ONE PHASE PROBLEM FOR TWO POSITIVE HARMONIC
FUNCTION: BELOW THE CODIMENSION 1 THRESHOLD

ALEXANDER VOLBERG

Abstract. What can be said about the domain $\Omega$ in $\mathbb{R}^n$ for which its Green’s function $G(z)$ satisfies $G(z) \asymp \text{dist}(z, \partial \Omega)^\delta$? What can we say about $\Omega$ if the Boundary Harnack Principle holds in the form $u/v = \text{real analytic}$ on the part $E$ of its boundary? Here $u, v$ are positive harmonic functions on $\Omega$ vanishing on $E$. Is this part of the boundary also nice? We discuss these questions below and give answers in very special cases.

1. Preliminaries and notations

In recent years several outstanding problems at the edge of harmonic analysis and geometric measure theory in $\mathbb{R}^n$ were solved, see e.g. [ENV], [HMM], [NTV], [DM], [DFM] and the literature therein. However, these works revealed a new threshold: the technique works if the codimension of a (very irregular) set in question has codimension at most $1$ in $\mathbb{R}^n$.

Here we present several questions (some of them looking very simple) and a tiny bit of answers breaking down this threshold. We consider only $n = 2$ case, but most of the problems (and some solutions) presented below work for all $n$.

Two phase problem for positive harmonic function can be vaguely described as follows: there are two disjoint domains $U, V$ in $\mathbb{R}^n$ with a common part of their boundaries, $E$ called interface. Let $O$ be an open neighborhood of $E$. There are two positive harmonic functions: $u$ in $O \cap U$ and $v$ in $O \cap V$. Both functions vanish on $E$ (in some sense), but their ratio is well defined on $E$ (again in some sense) and this ratio is a nice function on $E$. One then concludes that then $E$ is nice (in some sense). The example of this kind of problem can be found in [KT1], [AMTV], where $u = G_U(\cdot, p), v = G_V(\cdot, q), p \in U, q \in V$, two Green functions of corresponding domains and the assumption on the ratio of $u$ and $v$ is formulated in terms of harmonic measures $\omega_U(\cdot, p), \omega_V(\cdot, q)$ of two disjoint domains being mutually absolutely continuous (and non-zero) on the interface $E$.

On the other hand, a one phase problem for positive harmonic function defined near the boundary and vanishing at the boundary (or for harmonic measure) usually compares the harmonic measure of one given domain with the Hausdorff measure on the boundary. The literature here is rather vast, see, for example, [AHM3TV], [KT2], [KT3], [KPT] and citation therein.

We wish to consider a one phase problem in domain $\Omega$, with two positive harmonic functions $u, v$ in $O \cap \partial \Omega$, where $O$ is just an open disc (ball), $u = 0 = v$ on $E := O \cap \partial \Omega$. The main assumption is that the ratio $u/v$ makes a rather nice function on $E$. We would wish to prove that then $E$ is nice. One can think about these two harmonic functions as two Green’s functions of the domain $\Omega$ but with different poles. Then we are considering the comparison of harmonic measures related to two different poles, rather than comparison of harmonic measure and Hausdorff measure on the boundary.

Now we need to explain what “nice” means and what kind of domains $\Omega \subset \mathbb{R}^n$ we consider. Let us remind that if $\Omega$ is Lipschitz domain then $\log \frac{1}{r}$ is Hölder continuous on $E = B(x, r) \cap \Omega, x \in \partial \Omega$, by Jerison–Kenig theorem [JK]. Moreover, by their result the same is true for NTA domains, see the definition in [JK].
We will deal mostly (but not exclusively) with plane domains, $n = 2$, but our $\Omega$ can very well be infinitely connected, with rather complicated (but compact) boundary. However, we will require several things from $\Omega$ to make it “one sided NTA domain”.

So, we require that $\partial \Omega$ has 1) capacity density condition (CDC): $\cap(B(\zeta, \delta)) \asymp \cap(B(z, \delta) \cap \partial \Omega)$ for all small $\delta$ and all $z \in \partial \Omega$; 2) corkscrew condition, see [JK]; 3) Harnack chain condition, see [JK].

The NTA domains of course satisfy 1), 2), 3). But another example that is important in what follows is the domain $\Omega$ whose boundary $\partial \Omega$ is a sufficiently nice Cantor set, for example a self-similar set in the sense of conformal dynamical systems.

