MODIFIED VERTEX FOLKMAN NUMBERS

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Abstract. Let $a_1, \ldots, a_s$ be positive integers. For a graph $G$ the expression
$G \rightarrow (a_1, \ldots, a_s)$
means that for every coloring of the vertices of $G$ in $s$ colors ($s$-coloring) there exists $i \in \{1, \ldots, s\}$, such that there is a monochromatic $a_i$-clique of color $i$. If $m$ and $p$ are positive integers, then
$G \rightarrow m|_p$
means that for arbitrary positive integers $a_1, \ldots, a_s$ ($s$ is not fixed), such that
$\sum_{i=1}^{s} (a_i - 1) + 1 = m \frac{\max\{a_1, \ldots, a_s\}}{m}$
we have $G \rightarrow (a_1, \ldots, a_s)$. Let
$\tilde{H}(m|_p; q) = \{G : G \rightarrow m|_p \text{ and } \omega(G) < q\}$.
The modified vertex Folkman numbers are defined by the equality
$\tilde{F}(m|_p; q) = \min\{|V(G)| : G \in \tilde{H}(m|_p; q)\}$.

If $q \geq m$ these numbers are known and they are easy to compute. In the case $q = m - 1$ we know all of the numbers when $p \leq 5$. In this work we consider the next unknown case $p = 6$ and we prove with the help of a computer that
$\tilde{F}(m|_6; m - 1) = m + 10$.

1. Introduction

In this paper only finite, non-oriented graphs without loops and multiple edges are considered. The following notations are used:
$V(G)$ - the vertex set of $G$;
$E(G)$ - the edge set of $G$;
$\overline{G}$ - the complement of $G$;
$\omega(G)$ - the clique number of $G$;
$\alpha(G)$ - the independence number of $G$;
$\chi(G)$ - the chromatic number of $G$;
$N(v), N_G(v), v \in V(G)$ - the set of all vertices of $G$ adjacent to $v$;
d$(v), v \in V(G)$ - the degree of the vertex $v$, i.e. $d(v) = |N(v)|$;
$G - v, v \in V(G)$ - subgraph of $G$ obtained from $G$ by deleting the vertex $v$ and all edges incident to $v$;
$G - e, e \in E(G)$ - subgraph of $G$ obtained from $G$ by deleting the edge $e$;
$G + e, e \in E(\overline{G})$ - supergraph of $G$ obtained by adding the edge $e$ to $E(G)$.
$K_n$ - complete graph on $n$ vertices;
$C_n$ - simple cycle on $n$ vertices;
$m_0 = m_0(p)$ - see Theorem 2.1.
$G_1 + G_2$ - a graph $G$ for which: $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y] : x \in V(G_1), y \in V(G_2)\}$, i.e. $G$ is obtained by connecting every vertex of $G_1$ to every vertex of $G_2$.

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All undefined terms can be found in [18].

Let \( a_1, \ldots, a_s \) be positive integers. The expression \( G \stackrel{v}{\rightarrow} (a_1, \ldots, a_s) \) means that for any coloring of \( V(G) \) in \( s \) colors (\( s \)-coloring) there exists \( i \in \{1, \ldots, s\} \) such that there is a monochromatic \( a_i \)-clique of color \( i \). In particular, \( G \stackrel{v}{\rightarrow} (a_1) \) means that \( \omega(G) \geq a_1 \).

Define:
\[
\mathcal{H}(a_1, \ldots, a_s; q) = \left\{ G : G \stackrel{v}{\rightarrow} (a_1, \ldots, a_s) \text{ and } \omega(G) < q \right\}.
\]
\[
\mathcal{H}(a_1, \ldots, a_s; q; n) = \left\{ G : G \in \mathcal{H}(a_1, \ldots, a_s; q) \text{ and } |V(G)| = n \right\}.
\]

The vertex Folkman number \( F_v(a_1, \ldots, a_s; q) \) is defined by the equality:
\[
F_v(a_1, \ldots, a_s; q) = \min \{|V(G)| : G \in \mathcal{H}(a_1, \ldots, a_s; q)\}.
\]

Folkman proves in [5] that:
\[
(1.1) \quad F_v(a_1, \ldots, a_s; q) \text{ exists } \iff q > \max \{a_1, \ldots, a_s\}.
\]

Other proofs of (1.1) are given in [4] and [9].

