Peak Estimation for Uncertain and Switched Systems

**Jared Miller**\(^1\), Didier Henrion\(^2, 3\), Mario Sznaier\(^1\), Milan Korda \(^2, 3\)

Conference on Decision and Control, Dec 13-17, 2021

\(^1\)Northeastern University, Boston
\(^2\)LAAS-CNRS Toulouse
\(^3\)Czech Technical University, Prague
Main Ideas

Occupation measures can be used for peak estimation

Formulate peak estimation problems with uncertain dynamics

Find convergent approximations to peak values
Peak Estimation Problem

Find maximum value of $p(x)$ along trajectories

$$P^* = \max_{t, x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \quad t \in [0, T]$$

$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3], \quad T = 5$$
Examples of Peak Estimation

Maximum infection rate in an epidemic
Maximum altitude of an aircraft
Maximum voltage on a powerline
Maximum speed of a motor
Maximum height of a shock wave
Maximum concentration of a chemical reagent
Peak and Measure Background
Measures

Nonnegative Borel Measure $\mu : \text{Set}(X) \to \mathbb{R}_+$

$\mu \in \mathcal{M}_+(X)$: space of measures on $X$

$f \in C(X)$: continuous function on $X$

Duality pairing by Lebesgue integration:

$$\langle f(x), \mu \rangle = \int_X f(x) d\mu(x)$$

$\mu(X) = \langle 1, \mu \rangle = 1$: Probability distribution
Occupation Measure

Time trajectories spend in set

Test function
\( v(t, x) \in C([0, T] \times X) \)

Single trajectory:
\[ \langle v, \mu \rangle = \int_0^T v(t, x(t \mid x_0)) \, dt \]

Averaged trajectory: \( \langle v, \mu \rangle = \int_X \left( \int_0^T v(t, x) \, dt \right) d\mu_0(x) \)

\[ x' = -x(x + 2)(x - 1) \]
Measures: Initial $\mu_0$, Peak $\mu_p$, Occupation $\mu$

For all functions $v(t, x) \in C([0, T] \times X)$

- $\mu_0^* : \langle v(0, x), \mu_0^* \rangle = v(0, x_0^*)$
- $\mu_p^* : \langle v(t, x), \mu_p^* \rangle = v(t_p^*, x_p^*)$
- $\mu^* : \langle v(t, x), \mu^* \rangle = \int_0^{t_p^*} v(t, x^*(t | x_0^*)) dt$
Infinite dimensional linear program (Cho, Stockbridge, 2002)

\[ p^* = \max \langle p(x), \mu_p \rangle \quad (1a) \]

\[ \langle v(t, x), \mu_p \rangle = \langle v(0, x), \mu_0 \rangle + \langle \mathcal{L}_f v(t, x), \mu \rangle \quad \forall v \quad (1b) \]

\[ \langle 1, \mu_0 \rangle = 1 \quad (1c) \]

\[ \mu, \mu_p \in \mathcal{M}_+(\lbrack 0, T \rbrack \times X) \quad (1d) \]

\[ \mu_0 \in \mathcal{M}_+(X_0) \quad (1e) \]

Test functions \( v(t, x) \in C^1(\lbrack 0, T \rbrack \times X) \).

Lie derivative \( \mathcal{L}_f v = \partial_t v(t, x) + f(t, x) \cdot \nabla_x v(t, x) \).

\( (\mu^*_0, \mu^*_p, \mu^*) \) is feasible with \( P^* = \langle p(x), \mu^*_p \rangle \).
Approximating Peak Estimation

Truncate infinite-dimensional LP

- Discretized (gridded) finite LPs
- Markov Chain Martingale
- **Moment-Sum-of-Squares Hierarchy**

Moment-SOS: LMIs in increasing degree $d$,

$$p_d^* \geq p_{d+1}^* \geq \cdots p^*,$$

$$\lim_{d \to \infty} p_d^* = p^*$$
Uncertainty
System with Uncertainty Example

Time-Independent Uncertainty

\[ \dot{x} = \begin{bmatrix} x_2 \\ -x_1 \theta - x_2 + x_1^3/3 \end{bmatrix} \]

