INITIAL-BOUNDARY VALUE PROBLEMS FOR THE COUPLED
MODIFIED KORTEWEG-DE VRIES EQUATION ON THE
INTERVAL

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ABSTRACT. In this paper, we study the initial-boundary value problems of the
coupled modified Korteweg-de Vries equation formulated on the finite interval
with Lax pairs involving $3 \times 3$ matrices via the Fokas method. We write the
solution in terms of the solution of a $3 \times 3$ Riemann-Hilbert problem. The
relevant jump matrices are explicitly expressed in terms of the three matrix-
value spectral functions $s(k)$, $S(k)$, and $S_L(k)$, which are determined by the
initial values, boundary values at $x = 0$, and at $x = L$, respectively. Some of
the boundary values are known for a well-posed problem, however, the remain-
ing boundary data are unknown. By using the so-called global relation, the
unknown boundary values can be expressed in terms of the given initial and
boundary data via a Gelfand-Levitan-Marchenko representation.

1. Introduction. Several important partial differential equations (PDEs) in mat-
hematics and physics are integrable, which can be rewritten in terms of two linear
eigenvalue equations, called a Lax pair [22]. In 1967, Gardner, Greene, Kruskal,
Miura [16] solved the initial value problems for Korteweg-de Vries (KdV) equation
by using the Inverse Scattering Transform (IST) formalism. From then on, the IST
method can be often used to study the initial value problems for integrable evolution
equations on the line [1, 5, 6, 16]. In this case, the solution at time $t$ can be
recovered by using the solution of an inverse problem. This inverse problem is most
conveniently formulated as a Riemann-Hilbert problem (RHP). However, boundary
conditions play a very important role in some specific mathematical physics pro-
blems. Therefore, it is more meaningful to study the initial-boundary value (IBV)
problem than simply studying the pure initial value problem.

In 1997, Fokas [7] (also see [8, 9]) introduced a new method, the so-called Fokas
method, for analyzing boundary value problems for linear and for integrable nonli-
ear PDEs. The Fokas method is the extension of the IST formalism from initial
value to IBV problems. It is well-known that the IST method is usually used to
find the scattering data via analyzing the $x$-part of the Lax pairs. It is based on
the $t$-part of the Lax pairs to recover the time evolution of the scattering data. However, the Fokas method is based on the simultaneous spectral analysis of the Lax pair, as well as on the analysis of an algebraic relation coupling the initial conditions with all boundary values, which is called by Fokas as the global relation. For a well-posed problem, some of the boundary values are known, however, the remaining boundary data are unknown. By using the so-called global relation, the unknown boundary values can be expressed in terms of the given initial and boundary data via a Gelfand-Levitan-Marchenko representation. The Fokas method can be used to study the boundary value problems of several important integrable equations including $2 \times 2$ Lax pair equations, such as the KdV equation [10, 11], the nonlinear Schrödinger equation [12, 13, 14, 15] and other PDEs [2, 3, 4, 17, 21, 23, 24, 25, 29, 30, 34].

In 2012, Lenells firstly extended the Fokas method to study the IBV problems of the integrable PDEs with $3 \times 3$ Lax pair equations on the half-line [26]. There are many novelties for the transition from $2 \times 2$ to $3 \times 3$ matrix Lax pair equations. Following Lenells’ work, the IBV problems of several integrable PDEs with $3 \times 3$ Lax pair were studied, such as the Degasperis-Procesi [27], Sasa-Satsuma [36], three wave [37], the two-component nonlinear Schrödinger equations [17, 31, 32, 38], and the coupled modified Korteweg-de Vries equation[33], etc.

In this paper, we will consider IBV problems of the following coupled modified Korteweg-de Vries (cmKdV) equation:

\[
\begin{align*}
    p_t + p_{xxx} - 3p_xq^2 - 3pqq_x - 6p^2 p_x &= 0, \\
    q_t + q_{xxx} - 3p^2 q_x - 3pp_xq - 6q^2 q_x &= 0,
\end{align*}
\]

(1.1)
on the plane wave background, where $p(x,t)$ and $q(x,t)$ are complex-valued functions of $(x,t) \in \Omega$, with $\Omega$ denoting the following domain

\[
\Omega = \{(x,t) \in \mathbb{R}^2 | 0 \leq x \leq L, 0 \leq t \leq T\};
\]

(1.2)

here $L > 0$ is a fixed constant and $T > 0$ is a fixed final time. The functions $p(x,t)$ and $q(x,t)$ are slowly varying pulse envelopes. The cmKdV equation (1.1) can be considered as the generalization of the modified KdV equation investigated by many researchers [19]. For instance, based on the bilinear method, Hirota and Iwao obtained ‘molecule solution’ and multi-soliton solution for the cmKdV equation [20]. Tsuchida and Wadati solved the initial value problem of the cmKdV equation by using the inverse scattering transformation [35]. Recently, Geng et al derived algebro-geometric solutions of the cmKdV hierarchy associated with a $3 \times 3$ matrix spectral problem on the basis of the theory of algebraic curves [18]. Moreover, the published information shows that equation (1.1) is among a soliton hierarchy, which is also a special reduction of the multiple-component AKNS systems or more generally, the multiple-component interaction systems [28, 39]. Recently, we studied the IBV problems of Eq. (1.1) on the half-line [33]. However, to the best of author’s knowledge, the IBV problems of Eq. (1.1) on the interval have not been investigated before.

In [31], the general coupled nonlinear Schrödinger equations was successfully studied on the interval. The main purpose of the present study is to try solve the IBV problems of Eq. (1.1) on the interval. Here, by using the Gelfand-Levitan-Marchenko representation, the global relation is analyzed to find the expressions of the unknown boundary values. Because the order of the derivative is higher than
the general coupled nonlinear Schrödinger equations case, it will be more complicate to analyze the global relation of the cmKdV equation (1.1).

Throughout this paper, we will consider the following IBV problems for the cmKdV equation

**Initial values:** \( p_0(x) = p_0(x, t = 0), \ q_0(x) = q_0(x, t = 0), \)

**Dirichlet boundary values:** \( g_{01}(t) = p(x = L, t), \ g_{02}(t) = q(x = L, t), \)

**First Neumann boundary values:** \( g_{11}(t) = p_x(x = 0, t), \ g_{12}(t) = q_x(x = 0, t), \)

**Second Neumann boundary values:** \( g_{21}(t) = p_{xx}(x = 0, t), \ g_{22}(t) = q_{xx}(x = 0, t), \)

\[ f_{01}(t) = p(x = L, t), \ f_{02}(t) = q(x = L, t), \]

\[ f_{11}(t) = p_x(x = L, t), \ f_{12}(t) = q_x(x = L, t), \]

\[ f_{21}(t) = p_{xx}(x = L, t), \ f_{22}(t) = q_{xx}(x = L, t). \] (1.3)

**Organization of this paper.** Section 2 contains the spectral analysis of the associated Lax pair equations. The main Riemann-Hilbert problem is formulated in Section 3. In Section 4, we study the the asymptotic analysis of the spectral functions. Finally, in Section 5, we derive the Gelfand-Levitan-Marchenko representations of \( \Psi_{ij} \)_{i,j=1} and \( \Phi_{ij} \)_{i,j=1}, based on which we further analyze the global relation to obtain the expressions of the unknown boundary values.

2. **Spectral analysis.** System (1.1) is still integrable. Its Lax pair reads

\[ \Psi_x = U \Psi, \quad \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, \]

(2.1a)

\[ \Psi_t = V \Psi, \]

(2.1b)

with

\[ U = ik \Lambda + V_1 \]

(2.2)

and

\[ V = 4ik^3 \Lambda + V_2, \]

(2.3)

here

\[ \Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ V_1 = \begin{pmatrix} 0 & p & q \\ p & 0 & 0 \\ q & 0 & 0 \end{pmatrix} \]

(2.4)

and

\[ V_2 = k^2 V_2^{(2)} + k V_2^{(1)} + V_2^{(0)}, \]

(2.5)

with \( V_2^{(2)} = 4V_1 \) and

\[ V_2^{(1)} = 2i \begin{pmatrix} -(p^2 + q^2) & p_x & q_x \\ -p_x & p^2 & pq \\ -q_x & pq & q^2 \end{pmatrix}, \]

\[ V_2^{(0)} = \begin{pmatrix} 0 & -p_{xx} + 2p^3 + 2pq^2 & -q_{xx} + 2p^2 q + 2q^3 \\ -p_{xx} + 2p^3 + 2pq^2 & 0 & p_x q - pq_x \\ -q_{xx} + 2p^2 q + 2q^3 & -p_x q + pq_x & 0 \end{pmatrix}. \] (2.6)
where \( k \) is a spectral parameter, and \( \Psi(x, t, k) \) is a vector or a matrix function. The compatibility condition of Lax pair \((2.1a)\) and \((2.1b)\) gives the cmKdV equation \((1.1)\).

2.1. The closed one-form. Let \( p(x, t) \) and \( q(x, t) \) be two sufficiently smooth functions of \((x, t)\) in the interval domain \( \Omega \), which decay as \( x \to \infty \). By introducing a new eigenfunction \( \mu(x, t, k) \)

\[
\Psi = \mu e^{ik\Lambda x + 4ik^3\Lambda t},
\]

one has the new Lax pair equations

\[
\begin{cases}
\mu_x - [ik\Lambda, \mu] = V_1\mu, \\
\mu_t - [4ik^3\Lambda, \mu] = V_2\mu.
\end{cases}
\]

(2.8)

Supposing that \( \hat{\Lambda} \) satisfies \( \hat{\Lambda}X = [\Lambda, X] \) which acts on a \( 3 \times 3 \) matrix \( X \), then one can rewrite the equations in \((2.8)\) as the following form

\[
d (e^{(-ikx - 4ik^3t)\hat{\Lambda}}\mu) = W,
\]

(2.9)

where the closed one-form \( W(x, t, k) \) can be defined as

\[
W = e^{(-ikx - 4ik^3t)\hat{\Lambda}} (V_1 dx + V_2 dt) \mu.
\]

(2.10)

2.2. The spectral function \( \mu_j's \) definition. Based on the Volterra integral equation, four eigenfunctions \( \{\mu_j\}^4_1 \) of \((2.8)\) can be defined as

\[
\mu_j(x, t, k) = I + \int_{\gamma_j} e^{(ikx + 4ik^3t)\hat{\Lambda}} W_j(x', t', k), \quad j = 1, 2, 3, 4,
\]

(2.11)

where \( W_j \) is determined by \((2.10)\) with \( \mu \) replaced with \( \mu_j \), and the contours \( \{\gamma_j\}^4_1 \) are shown in Figure 1. The first, second, and third columns of the matrix equation \((2.11)\) involves the exponentials

\[
\begin{align*}
[\mu_j]_1 &: e^{2ik(x-x')+8ik^3(t-t')}, \quad e^{2ik(x-x')+8ik^3(t-t')}, \\
[\mu_j]_2 &: e^{-2ik(x-x')-8ik^3(t-t')}, \\
[\mu_j]_3 &: e^{-2ik(x-x')-8ik^3(t-t')}.
\end{align*}
\]

(2.12)
The contours \( \{ \gamma_j \}_1^4 \) can be given by the following inequalities

\[
\begin{align*}
\gamma_1 : & \quad x - x' \geq 0, \quad t - t' \leq 0, \\
\gamma_2 : & \quad x - x' \geq 0, \quad t - t' \geq 0, \\
\gamma_3 : & \quad x - x' \leq 0, \quad t - t' \geq 0, \\
\gamma_4 : & \quad x - x' \leq 0, \quad t - t' \leq 0.
\end{align*}
\]  

(2.13)

From (2.13), one can show that the functions \( \{ \mu_j \}_1^4 \) are bounded and analytic for \( k \in \mathbb{C} \) such that \( k \) belongs to

\[
\begin{align*}
\mu_1 & \text{ is bounded and analytic for } k \in (D_2, D_4, D_4), \\
\mu_2 & \text{ is bounded and analytic for } k \in (D_1, D_3, D_3), \\
\mu_3 & \text{ is bounded and analytic for } k \in (D_4, D_2, D_2), \\
\mu_4 & \text{ is bounded and analytic for } k \in (D_3, D_1, D_1),
\end{align*}
\]  

(2.14)

where \( \{ D_n \}_1^4 \) denote four open, pairwisely disjoint subsets of the Riemann \( k \)-sphere shown in Figure 2.

It should notice that the sets \( \{ D_n \}_1^4 \) admit the following properties

\[
\begin{align*}
D_1 & = \{ k \in \mathbb{C} | \text{Rel}_1 > \text{Rel}_2 = \text{Rel}_3, \text{Rez}_1 > \text{Rez}_2 = \text{Rez}_3 \}, \\
D_2 & = \{ k \in \mathbb{C} | \text{Rel}_1 > \text{Rel}_2 = \text{Rel}_3, \text{Rez}_1 < \text{Rez}_2 = \text{Rez}_3 \}, \\
D_3 & = \{ k \in \mathbb{C} | \text{Rel}_1 < \text{Rel}_2 = \text{Rel}_3, \text{Rez}_1 < \text{Rez}_2 = \text{Rez}_3 \}, \\
D_4 & = \{ k \in \mathbb{C} | \text{Rel}_1 < \text{Rel}_2 = \text{Rel}_3, \text{Rez}_1 > \text{Rez}_2 = \text{Rez}_3 \},
\end{align*}
\]  

(2.15)

where \( l_i(k) \) and \( z_i(k) \) are the diagonal entries of matrices \( ik\Lambda \) and \( 4ik^3\Lambda \), respectively.

