Three-particle States in Nonrelativistic Four-fermion Model

A.N.Vall, S.E.Korenblit, V.M.Leviant, D.V.Naumov, A.V.Sinitskaya. 
Irkutsk State University, 664003, Gagarin blrd, 20, Irkutsk, Russia.

Abstract

On a nonrelativistic contact four-fermion model we have shown that the simple Λ-cut-off prescription together with definite fine-tuning of the Λ dependency of ”bare”quantities lead to self-adjoint semi-bounded Hamiltonian in one-, two- and three-particle sectors. The fixed self-adjoint extension and exact solutions in two-particle sector completely define three-particle problem. The renormalized Faddeev equations for the bound states with Fredholm properties are obtained and analyzed.

1. Introduction

Models with contact four-fermion interaction are considered in a wide range of problems both in solid medium and in quantum field theory. It is well known that in quantum field theory such interaction is nonrenormalizable in the frames of conventional perturbation approach. In our previous works [1], [2], [3], we have demonstrated that nonrelativistic four-fermion quantum field models possess exact two-particle solutions which clarify the meaning of renormalization in these models. In the present one we show how these solutions lead to correct definition of three-particle problem as well.

2. Contact four-fermion models

Let us consider the following Hamiltonian: 

\[ H = \int d^3x \left\{ \Psi^a_{\alpha}(x) \mathcal{E}(P) \Psi^a_{\alpha}(x) - \frac{\Lambda}{4} \left[ S^2(x) - \vec{J}^2(x) \right] \right\}, \quad \text{with the conventions:} \]

\[ \mathbf{P}_x = -i \nabla_x; \quad \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}; \quad S(x) = \Psi^a_{\alpha}(x) \Psi^a_{\alpha}(x); \quad \vec{J}(x) = (2mc)^{-1} \Psi^a_{\alpha}(x) \vec{P}^a \Psi^a_{\alpha}(x); \]

\[ \left\{ \Psi^a_{\alpha}(x), \Psi^b_{\beta}(y) \right\}_{x_0=y_0} = 0; \quad \left\{ \Psi^a_{\alpha}(x), \Psi^b_{\beta}(y) \right\}_{x_0=y_0} = \delta_{\alpha\beta} \delta^{ab} \delta_3(\vec{x} - \vec{y}) \implies \delta_{\alpha\beta} \delta^{ab} \frac{1}{V^*}. \]

Here \( \mathcal{E}(k) \) is arbitrary ”bare” one-particle spectrum, \( V^* \) has a meaning of excitation volume, which could be connected with momentum cut-off \( \Lambda: 6\pi^2/V^* = \Lambda^3. \) This Hamiltonian is invariant under (global) symmetry transformations \( SU_I(2) \times SU_A(2) \times U(1). \) Introducing Heisenberg fields (HF) in momentum representation

\[ \Psi^a_{\alpha}(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i(\vec{k}\vec{x})} b^a_{\alpha}(\vec{k}, t); \quad \left\{ b^a_{\alpha}(\vec{k}, t), b^b_{\beta}(\vec{q}, t) \right\} = \delta_{\alpha\beta} \delta^{ab} \delta_3(\vec{k} - \vec{q}); \quad \text{(2)} \]

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\(^2\text{E-mail KORENB@math.isu.runnet.ru}\)
we consider three different linear operator realizations of HF for \( t = 0 \) via physical fields, connected by Bogolubov rotations: \( u_a = \cos \vartheta^a; v_a = \sin \vartheta^a; (u_a)^2 + (v_a)^2 = 1, \)

\[
    b^a_\alpha(\vec{k},0) = e^G d^a_\alpha(\vec{k}) e^{-G} = u_\alpha d^a_\alpha(\vec{k}) - v_\alpha \varepsilon_{\alpha\beta} d^b_\beta(-\vec{k}),
    
    G = \frac{1}{2} \sum_{a=1,2} \vartheta^a \varepsilon_{\alpha\beta} \int d^3k \left[ d^a_\alpha(\vec{k}) d^a_\beta(-\vec{k}) + d^b_\alpha(\vec{k}) d^b_\beta(-\vec{k}) \right] = -G^\dagger,
\]

which under condition \( u_\alpha v_\alpha = 0 \) for \( a = 1, 2 \) lead to reduced Hamiltonians in normal form exactly diagonalizable above corresponding vacuums \( d^a_\alpha(\vec{k}) \mid 0 \rangle = 0; \)

\[
    H = Vw_0 + \hat{H}; \quad \hat{H} = \hat{H}_0 + \hat{H}_1; \quad \hat{H} \{d\} \mid 0 \rangle = 0; \quad \left[ H \{d\}, d^a_\alpha(\vec{k}) \right] \mid 0 \rangle = E^a(\vec{k}) d^a_\alpha(\vec{k}) \mid 0 \rangle; \quad (3)
    
    w_0 = \frac{1}{V^*} \left[ (2 < \mathcal{E}(k) > -4g \left(v_1^2 + v_2^2\right) - 8g(v_1v_2)) \right]; \quad < \mathcal{E}(k) > \overset{\text{def}}{=} V^* \int \frac{d^3k}{(2\pi)^3} \mathcal{E}(k); \quad (4)
    
    E^a(\vec{k}) = \frac{g}{(2mc)^2} \left(k^2_+ + k^2_-\right) + g + (1 - 2v^2_\alpha) \left[ \mathcal{E}(k) - 2g(1 + 2v^2_{\alpha-a}) \right]; \quad k^2_- = \frac{3}{5} \Lambda^2; \quad (5)
    
    \hat{H}_0 \{d\} = \sum_{a=1,2} \int d^3k E^a(\vec{k}) d^a_\alpha(\vec{k}) d^a_\beta(\vec{k}); \quad \hat{H}_1 \{d\} = \sum_{a,b} \int d^3k d^3k_1 d^3k_2 d^3k_3 d^3k_4 \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4)
    
    .K^{P(ab)} \left( \frac{\vec{k}_1 - \vec{k}_2}{2}; \frac{\vec{k}_1 - \vec{k}_3}{2} \right) \frac{d^a_\alpha(\vec{k}_1) d^b_\alpha(\vec{k}_2) d^b_\beta(\vec{k}_3) d^a_\beta(\vec{k}_4)}{d_\alpha(\vec{k})}; \quad g = \frac{\lambda}{4V^*}. \quad (6)
\]

