Probing CKM parameters through unitarity, $\varepsilon_K$ and $\Delta m_{B_{d,s}}$

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Abstract

A recently carried out unitarity based analysis \cite{1,2} involving the evaluation of Jarlskog's rephasing invariant parameter J as well as the evaluation of the angles $\alpha$, $\beta$ and $\gamma$ of the unitarity triangle is extended to include the effects of $K^o - \bar{K}^o$ as well as $B^o - \bar{B}^o$ mixings. The present analysis attempts to evaluate CP violating phase $\delta$, $|V_{td}|$ and the angles $\alpha$, $\beta$ and $\gamma$ by invoking the constraints imposed by unitarity, $K^o - \bar{K}^o$ and $B^o - \bar{B}^o$ mixings independently as well as collectively. By invoking the “full scanning” of input parameters, as used by Buras et al., our analysis yields the following results respectively for $\delta$ and $|V_{td}|$, unitarity: $50^o \pm 20^o$ (in I quadrant), $130^o \pm 20^o$ (in II quadrant) and $5.1 \times 10^{-3} \leq |V_{td}| \leq 13.8 \times 10^{-3}$, unitarity and $\varepsilon_K$: $33^o \leq \delta \leq 70^o$ (in I quadrant), $110^o \leq \delta \leq 150^o$ (in II quadrant), $5.8 \times 10^{-3} \leq |V_{td}| \leq 13.6 \times 10^{-3}$, unitarity and $\Delta m_d$: $30^o \leq \delta \leq 70^o$, $6.4 \times 10^{-3} \leq |V_{td}| \leq 8.9 \times 10^{-3}$. Incorporating all the constraints together, we get $37^o \leq \delta \leq 70^o$, $6.5 \times 10^{-3} \leq |V_{td}| \leq 8.9 \times 10^{-3}$, the corresponding ranges for $\alpha$, $\beta$ and $\gamma$ are, $80^o \leq \alpha \leq 124^o$, $15.1^o \leq \beta \leq 31^o$ and $37^o \leq \gamma \leq 70^o$, again in agreement with the recent analysis of Buras et al. as well as with other recent analyses. Our analysis yields $0.50 \leq \sin^2 \beta \leq 0.88$, in agreement with the recently updated BABAR results. This range of $\sin^2 \beta$ is in agreement with the earlier result of BELLE, however has marginal overlap only with the lower limit of recent BELLE results. Interestingly, our results indicate that the lower limit of $\delta$ is mostly determined by unitarity, while the upper limit is governed by constraints from $\Delta m_d$.

1 Introduction

In the context of CKM phenomenology several important developments have taken place in the last decade. On the one hand, there is considerable refinement in the
data pertaining to CKM parameters, on the other hand several phenomenological inputs have also refined the CKM elements further. However, despite a good deal of knowledge of CKM matrix elements, the CP violation still remains one of the least tested aspect of the CKM paradigm. In this context, an intense amount of experimental activity is being carried out in a variety of experiments at B-factories, KEK-B, SLAC-B, HERA-B and will be carried out at LHC-B and B-TeV, both of which will become operational by 2006. The main goal of the current phase of experiments will be to observe CP violation in the B system. It is expected that the data provided by these factories would throw new lights on the CP violating B decays, in particular it will lead to the construction of unitarity triangle involving first and third columns of the CKM matrix.

In this context, the recent flurry of activity created by the preliminary BABAR [3] and BELLE [4] results regarding the time dependent CP asymmetry $a_{\psi K_S}$ in $B_d^0(B_{s}^0) \rightarrow \psi K_S$, has given lot of impetus to the study of CKM phenomenology. Not withstanding the agreement of the latest BABAR results [3] with the SM (Standard Model) [6]-[8], the activity generated by their preliminary results has thrown open several interesting possibilities of discovering new physics beyond the SM. In this context, it has also been emphasized by several authors in their recent works [9]-[12] that there could be new physics in the loop dominated processes such as $K^0 - \bar{K}^0$ mixing and $B^0 - \bar{B}^0$ mixing, as well as in some of the CP violating amplitudes.

In view of the possibility of signals of new physics in $K^0 - \bar{K}^0$ mixing and $B^0 - \bar{B}^0$ mixing, it is desirable to sharpen their implications on the CKM parameters, particularly on the CP violating phase $\delta$ as well as on the angles $\alpha$, $\beta$ and $\gamma$ of the much talked about unitarity triangle. In this context, in the last few years some very detailed and extensive analyses of CKM phenomenology [13]-[16] have been carried out. These analyses primarily concentrate on finding CKM parameters in the Wolfenstein representation [17] and the angles of unitarity triangle. Apart from unitarity, usual inputs for such analyses are CP violating parameter $\varepsilon_K$ as well as the mass difference, $\Delta m_{d,s}$, between the light and heavy mass eigenstates of the $B^0_{d,s} - \bar{B}^0_{d,s}$ system. A careful perusal of these analyses indicates that these do not adequately emphasize the evaluation of Jarlskog’s rephasing invariant parameter $J$ [18], a key quantity measuring the CP violating effects, as well as these do not investigate independently the implications of unitarity, $\varepsilon_K$, $\Delta m_d$ and $\Delta m_s$ respectively. To closely scrutinize the effects of unitarity, $K^0 - \bar{K}^0$ and $B^0 - \bar{B}^0$ mixing on the CKM parameters, one should study their implications separately. We believe, this would facilitate deciphering the signals of new physics in the loop dominated CP violating amplitudes in the K as well as B mesons.

