NONRADIAL LEAST ENERGY SOLUTIONS OF THE 

\( p \)-LAPLACE ELLIPTIC EQUATIONS

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Abstract. We study the \( p \)-Laplace elliptic equations in the unit ball under the Dirichlet boundary condition. We call \( u \) a least energy solution if it is a minimizer of the Lagrangian functional on the Nehari manifold. A least energy solution becomes a positive solution. Assume that the nonlinear term is radial and it vanishes in \(|x| < a\) and it is positive in \( a < |x| < 1\). We prove that if \( a \) is close enough to \( 1 \), then no least energy solution is radial. Therefore there exist both a positive radial solution and a positive nonradial solution.

1. Introduction. We study nonradial positive solutions for the \( p \)-Laplace elliptic equation

\[
-\Delta_p u = f(x, u), \quad u > 0 \text{ in } B, \quad u = 0 \text{ on } \partial B,
\]

where \( \Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u) \) is the \( p \)-Laplacian with \( p \geq 2 \), \( B \) is the unit ball in \( \mathbb{R}^N \) and \( f(x, u) \) is a continuous function on \( \overline{B} \times [0, \infty) \) which is radially symmetric with respect to \( x \) and has a subcritical growth with respect to \( u \). In order to use the second derivative of the Lagrangian functional, we suppose that \( p \geq 2 \).

In the present paper, we shall show that no least energy solution is radial. To define a least energy solution, we first define the Lagrangian functional \( I(u) \) for (1.1) by

\[
I(u) := \int_B \left( \frac{1}{p} |\nabla u|^p - F(x, u) \right) dx, \quad F(x, u) := \int_0^u f(x, s) ds. \tag{1.2}
\]

Denote the Fréchet derivative of \( I(u) \) by \( I'(u) \), which is computed as

\[
I'(u)v = \int_B \left( |\nabla u|^{p-2}\nabla u \nabla v - f(x, u)v \right) dx \quad \text{for } u, v \in W_0^{1,p}(B).
\]

Here \( W_0^{1,p}(B) \) is the usual Sobolev space. We put

\[
J(u) := I'(u)u = \int_B (|\nabla u|^p - f(x, u)u) dx.
\]

We define the Nehari manifold \( \mathcal{N} \) and the least energy \( I_0 \) by

\[
\mathcal{N} := \{ u \in W_0^{1,p}(B) \setminus \{0\} : J(u) = 0 \}, \quad I_0 := \inf \{ I(u) : u \in \mathcal{N} \}. \tag{1.3}
\]

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We define a solution of (1.1) by $u$ if $u \in W_0^{1,p}(B)$ and it satisfies (1.1) in the distribution sense. Therefore a function $u$ in $W_0^{1,p}(B)$ is a solution of (1.1) if and only if $I'(u) = 0$. Since any solution $u$ fulfills $J(u) = I'(u)u = 0$, it belongs to the Nehari manifold, i.e., any solution $u$ of (1.1) satisfies

$$
\int_B |\nabla u|^p dx = \int_B f(x, u)u dx. \tag{1.4}
$$

We call $u$ a least energy solution if $u \in \mathcal{N}$ and $I(u) = I_0$. Such a minimizer exists and becomes a positive solution of (1.1) after replacing $u$ by $-u$ if necessary. This claim will be proved in Theorem 2.3.

On the other hand, we define

$$
\mathcal{N}_r := \{u \in \mathcal{N} : u \text{ is radial}\}, \quad I_r := \inf \{I(u) : u \in \mathcal{N}_r\}. \tag{1.5}
$$

We call $u$ a radial least energy solution if $u \in \mathcal{N}_r$ and $I(u) = I_r$. To avoid confusion, a usual least energy solution is called a global least energy solution.

To explain our purpose, we start with the result by Moore and Nehari [16]. They considered the ordinary differential equation

$$
u''(x) + h_a(x)u(x)^q = 0 \quad \text{in } (-1, 1), \quad u(-1) = u(1) = 0, \tag{1.6}
$$

where $1 < q < \infty$, $h_a(x) = 0$ for $|x| \leq a$ and $h_a(x) = 1$ for $a < |x| \leq 1$. Then they proved the next result.

**Theorem 1.1** (Moore and Nehari [16, pp.32-33]). For a suitable $a \in (0, 1)$ (a is close to 1), (1.6) has at least three positive solutions: the first one is even, the second one $u(x)$ is non-even and the third one is the reflection $u(-x)$.

On the other hand, Smets, Willem and Su [21] studied the Hénon equation

$$
-\Delta u = |x|^\lambda u^q, \quad u > 0 \text{ in } B, \quad u = 0, \text{ on } \partial B, \tag{1.7}
$$

where $B$ is the unit ball in $\mathbb{R}^N$, $\lambda > 0$, $1 < q < \infty$ if $N = 1, 2$ and $1 < q < (N + 2)/(N - 2)$ if $N \geq 3$. They proved the next result.

**Theorem 1.2** (Smets, Willem and Su [21]). If $\lambda > 0$ is large enough, then no least energy solution for (1.7) is radially symmetric.

Put $N = 1$ in (1.7). Then the weight function $|x|^\lambda$ with $\lambda$ large enough is similar to $h_a(x)$ with $a$ sufficiently close to 1, where $h_a(x)$ is defined after (1.6). Hence we expect that a noneven solution in Theorem 1.1 may be a least energy solution. Indeed, we have proved in [12, 13] that no least energy solution of (1.6) is even when $a$ is sufficiently close to 1. There are many contributions which have studied the Hénon equation ([1, 2, 3, 4, 5, 6, 7, 10, 11, 13, 18, 20]). The purpose of the present paper is to extend Theorem 1.1 to more general equation (1.1).

