Strangeness content of the pion in the U(3) Nambu-Jona-Lasinio model

Fábio L. Braghin
Instituto de Física, Federal University of Goias,
Av. Esperança, s/n, 74690-900, Goiânia, GO, Brazil

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Abstract

The Nambu-Jona-Lasinio model is considered with flavor-dependent coupling constants
\[ G_{ij} \left[ (\bar{\psi}_i \lambda_i \psi)(\bar{\psi}_j \lambda_j \psi) + (\bar{\psi}_i \gamma_5 \lambda_i \psi)(\bar{\psi}_j \gamma_5 \lambda_j \psi) \right] \] for \( i, j = 0, 1, \ldots, N_f - 1 \), and \( N_f = 3 \). A self consistent calculation of quark effective masses and coupling constants is performed making the strange quark effective mass to vary considerably. Quantum mechanical mixings between up, down and strange constituent quarks yields a strangeness content of the light u and d quarks constituent and of the pion. Different types of estimates for the strangeness contribution for the pion mass are provided. Mixing type interactions, \( G_{i\neq j} \), induce the light mesons mixings and estimates for the \( \pi^0 - \eta \) and \( \eta - \eta' \) mixings are provided. The \( \eta - \pi^0 \) mixing is argued to be an indication of the strangeness content of the pion.

1 Introduction

The detailed description of hadron spectra with analytical methods in Quantum Chromodynamics presents many difficulties. Usually it requires some approximate schemes being also possible to resort to effective models valid within a range of a variable, usually low energy, associated to some physical scale. In spite of the limitations of an effective model, when compared to first principles calculations, in many cases they manifest the most important degrees of freedom and allow for a deeper understanding of Strong Interactions. One also expects that improvements can be done and eventually may produce a framework hopefully comparable to effective field theories (EFT). This might be achieved if fundamental properties of QCD are taken into account by introducing the correct degrees of freedom in a suitable and correct way. Besides that, effective models can show very clearly the main connections between observable and the corresponding relevant degrees of freedom. The quark-level Nambu-Jona-Lasinio model (NJL) \[ \text{NJL} \] captures some important features of quark dynamics. It has shown to be appropriate to describe several aspects of hadrons dynamics, in particular the light hadron spectra, whenever Dynamical Chiral Symmetry Breaking (DChSB) plays an important role. It provides a framework, in general, consistent with the constituent quark model \[ \text{QCD} \]. Constituent, or dressed, quark masses are obtained with contributions of the chiral condensates that, added to the masses originated from the Higgs boson, provides the correct scale of magnitude of hadrons masses \[ \text{QCD} \]. Usually, DChSB is only produced as long as coupling constants are minimally strong and this imposes restrictions in coupling constant of the NJL model. A strongly interacting
A microscopic origin for the NJL coupling constant, as discussed above, must rely only on gluon exchange that is independent of flavor and this corresponds to chirally symmetric quark dynamics if quark loops are not included. Non-degeneracy of quark masses must however manifest on quark dynamics, and, the way it manifest at the hadron level might involve different effects. Being an effective model for QCD, it is reasonable to expect that all the (free) parameters of the model might be traced back to degrees of freedom of QCD. At the QCD level the quark current masses are the only parameters containing flavor symmetry breaking. Therefore, in an effective model, contributions of the current mass differences should be expected in all the free parameters of the effective model, similarly to the underlying ideas for an effective field theory (EFT) \cite{39, 40, 41}. In \cite{32, 33, 38, 44} the background field method was employed to calculate quark-antiquark effective interactions at the one loop level. Quark field is split in sea quarks and background quarks that might correspond to constituent quarks eventually. Both the NJL-model and the Global Color Model (GCM) were considered. The same structure obtained with the NJL model is recovered in the very long wavelength limit for the GCM, in particular when zero momentum exchange limit is taken. Preliminary perturbative estimation for pseudoscalar and scalar light mesons masses showed that flavor-dependent coupling constants change resulting mesons masses slightly less than the flavor-dependence of quark effective masses. The sizable corrections nevertheless improve the description of several observable. In the present work, a self consistent calculation of coupling constants and effective
masses will be done. The resulting s-content of u and d constituent quarks can be understood in terms of
the quantum mixings \[45\,7\]. The need to deal with eigenstates of two different flavor-U(N) representations
for quarks and quark-antiquark mesons generates mixings of quarks and mesons. Fundamental up-down-
strange quark mixings are given in terms of the Cabibbo angle for the Cabibbo-Kobayashi-Maskawa (CKM)
matrix \[46\,26\]. The parameterizations of quark and mesons mixings are therefore established.

Therefore, in this work, the strangeness content of the u and d constituent quarks and of the neutral and
charged pion masses will be analyzed in the NJL model with flavor-dependent coupling constants. Because
we consider a polarization process with the contribution of gluon dynamics by means of an effective gluon
propagator, there appears the need to normalize the resulting coupling constants with respect to the initial,
standard, NJL -coupling constant. This normalization will be different from the one adopted in Ref. \[38\]
and it will favor a faster convergence of the self consistent calculation of masses and coupling constants. As
we add different components of the coupling constants and perform a self consistent calculation for effective
masses and coupling constants the resulting effective masses will change considerably. Therefore the (input)
parameters of the model either must be redefined in a fitting procedure or the coupling constants in the BSE
might be eventually dressed differently from the one in the gap equation because of the non-renormalizability
of the model. Therefore in the BSE considered to calculate mesons bound state, one might need either to
perform a new fit of the current quark masses or to truncate the equations by keeping a constant \(G_0\) for
the part of the equation that contains the quadratic divergence. With this choice, results become similar
to the results obtained perturbatively in Ref. \[38\]. We choose the latter procedure and left the overall
complete new fit of parameters for another work. Different estimations of the strangeness contribution for
their masses will be provided. Estimates for the \(\eta - \eta'\) and \(\pi^0 - \eta\) angle mixings are also provided. For
these estimates, the masses of \(\eta\) and \(\eta'\) will not be computed, and the mixing angles will be computed in a
restricted way with the mixing interactions, \(G_{i\neq j}\). This can be done by imposing the corresponding meson
mass differences. For this, the auxiliary field method will be considered in a more general prescription than
adopted in \[38\] but results are very similar. Light mesons mixings must be proportional to the light quark
mass differences and therefore have small amplitudes \[47\,48\]. Moreover, it will be argued that the \(\eta - \pi^0\)
mixing can provide information about the strangeness-content of the pion, including a contribution for its
mass. A contribution for the pion mass will be computed by assuming sea strange quark masses to be of
the order of a constituent strange quark mass in a rest frame. Whereas the quantum mixing is considered
for the calculation of quark effective masses and coupling constants, the mixing type interactions, \(G_{i\neq j}\)
or \(G_{f_1\neq f_2}\), will only be considered for the estimation of mesons mixings.

The work is organized as follows. In the next section the whole framework will be reminded with
particular attention to the definition of the coupling constants. The logics of the self consistent calculation
of \(G_{ij}\) and quark effective masses will be emphasized. The bound state equation (BSE), a Bethe-Salpeter
equation at the Born level, for the quark-antiquark pseudoscalar mesons will be also briefly reminded. The
NJL model is a non renormalizable model intended to be valid for global properties of hadrons at lower
energies and, as such, its calculated observables do depend on a chosen ultraviolet (UV) cutoff. Moreover,
results from the NJL model are known to depend on the chosen regularization scheme. However, it has
been found in different works that the difference among the different schemes for many observables are not
really large for light hadrons \[49\,2\]. In the present work the three-dimensional (Euclidean) momentum
cutoff scheme is adopted. Numerical results will be presented in the following section for sets of coupling
constants generated by three different gluon propagators. Results will be compared with a calculation for
the flavor-independent NJL model with a coupling constant of reference \(G_0\). The neutral pion and kaon
masses, or conversely the charged pion and kaon masses, will be used to fix the set of parameters with
which further observables are also presented to assess the overall predictions of the model within the self
consistent calculation. After the self consistent calculation, that fixes the parameters, the strange quark
effective mass will be freely varied. Nevertheless the self consistency of the up and down constituent quark effective masses and the coupling constants is maintained. With this procedure one expects to understand the role of the quark-antiquark strange condensate on the up and down quark effective masses and pion masses. The dependence of the pion decay constant with the strange quark effective mass is also presented. Several observables, typically estimated within the NJL model, are also calculated. Among them, the angle associated to the \( \eta - \eta' \) and \( \eta - \pi^0 \) mixings are provided by considering the flavor-dependent interactions \( G_{i\neq j} \) \[ f = 0,8 \]. Approximate estimations are done to reproduce the \( \eta - \eta' \) and \( \eta - \pi^0 \) mass differences (not the complete set of neutral pseudoscalar masses \( \eta, \eta' \) and \( \pi^0 \) simultaneously) for which one needs \( G_{08} \) and \( G_{38} \) respectively. A strangeness-content of the pion will be obtained from the \( \pi^0 - \eta \) mixing. Particular values for the up and down constituent quarks and for the neutral pion will be also presented for particular contributions of the strange quark condensate (or effective mass). In the last section there is a Summary.