Recently Randhawa et al. [1, 2], have studied in detail the implications of unitarity on $J$ and on the CP violating phase $\delta$, as well as a “reference triangle” has been constructed purely from the considerations of unitarity of the CKM matrix. The purpose of the present communication is to extend the analysis carried in references [1, 2] by including independently the implications of $K^0 - \bar{K}^0$ and $B^0 - \bar{B}^0$ mixing, along with the unitarity on the CKM parameters. Besides comparing our results
with one of the most recent analyses [6], we also intend to study the effect of the future refinements in the data.

The plan of the paper is as follows. To facilitate the discussion as well as to make the mss. readable, in section 2 we detail the essentials of the unitarity based analysis carried out in reference [2]. In the section 3, we determine the CKM phase \( \delta \) from unitarity and the \( K^0 - \bar{K}^0 \) mixing parameter \( \varepsilon_K \) and then use it to evaluate CKM matrix elements and the angles \( \alpha, \beta \) and \( \gamma \) of the CKM triangle. Similarly, in the section 4 we evaluate above mentioned CKM parameters using unitarity and the mass differences \( \Delta m_d \) and \( \Delta m_s \). In section 5, we discuss our results obtained after incorporating all the constraints, for example, unitarity, \( \varepsilon_K \) and \( \Delta m_{d,s} \), simultaneously. A comparison of our results with those of Buras [6] is also included in this section. The implications of the future refinements in the data have been discussed in section 6. Finally, section 7 summarizes our conclusions.

2 Unitarity and its implications

Unitarity of the CKM matrix implies nine relations, three are diagonal and the six non-diagonal relations can be expressed through the six unitarity triangles in the complex plane, for example,

\[
uc \quad V_{ud}V_{cd}^* + V_{us}V_{cs}^* + V_{ub}V_{cb}^* = 0, \quad (1)
db \quad V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0, \quad (2)
ds \quad V_{us}V_{us}^* + V_{cs}V_{cs}^* + V_{ts}V_{ts}^* = 0, \quad (3)
sb \quad V_{us}V_{ub}^* + V_{cs}V_{cb}^* + V_{ts}V_{tb}^* = 0, \quad (4)
utt \quad V_{td}V_{ts}^* + V_{ub}V_{tb}^* + V_{ub}V_{cb}^* = 0, \quad (5)
cct \quad V_{cd}V_{td}^* + V_{cs}V_{ts}^* + V_{cb}V_{tb}^* = 0, \quad (6)
\]

where the letters ‘uc’ etc. represent the corresponding unitarity triangle for further discussions. The areas of all the six triangles (equations [1-6]) are equal and the area of any of the unitarity triangle is related to Jarlskog’s rephasing invariant parameter \( J \) as

\[
J = 2 \times \text{Area of any of the Unitarity Triangle}. \quad (7)
\]

In principle, \( J \) can be evaluated by using above relation through any of the six unitarity triangles, however in practice there are problems associated with the evaluation of area of the triangles. Apart from triangle \( uc \), all other triangles involve elements involving \( t \) quark, which are not experimentally measured as yet. Apparently, it seems that the triangle \( uc \) is ideal for evaluating \( J \) as it involves the elements known with fair degree of accuracy. However, on a closer look one finds that \( uc \) is a highly squashed triangle, for example, the sides \(|V_{ud}V_{cd}|\) and \(|V_{us}V_{cs}|\) are of comparable lengths while the third side \(|V_{ub}V_{cb}|\) is several orders of magnitude smaller compared to the other two, hence it is not easy to evaluate its area unambiguously. Assuming the existence of CP violation through the CKM mechanism, in reference
we have detailed the procedure for evaluation of $J$ through triangle $uc$. Using the experimental values of CKM matrix elements, listed in the Table 1, we obtain

$$J = (2.59 \pm 0.79) \times 10^{-5}. \quad (8)$$

In view of the fact that the triangle is highly squashed, therefore to ascertain whether $J$ given by equation $(8)$ faithfully incorporates the constraints of unitarity, it is very much desirable to compare our results in this regard with those of PDG [19], who have also calculated CKM matrix by incorporating unitarity and experimental data pertaining to $|V_{ud}|$, $|V_{us}|$, $|V_{cd}|$, $|V_{cs}|$ and $|V_{cb}|$. In the context of PDG CKM matrix, it may be noted that once CKM elements are known, one can calculate $J$ corresponding to any of the six triangles, however the two unsquashed triangles $db$ and $ut$ provide the best opportunity in this regard. The $J$ values evaluated from PDG CKM matrix corresponding to the triangles $db$ and $ut$ are given as

$$J_{db} = (2.51 \pm 0.87) \times 10^{-5}, \quad (9)$$

$$J_{ut} = (2.45 \pm 0.91) \times 10^{-5}. \quad (10)$$

The agreement between the equations $(8)$, $(9)$ and $(10)$ strongly supports our evaluation of $J$ through the triangle $uc$ as detailed in [1, 2].

To evaluate $\delta$ from $J$, we first consider the standard representation of CKM matrix [19], for example,

$$V_{\text{CKM}} = \begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
  -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
  s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}, \quad (11)$$

with $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$ for $i, j = 1, 2, 3$. In terms of the CKM elements, $J$ can be expressed as [18]

$$J = s_{12}s_{23}s_{13}c_{13}c_{23}e^{2i\delta} \sin \delta. \quad (12)$$

Calculating $s_{12}$, $s_{23}$ and $s_{13}$ from $|V_{us}|$, $|V_{cb}|$ and $|V_{cb}|$, listed in the table 1, one can plot a distribution for $\delta$, using equations $(8)$ and $(12)$ yielding

$$\delta = 50^\circ \pm 20^\circ, \quad (13)$$

which in the second quadrant translates to

$$\delta = 130^\circ \pm 20^\circ. \quad (14)$$

This value of $\delta$ apparently looks to be the consequence only of the unitarity relationship given by equation $(4)$. However, on further investigations, as shown by Branco and Lavoura [20], one finds that this $\delta$ range is a consequence of all the non trivial unitarity constraints. In this sense, the above range could be attributed to the unitarity of the CKM matrix.

