A note on partial isometries on pseudo-Hilbert spaces

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Abstract

The aim of this paper is to show that two accessible subspaces in the Loynes $\mathcal{Z}$ - space $\mathcal{H}$ are the initial and final space of a partial gramian isometry, respectively if the norm of the difference of the associated gramian selfadjoint projections is strictly less than 1.

1 Introduction

Generalizing the concept of pre-Hilbert or Hilbert space, R.M. Loynes introduced in [5] the $VE$ - spaces or $VH$ - spaces respectively. A $VH$ - space is characterized in [6] by the fact that the inner product takes values in a suitable ordered topological vector (admissible) space $\mathcal{Z}$, thus being also called Loynes $\mathcal{Z}$ - spaces. Many authors used these spaces in the study of abstract stochastic processes (see [7], [1], [11], [12]). In [11] these spaces are referred to as pseudo-Hilbert spaces. Spectral theory for some classes of operators on such spaces was developed initially by Loynes himself ([5], [6]) and later by the authors in [3], respectively [2] and by A. Gheondea and B. E. Ugurcan in [4].

In what follows $\mathcal{H}, \mathcal{K}$ will denote two pseudo-Hilbert spaces over the same admissible space $\mathcal{Z}$ and $\mathcal{L}(\mathcal{H}, \mathcal{K})$ the space of all linear operators from $\mathcal{H}$ to $\mathcal{K}$. Recall that an operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is bounded, if there exists a constant $M > 0$ such that

$$[Th, Th]_\mathcal{K} \leq M^2[h, h]_\mathcal{H}, \quad h \in \mathcal{H},$$

where $[\cdot, \cdot]_\mathcal{K}$ is the inner product (also referred to as gramian) of the Loynes $\mathcal{Z}$ - space $\mathcal{K}$, while “≤” means the order in $\mathcal{Z}$. We shall denote that by

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\( T \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \). As usually, for \( \mathcal{H} = \mathcal{K} \), we use the notations \( \mathcal{L}(\mathcal{H}) \) and \( \mathcal{B}(\mathcal{H}) \) respectively. Moreover \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) is a Banach space (algebra if \( \mathcal{H} = \mathcal{K} \)) with the norm defined by

\[
\| T \| = \| T \|_{\mathcal{B}(\mathcal{H}, \mathcal{K})} = \inf \{ M : (\|) \text{ holds } \}.
\]

The operators \( T \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) for which \( \| T \| \leq 1 \) will be called grammian contractions.

The adjoint \( T^* \) of an operator \( T \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) and the gramian orthogonal complement \( \mathcal{M}^\perp \) of a subspace \( \mathcal{M} \) of \( \mathcal{H} \) will be defined (if they exist) analogously as in the Hilbert space case, but with respect to the inner products of \( \mathcal{H} \) and \( \mathcal{K} \).

By \( \mathcal{L}^*(\mathcal{H}, \mathcal{K}), \mathcal{B}^*(\mathcal{H}, \mathcal{K}) \) will be denoted the set of all adjointable elements of \( \mathcal{L}(\mathcal{H}, \mathcal{K}) \) and \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \), respectively, whereas \( P_\mathcal{M} \) denotes the gramian selfadjoint projection associated to the complementable (accessible) subspace \( \mathcal{M} \) of \( \mathcal{H} \).

We also remark that \( \mathcal{L}^*(\mathcal{H}, \mathcal{K}) \cap \mathcal{B}(\mathcal{H}, \mathcal{K}) = \mathcal{B}^*(\mathcal{H}, \mathcal{K}) \) and \( \mathcal{B}^*(\mathcal{H}) \) is a \( C^* \)-algebra.

\( T \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) is called a grammian isometry (grammian co-isometry) if \( T \in \mathcal{B}^*(\mathcal{H}, \mathcal{K}) \) and \( T^*T = I_\mathcal{H} \) (\( TT^* = I_\mathcal{K} \), respectively) and \( T \) is grammian unitary if it is simultaneously a grammian isometry and a grammian co-isometry.

