Zero rest mass soliton solutions

M Mohammadi$^{1,3}$ and R Gheisar$^{1,2}$

$^1$Physics Department, Persian Gulf University, Bushehr 75169, Iran
$^2$Nuclear Energy Research Center, Persian Gulf University, Bushehr, 75169, Iran
$^3$Author to whom any correspondence should be addressed.

Abstract

In this paper, extended Klein-Gordon field systems will be introduced. Theoretically, it will be shown that for a special example of these systems, it is possible to have a single zero rest mass soliton solution, which is forced to move at the speed of light provided it is considered a non-deformed rigid object. This special soliton solution has the minimum energy among the other solutions, i.e. any arbitrary deformation in its internal structure leads to an increase in the total energy.

Keywords: zero rest mass, soliton, stability, extended Klein-Gordon, solitary wave

(Some figures may appear in colour only in the online journal)

1. Introduction

The classical relativistic field theory with soliton solutions is an attempt to model particles as some stable localized nonsingular objects. Solitons are the stable solitary wave solutions of the nonlinear field systems whose energy density functions are localized and satisfies the standard relativistic energy-rest-mass-momentum relations properly [1–4]. A well-known example for solitons in 1 + 1 dimensions are kink and anti-kink solutions of the real nonlinear Klein-Gordon (KG) systems [5–26]. For the complex nonlinear KG systems, it was shown that there are some localized soliton (-like) solutions that are called Q-balls [27–42]. Moreover, the Skyrme model [43, 44] and ’t Hooft-Polyakov model [45, 46] are also two well-known relativistic field models, which yield soliton solutions in 3 + 1 dimensions.

All well-known relativistic linear and nonlinear scalar field systems, which have been used in the classical or quantum field theory, are in the standard mathematical formats that can be called (nonlinear) KG (-like) systems. In this work, the extended KG systems for scalar fields are introduced as well. Theoretically, we show that for a special extended KG system in 1 + 1 dimensions, as a simple example, it is possible to have a soliton solution with zero rest mass (or zero energy). The other non-trivial solutions of this special system all have non-zero positive rest energies. In fact, the massless soliton solution is a special single solution among the others with the least amount of energy. This massless solution, if it is considered a non-deformed rigid object, responds to any amount of force, no matter how tiny, by immediately accelerating to the speed of light. This model can be simply extended to 3 + 1 dimensions to have a particle-like solution with zero rest mass. In general, if one considers such a massless particle in the space, since there is no area in the space with absolute zero interaction, it has to move at the speed of light, exactly according to the special relativity.

The organization of this paper is as follows. In section 2, the (nonlinear) KG systems and the extended KG systems are introduced. In section 3, a special extended KG system is introduced, which leads to a stable massless soliton solution. The last section is devoted to the summary and conclusions.

2. The (nonlinear) KG systems and the extended KG systems

In the standard relativistic field theory, different systems can be identified with different Lagrangian densities. The standard Lagrangian densities are considered to be functions of several fields $\phi_r$ ($r = 1, \ldots, N$) and their first derivatives $\left(\phi_{r,\alpha} = \frac{\partial \phi_r}{\partial \alpha}\right)$:

$$\mathcal{L} = \mathcal{L}(\phi_r, \phi_{r,\alpha}).$$

(1)

According to the principle of least action, the related equations of motion would be,

$$\frac{\partial \mathcal{L}}{\partial \phi_r} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial \mathcal{L}}{\partial (\partial_r \phi_{r,\mu})} \right) = 0.$$  

(2)
In general, since the Lagrangian density \( (1) \) is invariant under space-time translations \( x^\mu \rightarrow x^\mu + a^\mu \), four continuity equations \( \partial_\mu T^{\mu \nu} = 0 \) and hence four conserved quantities \( P^\mu = \int T^{\mu \nu} \, \mathrm{d}^4x \) are obtained where,

\[
T^{\mu \nu} = \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} - \frac{\partial \mathcal{L}}{\partial \phi} \partial_\nu \phi - \mathcal{L} g^{\mu \nu}
\]

