Kink motion for the one-dimensional stochastic Allen–Cahn equation

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Abstract
We study the kink motion for the one-dimensional stochastic Allen-Cahn equation and its mass conserving counterpart. Using a deterministic slow manifold, in the sharp interface limit for sufficiently small noise strength we derive an explicit stochastic differential equation for the motion of the interfaces, which is valid as long as the solution stays close to the manifold.

On a relevant time-scale, where interfaces move at most by the minimal allowed distance between interfaces, we show that the kinks behave approximately like the driving Wiener-process projected onto the slow manifold, while in the mass-conserving case they are additionally coupled via the mass constraint.

1 Introduction
We study the stochastic Allen–Cahn equation \(\text{AC}\) together with its mass conserving modification \(\text{mAC}\) posed on an one-dimensional domain driven by a small additive spatially smooth and white in time noise, which might depend on \(\varepsilon\), too.

As in the deterministic case (see [15]) we use a deterministic slow manifold, which is parametrized by the position of the interfaces. The same manifold was also used for the Cahn–Hilliard equation in [11, 12] and its stochastic counterpart in [7, 9]. Its key idea is to approximate an unknown true invariant manifold of the deterministic dynamic. See also [10] for a more recent result on approximately invariant manifolds. Here in Section 3 we introduce a different and simpler manifold, as due to the noise we observe a faster motion than the metastable motion of the deterministic case, where it is necessary to study in more detail also all exponentially small error terms.

In Section 6, for the stochastic stability of the manifold we show that with overwhelming probability a solution of the stochastic equation stays close to the manifold for extremely large times, unless the distance between two interfaces gets almost on the order of the atomistic interaction length \(\varepsilon\). We are based on the method proposed in [7] but modify it in order to show stability in \(L^2\) and \(L^4\) spaces, which is necessary for the analysis of the dynamics along the manifold. Crucial for stability is the spectral information of the linearized operator, which we recover from the single interface case on the whole real line in Section 4.

As long as the solutions of Allen–Cahn stay close to the manifold, we show that the motion of the interfaces is given by a stochastic differential equation (see (5.1)). We do not show that the projection onto the manifold is globally well defined, but verify that we can always split the solution into a well defined diffusion process for the position on the manifold and the distance orthogonal to the manifold. However, we do not show that these coordinates are uniquely defined nor that the projection onto the manifold is well defined, which might only be true closer to the manifold.
We analyze the motion and show for the Allen–Cahn equation that on timescales where the interfaces move on the order $\varepsilon$, the motion is given by the driving Wiener process projected onto the slow manifold. If the noise has no long-range correlation, then the motion of the interfaces are approximately independent. For the mass conserving Allen–Cahn equation we verify a similar result, but the interfaces are additionally coupled due to mass-conservation. The result for the Allen–Cahn equation was already studied in [21] using formal analysis and numerical experiments.

We do not study what happens if two interfaces get on the order $\varepsilon$ close to each other. The obvious conjecture is that both annihilate, as shown in the deterministic case in [16], which is heavily based on the maximum principle. Conjectures and some details in the stochastic case can be found in [23].

In the higher dimensional case the situation is more complicated, as the motion of the interface should be driven by a free interface problem, which in general cannot be approximated by a finite dimensional manifold. See [8,24] for partial results or [4] in the deterministic case.

In special cases slow manifolds were used to study the motion of droplets (or bubbles) in various settings of the stochastic mass conserving Allen–Cahn equation [6] (or [5] in the deterministic case) or the related Cahn–Hilliard equation [13] or [1,2] for the deterministic case. On the other hand, for the Allen–Cahn equation the motion of droplets was not studied, as due to a lack of mass-conservation these droplets should collapse immediately.

Note that all the previous examples of droplet motion study the case of a single droplet, where the slow manifold is parametrized only by the position, while mass-conservation fixes the radius. The case of many bubbles does not seem to be studied rigorously yet. Although the slow manifold can be constructed and parametrized by position and radius of the droplets, it seems that the error terms by glueing single droplet solutions together are not exponentially small and thus pose an obstacle in obtaining rigorous results.

Let us finally remark, that the method of proof is related to the motion of traveling waves, where the manifold is given by translates of the wave profile. The main difference here is that traveling waves do not move slowly on the manifold, but travel at a constant speed.

2 Setting

The stochastic Allen–Cahn equation on an one-dimensional domain driven by an additive spatially smooth and white in time noise $\partial_t W$ is given by

$$\begin{cases} \partial_t u = \varepsilon^2 u_{xx} - f(u) + \partial_t W, & 0 < x < 1, \ t > 0 \\ u_x = 0, & x \in \{0,1\}. \end{cases} \tag{AC}$$

Here, $0 < \varepsilon \ll 1$ is a small parameter measuring the typical width of a phase transition, and $f = F'$ is the derivative of a double well potential $F$. We assume that $F \in C^3(\mathbb{R})$ is a smooth, even potential satisfying

(S1) $F(u) \geq 0$ and $F(u) = 0$ if, and only if, $u = \pm 1$,

(S2) $F'$ has three zeros $\{0, \pm 1\}$ and $F''(0) < 0, F''(\pm 1) > 0$,

(S3) $F$ is symmetric: $F(u) = F(-u) \ \forall u \geq 0$.

The standard example is $F(u) = \frac{1}{4}(1-u^2)^2$ and thus $f(u) = u^3 - u$. For the simplicity of some arguments later, especially in determining the spectrum of the linearized operator, we focus for the most part of this paper on this standard quartic potential, although the results remain valid for potentials satisfying the conditions (S1)–(S3). For more details on this discussion, we refer to [20].
For the moment, let us assume that \( \int_0^1 \partial_t W(t,x) \, dx = 0 \) for all \( t \geq 0 \), i.e., in a Fourier series expansion there is no noise on the constant mode. In contrast to the Cahn–Hilliard equation, \( \text{(AC)} \) does not preserve mass as
\[
\partial_t \int_0^1 u(t,x) \, dx = \varepsilon^2 \int_0^1 u_{xx} \, dx - \int_0^1 f(u) \, dx + \int_0^1 \partial_t W(t,x) \, dx = -\int_0^1 f(u) \, dx.
\]
Throughout our analysis, we will therefore separately consider the mass conserving Allen–Cahn equation \( \text{(mAC)} \)
\[
\begin{cases}
\partial_t u = \varepsilon^2 u_{xx} - f(u) + \int_0^1 f(u) \, dx + \partial_t W, & 0 < x < 1, \ t > 0 \\
u_x = 0, & x \in \{0,1\},
\end{cases}
\]
where we added the integral of \( f \) over the interval \((0,1)\) to guarantee the conservation of mass. This can also be seen as an orthogonal projection of the right hand side onto the space orthogonal to the constants.

We denote the standard inner product in \( L^2(0,1) \) by \( \langle \cdot, \cdot \rangle \), i.e., \( \langle f,g \rangle = \int_0^1 f(x)g(x) \, dx \), and the \( L^2 \)-norm by \( \| \cdot \| \). Other scalar products and norms appearing in subsequent sections will be endowed with a subindex. Moreover, we denote the Allen–Cahn operator by
\[
\mathcal{L}(\psi) = \varepsilon^2 \psi_{xx} - f(\psi).
\]
We consider for a given ansatz function \( u^h \) (defined later in Definition 3.3) the Taylor expansion of \( \mathcal{L} \) around \( u^h \)
\[
\mathcal{L}(u^h + \psi) = \mathcal{L}(u^h) + \mathcal{L}^h \psi + N^h(\psi),
\]
where we define the linearization \( \mathcal{L}^h \) of \( \mathcal{L} \) at the ansatz function \( u^h \) and the remaining nonlinear terms \( N^h(\psi) \) by
\[
\mathcal{L}^h \psi := D\mathcal{L}(u^h)\psi = \varepsilon^2 \psi_{xx} - f'(u^h)\psi \quad \text{and} \quad N^h(\psi) := f(u^h) - f(u^h + \psi) + f'(u^h)\psi.
\]
In the prototypical case of the quartic potential, this leads to
\[
\mathcal{L}^h \psi = \varepsilon^2 \psi_{xx} + \psi - 3(u^h)^2 \psi \quad \text{and} \quad N^h(\psi) = -3u^h \psi^2 - \psi^3.
\]
In the case of the mass conserving Allen–Cahn equation, we have to assume that the Wiener process \( W \) has mean zero. Furthermore, in order to apply Itô-formula later, we need that solutions to both \( \text{(AC)} \) and \( \text{(mAC)} \) are sufficiently smooth in space and hence, we need that the stochastic forcing \( \partial_t W \) is sufficiently smooth in space, too. The existence of solutions to both \( \text{(AC)} \) and \( \text{(mAC)} \) at least in the case of a quartic potential is standard and we will not comment on this in more detail. See [17].

In the following we will assume that \( W \) which also depends on \( \varepsilon \) is given by a \( \mathcal{Q} \)-Wiener process satisfying the following regularity properties.

**Assumption 2.1** (Regularity of the Wiener process \( W \)). *Let \( W \) be a \( \mathcal{Q} \)-Wiener process in the underlying Hilbert space \( L^2(\Omega) \), \( \mathcal{Q} \) a symmetric operator, and \( \{e_k\}_{k \in \mathbb{N}} \) an orthonormal basis with corresponding eigenvalues \( \alpha_k^2 \) such that
\[
\mathcal{Q}e_k = \alpha_k^2 e_k \quad \text{and} \quad W(t) = \sum_{k \in \mathbb{N}} \alpha_k \beta_k(t) e_k,
\]
for a sequence of independent real-valued standard Brownian motions \( \{\beta_k\}_{k \in \mathbb{N}} \). We assume that the \( \mathcal{Q} \)-Wiener process \( W \) satisfies
\[
\text{trace}_{L^2(\mathcal{Q})} = \sum_{k \in \mathbb{N}} \alpha_k^2 =: \eta_\varepsilon < \infty.
\]
Moreover, in the case of the mass conserving Allen–Cahn equation \( m\text{AC} \), we suppose that \( W \) takes its values in \( L^2_0(\Omega) \), that is,
\[
\int_0^1 W(t, x) \, dx = 0 \quad \text{for all } t \geq 0.
\]

Note that our results will thus depend on the squared noise strength \( \eta_\varepsilon \), which also depends on the parameter \( \varepsilon > 0 \). The exact order in \( \varepsilon \) of \( \eta_\varepsilon \) will be fixed later in the main results.

## 3 Construction of the slow manifold

In this section, we construct the fundamental building block for our analysis, the slow manifolds \( \mathcal{M} \) for \( \text{AC} \) and \( \mathcal{M}_\mu \) for \( m\text{AC} \). Our construction of the slow manifolds is different to the deterministic case \[15\]. We do not introduce a cut-off function to glue together the profiles connecting the stable phases \( \pm1 \). With this cut-off function, the authors took extra care of the exponentially small error away from the interface positions, which is crucial as the motion of the kinks in the deterministic case is dominated by exponentially small terms.

In our stochastic case, however, the (polynomial in \( \varepsilon \)) noise strength dominates and hence, we are not concerned with these exponentially small terms. Thus we use our simplified manifolds, but we believe that as in the stochastic Cahn–Hilliard equation (see \[7\]) the original manifold of \[15\] should work in our case, too. The main idea in our construction goes as follows:

We start with a stationary solution \( U \) to \( \text{AC} \) on the whole line \( \mathbb{R} \), centered at 0 and connecting the stable phases \( \pm1 \) (Definition 3.1). Using the exponential decay of \( U \) (Proposition 3.2), we introduce a rescaled version in the domain \([0, 1]\) in order to construct an ansatz function \( u^h \), which jumps from \( \pm1 \) to \( \mp1 \) in an \( O(\varepsilon) \)-neighborhood of the zeros \( h_i \) (Definition 3.4).

Throughout our analysis, we fix the number \( N + 1 \) of transitions. The presented results hold up to times, where the distance between two neighboring interfaces gets too small and we thus cannot exclude the possibility of a collapse of two interfaces. This behavior of the stochastic equation was not studied in full detail yet. In the deterministic case, we refer to the nice work by X. Chen, \[16\]. For some ideas and conjectures in the stochastic case, see the thesis of S. Weber \[23\]. Essentially, after an annihilation the number of transitions is reduced to \( N - 1 \) and we can restart our analysis on a lower-dimensional slow manifold.

Let us now define the building block of our manifold, the heteroclinic connection on the whole real line connecting \(-1 \) and \(+1 \).

**Definition 3.1 (The heteroclinic).** Let \( U \) be the unique, increasing solution to
\[
U'' - f(U) = 0, \quad U(0) = 0, \quad \lim_{x \to \pm\infty} U(x) = \pm1.
\]  

In the prototypical case \( f(u) = u^3 - u \), we have the explicit solution \( U(x) = \tanh(x/\sqrt{2}) \).

The function \( U \) is the heteroclinic of the ODE connecting the stable points \(-1 \) and \(+1 \). For a later discussion of the spectrum of the linearized Allen–Cahn operator, we need some relations between the heteroclinic \( U \) and the potential \( F \). We observe that if \( U \) is a solution to (3.1), then
\[
\partial_x \left( U_x^2 - 2F(U) \right) = 2U_x \left( U_{xx} - F'(U) \right) = 0.
\]

From the boundary condition \( U(0) = 0 \), we conclude that solving equation (3.1) is equivalent to solving the first-order ODE
\[
U_x = \sqrt{2F(U)}, \quad U(0) = 0, \quad \lim_{x \to \pm\infty} U(x) = \pm1.
\]
By the assumptions on the potential $F$, we see that $\sqrt{F}$ is $C^1$ and hence, the solution to (3.2) is unique. Moreover, we observe that all derivatives of $U$ can be expressed as a function of $U$. For instance, we have $U'' = F'(U)$, $U'(3) = F''(U)\sqrt{2F(U)}$, and so on. Also note that, due to the symmetry of $F$, the mirrored function $-U$ solves the same differential equation, but transits from $U(-\infty) = +1$ to $U(+\infty) = -1$. For some fine properties of $U$, we refer to the work of Carr and Pego [15], which is based on [14]. Crucial for the construction of a slow manifold (cf. Definition 3.7) is that the heteroclinic $U$ together with its derivatives decay exponentially fast. The following proposition can be shown via phase plane analysis. For a proof we refer to [3].