2 Masses and coupling constants: a self consistent analysis

The generating functional of the NJL model with flavor dependent corrections to the coupling constants can be written as:

\[
Z[\eta, \bar{\eta}] = \int D\psi \exp \left[ i \int_x \left\{ \bar{\psi} S_0^{-1} \psi + G_{ij} \left( (\bar{\psi} \gamma_i \psi)(\bar{\psi} \gamma_j \psi) + (\bar{\psi} \gamma_5 \lambda_i \psi)(\bar{\psi} \gamma_5 \lambda_j \psi) \right) + L_s \right\} \right],
\]

where \( S_0^{-1} = (i\slashed{D} - m_f) \), where \( \slashed{D} \) is the U(1) covariant derivative, \( D_\psi = D[\psi, \bar{\psi}] \) is the functional measure, \( \int_x = \int d^4x \), the subscript \( f = u, d, s \) is used for the flavor \( SU(3) \) fundamental representation, \( i, j = 0, \ldots, N_f^2 - 1 \) is used for flavor indices in the adjoint representation, being \( N_f = 3 \) the number of flavors, and \( \lambda_i \) are the flavor Gell-Mann matrices with \( \lambda_0 = \sqrt{2/3} I \). Quark sources are encoded in \( L_s = \bar{\eta} \psi + \bar{\psi} \eta \). Usually to account for the axial anomaly the 't Hooft interaction is considered. It is a determinant of a \( N_f \times N_f \) matrix that can be written as: \( \mathcal{L}_{tH} = \kappa \left( \det(\bar{\psi} P_L \psi) + \det(\bar{\psi} P_R \psi) \right) \), where \( P_R/L \) are the chirality projectors and \( \kappa \) is a coupling constant taken as free parameter of the model. In the \( N_f = 3 \) model, this interaction is a 6th order quark self interaction that has been investigated in many works \[ 2, 3, 4, 17, 18, 19 \]. It is interesting to note that, in this \( N_f = 3 \) the same 6th order interaction, except for the value of the coupling constant, can be obtained from polarization correction for the NJL model by using the background field method \[ 42, 43 \]. In the present work all the flavor-dependent coupling constants will be given by \( G_{ij} \) only. The coupling constant has two components: \( G_{ij} = (G_0 + \tilde{G}_{ij}) \), where \( G_0 \) is a standard NJL-coupling constant. \( G_0 \) is flavor independent and therefore it must be due to gluon dynamics. This is a parameter of the model and a minimum critical value for it is required to provide DChSB in the NJL model \[ 2, 3 \]. As pointed out in the Introduction there are several estimations of \( G_0 \) from QCD degrees of freedom. By means of the background field method in the very long wavelength limit the flavor-dependent corrections were found to be given by \[ 38 \]:

\[
\tilde{G}_{ij} = d_2 N_c (\alpha g^2)^2 T_{\tau_D} T_{\tau_F} \int \frac{d^4k}{(2\pi)^4} S_{0f}(k) R(k) i \gamma_5 \lambda_i S_{0f}(-k) R(-k) i \gamma_5 \lambda_j,
\]

where \( T_{\tau_D}, T_{\tau_F} \) are the traces in Dirac and flavor indices, \( \alpha = 4/9 \), \( g^2 \) is the running quark-gluon coupling constant, \( d_n = (-1)^n \), \( S_{0f}(k) \) is the Fourier transform of the effective quark propagator \( S_{0f}(x - y) \) which account for the DChSB by means of the quark effective mass \( M_f \) or \( M_f^* \) as discussed below. In this last equation \( R(k) = 2 (R_{Tf}(k) + R_L(k)) \), where \( R_{Tf}(k) \) and \( R_L(k) \) are transversal and longitudinal components of an effective gluon propagator in a covariant gauge. Other types of contributions, due to gauge boson
dynamics and confinement, proportional to delta functions, \( \delta(p^2) \), provide smaller or vanishing contributions \[3.8\]. The corresponding Feynman diagrams of Eq. (2) are exhibited in Fig. (1) where the straight lines represent quarks and wiggly lines with a dot represents non perturbative gluon propagator. An alternative way of doing the calculation for \( G_{ij} \) - eq. (2) - would be the one-loop background field for the standard SU(3) NJL model, along the lines of Ref. [42]. However in the present version, we keep track of possible contributions of the specific (effective) gluon propagator making possible to compare the effects of different (effective) gluon propagators on constituent quark (or hadron) dynamics in an effective way.

Figure 1: Feynman diagrams that correspond to eq. (2), where the straight lines are quarks and wiggly lines with a dot represent a non perturbative (dressed) gluon propagator. The dots in the vertices represent the running quark-gluon coupling constant in the strong coupling limit.

The following important properties, due to CP and electromagnetic U(1) invariances, hold:

\[ G_{ij} = G_{ji}, \quad G_{22} = G_{11}, \quad G_{55} = G_{44}, \quad G_{77} = G_{66}. \]  

(3)

The mixing type interactions \( G_{i \neq j} \) are proportional to quark effective mass differences and therefore they have considerably smaller numerical values. As it can be seen from eq. (2) the flavor dependent coupling constants are not free parameters. The overall normalization in \( G_{ij} \), however, is arbitrary in the same way \( G_0 \) is. Within the usual approach for the NJL, scalar and pseudoscalar, \( S_i, P_i \), auxiliary fields are introduced by means of an unit integral multiplied in the generating functional with the corresponding shifts with quark currents that make possible the integration of the quark field. By considering the quark propagator with the electromagnetic quark coupling, the relations (3) are preserved. This gauge invariance has the same roots of the description in terms of auxiliary fields to describe electromagnetic couplings of charged mesons and their resulting couplings with (background) constituent quarks analyzed in [50, 44, 51, 52]. In the limit of degenerate quark effective masses, \( M_u = M_d = M_s \), the coupling constants reduce to a single constant that, as discussed below, will be normalized to be the coupling constant of reference \( G_0 \) such that: \( G_{ij} \rightarrow G_0 \delta_{ij} \) and the standard treatment of the model can be done. In this case the quark effective masses of constituent quarks are obtained with the contribution of the scalar-quark-antiquark condensate, \( M_f = m_f + S_f \), where \( S_f \) are the solutions of the auxiliary field gap equations. Gap equations might be found as saddle point equations for the scalar and pseudoscalar auxiliary fields \( S_i, P_i \) and non trivial solutions should emerge for the (neutral) scalar fields \( S_0, S_3, S_8 \). For the coupling constant of reference \( G_0 \) these equations can be written as:

\[ (G1) \quad M_f - m_f = G_0 Tr(S_0, f(0)). \]  

(4)

The gap equations for the model (1) however receive corrections from the flavor dependent coupling constants. The final values of the coupling constants \( G_{ij} \) therefore are obtained from a self consistent calculation with the corrected gap equations as discussed below such that:

\[ G_{ij} = G_{ij}(M_u^*, M_d^*, M_s^*). \]  

(5)
In these equations for $G_{ij}$ one has the first type of mixing interactions, i.e. $G_{i \neq j}$, that are numerically much smaller than the diagonal ones $G_{ii}$ because they depend on the differences between quark effective masses and they will not be considered in most part of this work.

Corrections to the coupling constant from polarization, however, might produce spurious increasing values of the NJL coupling constant that is a free parameters of the model. To make possible comparisons of numerical results from different choices of the gluon propagator for $G_{ij}$ with results from a coupling constant of reference, $G_0 = 10\text{GeV}^{-2}$, a normalization procedure will be adopted after the calculation of the integrals in eqs. (5). As discussed in the Introduction, lately one has associated $G_0 \sim 1/M_G^2$, where $M_G$ is an effective gluon mass. By neglecting further dimensionless constants, this would correspond to $M_G \approx 315\text{MeV}$, that is smaller than usual values obtained in lattice and SDE calculations. However this value for $M_G$ is close to the value considered in [53]. The larger value of $G_0$, when compared to usual NJL-model calculations, favors faster convergence of the self consistent numerical calculations. Since polarization process should produce corrections to an initial NJL-coupling constant, say $G_0$ the following resulting complete coupling constant should be obtained to compute observables:

$$G_{ij}^{\text{comp}} = \left( G_0 + \tilde{G}_{ij} \right) \bar{G}_0,$$

where $G_{ij}$ is obtained by eq. (2) and $\bar{G}_0$ is a renormalization factor that brings the resulting value to a value of reference whenever the symmetric limit is reached, i.e. $G_{ij}(M^*) = G_0 \delta_{ij} = 10\text{GeV}^{-2}$, being, in that limit, $M^* = M_u^* = M_d^* = M_s^*$. Besides that, the different effective gluon propagator with the running quark-gluon coupling constant, defined below, have different normalizations and it becomes important to normalize all the results by a common factor to make possible to understand the role of each of the variables in the set of parameters and effective gluon propagator. By choosing, for example, the charged pion mass to be a fitted parameter/observable for $G_{11}^{\text{comp}} = 10\text{ GeV}^{-2}$, the following normalization, written in the main text, can be used:

$$G_{i=j}^m \equiv G_{i=j}^{\text{comp}} = 10 \times \frac{G_{\text{sym}} \delta_{ij} + G_{ij}}{G_{\text{sym}} \delta_{ij} + G_{11}}.$$

Because the coupling constants $G_{11}$ is almost equal to $G_{33}$, it makes basically no difference to adopt neutral or charged pion mass to be a fitted parameter. In the flavor-symmetric calculation for the NJL model, polarization effect is also added to the original value of $G_0$ [42, 43]. For the mixing type interactions $G_{i \neq j}$ a similar reasoning is adopted, being that in the flavor symmetric limit and in the original NJL model $G_{i \neq j} = 0$, so that one can write:

$$G_{i \neq j}^m = 10 \times \frac{G_{ij}}{G_{\text{sym}}}.$$

This normalization is compatible with the one for diagonal $G_{ii}$ although it is somewhat arbitrary. This normalization (7) is different from the one considered in the perturbative investigation [38] and the numerical results for $G_{ij}$ are somewhat similar to the ones presented in the perturbative case just mentioned. Besides that, the present normalization was found to be more appropriate for the convergence of the self consistent numerical calculations. It will be discussed that this normalization might overestimate the role of flavor dependent interactions. The ’t Hooft interaction, however, has been neglected and results may, at the end, be reasonably close to realistic ones.

