THE RELATIVE BRUCE–ROBERTS NUMBER OF A FUNCTION ON A HYPERSURFACE

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Abstract We consider the relative Bruce–Roberts number $\mu_{BR}(f, X)$ of a function on an isolated hypersurface singularity $(X, 0)$. We show that $\mu_{BR}(f, X)$ is equal to the sum of the Milnor number of the fibre $\mu(f^{-1}(0) \cap X, 0)$ plus the difference $\mu(X, 0) - \tau(X, 0)$ between the Milnor and the Tjurina numbers of $(X, 0)$. As an application, we show that the usual Bruce–Roberts number $\mu_{BR}(f, X)$ is equal to $\mu(f) + \mu_{BR}(f, X)$. We also deduce that the relative logarithmic characteristic variety $LC(X)^{−}$, obtained from the logarithmic characteristic variety $LC(X)$ by eliminating the component corresponding to the complement of $X$ in the ambient space, is Cohen–Macaulay.

Keywords: isolated hypersurface singularity; Bruce–Roberts number; logarithmic characteristic variety

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1. Introduction

Let $(X, 0)$ be a germ of complex analytic set in $\mathbb{C}^n$ and $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a holomorphic function germ. The Bruce–Roberts number of $f$ with respect to $(X, 0)$ was introduced by Bruce and Roberts in [4] and is defined as

$$\mu_{BR}(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X)},$$

where $\mathcal{O}_n$ is the local ring of holomorphic functions $(\mathbb{C}^n, 0) \to \mathbb{C}$, $df$ is the differential of $f$ and $\Theta_X$ is the $\mathcal{O}_n$-submodule of $\Theta_n$ of vector fields on $(\mathbb{C}^n, 0)$ which are tangent

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to \((X, 0)\) at its regular points. If \(I_X\) is the ideal of \(\mathcal{O}_n\) of functions vanishing on \((X, 0)\), then
\[
\Theta_X = \{\xi \in \Theta_n \mid dh(\xi) \in I_X, \forall h \in I_X\}.
\]
In particular, when \(X = \mathbb{C}^n\), \(df(\Theta_X)\) is the Jacobian ideal of \(f\) and thus, \(\muBR(f, X)\) coincides with the classical Milnor number \(\mu(f)\). We remark that \(\Theta_X\) is also denoted in some papers by \(\text{Der}(- \log X)\), following Saito’s notation [11]. The main properties of \(\muBR(f, X)\) are the following (see [4]):

(a) \(\muBR(f, X)\) is invariant under the action of the group \(\mathcal{R}_X\) of diffeomorphisms \(\phi: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)\) which preserve \((X, 0)\);

(b) \(\muBR(f, X) < \infty\) if and only if \(f\) is finitely determined with respect to the \(\mathcal{R}_X\)-equivalence;

(c) \(\muBR(f, X) < \infty\) if and only if \(f\) restricted to each logarithmic stratum is a submersion in a punctured neighbourhood of the origin.

In general, \(\muBR(f, X)\) is not so easy to compute as the classical Milnor number. The main difficulty comes from the computation of the module \(\Theta_X\) and most of the times, it is necessary to use a symbolic computer system like SINGULAR [6]. If \((X, 0)\) is an isolated complete intersection singularity (ICIS) and \(\muBR(f, X)\) is finite, then \((f^{-1}(0) \cap X, 0)\) is an ICIS [2, Proposition 2.8], therefore it has well-defined Milnor number. In a previous paper, [9] we considered the case that \((X, 0)\) is an isolated hypersurface singularity (IHS). We showed that
\[
\muBR(f, X) = \mu(f) + \mu(f^{-1}(0) \cap X, 0) + \mu(X, 0) - \tau(X, 0),
\]
where \(\mu\) and \(\tau\) are the Milnor and the Tjurina numbers, respectively. Thus, (1) gives an easy way to compute \(\muBR(f, X)\) in terms of well-known invariants. The formula (1) was also obtained independently in [8] and previously in [10] when \((X, 0)\) is weighted homogeneous.

An important application of (1) allowed us to conclude in [9] that the logarithmic characteristic variety \(LC(X)\) is Cohen–Macaulay. We recall that \(LC(X)\) is the subvariety of the cotangent bundle \(T^*\mathbb{C}^n\) of pairs \((x, \alpha)\) such that \(\alpha(\xi_x) = 0\), for all \(\xi \in \Theta_X\) and for all \(x\) in a neighbourhood of 0. When \((X, 0)\) is holonomic, \(LC(X)\) is Cohen–Macaulay if and only if for any Morsification \(f_t\) of \(f\) we have
\[
\muBR(f, X) = \sum_\alpha m_\alpha n_\alpha,
\]
where \(n_\alpha\) is the number of critical points of \(f_t\) restricted to each logarithmic stratum \(X_\alpha\) and \(m_\alpha\) is the multiplicity of \(LC(X)\) along the irreducible component \(Y_\alpha\) associated with \(X_\alpha\) (see [4, Corollary 5.8]). When \((X, 0)\) is an IHS, it always has a finite number of logarithmic strata (i.e., it is holonomic in Saito’s terminology) given by \(X_0 = \mathbb{C}^n \setminus X, X_i \setminus \{0\}\), with \(i = 1, \ldots, k\) and \(X_{k+1} = \{0\}\), where \(X_1, \ldots, X_k\) are the irreducible components of \(X\) at 0.
In this paper, we are interested in another important invariant introduced in [4],