If a grammian contraction \( T \) is adjointable, then \( T^* \) is a grammian contraction too (see [5], [6]). Familiar examples of adjointable contractions are self-adjoint grammian projections and grammian partial isometries, the latter containing two remarkable subclasses: that of grammian isometries and of grammian co-isometries. The latter classes were studied by the first author in [3], where also a geometric proof of the existence of the grammian co-isometric extension of a grammian adjointable contraction is given.

In what follows some definitions and results from [3] are needed.

**Definition 1.1.** Let \( \mathcal{H} \) and \( \mathcal{K} \) be two Loynes \( \mathbb{Z} \)-spaces. A linear operator \( T \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) is a partial grammian isometry, if its kernel \( N(T) \) and its range \( \mathcal{R}(T) \) are accessible (i.e. they have grammian orthogonal complements) in \( \mathcal{H} \) and \( \mathcal{K} \), respectively and from \( N(T)^\perp \) to \( \mathcal{R}(T) \) it preserves the grammian (is grammian unitary). The spaces \( \mathcal{M} \) := \( N(T)^\perp \) and \( \mathcal{R}(T) \) are called the initial and the final space of \( T \), respectively. The set of all partial grammian isometries from \( \mathcal{H} \) to \( \mathcal{K} \) will be denoted by \( \mathcal{PI}(\mathcal{H}, \mathcal{K}) \).

It can be easily seen, that \( \mathcal{PI}(\mathcal{H}, \mathcal{K}) \subset \mathcal{B}^*(\mathcal{H}, \mathcal{K}) \). Observe that if \( N(T) = 0 \), then \( T \) is simply a grammian isometry.

**Proposition 1.1.** If \( T \in \mathcal{PI}(\mathcal{H}, \mathcal{K}) \), then \( T^*T \) and \( TT^* \) are grammian self-adjoint projections for which the following hold:
(i) \( T^*T = P_{M(T)} \);
(ii) \( P_{M(T)} = TT^* \);
(iii) If \( \mathcal{H} = \mathcal{K} \) then \( T \) is a partial isometry in \( B^*(\mathcal{H}) \) as a \( C^* \)-algebra.

**Proposition 1.2.** For \( T \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \), the following are equivalent:

(i) \( T \in \mathcal{PI}(\mathcal{H}, \mathcal{K}) \);
(ii) \( T \in B^*(\mathcal{H}, \mathcal{K}) \) and \( T^*T \) is a gramian self-adjoint projection on \( \mathcal{H} \);
(iii) \( T \in B^*(\mathcal{H}, \mathcal{K}) \) and \( TT^* \) is a gramian self-adjoint projection on \( \mathcal{K} \);
(iv) \( T \in L^*(\mathcal{H}, \mathcal{K}) \) and \( T^* \in \mathcal{PI}(\mathcal{K}, \mathcal{H}) \).

It is obvious that any gramian isometry or gramian co-isometry is a partial gramian isometry.

# 2 The result

Focusing on the case \( \mathcal{H} = \mathcal{K} \) and taking \( T \in \mathcal{PI}(\mathcal{H}) \), then the operators \( T^*T \) and \( TT^* \) will be two gramian self-adjoint projections in \( B^*(\mathcal{H}) \). It is thus interesting, as in the case of Hilbert space (see [10, pp. 266,267]), to find a sufficient condition on two gramian self-adjoint projections \( P \) and \( Q \) in order to have their ranges as initial and final space of a certain partial gramian isometry. Indeed the following assertion holds.

**Theorem 2.1.** If \( P \) and \( Q \) are gramian self-adjoint projections and

\[
\| P - Q \| < 1, \tag{2}
\]

then there exists \( T \in \mathcal{PI}(\mathcal{H}) \) such that \( P = T^*T \) and \( Q = TT^* \).

