Categorified skew Howe duality and comparison of knot homologies

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Abstract. In this paper, we show an isomorphism of homological knot invariants categorifying the Reshetikhin-Turaev invariants for \( \mathfrak{sl}_n \). Over the past decade, such invariants have been constructed in a variety of different ways, using matrix factorizations, category \( \mathcal{O} \), affine Grassmannians, and diagrammatic categorifications of tensor products.

While the definitions of these theories are quite different, there is a key commonality between them which makes it possible to prove that they are all isomorphic: they arise from a skew Howe dual action of \( \mathfrak{gl}_\ell \) for some \( \ell \). In this paper, we show that the construction of knot homology based on categorifying tensor products (from earlier work of the second author) fits into this framework, and thus agrees with other such homologies, such as Khovanov-Rozansky homology. We accomplish this by categorifying the action of \( \mathfrak{gl}_\ell \times \mathfrak{gl}_n \) on \( \wedge^p (\mathbb{C}^\ell \otimes \mathbb{C}^n) \) using diagrammatic bimodules. In this action, the functors corresponding to \( \mathfrak{gl}_\ell \) and \( \mathfrak{gl}_n \) are quite different in nature, but they will switch roles under Koszul duality.

1. Introduction

The field of knot homology has seen remarkable development over the past 15 years, since Khovanov’s groundbreaking categorification of the Jones polynomial [Kho00]. This paper, and the important work that came after it, led naturally to the question of which other knot invariants had similar categorifications. One natural generalization is provided by the Reshetikhin-Turaev invariants of type A quantum groups. These seem to be in an accessible middle ground, with a considerably more complex structure than Khovanov homology, but using more familiar tools and technology than the categorifications of general Reshetikhin-Turaev invariants defined by the second author [Webb]. In fact, the program of categorifying these invariants has been arguably too successful, in that a great variety of approaches led to defining different homology theories, but it has proved easier to give definitions than to prove equivalence with the other definitions.

- One theory was defined using matrix factorizations by Khovanov and Rozansky [KR08] for the vector representation, with an extension to arbitrary fundamentals by Wu [Wu] and Yonezawa [Yon].
One theory was defined using foams by [MSV09] for the vector representation, building on work in the case of $\mathfrak{sl}_3$ by Khovanov [Kho04]; this was shown to match Khovanov-Rozansky. Much later, a version of foams using only local relations and allowing the use of arbitrary fundamentals was given by Queffelec and Rose [QR].

One theory was defined using the categories of coherent sheaves on convolution varieties for the affine Grassmannian by Cautis and Kamnitzer [CK08a, CK08b].

One theory was defined using category $\mathcal{O}$ in type A by Mazorchuk-Stroppel [MS09] for the vector representation and by Sussan for arbitrary fundamentals [Sus07].

One theory $\mathcal{K}_n$ was defined using diagrammatic categorifications of tensor products by the second author [Webb].

Luckily, there has been excellent progress on resolving this quandary in recent years. Cautis (building on his previous work with Kamnitzer) gave a “universal” construction of type A knot homologies based purely on the structure of the action of the categorified quantum group $\mathfrak{sl}_\infty$ on a categorification of its Fock space representation. All the disparate theories mentioned above can be rephrased in terms of such a categorical action of $\mathfrak{sl}_\infty$.

Our aim in this paper is to explain how this is done for the last theory mentioned. This gives us the result:

**Theorem A.** The knot homology $\mathcal{K}_n$ for framed links labeled with representations of $\mathfrak{sl}_n$ coincides with that defined by Cautis in [Caua, 9.3], and thus with all other invariants mentioned above.

We show this theorem by understanding the relationship between Cautis’s construction (and thus other constructions based on skew Howe duality, such as the recent work of Queffelec and Rose [QR]) and the approach of [Webb], based on the direct categorification of tensor products and the maps between them. In the latter paper, the second author shows that given a sequence $p_i \in [1, n]$, we can define a categorical representation of $\mathfrak{sl}_n$ sending the weight $\mu$ to the finite dimensional representations of an algebra $T^{p_1, \ldots, p_{\ell}}_\mu$ with the Grothendieck group of the sum being $\bigwedge^{p_1} \mathbb{C}^{n} \otimes \cdots \otimes \bigwedge^{p_{\ell}} \mathbb{C}^{n}$.

The skew Howe duality approach requires thinking about several of these tensor products at once. Its most basic observation is that the vector space $\bigwedge^{p} (\mathbb{C}^{\ell} \otimes \mathbb{C}^{n})$ has a natural $\mathfrak{sl}_\ell \times \mathfrak{sl}_n$-action, and as an $\mathfrak{sl}_n$-module, we have an isomorphism

$$\bigoplus_{p_1 + \cdots + p_{\ell} = p} \bigwedge^{p_1} \mathbb{C}^{n} \otimes \cdots \otimes \bigwedge^{p_{\ell}} \mathbb{C}^{n} \cong \bigwedge^{p} (\mathbb{C}^{\ell} \otimes \mathbb{C}^{n}).$$

Defining the $\mathfrak{sl}_\ell$-action is highly non-trivial, and requires making one very serious sacrifice: we can no longer consider just the abelian category of $T^{p_1, \ldots, p_{\ell}}_\mu$-modules, but have to work with its bounded derived category.
**Theorem B.** There is a categorical $\mathfrak{sl}_\ell \times \mathfrak{sl}_n$-action which associates $D^b(T_{p_1,\ldots,p_\ell}^{\mu} - \text{mod})$ to the weight $(\mathbf{p}, \mu)$ if $p_1 + \cdots + p_\ell = p$ and the zero category to all other weights. This action categorifies the skew Howe dual action on $\bigwedge^\ell (\mathbb{C}^\ell \otimes \mathbb{C}^n)$.

This theorem has an obvious asymmetry: the $\mathfrak{sl}_\ell$ and $\mathfrak{sl}_n$ action are quite different. For example, the latter action is by exact functors and the former not. However, there is a non-obvious symmetry exchanging these two actions: the algebras $T_{p_1,\ldots,p_\ell}^{\mu}$ and $T_{m_1,\ldots,m_n}^{\nu}$ are Koszul dual since it is shown in [Webb, Prop. 9.11] that these are equivalent to blocks of parabolic category $O$ for $\mathfrak{gl}_n$ whose duality is proven by a result of Backelin [Bac99]. The corresponding equivalence of derived categories interchanges these two actions. This fact will not be surprising to readers familiar with the theory of category $O$, since the equivalence to blocks of $O$ sends the $\mathfrak{sl}_\ell$-functors to Zuckerman functors and the $\mathfrak{sl}_n$-functors to translation functors. The interchange of these under Koszul duality is proven by Ryom-Hansen [RH04]. We hope that the diagrammatic action of $\mathfrak{sl}_\ell$ we have described will be of some interest to representation theorists as a “hands-on” understanding of Zuckerman functors. We also intend it to point the way to understanding foam categories attached to other Lie algebras with well-understood spiders. The strategy it suggests is to define bimodules over tensor product algebras in other types attached to diagrams in the spider, and foams which correspond to morphisms between these bimodules (in the derived category), and relations between these foams that match the relations of these morphisms.

In general, we have striven to write this paper dealing solely with diagrammatics and not appealing too often to facts about category $O$, but using a couple of facts (the $t$-exactness of Zuckerman functors in the linear complex $t$-structure, and the “Struktursatz” of Soergel) considerably reduces the number of equalities we have to check, and we gave in to temptation on these points.

Now let us summarize the content of the paper. In Section 2, we discuss the necessary background for the paper from higher representation theory. Then we turn in Section 3 to constructing the bimodules needed for the $\mathfrak{sl}_\ell$-action, and probing their basic properties, including the connection to category $O$. In Section 4, we construct the action of Theorem 3 using the results from category $O$ mentioned above to reduce to checking the relations in a particular representation. In fact, this representation matches that defined by Khovanov and Lauda on the cohomology of flag varieties. Finally, in Section 5, we apply these results to knot homology to prove Theorem A.

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2. Background

2.1. Skew Howe duality and webs. One of the key ingredients in this paper will be a $q$-deformed version of skew Howe duality, which we discuss in this section. Let $n \in \mathbb{Z}_{>1}$ be arbitrary but fixed for the rest of this subsection and $\hat{\mathcal{U}}_q(\mathfrak{gl}_n)$ denote the idempotented version of the quantum algebra over $\mathbb{Q}(q)$ associated to $\mathfrak{gl}_n$. Let $C_q^n$ be the $q$-deformation of the vector representation of $\mathfrak{gl}_n$ and $\wedge^a q C_q^n$ the $a$-fold $q$-deformed wedge product. Fix another integer $\ell \in \mathbb{Z}_{>1}$. For any $n$-bounded $\mathfrak{gl}_\ell$ weight $p = (p_1, \ldots, p_\ell)$, i.e. $p_i \in [0, n]$ for all $i \in [1, \ell]$, we let $\wedge^p q C_q^n := \wedge^{p_1} q C_q^n \otimes \cdots \otimes \wedge^{p_\ell} q C_q^n$. Denote the set of all such $p$ by $\Gamma^n_\ell$ and let $|p| := p_1 + \ldots + p_\ell$. By quantum skew Howe duality, there is a surjective homomorphism

$$\hat{\mathcal{U}}_q(\mathfrak{gl}_\ell) \rightarrow \bigoplus_{p, p' \in \Gamma^n_\ell, |p| = |p'|} \text{Hom}_{\hat{\mathcal{U}}_q(\mathfrak{gl}_n)} \left( \wedge^p q C_q^n, \wedge^{p'} q C_q^n \right).$$

The kernel of this map is the ideal generated by all $\mathfrak{gl}_\ell$ weights which are not $n$-bounded. The quotient of $\hat{\mathcal{U}}_q(\mathfrak{gl}_\ell)$ by this kernel is denoted $\hat{\mathcal{U}}_n q (\mathfrak{gl}_\ell)$. This is all proved in [CKM14, Thm. 4.4.1]. Recall that both algebras above can be seen as categories whose objects are the weights, and the homomorphism above as a linear functor.

The elements of $\text{Hom}_{\hat{\mathcal{U}}_q(\mathfrak{gl}_n)} \left( \wedge^p q C_q^n, \wedge^{p'} q C_q^n \right)$ can be represented by $\mathfrak{gl}_n$ webs, whose definition we now briefly recall. We only need a restricted class of webs for our purposes, called $n$-ladders. The basic building blocks for these are the webs denoted $Y$ and $Y^*$:

$$Y := \begin{array}{cc} a & b \\ \hline & a+b \end{array} \quad Y^* := \begin{array}{cc} a & b \\ \hline & a \end{array}$$

with $a, b, a + b \in \{0, \ldots, n\}$. Let $p \in \Gamma^n_\ell$. We use $Y$ and $Y^*$ to define two basic $n$-ladders with $\ell$ uprights:

$$\begin{array}{c} p_i - c \\ \hline \vdots \end{array} \quad \begin{array}{c} \vdots \\ \hline p_i + 1 + c \end{array} \quad \begin{array}{c} p_i + c \\ \hline \vdots \end{array} \quad \begin{array}{c} \vdots \\ \hline p_i + 1 - c \end{array}$$

$$\begin{array}{cc} p_i \\ \hline c \end{array} \quad \begin{array}{cc} \vdots \\ \hline p_{i+1} \end{array} \quad \begin{array}{cc} c \\ \hline \vdots \end{array} \quad \begin{array}{cc} \vdots \\ \hline p_{i+1} \end{array}$$

In the figure we have suppressed $\ell - 2$ uprights (vertical strands from bottom to top). The subscript $i$ indicates that the rung is between the $i$th and the $i+1$st upright, with $i \in [1, \ell - 1]$. Again, we require all labels to be in the range from 0 to $n$. In practice, strands with label 0 act as a placeholder for when we keep the number of strands
fixed, and for most purposes they can be ignored. In this way we can recover \( Y \) and \( Y^* \) as special cases of \( f^{(c)}_i 1_p \) and \( f^{(c)}_j 1_{p'} \) with one of the labels being equal to 0.

An arbitrary \( n \)-ladder with \( \ell \) uprights is by definition a labeled trivalent graph obtained by vertically glueing basic ladders of type \( e^{(c)}_i 1_p \) and \( f^{(c)}_j 1_{p'} \), for any combination of \( i, j \in [1, \ell - 1], c \in [0, n] \) and \( p, p' \in \Gamma^n_\ell \).

**Definition 2.1.** Let \( \text{Lad}^n_\ell \) denote the category of all \( n \)-ladders with \( \ell \) uprights. The objects are all \( p \in \Gamma^n_\ell \) and \( \text{Hom}(p, p') \) is the \( \mathbb{Q}(q) \)-vector space freely generated by the isotopy classes of all \( n \)-ladders with \( \ell \) uprights whose lower and upper boundary are labeled \( p \) and \( p' \) respectively. Composition is defined by vertically glueing and the unit of \( p \), denoted \( 1_p \), is the ladder without rungs whose uprights are labeled \( p \).

In our notation, we will usually suppress the labels of the uprights. They will only be specified when needed.

Cautis, Kamnitzer and Morrison [CKM14, Prop. 5.1.2] defined an ideal generated by certain relations in \( \text{Lad}^n_\ell \), which we denote by \( \mathcal{I}^n_\ell \). Their main results (Thm. 3.3.1 and Prop. 5.1.2) can then be paraphrased as follows:

**Theorem 2.2** (Cautis-Kamnitzer-Morrison). There exists an equivalence of categories

\[
\text{Lad}^n_\ell / \mathcal{I}^n_\ell \cong \bigoplus_{p, p' \in \Gamma^n_\ell, |p| = |p'|} \text{Hom}_{U_q(\mathfrak{gl}_n)} \left( \bigwedge^p \mathbb{C}_q^n, \bigwedge^p \mathbb{C}_q^n \right)
\]

such that composition with the functor in (2.1) gives an essentially surjective full functor

\[
\tilde{U}_q(\mathfrak{gl}_\ell) \to \text{Lad}^n_\ell / \mathcal{I}^n_\ell
\]

defined by \( E^{(c)}_i 1_p \to e^{(c)}_i 1_p \) and \( F^{(c)}_j 1_p \to f^{(c)}_j 1_p \) for any \( i \in [1, \ell - 1], c \in [0, n] \) and \( p \in \Gamma^n_\ell \).

Khovanov and Lauda [KL10] gave a categorification \( \mathcal{U}_\ell \) of \( U_q(\mathfrak{sl}_\ell) \), which can easily be extended to \( \tilde{U}_q(\mathfrak{gl}_\ell) \) and \( \tilde{U}_q^\vee(\mathfrak{gl}_\ell) \). In this paper, we give a categorification \( \mathcal{L} \mathcal{B}^n_\ell \) of \( \text{Lad}^n_\ell / \mathcal{I}^n_\ell \) (introduced in Section 3.4). In this categorification we replace:

- \( p \) with the derived category \( \mathcal{D}^b(\mathcal{P} \text{-mod}) \) of modules over an algebra defined by the second author in [Webb].
- An \( n \)-ladder \( L \in \text{Hom}_{\text{Lad}^n_\ell}(p, p') \) with an associated \( \mathcal{P}' \cdot \mathcal{P} \)-bimodule \( W_L \). Tensor product with this bimodule induces a functor between derived categories. The space of 2-morphisms between two ladders is the morphism space between the bimodules in the derived category which are \( \mathfrak{gl}_n \) invariant in an appropriate sense.
- Each generating relation in \( \mathcal{I}^n_\ell \) with a natural isomorphism between the associated derived functors.

In Corollary 4.17 we give the categorification of the second functor in Theorem 2.2, that is, a 2-functor \( \mathcal{U}_\ell \to \mathcal{L} \mathcal{B}^n_\ell \) which is one of the main results of this paper. Our proof of the well-definedness of this 2-functor is a bit roundabout: we analyze the structure of \( \mathcal{L} \mathcal{B}^n_\ell \) enough to define candidate 2-morphisms, and use results from category \( \mathcal{O} \) to show it suffices to check the relations between these 2-morphisms on certain objects. We then show that the Hom spaces between these objects exactly
match Khovanov and Lauda’s 2-representation of \( \mathcal{U} \) using the cohomology rings of partial flag varieties \([\text{KL10}]\). The remaining relations, which are required for the well-definedness of our 2-functor, then follow from Khovanov and Lauda’s work.

### 2.2. Categorifications of tensor products.

#### 2.2.1. The 2-category \( \mathbb{U}_n \).

We fix an arbitrary \( n \in \mathbb{N}_{>1} \) in this section. The object of interest for this subsection is a strict 2-category \( \mathbb{U}_n \), due to Khovanov and Lauda \([\text{KL10}]\). We will give a more compact definition of this category, shown to be equivalent to that of earlier literature such as \([\text{KL10}, \text{CL}, \text{Webb}]\) in a recent paper of Brundan \([\text{Bru}]\).

In order to define it, we will need to define a class of diagrams. Consider the set of diagrams in the horizontal strip \( \mathbb{R} \times [0,1] \) composed of embedded oriented curves, whose endpoints lie on distinct points of \( \mathbb{R} \times \{0\} \) and \( \mathbb{R} \times \{1\} \). At each point, projection to the \( y \)-axis must locally be a diffeomorphism, unless at that point it looks like one of the diagrams:

\[
\iota = \begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}, \quad \\
\epsilon = \begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}, \quad \\
\psi = \begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}, \quad \\
y = \begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\]

We’ll consider labelings of the components of these diagrams by elements of \([1, n-1]\). The **top** of such a diagram is the sequence where we read off the label of each of the endpoints on \( \mathbb{R} \times \{1\} \) in order from left to right, taking them with positive sign if the curve is oriented upward there, and a negative sign if it is oriented downward. The **bottom** is defined similarly with the endpoints on \( \mathbb{R} \times \{0\} \). The **vertical composition** \( ab \) of two diagrams where the bottom of \( a \) matches the top of \( b \) is the stacking of \( a \) on top of \( b \) and then scaling the \( y \)-coordinate by \( 1/2 \) to lie again in \( \mathbb{R} \times [0,1] \). The horizontal composition of two diagrams \( a \circ b \) places \( a \) to the right of \( b \) in the plane, and thus has the effect of concatenating their tops and bottoms in the opposite of the usual order.

We think of elements of \( \mathbb{Z}^n \) as weights of \( \mathfrak{g} \mathfrak{l}_n \) in the usual way, and let \( \alpha_i = -\alpha_{-i} = (0, \ldots, 0, 1, -1, 0, \ldots, 0) \) with the non-zero entries in the \( i \)th and \( i + 1 \)st positions. We let \( \lambda^i = \alpha_i^\vee(\lambda) = \lambda_i - \lambda_{i-1} \) for any weight \( \lambda \).

**Definition 2.3.** Let \( \tilde{\mathbb{U}}_n \) be the strict 2-category where

- the set of objects is \( \mathbb{Z}^n \)
- 1-morphisms \( \mu \rightarrow \nu \) are sequences \( \mathbf{i} = (i_1, \ldots, i_m) \) with each \( i_j \in \pm[1, n-1] \), which we interpret as a list of simple roots and their negatives such that \( \mu + \sum_{j=1}^{m} \alpha_{i_j} = \nu \). Composition is given by concatenation.
- 2-morphisms \( h \rightarrow h' \) between sequences are \( \mathbb{K} \)-linear combinations of diagrams of the type defined above with \( h \) as bottom and \( h' \) as top.

Since the underlying objects in \( \tilde{\mathbb{U}}_n \) are fixed for any 2-morphism, we incorporate them into the diagram by labeling each region of the place with \( \mu \) at the far left, \( \nu \) at
the far right, and intermediate regions are labeled by the rule

\[
\begin{array}{c}
\mu \\
\downarrow \\
\mu - \alpha_i .
\end{array}
\]

We’ll typically use \( E_i \) to denote the 1-morphism \((i)\) (leaving the labeling of regions implicit) and \( F_i \) to denote \((-i)\).

We can define a degree function on diagrams. The degrees are given on elementary diagrams by

\[
\deg \begin{array}{c}
\times \\
i \quad j
\end{array} = \begin{cases}
-2 & i = j \\
1 & |i - j| = 1 \\
0 & |i - j| > 1
\end{cases}
\]

\[
\deg \begin{array}{c}
\lambda \\
i \quad \lambda
\end{array} = \langle \lambda, \alpha_i \rangle - 1
\]

For a general diagram, we sum together the degrees of the elementary diagrams it is constructed from. This defines a grading on the 2-morphism spaces of \( \tilde{\mathcal{U}} \).

**Definition 2.4.** Let \( \mathcal{U}_n \) be the quotient of \( \tilde{\mathcal{U}}_n \) by the following relations on 2-morphisms:

- \( e \) and \( i \) are the units and counits of an adjunction, i.e. critical points can cancel.
- the endomorphisms of words only using \( F_i \) (or by duality only \( E_i \)’s) satisfy the relations of the quiver Hecke algebra \( R \).

\[
(2.2a) \quad \begin{array}{c}
\times \\
i \quad j
\end{array} = \begin{array}{c}
\times \\
i \quad j
\end{array} = \begin{array}{c}
\times \\
i \quad j
\end{array} \quad \text{unless } i = j
\]

\[
(2.2b) \quad \begin{array}{c}
\times \\
i \quad i
\end{array} = \begin{array}{c}
\times \\
i \quad i
\end{array} \quad \begin{array}{c}
\times \\
i \quad i
\end{array} = \begin{array}{c}
\times \\
i \quad i
\end{array} \quad \begin{array}{c}
\times \\
i \quad i
\end{array} = \begin{array}{c}
\times \\
i \quad i
\end{array} \quad \begin{array}{c}
\times \\
i \quad i
\end{array}
\]

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(2.2c) \[
\begin{cases}
0 & i = j \\
i & i \\
i & j \\
i & j \\
i & j \\
i & j - 1 & i = j - 1 \\
i & j \\
i & j \\
i & j \\
i & j + 1 & i = j + 1 \\
i & j \\
i & j \\
i & j \\
i & j \\end{cases}
\]

(2.2d) \[
\begin{cases}
0 & i = k = j + 1 \\
i & j & k \\
i & j \\
i & j \\
i & j \\
i & j \\
i & j \\
i & j \\
0 & \text{otherwise}
\end{cases}
\]

- the composition

possesses an inverse.

- if \( \lambda^i \geq 0 \), then the map \( \sigma_{\lambda,i} : \mathcal{E}_i \mathcal{F}_i \to \mathcal{F}_i \mathcal{E}_i \oplus \text{id}_\lambda^{\otimes \lambda^i} \) given by

possesses an inverse.
• if \( \lambda^i \leq 0 \), then the map \( \sigma_{\lambda,i} : \mathcal{E}_i \mathcal{F}_i \oplus \text{id}_{\mathcal{F}_i}^{g_0 - \lambda^i} \to \mathcal{F}_i \mathcal{E}_i \) given by

\[
\begin{array}{c}
\lambda \\
\downarrow \downarrow \\
-\lambda^i - 1
\end{array}
\oplus
\begin{array}{c}
i \\
i
\oplus
i
\oplus
\cdots
\oplus
\end{array}
\]

possesses an inverse.

We can define an adjunction of \( \mathcal{F}_i \) and \( \mathcal{E}_i \) in the opposite order by defining

\[
\begin{array}{c}
i \\
\lambda + \alpha_i
\end{array}
\]

\[
\begin{array}{c}
\lambda \\
i
\end{array}
\]

according to the rule of [Bru, (1.14-18)].

2.2.2. The algebras \( T \) and \( \tilde{T} \). We first recall the definition of the \( \mathfrak{gl}_n \) tensor algebras \( T \) and \( \tilde{T} \) from [Webb]. In order to prove our results, we have also needed to add some new material, especially concerning the relationship between \( T \) and \( \tilde{T} \) and Hochschild homology.

