The phantom shell around a black hole and global geometry

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Abstract
We describe the possible scenarios for the evolution of a thin spherically symmetric self-gravitating phantom shell around the Schwarzschild black hole. The general equations describing the motion of the shell with a general form of the equation of state are derived and analysed. The different types of spacetime $R_\pm$ and $T_\pm$ regions and shell motion are classified depending on the parameters of the problem. It is shown that in the case of a positive shell mass there exist three scenarios for the shell evolution with an infinite motion and two distinctive types of collapse. Analogous scenarios were classified for the case of a negative shell mass. In particular, this classification shows that it is impossible for a physical observer to detect the phantom energy flow. We shortly discuss the importance of our results for astrophysical applications.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction
Recent observations of both type Ia high-redshift supernovas (SNs) and cosmic microwave background (CMB) anisotropy at small angular scales strongly indicate in favour of the acceleration of the universe expansion at the present epoch [1]. The simplest possibility for the acceleration of the universe expansion is the existence of the cosmological constant $\Lambda$ with an equation of state $w \equiv p/\rho = -1$ [2]. The cosmological constant provides a satisfactory explanation of the cosmic dynamics but encounters the fine-tuning problem. An alternative explanation is the existence of dark energy in the form of a specific scalar field (quintessence) whose equation of state varies with time (see, e.g., [3, 4]). Such a model allows us to construct the so-called ‘tracker’ or ‘attractor’ cosmological solutions which resolve, in particular, the cosmological fine-tuning problem [4].
One of the peculiar features of dark energy is the possibility of the phantom energy equation of state $\rho + p < 0$. This phantom energy violates the weak energy dominance condition. In the case of phantom energy the cosmological scenario with the ‘big rip’ is possible when cosmological phantom energy density grows at large times and disrupts finally all bounded objects up to a subnuclear scale [5]. The other peculiarity is the diminishing of black hole mass due to phantom energy accretion [6]. Phantom energy is usually associated with phantom or ghost fields: for example, scalar fields with a wrong sign kinetic term [5, 7]. Phantom energy has some peculiar properties in the framework of QFT in curved spacetime [8]. The thermodynamic properties of phantom energy are also rather unusual [9].

The existence of phantom energy is not excluded by the present-day observations. The measurements [10] of the distances and host extinctions of 230 SN Ia provide constraints on the dark energy equation of state, $-1.48 < w < -0.72$. In [11] the data sets containing 172 type Ia supernovas are analysed in a model-independent manner and it was shown that the presence of phantom energy with $-1.2 < w < -1$ is preferable for the recent epoch. The Chandra telescope observations [12] of the hot gas in the 26 x-ray luminous dynamically relaxed galaxy clusters provide $w = -1.20^{+0.34}_{-0.28}$, which is also in favour of phantom energy.

The evolution of dark energy is usually considered in relation to the cosmological problems. However, the local evolution of self-gravitating dark energy may be quite different from the cosmological one. This is because of the nonlinearity of the general relativity equations. The three-dimensional analytical treatment is possible only in very restrictive cases, e.g., in the case of the stationary accretion onto a black hole of the dark energy considered as a test fluid, i.e., with negligible self-gravitation [6]. To consider a self-gravitating fluid one must use some simplified models. One of the analytically treatable approaches with fluid self-gravitation is taken into account in the thin-shell model. The theory of thin shells in general relativity was developed by Israel [13] and then by many authors (see, e.g., [14] for a review and references). The problem of thin-shell analysis is greatly simplified in the case of spherical symmetry. The aim of this paper is to consider several scenarios for the thin spherically symmetric phantom shell evolution.

The paper is organized as follows. In section 2 the general concepts of spherically symmetric gravity are outlined with special attention to Schwarzschild spacetime. In section 3 the specific equations of motion for thin shells are obtained. In section 4 the evolution of a shell with the phantom equation of state is analysed and different types of motion are classified. In section 5 we briefly discuss the obtained results. Throughout the paper we use the units $\hbar = c = 1$.