The different realizations correspond to different systems when \( v_1, v_2 \) independently take values 0,1. B-system: \( v_1 = v_2 = 0, d^1_\alpha(\vec{k}) = B_\alpha(\vec{k}), d^2_\alpha(\vec{k}) = \bar{B}_\alpha(\vec{k}), \) \( E^{1,2}_B(k) = E_B(k). \) One can check, that corresponding vacuum state \( \mid 0 \rangle_B \) is singlet for both \( SU(2) \) and \( SU(2) \) groups and the one-particle excitations of \( B \) and \( \bar{B} \) form corresponding fundamental representations. C-system: \( v_1 = v_2 = 1, d^1_\alpha(\vec{k}) = \epsilon_{\alpha\beta} C_\beta(\vec{k}), d^2_\alpha(\vec{k}) = \epsilon_{\alpha\beta} \bar{C}_\beta(\vec{k}), E^{1,2}_C(k) = E_C(k). \) The symmetry structure of this system is similar to B-system. A-system: \( v_1 = 0, v_2 = 1 \) (or otherwise), which will be considered in more detail. Let \( d^1_\alpha(\vec{k}) = A_\alpha(\vec{k}), d^2_\alpha(\vec{k}) = \epsilon_{\alpha\beta} \bar{A}_\beta(\vec{k}), \) and let \( f^{ab} \) be an arbitrary constant \( SU_A(2) \) matrix, then for \( E^{2,1}_A(k) \equiv E_A^{(+,-)}(k) \equiv E_{\bar{A}A}(k) \) corresponding HF \( \overset{2}{[2]} \) resemble relativistic ones:

\[
    \Psi^a_\alpha(x)_A = \int \frac{d^3k}{(2\pi)^3/2} \left[ f^{a1} A_\alpha(\vec{k},t)e^{-iE^{(-)}_A(k)} + f^{a2} \bar{A}^\dagger_\alpha(-\vec{k},t)e^{iE^{(+)}_A(k)} \right] e^{i(\vec{k}\vec{x})},
    
    \text{where: } \left\{ \begin{array}{c}
    A_\alpha(\vec{k},t)e^{-iE^{(-)}_A(k)} \\
    \bar{A}^\dagger_\alpha(\vec{k},t)e^{-iE^{(+)}_A(k)}
\end{array} \right\} = e^{iHt} \left\{ \begin{array}{c}
    A_\alpha(\vec{k}) \\
    \bar{A}^\dagger_\alpha(\vec{k})
\end{array} \right\} e^{-iHt}. \quad (7)
\]

It is a simple matter to show that for system A the \( SU_A(2) \) and \( U(1) \) symmetries turn out to be spontaneously broken and there are four composite Goldstone states \( [1], [2]. \)

3. Two-particle eigenvalue problems

The interaction between all particles in the systems B and C is the same as for \( AA, \bar{A}\bar{A} \) in system A. So it is enough to consider the later one. Hereafter \( BB \) means \( \bar{B}\bar{B}, \bar{B}B \) and analogously for \( CC \). Defining the two-particle interaction kernels and energies as:

\[
    K^{P(QQ')}(\vec{s},\vec{k}) = \left\{ \begin{array}{c}
    K^{P(+)\{s,\vec{k}\}}, \quad \text{for} \quad QQ' = \bar{A}\bar{A}, AA, BB, CC \\
    K^{P(-)\{s,\vec{k}\}}, \quad \text{for} \quad QQ' = AA
\end{array} \right\}, \quad \text{where},
\]
\[-2K^{P(\pm)}(\vec{s}, \vec{k}) = \frac{V^*}{(2\pi)^3} \cdot \frac{2g}{(2mc)^2} [(\vec{s} + \vec{k})^2 - P^2 \pm (2mc)^2]; \quad \text{as well as:} \]

\[E_{2}^{QQ'}(\mathcal{P}, \vec{k}) \equiv E_{Q}(\frac{P}{2} + \vec{k}) + E_{Q'}(\frac{P}{2} - \vec{k}) = E_{2}^{(\pm)}(\mathcal{P}, \vec{k}); \quad \text{so:} \]

\[E_{2}^{(\pm)}(\mathcal{P}, \vec{k}) = \frac{2g}{(2mc)^2} \left[ <k^2 > + (2mc)^2 + k^2 + \frac{P^2}{4} \right] + \]

\[
\begin{align*}
\pm [4g - \mathcal{E}(\frac{P}{2} + \vec{k}) - \mathcal{E}(\frac{P}{2} - \vec{k})] \\
\pm [12g - \mathcal{E}(\frac{P}{2} + \vec{k}) - \mathcal{E}(\frac{P}{2} - \vec{k})]
\end{align*}
\]

\[E_{2}^{(-)}(\mathcal{P}, \vec{k}) = \frac{2g}{(2mc)^2} \left[ <k^2 > - (2mc)^2 + k^2 + \frac{P^2}{4} \right] + \left[ \mathcal{E}(\frac{P}{2} + \vec{k}) - \mathcal{E}(\frac{P}{2} - \vec{k}) \right], \quad \text{for } QQ' = AA, \]

we can formulate both scattering and bound state two-particle eigenvalue problems in the Fock eigenspace of kinetic part \(\hat{H}_0\) of the reduced Hamiltonian \(\hat{H}\) (3), (5):

\[
\hat{H} | R^{\pm QQ'}_{a\beta}(\mathcal{P}, \vec{q}) = E_{2}^{QQ'}(\mathcal{P}, \vec{q}) | R^{\pm QQ'}_{a\beta}(\mathcal{P}, \vec{q}); \quad \hat{H} | B^{P QQ'}_{a\beta}(\mathcal{P}) = M_{2}^{QQ'}(\mathcal{P}) | B^{P QQ'}_{a\beta}(\mathcal{P}); \]

\[
| R^{\pm QQ'}_{a\beta}(\mathcal{P}, \vec{q}) = \int d^3k \Phi^{\pm QQ'}_{pq}(\vec{k}) | R^{0 QQ'}_{a\beta}(\mathcal{P}, \vec{k}); \quad | B^{P QQ'}_{a\beta}(\mathcal{P}) = \int d^3k \Phi^{QQ'}_{py}(\vec{k}) | R^{0 QQ'}_{a\beta}(\mathcal{P}, \vec{k}); \]

\[| R^{0 QQ'}_{a\beta}(\mathcal{P}, \vec{k}) = \hat{Q}^{\dagger}_{aq}(\frac{P}{2} + \vec{k}) \hat{Q}^{\dagger}_{bq}(\frac{P}{2} - \vec{k}) | 0; \quad M_{2}^{QQ'}(\mathcal{P}) = E_{2}^{QQ'}(\mathcal{P}, q = ib); \]

