THE DEGREE COMPLEXITY OF SMOOTH SURFACES OF CODIMENSION 2

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ABSTRACT. For a given term order, the degree complexity of a projective scheme is defined by the maximal degree of the reduced Gröbner basis of its defining saturated ideal in generic coordinates [2]. It is well-known that the degree complexity with respect to the graded reverse lexicographic order is equal to the Castelnuovo-Mumford regularity [3]. However, much less is known if one uses the graded lexicographic order [1], [5].

In this paper, we study the degree complexity of a smooth irreducible surface in $\mathbb{P}^4$ with respect to the graded lexicographic order and its geometric meaning. Interestingly, this complexity is closely related to the invariants of the double curve of a surface under a generic projection. As results, we prove that except a few cases, the degree complexity of a smooth surface $S$ of degree $d$ with $h^0(\mathcal{I}_S(2)) \neq 0$ in $\mathbb{P}^4$ is given by

$$2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S)),$$

where $Y_1(S)$ is a double curve of degree $\binom{d - 1}{2} - g(S \cap H)$ under a generic projection of $S$. In particular, this complexity is actually obtained at the monomial $x_0x_1x_3^{(\deg Y_1(S) - 1)} - g(Y_1(S))$ where $k[x_0, x_1, x_2, x_3, x_4]$ is a polynomial ring defining $\mathbb{P}^4$. Exceptional cases are a rational normal scroll, a complete intersection surface of $(2, 2)$-type, or a Castelnuovo surface of degree 5 in $\mathbb{P}^4$ whose degree complexities are in fact equal to their degrees. This complexity can also be expressed in terms of degrees of defining equations of $\mathcal{I}_S$ in the same manner as the result of A. Conca and J. Sidman [5]. We also provide some illuminating examples of our results via calculations done with Macaulay 2 [10].

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1. Introduction

D. Bayer and D. Mumford in [2] have introduced the degree complexity of a homogeneous ideal $I$ with respect to a given term order $\tau$ as the maximal degree of the reduced Gröbner basis of $I$, and this is exactly the highest degree of minimal generators of the initial ideal of $I$. Even though degree complexity depends on the choice of coordinates, it is constant in generic coordinates since the initial ideal of $I$ is invariant under a generic change of coordinates, which is the so-called the generic initial ideal of $I$ [7].

For the graded lexicographic order (resp. the graded reverse lexicographic order), we denote by $M(I)$ (resp. $m(I)$) the degree complexity of $I$ in generic coordinates. For a projective scheme $X$, the degree complexity of $X$ can also be defined as $M(I_X)$ (resp. $m(I_X)$) for the graded lexicographic order (resp. the graded reverse lexicographic order) where $I_X$ is the defining saturated ideal of $X$.

D. Bayer and M. Stillman have shown in [3] that $m(I)$ is exactly equal to the Castelnuovo-Mumford regularity $\text{reg}(I)$. Then what can we say about $M(I)$? A. Conca and J. Sidman proved in [5] that if $I_C$ is the defining ideal of a smooth irreducible complete intersection curve $C$ of type $(a, b)$ in $\mathbb{P}^3$ then $M(I_C)$ is $1 + \frac{ab(a-1)(b-1)}{2}$ with the exception of the case $a = b = 2$, where $M(I_C)$ is 4. Recently, J. Ahn has shown in [1] that if $I_C$ is the defining ideal of a non-degenerate smooth integral curve of degree $d$ and genus $g(C)$ in $\mathbb{P}^r$ (for $r \geq 3$), then $M(I_C) = 1 + \binom{d-1}{2} - g(C)$ with two exceptional cases.

In this paper, we would like to compute the degree complexity of a smooth surface $S$ in $\mathbb{P}^4$ with respect to the graded lexicographic order. Interestingly, this complexity is closely related to the invariants of the double curve of $S$ under the generic projection. Our main results are: if $S \subset \mathbb{P}^4$ is a smooth non-degenerate surface of degree $d$ with $h^0(I_S(2)) \neq 0$, then the degree complexity $M(I_S)$ of $S$ is given by $2 + \left( \deg Y_1(S) - 1 \right) - g(Y_1(S))$ with three exceptional cases, where $Y_1(S)$ is a smooth double curve of $S$ in $\mathbb{P}^3$ under a generic projection and $\deg Y_1(S) = \binom{d-1}{2} - g(S \cap H)$. Moreover, this complexity is actually obtained at the monomial

$$x_0x_1x_3^{\left( \deg Y_1(S) - 1 \right)} - g(Y_1(S))$$

where $k[x_0, x_1, x_2, x_3, x_4]$ is a polynomial ring defining $\mathbb{P}^4$.

On the other hand, $M(I_S)$ can also be expressed in terms of degrees of defining equations of $I_S$ in the same manner as the result of A. Conca and J. Sidman [5] (see Theorem 1.9). Note that if $S$ is a locally Cohen-Macaulay surface with $h^0(I_S(2)) \neq 0$ then there are two types of $S$. One is a complete intersection of $(2, \alpha)$-type and the other is projectively Cohen-Macaulay of degree $2\alpha - 1$. For those cases, $\deg Y_1(S)$, $g(Y_1(S))$ and $g(S \cap H)$ can be obtained in terms of $\alpha$.

Consequently, if $S$ is a complete intersection of $(2, \alpha)$-type for some $\alpha \geq 3$ then $M(I_S) = \frac{1}{2} \left( \alpha^4 - 4\alpha^3 + 5\alpha^2 - 2\alpha + 4 \right)$. If $S$ is projectively Cohen-Macaulay of degree $2\alpha - 1$, $\alpha \geq 4$, then $M(I_S) = \frac{1}{7} \left( \alpha^4 - 6\alpha^3 + 13\alpha^2 - 12\alpha + 8 \right)$ (see...
Theorem 4.9). Exceptional cases are a rational normal scroll, a complete intersection surface of $(2,2)$-type, or a Castelnuovo surface of degree 5 in $\mathbb{P}^4$. In these cases, $M(I_S) = \deg(S)$ (see Theorem 4.5).

The main ideas are divided into two parts: one is to show that the degree complexity $M(I_S)$ is given by the maximum of $\text{reg}(\text{Gin}_{\text{GLex}}(K_i(I_S))) + i$ for $i = 0, 1$ and the other part is to compare the schemes of multiple loci defined by partial elimination ideals and their classical scheme structures defined by the Fitting ideals of an $\mathcal{O}_{\mathbb{P}^3}$-module $\pi_*\mathcal{O}_S$ where $\pi$ is a generic projection of $S$ to $\mathbb{P}^3$.

