1. INTRODUCTION

We’ve been working on the crackling noise in hysteresis loops [1, 2, 3, 4, 5, 6]. Hysteresis occurs when you push and pull on a system with an external force, and the response lags behind the force. The hysteresis loop is the graph of force (say, an external magnetic field $H$) versus the response (say, the magnetization $M$ of the material). In many materials, the hysteresis loop is not actually microscopically smooth: it is composed of small bursts, or avalanches (figure 1). In many first-order phase transitions, these bursts cause acoustic emission (crackling noise); in magnets, they are called Barkhausen noise.

Naturally, these pulses are associated with some kind of inhomogeneity or disorder in the material. Magnetic tapes are composed of small grains of iron oxide, and individual grains (when small enough) flip over as a unit, leading to a pulse in the magnetization. However, the pulses observed have a wide range of sizes: they can range over three to six decades in size in a typical experiment (figure 2). Since the grains in the material don’t come in such a variety of sizes, one can conclude that many grains must be flipping at once, coupled together in a kind of avalanche.

Having events of all sizes is not trivial! If the coupling between grains is weak compared to the disorder, the grains will tend to flip independently, leading to small avalanches. If the coupling is strong, a grain which flips will give a
Fig. 1. — Left: A hysteresis loop in our model, showing the subloops. If you look carefully, you should be able to see small irregularities in the curve: these correspond to the avalanches causing the crackling noise. Right: The pulses in the upper branch of the outer hysteresis loop. Notice that the pulses become larger near $H = 1$ where the magnetization changes fastest with external field.

Fig. 2. — Left: The avalanche size distribution for our model in three dimensions, for disorders $R = 4, 3.2, 2.6$, and 2.25. The dashed line shows the expected behavior at the critical disorder $R_c \sim 2.16$ (a power law, $D_{int}(s,R_c) \propto s^{-(\sigma+\beta\delta)}$). The curves are cut off after a few decades of scaling, at a size $S_{cut\_ff} \propto (R-R_c)^{-1/\sigma}$. Right: The jump in the avalanche size distribution ends at the critical disorder $R_c = 2.16$. The size of the jump scales as $\Delta M \propto (R-R_c)^{\beta}$. At $R_c$, the magnetization has a power-law form $H(M) \propto (M-M_c)^{\delta}$.

large kick to its neighbors, likely flipping several of them, who will flip several more — leading to one large avalanche. On the right in figure 2, we see that this is precisely what happens in our model. The large range of avalanches is associated with a critical value of the disorder $R_c$ relative to the coupling: when the avalanches can’t decide whether to be huge or small, they come in all sizes!
2. THE MODEL

To model these magnetic systems, we use a lattice of “spins” $S_i = \pm 1$, pointing up or down: each spin represents a domain or particle in the material. They are attached to their nearest neighbor (n.n.) spins by bonds of uniform strength $J$ which we set for convenience to one; they are coupled to an external field $H(t)$ which we sweep from $-\infty$ to $\infty$. Finally, we model the inhomogeneities in the material with a random bias for up or down: at each site, we pick a random field $h_i$, distributed with a normal probability distribution $P(h) = \exp\left(-\frac{h^2}{2R^2}\right)/\sqrt{2\pi}R$. We call $R$ the disorder: large $R$ makes the distribution of $h_i$ broad, makes the coupling between spins unimportant, and leads to smooth hysteresis loops and small avalanches. The energy of our system thus is

$$H = -\sum_{ij \text{ n.n.}} JS_i S_j - \sum_i (H(t) + h_i) s_i. \quad (1)$$

Each spin flips as soon as its local external field $J\sum_{n.n.} s_j + H(t) + h_i$ changes sign: it then can kick over its neighbors if the resulting $2J$ change in their local fields is big enough. Thermal (and quantum) fluctuations aren’t important for us, because the particles are large and (often by design) don’t flip over spontaneously. Our model is called the random-field Ising model, and we simulate it at zero temperature [7, 8].

