Almost-Reed–Muller Codes Achieve Constant Rates for Random Errors

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Abstract

This paper considers “\(\delta\)-almost Reed–Muller codes”, i.e., linear codes spanned by evaluations of all but a \(\delta\) fraction of monomials of degree at most \(d\). It is shown that for any \(\delta > 0\) and any \(\varepsilon > 0\), there exists a family of \(\delta\)-almost Reed–Muller codes of constant rate that correct \(1/2 - \varepsilon\) fraction of random errors with high probability. For exact Reed–Muller codes, the analogous result is not known and represents a weaker version of the longstanding conjecture that Reed–Muller codes achieve capacity for random errors (Abbe-Shpilka-Wigderson STOC ’15). Our proof is based on the recent polarization result for Reed–Muller codes, combined with a combinatorial approach to establishing inequalities between the Reed–Muller code entropies.
1 Introduction

Reed–Muller (RM) codes [Ree54, Mul54] have long been conjectured to achieve the Shannon capacity for symmetric channels. Traces of this conjecture date back to the 1960s, as discussed recently in [ASY20] (we refer to [KKM+17, ASY20] for further references).

In the recent years, significant progress has been made on this conjecture for both the binary erasure channel (BEC), a.k.a. random erasures, and for the binary symmetric channel (BSC), a.k.a. random errors, using a variety of approaches. These are based on (i) estimating the weight enumerator, verifying the conjecture for some vanishing rates on the BEC and BSC [ASW15, SS20] — significant improvements on the weight enumerator were also recently obtained for certain rates using contractivity arguments [Sam19]; (ii) studying common zeros of bounded degree polynomials, verifying the conjecture for some rates tending to one on the BEC [ASW15]; (iii) using sharp threshold results for monotone Boolean functions [KKL88, BKK+92], settling the conjecture for the BEC at constant rate [KKM+17], a major step towards the general conjecture.

The main conjecture of achieving capacity on the BSC in the constant rate regime remains nonetheless open, and currently this does not seem reachable from the above developments. In fact, the weaker conjecture of achieving both a constant code rate and a constant error rate (i.e., showing a “good code” property in the average-case sense) had been until recently still open [ASW15, SS20]. More specifically:

Can RM codes of rate $\Omega(1)$ decode an $\Omega(1)$ fraction of random errors with high probability?

In this paper, we prove the above for almost-RM codes. To make this statement precise, let us make some definitions. Let $m \in \mathbb{N}$ and $n := 2^m$. Consider the $n \times n$ Reed–Muller matrix $M$ where rows are indexed by subsets $A \subseteq [m]$ and columns are indexed by Boolean vectors $z \in \{0, 1\}^m$, and the entries are given by:

$$M_{A,z} := \prod_{i \in A} z_i,$$

i.e., they are evaluations of monomials with variables indexed by $A$. We also use the notation $v_A$ for the row $M_{(A,.)}$. Let $\mathcal{A}$ be a collection of subsets of $[m]$. Define the linear code

$$\text{RM}(m, \mathcal{A}) := \text{span}\{v_A : A \in \mathcal{A}\}.$$

The special case of $\mathcal{A}$ corresponding to all subsets of size at most $r$ gives the Reed–Muller code $\text{RM}(m, r)$ of degree $r$ on $m$ variables.

**Definition 1.** We say that a code $\text{RM}(m, \mathcal{A})$ is a $\delta$-almost Reed–Muller code if there exists $r$ such that $\text{RM}(m, \mathcal{A}) \subseteq \text{RM}(m, r)$ and $|A| \geq (1 - \delta) \sum_{i=0}^{r} \binom{m}{i}$.

In other words, a $\delta$-almost RM code can be constructed by deleting up to $\delta$ fraction of vectors from the standard basis of an RM code. Note that for an RM code of rate $R$, a $\delta$-almost RM code has rate at least $(1 - \delta)R$. Our result can be stated as:

**Theorem 2** (Cf. Theorem 10). For any $\delta > 0$ and any $\varepsilon > 0$, there exists $R > 0$ and a family of $\delta$-almost Reed–Muller codes of rate $R$ that correct $1/2 - \varepsilon$ fraction of random errors with high probability.

To the best of our knowledge, such a result cannot be deduced from the known relationship between RM codes and polar codes [Has13], nor from the previous work on polarization of RM codes [AY20] (see Section 1.1 for more details). For exact RM codes, the closest result comes from [SS20] (improving upon [ASW15]), which shows that for $r < m/2 - \Omega(\sqrt{m \log m})$, corresponding to rates $R = 1/\text{polylog}(n)$, the RM$(m, r)$ codes decode a fraction $1/2 - o(1)$ of random errors with high probability.

We emphasize that when we say that the codes “correct” a fraction of errors we mean that there exists a decoding algorithm that succeeds with high probability and we do not claim that this algorithm is efficient. Efficiency of decoding of Reed–Muller codes is a research area of its own (see, e.g., Section V of [ASY20]).

Our approach is based on the polarization theory for RM codes [AY20]. Polarization theory emerged from the development of polar codes [Ari09], and consists in tracking the conditional entropies of the information bits rather than the block error probability directly. We refer to Sections 2 and 3 for a precise description.

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1 All algebraic operations are over $\mathbb{F}_2$ unless otherwise specified.
of concepts and results that are used in this work, including successive decoding and the relevant decoding order (and [Arı09] [AT09] [Arı10] BGN+18 for references on polarization theory). Now we provide a brief description of the approach.

For RM codes on the BSC, the channel output is $Y = UM + E$ where $U$ contains $n$ i.i.d. uniform bits (the information bits) and $E$ contains $n$ i.i.d. Bernoulli($p$) bits (the noise). Recall that the rows of $M$, and therefore also the coordinates of $U$, are indexed by subsets $A \subseteq [m]$. In Definition 3 we define a certain total ordering on the subsets, basically from the sparsest to the heaviest rows of $M$. Then, the conditional entropies are defined using this ordering by $H_A := H(U_A|Y,(U_B)_{B \subset A})$. To decode with a small error probability, these conditional entropies should be low for the components selected by the code, i.e., those indexed by sets $A \in A$ in the $Rm(m,A)$ code. In particular, to prove that successive decoding of $Rm(m,r)$ succeeds, we would like to show that all components indexed by sets with size up to $r$ have low entropies.

In the case of polar codes, a different decoding order is used (in other words, rows of the matrix $M$ are permuted), which allows for a simpler recursive analysis. In [Arı09], the polarization property is shown for the polar ordering, i.e., it is shown that most conditional entropies are tending to either 0 or 1, and thus that the code induced by the low entropy components achieves capacity. In [AY20], the same property is shown for the RM code ordering, and thus, the code induced by the low entropy components in this ordering also achieves capacity. This code is called the twin-RM code in [AY20].

To prove the capacity conjecture for RM codes, it remains to show that the twin-RM code is in fact the RM code for any symmetric channel. In other words, we would like to prove that the components with low entropies correspond in fact to the small subsets $A$. In particular, showing that

$$|A| > |B| \implies H_A \geq H_B$$

(2)

would be sufficient to prove that the two codes are the same. The ‘twin’ terminology is used in [AY20] mainly based on numerical simulations, which provide some evidence that the codes are indeed the same, and on a proof of their equality up to blocklength $n = 16$. Moreover, [AY20] shows a weaker property

$$A \supseteq B \implies H_A \geq H_B$$

(3)

indicating that at least in general the entropies increase with the set size. However, [AY20] does not prove that the twin-RM code is in fact the RM code and [3] does not seem to be enough for a good bound on their similarity.

In this paper, we push this approach and study how close the twin-RM code is to the RM code, investigating what rates can be achieved using the level of similarity that we can quantify. Rewriting [2] as $H_A \geq H_{A \setminus \{a\}}$ for any $a$, one of our contributions is to establish a stronger property

$$B = (A \cup \{b\}) \setminus \{a,a'\} \implies H_A \geq H_B$$

(4)

for any $a,a' \in A$ and any $b$. Subsequently, [4] is combined with other inequalities that follow from symmetries of RM codes and the decoding order. By arguments featuring deviation bounds on integer random walks, we then show that while we are still short of [2], we have that $H_A \geq H_B$ holds for almost all pairs of sets $(A,B)$ with $|A| = r + k, |B| = r$ for some relevant values of $r$ and $k$. This allows us to use a standard polarization argument based on the relation $\sum_A H_A = h(p) \cdot n$, where $1 - h(p)$ is the channel capacity, to conclude that for the BSC and certain rates twin-RM codes and RM codes are indeed “close cousins”, resulting in Theorem 2.

Consider the channel naturally given by a conditional entropy $H_A$. That is, consider the channel $Y,U_{\leq A} \rightarrow U_A$ which takes a noisy codeword $Y$ and information bits $U_C$ for sets $C$ preceding $A$ in the decoding order and outputs the information bit $U_A$. The proof of $H_A \geq H_B$ in [4] in fact establishes that such channel for set $A$ is a degradation [Cov72] [Ber73] [MP18] of the channel for $B$. More specifically, we use the relation between channel and source coding for BSC to show that the channel for $A$ can be obtained by applying a linear isomorphism on the inputs of the channel for $B$ and subsequently dropping some information bits. This isomorphism is induced by a permutation of columns of the RM matrix $M$.

We conclude this discussion by noting that some of the mechanisms we develop in this paper (e.g., Lemma 19) could be reused to further improve the rate/edge-closeness tradeoff if any of the inequalities on the RM

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2One also needs to show that the decay to 0 is fast enough. With polar codes, capacity is obtained for any binary input symmetric output channel, including the BEC and BSC.
entropies were to be obtained. In Section 7 we show that certain aspects of our analysis are tight, suggesting that more inequalities are also necessary in order to advance our approach. A natural next step would be to extend [1] to \( H_A \geq H_B \) in case where \( B \) can be constructed from \( A \) by removing three and adding two elements. However, at the moment we do not know how to do that. We stress that the full inequality (2) might well be true. Circumstantial evidence in its favor is provided by simulations conducted in [AY20] in the context of erasure channel, see Figures 11–13 therein.

Another natural question to investigate is if there exists a way to “boost” a good \( \delta \)-almost RM code and obtain a statement about actual RM codes. For example, using symmetry arguments we can show that if an RM code with two vectors deleted from the standard basis is good for a BSC channel, then the full RM code is also good. However, this is a far cry from a constant \( \delta \) fraction that appears in our results.

### 1.1 Quantifying closeness to Reed–Muller codes

As discussed, both polar codes and RM codes are instances of RM(\( m, A \)) codes, in other words they are both generated by some of the rows of the RM matrix \( M \). In this section we briefly discuss some results on \( \delta \)-almost RM codes that are implicit in existing literature on polarization theory [Has13, AY20]. The codes obtained from these works can be described as “distant cousins” (constant rate for some \( \delta \)), as opposed to our “close cousins” (constant rate for any \( \delta > 0 \)).

There is a wealth of literature on performance of RM codes on different channels [HKL04, HKL05, ASW15, KKM+17, SSV17, SS20, Sam19, AY20, ASY20] and our foregoing discussion already touched the elements. However, at the moment we do not know how to do that. We stress that the full inequality (2) was to be obtained. In Section 7 we show that certain aspects of our analysis are tight, suggesting that more inequalities are also necessary in order to advance our approach. A natural next step would be to extend [1] to \( H_A \geq H_B \) in case where \( B \) can be constructed from \( A \) by removing three and adding two elements. However, at the moment we do not know how to do that. We stress that the full inequality (2) might well be true. Circumstantial evidence in its favor is provided by simulations conducted in [AY20] in the context of erasure channel, see Figures 11–13 therein.

Subsequent work A subsequent work by one of the authors, Samorodnitsky and Sberlo [HSS21] shows that Reed–Muller codes of constant rate \( R \) decode errors on BSC(\( p \)) for \( p < \frac{1}{2} - \sqrt{2^{-R}(1 - 2^{-R})} \), achieving the positive rate conjecture with techniques building on papers by Samorodnitsky [Sam19, Sam20]. This approach is disjoint from our approach, has the advantage of dealing with the exact Reed–Muller code, but is no longer related to the successive decoder and to polarization. It remains open to show that constant-rate Reed–Muller codes achieve capacity on the binary symmetric channel.

### 2 Background and Notation

#### 2.1 Our setting

Let \( 0 < p < 1/2 \) be a parameter of the BSC. The basic elements of our probability space are two random (row) vectors

\[
U = (U_A)_{A \subseteq [m]} , \quad E = (E_z)_{z \in \{0, 1\}^m} .
\]
Vector $U$ consists of $n = 2^m$ i.i.d. uniform $\{0,1\}$ random variables $U_A$ indexed by subsets of $[m]$. The components of vector $E$ are $n$ i.i.d. Ber$(p)$ random variables $E_z$ indexed by bitstrings $z = (z_1, \ldots, z_m) \in \{0,1\}^m$. Furthermore, $U$ and $E$ are independent.