These corrections for the NJL-coupling constants can re-arrange quark effective masses. Restricting to the diagonal generators, $i, j = 0, 3, 8$, the coupling constants for the diagonal flavor singlet quark currents,
G_{ff} (f = u, d, s), can be defined in the following way:
\[ G_{ff}^{ij}(\bar{\psi}_i\gamma_5\lambda_1\psi)(\bar{\psi}_j\gamma_5\lambda_1\psi) = 2 \, G_{f_1f_2}(\bar{\psi}_i\gamma_5\lambda_1\psi)(\bar{\psi}_j\gamma_5\lambda_1\psi), \]
(9)
where \( \lambda_{f1} \) are three single-entry matrices obtained from combinations of diagonal Gell-Mann matrices with a single non zero matrix element that are: \( \lambda_u = 1e_{11}, \lambda_d = 1e_{22} \) and \( \lambda_s = 1e_{33} \), where \( e_{ii} \) is a diagonal matrix element. The following relations between the coupling constants \( G_{ii} \) and \( G_{ff} \) in the absence of the (numerically smaller) mixing-type interactions, \( G_{i\neq j} = G_{f_1\neq f_2} = 0 \), are obtained:
\[
\begin{align*}
2G_{uu} &= 2\frac{G_{00}}{3} + G_{33}^n + x_s \frac{G_{88}^n}{3}, \\
2G_{dd} &= 2\frac{G_{00}}{3} + G_{33}^n + x_s \frac{G_{88}^n}{3}, \\
2G_{ss} &= 2\frac{G_{00}}{3} + 4x_s \frac{G_{88}^n}{3},
\end{align*}
\tag{10}
\]
where \( x_s \) is an ad hoc parameter to control the strength of \( G_{88} \). For equal quark masses the flavor independent coupling constants reduce to an unique constant \( G_{ij} = G_{sym} \delta_{ij} \) and \( G_{f_1f_2} = G_{sym} \delta_{f_1f_2} \). Note that for the diagonal interactions \( i, j = 0, 3, 8 \) one has \( G_{uu} = G_{dd} \). Two cases for the strangeness content of the coupling constants will be considered by introducing a parameter \( x_s \) in \( G_{88} \) that provide the a contributions from the asymmetry of strange to up and down quark dynamics. Therefore an ad hoc parameter \( x_s \), that controls its strength will be introduced by multiplying \( G_{88} \) and it will be made variable to test with more details the contribution of the strangeness in the u and d sector. This parameter can be set \( x_s = 1 \) at any time, without loss of generality, in which case one can expect to reach a physical point that describes mesons masses. Also, it can be used to make the flavor-breaking content of \( G_{88} \) to be suppressed whenever \( x_s G_{88} = 10 \text{ GeV}^{-2} \), that is the value of reference for the flavor symmetric point. The following cases will be considered into the equations of \( G_{ff} \) as written above:
\[
\begin{align*}
(M2) \quad x_s G_{88} &= 10 \text{ GeV}^{-2}, \\
(M3) \quad x_s G_{88} &= G_{88} \text{ GeV}^{-2},
\end{align*}
\tag{11}
\]
In the first case, \( M2 \), the role of strangeness does not take into account the flavor- asymmetry interaction \( G_{88} \) which is obtained from the eighth flavor generator \( \lambda_8/2 \). The second case, \( M3 \), is obtained with a more realistic account of the strange quark content.

The gap equations for the flavor dependent coupling constants in the absence of mixing-type interactions can be written as:
\[
(G2) \quad M_f^2 - m_f = G_{ff} \, Tr \, (S_{0,f}(0)),
\tag{12}
\]
where \( Tr \) includes traces in color, flavor and Dirac indices and momentum integral, and \( S_{0,f}(x - y) \) is the quark propagator in terms of the quark effective masses \( M_f^* \).

2.1 Mesons bound state equation

Pseudoscalar auxiliary fields for the composite quark-antiquark states can describe pseudoscalar mesons. In particular for the case of the pseudoscalar mesons, the two point Green’s function have pole at a time-like momentum at zero tridimensional (Euclidean) momentum \( \vec{P} = 0 \). The NJL- model condition for the quark-antiquark pseudoscalar BSE can be written as:
\[
1 - 2G_{ij} I_{f_1f_2}^{ij} (P_0^2 = -M_{PS}^2, \vec{P}^2 = 0) = 0,
\tag{13}
\]
where

\[
I_{f_1f_2}^{ij}(P_0, \vec{P}) = iTr_{D,F,G} \int \frac{d^4k}{(2\pi)^4} \lambda_i i\gamma_5 S_{0,f_1}(k + P/2)\lambda_j i\gamma_5 S_{0,f_2}(k - P/2),
\]

where the different flavor indices for the case of the pion bound states are the following: \(\pi^0\) with \(i, j = 3\) and \(f_1, f_2 = u, d\) and \(\pi^\pm\) with \(i, j = 1, 2\) and \(f_1, f_2 = u, d\). Kaons and some of the scalar mesons were discussed in [38]. After the traces in Dirac, color and flavor indices have been calculated the equation is Wick rotated to the Euclidean momentum space-time and the condition for obtaining the mesons masses become \(P_0^2 = -M_{PS}^2\), where \(M_{PS}\) is the mass of the pseudoscalar meson.

The gap equations can be used to eliminate the quadratic divergence of \(I_{f_1f_2}^{ij}\). In particular for the case of the pions and kaons, one can write the following reduced equation:

\[
(M_{PS}^2 - (M_{f_1}^* - M_{f_2}^*)^2)G_{ij} I_{f_1f_2}^{ij} = \frac{G_{ij}}{2} \left( \frac{m_{f_1}}{G_{f_1f_1} M_{f_1}^*} + \frac{m_{f_2}}{G_{f_2f_2} M_{f_2}^*} \right) + 1
- \frac{1}{2} \left( \frac{G_{ij}}{G_{f_1f_1}} + \frac{G_{ij}}{G_{f_2f_2}} \right),
\]

where \(G_{f_1f_2}\) is the normalized coupling constant from the gap equations. To cope with the need of different renormalizations for the gap eqs. and BSE, the quark condensate from the gap eqs. will be renormalized by \(G_{f_1f_2}/G_0\) corresponding to the choice: \(G_{f_1f_2} \to G_0\) in these BSE. This guarantees the correct order of magnitude of the resulting neutral and charged pions and kaons masses. In this equation there as a logarithmic divergent integral given by:

\[
I_{f_1f_2}^{ij} = 4N_c \int \frac{d^3k}{(2\pi)^3} \frac{(E_{f_1} + E_{f_2})}{E_{f_1} E_{f_2} (M_{PS}^2 - (E_{f_1} + E_{f_2})^2)},
\]

where \(E_f = \sqrt{k^2 + M_f^2}\) in Euclidean momentum space. These integrals are solved with the same 3-dim cutoff \(\Lambda\) of the gap equations.

### 3 Numerical results

Flavor dependent coupling constants were calculated by considering three different effective gluon propagators each of the two different sets of current quark masses and UV cutoff: \(S\) and \(V\). These effective gluon propagators will be labeled by: \(\alpha = 2, 5\) and 6. They incorporate the quark-gluon running coupling constant \(g^2\) as shown below. The first effective gluon propagator (2) is a transversal one extracted from Schwinger Dyson equations calculations [54, 55]. It can be written as:

\[
(S_2, V_2): \quad D_2(k) = g^2 R_T(k) = \frac{8\pi^2}{\omega^4} D e^{-k^2/\omega^2} + \frac{8\pi^2 \gamma_m E(k^2)}{\ln \left[ \tau + (1 + k^2/\Lambda^2_{QCD})^2 \right]},
\]

where \(\gamma_m = 12/(33 - 2N_f)\), \(N_f = 4\), \(\Lambda_{QCD} = 0.234\) GeV, \(\tau = e^2 - 1\), \(E(k^2) = [1 - \exp(-k^2/[4m_t^2])]/k^2\), \(m_t = 0.5\) GeV, \(D = 0.55^3/\omega\) (GeV$^2$) and \(\omega = 0.5\) GeV.

The second type of effective gluon propagator is based in a longitudinal effective confining parameterization [53] that can be written as:

\[
(S_{\alpha=5,6}, V_{\alpha=5,6}): \quad D_{\alpha=5,6}(k) = g^2 R_{L,\alpha}(k) = \frac{K_F}{(k^2 + M_{\alpha}^2)^2},
\]

\(\alpha = 5, 6\)
where $K_F = (0.5\sqrt{2\pi})^2/0.6$ GeV$^2$, as considered in previous works [56] to describe several mesons-constituent quark effective coupling constants and form factors. However different effective gluon masses can be tested [57] such as a constant one: $(M_5 = 0.8$ GeV) or a running effective mass given by: $M_6 = M_6(k^2) = 0.5\frac{\sqrt{2\pi}}{1+k^2/\omega_6}$ GeV for $\omega_6 = 1$ GeV.

The sets of (free) parameters, that reproduce the neutral pion and kaon masses after the self consistent calculation, are given in Table 1: current quark masses $m_u, m_d, m_s$ and the ultraviolet (UV) cutoff $\Lambda$. In this Table the resulting effective masses from the gap equation (G1), for $G_0$, are also presented. It is important to emphasize that the only role of the effective gluon propagator is to produce numerical results for $G_{ij}$. The sets of parameters $S$ and $V$ yield the same overall behavior of results when $M_s^*$ is varied, therefore figures will be exhibited only for $S$. Having obtained these fittings from the self consistent calculation, for different sets of $G_{ij}$, kaons are neglected and the investigation of the contribution of the strangeness is done. For this, the free-variation of the effective mass, $M_s^*$, will be done, by keeping the effective masses of the up and down quarks calculated self consistently. The values of the mesons masses at the physical point, obtained from the sets of parameters of Table 1, will be shown below in Table 2. The strange quark current mass, $m_s$, is not really relevant for the pion observables, but it helps to define the physical point, where kaon masses are obtained, and to keep track of the value of the strange quark-antiquark condensate. For the chosen three-dim regularization scheme the resulting values of $\Lambda$ are not considerably larger than the resulting effective masses. Note however that the cutoff is used only for the three-momentum component, contrarily to the other regularization schemes for which the cutoff applies for the four-momenta [2, 49]. Therefore it is natural to expect a lower value for the cutoff in the three-dim regularization scheme.

