CLASSIFICATION OF THE ASYMPOTIC BEHAVIOUR OF GLOBALLY STABLE LINEAR DIFFERENTIAL EQUATIONS WITH RESPECT TO STATE–INDEPENDENT STOCHASTIC PERTURBATIONS

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Abstract. In this paper we consider the global stability of solutions of an affine stochastic differential equation. The differential equation is a perturbed version of a globally stable linear autonomous equation with unique zero equilibrium where the diffusion coefficient is independent of the state. We find necessary and sufficient conditions on the rate of decay of the noise intensity for the solution of the equation to be globally asymptotically stable, stable but not asymptotically stable, and unstable, each with probability one. In the case of stable or bounded solutions, or when solutions are a.s. unstable asymptotically stable in mean square, it follows that the norm of the solution has zero liminf, by virtue of the fact that $\|X\|^2$ has zero pathwise average a.s.s. Sufficient conditions guaranteeing the different types of asymptotic behaviour which are more readily checked are developed. It is also shown that noise cannot stabilise solutions, and that the results can be extended in all regards to affine stochastic differential equations with periodic coefficients.

1. Introduction

In this paper we analyse the asymptotic behaviour of finite–dimensional affine stochastic differential equations. We suppose that in the absence of a stochastic perturbation that there is unique and globally stable equilibrium at zero. The perturbation can be viewed as an external force, in the sense that the intensity of the entries in the diffusion matrix are independent of the state.

Therefore we may consider the underlying $d$–dimensional ordinary (deterministic) differential equation

$$x'(t) = Ax(t), \quad t \geq 0; \quad x(0) = \xi \in \mathbb{R}^d.$$  

Here we have that $A$ is a $d \times d$ real matrix. Since we are presuming that there is a unique equilibrium at zero, and that it is globally stable, we assume that all the eigenvalues of $A$ have negative real parts. One of the important tasks in this paper is to classify the asymptotic behaviour of the stochastic differential equation

$$dX(t) = AX(t) \, dt + \sigma(t) \, dB(t) \quad (1.1)$$

In this setting, $\sigma$ is a continuous and deterministic function and $B$ is a finite dimensional Brownian motion. Specifically, we let

$$\sigma \in C([0, \infty); \mathbb{R}^{d \times r}) \quad (1.2)$$

and $B$ be an $r$–dimensional standard Brownian motion.

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1
Since equations with state-independent noise should be in general simpler to analyse than the state-dependent case, and their applications are of interest, it is not surprising that such equations have attracted a lot of attention. Lyapunov function techniques have been applied to study their asymptotic stability in Khas'minski [12], with a lot of emphasis given to equations with perturbations $\sigma$ being in $L^2(0, \infty)$. However, in a pair of papers in 1989, Chan and Williams [10] and Chan [9] demonstrated that the stability of global equilibria in these systems could be preserved with a much slower rate of decay in $\sigma$: in fact, they showed that provided the noise perturbation decayed monotonically in its intensity, then solutions converged to the equilibrium with probability one if and only if
\[
\lim_{t \to \infty} \|\sigma(t)\|^2 \log t = 0.
\]
These results also required strong assumptions on the strength of the nonlinear feedback. Shortly thereafter, Rajeev [20] demonstrated that these results could be generalised to equations with some non-autonomous features, and some results on bounded solutions were obtained. In parallel, Mao demonstrated in [19] that a polynomial rate of decay of solutions was possible if the perturbation intensity decayed at a polynomial rate. These results were extended to neutral functional differential equations by Mao and Liao in [16], with exponential decaying upper bounds on the intensity giving rise to an exponential convergence rate in the solution.

After this, Appleby and his co-authors extended Chan and Williams’ results to stochastic functional differential equations [7] and to Volterra equations especially (see Appleby and Appleby and Riedle [3, 5]), with extensions to discrete Volterra equations appearing in Appleby, Riedle and Rodkina [6]. Necessary and sufficient conditions for exponential stability in linear Volterra equation in the presence of fading noise was studied in [4].

One of the papers which has most influence on this work is Appleby, Gleeson and Rodkina [2], which returns directly to the nonlinear equations studied by Chan and Williams in [10]. In it, the monotonicity assumptions on $\sigma$ were completely relaxed, and the mean reversion strength was also considerably weakened. Moreover, results on unbounded and unstable solutions also appeared for the first time. However, the finite dimensional case was not addressed, nor was a complete classification of the dynamics presented. The goal this paper is to address this of the thesis is to address each of these shortcomings.

An important idea which appears in [7, 5, 2] in various forms is that many facts about more complicated stochastic differential, functional or Volterra equations with state-independent noise can be inferred from a much simpler $d$-dimensional equation whose solution $Y$ which is given by
\[
dY(t) = -Y(t)dt + \sigma(t)dB(t), \quad t \geq 0; \quad Y(0) = 0.
\]
(1.3)
In fact, we demonstrate that $X$ and $Y$ have equivalent asymptotic behaviour, in the sense that $X$ converges to zero if and only if $Y$ does; is bounded but not convergent if and only if $Y$ is; and is unbounded if and only if $Y$ is.

Therefore, the question of analysing the asymptotic behaviour of the general linear equation reduces to that of studying the special linear equation (1.3). If $\sigma$ is identically zero, it follows that the solution of
\[
y'(t) = -y(t), \quad t \geq 0; \quad y(0) = 0.
\]
obeyes $y(t) = 0$ for all $t \geq 0$ if $y(0) = 0$. The question naturally arises as under what condition on $\sigma$ does the solution $Y(t)$ obey
\[
\lim_{t \to \infty} Y(t) = 0, \quad a.s.
\]
(1.4)
It is shown in [10] that $Y(t)$ obeys (1.3) in the one-dimensional case if
\[ \lim_{t \to \infty} \sigma^2(t) \log t = 0. \]

Moreover in [10], it is shown that if $t \to \sigma^2(t)$ is decreasing to zero, and $Y(t)$ obeys (1.3), then we must have $\lim_{t \to \infty} \sigma^2(t) \log t = 0$. These results are extended to finite-dimensions in [9]. In [2], monotonicity assumptions on $\sigma$ are relaxed, and results for unbounded solutions for (1.3) are presented. However, none of these papers classify all the possible types of asymptotic behaviour of $Y$. This situation was rectified in the scalar case ($d = 1$) in [11], in which the asymptotic behaviour of solutions of (1.3) are classified.

In this paper, we extend the classification of solutions to the general finite-dimensional case. In fact, we characterise the convergence, boundedness and unboundedness of solutions of (1.3), and this leads in turn to a classification of the convergence, boundedness and unboundedness of solutions of (1.1). Moreover, it turns out that neither pointwise convergence rates nor pointwise monotonicity are needed in order to achieve this classification. Our main results show that neither pointwise convergence rates nor pointwise monotonicity are needed in order to achieve this classification. Our main results show that $X$ obeys $\lim_{t \to \infty} X(t) = 0$ a.s. if and only if
\[ S_h(\epsilon) = \sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{nh}^{(n+1)h} \|\sigma(s)\|^2 ds}} \right) \right\} < +\infty, \quad \text{for every } \epsilon > 0, \quad (1.5) \]

where $\Phi$ is the distribution function of a standardised normal variable and $h$ is any positive constant. We also show that in contrast to (1.3), if $S_h(\epsilon)$ is infinite for all $\epsilon$, we have that $\lim \sup_{t \to \infty} \|X(t)\| = +\infty$; while if the sum is finite for some $\epsilon$ and infinite for others, then $c_1 \leq \lim \sup_{t \to \infty} \|X(t)\| \leq c_2$ a.s., where $0 < c_1 \leq c_2 < +\infty$ are deterministic and $\lim \inf_{t \to \infty} \|X(t)\| = 0$ a.s. In this last case, when $X$ is bounded, the solution spends most of the time close to zero, because
\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \|X(s)\|^2 ds = 0, \quad \text{a.s.} \quad (1.6) \]

Since $S_h(\epsilon)$ is monotone in $\epsilon$, it can be seen that we can describe the asymptotic behaviour for every function $\sigma$, and that, moreover, the stability, boundedness or unboundedness of the solution depends on $\sigma$ only through the overall intensity of the perturbation through the Frobenius norm $\|\sigma\|_F$, and not through the configuration of the perturbation and its interaction with the matrix $A$. Moreover, it can be seen that these conditions which guarantee convergence, boundedness or unboundedness are independent of the matrix $A$. Also, by virtue of the form of $S_h(\epsilon)$ and the equivalence of all norms on $\mathbb{R}^{d \times r}$, it follows that the asymptotic behaviour relies only on $\|\sigma\|$, where $\|\cdot\|$ is any norm in $\mathbb{R}^{d \times r}$.

Since the underlying deterministic differential equations is assumed to be stable, it is of interest to determine its response to fading noise perturbations. In this case, we can find a quite general characterisation of “fading noise” which yields a more comprehensive picture about the asymptotic behaviour of $X$. If the fading noise condition is $\int_{nh}^{(n+1)h} \|\sigma(s)\|^2 ds \to 0$ as $n \to \infty$—which is automatically true in the case that $X$ is bounded or stable—is assumed in the case when $S_h(\epsilon) = +\infty$, then the process $\|X\|$ is recurrent on $(0, \infty)$, because $\lim \inf_{t \to \infty} \|X(t)\| = 0$ and $\lim \sup_{t \to \infty} \|X(t)\| = +\infty$ a.s. Furthermore $X$ spends most of the time close to zero in the sense that (1.6) holds. Hence, under the fading noise condition, we can see that we always have $\lim \inf_{t \to \infty} \|X(t)\| = 0$ and (1.6) holding, regardless of the finiteness of $S_h$ but that $\lim \sup_{t \to \infty} \|X(t)\|$ is zero, positive and finite, or infinite a.s., according as to whether $S_h$ is always finite, sometimes finite, or always infinite.
It is worth remarking that the fading noise condition we choose is precisely that which is necessary and sufficient for the mean square stability of solutions of (1.2).

Given that we are dealing with a continuous time equation, it seems appropriate that the conditions which enable us to characterise the asymptotic behaviour should be “continuous” rather than “discrete”. The finiteness condition on \(S_h(\epsilon)\), which relies on a particular partition of time, and the convergence of a sum, can certainly be seen as a “discrete” condition, in this sense. Therefore, we develop an integral condition on \(\sigma\) which is equivalent to the summation condition in (1.5). More precisely, we define

\[
I_c(\epsilon) = \int_0^\infty \sqrt{\int_t^{t+c} \|\sigma(s)\|_F^2 \, ds} \exp\left(-\frac{\epsilon^2/2}{\int_t^{t+c} \|\sigma(s)\|_F^2} \right) \chi(0,\infty) \left(\int_t^{t+c} \|\sigma(s)\|_F^2 \right) \, ds
\]

(1.7)

for arbitrary \(c > 0\). We then show that \(I_c(\epsilon)\) being finite for all \(\epsilon\) implies that \(X\) tends to 0; if \(I_c(\epsilon)\) is infinite for all \(\epsilon\) then \(X\) is unbounded; and if \(I_c(\epsilon)\) is finite for some \(\epsilon\) and infinite for others, then \(X\) is bounded but not convergent to zero.

The value of \(c\) turns out to be unimportant, and can be chosen to be unity for convenience. As might be guessed, the finiteness of \(I_c(\epsilon)\) for all \(\epsilon\) is equivalent to the finiteness of \(S_h(\epsilon)\) for all \(\epsilon\); \(I_c(\epsilon)\) being infinite for all \(\epsilon\) is equivalent to \(S_h(\epsilon)\) being infinite for all \(\epsilon\) and \(I_c(\epsilon)\) is finite for some \(\epsilon\) and infinite for others if and only if \(S_h(\epsilon)\) is.

Although (1.5) or \(I_c(\epsilon)\) being finite are necessary and sufficient for \(X\) to obey \(\lim_{t \to \infty} X(t) = 0\) a.s., these conditions may be hard to apply in practice. For this reason we also deduce sharp sufficient conditions on \(\sigma\) which enable us to determine for which value of \(c\) the functions \(S_h(\epsilon)\) or \(I_c(\epsilon)\) are finite. One such condition is the following: if it is known for some \(c > 0\) that

\[
\lim_{t \to \infty} \int_t^{t+c} \|\sigma(s)\|_F^2 \, ds \log t = L \in [0,\infty],
\]

then \(L = 0\) implies that \(X\) tends to zero a.s.; \(L\) being positive and finite implies \(X\) is bounded, but does not converge to zero; and \(L\) being infinite implies \(X\) is unbounded. In the case when \(t \mapsto \|\sigma(t)\|^2 =: \Sigma_1(t)^2\) or \(t \mapsto \int_t^{t+c} \|\sigma(s)\|^2 \, ds =: \Sigma_2(t)^2\) are nonincreasing functions, it can also be seen that \(X(t) \to 0\) as \(t \to \infty\) a.s.

One other result of note is established. We ask: is it possible for solutions of the unperturbed ODE \(x'(t) = Ax(t)\) to be unstable, but solutions of the SDE to be stable for some nontrivial \(\sigma\)? In other words, can the noise stabilise solutions? We prove that it cannot, in the sense that if there are a representative and finite collection of initial conditions \(\xi\) for which \(X(t, \xi)\) tends to zero with positive probability, then it must be the case that all the eigenvalues of \(A\) have negative real parts, and that \(S(\epsilon)\) is finite for all \(\epsilon > 0\). These conditions are therefore equivalent to \(\lim_{t \to \infty} X(t, \xi) = 0\) a.s. for each initial condition \(\xi\).

The results on the equation (1.3) are of more general utility than in the linear autonomous case. We give an example here of how they can be used to classify the asymptotic behaviour of a periodic linear ODE. We plan to show in other works that the asymptotic behaviour of \(Y\) can be used in both the scalar and finite–dimensional case to understand the asymptotic behaviour of the general nonlinear SDE

\[
dX(t) = -f(X(t)) \, dt + \sigma(t) \, dB(t)
\]

which, in the absence of a stochastic perturbation, has a unique globally asymptotically stable equilibrium at zero.
The next section states and discusses the main results, with proofs and supporting lemmata in the following section. Then we discuss the sufficient conditions on \( \sigma \) for stability with proofs and supporting lemmata.

2. Discussion and Statement of Main Results

2.1. Notation. In advance of stating and discussing our main results, we introduce some standard notation. Let \( d \) and \( r \) be integers. We denote by \( \mathbb{R}^d \) \( d \)-dimensional real-space, and by \( \mathbb{R}^{d \times r} \) the space of \( d \times r \) matrices with real entries. Here \( \mathbb{R} \) denotes the set of real numbers. We denote the maximum of the real numbers \( x \) and \( y \) by \( x \vee y \) and the minimum of \( x \) and \( y \) by \( x \wedge y \). If \( x \) and \( y \) are in \( \mathbb{R}^d \), the standard inner product of \( x \) and \( y \) is denoted by \( \langle x, y \rangle \). The standard Euclidean norm on \( \mathbb{R}^d \) induced by this inner product is denoted by \( \| \cdot \| \).

Let \( A \in \mathbb{R}^{d \times r} \), we denote the entry in the \( i \)-th row and \( j \)-th column by \( A_{ij} \). For \( A \in \mathbb{R}^{d \times r} \) we denote the Frobenius norm of \( A \) by

\[
\| A \|_F = \left( \sum_{j=1}^r \sum_{i=1}^d |A_{ij}|^2 \right)^{1/2}.
\]

Let \( C(I; J) \) denote the space of continuous functions \( f : I \rightarrow J \) where \( I \) is an interval contained in \( \mathbb{R} \) and \( J \) is a finite dimensional Banach space. We denote by \( L^2([0, \infty); \mathbb{R}^{d \times r}) \) the space of Lebesgue square integrable functions \( f : [0, \infty) \rightarrow \mathbb{R}^{d \times r} \) such that \( \int_0^\infty \| f(s) \|^2_2 \, ds < +\infty \).

2.2. Main results. Our first result demonstrates that it is necessary to classify completely the asymptotic behaviour of only a single affine stochastic differential equation in order to classify the asymptotic behaviour for all affine stochastic differential equations with the same diffusion coefficient, for which the underlying deterministic linear differential equation is asymptotically stable.

To make this precise, let \( d \) be an integer and \( A \) be a \( d \times d \) matrix with real entries, and consider the deterministic linear differential equation

\[
x'(t) = Ax(t), \quad t \geq 0; \quad x(0) = \xi \in \mathbb{R}^d, \tag{2.1}
\]

and also consider the stochastically perturbed version of (2.1), namely

\[
dX(t) = AX(t) \, dt + \sigma(t) \, dB(t), \quad t \geq 0; \quad X(0) = \xi \in \mathbb{R}^d. \tag{2.2}
\]

Our first main result states that if \( Y \) has certain types of almost sure asymptotic behaviour, then \( X \) inherits that almost sure asymptotic behaviour.

**Theorem 1.** Let \( A \) be a \( d \times d \) real matrix for which all eigenvalues have negative real parts. Let \( \sigma \) obeys (1.2), \( Y \) be the unique continuous adapted process which obeys (1.3), and \( X \) be the unique continuous adapted process which obeys (2.2). Then

(A) If \( \lim_{t \to \infty} Y(t) = 0 \) a.s., then \( \lim_{t \to \infty} X(t) = 0 \), a.s.

(B) If there exist \( 0 \leq c_1 \leq c_2 < +\infty \) such that

\[
c_1 \leq \liminf_{t \to \infty} \| Y(t) \| \leq \limsup_{t \to \infty} \| Y(t) \| \leq c_2, \quad \text{a.s.}
\]

then there exist \( 0 \leq c_3 \leq c_4 < +\infty \) such that

\[
c_3 \leq \liminf_{t \to \infty} \| X(t) \| \leq \limsup_{t \to \infty} \| X(t) \| \leq c_4, \quad \text{a.s.}
\]

(C) If \( \limsup_{t \to \infty} \| Y(t) \| = +\infty \) a.s., then \( \limsup_{t \to \infty} \| X(t) \| = +\infty \) a.s.
Therefore, the asymptotic behaviour of $X$ can be classified, provided the hypothesised asymptotic behaviour of $Y$ in Theorem 11 can be established. Our next result claims that such a classification can be achieved. Before it can be stated, we make some observations and fix notation. First, we see that $Y$ has the representation

$$Y(t) = e^{-t} \int_0^t e^{s \sigma(s)} dB(s), \quad t \geq 0. \quad (2.3)$$

Denote by $\Phi : \mathbb{R} \to \mathbb{R}$ the distribution function of a standard normal random variable

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du, \quad x \in \mathbb{R}. \quad (2.4)$$

We interpret $\Phi(-\infty) = 0$ and $\Phi(\infty) = 1$. Define $S_h$ by

$$S_h(\epsilon) = \sum_{n=0}^{\infty} \left\{ 1 - \Phi\left( \frac{\epsilon}{\sqrt{\int_{nh}^{(n+1)h} ||\sigma(s)||_F^2 \, ds}} \right) \right\}. \quad (2.5)$$

Since $S_h$ is a monotone function of $\epsilon$, it is the case that either (i) $S_h(\epsilon)$ is finite for all $\epsilon > 0$; (ii) there is $\epsilon' > 0$ such that for all $\epsilon > \epsilon'$ we have $S_h(\epsilon) < +\infty$ and $S_h(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$; and (iii) $S_h(\epsilon) = +\infty$ for all $\epsilon > 0$. The finiteness of the sum $S_h(\epsilon)$ may be hard to estimate because $\Phi$ is not known in closed form. However, the asymptotic behaviour of $1 - \Phi$ is well-known via Mill’s estimate cf., e.g., [15] Problem 2.9.22

$$\lim_{x \to \infty} \frac{1 - \Phi(x)}{x e^{-x^2/2}} = \frac{1}{\sqrt{2\pi}}, \quad (2.6)$$

so it is possible to determine whether $S_h(\epsilon)$ is finite according as to whether

$$S_h'(\epsilon) = \sum_{n=1}^{\infty} \sqrt{\int_{nh}^{(n+1)h} ||\sigma(s)||_F^2 \, ds} \cdot \exp\left( -\frac{\epsilon^2}{2 \int_{nh}^{(n+1)h} ||\sigma(s)||_F^2 \, ds} \right), \quad (2.7)$$

is finite.