Time-Dependent Uncertainty

\[ \dot{x} = \begin{bmatrix} x_2 \\ -x_1 w(t) - x_2 + x_1^3/3 \end{bmatrix} \]

\[ \theta, \ w(t) \in [0.5, 1.5], \ x_0 = [1; 0] \]
Peak Estimation with Uncertainty

Time-independent \( \theta \in \Theta \)

Time-dependent \( w(t) \in W, \ \forall t \in [0, T] \)

\[
P^* = \max_{t \in [0, T], x_0 \in X_0, \theta \in \Theta, w(t)} p(x(t | x_0, \theta, w(t)))
\]

\[
\dot{x}(t) = f(t, x(t), \theta, w(t)), \quad w(t) \in W \quad \forall t \in [0, T].
\]
Uncertain Peak Measure Program

Based on optimal control LP of Lewis, Vinter (1980)

\[ p^* = \max p(x, \mu_p) \]
\[ \langle v(t, x, \theta), \mu_p \rangle = \langle v(0, x, \theta), \mu_0 \rangle \langle L_f v(t, x, \theta), \mu \rangle \quad \forall v \]
\[ \langle 1, \mu_0 \rangle = 1 \]
\[ \mu \in \mathcal{M}_+([0, T] \times \mathcal{X} \times \mathcal{\Theta} \times \mathcal{W}) \]
\[ \mu_p \in \mathcal{M}_+([0, T] \times \mathcal{X} \times \mathcal{\Theta}) \]
\[ \mu_0 \in \mathcal{M}_+(\mathcal{X}_0 \times \mathcal{\Theta}) \]

\[ P^* = p^* \text{ when } f \text{ Lipschitz, } [0, T] \times \mathcal{X} \times \mathcal{\Theta} \times \mathcal{W} \text{ compact} \]
Continuous-Time Example

Maximize $x_1$ at order 4 with $w(t) \in [-0.2, 0.2]$

\[ \dot{x}(t) = \begin{bmatrix} -0.5x_1 - (0.5 + w(t))x_2 + 0.5 \\ -0.5x_2 + 1 + \theta \end{bmatrix} \]

\[ \theta = 0 \]

\[ \theta \in [-0.5, 0.5] \]
Distance Uncertainty

Time dependent uncertainty $w(t) \in W \ \forall t \in [0, T]$  

Uncertainty changes Liouville, Distance changes cost
Switching
Switching Dynamics

Closed cover \( \bigcup_{k=1}^{N_s} X^k = X \)

Dynamics \( \dot{x} = f_k(t, x) \) admissible in set \( X^k \)

Per-subsystem occupation measures \( \mu \in \mathcal{M}_+([0, T] \times X_k) \),

\[
\mu = \sum_k \mu_k \\
\langle \mathcal{L}_f v(t, x), \mu \rangle = \sum_k \langle \mathcal{L}_{f_k} v(t, x), \mu_k \rangle
\]
Impulse response, switching between 2 linear systems

Linear Switched System order = 4, peak estimate = 4.265
Discrete-Time Dynamics
Transition dynamics $x_{t+1} = f(x_t)$, $\forall t = 0, \ldots, T - 1$

Liouville equation

$$\langle v(x), \mu_p \rangle = \langle v(x), \mu_0 \rangle + \langle v(f(x)) - v(x), \mu \rangle \quad \forall v$$

Liouville with uncertainty $x_{t+1} = f(x_t, \theta, w_t)$

$$\langle v(x, \theta), \mu_p \rangle = \langle v(x, \theta), \mu_0 \rangle + \langle v(f(x, \theta, w)) - v(x), \mu \rangle \quad \forall v$$

Finite terminal time constraint

$$T \geq \langle 1, \mu \rangle$$
Discrete-Time Example

Discrete dynamics with switching and time-dependent uncertainty

\[ w_t = 0 \]

\[ w_t \in [-0.2, 0.2] \]
Conclusion
Conclusion

Extended peak estimation to problems with uncertainty

Time-independent, time-dependent, and switching processes

Continuous-time and discrete-time dynamics
Acknowledgements

National Science Foundation (NSF)
Air Force Office of Scientific Research (AFOSR)
Artificial and Natural Intelligence Toulouse Institute (ANITI)
Czech Science Foundation (GACR)
Marie Skłodowska-Curie Actions (POEMA)
The Audience

Code: http://github.com/jarmill/peak