Hubble selection of the weak scale from QCD quantum phase transition

Sunghoon Jung\textsuperscript{1,2} \textsuperscript{*} and TaeHun Kim\textsuperscript{1,2} \textsuperscript{†}

\textsuperscript{1}Center for Theoretical Physics, Department of Physics and Astronomy, Seoul National University, Seoul 08826, Korea
\textsuperscript{2}Astronomy Research Center, Seoul National University, Seoul 08826, Korea

If the strange quark were lighter, QCD phase transition could have been first order. This implies that QCD may have quantum critical points as the Higgs vev $v_h$ is varied from its Standard Model value. We show that inflationary quantum evolution of $v_h$ with the relaxion can drive our universe toward those critical points, realizing the weak scale close to the observed value while explaining its closeness to $\Lambda_{\text{QCD}}$. We first explore quantum critical points of $N_f=3$ QCD, parameterized by $v_h$ at $T=0$, and present a basic model for the weak scale. It results in a sharply localized probability distribution of the weak scale, which is critical not to the crossover at zero but to the quantum transition at $\sim \Lambda_{\text{QCD}}$.

I. INTRODUCTION

The Planck-weak scale hierarchy may be addressed by the near-criticality of the Higgs mass parameter $\mu_\text{H}$. As it crosses zero, the universe transitions between broken and unbroken phases of the electroweak symmetry. The crossing could generate some SM backreactions \textsuperscript{3,4,7} that allow dynamical selection of the critical weak scale. However, this transition itself is crossover at zero temperature $T$, providing only a smooth selection rule. Meanwhile, the Higgs mass near-criticality could also be a result of anthropic \textsuperscript{7–11} or entropic selections \textsuperscript{12–15}.

Recently, a powerful mechanism for dynamical selection of criticality was developed in \textsuperscript{16}, where inflationary quantum-dominated evolution of the relaxion inevitably drives a theory close to a quantum critical point. It is crucial to be the critical point between discrete phases, because large energy contrast there is what allows exponential localization by large Hubble rate difference – Hubble selection.

Then we ask: to what quantum critical points does the Higgs mass have relevance? \textsuperscript{16} discussed its renormalization scale dependence through the Higgs self-coupling, as the electroweak vacuum is barrier separated from the global one near the Planck scale \textsuperscript{17,18}; and \textsuperscript{5} studied a prototype model with multiple axions. We note that if the strange quark were lighter, QCD chiral phase transition (finite-$T$) could have been first order. Although it is not yet firmly established by lattice calculation \textsuperscript{19,20}, it has been argued based on the (non-)existence of IR fixed points in the 3D linear sigma model \textsuperscript{21,22}. In other words, QCD at $T = 0$ too may have rich vacuum structure, as a function of quark masses (hence, the Higgs vev $v_h$). But it is not well known currently.

In this work, we present a cosmological account of the weak scale criticality from possible quantum critical points of QCD. We first explore such critical points as a function of $v_h$ for the first time, and calculate their roles for the weak scale criticality that is realized through cosmological Hubble selection. As a result, the weak scale is near-critical, not to the electroweak crossover, but to the quantum phase transition of the chiral symmetry, whose generic scale $\Lambda_{\text{QCD}}$ happens to be close to the observed weak scale.

The paper discusses the following in order: basic model ingredients, QCD quantum critical points, Hubble selection mechanism, and realization of the weak scale criticality. Then we conclude with potential improvements of the scenario.

II. MODEL

The model consists of the relaxion $\phi$ \textsuperscript{3,7}, the Higgs $h$, and the meson field $\Sigma$: $V_{\text{tot}} = V_\phi + V_h + V_\Sigma$. The relaxion couples only to the Higgs sector, but the change of $v_h$ (scanned by $\phi$) induces a change in the $\Sigma$ sector. The desired quantum critical point $v_h^*$ is developed in the $\Sigma$ sector. Then Hubble selection (acting on $\phi$) self-organizes the universe to the critical point.

The relaxion potential is axion-like:

$$V_\phi = \Lambda_\phi^4 \cos \frac{\phi}{f_\phi}.$$  \hspace{1cm} (1)

For Hubble selection, its field range $f_\phi$ shall exceed the Planck scale (see later), which is possible with multiple axions \textsuperscript{29,30}.