We will consider a total order on sets that we call the \textit{decoding order}:

**Definition 3** (Decoding order). We say that $A < B$ if

- $|A| > |B|$, or
- $|A| = |B|$ and there exists $i$ such that $i \notin A$, $i \in B$ and $\forall j > i : j \in A \iff j \in B$.

Note that when $|A| = |B|$, the decoding order can be described as \textit{reverse lexicographic}: First, all sets that do not contain $m$ come before all sets that contain $m$. Then, inside of each group, sets that do not contain $m-1$ come before sets that contain $m-1$, and so on, recursively. To dispel doubts, let us write the decoding order in the case $m = 4$ (note the highlighted segment pointing out a difference from a more ordinary lexicographic ordering):

$$1234 < 123 < 124 < 134 < 12 < 13 < 23 < 14 < 24 < 34 < 1 < 2 < 3 < 4 < \emptyset.$$  

We define another random vector $X = (X_z)_{z \in \{0,1\}^m}$ as $X := UM$. More precisely,

$$X_z = \sum_{A \subseteq [m]} U_A \prod_{i \in A} z_i.$$  

(5)

We also let $Y := X + E$.

We are going to use vectors $U, E, X$ and $Y$ in the analysis of successive decoding of codes $\text{RM}(m, A)$. To this end, we will consider several information measures like conditional entropy $H(X \mid Y)$, Bhattacharyya parameter and MAP decoding error:

**Definition 4** (Bhattacharyya parameter). Let $(V, W)$ be discrete random variables such that $V$ is uniform in $\{0,1\}$ and $W \in \mathcal{W}$. The Bhattacharyya parameter $Z(V \mid W)$ is

$$Z(V \mid W) := \sum_{w \in \mathcal{W}} \sqrt{\Pr[W = w \mid V = 0] \Pr[W = w \mid V = 1]}.$$  

(6)

**Definition 5** (MAP error). Let $(V, W)$ be discrete random variables such that $V$ is uniform in $\{0,1\}$ and $W \in \mathcal{W}$. The maximum a posteriori probability (MAP) decoding error $P_e(V \mid W)$ is

$$P_e(V \mid W) := \frac{1}{2} \sum_{w \in \mathcal{W}} \min \{ \Pr[W = w \mid V = 0], \Pr[W = w \mid V = 1] \}.$$  

Note that $P_e(V \mid W)$ is the probability of error under an optimal scheme for guessing the value of $V$ given $W$.

Turning back to our setting, let $U_{<A} := (U_B)_{B < A}$. When analyzing codes $\text{RM}(m, A)$, we will be interested in values like

$$H_A := H(U_A \mid U_{<A}, Y), \quad Z_A := Z(U_A \mid U_{<A}, Y),$$  

i.e., in information measures encountered in the process of \textit{successive decoding}: Decoding the information bit $U_A$ given the noisy codeword $Y$ and previously decoded (in the decoding order) bits $U_{<A}$.

For a code $\text{RM}(m, A)$, the successive decoding algorithm decodes a noisy codeword $y$ to $\hat{u} = (\hat{u}_A)_{A \subseteq A}$, where each bit $\hat{u}_A$ is guessed according to the MAP formula, assuming that the preceding bits were decoded correctly and $U_A = 0$ for $A \notin A$:

$$\hat{u}_A(y) := \arg\max_{u \in \{0,1\}} \Pr[Y = y, (U_B = \hat{u}_B)_{B < A, B \notin A}, (U_B = 0)_{B < A, B \notin A} \mid U_A = u].$$  

**Definition 6** (Successive decoding under decoding order). Using the notation above, the decoding error of $\text{RM}(m, A)$ under successive decoding is given by

$$\Pr[\hat{u}(Y) \neq (U_A)_{A \subseteq A}].$$  

(7)

In particular, if for a given channel and a code family $\text{RM}(m, A)$ the probability in (7) vanishes, then the code $\text{RM}(m, A)$ corrects errors under this channel with high probability.
2.2 Miscellaneous notation

We specify some shorthand notation that we use throughout. \([m]\) denotes the set \(\{1,\ldots,m\}\) and \(\mathcal{P}(m)\) is a set of all subsets of \([m]\). Given \(A \subseteq [m]\) we write \(\overline{A} := [m] \setminus A\). We use binomial coefficients \(\binom{m}{k}\) and write \(\binom{s}{k}\) for the set of all subsets of \(S\) of size \(k\). To avoid clutter we abuse notation writing \(\binom{[m]}{k} := \binom{[m]}{k}\). From the context it should always be clear whether \(\binom{[m]}{k}\) is meant as a number or a set. We also write

\[
\binom{m}{\leq k} := \sum_{i=0}^{k} \binom{m}{i}, \quad \binom{m}{\leq k} := \left\{ \binom{m}{i} \right\} \cup \left\{ \binom{[m]}{k} \right\}
\]

and analogously for \(\binom{m}{\geq k}\). We use log to denote binary logarithm and \(h(p)\) for the binary entropy function

\[
h(p) = -p \log p - (1 - p) \log(1 - p).
\]

We also write \(\Phi\) and \(\Phi^{-1}\) for the standard Gaussian CDF and its inverse. Whenever we consider two sets \(A, B\), we try to stick to the convention that \(A\) precedes \(B\) in the decoding order, in particular \(|A| \geq |B|\). We sometimes drop parentheses for consecutive set operations. In that case we adopt left-to-right associativity, e.g., \(A \setminus \{a\} \cup \{b\} = (A \setminus \{a\}) \cup \{b\}\).

3 Preliminaries

In this section we list some known results that we use in our proofs. The most important one is the polarization theorem for RM codes from \cite{AY20}:

**Theorem 7** (Theorem 1 in \cite{AY20}). For every \(0 < p < 1/2, 0 < \varepsilon < 1/10, c \in \mathbb{N}\) and \(0 < \xi < 1/2\), there exists \(m_0 = m_0(p, \varepsilon, c, \xi)\) such that for \(m > m_0\),

\[
\left| \left\{ A \subseteq [m] : Z_A \geq 1/n^c \land H_A \leq 1 - \varepsilon \right\} \right| \leq \frac{n}{m^{1/2-\xi}}.
\]

Theorem 7 has the following interpretation: The fraction of subsets \(A\) for which their respective channels are not polarized (where “polarized” means that either \(Z_A \approx 0\) or \(H_A \approx 1\)) goes to 0 at a rate \(O(m^{-1/2+\xi})\). We also use an ingredient from the proof of Theorem 7.

**Lemma 8** (Lemma 3 in \cite{AY20}). If \(A \supseteq B\), then \(Z_A \geq Z_B\).

It is well-known that for symmetric channels (in particular for the BSC) the probability of error under successive decoding is controlled by the sum of the Bhattacharyya parameters:

**Theorem 9** (see Proposition 2 and Theorem 4 in \cite{Arı09}). The probability of error under successive decoding of RM\((m,A)\) is bounded by

\[
\Pr \left[ \hat{u}(Y) \neq (U_A)_{A \in A} \right] \leq \sum_{A \in A} Z_A.
\]

We also state some known (e.g., \cite{Arı09}) background facts that will be needed in our proofs:

**Fact 10.** The inverse (over \(\mathbb{F}_2\)) of the Reed–Muller matrix \(M\) is given by

\[
(M^{-1})_{z,A} = \prod_{i \notin A} (1 - z_i).
\]

**Proof.** We check directly that

\[
(MM^{-1})_{A,B} = \sum_{z \in \{0,1\}^m} \prod_{i \in A} z_i \prod_{i \notin B} (1 - z_i). \quad (8)
\]

First, if \(A = B\), then the sum \(\bigotimes\) has exactly one non-zero term with \(z\) being indicator of \(A\). On the other hand, if \(A \setminus B \neq \emptyset\), then all terms of the sum in \(\bigotimes\) are zero. Furthermore, if \(B \setminus A \neq \emptyset\), the number of non-zero terms in \(\bigotimes\) must be even. Hence,

\[
(MM^{-1})_{A,B} = 1 \iff A \setminus B = \emptyset \land B \setminus A = \emptyset \iff A = B.
\]
Fact 11. $\sum_{A \subseteq [m]} H_A = h(p) \cdot n$.

Proof. Recall that $X = U M$. By Fact 10 random vectors $U$ and $X$ are deterministic, invertible functions of each other. Furthermore, the collection of pairs $(X_z, Y_z)_{z \in \{0,1\}^n}$ is independent. Applying these observations and the chain rule,

$$\sum_{A \subseteq [m]} H_A = \sum_{A \subseteq [m]} H(U_A \mid U_{<A}, Y) = H(U \mid Y) = H(X \mid Y) = H(X_z \mid Y_z) \cdot n$$

$$= H(X_z \mid X_z + E_z) \cdot n = H(E_z) \cdot n = h(p) \cdot n . \quad \square$$

Fact 12. Let $U$ be uniform in $\{0,1\}$ and $X$, $Y$ be discrete random variables. We have:

1. $Z(U \mid XY) \leq Z(U \mid X)$.
2. If $X \in \mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$ is injective on the support of $X$, then $Z(U \mid X) = Z(U \mid f(X))$.
3. If $(U,X)$ is independent of $Y$, then $Z(U \mid XY) = Z(U \mid X)$.

We omit the proof of Fact 12, but all these basic properties are established by direct computations using (6). For more on the Bhattacharyya parameter in the context of polar codes, see, e.g., [Arı09]. We also state a property which follows by checking both cases in Definition 3:

Fact 13. For $A, B \subseteq [m]$:

1. If $A < B$ and $b \in B$, then $A \setminus \{b\} < B \setminus \{b\}$.
2. If $A < B$ and $a \notin A$, then $A \cup \{a\} < B \cup \{a\}$.

Finally, we make use of a standard CLT approximation of $\binom{m}{\leq r}$:

Fact 14. Let $r = r(m)$ be such that $r = \frac{m}{2} + \alpha \sqrt{m} + o(\sqrt{m})$. Then, we have

$$\lim_{m \to \infty} \frac{1}{n} \binom{m}{\leq r} = \Phi(2\alpha) .$$

Equivalently, if $r$ is the smallest integer such that $\binom{m}{\leq r} \geq R n$ for some $R > 0$, then $r = \frac{m}{2} + \alpha \sqrt{m} + o(\sqrt{m})$ for $\alpha := \Phi^{-1}(R)/2$.

4 Our Result

In our main result we prove that for a binary symmetric channel, a positive rate $\delta$-almost Reed–Muller code succeeds with high probability under successive decoding. Due to Theorem 9 to create such a code it makes sense to delete vectors with largest Bhattacharyya $Z_A$ values:

Definition 15. For $0 \leq r \leq m$ and $0 \leq \delta \leq 1$, we fix $\text{RM}(m, r, \delta)$ to be any code $\text{RM}(m, \mathcal{A})$ such that:

- $\mathcal{A} \subseteq \binom{m}{\leq r}$.
- $|\mathcal{A}| = |(1 - \delta) \binom{m}{\leq r}|$.
- For all $A \in \mathcal{A}$ and $B \in \binom{m}{\leq r} \setminus \mathcal{A}$, we have $Z_A \leq Z_B$.

In particular, $\text{RM}(m, r, \delta)$ is $\delta$-almost Reed–Muller and its rate is at least $(1 - \delta) R$, where $R$ is the rate of $\text{RM}(m, r)$. We can now state our main theorem. For simplicity, we focus only on “noisier” binary symmetric channels with $h(p) \geq 1/2$, i.e., $p \geq h^{-1}(1/2) \approx 0.11$. Since a code that corrects fraction $p$ of random errors also corrects a fraction $p' < p$ of errors, this is without loss of generality.

Theorem 16. Let $0 < p < 1/2$ and $\delta > 0$ be such that $0 < 1 - h(p) - 2\delta \leq 1/2$. Then, there exist $R > 0$ and $r = r(m)$ such that:
• Codes RM(m, r, δ) have rate at least R.

• For every c ∈ N, there exists m₀ = m₀(p, δ, c) such that for m > m₀ the error probability under successive decoding of RM(m, r, δ) is at most 1/α².

Furthermore, R can be set to R := (1 − δ)R₀, with R₀ given as

\[ R₀ = R₀(p, δ) := \Phi(2α), \]
\[ α = α(p, δ) := 2γ − \sqrt{\frac{9}{32} \ln(2/δ²)}, \]
\[ γ = γ(p, δ) := \frac{Φ⁻¹(1 − h(p) − 2δ)}{2}. \]

We believe the main interest of this result lies in the qualitative statement: For every p < 1/2 and δ > 0, there exists r corresponding to a constant rate R such that the successive decoding of RM(m, r, δ) corrects fraction p of random errors with high probability. In any case, we have an estimate

\[ R = δ^{9/8 + o(1)}, \tag{9} \]

where o(1) is a function that, for any fixed p, goes to 0 as δ goes to 0. The derivation of (9) is provided in Section 8.1.