**Proposition 1.** Suppose that $S_h$ is defined by (2.5) and $S_h'$ is defined by (2.7). Then for any $\epsilon > 0$ we have that $S_h(\epsilon)$ is finite if and only if $S_h'(\epsilon)$ is finite.

**Proof.** Define

$$\theta(n)^2 = \int_{nh}^{(n+1)h} ||\sigma(s)||_F^2 \, ds. \quad (2.7)$$

If $S_h(\epsilon)$ is finite, then $1 - \Phi(\epsilon/\theta(n)) \to 0$ as $n \to \infty$. This implies $\epsilon/\theta(n) \to \infty$ as $n \to \infty$. Therefore by (2.6), we have

$$\lim_{n \to \infty} \frac{1 - \Phi(\epsilon/\theta(n))}{\theta(n)/\epsilon \cdot \exp(-\epsilon^2/(2\theta^2(n))} = \frac{1}{\sqrt{2\pi}}. \quad (2.8)$$

Since $(1 - \Phi(\epsilon/\theta(n)))_{n \geq 1}$ is summable, it therefore follows that the sequence

$$(\theta(n)/\epsilon \cdot \exp(-\epsilon^2/(2\theta^2(n)))_{n \geq 1}$$

is summable, so $S_h'(\epsilon)$ is finite, by definition.

On the other hand, if $S_h'(\epsilon)$ is finite, and we define $\phi : [0, \infty) \to \mathbb{R}^d$ by

$$\phi(x) = \begin{cases} x \exp(-1/(2x^2)), & x > 0, \\ 0, & x = 0, \end{cases}$$

then as we have $\theta(n) \exp(-\epsilon^2/2\theta^2(n))$ summable, we have that $(\phi(\theta(n)/\epsilon))_{n \geq 1}$ is summable. Therefore $\phi(\theta(n)/\epsilon) \to 0$ as $n \to \infty$. Then, as $\phi$ is continuous and increasing on $[0, \infty)$, we have that $\theta(n)/\epsilon \to 0$ as $n \to \infty$, or $\epsilon/\theta(n) \to \infty$ as $n \to \infty$. Therefore (2.8) holds, and thus $(1 - \Phi(\epsilon/\theta(n)))_{n \geq 1}$ is summable, which implies that $S_h(\epsilon)$ is finite, as required.

$\square$
Armed with these observations, we see that the following theorem characterises
the pathwise asymptotic behaviour of solutions of (1.3). In the scalar case it yields
a result of Appleby, Cheng and Rodkina in \[1\] when \( h = 1 \). It is also of utility
when considering the relationship between the asymptotic behaviour of solutions of
stochastic differential equations and the asymptotic behaviour of uniform step-size
discretisations.

**Theorem 2.** Suppose that \( \sigma \) obeys \[12\] and \( Y \) is the unique continuous adapted
process which obeys (1.3). Suppose that \( S'_h \) is defined by (2.7).

(A) If

\[
S'_h(\epsilon) \text{ is finite for all } \epsilon > 0,
\]

then

\[
\lim_{t \to \infty} Y(t) = 0, \quad a.s.
\]  

(B) If there exists \( \epsilon' > 0 \) such that

\[
S'_h(\epsilon) \text{ is finite for all } \epsilon > \epsilon', \quad S'_h(\epsilon) = +\infty \text{ for all } \epsilon < \epsilon',
\]

then there exists deterministic \( 0 < c_1 \leq c_2 < +\infty \) such that

\[
c_1 \leq \limsup_{t \to \infty} \|Y(t)\| \leq c_2, \quad a.s.
\]  

Moreover

\[
\liminf_{t \to \infty} \|Y(t)\| = 0, \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t \|Y(s)\|^2 ds = 0, \quad a.s.
\]  

(C) If

\[
S'_h(\epsilon) = +\infty \text{ for all } \epsilon > 0,
\]

then

\[
\limsup_{t \to \infty} \|Y(t)\| = +\infty, \quad a.s.
\]  

The conditions and form of Theorem 2 as well as other theorems in this section,
are inspired by those of \[10, \text{Theorem 1}\] and by \[6, \text{Theorem 6, Corollary 7}\].

We next show that the parameter \( h > 0 \) in Theorem 2 while potentially of interest for
numerical simulations, plays no role in classifying the dynamics of (1.3).

Therefore, we may take \( h = 1 \) without loss of generality.

**Proposition 2.** Suppose that \( S'_h \) is defined by (2.4).

(i) If \( S'_h(\epsilon) < +\infty \) for all \( \epsilon > 0 \), then for each \( h > 0 \) we have \( S'_h(\epsilon) < +\infty \) for all \( \epsilon > 0 \).

(ii) If there exists \( \epsilon' > 0 \) such that \( S'_h(\epsilon) < +\infty \) for all \( \epsilon > \epsilon' \) and \( S'_h(\epsilon) = +\infty \) for all \( \epsilon < \epsilon' \), then for each \( h > 0 \) there exists \( \epsilon'_h > 0 \) such that \( S'_h(\epsilon) < +\infty \) for all \( \epsilon > \epsilon'_h \) and \( S'_h(\epsilon) = +\infty \) for all \( \epsilon < \epsilon'_h \).

(iii) If \( S'_h(\epsilon) = +\infty \) for all \( \epsilon > 0 \), then for each \( h > 0 \) we have \( S'_h(\epsilon) = +\infty \) for all \( \epsilon > 0 \).

**Proof.** To prove part (i), note by hypothesis that part (A) of Theorem 2 implies
\( Y(t) \to 0 \) as \( t \to \infty \) a.s. Now suppose that there is a \( h > 0 \) such that \( S_h(\epsilon') = +\infty \)
for some \( \epsilon' > 0 \). But by parts (B) and (C) we have that \( \mathbb{P}[Y(t) \to 0 \text{ as } t \to \infty] = 0 \),
a contradiction.

To prove part (iii), note by hypothesis that part (C) of Theorem 2 implies
\( \limsup_{t \to \infty} \|Y(t)\| = +\infty \) a.s. Now suppose that there is a \( h > 0 \) such that \( S'_h(\epsilon') < +\infty \) for some \( \epsilon' > 0 \). But by parts (A) and (B) we have that \( \limsup_{t \to \infty} \|Y(t)\| < +\infty \) a.s., a contradiction.

To prove part (ii), we note by hypothesis that part (B) of Theorem 2 implies
\( 0 < \limsup_{t \to \infty} \|Y(t)\| < +\infty \) a.s. Suppose now there exists \( h > 0 \) such that
\( S'_h(\epsilon) = +\infty \) for all \( \epsilon > 0 \). Then we have that \( \limsup_{t \to \infty} \|Y(t)\| = +\infty \) a.s., a
contradiction. Suppose on the other hand that there is $h > 0$ such that $S'_h(\epsilon) < +\infty$ for all $\epsilon > 0$. Then we have that $\limsup_{t \to \infty} ||Y(t)|| = 0$ a.s., a contradiction. Therefore it must follow that for each $h > 0$ there exist $\epsilon'_h, \epsilon''_h > 0$ such that $S'_h(\epsilon'_h) < +\infty$ and $S'_h(\epsilon''_h) = +\infty$. Then as $\epsilon \mapsto S'_h(\epsilon)$ is a non-increasing function, it follows that for each $h > 0$ there is an $\epsilon'_h > 0$ such that $S'_h(\epsilon) < +\infty$ for all $\epsilon > \epsilon'_h$ and $S'_h(\epsilon) + = +\infty$ for all $\epsilon < \epsilon'_h$.

Combining Theorems 1 and 2 we immediately get the following result concerning the solutions of the differential equation (2.2).

**Theorem 3.** Suppose that $\sigma$ obeys (1.2). Let $A$ be a $d \times d$ real matrix for which all eigenvalues have negative real parts. Let $X$ be the solution of (2.2) and suppose that $S'_h$ is defined by (2.7). Then the following holds:

(A) If $S'_h$ obeys (2.9), then $\lim_{t \to \infty} X(t, \xi) = 0$ a.s. for each $\xi \in \mathbb{R}^d$;

(B) If $S'_h$ obeys (2.11), then there exist deterministic $0 < c_1 \leq c_2 < \infty$ independent of $\xi$ such that

$$c_1 \leq \limsup_{t \to \infty} \|X(t, \xi)\| \leq c_2, \quad a.s.$$  

Moreover

$$\liminf_{t \to \infty} \|X(t, \xi)\| = 0, \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t \|X(s, \xi)\|^2 ds = 0, \quad a.s. \quad (2.16)$$

(C) If $S'_h$ obeys (2.14), then $\limsup_{t \to \infty} \|X(t, \xi)\| = +\infty$ a.s. for each $\xi \in \mathbb{R}^d$.

In the last case, when $S'_h(\epsilon) = +\infty$ for all $\epsilon > 0$, it is interesting to ask what is the limit inferior and ergodic behaviour of $\|X(t)\|$. It is very much in the spirit of this work to ask what happens when the noise intensity fades (in some sense) as $t \to \infty$. In cases (A) and (B), $S'_h(\epsilon) < +\infty$ for all $\epsilon$ sufficiently large. This implies that

$$\lim_{n \to \infty} \int_{nh}^{(n+1)h} \|\sigma(s)\|^2 ds = 0. \quad (2.17)$$

Making this additional fading noise hypothesis, we can describe more completely the limiting asymptotic behaviour of $X$ in the case when $S'_h(\epsilon) = +\infty$.

**Theorem 4.** Suppose that $\sigma$ obeys (1.2). Let $A$ be a $d \times d$ real matrix for which all eigenvalues have negative real parts. Let $X$ be the solution of (2.2) and suppose that $S'_h$ is defined by (2.7). Suppose further that (2.17) holds.

(A) If $S'_h$ obeys (2.9), then $\lim_{t \to \infty} X(t, \xi) = 0$ a.s. for each $\xi \in \mathbb{R}^d$;

(B) If $S'_h$ obeys (2.11), then there exist deterministic $0 < c_1 \leq c_2 < \infty$ independent of $\xi$ such that

$$c_1 \leq \limsup_{t \to \infty} \|X(t, \xi)\| \leq c_2, \quad a.s.$$  

Moreover, $X$ obeys (2.16).

(C) If $S'_h$ obeys (2.14), then

$$\limsup_{t \to \infty} \|X(t, \xi)\| = +\infty, \quad a.s. \quad (2.17)$$

Moreover, $X$ obeys (2.16).

The condition (2.17) is interesting because it is equivalent to asking that all solutions of (2.2) converge to zero in mean–square. Results yielding sufficient conditions for mean square stability of linear stochastic differential equations abound, and no claim is made for the novelty of the result below. However, we believe that the formulation of the result is of interest when placed in the context of our analysis of a.s. asymptotic behaviour.
Proposition 3. Suppose that \( \sigma \) obeys (1.2). Let \( A \) be a \( d \times d \) real matrix for which all eigenvalues have negative real parts. Let \( X \) be the solution of (2.2). Then the following are equivalent:

1. \( \sigma \) obeys (2.17) for some \( h > 0 \);
2. \( \sigma \) obeys (2.17) for all \( h > 0 \);
3. \( \lim_{t \to \infty} \int_{t}^{t+1} \|\sigma(s)\|_{F}^{2} \, ds = 0 \);
4. \( \lim_{t \to \infty} \mathbb{E}[\|X(t)\|^{2}] = 0 \).

Given that the equations studied are in continuous time, it is natural to ask whether the summation conditions can be replaced by integral conditions on \( \sigma \) instead. The answer is in the affirmative. To this end we introduce for fixed \( \epsilon > 0 \) the \( \epsilon \)-dependent integral
\[
I_{\epsilon}(\cdot) = I_{\epsilon}(\cdot) = \int_{0}^{\infty} \zeta_{\epsilon}(t) \exp \left( -\frac{\epsilon^{2}/2}{\zeta_{\epsilon}(t)^{2}} \right) \chi_{(0, \infty)}(\zeta_{\epsilon}(t)) \, dt,
\]
where we have defined
\[
\zeta_{\epsilon}(t) := \left( \int_{t}^{t+c \epsilon} \|\sigma(s)\|_{F}^{2} \, ds \right)^{1/2}, \quad t \geq 0.
\]
We notice that \( \epsilon \to I_{\epsilon}(\cdot) \) is a monotone function, and therefore \( I_{\epsilon}(\cdot) \) is either finite for all \( \epsilon > 0 \); infinite for all \( \epsilon > 0 \); or finite for all \( \epsilon > \epsilon' \) and infinite for all \( \epsilon < \epsilon' \). The following theorem is therefore seen to classify the asymptotic behaviour of (1.3).

**Theorem 5.** Suppose that \( \sigma \) obeys (1.2) and that \( Y \) is the unique continuous adapted process which obeys (1.3). Let \( c > 0 \), \( I_{\epsilon}(\cdot) \) be defined by (2.18), and \( \zeta_{\epsilon} \) by (2.19).

1. If \( I_{\epsilon}(\cdot) \) is finite for all \( \epsilon > 0 \), then \( \lim_{t \to \infty} Y(t) = 0 \) a.s.
2. If there exists \( \epsilon' > 0 \) such that \( I_{\epsilon}(\cdot) \) is finite for all \( \epsilon > \epsilon' \), \( I_{\epsilon}(\cdot) = +\infty \) for all \( \epsilon < \epsilon' \), then there exist deterministic \( 0 < c_{1} \leq c_{2} < +\infty \) such that
\[
c_{1} \leq \limsup_{t \to \infty} \|Y(t)\| \leq c_{2}, \quad \text{a.s.}
\]
Moreover, \( Y \) also obeys (2.13).

3. If \( I_{\epsilon}(\cdot) = +\infty \) for all \( \epsilon > 0 \), then \( \limsup_{t \to \infty} \|Y(t)\| = +\infty \) a.s.

Using this result and Theorem 4 we immediately arrive at a classification theorem for the solution of (2.2).

**Theorem 6.** Suppose that \( \sigma \) obeys (1.2) and that \( X \) is the unique continuous adapted process which obeys (2.2). Suppose all the eigenvalues of \( A \) have negative real parts. Let \( c > 0 \), \( I_{\epsilon}(\cdot) \) be defined by (2.18), and \( \zeta_{\epsilon} \) by (2.19).

1. If \( I_{\epsilon}(\cdot) \) obeys (2.20), then \( \lim_{t \to \infty} X(t, \xi) = 0 \) a.s.
2. If \( I_{\epsilon}(\cdot) \) obeys (2.21), then there exist deterministic \( 0 < c_{1} \leq c_{2} < +\infty \) independent of \( \xi \) such that
\[
c_{1} \leq \limsup_{t \to \infty} \|X(t, \xi)\| \leq c_{2}, \quad \text{a.s.}
\]
Moreover \( X \) obeys (2.10).

3. If \( I_{\epsilon}(\cdot) \) obeys (2.22) then \( \limsup_{t \to \infty} \|X(t, \xi)\| = +\infty \) a.s.
We can prove in a manner analogous to that used to establish Proposition 2 that we can take $c = 1$ without loss of generality in (2.18) and (2.19). It is therefore enough to consider the finiteness of $I_1(\varepsilon)$ in order to determine the asymptotic behaviour.

If we impose the fading noise condition $\varsigma_c(t) \to 0$ as $t \to \infty$ (which is equivalent to (2.17)), we can demonstrate in a manner analogous to the proof of Theorem 1 that $\liminf_{t \to \infty} \|X(t, \xi)\| = 0$ a.s. in the case when $I_c(\varepsilon) = +\infty$ for every $\varepsilon > 0$.

**Theorem 7.** Suppose that $\sigma$ obeys (1.2) and that $X$ is the unique continuous adapted process which obeys (2.2). Suppose all the eigenvalues of $A$ have negative real parts. Let $c > 0$, $I_c(\cdot)$ be defined by (2.18), and $\varsigma_c$ by (2.19). Suppose finally that $\varsigma_c(t) \to 0$ as $t \to \infty$.

(A) If $I_c$ obeys (2.20), then $\lim_{t \to \infty} X(t, \xi) = 0$, a.s.

(B) If $I_c$ obeys (2.21), then there exist deterministic $0 < c_1 \leq c_2 < +\infty$ independent of $\xi$ such that

$$c_1 \leq \limsup_{t \to \infty} \|X(t, \xi)\| \leq c_2, \quad \text{a.s.}$$

Moreover $X$ obeys (4.10).

(C) If $I_c$ obeys (2.22) then $\limsup_{t \to \infty} \|X(t, \xi)\| = +\infty$ a.s. Moreover $X$ obeys (4.10).

The result of Theorem 6 shows that $\liminf_{t \to \infty} \|X(t)\| = 0$ a.s. when $I_1(\varepsilon)$ is finite for some $\varepsilon > 0$ and infinite for others. In Theorem 7 we strengthened the condition on the smallness of the noise coefficient, enabling us to prove that when $I_1(\varepsilon) = +\infty$ for every $\varepsilon > 0$, we have $\limsup_{t \to \infty} \|X(t)\| = +\infty$ and $\liminf_{t \to \infty} \|X(t)\| = 0$ a.s. We now give an example which shows that this conclusion cannot be extended if the diffusion coefficient grows in intensity as $t \to \infty$, and that therefore part (C) of Theorem 6 is the most general conclusion that can be drawn without imposing more specific growth conditions on the diffusion coefficient.

