Probability Densities in Strong Turbulence.

Victor Yakhot
Department of Aerospace and Mechanical Engineering,
Boston University, Boston 02215
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Abstract

According to modern developments in turbulence theory, the "dissipation" scales (u.v. cut-offs) $\eta$ form a random field related to velocity increments $\delta_\eta u$. In this work we, using Mellin’s transform combined with the Gaussian large-scale boundary condition, calculate probability densities (PDFs) of velocity increments $P(\delta_r u, r)$ and the PDF of the dissipation scales $Q(\eta, Re)$, where $Re$ is the large-scale Reynolds number. The resulting expressions strongly deviate from the Log-normal PDF $P_L(\delta_r u, r)$ often quoted in the literature. It is shown that the probability density of the small-scale velocity fluctuations includes information about the large (integral) scale dynamics which is responsible for deviation of $P(\delta_r u, r)$ from $P_L(\delta_r u, r)$. A framework for evaluation of the PDFs of various turbulence characteristics involving spatial derivatives is developed. The exact relation, free of spurious Logarithms recently discussed in Frisch et al (J. Fluid Mech. 542, 97 (2005)), for the multifractal probability density of velocity increments, not based on the steepest descent evaluation of the integrals is obtained and the calculated function $D(h)$ is close to experimental data. A novel derivation (Polyakov, 2005), of a well-known result of the multi-fractal theory [Frisch, "Turbulence. Legacy of A.N.Kolmogorov", Cambridge University Press, 1995]), based on the concepts described in this paper, is also presented.
1 Introduction.

A reasonably well experimentally established anomalous (multi) scaling of the structure functions $S_{n,m} = \frac{(u(x+r) - u(x))^n(v(x+r) - v(x))^m}{n^m(m^m)}$, where $u$ and $v$ are components of the velocity field parallel and perpendicular to the displacement vector $r = ri$, respectively, is one of the properties of strong turbulence which makes it "the last unsolved problem of continuum mechanics". [1]-[4]. The anomalous dimension, a property of strongly interacting systems first introduced to the quantum field theory by Gribov and Migdal [5] and Polyakov [6], is a notoriously difficult from theoretical viewpoint concept and only recently, after many years of trying, the theory of anomalous scaling of the structure functions of a passive scalar advected by a white-in-time random velocity field, has been developed [7]-[8]. Almost simultaneously, the theory of bi-scaling in turbulence generated by the forced Burgers equation [9]-[10] was formulated. The attempts to explain anomalous dimensions in three dimensional turbulence were made in Refs.[11]-[12]. It was shown that the Navier-Stokes equations combined with a simple model for the pressure -velocity correlations , lead to homogeneous differential equations for the structure functions and, as a result, to anomalous scaling exponents which cannot be obtained on dimensional grounds.

During last forty-fifty years, a substantial effort was devoted to derivation or at least modeling of various probability densities in turbulence. The first attempts resulting in the Log-normal PDF of the dissipation rate fluctuations, consistent with the multi-scaling, were made by Kolmogorov [13] and Yaglom [14], using a simple cascade model. This result was later criticized by Orszag [15] as, in general, not realizable. Similar Log-normal PDF was obtained for the not too large magnitudes of velocity increment $\delta_u = u(x+r) - u(x)$ in Ref. [11] , which was later experimentally tested by Kurien and Sreenivasan [3]. Some other attempts based on analysis of experimental data led to various fits ranging from Log-normal and Log-Poisson expressions to exponential and "stretched exponential " PDFs $P(\delta_u)$. In this paper we, addressing this problem, will restrict ourselves by considerations based on equations of motion.

This paper is organized in a following way. In the Section 2 we introduce the Mellin trans-
form for the probability density of velocity increments and define the large-scale Gaussian boundary condition. We show that the "normal (linear) scaling" corresponds to the Gaussian PDF of the small-scale fluctuations. The PDF accounting for small deviations from the linear scaling is calculated and compared with traditional Log-normal expressions. In Section 3, we based on theoretical analysis of the Navier-Stokes equations and numerical simulations by Gotoh and Nakano [16], justify the boundary conditions used for the derivation of Sec 2. The random field of the dissipation scales $\eta$ (linear dimensions of the dissipative structures), derived from the expression for the dissipation anomaly, is introduced in Sec 4 and the PDF $Q(\eta, Re)$ is computed in Section 5. Discussions and conclusions are presented in Section 6. There, the exact expression for the probability density of velocity increments following the multifractal formalism is presented and one of the results of the multi-fractal theory is derived (Polyakov 2005) without the multifractal input. Some of the concepts and relations presented below have been reported in the previous publications [12], [17]-[18]. Wherever needed, we repeat them here for the sake of continuity and clarity.

2 The Mellin transform.

If the moments of velocity increments $\delta_r u$ are given by the scaling relations $S_{n,0} = (\delta_r u)^n = A(n) r^{\xi_n}$, then probability density function can be found from the Mellin transform:

$$P(\delta_r u, r) = \frac{1}{\delta_r u} \int_{-\infty}^{\infty} A(n) r^{\xi(n)} (\delta_r u)^{-n} dn$$

where we set the integral scale $L$ and the dissipation rate $\mathcal{E}$ equal to unity. Multiplying (1) by $(\delta_r u)^k$ and evaluating a simple integral, gives $S_{k,0} = A(k) r^{\xi_k}$.