(\(\hat{Q}, \hat{Q}'\) stands for creation operators \(A, \tilde{A}\), or \(B, \tilde{B}\), or \(C, \tilde{C}\)) in terms of Schroedinger equation for corresponding (scattering or bound state) wave function:

\[
\left[ E_{2}^{QQ'}(\mathcal{P}, \vec{k}) - M_{2}^{QQ'}(\mathcal{P}) \right] \Phi^{QQ'}_{pq}(\vec{k}) = -2 \int d^3s \Phi^{QQ'}_{pq}(\vec{s}) K^{QQ'}(\vec{s}, \vec{k}). \]

As was shown in [4], [3], this equation really has for \(-\) case exact simple solutions corresponding to the Goldstone states almost independently from the very form of ”bare” spectrum.

For the \(AA\) or \(\tilde{A}A\) two-particle states (case \(+\)) the quadratic form of ”bare” spectrum according to (4), (6) transforms to the renormalized one:

\[\mathcal{E}(k) = \frac{k^2}{2m} + \mathcal{E}_0 \rightarrow E^{(\pm)}_{A}(k) = \frac{k^2}{2\mathcal{M}(\pm)} + E^{(\pm)}_{A0}; \quad \frac{1}{2\mathcal{M}(\pm)} = \frac{g}{(2mc)^2} + \frac{1}{2m}; \]

\[E^{(\pm)}_{A0} = g \left( \frac{k^2}{(2mc)^2} - 1 \pm 4 \right) \mp \mathcal{E}_0; \quad \lambda_0 = \frac{\lambda \mathcal{M}(\pm)}{2}; \quad \mu_0 = \frac{\lambda_0}{(2mc)^2}. \]

The eq.(12) in configuration space reveals strongly singular interaction potential, studied in [3], [5], [6]:

\[\left( -\nabla_{\vec{x}}^2 - q^2 \right) \psi_{q}(\vec{x}) = \delta_{3}(\vec{x}) R_{1}(q) - \nabla_{\vec{x}}^2 \delta_{3}(\vec{x}) R_{2}(q) - 2\mu_0 \left( (\vec{\nabla} \psi_{q})(0) \cdot \vec{\nabla} \delta_{3}(\vec{x}) \right); \]

\[R_{1}(q) \equiv \lambda_0 - \mu_0 \mathcal{P}^2 \psi_{q}(0) - \mu_0 \mathcal{P}^2 \psi_{q}(0); \quad R_{2}(q) \equiv \mu_0 \psi_{q}(0). \]

The first and second terms in the R.H.S. of (15) represent interaction with the orbital momentum \(l = 0\), the third one gives interaction only for \(l = 1\) and disappear after integration over unit sphere of
\( \vec{q} \)-directions. Among the obtained in [1], [2] different solutions for the two-particle wave functions (14) which correspond to different self-adjoint extensions [3] of operator (13), (16), the use of \( \Lambda \)-cut-off regularization [4] together with simple subtraction procedure, as well as in [8], with \( \Lambda \to \infty \) pick out the following renormalized ones:

\[
| l, J, m; P, q \rangle = \int d^3k \phi_{Pq}^{+(l,1,m)}(k)_{\alpha\beta} | R^{0(QQ')}_{\alpha\beta}(P, k); \quad \phi_{Pq}^{+(l,1,m)}(k)_{\alpha\beta} = \chi_{\alpha\beta}^{(1,m)} \Phi_{Pq}^{+(l)}(k); \quad (17)
\]

where for \( Q = Q' \):

\[
\chi^{(0,0)}_{\alpha\beta} = \frac{1}{\sqrt{2}}(\delta_{\alpha1}\delta_{\beta2} - \delta_{\alpha2}\delta_{\beta1}); \quad \chi^{(1,0)}_{\alpha\beta} = \frac{1}{\sqrt{2}}(\delta_{\alpha1}\delta_{\beta2} + \delta_{\alpha2}\delta_{\beta1}); \quad \chi^{(1,\pm1)}_{\alpha\beta} = \left\{ \begin{array}{ll}
\delta_{\alpha1}\delta_{\beta1} & \\
\delta_{\alpha2}\delta_{\beta2} & \end{array} \right.; \quad (18)
\]

\[
\Phi_{Pq}^{+(l)}(k) = \frac{1}{2} \left[ \delta_3(k - \vec{q}) + (-1)^l \delta_3(k + \vec{q}) \right] + \frac{T^{(l)}_{P}(q; k)}{k^2 - q^2 + i0}; \quad \Phi_{Pq}^{(0)}(k) = \frac{\text{const}}{k^2 + b^2}; \quad (19)
\]

\[
T^{(0)}_{P}(q; k) = \mu_0 \left( \frac{(2mc)^2 + \gamma < k^2 > - P^2 + q^2 + (1 - \gamma)k^2}{(2\pi)^3 |D^P(\pm iq) - D^P(b)|} \right) \xrightarrow{\Lambda \to \infty} - \frac{(2\pi)^{-1}}{Y} \frac{1}{(Y + b \mp iq)(b \pm iq)}; \quad (20)
\]

\[
D^P(q) = (\gamma - 1)^2 - I_0(\gamma) \left[ (2mc)^2 + \gamma < k^2 > - P^2 - (2 - \gamma)q^2 \right]; \quad D^P(b) = 0; \quad (21)
\]

\[
T^{(1)}_{P}(q; k) = \frac{2\mu_0(k \cdot \vec{q})}{(2\pi)^3} \left[ 1 - \frac{2}{3} I_1(\pm iq) \right]^{-1} \xrightarrow{\Lambda \to \infty} 0; \quad I_n(q) = \mu_0 \int \frac{d^3k}{(2\pi)^3} \frac{(k^2)^n}{k^2 + q^2}; \quad (22)
\]

