THE RIGOROUS DERIVATION OF THE $\mathbb{T}^2$ FOCUSING CUBIC NLS FROM 3D

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Abstract. We derive rigorously the 2D periodic focusing cubic NLS as the mean-field limit of the 3D focusing quantum many-body dynamics describing a dilute Bose gas with periodic boundary condition in the $x$-direction and a well of infinite-depth in the $z$-direction. Physical experiments for these systems are scarce. We find that, to fulfill the empirical requirement for observing NLS dynamics in experiments, namely, the kinetic energy dominates the potential energy, it is necessary to impose an extra restriction on the system parameters. This restriction gives rise to an unusual coupling constant.

1. Introduction

Bose-Einstein condensate (BEC) is a state of matter occurring in a dilute gas of bosons, where all particles take the same quantum state. The first experimental observation of BEC in an interacting atomic gas occurred in 1995, using laser cooling techniques [2, 29].

Let $t \in \mathbb{R}$ be the time variable and $r_N = (r_1, r_2, ..., r_N) \in \mathbb{R}^{3N}$ be the position vector of $N$ particles in $\mathbb{R}^3$. Then BEC naively means that the $N$-body wave function $\psi_N(t, r_N)$ satisfies

$$\psi_N(t, r_N) \sim \prod_{j=1}^{N} \varphi(t, r_j)$$

up to a phase factor solely depending on $t$, for some one particle state $\varphi$. That is, every particle takes the same quantum state. Equivalently, there is the Penrose-Onsager formulation of BEC: if we take $\gamma_N^{(k)}$ be the $k$-particle marginal densities associated with $\psi_N$ by

$$\gamma_N^{(k)}(t, r_k, r'_k) = \int \psi_N(t, r_k, r_{N-k}) \overline{\psi_N(t, r'_k, r_{N-k})} d\mathbf{r}_{N-k}, \quad r_k, r'_k \in \mathbb{R}^{3k}.$$  

Then BEC equivalently means

$$\gamma_N^{(k)}(t, r_k, r'_k) \sim \prod_{j=1}^{k} \varphi(t, r_j) \overline{\varphi(t, r'_j)}.$$  

It is widely believed that the cubic nonlinear Schrödinger equation (NLS)

$$i\partial_t \varphi = -\Delta \varphi + \mu |\varphi|^2 \varphi,$$

which is called focusing if $\mu < 0$ and defocusing $\mu > 0$, describes BEC in the sense that $\varphi$ satisfies NLS. In this paper, we are interested in the focusing case. There have been many physical experiments [28, 30, 46, 65] and mathematical results [52, 21, 54, 23, 24, 55, 58] regarding the focusing case. However, from the experiment [28], one infers that not only it is very difficult to prove the 3D focusing NLS as the mean-field limit of a 3D focusing quantum many-body dynamic, but such a limit also may not be true. Thus, in focusing settings, both physical experiments and mathematical results emphasize one dimensional and two dimensional behaviours. To our knowledge, physical experiments regarding the two dimensional behavior in the real-world three dimensional setting are limited and the corresponding mathematical research only studies the two dimensional behaviour in 2D. Therefore, we turn our attention to the derivation of 2D focusing NLS from 3D. Interestingly, our analysis produces an unusual microscopic-to-macroscopic coupling constant and might provide some suggestions to the experiment. To expect a two-dimensional behaviour, we should confine a large number of bosons inside a trap with strong confinement in one direction. We consider a simple physical model, namely, quantum many-body dynamics with periodic boundary condition in the $x$-direction.

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and a well of infinite-depth in the z-direction\(^1\). Such model with strong restriction in one direction was first considered by Schnee and Yngvason\(^2\) for the defocusing time-independent problem. Then, the defocusing time-dependent 3D-to-2D program was studied by X. Chen and Holmer in\(^3\), in which they used the quadratic potential \(| \cdot |^2\) to represent the trap and considered the following Hamiltonian

\[
H_{N,\omega} = \sum_{j=1}^{N} (-\Delta_{r_j} + \omega^2 z_j^2) + \frac{1}{N \sqrt{\omega}} \sum_{1 \leq i < j \leq N} (N \sqrt{\omega})^\beta (\omega z_i - \omega z_j).
\]

(1.4)

Here, we model the trap by using a well of infinite-depth in the z-direction. That is, we consider the Hamiltonian

\[
H_{N,L} = \sum_{j=1}^{N} -\Delta_{r_j} + \frac{L}{N-1} \sum_{1 \leq i < j \leq N} V_{N,L}(r_i - r_j).
\]

(1.5)

acting on the Hilbert space \(L^2(\Omega^\otimes L)^N\), the subspace of \(L^2(\Omega^\otimes L)^N\) consisting of functions that are symmetric with respect to permutations of the \(N\) particles, where

\[
V_{N,L}(r_i - r_j) = (N/L)^{3\beta} V ((N/L)^\beta (r_i - r_j))
\]

and the domain\(^4\) \(\Omega_L = (-\pi, \pi)^2 \times (-L\pi/2, L\pi/2)\). As \(L \to 0\), we see that the particles are strongly confined in the \(z\)-direction. We remark that the system parameter \(L\) in (1.5) is corresponding to \(\omega^{-1/2}\) in (1.4). For more detailed analysis of system parameters, see\(^5\).

We take the periodic boundary condition\(^3\) in the \(x\)-direction and Dirichlet boundary condition in the \(z\)-direction. We will derive rigorously \(T^2\) focusing cubic NLS from the 3D quantum many-body dynamic. For simplicity, we take \(\cos_L(z) = (2/\pi)^{1/2} \cos(z/L)/L^2\), which is the normalized ground state eigenfunction. With the lowest energy, we notice that, as \(L \to 0\), \(\cos_L(z)\) has infinite energy. Thus, our main theorem is better to be stated regarding the renormalization.

Let \(\psi_{N,L}(t, \cdot) = e^{itH_{N,L}} \psi_{N,L}(0, \cdot)\) denote the evolution of this initial data corresponding to the Hamiltonian operator (1.5). Define the rescaled solution

\[
\tilde{\psi}_{N,L}(t, r_N) \overset{\text{def}}{=} L^{N/2} \psi_{N,L}(t, x_N, Lz_N), \quad r_N \in \mathbb{T}^{2N} \times (-\pi/2, \pi/2)^N,
\]

and the rescaled Hamiltonian

\[
\tilde{H}_{N,L} = \sum_{j=1}^{N} \left( -\Delta_{x_j} - \frac{1}{L^2} \tilde{\omega}^2_{z_j} \right) + \frac{1}{N-1} \sum_{1 \leq i < j \leq N} \tilde{V}_{N,L}(r_i - r_j),
\]

(1.7)

where

\[
\tilde{V}_{N,L}(r) = L (N/L)^{3\beta} V ((N/L)^\beta x, (N/L)^\beta z).
\]

(1.8)

Then

\[
\left( \tilde{H}_{N,L,\tilde{\psi}_{N,L}}(t, x_N, z_N) = L^{N/2} (H_{N,L,\psi_{N,L}}(t, x_N, Lz_N),
\]

and hence, we have

\[
\tilde{\psi}_{N,L}(t, r_N) = e^{it\tilde{H}_{N,L}} \tilde{\psi}_{N,L}(0, r_N).
\]

(1.9)

**Definition 1.1.** We denote \(C_{gn}\) the sharp constant of the 2D inhomogeneous Gagliardo-Nirenberg estimate\(^4\) on torus:

\[
\| \phi \|_{L^4(T^2)} \leq C_{gn} \| \phi \|^{\frac{2}{3}}_{L^2(T^2)} \| \sqrt{1 - \Delta} \phi \|^{\frac{2}{3}}_{L^2(T^2)}.
\]

(1.10)

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\(^1\)Our exact proof works for the case which puts \(\mathbb{R}^2\) in the \(x\)-direction. We choose \(\mathbb{T}^2\) here, considering all of these limits problem originated from the thermodynamic limit on \(\mathbb{T}^3\) (see a survey in\(^3\)).

\(^2\)When \(L = 1\), we take \(\Omega = \Omega_1\) for convenience.

\(^3\)To match the periodic condition, \(V_{N,L}(r)\) is considered as the periodic extension in the \(x\)-direction of the rescaled \(V\) which is compactly supported on \(\Omega\).

\(^4\)There are many versions of the Gagliardo-Nirenberg inequalities on \(\mathbb{T}^2\). Our proof works more or less the same.
Theorem 1.2. Assume $L(N/L)^{β} \to 1^{-\frac{3}{2}}$ and the pair interaction $V$ is an even nonpositive smooth function compactly support on $Ω$ such that $∥V∥_{L_{x}^{2}L_{r}^{∞}} \leq \frac{2ν}{C_{ν}}$ for some $α \in (0, 1)$. Let $\{\tilde{γ}_{N,L}^{(k)}(t, r_{k}, r_{k}′)\}$ be the family of marginal densities associated with the 3D rescaled Hamiltonian evolution $\tilde{ψ}_{N,L}(t) = e^{it\tilde{H}_{N,L}}\tilde{ψ}_{N,L}(0)$ for $β \in (0, 1/3)$. Suppose the initial datum $\tilde{ψ}_{N,L}(0)$ satisfies the following:

(i) $\tilde{ψ}_{N,L}(0)$ is normalized, that is, $∥\tilde{ψ}_{N,L}(0)∥_{L_{x}^{2}} = 1$,

(ii) $\tilde{ψ}_{N,L}(0)$ is asymptotically factorized in the sense that

$$\lim_{N,L \to ∞} Trs[\tilde{γ}_{N,L}^{(1)}(0, x_{1}, z_{1}; x_{1}′, z_{1}′)] = 0,$$

for some one particle state $φ_{0} \in H^{1}(T^{2})$.

(iii) Away from the $z$-direction ground state energy, $\tilde{ψ}_{N,L}(0)$ has finite energy per particle

$$\sup_{N,L} (\tilde{ψ}_{N,L}(0), (N^{-1}\tilde{H}_{N,L} - 1/L^{2})\tilde{ψ}_{N,L}(0)) \leq C.$$

Then $∀k ≥ 1, t ≥ 0$, we have the convergence in trace norm that

$$\lim_{N,L \to ∞} Trs[\tilde{γ}_{N,L}^{(k)}(t, x_{k}, z_{k}; x_{k}′, z_{k}′)] = 0,$$

where $φ(t, x)$ solves the 2D periodic focusing cubic NLS with constant $g_{0} = \frac{4}{π^{2}} \int \int V(x, z_{1} - z_{2})dx|cos(z_{1})cos(z_{2})|^{2}dz_{1}dz_{2} < 0$,

that is

$$i∂_{t}φ = -\Delta xφ + g_{0}|φ|^2φ,$$

with initial condition $φ(0, x) = φ_{0}(x)$.

It is well-known that Theorem 1.2 is equivalent to Theorem 1.3 by the method of Erdős, Schlein, and Yau [32, 33, 36, 34, 35].

Theorem 1.3. Assume $L(N/L)^{β} \to 1^{-\frac{3}{2}}$ and the pair interaction $V$ is an even nonpositive smooth function compactly support on $Ω$ such that $∥V∥_{L_{x}^{2}L_{r}^{∞}} \leq \frac{2ν}{C_{ν}}$ for some $α \in (0, 1)$. Let $\{\tilde{γ}_{N,L}^{(k)}(t, r_{k}, r_{k}′)\}$ be the family of marginal densities associated with the 3D rescaled Hamiltonian evolution $\tilde{ψ}_{N,L}(t) = e^{it\tilde{H}_{N,L}}\tilde{ψ}_{N,L}(0)$ for $β \in (0, 1/3)$. Suppose the initial datum $\tilde{ψ}_{N,L}(0)$ is normalized asymptotically factorized and satisfies the energy condition that

(iii') there is a constant $C > 0$ such that

$$\sup_{N,L} (\tilde{ψ}_{N,L}(0), (N^{-1}\tilde{H}_{N,L} - 1/L^{2})\tilde{ψ}_{N,L}(0)) \leq C^{k}, \quad ∀k ≥ 1.$$

Then $∀k ≥ 1, t ≥ 0$, we have the convergence in trace norm that

$$\lim_{N,L \to ∞} Trs[\tilde{γ}_{N,L}^{(k)}(t, x_{k}, z_{k}; x_{k}′, z_{k}′)] = 0,$$

where $φ(t, x)$ solves the 2D periodic focusing cubic NLS with the coupling constant $g_{0} = \frac{4}{π^{2}} \int \int V(x, z_{1} - z_{2})dx|cos(z_{1})cos(z_{2})|^{2}dz_{1}dz_{2} < 0$,

that is

$$i∂_{t}φ = -\Delta xφ + g_{0}|φ|^2φ,$$

with initial condition $φ(0, x) = φ_{0}(x)$.

We use the notation $L(N/L)^{β} \to 1^{-\frac{3}{2}}$ to denote $L(N/L)^{β} \leq 1$ and $L(N/L)^{β} \to 1$. 

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We notice that Theorems 1.2 - 1.3 carry an extra requirement \( L(N/L)^\beta \to 1^- \) and a different coupling constant\(^6\), if compared to the previous work, for example \([19, 23, 24]\), in which the constant is usually \( \int V \) or the scattering length of \( V \). It emerges from the empirical requirement for observing NLS dynamics in experiments, namely, the kinetic energy dominates the potential energy. We will certainly explain it in detail during the course of the proof. Due to the requirement, the limit of \( \tilde{V}_{N,L} \) defined by (1.15) is not a 3D \( \delta \)-function, though it scales like one.

There are two well-developed schemes to deal with this type of procedure. One is the Fock space method, while the other is the hierarchy approach. We take the hierarchy approach here\(^7\). The BBGKY hierarchy associated with \( \tilde{\psi}_{N,L} \) is

\[
(1.13) \quad i \partial_t \tilde{\gamma}^{(k)}_{N,L} = \sum_{j=1}^{k} \left[ -\Delta_{x_j}, \tilde{\gamma}^{(k)}_{N,L} \right] + \frac{1}{L^2} \sum_{j=1}^{k} \left[ -\partial^2_{x_j}, \tilde{\gamma}^{(k)}_{N,L} \right] + \frac{1}{N-1} \sum_{1 \leq i < j \leq k} \left[ \tilde{V}_{N,L}(r_i - r_j), \tilde{\gamma}^{(k)}_{N,L} \right] + \frac{N-k}{N-1} \sum_{j=1}^{k} T_{rr_{k+1}} \left[ \tilde{V}_{N,L}(r_j - r_{k+1}), \tilde{\gamma}^{(k+1)}_{N,L} \right].
\]

It was Erdös, Schlein, and Yau who first rigorously derived the 3D cubic defocusing NLS from a 3D quantum many-body dynamic in their fundamental papers \([32, 33, 34, 35]\). They proved a-priori \( L_\infty^2 H_x^1 \) bound to establish the compactness of BBGKY with respect to a topology on the trace class operators. Then, they showed that the limit point satisfies GP hierarchy. Finally, the proof for the uniqueness of GP hierarchy was the principal part and also surprisingly dedicate due to the fact that it is a system of infinitely many coupled equations over an unbounded number of variables. It motivated a large amount of works \([1, 9, 16, 17, 12, 15, 48, 47, 18, 10, 19, 8, 13, 25, 11, 59, 66]\).

Subsequently, with imposing an additional a-prior condition on space-time norm, Klainerman and Machedon \([48]\) gave another proof of the uniqueness of GP hierarchy in a different space of density matrices defined by Hilbert-Schmidt type Sobolev norms. Later, the approach of Klainerman and Machedon was used by Kirkpatrick, Schlein and Staffilani \([47]\) to derived the 2D cubic defocusing NLS from the 2D quantum many-body dynamic both on \( \mathbb{R}^2 \) and \( T^2 \); by T. Chen and Pavlović \([10]\) to derive the quintic NLS for \( d = 1, 2 \); by X. Chen \([18]\) to investigate the trapping problem in 2D and 3D; and by X. Chen and Holmer \([19]\) to derive 2D cubic defocusing NLS from the 3D quantum many-body dynamic.

Later on, T. Chen, Hainzl, Pavlović and Seiringer \([8]\), using the quantum de Finetti theorem from \([53]\), provided a simplified proof of the \( L_\infty^2 H_x^1 \)-type 3D cubic uniqueness theorem in \([33]\). This method in \([8]\) inspired the study for refined uniqueness theorems, such as \([26, 63, 45]\).

Using Fock space methods to study the convergence rate has also been worked on by many authors, for example, see \([60, 7, 39, 3, 40, 50, 51, 58]\), and the references within.

For the focusing setting, which is a natural continuation of the defocusing problem, X. Chen and Holmer \([21]\) first derived the 1D focusing cubic NLS and later a 3D-to-1D reduction in \([23]\). But the 2D cubic case did not see any process until \([54]\), in which Lewin, Nam, and Rougerie used a quantitative version of the quantum de Finetti theorem \([27]\) to show that the ground state energy of the 2D \( N \)-body was described by a NLS ground state energy. Using the finite-dimensional quantum of de Finetti theorem in \([54]\), X. Chen and Holmer \([21]\) derived 2D focusing cubic NLS from the 2D quantum many-body dynamic for \( \beta \in (0, 1/6) \). For higher \( \beta \), Lewin, Nam, and Rougerie \([55]\) used a bootstrapping argument to improve \( \beta \), which, together with the approach in \([24]\), implied the convergence of the quantum many-body dynamics to the focusing NLS for \( \beta \in (0, 3/4) \). Nam and Napiórkowski \([58]\) used \( H^1 \) regularity to extend the norm approximation to lower dimension in both focusing and defocusing cases. Recently, Bosmann \([5]\) has considered the defocusing 3D-to-2D program for \( \beta < 1 \). To our knowledge, the derivation of 2D focusing cubic NLS from 3D has not been completed before. In this paper, we follow the lead of the aforementioned focusing works \([52, 21, 54, 23, 24, 55]\) and pursue the treatment of 2D case from the 3D physical setting.

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\(^6\)This extra requirement and the coupling constant certainly give rises to a density condition for the gas. We do not compute this density as it is not our main goal here.

\(^7\)We believe the Fock space method will reach the same result. We just prefer a \( H^1 \) result here. In fact, some techniques we used come from the Fock space literatures \([52, 55]\).
1. Outline of the Proof of Theorem 1.2. We first establish in Section 2, under the assumption $L(N/L)^\beta \to 1^-$, that the renormalized kinetic energy controls the potential energy and hence yield an $H^1$ regularity bound to make the other parts of the paper work.

In section 2.1, we use scaling arguments to show why we choose the uncommon mixed norm $\|V\|_{L^\beta L^1}$ and we are bounded by the extra restriction $L(N/L)^\beta \to 1^-$. In fact, a similar requirement would also show up in the harmonic well case studied by X. Chen and Holmer [19] if one wants the renormalized kinetic energy to bound the potential energy instead of dropping it. Subsequently in Section 2.2, we prove the energy bound which is Theorem 2.1 when $k = 1$. Instead of taking the approach in X. Chen and Holmer [19, 23, 24], our proof improvises from Lewin, Nam, and Rougerie [55] and Lewin [52]. This proof does not use the finite-dimensional quantum de Finetti theorem and thus can be applied to the $\mathbb{R}^2$ case as well. In Section 2.3, we adapt the bootstrapping argument in [55] to reach $\beta < 1/3$ in case one would like to use the method in [24], which will give a $\beta < 1/5$ base case. Then in Section 2.4, we complete Theorem 2.1 when $k > 1$.

In Section 3, we show the compactness of the BBGKY sequence. Then, we use a modified version of the approximation of identity type lemma to show that limit points satisfy the GP hierarchy with $g_0$ being the coupling constant. The uniqueness for GP hierarchy on $\mathbb{T}^2$ has been well studied by Kirkpatrick, Schlein and Staffilani [17], Herr and Sohinger [42, 43]. We use their uniqueness theorems to conclude our proof.

2. Focusing energy estimates

In this section, we prove focusing energy estimates. Define

$$S_j := (1 - \Delta_{x_j} - 1/L^2)^{1/2},$$

and write

$$S^{(k)} = \prod_{j=1}^{k} S_j.$$

Theorem 2.1. Assume $L(N/L)^\beta \to 1^-$, $\beta < 1/3$, and $\|V\|_{L^\beta L^1} \leq \frac{c_0}{L^{\alpha}}$ for some $\alpha \in (0,1)$, then let $c_0 = \min \left(\frac{1}{\sqrt{2}}, \frac{1}{\pi}\right)$, we have $\forall k \geq 0$, there exists an $N_0(k) > 0$ such that

$$\langle \psi_{N,L}, (2 + N^{-1}H_{N,L} - 1/L^2)^k \psi_{N,L} \rangle \geq c_0 \|S^{(k)} \psi_{N,L}\|^2_{L^2}$$

for all $N > N_0$ and for all $\psi_{N,L} \in L^2(\Omega^k)$. $\Box$

Proof. For smoothness of presentation, we postpone the proof of Theorem 2.1 to Section 2.2-2.4.

Now we convert the conclusions of Theorem 2.1 into the statement about the rescaled solution, which we will use in the remainder of the paper.

Let $\tilde{P}_0$ denote the orthogonal projection onto the ground state of $-\partial_x^2 - 1$ on the region $(-\pi/2, \pi/2)$ with Dirichlet boundary condition and $\tilde{P}_{\geq 1} = I - \tilde{P}_0$. We define $\tilde{P}_0^\alpha$ and $\tilde{P}_1^\alpha$ to be respectively $\tilde{P}_0$ and $\tilde{P}_{\geq 1}$ acting on the $z_j$-variable, and

$$\tilde{P}_\alpha = \tilde{P}_0^\alpha \cdots \tilde{P}_1^\alpha$$

for a $k$-tuple $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_j \in \{0,1\}$ and adopt the notation $|\alpha| = \alpha_1 + \cdots + \alpha_k$. Then

$$I = \sum_\alpha \tilde{P}_\alpha,$$

where $I : L^2(\Omega^k) \to L^2(\Omega^k)$.

Corollary 2.2. Define

$$\tilde{S}_j = (1 - \Delta_{x_j} - \partial_{x_j}^2/L^2 - 1/L^2)^{1/2},$$

and write

$$\tilde{S}^{(k)} = \prod_{j=1}^{k} \tilde{S}_j, \quad \langle \nabla \rangle^{(k)} = \prod_{j=1}^{k} \sqrt{1 - \Delta_{x_j}}.$$
Assume $L(N/L)^\beta \to 1^-$. Let $\tilde{\psi}_{N,L}(t) = e^{i\tilde{H}_{N,L}}\tilde{\psi}_{N,L}(0)$ and $\left\{\tilde{\gamma}^{(k)}_{N,L}(t)\right\}$ be the associated marginal densities. Then for all $k \geq 0$, we have the uniform-in-time bound

\begin{equation}
(2.4) \quad Tr S^{(k)} \tilde{\gamma}^{(k)}_{N,L} \tilde{S}^{(k)} = \|S^{(k)} \tilde{\psi}_{N,L}(t)\|_{L^2}^2 \leq C^k.
\end{equation}

Consequently,

\begin{equation}
(2.5) \quad Tr (\nabla)^{(k)} \tilde{\gamma}^{(k)}_{N,L} (\nabla)^{(k)} = \| (\nabla)^{(k)} \tilde{\psi}_{N,L}(t)\|_{L^2}^2 \leq C^k,
\end{equation}

and

\begin{equation}
(2.6) \quad \| \tilde{P}_\alpha \tilde{\psi}_{N,L}\|_{L^2} \leq C^k L^{[\alpha]}, \quad |Tr \tilde{P}_\alpha \tilde{\gamma}^{(k)}_{N,L} \tilde{P}_\beta| \leq C^k L^{[\alpha]+[\beta]},
\end{equation}

where $\tilde{P}_\alpha$ and $\tilde{P}_\beta$ are defined as in (2.2).

Proof. We notice that

\begin{align*}
(S^2_j \tilde{\psi}_{N,L})(t, x_N, z_N) &= L^{N/2}(S^2_j \psi_{N,L})(t, x_N, Lz_N), \\
(H_{N,L} \tilde{\psi}_{N,L})(t, x_N, z_N) &= L^{N/2}(H_{N,L} \psi_{N,L})(t, x_N, Lz_N),
\end{align*}

where $\tilde{\psi}_{N,L}$ is defined by (1.6). Thus, we have

\begin{align*}
\|S^{(k)} \tilde{\psi}_{N,L}(t)\|_{L^2}^2 &= \|S^{(k)} \psi_{N,L}(t)\|_{L^2}^2, \\
\langle \tilde{\psi}_{N,L}, (2 + N^{-1} H_{N,L} - 1/L^2)^k \tilde{\psi}_{N,L} \rangle &= \langle \psi_{N,L}, (2 + N^{-1} H_{N,L} - 1/L^2)^k \psi_{N,L} \rangle.
\end{align*}

From estimate (2.1) in Theorem (2) we obtain

\begin{equation}
\|S^{(k)} \tilde{\psi}_{N,L}(t)\|_{L^2}^2 \leq C^k \langle \tilde{\psi}_{N,L}(t), (2 + N^{-1} H_{N,L} - 1/L^2)^k \tilde{\psi}_{N,L}(t) \rangle.
\end{equation}

The term on the right-hand side is conserved, so

\begin{equation}
\|S^{(k)} \tilde{\psi}_{N,L}(t)\|_{L^2}^2 \leq C^k \langle \tilde{\psi}_{N,L}(0), (2 + N^{-1} H_{N,L} - 1/L^2)^k \tilde{\psi}_{N,L}(0) \rangle.
\end{equation}

Applying the binomial theorem twice,

\begin{align*}
\|S^{(k)} \tilde{\psi}_{N,L}(t)\|_{L^2}^2 &\leq C^k \sum_{j=0}^{k} \binom{k}{j} 2^j \langle \tilde{\psi}_{N,L}(0), (N^{-1} H_{N,L} - 1/L^2)^{k-j} \tilde{\psi}_{N,L}(0) \rangle \\
&\leq C^k \sum_{j=0}^{k} \binom{k}{j} 2^j C^{k-j} \\
&= C^k (2 + C)^k \leq \tilde{C}^k,
\end{align*}

where we used initial condition in the second-to-last line. So we have established (2.4). Combining (2.4) and (4.31), estimate (2.5) then follows. By (2.4) and (4.30), we obtain the first inequality of (2.6). By Lemma 4.7

\begin{equation}
Tr \tilde{P}_\alpha \tilde{\gamma}^{(k)}_{N,L} \tilde{P}_\beta = \langle \tilde{P}_\alpha \tilde{\psi}_{N,L}, \tilde{P}_\beta \tilde{\psi}_{N,L} \rangle,
\end{equation}

so the second inequality of (2.6) follows by Cauchy-Schwarz inequality.

