On a possible new $R^2$ theory of supergravity. 

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Abstract

We consider a new MacDowell–Mansouri $R^2$–type of supergravity theory, an extension of conformal supergravity, based on the superalgebra $Osp(1|8)$. Invariance under local symmetries with negative Weyl weight is achieved by imposing chirality-duality and double-duality constraints on curvatures, along with the usual constraint of vanishing supertorsion. An analysis of the remaining gauge symmetries shows that those with vanishing Weyl weight are invariances of the action at the linearized level. For the symmetries with positive Weyl weight we find that invariance of the action would require further modifications of the transformation rules. This conclusion is supported by a kinematical analysis of the closure of the gauge algebra.

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1 Introduction and Review.

This article presents a study of a possible extension of conformal supergravity in four dimensions. The motivation of our work is the growing belief, due to string theory, that new supergravities in 10, 11 and or 12 dimensions may exist [1]-[5]. It is known [6] that the largest symmetry of the $S$–matrix in asymptotically flat spacetime, is the superconformal algebra, therefore, if the model were to exist as a quantum theory, the extra symmetries of the action would not manifest themselves as symmetries of the $S$–matrix.

In this article we try to construct a classical action, quadratic in curvatures, which is invariant under all local symmetries of the superalgebra $Osp(1|8)$. The algebra is given in (42) and (45)-(49), the action is given in (67) and the constraints on curvatures are given in (105)-(112). We use an approach based on the “gauging” of spacetime algebras by imposing constraints on curvatures which was published in this journal [7]. We have obtained invariance under most local symmetries, but further modifications of the Yang–Mills rules for gauge theories are found to be necessary, see the last section. A hint in this direction is the observation that if one uses the results of [8] that in ordinary conformal supergravity in $3 + 1$ dimensions the number of bosonic and fermionic states match, and notices that the fermionic spectrum of $Osp(1|8)$ is the same as in the superconformental case, it follows that there should not be any further bosonic states if there should be equal numbers of bosonic and fermionic states. We now begin with an introduction to the “gauging of spacetime algebras”; some of this material is new, while the rest is based on [1]-[15].

When supergravity was discovered, it was constructed as the gauge theory of supersymmetry, and contained only two physical fields: the spin 2 gravitational vierbein field $e^m_\mu$ and its superpartner, the spin 3/2 supersymmetry gauge field $\psi^\alpha_\mu$ [10]. The spin connection $\omega^{mn}_\mu$ (the gauge field for Lorentz symmetry) was expressed in terms of $e^m_\mu$, as usual in general relativity. The action consisted of three parts: the Einstein-Hilbert action (with vierbein dependent $\omega(e)^{mn}_\mu$), the Rarita-Schwinger action for the spin 3/2 gauge field $\psi^\alpha_\mu$ (with a Lorentz covariant Schwinger action for the spin 3/2 gauge field $\psi^\alpha_\mu$ (with a Lorentz covariant derivative containing $\omega(e)^{mn}_\mu$), and complicated four fermion terms. In a reformulation, the theory was written with an independent spin connection $\omega^{mn}_\mu$ and in this formulation there were no four-fermion terms [17]. However upon solving the algebraic field equation for $\omega^{mn}_\mu$ one finds $\omega^{mn}_\mu = \omega(e)^{mn}_\mu + \psi\bar{\psi}$ terms and substituting this solution
back into the action, the formulation with $\omega_{\mu}^{mn}(e)$ and four-fermion terms reappeared. As already anticipated in [18], the spin 2 and spin 3/2 curvatures which appear in the action are Yang–Mills curvatures for the super Poincaré algebra, but not anticipated in [18], is the fact that “gauging” of the super Poincaré algebra is not enough: in order to obtain invariance under local supersymmetry one must either use an independent spin connection whose transformation rule is not the same as obtained by gauging the super Poincaré algebra, or one should use the expression $\omega(e, \psi)_{\mu}^{mn}$ which results from solving the $\omega$ field equation. Only gradually realized after [16] and [17] is the fact that this result for $\omega(e, \psi)_{\mu}^{mn}$ can also be obtained by imposing the constraint that

$$R(P)_{\mu\nu}^{\ m} = 0,$$

where $R(P)_{\mu\nu}^{\ m}$ is the Yang–Mills curvature for the translation generator in the super Poincaré algebra. Thus the dynamical origin of the compositeness of $\omega(e, \psi)_{\mu}^{mn}$ was replaced by a kinematical origin. The relation between first-order formalism (with independent $\omega_{\mu}^{mn}$) and second-order formalism (with $\omega(e, \psi)_{\mu}^{mn}$) becomes very clear by writing the variation of the action under supersymmetry as

$$\delta S \sim \int \epsilon_{\mu\rho\sigma} \epsilon_{mnr\ell} R(P)_{\mu\nu}^{\ m} \left[ \delta \omega_{\rho}^{\ nr} - \Omega(e, \psi)_{\rho}^{\ nr} \right] e_{s}^{\ s}.$$

In this expression the variation of $\delta e_{\mu}^{\ m}$ and $\delta \psi_{\mu}^{\ \alpha}$ has already been performed, giving rise to $\Omega_{\rho}^{\ nr}$, which is a complicated function of $e_{\mu}^{\ m}$ and $\psi_{\mu}^{\ \alpha}$. Invariance under local supersymmetry ($\delta S = 0$) can be obtained in two (and not more) ways: either by setting $R(P)_{\mu\nu}^{\ m} = 0$ (second-order formalism), or by setting $\delta \omega_{\rho}^{\ nr} = \Omega_{\rho}^{\ nr}$ (first-order formalism). A very useful observation, sometimes called 1.5 order formalism, states that since in second-order formalism $\omega(e, \psi)$ satisfies its own field equation, it is not necessary to vary the composite object $\omega(e, \psi)$ (because if one were to vary it using the chain rule, the result would anyhow be multiplied by the $\omega$ field equation which vanishes identically in second-order formalism).

Another reformulation of this simplest supergravity theory ($N = 1$, or simple supergravity) was discovered by MacDowell and Mansouri [9]. Instead of the super Poincaré algebra, they considered the super-anti de Sitter algebra, with the same generators $P_{m}$ (for translations), $Q_{\alpha}$ (for supersymmetry) and $M_{mn}$ (for Lorentz symmetry), but with different curvatures: $R(P)_{\mu\nu}^{\ m}$ is the same, but $R(M)_{\mu\nu}^{\ mn}$ has two extra terms $-\lambda^{2} e_{[m}^{\ [\mu} e_{\nu]}^{\ ]} + \lambda \bar{\psi}_{\mu}^{\ [m} \gamma^{[n} \psi_{\nu]}^{\ ]}$. 
and \( R(Q)_{\mu\nu}^\alpha \) has an extra term \( \frac{1}{2} \lambda (\gamma_\mu \psi_\nu^\alpha - \gamma_\nu \psi_\mu^\alpha) \). The constant \( \lambda \) is the inverse of the radius of the anti de Sitter space. They proposed the following action

\[
S = \int d^4x \epsilon^{\mu\nu\rho\sigma} \left[ R(M)_{\mu\nu}^{\; mn} R(M)_{\rho\sigma}^{\; rs} \epsilon_{mnrs} \right. \\
\left. + 8\lambda R(Q)_{\mu\nu}^\alpha \gamma^{\beta}_{\alpha\beta} R(Q)_{\rho\sigma}^\beta \right].
\]

Expanding \( R(M)_{\mu\nu}^{\; mn} = R(M)^L_{\mu\nu}^{\; mn} + \lambda \)-dependent terms, where \( R(M)^L_{\mu\nu}^{\; mn} \) is the Lorentz curvature, they found that the leading term in the first square is the Gauss–Bonnet topological invariant \( (R(M)^L_{mn} \wedge R(M)^L_{rs} \epsilon_{mnrs}) \). The cross term between \( R(M)^L \) and \( \lambda \psi \psi \) cancels a similar term due to partially integrating one of the two covariant derivatives \( D(\omega) \) in the second square of curvatures. One is then left with: the Einstein-Hilbert action \( (R(M)^L \) contracted with the \( \lambda^2 \epsilon \epsilon \) term), the Rarita-Schwinger action \( (R(Q) \) contracted with the term \( \lambda \gamma \psi \) \) and a supercosmological constant \( (the \ square \ of \ \lambda^2 \epsilon \epsilon \ plus \ the \ square \ of \ \lambda \gamma \psi) \). By first dropping the topological term, then dividing by \( \lambda \) and finally taking the limit \( \lambda \to 0 \), the ordinary \( N = 1 \) supergravity theory was reobtained.

This approach was extended to \( N = 2 \) supergravity whose physical fields are all gauge fields \( (e^m_\mu, \psi_\mu^{\alpha a} \) with \( a = 1, 2, \) and \( A_\mu) \) \[10\]. However, for \( N \geq 3 \) supergravity no similar results could be obtained since these theories contain also scalar fields.

Subsequently, Kaku, Townsend and van Nieuwenhuizen studied supersymmetric conformal gauge theories, instead of supersymmetric Poincaré gauge theories \[11, 7\]. Since the conformal group contains dilations \( D \), the action should not contain dimensionful parameters, and since the dimension of the Lagrangian density is 4 while curvatures (corresponding to the zero dilaton weight gauge fields) have dimension 2, it was natural to construct an \( R^2 \) action for conformal supergravity along the lines of MacDowell and Mansouri. There are now \( 15 + 9 = 24 \) generators, which can be written in terms of increasing scale weight as

\[
P_m(e^m_\mu) \quad Q_\alpha(\psi_\mu^\alpha) \quad M_{mn}(\omega_\mu^{\; mn}) \quad S_\alpha(\phi_\mu^\alpha) \quad K_m(f_\mu^{\; m})
\]

\[
D(b_\mu) \quad A(a_\mu)
\]

The corresponding gauge fields are given in parentheses. The \( S_\alpha \) are conformal supersymmetry generators, the \( K_m \) denote conformal boosts while the
bosonic generator $A$ for chiral transformations is also needed to extend the ordinary conformal algebra $SU(2, 2)$ to the superconformal algebra $SU(2, 2|1)$. As action they took the most general dilaton-weight zero, Lorentz and Einstein scalar

$$f \, d^4x [a R(P) R(K) + b R(Q) \gamma_5 R(S) + c R(M) \bar{R}(M) + d R(D) R(A) + \alpha e g^{\mu\rho} g^{\nu\sigma} R(A)_{\mu\nu} R(A)_{\rho\sigma}].$$

(5)

(whence $a$, $b$, $c$, $d$ and $\alpha$ are constants and $\bar{R}(M)$ denotes $\frac{1}{2} \epsilon^{mnrs} R(M)^{\tau^\prime \sigma^\prime} \eta_{rr^\prime} \eta_{ss^\prime}$) All terms were affine (i.e., indices were contracted with constant Lorentz-invariant tensors) except the last term. The last term was added since by counting states one finds that the chiral gauge field $A_\mu$ should be physical, and the term $R(D) R(A)$ seemed not to lead to a propagator for $A_\mu$.

By construction the action was invariant under $M$, $D$, $A$ symmetries and under general coordinate transformations. Requiring invariance under the Yang–Mills symmetries corresponding to $S$ and $K$ led to the following information

1. constraints on the curvatures were needed: $R(P)_{\mu \nu}^{\mu \nu} = 0$ and the chirality-duality constraint $R(Q)_{\mu \nu} + \frac{1}{2} \gamma^5 \epsilon_{\mu \nu \rho \sigma} R(Q)^{\rho \sigma} = 0$.

2. all constants were fixed.

With hindsight, this was only one possible solution. When the analysis of invariance under $S$ and $K$ was performed, it was not yet known which further constraints on curvatures would be found. In particular, $R(A)_{\mu \nu}$ and $R(D)_{\mu \nu}$ were still considered as independent objects, and this fixed both $\alpha$ and $d$. Later, a constraint $R(A) = * R(D)$ was found (where $* R(D)_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} R(D)_{\rho \sigma} g^{\rho \sigma}$), and using this in the $K$ and $S$ variations, a one-parameter solution is found. Substituting $R(A)_{\mu \nu} = * R(D)_{\mu \nu}$ in the action, this freedom is clearly cancelled.

Requiring invariance under $Q$ (ordinary local supersymmetry) was a complicated task, but finally achieved. A further constraint $\gamma^{\mu \nu} R(Q)_{\mu \nu} = 0$ was found to be necessary, and the 1.5 order formalism for the conformal gauge field $f_\mu^m$ corresponding to $K_m$ was used: $f_\mu^m$ had an algebraic field equation, and upon solving it one found $f_\mu^m$ as a complicated composite field $f_\mu^m(e, \psi, \phi, \omega, b, A)$ whose variation was not needed for reasons explained.
above. The two constraints for $R(Q)_{\mu\nu}$ could be written together in the simple form

$$\gamma^\mu R(Q)_{\mu\nu} = 0$$

(6)

and these 16 constraints could be solved, yielding the conformal supersymmetry gauge field $\phi_{\mu}^\alpha$ as a composite object. As in ordinary supergravity (and gravity), the 24 constraints $R(P)_{\mu\nu}^m = 0$ yielded $\omega_{\mu}^{mn}$ as a composite object. This left only the fields $e_{\mu}^m$, $\psi_{\mu}^\alpha$, $A_\mu$ and $b_\mu$ as independent fields. However, as a result of $K$-gauge invariance, the dilaton gauge field $b_\mu$ dropped from the action, and one ended up with only $e_{\mu}^m$, $\psi_{\mu}^\alpha$ and $a_\mu$. (In later developments, Poincaré supergravity was obtained from conformal supergravity by coupling the latter to a WZ multiplet. The axial gauge field $a_\mu$ and the WZ auxiliary fields $F_{WZ}$ and $G_{WZ}$ became then the minimal auxiliary fields $S$, $P$, $A_\mu$ of $N = 1$ supergravity, while $D$, $A$ and $S$ symmetry was fixed by choosing the gauge $A_{WZ} = 1$ and $B_{WZ} = \chi_{WZ} = 0$. Since $K$-symmetry did not act on $e_{\mu}^m$, $\psi_{\mu}^\alpha$ or $A_\mu$ it did not need to be fixed [13]).

A simplification was obtained when it was realized [12] that the $f_{\mu}^m$ field equation could also be written as a constraint on the Ricci tensor $e_{\mu}^m R(M)_{\mu\nu}^m$, just as in ordinary supergravity where the $\omega$ field equation could be written as the constraint $R(P)_{\mu\nu}^m = 0$. At this point there were 3 constraints, $R(P)_{\mu\nu}^m = 0$, $\gamma^\mu R(Q)_{\mu\nu} = 0$ and $e_{\mu}^m R(M)_{\mu\nu}^m + \cdots = 0$ (where $\cdots$ contain supercovariantizations, see eqn. (91)); these constraints are all field equations in Poincaré supergravity.

A final simplification was obtained in [14], where only the pure affine part of the action was considered (so with $\alpha = 0$). It was found that the constraint on the Ricci tensor implied another duality constraint

$$R(A)_{\mu\nu} = \frac{1}{2} e_{\mu\nu\rho\sigma} R^{\rho\sigma}(D),$$

(7)

where $e = \det e_{\mu}^m$, so that the term $R(A) \wedge R(D)$ did yield, after all, the Maxwell action for the chiral field $a_\mu$. However, for the affine action, the constraint on the Ricci tensor is no longer a field equation.

Up till now we have reviewed and commented on the dynamical approach (requiring invariance of the action), because this is the approach we shall mainly follow below. However, in the original work [11, 7] and especially in [14], a kinematical approach was followed which directly yielded all constraints. We briefly review it here since we shall use it in the last section of this paper.
The basic idea is that when the (anti)commutator of two Yang–Mills symmetries (P–gauge Yang–Mills symmetries excluded) produces a P–gauge Yang–Mills symmetry, constraints on one or more curvatures are necessary. These constraints should be such that as a result, one or more fields become dependent fields and their transformation rules (obtained by applying the chain rule) should deviate from the Yang–Mills transformation rule (which uses only the structure constants of the superalgebra) in such a way that one finds on the right hand side a general coordinate transformation instead of a P–gauge Yang–Mills transformation. Starting with the gauge fields with highest dilaton weight ($e_{\mu}^{m}$), and then moving towards fields with decreasing dilaton weight, and requiring that the algebra now closes onto general coordinate rather than P–gauge transformations, one finds, in an unambiguous and methodical way, all constraints on curvatures.

For the conformal superalgebra, one only needs to consider the anticommutator of two Q’s since $\{Q, Q\} \sim P$, and one acts successively on $e_{\mu}^{m}$, $\psi_{\mu}^{\alpha}$, $b_{\mu}$ and $a_{\mu}$. This produces the constraints $R(P)_{\mu\nu}^{m} = 0$, $\gamma^{\mu} R(Q)_{\mu\nu} = 0$ and the constraint on the Ricci tensor, respectively, and solving these constraints, $\omega_{\mu}^{mn}$, $\phi_{\mu}^{\alpha}$ and $f_{\mu}^{m}$ become dependent fields.

Work on conformal supergravity continued in 3 dimensions where it turned out to yield Chern-Simons theory, but this will not concern us here.

