TERMINAL QUOTIENT SINGULARITIES IN DIMENSION THREE VIA VARIATION OF GIT

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Abstract. A 3-fold terminal quotient singularity $X = \mathbb{C}^3/G$ admits the economic resolution $Y \to X$, which is “close to being crepant”. This paper proves that the economic resolution $Y$ is isomorphic to a distinguished component of a moduli space of certain $G$-equivariant objects using the King stability condition $\theta$ introduced by Kedzierski [11].

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1. INTRODUCTION

The motivation of this work stems from the philosophy of the McKay correspondence, which says that if a finite group $G$ acts on a variety $M$, then a crepant resolution of the quotient $M/G$ can be realised as a moduli space of $G$-equivariant objects on $M$.

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Let $G \subset \text{GL}_n(\mathbb{C})$ be a finite group. A $G$-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^n$ is called a $G$-constellation if $H^0(\mathcal{F})$ is isomorphic to the regular representation $\mathbb{C}[G]$ of $G$ as a $\mathbb{C}[G]$-module. In particular, the structure sheaf of a $G$-invariant subscheme $Z \subset \mathbb{C}^n$ with $H^0(O_Z)$ isomorphic to $\mathbb{C}[G]$, which is called a $G$-cluster, is a $G$-constellation. Define the GIT stability parameter space

$$
\Theta = \{ \theta \in \text{Hom}_Z(R(G), \mathbb{Q}) \mid \theta(\mathbb{C}[G]) = 0 \},
$$

where $R(G)$ is the representation ring of $G$. For $\theta \in \Theta$, we say that a $G$-constellation $\mathcal{F}$ is $\theta$-(semi)stable if $\theta(\mathcal{G}) > 0$ ($\theta(\mathcal{G}) \geq 0$) for every nonzero proper subsheaf $\mathcal{G}$ of $\mathcal{F}$. A parameter $\theta$ is called generic if every $\theta$-semistable $G$-constellation is $\theta$-stable.

Let $M_\theta$ be the moduli space of $\theta$-semistable $G$-constellations. In the celebrated paper [1], Bridgeland, King and Reid proved that for a finite subgroup $G$ of $\text{SL}_3(\mathbb{C})$, $M_\theta$ is a crepant resolution of $\mathbb{C}^3/G$ if $\theta$ is generic. Craw and Ishii [2] showed that in the case of a finite abelian group $G \subset \text{SL}_3(\mathbb{C})$, any projective crepant resolution can be realised as $M_\theta$ for a suitable GIT parameter $\theta$.

While the moduli space $M_\theta$ need not be irreducible [4] in general, Craw, Maclagan and Thomas [3] showed that for generic $\theta$, $M_\theta$ has a unique irreducible component $Y_\theta$ containing the torus $(\mathbb{C}^3)^*/G$ if $G$ is abelian. The component $Y_\theta$ is birational to $\mathbb{C}^n/G$ and is called the birational component\(^1\) of $M_\theta$.

On the other hand, in the case of $G \subset \text{GL}_3(\mathbb{C})$ giving a terminal quotient singularity $X = \mathbb{C}^3/G$ in dimension 3, $X$ has the economic resolution $\phi: Y \to X$ satisfying

$$
K_Y = \varphi^*(K_X) + \sum_{1 \leq i < r} \frac{i}{r}E_i
$$

with $E_i$’s prime exceptional divisors. Kędzierski [11] proved that $Y$ is isomorphic to the normalization of $Y_\theta$ for some $\theta$. The main theorem of this paper is that the economic resolution $Y$ of $X$ can be interpreted as a component of a moduli space of $G$-constellations as follows.

**Theorem 1.1** (Theorem 4.19). *The economic resolution $Y$ of a 3-fold terminal quotient singularity $X = \mathbb{C}^3/G$ is isomorphic to the birational component $Y_\theta$ of the moduli space $M_\theta$ of $\theta$-stable $G$-constellations for a suitable parameter $\theta$.***

To prove the theorem, first we generalize Nakamura’s result [16]. Let $G \subset \text{GL}_n(\mathbb{C})$ be a finite diagonal group. Nakamura [16] introduced a $G$-graph which is a $\mathbb{C}$-basis of $O_Z$ for a torus invariant $G$-cluster $Z$. Using $G$-graphs, he described a local chart of $G$-Hilb. In this paper, we introduce a $G$-prebrick which is a $\mathbb{C}$-basis of $H^0(\mathcal{F}) \cong \mathbb{C}[G]$ for a torus invariant $G$-constellation $\mathcal{F}$.

---

\(^1\)This component is also called the coherent component.
For a $G$-prebrick $\Gamma$, by King [12], we have an affine scheme $D(\Gamma)$ parametrising $G$-constellations whose basis is $\Gamma$. The affine scheme $D(\Gamma)$ is not necessarily irreducible, but $D(\Gamma)$ has a distinguished component $U(\Gamma)$ containing the torus $T = (\mathbb{C}^\times)^n/G$. In addition, we can show that $U(\Gamma) = \text{Spec } \mathbb{C}[S(\Gamma)]$ for a semigroup $S(\Gamma)$. If the toric affine variety $U(\Gamma)$ has a torus fixed point, then $\Gamma$ is called a $G$-brick. We can prove that $Y_\theta$ is covered by $U(\Gamma)$‘s for suitable $G$-bricks $\Gamma$.

On the other hand, from [14, 17], we know that a 3-fold quotient singularity $X = \mathbb{C}^3/G$ has terminal singularities if and only if the group $G$ is of type $\frac{1}{r}(1, a, r - a)$ with $r$ coprime to $a$, i.e.

$$G = \langle \text{diag}(e^a, e^{r-a}) \mid e^r = 1 \rangle.$$ 

In this case, the quotient variety $X = \mathbb{C}^3/G$ is not Gorenstein. While $X$ does not admit a crepant resolution, $X$ has the economic resolution $\phi : Y \to X$ obtained by a toric method called weighted blowups (or Kawamata blowups). For each step of the weighted blowups, we define three round down functions, which are maps between monomial lattices.

As $Y$ is toric, $Y$ is determined by its associated toric fan $\Sigma$ with the lattice $M$ of $G$-invariant monomials. From toric geometry, note that $Y$ is covered by torus invariant affine open subsets $U_\sigma = \text{Spec } \mathbb{C}[\sigma^\vee \cap M]$ for $\sigma \in \Sigma_{\text{max}}$ where $\Sigma_{\text{max}}$ denotes the set of maximal cones in $\Sigma$.

Using the round down functions, we find a set $\mathcal{S}$ of $G$-bricks such that there exists a bijective map $\Sigma_{\text{max}} \to \mathcal{S}$ sending $\sigma$ to $\Gamma_\sigma$ with $U(\Gamma_\sigma) \cong U_\sigma$. We show that there exists a parameter $\theta \in \Theta$ such that $U(\Gamma_\sigma)$’s cover $Y_\theta$ for $\Gamma_\sigma \in \mathcal{S}$. This proves that the economic resolution $Y$ is isomorphic to the birational component $Y_\theta$ of $\mathcal{M}_\theta$.

Moreover, we further prove $D(\Gamma) \cong \mathbb{C}^3$ for $\Gamma \in \mathcal{S}$. So the irreducible component $Y_\theta$ is actually a connected component. We conjecture that the moduli space $\mathcal{M}_\theta$ is irreducible, which implies $Y \cong \mathcal{M}_\theta$.

**Layout of this article.** In Section 2, we define $G$-(pre)bricks and describe the birational component $Y_\theta$ using $G$-bricks. Section 3 explains how to obtain the economic resolutions using toric methods and defines round down functions. In Section 4, we explain how to find $G$-bricks and a parameter $\theta \in \Theta$ such that the economic resolution is isomorphic to the birational component $Y_\theta$. In Section 5 we describe Kedzierski’s GIT chamber using the $A_{r-1}$ root system. In Section 6, we calculate $G$-bricks and Kedzierski’s GIT chamber for the group of type $\frac{1}{12}(1, 7, 5)$.

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2. G-BRICKS AND MODULI SPACES OF G-CONSTELLATIONS

In this section we define a $G$-prebrick which is a generalized version of Nakamura’s $G$-graph from [16]. By using $G$-prebricks, we describe local charts of moduli spaces of $G$-constellations.

In this section, we restrict ourselves to the case where $G$ is a finite cyclic subgroup of $\text{GL}_3(\mathbb{C})$. It is possible to generalize part of the argument to include general finite small abelian groups in $\text{GL}_n(\mathbb{C})$ for any dimension $n$. However we prefer to focus on this case where we can avoid the difficulty of notation.

2.1. Moduli spaces of $G$-constellations. In this section, we review the construction of moduli spaces $\mathcal{M}_\theta$ of $\theta$-stable $G$-constellations as described in [2, 12].

Define the group $G = \langle \text{diag}(\epsilon^{\alpha_1}, \epsilon^{\alpha_2}, \epsilon^{\alpha_3}) \mid \epsilon^r = 1 \rangle \subset \text{GL}_3(\mathbb{C})$. We call $G$ the group of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$. We can identify the set of irreducible representations of $G$ with the character group $G^\vee := \text{Hom}(G, \mathbb{C}^\times)$ of $G$.

Note that the regular representation $\mathbb{C}[G]$ is isomorphic to $\bigoplus_{\rho \in G^\vee} \mathbb{C}\rho$.

Definition 2.1. A $G$-constellation on $\mathbb{C}^3$ is a $G$-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^3$ with $H^0(\mathcal{F})$ isomorphic to the regular representation $\mathbb{C}[G]$ of $G$ as a $\mathbb{C}[G]$-module.

The representation ring $R(G)$ of $G$ is $\bigoplus_{\rho \in G^\vee} \mathbb{Z} \cdot \rho$. Define the GIT stability parameter space

$$\Theta = \{ \theta \in \text{Hom}_\mathbb{Z}(R(G), \mathbb{Q}) \mid \theta(\mathbb{C}[G]) = 0 \}.$$

Definition 2.2. For a stability parameter $\theta \in \Theta$, we say that:

(i) a $G$-constellation $\mathcal{F}$ is $\theta$-semistable if $\theta(G) \geq 0$ for every proper submodule $\mathcal{G} \subset \mathcal{F}$.

(ii) a $G$-constellation $\mathcal{F}$ is $\theta$-stable if $\theta(G) > 0$ for every nonzero proper submodule $\mathcal{G} \subset \mathcal{F}$.

(iii) $\theta$ is generic if every $\theta$-semistable object is $\theta$-stable.

It is known [4] that the language of $G$-constellations is the same as the language of the McKay quiver representations. Thus by King [12], the moduli spaces of $G$-constellations can be constructed using Geometric Invariant Theory (GIT).

Let $\text{Rep} \, G$ be the affine scheme whose coordinate ring is

$$\mathbb{C}[\text{Rep} \, G] = \mathbb{C}[x_i, y_i, z_i \mid i \in G^\vee]/I_G$$

where $I_G$ is the ideal generated by the following quadrics:

$$\begin{align*}
&x_iy_i + \alpha_1 - y_ix_i + \alpha_2, \\
x_i z_i + \alpha_1 - z_ix_i + \alpha_3, \\
y_i z_i + \alpha_2 - z_i y_i + \alpha_3.
\end{align*}$$

(2.3)
Let $\delta = (1, \ldots, 1) \in \mathbb{Z}_{\geq 0}^r$. The group $\text{GL}(\delta) := \prod_{i \in G^\vee} \mathbb{C}^\times = (\mathbb{C}^\times)^r$ acts on $\text{Rep} \ G$ via change of basis. For a parameter $\theta \in \Theta$, define the GIT quotient with respect to $\theta$

$$\text{Rep} \ G / \theta \text{GL}(\delta) := \text{Rep}^{ss}_{\theta} G / \text{GL}(\delta)$$

parametrising closed $\text{GL}(\delta)$-orbits in $\text{Rep}^{ss}_{\theta} G$ where $\text{Rep}^{ss}_{\theta} G$ denotes the $\theta$-semistable locus in $\text{Rep} \ G$.

**Theorem 2.4** (King [12]). Let us define $\mathcal{M}_{\theta} := \text{Rep} \ G / \theta \text{GL}(\delta)$.

(i) The quasiprojective scheme $\mathcal{M}_{\theta}$ is a coarse moduli space of $\theta$-semistable $G$-constellations up to $S$-equivalence.

(ii) If $\theta$ is generic, the scheme $\mathcal{M}_{\theta}$ is a fine moduli space of $\theta$-stable $G$-constellations.

(iii) The scheme $\mathcal{M}_{\theta}$ is projective over $\mathcal{M}_0 = \text{Spec} \mathbb{C}[\text{Rep} \ G]^{\text{GL}(\delta)}$.

**Birational component $Y_{\theta}$ of the moduli space $\mathcal{M}_{\theta}$.** Let $\mathcal{M}_{\theta}$ denote the moduli space of $\theta$-semistable $G$-constellations. Note that the moduli space $\mathcal{M}_{\theta}$ need not be irreducible [4].

Note that for every parameter $\theta$, there exists a natural embedding of the torus $T := (\mathbb{C}^\times)^3/G$ into $\mathcal{M}_{\theta}$. Indeed, for a $G$-orbit $Z$ in the algebraic torus $T := (\mathbb{C}^\times)^3 \subset \mathbb{C}^3$, since $Z$ is a free $G$-orbit, $\mathcal{O}_Z$ has no nonzero proper submodules. Thus $\mathcal{O}_Z$ is a $\theta$-stable $G$-constellation. Hence it follows that the torus $T := (\mathbb{C}^\times)^3/G$ is the fine moduli space of $\theta$-stable $G$-constellations supported on $T$ because any $G$-constellation supporting on a free $G$-orbit $Z$ is isomorphic to $\mathcal{O}_Z$.

**Theorem 2.5** (Craw, Maclagan and Thomas [3]). Let $\theta \in \Theta$ be generic. Then $\mathcal{M}_{\theta}$ has a unique irreducible component $Y_{\theta}$ that contains the torus $T := (\mathbb{C}^\times)^n/G$. Moreover $Y_{\theta}$ satisfies the following properties:

$$
\begin{align*}
Y_{\theta} & \hookrightarrow \mathcal{M}_{\theta} \\
\mathbb{C}^3/G & \hookrightarrow \mathcal{M}_0
\end{align*}
$$

(i) $Y_{\theta}$ is a not-necessarily-normal toric variety which is birational to the quotient variety $\mathbb{C}^3/G$.

(ii) $Y_{\theta}$ is projective over the quotient variety $\mathbb{C}^3/G$.

**Definition 2.6.** The unique irreducible component $Y_{\theta}$ in Theorem 2.5 is called the **birational component** of $\mathcal{M}_{\theta}$.

Since Craw, Maclagan and Thomas [3] constructed $Y_{\theta}$ as GIT quotient of a reduced affine scheme, it follows that $Y_{\theta}$ is reduced.

**Remark 2.7.** Since $T = (\mathbb{C}^\times)^3$ acts on $\mathbb{C}^3$, the algebraic torus $T$ acts on the moduli space $\mathcal{M}_{\theta}$ naturally. Fixed points of the $T$-action play a crucial role in the study of the moduli space $\mathcal{M}_{\theta}$. ♦
2.2. \textit{G}-prebricks and local charts of $M_\theta$. Let $G \subset \text{GL}_3(\mathbb{C})$ be the finite group of type $\frac{1}{3} (\alpha_1, \alpha_2, \alpha_3)$. Define the lattice
\[ L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{3} (\alpha_1, \alpha_2, \alpha_3), \]
which is an overlattice of $\mathcal{L} = \mathbb{Z}^3$ of finite index. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{Z}^3$. Set $M_\theta = \text{Hom}_\mathbb{Z}(\mathcal{L}, \mathbb{Z})$ and $M = \text{Hom}_\mathbb{Z}(L, \mathbb{Z})$. The two dual lattices $M$ and $M_\theta$ can be identified with Laurent monomials and $G$-invariant Laurent monomials, respectively. The embedding of $G$ into the torus $(\mathbb{C}^\times)^3 \subset \text{GL}_3(\mathbb{C})$ induces a surjective homomorphism $\text{wt} : M_\theta \rightarrow G^\vee$ whose kernel is $M$.