**Proposition 3.2 (Exponential decay of $U$).** Let $U(x), x \in \mathbb{R}$, be the heteroclinic defined by (3.1). There exist constants $c, C > 0$ such that for $x \geq 0$

$$|1 - U(\pm x)| \leq Ce^{-cx}, \quad |U'(\pm x)| \leq Cc e^{-cx}, \quad \text{and} \quad |U''(\pm x)| \leq Cc^2 e^{-cx}.$$  

For $\xi \in \mathbb{R}$, we define a translated and rescaled version of $U$ by

$$U(x; \xi, \pm 1) := \pm U \left( \frac{x - \xi}{\varepsilon} \right). \quad (3.3)$$

One easily verifies that $U(\cdot; \xi, \pm 1)$ is a solution to the rescaled ODE $\varepsilon^2 U_{xx} - f(U) = 0$, centered at $U(\xi; \xi, \pm 1) = 0$ and going from $\mp 1$ to $\pm 1$. Due to the exponential decay of the heteroclinic, the rescaled profile $U(x; \xi, \pm 1)$ is exponentially close to the states $\pm 1$, if $x$ is at least $O(\varepsilon^{-1})$-away from the zero $\xi$.

**Lemma 3.3.** Let $\kappa > 0$ and $0 < \varepsilon < \varepsilon_0$. Then, uniformly for $|x - \xi| > \varepsilon^{1-\kappa}$

$$|U(x; \xi, \pm 1)| = 1 + O(\exp).$$

Similar exponential estimates hold for the derivatives of $U(\cdot; \xi, \pm 1)$.

Motivated by this lemma, we can construct for interface positions $h \in (0, 1)^{N+1}$ with $h_1 < h_2 < \ldots < h_{N+1}$ profiles $u^h : [0, 1) \to \mathbb{R}$ such that $u^h$ jumps from $\pm 1$ to $\mp 1$ in a small neighborhood around $h_i$ of size $O(\varepsilon)$. Locally around $h_i$, we prescribe

$$u^h(x) \approx U(x; h_i, (-1)^{i+1}).$$

If $x$ is of order $\varepsilon^{1-\kappa}$ away from $h_i$, we assured in Lemma 3.3 that each profile $U(x; h_i, \pm 1)$ is close to $\pm 1$ up to an exponentially small error. See Figure 1. Thus we assume that the distance between two neighboring interfaces and to the boundary is bounded from below by $\varepsilon^{1-\kappa}$ for some small $\kappa > 0$, and up to exponentially small error terms, we define $u^h$ as the sum of profiles given by (3.3). This leads to the following definition.

**Definition 3.4 (The profile $u^h$).** Fixing $\rho_\varepsilon = \varepsilon^\kappa$ for $\kappa > 0$ very small, we define the set $\Omega_{\rho_\varepsilon}$ of admissible interface positions in the interval $(0, 1)$ by

$$\Omega_{\rho_\varepsilon} := \left\{ h \in \mathbb{R}^{N+1} : 0 < h_1 < \ldots < h_{N+1} < 1, \max_{j=0,\ldots,N+1} |h_{j+1} - h_j| > \varepsilon/\rho_\varepsilon \right\},$$

where $h_0 := -h_1$ and $h_{N+2} := 2 - h_{N+1}$. For $x \in (0, 1)$ and $h \in \Omega_{\rho_\varepsilon}$, we define

$$u^h(x) := \sum_{j=1}^{N+1} U(x; h_j, (-1)^{j+1}) + \beta_N(x),$$

where the normalization function $\beta_N(x)$ satisfies $\beta_N(x) = (-1)^{N-1}/2 + O(\exp)$ and similarly for all derivatives. (cf. Remark 3.7).
Figure 1: A sketch of the profile $u^h$ for $N = 8$. The function is close to $\pm 1$ up to sharp transitions around the interface positions.

Note that the positions $h_0$ and $h_{N+2}$ were introduced to bound the distance of the interface positions from the boundary 0 and 1. Moreover, it is straightforward to check that the set $\Omega_{\rho\varepsilon}$ is convex, which is later used to bound the Lipschitz constant of the map $h \mapsto u^h$.

**Remark 3.5.** Let us comment on why we needed to add the normalization term $\beta_N(x)$ in the definition of $u^h$. Due to symmetry, we can assume that the multi-kink profile starts in the phase $u^h(0) = -1$. Depending on the parity of the number of transitions, we have to add a constant to assure this. As $h_j > \varepsilon/\rho\varepsilon$, we obtain by Lemma 3.3 in $x = 0$

\[ \sum_{j=1}^{N+1} U(0; h_j, (-1)^{j+1}) = \sum_{j=1}^{N+1} (-1)^j + O(\exp) = -1 + \frac{1 - (-1)^N}{2} + O(\exp). \]

Therefore, we have to add the correction $\frac{1}{2}((-1)^N - 1) + O(\exp)$ to obtain $u^h(0) = -1$.

Moreover, we need to assure that $u^h$ satisfies Neumann boundary conditions. By Lemma 3.3, the derivative of $U(x; h_j, \pm 1)$ is exponentially small at $x \in \{0, 1\}$. Hence, in order to correct the boundary condition, we additionally have to add a function of order $O(\exp)$.

Before we finally define the slow manifolds for the (mass conserving) Allen–Cahn equation (Definition 3.7), we collect some properties of the multi-kink configurations $u^h$.

**Proposition 3.6 (Properties of $u^h$).** The function $u^h$ is an almost stationary solution to (AC) in the sense that it satisfies the equation only up to an exponentially small error, that is,

\[ \varepsilon^2 u_{xx}^h - f(u^h) = O(\exp), \quad u^h_x(0) = 0, \quad u^h_x(1) = 0. \]  

(3.4)

For $i,j \in \{1, \ldots, N + 1\}$, we denote the partial derivatives of $u^h$ with respect to the $h$–variables by $u_i^h = \partial_{h_i} u^h$, $u_{ij}^h = \partial_{h_i} \partial_{h_j} u^h$, and third derivatives accordingly. We have

\[ u_i^h(x) = U'(x; h_i, (-1)^{i+1}) + O(\exp) = (-1)^i \frac{1}{\varepsilon} U' \left( \frac{x - h_i}{\varepsilon} \right) + O(\exp). \]  

(3.5)

Furthermore, the following estimates hold true in $L^2(0,1)$:

\[ \langle u_i^h, u_j^h \rangle = X \varepsilon^{-1} \delta_{ij} + O(\exp), \quad \| u_i^h \| = O(\varepsilon^{-3/2}) \delta_{ij} + O(\exp), \]

\[ \langle u_{kk}^h, u_k^h \rangle = O(\exp), \quad \text{and} \quad \| u_{kkk}^h \| = O(\varepsilon^{-5/2}), \]

where $X := \int_\mathbb{R} U'(y)^2 dy$. In $L^\infty(0,1)$, we have

\[ \| u^h \| = O(1) \quad \text{and} \quad \| u_i^h \| = O(\varepsilon^{-1}). \]

Moreover all higher derivatives of $u^h$ are uniformly exponentially small if the variables are mixed.
Proof. Equations (3.4) and (3.5) follow directly from Definition 3.4 and Lemma 3.3. By Lemma 3.3, we also see that $U'(x; h_i, (-1)^{i+1})$ is exponentially small for $|x - h_i| > \varepsilon/p_\mu$ and thus we obtain $\langle u^h_k, u^h_j \rangle = O(\exp)$ for $i \neq j$. Moreover, the same argument implies that higher derivatives with respect to different positions $h_i$ and $h_j$ are exponentially small. The uniform bounds are a direct consequence from the definition of $u^h$ and the $L^2$-norm of $u^h_k$ is given by

\[
\|u^h_k\|^2 = \varepsilon^{-2} \int_0^1 U' \left( \frac{x - h_k}{\varepsilon} \right)^2 dx + O(\exp)
\]

In the last step, we used that $h \in \Omega_{p_\mu}$ and $|U(x)| \leq ce^{-\varepsilon|x|}$ by Proposition 3.2. Thus, we obtain

\[
\int_{-\infty}^{-h_k/\varepsilon} U'(y)^2 dy \leq \int_{-\infty}^{-1/p_\mu} U'(y)^2 dy \leq c \int_{-\infty}^{1/p_\mu} e^{-c|y|} dy = O(\exp),
\]

and with the same argument the integral at $\infty$ is exponentially small as well. Analogously, the $n$-th derivative with respect to $h_k$ is then given by

\[
\|\partial^n_{h_k} u^h\|^2 = \varepsilon^{-2n+1} \int_R U^{(n)}(y)^2 dy + O(\exp).
\]

The mixed term can be estimated as follows:

\[
\langle u^h_k, u^h_k \rangle = \varepsilon^{-3} \int_0^1 U'' \left( \frac{x - h_k}{\varepsilon} \right) U' \left( \frac{x - h_k}{\varepsilon} \right) dx + O(\exp)
\]

\[
= \frac{1}{2} \varepsilon^{-2} \left[ U' \left( \frac{1 - h_k}{\varepsilon} \right)^2 - U' \left( \frac{-h_k}{\varepsilon} \right)^2 \right] + O(\exp) = O(\exp). \quad \Box
\]

We finally introduce the approximate slow manifolds for the stochastic (mass conserving) Allen–Cahn equation. The second manifold will play an important role in the study of the mass conserving Allen–Cahn equation (mAC), while the first one will be used for the analysis of (AC) without this constraint.

**Definition 3.7 (Slow manifolds).** For $\Omega_{p_\mu}$ and $u^h$ given by Definition 3.4, we define the approximate slow manifold by

\[
\mathcal{M} := \left\{ u^h : h \in \Omega_{p_\mu} \right\}.
\]

Fixing a mass $\mu \in (-1, 1)$, we define the mass conserving approximate manifold by

\[
\mathcal{M}_\mu := \left\{ u^h \in \mathcal{M} : \int_0^1 u^h(x) dx = \mu \right\}.
\]

We have a global chart for $\mathcal{M}$. Later in Lemma 6.3, we will see that this also holds true for $\mathcal{M}_\mu$, as it is the manifold $\mathcal{M}$ intersected by a vector space of codimension 1.

We have to compute the tangent vectors for $\mathcal{M}$ and $\mathcal{M}_\mu$, since we need them later in Definition 5.1 to define a coordinate system around the slow manifolds. We immediately see that the tangent space of the slow manifold $\mathcal{M}$ at $u^h$ with $h \in \Omega_{p_\mu}$ is given by

\[
\mathcal{T}_{u^h} \mathcal{M} = \text{span} \left\{ u^h_i : i = 1, \ldots, N + 1 \right\}.
\]
Lemma 4.1. Let $\lambda$ be a constant such that $L$ has full rank $N$. Consider the singular Sturm–Liouville problem 

$$
L u = -u'' + f'(U)u = \lambda u
$$

in $L^2(\mathbb{R})$, where $U$ is the heteroclinic solution defined by (3.1). Note that the ODE (3.1) directly implies that $U'$ is an eigenfunction of $L$ corresponding to the eigenvalue zero. As $U' > 0$, we also know that zero must be the largest eigenvalue. The following description of the spectral behavior of $L$ orthogonal to $U'$ is taken from [19], Proposition 3.2.

**Lemma 4.1 (Spectral gap of the Allen–Cahn operator, [19], Proposition 3.2).** There exists a constant $\lambda_0 > 0$ such that if $v \in H^1(\mathbb{R})$ satisfies

(i) $v(0) = 0$ or (ii) $\int_{\mathbb{R}} v(s)U'(s) \, ds = 0$,

then it holds true that

$$
\langle Lv, v \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left[ -v'(s)^2 - f'(U(s))v(s)^2 \right] \, ds \leq -\lambda_0 \|v\|_{L^2(\mathbb{R})}^2.
$$
For the simplicity of some arguments, we focus in the remainder on the classical cubic potential \( f(u) = u^3 - u \). In this case, one can show that \( \lambda_0 = 3/2 \) and \( U \sqrt{U^T} \) serves as corresponding eigenfunction of (4.1) (cf. \[13\]). The eigenfunction \( U \sqrt{U^T} \) has exactly one zero and hence it corresponds to the second largest eigenvalue. As \( \lim_{x \to -\infty} (1 - 3U^2) = -2, \) we also know by a standard argument for Schrödinger operators that the essential spectrum lies in the interval \((-\infty, -2]\). For more details on the spectrum of Schrödinger operators, we refer to \[13\]. The standard arguments for Sturm–Liouville problems can be found in \[22\].

With the spectral gap of Lemma 4.1 at hand, we consider the linearization of the Allen–Cahn operator at a multi-kink state \( u^h \in \mathcal{M} \). The following theorem gives a bound on the quadratic form orthogonal to the tangent space \( T_{h} \mathcal{M} \). Essentially, up to exponentially small terms, the support of the tangent vectors \( u^h \) is concentrated in a small neighborhood of width \( \varepsilon \) around the zero \( h_i \). Hence, it is sufficient to study the quadratic form locally around each \( h_i \). After rescaling, we essentially arrive at the setting of Theorem 4.1 and the spectral gap of order 1 is transferred to our problem.

**Theorem 4.2** (Spectral gap for \( (\mathcal{M}) \)). Let \( u^h \in \mathcal{M} \) and \( v \perp u^h \) for any \( i = 1, \ldots, N + 1 \). Then, for \( \lambda_0 \) given in Lemma 4.1, we have

\[
\langle \mathcal{L}^h v, v \rangle \leq \left( -\frac{1}{2} \lambda_0 + \mathcal{O}(\rho^2) \right) \| v \|^2.
\]

**Proof.** Since the minimal distance between the interfaces \( h_i \) is bounded from below by \( \varepsilon/\rho \) and the heteroclinic solution \( U \) goes exponentially fast to \( \pm 1 \) by Proposition \[22\], we find \( 0 < \delta \leq \varepsilon/\rho \) such that \( u^h = \pm 1 + \mathcal{O}(\exp) \) on \( \mathcal{R} := [0, 1] \setminus \bigcup B_{\delta_i}(h_i) \). On the set \( \mathcal{R} \) we have

\[
\langle \mathcal{L}^h v, v \rangle_{L^2(\mathcal{R})} = -\varepsilon^2 \int_\mathcal{R} v_x^2 - \int_\mathcal{R} f'(\pm 1)v^2 + \mathcal{O}(\exp)||v||^2_{L^2(\mathcal{R})} \leq \left( -f'(\pm 1) + \mathcal{O}(\exp) \right) \| v \|^2_{L^2(\mathcal{R})},
\]

which is strictly negative as \( f'(\pm 1) > 0 \).