is called the energy-momentum tensor and \( g^{\mu \nu} \) the 3 + 1 dimensional Minkowski metric. Note that, in this paper, for simplicity, we take the speed of light equal to one \( (c = 1) \). For any arbitrary solution for which the components of the energy-momentum tensor \( T^{\mu \nu} \) asymptotically approach zero at infinity, the four conserved quantities \( P^\mu = \int T^{\mu \nu} \, \mathrm{d}^4x \) form a four vector, which is called the energy-momentum four vector. So, for all localized solutions of equation (2), the standard relativistic energy-rest mass-momentum relations satisfied generally:

\[
E = m = P^0 = \int T^{00} \, \mathrm{d}^4x = \gamma E_0 = \gamma m_o,
\]

\[
P = (P^1, P^2, P^3) = \int (T^{01}, T^{02}, T^{03}) \, \mathrm{d}^4x = \gamma m_o \mathbf{v}.
\]

Note that, \( E_0 \) \((m_o)\) is the rest energy (mass) for a non-moving localized solution and \( E \) \((m)\) is the total energy (mass) for its moving version.

Many of the known standard Lagrangian densities, which are used in quantum or classical relativistic field theory, are in the same formats that can be called (nonlinear) KG (like) Lagrangian densities. The (nonlinear) KG Lagrangian densities can be defined as the linear functions of the kinetic scalar terms \( S_{ij} = \partial_i \phi_j \partial^i \phi_j, \) i.e.

\[
\mathcal{L} = \sum_{i,j=1}^n \alpha_{ij}(\phi_{1i}, \cdots, \phi_{N}) S_{ij} = V(\phi_{1i}, \cdots, \phi_{N}),
\]

where \( \phi_i \) must be some scalar fields (not a vector field \( \phi_{i\mu} \)), such as electromagnetic vector field \( A^\mu \), \( V \) \((\text{field potential})\) and coefficients \( \alpha_{ij} \) all are functions of scalar fields. For example, if one considers a complex scalar field \( \phi_1 = \phi (\phi_2 = \phi^*), \) the allowed kinetic scalar terms are \( S_{11} = \partial_\mu \phi \partial^\mu \phi, S_{22} = \partial_\mu \phi^* \partial^\mu \phi^* \) and \( S_{12} = \partial_\mu \phi^* \partial^\mu \phi. \) Note that the permissible combinations of the kinetic scalars must be ones for which the constructed Lagrangian density \((6)\) is a real functional. For example, the known complex nonlinear KG Lagrangian densities, which yield non-topological particle-like solutions \( (\text{Q-balls}) \) [27–42], are introduced as follows:

\[
\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(|\phi|) = S_{11} - V(|\phi|),
\]

for which \( \alpha_{11} = \alpha_{22} = 0 \) and \( \alpha_{12} = 1. \) Equivalently, one can decompose complex scalar field \( \phi \) into two distinct real and imaginary parts \( \phi_1 = \phi_i \) and \( \phi_2 = \phi_{\text{im}} \text{i.e.} \phi = \phi_i + \text{i} \phi_{\text{im}}. \) In this representation, the allowed kinetic scalar terms are \( S_{11} = \partial_\mu \phi_i \partial^\mu \phi_i, S_{22} = \partial_\mu \phi_{\text{im}} \partial^\mu \phi_{\text{im}} \) and \( S_{12} = \partial_\mu \phi_i \partial^\mu \phi_{\text{im}}, \) respectively. Therefore, the Lagrangian density \((7)\) becomes,

\[
\mathcal{L} = \partial_\mu \phi_i \partial^\mu \phi_i + \partial_\mu \phi_{\text{im}} \partial^\mu \phi_{\text{im}} - V(\sqrt{\phi_r^2 + \phi_i^2}),
\]

for which \( \alpha_{11} = \alpha_{22} = 1 \) and \( \alpha_{12} = 0. \) Moreover, instead of scalar fields \( \phi \) and \( \phi^* \), one can change the variables into the polar fields \( R(x^\mu) \) and \( \theta(x^\mu) \) as defined by

\[
\phi(x, y, z, t) = R(x, y, z, t) \exp[i\theta(x, y, z, t)].
\]

