Triangular symmetry in cluster nuclei

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Abstract. In this contribution, we present evidence for the occurrence of triangular symmetry in cluster nuclei. We discuss the structure of rotational bands for 3α and 3α + 1 configurations with triangular $D_{3h}$ symmetry by exploiting the double group $D'_{3h}$, and study the application to $^{12}$C and $^{13}$C. The structure of rotational bands can be used as a fingerprint of the underlying geometric configuration of α particles.

1. Introduction
The study of cluster degrees of freedom in light nuclei, in particular nuclei with $A = 4k$ and $4k + x$ nucleons goes back to early work by Wheeler [1], and Hafstad and Teller [2], followed by later work by Brink [3, 4] and Robson [5, 6]. Recently, there has been a lot of renewed interest in the structure of α-cluster nuclei, especially for the nucleus $^{12}$C [7]. The measurement of new rotational excitations of the ground state [8, 9, 10] and of the Hoyle state [11, 12, 13, 14] has stimulated a large theoretical effort to understand the structure of $^{12}$C (for a review see e.g. Refs. [7, 15, 16]). In addition, there are measurements of many new states in $^{13}$C [17].

In this contribution, we present evidence for triangular symmetry in both even- and odd-cluster nuclei, $^{12}$C and $^{13}$C, respectively.

2. Triangular symmetry in $^{12}$C
The symmetry group of the equilateral triangle is the point group $D_{3h}$. The properties of $D_{3h}$ and the double group $D'_{3h}$ are well-known in molecular physics [18] and crystal physics [19]. Here we summarize the results relevant for applications to α-cluster nuclei in nuclear physics [20, 21, 22, 23].

For even-cluster nuclei the states can be labeled by $|\Omega, K, L\rangle$ where $\Omega$ labels the tensor (or bosonic) representations of the $D_{3h}$ triangular symmetry, and $K$ and $L$ are integers representing the projection $K$ of the angular momentum $L$ on the symmetry axis

$$\begin{align*}
\Omega &= A'_1: & K^P &= 0^+, 3^-, 6^+, \ldots, \\
\Omega &= E': & K^P &= 1^-, 2^+, 4^+, 5^-, \ldots,
\end{align*}$$

(1)

The angular momentum content of each $K$ band is given by $L = 0, 2, 4, \ldots$, for $K = 0$ and $L = K, K + 1, K + 2, \ldots$, for $K > 0$. The rotational structure depends on the $D_{3h}$ point group symmetry of the equilateral triangle configuration and is summarized in Fig. 1.

The band with $A'_1$ symmetry is characterized by a rotational sequence involving both positive and negative parity states, $L^P = 0^+, 2^+, 3^-, 4^+, 5^-, \ldots$, all of which have been observed in the
Figure 1. Structure of rotational bands for a triangular configuration of $\alpha$ particles in even-cluster nuclei with $A_1'$ (left) and $E'$ symmetry (right). Each rotational band is labeled by $K^P$.

ground-state band of $^{12}$C. The so-called Hoyle band has the same structure, but so far only the positive parity states have been observed. There is some evidence for an excited band with $E'$ symmetry. The rotational bands in $^{12}$C are shown in Fig. 2.

Figure 2. Rotational bands in $^{12}$C [10].
3. Triangular symmetry in $^{13}$C

For odd-cluster nuclei the states can be labeled by $|\Omega, K, J \rangle$ where $\Omega$ now labels the spinor (or fermionic) representations of the double group $D'_{3h}$, and $K$ and $J$ are half integers representing the projection $K$ of the total angular momentum $J$ on the symmetry axis $[24]$

$$\Omega = E_{1/2}^{(+)}: \quad K^P = \frac{1^+}{2}, \frac{5^-}{2}, \frac{7^-}{2}, \frac{11^+}{2}, \frac{13^+}{2}, \ldots,$$
$$\Omega = E_{1/2}^{(-)}: \quad K^P = \frac{1^-}{2}, \frac{5^+}{2}, \frac{7^+}{2}, \frac{11^-}{2}, \frac{13^-}{2}, \ldots,$$
$$\Omega = E_{3/2}^{(+)}: \quad K^P = \frac{3^+}{2}, \frac{9^+}{2}, \frac{15^+}{2}, \ldots.$$  \hspace{1cm} (2)

The angular momenta of each $K$ band are given by $J = K, K + 1, K + 2, \ldots$. The angular momentum structure of each one of the representations of $D'_{3h}$ is shown in Fig. 3.