**Example 8.** Suppose that $d = r \geq 3$, that $A = -I_d$ and that $\sigma(t) = \eta(t)I_d$ for $t \geq 0$, where $\eta \in C([0, \infty); (0, \infty))$. Suppose also that

$$\lim_{t \to \infty} \int_0^t e^{2s} \eta^2(s) \, ds = +\infty.$$  

Then the $i$-th component of $X$ obeys

$$X_i(t) = \xi_i e^{-t} + e^{-t} \int_0^t e^s \eta(s) \, dB_i(s), \quad t \geq 0.$$  

Hence

$$e^{2t} \|X(t)\|^2 = \|\xi\|^2 + \sum_{i=1}^d \left( \int_0^t e^s \eta(s) \, dB_i(s) \right)^2, \quad t \geq 0.$$  

Define

$$T(t) := \int_0^t e^{2s} \eta^2(s) \, ds, \quad t \geq 0.$$  

Then $T : [0, \infty) \to [0, \infty)$ is an increasing and $C^1$ function with $T(t) \to \infty$ as $t \to \infty$. Define $\tau(t) = T^{-1}(t)$ for $t \geq 0$ and

$$U(t) = \|\xi\|^2 + \sum_{i=1}^d \left( \int_0^t e^s \eta(s) \, dB_i(s) \right)^2, \quad t \geq 0.$$  

Also define $\tilde{U}(t) = U(\tau(t))$ and

$$B_i^*(t) = \int_0^{\tau(t)} e^s \eta(s) \, dB_i(s), \quad t \geq 0.$$
Let $G(t) = \mathcal{F}^B(\tau(t))$. Then $\tilde{U}$ and $B^*_t$ are $G$–adapted and

$$\tilde{U}(t) = \|\xi\|_2^2 + \sum_{i=1}^d B^*_i(t)^2, \quad t \geq 0.$$ 

We now establish that $B^*_t$ is a $G$ standard Brownian motion. To do this we must check the conditions of Lévy’s theorem for characterising standard Brownian motion. First, we see that $B^*_t$ is $\mathcal{F}^B(\tau(t))$ measurable, and therefore $G(t)$ measurable. Since $\tau$ is increasing, $G$ is a filtration. Also because $\tau$ is continuous and $s \mapsto e^s\eta(s)$ is continuous, then $t \mapsto B^*_t(t)$ is continuous. Finally, if we let $I_i(t) = \int_0^t e^s\eta(s)dB_i(s)$, then $\mathbb{E}[I_i(t)^2] = \int_0^t e^{2s}\eta(s)^2ds = T(t)$. Thus

$$\mathbb{E}[B^*_t(t)^2] = \mathbb{E}[I_i(t)] = T(\tau(t)) = t < +\infty.$$ 

Therefore, we need only to check that $B^*_t$ obeys the projection property for martingales. Let $t > s \geq 0$. Then as $\tau$ is increasing, we have

$$\begin{align*}
\mathbb{E}[B^*_t(t)|G(s)] &= \mathbb{E}[I_i(t)|G(s)] \\
&= \mathbb{E}\left[\int_{\tau(s)}^{\tau(t)} e^s\eta(u)dB_i(u) + B^*_i(s)\right|G(s)] \\
&= \mathbb{E}\left[\int_{\tau(s)}^{\tau(t)} e^s\eta(u)dB_i(u)\right|G(s)] + B^*_i(s) \\
&= \mathbb{E}\left[\int_{\tau(s)}^{\tau(t)} e^s\eta(u)dB_i(u)\right] + B^*_i(s) = B^*_i(s).
\end{align*}$$

Hence $B^*_t$ is a $G(t)$–martingale. Finally, $\langle B^*_t(t) \rangle = \int_0^{\tau(t)} e^{2s}\eta(s)^2ds = T(\tau(t)) = t$. Therefore, by Lévy’s characterisation theorem, $B^*_t$ is a $G$ standard Brownian motion. Also, because the Brownian motions $B_1, \ldots, B_d$ are independent, it follows that $B^*_t, B^*_t, \ldots, B^*_t$ are independent $G$–adapted standard Brownian motions. Therefore $\tilde{U}$ is a $d$–dimensional square Bessel process starting at $\|\xi\|_2^2$, and indeed

$$e^{2\tau(t)}\|X(\tau(t))\|_2^2 = \tilde{U}(t), \quad t \geq 0.$$ 

Thus, $\tilde{U}_2 = \sqrt{\tilde{U}}$ is a $d$–dimensional Bessel process starting at $\|\xi\|_2$.

Now, if $\xi \neq 0$, it was proven in Appleby and Wu [5] that

$$\liminf_{t \to \infty} \frac{\log \tilde{U}_2(t)}{\log \log t} = -\frac{1}{d-2}, \quad \limsup_{t \to \infty} \frac{\tilde{U}_2(t)}{\sqrt{2e^{2\tau(t)}T(t)}} = 1, \quad \text{a.s.}$$

Hence

$$\liminf_{t \to \infty} \frac{\log \frac{e^{\tau(t)}\|X(\tau(t))\|_2}{\sqrt{T(t)}}}{\log \log t} = -\frac{1}{d-2}, \quad \text{a.s.}$$

which yields

$$\liminf_{t \to \infty} \frac{\log \frac{\|X(t)\|_2}{\sqrt{e^{-2T(t)}}}}{\log \log T(t)} = -\frac{1}{d-2}, \quad \limsup_{t \to \infty} \frac{\|X(t)\|_2}{\sqrt{2e^{-2T(t)}T(t)\log \log T(t)}} = 1, \quad \text{a.s.}$$

(2.23) If we suppose that $\eta$ is such that $\eta'(t)/\eta(t) \to 0$ as $t \to \infty$, so that $\eta$ neither decays nor grows at an exponential rate, we have by l’Hôpital’s rule that

$$\lim_{t \to \infty} \frac{T(t)}{e^{2\eta(t)^2}} = \frac{1}{2},$$
and because \( \lim_{t \to \infty} \log \eta(t)/t = 0 \), we have also that
\[
\lim_{t \to \infty} \frac{\log \log T(t)}{\log t} = 1.
\]
Therefore, from (2.23) we get
\[
\liminf_{t \to \infty} \frac{\log \|X(t)\|}{\log \eta(t)} = -\frac{1}{d-2}, \quad \limsup_{t \to \infty} \frac{\|X(t)\|}{\sqrt{\eta^2(t) \log t}} = 1, \quad \text{a.s.}
\]
Now, we suppose that \( \eta(t)/t^\alpha \to L \in (0, \infty) \) as \( t \to \infty \). If \( \alpha \geq 0 \), we can show that all the hypotheses hold and that \( I_1(\epsilon) = +\infty \) for all \( \epsilon > 0 \). Moreover, if \( \alpha > 1/(d-2) > 0 \), then
\[
\lim_{t \to \infty} \|X(t)\|_2 = +\infty, \quad \text{a.s.}
\]
while if \( 0 \leq \alpha < 1/(d-2) \), we have
\[
\liminf_{t \to \infty} \|X(t)\|_2 = 0, \quad \limsup_{t \to \infty} \|X(t)\|_2 = +\infty, \quad \text{a.s.}
\]
(In the case \( \alpha < 0 \), we have that \( X(t) \to 0 \) as \( t \to \infty \) a.s. because \( I_1(\epsilon) \) is finite for all \( \epsilon > 0 \).)

Therefore, it can be seen that without further information on the growth or decay rate of \( \|\sigma(t)\| \) as \( t \to \infty \), it is impossible to make a general conclusion about the size of \( \liminf_{t \to \infty} \|X(t)\| \). In this sense, the overall conclusions of Theorem 5 cannot be improved upon if \( d \geq 3 \) without further analysis.

However, in the case when \( d = 1 \) (and one can take \( r = 1 \) without loss of generality), we can show that \( \liminf_{t \to \infty} |X(t)| = 0 \) a.s. Suppose that \( I_1(\epsilon) = +\infty \) for all \( \epsilon > 0 \). Then \( \limsup_{t \to \infty} |X(t)| = +\infty \). By Theorem 5 it follows that \( \limsup_{t \to \infty} |Y(t)| = +\infty \) a.s. Then we know that \( S_1(\epsilon) = +\infty \) for all \( \epsilon > 0 \). Hence \( \sigma^2 \not\in L^1(0, \infty) \). By mimicking a proof of a result in Appleby, Cheng and Rodkina [11], it follows that we must have \( \liminf_{t \to \infty} |X(t)| = 0 \) a.s.

We now present a result concerning the inability of noise to stabilise the asymptotically stable differential equation \( x'(t) = Ax(t) \).

**Theorem 9.** Suppose that \( \sigma \) obeys (2.2) and that \( X(\cdot, \xi) \) is the unique continuous adapted process which obeys (2.2) with initial condition \( X(0) = \xi \). Then the following are equivalent:

(A) All the eigenvalues of \( A \) have negative real parts, and \( I \) defined by (2.15) obeys (2.20);

(B) There is a basis \((\xi_i)_{i=1}^d \) of \( \mathbb{R}^d \) and an event \( C \) with \( \mathbb{P}[C] > 0 \) given by
\[
C = \{ \omega : \lim_{t \to \infty} X(t, \xi_i, \omega) = 0, \text{ for } i = 1, \ldots, d, \lim_{t \to \infty} X(t, 0, \omega) = 0 \};
\]

(C) For each \( \xi \in \mathbb{R}^d \) we have \( \lim_{t \to \infty} X(t, \xi) = 0 \) a.s.

This section closes with one further remark. The classification of the asymptotic behaviour of (2.2) is achieved by means of summability or equivalent integrability conditions which are written in terms of the Frobenius norm of \( \sigma \). However, by norm equivalence, it can be shown that any norm on \( \mathbb{R}^{d \times r} \) can be used in place of the Frobenius norm. More precisely, the following holds.
Proposition 4. Let $\| \cdot \|$ be any norm on $\mathbb{R}^{d \times r}$, and define

$$J_1(\epsilon) = \int_0^\infty \sqrt{\int_t^{t+1} \|\sigma(s)\|^2 \exp\left(-\frac{\epsilon^2}{2 \int_t^{t+1} \|\sigma(s)\|^2 \, ds}\right) \, dt},$$

$$T_1'(\epsilon) = \sum_{n=1}^{\infty} \left\{ \sqrt{\int_{n}^{n+1} \|\sigma(s)\|^2 \exp\left(-\frac{\epsilon^2}{2 \int_{n}^{n+1} \|\sigma(s)\|^2 \, ds}\right) \, ds} \right\}.$$

Let $S_1'$ be defined by (2.17) and $I_1$ be defined by (2.15) and (2.19).

(A) The following statements are equivalent:

(i) $S_1'(\epsilon) < +\infty$ for all $\epsilon > 0$;
(ii) $T_1'(\epsilon) < +\infty$ for all $\epsilon > 0$;
(iii) $I_1(\epsilon) < +\infty$ for all $\epsilon > 0$;
(iv) $J_1(\epsilon) < +\infty$ for all $\epsilon > 0$.

(ii) The following statements are equivalent:

(i) There exists $\epsilon_1 > 0$ such that $S_1'(\epsilon) < +\infty$ for all $\epsilon > \epsilon_1$ and $S_1'(\epsilon) = +\infty$ for all $\epsilon < \epsilon_1$;
(ii) There exists $\epsilon_2 > 0$ such that $T_1'(\epsilon) < +\infty$ for all $\epsilon > \epsilon_2$ and $T_1'(\epsilon) = +\infty$ for all $\epsilon < \epsilon_2$;
(iii) There exists $\epsilon_3 > 0$ such that $I_1(\epsilon) < +\infty$ for all $\epsilon > \epsilon_3$ and $I_1(\epsilon) = +\infty$ for all $\epsilon < \epsilon_3$;
(iv) There exists $\epsilon_4 > 0$ such that $J_1(\epsilon) < +\infty$ for all $\epsilon > \epsilon_4$ and $J_1(\epsilon) = +\infty$ for all $\epsilon < \epsilon_4$.

(iii) The following statements are equivalent:

(i) $S_1'(\epsilon) = +\infty$ for all $\epsilon > 0$;
(ii) $T_1'(\epsilon) = +\infty$ for all $\epsilon > 0$;
(iii) $I_1(\epsilon) = +\infty$ for all $\epsilon > 0$;
(iv) $J_1(\epsilon) = +\infty$ for all $\epsilon > 0$.

The proof is straightforward and hence omitted.

3. Sufficient conditions on $\sigma$ for asymptotic classification

Although the summability conditions on (2.5) classify necessary and sufficient, it can be quite difficult to check in practice. We supply more easily-checked sufficient conditions on $\sigma$ for which the solution of (2.2) converges to zero, is bounded or is unbounded.

It is well-known in the case that all eigenvalues of $A$ have negative real parts that the solution of (2.2) is a.s. asymptotically stable in the case that $\sigma \in L^2([0, \infty); \mathbb{R}^{d \times r})$. We can see that this fact is a simple corollary of parts (A) of Theorem 1, Theorem 2 and the following observation.

Proposition 5. If $\sigma \in L^2([0, \infty); \mathbb{R}^{d \times r})$, then $S_1'(\epsilon) < +\infty$ for every $\epsilon > 0$.

Proof. By hypothesis, we have that $\int_n^{n+1} \|\sigma(s)\|^2 \, ds \to 0$ as $n \to \infty$. Since

$$\lim_{x \to \infty} \frac{x^{-1}e^{-x^2/2}}{x^{-2}} = 0,$$

we have for each $\epsilon > 0$ that

$$\lim_{n \to \infty} \frac{\sqrt{\int_n^{n+1} \|\sigma(s)\|^2 \, ds} \exp\left(-\frac{\epsilon^2}{2 \int_n^{n+1} \|\sigma(s)\|^2 \, ds}\right)}{\int_n^{n+1} \|\sigma(s)\|^2 \, ds} = 0.$$

Since the denominator is a summable sequence, the numerator must also be summable; and this is simply the statement that $S_1'(\epsilon) < +\infty$ for every $\epsilon > 0$, as required. \qed
The next result characterises the asymptotic behaviour of $X$, according as to whether a certain limit exists, and is zero, finite but non-zero, or infinite.

**Theorem 10.** Suppose that $\sigma$ obeys (1.2). Suppose that there exists $h > 0$ and $L_h \in [0, \infty]$ such that

$$\lim_{n \to \infty} \int_{nh}^{(n+1)h} \|\sigma(s)\|^2_F \, ds \cdot \log n = L_h. \quad (3.1)$$

Let $A$ be a $d \times d$ real matrix whose eigenvalues all have negative real parts, and suppose that $X$ is the unique continuous adapted process which obeys (2.2).

- (i) If $L_h = 0$, then $\lim_{t \to \infty} X(t, \xi) = 0$, a.s.
- (ii) If $L_h \in (0, \infty)$, then there exist $0 \leq c_1 < c_2 < \infty$ independent of $\xi$ such that
  $$c_2 \leq \limsup_{t \to \infty} \|X(t, \xi)\| \leq c_2, \quad \liminf_{t \to \infty} \|X(t, \xi)\| = 0, \quad \text{a.s.}$$

and

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \|X(s, \xi)\|^2 \, ds = 0, \quad \text{a.s.}$$

- (iii) If $L_h = +\infty$, then $\limsup_{t \to \infty} \|X(t, \xi)\| = +\infty$, a.s.

If pointwise conditions are preferred to (3.1) in Theorem 10 we may instead impose the condition

$$\lim_{t \to \infty} \|\sigma(t)\|^2_F \log t = L \in [0, \infty]$$
on $\sigma$. In this case, if $L = 0$, then $L_h = 0$ in (3.1), and part (i) of Theorem 10 applies; if $L \in (0, \infty)$, then $L_h = hL$ in (3.1) and part (ii) of Theorem 10 applies; and if $L = \infty$, then $L_h = +\infty$ in (3.1), and part (iii) of Theorem 10 applies.

We can also characterise the asymptotic stability of solutions of solutions with a very simple condition, contingent on a certain class of monotonicity conditions holding on $t \mapsto \|\sigma(t)\|_F$.

**Theorem 11.** Suppose that $\sigma$ obeys (1.2). Let $A$ be a $d \times d$ real matrix all of whose eigenvalues have negative real parts. Let $X$ be the unique continuous adapted process which obeys (2.2). Suppose that there is $h > 0$ such that the sequence

$$\lim_{n \to \infty} \int_{nh}^{(n+1)h} \|\sigma(s)\|^2_F \, ds \text{ is non-increasing.}$$

Then the following are equivalent.

- (A) $\lim_{t \to \infty} \int_{nh}^{(n+1)h} \|\sigma(s)\|^2_F \, ds \cdot \log n = 0$;
- (B) $\lim_{t \to \infty} X(t, \xi) = 0$ a.s. for each $\xi \in \mathbb{R}^d$;
- (C) There exists $\xi \in \mathbb{R}^d$ such that $\lim_{t \to \infty} X(t, \xi) = 0$ with positive probability.

Stronger monotonicity conditions which can be imposed are that

$$t \mapsto \Sigma_t^2 = \int_0^t \|\sigma(s)\|^2_F \, ds, \quad t \mapsto \Sigma_t^2 = \|\sigma(t)\|^2_F,$$

are non-increasing. In this case statement (A) in Theorem 11 can be replaced by

$$\lim_{t \to \infty} \Sigma_t^2 \log t = 0.$$

4. **Periodic Affine Equations**

We present one further application of our results, which enables a classification of the asymptotic behaviour of affine stochastic differential equations with periodic features to be analysed. Towards this end, suppose that

$$A \in C([0, \infty); \mathbb{R}^{d \times d}) \text{ is a } T \text{-periodic function} \quad (4.1)$$

and consider the stochastic differential equation

$$dX(t) = A(t)X(t) \, dt + \sigma(t) \, dB(t), \quad t \geq 0; \quad X(0) = \xi \in \mathbb{R}^d \quad (4.2)$$
where as before $\sigma \in C([0, \infty); \mathbb{R}^{d \times r})$ and $B$ is an $r$–dimensional standard Brownian motion. It is standard that there is a unique continuous adapted process which obeys (4.2).

The analysis of (4.2) is facilitated greatly by the introduction of the unique continuous $\mathbb{R}^{d \times d}$–valued solution of

$$\Psi'(t) = A(t)\Psi(t), \quad t \geq 0, \quad \Psi(0) = I_d.$$  \hspace{1cm} (4.3)

In general

$$\det(\Psi(t)) = \exp\left(\int_0^t \text{tr}(A(s)) \, ds\right) \neq 0, \quad t \geq 0,$$

so $\Psi(t)$ is invertible for all $t \geq 0$. The matrix $\Psi(T)$ plays a central role in the asymptotic theory of (4.3) and (4.2). It is called the Floquet multiplier. Let us assume that

$$\rho(\Psi(T)) < 1$$ \hspace{1cm} (4.4)

where $\rho(C)$ denotes the spectral radius of the square matrix $C$.

