ON GRÖBNER BASES OVER DEDEKIND DOMAINS

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Abstract. Gröbner bases are a fundamental tool when studying ideals in multivariate polynomial rings. More recently there has been a growing interest in transferring techniques from the field case to other coefficient rings, most notably Euclidean domains and principal ideal rings. In this paper we will consider multivariate polynomial rings over Dedekind domain. By generalizing methods from the theory of finitely generated projective modules, we show that it is possible to describe Gröbner bases over Dedekind domains in a way similar to the case of principal ideal domains, both from a theoretical and algorithmic point of view.

1. Introduction

The theory of Gröbner bases, initiated by Buchberger [Buc65] plays an important role not only in mathematical disciplines like algorithmic commutative algebra and algebraic geometry, but also in related areas of science and engineering. Although the original approach of Buchberger was restricted to multivariate polynomials with coefficients in a field, Trinks [Tri78] and Zacharias [Zac78] showed that by generalizing the notions of S-polynomials and reduction, Gröbner bases can also be constructed in the ring case. For coefficient rings that are principal ideal domains, the approach to constructing Gröbner bases is very close to the field case has attracted a lot of attention, see for example [PP88, KRK88, Möl88, Pan89], also [AL94, Chapter 4] or [BW93, Chapter 10].

In this paper, we will investigate Gröbner bases over Dedekind domains, that is, over integral domains which are locally discrete valuation rings. Despite the prominent role of Dedekind domains as coefficient rings for example in arithmetic geometry, not much is known in connection with the construction of Gröbner bases. Our aim is to show that it is possible to improve upon the generic algorithms for Noetherian domains. In particular, we will show that using the notions of pseudo-polynomials and pseudo-Gröbner bases the approach comes very close to that of principal ideal domains.

The idea of using so called pseudo-objects to interpolate between principal ideal domains and Dedekind domains has already been successfully applied to the theory of finitely generated projective modules. Recall that over a principal ideal domain such modules are in fact free of finite rank. By using the Hermite and Smith form, working with such modules is as easy as working with finite dimensional vector spaces over a field. If the ring is merely a Dedekind domain, such modules are in general not free, rendering the Hermite and Smith form useless. But since the work of Steinitz [Ste11, Ste12] it has been known that these modules are direct sums of projective submodules of rank one. In [Coh96] (see also [Coh00]), based upon ideas already present in [BP91], a theory of pseudo-elements has been developed, which enables an algorithmic treatment of this class of modules very close to the case of principal ideal domains. In particular, a generalized Hermite form algorithm is described, which allows for similar improvements as the classical Hermite form algorithm in the principal ideal case, see also [BFH17, FH14].
Now—in contrast to the setting of finitely generated projected modules just described—Gröbner bases do exist if the coefficient ring is a Dedekind domain. In [AL97] using a generalized version of Gröbner basis, the structure of ideals in univariate polynomial rings over Dedekind domains is studied. Apart from that, nothing is published on how to exploit the structure of Dedekind domains in the algorithmic study of ideals in multivariate polynomial rings. Building upon the notion of pseudo-objects, in this paper we will introduce pseudo-Gröbner bases, that will interpolate more smoothly between the theory of Gröbner bases for Dedekind domains and principal ideal domains. Of course the hope is that one can apply more sophisticated techniques from principal ideal domains to Dedekind domains, for example, signature-based algorithms as introduced in [EPP17]. As an illustration of this idea, we prove a simple generalization of the product criterion for pseudo-polynomials. We will also show how to use the pseudo-Gröbner basis to solve basic tasks from algorithmic commutative algebra, including the computation of primes of bad reduction.

The paper is organized as follows. In Section 2 we recall standard notions from multivariate polynomials and translate them to the context of pseudo-polynomials. This is followed by a generalization of Gröbner bases in Section 3, where we present various characterizations of the so called pseudo-Gröbner bases. In Section 4 by analyzing syzygies of pseudo-polynomials, we prove a variation of Buchberger’s criterion. As a result we obtain a simple to formulate algorithm for computing Gröbner bases. We also use this syzygy-based approach to prove the generalized product criterion. In Section 5 we consider the situation over a ring of integers of a number field and address the omnipresent problem of quickly growing coefficients by employing classical tools from algorithmic number theory. In the final section we give some applications to classical problems in algorithmic commutative algebra and the computation of primes of bad reduction.

Acknowledgments. The author was supported by Project II.2 of SFB-TRR 195 ‘Symbolic Tools in Mathematics and their Application’ of the German Research Foundation (DFG).

Notation. Throughout this paper, we will use \( R \) to denote a Dedekind domain, that is, a Noetherian integrally closed domain of Krull dimension one, and \( K \) to denote its total ring of fractions. Furthermore, we fix a multivariate ring \( R[x] = R[x_1, \ldots, x_n] \) and a monomial ordering \( < \) on \( R[x] \).

2. Pseudo-elements and pseudo-polynomials

In this section we recall basic notions from multivariate polynomials and generalize them in the context of pseudo-polynomials over Dedekind domains.

2.1. Multivariate polynomials. For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), we denote by \( x^\alpha \) the monomial \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \). We call \( \alpha \) the degree of \( f \) and denote it by \( \deg(f) \). A polynomial \( f = cx^\alpha \) with \( c \in R \) and \( \alpha \in \mathbb{N}^n \) is called a term. For an arbitrary multivariate polynomial \( f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha \) we denote by \( \deg(f) = \max_{\alpha \in \mathbb{N}^n} \{ \alpha \in \mathbb{N}^n \mid c_\alpha \neq 0 \} \) the degree of \( f \), by \( \text{lm}(f) = x^{\deg(f)} \) the leading monomial, by \( \text{lc}(f) = c_{\deg(f)} \) the leading coefficient and by \( \text{lt}(f) = c_{\deg(f)} x^{\deg(f)} \) the leading term of \( f \).

2.2. Pseudo-elements and pseudo-polynomials. A fractional ideal of \( R \) is a non-zero finitely generated \( R \)-submodule of \( K \). Let now \( V \) be a vector space over \( K \) and \( M \) an \( R \)-submodule of \( V \) such that \( KM = V \), that is, \( M \) contains a \( K \)-basis of \( V \). Given a fractional ideal \( a \) of \( R \) and an element \( v \in V \) we denote by \( av \) the set \( \{ \alpha v \mid \alpha \in a \} \subseteq V \), which is in fact an \( R \)-submodule of \( V \).
**Definition 2.1.** A pair \((v, a)\) consisting of an element \(v \in V\) and a fractional ideal \(a\) of \(R\) is called a pseudo-element of \(V\). In case \(av \subseteq M\), we call \((v, a)\) a pseudo-element of \(M\).

**Remark 2.2.** The notion of pseudo-objects goes back to Cohen [Coh96], who introduced them to compute with finitely generated projective modules over Dedekind domains. Note that in [Coh00] the \(R\)-submodule \(av\) itself is defined to be a pseudo-element, whereas with our definition, this \(R\)-submodule is only attached to the pseudo-element \((v, a)\). We choose the slightly modified version to simplify the exposition and to ease notation.

**Lemma 2.3.** Let \(V\) be a \(K\)-vector space.

(i) For \(v, w \in V\) and \(a, b, c\) fractional ideals of \(R\) we have \(a(bv) = (ab)v\) and \(c(av + bw) = (ca)v + (cb)vw\).

(ii) Let \((v, a), (v_i, a_i)_{1 \leq i \leq l}\) be pseudo-elements of \(V\). If \(av \subseteq \sum_{1 \leq i \leq l} a_i v_i\), then there exist \(a_i \in a_i a^{-1}, 1 \leq i \leq l\), such that \(v = \sum_{1 \leq i \leq l} a_i v_i\).

**Proof.** (i): Clear. (ii): Using (i) and by multiplying with \(a^{-1}\) we are reduced to the case where \(a = R\), that is, \(v \in \sum_{1 \leq i \leq l} a_i v_i\). But then the assertion is clear.

We will now specialize to the situation of multivariate polynomial rings, where additionally we have the \(R[x]\)-module structure. For a fractional ideal \(a\) of \(R\) we will denote by \(a[x]\) the ideal \(\{\sum_a c_a x^a \mid c_a \in a\}\) of \(R[x]\).

**Lemma 2.4.** The following hold:

(i) For fractional ideals \(a, b, c\) of \(R\) we have \(a(b[x]) = (ab)[x]\) and \(a(b[x] + c[x]) = (ab)[x] + (ac)[x]\).

(ii) If \(M\) is an \(R[x]\)-module and \((v, a)\) a pseudo-element, then \(\langle av \rangle_{R[x]} = a[x]v\).

(iii) Let \(M\) be an \(R[x]\)-module and \((v_i, a_i)_{1 \leq i \leq l}\) pseudo-elements of \(M\) with \(\langle a_i v_i \mid 1 \leq i \leq l \rangle_{R[x]} = M\). Given a pseudo-element \((v, a)\) of \(M\), there exist \(f_i \in a_i a^{-1}[x], 1 \leq i \leq l\), such that \(v = \sum_{1 \leq i \leq l} f_i v_i\).

