ANALYTIC TORSIONS ASSOCIATED WITH THE RUMIN COMPLEX ON CONTACT SPHERES

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Abstract. We explicitly write down all eigenvalues of the Rumin Laplacian on the standard contact spheres, and express the analytic torsion functions associated with the Rumin complex in terms of the Riemann zeta function. In particular, we find that the functions vanish at the origin and determine the analytic torsions.

Introduction. Let $(M, H)$ be a compact contact manifold of dimension $2n + 1$. Rumin [6] introduced a complex $(E^\bullet, d^E_\bullet)$, which is a resolution of the constant sheaf of $\mathbb{R}$ given by a subquotient of the de Rham complex. A specific feature of the complex is that the operator $D = d^\mathbb{R}_n : E^n \to E^{n+1}$ in ‘middle degree’ is a second-order, while $d^E_k : E^k \to E^{k+1}$ for $k \neq n$ are first order which are induced by the exterior derivatives. Let $a_k = 1/\sqrt{|n-k|}$ for $k \neq n$ and $a_n = 1$. Then, $(E^\bullet, d^E_\bullet)$ is also a complex. We call $(E^\bullet, d^E_\bullet)$ the Rumin complex. In virtue of the rescaling, $d^E_k$ satisfies Kähler-type identities on Sasakian manifolds [7], which include the case of spheres.

Let $\theta$ be a contact form of $H$ and $J$ be an almost complex structure on $H$. Then we may define a Riemann metric $g_\theta$ on $TM$ by extending the Levi metric $d\theta(-, J-)$ on $H$ (see §1.1). Following [6], we define the Rumin Laplacians $\Delta^E_k$ associated with $(E^\bullet, d^E_\bullet)$ and the metric $g_\theta$ by

$$\Delta^E_k := \begin{cases} (d_E d^\dagger_E)^2 + (d^\dagger_E d_E)^2, & k \neq n, n+1, \\ (d_E d^\dagger_E)^2 + D^1 D, & k = n, \\ D D^\dagger + (d^\dagger_E d_E)^2, & k = n + 1. \end{cases}$$

Rumin showed that $\Delta^E_k$ have discrete eigenvalues with finite multiplicities.

In this paper, we determine explicitly eigenvalues of $\Delta^E_k$ on the standard contact spheres $S^{2n+1} \subset \mathbb{C}^{n+1}$. The sphere $S^{2n+1}$ also admits an almost complex structure $J$ induced from the complex structure of $\mathbb{C}^{n+1}$. We call $(S^{2n+1}, H, J)$ the standard CR sphere and simply denoted by $S^{2n+1}$. To state our result we need to introduce notation for highest weight representations of the unitary group $U(n + 1)$ which acts on $S^{2n+1}$. The irreducible representations of $U(n + 1)$ are classified by the highest weights $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$; the corresponding representation will be denoted by $V(\lambda)$. Julg and Kasparov [4] showed that the complexification of $E^k(S^{2n+1})$, as a $U(n + 1)$-module, is decomposed into the irreducibles of the form

$$\Psi_{(q,j,i,p)} := V(q, 1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1, -p).$$
See Proposition 1.1 below for the relations between $k$ and $(q, i, j, p)$. Since $\Delta_E$ commutes with the $U(n+1)$-action, it acts as a scalar on each $\Psi_{(q,j,i,p)}$.

**Theorem 0.1.** Let $S^{2n+1}$ be the standard CR sphere with the contact from $\theta = \sqrt{-1}(\overline{\partial} - \partial)|z|^2$. Then, on the subspaces of the complexification of $\mathcal{E}^\bullet$ corresponding to the representations $\Psi_{(q,j,i,p)}$, the eigenvalue of $\Delta_E$ is

$$\frac{(p+i)(q+n-i)+(q+j)(p+n-j))^2}{4(n-i-j)^2}$$

This theorem claims that the eigenvalues of $\Delta_E$ are determined by the highest weight. This phenomenon also appears in the case of the Hodge-de Rham Laplacian $\Delta_{dR}$ on symmetric spaces $G/K$. Ikeda and Taniguchi [3] showed that on the subspaces of $k$-forms of $G/K$ corresponding to $\mathcal{V}(\lambda)$, the eigenvalue of $\Delta_{dR}$ is determined by $\lambda$. It is a natural question to ask whether the eigenvalues of $\Delta_E$ on a contact homogeneous space $G/K$ are determined by the highest weight of $G$.

Theorem 0.1 unifies the following results on the eigenvalues of Rumin Laplacians on the spheres. Julg and Kasparov [4] determined the eigenvalues of $D^\dagger D$. Folland [2] calculated the eigenvalue of the sub-Laplacian $\Delta_b$ on $\mathcal{E}^1$ in the case $S^3$. Ørsted and Zhang [5] determined eigenvalues of the Laplacian of the holomorphic and anti-holomorphic part of $d_R$ except for the ones containing $D$.

Note that Ørsted and Zhang used $d_R$ in place of $d_E$. As a result, the eigenvalues of the Laplacian in their paper are not determined by the highest weights. This also explains the importance of the scaling factor $a_k$.