Let $M_{\geq 0}$ denote genuine monomials in $M$, i.e.
\[ M_{\geq 0} = \{ x^{m_1} y^{m_2} z^{m_3} \in M_\theta \mid m_1, m_2, m_3 \geq 0 \}. \]

For a set $A \subset \mathbb{C}[x^\pm, y^\pm, z^\pm]$, let $\langle A \rangle$ denote the $\mathbb{C}[x, y, z]$-submodule of $\mathbb{C}[x^\pm, y^\pm, z^\pm]$ generated by $A$.

Let $\sigma_+$ be the cone in $L_\mathbb{R} := L \otimes \mathbb{R}$ generated by $e_1, e_2, e_3$. Note that the corresponding affine toric variety $U_{\sigma_+} = \text{Spec} \mathbb{C}[\sigma_+^\vee \cap M]$ is isomorphic to the quotient variety $\mathbb{C}^3/G = \text{Spec} \mathbb{C}[x, y, z]^G$.

\textbf{Definition 2.8.} A $G$-prebrick $\Gamma$ is a subset of Laurent monomials in $\mathbb{C}[x^\pm, y^\pm, z^\pm]$ satisfying:

(i) the monomial $1$ is in $\Gamma$.

(ii) for each weight $\rho \in G^\vee$, there exists a unique Laurent monomial $m_\rho \in \Gamma$ of weight $\rho$, i.e. $\text{wt} : \Gamma \rightarrow G^\vee$ is bijective.

(iii) if $n' \cdot n \cdot m_\rho \in \Gamma$ for $m_\rho \in \Gamma$ and $n, n' \in M_{\geq 0}$, then $n \cdot m_\rho \in \Gamma$.

(iv) the set $\Gamma$ is connected in the sense that for any element $m_\rho$, there is a (fractional) path in $\Gamma$ from $m_\rho$ to $1$ whose steps consist of multiplying or dividing by one of $x, y, z$.

For a Laurent monomial $m \in M$, let $\text{wt}_\Gamma(m)$ denote the unique element $m_\rho$ in $\Gamma$ of the same weight as $m$.

\textbf{Remark 2.9.} Nakamura’s $G$-graph $\Gamma$ in [16] is a $G$-prebrick because if a monomial $n' \cdot n$ is in $\Gamma$ for two monomials $n, n' \in M_{\geq 0}$, then $n$ is in $\Gamma$. The main difference between $G$-graphs and $G$-prebricks is that elements of $G$-prebricks are allowed to be Laurent monomials, not just genuine monomials.

\textbf{Example 2.10.} Let $G$ be the group of type $\frac{1}{7}(1, 3, 4)$. Then
\[ \Gamma_1 = \left\{ 1, y, y^2, z, z^2, z^2, \frac{y^2}{z^2} \right\}, \]
\[ \Gamma_2 = \left\{ 1, z, y, y^2, \frac{y^2}{z}, \frac{y^2}{z^2} \right\}. \]
are $G$-prebricks. For $\Gamma_1$, we have $\text{wt}_{\Gamma_1}(x) = \frac{x^2}{y}$ and $\text{wt}_{\Gamma_1}(y^3) = \frac{x^2}{y^3}$.

For a $G$-prebrick $\Gamma = \{m_\rho\}$, as an analogue of [16], define $S(\Gamma)$ to be the subsemigroup of $M$ generated by $\frac{n \cdot m_\rho}{\text{wt}_\Gamma(n \cdot m_\rho)}$ for all $n \in \overline{M}_{\geq 0}$, $m_\rho \in \Gamma$. Define a cone $\sigma(\Gamma)$ in $L_\mathbb{R} = \mathbb{R}^3$ as follows:

$$\sigma(\Gamma) = \left\{ u \in L_\mathbb{R} \mid \left\langle u, \frac{n \cdot m_\rho}{\text{wt}_\Gamma(n \cdot m_\rho)} \right\rangle \geq 0, \forall m_\rho \in \Gamma, \ n \in \overline{M}_{\geq 0} \right\}.$$ 

Observe that:

(i) $(\overline{M}_{\geq 0} \cap M) \subset S(\Gamma)$,

(ii) $\sigma(\Gamma) \subset \sigma^+$,

(iii) $S(\Gamma) \subset (\sigma(\Gamma)^\vee \cap M)$.

Lemma 2.11. Let $\Gamma$ be a $G$-prebrick. Define

$$B(\Gamma) := \left\{ f \cdot m_\rho \mid m_\rho \in \Gamma, \ f \in \{x, y, z\} \right\} \setminus \Gamma.$$ 

Then the semigroup $S(\Gamma)$ is generated by $\frac{b}{\text{wt}_\Gamma(b)}$ for all $b \in B(\Gamma)$ as a semigroup. In particular, $S(\Gamma)$ is finitely generated as a semigroup.

Proof. Let $S$ be the subsemigroup of $M$ generated by $\frac{b}{\text{wt}_\Gamma(b)}$ for all $b \in B(\Gamma)$. Clearly, $S \subset S(\Gamma)$. For the opposite inclusion, it is enough to show that the generators of $S(\Gamma)$ are in $S$.

An arbitrary generator of $S(\Gamma)$ is of the form $\frac{n \cdot m_\rho}{\text{wt}_\Gamma(n \cdot m_\rho)}$ for some $n \in \overline{M}_{\geq 0}$, $m_\rho \in \Gamma$. We may assume that $n \cdot m_\rho \not\in \Gamma$. In particular, $n \neq 1$. Since $n$ has positive degree, there exists $f \in \{x, y, z\}$ such that $f$ divides $n$, i.e. $\frac{f}{n} \in \overline{M}_{\geq 0}$ and $\deg(\frac{f}{n}) < \deg(n)$. Let $m_{\rho'}$ denote $\text{wt}_\Gamma(\frac{f}{n} \cdot m_\rho)$. Note that

$$\text{wt}_\Gamma(f \cdot m_{\rho'}) = \text{wt}_\Gamma(\frac{n}{f} \cdot m_\rho) = \text{wt}_\Gamma(n \cdot m_\rho).$$

Thus

$$\frac{n \cdot m_\rho}{\text{wt}_\Gamma(n \cdot m_\rho)} = \frac{\frac{n}{f} \cdot m_{\rho'}}{\text{wt}_\Gamma(\frac{n}{f} \cdot m_{\rho'})} \cdot \frac{f}{\text{wt}_\Gamma(\frac{n}{f} \cdot m_{\rho'})}.$$ 

By induction on the degree of monomial $n$, the assertion is proved. \qed

The set $B(\Gamma)$ in the lemma above is called the Border bases of $\Gamma$. As $B(\Gamma)$ is finite, the semigroup $S(\Gamma)$ is finitely generated as a semigroup.

Thus the semigroup $S(\Gamma)$ defines an affine toric variety. Define two affine toric varieties:

$$U(\Gamma) := \text{Spec} \mathbb{C}[S(\Gamma)],$$

$$U'(\Gamma) := \text{Spec} \mathbb{C}[\sigma(\Gamma)^\vee \cap M].$$
Note that the torus \( \text{Spec} \mathbb{C}[\mathcal{M}] \) of \( U(\Gamma) \) is isomorphic to \( T = (\mathbb{C}^\times)^3/G \) and that \( U^\nu(\Gamma) \) is the normalization of \( U(\Gamma) \).

Craw, Maclagan and Thomas [4] showed that there exists a torus invariant \( G \)-cluster which does not lie in the birational component \( Y_\theta \).

The following definition is implicit in [4].

**Definition 2.12.** A \( G \)-prebrick \( \Gamma \) is called a \( G \)-brick if the affine toric variety \( U(\Gamma) \) contains a torus fixed point.

From toric geometry, \( U(\Gamma) \) has a torus fixed point if and only if \( S(\Gamma) \cap (S(\Gamma))^{-1} = \{1\} \), i.e. the cone \( \sigma(\Gamma) \) is a 3-dimensional cone.

**Example 2.13.** Consider the \( G \)-prebricks \( \Gamma_1, \Gamma_2 \) in Example 2.10. By Lemma 2.11, \( S(\Gamma_1) \) is generated by \( y^5z^2, z^3y^4, xy^z \). We have

\[
\sigma(\Gamma_1) = \{ u \in \mathbb{R}^3 \mid \langle u, m \rangle \geq 0, \text{ for all } m \in \{ y^5, z^3, xz \} \},
\]

\[
= \text{Cone} \left( (1, 0, 0), \frac{1}{7}(3, 2, 5), \frac{1}{7}(1, 3, 4) \right).
\]

Similarly, we can see that \( \sigma(\Gamma_2) = \text{Cone} \left( (1, 0, 0), \frac{1}{7}(1, 3, 4), \frac{1}{7}(6, 4, 3) \right) \).

Since \( S(\Gamma_1) = \sigma(\Gamma_1)^\vee \cap \mathcal{M} \) and \( S(\Gamma_2) = \sigma(\Gamma_2)^\vee \cap \mathcal{M} \), the two \( G \)-prebricks \( \Gamma_1, \Gamma_2 \) are \( G \)-bricks. Moreover the two toric varieties \( U(\Gamma_1) \) and \( U(\Gamma_2) \) are smooth. \(\Box\)

Let \( \Gamma \) be a \( G \)-prebrick. Define

\[ C(\Gamma) := \langle \Gamma \rangle / \langle B(\Gamma) \rangle. \]

The module \( C(\Gamma) \) is a torus invariant \( G \)-constellation. A submodule \( \mathcal{G} \) of \( C(\Gamma) \) is determined by a subset \( A \subset \Gamma \), which forms a \( \mathbb{C} \)-basis of \( \mathcal{G} \).

**Lemma 2.14.** Let \( A \) be a subset of \( \Gamma \). The following are equivalent.

(i) The set \( A \) forms a \( \mathbb{C} \)-basis of a submodule of \( C(\Gamma) \).

(ii) If \( m_\rho \in A \) and \( f \in \{ x, y, z \} \), then \( f \cdot m_\rho \in \Gamma \) implies \( f \cdot m_\rho \in A \).

Let \( p \) be a point in \( U(\Gamma) \). Then the evaluation map

\[ \text{ev}_p : S(\Gamma) \rightarrow (\mathbb{C}, \times), \]

is a semigroup homomorphism.

To assign a \( G \)-constellation \( C(\Gamma)_p \) to the point \( p \) of \( U(\Gamma) \), first consider the \( \mathbb{C} \)-vector space with basis \( \Gamma \) whose \( G \)-action is induced by the \( G \)-action on \( \mathbb{C}[x, y, z] \). Endow it with the following \( \mathbb{C}[x, y, z] \)-action:

\[
n \ast m_\rho := \text{ev}_p \left( \frac{n \cdot m_\rho}{\text{wt}_\Gamma(n \cdot m_\rho)} \right) \text{wt}_\Gamma(n \cdot m_\rho),
\]

for a monomial \( n \in \mathcal{M}_{\geq 0} \) and \( m_\rho \in \Gamma \).
Lemma 2.16. Let $\Gamma$ be a $G$-prebrick.

(i) For every $p \in U(\Gamma)$, $C(\Gamma)_p$ is a $G$-constellation.

(ii) For every $p \in U(\Gamma)$, $\Gamma$ is a $\mathbb{C}$-basis of $C(\Gamma)_p$.

(iii) If $p$ and $q$ are different points in $U(\Gamma)$, then $C(\Gamma)_p \not\sim C(\Gamma)_q$.

(iv) Let $Z \subset T = (\mathbb{C}^\times)^3$ be a free $G$-orbit and $p$ the corresponding point in the torus $\text{Spec} \mathbb{C}[M]$ of $U(\Gamma)$. Then $C(\Gamma)_p \cong \mathcal{O}_Z$ as $G$-constellations.

(v) If $U(\Gamma)$ has a torus fixed point $p$, then $C(\Gamma)_p \cong C(\Gamma)$.

Proof. From the definition of $C(\Gamma)_p$, the assertions (i), (ii) and (v) follow immediately. The assertion (iii) follows from the fact that points on the affine toric variety $U(\Gamma)$ are in 1-to-1 correspondence with semigroup homomorphisms from $S(\Gamma)$ to $\mathbb{C}$.

It remains to show (iv). Let $Z \subset T = (\mathbb{C}^\times)^3$ be a free $G$-orbit and $p$ the corresponding point in $\text{Spec} \mathbb{C}[M] \subset U(\Gamma)$. There is a surjective $G$-equivariant $\mathbb{C}[x, y, z]$-module homomorphism

$$\mathbb{C}[x, y, z] \rightarrow C(\Gamma)_p \quad \text{given by } f \mapsto f * 1,$$

whose kernel is equal to the ideal of $Z$. This proves (iv). \qed

Definition 2.17. A $G$-prebrick is said to be $\theta$-stable if $C(\Gamma)$ is $\theta$-stable.

Defomation space $D(\Gamma)$. We introduce deformation theory of $C(\Gamma)$ for a $\theta$-stable $G$-prebrick $\Gamma$. We deform $C(\Gamma)$, keeping the same vector space structure, but perturbing the structure of $\mathbb{C}[x, y, z]$-module. Since we fix a $\mathbb{C}$-basis $\Gamma$ of $C(\Gamma)$, deforming $C(\Gamma)$ involves $3r$ parameters $\{x_\rho, y_\rho, z_\rho \mid \rho \in G^\vee\}$ with

$$\begin{align*}
x \cdot m_\rho &= x_\rho \text{wt}_G(x \cdot m_\rho), \\
y \cdot m_\rho &= y_\rho \text{wt}_G(y \cdot m_\rho), \\
z \cdot m_\rho &= z_\rho \text{wt}_G(z \cdot m_\rho),
\end{align*}$$

with the following commutation relations:

$$\begin{align*}
x_\rho y_\rho \text{wt}(x \cdot m_\rho) - y_\rho x_\rho \text{wt}(y \cdot m_\rho), \\
x_\rho z_\rho \text{wt}(x \cdot m_\rho) - z_\rho x_\rho \text{wt}(z \cdot m_\rho), \\
y_\rho z_\rho \text{wt}(y \cdot m_\rho) - z_\rho y_\rho \text{wt}(y \cdot m_\rho),
\end{align*}$$

(2.18)

Note that $\text{wt}_G(m) \in \Gamma$ is the base of the same weight as $m$. Fixing a basis $\Gamma$ means that we set $f_\rho = 1$ if $\text{wt}_G(f \cdot m_\rho) = f \cdot m_\rho$ for $f \in \{x, y, z\}$.

Define a subset of the $3r$ parameters

$$\Lambda(\Gamma) := \{f_\rho \mid \text{wt}_G(f \cdot m_\rho) = f \cdot m_\rho, \ f_\rho \in \{x_\rho, y_\rho, z_\rho\}\},$$

i.e. $\Lambda(\Gamma)$ is the set of parameters fixed to be 1. Define the affine scheme

$$D(\Gamma) := \text{Spec} \left(\mathbb{C}[x_\rho, y_\rho, z_\rho \mid \rho \in G^\vee]/I_\Gamma\right)$$

(2.19)

where $I_\Gamma = \langle \text{the quadrics in (2.18), } f_\rho - 1 \mid f_\rho \in \Lambda(\Gamma) \rangle$. By King [12], the affine scheme $D(\Gamma)$ is an open set of $\mathcal{M}_\theta$ containing the point
corresponding to \( C(\Gamma) \). More precisely, consider an affine open set \( \widetilde{U}_\Gamma \) in \( \operatorname{Rep} G \), which is defined by \( f_\rho \) to be nonzero for all \( f_\rho \in \Lambda(\Gamma) \). Note that \( \widetilde{U}_\Gamma \) is \( \operatorname{GL}(\delta) \)-invariant and that \( \widetilde{U}_\Gamma \) is in the \( \theta \)-stable locus. Since the quotient map \( \operatorname{Rep}_G^{ss} \to \mathcal{M}_\theta \) is a geometric quotient for generic \( \theta \), from GIT [15], it follows that \( \operatorname{Spec} \mathbb{C}[\widetilde{U}_\Gamma]^{\operatorname{GL}(\delta)} \) is an affine open subset of \( \mathcal{M}_\theta \). On the other hand, setting \( f_\rho \in \Lambda(\Gamma) \) to be 1 for all \( f_\rho \in \Lambda(\Gamma) \) gives a slice of the \( \operatorname{GL}(\delta) \)-action. Thus \( D(\Gamma) \) is isomorphic to \( \operatorname{Spec} \mathbb{C}[\widetilde{U}_\Gamma]^{\operatorname{GL}(\delta)} \).