The Higgs potential takes the SM form ($\lambda_h \simeq 0.13$) plus the coupling to the relaxion

$$V_h = \frac{1}{2}(M^2 - g_\phi^2)h^2 + \frac{\lambda_h}{4}h^4 \rightarrow -\frac{1}{2}(g_\phi^2)h^2 + \frac{\lambda_h}{4}h^4,$$  \hspace{1cm} (2)

where $h$ is the real Higgs field in unitary gauge. We shift $\phi$ such that the quadratic term $\mu_\phi^2 = -g_\phi v_h^2$ vanishes at $\phi=0$. $v_h^2 \equiv -\mu_\phi^2/\lambda_h$ is used to label the relaxion scanning ($v_{\text{EW}}=246$ GeV)\textsuperscript{11}. The required field range of $\phi$ to scan

\textsuperscript{1}The QCD backreaction $V_h \equiv y_h h(\bar{q}q)/\sqrt{2}$ is ignored, as it is sizable only from the $SU(3)_c$ vacuum for $v_h \lesssim \Lambda_{\text{QCD}}$ (which is not Hubble selected), while only deepening the $V_h$ minimum (increasing the critical energy drop).
\[ \mu^2 \text{ up to the cutoff } M^2 \text{ is } \delta \phi \sim M^2/g, \text{ thus we set } f_\phi = M^2/g. \text{ The dimensionful coupling } g \text{ is a spurion of the } \text{relaxion shift symmetry, thus can be small naturally.} \]

Below \( \Lambda_{QCD} = 200 \text{ MeV} \), scalar meson fields \( \Sigma_i(x) \) are relevant degrees of freedom, and their condensate is an order parameter for the chiral symmetry breaking. The linear sigma model (LSM) is given by \[ V_\Sigma = \mu^2 \text{Tr}[\Sigma \Sigma^\dagger] + \lambda_1 \text{Tr}[\Sigma \Sigma^\dagger]^2 + \lambda_2 \text{Tr}[(\Sigma \Sigma^\dagger)^2] - c(\det \Sigma + \det \Sigma^\dagger) - \text{Tr}[H(\Sigma + \Sigma^\dagger)], \] where fields and parameters are decomposed as \( \Sigma = (\sigma_v + i \sigma_s) \tau^a, H = h_a T^a \) with generators \( T^a \) satisfying \( \text{Tr}[T^a T^b] = \delta^{ab}/2 \) for \( a = 0, ..., N_f^2 - 1 \). Without losing generality, \( \lambda_{1,2}, h_a \) are real, \( c > 0 \), and \( \mu^2 \) can take either sign.

LSM describes the \( U(N_f) \times U(N_f) \) flavor symmetry of QCD, under which \( \Sigma \) is bi-fundamental. The first line conserves \( SU(N_f)_L \times SU(N_f)_R \); \( \lambda_2 \) is non-zero, otherwise symmetry is enhanced to \( O(2N_f^2) \). \( U(1)_V \) is the conserved baryon number; we omit to write it. The instanton contribution \( c \) breaks the anomalous \( U(1)_A \) down to \( Z_2 \times \langle N_f \rangle \). \( H \) is the leading chiral-symmetry breaking mass term. We fix \( N_f = 3 \) with the isospin symmetry \( m_u = m_d \); only \( h_0, h_8 \neq 0 \).

Remarkably, QCD LSM presents necessary features for quantum critical points at \( T = 0 \). \( c \) is a cubic term for \( N_f = 3 \), creating local vacua even with \( \mu^2 > 0 \); this would not have been possible for \( N_f = 2 \). In addition, the linear term \( H \) plays the role of external magnetic fields, as a function of the quark mass (or \( v_h \)), destabilizing the local vacuum at a critical point \( v_h^* \).