We next introduce the analytic torsions and metrics of the Rumin complex $(\mathcal{E}^\bullet, d_E^\bullet)$ by following [1, 8]. We define the contact analytic torsion function associated with $(\mathcal{E}^\bullet, d_E^\bullet)$ by

$$(0.1) \quad \kappa_E(s) := \sum_{k=0}^{n} (-1)^{k+1}(n+1-k)\zeta(\Delta_E^k)(s),$$

where $\zeta(\Delta_E^k)(s)$ is the spectral zeta function of $\Delta_E^k$, and the contact analytic torsion $T_E$ by

$$2 \log T_E = \kappa_E'(0).$$

Let $H^\bullet(\mathcal{E}^\bullet, d_E^\bullet)$ be the cohomology of the Rumin complex. We define the contact metric on $\det H^\bullet(\mathcal{E}^\bullet, d_E^\bullet)$ by

$$\| \|_{E} = T_E|_{L^2(\mathcal{E}^\bullet),}$$

where the metric $\| \|_{E}$ is induced by $L^2$ metric on $\mathcal{E}^\bullet$ via identification of the cohomology classes by the harmonic forms on $\mathcal{E}^\bullet$.

Rumin and Seshadri [8] defined another analytic torsion function $\kappa_R$ from $(\mathcal{E}^\bullet, d_R^\bullet)$, which is different from $\kappa_E$ except in dimension 3. In dimension 3, they showed that $\kappa_R(0)$ is a contact invariant; so is $\kappa_E(0)$. Moreover, on 3-dimensional Sasakian manifolds with $S^1$-action, $\kappa_R(0) = 0$ and the contact analytic torsion and metric coincides with the Ray-Singer torsion $T_{dR}$ and the metric $\| \|_{dR}$.

Our second main result is
Theorem 0.2. In the setting of Theorem 0.1, the contact analytic torsion function on $S^{2n+1}$ is given by

$$\kappa_E(s) = -(n+1)(1 + 2s + \zeta(2s)),$$

where $\zeta$ is the Riemann zeta function. In particular, we have

(0.2) \hspace{1cm} \kappa_E(0) = 0,

(0.3) \hspace{1cm} T_E = (4\pi)^{n+1}.

From (0.2), we see that the metric $\parallel_E$ on $S^{2n+1}$ is invariant under the constant rescaling $\theta \mapsto K\theta$. The argument is exactly the same as the one in [8].

The fact that the representations determines the eigenvalues of $\Delta_E$ causes several cancelations in the linear combination (0.1), which greatly simplifies the computation of $\kappa_E(s)$. We cannot get such a simple formula for the contact torsion function $\kappa_R$ of $(E^*, d^*)$ for dimensions higher than 3.

Let us compare the contact analytic torsion with the Ray-Singer torsion on the standard spheres given by Weng and You [10]. Let $g_{std}$ be the standard metric on $S^{2n+1}$. They showed that the Ray-Singer torsion of $(S^{2n+1}, 4g_{std})$ is $(4\pi)^{n+1}/n!$. The metric $4g_{std}$ agrees with the metric $g_{\theta}$ defined from the contact from $\theta = \sqrt{-1}(\bar{\partial} - \partial)|z|^2$. Since $(E^*, d_E^*)$ and $(\Omega^*, d)$ are resolutions of $\mathbb{R}$, there is a natural isomorphism $\det H^*(E^*, d_E^*) \cong \det H^*(\Omega^*, d)$, which turns out to be isometric for the $L^2$ metrics. Therefore (0.2) gives

Corollary 0.3. In the setting of Theorem 0.1, we have

$$T_E = n!T_{dR} \quad \text{and} \quad \parallel_E = n!\parallel_{dR}.$$ 

The paper is organized as follows. In §1, we recall the definition and properties of the Rumin complex on $S^{2n+1}$, and decompose $E^k$ as a direct sum of the irreducible representation of $U(n+1)$. In §2.1, we construct highest weight vectors, and compute the actions of $d_{dR}$ and the Lie derivative $L_T$ with respect to the Reeb vector field $T$ on these vectors. In §2.2, we calculate the $L^2$-norm of them. Then, in §2.3, we compute the eigenvalues of $\Delta_E$ for each irreducible component. In §3, we calculate the contact analytic torsion function $\kappa_E$.

Acknowledgement. The author is grateful to his supervisor Professor Kengo Hirachi for introducing this subject and for helpful comments. He would also like to thank Yuya Takeuchi for carefully reading an earlier draft of this paper. This work was supported by the program for Leading Graduate Schools, MEXT, Japan.

1. The Rumin complex

1.1. The Rumin complex on contact manifolds. We call $(M, H)$ an orientable contact manifold of dimension $2n+1$ if $H$ is a subbundle of $TM$ of codimension 1 and there exists a 1-form $\theta$, called a contact form, such that $\text{Ker}(\theta: TM \to \mathbb{R}) = H$ and $\theta \wedge (d\theta)^n \neq 0$. The Reeb vector field of $\theta$ is the unique vector field $T$ satisfying $\theta(T) = 1$ and $\text{Int}_T d\theta = 0$, where $\text{Int}_T$ is the interior product with respect to $T$.

For $H$ and $\theta$, we call $J \in \text{End}(TM)$ an almost complex structure associated with $\theta$ if $J^2 = -\text{Id}$ on $H$, $JT = 0$, and the Levi form $d\theta(-, J-)$ is
positive definite on $H$. Given $\theta$ and $J$, we define a Riemannian metric $g$ on $TM$ by

$$g(X,Y) := d\theta(X,JY) + \theta(X)\theta(Y) \quad \text{for } X,Y \in TM.$$  

Let $*$ be the Hodge star operator on $\bigwedge^*TM$ with respect to $g$.