Remark 2.20. The affine open subset \( D(\Gamma) \) of the moduli space \( \mathcal{M}_\theta \) parametrises \( G \)-constellations whose basis is \( \Gamma \). ♦

Proposition 2.21. For generic \( \theta \), let \( \Gamma \) be a \( \theta \)-stable \( G \)-brick and \( Y_\theta \) the birational component of \( \mathcal{M}_\theta \). Then \( C(\Gamma)_p \) is \( \theta \)-stable for every \( p \in U(\Gamma) \). Furthermore, there exists an open immersion 
\[
U(\Gamma) = \operatorname{Spec} \mathbb{C}[S(\Gamma)] \hookrightarrow Y_\theta \subset \mathcal{M}_\theta,
\]
which fits into the following commutative diagram: 
\[
\begin{array}{ccc}
U(\Gamma) & \hookrightarrow & Y_\theta \\
\downarrow & & \downarrow \\
D(\Gamma) & \hookrightarrow & \mathcal{M}_\theta
\end{array}
\]
where the vertical morphisms are closed embeddings.

Proof. Let us assume that the \( G \)-constellation \( C(\Gamma) \) is \( \theta \)-stable. Let \( p \) be an arbitrary point in \( U(\Gamma) \) and \( \mathcal{G} \) a submodule of \( C(\Gamma)_p \). By the definition of \( C(\Gamma)_p \), there is a submodule \( \mathcal{G}' \) of \( C(\Gamma) \) whose support is the same as \( \mathcal{G} \). Since \( C(\Gamma) \) is \( \theta \)-stable, \( \theta(\mathcal{G}) = \theta(\mathcal{G}') > 0 \). Thus \( C(\Gamma)_p \) is \( \theta \)-stable.

Since there is a \( \mathbb{C} \)-algebra epimorphism from \( \mathbb{C}[D(\Gamma)] \) to \( \mathbb{C}[S(\Gamma)] \) given by
\[
f_\rho \mapsto \frac{f \cdot m_\rho}{\operatorname{wt}_\Gamma(f \cdot m_\rho)}
\]
for \( f_\rho \in \{x_\rho, y_\rho, z_\rho\} \), it follows that \( U(\Gamma) \) is a closed subscheme of \( D(\Gamma) \).

As Craw, Maclagan, and Thomas [3] proved that the birational component \( Y_\theta \) is a unique irreducible component of \( \mathcal{M}_\theta \) containing the torus \( T = (\mathbb{C}^*)^3/G \), it follows that \( Y_\theta \cap D(\Gamma) \) is a unique irreducible component of \( D(\Gamma) \) containing the torus \( T \).

The morphism \( U(\Gamma) \hookrightarrow D(\Gamma) \subset \mathcal{M}_\theta \) induces an isomorphism between the torus \( \operatorname{Spec} \mathbb{C}[M] \) and the torus \( T \) of \( Y_\theta \) by Lemma 2.16 (iv). Note that \( U(\Gamma) \) is in the component \( Y_\theta \cap D(\Gamma) \) because \( U(\Gamma) \) is a closed subset of \( D(\Gamma) \) containing \( T \). Since both \( U(\Gamma) \) and \( Y_\theta \cap D(\Gamma) \) are reduced and of the same dimension, \( U(\Gamma) \) is equal to \( Y_\theta \cap D(\Gamma) \). Thus there exists an open immersion from \( U(\Gamma) \) to \( Y_\theta \). □
2.3. G-bricks and the birational component \( Y_\theta \). In this section, we present a 1-to-1 correspondence between the set of torus fixed points in \( Y_\theta \) and the set of \( \theta \)-stable \( G \)-bricks.

**Proposition 2.22.** Let \( G \subset \text{GL}_3(\mathbb{C}) \) be the group of type \( \frac{1}{r}(\alpha_1, \alpha_2, \alpha_3) \). For a generic parameter \( \theta \), there is a 1-to-1 correspondence between the set of torus fixed points in the birational component \( Y_\theta \) of the moduli space \( \mathcal{M}_\theta \) and the set of \( \theta \)-stable \( G \)-bricks.

**Proof.** In Section 2.2, we have seen that if \( \Gamma \) is a \( \theta \)-stable \( G \)-brick, then \( C(\Gamma) \) is a torus invariant \( G \)-constellation corresponding to a torus fixed point in \( Y_\theta \). Let \( p \in Y_\theta \) be a torus fixed point and \( F \) the corresponding torus invariant \( G \)-constellation. For a one parameter subgroup \( \lambda : \mathbb{C}^\times \rightarrow T \subset Y_\theta \) with \( \lim_{t \to 0} \lambda(t) = p \), \( \lambda \) induces a flat family \( \mathcal{V} \) of \( G \)-constellations over \( A^1_\mathbb{C} \). Note that \( \mathcal{V} \) has generic support; for every nonzero \( t \in A^1_\mathbb{C} \), the \( G \)-constellation \( \mathcal{V}_t \) over \( t \) is isomorphic to \( \langle \Gamma \rangle / N_t \) for a submodule \( N_t \) of \( \langle \Gamma \rangle \), where \( \langle \Gamma \rangle \) denotes the \( \mathbb{C}[x, y, z] \)-module generated by \( \Gamma \). We prove that \( \Gamma \) is a \( G \)-prebrick and that \( F \cong C(\Gamma) \). Note that \( F \) can be written as \( \langle \Gamma \rangle / N_t \) for a submodule \( N_t \). For any \( m_\rho \in \Gamma \), since \( m_\rho \) is a base, \( m_\rho \) is not in \( N_t \). Moreover if \( n \cdot m_\rho \notin \Gamma \) for \( n \in \mathbb{M} \geq 0 \), \( m_\rho \in \Gamma \), then \( n \cdot m_\rho \in N_t \) because the dimension of \( \text{H}^0(\mathcal{F}) \) is \( r = |\Gamma| \). This proves that \( N = \langle B(\Gamma) \rangle \), where \( B(\Gamma) \) is the Border bases in Lemma 2.11. From this, it follows that \( \Gamma \) satisfies the conditions (i),(ii),(iii) in Definition 2.8. As \( \mathcal{F} \) is \( \theta \)-stable for generic \( \theta \), the connectedness condition (iv) follows.

To see that the \( G \)-prebrick \( \Gamma \) is a \( G \)-brick, note that the point \( p \in Y_\theta \) corresponds to the isomorphism class of \( C(\Gamma) \) so \( p \in D(\Gamma) \). Thus \( p \) is in \( U(\Gamma) = Y_\theta \cap D(\Gamma) \).

**Corollary 2.23.** Let \( \Gamma \) be a \( G \)-prebrick. Then \( C(\Gamma) \) lies in the birational component \( Y_\theta \) if and only if \( \Gamma \) is a \( G \)-brick.

**Theorem 2.24.** Let \( G \subset \text{GL}_3(\mathbb{C}) \) be a finite diagonal group and \( \theta \) a generic GIT parameter for \( G \)-constellations. Assume that \( \mathcal{S} \) is the set of all \( \theta \)-stable \( G \)-bricks.

(i) The birational component \( Y_\theta \) of \( \mathcal{M}_\theta \) is isomorphic to the not-necessarily-normal toric variety \( \bigcup_{\Gamma \in \mathcal{S}} U(\Gamma) \).

\(^{2}\)In [8], there is another proof using the language of the McKay quiver representations.
The normalization of \( Y_\theta \) is isomorphic to the normal toric variety whose toric fan consists of the 3-dimensional cones \( \sigma(\Gamma) \) for \( \Gamma \in \mathcal{S} \) and their faces.

**Proof.** Let \( G \) be the group of type \( \frac{1}{r}(1, a, r - a) \). Consider the lattice \( L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \alpha_2, \alpha_3) \).

Let \( Y_\theta \) be the birational component of the moduli space of \( \theta \)-stable \( G \)-constellations and \( Y_\nu^\theta \) the normalization of \( Y_\theta \). Let \( Y \) denote the not-necessarily-normal toric variety \( \bigcup_{\Gamma \in \mathcal{S}} U(\Gamma) \). Define the fan \( \Sigma \) in \( L_{\mathbb{R}} \) whose maximal cones are \( \sigma(\Gamma) \) for \( \Gamma \in \mathcal{S} \). Note that the corresponding toric variety \( Y_\nu := X_\Sigma \) is the normalization of \( Y \).

By Proposition 2.21, there is an open immersion \( \psi: Y \to Y_\theta \). From Proposition 2.22, it follows that the image \( \psi(Y) \) contains all torus fixed points of \( Y_\theta \). The induced morphism \( \psi^\nu: Y^\nu \to Y_\nu^\theta \) is an open embedding of normal toric varieties with the same number of torus fixed points. Thus the morphism \( \psi^\nu \) should be an isomorphism. This proves (ii).

To show (i), suppose that \( Y_\theta \setminus \psi(Y) \) is nonempty so it contains a torus orbit \( O \) of dimension \( d \geq 1 \). Since the normalization morphism is torus equivariant and surjective, there exists a torus orbit \( O' \) in \( Y^\nu \cong Y_\nu^\theta \) of dimension \( d \) which is mapped to the torus orbit \( O \). At the same time, from the fact that \( Y^\nu \) is the normalization of \( Y \) and that the normalization morphism is finite, it follows that the image of \( O' \) is a torus orbit of dimension \( d \), so the image is \( O \). Thus \( O \) is in \( \psi(Y) \), which is a contradiction. \( \square \)

**Corollary 2.25.** With the notation as in Theorem 2.24, \( Y_\theta \) is a normal toric variety if and only if \( S(\Gamma) = \sigma(\Gamma)^\vee \cap M \) for all \( \Gamma \in \mathcal{S} \).

### 3. Weighted blowups and economic resolutions

Let \( G \subset \text{GL}_3(\mathbb{C}) \) be the finite subgroup of type \( \frac{1}{r}(1, a, r - a) \) with \( r \) coprime to \( a \), i.e.

\[
G = \langle \text{diag}(e, e^a, e^{r-a}) \mid e^r = 1 \rangle.
\]

The quotient \( X = \mathbb{C}^3 / G \) has terminal singularities and has no crepant resolution. However there exist a special kind of toric resolutions, which can be obtained by a sequence of weighted blowups. In this section, we review the notion of toric weighted blowups and define round down functions which are used for finding admissible \( G \)-bricks.

#### 3.1. Terminal quotient singularities in dimension 3

In this section, we collect various facts from birational geometry. Most of these are taken from [17].

**Definition 3.1.** Let \( X \) be a normal quasiprojective variety and \( K_X \) the canonical divisor on \( X \). We say that \( X \) has *terminal singularities* (resp. *canonical singularities*) if it satisfies the following conditions:
(i) there is a positive integer $r$ such that $rK_X$ is a Cartier divisor.

(ii) if $\varphi: Y \to X$ is a resolution with $E_i$ prime exceptional divisors such that
\[
rK_Y = \varphi^*(rK_X) + r \sum a_i E_i,
\]
then $a_i > 0$ (resp. $\geq 0$) for all $i$.

In the definition above, $a_i$ is called the discrepancy of $E_i$. A crepant resolution $\varphi$ of $X$ is a resolution with all discrepancies zero.

**Birational geometry of toric varieties.** Let $L$ be a lattice of rank $n$ and $M$ the dual lattice of $L$. Let $\sigma$ be a cone in $L \otimes \mathbb{R}$. Fix a primitive element $v \in L \cap \sigma$. The barycentric subdivision of $\sigma$ at $v$ is the minimal fan containing all cones $\text{Cone}(\tau, v)$ where $\tau$ varies over all subcones of $\sigma$ with $v \not\in \tau$.

**Proposition 3.2** (see e.g. [17]). Let $\Sigma$ be the barycentric subdivision of an $n$-dimensional cone $\sigma$ at $v$. Let $X := U_\sigma$ be the affine toric variety corresponding to $\sigma$ and $Y$ the toric variety corresponding to the fan $\Sigma$.

(i) The barycentric subdivision induces a projective toric morphism $\varphi: Y \to X$.

(ii) The set of 1-dimensional cones of $\Sigma$ consists of 1-dimensional cones of $\sigma$ and $\text{Cone}(v)$.

(iii) The torus invariant prime divisor $D_v$ corresponding to the 1-dimensional cone $\text{Cone}(v)$ is a $\mathbb{Q}$-Cartier divisor on $Y$.

Furthermore if $v$ is an interior lattice point in $\sigma$, then
\[
K_Y = \varphi^*(K_X) + \langle x_1x_2 \cdots x_n, v \rangle - 1)D_v,
\]
i.e. the discrepancy of the exceptional divisor $D_v$ is $\langle x_1x_2 \cdots x_n, v \rangle - 1$.

**Example 3.3.** Define the lattice $L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1, a, r - a)$ with $r$ coprime to $a$ and $M = \text{Hom}_\mathbb{Z}(L, \mathbb{Z})$ the dual lattice. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{Z}^3$ and $\sigma_+$ the cone generated by $e_1, e_2, e_3$. As in Section 2.2, the toric variety $X := U_{\sigma_+}$ is the quotient variety $\mathbb{C}^3/G$ where $G$ is the group of type $\frac{1}{r}(1, a, r - a)$.

Set $v_i := \frac{1}{r}(i, ai, r - ai) \in L$ for each $1 \leq i < r - 1$ where $\bar{\cdot}$ denotes the residue modulo $r$. Let $E_i$ be the torus invariant prime divisor corresponding to $v_i$. From Proposition 3.2, the discrepancy of $E_i$ is
\[
i \frac{i}{r} + \frac{a_i}{r} + \frac{r - ai}{r} - 1 = \frac{i}{r} > 0.
\]
This shows that $X$ has only terminal singularities.

We have seen that the quotient singularity $X = \mathbb{C}^3/G$ has terminal singularities if $G$ is the group of type $\frac{1}{r}(1, a, r - a)$ with $r$ coprime to $a$. Conversely, these groups are essentially all the cases, by the following.
The induced toric morphism \( \varphi \) Let

\[
\{ \text{the toric variety corresponding to the fan } \Sigma \text{ together with the lattice of the smallest discrepancy (see Example 3.3). Define three cones } \\
\}
\]

Set \( v \) be the standard basis of \( L \) with weight \( (1, a, r - a) \). Consider the sublattice \( L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1, a, r - a) \). Set \( \overline{\Sigma} = \mathbb{Z}^3 \subset L \). Consider the two dual lattices \( M = \text{Hom}_\mathbb{Z}(L, \mathbb{Z}) \), \( \overline{M} = \text{Hom}_\mathbb{Z}(\overline{L}, \mathbb{Z}) \). Note that a (Laurent) monomial \( m \in \overline{M} \) is \( G \)-invariant if and only if \( m \) is in \( M \). Let \( \{e_1, e_2, e_3\} \) be the standard basis of \( \mathbb{Z}^3 \) and \( \sigma_+ \) the cone generated by \( e_1, e_2, e_3 \). Then \( \text{Spec} \mathbb{C}[\sigma_+^\vee \cap M] \) is the quotient variety \( X = \mathbb{C}^3/G \). Set \( v = \frac{1}{r}(1, a, r - a) \in L \), which corresponds to the exceptional divisor of the smallest discrepancy (see Example 3.3). Define three cones

\[
\sigma_1 = \text{Cone}(v, e_2, e_3), \quad \sigma_2 = \text{Cone}(e_1, v, e_3), \quad \sigma_3 = \text{Cone}(e_1, e_2, v).
\]

Define \( \Sigma \) to be the fan consisting of the three cones \( \sigma_1, \sigma_2, \sigma_3 \) and their faces. The fan \( \Sigma \) is the barycentric subdivision of \( \sigma_+ \) at \( v \). Let \( Y_1 \) be the toric variety corresponding to the fan \( \Sigma \) together with the lattice \( L \). The induced toric morphism \( \varphi: Y_1 \to X \) is called the weighted blowup of \( X \) with weight \( (1, a, r - a) \).

\[
\begin{array}{c}
\sigma_2 \\
v = \frac{1}{r}(1, a, r - a) \\
\sigma_3 \\
\end{array}
\]

\[
\begin{array}{c}
\sigma_1 \\
e_3 \\
e_2 \\
\end{array}
\]

\[
\uparrow e_1
\]

**Figure 3.1.** Weighted blowup of weight \( (1, a, r - a) \)

Let us consider the sublattice \( L_2 \) of \( L \) generated by \( e_1, v, e_3 \). Define \( M_2 := \text{Hom}_\mathbb{Z}(L_2, \mathbb{Z}) \) with the dual basis

\[
\xi_2 := xy^{-\frac{1}{r}}, \quad \eta_2 := y^\frac{a}{r}, \quad \zeta_2 := y^\frac{a}{r}z.
\]

The lattice inclusion \( L_2 \hookrightarrow L \) induces a toric morphism

\[
\varphi: \text{Spec} \mathbb{C}[^\vee_2 \cap M_2] \to U_2 := \text{Spec} \mathbb{C}[^\vee_2 \cap M] \]

Since \( \mathbb{C}[^\vee_2 \cap M_2] \cong \mathbb{C}[^\vee_2 \cap M_2] \) and the group \( G_2 := L/L_2 \) is of type \( \frac{1}{a}(1, -r, r) \) with eigencordinates \( \xi_2, \eta_2, \zeta_2 \), the open subset \( U_2 \) has a quotient singularity of type \( \frac{1}{a}(1, -r, r) \).

