}.$$

The Dunkl convolution $f \ast g$ of two reasonable functions (for instance Schwartz functions) is defined by

$$(f \ast g)(x) = c_k \mathcal{F}^{-1}[(\mathcal{F}f)(\mathcal{F}g)](x) = \int_{\mathbb{R}^N} (\mathcal{F}f)(\xi) (\mathcal{F}g)(\xi) E(x, i\xi) \, dw(\xi) \text{ for } x \in \mathbb{R}^N$$

or, equivalently, by

$$(f \ast g)(x) = \int_{\mathbb{R}^N} f(y) \tau_x g(-y) \, dw(y) = \int_{\mathbb{R}^N} f(y) g(x, y) \, dw(y) \text{ for all } x \in \mathbb{R}^N,$$

where, here and subsequently, $g(x, y) = \tau_x g(-y)$.

The Dunkl Laplacian associated with $R$ and $k$ is the differential-difference operator $\Delta = \sum_{j=1}^{N} T_j^2$, which acts on $C^2(\mathbb{R}^N)$-functions by
\begin{equation}
\Delta f(x) = \Delta_{\text{eucl}} f(x) + \sum_{\alpha \in R} k(\alpha) \delta_\alpha f(x),
\end{equation}
\begin{equation}
\delta_\alpha f(x) = \frac{\partial f(x)}{\langle \alpha, x \rangle} - \frac{1}{2} \frac{\|\alpha\|^2 f(x) - f(\sigma(x))}{\langle \alpha, x \rangle^2}.
\end{equation}

Obviously, \( F(\Delta f)(\xi) = -\|\xi\|^2 F_f(\xi) \). The operator \( \Delta \) is essentially self-adjoint on \( L^2(dw) \) (see for instance [2, Theorem 3.1]) and generates the semigroup \( e^{t\Delta} \) of linear self-adjoint contractions on \( L^2(dw) \). The semigroup has the form
\begin{equation}
e^{t\Delta} f(x) = F^{-1}(e^{-t\|\xi\|^2} F_f(\xi))(x) = \int_{\mathbb{R}^N} h_t(x,y) f(y) dw(y),
\end{equation}
where the heat kernel
\begin{equation}h_t(x,y) = \tau_x h_t(-y), \quad h_t(x) = F^{-1}(e^{-t\|\xi\|^2})(x) = c_k^{-1} (2t)^{-N/2} e^{-\|\xi\|^2/(4t)},\end{equation}
is a \( C^\infty \)-function of all variables \( x, y \in \mathbb{R}^N, t > 0 \) and satisfies
\begin{equation}
0 < h_t(x,y) = h_t(y,x),
\end{equation}
\begin{equation}
\int_{\mathbb{R}^N} h_t(x,y) dw(y) = 1.
\end{equation}

Set
\[ V(x, y, t) = \max(w(B(x, t)), w(B(y, t))). \]

The following theorem was proved in [1, Theorem 4.1].

**Theorem 2.2.** There are constants \( C, c > 0 \) such that for all \( x, y \in \mathbb{R}^N \) and \( t > 0 \) we have
\begin{equation}h_t(x,y) \leq C V(x, y, \sqrt{t})^{-1} e^{-c d(x,y)^2/t}.
\end{equation}

3. Weighted Hilbert spaces and bilinear forms

3.1. Definition and properties of exponential weight functions. For any \( s > 0 \) and \( x \in \mathbb{R}^N \) let us define
\begin{equation}
\eta(x) = \exp(\sqrt{1 + \|x\|^2}), \quad \eta(x, s) = \exp(\sqrt{1 + s \|x\|^2}).
\end{equation}
Clearly,
\begin{equation}e^{s \|x\|} \leq \eta(x, s) \leq e^{s \|x\| + 1}.
\end{equation}

**Lemma 3.1.** For every \( \beta \in \mathbb{N}_0^N \) there is a constant \( C_\beta > 0 \) such that for all \( x \in \mathbb{R}^N \) and \( s > 0 \) we have
\begin{equation}|
\partial^\beta_x \eta(x, s)| \leq C_\beta s^{|
\beta|} \eta(x, s),\end{equation}
where, here and subsequently, \( \partial^\beta_x \) denotes the partial derivative with respect to the variable \( x \).

**Proof.** The proof is straightforward. \( \square \)
Lemma 3.2. Suppose that $\phi : \mathbb{R}^N \times (0, \infty) \to \mathbb{R}$ is a $C^\infty(\mathbb{R}^N)$-function such that for any $\beta \in \mathbb{N}_0^N$ there is $C_\beta > 0$ such that

$$|\partial_x^\beta \phi(x, s)| \leq C_\beta s^{\frac{2|\beta|}{2}} \eta(x, s)$$

for all $x \in \mathbb{R}^N$ and $s > 1/4$.

Then for every $\zeta \neq 0$ and every $\alpha \in \mathbb{R}$ the functions

$$s^{-1}T_\zeta \phi(x, s) \quad \text{and} \quad \psi_\alpha(x, s) = s^{-1} \frac{\phi(x, s) - \phi(\sigma_\alpha(x), s)}{\langle x, \alpha \rangle}$$

satisfy (3.4).

**Proof.** Thanks to Lemma 3.1 and (2.7) it is enough to check the claim for $\psi_\alpha$ for all $\alpha \in \mathbb{R}$. Note that

$$\frac{\phi(x, s) - \phi(\sigma_\alpha(x), s)}{s \langle x, \alpha \rangle} = -\langle x, \alpha \rangle^{-1} s^{-1} \int_0^1 \frac{d}{dt} \phi\left(x - 2t \frac{\langle x, \alpha \rangle}{\|\alpha\|^2}, s\right) dt$$

$$= c_\alpha s^{-1} \int_0^1 \left(\int (\nabla_x \phi)\left(x - 2t \frac{\langle x, \alpha \rangle}{\|\alpha\|^2}, s\right), \alpha \right) dt,$$

therefore

$$\partial^\beta \left\{ \frac{\phi(x, s) - \phi(\sigma_\alpha(x), s)}{s \langle x, \alpha \rangle} \right\} = c_\alpha s^{-1} \int_0^1 \left(\int \left(\nabla_x \phi\right)\left(x - 2t \frac{\langle x, \alpha \rangle}{\|\alpha\|^2}, s\right), \alpha \right) dt,$$

so the claim is a consequence of (3.4) for $\phi$. \hfill $\square$

Lemma 3.3. Suppose that $C^\infty \ni \phi : \mathbb{R}^N \times (0, \infty) \to \mathbb{R}$ satisfies (3.4). Then for every $\zeta \neq 0$ there are $\phi_\zeta, \psi_{\alpha, \zeta} \in C^\infty(\mathbb{R}^N \times (0, \infty))$, $\alpha \in \mathbb{R}$, which satisfy (3.4), such that for all $f \in C^1(\mathbb{R}^N)$, $x \in \mathbb{R}^N$, and $s > 1/4$ we have

$$T_\zeta(f(\cdot)\phi(\cdot, s))(x) = \phi(x, s)T_\zeta f(x) + f(x)s\phi_\zeta(x, s) + s \sum_{\alpha \in \mathbb{R}} f(\sigma_\alpha(x)) \phi_{\alpha, \zeta}(x, s).$$

**Proof.** By (2.7) we have

$$T_\zeta(f(\cdot)\phi(\cdot, s))(x) = \partial_{x, \zeta}(f(\phi(\cdot, s)))(x) + \sum_{\alpha \in \mathbb{R}} \frac{k(\alpha)}{2} \langle \alpha, \zeta \rangle f(x) \phi(x, s) - f(\sigma_\alpha(x)) \phi(\sigma_\alpha(x), s) \langle x, \alpha \rangle$$

$$= f(x)\partial_{x, \zeta} \phi(x, s) + \phi(x, s)\partial_{x, \zeta} f(x) + \sum_{\alpha \in \mathbb{R}} \frac{k(\alpha)}{2} \langle \alpha, \zeta \rangle f(x) \phi(\sigma_\alpha(x), s) - f(\sigma_\alpha(x)) \phi(\sigma_\alpha(x), s) \langle x, \alpha \rangle$$

$$+ \sum_{\alpha \in \mathbb{R}} \frac{k(\alpha)}{2} \langle \alpha, \zeta \rangle f(\sigma_\alpha(x)) \phi(x, s) - f(\sigma_\alpha(x)) \phi(\sigma_\alpha(x), s) \langle x, \alpha \rangle.$$  

Setting

$$\phi_\zeta(x, s) = s^{-1} \partial_{x, \zeta} \phi(x, s), \quad \phi_{\alpha, \zeta}(x, s) = s^{-1} \frac{k(\alpha)}{2} \langle \alpha, \zeta \rangle \frac{\phi(x, s) - \phi(\sigma_\alpha(x), s)}{\langle x, \alpha \rangle}$$

and using Lemma 3.2 we get the claim. \hfill $\square$
Lemma 3.4. Suppose that \( C^\infty \ni \phi : \mathbb{R}^N \times (0, \infty) \to \mathbb{R} \) is a function such that \( \phi(x, s) = \phi(x', s) \) for all \( \|x\| = \|x'\| \) and it satisfies (3.4). Then for every \( \zeta \neq 0 \) there is \( \phi_\zeta \in C^\infty(\mathbb{R}^N \times (0, \infty)) \) which satisfies (3.4) such that for all \( f \in C^1(\mathbb{R}^N), x \in \mathbb{R}^N, \) and \( s > 1/4 \) we have

\[
T_\zeta(f(\cdot)\phi(\cdot, s))(x) = \phi(x, s)T_\zeta f(x) + s\phi_\zeta(x, s)f(x).
\]

Proof. The claim follows directly by (2.8) and (3.3). \( \square \)

For \( \sigma \in G \) let \( f^\sigma(x) = f(\sigma(x)) \). It is easy to check that for all \( \zeta \neq 0 \) we have

\[
(3.7) \quad T_\zeta f^\sigma(x) = (T_\zeta f)(\sigma(x)) \text{ for all } x \in \mathbb{R}^N.
\]

Iteration of Lemma 3.3 together with (3.7) and Lemma 3.4 gives the following proposition.