We note that \( \mu_1(x, t, k) \) and \( \mu_2(x, t, k) \) are entire functions of \( k \). Moreover, in their corresponding regions of boundedness,

\[
\mu_j(x, t, k) = 1 + O \left( \frac{1}{k} \right), \quad k \to \infty, \quad j = 1, 2, 3, 4.
\]  

(2.16)

In fact, for \( x = 0 \), the function \( \mu_1(0, t, k) \) can be enlarged the domain of boundedness: \( (D_2 \cup D_3, D_1 \cup D_4, D_1 \cup D_4) \), the function \( \mu_2(0, t, k) \) can be enlarged the domain of boundedness: \( (D_1 \cup D_3, D_2 \cup D_3, D_2 \cup D_3, D_2 \cup D_3) \), the function \( \mu_3(0, t, k) \) can be enlarged the domain of boundedness: \( (D_1 \cup D_4, D_2 \cup D_3, D_2 \cup D_3) \), and the function \( \mu_4(0, t, k) \) can be enlarged the domain of boundedness: \( (D_2 \cup D_3, D_1 \cup D_4, D_1 \cup D_4) \).
2.3. The $M'_n$’s definition. The solutions $\{M_n(x, t, k)\}^4_{n=1}$ of (2.8) can be defined by the following system of integral equations

$$(M_n)_{ij}(x, t, k) = \delta_{ij} + \int_{\gamma_{ij}^n} e^{i(\kappa x + 4\kappa^3 t)} W_n(x', t', k) \, dj, k \in D_n, i, j = 1, 2, 3,$$  \hspace{1cm} (2.17)

where $\{W_n\}^4_{n=1}$ are determined by (2.10) with $\mu$ replaced with $M_n$, and the contours $\gamma_{ij}^n, n = 1, \ldots, 4, i, j = 1, 2, 3$ are defined by

$$\gamma_{ij}^n = \begin{cases} 
\gamma_1 & \text{if } \text{Re}_1(k) < \text{Re}_j(k) \text{ and } \text{Re}_i(k) \geq \text{Re}_j(k), \\
\gamma_2 & \text{if } \text{Re}_i(k) < \text{Re}_j(k) \text{ and } \text{Re}_i(k) < \text{Re}_j(k), \\
\gamma_3 & \text{if } \text{Re}_i(k) \geq \text{Re}_j(k) \text{ and } \text{Re}_i(k) \leq \text{Re}_j(k), \\
\gamma_4 & \text{if } \text{Re}_i(k) \geq \text{Re}_j(k) \text{ and } \text{Re}_i(k) \geq \text{Re}_j(k), 
\end{cases} \quad \text{for } k \in D_n. \hspace{1cm} (2.18)$$

It is remarked that the distinction between the contours $\gamma_3$ and $\gamma_4$ is given by

$$\gamma_{ij}^n = \begin{cases} 
\gamma_3 & \text{if } \Pi_{1 \leq i < j \leq 3} (\text{Re}_i(k) - \text{Re}_j(k))(\text{Re}_i(k) - \text{Re}_j(k)) < 0, \\
\gamma_4 & \text{if } \Pi_{1 \leq i < j \leq 3} (\text{Re}_i(k) - \text{Re}_j(k))(\text{Re}_i(k) - \text{Re}_j(k)) > 0. 
\end{cases} \hspace{1cm} (2.19)$$

It implies that if $l_n = l_n, m$ may not equal $n$, one can just choose the subscript is smaller one by considering the rule (2.19).

Based on the definition of the $\gamma^n$, one can show that

$$\gamma^1 = \begin{pmatrix} \gamma_2 & \gamma_4 & \gamma_4 \\
\gamma_2 & \gamma_4 & \gamma_4 \\
\gamma_2 & \gamma_4 & \gamma_4 
\end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 \\
\gamma_3 & \gamma_3 & \gamma_3 \\
\gamma_3 & \gamma_3 & \gamma_3 
\end{pmatrix}, \hspace{1cm} (2.20)$$

$$\gamma^3 = \begin{pmatrix} \gamma_4 & \gamma_2 & \gamma_2 \\
\gamma_4 & \gamma_4 & \gamma_4 \\
\gamma_4 & \gamma_4 & \gamma_4 
\end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} \gamma_3 & \gamma_1 & \gamma_1 \\
\gamma_3 & \gamma_3 & \gamma_3 \\
\gamma_3 & \gamma_3 & \gamma_3 
\end{pmatrix}.$$

In order to present the formulation of a Riemann-Hilbert problem, we provide the following proposition ascertains that the $M'_n$’s defined in this way.

**Proposition 1.** The function $M_n(x, t, k)$ is well defined by Equation (2.17) with $k \in \mathcal{D}_n$ and $x, t \in \Omega$ for each $n = 1, \ldots, 4$. For any fixed point $(x, t)$, $\{M_n\}^4_{n=1}$ are bounded and analytic as functions of $k \in D_n$ away from a possible discrete set of singularities $\{k_j\}$ at which the Fredholm determinant vanishes. Moreover, $M_n$ admits a bounded and continuous extension to $\overline{\mathcal{D}_n}$ and

$$M_n(x, t, k) = I + O \left( \frac{1}{k} \right), \quad k \to \infty, k \in D_n. \hspace{1cm} (2.21)$$

**Proof.** The boundedness and analyticity properties are established in appendix B in [26]. Substituting the expansion

$$M = M_0 + \frac{M^{(1)}}{k} + \frac{M^{(2)}}{k^2} + \cdots, \quad k \to \infty \hspace{1cm} (2.22)$$

into the Lax pair equations (2.8) and comparing the terms of the same order of $k$ yield the result (2.21). \hfill \Box

**Remark 1.** Here, two sets of eigenfunctions: $\{\mu_j\}^3_{j=1}$ and $\{M_n\}^4_{n=1}$ are introduced. The two types of eigenfunctions are also used in the unified approach introduced by Fokas [7] for Lax pairs involving $2 \times 2$ matrices. The $\mu_j$ is introduced for the spectral analysis, whereas the other set of eigenfunctions can be used to formulate the Riemann-Hilbert problem. Here $M'_n$ is the analogues of eigenfunctions.
Remark 2. For given point \((x,t)\), the function \(M_n\) is bounded and analytic with \(k \in D_n\) away from a possible discrete set of singularities \(\{k_j\}\), where the Fredholm determinant vanishes. The boundedness and analyticity properties are provided in [26].

2.4. The jump matrices. The matrix-value functions \(\{S_n(k)\}_1^4\) can be defined by

\[
S_n(k) = M_n(0,0,k), \quad k \in D_n, \quad n = 1, \ldots, 4.
\]  

Let \(M\) be the sectionally analytic function on the Riemann \(k\)-sphere which equals \(M_n\) for \(k \in D_n\). Then \(M\) satisfies the following jump conditions

\[
M_n = M_m J_{m,n}, \quad k \in \overline{D_n} \cap \overline{D_m}, \quad n, m = 1, \ldots, 4, \quad n \neq m,
\]  

where \(J_{m,n} = J_{m,n}(x,t,k)\) are the jump matrices given by

\[
J_{m,n} = e^{(ikx+4ik^3)t} \tilde{\Lambda} \left( S_{m}^{-1} S_{n} \right).
\]  

Remark 3. The \(M_n(0,0,k)\) is defined by the integral equations (2.17) involving only integration along the boundary \(\{x = 0, 0 < t < T\}\) and the interval \(\{0 < x < L, t = 0\}\). Similarly, the \(S_n\) (and also the \(J_{m,n}\)) can be derived from the initial and boundary data alone. Thus, the solution \(\{p(x,t), q(x,t)\}\) from the initial and boundary data can be reconstructed by the jump condition provided by the relation (2.24) for a Riemann-Hilbert problem in the absence of singularities. However, if the \(M_n\) admit pole singularities at some points \(\{k_j\}, \quad k_j \in \mathbb{C}\), the Riemann-Hilbert problem needs to involve the residue conditions at these points. In order to determine the correct residue conditions (and also for analysing the nonlinearizable boundary conditions in Section 4), four eigenfunctions \(\{\mu_j(x,t,k)\}_{j=1}^4\) should be introduced in addition to the \(M_n\).

2.5. The adjugated eigenfunctions. In order to obtain the analyticity and boundedness properties of the minors of the matrices \(\{\mu_j(x,t,k)\}_1^4\). Let’s consider the cofactor matrix \(X^A\) of a \(3 \times 3\) matrix \(X\) given by

\[
X^A = \begin{pmatrix}
  m_{11}(X) & -m_{12}(X) & m_{13}(X) \\
  -m_{21}(X) & m_{22}(X) & -m_{23}(X) \\
  m_{31}(X) & -m_{32}(X) & m_{33}(X)
\end{pmatrix},
\]  

where \(m_{ij}(X)\) is the \((ij)\) th minor of \(X\).

From (2.8), one can show that the adjugated eigenfunction \(\mu^A\) admits the following Lax pair equations

\[
\begin{cases}
\mu_x^A + [ik\Lambda, \mu^A] = -V_1^T \mu^A, \\
\mu_t^A + [4ik^3\Lambda, \mu^A] = -V_2^T \mu^A,
\end{cases}
\]  

where \(V^T\) is the transform of a matrix \(V\). Thus, the eigenfunctions \(\{\mu_j\}_1^4\) satisfy the following integral equations

\[
\mu_j^A(x,t,k) = I - \int_{\gamma_j} e^{i[k(x-x') - 4ik^3(t-t')]} (V_1^T dx + V_2^T dt) \mu^A, \quad j = 1, \ldots, 4.
\]  

(2.28)
For \( \{\mu_j^4\}^4_{j=1} \), one can obtain the following analyticity and boundedness properties

\[
\begin{align*}
\mu_1^4 & \text{ is bounded and analytic for } k \in (D_4, D_2, D_2), \\
\mu_2^4 & \text{ is bounded and analytic for } k \in (D_3, D_1, D_1), \\
\mu_3^4 & \text{ is bounded and analytic for } k \in (D_2, D_4, D_4), \\
\mu_4^4 & \text{ is bounded and analytic for } k \in (D_1, D_3, D_3).
\end{align*}
\]  

(2.29)

In fact, for \( x = 0 \), \( \mu_1^4(0, t, k) \) can be enlarged the domain of boundedness: \( (D_1 \cup D_4, D_2 \cup D_3, D_2 \cup D_3, D_3) \), \( \mu_3^4(0, t, k) \) can be enlarged the domain of boundedness: \( (D_2 \cup D_3, D_1 \cup D_4, D_1 \cup D_4, D_1 \cup D_4) \), and \( \mu_3^4(0, t, k) \) can be enlarged the domain of boundedness: \( (D_1 \cup D_4, D_2 \cup D_3, D_2 \cup D_3) \).

2.6. Symmetries. By the following Lemma, we show that the eigenfunctions \( \mu_j(x, t, k) \) admit an important symmetry.

**Lemma 2.1.** The eigenfunction \( \Psi(x, t, k) \) of the Lax pair equations (2.1a) and (2.1b) admits the following symmetry

\[
\Psi^{-1}(x, t, k) = A \Psi(x, t, k)^T A,
\]

(2.30)

with

\[
A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}, \quad \sigma^2 = 1,
\]

(2.31)

where the superscript \( T \) denotes a matrix transpose.

**Proof.** From the following equations

\[
-\hat{A} U(x, t, k) A = U(x, t, k)^T, \quad -\hat{A} V(x, t, k) A = V(x, t, k)^T,
\]

(2.32)

and

\[
\begin{align*}
\Psi^A_\mu(x, t, k) &= -U(x, t, k)^T \Psi^A(x, t, k), \\
\Psi^A_\mu(x, t, k) &= -V(x, t, k)^T \Psi^A(x, t, k),
\end{align*}
\]

(2.33)

one can show that equation (2.30) holds. \( \square \)

**Remark 4.** From Lemma 2.1, one can show that the eigenfunctions \( \mu_j(x, t, k) \) of Lax pair equations (2.8) admit the same symmetry.

2.7. The \( J_{m,n,s} \) computation. The \( 3 \times 3 \)-matrix value spectral functions \( s(k), S(k), \) and \( S_L(k) \) can be defined by

\[
\begin{align*}
\mu_3(x, t, k) &= \mu_2(x, t, k) e^{ikx+4ik^3t} \hat{\lambda} s(k), \\
\mu_1(x, t, k) &= \mu_2(x, t, k) e^{ikx+4ik^3t} \hat{\lambda} S(k), \\
\mu_4(x, t, k) &= \mu_3(x, t, k) e^{ikx-L+4ik^3t} \hat{\lambda} S_L(k).
\end{align*}
\]

(2.34)

Thus,

\[
\begin{align*}
s(k) &= \mu_3(0, 0, k), \\
S(k) &= \mu_1(0, 0, k) = e^{-4ik^3T} \mu_2^{-1}(0, 0, k), \\
S_L(k) &= \mu_4(L, 0, k) = e^{-4ik^3T} \mu_3^{-1}(L, T, k).
\end{align*}
\]

(2.35)
From the properties of $\mu_j$ and $\mu_j^A$, one can derive that $\{s(k), S(k), S_L(k)\}$ and $\{s^A(k), S^A(k), S^A_L(k)\}$ admit the following boundedness properties

$$s(k) \text{ is bounded for } k \in (D_3 \cup D_4, D_1 \cup D_2, D_1 \cup D_2),$$
$$S(k) \text{ is bounded for } k \in (D_2 \cup D_3, D_1 \cup D_4, D_1 \cup D_4),$$
$$S_L(k) \text{ is bounded for } k \in (D_2 \cup D_3, D_1 \cup D_4, D_1 \cup D_4),$$
$$s^A(k) \text{ is bounded for } k \in (D_1 \cup D_2, D_3 \cup D_4, D_3 \cup D_4),$$
$$S^A(k) \text{ is bounded for } k \in (D_1 \cup D_4, D_2 \cup D_3, D_2 \cup D_3),$$
$$S^A_L(k) \text{ is bounded for } k \in (D_1 \cup D_4, D_2 \cup D_3, D_2 \cup D_3).$$

(2.36)

We also notice that

$$M_n(x, t, k) = \mu_2(x, t, k)e^{(ikx+4ik^3t)}\tilde{\Lambda}S_n(k), \quad k \in D_n. \quad (2.37)$$

**Proposition 2.** The functions $\{S_n\}^4_1$ can be expressed in terms of the entries of $s(k), S(k),$ and $S_L(k)$ as follows

$$S_1 = \begin{pmatrix}
\frac{1}{m_{11}(A)} & A_{12} & A_{13} \\
0 & A_{22} & A_{23} \\
0 & A_{32} & A_{33}
\end{pmatrix}, \quad (2.38a)$$

$$S_2 = \begin{pmatrix}
s_{11} & s_{12} & s_{13} \\
\left(\frac{s^T S^A}{m_{11}}\right)_{11} & s_{22} & s_{23} \\
\left(\frac{s^T S^A}{m_{11}}\right)_{11} & s_{32} & s_{33}
\end{pmatrix}, \quad (2.38b)$$

$$S_3 = \begin{pmatrix}
A_{11} & 0 & 0 \\
A_{21} & \frac{m_{33}(A)}{m_{23}(A)} & \frac{m_{22}(A)}{m_{23}(A)} \\
A_{31} & \frac{m_{33}(A)}{m_{23}(A)} & \frac{m_{22}(A)}{m_{23}(A)}
\end{pmatrix}, \quad (2.38c)$$

$$S_4 = \begin{pmatrix}
s_{11} & \frac{m_{23}(s)M_{22}(S) - m_{23}(s)M_{31}(S)}{(s^T S^A)_{11}} & \frac{m_{22}(s)M_{21}(S) - m_{22}(s)M_{31}(S)}{(s^T S^A)_{11}} \\
\left(\frac{s^T S^A}{m_{11}}\right)_{11} & m_{23}(s)M_{21}(S) - m_{23}(s)M_{31}(S) & m_{22}(s)M_{21}(S) - m_{22}(s)M_{31}(S) \\
\left(\frac{s^T S^A}{m_{11}}\right)_{11} & m_{23}(s)M_{11}(S) - m_{23}(s)M_{21}(S) & m_{22}(s)M_{11}(S) - m_{22}(s)M_{21}(S)
\end{pmatrix}, \quad (2.38d)$$

where the $3 \times 3$ matrix $\{A_{ij}\}^3_{i,j=1}$ is defined as $A = s(k)e^{ik\tilde{\Lambda}}S_L(k)$, and the functions $(s^T S^A)_{11}, (s^T S^A)_{11}$ are defined as follows

$$(s^T S^A)_{11} = S_{11}m_{11}(s) - S_{21}m_{21}(s) + S_{31}m_{31}(s),$$
$$(s^T S^A)_{11} = S_{11}M_{11}(S) - S_{21}M_{21}(S) + S_{31}M_{31}(S). \quad (2.39)$$

