Microextensive Chaos of a Spatially Extended System

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(June 4, 2001)

By analyzing chaotic states of the one-dimensional Kuramoto-Sivashinsky equation for system sizes \( L \) in the range \( 79 \leq L \leq 93 \), we show that the Lyapunov fractal dimension \( D \) scales microextensively, increasing linearly with \( L \) even for increments \( \Delta L \) that are small compared to the average cell size of 9 and to various correlation lengths. This suggests that a spatially homogeneous chaotic system does not have to increase its size by some characteristic amount to increase its dynamical complexity, nor is the increase in dimension related to the increase in the number of linearly unstable modes.

An important phenomenon associated with sustained nonequilibrium systems is spatiotemporal chaos, a chaotic dynamical state that is spatially disordered. An open question is how best to characterize spatiotemporal chaos so that theory can be quantitatively compared with experiment and experiment with simulation. Presently, there is no fundamental theory of nonequilibrium systems to indicate the appropriate quantities to measure and so researchers have borrowed ideas from condensed matter physics, fluid dynamics, nonlinear dynamics, and statistics. Commonly used ways to characterize spatiotemporal chaos include critical exponents, the two-point correlation time \( \tau \), the Lyapunov fractal dimension \( D \), the two-point correlation length \( \xi_2 \), the dimension length \( \xi_\delta \) and other lengths. However, calculations have shown that these quantities do not always lead to the same conclusions, e.g., there are systems for which the length \( \xi_2 \) diverges while the length \( \xi_\delta \) remains finite as some parameter is varied. Further research is therefore needed to understand the particular features of spatiotemporal chaos that are measured by any one of these quantities and how these quantities are related to one another.

In the following, we report results that provide new insights about how the dynamical complexity of a nonequilibrium system depends on the volume of the system, and about the interpretation of the dimension length \( \xi_\delta \). In 1982, Ruelle conjectured, and numerical calculations later confirmed, that the dimension \( D \) of a sufficiently large spatially homogeneous chaotic system should increase extensively, i.e., linearly with its volume \( V \). Using an argument similar to that used by Landau and Lifshitz to explain the extensivity of additive quantities in thermodynamics, this extensivity of \( D \) can be understood heuristically as a consequence of spatiotemporal disorder. If two subsystems of a spatiotemporal chaotic system are sufficiently far apart, their coupling is weak because of the disorder and so their dynamics contribute independently and additively to the overall fractal dimension.

This picture of weakly interacting subsystems raises the question of how precisely does the fractal dimension \( D \) increase with increasing system volume in the extensive regime. One possibility is that the curve \( D(V) \) may be linear only on average and has a staircase-like structure, with the steps corresponding to new degrees of freedom that appear once the system volume has increased sufficiently to include a new subsystem. The widths \( \Delta V \) of the steps would then define a length scale \( (\Delta V)^{1/d} \) (where \( d \) is the spatial dimensionality of the system) that would be interesting to compare with the lengths mentioned above (\( \xi_2 \), \( \xi_\delta \), etc.). Possible step-like features in the \( D(V) \) curve might also be associated with the appearance of new linearly unstable modes of the uniform state, since the number of such modes typically increases linearly on average with increasing volume. Another possibility is that the curve \( D(V) \) is extensive only on average but its deviation from linearity is too irregular to characterize by a single length scale. A fourth possibility is that there are no length scales associated with how \( D \) increases with \( V \) and the curve \( D(V) \) is exactly linear for arbitrarily small increases in \( V \), a situation that one could call microextensive chaos. In this case, it would be interesting to understand how the geometric structure of the chaotic attractor in phase space changes with \( V \) so as to produce such an exact linear behavior.

