Decentralized Event-Triggering for Control of Nonlinear Systems

Pavankumar Tallapragada and Nikhil Chopra

Abstract—This paper considers nonlinear control systems with full state feedback, a central controller and distributed sensors not co-located with the central controller. We present a methodology for designing decentralized asynchronous event-triggers, which utilize only locally available information, for determining the time instants of transmission from the sensors to the central controller. The proposed design guarantees a positive lower bound for the inter-transmission times of each sensor, while ensuring asymptotic stability of the origin of the system with an arbitrary, but priorly fixed, compact region of attraction. In the special case of Linear Time Invariant (LTI) systems, global asymptotic stability is guaranteed and scale invariance of inter-transmission times is preserved. A modified design method is also proposed for nonlinear systems, with event-triggered communication from the controller to the sensors that significantly increases the average sensor inter-transmission times. The proposed designs are illustrated through simulations of a linear and a nonlinear example.

I. INTRODUCTION

State based aperiodic event-triggering is receiving increased attention (a representative list of the recent literature includes [1]–[7]) as an alternative to the traditional time-triggering (example: periodic triggering) in sampled data control systems. In event based control systems, a state or data dependent event-triggering condition implicitly determines time instances at which control is updated or when a sensor transmits data to a controller. Such updates or transmissions are in general aperiodic and depend on the system state. Such a paradigm is particularly appealing in control systems with limited computational and/or communication resources.

Much of the literature on event-triggered control utilizes the full state information in the triggering conditions. However, in two very important class of problems full state information is not available to the event-triggers. These are systems with decentralized sensing and/or dynamic output feedback control. In the latter case, full state information is not available even when the sensors and the controller are centralized (co-located). In systems with decentralized sensing, each individual sensor has to base its decision to transmit data to a central controller only on locally available information. These two classes of problems are receiving attention in the community only recently - [8]–[12] (decentralized sensing) and [13]–[18] (output feedback control). This paper is an important addition to the limited literature on decentralized event-triggering in control systems with distributed sensors.

The basic contribution of this paper is a methodology for designing implicitly verified decentralized event-triggers for control of nonlinear systems. The system architecture we consider is one with full state feedback but with the sensors distributed and not co-located with a central controller. The proposed design methodology provides event-triggers that determine when each sensor transmits data to a central controller. The event-triggers are designed to utilize only locally available information, making the transmissions from the sensors asynchronous. The proposed design guarantees asymptotic stability of the origin of the system with an arbitrary, but fixed a priori, compact region of attraction. It also guarantees a positive lower bound for the inter-transmission times of each sensor individually. In the special case of Linear Time Invariant (LTI) systems, global asymptotic stability is guaranteed and scale invariance of inter-transmission times is preserved. For nonlinear systems, we also propose a variant with event-triggered communication from the central controller to the sensors that significantly increases the average sensor inter-transmission times.

In the literature, distributed event-triggered control was studied in [11], [12] with the assumption that the subsystems are weakly coupled, which allowed the design of event-triggers depending on only local information. Our proposed design method requires much less restrictive assumptions. In [8]–[10], each sensor checks a local condition (based on threshold crossing) that triggers asynchronous transmission of data by sensors to a central controller. However, this design guarantees only semi-global practical stability (even for linear systems) if the sensors do not listen to the central controller. Compared to this work, our proposed design guarantees semi-global asymptotic stability even when the sensors do not listen to the central controller. For linear systems, our proposed method guarantees global asymptotic stability without the sensors having to listen to the central controller. A similarity between the current paper and [8]–[10] is that both are partially motivated by the need to eliminate or drastically reduce the listening effort of the sensors to save energy.

In the dynamic output feedback control literature, [13]–[15] consider asynchronous and decentralized event-triggering for Linear Time Invariant (LTI) systems. Again, the method in [13] can guarantee only semi-global practical stability. In [14], [15], we have proposed a method that guarantees global asymptotic stability and positive minimum inter-transmission times. In fact, the basic principle behind the proposed design in the current paper is the same as in [14], [15], with additional
considerations for the nonlinear systems.

The rest of the paper is organized as follows. Section II describes and formally sets up the problem under consideration. In Section III, the design of asynchronous decentralized event-triggers for nonlinear systems is presented - without, and then with, feedback from the central controller. Section IV presents the special case of Linear Time Invariant (LTI) systems. The proposed design methodology is illustrated through simulations in Section V and finally Section VI provides some concluding remarks.

II. PROBLEM SETUP

Consider a nonlinear control system
\[ \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \]  
(1)
with the feedback control law
\[ u = k(x + x_e) \]  
(2)
where \( x_e \) is the error in the measurement of \( x \). In general, the measurement error can be due to many factors such as sensor noise and quantization. However, we consider measurement error that is purely a result of “sampling” of the sensor data \( x \). Before going into the precise definition of this measurement error, we first describe the broader problem. First, let us express (1) as a collection of \( n \) scalar differential equations
\[ \dot{x}_i = f_i(x, u), \quad x_i \in \mathbb{R}, \quad i \in \{1, 2, \ldots, n\} \]  
(3)
where \( x \) is \( [x_1, x_2, \ldots, x_n]^T \). In this paper we are concerned with a distributed sensing scenario where each component, \( x_i \), of the state vector \( x \) is sensed at a different location. Although the \( i \)th sensor senses \( x \), continuously in time, it transmits this data to a central controller only intermittently. In other words, the controller is a sampled-data controller that uses intermittently transmitted/sampled sensor data. In particular, we are interested in designing an asynchronous decentralized sensing mechanism based on local event-triggering that renders the origin of the closed loop system asymptotically stable.

To precisely describe the sampled-data nature of the problem, we now introduce the following notation. Let \( \{t_{j}^x\} \) be the increasing sequence of time instants at which \( x_i \) is sampled and transmitted to the controller. The resulting piecewise constant sampled signal is denoted by \( x_{i,s} \), that is,
\[ x_{i,s} \triangleq x(t_{j}^x), \quad \forall t \in [t_{j}^x, t_{j+1}^x), \quad \forall j \in \{0, 1, 2, \ldots\} \]  
(4)
As mentioned previously, the sampled data, \( x_{i,s} \), may also be viewed as resulting from an error in the measurement of the continuous-time signal, \( x_i \). This measurement error is denoted by
\[ x_{i,e} \triangleq x_{i,s} - x_i = x_i(t_{j}^x) - x_i, \quad \forall t \in [t_{j}^x, t_{j+1}^x) \]
Finally, we define the sampled-data vector and the measurement error vector as
\[ x_s \triangleq [x_{1,s}, x_{2,s}, \ldots, x_{n,s}]^T, \quad x_e \triangleq [x_{1,e}, x_{2,e}, \ldots, x_{n,e}]^T \]

Note that, in general, the components of the vector \( x_s \) are asynchronously sampled components of the plant state \( x \). The components of \( x_e \) are also defined accordingly.

In time-triggered implementations, the time instants \( t_{j}^x \) are pre-determined and are commonly a multiple of a fixed sampling period. However, in event-triggered implementations the time instants \( t_{j}^x \) are determined implicitly by a state/data based triggering condition at run-time. Consequently, an event-triggering condition may result in the inter-sample times \( t_{j+1}^x - t_{j}^x \) to be arbitrarily close to zero or it may even result in the limit of the sequence \( \{t_{j}^x\} \) to be a finite number (Zeno behavior). Thus for practical utility, an event-trigger has to ensure that these scenarios do not occur.