Let us explain briefly what is conformal self-similar sets (conformal repellers). We consider more general sets below, but to have in mind those special examples is a good idea. So, let $O$ be a topological discs and $O_1, \ldots, O_d$, $d \geq 2$, be topological discs with disjoint closures inside $O$, we require also that $\partial O$ were disjoint with closures of all $O_j$. Let $f = (f_1, \ldots, f_d)$ be a collection of conformal maps of $O_i$ onto $O$. By considering $f_{i,j}^{-1}(O_j)$, $i, j = 1, \ldots, d$ we see $d^2$ topological discs of the second generation ($d$ groups of $d$ discs each) that are mapped onto $O$ conformally by iterative power $f^2 = f \circ f$ (think about $f$ as a piecewise conformal map). By iterating this construction we have $d^k$ discs of the third generation, et cetera. Finally we can consider the unions of discs of generation $k$, call of $U_k$ and $J = \cap U_k$. Domain $\Omega = \mathbb{C} \setminus J$ is a typical example of domains we consider here, and $O$ itself is a typical example of a neighborhood of $J = \partial \Omega$.

It is easy to prove that such one sided NTA domains satisfy Jerison–Kenig boundary Harnack principle (BHP), meaning that for any two harmonic and positive $u, v$ in $\Omega \cap B(z, r)$ ($z \in \partial \Omega$, $r$ being small) that vanish on $B(z, r) \cap \partial \Omega$, one has the Hölder property for $u/v$ on $B(z, r) \cap \partial \Omega$ in the following sense:

$$\exists \varepsilon \in (0, 1): \forall z_1, z_2 \in \Omega \cap B(z, r) \quad |\log \frac{u(z_1)}{v(z_1)} - \log \frac{u(z_2)}{v(z_2)}| \leq C|z_1 - z_2|^\varepsilon. \quad (1.1)$$

Tending $z_i$ to $\zeta_i \in B(z, r) \cap \partial \Omega$, $i = 1, 2$ we get the Hölder property of $u/v$ on $B(z, r) \cap \partial \Omega$ (which is called BHP). By the way, we use complex notations because below we deal mostly with planar $\Omega \subset \mathbb{R}^2$, but all our questions and conjectures make sense for $n > 2$ as well.

Now let us explain what we mean by saying that $\frac{u}{v}$ is “nice” on $B(z, r) \cap \partial \Omega$. The above (1.1) always happen on one sided NTA, so it is normal and not sufficiently nice. By “nice” we understand the case a much stronger property: when in one sided NTA domain described by 1), 2), 3) above, on the top of (1.1), we have also

$$\exists \text{ real analytic } R \text{ on } B(z, r), z \in \partial \Omega: \forall \zeta \in B(z, r) \cap \partial \Omega, \quad \frac{u(\zeta)}{v(\zeta)} = R(\zeta). \quad (1.2)$$

We now list several conjectures in the order of decreasing difficulty.

**Conjecture 1.3.** Let $\Omega$ be our one sided NTA, $u, v$ be as above, in particular, let (1.2) holds. Then either $u = \lambda v$ for some constant $\lambda$ or $B(z, r) \cap \partial \Omega$ is real analytic maybe with the exception of a set of dimension $n - 2$.

**Conjecture 1.4.** Let $n = 2$. Let $\Omega$ be our one sided NTA, $u, v$ be as above, in particular, let (1.2) hold, but let also $R = |A|$, where $A$ is a non-constant holomorphic function on $B(z, r)$. Then either $u = \lambda v$ for some constant $\lambda$ or $B(z, r) \cap \partial \Omega$ is real analytic maybe with the exception of a set of isolated points.

This latter conjecture is proved in [VaVo], [VaVo1] for the case of simply connected $\Omega$. The paper uses a very powerful result of Sakai [Sa] that describes the so-called Schwarz functions in arbitrary planar domains. However, the reduction of Conjecture 1.4 to Sakai’s theorem works only in simply (or finitely) connected domains. Any solution of Conjecture 1.4 should first eliminate the possibility of infinitely connected domain. For the simply connected case Conjecture 1.4 turned
out to be closely related to the so-called Nevanlinna domains, which were ex-
tensively studied e.g. in [DK], [BaF], [BeF], [BeBoF], [MM] and the literature cited
therein.

**Conjecture 1.5.** Let \( n = 2 \). Let \( \Omega \) be our one sided \( \text{NTA} \), \( u, v \) be as above, in particular, let (1.2) hold, but let also \( R = |A| \), where \( A \) is a non-constant holomorphic function on \( B(z, r) \). Let, in addition, \( \partial \Omega \) be a regular Cantor set in the following sense:

\[
\exists a > 1 : \forall k \in \mathbb{Z}_+ \text{ the set } \{ z : \text{dist}(z, \partial \Omega) < a^{-k} \} = V_1 \cup \ldots \cup V_{m_k}, \tag{1.6}
\]

where \( \text{length}(V_i) \leq Ca^{-k} \) and \( m_k \leq Ca^{\delta k} \), for some \( \delta > 0 \). Then \( u = \lambda v \) for some constant \( \lambda \).