**Proof.** In order to derive the expressions of $\{S_n\}^4_1$, we can introduce three new functions $R_n(k), T_n(k)$ and $Q_n(k)$ given by

$$R_n(k) = e^{-4ik^3T}\tilde{\Lambda}M_n(0, T, k), \quad (2.40a)$$
$$T_n(k) = e^{-ikL\tilde{\Lambda}}M_n(L, 0, k), \quad (2.40b)$$
$$Q_n(k) = e^{-(ikL+4ik^3T)}\tilde{\Lambda}M_n(L, T, k). \quad (2.40c)$$
Then, one can derive the following relations

\[
\begin{align*}
M_n(x, t, k) &= \mu_1(x, t, k)e^{(ikx+4ik^3t)x}R_n(k), \\
M_n(x, t, k) &= \mu_2(x, t, k)e^{(ikx+4ik^3t)x}S_n(k), \\
M_n(x, t, k) &= \mu_3(x, t, k)e^{(ikx+4ik^3t)x}T_n(k), \\
M_n(x, t, k) &= \mu_4(x, t, k)e^{(ikx+4ik^3t)x}Q_n(k),
\end{align*}
\]

which show that

\[
\begin{align*}
\{s(k) &= S_n(k)T_n(k)^{-1}, \\
S(k) &= S_n(k)R_n(k)^{-1}, \\
A(k) &= S_n(k)Q_n(k)^{-1}.
\end{align*}
\]

From system (2.42), one can obtain a matrix factorization problem. For the given \{s(k), S(k), S_L(k)\}, they can be solved for the \{R_n, S_n, T_n, Q_n\}. From the definitions of \{R_n, S_n, T_n, Q_n\} and the integral equations (2.17), one has

\[
\begin{align*}
(R_n(k))_{ij} &= 0, \quad \text{if } \gamma^{n}_{ij} = \gamma_1, \\
(S_n(k))_{ij} &= 0, \quad \text{if } \gamma^{n}_{ij} = \gamma_2, \\
(T_n(k))_{ij} &= \delta_{ij}, \quad \text{if } \gamma^{n}_{ij} = \gamma_3, \\
(Q_n(k))_{ij} &= \delta_{ij}, \quad \text{if } \gamma^{n}_{ij} = \gamma_4.
\end{align*}
\]

From the results in (2.43), we have 27 scalar equations for 27 unknowns. By computing the explicit solution of this algebraic system, one can derive the expressions of \{S_n\}^4_1 in (2.38a)-(2.38d). \qed

**Remark 5.** Based on symmetry in Lemma 2.1, the representations of the functions \{S_n(k)\}^4_1 can be simple. It is easy to calculate the jump matrices \(M_{m,n}(x, t, k)\) by considering the simple expressions of \{S_n(k)\}^4_1.

### 2.8. The global relation.

In what follows, we will show that the spectral functions \(S(k), S_L(k)\) and \(s(k)\) admit a very important relationship. They are dependent with each other. From the system (2.34a)-(2.34c), one can show that

\[
\mu_1(x, t, k)e^{(ikx+4ik^3t)x} \left\{ S_n^{-1}(k)s(k)e^{-ikLx}S_L(k) \right\} = \mu_3. \tag{2.44}
\]

By considering the point \((0, T)\), we have the following global relationship

\[
S_n^{-1}(k)s(k)e^{-ikLx}S_L(k) = e^{-4ik^3T}c(T, k) \tag{2.45}
\]

since \(\mu_1(0, T, k) = 1\), where \(c(T, k) = \mu_4(0, T, k)\).

In fact, for each \(t \in (0, T)\), suppose that \(\mathcal{R}(x, t, k)\) is the solution of the \(x\)-part of the Lax pair of (1.1), such that \(\mathcal{R}(L, t, k) = 1\), i.e. \(\mathcal{R}\) is the unique solution of the Volterra integral equation

\[
\mathcal{R}(L, t, k) = 1 + \int_0^L e^{ik(x-x')} \left( V_1 \mathcal{R} \right) (x', t, k) dx', \quad 0 < x < L. \tag{2.46}
\]

It implies that \(\mathcal{R}\) can be connected with \(\mu_3\) by

\[
\mathcal{R}(x, t, k) = \mu_3(x, t, k)e^{ik(L-x)}\mu_3^{-1}(L, t, k). \tag{2.47}
\]

From the equations (2.34a) and (2.47), we have

\[
\mathcal{R}(x, t, k) = \mu_2(x, t, k) \left[ e^{(ikx+4ik^3t)x} s(k) \right] e^{ik(L-x)}\mu_3^{-1}(L, t, k). \tag{2.48}
\]
2.9. The residue conditions. From (2.37), one can show that $M$ only has singularities at the points where the $\{S'_n s\}^4$ have singularities since $\mu_2$ is an entire function. We introduce the symbols $\{k_j\}_1^N$ to denote the possible zeros and assume $\{k_j\}_1^N$ satisfy the following assumption.

**Assumption 2.2.** We suppose that
- $m_{11}(A(k))$ admits $n_0 \geq 0$ possible simple zeros in $D_1$ denoted by $\{k_j\}_1^{n_0}$;
- $(S^T S^A)_{11}(k)$ admits $n_1 - n_0 \geq 0$ possible simple zeros in $D_2$ denoted by $\{k_j\}_n^{n_1+1}$;
- $A_{11}(k)$ admits $n_2 - n_1 \geq 0$ possible simple zeros in $D_3$ denoted by $\{k_j\}_n^{n_2+1}$;
- $(s^T S^A)_{11}(k)$ admits $N - n_2 \geq 0$ possible simple zeros in $D_4$ denoted by $\{k_j\}_N^{n_2+1}$;

and none of $\{k_j\}_1^N$ coincide. Moreover, we suppose that none of such functions $m_{11}(A)(k)$, $(S^T s^A)_{11}(k)$, $A_{11}(k)$ and $(s^T S^A)_{11}(k)$ have zeros on the boundaries of the $\{D_n's\}^4$.

The residue conditions at these zeros $\{k_j\}_1^N$ are determined by the following Proposition.

**Proposition 3.** Suppose that $\{M_n\}_1^4$ are the eigenfunctions defined by (2.17) and the zeros $\{k_j\}_1^N$ admit the above assumption, we have the following residue conditions

$$Res_{k_j} [M]_1 = A_{23}(k_j)M(k_j)_2 - A_{22}(k_j)M(k_j)_3 e^{2i\kappa x + 2ik_j t}, 0 \leq j \leq n_0, k_j \in D_1,$$

(2.49a)

$$Res_{k_j} [M]_1 = \frac{S_{21}(k_j)s_{33}(k_j) - S_{31}(k_j)s_{23}(k_j)}{(S^T S^A)_{11}(k_j)m_{11}(k_j)} e^{2i\kappa x + 2ik_j t}[M(k_j)]_2$$

$$+ \frac{S_{31}(k_j)s_{32}(k_j) - S_{21}(k_j)s_{32}(k_j)}{(S^T S^A)_{11}(k_j)m_{11}(k_j)} e^{2i\kappa x + 2ik_j t}[M_3(k_j)]_3, n_0 + 1 \leq j \leq n_1, k_j \in D_2,$$

(2.49b)

$$Res_{k_j} [M]_2 = \frac{m_{33}(A(k_j))}{A_{11}(k_j)A_{21}(k_j)} e^{-2i\kappa x - 2ik_j t}[M(k_j)]_1, n_1 + 1 \leq j \leq n_2, k_j \in D_3,$$

(2.49c)

$$Res_{k_j} [M]_3 = \frac{m_{32}(A(k_j))}{A_{11}(k_j)A_{21}(k_j)} e^{-2i\kappa x - 2ik_j t}[M(k_j)]_1, n_1 + 1 \leq j \leq n_2, k_j \in D_3,$$

(2.49d)

$$Res_{k_j} [M]_2 = \frac{m_{33}(s(k_j))M_{21}(S(k_j)) - m_{23}(s(k_j))M_{31}(S(k_j))}{(s^T S^A)_{11}(k_j)s_{11}(k_j)} e^{-2i\kappa x - 2ik_j t}[M(k_j)]_1,$$

(2.49e)

$$n_2 + 1 \leq j \leq N, k_j \in D_4,$$

$$Res_{k_j} [M]_3 = \frac{m_{32}(s(k_j))M_{21}(S(k_j)) - m_{22}(s(k_j))M_{31}(S(k_j))}{(s^T S^A)_{11}(k_j)s_{11}(k_j)} e^{-2i\kappa x - 2ik_j t}[M(k_j)]_1,$$

(2.49f)

where $\dot{f} = \frac{df}{dt}$.

**Proof.** In what follows, we will derive the results (2.49a), (2.49b), (2.49c) and (2.49e), the other conditions follow similar arguments. From equation (2.37), we
have the following relations

\[ M_1 = \mu_2 e^{(i k x + 4 ik^3 t) \hat{A}} S_1, \]  
\[ M_2 = \mu_2 e^{(i k x + 4 ik^3 t) \hat{A}} S_2, \]  
\[ M_3 = \mu_2 e^{(i k x + 4 ik^3 t) \hat{A}} S_3, \]  
\[ M_4 = \mu_2 e^{(i k x + 4 ik^3 t) \hat{A}} S_4. \]  

By considering the expression for \( S_1 \) determined by (2.38a), the three columns of \( M_1 \) (2.50a) can be derived as

\[ [M_1]_1 = [\mu_2]_3 \frac{1}{m_{11}(A)}, \]  
\[ [M_1]_2 = [\mu_2]_1 A_{12} e^{-2ik_j x - 8ik^3 t} + [\mu_2]_3 A_{22} + [\mu_2]_3 A_{32}, \]  
\[ [M_1]_3 = [\mu_2]_1 A_{13} e^{-2ik_j x - 8ik^3 t} + [\mu_2]_3 A_{23} + [\mu_2]_3 A_{33}. \]

Similarly, one can also obtain the three columns of \( M_2 \) (2.50b)

\[ [M_2]_1 = [\mu_2]_1 S_{11} \left( \frac{s^7 S^4}{s^5 S^4} \right)_{11} + [\mu_2]_2 S_{21} \left( \frac{s^7 S^4}{s^5 S^4} \right)_{11} e^{2ik_j x + 8ik^3 t} + [\mu_2]_3 S_{31} \left( \frac{s^7 S^4}{s^5 S^4} \right)_{11} e^{2ik_j x + 8ik^3 t}, \]

\[ [M_2]_2 = [\mu_2]_1 s_{12} e^{-2ik_j x - 8ik^3 t} + [\mu_2]_2 s_{22} + [\mu_2]_3 s_{32}, \]
\[ [M_2]_3 = [\mu_2]_1 s_{13} e^{-2ik_j x - 8ik^3 t} + [\mu_2]_2 s_{23} + [\mu_2]_3 s_{33}, \]

the three columns of \( M_3 \) (2.50c)

\[ [M_3]_1 = [\mu_2]_1 A_{11} + [\mu_2]_2 A_{21} e^{2ik_j x + 8ik^3 t} + [\mu_2]_3 A_{31} e^{2ik_j x + 8ik^3 t}, \]
\[ [M_3]_2 = [\mu_2]_1 \frac{m_{33}(A)}{A_{11}} + [\mu_2]_3 \frac{m_{23}(A)}{A_{11}}, \]
\[ [M_3]_3 = [\mu_2]_1 \frac{m_{32}(A)}{A_{11}} + [\mu_2]_3 \frac{m_{22}(A)}{A_{11}}, \]

and the three columns of \( M_4 \) (2.50d)

\[ [M_4]_1 = [\mu_2]_1 s_{11} + [\mu_2]_2 s_{21} e^{2ik_j x + 8ik^3 t} + [\mu_2]_3 s_{31} e^{2ik_j x + 8ik^3 t}, \]
\[ [M_4]_2 = [\mu_2]_1 \frac{m_{33}(s) M_{21}(s) - m_{13}(s) M_{31}(s)}{(s^7 S^4)_{11}} e^{-2ik_j x - 8ik^3 t} \]
\[ + [\mu_2]_2 \frac{m_{33}(s) M_{11}(s) - m_{13}(s) M_{31}(s)}{(s^7 S^4)_{11}} + [\mu_2]_3 \frac{m_{23}(s) M_{11}(s) - m_{13}(s) M_{21}(s)}{(s^7 S^4)_{11}}, \]
\[ [M_4]_3 = [\mu_2]_1 \frac{m_{32}(s) M_{21}(s) - m_{22}(s) M_{31}(s)}{(s^7 S^4)_{11}} e^{-2ik_j x - 8ik^3 t} \]
\[ + [\mu_2]_2 \frac{m_{32}(s) M_{11}(s) - m_{12}(s) M_{31}(s)}{(s^7 S^4)_{11}} + [\mu_2]_3 \frac{m_{22}(s) M_{11}(s) - m_{12}(s) M_{21}(s)}{(s^7 S^4)_{11}}. \]

Let \( k_j \in D_1 \) be a simple zero of \( m_{11} \). Solving (2.51b) and (2.51c) for \([\mu_2]_1\) and \([\mu_2]_2\), and substituting the result of \([\mu_2]_1\) into (2.51a), one can obtain

\[ [M_1]_1 = \frac{A_{23}[M_1]_2 - A_{22}[M_1]_3}{m_{11}(A)m_{31}(A)} e^{2ik_j x + 8ik^3 t} + \frac{[\mu_2]_3}{m_{31}(A)} e^{2ik_j x + 8ik^3 t}. \]
Taking the residue of (2.55) at $k_j$, it implies that (2.49a) holds in the case when $k_j \in D_1$.

