Lagrangian Formulation of Connes’ Gauge Theory

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It is shown that Connes’ generalized gauge field in non-commutative geometry is derived by simply requiring that Dirac lagrangian be invariant under local transformations of the unitary elements of the algebra, which define the gauge group. The spontaneous breakdown of the gauge symmetry is guaranteed provided the chiral fermions exist in more than one generations as first observed by Connes-Lott. It is also pointed out that the most general gauge invariant lagrangian in the bosonic sector has two more parameters than in the original Connes-Lott scheme.

§1. Introduction

Connes’ gauge theory in non-commutative geometry (NCG) based on Euclidean 4-space times a 2-point space employs the free Dirac operator with mass term among others in the beginning and unifies the gauge and Higgs fields in an ingenious but highly mathematical way. A lot of works have been done along this line of thought. In view of so many papers in this field we may be excused to quote only one recent article.

On the other hand, Sogami proved that it is possible to describe the gauge and Higgs fields in a unified way using the generalized covariant derivative defined for Dirac lagrangian with given gauge-Yukawa interactions for chiral fermions.

One may then ask if there exists a lagrangian formulation of Connes’ gauge theory, which allows one to derive Connes’ generalized gauge field from a symmetry principle, determining the type of interactions for chiral fermions with gauge and Higgs fields. In the present paper we propose such a lagrangian formulation.

It is well known that the free Dirac lagrangian

\[ \mathcal{L}_D = \bar{\psi} D \psi, \quad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \bar{\psi} = (\bar{\psi}_L, \bar{\psi}_R) \]  

(1.1)

can be made gauge invariant if \( \psi \) is non-chiral, while it cannot be made chiral-gauge invariant if \( \psi \) is chiral; the reason is simply because the mass term is chiral-non-invariant. The free Dirac operator \( D \) consists of the free derivative operator and the mass matrix:

\[ D = D_0 + i\gamma_5 M, \quad D_0 = \begin{pmatrix} i\gamma^\mu \partial_\mu & 0 \\ 0 & i\gamma^\mu \partial_\mu \end{pmatrix}, \quad M = \begin{pmatrix} 0 & M_1 \\ M_1^\dagger & 0 \end{pmatrix}, \]  

(1.2)

One may add two arbitrary hermitian matrices to the diagonal blocks of the mass matrix. For simplicity we shall consider only the case indicated in the text.
meaning the hermitian conjugation. We take the Dirac matrices to satisfy \( \{ \gamma^\mu, \gamma^\nu \} = 2\eta^\mu\nu \), \( \eta^\mu\nu = \text{diag}(+1, -1, -1, -1) \) with \( \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 \). The \( \gamma_5 \) matrix in front of the mass matrix is only for later convenience. The chiral spinors in Eq. (1.1) are defined by \( \psi_{L,R} = 2^{-1}(1 \pm \gamma_5)\psi \).

Usually the gauge group is taken to be any Lie group. On the other hand, in the Connes’ approach\(^1\)\(^2\) the gauge group \( G \) is regarded as the unitary group of some (local) algebra \( \mathcal{A} \)

\[ G = \mathcal{U}(\mathcal{A}) = \{ g \in \mathcal{A}; gg^\dagger = g^\dagger g = 1 \}. \tag{1.3} \]

The exceptional Lie groups are then excluded from the gauge groups. Moreover, the well-known color gauge group \( SU(3) \) cannot be obtained from a single algebra. Instead Connes’ gauge theory can accommodate color symmetry only if it is combined with flavor symmetry\(^1\)\(^2\)\(^3\), which points to a unification of color and flavor, though the standard model gauge group is a product group. Thus considering an algebra and its associated unitary group seems to play a characteristic role in gauge theory.

The next section discusses a new style of symmetry principle which, starting from Eq. (1.1) for given gauge group (1.3), determines the type of interactions for chiral fermions with gauge and Higgs fields. Section 3 is devoted to discuss the spontaneous breakdown of the symmetry in the bosonic sector which is followed by Weinberg-Salam theory for leptons in section 4. The last section gives a brief prescription how to include quarks with color and a short summary.

