Enumeration on Trees under Relabelings

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Abstract

We study how to evaluate MSO queries with free variables on trees, within the framework of enumeration algorithms. Previous work has shown how to enumerate answers with linear-time preprocessing and delay linear in the size of each output, i.e., constant-delay for free first-order variables. We extend this result to support relabelings, a restricted kind of update operations on trees which allows us to change the node labels. Our main result shows that we can enumerate the answers of MSO queries on trees with linear-time preprocessing and linear delay, while supporting node relabelings in logarithmic time. To prove this, we reuse the circuit-based enumeration structure from our earlier work, and develop techniques to maintain its index under node relabelings. We also show how enumeration under relabelings can be applied to evaluate practical query languages, such as aggregate, group-by, and parameterized queries.

1 Introduction

Enumeration algorithms are a well-established way to compute large query results on databases, see e.g. the survey [27]. Instead of computing the complete set of results, these algorithms compute solutions one after the other, while ensuring that the time between two successive solutions (the delay) is as small as possible. Ideally, the delay should be linear in the size of each solution to be printed out, in particular independent of the size of the input database. To make this possible, enumeration algorithms are allowed to compute an index structure on the input database during a preprocessing phase that ideally runs in linear time.

Most enumeration algorithms assume that the input database will not change after the preprocessing. When we update the database, the costly preprocessing phase must be re-run from scratch, which is infeasible in practical settings. Losemann and Martens [23] proposed the first enumeration algorithm that addresses this issue: they study the evaluation of monadic second-order (MSO) queries on trees, and showed that it is possible to maintain an index structure for enumeration under updates. More precisely, their index can be updated in time polylogarithmic in the size of the input tree $T$ (which is much better than re-running the linear preprocessing). The tradeoff is that their enumeration delay is also polylogarithmic in $T$, whereas the delay can be independent of $T$ when there are no updates [8].

This result of [23] leads to a natural question: does the support for updates inherently increase the delay of enumeration algorithms? This is not the case with some query languages: for instance, when evaluating first-order queries (plus modulo-counting quantifiers) on bounded-degree databases, updates can be applied in constant time [9] while preserving the constant-delay bound of earlier works without updates [16, 21]. However, when evaluating conjunctive queries (CQs) on arbitrary databases, supporting updates has an inherent cost: under complexity-theoretic assumptions, the class of CQs allowing efficient enumeration under updates [10] is a strict subclass of the class of CQs for the case without updates. Could the same be true of MSO queries on trees, as [23] would suggest?

In this work, we answer this question in the negative, for a restricted update language. Specifically, we show an enumeration algorithm for MSO on trees with the same delay as in
the case without updates [8], while supporting updates with a better complexity than [23] (see detailed comparison of results in Section 3). The tradeoff is that we only allow updates that change the labels of nodes, called relabelings, unlike [23] where updates can also insert and delete leaves. We still show how these relabelings are useful to evaluate practical query languages, such as parameterized queries and group-by queries with aggregates. A parameterized query allows the user to specify some parameters for the evaluation (e.g., select some positions on the tree). Our results support such queries: we can model the parameters as labels and apply relabeling updates when the user changes the parameters. A group-by query with aggregates partitions the set of results into groups based on an attribute, and computes some aggregate quantity on each group (e.g., a sum). We show how to enumerate the results of such queries. For groups, our techniques can handle them with one single enumeration structure using relabelings to switch groups. For aggregates, we can efficiently compute and maintain them in arbitrary semirings; this problem was left open by [23] even for counting, and is practically relevant in its own right [25].

The proof of our main result follows the approach of [4] which is grounded in both the field of knowledge compilation in artificial intelligence and of factorized representations in database theory. The common idea of these two areas is to encode knowledge (in our case the query result) in the form of restricted classes of circuits, and use them for efficient reasoning. Circuits also support aggregates in a natural way [15]. In [4], we have used this circuit-based approach to recapture existing enumeration results for MSO on trees [8, 22]. In this work, we refine the approach and show that it can support updates in a natural way. Our key new ingredient are hybrid circuits: they have both set-valued gates that represent the values to enumerate, and Boolean gates that encode the tree labels which can be updated. We first show that we can efficiently compute such circuits to capture the possible results of an MSO query under all possible labellings of a tree. Second, we show how to efficiently enumerate the set of assignments captured by these circuits, also supporting updates that toggle the Boolean gates affected by a relabeling. We also introduce some standalone tools, e.g., a lemma to balance the input trees to MSO queries (Lemma 4.3), ensuring that hybrid circuits have logarithmic depth so that changes can be propagated quickly; and a constant-delay enumeration algorithm for reachability in forests under updates (Section 7).

Paper structure. We start with preliminaries in Section 2, and define our problem and give our main result in Section 3. In Section 4, we review provenance circuits in the form in which they were used in [4] to give enumeration algorithms. We also show our lemma to balance the input trees. In Section 5, we introduce hybrid circuits before showing in Section 6 how we can use these circuits for enumeration under updates, using a standalone reachability indexing scheme on forests given in Section 7. This constitutes the proof of our main result: we outline consequences of this result for application-oriented query languages in Section 8 and conclude in Section 9.

2 Preliminaries

Trees, queries, answers, assignments. In this work, unless otherwise specified, a tree is always binary, rooted, ordered, and full. Letting $\Gamma$ be a finite set called a tree alphabet, a $\Gamma$-tree $(T, \lambda)$ is a pair of a tree $T$ and of a labeling function $\lambda$ that maps each node $n$ of $T$ to a set of labels $\lambda(n) \subseteq \Gamma$. We will sometimes abuse notation and identify $T$ to its node set, e.g., write $\lambda$ as a function from $T$ to $\Gamma$; we may also omit $\lambda$ and write the $\Gamma$-tree as just $T$.

We consider queries in monadic second-order logic (MSO), the extension of first-order...
logic on the signature of $\Gamma$-trees, featuring two binary relations $E_1$ and $E_2$ denoting the first and second child of each internal node, and a unary relation $P_l$ for each $l \in \Gamma$ describing which nodes carry label $l$ (i.e., for which nodes $n$ we have $l \in \lambda(n)$). First-order logic builds formulas from atoms of this signature and from equality atoms, using the Boolean connectives and existential and universal quantification over nodes. Formulas in MSO can additionally use second-order quantification over sets of nodes, written as second-order variables. For instance, we can express with an MSO formula on $\Gamma = \{l_1, l_2, l_3\}$ that every node carrying labels $l_1$ and $l_2$ has a descendant carrying label $l_3$.

In this work, we study MSO queries, i.e., MSO formulas with free variables. The free variables can be first-order or second-order, but we can rewrite any MSO query $Q(x, Y)$ to ensure that all free variables are second-order: for instance as $Q'(X, Y) : \exists x \wedge \text{Sing}(X, x) \wedge Q(x, Y)$, where $\text{Sing}(X, x)$ asserts that $X$ is exactly the singleton $\{x\}$. Hence, we usually assume without loss of generality that MSO queries only have second-order free variables.

Given a $\Gamma$-tree $T$ and an MSO query $Q(X_1, \ldots, X_m)$, an $m$-tuple $B = B_1, \ldots, B_m$ of subsets of $T$ is an answer of $Q$ on $T$, written $T \models Q(B)$, if $T$ satisfies $Q(B)$ in the usual logical sense. It will be more convenient to represent each answer as an assignment, which is a set of pairs called singletons that indicate that an element is in the interpretation of a variable. Formally, given an $m$-tuple $B$ of subsets of $T$, the corresponding assignment is $\{\{X_i : n\} | 1 \leq i \leq m \text{ and } n \in B_i\}$. We can convert each assignment in linear time to the corresponding answer and vice-versa, so we will use the assignment representation throughout this work. Our goal is to compute the set of assignments of $Q$ on $T$, which we call the output of $Q$ on $T$; we abuse notation and write it $Q(T)$. We measure the complexity of this task in data complexity, i.e., as a function of the input tree $T$, with the query $Q$ being fixed.

**Enumeration.** The output of an MSO query can be huge, so we work in the setting of enumeration algorithms [30, 27] which we present following [4]. As usual for enumeration algorithms [27], we work in the RAM model with uniform cost measure (see, e.g., [2]), where pointers, numbers, labels for elements and facts, etc., have constant size.

An enumeration algorithm with linear-time preprocessing for a fixed MSO query $Q(X)$ on $\Gamma$-trees takes as input a $\Gamma$-tree $T$ and computes the output $Q(T)$ of $Q$ on $T$. It consists of two parts. First, the preprocessing phase takes $T$ as input and produces in linear time a data structure $J$ called the index, and an initial state $s$. Second, the enumeration phase repeatedly calls an algorithm $A$. Each call to $A$ takes as input the index $J$ and the current state $s$, and returns one assignment and a new state $s'$; a special state value indicates that the enumeration is over so $A$ should not be called again. The assignments produced by the successive calls to $A$ must be exactly the elements of $Q(T)$, with no duplicates.

We say that the enumeration algorithm has linear delay if the time to produce each new assignment $A$ is linear in its cardinality $|A|$, and is independent of $T$. In particular, if all answers to $Q$ are tuples of singletons (for instance, if $Q$ is the translation of a MSO query where all free variables are first-order), then the cardinality of each assignment is constant (it is the arity of $Q$). In this case, the enumeration algorithm must produce each assignment with constant delay: this is called constant-delay enumeration. The memory usage of an enumeration algorithm is the maximum number of memory cells used during the enumeration phase (not counting the index $J$, which resides in read-only memory), expressed as a function of the size of the largest assignment (as in [8]): we say that the enumeration algorithm has linear memory if its memory usage is linear in the size of the largest assignment.

Several previous works have studied enumeration algorithms for MSO queries on trees. Bagan [8] showed that for any fixed MSO query $Q(X)$, given a $\Gamma$-tree $T$, we can enumerate
the output $Q(T)$ of $Q$ on $T$ with linear delay and memory, amounting to constant delay and memory for MSO queries with only free first-order variables. This result was re-proven by Kazana and Segoufin [22] using a result of Colcombet [12], and a third proof via provenance circuits was recently proposed by the present authors [4].

3 Problem Statement and Main Result

Our goal in this work is to address a limitation of these existing results, namely, the assumption that the input $\Gamma$-tree $T$ will never change. Indeed, if $T$ is updated, these results must discard the index $J$ and re-run the preprocessing phase on the new tree. Our goal is to improve on this: we want our enumeration algorithm to support update operations on $T$, and to update $J$ accordingly instead of recomputing it from scratch. Specifically, an algorithm for enumeration under updates on a tree $T$ has a preprocessing phase that produces the index $J$ as usual, but has two algorithms during the enumeration phase: (i.) an enumeration algorithm $A$ as presented before, and (ii.) an update algorithm $U$. Whenever we want to change the tree $T$, we call $U$ with a description of the changes to make: $U$ modifies $T$ accordingly, updates $J$ for the new tree, and resets the state (so enumeration starts over on the new tree, in particular all working memory used during the enumeration phase is freed). The update time of the enumeration algorithm refers to the complexity of $U$: like preprocessing, but unlike delay, it is measured as a function of the size of the (current) tree $T$.

To our knowledge, the only published result on enumeration for MSO queries under updates is the work of Losemann and Martens [23], which applies to words and to trees, for MSO queries with only first-order variables. They show an enumeration algorithm with linear-time preprocessing: on words, the update complexity and delay is $O(\log |T|)$; on trees, these complexities become $O(\log_2 |T|)$. Thus the delay is worse than in the case without updates [8], and in particular it is no longer independent from $T$.

Main result. In this work, we show that enumeration under updates for MSO queries on trees can be performed with a better complexity that matches the case without updates: linear-time preprocessing, linear delay and memory (in the assignments), and update time in $O(\log |T|)$. This improves on the bounds of [23] (and uses entirely different techniques). However, in exchange for the better complexity, we only support a weaker update language: we can change the labels of tree nodes, called a relabeling, but we do not support inserting or deleting leaf nodes as in [23]. We show in Section 8 that relabelings are useful to derive results for some practical query languages. We believe that our approach may extend to insertions and deletions as in [23], but leave this for future work (see the conclusion in Section 9).

Formally, a relabeling on a $\Gamma$-tree $T$ is a pair of a node $n \in T$ and a label $l \in \Gamma$. To apply it, we change the label $\lambda(n)$ of $n$ by adding $l$ if $l \notin \lambda(n)$, and removing it if $l \in \lambda(n)$. In other words, the tree $T$ never changes, and updates only modify $\lambda$. Our main result is then:

$\triangleright$ Theorem 3.1. For any fixed tree alphabet $\Gamma$ and MSO query $Q(X)$ on $\Gamma$-trees, given a $\Gamma$-tree $T$, we can enumerate the output $Q(T)$ of $Q$ on $T$ with linear-time preprocessing, linear delay and memory, and logarithmic update time for relabelings.

In other words, after preprocessing $T$ in time $O(|T|)$ to compute the index $J$, we can:

-Enumerate the assignments of $Q$ on $T$, using $J$, with delay linear in the size of each assignment, so constant if the assignments to $Q$ have constant size.

-Toggle a label of a node of $T$, update $J$, and reset the enumeration, in time $O(\log |T|)$.

We show this result in Sections 4–7, and then give consequences of this result in Section 8.
Provenance Circuits

Our general technique for enumeration follows our earlier work [4]: from the query and input tree, we compute in linear time a structure called a provenance circuit to represent the results to enumerate, we observe that it falls in a restricted circuit class, and we conclude by showing a general enumeration result for circuits of this class. In this section, we review our construction of provenance circuits in [4], with some additional observations that will be useful for updates. In particular, we show an independent balancing lemma on input trees, which allows us to bound a parameter of the circuit called dependency size. We will extend the formalism of this section to so-called hybrid circuits in the next section; and we will show our enumeration result for such circuits in Sections 6 and 7.

Set circuits. We start with some preliminaries about circuits. A circuit $C = (G,W,g_0,\mu)$ is a directed acyclic graph $(G,W)$ whose vertices $G$ are called gates, whose edges $W$ are called wires, where $g_0 \in G$ is the output gate, and where $\mu$ is a function giving a type to each gate of $G$ (the possible types depend on the kind of circuit). The inputs to a gate $g \in G$ are $\text{inp}(g) := \{g' \in G \mid (g',g) \in W\}$, the fan-in of a gate is its number of inputs, and the fan-out of a gate is the number of gates to which it is an input.

We define set-valued circuits, which are an equivalent rephrasing of the circuits in zero-suppressed semantics used in [4]. They can also be seen to be isomorphic to arithmetic circuits, and generalize factorized representations used in database theory [26]. The type function $\mu$ of a set-valued circuit maps each gate to one of $\cup, \times, \var$. We require that $\times$-gates have fan-in 0 or 2, and that $\var$-gates have fan-in 0: the latter are called the variables of $C$, with $C_{\var}$ denoting the set of variables. Each gate $g$ of $C$ captures a set $S(g)$ of assignments, where each assignment is a subset of $C_{\var}$. These sets are defined bottom-up as follows:

- For a variable gate $g$, we have $S(g) := \{\{\}\}$. 
- For a $\cup$-gate $g$, we have $S(g) := \bigcup_{g' \in \text{inp}(g)} S(g')$. In particular, if $\text{inp}(g) = \emptyset$ then $S(g) = \emptyset$.
- For a $\times$-gate $g$ with no inputs, we have $S(g) := \{\{\}\}$.
- For a $\times$-gate $g$ with two inputs $g_1$ and $g_2$, we have $S(g) := \{A_1 \cup A_2 \mid (A_1,A_2) \in S(g_1) \times S(g_2)\}$, which we write $S(g_1) \times_{\text{rel}} S(g_2)$ (this is the relational product).

The set $S(C)$ captured by $C$ is $S(g_0)$ for $g_0$ the output gate of $C$.

Structural requirements. Before defining our provenance circuits, we introduce some structural restrictions that they will respect, and that will be useful for enumeration.

The first requirement is that the circuit is a $d$-DNNF [14], which we now define. For each gate $g$ of a set-valued circuit $C$, we define the domain $\text{dom}(g)$ of $g$ as the variable gates having a directed path to $g$. In particular, for $g \in C_{\var}$, we have $\text{dom}(g) = \{g\}$, and if $\text{inp}(g) = \emptyset$ then $\text{dom}(g) = \emptyset$. We now call a $\times$-gate $g$ decomposable if it has no inputs or if, letting $g_1' \neq g_2'$ be its two inputs, the domains $\text{dom}(g_1')$ and $\text{dom}(g_2')$ are disjoint. This ensures that no variable of $C$ occurs both in an assignment of $S(g_1')$ and in an assignment of $S(g_2')$. We call a $\cup$-gate $g$ deterministic if, for any two inputs $g_1' \neq g_2'$ of $g$, the sets $S(g_1')$ and $S(g_2')$ are disjoint, i.e., there is no assignment that occurs in both sets. We call $C$ a $d$-DNNF if every $\times$-gate $g$ is decomposable and every $\cup$-gate $g$ is deterministic.