where \( g = \Lambda^2 G(\Lambda), \quad (2mc)^2 = \Lambda^2 \nu(\Lambda), \quad \varepsilon_0 = \Lambda^2 \varepsilon(\Lambda), \quad (\gamma(\pm)) = \frac{\mu_0}{m} \); with

\[
G(\Lambda) = G_0 + G_1/\Lambda + G_2/\Lambda^2 + \ldots; \quad \text{and so for } \nu(\Lambda), \varepsilon(\Lambda), \gamma(\Lambda); \quad \text{and if } G_0, \nu_0 \neq 0, \text{ then one has: } \gamma_0(\pm) = 1; \quad \gamma_1(\pm) = \pm \sqrt{\nu_0 / G_0}; \quad \mathcal{M}(\pm) = 0 = \frac{\nu_0}{2G_0}; \quad \gamma = \frac{\pi}{2} \left( \frac{3}{5} \gamma_1 + \nu_1 \right); \quad \nu_0 = -\frac{3}{5}. \quad (23)
\]

The last equality in (24) reflects the bound state condition (21) which serves here as a dimensional transmutation condition [3], [4]. Thus \( Y \) and real \( b \) are arbitrary parameters of self-adjoint extension [3] which may be formally partially expressed via parameters of \( \Lambda \)-dependence of “bare” quantities by fine tuning relations (24). Strictly speaking, the solution (19), (20), (21) imply a self-adjoint extension for restricted on appropriate subspace of \( L^2 \) initial free Hamiltonianto to extended Hilbert space \( L^2 \oplus C^1 \). The additional discrete component of eigenfunctions “improves” their scalar product, it is completely defined by the same parameters of self-adjoint extension but does not affect on physical meaning of obtained solution in ordinary space [3], [4], [10].

Another extension corresponds to the choice of finite “bare” mass that is possible only for B-system and for (-) case of A-system. Thus \( G_{0,1} = \nu_{0,1} = 0 \), and (23) with transmutation condition (21) leads to the solution which coincides with the well known extension in \( L^2 \) [4] for operator (13) with \( \mu_0 \equiv 0 \), for which: \( \gamma_0(\pm) = 1 - (3/4)(3 \pm \sqrt{5}) < 1 \), and \( r = |\vec{x}| \)

\[
\mathcal{M}(-) = \frac{3}{4}(3 \pm \sqrt{5}); \quad T_{P}^{(0)}(q; k) |_{\Lambda \to \infty} = -\frac{(2\pi)^{-1}}{b \pm iq}; \quad \psi_b^{(0)}(\vec{x}) = \frac{\sqrt{8\pi}b e^{-br}}{4\pi r}. \quad (25)
\]

4. Three-particle eigenvalue problems

From Schroedinger equation with Hamiltonian (3) for eigenstate of three identical (A) particles with total momentum \( P \) follows equation for wave function satisfying Pauli principle: (further
\[ E_A(\vec{k}) = E(\vec{k}) \]
\[ |3, \mathcal{P} \rangle = \int d^3 q_1 d^3 q_2 d^3 q_3 D^{(P;J,m)}_{\alpha \beta \gamma}(\vec{q}_1, \vec{q}_2, \vec{q}_3) A^\dagger_{\alpha}(\vec{q}_1) A^\dagger_{\beta}(\vec{q}_2) A^\dagger_{\gamma}(\vec{q}_3) | 0 \rangle; \quad \hat{H}|3, \mathcal{P} \rangle = M_3(\mathcal{P}) |3, \mathcal{P} \rangle; \quad (26) \]
\[ D^{(P;J,m)}_{\alpha \beta \gamma}(\vec{q}_1, \vec{q}_2, \vec{q}_3) = -D^{(P;J,m)}_{\beta \gamma \alpha}(\vec{q}_2, \vec{q}_1, \vec{q}_3) \equiv \delta(\vec{q}_1 + \vec{q}_2 + \vec{q}_3 - \mathcal{P}) D^{(P;J,m)}_{\alpha \beta \gamma}(\vec{q}_1, \vec{q}_2, \vec{q}_3)|_{\vec{q}_1+\vec{q}_2+\vec{q}_3=\mathcal{P}}; \quad (27) \]
\[ D^{(P;J,m)}_{\alpha \beta \gamma}(\vec{q}_1, \vec{q}_2, \vec{q}_3) \equiv \frac{\sum_{i=1}^3 E(\vec{q}_i) - M_3(\mathcal{P})}{3}; \quad (28) \]
\[ \mathcal{H}(\vec{k}_1\vec{k}_2\vec{k}_3|\vec{q}_1\vec{q}_2\vec{q}_3) = \frac{-\lambda}{2(2\pi)^3} \delta \left( \sum_{i=1}^3 \vec{k}_i - \sum_{i=1}^3 \vec{q}_i \right) \left\{ \sum_{i=1}^3 \delta(\vec{k}_n - \vec{q}_n) \left[ 1 - \frac{(\vec{k}_j + \vec{q}_j) \cdot (\vec{k}_l + \vec{q}_l)}{(2mc)^2} \right] \right\}. \quad (29) \]

The kernel (29) obviously reproduces all permutation symmetries and momentum conservation. So, it seems convenient to simplify the study of spin-symmetry structure of wave function using the formal function of three (dependent) variables like \( \tilde{D}^{(P;J,m)}_{\alpha \beta \gamma}(\vec{q}_1, \vec{q}_2, \vec{q}_3) \) (27) and introducing corresponding "formfactors":
\[
\tilde{K}^{(P;J,m)}_{\alpha \beta \gamma}(\vec{q}_1, \vec{q}_2, \vec{q}_3)|_{\vec{q}_1+\vec{q}_2+\vec{q}_3=\mathcal{P}} = \left[ \sum_{i=1}^3 E(\vec{q}_i) - M_3(\mathcal{P}) \right] \tilde{D}^{(P;J,m)}_{\alpha \beta \gamma}(\vec{q}_1, \vec{q}_2, \vec{q}_3)|_{\vec{q}_1+\vec{q}_2+\vec{q}_3=\mathcal{P}}. \quad (30)
\]