\section*{§2. Connes’ generalized gauge field from symmetry principle}

We shall now demonstrate that the Dirac lagrangian (1.1) is easily made gauge invariant irrespective of the chiral property of the fermions if we employ the gauge group \( \mathcal{U}(\mathcal{A}) \). Namely, we require that the Dirac lagrangian be invariant under the local transformation \( \psi \rightarrow a\psi \) for any invertible element \( a \) of the algebra \( \mathcal{A} \). (The set of invertible elements of the algebra forms a group.) This is accomplished by replacing Eq. (1.1) with

\[ \mathcal{L}_D = \sum_i (a_i\psi)D(a_i\psi), \tag{2.1} \]

where \( a_i \in \mathcal{A} \) and the summation over the index \( i \) is to be taken for later reason. Since the transformation \( \psi \rightarrow a\psi \) is considered only for the invertible element \( a \), there exists \( a^{-1} \) that belongs to the algebra if \( a \in \mathcal{A} \). Consequently, the Dirac lagrangian Eq. (2.1) is invariant under the local transformation \( \psi \rightarrow a\psi \) for any invertible element \( a \) of the algebra \( \mathcal{A} \) provided that all \( a_i \) are transformed into \( a_ia_i^{-1} \) at the same time. We assume that any element of the algebra does not change the chirality so that it is regarded as \( 2 \times 2 \) block-diagonal matrix and the same is also true for \( g \).

To further convert Eq. (2.1) into gauge-invariant form, we next require that the Lorentz scalar \( \bar{\psi}\psi \) be also invariant under the replacement \( \psi \rightarrow a_i\psi \) together with the summation over \( i \). This leads to the condition

\[ \sum_i a_i^\dagger a_i = 1. \tag{2.2} \]

\(^{1)}\) Strictly speaking, we should write \( \rho(a)\psi \) for \( a\psi \) and \( \rho(a_i) \) for \( a_i \) in Eq. (2.1), where the notation \( \rho \) indicates the representation of the algebra \( \mathcal{A} \) on the Hilbert space of the spinors. For simplicity we omit the notation \( \rho \) in what follows unless necessary.
If the sum is reduced to a single term, the involved element is necessarily unitary. Since we can choose any element from the algebra to construct Eq. (2.1), this means that the algebra consists of unitary elements only, which is impossible. Hence we have to take the summation in Eq. (2.1).

The condition (2.2) which takes the same form as in Ref. 6) remains invariant under $a_i \rightarrow a_i g^\dagger$ if $g \in G = U(A)$. This implies that the theory is now invariant under the gauge transformation $\psi \rightarrow g \psi$ for $g \in G = U(A)$, which is simultaneously accompanied with the transformation $a_i \rightarrow a_i g^\dagger$, where the representation content of the fermions is limited as we shall see in §4.

Using Eq. (2.2) and the Leibniz rule we rewrite Eq. (2.1) as

$$L_D = \bar{\psi} D \psi,$$

where the generalized covariant derivative is given by

$$D = D + A, \quad A = \sum_i a_i^\dagger [D, a_i].$$

The generalized gauge field $A$ is essentially the same as Connes’ one [1–3]. We henceforth call it Connes’ generalized gauge field. We derived it from a new symmetry principle in the lagrangian formalism. We have developed a similar idea but in the extended differential formalism [7]. The theory is automatically gauge invariant, leading to the gauge transformation law

$$A \rightarrow gA = \sum_i g a_i^\dagger [D, a_i g^\dagger] = g Ag^\dagger + g [D, g^\dagger],$$

where use has been made of the condition (2.2).