The second requirement on circuits is called upwards-determinism and was introduced in [5]. In that paper, it was used to show an improved memory bound; in the present paper, we will always be able to enforce it. A wire $(g,g')$ in a set-valued circuit $C$ is called pure if:

- $g'$ is a $\cup$-gate; or
g’ is a ×-gate and, letting g'' be the other input of g’, we have \{\} \in S(g''), i.e., g'' captures the empty assignment.

We say that a gate g is upwards-deterministic if there is at most one gate g’ such that (g, g’) is pure. We call C upwards-deterministic if every gate of C is.

The third requirement concerns the maximal fan-in of circuits, which is simply defined for a set-valued circuit C as the maximal fan-in of a gate of C. We will require that the maximal fan-in is bounded by a constant.

The fourth and last requirement concerns a new parameter called dependency size. To introduce this, we define the dependent gates \(\Delta(g')\) of a gate g’ in a set-valued circuit C as the gates g such that there is a directed path from g’ to g. Intuitively, the set \(S(g)\) of g may then depend on the set \(S(g')\) of g’. The dependency size of C is \(\Delta(C) := \max_{g \in C} |\Delta(g)|\), i.e., the maximal number of gates that are dependent on any given gate g. We will require this parameter to be connected to the height of the input tree.

**Set-valued provenance circuits.** We can now define provenance circuits like in [4]. A set-valued circuit C is a provenance circuit of a MSO query \(Q(X_1, \ldots, X_m)\) on a Γ-tree T if:

- The variables of C correspond to the possible singletons, formally: \(C_{\text{var}} := \{(X_i : n) \mid 1 \leq i \leq m \text{ and } n \in T\}\); and
- The set of assignments captured by C is the output of Q on T, formally: \(S(C) = Q(T)\).

In other words, the second requirement says that for any tuple \(B = (B_1, \ldots, B_m)\) of subsets of T, we have \(T \models Q(B)\) if and only if there is an assignment \(\{(X_i : n) \mid 1 \leq i \leq m \text{ and } n \in B_i\}\) in \(S(C)\).

**Example 4.1.** Consider the unlabeled tree T of Figure (a), the alphabet \(\Gamma = \{B\}\), and the MSO query \(Q(x)\) with one free first-order variable asking for the leaf nodes whose B-annotation is different from that of its parent (i.e., the node carries label B and the parent does not, or vice-versa). Consider the labeling \(\lambda\) mapping 1 to \{B\} and 2 and 3 to \emptyset. A set-valued circuit capturing the provenance of Q on \((T, \lambda)\) is given in Figure (b).

We then know from [5] that provenance circuits can be computed efficiently, and they can be made to respect our structural requirements:

**Theorem 4.2 (from [4], Theorem 7.3).** For any fixed MSO query \(Q(X)\) on Γ-trees, given a Γ-tree T, we can compute in time \(O(|T|)\) a set-valued provenance circuit C of Q on T. Further, C is a d-DNNF, it is upwards-deterministic, its maximal fan-in is constant, and its dependency size is in \(O(h(T))\), where h denotes the height of T.

**Proof sketch.** We recall the main proof technique: we convert Q to a bottom-up deterministic tree automaton A on Γ-trees, and we nodes to T to describe the possible valuations of variables. The provenance circuit C then captures the possible ways that A can read T depending on the valuation: we compute it with the construction of [6], and is a d-DNNF thanks to automaton determinism (see [3]). Upwards-determinism is shown like in [5].
The bounds on fan-in and dependency size are not stated in [4, 5] but already hold there. Specifically, the maximal fan-in is a function of the transition function of $A$, i.e., it does not depend on $T$. As for dependency size, this is because $C$ is constructed following the structure of $T$: we create for each tree node a gadget whose size depends only on $A$, and the connections between these gadgets follow precisely the structure of $T$, so that $\Delta(g)$ for any gate $g$ of $C$ can only contain gates from the node of $n$ or from ancestors of $n$ in the tree.

In the context of updates, the bound of dependency size will be crucial: intuitively, it describes how many gates need to be updated when an update operation modifies a gate of the circuit. As this bound depends on the height of the input tree, we will conclude this section by a balancing lemma that ensures that this height is always logarithmic (which matches to our desired update complexity). We will then add support for updates in the next section by extending circuits to hybrid circuits.

Balancing lemma. Our balancing lemma is a general observation on MSO query evaluation on trees, and is in fact completely independent from provenance circuits. It essentially says that the input tree can be assumed to be balanced. Formally, we will show that we can rewrite any MSO query $Q$ on $\Gamma$-trees to an MSO query $Q'$ on a larger signature $\Gamma'$ so that any input tree $T$ for $Q$ can be rewritten in linear time to a tree $T'$ on which $Q'$ which returns exactly the same output. Because we intend to support update operations, the input tree $T$ will be unlabeled, and the rewritten tree $T'$ will work for any labeling of $T$. Formally:

For any tree alphabet $\Gamma$ and MSO query $Q(X)$ on $\Gamma$-trees, we can compute a tree alphabet $\Gamma' \supseteq \Gamma$ and MSO query $Q'(X)$ on $\Gamma'$-trees such that the following holds. Given any unlabeled tree $T$ with node set $N$, we can compute in linear time a $\Gamma'$-tree $(T', \lambda')$ with node set $N' \supseteq N$, such that $h(T') = O(\log |T|)$ and such that, for any labeling function $\lambda : T \to \Gamma$, we have $Q(\lambda(T)) = Q'(\lambda'(T'))$, where $\lambda''(n)$ maps $n \in T'$ to $\lambda(n)$ if $n \in T$ and $\lambda'(n)$ otherwise.

Proof sketch. Somewhat surprisingly, the only way we know to prove Lemma 4.3 is via a tree decomposition of the input tree $T$: we see it as a relational structure of treewidth 1, and we invoke the result by Bodlaender [11] to compute in linear time a constant-width tree decomposition of the structure which is of logarithmic height. We then translate the query $Q$ to a MSO query $Q'$ on tree encodings of this width, and compute from $T$ the tree encoding $T'$ corresponding to the tree decomposition (we rename some nodes of $T'$ to ensure that the nodes of $T$ are reflected in $T'$). We note that the balanced tree decompositions of [11] were already used for similar purposes in different contexts, e.g., in [17], end of Section 2.3.

5 Hybrid Circuits for Updates

In this section, we extend set-valued circuits to support updates, defining hybrid circuits. We then extend Theorem 4.2 for these circuits. Last, we introduce a new structural notion of homogenization of hybrid circuits and show how to enforce it. We close the section by stating our main enumeration result on hybrid circuits, which implies our main theorem (Theorem 3.1), and is proved in the two next sections.

Hybrid circuits. A hybrid circuit is intuitively similar to a set-valued circuit, but it additionally has Boolean variables (which can be toggled when updating), Boolean gates ($\land$, $\lor$, $\neg$), and gates labeled $\otimes$ which keep or discard a set of assignments depending on a Boolean value. Formally, a hybrid circuit $C = (G, W, g_0, \mu)$ is a circuit where the possible gate types
are svar (set-valued variables), bvar (Boolean variables), \(\cup, \times, \boxtimes, \land, \lor, \land, \lor\). We call a gate Boolean if its type is bvar, \(\land, \lor\), or \(\land\); and set-valued otherwise. We require that the output gate \(g_0\) is set-valued and that the following conditions hold:
- svar-gates and bvar-gates have fan-in exactly 0;
- All inputs to \(\land\)-gates, \(\lor\)-gates, and \(\land\)-gates are Boolean, and \(\land\)-gates have fan-in exactly 1;
- All inputs to \(\cup\) and \(\times\)-gates are set-valued, and \(\times\)-gates have fan-in either 0 or 2;
- \(\boxtimes\)-gates have one set-valued input and one Boolean input (so they have fan-in exactly 2).

We write \(C_{\text{bvar}}\) to denote the gates of \(C\) of type bvar, called the Boolean variables, and define likewise the set-valued variables \(C_{\text{svar}}\). An example hybrid circuit is illustrated in Figure (c).

Unlike set-valued circuits, which capture only one set of assignments, hybrid circuits capture several different sets of assignments, depending on the value of the Boolean variables (intuitively corresponding to the tree labels). This value is given by a valuation of \(C\), i.e., a function \(\nu : C_{\text{bvar}} \to \{0, 1\}\). Given such a valuation \(\nu\), each Boolean gate \(g\) captures a Boolean value \(V_\nu(g) \in \{0, 1\}\), computed bottom-up in the usual way: we set \(V_\nu(g) := \nu(g)\) for \(g \in C_{\text{bvar}}\), and otherwise \(V_\nu(g)\) is the result of the Boolean operation given by the type \(\mu_\nu(g)\) of \(g\), applied to the values captured by the inputs (in particular, a \(\land\)-gate with no inputs always has value 1, and a \(\lor\)-gate with no inputs always has value 0).

We then define the evaluation of \(C\) under \(\nu\) as the set-valued circuit \(\nu(C)\) on the same set of gates that we obtain as follows. First, replace each Boolean gate \(g\) of \(C\) by a \(\times\)-gate with no inputs (capturing \(\{\}\) if \(V_\nu(g) = 1\), and by a \(\cup\)-gate with no inputs (capturing \(\emptyset\)) if \(V_\nu(g) = 0\). Second, relabel each \(\boxtimes\)-gate \(g\) of \(C\) to be a \(\times\)-gate. We can then use \(\nu(C)\) to define the set captured by each set-valued gate \(g\) of \(C\) under \(\nu\): it is the set of assignments (subsets of \(C_{\text{svar}}\)) that \(g\) captures in \(\nu(C)\). The set \(S_\nu(C)\) captured by \(C\) under \(\nu\) is then \(S_\nu(g_0)\) for \(g_0\) the output gate of \(C\).

We have lifted the structural definitions from set-valued circuits to hybrid circuits. The maximal fan-in and dependency size of a hybrid circuit are defined like before (these definitions do not depend on the kind of circuit). A hybrid circuit \(C\) is a d-DNNF, resp. is upwards-deterministic, if for every valuation \(\nu\) of \(C\), the set-valued circuit \(\nu(C)\) has the same property. For instance, the hybrid circuit in Figure (c) is upwards-deterministic and is a d-DNNF.

Hybrid provenance circuits. We can now use hybrid circuits to define provenance with support for updates. The set-valued variables of the circuit will correspond to singletons as before, describing the interpretation of the free variables of the query; and the Boolean variables stand for a different kind of singletons, describing which labels are carried by each node. To describe this formally, we will consider an unlabeled tree \(T\), and define a labeling assignment of \(T\) for a tree alphabet \(\Gamma\) as a set of singletons of the form \(\langle l : n \rangle\) where \(l \in \Gamma\) and \(n \in T\). Given a labeling assignment \(A\), we can define a labeling function \(\lambda_A\) for \(T\), which maps each node \(n \in T\) to \(\lambda_A(n) := \{l \in \Gamma \mid \langle l : n \rangle \in A\}\). Now, we say that a hybrid circuit \(C\) is a provenance circuit of a MSO query \(Q(X_1, \ldots, X_m)\) on an unlabeled tree \(T\) if:
- The set-valued variables of \(C\) correspond to the possible singletons in an assignment, formally \(C_{\text{svar}} = \{\langle X_i : n \rangle \mid 1 \leq i \leq m \land n \in T\}\);
- The Boolean variables of \(C\) correspond to the possible singletons in an update assignment, formally \(C_{\text{bvar}} = \{\langle l : n \rangle \mid l \in \Gamma \land n \in T\}\);
- For any labeling assignment \(A\), let \(\nu_A\) be the Boolean valuation of \(C_{\text{bvar}}\) mapping each \(\langle l : n \rangle\) to 0 or 1 depending on whether \(\langle l : n \rangle \in A\) or not, and let \(\lambda_A\) be the labeling function on \(T\) defined as above. Then we require that the set of assignments \(S_{\nu_A}(C)\) captured by \(C\) under \(\nu_A\) is exactly the output of \(Q\) on \(\lambda_A(T)\), formally, \(S_{\nu_A}(C) = Q(\lambda_A(T))\).
In other words, for each labeling $\lambda$ of the tree $T$, considering the valuation $\nu$ that sets the Boolean variables of $C$ accordingly, then $\nu(C)$ is a provenance circuit for $Q$ on $\lambda(T)$.

**Example 5.1.** We reuse the query $Q(X)$ and alphabet $\Gamma = \{B\}$ of Example 4.1, and the tree $T$ of Figure (a). A hybrid circuit $C$ capturing the provenance of $Q$ on $T$ is given in Figure (c). Square leaves correspond to Boolean variables testing node labels, and circle leaves correspond to set-valued variables capturing a singleton of the form $(X : n)$ for some $n \in T$. For legibility, we deviate from our formalism and repeat the variable gates in multiple places in the figure. Notice that, for $\lambda$ the labeling of Example 4.1 and $\nu$ the corresponding valuation, the set-valued circuit $\nu(C)$ captures the same set as that of Figure (b).

We can now extend Theorem 4.2 to compute a hybrid provenance circuit as follows:

**Theorem 5.2.** For any fixed MSO query $Q(X)$ on $\Gamma$-trees, given an unlabeled tree $T$, we can compute in time $O(|T|)$ a hybrid provenance circuit $C$ which is a d-DNNF, is upwards-deterministic, has constant maximal fan-in, and has dependency size in $O(|\lambda(T)|)$.

**Proof sketch.** The proof is analogous to that of Theorem 4.2. The only difference is that the automaton now reads the label of each node as if it were a variable, so that the provenance circuit $C$ also reflects these label choices as Boolean variables.

**Homogenization.** We will make enumeration simpler by imposing one last requirement on hybrid circuits. A hybrid circuit $C$ is **homogenized** if there is no valuation $\nu$ of $C$ and set-valued gate $g$ of $C$ such that $\{\}$ $\in$ $S_\nu(g)$; equivalently, $C$ does not contain $\times$-gates with no inputs. (Note that the requirement does not apply to the Boolean gates of $C$, nor to the gates that replace them in evaluations $\nu(C)$ of $C$.)

Intuitively, set-valued gates in $C$ that capture the empty answer would waste time in the enumeration. We will show that we can rewrite circuits in linear-time to make them homogenized, while preserving our requirements; but we need to change our definitions slightly to ensure that the circuit can still capture the empty assignment overall. To do so, we add the possibility of distinguishing a Boolean gate $g_1$ of a hybrid circuit $C$ as its **secondary output**; in this case, given a valuation $\nu$ of $C$, the set $S_\nu(C)$ captured by $C$ under $\nu$ is $S_\nu(C)$ plus the empty assignment $\{\}$ if the secondary output $g_1$ evaluates to 1, i.e., if $V_\nu(g_1) = 1$. We say that two hybrid circuits $C$ and $C'$ (with or without secondary outputs) are **equivalent** if $C_{\text{bvar}} = C'_{\text{bvar}}$, $C_{\text{svar}} = C'_{\text{svar}}$, and for any valuation $\nu$ of $C$, we have $S_\nu(C) = S_\nu(C')$. We then have:

**Lemma 5.3.** For any hybrid circuit $C$, we can build in linear time a hybrid circuit $C'$ with a secondary output $g_1$, such that $C'$ is homogenized and it is equivalent to $C$. Further, if $C$ is a d-DNNF and is upwards-deterministic, then so is $C'$; if $C$ has bounded fan-in then the same holds of $C'$; and we have $\Delta(C') = O(\Delta(C))$.

**Proof sketch.** This is shown analogously to homogenization in [4], which follows the technique of Strassen [28] (only done for two “layers”, namely, empty and non-empty assignments).

Hence, up to linear-time processing, we can additionally assume that the circuits of Theorem 5.2 are homogenized. We can now use this theorem, the lemma above, and Lemma 4.3, to reduce enumeration for MSO on trees (as in our main theorem, Theorem 3.1) to the task of enumerating the set captured by a hybrid circuit satisfying some structural properties. The result that we need is the following (we prove it in the next two sections):

**Theorem 5.4.** Given an upwards-deterministic, d-DNNF, homogenized hybrid circuit $C$ with constant fan-in, given an initial Boolean valuation $\nu$ of $C_{\text{bvar}}$, there is an enumeration...
algorithm with linear-time preprocessing to enumerate the set $S_{\nu}(C)$ captured by $C$ under $\nu$, with linear delay and memory in each produced assignment, and with update time in $O(\Delta(C))$: an update consists here of toggling one value in $\nu$.

## 6 Enumerating Assignments of Hybrid Circuits

In this section and the next, we prove Theorem 5.4 by giving an algorithm for enumeration under updates. We start by describing the preprocessing phase, computing two simple structures: a shortcut function and a partial evaluation; we also explain how this index can be efficiently updated. We then describe an algorithm for the enumeration phase, which needs an additional index structure to achieve the required delay. We close the section by presenting the missing index, called a switchboard. The switchboard must support a kind of reachability queries with a specific algorithm for enumeration under updates: we give a self-contained presentation of this scheme in the next section.