In order to derive (2.49b), we solve (2.52b) and (2.52c) for $[\mu_2]_2$ and $[\mu_2]_3$. Substituting the results into (2.52a), one can obtain

$$
[M_2]_1 = \frac{1}{m_{11}(s)}[\mu_2]_1 + \frac{S_{21}s_{33} - S_{31}s_{23}}{(S^T S^A)_{11}m_{11}(s)}e^{2ik_jx+8ik_j^3t}[M_2]_2
+ \frac{S_{31}s_{22} - S_{21}s_{32}}{(S^T S^A)_{11}m_{11}(s)}e^{2ik_jx+8ik_j^3t}[M_2]_3.
$$

(2.56)

Taking the residue of Equation (2.56) at $k_j$, it implies that the condition (2.49b) holds in the case when $k_j \in D_2$.

Similarly, we solve (2.53a) for $[\mu_2]_2$. Substituting the result into (2.53b), one can obtain

$$
[M_3]_2 = \frac{m_{33}(A)}{A_{11}A_{21}}e^{-2ik_jx-8ik_j^3t}[M_3]_1 - \frac{m_{33}(A)}{A_{21}}e^{-2ik_jx-8ik_j^3t}[\mu_2]_1 + \frac{m_{13}(A)}{A_{21}}[\mu_2]_3.
$$

(2.57)

Taking the residue of Equation (2.57) at $k_j$, one can show that the condition (2.49c) holds in the case when $k_j \in D_3$.

By similar arguments, solving (2.54a) for $[\mu_2]_1$, and substituting the result into (2.54b), one can obtain

$$
[M_4]_2 = \frac{m_{33}(s)M_{21}(S) - m_{23}(s)M_{31}(S)}{(s^T S^A)_{11}s_{11}}e^{-2ik_jx-8ik_j^3t}[M_4]_1
+ \frac{m_{33}(s)}{s_{11}}[\mu_2]_2 + \frac{m_{23}(s)}{s_{11}}[\mu_2]_3.
$$

(2.58)

Taking the residue of Equation (2.58) at $k_j$, one can show that the condition (2.49c) hold in the case when $k_j \in D_4$. 

3. The Riemann-Hilbert problem. In Section 2, we introduce the sectionally analytic function $M(x,t,k)$. It admits a Riemann-Hilbert problem, which can be formulated in terms of the initial and boundary values of the functions $p(x,t)$ and $q(x,t)$. By solving the Riemann-Hilbert problem, one can recover the solution of (1.1) for all values of the independent variables $x, t$.

**Theorem 3.1.** Let $p(x,t)$ and $q(x,t)$ be a pair of solutions of (1.1) in the interval domain $\Omega$. Then $p(x,t)$ and $q(x,t)$ can be reformulated by the initial values $\{p_0(x), q_0(x)\}$ and boundary values $\{g_{01}(t), g_{02}(t), g_{11}(t), g_{12}(t), g_{21}(t), g_{22}(t), f_{01}(t), f_{02}(t), f_{11}(t), f_{12}(t), f_{21}(t), f_{22}(t)\}$, which are defined by

$$
p_0(x) = p(x,t = 0), \quad q_0(x) = q(x,t = 0),
g_{01}(t) = p(x = 0,t), \quad g_{02}(t) = q(x = 0,t),
f_{01}(t) = p(x = L,t), \quad f_{02}(t) = q(x = L,t),
g_{11}(t) = p_x(x = 0,t), \quad g_{12}(t) = q_x(x = 0,t),
f_{11}(t) = p_x(x = L,t), \quad f_{12}(t) = q_x(x = L,t),
g_{21}(t) = p_{xx}(x = 0,t), \quad g_{22}(t) = q_{xx}(x = 0,t),
f_{21}(t) = p_{xx}(x = L,t), \quad f_{22}(t) = q_{xx}(x = L,t).
$$

(3.1)
By using the initial and boundary data, the jump matrices $J_{m,n}(x,t,k)$ can be defined in terms of the spectral functions $s(k)$ and $S(k)$, $S_L(k)$ by the system (2.34a)-(2.34c).

Supposing that the possible zeros $\{k_j\}_{j=1}^N$ of the functions $m_{11}(A)(k)$, $(S^tS^A)_{11}(k)$, $A_{11}(k)$ and $(S^tS^A)_{11}(k)$ satisfy Assumption 2.2 in $\{D_n\}_n$.

Then the solution $\{p(x,t), q(x,t)\}$ of (1.1) with initial and boundary values problem (3.1) is given by

$$p(x,t) = -2i \lim_{k \to \infty} (kM(x,t,k))_{21}, \quad q(x,t) = -2i \lim_{k \to \infty} (kM(x,t,k))_{31}, \quad (3.2)$$

where $M(x,t,k)$ admit the following $3 \times 3$ matrix Riemann-Hilbert problem:

- $M$ is sectionally meromorphic on the Riemann $k$-sphere with jumps across the contours $\overline{D}_n \cap \overline{D}_m$, $n, m = 1, \ldots, 4$, see Figure 2.
- Across the contours $\overline{D}_n \cap \overline{D}_m$, $M$ admits the jump condition

$$M_n(x,t,k) = M_m(x,t,k)J_{m,n}(x,t,k), \quad k \in \overline{D}_n \cap \overline{D}_m, \quad n, m = 1, 2, 3, 4. \quad (3.3)$$

- $M(x,t,k) = 1 + O\left(\frac{1}{k}\right)$, $k \to \infty$.
- $M$ admits the residue condition provided in Proposition 3.

Proof. From Section 2, it only remains to prove (3.2), which can follows from the large $k$ asymptotics of the eigenfunctions.

4. **Asymptotic analysis.** By analyzing Lax pair Equations (2.8), one can show that the eigenfunctions $\{\mu_j\}_j$ admit the following asymptotics as $k \to \infty$ (see [33])

$$\mu_j(x,t,k) = 1 + \frac{1}{k} \left( \begin{array}{c}
\frac{j}{2} \int_{(x,t)} \Delta^{(1)}_{11} - \frac{i}{2} p \Delta^{(1)}_{12} - \frac{i}{2} q \Delta^{(1)}_{13} \\
\frac{i}{2} \int_{(x,t)} \Delta^{(2)}_{21} + \frac{1}{k^2} \mu_{12}^{(2)} + \frac{1}{k^3} \mu_{13}^{(2)} \\
\frac{i}{2} \int_{(x,t)} \Delta^{(3)}_{31} + \frac{1}{k^2} \mu_{21}^{(3)} + \frac{1}{k^3} \mu_{22}^{(3)} + \frac{1}{k^2} \mu_{31}^{(3)} + \frac{1}{k^3} \mu_{32}^{(3)} + \frac{1}{k^3} \mu_{33}^{(3)} + O\left(\frac{1}{k^4}\right) 
\end{array} \right), \quad (4.1)$$

where

$$\begin{align*}
\Delta^{(1)}_{11} &= \left(p^2 + q^2\right) dx + \left[-2pp_{xx} - 2qq_{xx} + p_x^2 + q_x^2 + 3 \left(p^2 + q^2\right)^2\right] dt, \\
\Delta^{(1)}_{22} &= -p dx + \left[2pp_{xx} - p_x^2 - 2p^2 (p^2 + q^2)\right] dt, \\
\Delta^{(1)}_{33} &= -pq dx + \left[2pp_{xx} - pq_{xx} - 3pq (p^2 + q^2)\right] dt, \\
\Delta^{(1)}_{23} &= -pq dx + \left[2pp_{xx} - p_x q_x - 3pq (p^2 + q^2)\right] dt, \\
\Delta^{(1)}_{32} &= -pq dx + \left[2pp_{xx} - p_x q_x - 3pq (p^2 + q^2)\right] dt, \\
\end{align*} \quad (4.2a)$$
\[
\begin{align*}
\mu_{12}^{(2)} &= \frac{1}{4} \left( p \int_{(x_j, t_j)}^{(x, t)} \Delta_{22}^{(1)} + q \int_{(x_j, t_j)}^{(x, t)} \Delta_{32}^{(1)} \right) + \frac{1}{4} p_x, \\
\mu_{13}^{(2)} &= \frac{1}{4} \left( p \int_{(x_j, t_j)}^{(x, t)} \Delta_{23}^{(1)} + q \int_{(x_j, t_j)}^{(x, t)} \Delta_{33}^{(1)} \right) + \frac{1}{4} q_x, \\
\mu_{21}^{(2)} &= -\frac{1}{4} p \int_{(x_j, t_j)}^{(x, t)} \Delta_{11}^{(1)} + \frac{1}{4} p_x, \\
\mu_{31}^{(2)} &= -\frac{1}{4} q \int_{(x_j, t_j)}^{(x, t)} \Delta_{11}^{(1)} + \frac{1}{4} q_x. 
\end{align*}
\]

\[
\begin{align*}
\Delta_{11}^{(2)} &= -(p^2 + q^2) \int_{(x_j, t_j)}^{(x, t)} \Delta_{11}^{(1)} dx + \{pp_t + qq_t + pp_x + qq_x - 3i (p^2 + q^2) - i (p^2 + q^2)\} \int_{(x_j, t_j)}^{(x, t)} \Delta_{11}^{(1)} dt, \\
\Delta_{22}^{(2)} &= \left[ 2p^2 \int_{(x_j, t_j)}^{(x, t)} \Delta_{22}^{(1)} + 2pq \int_{(x_j, t_j)}^{(x, t)} \Delta_{32}^{(1)} + 2pp_x \right] dx \\
&+ \left\{ 2pp_t - [3pp_{xx} - 2p_{x}^2 + 6p^2 (p^2 + q^2)] \int_{(x_j, t_j)}^{(x, t)} \Delta_{22}^{(1)} - [pq_{xx} + 2p_{xx}q - 2p_x q_x + 6pq (p^2 + q^2)] \int_{(x_j, t_j)}^{(x, t)} \Delta_{32}^{(1)} \right\} dt, \\
\Delta_{23}^{(2)} &= \left[ 2p^2 \int_{(x_j, t_j)}^{(x, t)} \Delta_{23}^{(1)} + 2pq \int_{(x_j, t_j)}^{(x, t)} \Delta_{33}^{(1)} + 2p_{xx} \right] dx \\
&+ \left\{ 2pp_t + 2(p^2 + q^2)(pq_{xx} - p_{x} q_x) - [3pp_{xx} - 2p_{x}^2 + 6p^2 (p^2 + q^2)] \int_{(x_j, t_j)}^{(x, t)} \Delta_{33}^{(1)} \right\} dt, \\
\Delta_{32}^{(2)} &= \left[ 2pq \int_{(x_j, t_j)}^{(x, t)} \Delta_{32}^{(1)} + 2q^2 \int_{(x_j, t_j)}^{(x, t)} \Delta_{33}^{(1)} + 2p_{x} q \right] dx \\
&+ \left\{ 2qq_t + 2(p^2 + q^2)(pq_{xx} - p_{x} q_x) - [pq_{xx} + 2pq_{xx} - 2p_{x} q_x + 6pq (p^2 + q^2)] \int_{(x_j, t_j)}^{(x, t)} \Delta_{33}^{(1)} \right\} dt, \\
\Delta_{33}^{(2)} &= \left[ 2pq \int_{(x_j, t_j)}^{(x, t)} \Delta_{33}^{(1)} + 2q^2 \int_{(x_j, t_j)}^{(x, t)} \Delta_{33}^{(1)} + 2qq_x \right] dx \\
&+ \left\{ 2qq_t - [p_{xx} q + 2pq_{xx} - 2p_{x} q_x + 6pq (p^2 + q^2)] \int_{(x_j, t_j)}^{(x, t)} \Delta_{33}^{(1)} \right\} dt, \\
&- [3qq_{xx} - 2q_{x}^2 + 6q^2 (p^2 + q^2)] \int_{(x_j, t_j)}^{(x, t)} \Delta_{33}^{(1)} \right\} dt, 
\end{align*}
\]

\[(4.2b)\]
and the functions \( \{ \mu_{ij}^{(3)} \}_{j=2}^3 \) and \( \{ \mu_{j1}^{(3)} \}_{j=2}^3 \) are given by

\[
\mu_{12}^{(3)} = -\frac{i}{16} \left( p \int_{(x,t)} \Delta_{22}^{(2)} dx + q \int_{(x,t)} \Delta_{32}^{(2)} dx \right) + \frac{i}{8} \left( p x \int_{(x,t)} \Delta_{22}^{(1)} dx + q x \int_{(x,t)} \Delta_{32}^{(1)} dx + p_{xx} \right), \\
\mu_{13}^{(3)} = -\frac{i}{16} \left( p \int_{(x,t)} \Delta_{33}^{(2)} dx + q \int_{(x,t)} \Delta_{33}^{(2)} dx \right) + \frac{i}{8} \left( p x \int_{(x,t)} \Delta_{23}^{(1)} dx + q x \int_{(x,t)} \Delta_{33}^{(1)} dx + p_{xx} \right),
\]

\[
\mu_{21}^{(3)} = \frac{i}{8} \left[ p \int_{(x,t)} \Delta_{11}^{(2)} dx + p x \int_{(x,t)} \Delta_{11}^{(1)} dx + p(p^2 + q^2) - p_{xx} \right], \\
\mu_{31}^{(3)} = \frac{i}{8} \left[ q \int_{(x,t)} \Delta_{11}^{(2)} dx + q x \int_{(x,t)} \Delta_{11}^{(1)} dx + q(p^2 + q^2) - q_{xx} \right].
\]

Next, we introduce the following new functions \( \{ \Phi_{ij}(t, k) \}_{i,j=1}^3 \) and \( \{ \phi_{ij}(t, k) \}_{i,j=1}^3 \)

\[
\mu_2(0, t, k) = \begin{pmatrix} \Phi_{11}(t, k) & \Phi_{12}(t, k) & \Phi_{13}(t, k) \\ \Phi_{21}(t, k) & \Phi_{22}(t, k) & \Phi_{23}(t, k) \\ \Phi_{31}(t, k) & \Phi_{32}(t, k) & \Phi_{33}(t, k) \end{pmatrix}, \tag{4.3}
\]

\[
\mu_3(L, t, k) = \begin{pmatrix} \phi_{11}(t, k) & \phi_{12}(t, k) & \phi_{13}(t, k) \\ \phi_{21}(t, k) & \phi_{22}(t, k) & \phi_{23}(t, k) \\ \phi_{31}(t, k) & \phi_{32}(t, k) & \phi_{33}(t, k) \end{pmatrix}. \tag{4.4}
\]