2. General theory

2.1. Spherically symmetric gravity

The line element of any spherically symmetric spacetime can always be written in the form

$$ds^2 = g_{\alpha \beta} dx^\alpha dx^\beta = A dt^2 + 2H dt dq + B dq^2 - R^2(t, q) d\Omega^2$$

(1)

with the signature $(+,-,-,-)$. Here $t$ and $q$ are correspondingly the timelike and spacelike coordinates, $A$, $H$ and $B$ are functions of $t$ and $q$ only, and $R(t, q)$ is the radius of a two-dimensional sphere (in the sense that the area of the sphere is equal to $4\pi R^2$),

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$$

(2)
being the line element of the unit sphere. For the given spacetime the coefficients $A$, $H$ and $B$ are not uniquely defined. One can transform the line element (1) to the new coordinate system which explicitly conserves the spherically symmetric form of the metric:

$$\tilde{t} = \tilde{t}(t, q), \quad \tilde{q} = \tilde{q}(t, q).$$

(3)

Unlike the metric coefficients in $M_2$, the radius $R(t, q)$ is invariant under the transformation (3). The other very important invariant is constructed from the partial derivatives of $R$ as follows,

$$\Delta = \gamma^{ab} R_\alpha R_\beta,$$

(4)

where $\gamma^{ab}$ is inverse to the two-dimensional metric tensor $\gamma_{ab}$. This invariant is nothing but the square of the normal vector to the surface $R = \text{const}$.

If we know two invariant functions $R(t, q)$ and $\Delta(t, q)$ we know the line element of the spherically symmetric spacetime up to the gauge transformations and, therefore, its local structure. To construct the global manifold we need some additional principle. Physics provides us with it. From the point of view of physicists any spacetime should be geodesically complete [2], that is, timelike and null geodesics have unbounded affine parameters.

The function $\Delta(t, q)$ gives nontrivial information about a spacetime structure. Indeed, in the flat Minkowskian spacetime $\Delta \equiv -1$, all the surfaces $R = \text{const}$ are timelike and therefore, $R$ can be chosen as a spatial coordinate $q = R$ on the whole manifold. But in the curved spacetime $\Delta$ is no longer constant and can, in general, be both positive and negative. The region with $\Delta < 0$ is called the $R$ region, and the radius can be chosen as a radial coordinate $q$.

In the region with $\Delta > 0$ the surfaces $R = \text{const}$ are spacelike (the normal vector is timelike), and the radius $R$ can be chosen as a time coordinate $t$. Such regions are called $T$ regions. The $R$ and $T$ regions were introduced by Igor Novikov. But this is not the whole story. It is easy to show that we cannot get $\dot{R} = 0$ (the ‘dot’ means a time derivative) in a $T$ region. Hence either $\dot{R} > 0$ (such a region of inevitable expansion is called the $T_+$ region) or $\dot{R} < 0$ (inevitable contraction, a $T_-$ region). The same holds for $R$ regions. They are divided into two classes, those with $R' > 0$ (the ‘prime’ stands for a spatial derivative) which are called $R_+$ regions and $R_-$ regions with $R' < 0$. These $R$ and $T$ regions are separated by the surfaces $\Delta = 0$ which are called the apparent horizons. The apparent horizons can be null, timelike or spacelike.

Thus, the curved spherically symmetric spacetimes may, in general, have a rather complex structure, a set of $R_\pm$ and $T_\pm$ regions separated by apparent horizons $\Delta = 0$. In the next subsection, we consider one of the important examples of spherically symmetric manifolds.

2.2. Schwarzschild spacetime

The solutions to the vacuum Einstein equations consist of only one-parameter family. In the curvature coordinates ($q = R$) the metric of a Schwarzschild spacetime has the form

$$ds^2 = F(R) dt^2 - F^{-1}(R) dR^2 - R^2 d\Omega^2,$$

(5)

where

$$F(R) = 1 - \frac{2Gm}{R}, \quad m > 0$$

(6)

and $G$ is the Newton gravitational constant. This static metric describes a spacetime outside a spherically symmetric body with a mass $m$ also called a Schwarzschild mass. The metric (5) has a coordinate singularity at $R = 2Gm$. It is related to the static nature of the line element (5) and impossibility of synchronizing the clocks of the observers who are static at the spatial infinity ($R = \text{const} \rightarrow \infty$) and those (who are nonstatic) in the region $R < 2Gm$. 
Figure 1. The Carter–Penrose diagram for Schwarzschild spacetime. Every point represents a sphere. Here $I_-$ and $I_+$ mean past and future null infinities respectively. The past and future timelike and spacelike infinities are $i_-$ and $i_0$ respectively. Regions $R$ and $T$ are separated by the two apparent horizons (future and past ones) which are null surfaces. In our case, the future horizon coincides with the event horizon defined as the first null geodesics which does not reach infinity. Its time reversal is called a particle horizon. The $T_+$ region is called the black hole. The $T_-$ region is called the white hole, and the $R_-$ region is called the wormhole respectively. The value of $\sigma$ is equal to $+1$ in the $R_+$ region and it is equal to $-1$ in the $R_-$ region. The sign of $\sigma$ could be changed only at $T$ regions.