Having found $\delta$, it is desirable to examine whether its above range and the mixing angles $s_{12}$, $s_{23}$ and $s_{13}$, as calculated from the experimental values of $|V_{us}|$, $|V_{cb}|$ and
\(|V_{ub}|\), are able to reproduce the CKM matrix given by PDG. The CKM matrix thus evaluated is given as

\[
\begin{pmatrix}
0.9751 - 0.9761 & 0.2173 - 0.2219 & 0.0025 - 0.0048 \\
0.2169 - 0.2219 & 0.9742 - 0.9754 & 0.0383 - 0.0421 \\
0.005 - 0.0138 & 0.0365 - 0.0420 & 0.9991 - 0.9993
\end{pmatrix}.
\] (15)

The above matrix has a good deal of overlap with that of PDG matrix. To facilitate a more detailed comparison, we present below the PDG matrix \([19]\),

\[
\begin{pmatrix}
0.9742 - 0.9757 & 0.219 - 0.226 & 0.002 - 0.005 \\
0.219 - 0.225 & 0.9734 - 0.9749 & 0.037 - 0.043 \\
0.004 - 0.014 & 0.035 - 0.043 & 0.9990 - 0.9993
\end{pmatrix}.
\] (16)

Comparison of 15 and 16 reveals that for most of the elements, we have an excellent overlap, justifying the procedure followed in carrying out the present analysis. It should be noted that there is an excellent agreement in the case of \(|V_{td}|\), the element having a sensitive dependence on \(\delta\).

Having found \(\delta\), one can evaluate the angles \(\alpha\), \(\beta\) and \(\gamma\) of the unitarity triangle \(db\), likely to be measured in the near future. The angles of the triangle can be expressed in terms of the CKM elements as

\[
\alpha = \arg \left( \frac{-V_{td}V_{tb}^*}{V_{ud}V_{ub}^*} \right),
\] (17)

\[
\beta = \arg \left( \frac{-V_{cd}V_{cb}^*}{V_{td}V_{tb}^*} \right),
\] (18)

\[
\gamma = \arg \left( \frac{-V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} \right).
\] (19)

Making use of the PDG representation of CKM matrix, experimental values of \(|V_{us}|\), \(|V_{cb}|\) and \(|V_{ub}|\) from table [I] and the range of \(\delta\), given by equations [13] and [14], one can easily find out the corresponding ranges for the three angles, listed in column II of Table [I].

The range of \(\sin^2\beta\), corresponding to \(\delta\) given by equations [13] and [14], is

\[
\sin^2\beta = 0.28 \text{ to } 0.88,
\] (20)

which is in good agreement with the latest BABAR results [3], for example,

\[
\sin^2\beta = 0.59 \pm 0.14 \pm 0.05 \quad \text{BABAR.}
\] (21)

This range of \(\sin^2\beta\) is in agreement with the earlier result of BELLE, however has marginal overlap only with the lower limit of recent BELLE results [21], for example,

\[
\sin^2\beta = 0.99 \pm 0.14 \pm 0.06 \quad \text{BELLE.}
\] (22)
Most of the analyses of the unitarity triangle have been carried out using the Wolfenstein parametrization \[^{[7]}\], therefore to facilitate the comparison of our results with other contemporary analyses, we have also evaluated $\bar{\rho}$ and $\bar{\eta}$ defined as

$$\bar{\rho} = \rho \left(1 - \frac{\lambda^2}{2}\right), \quad \bar{\eta} = \eta \left(1 - \frac{\lambda^2}{2}\right)$$

where $\rho = \frac{s_{12}}{s_{12}s_{23}} \cos \delta$, $\eta = \frac{s_{12}}{s_{12}s_{23}} \sin \delta$ and $\lambda \simeq s_{12}$. It may be mentioned that while evaluating $\alpha$, $\beta$, $\gamma$, $\bar{\rho}$ and $\bar{\eta}$, we have used the method of “full scanning” of the input parameters as employed by Buras et al. \[^{[8]}\]. Our results in this regard are listed in the Table\[^{[3]}\].

### 3 $K^0 - \bar{K}^0$ mixing and its implications

From the previous section, it is clear that unitarity alone doesn’t give any significant constraints on $\delta$ and $|V_{td}|$. However, in view of the fact that the CP violating parameter $\varepsilon_K$ is experimentally very well determined, it is therefore interesting to examine its implications on $\delta$ and $|V_{td}|$. To this end, we consider the following QCD corrected and quark loop dominated expression for $\varepsilon_K$ \[^{[22]}\],

$$|\varepsilon_K| = \frac{G_F^2 F_K^2 m_K m_{\bar{K}}^2}{6\sqrt{2}\pi^2 \Delta m_k} B_K \text{Im} \lambda_t [\text{Re} \lambda_c (\eta_1 F(x_c) - \eta_3 F(x_c, x_t)) - \text{Re} \lambda_t \eta_2 F(x_t)],$$

where $F(x_i)$ are Inami-Lim functions, $x_i = m_i^2/M_W^2$ and $\lambda_i = V_{id}V_{is}^*$, $i = c, t$. In terms of the mixing angles and the phase $\delta$, $\text{Im} \lambda_t$, $\text{Re} \lambda_t$ and $\text{Re} \lambda_c$ can be expressed as

$$\text{Im} \lambda_t = s_{23}s_{13}c_{23}\sin\delta, \quad \text{Re} \lambda_t = s_{23}s_{13}c_{23}(c_{12}^2 - s_{12}^2)\cos\delta - s_{12}c_{12}(s_{23}^2 - c_{23}s_{13}^2), \quad \text{Re} \lambda_c = s_{23}s_{13}c_{23}(s_{12}^2 - e_{12}^2)\cos\delta - s_{12}c_{12}(c_{23}^2 - s_{23}s_{13}^2).$$

