Nonlinear Competition Between Small and Large Hexagonal Patterns

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Recent experiments by Kudrolli, Pier and Gollub \(^1\) on surface waves, parametrically excited by two-frequency forcing, show a transition from a small hexagonal standing wave pattern to a triangular “superlattice” pattern. We show that generically the hexagons and the superlattice wave patterns bifurcate simultaneously from the flat surface state as the forcing amplitude is increased, and that the experimentally-observed transition can be described by considering a low-dimensional bifurcation problem. A number of predictions come out of this general analysis.

In recent years there has been considerable interest in pattern formation in spatially extended nonequilibrium systems \(^2\). These studies have identified universal mechanisms for the formation of certain patterns. For instance, the origin of the ubiquitous hexagonal pattern can be traced to a symmetry-breaking instability of a spatially homogeneous state that occurs in the presence of certain Euclidean symmetries, which play a central role in determining the possible nonlinear evolution of the instability. In this letter we use similar ideas to investigate a family of hexagonal and triangular patterns which are born in the same symmetry-breaking instability that gives rise to simple hexagons, but have structure on two disparate length scales \(^3\). Kudrolli et al. \(^1\) call such structures “superlattice patterns”; Fig. 1 gives two examples. Current interest in superlattice patterns is sparked by recent observations of their formation in experiments on parametrically excited surface waves \(^1\) and in nonlinear optics experiments \(^4\), as well as in a study of Turing patterns in reaction-diffusion systems \(^3\).

In this letter we show that superlattice patterns can arise directly from the spatially homogeneous state via a transcritical bifurcation. Moreover, this occurs in generic pattern-forming systems for which there is only one unstable wavenumber. We show that the triangular superlattice pattern differs from simple patterns such as hexagons and stripes, and complex quasi-patterns, because it is characterized by both an amplitude and a phase which depend on the distance from the bifurcation point. In order to investigate the phase associated with this state we find that it is necessary to include high order resonant interaction terms in the amplitude equations. When these terms are neglected the problem becomes degenerate and superlattice patterns such as those in Fig. 1 become just two isolated examples in a continuum of states that have varying degrees of symmetry. Thus high order terms are essential to understanding the formation of superlattice patterns.

While much of our analysis is quite general, our presentation will focus on the recent experiments of \(^1\), in which surface waves are parametrically excited by subjecting a fluid layer to a periodic vertical acceleration that involves two rationally-related frequencies. In the classic Faraday problem, which has single frequency forcing, the first modes to lose stability with increased acceleration are subharmonic with respect to the vibration frequency \(^3\). With two frequencies, harmonic response can occur \(^4\). In particular, the triangular superlattice pattern in Fig. 1 was obtained with an acceleration of the form \(f(t) = a(\cos(\chi)\cos(6\omega t) + \sin(\chi)\cos(7\omega t + \phi))\). In the experiments, a hexagonal standing wave pattern is produced at the onset of instability, then, as the acceleration is increased, there is a transition to the superlattice pattern depicted in Fig. 1 \(^1\). Comparison of the spatial Fourier transform of the onset hexagons and the superlattice patterns reveals that the new state is formed from the nonlinear interaction of twelve prominent Fourier modes whose wavenumbers have the same modulus \(k_c\) and lie on a hexagonal lattice with fundamental wavenumber \(k_c/\sqrt{7}\). In this letter we address the
transition between hexagons and superlattice patterns by considering a degenerate bifurcation problem akin to that which successfully explains the transition between hexagons and rolls in a variety of systems. A number of concrete predictions about the form, origin and stability properties of the superlattice patterns come out of this general symmetry-based analysis.

FIG. 2. Intersection of the critical circle $|k| = k_c$ with the $k$-space hexagonal lattice generated by $k_1, k_2$. In this example $\alpha = 3$, $\beta = 2$ in (3). The wave vectors $\pm K_1, \ldots, \pm K_6$ lie at the vertices of a hexagon, as do $\pm K_4, \ldots, \pm K_6$.