**Definition 2.5.** A Stendhal diagram is a collection of finitely many oriented curves in \( \mathbb{R} \times [0, 1] \). Each curve is either

- colored red and labeled with a dominant weight of \( \mathfrak{gl}_n \), or
- colored black and labeled with \( i \in [1, n-1] \) and decorated with finitely many dots.

The diagram must be locally of the form

\[
\begin{array}{c}
\text{\textbullet}\text{\textbullet}\text{\textbullet}
\end{array}
\]

with each curve oriented in the negative direction. In particular, no red strands can ever cross. Each curve must meet both \( y = 0 \) and \( y = 1 \) at distinct points from the other curves.

We’ll typically only consider Stendhal diagrams up to isotopy. Since the orientation on a diagram is clear, we typically won’t draw it.

We call the lines \( y = 0, 1 \) the **bottom** and **top** of the diagram. Reading across the bottom and top from left to right, we obtain a sequence of dominant weights and elements of \([1, n-1]\). We record this data as

- the list \( i = (i_1, \ldots, i_n) \) of elements of \([1, n-1]\), read from the left;
- the list \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) of fundamental \( \mathfrak{gl}_n \) weights, read from the left;
- the weakly increasing function \( \kappa : [1, \ell] \to [0, n] \) such that \( \kappa(m) \) is the position of the rightmost black strand which is left of \( m \)th red strand (both counted
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from the left). By convention, we write \( \kappa(i) = 0 \) if the \( i \)th red strand is left of all black strands.

We call such a triple of data a **Stendhal triple**. We will often want to partition the sequence \( i \) in the groups of black strands between two consecutive reds, that is, the groups

\[
i_0 = (i_1, \ldots, i_{\kappa(1)}), \quad i_1 = (i_{\kappa(1)+1}, \ldots, i_{\kappa(2)}), \ldots, \quad i_\ell = (i_{\kappa(\ell)+1}, \ldots, i_n).
\]

We call these **black blocks**.

Here are two examples of Stendhal diagrams:

\[
a = \begin{array}{ccc}
\lambda_1 & i & \lambda_2 & j & i \\
\end{array}
b = \begin{array}{ccc}
\lambda_1 & i & \lambda_2 & j & i \\
\end{array}
\]

- At the top of \( a \), we have \( i = (i, i, j), \quad A = (\lambda_1, \lambda_2) \) and \( \kappa = (1 \mapsto 0, 2 \mapsto 0) \).
- At the top of \( b \) and bottom of \( a \) and \( ba = 0 \), we have \( i = (i, j, i), \quad A = (\lambda_1, \lambda_2) \) and \( \kappa = (1 \mapsto 0, 2 \mapsto 1) \).

**Definition 2.6.** Given Stendhal diagrams \( a \) and \( b \), their composition \( ab \) is given by stacking \( a \) on top of \( b \) and attempting to join the bottom of \( a \) and top of \( b \). If the Stendhal triples from the bottom of \( a \) and top of \( b \) don’t match, then the composition is not defined and by convention is 0, which is not a Stendhal diagram, just a formal symbol.

\[
ab = \begin{array}{ccc}
\lambda_1 & i & \lambda_2 & j & i \\
\end{array}
\]

Fix a field \( \mathbb{k} \) and let \( \hat{T} \) be the formal span over \( \mathbb{k} \) of Stendhal diagrams (up to isotopy). The composition law induces an algebra structure on \( \hat{T} \).

Let \( e(i, A, \kappa) \) be the unique crossingless, dotless diagram where the triple read off from both top and bottom is \( (i, A, \kappa) \). Composition on the left/right with \( e(i, A, \kappa) \) is an idempotent operation; it sends a diagram \( a \) to itself if the top/bottom of \( a \) matches \( (i, A, \kappa) \) and to 0 otherwise. We’ll often fix \( A \) and thus leave it out from the notation, just writing \( e(i, \kappa) \) for this diagram. Let \( \hat{P}_i^\kappa = T^\Delta e(i, \kappa) \) and \( \hat{P}_i^\kappa = T^\Delta e(i, \kappa) \).

Considered as elements of \( \hat{T} \), the diagrams \( e(i, A, \kappa) \) are orthogonal idempotents. The algebra \( \hat{T} \) is not unital, but it is **locally unital**, that is for any element \( a \), there is an idempotent such that \( ea = ae = a \). This can be taken to be the sum of \( e(i, A, \kappa) \) for all triples that occur at the top or bottom of one of the diagrams in \( a \).

Alternatively, we can organize these diagrams into a category whose objects are Stendhal triples \( (i, A, \kappa) \) and whose morphisms are Stendhal diagrams read from bottom to top. In this perspective, the idempotents \( e(i, A, \kappa) \) are the identity morphisms of different objects.
Definition 2.7. We call a black strand in a Stendhal diagram \textbf{violating} if at some horizontal slice \( y = c \) for \( c \in [0,1] \), it is the leftmost strand. A Stendhal diagram which possesses a violating strand is called \textbf{violated}.

Both the diagrams \( a \) and \( b \) above are violated. The diagrams

\[
\begin{align*}
c &= \begin{array}{ccc}
\lambda_1 & i & \lambda_2 & j & i \\
\end{array} \\
d &= \begin{array}{ccc}
\lambda_1 & i & \lambda_2 & j & i \\
\end{array}
\end{align*}
\]

are not violated. The diagram \( e(i, \lambda, \kappa) \) is violated if and only if \( \kappa(1) > 0 \).

Definition 2.8. The \textbf{degree} of a Stendhal diagram is the sum over crossings and dots in the diagram of

\begin{itemize}
\item \( -\langle \alpha_i, \alpha_j \rangle \) for each crossing of a black strand labeled \( i \) with one labeled \( j \);
\item \( \langle \alpha_i, \alpha_i \rangle = 2 \) for each dot on a black strand labeled \( i \);
\item \( \langle \alpha_i, \lambda \rangle = \lambda^i \) for each crossing of a black strand labeled \( i \) with a red strand labeled \( \lambda \).
\end{itemize}

The degree of diagrams is additive under composition. Thus, the algebra \( \tilde{T} \) inherits a grading from this degree function.

The reflection through the horizontal axis of a Stendhal diagram \( a \) is again a Stendhal diagram, which we denote \( \dot{a} \). Note that \( \dot{ab} = \dot{b}\dot{a} \), so \( \dot{a} \) induces an anti-automorphism of \( \tilde{T} \).

Definition 2.9. Let \( \tilde{T} \) be the quotient of \( \tilde{T} \) by the following local relations between Stendhal diagrams:

\begin{itemize}
\item the KLR relations (2.2a–2.2d)
\item All black crossings and dots can pass through red lines, with a correction term similar to Khovanov and Lauda’s (for the latter two relations, we also include their mirror images). By convention, the term with the summation below is taken to be zero if \( \lambda^i = 0 \):
\end{itemize}

\[
\begin{align*}
\text{(2.3a)} & \quad \begin{array}{cccc}
j & \lambda & i \\
\end{array} = \begin{array}{cccc}
j & \lambda & i \\
\end{array} + \delta_{i,j} \sum_{a+b+1=\lambda^i} b \\
\end{align*}
\]

\[
\begin{align*}
\text{(2.3b)} & \quad \begin{array}{cccc}
\text{ } & \text{ } & \text{ } \\
\end{array} = \begin{array}{cccc}
\text{ } & \text{ } & \text{ } \\
\end{array}
\end{align*}
\]
The “cost” of separating a red and a black line is adding \( \lambda^i = \alpha^\vee_i(\lambda) \) dots to the black strand.

\[ \begin{array}{c}
\lambda & i \\
\downarrow & \downarrow \\
\lambda & \lambda^i \\
\end{array} = \begin{array}{c}
\lambda & i \\
\\| \quad \| \quad \| \\
\lambda & \lambda^i \\
\end{array} \]

**Definition 2.10.** Let \( T \) be the quotient of \( \hat{T} \) by the 2-sided ideal \( K \) generated by all violated diagrams.

Now, as before, fix a sequence of dominant weights \( \underline{\lambda} = (\lambda_1, \ldots, \lambda_\ell) \) and let \( \lambda = \sum_{i=1}^\ell \lambda_i \).

**Definition 2.11.** We let \( T^\underline{\lambda} \) (resp. \( \hat{T}^\underline{\lambda} \)) be the subalgebra of \( T \) (resp. \( \hat{T} \)) where the red lines are labeled, from left to right, with the elements of \( \underline{\lambda} \). Let \( T^\underline{\lambda}_m \) (resp. \( \hat{T}^\underline{\lambda}_m \)) for any element of the \( \mathfrak{gl}_n \) weight lattice \( \alpha \), be the subalgebra of \( T^\underline{\lambda} \) (resp. \( \hat{T}^\underline{\lambda} \)) where the sum of the roots associated to the black strands is \( \lambda - \alpha \), and let \( T^\underline{\lambda}_m \) (resp. \( \hat{T}^\underline{\lambda}_m \)) be the subalgebra of diagrams with \( m \) black strands.

To give a simple illustration of the behavior of our algebra, let us consider \( \mathfrak{gl}_2 = \mathfrak{gl}_2 \), and \( \underline{\lambda} = (\omega_1, \omega_1) \) where \( \omega_1 = (1,0) \). Thus, our diagrams have 2 red lines, both labeled with \( \omega_1 \)'s. In this case, the algebras \( T^\underline{\lambda}_m \) are easily described as follows:

- \( T^\underline{\lambda}_{(2,0)} = k \): it is spanned by the diagram \( \| \| \).
- \( T^\underline{\lambda}_{(1,1)} \) is spanned by

\[
\| \|, \| \|, \| \|, \| \|, \| \|, \| \|, \| \|, \| \|, \| \|, \| \|, \| \|, \| \|, \| \|, \| \|.
\]

One can easily check that this is the standard presentation of a regular block of category \( \hat{O} \) for \( \mathfrak{gl}_2 \) as a quotient of the path algebra of a quiver (see, for example, [Str03]).

- \( T^\underline{\lambda}_{(0,2)} = \text{End}(k^3) \): The algebra is spanned by the diagrams

\[
\|, \|, \|, \|, \|, \|, \|, \|, \|, \|, \|, \|, \|, \|, \|, \|
\]
which one can easily check multiply (up to sign) as the elementary generators of \( \text{End}(k^3) \).

Perhaps a more interesting example is the case of \( \mathfrak{g}l_3 \) with \( \lambda = (\omega_1, \omega_2) \) and \( \mu = (1, 1, 1) \). Based on the construction of a cellular basis in [SW], we can calculate that this algebra is 19 dimensional, with a basis given by

\[
\begin{align*}
&\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array},
\end{align*}
\]

We leave the calculation of the multiplication in this basis to the reader; it is a useful exercise for those wishing to become more comfortable with these sorts of calculations. For example, when we multiply the last two vectors in the basis above, we get that (for \( Q_{21}(u, v) = u - v \))

\[
\begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array} \cdot \begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array} = \begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array} = \begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array} = \begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array} = \begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array} = \begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array} = \begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array} = \begin{array}{cccc}
\times & \times & \times & \times \\
1 & 1 & 2 & 2
\end{array}.
\]

**Definition 2.12.** Let \( B^\lambda \) (resp. \( \bar{B}^\lambda \)) be the abelian category of finitely generated graded \( T^\lambda \) (resp. \( \bar{T}^\lambda \)) modules. Let \( \mathcal{V}^\lambda = D^b(B^\lambda) \) and \( \bar{\mathcal{V}}^\lambda = D^b(\bar{B}^\lambda) \) be the bounded derived categories of these categories of modules.

We let \( [n] \) be the homological shift in the derived category, reindexing a complex so that the \( k \)th term of \( C^*[n] \) is \( C^n+k \), that is, decreasing the homological degree of each term by \( n \).

Let \( (n) \) be the shift of internal grading on \( B^\lambda \) and its related categories. As with the homological shift, this decreases the grading by \( n \).

Finally, we let \( \langle n \rangle \) be the “Tate twist” which is the composition \( [n](-n) \).

2.2.3. *Induction and restriction functors.* There are maps \( i^k_L, i^k_R: \bar{T}^\lambda \rightarrow T^\lambda \) which act by adding a black strand at the left (resp. right) side of the diagram.

**Definition 2.13.** We’ll let \( \bar{\delta}^*_k, \delta_k: B^\lambda \rightarrow \bar{B}^\lambda \) denote the extension of scalars (that is, induction) by the maps \( i^k_L, i^k_R \) (respectively). After imposing the violating relation, the map induced by \( i^k_L \) is 0, but we can still consider the map induced by \( i^k_R \), and let \( \bar{\delta}^*_k: B^\lambda \rightarrow \bar{B}^\lambda \) be the extension of scalars along this induced map. We let \( \delta_k: B^\lambda \rightarrow \bar{B}^\lambda \) be the adjoint of \( \bar{\delta}^*_k \).
These functors give the compatibility of the categories $\mathcal{H}^\Delta$ and $\tilde{\mathcal{H}}^\Delta$ with our constructions from the previous sections as follows: the category of $\mathcal{H}^\Delta$ carries a representation of $\mathcal{U}_n$ via the functors $\mathcal{C}_i$ and $\tilde{\mathcal{C}}_i$, as shown in [Webb, 4.31], whereas $\tilde{\mathcal{H}}^\Delta$ carries an action of two commuting copies of the lower half of this category $\mathcal{U}_n \times \mathcal{U}_n$ via $\tilde{\delta}_k^\ast, \tilde{\delta}_k$. While these functors possess left and right adjoints, these adjoints are not isomorphic and do not preserve the category of finitely generated modules, so there is no hope of extending this to a $\mathcal{U}_n \times \mathcal{U}_n$ action on this category.

For any highest weight $\mu$, we also have a map $\tilde{I}_\mu : \tilde{T}^\lambda \to \tilde{T}^{(\lambda_1, \ldots, \lambda_\ell, \mu)}$ which adds a red line at the far right. This preserves violated diagrams, and thus induces a map $I_\mu : T^\lambda \to T^{(\lambda_1, \ldots, \lambda_\ell, \mu)}$.

**Definition 2.14.** Let $\tilde{\mathcal{F}}, \mathcal{F}$ denote extension of scalars along the maps $\tilde{I}_\mu, I_\mu$. That is:

$$\tilde{\mathcal{F}}_\mu(M) := \tilde{T}^{(\lambda_1, \ldots, \lambda_\ell, \mu)} \otimes_{\mathcal{H}^\Delta} M \quad \mathcal{F}_\mu(M) := T^{(\lambda_1, \ldots, \lambda_\ell, \mu)} \otimes_{\mathcal{H}^\Delta} M.$$  

These functors have several important homological properties:

**Proposition 2.15.** The functors $\tilde{\mathcal{F}}^\ast_\mu, \mathcal{F}^\ast_\mu, \mathcal{F}_\mu, \mathcal{E}_\mu, \tilde{\mathcal{F}}_\mu, \mathcal{F}_\mu$ are all exact and send projectives to projectives.

**Proof.** Tensor with a bimodule will be exact and send projectives to projectives if and only if the bimodule is sweet, that is projective as a left module and as a right module. For $\tilde{\delta}_k^\ast, \delta_k^\ast, \tilde{\delta}_k, \mathcal{E}_k, \tilde{\mathcal{F}}_\mu, \mathcal{F}_\mu$, the bimodule under consideration is of the form $T^e\mathcal{C}$ or $\tilde{T}^e\mathcal{C}$ for some idempotent $e$, and thus is obviously projective as a left module. After applying the Morita equivalence of $T^\Delta$ with the double Stendhal algebra $DT^\Delta$ [Webb, 4.29] the bimodule $\mathcal{C}_k$ has the same property. For $\tilde{\delta}_k^\ast, \delta_k^\ast, \tilde{\delta}_k, \tilde{\mathcal{F}}_\mu, \mathcal{F}_\mu$, projectivity as a right module follows immediately from [Webb, Prop. 4.16]. For $\tilde{\mathcal{F}}_\mu, \mathcal{F}_\mu$, projectivity on the right follows from [Webb, Prop. 4.32], since it shows that $e(i, \kappa)\tilde{\delta}_k$ is isomorphic to $e(i, \kappa')T$ where $\kappa'$ is $\kappa$ restricted to $[1, \ell - 1]$. Finally, for $\tilde{\delta}_k, \delta_k$, this follows since these functors are biadjoint, and the right adjoint of a functor sending projectives to projectives is exact. □

### 2.2.4. Derived pushforward and pullback

When we have a $T^\Delta$-module $M$, we can abuse notation and use $M$ to also denote the pullback $\tilde{T}^\Delta$-module. In this case, it is worth noting that the pullback of $\tilde{\delta}_k M$ is not the same as the $\tilde{T}^\Delta$-module $\tilde{\delta}_k M$, but we have a surjective map $\tilde{\delta}_k M \to \tilde{\delta}_k M$. The difference is that in $\tilde{\delta}_k M$, we have not killed the submodule of elements where the new strand we’ve added passes through the region at the far left, whereas in $\tilde{\delta}_k M$, we have.

Given a sequence $i$, we have a surjective map $P^k_\mu \to P^k_\mu$, whose kernel is the submodule generated by the elements $c_m$
Let \( P_i^c(s) \) for \( s \in [1, p] \) be the quotient of \( \tilde{P}_i^c \) by the submodule generated by \( c_m \) for \( m \leq s \); in particular, \( P_i^c(0) = \tilde{P}_i^c \) and \( P_i^c(p) = \tilde{P}_i^c \). Note that

\[
(2.5) \quad \tilde{\delta}_i P_i^c(s) \cong P_i^c_{\iota \cup \iota}(s) \quad \mathfrak{A}_i P_i^c(s) \cong P_i^{c \cup (\iota)}(s).
\]

Fix \( s \in [1, p] \). Given a Stendhal triple \((\iota, \iota, \kappa)\), let \( \iota^- \) be the sequence given by removing \( i_s, \kappa^- \) the corresponding function \( \kappa^-(q) = \kappa(q) \) if \( \kappa(q) \leq s \) and \( \kappa^-(q) = \kappa(q) - 1 \) if \( \kappa(q) > s \).

**Lemma 2.16.** If \( s \in S \), then we have a surjective map \( \tilde{P}_i^c(s - 1) \to P_i^c(s) \) with kernel isomorphic to \( \tilde{\delta}_i^s P_i^c(s - 1) \), via the map attaching the diagram \( c_s \) to the bottom of the diagram.

**Proof.** First note that this attachment map is in fact a map of modules: that if we take \( c_m \) for \( m < s \), add a black strand at the far left, and then attach \( c_s \) to its bottom, we obtain an element of the submodule generated by \( c_q \) for \( q < s \).

In this diagram, we slide the crossing between the \( m \)th and \( s \)th strands (currently between the \( m + 1 \)st and \( m + 2 \)nd strands) to the far left, so that it is the crossing closest to the top. In reduced words for permutations (reading from bottom), we go from \( s_2 \cdots s_{m+1} s_1 \cdots s_s \) to \( s_1 \cdots s_s s_1 \cdots s_m \).

This may have correction terms, but these will all lie in the submodule generated by \( c_q \) for \( q < s \), as does this new leading term diagram. Thus the map \( \tilde{\delta}_i^s P_i^c(s - 1) \to \tilde{P}_i^c \) given by attaching \( c_s \) at the bottom induces a map \( \tilde{\delta}_i^s P_i^c(s - 1) \to P_i^c(s) \) whose image is clearly the submodule generated by \( c_s \), and thus has cokernel \( P_i^c(s) \). Thus, we need only prove that this map is injective.

Since the functors \( \tilde{\delta}_i \) and \( \mathfrak{A}_i \) are both exact, using (2.5), we can reduce to the case where \( s = p \). In this case, both \( P_i^c(p - 1) \cong \tilde{\delta}_i^p P_i^c(p - 1) \) and \( \tilde{\delta}_i^p P_i^c(p - 1) \) are free modules over \( k[e_i(y)] \) of rank \( (p - 1 + \ell) \dim(P_i^c(p - 1)) \). Here \( e_i(y) = y_1 + \ldots + y_p \), with \( y_k \) being given by the sum of all \( e(j, \kappa') \) with one dot on the \( k \)th black strand, where \( j \) is a permutation of \( i \) and \( \kappa' \) is arbitrary. Note that \( p - 1 + \ell \) is the number of possible endpoints at the top of the diagram for the \( p \)th black strand (i.e. the number of shuffles). The claim follows from the basis theorem [Webb 4.16] by the same argument as in [KL09, Prop. 2.16-8].

A map between free modules of the same rank of a polynomial ring is injective if and only if the cokernel is finite dimensional. So the map is injective by the finite dimensionality of \( P_i^c(s) \).

**Lemma 2.17.** We have an isomorphism of bimodules \( T^{A \otimes L}_{\mathfrak{A}_i} T^A \cong T^A \).

**Proof.** This is clear for the naive tensor product, so we only need to check that higher Tor’s vanish. That is, we need to check that \( \text{Tor}^i_{\mathfrak{A}_i}(T^A, P_i^c) = 0 \) for \( i > 0 \). We’ll prove the stronger statement that \( \text{Tor}^i_{\mathfrak{A}_i}(T^A, P_i^c(s)) = 0 \) for all \( s \) by induction on increasing \( s \). Thus, the base case is when \( s = 0 \); in this case \( P_i^c(0) \) is projective over \( T^A \), so the statement is clear. Otherwise, we have a long exact sequence on Tor given by

\[
\cdots \to \text{Tor}^i_{\mathfrak{A}_i}(T^A, \tilde{\delta}_i^s P_i^c(s - 1)) \to \text{Tor}^i_{\mathfrak{A}_i}(T^A, P_i^c(s)) \to \text{Tor}^i_{\mathfrak{A}_i}(T^A, P_i^c(s - 1)) \to \cdots
\]
By the inductive assumption, the middle term vanishes when \( i > 0 \). On the other hand, for all \( i \geq 0 \), we have

\[
\text{Tor}_{T_\lambda}^i(T, \tilde{\mathcal{E}}_{i_\lambda} P_{i_\lambda}^e (s - 1)) \cong \text{Tor}_{T_\lambda}^i(\mathcal{E}_{i_\lambda} T, P_{i_\lambda}^e (s - 1)) \cong 0
\]

since \( \mathcal{E}_{i_\lambda} T = 0 \).

Applied to the long exact sequence, this shows that \( \text{Tor}_{T_\lambda}^i(T, P_{i_\lambda}^e(s)) \) for \( i > 0 \) is trapped between two 0’s and thus itself 0.

2.2.6. Standard modules. We have isomorphisms

\[
\mathcal{K}_{q}(\tilde{\mathfrak{g}}^{\lambda}) \cong U_q \otimes V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell} \quad \mathcal{K}_{q}(\mathfrak{g}^{\lambda}) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell}.
\]

This isomorphism sends

\[
[\tilde{\mathfrak{g}}_{i_\lambda}^{\lambda}](u \otimes w) \mapsto F_i \cdot (u \otimes w) \quad [\mathfrak{g}_{i_\lambda}^{\lambda}](u \otimes w) \mapsto (u \otimes w) \cdot F_i
\]

\[
[\tilde{\mathfrak{g}}_{\lambda}]^\lambda(u \otimes w) \mapsto u(1) \otimes (u(2) v_\lambda \otimes w) \quad [\mathfrak{g}_{\lambda}]^\lambda(u \otimes w) \mapsto u \otimes w \otimes v_\lambda
\]

These are compatible with pullback from \( T^p \)- to \( \tilde{T}^p \)-modules (resp. its adjoint pushforward), in the sense that these functors categorify the inclusion

\[
w_1 \otimes \cdots \otimes w_\ell \mapsto 1 \otimes w_1 \otimes \cdots \otimes w_\ell
\]

(resp. its adjoint projection, induced by the counit map of \( U_q \)).