Thus, the problem under consideration may be stated more precisely as follows. For the \( n \) sensors, design event-triggers that depend only on local information and implicitly define the non-identical sequences \( \{t_{j}^x\} \) such that (i) the origin of the closed loop system is rendered asymptotically stable and (ii) inter-sample (inter-transmission) times \( t_{j+1}^x - t_{j}^x \) are lower bounded by a positive constant.

Finally, a point regarding the notation in the paper is that the notation \(|.|\) denotes the Euclidean norm of a vector. In the next section, the main assumptions are introduced and the event-triggering conditions for the decentralized sensing architecture is developed.

III. DECENTRALIZED ASYNCHRONOUS EVENT-TRIGGERING

In this section, the main assumptions are introduced and the event-triggers for the decentralized asynchronous sensing problem are developed.

(A1) The closed loop system (1)-(2) is Input-to-State Stable (ISS) with respect to measurement error \( x_e \). That is, there exists a smooth function \( V : \mathbb{R}^n \rightarrow \mathbb{R} \) as well as class \( \mathcal{K}_\infty \) functions\(^1\) \( \alpha_1, \alpha_2, \alpha \) and \( \gamma_i \) for each \( i \in \{1, \ldots, n\} \), such that
\[ \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \]
\[ \frac{\partial V}{\partial x} f(x, k(x + x_e)) \leq -\alpha(|x|), \text{ if } \gamma_i(|x_{i,e}|) \leq |x|, \forall i. \]  
(A2) The functions \( f, k \) and \( \gamma_i \), for each \( i \in \{1, \ldots, n\} \), are Lipschitz on compact sets.

Note that the standard ISS assumption involves a single condition \( \gamma(|x_e|) \leq |x| \) instead of the \( n \) conditions: \( \gamma_i(|x_{i,e}|) \leq |x| \), for \( i \in \{1, \ldots, n\} \), in (A1). Given a function \( \gamma(\cdot) \) in the standard ISS assumption, one may define \( \gamma_i(\cdot) \) as
\[ \gamma_i(|x_{i,e}|) = \gamma\left(\frac{|x_{i,e}|}{\theta_i}\right), \quad i \in \{1, \ldots, n\} \]
where \( \theta_i \in (0, 1) \) such that \( \theta^2 = \sum_{i=1}^{n} \theta_i^2 \leq 1 \). Then, the \( n \) conditions in (A1) are equivalent to \( |x_{i,e}| \leq \theta_i \gamma^{-1}(|x|) \). Thus,
\[ |x_e| = \sum_{i=1}^{n} |x_{i,e}|^2 \leq \sum_{i=1}^{n} \theta_i^2 \gamma^{-1}(|x|) \leq \gamma^{-1}(|x|) \]
which is the condition in the standard ISS assumption. Similarly, given (A1) one may pick \( \gamma(\cdot) = \gamma_i(\cdot) \) for any \( i \) to get

\(^1\)A continuous function \( \alpha : [0, \infty) \rightarrow [0, \infty) \) is said to belong to the class \( \mathcal{K}_\infty \) if it is strictly increasing, \( \alpha(0) = 0 \) and \( \alpha(r) \rightarrow \infty \) as \( r \rightarrow \infty \) [19].
the standard ISS assumption, although in practice it may be possible to choose a less conservative $\gamma(.)$.

In this section, our aim is to constructively show that decentralized asynchronous event-triggering can be used to asymptotically stabilize $x = 0$ (the trivial solution or the origin) with a desired region of attraction while also guaranteeing positive minimum inter-sample times. Further, without loss of generality, the desired region of attraction may be assumed to be a compact sub-level set $S(c)$ of the Lyapunov-like function $V$ in (A1). Specifically, $S(c)$ is defined as

$$S(c) = \{ x \in \mathbb{R}^n : V(x) \leq c \}$$  \hspace{1cm} (5)

A. Centralized Asynchronous Event-Triggering

The proposed design of decentralized asynchronous event-triggering progresses in stages. In the first stage, centralized event-triggers for asynchronous sampling of the sensors are proposed in the following lemma. One of the key steps in the result is choosing linear bounds on the functions $\gamma_i(.)$ on appropriately defined sets $E_i$. Given that $x \in S(c)$, we define the sets $E_i$ over which the error bounds in (A1) are still satisfied, that is,

$$E_i(c) = \{ x_{i,e} \in \mathbb{R} : |x_{i,e}| \leq \gamma_i^{-1}(|x|), \ x \in S(c) \} = \{ x_{i,e} \in \mathbb{R} : |x_{i,e}| \leq \max_{x \in S(c)} \gamma_i^{-1}(|x|) \}$$  \hspace{1cm} (6)

Then, by (A2), for each $c > 0$ and each $i \in \{1, \ldots, n\}$, there exist positive constants $M_i(c)$ such that

$$\gamma_i(|x_{i,e}|) \leq \frac{1}{M_i(c)} |x_{i,e}|, \ \forall x_{i,e} \in E_i(c)$$  \hspace{1cm} (7)

Lemma 1. Consider the closed loop system (1)-(2) and assume (A1) and (A2) hold. Suppose for each $i \in \{1, \ldots, n\}$, the sampling instants, $\{\tau_i^c\}$ ensure $|x_{i,e}| \leq M_i(c)|x|$ for all time $t \geq 0$, where $M_i(c)$ are given by (7) and $c \geq 0$ is an arbitrary constant. Then, the origin is asymptotically stable with $S(c)$, given by (5), as the region of attraction.

Proof: Suppose $x(0) \in S(c)$ is an arbitrary point, we have to show that the trajectory $x(.)$ asymptotically converges to zero. Next, by assumption, the sampling instants and $\{\tau_i^c\}$ are such that for each $i \in \{1, \ldots, n\}$, $\frac{|x_{i,e}|}{M_i(c)} \leq |x|$ for all time $t \geq 0$. Then, for all time $t \geq 0$, (7) implies

$$\gamma_i(|x_{i,e}|) \leq \frac{1}{M_i(c)} |x_{i,e}| \leq |x|, \ \forall x \in S(c)$$

Consider the ISS Lyapunov function $V(.)$ in (A1), which is a function of the state $x$. Letting $E(c) \triangleq E_1(c) \times E_2(c) \times \ldots \times E_n(c)$, the time derivative of the function $V$ along the flow of the closed loop system, with a restricted domain, $V(x, x_e) : S(c) \times E(c) \rightarrow \mathbb{R}$ can be upper-bounded as

$$\dot{V}(x, x_e) \leq -\alpha(|x|), \ \forall x \in S(c), \ \forall x_e \in E(c)$$

Thus, the flow of the closed loop system is dissipative on the sub-level set, $S(c)$, of the Lyapunov function $V$. Therefore, the origin is asymptotically stable with $S(c)$ as the region of attraction. 

The lemma does not mention a specific choice of event-triggers but rather a family of them - all those that ensure the conditions $|x_{i,e}| \leq M_i(c)|x|$ are satisfied. Thus, any decentralized event-triggers in this family automatically guarantee asymptotic stability with the desired region of attraction. To enforce the conditions $|x_{i,e}| \leq M_i(c)|x|$ strictly, event-triggers at each sensor would need to know $|x|$, which is possible only if we have centralized information. One obvious way to decentralize these conditions is to enforce

$$|x_{i,e}| \leq M_i(c)|x_i|.$$ 

However, such event-triggers cannot guarantee any positive lower bound for the inter-transmission times, which is not acceptable. So, we take an alternative approach, in which the next step is to derive lower bounds for the inter-transmission times when the conditions in Lemma 1 are enforced strictly.