In our paper [14], we studied the equation

$$
-\Delta_p u = f(x)u^{q-1}, \quad u > 0 \text{ in } B, \quad u = 0 \text{ on } \partial B, \tag{1.8}
$$

where $B$ is the unit ball in $\mathbb{R}^N$, $2 \leq p < q < p^*$ and $f(x)$ is radial. Here $p^*$ is the critical exponent defined by

$$
p^* := \frac{Np}{(N - p)} \quad \text{when } p < N, \quad p^* := \infty \quad \text{when } p \geq N. \tag{1.9}
$$

We assume that

$$
f(x) \leq 0 \quad \text{for } |x| \leq a, \quad f_+(x) \neq 0 \quad \text{for } a < |x| \leq 1, \tag{1.10}
$$
with some \( a \in (0, 1) \). Here \( f_+(x) := \max\{f(x), 0\} \). Since the nonlinear term \( f(x)u^{q-1} \) in (1.8) has homogeneity with degree \( q - 1 \), we can define the Rayleigh quotient

\[
R(u) := \int_{\Omega} |\nabla u|^p dx \left( \int_{\Omega} f(x)|u|^q dx \right)^{-p/q}.
\]

Instead of the Lagrangian functional \( I(u) \), we use the Rayleigh quotient \( R(u) \) and define a least energy solution by a minimizer of \( R \) in \( \mathcal{N} \). Then we proved the next result.

**Theorem 1.3** ([14]). Let \( 2 \leq p < q < p^* \), where \( p^* \) is defined by (1.9). If \( f(x) \) is a radial function which satisfies (1.10) with \( a \in (0, 1) \) close enough to 1, then no global least energy solution of (1.8) is radial.

To prove the theorem above, we used the homogeneity of the nonlinear term in the paper [14], however (1.1) has no homogeneity. This makes our problem complicated and we cannot apply the method used in [14] to problem (1.1). Another purpose of the present paper is to extend the result in [14] to more general nonlinear term \( f(x, u) \).

This paper is organized into five sections. In Section 2, we state the main results and give an example of \( f(x, u) \). In Section 3, we show the existence of a global (or radial) least energy solution and prove that it becomes a positive solution of (1.1). In Section 4, we give an estimate of a positive radial solution. In Section 5, we show that no global least energy solution is radial.

2. Main results. In this section, we state main results.

**Assumption 2.1.** We assume that \( p \geq 2 \) and \( f(x, u) \) is a continuous function on \( \overline{B} \times [0, \infty) \) which satisfies the following conditions.

(i) \( f(x, u) \) is a radial function of \( x \).

(ii) \( f(x, u) \) has a partial derivative \( f_u(x, u) \) with respect to \( u \) which is continuous on \( \overline{B} \times [0, \infty) \).

(iii) Let \( p^* \) be defined by (1.9). There exist \( a \in (0, 1) \) and \( q \in (p, p^*) \) such that \( f(x, u) = 0 \) when \( |x| \leq a \) and \( u \geq 0 \), \( f(x, u) > 0 \) when \( a < |x| \leq 1 \) and \( u > 0 \), \( f(x, 0) = 0 \) for all \( x \) and

\[
\frac{\partial}{\partial u} \left( \frac{f(x, u)}{u^{q-1}} \right) = \frac{f_u(x, u)u - (q-1)f(x, u)}{u^q} > 0 \tag{2.1}
\]

when \( a < |x| \leq 1 \) and \( u > 0 \).

(iv) Let \( q \) be the exponent given in (iii). There exist \( r \in (q, p^*) \) and \( C > 0 \) such that

\[
|f(x, u)| \leq C(|u|^{-1} + 1) \quad \text{for } x \in B, u \geq 0. \tag{2.2}
\]

**Example 2.2.** We shall give examples of \( f(x, u) \) which satisfies Assumption 2.1. Let \( f(x, u) = h(x)g(u) \). Here \( h(x) \) is a continuous radial function such that \( h(x) = 0 \) for \( |x| \leq a \), \( h(x) > 0 \) for \( a < |x| \leq 1 \), and \( g(u) \) is one of the following functions:

\[ u^{\lambda-1}, \quad u^{\lambda-1} \log(1 + u), \quad u^{\lambda-1} \tanh u, \quad u^{\lambda-1}/(1 + u^{\mu-1}), \]

where \( \lambda \in (p, p^*) \) and \( 1 < \mu < \lambda - p + 1 \).

Let \( I_0 \) and \( I_r \) be defined by (1.3) and (1.5), respectively. We state the first main result, which ensures the existence of positive solutions for (1.1).
Theorem 2.3. Let Assumption 2.1 hold. Then there exist a global least energy solution $u$ and a radial least energy solution $v$, that is, $v \in \mathcal{N}$ and $I(v) = I_0$, $v \in \mathcal{N}_r$ and $I(v) = I_r$. Moreover, $u$ is a positive solution of (1.1) and $v$ is a positive radial solution of (1.1) after replacing $u$ by $-u$ and $v$ by $-v$ if necessary.

In the theorem above, $v$ is radial. However, we do not know whether $u$ is radial or not. There is a possibility that $u = v$ and $I_0 = I_r$ (see Remark 2.5). To guarantee that $u$ is not radial, we need the assumption that $a$ is close to 1, where $a$ is the constant given in (iii) of Assumption 2.1. Recall that in Assumption 2.1, $f(x, u) \equiv 0$ in $|x| \leq a$ and $f(x, u) > 0$ in $a < |x| < 1$. The closeness of $a$ to 1 makes the symmetry breaking. Indeed, we have the next theorem.

Theorem 2.4. There exists a constant $a_0 \in (0, 1)$ such that if $f(x, u)$ satisfies Assumption 2.1 with $a \in (a_0, 1)$, then $I_0 < I_r$ and hence no global least energy solution is radial. Therefore (1.1) has both a positive radial solution and a positive nonradial solution.

We supposed in Assumption 2.1 that $f(x, u)$ is continuous in $x$ for the sake of simplicity. Observing the proof of Theorem 2.4, we see that Theorem 2.4 is applicable to $f(x, u) = h(x)u^q$ with $h(x) = 0$ for $|x| \leq a$ and $h(x) = 1$ for $a < |x| < 1$. Therefore our theorem is a generalization of the result by Moore and Nehari [16].

Remark 2.5. In Theorem 2.4, we suppose that $a$ is sufficiently close to 1. This assumption is essential for the symmetry breaking of solutions. If $a$ is close to 0, then Theorem 2.4 is not necessary valid. Indeed, put $N = 1$, $p = 2$ and $f(x, u) = h_0(x)u^q$ and consider (1.6). Then we proved in our paper [14, Theorem 1.2] that if $a > 0$ is small enough, then (1.6) has a unique positive solution and it is even. Since $N = 1$, a radial solution means an even solution. In this case, it happens that $u = v$ and $I_0 = I_r$, where $u$ and $v$ are given in Theorem 2.3.