2.2.6. Standard modules. In this section, we summarize results from Section 5.4, specialized to the case where \( \mathfrak{g} = \mathfrak{gl}_n \) and all highest weights are fundamental. By [Webb, 5.22], the algebra \( T^p \) is quasi-hereditary, that is, the category \( T^p \)-mod is highest weight.

In this case, the labeling set for simple and projective modules is given by \( \ell \)-tuples \( \mu = (\mu_1, \ldots, \mu_\ell) \) where \( \mu_i \) is a weight of \( \wedge^p \mathbb{C}^n \). We order these elements according to reduced dominance order

\[
(\mu_1, \ldots, \mu_\ell) \geq (\mu_1', \ldots, \mu_\ell') \quad \text{if} \quad \sum_{i=j}^{\ell} \mu_i \geq \sum_{i=j}^{\ell} \mu_i' \quad \text{for all} \ j \in [1, \ell].
\]

Each weight \( \mu \) of the representation \( \wedge^p \mathbb{C}^n \) corresponds to a \( p \)-element subset \( J_\mu \) of \( [1,n] \), and this weight space is spanned by the wedge \( v_\mu := v_{j_1} \wedge \cdots \wedge v_{j_p} \) where \( j_1 < \cdots < j_p \) are the elements of \( J_\mu \) in order. We let \( v_\mu = v_{\mu_1} \otimes \cdots \otimes v_{\mu_\ell} \). We can characterize
$P_\mu$ as the unique indecomposable projective of the form $[P_\mu] = v_\mu + \sum_{\mu' > \mu} m_i(q)v_{\mu'}$; similarly, $L_\mu$ is the unique simple such that $[L_\mu] = v_\mu + \sum_{\mu' < \mu} n_i(q)v_{\mu'}$.

In brief, this means that $T^p$-mod contains a distinguished collection of objects $\nabla_\mu$ which are “intermediate” between the simples and projectives. In this case, the labeling is fixed by the fact that $[\nabla_\mu] = v_\mu$.

The induced structure on modules over $\tilde{T}^p$ is slightly more complicated. By [Webb, 5.23], that category is standardly stratified and the standard modules are precisely the summands of $\nabla_i^*\nabla_\mu$ for all different $i$ and $\mu$ (where as always, we use the same symbol for a $T^p$-module and its pullback to $\tilde{T}^p$).

2.3. $A_\infty$ algebras. While we will not use them in an extremely deep way, it will be useful to us to think about an action of $A$ in the $A_\infty$ sense. The basic situation in which this comes up for us is this: assume $M$ is an object in a $k$-linear abelian category, and consider a projective resolution $\cdots \to M_{i-1} \to M_i \to \cdots \to M_0$, where all indices are non-positive. If a $k$-algebra $A$ acts on $M$ on the right, then every element $a \in A$ induces an endomorphism of the chain complex $M$, which is unique up to homotopy. Thus, one can easily see that these endomorphisms will induce an action of $A$ on this complex if we strictly identify homotopic maps.

This sort of crude identification of homotopic maps is impractical for actually thinking of $M$ as an $A$-module. A context that allows us more control over the situation is to instead define an action of $A$ on $M$ in the $A_\infty$-sense. That is, we define higher multiplications $\alpha_i: M_i \otimes A^{\otimes (n-1)} \to M_{i+n}$ for all $i > 0$; by convention $\alpha_1 = \partial$. Since $A$ is an associative algebra, the relations these must satisfy take the form:

\[
\sum_{s+t=u} (-1)^{st} \alpha_{t+1}(\alpha_s(m, a_1, \ldots, a_{s-1}), a_s, \ldots, a_{u-1}) + \sum_{r=1}^{u-2} (-1)^r \alpha_{u-1}(m, a_1, \ldots, a_r a_{r+1}, \ldots, a_{u-1}) = 0 \tag{2.7}
\]

One can prove that such an action always exists since the sum of terms above when $s > 1$ and $t > 0$ acts trivially on the module $M$, and thus is nullhomotopic. The equation (2.7) then says that $\alpha_1$ can be chosen to be any choice of null-homotopy. This is one low-tech way of understanding the fact that the $A_\infty$ operad is a cofibrant replacement of the associative algebra operad.

We can always turn a module or complex of modules in the $A_\infty$ sense into an honest module by taking the $A_\infty$ tensor product with the bimodule $A$ itself. Let $\text{SA} := A[-1]$ denote the suspension of $A$. For simplicity, let $m := m \otimes a_1 \otimes \cdots \otimes a_{u-1} \otimes a_u$. 

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Definition 2.19. The $A_\infty$ tensor product $M \otimes A$ is given by the complex $M \otimes TSA \otimes A$ whose $u$-th term is given by $\bigoplus_{i+j=u} (M_i \otimes (A[-1])^j \otimes A)$ with the differential

\[(2.8) \quad \partial(m) = \sum_{s=1}^{u} (-1)^{s(u-s)} \alpha_s(m, a_1, \ldots, a_{s-1}) \otimes a_s \otimes \cdots \otimes a_u \]

$$+ \sum_{r=1}^{u-1} (-1)^{r} m \otimes a_1 \otimes \cdots \otimes a_r a_{r+1} \otimes \cdots \otimes a_u.$$ 

The relations (2.7) are equivalent to this differential squaring to 0.

Lemma 2.20. The map sending $m \mapsto (-1)^{u+1} \alpha_{u+1}(m, a_1, \ldots, a_{u-1}, a_u)$ gives a homotopy equivalence between the complexes $M \otimes A$ and $M$ with homotopy inverse $m \mapsto m \otimes 1$.

Proof. To show this, we need to show that there is a homotopy $h$ such that

$$(\partial \circ h + h \circ \partial)(m) = m + (-1)^u \alpha_{u+1}(m, a_1, \ldots, a_{u-1}, a_u) \otimes 1$$

We claim that this homotopy is given by

$$h(m) = (-1)^u m \otimes a_1 \otimes \cdots \otimes a_{u-1} \otimes a \otimes 1,$$

since we have

$$(\partial \circ h + h \circ \partial)(m) = \sum_{s=1}^{u+1} (-1)^{(u+1-s)+u} \alpha_s(m, a_1, \ldots, a_{s-1}) \otimes a_s \otimes \cdots \otimes a_u \otimes 1$$

$$+ \sum_{r=1}^{u} (-1)^{r+u} m \otimes a_1 \otimes \cdots \otimes a_r a_{r+1} \otimes \cdots \otimes a_u \otimes 1$$

$$+ \sum_{s=1}^{u} (-1)^{s(u-s)+u-1} \alpha_s(m, a_1, \ldots, a_{s-1}) \otimes a_s \otimes \cdots \otimes a_u \otimes 1$$

$$+ \sum_{r=1}^{u-1} (-1)^{r+u-1} m \otimes a_1 \otimes \cdots \otimes a_r a_{r+1} \otimes \cdots \otimes a_u \otimes 1$$

$$= m + (-1)^u \alpha_{u+1}(m, a_1, \ldots, a_{u-1}, a_u) \otimes 1. \quad \Box$$

2.4. Hochschild cohomology. For any algebra $A$ the Hochschild cohomology is the Ext algebra $HH^*(A) \cong Ext^*(A \otimes A^\text{op})(A, A)$ of the diagonal bimodule with itself. It’s the “center” of this algebra, computed in the derived category.

For us, the most important thing about Hochschild cohomology is its connection to deformations. For a $k$-algebra $A$, a deformation of $A$ is a free $k[h]/(h^2)$-algebra $\tilde{A}$ with a fixed isomorphism $A \cong \tilde{A}/h\tilde{A}$ (up to the obvious equivalence). A classical theorem of Hochschild relates these to Hochschild cohomology:

Theorem 2.21 ([Hoc45, 6.2]). There is a bijection between deformations of $A$ and $HH^2(A)$.

For a more modern treatment of this theorem, see [Wei94, 9.3.1]. The important point for us is that a Hochschild class $s$ can be represented by a 2-cocycle $r: A \otimes A \to A$,
and the associated deformation is the product on \( A[h]/(h^2) \) defined by \( a \star b = ab + hr(a, b) \).

The algebras \( \tilde{T}^p \) have a natural deformation coming from changing the relation (2.4); we consider the span of Stendhal diagrams over the polynomial ring \( \mathbb{k}[z_1, \ldots, z_\ell] \), and then impose the relations as before, except replacing (2.4) with the relation

\[
\begin{array}{c}
\includegraphics{diagram.png}
\end{array}
\]

where the red strand shown is the \( k \)th from the left.

Since \( \tilde{T} \) has a basis given by diagrams in [Webb, 4.16], we can use this basis to define a splitting from \( \tilde{T}^p \) into its deformation. We can then use this splitting to compute a 2-cocycle for this deformation. By definition, we take the product of two diagrams in this basis, and expanding them in terms of the basis using the deformed relations; the sum of the terms divisible by \( h \) with the \( h \) removed is \( r \) applied to these diagrams.

This is a special case of the canonical deformation discussed in [Webd, §2.6], since by [Webd, §3.5], the algebra \( \tilde{T}^p \) is a reduced weighted KLR algebra. The deformation is induced by deforming the polynomials \( Q_\ast \) from the definition of the weighted KLR algebra, and thus [Webd, 2.7] shows that the same set of diagrams provides a basis for the deformed and undeformed algebras. The key point is that the polynomial representation of \( \tilde{T} \) can easily be deformed compatibly with this deformation, which allows to show that these diagrams remain linearly independent.

Thus, this deformation defines a map \( y: \mathbb{k}^\ell \to HH^2(\tilde{T}^p) \). We’ll denote the images of the coordinate vectors by \( y_k \); these correspond to the specialization \( z_k = h, z_j = 0 \) for \( j \neq k \). We’ll choose a specific Hochschild cochain \( r_k: \tilde{T}^p \otimes \tilde{T}^p \to \tilde{T}^p \) which represents this deformation.

Another important issue we’ll consider is how these classes act on \( \text{Ext}(M, M) \) for an \( A \)-module \( M \). Let \( A_h \) be a deformation of \( A \) over \( \mathbb{k}[h]/(h^2) \) and let \( s \in HH^2(A) \) be the corresponding Hochschild class, and \( s_M \in \text{Ext}^2(M, M) \) the induced action on \( M \).

**Proposition 2.22** (Gerstenhaber and Schack, [GS83, §3]). The module \( M \) has a deformation to a \( \mathbb{k}[h]/(h^2) \)-free module over \( A_h \) if and only if \( s_M = 0 \).

**Proof.** Let \( \varphi: A \to \text{End}_\mathbb{k}(M) \) be the action map. We make \( \text{End}_\mathbb{k}(M) \) into an \( A-A \)-bimodule, with the left action being precomposition with \( \varphi(a) \) and the right action post-composition. If we freely resolve \( M \) using the tensor product with the Hochschild resolution, we can identify \( \text{Ext}^2(M, M) \) with the Hochschild cohomology \( HH^2(A, \text{End}_\mathbb{k}(M)) \) of the bimodule \( \text{End}_\mathbb{k}(M) \). More concretely, this group is the subquotient of \( \text{Hom}_\mathbb{k}(M \otimes A^{\otimes 2}, M) \cong \text{Hom}_\mathbb{k}(A^{\otimes 2}, \text{End}_\mathbb{k}(M)) \) of cocycles \( \xi \) satisfying

\[
\varphi(a_1)\xi(a_2 \otimes a_3) - \xi(a_1 a_2 \otimes a_3) + \xi(a_1 \otimes a_2 a_3) - \xi(a_1 \otimes a_2)\varphi(a_3),
\]
modulo those of the form
\[ \partial \eta(a_1 \otimes a_2) = \varphi(a_1)\eta(a_2) - \eta(a_1a_2) + \eta(a_1)\varphi(a_2). \]

If \( r \) is a Hochschild 2-cocycle representing \( s \), then the induced cocycle \( r_M \) is \( r_M(a_1 \otimes a_2) = \varphi(r(a_1, a_2)) \).

The module \( M \) has a flat deformation if the map \( \varphi: A \rightarrow \text{End}_k(M) \) has a deformation to a map \( \varphi_h = \varphi + h\varphi_1: A_h \rightarrow \text{End}_k(M)[h]/(h^2) \). The argument of \cite{GS83} shows that this happens when \( r_M(a_1, a_2) = \partial(\varphi_1) \). Thus, we have \( s_M = 0 \). \( \square \)

In general we can compute the class \( s_M \) by studying how a module fails to deform flatly. Let us make this more precise. Consider a complex
\[ \cdots \rightarrow M_i \xrightarrow{\partial} M_{i+1} \xrightarrow{\partial} \cdots \rightarrow M_0 \]
of \( A \)-modules. Assume there exists a precomplex (that is, a sequence of maps where the differential is not assumed to square to 0.)
\[ \cdots \xrightarrow{\hat{\partial}} \tilde{M}_i \xrightarrow{\hat{\partial}} \tilde{M}_{i+1} \xrightarrow{\hat{\partial}} \cdots \xrightarrow{\hat{\partial}} \tilde{M}_0 \]
of \( \mathbb{k}[h]/(h^2) \)-free modules over the deformation \( A_h \), such that \( \tilde{M}_i/h\tilde{M}_i \cong M_i \) and \( \hat{\partial} \) is the reduction of \( \tilde{\partial} \) modulo \( h \). For example, if each \( M_i \) is projective, then such a precomplex always exists. If this precomplex is actually a complex, that is, \( \tilde{\partial}^2 = 0 \), then this is a deformation in the derived category; if the \( M_i \)'s are a projective resolution of a module \( N \), then the cohomology of the deformation will be a flat deformation of \( N \) and \( s_N \) must be 0.

Even if \( \tilde{\partial}^2 \neq 0 \), then we still have that \( \hat{\partial}^2 \) is equal to 0 (mod \( h \)), so we have an induced map \( \hat{\partial}^2/h: M_i \rightarrow M_{i+2} \). This is a chain map, since
\[ \partial \circ \hat{\partial}^2/h = \hat{\partial}^2/h = \tilde{\partial}^2/h \circ \partial, \]
and we can think of its class up to homotopy as an obstruction to deforming the complex. Since \( s_M \) is another such obstruction, it seems logical they would agree, as indeed is true:

**Lemma 2.23.** The class of \( s_M \in \text{Ext}^2(M, M) \) agrees with the action induced by \( \hat{\partial}^2/h \) on the complex \( M \).

**Proof.** First, we will prove that this induced map commutes with all chain maps, and in particular, its action on an object in the derived category is well defined and independent of the choice of precomplex.

Assume that \( M \) and \( N \) are two complexes of \( A \)-modules, and we deform them to precomplexes \( \tilde{M} \) and \( \tilde{N} \) with “differentials” \( \tilde{\partial} \) and \( \tilde{\partial}' \). Assume that \( f: M \rightarrow N \) is a chain map. Then, modulo \( h \), we have
\[ f \circ \hat{\partial}^2/h - (\hat{\partial}')^2/h \circ f = \hat{\partial}' \circ (f \circ \hat{\partial} - \hat{\partial}' \circ f)/h + (f \circ \hat{\partial} - \hat{\partial}' \circ f)/h \circ \partial, \]
so \( (f \circ \hat{\partial} - \hat{\partial}' \circ f)/h \) exhibits a homotopy between the two maps. Most importantly, this shows that the answer will be the same up to homotopy for any deformation of the same differential on a single complex. In fact, this shows that \( \hat{\partial}^2/h \) must be the action of some element of Hochschild cohomology.
Thus, we only need to show that it agrees with $s_M$ for some quasi-isomorphic complex and one particular choice of deformed differential. Choose a splitting $\phi: A \to A_h$, and consider the deformed Hochschild complex whose $k$th term is $A_h \otimes A^{\otimes k} \otimes M$. The deformed differential on this is given by

$$\tilde{\partial}(a' \otimes a_1 \otimes \cdots \otimes a_k \otimes m) = a' \phi(a_1) \otimes a_2 \otimes \cdots \otimes m - a' \otimes a_1a_2 \otimes \cdots \otimes m + \cdots$$

One can easily check that

$$\tilde{\partial}^2(a' \otimes a_1 \otimes \cdots \otimes a_k \otimes m) = ha'r(a_1, a_2) \otimes \cdots \otimes m.$$

Thus, the resulting endomorphism agrees with that induced by the cocycle $r$. \qed

We note that if the action of $A$ on $M$ is an $A_\infty$ action, then we can consider any deformation of the $A_\infty$ operations: these induce a precomplex structure on $\tilde{M}\otimes TSA\otimes A$ whose square calculates the same Hochschild homology class.

3. Ladder bimodules

3.1. The case of Y-ladders. Fix a triple of integers $a + b = c \leq n$.

Let $L_{a,b}$ be the unique highest weight simple module over $T_{\omega c}^{a,b}$. We’ll instead want to think of this simple as a module over $T_{\omega c}^{a,b}$. Let $e_{a,b}$ be the idempotent in $T_{\omega c}^{a,b}$ summing straight line diagrams with the red strands at far left and far right, and all black strands between them.

By [Webb 4.19], we have the isomorphism $e_{a,b} T_{\omega c}^{a,b} e_{a,b} \cong T_{\omega c-a,b}^{a}$, and thus a surjective map $e_{a,b} T_{\omega c}^{a,b} e_{a,b} \to T_{\omega c-a,b}^{a}$. Since we have that $\omega_c - \omega_b = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$ with $b$ 0’s, a 1’s and $n - c$ 0’s, the corresponding weight space in $\wedge^a \mathbb{C}^n$ is 1-dimensional and extremal. The algebra $T_{\omega c-a,b}^{a}$ is thus a matrix algebra by [Webb 7.2].

**Proposition 3.1.** We have that $L_{a,b} = e_{a,b} L_{a,b}$ and $L_{a,b}$ is thus isomorphic to the unique simple module over $e_{a,b} T_{\omega c}^{a,b} e_{a,b} \cong T_{\omega c-a,b}^{a}$.

**Proof.** Since the simple module $L_{a,b}$ satisfies $\mathcal{E}_i L_{a,b} = 0$, we have that this simple is killed by the idempotent $e_{c,i}$ if the second red line is not at the far right. Thus, by process of elimination, $e_{a,b}$ acts by the identity. \qed

We can describe this simple more explicitly using the fact that $T_{\omega c-a,b}^{a}$ has a graded cellular basis by work of Hu and Mathas [HM10]. This basis is indexed by pairs of standard tableaux on an $a \times b$ rectangle.

In this paper, we read Young tableaux from the bottom left corner to the bottom right corner, and then in rows from left to right up the diagram. The content of a box in a Young tableau we define to be equal to $a + j - i$, where $i$ is the row number of the box and $j$ its column number and $a$ is the number of rows in $S$. We further identify this content with the corresponding root $\alpha_{a+i-j}$ of $\mathfrak{gl}_n$.

**Definition 3.2.** Given a standard tableau $S$, we let the **content reading word** of $S$ be the word whose $i$th entry is the content of the box containing $i$ in $S$. The **row reading word** of $S$ is the entries of $S$ read in the natural order explained above.
Example 3.3. Consider the following two tableaux:

\[
S_1 = \begin{array}{cc}
3 & 4 \\
1 & 2 \\
\end{array} \quad \quad S_2 = \begin{array}{cc}
2 & 4 \\
1 & 3 \\
\end{array}
\]

The content reading word of \(S_1\) is equal to \((2, 3, 1, 2)\) and its row reading word is equal to \((1, 2, 3, 4)\). The content reading word of \(S_2\) is equal to \((2, 1, 3, 2)\) and its row reading word is equal to \((1, 3, 2, 4)\).

Let \(R\) be the tableau in which the box \((i, j)\) has filling \((i - 1)b + j\), that is, with row reading \((1, 2, 3, \ldots, ab)\). Attached to \(S\), there is a unique permutation \(w_S\) sending \((1, 2, 3, \ldots)\) to the row reading word of \(S\). Note that \(w_R\) is the identity.

The basis vector \(C_{S, T}\) for a pair \(S, T\) is given by the diagram that traces out a string diagram for \(w_S\) read upward from \(y = 1/2\) to \(y = 1\) and a string diagram for \(w_T\) read downward from \(y = 1/2\) to \(y = 0\). The labels on strands are determined by the property that

- at \(y = 1\), reading left to right gives the content reading word of \(S\).
- at \(y = 1/2\), reading left to right gives the content reading word of \(R\), that is, \((\alpha_{a}, \alpha_{a+1}, \ldots, \alpha_{c-1}, \alpha_{a-1}, \ldots, \alpha_{c-2}, \ldots, \alpha_{1}, \ldots, \alpha_{b})\).
- at \(y = 0\), reading left to right gives the content reading word of \(T\).

This follows the general principle that each strand is attached to a box in the tableau, and the strand connects the entries of the sequences at \(y = 0, 1/2, 1\) associated to the same box of the diagrams.

Proposition 3.4 (Hu-Mathas). The vectors \(C_{S, T}\) form a basis of \(T_{\omega_c - \omega_b}^a\) with the multiplication rule

\[
C_{S, T}C_{S', T'} = \begin{cases} 
C_{S, T'} & S' = T \\
0 & S' \neq T
\end{cases}
\]

This is a special case of a much more general result for all type A cyclotomic quotients [HM10 Main Theorem].

In particular \(C_{S, S}\) is an idempotent. Note that all the vectors \(C_{S, T}\) are of degree 0, showing that all elements of \(T_{\omega_c - \omega_b}^a\) of positive or negative degree vanish.

Example 3.5. Consider the case where \(a = b = 2\), so \(c = 4\). We’re considering tableaux on a \(2 \times 2\) rectangle, of which there are two:

\[
\begin{array}{cc}
3 & 4 \\
1 & 2 \\
\end{array} \quad \quad \begin{array}{cc}
2 & 4 \\
1 & 3 \\
\end{array}
\]

Thus, we have 4 basis vectors in \(e_{a, b}T_{a, b}^a e_{a, b}\), one for each pair of these tableaux:

\[
\begin{array}{cccc}
3 & 4 & 3 & 4 \\
1 & 2 & 1 & 2 \\
\end{array} \quad \quad \begin{array}{cccc}
2 & 4 & 3 & 4 \\
1 & 3 & 1 & 2 \\
\end{array}
\]

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The relation (2.2c) shows that this is isomorphic to a $2 \times 2$ matrix algebra, as expected.

As with any matrix algebra, there is a single irreducible representation, given by the left ideal $L_{a,b} \cong T_{\omega_c-\omega_b} C_{R,R}$; in fact, we could replace $R$ by any $S$, but we prefer to have a fixed choice.

We will use a graphical notation for elements of this representation. To avoid confusion with the algebra, we’ll represent $C_{R,R}$ thought of as an element of $L_{a,b}$ as the diagram

$$C_{R,R} = \begin{array}{c}
\begin{array}{ccc}
 a & a & a+1 \\
 b & b & b
\end{array}
\end{array}$$

By our usual conventions, the labelling of the strands reminds us that all idempotents corresponding to other tableaux act by zero, since the labels on strands won’t match when we compose the diagrams. If, as before, we take $a = b = 2$, then this module is 2-dimensional, with basis given by

$$\begin{array}{c}
\begin{array}{cccc}
 2 & 2 & 3 & 1 \\
 2 & 2 & 2 & 2
\end{array}
\end{array}$$

All the relations of this module are encapsulated in the fact that $(1 - e_{a,b})L_{a,b} = 0$ and the relations in $T_{\omega_c-\omega_b}$. However, it will be useful to record some of them here for later use:

$$\begin{array}{c}
\begin{array}{cc}
 a & a \\
 b & b
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
 a \\
 b
\end{array}
\end{array} = 0$$

(3.10a)

$$\begin{array}{c}
\begin{array}{ccc}
 i & i-1 & \cdots \\
 \cdots & \cdots & \cdots
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{ccc}
 i & i+1 & \cdots \\
 \cdots & \cdots & \cdots
\end{array}
\end{array} = 0$$

(3.10b)

Note that (3.10a) and (3.10b) imply that all diagrams of positive or negative degree are zero in $L_{a,b}$; this also follows from the basis given by Proposition 3.4. In particular, this also kills any diagram which contains a dot. If we let $d$ denote the portion of
the diagram including the dot and things below it and \( d' \) denote the portion of the
diagram strictly below the dot, then \( \deg(d') = \deg(d) + 2 \). However this means that
one of these diagrams has non-zero degree, and thus is 0.