In [10] for arbitrary positive integers \( a_1, \ldots, a_s \) the following are defined
\[
(1.2) \quad m(a_1, \ldots, a_s) = m = \sum_{i=1}^{s} (a_i - 1) + 1 \quad \text{ and } \quad p = \max \{a_1, \ldots, a_s\}.
\]

Obviously, \( K_m \stackrel{v}{\rightarrow} (a_1, \ldots, a_s) \) and \( K_{m-1} \stackrel{v}{\rightarrow} (a_1, \ldots, a_s) \). Therefore,
\[
F_v(a_1, \ldots, a_s; q) = m, \quad q \geq m + 1.
\]

The following theorem for the numbers \( F_v(a_1, \ldots, a_s; m) \) is true:

**Theorem 1.1.** Let \( a_1, \ldots, a_s \) be positive integers and \( m \) and \( p \) are defined by (1.2). If \( m \geq p + 1 \), then:

(a) \( F_v(a_1, \ldots, a_s; m) = m + p \). [10], [9].

(b) \( K_{m+p} - C_{2p+1} = K_{m-p-1} + C_{2p+1} \)

is the only extremal graph in \( \mathcal{H}(a_1, \ldots, a_s; m) \). [9].

The condition \( m \geq p + 1 \) is necessary according to (1.1). Other proofs of Theorem 1.1 are given in [12] and [13].

Very little is known about the numbers \( F_v(a_1, \ldots, a_s; q) \), \( q \leq m - 1 \). In this work we suggest a method to bound these numbers with the help of the modified vertex Folkman numbers \( \tilde{F}_v(m|_p; q) \), which are defined below.

**Definition 1.2.** Let \( G \) be a graph and \( m \) and \( p \) be positive integers. The expression
\[
G \stackrel{v}{\rightarrow} m|_p
\]
means that for any choice of positive integers \( a_1, \ldots, a_s \) (\( s \) is not fixed), such that
\[
m = \sum_{i=1}^{s} (a_i - 1) + 1 \quad \text{and} \quad \max \{a_1, \ldots, a_s\} \leq p,
\]
we have
\[
G \stackrel{v}{\rightarrow} (a_1, \ldots, a_s).
\]

Define:
\[
\tilde{\mathcal{H}}(m|_p; q) = \left\{ G : G \stackrel{v}{\rightarrow} m|_p \text{ and } \omega(G) < q \right\}.
\]
\[
\tilde{\mathcal{H}}(m|_p; q; n) = \left\{ G : G \in \tilde{\mathcal{H}}(m|_p; q) \text{ and } |V(G)| = n \right\}.
\]
The modified vertex Folkman numbers are defined by the equality:

\[ \tilde{F}_v(m_p|q) = \min \left\{ |V(G)| : G \in \tilde{H}(m_p|q) \right\}. \]

The graph \( G \) is called a maximal graph in \( \tilde{H}(m_p|q) \) if \( G \in \tilde{H}(m_p|q) \), but \( G + e \not\in \tilde{H}(m_p|q), \forall e \in E(G) \), i.e. \( \omega(G + e) \geq q, \forall e \in E(G) \).

For convenience, we will also define the following term:

**Definition 1.3.** The graph \( G \) is called a \((+K_t)\)-graph if \( G + e \) contains a new \( t \)-clique for all \( e \in E(G) \).

Obviously, \( G \in \tilde{H}(m_p|q) \) is a maximal graph in \( \tilde{H}(m_p|q) \) if and only if \( G \) is a \((+K_q)\)-graph.

From the definition of the modified Folkman numbers, it becomes clear that if \( a_1, \ldots, a_s \) are positive integers and \( m \) and \( p \) are defined by (1.2), then

\[ F_v(a_1, \ldots, a_s; q) \leq \tilde{F}_v(m_p|q). \]

Defining and computing the modified Folkman numbers is appropriate because of the following reasons:

1) On the left side of (1.3), there is actually a whole class of numbers, which are bound by only one number \( \tilde{F}_v(m_p|q) \).