The conclusion is natural. Cantor structure of the boundary unequivocally says that boundary is not nice and definitely is not real analytic. Therefore, the only option that should exist for \( u, v \) is to be proportional.

**Conjecture 1.7.** Let \( n = 2 \). Let \( \Omega \) be our one sided \( \text{NTA} \), \( u, v \) be as above, in particular, let (1.2) hold, but let also \( R = |A| \), where \( A \) is a holomorphic function on \( B(z, r) \). Let, in addition, \( \Omega \) be such that its Green’s function (with some pole) satisfies

\[
c_1 \text{dist}(z, \partial \Omega)^\delta \leq G(z, p) \leq c_2 \text{dist}(z, \partial \Omega)^\delta, \quad \text{for some } \delta \in (0, 1] \tag{1.8}
\]

for all \( z \in \Omega \) sufficiently close to \( \partial \Omega \) and for some \( \delta > 0 \). Then either \( u = \lambda v \) for some constant \( \lambda \) or \( \delta = 1 \) and \( B(z, r) \cap \partial \Omega \) is piecewise real analytic.

It is easy to see that assumption (1.8) (if \( \delta < 1 \)) implies that \( \partial \Omega \) is a regular Cantor set if \( \partial \Omega \) is a conformal self-similar set, see [AV]. So the proof of Conjecture 1.5 implies Conjecture (1.7) at least for conformal self-similar sets. But we want to formulate a very "easy" conjecture that claims part of Conjecture 1.7. The following statement looks very easy, but I can prove it only in one special case (see below). It is interesting in all dimensions \( n \geq 2 \).

**Conjecture 1.9.** Let for \( \Omega \subset \mathbb{R}^n \) we have (1.8) with \( \delta > 0 \). Then 1) \( \delta = 1 \) necessarily; 2) \( \partial \Omega \) has a certain weak smoothness, at least it is rectifiable.

Those conjectures appear naturally in a certain unsolved problem in complex dynamics. Conjecture 1.9 is especially intriguing because it looks so simple. Papers [DM] and [DFM] consider a very closely related problem of prevalent approximation of Green’s function by the distance. But for co-dimensions bigger than one cases the Green’s function is pertinent to a certain degenerate elliptic equation as far as I understand. Still the technique of these papers can probably be useful for solving this conjecture.

2. Warming up

To warm up we consider here Conjecture 1.9 in an extremely special case. We assume that (1.8) holds with \( \delta = 1 \):

\[
c_1 \text{dist}(z, \partial \Omega) \leq G(z, p) \leq c_2 \text{dist}(z, \partial \Omega), \tag{2.1}
\]

but \( \partial \Omega \) is a conformal self-similar Cantor set (conformal repeller) of Hausdorff dimension 1 in the sense of complex dynamics, for example, the reader can think that it is a 1/4 corner Cantor set. We wish to lead (2.1) to contradiction. For brevity we adopt the notation

\[
J := \partial \Omega.
\]

Notice that (2.1) immediately implies \( \partial G| \leq C \), hence the following Cauchy integral is bounded:

\[
C^\omega(z) := \int_J \frac{d\omega(\zeta)}{z - \zeta} \in L^\infty(\mathbb{C} \setminus J). \tag{2.2}
\]
From here we can standardly deduce that maximal Cauchy integral
\[
C^ω_r(z) := \sup_{r > 0} \int_{J \setminus b(z,r)} \left| \frac{d\omega(ζ)}{z - ζ} \right| \leq C \quad ∀z ∈ J.
\]  
(2.3)

In fact, (2.3) follows from (2.2) by mean value theorem for subharmonic functions if we use one more interesting inequality about harmonic measure (the pole is immaterial) of Ω:
\[
ω(B(z,r)) ≍ r, \quad ∀z ∈ J.
\]  
(2.4)
The latter inequality follows from CDC assumption on J, from (1.8) and from corkscrew assumption on Ω. Such folklore estimates can be found in many places, e.g. in [AHM3TV].