Before analyzing the lower bounds for the inter-transmission times that emerge from the event-triggers in Lemma 1, we introduce some notation. Noting that for each $c \geq 0$ the set $S(c)$ contains the origin, Assumption (A2) implies that there exist Lipschitz constants $L(c)$ and $D(c)$ such that

$$|f(x, k(x + x_e))| \leq L(c)|x| + D(c)|x_e|$$  \hspace{1cm} (8)

for all $x \in S(c)$ and for all $x_e$ satisfying $|x_{i,e}|/|x| \leq M_i(c)$, for each $i$. Similarly, there exist constants $L_i(c)$ and $D_i(c)$ for $i \in \{1, 2, \ldots, n\}$ such that

$$|f_i(x, k(x + x_e))| \leq L_i(c)|x| + D_i(c)|x_e|$$  \hspace{1cm} (9)

for all $x \in S(c)$ and for all $x_e$ satisfying $|x_{i,e}|/|x| \leq M_i(c)$, for each $i$. Lastly, we introduce a function $\tau$ defined as

$$\tau(w, a_0, a_1, a_2) = \{ t \geq 0 : \phi(t, 0) = w \}$$  \hspace{1cm} (10)

where $a_0$, $a_1$, $a_2$ are non-negative constants and $\phi(t, 0)$ is the solution of

$$\dot{\phi} = a_0 + a_1 \phi + a_2 \phi^2, \ \phi(0, 0) = \phi_0$$

Lemma 2. Consider the closed loop system (1)-(2) and assume (A2) holds. Let $c > 0$ be any arbitrary known constant. For $i \in \{1, \ldots, n\}$, let $0 \leq w_i \leq M_i(c)$ be any arbitrary constants and let $W_i = \sqrt{\sum_{j=1}^{n} w_j^2 - w_i^2}$. Suppose the sampling instants are such that $|x_{i,e}|/|x| \leq w_i$ for each $i \in \{1, \ldots, n\}$ for all time $t \geq t_0$. Finally, assume that for all $t \geq t_0$, $x$ belongs to the compact set $S(c)$. Then, for all $t \geq t_0$, the time required for $|x_{i,e}|/|x|$ to evolve from 0 to $w_i$ is lower bounded by $T_i > 0$, given by

$$T_i = \tau(w_i, a_{0,i}, a_{1,i}, a_{2,i})$$  \hspace{1cm} (11)

where the function $\tau$ is given by (10) and

$$a_{0,i} = L_i(c) + D_i(c)W_i, \ a_{1,i} = L(c) + D_i(c) + D(c)W_i, \ a_{2,i} = D(c)$$

Proof: By assumption, for all $t \geq t_0$, $x$ belongs to a known compact set $S(c)$ and $|x_{i,e}|/|x| \leq w_i \leq M_i(c)$ for each $i$. Thus, (8) and (9) hold for all $t \geq t_0$. Now, letting
\[\nu_i \triangleq \frac{|x_{i,e}|}{|x|}\] and by direct calculation we see that for \(i \in \{1, \ldots, n\}\)

\[
\frac{d\nu_i}{dt} = \left(\frac{T^x_i x_{i,e}}{|x|}\right)^{1/2} x^T_i \dot{x}_{i,e} - \frac{T^x_i x_{i,e}}{|x|^3} \leq \frac{|x_{i,e}| |\dot{x}_{i,e}|}{|x|} + \frac{|x| |\dot{x}| |x_{i,e}|}{|x|^3}
\]

\[
\leq \frac{L_i(c) |x| + D_i(c) |x_{i,e}| + (L(c) |x| + D(c) x_{i,e}) |x_{i,e}|}{|x|^2}
\]

where for \(x_{i,e} = 0\) the relation holds for all directional derivatives. Next, notice that

\[
\frac{|x|}{|x|} = \sqrt{\sum_{j=1}^{j=n} \nu_j^2} \leq \sqrt{\left(\sum_{j=1}^{j=n} w_j^2\right)} - w_i^2 + \nu_i \leq W_i + \nu_i
\]

where the condition that \(\nu_i \leq w_i\), the definition of \(W_i\) and the triangle inequality property have been utilized. Thus,

\[
\frac{d\nu_i}{dt} \leq L_i(c) + L(c)\nu_i + (D_i(c) + D(c)\nu_i) (W_i + \nu_i)
\]

\[
= a_{0,i} + a_1\nu_i + a_2\nu_i^2
\]

The claim of the Lemma now directly follows.

Now, by combining Lemmas 1 and 2, we get the following result for the centralized asynchronous event-triggering.

**Theorem 1.** Consider the closed loop system (1)-(2) and assume (A1)-(A2) hold. Suppose the \(i\)th sensor transmits its measurement to the controller whenever \(|x_{i,e}|/|x| \geq w_i\), where \(0 < w_i \leq M_i(c)\), with \(M_i(c)\) given by (7) and \(c \geq 0\) any arbitrary constant. Then, the origin is asymptotically stable with \(S(c)\) as the region of attraction and the inter-transmission times of each sensor have a positive lower bound given by \(T_i\) in (11).

**Proof:** The triggering conditions ensure that \(|x_{i,e}|/|x| \leq w_i \leq M_i(c)\) for all \(t > 0\). Thus, Lemma 1 guarantees \(x \in S(c)\) for all \(t \geq 0\) and that the origin is asymptotically stable with \(S(c)\) included in the region of attraction. Since \(S(c)\) is positively invariant, Lemma 2 guarantees a positive lower bound for the inter-transmission times.

**B. Decentralized Asynchronous Event-Triggering**

Now, turning to the main subject of this paper, in the decentralized sensing case, unlike in the centralized sensing case, no single sensor knows the exact value of \(|x|\) from the locally sensed data. We may let the event-trigger at the \(i\)th sensor enforce the more conservative condition \(|x_{i,e}|/|x| \leq w_i\) and still satisfy the assumptions of Lemma 1, though such a choice cannot guarantee a positive minimum inter-sample time. At this stage, it might seem that Lemma 2 cannot be used to design an implicitly verified event-triggering in the decentralized sensing case. However, the lemma can be interpreted in an alternative way, which would aid in our design goal.

Rather than providing a minimum inter-sampling time for an event-triggering mechanism, Lemma 2 can be interpreted as providing a minimum time threshold only after which it is necessary to check a data based event-triggering condition. For example, the event-triggers in Theorem 1,

\[
t_x^{i+1} = \min\{t \geq t_x^{i} : |x_{i,e}|/|x| \geq w_i\}, \ i \in \{1, \ldots, n\}
\]

can be equivalently expressed as

\[
t_x^{i+1} = \min\{t \geq t_x^{i} + T_i : |x_{i,e}|/|x| \geq w_i\}
\]

where \(T_i\) are the estimates of positive inter-sample times provided by Lemma 2 in (11). In the latter interpretation, a minimum threshold for inter-sample times is explicitly enforced, only after which, the state based condition is checked. Now, in order to let the event-triggers depend only on locally sensed data, one can let the sampling times, for \(i \in \{1, \ldots, n\}\), be determined as

\[
t_x^{i+1} = \min\{t \geq t_x^{i} + T_i : |x_{i,e}| \geq w_i |x_{i}|\}
\]

where \(T_i\) are given by (11). This allows us to implement decentralized asynchronous event-triggering. The following theorem is the core result of this paper and it shows that by appropriately choosing the constants \(T_i\) and \(w_i\), the event triggers, (14), guarantee asymptotic stability of the origin while also explicitly enforcing a positive minimum inter-sample time.