\hfill \Box

2.1. Explanations on the assumptions. We will explain the idea that we choose the mixed norm $\|V\|_{L^\infty_x L^1}$ and the relationship $L(N/L)^\beta \to 1^-$, both of which are different from the previous work, such as [19, 23, 24]. In fact, to derive the 2D focusing NLS equations, the key point is that the interaction energy can be controlled by the kinetic energy, which is described by Theorem (2.1) when $k = 1$. By a scaling, we can see that the mixed norm $\|V\|_{L^\infty_x L^1}$ is reasonable and $L(N/L)^\beta$ should be bounded.
2.1.1. We begin by setting up some notations for simplicity. Let
\[ H_{ij} = S_i^2 + S_j^2 + H_{1ij}, \]
and
\[ H_{1ij} = LV_{N,L}(r_i - r_j), \]
where the subscript \( I \) represents the interaction energy. Then, we can rewrite
\[ 1 + N^{-1}H_{N,L} - 1/L^2 = \frac{1}{N(N-1)} \sum_{1 \leq i < j \leq N} H_{ij}. \]
When we take \( \psi_{N,L} = \phi_L^{\otimes N} \) with \( \|\phi_L\|_{L^2} = 1 \), we find
\[ \langle \psi_{N,L}, (1 + N^{-1}H_{N,L} - 1/L^2) \psi_{N,L} \rangle = \frac{1}{2} \langle \phi_L^{\otimes 2}, H_{12}\phi_L^{\otimes 2} \rangle. \]
If \( H_{12} \geq 0 \), we can deduce that
\[ -L \int_{\Omega_L^2} V_{N,L}(r_1 - r_2)|\phi_L(r_1)\phi_L(r_2)|^2 dr_1 dr_2 \leq C(V)(S_1^2\phi_L(r_1)\phi_L(r_2), \phi_L(r_1)\phi_L(r_2)) \]
\[ \leq C(V)((1 - \Delta_{r_1})\phi_L(r_1)\phi_L(r_2), \phi_L(r_1)\phi_L(r_2)), \]
where \( C(V) \) depends on \( V \). Moreover, if we assume \( C(V) = C\|V\|_X \) where \( \|\cdot\|_X \) is a norm, by a scaling argument, it should satisfy
\[ \|V\|_X \leq \lambda^{-2}\|V(\cdot/\lambda)\|_X, \quad \forall \lambda \in (0, 1). \]
Indeed, we take \( V^\lambda(\cdot) = V(\cdot/\lambda) \) and \( \phi_L^\lambda(\cdot) = \lambda^{-3/2}\phi_L(\cdot/\lambda) \) to replace \( V \) and \( \phi_L \) respectively. Since we take the periodic condition in the \( x \)-direction, a scaling argument can only be used for the function supported in the interior of the domain. Thus, we consider the test function \( \phi_L \in C_c^\infty(\Omega_L) \), the space of smooth functions compactly supported in \((-\pi, \pi)^2 \times (-L\pi/2, L\pi/2)\). For every \( \phi_L \in C_c^\infty(\Omega_L) \), we have
\[ -L \int_{\Omega_L^2} V_{N,L}(r_1 - r_2)|\phi_L(r_1)\phi_L(r_2)|^2 dr_1 dr_2 = -L \int_{\Omega_L^2} V_{N,L}^\lambda(r_1 - r_2)|\phi_L^\lambda(r_1)\phi_L^\lambda(r_2)|^2 dr_1 dr_2, \]
and
\[ C\|V^\lambda\|_X ((1 - \Delta_{r_1})\phi_L^\lambda(r_1)\phi_L^\lambda(r_2), \phi_L^\lambda(r_1)\phi_L^\lambda(r_2)) \leq C\|V^\lambda\|_X \lambda^{-2}((1 - \Delta_{r_1})\phi_L(r_1)\phi_L(r_2), \phi_L(r_1)\phi_L(r_2)). \]
If there exists a \( \lambda_0 \in (0, 1) \) such that \( \lambda_0^{-2}\|V(\cdot/\lambda_0)\|_X \leq q_0\|V\|_X \) for some \( q_0 \in (0, 1) \), we take \( \lambda = \lambda_0 \). Putting (2.9), (2.10) and (2.11) together, we get
\[ -L \int_{\Omega_L^2} V_{N,L}(r_1 - r_2)|\phi_L(r_1)\phi_L(r_2)|^2 dr_1 dr_2 \leq q_0C\|V\|_X ((1 - \Delta_{r_1})\phi_L(r_1)\phi_L(r_2), \phi_L(r_1)\phi_L(r_2)), \]
for all \( \phi_L \in C_c^\infty(\Omega_L) \). Iterating the process, it will lead to a contradiction for \( q_0 < 1 \).
On the one hand, the common norm \( \|\cdot\|_{L^1} \) cannot satisfy the above requirement, since
\[ \|V\|_L^1 = \lambda^{-3}\|V^\lambda\|_{L^1} \geq \lambda^{-2}\|V^\lambda\|_{L^1}, \quad \forall \lambda \in (0, 1). \]
On the other hand, we note that
\[ \|V\|_{L_2^\infty L_1^\infty} = \lambda^{-2}\|V(\cdot/\lambda)\|_{L_2^\infty L_1^\infty}, \quad \forall \lambda \in (0, 1). \]
That is, the mixed norm \( L_2^\infty L_1^\infty \) satisfies the requirement. Indeed, we can establish a general Lemma (2.6) in Section 2.2.
2.1.2. To derive the relationship between \( N \) and \( L \), the key point is also that the interaction energy can be controlled by the kinetic energy. More precisely, let us consider the rescaled system. We take the test function \( \tilde{\phi}(r) = f(x)\tilde{g}(z) \), where \( f(x) \in C^\infty_c((-\pi, \pi)^2) \), \( \tilde{g}(z) \in C^\infty_c(-\pi/2, \pi/2) \) and 
\[
\|f\|_{L^2} = \|\tilde{g}\|_{L^2} = 1.
\]

Define
\[
\begin{align*}
(2.12) & \quad f_\varepsilon(x) = \frac{1}{\varepsilon} f(x/\varepsilon) \in C^\infty_c((-\pi, \pi)^2), \quad \varepsilon \in (0, 1), \\
(2.13) & \quad \tilde{g}_\lambda(z) = \frac{1}{\sqrt{\lambda}} \tilde{g}(z/\lambda) \in C^\infty_c(-\pi/2, \pi/2), \quad \lambda \in (0, 1), \\
(2.14) & \quad \tilde{\phi}_{\varepsilon, \lambda}(r) = f_\varepsilon(x)\tilde{g}_\lambda(z), \\
(2.15) & \quad \tilde{\psi}_{\varepsilon, \lambda}(r_1, r_2) = \tilde{\phi}_{\varepsilon, \lambda}(r_1)\tilde{\phi}_{\varepsilon, \lambda}(r_2).
\end{align*}
\]

where \( f_\varepsilon(x) \) should be considered as a periodic extension and \( f_\varepsilon(x) \in C^\infty(T^2) \).

The interaction energy is
\[
\int V_{N,L}(r_1 - r_2)|f_\varepsilon(x_1)\tilde{g}_\lambda(z_1)f_\varepsilon(x_2)\tilde{g}_\lambda(z_2)|^2 dr_1 dr_2
\]

The kinetic energy is
\[
\begin{align*}
(2.16) & \quad \langle \tilde{S}_{\varepsilon, \lambda}^2 \tilde{\psi}_{\varepsilon, \lambda}, \tilde{\psi}_{\varepsilon, \lambda} \rangle \\
= & \|f_\varepsilon\|_{L^2}^2 \|\tilde{g}_\lambda\|_{L^2}^2 \left( (1 - \Delta_x)f_\varepsilon(x)\tilde{g}_\lambda(z), f_\varepsilon(x)\tilde{g}_\lambda(z) \right) \\
= & \|f_\varepsilon\|_{L^2}^2 \|\tilde{g}_\lambda\|_{L^2}^2 \left( \|\nabla_x f_\varepsilon\|_{L^2}^2 \|\tilde{g}_\lambda\|_{L^2}^2 + \|f_\varepsilon\|_{L^2}^2 \|\tilde{g}_\lambda\|_{L^2}^2 + \frac{1}{L^2} \|f_\varepsilon\|_{L^2}^2 \|\partial_z \tilde{g}_\lambda\|_{L^2}^2 - \frac{1}{L^2} \|f_\varepsilon\|_{L^2}^2 \|\tilde{g}_\lambda\|_{L^2}^2 \right)
\end{align*}
\]

When we take \( \lambda^{-1} = L(N/L)^{\beta}, \varepsilon^{-1} = (N/L)^{\beta} \), the interaction energy is equal to
\[
L(N/L)^{3\beta} \int V(x_1 - x_2, z_1 - z_2)|f_\varepsilon(x_1)\tilde{g}_\lambda(z_1)f_\varepsilon(x_2)\tilde{g}_\lambda(z_2)|^2 dr_1 dr_2
\]

and the kinetic energy is controlled by
\[
(N/L)^{2\beta} + \frac{L^2(N/L)^{2\beta}}{L^2} = 2(N/L)^{2\beta}.
\]

Since \( L(N/L)^{\beta} \to \infty \), the interaction energy cannot be controlled by the kinetic energy. Therefore, it implies that we should consider the case \( L(N/L)^{\beta} \leq C \). On the other hand, to make the limit of \( V_{N,L} \) exist, we should take \( L(N/L)^{\beta} \) to be a constant or tend to 0. For the case \( L(N/L)^{\beta} \to 0 \), we note that the limit of \( V_{N,L} \) equals to 0, which is not sufficient to derive the cubic NLS equation. Hence, we only consider the case \( L(N/L)^{\beta} \to 1 \) and it works the same for \( L(N/L)^{\beta} \to R_0 \).

Remark 2.3. A similar argument can also apply to [19] in the focusing setting. To control the interaction energy instead of dropping it like the defocusing case, it also needs an extra condition \( (N\sqrt{\omega})^\beta \omega^{-1/2} \leq C \).

2.2. Focusing energy estimates when \( k = 1 \). In this section, we will prove Theorem 2.4 when \( k = 1 \) as follows.

**Theorem 2.4.** Assume \( L(N/L)^{\beta} \to 1^- \), \( \beta < \frac{1}{3} \), and \( \|V\|_{L^\infty L^1} \leq \frac{\alpha}{C_{0}} \), for some \( \alpha \in (0, 1) \), then \( \forall C_0 > 0 \), there exists an \( N_0 > 0 \) such that
\[
(2.18) \quad \langle \psi_{N,L}, (C_0 + 1 + N^{-1}H_{N,L} - 1/L^2)\psi_{N,L} \rangle \geq (1 - \alpha)\|S_1 \psi_{N,L}\|_{L^2}^2,
\]
for all \( N > N_0 \) and for all \( \psi_{N,L} \in L^2_q(Q_{T,N}^N) \).

The key of the proof of Theorem 2.4 is the following theorem.
Theorem 2.5. Assume $L(N/L)^{\beta} \rightarrow 1^{-}$, $\beta < \frac{1}{4}$, and $\|V\|_{L^{\infty}_x L^1_x} \leq \frac{2\alpha}{C_{gn}}$, for some $\alpha \in (0,1)$, define the operator
\begin{equation}
H_{i,j} = \alpha S_{i}^2 + \alpha S_{j}^2 + H_{i,j}.
\end{equation}
Then $\forall C_0 > 0$, there exists an $N_0 > 0$ such that
\begin{equation}
\langle \psi_{N,L}, (2C_0 + H_{12}) \psi_{N,L} \rangle \geq 0,
\end{equation}
for all $N > N_0$ and for all $\psi_{N,L} \in L^2(\Omega_L^{\otimes N})$.

Proof of Theorem 2.3 assuming Theorem 2.5: Using formula (2.7) and the symmetry of $\psi_{N,L}$, we have
\begin{align*}
\langle \psi_{N,L}, (C_0 + 1 + N^{-1} H_{N,L} - 1/L^2) \psi_{N,L} \rangle \\
= \frac{1}{N(N-1)} \sum_{1 \leq i < j \leq N} \langle \psi_{N,L}, (2C_0 + H_{i,j}) \psi_{N,L} \rangle \\
= \frac{1}{2} \langle \psi_{N,L}, (2C_0 + H_{12}) \psi_{N,L} \rangle \\
= \frac{1}{2} \langle \psi_{N,L}, (2C_0 + H_{12}) \psi_{N,L} \rangle + \frac{1}{2} \langle \psi_{N,L}, (S_{i}^2 + S_{j}^2) \psi_{N,L} \rangle \\
\geq (1 - \alpha) \|S_1 \psi_{N,L}\|_{L^2}^2.
\end{align*}

\hfill \square

Next, we turn our attention onto the proof of Theorem 2.5. Under the assumption $L(N/L)^{\beta} \rightarrow 1^{-}$, the renormalized kinetic energy can control the potential energy.

Lemma 2.6. Assume $L(N/L)^{\beta} \rightarrow 1^{-}$, $M \geq 1$, then for all $\psi_{M,L} \in L^2(\Omega_L^{\otimes M})$ with $\|\psi_{M,L}\|_{L^2} = 1$, we have
\begin{equation}
L \int_{\Omega_L^2} |V_{N,L}(r_1 - r_2)| \rho_{M,L}(r_1) \rho_{M,L}(r_2) dr_1 dr_2 \leq C_{gn}^4 \|V\|_{L^\infty_x L^1_x} \langle S_1^2 \phi_{L}, \phi_{M,L} \rangle,
\end{equation}
where density function $\rho_{M,L}(r_1) := \int \cdots \int |\psi_{M,L}|^2(r_1, \ldots, r_M) dr_2 \cdots dr_M$. Especially, if $\psi_{M,L} = \phi_L^{\otimes M}$ with $\|\phi_L\|_{L^2} = 1$, then
\begin{equation}
L \int_{\Omega_L^2} |V_{N,L}(r_1 - r_2)| \phi_L(r_1) \phi_L(r_2)^2 dr_1 dr_2 \leq C_{gn}^4 \|V\|_{L^\infty_x L^1_x} \langle S_1^2 \phi_L, \phi_L \rangle.
\end{equation}

Proof. Using Cauchy-Schwarz inequality and Young’s convolution inequality, we get
\begin{equation}
L \int_{\Omega_L^2} |V_{N,L}(r_1 - r_2)| \rho_{M,L}(r_1) \rho_{M,L}(r_2) dr_1 dr_2 \leq \|V_{N,L} \ast \rho_{M,L}\|_{L^\infty_x L^1_x} \|\rho_{M,L}\|_{L^1_x L^2_x} \leq \|V_{N,L}\|_{L^\infty_x L^1_x} \|\rho_{M,L}\|_{L^1_x L^2_x}^2.
\end{equation}

For $\|\rho_{M,L}\|_{L^1_x L^2_x}^2$ on the right-hand side of (2.22), we use 2D inhomogeneous Gagliardo-Nirenberg inequality (1.11),
\begin{equation}
\|\rho_{M,L}\|_{L^1_x L^2_x}^2 = \|\sqrt{\rho_{M,L}}\|_{L^2_x L^1_x}^2 \leq C_{gn}^4 \|\sqrt{\rho_{M,L}}\|_{L^2_x L^1_x}^2 \|1 - \Delta_x \sqrt{\rho_{M,L}}\|_{L^2_x L^1_x}^2
\end{equation}
\begin{equation}
= C_{gn}^4 \|\sqrt{\rho_{M,L}}\|_{L^2_x L^1_x}^2 \left(\|\Delta_x \sqrt{\rho_{M,L}}\|_{L^1_x L^2_x}^2 + \|\nabla_x \sqrt{\rho_{M,L}}\|_{L^2_x L^1_x}^2\right).
\end{equation}

By the Hoffman-Ostenhof inequality (A.1), we have
\begin{equation}
\|\nabla_x \sqrt{\rho_{M,L}}\|_{L^2_x} \leq \langle -\Delta_x, \psi_{M,L}, \psi_{M,L} \rangle.
\end{equation}
Now, putting (2.23) into (2.22) and (2.23) into (2.22), we have
\begin{equation}
L \int_{\Omega_L^2} |V_{N,L}(r_1 - r_2)| \rho_{M,L}(r_1) \rho_{M,L}(r_2) dr_1 dr_2 \leq C_{gn}^4 \|V_{N,L}\|_{L^\infty_x L^1_x} \langle (1 - \Delta_x), \psi_{M,L}, \psi_{M,L} \rangle.
\end{equation}

Noting that $\|V_{N,L}\|_{L^\infty_x L^1_x} = (N/L)^{\beta} \|V\|_{L^\infty_x L^1_x}$, with the assumption $L(N/L)^{\beta} \rightarrow 1^{-}$, we conclude that
\begin{equation}
L \int_{\Omega_L^2} |V_{N,L}(r_1 - r_2)| \rho_{M,L}(r_1) \rho_{M,L}(r_2) dr_1 dr_2 \leq C_{gn}^4 \|V\|_{L^\infty_x L^1_x} \langle (1 - \Delta_x), \psi_{M,L}, \psi_{M,L} \rangle.
\end{equation}
Since $S_1^2 \geq (1 - \Delta_{x_1})$, we arrive at the estimate (2.20) from estimate (2.26).

KLASSICAL
The following lemma is used to estimate the two-body interaction energy by a one-body term.

**Lemma 2.7.** If $V \in C_c^\infty(T^2 \times \mathbb{R})$ and has a positive Fourier transform $\hat{V} \geq 0$, then for all real function $\eta \in L^1(T^2 \times \mathbb{R})$

\[(2.27) \sum_{1 \leq j < k \leq N} V(r_j - r_k) \geq \frac{1}{(2\pi)^3} \sum_{j=1}^{N} \eta * V(r_j) - \frac{1}{2(2\pi)^6} \int_{T^2 \times \mathbb{R}} \int_{T^2 \times \mathbb{R}} V(r_1 - r_2)\eta(r_1)\eta(r_2)d_1d_2 - \frac{N}{2}V(0).\]

**Proof.** With the Fourier inversion formula,

\[(2.28) \sum_{1 \leq j < k \leq N} V(r_j - r_k) = \int \sum_{1 \leq j < k \leq N} e^{i\xi (r_j - r_k)} \hat{V}(\xi)d\xi \]

\[= \frac{1}{2} \int \left| \sum_{j=1}^{N} e^{i\xi r_j} - \hat{\eta}(\xi) \right|^2 \hat{V}(\xi)d\xi + \frac{1}{(2\pi)^3} \sum_{j=1}^{N} \eta * V(r_j)
- \frac{1}{2(2\pi)^6} \int_{T^2 \times \mathbb{R}} \int_{T^2 \times \mathbb{R}} V(r_1 - r_2)\eta(r_1)\eta(r_2)d_1d_2 - \frac{N}{2}V(0)\]

\[\geq \frac{1}{(2\pi)^3} \sum_{j=1}^{N} \eta * V(r_j) - \frac{1}{2(2\pi)^6} \int_{T^2 \times \mathbb{R}} \int_{T^2 \times \mathbb{R}} V(r_1 - r_2)\eta(r_1)\eta(r_2)d_1d_2 - \frac{N}{2}V(0),\]

where $\xi = (n_1, n_2, \tau)$ and $\int \cdot d\xi$ is short for $\int_{\mathbb{R}} \sum_{n_1, n_2} \cdot d\tau$. \qed

**Remark 2.8.** In our setting, the integral region is $\Omega_L$. To use Lemma 2.7, $V_{N,L}$ should be understood as the periodic extension in the $x$-direction and zero extension in the $z$-direction of the rescaled $V$ which is compactly supported on $\Omega$. That is, $V_{N,L} \in C_c^\infty(T^2 \times \mathbb{R})$. Similarly, $\rho_{N,L}(r)$ and $\psi_{N,L}$ should be seen as $\rho_{N,L}(r)1_{\Omega_L}(r) \in L^1(T^2 \times \mathbb{R})$ and $\psi_{N,L}1_{\Omega_L}$ respectively.

In case $\hat{V}_{N,L} \geq 0$, if we take $V = V_{N,L}$ and $\eta = (2\pi)^3N\rho_{N,L}$, we obtain

\[(2.29) \sum_{1 \leq j < k \leq N} \langle \psi_{N,L}, V_{N,L}(r_j - r_k)\psi_{N,L} \rangle \]

\[\geq N \sum_{j=1}^{N} \int \rho_{N,L} * V_{N,L}(r_1)|\psi_{N,L}|^2(r_N)dr_N - \frac{N^2}{2}(\rho_{N,L} * V_{N,L}, \rho_{N,L}) - \frac{N}{2}V_{N,L}(0)\]

\[= \frac{N^2}{2} \int \rho_{N,L}(r_1)\rho_{N,L}(r_2)V_{N,L}(r_1 - r_2)dr_1dr_2 - \frac{N(N/L)^{3\beta}V(0)}{2}\]

\[\geq -\frac{N(N/L)^{3\beta}V(0)}{2},\]

where we have used

\[\int \rho_{N,L}(r_1)\rho_{N,L}(r_2)V_{N,L}(r_1 - r_2)dr_1dr_2 = (2\pi)^6 \int |\hat{\rho}_{N,L}(\xi)|^2\hat{V}_{N,L}(\xi)d\xi\]

and $\hat{V}_{N,L} \geq 0$ in the last inequality.

By estimate 2.29, we have

\[\langle \psi_{N,L}, (2C_0 + H_{12,\alpha})\psi_{N,L} \rangle \geq 2C_0 - \frac{(N/L)^{3\beta}V(0)}{(N - 1)/L} \geq 0,\]

as long as $\beta < 1/3$ and $N/L$ is large enough.

Hence, we have established Theorem 2.5 if $\hat{V}_{N,L} \geq 0$. Next, we will use Lemma 2.7 to deal with a general interaction function $V$. 

\[\square\]
Proof of Theorem 2.35 For general $V$, we consider $N = 2M$ particles which we split into two groups of $M$. For the case $N = 2M + 1$, the proof works the same if we split the system into two groups of $M$ and $M + 1$. We denote the first $M$ variables by $r_1, ..., r_M$ and the others by $r'_1 = r_{M+1}, ..., r'_M = r_{2M}$. We decompose $V_{N,L} = V_{N,L}^+ - V_{N,L}^-$ on the Fourier side where $V_{N,L}^+ = (V_{N,L})_+ \geq 0$ and $V_{N,L}^- = (V_{N,L})_- \geq 0$. By its symmetry in the $2M$ variables, we rewrite

$$\begin{align*}
\frac{1}{2} \langle \psi_{2M,L}, H_{12}\psi_{2M,L} \rangle &= \frac{L}{2M(2M-1)} \langle \psi_{2M,L}, \sum_{1 \leq j < k \leq 2M} V_{N,L}(r_j - r_k) \psi_{2M,L} \rangle \\
&= \frac{L}{M(M-1)} \langle \psi_{2M,L}, \sum_{1 \leq j < k \leq M} V_{N,L}^+(r_j - r_k) \psi_{2M,L} \rangle \\
&\quad + \frac{L}{M(M-1)} \langle \psi_{2M,L}, \sum_{1 \leq j < m \leq M} V_{N,L}^-(r'_j - r'_m) \psi_{2M,L} \rangle - \frac{L}{M^2} \left\langle \psi_{2M,L}, \sum_{j=1}^M \sum_{i=1}^M V_{N,L}(r_j - r'_i) \psi_{2M,L} \right\rangle.
\end{align*}$$

This means that

$$\begin{align*}
(2.30) \\
\frac{1}{2} \langle \psi_{2M,L}, H_{12}\psi_{2M,L} \rangle &= \langle \psi_{2M,L}, I_{M,L}\psi_{2M,L} \rangle,
\end{align*}$$

where

$$I_{M,L} = \frac{L}{M(M-1)} \sum_{1 \leq j < k \leq M} V_{N,L}^+(r_j - r_k) + \frac{L}{M(M-1)} \sum_{1 \leq j < m \leq M} V_{N,L}^-(r'_j - r'_m) - \frac{L}{M^2} \sum_{j=1}^M \sum_{i=1}^M V_{N,L}(r_j - r'_i).$$

Then we have

$$\begin{align*}
(2.32) \\
\langle \psi_{2M,L}, (2C_0 + H_{12,\alpha})\psi_{2M,L} \rangle &= \langle \psi_{2M,L}, (2C_0 + 2I_{M,L} + \alpha(S_1^2 + S_2^2))\psi_{2M,L} \rangle.
\end{align*}$$

Thus, in order to bound the interaction $2C_0 + H_{12,\alpha}$ from below, it suffices to consider $I_{M,L}$. We fix the variables $r'_1, ..., r'_M$ in the second group. For simplicity, we use the notation $\langle \cdot, \cdot \rangle_{r_M}$ to denote the integral only in the variables $r_M := (r_1, ..., r_M)$. We denote the one-particle density by

$$\begin{align*}
(2.33) \\
\rho_{M,L}(r, r_M^c) := &\int_{\Omega_L} \cdots \int_{\Omega_L} |\psi_{2M,L}|^2(r, r_2, ..., r_M, r_M^c) dr_2 \cdots dr_M.
\end{align*}$$

Our goal is to get

$$\begin{align*}
(2.34) \\
\langle \psi_{2M,L}, (2C_0 + 2I_{M,L} + \alpha(S_1^2 + S_2^2))\psi_{2M,L} \rangle_{r_M} \\
\geq (2\alpha - C_{sgn}\|V\|_{L^\infty_\Omega}) \langle \psi_{2M,L}, S_1^2 S_2^2 \psi_{2M,L} \rangle_{r_M} + \int \rho_{M,L}(r, r_M^c) dr \left(2C_0 - \frac{LV_{N,L}^+(0) + LV_{N,L}^-(0)}{(M-1)} \right).
\end{align*}$$

First, we may assume $\int \rho_{M,L}(r, r_M^c) dr > 0$. The case $\int \rho_{M,L}(r, r_M^c) dr = 0$ is easier and will be presented later. By using Lemma 2.30 with

$$V = V_{N,L}^+, \quad \eta = \frac{(2\pi)^3 M \rho_{M,L}(r, r_M^c)}{\int \rho_{M,L}(r, r_M^c) dr},$$

we get...
we have
\[ \langle \psi_{2M,L}, I_{M,L} \psi_{2M,L} \rangle_{rM} \geq \frac{ML}{2(M-1)} \int \rho_{M,L}(r, r'_M) dr \int V^+_{N,L}(r_1 - r_2) \rho_{M,L}(r_1, r'_M) \rho_{M,L}(r_2, r'_M) dr_1 dr_2 \]
\[ - \frac{LV^+_{N,L}(0)}{2(M-1)} \int \rho_{M,L}(r, r'_M) dr + \frac{L}{M(M-1)} \sum_{1 \leq l < m \leq M} V^-_{N,L}(r'_l - r'_m) \int \rho_{M,L}(r, r'_M) dr \]
\[ - \frac{L}{M} \sum_{l=1}^M \rho_{M,L} \ast V^-_{N,L}(r'_l). \]