The boundedness of Maximal Cauchy operator obtained in (2.3) implies that the Cauchy integral operator \( f \to C^fω \) is a bounded operator in \( L^2(ω) \). But then (2.4) implies that he Cauchy integral operator \( f \to C^f\mathcal{H}^1 \) is a bounded operator in \( L^2(\mathcal{H}^1) \), where \( \mathcal{H}^1 \) is the Hausdorff measure of dimension 1 (length):
\[
\int_J \left| \int_J \frac{f(ζ)d\mathcal{H}^1(z)}{z - ζ} \right|^2 d\mathcal{H}^1(ζ) \leq C \int_J |f(ζ)|^2 d\mathcal{H}^1(ζ), \quad ∀f ∈ L^2(J, \mathcal{H}^1).
\]  
(2.5)
Melnikov found out a beautiful symmetrization trick (see [XT]) that deduces from (2.5) that Menger’s curvature \( c_2(μ) \) of measure \( μ = d\mathcal{H}^1|J \) is finite meaning that
\[
c_2(μ) := \int_J \int_J \int_J \frac{dμ(z_1)sμ(z_2)dμ(z_3)}{R^2(z_1, z_2, z_3)} < ∞,
\]  
(2.6)
where \( R(z_1, z_2, z_3) \) is the radius of a circle passing through \( z_1, z_2, z_3 \).

It is left to refer to [XT]: for 1/4 corner Cantor set this Menger’s curvature is infinite. Exactly the same proof shows that Menger’s curvature is infinite for any dimension 1 conformal self-similar Cantor set (conformal repeller).

So Conjecture 1.9 is proved for a very special case. And the proof uses several rather sophisticated results. There should be easier proof that works in general situation. We are unaware of it.

However, it is interesting to notice that basically the same claim and the same proof will work for domains in \( \mathbb{R}^n, n > 2 \). Let us give a brief explanation: of course Melnikov’s symmetrization and his reduction to Menger’s curvature does not work in \( \mathbb{R}^n, n > 2 \). This is a very well-known difficulty. But, for example, paper [NTV] would circumvent this difficulty. The fact is that the solution of David–Semmes conjecture implies that domain satisfying (2.1) should be rectifiable, the proof will be exactly as above, but instead of Cauchy transform, one would need to use singular Riesz transforms of singularity \( n - 1 \) and use [NTV]. One can be referred to [JA], where another proof is given.

**Corollary 2.7** (Carleson, 1985). Let J be conformal self-similar Cantor set (conformal repeller) of dimension 1. Then the dimension of harmonic measure of \( Ω := \mathbb{C} \setminus J \) is strictly less than 1: \( \dim ω < 1 \).

**Proof.** First one can use the theory of Gibbs measures (see [RB]) and notice that on conformal repeller the Hausdorff measure \( \mathcal{H}^1|J \) and \( ω \) are Gibbs measures (meaning that their Jacobians are Hölder continuous on J). So they are either bondedly mutually absolutely continuous with bounded density or they are mutually singular. In the former case one has (2.4), and, hence, (2.1), which brings us to contradiction–see above. In the case of singularity, we consider the invariant ergodic measure \( ν \) equivalent to \( ω \), and then variational principle for Gibbs measure \( \mathcal{H}^1|J \) (or rather for its invariant counterpart) implies that
\[
h_ν - \int_J \log |f'(ζ)|dν(ζ) < 0,
\]
where \( f \) is the conformal dynamical system that created \( J \) and \( h_\nu \) is the entropy of \( \nu \). By Manning’s formula [AM] the previous inequality implies
\[
\dim \omega = \dim \nu = \frac{h_\nu}{\int_J \log |f'(\zeta)| \, d\nu(\zeta)} < 1.
\]

\[ \square \]

Carleson’s proof [LC] was much more “hands on”. The above proof is in [AV] with more details.

The following statement would be an extension of the result of Carleson, but it is still a conjecture. It would be solved if Conjecture 1.7 were solved.

**Conjecture 2.8.** For any conformal self-similar set \( J \) of dimension \( \delta < 1 \) its harmonic measure has strictly smaller dimension:
\[
\dim \omega_{\mathbb{C}\setminus J} < \delta.
\]

We intentionally restricted to the case \( \delta < 1 \). In truth, if \( \delta = 1 \) this is proved by Carleson, see above, and if \( \delta > 1 \), one just uses a celebrated result of Jones–Wolff [JW] that for any planar domain harmonic measure \( \dim \omega \leq 1 \). So, we would get \( \dim \omega < \dim J \) for all possible conformal self-similar sets \( J \). In higher dimensions \( n \) one can still compare \( \dim \omega \) and \( \dim \partial \Omega \) and [JA] proves \( \dim \omega < \dim \partial \Omega \) when \( \partial \Omega \) is \( s \)-Ahlfors–David regular, \( n - 1 \leq s \leq n \), and \( \partial \Omega \) is “uniformly non-flat”. See the definition of uniformly non-flat in [JA], fractal boundaries are uniformly non-flat.