Figure 2. Schwarzschild spacetime at fixed time and $\theta$ and its embedding diagram. The throat width is proportional to the Schwarzschild mass.

Moreover, it appears that the manifold described by the line interval (5) is not geodesically complete. It can be shown that the maximally extended Schwarzschild manifold consists of four parts. Furthermore, it is possible to choose such coordinates, which put infinities at finite distances. Schwarzschild geometry in such coordinates is represented by the Carter–Penrose diagram shown in figure 1. Another useful representation of the Schwarzschild spacetime is the so-called embedding diagrams. One can consider the Schwarzschild metric with $t = \text{const}$ and $\theta = \pi/2$. It is easy to show that this is the metric on a paraboloid embedded in the three-dimensional flat space

$$r = \frac{z^2}{8Gm} + 2Gm.$$  

(7)

So, an embedding looks like that shown in figure 2. In the following we will use a schematic version of embedding which is called the embedding diagram (the right panel in figure 2). As one can see from figure 2 the wormhole is separated from the $R_+$ region (or the black hole exterior) by the throat also called the Einstein–Rosen bridge. Of course, the above definitions are by no means general, but they are sufficient for our purposes. (The general definitions require powerful mathematical tools and an enormous number of predefinitions.) With the
inclusion of matter sources the Schwarzschild solution is valid only outside their boundaries. In the case of a complete Schwarzschild manifold the sources are considered as concentrated at the past and future singularities at \( R = 0 \). The manifold is called the eternal black hole.

3. Thin shells

One of the most important features of general relativity is that the equations of motion of matter fields are incorporated into the Einstein equations. The Einstein equations of general relativity are nonlinear partial differential equations. This means that the motion of test particles or fields on the given background will, in general, be different from that of the matter for the self-consistent solutions. It makes analysis very complicated. To obtain some definite results we have to choose the simplest possible model. This is a self-gravitating thin shell. In this section we derive equations of motion for a thin shell.

Let us now introduce the notion of a thin shell. Because we will deal only with timelike spherically symmetric thin shells we adjust the very nice generally covariant formalism derived by Israel [13] to the case of interest (see, e.g., [14]). Let us choose some hypersurface \( \Sigma \) dividing the whole spacetime into two parts, ‘in’ and ‘out’. With this hypersurface \( \Sigma \) can be connected to some special coordinate system called Gauss normal coordinates. In our simple case, the line element written in these coordinates takes the form

\[
\text{d}s^2 = \text{d}\tau^2 - R^2(\tau, n) \text{d}\Omega^2,
\]

where \( \tau \) is the proper time of the observer sitting fixed on \( \Sigma \), the coordinate \( n \) grows from the ‘in’ to the ‘out’ region in the outer normal direction to the hypersurface \( \Sigma \), \( R(\tau, n) \) is the radius of the sphere (in the sense that a sphere area equals \( 4\pi R^2 \)) and \( \text{d}\Omega^2 \) is the line element of the unit sphere, \( \text{d}\Omega^2 = \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2 \). The hypersurface is situated at \( n = 0 \), and

\[
\text{d}s^2_\Sigma = \text{d}\tau^2 - \rho^2(\tau) \text{d}\Omega^2, \quad \rho(\tau) = R(\tau, 0).
\]

The hypersurface \( \Sigma \) is called the singular shell if some energy–momentum tensor is concentrated on it, namely \( T^i_k = S^i_k \delta(n) + \cdots \), where \( S^i_k \) is the surface energy–momentum tensor of the shell \((i, k = 0, 2, 3)\). Otherwise the hypersurface is nonsingular. In our case, due to the spherical symmetry the only nonzero components of \( S^i_k \) are \( S^0_0 \) and \( S^2_2 = S^3_3 \).