Next we use again Lemma 2.3 with
\[ V = V^-_{N,L}, \quad \eta = \frac{(2\pi)^3}{\int \rho_{M,L}(r, r'_M) dr}, \]
and obtain
\[ \frac{L}{M(M-1)} \sum_{1 \leq l < m \leq M} V^-_{N,L}(r'_l - r'_m) \int \rho_{M,L}(r, r'_M) dr - \frac{L}{M} \sum_{l=1}^M \rho_{M,L} \ast V^-_{N,L}(r'_l) \geq - \frac{(M-1)}{2M} \int \rho_{M,L}(r, r'_M) dr \int V^-_{N,L}(r_1 - r_2) \rho_{M,L}(r_1, r'_M) \rho_{M,L}(r_2, r'_M) dr_1 dr_2 \]
\[ - \frac{LV^-_{N,L}(0) \int \rho_{M,L}(r, r'_M) dr}{2(M-1)}. \]

Thus, we have
\[ \langle \psi_{2M,L}, I_{M,L} \psi_{2M,L} \rangle_{rM} \geq \int \rho_{M,L}(r, r'_M) dr \int \frac{L}{2} V^-_{N,L}(r_1 - r_2) \rho_{M,L}(r_1, r'_M) \rho_{M,L}(r_2, r'_M) dr_1 dr_2 \]
\[ - \int \rho_{M,L}(r, r'_M) dr \frac{LV^+_{N,L}(0) + LV^-_{N,L}(0)}{2(M-1)}. \]

By Lemma 2.3 we obtain
\[ \langle \psi_{2M,L}, (2C_0 + 2I_{M,L} + \alpha(S_1^2 + S_2^2)) \psi_{2M,L} \rangle_{rM} \geq (2\alpha - C_{gm} \| V \|_{L^2}) \langle \psi_{2M,L}, S_1^2 \psi_{2M,L} \rangle_{rM} + \int \rho_{M,L}(r, r'_M) dr \left( 2C_0 - \frac{LV^+_{N,L}(0) + LV^-_{N,L}(0)}{M-1} \right). \]

We arrive at the estimate 2.34 for the case \( \int \rho_{M,L}(r, r'_M) dr > 0. \)

Next, if \( \int \rho_{M,L}(r, r'_M) dr = 0 \), we can deduce that \( \rho_{M,L}(r, r'_M) = 0 \) due to the nonnegativity and smoothness of \( \rho_{M,L} \). Then, we have
\[ \langle \psi_{2M,L}, I_{M,L} \psi_{2M,L} \rangle_{rM} \]
\[ = \frac{L}{M(M-1)} \sum_{1 \leq j < k \leq M} \int V^+_{N,L}(r_j - r_k) |\psi_{M,L}(r, r'_M)|^2 dr_M \]
\[ + \frac{L}{M(M-1)} \sum_{1 \leq l < m \leq M} V^-_{N,L}(r'_l - r'_m) \int \rho_{M,L}(r, r'_M) dr \]
\[ - \frac{L}{M^2} \sum_{j=1}^M \sum_{l=1}^M \int V^-_{N,L}(r_j - r'_l) \rho_{M,L}(r, r'_M) dr_j \]
\[ = I + II + III. \]

Since \( V^+_{N,L} \) has a positive Fourier transform, we have \( I \geq 0 \). By the fact that \( \rho_{M,L}(r, r'_M) = 0 \), we obtain \( II = III = 0 \). That is, the estimate 2.34 still holds.
Hence, when $\|V\|_{L^\infty} \leq \frac{3\pi}{y_n}$, we have

\begin{equation}
\langle \psi_{2M,L}, (2C_0 + H_{12,\alpha})\psi_{2M,L} \rangle = \langle \psi_{2M,L}, (2C_0 + 2I_{M,L} + \alpha(S_1^2 + S_2^2))\psi_{2M,L} \rangle \\
\geq 2C_0 - \frac{L(N/L)^{3\beta}(V^+(0) + V^-(0))}{2(M-1)}.
\end{equation}

Since $\beta < \frac{1}{3}$, there exists an $N_0 > 0$ such that

\begin{equation}
\langle \psi_{2M,L}, (2C_0 + H_{12,\alpha})\psi_{2M,L} \rangle \geq 0,
\end{equation}

for all $N = 2M \geq N_0$. \hfill \Box

### 2.3. Bootstrapping argument.

Let us define

\begin{equation}
E_{N,L} := \inf_{\|\psi_{N,L}\|_{L^2}=1} \langle \psi_{N,L}, (1 + N^{-1}H_{N,L} - 1/L^2) \psi_{N,L} \rangle,
\end{equation}

\begin{equation}
E_{N,L,\varepsilon} := \inf_{\|\psi_{N,L}\|_{L^2}=1} \langle \psi_{N,L}, (1 + N^{-1}H_{N,L} - 1/L^2 - \varepsilon N^{-1} \sum_{j=1}^N S_j^2) \psi_{N,L} \rangle,
\end{equation}

where $E_{N,L}$ denotes the many-body ground state energy per particle. From the definition (2.43), estimate (2.18) in Theorem 2.3 is equivalent to prove $E_{N,L,1-\alpha} \geq -C_0$ for $N \geq N_0$. Indeed, if $E_{N,L,1-\alpha} \geq -C_0$ for $N \geq N_0$, it means that

\begin{equation}
C_0 + 1 + N^{-1}H_{N,L} - 1/L^2 - (1 - \alpha) S_j^2 \geq 0,
\end{equation}

for $N \geq N_0$, which is the estimate (2.18).

Thus, our goal is to bound $E_{N,L,\varepsilon}$ from below. We note that $E_{N,L,\varepsilon} = (1 - \varepsilon) E_{N,L}^\varepsilon$, where $E_{N,L}^\varepsilon$ is the ground state energy with interaction function $V^\varepsilon = (1 - \varepsilon)^{-1}V$. So we only need to deal with $E_{N,L}^\varepsilon$ or $E_{N,L}$. A main tool is the finite-dimensional quantum de Finetti theorem (Lemma 2.10). Then we can give a lower bound on the Hamiltonian energy in Lemma 2.12 that is,

\begin{equation}
\langle \psi_{N,L}, (1 + N^{-1}H_{N,L} - 1/L^2) \psi_{N,L} \rangle \geq C(V, N, L, \lambda_x, \lambda_z, \|S_1\psi_{N,L}\|_{L^2}, \|S_1S_2\psi_{N,L}\|_{L^2}),
\end{equation}

where $\lambda_x$ and $\lambda_z$ are cut-off parameters.

Subsequently in Lemma 2.13, we will control $\|S_1\psi_{N,L}\|_{L^2}$ and $\|S_1S_2\psi_{N,L}\|_{L^2}$ for the ground state $\psi_{N,L}$. More precisely,

\begin{equation}
Tr \left(S_1^2 \gamma_{N,L}^{(1)} \right) \lesssim C_{up} \frac{1 + |E_{N,L,\theta}|}{\theta},
\end{equation}

\begin{equation}
Tr \left(S_1^2 S_2^2 \gamma_{N,L}^{(2)} \right) \lesssim C_{up} \left(1 + |E_{N,L,\theta}| \right)^2 \frac{1}{\theta}.
\end{equation}

where $C_{up}$ is a upper bound constant defined by (2.49).

With Lemma 2.12 and 2.13 we arrive at a closed control relationship, namely,

\begin{equation}
|E_{N,L,\varepsilon}| \leq C(V, N, L, \lambda_x, \lambda_z, \|E_{N,L,\theta}\|).
\end{equation}

Thus we can use the bootstrapping argument as long as there exists a starting point.

Now, we present the above procedure in detail. First, we take $\psi_{N,L} = \phi_L^{\otimes N}$ with $\|\phi_L\|_{L^2} = 1$ and obtain the NLS energy functional

\begin{equation}
E_{N,L}(\phi_L) = \langle \phi_L^{\otimes N}, (1 + N^{-1}H_{N,L} - 1/L^2) \phi_L^{\otimes N} \rangle \\
= \frac{1}{2} \langle \phi_L^{\otimes 2}, H_{12} \phi_L^{\otimes 2} \rangle = \langle S_1^2 \phi_L + \frac{L}{2} \int_{\Omega_2} V_{N,L}(r_1 - r_2) \phi_L(r_1) \phi_L(r_2) \rangle^2 dr_1 dr_2.
\end{equation}

Define

\begin{equation}
\varepsilon_{N,L} := \inf_{\|\phi_L\|_{L^2}=1} E_{N,L}(\phi_L),
\end{equation}

\footnote{The finite dimensional quantum de Finetti theorem is only used in this section. In Section 2.2 we have already reached $\beta < 1/3$ without it. Hence, the main theorem works the same for $\mathbb{R}^2$.}
where \( e_{N,L} \) stands for the ground state energy of the NLS energy functional. From the above definition, we know \( e_{N,L} \geq E_{N,L} \geq E_{N,L,c} \). To bound \( E_{N,L} \) from below, it is necessary to bound \( e_{N,L} \) from below. Here, we first give a sufficient condition to bound \( e_{N,L} \) as follows.

**Lemma 2.9.** Assume \( (N/L)^\beta \to 1^- \), then \( 0 \leq e_{N,L} \leq C_{up} \), where

\[
C_{up} = 1 + \frac{C_{gn}^4 \|V\|_{L^\infty L^1}}{2}.
\]

**Proof.** For the lower bound, it suffices to prove \( E_{N,L}(\phi_L) \geq 0 \). From estimate (2.21) in Lemma 2.6 we have

\[
E_{N,L}(\phi_L) = \langle S^2_L \phi_L, \phi_L \rangle + \frac{L}{2} \int_{\Omega^2_L} V_{N,L}(r_1 - r_2) |\phi_L(r_1) \phi_L(r_2)|^2 \, dr_1 dr_2
\]

\[
\geq \langle S^2_L \phi_L, \phi_L \rangle - \frac{C_{gn}^4 \|V\|_{L^\infty L^1}}{2} \langle S^2_L \phi_L, \phi_L \rangle \geq 0,
\]

as long as \( \|V\|_{L^\infty L^1} \leq \frac{2}{C_{gn}} \).

For the upper bound, we use estimate (2.21) again and obtain

\[
E_{N,L}(\phi_L) = \langle S^2_L \phi_L, \phi_L \rangle + \frac{L}{2} \int_{\Omega^2_L} V_{N,L}(r_1 - r_2) |\phi_L(r_1) \phi_L(r_2)|^2 \, dr_1 dr_2
\]

\[
\leq \langle S^2_L \phi_L, \phi_L \rangle + \frac{C_{gn}^4 \|V\|_{L^\infty L^1}}{2} \langle S^2_L \phi_L, \phi_L \rangle.
\]

When we take \( \phi_L(x,z) = \cos_1(z) \), which is the \( L^2 \) normalized ground state function of \( S^2_L \), then \( \langle S^2_L \phi_L, \phi_L \rangle = 1 \). Hence, we have

\[
e_{N,L} \leq 1 + \frac{C_{gn}^4 \|V\|_{L^\infty L^1}}{2}.
\]

To establish the lower bound estimate for \( E_{N,L} \), we need the finite-dimensional quantum de Finetti theorem. We define the Littlewood-Paley projectors (eigenspace projector) by

\[
P_{x_j,m} = \chi(m)(-\Delta x_j), \quad m \geq 0,
\]

\[
P_{x_j,\leq \lambda} = \chi(0,\lambda^2](-\Delta x_j), \quad P_{x_j,\leq \lambda} = \chi[1/L^2,\lambda^2/L^2](-\partial^2 x_j),
\]

\[
P_{x_j,> \lambda} = I - P_{x_j,\leq \lambda}, \quad P_{x_j,> \lambda} = I - P_{x_j,\leq \lambda}.
\]

From the definition, we notice that \( P_{x_j,\leq \lambda} \) and \( P_{x_j,> \lambda} \) are \( L \) dependent. However, we omit it for simplicity.

**Lemma 2.10** (Finite-dimensional quantum de Finetti [54]). Assume \( \{\gamma_{N,L}^{(k)}\}_{k=1}^N \) is the marginal density generated by an \( N \)-body wave function \( \psi_{N,L} \in L^2_\phi(\Omega^N_L) \) and \( P_L \) be a finite-rank orthogonal projector with

\[
dim(P_L(L^2(\Omega_L))) = d < \infty,
\]

where \( d \) can be independent of \( L \). Then, there is a positive Borel measure \( d\mu_{N,L} \) supported on the unit sphere \( S_{P_L}(L^2(\Omega_L)) \) such that

\[
\text{Tr}[P_L^{\otimes 2} \gamma_{N,L}^{(2)} P_L^{\otimes 2} - \int_{S(P_L(L^2(\Omega_L)))} \langle \phi_L^{(2)} | \phi_L^{(2)} \rangle d\mu_{N,L}(\phi_L)] \leq \frac{8d}{N}.
\]

Moreover, we will need operator inequalities for two-body interaction as follows.

**Lemma 2.11.** Assume \( (N/L)^\beta \to 1^- \). For \( \delta \in (0,1) \), the multiplication operator \( V_{N,L}(r_1 - r_2) \) on \( L^2(\Omega_L^{\otimes 2}) \) satisfies

\[
L|V_{N,L}(r_1 - r_2)| \leq C_\delta (N/L)^\delta \|V\|_{L^\infty L^1} (1 - \Delta x_1),
\]

\[
L|V_{N,L}(r_1 - r_2)| \leq C_\delta \|V\|_{L^\infty L^1} (1 - \Delta x_1)^{1/2+\delta} (1 - \Delta x_2)^{1/2+\delta},
\]

\[
\|\langle \nabla x_1 \rangle^{-1}\langle \nabla x_2 \rangle^{-1} V(r_1 - r_2) \langle \nabla x_1 \rangle^{-1}\langle \nabla x_2 \rangle^{-1} \|_op \leq C \|V\|_{L^\infty L^1},
\]

\[
S^2_L LV_{N,L}(r_1 - r_2) + LV_{N,L}(r_1 - r_2)S^2_L \geq -C_\delta (N/L)^{\delta+3} S^2_L S^2_L,
\]

where \( C_\delta(V) \) is dependent on \( V \).
Proof. For smoothness of presentation, we put the proof in the Appendix.

Now, along with Lemma 2.10 we can establish the following lower bound estimate.

**Lemma 2.12.** Assume \( L(N/L)^2 \to 1^- \) and \( \|V\|_{L^\infty_L L^1_z} \leq \frac{2}{L^2} \). Then for every \( \delta \in (0, 1/2) \), there exists a constant \( C_\delta > 0 \) such that for all \( N \geq 2, \lambda_z \geq 1, \lambda_x \geq 0 \) and for all wave functions \( \psi_{N,L} \)

\[
\left\langle \psi_{N,L}, \left( 1 + N^{-1} H_{N,L} - 1/L^2 \right) \psi_{N,L} \right\rangle 
\geq - (C_\delta \|V\|_{L^\infty_L L^1_z} (1 + \lambda_z^2)^{1+2\delta} + (1 + \lambda_x^2) + (\lambda_x^2 - 1)/L^2) \frac{\lambda_z^2 \lambda_x}{N} 
- \frac{C_\delta \|V\|_{L^\infty_L L^1_z}}{\min \{1 + \lambda_z^2, 1 + (\lambda_x^2 - 1)/L^2\}^{1/4-\delta/2}} \left( \text{Tr} \left( S_{1}^2 (\gamma_{N,L}^{(1)}) \right) \right)^{1/4-\delta/2} \left( \text{Tr} \left( S_{2}^2 (\gamma_{N,L}^{(2)}) \right) \right)^{1/2+\delta}.
\]

**Proof.** For simplicity, we adopt the notation

\[
P := P_{x_1 \leq \lambda_x} P_{x_2 \leq \lambda_z}, \quad Q = 1 - P,
\]

\[
P^j := P_{x_1 \leq \lambda_x} P_{x_2 \leq \lambda_z}, \quad Q^j := 1 - P^j,
\]

\[
P^{(k)} := \prod_{j=1}^k P_{x_1 \leq \lambda_x}, \quad P^{(k)} := \prod_{j=1}^k P^j.
\]

Using Lemma 2.10, we write

\[
(2.61) \quad \left\langle \psi_{N,L}, \left( 1 + N^{-1} H_{N,L} - 1/L^2 \right) \psi_{N,L} \right\rangle 
= \frac{1}{2} \int \left\langle \phi_L^{(2)} H_{12} \phi_L^{(2)} \right\rangle d\mu_{N,L}(\phi_L) + \frac{1}{2} \text{Tr} \left( \left[ H_{12} - P^{(2)} H_{12} P^{(2)} \right] \gamma_{N,L}^{(2)} \right)
+ \frac{1}{2} \text{Tr} \left( H_{12} \left( P^{(2)} \gamma_{N,L}^{(2)} P^{(2)} - \int_{S(P(L^2(\Omega_{L})) \left\langle \phi_L^{(2)} \phi_L^{(2)} \right\rangle d\mu_{N,L}(\phi_L) \right) \right)
= I + II + III.
\]

we only need to bound these terms from below.

We first handle \( I \). Since \( \mu_{N,L} \) is a positive measure and \( e_{N,L} \geq 0 \) via Lemma 2.9, we obtain

\[
(2.62) \quad I = \int \mathcal{E}_{N,L}(\phi_L) d\mu_{N,L}(\phi_L) \geq e_{N,L} \int d\mu_{N,L} \geq 0.
\]

Hence we can discard \( I \).

We now deal with \( II \). Expanding \( H_{12} \) gives

\[
(2.63) \quad II = \frac{1}{2} \text{Tr} \left( \left( S_1^2 + S_2^2 \right) (\gamma_{N,L}^{(2)} - P^{(2)} \gamma_{N,L}^{(2)} P^{(2)}) \right) + \frac{1}{2} \text{Tr} \left( LV_{N,L}(r_1 - r_2) \left( \gamma_{N,L}^{(2)} - P^{(2)} \gamma_{N,L}^{(2)} P^{(2)} \right) \right) 
= II_K + II_V.
\]

Since \( S_j^2 \geq S_j^2 P \), with Lemma 4.7, we obtain

\[
(2.64) \quad II_K = \frac{1}{2} \text{Tr} \left( \left( S_1^2 + S_2^2 \right) (\gamma_{N,L}^{(2)} - P^{(2)} \gamma_{N,L}^{(2)} P^{(2)}) \right) 
= \frac{1}{2} \left\langle (S_1^2 + S_2^2) \psi_{N,L}, \psi_{N,L} \right\rangle + \frac{1}{2} \left\langle (S_1^2 - S_2^2 P) \psi_{N,L}, \psi_{N,L} \right\rangle \geq 0.
\]

For \( II_V \), we expand

\[
(2.65) \quad 2 \left( LV_{N,L}(r_1 - r_2) - P^{(2)} LV_{N,L}(r_1 - r_2) P^{(2)} \right) 
= LV_{N,L}(r_1 - r_2) (1 - P^{(2)}) + (1 - P^{(2)}) LV_{N,L}(r_1 - r_2) 
+ (1 - P^{(2)}) LV_{N,L}(r_1 - r_2) P^{(2)} + P^{(2)} LV_{N,L}(r_1 - r_2) (1 - P^{(2)}).
\]
By Lemma A.7 and Cauchy-Schwarz inequality,

\begin{align}
&\quad \frac{1}{2} \left| \text{Tr} \left( \left( LV_{N,L}(r_1 - r_2)(1 - P^{(2)}) + (1 - P^{(2)}) LV_{N,L}(r_1 - r_2) \right) \gamma_{N,L}^{(2)} \right) \right| \\
&= \frac{1}{2} \left| \langle LV_{N,L}(r_1 - r_2)(1 - P^{(2)}) \psi_{N,L}, \psi_{N,L} \rangle + \langle LV_{N,L}(r_1 - r_2)(1 - P^{(2)}) \psi_{N,L}, (1 - P^{(2)}) \psi_{N,L} \rangle \right| \\
&\leq L \left\| V_{N,L}(r_1 - r_2) \right\|_{L^2}^{1/2} \left\| \psi_{N,L}(r_1 - r_2) \right\|_{L^2}^{1/2} (1 - P^{(2)}) \psi_{N,L} \| L^2. \\
\end{align}

Computing in the same way, we have

\begin{align}
&\quad \frac{1}{2} \left| \text{Tr} \left( (1 - P^{(2)}) LV_{N,L}(r_1 - r_2) P^{(2)} + P^{(2)} LV_{N,L}(r_1 - r_2)(1 - P^{(2)}) \right) \gamma_{N,L}^{(2)} \right| \\
&= \frac{1}{2} \left| \langle LV_{N,L}(r_1 - r_2) P^{(2)} \psi_{N,L}, (1 - P^{(2)}) \psi_{N,L} \rangle + \langle LV_{N,L}(r_1 - r_2)(1 - P^{(2)}) \psi_{N,L}, P^{(2)} \psi_{N,L} \rangle \right| \\
&\leq L \left\| V_{N,L}(r_1 - r_2) \right\|_{L^2}^{1/2} P^{(2)} \left\| \psi_{N,L} \right\|_{L^2} \left\| V_{N,L}(r_1 - r_2) \right\|_{L^2}^{1/2} (1 - P^{(2)}) \psi_{N,L} \| L^2. \\
\end{align}

Combining estimates (2.66) and (2.67), we obtain

\begin{align}
&\quad \left| I_{1V} \right| \leq L \left( \left\| V_{N,L}(r_1 - r_2) \right\|_{L^2}^{1/2} \left\| \psi_{N,L} \right\|_{L^2} + \left\| V_{N,L}(r_1 - r_2) \right\|_{L^2}^{1/2} P^{(2)} \left\| \psi_{N,L} \right\|_{L^2} \right) \\
&\quad \times \left\| V_{N,L}(r_1 - r_2) \right\|_{L^2}^{1/2} (1 - P^{(2)}) \psi_{N,L} \| L^2. \\
\end{align}

Next, we need to bound the right side terms. From estimate (2.58), we obtain

\begin{align}
&\quad \left\| LV_{N,L}(r_1 - r_2) \right\|_{L^2}^{1/2} \left\| \psi_{N,L} \right\|_{L^2} \left( (1 - \Delta_{x_1})(1 - \Delta_{x_2}) \right)^{1/2 + \delta} \\
&\quad \leq C_6 \left\| V \right\|_{L^\infty} \left( \eta^{-1}(1 - \Delta_{x_1})(1 - \Delta_{x_2}) + \eta^{\frac{1 + 2\delta}{2}} \right),
\end{align}

where we have used the interpolation inequality for fractional powers in Lemma A.3 in the last line. By optimizing over \( \eta > 0 \), we have

\begin{align}
&\quad L \left\| V_{N,L}(r_1 - r_2) \right\|_{L^2}^{1/2} \left\| \psi_{N,L} \right\|_{L^2} \leq C_6 \left\| V \right\|_{L^\infty} \left( (1 - \Delta_{x_1})(1 - \Delta_{x_2}) \psi_{N,L}, \psi_{N,L} \right)^{1/2 + \delta}.
\end{align}

Similarly,

\begin{align}
&\quad L \left\| V_{N,L}(r_1 - r_2) \right\|_{L^2}^{1/2} P^{(2)} \left\| \psi_{N,L} \right\|_{L^2} \leq C_6 \left\| V \right\|_{L^\infty} \left( (1 - \Delta_{x_1})(1 - \Delta_{x_2}) \psi_{N,L}, \psi_{N,L} \right)^{1/2 + \delta}.
\end{align}

Using estimate (2.58) again, we get

\begin{align}
&\quad (1 - P^{(2)}) \left| LV_{N,L}(r_1 - r_2)(1 - P^{(2)}) \right| \\
&\quad \leq C_6 \left\| V \right\|_{L^\infty} \left( (1 - P^{(2)})(1 - \Delta_{x_1})(1 - \Delta_{x_2}) \right)^{1/2 + \delta} (1 - P^{(2)}) \\
&\quad = C_6 \left\| V \right\|_{L^\infty} \left( (1 - \Delta_{x_1})(1 - \Delta_{x_2}) \right)^{1/2 + \delta},
\end{align}

where we used \((1 - \Delta_{x_1}) P^{(2)} = P^{(2)}(1 - \Delta_{x_1})\) in the last equality.

