Book Embeddings of Infinite Sequences of Extended Periodic Regular Graphs

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Abstract. A graph is extended periodic if it can be made up by gluing identical pieces together with a graph which is isomorphic to or subgraph of that piece, in a circular fashion. We consider an infinite sequence $H_n$ of extended periodic 4-regular graphs and an infinite sequence $M_n$ of extended periodic 5-regular graphs. A book is a collection of half-planes called pages, all having a common boundary called the spine. A book embedding of a graph $G$ is to put the vertices of $G$ on the spine and each edge of $G$ is contained in one page. The minimum pages needed to embed a graph $G$ into a book is called pagenumber of $G$. We determine the pagenumbers of $H_n$ and $M_n$ to be three.

Keywords: book embedding, pagenumber, extended periodic graph.

1. Introduction

A graph is an ordered pair $(N, E)$ where $N$ is a non-empty set of vertices and $E$ is a set of edges. We consider only simple graphs, i.e. without parallel edges and loops. A graph can be drawn in many ways on the plane. A good drawing has the following properties: no edge crosses itself, no pair of adjacent edges cross, two edges cross at most once, and no more than two edges cross at one point. The crossing number $cr(G)$ of a graph $G$ is the minimum number of crossings among all good drawings of $G$ in the plane.

Mohar and Dvořák [5] defined that a graph is periodic if it can be obtained by joining identical pieces in a circular fashion. These small pieces are identified as tiles by Pinontoan and Richter [6]. We define a graph to be extended periodic if it can be made up by joining identical tiles $T$ together with a tile $T'$ which is isomorphic to or subgraph of $T$, in a circular fashion. Pinontoan and Richter [6] found out that graphs made up by tiles this way, in particular for planar tiles $T$, then the crossing number of the graph can just be determined by the number of pair-disjoint paths in tile $T'$. Richter dan Thomasen [7] introduced an infinite sequence $H_n (n \geq 3)$ of 4-regular graphs that can be arranged this way. Pinontoan and Richter [6] also studied an infinite sequence $M_n (n \geq 3)$ of 5-regular graphs which also can be arranged this way.

A book consists of half-planes called pages, all having common boundary called the spine. A book embedding of a graph $G$ is an embedding of $G$ to a book such that the vertices of $G$ lie on the spine and each edge of $G$ lie on a single page. We consider the combinatorial embedding, i.e, no edge crosses the spine. The pagenumber $\rho(G)$ of graph $G$ is the minimal number of pages needed to embed a graph $G$ to a book. Figure 1(a) shows the complete graph $K_5$ which is already known having crossing number one and Figure 1(b) shows a book embedding of $K_5$ which needs three pages; this is also the minimal number of pages, and hence $\rho(K_5) = 3$. 
In this paper, we determine the pagenumbers of the infinite sequence \(H_n(n \geq 3)\) of extended periodic 4-regular graphs and the infinite sequence \(M_n(n \geq 3)\) of extended periodic 5-regular graphs.

2. Infinite Sequences of Extended Periodic 4-regular and 5-regular Graphs

In this section we recall the notion of tile, the infinite sequence \(H_n(n \geq 3)\) of extended periodic 4-regular graphs, and the infinite sequence \(M_n(n \geq 3)\) of extended periodic 5-regular graphs.

2.1. Tiles

The concept of tile was introduced by Pinontoan and Richter [6]. A tile \(T = (G, L, R)\) is a graph connected \(G\) with specified two finite sequences of distinct vertices of \(G\), called the left-wall \(L\) and right-wall \(R\). The tile crossing number \(tcr(T)\) of \(T = (G, L, R)\) is the minimum number of crossings among all drawings of \(G\) in the unit square \([0, 1] \times [0, 1]\) on the plane such that the vertices of \(L\) appear on \([0] \times [0, 1]\) in the order of decreasing \(y\)-coordinate and the vertices of \(R\) appear on \([1] \times [0, 1]\) also in the order of decreasing \(y\)-coordinate. A tile \(T\) is planar if \(tcr(T) = 0\). The tile \(H = ((\{a, b, c, d, e\} \{ac, ad, cb, be, ce, cd\}), (a, b), (d, e))\) in Figure 2(a) is planar whereas the tile \(H' = ((\{a, b, c, d, e\} \{ac, ad, cb, be, ce, cd\}), (a, b), (e, d))\) in Figure 2(b) is not planar.