It remains to control the quadratic form on each \( B_{\delta_i}(h_i) \) and, without loss of generality, we may shift it to \( h_i = 0 \). Note that \( u^h(x) = U(\varepsilon x/\rho) + \mathcal{O}(\exp) \) on the set \( B_{\delta_i}(h_i) \) by Proposition \[22\]. Defining \( \tilde{v}(x) := v(\varepsilon x, y) \), one easily computes for \( h_i = 0 \)

\[
\langle \mathcal{L}^h v, v \rangle_{L^2(B_{\delta_i})} = \varepsilon \langle \mathcal{L} \tilde{v}, \tilde{v} \rangle_{L^2(B_{\delta_i/\rho})} + \mathcal{O}(\exp)||v||^2_{L^2(B_{\delta_i})}.
\]

Here, \( L \) denotes the singular Sturm–Liouville operator defined by (4.1). After rescaling, we essentially have to bound the quadratic form \( \langle \mathcal{L} \tilde{v}, \tilde{v} \rangle \) on the interval \((-\delta_i/\varepsilon, \delta_i/\varepsilon) =: D_\varepsilon \), which is a set of length of order \( \mathcal{O}(\varepsilon^{-\kappa}) \). To compare with the spectrum on the whole line, we define a cut-off function \( \phi \in C^\infty_c(D_\varepsilon) \) such that \( 0 \leq \phi \leq 1 \) and \( \phi \equiv 1 \) on the set \( \{ f'(U) < C \} \) for some \( \lambda_0 < C < \sup_{-1< \varepsilon<1} f'(x) \). As \( |D_\varepsilon| = \mathcal{O}(\rho^{-1}) \), we can also assume that uniformly \( |\phi_x| \leq C \rho \) and \( |\phi_{xx}| \leq C \rho^2 \). We obtain

\[
\langle \mathcal{L} \tilde{v}, \tilde{v} \rangle_{L^2(D_\varepsilon)} = -\int_{D_\varepsilon} \tilde{v}_x^2 \phi^2 - \int_{D_\varepsilon} (1 - \phi^2)\tilde{v}_x^2 - \int_{D_\varepsilon} \phi^2 f'(U)\tilde{v}^2 - \int_{D_\varepsilon} (1 - \phi^2) f'(U)\tilde{v}^2
\]

\[
\leq -\int_{D_\varepsilon} \tilde{v}_x^2 \phi^2 - \int_{D_\varepsilon} \phi^2 f'(U)\tilde{v}^2 - C \int_{D_\varepsilon} (1 - \phi^2)\tilde{v}^2
\]

\[
= \int_{D_\varepsilon} -((\phi \tilde{v})_x)^2 + f'(U)(\phi \tilde{v})^2 + 2\int_{D_\varepsilon} \phi \tilde{v}_x \phi \tilde{v} + \int_{D_\varepsilon} \tilde{v}_x^2 \phi^2 - C \int_{D_\varepsilon} (1 - \phi^2)\tilde{v}^2
\]

\[
= \langle \mathcal{L} \phi \tilde{v}, \phi \tilde{v} \rangle_{L^2(\mathcal{R})} - \int_{D_\varepsilon} \tilde{v}^2 (\phi \tilde{v}_x) + \int_{D_\varepsilon} \tilde{v}_x^2 \phi^2 - C \int_{D_\varepsilon} (1 - \phi^2)\tilde{v}^2
\]

\[
\leq \frac{1}{2} \lambda_0 \int_{D_\varepsilon} \phi^2 \tilde{v}^2 - \int_{D_\varepsilon} \tilde{v}^2 (\phi \tilde{v}_x) + \int_{D_\varepsilon} \tilde{v}_x^2 \phi^2 - C \int_{D_\varepsilon} (1 - \phi^2)\tilde{v}^2 + \mathcal{O}(\exp).
\]
Motivated by \( L \) we observe that it is sufficient to consider the same operator equation, but restricted to space \( H \). Then, Theorem 4.3

\[
\langle L\phi\tilde{v},\phi\tilde{v}\rangle = -\int_{D_\epsilon} ((\phi\tilde{v})_x)^2 + \int_{D_\epsilon} f'(U)(\phi\tilde{v})^2 \leq -\frac{1}{2} \lambda_0 \int_{D_\epsilon} \phi^2 \tilde{v}^2 + O(\exp).
\]

Now, we observe that

\[
\left\| \tilde{v} \right\|^2_{L^2(D_\epsilon)} = \int_{D_\epsilon} v(\varepsilon x)^2 \, dx = \varepsilon^{-1} \int_{B_{\varepsilon}} v(y)^2 \, dy = \varepsilon^{-1} \left\| v \right\|^2_{L^2(B_{\varepsilon})},
\]

and therefore, combining (4.2) and (4.3) yields

\[
\langle L^h v, v \rangle_{L^2(B_{\varepsilon})} \leq \left( -\frac{1}{2} \lambda_0 + O(\rho_\varepsilon^2) \right) \left\| v \right\|^2_{L^2(B_{\varepsilon})} + O(\exp) \left\| v \right\|^2_{L^2(B_{\varepsilon})}.
\]

As a next step, we analyze the spectral gap in the mass conserving case. For this purpose, we denote by \( P \) be the projection of \( L^2 \) onto the linear subspace \( L^2_0 = \{ f \in L^2 : \int_0^1 f(x) \, dx = 0 \} \). Motivated by

\[
\langle P L^h P v, v \rangle_{L^2} = \langle L^h v, v \rangle_{L^2} = \langle L^h v, v \rangle_{L^2_0}, \quad \text{for } v \in L^2_0,
\]

we observe that it is sufficient to consider the same operator \( L^h \) as for the classical Allen–Cahn equation, but restricted to \( L^2_0 \), the linear subspace of \( L^2 \) containing functions with mean zero. This constraint leads to a subspace of codimension 1,

\[
L^2_0 = \{ v \in L^2 : \langle v, 1 \rangle = 0 \} = 1^\perp = PL^2,
\]

and therefore, we need to control the quadratic form on this subspace. First, we will formulate the problem in general and only after that consider the special case for the mass conserving Allen–Cahn equation. The following theorem deals with establishing a spectral gap on a subspace of codimension 1. The following simple argument shows that, under a suitable angle condition, the Rayleigh quotient can be bounded from above. This yields a bound on the spectral gap.

**Theorem 4.3** (Spectral gap on subspaces). Consider a self-adjoint operator \( L \) on a Hilbert space \( \mathcal{H} \) with an orthonormal basis of eigenfunctions \( L f_k = \lambda_k f_k \) and assume that

\[
\delta \geq \lambda_1, \ldots, \lambda_{N+1} \geq -\delta > -\lambda \geq \lambda_{N+2} \geq \ldots \quad (4.4)
\]

for some \( 0 < \delta < \lambda \). For \( u \in \mathcal{H} \), we define

\[
u^\perp := \left\{ f \in \mathcal{H} : \langle f, u \rangle = 0 \right\} \quad \text{and} \quad F_u := \frac{1}{\langle f_{N+1}, u \rangle} \sum_{i=1}^N \langle f_i, u \rangle f_i + f_{N+1}.
\]

Then,

i) there exists an \( N \)-dimensional subspace \( \mathcal{U} \) of \( \nu^\perp \) such that

\[
\left| \langle L^h, h \rangle \right| \leq \delta \| h \|^2 \quad \forall h \in \mathcal{U}.
\]
ii) the condition \( |\cos \frac{\phi}{2} (F_u, u)| \geq \sqrt{\delta/\lambda} \) implies that for \( h \perp u, f_1, \ldots, f_{N+1} \)

\[
\frac{\langle Lh, h \rangle}{\|h\|^2} \leq \frac{\delta - \lambda \cos^2 \frac{\phi}{2} (F_u, u)}{\cos^2 \frac{\phi}{2} (F_u, u) + 1}.
\]

Proof. First, we construct an \( N \)-dimensional subspace corresponding to the small eigenvalues in the interval \([-\delta, \delta] \). For \( i = 1, \ldots, N \) define

\[
g_i := f_i + c_i f_{N+1} \quad \text{with} \quad c_i := -\frac{\langle f_i, u \rangle}{\langle f_{N+1}, u \rangle}.
\]

Obviously, we have \( g_1, \ldots, g_N \in \text{span}\{f_1, \ldots, f_{N+1}\} \) and \( g_1, \ldots, g_N \perp u \) by the definition of the constant \( c_i \). It is also straightforward to check that the functions \( g_i \) span an \( N \)-dimensional space. This yields directly

\[
-\delta \|h\|^2 \leq \langle Lh, h \rangle \leq \delta \|h\|^2 \quad \text{for} \quad h \in \text{span}\{g_1, \ldots, g_N\} =: U.
\]

Define \( V := \text{span}\{g_1, \ldots, g_N\}^\perp \cap u^\perp = \text{span}\{u, g_1, \ldots, g_N\}^\perp \). For \( h \in V \) we can then write

\[
h = \sum_{i=1}^{N+1} \alpha_i f_i + r, \quad \text{with} \quad r \perp f_i \quad \forall i = 1, \ldots, N+1.
\]

We have \( r, h \perp g_j \) for any \( j = 1, \ldots, N \) and thereby

\[
\sum_{i=1}^{N+1} \alpha_i \langle f_i, g_j \rangle = 0. \tag{4.7}
\]

With \( \ref{eq:4.5} \) and \( f_i \perp f_j \) for \( i \neq j \), we easily compute that

\[
\langle f_i, g_j \rangle_{i,j} = \begin{pmatrix}
1 & 0 & \cdots & 0 & \alpha_1 \\
0 & \ddots & \vdots & 0 & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & \alpha_N
\end{pmatrix} \in \mathbb{R}^{N \times (N+1)}.
\]

The kernel of this matrix is one-dimensional and spanned by a vector \( \beta \in \mathbb{R}^{N+1} \) with \( \beta_i = -\alpha_i \) for \( 1 \leq i \leq N \) and \( \beta_{N+1} = 1 \). By \( \ref{eq:4.7} \), \( \alpha \) lies in the kernel and we can rewrite \( \ref{eq:4.6} \) as

\[
h = \gamma \sum_{i=1}^{N+1} \beta_i f_i + r = \gamma F_u + r, \quad \gamma \in \mathbb{R}.
\]

Since \( h \in V \subset u^\perp \), we have \( 0 = \langle h, u \rangle = \gamma \langle F_u, u \rangle + \langle r, u \rangle \). This implies immediately that

\[
\gamma^2 = \frac{\langle r, u \rangle^2}{\langle F_u, u \rangle^2} \leq \frac{\|r\|^2 \|u\|^2}{\langle F_u, u \rangle^2}.
\]

Thus, we compute

\[
\frac{\langle Lh, h \rangle}{\|h\|^2} = \frac{\sum_{j=1}^{N+1} \alpha_j^2 \langle Lf_j, f_j \rangle + \langle Lr, r \rangle}{\gamma^2 \|F_u\|^2 + \|r\|^2} \leq \frac{\delta \sum \alpha_i^2 - \lambda \|r\|^2}{\gamma^2 \|F_u\|^2 + \|r\|^2}
\]

\[
\leq \frac{\delta \gamma^2 \sum \beta_i^2 - \lambda \|r\|^2}{\gamma^2 \|F_u\|^2 + \|r\|^2} = \frac{\delta \gamma^2 \|F_u\|^2 - \lambda \|r\|^2}{\gamma^2 \|F_u\|^2 + \|r\|^2} \leq \left( \frac{\delta \|u\|^2 \|F_u\|^2}{\gamma^2 \|F_u\|^2 + \|r\|^2} - \lambda \right) \|r\|^2. \tag{4.8}
\]
At this point, we need the angle condition

$$\cos \beta(F_u, u) = \frac{\langle F_u, u \rangle}{\|F_u\|\|u\|} \geq \frac{\sqrt{\delta}}{\lambda}$$

to guarantee that the numerator is negative. Under this assumption, we can continue estimating (4.8) and derive

$$\langle L^\epsilon h, h \rangle \leq \delta \frac{\|u\|^2\|F_u\|^2 - \lambda}{\|u\|^2\|F_u\|^2 + 1} = \delta - \lambda \cos^2 \beta(F_u, u) \frac{\sqrt{\delta}}{1 + \cos^2 \beta(F_u, u)}.$$

Finally, we can apply Theorem 4.3 to analyze the spectrum of the linearized mass conserving Allen–Cahn operator. Recall that it is crucial to have a good negative upper bound of the quadratic form orthogonal to the tangent space. We show that in this case the spectral gap is of order $\varepsilon$. This is quite different to the spectral gap for the Allen–Cahn equation (AC) without the mass constraint, which by Theorem 4.2 is of order 1.

**Theorem 4.4** (Spectral gap for (mAC)). Let $v \in L^2_0(0,1)$ with $v \perp u_i^h$ for $i = 1, \ldots, N$. Then, we have

$$\langle L^\epsilon v, v \rangle \leq \left( - \lambda_0 \varepsilon + O(\exp) \right) \|v\|^2,$$

where $\lambda_0$ is the same constant as in Theorem 4.2.