3. THE CRITICAL EXPONENTS

It’s a remarkable fact that models like ours can accurately describe real systems near their transitions. The basic idea is a lot like hydrodynamics. All kinds of fluids look alike at long lengths and times, apart from their viscosity, density, and surface tension: despite rather different molecular structures and interactions, they all are described by the same equations for long distances and long times compared to the atomic scales. Similarly, a variety of hysteretic systems near the onset of a big jump (the “infinite avalanche”) should be quantitatively describable by our simple model. This amazing property is called universality, and the family of models with common descriptions is called the universality class.

The most commonly measured universal quantities are the critical exponents. Many properties near the critical point have power-law scaling. This can be understood as a kind of self-universality: the system on one scale is quantitatively described by the same system at a different scale. Thus the properties of the system become scale invariant, and (in the usual way) develop power laws.

There are several critical exponents for our system that we measure and calculate. The most common is the power law giving the number of avalanches of a given size. This power law depends on whether you count all the avalanches in the hysteresis loop (the integrated avalanche size distribution plotted on the
left in figure 2, whose power law is \( \tau + \sigma \beta \delta \), or just measure them near the incipient jump in \( M(H) \) (near which point the power law is \( \tau \)). The exponents \( \beta, \delta, \text{ and } \sigma \) are also described in figure 2: they describe the shapes of the hysteresis loops and the cutoff in the avalanche size distribution as the disorder \( R \) is varied. The exponent \( \nu \) describes the dependence of the size of the large avalanches on the distance from the critical point; the exponent \( z \) describes the lifetime of the large avalanches.

The first two columns of the left of figure 3 compare the two avalanche-size power-laws with experiments on a variety of materials [9]. While the scatter is large, the theory is well within the range of exponents measured. That doesn’t mean we know the experiments are described by our theory: there might well be other universality classes with exponents not so far from ours... The other columns on the left of figure 3 represent combinations of exponents derived from different kinds of measurements: for example, the power spectrum of the noise from the hysteresis loops.

On the right-hand side of figure 3 we see the critical exponents in different spatial dimensions. Two dimensions might describe the behavior of magnetic tapes. Of course, dimensions greater than three have only theoretical interest! We’ll see that we can learn something from high dimensions anyway...

4. THE EPSILON EXPANSION

What justifies us in thinking that our exponents and scaling is universal? How can we explain why we expect many systems to have exactly the same critical exponents (and scaling functions, and amplitude ratios, ...)? The theoretical justification of this was given by Leo Kadanoff, Ken Wilson, and Michael Fisher
in the early 1970’s, using what is called the renormalization group [11]. It’s a bit technical, but theorists get unhappy unless they can point to a difficult calculation underlying their work.

The basic idea of the renormalization group is to think of the process of rescaling a system as a mapping from the space of all systems to itself! In statistical mechanics, one describes a system with a Hamiltonian: coarse-graining from one length scale to another maps Hamiltonian space into itself. In studying the period-doubling route for the onset of chaos, one describes the system with a mapping: rescaling the time by a factor of two is done by composing the mapping with itself. In our problem, we write a path integral for the probability of all histories for the system, and consider the effect of coarse-graining from one length-scale to another as a mapping taking one path-probability functional onto another [11].

This leap of abstraction is more useful than you might imagine. The subspace of all systems at their critical points must map onto itself. (If a system is on the verge of having an infinite avalanche, looking at it on a longer length scale won’t change that.) Suppose one of the critical systems is a fixed-point under the mapping: suppose it has a “basin of attraction” of critical systems which flow towards it under coarse-graining. Then, on long length scales, all of these systems should look like the fixed point: the basin of attraction becomes the universality class.

