Universal statistics of wave functions in chaotic and disordered systems

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Abstract. – We study a new statistics of wave functions in several chaotic and disordered systems: the random matrix model, band random matrix model, the Lipkin model, chaotic quantum billiard and the 1D tight-binding model. Both numerical and analytical results show that the distribution function of a generalized Riccati variable, defined as the ratio of components of eigenfunctions on basis states coupled by perturbation, is universal, and has the form of Lorentzian distribution.

Statistics of eigenfunctions in disordered and chaotic systems are of great interest in many branches of physics such as condensed matter physics, nuclear physics, chemical physics, and in particular, in the new developing field of quantum chaos[1,2,3]. However, although eigenfunctions contain more information than eigenenergies, much less has been studied compared with the eigenenergies.

The mostly studied property is the statistics of wave function amplitude $|\psi(r)|^2$ and two point correlation function $\langle \psi(r_1)\psi(r_2) \rangle$. The former one has been found to be the Porter-Thomas distribution for time-reversal invariant systems whose classical counterpart is chaotic. This is predicted by the Random Matrix Theory[1,2,3], and confirmed numerically[4,5] and experimentally[6]. The later one, which is proportional to $J_0(kd)$ ($J_0$ is the Bessel function of the zeroth order, $k$ the wave vector and $d$ the distance between the two points), was proposed by Berry[6] for a chaotic billiard within an assumption that the wave function is a superposition of plane waves with random coefficients. The two point correlation function has been extended to large separation by Hortikar and Srednicki[7] recently. A beautiful review for the correlations of wave functions in disordered systems has been given recently by Mirlin[8].

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In this Letter, we would like study another interesting statistics, namely the statistics of the Riccati variable of wave functions (see, e.g., [12]). The Riccati variable has been studied in one-dimensional disordered system, where it is related to the reflection coefficient of waves. As we shall see later, the Riccati variable can be easily related to the two-point correlation function, but it contains more information than the correlation function. Furthermore, the Riccati variable gives information about local fluctuation properties of wave functions, and the form of its distribution is related to properties of the perturbation of the physical models. To seek a general universality of this quantity among different quantum systems, our study will be extended to a wide range of random matrix model, disordered and chaotic systems such as the Random Matrix Model (RMM) [13], Band Random Matrix Model (BRMM) [14], the Lipkin model [15] with chaotic behavior in the classical limit, a chaotic quantum billiard model [16] and a 1D tight-binding model in the case of weak disorder [17].

In order to study the chaotic systems, the Riccati variable for the 1D tight-binding model is first extended to a series types of Riccati variable, namely, type I, type II and so on. Then, the statistics for type I Riccati variable is found analytically to be of a Lorentzian form for the type I Riccati variable gives information on properties of chaotic eigenfunctions which can not be supplied by, e.g., the statistics of intensity of wave functions. Usually the perturbation $V$ will be denoted by $H$. Let’s first consider a general case in which the Hamiltonians can be written in the form

$$H = H_0 + \lambda V,$$

where $\lambda$ is a parameter for adjusting the strength of the perturbation. Eigenvectors of $H_0$ and $H$ will be denoted by $|i\rangle$ and $|\alpha\rangle$ in what follows, respectively, i.e.,

$$H_0|i\rangle = E_0^i|i\rangle, \quad H|\alpha\rangle = E_\alpha|\alpha\rangle.$$  \hfill (2)

Usually the perturbation $V$ can be written in the form of $V = \sum_i V_i$ with $V_i = U_i + U_i^\dagger$ and $U_i$ coupling a basis state to one of the other basis states only. In this Letter, we choose the strength of $V_i$ comparable with each other. The component of an eigenstate $|\alpha\rangle$ on a basis state $|i\rangle$ will be denoted by $C_{\alpha i}(= \langle i|\alpha\rangle)$. For simplicity, we consider the case that both $H$ and $H_0$ are time-reversal invariant only, so that $C_{\alpha i}$ are real.

In the 1D tight-binding model the so-called Riccati variable is defined as the ratio of components of eigenfunctions of the nearest sites, where the off-diagonal elements of the Hamiltonian are non-zero for the nearest sits only. A natural extension of the concept to the general case of $H$ in Eq. (1) is to define the Riccati variable as $p_{\alpha,i,j} = C_{\alpha i}/C_{\alpha j}$ for $|i\rangle$ and $|j\rangle$ coupled by a $V_i$ ($\langle i|V_i|j\rangle \neq 0$). Note that the label $j$ is determined by the label $i$ due to $V_i$. The distribution of $p_{\alpha,i,j}$ for all possible $i$ of an eigenfunction, or of eigenfunctions with eigenenergies in some energy range, will be denoted by $f_E(p)$ (the subscript $E$ will be omitted for brevity in what follows).