To make use of the relation (1) the dynamic information about both the amplitudes $A(n)$ and the exponents $\xi_n$ is needed. In order to obtain an expression for $A(n)$ from a large scale boundary condition on the PDF, we first have to define the integral scale $L$. Based on experimental data and theoretical consideration (see below), we choose $L$ as a scale at which the energy flux toward small scales changes sign or tends to zero. This means that at the
small scales $r < L$ the structure function $S_3(r) < 0$ and at the larger scales $r > L$, $S_3 \geq 0$. 
Typically, at this scale which depends upon the geometric details of the flow, the odd-order moments $S_{2n+1}(L) = 0$ and the even-order ones saturate: $\partial_r S_{2n,0}(L) = 0$. This property of turbulence has recently been examined both numerically and experimentally [19]-[20] (a theoretical argument will be given below). The scale $L$ appears naturally in a simple case of the Navier-Stokes equations on an infinite domain driven by the white-in-time forcing function with the variance:

$$\overline{f^2(k)} = \frac{\mathcal{P}}{2(2\pi)^4} \delta(k - k_f)/k^2$$

where $\mathcal{P}$ is the forcing power. The exact calculation of the relation for the third-order structure function $S_3(r)$ gives an oscillating expression

$$S_3 = -\mathcal{P} \frac{-36r \cos r + 12 \sin r - 12(-2 + r^2) \sin r}{r^4}$$

In the limit $r \to 0$, we have the Kolomogorov relation $S_3 = -\frac{4}{3} \mathcal{P} r$. At the large scale $L \approx 5.88/k_f$, $S_3(L) = 0$ [21]. In all flows studied, the probability density $P(\delta_L u, L)$ was extremely close to the Gaussian. We would like to stress that the integral scale defined this way is not the largest scale (size) a system but rather corresponds to the top of the inertial range where the constant energy flux toward small scales sets up. The Gaussian boundary condition at $r = 1$ leads to a plausible and well-tested (both experimentally and numerically), expression $A(n) = (2n-1)!!$ which will be used in all calculations of this paper. In what follows a dynamic and numerical justification for this result will be presented.

First we consider the case of "normal scaling" $\xi = an$. Writing $(2n-1)!! = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} x^{2n} dx$ and rotating the integration axis by $90^\circ$, we have setting $\delta_r u \equiv u$:

$$P(u, r) = \frac{1}{\sqrt{\pi u}} \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{in \left( \ln \frac{x}{\sqrt{2u}} + \ln x \right)} dn = \frac{1}{\sqrt{\pi u}} \int_{-\infty}^{\infty} e^{-x^2} \delta \left( \ln \frac{r a \sqrt{2}}{u} + \ln x \right) dx$$

This integral is evaluated readily with the result:

$$P(u) = \frac{1}{\sqrt{2\pi r^a}} e^{-\left( \frac{u^2}{2r^a} \right)}$$
Now we consider anomalous scaling by introducing small deviations from the linear relation:

\[ \xi(n) = an - bn^2 \]  

which for not too large moment numbers \( n \) can be perceived as first two terms of the Taylor expansion of \( \xi_n \) in the vicinity of \( n = 0 \). The formula (3) is a generic perturbative expression for the exponents \( \xi_n \) independent upon the nature of the problem and the relations similar to (3) have resulted from the recent perturbative theories of a passive scalar in a random velocity field. [7]-[8]. Using the Kolmogorov constraint \( \xi_3 = 1 \), gives: \( b = (3a - 1)/9 \). It is clear that the expression (3) cannot be correct for all values of the moment order \( n \). Indeed, in accord with Holder’s inequality, \( \xi_n \), is a concave and non-decreasing function of \( n \) or in other words as \( n \to \infty \), \( \xi(n)/n \to 0 \ (\xi_n/n \gg 1/n) \). Still for \( n \leq 10 - 15 \), the experimental data on strong turbulence are consistent with \( a \approx 0.383 \) and \( b \approx 0.0166 \) and the expressions derived below can be accurate only for not-too-large values of velocity increment \( \delta_r u \). Thus, the probability density is given by the integral:

\[
P(u, r) = \frac{1}{\sqrt{\pi}u} \int_{-\infty}^{\infty} e^{-x^2} \int_{-\infty}^{\infty} e^{in \ln \frac{\xi}{\sqrt{2}}} e^{-bn^2 \ln r} dn
\]

which is reduced to:

\[
P(u, r) = \frac{2}{\pi u \sqrt{4 \ln r b}} \int_{-\infty}^{\infty} e^{-x^2} \exp\left[-\frac{(\ln \frac{u}{r^{1/2}})^2}{4b \ln r}\right] dx
\]

The integration over \( n \) leading from (4) to (5) was based on the following estimate. Since \( e^{in \ln(u/r^a)} \) is an oscillating function, the main contribution to the integral over \( n \) comes from the interval \( n \approx 1/\sqrt{b|\ln r|} \) and in order for the relation (3) to be valid in the integration interval, the following condition must be satisfied: \( 1/\sqrt{b|\ln r|} \leq 10 - 15 \) which is satisfied when the displacement \( r \) is small enough.

The PDF \( P(\frac{u}{r^a}, r) \) numerically evaluated from equation (5), is shown on Fig. 1. for a few values of displacement \( r \). We can see that the tails of the PDF strongly depend upon \( r \) which is a sign of intermittency and anomalous scaling.

In the range \( u/r^a \approx 1 \), the integral is dominated by the interval \( x \approx 1 \) leading to the log-normal result:
\[ P_L = \frac{1}{u\sqrt{4\pi b \ln r}} \exp\left[-\frac{\ln^2(r)}{4b \ln r}\right] \]  

Both PDFs (5) and (6) are normalized to two. The expression (6), which has also been derived directly from the Navier-Stokes equations combined with a simple model for the pressure-velocity correlation function in Ref. [11], has been experimentally verified by Kurien and Sreenivasan [3]. The comparison between (5) and (6) showing a surprisingly large difference between the approximately and numerically calculated integral (5) is presented on Fig. 2. In addition, it follows from Figs. 1 and 2 that while \( P(\delta u, r) \) has a maximum at \( \delta u = 0 \), in the limit \( \delta u \to 0 \), the Log-normal PDF \( P_L \), given by (6) rapidly decreases to zero, meaning that it may be a reasonable approximation not too close to the origin at \( \delta u = 0 \).