**Proof.** In the notation of Theorem 4.3 we take $u = 1 \in L^2(0,1)$ such that $L^2_0 = \text{span}\{u\}$, and $f_i = u_i^h$. Furthermore, we compute

$$\langle f_i, 1 \rangle = \int_0^1 u_i^h(x) \, dx = O(\exp) + \int_0^1 U'(x; h_i, (-1)^{i+1}) \, dx = 2(-1)^i + O(\exp).$$

With $F_u$ defined as before in Theorem 4.3 this yields

$$\langle F_u, u \rangle = \frac{1}{\langle f_{N+1}, 1 \rangle} \sum_{j=1}^{N+1} \langle f_j, 1 \rangle^2 = 2(N + 1)(-1)^{N+1} + O(\exp).$$

We have $\|f_i\| = O(\varepsilon^{-1/2})$ by Proposition 3.6 and thus $\|F_u\| = (N + 1) \cdot O(\varepsilon^{-1/2})$. Combined we obtain

$$\cos \beta(F_u, u) = O(\varepsilon^{1/2}).$$

By Proposition 3.6 we have $L(u^h) = O(\exp)$ and thus differentiating with respect to $h_i$ leads to $L^h u_i^h = O(\exp)$. Hence, the first $N + 1$ eigenvalues are exponentially small. This shows that we can choose $\delta = O(\exp)$. Plugging this observation into Theorem 4.3 yields

$$\frac{\langle L^\epsilon v, v \rangle}{\|v\|^2} \leq \frac{\delta - \lambda_0 \varepsilon}{1 + \varepsilon} \leq -\lambda_0 \varepsilon + O(\exp).$$

For the classical Allen–Cahn equation we established a spectral gap of order 1, whereas due to mass conservation the gap shrinks to $O(\varepsilon)$ for (mAC). As we will see later in Theorems 6.9 and 6.10 this heavily influences the maximal radius and noise strength that we can treat in our stability analysis.
5 Analysis of the stochastic ODE along the slow manifold

In this section, we give the stochastic ODEs governing the motion of the kinks for both cases. We show that for the non-massconserving Allen–Cahn equation the \(N+1\) interfaces move—up to the time scale where a collision is likely to occur—independently according to Brownian motions projected onto the slow manifold. This is quite different to the mass-conserving case where (as one would expect) the dynamics is coupled through the mass constraint.

Before we analyze the stochastic ODEs for the interface motion, we have to introduce a new coordinate frame, in which we derive the differential equations for the shape variable \(h\) and the normal component \(v\). Due to Theorems 4.2 and 4.4, we established good control of the quadratic form orthogonal to the tangent space \(T_h \mathcal{M}\), or \(T_{\xi} \mathcal{M}_{\mu}\), respectively. Therefore, it is fruitful to split the solution to the Allen–Cahn equation into a component on the slow manifold and the orthogonal direction. This leads to the following definition of the Fermi coordinates.

**Definition 5.1** (Fermi coordinates). Let \(u(t)\) be the solution to (AC). For a fixed time \(t > 0\), we define the pair of coordinates \((h(t), v(t)) \in \Omega_{\rho \varepsilon} \times L^2(0, 1)\) such that

\[
    u(t) = u^h(t) + v(t), \quad v(t) \perp T_{h(t)} \mathcal{M},
\]

as Fermi coordinates of \(u(t)\).

In case of the mass conserving equation (mAC), the definition works analogously. One only has to replace the set of admissible interface positions \(\Omega_{\rho \varepsilon}\) by the set \(A_{\rho} \varepsilon\) (given by (3.6)) and the slow manifold \(\mathcal{M}\) by its mass conserving counterpart \(\mathcal{M}_{\mu}\).

Unless we are close to the boundary of the slow manifold, for the initial condition \(u(0)\) we always find Fermi coordinates by considering the point of smallest distance. Also note that we do not assume that the Fermi coordinates are uniquely determined, or that the map \(u \mapsto u^h\) is a well defined projection. Later in Lemma 5.5 and Remark 5.6 we show that sufficiently close to the slow manifold the Fermi coordinates are at least always defined. We find one possible choice being as smooth as \(u\) in time.

For now, we first assume that the coordinate system is well-defined in order to derive an equation governing the motion of the kink positions \(h\). Under the assumption that \(h\) performs a diffusion process given by

\[
    dh = b(h, v) dt + \langle \sigma(h, v), dW \rangle,
\]

one can compute the drift \(b\) and diffusion \(\sigma\) explicitly by applying Itô formula to the orthogonality condition of Definition 5.1. The computation is straightforward, but quite lengthy. For details see [7] or [20]. The diffusion term \(\sigma\) is given by

\[
    \sigma_r(h, v) = \sum_i A^{-1}_{ri} u^h_i,
\]

and for the drift \(b\) we obtain

\[
    b_r(h, v) = \sum_i A^{-1}_{ri} \langle u^h_i, L(u^h + v) \rangle + \sum_i A^{-1}_{ri} \sum_j \langle u^h_j, Q_{\sigma j} \rangle + \sum_{i,j,k} A^{-1}_{ri} \left[ \frac{1}{2} \langle u^h_{ij}, v \rangle - \langle u^h_{ij}, u^h_k \rangle - \frac{1}{2} \langle u^h_i, u^h_{jk} \rangle \right] \langle Q_{\sigma j}, \sigma_k \rangle.
\]

Note that, for the sake of simplicity, we expressed everything with respect to the coordinate \(h\), although we introduced the coordinate \(\xi\) for the mass conserving equation. An essential point in the computation and analysis of the SDE is the invertibility of the matrix \(A\) given by

\[
    A_{kj}(h, v) = \langle u^h_k, u^h_j \rangle - \langle u^h_{kj}, v \rangle
\]

which we will discuss in the next section.
Remark 5.2. Given a solution $u$, we can replace $v$ in (5.1) by $u - u^h$, and obtain an equation on the slow manifold for the position of the interfaces $h$. Note that this equation is not an approximation.

Moreover, by exactly reverting the calculation that leads to (5.1) one can verify the following result. See [13, 20].

Proposition 5.3. Given a solution $u$ of (AC) or (mAC) let $h$ be a solution of (5.1) with $v$ replaced by $u - u^h$, i.e.,

$$\frac{dh}{dt} = b(h, u - u^h) dt + \langle \sigma(h, u - u^h), dW \rangle,$$

then $(h, u - u^h)$ are Fermi coordinates for $u$.

5.1 Analysis of the stochastic ODE for (AC)

We start with the non-massconserving case. Here, we will see that $A$ and its inverse are diagonal matrices up to terms being small in $\|v\|$.

Lemma 5.4. For $h \in \Omega_{pe}$ consider the matrix $A \in \mathbb{R}^{(N+1)\times(N+1)}$ defined by

$$A_{kj} := \langle u^h_k, u^h_j \rangle - \langle u^h_k, v \rangle.$$

We obtain

$$A_{kj} = \varepsilon^{-1} \left[ X + O(\varepsilon^{-1/2})\|v\| \right] \delta_{kj} + O(\exp)\|v\|. $$

Moreover, as long as $\|v\| < C\varepsilon^{1/2+m}$ for some $m > 0$, the inverse $A^{-1}$ is given by

$$A^{-1} = \varepsilon \left[ X^{-1} + O(\varepsilon^m) \right] I_{N+1} + O(\exp),$$

where $X$ is the constant given in Proposition 5.3.

Proof. We obtain $\langle u^h_k, u^h_j \rangle = X\varepsilon^{-1}\delta_{kj} + O(\exp)$ and $\|u^h_k\| = O(\varepsilon^{-3/2})\delta_{kj} + O(\exp)$ by Proposition 5.3. Thus, the bound on $A_{kj}$ follows directly by applying the Cauchy–Schwarz inequality. Using geometric series, this yields for $\|v\|$ sufficiently small

$$A^{-1} = \left[ X\varepsilon^{-1} + O(\varepsilon^{-3/2})\|v\| \right]^{-1} I_{N+1} + O(\exp)$$

$$= \varepsilon^{-1} \left[ 1 + O(\varepsilon^{-1/2})\|v\| \right]^{-1} I_{N+1} + O(\exp)$$

$$= \left[ X^{-1}\varepsilon + O(\varepsilon^{1+m}) \right] I_{N+1} + O(\exp).$$

Before we continue analyzing the stochastic ODE, let us first show that the coordinate frame around $\mathcal{M}$ given by Definition 5.1 is well-defined. We prove that, as long as the matrix $A$ is invertible, i.e., $\|v\| < C\varepsilon^{1/2+m}$, and the nonlinearity is bounded, i.e., $v \in L^4$, the coefficients $b$ and $\sigma$ defined by (5.3) and (5.2) are Lipschitz continuous with respect to $h$. Note that we will only compute the Lipschitz constant for $\sigma$ explicitly, as we need it for the analysis of the stochastic ODE.

Lemma 5.5 (Lipschitz continuity of $b$ and $\sigma$). Let $h, \bar{h} \in \Omega_{pe}$ and $v \in L^4(0, 1)$ satisfying $\|v\| < C\varepsilon^{1/2+m}$ for some $m > 0$. Then, there exist constants $C > 0$ and $C_\varepsilon > 0$ (depending on $\varepsilon$ and $\|v\|_{L^4}$) such that

$$\|\sigma(h, v) - \sigma(\bar{h}, v)\| \leq C\varepsilon^{-1/2}\|h - \bar{h}\| \quad \text{and} \quad \|b(h, v) - b(\bar{h}, v)\| \leq C_\varepsilon|h - \bar{h}|.$$  (5.5)
Proof. Note that in the following computation the pair \((h, v)\) does not denote the Fermi coordinate defined in Definition 5.1 and therefore, \(v\) does not depend on \(h\). We start with estimating the derivative of the inverse \(A^{-1}(h, v)\). By construction of \(u^h\), the matrix \(A(h, v)\) is smooth in \(h\) and we compute
\[
\partial_h A_{ij} = \frac{\partial \langle u^h_i, u^h_j \rangle - \langle u^h_i, v \rangle}{\partial h_k} = \langle u^h_{ik}, u^h_j \rangle + \langle u^h_i, u^h_{jk} \rangle - \langle u^h_{ij}, v \rangle,
\]
which by Proposition 3.6 is exponentially small unless \(i = j = k\). In the latter case, we have
\[
\partial_h A_{kk} = 2\langle u^h_{kk}, u^h_k \rangle - \langle u^h_{kkk}, v \rangle = O(\exp) + O(\varepsilon^{-5/2}\|v\|).
\]
By virtue of \(D_h A^{-1} = -A^{-1}(D_h A)A^{-1}\) and \(A^{-1} = O(\varepsilon)\) (cf. Lemma 5.4), this yields
\[
D_h A^{-1} = O(\varepsilon^{-1/2}\|v\|) = O(\varepsilon^m).
\]
Recall that \(\sigma(h, v) = A^{-1} \cdot \partial_h u^h\). Differentiating with respect to \(h\) yields
\[
D_h \sigma = D_h A^{-1} \partial_h u^h + A^{-1} \partial_h^2 u^h
\]
and thus, by the previous bound on \(D_h A^{-1}\) and Proposition 3.6 \(\|D_h \sigma(h, v)\| = O(\varepsilon^{-1/2})\).
Since the set \(\Omega_{\rho_i}\) of admissible interface positions is convex, we have
\[
\sigma(h, v) - \sigma(h, v) = \int_0^1 D_h \sigma(h + s(h - h), v) \, ds
\]
and with that we easily obtain (5.5).

In order to derive the Lipschitz continuity of \(b\), one can analogously verify that \(b(h, v)\) is differentiable with respect to \(h\) and the derivative is bounded. Note that only here we need the condition \(v \in L^4\) to control the nonlinearity \(\langle N^h(v), v \rangle\) appearing in the definition (5.3) of \(b\). The careful analysis of the Lipschitz constant can be carried out after some lengthy calculation. We omit the details here.

\[\text{Remark 5.6.}\] We can use the Lipschitz continuity of the coefficients to show that for a solution \(u\) to (AC) the Fermi coordinates given by Definition 5.1 are locally well defined. Since the multi-kink profiles \(u^h\) define smooth functions in \(h\), we see that by Lemma 5.3 the map \(h \mapsto b(h, u(t) - u^h)\) and \(h \mapsto \sigma(h, u(t) - u^h)\) are locally Lipschitz continuous in \(h\). Thus, as long as \(h(t)\) lies in \(\Omega_{\rho_i}\) and \(u - u^h\) is sufficiently small (see Lemma 5.2) a unique (local) solution \(h(t)\) to (5.1) with \(v\) replaced by \(u - u^h\) exists. The pair \((h, v)\) satisfies the Definition 5.1 of the Fermi coordinates. See proposition 5.3.

As the matrix \(A\) and its inverse are (up to exponentially small terms) diagonal matrices, we can show that the stochastic ODE in the non-massconserving case essentially decouples fully. We split equation (5.1) into its deterministic part and a remainder \(\mathcal{A}\), where we collect all terms depending on stochastics, i.e., we write
\[
dh = \sum_i A^{-1}_{ii} \langle u^h_i, \mathcal{L}(u^h + v) \rangle \, dt + d\mathcal{A},
\]
where by (5.2) and (5.3)
\[
\sum_j A_{kj} d\mathcal{A}^{(j)} = \sum_j \langle u^h_{kj}, Q\sigma_j \rangle \, dt + \sum_{i,j} \left[ \frac{1}{2} \langle u^h_{ik}, v \rangle - \langle u^h_{kj}, u^h_i \rangle - \frac{1}{2} \langle u^h_k, u^h_{ij} \rangle \right] (Q\sigma_i, \sigma_j) \, dt
\]
where \(\mathcal{A}\) and \(\eta_{\varepsilon}\).
Lemma 5.7. As long as \( \|v(t)\| < \varepsilon^{1/2+m} \) for some \( m > 0 \), we have

\[
dA^{(k)} = A_{kk}^{-2} \langle u_{kk}^h, Qu_k^h \rangle dt + A_{kk}^{-1} \langle u_k^h, dW \rangle + \mathcal{O}(\varepsilon^m \eta_k) dt + \langle \mathcal{O}_L^2(\varepsilon^{1/2+m}), dW \rangle.
\]

Moreover, the dominating term can be estimated by

\[
A_{kk}^{-2} \langle u_{kk}^h, Qu_k^h \rangle dt + A_{kk}^{-1} \langle u_k^h, dW \rangle = \mathcal{O}(\eta_k) dt + \langle \mathcal{O}_L^2(\varepsilon^{1/2}), dW \rangle.
\]

Proof. Lemma 5.4 and (5.2) imply directly that

\[
\sigma_r(h, v) = [X^{-1} \varepsilon + \mathcal{O}(\varepsilon^{1+m})] u_r^h + \mathcal{O}(\exp).
\]