3. Least energy solution. In this section, we shall show that a least energy solution exists and becomes a positive solution of (1.1). Hereafter we always suppose Assumption 2.1 and define an annulus $A$ by

$$A := \{ x \in \mathbb{R}^N : a < |x| < 1 \},$$

(3.1)

where $a$ is the constant given by (iii) of Assumption 2.1. We extend $f(x, u)$ as an odd function by putting $f(x, -u) = -f(x, u)$.

We first study the Nehari manifold $\mathcal{N}$. Let $F(x, u)$ be given by (1.2). We note that $F(x, u) = 0$ for $|x| \leq a$ and $u \in \mathbb{R}$ because $f(x, u) = 0$ for $|x| \leq a$ and $u \in \mathbb{R}$. Let $A$ be defined by (3.1) and let $u \in W_0^{1,p}(B)$. By the notation, $u \neq 0$ in $A$, we mean that the set of $x \in A$ satisfying $u(x) \neq 0$ has positive Lebesgue measure. Therefore $u \neq 0$ in $A$ if and only if $\int_A |u(x)|^p dx > 0$. If $u \in \mathcal{N}$, then we have

$$0 < \int_B |\nabla u|^p dx = \int_B f(x, u)udx = \int_A f(x, u)udx,$$

because $f(x, u) = 0$ for $|x| \leq a$. Thus, if $u \in \mathcal{N}$, then $u \neq 0$ in $A$. Conversely, if $u \neq 0$ in $A$, then there exists a unique $\lambda > 0$ such that $\lambda u \in \mathcal{N}$. Indeed, we have the next lemma.

Lemma 3.1. Let $u \in W_0^{1,p}(B)$ satisfy that $u \neq 0$ in $A$. Then there exists a unique number $\lambda > 0$ such that $J(tu) > 0$ for $t \in (0, \lambda)$, $J(\lambda u) = 0$ and $J(tu) < 0$ for $t \in (\lambda, \infty)$. 

Proof. Since \( f(x, u) = 0 \) for \(|x| \leq a\), the definition of \( J \) implies that for \( t > 0\),
\[
J(tu) = t^p \int_B |\nabla u|^p dx - \int_A f(x, tu) tudx.
\]
Since \( f(x, u)u \) is even in \( u \), we have \( f(x, tu)tu = f(x, t|u|)|u| \). Then the equation above is rewritten as
\[
t^{-p}J(tu) = \int_B |\nabla u|^p dx - \int_A \frac{f(x, t|u|)}{|tu|^{p-1}}|u|^p dx.
\]
We define
\[
L(t) := \int_A \frac{f(x, t|u|)}{|tu|^{p-1}}|u|^p dx.
\]
Since \( u \neq 0 \) in \( A \), \( L(t) \) is positive. We shall show that \( L(t) \) is strictly increasing and
\[
\lim_{t \to +0} L(t) = 0, \quad \lim_{t \to \infty} L(t) = \infty. \tag{3.2}
\]
If these assertions would be proved, then \( t^{-p}J(tu) \) has a unique zero and the lemma follows. By (2.1), we have
\[
\frac{\partial}{\partial u} \left( f(x, u)/u^{p-1} \right) = u^{-p} \{ f_u(x, u)u - (p-1)f(x, u) \} \geq u^{-p} \{ (q-1)f(x, u) - (p-1)f(x, u) \} > 0,
\]
for \( x \in A \) and \( u > 0 \). Therefore \( L(t) \) is strictly increasing.

By (2.1), \( f(x, u)/u^{q-1} \) is increasing in \( u > 0 \) for each \( x \in A \). Hence we have
\[
0 < \frac{f(x, u)}{u^{q-1}} \leq f(x, 1) \leq C \quad \text{for} \ 0 < u < 1, \ x \in A,
\]
with a certain constant \( C > 0 \). Therefore \( |f(x, u)| \leq C|u|^{q-1} \) when \( |u| \leq 1 \) and \( x \in A \) because \( f(x, u) \) is odd in \( u \). This inequality with (2.2) gives us
\[
|f(x, u)| \leq C(|u|^{q-1} + |u|^{r-1}) \quad \text{for} \ u \in \mathbb{R}, \ x \in A, \tag{3.3}
\]
with some \( C > 0 \). Accordingly,
\[
0 < L(t) \leq C \int_A (t^{q-p}|u|^q + t^{r-p}|u|^r)dx \to 0,
\]
as \( t \to +0 \). Thus the first assertion in (3.2) holds.

We shall show the second assertion in (3.2). For \( \delta > 0 \), we define
\[
D := \{ x \in A : |u(x)| > \delta \}.
\]
Since \( u \neq 0 \) in \( A \), we choose \( \delta > 0 \) so small that \(|D| > 0\). Here \( |D| \) denotes the Lebesgue measure of \( D \). For \( \varepsilon > 0 \), we put
\[
E := \{ x \in D : f(x, \delta) \geq \varepsilon \}.
\]
We can choose \( \varepsilon > 0 \) so small that \(|E| > 0\). Indeed, if \(|E| = 0\) for any \( \varepsilon > 0 \), then the set of \( x \in D \) satisfying \( f(x, \delta) = 0 \) has Lebesgue measure zero. This contradicts the fact that \( D \) has a positive Lebesgue measure. Therefore \(|E| > 0\) for small \( \varepsilon > 0 \). We fix such an \( \varepsilon > 0 \). By (2.1), we have
\[
\frac{f(x, s)}{s^{q-1}} \geq \frac{f(x, \delta)}{\delta^{q-1}} \quad \text{when} \ s > \delta, \ x \in A.
\]
Hence we obtain
\[
f(x, t|u(x)|) \geq f(x, \delta)\delta^{-(q-1)}|u(x)|^{q-1} \geq \varepsilon\delta^{-(q-1)}t^{q-1}|u(x)|^{q-1},
\]
for $x \in E$ and $t > 1$. Then $L(t)$ is estimated as
\[ L(t) \geq \varepsilon t^{-p} \int_{E} |u|^q dx \geq \varepsilon t^{-p} |E| \to \infty \quad \text{as } t \to \infty. \]

The proof is complete. \hfill \Box

Lemma 3.1 ensures the next result.

**Lemma 3.2.** $\mathcal{N} \neq \emptyset$ and $\mathcal{N}_r \neq \emptyset$.

By Lemma 3.1, we can define $\lambda(u)$ as below.