In the most interesting case \( \xi(n) = \frac{an}{1+bn} \), the integral (1) with \( A(n) = (2n - 1)! \) can be evaluated both exactly and by the steepest descent method leading at the large magnitudes of the argument \( u/r^a \) to the algebraically decreasing probability density consistent with the saturation of the exponents \( \xi_n \to a/\beta \).

3 The relations between moments. Simulations by Gotoh and Nakano (Ref. [16].)

In this Section we would like to develop some dynamic arguments justifying the choice of the structure functions amplitudes \( A_{2n,0} = (2n - 1)! \) used in the above calculations. The relations for the moments of velocity difference \( S_{n,m} \) were derived in Refs. [11], [12] and later in [22] using an alternative approach. In particular, the equation for the even-order moments is:

\[ \frac{\partial S_{2n,0}}{\partial r} + \frac{d - 1}{r} S_{2n,0} - \frac{(d - 1)(2n - 1)}{r} S_{2n-2,2} = (2n - 1) < \delta u r^{n-2} > \] (7)
and taking into account that the dissipation contribution can be neglected \[11\], \[12\], \[3\], \[22\]: we have

\[ \frac{\partial S_{2n,0}}{\partial r} + \frac{d-1}{r} S_{2n,0} - \frac{(d-1)(2n-1)}{r} S_{2n-2,2} = -(2n-1) < \delta_r(\partial_x p)u^{2n-2} > \quad (8) \]

This relation is not closed. In the vicinity of the integral scale where the odd-order moments \( S_{2n+1}(L) \) and derivatives of the even order moments \( \partial_r S_{2n,0}(L) \) disappear, the pressure gradient terms can be neglected. Here we would like to present a possible physical explanation of this effect. If geometry of the large-scale unstable structures leading to the energy cascade toward small scales is defined by the condition \( S_3(L(x, y, z)) = 0 \), then associating the instability with a flow separation phenomenon, gives the standard relation (see any textbook on hydrodynamics): \( \nabla p = \nu \nabla^2 u \), meaning that at these points \( a = 0 \) and \( \delta_L a = 0 \). This mechanism, if locally correct allows one to neglect acceleration contributions to the equation (7) (or pressure gradients in (8)) with the remaining terms giving \( S_{2n,0} = (2n-1)S_{2n-2,2} \), consistent with the Gaussian distribution. Thus, extrapolating this result into the “inertial range”, we assume that \( (2n-1)S_{2n-2,2} = S_{2n,0}S_{0,2}/S_{2,0} = (1 + \frac{\xi_2}{2})S_{2n,0} \) which is consistent with the Gaussian distribution at \( r = 1 \). This relation can be proven as follows: since \( \overline{uv} = 0 \), at a gaussian point \( r = 1 \) we have \( S_{2n-2,2} = S_{0,2}S_{2n-2,0} = \frac{1}{2n-1} \frac{S_{0,2}}{S_{2,0}} S_{2n,0} = \frac{1+\xi_2/2}{2n-1} S_{2n,0} \). One remark is in order: in the inertial range \( \xi_2 \neq 0 \) while in the limit \( r \to L \), the \( r \)-derivatives disappear or in other words loosely defined in this limit ”exponent” \( \xi_2 \to 0 \). Thus the above relation has to be treated as a parametrization valid in both limits \( r \to L \) and \( r/L \ll 1 \). In their remarkable paper Gotoh and Nakano \[16\] made a detailed examination of the various terms in the equations (7) and (8). Some of their results are presented on Fig. 3, where the left side of equations (7)-(8) denoted as ”lhs” is compared with the right side (”rhs” minus pressure-gradient terms) for \( n = 4; 6; 8 \). On the Fig. 3 \( \lambda \) stands for the Taylor micro-scale. They demonstrated that : 1. indeed , the dissipation contributions to the equations for the even order moments are negligibly small; 2. the relation \( (2n-1)S_{2n-2,2} \approx 1.35S_{2n,0} \), independently upon the displacement magnitude \( r \) (not shown on Fig. 3); 3. The pressure contributions \( (2n-1)\overline{\delta_r(\partial_x p)(\delta_r u)^{2n-2}/lhs} \approx const \) in the entire range of the displacement \( r \)-variation. Substituting \( (2n-1)S_{2n-2,2} \approx 1.35S_{2n,0} \) and the algebraic relation \( S_{2n,0} = A_{2n,0}r^{2n-1} \)
into (8) gives \( (d = 3) \):

\[
R_{2n} = \frac{\partial_r + \frac{2}{r}S_{2n,0} - \frac{2(2n-1)}{r}S_{2n-2,2}}{\partial_r + \frac{2}{r}S_{2n,0}} = \frac{\xi_{2n} - \xi_2}{\xi_{2n} + 2} = -\frac{(2n - 1) < \delta_r (\partial_x p) u^{2n - 2} >}{S_{2n,0} + \frac{2}{r}S_{2n,0} (9)}
\]

and for \( \xi_2 \approx 0.7, \xi_4 \approx 1.27, \xi_6 \approx 1.77 \) and \( \xi_8 \approx 2.15 \) the magnitudes of the ratio: \( R_4 = 0.18; \ R_6 = 0.28 \) and \( R_8 = 0.35 \) in a close agreement with numerical data of Ref. [16] (For comparison, see Fig.3). Moreover, one can see form Fig. 3 that all pressure-gradient velocity correlations become negligibly small in the vicinity of a single length scale \( (L \approx 1000\eta_K) \) consistent with the above introduced definition of integral scale \( L \). Here \( \eta_K \) is the Kolmogorov scale. The equations (8) combined with the above considerations yield another interesting relation for the pressure gradient-velocity correlations:

\[
(\xi_{2n} - \xi_2)S_{2n,0} = -(2n - 1)r\delta_r(\partial_x p)\delta_r u^{2n - 2} \geq 0
\]

The most important outcome of the formula (9) and numerical data of Ref. [16] is that the ratios \( R_{2n} = const \) are independent upon the magnitude of displacement \( r \). This means that the Lagrangian acceleration contribution to (8) can be expressed as a linear combination:

\[
-< \delta_r \partial_x p | \delta u, \delta v >= \alpha_p \partial_r(\delta_r u)^2 + \beta_p(\delta_r u)^2/r + \gamma_p(\delta_r v)^2/r + \kappa_p \partial_r(\delta_r v)^2 (10)
\]

plus sub-leading terms which are irrelevant in the small-scale limit \( r \to 0 \).