Since the momentum conservation condition is totally symmetrical over \( \vec{q}_j \), the \( \tilde{K} \) and \( \tilde{D} \) have the same spin-symmetry structure as of \( D \). Namely, let \( (\ldots)|\ldots\rangle \) means hereafter (anti) symmetrization over internal indices, then one has:
\[
\tilde{K}^{(P;1/2,m)}_{\gamma \alpha \beta}(\vec{q}_1, \vec{q}_2, \vec{q}_3) = \Gamma^{1/2,m}_{\gamma \alpha \beta} X(\vec{q}_1|\vec{q}_2, \vec{q}_3) + \Gamma^{1/2,m}_{\gamma \beta \alpha} X(\vec{q}_2|\vec{q}_1, \vec{q}_3) + \Gamma^{1/2,m}_{\gamma \beta \alpha} X(\vec{q}_3|\vec{q}_1, \vec{q}_2) = (31)
\]
\[
\Gamma^{1/2,m}_{\alpha \beta \gamma} K^{(P)}_{\alpha \beta \gamma}(\{\vec{q}_1, \vec{q}_2\}) = \Gamma^{1/2,m}_{\alpha \beta \gamma} K^{(P)}_{\alpha \beta \gamma}(\{\vec{q}_2, \vec{q}_1\}) = \Gamma^{1/2,m}_{\gamma \alpha \beta} Y(\vec{q}_1|\vec{q}_2, \vec{q}_3) + \Gamma^{1/2,m}_{\gamma \beta \alpha} Y(\vec{q}_1|\vec{q}_3, \vec{q}_2) + \Gamma^{1/2,m}\gamma \gamma \gamma \alpha \beta \gamma \gamma Y(\vec{q}_1|\vec{q}_2, \vec{q}_3) \equiv Y(\vec{q}_1|\vec{q}_2, \vec{q}_3) - Y(\vec{q}_2|\vec{q}_1, \vec{q}_3) = (32)
\]
\[
\tilde{K}^{(P;3/2,m)}_{\alpha \beta \gamma}(\vec{q}_1, \vec{q}_2, \vec{q}_3) = \Gamma^{3/2,m}_{\alpha \beta \gamma} K^{(P)}_{\alpha \beta \gamma}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}). \quad (33)
\]

Here the following properties of three-spin-wave functions were used:
\[
\Gamma^{1/2,1/2}_{\alpha \beta \gamma} = a \delta_{\alpha \beta} \delta_{\beta \gamma} \delta_{\gamma \alpha} + b \delta_{\alpha \beta} \delta_{\beta \gamma} \delta_{\gamma \alpha} + c \delta_{\gamma \alpha} \delta_{\alpha \beta} \delta_{\beta \gamma}, \quad \text{with } a + b + c = 0, \quad \text{what means, that} \quad (34)
\]
\[
\Gamma^{1/2,m}_{\alpha \beta \gamma} + \Gamma^{1/2,m}_{\gamma \alpha \beta} + \Gamma^{1/2,m}_{\alpha \beta \gamma} = 0; \quad \Gamma^{3/2,3/2}_{\alpha \beta \gamma} = \delta_{\alpha \beta} \delta_{\beta \gamma} \delta_{\gamma \alpha} + \delta_{\alpha \beta} \delta_{\beta \gamma} \delta_{\gamma \alpha} + \delta_{\gamma \alpha} \delta_{\alpha \beta} \delta_{\beta \gamma}. \quad (34)
\]

To change projection \( m \) on \(-m\) it is enough to replace indices 1 \( \leftrightarrow \) 2. For case \( J=1/2 \) three-spin-functions with definite partial symmetry correspond to eigenvalue of definite spin-permutation operator: \( \Sigma_{23} = +1, \quad (X), \quad b = c, \quad a = -2c \), for symmetrical function \( \Gamma^{1/2,m}_{\alpha \beta \gamma} \); and \( \Sigma_{23} = -1, \quad (Y), \quad b = -c, \quad a = 0 \), for antisymmetrical one \( \Gamma^{1/2,m}_{\alpha \beta \gamma} \). All the "formfactors" satisfy the same equation and differ only by the symmetry type \( S = X, Y, Z \):
\[
K^{(P)}_{S}(\vec{q}_1, \vec{q}_2, \vec{q}_3) = -\int d^3 k_1 d^3 k_2 d^3 k_3 \frac{K^{(P)}_{S}(\vec{k}_1, \vec{k}_2, \vec{k}_3)}{\sum_{i=1}^3 E(\vec{k}_i) - M_3(\mathcal{P})} \mathcal{H}(\vec{k}_1, \vec{k}_2, \vec{k}_3|\vec{q}_1, \vec{q}_2, \vec{q}_3). \quad (35)
\]
Now we put consequently for every term of the kernel (29):

\[ 1, 2, 3 = n \neq j \neq l, \quad j < l; \quad \vec{k}_j - \vec{k}_l = 2\vec{s}, \quad \vec{k}_j + \vec{k}_l = \vec{r}_n, \quad \text{with} \quad \vec{q}_1 + \vec{q}_2 + \vec{q}_3 = \mathcal{P}, \]

obtaining: \( \vec{r}_n = \mathcal{P} - \vec{q}_n; \quad (\vec{k}_j + \vec{q}_j) \cdot (\vec{k}_j + \vec{q}_j) = (\mathcal{P} - \vec{q}_n)^2 - \left( \vec{s} + \frac{\vec{q}_j - \vec{q}_l}{2} \right)^2, \)

and thus immediately find out the general structure of "formfactors" in (35):

\[ K^{(P)}_S(\vec{q}_1, \vec{q}_2, \vec{q}_3) = \sum_{1=n\neq j<l}^3 \left[ \vec{C}_{Sn}(\vec{q}_n) \cdot (\vec{q}_j - \vec{q}_l) + A_{Sn}(\vec{q}_n) + B_{Sn}(\vec{q}_n)(\vec{q}_j - \vec{q}_l)^2 \right], \quad \text{with (36)} \]

\[
\begin{align*}
\frac{A_{Sn}(\vec{q})}{B_{Sn}(\vec{q})} \cdot \frac{1}{\vec{C}_{Sn}(\vec{q})} = & \frac{\lambda}{2(2mc)^2} \int \frac{d^3s}{(2\pi)^3} \cdot \frac{K^{(P)}_S(\vec{k}_1\vec{k}_2\vec{k}_3)}{E(\vec{k}_n) + E(\vec{k}_j) + E(\vec{k}_l) - M_3(\mathcal{P})} \cdot \left\{ \frac{(2mc)^2 + \vec{s}^2 - (\mathcal{P} - \vec{q})^2}{\vec{s}} \right\},
\end{align*}
\]

where for \( 1, 2, 3 = n \neq j \neq l, \quad j < l: \quad \vec{k}_n = \vec{q}_j; \quad \vec{k}_j = \frac{\mathcal{P} - \vec{q}_l}{2} + \vec{s} \equiv \vec{k}_+; \quad \vec{k}_l = \frac{\mathcal{P} - \vec{q}_l}{2} - \vec{s} \equiv \vec{k}_-. \quad (37) \]