Recently, there have been speculations that further supergravity theories might exist in $d = 11$ and $d = 12$ dimensions. These ideas are based on the fact that p-branes couple naturally to (p+1)-form gauge potentials via the currents:

$$J^{\mu_1 \ldots \mu_{p+1}}(x) = \frac{1}{\sqrt{g}} \int d\tau d\sigma \delta^{d}(x-X(\tau, \sigma)) e^{i_1 \ldots i_{p+1}}\partial_{i_1} X^{\mu_1}(\tau, \sigma) \ldots \partial_{i_{p+1}} X^{\mu_{p+1}}(\tau, \sigma).$$

These currents are conserved:

$$\partial_{\nu} (\sqrt{g} J^{\nu \mu_1 \ldots \mu_p}(x)) = 0$$

and give rise to tensor charges

$$Z^{\mu_1 \ldots \mu_p} = \int d^{d-1}x J_{0 \mu_1 \ldots \mu_p}(x).$$

They appear then in (maximally) extended supersymmetry algebras as follows: for the case of IIA supersymmetry in d=(1,9) one has

$$\{Q_{a}, Q_{b}\} = \Gamma_{ab}^{\mu} P_{\mu} + \Gamma_{ab}^{\mu} Z_{\mu} + \Gamma_{\mu_1 \ldots \mu_5}^{\mu_1 \ldots \mu_5} Z_{\mu_1 \ldots \mu_5}.$$
\[ \{ Q^a, Q^b \} = \Gamma^{\mu ab} P_\mu - \Gamma^{\mu ab} Z_\mu + \Gamma^{\mu_1 \ldots \mu_5 ab} Z_{\mu_1 \ldots \mu_5} \]  
(12)

\[ \{ Q_a, Q_b \} = \delta^b_a Z + \Gamma^{\mu \nu} a^b Z_{\mu \nu} + \Gamma^{\mu_1 \ldots \mu_4} b^b Z_{\mu_1 \ldots \mu_4} \]  
(13)

where we have 16-dimensional Majorana-Weyl spinors \( Q^a \) and \( Q^a \) of opposite chirality. This algebra may be given a (1,10)-d interpretation in terms of real 32-component spinors:

\[ \{ Q^i, Q^j \} = \Gamma^{iM} P_M + \Gamma^{iM} Z_{MN} + \Gamma^{M_1 \ldots M_5} Z_{M_1 \ldots M_5}, \]  
(14)

or even a (2,10)-d one in terms of 32-component Majorana-Weyl spinors:

\[ \{ Q^i, Q^j \} = \Gamma^{iM} M_M + \Gamma^{iM} M_{MN} Z_{M_1 \ldots M_6}. \]  
(15)

The type IIB algebra in d=(1,9) reads

\[ \{ Q_{ai}, Q_{bj} \} = \Gamma^{ij} a^i b^j Z_{j \mu} + \Gamma^{iM} a^i b^j Z_{j \mu} + \Gamma^{iM} a^i b^j Z_{j \mu}, \]  
(16)

where we use the conventions \( \Sigma_{(ij)} = \epsilon_i \Sigma^j, \Sigma_0 = -i \sigma^2, \Sigma_1 = -\sigma^1 \) and \( \Sigma_2 = \sigma^3 \). These matrices satisfy \( \Sigma_i \Sigma_j = \eta_{ij} \Sigma_0 + \eta_{ij} \Sigma_K \eta^{LK} \Sigma_K \), with \( \eta_{ij} = (- + +) \), \( \epsilon_{012} = 1 \), and hence generate \( SL(2,R) \). In all cases the Z-charges fit the respective brane-scan, and all cases form some decomposition of the Q-Q part of \( OSp(1\mid32) \). It seems plausible that the rest of the algebra completes (possibly some contraction of) the extended (1,9)-d superconformal algebra of van Holten and van Proeyen [21], namely

\[ \{ Q^a, Q_b \} = -\frac{1}{128} \Gamma^{M} a^a b^b Q_M - \frac{1}{256} \Gamma^{M} a^a b^b J_{M_1 \ldots M_6}, \]  
(17)

In the (1,10)-d context this algebra was studied by D’Auria and Fré [25]. We will try to take some first steps towards constructing a supergravity theory based on that algebra.
The signature of the vector space that appears in the above algebra is (2, 10). This provides another hint of a connection to string-theoretic ideas, as Vafa’s argument shows: \( SL(2, Z) \)-duality of type IIB strings may be explained via D-strings. The zero-modes of the open strings stretched between such D-strings determine the worldsheet fields of the latter. We have

\[
\Psi_{\mu}^{\mu} | k > \quad \mu = 0, 1 \quad 2 \text{-d gauge fields} \quad (18)
\]

\[
\Psi_{m}^{m} | k > \quad m = 2, \cdots, 9 \quad \text{transverse fluctuations}, \quad (19)
\]

and hence we find on the D-string an extra \( U(1) \) gauge field. In \( d=2 \) this is nondynamical, of course, but it leaves, after gauge fixing, a pair of ghosts \( B, C \) with central charge \( c = -2 \). The critical dimension is hence raised by two, and the no-ghost theorem \([22, 23]\), which states that the BRST cohomology effectively eliminates those extra dimensions, forces us to assume the existence of a nullvector in the extra dimensions, and that means they must have signature (1,1).

Taking the idea of strings moving in a 12-dimensional target space more seriously, we are immediately led to the puzzle of why strings oscillate in only 10 of these dimensions, but never in the extra 2. If one has conformal symmetry in mind, there is a natural answer: the 12 dimensions are those in which the conformal group is linearly realized, but only a 10-dimensional null hypersurface in real projective classes of these coordinates is physical. The extra two dimensions “don’t really exist”. The idea that strings might have some sort of target space conformal symmetry is not new \([24]\), but as of now no model exists that can be convincingly linked to the string theories known today.

The idea of supergravities beyond \( d=11 \) was already explored fifteen years ago, but no conventional supergravity theory was found \([19]\), even though in \( d = (10, 2) \) dimensions Majorana-Weyl spinors exist, and dimensional reduction to \( d = (10, 1) \) would therefore lead to a \( N = 1 \) supergravity theory. One of us has been pursuing this problem further over the years, also studying a possible gravitational Chern-Simons supergravity theory in \( d = 11 \). In fact, we might speculate that an unconventional (topological?) supergravity theory in \( d = (10, 2) \) might lead to a new \( d = 11 \) supergravity theory (Chern-Simons ?) in which the usual problems with Kaluza-Klein compactification might go away. However, if new theories do exist in \( d = 12 \) and/or \( d = 11 \), there might perhaps also be new theories in \( d = 4 \), and conversely, finding
new theories in $d = 4$ might show the way to new theories in $d = 11$ and/or $d = 12$.

With this motivation in mind, it is natural to ask for a modification of the superconformal algebra in $d = 4$, and to try to construct a corresponding gauge theory. A natural candidate is the four-dimensional version of $\mathfrak{osp}(1|8)$, namely $\mathfrak{osp}(1|8)$. It resembles $SU(2,2|1)$ in that the two spinors $Q_\alpha$ and $S_\alpha$ form an 8-real-component spinor in the fundamental representation, but now of course of $Sp(8,\mathbb{R})$, which contains as a maximal subalgebra $U(2,2) = SU(2,2) \times U(1)$, where $SU(2,2)$ is the conformal algebra and $U(1)$ corresponds to the chiral generator $A$. In fact, since the bosonic subalgebra of $\mathfrak{osp}(1|8)$ is simple, in contrast to the superconformal case, one would be inclined to believe that the geometric approach ought to work even better for $\mathfrak{osp}(1|8)$.

This is certainly true for $\mathfrak{osp}(1|4)$: since $Sp(4)$ is isomorphic to the anti de Sitter algebra $SO(2,3)$, we can view it as the conformal algebra in $d = (1,2)$. (Recall that $SU(2,2)$ is isomorphic to $SO(2,4)$.) Thus the conformal algebra in $d = (1,2)$ corresponds to an ordinary anti de Sitter algebra in $d = (1,3)$, and both algebras can be supersymmetrized, and lead to gauge theories. In a similar manner, one might expect that if an extended superconformal gauge theory for $\mathfrak{osp}(1|8)$ can be found, it will correspond to a non-conformal supergravity theory in 5-dimensional Minkowski spacetime. Jumping ahead: perhaps $\mathfrak{osp}(1|32)$ yields an extended superconformal theory in 10-dimensional Minkowski spacetime, and perhaps this would lead to a new theory in $d = 11$.

Using the $R^2$-method, we will now start constructing a field theory based on $\mathfrak{osp}(1|8)$ in $d = (1,3)$ spacetime. The harder problem of $\mathfrak{osp}(1|32)$ we temporarily set aside. In section 2 we present a real representation for $\mathfrak{osp}(1|8)$ which at once identifies the gauge fields, curvatures and the Yang–Mills transformation rules. In section 3 we write down the most general affine $R^2$ action invariant under all symmetries with positive dilaton weight. (We call them “sure symmetries”. They are generalizations of $K_\alpha$ and $S_\alpha$.) By construction this action is also Lorentz, dilation and chiral invariant (“automatic symmetries”). This leaves us with the symmetries corresponding to two new scale-zero generators ($V_m$ and $Z_m$), ordinary supersymmetry ($Q_\alpha$), and the local symmetries corresponding to two scale-2 generators: $P_m$ and $E_{mn}$. As before, we exchange $P_m$ Yang–Mills symmetry with general coordinate invariance so that we are left with the problem of obtaining local
$V_m$, $Z_m$, $Q_a$ and $E_{mn}$ symmetry of the action (we call these “unsure” symmetries). As in the case of conformal supergravity, we anticipate that we will need further constraints, and with this possibility in mind, in sections 4 and 5 we construct a list of the maximal set of constraints on curvatures which can be solved by expressing one (or more) field(s) in terms of others. It is not obvious at this point that all these constraints must actually be imposed, but it is clear that only constraints from this set can play a rôle. In sections 6 and 7, we make an analysis of all unsure symmetries at leading order. We find that $V_m$ and $Z_m$ invariance in fact requires the maximal set of solvable constraints at the linearized level and we do find invariance under these symmetries at the linearized level. In section 8, we give our conclusions, and make a purely kinematical analysis of $Q$ and $E_{mn}$ symmetry by studying the conditions under which $P_m$ gauge transformations turn into general coordinate transformations.

2 The Algebra and Derivation of the Curvatures.

Our first task is to construct a set of curvature two-forms representing the superalgebra $Osp(1|8)$. We begin with an explicit $8 \times 8$ matrix representation of the bosonic algebra $Sp(8, \mathbb{R})$:

$$
\begin{align*}
V_m &= -\gamma^m \otimes \sigma^3 \\
Z_m &= \gamma_5 \tau^m \otimes \sigma^3
\end{align*}
$$

The matrices $\sigma^\pm \equiv \sigma^1 \pm i\sigma^2$ where $\sigma^1$, $\sigma^2$ and $\sigma^3$ are the usual Pauli matrices. Hence $\sigma^+$, $\sigma^-$ and $\sigma^3$ are real and generate $SO(2, 1) = Sl(2, \mathbb{R})$. The matrices $\gamma_m$ are four real Dirac matrices satisfying

$$
\{\gamma_m, \gamma_n\} = 2\eta_{mn} = 2\text{diag}(-1, 1, 1, 1)_{mn},
$$

and $\gamma^5 \equiv \gamma^0\gamma^1\gamma^2\gamma^3$ is also real and satisfies $(\gamma^5)^2 = -1$, $\gamma^5 = -\gamma^{5\top}$. Further, $\gamma^{mn} \equiv \gamma^{[m}\gamma^{n]} = \frac{1}{2}(\gamma^m\gamma^n - \gamma^n\gamma^m)$. It is easy to check that the
matrices in (20) close under commutation. All matrices $M$ in (20) are real and since they satisfy $M^\top C + CM = 0$ where $C = \gamma^0 \otimes \sigma^1 = -C^\top$ is the direct product of the four and two-dimensional charge conjugation matrices $C_4 = \gamma^0$ (where $C_4 \gamma_m = -\gamma^m_0 C_4$) and $C_2 = (-i\sigma^2)\sigma^3$ (where $C_2 \sigma^\pm = (\sigma^\pm)^\top C_2$), they generate $Sp(8, \mathbb{R})$.

The supersymmetric extension of the above matrix algebra is obtained by introducing fermionic generators $Q_\alpha$ and $S_\alpha$ with $\alpha = 1,\ldots, 4$, both of which are Majorana spinors, possessing dilaton weights $+1$ and $-1$, respectively (i.e., $[D, Q] = Q$, $[D, S] = -S$). The commutation relations between fermionic and bosonic generators (which may be obtained by requiring that $[\text{bose}, \{\text{fermi}, \text{fermi}\}]$ and $[\text{fermi}, \{\text{fermi}, \text{fermi}\}]$ Jacobi identities hold, or directly from the explicit $SO(4, 2)$–covariant expressions for the $Osp(1|8)$ algebra given below) imply that the explicit matrix representation (50) acts on the eight-dimensional real $SO(2, 4)$-spinor $Q_\alpha = (S_\alpha, Q_\alpha)$:

$$\left[ P_m, \begin{pmatrix} S \\ Q \end{pmatrix} \right] = -\sqrt{2} \begin{pmatrix} \gamma_m Q_0 \\ 0 \end{pmatrix}. \quad (22)$$

To each of the generators

$$T_A = \{ P_m, E_{mn}, Q, M_{mn}, D, A, V_m, Z_m, S, K_m, F_{mn} \} \quad (23)$$

with (anti)commutation relations

$$[T_A, T_B] = f_{AB}^C T_C, \quad (24)$$

we now associate a gauge field one-form

$$h^A = \{ e^m, E^{mn}, \psi, \omega^{mn}, b, a, v^m, z^m, \phi, f^m, F^{mn} \} \quad (25)$$

(whose, to avoid overcounting, for the generators $E_{mn}$, $M_{mn}$ and $F_{mn}$ the indices take values $m < n$ only). In particular we interpret the $P_m$ gauge field $e^m_\mu e_{\nu m} = g_{\mu \nu}$. The curvature two-form is then given by (for brevity we omit wedge symbols)

$$R = dh + hh = \left( dh^A - \frac{1}{2} h^C h^B f_{BC}^A \right) T_A = R^A T_A \equiv \frac{1}{2} R^A_{\mu \nu} T_A dx^\mu dx^\nu, \quad (26)$$

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6The gauge fields have mass dimensions $[e^m_\mu] = [E^{mn}_\mu] = 0$, $[\psi_\mu] = 1/2$, $[\omega^{mn}_\mu] = [b_\mu] = [a_\mu] = [e^m_\mu] = [z^m_\mu] = 1$, $[\phi_\mu] = 3/2$, $[f^m_\mu] = [F^{mn}_\mu] = 2$. 

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and \( h = h^A_\mu T_A dx^\mu \).

The fermionic terms in the bosonic curvatures may be obtained independently by a method which we shall explain below. The results for the curvatures are

\[
R(P)^m = de^m + \omega^m n e^n + 2be^m - 2E^m_\nu v^n - 2\tilde{E}^m_\nu z^n - \frac{1}{4\sqrt{2}} \overline{\psi} \gamma^m \psi \quad (27)
\]

\[
R(E)^{mn} = dE^{mn} - 2\omega^m [k E^n]^k + 2bE^{mn} + 2\tilde{E}^{mn} a - 4e^{[m} v^{n]} + 2e^{[mnpq} e_{pq} z_q + \frac{1}{4\sqrt{2}} \overline{\psi} \gamma^{mn} \psi \quad (28)
\]

\[
R(Q) = d\psi + \left(-a \gamma^5 + b + \frac{1}{4} \omega^m \xi_{mn} - v^m \gamma_m + z^m \gamma^5 \gamma_m\right) \psi + \left(\sqrt{2} e^m \gamma_m + \frac{1}{4\sqrt{2}} E^{mn} \gamma_{mn}\right) \phi \quad (29)
\]

\[
R(M)^{mn} = d\omega^{mn} - \omega^m [k \omega^n]^k + 4v^{[m} v^{n]} + 4z^{[m} z^{n]} - 8e^{[m} f^{n]} - 8\tilde{E}^{mn} F^n[k] + \frac{1}{2} \overline{\psi} \gamma^{mn} \phi \quad (30)
\]

\[
R(D) = db - 2e^m f_m - E^{mn} F_{mn} + \frac{1}{4} \overline{\psi} \phi \quad (31)
\]

\[
R(A) = da - 2v^m z_m - \tilde{E}^{mn} F_{mn} + \frac{1}{4} \overline{\psi} \gamma^5 \phi \quad (32)
\]

\[
R(V)^m = dv^m + \omega^m n v^n + 2z^m a + 2E^m_\nu f^n - 2F^m_\nu e^n + \frac{1}{4} \overline{\psi} \gamma^m \phi \quad (33)
\]

\[
R(Z)^m = dz^m + \omega^m n z^n - 2v^m a + 2\tilde{E}^m_\nu f^n + 2\tilde{F}^{mn} e_n + \frac{1}{4} \overline{\psi} \gamma^5 \gamma_m \phi \quad (34)
\]

\[
R(S) = d\phi + \left(a \gamma^5 - b + \frac{1}{4} \omega^m \xi_{mn} - v^m \gamma_m - z^m \gamma^5 \gamma_m\right) \phi + \left(-\sqrt{2} f^m \gamma_m + \frac{1}{4\sqrt{2}} F^{mn} \gamma_{mn}\right) \psi \quad (35)
\]

\[
R(K)^m = df^m + \omega^m n f^n - 2bf^m + 2F^m_\nu v^n - 2\tilde{F}^m_\nu z_n + \frac{1}{4\sqrt{2}} \overline{\phi} \gamma^m \phi \quad (36)
\]

\[
R(F)^{mn} = dF^{mn} - 2\omega^m [k F^n]^k - 2bf^{mn} + 2\tilde{F}^{mn} a + 4f^{[m} v^{n]} + 2e^{mnpq} f_{pq} z_q + \frac{1}{4\sqrt{2}} \overline{\phi} \gamma^{mn} \phi. \quad (37)
\]

The duals are defined by \( \tilde{X}^{mn} \equiv (1/2) \epsilon^{mnpq} X_{pq} \), and the bars on the Majorana fermions are defined by \( \bar{\psi} = \psi^\dagger C_4 = \psi^\dagger \gamma^0 \). To fix the fermionic terms in the bosonic curvatures, we used the Bianchi identities

\[
dR^A = -R^C h^B f_{BC}^A, \quad (38)
\]

Acting on any given curvature with an exterior derivative \( d \) yields terms of the form \( dh^A h^B \). Replacing \( dh^A \) by \( R^A \) minus its field–field terms, and setting all curvatures \( R^A = 0 \), all remaining terms, which are cubic in fields, which are cubic in fields,
must cancel amongst themselves as a consequence of the Jacobi identities. This fixes the coefficients of the fermionic terms in the bosonic curvatures, for example the coefficient of the $\bar{\psi}\gamma^m\psi$ term in $R(P)^m$. This cancellation of combinations of terms cubic in fields will also be highly useful when we consider variations of actions.