**Proof.** Item (i) follows from the distributive properties of ideal multiplication. Proving (ii), (iii) is analogous to Lemma 2.3. □

**Definition 2.5.** A pseudo-polynomial of \(R[x]\) is a pseudo-element of \(R[x]\), that is, a pair \((f, \mathfrak{f})\) consisting of a polynomial \(f \in K[x]\) and a fractional ideal \(\mathfrak{f}\) of \(R\) such that \(f \cdot \mathfrak{f} \subseteq R[x]\). We call \(\mathfrak{f}c(f) \subseteq R\) the leading coefficient of \((f, \mathfrak{f})\) and denote it by \(\text{lc}(f, \mathfrak{f})\). The set \(\mathfrak{f}[x]f \subseteq R[x]\) is called the ideal generated by \((f, \mathfrak{f})\) and is denoted by \(\langle (f, \mathfrak{f}) \rangle\). We say that the pseudo-polynomial \((f, \mathfrak{f})\) is zero, if \(f = 0\).

**Lemma 2.6.** Let \((f, \mathfrak{f})\) be a pseudo-polynomial of \(R[x]\). Then the following hold:

(i) The leading coefficient \(\text{lc}(f, \mathfrak{f})\) is an integral ideal of \(R\).

(ii) We have \(\langle f \rangle_{R[x]} = f[x]f\).

**Proof.** Clear. □

### 3. REDUCTION AND PSEUDO-GRÖBNER BASES

At the heart of the construction of Gröbner bases lies a generalization of the Euclidean division in univariate polynomial rings. In the context of pseudo-polynomials this takes the following form.

**Definition 3.1** (Reduction). Let \((f, \mathfrak{f})\) and \(G = \{(g_i, \mathfrak{g}_i) \mid 1 \leq i \leq l\}\) be set of non-zero pseudo-polynomials of \(R[x]\) and \(J = \{1 \leq i \leq l \mid \text{Im}(g_i) \text{ divides } \text{Im}(f)\}\). We say that
(f, f) can be reduced modulo G if \( \text{lc}(f, f) \subseteq \sum_{i \in J} \text{lc}(g_i, g_i) \). In case \( G = \{(g, g)\} \) consists of a single pseudo-polynomial, we say that \((f, f)\) can be reduced modulo \((g, g)\). We define \((f, f)\) to be minimal with respect to \( G \), if it cannot be reduced modulo \( G \).

**Lemma 3.2.** Let \((f, f)\) and \( G = \{(g_i, g_i) \mid 1 \leq i \leq l\} \) be non-zero pseudo-polynomials of \( R[x] \) and \( J = \{1 \leq i \leq l \mid \text{lm}(g_i) \text{ divides } \text{lm}(f)\} \). Then \((f, f)\) can be reduced modulo \( G \) if and only if there exist \( a_i \in g_i f^{-1}, i \in J \), such that \( \text{lc}(f) = \sum_{i \in J} a_i \text{lc}(g_i) \).

**Proof.** Set \( c = \sum_{i \in J} \text{lc}(g_i, g_i) \). First assume that \((f, f)\) can be reduced modulo \( G \), that is \( \text{lc}(f, f) = f \text{lc}(f) \subseteq c \). Hence \( \text{lc}(f) = c f^{-1} \) and there exist \( b_i \in g_i f^{-1} \text{lc}(g_i), i \in J \), such that \( \text{lc}(f) = \sum_{i \in J} b_i \). Then the elements \( a_i = b_i \text{lc}(g_i) \in g_i f^{-1}, i \in J \), satisfy the claim.

On the other hand, if \( \text{lc}(f) = \sum_{i \in J} \alpha \text{lc}(g_i) \) for \( a_i \in g_i f^{-1} \), then
\[
\text{lc}(f, f) = f \text{lc}(f) \subseteq \sum_{i \in J} f a_i \text{lc}(g_i) \subseteq \sum_{i \in J} \text{lc}(g_i, g_i). \quad \square
\]

**Lemma 3.3.** Let \((f, f)\) and \((g, g)\) be two non-zero pseudo-polynomials of \( R[x] \). Then the following are equivalent:

(i) \((f, f)\) can be reduced modulo \((g, g)\).
(ii) \( f[x] \text{lt}(f) \subseteq g[x] \text{lt}(g) \).
(iii) \( f \text{lc}(f) \subseteq g \text{lc}(g) \) and \( \text{lm}(f) \) divides \( \text{lm}(g) \).

**Proof.** (i) \( \Rightarrow \) (ii): By assumption \( \text{lm}(g) \) divides \( \text{lm}(f) \) and \( \text{lc}(f) = \alpha \text{lc}(g) \) for some \( \alpha \in g f^{-1} \). Hence
\[
f[x] \text{lt}(f) = f[x] \text{lc}(f)x^{-\deg(f)} = f[x] \alpha x^{-\deg(f) - \deg(g)} \text{lt}(g) \subseteq f[x] \alpha \text{lt}(g) \subseteq g[x] \text{lt}(g). 
\]

(ii) \( \Rightarrow \) (iii): Let \( \mu \in g f[x] \). Since \( \mu \text{lt}(f) \in g[x] \text{lt}(g) \) it follows that \( \text{lm}(g) \) divides \( \text{lm}(f) \) and \( f \text{lc}(f) \subseteq g \text{lc}(g) \). (iii) \( \Rightarrow \) (i): Clear. \( \square \)

**Definition 3.4.** Let \((f, f)\) and \( G = \{(g_i, g_i) \mid 1 \leq i \leq l\} \) be pseudo-polynomials of \( R[x] \) and assume that \((f, f)\) can be reduced modulo \( G \) and \((a_i)_{i \in J}\) are as in Lemma 3.2. Then we call \((f - \sum_{i \in J} a_i g_i, f)\) a one step reduction of \((f, f)\) with respect to \( G \) and we write
\[
(f, f) \xrightarrow{G} (f - \sum_{i \in J} a_i x^{-\deg(f) - \deg(g)} g_i, f). 
\]

**Lemma 3.5.** Let \((h, f)\) be a one step reduction of \((f, f)\) with respect to \( G = \{(g_i, g_i) \mid 1 \leq i \leq l\} \). Denote by \( I = \langle G \rangle \) the ideal of \( R[x] \) generated by \( G \). Then the following hold:

(i) \( \langle f, h \rangle \) is a pseudo-polynomial of \( R[x] \).
(ii) We have \( f[x] (f - h) \subseteq I \).
(iii) We have \( \langle (f, f) \rangle \subseteq I \) if and only if \( \langle (h, f) \rangle \subseteq I \).

**Proof.** By definition there exists \( J \subseteq \{1, \ldots, r\}, a_i \in g_i f^{-1}, i \in J \), with \( \text{lc}(f) = \sum_{i \in J} a_i \text{lc}(g_i) \).

(i) We have
\[
f h = f \left( f - \sum_{i \in J} a_i g_i \right) \subseteq f f + \sum_{i \in J} f a_i g_i \subseteq ff + \sum_{i \in J} g_i g_i \subseteq R[x].
\]

(ii) Since \( f - h = \sum_{i \in J} a_i g_i \) and \( a_i \in g_i f^{-1} \), it is clear that \( f a_i g_i \subseteq I \).
(iii): If \( f[x]f \subseteq I \), then \( f[x](f - \sum_{i \in J} a_i g_i) \subseteq I \), since \( f a_i \subseteq g_i \). On the other hand, if \( f[x]f \subseteq I \), then
\[
f[x]f = f[x]\left(f - \sum_{i \in J} a_i g_i + \sum_{i \in J} a_i g_i\right) \subseteq f[x]f + f[x]\left(\sum_{i \in J} a_i g_i\right) \subseteq I.
\]

**Definition 3.6.** Let \( (f, f) \) and \( G = \{(g_i, g_i) \mid 1 \leq i \leq l\} \) be non-zero pseudo-polynomials of \( R[x] \). We say that \( (f, f) \) reduces to \( (h, f) \) modulo \( G \) if there exist pseudo-polynomials \( (h_i, f) \), \( 1 \leq i \leq l \) such that
\[
(f, f) = (h_1, f) \rightarrow (h_2, f) \rightarrow \cdots \rightarrow (h_l, f) = (h, f).
\]
In this case we write \( (f, f) \nleftrightarrow (h, f) \). (The relation \( \nleftrightarrow \) is thus the reflexive closure of \( \rightarrow \).

**Lemma 3.7.** If \( (f, f) \) and \( (h, f) \) and \( G = \{(g_i, g_i) \mid 1 \leq i \leq l\} \) are non-zero pseudo-polynomials with \( (f, f) \rightarrow (h, f) \), then \( f[x](f - h) \subseteq I \). Moreover \( \langle f, f \rangle \subseteq I \) if and only if \( \langle (h, f) \rangle \subseteq I \).

**Proof.** Note that \( f - h = f - h_1 + h_1 - h_2 + \cdots - h_l - h \). Hence the claim follows from Lemma 3.5 (ii). \( \square \)

**Remark 3.8.** If \( (h, f) \) is a one step reduction of \( (f, f) \), then \( \deg(h) < \deg(f) \) and there exist \( h_i \in (g_i f^{-1})[x] \), \( i \in I \), such that \( f - h = \sum_{1 \leq i \leq I} h_i g_i \). Applying this iteratively we see that if \( (h, f) \) is a pseudo-polynomial of \( R[x] \) with \( (f, f) \rightarrow (h, f) \), then \( \deg(h) < \deg(f) \) and there exists \( h_i \in (g_i f^{-1})[x] \), \( i \in I \), such that \( f - h = \sum_{1 \leq i \leq I} h_i g_i \). Moreover in both cases we have \( \deg(f) = \max_{i \in I} (\deg(h_i, g_i)) \).