To examine the exclusive role played by $\varepsilon_K$ in restricting $\delta$, we have calculated $\varepsilon_K$ by varying $\delta$ from $0^\circ$ to $180^\circ$ and by “full scanning” of the various input parameters. The experimental limits on $\varepsilon_K$, for example

$$\varepsilon_K = (2.280 \pm 0.013) \times 10^{-3},$$

yield $\delta$ in the range

$$\delta = 33^\circ$ to $163^\circ.$$

The above range of $\delta$ when combined with the constraints obtained in the last section from unitarity becomes

$$\delta = \begin{cases} 33^\circ$ to $70^\circ & \text{(in I quadrant)} \vphantom{100^\circ} \\ 110^\circ$ to $150^\circ & \text{(in II quadrant)}. \end{cases}$$
In the Table 2, we have listed the ranges for $|V_{td}|$, $\alpha$, $\beta$, $\gamma$, $\bar{\rho}$ and $\bar{\eta}$ obtained from the unitarity constraints, the results corresponding to the combined constraints of unitarity and $\varepsilon_K$ are presented in column III of the same table.

There are several important points which need to be discussed further. It is interesting to note that even a well determined $\varepsilon_K$, despite its direct proportionality to $\sin \delta$, is not able to improve upon the range of $\delta$ implied by unitarity. This could be understood presumably by closely examining the expression for $\varepsilon_K$, which involves parameters such as $B_K$, $|\frac{V_{ub}}{V_{cb}}|$ and $|V_{cb}|$, having much larger uncertainties than that of $\varepsilon_K$.

Similarly, one finds that $\varepsilon_K$ is not able to constrain $|V_{td}|$ much in comparison to the unitarity. This looks to be a consequence of not so improved constraints on $\delta$ in comparison with the unitarity. To examine this more closely, we consider the expression for $V_{td}$ in the standard parametrization, for example,

$$V_{td} = s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i \delta}.$$  \hspace{1cm} (32)

In figure 4, we have plotted $\delta$ versus $|V_{td}|$, depending upon $s_{12}$, $s_{23}$, $s_{13}$ and $\delta$, the solid curve corresponding to maximum $|V_{td}|$ and the other one corresponding to minimum $|V_{td}|$. It is clear from the figure that for low values of $\delta$ in the first quadrant and correspondingly for large values of $\delta$ in the second quadrant, $|V_{td}|$ is not very sensitive to variations in $\delta$. For example, examining the solid curve, one finds that variation in $\delta$ from 0° to 20° corresponds to $|V_{td}|$ from 0.0046 to 0.0049, while in the case when $\delta$ varies from 70° to 90°, there is a sharp variation in $|V_{td}|$, for example it varies from 0.0078 to 0.0094. Similar conclusions follow from the other curve. Hence $\varepsilon_K$ constraint is not able to register any noticeable improvement in the range of $|V_{td}|$ in comparison to the unitarity constraints.

After having examined the reasons for not so sharp limits on $\delta$ and $|V_{td}|$ by $\varepsilon_K$, it may be of interest to investigate which parameters are primarily responsible for this. We have already noted that $\varepsilon_K$ depends on $B_K$, $|\frac{V_{ub}}{V_{cb}}|$ and $|V_{cb}|$, etc., it is therefore likely that the uncertainties in $B_K$ and the CKM elements $|\frac{V_{ub}}{V_{cb}}|$ and $|V_{cb}|$ might be playing an important role in giving not so sharp limits on $\delta$. To examine this point further, in figure 4 we have plotted $\varepsilon_K$ versus $\delta$ scanning full range of $|\frac{V_{ub}}{V_{cb}}|$, keeping other parameters fixed. The uppermost curve is for $|\frac{V_{ub}}{V_{cb}}| = 0.115$ and the lowermost curve corresponds to $|\frac{V_{ub}}{V_{cb}}| = 0.065$, the curves corresponding to other values of $|\frac{V_{ub}}{V_{cb}}|$ lying in between. Horizontal lines represent the experimental range for $\varepsilon_K$. From the figure, it is evident that there is no value of $|\frac{V_{ub}}{V_{cb}}|$ which is excluded by $\varepsilon_K$ constraints. Further, it is also clear from the figure that the refinements in $|\frac{V_{ub}}{V_{cb}}|$ would not lead to any significant improvements in the range of $\delta$ beyond that given in equation 29, however if the upper limit of $|\frac{V_{ub}}{V_{cb}}|$ comes down, then the lower limit of $\delta$ will go up slightly. In the same manner, one can find that for the present experimental range of $|V_{cb}|$, no value of $|V_{cb}|$ is excluded by the $\varepsilon_K$ constraints, as well as further refinements in $|V_{cb}|$ will not affect the $\delta$ range significantly. From the above discussion, one may conclude that the reason for not so sharp cuts on
basically emanates from the ‘large uncertainties’ in \( B_K \).