In the representation of polar fields, \( \phi_i = R = |\phi| = \sqrt{\phi \phi^*} \) is the module scalar field and \( \phi_2 = \theta \) is the phase scalar field, while the allowed kinetic scalar terms are \( S_{11} = \partial_\mu R R^\mu, S_{22} = \partial_\mu \theta \partial^\mu \theta, S_{12} = \partial_\mu R \partial^\mu \theta. \) In terms of polar fields, the Lagrangian density \((7)\) becomes,

\[
\mathcal{L} = (\partial_\mu R \partial^\mu \theta) (\partial_\mu \theta \partial^\mu \theta) - V(R) = (S_{11}) + R^2 (S_{22}) - V(R),
\]

\[
\mathcal{L} = (\partial_\mu R \partial^\mu \theta) (\partial_\mu \theta \partial^\mu \theta)^2 - V(R) = (S_{11}) + R^2 (S_{22})^2 - V(R),
\]

\[
\mathcal{L} = [(\partial_\mu R \partial^\mu \theta) + R^2 (\partial_\mu \theta \partial^\mu \theta)^2 - V(R)]^2 = [(S_{11}) + R^2 (S_{22})^2 - V(R)]^2,
\]

\[
\mathcal{L} = [(\partial_\mu R \partial^\mu \theta) + R^2 (\partial_\mu \theta \partial^\mu \theta)]^2 - V(R) = [(S_{11}) + R^2 (S_{22})^2]^2 - V(R).
\]

For an extended one, it is expected to encounter very complicated equations and other properties. Although it is not usual to use extended KG systems in standard (quantum or classical) field theory, in this paper it will be shown that using some special extended KG Lagrangian densities can lead to some interesting particle-like solutions with zero rest masses.

3. An extended KG system with a single massless soliton solution in 1 + 1 dimensions

In this section, we are going to show that it is possible to have an extended KG system in 1 + 1 dimensions with a single non-trivial massless stable particle-like solution. A similar model can be introduced in 3 + 1 dimensions. However, for simplicity, we restrict ourselves to 1 + 1 dimensions. To achieve this goal, we can consider an extended KG system for a single complex scalar field \( \phi = R e^{i\theta} \), or equivalently for...
two independent scalar fields \( \phi_1 = R \) and \( \phi_2 = \theta \), in the following form:

\[
\mathcal{L} = \sum_{i=1}^{3} \mathcal{K}_i^3,
\]

where

\[
\mathcal{K}_1 = R^2[S_{22} - 2],
\]

\[
\mathcal{K}_2 = R^2[S_{22} - 21 + [S_{11} - 4R^2 + 4R^3]],
\]

\[
\mathcal{K}_3 = R^2[S_{22} - 2][S_{11} - 4R^2 + 4R^3] + 2R[S_{12}],
\]

in which, \( S_{11} = \partial_x R \partial_x R, S_{22} = \partial_x \theta \partial_x \theta, S_{12} = \partial_x R \partial_x \theta \).

The related equations of motion can be obtained easily:

\[
\sum_{i=1}^{3} \mathcal{K}_i \left[ 2(\partial_x \mathcal{K}_i) \frac{\partial \mathcal{K}_i}{\partial (\partial_x R)} + \mathcal{K}_i \partial_x \left( \frac{\partial \mathcal{K}_i}{\partial (\partial_x R)} \right) - \mathcal{K}_i \frac{\partial \mathcal{K}_i}{\partial R} \right] = 0,
\]

\[
\sum_{i=1}^{3} \mathcal{K}_i \left[ 2(\partial_x \mathcal{K}_i) \frac{\partial \mathcal{K}_i}{\partial (\partial_x \theta)} + \mathcal{K}_i \partial_x \left( \frac{\partial \mathcal{K}_i}{\partial (\partial_x \theta)} \right) \right] = 0.
\]

The energy density that belongs to the new extended Lagrangian density (15), would be,

\[
\varepsilon(x, t) = T^{00} = \sum_{i=1}^{3} \mathcal{K}_i^3 [3C_i - \mathcal{K}_i] = \varepsilon_1 + \varepsilon_2 + \varepsilon_3,
\]

which is divided into three distinct parts, in which,

\[
C_i = \frac{\partial \mathcal{K}_i}{\partial \theta} \partial_t \theta + \frac{\partial \mathcal{K}_i}{\partial R} \partial_t R = \begin{cases} 2R \partial \theta^2 & i = 1 \\ 2(R^2 + R \partial \theta^2) & i = 2 \\ 2(2R^2 + R \partial \theta^2) & i = 3. \end{cases}
\]