Fig. 4. — Left: Schematic phase diagram for our model, with arrows showing flows under coarse-graining. The dark line is $H_c(R)$, the external field at which the infinite avalanche occurs when the system is swept upwards from $H = -\infty$. Under coarse-graining, the effective external field $h = (H - H_c)/H$ grows fastest, and the effective disorder $r = (R - R_c)/R$ grows more slowly: all other directions are stable under coarse-graining. Right: Flow diagram near six dimensions, showing the mean-field fixed-point and the new, Wilson-Fisher fixed point. The mean-field fixed point is unstable below $d = 6$; at six dimensions, these two points merge.
Karin Dahmen [2, 4] figured out how to implement this in a tangible calculation. In dimension $d$, each spin on a hypercubic lattice has $2^d$ neighbors: in high dimensions, there are so many neighbors that one may just as well assume that every spin sees an average environment. This mean-field theory can be solved; it has a transition where the infinite avalanche first occurs, and that point leads to a fixed-point under the rescaling. In high dimensions (for us, dimensions $d > 6$) this fixed point is stable: all systems have the same critical properties as the mean-field fixed point. As the dimension decreases below $d = 6$, the mean-field fixed-point becomes unstable, but Karin did a perturbation theory in the dimension $\epsilon = 6 - d$ to find the new fixed-point to first order in $\epsilon$ (figure 4). The calculation is completely analogous to Ken Wilson’s original calculation of the thermal, pure Ising model in $d = 4 - \epsilon$.

The results of Karin’s calculation are shown in the right-hand part of figure 3. It’s nice to see that the critical exponents approach their mean-field values as $d$ approaches six, and that the $\epsilon$-expansion captures the first corrections rather well. Indeed, the method works amazingly well, considering $\epsilon \geq 3$ in realistic problems! It so happens that the analogy to Ken Wilson’s calculation is embarrassingly good: our calculation agrees with his to all orders in $\epsilon$. Other people have calculated terms up to order $\epsilon^5$ for the exponent $\nu$ in the pure, thermal Ising model [10]; we use their coefficients (in two higher dimensions) to predict $1/\nu$ in figure 3 right. There is one subtlety: the series for $\nu(\epsilon)$ doesn’t converge without help (it’s an asymptotic series), so we plot three different Borel resummations of the series for $1/\nu$ [10].

The fact that our series in $6 - \epsilon$ maps to Wilson’s calculation in $4 - \epsilon$ is embarrassing for another reason. Our values for $\nu$ in $d = 3$ definitely do not agree with the pure, thermal Ising model in $d = 1$! This caused great anguish when it was first discovered in another context [10]. It’s of course possible that we’ve erred in trying to perturb in a discrete variable like the dimension. The method seems to be working rather well, though, from figure 3. I personally believe it has something to do with the fact that the series doesn’t converge for either problem: maybe we have one series trying to describe two different functions [3, 4]?

5. WIDOM SCALING

It’s important to stress that critical exponents are not the only predictions of the theory. I’d like to briefly discuss Ben Widom’s discovery (later explained using the renormalization group): data for systems near criticality can be collapsed onto one another.

Consider again the avalanche size distribution (left figure 5, the same data as in left figure 2). Notice how the curves never quite lie along the dashed line, which I claimed was the power-law you would see if you were exactly at the critical point. Even when there are avalanches of size $10^6$, the system still isn’t exhibiting the critical exponents predicted! No wonder the experiments on the
left of figure 3 don’t agree with our theory: if you fit a power law to three
decades of data, the exponent you get depends on $R - R_c$ and on which three
decades you measure.

Does this mean our theory is useless? Not at all: our theory not only predicts
the power laws, it also predicts the shape of the curves and the way they cut off.
In particular, the avalanche size distribution is predicted to have the following
form as $r = (R - R_c)/R \to 0$:

$$D_{int}(S, R) \sim S^{-(\tau + \sigma \beta \delta)} D_{int}(s^\sigma r). \quad \text{(2)}$$

At the critical point $r = 0$: so long as $D(0) \neq 0$, the distribution is a pure
power law with power $\tau + \sigma \beta \delta$. Near the critical point, it starts to deviate
from a pure power law when the argument of $D$ becomes near one — that is,
when the avalanche size $s \sim r^{-1/\sigma}$. Usually, the scaling function $D$ is constant
for small arguments, and dies away exponentially for large arguments. Again,
we emphasize that the whole function $D(x)$ is a universal property just as the
critical exponents are.

Fig. 5. — Left, main: The avalanche size distribution for our model in three dimen-
sions, for disorders $R = 4, 3.2, 2.6,$ and $2.25$. The smooth curves going through the
data are the scaling predictions of the theory. The dashed line shows the expected
behavior at the critical disorder $R_c \sim 2.16$. Left, inset: The scaling collapses of these
curves. The reason the slope of the avalanche size distribution converges so slowly
is the large bump in this curve: it grows by an order of magnitude from its value at
zero to the peak value. Right: The universal scaling curves in different dimensions.
The explanation for the large bump in three dimensions is that the scaling curve
$D(0) = 0$ in two dimensions. We believe the exponent for the decay of the avalanche
size distribution in two dimensions is shifted by one, because of this zero [5].