Since \(1 - P^{(2)} \leq Q_1 + Q_2\), we use Property 3 in Lemma A.6 to get

\begin{align}
&\quad (1 - P^{(2)})(1 - \Delta_{x_1})(1 - \Delta_{x_2})^{1/2 + \delta} \leq (Q_1 + Q_2) ((1 - \Delta_{x_1})(1 - \Delta_{x_2}))^{1/2 + \delta}.
\end{align}

By min \( \left\{ 1 + \lambda_2^2, 1 + (\lambda_2^2 - 1)/L^2 \right\} \leq S^2\), we obtain

\begin{align}
&\quad (Q_1 + Q_2) ((1 - \Delta_{x_1})(1 - \Delta_{x_2}))^{1/2 + \delta} \\
&\quad \leq (Q_1 + Q_2)(S_1^2 S_2^2)^{1/2 + \delta} \\
&\quad \leq \frac{1}{\min \left\{ 1 + \lambda_2^2, 1 + (\lambda_2^2 - 1)/L^2 \right\}^{1/2 + \delta}} \left( S_1^2 S_2^2 \right)^{1/2 + \delta} + \left( S_2^2 \right)^{1/2 + \delta} S_2^2 \\
&\quad \leq \frac{1}{\min \left\{ 1 + \lambda_2^2, 1 + (\lambda_2^2 - 1)/L^2 \right\}^{1/2 + \delta}} \left[ \eta^{-1} S_1^2 S_2^2 + \eta^{1 + 2\delta} \left( S_1^2 + S_2^2 \right) \right],
\end{align}

where we have used the interpolation inequality for fractional powers in Lemma A.3 in the last line.
Putting estimates (2.71), (2.72) and (2.73) together, we have

\[
(1 - P^{(2)}) |L_{V,N,L}(r_1 - r_2)| (1 - P^{(2)}) \leq \frac{C \delta \|V\|_{L^{\infty}_x L^1_t}}{\min \{1 + \lambda_2^2, 1 + (\lambda_2^2 - 1)/L^2\}^{1/2 - \delta}} \left[ \eta^{-1} S_1^2 S_2^2 + \eta \frac{1 + \delta}{1 - \delta} (S_1^2 + S_2^2) \right].
\]

By optimizing over \( \eta \), we deduce that

\[
L \|V_{N,L}(r_1 - r_2)\|_{L^2}^{1/2} (1 - P^{(2)}) \psi_{N,L}^2 \leq (L \|V_{N,L}(r_1 - r_2)\|_{L^2} (1 - P^{(2)}) \psi_{N,L}) \leq \frac{C \delta \|V\|_{L^{\infty}_x L^1_t}}{\min \{1 + \lambda_2^2, 1 + (\lambda_2^2 - 1)/L^2\}^{1/2 - \delta}} \left( \text{Tr} \left(S_1^2 \gamma_{N,L}^1\right) \right)^{1/2 - \delta} \left( \text{Tr} \left(S_1^2 S_2^2 \gamma_{N,L}^2\right) \right)^{1/2 + \delta}.
\]

Combining estimates (2.68), (2.69), (2.70) and (2.73), we get

\[
II \geq - \frac{C \delta \|V\|_{L^{\infty}_x L^1_t}}{\min \{1 + \lambda_2^2, 1 + (\lambda_2^2 - 1)/L^2\}^{1/4 - \delta/2}} \left( \text{Tr} \left(S_1^2 \gamma_{N,L}^1\right) \right)^{1/4 - \delta/2} \left( \text{Tr} \left(S_1^2 S_2^2 \gamma_{N,L}^2\right) \right)^{1/2 + \delta}.
\]

For \( III \), expanding \( H_{12} \) yields

\[
III = \frac{1}{2} \text{Tr} \left( S_1^2 + S_2^2 \right) \left( P^{(2)} \gamma_{N,L}^2 P^{(2)} - \int_{SP(L^2(\Omega_L))} |\phi_L^{(2)} \rangle \langle \phi_L^{(2)}| d\mu_{N,L}(\phi_L) \right) + \frac{1}{2} \text{Tr} \left( L_{V,N,L}(r_1 - r_2) \left( P^{(2)} \gamma_{N,L}^2 P^{(2)} - \int |\phi_L^{(2)} \rangle \langle \phi_L^{(2)}| d\mu_{N,L}(\phi_L) \right) \right)
\]

\[
= III_K + III_V.
\]

For the first term \( III_K \), we can use the inequality \( |\text{Tr} AB| \leq \|A\|_{op} |\text{Tr} B| \) to get

\[
|III_K| \leq \|P^{(2)} L_{V,N,L}(r_1 - r_2)|P^{(2)}|_{op} |\text{Tr} P^{(2)} \gamma_{N,L}^2 P^{(2)} - \int_{SP(L^2(\Omega_L))} |\phi_L^{(2)} \rangle \langle \phi_L^{(2)}| d\mu_{N,L}(\phi_L) \right)|.
\]

Using \( P S^2 \leq (1 + \lambda_2^2 + (\lambda_2^2 - 1)/L^2) P \) and Lemma 2.10 with \( d \leq \lambda_2^2 \lambda_2 \), we have

\[
|III_K| \leq (1 + \lambda_2^2 + (\lambda_2^2 - 1)/L^2) \lambda_2^2 \lambda_2 \frac{\lambda_2}{N}.
\]

For the second term \( III_V \), we use the inequality \( |\text{Tr} AB| \leq \|A\|_{op} |\text{Tr} B| \) again to get

\[
|III_V| \leq \|P^{(2)} L_{V,N,L}(r_1 - r_2)|P^{(2)}|_{op} |\text{Tr} P^{(2)} \gamma_{N,L}^2 P^{(2)} - \int_{SP(L^2(\Omega_L))} |\phi_L^{(2)} \rangle \langle \phi_L^{(2)}| d\mu_{N,L}(\phi_L) \right)|.
\]

By estimate (2.58) and \( P^j (1 - \Delta x) \leq (1 + \lambda_2^2) P^j \), we have

\[
P^{(2)} |L_{V,N,L}(r_1 - r_2)|P^{(2)} \leq C \delta \|V\|_{L^\infty_x L^1_t} \left( (P^j (1 - \Delta x)) \otimes (P^2 (1 - \Delta x)) \right)^{1/2 + \delta} \leq C \delta \|V\|_{L^\infty_x L^1_t} (1 + \lambda_2^2)^{1 + 2\delta / N}.
\]

Noting that \( |P^{(2)} L_{V,N,L}(r_1 - r_2)|P^{(2)}, P^{(2)}| = 0 \), by Property 2 in Lemma 2.6, we deduce that

\[
\|P^{(2)} L_{V,N,L}(r_1 - r_2)|P^{(2)}|_{op} \leq C \delta \|V\|_{L^\infty_x L^1_t} (1 + \lambda_2^2)^{1 + 2\delta / N}.
\]

Therefore, using Lemma 2.10 again, we get

\[
|\text{Tr} \left( L_{V,N,L}(r_1 - r_2) \left( P^{(2)} \gamma_{N,L}^2 P^{(2)} - \int_{SP(L^2(\Omega_L))} |\phi_L^{(2)} \rangle \langle \phi_L^{(2)}| d\mu_{N,L}(\phi_L) \right) \right) | \leq C \delta \|V\|_{L^\infty_x L^1_t} (1 + \lambda_2^2)^{1 + 2\delta / N} \frac{\lambda_2^2 \lambda_2}{N}.
\]

Thus, for all \( \delta > 0 \),

\[
III \gtrsim -(C \delta \|V\|_{L^\infty_x L^1_t} (1 + \lambda_2^2)^{1 + 2\delta} + (1 + \lambda_2^2 + (\lambda_2^2 - 1)/L^2) \lambda_2^2 \lambda_2 \frac{\lambda_2}{N}).
\]
Combining estimates (2.82) and (2.76), we get the desired lower bound, that is,
\[
\langle \psi_{N,L}, (1 + N^{-1}H_{N,L} - 1/L^2) \psi_{N,L} \rangle 
\geq - (C_5 ||V||_{L^2} L^2_\gamma (1 + \lambda_2^N)^{1+2\theta} + (1 + \lambda_2^N) + (\lambda_2^N - 1)/L^2) \frac{\lambda_2^N}{N} 
- \frac{C_5 ||V||_{L^2} L^2_\gamma}{\min \{1 + \lambda_2^N, 1 + (\lambda_2^N - 1)/L^2\}^{1/4-\delta/2}} \left( Tr \left( S_1^2 \gamma_{N,L}^{(1)} \right) \right)^{1/4-\delta/2} \left( Tr \left( S_1^2 S_2^2 \gamma_{N,L}^{(2)} \right) \right)^{1/2+\delta}.
\]

From Lemma 2.12 to get the lower bound estimate for $E_{N,L}$, we are left to control $Tr(s_1^2 \gamma_{N,L}^{(1)})$ and $Tr(s_1^2 s_2^2 \gamma_{N,L}^{(2)})$ for the ground state $\psi_{N,L}$.

**Lemma 2.13.** Assume $L(N/L)^{3} \to 1^{-},$ $0 < \beta < \frac{1}{4}$. Let $\psi_{N,L}$ be a ground state of $1 + H_{N,L} - 1/L^2$. For all $\theta \in (0, 1)$, we have

\begin{align}
(2.83) & \quad Tr \left( S_1^2 \gamma_{N,L}^{(1)} \right) \lesssim C_{up} \frac{1 + |E_{N,L,\theta}|}{\theta}, \\
(2.84) & \quad Tr \left( S_1^2 S_2^2 \gamma_{N,L}^{(2)} \right) \lesssim C_{up} \left( \frac{1 + |E_{N,L,\theta}|}{\theta} \right)^2.
\end{align}

**Proof.** For estimate (2.83), from the definition of $E_{N,L,\theta}$, we have

\[
(2.85) \quad 1 + N^{-1}H_{N,L} - 1/L^2 - \theta N^{-1} \sum_{j=1}^{N} S_j^2 \geq E_{N,L,\theta}.
\]

Let $\psi_{N,L}$ be a ground state of $1 + N^{-1}H_{N,L} - 1/L^2$. By taking the expectation against $\psi_{N,L}$ on both sides of (2.85), we obtain

\[
Tr \left( S_1^2 \gamma_{N,L}^{(1)} \right) \leq \frac{1}{\theta} \left( \langle \psi_{N,L}, (1 + N^{-1}H_{N,L} - 1/L^2) \psi_{N,L} \rangle - E_{N,L,\theta} \right) = \frac{E_{N,L} - E_{N,L,\theta}}{\theta}.
\]

From Lemma 2.7, we see $C_{up} \geq e_{N,L} \geq E_{N,L} \geq E_{N,L,\theta}$. Therefore, $|E_{N,L}| \leq C_{up} + |E_{N,L,\theta}|$ and

\[
(2.86) \quad Tr \left( S_1^2 \gamma_{N,L}^{(1)} \right) \leq 2C_{up} (1 + \frac{|E_{N,L,\theta}|}{\theta}).
\]

For estimate (2.84), we notice that

\[
(2.87) \quad Tr \left( S_1^2 S_2^2 \gamma_{N,L}^{(2)} \right) \leq \frac{2}{N^2} \langle \psi_{N,L}, \left( \sum_{j=1}^{N} S_j^2 \right)^2 \psi_{N,L} \rangle.
\]

Thus, it needs only to control the right term of (2.87). We rewrite

\[
(2.88) \quad \frac{2}{N^2} \left( \sum_{j=1}^{N} S_j^2 \right)^2 = (1 + N^{-1}H_{N,L} - 1/L^2) \frac{1}{N} \sum_{j=1}^{N} S_j^2 + \frac{1}{N} \sum_{j=1}^{N} S_j^2 (1 + N^{-1}H_{N,L} - 1/L^2)
- \frac{1}{N^2(N-1)} \sum_{i=1}^{N} \sum_{j<k} \left(S_j^2 L V_{N,L} (r_j - r_k) + L V_{N,L} (r_j - r_k) S_i^2 \right).
\]

We need to control the right side terms of (2.88).

For the first and second terms, by using the ground state $\psi_{N,L}$ and estimate (2.86), we have

\[
(2.89) \quad \langle \psi_{N,L}, \left( (1 + N^{-1}H_{N,L} - 1/L^2) \frac{1}{N} \sum_{j=1}^{N} S_j^2 + \frac{1}{N} \sum_{j=1}^{N} S_j^2 (1 + N^{-1}H_{N,L} - 1/L^2) \right) \psi_{N,L} \rangle
= \frac{2E_{N,L}}{N} \langle \psi_{N,L}, \sum_{j=1}^{N} S_j^2 \psi_{N,L} \rangle \leq \frac{4C_{up}^2 (1 + |E_{N,L,\theta}|)}{\theta} \left( 1 + \frac{|E_{N,L,\theta}|}{\theta} \right).
\]
Therefore, we decompose it into two cases. On the one hand, we consider the case $j \neq i$ and $k \neq i$. By estimate (2.57), we have

$$
\frac{1}{N(N-1)} \sum_{i \neq j < k \neq i} LV_{N,L}(r_j - r_k) = 1 + N^{-1} H_{N,L} - 1/L^2 - \theta N^{-1} \sum_{j=1}^N S_j^2 - (1 - \theta) N^{-1} \sum_{j=1}^N S_j^2 - \frac{1}{N(N-1)} \sum_{j \neq i} LV_{N,L}(r_i - r_j)
$$

\begin{align*}
\geq & \ E_{N,L,\theta} - \left( \frac{1 - \theta}{N} + \frac{C_\delta(N/L)\|V\|_{L^\infty L_1^{\alpha+\delta}}}{N^2} \right) \sum_{j=1}^N S_j^2.
\end{align*}

Since $[V_{N,L}(r_j - r_k), S_i^2] = 0$, by summing over $i$, we obtain

$$
\frac{1}{N^2(N-1)} \sum_{i=1}^N \sum_{j \neq k \neq i} \left( S_i^2 LV_{N,L}(r_j - r_k) + LV_{N,L}(r_j - r_k) S_i^2 \right) \geq 2 E_{N,L,\theta} \sum_{j=1}^N S_j^2 - 2 \left( \frac{1 - \theta}{N} + \frac{C_\delta(N/L)\|V\|_{L^\infty L_1^{\alpha+\delta}}}{N^2} \right) \sum_{j=1}^N S_j^2.
$$

On the other hand, when $j = i$ or $k = i$, by estimate (2.60), we have

$$
S_i^2 LV_{N,L}(r_j - r_k) + LV_{N,L}(r_j - r_k) S_i^2 \geq -C_\delta(V)(N/L)^{\beta+\delta} S_i^2 S_k^2.
$$

Therefore,

$$
\frac{1}{N^2(N-1)} \sum_{j \neq k} \left( S_j^2 LV_{N,L}(r_j - r_k) + LV_{N,L}(r_j - r_k) S_j^2 \right) \geq -C_\delta(V)(N/L)^{\beta+\delta} \left( \sum_{j=1}^N S_j^2 \right)^2.
$$

Putting estimates (2.91) and (2.92) together, we obtain

$$
\frac{1}{N^2(N-1)} \sum_{i=1}^N \sum_{j < k} \left( S_i^2 LV_{N,L}(r_j - r_k) + LV_{N,L}(r_j - r_k) S_i^2 \right) \geq -\frac{2}{N^2} \left( 1 - \theta + \frac{C_\delta(V)(N/L)^{\beta+\delta}}{N} \right) \left( \sum_{j=1}^N S_j^2 \right)^2 + \frac{2E_{N,L,\theta}}{N} \sum_{j=1}^N S_j^2.
$$

Taking the expectation against the ground state $\psi_{N,L}$ on both sides of (2.93), we obtain

$$
\frac{1}{N^2(N-1)} \sum_{i=1}^N \sum_{j < k} \langle \psi_{N,L}, \left( S_i^2 LV_{N,L}(r_j - r_k) + LV_{N,L}(r_j - r_k) S_i^2 \right) \psi_{N,L} \rangle \geq -\frac{2}{N^2} \left( 1 - \theta + \frac{C_\delta(V)(N/L)^{\beta+\delta}}{N} \right) \left( \sum_{j=1}^N S_j^2 \right)^2 \langle \psi_{N,L}, \psi_{N,L} \rangle + \frac{2E_{N,L,\theta}}{N} \sum_{j=1}^N \langle \psi_{N,L}, S_j^2 \psi_{N,L} \rangle
$$

\begin{align*}
\geq & -\frac{2}{N^2} \left( 1 - \theta + \frac{C_\delta(V)(N/L)^{\beta+\delta}}{N} \right) \langle \psi_{N,L}, \left( \sum_{j=1}^N S_j^2 \right)^2 \psi_{N,L} \rangle - \frac{2C_{\text{up}}(1 + |E_{N,L,\theta}|^2)}{\theta},
\end{align*}

where we used estimate (2.80) in the last line.
Taking the expectation against the ground state $\psi_{N,L}$ on both sides of (2.88), we use estimates (2.89) and (2.91) to obtain

$$
(2.95) \quad \frac{2}{N^2} \left\langle \psi_{N,L}, \left( \sum_{j=1}^{N} S_j^2 \right)^2 \psi_{N,L} \right\rangle \leq \frac{8C_{up}^2(1 + |E_{N,L,\theta}|)^2}{\theta} + \frac{2}{N^2} \left( 1 - \theta + \frac{C_\delta(V)(N/L)^{\beta+\delta}}{N} \right) \left\langle \psi_{N,L}, \left( \sum_{j=1}^{N} S_j^2 \right)^2 \psi_{N,L} \right\rangle.
$$

Equivalently, we have

$$
(2.96) \quad \frac{2}{N^2} \left( \theta - \frac{C_\delta(V)(N/L)^{\beta+\delta}}{N} \right) \left\langle \psi_{N,L}, \left( \sum_{j=1}^{N} S_j^2 \right)^2 \psi_{N,L} \right\rangle \leq \frac{8C_{up}^2(1 + |E_{N,L,\theta}|)^2}{\theta}.
$$

Since $\beta < 1/2$, we can take $\delta$ such that $2\beta + \delta < 1$. Then, with $L(N/L)^{\beta} \to 1^-$, we deduce that

$$
\theta - \frac{C_\delta(V)(N/L)^{\beta+\delta}}{N} \geq \theta - \frac{C_\delta(V)(N/L)^{2\beta+\delta}}{N/L} \geq \theta/2,
$$

for large $N$ and $1/L$. With estimate (2.96), we conclude

$$
Tr \left( S_1^2 \delta E_{N,L}^{(2)} \right) \leq \frac{2}{N^2} \left\langle \left( \sum_{j=1}^{N} S_j^2 \right)^2 \psi_{N,L}, \psi_{N,L} \right\rangle \lesssim \frac{C_{up}^2(1 + |E_{N,L,\theta}|)^2}{\theta^2}.
$$

Then, we prove the following theorem with a bootstrapping argument.

**Theorem 2.14.** Assume $L(N/L)^{\beta} \to 1^-$ and $\|V\|_{L^\infty L^1} \leq \frac{2\pi}{\epsilon^2}$ for $\alpha \in (0, 1)$. Then for $\beta < 1/3$, $\epsilon \in [0, 1-\alpha)$

$$
E_{N,L,\epsilon} \geq -C.
$$

**Proof.** From Lemma 2.12 and Lemma 2.13

$$
(2.97) \quad E_{N,L} \geq - \left( C_{\delta}(\|V\|_{L^\infty L^1}) \lambda_x^{\alpha_{x+\delta}} + (1 + \lambda_x^2) + (\lambda_x^2 - 1)/L^2 \right) \frac{\lambda_x^2 \lambda_z}{N} \\
- \frac{C_{\delta}(\|V\|_{L^\infty L^1})}{\min \{1 + \lambda_x^2, 1 + (\lambda_x^2 - 1)/L^2\}^{1/4-\delta/2}} \left( \frac{C_{up}(1 + |E_{N,L,\theta}|)}{\theta} \right)^{5/4+3\delta/2}.
$$

Similarly, we use $e_\epsilon^{N,L}$, $E_{N,L}^\epsilon$, $E_{N,L,\theta}$ to denote the ground state energy and $C_{up}$ to denote the upper bound in Lemma 2.3 with interaction function $V^\epsilon = (1 - \epsilon)^{-1}V$. Then, we obtain

$$
(2.98) \quad E_{N,L}^\epsilon \geq - \left( C_{\delta}(\|V\|_{L^\infty L^1}) \lambda_x^{\alpha_{x+\delta}} + (1 + \lambda_x^2) + (\lambda_x^2 - 1)/L^2 \right) \frac{\lambda_x^2 \lambda_z}{N} \\
- \frac{C_{\delta}(\|V\|_{L^\infty L^1})}{\min \{1 + \lambda_x^2, 1 + (\lambda_x^2 - 1)/L^2\}^{1/4-\delta/2}} \left( \frac{C_{up}(1 + |E_{N,L,\theta}|)}{\theta} \right)^{5/4+3\delta/2}.
$$

Noting that $E_{N,L,\epsilon} = (1 - \epsilon)E_{N,L}$, we take $\theta = (\epsilon' - \epsilon)/(1 - \epsilon)$ to obtain

$$
E_{N,L,\epsilon} \geq -C(V, \delta, \epsilon, \epsilon') \left( (\lambda_x^{\alpha_{x+\delta}} + (\lambda_x^2 - 1)/L^2) \frac{\lambda_x^2 \lambda_z}{N} + \frac{|E_{N,L,\epsilon}|^{5/4+3\delta/2}}{\min \{1 + \lambda_x^2, 1 + (\lambda_x^2 - 1)/L^2\}^{1/4-\delta/2}} \right).
$$

for all $0 < \epsilon < \epsilon' < 1/2$ and $\delta \in (0, 1/2)$, $N \geq N(\delta, \epsilon, \epsilon')$.

We make the induction hypothesis (labeled $I_\eta$)

$$
(2.99) \quad \limsup_{N \to \infty} \frac{|E_{N,L,\epsilon}|}{1 + (N/L)^{\eta}} < \infty \text{ for all } 0 < \epsilon < 1 - \alpha.
$$

$$
\text{for all } 0 < \epsilon < 1 - \alpha.
$$
By Theorem 2.3, \( I_\eta \) holds for \( \eta = 3\beta - 1 \) as a start point. Then, by choosing \( \lambda_z = 2 \) and \( \lambda_x = (N/L)\tau \) with \( \tau \leq \beta \), we deduce that \( I_{\eta'} \) holds provided that

\[
(2.100) \quad \eta' > \max\{2\tau + (3\beta - 1), 5\eta/4 - \tau/2\}.
\]

With the optimal choice \( \tau = \eta/2 + 2(1-3\beta)/5 \), we get

\[
(2.101) \quad \eta' > \eta - \frac{(1-3\beta)}{5}.
\]

When \( \beta < 1/3 \), we can choose a constant \( c \) such that

\[
0 < c < 1 - 3\beta
\]

and \( I_{\eta'} \) holds with \( \eta' = \eta - c \). Repeating the process, we finally deduce that \( I_0 \) holds. It means that \( |E_{N,L,\varepsilon}| \leq C_0 \) for \( N \geq N_0 \), which is equivalent to \( E_{N,L,\varepsilon} \geq -C_0 \) for \( N \geq N_0 \).

\[ \square \]

2.4. **High Energy estimates when \( k > 1 \).** Assuming (2.21) holds for \( k \), we now prove it for \( k + 2 \). By the induction hypothesis, we have

\[
\frac{1}{c_0^{k+2}} \langle \psi_{N,L}, (2 + N^{-1}H_{N,L} - 1/L^2)^{k+2} \psi_{N,L} \rangle
\geq \frac{1}{c_0^k} \| S^{(k)}(2 + N^{-1}H_{N,L} - 1/L^2)\psi_{N,L} \|_{L^2}^2
= MS + EC + EP
\]

where the main sum \( MS \) is

\[
MS = \frac{1}{c_0^2N^2(N-1)^2} \sum_{1 \leq i_1 < j_1 \leq N \atop i_1 > k, i_2 > k} \langle S^{(k)}(2 + H_{i_1,j_1})\psi_{N,L}, S^{(k)}(2 + H_{i_2,j_2})\psi_{N,L} \rangle
\]

the cross error term \( EC \) is

\[
EC = \frac{1}{c_0^2N^2(N-1)^2} \sum_{1 \leq i_1 < j_1 \leq N \atop 1 \leq i_2 < j_2 \leq N \atop i_1 < k, i_2 > k} 2Re\langle S^{(k)}(2 + H_{i_1,j_1})\psi_{N,L}, S^{(k)}(2 + H_{i_2,j_2})\psi_{N,L} \rangle
\]

and the nonnegative error term \( EP \) is

\[
EP = \frac{1}{c_0^2N^2(N-1)^2} \sum_{1 \leq i_1 < j_1 \leq N \atop 1 \leq i_2 < j_2 \leq N \atop i_1, i_2 \leq k} \langle S^{(k)}(2 + H_{i_1,j_1})\psi_{N,L}, S^{(k)}(2 + H_{i_2,j_2})\psi_{N,L} \rangle
\]

\[
= \frac{1}{c_0^2N^2(N-1)^2} \left( \sum_{1 \leq i_1 < j_1 \leq N \atop i_1 \leq k} S^{(k)}(2 + H_{i,j})\psi_{N,L} \right) \left( \sum_{1 \leq i_1 < j_1 \leq N \atop i_1 \leq k} S^{(k)}(2 + H_{i,j})\psi_{N,L} \right) \geq 0.
\]

2.4.1. **Handling the main sum.** Commuting \( (1 + H_{i_1,j_1}) \) and \( (1 + H_{i_2,j_2}) \) with \( S^{(k)} \) in \( MS \), we get

\[
MS = M_1 + M_2 + M_3,
\]

where \( M_1 \) consists of the terms with

\[
\{i_1, j_1\} \cap \{i_2, j_2\} = \emptyset,
\]

\( M_2 \) consists of the terms with

\[
|\{i_1, j_1\} \cap \{i_2, j_2\}| = 1,
\]

and \( M_3 \) consists of the terms with

\[
|\{i_1, j_1\} \cap \{i_2, j_2\}| = 2.
\]
By symmetric of $\psi_{N,L}$, we have

$$M_1 = \frac{1}{4c_0} \langle S^{(k)}(2 + H_{(k+1)(k+2)})\psi_{N,L}, S^{(k)}(2 + H_{(k+3)(k+4)})\psi_{N,L} \rangle,$$

$$M_2 = \frac{1}{2c_0} N^{-1} \langle S^{(k)}(2 + H_{(k+1)(k+2)})\psi_{N,L}, S^{(k)}(2 + H_{(k+2)(k+3)})\psi_{N,L} \rangle,$$

$$M_3 = \frac{1}{2c_0} N^{-2} \langle S^{(k)}(2 + H_{(k+1)(k+2)})\psi_{N,L}, S^{(k)}(2 + H_{(k+1)(k+2)})\psi_{N,L} \rangle,$$

up to an unimportant combination number.

Since $M_3 \geq 0$, we drop it. By the fact that

$$[2 + H_{(k+1)(k+2)}, 2 + H_{(k+3)(k+4)}] = 0,$$

we have

$$M_1 \geq \frac{4(1 - \alpha)^2}{4c_0^2} \langle S^{(k)}\psi_{N,L}, S_{k+1}^2 S_{k+2}^2 S^{(k)}\psi_{N,L} \rangle$$

using Theorem 2.14 and Lemma A.6. Recall $c_0 = \min \left( \frac{1}{\sqrt{2}}, \frac{1}{2} \right)$, hence

$$(2.102) \quad M_1 \geq 2 \langle S^{(k+2)}\psi_{N,L}, S^{(k+2)}\psi_{N,L} \rangle = 2\|S^{(k+2)}\psi_{N,L}\|_{L^2}^2.$$

To deal with $M_2$, we expand

$$M_2 = M_{21} + M_{22} + M_{23},$$

where

$$M_{21} = \frac{N^{-1}}{2c_0} \langle (2 + S^{(k+1)} S^{(k+2)})(2 + S^{(k+2)} S^{(k+3)})\psi_{N,L}, \psi_{N,L} \rangle,$$

$$M_{22} = \frac{N^{-1}}{c_0} \text{Re} \langle (2 + S^{(k+1)} S^{(k+2)})(2 + S^{(k+2)} S^{(k+3)})\psi_{N,L}, \psi_{N,L} \rangle,$$

$$M_{23} = \frac{N^{-1}}{2c_0} \langle \text{LV}_{N,L}(r_{k+1} - r_{k+2}) S^{(k)} \psi_{N,L}, \text{LV}_{N,L}(r_{k+1} - r_{k+2}) S^{(k)} \psi_{N,L} \rangle.$$

We keep only the $S_{k+2}^2$ terms inside $M_{21}$, which is the main contribution. That is,

$$(2.103) \quad M_{21} \geq \frac{N^{-1}}{2c_0} \langle S_{k+2}^2 S_{k+2}^2 S^{(k)} \psi_{N,L}, S^{(k)} \psi_{N,L} \rangle \geq 2N^{-1} \langle S_{k+1}^4 \psi_{N,L}, S^{(k)} \psi_{N,L} \rangle$$

$$= 2N^{-1} \|S_1 S^{(k+1)} \psi_{N,L}\|_{L^2}^2.$$ 

For $M_{22}$, we expand

$$M_{22} = \frac{2N^{-1}}{c_0} \langle S^{(k)} \psi_{N,L}, \text{LV}_{N,L}(r_{k+2} - r_{k+3}) S^{(k)} \psi_{N,L} \rangle$$

$$+ \langle \frac{N^{-1}}{c_0} \langle S^{(k+1)} \psi_{N,L}, \text{LV}_{N,L}(r_{k+2} - r_{k+3}) S^{(k+1)} \psi_{N,L} \rangle$$

$$+ \langle \frac{N^{-1}}{c_0} \langle S^{(k)} \psi_{N,L}, S_{k+2}^2 \text{LV}_{N,L}(r_{k+2} - r_{k+3}) S^{(k)} \psi_{N,L} \rangle$$

$$= M_{221} + M_{222} + M_{223}.$$ 

By estimate (2.97),

$$(2.104) \quad |M_{221}| + |M_{222}| \lesssim \frac{L(N/L)^{3+}}{N} \left( \|S^{(k+1)} \psi_{N,L}\|_{L^2}^2 + \|S^{(k+2)} \psi_{N,L}\|_{L^2}^2 \right).$$

By estimate (2.60),

$$(2.105) \quad |M_{223}| \lesssim \frac{(N/L)^{3+}}{N} \|S^{(k+2)} \psi_{N,L}\|_{L^2}^2.$$ 

This requires $\beta < \frac{1}{2}$. 