**Preprocessing phase: shortcuts and partial evaluation.** The first index structure that we precompute on our hybrid circuit $C$ consists of a shortcut function to avoid wasting time in chains of $\times$-gates. For each $\times$-gate $g$, we precompute the one set-valued gate, called $\delta(g)$ which is not a $\times$-gate and which has a directed path to $g$ going only through $\times$-gates. The function $\delta$ can clearly be computed in a linear-time bottom-up pass during the preprocessing, and it will never need to be updated (it does not depend on $\nu$). For notational convenience, we extend $\delta$ by setting $\delta(g) := g$ for any set-valued gate $g$ which is not a $\times$-gate.

The second index structure that we precompute is a partial evaluation, which depends on the valuation $\nu$: it is a function $\omega_{\nu}$ from the gates of $C$ to $\{0, 1\}$ satisfying the following:
- For every Boolean gate $g$, we have $\omega_{\nu}(g) = V_{\nu}(g)$.
- For every set-valued gate $g$, we have $\omega_{\nu}(g) = 1$ iff $S_{\nu}(g)$ is non-empty.

The function $\omega_{\nu}$ is intuitively an evaluation of the Boolean gates in the circuit, extended to the set-valued gates to determine whether their set is empty or not. We can easily compute $\omega_{\nu}$ bottom-up from $\nu$. Further, whenever $\nu$ is changed on a Boolean variable gate $g$, we can update $\omega_{\nu}$ by recomputing it bottom-up on $\Delta(g)$. Formally:

**Lemma 6.1.** Given a hybrid circuit $C$ and valuation $\nu$ of $C$, we can compute $\omega_{\nu}$ in linear time from $\nu$ and $C$. Further, for any $g \in C_{\text{bool}}$, letting $\nu'$ be the result of toggling the value of $\nu$ on $g$, we can update $\omega_{\nu}$ to $\omega_{\nu'}$ in time $O(\Delta(g))$.

Hence, we can compute $\omega_{\nu}$ and $\delta$ in the preprocessing and maintain them under updates.

**Enumeration phase.** We can use the shortcut function and partial evaluation to enumerate the assignments in the set $S_{\nu}(C)$ of our hybrid circuit $C$. Of course, if $\omega_{\nu}(g_0) = 0$ then we detect in constant time that there is nothing to enumerate. Otherwise, the enumeration scheme proceeds essentially like in [4]: to achieve the right delay bounds, it will need an additional index that we present later. We start enumerating $S_{\nu}(g_0)$, and describe what happens when we try to enumerate $S_{\nu}(g)$ for a set-valued gate $g$; we will always ensure that $\omega_{\nu}(g) = 1$. The base case is when $g$ is a set-valued variable, in which case the only assignment to enumerate is $\{g\}$. There are three induction cases: $\times$-gates, $\oplus$-gates, and $\cup$-gates.

First, assume that $g$ is a $\times$-gate. As $C$ is homogenized, $g$ has two inputs $g_1$ and $g_2$. Then we have $S_{\nu}(g) = S_{\nu}(g_1) \times_{\text{def}} S_{\nu}(g_2)$. Hence, we can simply enumerate $S_{\nu}(g)$ as the lexicographic product of $S_{\nu}(g_1)$ and $S_{\nu}(g_2)$. In particular, as $\omega_{\nu}(g) = 1$, we have $\omega_{\nu}(g_1) = \omega_{\nu}(g_2) = 1$, so neither set is empty. Further, $g$ is decomposable in $\nu(C)$, so the assignments in these two sets do not share any variable. Formally, we have the following lemma.
Lemma 6.2. For any $\times$-gate $g$ with inputs $g_1$ and $g_2$, if we can enumerate $S_{\nu}(g_1)$ and $S_{\nu}(g_2)$ with delay and memory respectively $\theta_1$ and $\theta_2$, then we can enumerate $S_{\nu}(g)$ with delay and memory $\theta_1 + \theta_2 + c$ for some constant $c$.

Note that the constant $c$ paid at the $\times$-gate is not a problem to achieve linear delay and memory, because it is paid at most $n - 1$ times when enumerating an assignment $A$ of size $n$; this is because $C$ is homogenized so $A$ is always split non-trivially at each $\times$-gate.

Second, assume that $g$ is a $\ominus$-gate. As $\omega_{\nu}(g) = 1$, it is easy to see that we have $S_{\nu}(g) = S_{\nu}(\delta(g))$. Hence, we can simply follow the pointer to $\delta(g)$ and enumerate $S_{\nu}(\delta(g))$. Intuitively, the cost of this operation can be amortized by that of $g$, because $\delta(g)$ can no longer be a $\ominus$-gate.

Lemma 6.3. For any $\ominus$-gate $g$, if we can enumerate $S_{\nu}(\delta(g))$ with delay and memory $\theta$, then we can enumerate $S_{\nu}(g)$ with delay and memory $\theta + c$ for some constant $c$.

Third, assume that $g$ is a $\cup$-gate $g$. Naively, we can enumerate $S_{\nu}(g)$ as the union of the $S_{\nu}(g')$ for the inputs $g'$ of $g$ for which $\omega_{\nu}(g') = 1$ (this union is disjoint thanks to determinism). This is correct, but does not satisfy the delay bounds, because $g'$ may be another $\cup$-gate. A more clever scheme is to “jump” to the $\times$-gate or set-valued variable gates on which $S_{\nu}(g)$ depends. Let us accordingly call exits the gates of these two types.

The set $S_{\nu}(g)$ can then be expressed as a union of $S_{\nu}(g')$ for the exits $g'$ that have a directed path of $\cup$-gates and $\ominus$-gates to $g$. We introduce definitions to “collapse” these paths.

The first definition collapses paths of $\ominus$-gates. We say that there is a $\ominus$-path from a set-valued gate $g'$ to a set-valued gate $g \neq g'$, written $g' \rightarrow_{\ominus} g$, if there is a directed path $g' = g_1 \rightarrow \cdots \rightarrow g_n = g$ in $C$ such that $g_2, \ldots, g_{n-1}$ are all $\ominus$-gates. In particular, if $\nu$ depends. Let us accordingly call exits the gates of these two types.

The second definition collapses paths of $\cup$-gates. An $\cup$-path from a set-valued gate $g'$ to a set-valued gate $g \neq g'$ is a sequence $g' = g_1 \rightarrow_{\cup} g \rightarrow_{\cup} \cdots \rightarrow_{\cup} g_n = g$ in $C$, where there is a $\ominus$-path between any two consecutive gates. The path is live under $\nu$ if there is a live $\ominus$-path under $\nu$ between any two consecutive gates, and dead under $\nu$ otherwise.

We now use these definitions to express $S_{\nu}(g)$ as a function of the set of exits under $\nu$ of $g$ in $C$, written $D_{\nu}^g$, which is the set of exits $g'$ having a live $\cup$-path to $g$ under $\nu$ in $C$:

Lemma 6.4. For any valuation $\nu$ and $\cup$-gate $g$, we have $S_{\nu}(g) = \bigcup_{g' \in D_{\nu}^g} S_{\nu}(g')$. Further, this union is disjoint and all its terms are nonempty.

Hence, we can enumerate $S_{\nu}(g)$ for a $\cup$-gate $g$ by enumerating $D_{\nu}^g$ and the set $S_{\nu}(g')$ for each $g' \in D_{\nu}^g$. Note that $g'$ is an exit, i.e., a variable or a $\times$-gate; so we make progress.

Lemma 6.5. For any $\cup$-gate $g$, if we can enumerate $D_{\nu}^g$ with delay and memory $c$, and can enumerate $S_{\nu}(g')$ for every $g' \in D_{\nu}^g$ with delay and memory $\theta$, then we can enumerate $S_{\nu}(g)$ with delay and memory $\theta + c + c'$ for some constant $c'$.

We have described our enumeration scheme in Lemmas 6.2, 6.3, and 6.5. The only missing piece is to enumerate the sets $D_{\nu}^g$ of exits under $\nu$ of $\cup$-gates $g$, with constant delay and memory. To do so, we will need additional computation in our preprocessing. We will rely on upwards-determinism, and extend the tree-based index of [5] to support updates. We first present an additional structure, called the switchboard, that we compute in the preprocessing; and we explain in the next section an indexing scheme that we perform on this structure.
Switchboard.} Our third index component in the preprocessing is called the switchboard. It consists of a directed graph \( B = (V, E) \) called the panel, which does not depend on \( \nu \) (so it does not need to be updated), and a valuation \( \beta_\nu : E \rightarrow \{0, 1\} \) called the wiring. The panel \( B = (V, E) \) is defined as follows: \( V \) consists of all \( \sqcup \)-gates, \( \times \)-gates, and \( \sqcap \)-gates, and \( E \subseteq V \times V \) contains the edge \( (\delta(g'), g) \) for each wire \( (g', g) \) of \( C \) such that \( g \) is a \( \sqcup \)-gate. This implies that the maximal fan-in of \( B \) is no greater than that of \( C \), and it implies that \( B \) is a DAG. The wiring \( \beta_\nu \) maps every edge \( (g', g) \) of \( B \) to 1 if there is a \( \sqsubseteq \)-path from \( g' \) to \( g \) in \( C \) which is live under \( \nu \), and 0 otherwise. We can use \( \omega_\nu \) to compute the switchboard, and to update it in time \( O(\Delta(C)) \) whenever \( \nu \) is updated by toggling a gate of \( C_{\text{bare}} \). Formally:

\[ \textbf{Lemma 6.6.} \] The switchboard can be computed in linear time given \( C \) and \( \nu \), and we can update it in time \( O(\Delta(C)) \) when toggling any gate in \( \nu \).

We now explain how we use the switchboard to enumerate, given a \( \sqcup \)-gate \( g \), the set \( D_g^\nu \) of the exits \( g' \) having a live \( \sqcup \)-path to \( g \) under \( \nu \). In terms of the switchboard, we must enumerate the exits \( g' \) that have a path to \( g \) in \( B \) whose edges are all mapped to 1 by \( \beta_\nu \). Hence, we must solve the following enumeration task on the switchboard: letting \( \beta_\nu(B) \) be the DAG of edges of \( B \) mapped to 1 by \( \beta_\nu \), we are given a gate \( g \) of \( B \), and we must enumerate all exit gates \( g' \) of \( B \) (i.e., the \( \times \)-gates or \( \sqcap \)-gates) that have a directed path to \( g \) in \( \beta_\nu \). Further, we must be able to handle updates on \( \beta_\nu(B) \), as given by updates on \( \nu \). Fortunately, thanks to upwards-determinism, this problem is easier than it looks:

\[ \textbf{Claim 6.7.} \] For any valuation \( \nu \) of the hybrid circuit \( C \), the DAG \( \beta_\nu(B) \) is a forest.

\[ \textbf{Example 6.8.} \] Figure (d) describes the switchboard for the hybrid circuit \( C \). The edges of the switchboard correspond to \( \sqsubseteq \)-paths. The switchboard itself is not a forest; however, for every valuation of \( C \), the \( \sqsubseteq \)-paths that are live must always form a forest.

Thus, what we need is a constant-delay reachability index on forests that can be updated efficiently when adding and removing edges to the forest. This is the focus of the next section.

7 Reachability Indexing under Updates

In this section, we present our indexing scheme for reachability on forests under updates. The construction in this section is independent from what precedes. For convenience, we will orient the edges of the forest downwards, i.e., the reverse of the previous section (so \( g \) is the parent of \( g' \) in the forest if there is an edge from \( g' \) to \( g \) in the switchboard). We first define the problem and state the enumeration result, and then sketch the proof.

\[ \textbf{Definitions and main result.} \] A reachability forest \( F = (V, E, X) \) is a directed graph \( (V, E) \) where \( V \) is the vertex set, \( E \subseteq V \times V \) are the edges, and \( X \subseteq V \) is a subset of vertices called exits. When \( (v, v') \in E \), we call \( v \) a parent of \( v' \), and \( v' \) a child of \( v \). We impose three requirements on \( F \): (i.) the graph \( (V, E) \) is a forest, i.e., each vertex of \( V \) has at most one parent; (ii.) there is a constant degree bound \( c \in \mathbb{N} \) such that every vertex has at most \( c \) children; (iii.) every exit \( v \in X \) is a leaf, i.e., a vertex with no children. We will call trees the connected components of \( F \). For convenience, we assume that \( F \) is ordered, i.e., there is some total order \( < \) on the children of every node.

Given a reachability forest \( F = (V, E, X) \) and a vertex \( v \in V \), we write \( \text{reach}(v) \) for the set of exits reachable in \( F \) from \( V \), i.e., the vertices of \( X \) to which \( v \) has a directed path. These are the sets that we wish to enumerate efficiently, allowing two kinds of updates on the edges \( E \) of \( F \). First, a delete operation is written \( -E' \) for a set \( E' \subseteq E \), and
\(F = (V, E, X)\) is updated to \(F - E' := (V, E \setminus E', X)\); it is still a reachability forest. Second, an insert operation is written \(+E'\) for some \(E' \subseteq V \times V\), and we require that the update result \(F + E' := (V, E \cup E', X)\) still satisfies the three requirements above (with the same degree bound). In terms of the order \(<\) on children, when we remove edges, we take the restriction of \(<\) in the expected way, and when we insert edges, we add each new child at an arbitrary position in \(<\). We introduce a notion of ancestry to measure the impact of updates (analogously to dependency size): the ancestry \(A_F(v)\) of \(v \in V\) is the set of vertices of \(F\) that have a directed path to \(v\), and the ancestry \(A_F(E')\) for \(E' \subseteq V \times V\) is \(\bigcup \{(v, w) \in E' \mid A_F(v)\}\). We then have:

**Theorem 7.1.** Given a reachability forest \(F\), there is an enumeration algorithm with linear-time preprocessing such that: (i.) given any \(v \in V\), we can enumerate \(\text{reach}(v)\) with constant delay and memory; (ii.) given an update \(\pm E'\), we can apply it (replacing \(F\) by \(F \pm E'\) and updating the index) with update time in \(O(A_F(E'))\).

Note how we can insert (or delete) many edges at the same time, paying only once the price \(A_F(E')\): this point is used in the proof of Theorem 5.4 to bound the total cost of each update on the circuit. We sketch the proof of Theorem 7.1 in the rest of this section.

**Construction for Theorem 7.1.** Our index structure follows the one used to prove Proposition F.4 of [5]: it maps every \(v \in V\) to a pointer \(\text{first}_F(v)\) and a pointer \(\text{last}_F(v)\), called the first and last pointer; and maps every exit \(v \in V\) to a pointer \(\text{next}_F(v)\) called the next pointer. These pointers are defined using the order \(<'\) given by a preorder traversal of \(F\) following \(<\). Specifically, \(\text{first}_F(v)\) is the first exit \(v' \in \text{reach}_F(v)\) according to \(<'\), and \(\text{last}_F(v)\) is the last such exit; if \(\text{reach}_F(v) = \emptyset\) then both pointers are null. Now, \(\text{next}_F(v)\) for \(v \in X\) is the exit \(v' \in X\) in the tree of \(v\) which is the successor of \(v\) according to \(<'\); if \(v\) is the last exit of its tree, then \(\text{next}_F(v)\) is null. If we know these pointers, we can enumerate \(\text{reach}_F(v)\) for any \(v \in V\) with constant delay and memory as in [5]: if \(\text{first}_F(v)\) is null then there is nothing to enumerate, otherwise start at \(v_- := \text{first}_F(v)\), memorize \(v_+ := \text{last}_F(v)\), and enumerate the reachable exits following the next pointers from \(v_-\) until reaching \(v_+\). Hence, to conclude the proof of Theorem 7.1, it suffices to show how to compute and update these pointers efficiently:

**Lemma 7.2.** Given a reachability forest \(F\), we can compute the first, last, and next pointers of all vertices in time \(O(|F|)\). Further, for any update \(\pm E'\), we can apply it and update the pointers in time \(O(\mathcal{A}_F(E'))\).

**Proof sketch.** The first and last pointers are computed bottom-up in linear time: for a leaf \(v\), they either point to \(v\) if \(v \in X\) or to null otherwise; for an internal vertex \(v\), we set \(\text{first}_F(v)\) as \(\text{first}_F(v')\) for the smallest child \(v'\) of \(v\) with a non-null first pointer (or null if all first pointers of children are null), and we set \(\text{last}_F(v)\) analogously, using the last pointer of the largest child of \(v\) for which the last pointer is non-null. Further, given an update \(\pm E'\), the first and last pointers need only to be updated in \(\mathcal{A}_F(E')\), and we can recompute them there with the same bottom-up scheme.

The next pointers are also computed bottom-up in linear time: at each internal vertex \(v\), we go over its children and stitch together the sequences of next pointers of their subtrees. Specifically, when \(\text{last}_F(v_1)\) is not null for a child \(v_1\), we find the next child \(v_2\) for which \(\text{first}_F(v_2)\) is not null, and set \(\text{next}_F(\text{last}_F(v_1)) := \text{first}_F(v_2)\). Again, for an update \(\pm E'\), we recompute the next pointers by processing \(\mathcal{A}_F(E')\) bottom-up in a similar fashion.
8 Applications

We have finished the proof of our main result (Theorem 3.1), and now explain how it applies to query languages motivated by applications. Specifically, we show how to extend our techniques to support aggregate queries in arbitrary semirings, following the ideas of semiring provenance [20] and provenance circuits [15]. We then extend this to group-by queries, and last explain how updates are useful to support parameterized queries. Throughout this section, unlike the rest of the paper, we only study MSO queries with free first-order variables.