Proposition 3.5. For every \( \beta \in \mathbb{N}_0^N \) there are functions \( \phi_{\beta, \beta', \sigma}(x, s) \) which satisfy (3.4) such that for all \( f \in C^\infty_c(\mathbb{R}^N), x \in \mathbb{R}^N, \) and \( s > 1/4 \) we have

\[
T^\beta(f(\cdot)\eta(\cdot, s))(x) = T^\beta f(x)\eta(x, s) + \sum_{\sigma \in G} \sum_{\beta' = \beta, |\beta'| < |\beta|} s^{\beta - |\beta'|} (T^{\beta'} f)(\sigma(x)) \phi_{\beta, \beta', \sigma}(x, s).
\]

3.2. Weighted Hilbert spaces. We define a family \( \{\mathcal{H}_s\}_{s > 0} \) of weighted \( L^2 \)-spaces by

\[
\mathcal{H}_s = \left\{ f \in L^2(dw) : \|f\|^2_{\mathcal{H}_s} := \int_{\mathbb{R}^N} |f(x)|^2 \eta(x, s) \, dw(x) < \infty \right\}.
\]

To unify our notation we write

\[
\mathcal{H}_0 = L^2(dw).
\]

Clearly, for \( s_1 \leq s_2 \) we have

\[
(3.9) \quad \mathcal{H}_{s_2} \subset \mathcal{H}_{s_1} \text{ and } \|f\|_{\mathcal{H}_{s_1}} \leq \|f\|_{\mathcal{H}_{s_2}}.
\]

Let us note that for all \( x \in \mathbb{R}^N \) and \( s > 0 \) we have

\[
(3.10) \quad \eta(x, 2s) \leq \eta^2(x, s) \leq e^2 \eta(x, 2s).
\]

Therefore,

\[
(3.11) \quad \|f\|^2_{\mathcal{H}_{2s}} \leq \int_{\mathbb{R}^N} |f(x)|^2 \eta^2(x, s) \, dw(x) \leq e^2 \|f\|^2_{\mathcal{H}_s}.
\]

Let us recall that \( \eta(x) = \eta(x, 1) \). The following corollary in a consequence of (3.8) and (3.11).

Corollary 3.6. For every \( \beta \in \mathbb{N}_0^N \) there is a constant \( C_\beta > 0 \) such that for every \( f \in C^\infty_c(\mathbb{R}^N) \) we have

\[
(3.12) \quad \|T^\beta f\|^2_{L^2(\eta^2 dw)} \leq C_\beta \|T^\beta f\eta\|^2_{L^2(dw)} + C_\beta \sum_{|\beta'| < |\beta|} \|T^{\beta'} f\|^2_{L^2(\eta^2 dw)}.
\]
\[(3.13) \quad \left\| T^\beta (f\eta) \right\|^2_{L^2(dw)} \leq C_\beta \left\| T^\beta f \right\|^2_{L^2(\eta^2 dw)} + C_\beta \sum_{|\beta'| < |\beta|} \left\| T^{\beta'} f \right\|^2_{L^2(\eta^2 dw)}. \]

**Proposition 3.7.** For every \( \delta > 0 \) and \( \ell_1 \in \mathbb{N} \) (in particular, for \( \ell_1 = \ell \)) there is a constant \( C_{\delta, \ell_1} > 0 \) such that for all \( f \in C_c^\infty (\mathbb{R}^N) \) we have

\[(3.14) \quad \sum_{|\beta| < \ell_1} \left\| T^\beta f \right\|^2_{L^2(\eta^2 dw)} \leq \delta \sum_{j=1}^m \left\| T^\beta_{\zeta_j} f \right\|^2_{L^2(\eta^2 dw)} + C_{\delta, \ell_1} \left\| f \right\|^2_{L^2(\eta^2 dw)}. \]

**Proof.** Thanks to (2.11), (2.13), (2.14), and the fact that \( \zeta_1, ..., \zeta_m \) span \( \mathbb{R}^N \), we get that for every \( \beta \in \mathbb{N}_0^N \) there is a constant \( C_\beta > 0 \) such that

\[(3.15) \quad \left\| T^\beta f \right\|^2_{L^2(dw)} \leq C_\beta \sum_{j=1}^m \left\| T^\beta_{\zeta_j} f \right\|^2_{L^2(dw)}. \]

Moreover, for every \( \ell_1 \in \mathbb{N}_0 \), \( \beta \in \mathbb{N}_0^N \) such that \( |\beta| < \ell_1 \), and every \( \delta > 0 \) there is a constant \( C_{\beta, \delta} > 0 \) such that

\[(3.16) \quad \left\| T^\beta f \right\|^2_{L^2(dw)} \leq \delta \sum_{j=1}^m \left\| T^\beta_{\zeta_j} f \right\|^2_{L^2(dw)} + C_{\beta, \delta} \left\| f \right\|^2_{L^2(dw)}. \]

The proof of (3.14) is by induction on \( \ell_1 \). Assume that (3.14) holds for \( \ell_1 \). Using (3.12) we have

\[(3.17) \quad \sum_{|\beta| < \ell_1} \left\| T^\beta f \right\|^2_{L^2(\eta^2 dw)} \leq C \sum_{|\beta| < \ell_1} \left\| T^\beta (f\eta) \right\|^2_{L^2(dw)} + C \sum_{|\beta'| < \ell_1} \left\| T^{\beta'} f \right\|^2_{L^2(\eta^2 dw)}. \]

Then, by (3.16) (for the first summand) and induction hypothesis (3.14) (for the second summand) for any \( \varepsilon > 0 \) we get

\[(3.18) \quad C \sum_{|\beta| < \ell_1} \left\| T^\beta (f\eta) \right\|^2_{L^2(dw)} + C \sum_{|\beta'| < \ell_1} \left\| T^{\beta'} f \right\|^2_{L^2(\eta^2 dw)} \]

\[\leq \varepsilon C \sum_{j=1}^m \left\| T^\beta_{\zeta_j} f \right\|^2_{L^2(dw)} + C' \left\| f \right\|^2_{L^2(dw)} + \varepsilon' \sum_{j=1}^m \left\| T^{\beta'}_{\zeta_j} f \right\|^2_{L^2(\eta^2 dw)} + C' \left\| f \right\|^2_{L^2(\eta^2 dw)}. \]

Finally, joining (3.17) and (3.18) and applying (3.13) we get

\[\sum_{|\beta| < \ell_1} \left\| T^\beta f \right\|^2_{L^2(\eta^2 dw)} \leq \varepsilon C_{\ell_1} \sum_{j=1}^m \left\| T^\beta_{\zeta_j} f \right\|^2_{L^2(\eta^2 dw)} + \varepsilon C_{\ell_1} \sum_{|\beta'| < \ell_1} \left\| T^{\beta'} f \right\|^2_{L^2(\eta^2 dw)} \]

\[+ \varepsilon C \sum_{j=1}^m \left\| T^\beta_{\zeta_j} f \right\|^2_{L^2(\eta^2 dw)} + C' \varepsilon C \left\| f \right\|^2_{L^2(\eta^2 dw)}. \]

The proof is finished by taking \( \varepsilon = \frac{1}{4} \min \{ \delta, 1 \} (C_{\ell_1} + C)^{-1} \). \( \square \)
Proposition 3.8. Let $\beta \in \mathbb{N}_0^N$. There is a constant $C_\beta > 0$ such that for all $f \in C_c^\infty(\mathbb{R}^N)$ we have

$$
(3.19) \quad \|T^\beta f\|^2_{L^2(\eta^2 dw)} \leq C_\beta \left( \sum_{j=1}^m \|T_{\xi_j}^{[\beta]} f\|^2_{L^2(\eta^2 dw)} + \|f\|^2_{L^2(\eta^2 dw)} \right).
$$

**Proof.** Thanks to (3.12) and Proposition 3.7 with $\delta = 1$ we get

$$
\|T^\beta f\|^2_{L^2(\eta^2 dw)} \leq C_\beta \|T^\beta (f\eta)\|^2_{L^2(\eta^2 dw)} + C_\beta \sum_{j=1}^m \|T_{\xi_j}^{[\beta]} f\|^2_{L^2(\eta^2 dw)} + C_\beta \|f\|^2_{L^2(\eta^2 dw)}.
$$

In order to estimate $\|T^\beta (f\eta)\|^2_{L^2(\eta^2 dw)}$, we use (3.15), then (3.13), which lead to

$$
\|T^\beta (f\eta)\|^2_{L^2(\eta^2 dw)} \leq C'_\beta \sum_{j=1}^m \|T_{\xi_j}^{[\beta]} (f\eta)\|^2_{L^2(\eta^2 dw)}
$$

$$
\leq C''_\beta \sum_{j=1}^m \|T_{\xi_j}^{[\beta]} f\|^2_{L^2(\eta^2 dw)} + C'''' \sum_{|\beta'|<|\beta|} \|T_{\xi_j}^{[\beta']} f\|^2_{L^2(\eta^2 dw)}.
$$

The claim follows by Proposition 3.7 with $\delta = 1$ applied to $\sum_{|\beta'|<|\beta|} \|T_{\xi_j}^{[\beta']} f\|^2_{L^2(\eta^2 dw)}$. \hfill \Box

**Corollary 3.9.** Let $n < \ell$ be a positive integer. For every $\delta > 0$ there is a constant $C = C_\delta > 0$ such that for all $f \in C_c^\infty(\mathbb{R}^N)$ and for all $s > 1/4$ we have

$$
(3.20) \quad s^{2(\ell-n)} \sum_{|\beta|=n} \|T^\beta f\|^2_{L^2(\eta^{2\cdot(s)} dw)} \leq \delta \sum_{j=1}^m \|T_{\xi_j}^{\ell} f\|^2_{L^2(\eta^{2\cdot(s)} dw)} + Cs^{2\ell} \|f\|^2_{L^2(\eta^{2\cdot(s)} dw)},
$$

$$
(3.21) \quad s^{2(\ell-n)} \sum_{|\beta|=n} \|T^\beta f\|^2_{\mathcal{H}_s} \leq \delta \sum_{j=1}^m \|T_{\xi_j}^{\ell} f\|^2_{\mathcal{H}_s} + Cs^{2\ell} \|f\|^2_{\mathcal{H}_s}.
$$

**Proof.** Let us apply (3.14) to $f_{\{s\}}(x) = \frac{1}{\pi N^s} f(x/s)$. Then (3.20) follows from the fact that

$$
\|T^\beta f_{\{s\}}\|^2_{L^2(\eta^{2\cdot(s)} dw)} = s^{-2|\beta|} \|T^\beta f\|^2_{L^2(\eta^{2\cdot(s)} dw)}.
$$

Finally, (3.21) is a consequence of (3.20) and (3.11). \hfill \Box

**Corollary 3.10.** There is a constant $C > 0$ such that for all $f \in C_c^\infty(\mathbb{R}^N)$ and $s > 1/4$ we have

$$
(3.22) \quad \sum_{|\beta|=\ell} \|T^\beta f\|^2_{\mathcal{H}_s} \leq C \sum_{j=1}^m \|T_{\xi_j}^{\ell} f\|^2_{\mathcal{H}_s} + Cs^{2\ell} \|f\|^2_{\mathcal{H}_s}.
$$

**Proof.** The proof is the same as the proof of Corollary 3.9, but instead of (3.14) we use (3.19). \hfill \Box
3.3. Weighted Sobolev spaces. For \( s > 0 \) we define the weighted Sobolev space \( V_{\ell,s} \) as the completion of \( C^\infty_c(\mathbb{R}^N) \)-functions in the norm

\[
\|f\|_{V_{\ell,s}}^2 = \|f\|_{\mathcal{H}_s}^2 + \sum_{j=1}^m \|\zeta_j f\|_{\mathcal{H}_s}^2.
\]

Clearly, \( V_{\ell,s} \subset \mathcal{H}_s \). Moreover, \( V_{\ell,s} \) is a dense subspace of \( \mathcal{H}_s \).