It is interesting to recast the above results for the $Osp(1|8)$ generators and curvatures into an $SO(2,4)$-covariant form. First, we define the supersymmetry generators $Q_a = a_a a_b$, with $[a_a, a_b] = -C_{ab}$ and $\{a, a\} = 1$ with indices $a, b = 1, ..., 8$. We obtain

$$\{Q_a, Q_b\} = a(a_{a_b})$$  \hspace{1cm} (39)

and use the Fierz identity

$$\delta^c_a \delta^d_b = -\frac{1}{8} \left\{ \Gamma^7_{a_b} \Gamma^{7cd} + \frac{1}{2} \Gamma^{MN}_{a_b} \Gamma^{cd}_{MN} + \frac{1}{6} \Gamma^{LMN}_{a_b} \Gamma_{LMN}^{cd} \right\}$$  \hspace{1cm} (40)

to rewrite this as

$$\{Q_a, Q_b\} = \frac{1}{8} \left\{ \Gamma^7_{a_b} \Gamma^7_{c_d} a^c d^d + \frac{1}{2} \Gamma^M_{a_b} \Gamma^c_{MN} c^d a^d \\
+ \frac{1}{6} \Gamma^L_{a_b} \Gamma^c_{LMN} c^d a^d \right\}$$

$$\equiv \frac{1}{4} \left\{ \Gamma^7_{a_b} J_7 + \frac{1}{2} \Gamma^M_{a_b} J_{MN} + \frac{1}{6} \Gamma^{LMN}_{a_b} J_{LMN} \right\}.$$  \hspace{1cm} (41)

The $SO(1,3)$-decomposition of the Gamma-matrices we use reads

$$\Gamma^m = -\gamma^m \otimes \sigma^3, \quad \Gamma^\Xi = \frac{1}{\sqrt{2}} \mathbf{1} \otimes \sigma^+, \quad \Gamma^\Theta = \frac{1}{\sqrt{2}} \mathbf{1} \otimes \sigma^-$$

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN} = \text{diag}(-+--++)^{MN}, \quad \Gamma^7 = -\gamma^5 \otimes \sigma^3$$  \hspace{1cm} (43)

where $\Gamma^M = \Gamma^M_{a_b}$ and we have chosen $\eta_{\Theta\Theta} = 1$. We raise and lower indices as follows: $a^a = C^{ab}_{a_b}$, $\Gamma^{*ab} = \Gamma^{*a}_{b} C^{cb} = C^{ac}_{b} \Gamma^{*c}_{b}$, $\Gamma^{*ab} = \Gamma^{*a}_{b} C^{cb} = C^{ac}_{b} \Gamma^{*c}_{b}$, $C^{ab}_{c} C^{cb}_{d} = \delta_{a}^{c}$. $C^{ab}_{c} = (\gamma^0 \otimes \sigma^1)^{ab}$ is the $8 \times 8$ charge conjugation matrix introduced above. With these conventions, among the matrices $\Gamma^{*ab}$ we find $\Gamma^7, \Gamma^{MN}$ and $\Gamma^{MNP}$ symmetric under interchange of $a$ and $b$, while $C, \Gamma^M, \Gamma^{MNPQ}$ and $\Gamma^{MNPQR}$ are antisymmetric. Similarly, the real $4 \times 4$ matrices $\gamma_{a_b}^{*}$ are split into the symmetric $\gamma^{m}, \gamma^{mn}$ and the antisymmetric $C^4, \gamma^{mpn}$ and $\gamma^5$. The remaining sectors of $Osp(1|8)$ now read:

$$[J^*, Q_a] = -\Gamma^{*a}_{b} Q_b$$  \hspace{1cm} (45)
\[ [J^7, J^{MNP}] = \frac{1}{3} \epsilon^{MNPQRST} J_{RST} \]
\[ [J^{MN}, J_{RS}] = 8 \delta_{[R}^{[N} J_{M]S]} \]
\[ [J^{MN}, J_{RST}] = 12 \delta_{[R}^{[N} J_{M]ST]} \]
\[ [J^{MNP}, J_{RST}] = 2 \epsilon^{MNP} RST J^7 - 36 \delta_{[R}^{[M} \delta_{S]}^{N} J_{P]T} \]

The bosonic generators decompose under SO(1,3) as
\[ P^m = \Gamma^{\oplus m} \quad M^{mn} = \frac{1}{2} \Gamma^{mn} \quad K^m = \Gamma^{\oplus m} \]
\[ E^{mn} = \Gamma^{\oplus mn} \quad D = \Gamma^{\otimes} \quad F^{mn} = \Gamma^{\oplus mn} \]
\[ A = \Gamma^7 \quad V^m = \Gamma^{\oplus m} \]
\[ Z^m = -\frac{1}{3!} \epsilon^{mnq} \Gamma_{npq} \]

where the index structure of the matrices is \( \Gamma^* = \Gamma^{* \, a \, b} \).

In SO(2,4)-covariant language the connection 1-forms are written as
\[ h = h_7 J^7 + \frac{1}{2} h_{MN} J^{MN} + \frac{1}{3!} h_{MNP} J^{MNP} + \psi^a Q_a, \]
with \( \psi^a = (\phi^a, \psi^a) \), and the curvatures \( R = dh + hh \) are given by
\[ R = \left\{ dh_7 + \frac{1}{3!} \epsilon^{MNPQRST} h_{MNP} h_{RST} + \frac{1}{8} \psi^a \Gamma_{ab} \psi^b \right\} J^7 \]
\[ + \frac{1}{2} \left\{ dh_{MN} + 2 h_{MK} h^K_N - h^{RS}_{\, M} h_{RSN} + \frac{1}{8} \psi^a \Gamma_{MNab} \psi^b \right\} J^{MN} \]
\[ + \frac{1}{6} \left\{ dh_{MNP} + \frac{1}{3} \epsilon^{MNPQRST} h_{RST} h_7 + 6 h_{MK} h^K_{NP} + \frac{1}{8} \psi^a \Gamma_{MNPab} \psi^b \right\} J^{MNP} \]
\[ + \left\{ d \psi^a + h_7 \Gamma^7_{ab} \psi^b + \frac{1}{2} h_{MN} \Gamma^{MNab} \psi^b + \frac{1}{6} h_{MNP} \Gamma^{MNPab} \psi^b \right\} Q_a \]

The gauge transformations \( \delta h = d \lambda + [h, \lambda] \) imply \( \delta R = [R, \lambda] \), i.e.
\[ \delta R = \left\{ \frac{1}{18} \epsilon^{MNPQRST} R_{MNP} \lambda_{RST} - \frac{1}{4} R^{a} \Gamma_{ab} \lambda^{b} \right\} J^7 \]
\[ + \frac{1}{2} \left\{ 4 R_{MK} \lambda^K_{\, N} - 2 R_{RS}^{\, M} \lambda_{RSN} - \frac{1}{4} R^{a} \Gamma_{MNab} \lambda^{b} \right\} J^{MN} \]
\[ + \frac{1}{6} \left\{ - \frac{1}{3} \epsilon^{MNPQRST} R_{7\lambda}^{\, RST} + 6 R_{MK}^{\, K} \lambda_{KNP}^{\, K} + \frac{1}{3} \epsilon^{MNPQRST} R_{7}^{\, RST} \lambda_{7} - 6 R_{MN}^{\, K} \lambda_{P}^{\, K} \right\} J^{MNP} \]
\[ + \left\{ R_{7}^{\, a} \lambda^{b} + \frac{1}{2} R_{MN} \Gamma^{MNab} \lambda^{b} + \frac{1}{6} R_{MNP} \Gamma^{MNPab} \lambda^{b} \right\} Q_a \]
Using the explicit expressions for the $Osp(1|8)$-generators (50) it is straightforward to recover the corresponding $SO(1,3)$-decomposition of the curvatures and gauge transformations.

According to the algebraic program described in [14] we split the symmetries generated by the $T_{A}$ into three classes: “sure”, “automatic” and “unsure” symmetries, which we now explain. The action that we consider will be constructed such that it is manifestly invariant under local Lorentz ($M_{mn}$), dilation ($D$), axial ($A$) and general coordinate symmetries. These symmetries we call “automatic” symmetries. However, the usual gauge transformations

$$\delta h^{A} = d\epsilon^{A} + \epsilon^{C} h^{B} f_{BC}^{A},$$

(53)

for any gauge field $h^{A}$, specialized to the case of $P^{m}$ gauge transformations (so that $\epsilon^{C}$ is proportional to the parameter for diffeomorphisms), do not coincide with general coordinate transformations. Therefore, $P^{m}$ gauge transformations are no longer a symmetry of the action, but are replaced by general coordinate transformations. Hence, one no longer considers $P^{m}$ gauge transformations on the left hand side of gauge commutators. However, in order that the symmetry algebra still closes, symmetries whose commutators produce $P^{m}$ gauge transformations must be modified. This is achieved by imposing appropriate constraints on the curvatures. The symmetries $K_{m}$, $F_{mn}$ and $S$ never produce a $P_{m}$ in commutators because they have negative dilaton weights, and are therefore unmodified. We call them “sure” symmetries since they must act on all fields according to the group law (53). (This is, of course, an assumption, to be justified by the results.) All constraints introduced into the theory must be invariant under all symmetries, in particular the sure and automatic symmetries. The remaining gauge transformations for the symmetries $V_{m}$, $Z_{m}$, $Q$ and $E_{mn}$ will, in general, be modified and are called “unsure” symmetries.

In more detail the way that constraints imply the modification of the transformation rules of the unsure symmetries, in contrast to the case of $P_{m}$ gauge transformations which are simply exchanged for general coordinate transformations, is as follows. The introduction of constraints implies that by solving these constraints, certain fields are expressed in terms of other fields; such fields we call dependent fields. In general, these dependent fields no longer transform according to the group law (53), but rather, their transformation rules are obtained by varying their constituent independent fields according
to (53) (the “chain rule”). Of course, in the case of the sure and automatic symmetries which leave the constraints invariant, the transformations of the dependent fields are left unmodified.

We shall determine these constraints first dynamically and then kinematically: first we shall require invariance of the action, and later we shall study the implications for the relation between $P^m$ gauge transformations and general coordinate transformations. As we shall show later, because curvatures transform homogeneously under the group law, the modifications in the transformation rules will contain terms proportional to curvatures. However, further modifications, beyond those implied by constraints, are, in principle, not ruled out.

3 The Set of Sure Constraints from Invariance of the Action under Sure Symmetries.

We now construct an affine action quadratic in curvatures and invariant under the sure symmetries $K_m$, $F_{mn}$ and $S$. By affine we mean that no vierbeins are used to contract indices, but only constant Lorentz tensors such as $\epsilon^{\mu\nu\rho\sigma}$, $\eta_{\mu\nu}$ and Dirac matrices. The most general parity-even, Lorentz-invariant, dilaton-weight zero, mass dimension zero affine action ($S = \int_M \mathcal{L}$ for some four–manifold $M$) reads

$$-\mathcal{L} = \alpha_0 \epsilon_{mnpq} R(M)^{mn} R(M)^{pq} + \alpha_1 R(A) R(D) + \alpha_2 R(V)^m R(Z)_m + \alpha_3 \epsilon_{mnpq} R(E)^{mn} R(F)^{pq} + \beta R(Q) \gamma^5 R(S).$$

(54)

This action is, of course, manifestly general coordinate invariant since the integration measure $\epsilon^{\mu\nu\rho\sigma}$ is a tensor density under general coordinate transformations. The term $\alpha_2 R(V)^m R(Z)_m$ is not chirally invariant (i.e., w.r.t. the axial gauge symmetry $A$), since $R(V)^m$ and $R(Z)^m$ undergo an infinitesimal $SO(2)$ rotation, but all the other terms are chirally invariant ($R(E)^{mn}$ and $R(F)^{mn}$ transform into plus or minus their duals, respectively). One

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8 The action is Hermitean and the curvatures are real if one takes the reality condition for Majorana spinors $\overline{\psi} = \psi^\dagger C_4 = \psi^\dagger i\gamma^0$. We denote the left hand side of the Minkowski action in (54) by $-\mathcal{L}$ to stress that we are using the metric $(-++++)$ rather than the Euclidean notation of [7]. The sign $-\mathcal{L}$ ensures that the kinetic terms for the vierbein have the correct sign, see, for example, reference [27].
could therefore consider setting the coefficient \( \alpha_2 = 0 \) at this point. However, for now, we will leave \( \alpha_2 \neq 0 \), but requiring invariance under sure \( K^m \), \( F^{mn} \) and \( S \) symmetries will in any case imply \( \alpha_2 = 0 \). Since we are interested in a theory of gravity we set \( \alpha_0 = 1 \). (In fact, no nontrivial solution exists for \( \alpha_0 = 0 \).

The requirement that the action in (54) be invariant under the sure symmetries yields

\[
\alpha_0 = 1, \alpha_1 = -32, \alpha_2 = 0, \alpha_3 = 8, \beta = -8,
\]

as we now explain. Under the sure \( K^m \), \( F^{mn} \) and \( S \) symmetries, curvatures simply rotate into curvatures according to the group law,

\[
\delta R^A = -R^C \epsilon^B f_{BC}^A.
\]

One may therefore readily verify\(^9\) that the variation of the action (54) under the sure symmetries \( K^m \), \( F^{mn} \) and \( S \) with parameters \( \epsilon^m \), \( \epsilon^{mn} \) and \( \epsilon^\alpha \), respectively, is

\[
- \delta_K \mathcal{L} = -32 R(P)_m \left[ \tilde{R}(M)^{mn} + \frac{\alpha_1}{16} R(A) \eta^{mn} \right] \epsilon_n + \sqrt{2} \beta R(Q) \gamma^5 \gamma^m R(Q) \epsilon_m \tag{57}
\]

\[
+ 2 \left( (\alpha_2 + 4\alpha_3) \tilde{R}(E)^{mn} R(V)_m + (\alpha_2 - 4\alpha_3) \tilde{R}(E)^{mn} R(Z)_m \right) \epsilon_n
\]

\[
- \delta_F \mathcal{L} = \tilde{R}(E)_m \left[ (4\alpha_3 - 32) R(M)^{m} k_r^{kn} - (4\alpha_3 + \alpha_1) (R(D) \epsilon^{mn} - R(A) \epsilon^{mn}) \right] \tilde{R}(V)_n \tag{58}
\]

\[
- \delta_S \mathcal{L} = -\tilde{R}(Q) \left[ \left( \frac{\alpha_1}{4} - \beta \right) (R(A) + \gamma^5 R(D)) + \left( \frac{\alpha_2}{4} + \beta \right) \gamma^m R(Z)_m \right] \epsilon
\]

\[
+ \tilde{R}(S) \left[ \sqrt{2} \beta \gamma^5 \gamma^m R(P)_m + \left( \frac{\beta}{\sqrt{2}} + \frac{\alpha_3}{\sqrt{2}} \right) \gamma^{mn} \tilde{R}(E)_m \epsilon \right].
\]

We now must find constraints on the curvatures and values for the coefficients \( \alpha_1 \), \( \alpha_2 \), \( \alpha_3 \) and \( \beta \) such that the variations (57)-(59) vanish.

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\(^9\)To bring this result into the form quoted above one needs the Schouten identity

\[
\epsilon_{mnpq} X_r = \epsilon_{rnqp} X_m + \epsilon_{mrpq} X_n + \epsilon_{mrq} X_p + \epsilon_{mnpq} X_q,
\]

from which the following most useful identity for any pair of antisymmetric \( 4 \times 4 \) matrices \( X \) and \( Y \) may be derived

\[
X Y^{mn} \equiv \frac{1}{2} \epsilon_{mnpq} X_{pk} Y_{q}^{k} = \frac{1}{2} [X Y - Y X]^{mn} \equiv X_k [m] Y_n [k].
\]
Consider first $\delta K$. The fermionic term must vanish by itself and this is achieved via the constraint

$$R(Q) = -\gamma^5 \ast R(Q), \quad (60)$$

where we denote the Hodge dual on curved indices by a star $\ast R_{\mu\nu} = (1/2) e \epsilon_{\mu\nu\rho\sigma} R^{\rho\sigma}$ and $e = \det(e^m_\mu)$. (Note that $\bar{R}(Q)_{\mu\nu} \gamma^m R(Q)^{\mu\nu}$ vanishes for Majorana spinors). In principle the sign in this constraint is at this point arbitrary. However, we now impose the additional requirement that all constraints should be solvable. By solvable we mean that one can solve the constraint for some field(s) algebraically by using the invertibility of the vierbein. We shall discuss this issue in detail later at the end of this section, but at this point we observe the following: In (60) we have 12 constraints and in order that we can solve for the 12 gamma-traceless components of $\phi_\mu - \frac{1}{4} \gamma_\mu \gamma \cdot \phi$ instead of the insufficient 4 gamma-trace components $\gamma \cdot \phi$, we need the minus sign in (60). The definition\footnote{Recall also that in Minkowski space, $\epsilon_{\mu\nu\rho\sigma} \epsilon^{mnrs} = -\epsilon_{\mu}^m \epsilon_{\nu}^m \epsilon_{\rho}^m \epsilon_{\sigma}^s + \cdots$} of $\epsilon_{\mu\nu\rho}\sigma$ which achieves this is such that $\gamma_5 \gamma_{\mu\nu} = \frac{1}{2} e \epsilon_{\mu\nu\rho\sigma} \gamma^{\rho\sigma}$.