**Definition 3.9.** Let \( (f, f) \) and \( G = \{(g_i, g_i) \mid 1 \leq i \leq l\} \) be pseudo-polynomials. The leading term \( \text{lt}(f, f) \) is defined to be \( \text{lt}(f) \). Moreover we define the leading term ideal of \( (f, f) \) and \( G \) as \( \text{Lt}(f, f) = \langle \text{lt}(f), f \rangle_{R[x]} \) and \( \text{Lt}(G) = \sum_{i=1}^{l} \text{Lt}(g_i, g_i)_{R[x]} \) respectively. If \( F \subseteq R[x] \) is a set of polynomials, then we define \( \text{Lt}(F) = \langle \text{lt}(f) \mid f \in F \rangle_{R[x]} \).

We can now characterize minimality in terms of leading term ideals.

**Lemma 3.10.** Let \( G = \{(g_i, g_i) \mid 1 \leq i \leq l\} \) be non-zero pseudo-polynomials of \( R[x] \). A non-zero pseudo-polynomial \( (f, f) \) is minimal with respect to \( G \), if and only if \( \text{Lt}(f, f) \not\subseteq \text{Lt}(G) \).

**Proof.** Denote by \( J = \{i \in \{1, \ldots, r\} \mid \text{lm}(g_i) \text{ divides } \text{lm}(f)\} \). Assume first that \( (f, f) \) is non-minimal, that is, the pseudo-polynomial can be reduced modulo \( G \). Then there exist \( a_i \in g_i f^{-1} \), \( i \in J \), such that \( \text{lc}(f) = \sum_{i \in J} a_i \text{lc}(g_i) \). For every \( i \in J \) there exists a monomial \( x^a \) with \( \text{lm}(g_i) x^a = \text{lm}(f) \). Hence
\[
\text{lt}(f) = \text{lm}(f) \text{lc}(f) = \sum_{i \in J} a_i \text{lm}(f) \text{lc}(g_i) = \sum_{i \in J} a_i x^{a_i} \text{lt}(g_i).
\]
Thus it holds that \( \text{lt}(f) = \sum_{i \in J} f a_i x^{a_i} \text{lt}(g_i) \in \sum_{i \in J} g_i[x] \text{lt}(g_i) \subseteq \text{Lt}(G) \). This implies \( \text{Lt}(f, f) = f[x]f \subseteq \text{Lt}(G) \), as claimed.

Now assume that \( \text{Lt}(f, f) \subseteq \text{Lt}(G) \). Let \( \alpha \in f \). Since \( \alpha \text{lt}(f) \subseteq \text{Lt}(f, f) \subseteq \text{Lt}(G) \), there exist \( h_i \in g_i[x] \), \( 1 \leq i \leq l \), with \( \alpha \text{lt}(f) = \sum_{i=1}^{l} h_i \text{lt}(g_i) \). Without loss of generality we may assume that \( h_i \) is a term, say, \( h_i = a_i x^{a_i} \), where \( a_i \in g_i \). Denote by \( J' \) the set \( \{i \in \{1, \ldots, r\} \mid x^{a_i} \text{lm}(g_i) = \text{lm}(f)\} \). Hence we have
\[
\alpha \text{lm}(f) = \sum_{i \in J'} a_i x^{a_i} \text{lt}(g_i) = \sum_{i \in J'} a_i x^{a_i} \text{lm}(g_i) \text{lc}(g_i).
\]
Comparing coefficients this yields \( \alpha \text{lc}(f) = \sum_{i \in J'} \alpha_i \text{lc}(g_i) \). Thus \( \text{lc}(f, f) = f \text{lc}(f) \subseteq \sum_{i \in J'} g_i \text{lc}(g_i) = \sum_{i \in J'} \text{lc}(g_i, g_i) \). As \( J' \subseteq J \), it follows that \((f, f)\) can be reduced modulo \( G \).

**Theorem 3.11.** Let \((f, f)\) and \( G = \{(g_i, g_i) \mid 1 \leq i \leq l\} \) be pseudo-polynomials. There exists a pseudo-polynomial \((h, f)\) which is minimal with respect to \( G \) and \( h_i \in (g_if^{-1})[x], \) \( 1 \leq i \leq l \), such that \((f, f) \xrightarrow{G} (r, f)\),

\[
f - r = \sum_{i=1}^{r} h_i g_i,
\]

and \( \text{deg}(f) = \max((\max_{1 \leq i \leq l} \text{deg}(h_i g_i)), \text{deg}(r)) \).

**Proof.** Follows immediately from Lemma 3.10 and Remark 3.8.

We can now generalize the characterization of Gröbner bases to pseudo-Gröbner bases.

**Theorem 3.12.** Let \( I \) be an ideal of \( R[x] \) and \( G = \{(g_i, g_i) \mid 1 \leq i \leq l\} \) non-zero pseudo-polynomials of \( I \). Then the following are equivalent:

(i) \( \text{Lt}(I) = \text{Lt}(G) \);

(ii) For a pseudo-polynomial \((f, f)\) of \( R[x] \) we have \( \langle (f, f) \rangle \subseteq I \) if and only if \((f, f)\) reduces to 0 modulo \( G \).

(iii) For every pseudo-polynomial \((f, f)\) of \( R[x] \) with \( \langle (f, f) \rangle \subseteq I \) there exist \( h_i \in (g_i f^{-1})[x], 1 \leq i \leq l \), such that \( f = \sum_{i=1}^{l} h_i g_i \) and \( \text{lm}(f) = \max_{1 \leq i \leq l}(\text{lm}(h_i g_i)) \).

(iv) If \((a_{ij})_{1 \leq j \leq n_i}\) are ideal generators of \( g_i \) for \( 1 \leq i \leq l \), then the set

\[
\{a_{ij} g_i \mid 1 \leq i \leq l, 1 \leq j \leq n_i\}
\]

is a Gröbner basis of \( I \).

**Proof.** (i) \( \Rightarrow \) (ii): If \((f, f) \xrightarrow{G} 0\), then Lemma 3.7 implies that \( \langle (f, f) \rangle \subseteq I \). Now assume \( \langle (f, f) \rangle \subseteq I \). Theorem 3.11 there exists a non-zero pseudo-polynomial \((r, f)\), which is minimal with respect to \( G \) such that \((f, f) \xrightarrow{G} (r, f)\). If \( r \neq 0 \), then Lemma 3.10 shows that \( \text{Lt}(r, f) \subseteq \text{Lt}(G) = \text{Lt}(I) \). As \( \langle (f, f) \rangle \subseteq I \) we also have \( \langle (r, f) \rangle \subseteq I \) by Lemma 3.7 and hence \( \text{Lt}(r, f) \subseteq \text{Lt}(I) \), a contradiction. Thus \((f, f) \xrightarrow{G} 0\).

(ii) \( \Rightarrow \) (iii): Clear from Remark 3.8.

(iii) \( \Rightarrow \) (i): We just have to show that \( \text{Lt}(I) \subseteq \text{Lt}(G) \). Let \( \langle (f, f) \rangle \subseteq I \) and write \( f = \sum_{i=1}^{l} h_i g_i \) with \( h_i \in (g_i f^{-1})[x] \) and \( \text{lm}(f) = \max_{1 \leq i \leq l}(\text{lm}(h_i g_i)) \). Thus \( \text{lt}(f) = \sum_{i \in J} \text{lt}(h_i) \text{lt}(g_i) \), where \( J = \{i \in J \mid \text{lm}(g_i h_i) = \text{lm}(f_i)\} \). Since \( \text{lt}(h_i) \in g_i f^{-1}[x] \), for every \( \alpha \in \mathfrak{f} \) we therefore have

\[
\alpha \text{lt}(f) \subseteq \sum_{i \in J} g_i[x] \text{lt}(g_i),
\]

that is, \( \text{Lt}(f, f) = f[x] \text{lt}(f) \subseteq \sum_{i \in I} \text{Lt}(g_i, g_i) = \text{Lt}(G) \).

(iv) \( \Leftrightarrow \) (i): This follows from the fact that

\[
\text{Lt}(G) = \text{Lt}(\{a_{ij} g_i \mid 1 \leq i \leq l, 1 \leq j \leq n_i\}).
\]

**Definition 3.13.** Let \( I \) be an ideal of \( R[x] \). A family \( G \) of pseudo-polynomials of \( R[x] \) is called a pseudo-Gröbner basis of \( I \) (with respect to \( < \)), if \( G \) satisfies any of the equivalent conditions of Theorem 3.12.

**Remark 3.14.**

(i) If one replaces pseudo-polynomials by ordinary polynomials in Theorem 3.12, one recovers the notion of Gröbner basis of an ideal \( I \subseteq G \).
(ii) Since $R$ is Noetherian, an ideal $I$ of $R[x]$ has a Gröbner basis $\{g_1, \ldots, g_l\}$ in the ordinary sense [AL94, Corollary 4.1.17]. Recall that his means that $\text{Lt}(g_1, \ldots, g_l) = \langle \text{lt}(g_1), \ldots, \text{lt}(g_l) \rangle = \text{Lt}(I)$. As $\text{Lt}(g_1, \ldots, g_l)$ is equal to the leading term ideal of $G = \{(g_i, R) \mid 1 \leq i \leq l\}$, we see at once that $I$ also has a pseudo-Gröbner basis.