Similarly, consider the sublattice \( L_3 \) of \( L \) generated by \( e_1, e_2, v \). Let us define the lattice \( M_3 := \text{Hom}_\mathbb{Z}(L_3, \mathbb{Z}) \) with basis

\[
\xi_3 := xz^{-\frac{1}{r-a}}, \quad \eta_3 := yz^{-\frac{a}{r-a}}, \quad \zeta_3 := z^\frac{r}{r-a}.
\]

The open set \( U_3 = \text{Spec} \mathbb{C}[\xi_3, \eta_3, \zeta_3] \) has a quotient singularity of type \( \frac{1}{r-a}(1, r, -r) \) with eigencordinates \( \xi_3, \eta_3, \zeta_3 \). Set \( G_3 := L/L_3 \).
Lastly, consider the sublattice $L_1$ of $L$ generated by $v, e_2, e_3$. Let us define $M_1 := \text{Hom}_\mathbb{Z}(L_1, \mathbb{Z})$ with the dual basis 
\[ \xi_1 := x^1, \quad \eta_1 := x^{-r}y, \quad \zeta_1 := x^{-r}z. \]
Since $\{v, e_2, e_3\}$ forms a $\mathbb{Z}$-basis of $L$, i.e. $G_1 = L/L_1$ is the trivial group, the open set $U_1 = \text{Spec} \mathbb{C}[\xi_1, \eta_1, \zeta_1]$ is smooth.

**Example 3.5.** Let $G$ be the group of type $\frac{1}{7}(1, 3, 4)$ as in Example 2.10. The toric fan of the weighted blowup with weight $(1, 3, 4)$ is shown in Figure 3.2.

![Figure 3.2. Weighted blowup of weight $(1, 3, 4)$](image)

The affine toric variety corresponding to the cone $\sigma_2$ on the left side of $v = \frac{1}{7}(1, 3, 4)$ has a quotient singularity of type $\frac{1}{7}(1, 2, 1)$ with eigencoordinates $xy^{-\frac{r}{7}}, yz^{-\frac{r}{7}}, yz^{-\frac{r}{7}}z$. The affine toric variety corresponding to the cone $\sigma_3$ on the right side of $v$ has a singularity of type $\frac{1}{7}(1, 3, 1)$ with eigencoordinates $xz^{-\frac{r}{7}}, yz^{-\frac{r}{7}}, z^7$. On the other hand, the affine toric variety corresponding to the cone $\sigma_1 = \text{Cone}(e_2, e_3, v)$ is smooth as $e_2, e_3, v$ form a $\mathbb{Z}$-basis of $L$.

**Definition 3.6 (Round down functions).** With the notation above, the left round down function $\phi_2: \overline{M} \to M_2$ of the weighted blowup with weight $(1, a, r - a)$ is defined by
\[ \phi_2(x^{m_1}y^{m_2}z^{m_3}) = \xi_2^{m_1}, \eta_2^{\lfloor m_1 + r - m_2 + r - m_3 \rfloor}, \zeta_2^{m_3}. \]
where $\lfloor \rfloor$ is the floor function. In a similar manner, the right round down function $\phi_3: \overline{M} \to M_3$ of the weighted blowup with weight...
(1, a, r − a) is defined by
\[ \phi_3(x^{m_1} y^{m_2} z^{m_3}) = \xi_3^{m_1} \eta_3^{m_2} \zeta_3^{\lfloor m_1 + \frac{a}{r} m_2 + \frac{r-a}{r} m_3 \rfloor}, \]
and the central round down function \(\phi_1: \mathcal{M} \to M_1\) of the weighted blowup with weight \(1, a, r - a\) by
\[ \phi_1(x^{m_1} y^{m_2} z^{m_3}) = \xi_1^{\lfloor m_1 + \frac{a}{r} m_2 + \frac{r-a}{r} m_3 \rfloor} \eta_1^{m_2} \zeta_1^{m_3}. \]

**Lemma 3.7.** For each \(k = 1, 2, 3\), let \(\phi_k\) be the round down function of the weighted blowup with weight \((1, a, r - a)\). For a monomial \(m \in \mathcal{M}\) and a \(G\)-invariant monomial \(n \in M\),
\[ \phi_k(m \cdot n) = \phi_k(m) \cdot n. \]
Thus the weight of \(\phi_k(m \cdot n)\) and the weight of \(\phi_k(m)\) are the same in terms of the \(G_k\)-action. Therefore \(\phi_k\) induces a surjective map
\[ \phi_k: G^j \to G^j, \quad \rho \mapsto \phi_k(\rho), \]
where \(\phi_k(\rho)\) is the weight of \(\phi_k(m)\) for a monomial \(m \in \mathcal{M}\) of weight \(\rho\).

**Proof.** Since \(M_k\) contains \(M\) as the lattice of \(G_k\)-invariant monomials, \(n\) is in \(M_k\). By definition, the assertions follow. \(\Box\)

**Remark 3.8.** Davis, Logvinenko, and Reid [6] introduced a related construction in a more general setting.

**Lemma 3.9.** For each \(k = 1, 2, 3\), let \(\phi_k\) be the round down function of the weighted blowup with weight \((1, a, r - a)\). Let \(m \in \mathcal{M}\) be a Laurent monomial of weight \(j\).

(i) If \(0 \leq j < r - a\), then \(\phi_2(y \cdot m) = \phi_2(m)\).
(ii) If \(0 \leq j < a\), then \(\phi_3(z \cdot m) = \phi_3(m)\).
(iii) If \(0 \leq j < r - 1\), then \(\phi_1(x \cdot m) = \phi_1(m)\).
(iv) If \(\phi_k(m) = \phi_k(m')\), then \(m = n \cdot m'\) or \(m' = n \cdot m\) for some \(n \in M_{\geq 0}\).

**Proof.** Let \(m = x^{m_1} y^{m_2} z^{m_3}\) be a Laurent monomial of weight \(j\). To prove (i), assume that \(0 \leq j < r - a\). This means that
\[ 0 \leq \frac{1}{r} m_1 + \frac{a}{r} m_2 + \frac{r-a}{r} m_3 - \frac{1}{r} m_1 + \frac{a}{r} m_2 + \frac{r-a}{r} m_3 \leq \frac{r-a}{r}. \]
Thus \(\phi_2(y \cdot m) = \phi_2(x^{m_1} y^{m_2+1} z^{m_3}) = \phi_2(x^{m_1} y^{m_2} z^{m_3}) = \phi_2(m)\). The assertions (ii) and (iii) can be proved similarly. The definition of \(\phi_k\) implies (iv). \(\Box\)

**Lemma 3.10.** For each \(k = 1, 2, 3\), let \(\phi_k\) be the round down function of the weighted blowup with weight \((1, a, r - a)\). Let \(k\) be a lattice point in the monomial lattice \(M_k\) and \(g\) a monomial of degree 1 in \(M_k\). There exist a monomial \(f \in \{x, y, z\}\) of degree 1 and \(m \in \mathcal{M}\) such that
\[ \phi_k(f \cdot m) = g \cdot k \]
satisfying \(\phi_k(m) = k\).
Proof. Here we prove the assertion for the left round down function. Let $\xi, \eta, \zeta$ denote the eigencoordinates for the $G_2$-action. Let $k$ be a monomial in $M_2$ and $g \in \{\xi, \eta, \zeta\}$.

Consider the case where $g = \zeta$. Since $\phi^2$ is surjective, there exists $m = x^{m_1}y^{m_2}z^{m_3} \in \bar{M}$ such that $\phi^2(m) = k$. If $\zeta \cdot k = \phi^2(z \cdot m)$, then we are done.

Suppose $\zeta \cdot k \neq \phi^2(z \cdot m)$. This means that
\[
\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 + \frac{r-a}{r} \geq \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor + 1.
\]
There is a positive integer $l_0^3$ such that $\phi^2(m_{y^l}) = k$ for all $0 \leq l \leq l_0$ with $\phi^2(m_{y^{l_0+1}}) \neq k$. Since $\phi^2(z \cdot m_{y^l}) = \zeta \cdot k$, the assertion follows.

For the other cases, we can prove the assertion similarly. □

3.3. Economic resolutions. By the fact that the quotient variety $X = \mathbb{C}^3/G$ has terminal singularities, $X$ does not admit crepant resolutions. However $X$ has a certain toric resolution introduced by Danilov [5] (see also [17]).

Definition 3.11. Let $G \subset \text{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{r}(1, a, r-a)$. For each $1 \leq i < r$, let $v_i := \frac{1}{r}(i, a, r-a) \in L$ where $\bar{\cdot}$ denotes the residue modulo $r$. The economic resolution of $\mathbb{C}^3/G$ is the toric variety obtained by the consecutive weighted blowups at $v_1, v_2, \ldots, v_{r-1}$ from $\mathbb{C}^3/G$.

Proposition 3.12 (see [17]). Let $\varphi : Y \to X = \mathbb{C}^3/G$ be the economic resolution of $\mathbb{C}^3/G$. For each $1 \leq i < r$, let $E_i$ denote the exceptional divisor of $\varphi$ corresponding to the lattice point $v_i$.

(i) The toric variety $Y$ is smooth and projective over $X$.

(ii) The morphism $\varphi$ satisfies
\[
K_Y = \varphi^*(K_X) + \sum_{1 \leq i < r} \frac{i}{r}E_i.
\]
In particular, each discrepancy is $0 < \frac{i}{r} < 1$.

From the fan of $Y$, we can see that $Y$ can be covered by three open sets $U_2, U_3$ and $U_1$, which are the unions of the affine toric varieties corresponding to the cones on the left side of, the right side of, and below the vector $v = \frac{1}{r}(1, a, r-a)$, respectively. Note that $U_2$ and $U_3$ are isomorphic to the economic resolutions for the singularity of type $\frac{1}{a}(1, -r, r)$ and of type $\frac{1}{r-a}(1, r, -r)$, respectively.

This integer $l_0$ is the largest integer satisfying
\[
\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \frac{a}{r}l \geq \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor.
\]
Remark 3.13. Let $\Sigma$ be the toric fan of the economic resolution $Y$. Note that the number of 3-dimensional cones in $\Sigma$ is $2r - 1$ and that the number of 3-dimensional cones containing $e_1$ is $r$.

Example 3.14. Let $G$ be the group of type $\frac{1}{r}(1, 3, 4)$ as in Example 2.10. The fan of the economic resolution of the quotient variety $\mathbb{C}^3/G$ is shown in Figure 3.3.

![Figure 3.3. Fan of the economic resolution for $\frac{1}{r}(1, 3, 4)$](image)

The toric variety corresponding to the fan consisting of the cones on the left side of $v = \frac{1}{r}(1, 3, 4)$ is the economic resolution of the quotient $\frac{1}{r}(1, 2, 1)$. On the other hand, the toric variety corresponding to the fan consisting of the cones on the right side of $v$ is the economic resolution of the quotient $\frac{1}{r}(1, 3, 1)$.

4. Moduli interpretations of economic resolutions

This section contains our main theorem. First, we explain how to find a set $\mathcal{S}(r, a)$ of $G$-bricks using the round down functions and a recursion process. In Section 4.3, we show that there exists a stability parameter $\theta$ such that every $G$-brick in $\mathcal{S}(r, a)$ is $\theta$-stable.

4.1. $G$-bricks and stability parameters for $\frac{1}{r}(1, r - 1, 1)$. Let $G$ be the group of $\frac{1}{r}(1, r - 1, 1)$ type, i.e. $a = 1$ or $r - 1$. In this case, Kędzierski [10] proved that $G$-Hilb $\mathbb{C}^3$ is isomorphic to the economic resolution of $\mathbb{C}^3/G$. 
4.2. Nakamura's G-graphs

Theorem 4.1 (Kędzierski [10]). Let $G \subset \text{GL}_3(\mathbb{C})$ be the finite group of type $\frac{1}{3}(1, a, r - a)$ with $a = 1$ or $r - 1$. Then $G$-Hilb $\mathbb{C}^3$ is isomorphic to the economic resolution of the quotient variety $\mathbb{C}^3/G$.

For each $1 \leq i < r$, set $v_i = \frac{1}{r}(i, r - i, i)$. Set $v_0 = e_2$ and $v_r = e_3$. The toric fan corresponding to $G$-Hilb $\mathbb{C}^3$ consists of the following $2r - 1$ maximal cones and their faces:

$$\sigma_i = \text{Cone}(e_1, v_{i-1}, v_i) \quad \text{for } 1 \leq i \leq r,$$
$$\sigma_{r+i} = \text{Cone}(e_3, v_{i-1}, v_i) \quad \text{for } 1 \leq i \leq r - 1.$$

Each 3-dimensional cone has a corresponding (Nakamura's) $G$-graph:

$$\Gamma_i = \{1, y, y^2, \ldots, y^{i-1}, z, z^2, \ldots, z^{r-i}\} \quad \text{for } 1 \leq i \leq r,$$
$$\Gamma_{r+i} = \{1, y, y^2, \ldots, y^{i-1}, x, x^2, \ldots, x^{r-i}\} \quad \text{for } 1 \leq i \leq r - 1,$$

with $S(\Gamma_j) = \sigma_j^\vee \cap M$. As the cone $\sigma_j$ is 3-dimensional, the $G$-prebrick $\Gamma_j$ is a $G$-brick. Furthermore, $U(\Gamma_j) = D(\Gamma_j) \cong \mathbb{C}^3$.

By Ito-Nakajima [7], all $G$-bricks in (4.2) are $\theta$-stable for any $\theta \in \Theta_+$, where

$$\Theta_+ := \{\theta \in \Theta \mid \theta(\rho) > 0 \text{ for } \rho \neq \rho_0\}.$$

Example 4.4. Let $G$ be the finite group of type $\frac{1}{3}(1, 2, 1)$ with eigencoordinates $\xi$, $\eta$, $\zeta$. Set $v_1 = \frac{1}{3}(1, 2, 1)$ and $v_2 = \frac{1}{3}(2, 1, 2)$. Recall that the economic resolution $Y$ of $X = \mathbb{C}^3/G$ can be obtained by the sequence of the weighted blowups:

$$Y \xrightarrow{\varphi_2} Y_1 \xrightarrow{\varphi_1} X,$$

where $\varphi_1$ is the weighted blowup with weight $(1, 2, 1)$ and $\varphi_2$ is the toric morphism induced by the weighted blowup with weight $(2, 1, 2)$. The fan corresponding to $Y$ consists of the following five 3-dimensional cones and their faces:

$$\sigma_1 = \text{Cone}(e_1, e_2, v_1), \quad \sigma_2 = \text{Cone}(e_1, v_1, v_2), \quad \sigma_3 = \text{Cone}(e_1, v_2, e_3),$$
$$\sigma_4 = \text{Cone}(e_3, e_2, v_1), \quad \sigma_5 = \text{Cone}(e_3, v_1, v_2).$$

The following

$$\Gamma_1 = \{1, \zeta, \zeta^2\}, \quad \Gamma_2 = \{1, \eta, \zeta\}, \quad \Gamma_3 = \{1, \eta, \eta^2\},$$
$$\Gamma_4 = \{1, \xi, \xi^2\}, \quad \Gamma_5 = \{1, \xi, \eta\}.$$

are their corresponding $G$-bricks. \hfill \Box

4.2. $G$-bricks for $\frac{1}{3}(1, a, r - a)$. In this section, we assign a $G$-brick $\Gamma_\sigma$ with $S(\Gamma_\sigma) = \sigma^\vee \cap M$ to each maximal cone $\sigma$ in the fan of the economic resolution $Y$.