### Table 1: Sets of parameters: Lagrangian quark masses, ultraviolet cutoff and the quark effective masses obtained from an initial NJL-gap equation (G1) for $G_0 = 10$ GeV$^{-2}$.

| set of parameters | $m_u$ (MeV) | $m_d$ (MeV) | $m_s$ (MeV) | $\Lambda$ (MeV) | $M_u$ (MeV) | $M_d$ (MeV) | $M_s$ (MeV) |
|------------------|-------------|-------------|-------------|----------------|-------------|-------------|-------------|
| $S$              | 3           | 7           | 133         | 680           | 405         | 415         | 612         |
| $V$              | 3           | 7           | 133         | 685           | 422         | 431         | 625         |

#### 3.1 Up and down quark effective masses and quark-antiquark coupling constants dependencies on $M_s^*$

In figures 2 and 3 results for the self consistent calculation for the up and strange flavor-dependent coupling constants, $G_{uu}$ and $G_{ss}$, are presented as functions of the (freely-varied) strange quark effective mass $M_s^*$ for the sets $S_2, S_5$ and $S_6$ and for the parameterizations $M2$ and $M3$. All the resulting $G_{ff}$ are normalized in the flavor symmetric point according to eq. (7), i.e. $G_{ff}(M^* = m^*) = 10$ GeV$^{-2}$ when $M_u^* = M_d^* = M_s^* = m^*$. Results are sounder physically for $M_s^* \geq m_s \simeq 0.133$ GeV, below this value, $M_s^*$ represents nearly a variable strange quark current mass in the absence of self-consistency. Result for the down quark is the same as $G_{uu}$, according to eqs. (10). The different coupling constants $G_{ij}$ obtained for the different effective gluon propagators ($S_2, S_5$ and $S_6$) may produce quite different numerical results (in particular for $M3$) although the overall behavior is basically the same. The behavior for small and large strange quark effective mass limits are quite different depending on the set $S_2, S_5$ or $S_6$. For lower values of $M_s^*$, the sets $S_2$ and $S_6$ that have larger variations than $S_5$. For larger strange quark effective masses the
quantities $G_{ff}$ become smaller and tend to reach finite definite values at $M^*_s \to \infty$ that tend to be nearly independent of the gluon propagator. Note that the strange quark effective mass that produces correct values for the kaon masses is around $0.550 - 0.580$ GeV (for $M^*3$), shown in the Table 4 below, for all the sets $S_2, S_5, S_6$ and for $V_2, V_5, V_6$. These large variations of $G_{ff}$ with $M^*_s$ may be indication that the (re)normalization prescription in eq. (7) overestimates the role of the flavor symmetry breaking.

Figure 2: The up quark coupling constant $G_{uu}$, eq. [10], as a function of $M^*_s$, arbitrarily varied. Up and down quark masses are obtained self consistently from their gap equations (G2).
Figure 3: The strange quark coupling constant $G_{ss}$, eq. (10), as a function of $M_s^*$, arbitrarily varied. Up and down quark effective masses are obtained self consistently from their gap equations (G2).

In figure (4) the up quark effective mass, as self consistent solution to the gap (G2), eq. (12), is presented as a function of the strange quark effective mass $M_s^*$ that is made to vary freely. Again, the figure has a clear meaning for $M_s^* > m_s$. The point in which all the cases coincide is the symmetric point $G_{ij} = G_{sym} \delta_{ij}$ due to the normalization adopted.

The self consistency is implemented for both cases $M2$ and $M3$ that controls the strangeness dependence of $G_{ss}$. The "physical value" of $M_s^*$, i.e., the value that reproduces the correct kaons masses, being solution of the gap equation $G2$, is around $0.500 - 0.600 \text{GeV}$ depending on set $S2, S5, S6$, as presented in Table (2). This self consistence procedure lowers the values of the quark effective masses. It is interesting to note that both limits, zero and very large strange quark effective mass, might be, in different ways, somehow associated to absence of strangeness in the up and down quark dynamics.
Figure 4: Self consistent solutions for the up quark effective mass, \( M_u^* \), obtained from the gap equation (G2) \(^{(12)}\), as a function of \( M_s^* \) that is arbitrarily varied.

In figure (5) the difference between the self consistent solutions of down and up quark effective masses, \( M_d^* - M_u^* \), is presented as a function of the strange quark effective mass \( M_s^* \) that is freely varied. Again it is important to stress that the quantity \( M_s^* \geq m_s \) corresponds to varying the strange chiral condensate arbitrarily. It can be seen that the parameterizations \( M3 \) (smaller symbols) yield much larger variation of \( M_d^* - M_u^* \) mainly for the case of smaller strange quark masses. The behavior with \( M_s^* \) is the opposite of the individual quark effective masses \( M_u^* \), \( M_d^* \). The difference in the results between the sets \( S2, S5, S6 \) (i.e. effective gluon propagator) reaches around only 1 MeV either for \( M2 \) or \( M3 \), that is of the order of 10% of the effective mass difference.
3.2 Pion mass dependence on $M_s^*$

In figure 6, the neutral pion mass as a function of the strange quark effective mass is exhibited for the self consistent values of $M_u^*$, $M_d^*$ and coupling constants and for $M_s^*$ freely varying. Because of the normalization adopted, the point in which all the cases coincide is the symmetric point for which $G_{ij} = G_{sym} \delta_{ij}$. The same sets of parameters S2, S5 and S6 were considered for the two parameterizations M2 and M3. Both limits of strange quark mass going to zero and going to infinite are well defined, although some points were left out of the figure to emphasize the behavior for $M_s^* > m_s$, i.e. for the strange quark condensate. It is seen that the variation of the pion mass with the strange quark effective mass is larger for smaller strange quark masses, i.e. smaller or vanishing strange quark condensate.
Figure 6: $M_{\pi_0}$ as a function of $M_{s}^*$, arbitrarily varied. All the other parameters $M_s^*, M_d^*$ and coupling constants are obtained self consistently.

Finally, in the figure (7) the mass difference of charged and neutral pions, $\Delta M_\pi = M_{\pi\pm} - M_{\pi^0}$, is exhibited as a function of the strange quark effective mass, arbitrarily varied. The neutral and charge pion mass difference is known to have a larger contribution from electromagnetic interactions and only a small counterpart from strong interactions. The value obtained in quite in agreement with known values $[58, 59, 60, 61, 62]$. Whereas parameterizations $M_3$, smaller symbols, provide small values of $\Delta M_\pi$ for smaller $M_s^*$, for large strange quark masses parameterizations $M_3$ tends however to produce an increase considerably larger than $M_2$. $M_2$ ($M_3$) makes the mass difference to reach a maximum value close to 0.15 $-$ 0.20MeV (0.15 $-$ 0.27MeV) for $M_s^* > 0.8$GeV.
Figure 7: $\Delta M_\pi = M_{\pi\pm} - M_{\pi\pi}$ as a function of $M_\pi^*$, arbitrarily varied. All the other parameters $M_u^*, M_d^*$ and coupling constants are obtained self consistently.

### 3.3 $\eta - \eta'$ and $\eta - \pi^0$ mixings

The pseudoscalar mesons mixings will be discussed next. For this, the explicit mixing interaction $G_{i\neq j}$ will be considered. The $\eta - \eta'$ mass difference will be obtained by means of the flavor-dependent coupling constants $G_{08}$. The auxiliary fields can be introduced by means of functional delta functions in the generating functional \[63, 64\], for the case of the pseudoscalar fields one can write:

$$1 = \int D[P_i] \delta \left( P_i - G_{ik} j^{k}_{ps} \right),$$  \hspace{1cm} \text{(19)}

where $j^k_{ps} = \bar{\psi} \lambda^k i \gamma_5 \psi$, $i, k = 0, 3, 8$ provides the needed components to describe the mesons $\eta, \eta'$ and $\pi_0$, and where the fields dimensions are properly taken into account. This method neglects possible non factorizations \[65\] which, nevertheless, can be expected to be small. This definition reduces to the usual auxiliary fields when mixing interactions are neglected. For the mixing-type interactions $G_{i\neq j}$, for $i, j = 0, 3, 8$, one can neglect the smaller one, $G_{03}$. Next the corresponding quark-antiquark states masses and mixings can be written in the adjoint representation, $M^2_{ii} P^2_i$, in a diagonalized form. In general, the following quadratic terms from the pseudoscalar auxiliary fields with the mixing interactions $G_{ij}$ can be written:

$$L_{\text{mix}} = -\frac{M^2_{88} P^2_8}{2} - \frac{M^2_{00} P^2_0}{2} + 2G_{08} \bar{G}_{08} P_0 P_8 + O(P_3, P^2_3) \ldots$$  \hspace{1cm} \text{(20)}

where $M^2_{ii}$ include the contributions from $G_{i=j}$ derived above, and

$$\bar{G}_{08} = \frac{2}{G^m_{00} \left( C^m_{88} - \frac{G^m_{08} - G^{m2}_{08}}{C^m_{00}} \right)},$$  \hspace{1cm} \text{(21)}
where the mixing terms $G_{i\neq j}$ are exclusively obtained from the one-loop polarization. As seen in eq. \cite{21} and the flavor dependent coupling constants $G_{ij} \propto N_c$, as $N_c \to \infty$ one has degenerate $\eta$ and $\eta'$ \cite{20}.