$$\mu_{BR}(f, X) = \dim C \frac{\mathcal{O}_n}{df(\Theta_X) + I_X},$$

which we call here the relative Bruce–Roberts number. This is an invariant of the restricted function $f : (X, 0) \to (\mathbb{C}, 0)$ under the induced $\mathcal{R}_X$-action. In fact, as commented in [4], it is equal to the codimension of the $\mathcal{R}_X$-orbit. Moreover, $\mu_{BR}(f, X)$ is finite if and only if $f$ restricted to each logarithmic stratum (excluding $X_0$) is a submersion in a punctured neighbourhood of the origin.

A natural question is about the relationship between $\mu_{BR}(f, X)$ and $\mu_{BR}^-(f, X)$. It is shown in [4] that if $(X, 0)$ is a weighted homogeneous ICIS then

$$\mu_{BR}(f, X) = \mu(f^{-1}(0) \cap X, 0).$$

This, combined with (1) when $(X, 0)$ is a weighted homogeneous IHS, gives that

$$\mu_{BR}(f, X) = \mu(f) + \mu_{BR}^-(f, X).$$

(2)

Our main result in § 2 is that if $(X, 0)$ is any IHS and $\mu_{BR}^-(f, X)$ is finite, then

$$\mu_{BR}^-(f, X) = \mu(f^{-1}(0) \cap X, 0) + \mu(X, 0) - \tau(X, 0).$$

(3)

In particular, (2) also holds when $\mu_{BR}(f, X)$ is finite, even when $(X, 0)$ is not weighted homogeneous. We also show in Example 3.1 that (2) is not true for higher codimension ICIS.

The relative logarithmic characteristic variety $LC(X)^-$ is obtained from $LC(X)$ by eliminating the component $Y_0$ associated with the stratum $X_0 = \mathbb{C}^n \setminus X$. In [4], they showed that $LC(X)$ is never Cohen–Macaulay when $(X, 0)$ has codimension $> 1$ along the points on $X_0$, but $LC(X)^-$ is always Cohen–Macaulay when $(X, 0)$ is a weighted homogeneous ICIS (of any codimension). Again, Cohen–Macaulayness of $LC(X)^-$ is interesting since it implies that

$$\mu_{BR}^-(f, X) = \sum_{\alpha \neq 0} m_{\alpha} n_{\alpha},$$

for any Morsification $f_t$ of $f$. As an application of (3), we show in § 3 that $LC(X)^-$ is also Cohen–Macaulay for any IHS $(X, 0)$ (not necessarily weighted homogeneous).

In § 4, we consider any holonomic variety $(X, 0)$ and study characterizations of Cohen–Macaulayness of $LC(X)$ and $LC(X)^-$ in terms of the relative polar curve associated with a Morsification $f_t$ of $f$. Finally, in § 5, we give a formula which generalizes the classical Thom–Sebastiani formula for the Milnor number of a function defined as a sum of functions with separated variables.

2. The relative Bruce–Roberts number

The main goal of this section is to prove the equality (3). The next lemma is inspired by [2, Proposition 2.8].
Lemma 2.1. Let \((X, 0)\) be an IHS determined by \(\phi: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)\) and \(f \in \mathcal{O}_n\). The map \((\phi, f): (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^2, 0)\) defines an ICIS if and only if \(\mu_{BR}(f, X) < \infty\).

Proof. If \((\phi, f): (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^2, 0)\) defines an ICIS then \(\mu_{BR}(f, X)\) is finite because

\[
V(df(\partial f/\partial x)) \subset V(J(f, \phi) + I_X) \subset \{0\}.
\]

For the converse, if \(\mu_{BR}(f, X) < \infty\) then the restriction of \(f\) to each logarithmic stratum, excluding \(\mathbb{C}^n \setminus X\) is non-singular. The proof is now the same of Proposition 2.8 in [2]. \(\square\)

The following technical lemma will be used in the proof of the next theorem. Given a matrix \(A\) with entries in a ring \(R\), we denote by \(I_k(A)\) the ideal in \(R\) generated the \(k \times k\) minors of \(A\).

Lemma 2.2. Let \(f, g \in \mathcal{O}_n\) be such that \(\dim V(J(f, g)) = 1\) and \(V(Jf) = \{0\}\), and consider the following matrices

\[
A = \begin{pmatrix}
\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n}
\frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n}
\end{pmatrix},
A' = \begin{pmatrix}
\mu \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n}
\lambda \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n}
\end{pmatrix},
\]

where \(\lambda, \mu \in \mathcal{O}_n\). Let \(M, M'\) be the submodules of \(\mathcal{O}_n^2\) generated by the columns of \(A, A'\) respectively. If \(I_2(A) = I_2(A')\) then \(M = M'\).