Aggregate queries. We will describe aggregation operators using a general structure called a semiring (always assumed to be commutative). It consists of a set $K$ (finite or infinite), two binary operations $\oplus$ and $\otimes$, and distinguished elements $0_K, 1_K \in K$. We require that $(K, \oplus)$ and $(K, \otimes)$ are commutative monoids with neutral elements respectively $0_K$ and $1_K$; that $\otimes$ distributes over $\oplus$, and that $0_K$ is absorptive for $\otimes$, i.e., $0_K \otimes a = 0_K$ for all $a \in K$.

We always assume that evaluating $\oplus$ or $\otimes$ take constant time, and that elements from $K$ take constant space. Examples of semirings include the natural numbers $\mathbb{N}$ with usual addition and product (assumed to take unit time in the RAM model); or the security semiring [19], the tropical semiring [15], etc. Note that sets of assignments with union and relational product are also a semiring, but one that does not satisfy our constant-space assumption.

To define aggregation in a semiring $K$ on a tree $T$, we consider a mapping $\rho : T \to K$ giving a value in $K$ to each node. We extend $\rho$ to tuples $B$ of $T$ by setting $\rho(B) := \bigotimes_{n \in B} \rho(n)$; to assignments $A$ on some first-order variable set $\mathbf{x}$ by setting $\rho(A) := \bigotimes_{(x, n) \in A} \rho(n)$; and to sets $S$ of assignments by setting $\rho(S) := \bigoplus_{A \in S} \rho(A)$. An aggregate query on $\Gamma$-trees consists of a semiring $K$ (satisfying our assumptions) and of a MSO query $Q(\mathbf{x})$ on $\Gamma$-trees. Given a $\Gamma$-tree $T$ and a mapping $\rho : T \to K$, the aggregate output $Q_\rho(T)$ of $Q$ on $T$ under $\rho$ is $\rho(Q(T))$, where $Q(T)$ is the output of $Q$ on $T$ as we studied so far, i.e., the set of assignments $A$ such that $T \models Q(A)$. Aggregate MSO queries on trees were already studied, e.g., by Arnborg and Lagergren [7], but our techniques allow us to handle updates:

► Theorem 8.1. For any aggregate query $Q(\mathbf{x})$ on $\Gamma$-trees with semiring $K$, given a $\Gamma$-tree $T$ and mapping $\rho : T \to K$, we can compute $Q_\rho(T)$ in time $O(|T|)$, and recompute it in time $O(|\log |T||)$ after any update that relabels a node of $T$ or changes a value of $\rho$.

Proof sketch. We adapt hybrid circuits by replacing set-valued gates by $K$-valued gates. Now, the set $S_\nu(g)$ captured by a gate $g$ under a Boolean valuation $\nu$ is an element of $K$, so we can simplify our linear-time preprocessing by making $\omega_\nu$ compute exactly $S_\nu(g)$ for each gate $g$. We can then handle updates to $\nu$ as before, and handle updates to $\rho$ by recomputing $\omega_\nu$, bottom-up. All of this still relies on the balancing lemma (Lemma 4.3). ▶

One important application of this result is maintaining the number of query answers under updates, a question left open by [23]. We answer the question for relabeling updates (and in the set semantics), using the semiring $\mathbb{N}$ and mapping each node to 1 with $\rho$:

► Corollary 8.2. For any MSO query $Q(\mathbf{x})$ on $\Gamma$-trees, given a $\Gamma$-tree $T$, we can compute the number $|Q(T)|$ of answers of $Q$ on $T$ in time $O(|T|)$, and we can update it in time $O(|\log |T||)$ after a relabeling of $T$.

However, we can also use Theorem 8.1 for more complex aggregation semirings:

► Example 8.3. Let $Q(x)$ be a MSO query with one variable that selects some tree nodes, and $\chi$ mapping each node of a tree $T$ to an element of the set $\mathbb{D}$ of floating-point numbers.
(with fixed precision). We can compute in linear time the average of $\chi(n)$ for the nodes $n$ such that $T \models Q(n)$, and update it in logarithmic time when relabeling a node of $T$ or changing a value of $\chi$. This follows from Theorem 8.1: we use the semiring of pairs in $\mathbb{N} \times \mathbb{D}$ and the mapping $\rho : n \mapsto (1, \chi(n))$ to compute and maintain the number of selected nodes and the sum of their $\chi$-images, from which we can deduce the average in constant time.

**Group-by.** We have adapted our techniques to show results for aggregate queries under updates. However, supporting updates is also useful for group-by queries. A group-by query consists of a MSO query $Q(x,y)$ on $\Gamma$-trees with two tuples of first-order variables, and of a semiring $K$. A group on a $\Gamma$-tree $T$ is a set of tuples $G(B) := \{(B,C) \mid T \models Q(B,C)\}$ for some tuple $B$ of nodes of $T$. The output $Q_\rho(T)$ of $Q$ on $T$ under a mapping $\rho : T \rightarrow K$ contains one pair $(B, \rho(G(B)))$ for each tuple $B$ such that $G(B)$ is non-empty.

**Example 8.4.** Consider a MSO query $Q(x,y)$ and the semiring $\mathbb{N}$. The output of $Q$ on a $\Gamma$-tree $T$ under a mapping $\rho$ contains one pair per $n \in T$, annotated with the sum of $\rho(n')$ for $n' \in T$ such that $T \models Q(n,n')$, where we exclude the nodes $n$ for which the sum is empty.

**Theorem 8.5.** For any group-by query $Q(x,y)$ and semiring $K$, given a $\Gamma$-tree $T$ and $\rho : T \rightarrow K$, we can enumerate $Q_\rho(T)$ with linear-time preprocessing and delay in $O(\log |T|)$

**Proof sketch.** We use two enumeration structures. First, we prepare the structure of Theorem 8.1 for $Q(x,y)$ but writing the valuation of $x$ as part of the tree label. Second, we enumerate the non-empty groups with constant delay using Theorem 3.1 on $\exists x \ Q(x,y)$. For each group $G(B)$, we update the first structure to compute $\rho(G(B))$ in time $O(\log |T|)$.

**Parameterized queries.** We conclude by presenting another kind of practical queries that we can support thanks to updates. A parameterized MSO query $Q(x,y)$ on $\Gamma$-trees has two kinds of first-order variables, like group-by: we call $x$ the parameters. The idea is that, given a $\Gamma$-tree $T$, the user chooses a tuple $B$ to instantiate the parameters $x$, and we must enumerate efficiently the results of $Q(B,y)$; however the user can change their mind and modify $B$ to change the value of the parameters. We know by Theorem 3.1 that we can support these queries efficiently: after a linear-time preprocessing of $T$, we can enumerate the results of $Q(B,y)$ with constant delay; and we can react to changes to $B$ in time $O(\log |T|)$ by performing an update on the enumeration structure.

### 9 Conclusion

We have studied MSO queries on trees under relabeling updates, and shown how to enumerate their answers with linear-time preprocessing, delay and memory linear in each valuation, and update time logarithmic in the input tree. We have shown this by extending our circuit-based approach [4] to hybrid circuits, and we have deduced consequences for practical query languages, in particular for efficient aggregation. Our results have another technical property that we have not presented in the main text: like those of [23], they are also tractable in the size of the query when representing it as a deterministic automaton.

The main direction for future work would be to extend our result to support insertions and deletions of leaves, like [23], hopefully while preserving our improved bounds. Such a result was very recently shown in [24] for the case of words rather than trees. We believe that many of our constructions should adapt to insertions and deletions; the main challenge is to extend Lemma 4.3, which we believe to be an interesting question in its own right.
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A Proofs for Section 3 (Problem Statement and Main Result)

▸ Theorem 3.1. For any fixed tree alphabet $\Gamma$ and MSO query $Q(X)$ on $\Gamma$-trees, given a $\Gamma$-tree $T$, we can enumerate the output $Q(T)$ of $Q$ on $T$ with linear-time preprocessing, linear delay and memory, and logarithmic update time for relabelings.

Proof. See Appendix E.2 for the proof of this result.

B Proofs for Section 4 (Provenance Circuits)

In this appendix, we prove Lemma 4.3:

▸ Lemma 4.3. For any fixed tree alphabet $\Gamma$ and MSO query $Q(X)$ on $\Gamma$-trees, we can compute a tree alphabet $\Gamma' \supseteq \Gamma$ and MSO query $Q'(X)$ on $\Gamma'$-trees such that the following holds. Given any unlabeled tree $T$ with node set $N$, we can compute in linear time a $\Gamma'$-tree $(T', \lambda')$ with node set $N' \supseteq N$, such that $h(T') = O(\log |T|)$ and such that, for any labeling function $\lambda : T \rightarrow \Gamma$, we have $Q(\lambda(T)) = Q'(\lambda''(T'))$, where $\lambda''(n)$ maps $n \in T'$ to $\lambda(n)$ if $n \in T$ and $\lambda'(n)$ otherwise.

To prove Lemma 4.3, we will need to introduce preliminaries about relational instances [1], tree decompositions, and tree encodings.

Instances. A relational signature is a set of relation names together with an associated arity (a non-zero natural number). We fix a relational signature $\sigma_2$ that codes unlabeled trees, consisting of two binary relations $E_1$ and $E_2$ indicating the first and second child of each internal node. For any tree alphabet $\Gamma$, we let $\sigma_T$ denote a signature to represent labels of $\Gamma$, i.e., one unary relation $P_l$ for each $l \in \Gamma$. Last, for a tuple $X = X_1, \ldots, X_m$ of second-order variables, we let $\sigma_X$ denote a signature to represent the interpretation of these variables, i.e., one unary relation $B_i$ for each $1 \leq i \leq m$. By monadic second-order logic (MSO) over $\sigma$, we denote MSO with the relations of $\sigma$ and equality in the usual way.

A relational instance of a relational signature $\sigma$ is a set $I$ of $\sigma$-facts of the form $R(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are elements, $R$ is a relation in $\sigma$, and $n$ is the arity of $R$. The domain $\text{dom}(I)$ of $I$ is the set of elements that occur in $I$.

Given a $\Gamma$-tree $(T, \lambda)$, we can easily compute in linear time a couple $(I, I_\lambda)$ where $I$ is a $\sigma_T$-instance describing the unlabeled tree $T$ in the expected way (in particular, $\text{dom}(I)$ is exactly the set of nodes of $T$), and $I_\lambda$ is the $\sigma_T$-instance $\{P_l(n) \mid n \in T$ and $l \in \lambda(n)\}$.

Tree decompositions. A tree decomposition of an undirected graph $G = (V, E)$ is a tree $\Theta$ (whose nodes are called bags) and a labeling function $\text{dom} : \Theta \rightarrow 2^V$ such that:

- For every $e \in E$, there is $b \in \Theta$ such that $e \subseteq \text{dom}(b)$
- For every $v \in V$, the set $T_v := \{b \in \Theta \mid v \in \text{dom}(b)\}$ is a connected subtree of $\Theta$.

We still assume for convenience that tree decompositions are rooted, ordered, binary, and full trees. Specifically, they will be computed as rooted binary trees by [11], they can be made full without loss of generality (in linear time and without impacting the height) by adding empty bags, and we can add an arbitrary order on the children of each internal bag to make them ordered. The width of $\Theta$ is $\max_{b \in \Theta} |\text{dom}(b)| - 1$, and the treewidth of $G$ is the smallest width of a tree decomposition of $G$.

A tree decomposition of a relational instance $I$ is a tree decomposition of its Gaifman graph, i.e., the graph on vertex set $\text{dom}(I)$ where there is an edge between any two elements $a, a'$ that occur together in some fact. The treewidth of $I$ is that of its Gaifman graph.
The definition of tree decompositions ensures that, for any relational instance $I$ and tree decomposition $\Theta$, for any $a \in \text{dom}(I)$, we can talk of the topmost bag $b$ of $\Theta$ such that $a \in \text{dom}(b)$; we write this bag node$(a)$. This mapping node can be computed explicitly in linear time given $I$ and $\Theta$ by [18, Lemma 3.1].

We will make a standard assumption on our tree decompositions, namely, that the function node is an injective function: in other words, the root bag contains only one element, and for any non-root bag $b$ with parent bag $b'$, we have $|\text{dom}(b) \setminus \text{dom}(b')| \leq 1$. This requirement can be enforced on a tree decomposition $\Theta$ in linear time using standard techniques, without impacting the width of $\Theta$, and only multiplying the height of $\Theta$ by a constant (assuming that the width is constant): specifically, we replace each bag violating the condition by a chain of bags where the new elements are introduced one after the other. Hence, we will always make this assumption.

We now recall the result of Bodlaender [11], which is the key to our construction:

> **Theorem B.1 from [11].** For any relational signature $\sigma$, given a relational instance $I$ on $\sigma$ of width $w \in \mathbb{N}$, we can compute in linear time in $I$ a tree decomposition $\Theta$ of $I$ of width $O(w)$, such that $h(\Theta)$ is in $O(\log(|I|))$.

Specifically, the algorithm of [11] is described for a parallel machine, but can be run sequentially in linear time, as explained in [17], end of Section 2.3.

**Tree encodings.** If we fix a relational signature $\sigma$ and a treewidth bound $k \in \mathbb{N}$, we can compute an alphabet $\Gamma^k_\sigma$, called the alphabet of tree encodings for $\sigma$ and $k$, which ensures the following: given any $\sigma$-instance $I$ with a tree decomposition $\Theta$ of width $k$, we can translate them in linear time to a $\Gamma^k_\sigma$-tree $E$ (called a tree encoding of $I$) that can be decoded back in linear time to an instance isomorphic to $I$. What is more, Boolean MSO formulas on $\sigma$-instances (i.e., MSO formulas without free variables) can be translated to Boolean MSO formulas on $\Gamma^k_\sigma$-trees that are equivalent through encoding and decoding. An example of such a scheme is given in [18]; we will use a different scheme, detailed in [3], which ensures a property dubbed subinstance-compatibility: intuitively, removing a fact $F$ from $I$ amounts to toggling labels on a node of the tree encoding that corresponds to $F$ (without changing the skeleton of the tree encoding).

We are now ready to conclude the proof of Lemma 4.3:

**Proof of Lemma 4.3.** Let $Q(X)$ be the input query on $\Gamma$-trees. Let $\sigma := \sigma_2 \cup \sigma_T \cup \sigma_X$. We let $Q'$ be the Boolean MSO query on $\sigma$-instances obtained from $Q$ in the expected way, making it Boolean by replacing each second-order variable $X_i$ with the unary relation $B_i$. Given an input tree $T$, we compute in linear time the $\sigma_2$-instance $I$ which represents it. It is clear that, given a labeling $\lambda : T \rightarrow \Gamma$, the output $Q(\lambda(T))$ of $Q$ on $\lambda(T)$ is equal to the set of $\sigma_X$-instances $I'$ of $B_i$-facts on $\text{dom}(I)$ (seeing each such instance $I'$ as a set of singletons of the form $(X_i : a)$) such that $I \cup I_\lambda \cup I'$ satisfies $Q'$.

Let $w$ be the width of the tree decomposition obtained when applying Theorem B.1 to an input tree decomposition of width 1. Let us compute from $Q'$ the Boolean MSO query $Q''$ on the alphabet $\Gamma^w_\sigma$ of tree encodings for width $w$ which is equivalent to $Q'$ on $\sigma$-instances (up to encoding and decoding), i.e., an instance $I$ on $\sigma$ satisfies $Q'$ iff its encoding $E$ as a $\Gamma^w_\sigma$-tree satisfies $Q''$.

Now, as $T$ is a tree, the treewidth of $I$ is 1. Let us define an instance $I^+$ by adding to $I$ the instance $I^+$ of all possible $\sigma_2$-facts on $\text{dom}(I)$, plus the instance $I^X$ of all possible $\sigma_X$-facts on $\text{dom}(I)$. As all these additional facts are unary, the instance $I^+$ still has treewidth 1. Hence, by Theorem B.1, we can compute in linear time in $I_+$, a tree decomposition $\Theta$ of $I_+$.
of treewidth $w$ and logarithmic height, for some fixed constant $w \in \mathbb{N}$. We also compute in linear time the mapping $\text{node} : I_+ \rightarrow \emptyset$, and a tree encoding $E$ of $I_+$, i.e., a $\Gamma^w_\sigma$-tree.

Thanks to subinstance-compatibility, we know that, for any labeling $\lambda : T \rightarrow \Gamma$ and answer tuple $B$ of subsets of $I$, letting $I_1 \subseteq I^T$ and $I_B \subseteq I^X$ be the $\sigma_1$- and $\sigma_X$-instances that respectively denote it, then we can obtain a tree encoding of $I \cup I_1 \cup I_B \subseteq I^+$ by toggling the labels of some nodes of $E$. Specifically, each fact of $I^T \cup I^X$ corresponds to one node of $E$ whose label has to be changed.