Ultimately, we’ll attach a bimodule to any ladder, but let us start with the case of
a single trivalent vertex. Recall \( Y_i \) and \( Y_i^* \):

![Diagram](image)

**Definition 3.6.** A ladder diagram for a ladder with a single trivalent vertex is a diagram
with

- red lines that trace out the ladder, labeled with the appropriate fundamental weights
  (which we’ll just denote with their number)
- black lines that map immersively to \([0, 1]\); as usual, these are labeled with simple roots
  and constrained intersect generically, that is, with no triple points or tangencies. These can carry any number of dots, which must avoid intersection points.
- over the triple point of the ladder, we include a box which contains an element of \( L_{a,b} \).
  If the \( Y \) opens upward, this box connects with 2 red and \( ab \) different black strands if
  \( a + b = c \) at distinct points on the top of the box, and connects with one red strand
  labeled \( c \) at the bottom. If the \( Y \) opens downwards, we reflect this configuration
  through a horizontal axis.
- every black strand is constrained to have its endpoints on \( y = 1, y = 0 \) or on the box.
  By the requirement that the projection to the \( y \)-axis is an immersion, we see that the
  only possible combinations are one endpoint at \( y = 0 \) and the other at \( y = 1 \), or one at
  \( y = 1 \) (resp. \( y = 0 \)) and the other on the box if the \( Y \) opens upward (resp. downward).

Using the notation introduced above for elements of \( L_{a,b} \), we can represent a ladder
diagram as a Stendhal diagram where at the trivalent vertex, we have inserted a
picture which looks like the diagram (3.9). The relation moving a crossing or dot
in or out of the box just becomes an isotopy in this schema; the relations of \( L_{a,b} \) are
encapsulated in the local relations (3.10a–3.10b).

An example of such a diagram is

![Diagram](image)

Note that the \( k \)-span of these diagrams is naturally a bimodule over the algebras
\( \tilde{T} \) freely spanned by Stendhal diagrams for the sequences at top and bottom of the
ladder.

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**Definition 3.7.** The ladder bimodule $W_{Y_i}$ of a ladder with one trivalent vertex as above is the quotient of the $k$-span of ladder diagrams by the local relations (2.2a–2.2d) and (2.3a–2.4) as well as:

- for any element of $\tilde{T}^{a,b}$, we obtain the same result by tracing out the Stendhal diagrams with the strands leading to the top of the box, or by acting on the element of $L_{a,b}$ in the box.
- black strands can pass through the ladder triple point up to a scalar. Let the scalar $\sigma(m,a,b)$ be $1$ if the number of elements of $\{a, b, c\}$ less than $m$ is odd and $-1$ if it is even. That is, $\sigma(m,a,b) = 1$ if $a \leq m < b$ or $b \leq m < a$ or $m \geq c$ and $-1$ otherwise.

\[ (3.11a) \]

\[ (3.11b) \]

We let $W_{Y_i}^\ast$ be the bimodule $W_{Y_i}$ with left and right actions exchanged by the anti-automorphism of $\tilde{T}^p$ and $\tilde{T}^{p'}$ reflecting diagrams through the x-axis with internal and homological grading shifted downward by the quantity $\eta(Y_i) = -ab$. Pictorially, we can think of this as a bimodule with the same definition as $W_{Y_i}$ but with the $Y$ upside down (and a grading shift).

We fix $p$ to be a sequence of indices given by the bottom of $Y_i$ (resp. top of $Y_i^\ast$), with $p_i = c$ and $p'_i$ to be the sequence at the top of $Y_i$ (resp. bottom of $Y_i^\ast$). That is, we have that $p' = (\ldots, p_{i-1}, a, b, p_{i+1}, \ldots)$. We can attach a Stendhal diagram from $\tilde{T}^{p'}$ to the top of $Y_i$ or bottom of $Y_i^\ast$; similarly, a diagram from $\tilde{T}^p$ can be attached to the bottom of $Y_i$ or top of $Y_i^\ast$. This endows $W_{Y_i}$ (resp. $W_{Y_i^\ast}$) with a $\tilde{T}^{p'}$-$\tilde{T}^p$-bimodule structure (resp. $\tilde{T}^p$-$\tilde{T}^{p'}$-bimodule structure) by the stacking of diagrams.

If $X$ and $Y$ are functors between categories with a categorical action of $U_n$, then a **strong equivariant** structure on $X$ is a system of natural isomorphisms $u \circ X \cong X \circ u$ for every 1-morphism $u$ which satisfies the obvious commutative square for every 2-morphism. We say that a natural transformation $X \to Y$ of strongly equivariant functors **commutes with the action of** $U_n$ (resp. $U_n^\ast$) if the obvious square commutes.

**Proposition 3.8.** The ladder bimodules $W_{Y_i}, W_{Y_i^\ast}$ commute with induction functors, that is, they have a strongly equivariant structure for the action of $U_n$ (resp. $U_n^\ast$) if the obvious square commutes.
Categorified skew Howe duality and comparison of knot homologies

(3.11a–3.11b) to slide the new strands above the $Y$. Thus this diagram is in the image of this map.

Now, consider an element of its kernel. This is a sum of diagrams which become 0 if we allow relations where the new strands can pass below the $Y$, but not if they must stay above. Thus, in $W_A \otimes \mathfrak{S}_N$, we can write it as a sum of relations. Now, we take these relations and “unzip the $Y$.” That is, in each relation, we push the branch point in the $Y$ further down. When we push through a black strand, we include the sign in (3.11a–3.11b); we are not using this relation, but rather we wish to show that doing this on both sides of a relation will result in a new relation, by the locality of relations, or will simply result in the two sides coinciding if the relation is (3.11a–3.11b).

This requires some care about the relations (2.3a,2.4). For (2.3a), we wish to show that

$$b_{k} a_{k} b_{k} \cdots - b_{k} a_{k} b_{k} \cdots = \delta_{k,c} e_{k,\omega_{a_{1}},\ldots,b_{1},\omega_{b_{1}},k}$$

In order to do this calculation, we apply the relations (2.2d, 2.3a) successively. We leave the reader to check that whenever $c \neq k$, the correction terms which appear create a bigon with both sides labeled $k$, and thus are 0 by (2.2c). If $k = c$, then the crossing commutes past every strand in the diagram except the one labeled $c - 1$. In moving past this strand, we apply the relation (2.2d); now, we have a non-zero correction term which breaks open the crossing. The relation (2.2c) shows that the strands labeled $c$ can be pulled straight, and thus give the idempotent attached to $c, \omega_{a_{1}}, \ldots, b, \omega_{b_{1}}, c$.

Now we return to (2.4). What we must show is that

$$a_{a} a_{a+1} b_{k} k = \sigma(k,a,b)$$

In order to do these calculations, let us note the following relations:

$$k \pm 1 \quad k \mp 1 \quad k $$

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In order to calculate the LHS of (3.13), we use the equations above. To organize this calculation, recall that the strands that interact with one labeled $k$ correspond to boxes of content $k, k \pm 1$ in a $a \times b$ box. We can think about computing the left hand side as a process of killing bigons and stripping off the corresponding boxes. The equations (3.14a–3.14d) show how this process can be systematized into removing chunks which look like 3-box tetris pieces. In each case, the second term on the RHS contributes 0 because its product with $C_{RR}$ is 0 by (3.10a).

These pieces can be categorized as positive or negative depending on how they affect the sign of the remaining term:

- **Positive pieces:**
  \[
  \begin{array}{cccc}
  i-1 & i & i+1 & a +1 \\
  \end{array}
  \]

- **Negative pieces:**
  \[
  \begin{array}{cccc}
  i-1 & i & i+1 & a -1 \\
  \end{array}
  \]

Assume for simplicity that $a < b$; if $k < a$, then the boxes of content $k, k \pm 1$ can broken into one negative $L$ and then $k - 1$ positive pieces. This matches $\sigma(k, a, b) = -1$.

If $k = a$, we can use the postive piece with $a$ at the corner, and then positive pieces. If $a < k \leq b$, then we can use all positive pieces. This matches $\sigma(k, a, b) = 1$. If $b < k < c$, then as when $k < a$, we use one $L$ and then positive pieces, matching $\sigma(k, a, b) = -1$. If $k > c$, then, there is no interaction between any strands, matching $\sigma(k, a, b) = 1$.

The remaining case of $k = c$ follows from 2.2c; the $c$-labeled strand only interacts with the sole strand labeled $c - 1$, producing the desired dot on the strand labeled $c$. The other term, that with a dot on the strand labeled $c - 1$, is 0 since any dot kills $L_{a,b}$.

The ultimate result is a series of relations, all of which keep the new strands above the $Y$, which shows that this element of the kernel is 0. □
The right $U_\mathcal{U}$-equivariant structure also induces a $U_\mathcal{U}$-equivariant structure on the ladder bimodules over $T$. Let $\phi: \mathcal{F}_j W_{Y_i} \to W_{Y_i} \mathcal{F}_j$ denote the commutation isomorphism. Note that $\varepsilon_j W_{Y_i}$ and $W_{Y_i} \varepsilon_j$ are both adjoints of $W_{Y_1} \mathcal{F}_j \cong \mathcal{F}_j W_{Y_1}$ and therefore have to be naturally isomorphic. The unit and counit we pick are of course the ones induced by the units and counits of the adjunctions between $W_{Y_i}$ and $W_{Y_1}$, and $\varepsilon_j$ and $\mathcal{F}_j$. Since the unit and counit of the first adjunction are $U_\mathcal{U}$ equivariant, the induced units and counits for $\varepsilon_j W_{Y_i} \cong W_{Y_i} \varepsilon_j$ and $W_{Y_1} \mathcal{F}_j \cong \mathcal{F}_j W_{Y_1}$ are independent of the possible choices in their definition and they are compatible with the commutation isomorphisms. By the zig-zag relation for the unit and counit of $W_{Y_i}$ and $W_{Y_1}$, we can simplify the definition of the commutation map $\phi': W_{Y_i} \varepsilon_j \to \varepsilon_j W_{Y_i}$: It is given by the composite

$$W_{Y_i} \varepsilon_j \to \varepsilon_j \mathcal{F}_j W_{Y_i} \varepsilon_j \cong \varepsilon_j W_{Y_i} \mathcal{F}_j \varepsilon_j \to \varepsilon_j W_{Y_i}.$$ 

This isomorphism is the “vertical reflection” of the commutation map $\mathcal{F}_j W_{Y_i} \to W_{Y_1} \mathcal{F}_j$.

**Lemma 3.9.** The maps $\phi$ and $(\phi')^{-1}$ define an equivariant structure over $U_\mathcal{U}$.

**Proof.** Since the morphisms in $U_\mathcal{U}$ are generated by the KLR endomorphisms of $\mathcal{F}_n$, the unit and counit of the adjunction $(\mathcal{F}_j, \varepsilon_j)$ and the inverse of a map build of these. Thus commutation with the KLR endomorphisms and the unit and counit suffice. The former follows by $U_\mathcal{U}$-equivariance, and the latter is automatic from the definition of $\phi'$.

Note that from the definition of $\phi'$ above, it is also clear that if a given natural map between two ladder bimodules commutes with $U_\mathcal{U}$, then it commutes with the whole $U$.

The bimodule $e_{i_b} W_{Y_i} e_{j_c}$ has a basis in bijection with quadruples consisting of

1. a permutation $\pi \in S_n$ acting on the entries of $j$,
2. a vector $a \in \mathbb{Z}_{\geq 0}^m$
3. a tableau $S$ on $a \times c$ rectangle together with
4. a shuffle $\sigma$ of $\pi \cdot j$ and the content word of $S$ which is equal to $i$, such that all entries from the content word lie in the segment $[\kappa(i) + 1, \kappa(i + 1)]$.

We construct a basis vector from this data by taking the basis vector of $L_{a,b}$ associated to the tableau in the box, connecting the strands from the top of the box to $y = 1$ without crossings, and then adding in strands starting at $y = 0$ corresponding to $j$. The strands trace out a string diagram of the permutation $\pi$, and the interlace with those from the top of the box according to the shuffle $\sigma$ while creating a minimal number of crossings of all types. We then multiply at the bottom by $y_1^{\sigma_1} \cdots y_m^{\sigma_m}$. There is not a unique such diagram, but for each set of data, we simply choose one.

For $W_{Y_1}$ we obtain a similar basis by reflecting the basis elements above in a horizontal axis.

**Proposition 3.10.** The diagrams as above for all appropriate triples form a basis of $W_{Y_1}$.

**Proof.** Consider $W_{Y_i} e_{i_b}$, the other case can be proved similarly. We need to show that this has the correct basis. If $i = \emptyset$ and $\ell = 1$, then this is just the basis of $L_{a,b}$ we already
know and the result holds. Otherwise, by Proposition 3.8, the module $W_Y e_{t^x}$ is an induction, and the result follows since an induction has a basis given by all ways of shuffling together basis vectors of the two modules.

This basis further shows that $e_{t^x} W_Y$ is projective as a right module:

**Corollary 3.11.** The right module $e_{t^x} W_Y$ is isomorphic to the sum of $e_{t^{x'}} \tilde{T}$ with multiplicity given by the number of tableaux $S$ and shuffles $\sigma$ of $1'$ with the content word of $S$ which are equal to $i$. 

3.2. The case of $a = 1$. We’ll consider further the structure of $\tilde{T}^{|1,c|}$. Every sequence that appears in this algebra is a permutation of $(\omega_1, 1, \ldots, c - 1, \omega_{c-1})$. Note that the space of degree $0$ endomorphisms of such a projective is $1$-dimensional in each case, so these projectives are all indecomposable. Conversely, every indecomposable is necessarily of this form. Furthermore, if two such sequences differ by permuting symbols which are not consecutive in the sequence above, they are isomorphic by (2.2c) or (2.4).

Consider the collection of proper subsets $S \subseteq [1,c]$. Let $w_S$ be the longest permutation of $[0,c]$ such that we have $m < k$ and $w_S(m) > w_S(k)$ if and only if $[m + 1, k] \in S$. Let $P_S$ be the projective over $\tilde{T}^{|1,c|}$ associated to the unique sequence obtained by applying $w_S$ to $(\omega_1, 1, \ldots, c - 1, \omega_{c-1})$.

**Lemma 3.12.** The projectives $P_S$ are a complete irredundant collection of indecomposable projectives for $\tilde{T}^{|1,c|}$.

**Proof.** We’ve already observed that every indecomposable projective must be isomorphic to one of these, by a diagram crossing strands labeled with distant roots.

On the other hand, if $S \neq S'$, there are no diagrams of degree $0$ joining them, so the corresponding projectives are not isomorphic. □

Let $S \triangle S' = (S \cup S') \setminus (S \cap S')$ be the operation of symmetric difference. For every $k$, we have a unique dotless diagram $D_{S,k}$ with a minimal number of crossings with bottom $P_S$ and top $P_{S \triangle[k]}$. One can easily check that this diagram has degree $1$. We’ll define a map $x_{S,k}: P_S \to P_{S \triangle[k]}$ given by $D_{S,k}$ unless $k = c \in S$, in which case we take $-D_{S,k}$. One can easily calculate that:

**Lemma 3.13.** For all $k, m \in [1, c]$, we have

\[
(3.15) \quad x_{S \triangle[k],m} x_{S,k} = x_{S \triangle[m],k} x_{S,m} \quad \sum_k x_{S \triangle[k],k} x_{S,k} = 0
\]

**Proof.** The products $x_{S \triangle[k],m} x_{S,k}$ and $x_{S \triangle[m],k} x_{S,m}$ are both the unique diagram with a minimal number of crossings and no dots from $P_S$ to $P_{S \triangle[k,m]}$, unless $k = c$ or $m = c$, in which case both are the negatives of this diagram. In either case, they are equal.

The product $x_{S \triangle[k],k} x_{S,k}$ is the identity on $P_S$ times a dot on the strand labeled $k + 1$ minus a dot on that labeled $k$ if $0 < k < c$. If $k = c$, we only get the negative term due to the sign change $x_{S,k} = -D_{S,k}$, and if $k = 0$ only the positive. Thus, in the sum of (3.15), all terms appear twice with opposite signs, and we get that the sum is $0$. □
Proposition 3.14. The endomorphism algebra $A = \text{End}(\oplus S P_s)$ is a quadratic algebra generated by the idempotents $e_s$, the degree 1 morphisms $x_{S,k}$, and with all relations given by (3.15).

The algebra $A$ is isomorphic to the algebra $A_{\text{pol}}^1(\eta, -)$ defined in [BLPW12 §8.5] corresponding to the intersection of the coordinate arrangement in $\mathbb{R}^n$ with the affine subspace $\{(z_1, \ldots, z_n) | z_1 + \cdots + z_n = 1\}$.

Proof. First, note that the morphisms $x_{S,k}$ generate $A$; obviously, any diagram without dots can be factored into degree 1 morphisms, which must correspond to a factorization as $x_{S,k}$. Thus, we need only show we can get all dots. However, the products $x_{S_{\Delta[k]}, k} x_{S,k}$ span the degree 2 dots, with one relation (given by (3.15)).

The equations (3.15) correspond to the relations of $A_{\text{pol}}^1(\eta, -)$ as described in [BLPW12 §3.3]. The first set to the relation that the length two paths between $\alpha$ and $\beta$ are the same, and the second to the relation killing the image of $\delta(t)$, since in this case, $t$ corresponds to the diagonal $\mathbb{C}^*$-action. The fact that we only consider proper subsets matches the fact that the subspace $\{(z_1, \ldots, z_n) | z_1 + \cdots + z_n = 1\}$ misses one of the chambers of $\mathbb{R}^n$, that where all $z_i < 0$.

Thus, we have an algebra map $A_{\text{pol}}^1(\eta, -) \to A$. This map is surjective since the $x_{S,k}$ generate. On the other hand, $A$ and $A_{\text{pol}}^1(\eta, -)$ have the same Hilbert series since in both cases, the morphism space between two projectives is a free module over polynomials in $c - 1$ variables generated by a morphism of minimal degree equal to the length of the shortest path between the two idempotents. \hfill \Box

By [BLPW12 8.25], this shows that $A$ is Koszul, and we can construct a canonical projective resolution of every simple using the Koszul resolution from [BGS96]. Let $B$ be the quadratic dual of $A$. The Koszul resolution of the simple quotient of $P_s$ has the form

$$\cdots \to A \otimes_{A_0} B^*_2 e_S \to A \otimes_{A_0} B^*_1 e_S \to A \otimes_{A_0} B^*_0 e_S.$$ 

If we identify $B^*_n$ with $A_1 \otimes_{A_0} \cdots \otimes_{A_0} R \otimes_{A_0} \cdots \otimes_{A_0} A_1$ where $R \subset A_1 \otimes_{A_0} A_1$ is the span of the relations (3.15), then the differential just becomes the usual Hochschild differential.

Let $x'_{S,k} = (-1)^{\#(S \cap [1\ldots k-1])} x_{S,k}$. A wall path of length $m$ from $S$ to $S'$ is a sequence $S' = S_0, S_1, \ldots, S_m = S$ such that $S_p = S_{p-1} \Delta[k_p]$ for some sequence $k_1, \ldots, k_m$.

Proposition 3.15. The space $e_S^* B^* e_S$ has a basis given by the sums

$$b_m = \sum_{S=S_0, S_1, \ldots, S_m} x'_{S_{m-1}, k_m} \otimes \cdots \otimes x'_{S_0, k_1}$$

ever wall paths of length $m = \#(S \Delta S') + 2p$ for some $0 \leq p < c - \#(S \cup S')$.

Proof. By [BLPW12 8.25], the dimension of $e_S^* B^*_0 e_S$ is $c - \#(S \cup S')$, the number of vertices of the intersection of the corresponding closed chambers. Thus, we need only show that $b_m$ lies in this space (since these are obviously non-zero and linearly independent).

Of course, a wall paths of length $m$ from $S$ to $S'$ can be factored into paths from $S$ to $S_1$ of length $g$, from $S_1$ to $S_2$ of length 2, and from $S_2$ to $S'$ of length $m - g - 2$. Fixing the first and last of these paths, we can break $b_m$ into a sum over pairs $S_1$ and $S_2$ and length
two paths from $S_1$ to $S_2$. If $S_1 = S_2$, then we obtain

\[
\sum_k x_{S_1 \Delta |k|, k}^' \otimes x_{S_1, k}^' = \sum_k x_{S_1 \Delta |k|, k} \otimes x_{S_1, k}
\]

which lies in $R$ by (3.15). If $S_1 \neq S_2$ then as long as $S_1 \cup S_2 \neq [1, c]$, we arrive at

\[
x_{S_1 \Delta |k|, k}^' x_{S_1, k}^' + x_{S_2 \Delta |k'|, k'}^' x_{S_2, k'}^' = \pm (x_{S_1 \Delta |k|, k} x_{S_1, k} - x_{S_2 \Delta |k'|, k'} x_{S_2, k'})
\]

where $\{k, k'\} = S_1 \Delta S_2$. This is in $R$ by (3.15), completing the proof. Note that we need an upper bound on $m$ precisely to avoid the case where $S_1 \cup S_2 = [1, c]$. □

If $S = \emptyset$, then this is particularly simple; $e_S B^e_\emptyset$ is spanned by $b_m$ for $m = \#S', \#S' + 2, \ldots, 2c - 2 - \#S'$. Let $Q_{-k}$ be the sum of $P_{S'}(-k)$ for all $S'$ such that $\#S' = k \pmod{2}$ and $\#S' \leq k \leq 2c - 2 - \#S'$. We have a differential $Q_{-k} \to Q_{-k+1}$ where the component $P_S \to P_{S \Delta |k|}$ is just $x_{S, k}^'$.

**Corollary 3.16.** As a left module $L_{1,c-1}$ has a resolution as a left module of the form:

\[
Q_{-(2c-2)} \cong P_{0}(-2c + 2) \to \cdots \to Q_{-k} \to \cdots \to Q_{-1} \cong \bigoplus_{s \in [1,c]} P_{[s]} \to Q_{0} \cong P_{\emptyset}.
\]

For example, if $c = 2$, then the resolution is 3 step, of the form:

\[
\begin{array}{c}
X \\
| | \\
\pi_{1,1} \Rightarrow L_{1,1}
\end{array}
\]

where

\[
\begin{array}{c}
| | = P_0 \\
| | = P_{[1]} \\
| | = P_{[2]}
\end{array}
\]

For $c = 3$, we have a slightly more complicated example, which is illustrative:
Reading from top to bottom, we have: in the first and the last column $P_0$; in the second and the fourth column, the sum $P_{[1]} \oplus P_{[2]} \oplus P_{[3]}$; and in the middle column $P_{[1,2]} \oplus P_0 \oplus P_{[1,3]} \oplus P_{[2,3]}$. The differentials are defined by the diagrams $\pm D_{S,k}$.

Using induction, this is easily extended to a resolution of $W_{Y_i}$ as a left module.

As explained in Sect. 2.3, this resolution has a right $A_\infty$ action, whose existence is important to us but not its precise definition, which we therefore omit.