We turn now to the bosonic terms in $\delta K$. Solvable constraints contain a term linear in curvatures and all terms in the constraint must have the same dilaton weight. Hence the first and last pair of terms in $\delta K$ must cancel independently. Since the constraints $R(M)^{mn} = 0 = R(A)$ are not solvable (because $R(M)^{mn}$ has 36 components but one can at best solve for the 16 components of $f^m_\mu$ whilst $R(A)$ contains no terms with a vierbein), we are forced to impose the constraint

$$R(P)^m = 0. \quad (61)$$

We note that both constraints (60) and (61) also appeared in the original conformal supergravity case [11, 7]. Let us now observe that the constraint (60) should be invariant under the sure symmetry $S$, while on the other hand we find that its variation is given by

$$\delta_S \left[ R(Q) + \gamma^5 \ast R(Q) \right] = \frac{1}{\sqrt{2}} \left[ R(E)^{mn} + \ast \bar{R}(E)^{mn} \right] \gamma_{mn} e. \quad (62)$$

We must therefore impose the additional (solvable) constraint

$$R(E)^{mn} = -\ast \bar{R}(E)^{mn}. \quad (63)$$
We then find that the remaining terms in \(-\delta_{KL}\) may be written as
\[
2R(E)^{mn}[\alpha_2(\ast R(V)_m + R(Z)_m) + 4\alpha_3(\ast R(V)_m - R(Z)_m)]\varepsilon_n. \tag{64}
\]
The curvature \(R(E)^{mn}\) cannot be constrained to vanish as this constraint would not be solvable. However the constraint
\[
\ast R(V)^m = R(Z)^m. \tag{65}
\]
is solvable but is not solvable with the opposite sign. Therefore we must take \(\alpha_2 = 0\) (as predicted already by chiral invariance) and impose the constraint \((65)\) (we will find later that \(\alpha_3 \neq 0\)). One may verify that the set of constraints \((60), (61), (63)\) and \((65)\) is invariant under the sure and automatic symmetries.

Let us now consider \(\delta_{FL}\). Again the first term with \(R(M)^{mn}\) should vanish by itself because the traceless parts of \(R(M)^{mn}\) and \(R(E)^{mn}\) (“the Weyl parts”, see below) cannot be set to zero as these constraints cannot be solved. So we learn that, as promised, \(\alpha_3 = 8 \neq 0\). The constraint \(\ast R(A) = R(D)\) would kill the terms with coefficients \(4\alpha_3 + \alpha_1\), but although solvable, it produces the wrong sign in \((64)\) when varied under the sure symmetry \(S\). (Further, under the sure symmetry \(F_{mn}\), this constraint \(\ast R(A) = R(D)\) rotates into a double self-dual constraint for \(R(E)^{mn}\) instead of the required double anti-self-dual constraint in \((63)\). Under \(K_m\), \(\ast R(A) = R(D)\) is invariant.) Therefore we take \(\alpha_1 = -4\alpha_3 = -32\).

Finally let us study \(\delta_S\mathcal{L}\). There is no solvable constraint on \(R(S)\) so that the last term must be zero by itself. This yields \(\beta = -\alpha_3 = -8\). At this point the first two terms in \((53)\) cancel. Also the term with \(\bar{R}(M)^{mn}\) cancels since \(\beta = -8\) as in \((53)\). We are therefore left only with the third and fourth terms depending on \(R(Z)^m\) and \(R(V)^m\), respectively, which may be rewritten using \((60)\) as
\[
\bar{R}(Q)\gamma^m \left\{ \frac{\alpha_2}{4} [R(Z)_m + \ast R(V)_m] + \beta [R(Z)_m - \ast R(V)_m] \right\}. \tag{66}
\]
The term with \(\beta\) vanishes due to the constraint in \((53)\). Hence again we find \(\alpha_2 = 0\). In summary, requiring invariance under the sure symmetries \(S, K_m\) and \(F_{mn}\) has unambiguously led us to the following action and set of solvable
The constraints on $R(P)^m$ and $R(Q)$ were found already in conformal supergravity, and were not unexpected. The new constraint in (69) is a direct consequence of the $R(Q)$ constraint in (71). The constraint in (70) rotates into the $R(Q)$ chiral self-dual constraint under $S$, hence it is compatible with this constraint. As the reader may verify from the above pages, not only are all coefficients in the action fixed, but several reconfirmations of our results were found in other sectors. The fact that one finds over and over the same conditions on constraints and parameters yields confidence in the results obtained so far.

We call the above set of constraints sure constraints since they are necessary in order that the action is invariant under the sure symmetries. They are, however, not the maximal set of solvable constraints that one could write down. We will consider the maximal set of constraints in the next section, but, as promised above, let us now discuss precisely what is meant by the term “solvable constraint”. A constraint is algebraically solvable only if it depends on the combination (vierbein)$\times$(field) and one can solve the constraint by expressing one or more fields in terms of other fields. (The condition that the constraint depends on the combination (vierbein)$\times$(field) is of course only necessary but not sufficient.) For example, the constraint $R(P)^m = 0$ is solved for the spin connection $\omega_{\mu mn}$ via

$$\omega_{\mu mn} = \frac{1}{2}(-\tilde{R}(P)_{mn} + \tilde{R}(P)_{mn} - \tilde{R}(P)_{nm});$$

$$\tilde{R}(P)^{\mu mn} \equiv R(P; \omega_{\mu mn} = 0)\mu^m$$  

(To avoid confusion we always write indices that were originally curved to the left, for example we denote $R(P)^{\mu mn}e_a^\mu e_b^\nu\eta_{mc}$ by $R(P)_{abc}$.) In the case of
conformal supergravity it was possible in this way to find explicit expressions for dependent fields in terms of the remaining independent fields only. However, notice now that the above solution for the spin connection in (72) is really only an expression for the spin connection in terms of other dependent fields ($v^m$ and $z^m$). In fact, in distinction to the conformal supergravity case, our solutions to the constraints only provide a set of coupled equations for the dependent fields. In principle one could iterate this set of equations and it is even possible that an iterated series solution could terminate at some order. In any case, however, this is a new feature of our extended conformal supergravity model which we shall discuss in more depth below.

4 The Maximal Set of Solvable Constraints.

Let us now study the maximal set of constraints which are solvable in the sense defined above. Clearly only curvatures depending explicitly on undifferentiated vierbeins, namely the non-negative dilaton weight curvatures $R(P)^m, R(E)^{mn}, R(Q), R(M)^{mn}, R(D), R(V)^m$ and $R(Z)^m$, can be solvably constrained, and conversely only the non-positive weight gauge fields $f^m, F^{mn}, \phi, \omega^{mn}, b, v^m$ and $z^m$ can appear in combination with a vierbein and possibly be dependent. To determine exactly which constraints are allowed we must analyse which Lorentz irreducible pieces of the curvatures can be constrained. The results are summarized in figure 1 which we now explain in more detail.

The curvature $R(P)_{\mu\nu}^m$ has 24 independent components and can be decomposed into a trace $R(P)_{\mu\nu}^\rho = \frac{1}{4}$, a totally antisymmetric part $\epsilon^{\mu\nu\rho\sigma} R(P)_{\nu\rho\sigma} = \frac{1}{4}$ and the remaining traceless piece (16) all of which may be constrained and solved for in terms of the 24 components of $\omega_{\mu}^{mn}$. The constraint (18) is clearly already the maximal constraint possible for $R(P)_{\mu\nu}^m$. Note therefore, in particular, that some combination of the Lorentz connection and the dilaton connection is a dependent field, but whether the dilaton field or the trace over the spin connection, or some combination of the two, is the independent part is at this point not yet settled.

The curvatures $R(V)_{\mu\nu}^m$ and $R(Z)_{\mu\nu}^m$ have the same Lorentz decomposition as $R(P)_{\mu\nu}^m$. The constraint $R(Z)^m = * R(V)^m$ in (74) yields 24 equations which can be solved by expressing the 24 components of $F_{\mu}^{mn}$ in terms of other fields. Hence no further constraints involving only $R(V)^m$ and $R(Z)^m$
\begin{table}
\begin{align*}
R(P)_{\mu\nu}^m &= 24 = \begin{array}{c} 16 \end{array} + \begin{array}{c} 4 \end{array} + \begin{array}{c} 4 \end{array} \\
R(E)_{\mu\nu}^{mn} &= 36 = \begin{array}{c} 1 \end{array} + \begin{array}{c} 10 \end{array} + \begin{array}{c} 9 \end{array} + \begin{array}{c} 9 \end{array} + \begin{array}{c} 6 \end{array} + \begin{array}{c} 1 \end{array} \\
R(Q)_{\mu\nu} &= 24 = \begin{array}{c} 8 \end{array} + \begin{array}{c} 12 \end{array} + \begin{array}{c} 4 \end{array} \\
R(M)_{\mu\nu}^{mn} &= 36 = \begin{array}{c} 1 \end{array} + \begin{array}{c} 10 \end{array} + \begin{array}{c} 9 \end{array} + \begin{array}{c} 9 \end{array} + \begin{array}{c} 6 \end{array} + \begin{array}{c} 1 \end{array} \\
R(D)_{\mu\nu} &= 6 \\
R(V)_{\mu\nu}^m + *R(Z)_{\mu\nu}^m &= 24 = \begin{array}{c} 16 \end{array} + \begin{array}{c} 4 \end{array} + \begin{array}{c} 4 \end{array} \\
R(V)_{\mu\nu}^m - *R(Z)_{\mu\nu}^m &= 24 = \begin{array}{c} 16 \end{array} + \begin{array}{c} 4 \end{array} + \begin{array}{c} 4 \end{array}
\end{align*}
\end{table}

Figure 1: Lorentz irreducible pieces of the “solvable” curvatures. The ticks “√” and crosses “×” indicate those Lorentz irreducible pieces of curvatures that may or may not, respectively, be solvably constrained.

are possible.

The curvature \( R(Q)_{\mu\nu} \) is a Majorana spinor and hence has 24 real components which are decomposed into a single gamma-trace \( \gamma^\nu R(Q)_{\mu\nu} - \frac{1}{4} \gamma_\mu \gamma^{\alpha\beta} R(Q)_{\alpha\beta} = 12 \) (which vanishes if traced a second time), a double gamma-traceless part \( 8 \) and a gamma-traceless part \( 8 \). The constraint that the 8 vanishes cannot be solved so it must be unconstrained. So far only the 12 is constrained by the chiral self-dual constraint \((71)\). However the maximal constraint

\[
\gamma^\nu R(Q)_{\mu\nu} = 0
\]

was found to be necessary in the case of conformal supergravity in order that the action be invariant under supersymmetry. It is solved for in terms of all 16 components of \( \phi_\mu \) and is equivalent to the sum of two constraints: the chiral self-dual constraint in \((71)\) and the double gamma-trace constraint

\[
\gamma^{\mu\nu} R(Q)_{\mu\nu} = 0.
\]
We shall later argue that also in our case the full constraint in (73) must be imposed.

The curvature $R(E)_{\mu\nu}{}^{mn}$ has 36 components. Taking a single trace yields the 16 component Ricci $R(E)_{\mu\nu}{}^{mn}$ which may be further decomposed into its antisymmetric (6), trace (1) and symmetric traceless parts (9) which may be solved for in terms of the combinations $z_{[\mu\nu]} + *v_{[\mu\nu]}$, $v_{\mu}$ and $v(\mu\nu) - \frac{1}{4}g_{\mu\nu}v_\rho^\rho$, respectively. The remaining 20 trace free components decompose further into a piece antisymmetric in the interchange of the first and last pair of indices ($\bar{9}$), a piece totally antisymmetric in all four indices (1) and a 10 which is the traceless piece symmetric in pairwise interchange whose totally antisymmetric part (1) has been subtracted out. The 10 cannot be solved for but the $\bar{9}$ and 1 may be solved for in terms of $z(\mu\nu) - \frac{1}{4}g_{\mu\nu}z_\rho^\rho$ and $z_\mu^\mu$ respectively. If one now considers all combinations of $v_{\mu}^m$ and $z_{\mu}^m$ which can be solved for in all these constraints, one sees that only the combination $z_{[\mu\nu]} - *v_{[\mu\nu]}$ does not appear in any constraint. There remains, of course, a freedom to choose which combination of $z_{[mn]}$ and $*v_{[\mu\nu]}$ one takes to be dependent, but there will always exist a second combination that remains independent. We will investigate this freedom along with the freedom to choose which combination of the dilaton field and trace of the spin connection remains independent in the following sections.

The double anti-self-dual constraint $R(E)^{mn} = -*\tilde{R}(E)^{mn}$ in (69) contains 18 equations. There are two ways to obtain 18 by combining the dimensions 1, 1, 6, 9, $\bar{9}$ and 10. One combination corresponds to 1, 1, 6 and 10, but since the constraint 10 cannot be solved it would be a disaster if (69) would correspond to this combination of constraints. Fortunately, the sign in (69) is precisely such that it corresponds to the solvable combination of constraints with dimensions 9 and $\bar{9}$. Our discussion implies that further solvable constraints on $R(E)^{mn}$ are possible, namely those with dimensions 1, 1 and $\bar{9}$, which can be written explicitly as follows

$$R(E)_{\nu[\mu\nu]} = 0$$

(75)

$$R(E)_{\mu\nu}{}^{\mu\nu} = 0$$

(76)

$$\epsilon^{\mu\nu\rho\sigma} R(E)_{\mu\rho\sigma} = 0.$$  

(77)

Note that the constraints (75)-(77) also follow as a consequence of the constraint (74) by requiring invariance under the sure symmetry $S$. (The variation of $\gamma^{\mu\nu} R(Q)_{\mu\nu} = 0$ leads to 16 conditions, 8 of which are represented
by \((74)-(77))\).

It is easy to check that the complete set of solvable constraints for \(R(E)^{mn}\), \(R(Q)\), \(R(V)^m\) and \(R(Z)^m\) are invariant under the sure and automatic symmetries. We shall later see that the additional constraints \((74)\), \((75)\), \((76)\) and \((77)\) (i.e., those constraints which were not required for the invariance of the action under the sure symmetries) must be imposed in order to obtain invariance under \(V_m\) and \(Z_m\) symmetries.

The Riemann tensor \(R(M)_{\mu\nu}^{mn}\) has the same decomposition as \(R(E)_{\mu\nu}^{mn}\), however, only the 16 components of the Ricci tensor \(R(M)_{\mu\nu}^{\nu m}\) can be constrained and solved for in terms of the field \(f_{\mu}^m\). Therefore, a constraint on the Ricci tensor along with all constraints given above, represents the maximal set of solvable constraints. The construction of such a Ricci constraint is the subject of the next section.

5 The Complete Ricci Constraint

The construction of a constraint on the Ricci curvature, which is solvable in terms of \(f_{\mu}^m\) and is invariant under the sure symmetries, is somewhat subtle. Moreover we observe that the antisymmetric part of \(f_{\mu\nu}\) (a \(\delta\)) occurs not only in the \(\delta\) of \(R(M)_{\mu\nu}^{mn}\) but also in the \(\delta\) of \(R(D)_{\mu\nu}\). Hence, it is not clear at this point whether one should constrain the \(\delta\) of \(R(M)_{\mu\nu}^{mn}\), or \(R(D)_{\mu\nu}\), or perhaps a linear combination. In fact, as we shall argue, both should be constrained. Since the same situation occurred in conformal supergravity, we temporarily digress to the latter model. In that model we shall obtain a useful new interpretation of the maximal solvable \(R(M)_{\mu\nu}^{mn}\) constraint which we will generalize to our model of extended conformal supergravity.

Intermezzo: Conformal Supergravity.