3. The proof of Conjecture 1.7 in a special case

**Theorem 3.1.** Let \( n = 2 \). Let \( \Omega \) is a one sided NTA domain as above and let \( \partial \Omega \) be a regular Cantor set. Let \( n = 2 \). Let (1.2) hold. Let, in addition, \( \Omega \) be such that its Green’s function (with some pole) satisfies
\[
c_1 \operatorname{dist}(z, \partial \Omega)^\delta \leq G(z, p) \leq c_2 \operatorname{dist}(z, \partial \Omega)^\delta
\]
for all \( z \in \Omega \) sufficiently close to \( \partial \Omega \) and for some \( \delta > 0 \). In addition we require
\[
\partial \Omega \subset \mathbb{R}.
\]

Then \( u = \lambda v \) for some constant \( \lambda \).

The assumption (3.3) is decisively restrictive and I do not know how to get rid of it.

**Proof.** For brevity again denote \( J = \partial \Omega \). We may think that \( J \subset [-1, 1] =: I \). First use of (3.3) is that \( R[I] = A[I] > 0 \) and \( A[I] \) is the trace of analytic function in the neighborhood of \( J \). Of course \( A = u/v > 0 \) on \( J \), which of course means \( A \) is symmetric. We can write
\[
A = e^{\alpha + i\beta}.
\]

The second use of (3.3) (but not the last one) we can think that \( u \) and \( v \) are symmetric:
\[
u(z) = \overline{u(z)}; \quad v(z) = \overline{v(z)}.
\]

**Lemma 3.4.** Let \( d(z) := \operatorname{dist}(z, J) \). Let \( U \) be a neighborhood of \( J \). Regularity of \( J \) gives that there exists \( \delta > 0 \) such that
\[
\frac{1}{d^{1-\delta}} \in L^{2+\delta}(U, dm_2).
\]

**Proof.** Let \( D_k = \{ z : d(z) \leq a^{-k} \} \). Then \( D_k = V_1 \cup \cdots \cup V_{m_k} \), \( m_k \leq a^{k\delta} \).
\[
f \left( \frac{1}{d^{1-\delta}} \right)^{2+\delta} \, dm_2 = \sum_{k=0}^\infty \sum_{i=1}^{m_k} m_2(V_i) \left( \frac{1}{a^{-k(1-\delta)}} \right)^{2+\delta} \leq
\]
\[
\sum_k m_k \cdot a^{2k} \text{Big} \left( \frac{1}{a^{-k(1-\delta)}} \right)^{2+\delta} \leq \sum_k a^{k\delta} \cdot a^{-2k} \cdot a^{k(2+\delta-2\delta-\delta^2)} =
\]
\[
\sum_k a^{-k\delta^2} < \infty.
\]

\[ \square \]
We wish to prove that $\partial \Phi \in C^{1+\frac{\delta}{2}+\varepsilon}$. This will require a bit of work. We know $A = e^{\alpha+i\beta}$ and $\frac{\partial}{\partial z} A = |A| = e^{\alpha}$ on $J$. Then using (1.1) and (3.2) we obtain
\[
|e^{\alpha} v - u| = |v e^{\alpha} - \frac{u}{v}| \lesssim d^{\delta+\varepsilon}. \tag{3.5}
\]

**Lemma 3.6.** Let $U$ be a neighborhood of $J$ and $\alpha$ be a harmonic function on $U$. Let $u, v$ be two harmonic functions in $U \setminus J$ and $|u| + |v| \lesssim d^\delta$, and $e^{\alpha} v - u \lesssim d^\gamma$ with $\gamma - 1 \leq \delta$. Then
\[
|\nabla (e^{\alpha} v - u)| \lesssim d^{\gamma-1}. \tag{3.6}
\]

**Proof.** Fix $z_0 \in U \setminus J$, let $d_0 = d(z_0)$. Consider disc $D := D(z_0, \frac{1}{2} d_0)$. Let $\alpha_0 = \alpha(z_0)$. On $D$ we have $|e^{\alpha_0} u - v| \lesssim d_0^\delta + d_0^\delta \leq 2d_0^\delta$. As $e^{\alpha_0} u - v$ is harmonic on $D$ we get its gradient estimated:
\[
|\nabla (e^{\alpha_0} u - v)(z_0)| \leq C d_0^{-1}. \tag{3.6}
\]

On the other hand $|\nabla [(e^{\alpha} - e^{\alpha_0})v]| \lesssim d^{-1+\delta} \cdot d + d^\delta \lesssim 2d^\delta$. □

Put $\Phi := e^{\alpha} v - u$, $\partial \Phi = e^{\alpha} \partial v - \partial u + v \partial e^\alpha$, so, by Lemma 3.6 with $\gamma = \delta + \varepsilon$ we have
\[
|\partial \Phi| \lesssim \frac{1}{d^{1-\delta-\varepsilon}}. \tag{3.7}
\]

Now we write
\[
\partial \Phi = \int_{\partial U} \frac{\partial \Phi(\zeta)}{z - \zeta} d\zeta + \int_{U \setminus D_k} \frac{\partial \Phi(\zeta)}{z - \zeta} dm_2(\zeta) + \int_{\partial D_k} \frac{\partial \Phi(\zeta)}{z - \zeta} d\zeta =: I + II + III. \tag{3.8}
\]

By (3.7) $III \lesssim C m_k \cdot a^{-k} \cdot a^{-k(1+\delta+\varepsilon)} = a^{-\delta k^2} \to 0$, $k \to \infty$.