Lemma 3.17. If $a = 1$ or $b = 1$, we have

$$\mathbb{R} \text{Hom}_{\tilde{T}^p}(W_{Y_i}, \tilde{T}^p') \cong W_{Y_i}^* \langle c - 1 \rangle \quad \mathbb{R} \text{Hom}_{\tilde{T}^p}(W_{Y_i}^*, \tilde{T}^p) \cong W_{Y_i} \langle 1 - c \rangle.$$  

This defines adjunction maps

$$\epsilon_{Y}: W_{Y_i} \overset{L}{\otimes}_{\tilde{T}^p} W_{Y_i}^* \langle c - 1 \rangle \rightarrow \tilde{T}^p'$$

$$\iota_{Y}: \tilde{T}^p \rightarrow W_{Y_i} \overset{L}{\otimes}_{\tilde{T}^p'} W_{Y_i} \langle c - 1 \rangle$$

and

$$\epsilon_{Y^*}: W_{Y_i}^* \overset{L}{\otimes}_{\tilde{T}^p} W_{Y_i} \langle 1 - c \rangle \rightarrow \tilde{T}^p'$$

$$\iota_{Y^*}: \tilde{T}^p' \rightarrow W_{Y_i}^* \overset{L}{\otimes}_{\tilde{T}^p} W_{Y_i} \langle 1 - c \rangle$$

Proof. In both cases, we can reduce to assuming that there are no red or black strands but those coming from the trivalent vertex. Also, the cases $a = 1, b = c - 1$ and $a = c - 1, b = 1$ are completely symmetric, so we can assume we are in the former case.

Applying $\mathbb{R} \text{Hom}_{\tilde{T}^p}(-, \tilde{T}^p)$ to the resolution of Lemma 3.16 we get $\text{Hom}(Q_k, \tilde{T}^p) \cong Q_{2c-2-k}(2 - 2c)$. The part of the differential $P_S \rightarrow P_{S \cup k}$ is given by $x_{S,k}$ and its dual is the mirror image of this through the $x$-axis, which is $x'_{S,k}$ (unless $k = 0$, in which case we must fix the sign) and thus this isomorphism matches $\partial^*_k$ with $\partial_{2c-k}$. Thus, we arrive at the same complex, but with degrees shifted upward by $2c - 2$, since taking $\mathbb{R} \text{Hom}$ negates the degrees of the complex.
On the other hand, as a left module $Y_i^*$ is projective, so its dual is again a bimodule. In fact, we simply take the vector space dual of $L_{a,b}$, the same underlying simple but with the action switched from left to right using reflection. This dual is just $L_{a,b}$. □

Thus, we have three functors, which are isomorphic up to shift, but not canonically so: $Y_i^*$, the left adjoint $Y_i^L(c-1)$ and right adjoint $Y_i^R(1-c)$.

In order to fix an isomorphism $Y_i^* \equiv Y_i^L(c-1)$, we should define a map $\iota_{Y_i} : \text{id} \rightarrow Y_i^* Y_i(c-1)$ which we identify with the unit of the adjunction. We let $\epsilon_{Y_i}$ be the induced counit, which is unique. Similarly, an isomorphism $Y_i^* \equiv Y_i^R(1-c)$ will be fixed by choosing a map $\epsilon_{Y_i'} : Y_i'^* Y_i(1-c) \rightarrow \text{id}$, which we match with the counit of the other adjunction, and let $\iota_{Y_i'}$ be the unique induced unit.

**Lemma 3.18.** If $a = 1$ (resp. $b = 1$), then $y_i^b$ (resp. $y_{i+1}^a$) induces an isomorphism from the $-c + 1$st homological degree to the $c - 1$st homological degree.

**Proof.** We’ll again apply Lemma 2.23. Assume that $a = 1$. We realize $Y_i^* \otimes_{\mathbb{T}^a} Y_i$ as the tensor product of two copies of the resolution Corollary 3.16. By general results, this is an $A_\infty$-bimodule over $\mathbb{T}^a$; the structure of the higher products will be irrelevant for us.

There is one homotopy representative of $\iota_{Y_i}$ which sends $1 \mapsto X_{2c-2} \otimes X_0 + \cdots + X_0 \otimes X_{2c-2}$, where $X_0$ is the usual generator of $Q_0$, and $X_{2c-2}$ the usual generator of $Q_{2c-2}$.

We consider the precomplex over the deformation associated to $y_i$ given by the same diagrams as the complex $Q_{2c-2} \rightarrow \cdots \rightarrow Q_0$. The differential is no longer 0. Instead, the map $\delta^2/h$ induces the map $Q_j \rightarrow Q_{j+2}$ which acts by the identity on $P_\delta$ if $\#S \leq j - 2 < j \leq 2p_i - \#S$, and 0 on all other summands. Thus, the $p_i$th power of this chain map is just the identity map on $Q_{2c-2} \equiv P_0 \equiv Q_0$. Thus, $(y_{i+1}^a \otimes 1)\iota_{Y_i}$ maps $1 \mapsto X_0 \otimes X_0$. The result then follows from the fact that $\epsilon_{Y_i'}(X_0 \otimes X_0)$ is a non-zero scalar, which by definition we take equal to 1. Note that this uniquely determines $\epsilon_{Y_i'}$.

In the case where $b = 1$, the argument is almost the same, except that the signs in the definition of the complex $Q_*$ are slightly different, since $x_{S_\delta c}$ is minus the corresponding diagram if $c \in S$. Thus, $\delta^2/h$ gives the map $Q_j \rightarrow Q_{j+2}$ given by multiplication by $-1$. This also gives an isomorphism, so it completes the proof. □

From the proof of Lemma 3.18 we immediately get the following result.

**Lemma 3.19.** We can fix the biadjunction of $Y_i$ and $Y_i^*$ utilizing the isomorphism in Lemma 3.18, such that: if $a = 1$, then

$$
\epsilon_{Y_i'}(y_i^b \otimes 1)\iota_{Y_i} = \begin{cases} 
0 & q < b \\
1 & q = b 
\end{cases};
$$

if $b = 1$, then

$$
\epsilon_{Y_i'}(1 \otimes y_{i+1}^a)\iota_{Y_i} = \begin{cases} 
0 & q < a \\
(-1)^a & q = a
\end{cases}.
$$
As remarked above, this also fixes $t_{Y_i}$ and $e_{Y_i}$ uniquely by the usual zig-zag relations for units and counits.

Note, for the case where $a = b = 1$, these conditions define the same adjunction: as shown in the proof of Lemma 3.18, the action of $y_i$ and $-y_{i+1}$ agree when we identify $e_{i-1}$ and $f_i$ (both of which are $Y_i$ when thought of as webs, rather than ladders).

Finally, we wish to note the categorical version of the bigon relation: let $Y_{i+1}$ be the bigon ladder on the $i$th red strand when $p_i = c$, and the sides of the bigon are labeled with $c - 1$ and 1.

**Proposition 3.20.** $W_β \cong \bar{T} \langle c - 1 \rangle \oplus \bar{T} \langle c - 3 \rangle \oplus \cdots \oplus \bar{T} \langle 1 - c \rangle$.

**Proof.** This is immediate from Lemma 3.16. As usual, we can reduce to the case where at the outside there are no black strands. If we resolve $W_{Y_i}$ as a left module, and then tensor with $Y_i^\star$, then the terms corresponding to $P_S$ for $S \neq 0$ all vanish, and we are left with a 1-dimensional contribution from $P_1$ in the degrees $c - 1, \ldots - (c - 1)$ and thus isomorphic to $\bar{T} \rho$ as a bimodule.

Note that in particular, if $a = b = 1$, this implies that $\text{Ext}^{-2}(W_β, W_β) \cong Z(\bar{T} \rho)(-2)$. In particular, the space of elements of degree $-2$ in this space is 1-dimensional, spanned by $\iota \circ \epsilon$.

### 3.3. General ladders.

**Definition 3.21.** The ladder bimodule for a general ladder is the composition of the $\bar{T}$-bimodules attached to a slicing of the ladder into trivalent ladders.

**Proposition 3.22.** For every ladder bimodule $W$ over $\bar{T} \rho^{'} - \bar{T} \rho$, we have that $T \rho^{'} \otimes_{\bar{T} \rho^{'}} W \cong W \otimes_{\bar{T} \rho^{'}} T \rho$ as $T \rho^{'} - T \rho$ bimodules and all higher Tors vanish.

**Proof.** We can reduce to the case of the ladder $Y_i$. As a right module, $W_{Y_i}$ is projective, and thus the higher Tors always vanish. On the other hand, as a left module $W_{Y_i}$ is the simple module $L_{a,b}$ induced on the left with $\bar{T} \rho^1$ and on the right with $\bar{T} \rho^1$ where $p'$ is the concatenation $(p_1, a, b, p_2)$. We have that $T \rho^{a,b} \otimes_{\bar{T} \rho^{a,b}} L_{a,b} \cong L_{a,b}$ by Lemma 2.17. Thus, we have vanishing higher Tors in general.

Thus, it only remains to prove the first claim of the proposition. The module $T \rho \otimes_{T \rho^{'}} W$ kills violating strands above the $Y$ and $W \otimes_{T \rho^{'}} T \rho$ kills them below it. Thus, we need only show that these are the same subbimodule.

Consider a diagram with a violating strand above the $Y$, and let $y_0$ be the lowest value of $y$ where a black strand crosses from right of the red strand (not violating) to left of it (violating) when reading bottom to top. If this is below the $Y$, then we are done. Otherwise, follow the course of this black strand toward the bottom of the diagram. If it joins the top of the box, then we push this strand through any crossings separating it from the box. Once there are no such crossings and dots, we will have a diagram which is 0 by (3.10a), leaving only correction terms with fewer crossings. Eventually, we can rewrite this diagram as a sum of others with fewer crossings where this strand is still violating and doesn’t reach the top of the box. Thus, we can now use relation (3.11a–3.11b) to move the violating strand partially below the $Y$. 

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Similarly, any diagram with a violating strand below the $Y$ can be rewritten as one with such a point above the $Y$, using \textcolor{red}{(3.11a-3.11b)}. This completes the proof. \hfill \Box

This result shows that tensoring with the cyclotomic quotients on the left or right defines a 2-functor from the 2-category above to the 2-category with the same objects and 1-morphisms, but 2-morphisms given by maps between ladder bimodules over $T$. In particular, together with Lemma \textcolor{red}{2.17}, it shows that:

**Corollary 3.23.** For any ladder bimodules $W_1, W_2$ over $T$, we have that

$$TP_1 \otimes_{T^p_1} (W_1 \otimes_{T^p_2} W_2) \equiv (W_1 \otimes_{T^p_2} T^{p_1}) \otimes_{T^p_2} (T^{p_2} \otimes_{T^p_2} W_2) \equiv (W_1 \otimes_{T^p_2} W_2) \otimes_{T^p_3} T^{p_3}.$$

Assume that $c_1 + c_2 + c_3 = c$. We have two different ways of branching from a strand labeled $c$ to $c_1, c_2, c_3$, depending on whether we split off the left or right strand first:

\begin{center}
\begin{tikzpicture}
\draw (0,0)--(0,1);
\draw (1,0)--(1,1);
\end{tikzpicture}
\end{center}

(3.16)

**Proposition 3.24.** The bimodules attached to the ladders in \textcolor{red}{(3.16)} are canonically isomorphic.

**Proof.** Since the $Y$-ladder is projective on the bottom, both of these ladders are attached to an honest bimodule. First we check that these bimodules are isomorphic. Commutation with induction functors means that we only need to show this for the block of $\tilde{T}^c$ with no black lines. That is, we must show that applying these two ladders results in the same module over $\tilde{T}^{(c_1,c_2,c_3)}$. Let $L_1$ be the module resulting from the first ladder, and $L_2$ from the second. The resulting module in both cases is killed by all restriction functors. Since $\Lambda^c C^n \otimes \Lambda^c C^n \otimes \Lambda^c C^n$ contains $\Lambda^c C^n$ with multiplicity 1 (as is easily shown, for example, with jeu de taquin), there is a unique such simple $L$. Both $L_1$ and $L_2$ must be iterated extensions of this simple. Furthermore, if we let $i_1$ be the idempotent for the sequence

$$(\omega_{c_3}, c_1 + 1, \ldots, c_1 + c_3)$$

then $\dim e_{i_1} L_1 = 1$, so $L_1$ must itself be simple. Similarly, if $i_2$ corresponds to

$$(\omega_{c_3}, c_1 + 1, \ldots, c - 1, c_1 - 1, \ldots, c - 2, \ldots, 1, \ldots, c_3)$$

then $\dim e_{i_2} L_2 = 1$, so $L_2$ must be simple as well. Thus, these modules are isomorphic.

Thus, we can fix an isomorphism between these modules by producing an isomorphism between the 1-dimensional spaces $e_{i_1} L_1$ and $e_{i_2} L_2$. The former space is generated by a straight line diagram, so we only describe the image of this vector in $e_1 L_2$. First, let $v$ be the vector in $L_{c_1,c_2+1,c_3}$ attached to the standard tableau on a $c_1 \times (c_2 + c_3)$ box where we fill the first $c_2$ boxes in all the rows in order, and then the
last $c_3$. That is, fills $(i, j)$ with $j + c_3(i - 1)$ if $j \leq c_2$ and with $j + (c_1 - 1)c_2 + c_3(i - 1)$. If $c_1 = c_3 = 2$ and $c_2 = 1$, then this tableau is

$$
\begin{array}{ccc}
2 & 5 & 6 \\
1 & 3 & 4 \\
\end{array}
$$

Now apply the second split (that is, tensor with $Y_2$), taking the straight line diagram in the smaller $Y$. Finally, we act on this diagram by pulling the rightmost $c_1c_3$ strands in the large $Y$ into the smaller $Y$ without introducing any additional crossings between black strands. This has the correct top, and a minimal number of crossings in order to obtain this top. Thus, it defines an isomorphism. In fact, one can easily check that the map defined symmetrically through a horizontal axis is inverse to this one, since in both cases, all crossings introduced are between labels that commute. □

For example, if $c_1 = c_3 = 2$ and $c_2 = 1$, then $i_1 = (ω_2, 2, 1, ω_1, 3, 4, 2, 3, 1, 2, ω_2)$. The image of the straight-line diagram in $e_iL_2$ is

One can easily extend this argument to show that any two ladders that start at a single red line and branch out to the same top have canonically isomorphic bimodules. By Mac Lane’s coherence theorem, it suffices to show that the following diagram commutes on the nose:

We number the ladders in this coherence diagram 1 to 5 as indicated. Note that in the proof of Proposition 3.24, there is a canonical degree-zero isomorphism $e_{i_1}L_2 \cong e_{i_2}L_2$ induced by the permutation diagram with bottom labeled $i_1$ and top labeled $i_2$. Similarly, all the ladders in the coherence diagram above have a natural idempotent
associated to their top boundary, denoted $e_i, \ldots, e_k$. And there is a canonical degree-
zero isomorphism $e_i L_j \cong e_i L_{j'}$ for every $j = 1, \ldots, 5$, where $L_j$ is the simple module
associated to the $j$th ladder as in the proof of Proposition 3.24. Furthermore, there are
canonical degree-zero isomorphisms $e_i L_{j+1} \cong e_i L_{j+1}$, for $j = 1, \ldots, 3$, and $e_i L_4 \cong e_i L_4$.
Using the above canonical isomorphisms, one can easily compute the image of the
generator of $e_i L_1$ in $e_i L_{j'}$, for any $j = 2, \ldots, 5$: it is always given by the degree-zero
diagram with a minimal number of crossings whose top is $i_1$ and which at each vertex
starts with the canonical configuration of black strands. This shows in particular that
the two isomorphisms $L_1 \to L_4$ in the coherence diagram, the one composed by the
isomorphisms over the top and the other composed by the isomorphisms over the
bottom, are equal.

We will use this associativity isomorphism many times over the course of the paper.
Note that this associativity can be used to show the invariance of the corresponding
bimodule under changing slicing.

For any ladder bimodule $W$, we can consider the bimodule $W^*$ attached to the
ladder given by its reflection through the $x$-axis. This coincides with the bimodule
given by switching the left and right actions on $W$ via the reflection automorphism
on $\hat{T}$. Let $\eta(W)$ be the sum over trivalent vertices in $W$ of the quantity $\eta$ defined as follows:
for $Y_i$ or $Y_i^*$ labeled $a, b, a + b$, let $\eta(Y_i) = -ab$.

**Proposition 3.25.** For every ladder bimodule $W$, we have $\mathbb{R} \text{Hom}_{\hat{T}^p}(W, \hat{T}^p) \cong W^* \langle \eta(W) \rangle$.
This defines adjunction maps

$$
\varepsilon_W : W \otimes_{\hat{T}^p} W^* \langle \eta(W) \rangle \to R \\
\iota_W : R \to W^* \otimes_{\hat{T}^p} W \langle \eta(W) \rangle
$$

**Proof.** Of course, it’s sufficient to prove this for ladders with one trivalent vertex.
We can achieve this by induction on $\min(a, b)$. The case $\min(a, b) = 1$ was proved in
Lemma 3.17. For the induction step we can assume that $1 < a \leq b$. Then we can
attach the ladder which splits the first strand to arrive at $(a - 1, 1, b)$. By Proposition
3.24, this is the same as starting with $(a - 1, b + 1)$ and splitting the second strand. By
induction, both these ladders have the desired adjoints, so the same is true of their
composite. This shows that the ladder $W'$ given by our original trivalent vertex with
a bigon attached to the first branch is adjoint to its reflection. By Proposition 3.20,
$W' \cong W^* \otimes_{[a]}$, so the same is true of $W$. \hfill \Box

Note that by symmetry, this means that $W$ and $W^*$ are biadjoint up to shift.

### 3.4. 2-categories

The material from the previous sections can be encapsulated into the definition of two 2-categories:

**Definition 3.26.** Let $\mathcal{LB}_l^n$ (resp. $\mathcal{LB}_l^n$) be the 2-category with:
- objects given by sequences of integers $(p_1, \ldots, p_\ell)$, with $p_i \in [0, n]$ for $i \in [1, \ell]$.
- morphisms given formal sums of $n$-ladders with $\ell$ uprights.
- 2-morphisms $A \to A'$ between two ladders is given by the space of morphisms between
the ladder bimodules in the derived category of bimodules over $\hat{T}^p - \hat{T}^p$ (resp. $T^p - T^p$)
that commute with the \( \mathcal{U}_n^* \times \mathcal{U}_n^* \)-action (resp. \( \mathcal{U}_n^* \)-action):

\[
\text{Hom}_{\mathcal{L}B}(A, A') := \mathbb{R} \text{Hom}_{\mathcal{T} \otimes \mathcal{T}'}(W_A, W_{A'})^{\mathcal{U}_n^* \times \mathcal{U}_n^*}
\]

\[
\text{Hom}_{\mathcal{L}B}(A, A') := \mathbb{R} \text{Hom}_{\mathcal{T} \otimes \mathcal{T}'}(W_A \otimes \mathcal{T}^p, W_{A'} \otimes \mathcal{T}'^p)^{\mathcal{U}_n^*}
\]

There is a 2-functor \( \mathcal{L}B \rightarrow \mathcal{L}B \) which is induced by the tensor product functor \( \otimes \mathcal{T}'^p \mathcal{T}^p \). Corollary 3.23 shows the compatibility with composition of 1-morphisms, and the discussion before Proposition 3.8 shows that this functor sends natural transformations commuting with \( \mathcal{U}_n^* \times \mathcal{U}_n^* \) to those commuting with \( \mathcal{U}_n^* \). In particular, it is the identity on objects and 1-morphisms. Proposition 3.25 shows that every 1-morphism in this category has a left and right adjoint, isomorphic to the reflection of that ladder. These 2-categories act in an obvious way on the derived categories of modules over the rings \( \mathcal{T}^p \) and \( \mathcal{T}'^p \) respectively. The isomorphisms of Propositions 3.20 and 3.24 can be regarded as isomorphisms on 1-morphisms in the category \( \mathcal{L}B \) or \( \mathcal{L}B \).

**3.5. Grothendieck groups.** As we mentioned in Section 2.1, we wish to argue that the 2-categories \( \mathcal{L}B \) and \( \mathcal{L}B \) give categorifications of the category \( \text{Lad}^n_{\mathcal{T}} \langle I \rangle^n_{\mathcal{T}} \). One of the ways we will make this precise is to compare the action of a ladder on \( \bigwedge_q^a \mathbb{C}_q^n \) with that of a ladder bimodule on the corresponding Grothendieck group. We’ll prove a slightly stronger statement using the algebra \( \mathcal{T} \).

**Theorem 3.27.** The induced action of a ladder bimodule on \( \bigoplus_{|p|=p} K_q^0(T^p) \) (resp. \( \bigoplus_{|p|=p} K_q^0(\mathcal{T}^p) \)) agrees with the action of the corresponding ladder on \( \bigwedge_q^a \mathbb{C}_q^n \otimes \bigwedge_q^b \mathbb{C}_q^n \) (resp. \( U_q^a \otimes U_q^b \)).

**Proof.** First, by the compatibility of the maps in Proposition 2.18 is suffices to check this for \( \mathcal{T} \), since this will imply the statement for \( \mathcal{T}^p \).

We also only need to check this in the case of \( Y_i \) and \( Y_i^\ast \). Furthermore, by adjunction, it suffices to only check \( Y_i \). Using induction functors, we can reduce to the case where there are no strands other than those joining the trivalent vertex. That is, we need only check that these maps are correct on the \( \omega_c \) weight space.

In the case of \( Y_i \), the functor sends \( \mathcal{T}^a \mathcal{W} \cong \mathbb{K} \) to the module \( L_{a,b} \) over \( \mathcal{T}^{a,b} \). Under the identification \( K_q^0(T^{a,b}) \cong \bigwedge_q^a \mathbb{C}_q^n \otimes \bigwedge_q^b \mathbb{C}_q^n \), the class of the simple \( L_{a,b} \) considered as a \( T^{a,b} \)-module and denoted \( v_{a,b} := [L_{a,b}] \), spans the space of highest weight vectors of weight \( \omega_c \). In fact, it is the unique highest weight vector of the form \( [L_{a,b}] = v_{a,-\omega_b} \otimes v_{a,b} + \cdots \), since \( L_{a,b} \) is the unique simple quotient of the standard \( \Delta_{\omega_{-\omega_b,\omega_b}} \). This is also how the ladder \( Y_i \) acts on the vector \( v_c \) (see [CKM14, Sect. 3.1]).

Considered over \( \mathcal{T}^{a,b} \), the class of \( [L_{a,b}] \) is thus \( 1 \otimes v_{a,b} \), which is, indeed, how the foam \( Y_i \) acts on the vector \( 1 \otimes v_c \). \( \square \)

**3.6. Comparison to category \( \mathcal{O} \).** There is an equivalence of categories \( T^p_{\mu} \)-mod to a block of parabolic category \( \mathcal{O}^p \) for the Lie algebra \( \mathfrak{g} \mathfrak{l}_p \) and the parabolic \( \mathfrak{p}_p \) of block upper triangular matrices with blocks given by \( \mathfrak{p} \), which we denote \( \xi : T^p_{\mu} \)-mod \( \rightarrow \mathcal{O}^p \). If we let \( \mathfrak{p}' = (p_1, \ldots, p_j + p_{j+1}, \ldots, p_\ell) \), then we have an inclusion functor \( I^p_{\mathfrak{p}'} : \mathcal{O}^p \rightarrow \mathcal{O}^p \) and its right adjoint, the **Zuckerman functor** \( Z^p_{\mathfrak{p}'} : D^b(\mathcal{O}^p) \rightarrow D^b(\mathcal{O}^p) \).
Proposition 3.28. We have isomorphisms $\xi_p \circ W_{Y_i} \cong I_p^\ell \circ \xi_{P'}$ and $\xi_p \circ W_{Y_i}^\ell \cong Z_{P'}^\ell \circ \xi_p (p; p_{i+1})$.