### III. QCD Quantum Critical Points

How does QCD vacuum structure respond to the relaxation scanning of \( v_h \)? The partially conserved axial-vector currents yield
\[
\partial_\mu j_{\mu}^{a5} = m_\pi^2 f_{\pi}, \pi_a = \pi_b h_c d_{abc},
\]
where the last equality is from the variation under chiral transformations. For the pion \( \pi^0 = \pi_3 \) with \( d_{360} = \sqrt{2/3} \delta_{b3}, d_{368} = 1/\sqrt{3} \delta_{b3}, \) and for \( K^0 = (\pi_6 + i \pi_7)/\sqrt{2} \) with \( d_{K60} = \sqrt{2/3} \delta_{bK}, d_{K68} = -1/\sqrt{3} \delta_{bK}, \) we have
\[
m_\pi^2 f_\pi = \sqrt{2} h_0 + h_8/\sqrt{3}, \quad m_K f_K = \sqrt{2} h_0 - h_8/2\sqrt{3}.
\]
Using these most precisely measured properties (Table. I), we fix the “SM point” as \[ h_0(v_{EW}) = (287 \text{ MeV})^3, \quad h_8(v_{EW}) = -(312 \text{ MeV})^3. \]

From the chiral Lagrangian, \( m_\pi^2 f_\pi \propto m_q \). The comparison with current mass terms, \( L \propto -m_q(\bar{u}u + \bar{d}d) - m_s \bar{s}s \), also yields the same ratio \( h_s/h_0 \approx (m_u - m_d)/\sqrt{3} \). Using \( m_q = (m_u + m_d)/2 = 3.45 \text{ MeV}, \ m_s = 93 \text{ MeV} \), we assume \( H \) is linear to \( v_h \) as
\[
h_0(v_h) = h_0(v_{EW}) + \frac{v_h}{v_{EW}},
\]
\[
m_q/2 = 3.45 \text{ MeV}, \ m_s = 93 \text{ MeV} \]

This is the main response of QCD vacua to the relaxation. QCD vacuum at \( T = 0 \) are found by minimizing \( V_\Sigma \) and considering stability along all 18 field directions. It is known that LSM can have three types of vacua at \( H = 0 \): \( SU(3)_L \times SU(3)_R \) \( (s_1 = s_3 = 0) \), \( SU(3)_V \) \( (s_1 = s_3 \neq 0) \), and \( SU(2)_L \times SU(2)_R \times U(1)_V \) \( (s_1 = 0, s_3 \neq 0) \). In particular, the \( SU(3)_V \) global vacuum (that we live today) and the \( SU(3)_V \times SU(3)_V \) local vacuum coexist if \( \mu^2 > 0 \) and \( K \equiv \frac{c^2}{2\pi^2(3\lambda_1 + \lambda_2)} > 4.5 \) with \( 3\lambda_1 + \lambda_2 > 0 \). This parameter space is our focus. Currently, it is not known whether this can be realized near the SM.

By scanning LSM parameters with \( K > 4.5 \), we have successfully found a range of desired possibilities (App. A). The following benchmark SM point (with Eq. (6))
\[
m^2 = (60 \text{ MeV})^2, \ c = 4800 \text{ MeV}, \ \lambda_1 = 7, \ \lambda_2 = 46 \]
has \( K = 47.8 \), and the tree-level description of meson spectrum (Table. I) yields the goodness-of-fit \( \chi^2/\text{dof} = 0.44 \) with the first 7 observables and 3.11 including all. The first 7 are most reliable, while the last 3 are under dispute; their measurements are not as precise and identities are unclear. 5% theoretical uncertainties are assumed, as typical size of perturbative corrections. It is as good as example benchmarks’ in the literature: \( \chi^2 \) with \( \mu^2 > 0 \) yields 0.20 and 2.84, respectively, and \( \chi^2 \) with \( \mu^2 < 0 \) yields 1.22 and 4.64.

As \( H \) or \( v_h \) increases from zero with Eq. (7), the metastable \( SU(3)_V \times SU(3)_V \) vacuum becomes shallower and unstable at the critical point, which is found at \( v_h^* \approx 20 \text{ MeV} \) for the benchmark. In fact, a wider range of \( v_h^* \), at least \( \mathcal{O}(1 \sim 100) \text{ MeV} \), is found (see App. A).

The vacuum energies of coexisting QCD vacua are shown in Fig. 1 parameterized by \( v_h \) as Eq. (14) and (15). The critical energy contrast between those vacua is \( 93 \text{ MeV} \), comparable to \( \Lambda_{QCD} \).
Actually, LSM is over-constrained at tree-level, so it does not yield perfect, single best-fit. As LSM (and essentially all models) is an effective model for non-perturbative effects, lattice calculation is eventually needed to verify QCD quantum critical points.