2) The upper bound for \( \tilde{F}_v(m_p|q) \) is easier to compute than the numbers \( F_v(a_1, \ldots, a_s) \) because of the following

**Theorem 1.4.** ([1]), Theorem 7.2) Let \( m, m_0, p \) and \( q \) be positive integers, \( m \geq m_0 \) and \( q > \min \{m_0, p\} \). Then

\[ \tilde{F}_v(m_p|m_0 + q) \leq \tilde{F}_v(m_0_p|q) + m - m_0. \]

Therefore, if we know the value of one number \( \tilde{F}_v(m_p|q) \) we can obtain an upper bound for \( \tilde{F}_v(m_p|q) \) where \( m \geq m' \).

3) As we will see below (Theorem 2.1), the computation of the numbers \( \tilde{F}_v(m_p|m - 1) \) is reduced to finding the exact values of the first several of these numbers (bounds for the number of exact values needed are given in 2.1 (c)).

Let \( A \) be an independent set of vertices in \( G \). If \( V_1 \cup \ldots \cup V_s \) is \((a_1, \ldots, a_s)\)-free \( s \)-coloring of \( V(G - A) \) (i.e. \( V_i \) does not contain an \( a_i \)-clique, \( i = 1, \ldots, s \)), then \( A \cup V_1 \cup \ldots \cup V_s \) is \((2, a_1, \ldots, a_s)\)-free \((s + 1)\)-coloring of \( V(G) \). Therefore

\[ G \rightarrow (2, a_1, \ldots, a_s) \Rightarrow G - A \rightarrow (a_1, \ldots, a_s). \]

Further we will need the following

**Proposition 1.5.** Let \( G \rightarrow (m_p|m) \) and \( A \) is an independent set of vertices in \( G \). Then \( G - A \rightarrow (m - 1)|m \).

**Proof.** Let \( a_1, \ldots, a_s \) be positive integers, such that

\[ m - 1 = \sum_{i=1}^{s} (a_i - 1) + 1 \quad \text{and} \quad 2 \leq a_i \leq p. \]

Then

\[ m = (2 - 1) + \sum_{i=1}^{s} (a_i - 1) + 1. \]

It follows that \( G \rightarrow (2, a_1, \ldots, a_s) \) and from (1.4) we obtain \( G - A \rightarrow (a_1, \ldots, a_s). \) □
It is easy to see that if \( q > m \), then \( F_v(a_1, ..., a_s; q) = \tilde{F}_v(m_{|p}; q) = m \). From Theorem 1.1 it follows that \( F_v(a_1, ..., a_s; m) = \tilde{F}_v(m_{|p}; m) = m + p \). In the case \( q = m - 1 \) the following general bounds are known:

\[
(1.5) \quad m + p + 2 \leq \tilde{F}_v(m_{|p}; m - 1) \leq m + 3p, \quad m \geq p + 2.
\]

The upper bound follows from the proof of the Main Theorem from [7] and the lower bound follows from (1.3) and \( F_v(a_1, ..., a_s; q) \geq m + p + 2, \quad [12] \).

We know all the numbers \( \tilde{F}_v(m_{|p}; m - 1) \) where \( p \leq 5 \) (in the cases \( p \leq 4 \) see the Remark after Theorem 4.5 and (1.5) from [1], and in the case \( p = 5 \) see Theorem 7.4 also from [1]). It is also known that

\[
m + 9 \leq \tilde{F}_v(m_{|p}; m - 1) \leq m + 10, \quad [1]
\]

In this work we complete the computation of the numbers \( \tilde{F}_v(m_{|p}; m - 1) \) by proving

**Main Theorem 1.** \( \tilde{F}_v(m_{|p}; m - 1) = m + 10, \quad m \geq 8 \).

2. A theorem for the numbers \( \tilde{F}_v(m_{|p}; m - 1) \)

We will need the following fact:

\[
(2.1) \quad G \rightarrow (a_1, ..., a_s) \Rightarrow \chi(G) \geq m, \quad [13] \quad (\text{see also [1]}).
\]

It is easy to prove (see Proposition 4.4 from [1]) that

\[
(2.2) \quad \tilde{F}_v(m_{|p}; m - 1) \exists \quad m \geq p + 2.
\]

In [1] (version 1) we formulate without proof the following

**Theorem 2.1.** Let \( m_0(p) = m_0 \) be the smallest positive integer for which

\[
\min_{m \geq p+2} \{ \tilde{F}_v(m_{|p}; m - 1) - m \} = \tilde{F}_v(m_0_{|p}; m_0 - 1) - m_0.
\]

Then:

(a) \( \tilde{F}_v(m_{|p}; m - 1) = \tilde{F}_v(m_0_{|p}; m_0 - 1) + m - m_0, \quad m \geq m_0 \).