Furthermore, the ratio $p_{\alpha,i,j}$ for $|i\rangle$ and $|j\rangle$ not coupled by $V$ is also of interest here. When $\langle i|V|j\rangle = 0$ but $\langle i|V^2|j\rangle \neq 0$, the ratio $p_{\alpha,i,j}$ will be called type II Riccati variable. The Riccati variable mentioned above for $|i\rangle$ and $|j\rangle$ coupled by $V$, is type I. For $|i\rangle$ and $|j\rangle$ satisfying $\langle i|V^n|j\rangle \neq 0$ but $\langle i|V^{n+1}|j\rangle = 0$, the ratio $p_{\alpha,i,j}$ will be called type $n$ Riccati variable. The distribution of type $n$ Riccati variable is also denoted by $f(p)$. Expressing $C_{\alpha i}C_{\alpha j}$ as $p_{\alpha,i,j}C_{\alpha j}^2$, one can see that for chaotic eigenfunctions, whose components and ratios of components can be regarded as statistically independent, the average value of $p_{\alpha,i,j}$ is directly related to the two-point correlation function of a fixed distance by the relation

$$\langle p_{\alpha,i,j}\rangle = \langle C_{\alpha i}C_{\alpha j}\rangle$$

where $\langle C_{\alpha j}^2\rangle = 1/N$. 


**Random Matrix Model.** – We start our discussion with the RMM\[13\]. Choosing \( \lambda = 1 \) in Eq. (4) and taking the matrix elements of \( H_0 \) and \( V \) in the \( H_0 \) representation to be real random numbers with Gaussian distribution (\( \langle H_{ik}^2 \rangle = 1 \)), one has the Gaussian orthogonal ensemble (GOE). The joint distribution of two arbitrary components \( C_i \) and \( C_j \) for an arbitrary eigenfunction is (subscript \( \alpha \) omitted for brevity)

\[
P(C_i, C_j) = \frac{\Gamma((N + 1)/2)}{\pi \Gamma((N - 1)/2)}(1 - C_i^2 - C_j^2)^{(N-3)/2}
\]

(see ref. [3]). Then, the distribution \( f(p) \) can be readily obtained

\[
f_{\text{GOE}}(p) = \int dC_i dC_j \delta(p - C_i/C_j) P(C_i, C_j) = \frac{1}{p^2 + 1},
\]

which is a Lorentzian distribution centered at \( p = 0 \). Notice that the ratio of any two components of eigenfunctions of the GOE is a type I Riccati variable.

It is then natural to conjecture that the distribution of type I Riccati variable for chaotic eigenfunctions also has a Lorentzian form. As will be shown below numerically for other models, the distribution of the Riccati variable of type I for chaotic eigenfunctions is in fact of a more general Lorentzian form, denoted by \( f_L(p) \) in order to be distinguished from \( f_{\text{GOE}}(p) \), which may center at \( p \neq 0 \),

\[
f_L(p) = \frac{a/\pi}{(p - s)^2 + a^2},
\]

where \( a = \sqrt{1 - s^2} \) and \( s = \langle p \rangle \) which is proportional to the value of two point correlation function for a fixed \( |i - j| \). In fact, \( f_L(p) \) can be obtained under a condition more general than the RMM. Consider a distribution \( f(p) \) for \( p \) and the distribution \( F(q) \) for \( q = 1/p \). If \( F(q) \) has the same form as \( f(p) \), i.e., \( F(q) = f(q) \), and \( 1/f(p) \) can be expanded as a Taylor series \( 1/f(p) = d_0 + d_1 p + d_2 p^2 + \cdots \), then \( f(p) \) must has the form of \( f_L(p) \). This can be proved easily by making use of the relation between \( F(q) \) and \( f(p) \) required by \( q = 1/p \), i.e., \( F(q) = f(1/q)/q^2 \).

**Lipkin Model.** – The first dynamical model employed in the Letter is a three-orbital schematic shell model, called the Lipkin model, which is bounded and conservative. In this model there are totally \( \Omega \) particles distributed in three orbitals. Both the quantum and the corresponding classical dynamics of the model have already been known (see, e.g., Refs. [8, 14]). The Hamiltonian used in Ref. [4] has the form of Eq. (4) with

\[
H_0 = \epsilon_1 K_{11} + \epsilon_2 K_{22}, \quad V = \sum_{i=1}^{4} \mu_i V_i,
\]

(6)

where

\[
V_1 = K_{10}K_{10} + K_{01}K_{01}, \quad V_2 = K_{20}K_{20} + K_{02}K_{02}, \quad V_3 = K_{21}K_{21} + K_{02}K_{12}, \quad V_4 = K_{12}K_{10} + K_{01}K_{21}.
\]