**Theorem 12.** Suppose that $\sigma$ obeys (1.2) and $A$ obeys (1.1). Suppose that the fundamental solution $\Psi$ of (4.3) is such that $\rho(\Psi(T)) < 1$. Let $X$ be the solution of (4.2) and suppose that $S_h$ is defined by (2.7). Then the following holds:

(A) If $S_h$ obeys (2.9), then $\lim_{t \to \infty} X(t, \xi) = 0$ a.s. for each $\xi \in \mathbb{R}^d$;

(B) If $S_h$ obeys (2.11), then there exist deterministic $0 < c_1 \leq c_2 < \infty$ independent of $\xi$ such that

$$c_1 \leq \limsup_{t \to \infty} \|X(t, \xi)\| \leq c_2, \quad a.s.$$

Moreover

$$\liminf_{t \to \infty} \|X(t, \xi)\| = 0, \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t \|X(s, \xi)\|^2 \, ds = 0, \quad a.s.$$

(C) If $S_h$ obeys (2.14), then $\limsup_{t \to \infty} \|X(t, \xi)\| = +\infty$ a.s. for each $\xi \in \mathbb{R}^d$.

In the case that $S_h(\epsilon) = +\infty$ for all $\epsilon > 0$, but $\sigma$ obeys the fading noise condition (2.17), we can refine the asymptotic result in a manner identical to that in Theorem 3 in the autonomous case.

**Theorem 13.** Suppose that $\sigma$ obeys (1.2) and $A$ obeys (1.1). Suppose that the fundamental solution $\Psi$ of (4.3) is such that $\rho(\Psi(T)) < 1$. Let $X$ be the solution of (4.2) and suppose that $S_h$ is defined by (2.7). Suppose further that $\sigma$ obeys (2.17). Then the following holds:

(A) If $S_h$ obeys (2.9), then $\lim_{t \to \infty} X(t, \xi) = 0$ a.s. for each $\xi \in \mathbb{R}^d$;

(B) If $S_h$ obeys (2.11), then there exist deterministic $0 < c_1 \leq c_2 < \infty$ independent of $\xi$ such that

$$c_1 \leq \limsup_{t \to \infty} \|X(t, \xi)\| \leq c_2, \quad a.s.$$

Moreover

$$\liminf_{t \to \infty} \|X(t, \xi)\| = 0, \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t \|X(s, \xi)\|^2 \, ds = 0, \quad a.s.$$

(C) If $S_h$ obeys (2.14), then $\limsup_{t \to \infty} \|X(t, \xi)\| = +\infty$ a.s. for each $\xi \in \mathbb{R}^d$.

Moreover

$$\liminf_{t \to \infty} \|X(t, \xi)\| = 0, \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t \|X(s, \xi)\|^2 \, ds = 0, \quad a.s.$$
5. A Key Theorem

The main results concerning the asymptotic behaviour of $Y$ in this paper (namely Theorems 2 and 5) are corollaries of a key technical result, which is stated and proven in this section.

Suppose that $(t_n)_{n \geq 0}$ is an increasing sequence with $t_0 = 0$ and $\lim_{n \to \infty} t_n = +\infty$. Define

$$S_t(\epsilon) = \sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|^2 ds}} \right) \right\}. \quad (5.1)$$

If there are uniform upper and lower bounds on the spacing of the sequence, it transpires that the finiteness (or otherwise) of the sum enables us to characterise the long run behaviour of (1.3). The following theorem then characterises the pathwise asymptotic behaviour of solutions of (1.3).

**Theorem 14.** Suppose that $\sigma$ obeys (1.2) and that $Y$ is the unique continuous adapted process which obeys (1.3). Let $S_t(\epsilon)$ be defined by (5.1) where $t$ is any $\epsilon$–independent sequence obeying

$$t_0 = 0, \quad 0 < \alpha \leq t_{n+1} - t_n \leq \beta < +\infty, \quad \lim_{n \to \infty} t_n = +\infty \quad (5.2)$$

for some $0 < \alpha \leq \beta < +\infty$.

(A) If

$$S_t(\epsilon) \text{ is finite for all } \epsilon > 0, \quad (5.3)$$

then $\lim_{t \to \infty} Y(t) = 0$ a.s.

(B) (i) If there exists $\epsilon' > 0$ such that

$$S_t(\epsilon) \text{ is finite for all } \epsilon > \epsilon', \quad (5.4)$$

then there exists a deterministic $0 < c_2 < +\infty$ such that

$$\limsup_{t \to \infty} \|Y(t)\| \leq c_2, \quad \text{a.s.}$$

(ii) On the other hand, if there exists $\epsilon'' > 0$ such that

$$S_{\tau}(\epsilon) = +\infty \text{ for all } \epsilon < \epsilon'' \quad (5.5)$$

where $\tau$ is any $\epsilon$–independent sequence obeying (1.2), then there exists a deterministic $0 < c_1 < +\infty$ such that

$$\limsup_{t \to \infty} \|Y(t)\| \geq c_1, \quad \text{a.s.}$$

(C) If

$$S_t(\epsilon) = +\infty \text{ for all } \epsilon > 0, \quad (5.6)$$

then $\limsup_{t \to \infty} \|Y(t)\| = +\infty$ a.s.

5.1. **Proof of Theorem 14:** preliminary estimates. We start by showing how estimates on the rows of the matrix $\sigma$ relate to its Frobenius norm. Let $(t_n)_{n \geq 0}$ be an increasing sequence with $t_0 = 0$ and $\lim_{n \to \infty} t_n = +\infty$ and define, by analogy to (5.1),

$$S_{t}^1(\epsilon) = \sum_{n=0}^{\infty} \sum_{i=1}^{d} \left\{ 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma_j(s)\|^2 ds}} \right) \right\}. \quad (5.7)$$
We can see that as $S_t^i$ is a monotone function of $\epsilon$, it is the case that either (i) $S_t^i(\epsilon)$ is finite for all $\epsilon > 0$; (ii) there is $\epsilon_1' > 0$ such that for all $\epsilon > \epsilon_1'$ we have $S_t^i(\epsilon) < +\infty$ and $S_t^i(\epsilon) = +\infty$ for all $\epsilon < \epsilon_1'$; and (iii) $S_t^i(\epsilon) = +\infty$ for all $\epsilon > 0$. In the next lemma, we show that $S_t$ defined by (5.1) is always finite if and only if $S_t^i$ is; that $S_t$ is infinite if and only if $S_t^i$ is; and that $S_t$ and $S_t^i$ are sometimes finite and sometimes infinite only if the other is.

**Lemma 1.** Let $(t_n)_{n \geq 0}$ be an increasing sequence with $t_0 = 0$ and $\lim_{n \to \infty} t_n = +\infty$. Suppose that $S_t$ is defined by (5.1) and that $S_t^i$ is defined by (5.7).

(a) The following are equivalent:
   (i) $S_t(\epsilon) < +\infty$ for all $\epsilon > 0$;
   (ii) $S_t^i(\epsilon) < +\infty$ for all $\epsilon > 0$.

(b) The following are equivalent:
   (i) There exists $\epsilon' > 0$ such that for all $\epsilon > \epsilon'$ we have $S_t(\epsilon) < +\infty$ and $S_t^i(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$;
   (ii) There exists $\epsilon_1' > 0$ such that for all $\epsilon > \epsilon_1'$ we have $S_t^i(\epsilon) < +\infty$ and $S_t^i(\epsilon) = +\infty$ for all $\epsilon < \epsilon_1'$;

(c) The following are equivalent:
   (i) $S_t(\epsilon) = +\infty$ for all $\epsilon > 0$;
   (ii) $S_t^i(\epsilon) = +\infty$ for all $\epsilon > 0$.

*Proof.* With $\theta$ and $\theta_i$ defined by (5.8) and (5.9), we have $\theta^2(n) \geq \theta_i(n)^2$ for each $i = 1, \ldots, d$. Thus

$$\sum_{i=1}^d \left\{ 1 - \Phi \left( \frac{\epsilon}{\theta_i(n)} \right) \right\} \leq d \left( 1 - \Phi \left( \frac{\epsilon}{\theta(n)} \right) \right).$$  \hfill (5.10)

Suppose, for each $n$, that $Z_i(n)$ for $i = 1, \ldots, d$ are independent standard normal random variables. Define $Z(n) = (Z_1(n), Z_2(n), \ldots, Z_d(n))$ and suppose that $(Z(n))_{n \geq 0}$ are a sequence of independent normal vectors. Define finally

$$X_i(n) = \theta_i(n)Z_i(n), \quad X(n) = \sum_{i=1}^d X_i(n), \quad n \geq 0.$$  

Then we have that $X_i$ is a zero mean normal with variance $\theta_i^2$ and $X$ is a zero mean normal with variance $\theta^2$. Define $Z^*(n) = X(n)/\theta(n)$ is a standard normal random variable. Therefore we have that

$$\mathbb{P}[|X(n)| > \epsilon] = \mathbb{P}[|Z^*(n)| \geq \epsilon/\theta(n)] = 2\mathbb{P}[Z^*(n) \geq \epsilon/\theta(n)] = 2 \left( 1 - \Phi \left( \frac{\epsilon}{\theta(n)} \right) \right).$$  \hfill (5.11)

With $A_i(n) = \{|X_i(n)| \leq \epsilon/d\}$, $B(n) = \{\sum_{i=1}^d |X_i(n)| \leq \epsilon\}$, then $\bigcup_{i=1}^d A_i(n) \subseteq B(n)$, so

$$\mathbb{P}[|X(n)| > \epsilon] \leq \mathbb{P}[B(n)] \leq \mathbb{P}[\bigcup_{i=1}^d A_i(n)] = \mathbb{P} \left[ \bigcup_{i=1}^d A_i(n) \right] \leq \sum_{i=1}^d \mathbb{P} \left[ A_i(n) \right].$$
Since $X_i = \theta_i Z_i$, we have

$$P(|X(n)| > \epsilon) \leq 2 \sum_{i=1}^{d} P[X_i(n) \geq \epsilon/d] = 2 \sum_{i=1}^{d} \left\{ 1 - \Phi \left( \frac{\epsilon/d}{\theta_i(n)} \right) \right\}.$$ (5.12)

By (5.11) and (5.12), we get

$$1 - \Phi \left( \frac{\epsilon}{\theta(n)} \right) \leq \sum_{i=1}^{d} \left\{ 1 - \Phi \left( \frac{\epsilon/d}{\theta_i(n)} \right) \right\}.$$ (5.13)

From (5.11), we can see that $S_t(\epsilon) < +\infty$ implies that $S^1_t(\epsilon) < +\infty$ and from (5.13) that $S^1_t(\epsilon/d) < +\infty$ implies $S_t(\epsilon) < +\infty$. Therefore, we have that part (a) holds.

Part (c) holds similarly, because from (5.11) we have that $S^1_t(\epsilon) = +\infty$ implies $S_t(\epsilon) = +\infty$, and from (5.13) we have that $S^1_t(\epsilon/d) = +\infty$ implies $S_t(\epsilon) = +\infty$. As to the proof of part (b), suppose that (i) holds. Then by (5.10), we can see that $S^1_t(\epsilon) \leq S_t(\epsilon) < +\infty$ for all $\epsilon < \epsilon'$, and by (5.13) that $S^1_t(\epsilon/d) \geq S(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$. Therefore, there exists $\epsilon'_t \in [\epsilon', \epsilon'/d]$ such that (ii) holds. The proof that (ii) implies (i) is similar. □

5.2. Organisation of the proof of Theorem 14. The proof is divided into four parts: we first derive estimates and identities common to parts (A)–(C) of Theorem 14. Second, we prove (2.15), which yields (C). Next, we obtain the lower bound on the limit superior in (2.12), which is part of (B). Finally, we find the upper bound on the limit superior in (2.12), which completes the proof of the limsup in (B). We also prove (2.10), which proves (A).

The proof of the liminf in (B) and the ergodic–type result in part (B) are not given at this point. Instead, we prove them independently for the solution of the general equation (2.2). The results for $Y$ are then simply corollaries of this general result, with $A = -I_d$.

5.3. Preliminary estimates. Let $V(j) := \int_{t_{j-1}}^{t_j} e^{s-t_j} \sigma(s) dB(s), j \geq 1$. Define $V_i(j) = \langle V(j), e_i \rangle$. Then

$$V_i(j) = \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} e^{s-t_j} \sigma_{il}(s) dB_l(s).$$

For each fixed $i$, then $(V_i(j))_{j \geq 1}$ is a sequence of independently and normally distributed random variables with mean zero and variance

$$\nu^2_i(j-1) := \text{Var}[V_i(j)] = \int_{t_{j-1}}^{t_j} e^{2s-2t_j} \sigma_{il}^2(s) ds \leq \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} \sigma_{il}^2(s) ds = \theta^2_i(j-1).$$

Similarly, $\nu^2_i(j-1) \geq e^{2(t_j-t_{j-1})} \theta^2_i(j-1) \geq e^{-2\beta} \theta^2_i(j-1)$, so $v_i(j-1) = 0$ if and only if $\theta_i(j-1) = 0$. Also, by (2.3), we get

$$Y(t_n) = e^{-t_n} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} e^{s-t_j} \sigma(s) dB(s) = \sum_{j=1}^{n} e^{-(t_n-t_j)} V(j), \quad n \geq 1.$$ (5.14)

This also implies that for $n \geq 1$ we have

$$Y(t_{n+1}) = V(n+1) + \sum_{j=1}^{n} e^{-(t_{n+1}-t_j)} V(j) = V(n+1) + e^{-(t_{n+1}-t_n)} Y(t_n).$$ (5.15)

Next, as $V_i(j)$ is normally distributed, we have $P[|V_i(j)| > \epsilon] = 2(1 - \Phi(\epsilon/v_i(j-1))$. However, as $\Phi$ is increasing, and $e^{-\beta} \theta_i(j-1) \leq v_i(j-1) \leq \theta_i(j-1)$, we have
1 - \Phi(\epsilon/e^{-\beta}t_i(j-1)) \leq 1 - \Phi(\epsilon/v_i(j-1)) \leq 1 - \Phi(\epsilon/\theta_i(j-1)), \quad \text{so}
2(1 - \Phi(\epsilon/e^{-\beta}t_i(j-1))) \leq \mathbb{P}[\|V(j)\| > \epsilon] \leq 2(1 - \Phi(\epsilon/\theta_i(j-1))), \quad j \geq 1.

(5.16)

Note that \|V(j)\| = \sum_{i=1}^{d} |V_i(j)|. Thus, as \|V(j)\| \geq |V_i(j)|, we have that \mathbb{P}[\|V(j)\| \geq \epsilon] \geq \mathbb{P}[|V_i(j)| \geq \epsilon] for each i = 1, \ldots, d. Therefore
\[d\mathbb{P}[\|V(j)\| \geq \epsilon] \geq \sum_{i=1}^{d} \mathbb{P}[|V_i(j)| \geq \epsilon].\]

(5.17)

On the other hand, defining \mathcal{A}(j) = \{ |V_i(j)| \leq \epsilon/d \} and \mathcal{B}(j) = \{ |V_i(j)| \leq \epsilon \}, we see that \bigcap_{i=1}^{d} \mathcal{A}(j) \subseteq \mathcal{B}(j). Then
\[\mathbb{P}[\|V(j)\| \geq \epsilon] = \mathbb{P}[\mathcal{B}(j)] \leq \mathbb{P}\left[\bigcup_{i=1}^{d} \mathcal{A}(j)\right] = \mathbb{P}\left[\bigcap_{i=1}^{d} \mathcal{A}(j)\right] \leq \sum_{i=1}^{d} \mathbb{P}[|V_i(j)| \geq \epsilon/d].\]

(5.18)

5.4. **Proof of part (C).** Suppose \(S_t\) obeys (5.6). Then by Lemma 1 we have that \(S_t^1(\epsilon) = +\infty\) for every \(\epsilon > 0\). Therefore by (5.16), \(\sum_{j=1}^{\infty} \sum_{i=1}^{d} \mathbb{P}[|V_i(j)| > \epsilon] = +\infty\) for every \(\epsilon > 0\). By (5.17), we have \(\sum_{j=1}^{\infty} \mathbb{P}[\|V(j)\| \geq \epsilon] = +\infty\) for all \(\epsilon > 0\). Since \((V(j))_{j \geq 1}\) are independent, it follows from the Borel–Cantelli Lemma that for every \(\epsilon > 0\) \(\limsup_{n \to \infty} \|V(n)\|_1 > \epsilon\) a.s. Letting \(\epsilon \to \infty\) through the integers, we have \(\limsup_{n \to \infty} \|V(n)\|_1 = +\infty\) a.s. Thus by (5.15), we obtain \(\limsup_{n \to \infty} \|Y(t_n)\|_1 = +\infty\) a.s., which implies that \(\limsup_{n \to \infty} \|Y(t)\|_1 = +\infty\) a.s.