Proof. By [Webb 9.12], the action of $\mathcal{U}_\ell$ is intertwined by $\xi_p$ with the action of certain projective functors. By the commutation of Zuckerman and projective functors, we have that the functor $\xi_p^{-1} \circ Z_{P'}^\ell \circ \xi_p$ commutes with the action of $\mathcal{U}_\ell$, just as $W_{Y_i}$ does. Similarly, if we have a larger algebra $g$, a Levi $l$ for a parabolic $p$, and two parabolics $q' \subset q \subset l$, then we have that the Zuckerman functor for the pair $q' \subset q$ intertwines under parabolic induction with that of $q' + \text{rad}(p) \subset q + \text{rad}(p)$. The description of standardization given in [Webb 9.13] shows that $\xi_{P'}^{-1} \circ Z_{P'}^\ell \circ \xi_p$ thus commutes with the functor $\mathcal{I}_{OP}$, adding a new red strand at the right.

Thus, if we establish the theorem when $\ell = i + 1$, it will follow in all other cases, since we can build all other projective modules using $\mathcal{I}_{OP}$ and the action of $\mathcal{U}_n$, which commute with both functors. Thus, let us restrict to that case.

As usual, we let $a = p_i, b = p_{i+1}, c = a + b$. One consequence of the standard stratification shown in [Webb 5.22] is that every projective is a summand of a module obtained by applying the action of $\mathcal{U}_\ell$ to $\text{Ind}_{\mathcal{I}_{OP} \otimes \mathbb{C}^c} (\xi(P) \otimes \mathbb{C}^c) = \xi(\mathcal{I}_{OP}(P))$, for $P$ a projective module in $T^\ell_{\mu} \text{-mod}$ with $p' = (p_1, \ldots, p_{\ell-1})$. Thus, it suffices to construct the isomorphism on $\xi(\mathcal{I}_{OP}(P))$ and extend it using the action of $\mathcal{U}_n$.

As in [Webb 7.19], for an arbitrary $T^\ell_{\mu} \text{-module}$, we have that $W_{Y_i} \otimes \mathcal{I}_{OP}(P)$ is the standardization of the $T\langle p_1, \ldots, p_{\ell-1} \rangle \otimes T^{\mu, b} \text{-module} P \otimes L_{a, b}$. Furthermore, we have an isomorphism $\xi_{(a, b)} (L_{a, b}) \cong \mathbb{C}^c$ since in both cases, these are the unique simples killed by translation to any singular central character (which are intertwined with the functors $\mathcal{I}_i$ on $T\langle a, b \rangle \text{-modules}$). By [Webb 9.13], we thus have an isomorphism

$$
\xi_p (W_{Y_i} \otimes \mathcal{I}_{OP}(P)) \cong \text{Ind}_{\mathcal{I}_{OP} \otimes \mathbb{C}^c} (\xi(P) \otimes \mathbb{C}^c) \cong I_p^\ell \xi_p (\mathcal{I}_{OP}(P))
$$

that is natural in $P$. This completes the proof. \qed

Due to the difficulty of calculating in derived categories, we will now smuggle certain results from category $O$ into our picture. We apologize to those readers who don’t like such techniques, but they should rest assured that these techniques are used in a minor way, have a big payoff, and facilitate rather than replace computation.

We call a complex of projective modules over $\hat{T}^\Lambda$ linear if every homogeneous summand $P$ in homological degree $j$ is isomorphic to $P^\otimes (-2j)$. If we replace $\hat{T}^\Lambda$ with a Morita equivalent algebra which is positively graded, then this condition will be equivalent to requiring that the $j$th grade is generated in degree $-j$. Thus this definition agrees with the usual one (see, for example, [MOS09]) for a positively graded algebra. This more general definition allows us to speak of a linear complex of projectives in any humorous category in the sense of [Webb]. The algebra $\hat{T}^\Lambda$ is called Koszul if every self-dual simple module has a linear resolution. Any algebra which is Koszul in this sense satisfies a numerical criterion of Koszulity: the matrices whose entries are the graded dimension of $\text{Hom}(P_{\mu}, P_{\nu})$ where $P_{\mu}$ denote the self-dual projectives, and the graded Euler characteristic of $\text{Ext}(L_{\mu}, L_{\xi})$ with $L_{\nu}$ the simple quotient of $P_{\nu}$ are inverse (since their product computes the graded multiplicity of $L_{\xi}$ in the projective resolution of $L_{\mu}$). By linearity, $\text{Ext}(L_{\mu}, L_{\xi})$ is non-positively graded, so
Hom$(P_{\mu}, P_{\nu})$ is non-negatively graded. Thus, Hom$(\oplus P_{\mu}, \oplus P_{\mu})$ is a Morita equivalent algebra which is Koszul in the usual sense of [BGS96].

**Theorem 3.29.** The algebras $T^p$ are Koszul. Any ladder bimodule preserves the category of linear complexes of projectives over $T^p$.

**Proof.** Koszulity follows from [Webb 9.14]. By [MOS09, Cor. 36], the Zuckerman and inclusion functors are Koszul dual to projective functors, which are exact. Thus the Zuckerman and inclusion functors are $t$-exact in the $t$-structure which is the image of the usual $t$-structure on the Koszul dual. This is the $t$-structure whose heart is linear complexes. \[\square\]

We also expect that in most cases, the category $\tilde{T}^p$ is Koszul. In certain degenerate cases, it seems to be necessary to consider its quotient by certain central elements. For example, if there are no red strands and one black one, the corresponding algebra is a polynomial ring with a generator in degree 2, which is thus not Koszul. However, if we add a red strand with the same label, then the resulting algebra is easily shown by hand to be Koszul; this is, for example, a consequence of Proposition 3.14.

# 4. The dual categorical action

We now wish to show that there is a 2-functor $U_\ell \to \tilde{\mathcal{L}B}$, which thus induces a categorical action on the module categories $\tilde{T}^p$. We first turn to defining this 2-functor on 1-morphisms.

**Definition 4.1.** Let $F_i^{(n)}$ be the functor of tensor product by the ladder bimodule associated to $f_i^{(n)}$. Let $E_i^{(n)}$ be the functor of tensor product by the ladder bimodule associated to $e_i^{(n)}$.

Note that our notation suppresses the labels of the vertical strands, as usual. Only when needed will they be specified.

**Proposition 4.2.** We have

$$E_i^{[n]} \cong (E_i^{(n)})^{\oplus [n]_q!} \quad \text{and} \quad F_i^{[n]} \cong (F_i^{(n)})^{\oplus [n]_q!}.$$

Here we use $[p]_q = \frac{q^p - q^{-p}}{q - q^{-1}}$ as a multiplicity to indicate the sum of $p$ copies of a module, with grading shifts that match the quantum integer.

**Proof.** We prove the result for $F_i^{(n)}$, for $E_i^{(n)}$ it can be proved similarly. By associativity and Proposition 3.20 we have

$\begin{align*}
\begin{array}{c}
1 \\
\downarrow
\end{array} & \cong \\
\begin{array}{c}
1 \\
\downarrow
\end{array} & \cong [n]_q \\
\begin{array}{c}
\downarrow \\
(n-1)
\end{array} & \cong \\
\begin{array}{c}
\downarrow \\
(n-1)
\end{array} & \cong [n]_q
\end{align*}$

The result follows by induction on $n \geq 1$. \[\square\]
The functors $E_i$ and $F_i$ are biadjoint up to shift. By Lemma 3.17, we have that left and right adjoints of $Y_i$ with bottom given by $p$ and top by $(p_1, \ldots, p_i-1, 1, p_{i+1}, \ldots, p_\ell)$ are given by $Y_i^L = Y_i^*(1-p_i)$ and $Y_i^R = Y_i^*(p_i-1)$. Similarly, for $Y_{i+1}$ with top given by $p-\alpha_i$ and bottom by $(p_1, \ldots, p_i-1, 1, p_{i+1}, \ldots, p_\ell)$, the adjoints are $(Y_{i+1}^*)^L = Y_{i+1}(p_{i+1})$ and $(Y_{i+1}^*)^R = Y_{i+1}(-p_{i+1})$. For convenience, let $\pi_i = p_i - p_{i+1} - 1$. Thus, the left and right adjoints of $F_i$ are given by

$$F_i^L = E_i(-\pi_i) \quad F_i^R = E_i(\pi_i).$$

We consider the adjunctions (co)units for the left and right adjunctions, respectively:

$$\epsilon_i := \epsilon_{Y_i^*} \circ \epsilon_{Y_{i+1}} : E_i F_i(-\pi_i) \rightarrow \tilde{T}^p$$

$$\epsilon'_i := (-1)^{p-1} \epsilon_{Y_i^*} \circ \epsilon_{Y_i} : F_i E_i(\pi_i) \rightarrow \tilde{T}'^p$$

$$\iota_i := \iota_{Y_i^*} \circ \iota_{Y_{i+1}} : F_i E_i(-\pi_i) \rightarrow F_i F_i$$

$$\iota'_i := (-1)^{p-1} \iota_{Y_i^*} \circ \iota_{Y_i} : F_i E_i(\pi_i) \rightarrow F_i F_i$$

where $p, p' \in \Gamma^n_\ell$ are arbitrary such that $p_i' = p_i - 1, p_{i+1}' = p_{i+1} + 1$ and $p_j' = p_j$ for all $j \neq i, i + 1$. Note that our grading shifts look a little different from those in [KL10]; in part this is because we have fixed $p$ to be the list of weights at the bottom of $F_i$, not the bottom of the picture, amongst other differences in convention. For the reader to "get the picture", we draw the maps needed for $\epsilon_i'$ and $\iota_i$. The maps for $\epsilon_i$ and $\iota_i'$ come from the reflection of this diagram through the $y$-axis.

**Definition 4.3.** Let $y$ be the natural transformation of $F_i$ given by $1 \otimes y_{i+1} \otimes 1$. Let $\psi : F_i F_j \rightarrow F_j F_i$ be the map defined by

- isotopy if $|i - j| > 1$

- the adjoint of the map of the associativity isomorphism $F_{i \pm 1} E_i \rightarrow E_i F_{i \pm 1}$ if $j = i \pm 1$

- the natural transformation induced by $\iota_Y \circ \epsilon_Y^*$ after using the associativity isomorphism if $i = j$
Our aim is to show that these natural transformations define a categorical action of the 2-category $\mathcal{U}_t$ sending $\mathcal{F}_i \to \mathcal{F}_i$ and $\mathcal{E}_i \to \mathcal{E}_i$.

**Remark 4.4.** Note that this functor follows the "Koszul dual" grading convention. It sends grading shift $(i)$ to the Tate twist $\langle i \rangle = [i](-i)$ and vice versa; in particular, a map of degree $k$ in the 2-category $\mathcal{U}$ is sent to an element of $\text{Ext}^k$ of internal degree $-k$.

### 4.1. NilHecke relations

We wish to establish that we have an action of the nilHecke algebra on $\mathcal{F}^i$ defined by.

**Lemma 4.5.** On $\mathcal{F}^2_i$, we have the relation

\[(4.17) \quad (y \otimes 1)\psi - \psi(1 \otimes y) = \psi(y \otimes 1) - (1 \otimes y)\psi = 1.\]

**Proof.** In terms of our usual notation for Hochschild cohomology classes, we must show that

\[(4.18) \quad y_1\psi - \psi y_{i+1} = \psi y_i - y_{i+1}\psi = 1.\]

Recall that, by associativity, we have

Therefore, we can prove (4.18) on the bigon web $\beta = Y^*Y$ (we suppress the subscript $i$ here) with $a = b = 1$ and $c = 2$.

The associated bimodule resolutions from Corollary 3.16 are both honest projective modules on the sides where the tensors occur. Let $A$ and $D$ be the generator of $P_0$ in homological degree $0$ and $-2$, respectively, $B$ the generator of $P_1$ and $C$ the generator of $P_2$. Thus, we can realize the tensor product as an $A_\infty$ bimodule, generated by the tensors $A \otimes A$, etc. The counit $\epsilon_{Y^*}$ is defined by the pairing

\[(4.19) \quad \langle A, D \rangle = \langle B, B \rangle = -\langle C, C \rangle = \langle D, A \rangle = 1\]

with all other pairings $0$. One can easily check that $\langle \partial x, y \rangle = (x, \partial y)$.

The map $\psi$ sends $A \otimes A$ to the canonical element of the pairing given in (4.19). That is:

\[A \otimes A \mapsto k := A \otimes D + B \otimes B - C \otimes C + D \otimes A\]

Technically, we should use the $A_\infty$ tensor product $\tilde{T}P \otimes W_\beta \otimes \tilde{T}P$; however, the map $W_\beta \mapsto \tilde{T}P \otimes W_\beta \otimes \tilde{T}P$ sending $w \mapsto 1 \otimes w \otimes 1$ is a quasi-isomorphism of complexes by so we can always transfer our maps using this quasi-isomorphism.
Note that since $W_{β} ≅ TP(1) ⊕ TP(−1)$ with these copies generated by $A ⊗ A$ and $k$, in order to check an equality of chain maps up to homotopy, it suffices to check that it holds on $A ⊗ A$ and $k$ modulo boundaries.

Let us note several important equalities. Using the associated deformation, we can calculate $y_i(k)$ by using $y_i$ to deform the differential on the left hand side of the complex (see Lemma 2.23 and the text above it). Thus, taking the square of the deformed differential, we have that

$$y_i(D ⊗ A) = \frac{1}{h} A ⊗ A = \frac{1}{h} (hA ⊗ A) = A ⊗ A.$$  

Since the other terms in $k$ are killed, we have that $y_i(k) = A ⊗ A$. Similarly, $y_{i+1}(k) = −A ⊗ A$ since

$$y_{i+1}(D ⊗ A) = \frac{1}{h} A ⊗ A = \frac{1}{h} (hA ⊗ A) = −A ⊗ A.$$  

On the other hand, $ψ(k)$ is a boundary since $ψ^2 = 0$, and $y_i(A ⊗ A) = y_{i+1}(A ⊗ A) = 0$ for degree reasons.

Thus, we have that (modulo boundaries)

$$y_i ψ(A ⊗ A) = A ⊗ A \quad \quad y_i(A ⊗ A) = 0$$  
$$y_{i+1} ψ(A ⊗ A) = −A ⊗ A \quad \quad ψ y_{i+1}(A ⊗ A) = 0$$  
$$y_i ψ(k) = 0 \quad \quad ψ y_i(k) = k$$  
$$y_{i+1} ψ(k) = 0 \quad \quad ψ y_{i+1}(k) = −k$$  

This shows that up to homotopy

$$y_i ψ − ψ y_{i+1} = ψ y_i − y_{i+1} ψ = 1,$$

thus giving the relation (4.18).

□

Lemma 4.6. On $F^3_i$, we have the relation

(4.20) \quad (ψ ⊗ 1)(1 ⊗ ψ)(ψ ⊗ 1) = (1 ⊗ ψ)(ψ ⊗ 1)(1 ⊗ ψ)

Proof. Note that the bidegree of both maps is $(-6, 6)$, where the first entry is the homological degree and the second the internal degree. Note also that, by associativity, we have

\[
\begin{array}{c|c|c}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array}
\]

\[
\begin{array}{c|c|c}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array}
\]  

and

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Applying Proposition 3.20 twice gives

\[
\begin{array}{ccc}
1 & 1 \\ 1 & 1 \\ 1 & 1
\end{array} \cong \begin{array}{ccc}
1 & 1 \\ 1 & 1
\end{array} \cong \langle 6 \rangle \oplus \cdots \oplus 3 \\ 3
\end{array}
\]

Let \( W_1 \) and \( W_2 \) be the bimodules associated to the first and the second web in this picture, respectively. Then

\[
\text{Ext}^{-6}(W_1, W_1) \cong Z(\tilde{T}_{\omega^3})(-6) \cong \text{Ext}^{-6}(W_2, W_2)
\]

and the composite of the two isomorphisms is equal to the associativity isomorphism in homological degree \(-6\). The remarks below Proposition 3.20 show that both \((\psi \otimes 1)(1 \otimes \psi)(\psi \otimes 1)\) and \((1 \otimes \psi)(\psi \otimes 1)(1 \otimes \psi)\) correspond to \(1 \in Z(\tilde{T}_{\omega^3})(-6)\). \(\square\)

This action gives us an alternate way of defining the divided power functor \(F^{(n)}_i\) as the image of a primitive idempotent in \(NH_n\).

In particular, this shows that:

**Corollary 4.7.** The maps \(y\) and \(\psi\) define an action of the nilHecke algebra \(NH_n\) on \(F^{(n)}_i\), and thus of the symmetric polynomials \(\mathbb{K}[y_1, \ldots, y_n]^{S_n} \cong Z(NH_n)\) on the summand \(F^{(n)}_i\).

### 4.2. Reduction to trees.

**Definition 4.8.** We call a ladder a **tree** if at \(y = 0\) there is a single red strand (necessarily labeled \(p\)) and there are no cycles in its underlying graph (that is, the underlying graph is a tree in the usual sense). We call a ladder a **rootstock** if its reflection is a tree.

Note that up to isotopy, if we fix the sequence at \(y = 1\) to be \(p\), there is a single tree with this top, which we denote \(\tau_p\). Note that every tree is generated from \(\tau_{(p,0,\ldots,0)}\) by applying categorification functors:

\[
\tau_p \cong f^{(p_2)}_1 \circ f^{(p_3)}_2 \circ f^{(p_3)}_1 \circ \cdots \circ f^{(p_{\ell-2})}_2 \circ f^{(p_{\ell-1})}_1 \circ f^{(p_{\ell-1})}_2 \circ \cdots \circ f^{(p_2)}_1 \circ f^{(p_1)}_2 \circ f^{(p_1)}_1 \tau_{(p,0,\ldots,0)}.
\]

The action of symmetric polynomials on \(F^{(n)}_i\) as the center of the nilHecke algebra defines an action of \(R_p \cong \mathbb{K}[y_1, \ldots, y_n]^{S_p \times \cdots \times S_p} \) on \(\tau_p\). In terms of ladders, we can think of this as the branch of the tree with label \(p_i\) having an action of symmetric polynomials on an alphabet of \(p_i\) variables.

We let \(P_p \subset G := GL(\mathbb{C}^p)\) be the block upper-triangular matrices with block sizes \(p\). Note that \(H'(G/P_p; \mathbb{C})\) is naturally a quotient of \(R_p\) identifying \(y_i\) with the Chern
classes of tautological line bundles on the full flag variety; the kernel of this map is the ideal generated by positive degree elements in the full ring of $S_p$ invariant polynomials $R_{(p,0,0,...)}$. Throughout what follows, we’ll use $\dagger$ to denote either $\mathcal{L}B$ or $\mathcal{L}B$ when a result holds in both.

**Proposition 4.9.** The map $R_p \to \text{Hom}_t(\tau_p, \tau_p)$ induces an isomorphism

$$\text{Hom}_t(\tau_p, \tau_p) \cong H^*(G/P_p; \mathbb{C}).$$

**Proof.** By Proposition 3.25, we have that

$$\text{Hom}_{\mathcal{L}B}(\tau_p, \tau_p) \cong \text{Hom}_{\mathcal{L}B}(\text{id}_{\tau(p)}, \tau_p^* \circ \tau_p).$$

Applying the bigon relation (Proposition 3.20) shows that

$$W_{(\ell,0,0,...,\ell)}^* \otimes_{\mathcal{T}(\ell,0,0,...,\ell)} W_{(\ell,0,...,\ell)}^* \cong (p_1 + p_2)! \cdot \left(\frac{(p_1 + p_2)!}{p_1!p_2!} \cdot \left(\frac{(p_1 + p_2 + p_3)!}{p_1!p_2!p_3!} \cdot \frac{n!}{p_1!...p_\ell!} \cdot \mathcal{T}^p\right) \right)$$

By Corollary 3.23, the same holds over $\mathcal{T}^p$. Note that we have $\dim \text{Hom}_t(\text{id}_{\tau(p)}, \text{id}_{\tau(p)}) = 1$, since every projective module over $\mathcal{T}^p$ is a summand of an induction of the simple with no black strands. Thus, any endomorphism commuting with induction is a scalar multiplication. Thus, we have that

$$\dim \text{Hom}_t(\tau_p, \tau_p) = \dim H^*(G/P_p; \mathbb{C}) \cdot \dim \text{Hom}_t(\text{id}_{\tau(p)}, \text{id}_{\tau(p)}) = \dim H^*(G/P_p; \mathbb{C}).$$

First, we specialize to the case where $p = (1,...,1)$. Note that any element of the kernel of $R_{(1,...,1)}$ acting on $W_{\tau_p}$ must lie in the kernel of the action of $NH_p$ on $W_{(0,...,0)}^* \otimes_{\mathcal{T}(0,...,0)} W_{(0,...,0)}$. Since $NH_p$ is Morita equivalent to $\mathbb{C}[y_1,...,y_p]^{S_p}$, the kernel of any action of $NH_p$ must be generated by an ideal of $\mathbb{C}[y_1,...,y_p]^{S_p}$. For any homogeneous action, this kernel must lie in the ideal $I$ generated by positive degree symmetric functions. This shows that for general $p$, the kernel is still contained in $I$.

On the other hand, the codimension of the kernel can be no greater than $\frac{n!}{p_1!...p_\ell!} = \dim R_p/I$. This is only possible if the map $R_p \to \text{Hom}_t(\tau_p, \tau_p)$ induces an isomorphism

$$\text{Hom}_t(\tau_p, \tau_p) \cong R_p/I \cong H^*(G/P_p; \mathbb{C}).$$

Recall that the space of 2-morphisms between two ladders in $\mathcal{L}B$ or $\mathcal{L}B$ carries both an internal and a homological grading. We call a 2-morphism $N$ pure if its internal and homological gradings sum to 0. Note that all of our 2-morphisms for the proposed dual action of $\mathcal{U}_t$ are pure by construction.

For technical reasons, we will restrict ourselves for now to constructing a 2-functor $\mathcal{U}_t \to \mathcal{L}B$; we will turn later to the stronger statement that we can lift this to a
2-functor $\mathcal{U}_p \to \widetilde{\text{LB}}$. Thus, until stated otherwise, all ladder bimodules will be cyclotomically reduced.

**Lemma 4.10.** Assume $N : W_{\beta_1} \to W_{\beta_2}$ is a pure 2-morphism in $\mathcal{LB}$ for $\beta_1$ a 1-morphism $p \to p'$. Then $N$ is 0 if and only if it induces the trivial map on modules whose weight is $(1, \ldots, 1)$.

**Proof.** The algebra $T_p$ for any weight $m = (m_1, \ldots, m_n)$ of $gl_n$ can be thought of as the same algebra for $gl_{n+1}$ and the weight $(m_1, \ldots, m_n, 0)$ where we simply never use the label we have added. By adding additional labels we never use, we may assume that $p \leq n$.

Assume that $P$ is a projective module with weight $m \neq (1, \ldots, 1, 0, \ldots, 0)$. There must be some $m_q$ with $q \leq p$ such that $m_q = 0$. Note that in this case, $F_{q-1}^{(m_q-1)}$ and $E_{q}^{(m_q+1)}$ are equivalences of categories from $T_{m} \text{-mod}$ to $T_{s_{q-1}m} \text{-mod}$ and $T_{s_{q}m} \text{-mod}$, respectively. Thus, we can move any zero further right in the dimension vector without affecting whether $N$ is 0. Thus,

1. we may assume that $p \leq n$.

2. we can assume that the last $n-p$ entries of the weight are 0.

Since we have not used the strands with labels $>p$, we can forget them and so

3. we may assume that $n = p$.