(iii) In view of Theorem 3.12 (iv), the notion of pseudo-Gröbner basis is a generalization of [AL97] from the univariate to the multivariate case.

Recall that a generating set $G$ of an ideal $I$ in $R[x]$ is called a strong Gröbner basis, if for every $f \in I$ there exists $g \in G$ such that $\text{lt}(g)$ divides $\text{lt}(f)$. It is well known, that in case of principal ideal rings, a strong Gröbner basis always exists. We show that when passing to pseudo-Gröbner bases, we can recover this property for Dedekind domains.

**Definition 3.15.** Let $(f, \bar{f})$ and $(g, \bar{g})$ be two non-zero pseudo-polynomials in $R[x]$. We say that $(f, \bar{f})$ divides $(g, \bar{g})$ if $g \bar{f}[x] \subseteq \bar{f}[x]$. Let $I \subseteq R[x]$ be an ideal. A set $G = \{(g_i, \bar{g}_i) \mid 1 \leq i \leq l\}$ of pseudo-polynomials in $I$ is a strong pseudo-Gröbner basis, if for every pseudo-polynomial $(f, \bar{f})$ in $I$ there exists $i \in \{1, \ldots, r\}$ such that $\text{Lt}(g_i, \bar{g}_i)$ divides $\text{Lt}(f, \bar{f})$.

We now fix non-zero pseudo-polynomials $G = \{(g_i, \bar{g}_i) \mid 1 \leq i \leq l\}$. For a subset $J \subseteq \{1, \ldots, r\}$ we define $x_J = \text{lcm}(\text{lm}(g_i) \mid i \in J)$ and $c_J = \sum_{i \in J} \bar{g}_i \text{lc}(g_i)$. Let $1 = \sum_{i \in J} a_i \text{lc}(g_i)$ with $a_i \in \mathbb{C}_J^{-1} \bar{g}_i$ for $i \in J$ and define $f_J = \sum_{i \in J} a_i \frac{x_J}{\text{lm}(g_i)} g_i$. Note that by construction $\text{lt}(f_J) = x_J$. Finally recall that $J \subseteq \{1, \ldots, r\}$ is saturated, if for $i \in \{1, \ldots, r\}$ with $\text{lm}(g_i) \mid x_J$ we have $i \in J$.

**Theorem 3.16.** Assume that $G = \{(g_i, \bar{g}_i) \mid 1 \leq i \leq l\}$ is a pseudo-Gröbner basis of the ideal $I \subseteq R[x]$. Then

$\{(f_J, c_J) \mid J \subseteq \{1, \ldots, r\} \text{ saturated}\}$

is a strong pseudo-Gröbner basis of $I$.

**Proof.** Let $(f, \bar{f})$ be a non-zero pseudo-polynomial in $I$ and let $J = \{i \in \{1, \ldots, r\} \mid \text{lm}(g_i) \text{ divides } \text{lm}(f)\}$. Then $J$ is saturated and since $G$ is a pseudo-Gröbner basis of $I$ we have

$\text{lc}(f, \bar{f}) = \text{lc}(f) \bar{f} \subseteq \sum_{i \in J} \text{lc}(g_i) \bar{g}_i = c_J = \text{lc}(f_J c_J)$.

Furthermore $\text{lm}(f_J) = x_J \mid \text{lm}(f)$ and thus $(f_J, c_J)$ divides $(f, \bar{f})$ by Lemma 3.3. \qed

**Corollary 3.17.** Every ideal $I$ of $R[x]$ has a strong pseudo-Gröbner basis.

4. **SYZYGIES**

We already saw in Remark 3.14 (ii), that the existence of pseudo-Gröbner basis is a trivial consequence of the fact the Gröbner bases exists whenever the coefficient ring is Noetherian. The actual usefulness of pseudo-polynomials come from the richer structure of their syzygies, which can be used to characterize and compute Gröbner bases (see [Möll88]). In this section we will show that, similar to the case of principal ideal rings, the syzygy modules of pseudo-polynomials have a basis corresponding to generalized $S$-polynomials.

4.1. **Generating sets.** Consider a family $G = \{(g_i, \bar{g}_i) \mid 1 \leq i \leq l\}$ of non-zero pseudo-polynomials. As $G = \sum_{1 \leq i \leq l} \bar{g}_i[x] g_i$, the map

$\varphi: \bar{g}_1[x] \times \cdots \times \bar{g}_l[x] \to I, \quad (h_1, \ldots, h_n) \mapsto \sum_{i=1}^{l} h_i g_i$
is a well-defined surjective morphism of $R[x]$-modules.

**Definition 4.1.** With the notation of the preceding paragraph we call ker$(\varphi)$ the syzygies of $G$ and denote it by Syz$(G)$. A pseudo-syzygy of $G$ is a pseudo-element of Syz$(G)$, that is, a pair $((h_1, \ldots, h_l), \mathbf{h})$ consisting of polynomials $(h_1, \ldots, h_n) \in K[x]^l$ such that $\mathbf{h} \cdot (h_1, \ldots, h_l) \subseteq$ Syz$(G)$. Equivalently, $\sum_{1 \leq i \leq l} h_i g_i = 0$ and $\mathbf{h} g_i \subseteq g_i[x]$ for all $1 \leq i \leq l$.

Assume that the polynomials $g_1, \ldots, g_l$ are terms. Then we call the pseudo-syzygy $((h_1, \ldots, h_l), \mathbf{h})$ homogeneous if $h_i$ is a term for $1 \leq i \leq l$ and there exists $\alpha \in \mathbb{N}^n$ with $\text{lm}(h_i g_i) = x^\alpha$ for all $1 \leq i \leq l$.

In the following we will denote by $e_i \in K[x]^l$ the element with components $(\delta_{ij})_{1 \leq j \leq l}$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $j \neq i$.

**Lemma 4.2.** Let $G = \{(g_i, g_i) \mid 1 \leq i \leq l\}$ be non-zero pseudo-polynomials. Then Syz$(G)$ has a finite generating set of homogeneous pseudo-syzygies.

**Proof.** Since $R$ is Noetherian, so is $R[x]$ by Hilbert’s basis theorem. In particular $R[x]^l$ is a Noetherian $R[x]$-module. Since the $g_i$ are fractional $R$-ideals, there exists $\alpha \in R$ such that $\alpha g_i \subseteq R$ for all $1 \leq i \leq l$. In particular $g_i[x] \subseteq (1/\alpha)(R[x])^l \cong R[x]^l$ is a Noetherian $R[x]$-module as well. Thus the $R[x]$-submodule Syz$(G)$ is finitely generated. A standard argument shows that Syz$(G)$ is generated by finitely many homogeneous syzygies $v_1, \ldots, v_m \in$ Syz$(G)$. Hence Syz$(G) = \langle (v_1, R), \ldots, (v_m, R) \rangle$ is generated by finitely many homogeneous pseudo-syzygies.

We can now characterize pseudo-Gröbner bases in terms of syzygies.

**Theorem 4.3.** Let $G = \{(g_i, g_i) \mid 1 \leq i \leq l\}$ be non-zero pseudo-polynomials of $R[x]$ and $B$ a finite generating set of homogeneous syzygies of Syz$(\text{lt}(g_1, g_1), \ldots, \text{lt}(g_l, g_l))$. Then the following are equivalent:

(i) $G$ is a Gröbner basis of $(G)$.

(ii) For all $((h_1, \ldots, h_l), \mathbf{h}) \in B$ we have $(\sum_{1 \leq i \leq l} h_i g_i, \mathbf{h}) \xrightarrow{G} (0, \mathbf{h})$.

**Proof.** (i) $\Rightarrow$ (ii): Since $\mathbf{h} g_i \subseteq g_i[x]$ by definition, we know that $\mathbf{h}(\sum_{1 \leq i \leq l} h_i g_i) \subseteq \sum_{1 \leq i \leq l} g_i[x] \cdot g_i = \langle G \rangle$. Hence the element reduces to zero by Theorem 3.12 (ii).

(ii) $\Rightarrow$ (i): We show that $G$ is a Gröbner basis by verifying Theorem 3.12 (iii). To this end, let $(f, f)$ be a pseudo-polynomial contained in $\langle G \rangle$. By Lemma 2.4 there exist elements $u_i \in (g_i f^{-1})[x]$, $1 \leq i \leq l$, such that $f = \sum_{1 \leq i \leq l} u_i g_i$. We need to show that there exists such a linear combination with $\text{lm}(f) = \text{max}_{1 \leq i \leq l} \text{lm}(u_i g_i)$. Let $x^\alpha = \text{max}_{1 \leq i \leq l} \text{lm}(u_i g_i)$ with $\alpha \in \mathbb{N}^n$, and assume that $x^\alpha > \text{lm}(f)$. We will show that $f$ has a representation with strictly smaller degree. Denote by $S$ the set $\{1 \leq i \leq l \mid \text{lm}(u_i g_i) = x^\alpha\}$. As $x^\alpha > \text{lm}(f)$ we necessarily have $\sum_{1 \leq i \leq l} \text{lt}(u_i) \cdot \text{lt}(g_i) = 0$. In particular $(\sum_{i \in S} e_i \text{lt}(u_i), f)$ is a homogeneous pseudo-syzygy of Syz$((\text{lt}(g_1, g_1), \ldots, \text{lt}(g_l, g_l))$ (since $f \cdot \text{lt}(u_i) \subseteq g_i[x]$).