Conformal supergravity is the gauge theory of the superconformal algebra \(SU(2,2|1)\). An explicit (reducible) representation of the bosonic conformal

\[\text{To avoid confusion, note that, in contrast to torsionless Riemannian general relativity, the Ricci tensor here has also an antisymmetric part (a \(\delta\)) and is, therefore, a completely general two index tensor.}\]

\[\text{The terminology “extended” as used here, should, of course, not be confused with the more common usage referring to } N \geq 1 \text{ supersymmetries.}\]
algebra \( SU(2, 2) \subset Sp(8) \) is obtained from the bosonic representation in \([20]\) by dropping the extensions \( E_{mn}, V_m, Z_m \) and \( F_{mn} \). Its supersymmetric extension \( SU(2, 2|1) \) is summarized by the following representation in terms of curvature two-forms

\[
R(P)^m = de^m + \omega^m e^n + 2be^m - \frac{1}{4\sqrt{2}} \bar{\psi} \gamma^m \psi
\]

\[
R(Q) = d\psi + (3a\gamma^5 + b + \frac{1}{4} \omega^{mn} \gamma_{mn}) \psi + \sqrt{2} e^m \gamma_m \phi
\]

\[
R(M)^{mn} = d\omega^{mn} - \omega^{[m} \omega^{n]k} - 8e^{[m} f^{n]} + \frac{1}{2} \bar{\psi} \gamma^{mn} \phi
\]

\[
R(D) = db - 2e^m f_m + \frac{1}{4} \bar{\psi} \phi
\]

\[
R(A) = da + \frac{1}{4} \bar{\psi} \gamma^5 \phi
\]

\[
R(S) = d\phi + (-3a\gamma^5 - b + \frac{1}{4} \omega^{mn} \gamma_{mn}) \phi - \sqrt{2} f^m \gamma_m \psi
\]

\[
R(K)^m = df^m + \omega^m f^n - 2b f^m + \frac{1}{4\sqrt{2}} \bar{\phi} \gamma^m \phi.
\]

Note that the superalgebra \( SU(2, 2|1) \) is not a subalgebra of \( Osp(1|8) \). This is most clearly seen by analyzing their embedding in \( Osp(2|8) \) \([26]\): let the oscillators \( a_A = (a^K, \pi_K, a, \bar{a}) \) have the (anti)commutation relations \( [a^K, \pi_L] = \delta^K_L, \{a, \bar{a}\} = 1 \). Here \( \pi_K = \eta_{KL} a^K \) is up to the \( SU(2, 2) \)-metric \( \eta_{KL} \) the complex conjugate of \( a_K \). A real \( Sp(8) \)-spinor is represented by the complex pair \( (a^K, \pi_K) = a_a \). \( Osp(2|8) \) has a total of 16 real supersymmetry charges, namely the two \( Sp(8) \)-multiplets \( Q_a^+ = a_a a = (a^K a, \pi_K a) \) and \( Q_a^- = a_a \bar{a} = (a^K \pi, \pi_K \bar{a}) \). The subalgebra \( SU(2, 2|1) \) is obtained by selecting the supercharges \( Q^K = a^K \pi \) and \( \overline{Q}_K = \pi_K a \), while \( Osp(1|8) \) contains eight different supercharges, namely \( Q_a = a_a (a + \bar{a})/\sqrt{2} \). As a consequence, the \( Q_a Q_b \) anticommutator produces, among other things, an \( E_{mn} \) generator. The \( SU(2, 2|1) \)-curvatures thus are not obtained by setting the new fields \( E^{mn}, v^m, z^m \) and \( F_{mn} \) to zero. Note, however, that one only needs to give the the \( a\gamma^5 \psi \) and \( a\gamma^5 \phi \) terms in the \( R(Q) \) and \( R(S) \) curvatures, respectively, an additional factor \(-3\). One may check that the Fierz identities required to obtain the Bianchi identity for the curvatures hold only with the correct factor \(-3\) given above. In the oscillator representation one readily sees this factor: the bosonic generators of \( SU(2, 2|1) \) are given by
\( J^k_L = \frac{1}{2} \{ a^K, \bar{\alpha}_L \} - \frac{1}{8} \delta^k_L \{ a^N, \bar{\alpha}_N \} \) and \( J = \frac{1}{2} \{ a^K, \bar{\alpha}_K \} = \frac{1}{2} [a, \bar{\alpha}] \) (which implies a nontrivial trace condition on the total Hilbert space) and hence

\[
\{ Q^K, \bar{Q}_L \} = \frac{1}{2} \{ a^K, \bar{\alpha}_L \} - \frac{1}{2} \delta^k_L [a, \bar{\alpha}] = J^k_L - \frac{3}{4} \delta^k_L J ,
\]  

(84)

while for \( Osp(1|8) \) we obtain

\[
\{ Q^K, \bar{Q}_L \} = \frac{1}{2} \{ a^K, \bar{\alpha}_L \} = J^k_L + \frac{1}{4} \delta^k_L J ,
\]

(85)

where we have defined \( J = \frac{1}{2} \{ a^K, \bar{\alpha}_K \} \) in the same fashion. Apart from this factor, and of course the generators \( J^k_L = a^{(K} \alpha_L^{L)} \) and \( J_{KL} = \bar{\alpha}(K \bar{\alpha}_L) \), the two algebras are identical.

One can give an explicit \( 5 \times 5 \) matrix representation \([11]\) of \( SU(2, 2|1) \), with the axial generator \( A \) represented by a supertraceless diagonal matrix with entries proportional to \( (1, 1, 1, 1, -4) \).

Again one may write down the most general parity even, dilaton weight zero, affine action and fix the coefficients and sure constraints by requiring invariance with respect to the sure symmetries \( K_m \) and \( S \). The results are

\[
- \mathcal{L} = \epsilon_{mnpq} R(M)^{mn} R(M)^{pq} + 32 R(A) R(D) - 8 \gamma \gamma^5 R(S) ,
\]  

(86)

\[
R(P)^m = 0
\]

(87)

\[
R(Q) = -\gamma^5 \gamma^5 R(Q),
\]

(88)

\[
R(A) = \gamma^5 \gamma^5 R(D).
\]

(89)

\(^{13}\)Observe that the term \( 32 R(A) R(D) \) appears with an opposite sign from the \( Osp(1|8) \) case. To study the sign of the kinetic terms of the axial \( a_\mu \) and vierbein \( e_\mu^m \) fields, one must substitute the leading terms of the solution of the constraint \([12]\) below for the conformal boost gauge field \( f_\mu^m \) into (86). The result is

\[
- \mathcal{L}_{\text{Kin}} = -2 [R(\omega)_{\mu \nu} R(\omega)^{\nu \mu} - \frac{1}{3} R(\omega)^2] + 24 R(A)_{\mu \nu} R(A)^{\mu \nu} ,
\]

where all terms have the required sign and \( R(\omega)_{\mu \nu} \equiv R(\omega)_{\alpha \mu \nu}^\alpha \) and \( R(\omega) \equiv R(\omega)_{\mu \nu}^\mu \) are the Ricci and scalar curvatures, respectively, of the usual Riemann curvature \( R(\omega)_{\mu \nu}^m = 2 \partial_{[\mu} \omega_{\nu]}^m + 2 \omega_{[k}^m \omega_{\nu] n]} k. \) Note, however, that the analogous calculation (using now the constraint \([17]\) derived below) in the same sector of the \( Osp(1|8) \) model yields

\[
- \mathcal{L}_{\text{Kin}} = -2 [R(\omega)_{\mu \nu} R(\omega)^{\nu \mu} - \frac{1}{3} R(\omega)^2] - 8 R(A)_{\mu \nu} R(A)^{\mu \nu} ,
\]

so that the kinetic terms of the axial gauge field now appear with the opposite (unphysical) sign.
These constraints are themselves invariant under the sure symmetries but do not represent a maximal set of constraints. In the early work of Kaku et. al. [11, 7], the constraint (89) was not yet known (but rather was discovered later in the algebraic approach in [14]). Furthermore, instead of the affine action in (88) a non-affine action was employed which is clearly equivalent to (88) since it differs only by the constraint (89). Consequently, also (90) is invariant under \( K \) and \( S \) symmetry. The non-geometric action is interesting for two reasons: firstly, it is invariant under the sure symmetries without having to use (89), and that was how it was found in [7]. Second, if one considers the algebraic field equation for the \( f^m \) field in the non-affine action

\[
\mathcal{L}_{\text{non-affine}} = \mathcal{L} + 64R(A)(R(D) + \ast R(A)),
\]

was employed which is clearly equivalent to (88) since it differs only by the constraint (89). Therefore, (91) also represents a possible constraint on \( R(M)^{mn} \) which may be solved in terms of \( f^m \). Imposing (91) as a constraint implies, of course, that the non-geometric action is invariant under arbitrary variations of the field \( f^m \) (the so-called 1.5 order formalism in supergravity [28]). Rather than imposing (91) as a field equation, it would be preferable to consider it as a constraint, on a par with (87)-(89). Therefore, in [14] the action (86) was chosen and (91) imposed. Of course, in the affine action (86) the field equation for \( f^m \) no longer coincides with the constraint (91).

The action (86) (and clearly also (90)) is obviously Lorentz \( (M_{mn}) \), dilation \( (D) \), axial \( (A) \) and general coordinate invariant so one only needs to check invariance under the unsure symmetry \( Q \). The fields \( e^m, \psi, b \) and \( a \) transform under \( Q \) according to the group law (i.e. as ordinary gauge fields), but the dependent fields \( \omega^{mn}, \phi \) and \( f^m \) get extra transformations in order that the constraints remain valid. It is therefore convenient to work with the non-affine action (which is, at this point, completely equivalent to the affine action because the constraint \( R(A) = \ast R(D) \) holds) since one therefore need not calculate the extra transformations of the field \( f^m \). In [7] it is shown that

\[\text{Note that } R(A) = \ast R(D) \text{ implies } R(D) = - \ast R(A).\]
the theory is supersymmetry \((Q)\) invariant if one extends the constraint (88) to read

\[
\gamma^\mu R(Q)_{\mu\nu} = 0. \tag{92}
\]

The constraints (87), (91) and (92) represent the maximal set of solvable constraints in this model.

Finally, let us complete this section by showing that the constraint (89) follows as a consequence of the constraints (91) and (92). The Bianchi identity for the constraint (87) written in the form (38) yields a new constraint

\[
0 = dR(P)^m = R(M)^{mn}e_n + 2R(D)e_m + \frac{1}{2\sqrt{2}}\bar{\psi}\gamma^m R(Q). \tag{93}
\]

Although the form of the two constraints (91) and (93) appears similar, in fact they are inequivalent, since the former is a constraint on the Ricci (1, 6 and 9) of \(R(M)^{mn}\) whereas the Bianchi identity (93) constrains the \(\tilde{1}, 6\) and \(\tilde{9}\). There is no contradiction with the claim above that only the Ricci part of \(R(M)_{\mu\nu}^{mn}\) could be solvably constrained since the Bianchi identity is an identity so that (93) holds simply because the dependent field \(\omega_{\mu}^{mn}\) is the solution to \(R(P)^m = 0\). The two constraints overlap on the \(6\) of \(R(M)_{\mu\nu}^{mn}\) and using the constraint \(\gamma^\mu R(Q)_{\mu\nu} = 0\) in (92) they may be rewritten as the following constraints on the \(6\)

\[
0 = R(M)_{\rho[\mu\nu]}\cdot + 2sR(A)_{\mu\nu} + \frac{1}{4\sqrt{2}}\bar{R(Q)}_{\mu\nu}^\gamma \cdot \psi \{f_m \text{ equation}\} \tag{94}
\]

\[
0 = R(M)_{\rho[\mu\nu]}\cdot - 2R(D)_{\mu\nu} + \frac{1}{4\sqrt{2}}\bar{R(Q)}_{\mu\nu}^\gamma \cdot \psi \{\text{Bianchi}\}. \tag{95}
\]

The difference is clearly given by (89).

The reader may feel that the way we have arrived at the constraint in (91) was rather indirect (a field equation in one model was turned into a constraint of another model), but a completely kinematical approach yields all these constraints directly. Requiring compatibility between \(P_m\) gauge transformations and general coordinate transformations as discussed in section 4 leads directly to the constraints (87), (88) and (91) from which (88) follows in the manner indicated [14].
Motivated by the above discussion, we consider a one parameter \( \alpha \) family of non-affine actions differing from \((67)\) only by terms that vanish due to the constraint \( R(V)^m = - * R(Z)^m \)

\[
\mathcal{L}_{\text{non-affine}} = \mathcal{L} + 32(1 - 2\alpha)R(V)^m R(Z)_m + 32\alpha R(V)^m * R(V)_m + 32(1 - \alpha)R(Z)^m * R(Z)_m.
\]  

One may consider more general non-affine actions equivalent to the action \((67)\) but the requirement that the \( f^m \) field equation be invariant under the sure symmetries leads one to \((96)\). We propose the \( f^m \) field equation of these non-affine actions as the 16 constraint for \( R(M)^{mn} \) which reads

\[
0 = \tilde{R}(M)^{mn} e_n + 2R(A) e^m - \frac{1}{2\sqrt{2}} \overline{\psi} \gamma^5 \gamma^m R(Q) - 2\tilde{R}(E)^{mn} v_n \\
- 2\tilde{E}^{mn} R(V)_n + 2R(E)^{mn} z_n + 2E^{mn} R(Z)_n,
\]  

where after varying the non-affine action in \((86)\) under \( f^m \to f^m + \delta f^m \), we imposed the constraint \( R(V)^m = - * R(Z)^m \). The new constraint in \((97)\) is invariant under the sure and automatic symmetries and is to be compared with the following Bianchi identity

\[
0 = dR(P)^m = R(M)^{mn} e_n + 2R(D) e^m + \frac{1}{2\sqrt{2}} \overline{\psi} \gamma^5 \gamma^m R(Q) - 2R(E)^{mn} v_n \\
+ 2E^{mn} R(V)_n - 2\tilde{R}(E)^{mn} z_n + 2\tilde{E}^{mn} R(Z)_n.
\]  

The constraint \((97)\) may also be found by writing down the most general three form constraint \( 0 = \tilde{R}(M)^{mn} e_n + \ldots \) with fixed parity and dilaton weight two, and fixing the coefficients by requiring invariance under the sure symmetries. We note also that, just like the Bianchi identity, to achieve invariance of the constraint \((17)\) under sure symmetries, one needs only use that \( \tilde{R}(P)^m = 0 \).

\[\text{Back to } Osp(1|8).\]

\[\text{Also in the conformal supergravity case, without altering any of the conclusions outlined in the above intermezzo, one can consider a one parameter family of additional non-affine terms in the action that vanish modulo the constraint } \tilde{R}(D) = - * R(A), \text{ namely } 0 = 64(1 - 2\alpha)\tilde{R}(D) R(A) 64\alpha R(D)_s R(D) 64(1 - \alpha)R(A)_s R(A), \text{ although in } \text{(f)} \text{ only the case with no } R(D)_s R(D) \text{ term (} \alpha = 0 \text{) was considered.}\]
Again, the constraints (97) and (98) overlap on the 6 of \( R(M)_{\mu\nu} \) so let us compute the difference of these two constraints which will, as a consequence, lead to another useful constraint.

The 6 of (97) and (98) is given, respectively, by

\[
0 = \star R(M)_{\mu|\rho\nu\sigma} - 2\star R(A)_{\mu\nu} + \frac{1}{4\sqrt{2}} \psi \cdot \gamma R(Q)_{\mu
u}
- R(E)_{\mu\nu\rho\sigma} v^{\rho\sigma} + 2\bar{E}^{\rho\sigma}_{\mu\nu} R(Z)_{\nu|\rho\sigma} - \star R(E)_{\mu\nu\rho\sigma} z^{\rho\sigma} + 2E^{\rho\sigma}_{\mu\nu} R(V)_{\nu|\rho\sigma}.
\]

(99)

To obtain (99) and (100) from (97) and (98), respectively, along with the sure constraints (69), (70) and (71), we used the following cyclicity relations:

\[
R(E)_{\mu\nu\rho\sigma} = 0
\]

(101)

which constrains the 1, 9 and 6 of \( R(E)_{\mu\nu} \) to zero as in (77), half of (69) and (75), respectively,

\[
\star R(E)_{\mu\nu\rho\sigma} = 0
\]

(102)

which constrains the 1, 9 and 6 of \( R(E)_{\mu\nu} \) to zero as in (76), the other half of (69) and again (75), respectively, and

\[
\gamma_{\mu R(Q)_{\rho\sigma}} = 0
\]

(103)

which follows from the constraint (73). Dualizing (99) in the indices \( \mu \) and \( \nu \) and then adding this result to (100) yields (dividing by an overall factor 2)

\[
0 = R(A)_{\mu\nu} - \star R(D)_{\mu\nu} + \bar{E}^{\rho\sigma}_{\mu\nu} R(V)_{\nu|\rho\sigma} - E^{\rho\sigma}_{\mu\nu} R(Z)_{\nu|\rho\sigma}
+ \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \left[ E^{\rho\sigma\alpha} R(V)_{\beta\rho\sigma} + \bar{E}^{\rho\sigma\alpha} R(Z)_{\beta\rho\sigma} \right].
\]

(104)

To see that the higher order terms in (104) do not cancel one can use \( R(V)^{m} = -\star R(Z)^{m} \) to write the result in terms of \( R(Z)^{m} \) only.
In summary, we have found the following maximal set of solvable constraints

\begin{align}
0 &= R(P)_{\mu\nu}^m \\
0 &= R(E)_{\rho[\mu\nu]}^\rho \\
0 &= R(E)_{\mu\nu}^{\nu\mu} \\
0 &= \epsilon^{\mu\nu\rho\sigma} R(E)_{\mu\nu\rho\sigma} \\
0 &= R(E)_{\mu\nu}^{mn} + \bar{R}(E)_{\mu\nu}^{mn} \\
0 &= R(Z)_{\mu\nu}^m - \ast R(V)_{\mu\nu}^m \\
0 &= \gamma^\mu R(Q)_{\mu\nu}
\end{align}

\begin{align}
0 &= R(M)_{\rho\mu\nu}^\rho - \frac{1}{2} g_{\mu\nu} R(M)_{\rho\sigma}^{\sigma\rho} + 2 s R(A)_{\mu\nu} + \frac{1}{2\sqrt{2}} R(Q)_{\rho\nu} \gamma_\mu \psi^\rho \\
&\quad - 2 s R(E)_{\rho\sigma\mu} z^{\rho\sigma} - 2 R(E)_{\rho\sigma\mu} t^{\rho\sigma} + 2 R(V)_{\rho\mu\sigma} E^{\rho\sigma}_{\mu} + 2 R(Z)_{\rho\mu\sigma} \tilde{E}^{\rho\sigma}_{\mu}.
\end{align}

All further constraints must follow from this set, either algebraically or, for example, by Bianchi identities.