While these rates are much smaller compared to R = (1 − h(p))δ obtainable from [Has13] or [AY20] for δ ≥ h(p) (cf. Section 1.1), they hold for values of δ arbitrarily close to zero.

5 Proof Outline

Our strategy for proving Theorem 16 focuses on inequalities between values of Z_A for different sets A. In particular, as explained in [AY20], if, for a given symmetric channel, we could prove that

\[ |A| > |B| \implies Z_A ≥ Z_B \tag{10} \]

it would follow that the twin-RM code for that channel and RM code are equal and, since twin-RM codes achieve capacity, that Reed–Muller codes achieve capacity on that channel. Instead, we rely on a weaker property

\[ B = A \cup \{ b \} \setminus \{ a, a' \} \implies Z_A ≥ Z_B \tag{11} \]

for \( a, a' ∈ A \).

5.1 Warm-up: \( Z_A ≥ Z_B \) for m = 4

Before presenting our general approach, let us consider the case m = 4. The main ideas required to establish (11) can be observed here. In this small case, we can actually show (10).

We analyze the task of decoding the message \( U \) from \( Y \). We have an under-determined linear system of equations \( Y = UM + E = \{ U, E \} \cdot \{ M; I \} \). In this notation \( \{ U, E \} \) (the unknowns vector) is a concatenation of vectors and \{ M; I \} (the coefficient matrix) is the RM matrix M with the identity matrix I underneath it.

We are interested in the Bhattacharyya parameter Z_A, so let us focus on the process of decoding information bit \( U_A \) given the preceding\(^3\) bits \( U_{<A} \) and the noisy codeword \( Y \). This means that in the system \( Y = UM + E \) we can substitute all values \( \hat{U}_B \) for sets \( B < A \). Therefore, the only remaining unknowns are \( U_B \) for \( B ≥ A \) and \( E_z \) for all \( z ∈ \{ 0, 1 \}^m \).

We consider all linear combinations of these equations, dividing them into three types:

[^3]: Throughout the paper we discuss and establish inequalities between Bhattacharyya parameters Z_A, but our technique uses only basic properties listed in Fact 12. Hence, it is applicable to any measure of information satisfying those properties, including conditional entropy H_A and MAP decoding error.

[^4]: Technically, we should always assume \( U_B = 0 \) for \( B < A \). However, it is known [Ari09] that for any fixed value of \( u_{<A} \) it holds that \( Z(U_A | U_{<A}, Y) = Z(U_A | U_{<A} = u_{<A}, Y) \) (and the same holds for other statistics like the entropy). Therefore, we do not need to worry about remembering that \( U_B = 0 \) for \( B ∈ A \).
1. Some $U_B$ appears in the equation for $B > A$.

2. No coordinate of $U$ appears in the equation. This tells us the exact value of $E_{i_1} + \ldots + E_{i_k}$, a sum of a subset of components of $E$ (the components of $E$ that appear in the equation).

3. Out of $U$, only the coordinate $U_A$ appears in the equation. Ideally, we want $U_A$ to appear alone because then we would know its exact value. In general, it will be accompanied by a sum of error terms $E_{i_1} + \ldots + E_{i_k}$.

Intuitively, all the information useful for decoding $U_A$ is contained in equations of types 2 and 3. Lemma 25 makes this intuition precise. The set of all equations of the second type can be thought of as a vector subspace $\mathcal{H}_A$ of $\mathbb{F}_2^{(0,1)^m}$. This holds since for each equation we can think of it as a binary vector with ones in positions indexed by variables that occur in the equation. Furthermore, the difference between two equations of the third type is an equation of the second type. Therefore, if we also treat the set of all equations of the third type as a subset of $\mathbb{F}_2^{(0,1)^m}$ (ignoring the variable $U_A$), this set is a coset (an affine space) of $\mathcal{H}_A$ in the vector space.

To summarize all of the above, the information regarding $U_A$ is represented by a coset of $\mathbb{F}_2^{(0,1)^m}$. This affine subspace is given as $W_A + \mathcal{H}_A$, where $W_A$ is a translation vector corresponding to an equation of the third type. With some thought, it can be seen that this gives a natural criterion for comparing $Z_A$ and $Z_B$. If we show that the coset of set $A$ is contained in the coset of set $B$, this means that information available to us when decoding $U_A$ is a subset of the information available for $U_B$, and therefore by Fact 12.1 we have $Z_A \geq Z_B$.

Since the vectors $W_A$ can be interpreted as elements of $\mathbb{F}_2^{(0,1)^m}$, we can also think of them as evaluation vectors of functions from $\mathbb{F}_2^n$ to $\mathbb{F}_2$. Hence, each of them can be identified with such a function, or in other words with a polynomial on $m$ variables. More so, it turns out that (after a permutation of coordinates, see Section 5.1 for details) each vector $W_A$ becomes an evaluation vector of the monomial $x_A = \prod_{a \in A} x_a$ (this follows from the self-duality of RM codes). Furthermore, we have that the subspace $\mathcal{H}_A$ is spanned by all previous vectors $(W_B)_{B < A}$. Therefore, denoting our coset as $C_A$, it can be written as

$$C_A = x_A + \mathcal{H}_A = x_A + \text{span}\{x_B\}_{B < A}.$$  \hfill (12)

Let us now focus back on $m = 4$ and recall the decoding order in this case:

$$1234 < 123 < 124 < 134 < 234 < 12 < 13 < 23 < 14 < 24 < 34 < 1 < 2 < 3 < 4 < \emptyset.$$  

Writing down the cosets in this example, we have

$$C_{1234} = x_0 + \{0\}, \quad C_{123} = x_4 + \text{span}\{x_0\}, \quad C_{124} = x_3 + \text{span}\{x_0, x_4\},$$

$$C_{134} = x_2 + \text{span}\{x_0, x_4, x_3\}, \quad C_{234} = x_1 + \text{span}\{x_0, x_4, x_3, x_2\},$$

$$C_{12} = x_{34} + \text{span}\{x_0, x_4, x_3, x_2, x_1\}, \quad \ldots$$

We are hoping to establish inequalities between the $Z_A$ values by analyzing a pure algebraic question of comparing the affine spaces $C_A$. As we said, $C_A \subseteq C_B$ would imply $Z_A \geq Z_B$. However, from (12) it follows that if $A < B$, then, on the one hand, $x_A \in C_A$, but on the other hand, $x_A \in \mathcal{H}_B$ and therefore $x_A \notin C_B$. Hence, $C_A \subseteq C_B$ is never the case if $A < B$. To bypass this issue, we use the fact that, since the components of the vector $E$ represent iid noise, permuting its indices does not change the underlying probability distribution. If we think of our vectors as polynomials, each such permutation $\tau$ of $\{0, 1\}^m$ induces a linear isomorphism of the polynomial space, and the effects of this isomorphism on the polynomials can be written as a change of variables. Therefore, it can be checked that also $\tau(C_A) \subseteq C_B$ ensures that $Z_A \geq Z_B$ holds. We illustrate this idea on two examples, corresponding to two types of permutations that we use throughout this paper.

1. If we start with

$$C_{123} = x_4 + \text{span}\{x_0\}$$

and apply the transposition of coordinates $x_3 \leftrightarrow x_4$, we get

$$\tau(C_{123}) = \tau(x_4 + \text{span}\{x_0\}) = x_3 + \text{span}\{x_0\} \subseteq x_3 + \text{span}\{x_0, x_4\} = C_{124}.$$  

Therefore, we established $\tau(C_{123}) \subseteq C_{124}$ and $Z_{123} \geq Z_{124}$.
2. Starting with \( C_{234} = x_1 + \text{span}\{x_0, x_4, x_3, x_2\} \) and applying the permutation that maps \( x_1 \rightarrow x_1 + x_{34} \) and leaves other coordinates unchanged, we get

\[
\tau(C_{234}) = x_1 + x_{34} + \text{span}\{x_0, x_4, x_3, x_2\} \subseteq x_{34} + \text{span}\{x_0, x_4, x_3, x_2, x_1\} = C_{12},
\]

establishing \( Z_{234} \geq Z_{12} \).

These two examples correspond to two types of permutations that we use throughout: First, transpositions \( x_b \leftrightarrow x_a \) will give inequalities \( Z_A \geq Z_{A \setminus \{a\} \cup \{b\}} \) for \( a < b \). Second, permutations \( x_b \rightarrow x_b + x_{a,a'} \) will give \( Z_A \geq Z_{A \setminus \{a,a'\} \cup \{b\}} \). In general, they correspond to Rule 1 and Rule 2 from Definition 20.

In our toy case \( m = 4 \), all other inequalities \( Z_A \geq Z_B \) for \( |A| > |B| \) follow in a similar way. For \( m = 5 \), this approach proves all \( Z_A \geq Z_B \) for \( |A| > |B| \) except for a single case of \( Z_{345} \) vs. \( Z_{12} \). For larger \( m \), we get more and more cases not covered by our rules, requiring us to resort to additional techniques.

5.2 Proof sketch

In this section we present the main ingredients in the proof of Theorem 16. Since we are unable to prove (10), we end up with Theorem 16 as a consequence of a weaker set of inequalities. Using our operations in a manner that we sketched in the previous section, we can show \( Z_A \geq Z_B \) at least for some sets with \( |A| = |B| + 1 \). This can be expanded inductively into inequalities with a larger gap between the sizes of \( A \) and \( B \). Ultimately, we take \( r = m/2 + O(\sqrt{m}) \) and some \( k = O(\sqrt{m}) \) and show \( Z_A \geq Z_B \) for almost every set \( A \) of size \( r + k \) and almost every set \( B \) of size \( r \).

As a consequence of this, imagine that a relatively small fraction of sets \( B \) with \( |B| \leq r \) has non-negligible \( Z_B \) values. Using our operations, it will turn out that almost all sets \( A \) with \( |A| \geq r + k \) also have non-negligible \( Z_A \) values. By Theorem 4, almost all of those sets \( A \) must in fact have \( H_A \) close to 1. Ultimately, \( r \) and \( k \) are chosen so that we obtain a contradiction with Fact 11. The final conclusion is that only a very small fraction of sets \( B \) with \( |B| \leq r \) can have non-negligible \( Z_B \) values. By deleting basis codewords corresponding to those sets, by Theorem 9 we obtain a \( \delta \)-almost Reed–Muller code that is amenable to successive decoding.

Let us expand on this explanation, starting with a general framework for passing between sets of size \( r + k \) and \( r \) as described in the previous paragraph. We work with orderings on the subsets of \( [m] \) (where it is a good idea to think about them as suborders of the decoding order from Definition 3):

**Definition 17.** We say that a partial order \( \ll \) on \( \mathcal{P}(m) \) is information-consistent if

\[
A \ll B \implies Z_A \geq Z_B
\]

for all sets \( A, B \subseteq [m] \).

Note that in general whether an order is information-consistent might depend on the channel (i.e., on \( p \)). The specific order we introduce later is information-consistent for every \( p \).

**Definition 18.** Let \( \ll \) be a partial order on \( \mathcal{P}(m) \). We say that \( \ll \) is \((\delta, r, k)\)-expanding if, for every collection \( B \subseteq \binom{m}{r} \) of subsets of size \( r \), letting

\[
A := \mathcal{A}(B) = \{ A : |A| = r + k \land \exists B \in B : A \ll B \},
\]

we have

\[
|B| \geq \delta \binom{m}{r} \implies |A| \geq (1 - \delta) \binom{m}{r + k}.
\]

Considering the bipartite graph with one group of vertices being sets of size \( r \), the other group sets of size \( r + k \) and edges according to the relation \( A \ll B \), Definition 18 states a strong expansion property of this graph. The following lemma makes use of Definitions 17 and 18 to formalize our strategy for this part of the proof:

---

Except for the special cases where \( A = [4] \) or \( B = \emptyset \), where there is no permutation with \( \tau(C_A) \subseteq C_B \). It is not hard to prove these by another argument.
Lemma 19. Let \(0 < p < 1/2\), let \(\ll\) be an information-consistent family of partial orders and let \(\delta > 0\), \(r = r(m)\), \(k = k(m)\) satisfy

\[
\liminf_{m \to \infty} \frac{1}{n} \left( \frac{m}{(r + k)} \right) > \frac{h(p)}{(1 - \delta)}.
\]

If \(\ll\) is \((\delta, r, k)\)-expanding, then, for every \(c \in \mathbb{N}\), there exists \(m_0\) such that for \(m > m_0\) the error probability of successive decoding of \(\text{RM}(m, r, \delta)\) is less than \(1/n^c\).

\[
\begin{align*}
H_A & \approx 1 \\
|\{A : |A| \geq r + k\}| & \approx (1 - h(p))n \\
H_B & > 0 \\
|\{B : |B| \leq r\}| & \approx Rn
\end{align*}
\]

Figure 1: An illustration of the expansion property from Lemma 19. For the RM code \(\text{RM}(m, r)\) with rate \(R\), the \((\delta, r, k)\)-expansion property together with RM code polarization imply that even a relatively small number of sets of size at most \(r\) with \(H_B\) significantly larger than 0 induces a very large number of sets of size at least \(r + k\) with \(H_A \approx 1\). This is in contradiction with \(\sum_A H_A = (1 - h(p))n\).