**Proposition 3.11.** Assume that \( f \in \mathcal{H}_s \). Then the following statements are equivalent:

(a) \( f \in V_{\ell,s} \);

(b) for any \( \beta \in \mathbb{N}_0^N \) such that \( |\beta| \leq \ell \) there is a function \( f_{\beta,s} \in \mathcal{H}_s \) such that for every \( \varphi \in C^\infty_c(\mathbb{R}^N) \) we have

\[
(-1)^{|\beta|} \int_{\mathbb{R}^N} f(x) T_\beta \varphi(x) \, dw(x) = \int_{\mathbb{R}^N} f_{\beta,s}(x) \varphi(x) \, dw(x).
\]

**Proof.** See Appendix A. \(\square\)

**Remark 3.12.** If \( 0 < s_1 < s_2 \) and \( f \in V_{\ell,s_2} \), then \( f \in V_{\ell,s_1} \) and the functions \( f_{\beta,s_1} \) and \( f_{\beta,s_2} \) from Proposition 3.11 coincide. They will be denoted by \( T^\beta f \).

3.4. Bilinear forms.

**Definition 3.13.** For \( s > 1/4 \) we define the bilinear form \( a_s(\cdot, \cdot) \) with the domain \( V_{\ell,s} \) by

\[
a_s(f, g) = -\sum_{j=1}^m \int_{\mathbb{R}^N} T_{\zeta_j}^\ell f(x) T_{\zeta_j}^\ell \{ \overline{g}(x) \eta(x, s) \} \, dw(x).
\]

**Proposition 3.14.** The form \( a_s(f, g) \) is bounded on \( V_{\ell,s} \). More precisely, there is a constant \( C > 0 \) such that for every \( s > 1/4 \) and every \( f, g \in V_{\ell,s} \) we have

\[
|a_s(f, g)| \leq C \left( s^2 \|f\|_{\mathcal{H}_s}^2 + \sum_{j=1}^m \|T_{\zeta_j}^\ell f\|_{\mathcal{H}_s}^2 \right)^{1/2} \left( s^2 \|g\|_{\mathcal{H}_s}^2 + \sum_{j=1}^m \|T_{\zeta_j}^\ell g\|_{\mathcal{H}_s}^2 \right)^{1/2}.
\]

**Proof.** By Proposition 3.5 there are functions \( \phi_{j,1/\sigma}(x, s) \), \( \beta' \in \mathbb{N}_0^N \) and \( \sigma \in G \), such that

\[
|\phi_{j,1/\sigma}(x, s)| \leq C_{j,1/\sigma} \eta(x, s) \text{ for all } x \in \mathbb{R}^N,
\]

and

\[
|a_s(f, g)| \leq \left| \sum_{j=1}^m \int_{\mathbb{R}^N} T_{\zeta_j}^\ell f(x) T_{\zeta_j}^\ell \overline{g}(x) \eta(x, s) \, dw(x) \right|
\]

\[
+ \left| \sum_{j=1}^m \sum_{\sigma \in G} \sum_{|\beta'| < \ell} \int_{\mathbb{R}^N} T_{\zeta_j}^\ell f(x) s^{\ell - |\beta'|}(T_{\beta'}^\sigma \overline{g} \eta(x, s) \phi_{j,1/\sigma}(x, s) \, dw(x) \right|.
\]
Hence, using the Cauchy-Schwarz inequality we obtain
\[ |a_s(f, g)| \leq \sum_{j=1}^{m} ||T_{\zeta_j}\ell f||_{H_s} ||T_{\zeta_j}\ell g||_{H_s} + C \sum_{j=1}^{m} \sum_{|\beta|<\ell} ||T_{\zeta_j}\ell f||_{H_s} s^{-|\beta|} ||T^{\beta}\ell g||_{H_s}, \]
Now, applying (3.21) we get
(3.25)
\[ |a_s(f, g)| \leq \sum_{j=1}^{m} ||T_{\zeta_j}\ell f||_{H_s} ||T_{\zeta_j}\ell g||_{H_s} + C \left( \sum_{j=1}^{m} ||T_{\zeta_j}\ell f||_{H_s} \right) \left( \sum_{j=1}^{m} ||T_{\zeta_j}\ell g||_{H_s} + s^{\ell} ||g||_{H_s} \right). \]
The proposition is a direct consequence of (3.25).

**Proposition 3.15** (Gårding inequality). There are constants $\alpha, C_\alpha > 0$ such that for all $s > 1/4$ and $f \in V_{\ell,s}$ we have
\[ -\text{Re} a_s(f, f) + C_\alpha s^{2\ell} ||f||_{H_s}^2 \geq \alpha ||f||_{V_{\ell,s}}^2. \]

**Proof.** Similarly to the proof of Proposition 3.14, applying Proposition 3.5, we have
(3.26)
\[ -\text{Re} a_s(f, f) \geq \sum_{j=1}^{m} \int_{\mathbb{R}^N} T_{\zeta_j}\ell f(x) T_{\zeta_j}\ell f(x) \eta(x, s) dw(x) \]
\[ - \left| \sum_{j=1}^{m} \sum_{\sigma \in G} \sum_{|\beta|<\ell} \int_{\mathbb{R}^N} T_{\zeta_j}\ell f(x) s^{-|\beta|} (T^{\beta}\ell) \eta(x, s) dw(x) \right| \]
\[ \geq \sum_{j=1}^{m} ||T_{\zeta_j}\ell f||_{H_s}^2 - C \sum_{j=1}^{m} \sum_{|\beta|<\ell} ||T_{\zeta_j}\ell f||_{H_s} s^{-|\beta|} ||T^{\beta}\ell f||_{H_s}. \]
Using (3.21) and the Cauchy-Schwarz inequality for any $\delta > 0$ there is a constant $C_\delta > 0$ such that for any $\varepsilon > 0$ we have
(3.27)
\[ C \sum_{j=1}^{m} \sum_{|\beta|<\ell} ||T_{\zeta_j}\ell f||_{H_s} s^{-|\beta|} ||T^{\beta}\ell f||_{H_s} \]
\[ \leq C \left( \sum_{j=1}^{m} ||T_{\zeta_j}\ell f||_{H_s} \right) \left( \delta \sum_{j=1}^{m} ||T_{\zeta_j}\ell f||_{H_s} + C_\delta s^{\ell} ||f||_{H_s} \right) \]
\[ \leq C \delta m \sum_{j=1}^{m} ||T_{\zeta_j}\ell f||_{H_s}^2 + CC'_\delta \left( \sum_{j=1}^{m} ||T_{\zeta_j}\ell f||_{H_s} s^{\ell} ||f||_{H_s} \right) \]
\[ \leq C \delta m \sum_{j=1}^{m} ||T_{\zeta_j}\ell f||_{H_s}^2 + CC'_\delta \left( \sum_{j=1}^{m} \varepsilon ||T_{\zeta_j}\ell f||_{H_s}^2 + \frac{ms^{2\ell}}{4\varepsilon} ||f||_{H_s}^2 \right). \]
Taking $\delta, \varepsilon > 0$ small enough such that $C_\delta m + CC'_\delta \varepsilon < 1/2$ we conclude the proposition from (3.26) and (3.27).
### 4. Perturbations of the bilinear form

For \( \varepsilon \geq 0 \) and \( s > 1/4 \) we consider the following bilinear form

\[
b_{s,\varepsilon}(f, g) = a_s(f, g) + \varepsilon \sum_{j=1}^{N} \int_{\mathbb{R}^N} T_j f(x) T_j \{ \overline{g}(\cdot) \eta(\cdot, s) \}(x) \, dw(x)
\]

with the domain \( V_{\ell,s} \). Let us note that \( b_{s,0}(f, g) = a_s(f, g) \).

**Proposition 4.1.** For every \( \varepsilon \geq 0 \) and \( s > 1/4 \) the form \( b_{s,\varepsilon} \) is bounded on \( V_{\ell,s} \).