The Rumin complex [6] is defined on contact manifolds as follows. We set $L := d\theta \wedge$ and $\Lambda := *^{-1}L*$, which is the adjoint operator of $L$ with respect to the metric $g$ at each point. We set

$$\bigwedge_{\text{prim}}^k H^* := \{ v \in \bigwedge^k H^* \mid \Lambda v = 0 \},$$

$$\bigwedge_L^k H^* := \{ v \in \bigwedge^k H^* \mid Lv = 0 \},$$

$$\mathcal{E}^k := \begin{cases} C^\infty(M,\bigwedge_{\text{prim}}^k H^*), & k \leq n, \\ C^\infty(M,\theta \wedge \bigwedge_L^{k-1} H^*), & k \geq n + 1. \end{cases}$$

We embed $H^*$ into $T^*M$ as the subbundle $\{ \phi \in T^*M \mid \phi(T) = 0 \}$ so that we can regard

$$\Omega^k_H := C^\infty(M,\bigwedge^k H^*)$$

as a subspace of $\Omega^k$, the space of $k$-forms. We define $d_b: \Omega^k_H \rightarrow \Omega^{k+1}_H$ by

$$(1.1) \quad d_b\phi := d\phi - \theta \wedge (\text{Int}_T d\phi).$$

and then $D: \mathcal{E}^n \rightarrow \mathcal{E}^{n+1}$ by

$$(1.2) \quad D = \theta \wedge (\mathcal{L}_T + d_b L^{-1} d_b),$$

where $\mathcal{L}_T$ is the Lie derivative with respect to $T$, and we use the fact that $L: \bigwedge^{n-1} H^* \rightarrow \bigwedge^{n+1} H^*$ is an isomorphism.

Let $P: \bigwedge^k H^* \rightarrow \bigwedge_{\text{prim}}^k H^*$ be the fiberwise orthogonal projection with respect to $g$, which also defines a projection $P: \Omega^k \rightarrow \mathcal{E}^k$. We set

$$d_R^k := \begin{cases} P \circ d & \text{on } \mathcal{E}^k, \\ \text{Id} & \text{on } \mathcal{E}^n, \\ d & \text{on } \mathcal{E}^k, \end{cases}$$

Then $(\mathcal{E}^\bullet, d_R^\bullet)$ is a complex. Let $d_{\mathcal{E}}^k = a_k d_R^k$, where $a_k = 1/\sqrt{|n-k|}$ for $k \neq n$ and $a_n = 1$. We call $(\mathcal{E}^\bullet, d_{\mathcal{E}}^\bullet)$ the Rumin complex.

We define the $L^2$-inner product on $\Omega^k$ by

$$(\phi, \psi) := \int_M g(\phi, \psi) d\text{vol}_g$$

and the $L^2$-norm on $\Omega^k$ by $\|\phi\| := \sqrt{(\phi, \phi)}$. Since the Hodge star operator $*$ induces a bundle isomorphism from $\bigwedge_{\text{prim}}^k H^*$ to $\theta \wedge \bigwedge_L^{2n-k} H^*$, it also induces a map $\mathcal{E}^k \rightarrow \mathcal{E}^{2n+1-k}$.  

Let $d_E^k$ and $D^\dagger$ denote the formal adjoint of $d_E$ and $D$, respectively for the $L^2$-inner product. We define the forth-order Laplacians $\Delta_E$ on $E^k$ by

$$
\Delta_E^k := \begin{cases} 
(d_E^{k-1}d_E^{k-1})^2 + (d_E^{k-1}d_E^{k})^2, & k \neq n, n + 1, \\
(d_E^{n-1}d_E^{n-1})^2 + D^\dagger D, & k = n, \\
DD^\dagger + (d_E^{n+1}d_E^{n+1})^2, & k = n + 1. 
\end{cases}
$$

We call it the Rumin Laplacian [6]. Since $\ast$ and $\Delta_E$ commute, to determine the eigenvalue on $E^\bullet$, it is enough to compute them on $E^k$ for $k \leq n$.

### 1.2. The Rumin complex on the CR spheres

Let $S := \{z \in \mathbb{C}^{n+1} \mid |z|^2 = 1\}$ and $\theta := \sqrt{-1}(\partial - \bar{\partial})|z|^2$. (We will omit the dimension from $S^{2n+1}$ for the simplicity of the notation.) The Reeb vector filed of $\theta$ is

$$T = \frac{\sqrt{-1}}{2} \sum_{l=1}^{n+1} \left( z_l \frac{\partial}{\partial z_l} - \bar{z}_l \frac{\partial}{\partial \bar{z}_l} \right).$$

With respect to the standard almost complex structure $J$, we decompose the bundles defined in the previous subsection as follows:

$$H^{1,0} := \{v \in \mathcal{C}H^* \mid Jv = \sqrt{-1}v\},$$

$$H^{0,1} := \{v \in \mathcal{C}H^* \mid Jv = -\sqrt{-1}v\},$$

$$\Lambda^{i,j}H^* := \Lambda^{i}H^{1,0} \otimes \Lambda^{j}H^{0,1},$$

$$\Lambda^{i,j}_{\text{prim}}H^* := \{ \phi \in \Lambda^{i,j}H^* \mid \Lambda_0 \phi = 0 \},$$

$$\Omega^{i,j}_{H^+} := C^\infty \left( S, \Lambda^{i,j}H^* \right),$$

$$\mathcal{E}^{i,j} := C^\infty \left( S, \Lambda^{i,j}_{\text{prim}}H^* \right).$$

