On the excursions of drifted Brownian motion and the successive passage times of Brownian motion

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Abstract

By using the law of the excursions of Brownian motion with drift, we find the distribution of the $n$-th passage time of Brownian motion through a straight line $S(t) = a + bt$. In the special case when $b = 0$, we extend the result to a space-time transformation of Brownian motion.

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1 Introduction

We consider a drifted Brownian motion of the form

$$X(t) = x + \mu t + B_t, \quad t \geq 0,$$

where $x, \mu \in \mathbb{R}$ and $B_t$ is standard Brownian motion (BM). When $X(t)$ is entirely positive or entirely negative on the time interval $(a, b)$, it is said that it is an excursion of drifted BM; this means that $B_t$ remains above or below the straight line $-x - \mu t$, for all $t \in (a, b)$. Excursions of drifted BM have interesting applications in Biology, Economics, and other applied sciences. As an example in Economics, if we admit that the time evolution of the gross domestic product is described by a BM with drift $\mu$, starting from $x$, then the downward and upward movement of it around its long-term growth trend (i.e. the straight line $x + \mu t$), gives rise to an economic cycle. These fluctuations typically involve shifts over time between periods of relatively rapid economic growth (expansions or booms), and periods of relative stagnation or decline (contractions or recessions) (see e.g. [15]). Excursions of drifted BM are also related to the last passage time of BM through a linear boundary; actually, last passage times of continuous martingales play an important role in Finance, for instance, in models of default risk (see e.g. [8], [9]).

When the drift $\mu$ is zero, $X(t)$ becomes BM and it is well-known that the excursions of BM have the arcsine law, namely the probability that BM has no zeros in the time interval $(a, b)$ is given by $\frac{2}{\pi} \arcsin \sqrt{a/b}$ (see e.g. [12]). By using Salminen’s formula for the last passage time of BM through a linear boundary (see [11]), we find the law of the excursions of drifted BM, namely the probability that $X(t) = x + \mu t + B_t$ has no zeros in the interval $[a, b]$. 

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\((a, b)\). From this, we derive the distribution of the \(n\)-th passage time of BM through the linear boundary \(S(t) = a + bt, \ t \geq 0\). We recall that the first-passage time of BM through \(S(t)\), when starting from \(x\), is defined by \(\tau_1(x) = \inf\{t > 0 : x + B_t = a + bt\}\) and the Bachelier-Levy formula holds:

\[
P(\tau_1(x) \leq t) = 1 - \Phi((a - x)/\sqrt{t} + b\sqrt{t}) + \exp(-2b(a - x))\Phi(b\sqrt{t} - (a - x)/\sqrt{t}),
\]

where \(\Phi(y) = \int_{-\infty}^{y} \phi(z)dz\), with \(\phi(z) = e^{-z^2/2}/\sqrt{2\pi}\), is the cumulative distribution function of the standard Gaussian variable. If \((a - x)b > 0\), then \(P(\tau_1(x) < \infty) = e^{-2b(a-x)}\), whereas, if \((a-x)b \leq 0\), \(\tau_1(x)\) is finite with probability one and it has the following Inverse Gaussian density, which is non-defective (see e.g. \([10]\)):

\[
f_{\tau_1}(t) = f_{\tau_1}(t|x) = \frac{d}{dt}P(\tau_1(x) \leq t) = \frac{|a - x|}{t^{3/2}} \phi\left(\frac{a + bt - x}{\sqrt{t}}\right), \ t > 0;
\]

moreover, if \(b \neq 0\), the expectation of \(\tau_1(x)\) is finite, being \(E(\tau_1(x)) = \frac{|a-x|}{|b|}\).

The second-passage time of BM through \(S(t)\), when starting from \(x\), is defined by \(\tau_2(x) = \inf\{t > \tau_1(x) : x + B_t = a + bt\}\), and generally, for \(n \geq 1\), \(\tau_n(x) = \inf\{t > \tau_{n-1}(x) : x + B_t = a + bt\}\) denotes the \(n\)-th passage time of BM through \(S(t)\). Our aim is to study its distribution.