**Proof.** Thanks to (3.21) and (3.22) there is a constant \( C > 0 \) such for all \( s > 1/4 \) we have

\[
\sum_{j=1}^{N} \| T_j f \|_{H_s}^2 \leq C(\| f \|_{V_{\ell,s}}^2 + s^{2\ell} \| f \|_{H_s}^2).
\]

Hence, using Lemma 3.4 and then either (3.21) or (3.22), we obtain

\[
\left| \sum_{j=1}^{N} \int_{\mathbb{R}^N} T_j f(x) T_j \{ \overline{g}(\cdot) \eta(\cdot, s) \}(x) \, dw(x) \right|
\]

\[
\leq \sum_{j=1}^{N} \| T_j f \|_{H_s} \| T_j g \|_{H_s} + C \sum_{j=1}^{N} \| T_j f \|_{H_s} s \| g \|_{H_s}
\]

\[
\leq C\left(\| f \|_{V_{\ell,s}}^2 + s^{2\ell} \| f \|_{H_s}^2\right)^{1/2} \left(\| g \|_{V_{\ell,s}}^2 + s^{2\ell} \| g \|_{H_s}^2\right)^{1/2}.
\]

Now Proposition 4.1 follows from (4.1) and Proposition 3.14. \( \square \)

**Proposition 4.2** (Gårding inequality for the perturbed bilinear form). There are \( \varepsilon_0 > 0 \) and \( \alpha, C_\alpha > 0 \) such that for all \( 0 \leq \varepsilon \leq \varepsilon_0 \), \( f \in V_{\ell,s} \), and every \( s > 1/4 \) we have

\[-\text{Re} b_{s,\varepsilon}(f, f) + C_\alpha s^{2\ell} \| f \|_{H_s}^2 \geq \alpha \| f \|_{V_{\ell,s}}^2.\]

**Proof.** It suffices to take \( \varepsilon_0 > 0 \) small enough and apply Proposition 3.15 together with (4.1). \( \square \)

The number \( \varepsilon_0 \) from Proposition 4.2 will be fixed throughout the remaining part of the paper.

### 5. Semigroups of operators and Lions theorem

**5.1. Lions theorem.** The following theorem is essentially due to J.-L. Lions [13]. Its proof, which includes holomorphy of the semigroup under consideration, and which is a combination of a number of propositions from [13] and [14], can be found in [9, Proposition (1.1)].
Theorem 5.1. Let $\mathcal{H}$ be a Hilbert space and $V$ be a dense subspace of $\mathcal{H}$ such that $V$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_V$ and the norm $\| \cdot \|_V$, and for some constant $c > 0$ we have $\| f \|_\mathcal{H} \leq c \| f \|_V$ for all $f \in V$. Let $b(\cdot, \cdot)$ be a bounded bilinear form on $V$. It defines an operator $A : D(A) \mapsto \mathcal{H}$ as follows

$$D(A) = \{ f \in V : |b(f, g)| \leq C_\ell \| f \|_\mathcal{H} \text{ for all } g \in V \}, \quad \langle Af, g \rangle_\mathcal{H} = b(f, g).$$

Suppose that for some $\alpha > 0$ and $\lambda_0 \in \mathbb{R}$ we have

$$(5.1) \quad \alpha \| f \|^2_V \leq -\Re b(f, f) + \lambda_0 \| f \|^2_\mathcal{H} \quad \text{for all } f \in V.$$

Then $A$ is the infinitesimal generator of a strongly continuous semigroup $\{ T_t \}_{t \geq 0}$ of operators on $\mathcal{H}$ which is holomorphic in a sector

$$S_\kappa = \{ z \in \mathbb{C} : |\arg z| < \kappa \}$$

for some $\kappa > 0$. Moreover,

$$(5.2) \quad \| T_t f \|_\mathcal{H} \leq \exp(\lambda_0 t) \| f \|_\mathcal{H} \quad \text{for all } t \geq 0 \text{ and } f \in \mathcal{H}.$$

5.2. Semigroup $\{ S_t \}_{t \geq 0}$ of operators on $L^2(dw)$. For $\varepsilon \in \{ 0, \varepsilon_0 \}$ let us define the symmetric bilinear form

$$b_{0, \varepsilon}(f, g) = -\sum_{j=1}^m \int_{\mathbb{R}^N} T_{\xi_j}^\ell f(x) T_{\xi_j}^\ell g(x) \, dw(x) + \varepsilon \sum_{j=1}^N \int_{\mathbb{R}^N} T_j f(x) T_j g(x) \, dw(x)$$

with the domain $V_{\varepsilon, 0} = \{ f \in L^2(dw) : (1 + \| \xi \|)^\ell \mathcal{F} f(\xi) \in L^2(dw) \}$ and the norm

$$\| f \|^2_{V_{\varepsilon, 0}} = \sum_{j=1}^m \| T_{\xi_j}^\ell f \|_{\mathcal{H}_0}^2 + \| f \|^2_{\mathcal{H}_0}.$$

The form can be written by means of the Dunkl transform as

$$b_{0, \varepsilon}(f, g) = \int_{\mathbb{R}^N} \mathcal{F} f(\xi) \mathcal{F} g(\xi) \left( -\sum_{j=1}^m \langle \xi_j, \xi \rangle^{2\ell} + \varepsilon \| \xi \|^2 \right) \, dw(\xi).$$

Proposition 5.2. Let $\varepsilon \in \{ 0, \varepsilon_0 \}$. The form $b_{0, \varepsilon}$ is bounded on $V_{\varepsilon, 0}$. Moreover, it satisfies the following Gårding inequality: there are $\lambda_0, \alpha > 0$ such that

$$(5.3) \quad \alpha \| f \|^2_{V_{\varepsilon, 0}} \leq -\Re b_{0, \varepsilon}(f, f) + \lambda_0 \| f \|^2_{\mathcal{H}_0} \quad \text{for all } f \in V_{\varepsilon, 0}.$$
Proof. By the Cauchy–Schwarz inequality and (2.11) we have
\[
|b_{0,\varepsilon}(f,g)| \leq \sum_{j=1}^{m} \|T_{\zeta_j}^\ell f\|_{L^2(dw)} \|T_{\zeta_j} g\|_{L^2(dw)} + \varepsilon \sum_{j=1}^{N} \|T_j f\|_{L^2(dw)} \|T_j g\|_{L^2(dw)}
\]
\[
\leq C \sum_{j=1}^{m} \|\xi^\ell \mathcal{F}f(\xi)\|_{L^2(\mathbb{R}^N, dw(\xi))} \|\xi^\ell \mathcal{F}g(\xi)\|_{L^2(\mathbb{R}^N, dw(\xi))}
\]
\[
+ \varepsilon C \sum_{j=1}^{N} \|\xi \mathcal{F}f(\xi)\|_{L^2(\mathbb{R}^N, dw(\xi))} \|\xi \mathcal{F}g(\xi)\|_{L^2(\mathbb{R}^N, dw(\xi))}
\]
\[
\leq C\|1 + |\xi|^\ell \mathcal{F}f(\xi)\|_{L^2(\mathbb{R}^N, dw(\xi))} \|1 + |\xi|^\ell \mathcal{F}g(\xi)\|_{L^2(\mathbb{R}^N, dw(\xi))},
\]
which implies that the form $b_{0,\varepsilon}$ is bounded on $V_{\ell,0}$. The Gårding inequality can be verified in the same way. □

As the consequence of the boundedness of $b_{0,\varepsilon}$, we conclude that it defines a self-adjoint linear operator $A^{(\varepsilon)}$, which, thanks to Theorem 5.1 and the Gårding inequality (5.3), generates a strongly continuous semigroup $\{S_t^{(\varepsilon)}\}_{t \geq 0}$ of bounded self-adjoint linear operators on $H_0 = L^2(dw)$, which has the form
\[
S_t^{(\varepsilon)} f(x) = f \ast q_t^{(\varepsilon)}(x),
\]
where
\[
q_t^{(\varepsilon)}(x) = \mathcal{F}^{-1}\left(\exp\left(-t\left(\sum_{j=1}^{m} \langle \zeta_j, \cdot \rangle^{2\ell} - \varepsilon \| \cdot \|^{2}\right)\right)\right)(x).
\]

Let us also remark (see Proposition 5.5) that the operator $A^{(\varepsilon)}$ is the closure in the space $H_0$ of
\[
L^{(\varepsilon)} = \sum_{j=1}^{m} T_{\zeta_j}^{2\ell} - \varepsilon \Delta,
\]
initially defined on $C_c^\infty(\mathbb{R}^N)$ (for the proof see Appendix C with $s = 0$).

5.3. Semigroups on weighted Hilbert spaces. We are in a position to apply Theorem 5.3 to the weighted bilinear forms $b_{s,\varepsilon}$, where $\varepsilon \in \{0, \varepsilon_0\}$ and $s > 1/4$. Let us remind that the forms $b_{s,\varepsilon}$ are bounded (see Propositions 3.14 and 4.1). Let $A_s^{(\varepsilon)}$ be the operator associated with the form $b_{s,\varepsilon}$ with its domain $D(A_s^{(\varepsilon)}) \subset V_{\ell,s} \subset H_s$. The following theorem is a direct consequence of Propositions 4.1, 4.2, and Theorem 5.1.

Theorem 5.3. Let $\varepsilon \in \{0, \varepsilon_0\}$. There are constants $c_0, \kappa > 0$ such that for all $s > 1/4$ the operator $A_s^{(\varepsilon)}$ is the infinitesimal generator of a strongly continuous semigroup $\{S_t^{(\varepsilon,s)}\}_{t \geq 0}$ of operators on $H_s$ which is holomorphic in a sector
\[
\{ z \in \mathbb{C} : |\text{Arg} z| < \kappa \},
\]
which for all \( f \in \mathcal{H}_s \) satisfies
\[
\|S_t^{(\varepsilon,s)}f\|_{\mathcal{H}_s} \leq \exp(c_0 s^{2t})\|f\|_{\mathcal{H}_s}
\]
for all \( t \geq 0 \) and for all \( f \in \mathcal{H}_s \).

Clearly, \( \mathcal{H}_{s_1} \subset \mathcal{H}_{s_2} \subset \mathcal{H}_0 \) and \( V_{t,s_1} \subset V_{t,s_2} \subset V_{t,0} \) for \( s_1 \geq s_2 \geq 0 \). The next theorem asserts that the semigroups \( \{S_t^{(\varepsilon,s)}\}_{t \geq 0} \) can be thought as the semigroup \( \{S_t\}_{t \geq 0} \) acting on \( \mathcal{H}_s \).

**Theorem 5.4.** Let \( \varepsilon \in \{0,\varepsilon_0\} \). For all \( s > 1/4 \) and \( f \in \mathcal{H}_s \subset L^2(dw) \) we have
\[
S_t^{(\varepsilon,s)}f = S_t f = f \ast q_t^{(\varepsilon)} \quad \text{for all } t \geq 0.
\]

**Proof.** See Appendix B. \( \square \)

**Proposition 5.5.** Let \( \varepsilon \in \{0,\varepsilon_0\} \). For \( s > 1/4 \) let \( \lambda > c_0 s^{2t} \), where \( c_0 > 0 \) is the constant from (5.6). Then for every \( n \in \mathbb{N} \) the space \( C_c^\infty(\mathbb{R}^N) \) is a core for \( (\lambda I - A_s^{(\varepsilon)})^n \).