| $D'_{3h} : E_{1/2}^{(-)}$ | $D'_{3h} : E_{1/2}^{(+)}$ | $D'_{3h} : E_{3/2}^{(+)}$ |
|------------------------|------------------------|------------------------|
| $\frac{9^-}{2} - \frac{9^+}{2} - \frac{9^+}{2}$ | $\frac{9^+}{2} - \frac{9^-}{2} - \frac{9^-}{2}$ | $\frac{9^\pm}{2} - \frac{9^\pm}{2}$ |
| $\frac{7^-}{2} - \frac{7^+}{2} - \frac{7^+}{2}$ | $\frac{7^+}{2} - \frac{7^-}{2} - \frac{7^-}{2}$ | $\frac{7^\pm}{2}$ |
| $\frac{5^-}{2} - \frac{5^+}{2}$ | $\frac{5^+}{2} - \frac{5^-}{2}$ | $\frac{5^\pm}{2}$ |
| $\frac{3^-}{2}$ | $\frac{3^+}{2}$ | $\frac{3^\pm}{2}$ |
| $\frac{1^-}{2}$ | $\frac{1^+}{2}$ | $\frac{1^+}{2}$ |
| $\frac{1^-}{2} - \frac{5^+}{2} - \frac{7^+}{2}$ | $\frac{1^+}{2} - \frac{5^+}{2} - \frac{7^+}{2}$ | $\frac{3^\pm}{2} + \frac{9^\pm}{2}$ |

**Figure 3.** Structure of rotational bands for a triangular configuration of $\alpha$ particles in odd-cluster nuclei with $E_{1/2}^{(-)}$, $E_{1/2}^{(+)}$ and $E_{3/2}^{(+)}$ symmetry. Each rotational band is labeled by $K^P$.

4. The cluster shell model

The structure of single-particle levels moving in the deformed field of the cluster potential has been studied recently in the context of the Cluster Shell Model (CSM) $[23, 25, 26]$. The CSM combines cluster and single-particle degrees of freedom, and is very similar in spirit as the Nilsson model $[27]$, but in the CSM the odd nucleon moves in the deformed field generated by the (collective) cluster degrees of freedom. For a cluster potential with triangular symmetry the single-particles levels of a neutron split according to the irreducible representations of the double group $D'_{3h}$, $\Omega = E_{1/2}^{(-)}$, $E_{1/2}^{(+)}$ and $E_{3/2}^{(+)}$, each of which is doubly degenerate. The resolution of single-particle levels into representations of $D'_{3h}$ is shown in Table 1.

A study of the neutron levels for $^{12}$C shows that the first six neutrons occupy the intrinsic states with $\Omega = E_{1/2}^{(-)}$ (arising from the $1s_{1/2}$ orbit), $E_{3/2}$ and $E_{1/2}^{(-)}$ (from the $1p_{3/2}$ orbit), so that the extra neutron in $^{13}$C occupies the intrinsic state with $E_{1/2}^{(-)}$ (from the $1p_{1/2}$ orbit), followed by $E_{1/2}^{(+)}$, $E_{1/2}^{(-)}$ and $E_{3/2}$ associated with the orbits from the $s$-$d$ shell $[23, 25, 28]$. 


Table 1. Resolution of single-particle levels into irreducible representations of $D_{3h}'$. Each $E$ level is doubly degenerate.

|        | $E_{1/2}^+$ | $E_{1/2}^-$ | $E_{3/2}$ | $E_{1/2}^+$ | $E_{1/2}^-$ | $E_{3/2}$ |
|--------|-------------|-------------|-----------|-------------|-------------|-----------|
| $1s_{1/2}$ | 1           | 0           | 0         | 2$s_{1/2}$  | 1           | 0         |
| $1p_{1/2}$ | 0           | 1           | 0         | 1$d_{3/2}$  | 1           | 0         |
| $1p_{3/2}$ | 0           | 1           | 1         | 1$d_{5/2}$  | 1           | 1         |

