ARITHMETIC THREE-SPHERES THEOREMS FOR QUASILINEAR RICCATI TYPE INEQUALITIES

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Abstract. We consider arithmetic three-spheres inequalities to solutions of certain second order quasilinear elliptic differential equations and inequalities with a Riccati-type drift term.

1. Introduction and main results

In this paper we consider the following second order quasilinear elliptic differential equation with a Riccati-type drift term,

\[(1.1) \quad -\nabla \cdot A(x, u, \nabla u) + B(x, u, \nabla u) = 0,\]

in a domain \(G\) of \(\mathbb{R}^n\), where \(n \geq 2\), and the related pair of differential inequalities

\[-\nabla \cdot A(x, u, \nabla u) + B(x, u, \nabla u) \leq 0,\]

\[-\nabla \cdot A(x, u, \nabla u) + B(x, u, \nabla u) \geq 0.\]

In some sense, solutions to these differential inequalities can be considered to be sub- and supersolutions to the equation (1.1). Here \(A\) and \(B\) are assumed to satisfy certain growth conditions which we specify in Section 2; see also Section 4. We shall prove an arithmetic version of the Hadamard three-circles/spheres theorem for sub- and supersolutions to the equation (1.1). Three-spheres inequalities are central in the qualitative theory of partial differential equations. As an application, we obtain a Cauchy–Liouville-type theorem for solutions to (1.1) under certain structural assumptions.

The classical Hadamard three-circles theorem for analytic functions has counterparts for solutions to elliptic differential equations and inequalities. For instance, suppose that \(u\) is a non-constant \(C^2\)-smooth 2-subharmonic function, i.e. \(\Delta u \geq 0\), in the set \(\{x \in \mathbb{R}^n : |x| < R\}\)

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and let $M(r) = \sup\{u(x) : |x| = r\}$ for every $0 < r \leq R$. Then by
the strong maximum principle $M(r)$ is a strictly increasing function of $r$. The three-circles theorem, on the other hand, tells that $M(r)$ is a convex function of $\log r$ when $n = 2$, and a convex function of $-r^{2-n}$ when $n > 2$; see Protter–Weinberger [19]. One application of the theorem is Liouville's theorem; a uniformly bounded subharmonic function in the whole $(x, y)$-plane, except possibly at one point, is a constant [19].

There are counterparts of the three-circles inequality for solutions to other elliptic equations. To name a few, we refer to Dow [5], Landis [9, 10], Výborný [21, 22], and to a more recent paper [6] by Fraas and Pinchover and to Miklyukov et al. [16], see also the references in these papers.

It is interesting to note that the aforementioned result for subharmonic functions holds in verbatim also in the case of the non-linear $p$-Laplace equation,

$$-\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0,$$

where $1 < p < \infty$. This is simply due to the existence of the radial fundamental solutions $|x|^{(p-n)/(p-1)}$ whenever $1 < p < n$, and $-\log |x|$ for the borderline case $p = n$. Then any function of the form $a + b|x|^{(p-n)/(p-1)}$, $a, b \in \mathbb{R}$, is a solution to the $p$-Laplace equation in punctured neighborhoods of the origin. Then the comparison principle gives the following arithmetic version of the Hadamard three-circles theorem: Suppose that $u$ is a $p$-subharmonic in a domain containing concentric circles of radii $r_1$ and $r_3$ and the region between them, then for each $r_1 < r_2 < r_3$

$$M(r_2) \leq \frac{M(r_1) \log(r_3/r_2) + M(r_3) \log(r_2/r_1)}{\log(r_3/r_1)},$$

when $p = n$, and in the case in which $1 < p < n$ we have

$$M(r_2) \leq \frac{M(r_1) (r_2^{\alpha} - r_3^{\alpha}) + M(r_3) (r_1^{\alpha} - r_2^{\alpha})}{r_1^{\alpha} - r_3^{\alpha}},$$

where $\alpha = (p - n)/(p - 1)$.

Let us also discuss another form of the three-spheres inequality for solutions to elliptic equations. Namely, inequalities of the type

(1.2) \quad $\|u\|_{B_2} \leq C \|u\|_{B_1}^{\tau} \|u\|_{B_3}^{1-\tau}$,

where $0 < r_1 < r_2 < r_3$. Usually, the radius $r_3$ has to be sufficiently small and $u$ is a $C^2$-solution to an elliptic differential equation or inequality, and $\|u\|$ is the $L^2$ or $L^\infty$-norm of $u$ on concentric spheres. The constants $C$ and $\tau \in (0, 1)$ depend only on the given elliptic operator.
and on the ratios $r_1/r_2$, $r_1/r_3$, and $r_2/r_3$. The classical Hadamard inequality for analytic functions in an annulus is the inequality (1.2) with $C = 1$ and $\tau = \log(r_3/r_2)/\log(r_3/r_1)$. For counterparts of the inequality (1.2) to elliptic equations, see e.g. Korevaar-Meyers [7], Brummenhuis [3], and Lin et al. [11] [12]. We also refer to Alessandrini et al. [1] for an exhaustive reference list and discussion on the topic.

In general, three-spheres inequalities of the form (1.2) do not hold for second order quasilinear elliptic equations of divergence form. This can be seen from the counterexamples in [13] which concern solutions to the equation (1.1) with $B = 0$, $p = n$, and $n \geq 3$ in (2.2). This phenomenon occurs already in the linear case as can be deduced from the counterexamples by Plis [18]; in [18] the reader can find counterexamples for solutions to certain second order linear elliptic equations of divergence form with Hölder continuous coefficients (with any exponent less than one) in $\mathbb{R}^3$.