**Proof.** See Appendix C. \( \square \)

6. **Pointwise estimates for integral kernel of \( S_t \)**

We define the sequence \( \{d(n)\}_{n \in \mathbb{N}} \) inductively by
\[
\begin{cases}
  d(1) = 2, \\
  d(n + 1) = 2d(n) + 2 \quad \text{for } n \geq 2.
\end{cases}
\]

**Lemma 6.1.** For every \( \beta \in \mathbb{N}_0^N \) there is a constant \( C > 0 \) such that for every \( s > 1/4 \) and every \( f \in C_c^\infty(\mathbb{R}^N) \) we have
\[
\|T^\beta f\|^2_{\mathcal{H}_s} \leq C \left( s^{d(|\beta|)}\|f\|^2_{\mathcal{H}_{2|\beta|}} + \sum_{|\beta'| \leq |\beta| + 1} \|T^{\beta'} f\|^2_{L^2(dw)} \right).
\]

**Proof.** The proof goes by induction on \( |\beta| \). First, let us note that \( \eta(x,s) = \eta(x',s) \) for \( \|x\| = \|x'\| \), so for any function \( f \in C_c^\infty(\mathbb{R}^N) \) integration by parts (see (2.8)) gives
\[
\int_{\mathbb{R}^N} T_j f(x)T_j \overline{f}(x) \eta(x,s) \, dw(x)
\]
\[
= -\int_{\mathbb{R}^N} f(x)T_j \overline{f}(x) \eta(x,s) \, dw(x) - \int_{\mathbb{R}^N} f(x)T_j \overline{f}(x)(\partial_{x,j} \eta)(x,s) \, dw(x).
\]

The claim for \( |\beta| = 1 \) follows from (6.2), the Cauchy–Schwarz inequality, and Lemma 3.1, because
\[
\left| \int_{\mathbb{R}^N} f(x)T_j \overline{f}(x) \eta(x,s) \, dw(x) \right| \leq \|T_j^2 f\|^2_{L^2(dw)} + C\|f\|^2_{\mathcal{H}_{2s}},
\]
\[
\left| \int_{\mathbb{R}^N} f(x)T_j \overline{f}(x) \partial_{x,j} \eta(x,s) \, dw(x) \right| \leq \|T_j f\|^2_{L^2(dw)} + C s^2 \|f\|^2_{\mathcal{H}_{2s}}.
\]
Assume that (6.1) is satisfied for $\beta \in \mathbb{N}_0^N$ such that $|\beta| = n$. Consider multi-index $\beta + \epsilon_j$, where $|\beta| = n$. Then by (6.1) with $f$ replaced by $T_j f$ we get

$$
\|T^{\beta + \epsilon_j} f\|_{\mathcal{H}_s}^2 \leq C s^{d(|\beta|)} \|T_j f\|_{\mathcal{H}_{2|\beta|+s}}^2 + C \sum_{|\beta'| \leq |\beta| + 1} \|T^{\beta'} T_j f\|_{L^2(\mathbb{R}^N)}^2
$$

(6.3)

Using again the Cauchy-Schwarz inequality together with Lemma 3.1, we obtain

$$
\int_{\mathbb{R}^N} f(x) T_j^2 \overline{f}(x) \frac{\partial \eta(x, 2^{|\beta|} s)}{\partial x} \, dw(x) \leq \|T_j^2 f\|_{L^2(\mathbb{R}^N)}^2 + C s^{2d(|\beta|)} \|f\|_{\mathcal{H}_{2|\beta|+1_s}}^2.
$$

and

$$
\int_{\mathbb{R}^N} f(x) T_j^2 \overline{f}(x) \frac{\partial \eta(x, 2^{|\beta|} s)}{\partial x} \, dw(x) \leq \|T_j^2 f\|_{L^2(\mathbb{R}^N)}^2 + C s^{2d(|\beta|)+2} \|f\|_{\mathcal{H}_{2|\beta|+1_s}}^2.
$$

Hence, repeating the calculation presented in (6.2) we get

$$
\|T^\beta f\|_{L^2(\mathbb{R}^N)} \leq C', s^{2d(|\beta|)+2} \|f\|_{\mathcal{H}_{2|\beta|+1_s}}^2 + \|T_j f\|_{L^2(\mathbb{R}^N)} + \|T_j f\|_{L^2(\mathbb{R}^N)}.
$$

Now (6.4) together with (6.3) completes the proof. $\square$

**Lemma 6.2.** Let $\varepsilon \in \{0, \varepsilon_0\}$ and $\beta \in \mathbb{N}_0^N$. There are constants $C, \lambda_0 > 0$, and $M \in \mathbb{N}$ such that for all $\lambda > \lambda_0$ and $f \in C_0^\infty(\mathbb{R}^N)$, we have

$$
\|T^\beta f\|_{L^2(\mathbb{R}^N)} \leq C \|\lambda I - A^{(\varepsilon)} f\|_{L^2(\mathbb{R}^N)}.
$$

*Proof.* Let us recall that $(\lambda I - A^{(\varepsilon)}) M f = (\lambda I - L^{(\varepsilon)}) M f$ for $f \in C_0^\infty(\mathbb{R}^N)$. The lemma is a consequence of (2.11) and (2.13). $\square$

Combination of Lemma 6.1 and Lemma 6.2 leads to the following corollary.

**Corollary 6.3.** Let $\varepsilon \in \{0, \varepsilon_0\}$ and $\beta \in \mathbb{N}_0^N$. There are constants $C, \lambda_0 > 0$, and $M = M_\beta \in \mathbb{N}$ such that for all $\lambda \geq \lambda_0$, and $s > 1/4$, we have

$$
\|T^\beta f\|_{\mathcal{H}_s}^2 \leq C \|\lambda I - A^{(\varepsilon)} M f\|_{L^2(\mathbb{R}^N)}^2 + C s^{d(|\beta|)} \|f\|_{\mathcal{H}_{2|\beta|+1_s}}^2.
$$

**Lemma 6.4.** Let $\varepsilon \in \{0, \varepsilon_0\}$ and $\beta \in \mathbb{N}_0^N$. There are constants $C, \lambda_0 > 0$, and $M = M_\beta > 0$ such that for all $s > 1/4$, $\lambda \geq \lambda_0$, and $f \in C_0^\infty(\mathbb{R}^N)$ we have

$$
\|T^\beta (f(\cdot) \eta(\cdot, s))\|_{L^2(\mathbb{R}^N)} \leq C s^{2|\beta|} \|\lambda I - A^{(\varepsilon)} M f\|_{L^2(\mathbb{R}^N)}^2 + C s^{2|\beta|+d(|\beta|)} \|f\|_{\mathcal{H}_{2|\beta|+1_s}}^2.
$$

*Proof.* By Proposition 3 we get

$$
\|T^\beta (f(\cdot) \eta(\cdot, s))\|_{L^2(\mathbb{R}^N)} \leq C \sum_{|\beta'| \leq |\beta|} s^{2|\beta'|} \|T^{\beta'} f\|_{\mathcal{H}_{2s}}^2.
$$

Then, applying Corollary 6.3 to each term of the sum we obtain the claim. $\square$

**Lemma 6.5.** Let $\varepsilon \in \{0, \varepsilon_0\}$ and $\beta \in \mathbb{N}_0^N$. There are constants $C, c > 0$ such that for all $s > 1/4$, $f \in \mathcal{H}_{2|\beta|+1_s}$, and $1/2 < t < 2$ we have

$$
\|T^\beta (S_t^\varepsilon f(\cdot)) \eta(\cdot, s))\|_{L^2(\mathbb{R}^N)} \leq C \exp(cs^t) \|f\|_{\mathcal{H}_{2|\beta|+1_s}}.
$$
Proof. Let $M = M_\beta$ be as in Lemma 6.4 and let $c_0$ be the constant from (5.6). We claim that Lemma 6.4 is satisfied if $f \in D((A_{s_1}^{(e)})^M)$, where $s_1 = 2|\beta|+1s$, and $\lambda > \max(\lambda_0, c_0 s_1^2)$. Indeed, since $C_c^\infty(\mathbb{R}^N)$ is a core for $(\lambda I - A_{s_1}^{(e)})^M$ (see Proposition 5.5) and $D((A_{s_1}^{(e)})^M) \subset D((\lambda I - A_{s_1}^{(e)})^M)$, there are $f_n \in C_c^\infty(\mathbb{R}^N)$ such that
\[
\lim_{n \to \infty} \|f_n - f\|_{\mathcal{H}_{s_1}} + \|\lambda I - A_{s_1}^{(e)}f_n - (\lambda I - A_{s_1}^{(e)})^M f\|_{\mathcal{H}_{s_1}} = 0.
\]

Consequently, by (3.9), (3.11), and Corollary 7.3 in Appendix B we have
\[
\lim_{n \to \infty} \|f_n \eta(\cdot, t) - f \eta(\cdot, t)\|_{L^2(dw)} + \|\lambda I - A^{(e)}f_n - (\lambda I - A^{(e)})^M f\|_{L^2(dw)} = 0.
\]

Now the claim follows, because $T^\beta$ is closed on $L^2(dw)$.