The last thing to argue is that we can change the format of $E$ by renaming the nodes to ensure that the nodes to relabel have the same nodes as the corresponding nodes in $T$, and that the relabeling can be performed by changing the $\Gamma$-label and by mapping the variables of $X$. (This is in contrast to the original construction of [3], where each fact correspond to one specific node whose label should be toggled.) To do so, we modify $E$ in linear time to another $\Gamma^k_\sigma$-tree $E'$: for each $n \in \text{dom}(I)$, letting $b := \text{node}(a)$, we replace $b$ by a gadget with two copies $b_1$ and $b_2$ of $b$ carrying the same label, $b_2$ being the left child of $b_1$, and we add a right child to $b_1$ which is a new node $n$ identified to the element $n \in \text{dom}(I)$, which itself corresponds to the node $n$ in $T$. This construction is well-defined because the function $\text{node}$ is injective. We must now argue that the query $Q''$ can be modified (independently from $I$) to a MSO query $Q'''$ on $(\Gamma \cup \Gamma^k_\sigma)$-trees to read labels and variable assignments from these new nodes: specifically, instead of reading (the encodings of) the $\sigma_1$-facts about (the encoding of) an element $n \in \text{dom}(I)$, the query $Q'''$ should read the label in $\Gamma$ of the new node of $E'$ identified with $n$; likewise, instead of reading (the encodings of) the $\sigma_X$-facts on an element $n \in \text{dom}(I)$ directly from $E$, the query should read the $X$-annotation of this same new node in $E'$ identified with $n$. To do this, the translations of the atoms from $\sigma_1$ and $\sigma_X$ in $Q''$ are replaced in $Q'''$ by a gadget which finds the bag where the corresponding element was introduced (i.e., the one for which it is in the image of $\text{node}$), finds the new node that we added, and reads the label and annotation of this node. We also add a conjunct to $Q'''$ to assert that the only nodes that can be part of the interpretation of the $X$ are the new nodes in $E'$ that correspond to nodes of $T$, thus ensuring that the set of answers of $Q'''$ on any labeling $\lambda''(E')$ of $E'$ is correct. This concludes the proof.

\section{Proofs for Section 5 (Hybrid Circuits for Updates)}

\subsection{Proof of the Provenance Circuit Theorem}

In this appendix, we prove Theorem 5.2:

\begin{theorem}
For any fixed MSO query $Q(X)$ on $\Gamma$-trees, given an unlabeled tree $T$, we can compute in time $O(|T|)$ a hybrid provenance circuit $C$ which is a $d$-DNNF, is upwards-deterministic, has constant maximal fan-in, and has dependency size in $O(\log(T))$.
\end{theorem}

The general idea is that, given the MSO query $Q(X)$ on $\Gamma$-trees, writing $X = X_1, \ldots, X_m$, we define a query $Q'(X, Y)$ on unlabeled trees, where $|Y| = |\Gamma|$, with one second-order variable $Y_l$ corresponding to each $l \in \Gamma$. The construction is simply that we replace each unary predicate $P_l$ in $Q$ by the corresponding second-order variable $Y_l$. It is now obvious that, for any labeled tree $(T, \lambda)$, defining $D_l := \{n \in T \mid l \in \lambda(n)\}$ for each $l \in \Gamma$, for any set $B = B_1, \ldots, B_m$ of subsets of $T$, we have $(T, \lambda) \models Q(B)$ iff $T \models Q(B, D)$. In other words, we have simply turned node labels into variables.

Now, at a high level, we can simply construct a provenance circuit of $Q'$ on $T$ in the sense of Theorem 4.2, replace the input gates corresponding to the $Y_l$ variables by a Boolean input gate, and observe that the desired properties hold. However, we give a self-contained
proof of the construction, to make sure that we reflect the change in definitions between the present work and [4, 5].

**Tree automata.** We will need to introduce some prerequisites about tree automata. Given a tree alphabet \( \Lambda \), a bottom-up deterministic tree automaton on \( \Lambda \), or \( \Lambda \)-bDTA, is a tuple \( A = (Q, F, \iota, \delta) \) where \( Q \) is a finite set of states, \( F \subseteq Q \) are the final states, \( \iota : \Lambda \rightarrow Q \) is the initial function, and \( \delta : Q^2 \times \Lambda \rightarrow Q \) is the transition function. The run of a \( \Lambda \)-tree \( T \) is the function \( \rho : T \rightarrow Q \) defined inductively as \( \rho(n) := \iota(\lambda(n)) \) when \( n \) is a leaf, and \( \rho(n) := \delta(\rho(n_1), \rho(n_2), \lambda(n)) \) when \( n \) is an internal node with children \( l_1 \) and \( l_2 \). We say that \( A \) accepts the tree \( T \) if the run \( \rho \) of \( A \) on \( T \) maps the root of \( T \) to a final state.

We will be interested in bDTAs to capture our non-Boolean query \( Q' \) on unlabeled trees. Let \( Z := X \cup Y \) be the set of variables, and let \( \Lambda := 2^Z \), where \( 2^Z \) denotes the powerset of \( Z \). Letting \( T \) be an unlabeled tree, we call a \( Z \)-annotation of \( T \) a function \( \nu : T \rightarrow 2^Z \): the annotation intuitively describes the interpretation of the variables of \( Z \) by annotating each node with the set of variables to which it belongs. Letting \( A \) be a \( 2^Z \)-bDTA, \( T \) be an unlabeled tree, and \( \nu \) be a \( Z \)-annotation of \( T \), we say that \( \nu \) is a satisfying annotation of \( A \) on \( T \) if \( A \) accepts \( \nu(T) \). In this case, we see \( \nu \) as defining an assignment \( \alpha_\nu \), which is the set \( \{(Z_i : n) | 1 \leq i \leq |Z| \text{ and } n \in T\} \) The output of \( A \) on \( T \), written \( A(T) \), is the set of assignments corresponding to its satisfying annotations. Following Thatcher and Wright [29], and determining the automaton using standard techniques [13], the output of an MSO query (here, on an unlabeled tree) can be computed as the output of an automata for that query. Formally:

▶ Lemma C.1 [29, 13]. Given a MSO query \( Q(Z) \) on unlabeled trees, we can compute a \( 2^Z \)-bDTA \( A \) such that, for any unlabeled tree \( T \), we have \( Q(T) = A(T) \).

**Restricting to Boolean annotations.** It will be more convenient in the sequel to assume that each tree node carries one single Boolean annotation rather than many, and to distinguish the annotations corresponding to \( X \) (original variables of \( Q \), called enumerable), and those corresponding to \( Y \) (labels of the input tree, called updatable). We will do this by creating \( |Z| \)-copies of each tree node, to stand for each separate variable. To do this, we will consider the fixed alphabet \( \Sigma_{e,u,f} \) comprising three labels: \( \text{enu}, \text{upd}, \text{fix} \). Intuitively, \( \text{enu} \) applies to nodes whose annotation corresponds to a variable of \( X \), \( \text{upd} \) applies to nodes whose annotation corresponds to a variable of \( Y \), and \( \text{fix} \) applies to nodes whose annotation does not code any variable and should be ignored. Given a \( \Sigma_{e,u,f} \)-tree \( T \), we will write \( T_{\text{enu}}, T_{\text{upd}}, \text{and } T_{\text{fix}} \) to refer to the set of nodes carrying each label. We will then consider \( \Sigma_{e,u,f} \)-trees, where \( \Sigma_{e,u,f} := \Sigma_{e,u,f} \times \{0,1\} \), the alphabet of \( \Sigma_{e,u,f} \)-trees annotated with a Boolean value at each node (as promised, each node carries one single value). Now, a Boolean annotation of a \( \Sigma_{e,u,f} \)-tree \( T \) is a function \( \nu : T \rightarrow \{0,1\} \), and we see \( \nu(T) \) as a \( \Sigma_{e,u,f} \)-tree defined in the expected way.

We want to rephrase the evaluation of \( Q' \) on an unlabeled tree \( T \) to a problem on \( \Sigma_{e,u,f} \)-trees, where variable valuations are coded in Boolean annotations. This process is formalized in the following lemma, whose construction is illustrated in Figure (e); it is analogous to Lemma E.2 of [4]:

▶ Lemma C.2. For any variable set \( Z = X \cup Y \), given a \( 2^Z \)-bDTA \( A \), we can compute a \( \Sigma_{e,u,f} \)-bDTA \( A' \) such that the following holds: given an unlabeled tree \( T \), we can compute in linear time a \( \Sigma_{e,u,f} \) tree \( T' \) of height \( O(h(T)) \) and an injective function \( \varphi : T \times Z \rightarrow T' \) such that:
(e) Illustration of the construction of Lemma C.2, with $Z = (X_1, X_2, Y_1, Y_2)$, on an internal leaf $n$ of a tree, and for a valuation mapping $n$ to $\{X_1, Y_2\}$. The node $n$ is replaced by a gadget of nodes with fixed labels $\text{fix}$. Two $\text{enu}$-labeled descendants indicate the annotation for the $X_i$, and two $\text{upd}$-labeled descendants indicate the annotation for the $Y_i$.

- $T'_{\text{enu}}$ is exactly the set of nodes $n'$ such that $\varphi(n, X_i) = n'$ for some $n \in T$ and $X_i \in X$;
- $T'_{\text{upd}}$ is exactly the set of nodes $n'$ such that $\varphi(n, Y_i) = n'$ for some $n \in T$ and $Y_i \in X$;
- $T'_{\text{fix}}$ is exactly the set of nodes not in the image of $\varphi$, and it includes all internal nodes.

Further, for any $Z$-annotation $\nu : T \rightarrow Z$, let $\nu'$ be the Boolean valuation of $T'$ defined by:

- If $n' \in T'$ is in the image of $\varphi$, then letting $(n, Z_i) := \varphi^{-1}(n')$, we set $\nu'(n') := 1$ iff $Z_i \in \nu(n)$;
- If $n' \in T'$ is not in the image of $\varphi$, we set $\nu'(n') := 0$.

Then $A$ accepts $\nu(T)$ iff $A'$ accepts $\nu'(T')$.

**Proof.** Given an input tree $T$, we change it following the idea of Figure (e): we replace each node $n$ by a gadget of nodes labeled with $\text{fix}$, having two subtrees: one whose leaves are labeled $\text{enu}$ and code the variables $X_i$ in order, and another whose leaves are labeled $\text{upd}$ and code the variables $Y_j$ in order. This tree can be completed to a full binary tree by adding leaves labeled $\text{fix}$ as necessary. Now we can clearly rewrite the $2^Z$-$\text{bDTA}$ to a $\Sigma_{\text{enu,fix}}$-$\text{bDTA}$ $A'$ which is equivalent in the sense required by the lemma. The states of $A'$ consist of the states of $A$, the pairs of states of $A$, and annotation states which consist of binary sequences of length up to $|Z|$. The final states are the final states of $A$. The initialization function $\iota'$ and transition function $\delta'$ are informally coded as follows. The initial function $\iota'$ maps nodes labeled (enu, b) or (fix, b) for $b \in \{0, 1\}$ to the singleton binary sequence $b$ formed of its Boolean value, and it maps nodes labeled (fix, $b$) for $b \in \{0, 1\}$ to the empty binary sequence. The transition function $|\text{delta}'|$ is defined only on nodes labeled (fix, $b$) for $b \in \{0, 1\}$, because all internal nodes of $T'$ carry such a label (as required); and it is defined as follows (where we ignore the Boolean annotation $b$):

- Given two states $q_1$ and $q_2$ of $A$, the new state is the pair $(q_1, q_2)$;
- Given two states that are binary sequences of length $< |Z|$, the new state is their concatenation;
- Given a binary sequence $s$ of length $|Z|$ and a pair of states $(q_1, q_2)$, the new state is the state $\delta(q_1, q_2, s)$ of $A$, where $\delta$ is the transition function of $A$;
- Given a binary sequence $s$ of length $|Z|$ and an empty binary sequence, the new state is the state $\iota(s)$.

Note that the last bullet point describes what happens when $A'$ reads the coding in $T'$ of a leaf $n$ of $T$, which is analogous to what is illustrated in Figure (e): the leaf $n$ in $T$ is translated to a node $n$ in $T'$ with one left child $n'$ that is the root of the tree describing the valuation of $Z$, and one right child $n''$ labeled fix which is a leaf of $T'$.

Now, it is easy to show that $A'$ is equivalent to $A$ in the sense of the lemma statement, which concludes the proof.
We now have a $\Sigma_{e,u,f}$-bDTA to run on a $\Sigma_{e,u,f}$-tree $T$. We can now rephrase our desired provenance result as a provenance result on such automata. We say that a hybrid circuit $C$ is a provenance circuit of a $\Sigma_{e,u,f}$-bDTA $A$ on a $\Sigma_{e,u,f}$-tree $T$ if:

- The set-valued variables of $C$ correspond to the nodes of $T$ with label $enu$, formally, $C_{\text{var}} = T_{\text{enu}}$.
- The Boolean variables of $C$ correspond to the nodes of $T$ with label $upd$, formally, $C_{\text{bvar}} = T_{\text{upd}}$.
- For any Boolean valuation $\nu$ of $T$ such that $\nu(n) = 0$ for each $n \in T_{\text{fix}}$, the automaton $A$ accepts $\nu(T)$ iff, letting $\nu'$ be the restriction of $\nu$ to $T_{\text{upd}}$, and letting $B$ be the set of nodes of $T$ corresponding to the restriction of $\nu$ to $T_{\text{enu}}$, we have $B \in \nu'(C)$.

We can now rephrase our desired result. Note that the statement of this result implies that our construction is also tractable in the automaton:

**Theorem C.3.** Given a $\Sigma_{e,u,f}$-bDTA $A$ and a $\Sigma_{e,u,f}$-tree $T$ where all internal nodes are labeled fix, we can compute in time $O(|T| \times |A|)$ a hybrid circuit $C$ which is a provenance circuit of $A$ on $T$. Further, $C$ is a d-DNNF, it is upwards-deterministic, its maximal fan-in is in $O(|A|)$, and its dependency size is in $O(|A| \times h(T))$, where $h(T)$ is the height of $T$.

**Proof.** We adapt the proof of Proposition E.8 from [5]. Let us write the $\Sigma_{e,u,f}$-bDTA $A = (Q,F,r,\delta)$. We will construct the circuit $C$ in a bottom-up fashion from $T$. We consider every node $n$ of $T$ with label $\lambda(n) \in \Sigma_{e,u,f}$.

If $n$ is a leaf node, for $b \in \{0,1\}$, we let $q_b := \iota((\lambda(n),b))$, and we create the following gates in $C$:

- One set-valued gate $g_n^b$ and one set-valued gate $g_{-n}$, defined as follows:
  - If $n \in T_{\text{fix}}$, the gate $g_n^b$ is a $\cup$-gate with no inputs (i.e., the annotation at $n$ is always 0), and the gate $g_{-n}$ is a $\times$-gate with no inputs;
  - If $n \in T_{\text{upd}}$, the gate $g_n$ is a $\exists$-gate of a $\times$-gate with no inputs and of a Boolean variable gate identified to $n$; and the gate $g_{-n}$ is a $\exists$-gate of a $\times$-gate with no inputs and of the negation of the Boolean variable gate previously mentioned;
  - If $n \in T_{\text{enu}}$, the gate $g_n$ is a set-valued input gate identified to $n$; and the gate $g_{-n}$ is a $\times$-gate with no inputs.

- One $\cup$-gate $g_n^q$ for each state $q \in Q$ with the following inputs:
  - If $q = q_1$, the gate $g_n$;
  - If $q = q_0$, the gate $g_{-n}$.

If $n$ is an internal node with children $n_1$ and $n_2$, remembering that necessarily $n \in T_{\text{fix}}$, we create the following gates in $C$:

- One $\times$-gate $g_{n_1,q_2}$ for each $q_1,q_2 \in Q$ whose inputs are $g_{n_1}^{q_1}$ and $g_{n_2}^{q_2}$;
- One $\cup$-gate $g_n^q$ for each $q \in Q$ whose inputs are all the $g_{n_1}^{q_1}$, $g_{n_2}^{q_2}$ such that $\delta((\lambda(n),0),q_1,q_2) = q$.

The output gate $g_0$ is a $\cup$-gate of the $g_{n}^{q}$ for all $q \in F$, where $n_r$ is the root of $T$.

It is clear that the construction satisfies the requirements of a hybrid circuit. It is also clear that the construction obeys the prescribed time bounds. The only gates in the construction whose arities are not obviously bounded are the $\cup$-gates, and they always have at most $O(|A|)$-transitions (the bound is in the size of the transition table of $A$), so the fan-in bound is respected. For the dependency size, if we consider an arbitrary gate $g$ of the circuit, let $n$ be the node of $T$ for which it was created. It is clear that $\Delta(g)$ is a subset of the set of
all gates created for a node $n'$ which is an ancestor of $T$ in $n$. Now, we create $O(|A|)$-gates for each node of $T$, so indeed the dependency size is bounded by $O(|A| \times b(T))$. We must now show that the circuit has the correct semantics, that it is a $d$-DNNF, and that it is upwards-deterministic.