(b) if \( m_0 > p + 2 \) and \( G \) is an extremal graph in \( \tilde{H}(m_0_{|p}; m_0 - 1) \), then

\( G \rightarrow (2, m_0 - 2) \).

(c) \( m_0 < \tilde{F}_v((p+2)_{|p}; p + 1) - p \).

In this section we present a proof of Theorem 2.1.

The condition \( m \geq p + 2 \) is necessary according to (2.2).

**Proof.** (a) According to the definition of \( m_0(p) = m_0 \) we have

\[
\tilde{F}_v(m_{|p}; m - 1) \geq \tilde{F}_v(m_0_{|p}; m_0 - 1) + m - m_0, \quad m \geq p + 2.
\]

According to Theorem 1.4 if \( m \geq m_0 \) the opposite inequality is also true.

(b) Assume the opposite is true and let

\[
V(G) = V_1 \cup V_2, \quad V_1 \cap V_2 = \emptyset,
\]

where \( V_1 \) is an independent set and \( V_2 \) does not contain an \((m_0 - 2)\)-clique. Let \( G_1 = G[V_2] = G - V_1 \). According to Proposition 1.5, from \( G \rightarrow m_0_{|p} \) it follows

\[
G_1 \rightarrow (m_0 - 1)_{|p}. \quad \text{Since} \quad \omega(G_1) < m_0 - 2, \quad G_1 \in \tilde{H}((m_0 - 1)_{|p}; m_0 - 2). \quad \text{Therefore}
\]

\[
|V(G)| - 1 \geq |V(G_1)| \geq \tilde{F}_v((m_0 - 1)_{|p}; m_0 - 2).
\]

Since \( |V(G)| = \tilde{F}_v(m_0_{|p}; m_0 - 1) \), from these inequalities it follows that
From these inequalities the inequality (c) follows easily.

Since

Therefore

The following proposition for maximal graphs in $\tilde{\mathcal{H}}(m_0; m_0 - 1)$, with the help of the computer, results for Folkman numbers are obtained in [14] (see Algorithm 1). Similar algorithms are used in [1], [2], [19], [8], [15].

Also with the help of a computer. The remaining graphs in this set can be obtained by

and therefore in this case the inequality (c) is true.

Let $m_0 > p + 2$ and $G$ be an extremal graph in $\tilde{\mathcal{H}}(m_0; m_0 - 1)$. If $a_1, \ldots, a_s$ are positive integers, such that $m = \sum_{i=1}^{s} (a_i - 1) + 1$ and $\max \{a_1, \ldots, a_s\} \leq p$, then $G \rightarrow^a (a_1, \ldots, a_s)$ and according to (2.1), $\chi(G) \geq m_0$. From (b) and Theorem 1.1 we see that $|V(G)| \geq 2m_0 - 3$ and $|V(G)| = 2m_0 - 3$ only if $G = \overline{C_{2m_0-3}}$. However, the last equality is not possible because $\chi(G) \geq m_0$ and $\chi(\overline{C_{2m_0-3}}) = m_0 - 1$.

Therefore

Since $m_0 > p + 2$ from the definition of $m_0$ we have

$F_{\varepsilon}(m_0; m_0 - 1) - m_0 < F_{\varepsilon}((p + 2); p + 1) - p - 2$.

From these inequalities the inequality (c) follows easily. 

3. Algorithms

In this section we present algorithms for finding all maximal graphs in $\tilde{\mathcal{H}}(m_0; q; n)$ with the help of a computer. The remaining graphs in this set can be obtained by removing edges from the maximal graphs. The idea for these algorithms comes from [13] (see Algorithm 1). Similar algorithms are used in [1], [2], [19], [8], [15].

Also with the help of the computer, results for Folkman numbers are obtained in [6], [17], [16] and [3].