6 Invariance of the Action under $V_m$ and $Z_m$

Symmetries at the Linearized Level.

At this point we have found the maximal set of solvable constraints and an action, both of which are invariant under sure and automatic symmetries. In principle, therefore, we should now simply vary the action with respect to the remaining unsure symmetries taking into account that the constraints endow the dependent fields with extra transformations over and above the usual group law gauge transformations. However, unlike the conformal supergravity case, where one had explicit solutions for the dependent fields in terms of independent fields so that the above program could be (and was) directly carried out, we must now grapple with the fact that we only have a coupled set of equations for the dependent fields whose iterative solution, in general, is an infinite series in independent fields.

Of course one may argue that to investigate the invariance of the action under the remaining symmetries one needs only the extra transformations
of dependent fields rather than explicit solutions for the fields themselves. However again the same problem arises, namely that the constraints provide only a coupled set of equations for the extra transformations of dependent fields. In order to calculate further, we make a consistent expansion in the number of fields and study the model in the lowest order in this expansion.

Consider, for example, the constraint \( R(P)^m = 0 \) and some unsure symmetry which we denote by \( \delta \). On independent fields \( \delta \) acts simply as a gauge transformation

\[
\delta h^A_{\text{Indep}} = d \epsilon^A + \epsilon^C h^B f_{BC}^A = \delta_{\text{Group}} h^A_{\text{Indep}}. \tag{113}
\]

However acting on dependent fields we have

\[
\delta h^A_{\text{Dep}} = d \epsilon^A + \epsilon^C h^B f_{BC}^A + \hat{\delta} h^A_{\text{Dep}} \equiv \delta_{\text{Group}} h^A_{\text{Dep}} + \hat{\delta} h^A_{\text{Dep}}. \tag{114}
\]

where the extra transformations \( \hat{\delta} h^A_{\text{Dep}} \) are determined by requiring that the constraints are invariant under the unsure symmetry \( \delta \), for example

\[
0 = \delta R(P)^m = \delta_{\text{Group}} R(P)^m + \hat{\delta} R(P)^m = \delta_{\text{Group}} R(P)^m + \hat{\delta} \omega^{mn} e_n - 2E^{mn} \hat{\delta} v_n - 2\tilde{E}^{mn} \hat{\delta} z_n. \tag{115}
\]

Note that in (115), the extra transformations of three dependent fields appear, as opposed to conformal supergravity where only \( \hat{\delta} \omega^{mn} \) was present. One may write down similar expressions for all other constraints which in principle uniquely determine all extra transformations of the dependent fields.

In practice, to write down a solution for the extra transformations of the dependent fields, we solve the set of coupled equations for the extra transformations given by requiring invariance of the constraints iteratively. Namely, we make an expansion order by order in the number of independent fields, where one counts the vierbein as a Kronecker delta (i.e. field number zero). In this section, we show that the action is invariant under \( V_m \) and \( Z_m \) symmetries at the leading order of this expansion.

For example, in (115) one first ignores the terms \( -2E^{mn} \hat{\delta} v_n - 2\tilde{E}^{mn} \hat{\delta} z_n \) since they are next to leading order and solves for the lowest order contribu-
tion to \( \tilde{\delta}\omega^{mn} \) which is given by

\[
\tilde{\delta}\omega_{\mu mn} = \frac{1}{2}(-\delta_{\text{Group}} R(P)_{m\mu} + \delta_{\text{Group}} R(P)_{\mu mn} - \delta_{\text{Group}} R(P)_{\mu mn}) + \cdots. \tag{116}
\]

One can proceed similarly for all other constraints and then insert the leading order results in the next order contributions thereby generating expressions of the form

\[
\tilde{h}^{A}_{\text{Dept.}} \sim (\text{curvatures}) + (\text{curvatures}) \times (\text{fields}) + (\text{curvatures}) \times (\text{fields})^2 + \cdots. \tag{117}
\]

Note that one must be somewhat careful when dealing with derivatives on dependent fields (for example the constraint \( R(Z)^m = \ast R(V)^m \) produces terms \( d\tilde{\delta}z^m \) and \( d\tilde{\delta}v^m \)). We count a derivative as field number zero so that at each order we cancel all derivative terms. Note, however, that often one can convert expressions involving explicit derivatives into expressions involving only curvatures and fields. However, as we shall soon see, at the lowest order we are able to remove all terms involving explicit derivatives.

We have explicitly checked the consistency of the expansion described above by carrying out the analogous calculation in the completely understood context of conformal supergravity, in order to verify "unsure" local supersymmetry at leading order.

Let us begin our lowest order ("linear") analysis. To compute the variation of the action with respect to \( \delta \) we need expressions for both \( \delta_{\text{Group}} \mathcal{L} \) and \( \tilde{\delta}\mathcal{L} \). In appendix A we compute expressions for both \( \delta_{\text{Group}} \mathcal{L} \) and \( \tilde{\delta}\mathcal{L} \) valid to all orders in our iterative expansion. All terms in \( \delta_{\text{Group}} \mathcal{L} \) (see (173)-(177)) are of the form \( (\text{curvature}) \times (\text{curvature}) \), since curvatures rotate homogeneously under the group law, and are of leading order. The lowest order terms in \( \tilde{\delta}\mathcal{L} \) (see (183)) are those of the form \( (\text{curvature}) \times (\text{vierbein}) \times \tilde{\delta}(\text{dept. field}) \), where, to linear order, the extra transformations of dependent fields yield curvatures (as in (117) above). In order that the action be invariant at leading order, terms quadratic in curvatures in \( \tilde{\delta}\mathcal{L} \) in (183) must cancel the group transformations of the action \( \delta_{\text{Group}} \mathcal{L} \) in (173)-(177). To lowest order, the variation of the action is given by

\[
-\delta \mathcal{L} = -\delta_{\text{Group}} \mathcal{L} + 32R(V)^m \delta_{\text{Group}} [R(Z)_m - \ast R(V)_m] - 16\sqrt{2} \ast R(S) \gamma^\mu \gamma^m e_m \tilde{\delta}\phi + 32R(K)^m e^n \tilde{\delta}\omega_{mn} - 128\tilde{R}(F)_{mn} e^m \tilde{\delta}v^n - 128\tilde{R}(F)_{mn} e^m \tilde{\delta}z^n + O \left( [\text{curvature}]^2 \times \text{(field)} \right) \tag{118}
\]
which is obtained by keeping only the terms with an explicit vierbein, along
with the first two terms in (181). Notice, as promised, by virtue of various
manipulations made on the expression for $\hat{\delta}L$ as given in appendix A, only
extra transformations of dependent fields arising from constraints without
derivatives on dependent fields are needed (i.e., at linear order, $\hat{\delta}v^m$ and
$\hat{\delta}z^m$ are determined from the $R(E)^{mn}$ constraints [106]-[109], $\hat{\delta}\omega^{mn}$ from
$R(P)^m = 0$ in [105], and $\hat{\delta}\phi$ from $\gamma^\mu R(Q)_{\mu\nu} = 0$ in [111]). Also observe that
even though the dilation gauge field $b_\mu$ may be dependent, at linear level it
makes no contribution to the extra variations of the action.

To verify that the action is invariant at leading order under $V_m$ and $Z_m$
symmetries, it only remains to insert the explicit leading order results for
the extra transformations of dependent fields into $\hat{\delta}L$ in (118) and calculate
the group variation of the constraint $R(Z)_m - \star R(V)_m$ appearing in the first
two terms of (118) and then compute the sum $\delta_{\text{Group}}L + \hat{\delta}L$. To this end
let us give the general, leading order results for the extra transformations of
the fields $v_\mu^m, z_\mu^m, \omega_\mu^{mn}, b_\mu$ and $\phi_\mu$ obtained by requiring invariance of the
maximal set of constraints. (The corresponding results for $f_\mu^m$ and $F_\mu^{mn}$ are
easily calculated but are not needed for our linear analysis.)

\begin{align}
\hat{\delta}v_{(\mu\nu)} & = \frac{1}{4} \delta_{\text{Group}} \left[ R(E)^{\rho(\mu\nu)} - \frac{1}{6} g_{\mu\nu} R(E)^{\sigma\rho} \right] \\
\hat{\delta}z_{(\mu\nu)} & = \frac{1}{4} \delta_{\text{Group}} \left[ \bar{R}(E)^{\rho(\mu\nu)} g^{\rho\nu} - \frac{1}{6} g_{\mu\nu} \bar{R}(E)^{\sigma\rho} g^{\rho\sigma} \right] \\
\hat{\delta}z_{[\mu\nu]} & = \frac{1}{4} \alpha_\ast (\delta_{\text{Group}} R(E)^{\rho(\mu\nu)} - \frac{1}{6} g_{\mu\nu} R(E)^{\sigma\rho}) \\
\hat{\delta}v_{[\mu\nu]} & = \frac{1}{4} (1 - \alpha) \delta_{\text{Group}} R(E)^{\rho(\mu\nu)} \\
\hat{\delta}\omega^{0}_{0mn} & = \frac{1}{2} \delta_{\text{Group}} \left[ R(P)^0_{mn\mu} - R(P)^0_{\mu mn} + R(P)^0_{\mu mn} \right] \\
\hat{\delta}\omega_m & = (1 - \beta) \delta_{\text{Group}} R(P)_m \\
\hat{\delta}b_\mu & = \frac{1}{6} \beta \delta_{\text{Group}} R(P)_\mu \\
\hat{\delta}\phi_\mu & = \frac{1}{2 \sqrt{2}} \delta_{\text{Group}} \left[ \gamma^\rho R(Q)_{\mu\rho} + \frac{1}{6} \gamma_\mu \gamma^{\sigma\rho} R(Q)_{\rho\sigma} \right].
\end{align}

Here we have denoted the traceless parts of $R(P)^{\mu\nu}$ and $\omega^{\mu mn}$ by $R(P)_0^{\mu\nu} =
R(P)^{\mu\nu} + \frac{2}{3} R(P)^{\mu e} v_{[m} e_{n]}$ and $\omega_0^{\mu mn} = \omega^{\mu mn} + \frac{2}{3} e^{[\mu}[n} \omega_{m]}$, respectively, where the
traces are denoted by $R(P)_\mu = R(P)^{\mu\mu}$ and $\omega_m = \omega_{pm}$. Furthermore, since
we are working at the linear level, we can ignore the action of $\delta_{\text{Group}}$ on the vierbein and metric because this produces terms that are of higher order in our expansion. Finally note that we have introduced two free parameters $\alpha$ and $\beta$ into the results for the 6 of $v^m_{\mu}$ and $z^m_{\mu}$ and into the trace of the spin connection and dilaton, respectively. Since the constraints only specify the extra transformations of the combinations $z_{[\mu\nu]} + *v_{[\mu\nu]}$ and $b_{\mu} + \frac{1}{6}\omega_{\mu}$ the results $\langle 121 \rangle$, $\langle 122 \rangle$, $\langle 124 \rangle$ and $\langle 125 \rangle$ represent the general solutions for the extra transformations of these fields. Note that the combinations $\alpha v_{[\mu\nu]} + (1 - \alpha) *z_{[\mu\nu]}$ and $(1 - \beta)b_{\mu} - \frac{1}{6}\beta\omega_{\mu}$ are independent fields. For example, in conformal supergravity, the choice $\beta = 0$ was taken, although this freedom, in any case, cancelled completely in that model. We shall study the dependence of the model on these parameters in the following calculations.

Applying these results to $V_m$ and $Z_m$ symmetry we find

\[ \hat{\delta}_V \omega_{\mu mn}^0 = 2 R(E)_{mn\mu k} \epsilon^k ; \quad \hat{\delta}_Z \omega_{\mu mn}^0 = 2 \tilde{R}(E)_{mn\mu k} \epsilon^k \]  
\[ \hat{\delta}_V \omega_{m} = 0 = \hat{\delta}_V b_{\mu} ; \quad \hat{\delta}_Z \omega_{m} = 0 = \hat{\delta}_Z b_{\mu} \]  
\[ \hat{\delta}_V v_{\mu}^{m} = 0 = \hat{\delta}_V z_{\mu}^{m} ; \quad \hat{\delta}_Z v_{\mu}^{m} = 0 = \hat{\delta}_Z z_{\mu}^{m} \]  
\[ \hat{\delta}_V \phi_{\mu} = \frac{1}{\sqrt{2}} R(Q)_{\mu\nu} \epsilon^{\nu} ; \quad \hat{\delta}_Z \phi_{\mu} = \frac{1}{\sqrt{2}} \gamma^5 R(Q)_{\mu\nu} \epsilon^{\nu}, \]

Observe that all dependence on the parameters $\alpha$ and $\beta$ has dropped out. To obtain $\langle 127 \rangle$ we used the cyclicity relation $R(E)_{[\mu\nu\rho\sigma]} = 0$ in $\langle 101 \rangle$ and $\tilde{R}(E)_{[\mu\nu\rho\sigma]} = 0$ from $\langle 102 \rangle$. Reusing these cyclicity relations and the additional cyclicity relation $\gamma_{[\mu} R(Q)_{\rho\sigma]} = 0$ in $\langle 103 \rangle$, it is easy to recast the results $\langle 127 \rangle$-$\langle 130 \rangle$ into form notation

\[ e^n \hat{\delta}_V \omega_{mn} = -2 \tilde{R}(E)_{mn} \epsilon^n \]  
\[ e^n \hat{\delta}_Z \omega_{mn} = 2 R(E)_{mn} \epsilon^n \]  
\[ e^m \gamma_{\mu} \hat{\delta}_V \phi = - \frac{1}{\sqrt{2}} \gamma_{\mu} R(Q) \epsilon^m \]  
\[ e^m \gamma_{\mu} \hat{\delta}_Z \phi = + \frac{1}{\sqrt{2}} \gamma^5 \gamma_{\mu} R(Q) \epsilon^m. \]

\footnote{Of course, one must nonetheless still resist the temptation to impose constraints “under” the $\delta_{\text{Group}}$ sign. For example, $\delta_{\text{Group}} R(E)_{\mu\nu\rho\sigma} = \frac{1}{2} \epsilon_{\rho\sigma\alpha\beta} \delta_{\text{Group}} R(E)_{\mu\nu}^{\alpha\beta} \neq \delta_{\text{Group}} * R(E)_{\mu\nu\rho\sigma} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \delta_{\text{Group}} R(E)^{\alpha\beta}_{\rho\sigma}.}$
Let us orchestrate the above formulæ. Using (173), (174) and (131)-(134) in (118) we have

\[-\delta_V \mathcal{L} = 32[\bar{R}(M)^{mn}R(V)_m - 2R(D)R(Z)^n]\epsilon_n + 32R(V)^m\delta_{V,\text{Group}}[R(Z)_m - *R(V)_m]\]

\[-\delta_Z \mathcal{L} = 32[\bar{R}(M)^{mn}R(Z)_m + 2R(D)R(V)^n]\epsilon_n + 32R(V)^m\delta_{Z,\text{Group}}[R(Z)_m - *R(V)_m].\]

Now it only remains to calculate the group transformations of the constraint (110) which, to leading order, yields

\[\delta_{V,\text{Group}}[R(Z)_m - *R(V)_m] = -\bar{R}(M)^{mn}\epsilon^n + 2*R(D)\epsilon_m\]

\[\delta_{Z,\text{Group}}[R(Z)_m - *R(V)_m] = R(M)^{mn}\epsilon^n - 2R(D)\epsilon_m\]

To obtain these results we have used that, to lowest order, the following constraints hold

\[R(M)^{mn} = -*\bar{R}(M)^{mn}\]

\[R(A) = *R(D)\]

The first may be derived by noting that the $\vartheta$ and $\bar{\vartheta}$ of $R(M)_{\mu\nu}{}^{mn}$ vanish at lowest order (although one may verify that at the next to leading order the quadratic terms involving $R(E)^{mn}$, $v^m$, $z^m$ and $E^{mn}$, $R(V)^n$, $R(Z)^m$ in (112) and (98) do contribute to the $\vartheta$ and $\bar{\vartheta}$). Inserting (137) and (138) in (135) and (136), respectively, and using (139) and (110), we see that the lowest order contributions to $\delta_V \mathcal{L}$ and $\delta_Z \mathcal{L}$ do indeed vanish.

The invariance of the action under $V_m$ and $Z_m$ symmetries confirms our proposed maximal set of constraints. Of course, our linear analysis does not probe the terms in the constraints of higher order in fields. In the case of the sure constraints however, such additional terms are severely restricted by the requirement that the action is invariant under the sure symmetries.

In general, they are also further restricted by requiring that the constraints themselves are invariant under the sure symmetries. Further, let us note that the only (linearized) constraint, which was not needed in the above analysis is the constraint on the scalar curvature (1) of $R(E)_{\mu\nu}{}^{mn}$ (i.e. $R(E)_{\mu\nu}{}^{\nu\mu} = 0$).
although even this constraint is necessary since it follows from $\gamma^\mu R(Q)_{\mu\nu} = 0$ by varying with respect to the sure symmetry $S$. We also observe that the freedom to choose the dependent combinations of $z_{[\mu\nu]}$ and $\ast v_{[\mu\nu]}$, and $b_\mu$ and $\omega_\mu$ (denoted by the parameters $\alpha$ and $\beta$, respectively) cancels in the $V_m$ and $Z_m$ variation of the action at linear order. We now turn to the remaining unsure symmetries $Q$ and $E_{mn}$.