The experimentally observed patterns are put into a theoretical framework by restricting to solutions that tile the plane in a hexagonal fashion. The time-periodic forcing in the Faraday problem leads us to a formulation in terms of a stroboscopic map. Since we seek spatially periodic solutions, we express all fields in terms of double a stroboscopic map. Since we seek spatially periodic solutions, we express all fields in terms of 

$$z(m + 1) = f(z(m)), z = (z_1, \ldots, z_6),$$

are restricted by the symmetries of the problem. The first component of $f$ has the general form

$$f_1 = h_1(u, q, q)z_1 + h_2(u, q, q)z_2z_3 + c_1z_2^2z_6^2 + c_2z_2z_4z_5 + O(|z|^6),$$

where $u \equiv (|z|^2, |z_2|^2, |z_3|^2, |z_4|^2, |z_5|^2, |z_6|^2)$, and $q \equiv (z_1 z_2 z_3, z_4 z_5 z_6)$. The discrete hexagonal symmetries place further restrictions on the functions $h_1, h_2$ and also determine the other components of $f$ from the diagonal entries of $f_1$. At this point we note that if the resonant interaction terms, with coefficients $c_1$ and $c_2$, are neglected, then the phases $\phi_j$ associated with each $z_j$ restrict to $R_j e^{i\phi_j}$ only enter the problem through the values of the total phases $\Phi_1 \equiv \phi_1 + \phi_2 + \phi_3$ and $\Phi_2 \equiv \phi_4 + \phi_5 + \phi_6$. As a result the cubic truncation of (3) admits solutions in the form of two hexagons rotated relative to each other in the form $\theta \approx 22^\circ$ (for $\alpha = 3$, $\beta = 2$), and translated relative to each other by arbitrary amounts. Kudrolli, et al. noted that only a restricted set of relative translations of the two hexagons leads to a superlattice pattern which retains triangular symmetry. Their phenomenological description did not, however, give a mechanism for the selection of a particular shift. We now investigate an alternative description of the triangular superlattice pattern which has the advantage that is in fact consistent with the inclusion of the high order resonant interaction terms.

We focus on patterns that have at least three-fold rotational symmetry by restricting the bifurcation problem to the subspace $z(m) = (u_m, u_m, u_m, v_m, v_m, v_m)$, where $u_m$ and $v_m$ are complex Fourier amplitudes measured at time $t = mT$. First we recall some results about period-
one simple hexagons which satisfy \((u_m,v_m) = (Re^{i\varphi},0)\) or \((0,Re^{i\varphi})\). The amplitude \(R\) and phase \(\varphi\) obey
\[
0 = \lambda + \epsilon Rc\cos(3\varphi) + AR^2 + \cdots \\
0 = \sin(3\varphi)\left(-\epsilon + BR^2 + \cdots\right),
\]
where \(\epsilon, A\) and \(B\) are nonlinear coefficients that arise from the Taylor expansions of \(h_k\) and \(h_2\) in (4). The solutions of (1) that bifurcate from the origin at \(\lambda = 0\) satisfy \(\sin(3\varphi) = 0\); the hexagons with \(\varphi = 0, \pm \frac{2\pi}{3}\) are related by translations, as are the hexagons with \(\varphi = \pi, \pm \frac{\pi}{3}\). These two sets of hexagons, ‘up-hexagons’ \((H^+)\) and ‘down-hexagons’ \((H^-)\), form the two branches associated with a transcritical bifurcation. In non-Boussinesq convection, these states correspond to ones in which fluid is rising or falling at the centers of convection cells [10].

Next we consider patterns that involve all six complex Fourier modes \(z_1, \ldots, z_6\), with \(z_j = \rho e^{i\psi}\). We find that the amplitude \(\rho\) and phase \(\psi\) satisfy equations of the form, cf. (1),
\[
0 = \lambda + \epsilon\rho\cos(3\psi) + A\rho^2 + \cdots \\
0 = \sin(3\psi)\left(-\epsilon + B\rho^2 + \cdots\right) + (c_1 - c_2)\rho^3\sin(2\psi) + \cdots
\]
In this case there are two distinct types of solutions that bifurcate from the origin at \(\lambda = 0\); those with phase \(\psi = 0, \pi\) and those with \(\psi \approx \pm \pi/3, \pm 2\pi/3\). Note that the phase \(\psi\) associated with the latter solutions depends on the amplitude \(\rho\), which in turn depends on the distance from the bifurcation point. The solutions with \(\psi = 0, \pi\) have hexagonal symmetry; in [1] they are referred to as “super hexagons”. As with simple hexagons, super hexagons bifurcate transcritically with the two parts of the branch satisfying \(\psi = 0\) and \(\psi = \pi\), respectively. The solutions satisfying \(\psi \approx \pm \pi/3, \pm 2\pi/3\) have only triangular symmetry, and are new. The triangular solutions with \(\psi \approx 2\pi/3\) and \(\psi \approx -\pi/3\) form the two parts of a transcritical branch; a rotation of these patterns by \(\pi\) changes the sign of \(\psi\). These solutions have structure identical to the superlattice harmonic state described in [1]: compare the triangular superlattice pattern in Fig. 1 with that in Fig. 3. Moreover, we know that the solutions with \(\cos(3\varphi) > 0\) in (1) and \(\cos(3\psi) > 0\) in (3) bifurcate in the same direction from the origin, as do the branches with \(\cos(3\varphi), \cos(3\psi) < 0\).