The invariant \( \Delta \), introduced earlier, equals

\[
\Delta = R^2(\tau) - R^2_\Sigma
\]

and

\[
R_{\Sigma n} = \sigma \sqrt{\dot{\rho}^2 - \Delta},
\]

where \( \sigma = +1 \) if radii increase in the direction of the outer normal, and \( \sigma = -1 \) if radii decrease. Evidently, \( \sigma = +1 \) in the \( R_+ \) region and \( \sigma = -1 \) in the \( R_- \) region. Thus on the equation of motion of the shell \( \sigma \) can change sign only in the \( T \) region, or in the region where its motion is forbidden.

The subsequent procedure is very simple. Keeping in mind that the metric itself is continuous but some of its derivatives could undergo a jump across the shell, we integrate the Einstein equations and obtain (the nontrivial result is only for the \((0)\) and \((2)\) components) after some algebra

\[
\frac{2\sigma_{\text{in}}}{\rho} \sqrt{\dot{\rho}^2 - \Delta_{\text{in}}} - \frac{2\sigma_{\text{out}}}{\rho} \sqrt{\dot{\rho}^2 - \Delta_{\text{out}}} = 8\pi G \rho S^0_0,
\]

### References

[13] Israel, W. (1966). *General Relativity*. Cambridge University Press.

[14] Misner, C. W., Thorne, K. S., & Wheeler, J. A. (1973). *Gravitation*. W. H. Freeman and Company.
The continuity equation for the energy–momentum tensor is transformed to
\[
\frac{2}{\rho} \frac{\sqrt{\rho^2 - \Delta_{\text{in}}}}{\Delta_{\text{in}}} - \Delta_{\text{out}} \frac{\sigma_{\text{out}}}{\sqrt{\rho^2 - \Delta_{\text{out}}}} + \frac{\sigma_{\text{in}}}{\Delta_{\text{in}}} \frac{\rho}{\sqrt{\rho^2 - \Delta_{\text{in}}} \Delta_{\text{in}}} - \frac{\sigma_{\text{out}}}{\Delta_{\text{out}}} \frac{\rho}{\sqrt{\rho^2 - \Delta_{\text{out}}} \Delta_{\text{out}}}
+ \frac{\sigma_{\text{in}}}{2\rho} (1 + \Delta_{\text{in}}) - \frac{\sigma_{\text{out}}}{2\rho} (1 + \Delta_{\text{out}})
+ 4\pi G \rho \left( \frac{T_{\text{out}}}{\rho} - \frac{T_{\text{in}}}{\rho} \right) = 8\pi G S_2^2.
\] (13)

The third equation is a differential consequence of the first two. But very often it is convenient to use all three of them.

In what follows, we will consider the thin shell in vacuum. So, both inside and outside the shell we will have the Schwarzschild metric with different masses (because the shell has its own mass which is added to the inner mass) and our equations become
\[
\frac{2}{\rho} \frac{\sqrt{\rho^2 - \Delta_{\text{in}}}}{\Delta_{\text{in}}} - \Delta_{\text{out}} \frac{\sigma_{\text{out}}}{\sqrt{\rho^2 - \Delta_{\text{out}}}} + \frac{\sigma_{\text{in}}}{\Delta_{\text{in}}} \frac{\rho}{\sqrt{\rho^2 - \Delta_{\text{in}}} \Delta_{\text{in}}} - \frac{\sigma_{\text{out}}}{\Delta_{\text{out}}} \frac{\rho}{\sqrt{\rho^2 - \Delta_{\text{out}}} \Delta_{\text{out}}}
+ 4\pi G \rho \left( \frac{T_{\text{out}}}{\rho} - \frac{T_{\text{in}}}{\rho} \right) = 4\pi G \rho \left( \frac{S_{\text{out}}^0}{\rho} - \frac{S_{\text{in}}^0}{\rho} \right). \] (15)

The third equation is a differential consequence of the first two. But very often it is convenient to use all three of them.

In what follows, we will consider the thin shell in vacuum. So, both inside and outside the shell we will have the Schwarzschild metric with different masses (because the shell has its own mass which is added to the inner mass) and our equations become
\[
\sigma_{\text{in}} \frac{\sqrt{\rho^2 + F_{\text{in}}}}{\rho^2} - \sigma_{\text{out}} \frac{\sqrt{\rho^2 + F_{\text{out}}}}{\rho^2} = 4\pi G \rho S_{\text{in}}^0, \] (15)
\[
\dot{\rho} = -4\pi^2 G^2 \rho (S_0^0)^2 + 8\pi^2 G^2 \rho S_0^0 S_2^2 - \frac{G(m_{\text{in}} + m_{\text{out}})}{2\rho^2} - \frac{\Delta m^2 S_2^2}{8\pi^2 \rho^5 (S_0^0)^3} \] (16)
\[
\dot{S}_0^0 + \frac{2\dot{\rho}}{\rho} \left( S_0^0 - S_2^2 \right) = 0, \] (17)