In this paper, we are concerned with equations of the general form (1.1), and hence we confine ourselves to the study of an arithmetic version of the three-spheres theorem. More precisely, we study the growth of sub- and supersolutions to (1.1) in terms of the functions

\[ \mathcal{M}(r) = \operatorname{ess sup} \{ u(x) : x \in B_r \} \]

and

\[ m(r) = \operatorname{ess inf} \{ u(x) : x \in B_r \}, \]

where $\overline{B}_r \subset G$ is a ball centered at some point in $G$ and with radius $r$. We discuss the finiteness of these functions in Remark 2.9. The main result in this paper is the following theorem.

**Theorem 1.3** (Global arithmetic three-spheres inequality: $1 < p < n$). Suppose that $u$ is a subsolution to (1.1) in $G$ under the structural assumptions (2.2) and (2.3) for $1 < p < n$. Assume further that there is a positive number $\tau$ so that for every $0 < r_1 < r_2 < r_3$, such that $\overline{B}_{r_3} \subset G$, and the inequalities $0 < \tau \leq r_1/r_2 < r_2/r_3 < 1$ hold. Then there exists a constant $0 < \lambda < 1$, depending only on $n$, $p$, $a_0$, $a_1$, $b_1$, and on the ratios $r_1/r_2$, $r_1/r_3$, and $r_2/r_3$, for which the inequality

\[ \mathcal{M}(r_2) \leq \lambda \mathcal{M}(r_1) + (1 - \lambda) \mathcal{M}(r_3) \]

holds. Also, we obtain the following dual result. If $u$ is a supersolution to (1.1) in $G$ the inequality

\[ m(r_2) \geq \lambda m(r_1) + (1 - \lambda) m(r_3) \]

holds. The balls $B_{r_1}$, $B_{r_2}$, and $B_{r_3}$ are concentric.
After discussing certain preliminary estimates in Section 2, we shall prove Theorem 1.3 in Section 3.

We obtain a local version of the arithmetic three-spheres inequality for sub- and supersolutions to the equation which has slightly different structure as the one in Theorem 1.3, see Theorem 4.3.

In Section 5 we consider the so-called borderline case, i.e. the case in which $p = n$ in (2.2) and (2.3). We obtain a global arithmetic three-spheres theorem for solutions. However, the method is slightly different to the one used in connection with Theorem 1.3 in the case $1 < p < n$. In the borderline case De Giorgi–Ladyzhenskaya–Ural’tseva-type $L^\infty - L^p$-estimates are replaced by an oscillation lemma due to Gehring and Mostow. With this replacement we obtain an explicit formula for the convexity parameter $\lambda$ as a function of the ratios $r_1/r_2$, $r_1/r_3$, and $r_2/r_3$.

In the borderline case, as an application of the global three-spheres inequality with an explicit convexity parameter $\lambda$, we obtain the following Cauchy–Liouville-type result in Theorem 5.6: a bounded entire solution to (1.1) under the structure presented in (2.2) and (2.3) must be constant.

2. Preliminary estimates

Let $G$ be a domain in $\mathbb{R}^n$, possibly unbounded, and $n \geq 2$. For simplicity and for notational purposes, we assume that $G$ contains the origin 0 and that open balls of radius $r$, written as $B_r$, are centered at 0. We write the closure of a ball $B_r$ as $\overline{B}_r$.

The structural conditions (2.2) and (2.3) below ensure that we can consider (1.1) in weak form as follows. A function $u \in W^{1,p}_{\text{loc}}(G)$ is a subsolution [supersolution] of the equation (1.1) in $G$ if, and only if, for any relatively compact $D \subset G$ and all $\eta \in W^{1,p}_0(D)$ with $\eta \geq 0$ in $D$ the inequality

$$ \int_D A(x,u,\nabla u) \cdot \nabla \eta \, dx + \int_D B(x,u,\nabla u) \eta \, dx \leq \geq 0 $$

holds; $u$ is a solution in $G$ if the equality holds in (2.1) for any relatively compact $D \subset G$ and all $\eta \in W^{1,p}_0(D)$. Here $A : G \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $B : G \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are assumed to satisfy the Carathéodory condition, i.e. each $A_i(x,t,h) (i = 1, \ldots, n)$ is measurable in $x \in G$ for every fixed $(t,h) \in \mathbb{R} \times \mathbb{R}^n$ and continuous in $(t,h)$ for almost every fixed $x$ and $B(x,t,h)$ is measurable in $x \in G$ for every fixed $(t,h) \in \mathbb{R} \times \mathbb{R}^n$ and continuous in $(t,h)$ for almost every fixed $x$. In addition, we shall assume that there are constants $1 < p < n$ and $0 < a_0 \leq a_1 < \infty$ such

"
that for all \((t, h)\) in \(\mathbb{R} \times \mathbb{R}^n\) and for almost every \(x \in G\) the following structural assumptions apply

\[(2.2)\quad A(x, t, h) \cdot h \geq a_0|h|^p, \quad |A(x, t, h)| \leq a_1|h|^{p-1}.
\]

For the drift term \(B\) we require it satisfies the following growth condition: there is a constant \(0 < b_1 < \infty\) such that for all \((t, h)\) in \(\mathbb{R} \times \mathbb{R}^n\) and for almost every \(x \in G\)

\[(2.3)\quad |B(x, t, h)| \leq g(x)|h|^{p-1},
\]

where the function \(g : G \to \mathbb{R}\) is defined as

\[g(x) = \begin{cases} b_1 & \text{if } |x| \leq 1, \\ \frac{b_1}{|x|} & \text{if } |x| > 1. \end{cases}\]

By imposing the asymptotic behavior as in \((2.3)\) on \(B\) we are able to obtain a global three-spheres inequality in the case of the general form of an elliptic quasilinear equation.