For $M_{23}$, with Hölder inequality, we have
\begin{equation}
\tag{2.106}
|M_{23}| \leq N^{-1}\|LV_{N,L}(r_{k+1} - r_{k+2})\|_{L^\infty_k L^1_k} + \|LV_{N,L}(r_{k+1} - r_{k+2})\|_{L^\infty_k L^1_k} + \|S_{(k)}^2\psi_{N,L}\|_{L^2_k L^\infty_k}^2 \lesssim N^{-1}L^2(N/L)^{2\beta+24}||S_{(k+2)}^2\psi_{N,L}\|_{L^2}^2.
\end{equation}

Putting (2.102)–(2.106) together, with the assumption $L(N/L)\beta \to 1^-$, we arrive at the following estimate for $MS$:
\begin{equation}
\tag{2.107}
MS \geq (2 - C(N/L)^{2\beta-1+\epsilon})(\|S_{(k+2)}\psi_{N,L}\|_{L^2}^2 + N^{-1}\|S_{(k+1)}^2\psi_{N,L}\|_{L^2}^2).
\end{equation}

2.4.2. Handling the cross error term. Next, we turn our attention to estimate $E_C$. We will prove that
\begin{equation}
\tag{2.108}
E_C \geq -C\max\left(\frac{N^{-\frac{2}{3}}(N/L)^{2\beta+1}}{1}, \frac{N^{-\frac{1}{2}}(N/L)^{2\beta+1}}{1}\right)(\|S_{(k+2)}\psi_{N,L}\|_{L^2}^2 + N^{-1}\|S_{(k+1)}^2\psi_{N,L}\|_{L^2}^2).
\end{equation}

Since $L(N/L)\beta \to 1^-$, (2.108) is equivalent to
\begin{equation}
\tag{2.109}
E_C \geq -C(N/L)^{(7\beta-3)/2}\|S_{(k+2)}\psi_{N,L}\|_{L^2}^2 + N^{-1}\|S_{(k+1)}^2\psi_{N,L}\|_{L^2}^2.
\end{equation}

That is, when $\beta < 3/7$, $E_C$ can be absorbable if added into 2.107.

We assume $k \geq 1$, since $E_C = 0$ when $k = 0$. We decompose the sum into three parts
\[ E_C = E_1 + E_2 + E_3, \]
where $E_1$ contains the terms with $j_1 \leq k$, $E_2$ contains the terms with $j_1 > k$ and $j_1 \in \{i_2, j_2\}$ and $E_3$ contains those term with $j_1 > k$, $j_1 \neq j_2$.

Since $H_{i_2j} = H_{j_1}$, by symmetry of $\psi_{N,L}$, we have
\begin{align*}
E_1 &= N^{-2}\langle S_{(k)}^2(2 + H_{i_2j})\rangle \psi_{N,L}, \\
E_2 &= N^{-2}\langle S_{(k)}^2(2 + H_{j_1})\rangle \psi_{N,L}, \\
E_3 &= N^{-1}\langle S_{(k)}^2(2 + H_{j_1})\rangle \psi_{N,L}.
\end{align*}

up to an unimportant combination number.

when $k = 1$, $E_1 = 0$. Therefore, we address $E_1$ for $k \geq 2$.
\[ E_1 = E_{11} + E_{12} + E_{13} + E_{14}, \]
where
\begin{align*}
E_{11} &= N^{-2}\langle S_{(k)}^2(2 + S_{(k)}^2 + S_{(k)}^2)\rangle \psi_{N,L}, \\
E_{12} &= N^{-2}\langle S_{(k)}^2(2 + S_{(k)}^2 + S_{(k)}^2)\rangle \psi_{N,L}, \\
E_{13} &= N^{-2}\langle S_{(k)}^2LV_{N,L}(r_1 - r_2)\rangle \psi_{N,L}, \\
E_{14} &= N^{-2}\langle S_{(k)}^2LV_{N,L}(r_1 - r_2)\rangle \psi_{N,L}.
\end{align*}

Since $E_{11} \geq 0$, we discard it. For $E_{12}$, by symmetry of $\psi_{N,L}$, we need to only consider
\[ N^{-2}\langle S_{(k)}^2S_{(k)}^2\rangle \psi_{N,L}, \]

Since \[ [V_{N,L}(r_{k+1} - r_{k+2}), S_1] = 0, \] we use estimate (2.57) to obtain
\[ N^{-2}\langle S_{(k)}^2S_{(k)}^2\rangle \psi_{N,L} = N^{-2}\langle S_{(k)}^2S_{(k)}^2\rangle \psi_{N,L} \lesssim N^{-2}L(N/L)^{2\beta}||S_{(k+1)}^2\psi_{N,L}|^2_{L^2}.
\]

Hence
\begin{equation}
\tag{2.110}
|E_{12}| \lesssim N^{-2}L(N/L)^{2\beta}||S_{(k+1)}^2\psi_{N,L}|^2_{L^2}.
\end{equation}

For $E_{13}$, we decompose
\[ E_{13} = E_{131} + E_{132} + E_{133}, \]
where

\[
E_{131} = N^{-2} \langle S^{(k)} LV_{N,L}(r_1 - r_2) \psi_{N,L}, S^{(k)} S_{k+1}^2 \psi_{N,L} \rangle,
\]

\[
E_{132} = N^{-2} \langle S^{(k)} LV_{N,L}(r_1 - r_2) \psi_{N,L}, S^{(k)} S_{k+2}^2 \psi_{N,L} \rangle,
\]

\[
E_{133} = N^{-2} \langle S^{(k)} LV_{N,L}(r_1 - r_2) \psi_{N,L}, 2S^{(k)} \psi_{N,L} \rangle.
\]

For \(E_{131}\), we expand

\[
E_{131} = N^{-2} \langle LV_{N,L}(r_1 - r_2) \prod_{j=3}^{k+1} S_j \psi_{N,L}, S_j^2 S_{k+1}^2 \prod_{j=3}^{k+1} S_j P_{z_1,>1} \psi_{N,L} \rangle
\]

\[
= N^{-2} \langle LV_{N,L}(r_1 - r_2) \prod_{j=3}^{k+1} S_j \psi_{N,L}, (1 - \Delta_{r_1} - 1/L^2) S_j^2 \prod_{j=3}^{k+1} S_j P_{z_1,>1} \psi_{N,L} \rangle
\]

\[
= E_{1311} + E_{1312},
\]

where

\[
E_{1311} = N^{-2} \langle LV_{N,L}(r_1 - r_2) \prod_{j=3}^{k+1} S_j \psi_{N,L}, -
\Delta_{r_1} S_j^2 \prod_{j=3}^{k+1} S_j P_{z_1,>1} \psi_{N,L} \rangle,
\]

\[
E_{1312} = N^{-2} \langle LV_{N,L}(r_1 - r_2) \prod_{j=3}^{k+1} S_j \psi_{N,L}, (1 - 1/L^2) S_j^2 \prod_{j=3}^{k+1} S_j P_{z_1,>1} \psi_{N,L} \rangle.
\]

Using integration by parts for \(E_{1311}\),

\[
E_{1311} = N^{-2} \langle \nabla_{r_1}(LV_{N,L}(r_1 - r_2) \prod_{j=3}^{k+1} S_j \psi_{N,L}), \nabla_{r_1} S_j^2 \prod_{j=3}^{k+1} S_j P_{z_1,>1} \psi_{N,L} \rangle
\]

\[
= N^{-2} L(N/L)^\beta \langle \langle \nabla_{r_1} V \rangle_{N,L}(r_1 - r_2) \prod_{j=3}^{k+1} S_j \psi_{N,L}), \nabla_{r_1} S_j^2 \prod_{j=3}^{k+1} S_j P_{z_1,>1} \psi_{N,L} \rangle
\]

\[+
N^{-2} L(V_{N,L}(r_1 - r_2) \nabla_{r_1} \prod_{j=3}^{k+1} S_j \psi_{N,L}, \nabla_{r_1} S_j^2 \prod_{j=3}^{k+1} S_j P_{z_1,>1} \psi_{N,L}).
\]

Using Hölder and Sobolev inequality,

\[
|E_{1311}| \leq N^{-2} L(N/L)^\beta \|\nabla V\|_{L^\infty_{r_1}} \|\psi_{N,L}\|_{L^\infty_{r_1} L^2_{r_2}} \|\prod_{j=3}^{k+1} S_j \psi_{N,L}\|_{L^2_{r_1}} \|\nabla_{r_1} S_j^2 \prod_{j=3}^{k+1} S_j P_{z_1,>1} \psi_{N,L}\|_{L^2_{r_1}}
\]

\[+
N^{-2} L \|V_{N,L}\|_{L^\infty_{r_1} L^2_{r_2}} \|\nabla_{r_1} \prod_{j=3}^{k+1} S_j \psi_{N,L}\|_{L^2_{r_1} L^\infty_{r_2}} \|\nabla_{r_1} S_j^2 \prod_{j=3}^{k+1} S_j P_{z_1,>1} \psi_{N,L}\|_{L^2_{r_1}}
\]

\[+
L(N/L)^\beta \|\nabla_{r_1} \prod_{j=2}^{k+1} S_j \psi_{N,L}\|_{L^2_{r_1}} \|\nabla_{r_1} S_j^2 \prod_{j=3}^{k+1} S_j P_{z_1,>1} \psi_{N,L}\|_{L^2_{r_1}}
\]

By estimates (A.25) and (A.29), with \(L(N/L)^\beta \rightarrow 1^-\), we have

\[
|E_{1311}| \leq N^{-2} L(N/L)^{2\beta+1} \|\prod_{j=2}^{k+1} S_j \psi_{N,L}\|_{L^2} \|\nabla_{r_1} S_j^2 \prod_{j=2}^{k+1} S_j P_{z_1,>1} \psi_{N,L}\|_{L^2}.
\]

(2.111)
For $E_{1312}$, with H"older and Sobolev inequality, we obtain

$$|E_{1312}| \lesssim N^{-2} L^{-1} \| V_{N,L} \|_{L^\infty_x L^2_t} \| \prod_{j=3}^{k+1} S_j \psi_{N,L} \|_{L^2_x L^\infty_t} \| S_2 \| \prod_{j=3}^{k+1} S_j P_{z_1,>1} \psi_{N,L} \|_{L^2}$$

$$\lesssim N^{-2} L^{-1} (N/L)^{2\beta^+} \| S_1 \prod_{j=3}^{k+1} S_j \psi_{N,L} \|_{L^2} \| S_2 \prod_{j=3}^{k+1} S_j P_{z_1,>1} \psi_{N,L} \|_{L^2}$$

By $L^{-2} P_{z_1,>1} \leq S_1^2 P_{z_1,>1}$, we get

$$|E_{1312}| \lesssim N^{-2} (N/L)^{2\beta^+} \| S^{(k)} \psi_{N,L} \|_{L^2_x} \| S_1 S^{(k+1)} \psi_{N,L} \|_{L^2}$$

$$\leq N^{-\frac{2}{3}} (N/L)^{2\beta^+} \{ \| S^{(k)} \psi_{N,L} \|_{L^2_x}^2 + N^{-1} \| S_1 S^{(k+1)} \psi_{N,L} \|_{L^2_x}^2 \}.$$ 

Estimated in the same way as $E_{1311}$.

$$|E_{132}| \lesssim N^{-\frac{2}{3}} (N/L)^{2\beta^+} \{ \| S^{(k)} \psi_{N,L} \|_{L^2_x}^2 + N^{-1} \| S_1 S^{(k+1)} \psi_{N,L} \|_{L^2_x}^2 \}.$$ 

For $E_{14}$, we decompose

$$E_{14} = N^{-2} (L V_{N,L} (r_1 - r_2) \| S_j \psi_{N,L}, S_1^2 S_2^2 L V_{N,L} (r_{k+1} - r_{k+2}) | S_j P_{z_1,>1} \psi_{N,L} \rangle = E_{141} + E_{142},$$

where

$$E_{141} = N^{-2} (L V_{N,L} (r_1 - r_2) \| S_j \psi_{N,L}, -\Delta r_1 L V_{N,L} (r_{k+1} - r_{k+2}) S_1^2 \prod_{j=3}^{k+1} S_j P_{z_1,>1} \psi_{N,L},$$

$$E_{142} = N^{-2} (L V_{N,L} (r_1 - r_2) \| S_j \psi_{N,L}, (1 - 1/L^2) L V_{N,L} (r_{k+1} - r_{k+2}) S_1^2 \prod_{j=3}^{k+1} S_j P_{z_1,>1} \psi_{N,L}.$$
By estimates (A.28) and (A.29), with \(L(N/L)^{\beta} \to 1^{-}\), we have

\[
|E_{141}| \lesssim N^{-2}(N/L)^{2\beta} \|\prod_{j=2}^{k+1} S_j \psi_{N,L}\|_{L^2} \|S_1 S^{(k+1)} \psi_{N,L}\|_{L^2} + N^{-2}(N/L)^{2\beta} \|S^{(k+1)} \psi_{N,L}\|_{L^2} \|S_1 S^{(k+1)} \psi_{N,L}\|_{L^2} \\
\leq N^{-\frac{1}{2}}(N/L)^{2\beta} (\|S^{(k+1)} \psi_{N,L}\|_{L^2} + N^{-1} \|S_1 S^{(k+1)} \psi_{N,L}\|_{L^2}^2).
\]

For \(E_{142}\), with Hölder and Sobolev inequality, we obtain

\[
|E_{142}| \lesssim N^{-2} \|V_{N,L}\|_{L^\infty} \left(\prod_{j=3}^{k} S_j \psi_{N,L}\right)_{L^\infty_{k+1}} \|V_{N,L}\|_{L^\infty_{k+2}} \|S_2 \prod_{j=3}^{k} S_j P_{>1} \psi_{N,L}\|_{L^2} \|S_1 S^{(1)} \psi_{N,L}\|_{L^2}^2 \\
\leq N^{-2} (N/L)^{2\beta} \|S^{(k)} \psi_{N,L}\|_{L^2} \|S_1 S^{(k+1)} \psi_{N,L}\|_{L^2} \leq N^{-\frac{1}{2}} (N/L)^{2\beta} (\|S^{(k+1)} \psi_{N,L}\|_{L^2}^2 + N^{-1} \|S_1 S^{(k+1)} \psi_{N,L}\|_{L^2}^2).
\]

Hence, we obtain

\[
E_1 \geq -CN^{-\frac{1}{2}}(N/L)^{2\beta} (\|S^{(k+1)} \psi_{N,L}\|_{L^2} + N^{-1} \|S_1 S^{(k+1)} \psi_{N,L}\|_{L^2}^2).
\]

Next, we deal with \(E_2\). We write

\[
E_2 = E_{21} + E_{22} + E_{23} + E_{24},
\]

where

\[
E_{21} = N^{-2} \langle S^{(k)}(2 + 2S_2 + S_{k+1}) \psi_{N,L}, S^{(k)}(2 + S_{k+1}^2 + S_{k+2}^2) \psi_{N,L} \rangle, \\
E_{22} = N^{-2} \langle S^{(k)}(2 + 2S_2 + S_{k+1}) \psi_{N,L}, S^{(k)} L V_{N,L}(r_{k+1} - r_{k+2}) \psi_{N,L} \rangle, \\
E_{23} = N^{-2} \langle S^{(k)} L V_{N,L}(r_1 - r_{k+1}) \psi_{N,L}, S^{(k+1)}(2 + S_{k+1} + S_{k+2}) \psi_{N,L} \rangle, \\
E_{24} = N^{-2} \langle S^{(k)} L V_{N,L}(r_1 - r_{k+1}) \psi_{N,L}, S^{(k)} L V_{N,L}(r_{k+1} - r_{k+2}) \psi_{N,L} \rangle.
\]

Since \(E_{21} \geq 0\), we can discard it. For \(E_{22}\), we decompose

\[
E_{22} = E_{221} + E_{222} + E_{223},
\]

where

\[
E_{221} = 2N^{-2} \langle S^{(k)} \psi_{N,L}, S^{(k)} L V_{N,L}(r_{k+1} - r_{k+2}) \psi_{N,L} \rangle, \\
E_{222} = N^{-2} \langle S^{(k)} S_{k+1}^2 \psi_{N,L}, S^{(k)} L V_{N,L}(r_{k+1} - r_{k+2}) \psi_{N,L} \rangle, \\
E_{223} = N^{-2} \langle S^{(k)} S_{k+1}^2 \psi_{N,L}, S^{(k)} L V_{N,L}(r_{k+1} - r_{k+2}) \psi_{N,L} \rangle.
\]

By estimate (2.57), we obtain

\[
|E_{221}| \lesssim N^{-2} L(N/L)^{\beta+} \|S^{(k+1)} \psi_{N,L}\|_{L^2}^2, \\
|E_{222}| \lesssim N^{-2} L(N/L)^{\beta+} \|S_1 S^{(k+1)} \psi_{N,L}\|_{L^2}^2.
\]

For \(E_{223}\), by Hölder and Sobolev inequality, we have

\[
|E_{223}| \lesssim N^{-2} \|S_1 S^{(k+1)} \psi_{N,L}\|_{L^2} \|L V_{N,L}\|_{L^\infty} \|S^{(k)} \psi_{N,L}\|_{L^2} \|L V_{N,L}\|_{L^\infty} \|S^{(k+1)} \psi_{N,L}\|_{L^2}^2 \\
\lesssim N^{-2} L(N/L)^{2\beta} \|S_1 S^{(k+1)} \psi_{N,L}\|_{L^2}^2.
\]

For \(E_{23}\), we expand

\[
E_{23} = E_{231} + E_{232} + E_{233},
\]

where

\[
E_{231} = N^{-2} \langle S^{(k)} L V_{N,L}(r_1 - r_{k+1}) \psi_{N,L}, S^{(k)} S_{k+2}^2 \psi_{N,L} \rangle, \\
E_{232} = N^{-2} \langle S^{(k)} L V_{N,L}(r_1 - r_{k+1}) \psi_{N,L}, S^{(k)} S_{k+1}^2 \psi_{N,L} \rangle, \\
E_{233} = N^{-2} \langle S^{(k)} L V_{N,L}(r_1 - r_{k+1}) \psi_{N,L}, 2S^{(k)} \psi_{N,L} \rangle.
\]
For $E_{231}$, we expand

$$E_{231} = N^{-2} \left( \prod_{j=2}^{k} S_j S_{k+2} \mathcal{L} V_{N,L} (r_1 - r_{k+1}) \psi_{N,L} S_1 \prod_{j=2}^{k} S_j S_{k+2} \Delta r_1 \prod_{j=2}^{k} S_j S_{k+2} P_{z_1, > 1} \psi_{N,L} \right)$$

$$= E_{2311} + E_{2312},$$

where

$$E_{2311} = N^{-2} \left( \prod_{j=2}^{k} S_j S_{k+2} \mathcal{L} V_{N,L} (r_1 - r_{k+1}) \psi_{N,L} - \Delta r_1 \prod_{j=2}^{k} S_j S_{k+2} P_{z_1, > 1} \psi_{N,L} \right),$$

$$E_{2312} = N^{-2} \left( \prod_{j=2}^{k} S_j S_{k+2} \mathcal{L} V_{N,L} (r_1 - r_{k+1}) \psi_{N,L} (1 - 1/L^2) \prod_{j=2}^{k} S_j S_{k+2} P_{z_1, > 1} \psi_{N,L} \right).$$

For $E_{2311}$, using integration by parts, we have

$$E_{2311} = N^{-2} \left( \nabla_{r_1} (\mathcal{L} V_{N,L} (r_1 - r_{k+1}) \prod_{j=2}^{k} S_j S_{k+2} \psi_{N,L} \right), \nabla_{r_1} \prod_{j=2}^{k} S_j S_{k+2} P_{z_1, > 1} \psi_{N,L} \right)$$

$$= N^{-2} L(N/L)^\beta \left( \nabla V_{N,L} (r_1 - r_{k+1}) \prod_{j=2}^{k} S_j S_{k+2} \psi_{N,L} \right), \nabla_{r_1} \prod_{j=2}^{k} S_j S_{k+2} P_{z_1, > 1} \psi_{N,L} \right)$$

$$+ N^{-2} L V_{N,L} (r_1 - r_{k+1}) \nabla_{r_1} \prod_{j=2}^{k} S_j S_{k+2} \psi_{N,L} \right), \nabla_{r_1} \prod_{j=2}^{k} S_j S_{k+2} P_{z_1, > 1} \psi_{N,L} \right).$$

Using Hölder and Sobolev inequality, we obtain

$$|E_{2311}| \leq N^{-2} L(N/L)^\beta \left\| (\nabla_{r_1} V_{N,L} \right\|_{L^\infty_{x_k} L^1_x} \left\| \prod_{j=2}^{k} S_j S_{k+2} \psi_{N,L} \right\|_{L^2_{L^\infty_{x_{k+1}}} L^1_{x_k}} \left\| \nabla_{r_1} \prod_{j=2}^{k} S_j S_{k+2} P_{z_1, > 1} \psi_{N,L} \right\|_{L^2_{L^\infty_{x_{k+1}}} L^1_{x_k}}$$

$$+ N^{-2} L V_{N,L} \left\| (\nabla V_{N,L} \right\|_{L^\infty_{x_k} L^1_x} \left\| \nabla_{r_1} \prod_{j=2}^{k} S_j S_{k+2} \psi_{N,L} \right\|_{L^2_{L^\infty_{x_{k+1}}} L^1_{x_k}} \left\| \nabla_{r_1} \prod_{j=2}^{k} S_j S_{k+2} P_{z_1, > 1} \psi_{N,L} \right\|_{L^2_{L^\infty_{x_{k+1}}} L^1_{x_k}}$$

$$\lesssim N^{-2} L(N/L)^{2\beta + \gamma} \left\| \prod_{j=2}^{k} S_j \psi_{N,L} \right\|_{L^2} \left\| \nabla_{r_1} \prod_{j=2}^{k} S_j P_{z_1, > 1} \psi_{N,L} \right\|_{L^2}$$

$$+ N^{-2} L(N/L)^{\beta + \gamma} \left\| \nabla_{r_1} \prod_{j=2}^{k} S_j \psi_{N,L} \right\|_{L^2} \left\| \nabla_{r_1} \prod_{j=2}^{k} S_j P_{z_1, > 1} \psi_{N,L} \right\|_{L^2}.$$
Finally, we handle \( E_{244} \), and expand

\[
E_{244} = N^{-2} \langle LV_{N,L}(r_1 - r_{k+1}) \prod_{j=2}^k S_j \psi_{N,L}, S_1^2 LV_{N,L}(r_{k+1} - r_{k+2}) \prod_{j=2}^k S_j P_{z_1, > 1} \psi_{N,L} \rangle
= E_{2441} + E_{2442},
\]
where

\[
E_{2441} = N^{-2} \langle LV_{N,L}(r_1 - r_{k+1}) \prod_{j=2}^k S_j \psi_{N,L}, -\Delta r_1 LV_{N,L}(r_{k+1} - r_{k+2}) \prod_{j=2}^k S_j P_{z_1, > 1} \psi_{N,L} \rangle,
\]

\[
E_{2442} = N^{-2} \langle LV_{N,L}(r_1 - r_{k+1}) \prod_{j=2}^k S_j \psi_{N,L}, (1 - 1/L^2) LV_{N,L}(r_{k+1} - r_{k+2}) \prod_{j=2}^k S_j P_{z_1, > 1} \psi_{N,L} \rangle.
\]

For \( E_{2441} \), with Hölder and Sobolev inequality, we obtain

\[
|E_{2441}| \leq N^{-2} \|LV_{N,L}\|_{L^2} \prod_{j=2}^k \|S_j \psi_{N,L}\|_{L^2} \|\Delta r_1 LV_{N,L}\|_{L^2} \prod_{j=2}^k \|S_j P_{z_1, > 1} \psi_{N,L}\|_{L^2}
\]

\[
\lesssim N^{-2} L^2 (N/L)^{2\beta} \|S^{(k+1)} \psi_{N,L}\|_{L^2} \lesssim N^{-2} L^2 (N/L)^{2\beta} \|S^{(k+1)} \psi_{N,L}\|_{L^2}.
\]

For \( E_{2442} \), with Hölder and Sobolev inequality, we have

\[
|E_{2442}| \leq N^{-2} \|LV_{N,L}\|_{L^2} \prod_{j=2}^k \|S_j \psi_{N,L}\|_{L^2} \|LV_{N,L}\|_{L^2} \prod_{j=2}^k \|S_j P_{z_1, > 1} \psi_{N,L}\|_{L^2}
\]

\[
\lesssim N^{-2} L^2 (N/L)^{2\beta} \|S^{(k+1)} \psi_{N,L}\|_{L^2}^2.
\]

Hence, we get

\[
E_2 \gtrsim - \max \left( N^{-2} (N/L)^{2\beta}, N^{-2} (N/L)^{2\beta} \right) \left( \|S^{(k+2)} \psi_{N,L}\|_{L^2}^2 + N^{-1} \|S_1 S^{(k+1)} \psi_{N,L}\|_{L^2}^2 \right).
\]

Finally, we handle \( E_3 \) and expand

\[
E_3 = E_{31} + E_{32} + E_{33},
\]

where

\[
E_{31} = N^{-1} \langle S^{(k)} (2 + S^2 + S_{k+1}^2) \psi_{N,L}, S^{(k)} (2 + H_{(k+2)(k+3)}) \psi_{N,L} \rangle,
\]

\[
E_{32} = N^{-1} \langle S^{(k)} LV_{N,L}(r_1 - r_{k+1}) \psi_{N,L}, S^{(k)} (2 + S_{k+2}^2 + S_{k+3}^2) \psi_{N,L} \rangle,
\]

\[
E_{33} = N^{-1} \langle S^{(k)} LV_{N,L}(r_1 - r_{k+1}) \psi_{N,L}, S^{(k)} LV_{N,L}(r_{k+2} - r_{k+3}) \psi_{N,L} \rangle.
\]

We first discard \( E_{31} \), since \( E_{31} \geq 0 \) by Theorem \( 2.14 \) and Lemma \( 4.6 \). For \( E_{32} \), we expand

\[
E_{32} = E_{321} + E_{322} + E_{323},
\]

where

\[
E_{321} = N^{-1} \langle S^{(k)} LV_{N,L}(r_1 - r_{k+1}) \psi_{N,L}, S^{(k)} S_{k+2}^2 \psi_{N,L} \rangle,
\]

\[
E_{322} = N^{-1} \langle S^{(k)} LV_{N,L}(r_1 - r_{k+1}) \psi_{N,L}, S^{(k)} S_{k+3}^2 \psi_{N,L} \rangle,
\]

\[
E_{323} = N^{-1} \langle S^{(k)} LV_{N,L}(r_1 - r_{k+1}) \psi_{N,L}, 2S^{(k)} \psi_{N,L} \rangle.
\]
Estimated in the same way as $E_{231}$,

\begin{align}
|E_{321}| & \lesssim N^{-1}(N/L)^{\beta+}\|S^{(k+2)}\psi_{N,L}\|_{L^2}, \\
|E_{322}| & \lesssim N^{-1}(N/L)^{\beta+}\|S^{(k+2)}\psi_{N,L}\|_{L^2}, \\
|E_{323}| & \lesssim N^{-1}(N/L)^{\beta+}\|S^{(k+2)}\psi_{N,L}\|_{L^2}.
\end{align}