Proof. We see \(A\) and \(A'\) as homomorphisms of modules over \(R := \mathcal{O}_n\):

\[
A: R^n \rightarrow R^2, \quad A': R^{n+1} \rightarrow R^2.
\]

We consider the \(R\)-module \(R^2/M = \text{coker}(A)\), which has support \(V(I_2(A)) = V(J(f, g))\). Therefore, \(\dim(R^2/M) = 1 = n - (n - 2 + 1)\) and hence it is Cohen–Macaulay (see [5]). In particular, it is unmixed. Now, \(M'/M\) is a submodule of \(R^2/M\), so the associated primes \(\text{Ass}(M'/M)\) are included in \(\text{Ass}(R^2/M)\). If \(M'/M \neq 0\) then \(\text{Ass}(M'/M) \neq \emptyset\) and it follows that \(\dim(M'/M) = 1\).

Let \(U\) be a neighbourhood of 0 in \(\mathbb{C}^n\) such that 0 is the only critical point of \(f\). For all \(x \in U \setminus \{0\}\), there exist \(i_0 \in \{1, \ldots, n\}\), such that \(\partial f/\partial x_{i_0}(x) \neq 0\). We may suppose \(i_0 = 1\). Making elementary column operations in the matrices \(A\) and \(A'\), we obtain

\[
B = \begin{pmatrix}
1 & 0 & \cdots & 0
\end{pmatrix}, \quad B' = \begin{pmatrix}
\mu & 1 & 0 & \cdots & 0
\end{pmatrix}
\lambda & c_1 & c_2 & \cdots & c_n
\lambda & c_1 & c_2 & \cdots & c_n
\]

such that

\[
I_2(A) = I_2(B), \quad I_2(A') = I_2(B'), \quad \text{Im}(A') = \text{Im}(B).
\]

By hypothesis \(I_2(A) = I_2(A')\) and consequently \(\langle c_2, \ldots, c_n \rangle = \langle \mu c_1 - \lambda, c_2, \ldots, c_n \rangle\).

This implies \(\lambda = \mu c_1 + \alpha_2 c_2 + \cdots + \alpha_n c_n\), for some \(\alpha_2, \ldots, \alpha_n \in R\). Thus,

\[
\begin{pmatrix}
\mu \\
\lambda
\end{pmatrix} = \mu \begin{pmatrix}
1 \\
c_1
\end{pmatrix} + \alpha_2 \begin{pmatrix}
0 \\
c_2
\end{pmatrix} + \cdots + \alpha_n \begin{pmatrix}
0 \\
c_n
\end{pmatrix},
\]

and hence \((M'/M)_x = 0\). This shows that \(\text{Supp}(M'/M) \subset \{0\}\) and hence, \(M' = M\). \(\square\)
Given an IHS \((X, 0)\) defined by a holomorphic function germ \(\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\), we consider the \(O_n\)-submodule of the trivial vectors fields, denoted by \(\Theta^T_X\), generated by 

\[
\frac{\partial \phi}{\partial x_i}, \frac{\partial \phi}{\partial x_j}, -\frac{\partial \phi}{\partial x_k}, \text{ with } i, j, k = 1, \ldots, n; k \neq j.
\]

This module was related to the Tjurina number of \((X, 0)\) in [9, 13]. By using different approaches, it is shown that

\[
\tau(X, 0) = \dim \mathbb{C} \Theta^T_X / \Theta^T_X.
\]

Moreover, in [9], we also proved that

\[
\tau(X, 0) = \dim \mathbb{C} df(\Theta_X) / df(\Theta^T_X),
\]

where \(f\) is any \(R_X\)-finitely determined function germ. The following result generalizes this equality with a weaker hypothesis on \(f\).

**Theorem 2.3.** Let \((X, 0)\) be an IHS determined by \(\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) and \(f \in O_n\) such that \(\mu_{BR}(f, X) < \infty\), then:

(i) \(\Theta_X / \Theta^T_X \approx df(\Theta_X) + I_X\);

(ii) \(\Theta_X / \Theta^T_X \approx df(\Theta_X) / df(\Theta^T_X)\);

(iii) \(df(\Theta_X) \cap I_X = Jf I_X\);

(iv) \(df(\Theta_X) / df(\Theta^T_X) \approx \Theta_X / \Theta^T_X\);

(v) \(df(\Theta_X) : I_X = Jf I_X\);

(vi) \(df(\Theta^T_X) : I_X = Jf\),

where \(I_X\) is the ideal generated by \(\phi\).