4 Dissipation anomaly. Calculation of spatial derivatives

Similar result is obtained if we consider the dissipation anomaly introduced by Polyakov [9] who was interested in turbulence generated by the random-force-driven Burgers equation. Later Polyakov’s results have been generalized to the Navier-Stokes equations by Duchon and Robert [24], Eyink [24] and Yakhot and Sreenivasan [18]. Interested in various characteristics of the velocity field, one often needs to calculate spatial derivatives defined as usual:
\[
\frac{\partial_x u}{y} = \lim_{y \to 0} \frac{u(x + y) - u(x)}{y}
\]

The problem is that if the velocity field is differentiable, then \(S_3 = O(y^3)\) and \(\partial_y S_3 \to 0\).

This is in contradiction with the Kolmogorov relation stating \(\partial_y S_3 \propto \mathcal{E} = O(1)\). This means that in the "inertial range" the velocity field is singular. To resolve this problem, Polyakov [9] introduced a limit-ordering procedure: 1. \(\nu \to 0\); 2. \(y \to 0\). Assuming that there exist a scale ("dissipation") separating analytic and singular ranges, we can approximately redefine derivative as [18]:

\[
\partial_x u(x) = \lim_{y \to \eta \to 0} \frac{(u(x + y) - u(x - y))/2y \approx (u(x + \eta) - u(x - \eta))/2\eta}{2y} \approx \frac{(u(x + \eta) - u(x - \eta))/2\eta}{2y} \approx \frac{(u(x + \eta) - u(x - \eta))/2\eta}{2y}
\]

This definition leads to an interesting consequence. In the analytic interval \(u(x + \eta) \approx u(x - \eta) + 2\eta \partial_x u\) the matching condition at the scale \(\eta\) reads: \(\delta \eta u \approx 2\eta \partial_x u\) thus defining \(\eta\) as a random field. Therefore, to calculate the moments of derivatives, one needs either the probability density \(P(\delta \eta u, \eta)\) or the relation coupling \(\eta\) and \(\delta \eta u\). The dissipation anomaly enables one to calculate this relation. Here we present the main steps of derivation of Refs. [9], [24], [18].

We are interested in the Navier-Stokes dynamics of incompressible fluids, for which the energy balance equation (with the density \(\rho = 1\)) is written as

\[
\frac{1}{2} \frac{\partial u^2}{\partial t} + \frac{1}{2} \mathbf{u} \cdot \nabla u^2 = -\nabla p \cdot \mathbf{u} + \nu \mathbf{u} \cdot \frac{\partial^2 \mathbf{u}}{\partial x^2},
\]

The differential equation for the scalar product \(\mathbf{u}(x + \frac{\gamma}{2}) \cdot \mathbf{u}(x - \frac{\gamma}{2}) \equiv \mathbf{u}(+) \cdot \mathbf{u}(-)\) can be written as

\[
\frac{\partial \mathbf{u}(+)}{\partial t} \cdot \mathbf{u}(-) + \mathbf{u}(+) \cdot \frac{\partial \mathbf{u}(+)}{\partial x_+} \cdot \mathbf{u}(+) = -\mathbf{u}(-) \cdot \frac{\partial \mathbf{u}(+)}{\partial x_+} + \nu \mathbf{u}(+) \cdot \frac{\partial^2 \mathbf{u}(+)}{\partial x^2} + \mathbf{u}(+) \cdot \frac{\partial^2 \mathbf{u}(-)}{\partial x^2}.
\]

It is clear that in the limit \(y \to 0\), for which \(x_+ \to x\), this equation gives the energy balance.

In the limit \(y \to 0\), the equation (12) has two kinds of terms: the regular ones which
disappear by the virtue of the energy balance and a few singular terms balancing each other. The calculation presented in detail in Ref. [18] leads to the equation (See also Duchon and Robert [23], Eyink [24]):

\[
\lim_{y \to 0} \left[ -\frac{\partial}{\partial y_i} \delta u_i |\delta y| \right]^2 + \frac{1}{2} \left( \frac{\partial}{\partial x_{+,i}} u_i(+) u_j(-) \right)^2 + \frac{\partial}{\partial x_{-,i}} u_i(-) u_j(+) = -2 \delta y \mathbf{u} \cdot \delta \mathbf{a},
\]

where \( \mathbf{a} = -\nabla p + \nu \nabla^2 \mathbf{u} \) is the lagrangian acceleration. The above equation, not involving time derivatives, is exact at each point in the flow at each instant of time. Choosing the displacement vector along one of the coordinate axes and averaging, one obtains [24]:

\[
\frac{\partial}{\partial y} \delta \mathbf{u} |\delta \mathbf{u}|^2 = 8 \nu \delta \mathbf{u} \frac{\partial^2}{\partial y^2} \delta u_i = 2 \nu (\delta y u_i) \frac{\partial^2}{\partial x^2} (\delta y u_i) = -\frac{4}{3} \xi,
\]

where \( \delta y = \delta \mathbf{u} \cdot \mathbf{y}/y \). The pressure terms in and the second contribution disappeared by the averaging procedure.