With \( \|u_r^h\| = \mathcal{O}(\varepsilon^{-1/2}) \) (cf. Proposition 3.6), this yields \( \|\sigma_r\| = \mathcal{O}(\varepsilon^{1/2}) \). The Cauchy–Schwarz inequality implies for the remaining terms of (5.7)

\[
|\langle u_{ij}^h, Q\sigma_j \rangle| \leq \|u_{ij}^h\| \|Q\| \|\sigma_j\| \leq C \varepsilon^{-3/2} \eta_i \varepsilon^{1/2} = C \varepsilon^{-1} \eta_i
\]

and

\[
|\langle u_{ij}^h, v \rangle \langle Q\sigma_i, \sigma_j \rangle| \leq C \varepsilon^{-5/2} \|v\| \eta_i \varepsilon^{1/2} \varepsilon^{1/2} = C \varepsilon^{-3/2} \eta_i \|v\|.
\]

Moreover by Proposition 3.6, the terms involving inner products of first and second derivatives of \( u^h \) are exponentially small. Plugging these estimates into (5.7) yields

\[
\sum_j A_{kj} dA^{(j)} = \langle u_{kk}^h, Q\sigma_k \rangle dt + \langle u_k^h, dW \rangle + \mathcal{O}(\varepsilon^{-1+m} \eta_k) + \mathcal{O}(\exp).
\]

By using Lemma 5.4, we obtain

\[
dA^{(k)} = A_{kk}^{-1} \left[ \langle u_{kk}^h, Q\sigma_k \rangle dt + \langle u_k^h + \mathcal{O}(\exp), dW \rangle + \mathcal{O}(\varepsilon^{-1+m} \eta_k) dt + \mathcal{O}(\exp) dt \right]
\]

\[
= A_{kk}^{-2} \langle u_{kk}^h, Qu_k^h \rangle dt + A_{kk}^{-1} \langle u_k^h + \mathcal{O}(\exp), dW \rangle + \mathcal{O}(\varepsilon^m \eta_k) dt + \mathcal{O}(\exp) dt.
\]

As a next step, we investigate the deterministic part. As we cannot control the nonlinearity in terms of the \( L^2 \)-norm, we additionally assume smallness of the normal component \( v \) in \( L^4 \). In the stability result of Section 6.3, the maximal \( L^4 \)-radius that we can treat is of order \( \varepsilon^{1/4+m/2-\kappa} \) for small \( \kappa > 0 \).

Lemma 5.8. Let \( m > 0 \) and \( \kappa > 0 \) be very small. For \( h \in \Omega_{P_h} \) and \( v \perp u_i^h, i = 1, \ldots, N+1 \), assume that \( \|v\| < \varepsilon^{1/2+m} \) and \( \|v\|_{L^4} < \varepsilon^{1/4+m/2-\kappa} \). Then, we have

\[
\sum_i A_{ri}^{-1} \langle u_i^h, \mathcal{L}(u_i^h + v) \rangle \leq C \varepsilon^{2m+1-2\kappa}.
\]

Proof. Expanding \( \mathcal{L} \) yields \( \mathcal{L}(u_i^h + v) = \mathcal{L}(u_i^h) + \mathcal{L}^h v + \mathcal{X}^h(v) \). We observe that \( \mathcal{L}(u_i^h) = \mathcal{O}(\exp) \) by Proposition 5.8. Differentiating with respect to \( h_i \) yields \( \mathcal{L}^h u_i^h = \mathcal{O}(\exp) \) and hence, \( \langle \mathcal{L}^h v, u_i^h \rangle = \langle v, \mathcal{L}^h[u_i^h] \rangle = \mathcal{O}(\exp) \), since \( \mathcal{L}^h \) is self-adjoint. The remaining nonlinear term is estimated by

\[
\langle \mathcal{X}^h(v), u_i^h \rangle = \int_0^1 3u_i^h v^2 - u_i^h v^3 \, dx \leq C \varepsilon^{-1} \left[ \|v\|^2 + \|v\|_{L^3}^3 \right]
\]

\[
\leq C \varepsilon^{-1} \left[ \|v\|^2 + \|v\|_{L^4}^2 \right] \leq C \varepsilon^{2m-2\kappa},
\]

where we interpolated the \( L^3 \)-term by Hölder’s inequality. Applying Lemma 5.4 concludes the proof.
We can finally show that, up to times of order $O(\varepsilon \eta_e^{-1})$, the motion of the kinks is approximately given by the projection of the Wiener process onto the slow manifold $\mathcal{M}$, that is, for $k = 1, \ldots, N + 1$

$$
\dot{h}_k = \frac{1}{||u_k^h||^2}(u_k^h, 0 \, dW).
$$

(5.8)

At times of order $O(\varepsilon \eta_e^{-1})$ the droplet is expected to move by the magnitude of $\varepsilon$ and hence, we treat the relevant time scale in our analysis, since we have to assure that the distance between two kinks is at least $\varepsilon^{-1}$ (cf. Remark 5.10). For a sufficiently large noise strength, the stochastic effects dominate the dynamics and hence, as expected, the approximation by the purely stochastic process is better for a larger noise strength $\eta_e$. In our main stability result (see Theorem 5.9) the maximal strength we can treat is of order $\varepsilon^{1+2\kappa}$.

**Theorem 5.9 (Approximation of the exact dynamics).** Let $h(t)$ be a solution to (5.1) and $\tilde{h}(t)$ be a solution to (5.8). For $m > 0$ and small $\kappa > 0$, define the stopping time

$$
\tau := \inf \left\{ t \geq 0 : h(t) \notin \Omega_{\nu \varepsilon} \text{ or } ||v|| > \varepsilon^{1/2+m} \text{ or } ||v||_{L^1} > \varepsilon^{1/4+m/2-\kappa} \right\}.
$$

Then, for a stopping time $T \leq \varepsilon \eta_e^{-1} \wedge \tau$, we obtain

$$
\mathbb{E} \sup_{0 \leq t \leq T} |h(t) - \tilde{h}(t)| \leq C\varepsilon + C\varepsilon^{2m+2-2\kappa} \eta_e^{-1}.
$$

**Proof.** For notational convenience, we define for $h, \tilde{h} \in \Omega_{\nu \varepsilon}$ the maps

$$
\gamma_r(h) := \frac{u^h_r}{||u^h_r||^2} \text{ and } \Delta(h, \tilde{h}) := \gamma_r(h) - \gamma_r(\tilde{h}).
$$

By (5.1), (5.2), and Lemma 5.2, we derive for $t \leq T$

$$
h_r(t) - \tilde{h}_r(t) \leq \int_0^t b_r(s) + I_r(\tilde{h}(s)) \, ds + \int_0^t \langle \Delta(h, \tilde{h}) + O(\varepsilon^{1+m})u^h_r, \, dW \rangle.
$$

Here, $I(\tilde{h})$ collects all the terms that appear after a conversion of the Stratonovich SDE (5.5) into an Itô SDE. This is important, as we need the stochastic integral to be a martingale. These Itô-Stratonovich correction terms are essentially identical to the terms in (5.3), where we set $v = 0$ and replace the matrix $A(\tilde{h}, v)$ by $S_{kr}(\tilde{h}) = A(\tilde{h}, 0) = \langle u^h_k, u^h_j \rangle$. In more detail, one easily computes that

$$
I_r(\tilde{h}) = \sum_i S_{ri}^{-1} \sum_j \langle u^h_{ij}, Q_{ij}(\tilde{h}, 0) \rangle + \sum_i S_{ri}^{-1} \left[ -\langle u^h_{ij}, u^h_k \rangle - \frac{1}{2} \langle u^h_{ij}, u^h_{jk} \rangle \right] \langle Q_{ij}(\tilde{h}, 0), \sigma_k(\tilde{h}, 0) \rangle
$$

$$
= S_{rr}^{-2} \langle u^h_r, Q_{rr}u^h_r \rangle + O(\exp) = O(\eta_e).
$$

Here, we utilized that $S_{ri} = \langle u^h_r, u^h_i \rangle = \mathcal{N}^{-1} \varepsilon \delta_{ri} + O(\exp)$ by Proposition 3.6 and, as $v = 0$, $\sigma_r(0, 0) = \sum S_{ri}^{-1} u^h_i = S_{rr}^{-1} u^h_r + O(\exp)$. Moreover, the inner product of first derivatives with second derivatives of $u^h$ is exponentially small due to Proposition 3.6.

In Lemmata 5.7 and 5.8 we established an $L^\infty$-bound for $b$ up to the stopping time $\tau$, namely,

$$
\sup_{0 \leq t \leq \tau} |b| \leq c(\eta_e + \varepsilon^{2m+1-2\kappa})
$$

Combining this with the bound of the Ito-Stratonovich correction term $I$ yields

$$
\mathbb{E} \sup_{0 \leq t \leq T} |h_r(t) - \tilde{h}_r(t)| \leq c(\eta_e + \varepsilon^{2m+1-2\kappa})T + \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \langle \Delta(h, \tilde{h}) + O(\varepsilon^{1+m})u^h_r, \, dW \rangle \right|.
$$
By Burkholder’s inequality and Lipschitz continuity of $\gamma$ with Lipschitz constant of order $O(\varepsilon^{-1/2})$ (cf. Lemma 5.5), the martingale term is estimated by

$$E \sup_{0 \leq t \leq T} \left| \int_0^t \langle \Delta(h, \tilde{h}) + O(\varepsilon^{1+m}) u^{h^1}_r, dW \rangle \right|$$

$$\leq C E \left[ \int_0^T \langle \Delta(h, \tilde{h}) + O(\varepsilon^{1+m}) u^{h^1}_r, \mathcal{Q}(\Delta(h, \tilde{h}) + O(\varepsilon^{1+m}) u^{h^1}_r) \rangle \right]^{1/2}$$

$$= C E \left[ \int_0^T \langle \Delta(h, \tilde{h}) + O(\varepsilon^{1+m}) (\mathcal{Q}u^{h^1}_r, \Delta(h, \tilde{h})) + O(\varepsilon^{2+2m}) (u^{h^1}_r, Qu^{h^1}_r) \rangle \right]^{1/2}$$

$$\leq C E \left[ \int_0^T \eta \|\Delta(h, \tilde{h})\|^2 + \varepsilon^{1/2+m} \eta \|\Delta(h, \tilde{h})\| + \varepsilon^{1+2m} \eta \right]^{1/2}$$

$$\leq C E \left[ \int_0^T \eta \|\Delta(h, \tilde{h})\|^2 + \varepsilon^{1+2m} \eta \right]^{1/2}$$

$$\leq C E \left[ \int_0^T \eta \|\Delta(h, \tilde{h})\|^2 + \varepsilon^{1+2m} \eta T \right]^{1/2}$$

$$\leq C \eta^{1/2} \varepsilon^{-1/2} T^{1/2} E \sup_{0 \leq t \leq T} \|h(t) - \tilde{h}(t)\|^2 + C \varepsilon^{1/2+m} \eta^{1/2} T^{1/2}.$$

With the assumption $T < c \eta^{-1}$, this implies

$$E \sup_{0 \leq t \leq T} |h(t) - \tilde{h}(t)| \leq \frac{C \eta^{1/2} \varepsilon^{-1/2} T^{1/2} + \varepsilon^{1+2m} \eta^{1/2} T^{1/2}}{1 - \eta^{1/2} \varepsilon^{-1/2} T^{1/2}} \leq C \varepsilon + C \varepsilon^{2m+2-2n} \eta^{-1} \eta^{-1}.$$

**Remark 5.10.** In the definition of admissible parameters $\Omega_{\rho_\varepsilon}$, we had to assume that the distance between two interfaces is bounded from below by $O(\varepsilon^{-1})$. Since $E h(\varepsilon^{1/2}) = h(0) + O(\varepsilon)$, the interface positions $h(t)$ might have moved by order $\varepsilon$ and thus, a collision of two interfaces can occur, which we cannot treat in our analysis. Therefore, up to the relevant time, the motion of the kinks behaves approximately like a Wiener process projected onto the slow manifold. After a breakdown of two interfaces, we could restart our analysis on a lower-dimensional slow manifold, where the number of kinks is reduced by two, or one if a kink is annihilated at the boundary. We do not cover this annihilation here. In the case when our analysis breaks down at the boundary of the slow manifold, we are still too far away from the one with less kinks.

### 5.2 Analysis of the stochastic ODE for [mac]

To conclude our study of the kink motion, we analyze the mass conserving Allen–Cahn equation. Recall that in this case, due to mass conservation, we reduced the parameter space $\Omega_{\rho_\varepsilon}$ via $h_{N+1}(h_1, \ldots, h_N)$ by one dimension and therefore obtain by chain rule and Lemma 5.8

$$u^\xi_k = u^{h_1}_k + (-1)^{N-k} u^{h_{N+1}} + O(\exp). \quad (5.9)$$

**Remark 5.11.** Analogously to Remark 5.6, we can verify that the Fermi coordinates $(\xi, \gamma)$ around $\mathcal{M}_\mu$ are locally well-defined (cf. Definition 5.7). The crucial point is that the maps $\xi \mapsto b(\xi, u - u^\xi)$ and $\xi \mapsto \sigma(\xi, u - u^\xi)$ are sufficiently smooth. In Lemma 5.7, we proved the local Lipschitz continuity of the corresponding maps in the non-mass conserving case. In fact, let us show that these maps are even smoother. By the expressions in (5.2) and (5.8), the coefficients $\sigma$ and $b$ depend on $\xi$ via various derivatives of $u^\xi$ (up to the third order). Note that also the matrix $A$ only depends on derivatives of $u^\xi$. Hence, if the profiles $u^\xi$ are sufficiently
smooth, the smoothness is directly inherited to the coefficients of the stochastic ODE and we then obtain a unique local solution to \( d\xi = b(\xi, u - u^\xi) \, dt + \langle \sigma(\xi, u - u^\xi), dW \rangle \).

In our construction of the slow manifold, we summed up rescaled and translated solutions to the ODE \( F''(U) = 0 \). In the toy case \( F(u) = \frac{1}{4}(u^2 - 1)^2 \), one obtains the explicit solution \( \tanh(x/\sqrt{2}) \), which is of course \( C^\infty \)-smooth. Thereby, we see that the multi-kink configuration \( u^\xi \) is sufficiently smooth with respect to \( \xi \), which shows that the aforementioned maps are at least \( C^1 \)-functions. For details on how to obtain the well-definedness of the Fermi coordinates, we refer to Remark 5.12.