We do not assume monotonicity of \(A\) since we do not deal with existence problems. The prototype operator of \(A\) is the \(p\)-Laplacian. A growth condition in some ways similar to \((2.3)\) can be found in [6], see also Lin et al. [12] and [11].

We shall use a method exploiting certain convex functions and which has been previously used to prove oscillation lemmas and Hölder continuity for solutions to quasilinear elliptic equations [8], [4], but which has not been applied to prove three-spheres inequalities to quasilinear equations with a Riccati-type drift term \((1.1)\).

In what follows, \(\phi : \mathbb{R} \to \mathbb{R}\) shall be a convex function satisfying the following conditions: there exists a sub-interval \(I\) of \(\mathbb{R}\) such that

\[(C-1)\quad \phi \in C^2(I);
(C-2)\quad \phi'(t)^2 \leq \phi''(t) \text{ for all } t \in I;
(C-3)\quad \text{either (i) } \phi'(t) > 0 \text{ or (ii) } \phi'(t) < 0 \text{ for all } t \in I.
\]

Positive constants \(C, C_1, C_2, C_3,\) and \(C_4\) depend only on \(n, p, a_0, a_1,\) and \(b_1,\) and may vary from line to line. Dependence on other parameters than the aforementioned is written as \(C(\varepsilon, \delta)\).

Let us consider some estimates for the composite function \(\phi(u),\) where \(u\) is a solution to a differential inequality related to the equation \((1.1)\).

In regard to the growth condition \((2.3)\) and the function \(g,\) we shall next treat only the case in which \(|x| > 1\) as the case \(|x| \leq 1\) follows easily by modifying the proofs for the former case; when deriving these a priori estimates in Lemma [2.4] and Lemma [2.6] and restricting ourselves to those balls whose radius \(r \leq 1\) we can embed the terms caused by the
drift function $\mathcal{B}$ to the corresponding terms produced by the operator $\mathcal{A}$. See also the discussion in Section 4.

**Lemma 2.4.** Suppose that $u$ is a subsolution to (1.1) in $G$, with the structural assumptions (2.2) and (2.3) for fixed $1 < p < n$, and that the conditions (C-1)–(C-3)(i) are valid for a convex function $\phi : I \to \mathbb{R}$. There exists a constant $C > 0$ such that for every $\bar{B}_r \subset G$, and $0 < \delta < 1$, the inequality

\begin{equation}
\int_{B_{(1-\delta)r}} |\nabla \phi(u(x))|^p \, dx \leq \frac{C}{\delta^p} r^{n-p}
\end{equation}

holds. We obtain an analogous inequality in the case in which $u$ is a supersolution to (1.1) and the conditions (C-1)–(C-3)(ii) are valid for a convex function $\phi : I \to \mathbb{R}$.

**Proof.** Suppose that $u$ is a subsolution. Let $\xi \in C_0^\infty(B_r)$ be non-negative and $\eta(x) = \phi'(u(x))^{p-1} \xi^p(x)$ at $x \in B_r$, here $\phi'(t) > 0$ for all $t \in I$. By plugging $\eta$ into the inequality (2.1) and using the structural assumptions (2.2) and (2.3) we have

\[
a_0(p-1) \int_{B_r} |\nabla u|^p \phi'(u(x))^{p-2} \phi''(u(x)) \xi^p \, dx \\
\leq a_1 p \int_{B_r} |\nabla u|^{p-1} \phi'(u(x))^{p-1} \xi^{p-1} |\nabla \xi| \, dx \\
+ b_1 \int_{B_r} |x|^{-1} |\nabla u|^{p-1} \phi'(u(x))^{p-1} \xi^p \, dx.
\]

Applying the condition (C-2) on the left-hand side, the preceding inequality becomes

\[
\int_{B_r} |\nabla u|^p \phi'(u(x))^{p} \xi^p \, dx \\
\leq \frac{a_1 p}{a_0(p-1)} \int_{B_r} |\nabla u|^{p-1} \phi'(u(x))^{p-1} \xi^{p-1} |\nabla \xi| \, dx \\
+ \frac{b_1}{a_0(p-1)} \int_{B_r} |x|^{-1} |\nabla u|^{p-1} \phi'(u(x))^{p-1} \xi^p \, dx.
\]
From Young’s inequality with positive \( \varepsilon \) we get

\[
\int_{B_r} |\nabla u|^p \phi' (u(x))^p \xi^p \, dx \\
\leq \frac{a_1 p \varepsilon}{a_0 (p-1)} \int_{B_r} |\nabla u|^p \phi' (u(x))^p \xi^p \, dx + C_1 (\varepsilon) \int_{B_r} |\nabla \xi|^p \, dx \\
+ \frac{b_1 \varepsilon}{a_0 (p-1)} \int_{B_r} |\nabla u|^p \phi' (u(x))^p \xi^p \, dx + C_2 (\varepsilon) \int_{B_r} |x|^{-p} \xi^p \, dx \\
\leq C_1 \int_{B_r} |\nabla \xi|^p \, dx + C_2 r^{n-p},
\]

where the last inequality is obtained by choosing \( \varepsilon \) sufficiently small so that the first and the third term on the right are absorbed by the term on the left. The preceding inequality leads to (2.5) by choosing \( \xi = 1 \) on \( B_r(1 - \delta) \), \( 0 \leq \xi \leq 1 \), such that \( |\nabla \xi| \leq C/\delta r \) in \( B_r \).