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2. Notations and basic facts

- We work over an algebraically closed field $k$ of characteristic zero.
- Let $R = k[x_0, \ldots, x_r]$ be a polynomial ring over $k$. For a closed subscheme $X$ in $\mathbb{P}^r$, we denote the defining saturated ideal of $X$ by $I_X = \bigoplus_{m=0}^{\infty} H^0(I_X(m))$.
- For a homogeneous ideal $I$, the Hilbert function of $R/I$ is defined by $H(R/I, m) := \dim_k(R/I)_m$ for any non-negative integer $m$. We denote its corresponding Hilbert polynomial by $P_{R/I}(z) \in \mathbb{Q}[z]$. If $I = I_X$ then we simply write $P_X(z)$ instead of $P_{R/I_X}(z)$.
- We write $\rho_a(X) = (-1)^{\dim(X)}(P_X(0) - 1)$ for the arithmetic genus of $X$.
- For a homogeneous ideal $I \subset R$, consider a minimal free resolution

$$
\cdots \to \bigoplus_j R(-i - j)^{\beta_{i,j}(I)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(I)} \to I \to 0
$$

of $I$ as a graded $R$-modules. We say that $I$ is $m$-regular if $\beta_{i,j}(I) = 0$ for all $i \geq 0$ and $j \geq m$. The Castelnuovo-Mumford regularity of $I$ is defined by

$$\text{reg}(I) := \min\{ m \mid I \text{ is } m\text{-regular}\}.$$

- Given a term order $\tau$, we define the initial term $\text{in}_\tau(f)$ of a homogeneous polynomial $f \in R$ to be the greatest monomial of $f$ with respect to $\tau$. If $I \subset R$ is a homogeneous ideal, we also define the initial ideal $\text{in}_\tau(I)$ to be the ideal generated by $\{\text{in}_\tau(f) \mid f \in I\}$. A set $G = \{g_1, \ldots, g_n\} \subset I$ is said to be a Gröbner basis if

$$\text{in}_\tau(g_1), \ldots, \text{in}_\tau(g_n) = \text{in}_\tau(I).$$
• For an element $\alpha = (\alpha_0, \ldots, \alpha_r) \in \mathbb{N}^r$ we define the notation $x^\alpha = x_0^{\alpha_0} \cdots x_r^{\alpha_r}$ for monomials. Its degree is $|\alpha| = \sum_{i=0}^{r} \alpha_i$.

  For two monomial terms $x^\alpha$ and $x^\beta$, the graded lexicographic order is defined by $x^\alpha \geq_{\text{GLex}} x^\beta$ if and only if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and if the left most nonzero entry of $\alpha - \beta$ is positive. The graded reverse lexicographic order is defined by $x^\alpha \geq_{\text{GRLex}} x^\beta$ if and only if we have $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and if the right most nonzero entry of $\alpha - \beta$ is negative.

• In characteristic 0, we say that a monomial ideal $I$ has the Borel-fixed property if, for some monomial $m$, we have $x_{\beta} m \in I$ then $x_{\beta} m \in I$ for all $j \leq i$.

• Given a homogeneous ideal $I \subset R$ and a term order $\tau$, there is a Zariski open subset $U \subset GL_{r+1}(k)$ such that $\tau(g(I))$ is constant. We will call $\tau(g(I))$ for $g \in U$ the generic initial ideal of $I$ and denote it by $\text{Gin}_\tau(I)$. Generic initial ideals have the Borel-fixed property (see [7], [8]).

• For a homogeneous ideal $I \subset R$, let $m(I)$ and $M(I)$ denote the maximum of the degrees of minimal generators of $\text{Gin}_{\text{GRLex}}(I)$ and $\text{Gin}_{\text{GLex}}(I)$ respectively.

• If $I$ is a Borel fixed monomial ideal then $\text{reg}(I)$ is exactly the maximal degree of minimal generators of $I$ (see [3], [8]). This implies that $m(I) = \text{reg}(\text{Gin}_{\text{GRLex}}(I))$ and $M(I) = \text{reg}(\text{Gin}_{\text{GLex}}(I))$.

3. Gröbner bases of partial elimination ideals

**Definition 3.1.** Let $I$ be a homogeneous ideal in $R$. If $f \in I_d$ has leading term $\text{lt}(f) = x_0^{d_0} \cdots x_r^{d_r}$, we will set $d_0(f) = d_0$, the leading power of $x_0$ in $f$. We let

$$\tilde{K}_i(I) = \bigoplus_{d \geq 0} \{ f \in I_d \mid d_0(f) \leq i \}.$$  

If $f \in \tilde{K}_i(I)$, we may write uniquely

$$f = x_0^{i_0} f + g,$$

where $d_0(g) < i$. Now we define $K_i(I)$ as the image of $\tilde{K}_i(I)$ in $\tilde{R} = k[x_1 \ldots x_r]$ under the map $f \to \overline{f}$ and we call $K_i(I)$ the $i$-th partial elimination ideal of $I$.

**Remark 3.1.** We have an inclusion of the partial elimination ideals of $I$:

$$I \cap \tilde{R} = K_0(I) \subset K_1(I) \subset \cdots \subset K_i(I) \subset K_{i+1}(I) \subset \cdots \subset \tilde{R}.$$  

Note that if $I$ is in generic coordinates and $i_0 = \min\{i \mid I_i \neq 0\}$ then $K_i(I) = \tilde{R}$ for all $i \geq i_0$. 
The following result gives the precise relationship between partial elimination ideals and the geometry of the projection map from \( \mathbb{P}^r \) to \( \mathbb{P}^{r-1} \). For a proof of this proposition, see [5] Proposition 6.2.

**Proposition 3.2.** Let \( X \subset \mathbb{P}^r \) be a reduced closed subscheme and let \( I_X \) be the defining ideal of \( X \). Suppose \( p = [1,0,\ldots,0] \in \mathbb{P}^r \setminus X \) and that \( \pi : X \to \mathbb{P}^{r-1} \) is the projection from the point \( p \in \mathbb{P}^r \) to \( x_0 = 0 \). Then, set-theoretically, \( K_i(I_X) \) is the ideal of \( \{ q \in \pi(X) \mid \text{mult}_q(\pi(X)) > i \} \).

For each \( i \geq 0 \), note that we can give a scheme structure on the set

\[ Y_i(X) := \{ q \in \pi(X) \mid \text{mult}_q(\pi(X)) > i \} \]

from the \( i \)-th partial elimination ideal \( K_i(I) \). Let

\[ Z_i(X) := \text{Proj}(\bar{R}/K_i(I_X)), \]

where \( \bar{R} = k[x_1 \ldots x_r] \). Then it follows from Proposition 3.2 that \( Z_i(X)_{\text{red}} = Y_i(X) \).

**Remark 3.3.** Let \( X \subset \mathbb{P}^r \) be a smooth variety of codimension two and let \( \pi : X \to \mathbb{P}^{r-1} \) be a generic projection of \( X \). A classical scheme structure on the set \( Y_i(X) \) is given by \( i \)-th Fitting ideal of the \( \mathcal{O}_{\mathbb{P}^{r-1}} \)-module \( \pi_*\mathcal{O}_X \) (see [12], [14]). Throughout this paper, we use the notation \( Y_i(X) \) in the sense that it is a closed subscheme defined by Fitting ideal of \( \pi_*\mathcal{O}_X \), as distinguished from the notation \( Z_i(X) \). We show that if \( S \subset \mathbb{P}^4 \) is a smooth surface lying in a quadric surface then \( Y_i(S) \) and \( Z_i(S) \) have the same reduced scheme structure (see Theorem 4.2), which will be used in the proof of Proposition 4.5.