5.5. **Proof of lower bound in part (B).** Suppose that \(S_t\) obeys (5.5). There exists an \(\epsilon < \epsilon'\) such that \(\sum_{j=1}^{\infty} \{1 - \Phi(\epsilon/\theta(j))\} = +\infty\). Therefore, by Lemma 1 it follows that there exists \(\epsilon'_1 > 0\) such that for all \(\epsilon/e^{-\beta} < \epsilon'_1\) we have
\[\sum_{j=1}^{\infty} \sum_{i=1}^{d} \left\{1 - \Phi\left(\frac{\epsilon}{e^{-\beta}t_i(j)}\right)\right\} = +\infty.
\]

By (5.16) we therefore have
\[\sum_{j=1}^{\infty} \sum_{i=1}^{d} \mathbb{P}[|V_i(j)| > \epsilon] \geq \sum_{j=1}^{\infty} 2\left\{1 - \Phi\left(\frac{\epsilon/e^{-\beta}}{e^{-\beta}t_i(j)}\right)\right\} = +\infty.
\]

Therefore by (5.17) we have
\[\sum_{j=1}^{\infty} \mathbb{P}[\|V(j)\|_1 > \epsilon] = +\infty.
\]

By the independence of \((V(j))\) together with the Borel–Cantelli Lemma, it follows that \(\limsup_{n \to \infty} \|V(n)\|_1 \geq \epsilon\) a.s. Letting \(\epsilon \uparrow \epsilon'_1 e^{-\beta}\) through the rational numbers gives \(\limsup_{n \to \infty} \|V(n)\|_1 \geq \epsilon'_1 e^{-\beta}\) on \(\Omega_1\), an a.s. event. By (5.18), \(V(n+1) = Y(t_{n+1}) - e^{-(t_{n+1}-t_n)}Y(t_n)\), so we have
\[\epsilon'_1 e^{-\beta} \leq \limsup_{n \to \infty} \|V(n, \omega)\|_1 \leq (1 + e^{-\alpha}) \limsup_{n \to \infty} \|Y(t_n, \omega)\|_1, \quad \text{for} \ \omega \in \Omega_1.
\]

Thus
\[\limsup_{n \to \infty} \|Y(t_n)\|_1 \geq \epsilon'_1 e^{-\beta}/(1 + e^{-\alpha}), \quad \text{a.s.},
\]

which implies \(\limsup_{n \to \infty} \|Y(t)\|_1 \geq \epsilon'_1 e^{-\beta}/(1 + e^{-\alpha}) =: c_1\), a.s.
5.6. Proof of upper bounds in parts (A) and (B). Suppose that
\[
\sum_{j=1}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\theta(j)} \right) \right\} < +\infty.
\] (5.19)
In part (A), (5.19) holds for all \(\epsilon > 0\), while in part (B) it holds for all \(\epsilon > \epsilon'\). By (5.19) and (5.10) we have
\[
\sum_{j=1}^{\infty} \sum_{i=1}^{d} \left\{ 1 - \Phi \left( \frac{\epsilon}{\theta(i)} \right) \right\} < +\infty,
\] and hence by (5.10) we have
\[
\sum_{j=1}^{\infty} \sum_{i=1}^{d} \mathbb{P}[|V_i(j)| \geq \epsilon] < +\infty.
\]
By the Borel–Cantelli lemma, it follows that \(\limsup_{n \to \infty} |V_i(n)| \leq \epsilon\) a.s. Now from (5.14), we have that
\[
Y_i(t_n) = \sum_{j=1}^{n} e^{-(t_n-t_j)} V_i(j),
\]
so therefore, as \(t_n - t_j \geq \alpha(n-j)\) for \(j = 1, \ldots, n\), we have that
\[
|Y_i(t_n)| \leq \sum_{j=1}^{n} e^{-(t_n-t_j)} |V_i(j)| \leq \sum_{j=1}^{n} e^{-\alpha(n-j)} |V_i(j)|,
\]
so
\[
\limsup_{n \to \infty} |Y_i(t_n)| \leq \epsilon \sum_{j=0}^{\infty} e^{-\alpha j} = \epsilon \frac{1}{1 - e^{-\alpha}}, \quad \text{a.s.} \quad (5.20)
\]
Next let \(t \in [t_n, t_{n+1})\). Therefore, from (1.3) we have
\[
Y_i(t) = Y_i(t_n)e^{-(t-t_n)} + \sum_{l=1}^{r} e^{-t} \int_{t_n}^{t} e^{s}(s) dB_l(s), \quad t \in [t_n, t_{n+1}).
\]
Therefore
\[
\max_{t \in [t_n, t_{n+1})} |Y_i(t)| \leq |Y_i(t_n)| + \max_{t \in [t_n, t_{n+1})} e^{-t} \left| \sum_{l=1}^{r} \int_{t_n}^{t} e^{s}(s) dB_l(s) \right| \leq |Y_i(t_n)| + Z_i(n), \quad (5.21)
\]
where
\[
Z_i(n) := e^{-t_n} \max_{t \in [t_n, t_{n+1})} \left| \sum_{l=1}^{r} \int_{t_n}^{t} e^{s}(s) dB_l(s) \right|, \quad n \geq 1.
\]
Fix \(n \in \mathbb{N}\). Now
\[
\mathbb{P}[Z_i(n) > \epsilon] = \mathbb{P} \left[ \max_{t \in [t_n, t_{n+1})} \left| \sum_{l=1}^{r} \int_{t_n}^{t} e^{s}(s) dB_l(s) \right| > \epsilon e^{t_n} \right]
\]
Define \(\tau_i(t) := \sum_{l=1}^{r} \int_{t_n}^{t} e^{2s}(s) dB_l(s) ds\) for \(t \in [n, n+1]\). Consider
\[
C_{in}(t) = \sum_{l=1}^{r} \int_{t_n}^{t} e^{s}(s) dB_l(s), \quad t \in [t_n, t_{n+1}].
\]
Then \(C_{in} = \{C_{in}(t) : t_n \leq t \leq t_{n+1}\}\) is a continuous martingale with \((C_{in})(t) = \tau_i(t)\) for \(t \in [t_n, t_{n+1}]\). Therefore, by the martingale time change theorem [21]
Since $\Phi$ is increasing, we have

$$P[Z_i(n) > \epsilon] = P \left[ \max_{t \in [t_n, t_n+1]} B^*_i(t) > e^{\epsilon t_n} \right]$$

$$= P \left[ \max_{u \in [0, \tau_i(n+1)]} |B^*_i(u)| > e^{\epsilon t_n} \right]$$

$$= P \left\{ \max_{u \in [0, \tau_i(n+1)]} B^*_i(u) > e^{\epsilon t_n} \right\} + P \left\{ -B^*_i(u) > e^{\epsilon t_n} \right\}$$

$$\leq P \left[ \max_{u \in [0, \tau_i(n+1)]} B^*_i(u) > e^{\epsilon t_n} \right] + P \left[ -B^*_i(u) > e^{\epsilon t_n} \right]$$

$$= P \left[ |B^*_i(\tau_i(n+1))| > e^{\epsilon t_n} \right] + P \left[ |B^*_i(\tau_i(n+1))| > e^{\epsilon t_n} \right],$$

where $B^*_i = -B^*_i$ is a standard Brownian motion. Recall that if $W$ is a standard Brownian motion that $max_{s \in [0,t]} W(s)$ has the same distribution as $|W(t)|$. Therefore, as $B^*_i(\tau_i(n+1))$ is normally distributed with zero mean we have

$$P[Z_i(n) > \epsilon] \leq 2 P \left[ |B^*_i(\tau_i(n+1))| > e^{\epsilon t_n} \right] = 4 P \left[ |B^*_i(\tau_i(n+1))| > e^{\epsilon t_n} \right]$$

$$= 4 \left( 1 - \Phi \left( \frac{e^{\epsilon t_n}}{\sqrt{\tau_i(n+1)}} \right) \right) = 4 \left( 1 - \Phi \left( \frac{\epsilon}{\theta_i(n)} \right) \right).$$

If we interpret $\Phi(\infty) = 1$, this formula holds valid in the case when $\tau_i(n+1) = 0$, because in this situation $Z_i(n) = 0$ a.s. Now

$$e^{-2\tau_i n} \tau_i(n+1) = e^{-2\tau_i} \int_{t_n}^{t_{n+1}} e^{2s} \sigma^2(s) ds$$

$$\leq e^{2(t_{n+1}-t_n)} \int_{t_n}^{t_{n+1}} \sigma^2(s) ds \leq e^{2\beta \theta_i(n)}.$$

Since $\Phi$ is increasing, we have

$$P[Z_i(n) > \epsilon] \leq 4 \left( 1 - \Phi \left( \frac{\epsilon}{\theta_i(n)} \right) \right).$$

Therefore we have

$$P[Z_i(n) > e^{\beta}] \leq 4 \left( 1 - \Phi \left( \frac{\epsilon}{\theta_i(n)} \right) \right), \quad n \geq 0. \quad (5.22)$$

Hence

$$\sum_{n=1}^{\infty} P[Z_i(n) > e^{\beta}] < +\infty,$$

so by the Borel–Cantelli lemma, we have that

$$\lim_{n \to \infty} P[Z_i(n)] \leq e^{\beta}, \quad a.s. \quad (5.23)$$

Therefore by (5.21), (5.20) and (5.23), we have that

$$\lim_{t \to \infty} \sup_{n \to \infty} |Y_i(t)| \leq (1/(1 - e^{-\alpha}) + e^{\beta})\epsilon, \quad a.s. \quad (5.24)$$

and so

$$\lim_{t \to \infty} \sup_{n \to \infty} \|Y_i(t)\|_1 \leq d(1/(1 - e^{-\alpha}) + e^{\beta})\epsilon, \quad a.s. \quad (5.24)$$

If (5.23) holds, (5.24) implies that $Y(t) \to 0$ as $t \to \infty$ a.s.

If the first part of (2.11) holds, then (5.24) holds for every $\epsilon > \epsilon'$. Thus, letting $\epsilon \downarrow \epsilon'$ through the rational numbers we have $\lim_{t \to \infty} \|Y(t)\|_1 \leq d(1/(1 - e^{-\alpha}) + e^{\beta})\epsilon' =: c_2$ a.s., proving (2.12).
We start by proving a preliminary lemma.

**Lemma 2.** Suppose \( x \in C([0, \infty); [0, \infty)). \)

(i) If \( \int_0^\infty x(t) \, dt = +\infty \), then for every \( h > 0 \) there exists a sequence \((t_n)_{n \geq 0}\) obeying

\[
t_0 = 0, \quad h \leq t_{n+1} - t_n \leq 3h, \quad n \geq 0
\]

such that

\[
\sum_{n=0}^{\infty} x(t_n) = +\infty \tag{6.1}
\]

(ii) If \( \int_0^\infty x(t) \, dt < +\infty \), then for every \( h > 0 \) there exists a sequence \((t_n)_{n \geq 0}\) obeying

\[
t_0 = 0, \quad h \leq t_{n+1} - t_n \leq 3h, \quad n \geq 0
\]

such that

\[
\sum_{n=0}^{\infty} x(t_n) < +\infty \tag{6.2}
\]

**Proof.** We start by proving part (i). Let \( s_0 = 0 \) and define for \( n \geq 1 \)

\[
s_n = \inf\{ t \in [nh, (n+1)h] : x(t) = \max_{s \in [nh, (n+1)h]} x(s) \}. \tag{6.3}
\]

Clearly \( s_n \in [nh, (n+1)h] \). Thus

\[
+\infty = \int_0^\infty x(t) \, dt = \int_0^h x(s) \, ds + \sum_{n=1}^{\infty} \int_{nh}^{(n+1)h} x(s) \, ds \leq h \max_{s \in [0, h]} x(s) + \sum_{n=1}^{\infty} h x(s_n).
\]

Therefore we have

\[
\sum_{n=1}^{\infty} x(s_{2n}) + \sum_{n=0}^{\infty} x(s_{2n+1}) = +\infty.
\]

Hence we have that either (I) \( \sum_{n=1}^{\infty} x(s_{2n}) = +\infty \) or (II) \( \sum_{n=0}^{\infty} x(s_{2n+1}) = +\infty \).

If case (I) holds, let \( t_n = s_{2n} \) for \( n \geq 0 \). Then \( t_0 = 0 \) and \((t_n)_{n \geq 0}\) obeys (6.1). Note that \( t_1 - t_0 = t_1 = s_2 \in [2h, 3h] \). For \( n \geq 1 \), we have \( t_{n+1} - t_n = s_{2n+2} - s_{2n} \). Hence \( t_{n+1} - t_n \leq (2n+3)h - 2nh = 3h \). Also \( t_{n+1} - t_n \geq (2n+2)h - (2n+1)h = h \). Therefore \( t_n \) obeys all the required properties.

If case (II) holds, let \( t_n = s_{2n-1} \) for \( n \geq 1 \) and \( t_0 = 0 \). Then \( t_0 = 0 \) and \((t_n)_{n \geq 0}\) obeys (6.1). Note that \( t_1 - t_0 = t_1 = s_1 \in [h, 2h] \). Therefore \( h \leq t_1 - t_0 \leq 2h < 3h \).

For \( n \geq 1 \), we have \( t_{n+1} - t_n = s_{2n+1} - s_{2n-1} \). Hence \( t_{n+1} - t_n \leq (2n+2)h - (2n-1)h = 3h \). Also \( t_{n+1} - t_n \geq (2n+1)h - (2n-1+1)h = h \). Therefore \( t_n \) obeys all the required properties.

We now turn to the proof of part (ii). Construct \((t_n)_{n=0}^{\infty}\) recursively as follows: let \( t_0 = 0 \), and for \( n \in \mathbb{N} \)

\[
t_{n+1} = \inf\{ t \in [t_n + h, t_n + 2h] : x(t) = \min_{t_{n+1} + h \leq s \leq t_{n+2}} x(s) \}. \tag{6.4}
\]

The existence of such a sequence can be proved by induction on \( n \), taking note that \( x \) is continuous on the compact interval \([t_n + h, t_n + 2h]\), and hence attains its minimum. By construction, we have

\[
t_{n+1} - t_n \geq h > 0, \tag{6.5}
\]

and \( t_{n+1} - t_n \leq 2h \). To prove (6.2), note that \( x(t_{n+1}) \leq x(t) \) for \( t_n + h \leq t \leq t_n + 2h \), so by integrating both sides of this inequality over \([t_n + h, t_n + 2h]\), using the non-negativity of \( x(\cdot) \) and \( t_n + 2h \leq t_{n+1} - h + 2h = t_{n+1} + h \) (which follows from (6.5)),
we get
\[ hx(t_{n+1}) \leq \int_{t_n+h}^{t_{n+1}+h} x(t) \, dt \leq \int_{t_n+h}^{t_{n+1}+2h} x(t) \, dt. \]
Summing both sides of this inequality establishes (6.2). \hfill \square

**Lemma 3.** Suppose that \( I \) is defined by (2.18).

(i) Suppose that \( I(\epsilon) = +\infty \). Then there exists \( (t_n)_{n \geq 0} \) independent of \( \epsilon > 0 \) such that
\[ t_0 = 0, \quad 0 < h \leq t_{n+1} - t_n \leq 3h < +\infty, \quad n \geq 0, \]
and
\[ \sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \sqrt{\frac{\epsilon \sigma(s)}{\|\sigma(s)\|^2}} \right) \right\} = +\infty. \]

(ii) Suppose that \( I(\epsilon) < +\infty \). Then there exists \( (t_n)_{n \geq 0} \) independent of \( \epsilon > 0 \) such that
\[ t_0 = 0, \quad 0 < h \leq t_{n+1} - t_n \leq 3h < +\infty, \quad n \geq 0, \]
and
\[ \sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \sqrt{\frac{\epsilon \sigma(s)}{\|\sigma(s)\|^2}} \right) \right\} < +\infty. \]

**Proof.** Define
\[ \zeta^2(t) = \int_t^{t+c} \|\sigma(s)\|^2 \, ds, \quad t \geq 0, \]
and \( \phi_\epsilon(x) = xe^{-\epsilon^2/(2\epsilon^2)} \chi_{[0,\infty)}(x) \) for \( x \geq 0 \). Therefore for \( x \geq 0 \) we have
\[ \frac{1}{\epsilon} \phi_\epsilon(x) = \frac{1}{\epsilon} xe^{-\epsilon^2/(2\epsilon^2)} \chi_{[0,\infty)}(x/\epsilon) = \phi_1(x/\epsilon). \]
Then
\[ I(\epsilon)/\epsilon = \int_0^{\infty} \phi_\epsilon (\zeta(t)) / \epsilon \, dt = \int_0^{\infty} \phi_1 (\zeta(t)/\epsilon) \, dt. \]
Let \( x_\epsilon(t) = \phi_1 (\zeta(t)/\epsilon) \) for \( t \geq 0 \). Clearly \( x \) is a non–negative function on \([0,\infty)\), and as \( \lim_{x \to 0^+} \phi_1(x) = 0 = \phi_1(0) \), we have that \( \phi_1 \) is continuous and increasing on \([0,\infty)\). Hence \( x_\epsilon \) is continuous on \([0,\infty)\). Note therefore that \( I(\epsilon)/\epsilon = \int_0^{\infty} x_\epsilon(t) \, dt \).

We are now in a position to prove part (ii). Suppose that \( I(\epsilon) < +\infty \). Let \( 0 < h \leq c/3 \). Then by Lemma 2 part (ii) there exists \( (t_n)_{n \geq 0} \) such that \( h \leq t_{n+1} - t_n \leq 3h \) and \( \sum_{n=0}^{\infty} \phi_\epsilon (\zeta(t_n)) < +\infty \). Recall that \( t_n \) are defined by (6.4) i.e., \( t_0 = 0 \), and for \( n \in \mathbb{N} \) we have
\[ t_{n+1} = \inf \{ t \in [t_n + h, t_n + 2h] : x_\epsilon(t) = \min_{t_n + h \leq s \leq t_n + 2h} x_\epsilon(s) \}. \]
Since \( x_\epsilon(t) = \phi_1 (\zeta(t)/\epsilon) \) and \( \phi_1 \) is increasing, it follows that
\[ t_{n+1} = \inf \{ t \in [t_n + h, t_n + 2h] : \zeta(t) = \min_{t_n + h \leq s \leq t_n + 2h} \zeta(s) \}, \]
and since \( \zeta \) is independent of \( \epsilon \), it follows that \( (t_n) \) is independent of \( \epsilon \).
\[ \sum_{n=0}^{\infty} \phi_\epsilon (\zeta(t_n)) < +\infty \] is therefore equivalent to
\[ \sum_{n=0}^{\infty} \zeta(t_n) \exp \left( -\frac{\epsilon^2}{2} \frac{1}{\zeta(t_n)^2} \right) < +\infty. \]
This implies that $\zeta(t_n) \to 0$ as $n \to \infty$, and by (2.6) we have that
\[
\lim_{n \to \infty} \frac{1 - \Phi(\epsilon/\zeta(t_n))}{\zeta(t_n)} \exp \left( -\frac{\epsilon^2}{2} \frac{1}{\zeta^2(t_n)} \right) = \frac{1}{\sqrt{2\pi}}
\]
Hence we have
\[
\sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|^2 \, ds}} \right) \right\} = \sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\zeta(t_n)} \right) \right\} < +\infty. \tag{6.7}
\]
Since $t_{n+1} \leq t_n + 3h$, and $3h \leq c$, we have
\[
\int_{t_n}^{t_{n+1}} \|\sigma(s)\|^2 \, ds \geq \int_{t_n}^{t_n+3h} \|\sigma(s)\|^2 \, ds \geq \int_{t_n}^{t_{n+1}} \|\sigma(s)\|^2 \, ds.
\]
Since $\Phi$ is increasing, we have
\[
1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|^2 \, ds}} \right) \geq 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|^2 \, ds}} \right).
\]
By (6.7) we have
\[
\sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|^2 \, ds}} \right) \right\} \leq \sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|^2 \, ds}} \right) \right\} < +\infty,
\]
which proves part (ii).

We are now in a position to prove part (i). Suppose that $I(\epsilon) = +\infty$. Let $h \in [c, \infty)$. Then by part (i) of Lemma 2 there exists $(t_n)_{n \geq 0}$ such that $h \leq t_{n+1} - t_n \leq 3h$ and $\sum_{n=0}^{\infty} \phi_1(\zeta(t_n)) = +\infty$. We now wish to show that the $(t_n)$ are independent of $\epsilon > 0$. Since they depend directly on the sequence $(s_n)$ defined by (6.3), we must simply show that the sequence $(s_n)$ is independent of $\epsilon$. By (6.3) we have
\[
s_n = \inf \{ t \in [nh, (n+1)h) : x_\epsilon(t) = \max_{s \in [nh, (n+1)h]} x_\epsilon(s) \}.
\]
Since $x_\epsilon(t) = \phi_1(\zeta(t)/\epsilon)$ and $\phi_1$ is increasing, it follows that
\[
s_n = \inf \{ t \in [nh, (n+1)h) : \zeta(t) = \max_{s \in [nh, (n+1)h]} \zeta(s) \},
\]
and since $\zeta$ is independent of $\epsilon$, so are $(s_n)$ and therefore $(t_n)$.