The system of coupled integral equations (36), (37) may be essentially simplified by utilizing the symmetry of functions \( K^{(P)}_S \) (31), (32), (33), that demands for example:

\[
\begin{align*}
\vec{C}_{Z1}(\vec{q}) = -\vec{C}_{Z2}(\vec{q}) = \vec{C}_{Z3}(\vec{q}) \equiv \vec{C}_Z(\vec{q}); \quad A_{Z1}(\vec{q}) = B_{Z1}(\vec{q}) = \vec{0}; \quad \text{and thus:} \\
K^{(P)}_Z([\vec{q}_1, \vec{q}_2, \vec{q}_3]) = \vec{C}_Z(\vec{q}_1) \cdot (\vec{q}_2 - \vec{q}_3) + \vec{C}_Z(\vec{q}_2) \cdot (\vec{q}_3 - \vec{q}_1) + \vec{C}_Z(\vec{q}_3) \cdot (\vec{q}_1 - \vec{q}_2). \\
\end{align*}
\]

Analogously: \( A_{X1}(\vec{q}) = A_{X2}(\vec{q}) \equiv A_X(\vec{q}); \quad B_{X1}(\vec{q}) = B_{X2}(\vec{q}) \equiv B_X(\vec{q}); \quad \vec{C}_{X1}(\vec{q}) = \vec{C}_{X2}(\vec{q}) \equiv \vec{C}_X(\vec{q}); \quad \vec{C}_{X3}(\vec{q}) = 0; \quad Q_{X1}(\vec{q}; \vec{p}) = A_{X1}(\vec{q}) + \vec{p}^2 B_{X1}(\vec{q}); \quad K^{(P)}_X([\vec{q}_1, \vec{q}_2, \vec{q}_3]) = Q_X(\vec{q}_1; \vec{q}_2 - \vec{q}_3) + Q_X(\vec{q}_2; \vec{q}_3 - \vec{q}_1) + Q_X(\vec{q}_3; \vec{q}_1 - \vec{q}_2) + \\
+ \vec{C}_X(\vec{q}_1) \cdot (\vec{q}_2 - \vec{q}_3) + \vec{C}_X(\vec{q}_2) \cdot (\vec{q}_3 - \vec{q}_1) + \vec{C}_X(\vec{q}_3) \cdot (\vec{q}_1 - \vec{q}_2) \quad (38) \\
\end{align*}
\]

\[
\begin{align*}
A_{Y1}(\vec{q}) = -A_{Y2}(\vec{q}) \equiv A_Y(\vec{q}); \quad B_{Y1}(\vec{q}) = -B_{Y2}(\vec{q}) \equiv B_Y(\vec{q}); \quad A_{Y3}(\vec{q}) = B_{Y3}(\vec{q}) = 0; \\
\vec{C}_{Y1}(\vec{q}) = -\vec{C}_{Y2}(\vec{q}) \equiv \vec{C}_Y(\vec{q}); \quad \vec{C}_{Y3}(\vec{q}) \equiv 2 \left( \vec{C}_{Y0}(\vec{q}) - \vec{C}_Y(\vec{q}) \right); \\
K^{(P)}_Y([\vec{q}_1, \vec{q}_2, \vec{q}_3]) = Q_Y(\vec{q}_1; \vec{q}_2 - \vec{q}_3) - Q_Y(\vec{q}_2; \vec{q}_3 - \vec{q}_1) + \vec{C}_{Y3}(\vec{q}_3) \cdot (\vec{q}_1 - \vec{q}_2) + \\
+ \vec{C}_Y(\vec{q}_1) \cdot (\vec{q}_2 - \vec{q}_3) - \vec{C}_Y(\vec{q}_2) \cdot (\vec{q}_1 - \vec{q}_3). \quad (39) \\
\end{align*}
\]

Solving now every of these systems (37) with (38), (39), (40), as nonhomogeneous algebraic one, where unknown integral terms would be considered as free ones, we come to corresponding homogeneous systems of Faddeev integral equations:

\[
\begin{align*}
\vec{C}_Z(\vec{q}) = & \int \frac{d^3s}{(2\pi)^3} \cdot \frac{1}{(s^2 + \vec{q}_l^2)} \left[ -\frac{2\mu_0}{\frac{3}{2}I_1(\vec{q})} \right] \vec{C}_Z(\vec{k}_+ \cdot (\vec{q}_+ - \vec{k}_-)); \quad \text{and the same eq. for } \vec{C}_{Y0}(\vec{q}). \quad (41) \\
\text{Let: } Q_{X0}(\vec{q}; \vec{p}) = Q_X(\vec{q}; \vec{p}) + 2Q_X(\vec{q}; \vec{p}); \quad & \Delta_X(\vec{q}; \vec{p}) = Q_X(\vec{q}; \vec{p}) - Q_X(\vec{q}; \vec{p}) \quad \text{then:} \\
Q_{X0}(\vec{q}; \vec{r} = 2\vec{r}) = & \int \frac{d^3s}{(2\pi)^3} \cdot \frac{2\mu_0}{(s^2 + \vec{q}_l^2)} \left[ \frac{\mathcal{O}^{(P)}(\vec{q}; \vec{s}, \vec{r})}{\mathcal{D}^{P-q}(\vec{q})} \right] Q_{X0}(\vec{k}_+ \cdot (\vec{q}_+ - \vec{k}_-)); \\
\Delta_X(\vec{q}; \vec{r} = 2\vec{r}) = & \int \frac{d^3s}{(2\pi)^3} \cdot \frac{(s^2 + \vec{q}_l^2)}{(s^2 + \vec{q}_l^2)} \left[ \frac{\mathcal{O}^{(P)}(\vec{q}; \vec{s}, \vec{r})}{\mathcal{D}^{P-q}(\vec{q})} \right] \left( \Delta_X(\vec{k}_+ \cdot (\vec{q}_+ - \vec{k}_-) - \vec{C}_X(\vec{k}_+ \cdot (\vec{q}_+ - \vec{k}_-)) \right); \quad (42) \\
\end{align*}
\]
\[ \tilde{C}_X(\vec{q}) = \int \frac{d^3 s}{(2\pi)^3} \cdot \frac{1}{s^2 + \vec{q}^2} \left[ \frac{\mu_0 \vec{s}}{1 - \frac{2}{3} I_1(\vec{q})} \right] \left( \tilde{C}_X(\vec{k}_+) \cdot (\vec{q} - \vec{k}_-) + \Delta_X(\vec{k}_+; \vec{q} - \vec{k}_-) \right). \]