Finally, we address the experimentally observed transition between the patterns in Fig. 3. Since all of the known primary solution branches are unstable at bifurcation, due to the presence of the quadratic nonlinear term in (3), we assume that the coefficient \(\epsilon\) of the quadratic term satisfies \(|\epsilon| \ll 1\). This allows us to investigate stable small amplitude solutions, and the transitions between them. If we truncate (3) at cubic order, thereby neglecting the high-order resonant terms with coefficients \(c_1, c_2\), then the superlattice patterns are at best neutrally stable. Specifically, while we expect a (multiplicity two) unit multiplier associated with translations of the pattern, the extra multiplier (also of multiplicity two) results from a symmetry of the truncated equations that allows a relative translation of the two rotated hexagons that make up the pattern. As noted above, this symmetry is broken when the resonant terms are included; the Floquet multiplier \(\mu\) then moves off the unit circle. In particular, \(\mu > 1\) for the super hexagons if \(4c_1 + 5c_2 < 0\). Analogously, we can show that \(\mu < 1\) for the triangular superlattice pattern if \(4c_1 + 5c_2 < 0\). Thus if the superlattice patterns are neutrally stable when the high-order resonant terms are neglected, then we expect one of them is stable and the other is in fact unstable. This is consistent with the experimental observations of Kudrolli, et al. [1] who only observe the triangular superlattice pattern.

In Fig. 4, we present part of a bifurcation diagram associated with the stroboscopic map. Specifically, we keep all terms through cubic and the essential quintic terms:
\[
f_1 = (1 + \lambda)z_1 + \epsilon\tau z_2 z_3 + (a_1|z_1|^2 + a_2|z_2|^2 + a_2|z_3|^2)z_1 \\
+ (a_4|z_4|^2 + a_5|z_5|^2 + a_6|z_6|^2)z_1 \\
+ c_1z_1^2z_3^2 + c_2z_1z_2z_4^2,
\]
and assume that the cubic coefficients satisfy the following two inequalities:
\[
a_1 + 2a_2 < -|a_4 + a_5 + a_6| \\
a_2 - a_1 > \sqrt{(a_4 - a_5)^2 + (a_4 - a_6)^2 + (a_5 - a_6)^2}/2
\]
with \(4c_1 + 5c_2 < 0\). Conditions (7) ensure that superlattice patterns eventually supersede the simple hexagons, and that the stripe pattern is never stabilized.
be very interesting to calculate the nonlinear coefficients in $P$ from the hydrodynamic equations to determine whether the minimal inequalities $\tilde{q}$ are satisfied for the experimental parameters. Such computations are quite involved in the case of two-frequency forcing of viscous fluids. However, much progress has been made in the case of very low viscosity fluids by Zhang and Viñals [11], who employ a quasi-potential approximation to the Navier-Stokes equations to simplify the calculations.

Finally, we note that all our calculations have assumed that the solutions are strictly periodic on a hexagonal lattice. This is certainly an appropriate model for observations of superlattice patterns, but as noted in [1] quasi-patterns are observed for other experimental parameters. A mechanism for favoring quasi-patterns by a resonant interaction between two bifurcating states with different horizontal wavenumbers was proposed by Edwards and Fauve [8], and investigated for a Swift-Hohenberg type model by Lifshitz and Petrich [12]. Whether a similar resonance mechanism is responsible for the spatially-periodic superlattice patterns is an intriguing, open question.

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**FIG. 4.** Part of the bifurcation diagram associated with the bifurcation problem $P$ for $-1 < \epsilon < 0$. Bifurcation points are indicated by solid circles. Only branches of simple hexagons ($H^\pm$) and superlattice triangles ($T^\pm$) are indicated; solid lines correspond to stable solutions. Branches produced in secondary bifurcations, and unstable primary branches are not shown.