Set $\lambda = \max(\lambda_0, 2c_0 s_1^2)$. If $f \in \mathcal{H}_{s_1}$, then $S_t^{(e)}f \in D((A_{s_1}^{(e)})^M)$, because $\{S_t^{(e)}\}_{t \geq 0}$ is analytic. Hence, by Lemma 6.4, we get
\[
\|T^\beta((S_t^{(e)}f)(\cdot)\eta(\cdot, s))\|_{L^2(dw)}^2 \leq C_{\beta,M} s_1^{2|\beta|}\|\lambda I - A_{s_1}^{(e)}S_t^{(e)}f\|_{L^2(dw)}^2 + C s_1^{2|\beta|+d(||\beta||)}\|S_t^{(e)}f\|_{\mathcal{H}_{s_1}}^2.
\]

By Proposition 5.2 and Theorem 5.1 we have that $A^{(e)}$ is the generator of the semigroup $\{S_t^{(e)}\}_{t \geq 0}$ of self-adjoint linear operators on $L^2(dw)$. Therefore, since $1/2 < t < 2$, by the spectral theorem (or Cauchy integral formula) we obtain
\[
s_1^{2|\beta|}\|\lambda I - A_{s_1}^{(e)}S_t^{(e)}f\|_{L^2(dw)}^2 = s_1^{2|\beta|}\left|\left(\lambda I - \frac{d}{dt}\right)^M S_t^{(e)}f\right|_{L^2(dw)}^2 \leq C s_1^{2|\beta|}\lambda^M\|f\|_{L^2(dw)}^2 \leq C \exp(cs_1^{2\beta})\|f\|_{\mathcal{H}_{s_1}}^2.
\]

Moreover, by Theorem 5.3 and the fact that $1/2 < t < 2$ we have
\[
s_1^{2|\beta|+d(||\beta||)}\|S_t^{(e)}f\|_{\mathcal{H}_{s_1}}^2 \leq C s_1^{2|\beta|+d(||\beta||)}\exp(cs_1^{2\beta})\|f\|_{\mathcal{H}_{s_1}}^2 \leq C' \exp(cs_1^{2\beta})\|f\|_{\mathcal{H}_{s_1}}^2,
\]

which completes the proof. \qed

6.1. Pointwise estimate for convolution kernels of semigroups.

**Corollary 6.6.** Let $\varepsilon \in \{0, \varepsilon_0\}$. There are constants $C, c > 0$ and $M \in \mathbb{N}$ such that for all $s > 1/4$, $f \in \mathcal{H}_{2M+1+s}$, $x \in \mathbb{R}^N$, and $1/2 < t < 2$ we have
\[
|S_t^{(e)}f(x)| \leq C \exp(-s\|x\|) \exp(cs_1^{2\beta})\|f\|_{\mathcal{H}_{2M+1+s}}.
\]

**Proof.** By (2.12), the Cauchy-Schwarz inequality, and (2.11), for $M \in \mathbb{N}$ such that $M > N/2$ and for any function $g \in D(\Delta^M)$, we have
\[
|g(x)| = c_{k-1}^{-1} \left| \int_{\mathbb{R}^N} E(i\xi, x) F g(\xi) dw(\xi) \right|
\]
\[
= c_{k-1}^{-1} \left| \int_{\mathbb{R}^N} (1 + \|\xi\|^2)^{-M}(1 + \|\xi\|^2)^M E(i\xi, x) F g(\xi) dw(\xi) \right|
\]
\[
\leq C_M \|(1 + \|\xi\|^2)^M F g\|_{L^2(dw)}
\]
\[
= C_M'\|(I - \Delta)^M g\|_{L^2(dw)}.
\]
Therefore, if for \( f \in \mathcal{H}_{2M+1} \) we plug \( g(x) = \eta(x, s) S_i^{(s)} f(x) \) in (6.7) and use Lemma 6.5, we obtain the claim, because \( \exp(s \|x\|) \leq \eta(x, s) \) for all \( x \in \mathbb{R}^N \) and \( s > 1/4 \).

**Lemma 6.7.** There is a constant \( C > 0 \) such that for all \( x, \xi \in \mathbb{R}^N \) we have

\[
|E(i\xi, x) - 1| \leq C\|x\|\|\xi\|. \tag{6.8}
\]

**Proof.** For all \( x, \xi \in \mathbb{R}^N \) we have

\[
E(i\xi, x) - 1 = E(\xi, ix) - E(\xi, 0) = \int_0^1 \frac{d}{dt} E(\xi, itx) \, dt = i \int_0^1 \langle \nabla_x E(\xi, itx), x \rangle \, dt.
\]

Therefore, by Cauchy-Schwarz inequality and Lemma 2.1 we get

\[
|E(i\xi, x) - 1| \leq C \int_0^1 \| \nabla_x E(\xi, itx) \| \|x\| \, dt \leq C\|x\|\|\xi\|. \tag*{\Box}
\]

Recall that the kernel \( q^{(s)}_t(x) \) is given by (5.4). Our goal is to obtain pointwise estimates of \( q^{(s)}_t \) for \( t = 1 \).

**Lemma 6.8.** Let \( \varepsilon \in \{0, \varepsilon_0\} \). There is a constant \( C > 0 \) such that for all \( x \in \mathbb{R}^N \) we have

\[
\|\tau_x q^{(s)}_1 - q^{(s)}_1\|_{L^\infty} \leq C\|x\|. \tag{6.9}
\]

**Proof.** For any \( y \in \mathbb{R}^N \) we have

\[
\tau_x q^{(s)}_1(-y) - q^{(s)}_1(-y) = c_k^{-1} \int_{\mathbb{R}^N} \mathcal{F} q^{(s)}_1(\xi) E(i\xi, -y)[E(i\xi, x) - 1] \, dw(\xi).
\]

Therefore, the claim is a consequence of Lemma 6.7, Lemma 2.1, and the fact that \( q^{(s)}_1 \in \mathcal{S}(\mathbb{R}^N) \), so \( \mathcal{F}(q^{(s)}_1) \in \mathcal{S}(\mathbb{R}^N) \) as well. \( \Box \)

Theorem 1.1 is a special case (for \( \varepsilon = 0 \)) of the theorem below.

**Theorem 6.9.** Let \( \varepsilon \in \{0, \varepsilon_0\} \). There are constants \( C, c > 0 \) such that for all \( x \in \mathbb{R}^N \) we have

\[
|q^{(s)}_1(x)| \leq C \exp(-c\|x\|^{2\varepsilon/(\varepsilon - 1)}). \tag{6.10}
\]

**Proof.** Since \( q^{(s)}_1 \in \mathcal{S}(\mathbb{R}^N) \), it suffices to prove (6.9) for large \( \|x\| \). For any \( x \in \mathbb{R}^N \), \( s > 1 \) and \( r > 0 \) we write

\[
q^{(s)}_1(x) = \frac{1}{w(B(0, r))} \int_{B(0, r)} q^{(s)}_1(y) \, dw(y) = \frac{1}{w(B(0, r))} \int_{B(0, r)} [q^{(s)}_1(x) - \tau_y q^{(s)}_1(x)] \, dw(y) + \frac{1}{w(B(0, r))} \int_{B(0, r)} \tau_y q^{(s)}_1(x) \, dw(y) = J_1 + J_2.
\]

By Lemma 6.8 we have

\[
|J_1| \leq C \frac{1}{w(B(0, r))} \int_{B(0, r)} \|y\| \, dw(y) \leq Cr. \tag{6.10}
\]
Furthermore, it follows by the definition of the Dunkl translation that
\[
\int_{B(0,r)} \tau_y q_1^{(e)}(x) \, dw(y) = \int_{B(0,r)} \tau_x q_1^{(e)}(-y) \, dw(y) = S_1^{(e)} \chi_{B(0,r)}(x).
\]
Therefore, by Corollary 6.6 and (3.2) we get that there is \( M \in \mathbb{N} \) such that
\[
|J_2| = w(B(0, r))^{-1} |S_1^{(e)} \chi_{B(0,r)}(x)|
\leq C w(B(0, r))^{-1} \exp(cs^{2\ell}) \exp(-s\|x\|) \|\chi_{B(0,r)}\|_{H_{2M+1}^s}.
\]
\[
\leq C w(B(0, r))^{-1} \exp(cs^{2\ell}) \exp(-s\|x\|) w(B(0, r))^{1/2} \exp(2^{2M} s^r)
\leq C r^{-\frac{N}{2}} \exp(c's^{2\ell}) \exp(-s\|x\|),
\]
where in the last inequality we have used (2.5). Therefore, taking into account (6.10) and (6.11) we obtain
\[
|q_1^{(e)}(x)| \leq (r + r^{-\frac{N}{2}} \exp(c's^{2\ell}) \exp(-s\|x\|)).
\]
Set
\[
r = \left( \exp(c's^{2\ell}) \exp(-s\|x\|) \right)^{\frac{1}{N+2\ell}},
\]
then (6.12) reduces to
\[
|q_1^{(e)}(x)| \leq C \left( \exp(c's^{2\ell}) \exp(-s\|x\|) \right)^{\frac{1}{N+2\ell}}.
\]
Finally, setting \( s = \delta\|x\|^{1/(2\ell-1)} \) for \( \delta > 0 \) small enough we obtain the claim. \( \square \)

6.2. Pointwise estimations for the integral kernel of the semigroup. The following proposition was proved in [11, Proposition 4.4].

**Proposition 6.10.** There is a constant \( C > 0 \) such that for any \( r_1, r_2 > 0 \), any \( f \in L^1(dw) \) such that \( \text{supp} \, f \subseteq B(0, r_2) \), any continuous radial function \( \phi \) such that \( \text{supp} \, \phi \subseteq B(0, r_1) \), and for all \( y \in \mathbb{R}^N \) we have
\[
\|\tau_y (f * \phi)\|_{L^1(dw)} \leq C (r_1(r_1 + r_2))^\frac{N}{2} \|\phi\|_{L^\infty} \|f\|_{L^1(dw)}.
\]

The lemma below is a suitable adaptation of [11, Proposition 4.10].