Let $G$ be the group of type $\frac{1}{3}(1, a, r - a)$ with $r$ coprime to $a$, $X$ the quotient $\mathbb{C}^3/G$, and $\varphi: Y \to X$ the economic resolution of $X$. Then $Y$ can be covered by $U_2$, $U_3$ and $U_1$, which are the unions of the affine

\[ U_2 = \text{Cone}(e_1, e_2, e_3), \quad U_3 = \text{Cone}(e_1, v_1, v_2), \quad U_1 = \text{Cone}(e_3, e_2, v_1). \]
toric varieties corresponding to the cones on the left side of, the right side of, and below the lattice point \( v = \frac{1}{r}(1, a, r - a) \), respectively.

**Proposition 4.5.** For \( k = 1, 2, 3 \), let \( \Gamma' \) be a \( G_k \)-brick. Define

\[
\Gamma := \{ \mathbf{m} \in \overline{M} \mid \phi_k(\mathbf{m}) \in \Gamma' \}.
\]

The set \( \Gamma \) is a \( G \)-brick with \( S(\Gamma) = S(\Gamma') \).

**Proof.** Since \( \phi_k(\mathbf{1}) = \mathbf{1} \in \Gamma' \), we have \( \mathbf{1} \in \Gamma \). To show that \( \Gamma \) satisfies (ii) in Definition 2.8, we need to show that there exists a unique monomial of weight \( \rho \) in \( \Gamma \) for each \( \rho \in G' \). Fix a positive integer \( i \) such that the weight of \( x^i \) is \( \rho \). Consider the monomial \( \phi_k(x^i) \) in \( M_k \) and its weight \( \chi \in G'_k \). Since \( \Gamma' \) is a \( G_k \)-brick, there exists a unique element \( \mathbf{k}_\chi \) of weight \( \chi \). Since the \( G_k \)-invariant monomial \( \frac{\mathbf{k}_\chi}{\phi_k(x^i)} \) is in the lattice \( M \), it follows from Lemma 3.7 that

\[
\phi_k : x^i \cdot \left( \frac{1}{\phi_2(x^i)} \right) \mapsto \mathbf{k}_\chi,
\]

i.e. \( x^i \cdot \left( \frac{1}{\phi_2(x^i)} \right) \) is in \( \Gamma \). To show uniqueness, assume that two monomials \( \mathbf{m}, \mathbf{m}' \) of the same weight are mapped into \( \Gamma' \). As \( \phi_k(\mathbf{m}) \) and \( \phi_k(\mathbf{m}') \) are of the same weight, we have \( \phi_k(\mathbf{m}) = \phi_k(\mathbf{m}') \in \Gamma' \). From Lemma 3.7, it follows that \( \phi_k(\mathbf{m}) = \phi_k \left( \mathbf{m}' \cdot \frac{\mathbf{m}}{\mathbf{m}'} \right) = \phi_k(\mathbf{m}') \cdot \frac{\mathbf{m}}{\mathbf{m}'} \).

and hence \( \mathbf{m} = \mathbf{m}' \). From Lemma 3.10, it follows that \( \Gamma \) is connected as \( \Gamma' \) is connected.

For (iii) in Definition 2.8, assume that \( \mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma \) for \( \mathbf{m}_\rho \in \Gamma' \) and \( \mathbf{n} \in \overline{M} \). We need to show \( \phi_k(\mathbf{n} \cdot \mathbf{m}_\rho) \in \Gamma' \). From

\[
\phi_k(\mathbf{n} \cdot \mathbf{m}_\rho) = \phi_k \left( \mathbf{n} \cdot \frac{\mathbf{m}_\rho}{\mathbf{m}_\rho} \right) \cdot \phi_k(\mathbf{m}_\rho) = \phi_k(\mathbf{m}_\rho) \in \Gamma',
\]

it follows that \( \phi_k(\mathbf{n} \cdot \mathbf{m}_\rho) = \phi_k(\mathbf{m}_\rho) \) is in \( \Gamma' \) because \( \Gamma' \) is a \( G_k \)-prebrick. This proves that \( \Gamma \) is a \( G \)-prebrick.

It remains to prove that \( S(\Gamma) = S(\Gamma') \). Note that \( S(\Gamma) \) is generated by \( \frac{n \cdot m_\rho}{\text{wt}_\Gamma(n \cdot m_\rho)} \) for \( n \in \overline{M} \) and \( m_\rho \in \Gamma' \). Let \( \mathbf{k}_\chi \) denote \( \phi_k(\mathbf{m}_\rho) \in \Gamma' \). Define \( \mathbf{k} \) to be \( \frac{\phi_k(\mathbf{m}_\rho)}{\phi_k(\mathbf{m}_\rho)} \). From the definition of the round down functions, we know that \( \mathbf{k} \) is a genuine monomial in \( \xi, \eta, \zeta \). Since \( \frac{n \cdot m_\rho}{\text{wt}_\Gamma(n \cdot m_\rho)} \) is \( G \)-invariant, it follows that

\[
\frac{n \cdot m_\rho}{\text{wt}_\Gamma(n \cdot m_\rho)} = \phi_k(\frac{n \cdot m_\rho}{\text{wt}_\Gamma(n \cdot m_\rho)}) = \phi_k(\phi_k(\frac{n \cdot m_\rho}{\text{wt}_\Gamma(n \cdot m_\rho)})) = \frac{k \cdot \mathbf{k}_\chi}{\text{wt}_{\Gamma'}(k \cdot \mathbf{k}_\chi)}
\]

from Lemma 3.7. This proves \( S(\Gamma) \subset S(\Gamma') \).

For the opposite inclusion, by Lemma 2.11, it suffices to show that \( g \cdot \frac{\mathbf{k}_\chi}{\text{wt}_{\Gamma'}(g \cdot \mathbf{k}_\chi)} \) is in \( S(\Gamma) \) for all \( g \in \{ \xi, \eta, \zeta \} \) and \( \mathbf{k}_\chi \in \Gamma' \). By Lemma 3.10
there are \( n \in \overline{M}_{\geq 0}, m_\rho \in \Gamma \) such that \( \phi_k(n \cdot m_\rho) = g \cdot k_\chi \). Lemma 4.6 implies that \( \mathrm{wt}_\Gamma(g \cdot k_\chi) = \phi_k(\mathrm{wt}_\Gamma(n \cdot m_\rho)) \). Thus
\[
\frac{g \cdot k_\chi}{\mathrm{wt}_\Gamma(g \cdot k_\chi)} = \frac{\phi_k(n \cdot m_\rho)}{\mathrm{wt}_\Gamma(\phi_k(n \cdot m_\rho))} = \frac{\phi_k(n \cdot m_\rho)}{\phi_k(\mathrm{wt}_\Gamma(n \cdot m_\rho))} = \frac{n \cdot m_\rho}{\mathrm{wt}_\Gamma(n \cdot m_\rho)},
\]
and we proved the proposition. \( \square \)

**Lemma 4.6.** With the notation as in Proposition 4.5, if \( m \in \overline{M} \), then
\[
\mathrm{wt}_\Gamma(\phi_k(m)) = \phi_k(\mathrm{wt}_\Gamma(m)).
\]

*Proof.* Since \( \phi_k(m) \) is of the same weight as \( \phi_k(\mathrm{wt}_\Gamma(m)) \) by Lemma 3.7, the assertion follows from the fact that \( \phi_k(\mathrm{wt}_\Gamma(m)) \in \Gamma' \). \( \square \)

**Lemma 4.7.** With the notation as in Proposition 4.5, let \( m \) be the Laurent monomial of weight \( j \) in \( \Gamma = \{ m \in \overline{M} \mid \phi_k(m) \in \Gamma' \} \).

(i) If \( k = 2 \) and \( 0 \leq j < r - a \), then \( \phi_2(y \cdot m) \in \Gamma \).

(ii) If \( k = 3 \) and \( 0 \leq j < a \), then \( \phi_3(z \cdot m) \in \Gamma \).

(iii) If \( k = 1 \) and \( 0 \leq j < r - 1 \), then \( \phi_1(x \cdot m) \in \Gamma \).

*Proof.* Lemma 3.9 implies the assertion. \( \square \)

**Proposition 4.8.** Let \( G \) be the group of type \( \frac{1}{r}(1, a, r - a) \) with \( r \) coprime to \( a \). Let \( \Sigma_{\max} \) be the set of maximal cones in the fan of the economic resolution \( Y \) of \( X = \mathbb{C}^3/G \). Then there exists a set \( \Sigma(r, a) \) of \( G \)-bricks such that there is a bijective map \( \Sigma_{\max} \to \Sigma(r, a) \) sending \( \sigma \) to \( \Gamma_\sigma \) satisfying \( S(\Gamma_\sigma) = \sigma^\vee \cap M \). In particular, \( U(\Gamma_\sigma) \) is isomorphic to the smooth toric variety \( U_\sigma = \text{Spec} \mathbb{C}[\sigma^\vee \cap M] \) corresponding to \( \sigma \).

*Proof.* From Section 4.1, the assertion holds when \( a = 1 \) or \( r - 1 \). We use induction on \( r \) and \( a \).

Let \( \sigma \) be a 3-dimensional cone in the fan of the economic resolution \( Y \) of \( X = \mathbb{C}^3/G \). For \( v = \frac{1}{r}(1, a, r - a) \), we have three cases:

1. the cone \( \sigma \) is below the vector \( v \).
2. the cone \( \sigma \) is on the left side of the vector \( v \).
3. the cone \( \sigma \) is on the right side of the vector \( v \).

**Case (1) the cone \( \sigma \) is below the vector \( v \).** Since there is a unique 3-dimensional cone below \( v \), \( \sigma = \text{Cone}(v, e_2, e_3) \). Consider the central round down function \( \phi_1 \) of the weighted blowup with weight \( (1, a, r - a) \). For \( m = x^m_1 y^m_2 z^m_3 \in \overline{M} \), note that
\[
\phi_1(m) = 1 \quad \text{if and only if} \quad m_2 = m_3 = 0 \quad \text{and} \quad 0 \leq \frac{m_1}{r} < 1.
\]

The set \( \Gamma := \phi^{-1}_1(1) = \{ 1, x, x^2, \ldots, x^{r - 1} \} \) is a \( G \)-prebrick with the property \( S(\Gamma) = \sigma^\vee \cap M \). Since the corresponding cone \( \sigma(\Gamma) \) is equal to \( \sigma \), the \( G \)-prebrick \( \Gamma \) is a \( G \)-brick.
Case (2) the cone $\sigma$ is on the left side of $v$. From the fan of the economic resolution, it follows that $U_2$ is isomorphic to the economic resolution $Y_2$ of $\frac{1}{a}(1,-r,r)$ with eigencoordinates $\xi, \eta, \zeta$. There exists a unique 3-dimensional cone $\sigma'$ in the toric fan of $Y_2$ corresponding to $\sigma$. Let $G_2$ be the group of type $\frac{1}{a}(1,-r,r)$. Note that $a$ is strictly less than $r$ so that we can use induction.

Assume that there exists a $G_2$-brick $\Gamma'$ with $S(\Gamma') = (\sigma')^\vee \cap M$. By Proposition 4.5, there is a $G$-brick $\Gamma$ with $S(\Gamma) = S(\Gamma') = \sigma^\vee \cap M$.

Case (3) the cone $\sigma$ is on the right side of $v$. The case where the cone $\sigma$ is on the right side of $v$ can be proved similarly. □

Definition 4.9. A $G$-brick $\Gamma$ in $\mathcal{G}(r,a)$ described above is called a Danilov $G$-brick.

Proposition 4.10. With the notation as in Proposition 4.5, we have $D(\Gamma') \cong D(\Gamma)$. Moreover we have a commutative diagram

$$
\begin{array}{ccc}
U(\Gamma') & \xrightarrow{\cong} & U(\Gamma) \\
\downarrow & & \downarrow \\
D(\Gamma') & \xrightarrow{\cong} & D(\Gamma)
\end{array}
$$

with the vertical morphisms closed embeddings. Therefore for a $G$-brick $\Gamma \in \mathcal{G}(r,a)$, we have $U(\Gamma) = D(\Gamma) \cong \mathbb{C}^3$.

Proof. Let $\Gamma$ be a $G$-brick and $\Gamma'$ the corresponding $G_k$-brick. Let $\xi, \eta, \zeta$ denote the eigencoordinate for the $G_k$-action. From (2.19), the coordinate rings of the affine schemes $D(\Gamma)$, $D(\Gamma')$ are

$$
\mathbb{C}[D(\Gamma)] = \mathbb{C}[x_\rho, y_\rho, z_\rho \mid \rho \in G^\vee]/I_\Gamma,
\mathbb{C}[D(\Gamma')] = \mathbb{C}[\xi, \eta, \zeta \mid \chi \in G^\vee_k]/I_{\Gamma'}
$$

where the ideal $I_\Gamma$ is $\langle \text{the quadrics in (2.18), } f_\rho - 1 \mid f_\rho \in \Lambda(\Gamma) \rangle$ and the ideal $I_{\Gamma'}$ is $\langle \text{the commutative relations, } g_\chi - 1 \mid g_\chi \in \Lambda(\Gamma') \rangle$.

By Lemma 3.10, we have an algebra epimorphism

$$
\mu: \mathbb{C}[x_\rho, y_\rho, z_\rho \mid \rho \in G^\vee] \rightarrow \mathbb{C}[D(\Gamma)] \quad f_\rho \mapsto k(\chi)
$$

defined as follows on the $3r$ generators $f_\rho \in \{x_\rho, y_\rho, z_\rho\}$. Let $m_\rho$ be the unique element of weight $\rho$ in $\Gamma$ and $k$ the weight of $\phi_k(f \cdot m_\rho)$. Then $k := \frac{\phi_k(f \cdot m_\rho)}{\phi_k(m_\rho)}$ is a monomial, so $k$ induces a linear map $k(\chi)$ on the vector space $\mathbb{C} \cdot \phi_k(m_\rho)$. Then $\mu$ is the morphism sending $f_\rho$ to $k(\chi)$. Since the generators of $I_\Gamma$ are in ker $\mu$, $\mu$ induces an epimorphism $\overline{\mu}: \mathbb{C}[D(\Gamma)] \rightarrow \mathbb{C}[D(\Gamma')]$.

To construct the inverse of $\overline{\mu}$, first we show that if $\mu(f_\rho) = \mu(f'_\rho)$, then $f_\rho \equiv f'_\rho \mod I_\Gamma$. If $\mu(f_\rho) = \mu(f'_\rho)$, then $\phi_k(f \cdot m_\rho) = \phi_k(f' \cdot m_{\rho'})$ and $\phi_k(m_\rho) = \phi_k(m_{\rho'})$. Since both $f_\rho$ and $f'_\rho$ are degree of 1, $f = f'$. By (iv) in Lemma 3.9, we may assume that $m_{\rho'} = n \cdot m_\rho$ for some $n \in M_{\geq 0}$. Since $\phi_k(m_\rho) = \phi_k(m_{\rho'})$, $n$ induces a linear map equal to
Lemma 4.11. In the situation as in Proposition 4.10, define

\[ S := \left\{ f_\rho \in \{ x_\rho, y_\rho, z_\rho \} \mid \frac{\phi_k(f_{\rho m})}{\phi_k(m_{\rho})} \text{ is of degree } \leq 1 \right\}. \]

If \( \frac{\phi_k(f_{\rho m})}{\phi_k(m_{\rho})} \) is of degree \( \geq 2 \) for some \( f_\rho \in \{ x_\rho, y_\rho, z_\rho \} \), then \( f_\rho \) can be written as a multiple of some elements in \( S \) modulo \( I_\Gamma \).