In this paper, we numerically integrate the one-dimensional Kuramoto-Sivashinsky (KS) equation—a widely studied continuum model of spatiotemporal chaos—to investigate how the Lyapunov fractal dimension \( D(L) \) of a homogeneous chaotic system varies with the system size \( L \) for increments \( \Delta L \) that are small compared to the lengths mentioned above (\( \xi_2 \), \( \xi_\delta \), etc). With one exception, all prior numerical studies used increments \( \Delta L \) that were large compared to these lengths and the detailed form of \( D(L) \) was not determined. We show below that, in fact, the Lyapunov fractal dimension \( D \) increases linearly with \( L \) even for system incre-
ments $\Delta L$ that are tiny compared to the average cell size and to various correlation lengths. The spatiotemporal dynamics of the one-dimensional KS equation therefore provides an example of microextensive chaos. We conjecture that this will be a general property of chaotic homogeneous nonequilibria media.

Our calculations of $D$ versus $L$ yield an additional insight, namely that the onset of extensivity in $D$ is not sharp but occurs only asymptotically with increasing $L$, after a sequence of alternating windows of stationary, periodic, intermittent, and chaotic dynamics. (Such alternating windows have been noted before [7] but have not been studied with such fine resolution in $L$ as we do here.) These results suggest the possibility that windows of non-chaotic behavior may persist to arbitrarily large values of $L$ but become too narrow to be detected. If true, then the dimension $D(L)$ may not be a continuous curve and extensive behavior occurs only between the narrow windows of non-chaotic dynamics.

Our results were obtained by numerical integrations of the one-dimensional Kuramoto-Sivashinsky equation in the form

$$\partial_t u(t, x) = -\partial_x^2 u - \partial_x^4 u - u \partial_x u, \quad x \in [0, L],$$

on an interval of length $L$, with rigid boundary conditions $u = \partial_x u = 0$ at $x = 0$ and at $x = L$. (Fig. 1 shows a chaotic and periodic state for $L = 50$ and 54 respectively.) The spatial derivatives were approximated by second-order-accurate finite differences on a uniform spatial mesh, and a standard operator-splitting method was used for the time integration [8]. For given initial conditions and interval length $L$, we used the Kaplan-Yorke formula [9] to calculate the Lyapunov fractal dimension $D(L)$ in terms of all of the positive and some of the negative Lyapunov exponents $\lambda_i$. These exponents were obtained using a standard algorithm [10] in which many copies of the linearized KS equation were integrated, each with their own initial condition.

The demonstration of microextensive scaling by the above numerical methods was delicate since the Lyapunov exponents $\lambda_i$ and so $D$ converge noisily and slowly [21] toward their infinite time limits. As the increment $\Delta L$ in system size became smaller, the corresponding increment in dimension $\Delta D$ was more difficult to determine since $\Delta D$ became comparable to the fluctuations in the dimension curve $D(T)$ as a function of integration time. The exponents $\lambda_i$ and dimension $D(L)$ were also sensitive to the values of the spatial resolution $\Delta x$ and temporal resolution $\Delta t$, to the renormalization time $T_{\text{norm}}$ for the Lyapunov vectors, and to the total integration time. For nearly all runs reported below, we used values of $\Delta x = 0.167$, $\Delta t = 0.025$, and $T_{\text{norm}} = 10$ and confirmed the correctness of the corresponding results by comparing the values with spatial and temporal resolutions up to four times larger and for integration times as long as $10^6$ time units.

We now turn to our results. Our starting point was the pioneering calculation of Manneville [11], who used numerical integrations of Eq. (1) with rigid boundary conditions to demonstrate for the first time that the fractal dimension $D$ scaled extensively with the system size $L$. For $L \geq 50$, he found that $D = 0.230L - 2.70$, which implies a dimension length

$$\xi_\delta = \frac{dD}{dL}^{-1} = \frac{1}{0.230} \approx 4.4$$

This length is somewhat smaller than the average cellular size $\lambda = 2\pi/\eta_{\text{max}} = 2\sqrt{2}\pi \approx 8.8$ corresponding to the fastest growing linear mode $\eta_{\text{max}} = 1/\sqrt{2}$. Based on these results, we chose to calculate the fractal dimension $D(L)$ over the range $50 < L < 100$ in constant increments $\Delta L = 0.5$ that were much smaller than these lengths. In contrast, the smallest increment used by Manneville was $\Delta L = 50$ for which the fractal dimension changes by about 12.