Just like in the analysis of [AC], we first show the invertibility of the matrix \( A \). To start with, we consider the metric tensor \( S_{kj} = \langle u_k^\xi, u_j^\xi \rangle \), which does not depend on \( v \). Due to the coupling through the mass constraint, the matrix \( S \) and its inverse are no longer diagonal. As we will see, this has an impact on the stochastic ODE governing the motion of the kinks.

**Lemma 5.12.** For \( u^\xi \in M_\mu \) and \( j, k \in \{1, \ldots, N\} \) we have
\[
S_{kj} = \langle u_k^\xi, u_j^\xi \rangle = \mathcal{X}^{-1} \left[ \delta_{kj} + (-1)^{k+j} \right] + O(\exp),
\]
where \( \mathcal{X} \) is the constant given in Proposition 5.10.

**Proof.** With Proposition 3.6 and the chain rule (5.9), we compute
\[
\langle u_k^\xi, u_j^\xi \rangle = \langle u_k^\xi + (-1)^{N-k}u_{N+1}^h, u_j^h + (-1)^{N-j}u_{N+1}^h \rangle = \|u_k^h\|^2 \delta_{kj} + (-1)^{k+j}\|u_{N+1}^h\|^2 + O(\exp) = \mathcal{X}^{-1} \left[ \delta_{kj} + (-1)^{k+j} \right] + O(\exp).
\]

With the structure of the matrix at hand, we can easily invert \( S \).

**Lemma 5.13.** Let \( u^\xi \in M_\mu \). The matrix \( S \) is invertible with
\[
S_{kj}^{-1} = \frac{\varepsilon}{\mathcal{X}} \left[ \delta_{kj} + \frac{1}{N+1}(-1)^{k+j+1} \right] + O(\exp).
\]

**Proof.** We have (ignoring exponentially small terms)
\[
\sum_{j=1}^N S_{kj} S_{jl}^{-1} = \sum_{j=1}^N \left[ \delta_{kj} + (-1)^{k+j} \right] \left[ \delta_{jl} + \frac{1}{N+1}(-1)^{j+l+1} \right] = \delta_{jl} + \frac{1}{N+1}(-1)^{k+l+1} + (-1)^{k+l} + \frac{N}{N+1}(-1)^{k+l+1} = \delta_{kl}.
\]
Finally, we show that—as long as \( \|v\| \) stays sufficiently small—the full matrix \( A(\xi, v) \) given by (5.11) is invertible. With that, the coefficients of the Itô diffusion (10.1) (with \( h \) replaced by \( \xi \)) are well-defined and we can continue to study the dynamics of kinks for the mass conserving Allen–Cahn equation in more detail.

**Lemma 5.14.** Consider the matrix \( A_{kj}(\xi, v) = S_{kj} - \langle u_{kj}^\xi, v \rangle \), where \( S \) is given by Lemma 5.12. Then, as long as \( \|v\| < \varepsilon^{1/2+m} \) for some \( m > 0 \), \( A \) is invertible with
\[
A^{-1} = S^{-1} + O(\varepsilon^{m+1}).
\]

**Proof.** For a small perturbation \( S(v) \), given by \( S_{kj}(v) = \langle u_k^\xi, v \rangle \), of the matrix \( S \) we compute via geometric series
\[
A^{-1} = [S - S(v)]^{-1} = [I_N - S^{-1}S(v)]^{-1} S^{-1} = \sum_{j=1}^\infty [S^{-1}S(v)]^j S^{-1} = S^{-1} + O(\varepsilon^{m+1}),
\]
where we used that \( S(v) = O(\varepsilon^{-3/2}\|v\|) \) and the sum converges for \( \|v\| < \varepsilon^{1/2+m} \).
We continue with estimating the deterministic part of \([5.11]\). Similarly to Lemma \([5.8]\) we have to assume smallness of the normal component \(v\) in \(L^2\) and \(L^4\) to control the nonlinearity. In the following lemma, we consider the radii for which we show stochastic stability later in Sections \([6.2]\) and \([6.4]\).

**Lemma 5.15.** Let \(m > 0\), \(\xi \in \mathcal{A}_{p\xi}\), and \(v \perp \mathcal{M}_{\mu}\). Also, assume that \(\|v\| < \varepsilon^{3/2+m}\) and \(\|v\|_{L^4} < \varepsilon^{3/4+m/2-\kappa}\). Then, we obtain

\[
\langle u_1^\xi, L(u_1^\xi + v) \rangle \leq C \varepsilon^{2+2m-2\kappa}.
\]

**Proof.** We follow the proof of Lemma \([5.8]\). Only for the nonlinearity we have to take the different radii into account. By Hölder’s inequality we obtain

\[
\langle N^\xi(v), u_1^\xi \rangle \leq C \varepsilon^{-1} \|v\|^2 + \|v\|_L^3 \leq C \varepsilon^{-1} \|v\|^2 + \|v\|_{L^4}^2 \leq C \varepsilon^{2+2m-2\kappa}.
\]

In order to analyze the SDE governing the motion of kinks, it is more convenient to rewrite \([5.11]\) in the Stratonovich sense. By leaving out Itô corrections, Lemmata \([5.13]\) and \([5.14]\) imply

\[
d\xi_r = \sum_i A_{ri}^{-1}(L(u_i^\xi + v), u_i^\xi) dt + \sum_i A_{ri}^{-1}(u_i^\xi, \circ dW)
= \sum_i S_{ri}^{-1}(L(u_i^\xi + v), u_i^\xi) dt + \sum_i S_{ri}^{-1}(u_i^\xi, \circ dW) + O(\varepsilon^{4+3m}) dt + \langle O_{L^2}(\varepsilon^{5/2+m}), \circ dW \rangle
= \mathcal{X}^{-1}\varepsilon \langle L(u_i^\xi + v), u_i^\xi \rangle dt + \mathcal{X}^{-1}\varepsilon \langle u_i^\xi, \circ dW \rangle
+ \frac{(-1)^{r}\varepsilon}{\mathcal{X}(N+1)} \langle L(u_i^\xi + v), \sum_{i=1}^N (-1)^{i+1} u_i^\xi \rangle dt + \frac{(-1)^{r}\varepsilon}{\mathcal{X}(N+1)} \sum_{i=1}^N (-1)^{i+1} u_i^\xi, \circ dW \rangle
+ O(\varepsilon^{4+3m}) dt + \langle O_{L^2}(\varepsilon^{5/2+m}), \circ dW \rangle.
\]

The first two summands (depending only on \(u_i^\xi\)) are similar to the non-mass-conserving case, but—due to the mass constraint—we obtain additional terms, which do not only depend on the position \(\xi_r\) but rather on all positions \((\xi_1, \ldots, \xi_N)\). To give a better understanding of this equation—especially of the additional terms—let us express it in the original \(h\)-coordinates. Recall that by chain rule \(u_i^\xi = u_i^h + (-1)^{N-i} h_{N+1}^h + O(\exp)\). Thus we compute (ignoring exponentially small terms)

\[
u_i^\xi = \frac{(-1)^{r}}{N+1} \sum_{i=1}^N (-1)^{i+1} u_i^\xi = u_i^h + (-1)^{N-r} h_{N+1}^h + \frac{(-1)^{r+N+1} N}{N+1} u_{N+1}^h + \frac{(-1)^{r}}{N+1} \sum_{i=1}^N (-1)^{i+1} u_i^h
= u_i^h + (-1)^{r} h_{N+1}^h \left[ \frac{(-1)^{N+1} N}{N+1} - (-1)^{N+1} \right] + \frac{(-1)^{r}}{N+1} \sum_{i=1}^N (-1)^{i+1} u_i^h
= u_i^h + (-1)^{r} h_{N+1}^h + \frac{(-1)^{r}}{N+1} \sum_{i=1}^N (-1)^{i+1} u_i^h = u_i^h + \frac{(-1)^{r}}{N+1} \sum_{i=1}^{N+1} (-1)^{i+1} u_i^h.
\]

Plugging this into the Stratonovich SDE yields

\[
\frac{\partial}{\partial t} \xi_r = \|u_i^h\|^{-2} \langle L(u_i^h + v), u_i^h \rangle dt + \|u_i^h\|^{-2} \langle u_i^h, \circ dW \rangle
\]

Plugging this into the Stratonovich SDE yields

\[
\frac{\partial}{\partial t} \xi_r = \|u_i^h\|^{-2} \langle L(u_i^h + v), u_i^h \rangle dt + \|u_i^h\|^{-2} \langle u_i^h, \circ dW \rangle
= \frac{(-1)^{r}}{(N+1)} \sum_{i=1}^N (-1)^{i+1} \left[ \|u_i^h\|^{-2} \langle L(u_i^h + v), u_i^h \rangle + \frac{(-1)^{r}}{(N+1)} \|u_i^h\|^{-2} \langle u_i^h, \circ dW \rangle \right]
\]

\[
+ O(\varepsilon^{4+3m}) dt + \langle O_{L^2}(\varepsilon^{5/2+m}), \circ dW \rangle.
\]
We observe that all the terms appearing in this formula are up to an exponentially small error the right-hand side of the equation for $dh$ (see (5.6) and (5.7) with $A(h,v)$ a diagonal matrix). Thus, we have

$$d\xi_r \approx dh_r + \frac{(-1)^r}{(N+1)} \sum_{i=1}^{N+1} (-1)^{i+1} dh_i.$$  \hspace{1cm} (5.11)

Therefore, the kink motion for the mass conserving Allen–Cahn equation is approximately given by the independent motion of the position $h_r$, which is moving according to the non-massconserving case, plus a weighted motion of all interface positions $(h_1, \ldots, h_{N+1})$ that guarantees the conservation of mass.

**Remark 5.16.** In Theorem 5.9, we proved that up to times of order $\varepsilon \eta_e^{-1}$ the interface positions $h(t)$ behave approximately like the projection of the Wiener process onto the slow manifold $\mathcal{M}$. Using that $dh_r \approx \|u_r^h\|^{-2} \langle u_r^h, \circ dW \rangle$ and plugging this into (5.11), we obtain heuristically

$$d\xi_r \approx \sum_i S^{-1}_{ri} (\xi_i^r, \circ dW),$$ \hspace{1cm} (5.12)

where we essentially used the identity (5.10). Since the matrix $S$ is given by $S_{ri} = \langle u_r^\xi, u_i^\xi \rangle$, we expect that also the dynamics for the mass conserving Allen–Cahn equation behaves approximately like the projection of the Wiener process onto $\mathcal{M}_\mu$. Analogously to Theorem 5.9, we could make this rigorous and estimate the error for a given time scale. Opposed to the previous analysis of (5.2), we cannot quite reach a good error estimate up the relevant time scale of order $O(\varepsilon \eta_e^{-1})$, which corresponds to the time that a kink is likely to move by the order of $\varepsilon$ (see Remark 5.16). Basically, this deficiency stems from the worse spectral gap in Theorem 4.4, which leads to a smaller maximal noise strength that we can treat in our stability analysis. See Theorem 6.10, where we can allow only for $\eta_e \leq \varepsilon^{4+2m-\kappa}$, and Theorem 6.12 for the interplay between the spectral gap and the noise strength. For a reasonable result, we need that the error, which is linear in the time scale $T_e$, is smaller than the magnitude of the process $\xi$, which grows like $T_e^{3/2}$. In the case of the mass-conserving Allen–Cahn equation, we expect the following result to hold true but omit the details.

**Conjecture 5.17.** Let $\xi(t)$ be the solution to (5.1) with $b$ and $\sigma$ given by (5.3) and (5.2) and replaced by $\xi$. Furthermore, let $\tilde{\xi}(t)$ be the projection of the Wiener process $W$ onto the mass conserving manifold $\mathcal{M}_\mu$ given by (5.12). For $m > 0$ and small $\kappa > 0$, define the exit time

$$\tau := \inf \left\{ t \geq 0 : \xi \notin A_{\eta_e} \text{ or } \|v(t)\| > \varepsilon^{3/2+m} \text{ or } \|v(t)\|_{L^4} > \varepsilon^{3/4+m/2-\kappa} \right\}$$

Then, for $T_e \leq c \varepsilon \eta_e^{-1} \wedge \tau$, we obtain

$$\mathbb{E} \sup_{0 \leq t \leq T_e} |\xi(t) - \tilde{\xi}(t)| \leq c \left[ \eta_e + \varepsilon^{3+2m-2\kappa} \right] T_e.$$

**Remark 5.18.** In the proof of the analogous result in Theorem 5.2, it was crucial to explicitly know the Lipschitz constant of the map $\xi \mapsto \sigma(\xi, v)$ provided $v$ is sufficiently small. In establishing the Lipschitz continuity (Lemma 5.3), we relied on the convexity of the set of admissible interface positions. While this is straightforward for the set $\Omega_{\eta_e}$, this is not quite true in the mass conserving case, where the set of admissible positions is given by

$$A_{\eta_e} = \left\{ (h_1, \ldots, h_N, h_{N+1}(h_1, \ldots, h_N)) \in \Omega_{\eta_e} : (h_1, \ldots, h_N) \in [0,1]^N \right\}.$$
By Lemma 3.8, the map $h_{N+1}$ is explicitly given by

$$h_{N+1}(h_1, \ldots, h_N) = \sum_{i=1}^{N} (-1)^{N-i} h_i + c(\mu) + O(\exp),$$

where we have to introduce a constant $c(\mu)$ depending only on the mass $\mu$. With this expression, one readily computes that $h_{N+1}(\xi + \lambda(\xi - \bar{\xi}) = h_{N+1}(\xi) + \lambda h_{N+1}(\xi - \bar{\xi}) + O(\exp)$ for any $\xi, \bar{\xi} \in \mathbb{R}^N$ and $\lambda \in (0, 1)$. Combined with the convexity of $\Omega_{\rho_\varepsilon}$, this shows that the set $A_{\rho_\varepsilon}$ is not exactly convex, but the error is exponentially small. We obtain the following result:

$$h, \bar{h} \in A_{\rho_\varepsilon} \Rightarrow \lambda h + (1 - \lambda) \bar{h} \in A_{2\rho_\varepsilon} \quad \forall \lambda \in (0, 1).$$

With this property at hand, we expect to bound the Lipschitz constant in the mass conserving case. For some of the technical details, we follow closely the proof of Lemma 5.5.