At this point the theory is classified into two categories depending on whether the fermion is non-chiral or chiral. For non-chiral fermions the mass matrix commutes with the gauge transformations, which means that $M_1$ in Eq. (1.2) is proportional to the unit matrix. Hence it also commutes with any element of the algebra $A$, reducing $A$ to the usual Yang-Mills gauge field $A$,

$$A = \sum_i a_i^\dagger [D_0, a_i] = i \gamma^\mu A_\mu.$$  (2.6)

If, on the other hand, the fermions are chiral, $M$ no longer commutes with the element of the algebra $A$, and the decomposition (1.2) implies the corresponding decomposition

$$A = A + i \gamma_5 \Phi, \quad \Phi = \sum_i a_i^\dagger [M, a_i].$$  (2.7)

That is, Connes’ generalized gauge field $A$ unifies the ordinary gauge field $A$ with the shifted Higgs field $\Phi$ if the fermions are chiral. The usual Higgs field is defined by $H = \Phi + M$ which transforms like $H \rightarrow g H g^\dagger$. We found that this unification is already achieved at the lagrangian level without recourse to NCG based on Euclidean 4-space times a 2-point space. Thus the lagrangian (2.3) with Eq. (2.4) determines the type of interactions for chiral fermions from our symmetry principle [8].

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* The symmetry is broken down to the unbroken subgroup consisting of those gauge transformations which commute with the mass matrix. The breakdown should be spontaneous because the transformation law of $\Phi$ is inhomogeneous, $\phi \rightarrow g \phi g^\dagger + g [M, g^\dagger]$ unless $g$ commutes with the mass matrix.
The $\gamma_5$ matrix in Eq. (2.3) can easily be removed if we transform $\psi \to e^{i\pi \gamma_5/4} \psi$. It then reads

$$\mathcal{L}_D = \bar{\psi} D \psi, \quad D = D_0 + A - H.$$  

(2.8)

By writing $D = i\gamma^\mu D_\mu$ with use of the relation $\gamma^\mu \gamma^\nu = 4$ Sogami called $D_\mu$ (with the so-called Sogami’s term added) the generalized covariant derivative. He then proceeds to define the generalized field strength based on it to obtain the correct bosonic lagrangian. In our derivation which determines the underlying gauge-Yukawa interactions for chiral fermions, Sogami’s generalized covariant derivative is expressed in terms of the auxiliary objects $a_i$. Therefore, we cannot apply Sogami’s method directly. We shall instead follow the method developed in Ref. 6) where the $\gamma_5$ matrix in Connes’ generalized gauge field plays a crucial role in determining the bosonic lagrangian below.

§ 3. Bosonic sector

To complete the lagrangian formulation we should consider the bosonic sector as well. The field strength is defined by

$$F = dA + A^2, \quad dA \equiv \sum_i [D, a_i^\dagger][D, a_i],$$  

(3.1)

which is gauge covariant, since $d(gA) = \sum_i [D, ga_i^\dagger][D, a_i g^\dagger]$ from Eq. (2.5) and we have, again using the condition (2.2),

$$gF = d(gA) + (gA)^2$$

$$= \sum_i [D, ga_i^\dagger][D, a_i g^\dagger] - [D, g][D, g^\dagger] - [D, g]A g^\dagger + gA [D, g^\dagger] + gA^2 g^\dagger$$

$$= g(dA + A^2)g^\dagger = gF g^\dagger.$$  

(3.2)

More elaborate proof without using the condition (2.2) was presented by Connes.

It follows from Eqs. (2.7) and (3.1) that we obtain

$$F = F - i\gamma_5[D_0 + A, H] - Y,$$

$$Y = X + H^2 - M^2 - \sum_i a_i^\dagger [M^2, a_i],$$

$$X = -\sum_i a_i^\dagger \partial_\mu a_i + \partial_\mu A^\mu + A_\mu A^\mu,$$  

(3.3)

where $F = -(1/4)[\gamma^\mu, \gamma^\nu]F_{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The most general gauge-invariant lagrangian is given by

$$\mathcal{L} \B = -\frac{1}{4} \mathrm{Tr} \frac{1}{g^2} FF.$$  

(3.4)

Here the notation $\mathrm{Tr}$ means taking the trace over the 2-dimensional chiral space, Dirac matrices as well as internal symmetry matrices and the gauge coupling constants become a diagonal
matrix \(1/g^2\) commuting with the gauge transformations. Also, an associated field strength \(\tilde{F}\) is defined by

\[
\tilde{F} = \sum_{\alpha} h_\alpha^2 \Gamma_{\alpha} F^{\Gamma_{\alpha}},
\]