In the case in which \( u \) is a supersolution and \( \phi'(t) < 0 \) for all \( t \), at each \( x \in B_r \) we set \( \eta(x) = (-\phi'(u(x)))^{p-1} \xi^p(x) \). Then the inequality (2.5) is obtained in the same way as above.

For any \( k \geq 0 \), let us write \( A_{k,r} = \{ x \in B_r : \phi(u(x)) > k \} \). We first deduce a Caccioppoli-type estimate on such level sets of the composite function \( \phi(u) \). Then, by referring to a well known iteration argument, an \( L^\infty - L^p \) estimate is obtained.

**Lemma 2.6.** Suppose that \( u \) is a subsolution to (1.1) in \( G \) with the structural assumptions (2.2) and (2.3), and that the conditions (C-1)–(C-3)(i) are valid for a convex function \( \phi : I \to \mathbb{R} \). Let \( \overline{B_{r_0}} \subset G \) and \( 0 < \delta_0 < 1 \) be a fixed constant. Assume further that there exists a radius \( \hat{r} > 0 \) such that \( \phi(u(x)) \leq 0 \) at almost every \( x \in B_{\hat{r}} \) and there is \( \tau, 0 < \tau < 1 \), such that \( \hat{r} \geq \tau (1-\delta_0) r_0 \). Then for every \( k \geq 0 \) and for \( (1-\delta_0) r_0 \leq (1-\delta) r < r \leq r_0 \) the inequality

\[
\int_{A_{k,(1-\delta) r}} |\nabla \phi(u(x))|^p \, dx \leq C(\delta_0, \tau) \int_{A_{k,r}} (\phi(u(x)) - k)^p \, dx
\]

holds. In addition, the inequality

\[
\left( \text{ess sup}_{x \in \overline{B((1-\delta_0) r_0)}} \phi(u(x)) \right)^p \leq \frac{C(\delta_0, \tau)}{r_0^n} \int_{A_0, r_0} \phi(u(x))^p \, dx
\]

is valid. All the balls above are concentric.

We obtain analogous inequalities in the case in which \( u \) is a supersolution to (1.1) and the conditions (C-1)–(C-3)(ii) are valid for a convex function \( \phi : I \to \mathbb{R} \).
Proof. Suppose that \(u\) is a subsolution and let \(k \geq 0\) be arbitrary. Define \(\psi(x) = \max\{\phi(u(x)) - k, 0\}\) at every \(x \in B_r\), where \(r \leq r_0\) with \(\overline{B_{r_0}} \subset G\). Let us choose \(\eta(x) = \psi(x)\phi'(u(x))^{p-1}\xi^p(x)\), where \(\xi \in C^\infty_0(B_r)\) is non-negative.

By plugging \(\eta\) into the inequality (2.1), using the structural assumptions (2.2) and (2.3), and then (C-2) we have

\[
\int_{B_r} |\nabla u|^p \phi'(u(x)) \xi^p \, dx + (p - 1) \int_{B_r} |\nabla u|^p \psi(x) \phi'(u(x)) \xi^p \, dx \\
\leq \frac{a_1 p}{a_0} \int_{B_r} |\nabla u|^{p-1} \psi(x) \phi'(u(x))^{p-1} \xi^{p-1} |\nabla \xi| \, dx \\
+ \frac{b_1}{a_0} \int_{B_r} |x|^{-1} |\nabla u|^{p-1} \psi(x) \phi'(u(x))^{p-1} \xi^p \, dx.
\]

We drop the second term on the left-hand side and use Young’s inequality with \(\varepsilon > 0\) to the terms on the right-hand side in the preceding inequality; it becomes

\[
\int_{B_r} |\nabla u|^p \phi'(u(x)) \xi^p \, dx \leq \frac{a_1 p}{a_0} \varepsilon \int_{B_r} |\nabla u|^p \phi'(u(x)) \xi^p \, dx \\
+ C_1(\varepsilon) \int_{B_r} \psi(x)^p |\nabla \xi|^p \, dx + \frac{b_1}{a_0} \varepsilon \int_{B_r} |\nabla u|^p \phi'(u(x)) \xi^p \, dx \\
+ C_2(\varepsilon) \int_{B_r} |x|^{-p} \psi(x)^p \xi^p \, dx.
\]

We then choose \(\varepsilon\) small enough so that the first and the third term on the right will be absorbed by the term on the left. Then let us choose the cut-off \(\xi\) so that \(\xi(x) = 1\) for \(x \in B_{(1-\delta)r}\), \(0 \leq \xi \leq 1\), and \(|\nabla \xi| \leq C/\delta r\) on \(B_r\). Furthermore, we clearly have that both \(\psi(x) = 0\) and \(\nabla \psi(x) = 0\) at a.e. \(x \in B_r \setminus A_{k,r}\). Also, by the hypothesis, \(\psi(x) = 0\) on \(B_r\). Altogether, we have

\[
\int_{A_{k,(1-\delta)r}} |\nabla u|^p \phi'(u(x))^p \, dx \leq \frac{C}{(\delta r)^p} \left(1 + \frac{\delta^p_0}{\tau^{p(1-\delta_0)^p}}\right) \int_{A_{k,r}} \psi(x)^p \, dx,
\]

and hence the desired inequality (2.7).