**Definition 3.3.** For $u \in W^{1,p}_0(B)$ satisfying $u \neq 0$ in $A$, we define $\lambda(u)$ by a unique number $\lambda > 0$ satisfying $\lambda u \in \mathcal{N}$.

By Lemma 3.1, we see that if $J(u) > 0$, then $J(t_0 u) = 0$ at some $t_0 > 1$, and hence $\lambda(u) = t_0 > 1$. This result is written as

**Lemma 3.4.** If $J(u) > 0$, then $\lambda(u) > 1$. If $J(u) < 0$, then $\lambda(u) < 1$.

Note that $\lambda(u)$ is defined for $u \in W^{1,p}_0(B)$ satisfying $u \neq 0$ in $A$. The set of $u$ satisfying this condition is an open subset of $W^{1,p}_0(B)$.

**Lemma 3.5.** $\lambda(\cdot)$ is continuous.

**Proof.** Let $u_n, u \in W^{1,p}_0(B)$, $u_n, u \neq 0$ in $A$ and suppose that $u_n$ converges to $u$.

We shall show that $\lambda(u_n) \to \lambda(u)$ as $n \to \infty$. Put $\frac{\lambda}{\gamma} := \liminf_{n \to \infty} \lambda(u_n)$. If $0 < t < \frac{\lambda}{\gamma}$, then $J(t u_n) > 0$ for all large $n$ by Lemma 3.1. As $n \to \infty$, it follows that $J(tu) \geq 0$ for $0 < t < \frac{\lambda}{\gamma}$. If $t > \frac{\lambda}{\gamma}$, then $J(tu_n) < 0$ along a subsequence $n_j$. As $j \to \infty$, $J(tu) \leq 0$ for $t > \frac{\lambda}{\gamma}$. Consequently, $J(tu) \geq 0$ in $(0, \frac{\lambda}{\gamma})$ and $J(tu) \leq 0$ in $(\frac{\lambda}{\gamma}, \infty)$.

By Lemma 3.1, $\lambda = \lambda(u)$. Put $\bar{\lambda} := \sup_{n \to \infty} \lambda(u_n)$. In the same way as above, we can show that $\bar{\lambda} = \lambda(u)$. Therefore $\lim_{n \to \infty} \lambda(u_n) = \lambda(u)$. The proof is complete. \hfill \Box

We shall show that the global least energy $I_0$ is positive.

**Lemma 3.6.** $I_0 := \inf_{u \in \mathcal{N}} I(u) > 0$.

**Proof.** Inequality (3.3) is valid for all $x \in B$ because $f(x, u) = 0$ for $x \in B \setminus A$.

Thus we have
\[ |f(x, u)| \leq C(|u|^{q-1} + |u|^{r-1}) \quad \text{for } u \in \mathbb{R}, \ x \in B. \]

Since $p < q < r$, for any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that
\[ |f(x, u)| \leq \varepsilon |u|^{p-1} + C_\varepsilon |u|^{r-1} \quad \text{for } u \in \mathbb{R}, \ x \in B. \tag{3.4} \]

Let $u \in \mathcal{N}$. The definition of $\mathcal{N}$ ensures that
\[ \int_B |\nabla u|^p dx = \int_B f(x, u) u dx. \tag{3.5} \]

In what follows, we denote the $L^p(B)$ norm of $u$ by $\|u\|_p$. Combining (3.4), (3.5) and using the Sobolev embedding, we have
\[ \|\nabla u\|_p^p \leq \varepsilon \|u\|_p^p + C_\varepsilon \|u\|_r^r \leq \varepsilon C_0 \|\nabla u\|_p^p + C_\varepsilon C_0 \|\nabla u\|_r^r, \]

where $C_0$ is an embedding constant independent of $\varepsilon$. Choose $\varepsilon > 0$ so small that $\varepsilon C_0 < 1/2$, which shows that
Hence there exists a constant $c_0 > 0$ such that
\[
\inf_{u \in \mathcal{N}} \|\nabla u\|_p \geq c_0 > 0.
\] (3.6)

By (2.1), we have $f_u(x, u) > (q - 1)f(x, u)$. Integrating this inequality with respect to $u$, we get $f(x, u)u \geq qF(x, u)$ for $x \in A$ and $u > 0$. Since the both sides are even in $u$, this inequality is valid for all $u \in \mathbb{R}$ and $x \in B$. Using this inequality with (3.5), we estimate $I(u)$ for $u \in \mathcal{N}$ as
\[
I(u) = \frac{1}{p} \int_B |\nabla u|^p dx - \int_B F(x, u) dx
\geq \frac{1}{p} \int_B |\nabla u|^p dx - \frac{1}{q} \int_B f(x, u) u dx
= \frac{q - p}{pq} \int_B |\nabla u|^p dx.
\] (3.7)

Combining the inequality above and (3.6), we get $\inf_{u \in \mathcal{N}} I(u) > 0$. The proof is complete.

Let $I(u)$ and $J(u)$ be defined in Section 1. Furthermore, we define
\[
K(u) := \int_B \left( \frac{1}{p} uf(x, u) - F(x, u) \right) dx.
\]
Since $I(u) = K(u)$ for $u \in \mathcal{N}$ because of (3.5), we have
\[
I_0 := \inf_{u \in \mathcal{N}} I(u) = \inf_{u \in \mathcal{N}} K(u).
\]

Lemma 3.7. If $J(u) < 0$, then $K(u) > I_0$.

Proof. Since $uf(x, u) \geq 0$ for $u \in \mathbb{R}$, it follows from (2.1) that
\[
\frac{d}{dt} K(tu) = \frac{1}{pt} \int_B \left( (tu)^2 f_u(x, tu) - (p - 1) tu f(x, tu) \right) dx
\geq \frac{q - p}{pt} \int_A tu f(x, tu) dx \geq 0,
\]
for $t > 0$. If $u \neq 0$ in $A$, $K(tu)$ is strictly increasing in $t \in [0, \infty)$. Let $J(u) < 0$. Then $u \neq 0$ in $A$. Furthermore, $\lambda(u) < 1$ by Lemma 3.4. Hence $K(tu)$ is strictly increasing and $\lambda(u) < 1$. Therefore
\[
I_0 = \inf_{v \in \mathcal{N}} K(v) \leq K(\lambda(u)u) < K(u).
\]
The proof is complete.

In the next lemma, we shall show the existence of a global least energy solution. Our method remains valid for the existence of a radial least energy solution.