(3.5)

where \(\alpha = S, V, A, T, P\) corresponding to \(\Gamma_{\alpha} = 1, \gamma_\mu, \gamma_5 \gamma_\mu, \sigma_{\mu\nu}, i\gamma_5\). We once proposed \(8\) similar extension like Eqs. (3.4) and (3.5) in Sogami’s method. Setting \(\sum_{\alpha} h_\alpha^2 \Gamma_{\alpha} \sigma_{\mu\nu} F^{\Gamma_{\alpha}} = \sigma_{\mu\nu}\) without loss of generality, we can see that \(\tilde{F}\) has two more parameters than \(F\).

\[
\tilde{F} = F - \xi^2 i\gamma_5 [D_0 + A, H] - \kappa^2 Y
\]

(3.6)

with \(\xi^2\) and \(\kappa^2\) being assumed to be positive. (This is always possible since we have 5 parameters in Eq. (3.5).) Needless to say, \(\xi^2 = \kappa^2 = 1\) if we put \(h_S^2 = 1, h_\alpha^2 = 0\) for \(\alpha \neq S\). Since the gauge transformation property is not changed for the sum (3.5), \(\tilde{F}\) takes the same role as the field strength \(F\) in constructing the lagrangian. Hence we get Eq. (3.4).

Substituting Eqs. (3.3) and (3.6) into Eq. (3.4) and using the property of the trace of Dirac matrices, we find that

\[
\mathcal{L}_B = \mathcal{L}_{YM} + \xi^2 \frac{1}{g^2} (D^\mu H)(D_\mu H) - \kappa^2 \frac{1}{g^2} Y^2,
\]

(3.7)

where \(\mathcal{L}_{YM}\) is the Yang-Mills lagrangian, \(D_\mu H = [\partial_\mu + A_\mu, H]\) and the notation \(\text{tr}\) now means taking the trace over the 2-dimensional chiral space and internal symmetry matrices. Equation of motion for the auxiliary field in \(Y\) reads \(Y = 0\) and the last term in Eq. (3.7) vanishes identically. This result is well known \(6\), producing no Higgs potential upon elimination of the auxiliary field. The authors in Ref. 6) evaded this unpleasant situation by assuming that fermions exist in generation with nontrivial generation mixing.

This observation should be linked with the fact that the representation content of the fermions is limited by an underlying algebra representation.

§4. Weinberg-Salam theory in the leptonic sector

As an illustration let us consider Weinberg-Salam theory in the leptonic sector. We shall see that the flavor algebra to be considered below allows only doublets and singlets and, moreover, quarks with fractional charges cannot be included unless color is taken into account.

The flavor algebra for leptons is taken to be \(A = C^\infty(M_4) \otimes (H \oplus C)\), where \(C^\infty(M_4)\) denotes the set of infinitely many differentiable functions over Minkowski space-time \(M_4\), \(H\) the real quaternion and \(C\) the complex field. The unitary group is \(G = U(C^\infty(M_4) \otimes (H + C)) = \text{Map}(M_4, SU(2) \times U(1))\). Choosing the mass-eigenstates basis

\[
\psi_L = \begin{pmatrix} \nu_e \\ U_1 e \end{pmatrix}_L, \quad \psi_e_R = \begin{pmatrix} \nu_e R \\ e_R \end{pmatrix}_R,
\]

(4.1)

where generation indices are omitted and \(U_1\) is the leptonic Kobayashi-Maskawa matrix (we assume massive neutrinos), we consider the representation \(1\)

\[
\rho(a, b) = \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} \otimes 1_{N_g}, \quad B = \begin{pmatrix} b & 0 \\ 0 & b^* \end{pmatrix},
\]