Let now $B = ((h_{ij}, \ldots, h_{ij}), \mathbf{h})$, $1 \leq j \leq r$ be the finite generating set of homogeneous pseudo-syzygies. By Lemma 2.4 we can find $f_j \in (\mathbf{h} f^{-1})[x]$ with

$$\sum_{i \in S} e_i \text{lt}(u_i) = \sum_{j=1}^r f_j \sum_{i=1}^l e_i h_{ij}.$$
Since each \( \text{lt}(u_i) \) is a term, we may assume that each \( f_j \) is also a term. Thus for all \( 1 \leq i \leq l, 1 \leq j \leq r \) we also have

\[
x^\alpha = \text{lm}(u_i g_i) = \text{lm}(u_i) \text{lm}(g_i) = \text{lm}(f_j) \text{lm}(h_{ij}) \text{lm}(g_i).
\]

whenever \( f_j h_{ij} \) is non-zero. By assumption, for all \( 1 \leq j \leq r \) the pseudo-polynomial \( (\sum_{1 \leq i \leq l} h_{ij} g_i, b_j) \) reduces to zero with respect to \( G \). Hence by Theorem 3.11 we can find \( v_{ij} \in (g, h_j^{-1})[x] \), \( 1 \leq i \leq l \), such that \( \sum_{1 \leq i \leq l} h_{ij} g_i = \sum_{1 \leq i \leq l} v_{ij} g_i \) and

\[
\max_{1 \leq i \leq l} \text{lm}(v_{ij} g_i) = \text{lm} \left( \sum_{i=1}^{l} h_{ij} g_i \right) < \max_{1 \leq i \leq l} \text{lm}(h_{ij} g_i).
\]

The last inequality follows from \( \sum_{1 \leq i \leq l} h_{ij} \text{lt}(g_i) = 0 \). For the element \( f \) we started with this implies

\[
f = \sum_{i=1}^{l} u_i g_i = \sum_{i \in S} \text{lt}(u_i) g_i + \sum_{i \in S} (u_i - \text{lt}(u_i)) g_i + \sum_{i \notin S} u_i g_i.
\]

The first term is equal to

\[
\sum_{i \in S} \text{lt}(u_i) g_i = \sum_{j=1}^{r} f_j \sum_{i=1}^{l} h_{ij} g_i = \sum_{j=1}^{r} \sum_{i=1}^{l} f_j h_{ij} g_i = \sum_{j=1}^{r} \sum_{i=1}^{l} f_j v_{ij} g_i = \sum_{i=1}^{l} \sum_{j=1}^{r} f_j v_{ij} g_i.
\]

Now \( f_j \in (h_j f^{-1})[x] \), \( v_{ij} \in (g, h_j^{-1})[x] \), hence \( f_j v_{ij} \in (g, f^{-1})[x] \). Moreover from (1) we have

\[
\max_{i,j} \text{lm}(f_j) \text{lm}(v_{ij}) \text{lm}(g_i) < \max_{i} \max_{j} \text{lm}(f_j) \text{lm}(h_{ij} g_i) = x^\alpha.
\]

Thus we have found polynomials \( \tilde{u}_i \in (g, f^{-1})[x], 1 \leq i \leq l \), such that \( \max_{1 \leq i \leq l} \text{lm}(\tilde{u}_i g_i) < \max_{1 \leq i \leq l} \text{lm}(u_i g_i) \) and \( f = \sum_{1 \leq i \leq l} \tilde{u}_i g_i \). \( \square \)

**Proposition 4.4.** Let \( (g_i, \tilde{g}_i)_{1 \leq i \leq l} \) be non-zero pseudo-polynomials of \( R[x] \) and \( (a_i)_{1 \leq i \leq l} \in (K^\times)^l \). Consider the map \( \Phi: K[x]^l \to K[x]^l, \quad \sum_{1 \leq i \leq l} a_i h_i \mapsto \sum_{1 \leq i \leq l} e_i \frac{h_i}{a_i} \). Then the following hold:

(i) The restriction of \( \Phi \) induces an isomorphism

\[
\text{Syz}((g_1, \tilde{g}_1), \ldots, (g_l, \tilde{g}_l)) \to \text{Syz} \left( (a_1 g_1, \frac{g_1}{a_1}), \ldots, (a_l g_l, \frac{g_l}{a_l}) \right)
\]

of \( R[x] \)-modules.

(ii) If \((h, \tilde{h})\) is a pseudo-syzygy of \( \text{Syz}((g_i, \tilde{g}_i) | 1 \leq i \leq l) \), then \( \Phi(h, \tilde{h}) \) is a pseudo-syzygy of \( \text{Syz}((a_i g_i, \frac{g_i}{a_i}) | 1 \leq i \leq l) \) and \( \Phi(\tilde{h}, h) \) is \( \Phi(h, \tilde{h}) \).

**Proof.** (i): The map \( \Phi \) is clearly \( K[x] \)-linear. We now show that the image of the syzygies \( \text{Syz}((g_i, \tilde{g}_i)_{1 \leq i \leq l}) \) under \( \Phi \) is contained in \( \text{Syz}((a_i g_i, \frac{g_i}{a_i})_{1 \leq i \leq l}) \). To this end let \( (h_1, \ldots, h_l) \in \text{Syz}((g_i, \tilde{g}_i)_{1 \leq i \leq l}) \), that is, \( \sum_{1 \leq i \leq l} h_i g_i = 0 \) and \( h_i \in g_i[x] \). But then \( \sum_{1 \leq i \leq l} \frac{h_i}{a_i} a_i g_i = 0 \) and \( \frac{h_i}{a_i} \in (\frac{g_i}{a_i})[x] \), that is, \( \frac{h_1}{a_1}, \ldots, \frac{h_l}{a_l} \in \text{Syz}((a_i g_i, \frac{g_i}{a_i})_{1 \leq i \leq l}) \). As the inverse map is given by \( (h_1, \ldots, h_l) \mapsto (a_1 h_1, \ldots, a_l h_l) \), the claim follows. (ii): Follows at one from (i)\. \( \square \)
4.2. Buchberger’s algorithm.

**Theorem 4.5.** Let \((a_i x^{a_i}, g_i)_{1 \leq i \leq l}\) be non-zero pseudo-polynomials, where each polynomial is a term. For \(1 \leq i, j \leq l\) we define the pseudo-element

\[
s_{ij} = \left( \frac{\lcm(x^{a_i}, x^{a_j})}{x^{a_i}} e_i - \frac{\lcm(x^{a_i}, x^{a_j})}{x^{a_j}} e_j \right), (a_i g_i \cap \alpha_j g_j)
\]

of \(K[x]\) and for \(1 \leq k \leq l\) we set \(S_k = \text{Syz}((a_i x^{a_i}, g_i)_{1 \leq i \leq k})\). Then the following hold:

(i) For \(1 \leq i, j \leq l, i \neq j\), the syzygies \(\text{Syz}((a_i x^{a_i}, g_i), (\alpha_j x^{a_j}, g_j))\) are generated by \(s_{ij}\).

(ii) If \(B_{k-1}\) is a generating set of pseudo-generators for \(S_{k-1}\), then

\[
B = \{ ((h, 0), h) | (h, h) \in B_{k-1} \} \cup \{ s_{ik} | 1 \leq i \leq k - 1 \}
\]

is a generating set of pseudo-generators for \(S_k\).

**Proof.** By Proposition 4.4 we are reduced to the monic case, that is, \(a_i = 1\) for \(1 \leq i \leq l\).

(i): It is clear that \(s_{ij}\) is a pseudo-syzygy of \((x^{a_i}, g_i), (x^{a_j}, g_j)\). Let now \(((h_i, h_j), h)\) be a homogeneous pseudo-syzygy with \(h_i = b_i x^{\beta_i}, h_j = b_j x^{\beta_j}, h h_i \subseteq g_i\) and \(h h_j \subseteq g_j\).

We may further assume that \(b_i \neq b_j\). In particular \(x^{a_i} x^{x^\beta_i} = x^{a_j} x^{\beta_j}\) and we can write \(x^{\beta_i} = \lcm(x^{a_i}, x^{a_j})/x^{a_i} \cdot x^\beta, x^{\beta_j} = \lcm(x^{a_i}, x^{a_j})/x^{a_j} \cdot x^\beta\) for some monomial \(x^\beta\). We obtain

\[
(h_i, h_j) = x^\beta \left( b_i \frac{\lcm(x^{a_i}, x^{a_j})}{x^{a_i}}, b_j \frac{\lcm(x^{a_i}, x^{a_j})}{x^{a_j}} \right) = x^\beta b_i \left( \frac{\lcm(x^{a_i}, x^{a_j})}{x^{a_i}}, \frac{\lcm(x^{a_i}, x^{a_j})}{x^{a_j}} \right),
\]

where the last equality follows from \(b_i + b_j = 0\). As \(b_i h \subseteq g_i, b_j h \subseteq g_j\), we obtain \(b_i h \subseteq g_i \cap g_j\). Thus \(\{(h_i, h_j), h\}\) is a pseudo-syzygy of \(S_{k-1}\), we can assume that \(b_k \neq 0\). Let \(J = \{i | 1 \leq i \leq k - 1, b_i \neq 0\}\). Since \((h, h)\) is homogeneous, we have \(x^{\beta_i} x^{a_i} = x^\beta\) for all \(i \in J \cup \{k\}\) and in particular \(\lcm(x^{a_i}, x^{a_k}) = x^\beta\) for all \(i \in J\).