4 \( B^0 - \bar{B}^0 \) mixing and its implications

We have already seen that \( \varepsilon_K \) and unitarity are not able to constrain \( \delta \) much. However, it is well known that the \( B^0 - \bar{B}^0 \) mixing is a loop dominated process with major contributions coming from the \( t \) quark, consequently from \( V_{td} \). Therefore \( B_d^0 - \bar{B}_d^0 \) mixing parameter \( x_d \), related to the mass difference \( \Delta m_d \), is expected to yield significant constraints on \( |V_{td}| \). To this end, we consider the QCD corrected expression for \( \Delta m_d \)\[22\], for example

\[
\Delta m_d = \frac{G_F^2}{6\pi^2} \eta_{QCD} m_{B_d} B_{B_d} F_{B_d}^2 M_W^2 F(x_t) |V_{td}|^2.
\] (33)

Using the input values of various parameters listed in table \[1\], and carrying out full scanning of all the parameters, we obtain

\[
|V_{td}| = 0.0064 \text{ to } 0.0089.
\] (34)

Comparing this range with that obtained from unitarity and \( \varepsilon_K \) given in table \[2\], we find that \( \Delta m_d \) considerably improves the constraints on \( |V_{td}| \). The above improved constraints on \( |V_{td}| \) can be used to yield constraints on \( \delta \). Using \( |V_{us}|, |V_{cb}| \) and \( |\frac{V_{td}}{V_{td}}| \), listed in the table \[1\], along with the equation \[32\] and above range of \( |V_{td}| \), we obtain the following limits on \( \delta \)

\[
\delta = 1^\circ \text{ to } 104^\circ.
\] (35)

It may be noted that the above range of \( \delta \) is exclusively a consequence of \( \Delta m_d \) constraint. This range when combined with the unitarity constraints, obtained in section 2, yields

\[
\delta = 30^\circ \text{ to } 70^\circ.
\] (36)

Thus, the combined constraints from \( \Delta m_d \) and unitarity limit \( \delta \) to the first quadrant. The corresponding ranges of \( \alpha, \beta, \gamma, \bar{\rho} \) and \( \bar{\eta} \) are listed in the column IV of Table \[2\].

For the sake of completeness, we have also investigated whether \( \Delta m_s \) could also constrain \( \delta \). In this regard, we find that the presently available lower limit on \( \Delta m_s \) (\( \Delta m_s > 15.0 \text{ ps}^{-1} \)) does not lead to any appreciable constraints on \( \delta \) and other quantities mentioned above. This is because of the fact that in \( \Delta m_s \), the \( \delta \) dependence comes through \( V_{is} \), an element which is not very sensitive to variations in \( \delta \). However, our calculations reveal that the lower bound on \( \Delta m_s \) yields a constraint on the upper bound of \( \delta \), for example, if in future \( \Delta m_s \) is found to be greater than \( 27 \text{ ps}^{-1} \), then one can show that this would imply \( \delta \leq 138^\circ \). Therefore, until and unless one finds a remarkable change in the lower limit of \( \Delta m_s \), one may not be able to improve upon a much better bound on \( \delta \) due to \( \Delta m_d \).
5 Unitarity, $\varepsilon_K$ and $\Delta m_{d,s}$ together

In most of the past analyses of the CKM phenomenology the constraints from unitarity, $\varepsilon_K$, $\Delta m_d$ and $\Delta m_s$ have been simultaneously applied to obtain $\bar{\rho}$, $\bar{\eta}$ and the CP angles $\alpha$, $\beta$ and $\gamma$. To facilitate comparison of our results with other recent analyses [4, 5, 6] as well as to construct the unitarity triangle by incorporating all the available constraints, we have also carried out an analysis wherein the implications of unitarity, $K^o - \bar{K}^o$ mixing and $B^o - \bar{B}^o$ mixing have been studied together. The corresponding results have been listed in the last column of the Table 2. A closer look at the Table 2 reveals several interesting points. The combined constraints of unitarity, $K^o - \bar{K}^o$ mixing and $B^o - \bar{B}^o$ mixing force $\delta$ to lie in the range $37^o$ to $70^o$, implying a very restricted range for $|V_{td}|$, for example

$$|V_{td}| = 0.0065 \text{ to } 0.0089.$$  \hspace{1cm} (37)

The corresponding range of $\sin2\beta$ is

$$\sin2\beta = 0.50 \text{ to } 0.88,$$  \hspace{1cm} (38)

which is almost within the latest BABAR results, however the lower limit is slightly above that of BABAR. In case the recent BELLE result (equation (22)) is confirmed, it will have serious implications for CKM paradigm as well as the origin of CP violation in the Standard Model. These issues will be discussed in detail elsewhere.

6 Implications of refinements in data

In the last section, we have seen that the combined constraints of unitarity, $K^o - \bar{K}^o$ mixing and $B^o - \bar{B}^o$ mixing have significant impact in constraining $\delta$ and consequently other CKM elements, particularly $|V_{td}|$. It, therefore, becomes interesting to
examine the implications of future refinements in the data on $\delta$ and other quantities mentioned in table 2.

The unitarity analysis involves the elements of first two rows of CKM matrix, all of which are known with the reasonable accuracy except for $|V_{ub}|$ and $|V_{cb}|$, where the uncertainties are more than 15%. Therefore, it would be interesting to examine the consequences of the future refinements in these two elements. In this context, we have repeated the unitarity based analysis with the future value of $|V_{ub}|$, $|V_{cb}|$ = 0.090 ± 0.010, whereas for $|V_{cs}|$, we consider the latest LEP value $|V_{cs}| = 0.996 ± 0.013$ [23]. The corresponding results for $\delta$, $|V_{td}|$, $\alpha$, $\beta$, $\gamma$, $\bar{\rho}$ and $\bar{\eta}$ are summarized in the Table 3. From the table, one finds that the future refinements in $|V_{ub}|$ and $|V_{cs}|$ may result in the relatively stronger bounds on the quantities mentioned above. In particular, the lower limit of $|V_{td}|$ and $\sin 2\beta$ goes up noticeably.