**Theorem 2.** Consider the closed loop system (1)-(2) and assume (A1) and (A2) hold. Let \(c \geq 0\) be an arbitrary known constant. For each \(i \in \{1, 2, \ldots, n\}\), let \(w_i\) be a positive constant such that \(w_i \leq M_i(c)\), where \(M_i(c)\) given by (7) and \(T_i\) be given by (11). Suppose the sensors asynchronously transmit the measured data at time instants determined by (14) and that \(t_x^{i} \leq 0\) for each \(i \in \{1, 2, \ldots, n\}\). Then, the origin is asymptotically stable with \(S(c)\) as the region of attraction and the inter-transmission times of each sensor are explicitly enforced to have a positive lower threshold.

**Proof:** The statement about the positive lower threshold for inter-transmission times is obvious from (14) and only asymptotic stability remains to be proven. This can be done by showing that the event-triggers (14) are included in the family of event-triggers considered in Lemma 1. From the equivalence of (12) and (13), it is clearly true that \(|x_{i,e}|/|x| \leq w_i\) for \(t \in [t_x^{i} + T_i, t_x^{i} + T_i + 1]\), for each \(i \in \{1, 2, \ldots, n\}\) and each \(j\). Next, for \(t \in [t_x^{i} + T_i, t_x^{i} + T_i + 1]\), (14) enforces \(|x_{i,e}|/|x_{i}| \leq w_i\), which implies \(|x_{i,e}|/|x| \leq w_i |x_{i}|\). Therefore, the event-triggers in (14) are included in the family of event triggers considered in Lemma 1. Hence, \(x \equiv 0\) (the origin) is asymptotically stable with \(S(c)\) as the region of attraction.

**Remark 1.** Although the assumption that \(t_x^{i} \leq 0\), for each \(i\), in Theorem 2 has not been used in the proof explicitly, it serves two key purposes - avoiding having the sensors send their first transmissions of data synchronously; and for the controller to have some latest sensor data to compute the controller output at \(t = 0\).

**Remark 2.** In Theorem 2, the parameters \(w_i\) cannot be chosen in a decentralized manner unless \(M_i(c)\) and hence \(c\) is fixed a priori. In other words, the desired region of attraction
C. Decentralized Asynchronous Event-Triggering with Intermittent Communication from the Central Controller

Apart from the fact that the set \( S(c) \) is chosen \textit{a priori}, conservativeness in transmission frequency may also be introduced because the Lipschitz constants of the nonlinear functions \( \gamma_i(\cdot), (7) \), are not updated after their initialization despite knowing that the system state is progressively restricted to smaller and smaller subsets of \( S(c) \). Although we started from the idea that energy may be saved by making sure that sensors do not have to listen, the cost of increased transmissions may not be in its favor. Thus, we now describe a design where the central controller intermittently communicates updated \( w_i \) and \( T_i \) to the event-triggers.

The first step in this design process is to characterize the region in which the system state lies, from the asynchronously transmitted data, \( x_s \), that is available at the central controller. Since the central controller knows the parameters used by each event-trigger, it may compute an estimate of \( |x| \) based on the centralized asynchronous event-triggering of Theorem 1, of which (14) is an under-approximation. Thus, we have that

\[
|x_{i,s} - x_i| = |x_{i,c}| \leq w_i|x|, \quad \forall i \in \{1, \ldots, n\}
\]

from which we obtain

\[
\sum_{i=1}^{n} |x_{i,s} - x_i|^2 \leq W^2 \sum_{i=1}^{n} |x_i|^2, \quad \text{where } W = \sqrt{\sum_{i=1}^{n} w_i^2}
\]

\[
\implies (1 - W^2) \sum_{i=1}^{n} |x_i|^2 - 2 \sum_{i=1}^{n} |x_{i,s}| |x_i| + \sum_{i=1}^{n} |x_{i,s}|^2 \leq 0
\]

which is the equation of an \( n \)-sphere. Thus, the system state is in the \( n \)-sphere given by

\[
|x - x_c| \leq R
\]

where \( x_c = \frac{1}{1 - W^2} x_s, \quad R = \frac{W}{1 - W^2} |x_s| \) (16)

Obviously, for these equations to make sense, \( W^2 \) has to be strictly less than 1. However, this is not a restriction at all. Notice that, by definition, a centralized event-trigger that enforces \( |x_c| = |x - x_s| \leq W|x| \) asymptotically stabilizes the origin of the system with the required convergence rate. Further, if \( W \geq 1 \) then \( |x - 0| \leq W|x| \) for all \( x \in \mathbb{R}^n \). The implication is that the constant control \( u = k(0) \) is sufficient to asymptotically stabilize the origin with required convergence rate. In that case, there is no need for event-triggered control. Thus, without loss of generality, we assume that \( W < 1 \).

The next idea is to estimate an upper bound on the value of \( V(x) \). From (15), we know that \( |x| \leq |x_c| + R \) and hence that \( V(x) \leq \alpha_2(|x_c| + R) \). However, this may be conservative and a better estimate may be obtained by maximizing \( V(x) \) on the set given by (15). In fact, on this set, \( V(x) \) is maximized on the boundary of the \( n \)-sphere. This is because if the maximum does not occur on the boundary and instead occurs only in the interior of the \( n \)-sphere (15), then the maximizing sub-level set, \( S_M \), of \( V \) lies strictly and completely in the interior of the \( n \)-sphere, which means \( S_M \) is not the smallest sub-level set of \( V \) that contains the complete \( n \)-sphere. Thus, an upper bound on the value of \( V(x) \) is provided by

\[
V \geq \max \{ \forall V(x) : |x - x_c| = R \} \tag{17}
\]

The final idea is to update sensor event-trigger parameters \( w_i \) and \( T_i \) at time instants determined by an event-trigger running at the central controller, namely,

\[
t_{j+1}^V = \min \{ t \geq t_j^V + T : V \leq \rho V(t_j^V) \}
\]

where \( T > 0 \) and \( 0 < \rho < 1 \) are arbitrary constants. To be precise, \( t_{j+1}^V \) are the time instants at which \( V \) is updated. In this paper, we assume that these are also the time instants at which new values of \( w_i \) and \( T_i \) are communicated to the sensors as well as updated by the sensors in (14). The initial condition \( V(t_0^V) = V(0) = c \) may be chosen, where \( c \) determines the region of attraction \( S(c) \). Thus the ‘sampled’ version of \( V \) is denoted by

\[
V_s \triangleq \forall V(t_j^V), \quad \forall t \in [t_j^V, t_{j+1}^V), \quad V_s(t_j^V) = V_s(0) = c \tag{19}
\]

where \( c > 0 \) is an arbitrary constant, \( t_j^V \) are given by (18) and \( V \) is given by (17). Now, the ideas in this subsection are formalized in the following result.

Theorem 3. Consider the closed loop system (1)-(2) and assume (A1) and (A2) hold. Let \( M_i(.) \) and \( V_s \) be given by (7) and (19), respectively. For each \( i \in \{1, 2, \ldots, n\} \), let \( w_i \) and \( T_i \) be positive piecewise-constant signals given by \( w_i = M_i(V_s) \) and (11) (with \( c = V_s \)), respectively. Suppose the sensors asynchronously transmit the measured data at time instants determined by (14) and that \( t_i^s \leq 0 \) for each \( i \in \{1, 2, \ldots, n\} \). Then, the origin is asymptotically stable with \( S(c) \) as the region of attraction and the inter-execution times of each event-trigger have a positive lower bound.

Proof: Clearly, the Lyapunov function evaluated at the state of the system is at all times lesser than the piecewise constant and non-increasing signal \( V_s \). Thus, \( x \in S(V_s) \) at all times, where \( S(.) \) is given by (5). Hence, \( w_i = M_i(V_s) \) and \( T_i \) given by (11) guarantee asymptotic stability of the origin of the closed loop system, with \( S(V_s(0)) \) as the region of attraction.