In this paper, we take \(\ll\) to be a specific order created by taking the transitive closure of two “rules”:

Definition 20. Let \(A, B \subseteq [m]\). We say that \(B\) was obtained from \(A\) by application of Rule 1 if there exist \(a < b, a \in A, b \notin A\) such that \(B = A \setminus \{a\} \cup \{b\}\).

We say that \(B\) was obtained from \(A\) by application of Rule 2 if there exist \(a, a' \in A, b \notin A \setminus \{a, a'\}\) such that \(B = A \setminus \{a, a'\} \cup \{b\}\).

For sets \(A, B \subseteq [m]\) we say that \(B\) can be constructed from \(A\) and write \(A \ll B\) if \(B\) can be obtained from \(A\) by a finite number of applications of Rules 1 and 2.

The bulk of our argument consists of proving that the “can be constructed from” \(\ll\) relation satisfies the assumptions of Lemma 19.

**Lemma 21.** The \(\ll\) relation is information-consistent for every \(0 < p < 1/2\).

We presented the most important ideas used to prove Lemma 21 in Section 5.1. We then show that the \(\ll\) relation indeed is \((\delta, k, r)\)-expanding for an appropriate choice of parameters.
Lemma 22. Let \( r = m/2 + \alpha \sqrt{m} \) and \( k = \beta \sqrt{m} \) such that \(|\alpha| + \beta \leq m^{1/12} \) and \( \beta \geq \max(\alpha, -\alpha/2) \). Then, the relation \( \ll \) is \((\delta, r, k)\)-expanding for

\[
\delta = \sqrt{2} \exp \left( -\frac{8}{9} (\beta - \alpha)(2\beta + \alpha) + \frac{C}{m^{1/4}} \right)
\]

for some universal constant \( C > 0 \).

Finally, let us say a few words about proving Lemma 22. An important intermediate step in its proof is a sufficient condition for \( A \ll B \):

Lemma 23. Let \( A = \{a_1, \ldots, a_{r+k}\}, B = \{b_1, \ldots, b_r\} \), with \( a_1 < \ldots < a_{r+k}, b_1 < \ldots < b_r \). Then,

\[
\forall 1 \leq i \leq r - k : a_i \leq b_{i+k} \implies A \ll B . \tag{15}
\]

With the benefit of Lemma 23, consider sampling a uniform set \( A \) of a given size \( r \) as a standard \( m \)-step \( \{\pm 1\} \) random walk conditioned on the endpoint \( S = 2r - m \). If we take two random sets \( A, B \) with \(|A| = r + k, |B| = k \), it can be shown that (15) is implied by the event that maximum deviations of respective random walks for both \( A \) and \( B \) were not too large. We can then compute exact tails of these deviations using standard symmetry arguments and conclude that, in fact, for a random choice of sets \( A \) and \( B \) relation \( A \ll B \) holds with probability \( 1 - \delta^2 \) in certain range of \( r \) and \( k \).

In terms of the bipartite graph between sets of size \( r + k \) and \( r \), it means that it is a full bipartite graph except for \( \delta^2 \) fraction of the edges. But we will show in Lemma 23 that such a graph must be \((\delta, r, k)\)-expanding, which implies Lemma 22. Essentially, the property of \((\delta, r, k)\)-expansion follows since the full bipartite graph where two vertex sets \( A \subseteq \binom{m}{r+k} \) and \( B \subseteq \binom{m}{r} \) of density \( \delta \) each are designated, and all edges between \( A \) and \( B \) deleted, is an extremal example: It minimizes our notion of expansion among the graphs with \( 1 - \delta^2 \) fraction of edges.

We note that a substantial improvement to our \( \ll \) relation could lead to a better result (in terms of larger rates or smaller \( \delta \)) via Lemma 19 or its variant. We also remark that while the rest of the paper concerns only binary symmetric channel, it can be checked that Lemma 19 holds for any binary memoryless symmetric channel with \((1 - h(p)) \) substituted by the respective channel capacity.

6 Proof of Theorem 16

Our proof can be divided into several parts, corresponding to the lemmas stated in Section 5. We start with Lemma 21 and then move on to Lemma 22. Finally, we prove Lemma 19 and put together the proof of Theorem 16.

6.1 Proof of Lemma 21

For a start, the fact that \( \ll \) is a partial order is easy to see from Definition 20. What remains is the following property:

Lemma 24. If \( B \) can be constructed from \( A \), then \( Z_A \geq Z_B \).

There are two observations underlying the proof of Lemma 24 both already used in [AY20] and earlier works. The first one utilizes the algebraic structure of the BSC to simplify the expression for \( Z_A \). Recall the Ber\( (p) \) random vector \( E = (E_z) \) and the Reed-Muller matrix \( M = (M_{A,z}) \):

Lemma 25. Let \( W' = (W'_A)_{A \subseteq \binom{m}{r}} \) be the random vector given by

\[
W' := EM^{-1} .
\]

Then, we have

\[
Z_A = Z(U_A \mid U_{<A}, Y) = Z(U_A \mid U_A + W'_A, (W'_B)_{B < A}) . \tag{16}
\]
Proof. Recalling (1) and (5) and repeatedly applying Facts 12 and 13,
\[
Z_A = Z(U \mid U_{<A}, Y) = Z(U \mid Y, U_{<A}, YM^{-1}) = Z(U \mid U_{<A}, U + EM^{-1})
\]
\[
= Z(U \mid U + W'_A, U_{<A}, (U_B + W'_B)_{<B}, U_B + W'_B)_{<B}
\]
\[
= Z(U \mid U + W'_A, U_{<A}, (U_B + W'_B)_{<B})
\]
\[
= Z(U \mid U + W'_A, W_B'_{<B}, U_{<A}) = Z(U \mid U + W'_A, W_B'_{<B}).
\]

The second observation is a formalization of a simple fact that relabeling random variables \( E_z \) does not change the right-hand side value in (10).

**Fact 26.** Let \( \tau : \{0,1\}^m \to \{0,1\}^m \) be a permutation and let \( P \) be the relevant permutation matrix given by
\[
P_{z,z'} = 1 \iff z = \tau(z').
\]
Furthermore, let \( v_1, \ldots, v_k, \tilde{v}_1, \ldots, \tilde{v}_{k'} \in \mathbb{F}_2^{\{0,1\}^m} \). Letting \( U := U_A \),
\[
W := \{ U + Ev_i^T : i = 1, \ldots, k \} \cup \{ Ev_i^T : i = 1, \ldots, k' \},
\]
\[
\tau W := \{ U + EPv_i^T : i = 1, \ldots, k \} \cup \{ EP\tilde{v}_i^T : i = 1, \ldots, k' \},
\]
we have
\[
Z(U \mid W) = Z(U \mid \tau W).
\]

Proof. Clear, since \( U \) is independent of \( E \) and random variables \( E_z \) are i.i.d. \( \square \)

Given \( A \subseteq [m] \), using Fact 10 we see that
\[
W_A' = \sum_z E_z \prod_{i \notin A} (1 - z_i).
\]

To simplify notation, let us define
\[
W_A := \sum_{z \in \{0,1\}^m} E_z \prod_{i \notin A} z_i, \quad W_{<A} := (W_B)_{<B}.
\]

In Section 8.2 we apply Fact 26 to the random vector \( W' \) to establish

**Corollary 27.** \( Z_A = Z(U_A \mid U_A + W_A, W_{<A}) \).

Note that \( U_A \) on the right-hand side in Corollary 27 is just a uniform bit independent of everything else, so we might just as well rename it \( U := U_A \). With Corollary 27 at hand, we use an elementary strategy to establish inequalities \( Z_A \geq Z_B \). We demonstrate this by showing that the information contained in \( U + W_A, W_{<A} \) is a “subset of” information contained in \( U + W_B, W_{<B} \) (technically, we show a type of channel degradation \( \text{MP18} \)). Informally, we look for permutations \( \tau \) of \( \{0,1\}^m \) such that
\[
\text{span} \left( U + \tau W_A, \tau W_{<A} \right) \subseteq \text{span} \left( U + W_B, W_{<B} \right).
\]
Facts 26 and 12 can then be used to conclude \( Z_A \geq Z_B \). It is worth noting (and keeping in mind) that a permutation of \( \{0,1\}^m \) can be thought of as an action permuting columns of the Reed–Muller matrix \( M \).

More precisely, we identify random vectors \( W_A \) with monomials, letting
\[
P_A \in \mathbb{F}_2[Z_1, \ldots, Z_m], \quad P_A(Z_1, \ldots, Z_m) := \prod_{i \notin A} Z_i,
\]
and their linear combinations with multilinear polynomials in \( \mathbb{F}_2[Z_1, \ldots, Z_m] \). Then, letting also
\[
P_{<A} := \{ P_B : B < A \}, \quad (P \circ \tau)(Z_1, \ldots, Z_m) := P(\tau(Z_1, \ldots, Z_m)),
\]
in Section 8.2 we prove (where \( P_{<A} \circ \tau = \{ P \circ \tau : P \in P_{<A} \} \))
Lemma 29. Let \( \tau \) be a permutation on \( \{0,1\}^m \) and consider \( A, B \subseteq [m] \). If

1. \( P_A \circ \tau \in P_B + \text{span}\{P_{<B}\} \); and
2. \( P_{<A} \circ \tau \subseteq \text{span}\{P_{<B}\} \)

both hold, then \( Z_A \geq Z_B \).

Lemma 28 provides a template for proving \( Z_A \geq Z_B \). In particular, Lemma 24 follows by induction from the two immediately following lemmas. More precisely, Lemma 29 covers Rule 1, and Lemma 30 together with Lemma 8 cover Rule 2. The lemmas are proved now by choosing appropriate \( \tau \).

6.2 Proofs of Lemmas 29 and 30

Proof of Lemma 29. Recall that \( \mathcal{A} = [m] \setminus A \) and note that we have

\[ a \notin \mathcal{A}, \quad b \in \mathcal{A}, \quad \mathcal{B} = \mathcal{A} \setminus \{b\} \cup \{a\} \].

Let \( \tau :\{0,1\}^m \rightarrow \{0,1\}^m \) be given as

\[
\tau(z_1,\ldots,z_m)_i := \begin{cases} 
  z_b & \text{if } i = a, \\
  z_a & \text{if } i = b, \\
  z_i & \text{otherwise}.
\end{cases}
\]  

(19)

Since clearly \( \tau \) is a permutation, to conclude that \( Z_A \geq Z_B \) we only need to check that the conditions from Lemma 28 apply. For a start, indeed we have

\[
P_A \circ \tau = \prod_{i \in \mathcal{A}} \tau(Z)_i = \prod_{i \in \mathcal{B}} Z_i = P_B \in P_B + \text{span}\{P_{<B}\}. \]

As for the second condition, we start by observing that \( A < B \) in the decoding order. Let us take \( P_C \in P_{<A} \) and proceed by case analysis:

- If \( a, b \in C \) or \( a, b \notin C \), then we have
  
  \[
P_C \circ \tau = P_C \in P_{<A} \subseteq P_{<B} \subseteq \text{span}\{P_{<B}\}. \]

- If \( a \in C \) and \( b \notin C \), then

  \[
P_C \circ \tau = \prod_{i \in \mathcal{C}} \tau(Z)_i = \prod_{i \in \mathcal{C} \setminus \{a\} \cup \{b\}} Z_i = P_{C \setminus \{a\} \cup \{b\}} \]

  and, since \( C < A, a \in A, b \notin C \), by Fact 13 we get \( C \setminus \{a\} \cup \{b\} < B \) and \( P_{C \setminus \{a\} \cup \{b\}} \in P_{<B} \).