Next, $\sum_{n=0}^{\infty} \phi_1(\zeta(t_n)) = +\infty$ is equivalent to
\[
\sum_{n=0}^{\infty} \zeta(t_n) \exp \left( -\frac{\epsilon^2}{2} \frac{1}{\zeta^2(t_n)} \right) = +\infty.
\]
Suppose that
\[
\sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\zeta(t_n)} \right) \right\} < +\infty.
\]
Then $\zeta(t_n) \to 0$ as $n \to \infty$, and by (2.6) we have
\[
\lim_{n \to \infty} \frac{1 - \Phi(\epsilon/\zeta(t_n))}{\zeta(t_n)} \exp \left( -\frac{\epsilon^2}{2} \frac{1}{\zeta^2(t_n)} \right) = \frac{1}{\sqrt{2\pi}}
\]
Hence we have that
\[ \sum_{n=0}^{\infty} \zeta(t_n) \exp \left( -\frac{\epsilon^2}{2} \zeta(t_n)^2 \right) < +\infty, \]
a contradiction. Therefore we have
\[ \sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2 \, ds}} \right) \right\} = \sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\zeta(t_n)} \right) \right\} = +\infty. \quad (6.8) \]
Next, as \( c \leq h \) and \( t_{n+1} \geq t_n + h \) we have
\[ \int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2 \, ds \leq \int_{t_n}^{t_n+h} \|\sigma(s)\|_F^2 \, ds \leq \int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2 \, ds. \]
Since \( \Phi \) is increasing, we have
\[ 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2 \, ds}} \right) \leq 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2 \, ds}} \right). \]
By \((6.8)\) we have
\[ \sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2 \, ds}} \right) \right\} \geq \sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2 \, ds}} \right) \right\} = +\infty, \]
which proves part (i). \( \square \)

**Proof of Theorem 5.** To prove part (A), we have by hypothesis that \( I(\epsilon) < +\infty \) for all \( \epsilon > 0 \). Then, by Lemma 3 part (ii), for every \( h \leq c/3 \), there exists \( (t_n)_{n \geq 0} \) independent of \( \epsilon \) for which \( h \leq t_{n+1} - t_n \leq 3h \) and
\[ \sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2 \, ds}} \right) \right\} < +\infty. \]
Therefore by Theorem 14 part (A), it follows that \( Y(t) \to 0 \) as \( t \to \infty \) a.s.

To prove part (C), we have by hypothesis that \( I(\epsilon) = +\infty \) for all \( \epsilon > 0 \). Then, by Lemma 3 part (ii), for every \( h \geq c \), there exists \( (t_n)_{n \geq 0} \) independent of \( \epsilon \) for which \( h \leq t_{n+1} - t_n \leq 3h \) and
\[ \sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2 \, ds}} \right) \right\} = +\infty. \]
Therefore by Theorem 14 part (C), it follows that \( \limsup_{t \to \infty} \|Y(t)\| = +\infty \) a.s.

To prove part (B), we have by hypothesis that \( I(\epsilon) < +\infty \) for all \( \epsilon > \epsilon' \). Then, by Lemma 3 part (ii), for every \( h \leq c/3 \), there exists \( (t_n)_{n \geq 0} \) independent of \( \epsilon \) for which \( h \leq t_{n+1} - t_n \leq 3h \) and
\[ \sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2 \, ds}} \right) \right\} < +\infty. \]
Therefore by Theorem 14 part (B), it follows that \( \limsup_{t \to \infty} \|Y(t)\| \leq c_2 \) a.s.
On the other hand, we have by hypothesis that $I(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$. Then, by Lemma 2 part (i), for every $h \geq \epsilon$, there exists $(\tau_n)_{n \geq 0}$ independent of $\epsilon$ for which $h \leq \tau_{n+1} - \tau_n \leq 3h$ and

$$\sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\epsilon}{\sqrt{\int_{\tau_n}^{\tau_{n+1}} \| \sigma(s) \|_F^2 \, ds}} \right) \right\} = +\infty.$$  

Therefore by Theorem 1 part (B), it follows that $\limsup_{t \to \infty} \|Y(t)\| \geq c_1$ a.s. $\square$

7. Proofs of Theorem 1 and 9

7.1. Proof of Theorem 1

Let $z(t, \omega) = X(t, \omega) - Y(t, \omega)$ for $t \geq 0$. Then $z(0) = \xi$ and

$$z'(t, \omega) = AX(t, \omega) + Y(t, \omega) = Az(t, \omega) + g(t, \omega), \quad t \geq 0$$

where

$$g(t, \omega) = AY(t, \omega) + Y(t, \omega), \quad t \geq 0. \tag{7.1}$$

Let $\Psi$ be the unique continuous $d \times d$-valued matrix solution of

$$\Psi'(t) = A\Psi(t), \quad t \geq 0; \quad \Psi(0) = I_d.$$ 

Since all eigenvalues of $A$ have negative real parts, there exist $K > 0$ and $\lambda > 0$ such that

$$\|\Psi(t)\|_2 \leq Ke^{-\lambda t}, \quad t \geq 0. \tag{7.2}$$

Now by variation of constants, $z$ is given by

$$z(t, \omega) = \Psi(t)\xi + \int_0^t \Psi(s, \omega)g(s, \omega) \, ds, \quad t \geq 0. \tag{7.3}$$

To prove statement (A), suppose that $Y(t, \omega) \to 0$ as $t \to \infty$ for all $\omega \in \Omega^*$ where $\Omega^*$ is an a.s. event. We show now that $X(t, \xi, \omega) \to 0$ as $t \to \infty$ for every $\xi \in \mathbb{R}^d$ and every $\omega \in \Omega^*$, which would prove statement (A). Since $Y(t, \omega) \to 0$ as $t \to \infty$ we have $g(t, \omega) \to 0$ as $t \to \infty$. Therefore by (7.3), we have $z(t, \omega) \to 0$ as $t \to \infty$. Since $Y(t, \omega) \to 0$ as $t \to \infty$ and $\Psi(t) \to 0$ as $t \to \infty$, it follows that $X(t, \omega) \to 0$ as $t \to \infty$.

To prove the upper bound in part (B), note that because there is a deterministic $c_2 > 0$ such that $\limsup_{t \to \infty} \|Y(t)\|_2 \leq c_2$ a.s., we have

$$\limsup_{t \to \infty} \|g(t)\|_2 \leq \|I + A\|_2c_2, \quad \text{a.s.}$$

Using this estimate, the fact that $\Psi(t) \to 0$ as $t \to \infty$, and (7.2) we get

$$\limsup_{t \to \infty} \|z(t)\|_2 \leq \int_0^\infty \|\Psi(s)\|_2 \, ds \cdot \|I + A\|_2c_2 =: c_4, \quad \text{a.s.}$$

Hence we have $\limsup_{t \to \infty} \|X(t)\|_2 \leq c_2 + c_4 =: c_5$ a.s., which proves the upper estimate in (B).

To prove the lower bound in part (B), notice that by rewriting (2.2) in the form

$$dX(t) = (-X(t) + \{AX(t) + X(t)\}) \, dt + \sigma(t) \, dB(t),$$

and by using stochastic integration by parts and deterministic variation of constants, we arrive at

$$X(t) = \xi e^{-t} + \int_0^t e^{-(t-s)}(I + A)X(s) \, ds + Y(t), \quad t \geq 0.$$ 

Therefore, we have that

$$Y(t) = X(t) - \xi e^{-t} - \int_0^t e^{-(t-s)}(I + A)X(s) \, ds, \quad t \geq 0. \tag{7.4}$$
Suppose now that \( \Omega_1 = \{ \omega \colon \limsup_{t \to \infty} \|Y(t, \omega)\|_2 \geq c_1 \} \), where it is already known that \( \Omega_1 \) is an a.s. event. Then for \( \omega \in \Omega_1 \), we have
\[
c_1 \leq \limsup_{t \to \infty} \|Y(t, \omega)\|_2 \\
\leq \limsup_{t \to \infty} \|X(t, \omega)\|_2 + \|I + A\|_2 \limsup_{t \to \infty} \int_0^t e^{-t-s} \|X(s, \omega)\|_2 \, ds.
\]
Therefore we arrive at
\[
c_1 \leq (1 + \|I + A\|_2) \limsup_{t \to \infty} \|X(t, \omega)\|_2,
\]
for each \( \omega \in \Omega_1 \), and so
\[
\limsup_{t \to \infty} \|X(t)\|_2 \geq c_3 := \frac{c_1}{1 + \|I + A\|_2}, \quad \text{a.s.,}
\]
as required.

To prove statement (C), we start by noting by hypothesis that the event \( \Omega_2 = \{ \omega \colon \limsup_{t \to \infty} \|Y(t, \omega)\|_2 = +\infty \} \) is almost sure. Now suppose that there is an event \( C = \{ \omega \colon \limsup_{t \to \infty} \|X(t, \omega)\| < +\infty \} \cap \Omega_2 \) such that \( P[C] > 0 \). Taking norms on both sides of (7.4) for \( \omega \in C \) yields
\[
\|Y(t, \omega)\|_2 \leq \|X(t, \omega)\|_2 + \|\xi\|_2 e^{-t} + \|I + A\|_2 \int_0^t e^{-(t-s)} \|X(s, \omega)\|_2 \, ds.
\]
Define for \( \omega \in C \) the finite \( c(\omega) := \limsup_{t \to \infty} \|X(t, \omega)\|_2 \). Then
\[
+\infty = \limsup_{t \to \infty} \|Y(t, \omega)\|_2 \leq c(\omega) + \|I + A\|_2 c(\omega) < +\infty,
\]
a contradiction. Therefore, we must have that \( \limsup_{t \to \infty} \|X(t)\| = +\infty \) a.s. as required.

7.2. Proof of Theorem 9. Theorem 8 shows that (A) implies (C), and (C) clearly implies (B). It remains to prove that (B) implies (A). Define \( \xi_0 = 0 \) and for \( i = 1, \ldots, d \) set \( \xi_i = \xi_i - \xi_{i-1} \). Next, for \( \omega \in C \), define \( V_i(t, \omega) = X(t, \xi_i, \omega) - X(t, \xi_{i-1}, \omega) \) for \( i = 1, \ldots, d \). Therefore by hypothesis we have that \( V_i(t, \omega) \to 0 \) as \( t \to \infty \). Moreover, we see that \( V_i \) obeys the differential equation
\[
V_i(t, \omega) = AV_i(t, \omega), \quad t \geq 0, \quad V_i(0, \omega) = \xi_i - \xi_{i-1} = \xi_i.
\]
If \( \Psi \in \mathbb{R}^{d \times d} \) is the principal matrix solution given by \( \Psi'(t) = A\Psi(t) \) with \( \Psi(0) = I_d \), then \( V_i(t, \omega) = \Psi(t)\xi_i \). Therefore we have that \( \Psi(t)\xi_i \to 0 \) as \( t \to \infty \) for each \( i = 1, \ldots, d \). Since \( \xi_i \) are linearly independent, we have that \( \Psi(t) \to 0 \) as \( t \to \infty \). Hence it follows that all the eigenvalues of \( A \) have negative real parts.

Let \( Y \) be the solution of (1.3). Writing \( X \) as
\[
dX(t) = (-X(t) + \{X(t) + AX(t)\}) \, dt + \sigma(t) \, dB(t),
\]
by variation of constants, we see that
\[
X(t) = X(0)e^{-t} + \int_0^t e^{-(t-s)} \{X(s) + AX(s)\} \, ds + Y(t), \quad t \geq 0.
\]
Therefore, we see that \( Y(t, \omega) \to 0 \) as \( t \to \infty \) for each \( \omega \in C \). Since \( C \) is an event of positive probability, we see from Theorem 8 that \( Y(t) \to 0 \) as \( t \to \infty \) a.s., and that therefore \( I(\epsilon) \) is finite for all \( \epsilon > 0 \). We have therefore shown that (B) implies both conditions in (A), as required.
8. Proof of (2.13) in part (B) of Theorems 2 and 5 and of (2.16) in part (B) of Theorems 3, 6

We note first that the proof of (2.13) in part (B) of Theorem 2 is a direct corollary of part (B) in Theorem 3 where \( A = -I_d \). Similarly, the proof of (2.13) in part (B) of Theorem 5 is a corollary of part (B) in Theorem 6.

To prove (2.16) in Theorem 3 it suffices to show that \( \|X\| \) being bounded a.s. and \( S_1(\epsilon) < +\infty \) for some \( \epsilon > 0 \) implies (2.16); on the other hand, to prove (2.16) in Theorem 6, it suffices to show that \( \|X\| \) being bounded a.s. and \( I(\epsilon) < +\infty \) for some \( \epsilon > 0 \) implies (2.16). We note that if there is an \( \epsilon^* > 0 \) such that \( I(\epsilon^*) < +\infty \), then there is an \( \epsilon > 0 \) such that \( S_1(\epsilon) < +\infty \). Hence it only remains to prove that \( \|X\| \) being bounded a.s. and \( S_1(\epsilon) < +\infty \) for some \( \epsilon > 0 \) implies (2.16).

To do this, we note that \( S_1(\epsilon) < +\infty \) for some \( \epsilon > 0 \) implies \( \int_0^{n+1} \|\sigma(s)\|^2 \, ds \to 0 \) as \( n \to \infty \). In turn, this implies

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \|\sigma(s)\|^2 \, ds = 0. \tag{8.1}
\]

Since all the eigenvalues of \( A \) have negative real part, there exists a \( d \times d \) positive definite matrix \( M \) such that

\[
A^T M + MA = -I_d. \tag{8.2}
\]

(see for example Horn and Johnson [14] or Rugh [23]). Define \( V(x) = x^T M x \) for all \( x \in \mathbb{R}^d \). Notice that

\[
\frac{\partial V}{\partial x_i} = [2Mx]_i = \sum_{k=1}^d 2M_{ik}x_k.
\]

Therefore we have

\[
\frac{\partial^2 V}{\partial x_i \partial x_j}(x) = 2M_{ij}.
\]

Let \( X_i(t) = \langle X(t), e_i \rangle \). Notice that the cross–variation of \( X_i \) and \( X_j \) obeys

\[
d\langle X_i, X_j \rangle(t) = \sum_{k=1}^r \sigma_{ik}(t)\sigma_{jk}(t) \, dt.
\]

Therefore, as \( V \) is a \( C^2 \) function, by the multidimensional version of Itô’s formula, we have

\[
dV(X(t)) = \sum_{i=1}^d \frac{\partial V}{\partial x_i}(X(t))dX_i(t) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial^2 V}{\partial x_i \partial x_j}(X(t))d\langle X_i, X_j \rangle(t).
\]

Hence

\[
dV(X(t)) = \langle 2MX(t), AX(t) \rangle \, dt + \sum_{i=1}^d \sum_{j=1}^d M_{ij} \sum_{k=1}^r \sigma_{ik}(t)\sigma_{jk}(t) \, dt
\]

\[+ \langle 2MX(t), \sigma(t) dB(t) \rangle.
\]

Next, we note that because \( M = M^T \) and \( A^T M + MA = -I_d \), we have

\[
\langle 2Mx, Ax \rangle = \langle (M + M^T)x, Ax \rangle = \langle Mx, Ax \rangle + \langle Ax, M^T x \rangle
\]

\[= \langle Mx \rangle^T A x + \langle Ax \rangle^T M^T x = x^T M^T A x + x^T A^T M x = -x^T x.
\]
Also, since $M$ is positive definite, there exists a $d \times d$ matrix $P$ such that $M = PP^T$, so we have
\[
\sum_{i=1}^{d} \sum_{j=1}^{d} M_{ij} \sum_{k=1}^{r} \sigma_{ik}(t)\sigma_{jk}(t) = \sum_{i=1}^{d} \sum_{k=1}^{r} \left( \sum_{j=1}^{d} M_{ij} \sigma_{jk}(t) \right) \sigma_{ik}(t)
\]
\[
= \sum_{i=1}^{d} \sum_{k=1}^{r} [M\sigma(t)]_{ik}\sigma_{ik}^T(t) = \sum_{i=1}^{d} [M\sigma(t)\sigma(t)^T]_{ii}
\]
\[
= \sum_{i=1}^{d} [PP^T\sigma(t)\sigma(t)^T]_{ii} = \text{tr}(PP^T\sigma(t)\sigma(t)^T)
\]
\[
= \text{tr}(PP^T\sigma(t)\sigma(t)^T P) = \|PP^T\sigma(t)\|^2_F.
\]
where we have used the fact that $\|C\|^2_F = \text{tr}(CC^T)$ for any matrix $C$ and that $\text{tr}(CD) = \text{tr}(DC)$ for square matrices $C$ and $D$. Thus
\[
V(X(t)) = V(\xi) - \int_0^t X(s)^T X(s) \, ds + \int_0^t \|PP^T\sigma(s)\|^2_F \, ds + K(t), \quad t \geq 0, \quad (8.3)
\]
where
\[
K(t) = \sum_{j=1}^{r} \int_0^t \left\{ \sum_{i=1}^{d} 2M X(s)_{ij} \sigma_{ij}(s) \right\} dB_j(s), \quad t \geq 0. \quad (8.4)
\]
We consider the third term on the righthand side of (8.3). Since $\|PP^T\sigma(s)\|_F \leq \|P^T\|_F\|\sigma(s)\|_F$, from (8.1), we have that
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \|PP^T\sigma(s)\|_F^2 \, ds = 0. \quad (8.5)
\]
As to $K$, the fourth term on the righthand side of (8.3), we see that $K$ is a local martingale with quadratic variation given by
\[
\langle K \rangle(t) = \sum_{j=1}^{r} \int_0^t \left\{ \sum_{i=1}^{d} 2M X(s)_{ij} \sigma_{ij}(s) \right\}^2 ds.
\]
Hence by the Cauchy–Schwarz inequality, we have
\[
\langle K \rangle(t) \leq \sum_{j=1}^{r} \int_0^t \sum_{i=1}^{d} [2M X(s)]_{ij}^2 \sum_{i=1}^{d} \sigma_{ij}^2(s) ds = 4 \int_0^t \|MX(s)\|_2^2\|\sigma(s)\|_F^2 ds. \quad (8.6)
\]
Since $t \mapsto \|X(t)\|$ is bounded a.s., we may use (8.1) to get
\[
\lim_{t \to \infty} \frac{1}{t} \langle K \rangle(t) = 0, \quad \text{a.s.}
\]
Hence by the strong law of large numbers for martingales, we have that $K(t)/t \to 0$ as $t \to \infty$ a.s. Since $t \mapsto \|X(t)\|$ is bounded a.s. we have that $V(X(t))/t \to 0$ as $t \to \infty$ a.s. Therefore, returning to (8.3), we get
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)^T X(s) \, ds = 0, \quad \text{a.s.} \quad (8.7)
\]
Suppose now that there is an event $A_1$ with $P[A_1] > 0$ such that
\[
A_1 = \{ \omega : \liminf_{t \to \infty} \|X(t, \omega)\| > 0 \}.
\]
Since $t \mapsto \|X(t)\|$ is bounded, it follows that for each $\omega \in A_1$, there is a positive and finite $\bar{x}(\omega)$ such that
\[
\liminf_{t \to \infty} \|X(t, \omega)\|_2 =: \bar{x}(\omega).
\]
Therefore for \( \omega \in A_1 \) we have
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t X(s, \omega)^T X(s, \omega) \, ds \geq \hat{\alpha}(\omega) > 0.
\]

Therefore
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t X(s)^T X(s) \, ds > 0, \quad \text{a.s. on } A_1,
\]
which contradicts \( \text{[5.7]} \), because \( P[A_1] > 0 \). Therefore, it must be the case that \( P[A_1] = 0 \), which implies that \( P'[A_1] = 1 \), or \( \liminf_{t \to \infty} \|X(t)\| = 0 \) a.s. as required.