The following proposition for maximal graphs in $\tilde{\mathcal{H}}(m_0; q; n)$ will be useful

**Proposition 3.1.** Let $G$ be a maximal graph in $\tilde{\mathcal{H}}(m_0; q; n)$. Let $v_1, v_2, \ldots, v_k$ be independent vertices of $G$ and $H = G - \{v_1, v_2, \ldots, v_k\}$. Then:

(a) $H \in \tilde{\mathcal{H}}((m-1); q; n-k)$

(b) $H$ is a $(+K_{q-1})$-graph

(c) $N_G(v_i)$ is a maximal $K_{q-1}$-free subset of $V(H)$, $i = 1, \ldots, k$

**Proof.** The proposition (a) follows from Proposition 1.1 (b) and (c) follow from the maximality of $G$. 

We will define an algorithm, which is based on Proposition 3.1 and generates all maximal graphs in $\tilde{\mathcal{H}}(m_0; q; n)$ with independence number at least $k$.

**Algorithm 3.2.** Finding all maximal graphs in $\tilde{\mathcal{H}}(m_0; q; n)$ with independence number at least $k$ by adding $k$ independent vertices to the $(+K_{q-1})$-graphs in $\tilde{\mathcal{H}}((m-1); q; n-k)$.

1. Denote by $\mathcal{A}$ the set of all $(+K_{q-1})$-graphs in $\tilde{\mathcal{H}}((m-1); q; n-k)$. The obtained maximal graphs in $\tilde{\mathcal{H}}(m_0; q; n)$ will be output in $\mathcal{B}$, let $\mathcal{B} = \emptyset$.

2. For each graph $H \in \mathcal{A}$:

2.1. Find the family $\mathcal{M}(H) = \{M_1, \ldots, M_t\}$ of all maximal $K_{q-1}$-free subsets of $V(H)$.

2.2. Consider all the $k$-tuples $(M_{i_1}, M_{i_2}, \ldots, M_{i_k})$ of elements of $\mathcal{M}(H)$, for which $1 \leq i_1 \leq \ldots \leq i_k \leq t$ (in these $k$-tuples some subsets $M_i$ can coincide). For every such $k$-tuple construct the graph $G = G(M_{i_1}, M_{i_2}, \ldots, M_{i_k})$ by adding to
V(H) new independent vertices \(v_1, v_2, \ldots, v_k\), so that \(N_G(v_j) = M_{ij}, j = 1, \ldots, k\) (see Proposition 3.1(c)). If \(\omega(G + e) = q, \forall e \in E(\overline{G})\), then add \(G\) to \(B\).

3. Exclude the isomorph copies of graphs from \(B\).

4. Exclude from \(B\) all graphs which are not in \(\hat{H}(m_i, q; n)\).

**Theorem 3.3.** Upon completion of Algorithm 3.2 the obtained set \(B\) is equal to the set of all maximal graphs in \(\hat{H}(m_i, q; n)\) with independence number at least \(k\).

**Proof.** From step 4 we see that \(B \subseteq \hat{H}(m_i, q; n)\) and from step 2.2 it becomes clear, that \(B\) contains only maximal graphs in \(\hat{H}(m_i, q; n)\) with independence number at least \(k\). Let \(G\) be an arbitrary maximal graph in \(\hat{H}(m_i, q; n)\) with independence number in \(k\). We will prove that \(G \in B\). Let \(v_1, \ldots, v_k\) be independent vertices of \(G\) and \(H = G - \{v_1, \ldots, v_k\}\). According to Proposition 3.1(a) and (b), \(H \in \hat{H}((m - 1)i, q; n - k)\) and \(H\) is a \((+K_{q-1})\)-graph. Therefore in step 1 we have \(H \in A\). According to Proposition 3.1(c), \(N_G(v_i) \in M(H)\) for all \(i \in \{1, \ldots, k\}\), hence in step 2 \(G\) is added to \(B\).

Let us note that if \(G \in \hat{H}(m_i, q; n)\) and \(n \geq q\), then \(G \neq K_n\) and therefore \(\alpha(G) \geq 2\). In this case, with the help of Algorithm 3.2 we can obtain all maximal graphs in \(\hat{H}(m_i, q; n)\) by adding to independent vertices to the \((+K_{q-1})\)-graphs in \(\hat{H}((m - 1)i, q; n - 2)\).