**Proof.** (i) The homomorphism \(\Psi : \Theta_X \to df(\Theta_X) + I_X\) defined by \(\Psi(\xi) = df(\xi)\) induces the isomorphism

\[
\overline{\Psi} : \frac{\Theta_X}{\Theta^T_X} \to \frac{df(\Theta_X) + I_X}{df(\Theta^T_X) + I_X}.
\]

In fact, it is enough to show that \(\Psi^{-1}(df(\Theta^T_X) + I_X) \subset \Theta^T_X\). Let \(\xi \in \Psi^{-1}(df(\Theta^T_X) + I_X)\) then \(\Psi(\xi) \in df(\Theta^T_X) + I_X\), that is, there exist \(\eta \in \Theta^T_X\) and \(\mu, \lambda \in O_n\), such that

\[
\begin{aligned}
\frac{df(\xi - \eta) = \mu \phi}{d\phi(\xi - \eta) = \lambda \phi}
\end{aligned}
\]

then

\[
\begin{pmatrix}
\mu \\
\lambda
\end{pmatrix}
\in
\begin{pmatrix}
\frac{\partial f}{\partial x_i} \\
\frac{\partial \phi}{\partial x_i}
\end{pmatrix},
\]
and

\[
I_2 \left( \begin{array}{cccc}
\mu \phi & \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\
\lambda \phi & \frac{\partial \phi}{\partial x_1} & \cdots & \frac{\partial \phi}{\partial x_n}
\end{array} \right) = I_2 \left( \begin{array}{cccc}
\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\
\frac{\partial \phi}{\partial x_1} & \cdots & \frac{\partial \phi}{\partial x_n}
\end{array} \right) = J(f, \phi).
\]

Therefore

\[
\left| \begin{array}{ccc}
\mu & \frac{\partial f}{\partial x_i} \\
\lambda & \frac{\partial \phi}{\partial x_i}
\end{array} \right| \phi \in J(f, \phi)
\]

and since \( \phi \) is regular in \( \mathcal{O}_n^{\mathcal{O}_n} \) then

\[
\left| \begin{array}{ccc}
\mu & \frac{\partial f}{\partial x_i} \\
\lambda & \frac{\partial \phi}{\partial x_i}
\end{array} \right| \in J(f, \phi), \quad i = 1, \ldots, n.
\]

By Lemma 2.2, \( \lambda \in J\phi \) and using [9, Lemma 3.1], \( \xi \in \Theta_T X \).

(ii) This equality also was proved in [9] with the additional hypothesis that \( f \) is \( \mathcal{R}_X \)-finitely determined.

The epimorphism \( \psi : \Theta_X \to df(\Theta_X) \) defined by \( \psi(\xi) = df(\xi) \) induces the isomorphism

\[
\psi : \Theta_X \to df(\Theta_X),
\]

In fact, let \( \xi \in \ker(\psi) \), then there exist \( \lambda \in \mathcal{O}_n \), such that

\[
\begin{cases}
df(\xi) = 0 \\
d\phi(\xi) = \lambda \phi
\end{cases}
\]

The rest is similar to the proof of (i).

(iii) Let \( \xi \in \Theta_X \) be such that \( df(\xi) \in I_X \), then there exist \( \mu, \lambda \in \mathcal{O}_n \), such that

\[
\begin{cases}
df(\xi) = \mu \phi \\
d\phi(\xi) = \lambda \phi
\end{cases}
\]

Using the same techniques of the proof of (i), we have

\[
df(\Theta_X) \cap I_X \subset Jf I_X.
\]

The other inclusion is immediate.

(iv) It follows from the isomorphisms

\[
\frac{df(\Theta_X)}{df(\Theta_X)} = \frac{df(\Theta_X) + I_X}{df(\Theta_X)} \approx \frac{I_X}{df(\Theta_X) \cap I_X} \approx \frac{I_X}{Jf I_X} \approx \mathcal{O}_n.
\]
(v) It follows from (iii).

(vi) It follows from (v) and \( Jf \subset df(\Theta_X^T) : I_X \).

\[ \square \]

**Remark 2.4.** The items (ii) and (iv) of Theorem 2.3 seem a bit peculiar since from (iv) the quotient \( df(\Theta_X^T)/df(\Theta_X) \) does not depend on \((X, 0)\) while from (ii), \( df(\Theta_X)/df(\Theta_X^T) \) does not depend on \( f \). Moreover by [9, 13] if \((X, 0)\) is an IHS determined by \( \phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \), then \( \dim \mathbb{C} \frac{\Theta_X}{\Theta_X^T} = \tau(X, 0) \), therefore

\[
\dim \mathbb{C} \frac{df(\Theta_X) + I_X}{df(\Theta_X^T) + I_X} = \dim \mathbb{C} \frac{df(\Theta_X)}{df(\Theta_X^T)} = \tau(X, 0).
\]

The next theorem is one of the main results of this work.