**Dissipation anomaly as a closure.** The relations (7)-(8) are valid for all magnitudes of the displacement vector \( r \) including \( r \to \eta \). Substituting the expression for the dissipation anomaly (13) into the right-side of (7) gives:

\[
\frac{\partial S_{2n,0}}{\partial r} + \frac{d-1}{r} S_{2n,0} - \frac{(d-1)(2n-1)}{r} S_{2n-2,2} = -(2n-1) < \delta (\partial_r p) u^{2n-2} > \nabla
\frac{3(2n-1)}{4} < \left( \frac{\partial (\delta \mathbf{u})^3}{\partial r} \right) + (d-1) \left( \frac{\partial (\delta \mathbf{u})^2}{\partial r} \right) (\delta_r \mathbf{u})^{2n-3} >
\]

The velocity-kinetic energy product, which appears in equation (13) has not been included in this relation as small. Using the gaussian boundary conditions implying \( (2n-1)S_{2n-2,2} = (1 + \xi_2/2)S_{2n,0} \) this expression can be somewhat simplified. The full closure is achieved if we express the second contribution to the right side of (14) in terms of \( \delta_r \mathbf{u} \). At this point we cannot do it rigorously but it is plausible to assume that it is \( O((\delta_r \mathbf{u})^3) \) leading to a generic relation (10). A simple closure based on the Bernoulli-like relation \( \delta_r (\partial_r p) = O(\delta_r (\delta \mathbf{u})^2) \) leading to the one-parametrical expression for the exponents \( \xi_n \) in a close agreement with available experimental data has been proposed in Refs. [12], [16].
5 Dissipation structures. PDF of dissipation scales.

On the dissipation scale $\eta$ all contributions from pressure, advection and dissipation terms are of the same order and the relation for the dissipation anomaly allows the estimate [18]:

$$\eta \delta_\eta u \approx \nu \quad (15)$$

which means that the dissipative structures correspond to the local magnitude of the Reynolds number

$$Re_\eta = \eta \delta_\eta u / \nu = O(1) \quad (16)$$

This defines $\eta$ as a random field. We would like to stress that the relation (15) has been obtained by balancing various contributions to the locally exact expression for the dissipation anomaly and by establishing the relation between the dissipation scale and velocity increment, this formula enables one to evaluate spatial derivatives and compute various correlation functions. If the displacement $y$ is in the analytic range, then $u(x+y) - u(x) \approx \partial_x u(x)y$ and extrapolating $y \rightarrow \eta$ where $\eta$ is the scale where the analytic and singular ranges overlap, we obtain [18]:

$$\partial_x u \approx \delta_\eta u / \eta \approx (\delta_\eta u)^2 / \nu \quad (17)$$

acceleration

$$a \approx \frac{\delta_\eta u}{\tau} \approx \frac{(\delta_\eta u)^2}{\eta} \approx \frac{(\delta_\eta u)^3}{\nu} \quad (18)$$

and the dissipation rate

$$\mathcal{E} = \nu \delta_\eta u \nabla^2 \delta_\eta u \approx (\delta_\eta u)^4 / \nu \quad (19)$$

It follows from (17) (See also Ref. [18]) that for each moment $S_{n,m}$ one can define a "dissipation scale $\eta_{n+m}$ separating analytic and singular ranges and

$$\eta_n \approx LR_{e^{\frac{1}{\xi_{n+1}}}} \quad (20)$$
Using the relations (17)- (20), we can develop the multi-scaling algebra. For example,
\[ a_{2n} \approx (Re_{\text{rms}})^{2n} S_{6n}(\eta_{6n}) \times (Re_{\text{rms}})^{2n} \eta_{6n}^{d_{6n}} \approx (\frac{u_{2n}^2}{L})^{2n} Re^{a_{2n}}, \] (21)
with \( a_{2n} = 2n + \frac{\xi_{6n}}{\xi_{6n+1} - 1} \). With \( \xi_6 = 2 \) and \( \xi_7 = 7/3 \), we recover Yaglom’s result [24] \( \overline{a^2} \approx \frac{u_{2n}^2}{\nu} \). If evaluating the exponents, one uses in (21) the anomalous exponents \( \xi_6 \approx 1.77 \) and \( \xi_7 \approx 1.99 \) from Ref.[12], then \( a_2 \approx 0.55 \) meaning that the intermittency correction is \( \approx 0.05 \). Recent experiments by Reynolds et al. [25] were in a close agreement with (21). A similar formula has recently been obtained from the multi-fractal formalism by Biferale et al. [26]. Formula (21) shows that the second moment of Lagrangian acceleration is expressed in terms of the sixth-order structure function evaluated on its dissipation scale \( \eta_6 \). To extract information about the fourth moment \( \overline{a^4} \), we have to have accurate data on \( S_{12}(\eta_{12}) \) which, in high-Reynolds-number flows, is very difficult to obtain from both physical and, especially, numerical experiments.

The moments of velocity derivatives evaluated easily. In accord with (17) and (20):
\[ \overline{(\partial_x u)^{2n}} \approx (\frac{\delta u}{\eta})^{2n} \approx (\frac{\delta u}{\nu})^{2n} \approx Re^{2n} S_{4n}(\eta_{4n}) \approx Re^{d_{2n}}, \] (22)
where \( d_{2n} = 2n + \frac{\xi_{4n}}{\xi_{4n+1} - 1} \).