- Similarly, if \( a \notin C \) and \( b \in C \), then \( P_C \circ \tau = P_{C \setminus \{b\} \cup \{a\}} \). But now it is enough to observe that \( C \setminus \{b\} \cup \{a\} < C < A < B \) and therefore \( P_{C \setminus \{b\} \cup \{a\}} \in P_{<A} \subseteq P_{<B} \).

\[\square\]

\footnote{Note that in case \( B = A \setminus \{a\}, a \in A \), the inequality \( Z_A \geq Z_B \) follows directly from Lemmas 29 and 30 in most cases: One can use Rule 2 to delete from \( A \) its maximum \( a' \) and \( a \) and insert minimum element \( b' \) not in \( A \), and follow up applying Rule 1 to replace back \( b' \) with \( a' \). This \textit{almost always} works, but we defer to Lemma 8 to avoid cumbersome special cases later on.}
Lemma 33. Let \( A \subseteq [m] \) be such that \( A \cap [i] \) contains significantly more elements than the number expected based just on the size of \( A \) (we will always use this definition for \( |A| \approx m/2 \)). The result we prove in this section is:

**Lemma 32.** Let \( A, B \) be sets with \( |A| = r + k \), \( |B| = r \) and \( d_1, d_2 \geq 0 \) be such that \( d_1 + d_2 \leq k \). If \( \overline{A} \) is \( d_1 \)-good and \( B \) is \( d_2 \)-good, then \( A \ll B \).

In order to prove Lemma 32, we start with an alternative characterization of the \( \ll \) relation, divided into two cases \( |A| = |B| \) and \( |A| > |B| \).

**Lemma 33.**

1. Let \( |A| = |B| \) with \( A = \{a_1, \ldots, a_r\}, B = \{b_1, \ldots, b_r\} \), \( a_1 < a_2 < \ldots < a_r, b_1 < \ldots < b_r \). Then, \( A \ll B \) if and only if \( a_i \leq b_i \) for every \( i \).

2. Given \( A, |A| \geq 2 \), let \( a < a' \) denote the two largest elements of \( A \) and \( b \) the smallest element of \( \overline{A} \cup \{a, a'\} \). Accordingly, let
\[
\overline{A} := A \setminus \{a, a'\} \cup \{b\}
\] (21)

and \( \overline{A}^{(k)} \) to be the result of \( k \) consecutive applications to set \( A \) of the operation defined in (21). Let \( A \) be such that \( |A| = |B| + k \), \( |B| \geq 1 \). Then, \( A \ll B \) if and only if \( \overline{A}^{(k)} \ll B \).
Lemma 33 is proved in Section 8.3 by elementary case analysis. For us its most important consequence is a sufficient condition for $A \ll B$ that we already pointed out in Section 5

Lemma 23. Let $A = \{a_1, \ldots, a_{r+k}\}$, $B = \{b_1, \ldots, b_r\}$, with $a_1 < \ldots < a_{r+k}$, $b_1 < \ldots < b_r$. Then,

$$\forall 1 \leq i \leq r - k : a_i \leq b_{i+k} \implies A \ll B.$$  

(15)

Proof. First, if $k \geq r$, then $B$ can be constructed from $A$ by applying Rule 2 only. Therefore, assume $k < r$ and consider $A^{(k)}$. By Lemma 33.2, we only need to establish that $B$ can be constructed from $A^{(k)}$. We do this by checking the condition from Lemma 33.1.

If $A^{(k)} = \{1, \ldots, r\}$, then clearly $A^{(k)} \ll B$ and we are done. Otherwise, we can write $A^{(k)} = \{\tilde{a}_1, \ldots, \tilde{a}_k, a_1, \ldots, a_{r-k}\}$, where $\tilde{a}_1, \ldots, \tilde{a}_k$ are $k$ elements added in the applications of Rule 2. Let $a' := \min(|m| \setminus A^{(k)})$ be the smallest element not in $A^{(k)}$ and let $c_i$ be the $i$-th smallest element of $A^{(k)}$. We consider two cases. First, if $c_i < a'$, then clearly $c_i = i \leq b_i$. Second, if $a' < c_i$, then note that also $\max\{\tilde{a}_1, \ldots, \tilde{a}_k\} < c_i$ (in particular $i > k$) and therefore $c_i = a_{i-k}$ and $a_{i-k} \leq b_i$ holds by assumption.

The definition of a good set can be linked to Lemma 23 by

Fact 34. Let $A = \{a_1, \ldots, a_r\}, a_1 < \ldots < a_r$. If $A$ is $d$-good, then for all $i \in [r],$

$$a_i \geq 2i - 2d.$$  

(22)

Similarly, if $A$ is $d$-good, then $a_i \leq 2i + 2d.$

Proof. To prove (22), assume otherwise, i.e., that there exists $i$ such that $a_i < 2i - 2d$ and consequently $a_i \leq 2i - 2d - 1$. This is equivalent to saying that (note that we can assume wlog that $2i - 2d > 1$)

$$|A \cap \{1, \ldots, 2i - 2d - 1\}| \geq i,$$

however since $A$ is $d$-good, we also have

$$|A \cap \{1, \ldots, 2i - 2d - 1\}| \leq \frac{2i - 2d - 1}{2} + d < i,$$

a contradiction.

As for the second statement, the proof is similar: Assuming wlog $2i + 2d < m$, if there exists $i$ with $a_i > 2i + 2d$, then

$$|A \cap \{1, \ldots, 2i + 2d\}| \leq i - 1,$$

but since $A$ is $d$-good, we have $|A \cap \{1, \ldots, i\}| \geq i/2 - d$ for every $i$, in particular

$$|A \cap \{1, \ldots, 2i + 2d\}| \geq \frac{2i + 2d}{2} - d = i,$$

another contradiction.

Now we are ready to complete the proof of Lemma 32.

Proof of Lemma 32. Assume that $A$ is $d_1$-good and $B$ is $d_2$-good. Then, by Fact 34 $a_i \leq 2i + 2d_1$ and $b_i \geq 2i - 2d_2$. In particular, for $1 \leq i \leq r - k,$

$$a_i \leq 2i + 2d_1 \leq 2(i + k) - 2d_2 \leq b_{i+k},$$

and, by Lemma 23 $A \ll B.$
6.4 Proof of Lemma \textbf{22}

As we indicated, our strategy to obtain Lemma \textbf{22} is to establish that, for \( r = m/2 + \alpha \sqrt{m} \) and \( k = \beta \sqrt{m} \) for \( \beta > 0 \) large enough compared to \( |\alpha| \), for almost every pair of sets \((A, B)\) with \( A \) of size \( r + k \) and \( B \) of size \( r \), \( B \) can be constructed from \( A \). The precise form of this statement is:

**Lemma 35.** Let \( r = m/2 + \alpha \sqrt{m} \) and \( k = \beta \sqrt{m} \) such that \( |\alpha| + \beta \leq m^{1/2} \) and \( \beta \geq \max(\alpha, -\alpha/2) \). Consider a random choice of two independent uniform sets \((A, B)\) conditioned on \( |A| = r + k \) and \( |B| = r \). Then, we have

\[
\Pr[|A - B| \leq 2 \exp \left( -\frac{16}{9} (\beta - \alpha)(2\beta + \alpha) + \frac{C}{m^{1/4}} \right)]
\]

for some universal \( C > 0 \).

Lemma \textbf{35} is proved by showing that typical sets of sizes \( r + k \) and \( r \) are likely to be \( d \)-good for appropriate \( d \), and therefore susceptible to applying Lemma \textbf{32}. In order to do that, we need a variant of a classic tail bound on the maximum of a simple integer random walk (for some background on this technique see, e.g., Chapter III in \cite{Fei83}):

**Lemma 36.** Let \( X_1, X_2, \ldots, X_m \) be a uniform i.i.d. sequence with \( X_i \in \{-1, 1\} \). Letting \( S_i := \sum_{j=1}^{i} X_j \), \( M := \max_{0 \leq i \leq m} S_i \) we have, for any \( s, d, \alpha \geq 0 \) such that \( d \geq \max(s, 0) \), \( r := (m + s)/2 \in \mathbb{Z} \) and \( 0 \leq r \leq m \),

\[
\Pr[M \geq d \mid S_m = s] = \frac{\binom{m-d}{m}}{\binom{m}{r}}.
\]

For example, for \( m = 2k \) and \( s = 0 \) we get \( \Pr[M \geq d \mid S_m = 0] = \frac{2^k}{\binom{k}{k-d}} \).

**Proof.** Consider a realization of the sequence \( x = (x_1, \ldots, x_m) \in \{-1, 1\}^m \) such that \( M(x) \geq d \) and \( S_m(x) = s \). Let \( T \) be the largest index such that \( S_T = d \) and take \( y = (y_1, \ldots, y_m) \) to be the mirror image of \( x \) after time \( T \), i.e.,

\[
y_i := \begin{cases} x_i & \text{if } i \leq T, \\ -x_i & \text{if } i > T. \end{cases}
\]

Note that this operation creates a bijection between sequences \( x \) such that \( M \geq d \) and \( S_m = s \) and sequences such that \( S_m = 2d - s \). Indeed, sequence \( y \) has \( S_m(y) = d - (s - d) = 2d - s \). On the other hand, since \( s \geq d \), we have \( 2d - s \geq d \) and therefore any walk that ends with \( S_m = 2d - s \) must achieve \( S_T = d \) at some point. But knowing that it is easy to find the inverse of \( y \), establishing the bijection.

Since a priori every walk has the same probability \( 2^{-m} \), we have

\[
\Pr[M \geq d \mid S_m = s] = \frac{\Pr[M \geq d \land S_m = s]}{\Pr[S_m = s]} = \frac{\Pr[S_m = 2d - s]}{\Pr[S_m = s]} = \frac{\binom{m-d}{m}}{\binom{m}{r}}.
\]

We need to connect Lemma \textbf{36} to the notion of good sets. This is done by

**Corollary 37.** Let \( A \) be a random uniform set of size \( r \) and \( d \in \mathbb{N} \) s.t. \( 2d + 1 \geq s := 2r - m \). Then,

\[
\Pr[A \not \text{ d-good}] = \frac{\binom{m-d-1}{r}}{\binom{m}{r}}.
\]

**Proof.** Define random variables \( X_1, \ldots, X_m \) as

\[
X_i = X_i(A) := \begin{cases} 1 & \text{if } i \in A, \\ -1 & \text{if } i \notin A, \end{cases} \quad S_i = S_i(A) := \sum_{j=1}^{i} X_j, \quad M = M(A) := \max_{0 \leq i \leq m} S_i.
\]
Recalling the setting of Lemma\textsuperscript{36} note that $X_1, \ldots, X_m$ are distributed as an i.i.d.
uniform \{-1, 1\}\textsuperscript{m} sequence conditioned on $S_m = 2r - m = s$. Furthermore, we have
\[ S_i = 2|A \cap [i]| - i , \]
and therefore
\[ A \text{ is } d\text{-good} \iff \forall 1 \leq i \leq m : S_i \leq 2d \iff M(A) \leq 2d . \]
Therefore, we can apply Lemma\textsuperscript{36} and get
\[ \Pr[A \text{ not } d\text{-good}] = \Pr[M \geq 2d + 1 | S_m = s] = \frac{\left(\binom{m}{r}^{-1}\right)}{\binom{m}{r} - \delta^2} . \]

The proof of Lemma\textsuperscript{35} chooses appropriate $d_1, d_2 = \Theta(\sqrt{m})$ and uses Lemma\textsuperscript{32} and union bound to show that
\[ \Pr[\neg(A \ll B)] \leq \Pr[A \not\text{ not } d_1\text{-good}] + \Pr[B \not\text{ not } d_2\text{-good}] \]
is small by approximating the expression in (25). The details are provided in Section\textsuperscript{8.4}.

The final ingredient of the proof of Lemma\textsuperscript{22} consists of connecting Lemma\textsuperscript{35} with our notion of expansion:

**Lemma 38.** Consider two independent random sets $(A,B)$ such that $|A| = r + k$ and $|B| = r$ and assume that we have $\Pr[A \ll B] \geq 1 - \delta^2$. Then, the order $\ll$ is $(\delta, r, k)$-expanding.

**Proof.** Let $B \subseteq \binom{m}{r}$ such that $|B| \geq \delta \binom{m}{r}$ and let $A := \{A \in \binom{m}{r+k} : \exists B \in B : A \ll B\}$. By definition, $A \ll B$ and $B \in B$ implies $A \in A$. Hence,
\[ \Pr[A \in A] \Pr[B \in B] = \Pr[A \in A \land B \in B] \geq \Pr[A \ll B \land B \in B] \]
\[ \geq \Pr[B \in B] - \Pr[\neg(A \ll B)] \geq \Pr[B \in B] - \delta^2 . \]
Consequently,
\[ \frac{|A|}{\binom{m}{r+k}} = \Pr[A \in A] \geq 1 - \frac{\delta^2}{\Pr[B \in B]} \geq 1 - \delta . \]

**Proof of Lemma 22** Immediately from Lemma\textsuperscript{35} and Lemma\textsuperscript{38}.

### 6.5 Proof of Lemma\textsuperscript{19}

Before we prove Lemma\textsuperscript{19} we need a simple fact stating that in expectation the density of sets with high $Z_A$ increases with the size $r$.