**Theorem 2.5.** Let \((X, 0)\) is an IHS determined by \( \phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) and \( f \in \mathcal{O}_n \) be a function germ such that \( \mu_{\text{BR}}^{-}(f, X) < \infty \). Then \((\phi, f)\) defines an ICIS and

\[
\mu(f^{-1}(0) \cap X, 0) = \mu_{\text{BR}}^{-}(f, X) + \tau(X, 0) - \mu(X, 0).
\]

**Proof.** We consider the exact sequence

\[
0 \longrightarrow \frac{df(\Theta_X^T)}{df(\Theta_X) + I_X} \overset{i}{\longrightarrow} \frac{\mathcal{O}_n}{df(\Theta_X^T) + I_X} \overset{\pi}{\longrightarrow} \frac{\mathcal{O}_n}{df(\Theta_X)} \longrightarrow 0.
\]

Since \((X, 0)\) is an IHS

\[
df(\Theta_X^T) = J(f, \phi) + JfI_X,
\]

hence

\[
\mu_{\text{BR}}^{-}(f, X) = \dim \mathbb{C} \frac{\mathcal{O}_n}{J(f, \phi) + I_X} - \dim \mathbb{C} \frac{df(\Theta_X) + I_X}{df(\Theta_X^T) + I_X}
\]

\[
= \mu(f^{-1}(0) \cap X, 0) + \mu(X, 0) - \tau(X, 0).
\]

The last equality is a consequence of the Lê-Greuel formula [3] and Theorem 2.3 (i). \( \square \)

3. The relative Bruce–Roberts number of a function with isolated singularity

In this section, \((X, 0)\) is an IHS and \( f \in \mathcal{O}_n \) is a function germ \( R_X \)-finitely determined, then all the results in the previous section are true in this case. In particular from (iv)
of Theorem 2.3
\[ \mu(f) = \dim_{\mathbb{C}} \frac{df(\Theta_X)}{df(\Theta_X)}. \] (4)

Therefore, by the exact sequence
\[ 0 \longrightarrow \frac{df(\Theta_X)}{df(\Theta_X)} \xrightarrow{i} \mathcal{O}_n \xrightarrow{\pi} \mathcal{O}_n \longrightarrow 0, \]
we conclude that
\[ \mu_{BR}(f, X) = \mu(f) + \mu_{BR}(f, X). \]

The following example shows that the characterization of the Milnor number (4) is not true anymore when \((X, 0)\) is an ICIS with codimension higher than one.

**Example 3.1.** Let \((X, 0)\) be an ICIS determined by \(\phi(x, y, z) = (x^3 + x^2y^2 + y^7 + z^3, xyz)\), and \(f(x, y, z) = xy - z^4, f\) is a \(R_X\)-finitely determined and

\[ 3 = \mu(f) \neq \dim_{\mathbb{C}} \frac{df(\Theta_X)}{df(\Theta_X)} = 6. \]

As a consequence of the characterization of the Milnor number (4), we prove that \(LC(X)^-\) is Cohen–Macaulay when \((X, 0)\) is an IHS.

The logarithmic characteristic variety, \(LC(X)\), is defined as follows. Suppose the vector fields \(\delta_1, \ldots, \delta_m\) generate \(\Theta_X\) on some neighbourhood \(U\) of 0 in \(\mathbb{C}^n\). Let \(T^*_U\mathbb{C}^n\) be the restriction of the cotangent bundle of \(\mathbb{C}^n\) to \(U\). We define \(LC_U(X)\) to be
\[ LC_U(X) = \{(x, \xi) \in T^*_U\mathbb{C}^n : \xi(\delta_i(x)) = 0, i = 1, \ldots, m\}. \]

Then \(LC(X)\) is the germ of \(LC_U(X)\) in \(T^*\mathbb{C}^n\) along \(T^*_0\mathbb{C}^n\), the cotangent space to \(\mathbb{C}^n\) at 0. As \(LC(X)\) is independent of the choice of the vector fields \(\delta_i\) then it is a well-defined germ of analytic subvariety in \(T^*\mathbb{C}^n\) (see [4, 11]).

If \((X, 0)\) is holonomic with logarithmic strata \(X_0, \ldots, X_k\) then \(LC(X)\) has dimension \(n\), and its irreducible components are \(Y_0, \ldots, Y_k\), with \(Y_i = N^*X_i\) as set-germs, where \(N^*X_i\) is the closure of the conormal bundle \(N^*X_i\) of \(X_i\) in \(\mathbb{C}^n\) (see [4, Proposition 1.14]).

When \((X, 0)\) has codimension higher than one, Bruce and Roberts proved that \(LC(X)\) is not Cohen–Macaulay. Then they consider the subspace of \(LC(X)\) obtained by deleting the component \(Y_0\) that corresponds to the stratum \(X_0 = \mathbb{C}^n \setminus X\), that is
\[ LC(X)^- = \bigcup_{i=1}^{k+1} Y_i \]
and as set-germs,
\[ LC(X)^- = \bigcup_{i=1}^{k+1} N^*X_i. \]

An interesting fact about \(LC(X)^-\) is that it may be Cohen–Macaulay even when \(LC(X)\) is not Cohen–Macaulay, for example, if \((X, 0)\) is a weighted homogeneous ICIS, then \(LC(X)^-\) is Cohen–Macaulay, [4].
**Proposition 3.2.** Let \((X, 0)\) be an IHS, then \(LC(X)^-\) is Cohen–Macaulay.