Manneville’s linear dependence of $D$ on $L$ suggested that for $L \geq 50$, only spatiotemporal chaos exists. In contrast, we find that there is a complicated sequence of different dynamical states over the range $50 < L \leq 75$ and then only chaotic states for $75 < L < 93$ [23]. Fig. 2 summarizes our results for the range $50 \leq L \leq 75$ by plotting the period of each state as a function of $L$. We observe four kinds of states: fixed points, time-periodic states, chaos, and intermittent states in which one kind of time dependence alternates irregularly with a different kind of time dependence. In most cases, these different categories were easily identifiable to the eye by looking at time series. To combine all the results on a single plot, chaotic states were arbitrarily assigned a period of -200, intermittent states a period of -100, fixed points a period of 0, and periodic states a direct estimate of their period based on repeating features of the time series.

There are two interesting features of the dynamical states of Fig. 2 in addition to the unexpected occurrence of many windows of alternating dynamics for this range of $L$. First, we found that for a given system size $L$, numerical integrations using up to seven different random initial conditions (each consisting of uniformly distributed numbers in the interval $[-0.1, 0.1]$) led to only one state. Thus empirically there seems to be only one basin of attraction for each system size and we do not expect hysteresis in the range $50 < L < 75$. Second, we found rather remarkably that the fractal dimension $D$ of each chaotic state in Fig. 2 lay on Manneville’s extensive curve $D(L) = 0.230L - 2.70$ with $D \geq D(50) = 8.8$ (A least-squares fit of our chaotic states gave the almost identical curve $0.227L - 2.85$). Thus the states jump abruptly from low-dimensional $D = 1$ periodic states to high-dimensional chaotic states that are scaling extensively with the system size. We did not try to characterize the intermittent states, e.g., by their fractal dimension or by the scaling properties of the fractional duration of a particular phase [21].

Over the range $78 < L < 93$, only chaotic states were observed. Fig. 3 shows that the corresponding values
of the Lyapunov fractal dimension $D$ lie on a straight line that has the same slope (to two digits) and intercept as that found by Manneville over the much larger range $50 < L < 400$. Thus the fractal dimension shows microextensive scaling: a linear dependence on $L$ for system increments $\Delta L = 0.5$ that are much smaller than any characteristic length scale such as the average cell size or various correlation lengths. Given the similar result obtained by Xi et al [13] for a different mathematical model of spatiotemporal chaos, we conjecture that microextensive scaling will be a general feature of spatiotemporal chaos in sufficiently large, approximately homogeneous nonequilibrium systems.

The linear behavior of Fig. 3 rules out a simple relation between the Lyapunov fractal dimension and the number of linearly unstable modes. For both periodic and rigid boundary conditions, the wave numbers are quantized in units of $2\pi/L$ and so the number of unstable modes increases linearly as $L/\pi \approx 0.32L$, but in discrete jumps when $L$ changes by about $\pi$. Over the range $78 < L < 93$, we would expect $(98 - 78)/\pi \approx 4$ new linearly unstable modes to appear but there are not correspondingly four step-like features in the figure.

Fig. 3 shows how the first sixteen exponents $\lambda_i$ vary with system size $L$ over the same range of system sizes as Fig. 2. The largest exponent $\lambda_1$ is approximately independent of $L$ and so is an intensive quantity; an increase of system size therefore does not change the forecasting time $1/\lambda_1$ over which the future behavior of the field $u$ becomes unpredictable, despite the fact that the fractal dimension is becoming correspondingly larger. The other exponents increase roughly linearly with increasing $L$, with the negative exponents having the largest slope and showing substantial deviations from a strictly linear dependence on $L$. Comparing Fig. 3 with the Kaplan-Yorke formula for the dimension

\[ D = K + \frac{1}{|\lambda_{K+1}|} \sum_{i=1}^{K} \lambda_i, \tag{2} \]

where $K$ is the largest integer such that $\sum_{i=1}^{K} \lambda_i \geq 0$, we see that the linear behavior of $D$ with $L$ in Fig. 3 is a subtle consequence of how the $\lambda_i$ vary with $L$.