### 6 Stochastic Stability

In this section, we discuss stochastic stability, both for (AC) and (mAC). The first part is concerned with establishing stability in $L^2$ which is crucial for defining the Fermi coordinates (cf. Definition 5.1). Note that this is not sufficient for the analysis of the SDE, where we additionally assumed that $v$ is small in $L^4$ in order to handle the nonlinear terms. Hence, the second part of this section is devoted to stochastic stability in $L^4$. The underlying problem is to prove that the normal component $v$ measuring the distance to the slow manifold, which satisfies the stochastic PDE

$$dv = \left[ L(u^h) + \mathcal{L}h v + \mathcal{N}^h(v) \right] dt + dW - \frac{1}{2} \sum_{i,j} u^h_{ij} dh_j - \sum_j u^h_j \langle Q_{\sigma_i}, \sigma_j \rangle dt,$$

remains small in various norms. For this purpose, we aim to show a stochastic differential inequality of the type

$$d\|v\| = -a_\varepsilon \|v\| dt + O(K_\varepsilon) dt + \langle O(c_\varepsilon \|v\|^\alpha), dW_\varepsilon \rangle \quad (6.1)$$

for some positive $\varepsilon$-dependent constants $a_\varepsilon$ and $K_\varepsilon$. With (6.1) at hand, one can prove the following theorem which serves as our main tool in the upcoming stability discussions. For more details and a proof, we refer to [7]. Here, this method was used for the stability analysis of the one-dimensional stochastic Cahn–Hilliard equation. See also [5, 13, 20].

**Theorem 6.1.** Define the stopping time

$$\tau^* = \inf \{ t \in [0, T_\varepsilon \wedge \tau_0] : \|v(t)\| > R_\varepsilon \},$$

where the deterministic cut-off $T_\varepsilon$ satisfies $T_\varepsilon = \varepsilon^{-M}$ for any fixed large $M > 0$ and $\tau_0$ denotes the first exit time from $\Omega_{\rho_\varepsilon}$, the set of admissible kink positions. Assume that for $t \leq \tau^*$ equation (6.1) is satisfied with some positive constants $a_\varepsilon, K_\varepsilon, c_\varepsilon$ and $\alpha$. Furthermore, assume that for some small $\varepsilon > 0$

$$\frac{K_\varepsilon + c_\varepsilon^2 \eta_\varepsilon R_\varepsilon^{2\alpha - 1}}{a_\varepsilon R_\varepsilon} = O(\varepsilon^\kappa) \quad \text{and} \quad \|v(0)\| \leq \frac{K_\varepsilon + c_\varepsilon^2 \eta_\varepsilon R_\varepsilon^{2\alpha - 1}}{a_\varepsilon}.$$

Then, the probability $\mathbb{P}(\tau^* < T_\varepsilon \wedge \tau_0)$ is smaller than any power of $\varepsilon$, as $\varepsilon$ tends to zero.
6.1 $L^2$-Stability for (AC)

We start with the analysis of (AC) without mass conservation. Crucial for establishing stochastic stability is the following theorem, which relies on the spectral gap derived in Theorem 4.2. As long as the $L^2$-norm of the normal component $v$ stays sufficiently small, the nonlinear term does not destroy the spectral estimate.

**Theorem 6.2.** Let $u^h \in M$ and $v \perp u^h_i, \ i = 1, \ldots, N + 1$. Assume that $\|v\| < \varepsilon^{1/2+m}$ for some $m > 0$. Then, for $\lambda_0$ the constant given in the spectral bound of Theorem 4.2, we obtain

$$\langle L^h v + N^h(v), v \rangle \leq -\frac{1}{2} \lambda_0 \|v\|^2.$$

**Proof.** Let $v \perp u^h_i \ \forall i = 1, \ldots, N + 1$. By the main spectral result of Theorem 4.2, we have

$$\langle L^h v, v \rangle \leq -\lambda_0 \|v\|^2.$$

Therefore, for $\gamma_1, \gamma_2 > 0$ with $\gamma_1 + \gamma_2 = 1$, we compute

$$\langle L^h v, v \rangle \leq -\gamma_1 \lambda_0 \|v\|^2 + \gamma_2 \varepsilon^2 \int_0^1 v_{xx} v \, dx + \gamma_2 \int_0^1 f'(u^h) v^2 \, dx$$

$$\leq -\gamma_1 \lambda_0 \|v\|^2 - \varepsilon^2 \gamma_2 \|v_x\|^2 + \gamma_2 \|f'(u^h)\|_{L^\infty} \|v\|^2.$$  (6.0)

By Gagliardo–Nirenberg and Young’s inequality we obtain

$$\langle N^h(v), v \rangle = \int_0^1 3(u^h)^2 v^3 - v^4 \leq 3 \|v_x\|^3 \leq C \|v_x\|^{1/2} \|v\|^{5/2}$$

$$\leq \varepsilon^2 \gamma_2 \|v_x\|^2 + C \varepsilon^{-2/3} \gamma_2^{-1/3} \|v\|^{4/3} \|v\|^2,$$  (6.3)

where we interpolated the $L^3$-norm between $H^1$ and $L^2$. Combining (6.2) and (6.3) yields

$$\langle L^h v + N^h(v), v \rangle \leq -\gamma_1 \lambda_0 \|v\|^2 + \left[ \gamma_2 \|f'(u^h)\|_{L^\infty} + C \varepsilon^{-2/3} \gamma_2^{-1/3} \|v\|^{4/3} \right] \|v\|^2$$

$$= \left[ -\lambda_0 + \gamma_2 \lambda_0 + \gamma_2 \|f'(u^h)\|_{L^\infty} + C \varepsilon^{-2/3} \gamma_2^{-1/3} \|v\|^{4/3} \right] \|v\|^2.$$

Fixing $\gamma_2 = \varepsilon^m$, we obtain for $\|v\| < \varepsilon^{1/2+m}$

$$\langle L^h v + N^h(v), v \rangle \leq \left[ -\lambda_0 + \varepsilon^m \left( \lambda_0 + \|f'(u^h)\|_{L^\infty} \right) \right] \|v\|^2.$$  \[ \square \]

As a next step, we need to analyze the remaining terms of $d\|v\|^2$. We show that, provided $\|v\|$ is sufficiently small, they are of order $O(\eta_c)$.

**Lemma 6.3.** Under the same assumptions as in Theorem 6.2, we obtain

$$\langle L(u^h), v \rangle \ dt - \frac{1}{2} \sum_{i,j} \langle u^h_{ij}, v \rangle \langle Q \sigma_i, \sigma_j \rangle \ dt + \langle dv, dv \rangle = O(\eta_c) \ dt.$$

**Proof.** We have $L(u^h) = O(\text{exp})$ and, as $\|v\| < \varepsilon^{1/2+m}$,

$$\langle u^h_{ij}, v \rangle \langle Q \sigma_i, \sigma_j \rangle \leq C \varepsilon^{-3/2} \|v\| \varepsilon^{1/2} \varepsilon^{1/2} = O(\varepsilon^m \eta_c).$$

For the Itô correction term $\langle dv, dv \rangle$ we see that

$$\langle dv, dv \rangle = \eta_c \ dt + \sum_{i,j} \left[ \langle u^h_{ij}, u^h_{ij} \rangle - 2 \langle Q u^h_{ij}, \sigma_j \rangle \right] \ dt = O(\eta_c) \ dt.$$

Here, we utilized that by Proposition 5.6 and Lemma 5.7 $\|u^h_{ij}\| = O(\varepsilon^{-1/2})$, $\|u^h_{ij}\| = O(\varepsilon^{-3/2})$, and $\|\sigma_i\| = O(\varepsilon^{1/2})$.  \[ \square \]
Combining the estimates of Theorem 6.2 and Lemma 6.3, we fully estimated the stochastic differential \( d\|v\|^2 \). This provides us with the following result, which is essential for proving stability in \( L^2 \).

**Corollary 6.4.** Let \( u^h \in \mathcal{M} \). If \( v \perp u_i^h \) for \( i = 1, \ldots, N+1 \) and \( \|v\| < \varepsilon^{1/2+m} \) for some \( m > 0 \), we obtain

\[
d\|v\|^2 \leq \left[ -\frac{1}{2}\lambda_0\|v\|^2 + \mathcal{O}(\eta_\varepsilon) \right] dt + 2\langle v, dW \rangle.
\]

We can finally show that the \( L^2 \)-norm of \( v \) stays small for very long times under small stochastic perturbations. Since the following stability results can only hold as long as \( h(t) \in \Omega_{\rho_\varepsilon} \), we define the first exit time from the open set \( \Omega_{\rho_\varepsilon} \) by

\[
\tau_0 := \left\{ t \geq 0 : h(t) \notin \Omega_{\rho_\varepsilon} \right\}.
\]

Note that we have seen in Remark 5.10 that at times of order \( \mathcal{O}(\varepsilon\eta_\varepsilon^{-1}) \) the interface positions are likely to move by the magnitude of \( \varepsilon \) and thus exit the set of admissible positions \( \Omega_{\rho_\varepsilon} \). This suggests that—if stability holds—the exit time \( \tau_0 \) is with high probability of order \( \varepsilon\eta_\varepsilon^{-1} \).

**Theorem 6.5 (\( L^2 \)-Stability for (AC)).** For \( m > 0 \) define the stopping time

\[
\tau^* := \inf \left\{ t \in [0, T_\varepsilon \wedge \tau_0] : \|v(t)\| > \varepsilon^{1/2+m} \right\},
\]

where the deterministic cut-off satisfies \( T_\varepsilon = \varepsilon^{-M} \) for fixed large \( M > 0 \) and \( \tau_0 \) is given by (6.4). Also, assume that for some \( \nu \in (0, 1) \)

\[
\|v(0)\| \leq \nu\varepsilon^{1/2+m} \quad \text{and} \quad \eta_\varepsilon \leq \varepsilon^{1+2m}.
\]

Then, the probability \( \mathbb{P}(\tau^* < T_\varepsilon \wedge \tau_0) \) is smaller than any power of \( \varepsilon \), as \( \varepsilon \) tends to zero.

**Proof.** The statement follows directly by combining the estimate of Corollary 6.4 with the general stability result of Theorem 6.4.

\[ \square \]

### 6.2 \( L^2 \)-Stability for (mAAC)

As a next step, we study the \( L^2 \)-stability for the mass conserving Allen–Cahn equation. Since the method of establishing stability works in a similar fashion to the preceding section, we will only state the main result here. Essentially, the main difference lies in the fact that the spectral gap is only of order \( \varepsilon \) opposed to a gap of order one in the previous case. To compensate this, we need to decrease the region of stability in order to absorb the nonlinear terms (compare to Theorem 6.2) and thereby, we can only allow for a smaller noise strength (cf. Theorem 6.1). For more details, we refer to [20].

**Theorem 6.6 (\( L^2 \)-Stability for (mAAC)).** For \( m > 0 \) define the stopping time

\[
\tau^* := \inf \left\{ t \in [0, T_\varepsilon \wedge \tau_0] : \|v(t)\| > \varepsilon^{3/2+m} \right\},
\]

where \( T_\varepsilon = \varepsilon^{-M} \) for fixed large \( M > 0 \) and \( \tau_0 \) denotes the first exit time from \( \mathcal{A}_{\rho_\varepsilon} \).

Also, assume that

\[
\|v(0)\| \leq \varepsilon^{3/2+m} \quad \text{and} \quad \eta_\varepsilon \leq \varepsilon^{4+2m}.
\]

Then, the probability \( \mathbb{P}(\|v(\tau^*)\| > \varepsilon^{3/2+m}) \) is smaller than any power of \( \varepsilon \), as \( \varepsilon \) tends to zero.
6.3 \( L^4 \)-Stability for (AC)

For controlling the stochastic ODE of the interface positions, we need to establish bounds on the nonlinear term

\[
\langle N^h(v), u^h_j \rangle = \int_0^1 (3u^h v^2 - v^3) u^h_i \, dx.
\]

Since smallness in \( L^2 \) is not sufficient to control the cubic term, we will prove that the \( L^4 \)-norm of \( v \) stays small for very long times with high probability. In our analysis, we rely on the results of the preceding section. There, we established stochastic stability in \( L^2 \) and hence, all constants which appear in the following computations may depend on \( \|v\|_{L^2} \) which—provided the assumptions of Theorem 6.5 hold true—is smaller than \( \varepsilon^{1/2+m} \) for polynomial times in \( \varepsilon^{-1} \).

We begin with the classical Allen–Cahn equation (AC) without mass conservation. By the Itô formula we have

\[
\frac{1}{4} \frac{d}{dt} \|v\|_{L^4}^4 = \langle v^3, dv \rangle + 3 \int_0^1 v^2(dv)^2 \, dx.
\]

Again, recall that the flow orthogonal to the slow manifold is given by

\[
dv = \left[ \mathcal{L}(u^h) + \mathcal{L}^h v + N^h(v) \right] \, dt + dW - \sum_j u^h_j dh_j - \frac{1}{2} \sum_{i,j} u^h_{ij} \langle Q\sigma_i, \sigma_j \rangle \, dt.
\]

First, let us estimate the Itô correction term \( \int_0^1 v^2(dv)^2 \, dx \).