The inter-transmission times \( t_j^V \) are clearly lower bounded by \( T > 0 \). Note that given \( V_s \), the different parameters in Lemma 2 are clearly determined, as is \( T_i \) in (14). Thus, the inter-transmission times of the \( i^{th} \) sensor in the interval \( [t_j^V, t_{j+1}^V) \) are lower bounded by \( T_i \) calculated with \( V(t_j^V) \), which are guaranteed to be positive by Lemma 2. The different parameters in Lemma 2 are upper and lower bounded by positive constants determined by \( V_s(0) \). Thus, \( T_i \) for all time have positive lower bounds \( \Gamma_i \). Each inter-transmission time of the \( i^{th} \) sensor is thus lower bounded by \( \Gamma_i > 0 \).
Remark 3. As \( S(c_1) \subseteq S(c_2) \) if \( c_1 \leq c_2 \). \( M_i(\cdot) \) in (7) can be assumed to be non-increasing functions of \( c \). Since the signal \( V_s \) is non-increasing, \( w_i = M_i(\cdot) \) are non-decreasing in time. Further, note that the aim of the event-triggers (14) is to enforce the conditions \( |x_{i,e}| \leq w_i|x| \). Thus whenever \( w_i \) and \( T_i \) are updated, the new parameters in the event-triggers are consistent with and an improvement over the previous parameters. Although \( w_i \) are non-decreasing in time, the same cannot be said about \( T_i \). However, it is not a restriction and the inter-transmission times are still lower bounded.

Remark 4. Computing the upper bound on \( V \), (17), may be computationally intensive depending on the Lyapunov function and the dimension of the system. However, since the Lyapunov function is guaranteed to decrease even with no updates to \( w_i \) and \( T_i \), there is no restriction on the time needed to compute the upper bound on \( V \) and to update the parameters of the event-triggers. On the other hand, it is true that the updates to all the event-triggers have to occur synchronously.

IV. LINEAR TIME INVARIANT SYSTEMS

Now, let us consider the special case of Linear Time Invariant (LTI) systems with quadratic Lyapunov functions. Thus, the system dynamics may be written as

\[
\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m
\]

where \( A, B \) and \( K \) are matrices of appropriate dimensions. As in the general case, let us assume that for each \( i \in \{1, 2, \ldots, n\} \), \( x_i \) is sensed by the \( i \)-th sensor. Comparing with (20)-(21) we see that \( x_i \) evolves as

\[
\dot{x}_i = r_i(A)x + r_i(BK)(x + x_e)
\]

where the notation \( r_i(H) \) denotes the \( i \)-th row of the matrix \( H \). Also note that \( x_e \) and \( x_{i,e} \) are defined just as in Section II.

Now, suppose the matrix \( (A + BK) \) is Hurwitz, which is equivalent to the following statement.

(A3) Suppose that for any given symmetric positive definite matrix \( Q \), there exists a symmetric positive definite matrix \( P \) such that \( P(A + BK) + (A + BK)^TP = -Q \).

Then, the following Lemma describes a centralized asynchronous sensing mechanism for linear systems.

Lemma 3. Consider the closed loop system (20)-(21) and assume (A3) holds. Let \( Q \) be any symmetric positive definite matrix and let \( Q_m \) be the smallest eigenvalue of \( Q \). For each \( i \in \{1, 2, \ldots, n\} \), let

\[
\begin{align*}
\theta_i & \in (0, 1) \quad \text{s.t.} \quad \theta = \sum_{i=1}^n \theta_i \leq 1 \\
\sigma \theta_i Q_m & = \frac{c_i(2PBK)}{|c_i(2PBK)|}
\end{align*}
\]

where \( 0 < \sigma < 1 \) is a constant and \( c_i(2PBK) \) is the \( i \)-th column of the matrix \( (2PBK) \). Suppose the sampling instants are such that for each \( i \in \{1, 2, \ldots, n\} \), \( |x_{i,e}|/|x| \leq w_i \) for all time \( t \geq 0 \). Then, the origin is globally asymptotically stable.

Proof: Consider the candidate Lyapunov function \( V(x) = x^TPx \) where \( P \) satisfies (A3). The derivative of the function \( V \) along the flow of the closed loop system satisfies

\[
\dot{V} = x^T[P(A + BK) + (A + BK)^TP]x + 2x^TPBKx_{c_e}
\]

\[
\leq -(1 - \sigma)x^TQx + x\left[\sum_{i=1}^n |c_i(2PBK)x_{i,e}| - \sigma Q_m|x|ight]
\]

\[
\leq -(1 - \sigma)x^TQx + x\left[\sum_{i=1}^n |c_i(2PBK)||x_{i,e}| - \sigma Q_m|x|ight]
\]

The sensor update instants have been assumed to be such that \( |x_{i,e}|/|x| \leq w_i = \frac{\sigma \theta_i Q_m}{|c_i(2PBK)|} \) for each \( i \) and for all time \( t \geq 0 \). Thus,

\[
\dot{V} \leq -(1 - \sigma)x^TQx
\]

which implies that the origin is globally asymptotically stable.

Upper bounds for the inter-sample times can be found in a manner analogous to the general nonlinear case in Lemma 2.

Lemma 4. Consider the closed loop system (20)-(21). For each \( i \in \{1, 2, \ldots, n\} \), let \( \theta_i, w_i \) be defined as in (23)-(24) and let \( W_i = \left( \sum_{j=1}^n w_j^2 \right)^{1/2} - w_i^2 \). Suppose the sampling instants are such that \( |x_{i,e}|/|x| \leq w_i \) for each \( i \in \{1, 2, \ldots, n\} \) for all time \( t \geq 0 \). Then, for all \( t \geq 0 \), the time required for \( |x_{i,e}|/|x| \) to evolve from 0 to \( w_i \) is lower bounded by \( T_i > 0 \), where

\[
T_i = \tau(w_i, a_0, a_1, a_2)
\]

where the function \( \tau \) is given by (10) and

\[
\begin{align*}
a_0 & = r_i(A + BK) + r_i(BK)|W_i| \\
a_1 & = |A + BK| + |r_i(BK)| + |BK||W_i| \quad a_2 = |BK|
\end{align*}
\]

Proof: Letting \( \nu_i \triangleq |x_{i,e}|/|x| \), for \( i \in \{1, 2, \ldots, n\} \), the upper bound for the time derivative of \( \nu_i \) can be found by direct calculation.

\[
\begin{align*}
\frac{d\nu_i}{dt} & = \frac{-x_i^T|x_{i,e}|^{-1/2}x_{i,e}}{|x|^3}x_i^T|x_{i,e}| - \frac{x_i^T|x_{i,e}|}{|x|^3} \\
& \leq \frac{|x_{i,e}|}{|x|}\left[\frac{|x_{i,e}|}{|x|}\right] - \frac{|x_{i,e}|}{|x|}\left[\frac{|x_{i,e}|}{|x|}\right] \\
& \leq \frac{|x_i(A + BK)|}{|x|} + \frac{|r_i(BK)|}{|x|} \\
& + \frac{(|A + BK||x| + |BK||x|)}{|x|}|x_{i,e}|
\end{align*}
\]

where for \( x_{i,e} = 0 \) the relation holds for all directional derivatives while the notation \( r_i(H) \) denotes the \( i \)-th row of the matrix \( H \). Next, notice that

\[
\begin{align*}
\frac{|x_{i,e}|}{|x|} & \leq \sum_{j=1}^{j=n} \nu_j^2 \leq \left(\sum_{j=1}^{j=n} w_j^2\right) - w_i^2 + \nu_i^2 \leq W_i + \nu_i
\end{align*}
\]
where the condition that \( \nu_i \leq w_i \), the definition of \( W_i \) and the triangle inequality property have been utilized. Thus,

\[
\frac{d\nu_i}{dt} \leq |r_i(A + BK)| + |A + BK|\nu_i + \left(|r_i(BK)| + |BK|\nu_i\right)(W_i + \nu_i) = a_0 + a_1\nu_i + a_2\nu_i^2
\]

The claim of the Lemma now directly follows.

