Optimal function spaces for the weak continuity of the distributional $k$-Hessian

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Abstract: In this paper we introduce the notion of distributional $k$-Hessian associated with Besov type functions in Euclidean $n$-space, $k = 2, \ldots, n$. Particularly, inspired by recent work of Baer and Jerison on distributional Hessian determinant, we show that the distributional $k$-Hessian is weak continuous on the Besov space $B(2 - \frac{2}{k}, k)$, and the result is optimal in the framework of the space $B(s, p)$, i.e., the distributional $k$-Hessian is well defined in $B(s, p)$ if and only if $B(s, p) \subset B_{loc}(2 - \frac{2}{k}, k)$.

Key words: $k$-Hessian; Minor; Besov space; Distribution.

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1 Introduction and main results

For $k = 1, \ldots, n$ and $u \in C_\infty^2(\mathbb{R}^n)$, the $k$-Hessian operator $F_k$ is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of the Hessian matrix of second derivatives $D^2u$, and $S_k$ is the $k$-th elementary symmetric function on $\mathbb{R}^n$, given by

$$S_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

Alternatively we may write

$$F_k[u] = [D^2u]_k,$$

where $[A]_k$ denotes the sum of the $k \times k$-principal minors of an $n \times n$ matrix $A$, which may also be called the $k$-trace of $A$. It is well known that the $k$-Hessian is the Laplace operator when $k = 1$ and the Monge-Ampère operator when $k = n$.

This paper is devoted to the study of the $k$-Hessian of a nonsmooth map $u$ from $\mathbb{R}^n$ into $\mathbb{R}$, with $2 \leq k \leq n$. Starting with the seminal work of Trudinger and Wang (see [19, 20, 21, 22]), it has been known that the $k$-Hessian makes sense as a Radon measure and enjoys the weak continuity property for $k$-admissible functions. In [7, 8], Fu introduced the space of Monge-Ampère functions for which all minors of the Hessian matrices, including in particular the Hessian determinant, are well defined as signed Radon measures and weakly continuous in a certain natural sense. Jerrard [12, 13] extended the notion of Monge-Ampère functions and showed analogous continuous property and other

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structural properties. Moreover other generalized notion of the $k$-Hessian measure are considered in [4, 5, 6]. Our purpose in this thesis is to extend the definition of the $F_k$ to corresponding classes of functions so that the $k$-Hessian $F_k[u]$ is a distribution on $\mathbb{R}^n$. In the case $k = 2$, inspired by the results of [11] characterizing the Hessian determinant on the space $W^{1,2}(\mathbb{R}^2)$, the 2-Hessian is well defined and continuous on $W^{1,2}(\mathbb{R}^n)$. More precisely, the 2-Hessian $F_2[u]$ is defined for all $u \in W^{1,2}(\mathbb{R}^n)$ by

$$\langle F_2[u], \varphi \rangle := \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^n} \partial_i u \partial_j u \partial_{i,j} \varphi - \frac{1}{2} \partial_i u \partial_i u \partial_{i,j} \varphi - \frac{1}{2} \partial_j u \partial_j u \partial_{i,j} \varphi dx$$  \hspace{1cm} (1.1)$$

for any $\varphi \in C_c^\infty(\mathbb{R}^n)$, where $\partial_i := \frac{\partial}{\partial x_i}$. It is obvious to show the weak continuous results by Hölder inequality.

In the case $3 \leq k \leq n$, we consider the $k$-Hessian operator on a class of Besov spaces on $\mathbb{R}^n$, denote by $B(s,p) = B^p_s$. In particular, we will show that the $k$-Hessian $F_k[u]$ is well defined and continuous from the Besov space $B(2 - \frac{2}{k}, k)$, into the space of distribution. Moreover, the definition and continuity property is optimal in the framework of the space of Besov type space.

The initial motivation of our work is the following: Baer and Jerison [1] showed that the Hessian determinant operator $u \mapsto \det(D^2u) : C^2_c(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ admits a unique continuous extension, which they denote by $\mathcal{H}$, from the Besov space $B(2 - \frac{2}{n}, n)$ to the space of distributions $\mathcal{D}'(\mathbb{R}^n)$, and the continuity property fails for any space in the framework of Besov space for which the inclusion $B(s,p) \subset B_{loc}(2 - \frac{2}{k}, k)$ and is not continuous on any other space in the framework of Besov type space.

We recall that for $1 < s < 2$ and $1 \leq p < \infty$, the Besov space $B(s,p)$ is defined by

$$B(s,p) := \left\{ u \in W^{1,p}(\mathbb{R}^n) \mid \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|Du(x) - Du(y)|^p}{|x-y|^{n+(s-1)p}} dxdy \right)^\frac{1}{p} < \infty \right\} ,$$

and the norm

$$\|u\|_{s,p} := \|u\|_{W^{1,p}} + \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|Du(x) - Du(y)|^p}{|x-y|^{n+(s-1)p}} dxdy \right)^\frac{1}{p} .$$

Then our first result is the following.

**Theorem 1.1.** For $2 \leq k \leq n$, the $k$-Hessian operator $u \mapsto F_k[u] : C^2_c(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ can be extended uniquely as a continuous mapping $u \mapsto F_k[u] : B(2 - \frac{2}{k}, k) \to \mathcal{D}'(\mathbb{R}^n)$. Moreover, for all $u_1, u_2 \in B(2 - \frac{2}{k}, k)$ and $\varphi \in C^2_c(\mathbb{R}^n)$, we have

$$|\langle F_k[u_1] - F_k[u_2], \varphi \rangle| \leq C \|u_1 - u_2\|_{2 - \frac{2}{k}, k} \|D^2 \varphi\|_{L^\infty} .$$

In the case $k = 2$, the results of Theorem [11] can be easily deduced by [11], in which case the regularity index becomes integer and the Besov function space is the usual Sobolev space $W^{1,2}$. In the case $k = n$, the $k$-Hessian operator in fact is the Hessian determinant operator, i.e. $F_n = \mathcal{H}$, and the analogous results were already established in [1]. Moreover, in analogy with [1], Theorem [11] immediately gives several consequences: in particular, the $k$-Hessian as a distribution is continuous in spaces $W^{1,p}(\mathbb{R}^n) \cap W^{2,q}(\mathbb{R}^n)$ with $1 < p, q < \infty$, $\frac{2}{p} + \frac{k-2}{q} = 1$ and $k \geq 3$.

Now we turn to the optimality result. More precisely, the distributional $k$-Hessian is well defined in $B(s,p)$ if and only if $B(s,p) \subset B_{loc}(2 - \frac{2}{k}, k)$. 

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Theorem 1.2. Let $3 \leq k \leq n$, $1 < p < \infty$ and $1 < s < 2$ be such that $B(s, p) \nsubseteq B_{loc}(2 - \frac{2}{k}, k)$. Then there exist a sequence $\{u_m\} \subset C_c^\infty(\mathbb{R}^n)$ and a function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that
$$\lim_{m \to \infty} \|u_m\|_{s,p} = 0$$
and
$$\lim_{m \to \infty} \int F_k[u_m] \varphi dx = \infty.$$ 

Remark 1.3. We recall the embedding properties of the Besov spaces $B(s, p) (1 < s < 2, 1 < p < \infty)$ into the space $B_{loc}(2 - \frac{2}{k}, k)$, more details see [17] or [18, page 196]:

(i) $s + \frac{2}{k} > 2 + \max\{0, \frac{n}{p} - \frac{2}{k}\}$, the embedding $B(s, p) \subset B_{loc}(2 - \frac{2}{k}, k)$ holds;

(ii) $s + \frac{2}{k} < 2 + \max\{0, \frac{n}{p} - \frac{2}{k}\}$, the embedding fails;

(iii) $s + \frac{2}{k} = 2 + \max\{0, \frac{n}{p} - \frac{2}{k}\}$, there are two sub-cases:

(a) if $p \leq k$, then the embedding $B(s, p) \subset B_{loc}(2 - \frac{2}{k}, k)$ holds;

(b) if $p > k$, the embedding fails.

In order to prove Theorem 1.2, we just consider three cases:

I: $1 < p \leq k$ and $s + \frac{2}{k} < 2 + \frac{n}{p} - \frac{2}{k}$;

II: $k < p$ and $0 < s < 2 - \frac{2}{k}$;

III: $k < p$ and $s = 2 - 2/k$.

This paper is organized as follows. Some notion about determinant and the proof of Theorem 1.1 are given in Section 2. In Section 3 we show Theorem 1.2 in the case I: $1 < p \leq k$ and $s + \frac{2}{k} < 2 + \frac{n}{p} - \frac{2}{k}$. Then we prove Theorem 1.2 in the case II: $k < p$ and $0 < s < 2 - \frac{2}{k}$ in Section 4. Finally in Section 5 we establish Theorem 1.2 in the remaining case: $k < p$ and $s = 2 - 2/k$.