The rotational energy spectra can be analyzed with

$$E_{\text{rot}}(\Omega, K, J) = \varepsilon_\Omega + A_\Omega \left[ J(J+1) + b_\Omega K^2 + a_\Omega (-1)^{J+1/2}(J+1/2)\delta_{K,1/2} \right], \quad (3)$$

where $\varepsilon_\Omega$ is the intrinsic energy, $A_\Omega = \hbar^2/2I$ the inertial parameter, $b_\Omega$ a Coriolis term, and $a_\Omega$ the decoupling parameter. The latter term applies only to representations $\Omega = E_{1/2}^{(\pm)}$ and $K^P = 1/2^\pm$. Eq. (3) is the same energy formula as used by Nuhn in Ref. [29] in a description of the bandheads of $^{13}$C in the context of a two-center shell model description of the system $^{13}$C+$^{16}$O $\rightarrow$ $^{29}$Si. Nuhn used an axially symmetric potential, in contrast to a cluster potential with triangular $D_{3h}$ symmetry in the CSM. As a consequence, in the CSM the $K^P = 1/2^-$ and $5/2^+$ bands belong to the same configuration $\Omega = E_{1/2}^-$ (see Eq. (2) and Fig. 3), whereas in the axially symmetric case they represent separate rotational bands.

![Figure 4. Left: Rotational bands in $^{13}$C [24]. Right: Comparison between calculated and experimental [30] longitudinal $E2$ form factors for the ground-state band of $^{13}$C, $F(q;1/2^+ \rightarrow 5/2^+)$ (black) and $F(q;1/2^+ \rightarrow 3/2^+)$ (red).](image)

Fig. 4 shows the rotational bands of $^{13}$C. The ground-state band has $K^P = 1/2^-$ and is assigned to the representation $\Omega = E_{1/2}^-$ of $D_{3h}'$ (blue lines and filled circles) arising from the coupling of the ground-state band in $^{12}$C to the intrinsic state with $E_{1/2}^-$. According to Eq. (2), this representation contains also $K^P = 5/2^+$ and $7/2^+$ bands, both of which appear to have been observed. In the shell model, positive parity states are expected to occur at much higher energies since they come from the $s$-$d$ shell. The first excited rotational band has $K^P = 1/2^+$.
which can be assigned to $\Omega = E^{(+)}_{1/2}$ (black line and filled squares) arising from the coupling of the ground-state band in $^{12}$C to the excited intrinsic state with $E^{(+)}_{1/2}$. In contrast to the ground-state band, this excited band has a large decoupling parameter. In addition, Fig. 4 shows evidence for the occurrence of a rotational band at an energy slightly higher than that of the Hoyle state in $^{12}$C which is interpreted as the coupling of the Hoyle band in $^{12}$C to the ground-state intrinsic state $E^{(-)}_{1/2}$ (red line and filled triangles).

Further evidence for the occurrence of $D_{3h}$ symmetry in $^{13}$C is provided by electromagnetic transition rates and form factors. As an example, we show in Fig. 4 the longitudinal $E2$ form factors of the states $5/2^+_{1}$ and $3/2^+_{1}$ of the ground-state rotational band. The two form factors are predicted to have identical shapes

$$F(q; 1/2^- \rightarrow 5/2^-) = F(q; 1/2^- \rightarrow 3/2^-),$$

and identical $B(E2; \uparrow)$ values: 9.6 W.U. This is verified to a very good approximation.

Finally, in Fig. 5 we show the expected structure of the vibrational spectrum of $^{13}$C.

Figure 5. Expected vibrational spectrum of $^{13}$C for the coupling of a single-particle level with $E^{(-)}_{1/2}$ (left), $E^{(-)}_{1/2}$ (middle) and $E_{3/2}$ symmetry (right) to the ground-state band, the Hoyle band and the bending band in $^{12}$C (see Fig. 2).

5. Summary and conclusions

In this contribution, we presented a discussion of triangular symmetry in cluster nuclei and studied the application to the even- and odd-cluster nuclei, $^{12}$C and $^{13}$C. A combined analysis of the rotation-vibration spectra and electromagnetic transition rates and form factors provides strong evidence for the occurrence of triangular symmetry in these nuclei. A characteristic feature of the triangular symmetry is the appearance of rotational bands consisting of both positive and negative parity states. As a consequence of the symmetry the form factors to the first excited state with $J^P = 3/2^-$ and $5/2^-$ are predicted to have the same shape and $B(E2; \uparrow)$ values. In addition, the quadrupole and octupole transitions in $^{12}$C and $^{13}$C are strongly correlated [24]. The good agreement between theory and experiment supports the interpretation of the nucleus $^{13}$C as a system of three $\alpha$-particles in a triangular configuration plus an additional neutron moving in the deformed field generated by the cluster (see Fig. 6).

Acknowledgments

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Figure 6. Molecular-like picture of $^{13}$C.

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