**Fact 39.**

1. Let $B := \{B \in \binom{m}{r} : Z_B \geq \varepsilon\}$ and let $B_r := B \cap \binom{m}{r}$. Then, $|B| \geq \delta \binom{m}{r}$ implies $|B_r| \geq \delta \binom{m}{r}$.

2. Similarly, let $A := \{A \in \binom{m}{r} : Z_A \geq \varepsilon\}$ and $A_r := A \cap \binom{m}{r}$. Then, $|A_r| \geq \delta \binom{m}{r}$ implies $|A| \geq \delta \binom{m}{r}$.

**Proof.** Let $0 \leq k < r$ and let $A, B$ be random subsets of $[m]$ chosen such that $B$ is uniform among sets of size $k$ and $A = B \cup \{a\}$, where $a$ is a uniform element not in $B$. Note that the marginal distribution of $A$ is uniform over sets of size $k + 1$. Furthermore, by Lemma\textsuperscript{3} in this random experiment we always have $Z_A \geq Z_B$. Therefore, $B \in B_k$ implies $A \in B_{k+1}$. Letting $B_k := \binom{m}{k}$, we have
\[ \frac{|B_k|}{\binom{m}{k}} = \Pr[B \in B_k] \leq \Pr[A \in B_{k+1}] = \frac{|B_{k+1}|}{\binom{m}{k+1}} , \]
which clearly implies
\[ \frac{|B_r|}{\binom{m}{r}} \geq \frac{|B|}{\binom{m}{r}} , \]
establishing the first point. The proof of the second point is symmetrical.
Proof of Lemma 19. Let

\[ B := \left\{ B \in \left( \frac{m}{r} \right) : Z_B \geq 1/n^{c+1} \right\}, \quad A := \left\{ A \in \left( \frac{m}{(r+k)} \right) : Z_A \geq 1/n^{c+1} \right\}. \]

Our objective is to show that, for \( m \) large enough, \( |B| < \delta^\left( \frac{m}{r} \right) \), since then by Theorem 9 we obtain that code RM\((m, r, \delta)\) has successive decoding error probability at most \( \left( \frac{m}{r} \right) \cdot n^{-(c+1)} \leq n^{-c} \).

Assume otherwise, i.e., \( |B| \geq \delta^\left( \frac{m}{r} \right) \). By Fact 39, we have \( B \cap \left( \frac{m}{r} \right) \geq \delta^\left( \frac{m}{r} \right) \). Since \( \ll \) is both information-consistent and \((\delta, r, k)\)-expanding, also \( A \cap \left( \frac{m}{r+k} \right) \geq (1-\delta)(\frac{m}{r+k}) \), and, applying the other part of Fact 39,

\[ |A| \geq (1-\delta)\left( \frac{m}{r+k} \right). \]

Recall 14 and choose \( \varepsilon > 0 \) such that

\[ \lim_{m \to \infty} \left( \frac{m}{r+k} \right) / n > \frac{h(p)}{(1-\varepsilon)(1-\delta)} . \]

Applying Theorem 4 for this \( \varepsilon \) and \( \xi = 1/4 \), we have, for \( m \) large enough,

\[ \left| \left\{ A \in \left( \frac{m}{r+k} \right) : H_A \geq 1-\varepsilon \right\} \right| \geq (1-\delta)\left( \frac{m}{r+k} \right) - \frac{n}{m^{1/4}} \]

\[ > \frac{h(p)}{(1-\varepsilon)} \cdot n . \]

But that implies \( \sum_{A \subseteq [m]} H_A > h(p) \cdot n \), which is in contradiction with Fact 11. \( \square \)

6.6 Proof of Theorem 16

Fix \( p \) and \( \delta \). Recall from the statement of the theorem how the values of \( \alpha, \gamma, R_0 \) and \( R \) are determined by \( p \) and \( \delta \). We choose \( r \) to be the smallest number such that the rate of the Reed–Muller code RM\((m, r)\) exceeds \( R_0 \). Recalling Definition 15, the rate of the code RM\((m, r, \delta)\) exceeds \( R_0(1-\delta) = R \), so the actual work lies in showing the successive decoding error bound.

To that end, consider the constructible \( \ll \) order from Definition 20. By Lemma 21, it is information-consistent. By Fact 14, we know that \( r = \frac{m}{2} + \alpha\sqrt{m} + o(\sqrt{m}) \). Furthermore, note that since we assumed \( 1-h(p)-2\delta \leq 1/2 \), we have \( \gamma \leq 0 \) and \( \alpha < 2\gamma \leq \gamma \) and accordingly we can let \( \beta := \gamma - \alpha > 0 \) and \( k := \beta \sqrt{m} = \beta \sqrt{m} + O(1) \).

What is more, \( \alpha < 2\gamma < 0 \) implies \( \beta > -\alpha/2 > 0 > \alpha \). Hence, letting \( \alpha' := (r-m/2)/\sqrt{m} \) and \( \beta' := k/\sqrt{m} \), we have \( \alpha' = \alpha + o(1) \) and \( \beta' = \beta + O(1/\sqrt{m}) \) and

\[ |\alpha'| + \beta' \leq m^{1/12} , \quad \beta' \geq \max(\alpha', -\alpha'/2) , \]

so that we can apply Lemma 22 and obtain that, for \( m \) large enough, the order \( \ll \) is \((\delta', r, k)\)-expanding for

\[ \delta' = \sqrt{2} \exp \left( -\frac{8}{9} (\beta' - \alpha')(2\beta' + \alpha') + \frac{C}{m^{1/4}} \right) = \sqrt{2} \exp \left( -\frac{8}{9} (\beta - \alpha)(2\beta + \alpha) \right) + o(1) \]

\[ = \sqrt{2} \exp \left( -\frac{8}{9} (\gamma - 2\alpha)(2\gamma - \alpha) \right) + o(1) , \]

and, since

\[ \frac{1}{2} (\gamma - 2\alpha)(2\gamma - \alpha) = \left( \frac{5}{4} \gamma - \alpha \right)^2 - \frac{9}{16} \gamma^2 = \left( \frac{9}{32} \ln(2/\delta^2) - \frac{3}{4} \gamma \right)^2 - \frac{9}{16} \gamma^2 > \frac{9}{32} \ln(2/\delta^2) , \]
In particular, could we improve upon Lemma 22 and get a constant rate \( R \) ?

One might wonder how tight is our analysis of the expansion properties of the constructible \( \ll \) order. Therefore, for large \( m \) we have that the order \( \ll \) is \((\delta, r, k)\)-expanding. Finally, we check that

\[
\liminf_{m \to \infty} \frac{\left(\sum_{\geq (r+k)} \frac{m}{n}\right)}{n} = \Phi(-2\gamma) = 1 - \Phi(2\gamma) = h(p) + 2\delta > \frac{h(p)}{1 - \delta},
\]

hence Lemma \[ applies and successive decoding of code \( \text{RM}(m, r, \delta) \) fails with probability at most \( n^{-c} \).

## 7 Lower Bound

One might wonder how tight is our analysis of the expansion properties of the constructible \( \ll \) order. In particular, could we improve upon Lemma 22 and get a constant rate \( R \) for codes \( \text{RM}(m, r, \delta) \) with \( \delta(m) = o(1) \)? In this section, we answer this question in the negative. We exhibit an assignment of “possible entropies” to sets \( A \to H(A) \) that respects the condition \( A \ll B \implies H(A) \geq H(B) \), satisfies \( \sum A H(A) = h(p)n \) and contains a \( \delta > 0 \) fraction of sets with \( |A| \leq r \) and \( H(A) = 1 \). If the actual entropies \( H_A \) behave similarly, then \( \delta \) fraction of components would have to be removed from \( \text{RM}(m, r) \) to ensure that the successive decoding corrects random errors.

Therefore, in order to make progress on the capacity conjecture using polarization theory, either more \( H_A \geq H_B \) inequalities need to be proved or new ingredients introduced to our approach. We now state our lower bound:

**Theorem 40.** Given \( 0 < R < 1 \), let \( r = r(R, m) \) be the smallest \( r \) such that \( m \geq Rn \). For every \( 0 < R, \varepsilon < 1 \), there exists \( \delta = \delta(\varepsilon, R) > 0 \), \( m_0 = m_0(\varepsilon, R) \) and a family of functions \( H = H^{(m)} : \mathcal{P}(m) \to [0, 1] \) such that for \( m > m_0 \):

1. \( A \ll B \) implies \( H(A) \geq H(B) \).
2. \( |\{ A \subseteq [m] : H(A) \notin \{0, 1\}\}| \leq 1 \).
3. \( \sum_{A \subseteq [m]} H(A) = \varepsilon n \).
4. \( \left|\left\{ A \in \binom{[m]}{\leq r} : H(A) = 1\right\}\right| \geq \delta \left(\binom{m}{\leq r}\right) \).

Intuitively, the parameter \( R \) corresponds to the rate of the code and \( \varepsilon \) to the noise entropy (i.e., 1 – capacity) of a binary input channel. Theorem 10 asserts that it is possible to “assign entropies” \( H_A \) such that: 1) They are consistent with the \( \ll \) relation. 2) In light of polarization (Theorem 7), almost all of them are zero or one. 3) The sum of entropies is \( \varepsilon n \) as required by Fact 11. 4) Yet, for any rate \( R \) Reed–Muller code \( \text{RM}(m, r) \), a \( \delta(\varepsilon, R) > 0 \) fraction of sets \( A \) of size at most \( r \) has \( H(A) = 1 \).

However, \( H_A \approx 1 \) implies that the bit corresponding to set \( A \) cannot be decoded under successive decoding. Consequently, if the entropy values are given by function \( H(\cdot) \), at least \( \delta \) fraction of basis codewords has to be deleted from \( \text{RM}(m, r) \) to make successive decoding work. Therefore, at least for the purposes of successive decoding, more constraints on function \( H(\cdot) \) are needed.

The main conceptual ingredient we need to prove Theorem 40 is the following easy lemma. Recall that for \( A \subseteq [m] \) we have \( S_i(A) = 2|A \cap [i]| - i \):

**Lemma 41.** Let \( \ell := \lfloor m/2 \rfloor \), \( 0 \leq r < m \), and \( k \in \mathbb{Z} \) and let

\[
\mathcal{B}_{m, r, k} := \left\{ B \in \binom{m}{r} : S_\ell(B) > k \right\},
\]

\[
\mathcal{A}_{m, r, k} := \left\{ A \in \binom{m}{r+1} : \exists B \in \mathcal{B}_{m, r, k} \text{ s.t. } A \ll B \right\}.
\]

Then, \( \mathcal{B}_{m, r+1, k} \subseteq \mathcal{A}_{m, r, k} \subseteq \mathcal{B}_{m, r+1, k-2} \), where the left containment holds if \( \ell + k < 2r \).
We will use this lemma for \( k \leq O(\sqrt{m}) \) and \( r \geq \ell - O(\sqrt{m}) \), so the condition \( \ell + k < 2r \) will be easily satisfied.

**Proof.** Fix \( m, r, k \) and let \( A := A_{m, r, k} \). For the first containment, let \( A \in B_{m, r+1, k} \), i.e., \( S_\ell(A) > k \) and let \( B := A \setminus \{a\} \), where \( a \) is the maximum element of \( A \). Since \( A \supseteq B \), we have \( A \ll B \). At the same time, \( S_\ell(B) > k \) (note that if \( a \leq \ell \), then \( \ell + k < 2r \) implies \( S_\ell(A) = 2(r+1) - \ell + k > 2 \)), hence \( B \in B_{m, r, k} \) and \( A \in A \).

For the second containment, let \( A \in A \), meaning \( A \ll B \) for some \( B \in B_{m, r, k} \). Recall from Lemma 33.1 that \( \tilde{A} = A \setminus \{a, a'\} \cup \{b\} \), where \( a, a' \) are two largest elements in \( A \) and \( b \) is the smallest element not in \( A \setminus \{a, a'\} \) and that \( A \ll \tilde{A} \ll B \).

Since clearly \( |A \cap \ell| \leq |A \cap [\ell]| + 1 \), we have
\[
S_\ell(A) = 2|A \cap \ell| - \ell \geq 2|\tilde{A} \cap \ell| - \ell - 2 = S_\ell(\tilde{A}) - 2.
\]

But recalling Lemma 33.1, it is also clear that \( S_\ell(\tilde{A}) \geq S_\ell(B) \) and hence
\[
S_\ell(A) \geq S_\ell(\tilde{A}) - 2 \geq S_\ell(B) - 2 > k - 2,
\]
hence \( A \in B_{m, r+1, k-2} \), as claimed.