2 Preliminaries and the proof of Theorem 1.1

In this section we prove the continuity results for the $k$-Hessian operator on spaces of Besov type into the space of distributions on $\mathbb{R}^n$. First we recall some notation and facts about determinant.

For integers $n \geq 2$, we shall use the standard notation for ordered multi-indices
$$I(k, n) := \{\alpha = (\alpha_1, \ldots, \alpha_k) \mid \alpha_i \text{ integers, } 1 \leq \alpha_1 < \cdots < \alpha_k \leq n\}.$$ 

Set $I(0, n) = \{0\}$ and $|\alpha| = k$ if $\alpha \in I(k, n)$. If $\alpha \in I(k, n)$, $k = 0, 1, \ldots, n$, $\alpha$ is the element in $I(n-k, n)$ which complements $\alpha$ in $\{1, 2, \ldots, n\}$ in the natural increasing order. So $\overline{0} = \{1, 2, \ldots, n\}$.

Given $\alpha = (\alpha_1, \ldots, \alpha_k) \in I(k, n)$, we say $i \in \alpha$ if $i$ is one of the indexes $\alpha_1, \ldots, \alpha_k$. For $i \in \alpha$, $\alpha - i$ means the multi-index of length $k - 1$ obtained by removing $i$ from $\alpha$. Similarly for $j \notin \alpha$, $\alpha + j$ means the multi-index of length $k + 1$ obtained by reordering naturally the multi-index $(\alpha_1, \ldots, \alpha_k, j)$.

Let $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ be $n \times n$ matrixes. Given two ordered multi-indices with $\alpha, \beta \in I(k, n)$, then $A^\beta_\alpha$ denotes the $k \times k$-submatrix of $A$ with rows $(\alpha_1, \ldots, \alpha_k)$ and columns $(\beta_1, \ldots, \beta_k)$. Its determinant will be denoted by
$$M^\alpha_\beta(A) := \det A^\beta_\alpha.$$
We denote $\sigma(\alpha, \beta)$ by the sign of the permutation which reorders $(\alpha, \beta)$ in the natural increasing order and $\sigma(0, 0) := 1$. The adjoint of $A_\alpha^\beta$ is defined by the formula

$$(\text{adj } A_\alpha^\beta)^i_j := \sigma(i, \beta - i)\sigma(j, \alpha - j) \det A_{\alpha-j}^\beta \quad i \in \beta, j \in \alpha.$$ 

So Laplace formulas can be written as

$$M_\alpha^\beta(A) = \sum_{j \in \alpha} a_{ij}(\text{adj } A_\alpha^\beta)^i_j.$$ 

And the Binet formulas can be written as (see [9 page 313])

$$M_\alpha^\beta(A + B) = \sum_{\alpha' + \alpha'' = \alpha; \beta' + \beta'' = \beta; |\alpha'| = |\beta'|} \sigma(\alpha', \alpha'')\sigma(\beta', \beta'') M_{\alpha'}^\beta'(A) M_{\alpha''}^\beta''(B). \quad (2.1)$$

Let $n \geq 2$ and $F: \mathbb{R}^n \to \mathbb{R}$ be given as

$$F(x) = \prod_{i=1}^n f_i(x_i), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where the function $f_i: \mathbb{R} \to \mathbb{R}$ for $i = 1, \ldots, n$. For any $\alpha \in I(k, n)$, it will be convenient to introduce the notation

$$F_\alpha(x_\alpha) := \prod_{i \in \alpha} f_i(x_i), \quad x_\alpha := (x_{\alpha_1}, \ldots, x_{\alpha_k}) \in \mathbb{R}^k,$$

$$F_{\overline{\alpha}}(x_{\overline{\alpha}}) := \prod_{i \in \overline{\alpha}} f_i(x_i), \quad x_{\overline{\alpha}} := (x_{\overline{\alpha_1}}, \ldots, x_{\overline{\alpha_{n-k}}}) \in \mathbb{R}^{n-k}.$$

We now turn to the proof of Theorem 2.1, which actually can be seen as an immediate consequence following from the standard approximation argument if we have proven the following result.

**Theorem 2.1.** Let $3 \leq k \leq n$. Then for all $u_1, u_2, \varphi \in C_c^2(\mathbb{R}^n)$, we have

$$\left| \int_{\mathbb{R}^n} (F_k[u_1] - F_k[u_2]) \varphi dx \right| \leq C \|u_1 - u_2\|_{2-\frac{2}{k}} \left(\|u_1\|_{2-\frac{2}{k}} + \|u_2\|_{2-\frac{2}{k}}\right) \|D^2 \varphi\|_{L^\infty}. \quad (2.2)$$

In order to prove the above theorem, we need the following extension result which is inspired from the work of Baer-Jerison [1].

**Lemma 2.2.** Let $3 \leq k \leq n$, $\alpha \in I(k, n)$ and $u, \varphi \in C_c^2(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} M_\alpha^\beta(D^2 u) \varphi dx = \sum_{i \in \alpha} \sum_{j \in \alpha} \int_{\mathbb{R}^n \times (0,1)^2} \text{adj} \left( (D^2 U)^{\alpha+(n+1)}_\alpha \right)^i_j \Psi d\bar{x}. \quad (2.3)$$

for any extensions $U$ and $\Phi \in C_c^2([0, 1] \times [0, 1])$ of $u$ and $\varphi$, respectively, here $\bar{x} = (x, x_{n+1}, x_{n+2})$.

**Proof.** Denote $V := U|_{x_{n+2} = 0}$, $\Psi := \Phi|_{x_{n+2} = 0}$ and $\partial_i := \frac{\partial}{\partial x_i}$. Then

$$\int_{\mathbb{R}^n} M_\alpha^\beta(D^2 u) \varphi dx = - \int_{\mathbb{R}^n \times (0,1)} \partial_{n+1} \left( M_\alpha^\beta(D^2 V) \Psi \right) dx dx_{n+1}$$

$$= - \int_{\mathbb{R}^n \times (0,1)} \partial_{n+1} \left( M_\alpha^\beta(D^2 V) \right) \Psi dx dx_{n+1} - \int_{\mathbb{R}^n \times (0,1)} M_\alpha^\beta(D^2 V) \partial_{n+1} \Psi dx dx_{n+1}.$$
We denote the first part integral on the right-hand side by $I$, using Laplace formulas we obtain

$$
I = - \sum_{i \in \alpha} \sum_{j \in \alpha} \int_{\mathbb{R}^n \times (0,1)} \sigma(i, \alpha - i) \sigma(j, \alpha - j) \partial_{n+1} \partial_j V M_{\alpha-j}^{\alpha-i}(D^2 V) \Psi dxdx_{n+1}
$$

$$
= \sum_{i \in \alpha} \sum_{j \in \alpha} \sigma(i, \alpha - i) \sigma(j, \alpha - j) \int_{\mathbb{R}^n \times (0,1)} \partial_{n+1} \partial_j V \left( \partial_i (M_{\alpha-j}^{\alpha-i}(D^2 V)) \Psi + M_{\alpha-j}^{\alpha-i}(D^2 V) \partial_i \Psi \right) dxdx_{n+1}.
$$

Since

$$
\sum_{i \in \alpha} \partial_i \left( (\text{adj}(D^2 V))_\alpha^i \right) = 0
$$

for any $j \in \alpha$, it follows that

$$
I = \sum_{i \in \alpha} \sum_{j \in \alpha} \int_{\mathbb{R}^n \times (0,1)} \sigma(i, \alpha - i) \sigma(j, \alpha - j) \partial_{n+1} \partial_j V M_{\alpha-j}^{\alpha-i}(D^2 V) \partial_i \Psi dxdx_{n+1}
$$

$$
= \sum_{i \in \alpha} \int_{\mathbb{R}^n \times (0,1)} \sigma(i, \alpha - i) \sigma(n + 1, \alpha - i) M_{\alpha}^{\alpha+(n+1)-i}(D^2 V) \partial_i \Psi dxdx_{n+1}
$$

$$
= \sum_{i \in \alpha} -\sigma(\alpha + (n + 1) - i, i) \int_{\mathbb{R}^n \times (0,1)} M_{\alpha}^{\alpha+(n+1)-i}(D^2 V) \partial_i \Psi dxdx_{n+1}.
$$