The paper is organized as follows: in Section 2 we will find explicitly the distribution of the \(n\)-th passage time of BM, in Section 3, we will extend the result to space-time transformations of BM, in the special case when \(b = 0\).

## 2 The \(n\)-th passage time of Brownian motion

In this section we suppose that \(b \leq 0\) and \(x < a\), or \(b \geq 0\) and \(x > a\), so that \(P(\tau_1(x) < \infty) = 1\). First, for fixed \(t > 0\), we consider the last-passage-time prior to \(t\) of BM, starting from \(x\), through the boundary \(S(t) = a + bt\), that is:

\[
\lambda^t_S = \sup\{0 \leq u \leq t : x + B_u = S(u)\}.
\]

The distribution of \(\lambda^t_S\) can be expressed in terms of the first-passage-time distribution of BM through the time-reversed boundary \(\hat{S}(u) = S(t-u)\) (see \([14]\)); in particular, we can derive from \([14]\) the following formula for the probability density, say \(\psi_t(u)\), of \(\lambda^t_S\):

\[
\psi_t(u) = \frac{d}{du}P(\lambda^t_S \leq u) = \frac{1}{\sqrt{2\pi u}} \exp(-b^2u/2) \int_{-\infty}^{+\infty} \nu_{x-a}(t-u, \hat{S})dx, \ u \leq t
\]

where:

\[
\nu_x(v, \hat{S}) = \exp\{-b(x - bt) - b^2v/2)\} \frac{|x-bt|}{\sqrt{2\pi v^3}} \exp\{-(x-bt)^2/2v\}
\]

Then, the following explicit formula is obtained, by calculation:

**Lemma 2.1** The probability density of \(\lambda^t_S\) is explicitly given by:

\[
\psi_t(u) = \frac{e^{-b^2u}}{\pi \sqrt{u(t-u)}} \left[ e^{\frac{b^2}{12(t-u)}} + \frac{b}{2} \sqrt{2\pi(t-u)} \left(2\Phi(b\sqrt{t-u}) - 1\right) \right], \ 0 < u < t
\]
In particular, if \( b \neq 0 \), one gets:

\[
\psi_t(u) = \frac{1}{\pi \sqrt{u(t-u)}} , \quad 0 < u < t, \tag{2.5}
\]

that is, the arc-sine law with support in \((0,t)\).

Proof. By using (2.2) and (2.3), we obtain:

\[
\psi_t(u) = \frac{e^{-b^2u/2}}{2\pi u} \cdot \frac{e^{-b^2(t-u)/2}}{t-u} \cdot J(u),
\]

where

\[
J(u) = \int_{-\infty}^{+\infty} \frac{e^{-by|y|}}{2\pi \sqrt{t-u}} e^{-y^2/2(t-u)} dy.
\]

Setting \( z = y/\sqrt{t-u} \), the integral \( J \) assumes the form:

\[
J(u) = \sqrt{t-u} \int_{-\infty}^{+\infty} \frac{|z|}{\sqrt{2\pi}} e^{-(z^2+2zb\sqrt{t-u})/2} dz
\]

\[
= \sqrt{t-u} \cdot e^{b^2(t-u)/2} \int_{-\infty}^{+\infty} \frac{|z|}{\sqrt{2\pi}} e^{-(z+b\sqrt{t-u})^2/2} dz
\]

\[
= \sqrt{t-u} \cdot e^{b^2(t-u)/2} \left[ \int_{-\infty}^{b\sqrt{t-u}} \frac{b\sqrt{t-u}-w}{\sqrt{2\pi}} e^{-w^2/2} dw + \int_{b\sqrt{t-u}}^{+\infty} \frac{w-b\sqrt{t-u}}{\sqrt{2\pi}} e^{-w^2/2} dw \right].
\]

By direct calculations, we get:

\[
J(u) = \sqrt{t-u} \cdot e^{b^2(t-u)/2} \left[ \frac{2}{\sqrt{2\pi}} \cdot e^{-b^2(t-u)/2} + 2b\sqrt{t-u} \cdot \Phi(b\sqrt{t-u}) - b\sqrt{t-u} \right].
\]

Finally, (2.3) soon follows, by using (2.5).