Lemma 3.8. There exists a minimizer of $I$ on $\mathcal{N}$.

Proof. Put $I_0 := \inf_{\mathcal{N}} I(u)$. Choose a minimizing sequence $u_n$, i.e., $u_n \in \mathcal{N}$ and $I(u_n) \to I_0$ as $n \to \infty$. Substituting $u_n$ in (3.7), we find that $u_n$ is bounded in $W_0^{1,p}(B)$. Hence it has a subsequence (denoted by $u_n$ again) converging to a weak limit $u \in W_0^{1,p}(B)$. By the compact embedding with (2.2), we get
\[
\int_B F(x, u_n) dx \to \int_B F(x, u) dx, \quad \int_B u_n f(x, u_n) dx \to \int_B u f(x, u) dx, \quad (3.8)
\]
as $n \to \infty$. Since $u_n \in \mathcal{N}$, it holds that $K(u_n) = I(u_n) \to I_0$ as $n \to \infty$. Then (3.8) shows that

$$K(u) = \int_B \left( \frac{1}{p}uf(x, u) - F(x, u) \right) dx = I_0 > 0. \quad (3.9)$$

We here note that $I_0 > 0$ by Lemma 3.6. Since $u_n \in \mathcal{N}$, it holds that

$$\int_B |\nabla u_n|^p dx = \int_B u_n f(x, u_n)dx.$$ 

Since $u_n$ weakly converges to $u$ in $W^{1, p}_0(B)$, we use the second convergence in (3.8) to get

$$\int_B |\nabla|^{\infty} \leq \liminf_{n \to \infty} \int_B |\nabla u_n|^p dx = \int_B u f(x, u)dx.$$ 

Thus $J(u) \leq 0$. If $J(u) < 0$, then (3.9) contradicts Lemma 3.7. Hence $J(u) = 0$. If $u = 0$, then $K(u) = 0$, which contradicts (3.9). Therefore $u \neq 0$ and so $u$ belongs to $\mathcal{N}$. Since $u \in \mathcal{N}$, $I(u)$ is equal to $K(u)$. By (3.9), $I(u) = K(u) = I_0$ and $u \in \mathcal{N}$.

The proof is complete. \qed

To prove Theorem 2.3, we need the Lagrange multiplier rule.

**Lemma 3.9** (Lagrange multiplier). Let $X$ be a Banach space, $u_0 \in X$ and let $U$ be an open neighborhood of $u_0$. Let $I$ and $J$ be continuously differentiable mappings from $U$ to $\mathbb{R}$. Suppose that $J'(u_0) : X \to \mathbb{R}$ is surjective and $u_0$ satisfies that $I'(u_0) = 0$ and

$$I(u_0) = \min \{ I(u) : J(u) = 0, \ u \in U \}.$$ 

Then there exists a $\lambda \in \mathbb{R}$ such that $I'(u_0) + \lambda J'(u_0) = 0$.

Proof of Theorem 2.3. By (2.1), we have

$$f_u(x, u)u^2 \geq (q - 1)f(x, u)u \geq 0 \quad \text{for} \ x \in B, \ u \in \mathbb{R}. \quad (3.10)$$

It is easy to see that

$$J'(u)v = \int_B \left( p|\nabla u|^{p-2}\nabla u \nabla v - f_u(x, u)uv - f(x, u)v \right) dx,$$

for $u, v \in W^{1, p}_0(B)$. In Lemma 3.8, we have already proved the existence of a global least energy solution. Let $v$ be a global least energy solution, i.e., $v \in \mathcal{N}$ and $I(v) = I_0$. Then it satisfies (3.5), that is,

$$\int_B |\nabla v|^p dx = \int_B f(x, v)dx.$$ 

Combining two equations above and using (3.10), we have

$$J'(v)v = \int_B \left( p|\nabla v|^p - f_u(x, v)v^2 - f(x, v)v \right) dx$$

$$= \int_B \left( (p - 1)f(x, v)v - f_u(x, v)v^2 \right) dx$$

$$\leq -(q - p) \int_B f(x, v)vdx < 0, \quad (3.11)$$
because $v \not\equiv 0$ in $A$ by $v \in \mathcal{N}$. Therefore $J'(v)$ is surjective from $W^{1,p}_0(B)$ onto $\mathbb{R}$. Since $v$ is a minimizer of $I$ under the restriction $J(v) = 0$ and $J'(v)$ is surjective, we use Lemma 3.9 to obtain a Lagrange multiplier $\lambda \in \mathbb{R}$ which satisfies

$$I'(v) + \lambda J'(v) = 0,$$

(3.12)

or equivalently

$$I'(v)w + \lambda J'(v)w = 0 \quad \text{for any } w \in W^{1,p}_0(B).$$

Substituting $w = v$ and noting $I'(v)v = 0$ by $v \in \mathcal{N}$, we have $\lambda J'(v)v = 0$. Since $J'(v)v < 0$ by (3.11), $\lambda$ is equal to 0. By (3.12) with $\lambda = 0$, $v$ is a critical point of $I$. Since $I(v) = I_0 > 0$, $v$ is a nontrivial solution of (1.1). Observe that $|v| \in \mathcal{N}$ and $I(v) = I(|v|)$. Accordingly, $|v|$ is also a minimizer of $I$ on $\mathcal{N}$. Hence $w := |v|$ is a solution of (1.1), which satisfies

$$-\Delta_p w = f(x, w) \geq 0 \quad \text{in } B.$$

We here need the strong maximum principle for the $p$-Laplacian. For its statement and the proof, we refer the readers to [19, p.34, Theorem 2.5.1], [22, p.801, Propositions 3.2.1, 3.2.2] or [23, p.200, Theorem 5]. By the strong maximum principle, we conclude that $w(x) > 0$. Hence $v$ is either a positive solution or a negative solution. Choose a positive one after replacing $v$ by $-v$ if necessary. Therefore a global least energy solution is a positive solution.