As has already been mentioned, the $\varepsilon_K$ analysis with the future refinements of data will not have significant implications. However, in the case of $\Delta m_d$ a refinement in $F_{B_d}/B_{B_d}$ to the value 0.230 ± 0.010 GeV, along with refinement in $|V_{ub}|/V_{cb}$ mentioned above, has quite marked implications on $|V_{td}|$ and $\delta$. In particular, $\delta$ range will improve from $1^\circ$ - $104^\circ$ to $38^\circ$ - $80^\circ$, the corresponding improved range for $|V_{td}|$ is 0.0070 to 0.0084. In the table 3 we have listed the ranges for $\delta$, $|V_{td}|$, $\alpha$, $\beta$, $\gamma$, $\bar{\rho}$ and $\bar{\eta}$ found by using future refinements of data corresponding to different combinations of constraints due to unitarity, $\varepsilon_K$ and $\Delta m_{d,s}$.

It is of interest to discuss the implications of the refinements in the measurement of $\sin 2\beta$. In this context, it may be noted that the lower and upper limit on $\delta$, given by equation 3, can be attributed respectively to unitarity and $\Delta m_d$. One can easily check that with $\delta$ in the first quadrant, the lower and upper limits of $\sin 2\beta$ correspond to the lower and upper limits of $\delta$ respectively. Therefore, refinements in the upper limit of $\sin 2\beta$ will have significant implications for $\Delta m_d$, whereas the lower limit will have implications for unitarity.

7 Summary and conclusions

A recently carried out unitarity based analysis [1, 2] involving the evaluation of Jarlskog’s rephasing invariant parameter $J$ as well as the CP violating phase $\delta$ is extended to include the effects of $K^0 - \bar{K}^0$ and $B^0 - \bar{B}^0$ mixings. The present analysis attempts to evaluate CP violating phase $\delta$, $|V_{td}|$ and the angles of the unitarity triangle $\alpha$, $\beta$ and $\gamma$ by invoking the constraints imposed by unitarity, $K^0 - \bar{K}^0$ and $B^0 - \bar{B}^0$ mixings independently as well as collectively. As discussed in references [1, 2], the unitarity constraints have been included via $J$ evaluated through the triangle $uc$. Using the $\delta$ range, found from the constraints of unitarity only, we are able to reproduce the PDG CKM matrix rather well, also found from the considerations of unitarity. It may be mentioned that we have used the method of “full scanning” of the input parameters, also employed by Buras et al. [3]. The $K^0 - \bar{K}^0$ mixing and $B^0 - \bar{B}^0$ mixing constraints have been incorporated through
The implications of above mentioned phenomena have been studied one by one on the CKM parameters, in particular on the CP violating phase $\delta$ and $|V_{td}|$. This enables one to explore their role clearly and independently of each other in constraining $\delta$ and consequently $|V_{td}|$ and the angles $\alpha$, $\beta$ and $\gamma$ of the unitarity triangle $db$. Finally we incorporate all the constraints together in order to compare our result with the other recent analyses, in particular with Buras et al.

Using the latest data listed in table I and incorporating the unitarity constraints, we get $\delta = 50^\circ \pm 20^\circ$ (in I quadrant), $130^\circ \pm 20^\circ$ (in II quadrant) and $5.1 \times 10^{-3} \leq |V_{td}| \leq 13.8 \times 10^{-3}$. The combined constraints of unitarity and $\varepsilon_K$ yield, $33^\circ \leq \delta \leq 70^\circ$ (in I quadrant), $110^\circ \leq \delta \leq 150^\circ$ (in II quadrant) and $5.8 \times 10^{-3} \leq |V_{td}| \leq 13.6 \times 10^{-3}$, thus slightly improving only the lower limit of $\delta$ over and above unitarity. However, the combined constraints of unitarity and $\Delta m_s$ restrict $\delta$ and $|V_{td}|$ significantly, for example, $30^\circ \leq \delta \leq 70^\circ$ and $6.4 \times 10^{-3} \leq |V_{td}| \leq 8.9 \times 10^{-3}$. We find that the presently available lower limit on $\Delta m_s$ ($\Delta m_s > 15.0 \text{ ps}^{-1}$) doesn't yield any significant constraints on $\delta$. Finally, incorporating all the constraints together we get, $37^\circ \leq \delta \leq 70^\circ$, $6.5 \times 10^{-3} \leq |V_{td}| \leq 8.9 \times 10^{-3}$, and the corresponding ranges for $\alpha$, $\beta$ and $\gamma$ are, $80^\circ \leq \alpha \leq 214^\circ$, $15.1^\circ \leq \beta \leq 31^\circ$ and $37^\circ \leq \gamma \leq 70^\circ$, which are in agreement with the recent results of Buras et al.. Also, the calculated range of $\sin^2 \beta$ is within the recently updated BABAR results. This range is in agreement with the earlier result of BELLE, however has marginal overlap only with the lower limit of recent BELLE results.

We have also examined the implications of future refinements in data and have found that the future refinements in $|V_{td}|$, $|V_{cb}|$, $|V_{cs}|$ and $F_{B_d} \sqrt{B_{B_d}}$ etc. would considerably narrow the range of $\delta$ and consequently of $|V_{td}|$. Since, with $\delta$ in the first quadrant, the lower and upper limits of $\sin^2 \beta$ correspond to the lower and upper limits of $\delta$, respectively, therefore, the refinements in the measurements of upper limit and the lower limit of $\sin^2 \beta$ will have implications for $\Delta m_d$ and unitarity of the CKM matrix respectively.

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**References**

[1] Monika Randhawa, V. Bhatnagar, P. S. Gill and M.Gupta, Mod. Phys. Letts A15, 2363(2000).