Suppose \(u\) is a supersolution. In this case, let us choose \(\eta(x) = \psi(x)(-\phi'(u(x)))^{p-1}\xi^p(x)\) and we shall proceed as above to obtain the inequality (2.7).

It is well known that an inequality of the form (2.8) follows from an inequality of the type (2.7) by a De Giorgi-type iteration argument, see [8, Lemma 5.4, page 76].
Remark 2.9. It follows by a similar argument as in Lemma 2.6 with \( \phi(t) = t \) that \( \mathcal{M}(r) < \infty \) whenever \( u \) is a subsolution to (1.1) in \( G \) and \( \overline{B}_r \subset G \) such that \( r \leq 1 \); in this case the extra hypotheses concerning the existence of the ball \( B_{\hat{r}} \) and the constant \( \tau \) in Lemma 2.6 become obsolete. We may, therefore, conclude by a covering argument that \( \mathcal{M}(r) < \infty \) for all \( r > 0 \) for which \( \overline{B}_r \subset G \).

It can be noted that under our structural assumptions (2.2) and (2.3) it does not necessarily hold that \(-u\) is a supersolution to (1.1) whenever \( u \) is a subsolution. Hence, to obtain (1.5) we assume that \( m(r) > -\infty \) for all \( r > 0 \) for which \( \overline{B}_r \subset G \). This extra hypothesis becomes void if we assumed certain homogeneity of the operator \( \mathcal{B} \). Since this would rule out an interesting set of equations, for instance the equation (4.2), we do not make such an assumption.

3. Theorem 1.3

Proof of Theorem 1.3. We consider first the inequality (1.4). Let \( u \) be a subsolution and \( \varepsilon > 0 \). We define a convex function \( \phi \) satisfying the conditions (C-1)--(C-3)(i) as

\[
\phi(t) = -\log \left( \frac{\mathcal{M}(r_3) - t + \varepsilon}{\mathcal{M}(r_3) - \mathcal{M}(r_1) + \varepsilon} \right)
\]

for \( t \in (-\infty, \mathcal{M}(r_3)] \). We consider the composite function \( \phi(u(x)) \). Define also \( \psi(x) = \max\{\phi(u(x)), 0\} \). In what follows, \( C \) is always a positive constant which may vary from line to line and depends only on \( n, p, a_0, a_1, b_1, r_1/r_2, r_1/r_3, \) and \( r_2/r_3 \).

Observe that since \( \psi(x) = 0 \) at each \( x \in B_{r_1} \), we have the Poincaré inequality

\[
\int_{B_{(r_2+r_3)/2}} \psi(x)^p \, dx \leq C \left( \frac{r_2 + r_3}{2} \right)^p \int_{B_{(r_2+r_3)/2}} |\nabla \psi(x)|^p \, dx.
\]

We obtain the following \( L^\infty \)-bound for \( \phi(u(x)) \) using first (2.8) with \( r_0 = (r_2 + r_3)/2, \ 0 < \delta_0 = (r_3 - r_2)/(r_2 + r_3) < 1, \hat{r} = r_1, \) and
\[ \tau \leq r_1/r_2, \] then the Poincaré inequality, and finally the inequality \( (2.5) \)

\[
\left( \text{ess sup}_{x \in B_{r_2}} \phi(u(x)) \right)^p \leq C \left( \frac{r_2 + r_3}{2} \right)^{-n} \int_{A_0,(r_2+r_3)/2} \phi(u(x))^p \, dx
\]

\[
= C \left( \frac{r_2 + r_3}{2} \right)^{-n} \int_{B(r_2+r_3)/2} \psi(x)^p \, dx
\]

\[
\leq C \left( \frac{r_2 + r_3}{2} \right)^{p-n} \int_{B(r_2+r_3)/2} |\nabla \phi(u(x))|^p \, dx
\]

\[
\leq C.
\]

The obtained upper bound \( C \) is independent of \( \varepsilon \). Since \( \phi \) is strictly increasing

\[
\text{ess sup}_{x \in B_{r_2}} \phi(u(x)) = -\log \left( \frac{\mathcal{M}(r_3) - \mathcal{M}(r_2) + \varepsilon}{\mathcal{M}(r_3) - \mathcal{M}(r_1) + \varepsilon} \right),
\]

and further we obtain

\[
\log \left( \frac{\mathcal{M}(r_3) - \mathcal{M}(r_2) + \varepsilon}{\mathcal{M}(r_3) - \mathcal{M}(r_1) + \varepsilon} \right) \geq -C.
\]

It follows that

\[
\mathcal{M}(r_3) - \mathcal{M}(r_2) + \varepsilon \geq e^{-C} (\mathcal{M}(r_3) - \mathcal{M}(r_1) + \varepsilon),
\]

or equivalently

\[
\mathcal{M}(r_2) \leq e^{-C} \mathcal{M}(r_1) + (1 - e^{-C}) \mathcal{M}(r_3) + (1 - e^{-C}) \varepsilon,
\]

from which the claim \( (1.4) \) follows by letting \( \varepsilon \to 0 \).