Hence

$$
\int_{\mathbb{R}^n} M_{\alpha}^{\alpha}(D^2 u) \varphi dx = \sum_{i \in \alpha+(n+1)} -\sigma(\alpha + (n + 1) - i, i) \int_{\mathbb{R}^n \times (0,1)} (M_{\alpha}^{\alpha-i+(n+1)}(D^2 U) \partial_i \Phi) |_{x_{n+2}=0} dxdx_{n+1}.
$$

It is well known consequence of integration by parts that the right-hand side of the above identity can be written as

$$
\sum_{i \in \alpha+(n+1)} \sigma(\alpha + (n + 1) - i, i) A(i),
$$

where

$$
A(i) := \int_{\mathbb{R}^n \times (0,1)^2} \partial_{n+2} \left( M_{\alpha}^{\alpha-i+(n+1)}(D^2 U) \partial_i \Phi \right) d\bar{x}
$$

$$
= \int_{\mathbb{R}^n \times (0,1)^2} \partial_{n+2} \left( M_{\alpha}^{\alpha-i+(n+1)}(D^2 U) \right) \partial_i \Phi d\bar{x} + \int_{\mathbb{R}^n \times (0,1)^2} M_{\alpha}^{\alpha-i+(n+1)}(D^2 U) \partial_i \partial_{n+2} \Phi d\bar{x}.
$$

For simplicity we may set $\beta := \alpha - i + (n + 1)$. Obviously,

$$
\partial_{n+2} \left( M_{\alpha}^{\beta}(D^2 U) \right) = \sum_{j \in \alpha} \sum_{t \in \beta} \sigma(j, \alpha - j) \sigma(t, \beta - t) \partial_{n+2} \partial_j \partial_t U M_{\alpha-j}^{\beta-t}(D^2 U),
$$

and for any $j \in \alpha$,

$$
\sum_{j \in \alpha} \partial_j \left( \sigma(\alpha - j, j) M_{\alpha-j+(n+2)}^{\beta}(D^2 U) \right)
$$

$$
= \sum_{j \in \alpha} \sigma(\alpha - j, j) \sum_{s \in \alpha - j} \sum_{t \in \beta} \sigma(s, \alpha - j + (n + 2) - s) \sigma(t, \beta - t) \partial_j \partial_s \partial_t U M_{\alpha-j+(n+2)-s}^{\beta-t}(D^2 U)
$$
+ \sum_{j \in \alpha} \sigma(\alpha - j, j) \sum_{t \in \beta} \sigma((n + 2), \alpha - j) \sigma(t, \beta - t) \partial_j \partial_{n+2} \partial_t U M_{\alpha-j}^{\beta-t}(D^2 U) \\
= \sum_{j \in \alpha} \sum_{s \in \alpha - j} \sigma(\alpha - j, j) \sigma(s, \alpha - j - s) \sum_{t \in \beta} \sigma(t, \beta - t) \partial_j \partial_s \partial_t U M_{\alpha-j+(n+2)-s}^{\beta-t}(D^2 U) \\
+ \sum_{j \in \alpha} \sum_{t \in \beta} \sigma(j, \alpha - j) \sigma(t, \beta - t) \partial_j \partial_{n+2} \partial_t U M_{\alpha-j}^{\beta-t}(D^2 U).

Note that for any \(i_1, i_2 \in \alpha\) with \(i_1 \neq i_2\)

\[
\sigma(\alpha - i_1, i_1) \sigma(i_2, \alpha - i_1 - i_2) = (-1)^{k-1} \sigma(i_1, \alpha - i_1 - i_2) \sigma(i_2, \alpha - i_1 - i_2)(-1)^{\tau(i_1, i_2)},
\]

where

\[
\tau(i_1, i_2) := \begin{cases} 
1, & i_1 > i_2, \\
0, & i_1 < i_2,
\end{cases}
\]

which implies that

\[
\sigma(\alpha - i_1, i_1) \sigma(i_2, \alpha - i_1 - i_2) = -\sigma(\alpha - i_2, i_2) \sigma(i_1, \alpha - i_1 - i_2).
\]

Combing with the above results, we can easily obtain

\[
\partial_{n+2} \left( M_\alpha^\beta (D^2 U) \right) = \sum_{j \in \alpha} \partial_j \left( \sigma(\alpha - j, j) M_{\alpha-j+(n+2)}^\beta (D^2 U) \right). \tag{2.6}
\]

Then taking the sum in \(i\) and recalling \(2.4\), we have

\[
\int_{\mathbb{R}^n} M_\alpha^\beta (D^2 u) \varphi dx = \int_{\mathbb{R}^n \times (0,1)^2} \sum_{i \in \alpha + (n+1)} \sigma(\alpha + (n + 1) - i, i) \\
\left\{ - \sum_{j \in \alpha} \sigma(\alpha - j, j) M_{\alpha-j+(n+2)}^\beta (D^2 U) \partial_{i,j} \Phi + M_\alpha^\beta (D^2 U) \partial_{i,n+2} \Phi \right\} d\bar{x},
\]

which completes the proof.

**Proof of Theorem 2.1.** According to a well known extension theorem of Stein in [15, 16], there is a bounded linear extension operator

\[
E : B(2 - \frac{2}{k}, k) \rightarrow W^{2,k}(\mathbb{R}^n \times [0,1)^2).
\]

Let \(U_1, U_2 \in C_c^2(\mathbb{R}^n)\) be extensions of \(u_1\) and \(u_2\) to \(\mathbb{R}^n \times (0,1)^2\), respectively, such that

\[
\|D^2 U_i\|_{L^k(\mathbb{R}^n \times (0,1)^2)} \leq C\|u_i\|_{2-\frac{2}{k},k}, \quad i = 1, 2,
\]

and

\[
\|D^2 U_1 - D^2 U_2\|_{L^k(\mathbb{R}^n \times (0,1)^2)} \leq C\|u_1 - u_2\|_{2-\frac{2}{k},k}.
\]

Let \(\Phi \in C_c^2(\mathbb{R}^n \times [0,1)^2)\) be an extension of \(\varphi\) such that

\[
\|D^2 \Phi\|_{L^\infty(\mathbb{n} \times (0,1)^2)} \leq C\|D^2 \varphi\|_{L^\infty(\mathbb{R}^n)}.
\]
Since
\[ |M^\beta_\alpha(A) - M^\beta_\alpha(B)| \leq C (|A| + |B|)^{k-1} |A - B| \]
for any \( \alpha, \beta \in I(k, n+2) \) and \((n+2) \times (n+2)\) matrixes \( A, B \). It follows from Lemma 2.2 and Hölder’s inequality that
\[
\left| \int_{\mathbb{R}^n} (F_k[u_1] - F_k[u_2]) \varphi dx \right| \\
\leq \sum_{\alpha \in I(k, n)} \sum_{i \in \alpha + (n+1)} \sum_{j \in \alpha + (n+2)} \int_{\mathbb{R}^n \times (0, 1)^2} |M^{\alpha-i+(n+1)}_{\alpha-j+(n+2)}(D^2U_1) - M^{\alpha-i+(n+1)}_{\alpha-j+(n+2)}(D^2U_2)| \varphi d\tilde{x}
\]
\[
\leq C \int_{\mathbb{R}^n \times (0, 1)^2} (|D^2U_1| + |D^2U_2|)^{k-1} |D^2(U_1 - U_2)| \varphi d\tilde{x}
\]
\[
\leq C \|u_1 - u_2\|_{2-k} \left( \|u_1\|_{2-k}^{k-1} + \|u_2\|_{2-k}^{k-1} \right) \|D^2\varphi\|_{L^\infty}.
\]
This completes the proof of Theorem 2.1. \( \square \)

3 Optimality results I: \( 1 < p \leq k, \ s + \frac{2}{k} < 2 + \frac{n}{p} - \frac{n}{k} \)

In this section we establish the optimality result of Theorem 1.2 in the case \( 1 < p \leq k \) and \( s + \frac{2}{k} < 2 + \frac{n}{p} - \frac{n}{k} \). For this, we need the following lemma

Lemma 3.1. Let \( g \in C^\infty_c(B(0, 1)) \) be given as
\[
g(x) = \int_0^{|x|} h(r) dr
\]
for any \( x \in \mathbb{R}^n \), where \( h \in C^\infty_c((0, 1)) \) and satisfies
\[
\int_0^1 h(r) dr = 0, \quad \int_0^1 h^k(r) r^{-k+n+1} dr \neq 0.
\]
Then
\[
\sum_{\alpha \in I(k, n)} \int_{B(0, 1)} |M^\alpha_{\alpha}(D^2g(x))|^2 dx \neq 0.
\]