Tile \(T_1 = (G_1, L_1, R_1)\) is compatible with tile \(T_2 = (G_2, L_2, R_2)\) if \(|R_1| = |L_2|\). If \(T_1 = (G_1, L_1, R_1)\) is compatible with \(T_2 = (G_2, L_2, R_2)\), then \(R_1\) and \(R_2\) can be glued together, i.e. by identifying them, to get a bigger tile \(T_1T_2 = (G_1 \cup G_2, L_1, R_2)\). Figure 2(c) shows tile \(HH'\) as result of gluing tile \(H\) in Figure 2(a) and tile \(H'\) in Figure 2(b), where the right wall-of \(H\) is identified with the left-wall of \(H'\), that is \(d = a\) and \(e = b\), then rename the inner vertices.

A tile is self-compatible if it is compatible to itself. If \(T\) is a self-compatible tile, then \(T'\) is obtained by gluing \(n\) copies of \(T\) in a linear fashion and \(o(T')\) is obtained by gluing the first and the last copies of \(T\) in \(T'\) together, to get a circular fashion of arrangement. Note that \(T'\) is a tile and \(o(T')\) is a graph. The twist \(T'\) of \(T = (G, L, R)\) is the tile obtained by reversing the order of \(R\). The tile \(H'\) is Figure 2(b) is the...
twist of the tile \( H \) in Figure 2(a). If \( T \) and \( S \) are compatible tiles, and \( T \) is a self-compatible tile, then we define \( T^\circ(n) = o(T' T) \) where \( T' \) is the twist of \( T \), and \( T(n, S) = o(T' S) \).

2.2. Infinite sequence \( H_n (n \geq 3) \) of extended periodic 4-regular graphs

Richter dan Thomassen [7] introduced the infinite sequence \( H_n (n \geq 3) \) of 4-regular graphs which were obtained from a cycle \( C_{2m} \) of length \( 2m \) by adding, and for each two diametrically opposite edges \( e \) and \( e' \) on \( C_{2m} \), a new vertex joined to the four ends of the edges \( e \) and \( e' \). The vertices of \( H_n \) are \( a_1, a_2, a_3, ..., a_{n+1}, b_1, b_2, b_3, ..., b_{n+1}, c_1, c_2, c_3, ..., \) and \( c_{n+1} \). The edges of \( H_1 \) are \( a_1 c_k, b_k c_k, 1 \leq k \leq n+1 \), \( a_k a_{k+1}, b_k b_{k+1}, c_k d_k, b_{k+1}c_{k+1} \) (where \( 1 \leq k \leq n \)), \( a_1 b_{n+1}, a_1 c_{n+1}, b_1 d_{n+1}, \) and \( b_1 c_{n+1} \).

Pinontoan and Richter [6] identified \( H_n \) as a graph made up by gluing \( n \) planar tiles \( H \) in Figure 2(a) and a single tile \( H' \) in Figure 2(b) which is the twist of \( H \) and isomorphic to \( H \), in circular fashion, that is, \( H_n = H^\circ(n) = o(H'H') \), and rename the vertices with appropriate names. Figure 3 shows \( B_5 \).

![Figure 3. Graph \( H_5 \).](image)

2.3. Infinite sequence \( M_n (n \geq 3) \) of extended periodic 5-regular graphs

Let \( n \geq 3 \). The graph \( M_n \) is the 5-regular graph consisting of \( 4n + 2 \) vertices \( a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, ..., a_n, b_n, c_n, d_n, \) \( e \), and \( f \), and containing the cycle \( a_1 c_1 d_1 c_2 a_2, a_n c_n d_n c_1 a_1, \) and also the edges \( a_k b_k, c_k d_k, a_k d_k, b_k c_k \) (where \( 1 \leq k \leq n \)), the edges \( c_k b_{k+1}, d_k a_{k+1}, b_k b_{k+1}, d_k d_{k+1} \) (where \( 1 \leq k \leq n-1 \)), the edges \( e f b_1, e f d_1, b_1 f c_1, f a_1, f d_1, \) and \( d_1 b_1 \). Figure 4 shows \( M_3 \).