**Lemma 6.11.** Let \( a, b > 1 \) and \( f, g \) be measurable functions such that \( g \) is radial and continuous, and there are constants \( C, c > 0 \) such that
\[
|f(x)| \leq C \exp(-c\|x\|^a) \quad \text{and} \quad |g(x)| \leq C \exp(-c\|x\|^b) \quad \text{for all} \ x \in \mathbb{R}^N.
\]
Then there are constants \( C', c' > 0 \) such that for all \( y \in \mathbb{R}^N \) we have
\[
\int_{\mathbb{R}^N} |\tau_y (f * g)(-x)| \exp(c'd(x, y)^{\min\{a, b\}}) \, dw(x) \leq C'.
\]

**Proof.** Let \( \Psi_0 \in C^\infty(-\frac{1}{2}, \frac{1}{2}) \) and \( \Psi \in C^\infty(\frac{1}{8}, 1) \) be such that
\[
1 = \Psi_0(\|x\|) + \sum_{n=1}^{\infty} \Psi(2^{-n}\|x\|) = \sum_{n=0}^{\infty} \Psi_n(\|x\|) \quad \text{for all} \ x \neq 0.
\]
Set \( f_n(x) = f(x)\Psi_n(||x||) \) and \( g_j(x) = g(x)\Psi_j(||x||) \), where \( n, j \geq 0 \). Clearly, 
\[
\tau_y(f * g) = \sum_{n,j=0}^{\infty} \tau_y(f_n \ast g_j) \quad \text{(see [11, Proposition 4.10] for details).}
\]
Since \( \text{supp} \ f_n \subseteq B(0,2^n) \) and \( \text{supp} \ g_j \subseteq B(0,2^j) \), we have 
\[
\text{supp} \ f_n \ast g_j \subseteq B(0,2^j + 2^n).
\]
By Proposition 6.10 we obtain 
\[
\int_{\mathbb{R}^N} |\tau_y(f_n \ast g_j)(-x)| \exp(c'(d(x,y)^\min\{a,b\})) \, dw(x)
\]
(6.15) 
\[
\leq \exp(c'(2^j + 2^n)^\min\{a,b\}) \int_{\mathbb{R}^N} |\tau_y(f_n \ast g_j)(-x)| \, dw(x)
\]
\[
\leq C_1 \exp(c'(2^j + 2^n)^\min\{a,b\}) 2^j 2^n (2^j + 2^n) \|f_n\|_{L^1(dw)} \|g_j\|_{L^\infty}.
\]
By (6.14) we have \( \|f_n\|_{L^1(dw)} \leq C \exp(-2^{na}c/2) \) and \( \|g_j\|_{L^\infty} \leq C \exp(-2^{jb}c) \), 
so (6.15) leads to 
\[
\int_{\mathbb{R}^N} |\tau_y(f \ast g)(-x)| \exp(c'(d(x,y)^\min\{a,b\})) \, dw(x)
\]
\[
\leq C_1 \sum_{n,j=0}^{\infty} \exp(c'(2^j + 2^n)^\min\{a,b\}) 2^j 2^n (2^j + 2^n) \exp(-c2^{na-1} - c2^{jb}).
\]
Finally, we see that if \( c' > 0 \) is small enough, then the double series above is convergent, so we are done. \qed

**Proof of Theorem 1.2.** We write 
\[
q_1 = F^{-1}(F q_1^{(0)}) = F^{-1}((F q_1^{(0)} e^\varepsilon_0 \|\cdot\|^2) e^{-\varepsilon_0 \|\cdot\|^2} e^{-\varepsilon_0 \|\cdot\|^2}) = q_1^{(e_0)} \ast h_{e_0/2} \ast h_{e_0/2},
\]
where \( h_{e_0/2} \) is the Dunkl heat kernel (see (2.20)). This gives 
\[
|q_1(x,y)| = |\tau_x((q_1^{(e_0)} \ast h_{e_0/2} \ast h_{e_0/2})(-y)| \leq \int_{\mathbb{R}^N} |\tau_y(q_1^{(e_0)} \ast h_{e_0/2})(z)| \|h_{e_0/2}(x,z)\| \, dw(z)
\]
\[
\leq \int_{d(x,y) \leq 2d(x,z)} + \int_{d(x,y) \leq 2d(y,z)} = J_1 + J_2.
\]
By Theorem 2.2 applied to \( h_{e_0/2}(x,z) \), we have 
\[
|J_1| \leq C \int_{d(x,y) \leq 2d(x,z)} |\tau_y(q_1^{(e_0)} \ast h_{e_0/2})(z)| w(B(x,\varepsilon_0))^{-1} \exp(-c_{e_0}d(x,z)^2) \, dw(z)
\]
\[
\leq Cw(B(x,1))^{-1} \exp(-c'd(x,y)^{2\ell-1}) \int_{\mathbb{R}^N} |\tau_y(q_1^{(e_0)} \ast h_{e_0/2})(z)| \, dw(z),
\]
where in the last inequality we have used the fact that the measure \( dw \) is doubling (see (2.6)). The functions \( f = q_1^{(e_0)} \) and \( g = h_{e_0/2} \) satisfy the assumptions of Lemma 6.11 with \( a = \frac{2\ell}{2\ell-1} \) and \( b = 2 \) respectively (see Theorems 6.9 and 2.2), so the last integral is bounded by a constant.
Thanks to the inequality $|h_{\varepsilon_0/2}(x, z)| \leq C w(B(x, \varepsilon_0))^{-1}$ (see Theorem 2.2), $|J_2|$ is less than

$$C w(B(x, \varepsilon_0))^{-1} \int_{d(x,y) \leq 2d(x,y)} |\tau_y(q^{(\varepsilon_0)}_1 * h_{\varepsilon_0/2}) (z)\exp(-cd(y, z)^{\frac{2d}{2d-1}})\exp(cd(y, z)^{\frac{2d}{2d-1}}) dw(z)| \leq C w(B(x, 1))^{-1}\exp(-c' d(x,y)^{\frac{2d}{2d-1}})\int_{\mathbb{R}^N} |\tau_y(q^{(\varepsilon_0)}_1 * h_{\varepsilon_0/2}) (z)\exp(cd(y, z)^{\frac{2d}{2d-1}}) dw(z)|.$$

Since the functions $f = q^{(\varepsilon_0)}_1$ and $g = h_{\varepsilon_0/2}$ satisfy the assumptions of Lemma 6.11 with $a = \frac{2d}{2d-1}$ and $b = 2$ respectively, the last integral is bounded by a constant independent of $y$, provided $c > 0$ is small enough. The proof is complete. \hfill \Box

7. Appendix

A. Proof of Proposition 3.11.

Lemma 7.1. Let $s > 1/4$ and let $\Phi$ be a radial $C^\infty_c(\mathbb{R}^N)$-function such that $\int \Phi \, dw = 1$ and supp $\Phi \subset B(0,1)$. There is a constant $C = C_\Phi > 0$ such that for all $f \in \mathcal{H}_s$ we have

$$(7.1) \quad \|\Phi_{1/n} * f\|_{\mathcal{H}_s} \leq C \|f\|_{\mathcal{H}_s}.$$ 

Moreover,

$$(7.2) \quad \lim_{n \to \infty} \|f - \Phi_{1/n} * f\|_{\mathcal{H}_s} = 0 \quad \text{for all } f \in \mathcal{H}_s.$$

Here and subsequently, $\Phi_{1/n}(x) = n^N \Phi(nx)$.

Proof. Let us note that by the definition of $\eta(x, s)$ (see (3.1)), there is a constant $C > 0$ such that for all $x, y \in \mathbb{R}^N$ and $s > 1/4$ we have

$$(7.3) \quad e^{s\|x\|} \leq \eta(x, s) \leq Ce^{s\|x\|} \leq Ce^{sd(x,y)+s\|y\|},$$

therefore, by the Cauchy–Schwarz inequality,

$$(7.4) \quad \|\Phi_{1/n} * f\|_{\mathcal{H}_s}^2 \leq C \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} \Phi_{1/n}(x, y) f(y) \, dw(y) \right|^2 e^{s\|x\|} \, dw(x) \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\Phi_{1/n}(x, y)|d(y) \int_{\mathbb{R}^N} |\Phi_{1/n}(x, y)||f(y)|^2 \, dw(y) e^{s\|x\|} \, dw(x).$$

Since $\Phi$ is radial, by (2.17) (see also (2.4)) we have

$$(7.5) \quad \int_{\mathbb{R}^N} |\Phi_{1/n}(x, y)| \, dw(y) \leq \int_{\mathbb{R}^N} |\Phi(y)| \, dw(y) \leq C.$$

Consequently, combining (7.3) and (7.4) we get

$$(7.6) \quad \|\Phi_{1/n} * f\|_{\mathcal{H}_s}^2 \leq C' \int_{\mathbb{R}^N} |f(y)|^2 e^{s\|y\|} \int_{\mathbb{R}^N} |\Phi_{1/n}(x, y)| e^{sd(x,y)} \, dw(x) \, dw(y).$$
Because supp $\Phi_{1/n} \subseteq B(0, 1)$ for all $n \in \mathbb{N}$ and $\Phi_{1/n}$ is radial, (2.16) implies that supp $\Phi_{1/n}(\cdot, y) \subseteq \mathcal{O}(B(y, 1))$ for all $y \in \mathbb{R}^N$. Therefore, $d(x, y) \leq 1$ for all $x \in \text{supp } \Phi_{1/n}(\cdot, y)$, so applying (7.5) to (7.6) we get
\[
\|\Phi_{1/n} * f\|_{H_s}^2 \leq C'e^{s} \int_{\mathbb{R}^N} |f(y)|^2 e^s \|y\| \, dw(y) \leq C'e^{s} \|f\|_{H_s}^2,
\]
where in the last inequality we have used the first inequality of (7.3).

To finish the proof it suffices to show that (7.2) holds for compactly supported $H_s$-functions, because they form a dense set there. Fix $f \in H_s$. Let $R > 0$ be such that supp $f \subseteq B(0, R)$. Then supp $f * \Phi_{1/n} \subseteq B(0, R + 1)$. By (3.2) we get
\[
\|f * \Phi_{1/n} - f\|_{H_s}^2 \leq e^{(R+1)s+1} \|f * \Phi_{1/n} - f\|_{L^2(dw)}^2.
\]
The right-hand side of the above inequality tends to zero, since one can easily prove (using the Dunkl transform) that $\Phi_{1/n}$ is an approximate of the identity on $L^2(dw)$.

**Proof of Proposition 3.11 (a)$\Rightarrow$(b).** Let $f = \{f_n\}_{n \in \mathbb{N}} \subseteq C_c^{\infty}(\mathbb{R}^N)$ be a Cauchy sequence in $V_{\ell,s}$. Clearly, by completeness of $H_s$, there is $f \in H_s \subseteq L^2(dw)$ such that $\lim_{n \to \infty} \|f_n - f\|_{H_s} = 0$. Let $|\beta| \leq \ell$, by Corollary 3.9 the sequence $\{T^\beta f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $H_s$, thus it converges to a function $f_{\beta,s}$ in $H_s$ and in $L^2(dw)$ as well. Let $\varphi \in C_c^{\infty}(\mathbb{R}^N)$. Integrating by parts we obtain
\[
(-1)^{|\beta|} \int_{\mathbb{R}^N} f(x) T^\beta \varphi(x) \, dw(x) = \lim_{n \to \infty} (-1)^{|\beta|} \int_{\mathbb{R}^N} f_n(x) T^\beta \varphi(x) \, dw(x)
\]
\[
= \lim_{n \to \infty} \int_{\mathbb{R}^N} T^\beta f_n(x) \varphi(x) \, dw(x)
\]
\[
= \int_{\mathbb{R}^N} f_{\beta,s}(x) \varphi(x) \, dw(x).
\]
Assume now that $g = \{g_n\}_{n \in \mathbb{N}}$ is another Cauchy sequence in $V_{\ell,s}$, such that $\{g_n\}_{n \in \mathbb{N}}$ converge to the $f$ in $H_s$. Then (7.8) implies that $g_{\beta,s} = f_{\beta,s}$ thus $\{g_n\}_{n \in \mathbb{N}}$ corresponds to the same element in $V_{\ell,s}$. Hence we have proved that for every element $f$ in $V_{\ell,s}$ we can find a unique element in $f \in H_s$ which satisfies (3.23).