\[ \tilde{C}_Y(\vec{q}) - \tilde{C}_{Y0}(\vec{q}) = \int \frac{d^3 s}{(2\pi)^3} \cdot \frac{1}{s^2 + \vec{q}^2} \left[ \frac{\mu_0 \vec{s}}{1 - \frac{2}{3} I_1(\vec{q})} \right] \left( \tilde{C}_Y(\vec{k}_+) \cdot (\vec{q} - \vec{k}_-) - Q_Y(\vec{k}_+; \vec{q} - \vec{k}_-) \right); \]

\[ Q_Y(\vec{q}; 2\vec{r}) = \int -\frac{d^3 s}{(2\pi)^3} \cdot \frac{(\mu_0)}{s^2 + \vec{q}^2} \left[ \frac{\Omega^P(\vec{q}; s, \vec{r})}{2\Omega^P(s, \vec{q})} \right] \left( Q_Y(\vec{k}_+; \vec{q} - \vec{k}_-) + 3\tilde{C}_Y(\vec{k}_+) \cdot (\vec{q} - \vec{k}_-) - 2\tilde{C}_{Y0}(\vec{k}_+) \cdot (\vec{q} - \vec{k}_-) \right). \]

Here: \( \Omega^P(\vec{q}; s, \vec{r}) \equiv (2mc)^2 + \gamma < k^2 > - (\vec{P} - \vec{q})^2 - \vec{q}^2 + (1 - \gamma)(s^2 + r^2 + \vec{g}^2) + I_0(\vec{g})(s^2 + \vec{g}^2)(r^2 + \vec{g}^2); \)

\[ \omega^2(\vec{P}) = M_0 \left( 3E_0 - M_3(\vec{P}) \right); \]

\[ E(\vec{q}) + E(\vec{k}_+) + E(\vec{k}_-) - M(\vec{P}) \equiv \frac{s^2 + \vec{q}^2}{M_0}; \quad \vec{g}^2 = \vec{g}^2(q) = \frac{3}{4}q^2 + \frac{\vec{T}^2}{4} - (\frac{\vec{q}P}{\vec{g}^2} + \omega^2(\vec{P}). \]

For finite \( \Lambda \) one can easily recognize the interior of square brackets in the kernels of that equations as off-shell extensions of (half-off-shell) two-particle T-matrices from L.H.S. of \( (20), (23) \). However, for \( \Lambda \to \infty \) the all these off-shell T-matrices obviously coincide with the corresponding on-shell renormalized ones given by R.H.S. of \( (20), (23) \). So, when \( \Lambda \to \infty \), one observes here, as well as in two-particle case \( [9] \), the restoration of Galileo invariance, and comes to further simplifications \( \tilde{C}_{X,Y,Z} = B_{X,Y} = 0 \) resulting to renormalized equations for the functions of one variable:

\[ Q_{X0}(\vec{q}) = \frac{T(q(q))}{\pi^2} \int d^3 s \frac{Q_{X0}(\vec{k}_+)}{(s^2 + \vec{q}^2(q))}; \quad \] (44)

\[ Q_Y(\vec{q}) \]

\[ \Delta_X(\vec{q}) \]

\[ T(q) \equiv 2\pi^2 \mathcal{T}_P(q; q) \bigg|_{q = \pm i\epsilon^\pm} = \frac{\Upsilon}{(\Upsilon + b + \vec{q})(\vec{q} + b)} \sim \frac{\Upsilon}{\vec{g}^2} \big|_{\vec{g} \to \infty}. \] (46)

Obviously, the eqs. (44) and (14) can not have nontrivial solutions simultaneously. Therefore, two different possibilities appear:

1) eq. (14) has nonzero solution. Then \( Q_Y(\vec{q}) = \Delta_X(\vec{q}) = 0, Q_X(\vec{q}) = A_X(\vec{q}); K^{(P)}_X = K^{(P)}_Y = 0, K^{(P)}_X(\vec{q}_1 \vec{q}_2 \vec{q}_3) = 0 \), where \( A_X(\vec{q}) \) satisfies the eq. (14) which coincides with Shondin’s equation (10).

2) eq. (13) has nonzero solution. Then \( Q_{X0} = 0 \), and coinciding equations on \( \Delta_X(\vec{q}) \) and \( Q_Y(\vec{q}) \) define in principle the coordinate wave function of one and the same bound state independently of its spin symmetry: \( Q_X(\vec{q}) = Q_Y(\vec{q}) = A(\vec{q}); K^{(P)}_X(\vec{q}_1 \vec{q}_2 \vec{q}_3) = 0 \), where \( A(\vec{q}) \) satisfies the eq. (13). As was shown in (10), (11), the asymptotic behavior (10) guarantees that for both cases we deal with self-adjoint semi-bounded bellow three-particle Hamiltonian. Whereas the Hamiltonians corresponding to more slowly vanishing T-matrix of another two-particle extensions (23) are unbounded and correspond to "collapse" in three-particle system.