Then $d_0\Omega^{i,j}_{H^+} \subset \Omega^{i+1,j}_{H^+} \oplus \Omega^{i,j+1}_{H^+}$. We define $\partial_0 : \Omega^{i,j}_{H^+} \to \Omega^{i+1,j}_{H^+}$ and $\bar{\partial}_0 : \Omega^{i,j}_{H^+} \to \Omega^{i,j+1}_{H^+}$ by

$$d_0 = \partial_0 + \bar{\partial}_0.$$

Similarly, we decompose

$$d_R = \partial_R + \bar{\partial}_R, \quad d_E = \partial_E + \bar{\partial}_E.$$

In view of the Lefshetz primitive decomposition, we may rewrite (1.2) as

$$(1.3) \quad D = \theta \wedge \left( \mathcal{L}_T - \sqrt{-1}(\partial_E + \bar{\partial}_E)(\partial_E^\dagger - \bar{\partial}_E^\dagger) \right)$$

by using $\partial_E^\dagger = \sqrt{-1}[\Lambda, \bar{\partial}_E]$ and $\bar{\partial}_E^\dagger = -\sqrt{-1}[\Lambda, \partial_E]$. Note that this equation holds on Sasakian manifolds.

We decompose $\mathcal{E}^{i,j}$ into a direct sum of irreducible representations of the unitary group $U(n+1)$. Recall that irreducible representations of $U(m)$ are parametrized by the highest weight $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Z}^m$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$; the representation corresponding to $\lambda$ will be denoted by $V(\lambda)$.

To simplify the notation, we introduce the following notation: for $a_1, \ldots, a_l \in \mathbb{Z}$ and $k_1, \ldots, k_l \in \mathbb{Z}$, $(a_1^\prime, \ldots, a_l^\prime, k_1, \ldots, k_l)$ denotes the $k_1 + \cdots + k_l$-tuple whose first $k_1$ entries are $a_1$, whose next $k_2$ entries are $a_2$, etc. For example,

$$(1, 1, 0, 0, -1, -1).$$
We note that $\omega_1$ is $a$ and $\omega_9$ is the zero tuple.

In [4], it is shown that the multiplicity of $V(q, \mathbb{1}_j, \mathbb{1}^n_{1-n-j}, -p, -1, -p)$ in $s,t$ is at most one. Thus we may set

$$\Psi^{(s,t)}_{(q,j,i,p)} := s,t \cap V(q, \mathbb{1}_j, \mathbb{1}^n_{1-n-j}, -p).$$

**Proposition 1.1.** ([4, Section 4(b)]) The irreducible decomposition of the $U(n+1)$-module $\mathcal{E}^{i,j}$ is given as follows:

**Case I:**

$$\mathcal{E}^{i,0} = \bigoplus_{q \geq 0, p \geq 0} \Psi^{(0,0)}_{(q,0,0,p)}.$$

**Case II:** For $i + j \leq n - 1$ with $i, j > 0$,

$$\mathcal{E}^{i,j} = \bigoplus_{q \geq 1, p \geq 1} (\Psi^{(i,j)}_{(q,j,i,p)} \oplus \Psi^{(i,j)}_{(q,j-1,i,p)} \oplus \Psi^{(i,j)}_{(q,j-1,i-1,p)}).$$

**Case III:** For $1 \leq i \leq n - 1$,

$$\mathcal{E}^{i,0} = \bigoplus_{q \geq 0, p \geq 1} (\Psi^{(i,0)}_{(q,0,i,p)} \oplus \Psi^{(i,0)}_{(q,0,i-1,p)}).$$

**Case IV:** For $1 \leq j \leq n - 1$,

$$\mathcal{E}^{0,j} = \bigoplus_{q \geq 1, p \geq 0} (\Psi^{(0,j)}_{(q,j,0,p)} \oplus \Psi^{(0,j)}_{(q,j-1,0,p)}).$$

**Case V:** For $i + j = n$ with $i, j > 0$,

$$\mathcal{E}^{i,j} = \bigoplus_{q \geq 1, p \geq 1} (\Psi^{(i,j)}_{(q,j,i-1,p)} \oplus \Psi^{(i,j)}_{(q,j-1,i,p)} \oplus \Psi^{(i,j)}_{(q,j-1,i-1,p)}).$$

**Case VI:**

$$\mathcal{E}^{n,0} = \bigoplus_{q \geq 0, p \geq 1} \Psi^{(n,0)}_{(q,0,n,p)}.$$

**Case VII:**

$$\mathcal{E}^{0,n} = \bigoplus_{q \geq 1, p \geq 0} \Psi^{(0,n)}_{(q,n-1,p)}.$$

2. The eigenvalues of the Rumin Laplacian

2.1. The action of $d_R$ and the Reeb vector field. Setting $\omega_i := dz_i - z_i \partial \lvert z \rvert^2$ and $\overline{\omega}_i := d\overline{z}_i - \overline{z}_i \partial \lvert z \rvert^2$, we define differential forms

$$\alpha^{(0,0)}_{(j,0)} := \sum_{\nu=1}^{j+1} (-1)^{\nu-1} z_{\nu} \overline{\omega}_1 \wedge \cdots \wedge \overline{\omega}_{\nu} \wedge \overline{\omega}_{j+1},$$

$$\alpha^{(0,1)}_{(j,0)} := \overline{\omega}_1 \wedge \cdots \wedge \overline{\omega}_{j+1},$$

$$\alpha^{(0)}_{(0,j)} := \sum_{\mu=n-i+1}^{n+1} (-1)^{\mu-(n-i+1)} z_{\mu} \omega_{n-i+1} \wedge \cdots \overline{\omega}_{\mu} \wedge \omega_{n+1},$$