On the other hand, $|\partial \partial \Phi| = |\partial \partial e^\alpha \cdot v + \partial e^\alpha \cdot \partial v + \partial \partial e^\alpha \cdot \partial v| \lesssim \frac{1}{d^{1-\varepsilon}}$ We use Lemma 3.4 and Sobolev inequality to get
\[
\partial \Phi \in L^{\frac{\delta}{2+\varepsilon}}, \tag{3.9}
\]

where $\Lambda^s$ stands for the Hölder class of order $s$. This is because $\frac{1}{\zeta} \ast L^p \subset L^{1 - \frac{s}{p}}$. But $\Phi$ is real-valued, so (3.9) implies that $\nabla \Phi(U \setminus J) \in L^{\frac{\delta}{2+\varepsilon}}(U \setminus J)$. Repeat the integral formula consideration (3.8), but now for $\Phi$ itself. As $|\Phi| \lesssim d^\delta$ we will get
\[
\forall z \in U \setminus J, \ \Phi(z) = \int_{\partial U} \frac{\Phi(\zeta)}{z - \zeta} d\zeta + \int_{U \setminus J} \frac{\partial \Phi(\zeta)}{z - \zeta} dm_2(\zeta). \tag{3.10}
\]

Notice that the right hand side belongs to $C^{1+\frac{\delta}{2+\varepsilon}}(U)$ by (3.9) and the fact that $\frac{1}{\zeta} \ast \Lambda^s \subset \Lambda^s$ if $s \in (0, 1)$. Also $\Phi$ is continuous up to $J$. Hence formula (3.10) holds for all $z \in U$.

Consequently $\Phi \vert U \in C^{1+\frac{\delta}{2+\varepsilon}}(U)$. Therefore, using that $\Phi = 0$ on $J$ we get
\[
|e^{\alpha} v - u| = |\Phi| \lesssim C d^{1+\frac{\delta}{2+\varepsilon}}. \tag{3.11}
\]

Notice that this inequality is an improvement upon inequality (3.5). Use Lemma 3.6 again, this time with $\gamma = 1 + \frac{\delta}{2+\varepsilon}$. The we get
\[
|\nabla \Phi| = |\nabla (e^{\alpha} v - u)| \lesssim d^{\frac{\delta}{2+\varepsilon}} \Rightarrow |e^{\alpha(x)} \nabla v(x) - \nabla u(x)| \lesssim d^{\frac{\delta}{2+\varepsilon}}, \ \forall x \in [-1, 1]. \tag{3.12}
\]

By symmetry $u_y = v_y = 0$ on $[-1, 1]$. Hence
\[
|e^{\alpha(x)} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}| \lesssim d^{\frac{\delta}{2+\varepsilon}}, \ \forall x \in [-1, 1]. \tag{3.13}
\]

Introduce a new function holomorphic in $U \setminus J$:
\[
H(z) := A(z) \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}
\]

We want to prove that it is holomorphic in $U$. Its restriction to $[-1, 1]$ satisfies (3.13). So if we prove that it is indeed holomorphic in $U$ we will prove that it is identically zero.
We know that a priori

$$|H(x + iy)| \leq \frac{C}{d(x + iy)^{1-\delta}} \leq \frac{C}{|y|^{1-\delta}}. \quad (3.14)$$

3.1. A punch line. Let $\mathbb{D}$ denote the unit disc.

**Lemma 3.15** (Hruschev, 1976). *Let $f$ be holomorphic in $\mathbb{D}_+ = \mathbb{D} \cap \mathbb{C}_+$ and let it satisfy*

1. $|f(z)| \leq C_1 e^{\frac{C_2}{|z|^\sigma}}$, $\sigma \in (0, 1)$;
2. for a $J \subset \mathbb{R}$, $f$ has boundary values $f^*(x) = \lim_{z \to x, z \in \mathbb{C}_+} f(z)$, $x \in \mathbb{R} \setminus J$;
3. $|f^*(x)| \leq 1$, for all $x \in \mathbb{R} \setminus J$;
4. $\mathcal{H}^{1-\sigma}(J) = 0$.