IV. HUBBLE SELECTION

Inflationary quantum fluctuations on the scalar field value allow to access higher potential regime. Although the field in each Hubble patch always rolls down in average, larger Hubble rates at higher potentials can make a difference in the global field-value distribution among patches.

The Hubble (volume) weighted field distribution \( \rho(\phi, t) \) obeys the modified Fokker-Planck equation (FPV) \[\frac{\partial \rho(\phi, t)}{\partial t} = \frac{\partial}{\partial \phi} \left( \frac{V'}{3H^2} \rho \right) + \frac{1}{8\pi^2} \frac{\partial^2(H^3\rho)}{\partial \phi^2} + 3\Delta H \rho. \] (9)

The \( \phi \) dependence of the Hubble constant \( H = H_0 + \Delta H(\phi) \) in the last term with \( \Delta H(\phi) = \frac{V(\phi)}{3H_0^2M^4} \ll H_0 \) accounts for the volume weights within the distribution. \( M_{Pl} \approx 2.4 \times 10^{18} \text{ GeV} \). It is solved by (for a linear potential without boundary conditions)

\[ \rho(\phi, t) \propto \exp \left\{ -\frac{1}{2\sigma^2} \left[ \phi - (\phi_0 + \dot{\phi}_0 t + \frac{3}{2} (\Delta H')^2) \right] \right\}, \] (10)

where \( \dot{\phi}_c = -V'/3H \) is classical rolling while quantum effects are captured by diffusion \( \sigma_\phi(t)^2 = \frac{H^2}{4\pi^2} H t \) due to the de-Sitter temperature \( H/2\pi \) \[51,52\] and additional velocity \( \dot{\phi}_H = 3(\Delta H')^2 \sigma_\phi^2 \) due to the Hubble volume effect. The terms inside the parenthesis describe the width of a peak.

“Hubble selection” starts to operate as the peak of the distribution starts to climb up the potential: \( \sigma_\phi^2 \approx \frac{4}{3} M_{Pl}^2 \). The width at this moment is always Planckian, reflecting quantum nature. The field excursion by this moment is non-negligible \( \Delta \phi \sim 4\pi^2 M_{Pl}^2/\Lambda^2_\phi \). Thus, for a peak to climb, the field range \( \delta \phi \sim M^2/g \) must accommodate both the field excursion (stronger condition) and the width, yielding respectively

\[ g \lesssim H \frac{H M^2}{M_{Pl}^2} \Lambda^2_\phi \lesssim M^2 M_{Pl}^{-2}. \] (11)

We call this condition global Quantum Beats Classical (QBC). It is stronger than the usual local QBC, \( V' \lesssim H^3 \) requiring \( g \lesssim H \frac{M^2}{M_{Pl}^2} \Lambda^2_\phi \), since \( \Lambda^2_\phi \lesssim H M_{Pl} \) from Condition 1 later. It also has different meanings as it involves the field range while the local one depends only on the potential slope. It turns out to be equivalent to the QV condition in \[10\] (App. B) that accounts for Hubble volume effects. If it is not satisfied, \( \rho \) makes an equilibrium at the bottom of a potential \[53\], with the width \( \sigma_\phi \sim H^2/m_\phi \sim H^2 M^2 \Lambda^2_\phi g \lesssim M_{Pl} \) being consistently sub-Planckian. Thus, we require the global QBC on a whole \( \phi \) field range, which reads Eq. (11).

The e-folding until this moment \( \Delta N \approx \frac{8\pi^2 M_{Pl}^2}{3H^2} \) already saturates the upper bound for finite inflation, \( \frac{8\pi^2 H^2}{H_{Pl}^2} \), given by the de-Sitter entropy \( \pi M_{Pl}^2 \). Thus, Hubble selection needs eternal inflation, and the universe eventually reaches a stationary state \[54,55\]. Probability distributions are to be defined among Hubble patches that have reached reheating \[61,63\]. Then since the latest patches dominate the ensemble with an exponentially larger number, only the stationary or equilibrium distribution can be relevant to us.

\[ \rho(\phi, t) \] makes an equilibrium somewhere near the top of a potential, which is the critical point \( \phi_* \) in this work. The distribution can be especially narrow, of the Planckian order (Eq. (19)), if the top is a critical point where \( \sigma_\phi \sim H^2/M_{Pl}^2 \). Then since the Hubble volume selection needs eternal inflation, and the universe eventually reaches a stationary state \[57,60\]. Probability distributions are to be defined among Hubble patches that have reached reheating \[61,63\]. Then since the latest patches dominate the ensemble with an exponentially larger number, only the stationary or equilibrium distribution can be relevant to us.