The change of basis from the singlet flavor states basis $|\bar{q}q \rangle$ ($q=u,d,s$), or correspondingly $P_3, P_8, P_0$, to the mass eigenstates $\pi^0, \eta, \eta'$ can be written as \cite{66,67}:

\[
\begin{pmatrix}
\pi^0 \\
\eta \\
\eta'
\end{pmatrix} = \begin{pmatrix}
1 & \sqrt{2} \epsilon_1 - \epsilon_2 \sin(\theta_{\pi\eta}) & \epsilon_1 + \epsilon_2 \cos(\theta_{\pi\eta}) \\
-\epsilon_2 - \epsilon_1 \left( \frac{\cos(\theta_{\pi\eta})}{\sqrt{3}} - \sqrt{\frac{2}{3}} \sin(\theta_{\pi\eta}) \right) & -\sin(\theta_{\pi\eta}) & \cos(\theta_{\pi\eta}) \\
-\epsilon_1 \left( \frac{\sqrt{2} \cos(\theta_{\pi\eta}) + \sin(\theta_{\pi\eta})}{\sqrt{3}} \right) & \cos(\theta_{\pi\eta}) & -\sin(\theta_{\pi\eta})
\end{pmatrix} \begin{pmatrix}
P_3 \\
P_0 \\
P_8
\end{pmatrix},
\]

where the parameters $\epsilon_1, \epsilon_2$ are mixing parameters from the Standard model. The two sectors with larger mixings will be addressed: the $\eta - \eta'$ mixing, that reduces to a rotation between $P_8$ and $P_0$, and the $\eta - \pi^0$ mixing. By performing the usual rotation to mass eigenstates $\eta, \eta'$, according to the convention from \cite{69}, it can be written:

\[
\begin{aligned}
|\eta \rangle &= \cos \theta_{\pi\eta} |P_8 \rangle - \sin \theta_{\pi\eta} |P_0 \rangle, \\
|\eta' \rangle &= \sin \theta_{\pi\eta} |P_8 \rangle + \cos \theta_{\pi\eta} |P_0 \rangle.
\end{aligned}
\]  

(22)

Although one needs two parameters/angles to describe both masses, $\eta, \eta'$ \cite{68}, in this work only the mass difference will be calculated. It is directly due to the mixing-type interaction $G_{08}$. By calculating $L_{\text{mix}}$ in this mass eigenstates basis, and comparing to the above $0-8$ mixing, the following $\eta - \eta'$ mixing angle is obtained:

\[
\theta_{\pi\eta} = \frac{1}{2} \arcsin \left( \frac{4G_{08}G_{80}}{(M_\eta^2 - M_{\eta'}^2)} \right).
\]  

(23)

This equation provides numerical results similar to the equation used in \cite{38} being however more complete.

Besides the (leading) mixing that describes $\eta - \eta'$ puzzle, the neutral pion also mixes with both $\eta, \eta'$ being the coupling to $\eta$ much larger than the coupling to $\eta'$ \cite{66,67}. The following rotation to define the physical meson fields will be considered:

\[
\begin{aligned}
|\eta \rangle &= \left( -\epsilon_2 - \epsilon_1 \left( \frac{\cos(\theta_{\pi\eta})}{\sqrt{3}} - \sqrt{\frac{2}{3}} \sin(\theta_{\pi\eta}) \right) \right) |P_3 \rangle + \cos(\theta_{\pi\eta}) |P_8 \rangle, \\
|\pi_0 \rangle &= |P_3 \rangle + \left( \frac{\epsilon_1}{\sqrt{3}} + \epsilon_2 \cos(\theta_{\pi\eta}) \right) |P_8 \rangle.
\end{aligned}
\]  

(24)

where $\epsilon_2 \simeq \epsilon$ is the usual parameter for this mixing when neglecting non leading mixing \cite{66,67,69}. The resulting mixing parameter can be written as:

\[
\epsilon_2 = -\frac{1}{2} \arcsin \left( \frac{4G_{38}G_{38}}{(M_\eta^2 - M_{\pi^0}^2) \sin(\theta_{\pi\eta})} \right),
\]  

(25)

where

\[
\bar{G}_{38} = \frac{2}{G_{88} (G_{38}^m - G_{38}^n C_{88}^n / C_{88}^m)}.
\]  

(26)

This eq. is analogous to the equations for the $\eta - \eta'$ channel above. The predictions for $\epsilon_2$ will be shown below being consistent with the estimation \cite{70}: $<\pi^0|H|\eta> \propto (m_u - m_d)$.

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3.4 Other observables

In this section some of the observables calculated with the resulting quark and mesons masses and coupling constants are described and their values are displayed in Table II.

The quark-antiquark scalar condensate, chiral condensate, is defined as:

$$< (\bar{q}q)_{f} > \equiv -Tr (S_{0,f}(k)),$$

and therefore is directly calculated by means of the solutions for the gap equations for the three flavors. Values in the Table correspond to the final self consistent solution of the (G2), i.e. eq. (12). These values improve initial estimation when calculated for $G_{0}$ and $M_{f}$.

The quark-meson, pion or kaon, coupling constants, or correspondingly the normalization of the field, obtained from the residue of the pole of the vertex can be written as [2, 3]:

$$G_{qqPS} = \left( \frac{\partial \Pi_{ij}(P^{2})}{\partial P_{0}^{2}} \right)^{-2}_{P_{0}^{2} = -M_{ps}^{2}},$$

where the values were calculated at the physical mesons masses $P_{0}^{2} = -M_{ps}^{2}$, and the polarization tensor was written in eq. (14). $\Pi_{ij}(P^{2}) = \Pi_{ij}^{f}(P_{0}^{2}, P^{2})$.

The weak decay constant of charged mesons, pion and kaon, $F_{ps} = F_{\pi}, F_{K}$, can be calculated as [2, 3]:

$$F_{ps} = \frac{N_{c} G_{qqPS}}{4} \int \frac{d^{4}q}{(2\pi)^{4}} Tr_{F,D} (\gamma_{\mu} \gamma_{5} \lambda_{i} S_{f_{1}}(q + P/2) \lambda_{j} S_{f_{2}}(q - P/2)) \bigg|_{P_{0}^{2} = -M_{ps}^{2}},$$

where $f_{1}, f_{2}$ correspond to the quark/antiquark of the meson and $i, j$ are the associated flavor indices as discussed for eq. (14).

In Table (2) several observables calculated for the sets of parameters shown above ($S$ and $V$ for the three different gluon propagators 2,5 and 6 and $G_{0}$) are exhibited. The neutral pion and kaon masses were fitted to values close to the experimental value, $M_{\pi^{0}} = 135$ MeV and $M_{K^{0}} = 498$ MeV. For the case of the pion, there are two estimates, one for the set M2 and the other from M3. The flavor-dependent coupling constants, however, tends to lower the mesons masses with respect to their values calculated with $G_{0}$. All the other observables, are obtained from the more complete calculation with M3. Self consistency, and a more complete account of strangeness in the coupling constants by means of M3, lead to lower values of the mesons masses and quark effective masses and the need of a larger value of the UV cutoff. The initial fit of the neutral pion mass, for the value of reference for the coupling constant $G_{0} = 10$ GeV$^{-2}$, was found to be $M_{\pi^{0}} = 136.4 - 137.1$ MeV. The final value with the flavor-dependent coupling constants and strange effective mass close to value that reproduce a physical point, for M3 goes to $M_{\pi^{0}} = 133 - 135$ GeV, whereas for M2 it goes to $M_{\pi^{0}} = 135.0 - 136.3$ GeV. Although the sets M2 pin down the correct (expected) value the idea is to show the effect of the mixing by comparing with the more complete result from M3. The estimation for the charged pion and kaon masses, $M_{\pi^{\pm}}, M_{K^{\pm}}$, are in quite good agreement with experimental or expected values. Note that, electromagnetic effects are not taken into account, and the expected mass differences due to strong interactions effects have opposite signs and they are respectively $M_{\pi^{\pm}} - M_{\pi^{0}} \simeq 0.1$ MeV and $M_{K^{\pm}} - M_{K^{0}} \simeq -5.3$ MeV [59, 60, 61]. The kaon masses are exhibited for the sake of completeness to show the entire set of observables used to fit parameters in the self consistent part of the calculation and the prediction for the neutral-charged meson mass difference.

The values of the charged pion and kaon decay constants $F_{\pi}, F_{K}$ are not far from the experimental/expected values (e.v.) although the choice made for the fitting yielded a much better value for the
kaon decay constant than $F_\pi$, differently from usual results in the literature, see e.g. in [14] and references therein. These results show an improvement with respect to the standard NJL model treatment. The up, down and strange quarks condensates, $<\bar{u}u>$, $<\bar{d}d>$ and $<\bar{s}s>$, and the pion (kaon)-quark coupling constants $G_{\pi qq}$ ($G_{K qq}$) are also presented.

The last observable shown in the Table are the mixing angles that were precisely the only quantities calculated with the mixing type interactions. The pseudoscalar mesons mixing angle $\theta_{ps}$ and the $\pi^0 - \eta$ mixing angle $\epsilon_2$. The following masses were considered for calculating the differences: $M_\eta = 548$ MeV, $M_{\eta'} = 958$ MeV and $M_\pi = 135$ MeV [69]. These values still eventually may change further by other effects, mainly for a complete self consistent calculation for all types of mixings and if one considers a 't Hooft type interaction that re-arranges the mixing-type interactions. The $\pi^0 - \eta$ mixing parameter, $\epsilon_2$, is exhibited for two situations: (I) $\theta_{ps} = 15^\circ$ [69] and (II) $\theta_{ps}$ as calculated from eq. (23). Whereas the estimate (I) is very close to other values found in the literature, the estimates (II) are considerably larger.