*Then $|f(z)| \leq 1$ on $\mathbb{D}_+$.***

This Lemma of S. Hruschev [SH] is a sophisticated variant of the Pfragmén–Lindelöf principle. In our case of regular Cantor set $J$ we can choose any $1-\delta < \sigma < 1$ to get

$$\mathcal{H}^{1-\sigma}(J) = 0.$$ 

All other assumptions hold with a good margin. Thus, our $H$ is bounded in $U_+ = U \cap \mathbb{C}_+$ (recall that $U$ is a neighborhood of $[-1, 1] \supset J$). Similarly $H$ is bounded in $U_- = U \cap \mathbb{C}_-$. But the boundary values of $H$ from $U_+$ coincide with the boundary values of $H$ from $U_-$ on $[-1, 1] \setminus J$ because $H$ is holomorphic in $U \setminus J$. So, those boundary values coincide Lebesgue almost everywhere on $[-1, 1]$. We also just established that $H$ is bounded in $U \setminus J \supset U \setminus [-1, 1]$. Therefore, $H$ removes the singularity $J$, meaning that $H$ is holomorphic in the whole $U$.

Now we can conclude from (3.13) combined with $u_y = v_y = 0$ on $[-1, 1]$ that $H$ vanishes on $J$. Hence,

$$H \equiv 0 \Rightarrow e^{\alpha(x)} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = 0, \quad x \in [-1, 1] \setminus J. \quad (3.16)$$

This is almost the end of the story, but not quite the end. Consider again

$$\Phi\left([-1, 1] \setminus J\right) = e^{\alpha(x)} v(x) - u(x).$$

By the vanishing assumption of $u, v$ on $J$ it vanishes at the end points of every complimentary interval $\ell$ of $J$ in $[-1, 1]$. So, for every complimentary interval $\ell$ there exists a point $c_\ell \in \ell$ such that

$$\frac{d}{dx}(e^{\alpha} v - u)(c_\ell) = 0.$$

Combined with (3.16) this gives us

$$\frac{d}{dx}(e^{\alpha})(c_\ell) = 0, \quad \forall \ell.$$

But $\alpha$ is a real analytic function and there are infinitely many $c_\ell$ ($J$ is a Cantor set). Therefore,

$$A = e^{\alpha} \equiv \text{const}.$$ 

Hence,

$$\frac{u(x)}{v(x)} = \lambda = \text{const}, \quad x \in J.$$

From here it follows easily that $u - \lambda v$ is harmonic not only in $U \setminus J$ but in $U$ as a whole. So $u - \lambda v$ is real analytic on $[-1, 1]$. Being zero on $J$ this harmonic function must be zero on real analytic arc containing $J$, thus

$$u - \lambda v \equiv 0 \text{ on } [-1, 1].$$

Inevitably

$$\frac{\partial v}{\partial x} - \lambda \frac{\partial u}{\partial x} \equiv 0, \quad x \in [-1, 1].$$
At the same time, by symmetry \( \frac{\partial v}{\partial y} - \lambda \frac{\partial u}{\partial y} \equiv 0 \), \( x \in [-1, 1] \). Hence, \( \frac{\partial v}{\partial z} - \lambda \frac{\partial u}{\partial z} \equiv 0 \) on \([-1, 1]\), and so, in \( U \). Function \( u - \lambda v \) is real valued, so the same is true for \( \bar{\partial} \)-derivative, and this means that \( u - \lambda v \equiv \text{const} \) in \( U \). But it vanishes on \( J \). Hence,

\[ u - \lambda v \equiv 0. \]

Theorem 3.1 is completely proved. \( \square \)

REFERENCES

[JA] J. Azzam, Dimension drop for harmonic measure on Ahlfors regular boundaries. Potential Anal. 53 (2020), no. 3, 1025–1041.

[AMTV] Azzam, Jonas Mourgoglou, Mihalis ; Tolsa, Xavier ; Volberg, Alexander On a two-phase problem for harmonic measure in general domains. Amer. J. Math. 141 (2019), no. 5, 1259–1279.

[AHM3TV] Azzam, Jonas; Hofmann, Steve; Martell, José María; Mayboroda, Svitlana; Mourgoglou, Mihalis; Tolsa, Xavier; Volberg, Alexander Rectifiability of harmonic measure. Geom. Funct. Anal. 26 (2016), no. 3, 703–728.

[RB] R. Bowen, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, Lecture Notes in Math., v 470, 1975.

[BaF] A. D. Baranov, K. Y. Fedorovskiy, “Boundary regularity of Nevanlinna domains and univalent functions in model subspaces”, Sb. Math. 202 (2011), no. 12, p. 1723–1740.