Suppose that \( u \) is a supersolution. To prove the inequality \( (1.5) \), let \( \varepsilon > 0 \) and choose a convex function satisfying the conditions \( \text{(C-1)}-(\text{C-3)}(\text{ii}) \) as

\[
\phi(t) = -\log \left( \frac{t - m(r_3) + \varepsilon}{m(r_1) - m(r_3) + \varepsilon} \right)
\]

for \( t \in [m(r_3), \infty) \). As above, we consider the composite function \( \phi(u(x)) \) and define \( \psi(x) = \max\{\phi(u(x)), 0\} \) which also vanishes at each \( x \in B_{r_1} \), and obtain by reasoning as above that for \( x \in B_{r_2} \)

\[
\phi(u(x)) \leq C,
\]

where the constant \( C \) is independent of \( \varepsilon \). Hence for each \( x \in B_{r_2} \)

\[
u(x) + (1 - e^{-C}) \varepsilon \geq e^{-C} m(r_1) + (1 - e^{-C}) m(r_3).
\]

We obtain the desired inequality \( (1.5) \) by letting \( \varepsilon \to 0 \).
4. Local three-spheres theorem

In this section, let us consider the equation (1.1) with the structural assumptions (2.2) and

\[ |\mathcal{B}(x, t, h)| \leq b_1|h|^{p-1} \]

for all \((t, h)\) in \(\mathbb{R} \times \mathbb{R}^n\) and for almost every \(x \in G\), where \(0 < b_1 < \infty\) and \(1 < p < n\) are fixed. The prototype equation becomes

\[ -\nabla \cdot \mathcal{A}(x, u, \nabla u) = b(x)|\nabla u|^{p-1}, \]

where \(b : \mathbb{R}^n \to \mathbb{R}\) is a bounded measurable function in \(G\). This equation has been under active consideration, see e.g. \([2]\), \([14]\), and \([15]\).

Suppose that \(u\) is either a sub- or supersolution to the equation (1.1) under the structural conditions (2.2) and (4.1). Then it is easy to verify that we are able to obtain an inequality similar to that in Lemma 2.4 by restricting ourselves to those balls \(B_r \subset G\) for which \(r \leq 1\). In addition, in this \(r \leq 1\) case the extra hypotheses concerning the existence of the ball \(B_{\hat{r}}\) and the constant \(\tau\) in Lemma 2.6 can be neglected and again we obtain two estimates similar to those in Lemma 2.6. We use the fact that \(r \leq 1\) when deriving these a priori estimates to embed the terms caused by the drift function \(B\) to the corresponding terms produced by the operator \(\mathcal{A}\).

We omit the proof of the following local result since it resembles that of Theorem 4.3.

**Theorem 4.3** (Local arithmetic three-spheres inequality: \(1 < p < n\)). *Suppose that \(u\) is a sub-solution to (1.1) in \(G\) with the structural assumptions (2.2) and (4.1) for \(1 < p < n\). For every \(0 < r_1 < r_2 < r_3 \leq 1\), such that \(\overline{B}_{r_3} \subset G\), there exists a constant \(0 < \lambda < 1\), depending only on \(n, p, a_0, a_1, b_1\), and on the ratios \(r_1/r_2, r_1/r_3,\) and \(r_2/r_3\), for which the inequality (1.4) holds. In the case in which \(u\) is a supersolution in \(G\) the inequality (1.5) is valid.*

5. Three-spheres theorem in the borderline case

In this section, we let \(p = n\) and consider solutions to the equation (1.1) under the structure (2.2) and (2.3). This is the so-called borderline case. We obtain a global arithmetic three-spheres theorem also in this case, however, the method is slightly different to the previously presented. De Giorgi–Ladyzhenskaya–Ural’tseva-type estimates are replaced by an oscillation lemma due to Gehring and Mostow. With this replacement we are able to obtain an explicit formula for the convexity parameter \(\lambda\) as a function of the ratios \(r_1/r_2, r_1/r_3,\) and \(r_2/r_3\).
As an application of the global arithmetic three-spheres theorem with
an explicit convexity parameter \( \lambda \), we obtain a Cauchy–Liouville-type
result in Theorem 5.6.

In what follows, if \( A \subset \mathbb{R}^n \) is a non-empty measurable set and
\( \sup_{x \in A} |u(x)| < \infty \), we let

\[
osc u = \sup_{A} u(x) - \inf_{A} u(x)
\]

denote the oscillation of \( u \) on the set \( A \).

We shall need the following variant of Lemma 2.4.

**Lemma 5.1.** Suppose that \( u \) is a solution to \((1.1)\) in \( G \) with the struc-
tural assumptions \((2.2)\) and \((2.3)\) for \( p = n, \) and that the conditions
\((C-1)-(C-3)\) are valid for a convex function \( \phi : I \to \mathbb{R} \). Then for every
\( B_{r_1} \subset B_{r_2} \subset B_{r_3} \subset G \) the inequality

\[
\int_{B_{(r_2+r_3)/2} \setminus B_{r_2}} |\nabla \phi(u(x))|^n \, dx \leq C \left( \left( \log \frac{2r_3}{r_2 + r_3} \right)^{1-n} + \left( \log \frac{r_2}{r_1} \right)^{1-n} + \log \frac{r_3}{r_1} \right)
\]

holds. The constant \( C > 0 \) depends only on \( n, a_0, a_1, \) and \( b_1 \).