**Moments of the dissipation rate. The exponent \( \mu_2 \).** Using (19) and (20), the moments of the dissipation rate are calculated readily: \( \overline{E^n} \approx Re^{\mu_n} \) where \( \mu_n = n + \frac{\xi_{4n}}{\xi_{4n+1} - 1} \). Taking in accord wit Refs. [[11],[12] \( \xi_n \approx 0.383n/(1 + 0.05n) \), gives
\[ \mu_2 \approx 0.16 \]
and, since according to (20) \( \eta_8 \approx Re^{-0.84} \), we derive \( \overline{E^2} \approx \eta_8^{-0.19} \). This result is based on the relation (19) telling us that the mean of the square dissipation rate must be evaluated in terms of finite difference (velocity increment) over the region of space having the linear dimension \( \eta_8 \) which is substantially smaller that the Kolmogorov scale \( O(Re^{-3/4}) \). Extrapolating this relation into an inertial range gives
\[ \overline{E(x)E(x+r)} \propto r^{-\mu} \]
with \( \mu \approx 0.19 \). The same expression, calculated on the Kolmogorov scale gives \( \mu \approx 0.21 \).

It is interesting that the exponents \( \mu_n \) for \( n < 1 \) are negative. This prediction is yet to be tested experimentally. Evaluation of the high-order moments of the dissipation rate involving the correlation functions \( S_{4n}(\eta_{4n}) \) in their respective analytic intervals requires very high resolution of the velocity field and is highly problematic.

**Dissipation scales.** According to its definition, the dissipation scale is a linear dimension of a structure defined by the local value of the Reynolds number \( Re_\eta = \eta \delta u/\nu = O(1) \). This introduces a random field. The probability density \( Q(\eta, Re, Re_\eta) \) is found by fixing the displacement \( r = \eta \) and counting the events with \( \eta \delta u/\nu = Re_\eta \) keeping the global and local Reynolds numbers \( Re = u_{rms} L/\nu \) and \( Re_\eta \) fixed as parameters. Defining \( \eta \) as a scale at which the analytic and singular parts of velocity field overlap, naturally leads to the matching condition: \( P(\delta u/\eta^a, \eta) = P_D(\delta u, \eta) \) where \( P_D \) is the PDF of \( \delta u \) for the displacement values from the ”dissipation” range \( r \leq \eta \) where the expression (3) does not work. On the matching scale however, the relation (3) is correct. Thus, the probability density \( Q(\eta, Re) \) can be found from \( P(\delta u|Re_\eta \approx 1) \) where \( Re_\eta = \eta \delta u/\nu \). From formula (1):

\[
P(\delta u) \equiv P(u) = \frac{1}{u_\eta} \int_{-\infty}^{\infty} A(n) n^{\xi(n)} \eta^{-\xi(n) - n} dn \tag{23}
\]

and for the scaling exponents given by (3), the result is derived readily. Introducing the large-scale Reynolds number \( \nu = 1/Re \) and taking into account that \( Re u_\eta \approx 1/\eta \) gives for the probability density \( Q(\eta) \):

\[
Q(\eta, Re) = \frac{1}{\eta} \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} dn e^{in \ln(\eta^{a+1} \sqrt{2Re}) - bn^2 \ln \eta} \tag{24}
\]

giving:

\[
Q(\eta, Re) = \frac{1}{\eta \sqrt{4b \ln \eta}} \int_{-\infty}^{\infty} e^{-x^2} dx e^{\ln^2(\eta^{a+1} \sqrt{2Re})/4b \ln \eta} \tag{25}
\]

The probability density \( Q(\eta, Re) \) is plotted on Fig.4a for \( Re = 1/\nu = 0.3 \) and the curves for \( Re = 0.1; 0.3 \) are compared on Fig.4b.
PDF of velocity derivative is computed from (23) combined with (17). The result is:

\[ P(u') = \frac{2}{u'} \int_{-\infty}^{\infty} A(n) \nu^{\frac{3 \xi(n)}{2}} u'^{-\frac{3 \xi(n)}{4}} \frac{2}{\pi} \, dn \]

The integral is evaluated using the procedure of Section 2. It is interesting that for a "normal" case \( \xi_n = n/3 \) we have:

\[ P(u') \propto e^{(\nu u'^2)^{\frac{2}{3}}} \]

This relation, which has been obtained by Benzi et al [27] from the multi-fractal formalism, cannot be a consequence of the Kolmogorov (K41) theory. Indeed, unlike the expression (17) based on the idea of the fluctuating u.v. cut-off, the K41 dissipation scale \( \eta_K = \text{const} \) is a number not related to \( \delta_n u \). Thus, since the fluctuating dissipation scale is not compatible with K41 and "normal scaling", this PDF is not realizable. If for consistency with K41, one uses the constant dissipation scale, then the PDF is simply the one of \( \delta_{\eta_K} u \) which, in accord with calculation of Sec 1, is a Gaussian.

The PDF of the dissipation rate can be easily calculated from the integral (23) with \( u_\eta \approx \langle E \nu \rangle^{\frac{1}{4}} \). The resulting expression gives \( P_e(\mathcal{E}, Re) \) with the broader tails than those of Kolmogorov’s Log-normal probability density.

6 Discussion and conclusions.

The theory presented in this paper is based on two principle assumptions: 1. The existence of a length-scale \( L \) such that for \( r = L \) the odd-order moments \( S_{2n+1}(L) \approx 0 \). At this scale the energy flux toward small scales sets up and the PDF \( P(\delta_r u, r = L) \) is close to the Gaussian. This statement, consistent with the Navier-Stokes equations, has been tested in both physical and numerical experiments [19]-[20]. The Gaussian PDF \( P(\delta_L u, L) \) is not unlike a large-scale boundary condition needed to solve the differential equations (7)-(8) for the moments \( S_{n,m} \). 2. The expression (3) for the first two terms of the Taylor expansion of the function \( \xi_n \), consistent with the Holder inequality, is a good approximation to the exponents of the first 10-15 moments of velocity increments. The existence of a small
parameter in turbulence theory is highly problematic and numerical smallness of deviations from the linear expression for not too large moment numbers $n$ in (3) was helpful for the theory developed above. The relations for the dissipation structures and "dissipation" scales $\eta$ coming from the order-of-magnitude balancing of the terms in the exact equations for the dissipation anomaly are well justified.