Furthermore we have \(b_k = -\sum_{i \in J} b_i \sum_{i \in J} b_i R\) and hence \(h b_k \subseteq \sum_{i \in J} h b_i \subseteq \sum_{i \in J} g_i\). Since at the same time it holds that \(h b_k \subseteq g_k\), we conclude that \(h b_k \subseteq (\sum_{i \in J} g_i) \cap g_k = \sum_{i \in J} (g_i \cap g_k)\). Hence there exist \(c_i \in (g_i \cap g_k) b^{-1}, i \in J\), such that \(b_k = -\sum_{i \in J} c_i\). For \(1 \leq i, j \leq k\) let us denote \(\lcm(x^{a_i}, x^{a_j})\) by \(x^{\alpha_{ij}}\). Now as \(x^\beta / x^{\alpha_{ik}} \cdot x^{a_k} / x^{a_i} = x^\beta\) we obtain

\[
b_k x^{\beta_k} e_k = \sum_{i \in J} c_i x^{\beta_{ik}} e_i x^{a_k} e_k = \sum_{i \in J} -c_i \frac{x^\beta}{x^{a_k}} \frac{x^{a_k}}{x^{\alpha_{ik}}} e_i x^{a_k} e_k
\]

\[
= \sum_{i \in J} c_i \frac{x^\beta}{x^{a_k}} \frac{x^{a_k}}{x^{a_i}} \left( e_i x^{a_k} - x^{a_k} e_k \right) - \sum_{i \in J} c_i x^{\beta_i} e_i.
\]

Hence

\[
h = \sum_{i=1}^l e_i b_i x^{\beta_i} = \sum_{i=1}^{l-1} e_i b_i x^{\beta_i} - \sum_{i \in J} c_i x^{\beta_i} e_i + \sum_{i \in J} c_i \frac{x^\beta}{x^{a_k}} \frac{x^{a_k}}{x^{a_i}} \left( e_i x^{a_k} - x^{a_k} e_k \right).
\]

We set

\[
\tilde{h} = \sum_{i=1}^{l-1} e_i b_i x^{\beta_i} - \sum_{i \in J} c_i x^{\beta_i} e_i \quad \text{and} \quad \tilde{z} = \sum_{i \in J} c_i \frac{x^\beta}{x^{a_k}} \frac{x^{a_k}}{x^{a_i}} \left( e_i x^{a_k} - x^{a_k} e_k \right).
\]
By construction, for all $i \in J$ we have $h_i \subseteq g_i \cap g_k$. Together with $J \subseteq \{1, \ldots, k-1\}$ this implies $\langle \tilde{h}, h \rangle_{R[x]} \subseteq \langle s_{ik} \mid 1 \leq i \leq k-1 \rangle_{R[x]}$. Let $\Phi: \sum_{1 \leq i \leq c} e_i h_i \mapsto \sum_{1 \leq i \leq I} h_i g_i$. As $h, \tilde{h} \in \text{ker}(\Phi)$, the same holds for $\tilde{h}$. Using again the property $h_i \subseteq g_i \cap g_k \subseteq g$, we conclude that $(\tilde{h}, h)$ is a pseudo-syzygy of $(g_i, g_k)_{1 \leq i \leq k-1}$. In particular $\langle (\tilde{h}, 0) \rangle_{R[x]} \subseteq R[x]$. Invoking again Lemma 4.2, this proves the claim.

**Definition 4.6.** Let $(f, f), (g, g)$ be two non-zero pseudo-polynomials of $R[x]$. We call

$$
\left( \frac{\text{lcm}(\text{lc}(f), \text{lc}(g))}{\text{lc}(f)} f - \frac{\text{lcm}(\text{lc}(f), \text{lc}(g))}{\text{lc}(g)} g, \text{lc}(f) g \right)
$$

the S-polynomial of $(f, g), (g, g)$ and denote it by spoly($((f, f), (g, g))$).

We can now give the analogue of the classical Buchberger criterion in the case of Dedekind domains.

**Corollary 4.7.** Let $G = \{(g_i, g_i) \mid 1 \leq i \leq l\}$ be non-zero pseudo-polynomials of $R[x]$. Then $G$ is a Gröbner basis of $\langle G \rangle$ if and only if spoly($((g_i, g_i))$) reduces to 0 modulo $G$ for all $1 \leq i < j \leq l$.

**Proof.** Applying Theorem 4.6 (ii) inductively using (i) as the base case shows that the set $\{s_{ij} \mid 1 \leq i < j \leq l\}$ is a of homogeneous pseudo-syzygies generating $\text{Syz}(G)$. The claim now follows from Theorem 4.3.

**Algorithm 4.8.** Given a family $F = (f_i, f_j)_{1 \leq i \leq j}$ of non-zero pseudo-polynomials, the following steps return a Gröbner basis $G$ of $\langle F \rangle$:

(i) We initialize $\tilde{G}$ as $\{(f_i, f_j) \mid 1 \leq i < j \leq l\}$ and $G = F$.

(ii) While $\tilde{G} \neq \emptyset$, repeat the following steps:

(a) Pick $((f, f), (g, g)) \in \tilde{G}$ and compute $(h, h)$ minimal with respect to $G$ such that spoly $((f, f), (g, g)) \rightarrow_{+} (h, h)$.

(b) If $h \neq 0$, set $G = G \cup \{(f, f), (h, h)\} \in G$ and $G = G \cup \{(h, h)\}$.

(iii) Return $G$.

**Algorithm 4.8 is correct.** By Corollary 4.7 it is to show that the algorithm terminates. But termination follows as in the field case by considering the ascending chain of leading term ideals $\text{Lt}(G)$ (in the Noetherian ring $R[x]$) and using Lemma 3.10.

4.3. **Product criterion.** For Gröbner basis computations a bottleneck of Buchberger’s algorithm is the reduction of the $S$-polynomials and the number of $S$-polynomials one has to consider. Buchberger himself gave criteria under which certain $S$-polynomials will reduce to 0. In [Möl88, Lic12] they have been adapted to coefficient rings that are principal ideal rings and Euclidean domains respectively. We will now show that the product criterion can be easily translated to the setting of pseudo-Gröbner bases. Recall that in the case $R$ is a principal ideal domain, the product criterion reads as follows: If $f, g$ are non-zero pseudo-polynomials in $R[x]$ such that $\text{GCD}(\text{lc}(f), \text{lc}(g)) = 1$ and $\text{GCD}(\text{lm}(f), \text{lm}(g)) = 1$, then the $S$-polynomial spoly($((f, f), (g, g))$ reduces to zero modulo $\{(f, f), (g, g)\}$.

**Theorem 4.9.** Let $(f, f), (g, g)$ be pseudo-polynomials of $R[x]$ such that $\text{lm}(f)$ and $\text{lm}(g)$ are coprime in $K[x]$, $\text{lc}(f)$ and $\text{lc}(g)$ are coprime ideals of $R$. Then the $S$-polynomial spoly($((f, f), (g, g))$ reduces to 0 modulo $\{(f, f), (g, g)\}$.

**Proof.** Denote by $f'$ and $g'$ the tails of $f$ and $g$ respectively. We consider three cases.

In the first case, let both $f$ and $g$ be terms. Then their $S$-polynomial will be 0 by definition.
Consider next the case in which \( f \) is a term and \( g \) is not. Then a quick calculation shows that
\[
(s, s) = \text{spoly}((f, f), (g, g)) = \left(-\frac{1}{\text{lc}(f) \text{lc}(g)} g f, \text{lc}(f)f \cdot \text{lc}(g)g\right).
\]
We want to show that \((s, s)\) reduces modulo \(\{(f, f)\}\). Since \(\text{lm}(f)\) divides \(\text{lm}(h)\) by definition it is sufficient to show that \(\text{lc}(s, s) \subseteq \text{lc}(f, f)\), which is equivalent to \(\text{lc}(g') \text{lc}(f)fg \subseteq \text{lc}(f)f\). But this follows from \(\text{lc}(g')g \subseteq \mathbb{R}\). Hence \((s, s)\) reduces modulo \((f, f)\) to
\[
\left(s - \frac{\text{lt}(g')}{\text{lc}(g) \text{lc}(f)} f, \text{lc}(f)f \cdot \text{lc}(g)g\right) = \left(-\frac{1}{\text{lc}(f) \text{lc}(g)} (g' - \text{lt}(g')) f, \text{lc}(f)f \cdot \text{lc}(g)g\right).
\]
Applying this procedure recursively, we see that \((s, s)\) reduces to 0 modulo \(\{(f, f)\}\).