4. Positive radial solution. In this section, we investigate a positive radial solution. Since $f(x, u)$ is a radial function of $x$, we denote it by $f(r, u)$ with $r = |x|$. Let $u = u(r)$ with $r = |x|$ be a positive radial solution of (1.1). Then it satisfies

$$(r^{N-1}|u'|^{p-2}u')' = -r^{N-1}f(r, u) \quad \text{in } (0, 1), \quad u'(0) = u(1) = 0. \quad (4.1)$$

Let $a \in (0, 1)$ be the constant given in (iii) of Assumption 2.1. Since $f(r, u) = 0$ for $r \leq a$, the function $r^{N-1}|u'|^{p-2}u'$ is constant by (4.1). Since $u$ is radial, $u'(0) = 0$. Hence $u'(r) \equiv 0$ in $[0, a]$. Since $f(r, u(r)) > 0$ in $(a, 1]$, $(r^{N-1}|u'|^{p-2}u')' < 0$, and so $r^{N-1}|u'|^{p-2}u'$ is decreasing. Since $u'(a) = 0$, $u'(r) < 0$ in $(a, 1]$. Thus we have

$$u'(r) = 0 \quad \text{on } [0, a], \quad u'(r) < 0 \quad \text{on } (a, 1]. \quad (4.2)$$

It is easy to prove the next lemma, but it plays an important role in the proof of Theorem 2.4.

Lemma 4.1. Let $u(r)$ be a positive radial solution of (1.1). Then

$$\int_0^1 u(r)p r^{N-1} dr \leq N^{-1}a^{-(N-1)}(1-a)^{p-1}\int_a^1 |u'(r)|^p r^{N-1} dr. \quad (4.3)$$

Proof. Define $\mu$ by $1/p + 1/\mu = 1$. We use the Hölder inequality with $u(1) = 0$ to obtain

$$u(a) = -\int_a^1 u'(r) dr \leq \int_a^1 |u'(r)|^{p(N-1)/p-r-(N-1)/p} dr$$

$$\leq \left( \int_a^1 |u'(r)|^{p} r^{N-1} dr \right)^{1/p} \left( \int_a^1 r^{-\mu(N-1)/p} dr \right)^{1/\mu}$$

$$\leq a^{-(N-1)/p}(1-a)^{1/\mu} \left( \int_a^1 |u'(r)|^{p} r^{N-1} dr \right)^{1/p}.$$

The inequality above is rewritten as
\[ u(a)^p \leq a^{-(N-1)}(1 - a)^{p-1} \int_0^1 |u'(r)|^p r^{N-1} dr. \]

By (4.2), \( u(0) = u(a) = \|u\|_{\infty} \). Therefore
\[ \int_0^1 u(r)^p r^{N-1} dr \leq N^{-1} \|u\|_p^p = N^{-1} u(a)^p. \]

Combining the two inequalities above, we obtain (4.3). The proof is complete. \( \square \)

5. Positive nonradial solution. In this section, we shall prove Theorem 2.4. Let 
\( B(x, r) \) denote a ball centered at \( x \) with radius \( r > 0 \). We observe the inclusion,
\[ \{ x \in \mathbb{R}^N : 3/4 \leq |x| \leq 1 \} \subset \bigcup_{|x|=1} B(x, 1/2). \]
Since the left hand side is compact, we have a finite covering,
\[ \{ x \in \mathbb{R}^N : 3/4 \leq |x| \leq 1 \} \subset \bigcup_{i=1}^M B(x_i, 1/2), \]  
(5.1)
with some \( x_1, \ldots, x_M \) satisfying \(|x_i| = 1\). We here choose \( x_1, \ldots, x_M \) and \( M \) suitably
so that \( M \) is the smallest positive integer satisfying the covering above. Then \( M \)
depends only on \( N \).

We define \( \Phi(r) \) by \( \Phi(r) := 1 \) for \( r \leq 1/2 \), \( \Phi(r) := -4r + 3 \) for \( 1/2 < r < 3/4 \) and
\( \Phi(r) := 0 \) for \( r \geq 3/4 \). Let \( z \in \mathbb{R}^N \) be a point satisfying \(|z| = 1\). Define
\[ \phi(x) := \Phi(|x - z|) - \Phi(|x + z|). \]  
(5.2)
Then \( \phi \in W^{1,\infty}(\mathbb{R}^N) \) and \( \phi \) is odd, i.e., \( \phi(-x) = -\phi(x) \) and the support of \( \phi \)
is \( \overline{B}(z, 3/4) \cup \overline{B}(-z, 3/4) \). Here \( \overline{B}(z, 3/4) \) is the closure of \( B(z, 3/4) \). Moreover,
\( \phi(x) = \pm 1 \) in \( B(\pm z, 1/2) \). We have the next lemma.

Lemma 5.1. Let \( u \) be a positive radial solution, let \( \phi \) be defined by (5.2) and let \( M \)
satisfy (5.1). Suppose that Assumption 2.1 holds with \( a \in (3/4, 1) \). Then we have
\[ \int_B |\nabla u|^p dx \leq M \int_B f(x, u)u\phi^2 dx, \]  
(5.3)
\[ \int_B u^p dx \leq MN^{-1}a^{-(N-1)}(1 - a)^{p-1} \int_B f(x, u)u\phi^2 dx. \]  
(5.4)

Proof. Let \( u \) be a positive radial solution. Put \( u(x) = 0 \) outside \( B \). Let \( x_i \) with
\( 1 \leq i \leq M \) satisfy (5.1). Since \( u \) is radial and \( |x_i| = 1 \), it holds that
\[ \int_{B(x_i, 1/2)} f(x, u)dx = \int_{B(y, 1/2)} f(x, u)dx, \]
for any \( i \) when \(|y| = 1\). Since \( f(x, u) \geq 0 \), it follows from (5.1) with the identity
above that
\[ \int_{3/4 \leq |x| \leq 1} f(x, u)dx \leq M \int_{B(y, 1/2)} f(x, u)dx, \]
for any \( y \) satisfying \(|y| = 1\). Since \(|z| = 1 \) and \( \phi(x) = 1 \) in \( B(z, 1/2) \), the inequality
above shows that
\[ \int_{3/4 \leq |x| \leq 1} f(x, u)dx \leq M \int_{B(z, 1/2)} f(x, u)u\phi^2 dx \leq M \int_B f(x, u)u\phi^2 dx. \]
Since \( f(x, u) = 0 \) for \(|x| \leq a\), the inequality above is reduced to
\[
\int_{a \leq |x| \leq 1} f(x, u) u dx \leq M \int_B f(x, u) u^2 dx,
\]
provided that \( a \in (3/4, 1)\).

