Vortices in attractive Bose-Einstein condensates in two dimensions

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The form and stability of quantum vortices in Bose-Einstein condensates with attractive interactions is elucidated. They appear as ring bright solitons, and are a generalization of the Townes soliton to nonzero winding number \( m \). An infinite sequence of radially excited stationary states appear for each value of \( m \), which are characterized by concentric matter-wave rings separated by nodes, in contrast to repulsive condensates, where no such set of states exists. It is shown that robustly stable as well as unstable regimes may be achieved in confined geometries, thereby suggesting that vortices and their radial excited states can be observed in experiments on attractive condensates in two dimensions.

PACS numbers:

The study of vortices has a long and illustrious scientific history reaching back to Helmholtz and Lord Kelvin in the nineteenth century. Vortices associated with quantized circulation are a central feature of superfluidity. Singly- and multiply-quantized vortices have been observed in Bose-Einstein condensates (BECs) with repulsive atomic interactions. Complex vortex structures have been shown to be stable in repulsive BEC’s, including vortex dipoles and vortex rings. The nonlinear Schrödinger equation (NLSE), which provides an excellent description of BECs at the mean-field level, supports vortex solutions, which have been studied extensively in the case of repulsive atomic interactions. The main goal of this Letter is to clarify the meaning and nature of single vortices and their excited states in BECs with attractive interactions, and thus encourage experimental exploration of stable and unstable two dimensional (2D) BECs.

The critical dimensionality for the NLSE is 2D. We will show that quantum vortices and their radially excited states can be made robustly stable in confined, attractive 2D condensates and are a generalization of the Townes soliton to nonzero winding number \( m \). The Townes soliton is fundamental to understanding the self-similar collapse of solutions to the 2D NLSE. Its generalization to winding number \( |m| \geq 1 \) has been studied in the context of optics, where such solutions are called “ring-profile solitary waves” or “spinning bright solitons.” This is in fact the attractive analog of the well-known single vortex solution in repulsive condensates, as we will show. An example contrasting the form of vortices in attractive and repulsive BECs is illustrated in Fig. 1a and 1b.

In previous studies of quantum vortices in condensed matter and optical systems, with attractive nonlinearity, the phase variation derived wholly from circulation of matter about the central vortex core. In this work, we investigate the most general type of single-vortex stationary solutions in attractive BECs for which the phase of the order parameter also alternates sign along radial lines. For each winding number \( m \), we find a denumerably infinite set of radially excited states characterized by the successive formation of nodes at \( r = \infty \). An example of such an excited state is illustrated in Fig. 1c. In contrast, Fig. 1d shows how extremely different is the repulsive case.

Consider an order parameter of the form

\[
\psi(r, \phi, t) = f_m(r) \exp(i m \phi) \exp(-i \mu t / \hbar) \exp(i \theta_0), \tag{1}
\]

which solves the 2D NLSE

\[
[-(i \hbar / 2M) \nabla_r^2 + g_{2D}] |\psi|^2 + V(r)] \psi = i \hbar \partial_t \psi, \tag{2}
\]

with \( V(r) \) a central potential in two dimensions, \( m \) the winding number, \( M \) the atomic mass, \( g_{2D} \equiv (4 \pi \hbar^2 a_s / M) \sqrt{M} \omega_z / 2 \hbar \) the 2D atomic interaction strength, \( a_s < 0 \) the s-wave scattering length, and \( \mu \) the chemical potential or eigenvalue. In Eq. 2, it was assumed that the BEC remains in the ground state in a harmonic trap of angular frequency \( \omega_z \) in the \( z \) direction, so that \( r = \sqrt{x^2 + y^2} \). Then, taking \( \theta_0 = 0 \) and defining \( \eta_m(\chi) \equiv \sqrt{|g_{2D}| / \mu} f_m(r) \), \( \chi \equiv (\sqrt{2M} \mu / \hbar) r \), one obtains a rescaled 2D NLSE of form

\[
\eta''_m + \chi^{-1} \eta'_m + m^2 \chi^{-2} \eta_m + \eta^{-2}_m - V(\chi) \eta_m + \sigma_m \eta_m = 0, \tag{3}
\]

where \( \sigma_m = \text{sgn}(\mu) = \pm 1 \). The solutions to this nonlinear second order ordinary differential equation describe quantum vortices and their radially excited states in an...
the nonlinearity regulates the divergence as unity, with the oscillations damping away as on the order of unity. It subsequently oscillates around $V_a$ for very small $a$ as a function of $K_m$. The case of $\sigma_m = -1$ and $V = 0$ are shown in Fig. 1(a) and 1(c).