**Proof.** We consider \((0, p) \in LC(X)^-\), then \((0, p) \in LC(X)\) and there exists \(f \in \mathcal{O}_n\) such that \(df(0) = p\). In [9], we proved that \(LC(X)\) is Cohen–Macaulay. Therefore, by [4, Proposition 5.8],

\[
\mu_{BR}(f, X) = \sum_{i=0}^{k+1} m_i n_i = m_0 n_0 + \sum_{i=1}^{k+1} m_i n_i = \mu(f) + \sum_{i=1}^{k+1} m_i n_i.
\]

where \(n_i\) is the number of critical points of a Morsification of \(f\) in \(X_i\) and \(m_i\) is the multiplicity of irreducible component \(Y_i\). Thus,

\[
\mu_{BR}(f, X) = \mu_{BR}(f, X) - \dim \mathbb{C} \frac{df(\Theta_X^{-})}{df(\Theta_X)} = \mu_{BR}(f, X) - \mu(f) = \sum_{i=1}^{k+1} m_i n_i.
\]

and by [4, Proposition 5.11], we obtain that \(LC(X)^-\) is Cohen–Macaulay. \(\square\)

**Remark 3.3.** We remark that in the proof of the previous proposition, we just used that if \((X, 0) \subset (\mathbb{C}^n, 0)\) is a hypersurface such that \(\dim \mathbb{C} \frac{df(\Theta_X^{-})}{df(\Theta_X)} = \mu(f)\) for all \(f \in \mathcal{R}_X\)-finitely determined then \(LC(X)^-\) is Cohen–Macaulay if and only if \(LC(X)\) is Cohen–Macaulay.

4. Polar curves and logarithmic characteristic varieties

It is important to know whether the logarithmic characteristic variety of an analytic variety is Cohen–Macaulay. In [9], we showed that this is the case for IHS. For non-isolated singularities, it is an open problem. In this section, we give one more step in order to solve it: we study the polar curve and the relative polar curve of a holomorphic function germ over a holonomic analytic variety. We show that these curves are Cohen–Macaulay if and only if the logarithmic characteristic variety and the relative logarithmic characteristic variety (respectively) are Cohen–Macaulay. As a consequence, we have the principle of conservation for the Bruce–Roberts number.

**Definition 4.1.** Let \(f \in \mathcal{O}_n\) be a \(\mathcal{R}_X\)-finitely determined function germ and \(F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0), F(t, x) = f_t(x)\),

a 1-parameter deformation of \(f\). The **polar curve** of \(F\) in \((X, 0)\) is

\[
C = \{(x, t) \in \mathbb{C}^n \times \mathbb{C}; df_t(\delta_i(x)) = 0, \forall i = 1, \ldots, m\},
\]

where \(\Theta_X = \langle \delta_1, \ldots, \delta_m \rangle\).

In [1], it was proved that if \(LC(X)\) is Cohen–Macaulay then the polar curve \(C\) is Cohen–Macaulay.

**Proposition 4.2.** Let \((X, 0)\) be a holonomic analytic variety. If any \(\mathcal{R}_X\)-finitely determined function germ has a Morsification whose polar curve is Cohen–Macaulay then \(LC(X)\) is Cohen–Macaulay.
Proof. Let \((0, p) \in LC(X)\), then there exists an \(R_X\)-finitely determined function germ \(f \in \mathcal{O}_n\), such that \(df(0) = p\). Let \(F : (\mathbb{C}^n \times \mathbb{C}) \to (\mathbb{C}, 0), F(x, t) = f_t(x)\), be a Morsification of \(f\). By hypothesis \(\mathcal{O}_{n+1}/df_t(\Theta_X)\) is Cohen–Macaulay of dimension 1, then by the principle of conservation of number

\[
\mu_{BR}(f, X) = \sum_{i=0}^{k+1} \sum_{x \in \Sigma f_t \cap X_i} \dim_C \frac{\mathcal{O}_{n,x}}{df_t(\Theta_{X,x})} = \sum_{i=0}^{k+1} \sum_{x \in \Sigma f_t \cap X_i} m_i = \sum_{i=0}^{k+1} n_i m_i
\]

because if \(x \in X_i\) is a Morse critical point of \(f_t\), then \(\mu_{BR}(f_t, X)_x = m_i\), and by [4, Proposition 5.8], \(LC(X)\) is Cohen–Macaulay. \(\square\)

When \(LC(X)\) is Cohen–Macaulay, we have

\[
\mu_{BR}(f, X) = \sum_{x \in \mathbb{C}^n} \mu_{BR}(f_t, X)_x,
\]

where \(f_t\) is any 1-parameter deformation of \(f\).