$$\alpha^{(1)}_{(0,j)} := \omega_{n-i+1} \wedge \cdots \wedge \omega_{n+1}.$$
Following [5], we see that \( \Psi(s,t) \) contains the following element \( \psi_{(q,j,i,p)}^{(s,t)} \), for \( p, q \geq 1 \), \( a, b \geq 0 \) and \( a + b \leq 1 \),
\[
\psi_{(0,0,0,0)}^{(0,0)} := 1,
\psi_{(q,j,i,p)}^{(i+a,j+b)} := \sqrt{-1} \alpha_{(0,0)}^{(a,b)} \wedge \alpha_{(j,i)}^{(0,1)}/\sqrt{2\pi n+1},
\psi_{(q,j,i,p)}^{(i+1,j+1)} := P \psi_{(q,j,i,p)}^{(i+1,j+1)},
\]
where \( \psi_{(q,j,i,p)}^{(i+1,j+1)} := \sqrt{-1} \alpha_{(0,0)}^{(a,b)} \wedge \alpha_{(j,i)}^{(0,1)}/\sqrt{2\pi n+1} \).

We have used the projection \( P \) in the definition of \( \psi_{(q,j,i,p)}^{(i+1,j+1)} \). Let us calculate \( P \) explicitly (see also Remark 2.5 below). Since
\[
2\Lambda(\omega_{\mu} \wedge \varpi_{\nu}) = -\sqrt{-1} \mu \nu \varpi_{\nu} \quad \text{for } \mu \neq \nu,
\]
we have
\[
2\Lambda \left( \alpha_{(0,i)}^{(1,0)} \wedge \alpha_{(j,0)}^{(0,1)} \right) = \sqrt{-1} (-1)^{i+1} \alpha_{(0,i)}^{(0,0)} \wedge \alpha_{(j,0)}^{(0,0)}.
\]
Thus
\[
2\Lambda^2 \psi_{(q,j,i,p)}^{(i+1,j+1)} = \sqrt{-1} (-1)^{i+1} \Lambda \psi_{(q,j,i,p)}^{(i,j)} = 0.
\]
By using the Lefschetz primitive decomposition, we get
\[
P|_{\psi_{(q,j,i,p)}^{(i+1,j+1)} + L \psi_{(q,j,i,p)}^{(i,j)}} = 1 + \frac{1}{n - i - j} \Lambda.
\]

**Lemma 2.1.** If \( i + j \leq n - 1 \) and \( p, q \geq 1 \) or \( i \leq n - 1, j = 0, p \geq 1 \) and \( q = 0 \),
\[
\begin{align*}
\partial_{\mathcal{R}} \psi_{(q,j,i,p)}^{(i,j)} &= (p+i) \psi_{(q,j,i,p)}^{(i+1,j+1)}, \\
\bar{\partial} \psi_{(p,i,j,q)}^{(j,i+1)} &= (-1)^{i}(p+i) \psi_{(p,i,j,q)}^{(j+1,i+1)},
\end{align*}
\]
If \( i + j \leq n - 2, p, q \geq 1 \),
\[
\begin{align*}
\partial_{\mathcal{R}} \psi_{(q,j,i,p)}^{(i,j+1)} &= (p+i) \psi_{(q,j,i,p)}^{(i+1,j+1)}, \\
\bar{\partial} \psi_{(p,i,j,q)}^{(j+1,i)} &= (-1)^{j+1}(p+i) \psi_{(p,i,j,q)}^{(j+1,i+1)}.
\end{align*}
\]
Otherwise, \( \partial_{\mathcal{R}} \psi_{(q,j,i,p)}^{(s,t)} = 0 \) and \( \bar{\partial} \psi_{(p,i,j,q)}^{(s,t)} = 0 \).

**Remark 2.2.** Since \( \Lambda \partial_{\mathcal{R}} \psi_{(q,j,i,p)}^{(i,j)} = 0 \) and \( \Lambda \bar{\partial} \psi_{(p,i,j,q)}^{(i,j)} = 0 \), the operators \( \partial_{\mathcal{R}} \) and \( \bar{\partial}_{\mathcal{R}} \) in (2.2) coincide with \( \partial_{\mathcal{R}} \) and \( \bar{\partial}_{\mathcal{R}} \). But, since \( \Lambda \partial_{\mathcal{R}} \psi_{(q,j,i,p)}^{(i+1,j)} \neq 0 \) and \( \Lambda \bar{\partial} \psi_{(p,i,j,q)}^{(i+1,j)} \neq 0 \), this is not the case for (2.3).
The action of $L_T$ on $\psi^{(s,t)}_{(q,i,j,p)}$ is also easy to compute. Since
\[
2L_Tz_i = \sqrt{-1}z_i, \quad 2L_T\omega_i = \sqrt{-1}\omega_i,
\]
we obtain
\[
(2.4) \quad 2L_T\psi^{(s,t)}_{(q,i,j,p)} = \sqrt{-1}(p + i - j - q)\psi^{(s,t)}_{(q,i,j,p)}.
\]

2.2. $L^2$-norms of highest weight vectors.