Thus, either the weight is $(1, \ldots, 1)$, or we have some largest entry $m_r > 1$. We can also assume that $m_{r+1} = 0$, since at least one entry must be 0, and we can move it freely using the Weyl group. In this case, the module $P$ must be highest weight for the categorical action of $E_r$ and $F_r$. Thus, we have that $E_r F_r P \cong P^{\text{inv}}$, by the commutator relation in $sl_2$. Thus, it suffices to check the result for $F_r P$. The resulting weight has fewer entries which are 0. Thus applying this result inductively, we can see that we can assume that $P$ has weight $(1, \ldots, 1)$, and the proof is done. $\square$

**Lemma 4.11.** Assume $N : W_{\beta_1} \to W_{\beta_2}$ is a pure 2-morphism in $\mathcal{LB}$ as before. Then $N$ is 0 if and only if the same is true for the induced morphism $N \circ \text{id}_{\tau_p} \in \text{Hom}_{\mathcal{LB}}(\beta_1 \circ \tau_p, \beta_2 \circ \tau_p)$ for $\tau_p$ a tree.

**Proof.** By [Webb 9.10], the weight $(1, \ldots, 1)$ subcategory is equivalent to a block of parabolic category $O$ for the parabolic $p$ containing the module corresponding to a tableau with entries given by $1, 2, 3, \ldots, n$ under the rule given by Brundan and Kleshchev in [BK08]. Applying their formulas, this corresponds to a block containing a finite dimensional representation; that is, a regular block $O^p_0$. This is thus Koszul dual to singular category $O$ for the corresponding singularity by the main theorem of [BGS96].

The equivalence of [Webb 9.10] sends the tree bimodule $W_{\tau_p}$ to the unique simple module killed by all translation functors to walls. This is, of course, the finite dimensional module in $O^p_0$.

This Koszul duality of [BGS96] by construction sends this finite dimensional simple (and thus the tree bimodule $W_{\tau_p}$) to the unique self-dual projective of the dual singular block of category $O$. Furthermore, it sends any ladder functor to a translation functor, and a pure natural transformation to a natural transformation of projective functors.
Lemma 4.12. For any tree \( \tau_p \), we have isomorphisms in the category \( \dagger \) (that is, in \( \widetilde{\mathcal{LB}} \) or \( \mathcal{LB} \)):
\[
e_i \circ \tau_p \cong \tau_{p+\alpha_i} \oplus [p_i]_l \quad f_i \circ \tau_p \cong \tau_{p-\alpha_i} \oplus [p_{i+1}]_l.
\]

Proof. Clear from Propositions 3.20 and 3.24.

Fix \( i \); for any \( p \), we let
\[
p^- = (p_1, \ldots, p_i, 1, p_{i+1} - 1, \ldots, p_\ell) \quad p^+ = (p_1, \ldots, p_i - 1, 1, p_{i+1}, \ldots, p_\ell).
\]
Note that \((p \pm \alpha_i)^\pm = p^\pm\). We have maps
\[
G/P_p \leftarrow G/P_{p^-} \rightarrow G/P_{p^+} \quad G/P_p \leftarrow G/P_{p^+} \rightarrow G/P_{p^-}.
\]
These maps induce a \( H'(G/P_p) \)-\( H'(G/P_{p^\pm}) \)-bimodule structure on \( H'(G/P_{p^\pm}) \).

Lemma 4.13. We have bimodule isomorphisms
\[
\text{Hom}_t(\tau_{p+\alpha_i}, e_i \circ \tau_p) \cong H'(G/P_{p^-}) \quad \text{Hom}_t(\tau_{p-\alpha_i}, f_i \circ \tau_p) \cong H'(G/P_{p^+}).
\]

Proof. We can write \( e_i \equiv Y^*_i \circ Y_{i+1} \). Thus, by adjunction, we have that
\[
\text{Hom}_t(\tau_{p+\alpha_i-1}, e_i \circ \tau_p) \cong \text{Hom}_t(Y_i \circ \tau_{p+\alpha_i-1}, Y_{i+1} \circ \tau_p).
\]
Both ladders that appear on the RHS are trees with top \( p^- \). Thus, the result follows from Proposition 4.9. The isomorphism for \( f_i \) follows by adjunction.

Since these isomorphisms involve a choice of isomorphism between \( Y^*_i \) and the adjoint of \( Y_i \), they are not canonical. We are using the adjunction \( \iota_Y \) chosen in Section 3.2 so that after identifying \( e_i \circ \tau_p \) with the tree \( \tau_{p+\alpha_i} \) with a bigon blown up in the \( i \)th strand, the element \( 1 \in H'(G/P_{p^\pm}) \) corresponds to \( \iota_Y \circ \text{id}_{\tau_{p+\alpha_i}} \).

Lemma 4.14. For any two ladders in \( \beta_1, \beta_2 \), we have that
\[
\text{Hom}_t(\tau_{p''}, \beta_1 \circ \beta_2 \circ \tau_p) \cong \text{Hom}_t(\tau_{p''}, \beta_1 \circ \tau_{p'}) \otimes \text{Hom}_t(\tau_{p'}, \beta_2 \circ \tau_p),
\]
where \( p \) is fixed, and \( p', p'' \) are the top \( gl \ell \) weights of \( \beta_2 \) and \( \beta_1 \) respectively.

Proof. Let \( \mathcal{T} \) be the dg-subcategory generated by the trees. The weight \( p \) part of \( \mathcal{T} \) is generated by \( \tau_p \), and thus is equivalent to the category of dg-modules over \( H'(G/P_p) \). Composition with the ladder \( Y_i \) or \( Y^*_i \) preserves \( \mathcal{T} \) and thus corresponds to a dg-bimodule over \( H'(G/P_p) \) and \( H'(G/P_{p^\pm}) \). In fact, we know that the corresponding bimodules are isomorphic to \( H'(G/P_{p^\pm}) \) with the trivial differential, since a tree is sent to a sum of shifts of trees (rather than a complicated cone of these). These bimodules are free both as left or right modules (i.e., they are sweet). Thus, the result follows from the fact that the dg-tensor product of any monomial in these bimodules is the same as the naive tensor product.

□
What we have now is that sending
\[ p \mapsto \text{Hom}_*(\tau_p, \tau_p) \mod \quad e_i \mapsto \text{Hom}_*(\tau_{p+1}, e_i \circ \tau_p) \quad F_i \mapsto \text{Hom}_*(\tau_{p-1}, f_i \circ \tau_p) \]
agrees on the level of 1-morphisms with the “quiver flag” construction of Khovanov and Lauda in [KL10]. More formally:

**Corollary 4.15.** If \( A \) is any monomial in \( e_i \)'s and \( f_i \)'s, then \( \text{Hom}_*(\tau_{p'}, A \circ \tau_p) \) is canonically isomorphic to the bimodule assigned to this monomial by the action of [KL10] §6.

Now, we need only check that our construction agrees with Khovanov and Lauda’s on the level of 2-morphisms as well. We’ll note an important relation for us, which is immediate from the choice of adjunctions we fixed in Section 3.2. Let \( \psi^\circ : E_i F_j \rightarrow F_j E_i \) be the adjoint of the map \( \psi : F_j F_i \rightarrow F_i F_j \).

**Lemma 4.16.** The action of \( y \) on \( F_i \), of \( \psi \) on \( F_i^2 \), of \( e, i, e', i' \), and \( \psi^\circ \) agrees with the formulas in [KL10].

**Proof.** The agreement of the action of \( y \) is essentially the definition of the isomorphism of Proposition 4.9 in both cases they are just multiplication operators. Similarly, the agreement of \( \psi \) follows from the commutation relations of (2.2b).

For \( j \neq i \), we have that \( E_i F_j W_{p_{1j}} = F_j E_i W_{p_{1j}} \). By definition, we have that \( \psi^\circ \) is the unique isomorphism that induces the identity after we identify \( \text{Ext}(W_{p_{1j}}, E_i F_j W_{p_{1j}}) \) with \( \text{H}^*(G/P_{p'}) \) where \( p' = (p_1, \ldots, p_{i-1}, p_i, 1, p_{i+1} - 1, \ldots, p_j - 1, 1, p_{j+1}, \ldots, p_r) \), which agrees with Khovanov and Lauda’s formulation.

Finally, we need to check that \( i, e, i', e' \) all induce the correct maps. We’ll need to consider maps \( \text{H}^*(G/P_{p'}) \otimes \text{H}^*(G/P_{p'}) \rightarrow \text{H}^*(G/P_{p}) \). Note that the bimodule on the LHS is generated by \( y_i^k \otimes 1 \) for \( k = 1, \ldots, \max(p_i, p_{i+1}) \).

First consider the special case where \( p_i = 0 \). In this case, \( F_i E_i \) is simply a bigon, and the sum of \( p_{i+1} \) many copies of the identity functor. In Khovanov and Lauda’s construction, we can identify the bimodule for this functor with \( \text{H}^*(G/P_{p_i}) \), thought of as a bimodule by the inclusion of \( \text{H}^*(G/P_{p_i}) \) via pullback. In Khovanov and Lauda’s construction, the map \( p_i \) is simply the pullback inclusion, sending \( 1 \mapsto 1 \), and the map \( e_i \) is integration along the fibers (with appropriately chosen relative orientation). Thus, our \( \iota, e_i \) corresponds to \( \iota_{Y_{p_{i+1}}} \) in this case, agrees with Khovanov and Lauda’s by the convention we have chosen for the isomorphism of Lemma 4.13.

Furthermore, our map \( e_i' : F_i E_i \rightarrow \text{id} \), which corresponds to \( e_{Y_{p_{i+1}}} \) in this case, is determined by the values of its compositions with \( (y_i^q \otimes 1)_i \) for \( q \leq p_{i+1} - 1 \). Lemma 3.19 allows us to compute these compositions; the result is 0 unless \( q = p_{i+1} - 1 \) in which case it is 1. The same relations hold in Khovanov and Lauda’s case by [KL10, (3.4)] and the unnumbered equation below.

The same argument shows that if \( p_{i+1} = 0 \), we have agreement of \( i_i' \) and \( e_i \) with Khovanov and Lauda’s, using the other case of Lemma 3.19. By definition, \( i_i' = (-1)^{p_i-1} \iota_{Y_{p_{i+1}}} \), so Lemma 3.19 shows that

\[
\epsilon_i(1 \otimes y_i^q) i_i' = (-1)^{p_i-1} \epsilon_i(1 \otimes y_i^q) i_i' = \begin{cases} 1 & q = p_i - 1 \\ 0 & q < p_i - 1. \end{cases}
\]
Now let $p_i = 1$. Then $\iota'$ corresponds to $\iota \gamma_{i+1}$, and $\epsilon$ to $\epsilon \gamma_{i+1}$, and the zig-zag relations of units and counits show that both are uniquely determined by $\iota$ and $\epsilon'$ for $p_i = 0$. So also in this case, we have agreement between our maps and Khovanov and Lauda’s $\iota', \epsilon$. Similarly, we have agreement with their maps for $\iota, \epsilon'$ when $p_{i+1} = 1$.

In terms of ladders, every map of the form $\iota, \epsilon, \iota'$ can be written as a composition of these, since $E_i$ and $F_i$ are both compositions of two ladders which involve splitting off a strand with label 1, and then joining a new one on. Thus, $\iota, \epsilon, \iota', \epsilon'$ can be written as composition of the special cases discussed above. Since the same is true for Khovanov and Lauda’s representation on the cohomology of flag varieties, we are done. □

By Lemma 4.11, we thus have that:

**Corollary 4.17.** There is a 2-functor $\mathbf{a} : U \rightarrow \mathcal{L} \mathcal{B}$ sending $E_i \mapsto \epsilon_i$, $F_i \mapsto \iota_i$ and 2-morphisms to 2-morphism as indicated above.

Applying the natural representation of $\mathcal{L} \mathcal{B}$, we find that the functors $E_i$ and $F_i$ define a categorical action of $\mathfrak{gl}_i$ sending the weight $(p_1, \ldots, p_i)$ to $D^h(T^{e_1,\ldots,e_i})$ which commutes with the action of $U$.

Combining this result with Theorem 3.27 concludes the proof of Theorem B.

### 4.3. Extension to $\tilde{\mathcal{L}} \mathcal{B}$

As noted before, we have a natural 2-functor $\tilde{\mathcal{L}} \mathcal{B} \rightarrow \mathcal{L} \mathcal{B}$. We wish to show that the 2-functor $\mathbf{a} : U \rightarrow \mathcal{L} \mathcal{B}$ defined in the previous section factors through this reduction. In order to show this, first we’ll prove the following lemma:

**Lemma 4.18.** Consider $N \in \text{Hom}_{\tilde{\mathcal{L}} \mathcal{B}}(\beta_1, \beta_2)$. Then $N$ is an isomorphism if and only if the induced natural transformation $\tilde{N} \in \text{Hom}_{\mathcal{L} \mathcal{B}}(\beta_1, \beta_2)$ is an isomorphism.

**Proof.** It suffices to check that $N(P)$ is an isomorphism for any projective $P$. By [Webb, 5.23], any projective module over $\bar{T}^p$ is filtered by standard modules, which are left inductions of standard modules pulled back from the cyclotomic quotients. The natural transformation $N$ on these subquotients is an isomorphism by assumption. Thus, $P$ has a filtration such that $N(\text{gr } P)$ is an isomorphism, which shows the same is true for $N(P)$. □

For each index $i$, we can define $h_{k,i}$ to be the sum of all diagrams with no crossings and exactly $k$ dots in total on the strands with label $i$ and no dots on other strands (this is a generalization of the complete symmetric polynomial).

**Lemma 4.19.** The center of $\bar{T}^\nu$ for $\nu = \sum \omega_p - \sum n_j \alpha_j$ is the polynomial ring $\mathbb{k}[[h_{k,i}]_{k \leq n_i}]$.

**Proof.** The proof directly follows [KL09, 2.9]. Obviously, these elements are central and the basis theorem [Webb, 4.16] for $\bar{T}^\nu$ shows that they freely generate a polynomial ring. Let $e$ be the sum of straight-line diagrams where the red stands are all at the far left and the black strands all at the far right. We have that $e \bar{T}^\nu e \cong R(\sum n_j \alpha_j)$ again by [Webb, 4.16], and [KL09, 2.9] shows that the center of the latter algebra is also a free polynomial ring $\mathbb{k}[[eh_{k,i}]_{k \leq n_i}]$. Thus, if there is an element of the center of $Z(\bar{T}^\nu)$ not in $\mathbb{k}[[h_{k,i}]_{k \leq n_i}]$, then there would have to be a non-zero central element $z$ such that $ezz = 0$. 49
On the other hand, we have composition with the diagram sweeping all strands to the far right is injective (by [Webb] 4.16), so every projective embeds as a submodule of $T^p e$. But on any element $a \in T^p e$, we have that $za = zae = ae e = 0$; the embedding above shows this is only possible if $z = 0$. Thus, we have arrived at a contradiction, and the $h_{k,i}$ with $k \leq n_j$ must freely generate the center. □

**Theorem 4.20.** There is a 2-functor $\mathfrak{a}: U_\ell \to \mathcal{LB}$ which lifts $a$. That is, the functors $E_i$ and $F_i$ define a categorical action of $\mathfrak{gl}_\ell$ sending the weight $(p_1, \ldots, p_\ell)$ to $D^b(T^{(p_1, \ldots, p_\ell)}\mathcal{-mod})$.

**Proof.** We’ll use Cautis’s rigidity theorem to show that the relations hold in this case. Consider $D^b(T^{(p_1, \ldots, p_\ell)}\mathcal{-mod})$ as a graded category with the grading shift given by the Tate twist $(\ell)$. We wish to show that this category carries a $(\mathfrak{g}, \theta)$ action in the sense of [Caub] §2.2. The basic data of such an action is given by the functors $E_i$ and $F_i$, and the map which Cautis denotes by $\theta$; note that since our grading shift is the Tate twist, $End^2(1_\ell)$ (in Cautis’s notation) is the degree $-\ell$ part of the Hochschild cohomology $HH^2(T^{(p_1, \ldots, p_\ell)})$, which is the target of $\gamma$.

Cautis’s conditions are all of the form that two functors are isomorphic, or that a given map is an isomorphism. Thus, they are left unchanged by taking the associated graded, and will hold for $D^b(T^{(p_1, \ldots, p_\ell)}\mathcal{-mod})$ if and only if they hold for $D^b(T^{(p_1, \ldots, p_\ell)}\mathcal{-mod})$. Let us discuss them in more detail.

Condition (i) is simply the fact that Hochschild cohomology of any algebra is trivial in negative degree, and the degree 0 part of the center of $T^{(p_1, \ldots, p_\ell)}$ is the scalars by Lemma [4.19]. The conditions (ii-v) follow from noting that we can construct the desired morphisms using combinations of the 2-morphisms in our prospective categorical action. While we cannot check the relations directly, Lemma [4.18] shows that the fact that they induce isomorphisms over $T^p$ (by Theorem B) suffices to show the same result for $T^p$. Condition (vi) is vacuous for $\mathfrak{gl}_\ell$, and conditions (vii,viii) are direct consequences of the fact that $T^p \neq 0$ if and only if we have $0 \leq p_i \leq n$. Thus we satisfy all of Cautis’s conditions and [Caub] 2.2 implies that we have the desired categorical action. □

By Proposition [2.18] and Theorem [3.27], this action categorifies the $\mathfrak{gl}_\ell$-action on $U_q^{\kappa} \otimes \wedge_{\mathfrak{q}}(\mathfrak{c}^{\prime}_{\mathfrak{q}} \otimes \mathfrak{c}^{\prime}_{\mathfrak{q}})$ which acts by the identity on the first factor.

### 4.4. Foams.

Quefellec and Rose [QR] define a foam 2-category $\mathcal{F} \text{Oam}(p)$, which is arguably a more natural object to consider in this case, and contains essentially the same information as the dual $\mathfrak{gl}_\ell$ action, without needing to fix a value of $\ell$. The objects of this category are tuples (of any length) of integers from $[1, n]$ whose sum is $p$. The 1-morphisms between $(p_1, \ldots, p_k)$ and $(p'_1, \ldots, p'_m)$ are sums of ladders whose top and bottom are labeled by $p'$ and $p$ respectively. Note that these are the same as the objects and 1-morphisms of the 2-categories $\mathcal{LB}, \mathcal{LB}$. However, 2-morphisms are defined quite differently; they are explicitly given by singular cobordisms between these ladders modulo relations given in [QR] §3.1. Recall that the facets of these foams can be decorated with symmetric polynomials.
**Proposition 4.21.** There is a 2-functor of the 2-category $\text{nFoam}(p)$ to $\tilde{\mathcal{L}}\mathcal{B}$ (and thus to $\mathcal{L}\mathcal{B}$) which is the identity on objects and 1-morphisms. This induces a representation of $\text{nFoam}(p)$ which sends the sequence $(p_1, \ldots, p_{\ell})$ to the derived category $\mathcal{D}^b(\tilde{T}^p\text{-mod})$, a 1-morphism to the derived tensor product with the corresponding ladder bimodule, and each foam to a natural transformation between these functors.

These natural transformations are uniquely determined by the fact that they send the foamation of a 2-morphism in $\mathcal{U}_{\ell}$ to its dual action as we have defined it. In particular:

- if $e_1$ denotes the first elementary symmetric polynomial in $p_i$ variables, then

\[
\cdots e_1 \cdots
\]

is sent to the action of the Hochschild homology class $y_i$,  

- The singular seams

\[
\begin{align*}
\text{and} \\
\text{are sent to the appropriate units and counits of adjunctions,}
\end{align*}
\]
Proof. First, we note that the functor \( \mathcal{U}_\ell \to \mathcal{LB} \) kills any weight space corresponding to a sequence \( (p_1, \ldots, p_\ell) \) with \( p_i > n \) or \( p_i < 0 \). Thus, this action factors through the quotient (following Queffelec and Rose’s notation) \( \mathcal{U}^{(n)}_\ell \). Note that as we increase \( \ell \), these actions are compatible with the inclusion that adds 0’s at the start or end of the sequence. We can now use [QR, 3.22] which writes the foam category as a direct limit of \( \mathcal{U}^{(n)}_\ell \) as \( \ell \to \infty \), and thus shows that the desired 2-functor exists. \( \Box \)

5. Knot homology

In [Webb], the second author defined equivalences of derived categories which correspond to the braid action on tensor products of representations of quantum groups. These equivalences are induced by derived tensor product with a bimodule \( \mathcal{B} \) over \( \mathcal{T}_p \) and \( \mathcal{T}^{s-p} \).

The reader can refer to [Webb, 6.3]; the bimodule is spanned by diagrams like Stendhal diagrams, but with a single crossing between the \( i \)th and \( i+1 \)st red strands, which satisfies the relations:

\[
\begin{align*}
\lambda_{i+1} & \quad \lambda_i \\
\lambda_i & \quad \lambda_{i+1}
\end{align*}
\]

(5.20a)

\[
\begin{align*}
\lambda_{i+1} & \quad \lambda_i \\
\lambda_i & \quad \lambda_{i+1}
\end{align*}
\]

(5.20b)

We grade this bimodule by giving this crossing the degree \( -\langle \lambda_{i+1}, \lambda_i \rangle \). Note that this is not an integer valued grading, but rather a \( \frac{1}{n}\mathbb{Z} \)-valued one. However, for fixed weights, the degrees of all elements lie in the same class in \( \frac{1}{n}\mathbb{Z}/\mathbb{Z} \). Let \( \xi(a, b) = (n - \max(a, b)) \min(a, b)/n \). Note that \( \langle \omega_a, \omega_b \rangle = \xi(a, b) \).
These functors induce an action of the braid group on the derived categories $D^b(\tilde{T}^p)$, i.e. they satisfy

\begin{equation}
\begin{array}{ccc}
\lambda_i & \lambda_{i+1} & \lambda_k^{-1} \\
\lambda_i & \lambda_{i+1} & \\
\end{array} \cong \begin{array}{ccc}
\lambda_i & \lambda_{i+1} & \\
\lambda_i & \lambda_{i+1} & \\
\end{array}
\end{equation}

and

\begin{equation}
\begin{array}{ccc}
\lambda_{i+2} & \lambda_{i+1} & \lambda_i \\
\lambda_i & \lambda_{i+1} & \\
\end{array} \cong \begin{array}{ccc}
\lambda_{i+2} & \lambda_{i+1} & \lambda_i \\
\lambda_i & \lambda_{i+1} & \\
\end{array}
\end{equation}

Therefore, they lead to the construction of a knot invariant. Now let us turn to comparing this action with the dual action of $U_\ell$.

**Proposition 5.1.** We have an isomorphism $Y_{i+1} \otimes \tilde{\mathcal{B}}_i(-\xi(p_1, p_2)) \cong \tilde{\mathcal{B}}_i \otimes \tilde{\mathcal{B}}_{i+1} \otimes Y_i(-\xi(a, p_2) - \xi(b, p_2))$. In terms of ladders, we have that

\begin{align*}
\begin{array}{ccc}
p_2 & a & b \\
p_1 & p_2 & \\
\end{array} = \begin{array}{ccc}
p_2 & a & b \\
p_1 & & p_2 \\
\end{array}
\end{align*}

Note that while this is one isomorphism of bimodules, applying the (anti-)automorphisms flipping diagrams through the $x$ or $y$ axes, together with taking adjoints of functors shows that this relation holds no matter how the diagram is rotated, and also if the positive crossing is replaced with a negative one. Thus, we can apply this relation locally, no matter where in the diagram we find it.

**Proof.** If we consider these as underived tensor products then the isomorphism between these spaces is clear: both spaces are spanned by Stendhal diagrams with the red/red crossing and triple point added, and the isomorphism simply shifts the triple point from one side of the red strand to the other. Thus, the result will follow if we can show that any higher Tor vanishes in both cases.

The tensor product $Y_{i+1} \otimes \tilde{\mathcal{B}}_i$ has no higher Tors since $Y_{i+1}$ is projective on the right.