Actually, LSM is over-constrained at tree-level, so it does not yield perfect, single best-fit. As LSM (and essentially all models) is an effective model for non-perturbative effects, lattice calculation is eventually needed to verify QCD quantum critical points.
A theory enters the $Q^2V$ regime when the balance width becomes larger than $M_{Pl}$:

$$g \lesssim H\frac{H^2M^2}{M_{Pl}^2} \frac{M^4}{\Lambda^4}. \tag{13}$$

This is equivalent to $V' \lesssim H^3/M_{Pl}$, thus field-range independent local balance near $\phi_*$. $Q^2V$ is typically stronger than the global QBC in Eq. (11), and is not absolutely needed for Hubble selection, but will be useful for efficient localization of $v_h$.

**V. THE WEAK SCALE CRITICALITY**

What is the final equilibrium distribution of $v_h$?

For $\mu_h^2 > 0$ ($\phi < 0$), $V_h = V_S = 0$ remain unchanged with $\phi$, so the $\phi$ dynamics is quantum dominated. But it must not dominate the inflation dynamics (Condition 1): $\Lambda^4_\phi \lesssim H^2M_{Pl}^2$.

As soon as $\mu_h^2 \lesssim 0$ ($\phi > 0$), the Higgs gets vev $v_h \geq 0$, and $V_h, V_S$ minima evolve with $\phi$: $V_h(h = v_h) = -\frac{\Lambda^4}{4}v_h^4$, and, at leading orders in $h$ with $\Theta(7)$,

$$V_{\Sigma(L\times R)} \simeq -a_1 \Lambda^4_{QCD} \frac{v_h^2}{v_{EW}}, \tag{14}$$

$$V_{\Sigma(V)} \simeq -a_2 \Lambda^4_{QCD} \frac{v_h^4}{v_{EW}}, \tag{15}$$

where $a_1 \simeq 114, a_2 \simeq 0.059, a_3 \simeq 1.38$ for the benchmark (Fig. 1). $\langle \sigma \phi \rangle_{(L \times R)} \simeq -\langle \sigma \phi \rangle \simeq a_4 v_h$ with $a_4 \simeq 0.025$. Note that these are decreasing with $\phi$, as opposed to $\Theta = \frac{g^4}{\mu_h^2} \phi$.

Above all, as $\phi$ slowly evolves, $h$ and $\Sigma$ must always follow their highest minima for given $\phi$. Since potential shapes change slowly with $\phi$, it only requires equilibrium widths to be small enough: $H^2/m_{h,\Sigma} \lesssim \Lambda_{QCD}$ with $m_h \sim v_h$ and $m_\Sigma \sim \Lambda_{QCD}$, where the Hubble-selected higher-energy $SU(3) \times SU(3)$ vacuum is used. For $v_h^* \lesssim \Lambda_{QCD}$ from Condition 3, we obtain (Condition 2): $H \lesssim v^*_h$.

For Hubble selection, the increasing $V_\Sigma$ needs to dominate the $\phi$ dynamics for $v_h \leq v^*_h$. Consider potential variations with respect to $\phi$

$$\frac{\delta V_\phi}{\delta \phi} \sim g^2 \Lambda^4_{QCD} \frac{v_h^2}{M^2}, \frac{\delta V_{\Sigma(L\times R)}}{\delta \phi} \sim -a_1 \frac{\Lambda^4_{QCD}}{\lambda_h} \frac{v_h^2}{v_{EW}}, \tag{16}$$

$$\frac{\delta V_{h}}{\delta \phi} \sim -\frac{g^2}{2} v_h^2, \frac{\delta V_{\Sigma(V)}}{\delta \phi} \sim -a_2 \frac{\Lambda^4_{QCD}}{\lambda_h} \frac{v_h^4}{v_{EW}}. \tag{15}$$