[BeBoF] Y. S. Belov, A. Borichev, K. Y. Fedorovskiy, “Nevanlinna domains with large boundaries”, J. Funct. Anal. 277 (2019), no. 8, p. 2617–2643.

[BeF] Y. S. Belov, K. Y. Fedorovskiy, Model spaces containing univalent functions, Russ. Math. Surv. 73(2018), no.1, p.172–174.

[LC] L. Carleson, On the support of harmonic measure for sets of Cantor type, Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985), 113-123.

[MC] M. Christ, A \( T(b) \) theorem with remarks on analytic capacity and the Cauchy integral. Colloquium Mathematicae 60-61.2 (1990): 601–628.

[DFM] G. David, J. Feneuil, S. Mayboroda, Green function estimates on complements of low-dimensional uniformly rectifiable sets, arXiv:2101.1164v1.

[DM] G. David, S. Mayboroda. Approximation of Green functions and domains with uniformly rectifiable boundaries of all dimensions Preprint, arXiv:2010.09793.

[DK] K. Dyakonov and D. Khavinson. Smooth functions in star-invariant sub- spaces. Recent advances in operator-related function theory, 59–66, Contemp. Math., 393, Amer. Math. Soc., Providence, RI, 2006. doi: 10.1090/conm/393/07371.

[ENV] V. Eiderman, F. Nazarov, A. Volberg, The s-Riesz transform of an s-dimensional measure in \( \mathbb{R}^2 \) is unbounded for \( 1 < s < 2 \). Journal d Analyse Mathématique 122(1) DOI: 10.1007/s11854-014-0001-1

[HMM] Hofmann S., Martell J. M., Mayboroda S.: Uniform rectifiability and harmonic measure III: Riesz transform bounds imply uniform rectifiability of boundaries of 1-sided NTA domains. Int. Math. Res. Not. IMRN 10 (2014), 2702–2729.

[SH] S. V. Hruschev, Simultaneous approximation and removal of singularities. Proc. Steklov Institute of Math., 1979, pp. 133–205.

[JK] D. Jerison, C. Kenig, Boundary behavior of harmonic functions in non-tangentially accessible domains, Adv. Math. 46 (1982), 80–147.

[JW] P. Jones, T. Wolff, Hausdorff dimension of harmonic measures in the plane, Acta Math. 161: 131-144 (1988). DOI: 10.1007/BF02392296.

[KT1] Kenig, Carlos; T. Toro Free boundary regularity below the continuous threshold: 2-phase problems. J. Reine Angew. Math. 596 (2006), 1-44.

[KT2] C. Kenig and T. Toro, Poisson kernel characterization of Reifenberg flat chord arc domains. Ann. Sci. Ecole Norm. Sup. (4) 36 (2003), no. 3, 323–401.

[KT3] C. Kenig and T. Toro, Free boundary regularity for harmonic measure and Poisson kernels. Annals of Mathematics (2) 150 (1999), 369–454.

[KPT] C. Kenig, D. Preiss, and T. Toro, Boundary structure and size in terms of interior and exterior harmonic measures in higher dimensions, J. Amer. Math. Soc. 22 (2009), no. 3, 771–796.

[AM] A. Manning, The dimension of the maximal measure for a polynomial map, Ann. of Math., (2) 119 (1984), 425–430.

[MM] M. Mazalov. Example of a non-rectifiable Nevanlinna contour. St. Petersburg Mathematical Journal 27 (2016), pp. 625–630. doi: 10.1090/spmj/1409.

[NTV] F. Nazarov, X. Tolsa, A. Volberg, On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1Acta Math. 213(2): 237-321 (2014). DOI: 10.1007/s11511-014-0120-7.

[Sa] M. Sakai, Regularity of a boundary having a Schwarz function, Acta Math. 166 (1991), no. 3-4, p. 263-297.
[XT] X. Tolsa, Analytic Capacity, the Cauchy Transform, and Non-Homogeneous Calderon Zygmund Theory, Birkhäuser, Progress in Mathematics, v. 307, 2014.

[VaVo] D. Vardakis, A. Volberg, Free boundary problems in the spirit of Sakai’s theorem, to appear in Comptes Rendus Mathématique, Volume 359 (2021) no. 10, pp. 1233–1238. https://doi.org/10.5802/crmath.259

[VaVo1] D. Vardakis, A. Volberg, Free boundary problems via Sakai’s theorem, to appear in St. Petersburg Math. J.

[AV] A. Volberg, On the dimension of harmonic measure of Cantor repellers, Mich. Math. J. 40 (1994), 239–258.

(A. Volberg) Department of Mathematics, Michigan State University, and Hausdorff Center, Universität Bonn

Email address: volberg@math.msu.edu