### 7 Local Supersymmetry and $E_{mn}$ Symmetry.

These symmetries are of vital importance for the perturbative consistency of our action, since in a flat gravitational and otherwise trivial background the fields $\psi^\alpha$ and $E_{mn}$ enter the quadratic part of the action only in terms of their linearized field strengths $d\psi^\alpha$ and $dE_{mn}$. If the associated gauge invariances do not survive at the interacting level (and they don’t), this kind of perturbation expansion will not lead to invertible kinetic terms. Conceivably there might exist a vacuum solution that does allow a perturbation expansion. However, unless modifications of the supersymmetry and $E_{mn}$ transformation rules exist under which the action is invariant, it is more likely that the theory simply does not exist as a gauge theory. In section 8 we examine in more detail some possibilities for such modifications.

The manipulations required to show that $E_{mn}$ and $Q$ are not linear order symmetries of the action are exactly the same as discussed in detail for $V_m$ and $Z_m$ symmetries, so let us just give the main results.

In fact many terms in both $\delta Q\mathcal{L}$ and $\delta E\mathcal{L}$ do cancel and the only terms remaining are given by (in form notation)

$$ -\delta Q\mathcal{L} = -16\sqrt{2} \bar{R}(S)\gamma^5\gamma^m e_m \delta Q\phi -128 \bar{R}(F)_{mn} e^m \delta Q\tilde{z}^n -128 R(F)_{mn} e^m \delta Q\tilde{z}^n $$

$$ -\delta E\mathcal{L} = 164 R(K)^m [e_{mn} R(V)^n - e_{mn} R(Z)^n + \frac{1}{2} e^n \delta E\tilde{\omega}_{mn}] $$

$$ -128 \bar{R}(F)_{mn} e^m \delta E\tilde{v}^n -128 R(F)_{mn} e^m \delta E\tilde{z}^n -16\sqrt{2} \bar{R}(S)\gamma^5\gamma^m e_m \delta \phi, $$

(141)

(142)

where the relevant extra transformations of dependent fields are given (in
explicit index notation) by

\[
\hat{\delta}_Q \phi_\mu = \frac{1}{3\sqrt{2}} [R(D)_{\rho\rho} + R(A)_{\rho\rho} \gamma^5 + \frac{1}{2} (R(V)_\rho + R(Z)_\rho \gamma^5) \gamma_\mu] \gamma^\rho \tag{143}
\]

\[
\hat{\delta}_Q v_{\mu\nu} = \frac{1}{8\sqrt{2}} (1 - \alpha) R(Q) \epsilon \tag{144}
\]

\[
\hat{\delta}_Q z_{\mu\nu} = \frac{1}{8\sqrt{2}} \alpha R(Q) \gamma^5 \epsilon \tag{145}
\]

\[
\hat{\delta}_E \phi_\mu = -\frac{1}{4} \gamma^\rho \gamma^{mn} R(S)_{\rho\mu} + \frac{1}{6} \gamma_\mu \gamma^{\rho \gamma^m n} R(S)_{\rho\sigma} \epsilon_{mn} \tag{146}
\]

\[
\hat{\delta}_E \omega^0_{\mu mn} = R(V)_{mn}^r \epsilon_{r\mu} - R(V)_{\mu mn}^r \epsilon_{rn} + R(V)_{\mu n}^r \epsilon_{rm} + R(Z)_{\mu n}^r \epsilon_{rn} - \frac{2}{3} \epsilon_{\mu [m} \tilde{\delta} \omega_{n]} \tag{147}
\]

\[
\hat{\delta}_E \omega_\mu = 2(1 - \beta) (\epsilon_{mn} R(Z)^n + \epsilon_{mn} R(V)^n) \tag{148}
\]

\[
\hat{\delta}_E v_{[\mu |\nu]} = \frac{1}{8} (1 - \alpha) [R(M)_{\rho\sigma\mu\nu} \epsilon^{\rho\sigma} + 8 R(D)_{\rho [\mu} \epsilon_{\nu]}] \tag{149}
\]

\[
\hat{\delta}_E z_{[\mu |\nu]} = \frac{1}{8} \alpha [R(M)_{\rho\sigma\mu|\nu} \epsilon^{\rho\sigma} + 8 R(D)_{\rho [\mu} \epsilon_{\nu]}] \tag{150}
\]

\[
\hat{\delta}_E v_{(\mu \nu)} = \frac{1}{6} g_{\mu\nu} R(D)_{\rho\sigma} \epsilon^{\rho\sigma} \tag{151}
\]

\[
\hat{\delta}_E z_{(\mu \nu)} = \frac{1}{6} g_{\mu\nu} R(D)_{\rho\sigma} \epsilon^{\rho\sigma}, \tag{152}
\]

where, as usual, we have denoted the traces \( R(V)_\nu = R(V)_\rho^\rho \) and \( R(Z)_\nu = R(Z)_\rho^\rho \).

In deriving the results (143)-(152) we used the following cyclicity relations which follow directly from

\[
R(V)_{\mu\rho\sigma} + R(V)_{\rho\sigma\mu} + R(V)_{\sigma\mu\rho} = -\epsilon_{\mu\rho\sigma\tau} R(Z)^\tau \tag{153}
\]

\[
R(Z)_{\mu\rho\sigma} + R(Z)_{\rho\sigma\mu} + R(Z)_{\sigma\mu\rho} = \epsilon_{\mu\rho\sigma\tau} R(V)^\tau \tag{154}
\]

It is tedious, but straightforward, to explicitly substitute the results (143)-(152) into (141) and (142) and verify that the result is indeed non-vanishing. We have checked this explicitly. Let us demonstrate this in a few examples, which, at the same time, will give us the dependence on the freedoms \( \alpha \) and \( \beta \).

The curvatures \( R(S)_{\mu\nu} \), \( R(K)_{\mu\nu}^m \) and \( R(F)_{\mu\nu}^{mn} \) are all unconstrained, so the coefficients of each of their Lorentz irreducible parts must vanish separately (modulo, of course, the maximal set of constraints).
Consider first the \( R(F)_{\mu\nu}^{mn} \) terms in \( \delta_Q \mathcal{L} \). Here only the coefficient of the antisymmetric Ricci of \( \ast \tilde{R}(F)_{\mu\nu}^{mn} \) is non-zero and is given by

\[
- \delta_Q \mathcal{L} = 8\sqrt{2} e \tilde{R} (Q)_{\mu\nu} \epsilon \left[ g_{\rho\sigma} \ast \tilde{R}(F)^{\rho\mu\nu\sigma} \right] + \cdots \tag{155}
\]

(The dots “\( \cdots \)” denote the remaining term proportional to \( R(S) \) in \( \delta_Q \mathcal{L} \).)

Note, in particular, that the freedom \( \alpha \) to choose which combination of \( z_{[\mu\nu]} \) and \( \ast v_{[\mu\nu]} \) is a dependent field, drops out. We note also that the same freedom \( \beta \) in the \( \omega_{\mu} \) and \( b_{\mu} \) sector also drops out at leading order (one finds that the extra transformation \( \tilde{\delta} \tilde{Q} \omega_{\mu} \) is identically zero). We observe that since there is no constraint on \( g_{\rho\sigma} \ast \tilde{R}(F)^{\rho\mu\nu\sigma} \), the expression (155) cannot vanish or cancel with any other terms in \( \delta_Q \mathcal{L} \).

Next consider the possibly \( \alpha \) dependent \( R(F)_{\mu\nu}^{mn} \) terms in (142), which are again proportional to the antisymmetric Ricci of \( \ast \tilde{R}(F)_{\mu\nu}^{mn} \). That \( \alpha \) again drops out holds, in fact, independently of the particular expression for \( \tilde{\delta} z_{[\mu\nu]} \) and \( \tilde{\delta} v_{[\mu\nu]} \), but rather since \( (1 - \alpha) \ast \tilde{\delta} z_{[\mu\nu]} = -\alpha \tilde{\delta} v_{[\mu\nu]} \). We are left then with the non-zero expression

\[
- \delta_E \mathcal{L} = -16 e (R(M)_{\mu\nu\alpha\beta} \epsilon ^{\alpha\beta} + 4 R(D)_{\alpha[\mu\nu] \epsilon _{\nu]}^{\alpha} ) \left[ g_{\rho\sigma} \ast \tilde{R}(F)^{\rho\mu\nu\sigma} \right] + \cdots . \tag{156}
\]

Our final example comprises the terms proportional to the trace \( R(K)^{\rho\mu\nu} \equiv R(K)^{\mu} \) in \( \delta_E \mathcal{L} \). Here we find that the model actually retains a dependence on the parameter \( \beta \). The result is

\[
- \delta_E \mathcal{L} = \frac{-64}{3} (2\beta - 3)e R(K)^{\mu} \epsilon _{\mu\nu} R(V)^{\nu} + \ast \epsilon _{\mu\nu} R(Z)^{\nu} \cdots , \tag{157}
\]

We note that, at leading order, this is the only instance in which the freedom \( \alpha \) or \( \beta \) does not cancel. Furthermore, for the value \( \beta = 3/2 \) these terms vanish. This corresponds to the case in which the combination \( (b_{\mu} + \frac{1}{2} \omega_{\mu}) \) is an independent field. Of course, even for this choice of \( \beta \), the remaining terms in \( \delta_E \mathcal{L} \) do not vanish. This concludes our demonstration that the gauge transformations \( Q \) and \( E_{mn} \) are not symmetries of the action at leading order.

Penultimately, we make the important remark that invariance at the linear level is a necessary but not sufficient condition that the action be invariant under some symmetry. Therefore our analysis shows that the symmetries \( V_m \) and \( Z_m \) could possibly be full symmetries of the action at all orders, but local supersymmetry \( Q \) and \( E_{mn} \) are not.
Finally, let us conclude this section by remarking that the sure symmetries act in a very simple way on the independent fields. For example, in [7] it was shown for the conformal supergravity model that the conformal boost symmetry $K_m$ acted only on the dilaton field $b$ as a translation $b_{\mu} \to b_{\mu} - 2\epsilon_{\mu}$ whereby it was argued that the action was independent of $b_{\mu}$. We may perform a similar analysis here also, but we now find that instead of being independent of the field $b_{\mu}$, the action depends only on a certain combination of the independent fields $a_\mu$, $b_\mu$ and the remaining independent components of the $v$ and $z$ fields $z_{[\mu\nu]} - *v_{[\mu\nu]}$. As shown in the discussion above, after imposing the maximal set of constraints the remaining independent fields are $e_m \, e_{\mu m}$, $E_{\mu mn}$, $E_{\mu mn}$, $b_{\mu}$, $a_\mu$ and the combination $z_{[\mu\nu]} - *v_{[\mu\nu]}$ so that after solving all constraints

$$L = L(e_m \, e_{\mu m}, E_{\mu mn}, \psi_{\mu}, b_{\mu}, a_\mu, z_{[\mu\nu]} - *v_{[\mu\nu]}).$$

(158)

Note for simplicity, and indeed without loss of generality that we have at this point chosen the dilaton $b_\mu$ and the combination $z_{[\mu\nu]} - *v_{[\mu\nu]}$ to be independent fields, although as discussed earlier, more general combinations are possible.

Under conformal boosts ($K_m$) the only independent fields which transform are $b$, $v^m$ and $z^m$,

$$b_{\mu} \to b_{\mu} - 2\epsilon_{\mu}$$

$$z_{[\mu\nu]} - *v_{[\mu\nu]} \to z_{[\mu\nu]} - *v_{[\mu\nu]} + 2e_{m[\nu}E_{\mu]}^{mn}\epsilon_n - 2*E_{[\mu\nu]}^n\epsilon_n.$$  

(159)

(160)

(Since two or more conformal boosts vanish when acting on the fields $b$, $v^m$ and $z^m$, the transformations given above are finite gauge transformations rather than just the infinitesimal transformations.) Similarly, $F_{mn}$ gauge transformations act only on the fields $a$, $b$, $v^m$ and $z^m$ and the finite results are given by

$$a_\mu \to a_\mu - \tilde{E}_{\mu}^{mn}\epsilon_{mn}$$

$$b_{\mu} \to b_{\mu} - E_{\mu}^{mn}\epsilon_{mn}$$

$$z_{[\mu\nu]} - *v_{[\mu\nu]} \to z_{[\mu\nu]} - *v_{[\mu\nu]} + 4*\epsilon_{\mu\nu}.$$  

(161)

(162)

(163)

The action (157) was constructed to be invariant under conformal boosts ($K_m$) and $F_{mn}$ gauge transformations. Therefore, one may now construct a $F_{mn}$ gauge transformation followed by an appropriate conformal boost ($K_m$) gauge transformation such that all dependence on the independent fields $b_{\mu}$ and
\[ z_{\mu\nu} - \ast v_{\mu\nu} \text{ appears only in the combination } a'_{\mu} = a_{\mu} + \Delta a_{\mu}(b, z - \ast v) \text{ so that } \mathcal{L}(a_{\mu}, b_{\mu}, z_{\mu\nu} - \ast v_{\mu\nu}) = \mathcal{L}(a'_{\mu}, 0, 0). \] The field equation equations of the \( b_{\mu} \) and \( z_{\mu\nu} - \ast v_{\mu\nu} \) fields, have therefore, no more content than that of the \( a_{\mu} \) field equation. We view this \( v^m, z^m \) and \( b \)-independent formulation as a kind of Wess Zumino gauge, in which we have gauged these fields away. Of course, to calculate the gauge transformations of the new field \( a'_{\mu} \), one must, in the usual fashion, perform appropriate compensating gauge transformations. Both formulations are, of course, entirely equivalent, and we found it more convenient to work in the formulation where the dependence on the fields \( v^m, z^m \) and \( b \) is kept explicit.

8 Kinematical Analysis and Conclusions.

Originally, conformal supergravity was discovered through a combination of a dynamical approach and a kinematical approach \[11, 7\] the former being similar to that which we have followed above. In the dynamical approach, an action was constructed, and constraints on curvatures were found such that the action was invariant under sure and automatic symmetries. In turn, these constraints endowed the transformations of dependent fields under “unsure” local supersymmetry with extra pieces and it was verified that the modified transformation rules were indeed an invariance of the action. The conformal gauge field \( f_{\mu}^m \) was solved from its own algebraic field equation, so that its variations did not need to be taken into account (1.5 order formalism). However, later it was shown that all the results of that model could also be obtained through an entirely kinematical approach \[14\], in which all constraints were found by requiring that the gauge algebra closed onto general coordinate transformations, rather than \( P_m \) gauge transformations.

We now apply this kinematical approach to our extended conformal supergravity model. We find that in order that the gauge algebra close onto general coordinate transformations, the symmetries \( Q \) and \( E_{mn} \) require extra modifications beyond that implied by the introduction of constraints on curvatures. This result is consistent with the findings of our linear analysis in which \( Q \) and \( E_{mn} \) were not invariances of the action.

We begin with the important observation \[9, 7\] that general coordinate transformations

\[ \delta_{\text{Gen. coord}.} h_{\mu}^A = \partial_{\mu} \xi^\rho h^A_\rho + \xi^\rho \partial_{\rho} h^A_{\mu}, \quad (164) \]
and $P_m$ gauge transformations with parameter $\epsilon^m = \xi^\rho e^m_\rho$ differ only by a sum of local gauge invariances of the theory and a curvature term

$$
\delta_{\text{Gen. coord.}} (\xi^\rho) h^A_\mu - \delta_{\text{Group}} (\epsilon_{B}^{\#} P_m = \xi^\rho h^B_\rho) h^A_\mu.
$$

(165)

The last term in (165) is a sum of group law gauge transformations whose field-dependent parameter is given by the contraction of the general coordinate parameter $\xi^\rho$ with each gauge field $h^A_\rho$, but where $P_m$ gauge transformations are omitted.

Therefore, for the gauge algebra to close, it is sufficient that whenever a $P_m$ transformation is produced on the right hand side of commutators acting on independent fields, the combination of any extra terms from the second gauge variation acting on a possibly dependent field should equal the curvature term on the left hand side of (165), modulo any constraints on curvatures.

For example, consider the commutator of two supersymmetry transformations acting on the independent vierbein field

$$
[\delta Q(\epsilon_1), \delta Q(\epsilon_2)] e^m_\mu = \delta_P \left( \frac{1}{2\sqrt{2}} \gamma^r \gamma^s \epsilon_1 \right) e^m_\mu + \delta_E \left( - \frac{1}{2\sqrt{2}} \gamma^r \gamma^s \epsilon_1 \right) e^m_\mu
$$

(166)

Observe that the first supersymmetry transformation produces an independent gravitino field $\psi_\mu$ so that the result above should coincide with the usual group law. Supposing that $E_{mn}$-gauge transformations were to be a symmetry of the theory, then the right hand side of would also be a symmetry if the constraint

$$
R(P)^{\mu\nu}_m = 0
$$

(167)

held (see [165]). We already found this constraint from the dynamical approach, so at this point we find a confirmation of our assumptions. Next we apply the same procedure to the $E^m_{\mu\nu}$ gauge field. Again the first supersymmetry transformation produces the independent gravitino field so that the commutator of two supersymmetry transformations yields the group law result only

$$
[\delta Q(\epsilon_1), \delta Q(\epsilon_2)] E^m_{\mu\nu} = \delta_P \left( \frac{1}{2\sqrt{2}} \gamma^r \gamma^s \epsilon_1 \right) E^m_{\mu\nu} + \delta_E \left( - \frac{1}{2\sqrt{2}} \gamma^r \gamma^s \epsilon_1 \right) E^m_{\mu\nu}
$$

(168)
from which we now conclude that the constraint

\[ R(E)_{\mu\nu}^{mn} = 0 \]  

should hold. However, as discussed in section 3, all of \( R(E)_{\mu\nu}^{mn} \) may be solvably constrained to vanish except the traceless 10. Therefore, we conclude that in order that the gauge algebra close, supersymmetry transformations on (some of) the independent fields should be explicitly modified. (One may also consider modifying \( E_{mn} \) transformations at this point, but since the parameter \( \frac{1}{\sqrt{2}} \tilde{e}_2 \gamma^r \epsilon_1 \) is independent from \( -\frac{1}{\sqrt{2}} \tilde{e}_2 \gamma^r \epsilon_1 \) it seems unlikely that a modified \( E_{mn} \) transformation on the right hand side of (168) could produce terms rendering the constraint (169) solvable).