\( \square \)

Remark 2.2 For \( z < t \), the event \( \{ \lambda^t_S \leq z \} \) is nothing but the event \( \{ x + B_u - S(u) \text{ has no zeros in the interval } (z,t) \} \).

Now, we go to consider the second-passage-time, \( \tau_2(x) \), of BM starting from \( x \), through the linear boundary \( S(t) = a + bt \), when \( x < a \) and \( b \leq 0 \), the case when \( b \geq 0 \) and \( x > a \) can be studied in a similar way. We set \( T_1(x) = \tau_1(x) \) and \( T_2(x) = \tau_2(x) - \tau_1(x) \). We will see that \( T_2(x) \) is finite with probability one only if \( b = 0 \). Conditionally to \( \tau_1(x) = s \), the event \( \{ \tau_2(x) > s+t \} \) \( (t > 0) \), is nothing but the event \( \{ x + B_u - S(u) \text{ has no zeros in the interval } (s,s+t) \} \). Therefore, from Lemma 2.1

\[
P(\tau_2(x) > \tau_1(x) + t | \tau_1(x) = s) = P(\lambda^{s+t}_S \leq s) = \int_{0}^{s} \psi_{s+t}(y) dy
\]
and so
\[ P(T_2(x) \leq t | \tau_1(x) = s) = 1 - \int_0^s \psi_{s+t}(y) dy. \]  
(2.7)

Then:
\[ P(T_2(x) \leq t) = \int_0^{+\infty} \left[ 1 - \int_0^s \psi_{s+t}(y) dy \right] f_{\tau_1}(s) ds \]
\[ = 1 - \int_0^{+\infty} f_{\tau_1}(s) ds \int_0^s \psi_{s+t}(y) dy. \]  
(2.8)

By taking the derivative with respect to \( t \), we obtain the density of \( T_2(x) \):
\[ f_{T_2}(t) = -\int_0^{+\infty} f_{\tau_1}(s) ds \frac{\partial}{\partial t} \int_0^s \psi_{s+t}(y) dy \]
\[ = \int_0^{+\infty} f_{\tau_1}(s) \left[ e^{-b^2(s+t)/2} \frac{\sqrt{s}}{\pi \sqrt{t(s+t)}} \right] ds. \]  
(2.9)

Notice that \( \int_0^s \psi_{s+t}(y) dy \) is decreasing in \( t \), that is, \( \frac{\partial}{\partial t} \int_0^s \psi_{s+t}(y) dy = \int_0^s \frac{\partial}{\partial t} \psi_{s+t}(y) dy < 0 \). As easily seen, \( f_{T_2}(t) \sim \text{const}/\sqrt{t} \), as \( t \to 0^+ \). Moreover, from (2.8) it follows that, if \( b \neq 0 \), the distribution of \( T_2(x) \) is defective, that is, \( P(T_2(x) = +\infty) > 0 \). In fact:
\[ \lim_{t \to +\infty} \int_0^s \psi_{s+t}(y) dy = \frac{|b|}{\sqrt{2\pi}} \int_0^s e^{-b^2 y/2} \sqrt{y} dy = 2 \text{sgn}(b) \left( \Phi(b\sqrt{s}) - \frac{1}{2} \right), \]
where \( \text{sgn}(b) = \begin{cases} 0 & \text{if } b = 0 \\ |b|/b & \text{if } b \neq 0 \end{cases} \).