9. Proofs of Proposition 3 and Part (C) of Theorem 4

We prove a simple lemma which will be of utility in the proof of each of these results.

**Lemma 4.** Suppose that \( f : [0, \infty) \to \mathbb{R} \) is a continuous function such that
\[
\lim_{n \to \infty} \int_{nh}^{(n+1)h} f^2(s) \, ds = 0.
\]
Then for any \( \lambda > 0 \) we have
\[
\lim_{t \to \infty} \int_0^t e^{-2\lambda(t-s)} f^2(s) \, ds = 0.
\]

**Proof.** For every \( t > 0 \) there exists \( n(t) \in \mathbb{N} \) such that \( n(t)h \leq t < (n(t)+1)h \). Then
\[
\int_0^t e^{-2\lambda(t-s)} f(s)^2 \, ds = \sum_{j=1}^{n(t)} \int_{(j-1)h}^{jh} e^{-2\lambda(t-s)} f^2(s) \, ds + \int_{n(t)h}^t e^{-2\lambda(t-s)} f^2(s) \, ds
\]
\[
\leq \sum_{j=1}^{n(t)} e^{-2\lambda h(n(t)-j)} \int_{(j-1)h}^{jh} f^2(s) \, ds + \int_{n(t)h}^{(n(t)+1)h} f^2(s) \, ds.
\]

Therefore, as the last term has zero limit because \( \int_{nh}^{(n+1)h} f^2(s) \, ds \to 0 \) as \( n \to \infty \), we have
\[
\limsup_{t \to \infty} \int_0^t e^{-2\lambda(t-s)} \|\sigma(s)\|^2_F \, ds \leq \limsup_{n \to \infty} \sum_{j=1}^{n} e^{-2\lambda h(n-j)} \int_{(j-1)h}^{jh} f^2(s) \, ds.
\]

We see that the righthand side is the discrete convolution of a summable and a null sequence. Hence the limit is zero, and the claim holds. \( \square \)

9.1. Proof of Proposition 3. It is easy to see that (A) implies (B), that (B) implies (C), and that (C) implies (A). Hence (A)–(C) are equivalent. We prove now (C) implies (D). Given that \( X(0) = \xi \) is independent of \( B \), Itô’s isometry yields
\[
\mathbb{E}[\|X(t)\|^2_F] = \mathbb{E}[\|\Psi(t)\xi\|^2_F] + \int_0^t \|\Psi(t-s)\sigma(s)\|^2_F \, ds, \quad t \geq 0.
\]

Since all the eigenvalues of \( A \) have negative real parts, it follows that there exists \( \lambda > 0 \) and \( K_2 > 0 \) such that \( \|\Psi(t)\|_2 \leq K_2 e^{-\lambda t} \) for all \( t \geq 0 \). Since there exists a \( c_1 > 0 \) such that \( \|C\|_F \leq c_1 \|C\|_2 \) for all \( C \in \mathbb{R}^{d \times d} \), we have
\[
\|\Psi(t)e^{\lambda t}\|_F \leq c_1 \|\Psi(t)e^{\lambda t}\|_2 \leq c_1 K_2, \quad t \geq 0,
\]
Define $\sigma$ by
$$
\int Y_i \text{ possible fluctuations of } t
$$
Notice also that for $t > 0$, hence using the submultiplicative property of the Frobenius norm, we have
$$
E[\|X(t)\|_F^2] \leq \|\Psi(t)\|_F^2 E[\|\xi\|_F^2] + \int_0^t \|\Psi(t-s)\|_F^2 \|\sigma(s)\|_F^2 \, ds
$$
$$
\leq K_2^2 e^{-2\lambda t} E[\|\xi\|_F^2] + c_1^2 K_2^2 \int_0^t e^{-2\lambda(t-s)} \|\sigma(s)\|_F^2 \, ds.
$$
By Lemma 4.14, the second term on the righthand side tends to zero as $t \to \infty$ when $\sigma$ obeys (2.17), which proves that statement (A) implies statement (D).

To prove that statement (D) implies statement (C), which will suffice to complete the proof, we start by writing
$$
\int_t^{t+1} \sigma(s) dB(s) = X(t + 1) - X(t) - \int_t^{t+1} AX(s) \, ds, \quad t \geq 0.
$$
Considering the expectation of $\| \cdot \|_2^2$ on both sides, and using Itô’s isometry on the left hand side, we deduce the identity
$$
\int_t^{t+1} \|\sigma(s)\|_F^2 \, ds = E \left[ \left\| X(t + 1) - X(t) - \int_t^{t+1} AX(s) \, ds \right\|_2^2 \right].
$$
Since (D) holds by hypothesis, the righthand side converges to zero as $t \to \infty$, completing the proof.

9.2. Proof of Part (C) of Theorem 4.4. In part (C), $\sigma$ is not in $L^2([0, \infty); \mathbb{R}^{d \times r})$. In this case, there exists a pair of integers $(i, j) \in \{1, \ldots, d\} \times \{1, \ldots, r\}$ such that $\sigma_{ij} \notin L^2([0, \infty); \mathbb{R})$. Note that $Y_i$ obeys
$$
dY_i(t) = -Y_i(t) \, dt + \sum_{j=1}^r \sigma_{ij}(t) \, dB_j(t), \quad t \geq 0.
$$
Thus there exists a standard Brownian motion $\bar{B}_i$ such that
$$
dY_i(t) = -Y_i(t) \, dt + \sqrt{\sum_{i=1}^r \sigma_{ii}^2(t)} \, d\bar{B}_i(t), \quad t \geq 0.
$$
Define
$$
\sigma_{ii}^2(t) = \sum_{i=1}^r \sigma_{ii}^2(t), \quad t \geq 0. \quad (9.1)
$$
Then $\sigma_i \notin L^2(0, \infty)$, and it is possible to define a number $T_i > 0$ such that $\int_0^t e^{2s} \sigma_{ii}^2(s) \, ds > e^c$ for $t > T_i$ and so one can define a function $\Sigma_i : [T_i, \infty) \to [0, \infty)$ by
$$
\Sigma_i(t) = \left( \int_0^t e^{-2(t-s)} \sigma_{ii}^2(s) \, ds \right)^{1/2} \left( \log \log \int_0^t e^{2s} \sigma_{ii}^2(s) \, ds \right)^{1/2}, \quad t \geq T_i. \quad (9.2)
$$
Notice also that for $t \geq T_i$ we have
$$
\Sigma_i(t)^2 \leq \int_0^t e^{-2(t-s)} \sigma_{ii}^2(s) \, ds \cdot \log \log \int_0^t e^{2s} \|\sigma(s)\|_F^2 \, ds
$$
$$
\leq \int_0^t e^{-2(t-s)} \|\sigma(s)\|_F^2 \, ds \cdot \log \log e^{2t} \int_0^t \|\sigma(s)\|_F^2 \, ds.
$$
The significance of the function $\Sigma_i$ defined in (9.2) is that it characterises the largest possible fluctuations of $Y_i$ when $\sigma_i \notin L^2(0, \infty)$. To do this we apply the Law of the
Hence for which ensures that By Lemma 4, we have that Since \( \Psi \) obeys the estimate (7.2), we have for

\[
\limsup_{t \to \infty} \frac{\|Y_i(t)\|^2}{\sum_i^N(t)} = 2, \quad \text{a.s. (9.3)}
\]

Let \( N = \{i = 1, \ldots, d : \sigma_i \notin L^2(0, \infty)\} \), and \( F = \{1, \ldots, d\} \setminus N \). Clearly, if \( i \in F \), we have that \( Y_i(t) \to 0 \) as \( t \to \infty \) a.s., so

\[
\lim_{t \to \infty} \sum_{i \in F} \|Y_i(t)\|^2 = 0, \quad \text{a.s.}
\]

By [9.3], for every \( i \in N \), there exist \( T'_i(\omega) > T_i \) such that \( \|Y_i(t, \omega)\|^2 \leq 4\Sigma_i^2(t) \) for \( t \geq T'_i(\omega) \). Define \( T^*(\omega) = \max_{i \in N} T_i(\omega) \). Then for \( t \geq T^*(\omega) \) we have

\[
\|Y(t, \omega)\|^2 \leq 4\Sigma^2(t) \leq 4 \int_0^t e^{-2(t-s)}\|\sigma(s)\|^2_F \, ds \cdot \log \log \left( e^{2t} \int_0^t \|\sigma(s)\|^2_F \, ds \right).
\]

Therefore for \( t \geq T^*(\omega) \) we get

\[
\sum_{i \in N} \|Y_i(t, \omega)\|^2 \leq 4d \int_0^t e^{-2(t-s)}\|\sigma(s)\|^2_F \, ds \cdot \log \log \left( e^{2t} \int_0^t \|\sigma(s)\|^2_F \, ds \right).
\]

Hence there exists \( T''(\omega) > 0 \) such that for all \( t \geq T''(\omega) \) we have

\[
\|Y(t, \omega)\|^2 \leq 1 + 4d \int_0^t e^{-2(t-s)}\|\sigma(s)\|^2_F \, ds \cdot \log \log \left( e^{2t} \int_0^t \|\sigma(s)\|^2_F \, ds \right).
\]

Now, because (2.17) holds, we have \( \int_0^t \|\sigma(s)\|^2_F \, ds/t \to 0 \) as \( t \to \infty \). Therefore

\[
\limsup_{t \to \infty} \frac{1}{\log t} \log \log \left( e^{2t} \int_0^t \|\sigma(s)\|^2_F \, ds \right) \leq 1.
\]

Hence there is \( T''''(\omega) > 0 \) such that for all \( t \geq T''''(\omega) \) we have

\[
\|Y(t, \omega)\|^2 \leq 1 + 8d \int_0^t e^{-2(t-s)}\|\sigma(s)\|^2_F \, ds \cdot \log t, \quad t \geq T''''(\omega).
\]

Hence

\[
\limsup_{t \to \infty} \frac{\|Y(t)\|^2}{\log t} \leq 8d \limsup_{t \to \infty} \int_0^t e^{-2(t-s)}\|\sigma(s)\|^2_F \, ds, \quad \text{a.s.}
\]

By Lemma 4, we have that

\[
\lim_{t \to \infty} \int_0^t e^{-2(t-s)}\|\sigma(s)\|^2_F \, ds = 0,
\]

which ensures that

\[
\lim_{t \to \infty} \frac{\|Y(t)\|^2}{\log t} = 0, \quad \text{a.s. (9.4)}
\]

Using the proof of part (A) of Theorem 1, we have from (7.3) and (7.4) that \( z(t) := X(t) - Y(t) \) for \( t \geq 0 \) obeys

\[
z(t) = \Psi(t)\xi + \int_0^t \Psi(t-s)(I_d + A)Y(s) \, ds, \quad t \geq 0.
\]

Since \( \Psi \) obeys the estimate (7.2), we have for \( t \geq 0 \)

\[
\|z(t)\|_2 \leq \|\Psi(t)\|_2 \|\xi\|_2 + \int_0^t \|\Psi(t-s)\|_2 \|I_d + A\|_2 \|Y(s)\|_2 \, ds
\]

\[
\leq Ke^{-\lambda t} \|\xi\|_2 + K \|I_d + A\|_2 \int_0^t e^{-\lambda(t-s)} \|Y(s)\|_2 \, ds.
\]
Therefore we have
\[ ||X(t)||_2 \leq Ke^{-\lambda t}||\xi||_2 + ||Y(t)||_2 + K||I_d + A||_2 \int_0^t e^{-\lambda(t-s)}||Y(s)||_2 ds, \quad t \geq 0.\]
Since \( Y \) obeys (9.4), it follows that
\[ \limsup_{t \to \infty} \frac{||X(t)||_2}{\sqrt{\log t}} = 0, \quad \text{a.s.} \tag{9.5} \]

Our strategy now is to return to the identity (8.3), and estimate the asymptotic behaviour of each of the terms. We start with the term on the left-hand side. Since all the eigenvalues of \( A \) have negative real parts, there exists a positive definite matrix \( M \) which satisfies (8.2). Then \( V(x) = \langle x, Mx \rangle \) obeys
\[
\frac{V(x)}{||x||^2} = \frac{x}{||x||^2} M \frac{x}{||x||^2} \leq \sup_{||u||^2 = 1} \langle u, Mu \rangle =: \mu_1 > 0
\]
for \( x \neq 0 \). Hence \( 0 \leq V(x) \leq \mu_1 ||x||^2 \) for all \( x \in \mathbb{R}^d \). Therefore by (9.5) we have
\[ \lim_{t \to \infty} \frac{V(X(t))}{\log t} = 0, \quad \text{a.s.} \tag{9.6} \]
The first term on the righthand side of (8.3) is constant. We wish to prove that the second term on the righthand side of (8.3) obeys (9.7), i.e.,
\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t ||X(s)||^2 ds = 0, \quad \text{a.s.} \]
We note that this limit automatically implies that \( \liminf_{t \to \infty} ||X(t, \xi)||_2 = 0 \) a.s.

The asymptotic behaviour of the third term on the righthand side of (8.3) is easily determined: since (2.17) holds, the limit (8.5) follows. It remains to estimate the asymptotic behaviour of the fourth term on the righthand side of (8.3), which is a local martingale with quadratic variation bounded by (8.0), i.e.,
\[ \langle K \rangle(t) \leq 4||M||^2 \int_0^t ||X(s)||^2 ||\sigma(s)||_F^2 ds, \quad t \geq 0. \]
By (9.5), it follows for every \( \epsilon > 0 \) and \( \omega \) in an a.s. event \( \Omega^* \) that there is a \( T_1(\epsilon, \omega) \) such that
\[ ||X(t, \omega)||^2 < \epsilon \log t, \quad t \geq T_1(\epsilon, \omega). \]
By (2.17), we have that there exists \( T_2(\epsilon) > 0 \) such that \( \int_0^t ||\sigma(s)||_F^2 ds < \epsilon t \) for \( t \geq T_2(\epsilon) \). Define \( T_3(\epsilon, \omega) = \max(T_1(\epsilon, \omega), T_2(\epsilon)) \). Then for \( t \geq T_3(\epsilon, \omega) \) we have
\[
\langle K \rangle(t) \leq D(\epsilon, \omega) + 4||M||^2 \int_{T_3(\epsilon, \omega)}^t ||X(s, \omega)||^2 ||\sigma(s)||_F^2 ds
\]
\[ \leq D(\epsilon, \omega) + 4||M||^2 \epsilon t \log t \int_{T_3(\epsilon, \omega)}^t ||\sigma(s)||_F^2 ds
\]
\[ \leq D(\epsilon, \omega) + 4||M||^2 \epsilon^2 t \log t, \]
where we have defined
\[ D(\epsilon, \omega) := 4||M||^2 \int_{T_3(\epsilon, \omega)}^\infty ||X(s, \omega)||^2 ||\sigma(s)||_F^2 ds. \]
Hence we have that
\[ \lim_{t \to \infty} \frac{\langle K \rangle(t)}{t \log t} = 0, \quad \text{a.s.} \tag{9.7} \]
Let \( A_1 := \{ \omega : \lim_{t \to \infty} \langle K \rangle(t, \omega) \text{ is finite} \} \), and \( A_2 := \{ \omega : \lim_{t \to \infty} \langle K \rangle(t, \omega) = +\infty \} \). Then \( K \) converges a.s. on \( A_1 \) and we have
\[ \lim_{t \to \infty} \frac{1}{t} \langle K \rangle(t) = 0, \quad \text{a.s. on } A_1. \]
On $A_2$, the Law of the iterated logarithm for martingales holds, namely
\[
\limsup_{t \to \infty} \frac{|K(t)|}{\sqrt{2(K(t) \log \log(K(t))}} = 1, \quad \text{a.s. on } A_2.
\]
By (9.7) we have
\[
\limsup_{t \to \infty} \frac{\log \log(K(t))}{\log t} \leq 1, \quad \text{a.s. on } A_2.
\] (9.8)
Therefore, we have
\[
\limsup_{t \to \infty} \frac{|K(t)|}{t} \leq \limsup_{t \to \infty} \sqrt{\frac{2(K(t) \log \log(K(t))}{t^2}}, \quad \text{a.s. on } A_2
\]
Now, we rewrite the quotient in the limit according to
\[
\frac{2(K(t) \log \log(K(t))}{t^2} = \frac{2(K(t))}{t \log t} \cdot \frac{\log \log(K(t))}{\log t},
\]
and so from (9.7) and (9.8), we have that
\[
\lim_{t \to \infty} \frac{K(t)}{t} = 0, \quad \text{a.s. on } A_2
\]
Since $A_1 \cup A_2$ is an a.s. event, it follows that $K(t)/t \to 0$ as $t \to \infty$ a.s. Using this limit, (8.5), and (9.6) in (8.3), we arrive at the desired limit (8.7).