where \( \Delta = -F = -1 + \frac{2Gm}{\rho} \). We wrote the second equation in a somewhat different (twice squared) form, which is more suitable for us. The information about a global geometry (signs of \( \sigma_{\text{in}} \) and \( \sigma_{\text{out}} \)) is already contained in equation (15).

4. Evolution of phantom shells

Now let us apply the above theory to phantom shells. Consider a simple (but a rather general!) linear equation of state \( S_0^0 = kS_2^2 \). In the phantom case \( k > 1 \). (Note that in the three-dimensional case the corresponding equation-of-state parameter \( p/\rho = w < -1 \).) The solution of (17) is
\[
S_0^0 = C \rho^{2(k-1)}, \] (18)
where we call the constant \( C \) the ‘shell power’. Denote also
\[
x = 4\pi^2 G C^2 \rho^{2k-1}. \] (19)

After some manipulations with (15) and taking into account (19) one gets the following two equations,
\[
\dot{\rho}^2 = -1 + \frac{G}{\rho} \left( (m_{\text{in}} + m_{\text{out}}) + \frac{\delta m^2}{4x} + x \right), \] (20)
\[
\dot{\rho} = -\frac{G}{2\rho^2} \left( (m_{\text{in}} + m_{\text{out}}) + \frac{k\delta m^2}{x} - 2(2k-1)x \right) \] (21)

and the sign conditions
\[
\sigma_{\text{in}} = \text{sign} \left[ \delta m + 8\pi^2 G \rho^3 (S_0^0) \right], \] (22)
\[
\sigma_{\text{out}} = \text{sign} \left[ \delta m - 8\pi^2 G \rho^3 (S_0^0) \right]. \] (23)
or in a more convenient form

\[ \sigma_{\text{in}} = \operatorname{sign}(\delta m + 2x), \]  
(24)

\[ \sigma_{\text{out}} = \operatorname{sign}(\delta m - 2x). \]  
(25)

Here, we denote \( \delta m \equiv m_{\text{out}} - m_{\text{in}} \).

The curves \( \dot{\rho}^2(\rho) \) and \( \ddot{\rho}(\rho) \) are shown in figure 3. There are several possible evolution scenarios for the shell. When the curve \( \dot{\rho}^2(\rho) \) lies completely above the \( \rho \) axis, there is only infinite motion. On the other hand, when the function \( \dot{\rho}^2(\rho) \) has roots, it is possible to have both the finite motion and the infinite one.

As one can see from equation (19), it is convenient to define the parameter space of the problem using \( m_{\text{in}}, m_{\text{out}}, k > 1/4 \) and \( \rho \) as free parameters. Then, instead of searching for conditions imposed on \( \rho \), it is enough to find conditions for the parameter \( C \). These conditions will define the global geometry. Now, let us construct these conditions. First of all, the change of sign of the acceleration \( \ddot{\rho} \) in (21) occurs when \( \ddot{\rho} = 0 \). This corresponds to the quadratic equation, whose positive root is

\[ x_0 = \frac{m_{\text{in}} + m_{\text{out}} + \sqrt{(4k - 1)^2 \delta m^2 + 4m_{\text{in}}m_{\text{out}}}}{4(2k - 1)}. \]  
(26)

The corresponding value of \( \rho \) according to (19), is denoted by \( \rho_0 \). Consider the sign of \( \dot{\rho}^2(\rho) \). Let us introduce the parameter \( C_0 \) such that \( \dot{\rho}^2(\rho_0) > 0 \) when

\[ C^2 > C_0^2 \]  
(27)

and \( \dot{\rho}^2(\rho_0) < 0 \) when

\[ C^2 < C_0^2. \]  
(28)
The explicit value for $C_0$ is
\[ C_0 = \left( \frac{x_0}{4\pi^2} \right)^{1/2} [8k(2k - 1)]^{\frac{d-1}{2}} \]
\[ \times \left\{ (4k - 1)G[(4k - 1)(m_{in} + m_{out}) + \sqrt{(4k - 1)^2\delta m^2 + 4m_{in}m_{out}}] \right\}^{\frac{1}{d-1}}. \]
(29)