![Figure 4. Graph \( M_3 \).](image)
The graph $M_n (n \geq 3)$ can also be made up by gluing $n$ identical planar tiles $M$ pictured in Figure 5(a) and one tile $S$ in Figure 5(b) which is a subgraph of $M$, in circular fashion, that is, $M_n = M(n, S) = o(M^n S)$, and then contract the vertices of degree two.

![Figure 5. Tile M and tile S.]

Figure 6 shows $M_3 = M(3, S) = o(M^3 S)$, that is obtained by gluing three tiles $M$ in Figure 5(a) and one tile $S$ in Figure 5(b) in a circular fashion.

![Figure 6. Graph M3.]

3. Book Embeddings of $H_n$ and $M_n$

In this section we show that $\rho(H_n) = 3$ and $\rho(M_n) = 3$, for $n \geq 3$.

**Theorem 1.** Let $n \geq 3$. Then $\rho(H_n) = 3$.

**Proof.** Richter dan Thomassen [7] determined that the crossing number of $H_n (n \geq 3)$ is 3, which was established by Pinontoan and Richter [6] using tile mechanism. So $M_n$ is not planar on the plane (two pages), it needs more than two pages to embed and hence $\rho(H_n) \geq 3$.

We show that $\rho(H_n) \leq 3$ by drawing an embedding to book. The graph $H_n$ is embeded to a book as follows. Put the vertices on the spine in the following order:

$$a_1, b_1, c_1, a_2, b_2, c_2, \ldots, a_{n+1}, b_{n+1}, c_{n+1}.$$  

Then on the first page, put the edges $b_1c_1 (1 \leq i \leq n + 1)$, $b_ib_{i+1} (1 \leq i \leq n)$, $b_{i+1} (1 \leq i \leq n)$, and $a_1b_{n+1}$ dan $a_{i+1} c_{n+1}$ without crossing. On the second page, put the edges $a_1c_1 (1 \leq i \leq n + 1)$, $a_1a_{i+1} (1 \leq i \leq n)$, and $c_1a_{i+1} (1 \leq i \leq n)$ without crossing, and on the third page, put the edges $b_1a_{n+1}$ and $b_{i+1} c_{n+1}$ without crossing. So we need at least three pages to put the edges and hence $\rho(H_n) \leq 3$. Therefore $\rho(H_n) = 3$. $\square$
Figure 7 shows a book embedding of $H_5$. The vertices are lined up on the spine, the edges drawn above the vertices are on page 1, the continuous edges under the vertices are on page 2, and the edges drawn with stipple lines under the vertices are on page 3.

**Figure 7.** A book embedding of $H_5$.

**Theorem 2.** Let $n \geq 3$. Then $\rho(M_n) = 3$.

**Proof.** Pinontoan and Richter [6] have determined that the crossing numbers of $M_n$ ($n \geq 3$) is 6. $M_n$ is not planar and so it needs at least three pages. Hence $\rho(M_n) \geq 3$.

To show that $\rho(M_n) \leq 3$ we draw an embedding of $M_n$ to book on three pages as follows. Put the vertices on the spine in the following order $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, ... , a_n, b_n, c_n, d_n, e, f$. Then put the following edges on the first page: $a_i b_i, b_i c_i, c_i d_i (1 \leq i \leq n)$, $b_i b_{i+1} (1 \leq i \leq n - 1)$, and the edges $c_n e, d_n e, e f, b_n f,$ and $c_n f$. On the second page, put the edges $a_i c_i, a_i d_i (1 \leq i \leq n)$, $d_i d_{i+1} (1 \leq i \leq n - 1)$, and the edge $d_1 f$. Finally, on the third page, put the edges $c_i b_{i+1} (1 \leq i \leq n - 1)$, and also the edges $a_1 e, a_1 f, b_1 e, and b_1 d_n$. So we need at least three pages to put the edges of $M_n$ and hence $\rho(M_n) \leq 3$. Therefore $\rho(M_n) = 3$. □

A book embedding of $M_3$ can be seen in Figure 8, where the edges shown in Figure 8(a) above the vertices are on the first page and the edges under the vertices are on the second page. Finally, the edges in Figure 8(b) are put on the third page.

**Figure 8.** A book embedding of $M_3$. 
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