By virtue of the asymptotic of \( \mu_j(x, t, k) \) in (4.1), one can show that

\[
\mu_2(0, t, k) = \mathbb{I} + \frac{1}{k} \begin{pmatrix} \Phi_{11}^{(1)}(t) & \Phi_{12}^{(1)}(t) & \Phi_{13}^{(1)}(t) \\ \Phi_{21}^{(1)}(t) & \Phi_{22}^{(1)}(t) & \Phi_{23}^{(1)}(t) \\ \Phi_{31}^{(1)}(t) & \Phi_{32}^{(1)}(t) & \Phi_{33}^{(1)}(t) \end{pmatrix} + O \left( \frac{1}{k^3} \right),
\]

\[
= \mathbb{I} + \frac{1}{k^2} \begin{pmatrix} \Phi_{11}^{(2)}(t) & \Phi_{12}^{(2)}(t) & \Phi_{13}^{(2)}(t) \\ \Phi_{21}^{(2)}(t) & \Phi_{22}^{(2)}(t) & \Phi_{23}^{(2)}(t) \\ \Phi_{31}^{(2)}(t) & \Phi_{32}^{(2)}(t) & \Phi_{33}^{(2)}(t) \end{pmatrix} + O \left( \frac{1}{k^3} \right) + \mathcal{O} \left( \frac{1}{k^3} \right), \tag{4.5}
\]

where \( \Phi_{ij}^{(m)}(t) = \left( \Phi_{12}^{(m)}(t), \Phi_{13}^{(m)}(t) \right), \Phi_{j1}^{(m)}(t) = \left( \Phi_{21}^{(m)}(t), \Phi_{31}^{(m)}(t) \right)^T, \) and \( \Phi_{2 \times 2}^{(m)} = \left( \Phi_{22}^{(m)}(t), \Phi_{23}^{(m)}(t) \right), m = 1, 2. \)

The definition of the boundary values at \( x = 0 \) yields

\[
\Phi_{21}^{(1)}(t) = \frac{i}{2} g_0(t), \quad \Phi_{31}^{(1)}(t) = \frac{i}{2} g_2(t),
\]
\[\Phi^{(1)}_{12}(t) = -\frac{i}{2}g_{01}(t), \quad \Phi^{(1)}_{13}(t) = -\frac{i}{2}g_{02}(t),\]
\[\Phi^{(2)}_{21}(t) = \frac{i}{2}g_{01}(t)\Phi^{(1)}_{11} + \frac{1}{4}g_{11}(t), \quad \Phi^{(2)}_{31}(t) = \frac{i}{2}g_{02}(t)\Phi^{(1)}_{11} + \frac{1}{4}g_{12}(t),\]
\[\Phi^{(3)}_{21}(t) = \frac{i}{8} \left[ 4g_{01}(t)\Phi^{(2)}_{11}(t) - 2ig_{11}(t)\Phi^{(1)}_{11}(t) + g_{01}(t) \left( g_{01}(t)^2 + g_{02}(t)^2 \right) - g_{21}(t) \right],\]
\[\Phi^{(3)}_{31}(t) = \frac{i}{8} \left[ 4g_{02}(t)\Phi^{(2)}_{11}(t) - 2ig_{12}(t)\Phi^{(1)}_{11}(t) + g_{02}(t) \left( g_{01}(t)^2 + g_{02}(t)^2 \right) - g_{22}(t) \right],\]
\[\Phi^{(1)}_{11}(t) = \frac{i}{2} \int_{0}^{t} \left[ -2g_{01}(t)g_{21}(t) - 2g_{02}(t)g_{22}(t) + g_{11}(t)^2 + g_{12}(t)^2 \right.\]
\[\left. + 3(g_{01}(t)^2 + g_{02}(t)^2)^2 \right] dt,\]
\[\Phi^{(2)}_{22}(t) = \frac{i}{2} \int_{0}^{t} \left[ 2g_{01}(t)g_{21}(t) - g_{11}(t)^2 - 3g_{01}(t)^2(g_{01}(t)^2 + g_{02}(t)^2) \right] dt,\]
\[\Phi^{(2)}_{23}(t) = \frac{i}{2} \int_{0}^{t} \left[ 2g_{01}(t)g_{22}(t) - g_{11}(t)g_{12}(t) - 3g_{01}(t)g_{02}(t)(g_{01}(t)^2 + g_{02}(t)^2) \right] dt,\]
\[\Phi^{(3)}_{33}(t) = \frac{i}{2} \int_{0}^{t} \left[ 2g_{21}(t)g_{22}(t) - g_{11}(t)g_{12}(t) - 3g_{01}(t)g_{02}(t)(g_{01}(t)^2 + g_{02}(t)^2) \right] dt,\]
\[\Phi^{(1)}_{33}(t) = \frac{i}{2} \int_{0}^{t} \left[ 2g_{02}(t)g_{22}(t) - g_{12}(t)^2 - 3g_{02}(t)^2(g_{01}(t)^2 + g_{02}(t)^2) \right] dt. \quad (4.6)\]

In particular, the expressions of the boundary data at \(x = 0\) are given as follows
\[g_{0}(t) = -2i\Phi^{(1)}_{11}(t), \quad (4.7a)\]
\[g_{1}(t) = 4\Phi^{(2)}_{11}(t) - 2ig_{0}(t)\Phi^{(1)}_{11}(t), \quad (4.7b)\]
\[g_{2}(t) = 8i\Phi^{(3)}_{11}(t) + 4g_{0}(t)\Phi^{(2)}_{11}(t) - 2g_{1}(t)\Phi^{(1)}_{11}(t) + g_{0}(t)g_{0}(t)T_{0}g_{0}(t), \quad (4.7c)\]
where \(g_{0}(t) = (g_{01}(t), g_{02}(t))^T, g_{1}(t) = (g_{11}(t), g_{12}(t))^T\) and \(g_{2}(t) = (g_{21}(t), g_{22}(t))^T\).

Similarly, the asymptotic formulas of \(\mu_{3}(L, t, k) = \{\phi_{ij}(t, k)\}_{i,j=1}^{3}\) is given by
\[\mu_{3}(L, t, k) = \mathbb{I} + \frac{1}{k} \begin{pmatrix} \phi^{(1)}_{11}(t) & \phi^{(1)}_{12}(t) & \phi^{(1)}_{13}(t) \\ \phi^{(1)}_{21}(t) & \phi^{(1)}_{22}(t) & \phi^{(1)}_{23}(t) \\ \phi^{(1)}_{31}(t) & \phi^{(1)}_{32}(t) & \phi^{(1)}_{33}(t) \end{pmatrix} + O \left( \frac{1}{k^3} \right)\]
\[\hat{\mathbb{I}} + \frac{1}{k} \begin{pmatrix} \phi^{(1)}_{11}(t) & \phi^{(1)}_{12}(t) \\ \phi^{(1)}_{11}(t) & \phi^{(1)}_{12}(t) \end{pmatrix} + O \left( \frac{1}{k^3} \right); \quad (4.8)\]
where \(\phi^{(m)}_{ij}(t) = \left( \phi^{(m)}_{12}(t), \phi^{(m)}_{13}(t) \right), \phi^{(m)}_{11}(t) = \left( \phi^{(m)}_{21}(t), \phi^{(m)}_{31}(t) \right)^T, \) and \(\phi_{2x2}^{(m)} = \left( \begin{array}{cc} \phi_{12}^{(m)}(t) & \phi_{13}^{(m)}(t) \\ \phi_{21}^{(m)}(t) & \phi_{31}^{(m)}(t) \end{array} \right), m = 1, 2.\)
By using the definition of the boundary values at $x = L$, we have

\[
\phi_{21}^{(1)}(t) = \frac{i}{2} f_{01}(t), \quad \phi_{31}^{(1)}(t) = \frac{i}{2} f_{02}(t),
\]
\[
\phi_{12}^{(1)}(t) = -\frac{i}{2} f_{01}(t), \quad \phi_{13}^{(1)}(t) = -\frac{i}{2} f_{02}(t),
\]
\[
\phi_{21}^{(2)}(t) = \frac{i}{2} f_{01}(t) \phi_{11}^{(1)} + \frac{1}{4} f_{11}(t), \quad \phi_{31}^{(2)}(t) = \frac{i}{2} f_{02}(t) \phi_{11}^{(1)} + \frac{1}{4} f_{12}(t),
\]
\[
\phi_{21}^{(3)}(t) = \frac{i}{8} \left[ 4 f_{01}(t) \phi_{11}^{(2)}(t) - 2 i f_{11}(t) \phi_{11}^{(1)}(t) + f_{01}(t) \left( f_{01}(t)^2 + f_{02}(t)^2 \right) - f_{21}(t) \right],
\]
\[
\phi_{31}^{(3)}(t) = \frac{i}{8} \left[ 4 f_{02}(t) \phi_{11}^{(2)}(t) - 2 i f_{12}(t) \phi_{11}^{(1)}(t) + f_{02}(t) \left( f_{01}(t)^2 + f_{02}(t)^2 \right) - f_{22}(t) \right],
\]
\[
\phi_{11}^{(1)}(t) = \frac{i}{2} \int_0^t \left[ -2 f_{01}(t) f_{21}(t) - 2 f_{02}(t) f_{22}(t) + f_{11}(t)^2 + f_{12}(t)^2 \right]
+ 3 \left( f_{01}(t)^2 + f_{02}(t)^2 \right) dt,
\]
\[
\phi_{22}^{(1)}(t) = \frac{i}{2} \int_0^t \left[ 2 f_{01}(t) f_{21}(t) - f_{11}(t)^2 - 3 f_{01}(t)^2 ( f_{01}(t)^2 + f_{02}(t)^2 ) \right] dt,
\]
\[
\phi_{22}^{(1)}(t) = \frac{i}{2} \int_0^t \left[ 2 f_{01}(t) f_{22}(t) - f_{11}(t) f_{12}(t) - 3 f_{01}(t) f_{02}(t) ( f_{01}(t)^2 + f_{02}(t)^2 ) \right] dt,
\]
\[
\phi_{32}^{(1)}(t) = \frac{i}{2} \int_0^t \left[ 2 f_{21}(t) f_{02}(t) - f_{11}(t) f_{12}(t) - 3 f_{01}(t) f_{02}(t) ( f_{01}(t)^2 + f_{02}(t)^2 ) \right] dt,
\]
\[
\phi_{32}^{(1)}(t) = \frac{i}{2} \int_0^t \left[ 2 f_{22}(t) f_{02}(t) - f_{12}(t)^2 - 3 f_{02}(t)^2 ( f_{01}(t)^2 + f_{02}(t)^2 ) \right] dt. \quad (4.9)
\]

In particular, the expressions of the boundary date at $x = L$ are given as follows

\[
f_0(t) = -2 i \phi_{11}^{(1)}(t), \quad (4.10a)
\]
\[
f_1(t) = 4 \phi_{11}^{(2)}(t) - 2 i f_0(t) \phi_{11}^{(1)}(t), \quad (4.10b)
\]
\[
f_2(t) = 8 i \phi_{11}^{(3)}(t) + 4 f_0(t) \phi_{11}^{(2)}(t) - 2 i f_1(t) \phi_{11}^{(1)}(t) + f_0(t) f_0(t) f_0(t), \quad (4.10c)
\]

where $f_0(t) = (f_{01}(t), f_{02}(t))^T$, $f_1(t) = (f_{11}(t), f_{12}(t))^T$ and $f_2(t) = (f_{21}(t), f_{22}(t))^T$.

Based on the global relation (2.45), one can obtain

\[
\mu_2(0, t, k) e^{i k L \Lambda} \left\{ s(k) e^{-i k L \Lambda} S_L(k) \right\} = c(t, k), \quad (4.11)
\]

by replacing $T$ by $t$.

From the symmetry (2.30) and the relation (2.35c), one can show that the spectral function $S_L(k)$ can be formulated in terms of $\{ \phi_{ij}(t, k) \}_{i,j=1}^3$. In what follows, let the matrix-value function $c(t, k)$ be $c(t, k) = (c_{ij}(t, k))_{i,j=1}^3$.

The functions $\{ c_{ij}(t, k) \}_{i=1,j=2}^3$ are analytic and bounded in $D_1$ far away from the possible zeros of $A_{11}(k)$ and of order $O(1 + e^{-2ikL})$ as $k \to \infty$.

For the vanishing initial value case, it is easy to study the asymptotic of $c_{11}(t, k)$, $j = 2, 3$.

**Lemma 4.1.** Let the initial date and boundary date be compatible at $x = 0$ and $x = L$ (i.e., at $x = 0$, $p_0(0) = g_{01}(0)$, $q_0(0) = g_{02}(0)$; at $x = L$, $p_0(L) = f_{01}(0)$, $q_0(L) = f_{02}(0)$). Then, for the vanishing initial date case, the global relation (4.11)
shows that the large \( k \) behavior of \( \{ c_{j1}(t, k) \}_{j=2}^3 \) admit

\[
c_{21}(t, k) = \frac{\Phi_{21}(t)}{k} + \frac{\Phi_{22}(t) + \Phi_{23}(t) \phi_{11}(t)}{k^2} + O \left( \frac{1}{k^3} \right)
+ \left[ \frac{\phi_{12}(t)}{k} + \frac{\phi_{11}(t)}{k^2} \right] e^{-2ikL}, \tag{4.12a}
\]

\[
k \to \infty,
\]

\[
c_{31}(t, k) = \frac{\Phi_{31}(t)}{k} + \frac{\Phi_{32}(t) + \Phi_{33}(t) \phi_{11}(t)}{k^2} + O \left( \frac{1}{k^3} \right)
+ \left[ \frac{\phi_{12}(t)}{k} + \frac{\phi_{11}(t)}{k^2} \right] e^{-2ikL}, \tag{4.12b}
\]

\[
k \to \infty.
\]

**Proof.** From the global relation, we have the \((2, 1)\)th and \((3, 1)\)th terms of \((4.11)\) as follows

\[
c_{21}(t, k) = \Phi_{21}(t, k) \phi_{11}(t, \bar{k}) + \Phi_{22}(t, k) \bar{\phi}_{12}(t, \bar{k}) e^{-2ikL} + \Phi_{23}(t, k) \bar{\phi}_{13}(t, \bar{k}) e^{-2ikL}, \tag{4.13a}
\]

\[
c_{31}(t, k) = \Phi_{31}(t, k) \phi_{11}(t, \bar{k}) + \Phi_{32}(t, k) \bar{\phi}_{12}(t, \bar{k}) e^{-2ikL} + \Phi_{33}(t, k) \bar{\phi}_{13}(t, \bar{k}) e^{-2ikL}, \tag{4.13b}
\]

under the assumption of vanishing initial date.