Proof. According to the symmetry of integral, it is sufficient to show
\[
\int_{B(0, 1)} |M^\alpha_{\alpha}(D^2g(x))|^2 dx \neq 0
\]
for any \( \alpha \in I(k, n) \). It is easy to see that
\[
D^2g = \frac{1}{|x|^3}(A + B),
\]
where \( A = (a_{ij})_{n \times n} \) and \( B = (b_{ij})_{n \times n} \) are \( n \times n \) matrices such that
\[
a_{i,j} = h(|x|)|x| |x_i x_j|, \quad b_{ij} = h(|x|)(\delta_i^j |x|^2 - x_i x_j), \quad i, j = 1, \ldots, n.
\]
Using Binet formula and the fact \(\text{rank}(A) = 1\), one has
\[
M_\alpha^\alpha(A + B) = M_\alpha^\alpha(B) + \sum_{i \in \alpha} \sum_{j \in \alpha} \sigma(i, \alpha - i) \sigma(j, \alpha - j) a_{ij} M_{\alpha-1}^\alpha(B)
\]
\[
= h^k(|x|)|x|^{2k-2}(|x|^2 - \sum_{i \in \alpha} x_i^2) + h'(|x|)h^{k-1}(|x|)|x| \cdot I,
\]
where
\[
I := \sum_{i \in \alpha} \sum_{j \in \alpha} \sigma(i, \alpha - i) \sigma(j, \alpha - j) x_i x_j M_{\alpha-1}^\alpha((|x|^2 \delta_i^j - x_i x_j)_{n \times n})
\]
\[
= \sum_{i \in \alpha} x_i^2 M^{\alpha-1}_{\alpha-1}((|x|^2 \delta_i^j - x_i x_j)_{n \times n}) + \sum_{i \in \alpha} \sum_{j \in \alpha - i} \sigma(i, \alpha - i) \sigma(j, \alpha - j) x_i x_j M_{\alpha-1}^\alpha((|x|^2 \delta_i^j - x_i x_j)_{n \times n})
\]
\[
= \sum_{i \in \alpha} x_i^2 |x|^{2k-4}(|x|^2 - \sum_{j \in \alpha - i} x_j^2) + \sum_{i \in \alpha} \sum_{j \in \alpha - i} x_i x_j x_i x_j |x|^{2k-4}
\]
\[
= |x|^{2k-2} \sum_{i \in \alpha} x_i^2.
\]
Hence
\[
\int_{B(0,1)} M_\alpha^\alpha(D^2 g)|x|^2 dx = \int_{B(0,1)} |x|^{-3k+2} M_\alpha^\alpha(A + B) dx = II - III + IV,
\]
where
\[
II := \int_{B(0,1)} h^k(|x|)|x|^{-k+2} dx,
\]
\[
III := \int_{B(0,1)} h^k(|x|)|x|^{-k} \sum_{i \in \alpha} x_i^2 dx,
\]
and
\[
IV := \int_{B(0,1)} h^{k-1}(|x|)h'(|x|)|x|^{-k+1} \sum_{i \in \alpha} x_i^2 dx.
\]
Then integration in polar coordinates gives
\[
IV = \frac{k-n-2}{n} 2\pi^{n-2} \prod_{i=1}^{n-2} I(i) \int_0^1 h^k(r)r^{-k+n+1} dr,
\]
where \(I(s) = \int_0^\pi \sin^s \theta d\theta\). Similarly,
\[
III = \frac{k}{n} 2\pi^{n-2} \prod_{i=1}^{n-2} I(i) \int_0^1 h^k(r)r^{-k+n+1} dr,
\]
and
\[
II = 2\pi^{n-2} \prod_{i=1}^{n-2} I(i) \int_0^1 h^k(r)r^{-k+n+1} dr,
\]
which implies \(3.3\), and then the proof is complete.
**Theorem 3.2.** Let \( 3 \leq k \leq n, 1 < p \leq k \) and \( 0 < s < 2 \) with \( s + \frac{2}{k} < 2 + \frac{n}{p} - \frac{n}{k} \). Then there exist a sequence \( \{u_m\} \subset C^\infty_c(\mathbb{R}^n) \) and \( \varphi \in C^\infty_c(\mathbb{R}^n) \) satisfying (1.2) and (1.3).

**Proof.** Consider \( u_m : \mathbb{R}^n \rightarrow \mathbb{R} \) defined by
\[
u_m(x) = m^{-\rho} g(mx), \quad m > 1,
\]
where \( g \) is given as (3.1) and \( \rho \) is a constant such that
\[
s - \frac{n}{p} < \rho < 2 - \frac{n}{k} - \frac{2}{k}.
\]
On the one hand, we have
\[
\|u_m\|_{s,p} \leq \|u_m\|_{L^p}^{1 - \frac{2}{p}} \|D^2 u_m\|_{L^p}^{\frac{2}{p}} \leq m^{s - \rho - \frac{2}{p}} \|g\|_{L^p}^{1 - \frac{2}{p}} \|D^2 g\|_{L^p}^{\frac{2}{p}},
\]
which implies (1.2). On the other hand, let \( \varphi \in C^\infty_c(\mathbb{R}^n) \) be such that \( \varphi(x) = |x|^2 + O(|x|^3) \) as \( x \to 0 \). Then
\[
\int_{\mathbb{R}^n} F_k[u_m] \varphi \, dx = \sum_{\alpha \in I(k,n)} m^{-(\rho - 2)k} \int_{\mathbb{R}^n} M_{\alpha}^\alpha(D^2 g(mx)) \varphi(x) \, dx \\
= \sum_{\alpha \in I(k,n)} m^{-(\rho - 2)k - n} \int_{B(0,1)} M_{\alpha}^\alpha(D^2 g) \varphi(\frac{x}{m}) \, dx \\
= m^{2k - \rho k - n - 2} \sum_{\alpha \in I(k,n)} \int_{B(0,1)} M_{\alpha}^\alpha(D^2 g) |x|^2 \, dx + O(m^{2k - \rho k - n - 3}).
\]
Collecting Lemma 3.1 and (3.4), it follows that
\[
\left| \int_{\mathbb{R}^n} F_k[u_m] \varphi \, dx \right| \geq C m^{2k - \rho k - n - 2} \to \infty \quad \text{as} \ m \to \infty.
\]
Hence the theorem is proved completely. \( \square \)

**4 Optimality results II: \( k < p, \ s < 2 - \frac{2}{k} \)**

In this section we consider the case \( p > k \) and \( 0 < s < \frac{2}{k} \) for Theorem 1.2. We begin with the following simple lemma which is a formula due to Chen \[3\] for the Hessian determinant of functions as tensor product.

**Lemma 4.1.** Let \( 2 \leq k \leq n, \alpha \in I(k,n) \) and \( F : \mathbb{R}^n \rightarrow \mathbb{R} \) be given by a tensor product
\[
F(x) = \prod_{i=1}^{n} f_i(x_i), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,
\]
where \( f_i \in C^2(\mathbb{R}), \ i = 1, \ldots, n \). Then
\[
M_{\alpha}^\alpha(D^2 F) = (F_{\alpha}(x_\alpha))^{k} (F_{\alpha}(x_\alpha))^{k-2} \left\{ \prod_{i \in \alpha} g_i(x_i) + \sum_{j \in \alpha} \left( \prod_{i \in \alpha - j} g_i(x_i) \right) [f_j'(x_j)]^2 \right\}
\]
with
\[
g_i(x_i) = f_i''(x_i) f_i(x_i) - [f_i'(x_i)]^2, \quad i = 1, \ldots, n.
\]
Let $\gamma = (1, \ldots, k-1) \in I(k-1, n)$ and $\Omega \subset \{ x \in \mathbb{R}^n \mid x_i > 0 \text{ for all } i \in \overline{\gamma} \}$ be a nonempty open set. For any $m \in \mathbb{N}^+$, define $u_m : \mathbb{R}^n \to \mathbb{R}$

$$u_m = m^{-\rho} \chi(x) P_\gamma(x_\gamma) \cdot Q_\gamma(x_\gamma),$$

where the functions $P, Q : \mathbb{R}^n \to \mathbb{R}$ are given by

$$P(x) := \prod_{i=1}^n \sin^2(mx_i), \quad Q(x) := \prod_{i=1}^n x_i.$$

Assume that $\max\{s, 2 - \frac{4}{k}\} < \rho < 2 - \frac{2}{k}$, and $\chi \in C_0^\infty(\mathbb{R}^n)$ is a smooth cutoff function with $\chi = 1$ on $\Omega$.

**Theorem 4.2.** Let $3 \leq k \leq n$, $k < p < \infty$ and $0 < s < 2 - \frac{2}{k}$. Let $u_m \in C_0^\infty(\mathbb{R}^n)$ and $\Omega$ be defined as above. Then for any $\Omega' \subset \subset \Omega$, $\varphi \in C_c(\Omega)$ with $\varphi \geq 0$ and $\varphi = 1$ on $\Omega'$, the functions $u_m$ satisfy \((L.2)\) and \((L.3)\).