Unless $v_h$ is too small, the $\delta V_\phi/\delta \phi$ dominance requires $\Lambda^4_\phi / M \gtrsim v^*_h$. Furthermore, the increase $\Delta V_\phi$ after $v_h \gtrsim v^*_h$ should never compensate the $\Lambda_{QCD}$-scale critical energy drop. As the decrease of $V_h \propto -\phi^2$ dominates for $v_h \gtrsim \sqrt{\Lambda_{QCD}}$, suppressing $\Delta V_\phi$ for $v_h \gtrsim v_h^* \gtrsim \Lambda_{QCD}$ yields $\Delta V_\phi \approx \frac{3g^2}{M^2} \Lambda^2_{QCD} \frac{\Lambda_{QCD}}{g} \lesssim \Lambda^2_{QCD}$. Thus, $V_\phi$ cannot be too flat nor steep relative to $V_{h,\Sigma}$ (Condition 3):

$$v^*_h \lesssim \Lambda^2_\phi / M \lesssim \Lambda_{QCD}. \tag{17}$$

Notably, it requires a mild separation of $\Lambda_\phi$ and $M$ (see later); if $M \sim \Lambda_\phi$ strictly, $M \lesssim \Lambda_{QCD}$. As $\Lambda_\phi$ and $g$ are only upper bounded, a large range of parameter space is consistent.

We choose a benchmark ($v_h^*$ from LSM)

$$v^*_h \simeq 20 \text{ MeV}, \ H = v_h^*, \ M = 3 \times 10^{-3} M_{Pl}, \quad \Lambda^2 = 10^{-2} H M_{Pl}, \ g = 10^{-3} H^2 / M_{Pl}, \tag{18}$$

satisfying the global QBC (Eq. (11)), $Q^2V$ (Eq. (13)) marginally, and Conditions 1 $\sim 3$. The total potential energy near $v^*_h$ is shown in Fig. 2. As desired, the energy peaks sharply at the critical point, drops significantly after that, and never comes back up; for much smaller or larger $V_\phi^*$, it would not sharply peak at the critical point.

In equilibrium, most Hubble patches need to have $v_h \leq v^*_h$ (or, $\phi$ near $\phi_* = \lambda_h v^*_h^2 / g$). Following (16) [23], we solve the FPV at large time, by approximating $V_\phi$ linearly and ignoring Hubble patches with $v_h > v^*_h$. Results are shown in Fig. 3. The width is estimated by

$$\sigma_\phi \simeq \frac{3g^2}{2M_{Pl}^2} v^*_h \sim M_{Pl}, \tag{19}$$

for global QBC, which is indeed Planckian, but is broader for $Q^2V$, as discussed. In any case, the width of $v_h$ is suppressed by $g$ as

$$\sigma_v \simeq \frac{g \sigma_\phi}{2 \lambda_h v^*_h} \tag{20}$$

thus can be arbitrarily narrow at the price of arbitrarily large $\phi^*$ range. For the benchmark satisfying
Q^2V marginally, \( \sigma_{v_h} \approx 0.5\,\text{MeV} \ll v_h^* \) is narrow enough; but for 10-100 times larger \( g \) with only global QBC, \( \sigma_{v_h} \approx v_h^* \).

Lastly, a small difference between \( v_h^* \) and \( v_{\text{EW}} \) remains. After inflation, the global QCD vacuum is reached, and \( \phi \) slow-rolls to the today’s \( v_h = 246\,\text{GeV} \) from the decreasing \( V_{\text{tot}}(\phi) \). \( \phi \) could be safely slow-rolling today or trapped by SM backreactions. Signals from phase transition or time-dependent \( v_h \) may be produced. In any case, \( v_{\text{EW}} \) is generically close to \( \Lambda_{\text{QCD}} \).

VI. DISCUSSION

The minimal model discussed does not solve the hierarchy problem even though \( v_h \) can be finely Hubble selected. The separation \( \Lambda_\phi \lesssim M \) needed in Eq. (17) is not stable against quantum corrections, from Higgs loop diagrams with the Higgs-relaxion interaction. Thus, it is another manifestation of the (little) hierarchy problem. Assuming a fine-tuning of order \( \epsilon = \Lambda_\phi/M < 1 \), the cut-off can be as high as \( M \gtrsim \Lambda_{\text{QCD}}/\epsilon^2 \). Once with this fine-tuning, small \( g \) is especially versatile for Hubble selection. As it appears all and only in \( \phi \)-dependent dynamics and is only upper bounded, small \( g \) makes it all possible that \( \phi \) is quantum dominated while \( h \) and \( \Sigma \) instantly follow their local minima. Only resulting hierarchy \( f_\phi \gg M_{\text{Pl}} \) needs to be generated consistently [29,31].