**Proof of Proposition 3.11 (b)$\Rightarrow$(a).** Let $\Phi$ be a radial $C_c^{\infty}(\mathbb{R}^N)$-function such that $\int \Phi \, dw = 1$ and supp $\Phi \subseteq B(0, 1)$. Let $\Psi$ be a radial $C_c^{\infty}(\mathbb{R}^N)$-function such that $\Psi \equiv 1$ on $B(0, 1)$ and $0 \leq \Psi \leq 1$. For $n \in \mathbb{N}$ we set
\[
f_n(x) = \Psi(x/n) \Phi_{1/n} * f(x).
\]
Since $f \in H_s$, we have $f_n \in C_c^{\infty}(\mathbb{R}^N)$ for all $n \in \mathbb{N}$. By iteration of (3.6), for all $\beta \in \mathbb{N}_0^N$ such that $|\beta| \leq \ell$, there are functions $\Psi_{\beta,\beta',\sigma} \in C_c^{\infty}(\mathbb{R}^N)$ such that
\[
T^\beta f_n(x) = T^\beta (\Phi_{1/n} * f)(x) \Psi(x/n)
\]
\[
+ \sum_{\sigma \in G} \sum_{\beta' \in \mathbb{N}_0^N \text{ with } |\beta'| < |\beta|} n^{|\beta'| - |\beta|} T^\beta' (\Phi_{1/n} * f)(\sigma(x)) \Psi_{\beta,\beta',\sigma}(x/n).
\]
We may assume that \( \psi \) is real-valued. Define \( \phi \in H_s \). The operator \( L^\varepsilon = (-1)^{\ell+1} \sum_{j=1}^{m} T_{\varepsilon}^{\ell} - \varepsilon \Delta \) is understood as a differential-difference operator acting on \( C^\infty(\mathbb{R}^N) \)-functions. We define its action on all \( L^2(dw) \)-functions by means of distributions, that is,

\[
\int_{\mathbb{R}^N} (L^\varepsilon f)(x) \varphi(x) \, dw(x) = \int_{\mathbb{R}^N} (L^\varepsilon \varphi)(x) f(x) \, dw(x) \quad \text{for all } \varphi \in C^\infty_c(\mathbb{R}^N).
\]

**Lemma 7.2.** Let \( f \in V_{\ell,s} \). Then \( f \in D(A_s^{(\varepsilon)}) \) if and only if \( L^\varepsilon f \) belongs to \( \mathcal{H}_s \) in the sense of distributions (cf. (7.10)).

**Proof.** Assume that \( f \in D(A_s^{(\varepsilon)}) \). Set \( g = A_s^{(\varepsilon)} f \in \mathcal{H}_s \). Fix \( \varphi \in C^\infty_c(\mathbb{R}^N) \). We may assume that \( \varphi \) is real-valued. Define \( \psi(x) = \varphi(x) \eta(x,s)^{-1} \). Then \( \psi \in C^\infty_c(\mathbb{R}^N) \subset V_{\ell,s} \). By the definition of \( A_s^{(\varepsilon)} \) (see Subsection 5.3) we get

\[
\int_{\mathbb{R}^N} g(x) \varphi(x) \, dw(x) = \int_{\mathbb{R}^N} g(x) \psi(x) \eta(x, s) \, dw(x) = b_{s,\varepsilon}(f, \psi)
\]

\[
= - \int_{\mathbb{R}^N} \sum_{j=1}^{m} T_{\varepsilon}^{\ell}(f(x)) T_{\varepsilon}^{\ell}(\psi(x) \eta(x, s)) \, dw(x)
\]

\[
+ \varepsilon \int_{\mathbb{R}^N} \sum_{j=1}^{N} T_j f(x) T_j(\psi(x) \eta(x, s)) \, dw(x)
\]

\[
= \int_{\mathbb{R}^N} f(x) L^\varepsilon \varphi(x) \, dw(x),
\]

which proves that \( g = L^\varepsilon f \) in the weak sense.

Conversely, assume that \( f \in V_{\ell,s} \) is such that \( L^\varepsilon f \in \mathcal{H}_s \) in the weak sense. Set \( g = L^\varepsilon f \). Take \( \varphi \in C^\infty_c(\mathbb{R}^N) \). Then \( \varphi(x) \eta(x,s) \in C^\infty_c(\mathbb{R}^N) \) and

\[
\int_{\mathbb{R}^N} g(x)(\varphi(x) \eta(x, s)) \, dw(x) = \int_{\mathbb{R}^N} f(x) L^\varepsilon(\varphi(x) \eta(x, s)) \, dw(x)
\]

\[
= b_{s,\varepsilon}(f, \varphi).
\]
By a density argument (see Subsection 3.3), the formula (7.11) holds for all \( \varphi \in V_{s_1,s_2} \), which implies that \( f \in D(A_s^{(e)}) \) and \( A_s^{(e)} f = g \). \( \square \)

**Corollary 7.3.** Let \( \varepsilon \in \{0, \varepsilon_0\} \) and \( c_0 \) be the constant from (5.6). For every \( s_1 > s_2 > 1/4 \) we have

(a) \( D(A_s^{(e)}) \subset D(A_s^{(e)}) \subset D(A^{(e)}) \) and \( A_s^{(e)} \subset A_s^{(e)} \subset A^{(e)} \);
(b) \( R(\lambda; A_s^{(e)}) \subset R(\lambda; A_s^{(e)}) \subset R(\lambda; A^{(e)}) \) for all \( \lambda > c_0 s_2^{2\varepsilon} \), where \( R(\lambda; A_s^{(e)}) \) denotes the resolvent operator, that is, \( R(\lambda; A_s^{(e)}) = (\lambda I - A_s^{(e)})^{-1} \)
(c) \( S_t^{(s_1,s_2)} \subset S_t^{(s_1,s_2)} \subset S_t^{(e)} \) for all \( t > 0 \).

**Proof.** The statements (a) and (b) are consequences of Lemma 7.2. To prove (c) we take \( \omega > 0 \) sufficiently large. Then, by the Lions theorem (see Theorem 5.1), the operators \( \tilde{A}_s^{(e)} = A_s^{(e)} - \omega I, \tilde{A}_s^{(e)} = A_s^{(e)} - \omega I \), and \( \tilde{A}^{(e)} = A^{(e)} - \omega I \), generate contraction semigroups \( \{e^{-t\omega S_t^{(s_1,s_2)}}\}_{t \geq 0}, \{e^{-t\omega S_t^{(s_1,s_2)}}\}_{t \geq 0} \), and \( \{e^{-t\omega S_t^{(e)}}\}_{t \geq 0} \) respectively (each semigroup acts on its corresponding Hilbert space \( \mathcal{H}_{s_2} \)). It follows from the statements (a) and (b) that the Yosida approximations of \( A_s^{(e)} \) (see [14, Section 3.1]) satisfy

\[
\lambda^2 R(\lambda; \tilde{A}_s^{(e)}) - \lambda I \subset \lambda^2 R(\lambda; \tilde{A}_s^{(e)}) - \lambda I \subset \lambda^2 R(\lambda; \tilde{A}^{(e)}) - \lambda I,
\]

for \( \lambda > 0 \), which implies (c), by the proof of the Hille-Yosida theorem (see [14]). \( \square \)

**C. Proof of Proposition 5.5.** Since \( \lambda > c_0 s_2^{2\varepsilon} \), the operator \( \lambda I - A_s^{(e)} \) is invertible on \( \mathcal{H}_{s_2} \). Let \( R(\lambda; A_s^{(e)}) \) denote its inverse. Since \( R(\lambda; A_s^{(e)}) \) is a bounded operator on \( \mathcal{H}_{s_2} \), it suffices to prove that \( (\lambda I - A_s^{(e)}) \) is a dense subspace in \( \mathcal{H}_{s_2} \). For this purpose let

\[
\mathcal{V}_s^\infty = \{ f \in C^\infty(\mathbb{R}^N) : T^\beta f \in \mathcal{H}_{s_2} \text{ for every } \beta \in \mathbb{N}_0^N \}.
\]

We claim that \( \mathcal{V}_s^\infty \) is a core for \( (\lambda I - A_s^{(e)}) \), because for \( f \in C^\infty(\mathbb{R}^N) \) we have \( T^\beta R(\lambda; A_s^{(e)}) f = R(\lambda; A_s^{(e)}) T^\beta f \in D((A_s^{(e)})^n) \subset \mathcal{H}_{s_2} \) and, consequently, \( R(\lambda; A_s^{(e)}) f \in \mathcal{V}_s^\infty \). Therefore \( C^\infty(\mathbb{R}^N) \subset (\lambda I - A_s^{(e)}) (\mathcal{V}_s^\infty), \) which proves the claim.

Let \( \Psi \) be as in Appendix A and let \( f \in \mathcal{V}_s^\infty \). Then \( f_j(x) = \Psi(x/j) f(x) \in C^\infty(\mathbb{R}^N) \) for all \( j \in \mathbb{N} \). It is not difficult to prove that \( \lim_{j \to \infty} \| T^\beta f_j - T^\beta f \|_{\mathcal{H}_{s_2}} = 0 \) for every multi-index \( \beta \in \mathbb{N}_0^N \), which finishes the proof of the proposition.

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J. Dziubański and A. Hejna, Uniwersytet Wrocławski, Instytut Matematyczny, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

E-mail address: jdziuban@math.uni.wroc.pl
E-mail address: hejna@math.uni.wroc.pl