We can make a plot of the scaling function, by taking our data and rescaling
it:

$$D_{int}(s^\sigma (R - R_c)/R) \sim S^{\tau + \sigma \beta \delta} D_{int}(S, R). \quad \text{(3)}$$
That is, if you plot $S^{\tau+\sigma\beta}D_{\text{int}}(S, R)$ versus $s^\sigma(R - R_c)/R$, data taken at different R will all collapse onto the same curve. This is what Widom discovered in the early days of critical phenomena. The data collapse is shown in the inset to the left side of figure 5. The theory tells us further that any other system governed by the same universality class (one hopes someday a real experiment) will also rescale onto this same curve.

Now we can understand why our pure power-law is so elusive. The scaling curve almost vanishes at $S^\sigma r = 0$: it rises by about an order of magnitude before dying exponentially. Indeed, each of the sets of data shows a bulge of about a factor of ten above the pure power law. (Why is the bulge so big? The right-hand side of figure 5 is our explanation: it’s because we’re so close to two dimensions, where $D_{\text{int}}(0) = 0$.) We can work backward from the scaling curve and make predictions for the avalanche size distribution for each of the different disorders: the smooth curves in figure 5 are predictions of the scaling theory. The power law still isn’t useful at $R = 2.25$, where $r = 0.04$, but the complete Widom scaling prediction is quite successful all the way out to $R = 4$, almost twice the critical value.

6. CONCLUSION

So, we have an understanding of why the noise pulses in magnets can span such a range of scales: they are near a critical point where the hysteresis loop develops a jump. We have a scaling description of the behavior near the critical point. We have a rough explanation of the experimental observations. We also think we understand why the measurements might fluctuate so far from our predictions: instead of varying a parameter and doing a scaling collapse, the experiments only measure an effective power-law. We recommend trying to get closer (or farther) from the critical point.

Why do the experiments see power laws? That is, why are the samples they use near the critical point? Unlike more traditional phase transitions, our model has a large critical range: 4% away from the critical point we have six decades of scaling, and a factor of two away we still have two decades. Perhaps there are mechanisms which tune the system precisely to the critical point, but it seems likely that the experimentalists could just be lucky and pick their sample inside this large range.

Finally, why is our critical range so large? This technically doesn’t have a clean answer: the size of the critical range isn’t a universal property! Universal questions aren’t the only important ones, of course. I think there are three contributing factors. (1) The critical exponent $\nu = 1.42$ in our model, where in the three-dimensional Ising model it is 0.63. That means that getting twice as close to $R_c$ makes the length spanned by an avalanche grow by a factor of 2.7, where getting twice as close to $T_c$ for the Ising model makes the correlation length grow only by a factor 1.55. (2) The size $S$ of an avalanche is more like a volume than a length. Six decades of scaling in $S$ should be thought of
as roughly two decades in length scale. (Actually, since the avalanches aren’t
space filling in three dimensions, the volume $S \sim \xi^{1/\sigma} \sim \xi^{2.6}$, so six decades
in size $S$ gives 2.3 decades in the length scale $\xi$.) (3) Three dimensions is
close to two dimensions. As you can see from the right side of figure five, the
behavior in three dimensions is far removed from the mean-field behavior of the
model in six and higher dimensions. The fluctuations are extremely important;
in two dimensions, we believe that they almost completely dominate, perhaps
even preventing an infinite avalanche from ever occurring! If $R_c = 0$ in two
dimensions, then the critical regime must span to $\infty \times R_c$; no wonder in three
dimensions that it spans to $2 \times R_c$.

There are still many important unsolved problems here, but it’s clear that
the traditional methods of critical phenomena — Widom scaling, the renor-
malization group, and the $\epsilon$ expansion — have been remarkably useful.

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lation, in Java) is available at
http://www.lassp.cornell.edu/sethna/hysteresis/hysteresis.html.

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