We have explored possible quantum critical points of the QCD \( \Sigma \) sector, resting on several assumptions. Most relevantly, \( N_f > 3 \) and other responses to the \( v_h \) scanning need to be studied. The vacuum structure with \( N_f > 3 \) is likely to be more rich and complicated; in this sense, our work is a conservative exploration. But these are not easy to improve just with perturbative calculations within LSM, whose parameters are not dictated by symmetries and inherently non-perturbative [42-46]. Eventually, lattice calculation is needed as a first-principle method [19,20] to verify and develop the scenario.

The near-criticality of the SM may have quantum cosmological origins. Using a toy model utilizing QCD critical points, we have shown how such can be realized in nature with Hubble selection. Having identified both potentials and problems, we await for further developments.

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Appendix A: Quantum critical points in LSM

We present our initial study of the $N_f = 3$ LSM parameter space at tree-level. Quantum critical points are found in a wide range of parameter space, at a range of $v_h^*$ values relevant to this work. More dedicated studies shall be done with lattice calculation.

We fix LSM parameters to the benchmark in Eq. (8) and (6), while varying $\lambda_1$ and $\mu^2 > 0$. It turns out that large uncertainties of the last 3 observables in Table I leave some freedom in choosing $\lambda_1$ and $\mu^2$. And surprisingly, these two free parameters are already good enough to find reasonable SM points with critical points. We should focus on the parameter space with $K > 4.5$ so that coexisting vacua are present at $\mathcal{H} = 0$; this roughly requires $\mu^2 \lesssim 10$ times the benchmark value.

Fig. 4 shows the numerical results of the critical point $v_h^*$ (upper panel) and $\chi^2$/dof (lower) as a function of $\lambda_1$ and $\mu^2$; the scale factors are relative to the benchmark values. We found that $v_h^*$ is mostly sensitive only to $\mu^2$, while $\chi^2$ only to $\lambda_1$. As alluded, $v_h^* = O(1 - 100)$ MeV can be obtained with the $\mu^2$ scale factor $= 0.3 \sim 3$, and $\chi^2$/dof can be minimized to $3 \sim 4$ with the $\lambda_1$ scale factor $= 0.5 \sim 1.5$. The energy contrast at the critical point does not vary much, 70$\sim$120 MeV close to $\Lambda_{QCD}$, in the parameter space shown in the figure.

![Figure 4](image_url)

**FIG. 4.** (Top): LSM quantum critical point $v_h^*$ with $\mu^2$. Other parameters are fixed to benchmark values. Scale factors are relative to the benchmark. (Bottom): $\chi^2$/dof with $\lambda_1$ using the same data as in Table I of the main text.

Appendix B: Quantum regimes

Quantum regimes for Hubble selection are categorized into two \cite{16}: the global QBC (or QV) and Q$^2$V. What distinguishes them? We discuss this by first describing the scaling behaviors of QBC and Q$^2$V; second, showing the equivalence between the global QBC and QV; and finally demonstrating the boundary effects in Q$^2$V. Refer to \cite{17 16 63} and references therein for further details on FPV solutions.

1. Scaling

Consider a linear potential

$$V = V' \phi \quad \text{for } |\phi| \leq \phi_0,$$

where the field range $\phi_0$ does not have to be the full range, and the slope is constant. QV and Q$^2$V conditions are expressed as $\alpha \beta \gtrsim 1$ and $\alpha^2 \beta \gtrsim 1$, respectively, in terms of \cite{16}

$$\alpha = \frac{3}{4\pi^2} \frac{H^4}{V'\phi_0}, \quad \beta = \frac{3}{2} \frac{\phi_0^2}{M_{Pl}^2}. \quad (B2)$$

First, scale the field range by $a$: $\phi_0 \rightarrow a\phi_0$. Under this, $\alpha \propto a^{-1}$, $\beta \propto a^2$. \hfill (B3)

Thus, the QV condition is not scale invariant, $\alpha \beta \propto a$, reflecting that it is equivalent to the global QBC (see below) which was derived by considering the necessary field range for Hubble selection. On the other hand, the Q$^2$V condition is invariant $\alpha^2 \beta \propto a^0$, as it measures only the local balance near a critical point (the upper boundary), as derived in the main text.