Our purpose now is to prove similar results for \(LC(X)\). We define the relative polar curve by

\[
C^- = \{(x, t) \in C; x \in X\},
\]

where \(C\) is the polar curve of \(F\) in \((X, 0)\).

The proof of the next proposition is similar to the one of [1, Theorem 3.7].

**Proposition 4.3.** Let \((X, 0)\) be a holonomic analytic variety. If \(LC(X^-)\) is Cohen–Macaulay then the relative polar curve of every 1-parameter deformation of any \(R_X\)-finitely determined function germ is Cohen–Macaulay.

For the converse, we need the following lemma, which is the analogous of [4, Proposition 5.12] for the relative Bruce–Roberts number.

**Lemma 4.4.** Let \((X, 0)\) be a holonomic analytic variety and \(f \in \mathcal{O}_n\). We assume that \(f\) restricted to \((X, 0)\) is a Morse function. If \(x \in X\) is a critical point of \(f\) then \(\mu_{BR}(f, X)^-_x = m_\alpha\), where \(m_\alpha\) is the multiplicity of the irreducible component \(Y_\alpha\) corresponding to the logarithmic stratum \(X_\alpha\) which contains \(x\).

**Proof.** Let \(Z_i = Y_i \setminus \bigcup_{j \neq i} Y_j\) where \(Y_i\) are the irreducible components of \(LC(X)\). We know from [4, Proposition 5.12] that \(LC(X)\) is Cohen–Macaulay at points in \(Z_i, i = 1, \ldots, k + 1\). We see that \(LC(X^-)\) coincides locally with \(LC(X)\) and hence, \(LC(X^-)\) is also Cohen–Macaulay at points in \(Z_i, i = 1, \ldots, k + 1\).

In fact, let \((0, p) \in Z_i\) with \(i \neq 0\), then \((x, p) \notin Y_0\). Let \(V := T^*\mathbb{C}^n \setminus Y_0\), which is an open neighbourhood of \((x, p)\). Obviously, we have the equality of sets

\[
LC(X) \cap V = LC(X^-) \cap V.
\]

Moreover, let \(I, I^-\) and \(I_j\) be the ideals which define \(LC(X)\), \(LC(X^-)\) and \(Y_j, j = 0, \ldots, k + 1\), respectively. Then,

\[
I = I_0 \cap I_1 \cap \cdots \cap I_{k+1}, \quad I^- = I_1 \cap \cdots \cap I_{k+1} \text{ and } I_0 = \langle p_1, \ldots, p_n \rangle.
\]
Since \( p \neq 0 \), \( I_0 \) is the total ring at \((x, p)\), so we have an equality between germs of complex spaces.

Finally, we have

\[
\mu_{BR}(f, X)_x = \sum_{i=1}^{k+1} m_i n_i = m_x.
\]

The equalities (*) and (**) are consequences of [4, Propositions 5.11 and 5.2], respectively.

We are ready now to prove the converse of Proposition 4.3.

**Proposition 4.5.** Let \((X, 0)\) be a holonomic analytic variety. If the relative polar curve of every 1-parameter deformation of any \(R_X\)-finitely determined function germ is Cohen–Macaulay then \(LC(X)^-\) is Cohen–Macaulay.

**Proof.** Let \((0, p) \in LC(X)^-\), then there exists an \(R_X\)-finitely determined function germ \(f \in O_n\), such that \(df(0) = p\). Let \(F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)\) be a Morsification of \(f\) and set \(f_t(x, t) = F(x, t)\).

By hypothesis \(O_{n+1}/df_t(\Theta_X^X)\) is Cohen–Macaulay of dimension 1. By the principle of the conservation of the multiplicity,

\[
\dim_{\mathbb{C}} \frac{O_n}{df(\Theta_X^X)} = \sum_{i=1}^{k+1} \sum_{x \in \Sigma f \cap X_i} \dim_{\mathbb{C}} \frac{O_{n,x}}{df_t(\Theta_X^X,x)} = \sum_{i=1}^{k+1} m_i = \sum_{i=1}^{k+1} n_i m_i,
\]

because if \(x \in X_i\) is a Morse critical point of \(f_t\), then \(\mu_{BR}(f_t, X)_x = m_i\) by Lemma 4.4.

By [4, Proposition 5.11], \(LC(X)^-\) is Cohen–Macaulay.

As a consequence of the previous result,

\[
\mu_{BR}(f, X) = \sum_{x \in \mathbb{C}^n} \mu_{BR}(f_t, X)_x,
\]

where \(f_t\) is any 1-parameter deformation of \(f\).

5. An example with non-isolated singularities

Given natural numbers \(0 < k \leq n\), we can see \(O_k\) as a subring of \(O_n\) and \(\Theta_k\) as a subset of \(\Theta_n\). We fix \((x_1, \ldots, x_n)\) as the system of coordinates in \(O_n\) and we use \((x_1, \ldots, x_k)\) as the coordinate system of \(O_k\) and \((x_{k+1}, \ldots, x_n)\) as the one in \(O_{n-k}\).