**Proof.** Denote \( A = I + P(Q - P)P \). Since \( \| I - A \| = \| P(Q - P)P \| \leq \| P - Q \| < 1 \), by using that \( B^*(\mathcal{H}) \) is a Banach algebra, it results that \( A \) is invertible with a bounded inverse. On the other hand the operator \( A \) is positive. Indeed

\[
[Ah, h] = [h, h] + [P(Q - P)Ph, h] = [h, h] + [QPh, h] - [P^3h, h] \\
= [(I - P)h, h] + [QPh, Ph] \geq 0,
\]

where we used the fact that \( I - P \) and \( Q \) are gramian self-adjoint projections. In this situation, there exists the square root of \( A \), which is also invertible.
The operator $T := QA^{-1/2}P$ satisfies the requirements of the statement. Indeed we have $T^* = PA^{-1/2}Q$ and further on $PT^* = T^* = A^{-1/2}PQ$. Since $PA = AP$, we infer that $PA^{1/2} = A^{1/2}P$ which implies $A^{-1/2}P = PA^{-1/2}$. Further we get $TP = T = QPA^{-1/2}$. Taking into account that $PA = PQP$ we infer

$$T^*T = A^{-1/2}PQQPA^{-1/2} = A^{-1/2}PQPA^{-1/2} = A^{-1/2}PAA^{-1/2} = P.$$  

Using Proposition 1.2 we infer that $T$ is a partial gramian isometry and $TT^* = P_{\mathcal{R}(T)}$. But, the calculation of $TT^*$ leads us to the equalities

$$TT^* = QA^{-1/2}PPA^{-1/2}Q = QA^{-1}PQ,$$

which imply $\mathcal{R}(T) = \mathcal{R}(TT^*) \subset \mathcal{R}(Q)$, i.e. $TT^* \leq Q$. Now, let us show that $I - TT^* \leq I - Q$. Let $h \in (I - TT^*)\mathcal{K}$. Then the next implications hold

$$h \in (I - TT^*)\mathcal{K} \Rightarrow h = (I - TT^*)h \Rightarrow TT^*h = 0 \Rightarrow R^*h \in \mathcal{N}(T) \cap \mathcal{R}(T^*) = \{0\} \Rightarrow T^*h = 0 \Rightarrow PA^{-1/2}Qh = 0 \Rightarrow PQh = 0 \Rightarrow (Q - P)Qh = Qh \Rightarrow Qh = 0 \Rightarrow (I - Q)h = h \Rightarrow h \in \mathcal{R}(I - Q).$$

This shows that $\mathcal{R}(I - TT^*) \subset \mathcal{R}(I - Q)$, which indicates that $I - TT^* \leq I - Q$. Hence the equality $TT^* = Q$ holds.

\begin{remark}
Our theorem can be applied in perturbation theory to treat the variation of the spectral measure of gramian selfadjoint operators on pseudo-Hilbert spaces in a limit taking process. For the Hilbert space case see [10] no. 135.
\end{remark}

\begin{remark}
Our theorem states that (2) is a sufficient condition on the two gramian selfadjoint projections $P$ and $Q$ in order to determine the initial and final space of a partial isometry. This condition isn’t however necessary, as the following example shows. For $V$ a gramian (non-unitary) isometry on $\mathcal{K}$ we have that $V^*V - VV^*$, being a gramian selfadjoint projection, has norm equal to 1. It would therefore be interesting to find a weaker condition that would still be sufficient.
\end{remark}

\begin{remark}
Our definition of the partial isometry on the pseudo-Hilbert space $\mathcal{K}$ as well as the statement of our result being given in the $C^*$- algebra $\mathcal{B}(\mathcal{K})$ let us observe that following [3] or [4] it is possible to define and characterize the notion of a partial isometry in the Banach algebra $\mathcal{B}(\mathcal{K})(\mathcal{S})$.
\end{remark}
It is then naturally to ask if there exist such partial isometries in $\mathcal{B}(\mathcal{H})$ which are not in $\mathcal{B}^*(\mathcal{H})$ and if this would be the case, would an analogue of Theorem 2.1 hold in $\mathcal{B}(\mathcal{H})$?

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