After a straightforward calculation, we obtain:

\[
\varepsilon_1 = \mathcal{K}_1^3[5R^2 \partial \theta^2 + R^2 \partial \theta^2 + 2R^2],
\]

\[
\varepsilon_2 = \mathcal{K}_2^3[5R^2 \partial \theta^2 + 5R \partial \theta^2 + R^2 + U(R)],
\]

\[
\varepsilon_3 = \mathcal{K}_3^3[5(R \partial \theta + \partial R) + (R \partial \theta + R \partial \theta) + R \partial \theta + U(R)],
\]

where the dot and prime denote differentiation with respect to \( t \) and \( x \) respectively, and

\[
U(R) = 4R^4 - 4R^3 + 2R^2,
\]

is an ascending function, which is bounded from below by zero. Therefore, all terms in equations (23), (24) and (25) are positive definite and then the energy density function (21) is bounded from below by zero too. Hence, at any arbitrary space-time point, the possible minimum value of the energy density function (21) is zero. For example, for the trivial solution \( R = 0 \), i.e. the vacuum state, the energy density function would be zero everywhere. Moreover, there is a single non-trivial localized solution for which the energy density function (21) is zero everywhere. In fact, it is a non-trivial localized solitary wave solution with zero energy (rest mass) as well.

In general, according to equations (19), (20), (23), (24) and (25), it is obvious that the special solutions with zero (rest) energies are ones for which \( \mathcal{K}_i \) are zero simultaneously. Therefore, to obtain such special solutions, in general, three conditions \( \mathcal{K}_i = 0 \) \( (i = 1, 2, 3) \) must be satisfied simultaneously, which lead to three independent nonlinear partial differential equations (PDEs) for two independent scalar fields \( R \) and \( \theta \) as follows:

\[
\partial_x \theta \partial \theta \partial \theta = 2,
\]

\[
\partial_x R \partial \theta \partial \theta = 0,
\]

\[
\partial_x R \partial \theta \partial \theta = 0.
\]

which do not have any common solution except a trivial solution \( R = 0 \) and a single non-trivial solitary wave solution in the following form:

\[
R_0(x, t) = \frac{1}{1 + \gamma^2 x^2}, \quad \theta_0(x, t) = \omega_0 t,
\]

in which \( \nu \) is the velocity, \( \gamma = 1/\sqrt{1 - \nu^2} \) and \( k^\mu \equiv (\omega, k) \) is a \( 1 \times 1 \) vector, provided

\[
k = \nu v,
\]

and

\[
\omega = \gamma \omega_0.
\]

Since the energy density function (21) is positive definite, the single non-trivial special solution \( (30) \), for which \( \mathcal{K}_i = 0 \) \( (i = 1, 2, 3) \), definitely has the minimum rest energy \( \mathcal{E}_m = 0 \) among the other solutions, i.e. it is a soliton solution. More precisely, for any non-trivial solution of the equations (19) and (20), except the special solution (30), three independent conditions (27), (28) and (29) as three independent PDEs, are not possible to satisfied simultaneously. Therefore, for other solutions of the dynamical equations (19) and (20), at least one of the independent scalars \( \mathcal{K}_i \) takes non-zero value, which leads to a non-zero positive energy density function (see equations (23), (24) and (25)), i.e. the rest energy is zero just for the single solitary wave solution (30), and for other non-trivial unknown solutions lead to non-zero positive values. In other words, for any arbitrary deformation above the background of the special solution \( (30) \) (i.e. \( R = R_0 + \delta R \) and \( \theta = \theta_0 + \delta \theta \)), the total energy always increases. For example, for eight different arbitrary small deformations, which are introduced as follows:

\[
R = \frac{1}{1 + \xi + \xi^2}, \quad \theta = \omega t,
\]

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R = \frac{1}{1 + \xi + \xi^2}, \quad \theta = \omega t,
\]