Proof. We prove this for the left round down function \( \phi_2 \). Note that \( \frac{\phi_2(y_{\rho m})}{\phi_2(m_{\rho})} \) is of degree \( \leq 1 \) for all \( \rho \in G' \). Thus \( y_\rho \)'s are in \( S \).

Suppose that \( \frac{\phi_k(f_{\rho m})}{\phi_k(m_{\rho})} \) is of degree \( \geq 2 \) with \( m_{\rho} = x_{m_1} y_{m_2} z_{m_3} \). Then the monomial \( f \) is either \( x \) or \( z \). In the case where \( f = z \), this means that

\[ \frac{1}{r} m_1 + \frac{a}{r} m_2 + \frac{r - a}{r} m_3 - \left( \frac{1}{r} m_1 + \frac{a}{r} m_2 + \frac{r - a}{r} m_3 \right) \geq a. \]

As in the proof of Lemma 3.10, there is a positive integer \( l \) such that \( \phi_2(\frac{m_{\rho}}{y^{l'}}) = \phi_2(m_{\rho}) \) for all \( 0 \leq l' \leq l \) with \( \phi_2(\frac{m_{\rho}}{y^{l+1}}) \neq \phi_2(m_{\rho}) \). Note
that \( \phi_2(f \cdot m_{\rho}) \) is of degree 1 where \( m_{\rho'} = \frac{m_{\rho}}{y'} \). Thus \( f_{\rho'} \in S \). From the commutation relations

\[
\begin{align*}
\mathbb{C} \cdot m_{\rho'} & \xrightarrow{y'} \mathbb{C} \cdot m_{\rho} \\
\mathbb{C} \cdot \text{wt}_\Gamma(f \cdot m_{\rho'}) & \xrightarrow{y'} \mathbb{C} \cdot \text{wt}_\Gamma(f \cdot m_{\rho}),
\end{align*}
\]

since \( y' \) induces a linear map on \( \mathbb{C} \cdot m_{\rho} \) set to be 1, we have

\[
f_{\rho} \equiv f_{\rho} \cdot y'_{(\rho')} \equiv y'_{(\rho')} \cdot f'_{\rho} \mod I_{\Gamma}.
\]

As all \( y_{\rho}' \)’s are in \( S \), the assertion follows. \( \square \)

**Example 4.12.** Let \( G \) be the group of type \( \frac{1}{3}(1,3,4) \) as in Example 2.10. The fan of the economic resolution of the quotient variety is shown in Figure 3.3.

We now calculate \( G \)-bricks associated to the following cones:

\[
\begin{align*}
\sigma_1 & := \text{Cone} \left( (1,0,0), \frac{1}{3}(1,3,4), \frac{1}{3}(3,2,5) \right), \\
\sigma_2 & := \text{Cone} \left( (1,0,0), \frac{1}{3}(6,4,3), \frac{1}{3}(1,3,4) \right).
\end{align*}
\]

Note that the left side of the fan corresponds to the economic resolution for the quotient singularity of type \( \frac{1}{3}(1,2,1) \), which is \( G_2 \)-Hilb \( \mathbb{C}^3 \), where \( G_2 \) is of type \( \frac{1}{3}(1,2,1) \). Let \( \xi, \eta, \zeta \) denote the eigencoordinates. Let \( \sigma_1' \) be the cone in the fan of \( G_2 \)-Hilb \( \mathbb{C}^3 \) which corresponds to \( \sigma_1 \). Observe that the corresponding \( G_2 \)-brick is

\[
\Gamma_1' = \{ 1, \zeta, \zeta^2 \}.
\]

Since the left round down function \( \phi_2 \) is

\[
\phi_2(x^{m_1}y^{m_2}z^{m_3}) = \xi^{m_1}\eta^{\frac{1}{3}m_1+\frac{2}{3}m_2+\frac{4}{3}m_3}\zeta^{m_3},
\]

**Figure 4.1.** Recursion process for \( \frac{1}{3}(1,3,4) \)
the $G$-brick corresponding to $\sigma_1$ is
\[
\Gamma_1 = \left\{ x^{m_1} y^{m_2} z^{3m_3} \in M \mid \phi_2(x^{m_1} y^{m_2} z^{3m_3}) \in \Gamma_1' \right\}
\]
\[
= \left\{ 1, y, y^2, z, y^2, \frac{y^2}{y^2} \right\}.
\]

On the other hand, the right side of the fan corresponds to the economic resolution of the quotient variety $\frac{1}{4}(1, 3, 1)$ which is $G_3\text{-}\text{Hilb} \mathbb{C}^3$, where $G_3$ is of type $\frac{1}{2}(1, 3, 1)$ with eigencoordinates $\alpha, \beta, \gamma$. Let $\sigma_2'$ be the cone in the fan of $G_3\text{-}\text{Hilb} \mathbb{C}^3$ which corresponds to $\sigma_2$. Observe that the corresponding $G_3$-brick is
\[
\Gamma_2' = \{ 1, \beta, \beta^2, \beta^3 \}.
\]
Since the right round down function $\phi_3$ is
\[
\phi_3(x^{m_1} y^{m_2} z^{3m_3}) = \alpha^{m_1} \beta^{m_2} \gamma^{m_3},
\]
the $G$-brick corresponding to $\sigma_2$ is
\[
\Gamma_2 = \left\{ x^{m_1} y^{m_2} z^{3m_3} \in M \mid \phi_2(x^{m_1} y^{m_2} z^{3m_3}) \in \Gamma_2' \right\}
\]
\[
= \left\{ 1, z, y, y^2, \frac{y^2}{y^2} \right\}.
\]

From Example 2.13, $\sigma(\Gamma_1) = \sigma_1$ and $\sigma(\Gamma_2) = \sigma_2$.

4.3. **Stability parameters for $\mathfrak{S}(r, a)$.** Let $G \subseteq \text{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{2}(1, a, r - a)$ with $r$ coprime to $a$. We may assume $2a < r$. Let $G_2$ and $G_3$ be the groups of type $\frac{1}{a}(1, -r, r)$ and of type $\frac{1}{r-a}(1, r, -r)$, respectively.

Given stability conditions $\theta^{(2)}$ for Danilov $G_2$-bricks and $\theta^{(3)}$ for Danilov $G_3$-bricks, take a GIT parameter $\theta_P \in \Theta$ satisfying the following system of linear equations:

\[
\begin{cases}
\theta^{(2)}(\chi) = \sum_{\phi_2(\rho) = \chi} \theta_P(\rho) & \text{for all } \chi \in G_2', \\
\theta^{(3)}(\chi') = \sum_{\phi_3(\rho) = \chi'} \theta_P(\rho) & \text{for all } \chi' \in G_3'.
\end{cases}
\]

Define the GIT parameter $\psi \in \Theta$ by

\[
\psi(\rho) = \begin{cases}
-1 & \text{if } 0 \leq \text{wt}(\rho) < a, \\
1 & \text{if } r - a \leq \text{wt}(\rho) < r, \\
0 & \text{otherwise}.
\end{cases}
\]

Observe that $\sum_{\phi_k(\rho) = \chi} \psi(\rho) = 0$ for all $\chi \in G_k'$. For a sufficiently large natural number $m$, set

\[
\theta := \theta_P + m \psi.
\]

\[\text{In addition, if any } \theta \in \Theta \text{ satisfies that } \sum_{\phi_k(\rho) = \chi} \theta(\rho) = 0 \text{ for all } \chi \in G_k' \text{ and } k = 2, 3, \text{ then } \theta \text{ must be a constant multiple of } \psi. \] This also explains the existence of a solution $\theta_P$ for (4.13).
We claim that every $\Gamma \in \mathcal{G}(r, a)$ is $\theta$-stable.

**Example 4.16.** As in Example 4.12, let $G$ be the group of type $\frac{1}{3}(1,3,4)$. For each $0 \leq i \leq 6$, let $\rho_i$ denote the irreducible representation of $G$ whose weight is $i$. We saw that the left side of the fan is $G_2$-Hilb $\mathbb{C}^3$, where $G_2$ is of type $\frac{1}{3}(1,2,1)$ and that the right side of the fan is $G_3$-Hilb $\mathbb{C}^3$, where $G_3$ is of type $\frac{1}{3}(1,3,1)$. Let $\{\chi_0, \chi_1, \chi_2\}$ and $\{\chi'_0, \chi'_1, \chi'_2, \chi'_3\}$ be the characters of $G_2$ and $G_3$, respectively. Take GIT parameters $\theta(2), \theta(3)$ corresponding to $G$-Hilb such as (see (4.3)):

$$\theta(2) = (-2, 1, 1), \quad \theta(3) = (-3, 1, 1, 1).$$

We have the following system of linear equations:

$$\begin{cases}
-2 = \theta_P(\rho_0) + \theta_P(\rho_3) + \theta_P(\rho_6), \\
1 = \theta_P(\rho_1) + \theta_P(\rho_4), \\
1 = \theta_P(\rho_2) + \theta_P(\rho_5), \\
-3 = \theta_P(\rho_0) + \theta_P(\rho_4), \\
1 = \theta_P(\rho_1) + \theta_P(\rho_5), \\
1 = \theta_P(\rho_2) + \theta_P(\rho_6), \\
1 = \theta_P(\rho_3).
\end{cases}$$

Take $\theta_P = (-1, 3, 3, 1, -2, -2, -2)$ as a partial solution. For the parameter $\psi = (-1, -1, -1, 0, 1, 1, 1)$, define $\theta = \theta_P + m\psi$ for large $m$.

Consider the following $G$-brick

$$\Gamma = \left\{1, y, y^2, z, \frac{x}{y}, \frac{x^2}{y}, \frac{x^2}{y} \right\}.$$

Let $\mathcal{F}$ be the submodule of $C(\Gamma)$ with basis $A = \{z, \frac{x}{y}, \frac{x^2}{y}\}$. Note that $\psi(\mathcal{F}) > 0$ and

$$\phi^{-1}_2(\phi_2(A)) = \left\{z, \frac{x}{y}, \frac{x^2}{y}, \frac{x^2}{y} \right\} \supseteq A.$$ 

Thus $\theta(\mathcal{F})$ is positive for large enough $m$. More precisely,

$$\theta(\mathcal{F}) = 3 - m + (-2 + m) + (-2 + m) = m - 1$$

is positive if $m > 1$.

On the other hand, consider the submodule $\mathcal{G}$ of $C(\Gamma)$ with basis $B = \{\frac{x}{y}, \frac{x^2}{y}\}$. Note that $\psi(\mathcal{G}) = 0$ and $\phi^{-1}_2(\phi_2(B)) = B$. In this case, the set $\phi_2(B)$ gives a submodule $\mathcal{G}'$ of $C(\Gamma')$ with

$$\theta(2)(\mathcal{G}') = \theta(\mathcal{G}).$$

Since $C(\Gamma')$ is $\theta(2)$-stable, $\theta(2)(\mathcal{G}')$ is positive. Hence $\theta(\mathcal{G})$ is positive. $\Diamond$

**Lemma 4.17.** Let $\theta$ be the parameter in (4.15). For the set $\mathcal{G}(r, a)$ in Proposition 4.8, if $\Gamma$ is in $\mathcal{G}(r, a)$, then $\Gamma$ is $\theta$-stable.

**Proof.** Let $\Gamma$ be a $G$-brick in $\mathcal{G}$ and $\sigma$ be the cone corresponding to $\Gamma$. We have the following three cases as in Section 4.2:

1. the cone $\sigma$ is below the vector $v$. 

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(2) the cone $\sigma$ is on the left side of the vector $v$.
(3) the cone $\sigma$ is on the right side of the vector $v$.

In Case (1), $\Gamma = \{1, x, x^2, \ldots, x^{r-2}, x^{r-1}\}$. By Lemma 2.14, any nonzero proper submodule $G$ of $C(\Gamma)$ is given by

$$A = \{x^j, x^{j+1}, \ldots, x^{r-2}, x^{r-1}\}$$

for some $1 \leq j \leq r - 1$. Since $\psi(G) > 0$, $\Gamma$ is $\theta$-stable for sufficiently large $m$.

We now consider Case (2). Let $\Gamma'$ be the $G_2$-brick corresponding to $\Gamma$. Let $G$ be a submodule of $C(\Gamma)$ with $\mathbb{C}$-basis $A \subset \Gamma$. Lemma 3.9 and Lemma 2.14 imply that if $m_\rho \in A$ for $0 \leq \text{wt}(m_\rho) < a$, then $\phi_2^{-1}(\phi_2(m_\rho)) \subset A$. Thus $\psi(G) \geq 0$ from the definition of $\psi$.

If $\psi(G) > 0$, then it follows that $\theta(G) > 0$ for sufficiently large $m$.

Let us assume that $\psi(G) = 0$. Note that $A = \phi_2^{-1}(\phi_2(A))$; otherwise there exists $m_\rho$ in $\phi_2^{-1}(\phi_2(A)) \setminus A$ with $0 \leq \text{wt}(m_\rho) < a$. To show that $\theta(G)$ is positive, we prove that $\phi_2(A)$ gives a submodule $G'$ of $C(\Gamma')$ and that $\theta(G) = \theta(G')$. Since $\theta$ satisfies the equations (4.13), it suffices to show that $\phi_2(A)$ gives a submodule of $C(\Gamma')$. Let $\xi, \eta, \zeta$ be the coordinates of $\mathbb{C}^3$ with respect to the action of $G_2$. By Lemma 2.14, it is enough to show that if $g \cdot \phi_2(m_\rho) \in \Gamma'$ for some $g \in \{\xi, \eta, \zeta\}$ and $m_\rho \in A$, then $g \cdot \phi_2(m_\rho) \in \phi_2(A)$. Suppose that $g \cdot \phi_2(m_\rho) \in \Gamma'$ for some $m_\rho \in A$. By Lemma 3.10, there exists $m_{\rho'}$ such that

$$\phi_2(f \cdot m_{\rho'}) = g \cdot \phi_2(m_\rho)$$

with $\phi_2(m_{\rho'}) = \phi_2(m_\rho)$ for some $f \in \{x, y, z\}$. In particular, $f \cdot m_{\rho'}$ is in $\Gamma$. Since $A = \phi_2^{-1}(\phi_2(A))$, we have $m_{\rho'} \in A$, which implies $f \cdot m_{\rho'} \in A$ as $A$ is a $\mathbb{C}$-basis of $G$. Thus $g \cdot \phi_2(m_\rho)$ is in $\phi_2(A)$. \qed

Remark 4.18. At this moment, our stability parameter $\theta$ in (4.15) has nothing to do with Kedzierski’s GIT chamber $\mathcal{C}(r, a)$ described in [11]. In Section 5, it is shown that the parameter $\theta$ is in $\mathcal{C}(r, a)$. \diamond

4.4. Main Theorem.

Theorem 4.19. The economic resolution $Y$ of a 3-fold terminal quotient singularity $X = \mathbb{C}^3/G$ is isomorphic to the birational component $Y_\theta$ of the moduli space $M_\theta$ of $\theta$-stable $G$-constellations for a suitable parameter $\theta$.