Recalling the Lax pair equation \((2.8)\), we have

\[
\mu_t - 4ik^3 [\lambda, \mu] = V_2 \mu. \tag{4.14}
\]

From the first column of Equation \((4.14)\), one can obtain

\[
\begin{align*}
\Phi_{11t} &= -2ik(g_{01}^3 - g_{02}^2) \Phi_{11} + (4k^2 g_{01} + 2ik g_{11} - g_{21} + 2g_{01}^3 + 2g_{01} g_{02}^2) \Phi_{21} \\
&+ (4k^2 g_{02} + 2ik g_{12} - g_{22} + 2g_{01} g_{02} + 2g_{02}^3) \Phi_{31},
\end{align*}
\]

\[
\begin{align*}
\Phi_{21t} &- 8ik^3 \Phi_{21} = (4k^2 g_{01} - 2ik g_{11} - g_{21} + 2g_{01}^3 + 2g_{01} g_{02}^2) \Phi_{11} + 2ik g_{02} \Phi_{21} \\
&+ (2ik g_{01} g_{02} + g_{11} g_{02} - g_{01} g_{12}) \Phi_{31},
\end{align*}
\]

\[
\begin{align*}
\Phi_{31t} &- 8ik^3 \Phi_{31} = (4k^2 g_{02} - 2ik g_{12} - g_{22} + 2g_{01} g_{02} + 2g_{02}^3) \Phi_{11} \\
&+ (2ik g_{01} g_{02} - g_{11} g_{02} + g_{01} g_{12}) \Phi_{21} + 2ik g_{02} \Phi_{31}.
\end{align*}
\]

\[\text{(4.15)}\]

From the second column of Equation \((4.14)\), one can obtain

\[
\begin{align*}
\Phi_{12t} + 8ik^3 \Phi_{12} &= -2ik(g_{01}^3 - g_{02}^2) \Phi_{12} + (4k^2 g_{01} + 2ik g_{11} - g_{21} + 2g_{01}^3 + 2g_{01} g_{02}^2) \Phi_{22} \\
&+ (4k^2 g_{02} + 2ik g_{12} - g_{22} + 2g_{01} g_{02} + 2g_{02}^3) \Phi_{32},
\end{align*}
\]

\[
\begin{align*}
\Phi_{22t} &= (4k^2 g_{01} - 2ik g_{11} - g_{21} + 2g_{01}^3 + 2g_{01} g_{02}^2) \Phi_{12} + 2ik g_{01} \Phi_{22} \\
&+ (2ik g_{01} g_{02} + g_{11} g_{02} - g_{01} g_{12}) \Phi_{32},
\end{align*}
\]

\[
\begin{align*}
\Phi_{32t} &= (4k^2 g_{02} - 2ik g_{12} - g_{22} + 2g_{01} g_{02} + 2g_{02}^3) \Phi_{12} \\
&+ (2ik g_{01} g_{02} - g_{11} g_{02} + g_{01} g_{12}) \Phi_{22} + 2ik g_{02} \Phi_{32}.
\end{align*}
\]

\[\text{(4.16)}\]
From the third column of Equation (4.14), one can obtain

\[
\begin{align*}
\Phi_{13t} + 8ik^3\Phi_{13} &= -2ik(g_{01}^3 - g_{02}^3)\Phi_{13} + (4k^2 g_{01} + 2ik g_{11} - g_{21} + 2g_{01}^3 + 2g_{01}g_{02}^2)\Phi_{23} \\
&\quad + (4k^2 g_{02} + 2ik g_{12} - g_{22} + 2g_{01}g_{02} + 2g_{02}^3)\Phi_{33}, \\
\Phi_{23t} &= (4k^2 g_{01} - 2ik g_{11} - g_{21} + 2g_{01}^3 + 2g_{01}g_{02}^2)\Phi_{13} + 2ik g_{01}\Phi_{23} \\
&\quad + (2ik g_{01}g_{02} + g_{11}g_{02} - g_{01}g_{12})\Phi_{33}, \\
\Phi_{33t} &= (4k^2 g_{02} - 2ik g_{12} - g_{22} + 2g_{01}g_{02} + 2g_{02}^3)\Phi_{13} \\
&\quad + (2ik g_{01}g_{02} - g_{11}g_{02} + g_{01}g_{12})\Phi_{23} + 2ik g_{02}\Phi_{33}.
\end{align*}
\]

(4.17)

Let \((\Phi_{11} \Phi_{21} \Phi_{31})^T\) be of the form

\[
\begin{pmatrix}
\Phi_{11} \\
\Phi_{21} \\
\Phi_{31}
\end{pmatrix} = \begin{pmatrix}
\alpha_0(t) + \frac{\alpha_1(t)}{k} + \frac{\alpha_2(t)}{k^2} + \cdots \\
\beta_0(t) + \frac{\beta_1(t)}{k} + \frac{\beta_2(t)}{k^2} + \cdots \\
\end{pmatrix} e^{8ik^3t},
\]

(4.18)

where the \(3 \times 1\) matrix functions \(\alpha_j(t)\) and \(\beta_j(t)\), \(j = 0, 1, 2, \ldots\), are independent of \(k\).

Substituting equation (4.18) into system (4.15) and using the initial conditions

\[
\begin{align*}
\alpha_0(0) + \beta_0(0) &= (1 \ 0 \ 0)^T, \\
\alpha_1(0) + \beta_1(0) &= (0 \ 0 \ 0)^T,
\end{align*}
\]

(4.19)

we can determine these coefficients \(\alpha_j(t)\) and \(\beta_j(t)\) by

\[
\begin{pmatrix}
\Phi_{11} \\
\Phi_{21} \\
\Phi_{31}
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} + \frac{1}{k} \begin{pmatrix}
\Phi_{11}^{(1)} \\
\Phi_{21}^{(1)} \\
\Phi_{31}^{(1)}
\end{pmatrix} + \frac{1}{k^2} \begin{pmatrix}
\Phi_{11}^{(2)} \\
\Phi_{21}^{(2)} \\
\Phi_{31}^{(2)}
\end{pmatrix} + O \left( \frac{1}{k^3} \right)
\]

\[
+ \left[ \frac{1}{k} \begin{pmatrix}
0 \\
-\Phi_{21}^{(1)}(0) \\
-\Phi_{31}^{(1)}(0)
\end{pmatrix} \right] + \frac{1}{k^2} \begin{pmatrix}
-\Phi_{11}^{(2)}(0) + \Phi_{11}^{(1)}(0)\Phi_{21}^{(1)} + \Phi_{11}^{(1)}(0)\Phi_{31}^{(1)} \\
\frac{1}{4}g_{01}\Phi_{11}^{(1)}(0) \\
\frac{1}{4}g_{02}\Phi_{11}^{(1)}(0)
\end{pmatrix}
\]

\[
+ O \left( \frac{1}{k^3} \right) \right] e^{8ik^3t}.
\]

(4.20)

By the same way to the derivation of \((\Phi_{11} \Phi_{21} \Phi_{31})^T\), under the following initial conditions

\[
\begin{align*}
\alpha_0(0) + \beta_0(0) &= (0 \ 1 \ 0)^T, \\
\alpha_1(0) + \beta_1(0) &= (0 \ 0 \ 0)^T,
\end{align*}
\]

(4.21)

we can determine the asymptotic formulas of \((\Phi_{12} \Phi_{22} \Phi_{32})^T\) by

\[
\begin{pmatrix}
\Phi_{12} \\
\Phi_{22} \\
\Phi_{32}
\end{pmatrix} = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} + \frac{1}{k} \begin{pmatrix}
\Phi_{12}^{(1)} \\
\Phi_{22}^{(1)} \\
\Phi_{32}^{(1)}
\end{pmatrix} + \frac{1}{k^2} \begin{pmatrix}
\Phi_{12}^{(2)} \\
\Phi_{22}^{(2)} \\
\Phi_{32}^{(2)}
\end{pmatrix} + O \left( \frac{1}{k^3} \right)
\]

\[
+ \left[ \frac{1}{k} \begin{pmatrix}
-\Phi_{12}^{(1)}(0) \\
0 \\
0
\end{pmatrix} \right] + \frac{1}{k^2} \begin{pmatrix}
-\Phi_{12}^{(2)}(0) + \Phi_{12}^{(1)}(0)\Phi_{22}^{(1)} + \Phi_{12}^{(1)}(0)\Phi_{32}^{(1)} \\
-\frac{1}{4}g_{01}\Phi_{12}^{(1)}(0) \\
-\frac{1}{4}g_{02}\Phi_{12}^{(1)}(0)
\end{pmatrix}
\]

\[
+ O \left( \frac{1}{k^3} \right) \right] e^{-8ik^3t}.
\]

(4.22)
from the system (4.16).

Similarly, let \( \begin{pmatrix} \Phi_{13} \\ \Phi_{23} \\ \Phi_{33} \end{pmatrix} \) be of the form

\[
\begin{pmatrix} \Phi_{13} \\ \Phi_{23} \\ \Phi_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{k} \begin{pmatrix} \Phi_{13}^{(1)} \\ \Phi_{23}^{(1)} \\ \Phi_{33}^{(1)} \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} \Phi_{13}^{(2)} \\ \Phi_{23}^{(2)} \\ \Phi_{33}^{(2)} \end{pmatrix} + O \left( \frac{1}{k^3} \right)
\]

\( + \frac{1}{k} \begin{pmatrix} -\Phi_{13}^{(1)}(0) \\ 0 \\ 0 \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} -\Phi_{13}^{(2)}(0) + \Phi_{13}^{(1)}(0) \Phi_{23}^{(1)} + \Phi_{13}^{(1)}(0) \Phi_{33}^{(1)} \\ -\frac{i}{2} g_{01} \Phi_{13}^{(1)}(0) \\ -\frac{i}{2} g_{02} \Phi_{13}^{(1)}(0) \end{pmatrix} + O \left( \frac{1}{k^3} \right) \) e\(-8ik^3t\), \hspace{1cm} (4.23)

where the \( 3 \times 1 \) matrix functions \( \alpha_j(t) \) and \( \beta_j(t) \), \( j = 0, 1, 2, \ldots \), are independent of \( k \).

Substituting the equation (4.23) into the system (4.17) and using the initial conditions

\[
\alpha_0(0) + \beta_0(0) = (0 \ 0 \ 1)^T, \quad \alpha_1(0) + \beta_1(0) = (0 \ 0 \ 0)^T,
\]  \hspace{1cm} (4.24)

we can determine these coefficients \( \alpha_j(t) \) and \( \beta_j(t) \) by

\[
\begin{pmatrix} \Phi_{13} \\ \Phi_{23} \\ \Phi_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{k} \begin{pmatrix} \Phi_{13}^{(1)} \\ \Phi_{23}^{(1)} \\ \Phi_{33}^{(1)} \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} \Phi_{13}^{(2)} \\ \Phi_{23}^{(2)} \\ \Phi_{33}^{(2)} \end{pmatrix} + O \left( \frac{1}{k^3} \right)
\]

\( + \frac{1}{k} \begin{pmatrix} -\Phi_{13}^{(1)}(0) \\ 0 \\ 0 \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} -\Phi_{13}^{(2)}(0) + \Phi_{13}^{(1)}(0) \Phi_{23}^{(1)} + \Phi_{13}^{(1)}(0) \Phi_{33}^{(1)} \\ -\frac{i}{2} g_{01} \Phi_{13}^{(1)}(0) \\ -\frac{i}{2} g_{02} \Phi_{13}^{(1)}(0) \end{pmatrix} + O \left( \frac{1}{k^3} \right) \) e\(-8ik^3t\). \hspace{1cm} (4.25)

From the systems (4.15), (4.16) and (4.17), we can show that \( \{ \phi_{ij} \}^3_{i,j=1} \) admit the similar partial derivative equations.

From the first column of Equation (4.14), one can obtain

\[
\begin{align*}
\phi_{11t} &= -2ik(f_{01}^3 - f_{02}^3)\phi_{11} + (4k^2 f_{01} + 2ik f_{11} - f_{21} + 2f_{01}^3 + 2f_{01} f_{02}^2)\phi_{21} \\
&\quad + (4k^2 f_{02} + 2ik f_{12} - f_{22} + 2f_{01} f_{02}^2 + 2f_{02}^3)\phi_{31}, \\
\phi_{21t} &= -8ik^3\phi_{21} = (4k^2 f_{01} - 2ik f_{11} - f_{21} + 2f_{01}^3 + 2f_{01} f_{02}^2)\phi_{11} + 2ik f_{01}^2 \phi_{21} \\
&\quad + (2ik f_{01} f_{02} + f_{11} f_{02} - f_{01} f_{12})\phi_{31}, \\
\phi_{31t} &= -8ik^3\phi_{31} = (4k^2 f_{02} - 2ik f_{12} - f_{22} + 2f_{01}^3 f_{02} + 2f_{02}^3)\phi_{11} \\
&\quad + (2ik f_{01} f_{02} - f_{11} f_{02} + f_{01} f_{12})\phi_{21} + 2ik f_{02}^2 \phi_{31}. \hspace{1cm} (4.26)
\end{align*}
\]

From the second column of Equation (4.14), one can obtain

\[
\begin{align*}
\phi_{12t} + 8ik^3\phi_{12} &= -2ik(f_{01}^3 - f_{02}^3)\phi_{12} + (4k^2 f_{01} + 2ik f_{11} - f_{21} + 2f_{01}^3 + 2f_{01} f_{02}^2)\phi_{22} \\
&\quad + (4k^2 f_{02} + 2ik f_{12} - f_{22} + 2f_{01} f_{02} + 2f_{02}^3)\phi_{32}, \\
\phi_{22t} &= (4k^2 f_{01} - 2ik f_{11} - f_{21} + 2f_{01}^3 + 2f_{01} f_{02}^2)\phi_{12} + 2ik f_{01}^2 \phi_{22} \\
&\quad + (2ik f_{01} f_{02} + f_{11} f_{02} - f_{01} f_{12})\phi_{32}, \\
\phi_{32t} &= (4k^2 f_{02} - 2ik f_{12} - f_{22} + 2f_{01}^3 f_{02} + 2f_{02}^3)\phi_{12} \\
&\quad + (2ik f_{01} f_{02} - f_{11} f_{02} + f_{01} f_{12})\phi_{22} + 2ik f_{02}^2 \phi_{32}. \hspace{1cm} (4.27)
\end{align*}
\]
From the third column of Equation (4.14), one can obtain

\[
\begin{aligned}
\phi_{13t} + 8ik^3\phi_{13} &= -2ik(f_{01}^2 - f_{02}^2)\phi_{13} + (4k^2f_{01} + 2ikf_{11} - f_{21} + 2f_{01}^3 + 2f_{01}f_{02}^2)\phi_{23} \\
&+ (4k^2f_{02} + 2ikf_{12} - f_{22} + 2f_{02}^3 + 2f_{02}f_{01}^2)\phi_{33}, \\
\phi_{23t} &= (4k^2f_{01} - 2ikf_{11} - f_{21} + 2f_{01}^3 + 2f_{01}f_{02}^2)\phi_{13} + 2ikf_{01}\phi_{23} \\
&+ (2ikf_{01}f_{02} + f_{11}f_{02} - f_{01}f_{12})\phi_{33}, \\
\phi_{33t} &= (4k^2f_{02} - 2ikf_{12} - f_{22} + 2f_{02}^3 + 2f_{02}f_{01}^2)\phi_{13} \\
&+ (2ikf_{01}f_{02} - f_{11}f_{02} + f_{01}f_{12})\phi_{23} + 2ikf_{02}^3\phi_{33}.
\end{aligned}
\]

(4.28)

Then, substituting these formulas into Equation (4.13) and noticing that we assume that the initial value and boundary value are compatible at \( x = 0 \) and \( x = L \), respectively, one can derive the asymptotic behavior (4.12a) of \( c_{j1}(t,k) \) as \( k \to \infty \). By the same way, one can show that the formula (4.12b) is also hold. \( \square \)

5. **Nonlinearizable boundary conditions.** It is very difficult to study IBV problems since some of the boundary values are unknown for a well-posed problem. All boundary dates are very important to define of the Riemann-Hilbert problem. In what follows, the effective characterizations of spectral functions \( S(k) \), \( S_L(k) \) can be derived in our main result, theorem 5.2, for the Dirichlet \( \{g_{01}(t),g_{02}(t)\} \) and \( \{f_{01}(t),f_{02}(t)\} \) prescribed, the first Neumann \( \{g_{11}(t),g_{12}(t)\} \) and \( \{f_{11}(t),f_{12}(t)\} \) prescribed) and the second Neumann \( \{g_{21}(t),g_{22}(t)\} \) and \( \{f_{21}(t),f_{22}(t)\} \) prescribed) problems.