**Proof.** The proof is similar to the proof of Lemma 2.4 apart from obvi-
ous modifications. We obtain, by assuming the condition (C-3)(i) and
choosing \( \eta \) as in the proof of Lemma 2.4

\[
\int_{B_{r_3}} |\nabla u|^{n} \phi'(u(x))^n \xi^n \, dx \leq C_1 \int_{B_{r_3}} |\nabla \xi|^n \, dx
\]

\[
+ C_2 \int_{B_{r_3}} |x|^{-n} \xi^n \, dx,
\]

where \( C_1 \) and \( C_2 \) are positive constants depending on \( n, a_0, a_1, \) and \( b_1 \).

Let us choose a non-negative cut-off function \( \xi \in C_0^\infty(B_{r_3}), 0 \leq \xi \leq 1, \)
as follows

\[
\xi(x) = \begin{cases} 
0 & \text{if } |x| \leq r_1, \\
1 & \text{if } r_2 < |x| < (r_2 + r_3)/2, \\
0 & \text{if } |x| \geq r_3,
\end{cases}
\]
and so that it is admissible for the conformal capacity. Then the inequality (5.3) becomes
\[ \int_{B_{r_2 + r_3}/2 \setminus B_{r_2}} |\nabla u|^n \phi'(u(x))^n \, dx \leq C_1 \left( \log \frac{2r_3}{r_2 + r_3} \right)^{1-n} + C_1 \left( \log \frac{r_2}{r_1} \right)^{1-n} + C_2 \log \frac{r_3}{r_1}, \]
and hence we obtain the inequality (5.2).

We have the following global three-spheres inequality with an explicit formula for the convexity parameter \( \lambda \).

**Theorem 5.4** (Global arithmetic three-spheres inequality: \( p = n \)).
Suppose that \( u \) is a solution to (1.1) in \( G \) with the structural assumptions (2.2) and (2.3) for \( p = n \). For every \( 0 < r_1 < r_2 < r_3 \), such that \( B_{r_3} \subset G \), there exists a constant \( 0 < \lambda < 1 \), depending only on \( n, a_0, a_1, b_1, \) and on the ratios \( r_1/r_2, r_1/r_3, \) and \( r_2/r_3 \), for which both the inequality (1.4) and (1.5) hold. Moreover, in both cases we have the formula
\[ \lambda = \exp \left( -C \left( \left( \log \frac{2r_3}{r_2 + r_3} \right)^{1-n} + \left( \log \frac{r_2}{r_1} \right)^{1-n} + \log \frac{r_3}{r_1} \right)^{1/2} \left( \left( \log \frac{r_2 + r_3}{2r_2} \right)^{-1/2} \right) \right), \]
where the constant \( C > 0 \) depends only on \( n, a_0, a_1, \) and \( b_1 \).

**Proof.** We first show that \( u \) is monotone in the sense of Lebesgue, i.e. that \( u \) reaches its extrema on the boundary of any relatively compact subdomain \( D \) of \( G \). We can consider only the maximum principle for \( u \) as the minimum principle is treated similarly. Assume, on the contrary, that \( u(x_0) = \max_{x \in D} u(x) > \max_{x \in \partial D} u(x) \) for some \( x_0 \in D \). It follows that \( L = u(x_0) = \max_{x \in B_{\rho}(x_0)} u(x) \) for some \( B_{\rho}(x_0) \subset D \), where \( \rho \leq 1 \). Then the function \( v = L - u \) is non-negative in \( B_{\rho} \), and \( v(x_0) = 0 \), and also \( v \) satisfies an equation similar to (1.1) with analogous structure conditions. Since the Harnack inequality holds for the solution \( v \) [20, Theorem 1.1], it follows that \( u = L \) on \( B_{\rho}(x_0) \). It is now easy to see that the set where \( u = L \) can be expanded to be the whole \( D \) which, in turn, leads to a contradiction.

Let us now turn to the proof of the inequality (1.4) by considering the increasing function \( \phi(t) \) as defined in the proof of Theorem 1.3 for \( t \in (-\infty, \mathcal{M}(r_3)] \). We consider the composite function \( \phi(u(x)) \) which
is both monotone and continuous since \( u \) can be shown to be locally Hölder continuous \([20]\). Then the Gehring–Mostow lemma, see e.g. \([17, \text{Lemma 4.3}]\), applied to \( \phi(u(x)) \) gives the inequality

\[
\int_{r_2}^{(r_2+r_3)/2} \left( \frac{\text{osc} \phi(u(x))}{\partial B_t} \right)^n \frac{dt}{t} \leq C \int_{B_{(r_2+r_3)/2}\setminus B_{r_2}} |\nabla \phi(u(x))|^n dx,
\]

where \( C > 0 \) is a constant depending only on \( n \). We have by monotonicity

\[
\left( \frac{\text{osc} \phi(u)}{B_{r_2}} \right)^n \leq \frac{C}{\log((r_2 + r_3)/2r_2)} \int_{B_{(r_2+r_3)/2}\setminus B_{r_2}} |\nabla \phi(u(x))|^n dx.
\]