For $E_{33}$, we expand

$$E_{33} = N^{-1}\langle LV_{N,L}(r_1 - r_{k+1}) \prod_{j=2}^{k} S_j \psi_{N,L}, S_1^2 LV_{N,L}(r_{k+2} - r_{k+3}) \prod_{j=2}^{k} S_j \psi_{N,L} \rangle$$

$$= E_{331} + E_{332},$$

where

$$E_{331} = N^{-1}\langle LV_{N,L}(r_1 - r_{k+1}) \prod_{j=2}^{k} S_j \psi_{N,L}, -\Delta r_1 LV_{N,L}(r_{k+2} - r_{k+3}) \prod_{j=2}^{k} S_j P_{z_1, > 1} \psi_{N,L} \rangle,$$

$$E_{332} = N^{-1}\langle LV_{N,L}(r_1 - r_{k+1}) \prod_{j=2}^{k} S_j \psi_{N,L}, (1 - 1/L^2) LV_{N,L}(r_{k+2} - r_{k+3}) \prod_{j=2}^{k} S_j P_{z_1, > 1} \psi_{N,L} \rangle.$$

Using integration by parts for $E_{331}$,

$$E_{331} = N^{-1}\langle \nabla r_1 LV_{N,L}(r_1 - r_{k+1}) \prod_{j=2}^{k} S_j \psi_{N,L}, \nabla r_1 LV_{N,L}(r_{k+2} - r_{k+3}) \prod_{j=2}^{k} S_j P_{z_1, > 1} \psi_{N,L} \rangle$$

$$= N^{-1}\langle (\nabla r_1 V)_{N,L}(r_1 - r_{k+1}) \prod_{j=2}^{k} S_j \psi_{N,L}, \nabla r_1 LV_{N,L}(r_{k+2} - r_{k+3}) \prod_{j=2}^{k} S_j P_{z_1, > 1} \psi_{N,L} \rangle$$

$$+ N^{-1} L \langle V_{N,L}(r_1 - r_{k+1}) \nabla r_1 \prod_{j=2}^{k} S_j \psi_{N,L}, \nabla r_1 LV_{N,L}(r_{k+2} - r_{k+3}) \prod_{j=2}^{k} S_j P_{z_1, > 1} \psi_{N,L} \rangle.$$
For $E_{332}$, with Hölder and Sobolev inequality, we obtain

$$|E_{332}| \lesssim N^{-1} \|V_{N,L}\|_{L^\infty_t L^1_x} \prod_{j=2}^k \|S_j \psi_{N,L}\|_{L^2_t L^\infty_x} \|V_{N,L}\|_{L^\infty_t L^1_x} \prod_{j=2}^{k+2} \|S_j P_{z_1,>1} \psi_{N,L}\|_{L^2_t L^\infty_x}$$

$$\lesssim N^{-1} (N/L)^{2\beta^+} \prod_{j=2}^{k+2} \|S_j \psi_{N,L}\|_{L^2} \prod_{j=2}^{k+2} \|S_j P_{z_1,>1} \psi_{N,L}\|_{L^2}.$$

By $L^{-2} P_{z_1,>1} \leq S^2 P_{z_1,>1}$, with $(N/L)^\beta \to 1^-$, we get

$$|E_{332}| \lesssim N^{-1} (N/L)^{2\beta^+} \|S^{(k+1)} \psi_{N,L}\|_{L^2} \|S^{(k+2)} \psi_{N,L}\|_{L^2}. \tag{2.132}$$

That is

$$E_3 \gtrsim -N^{-1} (N/L)^{2\beta^+} \|S^{(k+2)} \psi_{N,L}\|_{L^2}^2. \tag{2.133}$$

Putting (2.117), (2.127) and (2.133) together, we obtain the estimate for the cross error term

$$E_C \geq -C \max \left( N^{-2} \left( N/L \right)^{2\beta^+} + N^{-1} (N/L)^{2\beta^+} \right) \left( \|S^{(k+2)} \psi_{N,L}\|_{L^2}^2 + N^{-1} \|S^{(k+1)} \psi_{N,L}\|_{L^2}^2 \right). \tag{2.134}$$

Hence we have proved for all $k$ and established Theorem 2.1.

### 3. Compactness, Convergence, and Uniqueness

To work on compactness, convergence and uniqueness, we introduce an appropriate topology on the density matrices, as was previously done in [32,33,34,35,36,37,38,39,40,41,42,43,44,45,46]. Denote the spaces of compact operators and trace class operators on $L^2(\Omega^\otimes k)$ as $K_k$ and $\mathcal{L}_k^1$, respectively. Then $(\mathcal{K}_k)^\prime = L^1_k$.

By the fact that $K_k$ is separable, we select a dense countable subset $\{J_i^{(k)}\}_{i \geq 1} \subset K_k$ in the unit ball of $K_k$ (so $\|J_i^{(k)}\|_{op} \leq 1$ where $\|\cdot\|_{op}$ is the operator norm). For $\gamma^{(k)}, \tilde{\gamma}^{(k)} \in \mathcal{L}_k^1$, we then define a metric $d_k$ on $\mathcal{L}_k^1$ by

$$d_k(\gamma^{(k)}, \tilde{\gamma}^{(k)}) = \sum_{i=1}^\infty 2^{-i} \left| \text{Tr} J_i^{(k)} \left( \gamma^{(k)} - \tilde{\gamma}^{(k)} \right) \right|.$$  

A uniformly bounded sequence $\tilde{\gamma}^{(k)}_N \in \mathcal{L}_k^1$ converges to $\tilde{\gamma}^{(k)}$ with respect to the weak* topology if and only if

$$\lim_{N,1/L \to \infty} d_k(\gamma^{(k)}_N, \tilde{\gamma}^{(k)}) = 0.$$  

For fixed $T > 0$, let $C([0,T]; \mathcal{L}_k^1)$ be the space of functions of $t \in [0,T]$ with values in $\mathcal{L}_k^1$ that are continuous with respect to the metric $d_k$. On $C([0,T]; \mathcal{L}_k^1)$, we define the metric

$$d_k(\gamma^{(k)}(\cdot), \tilde{\gamma}^{(k)}(\cdot)) = \sup_{t \in [0,T]} d_k(\gamma^{(k)}(t), \tilde{\gamma}^{(k)}(t)),$$  

and denote by $\tau_{prod}$ the topology on the space $\bigotimes_{k \geq 1} C([0,T]; \mathcal{L}_k^1)$ given by the product of topologies generated by the metrics $d_k$ on $C([0,T], \mathcal{L}_k^1)$.

### 3.1. Compactness of the BBGKY sequence

**Theorem 3.1.** Assume $L(N/L)^\beta \to 1^-$. Then the sequence

$$\left\{ \Gamma_{N,L}(t) = \left\{ \tilde{\gamma}^{(k)}_{N,L} \right\}_{k=1}^N \right\} \subset \bigotimes_{k \geq 1} C([0,T]; \mathcal{L}_k^1), \tag{3.2}$$

which satisfies the BBGKY hierarchy, is compact with respect to the product topology $\tau_{prod}$. For any limit point $\Gamma(t) = \left\{ \tilde{\gamma}^{(k)}(t) \right\}_{k=1}^\infty$, we have $\tilde{\gamma}^{(k)}$ is a symmetric nonnegative trace class operator with trace bounded by $I$. 

Proof. By the standard diagonalization argument, it suffices to show the compactness of \( \tilde{\gamma}_{N,L}^{(k)} \) for fixed \( k \) with respect to the metric \( \tilde{d}_k \). By the Arzelà-Ascoli theorem, this is equivalent to the equicontinuity of \( \gamma_{N,L}^{(k)} \), and by, this is equivalent to the statement that for every observable \( J^{(k)} \) from a dense subset of \( K_k \) and for every \( \varepsilon > 0 \), there exists \( \delta(J^{(k)}, \varepsilon) \) such that for all \( t_1, t_2 \in [0, T] \) with \( |t_1 - t_2| \leq \delta \), we have

\[
(3.3) \quad \sup_{N,L} \left| \text{Tr} J^{(k)} \tilde{\gamma}_{N,L}^{(k)}(t_1) - \text{Tr} J^{(k)} \tilde{\gamma}_{N,L}^{(k)}(t_2) \right| \leq \varepsilon.
\]

We assume that compact operators \( J^{(k)} \) have been cutoff in Lemma \( \text{A.3} \). Since the observable \( J^{(k)} \) can be written as a sum of a self-adjoint operator and an anti-self-adjoint operator, we may assume \( J^{(k)} \) is self-adjoint. Inserting the decomposition \( (2.8) \), we have

\[
\tilde{\gamma}_{N,L}^{(k)} = \sum_{\alpha, \beta} \tilde{P}_\alpha \tilde{\gamma}_{N,L}^{(k)} \tilde{P}_\beta,
\]

where the sum is taken over all \( k \)-tuples \( \alpha \) and \( \beta \).

To establish \( (3.3) \), it suffices to prove that, for each \( \alpha \) and \( \beta \), we have

\[
(3.4) \quad \sup_{N,L} \left| \text{Tr} J^{(k)} \tilde{P}_\alpha \tilde{\gamma}_{N,L}^{(k)} \tilde{P}_\beta(t_1) - \text{Tr} J^{(k)} \tilde{P}_\alpha \tilde{\gamma}_{N,L}^{(k)} \tilde{P}_\beta(t_2) \right| \leq \varepsilon.
\]

To this end, we establish the estimate

\[
(3.5) \quad |\text{Tr} J^{(k)} \tilde{P}_\alpha \tilde{\gamma}_{N,L}^{(k)} \tilde{P}_\beta(t_1) - \text{Tr} J^{(k)} \tilde{P}_\alpha \tilde{\gamma}_{N,L}^{(k)} \tilde{P}_\beta(t_2)|
\leq C|t_2 - t_1| \left( \sum_{\alpha=0, \beta=0}^1 \left\{ 1, L^{\alpha+|\beta|-2} \right\} \cdot 1_{\{ \alpha \neq 0 \text{ or } \beta \neq 0 \}} \right).
\]

By \( (3.5) \), we can directly establish \( (3.4) \) except for the case \( |\alpha| + |\beta| = 1 \). However, from Corollary \( 2.2 \), we can also get a bound

\[
(3.6) \quad |\text{Tr} J^{(k)} \tilde{P}_\alpha \tilde{\gamma}_{N,L}^{(k)} \tilde{P}_\beta(t_1) - \text{Tr} J^{(k)} \tilde{P}_\alpha \tilde{\gamma}_{N,L}^{(k)} \tilde{P}_\beta(t_2)|
\leq 2 \sup_t |\text{Tr} J^{(k)} \tilde{P}_\alpha \tilde{\gamma}_{N,L}^{(k)} \tilde{P}_\beta \tilde{P}(t)|
\leq 2 \| J^{(k)} \|_{op} \| \tilde{P}_\alpha \tilde{\gamma}_{N,L}^{(k)} \tilde{P}_\beta \|_{L^2 \to L^2 \to L^2}
\leq L^{\alpha+|\beta|}.
\]

By averaging \( (3.5) \) and \( (3.6) \) in the case \( |\alpha| + |\beta| = 1 \), we obtain

\[
|\text{Tr} J^{(k)} \tilde{P}_\alpha \tilde{\gamma}_{N,L}^{(k)} \tilde{P}_\beta(t_1) - \text{Tr} J^{(k)} \tilde{P}_\alpha \tilde{\gamma}_{N,L}^{(k)} \tilde{P}_\beta(t_2)| \lesssim |t_2 - t_1|^{1/2},
\]

which suffices to establish \( (3.3) \).

Thus, we are left to prove \( (3.5) \). The BBGKY hierarchy \( (1.13) \) yields

\[
(3.7) \quad \partial_t \text{Tr} J^{(k)} \tilde{P}_\alpha \tilde{\gamma}_{N,L}^{(k)} \tilde{P}_\beta = I + II + III + IV,
\]

where

\[
I = -i \sum_{j=1}^k \text{Tr} J^{(k)} \left[ -\Delta_{x_j} \tilde{P}_\alpha \tilde{\gamma}_{N,L}^{(k)} \tilde{P}_\beta \right],
\]

\[
II = -\frac{i}{L} \sum_{j=1}^k \text{Tr} J^{(k)} \left[ -\partial^2_{x_j} \tilde{P}_\alpha \tilde{\gamma}_{N,L}^{(k)} \tilde{P}_\beta \right],
\]

\[
III = -i \frac{1}{N-1} \sum_{1 \leq i < j \leq k} \text{Tr} J^{(k)} \tilde{P}_\alpha \left[ \tilde{V}_{N,L}(r_i - r_j), \tilde{\gamma}_{N,L}^{(k)} \right] \tilde{P}_\beta,
\]

\[
IV = -\frac{i}{N-k} \sum_{j=1}^k \text{Tr} J^{(k)} \text{Tr}_{r_{k+1}} \tilde{P}_\alpha \left[ \tilde{V}_{N,L}(r_j - r_{k+1}), \tilde{\gamma}_{N,L}^{(k+1)} \right] \tilde{P}_\beta.
\]
First, we handle $I$. By Lemma A.7 and integration by parts, we have

$$I = \sum_{j=1}^{k} \left( (J^{(k)} \Delta x_j, \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L) - (J^{(k)} \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma \Delta x_j, \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L) \right)$$

$$= \sum_{j=1}^{k} \left( (J^{(k)} \Delta x_j, \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L) - (\Delta x_j, J^{(k)} \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L) \right).$$

Hence

$$(3.8) \quad |I| \leq \sum_{j=1}^{k} \left( \| J^{(k)} \Delta x_j \|_{op} + \| \Delta x_j, J^{(k)} \|_{op} \right) \| \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L \|_{L^2} \| \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L \|_{L^2} \leq C_{k,j^{(k)}},$$

where in the last step we used the energy estimate.

Next, we consider $II$. When $\alpha = \beta = 0$, we have

$$II = -\frac{1}{L^2} \sum_{j=1}^{k} Tr J^{(k)} \left[ -\partial^2_{x_j} - 1, \widetilde{\mathbf{P}}_0 \widetilde{\mathbf{P}}_\gamma N,L \right] \cdot \left[ \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L \right] = 0,$$

where we used $[1, \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L, \widetilde{\mathbf{P}}_\beta] = 0$ in the first equality. When $|\alpha| + |\beta| \geq 1$, applying Lemma A.7 and integration by parts again, we have

$$II = \frac{1}{L^2} \sum_{j=1}^{k} \left( (J^{(k)} \partial^2_{x_j} \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L, \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L) - (J^{(k)} \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L, \partial^2_{x_j} \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L) \right)$$

$$= \frac{1}{L^2} \sum_{j=1}^{k} \left( (J^{(k)} \partial^2_{x_j} \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L, \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L) - (\partial^2_{x_j} J^{(k)} \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L, \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L) \right).$$

Hence

$$(3.9) \quad |II| \leq \frac{1}{L^2} \sum_{j=1}^{k} \left( \| J^{(k)} \partial^2_{x_j} \|_{op} + \| \partial^2_{x_j} J^{(k)} \|_{op} \right) \| \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L \|_{L^2} \| \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L \|_{L^2}.$$ By the energy estimate (2.6),

$$(3.10) \quad |II| \lesssim C_{k,j^{(k)}} L^{[|\alpha| + |\beta|] - 2}, \quad |\alpha| + |\beta| \geq 1.$$

Next, we consider $III$. Similarly,

$$III = \frac{-i}{N-1} \sum_{1 \leq i < j \leq k} \langle J^{(k)} \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L(r_i - r_j) \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L \rangle$$

$$+ \frac{i}{N-1} \sum_{1 \leq i < j \leq k} \langle J^{(k)} \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L(r_i - r_j) \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L \rangle$$

$$= \frac{-i}{N-1} \sum_{1 \leq i < j \leq k} \langle J^{(k)} \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L(r_i - r_j) \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L \rangle$$

$$+ \frac{i}{N-1} \sum_{1 \leq i < j \leq k} \langle \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L, J^{(k)} \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L(r_i - r_j) \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L \rangle.$$

Let

$$W_{ij} = \langle \nabla r_i \rangle^{-1} \langle \nabla r_j \rangle^{-1} \widetilde{W}_{N,L}(r_i - r_j) \langle \nabla r_i \rangle^{-1} \langle \nabla r_j \rangle^{-1}.$$ 

Hence

$$(3.12) \quad |III| \lesssim N^{-1} \sum_{1 \leq i < j \leq k} \| J^{(k)} \langle \nabla r_i \rangle \langle \nabla r_j \rangle \|_{op} \| W_{ij} \|_{op} \| \langle \nabla r_i \rangle \langle \nabla r_j \rangle \widetilde{W}_{N,L} \|_{L^2} \| \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L \|_{L^2}$$

$$+ N^{-1} \sum_{1 \leq i < j \leq k} \| \langle \nabla r_i \rangle \langle \nabla r_j \rangle J^{(k)} \|_{op} \| W_{ij} \|_{op} \| \langle \nabla r_i \rangle \langle \nabla r_j \rangle \widetilde{W}_{N,L} \|_{L^2} \| \widetilde{\mathbf{P}}_\alpha \widetilde{\mathbf{P}}_\beta \widetilde{\mathbf{P}}_\gamma N,L \|_{L^2}.$$
Since $\|W_{ij}\|_{op} \lesssim \|\tilde{V}_{N,L}\|_{L^\infty_t L^4_x}$ by (2.59), the energy estimates (2.5) (2.6) imply that

\begin{equation}
(3.11)
\end{equation}

$$\|III\| \lesssim \frac{C_{k,J^{(k)}}}{N}.$$

For $IV$, we have

$$IV = -i \frac{N-k}{N-1} \sum_{j=1}^{k} \langle J^{(k)} \tilde{\mathcal{P}}_\alpha \tilde{V}_{N,L}(r_j - r_{k+1})\psi_{N,L}, \tilde{\mathcal{P}}_\beta \psi_{N,L} \rangle$$

$$+ i \frac{N-k}{N-1} \sum_{j=1}^{k} \langle J^{(k)} \tilde{\mathcal{P}}_\alpha \tilde{V}_{N,L}(r_j - r_{k+1})\psi_{N,L}, \tilde{\mathcal{P}}_\beta \psi_{N,L} \rangle.$$

Then, since $J^{(k)} \langle \nabla_{r_{k+1}} \rangle = \langle \nabla_{r_{k+1}} \rangle J^{(k)}$,

$$IV = -i \frac{N-k}{N-1} \sum_{j=1}^{k} \langle J^{(k)} \tilde{\mathcal{P}}_\alpha (\nabla_{r_j}) W_{j(k+1)} (\nabla_{r_{j+1}}) \tilde{\psi}_{N,L}, \tilde{\mathcal{P}}_\beta (\nabla_{r_{k+1}}) \tilde{\psi}_{N,L} \rangle$$

$$+ i \frac{N-k}{N-1} \sum_{j=1}^{k} \langle (\nabla_{r_j}) J^{(k)} \tilde{\mathcal{P}}_\alpha (\nabla_{r_{k+1}}) \tilde{\psi}_{N,L}, \tilde{\mathcal{P}}_\beta W_{j(k+1)} (\nabla_{r_j}) (\nabla_{r_{k+1}}) \tilde{\psi}_{N,L} \rangle.$$

Hence

$$\|IV\| \lesssim \sum_{j=1}^{k} \left( \|J^{(k)} \langle \nabla_{r_j} \rangle\|_{op} + \langle \nabla_{r_{j+1}} \rangle J^{(k)} \|_{op} \right) \|W_{j(k+1)}\|_{op} \|\nabla_{r_j} \|_{L^2} \|\nabla_{r_{k+1}}\|_{L^2}.$$

By energy estimate (2.5),

$$\|IV\| \lesssim \frac{C_{k,J^{(k)}}}{N}.$$

Integrating (3.7) from $t_1$ to $t_2$ and putting (3.5), (3.9), (3.10), (3.11) and (3.12) together, we obtain (3.5). \hfill \square

\textbf{Corollary 3.2.} Let $\Gamma(t) = \{\tilde{\gamma}^{(k)}(t)\}_{k=1}^\infty$ be a limit point of $\{\Gamma_{N,L}(t) = \{\tilde{\gamma}^{(k)}_N(t)\}_{k=1}^N\}$, with respect to the product topology $\tau_{prod}$. Then $\tilde{\gamma}^{(k)}$ satisfies the priori bound

\begin{equation}
(3.13)
\end{equation}

$$Tr(\nabla \tilde{\gamma}^{(k)}_x) \langle \nabla \tilde{\gamma}^{(k)} \rangle \leq C^k,$$

and takes the structure

\begin{equation}
(3.14)
\end{equation}

$$\tilde{\gamma}^{(k)}(t, (x_k, z_k); (x'_k, z'_k)) = \tilde{\gamma}^{(k)}_x(t, x_k; x'_k) \left( \prod_{j=1}^{k} \frac{1}{2\pi} \cos(z_j) \cos(z'_j) \right),$$

where $\tilde{\gamma}^{(k)}_x = Tr_x \tilde{\gamma}^{(k)}$.

\textbf{Proof.} The estimate (3.13) follows by (2.5) in Corollary 2.2 and Theorem 3.1. To establish the formula (3.14), it suffices to prove $\tilde{\mathcal{P}}_\alpha \tilde{\gamma}^{(k)} \tilde{\mathcal{P}}_\beta = 0$ if either $\alpha \neq 0$ or $\beta \neq 0$. This is equivalent to the statement that for any $J^{(k)} \in K_k$, $\text{Tr} J^{(k)} \tilde{\mathcal{P}}_\alpha \tilde{\gamma}^{(k)} \tilde{\mathcal{P}}_\beta = 0$. By Corollary 2.2, we obtain

\begin{equation}
(3.15)
\end{equation}

$$\text{Tr} J^{(k)} \tilde{\mathcal{P}}_\alpha \tilde{\gamma}^{(k)} \tilde{\mathcal{P}}_\beta = \lim_{N,L \to \infty} \text{Tr} J^{(k)} \tilde{\mathcal{P}}_\alpha \tilde{\gamma}^{(k)}_{N,L} \tilde{\mathcal{P}}_\beta.$$ 

By Lemma 3.7

$$\text{Tr} J^{(k)} \tilde{\mathcal{P}}_\alpha \tilde{\gamma}^{(k)}_{N,L} \tilde{\mathcal{P}}_\beta = \langle J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,L}, \tilde{\mathcal{P}}_\beta \tilde{\psi}_{N,L} \rangle,$$

and by Cauchy-Schwarz and (2.6),

$$\left| \text{Tr} J^{(k)} \tilde{\mathcal{P}}_\alpha \tilde{\gamma}^{(k)}_{N,L} \tilde{\mathcal{P}}_\beta \right| \leq \|J^{(k)}\|_{op} \|\mathcal{P}_\alpha \tilde{\psi}_{N,L}\|_{L^2} \|\tilde{\mathcal{P}}_\beta \tilde{\psi}_{N,L}\|_{L^2}$$

$$\leq C^k L^{|\alpha|+|\beta|}.$$

Hence the right side of (3.15) is 0. \hfill \square
Theorem 3.3. The sequence
\[ \Gamma_{x,N,L} = \{ \bar{\gamma}_{x,N,L} = Tr_x \gamma^{(k)}_{N,L} \}_{k=1}^N \in \bigotimes_{k \geq 1} C([0,T]; L^1_k(\mathbb{T}^{2k})) \]
is compact with respect to the 2D version of the product topology \( \tau_{prod} \) used in Theorem 3.1.

Proof. The proof is similar to the 3D case and we omit it. Also see [19, Theorem 5]. \( \square \)

3.2. Limit points satisfy GP hierarchy. To prove the limit points satisfy the GP hierarchy, a technical tool we need is the approximation of identity type lemma, which is used to compare the \( \delta \)-function and its approximation. Since we request \( L(N/L)^\beta \to 1^- \), we see that \( \bar{V}_{N,L}(x,z) \) defined by (1.3) formally converges to \( \delta(x) \int V(x,z)dx \). Thus, we need a modified version of this type lemma as follows.

Lemma 3.4. Let \( \rho \in L^1(\Omega) \) be a function compactly supported on \( \Omega \) such that
\[ \sup_z \int |\rho(x,z)||x|dx < \infty \]
and define \( \rho_{\varepsilon,\lambda}(x,z) = \varepsilon^{-2}\lambda^{-1}\rho(x/\varepsilon,z/\lambda) \) and \( g(z) = \int \rho(x,z)dx \). Then, for every \( \kappa \in [0,1) \), there exists \( C > 0 \), such that
\[ |Tr J^{(k)}(\rho_{\varepsilon,\lambda}(r_j - r_{k+1}) - \delta(x_j - x_{k+1})g(z_j - z_{k+1}))\gamma^{(k+1)}| \leq C\varepsilon \sup_z \int |\rho(x,z)||x|dx \left( \| \langle \nabla \rho_{\varepsilon,\lambda} \rangle^{-1} J^{(k)}(\nabla \rho_{\varepsilon,\lambda}) \| \right) \| \langle \nabla \rho_{\varepsilon,\lambda} \rangle J^{(k)}(\nabla \rho_{\varepsilon,\lambda})^{-1} \| \sup_z \int \rho(x,z)dx < \infty \]
and for all nonnegative \( \gamma^{(k+1)} \) in \( L^1_{k+1} \).

Proof. We will give a proof for Lemma 3.4 in the Appendix. \( \square \)

Theorem 3.5. Assume \( L(N/L)^\beta \to 1^- \), and let \( \Gamma(t) = \{ \bar{\gamma}^{(k)} \}_{k=1}^\infty \) be a limit point of
\[ \left\{ \Gamma_{N,L}(t) = \{ \bar{\gamma}_{N,L}^{(k)}(t) \}_{k=1}^N \right\} \]
with respect to the product topology \( \tau_{prod} \). Then \( \left\{ \bar{\gamma}^{(k)}_{x} = Tr_x \bar{\gamma}^{(k)} \right\}_{k=1}^\infty \) is a solution to the coupled focusing Gross-Pitaevskii hierarchy subject to initial data \( \bar{\gamma}^{(k)}_{x}(0) = |\phi_0\rangle\langle\phi_0| \otimes^k \), which, rewritten in integral form, is
\[ (3.16) \]
\[ \bar{\gamma}^{(k)}_{x}(t) = U^{(k)}(t)\bar{\gamma}^{(k)}_{x}(0) - i \sum_{j=1}^k \int_0^t U^{(k)}(t-s)Tr_{x_{k+1}}\left[ \delta(x_j - x_{k+1}) \int V(x,z_j - z_{k+1})dx, \bar{\gamma}^{(k+1)}(s) \right] ds, \]
where \( U^{(k)}(t) = \prod_{j=1}^k e^{it\Delta_{x_j}} e^{-it\Delta_{x_j'}} \).