We first show that claim for the semantics, by showing by bottom-up induction that for any valuation $\nu$ of $C$, for all $n \in T$, letting $T_n$ be the sub-tree of $T$ rooted at $n$, the set $S_\nu(g^{q_1}_{A_1}) \cap \mathcal{S}_\nu$ precisely denotes the set of assignments of $T_{nu} \cap T_n$ such that, letting $\nu_{nu} : T_{nu} \cap T_n \to \{0,1\}$ be the corresponding Boolean function, letting $\nu_{\text{fix}} : T_{\text{fix}} \cap T_n \to \{0,1\}$ be the constant-0 function, and letting $\nu^* := \nu \cup \nu_{\text{emu}} \cup \nu_{\text{fix}}$, the automaton $A_q$ accepts $\nu^*(T_n)$, where $A_q$ is the $\Sigma_{e,u,T}$-bDTA obtained from $A$ by setting $q$ as the only final state. This set of assignments is denoted $A_q(\nu, T_n)$ in what follows.

For the base case of a leaf $n \in T_{\text{fix}}$, we know that $A_q(\nu, T_n)$ is the empty set for all $q \neq \iota((\lambda(n), 0))$, and that it is the set $\{\} \cup \{q\}$ otherwise; this is what our construction ensures.

For the base case of a leaf $n \in T_{\text{upd}}$, we know that, for all $b \in \{0,1\}$, if $\nu(n) = b$, then $A_q(\nu, T_n)$ is the empty set for all $q \neq \iota((\lambda(n), b))$, and that it is the set $\{\} \cup \{q\}$ otherwise; again, this is exactly what we ensure.

For the base case of a leaf $n \in T_{\nu,T}$, letting $q_0 := \iota((\lambda(n), b))$ for all $b \in \{0,1\}$, we know that $A_{q_0}(\nu, T_n)$ contains the empty assignment $\{\}$, and $A_{q_1}(\nu, T_n)$ the singleton assignment $\{n\}$ (if $q_0 = q_1$ then $A_{q_0}(\nu, T_n)$ contains both), and that is all: this is what we compute.

For the induction case, letting $n$ be an internal node of $T$ with children $n_1$ and $n_2$, assuming by induction hypothesis that $S_\nu(g^{q_1}_{A_1}) = A_{q_1}(\nu, T_{n_1})$ and $S_\nu(g^{q_2}_{A_2}) = A_{q_2}(\nu, T_{n_2})$ for all $q_1, q_2 \in Q$, we know by definition that, for all $q \in Q$, the output $A_q(\nu, T_n)$ consists of the union, for $q_1, q_2 \in Q$ such that $\delta((\lambda(n), 0), q_1, q_2) = q$, of the relational product of the outputs $A_{q_1}(\nu, T_{n_1})$ and $A_{q_2}(\nu, T_{n_2})$. This is because is a bijection between the Boolean labelings of $T_n$ that are accepted by $A_{q_1}$ and the pairs of Boolean labelings of $T_{n_1}$ and of $T_{n_2}$ that are respectively accepted by $A_{q_1}$ and $A_{q_2}$ for some $q_1, q_2$ satisfying the condition. Again, this is precisely what we compute, so we have shown the invariant.

Now, as $A(\nu, T) = \bigcup_{q \in F} A_q(\nu, T)$, we have established that $S_\nu(C)$ is correct.

We now show that $C$ is a $d$-DNNF. For decomposability, we show the stronger claim that, for any $\times$-gate $g$ in $C$ (without choosing a valuation) with two inputs $q_1$ and $q_2$, the sets $\text{dom}(g^{q_1})$ and $\text{dom}(g^{q_2})$ in the circuit $C$ (again without choosing a valuation) are disjoint; it is clear that this implies the condition as stated in the main text. Now, the only $\times$-gates with two inputs are the $g^{q_1 \times q_2}_{A_1}$ whose inputs are $g^{q_1}_{A_1}$ and $g^{q_2}_{A_2}$ for the two children $n_1, n_2$ of $n$. Now, an immediate bottom-up induction shows that for any node $n'$ of $T$ and state $q$, we have $\text{dom}(g^{q}_{A_1}) \subseteq \{n' \in T_{\nu,T} \setminus T_n\}$. Hence, indeed, the domains are disjoint.

For determinism, the $\cup$-gates created for the leaf nodes $n \in T_{\text{upd}}$ may have two inputs, but in this case, the sets that they capture are clearly disjoint for any valuation $\nu$, because one is always empty depending on $\nu(n)$. For the $\cup$-gates $g^{q}_{A}$ created for a state $q$ of $A$ and an internal node $n$ of $T$ with children $n_1$ and $n_2$, assume by contradiction that there is a valuation $\nu$ of $C$ and some assignment $a$ such that $a \in S_\nu(g^{q_1}_{A_1})$ and $a \in S_\nu(g^{q_2}_{A_2})$ for $(q_1, q_2) \neq (q_1', q_2')$. Assume that $q_1 \neq q_1'$, the case $q_2 = q_2'$ is analogous. By our inductive invariant and the construction of the circuit, and by the definition of the output of automata, we know that for the valuation $\nu'$ of $T_n$ defined from $a$ and $\nu$, the automata $A_{q_2}$ and $A_{q_2'}$ both accept $\nu'(T_n)$, this contradicts the determinism of $A$, so we have a contradiction. Hence, $n$ is deterministic. We have thus shown that $C$ is a $d$-DNNF.

We must last show that $C$ is upwards-deterministic. We adapt the argument of Claim F.3
of [5]. As A is deterministic, each gate of the form $g_n^{q_1,q_2}$ is used as input to only one gate $g_n^q$, namely, the one defined according to the transition function. So the only set-valued gates in the construction which are used as inputs to multiple gates are the $g_n^q$ when $n$ is not the root of $T$. Fix a valuation $\nu$ of $C$. Let $n'$ be the parent of $n$, and assume that $n = n_1$ is the first child of $n'$; the second case is symmetric. Let $n_2$ be the other child of $n'$. Now, the gate $g_n^q$ is used as inputs to gates of the form $g_n^{q_1,q_2}$, and the other input is $g_n^{q_1'q_2}$. Let $\nu'$ be the extension of $\nu$ obtained by labeling all nodes of $T_{enu} \cap T_n$ and of $T_{fix} \cap T_n$ with $0$: it is a valuation of $T$. Now, by determinism of $A$, we know that there is exactly one state $q'$ such that $A_{q'}$ accepts $\nu'(T_{n_2})$. Hence, by our inductive invariant, the only $g_n^{q_1'q_2}$ such that $\nu'(T_{n_2})$ is $g_n^{q_1q_2}$; so for $\nu$ there is exactly one pure outgoing wire connecting $g_n^q$ to another gate, namely, the one connecting it to $g_n^{q_1'q_2}$. This shows that $g_n^q$ is upwards-deterministic. Hence, we have shown that $C$ is upwards-deterministic. This concludes the proof.

We can now recap the proof of Theorem 5.2:

**Proof of Theorem 5.2.** Given the MSO query $Q(X)$ on $\Gamma$-trees, writing $X = X_1, \ldots, X_m$, define a query $Q'(X,Y)$ on unlabeled trees as we explained initially, and write $Z = X \cup Y$.

Use Lemma C.1 to compute a $2^\mathcal{A}$-bDTA $A$ such that $Q'(T) = A(T)$. Now, use Lemma C.2 to compute the $\Sigma_{e,n,f}$-tree $T'$. All of this is independent from the input tree $T$.

Now, when we are given the unlabeled tree $T$ as input, we compute in linear time the $\Sigma_{e,n,f}$-tree $T'$ and the injective function $\varphi$ described in Lemma C.2. Now, we use Theorem C.3 to compute a provenance circuit $C'$ of $A'$ on $T'$. We know that $C'$ is a $d$-DNNF, that it is upwards-deterministic, and that its maximal fan-in depends only on $A'$, so it is constant. Further, its dependency size is in $O(h(T') \times |A|)$, i.e., it is in $O(h(T))$ (remember that $h(T') = O(h(T))$). Now, let us relabel the inputs of $C'$: for every $g \in C_{evar}$, remembering that it corresponds to a node $n \in T_{enu}$, letting $(n, X_i) := \varphi^{-1}(n')$, we relabel $g$ to $\langle X_i : n \rangle$. We also relabel $g \in C_{bvar}$ to $\langle Y_j : n \rangle$ in the same way. Let $C$ be the result of this renaming on $C'$; as this renaming is bijective, $C$ still has the same properties as before. Now, for any labeling assignment $A$ of $T$, letting $\nu$ be the corresponding Boolean labeling of $C_{bvar}$, this assignment corresponds to a value for the variables $Y$, hence, to a Boolean valuation of $T_{upd}$, hence, a Boolean valuation $\nu'$ of $C'_{bvar}$. Now, we know by definition of $C'$ being a provenance circuit of $A'$ that $\nu'(C')$ is the set of assignments $a$ such that $A'$ accepts the $\Sigma_{e,n,f}$-tree labeled according to $\nu'$ and $a$. By the conditions of Lemma C.2, this happens if $A$ accepts the corresponding labeling of $T$, so by definition of $Q'$ this happens if $a$ is an assignment for $Q$ on the labeling assignment $A$. This establishes correctness, and concludes the proof.

### C.2 Proof of the Homogenization Lemma

In this appendix, we prove Lemma 5.3:

**Lemma 5.3.** For any hybrid circuit $C$, we can build in linear time a hybrid circuit $C'$ with a secondary output $g_1$, such that $C'$ is homogenized and it is equivalent to $C$. Further, if $C$ is a $d$-DNNF and is upwards-deterministic, then so is $C'$; if $C$ has bounded fan-in then the same holds of $C'$; and we have $\Delta(C') = O(\Delta(C))$.

**Proof.** We first describe the construction. We will inductively rewrite each set-valued gate $g$ of $C$ to two gates $g_{=0}$ and $g_{\neq 0}$ of $C'$. The gate $g_{=0}$ will be set-valued and ensure that, for any valuation $\nu$ of $C$, we have $S_{\nu}(g_{=0}) = S_{\nu}(g) \setminus \{\}\}$. The gate $g_{=0}$ will be Boolean and ensure that, for any valuation $\nu$ of $C$, we have $V_{\nu}(g_{=0}) = 1$ iff $\{\} \in S_{\nu}(g)$. We first copy all
We conclude that the answers are indeed disjoint.

For the base case on a set-valued variable gate $g$, we identify $g_{>0}$ to $g$, and we let $g_{=0}$ be a $\lor$-gate with no inputs, so that $V_\nu(g_{=0}) = 0$ for each valuation $\nu$.

For the base case on a $\lor$-gate $g$ with no inputs, we define $g_{=0}$ to be a $\lor$-gate with no inputs, and let and $g_{>0}$ be a $\lor$-gate with no inputs.

For the base case on a $\times$-gate $g$ with no inputs, we define $g_{=0}$ to be a $\land$-gate with no inputs (which always evaluates to 1), and define $g_{>0}$ to be a $\lor$-gate with no inputs.

For the induction case on a $\lor$-gate $g$, letting $g'$ be its set-valued input and $g''$ its Boolean input, letting $g'_{=0}$ and $g'_{>0}$ the gates of $C'$ obtained by induction for $g'$, we define $g_{>0}$ as a $\lor$-gate of $g''$ and $g'_{>0}$, and define $g_{=0}$ as an $\land$-gate of $g''$ and $g'_{=0}$.

For the induction case on a $\lor$-gate $g$, let $g^1, \ldots, g^n$ be its inputs, with $n$ being a constant. We let $g_{=0}$ as an $\lor$-gate of $g^1_{=0}, \ldots, g^n_{=0}$, and $g_{>0}$ as an $\lor$-gate of $g^1_{>0}, \ldots, g^n_{>0}$.

For the induction case on a $\times$-gate $g$, let $g^1$ and $g^2$ be its two inputs. We let $g_{=0}$ be a $\lor$-gate of $g^1_{=0}$ and $g^2_{=0}$. We let $g_{>0}$ be a $\lor$-gate of:

- a $\times$-gate of $g^1_{>0}$ and $g^2_{>0}$
- a $\lor$-gate of $g^1_{=0}$ and $g^2_{=0}$
- a $\lor$-gate of $g^1_{=0}$ and $g^2_{>0}$

Finally, we let the output gate $g_0$ of $C'$ be $(g_{0})_{>0}$, and let the secondary output gate $g_1$ of $C'$ be $(g_{0})_{=0}$.

It is easy to check that $C'$ satisfies the conditions on hybrid circuits, and that the invariant is indeed equivalent to $C$. Further, the invariant ensures that no set-valued gate of $C'$ captures $\{\}$ under some valuation, so $C'$ is indeed homogenized.

It is clear that the construction is in linear time. It is also clear that the maximum fan-in of $C'$ is no bigger than that of $C$ (unless it is less than 3, in which case it is 3). Further, it is clear that if there is a directed path from a gate $g'_1$ to a gate $g'_2$ in $C'$, then letting $g_1$ and $g_2$ be the gates from which $g'_1$ and $g'_2$ were created, there is a directed path from $g_1$ to $g_2$ in $C$. As we create only constantly many gates in $C'$ for each gate of $C$, this ensures that the dependency of $C'$ is at most that of $C$ multiplied by a constant.

We now show that, if $C$ is a d-DNNF, then $C'$ also is. Specifically, letting $\nu$ be a valuation of $C$, we show that if $\nu(C)$ is a d-DNNF then so is $\nu(C')$. This is like in Proposition B.3 of [5] except it is simpler in our context because there are no range gates and the arities bounds are more convenient. Specifically, for decomposability, the only $\times$-gates in $\nu(C')$ are those introduced as the replacement of $\lor$-gates, which are clearly decomposable, and those created in the first bullet point of the list for the last induction case: now as $\text{dom}(g^1)$ and $\text{dom}(g^2)$ are disjoint in $\nu(C)$ because $g$ is decomposable, and as we clearly have by construction that $\text{dom}(g^2_{>0}) = \text{dom}(g_1)$ (identifying $g$ and $g_{>0}$ for $g \in C_{\text{var}}$), we can conclude. For determinism, we need to consider first the induction case for $\lor$, and second the induction case for $\times$. For $\lor$, the determinism of the $\lor$-gate $g_{>0}$ follows from that of $g$ in $\nu(C)$. For $\times$, the gate $g_{>0}$ is indeed deterministic because:

- Letting $g'$ be its first input, as $C'$ is homogenized, each answer in $S_\nu(g')$ must contain one variable from $\text{dom}(g^1_{>0})$ and one from $\text{dom}(g^2_{>0})$
- Letting $g''$ be its second input, the answers in $S_\nu(g'')$ contain only variables from $\text{dom}(g^1_{>0})$
- Letting $g'''$ be its second input, the answers in $S_\nu(g''')$ contain only variables from $\text{dom}(g^2_{>0})$

We conclude that the answers are indeed disjoint.
Last, we show that if $C$ is upwards-deterministic then so is $C'$; specifically, we show that for any valuation $\nu$ of $C$, if $\nu(C)$ is upwards-deterministic then so is $\nu(C')$. The proof is analogous to that of Claim F.7 in [5]. First note that the gates added when evaluating $C'$ to $\nu(C')$ cannot break upwards-determinism because they are used as the input to only one gate (the replacement of a $\exists$-gate), so we can ignore them. Like in the proof of [5], we define the \textit{original gate} $\omega_\nu(g)$ of any of the other set-valued gates $g$ of $\nu(C')$ to be the gate in $\nu(C)$ for which it was created: specifically, if $g$ is of the form $(g')_{>0}$ then $\omega_\nu(g) := g'$, and if $g$ is a fresh gate created for a $\cup$-gate $g'$ of $\nu(C)$ in the last induction case above, then $\omega_\nu(g) := g'$. Clearly the gates in $\nu(C')$ that come from fresh gates in $C'$ cannot violate upwards-determinism, because they are used as input to only one gate in $C'$, hence in $\nu(C')$. So it suffices to show that, for any gate $g$ of $\nu(C)$, the gate $g_{>0}$ in $\nu(C')$ is upwards-deterministic. We want to show that there is at most one gate $g'$ in $\nu(C')$ such that the wire $(g_{>0}, g')$ is pure in $\nu(C')$. First observe that, if $g'$ is a fresh gate from the first sub-item in the last induction case, then clearly this wire is not pure, because the other input is a set-valued gate and $C$ is homogenized so the other input cannot capture the empty assignment (neither in $C'$ nor in $\nu(C')$). Hence, we can exclude these wires from consideration. Now, in fact, for any wire $w = (g_{>0}, g')$ in $C'$, then $\omega_\nu(w) := (g, \omega_\nu(g'))$ is also a wire of $\nu(C)$, and this mapping is injective: there are no two wires $w \neq w'$ such that $\omega_\nu(w) = \omega_\nu(w')$. Indeed, for each gate $g$ of $C$ and outgoing wire of $g$ in $\nu(C)$, we create at most one wire from $g_{>0}$ to a gate of $\nu(C')$ among the wires that remain at this stage. Hence, using the upwards-determinism of $g$ in $\nu(C)$ it suffices to show that whenever a wire $w$ of $C'$ was not excluded yet and is pure, then $\omega_\nu(w)$ also is.