Let’s introduce the definitions of the functions \( \{\Phi_{ij}(t,k)\}_{i,j=1}^3 \) and \( \{\phi_{ij}(t,k)\}_{i,j=1}^3 \) by

\[
\begin{align*}
\mu_2(0,t,k) &= \begin{pmatrix}
\Phi_{11}(0,t,k) & \Phi_{12}(0,t,k) & \Phi_{13}(0,t,k) \\
\Phi_{21}(0,t,k) & \Phi_{22}(0,t,k) & \Phi_{23}(0,t,k) \\
\Phi_{31}(0,t,k) & \Phi_{32}(0,t,k) & \Phi_{33}(0,t,k)
\end{pmatrix}, \\
\mu_3(L,t,k) &= \begin{pmatrix}
\phi_{11}(L,t,k) & \phi_{12}(L,t,k) & \phi_{13}(L,t,k) \\
\phi_{21}(L,t,k) & \phi_{22}(L,t,k) & \phi_{23}(L,t,k) \\
\phi_{31}(L,t,k) & \phi_{32}(L,t,k) & \phi_{33}(L,t,k)
\end{pmatrix}.
\end{align*}
\]

For the eigenfunctions \( \Phi_{ij} \) and \( \phi_{ij} \), we will first derive their Gelfand-Levitan-Marchenko (GLM) representations, then analyze the solution of the global relation. By considering the analysis of the global relation, the unknown boundary values can be expressed in terms of the known ones via the GLM representations.

5.1. **The GLM representation.**

**Theorem 5.1.** The eigenfunctions \( \Phi_{ij}(t,k) \) and \( \phi_{ij}(t,k) \) have the following GLM representations

\[
\begin{aligned}
\Phi_{ij}(t,k) &= \delta_{ij} + \int_{-t}^t \left( \left[ \left( \tilde{L}(t,s) + \frac{1}{8}V_2^{(2)}M(t,s) + \frac{1}{16}V_2^{(2)}N(t,s) \right) e^{-4ik^3(t-s)} \right] ds, \\
n &+ k \left( \tilde{M}(t,s) - \frac{1}{8}V_2^{(2)}N(t,s) \right) + k^2 N(t,s) \right) e^{-4ik^3(t-s)} ds, \\
\phi_{ij}(t,k) &= \delta_{ij} + \int_{-t}^t \left( \left[ \left( \tilde{L}(t,s) + \frac{1}{8}V_2^{(2)}M(t,s) + \frac{1}{16}V_2^{(2)}N(t,s) \right) e^{-4ik^3(t-s)} \right] ds, \\
&+ k \left( \tilde{M}(t,s) - \frac{1}{8}V_2^{(2)}N(t,s) \right) + k^2 N(t,s) \right) e^{-4ik^3(t-s)} ds.
\end{aligned}
\]

(5.3a)

(5.3b)
where the symbol $\delta_{ij}$ given by

$$\delta_{ij} = \begin{cases} 
1, & i = j, \\
0, & i \neq j,
\end{cases} \quad (5.4)$$

and the $3 \times 3$ matrices $\tilde{L}(t, s)$, $\tilde{M}(t, s)$ and $N(t, s)$ admit the following initial conditions

$$\begin{cases}
L(t, -t) + \Lambda L(t, -t) \Lambda = 0, \\
M(t, -t) + \Lambda M(t, -t) \Lambda = 0, \\
N(t, -t) + \Lambda N(t, -t) \Lambda = 0,
\end{cases} \quad (5.5a)$$

$$\begin{cases}
N(t, t) - \Lambda N(t, t) \Lambda = V_2^{(2)}, \\
\tilde{M}(t, t) - \Lambda \tilde{M}(t, t) \Lambda = -\frac{1}{2} \Lambda V_2^{(2)}, \\
\tilde{L}(t, t) - \Lambda \tilde{L}(t, t) \Lambda = \frac{1}{32} \left( V_2^{(2)} \right)^3 - \frac{1}{4} V_2^{(2)} 2_{xx},
\end{cases} \quad (5.5b)$$

and the ODE systems

$$\begin{cases}
N_t(t, s) + \Lambda N_s(t, s) \Lambda = \left[ \frac{1}{64} \left( V_2^{(2)} \right)^3 - \frac{1}{4} V_2^{(2)} 2_{xx} \right] N(t, s) \\
&+ V_2^{(2)} \tilde{L}(t, s) + \frac{1}{2} \Lambda V_2^{(2)} \tilde{M}(t, s), \\
\tilde{M}_t(t, s) + \Lambda \tilde{M}_s(t, s) \Lambda = \left[ \frac{1}{64} \left( V_2^{(2)} \right)^3 - \frac{1}{4} V_2^{(2)} 2_{xx} \right] \tilde{M}(t, s) \\
&- \frac{1}{2} \Lambda V_2^{(2)} \tilde{L}(t, s) - \frac{\mathcal{A}}{8} \Lambda N(t, s), \\
\tilde{L}_t(t, s) + \Lambda \tilde{L}_s(t, s) \Lambda = \left[ \frac{1}{32} \left( V_2^{(2)} \right)^3 - \frac{1}{4} V_2^{(2)} 2_{xx} \right] \tilde{L}(t, s) \\
&- \frac{\mathcal{B}}{8} \tilde{M}(t, s) + \frac{\mathcal{C}}{16} N(t, s),
\end{cases} \quad (5.5c)$$

where the matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are determined by

$$\mathcal{A} = -V_2^{(2)} + \frac{3}{64} \left( V_2^{(2)} \right)^4 - \frac{1}{4} \left( V_2^{(2)} 2_{xx} V_2^{(2)} + V_2^{(2)} V_2^{(2)} \right) + \frac{1}{4} \left( V_2^{(2)} \right)^2$$

$$+ \frac{1}{16} \left[ \left( V_2^{(2)} \right)^2 V_2^{(2)} + V_2^{(2)} \left( V_2^{(2)} \right)^2 - V_2^{(2)} V_2^{(2)} V_2^{(2)} \right], \quad (5.6a)$$

$$\mathcal{B} = V_2^{(2)} + \frac{1}{2} \left( V_2^{(2)} V_2^{(2)} + V_2^{(2)} \right) - \frac{1}{4} \left( V_2^{(2)} \right)^2 - \frac{3}{64} \left( V_2^{(2)} \right)^4$$

$$- \frac{1}{16} \left[ V_2^{(2)} \left( V_2^{(2)} \right)^2 - V_2^{(2)} V_2^{(2)} V_2^{(2)} \right], \quad (5.6b)$$

$$\mathcal{C} = -V_2^{(2)} + \frac{3}{256} \left( V_2^{(2)} \right)^5 + \frac{1}{16} V_2^{(2)} V_2^{(2)} V_2^{(2)} - \frac{1}{16} \left[ \left( V_2^{(2)} \right)^2 V_2^{(2)} + V_2^{(2)} V_2^{(2)} V_2^{(2)} \right]$$

$$+ \frac{1}{4} \left( V_2^{(2)} \right)^2 V_2^{(2)} + \frac{1}{4} \left[ V_2^{(2)} \left( V_2^{(2)} \right)^2 - V_2^{(2)} V_2^{(2)} \right] + \frac{1}{256} \left[ \left( V_2^{(2)} \right)^3 V_2^{(2)} - V_2^{(2)} \left( V_2^{(2)} \right)^3 \right]. \quad (5.6c)$$

Analogously, the functions $\{ \tilde{L}(t, s), \tilde{M}(t, s), N(t, s) \}$ admit the similar system of the equations (5.5a)-(5.5c) with $\{g_0(t), g_0(t) + g_1(t), g_1(t), g_2(t), g_2(t)\}$ replaced by $\{f_0(t), f_0(t) + f_1(t), f_1(t), f_2(t), f_2(t)\}$, respectively.
Proof. We just show that the GLM representations of the eigenfunctions $\Phi_{ij}$ are hold. Following the same way, one can also derive the GLM representations of the eigenfunctions $\phi_{ij}$. Let's consider the following expression
\begin{equation}
\mu(t, k) = 1 + \int_{-t}^{t} \left[ L(t, s) + kM(t, s) + k^2 N(t, s) \right] e^{-4ik^3(t-s)\Lambda} ds,
\end{equation}
where $L$, $M$ and $N$ are $3 \times 3$ matrices. Substituting the above expression (5.7) into the $t$-part of the Lax pair (2.1b), one has the following systems
\begin{equation}
\begin{aligned}
N(t, t) + \Lambda N(t, t)\Lambda = 0, \\
M(t, t) - \Lambda M(t, t)\Lambda = \frac{1}{4} V_2^{(2)} N(t, t)\Lambda = 0, \\
L(t, t) + \Lambda L(t, t)\Lambda = \frac{1}{4} V_2^{(2)} M(t, t)\Lambda = \frac{1}{4} V_2^{(2)} N(t, t)\Lambda, \\
N(t, t) + \Lambda N(t, t)\Lambda = V_2^{(2)}, \\
M(t, t) - \Lambda M(t, t)\Lambda = i\Lambda V_2^{(1)} - \frac{1}{4} V_2^{(2)} N(t, t)\Lambda, \\
L(t, t) + \Lambda L(t, t)\Lambda = V_2^{(1)} + \frac{1}{4} V_2^{(2)} M(t, t)\Lambda = \frac{1}{4} V_2^{(2)} N(t, t)\Lambda,
\end{aligned}
\end{equation}
By introducing the following transformation
\begin{equation}
\begin{aligned}
M(t, s) &= \bar{M}(t, s) - \frac{1}{8} V_2^{(2)} \Lambda N(t, s), \\
L(t, s) &= \bar{L}(t, s) + \frac{1}{8} V_2^{(2)} \bar{M}(t, s) + \frac{1}{16} V_2^{(2)} N(t, s),
\end{aligned}
\end{equation}
we can derive the initial conditions (5.5a)-(5.5c) and the ODE systems (5.6a)-(5.6c).

5.2. The analysis of the global relation. In order to avoid routine technical complications, we will analyze the global relation in the case $s(k) = 1$ with respect to the zero initial conditions $p_0(x) = p(x, 0) = 0$ and $q_0(x) = q(x, 0) = 0$. For this case, the global relation (2.45) admits the following form
\begin{equation}
\Phi(t, k)e^{-ikL\bar{\Lambda}} \left( \bar{\phi}(t, k) \right)^T = c(t, k).
\end{equation}
From the Lemma 4.1, we have the following properties
\begin{equation}
c_{21}(t, k) = O \left( \frac{1}{k} \right), \quad c_{31}(t, k) = O \left( \frac{1}{k} \right), \quad k \in D_3, \\
e^{2ikL}c_{21}(t, k) = O \left( \frac{1}{k} \right), \quad e^{2ikL}c_{31}(t, k) = O \left( \frac{1}{k} \right), \quad k \in D_4.
\end{equation}
In what follows, we will consider the solution of the global relation. The expressions for $\{g_{11}(t), g_{21}(t), f_{21}(t)\}$ and $\{g_{12}(t), g_{22}(t), f_{22}(t)\}$ can be written in terms of the given boundary conditions $\{g_{01}(t), g_{02}(t), f_{01}(t), f_{02}(t), f_{11}(t), f_{12}(t)\}$, which are provided by Theorem 5.2 below in terms of the GLM representations.
In order to simplify our formulas, some notations are introduced. Let \( \mathcal{F}(t,k) \) be a scalar function, then we define the functions \( \mathcal{F}(t,k) \) and \( \mathcal{F}(t,k) \) as follows.

- Define \( \mathcal{F}(t,k) \) and \( \mathcal{F}(t,k) \) as follows
  \[
  \mathcal{F}(t,k) = \mathcal{F}(t,k) + \alpha \mathcal{F}(t,\alpha k) + \alpha^2 \mathcal{F}(t,\alpha^2 k),
  \]
  \[
  \mathcal{F}(t,k) = \mathcal{F}(t,k) + \alpha \mathcal{F}(t,\alpha k) + \alpha^2 \mathcal{F}(t,\alpha^2 k).
  \]

- Define \( \mathcal{F}(t,k) \) and \( \mathcal{F}(t,k) \) as follows
  \[
  \mathcal{F}(t,k) = \mathcal{F}(t,k) + \alpha^2 \mathcal{F}(t,\alpha k) + \alpha \mathcal{F}(t,\alpha^2 k),
  \]
  \[
  \mathcal{F}(t,k) = \mathcal{F}(t,k) + \alpha^2 \mathcal{F}(t,\alpha k) + \alpha \mathcal{F}(t,\alpha^2 k).
  \]

Here \( \alpha = e^{2\pi i} \).

From the above analysis, we have the following Theorem.