Proof. From Proposition 4.8 and Lemma 4.17, Proposition 2.21 implies that there exists an open immersion from $Y$ to $Y_\theta$ fitting in the following commutative diagram:

$$
\begin{array}{ccc}
Y & \rightarrow & Y_\theta \\
\downarrow \mathcal{M} & & \downarrow \\
X
\end{array}
$$
Since both $Y$ and $Y_\theta$ are projective over $X$, the open immersion $Y \to Y_\theta$ is a closed embedding. As both $Y$ and $Y_\theta$ are 3-dimensional and irreducible, this embedding is an isomorphism. □

**Conjecture 4.20.** The moduli space $\mathcal{M}_\theta$ is irreducible.

Proposition 4.10 implies that the irreducible component $Y_\theta$ is actually a connected component. In addition, if every torus invariant $\theta$-stable $G$-constellation lies over the birational component $Y_\theta$, then $\mathcal{M}_\theta$ is irreducible. For $a = 2$, we can prove Conjecture 4.20 so the economic resolution is isomorphic to $\mathcal{M}_\theta$ for $\theta \in \mathcal{G}(r, a)$ (See [8]). We hope to establish this more generally in future work.

**Remark 4.21.** By construction, $\mathcal{M}_0 = \text{Spec} \mathbb{C}[\text{Rep} G]^{GL(\delta)}$ is the moduli space of 0-semistable $G$-constellations up to $S$-equivalence. Since there exists an algebra isomorphism $\mathbb{C}[\text{Rep} G]^{GL(\delta)} \to \mathbb{C}[x, y, z]^G$, $\mathcal{M}_0$ is isomorphic to $\mathbb{C}^3/G$. In particular, $\mathcal{M}_0$ is irreducible. ♦

5. Kędzierski’s GIT chamber

Kędzierski [11] described his GIT cone in $\Theta$ using a set of inequalities. Using his lemma, we can prove further that the cone is actually a GIT chamber $\mathcal{C}$. In this section, we provide a description of $\mathcal{C}$ using the $A_{r-1}$ root system. Define

$$\mathcal{S}(r,a)_0 = \{ \Gamma \in \mathcal{S}(r,a) \mid x \not\in \Gamma \}.$$  

**Kędzierski’s lemma.** By the same argument as in Lemma 6.7 of [11], we can prove that it suffices to check the $\theta$-stability for $G$-bricks $\Gamma$ not containing $x$.

**Lemma 5.1** (Kędzierski’s lemma [11]). For a parameter $\theta \in \Theta$, the following are equivalent.

(i) Every $\Gamma \in \mathcal{S}(r,a)$ is $\theta$-stable.

(ii) Every $\Gamma \in \mathcal{S}(r,a)_0$ is $\theta$-stable.

Let $A$ be the finite group of type $\frac{1}{r}(a, r - a)$. Since $A \cong G$ as groups, the GIT parameter space $\Theta$ of $G$-constellations can be canonically identified with that of $A$-constellations.

Since $G$-constellations which $x$ acts trivially on are supported on the hyperplane $(x = 0) \subset \mathbb{C}^3$, they can be considered as $A$-constellations. As $\Gamma \in \mathcal{S}(r,a)_0$ is the set of $G$-bricks corresponding to $G$-constellations supported on $(x = 0) \subset \mathbb{C}^3$, Lemma 5.1 implies that the GIT chamber for $\mathcal{S}(r,a)$ is equal to a GIT chamber of $A$-constellations.

**Kędzierski’s GIT chamber.** We describe a set of simple roots $\Delta$ so that $Y_\theta$ is isomorphic to the economic resolution for $\theta \in \mathcal{C}(\Delta)$. After considering the case of $a = 1$, we describe simple roots for the case of $\frac{1}{r}(1, a, r - a)$ using a recursion process.
Root system $A_{r-1}$. Identify $I := \text{Irr}(G)$ with $\mathbb{Z}/r\mathbb{Z}$. Let $\{\varepsilon_i \mid i \in I\}$ be an orthonormal basis of $\mathbb{Q}^r$, i.e. $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$. Define

$$
\Phi := \{\varepsilon_i - \varepsilon_j \mid i, j \in I, i \neq j\}.
$$

Let $\mathfrak{h}^*$ be the subspace of $\mathbb{Q}^r$ generated by $\Phi$. Elements in $\Phi$ are called roots.

For each nonzero $i \in I$, set $\alpha_i = \varepsilon_i - \varepsilon_{i-a}$. Let $\rho_i$ denote the irreducible representation of $G$ of weight $i$. Note that each root $\alpha$ can be considered as the support of a submodule of a $G$-constellation. In other words, $\alpha_i$ corresponds to the dimension vector of $\rho_i$. In general we consider a root $\alpha = \sum n_i \alpha_i$ as the dimension vector of the representation $\oplus n_i \rho_i$. Abusing notation, let $\alpha = \sum n_i \alpha_i$ denote the corresponding representation $\oplus n_i \rho_i$.

Let $\Delta$ be a set of simple roots. Define $C(\Delta) \subset \Theta$ associated to $\Delta$ as

$$
C(\Delta) := \{\theta \in \Theta \mid \theta(\alpha) > 0 \ \forall \alpha \in \Delta\}.
$$

Note that for the cone $\Theta_+$ for $G$-Hilb in (4.3), the corresponding set of simple roots is

$$
\Delta_+ = \{\varepsilon_i - \varepsilon_{i-a} \in \Phi \mid i \in I, i \neq 0\} = \{\alpha_i \mid i \in I, i \neq 0\}.
$$

The case of $\frac{1}{r}(1, r - 1, 1)$. From Theorem 4.1, we know that the economic resolution of $X = \mathbb{C}^3/G$ is isomorphic to $G$-Hilb $\mathbb{C}^3$ if $G$ is of type $\frac{1}{r}(1, r - 1, 1)$. Thus in this case, the $G$-bricks are just Nakamura’s $G$-graphs, which are $\theta$-stable for $\theta \in \Theta_+$, where

$$
\Theta_+ := \{\theta \in \Theta \mid \theta(\rho) > 0 \ \forall \rho \neq \rho_0\}.
$$

The corresponding set of simple roots is

$$
\Delta = \{\varepsilon_i - \varepsilon_{i+1} \in \Phi \mid i \in I, i \neq 0\}

= \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{r-1} - \varepsilon_0\}.
$$

Example 5.2. For the group of type $\frac{1}{3}(1, 2, 1)$, let $\{\varepsilon_j^L \mid j = 0, 1, 2\}$ be the standard basis of $\mathbb{Q}^3$. The corresponding set of simple roots is

$$
\Delta^L = \{\varepsilon_1^L - \varepsilon_2^L, \varepsilon_2^L - \varepsilon_0^L\}.
$$

Similarly, for the group of type $\frac{1}{4}(1, 3, 1)$ with $\{\varepsilon_k^R \mid k = 0, 1, 2, 3\}$ the standard basis of $\mathbb{Q}^4$,

$$
\Delta^R = \{\varepsilon_1^R - \varepsilon_2^R, \varepsilon_2^R - \varepsilon_3^R, \varepsilon_3^R - \varepsilon_0^R\}
$$

is the corresponding set of simple roots for type $\frac{1}{4}(1, 3, 1)$.  
The case of $\frac{1}{a}(1, a, r - a)$. Let $G$ be the group of type $\frac{1}{a}(1, a, r - a)$. Let $\Delta^L$ and $\Delta^R$ denote the sets of simple roots for the types of $\frac{1}{a}(1, -r, r)$ and of $\frac{1}{a}(1, r, -r)$, respectively. As in Section 5, let

$$\{\varepsilon^L_l \mid l = 0, 1, \ldots, a - 1\}, \quad \{\varepsilon^R_k \mid k = 0, 1, \ldots, r - a - 1\}$$

be the standard basis of $\mathbb{Q}^a$ and $\mathbb{Q}^{r-a}$, respectively.

From the two sets $\Delta^L$ and $\Delta^R$, we construct a set $\Delta$ of simple roots in $A_{r-1}$ as follows. First, as in Section 5, let the standard basis $\{\varepsilon_i \mid i \in I\}$ of $\mathbb{Q}^r$ be identified with the union of the two sets

$$\{\varepsilon^L_l \mid l = 0, 1, \ldots, a - 1\} \text{ and } \{\varepsilon^R_k \mid k = 0, 1, \ldots, r - a - 1\}$$

using the following identification:

$$\varepsilon^L_l = \varepsilon_i \quad \text{with} \quad i \equiv l \mod a \quad \text{if} \quad r - a \leq i < r,$$

$$\varepsilon^R_k = \varepsilon_i \quad \text{with} \quad i \equiv k \mod (r-a) \quad \text{if} \quad 0 \leq i < r - a.$$

With the identification above, define $\Delta$ to be

$$\Delta = \Delta^L \cup \{\varepsilon_{\frac{r-1}{a}} \mid a - \varepsilon_{\frac{r-1}{a}}(r-a)-a\} \cup \Delta^R.$$

The root $\varepsilon_{\frac{r-1}{a}} \mid a - \varepsilon_{\frac{r-1}{a}}(r-a)-a$ is called the added root in $\Delta$. Note that $\Delta$ is actually a set of simple roots in $A_{r-1}$.

**Definition 5.5.** With $\Delta$ as above, the corresponding Weyl chamber

$$\mathcal{C}(r, a) := \mathcal{C}(\Delta) = \{\theta \in \Theta \mid \theta(\alpha) > 0 \quad \forall \alpha \in \Delta\}$$

is called Kêdzierski’s GIT chamber for $G = \frac{1}{a}(1, a, r - a)$.

**Proposition 5.6.** Let $\mathcal{C}(r, a)$ be Kêdzierski’s GIT chamber.

(i) The parameter $\psi$ in (4.14) is a ray of $\mathcal{C}(r, a)$.

(ii) Any $G$-brick in $\mathcal{G}(r, a)$ is $\theta$-stable for $\theta \in \mathcal{C}(r, a)$.

(iii) The cone $\mathcal{C}(r, a)$ is a full GIT chamber.

**Proof.** We may assume $a < r - a$. First, by construction, $\psi$ is zero on the sets $\Delta^L$ and $\Delta^R$ with the identification (5.4). To prove (i), it remains to show that $\psi(\alpha)$ is positive where $\alpha$ is the added root in $\Delta$. Since

$$\alpha = \varepsilon_{\frac{r-1}{a}} \mid a - \varepsilon_{\frac{r-1}{a}}(r-a)-a = \sum_{\phi_2(\rho_i) = \chi_0} a_i + \alpha r-a,$$

where $\chi_0$ is the trivial representation of $G_2$, (i) follows.

For $\theta$ defined by (4.15), every $\Gamma \in \mathcal{G}(r, a)_0$ is $\theta$-stable. For the group $A$ of type $\frac{1}{a}(a, -a)$, Kronheimer [13] showed that the chamber structure of the GIT parameter space of $A$-constellations is the same as the Weyl chamber structure of $A_{r-1}$.

Thus for $\mathcal{G}(r, a)_0$ considered as $A$-constellations, we have a Weyl chamber of the $A_{r-1}$ root system containing the parameter $\theta$.

By Kêdzierski’s lemma, to prove (ii), it suffices to show that $\mathcal{C}(r, a)$ contains the parameter $\theta$. Observe that every parameter in $\mathcal{C}(r, a)$

\({^5}\text{For an explicit description, see Section 5.1 in [8]}


satisfies the system of equations (4.13) for some \( \theta^{(2)} \in \mathcal{C}(a,-r) \) and \( \theta^{(3)} \in \mathcal{C}(r-a,r) \) by construction. Since \( \psi \) in (4.14) is a ray of the chamber \( \mathcal{C}(r,a) \), it follows that \( \theta \in \mathcal{C}(r,a) \).

It remains to prove (iii). By considering \( G \)-constellations supported on the hyperplane \( (x = 0) \subset \mathbb{C}^3 \), it follows that any facet of \( \mathcal{C}(r,a) \) is an actual GIT wall in \( \Theta \). Therefore Kędzierski’s GIT chamber \( \mathcal{C}(r,a) \) is a full GIT chamber in the stability parameter space \( \Theta \) (see [8,9]).

**Proposition 5.7.** Assume that \( a < r - a \). Let \( \theta \) be an element in \( \mathcal{C}(r,a) \). Then \( \theta(\alpha_i) \) is negative if and only if \( 0 \leq i < a \). Thus any \( \theta \)-stable \( G \)-constellation is generated by \( \rho_0, \rho_1, \ldots, \rho_{a-1} \).

**Proof.** Let \( \Delta \) be the set of simple roots corresponding to \( \mathcal{C}(r,a) \). Recall that any positive sum of simple roots is positive on \( \theta \).

Suppose that \( 0 \leq i < a \). From the identification (5.3), note that \( \varepsilon_i \) is identified with \( \varepsilon_k^R \) for some \( k \) and that \( \varepsilon_{i-a} = \varepsilon_{i+(r-a)} \) is identified with \( \varepsilon_l^L \) for some \( l \). Note that \( \varepsilon^{(\frac{r-i}{a})} \) is identified with a vector \( \varepsilon^L \) and that \( \varepsilon^{(\frac{r-i}{a})} \) is identified with a vector \( \varepsilon^R \). Since we added the root \( \varepsilon^{(\frac{r-i}{a})} \) to \( \Delta \), the root \( \alpha_i = \varepsilon_i - \varepsilon_{i-a} = \varepsilon_k^R - \varepsilon_l^L \) is a negative sum of simple roots in \( \Delta \).

Suppose that \( a \leq i < r - a \). The root \( \alpha_i = \varepsilon_i - \varepsilon_{i-a} \) is a sum of simple roots in \( \Delta^R \). A recursive argument yields that \( \alpha_i \) is a positive sum of simple roots in \( \Delta^R \). Thus \( \alpha_i \) is a positive sum of simple roots in \( \Delta \).

Consider the case where \( r - a \leq i < r \) and the root \( \alpha_i = \varepsilon_i - \varepsilon_{i-a} \). From the identification (5.3), \( \varepsilon_i \) is identified with \( \varepsilon_k^L \) for some \( k \) and \( \varepsilon_{i-a} \) is identified with \( \varepsilon_l^R \) for some \( l \). Thus \( \alpha_i = \varepsilon_k^L - \varepsilon_l^R \) is a positive sum of simple roots in \( \Delta \) with the same reason as the case where \( 0 \leq i < a \).

**Example 5.8.** Let \( G \) be the group of type \( \frac{1}{4}(1,3,4) \). From the fan of the economic resolution of this case (see Example 3.14), the left and right sides are the economic resolutions of singularities of \( \frac{1}{4}(1,2,1) \) and \( \frac{1}{4}(1,3,1) \), respectively. By Example 5.2, we have two sets

\[
\Delta^L = \{ \varepsilon_1^L - \varepsilon_2^L, \varepsilon_2^L - \varepsilon_0^L \} \quad \text{and} \quad \Delta^R = \{ \varepsilon_1^R - \varepsilon_2^R, \varepsilon_2^R - \varepsilon_3^R, \varepsilon_3^R - \varepsilon_0^R \}.
\]

As in the construction (5.4), the corresponding set of simple roots is

\[
\Delta = \{ \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_0, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3 - \varepsilon_0 \}
\]

\[
= \{ \alpha_1 + \alpha_3 + \alpha_2, -\alpha_1 - \alpha_3 - \alpha_2, \alpha_1 + \alpha_5, \alpha_2 + \alpha_6, \alpha_3 \},
\]

where the underlined root is the added root as in (5.4). Thus the set of parameters \( \theta \in \Theta \) satisfying

\[
\theta(\rho_4 \oplus \rho_1) > 0, \quad \theta(\rho_5 \oplus \rho_2) > 0, \quad \theta(\rho_1 \oplus \rho_5 \oplus \rho_2) < 0,
\]

\[
\theta(\rho_1 \oplus \rho_5) > 0, \quad \theta(\rho_2 \oplus \rho_6) > 0, \quad \theta(\rho_3) > 0
\]

is Kędzierski’s GIT chamber \( \mathcal{C}(r,a) \) where \( \rho_i \) is the irreducible representation of \( G \) of weight \( i \).
Figure 6.1. Toric fan of the economic resolution for $\frac{1}{12}(1, 7, 5)$

The rays of the chamber $\mathcal{C}(r, a)$ are the row vectors of the matrix

$$
\begin{pmatrix}
-1 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & -1 & 0 & 0 & 1 & 1 \\
-1 & -1 & -1 & 0 & 1 & 1 & 1 \\
-1 & -1 & 0 & 0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

with the dual basis $\{\theta_i\}$ with respect to $\{\rho_i\}$. Observe that for any $\theta \in \mathcal{C}(r, a)$, $\theta(\rho_i)$ is negative if and only if $0 \leq i < 3$.