10. Proof of Theorem 12

Under (4.4), By [11][Theorem 2.48], we have that there exists a continuously differentiable function such that $P(t) \in \mathbb{C}^{d \times d}$, $P(t)$ is invertible and $P$ is $T$–periodic, and a matrix $L \in \mathbb{C}^{d \times d}$ all of whose eigenvalues have negative real parts such that
\[
\Psi(t) = P(t)e^{Lt}.
\]
Notice also that $P^{-1}$ is continuously differentiable and $T$–periodic. Since all the eigenvalues of $L$ have negative real parts, there exists a Hermitian and positive definite matrix $Q \in \mathbb{C}^{d \times d}$ such that
\[
QL + L^*Q = \mathbb{I}_d.
\]
Also, as $P$ is periodic and continuous, and $P^{-1}$ is periodic and continuous, we have the estimate $\|P(t)\| \leq p_*$, $\|P(t)^{-1}\| \leq p_*$ for some $p_* > 0$. Also, as all eigenvalues of $L$ have negative real parts, we have the estimate
\[
\|\Psi(t)\| \leq p_*e^{-\lambda t}, \quad \|e^{Lt}\| \leq ce^{-\lambda t}.
\]
Define $z(t) = X(t) - Y(t)$ for $t \geq 0$. Then with $g(t) = (I_d + A(t))Y(t)$, we have
\[
z(t) = A(t)Z(t) + g(t), \quad t \geq 0; \quad z(0) = \xi.
\]
Hence for $t \geq 0$ we have the variation of constants formula
\[
z(t) = \Psi(t)\xi + \int_0^t \Psi(s)\Psi(s)^{-1}g(s)ds = \Psi(t)\xi + \int_0^t P(t)e^{L(t-s)}P(s)^{-1}g(s)ds.
\]
Therefore for $t \geq 0$ we have
\[
\|z(t)\| \leq p_*e^{-\lambda t}\|\xi\| + p_*^2c \max_{\sigma \in [0,T]} \|I + A(s)\| \cdot \int_0^t e^{-\lambda(t-s)}\|Y(s)\| ds.
\]
This leads to the estimate
\[
\|X(t)\| \leq p_*e^{-\lambda t}\|\xi\| + \|Y(t)\| + p_*^2c \max_{\sigma \in [0,T]} \|I + A(s)\| \cdot \int_0^t e^{-\lambda(t-s)}\|Y(s)\| ds. \quad (10.1)
\]
We see automatically that when $Y(t) \to 0$ as $t \to \infty$ a.s., then $X(t) \to 0$ as $t \to \infty$ a.s.; this proves part (A), because $S_0(t) < +\infty$ implies $\lim_{t \to \infty} Y(t) = 0$ a.s.
In the case that $S'_h(\epsilon) < +\infty$ for all $\epsilon > \epsilon'$ and $S'_h(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$, we have that $\limsup_{t \to \infty} \|Y(t)\| \leq c_2$ a.s. for some deterministic $c_2 > 0$. Therefore, from (10.1), we see that $\limsup_{t \to \infty} \|X(t)\| \leq c_4$ a.s., where $c_4$ is

$$c_4 = c_2 + p^2 e \max_{s \in [0, T]} \|I + A(s)\| \frac{1}{\epsilon} c_2,$$

which yields the desired upper bound in part (B).

To prove part (C), we start by noticing that $S'_h(\epsilon) = +\infty$ for all $\epsilon > 0$ implies $\limsup_{t \to \infty} \|Y(t)\| = +\infty$ a.s. Observing that the identity

$$Y(t) = X(t) - \xi e^{-t} - \int_0^t e^{-(t-s)}(I_d + A(s))X(s) \, ds, \quad t \geq 0,$$

holds, we see that if there is an event of positive probability for which the limit superior $\limsup_{t \to \infty} \|X(t)\|$ is finite, then $\limsup_{t \to \infty} \|Y(t)\| < +\infty$ on this event, which results in a contradiction.

The proof of the lower bound in part (B) is similar. Since $S'_h(\epsilon) < +\infty$ for all $\epsilon > \epsilon'$ and $S'_h(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$, it follows that there exists a deterministic $c_1 > 0$ such that $\limsup_{t \to \infty} \|Y(t)\| \geq c_1$ a.s. Suppose that there is an event of positive probability such that $\limsup_{t \to \infty} \|X(t)\| =: c(\omega) \leq c_1/(1 + \max_{t \in [0, T]} \|I_d + A(t)\|_2) =: c_3$. Then

$$c_1 \leq \limsup_{t \to \infty} \|Y(t)\| \leq c(\omega) \|X(t)\| + \sup_{t \in [0, T]} \|I_d + A(t)\|_2 \cdot c(\omega),$$

so $c_1/(1 + \max_{t \in [0, T]} \|I_d + A(t)\|_2) > c(\omega) \geq c_1/(1 + \max_{t \in [0, T]} \|I_d + A(t)\|_2)$, a contradiction. Therefore we have that $\limsup_{t \to \infty} \|X(t)\| \geq c_3$ a.s.

We now prove the ergodic result in part (B), from which $\liminf_{t \to \infty} \|X(t)\| = 0$ a.s. follows easily. To do this, we define for $x \in \mathbb{R}^d$ the function $V(t, x) = x^T (P(t)^{-1})^* Q P(t)^{-1} x$. Note that $V(\cdot, x)$ is $T$–periodic and real–valued, because $M(t) := (P(t)^{-1})^* Q P(t)^{-1}$ is Hermitian. We may now write $V(t, x) = x^T M(t)x$.

This function $V$ was used in Giesl and Hafstein [13, Theorem 6] as a strict Lyapunov function in proving that the zero solution of the unperturbed differential equation $x'(t) = A(t)x(t)$ is asymptotically stable.

We start by obtaining a $t$–uniform upper bound on $V$. Define $M_1(t) = M(t) + M(t)^T$. Suppressing $t$ dependence for a moment, we notice that $M_1 = M + M^T$ is symmetric. Also, if we define the matrices $G, H \in \mathbb{R}^{d \times d}$ so that $M = G + iH$, then $M^T = GT + iHT$ and $M^* = GT - iHT$. Therefore as $M = M^*$, we have $G = GT$ and $H = -HT$. Hence $M_1 = M + M^T = (G + GT) + i(H + HT) = 2G$ is a real–valued symmetric matrix.

For $x \neq 0$, we now have

$$\frac{V(t, x)}{\|x\|^2} = \frac{x^T M(t)x}{\|x\|^2} \leq \sup_{\|u\|^2 = 1; u \in \mathbb{R}^d} u^T M(t)u$$

$$= \sup_{\|u\|^2 = 1; u \in \mathbb{R}^d} \frac{1}{2} u^T M_1(t)u \leq \frac{1}{2} \|M_1(t)\|_2.$$  

Since $t \mapsto P(t)^{-1}$ is continuous and $T$–periodic, it follows that $t \mapsto M_1(t)$ is continuous, real–valued and $T$–periodic. Therefore, there exists $c_6 \in (0, \infty)$ defined by

$$c_6 := \max_{t \in [0, T]} \|M_1(t)\|_2/2$$

such that

$$V(t, x) \leq c_6 \|x\|^2, \quad \text{for all } (t, x) \in [0, \infty) \times \mathbb{R}^d.$$

Next, we notice that

$$\dot{P}^{-1}(t) = LP^{-1}(t) - P^{-1}(t)A(t).$$
Therefore
\[ M'(t) = (\hat{P}(t)^{-1})^*QP(t)^{-1} + (P(t)^{-1})^*Q\hat{P}(t)^{-1} \]
\[ = (LP^{-1}(t) - P^{-1}(t)A(t))^*QP(t)^{-1} + (P(t)^{-1})^*Q(LP^{-1}(t) - P^{-1}(t)A(t)). \]
Hence
\[ M'(t) = P^{-1}(t)^*L^*QP(t)^{-1} - A(t)^*P^{-1}(t)^*QP(t)^{-1} + (P(t)^{-1})^*QLP^{-1}(t) \]
\[ - (P(t)^{-1})^*QLP^{-1}(t)A(t). \]
Using the fact that QL + L^*Q = -I_d, and the definition of M(t) we get
\[ M'(t) = -P^{-1}(t)^*P(t)^{-1} - A(t)^*M(t) - M(t)A(t). \]
Hence
\[ \frac{\partial V}{\partial t}(t, x) = x^T M'(t)x = -x^T P^{-1}(t)^*P(t)^{-1}x - x^T A(t)^T M(t)x - x^T M(t)A(t)x. \]
Next, we notice that
\[ \frac{\partial^2 V}{\partial x_i \partial x_j}(t, x) = [(M(t) + M(t)^T)x]_i = \sum_{k=1}^d (M_{ik}(t) + M(t)_{ik}^T)x_k. \]
Therefore we have
\[ \frac{\partial^2 V}{\partial x_i \partial x_j}(t, x) = [M_1(t)]_{ij}. \]
Let \( X_i(t) = (X(t), e_i) \). Notice that the cross–variation of \( X_i \) and \( X_j \) obeys
\[ d\langle X_i, X_j \rangle(t) = \sum_{k=1}^r \sigma_{ik}(t)\sigma_{jk}(t) dt. \]
Therefore, as \( V \) is a \( C^{1,2} \) function, by the multidimensional version of Itô’s formula, we have
\[ dV(t, X(t)) = \left(-X(t)^TP^{-1}(t)^*P(t)^{-1}X(t) - X(t)^T A(t)^T M(t) X(t) - X(t)^T M(t) A(t) X(t) \right) \]
\[ + (M(t) + M(t)^T)X(t)^T A(t) X(t) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d [M_1(t)]_{ij} \sum_{k=1}^r \sigma_{ik}(t)\sigma_{jk}(t) \]
\[ + \langle (M(t) + M(t)^T)X(t), \sigma(t) dB(t) \rangle. \]
Since \( M_1 \) is a real–valued symmetric matrix, we may define the real–valued and deterministic function \( J : [0, \infty) \to \mathbb{R} \) by
\[ J(t) := \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d [M_{ij}(t) + M_{ji}(t)] \sum_{k=1}^r \sigma_{ik}(t)\sigma_{jk}(t), \quad (10.3) \]
and the real–valued continuous local martingale \( K \) by
\[ K(t) = \int_0^t \langle M_1(s)X(s), \sigma(s) dB(s) \rangle = \sum_{j=1}^r \int_0^t \left\{ \sum_{i=1}^d [M_1(s)X(s)]_i \sigma(s)_i \right\} dB_j(s), \quad (10.4) \]
and observe that

\[ V(t, X(t)) = V(0, \xi) - \int_0^t X(s)^T (P(s)^{-1})^* P(s)^{-1} X(s) \, ds \]

\[ + \int_0^t J(s) \, ds + K(t), \quad t \geq 0. \tag{10.5} \]

We now attempt to estimate each of the terms in \(10.5\). We start with \(J(t)\), observing that it can be written as

\[ J(t) = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d (M(t)^T + M(t))_{ji} (\sigma(t) \sigma(t)^T)_{ij} \]

\[ = \frac{1}{2} \sum_{j=1}^d \sum_{i=1}^d (M(t)^T + M(t))_{ji} (\sigma(t) \sigma(t)^T)_{ij} \]

\[ = \frac{1}{2} \text{tr}(M_1(t) \sigma(t) \sigma(t)^T). \]

Since \( t \mapsto P^{-1}(t) \) is \(T\)-periodic and continuous, it follows that \( t \mapsto M_1(t) \) is continuous and \(T\)-periodic. Therefore, using the fact that the Frobenius norm is subadditive and submultiplicative, \( \|D^T\|_F = \|D\|_F \) for every \(d \times r\) matrix \(D\), and that \( \text{tr}(C)^2 \leq d \|C\|_F^2 \) for every \(d \times d\) matrix \(C\), we have that

\[ |J(t)| = \frac{1}{2} \left| \text{tr}(M_1(t) \sigma(t) \sigma(t)^T) \right| \leq \frac{1}{2} \sqrt{d} \max_{t \in [0,T]} \|M_1(t)\|_F \|\sigma(t)^2\|_F \]

\[ \leq \frac{1}{2} \sqrt{d} \max_{t \in [0,T]} \|M_1(t)\|_F \|\sigma(t)\|_F^2. \tag{10.6} \]

Now, as \( S'_h(\epsilon) < +\infty \) for \( \epsilon > \epsilon' \), it follows that \( \int_{nt}^{(n+1)t} \|\sigma(s)\|_F^2 \, ds \to 0 \) as \( n \to \infty \). Therefore, it follows that

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t J(s) \, ds = 0. \tag{10.6} \]

Next we deal with the local martingale \(K\) defined in \(10.3\). We start by observing that it has quadratic variation given by

\[ \langle K \rangle(t) = \sum_{j=1}^r \int_0^t \left( \sum_{i=1}^d [M_1(s)X(s)]_i \sigma(s)_j \right)^2 \, ds. \]

Therefore applying the Cauchy–Schwarz inequality, we have

\[ \langle K \rangle(t) \leq \sum_{j=1}^r \int_0^t \sum_{i=1}^d [M_1(s)X(s)]_i^2 \sum_{i=1}^d \sigma(s)^2 \, ds \]

\[ = \int_0^t \|M_1(s)X(s)\|_F^2 \|\sigma(s)\|_F^2 \, ds \]

\[ \leq \int_0^t \|M_1(s)\|_F^2 \|X(s)\|_F^2 \|\sigma(s)\|_F^2 \, ds. \]

Now, as \( M_1 \) is continuous and \(T\)-periodic, it follows that

\[ \langle K \rangle(t) \leq \max_{t \in [0,T]} \|M_1(s)\|_F^2 \sup_{0 \leq s \leq t} \|X(s)\|_F^2 \cdot \int_0^t \|\sigma(s)\|_F^2 \, ds, \quad t \geq 0. \tag{10.7} \]
Therefore, as \( t \to \|X(t)\| \) is a.s. bounded, and \( \int_0^t \|\sigma(s)\|^2_P \, ds/t \to 0 \) as \( t \to \infty \), we have
\[
\lim_{t \to \infty} \frac{(K)(t)}{t} = 0, \quad \text{a.s.}
\]

In the case that \((K)(t)\) tends to a finite limit as \( t \to \infty \), we have that \( K(t) \) tends to a finite limit, and therefore that \( \lim_{t \to \infty} K(t)/t = 0 \). If on the other hand \((K)(t)\) \( \to \infty \) as \( t \to \infty \), by the strong law of large numbers for martingales we have that \( \lim_{t \to \infty} (K(t)/t) = 0 \). Therefore, in this case it follows that
\[
\limsup_{t \to \infty} \frac{|K(t)|}{t} = \limsup_{t \to \infty} \frac{|K(t)|}{(K(t)/t)} = 0.
\]

Therefore we have that
\[
\lim_{t \to \infty} \frac{-1}{t} K(t) = 0, \quad \text{a.s.} \quad (10.8)
\]

By (10.2) and the fact that \( X \) is bounded a.s. we have that
\[
\lim_{t \to \infty} \frac{1}{t} V(t, X(t)) = 0, \quad \text{a.s.} \quad (10.9)
\]

Therefore, inserting the estimates (10.9), (10.8) and (10.6) into (10.5), we get
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)^T (P(s)^{-1})^* P(s)^{-1} X(s) \, ds = 0, \quad \text{a.s.} \quad (10.10)
\]

For any \( F \in \mathbb{C}^{d \times d} \), we have that \( D = F^* F \) is Hermitian. Moreover, because \( z^* D z = (Fz)^* Fz \geq 0 \) for all \( z \in \mathbb{C}^d \), it follows not only that \( x^T Dx \) is real-valued for every \( x \in \mathbb{R}^d \), but also that \( x^T Dx \geq 0 \) for all \( x \in \mathbb{R}^d \) with equality only if \( Fx = 0 \). Specialising to the case that \( F = P(t)^{-1} \), we see that we have \( x^T (P(t)^{-1})^* P(t)^{-1} x \) for all \( x \neq 0 \). In fact, we have that
\[
\frac{x^T (P(t)^{-1})^* P(t)^{-1} x}{\|x\|^2_2} \geq \inf_{\|u\|_2 = 1, u \in \mathbb{R}^d} u^T (P(t)^{-1})^* P(t)^{-1} u \\
\geq \inf_{\|u\|_2 = 1, u \in \mathbb{C}^d} (P(t)^{-1} u)^* P(t)^{-1} u =: \lambda(t) > 0.
\]

Clearly, \( \lambda \) is \( T \)-periodic and \( \lambda(t) \) is the minimal eigenvalue of \((P(t)^{-1})^* P(t)^{-1}\). Since the matrix-valued function \((P(t)^{-1})^* P(t)^{-1}\) is continuous, \( t \mapsto \lambda(t) \) is continuous and attains its bounds on the compact interval \([0, T]\). Therefore for all \( x \in \mathbb{R}^d \) and \( t \geq 0 \), we have that there exists \( c_T > 0 \) such that
\[
x^T (P(t)^{-1})^* P(t)^{-1} x \geq \min_{s \in [0, T]} \lambda(s) \cdot \|x\|^2_2 =: c_T \|x\|^2_2. \quad (10.11)
\]

Therefore, applying this estimate in (10.10), we obtain
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \|X(s)\|^2_2 \, ds = 0, \quad \text{a.s.} \quad (10.12)
\]

from which we readily deduce \( \liminf_{t \to \infty} \|X(t)\|_2 = 0 \) a.s.

11. **Proof of Theorem 13**

In the case when \( \sigma \) obeys (2.17), we have already shown that \( Y \) obeys (9.4). Now, from (10.2), it follows that \( X \) obeys the limit (9.5). Due to (10.2) and (9.5), we have that (10.9) holds. By (2.17), \( J \) defined by (10.3) obeys (10.6). Next, the local martingale \( K \) defined by (10.4) has quadratic variation bounded by (10.7). Therefore from (10.7) we have
\[
\langle K \rangle(t) \leq \max_{t \in [0, T]} \|M_t(s)\|_2^2 \sup_{0 \leq s \leq t} \|X(s)\|^2_2 \frac{1}{t} \int_0^t \|\sigma(s)\|^2_P \, ds, \quad t \geq 1, \quad (11.1)
\]
Since $\sigma$ obeys (2.17), we have that $\int_0^t ||\sigma(s)||^2_F \, ds/t \to 0$ as $t \to \infty$. Combining this estimate with (9.5) and (11.1) we arrive at
\[
\lim_{t \to \infty} \frac{\langle K \rangle(t)}{t \log t} = 0, \quad \text{a.s.}
\]
Moreover, this implies
\[
\limsup_{t \to \infty} \frac{\log \log \langle K \rangle(t)}{\log \log t} \leq 1, \quad \text{a.s.}
\]
On the event on which $\langle K \rangle(t)$ tends to a finite limit as $t \to \infty$, it follows that $K$ tends to a finite limit a.s., and so we have that $K(t)/t \to 0$ as $t \to \infty$ a.s. on this event. On the other hand, consider the event on which $\langle K \rangle(t) \to \infty$ as $t \to \infty$. Then by the law of the iterated logarithm for martingales we have
\[
\limsup_{t \to \infty} \frac{|K(t)|}{t} = \limsup_{t \to \infty} \frac{|K(t)|}{\sqrt{2\langle K \rangle(t) \log \log \langle K \rangle(t)}} \sqrt{\frac{2\langle K \rangle(t) \log \log \langle K \rangle(t)}{t \log t} \cdot \frac{\log \log t \cdot \log t}{t} = 0
\]
a.s. on the event for which $\langle K \rangle(t) \to \infty$ as $t \to \infty$. Hence it follows that $K(t)/t \to 0$ as $t \to \infty$ a.s.

The representation \textbf{(10.5)} for $V(t, X(t))$ remains valid. Using the estimates \textbf{(10.6)}, \textbf{(10.8)}, and the fact that $K(t)/t \to 0$ as $t \to \infty$ a.s., we have that \textbf{(10.10)} is true. Since the estimate \textbf{(10.11)} is still valid, this together with \textbf{(10.11)} implies \textbf{(10.12)}, as required. The conclusion that $\liminf_{t \to \infty} \|X(t)\| = 0$ a.s. follows as before, completing the proof.

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