To do so, we consider the possible wires $w = (g_{>0}, g')$ in $\nu(C')$:

- If $g'$ is a $\cup$-gate, then $\omega_\nu(g')$ also was, which concludes.
- If $g'$ is a $\times$-gate coming from a $\times$-gate of $C'$, we have already excluded these wires.
- If $g'$ is the translation in $\nu(C')$ (as a $\times$-gate) of a $\exists$-gate of $C'$ created in the induction case for $\exists$, then its other input is the translation of a Boolean gate which existed also in $\nu(C)$ and had the same value, so the wire is pure iff the corresponding wire is pure in $\nu(C)$.
- If $G'$ is the translation in $\nu(C')$ of a $\exists$-gate of $C'$ created in the induction case for $\times$, then its second input in $\nu(C')$ captures the empty assignment iff it stands in $C'$ for a $\times$-gate of the form $g''_{>0}$ which evaluates to true under $\nu$, i.e., iff the original gate $g''$ captured the empty set in $\nu(C)$, so again we have an equivalence.

This concludes the proof of preservation of upwards-determinism, and concludes the proof.

### C.3 Proof of the Main Circuit Theorem

**Theorem 5.4.** Given an upwards-deterministic, $d$-DNNF, homogenized hybrid circuit $C$ with constant fan-in, given an initial Boolean valuation $\nu$ of $C_{Init}$, there is an enumeration algorithm with linear-time preprocessing to enumerate the set $S_\nu(C)$ captured by $C$ under $\nu$, with linear delay and memory in each produced assignment, and with update time in $O(\Delta(C))$: an update consists here of toggling one value in $\nu$.

**Proof.** See Appendix E.2 for the proof of this result.
**Proofs for Section 6 (Enumerating Assignments of Hybrid Circuits)**

**Lemma 6.1.** Given a hybrid circuit $C$ and valuation $\nu$ of $C$, we can compute $\omega_\nu$ in linear time from $\nu$ and $C$. Further, for any $g \in C_{\text{vvar}}$, letting $\nu'$ be the result of toggling the value of $\nu$ on $g$, we can update $\omega_\nu$ to $\omega_{\nu'}$ in time $O(\Delta(g))$.

**Proof.** We explain how to compute $\omega_\nu$ bottom-up in linear-time, in a way which is clearly correct by induction:

- For a Boolean variable gate $g$, we set $\omega_\nu(g) := \nu(g)$.
- For $\lor$-gates, $\land$-gates, and $\neg$-gates, we compute $\omega_\nu(g)$ from the $\omega_\nu$-value of the input gates with the Boolean operation indicated in the gate type.
- For a set-valued variable gate $g$, we set $\omega_\nu(g) := 1$.
- For a $\exists$-gate $g$, letting $g'$ and $g''$ be its two inputs, we set $\omega_\nu(g) := \omega_\nu(g') \land \omega_\nu(g'')$.
- For a $\times$-gate $g$ with two inputs $g'$ and $g''$, we set $\omega_\nu(g) := \omega_\nu(g') \lor \omega_\nu(g'')$.
- For a $\cup$-gate $g$, letting $g_1, \ldots, g_n$ be its inputs, we set $\omega_\nu(g) := \lor_i \omega_\nu(g_i)$.

For updates, whenever $\nu$ is toggled on $g \in C_{\text{vvar}}$, it is easy to see that, for any gate $g'$ of $C$, if $\omega_{\nu'}(g') \neq \omega_{\nu}(g')$ then $g' \in \Delta(g)$. Hence, we can update $\omega_{\nu'}$ to $\omega_{\nu'}$ within the prescribed time bound simply by taking $\omega_{\nu'}$ to be $\omega_{\nu}$ initially, and then recomputing $\omega_{\nu'}$ on $\Delta(g)$ according to the above scheme.

**Lemma 6.2.** For any $\times$-gate $g$ with inputs $g_1$ and $g_2$, if we can enumerate $S_\nu(g_1)$ and $S_\nu(g_2)$ with delay and memory respectively $\theta_1$ and $\theta_2$, then we can enumerate $S_\nu(g)$ with delay and memory $\theta_1 + \theta_2 + c$ for some constant $c$.

**Proof.** We enumerate all assignments for $g_1$, which is non-empty because $\omega_\nu(g_1) = 1$; further, every assignment is non-empty because $C$ is homogenized. For each assignment $a$, we enumerate all assignments for $g_2$, again a non-empty set of non-empty assignments, and for each assignment $a'$, we return the assignment $a \cup a'$, where the union is disjoint thanks to the determinism of $g$. The same is true for memory usage, because we only need to remember the state in the enumeration on $g_1$ and $g_2$ as well as a pointer to $g$.

**Lemma 6.3.** For any $\exists$-gate $g$, if we can enumerate $S_\nu(\delta(g))$ with delay and memory $\theta$, then we can enumerate $S_\nu(g)$ with delay and memory $\theta + c$ for some constant $c$.

**Proof.** There is nothing to explain beyond what is given in the main text before the lemma statement.

**Lemma 6.4.** For any valuation $\nu$ and $\cup$-gate $g$, we have $S_\nu(g) = \bigcup_{g' \in D_\nu} S_\nu(g')$. Further, this union is disjoint and all its terms are nonempty.

**Proof.** The first part of the result is easy to prove by bottom-up induction on the $\cup$-gates. Specifically, for any $\cup$-gate $g$, letting $g_1, \ldots, g_n$ be its inputs, we have by definition $S_\nu(g) := \bigcup S_\nu(g_i)$. For the $g_i$ which are $\exists$-gates, we can replace them by their one-set-valued child provided that the $\omega_\nu$-image of their input under $\nu$ is 1, otherwise we can remove them from consideration. Repeating this process on $\exists$-gates as long as possible, we eliminate some gates, and for those that remain, we reach a gate which is not a $\exists$-gate (specifically, the $\delta$-image of the corresponding $g_i$). By this reasoning, it is clear that $S_\nu(g)$ is the union of the $S_\nu(g_i)$ for the set $S_1$ of inputs $g_i$ of $g$ which are not $\exists$-gates, unioned to the union of the $S_\nu(g')$ for the set $S_2$ of gates $g'$ reached by going down live paths of $\exists$-gates as we
explained. Now, as $D'_g$ consist of the gates of $S_1$, plus the sets $D''_g$ for the gates $g''$ of the set $S_2$, we can conclude by induction that the claim made in the first part of the lemma holds for $g$, concluding the proof of the first part.

For the second part of the claim, the fact that the union is disjoint is thanks to determinism: assuming by contradiction that there is an assignment $a$ such that $a \in S_\nu(g')$ and $a \in S_\nu(g'')$ for two different gates $g' \neq g''$ of $D'_g$, consider a live $∪$-path $\pi' : g' = g'_1 \rightarrow g'_2 \cdots \rightarrow g'_n = g$ from $g'$ to $g$, and a live $∪$-path $\pi'' : g'' = g''_1 \rightarrow g''_2 \cdots \rightarrow g''_m = g$ from $g''$ to $g$. Let $g''' = g'_1 = g''_1$ be the first gate where these two paths join; we have $i > 1$ and $j > 1$ because $g' \neq g''$. Hence, we know that $g'''$ is a $∪$-gate such that $g'_{i-1}$ and $g''_{j-1}$ both have a live $\Xi$-path to $g'''$; further $g'_{i-1} \neq g''_{j-1}$ because $g'''$ is the first gate where the paths join. The paths $\pi'$ and $\pi''$ clearly witness that $a \in S_\nu(g'_{i-1})$ and $a \in S_\nu(g''_{j-1})$. Now, let $h'$ be the last gate of a witnessing live $\Xi$-path $\rho'$ from $g'_{i-1}$ to $g'''$, and $h''$ be the last gate of a witnessing live $\Xi$-path $\rho''$ from $g''_{j-1}$ to $g'''$. We have $h' \neq h''$ (otherwise we would have also $g'_{i-1} = \delta(h') = \delta(h'') = g''_{j-1}$, contradicting our earlier claim). Again $\rho'$ and $\rho''$ witness that $a \in S_\nu(h')$ and $a \in S_\nu(h'')$. But $h'$ and $h''$ are two different inputs to the $∪$-gate $g'''$, so we have witnessed a violation of determinism, a contradiction. Hence, indeed, the union is disjoint.

Last, the fact that none of the terms is empty is because the definition of live $∪$-paths enforces that $\omega_\nu(g') = 1$ for all $g' \in D'_g$, so we conclude by definition of $\omega_\nu$.

**Lemma 6.5.** For any $∪$-gate $g$, if we can enumerate $D'_g$ with delay and memory $c$, and can enumerate $S_\nu(g')$ for every $g' \in D'_g$ with delay and memory $θ$, then we can enumerate $S_\nu(g)$ with delay and memory $θ + c + c'$ for some constant $c'$.

**Proof.** This follows immediately from Lemma 6.4 and the explanations given in the main text before the lemma statement.

**Lemma 6.6.** The switchboard can be computed in linear time given $C$ and $ν$, and we can update it in time $O(Δ(C))$ when toggling any gate in $ν$.

**Proof.** We can compute $B$ in linear time during the preprocessing by processing all edges of $C$, and using the characterization: for each wire $(g', g)$ of $C$, if $g$ is a $∪$-gate, we add to $E$ the edge $(δ(g'), g)$, using $δ$ which has already been computed.

We can compute $β_ν$ in linear time during the preprocessing, from $ω_ν$ (which has already been computed): initialize $β_ν$ by mapping all edges of $B$ to 0, and for each wire $(g', g)$ of $C$ such that $g$ is a $∪$-gate, set $β_ν((δ(g'), g)) = 1$ if $ω_ν(g') = 1$.

For the claim on updating the wiring, let us define the dependent set $Δ_E(g)$ of a gate $g$ in $B$ like we did for circuits. Observe now that, by construction, for any gate $g$ of $C$, we have $Δ_E(g) \subseteq Δ(g)$. Now, when we update $ν$ by toggling the value of $g \in C_{var}$, then we can update $β_ν$ in $O(|Δ(g)|)$. Indeed, we know that we can update $ω_ν$ in this time, and that it only changes on gates in $Δ(g)$. Hence, we can map to 0 all edges $(g', g)$ of $E$ such that $g' \in Δ(g)$, and recompute $β_ν$ on these edges. As $B$ has constant fan-in, the number of such edges in $O(|Δ_E(g)|)$, which is $O(|Δ(g)|)$, achieving the bound.

**Claim 6.7.** For any valuation $ν$ of the hybrid circuit $C$, the DAG $β_ν(B)$ is a forest.

**Proof.** The claim equivalently states that no gate $g \in V$ such that $β_ν((g, g')) = β_ν((g, g'')) = 1$ for two different gates $g' \neq g''$. (Pay attention to the fact that the edges of the forest are oriented upwards rather than downwards, following the direction of the wires in circuits.)

To show this, let us assume to the contrary that there is a valuation $ν$ such that there are gates $g$ and $g' \neq g''$ with $β_ν((g, g')) = β_ν((g, g'')) = 1$, and let us conclude a violation of upwards-determinism on $g$. First, these $β_ν$-values imply in particular that $ω_ν(g) = 1$. Now,
(f) Illustration of the index structure for Theorem 7.1 on an example reachability forest (drawn with thick edges). Exits are drawn as squares, other vertices are drawn as circles. The next pointers are drawn in straight dashed red lines (from left to right), and the first and last pointers of each node are drawn as curved solid blue lines (from top to bottom), respectively at the left and right of the node. Pointers that are null are not drawn.

consider two witnessing live \( \boxplus \)-paths \( g = g'_1 \rightarrow \cdots \rightarrow g'_n = g' \) and \( g = g''_1 \rightarrow \cdots \rightarrow g''_m = g'' \) where \( g'_2, \ldots, g'_{n-1} \) and \( g''_2, \ldots, g''_{m-1} \) are \( \boxplus \)-gates whose \( \omega_v \)-image is 1. Let \( g'''' = g'_i = g''_j \) be the last common gate of these two paths; as \( g' \neq g'' \), we have \( i < n \) and \( j < m \). Consider the wires \( (g'''', g'_{i+1}) \) and \( (g'''', g''_{j+1}) \). The gate \( g'_{i+1} \) is either \( g' \), in which case it is a \( \cup \)-gate and the wire is pure, or it is a \( \boxplus \)-gate whose \( \omega_v \)-image is 1, i.e., its second input evaluates to 1 under \( v \). The same is true of \( g''_{j+1} \). Hence, these two wires witness a violation of the upwards-determinism condition on \( g'''' \) in \( v(C) \). This is a contradiction, which concludes the proof.

\[ \Box \]

### E Proofs for Section 7 (Reachability Indexing under Updates)

#### E.1 Proof of Theorem 7.1

In this appendix, we prove Theorem 7.1. Recall the result statement:

\[ \blacktriangleright \textbf{Theorem 7.1.} \text{Given a reachability forest } F, \text{ there is an enumeration algorithm with linear-time preprocessing such that: (i.) given any } v \in V, \text{ we can enumerate } \text{reach}(v) \text{ with constant delay and memory; (ii.) given an update } \pm E', \text{ we can apply it (replacing } F \text{ by } F \pm E' \text{ and updating the index) with update time in } O(A_F(E')). \]

To show this result, we only need to argue that we can compute and update the pointers in our index (illustrated on an example in Figure (f)). Indeed, as we have explained in the main text, when we have these pointers, we can perform enumeration with constant delay and memory using them.

Hence, the only thing to show is the following result:

\[ \blacktriangleright \textbf{Lemma 7.2.} \text{Given a reachability forest } F, \text{ we can compute the first, last, and next pointers of all vertices in time } O(|F|). \text{ Further, for any update } \pm E', \text{ we can apply it and update the pointers in time } O(A_F(E')). \]

Before we show the result, we make a simple observation on the complexity of updates. Remember that \( A_F(E') \) refers to the ancestry of the parent edges of the vertices of \( E' \) in \( F \), i.e., before the update is performed. We will sometimes want to process \( A_{F \pm E'}(E') \), i.e., the ancestry of \( E' \) in the forest after the update. However, the distinction between the two is inessential, because of the following result:

\[ \blacktriangleright \textbf{Claim E.1.} \text{For any reachability forest } F \text{ and update } \pm E', \text{ we have } A_F(E') = A_{F \pm E'}(E'). \]
Example for updating the index of Theorem 7.1 when inserting an edge (from the left forest to the right forest). We follow the same drawing conventions as in Figure (f).

Proof of Lemma 7.2. Let $F' := F \pm E'$. We first show the result for the first and last pointers. For the initial computation, we use the scheme described in the proof sketch; it clearly runs in linear time (it examines every edge of $F$ once), and it is correct by a straightforward induction.

To update these pointers, we observe that, for every vertex $v$ not in $\mathcal{A}_F(E')$, the pointers do not need to be changed: this is clear because, for every such $v$, the subtree in $F'$ rooted at $v$ is exactly the same as in $F$. Hence, outside of $\mathcal{A}_F(E')$, the first and last pointers are still correct, so it suffices to update the pointers in $\mathcal{A}_F(E')$. We do this by the same bottom-up scheme as for the initial computation. Specifically, for deletions, we process $\mathcal{A}_F(E')$ (i.e., the ancestry in the original $F$, before the update), but we perform the computation at each node based on its children after the update (i.e., ignoring children whose parent edge has just been deleted). For insertions, we process $\mathcal{A}_F(E')$, i.e., the ancestry in $F'$ after the update, and perform the computation at each node based on its children after the update (i.e., after all insertions have been performed): this gives the right complexity thanks to Claim E.1. The correctness of this update scheme is again shown by induction, using the additional base case that consists of the vertices outside of $\mathcal{A}_F(E')$, which are correct as we explained. The complexity is in $O(|\mathcal{A}_F(E')|)$, because we examine edges in the set $\mathcal{A}_F(E')$ and child edges of vertices of this set, so at most $(c + 1)|\mathcal{A}_F(E')|$ where $c$ is the constant degree bound. This concludes the proof for the first and last pointers.

For the next pointers, we give a more precise description of the scheme presented in the proof sketch. We process $F$ bottom-up and ensure that, whenever we are done processing a vertex $v \in F$, then the next pointers within the subtree rooted at $v$ are correct; but we do not specify anything about the next pointer of the last exit in this subtree. Initially, we
set all next pointers to null, which is correct as a base case for the leaves. Now, to process \( v \in F \) with children \( v_1 < \cdots < v_n \), assuming by induction that the next pointers within each subtree rooted at \( v_i \) are correct, we simply need to go over the \( v_i \) in order, maintaining a current last exit \( v' \) which denotes the last exit among all \( v_i \) seen so far, whose next pointer is currently null. The current last exit \( v' \) is initially null. When we look at \( v_i \), if \( \text{first}_F(v) = \text{null} \), we do nothing. Otherwise, if \( v' = \text{null} \), then we replace \( v' \) by \( \text{last}_F(v) \) and do nothing more. Otherwise, if \( v' \) is not null, then letting \( v'' := \text{first}_F(v_i) \), we set \( \text{next}_F(v') := v'' \) and we set \( v' := v'' \) as our new current last exit. It is clear that this process satisfies our invariant.