**Lemma 2.3.** ([5, Lemma 3.2]) Let $p, q \geq 1$ and set
\[
C(q, p) = 2^{n+1}\pi^{n+1}(q - 1)!(p - 1)!/(q + p + n)!, \quad D(q) = 2^{n+1}\pi^{n+1}(q - 1)!/(q + n)!.
\]
If $i + j \leq n - 1$,
\[
(2.5) \quad \left\| \psi^{(i,j)}_{(q,i,j,p)} \right\|^2 = \frac{C(q, p)}{2^{i+j}}(q + j)(p + i).
\]
If $j > 0$ and $i + j \leq n - 1$,
\[
(2.6) \quad \left\| \psi^{(i+1,j)}_{(q,i,j,p)} \right\|^2 = \left\| \psi^{(j+1,i)}_{(p,j,i,q)} \right\|^2 = \frac{C(q, p)}{2^{j+i+1}}(q + j)(q + n - i).
\]
If $i, j > 0$ and $i + j \leq n - 2$,
\[
(2.7) \quad \left\| \psi^{(i+1,j+1)}_{(q,i,j,p)} \right\|^2 = \frac{C(q, p)}{2^{j+i+2}}(q + n - i)(p + n - j)(n - 1 - i - j).
\]
If $0 \leq j \leq n - 1$,
\[
(2.8) \quad \left\| \psi^{(i,j)}_{(q,i,j,p)} \right\|^2 = \frac{D(q)}{2^j}(q + j),
\]
\[
(2.9) \quad \left\| \psi^{(i+1,j+1)}_{(q,i,j,p)} \right\|^2 = \frac{D(q)}{2^{j+1}}(n - j).
\]

**Remark 2.4.** These formulas are different from those in [5] by factors in powers of 2 due to the choice of the metric $g$.

**Proof.** We only prove (2.7) because others were proved in Lemma 3.2 in [5]; see also Remark 2.5. Since $P$ is the orthogonal projection and $\psi^{(i+1,j+1)}_{(q,i,j,p)} = P\tilde{\psi}^{(i+1,j+1)}_{(q,i,j,p)}$, the formula (2.1) gives
\[
\left\| \psi^{(i+1,j+1)}_{(q,i,j,p)} \right\|^2 = \left\| \tilde{\psi}^{(i+1,j+1)}_{(q,i,j,p)} \right\|^2 - \left\| (n - i - j)^{-1}L\tilde{\psi}^{(i+1,j+1)}_{(q,i,j,p)} \right\|^2.
\]
The first term of the right-hand side can be calculated by using the following facts: the squared norm of $\alpha_{(0,i)}^{(1,0)}$ in $g$ (see [2, Lemma 5]) is $\sum_{\mu=1}^{n-1} |z_\mu|^2/2^{i+1}$ and
\[
\int_S |z^n|^2 d\text{vol}_g = \frac{2^{n+1}\pi^{n+1}}{(|\alpha| + n)!}.
\]
For the second term, we can use
\[
L\tilde{\psi}^{(i+1,j+1)}_{(q,i,j,p)} = \frac{\sqrt{-1}}{2}(-1)^{i+1}\psi^{(i,j)}_{(q,i,j,p)}
\]
and
\[
\left\| Lf \right\|^2 = (n - i - j)\left\| f \right\|^2, \quad f \in \mathcal{E}^{i,j}
\]
to reduce it to
\[
\frac{1}{4(n-i-j)} \left\| \psi_{(q,j,i,p)}^{i,j} \right\|^2.
\]
This is given by (2.5). □

**Remark 2.5.** In [5], the formula of the projection \( P \), corresponding to our (2.1), is not correct. This result in errors in the evaluation of the norm corresponding to our (2.7) and the computations of the eigenvalues of the Laplacians using that formula.

### 2.3. Calculation of eigenvalues.

Given \((q, j, i, p)\), we list up all \((s, t)\) such that \(\psi_{(q,j,i,p)}^{(s,t)} \neq \{0\}\) and calculate the eigenvalues of \(\Delta_\xi\) on them. In this subsection, we omit the subscripts from \(\psi_{(q,j,i,p)}^{(s,t)}\), \(\psi_{(q,j,i,p)}^{(s,t)}\), and write \(\psi^{(s,t)}\), \(\psi^{(s,t)}\).

**Case I:** \(i = j = 0\) and \(p = q = 0\)

The spaces are \(\psi^{(0,0)}\) and we have \(\Delta_\xi \psi^{(0,0)} = 0\).

**Case II:** \(i + j \leq n - 2\), \(p \geq 1\) and \(q \geq 1\)

The spaces are \(\psi^{(i,j)}\), \(\psi^{(i+1,j)}\), \(\psi^{(i,j+1)}\) and \(\psi^{(i+1,j+1)}\). Let \(\|\partial_\xi\|\) and \(\|\bar{\partial}_\xi\|\) be the norm of bounded linear operators of \(\partial_\xi\) and \(\bar{\partial}_\xi\). By using Propositions 2.1 and 2.3, we have

\[
\|\partial_\xi \psi^{(i,j)}\|^2 = \frac{(p+i)^2}{n-i-j} \left\| \psi^{(i+1,j)} \right\|^2 = \frac{(p+i)(q+n-i)}{2(n-i-j)},
\]

\[
\|\bar{\partial}_\xi \psi^{(i,j)}\|^2 = \frac{(q+j)^2}{n-i-j} \left\| \psi^{(i,j+1)} \right\|^2 = \frac{(q+j)(p+n-j)}{2(n-i-j)},
\]

\[
\|\partial_\xi \psi^{(i,j+1)}\|^2 = \frac{(p+i)^2}{n-i-j-1} \left\| \psi^{(i+1,j+1)} \right\|^2 = \frac{(p+i)(q+n-i)}{2(n-i-j)},
\]

\[
\|\bar{\partial}_\xi \psi^{(i,j+1)}\|^2 = \frac{(q+j)^2}{n-i-j-1} \left\| \psi^{(i+1,j+1)} \right\|^2 = \frac{(q+j)(p+n-j)}{2(n-i-j)}.
\]