It is natural to ask: what is a Gröbner basis of \( K_i(I) \)? Recall that any non-zero polynomial \( f \) in \( R \) can be uniquely written as \( f = x^t \hat{f} + g \) where \( d_0(g) < t \). A. Conca and J. Sidman [5] show that if \( G \) is a Gröbner basis for an ideal \( I \) then the set

\[ G_i = \{ \hat{f} \mid f \in G \text{ with } d_0(f) \leq i \} \]

is a Gröbner basis for \( K_i(I) \). However if \( I \) is in generic coordinates then there is a more refined Gröbner basis for \( K_i(I) \), which plays an important role in this paper.

**Proposition 3.4.** Let \( I \) be a homogeneous ideal in generic coordinates and \( G \) be a Gröbner basis for \( I \) with respect to the graded lexicographic order. Then, for each \( i \geq 0 \),

(a) the \( i \)-th partial elimination ideal \( K_i(I) \) is in generic coordinates;

(b) \( G_i = \{ f \mid f \in G \text{ with } d_0(f) = \mu \} \) is a Gröbner basis for \( K_i(I) \).

**Proof.** (a) is in fact proved in Proposition 3.3 in [5]. For a proof of (b), it suffices to show that \( \langle \text{in}(G_i) \rangle = \text{in}(K_i(I)) \) by the definition of Gröbner bases. Since \( G_i \subset K_i(I) \), we only need to show that \( \langle \text{in}(G_i) \rangle \supset \text{in}(K_i(I)) \). Now, we denote \( \mathcal{G}(I) \) by the set of minimal generators of \( I \). Let \( m \in \text{in}(K_i(I)) \) be a
monomial. Then there is a monomial generator $M \in \mathcal{S}(\mathfrak{a}(K_i(I)))$ such that $M$ divide $m$.

We claim that $x_0^i M \in \mathcal{S}(\mathfrak{a}(I))$ if and only if $M \in \mathcal{S}(\mathfrak{a}(K_i(I)))$.

If the claim is proved then we will be done. Indeed, for $M \in \mathcal{S}(\mathfrak{a}(K_i(I)))$, we see that $x_0^i M \in \mathcal{S}(\mathfrak{a}(I))$. This implies that there exists a polynomial $f = x_0^i \bar{f} + g \in G$ with $d_0(g) < i$ such that

$$\mathfrak{a}(f) = x_0^i \mathfrak{a}(f) = x_0^i M.$$ 

This means that $M = \mathfrak{a}(f) \in (\mathfrak{a}(G_i))$. Thus we have $m \in (\mathfrak{a}(G_i))$.

Here is a proof of the claim: suppose that $x_0^i M \in \mathcal{S}(\mathfrak{a}(I))$ then we can say that $x_0^i M \in \mathfrak{a}(I)$. Thus there is a polynomial $f = x_0^i \bar{f} + g \in I$ such that $d_0(g) < i$ and $\mathfrak{a}(f) = x_0^i \mathfrak{a}(f) = x_0^i M$. By the definition of partial elimination ideals, we have that $f \in K_i(I)$, which means $M \in \mathfrak{a}(K_i(I))$. Assume that $M \notin \mathcal{S}(\mathfrak{a}(K_i(I)))$. Then for some monomial $N \in \mathcal{S}(\mathfrak{a}(K_i(I)))$ such that $N$ divide $M$. This implies that

$$x_0^i N \in \mathfrak{a}(I) \text{ and } x_0^i N | x_0^i M,$$

which contradicts the fact that $x_0^i M$ is a minimal generator of $\mathfrak{a}(I)$. Thus $M$ is contained in $\mathcal{S}(\mathfrak{a}(K_i(I)))$.

Conversely, suppose that there is $M \in \mathcal{S}(\mathfrak{a}(K_i(I)))$ such that $x_0^i M \notin \mathcal{S}(\mathfrak{a}(I))$. Then we may choose a monomial $x_0^j N \in \mathcal{S}(\mathfrak{a}(I))$ satisfying

$$(1) \quad x_0 \nmid N \text{ and } x_0^j N | x_0^i M.$$

Note that (1) implies that $i \geq j \geq 0$. Since $N \in \mathfrak{a}(K_j(I))$ and $K_0(I) \subset K_1(I) \subset \cdots$, it is obvious that $N \in \mathfrak{a}(K_i(I))$ and $N$ divides $M$. Now, we claim that $N$ can be chosen to be different from $M$. If $N = M$ then $j$ must be less than $i$. Denote $N$ by $x_{i_1}^j \cdots x_{i_t}^k$ and choose $j_t \neq 0$. By (a), note that $K_i(I)$ is in generic coordinates and so we may assume that $\mathfrak{a}(K_i(I))$ has the Borel-fixed property. Therefore, if we set $N' = N/x_{j_t}$ then $x_0^{j+1} N' \in \mathfrak{a}(I)$. Replace $x_0^{j} N$ by $N'' = x_0^{j+1} N'$. Then $N'' \in \mathfrak{a}(K_{j+1}(I))$. Since $j + 1 \leq i$, we can say that $N' \in \mathfrak{a}(K_i(I))$ and $N'$ divides $M$ with $N' \neq M$. This contradicts the assumption that $M \in \mathcal{S}(\mathfrak{a}(K_i(I)))$. 

**Remark 3.5.** The condition “in generic coordinates” is crucial in Proposition 3.4 (b) as the following example shows. Let $I = (x_0^3, x_0^2 x_1, x_0 x_2, x_3)$ be a monomial ideal. Then $G = \{x_0^3, x_0^2 x_1, x_0 x_2, x_3\}$ is a Gröbner basis for $I$.

Then we can easily check that

$$G_1 = \{\bar{f} \mid f \in G \text{ with } d_0(f) \leq 1\} = (x_1, x_2, x_3),$$

$$G_1' = \{\bar{f} \mid f \in G \text{ with } d_0(f) = 1\} = (x_1, x_2).$$

This shows that $G_1'$ is not a Gröbner basis for $K_1(I)$.

We have the following corollary from Proposition 3.4.
Corollary 3.6. For a homogeneous ideal \( I \subseteq R = k[x_0, \ldots, x_r] \) in generic coordinates, we have
\[
M(I) = \max \{ M(K_i(I)) + i \mid 0 \leq i \leq \beta \},
\]
where \( \beta = \min \{ j \mid I_j \neq 0 \} \).