Given \( R \) and \( \varepsilon \), Theorem 40 is proved by setting
\[
B_0 := B_{m, r, \gamma \sqrt{m}, \gamma' \sqrt{m}}
\]
for appropriately chosen \( \gamma \) and \( \gamma' \). Then, we set \( H(A) = 1 \) if and only if there exists \( B \in B_0 \) such that \( A \ll B \). Lemma 41 together with the CLT approximation from Fact 14 allow us to control both \( \sum_{A \subseteq [m]} H(A) \) and \( \sum_{A \in \binom{m}{\leq p}} H(A) \). Details are given in Section 8.5.

### 8 Remaining Proofs

#### 8.1 Approximation from (9)

We obtain the estimate stated in (9). We start with
\[
\frac{1}{C \max(x, 1)} \cdot \exp \left( -\frac{x^2}{2} \right) \leq \Phi(-x) \leq C \cdot \exp \left( -\frac{x^2}{2} \right),
\]
for every \( x \geq 0 \) and some universal \( C \geq 1 \). This is a standard Gaussian estimate that can be proved, e.g., by integration by parts. In particular, substituting in the right part \( y := C \cdot \exp(-x^2/2) \) and solving \( x = \sqrt{2\ln(C/y)} \) we get
\[
\Phi^{-1}(y) \geq -\sqrt{2\ln C} \over y
\]
for every \( 0 \leq y \leq 1 \). Now we can calculate, letting \( h := h(p) \),
\[
R = (1 - \delta) \cdot \Phi \left( 4\gamma - \sqrt{\frac{9}{8} \ln \frac{2}{\delta^2}} \right) = (1 - \delta) \cdot \Phi \left( 2\Phi^{-1}(1 - h - 2\delta) - \frac{9}{8} \ln \frac{2}{\delta^2} \right)
\]
\[
\geq (1 - \delta) \cdot \Phi \left( -2\sqrt{\frac{2\ln C}{1 - h - 2\delta}} - \frac{9}{8} \ln \frac{2}{\delta^2} \right)
\]
\[
= (1 - \delta) \cdot \Phi \left( -\sqrt{\frac{9}{8} \ln \frac{2}{\delta^2} + O(1)} \right) = \delta^{9/8 + o(1)},
\]
where the \( o(1) \) function goes to 0 as \( \delta \) goes to 0 for fixed \( p \).
8.2 Algebraic column permutations

In this section we provide two proofs omitted from Section 6.1: Corollary 27 and Lemma 28.

**Proof of Corollary 27.** By Lemma 25, 
\[ Z_A = Z(U_A | U_A + W'_A, (W'_B)_{B < A}) \]
Define a permutation \( \tau \) of \( \{0, 1\}^m \) as 
\[ \tau(z_1, \ldots, z_m) := (1 - z_1, \ldots, 1 - z_m) \]
and let \( P \) be the respective permutation matrix given by (17). Note that matrix \( P \) maps the random vector \( E = (E_z)_{z \in \{0, 1\}^m} \) to \( EP = (E_{\tau(z)})_{z \in \{0, 1\}^m} \). Therefore, using Fact 10, for any set \( B \)
\[ (EPM^{-1})_B = \sum_z E_{\tau(z)} \prod_{i \notin B} (1 - z_i) = \sum_z E_{\tau(z)} \prod_{i \notin B} \tau(z)_i = \sum_z E_z \prod_{i \notin B} z_i = W_B . \]
Applying Fact 26 for \( \tau \) and \( \tilde{v}_1^T, \tilde{v}_1^T, \ldots, \tilde{v}_k^T \) given as columns of \( M^{-1} \):
\[ \tilde{v}_1^T := (M^{-1})_{(.,A)} \]
\[ (\tilde{v}_1^T, \ldots, \tilde{v}_k^T) := (M^{-1})_{(.,B)} \]
we obtain
\[ Z(U_A | U_A + W'_A, (W'_B)_{B < A}) = Z(U_A | U_A + W_A, (W_B)_{B < A}) , \]
as claimed.

As for the proof of Lemma 28, we need some preparation first. Consider the \( n \)-dimensional vector space
\[ \mathcal{E} := \text{span}\{E_z\}_{z \in \{0, 1\}^m} ; \]
whose elements are random variables that can be created as linear combinations of \( E_z \) over \( \mathbb{F}_2 \). Similarly, let \( \mathcal{E}_U := \text{span}\{\mathcal{E}, U\} \) be the space of dimension \( n + 1 \) spanned by \( E_z \) and an additional independent uniform random variable \( U \). Since for a collection of random variables \( W \subseteq \mathcal{E}_U \) their values are uniquely determined by any subcollection that spans \( W \), the fact below is an immediate consequence of Fact 12:

**Fact 42.** Let \( W \subseteq \mathcal{E}_U \). Then, \( Z(U | W) = Z(U | \text{span}\{W\}) \).

Since their dimensions are equal, of course the space \( \mathcal{E} \) is isomorphic to the (multilinear) polynomial space \( \mathbb{F}_2[Z_1, \ldots, Z_m] \). Furthermore, it is easy to check that \( \{W_A : A \subseteq [m]\} \) is a basis of \( \mathcal{E} \) which lets us define a natural isomorphism \( \phi \) by
\[ \phi W_A := P_A . \]  
Furthermore, let \( \tau \) be a permutation of \( \{0, 1\}^m \). Permutation \( \tau \) naturally induces a linear operator on \( \mathcal{E} \), which we can give as
\[ \tau W := \phi^{-1}((\phi W) \circ \tau) , \]
in other words the linear operator \( \tau \) satisfies the relation \( \phi \tau W = (\phi W) \circ \tau \). The operator \( \tau \) can be extended to \( \mathcal{E}_U \) by adding
\[ \tau U := U . \]
Note that we have:

**Fact 43.** \( \tau E_z = E_{\tau^{-1}(z)} \).
Proof. First, we use (27) to verify that
\[(\phi E_z)(z') = 1 \iff z' = z .\]
But from this it follows that
\[(\phi \tau E_z)(z') = ((\phi E_z) \circ \tau)(z') = (\phi E_z)(\tau(z')) = 1 \iff \tau(z') = z \iff z' = \tau^{-1}(z) ,\]
and consequently \(\tau E_z = E_{\tau^{-1}(z)} .\)

Using Fact 43 we can prove a reformulation of Fact 26:

**Fact 44.** Let \(W \subseteq E\) and let \(\tau W := \{\tau W : W \in W\} .\) Then,

\[Z(U \mid W) = Z(U \mid \tau W) .\]

**Proof.** In our notation the conclusion looks identical to (18) in Fact 26 but we need to make sure the meanings of \(W\) and \(\tau W\) are the same in both settings.

Indeed, take \(W \in W \subseteq E\). Then, it must be
\[W = U + Ev^T \vee W = Ev^T\]
for some \(v \in \mathbb{F}_2^{(0,1)^m}\). On the other hand, taking matrix \(P\) from (17) for permutation \(\tau^{-1}\) and noting that
\[EP = (E_{\tau^{-1}(z)})_{z \in \{0,1\}^m} ,\]
from Fact 43 it follows that if \(W = U + Ev^T\), then
\[\tau W = U + \sum zE_{\tau^{-1}(z)} = U + EPv^T ,\]
and similarly for \(W = Ev^T\). Therefore, \(Z(U \mid W) = Z(U \mid \tau W)\) holds by an application of Fact 26 for permutation \(\tau^{-1} .\)

Finally, we are ready to prove Lemma 28:

**Proof of Lemma 28** Using (27) and (28), we can rewrite the assumptions:
\[P_A \circ \tau \in P_B + \text{span}\{P_{<B}\} \implies \tau W_A \in W_B + \text{span}\{W_{<B}\} ,\]
\[P_{<A} \circ \tau \subseteq \text{span}\{P_{<B}\} \implies \tau W_{<A} \subseteq \text{span}\{W_{<B}\} .\]
We also note that \(\tau W_A \in W_B + \text{span}\{W_{<B}\}\) implies
\[U + \tau W_A \in (U + W_B) + \text{span}\{W_{<B}\} .\]
But now, applying Corollary 27 (twice), Fact 42 (also twice), Fact 44 and Fact 12.1,
\[Z_A = Z(U \mid U + W_A, W_{<A})
\geq Z(U \mid \text{span}\{U + W_A, W_{<A}\})
\geq Z(U \mid \text{span}\{U + \tau W_A, \tau W_{<A}\})
= Z(U \mid \text{span}\{U + W_B, W_{<B}\})
= Z(U \mid U + W_B, W_{<B}) = Z_B .\]
8.3 Proof of Lemma 33

1. Assume first that \( a_i \leq b_i \) for every \( i \). Let \( A' := A \setminus (A \cap B) \), \( B' := B \setminus (A \cap B) \) and \( A' = \{a_1', \ldots, a_k'\}, B' = \{b_1', \ldots, b_k'\} \) with \( a_1' < \ldots < a_k' \) and \( b_1' < \ldots < b_k' \). It is not difficult to see that we have \( a_i' < b_i' \) for every \( i \). Therefore, \( B \) can be constructed from \( A \) by \( k \) applications of Rule 1.

On the other hand, assume that \( B \) can be constructed from \( A \). Clearly, \( B \) must have been obtained from \( A \) only by applications of Rule 1. Let us say that there have been \( k \) applications. If in the \( i \)-th application an element \( c \) was removed and \( d \) was inserted, we will say that \( c \) was mapped to \( d \).

By an easy inductive argument, whenever \( B \) can be constructed from \( A \) using a sequence of mappings \( (c_1, d_1, \ldots, c_k, d_k) \), it can also be constructed using another sequence \( (c_1', d_1', \ldots, c_k', d_k') \) such that no element is used more than once, i.e., \( \{|c_1', d_1', \ldots, c_k', d_k'|\} = 2k' \). Let us assume this property from now on.

This means that we can write \( A' = A \setminus (A \cap B) \), and \( B' \setminus (A \cap B) \) such that \( A' = \{c_1, \ldots, c_k\}, B' = \{d_1, \ldots, d_k\} \) and \( c_i < d_i \). But this implies that also after sorting \( A' = \{c_1' < \ldots < c_k'\}, B' = \{d_1' < \ldots < d_k'\} \) we have \( c_i' < d_i' \), which in turn implies \( a_i \leq b_i \) for every \( i \).

2. Clearly, \( \tilde{A}^{(k)} \) is constructed from \( A \) by \( k \) applications of Rule 2, so if \( B \) can be constructed from \( \tilde{A}^{(k)} \), then \( B \) can be constructed from \( A \).

In the other direction, first note that whenever \( B \) can be constructed from \( A \), there always exists a sequence of rule applications transforming \( A \) to \( B \) such that all applications of Rule 2 occur before any applications of Rule 1. We will now argue that if \( B \) can be constructed from \( A \) and the first applied rule is Rule 2, then \( B \) can be constructed from \( A \). That \( B \) can be constructed from \( A^{(k)} \) follows by induction.

To this end, assume that in this first Rule 2 application set \( A' := A \setminus \{c_1, c_2\} \cup \{c'\} \) was obtained from \( A \) with \( c_1 < c_2 \). Note that by definition, \( c_1 \leq a, c_2 \leq a' \) and \( b \leq c' \). Therefore \( A' \) (and therefore also \( B \)) can be constructed from \( A \) by applying Rule 1 three times: Mapping \( c_1 \) to \( a \), \( c_2 \) to \( a' \) (or possibly \( c_1 \) to \( a' \) if \( c_2 = a \)) and \( b \) to \( c' \). Special cases when \( b = a \) or \( c' = c_1 \) are handled in a similar way. \( \square \)

8.4 Proof of Lemma 35

Before we proceed to the main proof, let us make an approximation of \((25)\) for \( r = m/2 + \Theta(\sqrt{m}) \):

**Corollary 45.** Let \( A \) be a uniform random set of size \( r = m/2 + \alpha \sqrt{m} \) and \( d = \gamma \sqrt{m} \) such that \( |\alpha|, \gamma \leq m^{1/12} \). Then,

\[
\Pr[A \text{ not } d\text{-good}] \leq \exp \left( 8\alpha \gamma - 8\gamma^2 + \frac{C}{m^{1/2}} \right) \quad (29)
\]

for some universal constant \( C > 0 \).