**Proof.** According to the facts that $\|u_m\|_{L^\infty} \leq Cm^{-\rho}$ and $\|D^2 u_m\|_{L^\infty} \leq Cm^{2-\rho}$, where constants depending on the measure of spt$\chi$, it follows that

$$\|u_m\|_{s,p} \leq C \|u_m\|_{L^p}^{1-\frac{2}{p}} \|u_m\|_{W^{2,s}}^{\frac{2}{s}} \leq Cm^{s-\rho},$$

which implies \((L.2)\).

On the other hand, it follows from our hypotheses on the cutoff function $\chi$ that

$$u_m(x) = m^{-\rho} P_\gamma Q_\gamma \quad x \in \Omega.$$

For simplicity, we may set

$$I_c := \{ \alpha \in I(k, n) \mid \alpha = (\alpha', \alpha''), \alpha' \subset \gamma, \alpha'' \subset \overline{\gamma}, |\alpha'| = c \}.$$ 

Then

$$F_k[u_m] = \sum_{\alpha \in I(k, n)} M_\alpha^2(D^2 u_m) = \sum_{c=0}^{k-1} \sum_{\alpha \in I_c} M_\alpha(D^2 u_m).$$

Hence

$$\left| \int F_k[u_m] \varphi dx \right| \geq \sum_{\alpha \in I_{k-1}} \int M_\alpha(D^2 u_m) \varphi dx - \sum_{c=0}^{k-2} \sum_{\alpha \in I_c} \int M_\alpha(D^2 u_m) \varphi dx. \quad (4.3)$$

For any $\alpha \in I_{k-1}$, i.e., $\alpha = \gamma + j$ with $j \in \overline{\gamma}$, by using Lemma \ref{Lemma} we obtain that

$$M_\alpha(D^2 u_m) = m^{-\rho k}(Q_{\gamma-j})^k \det(D^2(P_\gamma \cdot x_j)) \left(-1\right)^{k-1} 2^k m^{2k-2-\rho k}(Q_{\gamma-j})^k(P_\gamma)^{k-1} x_j^{k-2} \left(\sum_{i \in \gamma} \cos^2(m x_i)\right).$$

So

$$\left| \sum_{\alpha \in I_{k-1}} \int M_\alpha(D^2 u_m) \varphi dx \right| = 2^k m^{2k-2-\rho k} \left| \sum_{j \in \gamma} \int (Q_{\gamma-j})^k(P_\gamma)^{k-1} x_j^{k-2} \left(\sum_{i \in \gamma} \cos^2(m x_i)\right) \varphi dx \right|$$
\[ \geq Cm^{2k-2-\rho} \sum_{i\in\gamma} \sum_{j\in\tau} \int_{\Omega} (Q_{\tau-j})^k (P_{\gamma})^{k-1} x_j^{k-2} \cos^2(mx_i) dx \]
\[ \geq Cm^{2k-2-\rho}. \]

For any \( \alpha \in I_c(0 \leq c \leq k-2) \), i.e., \( \alpha = (\alpha', \alpha'') \) with \( \alpha' \subset \gamma, \alpha'' \subset \tau \). Similarly, 
\[ M_{\alpha}^\alpha(D^2u_m) = (-1)^{k-1} 2^{k-\rho} m^{2c-\rho} (P_{\gamma-\alpha'}Q_{\tau-\alpha''})^k (P_{\alpha'}^{k-1} (Q_{\alpha''})^{k-2}(k-c-1+2 \sum_{i\in\alpha'} \cos^2(mx_i))). \]

Hence
\[ \sum_{c=0}^{k-2} \alpha \in I_{c} \int M_{\alpha}^\alpha(D^2u_m) \varphi dx \leq \sum_{c=0}^{k-2} 2^c m^{2c-\rho} \sum_{\alpha \in I_{c}} \int (P_{\gamma-\alpha'}Q_{\tau-\alpha''})^k (P_{\alpha'}^{k-1} (Q_{\alpha''})^{k-2}(k-c-1+2 \sum_{i\in\alpha'} \cos^2(mx_i))) \varphi dx \leq Cm^{2k-4-\rho}. \]

By the hypothesis \( \max\{s, 2 - \frac{4}{k}\} < \rho < 2 - \frac{2}{k} \), we may easily show \( (1.3) \). This completes the proof of the theorem. \( \square \)

## 5 Optimality results III: \( k < p, s = 2 - 2/k \)

We conclude the proof of Theorem 1.2 by showing the results in the remaining case \( p > k \) and \( s = 2 - \frac{2}{k} \).

**Theorem 5.1.** Let \( 3 \leq k \leq n, k < p \) and \( s = 2 - 2/k \). Then there exist a sequence \( \{u_m\} \subset C^\infty_c(\mathbb{R}^n) \) and a function \( \varphi \in C^\infty_c(\mathbb{R}^n) \) satisfying \( (1.2) \) and \( (1.3) \).

For \( m \in \mathbb{N} \) with \( m \geq 2 \), let 
\[ n_l = m^{k^l}, \quad l = 1, 2, \ldots, m. \]

Define \( g_l : \mathbb{R}^n \to \mathbb{R} \) as follows 
\[ g_l(x) = P_{l,\gamma}(x_\gamma) \cdot Q_{\tau}(x_\gamma), \]
where 
\[ P_l := \prod_{i=1}^{n} \sin^2(n_l x_i), \quad Q := \prod_{j=1}^{n} x_j, \quad \gamma = (1, \cdots, k-1) \in I(k-1, n). \]

Then define \( u_m : \mathbb{R}^n \to \mathbb{R} \) by 
\[ u_m(x) = \chi(x) \sum_{i=1}^{m} \frac{1}{n_l^{2 \frac{2}{k} l^k}} g_l(x), \quad (5.1) \]
where \( \chi(x) \in C^\infty_c(\mathbb{R}^n) \) is a smooth cutoff function satisfying \( \chi(x) = 1 \) for \( x \in (0, 2\pi)^n \). In order to end the proof, some results are introduced as follows.

**Lemma 5.2.** Let \( 3 \leq k \leq n, k < p < \infty \) and \( u_m \) defined by \( (5.1) \). Then 
\[ \sup_{m \in \mathbb{N}} \|u_m\|_{2 - \frac{2}{k} p} < \infty. \]
Proof. The proof is closely the same as the proof of boundedness (5.3) in [1]. According to the standard estimates for products in Besov space, it suffices to estimate \( \|w_m\|_{2-\frac{2}{k},p} \) on \([0,2\pi]^n\), where

\[
w_m = \sum_{l=1}^{m} \frac{1}{n_l^{2-\frac{2}{k}l^k}} P_{l,\gamma} = \sum_{l=1}^{m} \frac{1}{n_l^{2-\frac{2}{k}l^k}} \prod_{i=1}^{k-1} \sin^2(n_l x_i).
\]

The Littlewood-Paley characterization of the Besov space \(B(2-\frac{2}{k},p)([0,2\pi]^n)\) (see, e.g. [18]) implies

\[
\|w_m\|_{2-\frac{2}{k},p} \leq C \left( \|w_m\|_{L_p([0,2\pi]^n)}^p + \sum_{j=1}^{\infty} 2^{(2-\frac{2}{k})jp} \|T_j(w_m)\|_{L_p([0,2\pi]^n)}^p \right)^{\frac{1}{p}}.
\]

(5.2)

Here the operators \(T_j : L^p \rightarrow L^p\) are defined by

\[
T_j \left( \sum a_i e^{il\cdot x} \right) = \sum_{2^j \leq |l| \leq 2^{j+1}} \left( \rho \left( \frac{|l|}{2^{j+1}} \right) - \rho \left( \frac{|l|}{2^j} \right) \right) a_i e^{il\cdot x},
\]

where \(\rho \in C^\infty_c(\mathbb{R})\) is a suitably chosen bump function.

However, it is clear that \(\|w_m\|_{L^p}\) is uniformly bounded because of the definition of \(n_l\), while an argument similar to the one used to prove the (5.3) in [1] shows that

\[
\sum_{j=1}^{\infty} 2^{(2-\frac{2}{k})jp} \|T_j(w_m)\|_{L_p([0,2\pi]^n)}^p \leq C \sum_{l=1}^{\infty} \frac{1}{l^k},
\]

where \(C > 0\) is a constant only depending on \(k\). This gives the desired result.