It is clear that if \(G\) is a graph for which \(\alpha(G) = 2\) and \(H\) is a subgraph of \(G\) obtained by removing independent vertices, then \(\alpha(H) \leq 2\). We modify Algorithm 3.2 in the following way to obtain the maximal graphs in \(\hat{H}(m_i, q; n)\) with independence number 2:

**Algorithm 3.4.** A modification of Algorithm 3.2 for finding all maximal graphs in \(\hat{H}(m_i, q; n)\) with independence number 2 by adding 2 independent vertices to the \((+K_{q-1})\)-graphs in \(\hat{H}((m - 1)i, q; n - 2)\) with independence number not greater than 2.

In step 1 of Algorithm 3.2 we add the condition that the set \(A\) contains only the \((+K_{q-1})\)-graphs \(\hat{H}((m - 1)i, q; n - k)\) with independence number not greater than \(2\), and at the end of step 2.2 after the condition \(\omega(G + e) = q, \forall e \in E(\overline{G})\) we also add the condition \(\alpha(G) = 2\).

Thus, finding all maximal graphs in \(\hat{H}(m_i, q; n)\) with independence number 2 is reduced to finding all \((+K_{q-1})\)-graphs with independence number not greater than \(2\) in \(\hat{H}((m - 1)i, q; n - 2)\) and finding the remaining maximal graphs in \(\hat{H}(m_i, q; n)\) with independence number greater than or equal to 3 is reduced to finding all \((+K_{q-1})\)-graphs in \(\hat{H}((m - 1)i, q; n - 3)\). In this way we can obtain all maximal graphs in \(\hat{H}(m_i, q; n)\) in steps, starting from graphs with a small number of vertices.

The **nauty** programs [11] have an important role in this work. We use them for fast generation of non-isomorphic graphs and for graph isomorphism rejection.

4. **Computation of the number \(\tilde{F}_6(8i, 7)\)**

From Theorem 2.1 it becomes clear that in order to compute the numbers \(\tilde{F}_6(m_i, m - 1)\) we need the exact value of the number \(m_i(6)\). According to Theorem 2.1(c), to obtain an upper bound for this number we need to know \(\tilde{F}_6(8i, 7)\). In this section we compute this number by proving the following
Theorem 4.1. $\tilde{F}_v(8|_6;7) = 18$.

Proof. The inequality $\tilde{F}_v(8|_6;7) \leq 18$ is proved in [1] with the help of the graph $\Gamma_1$ which is given on Figure 1 (see the proof of Theorem 1.10 in version 1 or the proof of Theorem 1.9 in version 2). To obtain the lower bound we will prove with the help of a computer that $\tilde{H}(8|_6;7;17) = \emptyset$.

First, we search for maximal graphs in $\tilde{H}(8|_6;7;17)$ with independence number greater than 2. It is clear that $K_6$ and $K_6 - e$ are the only $(+K_6)$-graphs in $\tilde{H}(3|_6;7;6)$. With the help of Algorithm 3.2 we add 2 independent vertices to these graphs to find all maximal graphs in $\tilde{H}(4|_6;7;8)$. By removing edges from them we find all $(+K_6)$-graphs in $\tilde{H}(4|_6;7;8)$. In the same way, we successively obtain all maximal and all $(+K_6)$-graphs in the sets:

$\tilde{H}(5|_6;7;10), \tilde{H}(6|_6;7;12), \tilde{H}(7|_6;7;14)$.

In the end, with the help of Algorithm 3.2 we add 3 independent vertices to the obtained $(+K_6)$-graphs in $\tilde{H}(7|_6;7;14)$ to find all maximal graphs in $\tilde{H}(8|_6;7;17)$ with independence number greater than 2.

After that, we search for maximal graphs in $\tilde{H}(8|_6;7;17)$ with independence number 2. It is clear that $K_5$ is the only $(+K_6)$-graph in $\tilde{H}(2|_6;7;5)$. With the help of Algorithm 3.2 we add 2 independent vertices this graph to find all maximal graphs in $\tilde{H}(3|_6;7;7)$ with independence number 2. By removing edges from them we find all $(+K_6)$-graphs in $\tilde{H}(3|_6;7;7)$ with independence number 2. In the same way, we successively obtain all maximal and all $(+K_6)$-graphs with independence number 2 in the sets:

$\tilde{H}(4|_6;7;9), \tilde{H}(5|_6;7;11), \tilde{H}(6|_6;7;13), \tilde{H}(7|_6;7;15)$ and $\tilde{H}(8|_6;7;17)$. 