Proof. Passing to subsequences if necessary, we have
\[ (3.17) \]
\[ \lim_{N,1/L \to \infty} \sup_{L(N/L)^\beta \to 1^-} \lim_{t \to \infty} Tr J^{(k)}(\bar{\gamma}^{(k)}_{N,L} - \bar{\gamma}^{(k)}(t)) = 0, \quad \forall J^{(k)} \in \mathcal{K}(L^2(\Omega^k)) \]
\[ (3.18) \]
\[ \lim_{N,1/L \to \infty} \sup_{L(N/L)^\beta \to 1^-} \lim_{t \to \infty} Tr J^{(k)}(\bar{\gamma}^{(k)}_{x,N,L} - \bar{\gamma}^{(k)}_{x}(t)) = 0, \quad \forall J^{(k)} \in \mathcal{K}(L^2(\mathbb{T}^{2k})) \]
from Theorem 3.1 and 3.3.

It suffices to test the limit point against the test function \( J^{(k)}_{x} \in \mathcal{K}(L^2(\mathbb{T}^{2k})) \). We will prove that the limit point satisfies
\[ (3.19) \]
\[ Tr J^{(k)}_{x}\bar{\gamma}^{(k)}_{x}(0) = Tr J^{(k)}_{x}|\phi_0\rangle\langle\phi_0| \otimes^k, \]
Thus we have checked (3.19), where we adopt the notation $g(z) = \int V(x, z) dx$ for simplicity.

To end this, we use the coupled focusing BBGKY hierarchy, which is

\begin{equation}
\text{Tr} J_x^{(k)} \tilde{\gamma}^{(k)}(0) = \text{Tr} J_x^{(k)} U^{(k)}(t) \tilde{\gamma}^{(k)}(0)
\end{equation}

\begin{align*}
&+ i \sum_{j=1}^{N} \int_0^t U^{(k)}(t-s) \text{Tr} \left[ \delta(x_j - x_{k+1}) g(z_j - z_{k+1}), \tilde{\gamma}^{(k+1)}(s) \right] ds,
\end{align*}

where we adopt the notation $g(z) = \int V(x, z) dx$ for simplicity.

By (3.18), we have

\begin{align}
\lim_{N,1/L \to \infty} \lim_{L(N/L)^\beta \to 1^{-}} \text{Tr} J_x^{(k)} \tilde{\gamma}^{(k)}(t) & = \text{Tr} J_x^{(k)} \tilde{\gamma}^{(k)}(t), \\
\lim_{N,1/L \to \infty} \lim_{L(N/L)^\beta \to 1^{-}} \text{Tr} J_x^{(k)} U^{(k)}(t) \tilde{\gamma}^{(k)}(0) & = \text{Tr} J_x^{(k)} U^{(k)}(t) \tilde{\gamma}^{(k)}(0).
\end{align}

By the argument in [50], we know, from assumption (ii) in Theorem 1.2,

\begin{equation}
\tilde{\gamma}^{(1)}_{N,L}(0) \to \frac{2}{\pi} \phi_0(x_1) \overline{\phi_0} \cos(z_1) \cos(z_1'), \quad \text{strongly in trace norm;}
\end{equation}

that is,

\begin{equation}
\tilde{\gamma}^{(k)}_{N,L}(0) \to \prod_{j=1}^{k} \frac{2}{\pi} \phi_0(x_j) \overline{\phi_0}(x_j') \cos(z_j) \cos(z_j'), \quad \text{strongly in trace norm.}
\end{equation}

Thus we have checked (3.19), the left-hand side of (3.20), and the first term on the right-hand side of (3.20) for the limit point. We are left to prove that

\begin{align}
\lim_{N,1/L \to \infty} \lim_{L(N/L)^\beta \to 1^{-}} B & = 0, \\
\lim_{N,1/L \to \infty} \lim_{L(N/L)^\beta \to 1^{-}} \left( 1 - \frac{k+1}{N-1} \right) D & = \int_0^t \text{Tr} J_x^{(k)} U^{(k)}(t-s) \left[ \delta(x_j - x_{k+1}) g(z_j - z_{k+1}), \tilde{\gamma}^{(k+1)}(s) \right] ds.
\end{align}
First, we will show the boundedness of $|B|$ and $|D|$ for every finite time $t$. Noting that $[U^{(k)}, (\nabla_{x,z})] = 0$, we have

$$
|B| \leq \int_0^t \left| \text{Tr} J^{(k)}_x U^{(k)}(t-s) \left[ \tilde{V}_{N,L}(r_i - r_j), \tilde{\gamma}^{(k)}_{N,L}(s) \right] \right| ds \\
= \int_0^t ds \left| \text{Tr} (\nabla_{r_i}^{-1}(\nabla_{r_j})^{-1} J^{(k)}_x (\nabla_{r_i}^{-1}(\nabla_{r_j}) U^{(k)}(t-s) W_{ij} (\nabla_{r_i}^{-1}(\nabla_{r_j}) \tilde{\gamma}^{(k)}_{N,L}(s) \langle \nabla_{r_i}^{-1}(\nabla_{r_j}) W_{ij} \rangle) \right| \\
- \text{Tr} (\nabla_{r_i}^{-1}(\nabla_{r_j}) J^{(k)}_x (\nabla_{r_i}^{-1}(\nabla_{r_j}) U^{(k)}(t-s) (\nabla_{r_i}^{-1}(\nabla_{r_j}) \tilde{\gamma}^{(k)}_{N,L}(s) \langle \nabla_{r_i}^{-1}(\nabla_{r_j}) W_{ij} \rangle) \\
\leq \int_0^t ds \| \langle \nabla_{r_i}^{-1}(\nabla_{r_j})^{-1} J^{(k)}_x (\nabla_{r_i}^{-1}(\nabla_{r_j}) \|U^{(k)}\|_{\text{op}} W_{ij} \|_{\text{op}} \text{Tr} (\nabla_{r_i}^{-1}(\nabla_{r_j}) W_{ij} \rangle ^2 \tilde{\gamma}^{(k)}_{N,L}(s) \\
+ \int_0^t ds \| \langle \nabla_{r_i}^{-1}(\nabla_{r_j}) J^{(k)}_x \langle \nabla_{r_i}^{-1}(\nabla_{r_j})^{-1} \|U^{(k)}\|_{\text{op}} W_{ij} \|_{\text{op}} \text{Tr} (\nabla_{r_i}^{-1}(\nabla_{r_j}) W_{ij} \rangle ^2 \tilde{\gamma}^{(k)}_{N,L}(s) \\
\leq C_{ft}.
$$

$|D|$ can be estimated in the same way as $|B|$ and hence

$$
|D| \leq C_{ft}.
$$

That is,

$$
\lim_{N, 1/L \to \infty, L(N/L)^{\alpha} \to 1} \frac{B}{N-1} = \lim_{N, 1/L \to \infty, L(N/L)^{\alpha} \to 1} \frac{(k+1)D}{N-1} = 0.
$$

Next, we will use Lemma 3.3 to prove

$$
\lim_{N, 1/L \to \infty, L(N/L)^{\alpha} \to 1} D = \int_0^t \text{Tr} J^{(k)}_x U^{(k)}(t-s) \left[ \delta(x_j - x_{k+1}) g(z_j - z_{k+1}), \tilde{\gamma}^{(k+1)}(s) \right] ds
$$

Let $\eta \in L^1(T^2)$ be a smooth probability density function compactly supported on $(-\pi, \pi)^2$ and define $\eta_\varepsilon(x) = \varepsilon^{-2} \eta(x/\varepsilon)$. For simplicity, we adopt the notation $M^{(k)}_{x,t} = J^{(k)}_x U^{(k)}(t-s)$. Then, we expand

$$
\text{Tr} J^{(k)}_x M^{(k)}_{x,t} (t-s) \left[ \tilde{V}_{N,L}(r_j - r_{k+1}), \tilde{\gamma}^{(k+1)}(s) - \delta(x_j - x_{k+1}) g(z_j - z_{k+1}), \tilde{\gamma}^{(k+1)}(s) \right] = I + II + III + IV,
$$

where

$$
I = \text{Tr} M^{(k)}_{x,t} \left[ \tilde{V}_{N,L}(r_j - r_{k+1}) - \delta(x_j - x_{k+1}) g(z_j - z_{k+1}), \tilde{\gamma}^{(k+1)}(s) \right] \\
II = \text{Tr} M^{(k)}_{x,t} \delta(x_j - x_{k+1}) - \eta_\varepsilon(x_j - x_{k+1}) g(z_j - z_{k+1}), \tilde{\gamma}^{(k+1)}(s), \\
III = \text{Tr} M^{(k)}_{x,t} \eta_\varepsilon(x_j - x_{k+1}) g(z_j - z_{k+1}) \left( \tilde{\gamma}^{(k+1)}(s) - \tilde{\gamma}^{(k+1)}(s) \right), \\
IV = \text{Tr} M^{(k)}_{x,t} \eta_\varepsilon(x_j - x_{k+1}) - \delta(x_j - x_{k+1}) g(z_j - z_{k+1}), \tilde{\gamma}^{(k+1)}(s).
$$

It needs only to prove $I - IV$ converge to 0 as $N, 1/L \to \infty$. By Lemma 3.3 we have

$$
|I| \leq \left| \text{Tr} M^{(k)}_{x,t} \left[ \tilde{V}_{N,L}(r_j - r_{k+1}) - \delta(x_j - x_{k+1}) g(z_j - z_{k+1}), \tilde{\gamma}^{(k+1)}(s) \right] \tilde{\gamma}^{(k+1)}(s) \right| \\
\leq \frac{C}{(N/L)^{\alpha}} \sup_z \int |V(x,z)|^a dx \left( \|\langle \nabla_{r_j}, \tilde{\gamma}^{(k+1)}(s) \rangle \|_{\text{op}} + \|\langle \nabla_{r_j}, (\tilde{\gamma}^{(k+1)}(s) - \tilde{\gamma}^{(k+1)}(s)) \rangle \|_{\text{op}} \right) \\
\times \text{Tr} \langle \nabla_{r_j} \rangle (\tilde{\gamma}^{(k+1)}(s) \langle \nabla_{r_j} \rangle + C_j \|\tilde{V}_{1,\lambda} - \tilde{V}_{1,1} \|_{L_1} \text{Tr} \langle \nabla_{r_j} \rangle ^2 \tilde{\gamma}^{(k+1)}_{N,L} \\
\leq C_j \left[ \sup_z \int |V(x,z)|^a dx \right] (N/L)^{\delta} + \|L(N/L)^{\delta} \tilde{V}(x, L(N/L)^{\delta} z) - \tilde{V}(x, z) \|_{L_1} \right].
$$
Similarly, for $II$ and $IV$, via Lemma 3.4 we have

\begin{equation}
|II| = |Tr M_{s,t}^{(k)} (\delta(x_j - x_{k+1}) - \eta_c(x_j - x_{k+1})) g(z_j - z_{k+1}) \tilde{\gamma}_{N,L}^{(k+1)}(s)|
\leq C \varepsilon |V|_{L^\infty L^2_x} \int \eta(x) |x|^s dx \left( \| \langle \nabla x_j \rangle J_s^{(k)} \langle \nabla x_j \rangle^{-1} \|_{op} + \| \langle \nabla x_j \rangle^{-1} J_s^{(k)} \langle \nabla x_j \rangle \|_{op} \right)
\times \text{Tr} \langle \nabla x_j \rangle \langle \nabla x_{k+1} \rangle \tilde{\gamma}_{N,L}^{(k+1)}(s) \langle \nabla x_{k+1} \rangle \langle \nabla x_j \rangle
\leq C J \varepsilon \varepsilon,
\end{equation}

where the last inequality follows from Corollary 2.2 and

\begin{equation}
|IV| = |Tr M_{s,t}^{(k)} (\eta_c(x_j - x_{k+1}) - \delta(x_j - x_{k+1})) g(z_j - z_{k+1}) \tilde{\gamma}_{N,L}^{(k+1)}(s)|
\leq C \varepsilon |V|_{L^\infty L^2_x} \int \eta(x) |x|^s dx \left( \| \langle \nabla x_j \rangle J_s^{(k)} \langle \nabla x_j \rangle^{-1} \|_{op} + \| \langle \nabla x_j \rangle^{-1} J_s^{(k)} \langle \nabla x_j \rangle \|_{op} \right)
\times \text{Tr} \langle \nabla x_j \rangle \langle \nabla x_{k+1} \rangle \tilde{\gamma}_{N,L}^{(k+1)}(s) \langle \nabla x_{k+1} \rangle \langle \nabla x_j \rangle
\leq C J \varepsilon \varepsilon,
\end{equation}

where the last inequality follows from Corollary 3.2. That is,

$$|II| \leq C J \varepsilon \varepsilon \text{ and } |IV| \leq C J \varepsilon \varepsilon.$$ 

Hence $II$ and $IV$ converge to 0 as $\varepsilon \to 0$, uniformly in $N$ and $L$.

For $III$,

\begin{equation}
|III| = |Tr M_{s,t}^{(k)} \eta_c(x_j - x_{k+1}) g(z_j - z_{k+1}) \left( \tilde{\gamma}_{N,L}^{(k+1)}(s) - \tilde{\gamma}^{(k+1)}(s) \right) |
\leq \frac{1}{1 + \theta(\nabla x_j)} \left( \tilde{\gamma}_{N,L}^{(k+1)}(s) - \tilde{\gamma}^{(k+1)}(s) \right)
\left[ \frac{1}{1 + \theta(\nabla x_j)} \left( \tilde{\gamma}_{N,L}^{(k+1)}(s) - \tilde{\gamma}^{(k+1)}(s) \right) \right].
\end{equation}

The first term in the above estimate goes to zero as $N, 1/L \to \infty$ for every $\theta > 0$, since we have condition (3.17) and $M_{s,t}^{(k)} \eta_c(x_j - x_{k+1}) g(z_j - z_{k+1})$ is a compact operator. Due to the energy bounds on $\tilde{\gamma}_{N,L}^{(k+1)}$ and $\tilde{\gamma}^{(k+1)}$, the second term tends to zero as $\theta \to 0$, uniformly in $N$ and $L$.

Putting together the estimates for $I - IV$, we have established (3.20). Hence, we have obtained Theorem 3.6.

Combining Corollary 3.2 and Theorem 3.6, we see that $\tilde{\gamma}_{x}^{(k)}$ solves the 2D Gross-Pitaevskii hierarchy with the desired coupling constant $g_0 = \frac{1}{2 \pi} \int \int V(x, z_1 - z_2) dx |\cos(z_1)\cos(z_2)|^2 dz_1 dz_2$.

**Corollary 3.6.** Assume $L(N/L)^{\beta} \to 1^{-}$, and let $\Gamma(t) = \{ \tilde{\gamma}_{k}^{(k)} \}_{k=1}^{\infty}$ be a limit point of

$$\left\{ \Gamma_{N,L}(t) = \left\{ \tilde{\gamma}_{k, N,L}(t) \right\}_{k=1}^{N} \right\}$$

with respect to the product topology $\tau_{prod}$. Then $\tilde{\gamma}_{x}^{(k)} = Tr_{z} \tilde{\gamma}_{x}^{(k)}$ is a solution to the 2D Gross-Pitaevskii hierarchy subject to initial data $\tilde{\gamma}_{x}^{(k)}(0) = |\phi_0\rangle \langle \phi_0\rangle \otimes k$ with coupling constant $g_0 = \frac{1}{2 \pi} \int \int V(x, z_1 - z_2) dx |\cos(z_1)\cos(z_2)|^2 dz_1 dz_2$.

which, rewritten in integral form, is

\begin{equation}
\tilde{\gamma}_{x}^{(k)}(t) = U^{(k)}(t) \tilde{\gamma}_{x}^{(k)}(0) - ig_0 \sum_{k=1}^{k} U^{(k)}(t - s) Tr_{x_{k+1}} \left[ \delta(x_j - x_{k+1}), \tilde{\gamma}_{x}^{(k+1)}(s) \right] ds.
\end{equation}
Proof. The inhomogeneous term in hierarchy (3.10) is

\[ i \int_0^t U^{(k)}(t-s)Tr_{x_{k+1}}Tr_z \left[ \delta(x_j - x_{k+1}) \int V(x, z_j - z_{k+1})dx, \tilde{\gamma}^{(k+1)}(s) \right] ds \]

\[ = i \int_0^t U^{(k)}(t-s)Tr_{x_{k+1}}Tr_z \left[ \delta(x_j - x_{k+1}) \int V(x, z_j - z_{k+1})dx \tilde{\gamma}^{(k+1)}(s) \right] ds \]

\[ - i \int_0^t U^{(k)}(t-s)Tr_{x_{k+1}}Tr_z \left[ \delta(x_j' - x'_{k+1}) \int V(x, z_j' - z'_{k+1})dx \tilde{\gamma}^{(k+1)}(s) \right] ds \]

\[ = I - II. \]

From Corollary 3.2, we have

\[ I = i \int_0^t U^{(k)}(t-s)Tr_{x_{k+1}} \left[ \delta(x_j - x_{k+1})\tilde{\gamma}^{(k+1)}_x(s)Tr_z \left( \int V(x, z_j - z_{k+1})dx \prod_{j=1}^{k+1} \frac{2}{\pi} \cos(z_j) \cos(z_j') \right) \right] ds \]

\[ = i \int_0^t U^{(k)}(t-s)Tr_{x_{k+1}} \left[ \delta(x_j - x_{k+1})\tilde{\gamma}^{(k+1)}_x(s) \right] ds. \]

In the same manner we can see that

\[ II = i \int_0^t U^{(k)}(t-s)Tr_{x_{k+1}} \left[ \delta(x_j' - x'_{k+1})\tilde{\gamma}^{(k+1)}_x(s) \right] ds. \]

In summary, we have

\[ \tilde{\gamma}^{(k)}_x(t) = U^{(k)}(t)\tilde{\gamma}^{(k)}_x(0) - i \int_0^t U^{(k)}(t-s)Tr_{x_{k+1}} \left[ \delta(x_j - x_{k+1}), \tilde{\gamma}^{(k+1)}_x(s) \right] ds. \]

\[ \Box \]

3.3. Uniqueness of the 2D GP Hierarchy. By Bourgain [9], as we are below the Gagliardo-Nirenberg threshold here, we have the \( H^1 \) global wellposedness for the \( T^2 \) focusing cubic NLS (1.11). Thus, when \( \tilde{\gamma}^{(k)}_x(0) = |\phi_0\rangle \langle \phi_0| ^{\otimes k} \), we know one solution to the focusing GP hierarchy (1.13), namely \( |\phi\rangle \langle \phi| ^{\otimes k} \), where \( \phi \) solves (1.11).

**Theorem 3.7** ([42] [43]). There is at most one nonnegative operator sequence

\[ \left\{ \tilde{\gamma}^{(k)}_x \right\}_{k=1}^{\infty} \in \bigotimes_{k \geq 1} C \left( [0, T], L^1_k(T^2) \right) \]

that solves the 2D Gross-Pitaevskii hierarchy (3.31) subject to the energy condition

(3.32)

\[ Tr \left( \prod_{j=1}^{k} (1 - \Delta_{x_j}) \right) \tilde{\gamma}^{(k)}_x \leq C^k. \]

From Theorem 3.7, we conclude that the compact sequence

\[ \left\{ \Gamma_{N,L}(t) = \left\{ \tilde{\gamma}^{(k)}_{N,L} \right\}_{k=1}^{N} \right\} \]

has only one \( L(N/L)^{\beta} \to 1^- \) limit point, namely

\[ \tilde{\gamma}^{(k)} = \prod_{j=1}^{k} \frac{2}{\pi} \phi(t, x_j) \overline{\phi}(t, x_j') \cos(z_j) \cos(z_j'). \]

We then infer that as trace class operators

\[ \tilde{\gamma}^{(k)}_{N,L} \to \prod_{j=1}^{k} \frac{2}{\pi} \phi(t, x_j) \overline{\phi}(t, x_j') \cos(z_j) \cos(z_j') \text{ weak}^*. \]
Since the limit point $\tilde{\gamma}^{(k)}$ is an orthogonal projection, the well-known argument in [35; P296] upgrades the weak* convergence to strong, by using Grün's convergence theorem [62; Theorem 2.19].

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**Appendix A. Basic operator facts and Sobolev-type lemmas**

**Lemma A.1** (Hoffman-Ostenhof inequality[44]). For $\psi_N \in H^1(\mathbb{T}^dN)$, we have

\[(A.1)\]

$$\|\nabla_x \sqrt{\rho_N}\|_{L^2(\mathbb{T}^d)} \leq \|\nabla_{x_1} \psi_N\|_{L^2(\mathbb{T}^dN)}.$$  

with the one-particle density

$$\rho_N(x) = \int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} |\psi_N(x, x_2, \ldots, x_N)|^2 dx_2 \cdots dx_N.$$  

**Proof.** We may assume $\psi_N$ is a test function, so $\rho_N(x) \in C^\infty(\mathbb{T}^d)$. By Cauchy-Schwarz inequality, we have

$$\|\nabla_x \sqrt{\rho_N + \epsilon}\|_{L^2}^2 = \int \left| \frac{\nabla_x \rho_N}{2 \sqrt{\rho_N + \epsilon}} \right|^2 dx$$

$$= \int \left| \frac{\int \psi_N \nabla_x \psi_N dx_2 \cdots dx_N}{\rho_N(x) + \epsilon} \right|^2 \frac{\rho_N(x)}{\rho_N(x) + \epsilon} dx$$

$$\leq \int |\nabla_x \psi_N|^2 dx_2 \cdots dx_N \left( \frac{\rho_N(x)}{\rho_N(x) + \epsilon} \right) dx$$

$$\leq \int |\nabla_x \psi_N|^2 dx dx_2 \cdots dx_N.$$  

Thus, $\sqrt{\rho_N + \epsilon}$ is uniformly bounded in $H^1(\mathbb{T}^d)$ norm. We note that $-\Delta_x$ is a self-adjoint operator on $H^1(\mathbb{T}^d)$ and $\sqrt{\rho_N + \epsilon}$ converges to $\sqrt{\rho_N}$ in $L^2(\mathbb{T}^d)$ norm as $\epsilon \to 0$. From the definition of adjoint operator, we deduce that $\sqrt{\rho_N} \in H^1(\mathbb{T}^d)$ and

$$\|\nabla_x \sqrt{\rho_N}\|_{L^2}^2 \leq \int |\nabla_x \psi_N|^2 dx dx_2 \cdots dx_N.$$  

\[\square\]

**Lemma A.2.** Assume $L(N/L)^\beta \to 1^-$. For $\delta \in (0, 1)$, the multiplication operator $V_{N,L}(r_1 - r_2)$ on $L^2(\Omega_L^2)$ satisfies

\[(A.2)\]

$$L|V_{N,L}(r_1 - r_2)| \leq C_\delta(N/L)^\delta \|V\|_{L^\infty L^{1+s}_x}(1 - \Delta x_1),$$

\[(A.3)\]

$$L|V_{N,L}(r_1 - r_2)| \leq C_\delta \|V\|_{L^\infty L^1_t}(1 - \Delta x_1)^{1/2+s}(1 - \Delta x_2)^{1/2+s},$$

\[(A.4)\]

$$\|\langle \nabla_{x_1} \rangle^{-1} \langle \nabla_{x_2} \rangle^{-1} V(r_1 - r_2)(\nabla_{x_1})^{-1} \langle \nabla_{x_2} \rangle^{-1}\|_{op} \leq C \|V\|_{L^\infty L^1_t},$$

\[(A.5)\]

$$S^2_1LV_{N,L}(r_1 - r_2) + LV_{N,L}(r_1 - r_2)S^2_1 \geq -C_\delta(V)(N/L)^{\beta+\delta}S^2_1\phi_2^2,$$

where $C_\delta(V)$ is dependent on $V$.

**Proof.** For estimate (A.2), by Hölder and Sobolev inequality, we get

\[(A.6)\]

$$\langle LV_{N,L}(r_1 - r_2)\phi_L, \phi_L \rangle \leq \|LV_{N,L}\|_{L^\infty L^{1+s}_x} \|\phi_L\|^2_{L^2 L^{1+s}_x} \leq C_\delta(N/L)^{\beta+\delta} \|V\|_{L^\infty L^{1+s}_x} \|1 - \Delta x_1\phi_L\|^2_{L^2} \frac{1}{\rho_N} \leq C_\delta(N/L)^{\beta+\delta} \|V\|_{L^\infty L^{1+s}_x} \langle (1 - \Delta x_1)\phi_L, \phi_L \rangle.$$  

With $L(N/L)^\beta \to 1^-$, we obtain estimate (A.2).

For estimate (A.3), we recall Littlewood-Paley projectors defined by [29; 53] and decompose $\phi_L$ that

\[(A.7)\]

$$\phi_L = \phi_{L,1} + \phi_{L,2},$$

where

$$\phi_{L,1} = \sum_{0 \leq m_1 < m_2} P_{x_1,m_1} P_{x_2,m_2} \phi_L,$$

$$\phi_{L,2} = \sum_{0 \leq m_1 < m_2} P_{x_1,m_1} P_{x_2,m_2} \phi_L,$$
and 
\[ \varphi_{L,2} = \sum_{0 \leq m_2 \leq m_1} P_{x_1,m_1} P_{x_2,m_2} \varphi_L. \]

Then we have
\[ \|1 - \Delta x_1 \varphi_{L,1}\|_{L^2} \leq \|1 - \Delta x_2 \varphi_{L,1}\|_{L^2}, \]
\[ \|1 - \Delta x_2 \varphi_{L,2}\|_{L^2} \leq \|1 - \Delta x_1 \varphi_{L,2}\|_{L^2}. \]

By estimates (A.8), (A.9) and Sobolev inequality, we find that
\[ \|L V_{N,L}(r_1 - r_2)\varphi_{L,1}\varphi_L\| \leq \int L |V_{N,L}(r_1 - r_2)| (|\varphi_{L,1}|^2 + |\varphi_{L,2}|^2) dr_1 dr_2 \]
\[ \leq L\|V_{N,L}\|_{L^\infty L^1} (\|\varphi_{L,1}\|_{L^2}^2 + \|\varphi_{L,2}\|_{L^2}^2) \]
\[ \leq C_\delta L(N/L)^\beta \|V\|_{L^\infty L^1} (\|\varphi_{L,1}\|_{L^2}^2 + \|\varphi_{L,2}\|_{L^2}^2) \]
\[ \leq C_\delta L(N/L)^\beta \|V\|_{L^\infty L^1} (1 - \Delta x_1)^{1/2} (1 - \Delta x_2)^{1/2} \varphi_{L,1} \varphi_L. \]

With \(L(N/L)^\beta \to 1^-\), we obtain estimate (A.3).