**Lemma 6.7.** Let \( h \in \Omega_{\rho_\varepsilon} \). We obtain

\[
\int_0^1 v^2(dv)^2 \, dx = \mathcal{O}(\eta_\varepsilon)\|v\|^2 \, dt.
\]

**Proof.** Using the relation for \( dv \) we see that

\[
\text{trace}(Q) \int_0^1 v^2 \, dx - 2 \sum_j \int_0^1 v^2(u^h_j, Q\sigma_j) \, dx \, dt + \sum_{i,j} \int_0^1 v^2(u^h_i, u^h_j) \langle Q\sigma_i, \sigma_j \rangle \, dx \, dt
\]

\[
\leq \eta_\varepsilon\|v\|^2 \, dt + c\varepsilon^{-1/2}\varepsilon^{1/2}\eta_\varepsilon\|v\|^2 \, dt + c\varepsilon^{-1/2}\varepsilon^{1/2}\|v\|^2 \, dt = \mathcal{O}(\eta_\varepsilon)\|v\|^2 \, dt,
\]

where we utilized the estimates of Proposition 3.3 for the derivatives of \( u^h \) together with the bound on the diffusion \( \sigma \) by Lemma 5.7. \( \square \)

As a next step, we study the critical term \( \langle v^3, dv \rangle \). Expanding \( dv \) yields

\[
\langle v^3, dv \rangle = -3\varepsilon^2 \int_0^1 v^2 v^2_x \, dx \, dt - \|v\|_{L^6}^6 \, dt + \langle \mathcal{L}(u^h), v^3 \rangle \, dt + \int_0^1 \left( 1 - 3(u^h)^2 \right) v^4 \, dx \, dt
\]

\[
- \int_0^1 3u^h v^4 \, dx \, dt + \langle v^3, dW \rangle - \langle v^3, dv^h \rangle.
\]

We see that the good (negative) terms for our analysis are given by \( -\|v\|_{L^6}^6 \) and, due to integration by parts,

\[
\varepsilon^2 \int_0^1 v^3 v_{xx} \, dx = -3\varepsilon^2 \int_0^1 v^2 v^2_x \, dx = -\frac{3}{4} \varepsilon^2 \int_0^1 (v^2)^2_x \, dx = -\frac{3}{4} \varepsilon^2 \|v^2\|_{H^1}^2.
\]

Our strategy is to absorb as much as possible of the remaining terms into these negative ones, while also using that we can control the \( L^2 \)-norm by the preceding stability result. We begin
with analyzing the dominant term. Since $u^h$ is uniformly bounded, we obtain by interpolating the $L^4$-norm between the good terms

$$\int_0^1 \left( 1 - 3(u^h)^2 \right) v^4 \, dx \leq C \|v\|_{L^4}^4 \leq C \|v^2\|_{L^\infty} \int_0^1 v^2 \, dx \leq C \|v\|^2 \|v^2\|_{L^2} \|v^2\|_{L^2}^{1/2}$$

$$\leq \frac{1}{8} \epsilon \|v^2\|^2_{H^1} + c \epsilon^{-2/3} \|v\|_{L^3}^{8/3} \|v\|_{L^4}^{4/3}$$

Similarly, the $L^3$-term is estimated by

$$\int_0^1 3u^h v^5 \leq C \|v\|^2 \|v\|_{L^3} \leq C \|v^2\|_{H^1}^{1/2} \|v\|_{L^2}^{3/2} \|v\|_{L^2} \leq \frac{1}{8} \epsilon \|v^2\|^2_{H^1} + \frac{1}{4} \|v\|_{L^6}^6 + c \epsilon^{-4/3} \|v\|_{L^6}^{14/3},$$

where we used Hölder’s inequality to interpolate between $L^2$ and $L^6$. Combining the previous estimates, we derived so far

$$\langle v^3, dv \rangle \leq \left[ -\frac{1}{2} \epsilon \|v^2\|^2_{H^1} - \frac{1}{2} \|v\|_{L^6}^6 + c \epsilon^{-4/5} \|v\|_{L^5}^{18/5} + c \epsilon^{-4/3} \|v\|_{L^4}^{14/3} \right] dt + \langle v^3, dW \rangle - \langle v^3, du^h \rangle. \tag{6.5}$$

Note that we used $\mathcal{L}(u^h) = O(\exp)$ and thus $\langle \mathcal{L}(u^h), v^3 \rangle \leq O(\exp) + O(\exp) \|v\|_{L^5}^5$ by Hölder’s inequality. Finally, we estimate $\langle v^3, du^h \rangle$ given by

$$\langle v^3, du^h \rangle = \sum_j \langle v^3, u^h_j \rangle \, dh_j + \frac{1}{2} \sum_{i,j} \langle v^3, u^h_j \rangle \langle \mathcal{Q} \sigma_i, \sigma_j \rangle \, dt.$$

For the second summand we obtain

$$\langle v^3, u^h_j \rangle \langle \mathcal{Q} \sigma_i, \sigma_j \rangle \leq C \|v^2\|_{L^\infty} \|v\| \epsilon^{-3/2} \eta \epsilon^{1/2} \epsilon^{-1/2}$$

$$\leq C \|v^2\|_{H^1}^{1/2} \|v\|_{L^4} \|v\| \epsilon^{-3/2} \eta \epsilon^{1/2}$$

$$\leq C \|v^2\|_{H^1}^{1/2} \|v\|_{L^6}^{5/4} \|v\|_{L^3}^{1/4} \epsilon^{-1/2} \eta \epsilon^{1/2}$$

$$\leq C \|v^2\|_{H^1}^{1/2} \|v\|_{L^6}^{5/4} \epsilon^{-1/2} \eta \epsilon^{1/2}$$

$$\leq C \|v^2\|_{H^1}^{1/2} \|v\|_{L^6}^{5/4} \epsilon^{-1/2} \eta \epsilon^{1/2}$$

In addition to the specified inequalities, we used that $\|u^h_j\| = O(\epsilon^{-3/2})$ by Proposition 3.6 and $\|\sigma\| = O(\epsilon^{1/2})$ by Lemma 5.7.

We conclude by analyzing the term involving the stochastic differential $dh$. Recall that by (5.1)

$$dh_j = b_j(h, v) \, dt + \langle \sigma_j(h, v), dW \rangle,$$

where $b$ and $\sigma$ are given by (5.3) and (5.2), respectively. The diffusion term of $\langle v^3, u^h_j \rangle \, dh_j$ can be estimated as follows:

$$\langle v^3, u^h_j \rangle \langle \sigma_j, dW \rangle = \langle \mathcal{Q} \mathcal{L} \|v\|_{L^\infty} \|\sigma_j\| \|v\|_{L^2}^2, dW \rangle = \langle \mathcal{Q} \mathcal{L} \|v\|_{L^\infty} \|v\|_{L^4}^2, dW \rangle.$$ 

Estimating the drift of $\langle v^3, u^h_j \rangle$ is trickier, as we have to bound $b$ as well. By virtue of Lemmata 5.7 and 5.8 we can bound the drift $b$ up to a stopping time and obtain as long as $\|v(t)\| < \epsilon^{1/2 + m}$ for some $m > 0$

$$|b_j| = O(\eta \epsilon) + O(\epsilon) \|\mathcal{N}^h(v)\|_{L^2}^2.$$ 

Hence, this yields

$$\|\langle v^3, u^h_j \rangle \|_{L^2} \leq C \epsilon^{-1} \eta \epsilon^{1/2} \|v\|_{L^2}^2 \|v\|_{L^6}^3 + C \epsilon^{-1} \eta \epsilon^{1/2} \|v\|_{L^6}^3 \|v\|_{L^4}^2 \|v\|_{L^2}^2 \|v\|_{L^4}^3$$

$$\leq C \epsilon^{-1} \eta \epsilon^{1/2} \|v\|_{L^6}^3 \|v\|_{L^6}^3 + C \epsilon^{-1} \eta \epsilon^{1/2} \|v\|_{L^6}^3 \|v\|_{L^6}^3 + C \epsilon^{-1} \eta \epsilon^{1/2} \|v\|_{L^6}^3 \|v\|_{L^2}^2 \|v\|_{L^2}^2 \|v\|_{L^4}^3$$

$$\leq \frac{1}{8} \|v\|_{L^6}^6 + c \epsilon^{-3/4} \eta \epsilon^{1/3} \|v\|_{L^4}^2 + c \epsilon^{-1} \|v\|_{L^6}^4 + c \epsilon^{-2} \|v\|_{L^6}^6.$$
Finally, we estimated every term of $d\|v\|_{L^4}^4$. Plugging (6.6) and (6.7) into (5.5) furnishes

$$\langle v^3, dv \rangle \leq \left[ -\frac{1}{2} \epsilon^2 \|v^2\|^2_{H^1} - \frac{1}{4} \|v\|_{L^6}^6 + K_\epsilon(\|v\|) \right] dt + \langle O(\epsilon^{-1/2}\|v\|\|v\|_{L^4}^2), dW \rangle,$$

where $K_\epsilon$ is given by

$$K_\epsilon(\|v\|) = c\epsilon^{-4/3}\|v\|^{8/3} + c\epsilon^{-4/3}\|v\|^{14/3} + c\epsilon^{-4/3}\eta_\epsilon^{4/3}\|v\|^2 + c\epsilon^{-1}\|v\|^4 + \epsilon^{-2}\|v\|^6.$$

Thus far, the terms in $K_\epsilon$ depend on the $L^2$-norm of $v$. Under the assumptions of Theorem 6.9, i.e., a small noise strength $\eta_\epsilon$ and a suitable initial condition $v(0)$, we can bound $\|v\|_{L^2}$ by an $\epsilon$-dependent constant for long time scales. In more detail, we obtain for $\|v\| \leq \epsilon^{1/2+m}$ and $\eta_\epsilon \leq \epsilon^{1+2m}$ that

$$K_\epsilon(\|v\|) \leq \epsilon^{1+2m}.$$

Noticing now that under the same assumptions the bound in Lemma 6.7 provides us with an even smaller term and using the basic estimate $-\|v\|_{L^6}^6 \leq \|v\|^2 - \|v\|_{L^4}^4$, we proved the following inequality which is essential for deriving stochastic stability in $L^4$.

**Corollary 6.8.** As long as $\|v\| \leq \epsilon^{1/2+m}$ and $\eta_\epsilon \leq \epsilon^{1+2m}$ for some $m > 0$, we have

$$d\|v\|_{L^4}^4 \leq \left[ -\|v\|_{L^4}^4 + c\epsilon^{1+2m} \right] dt + \langle O(\epsilon^m\|v\|_{L^4}^2), dW \rangle.$$

With this inequality at hand, we can apply the main stability theorem 6.1. Bear in mind that in the derivation of Theorem 6.9 we presented only one technique and thus, we cannot guarantee the optimality of the radii.

**Theorem 6.9** ($L^4$-Stability for (AC)). For $m > 0$ and small $\kappa > 0$, consider the stopping time

$$\tau^* = \inf \left\{ t \in [0, T_\epsilon \wedge \tau_0] : \|v(t)\| > \epsilon^{1/2+m} \text{ or } \|v(t)\|_{L^4} > \epsilon^{1/4+m/2-\kappa} \right\},$$

where $T_\epsilon = \epsilon^{-M}$ for any fixed large $M > 0$ and $\tau_0$ denotes the first exit time from $\Omega_{\rho_\epsilon}$.

Also, assume that for some $\nu \in (0,1)$

$$\|v(0)\| \leq \nu\epsilon^{1/2+m} \quad \text{and} \quad \|v(0)\|_{L^4} \leq \nu\epsilon^{1/4+m/2-\kappa}$$

and that for the squared noise strength

$$\eta_\epsilon \leq \epsilon^{1+2m}.$$  

Then, the probability $\mathbb{P}(\tau^* < T_\epsilon \wedge \tau_0)$ is smaller than any power of $\varepsilon$, as $\varepsilon$ tends to zero.

**Proof.** To fit in the setting of our general stability result of Theorem 6.1, we set $x(t) = \|v(t)\|_{L^4}^4$. Utilizing the estimate of Corollary 6.8 then yields

$$dx(t) \leq [K_\epsilon(\eta_\epsilon) - a_\epsilon x(t)] dt + \langle O(\eta_\epsilon^m x(t)^9), dW \rangle,$$

where the constants are given by $a_\epsilon = 1$, $K_\epsilon(\eta_\epsilon) = O(\varepsilon^{2m+1})$, $c_\epsilon = O(\varepsilon^m)$, and $\alpha = \frac{1}{2}$. Note that due to the substitution and the definition of the stopping time $\tau^*$ the radius $R_\epsilon$ for the variable $x$ is now $\varepsilon^{1+2m-4\kappa}$. With that, one easily computes

$$\frac{K_\epsilon(\eta_\epsilon) + c_\epsilon^2 \eta_\epsilon}{a_\epsilon R_\epsilon} = O(\varepsilon^{4\kappa}).$$

By applying Theorem 6.1, this shows that $\mathbb{P}(\|v(\tau^*)\|_{L^4} > \varepsilon^{1/4+m/2-\kappa})$ is smaller than any power of $\varepsilon$. By the $L^2$-result of Theorem 6.5 and the basic inequality

$$\mathbb{P}(\tau_\epsilon < T_\epsilon \wedge \tau_0) \leq \mathbb{P} \left( \|v(\tau_\epsilon)\| > \varepsilon^{1/2+m} \right) + \mathbb{P} \left( \|v(\tau^*)\|_{L^4} > \varepsilon^{1/4+m/2-\kappa} \right),$$

the proof is complete. \qed
6.4 \( L^4 \)-Stability for \((mAC)\)

We conclude our analysis of the stochastic Allen–Cahn equation with stating the corresponding \( L^4 \)-stability result for the mass conserving case \((mAC)\). The proof is a straightforward adaption of the results presented in Section 6.3 and is stated in full detail in [20].

**Theorem 6.10** \((L^4\text{-Stability for \((mAC)\)})\). For \( m > 0 \) and small \( \kappa > 0 \), consider the stopping time

\[
\tau^* = \inf \left\{ t \in [0, T_\varepsilon \wedge \tau_0] : \|v(t)\| > \varepsilon^{5/2+m} \text{ or } \|v(t)\|_{L^4} > \varepsilon^{3/4+m/2-\kappa} \right\},
\]

where \( T_\varepsilon = \varepsilon^{-M} \) for fixed large \( M > 0 \) and \( \tau_0 \) denotes the first exit time from the set of admissible positions \( \mathcal{A}_{\rho_{\varepsilon}} \). Also, assume that for some \( \nu \in (0,1) \)

\[
\|v(0)\| \leq \nu \varepsilon^{3/2+m} \quad \text{and} \quad \|v(0)\|_{L^4} \leq \nu \varepsilon^{3/4+m/2-\kappa}
\]

and that the squared noise strength satisfies \( \eta_\varepsilon \leq \varepsilon^{4+2m} \).

Then, the probability \( \mathbb{P}(\tau^* < T_\varepsilon \wedge \tau_0) \) is smaller than any power of \( \varepsilon \), as \( \varepsilon \) tends to zero.

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