Therefore, we get:
\[ P(T_2(x) < +\infty) = 1 - \int_0^{+\infty} f_{\tau_1}(s) \cdot 2\text{sgn}(b) \left( \Phi(b\sqrt{s}) - \frac{1}{2} \right) ds \]
\[ = 1 - 2\text{sgn}(b)E \left( \Phi(b\sqrt{\tau_1(x)}) - \frac{1}{2} \right), \]
which is less than 1, if \( b \neq 0 \). Thus, in this case:
\[ P(T_2(x) = +\infty) = 2\text{sgn}(b)E \left( \Phi(b\sqrt{\tau_1(x)}) - \frac{1}{2} \right) > 0. \]  
(2.10)

Since the function \( g(s) = 2\text{sgn}(b) \left( \Phi(b\sqrt{s}) - \frac{1}{2} \right) \) is concave, by Jensen’s inequality we get:
\[ P((T_2(x) = +\infty) \leq \gamma(b), \]
where:
\[ \gamma(b) = 2\text{sgn}(b) \left( \Phi(b\sqrt{E(\tau_1(x))}) - \frac{1}{2} \right) = 2\text{sgn}(b) \left( \Phi(b\sqrt{(a-x)/|b|}) - \frac{1}{2} \right). \]  
(2.11)

Notice that \( \gamma \) is an even function of \( b \).
Figure 1: Comparison of the shapes of $P(\tau_2(x) = +\infty)$ (lower curve) and that of its upper bound $\gamma(b)$ (see (2.11)), as a function of $b \leq 0$, for $a = 1$ and $x = 0$.

On the contrary, if $b = 0$, from (2.10) we get that $P(T_2(x) = +\infty) = 0$, that is, $T_2(x)$ is a proper random variable, and also $\tau_2(x) = T_2(x) + \tau_1(x)$ is finite with probability one; precisely, by calculating the integral in (2.8) we have:

$$P(T_2(x) \leq t) = \int_0^{+\infty} \frac{2}{\pi} \arccos \sqrt{\frac{s}{s+t}} \frac{|a-x|}{\sqrt{2\pi s^{3/2}}} e^{-(a-x)^2/2s} ds$$ (2.12)

and

$$f_{T_2}(t) = \int_0^{+\infty} \frac{1}{\pi(s+t)\sqrt{t}} \frac{|a-x|}{\sqrt{2\pi s}} e^{-(a-x)^2/2s} ds.$$ (2.13)

In the Figure 1 we compare the plot of $P(\tau_2(x) = +\infty) = P(T_2(x) = +\infty)$ as a function of $b \leq 0$, for $a = 1$ and $x = 0$, with the plot of its upper bound $\gamma(b)$ given by (2.11). Notice that, as it must be, $\gamma(b)$ approaches 1, for large negative values of $b$.

In the Figure 2, we report the probability density of $T_2(x)$, obtained from (2.9), by calculating numerically the integral, for $x = 0$, $a = 1$, and various values of the parameter $b \leq 0$.

As far as the expectation of $\tau_2(x)$ is concerned, it is obviously infinite for $b \neq 0$, while $E(\tau_1(x))$ is finite. If $b = 0$, $E(\tau_1(x))$ and $E(\tau_2(x))$ are both infinite. As for $\tau_1(x)$ this is well-known, as for $\tau_2(x)$ it derives from the fact that $E(T_2(x)) = +\infty$. Indeed, since

$$P(T_2(x) > t) = 1 - \int_0^{+\infty} \frac{2}{\pi} \arccos \sqrt{\frac{s}{s+t}} \frac{|a-x|}{\sqrt{2\pi s^{3/2}}} e^{-(a-x)^2/2s} ds$$

$$= \int_0^{+\infty} \frac{2}{\pi} \arcsin \sqrt{\frac{s}{s+t}} \frac{|a-x|}{\sqrt{2\pi s^{3/2}}} e^{-(a-x)^2/2s} ds,$$

we have

$$E(T_2(x)) = \int_0^{+\infty} P(T_2(x) > t) dt = \int_0^{+\infty} ds \frac{2}{\pi} \frac{|a-x|}{\sqrt{2\pi s^{3/2}}} e^{-(a-x)^2/2s} \int_0^{+\infty} \arcsin \sqrt{\frac{s}{s+t}} dt ,$$
Figure 2: Approximate density of $T_2(x)$ for $x = 0$, $a = 1$ and various values of the parameter $b$; from top to the bottom: $b = 0$, $b = -0.5$, $b = -1$.