Hereafter \( \omega_N \) denotes the surface area of the unit sphere in \( \mathbb{R}^N \). Then any radial function \( v = v(r) \) with \( r = |x| \) satisfies
\[
\int_B |v|^p dx = \omega_N \int_0^1 |v(r)|^p r^{N-1} dr.
\]

Multiplying (4.3) by \( \omega_N \), we have
\[
\int_B u^p dx \leq N^{-1} a^{-(N-1)} (1 - a)^{p-1} \int_{a \leq |x| \leq 1} |\nabla u|^p dx.
\]

Multiplying the first equation of (4.1) by \( u \) and integrating it on \([a, 1]\) and using \( u'(a) = 0 \) by (4.2), we obtain
\[
\int_a^1 |u'| r^{N-1} dr = \int_a^1 f(r, u) ur^{N-1} dr,
\]
or equivalently
\[
\int_B |\nabla u|^p dx = \int_{a \leq |x| \leq 1} |\nabla u|^p dx = \int_{a \leq |x| \leq 1} f(x, u) u dx,
\]
because \( \nabla u(x) = 0 \) in \(|x| \leq a\) by (4.2). Combining this equation with (5.5), we obtain (5.3). Substituting the identity above in the right hand side of (5.6), we have
\[
\int_B u^p dx \leq N^{-1} a^{-(N-1)} (1 - a)^{p-1} \int_{a \leq |x| \leq 1} f(x, u) u dx.
\]

Using this inequality with (5.5), we obtain (5.4). The proof is complete.

We employ a method developed in our paper [15]. Let \( \phi \) be the function given by (5.2). For a radial least energy solution \( u \), we define
\[
g(t, s) := I((1 + t)(1 + s\phi)u) \quad \text{for } t, s \in \mathbb{R}. \tag{5.7}
\]
We shall show that
\[
(1 + t)(1 + s\phi)u \in \mathcal{N}, \quad I((1 + t)(1 + s\phi)u) < I(u),
\]
with some \( t, s \in \mathbb{R} \). If this claim would be proved, then
\[
I_0 \leq I((1 + t)(1 + s\phi)u) < I(u) = I_r.
\]
This inequality shows that no global least energy solution is radial, which is the desired conclusion. To prove our claim, we investigate \( g(t, s) \) near \((t, s) = (0, 0)\).

By Assumption 2.1 and \( p \geq 2 \), \( I(u) \) is twice continuously differentiable in the sense of the Fréchet derivative. The second derivative \( I''(u) \) is a bilinear form,
\[
I''(u)[v, w] = \int_B ((p - 1)|\nabla u|^{p-2}\nabla v \nabla w - f_u(x, u)v w) dx,
\]
for \( u, v, w \in W_0^{1,p}(B) \). Let \( u \) be a radial least energy solution. Define \( g(t, s) \) by (5.7). We compute the second derivative of \( g \) as below,
\[
g_t(t, s) = I'(((1 + t)(1 + s\phi)u)[(1 + s\phi)u],
g_s(t, s) = I'((1 + t)(1 + s\phi)u)[(1 + t)\phi u],
\]
\[
g_{tt}(t, s) = I''((1 + t)(1 + s\phi)u)[(1 + s\phi)u],
g_{ss}(t, s) = I''((1 + t)(1 + s\phi)u)[(1 + t)\phi u],
g_{ts}(t, s) = I''((1 + t)(1 + s\phi)u)[(1 + s\phi)u]
\]
\[
\int_B ((p - 1)|\nabla u|^{p-2}\nabla v \nabla w - f_u(x, u)v w) dx.
\]

Using this inequality with (5.5), we obtain (5.4). The proof is complete.
$$g_{tt}(t, s) = I''((1 + t)(1 + s\phi)u)[(1 + s\phi)u, (1 + s\phi)u],$$
$$g_{ss}(t, s) = I''((1 + t)(1 + s\phi)u)[(1 + t)\phi u, (1 + t)\phi u],$$
$$g_{ts}(t, s) = I''((1 + t)(1 + s\phi)u)[(1 + s\phi)u, (1 + t)\phi u]
+ I'((1 + t)(1 + s\phi)u)[\phi u].$$

Substituting $t = s = 0$ and using (5.8) and noting $I'(u) = 0$, we have
$$g_{tt}(0, 0) = 0, \quad g_{ss}(0, 0) = 0,$$
$$g_{ts}(0, 0) = I''(u)[u, \phi u] = \int_B ((p - 1)|\nabla u|^p - f_u(x, u)u^2)dx, \quad (5.9)$$
$$g_{ss}(0, 0) = I''(u)[\phi u, \phi u] = \int_B ((p - 1)|\nabla u|^p - 2|\nabla (\phi u)|^2 - f_u(x, u)u^2)dx, \quad (5.10)$$
$$g_{ts}(0, 0) = I''(u)[u, \phi u] = \int_B ((p - 1)|\nabla u|^p - 2\nabla u \nabla (\phi u) - f_u(x, u)u^2\phi)dx. \quad (5.11)$$

We shall show that $g_{ts}(0, 0) = 0$. Since $u$ is radial, it is even. However $\phi$ is odd. Accordingly, the integral of the second term on the right hand side of (5.11) vanishes, i.e.,
$$\int_B f_u(x, u)u^2\phi dx = 0.$$

The first term on the right hand side of (5.11) is rewritten as
$$(p - 1) \int_B |\nabla u|^{p-2}\nabla u \nabla (\phi u)dx = -(p - 1) \int_B \text{div}(|\nabla u|^{p-2}\nabla u)\phi u dx$$
$$= -(p - 1) \int_B (\Delta_p u)\phi u dx = 0,$$
because $\Delta_p u$ is radial and $\phi u$ is odd. Therefore $g_{ts}(0, 0) = 0$. Then the next lemma follows.

**Lemma 5.2.** Let $u$ be a radial least energy solution and define $g(t, s)$ by (5.7). Then
$$g(t, s) = g(0, 0) + \frac{t^2}{2} g_{tt}(0, 0) + \frac{s^2}{2} g_{ss}(0, 0) + o(t^2 + s^2), \quad (5.12)$$
as $t, s \to 0$. Here $o(t^2 + s^2)/(t^2 + s^2) \to 0$ as $t, s \to 0$ and $g_{tt}(0, 0)$ and $g_{ss}(0, 0)$ are written as
$$g_{tt}(0, 0) = \int_B ((p - 1)f(x, u)u - f_u(x, u)u^2) dx, \quad (5.13)$$
$$g_{ss}(0, 0) = (p - 1) \int_B (f(x, u)u\phi^2 + |\nabla u|^{p-2}|\nabla \phi|^2u^2) dx
- \int_B f_u(x, u)u^2\phi^2 dx. \quad (5.14)$$
Proof. Since \( g_t(0,0) = g_s(0,0) = g_{ss}(0,0) = 0 \), the Taylor theorem proves (5.12). Since \( u \) is a solution of (1.1), it satisfies (1.4), that is,
\[
\int_B |\nabla u|^p \, dx = \int_B f(x,u) \, u \, dx.
\]
Substituting this identity in (5.9), we have (5.13).