Using these assumptions we have shown that the calculated PDF of velocity differences strongly deviates from the Log-normal distribution, first obtained by Kolmogorov in 1962, which even today is widely used in the literature. This difference stems from contributions of the amplitudes $A(n)$ to the integral (1). The amplitudes $A(n)$ are fixed by the large-scale boundary conditions, leading to a natural conclusion: the small-scale dynamics are strongly coupled to the large-scale phenomena. This may be a reason for a serious reexamination of the very concept of the turbulence energy cascade which, within the framework of the present development, seem neither possible nor needed. An accurate experimental and numerical comparison of the measured and Log-normal PDFs of velocity increments may be extremely important.

It has been shown [18] (in a different way this is also an element of the multi-fractal theory) that the scales $\eta$ form a random field not necessarily related to the energy dissipation scales but rather to the linear dimensions of various dissipation structures defined by the local value of the Reynolds number $Re_\eta = O(1)$. Some of these structures are responsible for the energy and the second-order moment $S_{2,0}$ dissipation, while others, more powerful, for the dynamics of the higher -order moments. Thus, the scale $\eta$ must be perceived as a dynamic cut-off separating analytic and singular components of the velocity field. The probability density $Q(\eta, Re)$, calculated in Section 5 is an interesting and easily measurable quantity. A note of caution is in order: to make reliable calculations or measurements of the moments involving spatial derivatives of velocity field, one has to have a field resolved well enough to exhibit at least a fraction of the analytic range of the corresponding moments of velocity increment. For example, according to (18), the second order moment of Lagrangian acceleration is proportional to the sixth order structure function calculated on the scale $\eta_6$, while the fourth order one is expressed through $S_{12}(\eta_{12})$. In the high Reynolds number flows,
the measurements of the twelveth order structure function including analytic range where \( S_{12} \propto r^{12} \) do not exist. The situation with the moments of dissipation rate is even worse: fourth-order moment \( \mathcal{E}^4 \) is related to \( S_{16}(\eta_{16}) \). An interesting possibility is being explored by Schumacher \[28\] running very large \((1024^3)\) numerical simulations at reasonably low Reynolds numbers \( R_\lambda \approx 10 - 60 \). Analyzing the probability density of the dissipation scales Schumacher and Sreenivasan \[29\] obtained \( Q(\eta, Re) \) very similar to one shown on Fig. 3.

The results presented here were obtained from analysis, both theoretical and numerical, of the dynamic equations. No multi-fractal assumptions have been made. Still, it is interesting to compare the two approaches. In its present form, the multi-fractal (MF) theory consists of two parts \[1\]. The first one based on an idea of fractal dimension, attempts to explain the origin of anomalous scaling by assuming

\[
S_p(r) = \left( \frac{r}{L} \right)^{\xi_p} = \int d\mu(h) \left( \frac{r}{L} \right)^{ph+3-D(h)}
\]

(26)

The normalised structure functions \( S_p = (\frac{\delta u}{u_{rms}})^p \). The \( O((r/L)^p) \) term comes from the multi-fractal assumption

\[
\delta_r u = (r/L)^h
\]

(27)

defined on a set of fractal dimension \( D(h) \) where \( h \) is a value of the scaling exponent from the interval \( h_{\min} \geq h \geq h_{\max} \) and \( (r/L)^{3-D(h)} \) is the probability of being in the interval \( r \) in a volume of dimension \( 3 - D(h) \). Neglecting the Logs in the steepest descent evaluation of the integral \( (26) \), gives a relation between the fractal dimension and the exponents \( \xi_n \) (For the review see Frisch \[1\]).

**The multifractal PDF and Mellin transform.** Let us establish possible relations between the two theories. Using the gaussian expression for the amplitudes, we have:

\[
S_{2p}(r) = (2p-1)!! \left( \frac{r}{L} \right)^{\xi_{2p}} = (2p-1)!! \int d\mu(h) \left( \frac{r}{L} \right)^{2ph+3-D(h)}
\]

(28)

Substituting this into (1) and repeating the calculations of the section 2 gives \( (L = 1) \):

\[
P(u, r) = \frac{1}{2} \int e^{-x^2} dx \int_{-i\infty}^{i\infty} dn \int \frac{d\mu(h)}{dh} dh e^{-n(\ln \sqrt{2x} - h \ln r)} e^{(3-D(h)) \ln r}
\]

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Integration over \( n \) gives the delta-function \( \delta(h \ln r - \ln \frac{u}{\sqrt{2x}}) \) \( (u/x \propto r^h) \) with the final result:

\[
P(u, r) = \frac{1}{2u \ln r} \int dx e^{-x^2} \exp(3 - D(\ln \frac{u}{\sqrt{2x}})) \frac{d\mu(h_*)}{dh}
\]

where \( h_* = \frac{\ln \frac{u}{\sqrt{2x}}}{|\ln r|} \). Restricting ourselves by the relation (3) for the exponents and comparing this formula with the PDF (5) gives (neglecting the \( \mu \)-factor):

\[
[3 - D(\ln \frac{u}{\sqrt{2x}})] \ln r = -\frac{(\ln \frac{u}{\ln r})^2}{4b|\ln r|}
\]

No steepest descent approximation has been used in deriving this relation. If in accord with the MF theory, we set the amplitudes \( A(n) = 1 \) and \( \sqrt{2x} = 1 \) in (29) and taking into account that \( h = \ln u/\ln r \), the expression (29) gives:

\[
3 - D(h) = \frac{(\ln \frac{u}{\ln r})^2}{4b(\ln r)^2} = \frac{(h - a)^2}{4b}
\]