which is infinite, because, as easily seen, $\int_0^{+\infty} \arcsin \sqrt{\frac{s}{s+t}} dt = +\infty$, for any $s > 0$.

By taking the derivative with respect to $t$ in (2.7), we obtain the density of $T_2(x)$ conditional to $\tau_1(x) = s$, that is:

$$f_{T_2|\tau_1}(t|s) = -\frac{d}{dt} \int_0^s \psi_{s+t}(y) dy = e^{-b^2(s+t)/2} \frac{\sqrt{s}}{\pi(s+t)\sqrt{t}},$$

(2.14)

and, for $b = 0$ :

$$f_{T_2|\tau_1}(t|s) = \frac{\sqrt{s}}{\pi(s+t)\sqrt{t}}.$$

(2.15)

Since $\tau_2(x) = \tau_1(x) + T_2(x)$, by the convolution formula, we get the density of $\tau_2(x)$ :

$$f_{\tau_2}(t) = \int_0^t f_{T_2|\tau_1}(t-s|s)f_{\tau_1}(s) ds = \frac{e^{-b^2t/2}}{\pi t} \int_0^t \frac{|a-x|}{\sqrt{2\pi s}\sqrt{t-s}} e^{-(a+bs-x)^2/2s} ds.$$

(2.16)

Of course, the distribution of $\tau_2(x)$ is defective for $b \neq 0$, namely $\int_0^{+\infty} f_{\tau_2}(t) dt = 1 - P(\tau_2(x) = +\infty) < 1$, since $P(\tau_2(x) = +\infty) = P(T_2(x) = +\infty) > 0$.

If $b = 0$, we obtain:

$$f_{\tau_2}(t) = \frac{1}{\pi t} \int_0^t \frac{|a-x|}{\sqrt{2\pi s}\sqrt{t-s}} e^{-(a-x)^2/2s} ds,$$

(2.17)

which is non-defective.

In the Figure 3, we report the probability density of $\tau_2(x)$ obtained from (2.16) by calculating numerically the integral, for $x = 0$, $a = 1$, and various values of the parameter $b \leq 0$. Although the shapes appear to be similar to that of the inverse Gaussian density (1.2), the density of $\tau_2(x)$ is more concentrated around its maximum. In the Figure 4, we report the comparison between the probability density of $\tau_2(x)$ and the inverse Gaussian density, for $a = 1$, $b = 0$ and $x = 0$. 
Figure 3: Approximate density of $\tau_2(x)$ for $x = 0$, $a = 1$, and various values of the parameter $b$; from top to the bottom, with respect to the peak of the curve: $b = -2$, $b = -1$, $b = -0.5$, $b = 0$.

Figure 4: Comparison between the probability density of $\tau_2(x)$ (upper peak) and the inverse Gaussian density (lower peak), for $a = 1$, $b = 0$ and $x = 0$. 
By reasoning in analogous manner as above, we conclude:

**Proposition 2.3** Let be $T_1(x) = \tau_1(x)$, $T_n(x) = \tau_n(x) - \tau_{n-1}(x)$, $n = 2,\ldots$

Then:

\[ P(T_1(x) \leq t) = 2(1 - \Phi(a - x/\sqrt{t})), \]

\[ P(T_n(x) \leq t) = 1 - \int_0^{+\infty} f_{\tau_{n-1}}(s)ds \int_0^s \psi_{s+t}(y)dy, \quad n = 2,\ldots \]

Moreover, the density of $\tau_n(x)$ is:

\[ f_{\tau_n}(t) = \int_0^t f_{T_n|\tau_{n-1}}(t - s|s) f_{\tau_{n-1}}(s)ds, \]

where $f_{\tau_{n-1}}$ and $f_{T_n|\tau_{n-1}}$ can be calculated inductively, in a similar way, as done for $f_{\tau_2}$ and $f_{T_2|\tau_1}$.