Next, the result for the centralized asynchronous event-triggering is presented, whose proof is quite analogous to Theorem 1.

**Theorem 4.** Consider the closed loop system (20)-(21) and assume (A3) holds. Let \( Q \) be any symmetric positive definite matrix and let \( Q_m \) be the smallest eigenvalue of \( Q \). For each \( i \in \{1, 2, \ldots, n\} \), let \( \theta_i , w_i \) and \( T_i \) be defined as in (23), (24) and (25), respectively. Suppose the sensors asynchronously transmit the measured data at time instants determined by (14). Then, the origin is globally asymptotically stable and the inter-transmission times have a positive lower bound.

The following result is analogous to Theorem 2 and prescribes the constants \( T_i \) and \( w_i \) in the event triggers, (14), that guarantee global asymptotic stability of the origin while also explicitly enforcing a positive minimum inter-sample time.

**Theorem 5.** Consider the closed loop system (20)-(21) and assume (A3) holds. Let \( Q \) be any symmetric positive definite matrix and let \( Q_m \) be the smallest eigenvalue of \( Q \). For each \( i \in \{1, 2, \ldots, n\} \), let \( \theta_i , w_i \) and \( T_i \) be defined as in (23), (24) and (25), respectively. Suppose the sensors asynchronously transmit the measured data at time instants determined by (14). Then, the origin is globally asymptotically stable and the inter-transmission times are explicitly enforced to have a positive lower threshold.

In the context of the results for nonlinear systems in Section III, the reason we are able to achieve global asymptotic stability for LTI systems is because, the system dynamics, the functions \( \gamma_i(\cdot) \) are globally Lipschitz, thus giving us constants \( w_i \) and \( T_i \) that hold globally. In fact, for linear systems, something more is ensured - the proposed asynchronous event-triggers guarantee a type of scale invariance.

Scaling laws of inter-execution times for centralized synchronous event-triggering have been studied in [20]. In particular, Theorem 4.3 of [20], in the special case of linear systems, guarantees scale invariance of the inter-execution times determined by a centralized event-trigger \( |x_{r,i}| = |W_i| |x| \). The centralized and decentralized asynchronous event-triggers developed in this paper are under-approximations of this kind of central event-triggering. In the following, we show that the scale invariance is preserved in the asynchronous event-triggers. As an aside, we would like to point out that the decentralized event-triggers proposed in [8]-[10] are not scale invariant. In order to precisely state the notion of scale invariance and to state the result the following notation is useful. Let \( x(t) \) and \( z(t) \) be two solutions to the system: (20)-(21) along with the event-triggers (14).

**Theorem 6.** Consider the closed loop system (20)-(21) and assume (A3) holds. Let \( Q \) be any symmetric positive definite matrix and let \( Q_m \) be the smallest eigenvalue of \( Q \). For each \( i \in \{1, 2, \ldots, n\} \), let \( \theta_i , w_i \) and \( T_i \) be defined as in (23), (24) and (25), respectively. Suppose the sensors asynchronously transmit the measured data at time instants determined by (14). Assuming \( b \) is any scalar constant, let \( [z(0)^T, z_s(0)^T]^T = b[x(0)^T, x_s(0)^T]^T \in \mathbb{R}^n \times \mathbb{R}^n \) be two initial conditions for the system. Further let \( t^{z_i}_0 < t^{x_i}_0 \) for each \( i \in \{1, 2, \ldots, n\} \). Then, \( [z(t)^T, z_s(t)^T]^T = b[x(t)^T, x_s(t)^T]^T \) for all \( t \geq 0 \) and \( t^{z_i}_0 = t^{x_i}_j \) for each \( i \) and \( j \).

**Proof:** First of all, let us introduce two strictly increasing sequences of time, \( \{t^{z_i}_j\} \) and \( \{t^{x_i}_j\} \), at which one or more components of \( z_s \) and \( x_s \) are updated, respectively. Further, without loss of generality, assume \( t^{z_i}_0 = t^{x_i}_0 \). The proof proceeds by mathematical induction. Let us suppose that \( t^{z_i}_j = t^{x_i}_j \) for each \( j \in \{0, \ldots, k\} \) and that \( [z(t)^T, z_s(t)^T]^T = b[x(t)^T, x_s(t)^T]^T \) for all \( t \in [0, t^k] \). Then, letting \( t^{z_i}_{k+1} = \min\{t^{z_i}_{k+1}, t^{x_i}_{k+1}\} \) the solution, \( z \), in the time interval \( [t_k, t_{k+1}] \) satisfies

\[
z(t) = e^{A(t-t_k)}z(t_k) + \int_{t_k}^t e^{A(t-\sigma)}BKz_s(\sigma)d\sigma + bw^{A(t-t_k)}x(t_k) + b \int_{t_k}^t e^{A(t-\sigma)}BKx_s(\sigma)d\sigma
\]

Hence,

\[
z(t) = bx(t), \quad \forall t \in [t_k, t_{k+1}]
\]

Further, in the time interval \( [t_k, t_{k+1}] \)

\[
z_{i,c}(t) = z_i(t) - z_i(t) = b(x_i(t) - x_i(t)) = bx_{i,c}(t)
\]

Similarly, for all \( t \in [t_k, t_{k+1}] \),

\[
|z_{i,c}(t)| \leq |x_{i,c}(t)|
\]

Without loss of generality, assume \( z_{i,s} \) is updated at \( t_{k+1} \). Then, clearly, at least \( T_i \) amount of time elapsed since \( z_{i,s} \) was last updated. Next, by the assumption that \( t^{z_i}_0 = t^{x_i}_0 < 0 \) and the induction statement, it is clear that at least \( T_i \) amount of time has elapsed since \( x_{i,s} \) also was last updated. Further, it also means that \( |z_{i,s}(t_k) - z_{i,s}(t_{k+1})| \geq w_i |z_{i,s}(t_{k+1})| \). Then, (26)-(27) imply that \( |z_{i,s}(t_k) - x_{i,s}(t_{k+1})| \geq w_i |x_{i,s}(t_{k+1})| \), meaning \( t_{k+1} = t^{z_i}_{k+1} = t^{x_i}_{k+1} = t_{k+1} \). Arguments analogous to the preceding also hold for multiple \( z_{i,s} \) updated at \( t_{k+1} \) instead of one or even \( x_{i,s} \) instead of \( z_{i,s} \). Since the induction statement is true for \( k = 0 \), we conclude that the statement of theorem is true.

**Remark 5.** From the proof of Theorem 6, (28) specifically, it is clear that the centralized asynchronous event-triggers of Theorem 4 also guarantee scale invariance.

**Remark 6.** Scale invariance, as described in Theorem 6, means that the average inter-transmission times over an arbitrary length of time is independent of the scale (or the magnitude) of the initial condition of the system. Similarly for any given scalar, \( 0 < \delta < 1 \), the time and the number of transmissions it takes for \( |x(t)| \) to reduce to \( \delta|x(0)| \) is
independent of $|x(0)|$. So, the advantage is that the ‘average’ network usage remains the same over large portions of the state space.

V. Simulation Results

In this section, the proposed decentralized asynchronous event-triggered sensing mechanism is illustrated with two examples. The first is a linear system and the second a nonlinear system.