(7)

Here \( \epsilon_i \) and \( \mu_i \) are parameters given in [19], for which the system is almost chaotic when \( \lambda = 2 \). The operators \( K_{00}, K_{11} \) and \( K_{22} \) are particle number operators of the orbitals 0, 1 and 2, and \( K_{rs} \) for \( r \neq s \) are particle raising and lowering operators, respectively. The eigenstates of \( H_0 \) are \( |mn\rangle = A(m, n)K_{10}^mK_{20}^n|00\rangle \), where \( A(m, n) \) is the normalization coefficient, \( m \) and \( n \) are particle numbers of the orbitals 1 and 2, respectively. The total particle number is conserved, and as a result the particle number of the orbital 0 is \( \Omega = m - n \). In our calculations we take \( \Omega = 40 \) and the dimension of the Hilbert space is \( N = (\Omega + 1)(\Omega + 2)/2 = 861 \).
The basis states $|mn\rangle$ can also be labeled in energy order as $|i\rangle$. In $|i\rangle$-representation, the Hamiltonian matrix is banded \cite{19}. For convenience, we can put the two labels together with $m_i$ and $n_i$ indicating particle numbers of the orbital 1 and 2, respectively. Eq. (5) tells that the perturbation couples only four kinds of basis states with definite values of $\Delta m = m_i - m_j$ and $\Delta n = n_i - n_j$, e.g., $\Delta m = \pm 2$, $\Delta n = 0$ for $V_1$. That is, there are totally four kinds of Riccati variable of type one.

When $\lambda = 2$, in the middle energy region the nearest-level-spacing distribution of the Lipkin model is close to the Wigner distribution and the underlying classical dynamics is almost chaotic (except a few quite small regular islands). For this perturbation strength, it has been found that the distributions of all the four kinds of Riccati variable of type I are quite close to the distribution $f_L(p)$. As an example, in Fig. 1(a) we present the distribution $f(p)$ (dots, in the logarithm scale) for $p = C_{a}^{i}/C_{a}^{j}$ with $|i\rangle$ and $|j\rangle$ coupled by $V_1$. In order to have a good statistics we have diagonalized 51 Hamiltonian matrices with $|\lambda - 2| \leq 0.025$ and then put data obtained for eigenfunctions of $\alpha = 430 - 450$ together. The solid line in Fig. 1(a) is the best fitting curve of the Lorentzian form $f_L(p)$. The central parts of the $f(p)$ and the fitting curve are presented in the inset of Fig. 1(a) in the ordinary scale. We would like to mention that the center of the distribution $f(p)$ in Fig. 1(a) is not at $p = 0$.

On the other hand, for Riccati variable of type $n$ with $n \geq 2$ (also for $\lambda = 2$), it has been found that the distribution $f(p)$ deviates from the Lorentzian form $f_L(p)$, and the larger $n$ the larger the deviation is. In Fig. 1(b) an example is given for the distribution of type V Riccati variable with $m_i - m_j = \pm 7$ and $n_i - n_j = 0$, where deviation of $f(p)$ (dots) from the Lorentzian fitting curve (solid line) is quite obvious. However, when the perturbation strength $\lambda$ is increased further, the deviation from $f_L(p)$ will become smaller. In fact, for each type of Riccati variable, so long as the value of $\lambda$ is large enough, the distribution function will become Lorentzian. For example, when $\lambda = 6$ the form of $f(p)$ for type II Riccati variable can be fitted as well as the $f(p)$ in Fig. 1(a) by the Lorentzian distribution $f_L(p)$.

Finally, we would like to mention that if the system is not chaotic the distribution $f(p)$ for all types of Riccati variable deviates from the Lorentzian distribution $f_L(p)$. The deviation increases with decreasing $\lambda$. When $\lambda$ approaches to zero, namely, the system becomes near integrable, the distribution $f(p)$ will approach to a $\delta$ function.

**Band Random Matrix Model.** – Since the difference between the Lipkin model and the RMM lies in the band structure of the Hamiltonian matrix of the Lipkin model in $H_0$ representation, we have also studied the distribution $f(p)$ for the Band Random Matrix Model (BRMM). For this model we take $\lambda = 1$ in Eq. (4) and $H_{ik}$ being real random numbers with Gaussian distribution ($\langle H_{ik}^2 \rangle = 1$) for $|i - k| \leq b$ and zero otherwise. When $b > 1$, despite of whether the eigenfunctions are localized or extended, numerically it has been found that the distribution $f(p)$ for Riccati variable of type I ($|i - j| \leq b$) is very close to $f_{GOE}(p)$ (due to the randomness of $H_{ik}$ the center of $f(p)$ is at $p = 0$); while for the distribution of Riccati variable of type $n$ with $n > 1$, deviation of the $f(p)$ from $f_{GOE}(p)$ has been found which enlarges gradually as $n$ increases. An example is given in Fig. 1(c) for $i - j = \pm 4b$, i.e. type IV in which the difference between the distribution $f(p)$ (dots) and the fitting curve of the Lorentzian form (solid line) is quite clear. When $\lambda$ is increased, similar to the case of the Lipkin model, $f(p)$ distributions for Riccati variable of type $n$ with $n > 1$ will also approach to the Lorentzian form of $f_{GOE}(p)$.