At this point, it is already clear that the kinematical approach faces major difficulties, but one can consider also further commutators acting on independent fields. The remaining commutators producing \( P_m \) transformations are \([\delta_V, \delta_E]\) and \([\delta_Z, \delta_E]\). (It is not necessary to consider the commutator of general coordinate transformations with dilations or local Lorentz transformations, since the commutator of a general coordinate transformation, with a gauge transformation produces the same gauge transformation, but whose parameter is given by \( \epsilon^A_{[\text{Gen.coord,Group]}} = -\xi^\rho \partial_\rho \epsilon^A \)). Assuming that the constraint \( R(P)_{\mu\nu}^{m} = 0 \) holds and that \( E_{mn} \) gauge transformations are a symmetry, acting with commutators \([\delta_V, \delta_E]\) and \([\delta_Z, \delta_E]\) on the vierbein, one deduces that the extra transformations of the dependent \( v^m \) and \( z^m \) fields under \( V_m \) and \( Z_m \) symmetries should vanish. Therefore, the kinematical approach, produces the same results for the extra transformations of the spin connection, dilaton and \( v^m \) and \( z^m \) fields under \( V_m \) and \( Z_m \) symmetries as in the linear analysis of section 4. Note, however, that these extra transformations belong neither to the constraint (169) nor the \( R(E)^{mn} \) constraints found in the dynamical approach. We are therefore forced to conclude that \( E_{mn} \) gauge transformations should also be explicitly modified.

One might consider modifications in the transformation laws of independent gauge fields proportional to curvatures, so \( \delta h \sim \delta_{\text{Group}}(\epsilon) h + R\epsilon \). For dimensional reasons, such modifications cannot occur in \( \delta \epsilon^m \) and \( \delta E^{mn} \), but in \( \delta \psi \) one might try a term \( \delta Q \psi \sim R(E)_{\mu\nu}^{mn} \gamma_{mn} \gamma^\nu \epsilon_Q \). Since only the 10 in \( R(E)_{\mu\nu}^{mn} \) is nonvanishing, this term vanishes, hence also \( \delta Q \psi \mu \) is unmodified. Instead, one might study \( \delta E v_{\mu}^{m} \sim R(E)_{\mu\nu}^{mn} (\epsilon_{E})_{n}^{\nu} \) etcetera. However, rather than searching for modifications of \( Q \) and \( E_{mn} \) symmetries such that
the gauge algebra closes onto general coordinate transformations, one suspects that these symmetries should be dropped altogether, just as was the case for $P_m$ gauge transformations, and instead there should exist in their place a generalization of general coordinate transformations to $Q$ and $E_{mn}$ symmetries with parameters $\xi^\alpha$ and $\xi^{\mu\nu}$, respectively. One could then study generalizations of (165) to $Q$ and $E_{mn}$ symmetries in which the new “general coordinate” $Q$ and $E_{mn}$ transformations are given by $\delta_{\text{Group}(Q,E_{mn})}$ with a field dependent parameter $\epsilon \sim \xi h$ (which may allow for a more general mass dimension of the parameter $\xi$) plus curvature terms. Perhaps one can learn more about such a proposal by making a linearized analysis of the $P_m$ gauge symmetry in the dynamical approach, since we at least know for certain that the $Osp(1|8)$ model is general coordinate invariant. We feel that our dynamical approach has laid the groundwork for such investigations, which we, however, reserve for further study.

Having shown that the kinematical approach at least produces results consistent with the findings of the dynamical approach considered in the text of this paper, let us present our conclusions and some more speculative remarks.

A simple possibility is that there exists no $R^2$ type of supergravity based on $Osp(1|8)$. Let us proceed under the assumption that this is not the case. The most obvious way to proceed is then to attempt to combine the kinematical approach with the solely dynamical approach in the text in order to find explicit modifications or generalizations of the $Q$ and $E_{mn}$ transformation rules, as discussed above, such that the gauge algebra closes and invariance of the action is achieved. The fascinating possibility, that such generalizations of general coordinate transformations could exist, and can be probed via a dynamical analysis as given in the text, was a key motivation for us to study this model in detail, even though, a priori, from a kinematical standpoint, we suspected that new features would arise in the $Q$ and $E_{mn}$ symmetries.

In the limit that the symmetries become rigid and $e_\mu^m$ becomes a flat space delta function $\delta_\mu^m$, one would expect that one should add orbital parts to the transformation rules which are a consequence of the transformation of coordinates. The most general set of bosonic rigid symmetries leaving the line element $(dx^m)^2$ invariant is known to be the conformal group. Clearly, $E$ symmetry does not act on the coordinates $x^m$. An alternative which we have not pursued in this article at all, is to consider a space with bosonic coordinates $x^m$ and $y^{[mn]}$ (or a superspace with $x^m$, $\theta^\alpha$ and $y^{[omn]}$). One might then study dimensional reduction of a higher dimensional model.
For example, in order to construct rigid representations of the superalgebra $Osp(1|8)$, one can first consider the set of non-negative dilaton weight generators $\{P_m, E_{mn}, Q; M_{mn}, D, A, V_m, Z_m\}$ and consider a coset based on the reductive split where the positive dilaton weight generators are coset generators and the zero weight generators are subgroup generators. (Representations of the full $Osp(1|8)$ superalgebra should then be calculated using the theory of induced representations). One must consider then a superspace with coordinates $(x^m, y^{[mn]}, \theta^\alpha)$ where $x^m$ are the four spacetime coordinates, $\theta^\alpha$ ($\alpha = 1, 4$) are the usual anti-commuting coordinates and the $y^{[mn]}$ are six auxiliary coordinates. Notice that one does not have a ten dimensional Lorentz group, but rather, the auxiliary coordinates $y^{[mn]}$ transform as a rank two antisymmetric tensor under four dimensional Lorentz transformations. In two component notation the anti-commutator of two $Q$ transformations reads

$$\{Q_A, Q_B\} = E_{(AB)}$$ (170)
$$\{Q_A, \dot{Q}_B\} = P_{A\dot{B}}$$ (171)
$$\{Q_{A\dot{B}}, Q_{\dot{A}B}\} = E_{(A\dot{B})}$$ (172)

where $A, B, \dot{A}, \dot{B} = 1, 2$ and we have made the usual decomposition of the rank two antisymmetric tensor $E_{mn}$ into its self-dual and anti-self-dual parts. The algebra (170)-(172) represents a natural extension of the usual supersymmetry algebra but further possible constraints and/or dimensional reduction schemes are needed to control dependence on the six auxiliary coordinates $y^{[mn]}$ such that one regains new representations of $Osp(1|8)$ in four dimensions.

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A Group Law and Extra Transformations of the Action under Unsure Symmetries.

In this appendix we derive complete expressions for both $\delta_{\text{Group}}\mathcal{L}$ and $\tilde{\delta}\mathcal{L}$ valid to all orders in fields for some (unsure) symmetry $\delta$. The former is trivial to compute since curvatures just rotate homogeneously according to the group law (see (56)) and the results for all unsure symmetries are

$$-\delta_{V,\text{Group}}\mathcal{L} = 16 \left[ 2\tilde{R}(M)^{mn} R(V)_m - 4\tilde{R}(D) R(Z)^n + 4\tilde{R}(E)^{mn} R(K)_m - \tilde{R}(Q) \gamma^5 \gamma^n R(S) \right] \epsilon_n$$  
(173)

$$-\delta_{Z,\text{Group}}\mathcal{L} = 16 \left[ 2\tilde{R}(M)^{mn} R(Z)_m + 4\tilde{R}(D) R(V)^n - 4\tilde{R}(E)^{mn} R(K)_m + \tilde{R}(Q) \gamma^n R(S) \right] \epsilon_n$$  
(174)

$$-\delta_{Q,\text{Group}}\mathcal{L} = 8 \left[ \tilde{R}(S) \{ \gamma^5 \gamma_m R(V)^m + \gamma_m R(Z)^m \} + \sqrt{2} \tilde{R}(Q) \gamma^5 \gamma_m R(K)^m \right] \epsilon_n$$  
(175)

$$-\delta_{P,\text{Group}}\mathcal{L} = -32 \left[ \tilde{R}(M)^{mn} R(K)_m + 2\tilde{R}(A) R(K)^n + 2\tilde{R}(F)^{mn} R(V)_m + 2\tilde{R}(F)^{mn} R(Z)_m + 4\sqrt{2} \tilde{R}(S) \gamma^5 \gamma^n R(S) \right] \epsilon_n$$  
(176)

$$-\delta_{E,\text{Group}}\mathcal{L} = 0.$$  
(177)

For completeness we have included the result for $P_m$ gauge transformations, even though the action, being general coordinate invariant, will not, in general, be $P_m$ invariant. It is surprising that the action is invariant under $E_{mn}$ group transformations but note that we still have to compute the effect of the extra $E_{mn}$ transformations.

Let us now compute the variation of the action under extra transformations $\tilde{\delta}$. Under the most general variations of all fields one finds

$$-\tilde{\delta}\mathcal{L} = 32 \left[ v^m R(V)^n + z^m R(Z)^n - e^m R(K)^n \right] \tilde{\delta}\omega_{mn}$$  

$$+32 \left[ \tilde{R}(M)^{mn} v_m - 2\tilde{R}(F)^{mn} e_m + 2\tilde{R}(E)^{mn} f_m - 2\tilde{R}(D) z^n - \frac{1}{4} \tilde{\psi} \gamma^5 \gamma^n R(S) - \frac{1}{4} \tilde{R}(Q) \gamma^5 \gamma^n \phi \right] \tilde{\delta} v_n$$  

$$+32 \left[ \tilde{R}(M)^{mn} z_m - 2\tilde{R}(F)^{mn} e_m - 2\tilde{R}(E)^{mn} f_m + 2\tilde{R}(D) v^n + \frac{1}{4} \tilde{\psi} \gamma^n R(S) + \frac{1}{4} \tilde{R}(Q) \gamma^n \phi \right] \tilde{\delta} z_n$$  

$$+16 \left[ \tilde{R}(Q) (\gamma^5 v^m \gamma_m - z^m \gamma_m) - \frac{1}{2} \tilde{\psi} (\gamma^5 R(V)^m \gamma_m - R(Z)^m \gamma_m) - \sqrt{2} \tilde{R}(S) \gamma^5 e^m \gamma_m \right] \tilde{\delta} \phi$$
\[ +32 \left[ \tilde{R}(M)^{mn} e_n - 2 \tilde{R}(E)^{mn} v_n + 2 R(E)^{mn} z_n \right] \\
+ 2 R(A) e^m - \frac{1}{2 \sqrt{2}} \tilde{\psi} \gamma^5 \gamma^m R(Q) \delta f_m \]
\[ + 64 \left[ R(V)^m e_n \delta F_{mn} + R(Z)^m e^n \delta F_{mn} \right] \\
- 32 \left[ R(V)^m z_m - R(Z)^m v_m \right] \tilde{\delta} b + 64 R(K)^m e_m \tilde{\delta} a \]
\[ - 8 \left[ \tilde{\delta} (R(V)^m \gamma^5 \gamma_m + R(Z)^m \gamma_m) - 2 R(S)(v^m \gamma^5 \gamma_m + z^m \gamma_m) \right] \\
+ \sqrt{2} \sqrt{R(K)^m} \gamma^5 \gamma_m - 2 \sqrt{2} R(Q) f^m \gamma^5 \gamma_m \tilde{\delta} \psi \]
\[ - 64 \left[ (R(K)^m v^m - f m R(V)^n) \tilde{E} E_{mn} - (R(K)^m z^n - f m R(Z)^n) \tilde{E} E_{mn} \right] \\
+ 32 \left[ \tilde{R}(M)^{mn} f_n - 2 R(A) f^m + 2 \tilde{R}(F)^{mn} v_n \right] \\
+ 2 R(F)^{mn} z_n + \frac{1}{2 \sqrt{2}} R(S) \gamma^5 \gamma^m \delta \epsilon_m. \]

Note that the independent fields, of course, get no extra transformations so that \( 0 = \tilde{\delta} a = \tilde{\delta} \psi = \tilde{\delta} e^m = \tilde{\delta} E^{mn} \) but we have included these results here for completeness. The above formula was derived simply by varying all fields \( h^A \rightarrow h^A + \delta h^A \) as they appear in the explicit results for the curvatures (34)-(37). However whenever terms \( \tilde{\delta} h^A \) occurred, we integrated by parts and then used the Bianchi identity in the form (38) to convert the exterior derivative on curvatures into a sum of terms of the form (field) \times (curvature). One finds then many cancellations and the result for \( \tilde{\delta} \mathcal{L} \) has the form above involving only terms of the form (curvature) \times (field) \times (dept. field).

The result (178) can be brought into a more useful form by the following manipulations. First the coefficients of \( \tilde{\delta} f_m \) almost comprise the constraint (77) so that these terms may be rewritten as
\[-64 R(V)^m E^{mn} \tilde{\delta} f_n + 64 R(Z)^m E^{mn} \tilde{\delta} f_n. \]
However, if we now use the constraint \( R(Z)^m = * R(V)^m \) to convert all \( \tilde{R}(V)^m \) curvatures to \( R(V)^m \) curvatures, then the set of terms depending on \( R(V)^m \) is given by
\[-32 R(V)^m \left\{ 2 \tilde{E}_{mn} e^n + 2 \tilde{E}_{mn} \tilde{\delta} f^n + \frac{1}{2} \tilde{\psi} \gamma^5 \gamma_m \tilde{\delta} \phi - \tilde{\delta} \omega_{mn} v^n + z_m \tilde{\delta} b \right\} \]
\[ - \left[ -2 \tilde{E}_{mn} e^n + 2 E_{mn} \tilde{\delta} f^n + \frac{1}{4} \tilde{\psi} \gamma_m \tilde{\delta} \phi + \tilde{\delta} \omega_{mn} z^n + v_m \tilde{\delta} b \right] \]
(179)

Now, notice that the variation of the constraint \( R(Z)^m = * R(V)^m \) in (70) yields a similar set of terms to those appearing in (179)
\[ 0 = \delta [R(Z)_m - * R(V)_m] \]
\[ = \delta_{\text{Group}} [R(Z)_m - * R(V)_m] \]

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\begin{equation}
\left\{ \begin{array}{l}
d\delta z_m + \delta(\omega_{mn}z^n) - 2\delta v_m a + 2\delta F_{mn}e^n + 2\delta F_m e^n + \frac{1}{2}\gamma^5 \gamma_m \delta \phi \\
- \ast\left[ d\delta v_m + \delta(\omega_{mn}v^n) + 2\delta z_m a - 2\delta F_{mn}e^n + 2E_{mn}\delta f^n + \frac{1}{4}\gamma^5 \gamma_m \delta \phi \right] 
\end{array} \right.
\end{equation}

Therefore, we may rewrite (178) as

\begin{equation}
- \delta \mathcal{L} = 32R(V)^m \left\{ \delta_{\text{Group}}[R(Z)_m - \ast R(V)_m] \\
+ \delta\omega_{mn} v^n + \ast(\delta\omega_{mn} z^n) - z_m \delta b + \ast(v_m \delta b) \right\}
\end{equation}

\begin{equation}
- 128\tilde{R}(F)_{mn} e^m \delta v^n - 128\tilde{R}(F)_{mn} e^m \delta z^n
\end{equation}

\begin{equation}
- 64R(K)^m [E_{mn} \delta z^n - \tilde{E}_{mn} \delta v^n - \frac{1}{2} e^m \delta \omega_{mn}]
\end{equation}

\begin{equation}
+ 64R(D) [\delta v^m z_m + v^m \delta z_m] + 64R(A) [z^m \delta z_m + v^m \delta v_m]
\end{equation}

\begin{equation}
+ 32R(M)_{mn} [-\delta v_m z_n + v_m \delta z_n + \frac{1}{2} \epsilon_{mnpq}(v^p \delta v^q + z^p \delta z^q)]
\end{equation}

\begin{equation}
+ 16R(Q)[\gamma^5 \gamma^m v_m - \gamma^m z_m] \delta \phi - 16\sqrt{2} R(S) \gamma^5 \gamma^m e_m \delta \phi
\end{equation}

\begin{equation}
+ 16\psi[\gamma^m \delta z_m - \gamma^5 \gamma^m \delta v_m]R(S),
\end{equation}

where we have again integrated by parts in the terms involving \(d\delta v^m\) and \(d\delta z^m\) introduced in (180) and used the Bianchi identity. Furthermore we observe that the result (181) for \(\delta \mathcal{L}\) provides a possible avenue to avoid the problem of coupled expressions for the extra transformations of dependent fields. Indeed, if one were able to further rewrite (181) in terms only of the combinations of fields and extra transformations of dependent fields that appear in variations of the constraints, one could then express \(\delta \mathcal{L}\) in terms of readily calculable “group” transformations of curvatures. We have investigated this possibility further, but have found no obvious way in which this can be done.
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