If for \( P = 0 \) we consider zero orbital momentum subspace, \( A(\vec{q}) \Longrightarrow A(q) \), the corresponding equations read

\[ q A(q) = T(q(q)) \frac{\xi}{\pi} \int_0^\infty dk k A(k) \ln \left( \frac{k^2 + q^2 + kq + \omega^2}{k^2 + q^2 - kq + \omega^2} \right), \] (47)
where $\xi = 2$ for the case 1) and $\xi = -1$ for the case 2). A simple analysis, curried out in Appendix, shows that for appropriate conditions the integral operator written here is equivalent to symmetrical quite continuous positively defined operator of Hilbert-Schmidt type. Therefore nontrivial solutions of (47) may be take place only for case 1). Whereas for case 2) bound states are impossible. We conclude that the bound states of three identical particle may be appear in this model only for isospin 1/2 with $X$-type wave functions (31).

5. Conclusions

So, in \cite{[2]} and here we formulate unambiguous renormalization procedure to extract a renormalized dynamic from ”nonrenormalizable” contact four-fermion interaction that is selfconsistent in every $N$-particle sector and is intimately connected with construction of self-adjoint extension of corresponding quantum mechanical Hamiltonian and restoration of Galileo invariance. We have shown that the simple $\Lambda$-cut-off prescription with definite $\Lambda$ dependence of ”bare” quantities and fine-tuning relations lead for reduced field Hamiltonian (6) to the set of self-adjoint semi-bounded Hamiltonians in one-, two- and three-particle sectors with correctly defined solutions for scattering and bound states.

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Appendix

Using hyperbolic substitutions and natural odd continuation onto a whole real axis for the function

$$F(q) \equiv \frac{qA(q)}{T(\varphi(q))} = -F(-q) = \varphi(\vartheta) = -\varphi(-\vartheta); \quad q = \frac{2}{\sqrt{3}} \omega \sinh \vartheta,$$

the equation (47) may be reduced to the following convenient form:

$$\varphi(\vartheta) = \frac{2 \xi K}{\pi \sqrt{3}} \int_{-\infty}^{\infty} d\tau \ W(\cosh \tau) \varphi(\tau) \ln \left( \frac{2 \cosh(\tau - \vartheta) + 1}{2 \cosh(\tau - \vartheta) - 1} \right); \quad W(\cosh \tau) = \frac{\omega}{K} \cosh \tau \ \tilde{T}(\omega \cosh \tau),$$

where $W(\cosh \tau)$ is even function of $\tau$ and $K$ is appropriate positive constant introduced for convenience. Note that the last kernel has additional eigenfunctions with opposite parity.

According to general restrictions from two- and three-particle problem \cite{[12]} we suppose $\Upsilon > 0$, $\omega > b \geq 0$. Therefore, from (46) $T(\varphi(q)) > 0$ and tends to zero fast enough to make meaningful the next substitution:

$$\vartheta = \vartheta(\eta), \quad \tau = \tau(\zeta), \quad d\zeta = d\tau W(\cosh \tau), \quad \varphi(\vartheta) = \Phi(\eta) = -\Phi(-\eta);$$

$$\zeta(\tau) = -\zeta(-\tau) = \int_{-\infty}^{\tau} d\tau W(\cosh \tau) - \chi; \quad 2\chi \equiv \int_{-\infty}^{\infty} d\tau W(\cosh \tau); \quad \infty > 2\chi > 0.$$

It is obviously true for arbitrary $T(\varphi(q))$ with the above properties and it transforms (49) to equation with symmetrical quite continuous kernel:

$$\Phi(\eta) = \frac{4 \xi K}{\pi \sqrt{3}} \int_{-\chi}^{\chi} d\zeta \ \Phi(\zeta) \ln \left[ \frac{2 \cosh(\tau(\zeta) - \vartheta(\eta)) + 1}{2 \cosh(\tau(\zeta) - \vartheta(\eta)) - 1} \right] \equiv \frac{4 \xi K}{\pi \sqrt{3}} (\mathcal{L}\Phi)(\eta).$$

8
With the usual definition of scalar product in $L^2(-\chi, \chi)$ for arbitrary function $\Phi(\eta)$ from this space one has, using Fourier transformation:

$$\left( \hat{L}\Phi, \Phi \right) = \int_{-\infty}^{\infty} d\nu \left| g(\nu) \right|^2 \frac{\sinh(\pi\nu/6)}{\nu \cosh(\pi\nu/2)} > 0; \quad g(\nu) \equiv \int_{-\infty}^{\infty} d\tau e^{i\nu\tau} W(\cosh \tau) \varphi(\tau).$$

Therefore, all eigenvalues of operator $\hat{L}$ are positive.

At last, for $b = 0$, $\Upsilon = \omega \cosh 2\chi$, $K = 2 \coth 2\chi$, from (50) follow the manifest expressions

$$e^\vartheta = \frac{\sinh(\chi + \eta)}{\sinh(\chi - \eta)}; \quad e^\tau = \frac{\sinh(\chi + \zeta)}{\sinh(\chi - \zeta)}; \quad e^{2\eta} = \frac{\cosh(\chi + \vartheta/2)}{\cosh(\chi - \vartheta/2)}; \quad e^{2\xi} = \frac{\cosh(\chi + \tau/2)}{\cosh(\chi - \tau/2)},$$

which allow direct application of Faddeev consideration [12] to eq. (51) when $\omega \to 0$, $\chi \to \infty$. Thus $\cosh(\tau - \vartheta) \simeq \cosh 2(\zeta - \eta)$ and the seeking of coefficients $a_m$ of Fourier expansion

$$\Phi(\eta) = \sum_{m=-\infty}^{\infty} a_m e^{i\pi m \eta/\chi}; \quad a_m = \sum_{n=-\infty}^{\infty} C_{mn} a_n; \quad C_{mn} = \frac{4\xi K}{\pi^3} \left( \hat{L} e^{i\pi n \zeta/\chi}, e^{i\pi m \eta/\chi} \right),$$

leads to Faddeev’s like relation:

$$1 = \frac{2\xi K}{\pi^3 \sqrt{3}} \frac{\sinh(\pi\nu/6)}{\nu \cosh(\pi\nu/2)}; \quad \nu \equiv \frac{\pi m}{2\chi},$$

which has a solution $\nu = \nu_0 > 0$ indeed only for the case 1) ($\xi = 2$), giving asymptotic distribution of Efimov levels as: $\omega_m \simeq 2\Upsilon e^{-\pi m/\nu_0}$.

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