(4.2)
where $a = a(x) \in C^\infty(M_4) \otimes H$, $b = b(x) \in C^\infty(M_4) \otimes C$ with $b^*$ being the complex conjugate to $b$, so that left-handed and right-handed fermions in generation belong to the doublet and singlet of the gauge group, respectively. This statement is obtained by letting the element $(a, b)$ belong to the unitary group of the algebra. Remember that the real quaternion has only one irreducible representation of dimensions 2. The matrix $B$ cannot be equal to $a$ or $a^*$ because, in that case, both $\psi_{L,R}$ are doublets with no $U(1)$ charge, leading to vector-like theory. Moreover, it must take the form of Eq.(4.2) or $b \leftrightarrow b^*$, because the $2 \times 2$ matrix-valued Higgs field $h$ of Eq.(4.4) below receives the gauge transformation, $h \rightarrow ahB^\dagger$ for unitary matrices $(a, B)$. This automatically determines the correct hypercharge assignment of Higgs field. The representation (4.2), however, gives rise to wrong hypercharge assignment of leptons. It turns out that to remedy this point without changing the correct hypercharge of Higgs will require only a minor change in the theory. The matrix $1_{N_g}$ is the $N_g$-dimensional unit matrix here in generation space.

To show how Higgs potential naturally appears in the present formalism, we choose the mass matrix

$$M = \begin{pmatrix} 0 & M_1 \\ M_1^\dagger & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad m_{1,2} : N_g \times N_g$$  \hspace{1cm} (4.3)

to obtain Higgs field as

$$\Phi = \begin{pmatrix} \varphi M_1 \\ 0 \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_0^* & \varphi_+ \\ -\varphi_- & \varphi_0 \end{pmatrix},$$

$$H = \Phi + M = \begin{pmatrix} 0 & hM_1 \\ M_1^\dagger h^\dagger & 0 \end{pmatrix}, \quad h = \varphi + 1_2 = \begin{pmatrix} \phi_0^* & \phi_+ \\ -\phi_- & \phi_0 \end{pmatrix}. \hspace{1cm} (4.4)$$

It can be shown that Eq.(2.3) with (2.4) leads to, after transforming $\psi \rightarrow e^{i\pi\gamma_5/4}\psi$,

$$\mathcal{L}_D = \bar{\psi}_L i\gamma\mu (\partial_{\mu} - \frac{ig_2}{2} A_\mu^{(2)}) \psi_L + \bar{\psi}_R i\gamma\mu (\partial_{\mu} - \frac{ig_1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_\mu^{(1)}) \psi_R$$

$$-\bar{\psi}_L (m_1 \tilde{\phi}, m_2 \phi) \psi_R - \bar{\psi}_R \left( \tilde{\phi} ^\dagger m_1 \right) \psi_L, \hspace{1cm} \tilde{\phi} = \begin{pmatrix} \phi_+ \\ -\phi_- \end{pmatrix}, \hspace{1cm} (4.5)$$

where

$$\phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix}, \quad \langle \phi \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hspace{1cm} (4.6)$$
is the normalized Higgs field, and we set

$$A = \sum_i \rho(a_i^\dagger, b_i^\dagger) [D, \rho(a_i, b_i)]$$

\[
^*)\text{We have used the condition (2.2) in the present case to show that } \varphi^\dagger \text{ in Eq.(4.4) is the hermitian conjugate to } \varphi.
\]
Here, $A^{(2)}_\mu$ is an SU(2) gauge field, while $A^{(1)}_\mu$ is a U(1) gauge field. Both are hermitian. Also, $g_2$ and $g_1$ are the respective gauge coupling constants.

Let us now evaluate the bosonic lagrangian (3.7). Writing

$$\frac{1}{g^2} = \frac{4}{N_g} \begin{pmatrix} g_2^{-2} & 0 & 0 \\ 0 & g_1^{-2} & 0 \end{pmatrix} \otimes 1_{N_g},$$

(4.8)

we have, upon eliminating the auxiliary fields in tr$Y^2$ by the equation of motion and rescaling the Higgs field $\phi \to \frac{1}{\sqrt{\xi L}} \phi = \xi \phi$, which renders the Yukawa term in Eq. (4.5) multiplied by $\sqrt{2} v$.

$$\mathcal{L}_B = -\frac{1}{8} \text{tr} F_{\mu\nu}^{(2)} F^{(2)\mu\nu} - \frac{1}{4} F_{\mu\nu}^{(1)} F^{(1)\mu\nu} + (D^\mu \phi)^(D_\mu \phi) - \frac{\lambda}{4} (\phi^\dagger \phi - \frac{v^2}{2})^2,$$