If $b = 0$, $T_1(x)$, $T_2(x)$, .... are finite with probability one.

\[ \square \]

**Remark 2.4** The expression for $P(T_1(x) \leq t)$ is nothing but the Bachelier-Levy formula, written for $b = 0$.

### 3 The n-th passage time of space-time transformations of Brownian motion

The techniques of the previous section can be applied to get also results for time-changed BM. In fact, let be

\[ Z(t) = z + B(\rho(t)), \]

where $\rho(t) \geq 0$ is an increasing, differentiable function of $t \geq 0$, with $\rho(0) = 0$. Such kind of diffusion process $Z$ is a special case of Gauss-Markov process (see [1], [6], [12], [13]); in particular the form (3.1) is taken by certain integrated Gauss-Markov processes (see [2]).

Now, for $n \geq 1$ denote again by $\tau_n(z)$ the successive passage times of $Z(t)$ through the constant barrier $S = a$, and by $T_n(z) = \tau_n(z) - \tau_{n-1}(z)$ the inter-passage times; then, for $z < a$, we have $\tau_1(z) = \inf\{t > 0 : z + B(\rho(t)) = a\}$ and so $\rho(\tau_1(z)) = \inf\{s > 0 : z + B_s = a\} := \tau_1^B(z)$, where the superscript $B$ refers to BM. Therefore, $\tau_1(z) = \rho^{-1}(\tau_1^B(z))$, and in analogous way, for $n \geq 2$, we get $\rho(\tau_n(z)) = \inf\{s > \rho(\tau_{n-1}(z)) : z + B_s = a\} := \tau_n^B(z)$. In conclusion, we have:

\[ \tau_n(z) = \rho^{-1}(\tau_n^B(z)) \quad \text{and} \quad T_n(z) = \rho^{-1}(\tau_n^B(z)) - \rho^{-1}(\tau_{n-1}^B(z)), \quad n \geq 1, \]

where $\tau_n^B(z)$ is the n-th passage time of BM, starting from $z$, through $a$; thus, the calculations of the distributions of $\tau_n(z)$ and $T_n(z)$ are reduced to those of $\tau_n^B(z)$ and $T_n^B(z)$.

An analogous study concerning the successive spike (i.e. passage) times $\tau_n$ of a Gauss-Markov process $Z$ through a constant threshold $S$, was developed in [7], for a non homogeneous Leaky Integrate-and-Fire (LIF) neuronal model, in which the membrane potential of the neuron, $Z$, is instantaneously reset to its initial value, every time $S$ is attained.
The approach considered in the present paper can be also applied to one-dimensional diffusions which can be reduced to BM by a space transformation; let $Z(t)$ be the solution of the stochastic differential equation (SDE):

$$dZ(t) = \mu(Z(t))dt + \sigma(Z(t))dB_t, \ Z(0) = z,$$

where the coefficients $\mu(z)$ and $\sigma(z)$ are regular enough functions (see e.g. [3]), so that a unique strong solution exists. We consider the following:

**Definition 3.1** We say that $Z$ is conjugated to BM if there exists an increasing function $v$ with $v(0) = 0$, such that $Z(t) = v^{-1}(B_t + v(z))$, for any $t \geq 0$.