Figure 1. Graph $\Gamma_1 \in \tilde{H}(8|_6;7;18)$
The number of graphs found in each step is described in Table 1 in [1]. In both cases we do not obtain any maximal graphs in \( \mathcal{H}(8|_6; 7; 17) \), therefore \( \tilde{\mathcal{H}}(8|_6; 7; 17) = \emptyset \).

**Corollary 4.2.** \( 8 \leq m_0(6) \leq 11 \)

**Proof.** The inequality \( m_0(6) \geq 8 \) follows from the definition of \( m_0 \) and the upper bound follows from Theorem 2.1(c), \( p = 6 \). □

5. Proof of the Main Theorem

Since \( \tilde{F}_v(8|_6; 7) = 18 \), according to Theorem 2.1(a) it is enough to prove \( m_0(6) = 8 \). According to Corollary 4.2, this equality will be proved if we prove \( \tilde{F}_v(9|_6; 8) > 18 \), \( \tilde{F}_v(10|_6; 9) > 19 \) and \( \tilde{F}_v(11|_6; 10) > 20 \). The proof of these inequalities is similar to the proof of \( \tilde{F}_v(8|_6; 7) > 17 \) from Theorem 4.1. It is clear that it is enough to prove \( \tilde{F}(m|_6; m - 1; m + 9) = 0 \) for \( m = 9, 10, 11 \).

First, we search for maximal graphs in \( \mathcal{H}(m|_6; m - 1; m + 9) \) with independence number greater than 2. It is clear that \( K_{m-2} \) and \( K_{m-2} - e \) are the only (+\( K_{m-2} \))-graphs in \( \mathcal{H}(m-5|_6; m - 1; m - 2) \). With the help of Algorithm 3.2 we successively obtain all maximal graphs in \( \tilde{\mathcal{H}}(m|_6; m - 1; m + 6) \).

In the end, with the help of Algorithm 4.2 we add 3 independent vertices to the obtained (+\( K_{m-2} \))-graphs in \( \mathcal{H}(m - 1|_6; m - 1; m + 6) \) to find all maximal graphs in \( \tilde{\mathcal{H}}(m|_6; m - 1; m + 9) \) with independence number greater than 2.

After that, we search for maximal graphs in \( \mathcal{H}(m|_6; m - 1; m + 9) \) with independence number 2. It is clear that \( K_{m-3} \) is the only (+\( K_{m-2} \))-graph in \( \mathcal{H}(m - 6|_6; m - 1; m - 3) \). With the help of Algorithm 5.4 we successively obtain all maximal and all (+\( K_{m-2} \))-graphs with independence number 2 in the sets:

\[
\mathcal{H}(m - 5|_6; m - 1; m - 1) \\
\mathcal{H}(m - 4|_6; m - 1; m + 1) \\
\mathcal{H}(m - 3|_6; m - 1; m + 3) \\
\mathcal{H}(m - 2|_6; m - 1; m + 5) \\
\mathcal{H}(m - 1|_6; m - 1; m + 7) \\
\mathcal{H}(m|_6; m - 1; m + 9).
\]

The number of graphs found in each step is given in Table 2, Table 3 and Table 4 in [1]. In both cases we do not obtain any maximal graphs in the sets \( \tilde{\mathcal{H}}(m|_6; m - 1; m + 9) \), \( m = 9, 10, 11 \), hence it follows \( \tilde{F}_v(9|_6; 8) > 18 \), \( \tilde{F}_v(10|_6; 9) > 19 \), \( \tilde{F}_v(11|_6; 10) > 20 \) and \( m_0(6) = 8 \). Thus we finish the proof of the Main Theorem.