Now consider the case, where \(f\) and \(g\) are both not terms, that is, \(f' \neq 0 \neq g'\). Then the \(S\)-polynomial of \((f, f)\) and \((g, g)\) is equal to
\[
(s, s) = \left((\frac{\text{lt}(g)}{\text{lc}(f)} f - \frac{\text{lt}(f)}{\text{lc}(g)} g), \text{lc}(f)f \text{lc}(g)g\right) = \left(\frac{1}{\text{lc}(f) \text{lc}(g)} (f'g - g'f), \text{lc}(f)f \text{lc}(g)g\right).
\]
Since \(\text{lm}(f)\) and \(\text{lm}(g)\) are coprime, we have \(\text{lm}(f')g \neq \text{lm}(g')f\) and therefore \(\text{lm}(s)\) is either \(\text{lm}(f')g\) or \(\text{lm}(g')f\). In particular \(\text{lm}(s)\) is either a multiple of \(\text{lm}(f)\) or \(\text{lm}(g)\). If \(\text{lm}(s) = \text{lm}(f')g\) then \(\text{lc}(s) = \text{lc}(g')/\text{lc}(g)\) and \(\text{lc}(s, s) = \text{lc}(g')\text{lc}(f)\text{lc}(f)g\). As in third case, \((s, s)\) reduces to
\[
\left(-\frac{1}{\text{lc}(f) \text{lc}(g)} (f'g - g'f) - \frac{\text{lt}(g')}{\text{lc}(g) \text{lc}(f)} f, \text{lc}(f)f \text{lc}(g)g\right) = \left(-\frac{1}{\text{lc}(f) \text{lc}(g)} (f'g - (g' - \text{lt}(g'))f), \text{lc}(f)f \text{lc}(g)g\right),
\]
and similar in the other case. Note that again, the leading monomial of \((f'g - (g' - \text{lt}(g'))f)\) is a multiple of \(\text{lm}(f)\) and \(\text{lm}(g)\). Inductively this shows that \((s, s)\) \((f, f), (g, g)\), + 0. \(\square\)

5. COEFFICIENT REDUCTION

Although in contrast to \(\mathbb{Q}[x]\) the naive Gröbner basis computation of an ideal \(I\) of \(\mathbb{Z}[x]\) is free of denominators, the problem of quickly growing coefficients is still present. In case a non-zero element \(N \in I \cap \mathbb{Z}\) is known this problem can be avoided: By adding \(N\) to the generating set under consideration, all intermediate results can be reduced modulo \(N\), leading to tremendous improvements in runtime, see [EPP18].

In this section we will describe a similar strategy for the computation of pseudo-Gröbner bases in case the coefficient ring is the ring of integers of a finite number field. Although this is quite similar to the integer case, we now have to deal with the growing size of the coefficients of polynomials themselves as well as with the size of the coefficient ideals.

5.1. Admissible reductions. We first describe the reduction operations that are allowed during a Gröbner basis computation for arbitrary Dedekind domains.

Proposition 5.1. Let \(R\) be a Dedekind domain, and \((f, f)\) a non-zero pseudo-polynomials of \(R[x]\).

(i) If \((g, g)\) is a pseudo-polynomial of \(R[x]\) with \(ff = gg\), then \((f, f)\) reduces to 0 modulo \((g, g)\).

(ii) Write \(f = \sum_{1 \leq i \leq d} c_{\alpha_i} x^{\alpha_i}\) with \(c_{\alpha_i} \neq 0\). Assume that \(g = \sum_{1 \leq i \leq d} \tilde{c}_{\alpha_i} x^{\alpha_i} \in R[x]\) is a polynomial and \(\mathfrak{N}\) a fractional ideal of \(R\) such that \(c_{\alpha_i} - \tilde{c}_{\alpha_i} \in \mathfrak{N}f^{-1}\) for \(1 \leq i \leq d\). Then \(f\) reduces to 0 modulo \(((g, f), (1, \mathfrak{N}))\).
Proof. (i): By assumption $\text{lm}(f) = \text{lm}(g)$. Moreover, as $\frac{\text{lc}(f)}{\text{lc}(g)} \in \mathfrak{g}^{-1}$ we see that $(f, \bar{f})$ reduces to
\[
\left(f - \frac{\text{lc}(f) \text{lm}(f)}{\text{lc}(g) \text{lm}(g)}g, \bar{f}\right) = (0, \bar{f}).
\]

(ii): We first consider the case that $\text{lm}(f) \neq \text{lm}(g)$. By assumption this implies that $\text{lc}(f) \in \mathfrak{N}^{-1}$ and $(f, \bar{f})$ reduces to $(f - \text{lc}(f) \text{lm}(f), \bar{f})$ modulo $(1, \mathfrak{N})$. Since we also have $(f - \text{lc}(f) \text{lm}(f)) - g \in \mathfrak{N}^{-1}[x]$, we now may assume that $(f - \text{lc}(f) \text{lm}(f)) = 0$, in which case we are finished, or $\text{lm}(f) = \text{lm}(g)$. In the latter case, we use $\text{lc}(f) - \text{lc}(g) \in \mathfrak{N}^{-1}$ and $\text{lc}(f) = 1 \cdot \text{lc}(g) + (\text{lc}(f) - \text{lc}(g)) \cdot 1$ to conclude that $(f, \bar{f})$ reduces to $(\bar{f}, \bar{f})$ modulo $\{(g, \bar{f}), (1, \mathfrak{N})\}$, where $\bar{f} = f - g - (\text{lc}(f) - \text{lc}(g)) \text{lm}(f)$. Since the polynomial $\bar{f}$ satisfies $\bar{f} \in \mathfrak{N}^{-1}[x]$, it reduces to 0 modulo $(1, \mathfrak{N})$. $\square$

Since our version of Buchberger’s algorithm rests on S-polynomials reducing to 0 (see Corollary 4.7), the previous result immediately implies the correctness of the following modification of Algorithm 4.8.

Corollary 5.2. Assume that $F = (f_i, \bar{f}_i)_{1 \leq i \leq l}$ is family of pseudo-polynomials, such that $(F)$ contains a non-zero ideal $\mathfrak{N}$ of $R$. After adding $(1, \mathfrak{N})$ to $F$, in Algorithm 4.8 include the following Step after (a):

(a') Let $(g_1, \bar{g}_1)$ be a non-zero pseudo-polynomial with $g_1g_1 = h\bar{h}$. Now let $g_1 = \sum_i c_{\alpha_i}x^{\alpha_i}$ with $c_{\alpha_i} \neq 0$. Find a polynomial $g_2 = \sum_i c_{\alpha_i}x^{\alpha_i}$, with $c_{\alpha_i} - \bar{c}_{\alpha_i} \in \mathfrak{N}g^{-1}[x]$ for all $i$ and replace $(h, \bar{h})$ by $(g_2, \bar{g}_1)$.

Then the resulting algorithm is still correct.

5.2. The case of rings of integers. It remains to describe how to use the previous results to bound the size of the intermediate pseudo-polynomials. Since this question is meaningless in the general settings of Dedekind domains, we now restrict to the case where $R$ is the ring of integers of a finite number field $K/Q$. We assume that $I \subseteq R[x]$ is an ideal which contains non-zero ideal $\mathfrak{N}$ of $R$. In view of Proposition 5.1, we want to solve the following two problems for a given non-zero pseudo-polynomial $(f, \bar{f})$ of $R[x]$.

(i) Find a pseudo-polynomial $(g, \bar{g})$ of $R[x]$ with $g$ small such that $\bar{f}f = gg$.
(ii) Find a pseudo-polynomial $(g, \bar{f})$ of $R[x]$, such that $g$ has small coefficients, every monomial of $g$ is a monomial of $f$, and $\bar{f} - g \in \mathfrak{N}^{-1}[x]$.

We will now translate this problem to the setting of pseudo-elements in projective $R$-modules of finite rank, where the analogous problems are already solved in the context of generalized Hermite form algorithms. To this end, let $f = \sum_{1 \leq i \leq d} c_{\alpha_i} x^{\alpha_i}, c_{\alpha_i} \neq 0$, and consider
\[
\pi: K[x] \longrightarrow K^d, \quad \sum_{\alpha \in \mathbb{N}^n} c_{\alpha} x^{\alpha} \longmapsto (c_{\alpha_i})_{1 \leq i \leq d}, \quad \iota: K^d \longrightarrow K[x], \quad (c_{\alpha_i})_{1 \leq i \leq d} \longmapsto \sum_{i=1}^d c_{\alpha_i} x^{\alpha_i}.
\]

Using these $K$-linear maps, we can think of pseudo-polynomials having the same support as $f$ as projective $R$-submodules of $V$ of rank one, that is, as pseudo-elements in $K^d$. Moreover, if $f\pi(f) = gw$ for some $w \in K^d$ and fractional ideal $g$ of $R$, then $\bar{f}f = \bar{g}(w)$. In particular, by setting $v = (v_i)_{1 \leq i \leq d} = \pi(f) \in K^d$, problems (i) and (ii) are equivalent to the following two number theoretic problems:

(i') Find a pseudo-element $(w, \bar{g})$ of $K^d$ with $\bar{g}$ small such that $\bar{f}v = \bar{g}w$.
(ii') Find a pseudo-element $(w, \bar{f})$ of $K^d$, such that $w_i$ is small and $v_i - w_i \in \mathfrak{N}^{-1}$ for all $1 \leq i \leq d$. 