Although the values of all the observables are not as close as they could be to the experimental or expected values they are somewhat improved with respect to the flavor independent calculation which is represented in the Table by the column for the set of parameters with $G_0$. It is important to stress that, the coupling constants of reference is slightly larger than usual values and this makes the values of the condensates to be larger than they should. Large values of $G_0$, however, favor the convergence of the self consistent solutions for masses and coupling constants. A criterium for analyzing the s-content of the pion is the probability of finding a sea s-quark in it, denoted by Pr-s-content $\pi$. It can be approximately defined by means of the change in the pion normalization, $Z_\pi = G^2_{q\bar{q}\pi}$, with respect to the calculation with $G_0$. The most relevant reason for this change in the normalization is the variation of the strange quark effective mass (condensate). However, due to the self consistency of the problem, there might have other much smaller contributions due to up and down quarks. These values are considerably larger than the estimation from ref. [37] based on meson loops which have shown a probability of finding a s-quark in a up or down dressed quark to be of the order of $2 - 4\%$.

Finally, in the last two last lines the reduced chi-square for each of the set of parameters for two different situations are presented always for calculation with M3. Firstly the chiral condensates are taken into account, resulting in ten observables being two fitted observables, and secondly if the condensates are neglected, it provides seven observables with two fitted observables. The e.v. value for pseudoscalar mixing angle and quark-antiquark condensates were taken to be the average value of those shown in the last column of the Table. These two different estimations of the chi-square were done because, although the resulting values of the chiral condensates are improved with respect to the flavor-independent calculation, the deviations of their (corrected) values are still large with respect to the e.v. and this makes $\chi^2_{\text{red}}$ to be very large. Another source of increase of the chi-squared are the values of $F_\pi$. Although the pion decay constant was not really in agreement with experimental value, the important point is that the relative results $F_K - F_\pi$ is slightly improved with respect to standard NJL, that corresponds to the set of parameters for $G_0$. As discussed above, the relatively large value of $G_0$ may responsible for these discrepancies. Also, further terms in the gap equations, eventually due to higher order interactions or vector interactions may also be needed to pin down the corrected e.v. The numerical values of quark masses and meson masses needed to calculate the entries of Tables (2) and (4) are displayed in Table (3).
Table 2: Numerical results for some observables of the pion and the kaon. Where it has not been indicated, only the more complete set [M3] was considered. The estimation of the strangeness content of the pion is Pr.s-content π. e.v. refers to experimental or expected values. The π⁰ - η mixing angle (ϵ₂) is calculated in two cases (I) with θₚₜ = θₚₛ = 15° and (II) with θₚₛ from eq. (23) (its e.v. is the average value (*) from the references in the line below). Values (e.v.) of quark condensates from Refs. [71, 72, 73, 74], values for θₚₛ from [69] and for ϵ₂ * from [66, 67, 70, 75, 76].

| Observable                      | S₂ | S₅ | S₆ | S₆,ₐ₀ | V₂ | V₅ | V₆ | V,ₜₐ₀ | e.v. |
|---------------------------------|----|----|----|--------|----|----|----|--------|------|
| Mπ₀ (MeV) [M2]                 | 135.0 | 135.3 | 135.1 | 136.4 | 135.5 | 136.1 | 136.1 | 138.6 | 135 [69] |
| Mπ₀ (MeV) [M3]                 | 133.5 | 134.2 | 133.7 | 135.3 | 134.15 | 134.9 | 134.4 | 137.1 | 135.5 |
| Mπ± (MeV) [M2]                 | 135.2 | 135.4 | 135.3 | 136.7 | 135.7 | 136.2 | 135.8 | 137.4 | 136.1 |
| Mπ± (MeV) [M3]                 | 133.7 | 134.4 | 133.9 | 136.7 | 134.4 | 135.0 | 135.4 | 137.4 | 134.9 |
| Mπ± - Mπ₀ (MeV) [M3]           | 0.2 | 0.2 | 0.2 | 0.3 | 0.1 | 0.1 | 0.1 | 0.3 | 0.3 [69, 59, 62] |
| MΚ₀ (MeV) [M3]                 | 498.5 | 498.5 | 498.5 | 499 | 498 | 498 | 498 | 498 | 498 [69] |
| MΚ± (MeV) [M3]                 | 490 | 491 | 493 | 490 | 486 | 487 | 488 | 490 | 494 [69] |
| Fπ (MeV)                       | 99 | 99 | 99 | 102 | 100 | 101 | 101 | 103 | 92 |
| FΚ (MeV)                       | 111 | 111 | 111 | 112 | 112 | 112 | 113 | 111 | 111 |
| (−< q̅u >)¹/³ MeV              | 331 | 334 | 332 | 343 | 336 | 338 | 336 | 347 | 240-260 |
| (−< q̅d >)¹/³ MeV              | 333 | 335 | 333 | 344 | 338 | 339 | 338 | 349 | 240-260 |
| (−< q̅s >)¹/³ MeV              | 348 | 353 | 349 | 366 | 352 | 356 | 353 | 369 | 290-300 |
| Gqqπ                           | 3.3 | 3.2 | 3.2 | 3.4 | 3.3 | 3.3 | 3.3 | 3.6 | 3.3 |
| GqqΚ                           | 3.8 | 3.8 | 3.8 | 4.2 | 3.9 | 4.0 | 3.9 | 4.3 | 3.8 |
| θₚₛ                            | -3.7 | -2.7 | -3.6 | 0.0 | -3.4 | -2.6 | -3.4 | 0.0 | (−11°)−(−24°) |
| ϵ₂ (I)                         | -0.8 | -0.6 | -0.8 | -0.8 | -0.8 | -0.8 | -0.8 | - | (−1°)* |
| ϵ₂ (II)                        | -3.3 | -3.2 | -3.3 | - | -3.6 | -4.6 | -3.4 | - | |
| Pr.s-content π                 | 6% | 10% | 10% | 0 | 16% | 16% | 16% | 0 | |
| χ²_red (with < q̅q >)           | 103 | 110 | 103 | 146 | 122 | 129 | 123 | 164 | |
| χ²_red (without < q̅q >)        | 29 | 30 | 27 | 51 | 37 | 41 | 39 | 56 | |

In Figure (8) the charged pion decay constant, Fπ, is presented a function of the strange quark effective mass for the three sets S₂, S₅ and S₆ for the case of M3 defined in eq. (11). It is noticed a clear decrease of the pion decay constant with an increase of the strange quark (effective) mass. Some few results obtained from ChPT, however, indicate that the pion decay constant should actually increase with increasing strange quark mass mₛ or kaon mass M₂Κ [77, 78]. These available results from ChPT were obtained with quite large uncertainties in the knowledge of some lec’s, l₄, l₅ and l₆ and a more complete investigation about this issue is missing. There is not extensive specific results from lattice QCD that disentangle fully the dependence on the strange quark mass from other variables such as the pion mass.
3.5 Strangeness contribution for the masses of the u, d constituent quarks and pion

In the Table 3 further numerical results obtained from the change in $M_s^*$ are displayed. In the upper part of the Table there are particular values for the effective quark effective masses $M_u^*, M_d^*$ calculated self consistently (for M2 and M3). The values obtained from complete self consistent calculation - that reproduces the neutral and charged pion and kaon masses - are identified by G2. Both calculations for M2 and M3 correspond to different ways of taking into account $G_{S8}$ as responsible for a strange-up/down asymmetry. The resulting up and down quark effective masses when $M_s^* \to 0$, $M_s^* \to m_s$ and $M_s^* \to \infty$ are also shown. Note that the self consistent calculation of $M_{ch.L.}^*$, that is a flavor symmetric limit with degenerate quark masses, for the chiral limit, $m_u = m_d = m_s = 0$, provides a lower value for the quark effective masses than the value for $M_f^*(M_s^* \to 0)$ ($f=u$ and d). Note also that self consistent calculation for $M_{u,d}^*$ can easily provide values lower than their value in the chiral limit. Values of the neutral pion masses are shown in the same limits of M2 and M3 - self consistent results identified by (G2) - and also for $M_s^* = 0$, $M_s^* = m_s$ and $M_s^* \to \infty$. The analysis is basically the same as that for the up and down constituent or dressed quarks above, being however that in the chiral limit $M_{\pi^{ch.L.}}^* = 0$ since the pion is a Goldstone boson. The behavior of the charged pion mass is basically the same as the neutral pion mass as shown above.
Table 3: Numerical results for up and down quark effective masses and for the neutral pion mass, several of them obtained by varying freely the strange quark effective mass, $M_s^*$, and some of them obtained fully self consistently, identified by $G2$: $M_f^*$ and $M_{so}$ for $(M2)$ and $(M3)$ and $M^*$ (ch.lim.). Value for (e.v.) for $M_\pi^0$ from Refs. [69, 59].