Now, when we process the root \( v \) of a tree using this scheme, in the case where the last reachable exit \( v' := \text{last}_F(v) \) is not null at the end of the process, our invariant does not guarantee anything about \( \text{next}_F(v') \); but we can simply ensure that the next pointers in that tree are correct (including the next pointer of the last exit) by setting \( \text{next}_F(v') := \text{null} \). The overall scheme clearly runs in linear time for the initial computation, and it is inductively correct.

We conclude by explaining the update scheme for the next pointers. Note that, this time, it is no longer the case that the next pointers to be updated are all in \( A_F(E') \): see Figure (g) for an example. However, intuitively, the vertices outside of \( A_F(E') \) whose next-pointers need to be updated are all reachable as the value of a last pointer for a vertex of \( A_F(E') \), so we can fix all pointers by re-running our bottom-up computation scheme on \( A_F(E') \). Initially, we keep \( \text{next}_F(v) := \text{next}_F(v) \), which requires no modifications on the index. As in the preprocessing, we will ensure as an invariant when performing the update that, when we are done processing a vertex \( v \), the next pointers in the subtree rooted at \( v \) in \( F' \) are all correct (specifically, for every reachable exit of \( X \) in \( F' \), the next pointer correctly points to the next exit if it exists); but again we do not specify anything about the next pointer of the last exit of this subtree. Note that the invariant is already satisfied for all vertices not in \( A_F(\pm E') \): their reachable subtree is unchanged between \( F \) and \( F' \), so all next pointers within the subtree are still correct.

We now process \( A_F(\pm E') \) bottom-up, relying on the above observation for the base case, and doing the inductive case exactly as in the preprocessing algorithm above. Like in the update scheme for first and last pointers, we process \( F' \) in the case of insertions (using Claim E.1 to ensure that the size bound is correct), and process \( F \) in the case of deletions (but at each node we do not take taking into account the children corresponding to edges of \( F \) that are deleted in \( F' \)). This processing allows us to ensure that all nodes in \( F' \) satisfy the invariant. Now, as before, once we have processed a vertex which is the root of a tree in \( F' \), then we set the next pointer of the last reachable exit of the root of this tree to null to ensure that, in addition to the invariant, all next pointers in its tree are correct (including the last one).

At the end of this processing, the invariant is ensured on all trees, and further we know that the last exit of each tree correctly has null as its next pointer. Hence, we have correctly recomputed the next pointers in the prescribed time bound. Hence, we have explained how to handle updates for the next pointers, which concludes the proof of Lemma 7.2.

### E.2 Putting Everything Together

In this appendix section, we recap the proof of our main results. We first prove Theorem 5.4:

**Proof of Theorem 5.4.** We apply the scheme of Section 6. Given \( C \) and \( \nu \), we compute the shortcut function \( \delta \), the partial evaluation \( \omega_\nu \), and the switchboard composed of the panel \( B = (V,E) \) and its wiring \( \beta_\nu \). Further, we compute the index structure of Theorem 7.1 on
the DAG $\beta_\nu(B)$ of the edges of $B$ mapped to 1 by $\nu$ (choosing any arbitrary order on the children of each vertex), which is a forest by Claim 6.7. Note that the exits (non-$\sqcup$-gates) in the forest are the exits in the sense of Section 6 (i.e., the $\times$-gates and svar-gates in the panel), which are leaves by definition of the panel. Also note that, in the reachability forest, all nodes have degree no greater than the maximal fan-in of $C$ (so we can use it as degree bound $c$). We note again that the direction is reversed between $\beta_\nu(B)$ as defined in Section 6, and the reachability forest as studied in Section 7. Indeed, in the circuit, we want to enumerate the reachable exits of a $\sqcup$-gate $g$ in the sense of having a path (specifically, a live $\sqcup$-path) to $g$, whereas in the reachability forest, we enumerate the exits to which $g$ has a directed path. However, this is fine because upwards-determinism guarantees in Claim 6.7 that $\beta_\nu(B)$ is a forest with the edges are oriented upwards, so reversing the edges gives a forest in the sense of Section 7. This concludes the description of our preprocessing scheme, which runs in linear time.

To enumerate the assignments of the circuit, we use the scheme described by Lemmas 6.2, 6.3, and 6.5, as well as the explanations around them in the main text. To enumerate the set $D_\nu^\nu$ of the reachable exits of $g$ for Lemma 6.5, i.e., the exits having a directed path to $g$ in $\beta_\nu(B)$, we enumerate the set $reach_F(g)$ in the reachability forest, which is precisely what can be done with the index of Theorem 7.1. We summarize again why the enumeration is in delay linear in each produced assignment:

- Whenever we reach a $\boxtimes$-gate, we pay constant delay and memory and reach a gate which is not a $\boxtimes$-gate;
- Whenever we reach a $\sqcup$-gate, we pay constant delay and memory and reach a gate which is an exit, i.e., not a $\boxtimes$-gate or $\sqcup$-gate;
- Whenever we reach a $\times$-gate, we pay constant delay and memory to reach two other gates, and we will enumerate an assignment which is a disjoint union of the assignments enumerated at each gate, none of which is the empty assignment;
- Whenever we reach an svar-gate, we pay constant delay and memory to enumerate a singleton.

Hence, when enumerating an assignment $a$, we reach exactly $|a|$ svar-gates, and at most $|a| - 1$ $\times$-gates, so we reach at most $|a| + (|a| - 1)$ $\boxtimes$-gates and the same number of $\sqcup$-gates, hence the total delay is linear in the output assignment. This concludes the description of the enumeration scheme, which has delay and memory linear in each assignment.

We must now explain how updates are handled. Let $g$ be the Boolean variable whose value should be toggled in $\nu$. We modify $\nu$ to $\nu'$, use Lemma 6.1 to update $\omega_\nu$ to $\omega_{\nu'}$ in time $O(\Delta(C))$, use Lemma 6.6 to update $\beta_\nu$ to $\beta_{\nu'}$ in same time bound. Further, looking at the proof of Lemma 6.6, we know that the set $E'$ of edges $e = (g_1, g_2)$ of $E$ such that $\beta_\nu(e) \neq \beta_{\nu'}(e)$ must all be such that their second gate $g_2$ is in $\Delta(C)$ Let us split $E'$ into the edges $E_+ := \{ e \in E \mid \beta_{\nu'}(e) = 1 \land \beta_\nu(e) = 0 \}$ of that are added in $\beta_{\nu'}(B)$, and a set of edges $E_-$ analogously defined that are deleted in $\beta_{\nu'}(B)$; each of these edges satisfies that their second gate is in $\Delta(g)$. We update the indexing structure of Section 6 by first deleting $E_-$, and then adding $E_+$: the end result is still a forest by Claim 6.7, and the intermediate result is also a forest because we have performed deletions on a forest. We must now argue why each of these operations has the required complexity, i.e., $O(\Delta(C))$. To see why, observe that the ancestry of $E_-$ in the reachability forest before the deletions consists of $E_-$ plus edges where both endpoints are in $\Delta(g)$, so the ancestry has size $O(\Delta(C))$. Likewise, the ancestry of $E_+$ in the reachability forest before the insertions is a subset of the ancestry before the deletions, and in this case again it consists of $E_+$ plus edges where both endpoints are in $\Delta(g)$, hence again the ancestry has size $O(\Delta(C))$. Hence, the result
of Theorem 7.1 ensures that the complexity of updating the reachability structure is still in \(O(\Delta(C))\). This completes the description of updates, and the overall update complexity is indeed \(O(\Delta(C))\).

We can now show our main result:

**Proof of Theorem 3.1.** Given the tree alphabet \(\Gamma\) and MSO query \(Q(X)\), we use Lemma 4.3 to compute a tree alphabet \(\Gamma' \supseteq \Gamma\) and MSO query \(Q'(X)\). Now, given an input \(\Gamma\)-tree \((T, \lambda_0)\), we compute in linear time from \(T\) the \(\Gamma'\)-tree \(T'\) described by the lemma statement, and \(\lambda''_0\) as the valuation \(\lambda''\) in the lemma statement defined from the initial valuation \(\lambda_0\) of \(T\). Now, we can enumerate \(Q'\) on \(\lambda''(T')\) instead of \(Q\) on \(\lambda(T)\), and whenever an update operation changes \(\lambda\), then it takes constant time to translate it to an update on \(\lambda''\). Hence, we can work only with \(Q', T', \text{and } \lambda''\), without changing our bounds; and we know that \(T''\) is balanced, i.e., \(h(T'') = O(\log(|T'|))\).

We now use Theorem 5.2 to compute a hybrid circuit \(C\) capturing the provenance of \(Q''\) on the unlabeled tree \(T'\). We know that \(C\) is an upwards-deterministic d-DNNF with constant fan-in, and that its dependency size is in \(O(\log(|T'|))\). We do this as part of the linear-time preprocessing, computing also an initial Boolean valuation \(\nu\) of \(C_{\text{bvar}}\) from the initial valuation \(\lambda''\) of \(T'\). The definition of provenance circuits then ensures that we can enumerate \(Q'(\lambda''(T'))\) simply by enumerating \(\nu(C)\), and that we can reflect updates of \(\lambda''\) by translating them in constant time to an update on \(\nu\).

We now use Lemma 5.3 to make the circuit homogenized while preserving the d-DNNF, upwards-deterministic, and bounded fan-in as well as the dependency size requirements (and adding a secondary output gate). We now conclude our proof by appealing to Theorem 5.4: we can enumerate the assignments of \(C\) with linear-time preprocessing, delay and memory linear in each produced assignment, and handle updates to \(\nu\) in time linear in the dependency size of \(C\), that is, in \(O(\log(|T'|))\). This result ignores the secondary output added when homogenizing the circuit, so we may miss the empty assignment whenever it is captured, but we can simply extend Theorem 5.4 to handle the secondary output gate \(g_1\) by starting the enumeration with the empty assignment if we have \(\omega_\nu(g_1) = 1\). This achieves the desired bounds, and concludes the proof.

**F** Proofs for Section 8 (Applications)

**Theorem 8.1.** For any aggregate query \(Q(x)\) on \(\Gamma\)-trees with semiring \(K\), given a \(\Gamma\)-tree \(T\) and mapping \(\rho: T \rightarrow K\), we can compute \(Q_\rho(T)\) in time \(O(|T|)\), and recompute it in time \(O(\log |T|)\) after any update that relabels a node of \(T\) or changes a value of \(\rho\).

**Proof.** As explained in the sketch, the first step is to show the analogue of Theorem 5.4 where we want to compute \(\rho(S_\nu(C))\) instead of enumerating \(S_\nu(C)\), and where updates can additionally change \(\rho\). In this variant, we do not apply the homogenization result (Lemma 5.3), so we work with a circuit that directly captures the set of assignments of which we want to compute the \(\rho\)-image (i.e., the empty assignment is captured directly, without the need for a secondary output). We can then perform the initial computation with a much simpler variant of the preprocessing scheme, namely, we compute a function \(\omega'\) that maps each Boolean gate of \(C\) to its Boolean value \(V_\nu(y)\) under \(\nu\), and maps each set-valued gate of \(C\) to the value \(\rho(S_\nu(C)) \in K\). We compute \(\omega'\) bottom-up using the analogue of Lemma 6.1 (note that this did not depend on homogenization of the input circuit), changing the computation on set-valued gates as follows:
For a set-valued variable gate \( g \), we set \( \omega'_v(g) := \rho(g) \).

- For a \( \boxplus \)-gate \( g \) with no inputs, we set \( \omega'_v(g) := 1_k \).
- For a \( \boxtimes \)-gate \( g \) letting \( g' \) be its Boolean input and \( g'' \) be its set-valued input, we set \( \omega'_v(g) := \omega'_v(g') \) if \( \omega_v(g') = 1 \) and \( \omega'_v(g) := 0_k \) otherwise.
- For a \( \times \)-gate \( g \) with two inputs \( g' \) and \( g'' \), we set \( \omega'_v(g) := \omega'_v(g') \otimes \omega'_v(g'') \);
- For a \( \cup \)-gate \( g \), letting \( g_1, \ldots, g_n \) be its inputs, we set \( \omega'_v(g) := \bigoplus_i \omega'_v(g_i) \).

It is clear by induction that this computes the right value, and the computation takes linear time overall because semiring operations take constant time by our assumptions.

Now, whenever an update is performed on a variable gate \( g \) (either a set-valued gate, for updates to \( \rho \), or a Boolean gate, for updates to \( \nu \)), it is clear (like in Lemma 6.1) that the only gates whose \( \omega'_v \)-value may change are those of \( \Delta(g) \), so we can simply recompute \( \omega'_v \) on \( \Delta(g) \) in time \( O(|\Delta(g)|) \).

We can then conclude the proof using this variant of Theorem 5.4 like we proved Theorem 3.1 from Theorem 5.4, except that we do not apply Lemma 5.3. In particular, we make sure to apply Lemma 4.3 before invoking the enumeration result on circuits, to ensure that the height of the input tree, hence the dependency size of the circuit and the time bound on updates, are in \( O(\log |T|) \); we can do this because Lemma 4.3 preserves exactly the set of assignments, so it also preserves the \( \rho \)-image of this set.

**Corollary 8.2.** For any MSO query \( Q(x) \) on \( \Gamma \)-trees, given a \( \Gamma \)-tree \( T \), we can compute the number \( |Q(T)| \) of answers of \( Q \) on \( T \) in time \( O(|T|) \), and we can update it in time \( O(\log |T|) \) after a relabeling of \( T \).

**Proof.** We apply Theorem 8.1 using the semiring \( \mathbb{N} \) with usual addition and product (assumed to take unit time in the RAM model) and the mapping \( \rho \) that maps each node to 1. This ensures that, for each assignment \( A \) (including the empty assignment), we have \( \rho(A) = 1 \); hence, for each set \( S \) of assignments, we have \( \rho(S) = |S| \), the number of assignments in the set. Thus, Theorem 8.1 implies the desired result.

**Theorem 8.5.** For any group-by query \( Q(x, y) \) and semiring \( K \), given a \( \Gamma \)-tree \( T \) and \( \rho : T \rightarrow K \), we can enumerate \( Q_T(T) \) with linear-time preprocessing and delay in \( O(\log |T|) \)

**Proof.** Fix the group-by query \( Q(x, y) \) and the tree alphabet \( \Gamma \). Let \( \Gamma_x \) be the alphabet where we add one label \( l_x \) for each \( x_i \in x \). Let \( Q'(y) \) be the query obtained from \( Q \) by reading the valuation of \( x \) on the tree using the labels \( l_x \), i.e., we add a conjunct asserting that, for each \( i \), there is exactly one tree node carrying label \( l_x \), and we quantify over \( x \) so that \( x_i \) is interpreted as this one node. It is clear that for any unlabeled tree \( T \) and labeling \( \lambda : T \rightarrow \Gamma \), for each tuple \( B \) of nodes of \( T \), letting \( \lambda_B : T \rightarrow \Gamma_x \) be the valuation of \( T \) defined by \( \lambda_B : n \mapsto \lambda(n) \cup \{l_x \mid n = B_i \} \), we have that \( \rho(Q'((\lambda_B(T))) \) is equal to \( \rho(\mathcal{G}(B)) \) for the group \( \mathcal{G}(B) \) associated to \( B \) on \( \lambda(T) \). Hence, let us apply Theorem 8.1 to the query \( Q' \), the semiring \( K \), and the mapping \( \rho \), on the tree \( \lambda_B(T) \) for some arbitrary choice of \( B_0 \). We do this as part of our linear-time preprocessing, and this describes the first enumeration structure.

We now describe the second enumeration structure. We consider the query \( Q''(x) := \exists y Q(x, y) \). It is clear that, for any \( \Gamma \)-tree \( T \), the output \( Q''(T) \) of \( Q'' \) on \( T \) consists of the tuples \( B \) such that the group \( \mathcal{G}(B) \) of \( Q \) on \( T \) is non-empty. Hence, we apply Theorem 3.1 to this query, as part of our linear-time preprocessing.

We now enumerate the non-empty groups as follows. We first enumerate the output of \( Q''(\lambda(T)) \) in constant-delay using the second enumeration structure. Each produced tuple \( B \) corresponds to a non-empty group \( \mathcal{G}(B) \). We now modify the labeling function used in the
first enumeration structure to $\lambda_B$. To do so, we must change at most $2m$ labels, where $m$ is the arity of $x$; as $m$ is a constant, this is a constant number of updates, so the complexity of doing this update on the first structure is in $O(\log |T|)$. Now, the first structure gives us the aggregation value $\rho(\mathcal{G}(B))$, and we can produce the pair $(B, \rho(\mathcal{G}(B)))$ with delay $O(\log |T|)$. This concludes the description of the enumeration phase, and concludes the proof.