In order to compare with the result for the Lipkin model that centers of $f(p)$ are not necessarily at $p = 0$, we have tried to add a constant to all the non-zero off-diagonal elements of the BRMM. Then, it has been found that the peak of $f(p)$ could indeed be shifted (see Fig. 1(d)) to $s(\neq 0)$. The value of $s$ has been found numerically in a good agreement with two-point correlation function.

**Chaotic Quantum Billiard.** – Chaotic quantum billiards have been widely studied as a
Fig. 1. – (a) The distribution function \( f(p) \) (dots) of type I Riccati variable for the Lipkin model in chaotic region. (b) Same as (a) but for type V. (c) The \( f(p) \) of type IV Riccati variable for the Band Random Matrix model, i.e. \( i-j=20 \) band width \( b=5 \). (d) The \( f(p) \) (dots) obtained by adding a constant to all the non-zero off-diagonal matrix elements of BRMM. The solid curves in (a)-(d) are best fitting curves of the Lorentzian form.

Prototype of quantum chaos. We take the Bunimovich stadium\(^{[16]}\) as an example. A chaotic odd-odd wave function of a stadium with radius \( R=1 \) and \( a=1 \) has been calculated by plane wave decomposition method\(^{[6]}\). The wave function is shown in the inset of Fig. 2(a). For this wave function the statistics of the intensity \( |\psi|^2 \) is a Porter-Thomas distribution. The distribution \( f(p) \) for \( p=\psi(x,y)/\psi(x,y+\delta y) \) with a fixed \( \delta y \) larger than the de Broglie wavelength is shown in the figure, which is found to be a very good Lorentzian. The best fit by Eq. (5) gives rise to \( s=\langle p \rangle = 0 \). As was pointed out before, \( s \) is related to the two point correlation function at a fixed \( \delta y \). According to Berry's conjecture\(^{[3]}\),

\[
\langle \psi(x,y)\psi(x,y+\delta y) \rangle = J_0(k\delta y),
\]

where \( \mathcal{A} (= 1+\pi/4) \) is the area of the billiard, and \( k\delta y = 10\pi \) here. Thus, we have \( \langle p \rangle = \mathcal{A}J_0(k\delta y) = 0.18 \). This value is very close to the one obtained above.

1D Tight Binding Model. – Finally, although statistical properties of spectra and distribution of components of eigenfunctions for classically chaotic quantum systems are quite different from those of the 1D tight binding model (Anderson model) in the case of weak disorder, it has been found that for the tight-binding model the distribution of Riccati variable of type I is also close to the Lorentzian form. For this model we take \( v_{ik} = \delta_{i,k+1} + \delta_{i,k-1} \) and \( E^0_i \) being random numbers with flat distribution in the region \([-1/2,1/2]\). An example is given in Fig. 2(b) for the case of \( \lambda=1.0 \). The agreement between the \( f(p) \) distribution (dots) and the fitting curve of the Lorentzian form \( f_L(p) \) (solid lines) is quite obvious.

In conclusion, for all different chaotic and disordered models studied in this Letter, we have shown that the distribution \( f(p) \) for type I Riccati variable is universal and has the form of Lorentzian distribution. On the other hand, the distribution function for Riccati variable of type \( n \) with \( n > 1 \), although approaching to the Lorentzian form when perturbation is strong enough, shows different features in different stages of perturbation strength even though the systems are already chaotic. Therefore, statistics of Riccati variables of different types reflects statistical properties of eigenfunctions that can not be reflected by the statistics of wave function amplitude. Furthermore, the properties of Riccati variable may supply an alternative measure to probe information for coupling structure (the Hamiltonian structure) for real systems.

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Fig. 2. – (a) The distribution function \( f(p) \) (dots) of a chaotic eigenfunction \( (k = \sqrt{E} = 200.119 670) \) in stadium billiard. The eigenfunction is shown in the inset. (b) The \( f(p) \) distribution (dots) of the 1D Anderson model in the case of weak disorder. The solid curves in (a) and (b) are best fitting curves of the Lorentzian form.

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