In conclusion, we have demonstrated the occurrence of microextensive scaling of the Lyapunov fractal dimension with system size for the one-dimensional Kuramoto-Sivashinsky equation with rigid boundary conditions. This suggests that a spatially extended nonequilibrium dynamical system does not have to increase its volume by some minimal amount for the fractal dimension to increase. Correspondingly, the dimension length $\xi$ does not have some direct physical meaning as the characteristic size of a dynamical subsystem, it is simply determined by the linear growth of $D$ with $L$. Our calculations suggest two questions for further exploration. One is whether there is a cutoff system size $L_c$ above which only chaotic solutions are found for the KS eq. Second is to understand mathematically how the geometry of a strange attractor changes with system size $L$ such that that $D$ varies exactly linearly with $L$.

We thank Scott Zoldi for useful discussions and James Gunton for informing us of his related calculations on the Nicolaevski model. This work was supported by NSF grant DMS-9722814 and DOE grant DE-FG02-98ER14892.

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[22] It is interesting to note that the number of positive Lyapunov exponents increases more slowly than the dimension $D$ as $N_p = 0.14L - 1.5$, while the number of unstable modes increases more rapidly as $N_u \approx 0.32L$. [16]

[23] As summarized in Manneville’s review article [17], others have studied states and their bifurcations for the 1d KS equation with periodic boundary conditions over about the same range in length. A different sequence of different dynamical behaviors is observed compared to what we find in Fig. 2.

FIG. 1. Space-time evolution of the field $u(t, x)$ for two states of the Kuramoto-Sivashinsky equation Eq. (1) with rigid boundary conditions. The space-time resolution was $\Delta t = 0.025$ and $\Delta x = 0.166$ and the peak-to-peak amplitude is about 4. (a): Chaotic state for $L$ = 50. Spatial curves are plotted every $\Delta T = 1$ time units starting at time $t = 50,000$. (b): A periodic state for $L$ = 54 with period $\tau = 127.6$. Spatial curves are plotted every $\Delta T = 5$ time units starting at time $t = 80,000$.

FIG. 2. Periods of numerical solutions to Eq. (1) versus the system length $L$. Each integration was started from small-amplitude random initial condition and then integrated 500,000 time units. There is a complex sequence of windows corresponding to chaotic, constant, intermittent, and periodic dynamics. Chaotic and intermittent solutions have been assigned an arbitrary period of -200 and -100 respectively so that all the data could be compared on one plot.
FIG. 3. The Lyapunov fractal dimension $D$ of chaotic solutions to Eq. (1) versus system size $L$ for $79 \leq L \leq 93$. The dimension values accurately fall on a straight line, demonstrating the occurrence of microextensive scaling. The straight line $D = 0.227L - 2.85$ was obtained by a least-squares fit to the points and agrees well with Manneville’s result $D = 0.230L - 2.70$ over the much larger range $50 < L < 400$. The error bar for each point corresponds to a relative error of at most 0.05% in $D$. The error bar was determined by the peak-to-peak fluctuations of $D$ versus integration time $T$.

FIG. 4. The sixteen largest Lyapunov exponents $\lambda_1 > \cdots > \lambda_{16}$ versus system length $L$ for the same window of chaotic solutions discussed in Fig. 3. For each value of $L$, the $\lambda_i$ were estimated by estimating the asymptotic slope of the $\lambda_i(t)$ versus $t$ curve for $20,000 < t < 200,000$. All Lyapunov exponents increase with $L$ in this regime except $\lambda_1$, which is approximately independent of $L$ and so is an intensive quantity. The lines through the points for a given $\lambda_i$ are a guide to the eye and were determined by a least-squares fit. The exponents $\lambda_i$ increases approximately, but not exactly, as linear functions of $L$. 

The Lyapunov exponents $\lambda_i$ increased approximately, but not exactly, as linear functions of $L$. 

The error bar for each point corresponds to a relative error of at most 0.05% in $D$. The error bar was determined by the peak-to-peak fluctuations of $D$ versus integration time $T$. 

Lyapunov Exponents

$\lambda_1 > \cdots > \lambda_{16}$