Expanding the term \( |\nabla(\phi u)|^2 \) in (5.10), we obtain
\[
g_{ss}(0,0) = (p-1) \int_B (|\nabla u|^{p-2} |\nabla \phi| \nabla u \nabla \phi) \, dx + (p-1) \int_B |\nabla u|^{p-2} |\nabla \phi|^2 \, dx - \int_B f_u(x,u) u^2 \, dx.
\]
Multiplying (1.1) by \( u \phi^2 \) and integrating over \( B \), we have
\[
\int_B (|\nabla u|^{p-2} |\nabla \phi| u \phi) \, dx = \int_B f(x,u) u \phi^2 \, dx.
\]
Substituting this identity in (5.15), we obtain (5.14). The proof is complete. \( \square \)

Let \( u \) be a radial least energy solution and define \( g(t,s) \) by (5.7). Let \( a \) be the constant given in (iii) of Assumption 2.1. We shall show that the closeness of \( a \) to 1 ensures that \( g_t(0,0) \) and \( g_{ss}(0,0) \) are negative.

Lemma 5.3. There exists a constant \( a_0 \in (0,1) \) depending only on \( p, q \) and \( N \) such that if Assumption 2.1 holds with \( a \in (a_0,1) \), then \( g_t(0,0) \) and \( g_{ss}(0,0) \) are negative.

Proof. Let \( u \) be a positive radial solution. Since \( f(x,u(x)) = 0 \) in \( B \setminus A \), (5.13) is reduced to
\[
g_t(0,0) = \int_A ((p-1) f(x,u)u - f_u(x,u)u^2) \, dx \leq -(q-p) \int_A f(x,u)u \, dx < 0,
\]
where we have used (2.1). We shall show that \( g_{ss}(0,0) < 0 \). Since the support of \( \Phi(|x-z|) \) does not intersect that of \( \Phi(|x+z|) \), it follows from (5.2) that \( |\nabla \phi(x)| \leq 4 \).

We estimate the second term on the right hand side of (5.14) as
\[
\int_B |\nabla u|^{p-2} |\nabla \phi|^2 u^2 \, dx \leq 16 \int_B |\nabla u|^{p-2} u^2 \, dx \leq 16 \|\nabla u\|_{p-2} \|u\|_p^2,
\]
where we have used the Hölder inequality. Let \( a \in (3/4,1) \). By Lemma 5.1, we have a constant \( C > 0 \) such that
\[
\|\nabla u\|_{p-2} \leq C \left( \int_B f(x,u) \, u \phi^2 \, dx \right)^{(p-2)/p},
\]
\[
\|u\|_p^2 \leq C(1-a)^{2(p-1)/p} \left( \int_B f(x,u) \, u \phi^2 \, dx \right)^{2/p},
\]
where \( C \) depends only on \( N \) and \( p \). Using these two inequalities, we estimate (5.16) as
\[
\int_B |\nabla u|^{p-2} |\nabla \phi|^2 u^2 \, dx \leq 16 C^2 (1-a)^{2(p-1)/p} \int_B f(x,u) \, u \phi^2 \, dx.
\]
Employing this inequality and (2.1), we reduced (5.14) to

\[ g_{ss}(0, 0) \leq (p - 1) \left[ 1 + 16C^2(1 - u)^2(\gamma - 1/p) \right] \int_B f(x, u)u\phi^2 \ dx \]

\[ - (q - 1) \int_B f(x, u)u\phi^2 \ dx \]

\[ = -\gamma \int_B f(x, u)u\phi^2 \ dx, \]

where we have put

\[ \gamma := (q - 1) - (p - 1) \left[ 1 + 16C^2(1 - u_a)^2(\gamma - 1/p) \right]. \]

We choose \( a_0 \in (3/4, 1) \) sufficiently close to 1 such that

\[ (q - 1) > (p - 1) \left[ 1 + 16C^2(1 - a_0)^2(\gamma - 1/p) \right]. \]

Since \( C \) depends only on \( N \) and \( p \), the constant \( a_0 \) is determined by \( p, q \) and \( N \). For \( a \in (a_0, 1) \), it holds that \( \gamma > 0 \) and \( g_{ss}(0, 0) < 0 \). The proof is complete. \( \square \)

We conclude this paper by proving Theorem 2.4.

**Proof of Theorem 2.4.** Let \( a_0 \) be the constant given by Lemma 5.3. Suppose that Assumption 2.1 holds with \( a \in (a_0, 1) \). Let \( u \) be a radial least energy solution. Then \( u > 0 \) in \( B, u \in N_r, I'(u) = 0 \) and \( I(u) = I_r \). By Lemma 5.3, \( g_{ss}(0, 0) \) and \( g_{ss}(0, 0) \) are negative. Recall the definition of \( \lambda(v) \), i.e., \( \lambda(v) \) is a unique positive number satisfying \( \lambda(v)v \in N \) when \( v \neq 0 \) in \( A \). Since \( u \in N_r, \lambda(u) = 1 \). Since \( \lambda(\cdot) \) is continuous by Lemma 3.5, \( \lambda((1 + s\phi)u) \) converges to \( \lambda(u) = 1 \) as \( s \to 0 \). Define \( t(s) := \lambda((1 + s\phi)u) - 1 \), which converges to 0 as \( s \to 0 \). Then \( (1 + t(s))(1 + s\phi)u \in N \). When \( s > 0 \) is small enough, so is \( |t(s)| \). By Lemmas 5.2 and 5.3, it holds that \( g(t, s) < g(0, 0) \) for small \( |t|, |s| \) with \( (t, s) \neq (0, 0) \). Therefore, for small \( s > 0 \),

\[ I_0 \leq I((1 + t(s))(1 + s\phi)u) = g(t(s), s) < g(0, 0) = I(u) = I_r. \]

The proof is complete. \( \square \)

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