We wish to also show that $\tilde{\mathcal{B}}_i \otimes \tilde{\mathcal{B}}_{i+1} \otimes Y_i$ has no higher Tors. We can reduce to the case where there are no extraneous red strands (i.e. $\ell = 2$ on the right and $\ell = 3$ on
Proposition 3.24, we can reduce to the case where the sequence $(p_2, a, b)$ at the left has either $a = 1$ or $b = 1$. Every projective module is filtered by modules which are left and right inductions of standard modules $\mathcal{V}(\mu, \omega_{p_2})$ for various weights $\mu$. Thus, we need only check that the higher homology groups of $\widetilde{B}_i \otimes \widetilde{B}_{i+1} \otimes Y_i \otimes \mathcal{V}(\mu, \omega_{p_2})$ vanish.

The module $Y_i \otimes \mathcal{V}(\mu, \omega_{p_2}) \cong Y_i \otimes (\mu, \omega_{p_2})$ is given by $\mathcal{Z}_{\omega_{p_2}} \otimes \mathcal{Z}_{\omega_{1-\mu}L_{a,b}}$. Thus, if we replace $L_{a,b}$ with its projective resolution $Q_\bullet$, then $\mathcal{Z}_{\omega_{p_2}} \otimes \mathcal{Z}_{\omega_{1-\mu}Q_\bullet}$ is a projective resolution of $\mathcal{Z}_{\omega_{p_2}} \otimes \mathcal{Z}_{\omega_{1-\mu}L_{a,b}}$, because $\mathcal{Z}_{\omega_{p_2}}$ and $\mathcal{Z}_{\omega_{1-\mu}}$ are both exact and send projectives to projectives by Prop 2.15.

Recall that $\widetilde{B}_i \otimes \widetilde{B}_{i+1} \cong \widetilde{B}_{i+1} \otimes [\text{Webb, Lem. 6.11}]$, so $\widetilde{B}_i \otimes \widetilde{B}_{i+1} \otimes Y_i \otimes \mathcal{V}(\mu, \omega_{p_2})$ is given by the cohomology of the complex $\mathcal{B}_{i/\omega_{1+1}} \otimes \mathcal{B}_{\omega_{1+1}} \otimes Q_\bullet$. We claim that this complex only has non-zero cohomology in degree 0. If we use the basis for $\mathcal{B}_{i/\omega_{1+1}}$, given in [Webb 6.4], then we can write an element of the basis of any term in the resolution of $\mathcal{B}_{i/\omega_{1+1}} \otimes \mathcal{B}_{\omega_{1+1}} \otimes Q_\bullet$, uniquely as a sum of the products of:

- basis diagrams where all crossings and dots are on the strands coming from the terms in $Q_\bullet$, with all other strands remaining in place, times
- basis diagrams where the strands from $Q_\bullet$ have no crossings between them or dots (but they can cross the other strands). This piece includes the red/red crossings.

The differential in $Q_\bullet$ only changes the first type of diagram, so as a complex of vector spaces, our resolution is isomorphic to $Q_\bullet$ tensored with a fixed vector space. Since $Q_\bullet$ only has cohomology in degree 0, the same is true of this tensor product. This completes the proof. □

Chuang and Rouquier [CR08 §6.1] define a complex of 1-functors called the Rickard complex in the 2-quantum group $U_{\mathfrak{sl}_2}$ which we’ll denote $\Theta$. This is given by the sum of complexes of 1-morphisms:

$$\Theta_{1_n} := \left[ E^{(-n)} \rightarrow \cdots \rightarrow E^{(-n+s)}E^{(s)} \langle -n+s \rangle \rightarrow \cdots \right] 1_n \quad (n \leq 0)$$

$$\Theta_{1_n} := \left[ F^{(n)} \rightarrow \cdots \rightarrow F^{(n+s)}E^{(s)} \langle -n+s \rangle \rightarrow \cdots \right] 1_n \quad (n \geq 0)$$

**Remark 5.2.** As often happens in mathematics, different papers use different conventions for these due to incompatible choices in the past. Our conventions match those of Queffelec and Rose [QR (2.42-43)], and are the opposite of those in Cautis [Caua (3-4)], which uses the adjoint of these complexes:

$$\Theta^{-1}_{1_n} := \left[ \cdots \rightarrow E^{(-n+s)}E^{(s)} \langle n-s \rangle \rightarrow \cdots \rightarrow E^{(-n)} \right] 1_n \quad (n \leq 0)$$

$$\Theta^{-1}_{1_n} := \left[ \cdots \rightarrow F^{(n+s)}E^{(s)} \langle n-s \rangle \rightarrow \cdots \rightarrow F^{(n)} \right] 1_n \quad (n \geq 0)$$

This categorifies the quantum Weyl group inside $U_q(\mathfrak{sl}_2)$. For any higher rank group, we thus have a Rickard complex $\Theta$, for each simple root given by the image of $\Theta$ under the associated root embedding of $U_{\mathfrak{sl}_2}$. Note that according to this definition, the ladder bimodule $W_{[p_i-p_{i+1}]}$ is the degree 0 part of of $\Theta$ acting on $\mathcal{F}$-modules.
Theorem 5.3. There exists a quasi-isomorphism between the image of the Rickard complex $\Theta_i$ under the action on the derived categories $D^b(\tilde{T}^p)$ and the braiding functor $B_i(-\xi(p_i,p_{i+1}))$ switching the $i$th and $i+1$st strands.

Proof. First, we establish this in the case where $p_i = p_{i+1} = 1$. In this case, the complex $\Theta$ has two terms:

\[
\begin{array}{c}
0 \\
1 \\
1 \\
\end{array}
\rightarrow
\begin{array}{c}
1 \\
1 \\
1 \\
\end{array}
\]

The map is the unit of the adjunction between $Y_i$ and $Y_i^*$. Since this map commutes with the action of any element of the algebra, we only need to consider its effect on an idempotent; in fact, every diagram factors through idempotents where between two consecutive red strands labeled 1, there is either a single black strand labeled 1 or no black strands at all. The action of the map on these diagrams is given by

\[
\begin{array}{c}
1 \\
1 \\
\end{array}
\rightarrow
0
\quad\quad
\begin{array}{c}
1 \\
1 \\
1 \\
\end{array}
\rightarrow
\begin{array}{c}
1 \\
1 \\
1 \\
\end{array}
\]

This map is clearly surjective and we wish to show that its kernel is $\tilde{B}_i$. Here we include $\tilde{B}_i \rightarrow \tilde{T}^p$ sending

\[
\begin{array}{c}
1 \\
1 \\
1 \\
1 \\
\end{array}
\rightarrow
\begin{array}{c}
1 \\
1 \\
1 \\
1 \\
\end{array}
\]

Note that this map increases degree by $\xi(1,1) = (n-1)/n$, so if it is an isomorphism, the theorem will be established with the correct shift.

We only need to check this after it is applied to an arbitrary idempotent, i.e. sequence of black strands. As argued above, using commutation with induction, we can reduce to proving this when the two $i$-labeled red strands are the only red strands and there are no black strands, or one with label 1. The exactness of the sequence $0 \rightarrow \tilde{B}_i \rightarrow \tilde{T}^p \rightarrow W_{\beta_i}$ is clear in the former case, and proven in the latter case in [Webc, 4.11].

The above short exact sequence gives rise to a quasi-isomorphism of complexes

$\tilde{B}_i \rightarrow \text{Cone}(\tilde{T}^p \rightarrow W_{\beta_i})$.

This proves that in the case $p_i = p_{i+1} = 1$, the derived tensor product with $\tilde{B}_i$ and $\text{Cone}(\tilde{T}^p \rightarrow W_{\beta_i})$ gives rise to functors on $D^b(\tilde{T}^p)$ which are isomorphic up to the shift indicated in the theorem.
Now we turn to the general case. To make the pictures easier to draw, we first prove the case with \( p \) arbitrary and \( p + 1 = 1 \), by induction on \( p \). So let \( p = a > 1 \).

Since we aim to just prove the existence of an isomorphism, it suffices to prove this result after replacing both sides by the same number of summands of each functor. Thus, we multiply both functors by \([a]\). By the bigon relation Proposition 3.20 this is the same as creating an \((a - 1, 1)\)-bigon on the red \( a \)-strand at the bottom of the braid diagram. Then we use Proposition 5.1 and its analogue for \( \Theta_i \), given by \cite{Cau5a, 5.2},

and use induction. The general case now follows from this one by induction on \( p_i + 1 \), using similar arguments with a bigon on the other red strand.

\[ \begin{align*}
&\begin{array}{c}
1 \quad a \quad 1 \\
1 \quad 1 \quad a
\end{array}
\end{align*} \quad \equiv \begin{array}{c}
a - 1 \\
1 \quad 1
\end{array} \quad \equiv \begin{array}{c}
a - 1 \\
1
\end{array} \]

\[\begin{array}{c}
\text{[a]}
\end{array}\]

\[\begin{array}{c}
\xi'(a, b) = \frac{\min(a, b) \cdot \max(a, b)}{n}.
\end{array}\]

Note that \( \xi'(a, b) = \min(a, b) - \xi(a, b) \). Accounting for the grading shift in \cite{QR, 4.1-2}], we have that:

**Corollary 5.5.** The complex of functors \( \mathcal{T}_i = \Theta_i(\min(a, b)) \) acts on \( \tilde{T}_p \)-mod by \( \mathcal{B}_i(\xi'(a, b)) \).

There is another comparison result we’ll require. In \cite{Cau5a} §5,111, Cautis constructs idempotent complexes of functors \( P^\pm \). These complexes are constructed as a limit of powers of Rickard complexes. The element \((\Theta_1 \cdots \Theta_{l-1})^{\ell-1}_{p} \) corresponds to the action of the full twist braid in the quantum Weyl group action; Cautis shows, building on ideas of Rozansky \cite{Roz14}, that the functors \((\Theta_1 \cdots \Theta_{l-1})^{\ell_{p}(\ell-1)}_{p}\) stabilize to an idempotent functor \( P^\pm \) in appropriate sense as \( p \to \infty \).

We wish to understand how this complex acts in the dual action we have constructed. As in \cite{Webc}, we consider the subcategory \( J_{p}^\pm \) of \( D^\pm(\tilde{T}_p \)-mod) generated by projectives \( \tilde{T}_p e_{i, \kappa} \) with \( \kappa \) constant. That is, in terms of the sequence of red and black terminals, we have a group of black terminals, a group consisting of all red terminals, and then another black group. Let \( \tilde{P}^0 \) be the sum of \( \tilde{T}_p e_{i, \kappa} \) over all such \( \kappa \). By \cite{Webb 4.21], we have an isomorphism \( \tilde{T}_p^\lambda \equiv \End(\tilde{P}^0) \) where \( \lambda_p = \sum \omega_p \). Thus,
we have an equivalence $f_p^\pm \cong D^z(\bar{T}^\Lambda_p\text{-mod})$. The inclusion $i^\pm : f_p^\pm \rightarrow D^z(\bar{T}^\Lambda_p\text{-mod})$ has an exact right adjoint $\iota_\ast$, which we can interpret as the functor $\text{Hom}(\bar{P}^0, -)$, which satisfies $i^\ast i^\pm \cong \text{id}_{f_p}$. 

**Proposition 5.6.** The complex $P^\pm$ is well-defined on the category $D^z(\bar{T}^\Lambda_p\text{-mod})$ and acts in the dual action as the projection $i^\ast i^\pm$.

**Proof.** First, we note that $P^\pm$ preserves the category $D^z(\bar{T}^\Lambda_p\text{-mod})$, since the functors $\Theta_i = \mathbb{B}_i(-, \xi(a, b))$ (resp. $\Theta_i^{-1} = \mathbb{B}_i^{-1}(\xi(a, b))$) are right (resp. left) exact.

Let us specialize to the case of $P^+$; the case of $P^-$ is proved similarly. Note that we have an orthogonal decomposition of the category into $f_p$ and the kernel of $i^\ast$, which is generated by the simple modules whose projective cover is not a summand of $P_0$. Thus, to establish the equality $i^\ast i^\pm \cong P^+$, we must show that $P^+$ acts by the identity on $f_p$ and kills any simple module whose projective cover is not a summand of $P_0$.

Proving the first fact is easy. From the commutation of braiding and induction functors, it suffices to check that $P^+$ acts by the identity on $\bar{T}^\Lambda_p\Theta_0$, the subalgebra with no black strands. This is clear from the fact that a full twist does so.

For the second fact, let us first consider the case where $L$ is a simple factoring through $T^\Lambda_p$; it then suffices to check this result using braiding functors over $T^\Lambda_p$. If we let $\tau$ denote a half twist, then the bimodule associated to the bimodule $\mathcal{B}_\tau$ is tilting as a left or right module by $\text{[Webb] 6.14}$. In particular, for a simple $L$, we have that $\text{Hom}(\mathcal{B}_\tau, L) = 0$ unless $L$ is a quotient of a tilting module. Put differently, the module $\mathcal{B}_\tau^{-1}(L)$ has cohomology concentrated in degrees $\geq 1$ unless $L$ is a quotient of a tilting module. Since any tilting has a costandard filtration, any tilting receives a surjective map from the projective covers of some set of costandard modules. By $\text{[Webb] 5.30}$, the projective cover of any costandard (which is dual to the injective hull of a standard) is injective and thus a sum of summands of $P^0$.

That is, the module $\mathcal{B}_\tau^{-1}(L)$ has cohomology concentrated in degrees $\geq 1$ unless $L$ is a quotient of $P^0$.

Note that

$$\mathbb{R}\text{Hom}(P^0, \mathcal{B}_\tau^{-1}(L)) \cong \mathbb{R}\text{Hom}(\mathcal{B}_\tau(P^0), L) \cong \mathbb{R}\text{Hom}(P^0, L) = 0;$$

here we have abused notation and used $P^0$ to denote the corresponding module both over $\bar{T}^\Lambda_p$ and $\tilde{T}^\Lambda_p$. This shows that $\mathcal{B}_\tau^{-1}$ preserves the kernel of $i^\ast$. Thus, none of its cohomology groups have composition factors whose projective cover is a summand of $P_0$. It follows that $\mathcal{B}_\tau^{-2}(L)$ must be concentrated in degrees $\geq 2$, and have the same restriction on its composition factors.

Applying this argument inductively, the $n$-fold full twist $\mathcal{B}_\tau^{-2n}(L)$ is concentrated in degrees $\geq 2n$, so this complex vanishes in the limit.

Every other simple of $\bar{T}^\Lambda_p$ which is not a quotient of $\bar{P}^0$ is a quotient of $\tilde{\mathcal{B}}^0 L'$ where $L'$ is another simple which is not a quotient of $\bar{P}^0$. By induction on the number of black strands, we can assume that $P^+$ kills $L'$, and thus $\tilde{\mathcal{B}}^0 L'$. The exactness of $i^\ast$ shows that it also kills $L$. \qed

Using induction, we can extend this result further. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a sequence of dominant weights for $\mathfrak{gl}_n$ of total level $2w$. Then, we can choose a
sequence of fundamental weights \((p_1, \ldots, p_{2w})\) such that the first \(w_1\) add to \(\lambda_1\), the next \(w_2\) add to \(\lambda_2\), etc. We can define \(P^\pm_\Delta\) to be the idempotent sequence of functors given by horizontally composing \(P^-\) acting on the first \(w_1\) red strands, then on the next \(w_2\), etc. Just as before, we can define \(J_{\pm, P}\) to be the subcategory generated by projectives with \(\kappa\) constant on \([1, w_1]\), on \([w_1 + 1, w_1 + w_2]\), etc. and \(\iota^\pm_\Delta\) the corresponding inclusion and its right adjoint. As before, we have an equivalence \(J_{\pm, P} \cong T^\pm_{\lambda}\)-mod by [Webb, 4.21].

For any braid \(\sigma\) on \(m\) strands, we can arrive at an associated braid \(\sigma_{2w}\) on \(2w\) strands by taking the cabling with the blackboard framing where the \(i\)th strand from the left at the top of the braid into \(w_i\) strands. The arguments of Theorem 5.6 show that:

**Corollary 5.7.** The idempotent \(P^\pm_\Delta\) is isomorphic to \(\iota_{\Delta, \lambda}^\pm \iota_{\Delta, \lambda}^\ast\).

Furthermore, applying the argument of [Webb, 4.21] to the braiding bimodules shows that:

**Corollary 5.8.** The action of the cabled braiding functor \(P^\pm_{\sigma, \Delta} \circ \mathbb{B}_{\sigma_{2w}} \circ P^\pm_\Delta\) matches that of \(\mathbb{B}_\sigma\) under the equivalence \(J^\pm_{\Delta, P} \cong D^+ (T^\pm_{\lambda}-\text{mod})\).

While we have proved the theorems above for the category \(T_{\lambda}-\text{mod}\), the same results all hold for the category \(T_{\lambda}-\text{mod}\) embedded via pullback. In particular, Theorem 5.3 and Corollary 5.8 still hold for the braiding bimodule \(\mathbb{B}_i \cong T_{\lambda} \otimes_{T_{\lambda}} \mathbb{B}_i\) and the image of the Rouquier complex under the action \(a\).

**Remark 5.9.** Just so the reader does not think we are leaving a stone unturned: one can carry out the knot homology construction of [Webb, Sec. 8] over \(\hat{T}\). However, it does not result in anything new from the perspective of knot homology: the functors attached to a tangle using \(T\) are those constructed using \(\hat{T}\) when restricted to the pullback of \(T\)-modules. In particular, associated to a closed link, we have the functor of tensor product with a vector space, which remains tensor product with the same vector space when restricted to \(T\)-modules.

This allows us to complete the proof of Theorem A. We wish to consider link homologies attached to a link labeled with a representation of \(\mathfrak{sl}_n\). Each such representation has a level, given by \(\sum m_i\) if \(\lambda = \sum m_i \omega_i\). For a given labeled link projection, let \(\ell = \ell(L)\) be the sum of the levels of the representations labeling the minima of the projection.

In [Caua], Cautis defines a knot homology constructed in any additive categorification of the representation of \(\mathfrak{sl}_{2\ell}\) with highest weight \(n\omega_\ell\). These categorifications can be classified in terms by the data of the highest weight category which is additively generated by a unique indecomposable object \(C\), and an action of a polynomial ring \(\mathbb{k}[e_1, \ldots, e_n]\) on this object, given by the fake bubbles in the endomorphisms of the identity 1-morphism on \(n\omega_\ell\). Up to equivalence, this is controlled by the ring \(\text{End}(C)\) and the associated homomorphism \(S = \mathbb{k}[e_1, \ldots, e_n] \to \text{End}(C)\). This polynomial ring is graded with \(\text{deg}(e_i) = 2i\). The most important example of such a categorification is given by the cyclotomic quotient for \(n\omega_\ell\), which corresponds to the case where \(\text{End}(C) \cong \mathbb{k}\) with the map sending \(e_i \mapsto 0\).
In the construction of \([\text{Caua}]\), each link \(L\) labeled with representations of \(\mathfrak{sl}_n\) is translated into a complex \(\mathbb{Z}(L)\) of 1-morphisms in \(\mathcal{U}_{\mathfrak{sl}_2}\).

- For cups and caps, this follows the ladder and ladder formalism introduced earlier in this paper.
- A crossing is associated to the corresponding shifted Rickard complex \(T_i^{-1} = \Theta^{-1}(-\min(a, b))\).
- Finally, non-fundamental weights are dealt with using cabling and inserted a copy of the projector \(P_+\).

Queffelec and Rose follow the same recipe in the language of foams; their homology and ours differ from Cautis’s by taking the mirror. The result is a complex in the category of graded vector spaces. We think of this as a bigraded vector space \(C = \bigoplus_{i, j} C_{i, j}\) where \(i\) is the homological grading, and \(j\) the internal grading.

If one has a categorical module over \(\mathfrak{sl}_{2\ell}\), and a highest weight object \(C\) of weight \(\ell \omega\), the result \(\text{Hom}(C, \mathbb{Z}(L)C)\) is an invariant of the labeled link \(L\), and the \(S\)-algebra \(\text{End}(C)\). This allows one to identify knot homology theories categorifying Reshetikhin-Turaev invariants with seemingly disparate origins.

Let \(\mathcal{W}^{i,j}\) be the knot invariants attached to labeled knots by the second author in type A in \([\text{Webb}]\), \(\Omega \mathcal{R}^{i,j}\) those produced by Queffelec and Rose \([\text{QR}]\), and \(\mathcal{C}^{i,j}\) those produced by Cautis. Consider an oriented framed link, and let the \textbf{weighted writhe} \(\text{wwr}(L)\) be the sum over the positive/negative crossings in any diagram of \(L\) of \(\pm \xi'(a, b)\) where \(a\) and \(b\) are the labels on the crossing.

**Theorem 5.10.**

\[
\mathcal{W}^{i,j+\text{wwr}(L)}(L) \equiv \Omega \mathcal{R}^{i+j-\text{wwr}(L)}(L) \equiv \mathcal{C}^{i+j-\text{wwr}(L)}(L)
\]

**Proof.** It’s clear from Proposition 5.6 and Corollary 5.8 that our functors coincide with Cautis’s for the mirror up to grading shift, and the reindex of the grading. That is, for some invariant \(r(L)\) of a framed link, \(\mathcal{W}^{i,j+r(L)}(L) \equiv \Omega \mathcal{R}^{i+j-\text{wwr}(L)}(L)\).

Note that by Theorem 5.3 when we switch a crossing from positive to negative, \(r(L)\) must decrease by \(2\xi(a, b)\). Since \(\text{wwr}(L)\) has the same property, this allows us to reduce to the case of an unknot.

If this unknot is 0-framed, then the theorem simply asserts that the homologies are the same. Indeed,

\[
\sum_{i,j} t^i q^j \dim \Omega \mathcal{R}^{i,j}(L) = \binom{n}{p}_q \quad \sum_{i,j} t^i q^j \dim \mathcal{W}^{i+j-\text{wwr}(L)}(L) = \binom{n}{p}_q
\]

so the result follows in this case.

Now, note that a strand with a curl in it must correspond to a shift functor, so, we need only check that adding some nonzero power of it gives the same shift on both sides. However, if we do an even number of curls, we can get back to the 0-framed unknot by reversing crossings, so this follows from our reduction to the unknot. □

For completeness, let us note that these theories are already known to coincide with the other known examples of categorifications of the \(\mathfrak{sl}_n\) Reshetikhin-Turaev invariants.
Theorem 5.11. The following link homology theories for a labelling by representations of $sl_n$ are given by $\text{Hom}(C, Z(L)C)$ for a categorical action with $C$ a highest weight object of weight $\omega_\ell$ with endomorphism $S$-algebra $\text{End}(C) \cong k$:

1. Khovanov-Rozansky homology [KR08] (and its colored generalization [Wu, Yon],
2. Cautis-Kamnitzer homology [CK08b],
3. the foam-based invariants of Queffelec and Rose [QR],
4. the Mazorchuk-Stroppel-Sussan homology constructed from category $O$ [MS09, Sus07] and
5. the invariants $\mathcal{K}$ constructed in [Webb].

Thus, all these knot homologies coincide.

The items (1-3) are discussed in [LRQ, QR, Caub]; the new contribution of this paper is to add items (4-5) to this list.

Proof.

1. For Khovanov-Rozansky homology, this is based on the construction of a categorical action constructed in [MY] on matrix factorizations. This comparison is discussed in greater detail in [QR, §4].
2. For Cautis-Kamnitzer homology, this is essentially by the original definition given in [CK08b]; however, since it proved difficult to confirm all the relations between 2-morphisms, this required a strictification result [Caub, 14.9].
3. By construction, Queffelec and Rose’s invariant is the image under the formation 2-functor [QR, §3.2] of $Z(L)$.
4. The papers [MS09] and [Sus07] use different sides of Koszul duality; in the former case, the categorical action by translation functors is used, and in the latter, that by Zuckerman functors. While the comparison between the complex $Z(L)$ and the appropriate functors on category $O$’s can be done directly, we will take the short-cut of noting that we have an equivalence between the category $O$ picture and that of (5) by [Webb, 9.21].
5. This follows from Theorem 5.10.

This completes the proof of Theorem A. □

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