Therefore we can calculate the eigenvalue of \(\Delta_\xi\) on \(\psi^{(i,j)}\) and \(\psi^{(i+1,j+1)}\) are

\[
\Delta_\xi \psi^{(i,j)} = \frac{\partial_\xi \psi_{(i+1,j+1)} \partial_\xi \psi^{(i,j)} + \bar{\partial}_\xi \psi_{(i,j+1)} \bar{\partial}_\xi \psi^{(i,j)}}{4(n-i-j)^2},
\]

\[
\Delta_\xi \psi^{(i+1,j+1)} = \frac{\partial_\xi \psi_{(i+1,j+1)} \partial_\xi \psi^{(i+1,j+1)} + \bar{\partial}_\xi \psi_{(i+1,j+1)} \bar{\partial}_\xi \psi^{(i+1,j+1)}}{4(n-i-j)^2}.
\]

We consider \((i + j + 1)\)-form. Since \(\text{Im} \, d_\xi\) and \(\text{Im} \, d_\xi^\dagger\) are orthogonal, \(\psi^{(i+1,j+1)} \oplus \psi^{(i,j+1)} = d_\xi \psi^{(i,j)} \oplus d_\xi^\dagger \psi^{(i+1,j+1)}\). Since \(\Delta_\xi d_\xi\) and \(\Delta_\xi d_\xi^\dagger = d_\xi^\dagger \Delta_\xi\), the eigenvalues of \(\Delta_\xi\) on \(\psi^{(i+1,j)} \oplus \psi^{(i,j+1)}\) is

\[
\frac{((p+i)(q+n-i) + (q+j)(p+n-j))^2}{4(n-i-j)^2}.
\]
Case III: $i \leq n - 1$, $j = 0$, $p \geq 1$ and $q = 0$

The spaces are $\Psi^{(i,0)}$ and $\Psi^{(i+1,0)}$. We have

$$\|\partial\xi|_{\Psi^{(i,0)}}\|^2 = (p + i)/2.$$  

In the same way on Case II, we have the eigenvalue of $\Delta_{\xi}$ is

$$(p + i)^2/4.$$

Case IV: $i = 0$, $j \leq n - 1$, $p = 0$ and $q \geq 1$

The spaces are $\Psi^{(0,j)}$ and $\Psi^{(0,j+1)}$. Taking the conjugate of Case III, the eigenvalue of $\Delta_{\xi}$ is

$$(q + j)^2/4.$$

Case V: $i + j = n - 1$, $p \geq 1$ and $q \geq 1$

The spaces are $\Psi^{(i,j)}$, $\Psi^{(i+1,j)}$ and $\Psi^{(i,j+1)}$. We have

$$\|\partial\xi|_{\Psi^{(i,j)}}\|^2 = (p + i)(q + n - i)/2,$$

$$\|\partial\xi|_{\Psi^{(i,j)}}\|^2 = (q + j)(p + n - j)/2.$$  

Therefore, on $\Psi^{(i,j)}$, the eigenvalue of $\Delta_{\xi}$ is

$$((p + i)(q + n - i) + (q + j)(p + n - j))/4.$$  

Next we consider $W = \Psi^{(i+1,j)} \oplus \Psi^{(i,j+1)}$. We set

$$\psi^{(s,t)} = \psi^{(s,t)}/\|\psi^{(s,t)}\|.$$  

Let $A = \|\partial\xi|_{\Psi^{(i,j)}}\|$ and $B = \|\partial\xi|_{\Psi^{(i,j)}}\|$. Then, we have

$$d_{\xi} \psi^{(i,j)} = A \psi^{(i+1,j)} + B \psi^{(i,j+1)} \in \text{Im } d_{\xi},$$

and

$$d_{\xi} d_{\xi}^\dagger (A \psi^{(i+1,j)} + B \psi^{(i,j+1)}) = d_{\xi}(A^2 + B^2) \psi^{(i,j)} = (A^2 + B^2)(A \psi^{(i+1,j)} + B \psi^{(i,j+1)}).$$

Therefore, eigenvalue of $\Delta_{\xi}$ on $\text{Im } d_{\xi} \psi^{(i,j)}$ is

$$(A^2 + B^2)^2 = ((p + i)(q + n - i) + (q + j)(p + n - j))/4.$$  

Let us find the eigenvalue on $d_{\xi} \psi^{(i,j)} \perp$, which is the orthogonal complement in $W$. We note that

$$B \psi^{(i+1,j)} - A \psi^{(i,j+1)} \in d_{\xi} \psi^{(i,j)} \perp.$$  

Let $C = (p + i - j - q)/2$, $A' = C - 2A^2$ and $B' = C + 2B^2$. By (1.3) and (2.4),

$$D(B \psi^{(i+1,j)} - A \psi^{(i,j+1)}) = \sqrt{-1} \theta \wedge (A' B \psi^{(i+1,j)} - B' A \psi^{(i,j+1)}).$$

Since $D(A \psi^{(i+1,j)} + B \psi^{(i,j+1)}) = 0$, we have

$$D^\dagger D(B \psi^{(i+1,j)} - A \psi^{(i,j+1)}) = \frac{(A' B)^2 + (B' A)^2}{A^2 + B^2} (B \psi^{(i+1,j)} - A \psi^{(i,j+1)}).$$
We note that
\[
\frac{(A'B)^2 + (B'A)^2}{A^2 + B^2} = \frac{1}{4}(q + j - i - p)^2 + (p + i)(q + n - i)(q + j)(p + n - j).
\]
Under the condition \(i + j = n - 1\), it agrees with
\[
((p + i)(q + n - i) + (q + j)(p + n - j))^2/4.
\]
Therefore, we see that the eigenvalue on \(\text{Im} d_E \Psi^{(i,j)}_\perp\) is
\[
((p + i)(q + n - i) + (q + j)(p + n - j))^2/4.
\]