Proof. Note that \( K_\beta(I) = \bar{R} \) for \( \beta = \min \{ j \mid I_j \neq 0 \} \) by definition. We know that \( M(I) \) can be obtained from the maximal degree of generators in \( \text{Gin}(I) \). Remember that \( \text{Gin}(I) \) is the set of minimal generators of \( I \). Then by Proposition 3.4, every generator of \( \text{Gin}(I) \) is of the form \( x_i^0M \) where \( M \in \text{Gin}(K_i(I)) \) for some \( i \). This means that \( M(I) \leq M(K_i(I)) + i \) for some \( i \). On the other hand, if for each \( i \), we choose \( M \in \text{Gin}(K_i(I)) \), then by Proposition 3.4, \( x_i^0M \) is contained in \( \text{Gin}(I) \). Hence we conclude that
\[
M(I) = \max \{ M(K_i(I)) + i \mid 0 \leq i \leq \beta \}.
\]
\( \square \)

Corollary 3.6 with the following theorem can be used to obtain the degree-complexities of the smooth surface lying in a quadric hypersurface in \( \mathbb{P}^4 \). For a proof of this theorem, see [1, Theorem 4.4].

Theorem 3.7. Let \( C \) be a non-degenerate smooth curve of degree \( d \) and genus \( g(C) \) in \( \mathbb{P}^r \) for some \( r \geq 3 \). Then,
\[
M(I_C) = \max \{ d, 1 + \left( \frac{d - 1}{2} \right) - g(C) \}.
\]

4. Degree complexity of smooth irreducible surfaces in \( \mathbb{P}^4 \)

Let \( S \) be a non-degenerate smooth irreducible surface of degree \( d \) and arithmetic genus \( \rho_a(S) \) in \( \mathbb{P}^4 \) and let \( I_S \) be the defining ideal of \( S \) in \( R = k[x_0, \ldots, x_4] \). In this section, we study the scheme structure of
\[
Z_i(S) := \text{Proj}(\bar{R}/K_i(I_S)), \quad \text{where} \quad \bar{R} = k[x_1, x_2, x_3, x_4]
\]
arising from a generic projection in order to get a geometric interpretation of the degree-complexity \( M(I_S) \) of \( S \) in \( \mathbb{P}^4 \) with respect to the degree lexicographic order.

We recall without proof the standard facts concerning generic projections of surfaces in \( \mathbb{P}^4 \) to \( \mathbb{P}^3 \).

Let \( S \subseteq \mathbb{P}^4 \) be a non-degenerate smooth irreducible surface of degree \( d \) and arithmetic genus \( \rho_a(S) \) and \( \pi : S \to \pi(S) \subset \mathbb{P}^3 \) be a generic projection.

(a) The singular locus of \( \pi(S) \) is a curve \( Y_1(S) \) with only singularities a number \( t \) of ordinary triple points with transverse tangent directions. The inverse image \( \pi^{-1}(Y_1(S)) \) is a curve with only singularities \( 3t \) nodes, 3 nodes above each triple point of \( Y_1(S) \) (see [15]). This implies (using Proposition 3.2) that the ideals \( K_j(I_S) \) have finite colength if \( j > 2 \). This fact is used in the proofs of Proposition 4.6 and Theorem 4.3.
(b) If a smooth surface \( S \subset \mathbb{P}^4 \) is contained in a quadric hypersurface then there are no ordinary triple points in \( Y_1(S) \). This implies that the double curve \( Y_1(S) \) is smooth by (a).

(c) The double curve \( Y_1(S) \) is irreducible unless \( S \) is a projected Veronese surface in \( \mathbb{P}^4 \) (see [14]).

(d) The reduced induced scheme structure on \( Y_1(S) \) is defined by the first Fitting ideal of the \( \mathcal{O}_{\mathbb{P}^3} \)-module \( \pi_* \mathcal{O}_S \) (see [14]).

(e) The degree of \( Y_1(S) \) is \( \left(d - \frac{1}{2}\right) - g(S \cap H) \) where \( S \cap H \) is a general hyperplane section and the number of apparent triple points \( t \) is given in [13] by
\[
t = \left(d - \frac{1}{3}\right) - g(S \cap H)(d - 3) + 2\chi(\mathcal{O}_S) - 2.
\]

The following lemma shows that the Hilbert function of \( I_S \) can be obtained from those of partial elimination ideals \( K_i(I_S) \).

**Lemma 4.1.** Let \( S \subset \mathbb{P}^4 \) be a smooth surface with the defining ideal \( I_S \) in \( R = k[x_0, x_1, \ldots, x_4] \). Consider a projection \( \pi_q : S \to \mathbb{P}^3 \) from a general point \( q = [1, 0, 0, 0, 0] \notin S \). Then,
\[
H(R/I_S, m) = \sum_{i \geq 0} H(\bar{R}/K_i(I_S), m - i).
\]

In particular,
\[
P_S(z) = P_{Z_0(S)}(z) + P_{Z_1(S)}(z - 1) + P_{Z_2(S)}(z - 2).
\]

**Proof.** The equality on Hilbert functions basically comes from the following combinatorial identity
\[
\binom{m + d}{d} = \sum_{i=0}^{d} \binom{m - 1 + d - i}{d - i}.
\]

For a smooth surface \( S \subset \mathbb{P}^4 \), \( Z_i(S) = \emptyset \) for \( i \geq 3 \) by the (dimension +2)-secant lemma (see [16]) and so \( \bar{R}/K_i(I_S) \) is Artinian. Thus \( P_{Z_i(S)}(z) = 0 \) for \( i \geq 3 \) (see [1] Lemma 3.4) for details. \( \square \)

The following theorem says that the first partial elimination ideal \( K_1(I_S) \) gives the reduced induced scheme structure on the double curve \( Y_1(S) \) in \( \mathbb{P}^3 \) (i.e., \( I_{Z_1(S)} = I_{Y_1(S)} \)).

**Theorem 4.2.** Suppose that \( S \) is a reduced irreducible surface in \( \mathbb{P}^4 \). Then,

(a) the first partial elimination ideal \( K_1(I_S) \) is a saturated ideal, so we have \( K_1(I_S) = I_{Z_1(S)} \);

(b) if \( S \) is a smooth surface contained in a quadric hypersurface, then \( K_1(I_S) = I_{Y_1(S)} \), which implies that \( K_1(I_S) \) is a reduced ideal.

**Proof.** (a) Assume that \( S \) is a reduced irreducible surface in \( \mathbb{P}^4 \) of degree \( d \). Take a general point \( q \in \mathbb{P}^4 \); we may assume \( q = [1, 0, \ldots, 0] \). Then the generic projection of \( S \) into \( \mathbb{P}^3 \) from the point \( q \) is defined by a single
polynomial $F \in k[x_1, x_2, x_3, x_4]$ of degree $d$ and $K_0(I_S) = (F)$, which is a reduced ideal.