[2] Monika Randhawa and Manmohan Gupta, Phys. Letts. B516, 446(2001).
[3] David G. Hitlin, BABAR Collaboration, [hep-ex/0011024]; B. Aubert et al., BABAR Collaboration, Phys. Rev. Lett. 86, 2515(2001).

[4] Hiroaki Aihara, BELLE Collaboration, [hep-ex/0010008]; A. Abashian et al., BELLE Collaboration, Phys. Rev. Lett. 86, 2509(2001).

[5] B. Aubert et al., BABAR Collaboration, Phys. Rev. Lett. 87, 091801(2001).

[6] Andrzej J. Buras, [hep-ph/0101336] and references therein.

[7] M. Ciuchini, G. D’Agostini, E. Franco, V. Lubicz, G. Martinelli, F. Parodi, P. Roudeau and A. Stocchi, JHEP 0107, 013(2001).

[8] A. Hocker, H. Lacker, S. Laplace and F. Le Diberder, Eur. Phys. J. C21, 225(2001).

[9] A. L. Kagan and M. Neubert, Phys. Lett. B492, 115(2000).

[10] J. P. Silva and L. Wolfenstein, Phys. Rev. D63, 056001(2001).

[11] G. Eyal, Y. Nir and G. Perez, JHEP 0008, 028(2000); Z. Z. Xing, [hep-ph/0008018]; Y. Nir, [hep-ph/0008220]; A. J. Buras and R. Buras, Phys. Lett. B501, 223(2001); A. Masiero, M. Piai and O. Vives, Phys. Rev. D64, 055008(2001).

[12] T. Hurth, J. Phys. G27, 1277(2001); T. Hurth and T. mannel, [hep-ph/0109041].

[13] Manmohan Gupta and P. S. Gill, Pramana 38, 477(1992); P. S. Gill and Manmohan Gupta, Mod. Phys. Lett. A13, 2445(1998).

[14] M. Gronau and J. L. Rosner, Phys. Rev. Lett. 76, 1200(1996); J.L. Rosner, Braz. J. Phys. 31, 147(2001); J. Ellis, Nucl. Phys. Proc. Suppl. 99A, 331(2001); H. Fritzsch and Z. Z. Xing, Nucl. Phys. B556, 49(1999).

[15] R. Fleischer, [hep-ph/0011323]; A. Ali, in Proc. of the 13th Topical Conference on Hadron Collider Physics, TIFR, Mumbai, India (1999); Eur. Phys. J. C9, 687(1999); I. I. Bigi and A. I. Sanda, [hep-ph/9909479].

[16] R.D. Peccei, [hep-ph/9909236]; [hep-ph/0004152]; John Swain and Lucas Taylor, Phys. Rev. D58, 093006(1998); Stefan Herrlich and Ulrich Nierste, Phys. Rev. D52, 6505(1995); S. Mele, [hep-ph/9808411]; Proceedings of workshop on CP violation, Adelaide, Australia; Phys Rev. D59, 113011(1999).

[17] L. Wolfenstein, Phys. Rev. Lett. 51, 1945(1983).

[18] CP violation, Ed. L. wolfenstein, North Holland, elsevier Science Publishers B.V., 1989; CP violation, Ed. C. Jarlskog, World Scientific Publishing Co. Pte. Ltd, 1989.
[19] D.E. Groom et. al., Particle Data group, Euro. Phys. J. C15, 1(2000).
[20] G.C. Branco and L. Lavoura, Phys. Lett. B208, 123(1988).
[21] K. Abe, et al, BELLE Collaboration, Phys. Rev. Letts. 87, 091802(2001).
[22] G. Buchalla, Andrzej J. Buras, M. E. Lautenbacher, Rev. Mod. Phys. 68, 1125(1996).
[23] J. Drees, [hep-ex/0110077](http://arxiv.org/abs/hep-ex/0110077).
[24] L. Lellouch, Nucl. Phys. Proc. Suppl. 94, 142(2001).
[25] S. Herrlich and U. Nierste, Nucl. Phys. B419, 292(1994).
[26] The LEP B Oscillation Working Group, [http://lepbosc.web.cern.ch/LEPBOSC](http://lepbosc.web.cern.ch/LEPBOSC), LEPBOSC 98/3.

Figure 1: Plot of $\delta$ versus $|V_{td}|$. The solid line corresponds to maximum value of $|V_{td}|$ for a given $\delta$ and the broken line corresponds to minimum value of $|V_{td}|$ for a given $\delta$.  


Figure 2: Plot of $\delta$ versus $\varepsilon_K$ for $|{V_{ub}}/{V_{cb}}| = 0.065, 0.078, \ldots, 0.115$, keeping other parameters fixed. The horizontal lines represent the experimental limits of $\varepsilon_K$ given by equation 28.

| Parameter | Value | Reference |
|-----------|-------|-----------|
| $G_F^2 F_K^2 m_K m_W^2$ | $3.84 \times 10^4$ | [6] |
| $F_{B_d} B_{B_d}$ | $230 \pm 25 \pm 20$ MeV | [7] |
| $|V_{ud}|$ | $0.9735 \pm 0.0008$ | [19] |
| $|V_{us}|$ | $0.2196 \pm 0.0023$ | [19] |
| $|V_{cd}|$ | $0.224 \pm 0.016$ | [19] |
| $|V_{cs}|$ | $1.04 \pm 0.16$ | [19] |
| $|V_{cb}|$ | $0.0402 \pm 0.0019$ | [19] |
| $|{V_{ub}}/{V_{cb}}|$ | $0.090 \pm 0.025$ | [19] |
| $m_W$ | $80.419 \pm 0.056$ GeV | [19] |
| $m_t$ | $167 \pm 5$ GeV | [19] |
| $m_c$ | $1.3 \pm 0.1$ GeV | [19] |
| $m_{B_d}$ | $5279.2 \pm 1.8$ MeV | [19] |
| $B_K$ | $0.87 \pm 0.06 \pm 0.13$ | [24] |
| $\eta_1$ | $1.38 \pm 0.53$ | [25] |
| $\eta_2$ | $0.574 \pm 0.004$ | [25] |
| $\eta_3$ | $0.47 \pm 0.0$ | [25] |
| $\eta_{QCD}$ | $0.55 \pm 0.01$ | [25] |
| $\Delta m_d$ | $0.487 \pm 0.014 \pm 0.032$ ps$^{-1}$ | [26] |
| $\Delta m_s$ | $> 15.0$ ps$^{-1}$ at 95% C.L. | [26] |