The form of the radial wavefunction $\eta_m$ can be obtained from Eq. (3) by numerical shooting methods. This requires the initial conditions $\eta_m(\chi_0), \eta'_m(\chi_0)$, which can be obtained via a power series around $\chi = 0$:

$$\eta_m(\chi) = \sum_{j=0}^{\infty} a_j \chi^{2j+m},$$  

where the $a_j$ are coefficients. Note that $\eta_m(\chi) \to a_0 \chi^m$ as $\chi \to 0$. Upon substitution into Eq. (3) and solution of the resulting simultaneous equations, one finds that all coefficients $a_j$ for $j \neq 0$ can be expressed as a polynomial in powers of $a_0$. Thus the power series shows that $a_0$, the coefficient of the first nonzero term in the series, together with the winding number $m$, is sufficient to determine the entire solution. Note that we take $a_0 \geq 0$ and $m \geq 0$; solutions identical in form exist for $a_0 < 0$ and/or $m < 0$.

We first consider the case of no external potential, $V(\chi) = 0$. By following the form of the wavefunction as a function of $a_0$, one observes its entire development. The case of $m = 1$ and $\sigma_m = -1$ is illustrated in Fig. 2(a). For very small $a_0$, the linear, divergent Bessel function solution $K_m(\chi)$ is recovered for small $\eta_m(\chi)$. However, the nonlinearity regulates the divergence as $\eta_m$ becomes on the order of unity. It subsequently oscillates around unity, with the oscillations damping away as $\chi \to \infty$ (Fig. 2(a)). As $a_0$ approaches a critical value $a_0^{\text{vortex}}$, the oscillations are pushed out towards $\chi = \infty$ and a localized central peak appears in the wavefunction near the origin (Fig. 2(b)). For $a_0 = a_0^{\text{vortex}}$, precisely [23], a node forms at $\chi = \infty$ and a quantum vortex is obtained. As $a_0$ is increased, the node moves inwards and oscillations resume beyond it (Fig. 2(a)). To move the oscillations towards infinity again, one must increase $a_0$ towards a second critical value (Fig. 2(d)). When this value is reached precisely, one obtains the first excited state, with no oscillations past the first node, and a second node appears at $\chi = \infty$. In this way one can construct an infinite set of excited states by increasing $a_0$ to successive critical points. These critical points are always characterized by the formation of an additional node at $\chi = \infty$.

The appearance of a denumerably infinite set of critical points is in strong contrast to the case of a vortex in a repulsive condensate. As we have shown in previous work [23], for a repulsive BEC there is only one critical value of $a_0$, i.e., $a_0^{\text{vortex}}$. Larger values of $a_0$ result in an infinite number of nodes and the wavefunction asymptotically resembles the Coulomb function, as illustrated in Fig. 1(d). We called these ring solutions, in contrast to vortex solutions, as they have a different analytic structure and asymptotic behavior. Thus, in an infinitely extended system, vortices in repulsive condensates cannot be radially excited in a stationary way, whereas in attractive condensates they have a denumerably infinite set of excited states. We note that, for positive chemical potential, i.e., $\sigma_m = +1$, one can also find ring solutions in attractive condensates.

For winding number $m = 0$ and $a_0 = 2.20620086 \ldots$, one obtains the Townes soliton. Increasing $a_0$ results in
the formation of successive nodes at $\chi = \infty$, just as for
$m = 1$, i.e., one finds the radially excited states of the
Townes soliton. In this special case, it has been formally
proven that an infinite set of radially excited states corre-
spending to the formation of nodes exists, and that the
Townes soliton is the unique “ground state” in that se-
quence \cite{11}. It is in this sense that the vortex solutions we
have described are generalizations of the Townes soliton
to nonzero winding number. All such solutions wherein
a node has formed at infinity are normalizable.