| Observable/M3 | $S_2$ | $S_5$ | $S_6$ | $S.G_0$ | $V_2$ | $V_5$ | $V_6$ | $V.G_0$ | e.v. |
|--------------|-------|-------|-------|---------|-------|-------|-------|---------|------|
| $M_u^*$ M3-(G2) (MeV) | 367   | 377   | 368   | 405     | 385   | 393   | 387   | 422     |      |
| $M_d^*$ M2-(G2) (MeV) | 386   | 394   | 387   | 405     | 401   | 406   | 402   | 422     |      |
| $M_u^* (M^* \rightarrow 0)$ MeV | 618   | 512   | 595   | 405     | 651   | 537   | 626   | 422     |      |
| $M_d^* (M^* \rightarrow m_{0,s})$ MeV | 563   | 491   | 547   | 405     | 579   | 508   | 563   | 422     |      |
| $M_s^* (M^* \rightarrow \infty)$ MeV | 290   | 310   | 295   | 405     | 300   | 316   | 304   | 422     |      |
| $M_u^*$ M3-(G2) (MeV) | 375   | 384   | 378   | 415     | 394   | 402   | 395   | 431     |      |
| $M_d^*$ M2-(G2) (MeV) | 396   | 399   | 396   | 415     | 410   | 415   | 411   | 431     |      |
| $M_u^* (M^* \rightarrow 0)$ MeV | 625   | 520   | 602   | 415     | 657   | 544   | 632   | 431     |      |
| $M_d^* (M^* \rightarrow m_{0,s})$ MeV | 555   | 491   | 541   | 415     | 585   | 514   | 568   | 431     |      |
| $M_s^* (M^* \rightarrow \infty)$ MeV | 305   | 320   | 305   | 415     | 314   | 328   | 316   | 431     |      |
| $M_u^*$ M3-(G2) (MeV) | 555   | 567   | 558   | 612     | 566   | 581   | 569   | 625     |      |
| $M_d^*$ M2-(G2) (MeV) | 560   | 570   | 563   | 612     | 600   | 595   | 604   | 625     |      |
| ch.lim. $M_{ch.L.}$ (MeV) | 381   | 381   | 381   | 381     | 415   | 415   | 415   | 415     |      |
| $M_\pi^0$ M3-(G2) (MeV) | 133.5 | 134.2 | 133.7 | 136.7   | 134.2 | 134.9 | 134.4 | 137.4   | 0.135|
| $M_\pi^0$ M2-(G2) (MeV) | 135.0 | 135.3 | 135.1 | 136.7   | 135.5 | 136.1 | 135.6 | 137.4   |      |
| $M_\pi^0 (M^* \rightarrow 0)$ MeV | 158   | 147   | 156   | 136.7   | 162   | 149   | 159   | 137.4   |      |
| $M_\pi^0 (M^* \rightarrow m_{0,s})$ MeV | 150   | 144   | 149   | 136.7   | 150   | 144   | 149   | 137.4   |      |
| $M_\pi^0 (M^* \rightarrow \infty)$ MeV | 129   | 129   | 129   | 136.7   | 129   | 129   | 129   | 137.4   |      |

Different ways of defining strangeness and flavor asymmetry content of the constituent, or dressed, up and down quarks and of the pion can be envisaged. These mass differences discussed below are different from the usual strange-sigma terms, either for the constituent quarks u and d (as responsible for nearly 1/3 of the nucleon mass) and for the pion. It becomes useful to define particular differences of values that might correspond to variations with specific meanings. From here on, these quantities will be referred as to $T_{ff} = M_f^*$, $M_\pi$ and also $G_{ff}$. Furthermore these quantities defined below can also apply to the coupling constants $G_{ff}$ that present basically the same behavior of the up, down quark effective masses when varying $M_s^*$. We will make use of the following differences of a quantity $T_{ff}$ to characterize a specific s-content of the up and down quarks and of the pion:

$$
\Delta_s^{2,3} = T_{ff}(M3) - T_{ff}(M2),
$$

$$
\Delta_s^0 = T_{ff}(M_s^*) - T_{ff}(M_s^* = 0),
$$

$$
\Delta_s^{m_0} = T_{ff}(M_s^*) - T_{ff}(M_s^* = m_s),
$$

$$
\Delta_s^\infty = T_{ff}(M_s^*) - T_{ff}(M_s^* \rightarrow \infty).
$$

These mass differences will be exhibited in Table 3. First, note that the difference between the two curves, $M2$ and $M3$, can be considered as a first measure of the effect of the flavor-asymmetry (for the strange quark) for sea quarks in the coupling constants due to coupling $G_{ss}$. This quantity $\Delta_s^{2,3}$ is defined at the physical point.
Deviations with respect to the limit in which the strange quark effective mass is zero is encoded in $\Delta_0^s$, and it could be interpreted as the overall contribution of the strange quark effective mass to the observable $T_{ff}$. The deviation of the quantity $T_{ff}$ with respect to the point where strange quark condensate goes to zero was defined as $\Delta m^0_s$. This is an effective measure of the strange quark condensate in the constituent/dressed quark or pion. These mass differences, however, are not fully extracted in physical points in the sense that in these limits, $M^*_s = 0$ and $M^*_s = m_s$, there are no non trivial solutions for the full self consistent problem and kaons are not bound. The mass difference $\Delta_\infty s$ provides a dynamical way to measure the shift on the value of $T_{ff}$ due to the the strange quark effective mass. In its definition one considers the limit in which the strange quark effective mass goes to infinite $T_{ff}(M^*_s \to \infty)$. Curiously this is a well definite limit with interesting physical appeal, since in this limit the strange quark degrees of freedom should be frozen.

The same reasoning done above for mass differences, with $T_{ff}$, applies for the $G_{uu}$ and $G_{ss}$ because these coupling constants present a very similar behavior with the change in $M^*_s$.

### 3.6 Strangeness content of pion mass and the $\pi^0 - \eta$ mixing

To analyze further the mixing (24) in the physical pion state $|\pi^0>$, let us define the states:

$$ |P_3> = \frac{1}{\sqrt{2}}(|\bar{u}u> - |\bar{d}d>), \quad (34) $$

$$ |P_0> = \frac{1}{\sqrt{3}}(|\bar{u}u> + |\bar{d}d> + |\bar{s}s>), \quad (35) $$

$$ |P_8> = \frac{1}{\sqrt{6}}(|\bar{u}u> + |\bar{d}d> - 2|\bar{s}s>). \quad (36) $$

With the mixing (24) the physical pion state, for the $\pi^0 - \eta$ mixing, can be written as:

$$ |\pi^0> = \frac{1}{\sqrt{2}}[1 + a]|\bar{u}u> - \frac{1}{\sqrt{2}}[1 - a]|\bar{d}d> - \sqrt{2}a|\bar{s}s>, $$

$$ a = \frac{1}{3}[\epsilon_1 + \sqrt{3}\epsilon_2 \cos(\theta_{ps})], $$

The sea quark correction to the neutral pion state $|\bar{d}d>$ has the opposite sign of the one for the state $|\bar{u}u>$. As stated above, in phenomenology $\epsilon_1$ (or equivalently $\epsilon'$) is considerably smaller than $\epsilon_2$ and it will be neglected. Let us assume these states for a constituent quark model, that must be further specified, provide masses by applying energy operator in a rest frame. For orthonormal quark-antiquark singlet states, it may be considered to yield: $<\bar{q}q|\hat{H}|\bar{q}q> \simeq 2M_q$. The pion seems to be a particle for which the CQM does not provide good results given its very low mass in the hadron spectrum. What happens to the u-d sector with the valence quarks is not really important for this estimation of the $\pi^0 - \eta$ mixing. In what concerns a possible strange quark content for the pion in eqs. (37) a usual value for the strange quark mass can be assumed, $M^*_s \simeq 450$ MeV. These valence and sea quarks, however, do not really need to be quasi-particles.

---

1 It can be seen, however, in Table (3) that the quark-mixings (up, down and strange) can lower the quark effective masses. This also leads to additional strangeness content of up and down (constituent) quarks. So, one could even ask whether all these possible mixings, for some reason in the case of the pion, make the pion mass considerably smaller than the other pseudoscalar mesons down to nearly 135 MeV. The answer seems to be, of course, no, although one may ask the opposite question, i.e. why not? (why the neutral pion does not mix so strongly with the other neutral pseudoscalar). The neutral pion makes part of a iso-triplet state that for some reason (mainly isospin symmetry) may protect the neutral pion from strong mixings that do not take place for the charge pions. However these types of questions will not be addressed further in the present work. Goldstone boson masses are rather guided by the GellMann Oakes Renner relation.
since they are all confined. In this case the strange sea quark contribution for the pion mass must be proportional to

$$\Delta_\eta m_{\pi^0} \simeq 4a^2 M_s.$$  \hfill (37)

It can be written that the shift in the up or down constituent quark masses due to the mixing with the strange is of the order of

$$\Delta_{\eta} M_{u,d}^* \sim \frac{3}{4} \Delta_\eta m_{\pi^0}.$$  \hfill (38)

### 3.7 Mass differences: Numerical Results

Some mass differences for the up and down quarks, calculated according to eqs. (30-33), are exhibited in Table (4). Two different sources of changes in the up and down quark effective masses can be immediately identified. Firstly the shift in effective masses due to the explicit chiral symmetry breaking by mean of the quark current masses, and secondly the shift in the quark effective masses due to the flavor-dependent coupling constants by means of the self consistent procedure. By denoting the quark effective mass in the chiral limit \((m_0 = 0)\) that are degenerate - by \(M_{\text{ch.L.}}\) these two mass shifts can be written respectively as:

$$\Delta_{\text{ch.L.}}^{(f)} \simeq M_f - M_{\text{ch.L.}},$$

$$\Delta_{G_{ij}}^{(f)} \simeq M_{f}^* - M_f,$$

where \(M_f\) are solutions for the first gap equation \(G_1\) and \(M_f^*\) are solutions for the second gap equations \(G_2\). This second quantity, \(\Delta_{G_{ij}}^{(f)}\) for the up and down quarks, is mainly due to strangeness content of the coupling constant and, less importantly, also correspond to providing a u-content (d-content) for the d (u) quark effective mass. The numerical values for the neutral pion mass differences, eqs. (30-33), are also shown in Table (4), and they present the same relative behavior of the mass differences of up and down quarks. The mass differences are sizable are independent of the mixing interactions \(G_{i \neq j}\), that should, by the way, contribute as well. This complete calculation is not performed in the present work. It is interesting to compare these mass differences with results for the strangeness content of the pion of eqs. (37) due to the mixing with the \(\eta\) via interaction \(G_{38}\). The mass shift \(\Delta_\eta m_{\pi^0}\) is presented for two different pseudoscalar angle mixing \(\theta_{ps}\) (the one for the \(\eta - \eta'\) mixing). (I) for \(\theta_{ps} = 15^\circ\) \([69]\) and (II) for \(\theta_{ps}\) shown in Table (2) obtained in the calculations.