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Hence, we can reduce pseudo-polynomials by applying the following two algorithms to the coefficient ideal and the coefficients respectively. Both are standard tools in algorithmic algebraic number theory, see [BFH17] for a discussion including a complexity analysis.

**Lemma 5.3.** Let \( \mathfrak{N} \) be a non-zero ideal of \( R \).

(i) There exists an algorithm, that given a fractional ideal \( a \) of \( R \) and a vector \( v \in K^d \) determines an ideal \( b \) of \( R \) and a vector \( w \in K^d \) such that \( av = bw \) and the norm \( \#(R/b) \) can be bounded by a constant that depends only on the field \( K \) (and not on \( a \) or \( v \)).

(ii) There exists an algorithm, that given a non-zero ideal \( f \) of \( R \) and an element \( \alpha \) of \( K \), determines an element \( \beta \in K \) such that \( \alpha - \beta \in \mathfrak{N}^{-1} \) and the size of \( \beta \) can be bounded by a constant that depends only on the field \( K \) and the norms \( \#(R/\mathfrak{N}), \#(R/f) \).

**Remark 5.4.** Recall that \( \mathfrak{N} \) is a non-zero ideal of \( R \) such that \( \mathfrak{N} \subseteq I \), where \( I \) is the ideal of \( R[x] \) for which we want to find a pseudo-Gröbner basis. The preceding discussion together with Corollary 5.2 implies that during Buchberger’s algorithm (Algorithm 4.8), we can reduce the intermediate results so that the size of all pseudo-polynomials is bounded by a constant depending only on \( \mathfrak{N} \) and \( K \).

**Remark 5.5.** Assume that \( I \subseteq R[x] \) is an ideal. Then there exists a non-zero ideal \( \mathfrak{N} \) of \( R \) contained in \( I \) if and only if \( K[x] = (I)_{K[x]} \). In case, one can proceed as follows to find such an ideal \( \mathfrak{N} \). Let \( F = (f_i, 1 \leq i \leq l) \) be a generating set of pseudo-polynomials of \( I \). Using classical Gröbner basis computations and the fact that \( 1 \in (I)_{K[x]} \) we can compute \( a_i \in K \), \( 1 \leq i \leq l \), such that \( 1 = \sum_{1 \leq i \leq l} a_i f_i \). Next we determine \( d \in R \) such that \( da_i \in f_i \) for all \( 1 \leq i \leq l \). Then

\[
d = \sum_{i=1}^{l} da_i f_i \subseteq I
\]

and thus the non-zero ideal \( \mathfrak{N} = dR \) satisfies \( \mathfrak{N} \subseteq I \).

6. Applications

We give a few applications of pseudo-Gröbner bases to classical problems from algorithmic commutative algebra as well as to the problem of computing primes of bad reduction.

6.1. Ideal membership and intersections.

**Proposition 6.1.** Let \( I \) be an ideal of \( R[x] \) given by a finite generating set of non-zero (pseudo-)polynomials. There exists an algorithm, that given a polynomial \( f \) respectively a pseudo-polynomial \( (f, \beta) \) decides whether \( f \in I \) respectively \( ((f, \beta)) \subseteq I \).

**Proof.** Since \( f \in I \) if and only if \( ((f, R)) \subseteq I \), we can restrict to the case of pseudo-polynomials. After computing a pseudo-Gröbner basis of \( I \) using Algorithm 4.8, we can use Theorem 3.12 (ii) to decide membership. \( \square \)

Next we consider intersections of ideals, where as in the case of fields we use an elimination ordering.

**Proposition 6.2.** Consider \( R[x, y] \) with elimination order with the \( y \) variables larger than the \( x \) variables. Let \( G = \{(g_i, i) \mid 1 \leq i \leq l\} \) be a pseudo-Gröbner basis of an ideal \( I \subseteq R[x, y] \). Then \( \{(y_i, g_i) \mid g_i \in K[x]\} \) is a pseudo-Gröbner basis of \( I \cap R[x] \).
Proof. Follows from Theorem 3.12 (iv) and the corresponding result for Gröbner bases, see [AL94, Theorem 4.3.6]. □

Corollary 6.3. Let $I$, $J$ be two ideals of $R[x]$ given by finite generating sets of non-zero (pseudo-)polynomials. Then there exists an algorithm that computes a finite generating set of pseudo-polynomials of $I \cap J$.

Proof. This follows from Proposition 6.2 and the classical fact that $I \cap J = \langle wI, (1 - w)J \rangle_{R[x,w]} \cap R[x]$, where $w$ is an additional variable (see [AL94, Proposition 4.3.9]). □

Corollary 6.4. Let $I \subseteq R[x]$ be an ideal. Then there exists an algorithm for computing $I \cap R$.

6.2. Primes of bad reduction. It seems to be well known, that in the case where $R$ is $\mathbb{Z}$, the primes of bad reduction of a variety can be determined by computing Gröbner bases of ideals corresponding to singular loci. Due to the lack of references we give a proof of this folklore result and show how it relates to pseudo-Gröbner bases. Note that this does not determine the primes themselves, since one has to additionally determine the prime factors.

Example 6.5. Let $X = V(f_1, \ldots, f_l)$ and $I$ the ideal of $R[x]$ generated by $f_1, \ldots, f_l$ and the $(n-k)$ minors of $J$. Then $X_p \subseteq A^n_{pR}$ is smooth if and only if $p$ does not divide $I \cap R$.

Proof. The flatness condition implies that $X_p$ has dimension $k$. By the Jacobian criterion ([Liu02, Chapter 4, Theorem 2.14], $X_p$ is smooth if and only if $J_p(p)$ has rank $n-k$ for all $p \in X_p(k_p)$, where $J_p = (\frac{\partial f_i}{\partial x_j})_{1 \leq i \leq l, 1 \leq j \leq n}$ is the Jacobian of $f_1, \ldots, f_l$. Thus $X_p$ is smooth if and only if the ideal of $k_p[x]$ generated by $\bar{f}_1, \ldots, \bar{f}_l$ and the $(n-k)$-minors of $J_p$ is equal to $k_p[x]$. Hence $X_p$ is smooth if and only if the ideal $(I, p)$ of $R[x]$ is equal to $R[x]$. Now $(I, p) \subseteq R[x]$ if and only if there exists a maximal ideal $M$ of $R[x]$ containing $(I, p)$. But in this case the kernel $R \cap M$ of the projection $R \to R[x]/M$ contains $p$ and must therefore be equal to $p$. As $p \subseteq (I, p) \cap R \subseteq M \cap R = p$, the existence of $M$ is equivalent to $(I, p) \cap R = p$, that is, $I \cap R \subseteq p$. □

Combining this with the previous subsection, the primes of bad reduction can be easily characterized with pseudo-Gröbner bases. Note that this does not determine the primes themselves, since one has to additionally determine the prime ideal factors.

Corollary 6.6. Let $X = V(f_1, \ldots, f_k)$ and $I$ the ideal of $R[x]$ generated by $f_1, \ldots, f_l$ and the $(n-k)$ minors of the Jacobian matrix $J$. Let $\{(g_i, g_j) \mid 1 \leq i \leq l\}$ be a pseudo-Gröbner basis of $I$ and $\mathfrak{N} = \sum g_i g_i \subseteq R$, where the sum is over all $1 \leq i \leq l$ such that $g_i \in K$. Then $p$ is a prime of bad reduction of $X$ if and only if $p$ divides $\mathfrak{N}$.

Example 6.7. To have a small non-trivial example, we look at an elliptic curve defined over a number field. Although there are other techniques to determine the primes of bad reduction, we will do so using pseudo-Gröbner bases. Consider the number field $K = \mathbb{Q}(\sqrt{10})$ with ring of integers $\mathcal{O}_K = \mathbb{Z}[a]$, where $a = \sqrt{10}$. Let $E/K$ be the elliptic curve defined by

$$f = y^2 - x^2 + (1728a + 3348)x + (44928a - 324432) \in K[x, y].$$
Note that this is a short Weierstrass equation of the elliptic curve with label 6.1-a2 from the LMFDB ([LMF19]). To determine the places of bad reduction, we consider the ideal
\[ I = \left\langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \subseteq \mathcal{O}_K[x, y]. \]

Applying Algorithm 4.8 we obtain a pseudo-Gröbner basis \( G \), which together with Corollary 6.4 allows us to compute
\[ I \cap R = (940369969152, 437864693760a + 71663616) \subseteq \mathcal{O}_K. \]
The ideal \( I \cap R \) has norm \( 67390312367240773632 = 2^{31} \cdot 3^{22} \) and factors as
\[ I \cap R = (2, a)^{31} \cdot (3, a + 2)^{15} \cdot (3, a + 4)^{7}. \]
Thus the primes of bad reduction are \((2, a), (3, a + 2)\) and \((3, a + 4)\). Note that the conductor of \( E \) is divisible only by \((2, a)\) and \((3, a + 2)\) (the model we chose is not minimal at \((3, a + 4)\)). In fact this can be seen by determining the primes of bad reduction of the model \( y^2 = x^3 + \frac{1}{4}(64a + 124)x + \frac{1}{27}(1664a - 12016) \), which is minimal at \((3, a + 4)\) (computed with MAGMA [BCP97]).

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