(4.9)

where $D_\mu \phi = (\partial_\mu - \frac{ig_2}{2} A^{(2)}_\mu - \frac{ig_1}{2} A^{(1)}_\mu) \phi$, and we define

$$\lambda = \frac{4\kappa^2 K}{\xi^4 L^2}, \quad v^2 = 2\xi^2 L, \quad L = \frac{2}{N_g} \left( \frac{1}{g_2} + \frac{1}{g_1} \right) \text{tr}_g(M_1^\dagger M_1),$$

$$K = \frac{2}{N_g g_1^2} (\text{tr}_g(M_1^\dagger M_1)^2 - \frac{1}{N_g} (\text{tr}_g M_1^\dagger M_1)^2)$$

(4.10)

with tr$g$ denoting the trace in the generation space.

It follows that the symmetry breaking occurs only if $N_g > 1$ as in Ref.6). This result was first observed by Connes-Lott in connection with their reconstruction of the standard model in NCG. It should be recalled that our reconstruction has two more parameters than in Connes-Lott scheme, whence our lagrangian (4.9) is renormalizable with no constraint among the bare parameters.

As alluded to above the hypercharge assignment of leptons according to the present representation (4.2) is not phenomenologically correct: $Y = 0$ for $\psi_L$, $Y = +1$ for $\nu_R$, $Y = -1$ for $e_R$, leading to charge +1/2 for $\nu$ and −1/2 for $e$, but the hypercharge of Higgs is correctly given, $Y = 1$. To obtain the correct hypercharge assignment of leptons we simply double the chiral spinors by including the charge conjugate spinor $\psi^c$ as well

$$\Psi = \begin{pmatrix} \psi \\ \psi^c \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \psi_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L, \quad \psi_R = \begin{pmatrix} \nu_R \\ e_R \end{pmatrix}_R,$$

(4.11)

and consider the representation

$$\rho(a, b) = \begin{pmatrix} a & B & 0 \\ 0 & 0 & b_{14} \end{pmatrix} \otimes 1_{N_g} \equiv \rho(a, B; b_{12}, b_{14}).$$

(4.12)
This time the gauge transformation on the spinor $\Psi$ is not given by simply making the element $(a, b)$ in Eq. (4.12) unitary because the gauge transformation on $\psi$ determines that on $\psi^c$ by the charge conjugation. In fact, it is given by the product $\rho(a, B; b_{12}, b_{12})\rho(b^*_{12}, b^*_{12}; a^*, B^*)$ for unitary element $(a, b)$ of the flavor algebra. Although this is not an algebra representation, Connes’ generalized gauge field allows to construct gauge-invariant theory also in this case using the real structure. In what follows we shall simplify Connes’ presentation\(^3\) in the lagrangian formalism.

Since the additional 4-dimensional submatrix in Eq. (4.12) is proportional to the unit matrix, there is no change in the Higgs sector\(^\ast\). The fact that only the representation like $B$ or $b_{14}$ occurs implies the quantization of $U(1)$ charges to be $\pm 1$ in this model. Hence fractional charges can arise only from another stuff. In this case we have to replace $A$ in Eq.(4.7) with

$$A_c = -\frac{ig_1}{2} A^{(1)}_{\mu} 1_{4} \otimes 1_{N_g},$$ \hspace{1cm} (4.13)

Using the elementary formula $\bar{\psi}^c A^c \psi^c = \bar{\psi} A^c \psi$, where $A^c$ is complex conjugate to $A$, we obtain the following Dirac lagrangian instead of Eq.(4.4)

$$L_D = \bar{\psi}_L i \gamma^\mu (\partial_\mu - ig_2 A^{(2)}_{\mu}) - \bar{\psi}_R i \gamma^\mu (\partial_\mu - ig_1 A^{(1)}_{\mu}) \psi_L + \bar{\psi}_R i \gamma^\mu (\partial_\mu - ig_1 A^{(1)}_{\mu}) \psi_R + \text{Yukawa-terms.}$$ \hspace{1cm} (4.14)

We thus obtain the correct hypercharge assignment(delete $\nu_{e R}$ from Eq.(4.14) if necessary). The bosonic sector is untouched in this process because the hypercharge assignment of Higgs doublet is the same as before and no additional gauge bosons appear.