Recall that \( \sup_{x \in B_{r_1}} \phi(u(x)) = 0 \). Now, by combining (5.5) with the inequality (5.2) we have

\[
\sup_{x \in B_{r_2}} \phi(u(x)) = \sup_{x \in B_{r_2}} \phi(u(x)) - \sup_{x \in B_{r_1}} \phi(u(x)) \leq \text{osc} \phi(u)
\]

\[
\leq C \left( \log \frac{r_2 + r_3}{2r_2} \right)^{-1/n} \left( \left( \log \frac{2r_3}{r_2 + r_3} \right)^{1-n} \right.
\]

\[
\left. + \left( \log \frac{r_3}{r_1} + \log \frac{r_3}{r_2} \right)^{1/n} \right).
\]

Then as in the proof of Theorem 1.3, we obtain the inequality (1.4), and analogously the inequality (1.5), with the explicit convexity parameter \( \lambda \). \( \square \)

Let us state an application of the global arithmetic three-spheres inequality obtained in Theorem 5.4. We obtain the following Cauchy–Liouville-type result.

**Theorem 5.6.** Suppose that \( u \) is a bounded solution to (1.1) in \( \mathbb{R}^n \) with the structural assumptions (2.2) and (2.3) with \( p = n \). Then \( u \) is constant.

**Proof.** Suppose, on the contrary, that \( u \) is a non-constant bounded solution in \( \mathbb{R}^n \). We may assume that \( u \) is non-negative as \( u + a, a > 0 \), satisfies an equation similar to (1.1). By the hypothesis, there exists a constant \( M \) such that \( 0 \leq u(x) \leq M \) at every \( x \in B_r, r > 0 \); in addition, we can assume that \( \inf_{x \in \mathbb{R}^n} u(x) = 0 \). Then by letting \( r_3 \to \infty \) the convexity parameter \( \lambda \) in Theorem 5.4 tends to some number in \((0, 1)\), written as \( \lambda_\infty \), which depends only on \( n, a_0, a_1, \) and \( b_1 \). Moreover, the inequality (1.4) becomes

\[
\mathcal{M}(r_2) - (1 - \lambda_\infty)M \leq \lambda_\infty \mathcal{M}(r_1)
\]
and it holds for every $0 < r_1 < r_2 < \infty$.

We can choose $r_1$ in such a way that $\mathcal{M}(r_1)$ becomes arbitrarily small, possibly by transferring the origin to some point $x_0$. It is easy to check that in this new coordinate system we are able to obtain a growth condition similar to the condition (2.3) with a function $g(x-x_0)$ and the constant $b_1$ depending now on $x_0$.

We then may choose $r_2$ so that $\mathcal{M}(r_2)$ is close to $\mathcal{M}$; choose $r_2$ so that $\mathcal{M}(r_2) = \Theta M$, where $\Theta > (1 - \lambda_{\infty})$. Hence we reach a contradiction in (5.7). □

We remark that the preceding Cauchy–Liouville-type result for solutions to (1.1) cannot be obtained directly from the Harnack inequality since for general equations involving a drift term $\mathcal{B}$ a constant $C$ in the Harnack inequality depends on the radius of a ball [20, Theorem 1.1].

5.1. $A$-harmonic equation. We close the paper by considering solutions to the $A$-harmonic equation

$$-\nabla \cdot A(x, u, \nabla u) = 0$$

with the structural conditions (2.2) with $p = n$.

Suppose that $u$ is a solution to the $A$-harmonic equation in $G$, and that the conditions (C-1)–(C-3) are valid for a convex function $\phi : I \to \mathbb{R}$. As in the proof of Lemma 5.1, by choosing a non-negative cut-off function $\xi \in C_0^\infty(B_{r_3}), 0 \leq \xi \leq 1$, such that

$$\xi(x) = \begin{cases} 1 & \text{if } |x| < (r_2 + r_3)/2, \\ 0 & \text{if } |x| \geq r_3, \end{cases}$$

and so that it is admissible for the conformal capacity, we have that for every $B_{r_1} \subset B_{r_2} \subset B_{r_3} \subset G$ the inequality

$$\int_{B_{(r_2+r_3)/2}\setminus B_{r_2}} |\nabla \phi(u(x))|^n \, dx \leq C \left( \log \frac{2r_3}{r_2 + r_3} \right)^{1-n}$$

holds with a constant $C > 0$ depending only on $n$, $a_0$, and $a_1$. Then as in the proof of Theorem 5.4 it is straightforward to verify that for every $0 < r_1 < r_2 < r_3$, such that $\overline{B}_{r_3} \subset G$, there exists a constant $0 < \lambda < 1$, depending only on $n$, $a_0$, $a_1$, $r_1/r_2$, $r_1/r_3$, and $r_2/r_3$, for which both the inequality (1.4) and (1.5) hold. Moreover, the convexity parameter becomes

$$\lambda = \exp \left( -C \left( \log \frac{2r_3}{r_2 + r_3} \right)^{(1-n)/n} \left( \frac{r_2 + r_3}{2r_2} \right)^{-1/n} \right),$$

where the constant $C > 0$ depends only on $n$, $a_0$, and $a_1$. 

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Notice that if \( u \) is a bounded solution to the \( \mathcal{A} \)-harmonic equation in \( \mathbb{R}^n \), then by letting \( r_3 \to \infty \) in (1.4) it can be seen that \( \mathcal{M}(r_2) \leq \mathcal{M}(r_1) \) for all \( r_1 < r_2 \), and moreover that \( u \) must be constant.

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