The comparison with results for the strangeness content of the pion from other works that consider different frameworks may not be direct. In the following, we summarize some results found in the literature for the strangeness content of the pion. As discussed above, the contribution of the kaon cloud for the pion mass was found to be negligibly small, of the order of 1 MeV in Ref. \([37]\). Sigma terms have been calculated in different approaches from NJL, constituent quark models, Chiral Perturbation theory and more recently lattice QCD for example in: \([79, 80, 81, 82, 83, 84, 85]\). Specifically for the pion strange sigma term, \(\sigma_\pi^s\), lattice QCD calculations \([80]\) provided small values with large uncertainties being even compatible with zero. The pion-strange sigma term for the flavor dependent NJL model however is not exhibited in the present work because the normalization considered, eq. (7), leads to different interpretations and quite ambiguous results.

In ChPT and ChPT with unitarization there are two types of calculation. Firstly, the s-sigma term has been investigated as the zero momentum limit of the strange scalar form factor of the pion, up to \(p^6\) order, to be of the order of few percent of the pion mass \([83, 84, 85]\). There are imprecisions in the contributions.
of the fourth and sixth order contributions because of the imprecision in the knowledge of lecs \( (l_4, l_5, l_6) \). It is reasonably comparable to \( \Delta^0_s, \Delta^{2,3}_s \) or \( \Delta^\infty_s \) in Table 4. Secondly, the strange quark contribution for the pion mass was calculated in ChPT in \([56, 77]\) up to \( p^{(5)} \) order. It is not exactly the s-sigma term but it must contain part of the s-sigma term content. The resulting contribution can be written in a similar shape of the result found in the present work as:

\[
M^2_\pi = (1 + \Delta_s) m^2_\pi ,
\]

being \( \Delta_s \sim 0.14 \rightarrow 0.31 \). From this, the correction for the pion mass can be written as: \( M_\pi = m_\pi + \delta_s \). This contribution can be comparable to the mass-difference \( \Delta^0_s \) shown in the Table. To summarize these results the following values can be considered:

\[
LQCD \quad \sigma^p_\pi \quad 6(33) \text{ MeV at } m_\pi = 149.7 \text{MeV } [80] \]

\[
ChPT \quad F^p_{Ss}(t = 0) = \sigma^p_\pi \quad 0 - 12 \text{MeV } [83, 84, 85] \]

\[
\delta_s \quad 9 - 19 \text{MeV } \text{eq. (41)} [87] \]

Table 4: Numerical results for up and down quark effective mass differences and neutral pion mass differences defined in eqs. \([39, 40]\). In the last two lines the strange quark mass contribution for the pion mass by means of eq. (eq. (37)) for two different mixing angles \( \theta_{ps} \): (I) for \( \theta_{ps} = 15^\circ \) \([69]\) and (II) for \( \theta_{ps} \) shown in Table 2 obtained in the calculations.

| Observable \([\text{M3}]\) | S2 | S5 | S6 | S-G0 | V-2 | V-5 | V-6 | V-G0 |
|--------------------------|----|----|----|------|-----|-----|-----|------|
| \( \Delta^u_{chL} \) (MeV) | 24 | 24 | 24 | 24   | 7   | 7   | 7   | 7    |
| \( \Delta^d_{chL} \) (MeV) | 34 | 34 | 34 | 34   | 16  | 16  | 16  | 16   |
| \( \Delta^u_{Gij} \) (MeV) | -38 | -28 | -37 | 0    | -37 | -29 | -35 | 0    |
| \( \Delta^d_{Gij} \) (MeV) | -40 | -31 | -37 | 0    | -37 | -29 | -36 | 0    |
| \( \Delta^{2,3}_s(M^u_\pi) \) (MeV) | -19 | -17 | -19 | 0    | -16 | -13 | -15 | 0    |
| \( \Delta^0_s(M^u_\pi) \) (MeV) | -251 | -135 | -227 | 0    | -266 | -144 | -239 | 0    |
| \( \Delta^{ma}_s(M^u_\pi) \) (MeV) | -194 | -114 | -179 | 0    | -194 | -115 | -176 | 0    |
| \( \Delta^\infty_s(M^u_\pi) \) (MeV) | 77 | 67 | 73 | 0    | 85  | 77  | 83  | 0    |
| \( \Delta^{2,3}_s(M^\pi_\pi) \) (MeV) | -1.5 | -1.1 | -1.4 | 0    | -1.3 | -1.2 | -1.2 | 0    |
| \( \Delta^0_s(M^\pi_\pi) \) (MeV) | -24 | -13 | -22 | 0    | -28 | -14 | -25 | 0    |
| \( \Delta^{ma}_s(M^\pi_\pi) \) (MeV) | -16 | -10 | -16 | 0    | -16 | -9  | -14 | 0    |
| \( \Delta^\infty_s(M^\pi_\pi) \) (MeV) | 6 | 5 | 4 | 0    | 5   | 6   | 5   | 0    |
| \( \Delta_{m\pi} \) (I) (MeV) | 0.11 | 0.10 | 0.11 | 0    | 0.11 | 0.11 | 0.11 | 0    |
| \( \Delta_{m\pi} \) (II) (MeV) | 1.9 | 1.8 | 2.4 | 0    | 2.4 | 3.7 | 2.1 | 0    |

4 Summary

A self consistent calculation for the quark effective masses and flavor-dependent coupling constants of the NJL model was employed to investigate the role of the strange quark effective mass on the up and down constituent quarks and on the pion masses. The fully self consistent calculation was done to fit the parameters of the model to reproduce neutral (or charged) pion and kaon masses. In this step, several
observables were calculated to assess the reliability of the model for different sets of parameters. The strong value of the coupling constant of reference, $G_0 = 10 \text{ GeV}^{-2}$, in one hand, helps with the convergence of the self consistent numerical calculation but, in the other hand, induces considerable large values of the quark-antiquark condensates. Flavor dependent coupling constants, nevertheless, improve their values with respect to the ones obtained with $G_0$. Quark effective masses get lower and they require a slightly larger cutoff to make possible the description of light mesons masses. A further consequence of this larger value of the coupling constant of reference $G_0$, associated to the use of the three-dimensional cutoff, is the relatively low value of the cutoff which limits the large UV three-momenta instead of the four-momenta.

Because of the quantum mixing it was possible to estimate the s-content of the up and down constituent/dressed quarks as well as of the pion even in the absence of mixing-type interactions. Different ways of extracting the strangeness contributions for the up and down quark effective masses and pion masses were analyzed in this work by means of mass differences. Besides the resulting effect from the renormalization of the wave function $Z_\pi$, several mass differences were exhibited. The mass difference $\Delta_{s}^{2,3}$ provides the change in the pion (or up and down effective) mass when exchanging the value of the interaction $G_{88}$, without any flavor-dependence or mixing effect (M2), $G_{88} = 10 \text{ GeV}^{-2}$, by a value that takes into account the strange sea quark dynamics, (M3) with $G_{88} = G_{88}(M_{s}^*)$. The mass difference $\Delta_{s}^{\infty}$ corresponds to a sort of dynamical criterium to define a strangeness contribution for the pion mass (or up and down effective masses). The same analysis is valid for the flavor dependent coupling constants $G_{ij}$. It is important to note that the mixing investigated in the present work is responsible for the strangeness in the up and down quark sector corresponds to a different mechanism from the quark mixing induced usually given by the CKM matrix or from the instanton-induced determinantal effective interaction. These mechanisms should add to each other. The whole procedure does not necessarily lead to a simple rotation of dressed quarks but it might involve some other transformation, like a dilation. As a whole, results may overestimate the contribution of the strange condensate to the pion structure as compared to available lattice QCD, the kaon cloud for up and down constituent quarks and ChPT expectations [80, 87, 88, 83, 84, 85]. This can be attributed mostly to the normalization (7) adopted in this work that easily strengthen the flavor-symmetry breaking contributions from the polarization process. The chosen normalization guarantees, nevertheless, the strength of the flavor-dependent component was assessed with respect to the value of reference $G_0$.

The mesons mixings, $\pi^0 - \eta$ and $\eta - \eta'$, were the only observables for which the mixing type interactions $G_{i\neq j}$ were considered. Angle mixings were calculated to reproduce mesons mass differences. The $\eta - \pi^0$ mixing was associated to a strange quark content of the pion. It is interesting to compare resulting estimations with the mass differences for the strangeness content of the pion mass. These values have comparable order of magnitude, as seen in Table (4). However the physical origins are somewhat different. It is important to note that in the estimations of meson mixings, the binding energies and (valence) quark kinetic energies were neglected. It looks that the strangeness contribution for the pion mass extracted from the quantum mixing contributions (Table (4)) might be associated to an upper bound because of the normalization adopted for the coupling constants $G_{ij}^m$. The chosen normalization for the coupling constants, at the flavor symmetric point ($m_u = m_d = m_s$) eq. (7), provides quite similar results to the normalization adopted previously [38] although it provides a faster convergence of the self-consistent calculation. The charged pion decay constant was also calculated as a function of the strange quark effective mass. The resulting behavior was found to be the opposite of that obtained from ChPT [77, 78]. These results from ChPT are nevertheless preliminary because some of the lec’s involved in these calculations are not really well determined. Therefore it is not clear to what extent this behavior of $F_\pi$ with increasing $M_s$ is a shortcoming of the present model and further physical input is needed to correct this behavior as it occurs for NJL-calculations for heavy mesons decay constants. The mesons mixings induced by the mixing interactions ($\Delta_{m^0}$) might be understood as a minimum value because it neglects further mixing effects due to the...
quantum mechanical mixing and, eventually, other interactions such as the ’t Hooft interaction. Besides that, due to the self consistent character of the calculation, results contain mixings of the type $U_{us}, U_{ds}$ and $U_{ud}$. Although the NJL-model does not exhibit confinement of quark and gluons, it provides an interesting and quite appropriate effective way of investigating aspects of hadron structure and dynamics. It is not clear whether or how confinement would imply modification(s) in the different mixings interactions and mechanisms. A complete account of the flavor dependent - NJL model with all mixing type interactions, $G_{i\neq j}$ and $G_{f_{1}\neq f_{2}}$, will be reported in another work.

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