A. Linear System Example

We first present the mechanism for a linearized model of a batch reactor, [21]. The plant and the controller are given by (20)-(21) with

$$A = \begin{bmatrix} 1.38 & -0.20 & 6.71 & -5.67 \\ -0.58 & -4.29 & 0 & -0.67 \\ 1.06 & 4.27 & -6.65 & 5.89 \\ 0.04 & 4.27 & 1.34 & -2.10 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 5.67 \\ 1.13 -3.14 \\ 1.13 0 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.1006 & -0.2469 & 0 & -0.952 & -0.2447 \\ 1.4099 & -0.1966 & 0.0139 & 0.0823 \end{bmatrix}$$

which places the eigenvalues of the matrix $(A+BK)$ at around $\{-2.98 + 1.19i, -2.98 - 1.19i, -3.89, -3.62\}$. The matrix $Q$ was chosen as the identity matrix. The system matrices and $Q$ have been chosen to be the same as in [8]. Lastly, the controller parameters were chosen as $[\theta_1, \theta_2, \theta_3, \theta_4] = [0.6, 0.17, 0.08, 0.15]$ and $\sigma = 0.95$. For the simulations presented here, the initial condition of the plant was selected as $x(0) = [4, 7, -4, 3]^T$ and the initial sampled data that the controller used was $x_s(0) = [4.1, 7.2, -4.5, 2]^T$. The zeroth sampling instant was chosen as $t_0^x = -T_i$ for sensor $i$. This is to ensure sampling at $t = 0$ if the local triggering condition was satisfied. Finally the simulation time was chosen as 10s.

Figures 1a and 1b show the evolution of the Lyapunov function and its derivative along the flow of the closed loop system, respectively. Figures 1c and 1d show the inter-transmission times and the cumulative frequency distribution of the inter-transmission times for each of the sensor. The cumulative frequency distribution of the inter-transmission times is a measure of the performance of the event-triggers. A distribution that rises sharply to 100% indicates that event-trigger is not much better than a time-trigger. Thus, slower the rise of the cumulative distribution curves, greater is the justification for using the event-trigger instead of a time-trigger.

The minimum thresholds for the inter-transmission times $T_i$ for the example can be computed as in Lemma 4 and have been obtained as $[T_1, T_2, T_3, T_4] = [11, 15.4, 12.6, 19.9]$ms, which are also the minimum inter-transmission times in the simulations presented here. These numbers are a few orders of magnitude higher and an order higher than the guaranteed minimum inter-transmission times and the observed minimum inter-transmission times in [8], [9]. The average inter-transmission times obtained in the presented simulations were $[T_1, T_2, T_3, T_4] = [24.9, 27.7, 34.5, 34.2]$ms, which are about an order of magnitude lower than those reported in [8], [9]. A possible explanation for this phenomenon is that in [8], [9], the average inter-transmission times depends quite critically on the evolution of the threshold $\eta$. Although the controller gain matrix $K$ and the matrix $Q$ have been chosen to be the same, by inspection of the plots in [8], [9], it appears that the rate of decay of the Lyapunov function $V$ is roughly about half of that in our simulations. However, we would like to point out that our average inter-transmission times are of the same order as in [10] by the same authors. In any case, for LTI systems, our proposed method does not require communication from the controller to sensors to achieve global asymptotic stability. Lastly, as a measure of the usefulness of the event-triggering mechanism compared to a purely time-triggered mechanism, $T_i$/T was computed for each $i$ and were obtained as $[T_i/T_i T_2/T_2 T_3/T_3 T_4/T_4] = [0.44, 0.55, 0.36, 0.58]$. The lower these numbers are, the better it is.

$\eta \leq 0.5$

B. Nonlinear System Example

The general result for nonlinear systems is illustrated through simulations of the following second order nonlinear system.

$$\dot{x} = f(x, x_e) = \begin{bmatrix} f_1(x, x_e) \\ f_2(x, x_e) \end{bmatrix} = Ax + \begin{bmatrix} 0 \\ x_1 \end{bmatrix} + Bu \tag{29}$$

$$u = k(x + x_e) = K(x + x_e) - (x_1 + x_{1,e})^3 \tag{30}$$

where $x = [x_1, x_2]^T$ is a vector in $\mathbb{R}^2$ and the sampled data controller (in terms of the measurement error) is given as $u = k(x + x_e) = K(x + x_e) - (x_1 + x_{1,e})^3$ and $K = [k_1, k_2]$ is a $1 \times 2$ row vector such that $\bar{A} = (A + BK)$ is Hurwitz. Then, the closed-loop tracking error system with event-triggered control can be written as

$$\dot{x} = \dot{\bar{A}}x + BKx_1 + \begin{bmatrix} 0 \\ x_1^3 - (x_1 + x_{1,e})^3 \end{bmatrix}$$

$$= \bar{A}x + \begin{bmatrix} 0 \\ h_1 + h_2 \end{bmatrix} \tag{31}$$

where

$$h_1 = -\left(x_{1,e}^3 + 3x_{1,e}x_{1,e}^2 + (3x_1^2 - k_1)x_{1,e}\right) \tag{32}$$

$$h_2 = k_2x_{2,e} \tag{33}$$

Now, consider the quadratic Lyapunov function $V = x^TPx$ where $P$ is a symmetric positive definite matrix that satisfies the Lyapunov equation $PA + A^TP = -Q$, with $Q$ a symmetric positive definite matrix. Let $p_m$ and $p_M$ be the smallest and largest eigenvalues of the matrix $P$. Since $P$ is a symmetric positive definite matrix, $p_m$ and $p_M$ are each positive real numbers. Further, $\alpha_1(|x|) \equiv p_m|x|^2 \leq V(x) \leq p_M|x|^2 \equiv \alpha_2(|x|)$, $\forall x \in \mathbb{R}^2$.
The time derivative of $V$ along the flow of the closed loop system (31) can be shown to satisfy

$$\dot{V} = -x^T Q x + 2x^T PB(h_1 + h_2) \leq -(1 - \sigma)Q_m|x|^2 + |x|((2PB(h_1 + h_2)) - \sigma Q_m|x|)$$

where $Q_m$ is the smallest eigenvalue of the symmetric positive definite matrix $Q$ and $\sigma$ is a parameter satisfying $0 < \sigma < 1$.

Suppose that the desired region of attraction be $S(c)$, for some non-negative $c$ (see (5) for the definition of $S(c)$). Let $\mu_1$ be the maximum value of $x_1$ on the sub-level set $S(c)$. Then, we let

$$h_1^c = |x_{1,e}|^3 + 3\mu_1|x_{1,e}|^2 + \max_{|x_1| \leq \mu_1} \{3x_1^2 - k_1\}|x_{1,e}|$$

$$\gamma_1(|x_{1,e}|) \triangleq \frac{|2PB|h_1^c}{\sigma \theta_1 Q_m}, \quad \gamma_2(|x_{2,e}|) \triangleq \frac{|2PB_k2| |x_{2,e}|}{\sigma \theta_2 Q_m}$$

where $\theta_1$ and $\theta_2$ are positive constants such that $\theta_1 + \theta_2 = 1$.