Let \((X, 0) \subset (\mathbb{C}^k, 0)\) be an analytic variety. We denote by \((\mathcal{X}, 0) \subset (\mathcal{C}^n, 0)\) the inclusion of \((X, 0)\) in \((\mathbb{C}^n, 0)\). Then \(\Theta_X^X = O_n \Theta_X + \left\langle \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_n}\right\rangle\) and \(LC(\mathcal{X}) = LC(X) \times (\mathbb{C}^{n-k})\).

Consequently, if \(LC(X)\) is Cohen–Macaulay then \(LC(\mathcal{X})\) is Cohen–Macaulay.

In particular, if \((X, 0)\) is an IHS then \(LC(\mathcal{X})\) is Cohen–Macaulay.

Let \(F \in O_n\) a function germ with isolated singularity such that \(F = f + g\) with \(f \in O_k\) and \(g \in O_{n-k}\). It is known by Sebastiani and Thom [12] that \(\mu(F) = \mu(f)\mu(g)\). We prove a similar result for the Bruce–Roberts number,

\[
\mu_{BR}(F, X) = \mu(g)\mu_{BR}(f, X).
\]
**Proposition 5.1.** Let $I$ and $J$ be ideals in $O_k$ and $O_{n-k}$, respectively. If we denote by $I' = IO_n$ and $J' = JO_n$ the respective induced ideals in $O_n$, then

$$\dim_{\mathbb{C}} \frac{O_n}{I' + J'} < \infty \text{ if and only if } \dim_{\mathbb{C}} \frac{O_k}{I} < \infty \text{ and } \dim_{\mathbb{C}} \frac{O_{n-k}}{J} < \infty.$$ 

Moreover, if these dimensions are finite then

$$\dim_{\mathbb{C}} \frac{O_n}{I' + J'} = \left( \dim_{\mathbb{C}} \frac{O_k}{I} \right) \left( \dim_{\mathbb{C}} \frac{O_{n-k}}{J} \right).$$

**Proof.** The equivalence follows from

$$V(I') = V(I) \times \mathbb{C}^{n-t}, \quad V(J') = \mathbb{C}^t \times V(J) \text{ and } V(I' + J') = V(I) \times V(J).$$

For the equality, by hypothesis there exist positive integer numbers $k', k_i$ and $k_j$ such that

$$M_{n}^{k'} \subset I' + J', \quad M_k^{k_i} \subset I, \quad M_{n-k}^{k_j} \subset J,$$

where $M_\ell$ is the maximal ideal of $O_\ell$. Let $r = \max\{k', k_i, k_j\}$, then

$$\frac{O_n}{I' + J'} \approx \frac{O_n}{M_n^{r}} = \frac{\mathbb{C}[z_1, z_2]}{M_n^{r}} = \frac{\mathbb{C}[z_1, z_2]}{I'' + J''},$$

where $z_1 = (x_1, \ldots, x_k), z_2 = (x_{k+1}, \ldots, x_n)$ and $I''$ and $J''$ are the ideals in $\mathbb{C}[z_1, z_2]$ generated by the $r-1$-jets of the generators of $I$ and $J$, respectively. Analogously,

$$\frac{O_k}{I} \approx \frac{\mathbb{C}[z_1]}{I''} \text{ and } \frac{O_{n-t}}{J} \approx \frac{\mathbb{C}[z_2]}{J''},$$

where $I''$ and $J''$ are the ideals in $\mathbb{C}[z_1]$ and $\mathbb{C}[z_2]$ generated by the $r-1$-jets of the generators of $I$ and $J$, respectively. Finally, the equality follows from

$$\frac{\mathbb{C}[z_1]}{I''} \otimes_{\mathbb{C}} \frac{\mathbb{C}[z_2]}{J''} = \frac{\mathbb{C}[z_1, z_2]}{I'' + J''},$$

where $\otimes_{\mathbb{C}}$ denotes the tensor product, see [7, Proposition 2.7.13]. \hfill \Box

We observe that the previous result gives a simpler proof to the equality of [12] about the Milnor numbers. Finally, we relate the Bruce–Roberts numbers $\mu_{BR}(F, \tilde{X})$ and $\mu_{BR}(f, X)$.

**Corollary 5.2.** Let $(\tilde{X}, 0)$, and $(X, 0)$ as before, and

$$F : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0),$$

$$(z_1, z_2) \mapsto f(z_1) + g(z_2)$$

then:
(a) $F$ is $\mathcal{R}_X$-finitely determined if, and only if, $f$ is $\mathcal{R}_X$-finitely determined and $g$ has isolated singularity.

(b) If $F$ is $\mathcal{R}_X$-finitely determined, $\mu_{BR}(F, \tilde{X}) = \mu(g)\mu_{BR}(f, X)$.

**Proof.** It is a consequence of the characterization of $\Theta_X$ and the previous theorem. □

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