Examples of diffusions conjugated to BM are the following (for more, see e.g. [3]):

(i) (Feller process or Cox-Ingersoll-Ross (CIR) model)
the solution of the SDE

$$dZ(t) = \frac{1}{4} dt + \sqrt{Z(t)} dB_t, Z(0) = z,$$

which is conjugated to BM via the function $v(z) = 2\sqrt{z}$, i.e. $Z(t) = \frac{1}{4}(B_t + 2\sqrt{z})^2$;

(ii) (Wright & Fisher-like process)
the solution of the SDE

$$dZ(t) = \left(\frac{1}{4} - \frac{1}{2}Z(t)\right) dt + \sqrt{Z(t)(1 - Z(t))} dB_t, Z(0) = z \in [0,1],$$

which is conjugated to BM via the function $v(x) = 2\arcsin\sqrt{x}$, i.e. $Z(t) = \sin^2(B_t/2 + \arcsin\sqrt{x})$.

Let $Z$ be conjugated to BM, via the function $v$, and denote, as always, by $\tau_n(z)$ the n-th passage time of $Z$ through the constant barrier $S = a$, and by $T_n(z)$ the inter-passage time, with the condition that $Z(0) = z$; as easily seen, since now $\rho(t) = t$, we get:

$$\tau_n(z) = \tau_n^{B,a'}(z') \text{ and } T_n(z) = \tau_n^{B,a'}(z') - \tau_{n-1}^{B,a'}(z'), \ n \geq 1,$$

where $a' = v(a)$, $z' = v(z)$ and $\tau_n^{B,a'}(z')$ denotes the n-th passage time of BM through the barrier $a'$, when starting from $z'$. Again, the calculations of the distributions of $\tau_n(z)$ and $T_n(z)$ are reduced to those concerning BM.

This also works, for certain moving boundaries $S$, e.g. for Geometric Brownian motion, that is the solution of the SDE:

$$dZ(t) = rZ(t)dt + \sigma Z(t)dB_t, \ Z(0) = z > 0,$$

where $r$ and $\sigma$ are positive constant. This is a well-known equation in the framework of Mathematical Finance, since it describes the time evolution of a stock price $Z$; its explicit solution is $Z(t) = ze^{\mu t}e^{\sigma B_t}$, where $\mu = r - \sigma^2/2$, so $\ln Z/\sigma$ turns out to be BM with drift $\mu$, starting from $\ln z/\sigma$. If we consider the moving barrier $S(t) = e^{\sigma S_0 + \mu t}$, then the successive passages, $\tau_n(z)$, of $Z$ through $S$ are reduced to the successive passages, $\tau_n^{B}(z')$, of BM through
the linear boundary \( S_0 + (\mu' - \mu)t/\sigma \), with the condition that the starting point is \( z' = \frac{\ln z}{\sigma} \).

In fact:

\[
\tau_n(z) = \rho^{-1}(\tau_n^B(z')) \quad \text{and} \quad T_n(z) = \rho^{-1}(\tau_n^B(z')) - \rho^{-1}(\tau_{n-1}^B(z')).
\]

The same considerations apply to the Ornstein-Uhlenbeck process, which is solution of the SDE:

\[
dZ(t) = -\mu Z(t)dt + \sigma dB_t, \quad Z(0) = z,
\]

where \( \mu, \sigma \) are positive constants. By using a time-change (see e.g. [4]), the explicit solution assumes the form \( Z(t) = e^{-\mu t} (z + B(\rho(t))) \), where \( \rho(t) = \frac{\sigma^2}{2\mu} (e^{2\mu t} - 1) \). Let us consider the moving barrier \( S(t) = S_0 e^{-\mu t} \), with \( S_0 > z \); the first-passage time of \( Z(t) \) through \( S(t) \) is \( \tau_{S(t)} = \inf\{t > 0 : z + B(\rho(t)) = S_0\} \), and so \( \rho(\tau_{S(t)}) = \inf\{u > 0 : z + B_u = S_0\} \). Therefore, similarly to the case of Geometric BM, the successive passages of \( Z \) through \( S \) are reduced to those of BM through the constant boundary \( S_0 \).

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