It is clear that Assumption (A1) is satisfied and we have

$$\dot{V} \leq -(1 - \sigma)Q_m|x|^2, \quad \text{if } \gamma_i|x_{i,e}| \leq |x|, \quad i \in \{1, 2\}$$

Now, $\mu \triangleq \alpha_1^{-1}(c) = \sqrt{c/p_m}$ is the maximum value of $|x|$ on the set $S(c)$. Hence, $M_1(c)$ in (7) has to be defined for the set on which $|x_{1,e}| \leq R_1 \triangleq \gamma_1^{-1}(\mu)$. Thus, we have that

$$\frac{1}{M_1(c)} = \frac{|2PB|}{\sigma \theta_1 Q_m} \left( \mu^2 + 3\mu_1 \mu + \max_{|x_1| \leq \mu_1} \{3x_1^2 - k_1\} \right)$$

while

$$\frac{1}{M_2(c)} = \frac{|2PB_k2|}{\sigma \theta_2 Q_m}$$

Now, only $T_i$ for each $i$ need to be determined. To this end, the closed loop system dynamics (31) are bounded as in (8) and (9).

$$|f_1(x, x_e)| \leq L_1|x| + D_1|x_e|$$

$$|f_2(x, x_e)| \leq L_2|x| + D_{2,\mu}|x_e|, \quad \forall x \text{ s.t. } |x| \leq \mu$$

Comparing with (31) the following can be arrived at.

$$L_1 = |\gamma_1(\bar{A})|, \quad D_1 = 0, \quad L_2 = |\gamma_2(\bar{A})|$$

$$D_{2,\mu} = \left( \mu^2 + 3\mu_1 \mu + \max_{|x_1| \leq \mu_1} \{3x_1^2 - k_1\} \right)^2 + k_2^2$$

In the example simulation results presented here, the following gains and parameters were used.

$$K = \begin{bmatrix} 5 & 3 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \theta_1 = 0.9, \quad \theta_2 = 0.1$$

$$\sigma = 0.9, \quad c = 10, \quad \mu_1 = \mu$$

$$x(0) = [2.8, -2.6]^T, \quad x_s(0) = [2.9, -2.7]^T$$

Notice that $M_2(c)$ is a constant independent of $c$. That is why $\theta_2$ has been chosen much smaller than $\theta_1$. The parameter $\mu_1$ has been chosen to be equal to $\mu$. To be consistent with asynchronous transmissions, the initial value of $x_s(0)$ has been chosen to be different from $x(0)$.

For the chosen parameters and the initial conditions, the initial value of the Lyapunov function is $V(0) = 8.574$. Thus the initial state of the system is well within the region of attraction, given by $S(c) = S(10)$. The event-trigger parameters were obtained as $[w_1, w_2] = [0.0045, 0.0832]$ and $[T_1, T_2] = [4, 3.4]$ ms, which were also the minimum inter-transmission times. The average inter-transmission times of the sensors for the duration of the simulated time were obtained as $\bar{T}_1, \bar{T}_2 = [4.2, 26.2]$ ms. Thus for sensor 1, the average inter-transmission interval is only marginally better than the minimum. The number of transmissions by sensors 1 and 2 were 2366 and 382, respectively.

Figures 2a and 2b show the evolution of the Lyapunov function and its derivative along the flow of the closed loop system, respectively. Figures 2c and 2d show the inter-transmission times and the cumulative frequency distribution of the inter-transmission times for each of the sensor. The sharp rise of the cumulative distribution curve for Sensor 1 clearly indicates that the event-triggered transmission is nearly equivalent to time-triggered transmission. On the other hand, the slow rise of the cumulative distribution curve of Sensor 2 demonstrates the usefulness of event-triggering in its case.

Simulations were also performed for the case when the central controller intermittently sends updates to the parameters of the sensor event-triggers, as in Theorem 3. For the simulation results presented here, the controller gains, parameters and the initial conditions have been chosen the same as in (34). Additionally, the parameters in (18) were chosen as $T = 0.5$ and $\rho = 0.5$. The initial condition $V_s(0) = c = 10$ was chosen. For the 2 dimensional system in this example, $V$ in (17) is the...
maximum value of \( V \) along a circle. \( V \) was then found in MATLAB by maximization of \( V \) on the circle, which was parametrized by a single angle variable varying on the closed interval \([0, 2\pi]\).

In this case, the number of transmissions by Sensor 1 were much lower at 198 while those by Sensor 2 were 322. Notice that \( w_2 = M_2(c) \) is a constant, independent of the value of \( c \). Thus, we see that the reduction in the number of transmissions by Sensor 2 is only marginal while that of Sensor 1 is huge. The average inter-transmission times of the sensors for the duration of the simulated time were obtained as \([T_1, T_2] = [50.5, 31.1] \text{ms}\). The minimum inter-transmission times were observed as 4.2ms and 9ms for Sensors 1 and 2, respectively. The number of times the parameters of the sensor event-triggers were updated was 16.

The evolution of the Lyapunov function and its derivative along the flow of the closed loop system were very similar to that in Figures 2a and 2b, respectively. Hence, they have not been presented here again. Figures 3a and 3b show the inter-transmission times and the cumulative frequency distribution of the inter-transmission times for each of the sensors. These two plots clearly show the usefulness of the event-triggered transmissions. Figure 3c shows the evolution of the \( w_i \) parameters of the event-triggers at each of the sensors. As mentioned earlier, \( w_2 \) is independent of \( c \) and hence is a constant. The evolution of \( w_1 \) shows that it is a non-decreasing function of time. Finally, Figure 3d shows the evolution of the \( T_i \) parameters of the event-triggers at the sensors (for clarity \( T_2 \) has been scaled by 20 times). Although, \( T_1 \) evolves in a non-decreasing manner, the same is not the case with \( T_2 \). However, as mentioned in Remark 3, this does not pose any problem and the inter-transmission times of the sensor are still lower bounded by a positive constant.

VI. Conclusions

In this paper, we have developed a method for designing decentralized event-triggers for control of nonlinear systems. The architecture of the systems considered in this paper included full state feedback, a central controller and distributed sensors not co-located with the central controller. The aim was to develop event-triggers for determining the time instants of transmission from the sensors to the central controller. The proposed design ensures that the event-triggers at each sensor depend only on locally available information, thus allowing for asynchronous transmissions from the sensors to the central controller. Further, the design aimed at completely eliminating (or drastically reducing) the need for the sensors to listen to other sensors and/or the controller.

The proposed design was shown to guarantee a positive lower bound for inter-transmission times of each sensor (and of the controller in one of the special cases). The origin of the closed loop system is also guaranteed to be asymptotically stable with an arbitrary, but priorly fixed, region of attraction. In the special case of linear systems, the region of attraction was shown to be global with absolutely no need for the sensors to listen. Finally, the proposed design method was illustrated through simulations of a linear and a nonlinear example.

In the system architecture considered in this paper, although the control input to the plant is updated intermitently, it is not exactly event-triggered. In fact, in all the results the inter-transmission times of each sensor individually have been shown to have a positive lower bound. And the time interval between receptions of the central controller from two different sensors can be arbitrarily close to zero. Since the control input to the plant is updated each time the controller receives some information, no positive lower bound can be guaranteed for the inter-update times of the controller. However, it is not very tough to incorporate event-triggering (with guaranteed positive minimum inter-update times) or explicit thresholds on inter-update times of the control by choosing smaller \( \sigma \) values in the event-triggers for the sensors.

Next, although the transmissions of sensors have been designed to be asynchronous, the communication from the central controller to the sensors in Section III-C have been assumed to be synchronous. In future, we aim to allow these communications also to be asynchronous. Although time delays have not been considered explicitly, they may be handled as in most event-triggered control literature (see [1] for example). Finally, it is worthwhile to investigate more sophisticated triggers for updating the parameters \( w_i \) and \( T_i \) (Section III-C) as is a thorough study and quantification of sensor listening effort.
Fig. 3: Nonlinear system example with event-triggered communication from the controller to the sensor event-triggers: (a) Sensor inter-transmission times (b) cumulative frequency distribution of the sensor inter-transmission times. Evolution of (c) \( w_1 \), (d) \( T \), parameters of the sensor event-triggers.

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