But physical peak properties at equilibrium are also expressed in terms of $\alpha$ and $\beta$. The physical width and the peak location from the boundary must be scale invariant, which are indeed satisfied as

$$\sigma_\phi \sim \phi_0 \beta^{-1/2} \propto a^0, \quad \sigma_\phi \sim \phi_0 (\alpha/\beta)^{1/3} \propto a^0 \quad (B4)$$

for QV and Q$^2$V \cite{16}, and

$$\frac{\phi_0}{2a\beta} \sim \frac{4\pi^2 V' M_{Pl}^2}{9 H^4} \propto a^0 \quad (B5)$$

for QV. Note that the last expression is the minimum field excursion $\Delta \phi$ for a peak to start climbing up. This led to the global QBC in Eq. (11).

The scaling also dictates how the peak location and width change with, e.g. the potential slope (the $V'$ factor), in the global QBC regime. It can be convenient when numerically solving for equilibrium distributions since they can be localized in a very narrow range, away from the boundary.
2. Consistency

Restoring the factors of our model, the full field range \( \phi = M^2/g \) and \( V' = \Lambda^4/\phi \) with \( f_\phi = M^2/g \), we have

\[
\alpha = \frac{3}{4\pi^2} \frac{H^4}{\Lambda_\phi^2}, \quad \beta = \frac{3}{2} \frac{M^4}{g^2 M_{Pl}^2}.
\]

(B6)

Thus, \( QV \) and \( Q^{2V} \) read, respectively,

\[
g \lesssim H \frac{M^2}{M_{Pl} \Lambda_\phi^2}, \quad g \lesssim H \frac{H^2 M^2}{M_{Pl} \Lambda_\phi^4}.
\]

(B7)

They agree with our global QBC in Eq. (11) (not exactly with the local one) and the derivation in Eq. (13), respectively.

3. Boundary effects

Lastly, we demonstrate the boundary effects on \( Q^{2V} \) regime discussed heuristically in the main text. The absorbing boundary condition at \( \phi_0 \) is relevant to critical points. It means that any Hubble patches with \( \phi > \phi_0 \) are out of Hubble selection, hence discarded, as the potential energy sharply peaks at \( \phi_0 \).

The absorbing boundary condition can be imposed by the method of images. Putting two charges at \( \phi_0 \pm \Delta \phi \), and ignoring Hubble expansion for simplicity, the solution is of the form \([63]\)

\[
\rho(\phi) \propto e^{-\frac{1}{2\sigma_\phi^2} (\phi - (\phi_0 \pm \Delta \phi))^2} - e^{-\frac{1}{2\sigma_\phi^2} (\phi - (\phi_0 \pm \Delta \phi))^2},
\]

(B8)

where \( \phi < \phi_0 \) is the physical region. It describes that, as \( \Delta \phi \) and \( \sigma_\phi \) become comparable, or equivalently as the significant portion of the would-be distribution passes the boundary, the boundary becomes relevant, and the final distribution is displaced away from it. Fig. 5 demonstrates this behavior as a function of \( \sigma_\phi \) for fixed \( \Delta \phi \).

Physically, why/how does a boundary affect a peak, located away from it? The boundary condition generates the asymmetric absorbing probability, skewing the distribution, which otherwise would be symmetrically quantum diffused. The peak location \( \phi_{\text{peak}} \) is at the extremum satisfying

\[
\frac{2(\phi_0 - \phi_{\text{peak}}) \Delta \phi}{\sigma_\phi^2} = (\phi_0 - \phi_{\text{peak}}) + \Delta \phi.
\]

(B9)

When \( \sigma_\phi \gtrsim \Delta \phi \), the boundary is most important, and

\[
\phi_0 - \phi_{\text{peak}} \simeq \sigma_\phi.
\]

(B10)

This must be true as \( \sigma_\phi \) is the only dimensionful parameter in this limit. This effectively means repulsive motion

\[
\dot{\phi}_0 \sim -\frac{d\sigma_\phi}{dt} = -\frac{H^3}{8\pi^2 \sigma_\phi},
\]

(B11)

justifying the formula used in the main text and the peak location \( |\phi_{\text{peak}} - \phi_0| \sim \sigma_\phi \) in the \( Q^{2V} \) regime.

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