Table 1: Values of the input parameters used for the analysis.
Using only unitarity, $\varepsilon_K$ and $\Delta m_d$, Buras’s results

|      | Using only unitarity | Unitarity and $\varepsilon_K$ | Unitarity and $\Delta m_d$ | Unitarity, $\varepsilon_K$ and $\Delta m_d$ | Buras’s results |
|------|----------------------|-------------------------------|-----------------------------|---------------------------------|-----------------|
| $\delta$ | $50^\circ \pm 20^\circ$, $130^\circ \pm 20^\circ$ | $33^\circ$ to $70^\circ$, $110^\circ$ to $150^\circ$ | $30^\circ$ to $70^\circ$ | $37^\circ$ to $70^\circ$ | $\square$ |
| $|V_{td}|$ | $(5.1$ to $13.8) \times 10^{-3}$ | $(5.8$ to $13.6) \times 10^{-3}$ | $(6.4$ to $8.9) \times 10^{-3}$ | $(6.5$ to $8.9) \times 10^{-3}$ | $(6.7$ to $9.3) \times 10^{-3}$ |
| $\alpha$ | $20^\circ$ to $139^\circ$ | $20^\circ$ to $124^\circ$ | $80^\circ$ to $139^\circ$ | $80^\circ$ to $124^\circ$ | $78.8^\circ$ to $120^\circ$ |
| $\beta$ | $6.5^\circ$ to $31^\circ$ | $7.4^\circ$ to $31^\circ$ | $11^\circ$ to $31^\circ$ | $15^\circ$ to $31^\circ$ | $15.1^\circ$ to $28.6^\circ$ |
| $\gamma$ | $50^\circ \pm 20^\circ$, $130^\circ \pm 20^\circ$ | $33^\circ$ to $70^\circ$, $110^\circ$ to $150^\circ$ | $30^\circ$ to $70^\circ$ | $37^\circ$ to $70^\circ$ | $37.9^\circ$ to $76.5^\circ$ |
| $\bar{\rho}$ | $-0.447$ to $0.447$ | $-0.447$ to $0.433$ | $0.097$ to $0.387$ | $0.097$ to $0.387$ | $0.06$ - $0.34$ |
| $\bar{\eta}$ | $0.143$ to $0.486$ | $0.159$ to $0.486$ | $0.143$ to $0.485$ | $0.225$ to $0.486$ | $0.22$ - $0.46$ |

Table 2: Present results regarding $\delta$, $|V_{td}|$, $\alpha$, $\beta$, $\gamma$, $\bar{\rho}$ and $\bar{\eta}$ obtained by using the inputs given in Table I for different combinations of constraints.

|      | Using only unitarity | Unitarity and $\varepsilon_K$ | Unitarity and $\Delta m_d$ | Unitarity, $\varepsilon_K$ and $\Delta m_d$ |
|------|----------------------|-------------------------------|-----------------------------|---------------------------------|
| $\delta$ | $60^\circ \pm 18^\circ$, $120^\circ \pm 18^\circ$ | $42^\circ$ to $78^\circ$, $102^\circ$ to $138^\circ$ | $42^\circ$ to $78^\circ$ | $42^\circ$ to $78^\circ$ |
| $|V_{td}|$ | $(6.4$ to $12.9) \times 10^{-3}$ | $(6.6$ to $12.1) \times 10^{-3}$ | $(7.0$ to $8.4) \times 10^{-3}$ | $(7.0$ to $8.4) \times 10^{-3}$ |
| $\alpha$ | $29^\circ$ to $128^\circ$ | $29^\circ$ to $118^\circ$ | $76^\circ$ to $120^\circ$ | $76^\circ$ to $117^\circ$ |
| $\beta$ | $10^\circ$ to $27^\circ$ | $11^\circ$ to $27^\circ$ | $18^\circ$ to $27^\circ$ | $18^\circ$ to $27^\circ$ |
| $\gamma$ | $60^\circ \pm 18^\circ$, $120^\circ \pm 18^\circ$ | $42^\circ$ to $78^\circ$, $102^\circ$ to $138^\circ$ | $42^\circ$ to $78^\circ$ | $42^\circ$ to $78^\circ$ |
| $\bar{\rho}$ | $-0.334$ to $0.334$ | $-0.334$ to $0.334$ | $0.074$ to $0.311$ | $0.074$ to $0.311$ |
| $\bar{\eta}$ | $0.235$ to $0.439$ | $0.235$ to $0.439$ | $0.235$ to $0.439$ | $0.244$ to $0.439$ |

Table 3: The results regarding $\delta$, $|V_{td}|$, $\alpha$, $\beta$, $\gamma$, $\bar{\rho}$ and $\bar{\eta}$ using the "future" values: $|V_{cs}| = 0.996 \pm 0.013$, $|V_{ub}/V_{cb}| = 0.090 \pm 0.10$ and $F_{B_d}\sqrt{B_{B_d}} = 0.230 \pm 0.010 \text{GeV}$. 