\section*{5. Summary}

The Weinberg-Salam theory in the previous section is applicable only to leptons because $U(1)$ charge is quantized to be $\pm 1$. Quarks have fractional $U(1)$ charges and can only be incorporated into the scheme by taking account of color.

The color-flavor algebra\(^3\),\(^4\) is $A = C^\infty (M_4) \otimes (H \oplus C \oplus M_3(C))$, where $M_3(C)$ is the set of $3 \times 3$ complex matrices, which is represented on the doubled spinor (4.11) with

$$\psi_L = \begin{pmatrix} q_L \\ l_L \end{pmatrix}, \hspace{0.5cm} \psi_R = \begin{pmatrix} u_R \\ d_R \\ \nu_R \\ e_R \end{pmatrix}, \hspace{0.5cm} q_L = \begin{pmatrix} u \\ U_q d \end{pmatrix}, \hspace{0.5cm} l_L = \begin{pmatrix} \nu \\ U_{l e} \end{pmatrix}$$ \hspace{1cm} (5.1)

($U_q$ being the Kobayashi-Maskawa matrix) as\(^3\),\(^4\)

$$\rho(a, b, c) = \begin{pmatrix} \rho_u(a, b) & 0 \\ 0 & \rho_s(b, c) \end{pmatrix} \otimes 1_{N_g},$$

\hspace{1cm} \footnote{Conversely, there must be no change in the Higgs sector because the hypercharge is already correct, $Y = 1$ for Higgs. This necessarily implies that the additional 4-dimensional matrix is proportional to the unit matrix $b_{14}$ or $b^*_{14}$. What determines $b$ or $b^*$ is the requirement that $\nu_{e R}$ be $U(1)$-neutral, leading to $b$ for $B$ of Eq. (4.12).}
\[ \rho_w(a, b) = \begin{pmatrix} a_1 & 0 \\ 0 & B_1 \end{pmatrix}, \]

\[ \rho_s(b, c) = \begin{pmatrix} 1_2 \otimes c^* & 0 \\ b_1 & 1_2 \otimes c^* \\ 0 & b_1 \end{pmatrix}, \quad (5.2) \]

where \( c = c(x) \in C^\infty(M_4) \otimes M_3(C) \) and \((a, b)\) is the same as before. The hypercharges of the leptons remain unchanged, while those of quarks receive also from the phase of \( 3 \times 3 \) complex matrix \( c \) defined at each space-time point. One can assume \( \det \rho(a, b, c) = \det \rho_s(b, c) = 1 \) for unitary element \((a, b, c)\) of the color-flavor algebra in accordance with Connes’ unimodularity condition \([1, 3]\), which implies that the hypercharge coming from \( c \) is \( 1/3 \). Looking at the representations \([1, 13]\) as well as (5.2) the hypercharges of quarks are given by the sum of those of the corresponding leptons with \( 1 + 1/3 = 4/3 \). This gives the correct hypercharge assignment of quarks. For instance, \( Y = -1 + 4/3 = 1/3 \) for \( q_L \), \( Y = 0 + 4/3 = 4/3 \) for \( u_R \) and \( Y = -2 + 4/3 = -2/3 \) for \( d_R \), thereby assuring the anomaly cancellation in each generation.

We shall no longer dwell upon the details of the standard model in the present scheme since it has repeatedly been discussed in the literature. See, for instance, Refs. 4) and 7). We have to concede, however, that we have no quantitative idea yet about what the auxiliary objects \( a_i \) represent physically.

The present paper has introduced an elementary method of reformulating Connes’ gauge theory from the new style of symmetry principle in the lagrangian formalism. Using the present method, one can apply Connes’ gauge theory to the particle models without resort to NCG. This would greatly facilitate the model building along this line of thought. As an application we shall discuss gravity in the following paper.

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