Let $\mathcal{M} = (x_1, x_2, x_3, x_4)$ be the irrelevant maximal ideal of $\bar{R} = k[x_1, x_2, x_3, x_4]$. By the definition of saturated ideal, $K_1(I_S)$ is saturated if and only if

$$(K_1(I_S): \mathcal{M}) = K_1(I_S).$$

Hence it is enough to show that

$$(K_1(I_S): \mathcal{M})/K_1(I_S) = 0.$$ For the proof, consider the Koszul complex

$$\cdots \to \mathcal{K}^{-p-1}_m \to \mathcal{K}^{-p}_m \to \mathcal{K}^{-p+1}_m \to \cdots ,$$

where $\mathcal{K}^{-p}_m = \bigwedge^p \mathcal{M} \otimes K_0(I_S)_{m-p}$. From Corollary 6.7 in [8], the $\bar{R}$-module $(K_1(I_S): \mathcal{M})_d/K_1(I_S)_d$ injects into $H^{-1}(\mathcal{K}^{\bullet}_{d+3})$ for each $d$. Note that

$$H^{-1}(\mathcal{K}^{\bullet}_{d+3}) = H(\bigwedge^1 \mathcal{M} \otimes K_0(I_S)_{d+2}) = \text{Tor}_1^\bar{R}(\bar{R}/\mathcal{M}, K_0(I_S))_{d+3}.$$

Since the ideal $K_0(I_S)$ is generated by a single polynomial $F$, we have that

$$\text{Tor}_1^\bar{R}(\bar{R}/\mathcal{M}, K_0(I_S)) = 0.$$

This proves that $(K_1(I_S): \mathcal{M})/K_1(I_S) = 0$, as we wished.

(b) Since $S$ is contained in a quadric hypersurface and the center of projection is outside a quadric, we have a surjection $\varphi : \bar{R}(-1) \oplus \bar{R} \to \bar{R}/I_S$ as a $\bar{R}$-module homomorphism with the following diagram:

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \to & K_0(I_S) & \to & \bar{R} & \to & \bar{R}/K_0(I_S) & \to & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \to & \tilde{K}_1(I_S) & \to & \bar{R} \oplus \bar{R}(-1) & \varphi & R/I_S & \to & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \to & K_1(I_S)(-1) & \to & \bar{R}(-1) & \to & \bar{R}/K_1(I_S)(-1) & \to & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 \\
\end{array}
$$

where $\tilde{K}_1(I_S) = \{ f \in I_S \mid d_0(f) \leq 1 \}$ is an $\bar{R}$-module. Let $\mathcal{O}_{Z_1(S)}$ be the sheafification of $\bar{R}/K_1(I_S)$. By sheafifying the rightmost vertical sequence, we have

$$(2) \quad 0 \to \mathcal{O}_{\pi(S)} \to \pi_*\mathcal{O}_S \to \mathcal{O}_{Z_1(S)}(-1) \to 0.$$ 

Let $J_{Z_1(S)} = \mathcal{K}_1(I_S)$ be the sheafification of the ideal $K_1(I_S)$. In [12] (3.4.1), p. 302], S. Kleiman, J. Lipman and B. Ulrich proved that

$$J_{Y_1(S)} = \text{Fitt}_1^p(\pi_*\mathcal{O}_S) = \text{Fitt}_0^p(\pi_*\mathcal{O}_S/\mathcal{O}_{\pi(S)}) = \text{Ann}_p(\mathcal{O}_{Z_1(S)}(-1)),$$
and this defines the reduced scheme structure on $Y_1(S)$ (see [14, p. 3]).

On the other hand, from the sequence (2), we have

$$\mathcal{I}_{Y_1(S)} = \text{Ann}_{\mathbb{P}^3}(\mathcal{O}_{Z_1(S)}(-1)) = \mathcal{K}_1(I_S) = \mathcal{I}_{Z_1(S)}.$$  

Then it follows from (a) that

$$I_{Z_1(S)} = K_1(I_S)^{\text{sat}} = K_1(I_S) = I_{Y_1(S)}.$$  

Since $I_{Y_1(S)}$ is a reduced ideal, we conclude that $I_{Z_1(S)} = K_1(I_S)$ is also a reduced ideal.  

$\square$

If $S \subseteq \mathbb{P}^4$ is contained in a quadric hypersurface, then by Theorem 4.2, $K_1(I_S)$ is saturated and reduced. So, it defines the reduced scheme structure on $Y_1(S)$. Note also that the double curve $Y_1(S)$ is smooth (see the standard fact (b) in the beginning of this section). We use this fact to prove the following theorem.

**Theorem 4.3.** Let $S$ be a smooth irreducible surface of degree $d$ lying on a quadric hypersurface in $\mathbb{P}^4$. Let $Y_1(S)$ be the double curve of genus $g(Y_1(S))$ defined by a generic projection $\pi$ of $S$ to $\mathbb{P}^3$. Then, we have the following:

(a) $M(I_S) = \max\{d, 1 + \deg Y_1(S), 2 + (\deg Y_1(S) - 1) - g(Y_1(S))\}$;

(b) $M(I_S)$ can be obtained at one of monomials

$$x_1^d, x_0x_2^{\deg Y_1(S)}, x_0x_1x_3^{(\deg Y_1(S) - 1) - g(Y_1(S))}.$$  

**Proof.** Note that by Corollary 3.6

$$M(I_S) = \max_{0 \leq i \leq \beta} \{\text{reg}(\text{Gin}(K_i(I_S))) + i\},$$  

where $\beta = \min\{j \mid K_j(I_S) = \overline{R}\}$. Since $S$ is contained in a quadric hypersurface, $\text{Gin}(I_S)$ contains the monomial $x_0^2$. This means that $\text{Gin}(K_2(I_S)) = \overline{R}$. On the other hand, $\text{Gin}(K_0(I_S)) = (x_1^2)$ by the Borel fixed property because $\pi(S)$ is a hypersurface of degree $d$ in $\mathbb{P}^3$ and $I_{\pi(S)} = K_0(I_S)$. Thus $\text{Gin}(I_S)$ is of the form

$$(x_0^2, x_0g_1, x_0g_2, \ldots, x_0g_m, x_1^d).$$  

Note that $g_1, \ldots, g_m$ are monomial generators of $\text{Gin}(K_1(I_S)) = \text{Gin}(Y_1(S))$ by Proposition 3.4.