For estimate (A.4), in the same manner as estimate (A.3), we get
\[ \|V_{N,L}(r_1 - r_2)\varphi, \varphi\| \leq C\|V\|_{L^\infty L^1} (1 - \Delta x_1)(1 - \Delta x_2) \varphi, \varphi, \]
which is equivalent to
\[ \langle \nabla x_1 \rangle^{-1} \langle \nabla x_2 \rangle^{-1} |V_{N,L}(r_1 - r_2)||\nabla x_1 \rangle^{-1} \langle \nabla x_2 \rangle^{-1} \leq C\|V\|_{L^\infty L^1}. \]

By Property 2 in Lemma A.6 we obtain estimate (A.4).

For estimate (A.5), we decompose
\[ \langle S_2^2 L V_{N,L}(r_1 - r_2)\varphi_L, \varphi_L \rangle \]
\[ = (1 - \Delta x_1)(L V_{N,L}(r_1 - r_2)\varphi_L), \varphi_L) + ((-\Delta x_1 - 1/L^2)(L V_{N,L}(r_1 - r_2)\varphi_L), \varphi_L) \]
\[ = A + B. \]

It needs only to control \(A\) and \(B\). Using the identity \(\nabla x_1 V_{N,L}(r_1 - r_2) = -\nabla x_2 V_{N,L}(r_1 - r_2)\) and integration by parts, we get
\[ A = \langle \nabla x_1 (L V_{N,L}(r_1 - r_2)\varphi_L), \nabla x_2 \varphi_L \rangle \]
\[ = \langle L V_{N,L}(r_1 - r_2)\nabla x_1 \varphi_L, \nabla x_2 \varphi_L + \langle (L(\nabla x_1 V_{N,L})(r_1 - r_2)\varphi_L), \nabla x_2 \varphi_L \rangle \]
\[ = \langle L V_{N,L}(r_1 - r_2)\nabla x_1 \varphi_L, \nabla x_2 \varphi_L \rangle - \langle (L(\nabla x_2 V_{N,L})(r_1 - r_2)\varphi_L), \nabla x_2 \varphi_L \rangle \]
\[ = \langle L V_{N,L}(r_1 - r_2)\nabla x_1 \varphi_L, \nabla x_2 \varphi_L \rangle + \int L V_{N,L}(r_1 - r_2)\nabla x_2 \varphi_L \nabla x_2 \varphi_L dr_1 dr_2 \]
\[ = A_1 + A_2 + A_3. \]

By Hölder and Sobolev inequality, we have
\[ \|A_1\| \leq L \|V_{N,L}\|_{L^\infty L^1} \|\nabla x_1 \varphi_L\|_{L^2}^2 \]
\[ \leq L(N/L)^{2\beta} \|V\|_{L^\infty L^1} \|1 - \Delta x_2 \nabla x_1 \varphi_L\|_{L^2} \|\nabla x_1 \varphi_L\|_{L^2} \]
\[ \leq L(N/L)^{2\beta} \|V\|_{L^\infty L^1} (S_1^2 + S_2^2 \varphi_L, \varphi_L). \]

Estimated in the same way as \(A_1\),
\[ \|A_2\| \leq \int L |V_{N,L}(r_1 - r_2)| (|\nabla x_1 \varphi_L|^2 + |\nabla x_2 \varphi_L|^2) dr_1 dr_2 \]
\[ \leq L(N/L)^{2\beta} \|V\|_{L^\infty L^1} (S_1^2 + S_2^2 \varphi_L, \varphi_L). \]
For $A_3$, we have
\begin{equation}
|A_3| \leq \|L V_{N,L} (r_1 - r_2) \varphi_L\|_{L^2} \|\nabla_{x_1} \nabla_{x_2} \varphi_L\|_{L^2} \leq C \delta \frac{(N/L)^{2 \beta}}{\|V\|_{L^\infty_x L^2}} ((1 - \Delta_{x_1})(1 - \Delta_{x_2}) \varphi_L, \varphi_L),
\end{equation}
where we have used estimate \((A.3)\) with $\delta = 1/2$ and $V$ replaced by $V^2$ in the last inequality.
Hence we have
\begin{equation}
|A| \leq C \delta \frac{(N/L)^{2 \beta}}{\|V\|_{L^\infty_x L^2}} ((1 - \Delta_{x_1})(1 - \Delta_{x_2}) \varphi_L, \varphi_L).
\end{equation}

Next, we decompose $B$ into two terms
\begin{equation}
B = \langle - \partial_{x_1}^2 (L V_{N,L} (r_1 - r_2) \varphi_L, P_{z_1, > 1} \varphi_L) \\
= \langle - \partial_{x_1}^2 (L V_{N,L} (r_1 - r_2) \varphi_L), P_{z_1, > 1} \varphi_L \rangle - L^{-2} \langle L V_{N,L} (r_1 - r_2) \varphi_L, P_{z_1, > 1} \varphi_L \rangle \\
= B_1 + B_2.
\end{equation}

For $B_1$, we expand
\begin{equation}
B_1 = B_{11} + B_{12},
\end{equation}
where
\begin{align*}
B_{11} &= \langle L (\partial_{x_1} V_{N,L} (r_1 - r_2)) \varphi_L, \partial_{x_1} P_{z_1, > 1} \varphi_L \rangle, \\
B_{12} &= \langle L V_{N,L} (r_1 - r_2) \partial_{x_1} \varphi_L, \partial_{x_1} P_{z_1, > 1} \varphi_L \rangle.
\end{align*}

For $B_{11}$, applying Hölder inequality at $x_2$, we obtain
\begin{equation}
|B_{11}| \leq \|L \partial_{x_1} V_{N,L} (r_1 - r_2)\|_{L^\infty_x L^1} \|\varphi_L\|_{L^2_x L^\infty} \|\partial_{x_1} P_{z_1, > 1} \varphi_L\|_{L^2_x L^\infty} \leq C \delta \frac{(N/L)^{2 \beta + \delta}}{\|V\|_{L^\infty_x L^1}} \|\partial_{x_1} V\|_{L^\infty_x L^1} \|S \varphi_L\|_{L^2_x} \|S \partial_{x_1} P_{z_1, > 1} \varphi_L\|_{L^2_x} \leq C \delta \frac{(N/L)^{2 \beta + \delta}}{\|V\|_{L^\infty_x L^1}} \|\partial_{x_1} V\|_{L^\infty_x L^1} \|S \varphi_L\|_{L^2_x} \|S \partial_{x_1} P_{z_1, > 1} \varphi_L\|_{L^2_x},
\end{equation}
where we have used estimate \((A.29)\) in the last inequality.

Computing in the same way, we have
\begin{equation}
|B_{12}| \leq \|L \partial_{x_1} V_{N,L} (r_1 - r_2)\|_{L^\infty_x L^1} \|\varphi_L\|_{L^2_x L^\infty} \|\partial_{x_1} P_{z_1, > 1} \varphi_L\|_{L^2_x L^\infty} \leq C \delta \frac{(N/L)^{2 \beta + \delta}}{\|V\|_{L^\infty_x L^1}} \|\partial_{x_1} V\|_{L^\infty_x L^1} \|S \varphi_L\|_{L^2_x} \|S \partial_{x_1} P_{z_1, > 1} \varphi_L\|_{L^2_x},
\end{equation}
where we have used \((A.28)\) and \((A.29)\) in the last inequality.

Hence, with $L (N/L)^{\beta} \rightarrow 1^-$, we have
\begin{equation}
|B_{1}| \leq C \delta \frac{(N/L)^{2 \beta + \delta}}{\|V\|_{L^\infty_x L^1}} \|S \varphi_L\|_{L^2_x}^2.
\end{equation}

For $B_2$, we have
\begin{equation}
|B_2| \leq \|L^{-2} L V_{N,L}\|_{L^\infty_x L^1} \|\varphi_L\|_{L^2_x L^\infty} \|P_{z_1, > 1} \varphi_L\|_{L^2_x L^\infty} \leq C \delta \frac{(N/L)^{2 \beta + \delta}}{\|V\|_{L^\infty_x L^1}} \|S \varphi_L\|_{L^2_x} \|S \varphi_L\|_{L^2_x} \leq C \delta \frac{(N/L)^{2 \beta + \delta}}{\|V\|_{L^\infty_x L^1}} \|S \varphi_L\|_{L^2_x} \|S \varphi_L\|_{L^2_x}.
\end{equation}

where we used $L^{-2} P_{z_1, > 1} \leq S \|P_{z_1, > 1}\|_{L^2_x}$. In the last inequality.

Putting \((A.17)\), \((A.22)\) and \((A.23)\) together, with $L (N/L)^{\beta} \rightarrow 1^-$, we obtain estimate \((A.5)\). \hfill \Box

**Lemma A.3.** Let $A : D(A) \rightarrow H$ be a positive self-adjoint operator in the Hilbert space $H$, and let $0 < \alpha < 1$. Then
\[
A^\alpha \leq (1 - \alpha) \eta^{-1} A + \alpha \eta^{\alpha - \eta},
\]
for $\eta \in (0, 1)$.\hfill \Box
Proof. By the spectral representation and the inequality

\[ \lambda^\alpha \leq \alpha \eta^{-1} \lambda + (1 - \alpha) \eta^{\frac{\alpha}{\alpha + 1}}, \quad \forall \lambda \in (0, \infty), \]

we obtain

\[
\langle A^\alpha u, u \rangle_H = \int_0^\infty \lambda^\alpha d\|E_\lambda u\|^2_H \leq (1 - \alpha) \eta^{-1} \int_0^\infty \lambda d\|E_\lambda u\|^2_H + \alpha \eta^{\frac{\alpha}{\alpha + 1}} \int_0^\infty d\|E_\lambda u\|^2_H = (1 - \alpha) \eta^{-1} \langle Au, u \rangle_H + \alpha \eta^{\frac{\alpha}{\alpha + 1}} \langle u, u \rangle_H.
\]

\[ \square \]

Recall Lemma 4.4, we now give the proof as follows.

**Lemma A.4.** Let \( \rho \in L^1(\Omega) \) be a function compactly supported on \( \Omega \) such that

\[ \sup_z \int |\rho(x, z)||x|dx < \infty \]

and define \( \rho_{\varepsilon, \lambda}(x, z) = \varepsilon^{-2} \lambda^{-1}\rho(x/\varepsilon, z/\lambda) \) and \( g(z) = \int \rho(x, z)dx \). Then, for every \( \kappa \in [0,1) \), there exists \( C > 0 \), such that

\[
|\text{Tr}(J^{(k)}(\rho_{\varepsilon, \lambda}(r_j - r_{k+1}) - \delta(x_j - x_{k+1})g(z_j - z_{k+1}))\gamma^{(k+1)}| \leq C\varepsilon^\kappa \sup_z \int |\rho(x, z)||x|^{\kappa}dx \left( \|\langle \nabla_x \rangle^{-1} J^{(k)}(\nabla_x) \|_{op} + \|\langle \nabla_x \rangle J^{(k)}(\nabla_x) - 1\|_{op} \right) \text{Tr}(\nabla_{x_j})^2(\nabla_{x_{k+1}})^2 \gamma^{(k+1)}
\]

for all nonnegative \( \gamma^{(k+1)} \in L^1_{k+1} \).

**Proof.** We present a proof by modifying the proof in [47]. Such a method has been used by various authors, for example [19]. It suffices to prove the estimate for \( k = 1 \). Since the observable \( J^{(1)} \) can be written as a sum of a self-adjoint operator and an anti-self-adjoint operator, we may assume \( J^{(1)} \) is self-adjoint. We represent \( \gamma^{(2)} \) by \( \gamma^{(2)} = \sum_j \lambda_j |\varphi_j \rangle \langle \varphi_j| \), where \( \varphi_j \in L^2(\Omega^2) \) and \( \lambda_j \geq 0 \). Then, we have

\[
\text{Tr}(J^{(1)}(\rho_{\varepsilon, \lambda}(r_1 - r_2) - \delta(x_1 - x_2)g(z_1 - z_2))\gamma^{(2)} = \sum_j \lambda_j \langle \varphi_j, J^{(1)}(\rho_{\varepsilon, \lambda}(r_1 - r_2) - \delta(x_1 - x_2)g(z_1 - z_2))\varphi_j \rangle
\]

where \( \psi_j = (J^{(1)} \otimes 1)\varphi_j \). Then, we decompose

\[ \langle \psi_j, (\rho_{\varepsilon, \lambda}(r_1 - r_2) - \delta(x_1 - x_2)g(z_1 - z_2))\varphi_j \rangle = A_j + B_j, \]

where

\[ A_j = \langle \psi_j, (\rho_{\varepsilon, 1}(r_1 - r_2) - \delta(x_1 - x_2)g(z_1 - z_2))\varphi_j \rangle, \]

\[ B_j = \langle \psi_j, (\rho_{\varepsilon, \lambda}(r_1 - r_2) - \rho_{\varepsilon, 1}(r_1 - r_2))\varphi_j \rangle. \]

For \( A_j \), switching to Fourier space in the \( x \)-direction, we find

\[ |A_j| = |\int \hat{\psi}_j(n_1, n_2; z_1, z_2)\hat{\varphi}_j(m_1, m_2; z_1, z_2)\rho_{\varepsilon, \lambda}(x, z_1 - z_2)(e^{i\varepsilon \cdot (n_1 - m_1)} - 1) \times \delta(n_1 + n_2 - m_1 - m_2)dx dz_1 dz_2 dn_1 dn_2 dm_1 dm_2| \leq \int |\hat{\psi}_j(n_1, n_2; z_1, z_2)||\hat{\varphi}_j(m_1, m_2; z_1, z_2)|\delta(n_1 + n_2 - m_1 - m_2) \times |\int \rho(x, z_1 - z_2)(e^{i\varepsilon \cdot (n_1 - m_1)} - 1)dx|dz_1 dz_2 dn_1 dn_2 dm_1 dm_2. \]
Using the inequality that $\forall \kappa \in (0, 1)$

$$
|e^{ix((n_1 - m_1) - 1)}| \leq \varepsilon^\kappa |x|^\kappa |n_1 - m_1|^\kappa \\
\leq \varepsilon^\kappa |x|^\kappa (|n_1|^\kappa + |m_1|^\kappa),
$$

we get

$$
|\langle \psi_j, (\rho_{c,1}(r_1 - r_2) - \delta(x_1 - x_2)g(z_1 - z_2))\phi_j \rangle| \\
\leq \varepsilon^\kappa \sup_\mathcal{Z} |\rho(x, z)||x|^\kappa \int \delta(n_1 + n_2 - m_1 - m_2) \\
\times (|n_1|^\kappa + |m_1|^\kappa) |\hat{\psi}_j(n_1, n_2; z_1, z_2)||\hat{\phi}_j(m_1, m_2; z_1, z_2)|dz_1dz_2dn_1dn_2dm_1dm_2 \\
= \varepsilon^\kappa \sup_\mathcal{Z} \int |\rho(x, z)||x|^\kappa d(x + II),
$$

where

$$
I = \int \delta(n_1 + n_2 - m_1 - m_2)|n_1|^\kappa |\hat{\psi}_j(n_1, n_2; z_1, z_2)||\hat{\phi}_j(m_1, m_2; z_1, z_2)|dz_1dz_2dn_1dn_2dm_1dm_2, \\
II = \int \delta(n_1 + n_2 - m_1 - m_2)|m_1|^\kappa |\hat{\psi}_j(n_1, n_2; z_1, z_2)||\hat{\phi}_j(m_1, m_2; z_1, z_2)|dz_1dz_2dn_1dn_2dm_1dm_2.
$$

The estimates for $I$ and $II$ are similar, so we only deal with $I$ explicitly.

$$
I \leq \int \delta(n_1 + n_2 - m_1 - m_2) \frac{\langle n_1 \rangle \langle n_2 \rangle}{\langle m_1 \rangle \langle m_2 \rangle} |\hat{\psi}_j(n_1, n_2; z_1, z_2)| \\
\times \frac{\langle m_1 \rangle \langle m_2 \rangle}{\langle n_1 \rangle^{1-\kappa} \langle m_2 \rangle} |\hat{\phi}_j(m_1, m_2; z_1, z_2)|dz_1dz_2dn_1dn_2dm_1dm_2 \\
\leq \theta \int \delta(n_1 + n_2 - m_1 - m_2) \frac{(n_1)^2(n_2)^2}{\langle m_1 \rangle \langle m_2 \rangle^2} |\hat{\psi}_j(n_1, n_2; z_1, z_2)|^2dz_1dz_2dn_1dn_2dm_1dm_2 \\
+ \theta^{-1} \int \delta(n_1 + n_2 - m_1 - m_2) \frac{(m_1)^2(m_2)^2}{\langle n_1 \rangle^{2(1-\kappa)}} |\hat{\phi}_j(m_1, m_2; z_1, z_2)|^2dz_1dz_2dn_1dn_2dm_1dm_2 \\
\leq \theta \langle \psi_j, (1 - \Delta_{x_1})(1 - \Delta_{x_2})\psi_j \rangle \sup_n \sum_{m_2} \frac{1}{(m_2 - n)^2(m_2)} \\
+ \theta^{-1} \langle \phi_j, (1 - \Delta_{x_1})(1 - \Delta_{x_2})\phi_j \rangle \sup_n \sum_{m_2} \frac{1}{(m_2 - n)^2(1-\kappa)}
$$

Since

$$
\sup_n \sum_{m\in \mathbb{Z}^2} \frac{1}{(m - n)^2(1-\kappa)} < \infty
$$

for $\kappa \in [0, 1)$, we have

$$
(A.24) \quad |A_j| \leq \theta \langle \psi_j, (1 - \Delta_{x_1})(1 - \Delta_{x_2})\psi_j \rangle + \theta^{-1} \langle \phi_j, (1 - \Delta_{x_1})(1 - \Delta_{x_2})\phi_j \rangle.
$$
Hence, we obtain
\begin{equation}
\left| TrJ^{(1)}(\rho_{\varepsilon,\lambda}(r_1 - r_2) - \delta(x_1 - x_2)g(z_1 - z_2))\gamma^{(2)} \right|
\end{equation}
\begin{align*}
&\leq C\varepsilon \sup_x \int |\rho(x, z)||x|^\alpha dx \left( \theta Tr(\nabla_x)^2(\nabla_{x_2})^2\gamma^{(2)} + \theta^{-1} TrJ^{(1)}(\nabla_{x_1})^2(\nabla_{x_2})^2J^{(1)}\gamma^{(2)} \right)
\end{align*}
\begin{align*}
&= C\varepsilon \sup_x \int |\rho(x, z)||x|^\alpha dx \left( \theta Tr(\nabla_x)^2(\nabla_{x_2})^2\gamma^{(2)} 
+ \theta^{-1} Tr(\nabla_x)^{-1}(\nabla_{x_2})^{-1}J^{(1)}(\nabla_{x_1})^2J^{(1)}(\nabla_{x_2})^{-1}(\nabla_{x_1})(\nabla_{x_2})^2\gamma^{(2)}(\nabla_{x_1})(\nabla_{x_2}) \right) 
\end{align*}
\begin{align*}
&\leq C\varepsilon \sup_x \int |\rho(x, z)||x|^\alpha dx \left( \theta + \theta^{-1}\|\nabla_{x_1}\|^{-1}\|J^{(1)}(\nabla_{x_1})\|_{op}\|\nabla_{x_1}\|J^{(1)}(\nabla_{x_1})^{-1}\|_{op} Tr(\nabla_{x_1})^2(\nabla_{x_2})^2\gamma^{(2)} \right)
\end{align*}
\begin{align*}
&= C\varepsilon \sup_x \int |\rho(x, z)||x|^\alpha dx \left( \|\nabla_{x_1}\|^{-1}J^{(1)}(\nabla_{x_1})\|_{op} + \|\nabla_{x_1}\|J^{(1)}(\nabla_{x_1})^{-1}\|_{op} \right) Tr(\nabla_{x_1})^2(\nabla_{x_2})^2\gamma^{(2)},
\end{align*}
where we have taken $\theta = \|\nabla_{x_1}\|^{-1}J^{(1)}(\nabla_{x_1})^{-1}\|_{op}$ in the last line.

For $B_j$, we use the operator inequality (also see (31) (A.63))
\begin{equation}
|V(r_1 - r_2)| \leq C\|V\|_{L}\|1 - \Delta r_1\|\|1 - \Delta r_2\|
\end{equation}
which can be estimated in the same way as estimate (A.4). Then, we get
\begin{align*}
|B_j| &\leq \theta \langle \psi_i, (\rho_{\varepsilon,\lambda} - \rho_{\varepsilon,\lambda})(r_1 - r_2)\psi_i \rangle + \theta^{-1} \langle \varphi_i, (\rho_{\varepsilon,\lambda} - \rho_{\varepsilon,\lambda})(r_1 - r_2)\varphi_i \rangle 
\end{align*}
\begin{align*}
&\leq C\|\rho_{\varepsilon,\lambda} - \rho_{\varepsilon,\lambda}\|_{L^1\mathcal{L}^\infty} \|\theta \psi_i, (1 - \Delta r_1)(1 - \Delta r_2)\psi_i \rangle + \theta^{-1} \langle \varphi_i, (1 - \Delta r_1)(1 - \Delta r_2)\varphi_i \rangle 
\end{align*}
\begin{align*}
&= C\|\rho_{\varepsilon,\lambda} - \rho_{\varepsilon,\lambda}\|_{L^1\mathcal{L}^\infty} \|\theta \psi_i, (1 - \Delta r_1)(1 - \Delta r_2)\psi_i \rangle + \theta^{-1} \langle \varphi_i, (1 - \Delta r_1)(1 - \Delta r_2)\varphi_i \rangle .
\end{align*}
Similarly, we have
\begin{align*}
|TrJ^{(1)}(\rho_{\varepsilon,\lambda}(r_1 - r_2) - \rho_{\varepsilon,\lambda}(r_1 - r_2))\gamma^{(2)}| &\leq C\|\rho_{\varepsilon,\lambda} - \rho_{\varepsilon,\lambda}\|_{L^1\mathcal{L}^\infty} Tr(\nabla_{r_1})^2(\nabla_{r_2})^2\gamma^{(2)}.
\end{align*}

Together with estimate (A.25) and (A.27), we complete the proof.

\begin{lemma}
Recall
\begin{equation}
S^2 = 1 - \Delta x - 1/L^2,
\end{equation}
\begin{equation}
\tilde{S}^2 = 1 - \Delta x - \partial_z^2/L^2 - 1/L^2.
\end{equation}
We have
\begin{align*}
|1 - \Delta r| &\leq 2L^{-2}S^2,
|1 - \Delta r|P_{z,>1} &\leq 2S^2P_{z,>1},
\tilde{S}^2P_1 &\geq L^{-2}P_1,
2\tilde{S}^2 &\geq 1 - \Delta r.
\end{align*}
\end{lemma}
\begin{proof}
For (A.28), we have
\begin{equation}
1 - \Delta r = S^2 + 1/L^2 \leq 2L^{-2}S^2.
\end{equation}
For (A.29), we note that
\begin{equation}
L^{-2}P_{z,>1} \leq (-\partial_z^2 - 1/L^2)P_{z,>1} \leq S^2P_{z,>1}.
\end{equation}
\end{proof}
Thus, we obtain
\begin{equation}
(1 - \Delta r)P_{z,>1} = (S^2 + 1/L^2)P_{z,>1} \leq 2S^2P_{z,>1}.
\end{equation}
\begin{proof}
For (A.30), we have
\begin{equation}
L^{-2}P_1 \leq (-\partial_z^2/L^2 - 1/L^2)P_1 \leq \tilde{S}^2P_1.
\end{equation}
For (A.31), we note that
\begin{equation}
2(-\partial_z^2 - 1)\tilde{P}_1 = -\partial_z^2\tilde{P}_1 + (-\partial_z^2 - 2)\tilde{P}_1 \geq -\partial_z^2\tilde{P}_1.
\end{equation}
Then we have
\begin{align}
2\tilde{S}^2 &\geq 2(1 - \Delta_x) + 2L^{-2}(-\partial_x^2 - 1)\tilde{P}_1 \\
&\geq 2(1 - \Delta_x) - L^{-2}\partial_x^2 \tilde{P}_1 \\
&\geq 2(1 - \Delta_x) - \partial_x^2 \tilde{P}_1.
\end{align}

Noting that $-\partial_x^2 \tilde{P}_0 = \tilde{P}_0$, we obtain
\begin{equation}
1 - \Delta_x \geq 1 - \partial_x^2 \tilde{P}_0.
\end{equation}
Putting (A.33) and (A.34) together, we establish (A.31). \hfill \square

We need the following facts as well. The proofs are elementary and we omit them.

**Lemma A.6.** 1. Suppose that $A \geq 0$, $P_j = P_j^*$, and $I = P_0 + P_1$. Then $A \leq 2P_0AP_0 + 2P_1AP_1$. 2. If $A \geq B \geq 0$, and $AB = BA$, then $A^\alpha \geq B^\alpha$ for any $\alpha > 0$. Especially, if $\alpha = 2$, then $\|A\|_{op} \geq \|B\|_{op}$. 3. If $A_1 \geq A_2 \geq 0$, $B_1 \geq B_2 \geq 0$ and $A_1B_j = B_jA_1$ for all $1 \leq i, j \leq 2$, then $A_1B_1 \geq A_2B_2$. 4. If $A \geq 0$ and $AB = BA$, then $A^{1/2}B = BA^{1/2}$.

**Lemma A.7.** Suppose $\sigma : L^2(\Omega) \rightarrow L^2(\Omega)$ has kernel
\begin{equation}
\sigma(r, r') = \int \bar{\psi}(r, r)\psi(r', r) dr
\end{equation}
for some $\psi \in L^2(\Omega)$, and let $A, B : L^2(\Omega) \rightarrow L^2(\Omega)$. Then the composition $A\sigma B$ has kernel
\begin{equation}
(A\sigma B)(r, r') = \int (A\psi)(r, r)\bar{\psi}(r', r) dr.
\end{equation}

It follows that
\begin{equation}
\text{Tr} A\sigma B = \langle A\psi, B^*\psi \rangle.
\end{equation}

**Lemma A.8.** Let $P_M^j$ be the orthogonal projection onto the sum of the first $M$ eigenspaces with respect to the spectral decomposition of $L^2(\Omega)$ to the operator $-\Delta_{r_j}$ and
\begin{equation}
P_M^{(k)} = \prod_{j=1}^k P_M^j.
\end{equation}

1. Suppose that $J^{(k)}$ is a compact operator. Then $J_{M}^{(k)} := P_M^{(k)}J^{(k)}P_M^{(k)}$ converges to $J^{(k)}$ in the operator norm.
2. $\Delta_{r_j}J_{M}^{(k)}$ and $J_{M}^{(k)}\Delta_{r_j}$ are bounded operators.
3. There exists a countable dense subset $\{T_i\}$ of the closed unit ball in the space of bounded operators on $L^2(\Omega^k)$ such that $T_i$ is compact and in fact for each $i$ there exists $M$ (depending on $i$) such that $T_i = P_M^{(k)}T_i P_M^{(k)}$.

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