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**Appendix A. Results of the computations**

| set          | independence number | maximal graphs | (+$K_6$)-graphs |
|--------------|---------------------|----------------|-----------------|
| $\mathcal{H}(3)_{6;7;6}$ | -           | 2             | 2               |
| $\mathcal{H}(4)_{6;7;8}$ | -           | 8             | 13              |
| $\mathcal{H}(5)_{6;7;10}$ | -           | 56            | 324             |
| $\mathcal{H}(6)_{6;7;12}$ | -           | 18            | 104 271         |
| $\mathcal{H}(7)_{6;7;14}$ | $\geq 3$     | 0             | 1825            |
| $\mathcal{H}(8)_{6;7;17}$ | -           | 0             |                 |

Table 1. Steps in the search of all maximal graphs in $\mathcal{H}(8)_{6;7;17}$

| set          | independence number | maximal graphs | (+$K_7$)-graphs |
|--------------|---------------------|----------------|-----------------|
| $\mathcal{H}(4)_{6;8;7}$ | -           | 2             | 2               |
| $\mathcal{H}(5)_{6;8;9}$ | -           | 8             | 13              |
| $\mathcal{H}(6)_{6;8;11}$ | -           | 56            | 326             |
| $\mathcal{H}(7)_{6;8;13}$ | -           | 20            | 105 125         |
| $\mathcal{H}(8)_{6;8;15}$ | $\geq 3$     | 0             | 1844            |
| $\mathcal{H}(9)_{6;8;18}$ | = 2         | 1             |                 |
| $\mathcal{H}(3)_{6;8;6}$ | $\leq 2$     | 2             |                 |
| $\mathcal{H}(4)_{6;8;8}$ | = 2         | 2             |                 |
| $\mathcal{H}(5)_{6;8;10}$ | = 2         | 5             |                 |
| $\mathcal{H}(6)_{6;8;12}$ | = 2         | 25            |                 |
| $\mathcal{H}(7)_{6;8;14}$ | = 2         | 506           |                 |
| $\mathcal{H}(8)_{6;8;16}$ | = 2         | 0             |                 |
| $\mathcal{H}(9)_{6;8;18}$ | -           | 0             |                 |

Table 2. Steps in the search of all maximal graphs in $\mathcal{H}(9)_{6;8;18}$
Table 3. Steps in the search of all maximal graphs in $\tilde{H}(10|_6; 9; 19)$

| set       | independence number | maximal graphs | $(+K_8)$-graphs |
|-----------|---------------------|----------------|-----------------|
| $\tilde{H}(5|_6; 9; 8)$ | -                   | 2              | 13              |
| $\tilde{H}(6|_6; 9; 10)$  | -                   | 8              | 327             |
| $\tilde{H}(7|_6; 9; 12)$  | -                   | 56             | 105 281         |
| $\tilde{H}(8|_6; 9; 14)$  | $\geq 3$            | 20             | 1845            |
| $\tilde{H}(9|_6; 9; 16)$  | -                   | 0              |                 |
| $\tilde{H}(10|_6; 9; 19)$ | -                   | 0              |                 |

Table 4. Steps in the search of all maximal graphs in $\tilde{H}(11|_6; 10; 20)$

| set       | independence number | maximal graphs | $(+K_8)$-graphs |
|-----------|---------------------|----------------|-----------------|
| $\tilde{H}(6|_6; 10; 9)$ | -                   | 2              | 2               |
| $\tilde{H}(7|_6; 10; 11)$ | -                   | 8              | 327             |
| $\tilde{H}(8|_6; 10; 13)$ | -                   | 56             | 105 314         |
| $\tilde{H}(9|_6; 10; 15)$ | -                   | 20             | 1845            |
| $\tilde{H}(10|_6; 10; 17)$ | $\geq 3$            | 0              |                 |
| $\tilde{H}(11|_6; 10; 20)$ | $\leq 2$            | 1              | 1               |
| $\tilde{H}(6|_6; 10; 8)$ | = 2                 | 2              | 13              |
| $\tilde{H}(7|_6; 10; 10)$ | = 2                 | 5              | 498             |
| $\tilde{H}(8|_6; 10; 12)$ | = 2                 | 56             | 121 863         |
| $\tilde{H}(9|_6; 10; 14)$ | = 2                 | 20             | 2 749 171       |
| $\tilde{H}(10|_6; 10; 16)$ | = 2                 | 0              |                 |
| $\tilde{H}(11|_6; 10; 20)$ | -                   | 0              |                 |