Consider at first the case of $\delta m > 0$. According to (24) there must always be $\sigma_{in} = +1$. The $\sigma_{out}$ changes its sign at $x = x_1 \equiv \delta m/2$ and $\sigma_{out} = -1$ if $x > x_1$. Thus, in the case $x \to \infty$, one has $\sigma_{out} = -1$, and, if $x \to 0$, then $\sigma_{out} = +1$. The value of $\rho$ corresponding to $x_1$ is denoted by $\rho_1$. From (21) it is easy to see that
\[ \dot{\rho}(x_1) = -\frac{Gm_{out}}{\rho_1} < 0. \]
(30)

From this follows the important conclusion that $\rho_1 < \rho_0$. This can also be proved by the direct comparison of $x_0$ and $x_1$. From (20) one obtains
\[ \dot{\rho}^2(x_1) = -1 + \frac{2Gm_{out}}{\rho_1}. \]
(31)

Let us denote by $C_1$ such a value of $C$ that $\rho_1 = 2Gm_{out}$ at $C = C_1$:
\[ C_1^2 = \frac{\delta m}{2^{d/2+1}\pi^2m_{out}^{d-1}}. \]
(32)

In the case $\rho_1 > 2Gm_{out}$ one has
\[ C^2 < C_1^2 \]
(33)

and $\sigma_{out}$ changes its sign in the forbidden-for-motion part of the $R$ region. Conversely, if $\rho_1 < 2Gm_{out}$, then
\[ C^2 > C_1^2 \]
(34)

and $\sigma_{out}$ changes its sign in the $T$ region. Now we have to define which of the conditions $C_0^2 > C_1^2$ or $C_1^2 < C_0^2$ is true. This can be done by considering limits for the parameters in (29) and (32) or by direct numerical calculation of the fraction $(C_1/C_0)^2$ as a function of $\mu = \delta m/m_{out}$ and $k$. The results of such a calculation are shown in figure 4. It is clear now that $C_1^2 < C_0^2$. 

\[ \text{Figure 4. Fraction } (C_1/C_0)^2 \text{ as a function of } \mu \text{ and } k. \]
Figure 5. Infinite inflation or collapse (symmetric in time). The left throat is narrower than the right one because $\delta m > 0$.

Figure 6. The Carter–Penrose diagrams for a collapsing (left) and inflating (right) shell. The vertical line in this diagram represents the shell history separating the ‘in’ and ‘out’ regions.

Figure 7. The Carter–Penrose diagrams for a collapsing shell (left) and for a shell moving from infinity to infinity (right).

It is obvious from the above analysis that there exist three possible scenarios for the shell evolution in the case $\delta m > 0$. In short, these scenarios are as follows:

1. **Infinite motion.** The sign of $\sigma_{\text{out}}$ changes in the $T$ region. The shell power obeys the following inequality:

   $$C^2 > C_0^2.$$  \hfill (35)

   The embedding diagram for this scenario is shown in figure 5. The Carter–Penrose diagrams for the case of collapse and inflation are the same up to the time reversal (see figure 6).

2. There exist turning points and $\sigma_{\text{out}}$ changes its sign in the $T$ region:

   $$C^2_1 < C^2 < C_0^2.$$  \hfill (36)

   In this case the embedding diagrams are the same as in the previous case. The corresponding Carter–Penrose diagrams are shown in figure 7.

3. There exist turning points and $\sigma_{\text{out}}$ changes sign in the part of the $R$ region which is forbidden for the motion:

   $$C^2 < C_1^2.$$  \hfill (37)
Figure 8. The left diagram represents the case when the shell is at the left from the left turning point. In this case, the shell is collapsed finally. In the right diagram, the shell goes from past infinity to future infinity when it evolves towards the right from the right turning point.

Figure 9. The Carter–Penrose diagrams for the collapsing shell (left) and for the shell moving from infinity to infinity (right). In both cases, the shell evolves in the $T_\pm$ and $R_-$ regions.

In this case, we have two embedding diagrams (see figure 8). The Carter–Penrose diagrams are shown in figure 9. As follows from the above analysis if $\delta m > 0$, the right diagram in figure 9 is the only case when the shell shows itself in the $R_+$ region.