In order to study attractive BECs in experiments, it is
vital to consider stability properties in confined systems.
The special stability properties of 2D can be illustrated
by a simple variational study. Consider the variational
ansatz

$$
\psi(r, \phi, t) = A r^m e^{-r^2/2\alpha^2} e^{i m \phi} e^{-i \mu t/\hbar},
$$

(5)

where $r_0$ and $A$ are variational parameters, subject to
the power law potential $V(r) = V_0 r^j$, $j > 0$. Integrating
Eq. (2) over $\psi^*(r, \phi, t)$, one finds simultaneous equations
for the chemical potential and $q_{2D}$. Then, using the nor-
malization $\int d^2r |\psi|^2 = N$ to eliminate $A$, where $N$ is the
total number of atoms, one obtains $\mu(N)$ parametrically
in $r_0$:

$$
\mu = \frac{-\hbar^2}{2Mr_0^2} \Gamma(m+2) + \frac{V_0(1+j) \Gamma(m+\frac{j}{2}+1)}{\Gamma(m+2)} r_0^j,
$$

(6)

$$
\mathcal{N} = \frac{\Gamma(m+2)}{\pi^{-\frac{1}{2}} \Gamma(m+\frac{1}{2})} \frac{M\omega_0 r_0^j}{\pi^{-\frac{1}{2}} \hbar^2 \Gamma(m+\frac{1}{2})},
$$

(7)

where $\mathcal{N} \equiv MN|q_{2D}|/2\pi \hbar^2 = 2|a_s|NM \omega_z/\sqrt{2\pi \hbar}$ is the
dimensionless nonlinearity. The solution is radially sta-
bile when the Vakhitov-Kolokolokov (VK) criterion $2\pi$
$d\mu/d\mathcal{N} \leq 0$ is satisfied. One finds that the VK crite-
ron always holds for $\mathcal{N} < \mathcal{N}_c$, where

$$
\mathcal{N}_c = 2\sqrt{\pi} \Gamma(m+2)/\Gamma(m+1/2).
$$

(8)

Equation (8) is independent of both $V_0$ and $j$, i.e.,
radial stability does not depend on the details of any
positive power law potential. In the limit in which $j \to \infty$, one obtains the same result for a cylin-
deral hard-wall potential. When $\mathcal{N} \geq \mathcal{N}_c$, the total energy $E[\psi] = \mathcal{N}[\mu - (q_{2D}/2) \int d^2r |\psi|^2] \to -\infty$ as $r_0 \to 0$,
meaning that the wavefunction is unstable and implodes.
When $\mathcal{N} < \mathcal{N}_c$, the energy has a global minimum at some
finite $r_0$. In contrast, in 3D, the minimum, when it exists,
is always local, so that, at best, one obtains metastabi-

We now consider in detail the most experimentally rele-
vant case, i.e., $j = 2$ and $V_0 = M\omega^2/2$, a harmonic
trap. Note that typical trapping frequencies are on the
order of $\omega = 2\pi \times 100$ Hertz, which gives a time scale of
$T \equiv 2\pi/\omega = 10$ ms. For no external potential, $V(r) = 0$,
it is well known that the Townes soliton ($m = 0$) is radi-
ally unstable, while the quantum vortex of winding num-
ber $m = 1$ (Fig. 1(a)) is azimuthally unstable. The varia-
tional analysis of Eqs. (8)-9 suggests that the addition of a
confining potential stabilizes the solutions radially for
general $m$. However, this simple variational study
did not allow for azimuthal stability. Both radial and
azimuthal stability can be determined by linear stability
analysis, i.e., the Boguliubov-de Gennes equations
(BDGE) \cite{10}, a standard method which we do not re-
produce here, for the sake of brevity. The case of gen-
eral $m$ without radial excitations has been studied previ-
ously \cite{22, 23}. It was found that for sufficiently small
$\mathcal{N}$ the vortex with $m = 1$ is stable, while for $m \geq 2$ the
solution is unstable to quadrupole oscillations, though
instability times can be much greater than $T$. We can
therefore say that for small $\mathcal{N}$ vortices of winding num-
ber $m \geq 2$ are experimentally stable.