Therefore, by Theorem 3.7

$$\text{reg}(\text{Gin}(K_1(I_S))) = \max\{\deg Y_1(S), 1 + \left(\frac{\deg Y_1(S) - 1}{2}\right) - g(Y_1(S))\}$$  

and consequently,

$$M(I_S) = \max\{d, 1 + \deg Y_1(S), 2 + \left(\frac{\deg Y_1(S) - 1}{2}\right) - g(Y_1(S))\}.$$  

For a proof of (b), consider $\text{Gin}(K_1(I_S)) = \langle g_1, g_2, \ldots, g_m \rangle$ in (a). Note that the double curve $Y_1(S)$ is smooth in $\mathbb{P}^3$. By the similar argument used in (a), $\text{Gin}(K_1(I_S))$ contains $x_2^{\deg Y_1(S)}$ because the image of $Y_1(S)$ under a generic projection to $\mathbb{P}^2$ is a plane curve of degree $\deg Y_1(S)$. Finally, consider all
monomial generators of the form \( x_1 \cdot h_j(x_2, x_3, x_4) \) in \( \{g_1, g_2, \ldots, g_m\} \). Then, \( \{h_j(x_2, x_3, x_4) \mid 1 \leq j \leq m\} \) is a minimal generating set of \( \text{Gin}(K_1(I_{Y_1}(S))) \) by Proposition 3.4. Recall that \( K_1(I_{Y_1}(S)) \) defines \( \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S)) \) distinct nodes in \( \mathbb{P}^2 \). So, \( \text{Gin}(K_1(I_{Y_1}(S))) \) should contain the monomial \( x_3^{\binom{\deg Y_1(S) - 1}{2}} - g(Y_1(S)) \) (see also [5, Corollary 5.3]). Therefore, \( \text{Gin}(I_S) \) contains monomials \( x_1^d, x_0x_2^{\deg(Y_1(S))} \) and \( x_0x_1x_3^{\binom{\deg Y_1(S) - 1}{2}} - g(Y_1(S)) \). □

**Remark 4.4.** In the proof of Theorem 4.3 we showed that if a smooth irreducible surface \( S \) is contained in a quadric hypersurface then \( M(I_S) \) is determined by two partial elimination ideals \( K_0(I_S) \) and \( K_1(I_S) \) since \( K_i(I_S) = \bar{R} \) for all \( i \geq 2 \).

The following theorem shows that if \( d \geq 6 \) then \( M(I_S) \) is determined by the degree complexity of the first partial elimination ideal \( K_1(I_S) \).

**Proposition 4.5.** Let \( S \) be a smooth irreducible surface of degree \( d \) in \( \mathbb{P}^4 \). Suppose that \( S \) is contained in a quadric hypersurface. Then

\[
M(I_S) = \begin{cases} 
3 & \text{if } S \text{ is a rational normal scroll with } d = 3 \\
4 & \text{if } S \text{ is a complete intersection of (2,2)-type} \\
5 & \text{if } S \text{ is a Castelnuovo surface with } d = 5 \\
2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S)) & \text{for } d \geq 6
\end{cases}
\]

where \( Y_1(S) \subset \mathbb{P}^3 \) is a double curve of degree \( \binom{d - 1}{2} - g(S \cap H) \) under a generic projection of \( S \) to \( \mathbb{P}^3 \).

**Proof.** Since \( K_2(I_S) = \bar{R} \), Theorem 4.3 implies that

\[
M(I_S) = \max\{d, 1 + \deg Y_1(S), 2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S))\}.
\]

If \( \deg Y_1(S) \geq 5 \) then by the genus bound,

\[
1 + \deg Y_1(S) \leq 2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S)).
\]

We claim that if \( d \geq 6 \), then \( d \leq 1 + \deg Y_1(S) \). Notice that from our claim, we have the degree complexity of a surface lying on a quadric hypersurface in \( \mathbb{P}^4 \) for \( d \geq 6 \) as follows;

\[
M(I_S) = 2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S)).
\]

Note again that

\[
g(S \cap H) \leq \pi(d, 3) = \begin{cases} 
\left(\frac{d}{2} - 1\right)^2 & \text{if } d \text{ is even;} \\
\left(\frac{d-1}{2}\right)\left(\frac{d+3}{2}\right) & \text{if } d \text{ is odd.}
\end{cases}
\]
Then we can show that $\pi(d,3) \leq \binom{d-1}{2} - d + 1$ if $d = \deg(S \cap H) \geq 6$. Thus, if $d \geq 6$ then

$$d \leq 1 + \binom{d-1}{2} - g(S \cap H) = 1 + \deg Y_1(S).$$

So, our claim is proved and only three cases of $d = 3, 4, 5$ are remained.

Case 1: If $\deg S = 3$ then $S$ is a rational normal scroll with $g(S \cap H) = 0$ and the double curve $Y_1(S)$ is a line. So, by simple computation, $M(I_S) = 3$.

Case 2: If $\deg S = 4$ then $S$ is a complete intersection of $(2,2)$-type with $g(S \cap H) = 1$ and the double curve $Y_1(S)$ is a plane conic of $\deg Y_1(S) = 2$. So, by simple computation, $M(I_S) = 4$.

Case 3: If $\deg S = 5$ then $S$ is a Castelnuovo surface with $g(S \cap H) = 2$ and the double curve $Y_1(S) \subset \mathbb{P}^3$ is a smooth elliptic curve of degree $4$. In this case, we can also compute

$$M(I_S) = 5 = \deg S > 2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S)) = 4.$$

\[\square\]

**Proposition 4.6.** Let $S$ be a smooth irreducible surface of degree $d$ and arithmetic genus $\rho_a(S)$ in $\mathbb{P}^4$. Let $Y_i(S)$ be the multiple locus defined by a generic projection of $S$ to $\mathbb{P}^3$ for $i \geq 0$. Assume that $S$ is contained in a quadric hypersurface. Then, the following identity holds;

$$g(Y_1(S)) = \binom{d-1}{3} - \binom{d-1}{2} + g(S \cap H) - \rho_a(S) + 1.$$

**Proof.** Let $P_S(z)$ be the Hilbert polynomial of a smooth irreducible surface of degree $d$ and arithmetic genus $\rho_a(S)$. Since $Y_2(S) = \emptyset$, $P_{Y_2}(z) = 0$ and, by Lemma 4.1,

$$P_S(z) = P_{Y_0}(z) + P_{Y_1}(z)(z - 1).$$

Plugging $z = 0$, $P_S(0) = \rho_a(S) + 1$, $P_{Y_0}(0) = \binom{d-1}{3} + 1$, and

$$P_{Y_1}(1) = - \deg Y_1(S) + 1 - g(Y_1(S)) = - \binom{d-1}{2} + g(S \cap H) + 1 - g(Y_1(S)).$$

Therefore, we have the following identity:

$$g(Y_1(S)) = \binom{d-1}{3} - \binom{d-1}{2} + g(S \cap H) - \rho_a(S) + 1.$$

\[\square\]

**Remark 4.7.** By Proposition 4.6 when $d \geq 6$, $M(I_S)$ can be expressed with only three invariants of $S$: its degree, sectional genus, and arithmetic genus, as follows:

$$M(I_S) = \binom{d-1}{2} - g(S \cap H) - 1 - \binom{d-1}{3} + \binom{d-1}{2} - g(S \cap H) + \rho_a(S) + 1.$$
In order to compute $M(I_S)$ in terms of degrees of defining equations as A. Conca and J. Sidman did in [5], we need the following remark. This shows that a smooth surface in $\mathbb{P}^4$ has a nice algebraic structure when it is contained in a quadric hypersurface.

**Remark 4.8.** Let $S$ be a locally Cohen-Macaulay surface lying on a quadric hypersurface $Q$ in $\mathbb{P}^4$. Then $S$ satisfies one of following conditions (see [11, Theorem 2.1]):

(a) $S$ is a complete intersection of $(2, \alpha)$-type.
   (i) $I_S = (Q, F)$, where $F$ is a polynomial of degree $\alpha$.
   (ii) $\text{reg}(S) = \alpha + 1$.