Consider now the $\delta m < 0$ case. The ‘density’ $S^0_0$ is assumed to be always positive. So the negativity of $\delta m$ is caused exclusively by the gravitational mass defect. The $\sigma_{in}$ changes its sign at $x = x_2 \equiv -\delta m/(2\pi)$, and $\sigma_{out} = +1$ if $x > x_2$. At the same time $\sigma_{out} = -1$. Thus, we can conclude that in the case $x \to \infty$, one has $\sigma_{in} = +1$ and $\sigma_{out} = -1$ if $x \to 0$. Denote, according to (19), the value of $\rho$ corresponding to $x_2$ by $\rho_2$. From (21) it is easy to see that

$$\rho(x_2) = -\frac{Gm_{in}}{\rho_2^2} < 0. \quad (38)$$

From (20) one obtains

$$\dot{\rho}(x_2) = -1 + \frac{2Gm_{in}}{\rho_2}. \quad (39)$$

Let us also denote by $C_2$ such a value of $C$ that $\rho_2 = 2Gm_{in}$ at $C = C_2$:

$$C_2^2 = \frac{\delta m}{2^{2(2k+1)}\pi^{2}m_{out}^{4k-1}}. \quad (40)$$

In the case $\rho_2 > 2Gm_{in}$ one has

$$C^2 < C_2^2 \quad (41)$$

and the considered region is the $R$ region. Conversely, if $\rho_2 < 2Gm_{in}$ then

$$C^2 > C_2^2 \quad (42)$$
and one has the $T$ region here. Exactly as in the case $\delta m > 0$, the fraction $(C_2/C_0)^2 < 1$ because the 2D graph is an analogue of figure 4 with the replacement $\mu \rightarrow -\mu$. As a result, we can conclude that in the case $\delta m < 0$ there exist two evolution scenarios:

1) **Infinite motion.** The sign of $\sigma_{in}$ changes in the $T$ region. The inequality for $C$ parameters is

$$C_2 > C_0^2.$$  \hfill (43)

The embedding diagram for this scenario is shown in figure 10. The Carter–Penrose diagrams for the case of the collapse and inflation are the same up to the time reversal (see figure 11).

2) There exist turning points and $\sigma_{out}$ changes its sign in the $T$ region:

$$C_2^2 < C_2^2 < C_0^2.$$ \hfill (44)

In this case, the embedding diagrams are the same as in the previous case. The corresponding Carter–Penrose diagrams are shown in figure 12.
Figure 13. The left diagram represents the situation when the shell is on the left from the left turning point. In this case the shell is collapsed. In the right diagram, the shell goes from past infinity to future infinity when it evolves to the right from the right turning point.

Figure 14. The left Carter–Penrose diagram represents collapse of the shell. In the right diagram, the shell moves from infinity to infinity. In both cases the shell evolves in the $T_\pm$ and $R_-$ regions.

(3) There exist turning points and $\sigma_{out}$ changes its sign in the part of the $R$ region which is forbidden for motion:

$$C^2 < C_2^2.$$  (45)

For this case, we have two embedding diagrams (see figure 13). The Carter–Penrose diagrams are shown in figure 14.

We see again that for all the scenarios in the case of $\delta m < 0$ the shell evolves under horizons and cannot reach a distant observer living in the $R_+$ region.

5. Conclusion and discussion

We considered the dynamics of a phantom thin shell surrounding the Schwarzschild black hole. The motivation for this work comes from the fact that in many physically interesting situations in cosmology and astrophysics the essential role was played by the full account for gravitational backreaction. In our case of phantom shells such a backreaction may appear crucial for the formation of the global geometry of the spacetime. The matter is that in general relativity any type of energy is gravitating; that is, not only energy density but also the tension and pressure are gravitating. The pressure plays a twofold role. Positive pressure causes both repulsion and attraction, the latter is due to its contribution to the gravitating source. Negative pressure, in contrast, leads to gravitational repulsion (the famous example is the de Sitter spacetime). Hence the phantom shell is even more repulsive. Indeed, we show that the global geometry of the system consisting of the Schwarzschild black hole surrounded by the phantom shell is of the wormhole type in all but one case. In the wormhole-like geometry the distant observer cannot see the shells at all, they are separated by the throat (the Einstein–Rosen bridge). The only exception is the case of the bound motion with $\delta m > 0$. 
We are sure that despite the very simple character of our model the result obtained should be taken into account in doing calculation in cosmology and astrophysics when phantom energy is present.

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