We remind the reader that in accord with Ref. [11],[12], \( a \approx 0.383 \) and \( b \approx 0.0166 \). This derivation which did not involve the steepest descent evaluation of the integral does not have a "Log-problem", discussed in Ref. [30]. The experimental measurements of Cramer function \( f(\alpha) = D(h) + 2 \) for \( h = \alpha/3 \) by Meneveau and Sreenivasan [31] (See also Ref.[1]) are in an extremely close agreement with this expression. The quantitative differences are:

the maximum of the calculated curve (with \( \alpha = 3h \)) is at \( \alpha = 1.15 \) (instead of \( \alpha = 1 \) of Ref. [31]) and \( f(\alpha) = 0 \) at \( \alpha_1 = 0.369 \) and \( \alpha_2 = 1.92 \) compared with \( \alpha_1 \approx 0.5 \) and \( \alpha_2 \approx 1.8 \) of Ref.[31]. The small deviations come from the difference between \( h = \alpha/3 \) used for analysis of experimental data and the theoretically obtained \( h = a = 0.383 \). This difference decreases if the coordinates are rescaled by factor 0.383/0.333.

The second part of the MF theory, dealing with the small-scale properties of turbulence, is based on the relation (Paladin and Vulpiani [32] )

\[
\frac{\eta}{L} \approx Re^{-\frac{1}{1+\delta}}
\]

obtained by combining the MF assumption (27) and the outcome of the balance of the advective and viscous contributions to the Navier-Stokes equations. All small-scale results
derived using the MF formalism are numerically indistinguishable from the ones obtained both above and in the Ref. [18]. To illustrate this point, we present an alternative derivation of the moments of velocity derivative due to Polyakov [33].

**Polyakov’s derivation.** The probability density of velocity difference in the inertial range is:

\[ P(\delta_r u, r) = \langle \delta(\delta_r u - [u(x + r) - u(x)]) \rangle > \]  

(31)

and in the dissipation (analytic) range

\[ P_D(\delta_r u, r) = \langle \delta(\delta_r u - u'r) \rangle > \]  

(32)

Assuming the two PDFs match at the scale \( r = \eta = \nu/\delta_\eta u \equiv \nu/u \), introduced by (15), we have:

\[ P_D(u, \eta) = P(u, \eta) = \langle \delta(u - u'(x)\nu/u) \rangle = \int dnA(n)\nu^{\xi_n}u^{-\xi_n-n-1} \]  

(33)

Multiplying (33) by \( u^{2k} \) and integrating over \( u \) gives:

\[ \frac{1}{2}(\nu u')^k = \int dnA(n)\nu^{\xi_n} \frac{1}{2k - \xi_n - n} \]  

(34)

The integral is evaluated at a pole where \( \xi_n(k) + n = 2k \) giving

\[ (u'(x))^k \propto A(n(k))\nu^{\rho(k)} \]  

(35)

with

\[ \rho(k) = \xi_n(k) - k = k - n(k) \]  

(36)

which is identical to the formula (8.76) of Frisch’s book [1] obtained using multi-fractal theory. Comparing the relations (35) and (22) we find that, on the accepted magnitudes of exponents \( \xi_n \), numerically they are basically identical for not too large moment numbers \( n \). It is also easy to see that if in the limit \( n \rightarrow \infty \), \( \xi_n \propto n^{\alpha} \) with \( 0 \leq \alpha \leq 1 \), the two relations have the same asymptotics.
The only approximation involved in derivations of both relations (22) and (35), presented in this paper, is the choice of the cut-off $\eta = \nu/u$ instead of $\eta = O(\nu/u)$. In reality, there exist a random field of the dissipation scales described by the probability density $Q(\eta, Re)$ given by (25). Thus, a more accurate calculation of both (22) and (34) must involve averaging over the fluctuating cut-off $\eta$. However, we do not expect this procedure to introduce substantial modifications of the obtained results.

The theory presented here does not involve any multi-fractal assumptions. Still, the quantitative (numerical) agreements between the two approaches hints on the possibility of some qualitative connection. The essentially dynamic theory developed here couples the velocity fluctuations at the largest and smallest scales. One may speculate that if a typical structure is basically a strongly convoluted sheet with two $O(L)$ linear dimensions and $O(\eta)$ the third one, then these structures can loosely be identified with the multi-fractal sets of the MF theory.

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Figure 1: Probability density \( P(\frac{\delta u}{r^a}, r) \) vs \( \delta u/r^a \). a. \( r=0.01 \); b \( r=0.01, 0.1, 0.5 \).
Figure 2: a. Computed PDF $P(\frac{\delta u}{r_a}, r)$ vs $\delta, u/r^a$ (eq (5)) and Log-normal PDF $P_L(\frac{\delta u}{r_a}, r)$ (eq. (6)) for the same values of parameters. b. $(P - P_L)/P_L$ vs $\delta, u/r^a$. 

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Figure 3: From Gotoh and Nakano (Ref. [16]). The top curves show the right ("rhs") and left ("lhs") sides of equation (8) for $n = 4; 6; 8$. When the pressure contribution was added to the "rhs", very accurate equality $rhs = lhs$ has been reached (dotted line on top of the "lhs" curve). The bottom curves are $R_{2n} = \frac{(2n-1)\delta_{z}(\partial_z p)u^{2n-2}}{\partial_z S_{2n,0} + 2S_{2n,0}/r}$. The vertical axis of the Fig.3c is scaled by factor 10.
Figure 4: Probability density $Q(\eta/\eta_K, Re)$ vs $\eta/\eta_K$ where $\eta_K$ is the Kolmogorov scale.  

(a) The shape of the PDF $Q(\eta, Re)$ for $Re \approx 0.3$.  

(b) for $Re = 0.1; 0.3$. 

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