(b) $S$ is projectively Cohen-Macaulay of degree $2\alpha - 1$.
   (i) $I_S = (Q, F_1, F_2)$, where $F_1$ and $F_2$ are polynomials of degree $\alpha$.
   (ii) $\text{reg}(S) = \alpha$.

From the above Remark 4.8, we can compute $g(S \cap H)$ and $\rho_a(S)$ in terms of the degree of defining equations of $S$ by finding the Hilbert polynomial of $S$ in two ways. Therefore, we have the following Theorem.

**Theorem 4.9.** Let $S \subset \mathbb{P}^4$ be a smooth irreducible surface of degree $d$ and arithmetic genus $\rho_a(S)$, which is contained in a quadric hypersurface.

(a) Suppose $S$ is of degree $2\alpha, \alpha \geq 3$. Then,

$$M(I_S) = \frac{1}{2}(\alpha^4 - 4\alpha^3 + 5\alpha^2 - 2\alpha + 4).$$

(b) Suppose $S$ is of degree $2\alpha - 1, \alpha \geq 4$. Then

$$M(I_S) = \frac{1}{2}(\alpha^4 - 6\alpha^3 + 13\alpha^2 - 12\alpha + 8).$$

**Proof.** For a proof of (a), by Koszul complex we have the minimal free resolution of the defining ideal $I_S$ as follows:

$$0 \longrightarrow R(-\alpha - 2) \longrightarrow R(-2) \oplus R(-\alpha) \longrightarrow I_S \longrightarrow 0,$$

Hence the Hilbert function of $R/I_S$ is given by

$$H(R/I_S, m) = \alpha m^2 + (-\alpha^2 + 3\alpha)m + \frac{1}{6}\alpha(2\alpha^2 - 9\alpha + 13)$$

$$= \frac{2\alpha}{2} m^2 + (\alpha + 1 - g(S \cap H)) m + \rho_a(S) + 1.$$

Hence $g(S \cap H) = (\alpha - 1)^2$ and $\rho_a(S) = \frac{1}{6}\alpha(2\alpha^2 - 9\alpha + 13) - 1$.

If $Y_1(S)$ is the double curve of $S$ then

$$\deg Y_1(S) = \left(\frac{2\alpha - 1}{2}\right) - g(S \cap H) = \alpha(\alpha - 1).$$

By Remark 4.7

$$g(Y_1(S)) = \left(\frac{2\alpha - 1}{3}\right) - \left(\frac{2\alpha - 1}{2}\right) + g(S \cap H) - \rho_a(S) + 1.$$
Thus we conclude that
\[ M(I_S) = 2 + \binom{\alpha(\alpha - 1) - 1}{2} - g(Y_1(S)) \]
\[ = \left( \frac{\alpha(\alpha - 1) - 1}{2} \right) - \left( \frac{2\alpha - 1}{3} \right) + \left( \frac{2\alpha - 1}{2} \right) - (\alpha - 1)^2 + \rho_a(S) + 1 \]
\[ = \frac{1}{2}(\alpha^4 - 4\alpha^3 + 5\alpha^2 - 2\alpha + 4). \]

For a proof of (b), let \( S \) be a smooth surface of degree \( 2\alpha - 1 \) lying on a quadric hypersurface in \( \mathbb{P}^4 \). Note that \( S \) is arithmetically Cohen-Macaulay of codimension 2. By the Hilbert-Burch Theorem [6] we have the minimal free resolution of the defining ideal \( I_S \) as follows:

\[
\begin{align*}
0 & \longrightarrow R(-\alpha - 1)^2 \\
& \quad \longrightarrow R(-2) \oplus R(-\alpha)^2 \\
& \quad \longrightarrow I_S \\
& \quad \longrightarrow 0,
\end{align*}
\]

where \( L_1, L_2, L_3, L_4 \) are linear forms and \( F_5, F_6 \) are forms of degree \( \alpha - 1 \). Hence the Hilbert function of \( R/I_S \) is given by

\[
H(R/I_S, m) = \frac{1}{2}(2\alpha - 1)m^2 + \left( 4\alpha - \alpha^2 - \frac{3}{2} \right) m + \frac{1}{3}\alpha^3 - 2\alpha + \frac{11}{3}\alpha - 1
\]
\[ = \frac{1}{2}(2\alpha - 1)m^2 + \left( \frac{2\alpha - 1}{2} + 1 - g(S \cap H) \right) m + \rho_a(S) + 1. \]

Hence we have that \( g(S \cap H) = 2\left( \frac{\alpha - 1}{2} \right) \) and \( \rho_a(S) = 2\left( \frac{\alpha - 1}{3} \right) \).

If \( Y_1(S) \) be the double curve of \( S \) then

\[
\deg Y_1(S) = \left( \frac{2\alpha - 2}{2} \right) - g(S \cap H) = \left( \frac{2\alpha - 2}{2} \right) - 2\left( \frac{\alpha - 1}{2} \right).
\]

On the other hand, we have

\[
g(Y_1(S)) = \left( \frac{2\alpha - 2}{3} \right) - \left( \frac{2\alpha - 2}{2} \right) + g(S \cap H) - \rho_a(S) + 1
\]
\[ = (\alpha - 2)(\alpha^2 - 3\alpha + 1) \]

and thus we conclude that

\[
M(I_S) = 2 + \frac{\deg Y_1(S) - 1}{2} - g(Y_1(S)) \]
\[ = \frac{1}{2}(\alpha^4 - 6\alpha^3 + 13\alpha^2 - 12\alpha + 8). \]

\[ \square \]
Example 4.10 (Macaulay 2). We give some examples of \( \text{Gin}(I_S) \) and \( M(I_S) \) computed by using Macaulay 2.

(a) Let \( S \) be a rational normal scroll in \( \mathbb{P}^4 \) whose defining ideal is

\[
I_S = (x_0x_3 - x_1x_2, x_0x_1 - x_3x_4, x_0^2 - x_2x_4).
\]

Using Macaulay 2, we can compute the generic initial ideal of \( I_S \) with respect to GLex:

\[
\text{Gin}(I_S) = (x_0^2, x_0x_1, x_0x_2, x_1^3).
\]

Thus \( \text{reg}(\text{Gin}_{\text{GLex}}(K_0)) = 3 \) and \( \text{reg}(\text{Gin}_{\text{GLex}}(K_1)) = 1 \). Therefore,

\[
M(I_S) = \deg S = 3.
\]

(b) Let \( S \) be a complete intersection of \((2, 2)\)-type in \( \mathbb{P}^4 \). Then,

\[
\text{Gin}(I_S) = (x_0^2, x_0x_1, x_0x_2).
\]

Hence, we see \( M(I_S) = \deg S = 4 \).