Lemma 5.3. Let \(3 \leq k \leq n\), \(k < p < \infty\) and \(u_m\) defined by (5.7). And \(\varphi \in C^\infty_c(\mathbb{R}^n)\) is defined by

\[
\varphi(x) = \prod_{i=1}^{n} \varphi_i(x_i), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,
\]

where \(\varphi_i \geq 0\) and \(\text{spt} \varphi_i \subseteq (0,2\pi)\) for \(i = 1, \ldots, n\). Then there exist \(c > 0\) and \(K_0\) such that

\[
\left| \int_{\mathbb{R}^n} F_k[u_m] \varphi dx \right| \geq c (\log m - K_0).
\]

Proof. It follows from our hypotheses on the cutoff function \(\chi\) that

\[
u_m(x) = \sum_{l=1}^{m} \frac{1}{n_l^{2-\frac{2}{k}l^k}} g_l(x), \quad x \in (0,2\pi)^n.
\]

Fix \(c = 0, 1, \ldots, k - 1\), and set

\[
I_c := \{ \alpha \in I(k,n) \mid \alpha = (\alpha', \alpha''), \alpha' \subset \gamma, \alpha'' \subset \overline{\gamma}, |\alpha'| = c \}.
\]

Using the definition of \(k\)-Hessian we have

\[
\left| \int_{\mathbb{R}^n} F_k[u_m] \varphi dx \right| = \left| \sum_{\alpha \in I(k,n)} \int M_\alpha(D^2 u_m) \varphi dx \right|
\]

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\[ I := \left| \sum_{\alpha I k-1} \int M_\alpha^\alpha (D^2 u_m) \varphi dx \right| \]  

(5.3)

where

\[ I := \sum_{s=0}^{k-2} \sum_{\alpha I s} \int M_\alpha^\alpha (D^2 u_m) \varphi dx \]  

(5.4)

We shall divide the proof into seven steps.

Step 1: Estimate of I.

For any \( \alpha \in I k-1 \), we can write \( \alpha = \gamma + j = (1, 2, \ldots, k-1, j) \) with \( j \in (k, \ldots, n) \). According to the multilinearity of the determinant, we have

\[ M_\alpha^\alpha (D^2 u_m) = (Q_{\gamma-j})^k \sum_{L_\alpha} C(L_\alpha) \det(H(L_\alpha, j)), \]

where the sum is over \( L_\alpha = (l_1, \ldots, l_{k-1}, l_j) \in \{1, \ldots, m\}^k \), we set

\[ C(L_\alpha) = \prod_{i \in \gamma} \frac{1}{(n_l)^{2-\frac{j}{k}} (l_j)^{\frac{j}{k}}}, \]

and the \( k \times k \) matrix \( H(L_\alpha, j) \) is given by

\[ H(L_\alpha, j) = (\partial_{s,t}(x_j \Pi_{l,s,\gamma}))_{s,t \in \alpha}. \]

We denote \( J_0 \) the collection of all multi-indices \( L_\alpha = (l, l, \ldots, l) \) for \( l \in \{1, 2, \ldots, m\} \). Hence

\[ I = \left| \sum_{j \in J_0} \int M_{\gamma+j}^\gamma (D^2 u_m) \varphi dx \right| \]

\[ = \left| \sum_{j \in J_0} \int (Q_{\gamma-j})^k \sum_{L_\alpha} C(L_\alpha) \det(H(L_\alpha, j)) \varphi dx \right| \]

\[ \geq III - IV, \]

where

\[ III := \left| \sum_{j \in J_0} \int (Q_{\gamma-j})^k \sum_{L_\alpha \in J_0} C(L_\alpha) \det(H(L_\alpha, j)) \varphi dx \right|, \]

(5.5)

and

\[ IV := \left| \sum_{j \notin J_0} \int (Q_{\gamma-j})^k \sum_{L_\alpha \notin J_0} C(L_\alpha) \det(H(L_\alpha, j)) \varphi dx \right|. \]

(5.6)

Since

\[ \det(H(L_\alpha, j)) = (-1)^{k-1} 2^{k-1} (\prod_{i \in \gamma} \sin^2(n_i x_i))^{k-1} (x_j)^{k-2} (\sum_{i \in \gamma} \cos^2(n_i x_i)) \]
for any $L_\alpha = (l, l, \ldots, l) \in J_0$, then we have

$$III = \left| \sum_{j \in \Gamma} \sum_{l=1}^{m} (-1)^{k-1} 2^k \frac{1}{l} \int (Q_{\Gamma_{j}})^{k} \left( \prod_{i \in \gamma} \sin^2(n_i x_i) \right)^{k-1} x_j^{k-2} \left( \sum_{i \in \gamma} \cos^2(n_i x_i) \right) \varphi(x) dx \right|$$

$$= 2^k \sum_{j \in \Gamma} \sum_{l=1}^{m} \frac{1}{l} \int_{(0,2\pi)^n} (Q_{\Gamma_{j}})^{k} \left( \prod_{i \in \gamma} \sin^2(n_i x_i) \right)^{k-1} x_j^{k-2} \left( \sum_{i \in \gamma} \cos^2(n_i x_i) \right) \varphi(x) dx$$

$$\geq C_1 \log m,$$

where $C_1$ is a positive constant independent of $m$. Note that

$$IV \leq \sum_{j \in \Gamma} \sum_{L_\alpha \notin J_0} C(L_\alpha) \int (Q_{\Gamma_{j}})^{k} \det(H(L_\alpha, j)) \varphi dx$$

$$\leq \sum_{j \in \Gamma} \sum_{L_\alpha \notin J_0} C(L_\alpha) \int_{\mathbb{R}^n} \det(H(L_\alpha, j)) \prod_{i \in \alpha} \varphi_i(x_i) dx_{\alpha} \cdot \int_{\mathbb{R}^n} (Q_{\Gamma_{j}})^{k} \prod_{i \in \Gamma} \varphi_i(x_i) dx_{\alpha}$$

$$\leq C \| \varphi \|_{L^\infty} \sum_{j \in \Gamma} \sum_{L_\alpha \notin J_0} C(L_\alpha) \int_{\mathbb{R}^n} \det(H(L_\alpha, j)) \prod_{i \in \alpha} \varphi_i(x_i) dx_{\alpha}.$$

Then Proposition 5.2 in [1] implies that there exists a constant $C > 0$ such that

$$IV \leq C \sum_{L_\alpha \notin J_0} \frac{\| \varphi \|_{C^2}^2}{m^{2k}} \leq C \| \varphi \|_{C^2}.$$

Hence

$$I \geq C_1 \log m - C \| \varphi \|_{C^2}.$$

Step 2: Estimate of $\left| \sum_{\alpha \in I_0} \int M_{\alpha}^\alpha(D^2 u_m) \varphi dx \right|$.

Without loss of generality we can assume that $I_0 \neq \emptyset$. Then for any $\alpha \in I(k, n) \cap I_0$ we have

$$M_{\alpha}^\alpha(D^2 u_m) = \sum_{L_\alpha} C(L_\alpha) \prod_{i \in \gamma} \sin^2(n_i x_i) (Q_{\Gamma_{\alpha}})^{k} \det(G_\alpha),$$

where the $k \times k$ matrix $G_\alpha$ is given by

$$G_\alpha = \left( \partial_{st} \left( \prod_{i \in \alpha} x_i \right) \right)_{s,t \in \alpha}.$$

Therefore

$$\left| \sum_{\alpha \in I_0} \int M_{\alpha}^\alpha(D^2 u_m) \varphi dx \right| \leq \| \varphi \|_{L^\infty} \sum_{\alpha \in I_0} \sum_{L_\alpha} C(L_\alpha) \int_{(0,2\pi)^n} \prod_{i \in \gamma} \sin^2(n_i x_i) (Q_{\Gamma_{\alpha}})^{k} \det(G_\alpha) dx$$

$$\leq C \| \varphi \|_{L^\infty}.$$

Step 3: The first estimate of $| \sum_{\alpha \in I_c} \int M_{\alpha}^\alpha(D^2 u_m) \varphi dx |$ for $c = 1, \ldots, k - 2$.