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R = \frac{1}{1 + \xi + \xi^2}, \quad \theta = \omega t,
\]

\[
R = \frac{1}{1 + \xi + \xi^2}, \quad \theta = \omega t,
\]
Note that, the case can be considered as an indication of the order of deformations. Due to relativity, no particle can move faster than the speed of light. But, since the special solution is a stable soliton solution, which moves at a speed very close to the speed of light, and therefore this massless particle is actually possible to perceive it as a rigid object with absolute zero rest mass. In fact, Heisenberg’s uncertainty principle essentially does not let us have an ideal state for which the energy is absolute zero, and this statement is also valid for the special particle-like solution (30). Hence, physically, the particle-like solution (30) is always slightly deformed and then its rest mass is not absolute zero but is very small, i.e. it is a stable soliton solution, which moves at a speed very close to the speed of light, but not exactly equal to the speed of light.

To establish an extended KG model (15) for which the stability of the single non-trivial solitary wave solution (30) is guaranteed appreciably through a simple and straightforward conclusion, we select three independent linear combinations of $S_i$ in relations (16), (17) and (18) for which the functional coefficients $C_i (i = 1, 2, 3)$ would be definitely positive and finally lead to a positive definite energy density function (21). In general, it may be possible to choose other combinations of $S_i$ for this goal. However, we intentionally introduced these special combinations as a good example of extended KG systems for better and simpler conclusions.

4. Summary and conclusion

We have introduced, after reviewing some basic relations and equations of the standard relativistic classical field theory, (nonlinear) KG and the extended KG systems for the scalar fields $\phi_i$. The Lagrangian density of a (nonlinear) KG system is linear in the kinetic scalar terms, it can be called an extended KG system. If the Lagrangian density of the scalar fields is not linear in the kinetic scalar terms, it may be possible to choose other combinations of $S_i$ for this goal. However, we intentionally introduced these special combinations as a good example of extended KG systems for better and simpler conclusions.

Figure 1. Variations of the total energy $E$ versus small $\xi$ for different deformations (34)–(41) at $t = 0$. Various color curves of cyan, black, red, yellow, brown, purple, green, and blue are related to various deformations (34)–(41), respectively.

$$R = \frac{1}{1 + x^2}, \quad \theta = \omega_s t + \xi t,$$

$$R = \frac{1}{1 + x^2}, \quad \theta = \omega_s t + \xi e^{-x^2},$$

$$R = \frac{1}{1 + x^2}, \quad \theta = \omega_s t + \xi te^{-x^2},$$

where it is easy numerically to obtain the curves of the total energy $E$ versus $\xi$ (see figure 1). Here, $\xi$ is a small parameter and can be considered as an indication of the order of deformations. Note that, the case $\xi = 0$ leads to the same special solitary wave solution (30). Figure 1, just as some examples, properly shows how the special solution (30) is stable against any arbitrary deformation.

Due to relativity, no particle can move faster than the speed of light, and therefore this massless particle (30), if it is considered a rigid body, will respond to any amount of force (interaction), no matter how tiny, by immediately accelerating to the speed of light. In this paper, it was shown that for a special example of the extended KG systems (15), there is a single non-trivial localized solitary wave solution with zero rest mass (30). The energy density function of this special extended KG system is bounded from below by zero. Therefore, the single solitary wave solution (30) has the minimum energy among the other solutions, i.e. it is really a massless stable localized solution. In other words, it is a zero rest mass soliton solution. The other unknown solutions of these system, undoubtedly, have non-zero positive rest energies.
The single massless soliton solution (30), if considered a rigid body, responds to any amount of force, no matter how tiny, by immediately accelerating to the speed of light. Therefore, since there is no area in the space with zero interaction, it has to move at the speed of light exactly according to the special relativity. But, since the stable zero rest mass solution (30) has internal structure and is not essentially a rigid body, it would be deformed in collisions or in interactions with other particles. Hence, it is not physically possible to perceive it as an absolute zero rest energy (mass) object. In fact, the zero rest mass soliton solution (30) is just an ideal mathematical solution, which can be considered in a free space without any other particles and interactions, but is not an interesting physical case. Hence, physically, the deformed soliton solution (30) can move at a speed very close to the speed of light, but not exactly equal to the speed of light.

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ORCID iDs

M Mohammadi @ https://orcid.org/0000-0002-8057-237X

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