(c) Let \( S \) be a Castelnuovo surface of degree 5 in \( \mathbb{P}^4 \). Then, we can compute

\[
\text{Gin}(I_S) = (x_0^2, x_0x_1^5, x_0x_1x_2, x_0x_2^4, x_0x_1x_3^2).
\]

Hence, we see \( M(I_S) = \deg S = 5 \).

(d) Let \( S \) be a complete intersection of \((2, 3)\)-type in \( \mathbb{P}^4 \). Then, we see that \( M(I_S) = 8 \) from Theorem 4.9. On the other hand, we can compute the generic initial ideal:

\[
\text{Gin}(I_S) = (x_0^2, x_0x_1^2, x_1^6, x_0x_1x_2^2, x_0x_2, x_0x_1x_2x_3^2, x_0x_1x_2x_3^2, x_0x_1x_2x_4^4).
\]

This also shows \( M(I_S) = 8 \).

(e) Let \( S \) be a smooth surface of degree 7 lying on a quadric which is not a complete intersection in \( \mathbb{P}^4 \). Then, the minimal resolution of \( I_S \) is given by Hilbert-Burch Theorem and thus we have

\[
I_S = (L_1L_4 - L_2L_3, L_1F_5 - L_2F_6, L_3F_5 - L_4F_6),
\]

where \( L_i \) is a linear form and \( F_5, F_6 \) are forms of degree 3. This is the case of \( \alpha = 4 \) in Theorem 4.9 and we see \( M(I_S) = 20 \). This can also be obtained by the computation of generic initial ideal of \( I_S \) using Macaulay 2:

\[
\text{Gin}(I_S) = (x_0^2, x_0x_1^2, x_1^7, x_0x_1^7x_2, x_0x_1x_2^4, x_0x_2, x_0x_2^7x_3^2, x_0x_1x_3^3, x_0x_2x_3^5, x_0x_1x_2x_3^2, x_0x_1^3x_2x_3^18, x_0x_1x_2x_3^2x_4, x_0x_2^2x_3x_4^2, x_0x_1x_3^3x_4^2, x_0x_2^2x_3^2x_4^2, x_0x_1x_2^3x_3^2x_4^2, x_0x_1x_2x_3^2x_4^5, x_0x_1x_2x_3^2x_4^2, x_0x_1x_2x_3^2x_4, x_0x_1x_2x_3^2x_4^2, x_0x_1x_2x_3^2x_4^2, x_0x_1x_2x_3^2x_4^2, x_0x_1x_2x_3^2x_4^2, x_0x_1x_2x_3^2x_4^2, x_0x_1x_2x_3^2x_4^2, x_0x_1x_2x_3^2x_4^2).
\]
(f) Let $S$ be a complete intersection of $(2,4)$-type in $\mathbb{P}^4$. Then, we see that $M(I_S) = 38$ from Theorem 4.9. This can be given by the computation of generic initial ideal of $I_S$:

\[
\text{Gin}(I_S) = (x_0^2, x_0x_1^3, x_1^8, x_0x_1^2x_2, x_0x_1x_3^6, x_0x_2^2, x_0x_1x_2x_3^5, x_0x_1x_3^2x_4^7, x_0x_1x_2^2x_3^4, x_0x_1x_2x_3^3, x_0x_1x_2x_3^2, x_0x_1x_2x_3x_4, x_0x_1x_2x_3^2x_4, x_0x_1x_2x_3^3x_4, x_0x_1x_2x_3^4x_4, x_0x_1x_2x_3^5x_4, x_0x_1x_2x_3^6x_4, x_0x_1x_2x_3^7x_4, x_0x_1x_2x_3^8x_4, x_0x_1x_2x_3^9x_4, x_0x_1x_2x_3^{10}x_4, x_0x_1x_2x_3^{11}x_4, x_0x_1x_2x_3^{12}x_4, x_0x_1x_2x_3^{13}x_4, x_0x_1x_2x_3^{14}x_4, x_0x_1x_2x_3^{15}x_4, x_0x_1x_2x_3^{16}x_4, x_0x_1x_2x_3^{17}x_4, x_0x_1x_2x_3^{18}x_4, x_0x_1x_2x_3^{19}x_4, x_0x_1x_2x_3^{20}x_4, x_0x_1x_2x_3^{21}x_4, x_0x_1x_2x_3^{22}x_4, x_0x_1x_2x_3^{23}x_4, x_0x_1x_2x_3^{24}x_4, x_0x_1x_2x_3^{25}x_4, x_0x_1x_2x_3^{26}x_4, x_0x_1x_2x_3^{27}x_4, x_0x_1x_2x_3^{28}x_4, x_0x_1x_2x_3^{29}x_4, x_0x_1x_2x_3^{30}x_4, x_0x_1x_2x_3^{31}x_4, x_0x_1x_2x_3^{32}x_4, x_0x_1x_2x_3^{33}x_4, x_0x_1x_2x_3^{34}x_4).
\]

Even though we cannot compute the generic initial ideals for the cases $\alpha \geq 5$ by using computer algebra systems, we know the degree-complexity of smooth surfaces lying on a quadric by theoretical computations. We give the following tables:

| $\alpha$ | 5 | 6 | 7 | 8 | 9 | 10 | 20 | 50 | 100 |
|----------|---|---|---|---|---|----|----|----|----|
| $M(I_S)$ | 122 | 302 | 632 | 1178 | 2018 | 3242 | 64982 | 2881202 | 48024902 |
| $m(I_S)$ | 6 | 7 | 8 | 9 | 10 | 11 | 21 | 51 | 101 |

| $\alpha$ | 5 | 6 | 7 | 8 | 9 | 10 | 20 | 50 | 100 |
|----------|---|---|---|---|---|----|----|----|----|
| $M(I_S)$ | 74 | 202 | 452 | 884 | 1570 | 2594 | 58484 | 2765954 | 47064404 |
| $m(I_S)$ | 5 | 6 | 7 | 8 | 9 | 10 | 20 | 50 | 100 |

**Remark and Question 4.11.** Let $S$ be a non-degenerate smooth surface of degree $d$ and arithmetic genus $\rho_0(S)$, not necessarily contained in a quadric hypersurface in $\mathbb{P}^4$. Our question is: What can be the degree complexity
$M(I_S)$ of $S$? It is expected that $K_1(I_S)$ and $K_2(I_S)$ are reduced ideals and the degree-complexity $M(I_S)$ is given by

$$M(I_S) = \max \left\{ \begin{array}{l}
\deg(S) \\
\reg(G_{\text{GLex}}(K_1(I_S))) + 1 \\
\reg(G_{\text{GLex}}(K_2(I_S))) + 2
\end{array} \right\}$$

$$= \max \left\{ \begin{array}{l}
d \\
M(I_{Y_1}(S)) + 1 \\
t + 2
\end{array} \right\}.$$  

Note that $t$ is the number of apparent triple points of $S \subset \mathbb{P}^4$ and $Y_1(S)$ is the double curve (possibly singular with ordinary double points) under a generic projection.

\[\square\]

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