For any $\alpha \in I_c$, it can be written as $\alpha = (\alpha', \alpha'')$ with $\alpha' \subset \gamma$ and $\alpha'' \subset \overline{\gamma}$. Set

$$y_1 = x_{\alpha'_1}, \ y_2 = x_{\alpha'_2}, \ \ldots, \ y_c = x_{\alpha'_c}, \ y_{c+1} = x_{\alpha''_1}, \ y_{c+2} = x_{\alpha''_2}, \ \ldots, \ y_k = x_{\alpha''_{k-c}}.$$
Hence
\[ u_m = \sum_{l=1}^{m} \frac{1}{n_l^{2-\frac{2}{d}}} \left( \prod_{s=1}^{c} \sin^2(n_l y_s) \prod_{s=1}^{k-c} y_{c+s} \right) \prod_{i \in \gamma'-\gamma''} \sin^2(n_l x_i) \prod_{i \in \gamma'-\gamma''} x_i, \tag{5.7} \]
which implies
\[
\left| \int M^\alpha(D^2u_m) \varphi \, dx \right| \\
= \left| \int \left( \prod_{i \in \gamma'-\gamma''} \sin^2(n_l x_i) \prod_{i \in \gamma'-\gamma''} x_i \right)^k \det \left( D^2 \left( \sum_{l=1}^{m} \frac{1}{n_l^{2-\frac{2}{d}}} \left( \prod_{s=1}^{c} \sin^2(n_l y_s) \prod_{s=1}^{k-c} y_{c+s} \right) \right) \right) \varphi(x) \, dx \right| \\
\leq C \| \varphi \|_L^\infty \int \det \left( D^2 \left( \sum_{l=1}^{m} \frac{1}{n_l^{2-\frac{2}{d}}} \left( \prod_{s=1}^{c} \sin^2(n_l y_s) \prod_{s=1}^{k-c} y_{c+s} \right) \right) \right) \prod_{i \in \alpha} \varphi(x_i) \, dy,
\]
where \( y = (y_1, \ldots, y_k) \). So it is convenient to set
\[ v_m(y) = \sum_{l=1}^{m} \frac{1}{n_l^{2-\frac{2}{d}}} \prod_{s=1}^{c} \sin^2(n_l y_s) \prod_{s=1}^{n-c} y_{c+s}, \quad \psi(y) = \prod_{i \in \alpha} \varphi(x_i). \tag{5.8} \]
In order to estimate \( |\sum_{\alpha \in \Lambda} M^\alpha(D^2u_m) \varphi| \), it is sufficient to show that
\[ V := \left| \int \det(D^2v_m) \psi \, dy \right| \leq C \| \psi \|_{C^2}, \tag{5.9} \]
where \( C > 0 \) is a constant.

Step 4: The first estimate for \( V \).

Similarly, we define \( P_i : \mathbb{R}^c \to \mathbb{R} \) and \( Q : \mathbb{R}^{k-c} \to \mathbb{R} \) by
\[ P_i := \prod_{i=1}^{c} \sin^2(n_i y_i), \quad Q := \prod_{i=1}^{k-c} y_{c+i}. \]
Similar to Step 2, we have
\[ V = \left| \sum_{\mathcal{L}} C(\mathcal{L}) \int \det(H_{\mathcal{L}}) \psi \, dy \right|, \]
where \( \mathcal{L} = (l_1, \ldots, l_k) \in \{1, 2, \ldots, m\}^k \),
\[ C(\mathcal{L}) = \prod_{i=1}^{k} \frac{1}{n_l^{2-\frac{2}{d}}(l_i)^\frac{d}{2}}. \]
and the $k \times k$ matrix $H_L = H_L(y)$ is given by

$$(\partial_{ij}(P_iQ))_{i,j \in \{1,2,\ldots,k\}}.$$  

Fixing $L = (l_1,\ldots,l_k)$, denote

$$l_* := \max\{l_i \mid i = 1,\ldots,k\},$$

and define

$$\beta_L := \{i : l_i = l_*\}.$$  

Using Laplace formulas of the determinant we obtain

$$V = \left| \sum_{\rho=1}^{k} \sum_{|\beta_L| = \rho} C(L) \int \det(H_L) \psi dy \right|$$

$$= \left| \sum_{\rho=1}^{k} \sum_{|\beta_L| = \rho} C(L) \sum_{\xi \in I(\rho,k)} \sigma(\beta_L,\overline{\beta_L}) \sigma(\xi,\overline{\xi}) \int M_{\beta_L}^\xi(H_L) M_{\overline{\beta_L}}^\overline{\xi}(H_L) \psi dy \right|$$

$$\leq VI + VII,$$

where

$$VI := \sum_{\rho=1}^{c} \sum_{|\beta_L| = \rho} \sum_{\xi \in I(\rho,k)} \left| C(L) \int M_{\beta_L}^\xi(H_L) M_{\overline{\beta_L}}^\overline{\xi}(H_L) \psi dy \right|,$$

and

$$VII := \sum_{\rho=c+1}^{k} \sum_{|\beta_L| = \rho} \sum_{\xi \in I(\rho,k)} \left| C(L) \int M_{\beta_L}^\xi(H_L) M_{\overline{\beta_L}}^\overline{\xi}(H_L) \psi dy \right|.$$  

Note that we separated the determinant $\det(H_L)$ into two parts: $M_{\beta_L}^\xi(H_L)$ involves only frequencies of the highest order $n_{l_*}$ or 0, while $M_{\overline{\beta_L}}^\overline{\xi}(H_L)$ involves only frequencies of lower order $n_i$ with $l_i \leq n_{l_*-1}$.

If $\rho > c$, for any $L$ with $|\beta_L| = \rho$ and $\xi \in I(\rho,k)$, we set $|\beta_L \cap (1,2,\ldots,c)| = b$ and $|\xi \cap (1,2,\ldots,c)| = b'$. There is no loss of generality in assuming $\beta_L = (1,2,\ldots,b,\beta_{b+1},\ldots,\beta_{\rho})$, $\xi = (1,2,\ldots,b',\beta_{b'+1},\ldots,\beta_{\rho})$. Then

$$(H_L)_{\beta_L}^\xi = \begin{pmatrix} n_{l_*}^2 g_{1,1}, \ldots, n_{l_*}^2 g_{1,b}, n_{l_*} g_{1,b+1}, \ldots, n_{l_*} g_{1,a} \\ \vdots \\ n_{l_*}^2 g_{b,1}, \ldots, n_{l_*}^2 g_{b,b'}, n_{l_*} g_{b,b'+1}, \ldots, n_{l_*} g_{b,a} \\ n_{l_*} g_{b+1,1}, \ldots, n_{l_*} g_{b+1,b'}, g_{b+1,b'+1}, \ldots, g_{b+1,a} \\ \vdots \\ n_{l_*} g_{a,1}, \ldots, n_{l_*} g_{a,b'}, g_{a,b'+1}, \ldots, g_{a,a} \end{pmatrix}$$

where $g_{s,t}$ is a uniformly bounded function for $s \in \beta_L$, $t \in \xi$. It follows that

$$|M_{\beta_L}^\xi(H_L)| \leq C n_{l_*}^{b+b'} \leq C n_{l_*}^{2c}.$$  

The following result may be proved in much the same way as above:

$$|M_{\overline{\beta_L}}^\overline{\xi}(H_L)| \leq C n_{l_*-1}^{2(k-\rho)}.$$
Obviously we shall have established the theorem if we could estimate \( VI \).

Hence

\[
VII \leq C \sum_{i=1}^{k} \sum_{\rho=c+1}^{k} \sum_{\xi \in I(\rho,k)} \frac{1}{(2-\frac{3}{k})^\rho \rho} \int_{(0,2\pi)^k} |M_{\beta_{\xi}}^\xi(H_{\xi})| |M_{\beta_{\xi}}^{\xi}(H_{\xi})| |\psi| dy
\]

\[
\leq C \|\psi\|_{L^\infty} \sum_{\rho=c+1}^{k} \frac{1}{2^{2(k-\rho)}} \frac{1}{2^{k-2c}}
\]

\[
\leq C \|\psi\|_{L^\infty} \sum_{\rho=c+1}^{k} \frac{1}{2} m_{i}\]

\[
\leq C \|\psi\|_{L^\infty} m_{k}^{2k}.
\]

Obviously we shall have established the theorem if we could estimate \( VI \).

Step 5: Fix \( \beta \) such that \( |\beta| = \rho \leq c \), and we will prove that

\[
M_{\beta_{\xi}}^\xi(H_{\xi}) = 0
\]

for any \( \xi \in I(\rho,k) \) with \( |\beta_{\xi} \cap \xi| \leq \rho - 2 \).

Let \( i_1, i_2 \in \beta_{\xi} \setminus \xi \) be given with \( i_1 \neq i_2 \), and set

\[
h_i(y_i) := \begin{cases} \sin^2(n_i y_i), & i = 1, \ldots, c \\ y_i, & i = c + 1, \ldots, k \end{cases}
\]

and \( H(y) = \prod_{i=1}^{k} h_i(y_i) \),

\[
v_k(y) = (\partial_{i_k} H)_{j \in \xi} \in \mathbb{R}^\rho, \quad k = 1, 2.
\]

Since \( i_1, i_2 \notin \xi \), we have

\[
\begin{align*}
  v_1 &= (h_{i_1}^1 h_1^1 H_{1+1})_{j \in \xi} \\
  v_2 &= (h_{i_1}^2 h_1^2 H_{1+2})_{j \in \xi}
\end{align*}
\]

which immediately give (5.12).