**Proof.** First, we can assume wlog that \( \gamma > \alpha \): If \( \alpha < 0 \), then it holds since \( \gamma \geq 0 \), and if \( \alpha \geq 0 \), then, since \( 8\alpha \gamma - 8\gamma^2 = -8\gamma(\gamma - \alpha) \geq 0 \) for \( 0 \leq \gamma \leq \alpha \), the right-hand side of \((29)\) exceeds 1. Additionally, we can assume wlog that \( m \) is large enough (say, \( m \geq 100 \)), since for \( m < 100 \) the constant \( C \) in \((29)\) can be chosen large enough for the inequality to hold.

Since \( \gamma > \alpha \), we check that \( 2d + 1 > 2\sqrt{m} > 2\alpha \sqrt{m} = 2r - m \) and apply Corollary 37:

\[
\Pr[A \text{ not } d\text{-good}] = \frac{r - m}{m} = \frac{r!}{m} \frac{m!}{(r - 2d - 1)!} \frac{(m - r + 2d + 1)!}{(m - r + 1)!} = \prod_{i=0}^{2d} \frac{r - i}{m - r + i + 1} = \prod_{i=0}^{2\sqrt{m}} \frac{m/2 + \alpha \sqrt{m} - i}{m/2 - \alpha \sqrt{m} + i + 1} = \prod_{i=0}^{2\sqrt{m}} \left( 1 + 2\alpha/\sqrt{m} - 2i/m \right) \left( 1 - 2\alpha/\sqrt{m} + 2(i + 1)/m \right). \quad (30)
\]

We proceed to bound each term on the right-hand side of \((30)\). In the numerator we use \( 1 + 2\alpha/\sqrt{m} - 2i/m \leq \exp \left( 2\alpha/\sqrt{m} - 2i/m \right) \). In the denominator we first invoke \( m \geq 100 \) to observe that \( 1 - 2\alpha/\sqrt{m} + 2(i + 1)/m >
1 - 2/m^{5/12} > 1/2. Then, we apply the inequality 1 + x ≥ \exp(x - x^2), which holds for every x ≥ -1/2 to bound each of the terms 1 - 2α/√m + 2(i + 1)/m.

Plugging these inequalities into (30) results in

\[
\Pr[A \text{ not } d\text{-good}] \leq \exp \left( \sum_{i=0}^{\gamma \sqrt{m}/m} \frac{4\alpha - 2i + 2}{m} + \left( \frac{2\alpha}{\sqrt{m}} - \frac{2i}{m} \right)^2 \right)
\leq \exp \left( \sum_{i=0}^{\gamma \sqrt{m}/m} \frac{4\alpha - 2i + 2}{m} + O \left( \frac{\alpha^2}{m} + \frac{\gamma^2}{m} + \frac{1}{m^2} \right) \right)
\leq \exp \left( \sum_{i=0}^{\gamma \sqrt{m}/m} \frac{4\alpha - 2i + 2}{m} + O \left( \frac{1}{m^{5/6}} \right) \right) \leq \exp \left( 8\alpha \gamma - 8\gamma^2 + O \left( \frac{1}{m^{1/4}} \right) \right).
\]

Proof of Lemma 35 Let \( d_i := \lceil \gamma_i \sqrt{m} \rceil \) for

\[
\gamma_1 := \frac{\beta - \alpha}{3}, \quad \gamma_2 := \frac{2\beta + \alpha}{3}.
\]

Since \( d_1 + d_2 \leq (\gamma_1 + \gamma_2)\sqrt{m} = k \), by Lemma 32 and union bound we have

\[
\Pr[\neg(A \ll B)] \leq \Pr[A \text{ not } d_1\text{-good}] + \Pr[B \text{ not } d_2\text{-good}] = \exp \left( \frac{16}{9} (\beta - \alpha)(2\beta + \alpha) + \frac{C}{m^{1/4}} \right),
\]

and the result follows.

8.5 Proof of Theorem 40

In order to prove Theorem 40, we start with giving its reformulation which does not mention the \( H(\cdot) \) function and instead speaks only about collections of sets. Given 0 < \( R < 1 \), throughout we fix \( r = r(R, m) \) to be the smallest \( r \) such that \( m \geq Rn \).

**Theorem 46.** For every 0 < \( R, \varepsilon < 1 \), there exist \( \delta > 0 \) and \( m_0 \) such that for every \( m > m_0 \) there exists a collection of sets \( B \subseteq \binom{m}{\leq r} \) such that:

1. \( |B| \geq \delta \left( \frac{m}{2r} \right) \).
2. Letting \( A := \{ A \subseteq [m] : \exists B \in B \text{ s.t. } A \ll B \} \), we have \( |A| \leq \varepsilon n \).

We first prove that Theorem 46 implies Theorem 40 and then prove Theorem 46

Theorem 46 implies Theorem 40. Let 0 < \( R, \varepsilon < 1 \) and take \( \delta, m_0, B \) and \( A \) given by Theorem 46.

We define function \( H \) by induction, adding sets to a collection \( A' \) and maintaining an invariant \( A \in A' \) implies \( H(A) \neq 0 \). We start with setting \( A' := A \) and assigning \( H(A) := 1 \) for all \( A \in A \). Then, as long as \( \sum_{A \in A'} H(A) < \varepsilon n \), we:

- Pick a set \( A \) which is a minimal element of the partial order \( \ll \) restricted to \( \mathcal{P}(m) \setminus A \).
- Assign \( H(A) := \min \{ 1, \varepsilon n - \sum_{B \in A'} H(B) \} \).
• Add $A$ to $\mathcal{A}'$.

Since $\varepsilon n < n$, the algorithm described above terminates. After that happens, we assign $H(A) := 0$ to all remaining sets.

Let us verify the four conditions from Theorem 40 in turn. First, we need to show that $A \ll B$ implies $H(A) \geq H(B)$. We show it by induction, proving that this implication holds at all stages of building $\mathcal{A}'$ from $\mathcal{A}$. At the beginning we have $\mathcal{A}' = \mathcal{A}$, satisfying $H(A) = 1$ for $A \in \mathcal{A}$ and $H(A) = 0$ otherwise. If $A \ll B$ and $H(B) = 0$, then $H(A) \geq H(B)$ obviously holds. On the other hand if $A \ll B$ and $H(B) = 1$, then $B \in \mathcal{A}$ and from definition of $\mathcal{A}$ also $A \in \mathcal{A}$ and $H(A) = 1$, $H(B) \geq H(A)$.

Consider now a step where set $A$ is added to $\mathcal{A}'$ and the value $H(A)$ increases. By induction, we only need to check inequalities involving $A$. If $A \ll B$, then $H(A) \geq H(B)$ holds by induction, since $H(A)$ only increased. If $B \ll A$, then by the fact that $A$ is minimal when added to $\mathcal{A}'$ we have $B \in \mathcal{A}'$ and $H(B) \geq H(A)$.

The second condition that there is at most one set with $H(A) \notin \{0, 1\}$ follows by construction, and similarly with the third condition $\sum A H(A) = \varepsilon n$. Finally, the fourth condition $|\{A : H(A) = 1 \land |A| \leq r\}| \geq \delta \left(\frac{m}{\sqrt{r}}\right)$ follows since $|\mathcal{E}| \geq \delta \left(\frac{m}{\sqrt{r}}\right)$ and $\mathcal{E} \subseteq \mathcal{A}$ with $H(A) = 1$ for all $A \in \mathcal{A}$. 

**Proof of Theorem 40.** Let $0 < R, \varepsilon < 1$. Throughout the proof we assume that $m > m_0(R, \varepsilon)$ so that everything is well-defined. By Fact 14 we have

$$r = \frac{m}{2} + \alpha \sqrt{m} + o(\sqrt{m}) \quad \text{for} \quad \alpha := \frac{\Phi^{-1}(R, \varepsilon)}{2}.$$ 

Let $k := \lfloor 4\sqrt{m} \rfloor$ and recall the notation from Lemma 41: $S_i(A) = 2|A \cap [i]| - i$, $\ell = \lfloor m/2 \rfloor$ and

$$\mathcal{B}_{m, r, s} := \left\{ B \in \binom{m}{r} : S_i(B) > s \right\}.$$ 

Furthermore, choose some $\gamma', \gamma > 0$, such that

$$\Phi(-2\alpha + 8 - 2\gamma') \leq \frac{\varepsilon}{3}, \quad \Phi\left(\sqrt{2}(-\gamma + 2\gamma')\right) \leq \frac{\varepsilon}{3},$$

and let

$$s := \lfloor \gamma \sqrt{m} \rfloor, \quad \mathcal{B}_0 := \mathcal{B}_{m, r - k, s}, \quad \mathcal{B} := \left\{ A \in \binom{m}{\leq r} : \exists B \in \mathcal{B}_0 \text{ s.t. } A \ll B \right\}.$$ 

We are going to argue that $\mathcal{B} \subseteq \binom{m}{\leq r}$ is the collection of sets satisfying the two conditions of the theorem. To that end, we divide the rest of the proof into two claims.

**Claim 47.** There exists $\delta = \delta(R, \varepsilon) > 0$ such that $|\mathcal{E}| \geq \delta \left(\frac{m}{\sqrt{r}}\right)$.

**Proof.** Observe that if a set $B \subseteq [m]$ satisfies the conditions

$$\frac{\ell}{2} + \frac{s}{2} < |B \cap [\ell]| < \frac{\ell}{2} + \frac{s}{2} + \sqrt{m},$$

$$\frac{m - \ell}{2} - \frac{s}{2} + \alpha \sqrt{m} - 3\sqrt{m} < |B \cap \{\ell + 1, \ldots, m\}| < \frac{m - \ell}{2} - \frac{s}{2} + \alpha \sqrt{m} - 2\sqrt{m},$$

then, on the one hand, we have

$$S_i(B) = 2|B \cap [\ell]| - \ell > s,$$

and on the other hand

$$r - k < \frac{m}{2} + \alpha \sqrt{m} - 3\sqrt{m} < |B| < \frac{m}{2} + \alpha \sqrt{m} - \sqrt{m} < r,$$
therefore \( B \in \mathcal{B}_{m,r-k',s} \) for some \( 0 \leq k' \leq k \). Since, by multiple applications of the left containment in Lemma 41, we have

\[
\bigcup_{k'=0}^{k} \mathcal{B}_{m,r-k',s} \subseteq \mathcal{B} ,
\]

it also holds that \( B \in \mathcal{B} \). Therefore, letting \( C \) to be the collection of all sets satisfying (33) and (34), we can use Fact 14 to estimate

\[
|B| \geq |C| = \left\lfloor \frac{m^2 - \ell^2}{2} \right\rfloor - \left( \frac{m^2 - \ell^2}{2} - \alpha \sqrt{m} - 2 \sqrt{m} \right) .
\]

\[\sum_{i=0}^{m-r+k} \mathcal{B}_{m,r-k+i,s-2i} \]

Claim 48. Let \( A := \{ A \subseteq [m] : \exists B \in \mathcal{B} \text{ s.t. } A \ll B \} \). Then, \( |A| \leq \varepsilon n \).

Proof. By definition of \( \mathcal{B} \), we have

\[
A = \{ A \subseteq [m] : \exists B \in \mathcal{B}_0 \text{ s.t. } A \ll B \} .
\]

Observe that \( A \cap \mathcal{B}_{r-k} = \mathcal{B}_0 = \mathcal{B}_{m,r-k,s} \). Hence, by the right containment in Lemma 41, for every \( i \geq 0 \),

\[
A \cap \left( \mathcal{B}_{r-k+i,s-2i} \right) \subseteq \mathcal{B}_{m,r-k+i,s-2i}
\]

and consequently

\[
|A| \leq \sum_{i=0}^{m-r+k} |B_{m,r-k+i,s-2i}| .
\]

Letting \( i_0 := \lceil \gamma \sqrt{m} \rceil \), the last sum can be estimated as

\[
\sum_{i=0}^{m-r+k} |B_{m,r-k+i,s-2i}| \leq \left| \{ A : S_\ell(A) > s - 2i_0 \} \right| + \left| \{ A : |A| > r - k + i_0 \} \right| .
\]

To analyze the two terms above, we use Fact 14 for the last time:

\[
\left| \{ A : S_\ell(A) > s - 2i_0 \} \right| = \left( \frac{\ell}{2} + \frac{s}{2} - i_0 \right) 2^{m-\ell} = \Phi(\sqrt{2}(\gamma + 2)) \cdot n + o(n) ,
\]

\[
\left| \{ A : |A| > r - k + i_0 \} \right| = \left( \frac{m}{r - k + i_0} \right) = \Phi(-2\alpha + 8 - 2\gamma') \cdot n + o(n) .
\]

Recalling (32), we conclude

\[
|A| \leq \left( \Phi(\sqrt{2}(\gamma + 2)) + \Phi(-2\alpha + 8 - 2\gamma') \right) \cdot n + o(n) < \varepsilon n .
\]

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