6. Example: Type $\frac{1}{12}(1, 7, 5)$

In this section, as a concrete example, we calculate Danilov $G$-bricks and the corresponding set of simple roots $\Delta$ for the group $G$ of type $\frac{1}{12}(1, 7, 5)$. 
Let $G$ be the finite group of type $\frac{1}{12}(1,7,5)$ with eigencoordinates $x, y, z$ and $L$ the lattice $L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{12}(1,7,5)$. Let $X$ denote the quotient variety $\mathbb{C}^3/G$ and $Y$ the economic resolution of $X$. The toric fan $\Sigma$ of $Y$ is shown in Figure 6.1.

To use the recursion process in Section 4, first we need to investigate the cases of type $\frac{1}{4}(1,2,5)$ and of type $\frac{1}{3}(1,2,3)$. Let $G_2$ be the group of type $\frac{1}{4}(1,2,5)$ with eigencoordinates $\xi_2, \eta_2, \zeta_2$ and $G_3$ be the group of type $\frac{1}{3}(1,2,3)$ with eigencoordinates $\xi_3, \eta_3, \zeta_3$. Consider the toric fans $\Sigma_2$ and $\Sigma_3$ of the economic resolutions for the type $\frac{1}{4}(1,2,5)$ and the type $\frac{1}{3}(1,2,3)$, respectively.

**G-bricks.** We now calculate $G$-bricks corresponding to the following two maximal cones in $\Sigma$:

\[
\sigma_4 = \text{Cone} \left( \frac{1}{12}(12,0,0), \frac{1}{12}(3,9,3), \frac{1}{12}(8,8,4) \right), \\
\tau_3 = \text{Cone} \left( \frac{1}{12}(1,7,5), \frac{1}{12}(3,9,3), \frac{1}{12}(8,8,4) \right).
\]

The cones $\sigma_4, \tau_3$ are on the right side of the lowest vector $v = \frac{1}{12}(1,7,5)$. Their corresponding cones $\sigma'_3, \tau'_3$ in $\Sigma_3$, respectively, are

\[
\sigma'_3 = \text{Cone} \left( \frac{1}{5}(5,0,0), \frac{1}{5}(1,2,3), \frac{1}{5}(1,1,4) \right), \\
\tau'_3 = \text{Cone} \left( \frac{1}{5}(0,0,5), \frac{1}{5}(1,2,3), \frac{1}{5}(1,1,4) \right).
\]

Observe that the cones $\sigma'_4, \tau'_3$ are on the left side of $\Sigma_3$. To use the recursion, let $G_{32}$ be the group of type $\frac{1}{5}(1,1,1)$ with eigencoordinates $\xi_{32}, \eta_{32}, \zeta_{32}$. Let $\Sigma_{32}$ denote the fan of the economic resolution of the quotient $\mathbb{C}^3/G_{32}$. In $\Sigma_{32}$, there exist two cones $\sigma''_4, \tau''_3$ corresponding to $\sigma'_4, \tau'_3$, respectively:

\[
\sigma''_4 = \text{Cone} \left( \frac{1}{2}(2,0,0), \frac{1}{2}(0,2,0), \frac{1}{2}(1,1,1) \right), \\
\tau''_3 = \text{Cone} \left( \frac{1}{2}(0,0,2), \frac{1}{2}(0,2,0), \frac{1}{2}(1,1,1) \right).
\]

**Figure 6.2.** Recursion process for $\frac{1}{12}(1,7,5)$

As in Section 4.1, the $G_{32}$-bricks $\Gamma''_4, \Gamma''_3$ corresponding to $\sigma''_4, \tau''_3$ are

\[
\Gamma''_4 = \{1, \zeta_{23}\}, \\
\Gamma''_3 = \{1, \xi_{23}\}.
\]
Using the left round down function $\phi_3$ for $\frac{1}{7}(1,2,3)$

\[
\phi_3 : \xi_3^a y^b z^c \mapsto \xi_3^a y^{a+2b+3c} 
\]

we can see that the $G_3$-bricks $\Gamma_4', \Gamma_3'$ corresponding to $\sigma_4', \tau_3'$ are

\[
\Gamma_4' \overset{\text{def}}{=} \phi_3^{-1}(\Gamma_4') = \{1, \eta_3, \eta_3^2, \zeta_3, \xi_3 \}, \\
\Gamma_3' \overset{\text{def}}{=} \phi_3^{-1}(\Gamma_2') = \{1, \eta_3, \eta_3^2, \zeta_3, \xi_3 \eta_3 \}.
\]

To get the $G$-bricks $\Gamma_4$ and $\Gamma_3$ corresponding to $\sigma_4$ and $\tau_3$, respectively, we use the right round down function $\phi_3$ for $\frac{1}{12}(1,7,5)$:

\[
\phi_3 : x^a y^b z^c \mapsto \xi_3^a y^{a+7b+5c}.
\]

We get

\[
\Gamma_4 \overset{\text{def}}{=} \phi_3^{-1}(\Gamma_4') = \{1, y, y^2, z, z^2, z^3, z^4, z^5, z^6 \}, \\
\Gamma_3 \overset{\text{def}}{=} \phi_3^{-1}(\Gamma_2') = \{1, x, xz, xz^2, y, y, y, z^2, z^3 \}.
\]

Let us consider the following two cones in $\Sigma$:

\[
\sigma_9 = \text{Cone} \left( \frac{1}{12}(12,0,0), \frac{1}{12}(9,3,9), \frac{1}{12}(4,4,8) \right), \\
\tau_7 = \text{Cone} \left( \frac{1}{12}(2,2,10), \frac{1}{12}(9,3,9), \frac{1}{12}(4,4,8) \right).
\]

Observe that the cones $\sigma_9$, $\tau_7$ are on the left side of $v$. The cones in $\Sigma_2$ corresponding to $\sigma_9$, $\tau_7$ are

\[
\sigma_9' = \text{Cone} \left( \frac{1}{7}(12,0,0), \frac{1}{7}(5,3,4), \frac{1}{7}(2,4,3) \right), \\
\tau_7' = \text{Cone} \left( \frac{1}{7}(1,2,5), \frac{1}{7}(5,3,4), \frac{1}{7}(2,4,3) \right).
\]

Note that the cones $\sigma_9'$, $\tau_7'$ are on the right side of the fan $\Sigma_2$ and that the right side is equal to the fan $\Sigma_3$ of the economic resolution for $\frac{1}{5}(1,2,3)$. Moreover, the cones in $\Sigma_3$ corresponding to $\sigma_9'$, $\tau_7'$ are $\sigma_9$, $\tau_7$, respectively, in (6.1). Thus the corresponding $G_{23}$-bricks $\Gamma_9', \Gamma_7'$ are:

\[
\Gamma_9' = \{1, \eta_23, \eta_23^2, \zeta_23, \xi_23 \}, \\
\Gamma_7' = \{1, \xi_23, \xi_23^2, \eta_23, \eta_23^2 \},
\]

where $G_{23}$ denotes the group of type $\frac{1}{5}(1,2,3)$ with eigencoordinates $\xi_{23}, \eta_{23}, \zeta_{23}$. Using the right round down function $\phi_{23}$ for $\frac{1}{5}(1,2,5)$

\[
\phi_{23} : \xi_{23}^a y^{2b} z^c \mapsto \xi_{23}^a y^{a+2b+3c} 
\]

we can calculate the $G_2$-bricks corresponding to $\sigma_9$, $\tau_7'$:

\[
\Gamma_9' \overset{\text{def}}{=} \phi_{23}^{-1}(\Gamma_9') = \{1, \eta_2, \eta_2^2, \zeta_2, \xi_2, \xi_2 \}, \\
\Gamma_7' \overset{\text{def}}{=} \phi_{23}^{-1}(\Gamma_7') = \{1, \xi_2, \xi_2 \eta_2, \xi_2, \eta_2, \zeta_2, \eta_2, \zeta_2 \}.
\]
Lastly, from the left round down function $\phi_2$ for $\frac{1}{12}(1, 7, 5)$

$$\phi_2: x^a y^b z^c \mapsto \xi_2 \xi_{a+b+c}^{12} \xi_2,$$

it follows that the $G$-bricks $\Gamma_9, \Gamma_7$ corresponding to $\sigma_9, \tau_7$ are:

$$\Gamma_9 = \left\{ 1, y, y^2, y^3, y^4, z, z^2, z^2, z^2, z^2 \right\},$$

$$\Gamma_7 = \left\{ 1, x, xy, xy^2, xy^3, xz, y, y^2, y^3, y^4, y^5, z \right\}.$$
| Cone  | Generators | $G$-brick $\Gamma_\sigma$ | Coordinates on $U_\sigma$ |
|-------|------------|--------------------------|--------------------------|
| $\sigma_1$ | $e_1, e_2, v_{11}$ | $1, z, z^2, z^3, z^4, z^5, z^6, z^7, z^8, z^9, z^{10}, z^{11}$ | $\frac{z}{z^{11}}, \frac{y}{y}, \frac{z}{z^{10}}, \frac{z}{z^9}$ |
| $\sigma_2$ | $e_1, e_{10}, v_{11}$ | $1, y, y^2, z, z^2, z^3, z^4, z^5, z^6, z^7, z^8, z^9$ | $\frac{y^2}{y}, \frac{z^2}{z^{10}}, \frac{z^3}{z^9}, \frac{z^4}{z^8}$ |
| $\sigma_3$ | $e_1, v_9, v_{10}$ | $1, y, y^2, y^3, y^4, y^5, y^6, y^7, y^8, y^9, y^{10}, y^{11}$ | $\frac{y^5}{y}, \frac{y^4}{y^3}, \frac{y^3}{y^2}, \frac{y^2}{y}$ |
| $\sigma_4$ | $e_1, v_8, v_9$ | $1, y, y^2, y^3, y^4, y^5, y^6, y^7, y^8, y^9, y^{10}, y^{11}$ | $\frac{y^5}{y}, \frac{y^4}{y^3}, \frac{y^3}{y^2}, \frac{y^2}{y}$ |
| $\sigma_5$ | $e_1, v_7, v_8$ | $1, y, y^2, y^3, y^4, y^5, y^6, y^7, y^8, y^9, y^{10}, y^{11}$ | $\frac{y^5}{y}, \frac{y^4}{y^3}, \frac{y^3}{y^2}, \frac{y^2}{y}$ |
| $\sigma_6$ | $e_1, v_6, v_7$ | $1, y, y^2, y^3, y^4, y^5, y^6, y^7, y^8, y^9, y^{10}, y^{11}$ | $\frac{y^5}{y}, \frac{y^4}{y^3}, \frac{y^3}{y^2}, \frac{y^2}{y}$ |
| $\sigma_7$ | $e_1, v_5, v_6$ | $1, y, y^2, y^3, y^4, y^5, y^6, y^7, y^8, y^9, y^{10}, y^{11}$ | $\frac{y^5}{y}, \frac{y^4}{y^3}, \frac{y^3}{y^2}, \frac{y^2}{y}$ |
| $\sigma_8$ | $e_1, v_4, v_5$ | $1, y, y^2, y^3, y^4, y^5, y^6, y^7, y^8, y^9, y^{10}, y^{11}$ | $\frac{y^5}{y}, \frac{y^4}{y^3}, \frac{y^3}{y^2}, \frac{y^2}{y}$ |
| $\sigma_9$ | $e_1, v_3, v_4$ | $1, y, y^2, y^3, y^4, y^5, y^6, y^7, y^8, y^9, y^{10}, y^{11}$ | $\frac{y^5}{y}, \frac{y^4}{y^3}, \frac{y^3}{y^2}, \frac{y^2}{y}$ |
| $\sigma_{10}$ | $e_1, v_2, v_3$ | $1, y, y^2, y^3, y^4, y^5, y^6, y^7, y^8, y^9, y^{10}, y^{11}$ | $\frac{y^5}{y}, \frac{y^4}{y^3}, \frac{y^3}{y^2}, \frac{y^2}{y}$ |
| $\sigma_{11}$ | $e_1, v_1, v_2$ | $1, y, y^2, y^3, y^4, y^5, y^6, y^7, y^8, y^9, y^{10}, y^{11}$ | $\frac{y^5}{y}, \frac{y^4}{y^3}, \frac{y^3}{y^2}, \frac{y^2}{y}$ |
| $\sigma_{12}$ | $e_1, e_3, v_1$ | $1, y, y^2, y^3, y^4, y^5, y^6, y^7, y^8, y^9, y^{10}, y^{11}$ | $\frac{y^5}{y}, \frac{y^4}{y^3}, \frac{y^3}{y^2}, \frac{y^2}{y}$ |
| $\tau_1$ | $e_2, v_9, v_{11}$ | $1, x, xz, x^2z, x^3z, x^4z, x^5z, x^6z, x^7z, x^8z, x^9z, x^{10}z, x^{11}z$ | $\frac{z}{z^{11}}, \frac{x}{x^{10}}, \frac{z}{z^9}$ |
| $\tau_2$ | $v_9, v_{10}, v_{11}$ | $1, x, xz, x^2z, x^3z, x^4z, x^5z, x^6z, x^7z, x^8z, x^9z, x^{10}z, x^{11}z$ | $\frac{z}{z^{11}}, \frac{x}{x^{10}}, \frac{z}{z^9}$ |
| $\tau_3$ | $v_7, v_8, v_9$ | $1, x, xy, yz, xz, x^2z, x^3z, x^4z, x^5z, x^6z, x^7z, x^8z, x^9z, x^{10}z, x^{11}z$ | $\frac{z}{z^{11}}, \frac{x}{x^{10}}, \frac{z}{z^9}$ |
| $\tau_4$ | $e_2, v_7, v_9$ | $1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}z, x^{11}z$ | $\frac{z}{z^{11}}, \frac{x}{x^{10}}, \frac{z}{z^9}$ |
| $\tau_5$ | $v_4, v_6, v_7$ | $1, x, xy, xz, x^2z, x^3z, x^4z, x^5z, x^6z, x^7z, x^8z, x^9z, x^{10}z, x^{11}z$ | $\frac{z}{z^{11}}, \frac{x}{x^{10}}, \frac{z}{z^9}$ |
| $\tau_6$ | $v_4, v_5, v_6$ | $1, x, xy, xz, x^2z, x^3z, x^4z, x^5z, x^6z, x^7z, x^8z, x^9z, x^{10}z, x^{11}z$ | $\frac{z}{z^{11}}, \frac{x}{x^{10}}, \frac{z}{z^9}$ |
| $\tau_7$ | $v_2, v_3, v_4$ | $1, x, xy, x^2z, x^3z, x^4z, x^5z, x^6z, x^7z, x^8z, x^9z, x^{10}z, x^{11}z$ | $\frac{z}{z^{11}}, \frac{x}{x^{10}}, \frac{z}{z^9}$ |
| $\tau_8$ | $v_2, v_4, v_7$ | $1, x, xy, xz, x^2z, x^3z, x^4z, x^5z, x^6z, x^7z, x^8z, x^9z, x^{10}z, x^{11}z$ | $\frac{z}{z^{11}}, \frac{x}{x^{10}}, \frac{z}{z^9}$ |
| $\tau_9$ | $e_3, v_1, v_2$ | $1, x, xy, x^2z, x^3z, x^4z, x^5z, x^6z, x^7z, x^8z, x^9z, x^{10}z, x^{11}z$ | $\frac{z}{z^{11}}, \frac{x}{x^{10}}, \frac{z}{z^9}$ |
| $\tau_{10}$ | $e_3, v_2, v_7$ | $1, x, xy, x^2z, x^3z, x^4z, x^5z, x^6z, x^7z, x^8z, x^9z, x^{10}z, x^{11}z$ | $\frac{z}{z^{11}}, \frac{x}{x^{10}}, \frac{z}{z^9}$ |
| $\tau_{10}$ | $e_2, e_3, v_7$ | $1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}z, x^{11}z$ | $\frac{z}{z^{11}}, \frac{x}{x^{10}}, \frac{z}{z^9}$ |

**Table 6.1.** $G$-bricks for $G = \frac{1}{12}(1, 7, 5)$
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