Let \( \rho \leq c \), \( \beta_{\xi} \) and \( \xi \) be given. If either

(i) \( |\beta_{\xi} \cap \xi| = \rho \) such that \( \beta_{\xi} \cap (c+1, c+2, \ldots, k) \neq \emptyset \), or

(ii) \( |\beta_{\xi} \cap \xi| = \rho - 1 \) such that \( i^* := \beta_{\xi} \setminus \xi, j^* := \xi \setminus \beta_{\xi} \in (c+1, \ldots, k) \),

is satisfied, by the same method as in Step 4 and (5.24) in [11], it follows that

\[
\left| C_{\xi} \int M_{\beta_{\xi}}^\xi(H_{\xi}) M_{\beta_{\xi}}^{\xi}(H_{\xi}) \psi dy \right| \leq C \|\psi\|_{L^\infty} m_{k}^{2k}.
\]

Set

\[
S_{\rho} := \{(\xi, \xi) \mid \beta_{\xi}, \xi \in I(\rho,k), |\beta_{\xi} \cap \xi| = \rho, \beta_{\xi} \cap (c+1, \ldots, k) = \emptyset \}
\]

\[
\cup \{(\xi, \xi) \mid \beta_{\xi}, \xi \in I(\rho,k), |\beta_{\xi} \cap \xi| = \rho - 1, \beta_{\xi} \setminus \xi \notin (c+1, \ldots, k) \}
\]

\[
\cup \{(\xi, \xi) \mid \beta_{\xi}, \xi \in I(\rho,k), |\beta_{\xi} \cap \xi| = \rho - 1, \xi \setminus \beta_{\xi} \notin (c+1, \ldots, k) \},
\]

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then
\[ VI \leq C \frac{\| \psi \|_{L^\infty}}{m^{2k}} + \sum_{c=1}^{b_k} \sum_{\rho=1}^{b_k} \left| C(\mathcal{L}) \int M_{\rho_k}^\xi (H_L) M_{\rho_k}^{\bar{\xi}} (H_L) \psi dy \right|. \]

(5.13)

It is easy to see that for any \( \xi \in I(\rho, k) \) there exist integers \( b_{c+1}, b_{c+2}, \ldots, b_k \leq \rho \) and a sequence of coefficients \( \{ c_z \} \subset \mathbb{C} \) such that

\[ M_{\rho_k}^\xi (H_L) = \sum_{z \in \Lambda} c_z e^{2i\pi z} \gamma_{y_{c+1}}^{b_{c+1}} \gamma_{y_{c+2}}^{b_{c+2}} \cdots \gamma_{y_k}^{b_k}, \]

(5.14)

where \( \gamma = (y_1, y_2, \cdots, y_c) \),

\[ \Lambda = \{ z \in \mathbb{Z}^c \mid |z_i| \leq c \}, \]

and

\[ |c_z| \leq C n_i^{2b}. \]

In fact the proof of this statement follows in a similar manner in [1, Remark 5.5].

**Step 6:** Next we have to show that \( c_{(0,\ldots,0)} = 0 \) for any \( (\mathcal{L}, \xi) \in S_\rho \) where \( c_{(0,\ldots,0)} \) is defined in (5.14).

According to (5.14), it suffices to show that

\[ \int_{[0,2\pi]^c} M_{\rho_k}^\xi (H_L) d\gamma = 0 \]

(5.15)

for each \((\mathcal{L}, \xi) \in S_\rho\). Suppose that \( \beta_L = \xi \) and \( \beta_L \cap (c + 1, \ldots, k) = \emptyset \), and set \( \eta := (1, \cdots, \epsilon) \). Then

\[ \int_{[0,2\pi]^c} M_{\rho_k}^\xi (H_L) d\gamma = \int_{[0,2\pi]^c} (P_{\eta, \eta-\xi})^{\rho} \det \left( \prod_{i \in \xi} \sin^2(n_i y_i) \right) d\gamma \]

\[ = Q \int_{[0,2\pi]^{c-\rho}} (P_{\eta, \eta-\xi})^{\rho} dy_{\eta-\xi} \]

\[ \cdot (-2n_i^{2\rho})^{\rho} \int_{[0,2\pi]^\rho} \left( \prod_{i \in \xi} \sin(n_i y_i) \right)^{2\rho-2} \left( 1 - \sum_{i \in \xi} \cos^2(n_i y_i) \right) dy_{\xi}. \]

The equality (5.15) holds as desired due to the equality (5.34) in [1].

We now turn to the second case, suppose that \( |\beta_L \cap \xi| = \rho - 1 \) and \( j^* = \xi \setminus \beta_L \in (1, 2, \ldots, c) \). Using the Laplace formulas of determinant again we obtain

\[ M_{\rho_k}^\xi (H_L) = \sum_{i \in \beta_L} \sigma(i, \rho_k - i) \sigma(j^*, \xi - j^*) \partial_{i,j^*} (P_{\eta, \xi}) M_{\rho_k}^{\xi-j^*} (H_L) \]

\[ = n_{i_\eta} \sin(2n_i y_{j^*}) \sin^{2\rho-2} (n_i y_{j^*}) g(y), \]

where the function \( g : \mathbb{R}^k \to \mathbb{R} \) is independent of the variable \( y_{j^*} \). It follows that \( M_{\rho_k}^\xi (H_L) \) is an odd function in the variable \( y_{j^*} \), so the equality (5.15) is obtained.

The proof of the last case for this statement follows in a similar manner which implies \( c_{(0,\ldots,0)} = 0 \) for any \((\mathcal{L}, \xi) \in S_\rho\).

**Step 7:** Finally we have to estimate the second part on the right-hand side of (5.13) by integration by parts.
For any \((L, \xi) \in S_\rho\) we have \(c_{(0,\ldots,0)} = 0\), and then

\[
\text{VIII} := \sum_{\rho=1}^c \sum_{(L, \xi) \in S_\rho} \left| C(L) \int M_{\beta L}^\xi (H_L) M_{\beta L}^\xi (H_L) \psi dy \right|
\]

\[
\leq C \sum_{\rho=1}^c \sum_{(L, \xi) \in S_\rho} \left| \sup_{(L, \xi) \in S_\rho, \xi \in \Lambda \setminus \{0\}} \left| \int e^{2n_{t_*} iz \cdot \hat{y} + b_{c+1} y_{c+1} + b_{c+2} y_{c+2} + \cdots + b_k y_k} M_{\beta L}^\xi (H_L) \psi dy \right| \right|
\]

where \(C > 0\) is a constant. Let \(z = (z_1, \ldots, z_c) \in \Lambda \setminus \{0\}\) be given, there exists \(j \in (1, \ldots, c)\) such that \(z_j \neq 0\). Using the integration by parts two times in the \(y_j\) variable, we obtain

\[
\int e^{2n_{t_*} iz \cdot \hat{y} + b_{c+1} y_{c+1} + b_{c+2} y_{c+2} + \cdots + b_k y_k} M_{\beta L}^\xi (H_L) \psi dy = - \frac{1}{4(n_{t_*})^2(z_j)^2} \int e^{2n_{t_*} iz \cdot \hat{y} + b_{c+1} y_{c+1} + b_{c+2} y_{c+2} + \cdots + b_k y_k} \partial_j^2 \left( M_{\beta L}^\xi (H_L) \psi \right) dy.
\]

In fact

\[
\left| \partial_j^2 M_{\beta L}^\xi (H_L) \right| \leq C(n_{t_*} - 1)^{2(k-\rho)+2} \leq C(n_{t_*} - 1)^{2k}.
\]

It follows that

\[
\text{VIII} \leq C\|\psi\|_{C^2} \sum_{\rho=1}^c \sum_{(L, \xi) \in S_\rho} \left( n_{t_*} \right)^{2k} \left( n_{t_*} - 1 \right)^{2k} \frac{n_{t_*}}{m^{2k}} \leq C\|\psi\|_{C^2} \frac{m^{2k}}{n_{t_*}^{2k}}.
\]

Therefore we establish the estimate of I-VIII, which gives the desired result.

\[\square\]

**The proof of Theorem 5.1** Let

\[
\tilde{u}_m := \frac{u_m}{(\log m)^{1/4}},
\]

where \(u_m\) is defined in (5.1). Then by Lemma 5.2 and 5.3, Theorem 5.1 is established and hence Theorem 1.2 is completely proved.

\[\square\]

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