Applying the BDGE’s, we studied the azimuthal and
radial stability of the radially excited states of the Townes
soliton. The results for the first excited state, which has
a matter-wave ring separated from a soliton core by a
radial node, are illustrated in Fig. 1(a)-(b). The wind-
ing number of each Boguliubov mode is denoted by $q$.
The frequency of each mode is denoted by $\Omega_q$ with in-
stability time $t_q \equiv 2\pi/\text{Im}(\Omega_q)$. From Fig. 1(b), it is
apparent that a radial instability ($q = 0$) occurs for all
$\mathcal{N}$, since $\text{Im}(\Omega_0) \neq 0$. However, one finds experimen-
tal stability for small $\mathcal{N}$, since $t_q/T \gg 1$. Other modes
become unstable at higher $\mathcal{N}$, starting with the dipole,
$q = 1$. For large $\mathcal{N}$ the dominant instability occurs in the
quadrupole mode.

FIG. 3: (color online) Stable regimes of quantum vortices in
confined attractive BECs. Shown are the chemical potential
spectra and linear instability frequencies as a function of the
nonlinearity for (a)-(b) the first excited state of the Townes
soliton, and (c)-(d) the first excited state of a singly quant-
ized vortex $m = 1$, all in a harmonic potential. In (b),(d) the
winding number of the instability mode is denoted by $q$ and
the instability time given by $2\pi/\text{Im}(\Omega_q)$. Note that all axes
are scaled to harmonic oscillator units.
In Fig. 3(c)-(d) is shown the same stability analysis for the first excited state of the \( m = 1 \) vortex, which has two concentric matter-wave rings separated by a radial node. Unlike the excited states of the Townes soliton, here the solution is linearly stable for \(<5.5\). The solutions continue to be experimentally stable up to \( \sim10 \), although a dipole instability occurs for \( t_1 \gg T \). For large \( N \) the dominant instability is radial. For the radially excited states of both the Townes soliton and the \( m = 1 \) vortex, the VK criterion fails, as is apparent in Figs. 3(a) and 3(c). This is first instance we know of where this criterion, which so far as we know has never been proven formally, fails. For winding number \( m = 2 \) we find that the first excited state is linearly most unstable to quadrupolar excitations, although for \( N \lesssim 10 \) it is experimentally stable.

In experiments, one may access the stable regime of quantum vortices and their excited states via a Feshbach resonance, which allows for extremely small scattering lengths on the order of the Bohr radius; this technique was used to successfully create bright solitons \[16, 17\].

Vortices may be created by rotating the condensate; alternatively, one may start with a single vortex in a repulsive condensate and adiabatically switch the scattering length from positive to negative. Excited states may be made by phase imprinting through a pinhole mask, by passing a tightly focused laser pulse through the condensate center, or by utilizing a doughnut mode of a laser. It seems likely that the most useful approach will use these methods to first create dark ring solitons \[31\] on a repulsive condensate with a central vortex \[24\], and then tuning the scattering length to be negative. An important point in the assumption of the 2D regime is that, in order for our stability criterion to be valid, the condensate must remain oblate, so that excitations in \( z \) do not occur; this may be achieved by a correct choice of experimental parameters.

In conclusion, we have shown that quantum vortices and their radially excited states in attractive BECs can be created stably in confined systems. We contrasted vortices in attractive BECs, which can be thought of as ring bright solitons or spinning Townes solitons, to their counterparts in repulsive BECs. We showed that there exists a denumerably infinite set of excited states which, in an infinitely extended condensate, correspond to the creation of nodes at \( r = \infty \). In a harmonic trap, these are stable or experimentally stable for sufficiently small nonlinearity. In contrast to 3D, there is no metastability. We note that, in an optics context, vortices, or “spinning bright solitons,” can also be stabilized by competing nonlinearities, i.e., a defocusing quintic nonlinearity with a focusing cubic nonlinearity \[32\].

We thank Joachim Brand and William Reinhardt for useful discussions. LDC thanks the NSF for support. The work of CWC was partially supported by the Office of Naval Research.

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