**Case VI**: \(i = n - 1, j = 0, p \geq 1\) and \(q = -1\)

The space is \(\Psi^{(n,0)}\). Since there is no subspaces of \(\mathcal{E}^{n-1}(S)\) corresponding to the \(V(-1, -p)\), we conclude \(\tilde{\partial}_b \restriction \Psi^{(n,0)} = \partial_b \restriction \Psi^{(n,0)} = \{0\}\). By (1.3), we have
\[
D\psi^{(n,0)} = \theta \wedge L_T^T \psi^{(n,0)}.
\]
Therefore, we have
\[
\Delta_E \psi^{(n,0)} = (d_E d_E^\dagger)^2 \psi^{(n,0)} + D^\dagger D \psi^{(n,0)} = L_T^\dagger L_T \psi^{(n,0)},
\]
where \(L_T^\dagger\) is the formal adjoint of \(L_T\) for the \(L^2\)-inner product. By (2.4), we see that the eigenvalue of \(\Delta_E\) is
\[
(p + n)^2/4.
\]

**Case VII**: \(i = 0, j = n - 1, p = -1\) and \(q \geq 1\)

The space is \(\Psi^{(0,n)}\). Taking the conjugate of Case VI, the eigenvalue of \(\Delta_E\) is
\[
(q + n)^2/4.
\]

**Remark 2.6.** In Cases V-VII, the eigenvalues of \(D^\dagger D\) were determined by [4]. Their choice of highest weight vectors in \(\text{Ker} D\) and \(\text{Im} D^\dagger\) are different from ours.

### 3. Proof of Theorem 0.2

From Theorem 0.1, we see that the terms of \(\kappa_E(s)\) in Cases II and V in Proposition 1.1 cancel each other. Thus we get
\[
\kappa_E(s) = \kappa_1(s) + 2\kappa_2(s),
\]
where
\[
\kappa_1(s) = \sum_{k=0}^{n} (-1)^{k+1}(n + 1 - k) \dim \text{Ker} \Delta_E = -(n + 1),
\]
which is the sum of the terms of \(\kappa_E(s)\) in Case I, and
\[
\kappa_2(s) = \sum_{i=0}^{n} (-1)^{i+1} \sum_{p \geq 1} \dim V(0, -i, -1, -p) \frac{1}{((p + i)/2)^2s},
\]
which is the sum of the terms of $\kappa_E(s)$ in Cases III and VI. From Weyl’s dimensional formula, we have

$$\dim V(0, -i, -1, -p) = \frac{p}{p + i} \binom{n}{i} \left( \frac{p + n}{n} \right)$$

$$= \frac{1}{n!} \sum_{l=1}^{n+1} e_{n+1-l}(n-i, n-1-i, \ldots, -i)(p+i)^{l-1},$$

where $e_l(X_0, \ldots, X_n)$ are the elementary symmetric polynomials of $n + 1$ variables of degree $l$. Thus we get

$$\kappa_2(s) = \frac{2^{2s+1}}{n!} \sum_{i=0}^{n} (-1)^{i+1} \binom{n}{i} \sum_{p \geq 1}^{n+1} \sum_{l=1}^{n+1} e_{n+1-l}(n-i, \ldots, -i) \frac{1}{(p+i)^{2s-l+1}}$$

$$= \frac{2^{2s+1}}{n!} \sum_{i=0}^{n} (-1)^{i+1} \binom{n}{i} \sum_{l=1}^{n} e_{n+1-l}(n-i, \ldots, -i)
\cdot \left( \zeta(2s-l+1) - \sum_{k=1}^{i} k^{-(2s-l+1)} \right).$$

Since $\sum_{l=1}^{n} e_{n+1-l}(n-i, \ldots, -i) k^l = 0$ for $k \leq i$, the second sum in the last expression vanishes and we have

$$\kappa_2(s) = -\frac{2^{2s+1}}{n!} \sum_{l=1}^{n} c_l \zeta(2s-l+1),$$

where

$$c_l = \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} e_{n+1-l}(n-i, \ldots, -i).$$

If we set $\sigma(k) = \sum_{l=0}^{n+1} c_l k^l$ for $k \in \mathbb{N}$, then

$$\sigma(k) = \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} \prod_{l=0}^{n}(k+n-i-l)
= \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} \left. \frac{d^{n+1} t^{k+n-i}}{dt^{n+1}} \right|_{t=1} = \left. \frac{d^{n+1} t^{k}(t-1)^n}{dt^{n+1}} \right|_{t=1}.$$

It follows that $\sigma(k) = (n+1)! k$ and hence $c_l = 0$ except $c_1 = (n+1)!$. Summing up, we conclude

$$\kappa_E(s) = -(n+1)(1 + 2^{2s+1} \zeta(2s)).$$

Using $\zeta(0) = -1/2$ and $\zeta'(0) = -(log 2\pi)/2$, we get

$$\kappa_E(0) = -(n+1)(1 + 2\zeta(0)) = 0,$$

$$\kappa'_E(0) = 2(n+1) \log 4\pi$$

as claimed.
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