**Theorem 5.2.** Consider the initial-boundary value problem of the cmKdV equation (1.1) on the interval \( \Omega \). Let \( \rho_0(x) = q_0(x) = 0 \), \( 0 \leq x \leq L \) be two vanishing initial data with \( T < 1 \). For the boundary conditions \( \{g_{01}(t), g_{02}(t), f_{01}(t), f_{02}(t), f_{11}(t), f_{12}(t)\} \), we have the expressions for \( \{g_{11}(t), g_{21}(t), f_{21}(t)\} \) and \( \{g_{12}(t), g_{22}(t), f_{22}(t)\} \), respectively.

- For the given boundary conditions \( \{g_{01}(t), g_{02}(t), f_{01}(t), f_{02}(t), f_{11}(t), f_{12}(t)\} \), the expressions for \( g_{11}(t), g_{21}(t), \) and \( f_{21}(t) \) can be written as follows:

\[
\begin{align*}
g_{11}(t) &= \frac{g_{01}(t)}{\pi} \int_{\partial D^0} \Phi_{11}(t,k) dk - \frac{6}{\pi} \int_{\partial D^0} k \left[ \mathcal{E}^{-1}(k) \right]_1 \left( \begin{array}{c} \mathcal{G}_{11}(t,k) \\ \mathcal{G}(t,\alpha k) \\ e^{2\pi i k} \mathcal{G}(t,\alpha^2 k) \end{array} \right) dk \\
  &\quad - \frac{6}{\pi} \int_{\partial D^0} k \left[ \mathcal{E}_{11}(k) \right]_1 \left( \begin{array}{c} \mathcal{G}_{21}(t,k) \\ \mathcal{G}(t,\alpha k) \\ e^{2\pi i k} \mathcal{G}(t,\alpha^2 k) \end{array} \right) e^{ik(t-t')} dk, \\
\end{align*}
\]

\[
\begin{align*}
g_{21}(t) &= g_{01}(t) \left[ g_{01}(t) + g_{02}(t) \right] - \frac{2}{\pi} g_{01}(t) \int_{\partial D^0} k \Phi_{11}(t,k) dk + \frac{g_{11}(t)}{\pi} \int_{\partial D^0} \Phi_{11}(t,k) dk \\
  &\quad - \frac{12}{\pi} \int_{\partial D^0} k^2 \left[ \mathcal{E}_{11}(k) \right]_2 \left( \begin{array}{c} \mathcal{G}_{11}(t,k) \\ \mathcal{G}(t,\alpha k) \\ e^{2\pi i k} \mathcal{G}(t,\alpha^2 k) \end{array} \right) dk \\
  &\quad - \frac{12}{\pi} \int_{\partial D^0} k^2 \left[ \mathcal{E}_{21}(k) \right]_2 \left( \begin{array}{c} \mathcal{G}_{21}(t,k) \\ \mathcal{G}(t,\alpha k) \\ e^{2\pi i k} \mathcal{G}(t,\alpha^2 k) \end{array} \right) e^{ik(t-t')} dk, \\
\end{align*}
\]

\[
\begin{align*}
f_{21}(t) &= \frac{2}{\pi} f_{01}(t) \left[ f_{02}(t) + f_{02}(t) \right] - f_{01}(t) \left[ |f_{01}(t)|^2 + |f_{02}(t)|^2 \right] \\
  &\quad + \frac{2}{\pi} \left[ f_{01}(t) \int_{\partial D^0} k \Phi_{22}(t,k) dk + f_{02}(t) \int_{\partial D^0} k \Phi_{22}(t,k) dk \right] \\
  &\quad + \frac{1}{\pi} \int_{\partial D^0} \left[ f_{11}(t) \Phi_{22}(t,k) + f_{12}(t) \Phi_{22}(t,k) \right] dk \\
  &\quad - \frac{12}{\pi} \int_{\partial D^0} k^2 \left[ \mathcal{E}_{11}(k) \right]_1 \left( \begin{array}{c} \mathcal{G}_{11}(t,k) \\ \mathcal{G}(t,\alpha k) \\ e^{-2\pi i k} \mathcal{G}(t,\alpha^2 k) \end{array} \right) dk.
\end{align*}
\]
where \( \tilde{G}_{11}(t, k) \) and \( G_{21}(t, k) \) are given by

\[
\tilde{G}_{11}(t, k) = -e^{-2ikL} \left\{ \frac{1}{3} \tilde{\Phi}_{12}(t, k) - \frac{1}{4ik^2} \left[ \tilde{f}_{11}(t) - \frac{1}{\pi} \int_{\partial D^0} \left( f_{01}(t) \tilde{\Phi}_{22}(t, k) + f_{02}(t) \tilde{\Phi}_{33}(t, k) \right) dk \right] \right\} \\
\tilde{G}_{21}(t, k) = -e^{-2ikL} \left( (\Phi_{22}(t, k) - 1) \tilde{\Phi}_{12}(t, k) + \Phi_{23}(t, k) \tilde{\Phi}_{13}(t, k) \right) - \Phi_{21}(t, k)(\tilde{\Phi}_{11}(t, k) - 1).
\]

\begin{itemize}
  \item For the given boundary conditions \( \{g_{01}(t), g_{02}(t), f_{01}(t), f_{02}(t), f_{11}(t), f_{12}(t)\} \), the expressions for \( g_{12}(t), g_{22}(t), \) and \( f_{22}(t) \) can be written as follows:
\end{itemize}

\[
g_{12}(t) = \frac{g_{02}(t)}{\pi} \int_{\partial D^0} \tilde{\Phi}_{11}(t, k) dk - \frac{6}{\pi} \int_{\partial D^0} k [\mathcal{E}^{-1}(k)]_5 \left( \begin{array}{c} \frac{\tilde{G}_{12}(t, k)}{\tilde{G}_{12}(t, \alpha k)} \\ e^{2i\alpha^2kL} \tilde{G}_{12}(t, \alpha^2 k) \end{array} \right) dk,
\]

\[
g_{22}(t) = g_{02}(t) \left[ g_{01}(t) + g_{02}(t) \right] - \frac{2}{\pi} \frac{g_{02}(t)}{\pi} \int_{\partial D^0} \tilde{k}\tilde{\Phi}_{11}(t, k) dk + \frac{g_{12}(t)}{\pi} \int_{\partial D^0} \tilde{k}\tilde{\Phi}_{11}(t, k) dk,
\]

\[
f_{22}(t) = 2f_{02}(t) \left[ f_{01}(t) + f_{02}(t) \right] - f_{02}(t) \left[ |f_{01}(t)|^2 + |f_{02}(t)|^2 \right] + \frac{2}{\pi} \left[ f_{01}(t) \int_{\partial D^0} k\tilde{\Phi}_{23}(t, k) dk + f_{02}(t) \int_{\partial D^0} k\tilde{\Phi}_{33}(t, k) dk \right] + \frac{1}{\pi} \int_{\partial D^0} \left[ \tilde{f}_{11}(t) \tilde{\Phi}_{23}(t, k) + \tilde{f}_{12}(t) \tilde{\Phi}_{33}(t, k) \right] dk,
\]

\[
- \frac{12}{\pi} \int_{\partial D^0} k^2 [\mathcal{E}^{-1}(k)]_1 \left( \begin{array}{c} \frac{\tilde{G}_{12}(t, k)}{\tilde{G}_{12}(t, \alpha k)} \\ e^{-2i\alpha^2kL} \tilde{G}_{12}(t, \alpha^2 k) \end{array} \right) dk,
\]

\[
- \frac{12}{\pi} \int_{\partial D^0} k^2 [\mathcal{E}^{-1}(k)]_2 \left( \begin{array}{c} \frac{\tilde{G}_{22}(t, k)}{\tilde{G}_{22}(t, \alpha k)} \\ e^{-2i\alpha^2kL} \tilde{G}_{22}(t, \alpha^2 k) \end{array} \right) e^{-8i\alpha^2kL} \tilde{G}_{12}(t, \alpha^2 k) dk,
\]

\( (5.14c) \)
where \( \tilde{G}_{12}(t,k) \) and \( G_{22}(t,k) \) are given by

\[
\tilde{G}_{12}(t,k) = -e^{-2ikL} \left\{ \frac{1}{3} \hat{g}_{12}(t) \frac{1}{2ikL} \left( \frac{1}{3} \hat{g}_{12}(t) - \frac{\hat{f}_{12}(t)}{2ikL} \right) \right\} - \frac{1}{\pi} \int_{\partial D^{0}} \left( \hat{f}_{01}(t)\hat{\Phi}_{23}(t,k) + \hat{f}_{02}(t)\hat{\Phi}_{33}(t,k) \right) dk \right] \}

\[
- \left[ \frac{1}{3} \hat{\Phi}_{31}(t,k) - \frac{g_{02}(t)}{2ikL} \right] - e^{-2ikL} \left( \frac{1}{3} \hat{\Phi}_{13}(t,k) - \frac{\hat{f}_{02}(t)}{2ikL} \right),
\]

\[
G_{22}(t,k) = -e^{-2ikL} \left[ \hat{\Phi}_{32}(t,k)\hat{\Phi}_{12}(t,k) + (\hat{\Phi}_{33}(t,k) - 1)\hat{\Phi}_{13}(t,k) \right] - \hat{\Phi}_{31}(t,k)(\hat{\Phi}_{21}(t,k) - 1).
\]

Here \( [E^{-1}(k)]_{j} \), \( j = 1, 2, 3 \) imply the \( j \)-th row of the inverse matrix of \( E(k) \), which is of the form

\[
E(k) = \begin{pmatrix}
1 & e^{-2ikL} & e^{-2\alpha kL} \\
e^{-2\alpha kL} & 1 & \alpha \\
e^{-2\alpha kL} & \alpha & e^{2\alpha kL}
\end{pmatrix}.
\]

The symbol \( \partial D^{0} \) is the boundary contour of \( D \) deformed to pass the zeros of the \( \det E(k) \), where \( D = \{ k | \pi < \arg k < \frac{4\pi}{3} \} \).

Proof. By the substitution of the GLM representations of \( \Phi_{ij} \) and \( \phi_{ij} \) into \( c_{21}(t,k) (4.13a) \) and \( c_{31}(t,k) (4.13b) \), we have the following system

\[
e^{-2ikL} \int_{-t}^{t} \mathcal{L}_{12}(t,s)e^{4ik^{3}(t-s)}ds + \int_{-t}^{t} \mathcal{L}_{21}(t,s)e^{-4ik^{3}(t-s)}ds
\]

\[
+ k \int_{-t}^{t} M_{21}(t,s)e^{-4ik^{3}(t-s)}ds = G_{11}(t,k) + G_{21}(t,k) + c_{21}(t,k),
\]

\[
e^{-2ikL} \int_{-t}^{t} \mathcal{L}_{13}(t,s)e^{4ik^{3}(t-s)}ds + \int_{-t}^{t} \mathcal{L}_{31}(t,s)e^{-4ik^{3}(t-s)}ds
\]

\[
+ k \int_{-t}^{t} M_{31}(t,s)e^{-4ik^{3}(t-s)}ds = G_{12}(t,k) + G_{22}(t,k) + c_{31}(t,k),
\]

where \( \{G_{1j}\}_{j=1}^{2} \) and \( \{G_{2j}\}_{j=1}^{2} \) are given by

\[
G_{1j} = -e^{-2ikL} \left\{ k \int_{-t}^{t} \mathcal{M}_{1,j+1}(t,s)e^{4ik^{3}(t-s)}ds + k^{2} \int_{-t}^{t} \mathcal{N}_{1,j+1}(t,s)e^{4ik^{3}(t-s)}ds \right\}
\]

\[
- k^{2} \int_{-t}^{t} N_{j+1,1}(t,s)e^{-4ik^{3}(t-s)}ds,
\]

\[
G_{2j} = - \left\{ \int_{-t}^{t} \left[ L_{j+1,1}(t,s) + kM_{j+1,1}(t,s) + k^{2}N_{j+1,1}(t,s) \right] e^{-4ik^{3}(t-s)}ds \right\}
\]

\[
\times \left\{ \int_{-t}^{t} \left[ L_{11}(t,s) + kM_{11}(t,s) + k^{2}N_{11}(t,s) \right] e^{4ik^{3}(t-s)}ds \right\}
\]

\[
- e^{-2ikL} \left\{ \int_{-t}^{t} \left[ L_{j+1,2}(t,s) + kM_{j+1,2}(t,s) + k^{2}N_{j+1,2}(t,s) \right] e^{-4ik^{3}(t-s)}ds \right\}
\]

\[
\times \left\{ \int_{-t}^{t} \left[ L_{12}(t,s) + kM_{12}(t,s) + k^{2}N_{12}(t,s) \right] e^{4ik^{3}(t-s)}ds \right\}.
\]
\[-e^{-2iKL} \left\{ \int_{-t}^{t} \left[ L_{j+1,3}(t,s) + kM_{j+1,3}(t,s) + k^{2}N_{j+1,3}(t,s) \right] e^{-4ik^{3}(t-s)} ds \right\} \]
\[\times \left\{ \int_{-t}^{t} \left[ \mathcal{Z}_{13}(t,s) + k\mathcal{M}_{13}(t,s) + k^{2}\mathcal{N}_{13}(t,s) \right] e^{4ik^{3}(t-s)} ds \right\}, \quad (5.18b)\]

Replacing \( k \) by \( \alpha k \) and \( \alpha^{2}k \), respectively, in (5.17a) and (5.17b) for \( k \in D \), one has six equations. One can further write these equations in the following vectors

\[\mathcal{E}(k)U_{j}(t,k) = \mathcal{H}_{1j}(t,k) + \mathcal{H}_{2j}(t,k) + \mathcal{H}_{c_{j}}(t,k), \quad k \in D, \quad j = 1, 2, \quad (5.19)\]

where

\[
\mathcal{E}(k) = \begin{pmatrix}
-e^{-2iKL} & 1 & 1 \\
e^{-2i\alpha kL} & 1 & \alpha \\
1 & e^{2i\alpha^{2}kL} & \alpha^{2}e^{2i\alpha^{2}kL}
\end{pmatrix}, \quad \mathcal{H}_{1j}(t,k) = \begin{pmatrix}
\mathcal{G}_{1j}(t,k) \\
\mathcal{G}_{1j}(t,k) \\
\mathcal{G}_{1j}(t,k)
\end{pmatrix}, \quad (5.20)
\]

with \( l = 1, 2 \). From above analysis, one can show that \( \det \mathcal{E}(k) \to \alpha - 1 \neq 0 \), for \( |k| \to \infty \) and \( k \in \mathbb{D} \).

By multiplying the following factor

\[
\begin{pmatrix}
k^{2} & 0 & 0 \\
0 & k^{2} & 0 \\
0 & 0 & k
\end{pmatrix} \mathcal{E}^{-1}(k)e^{8ik^{3}(t-t')}, \quad 0 < t' < t,
\]

and by integrating along the contour \( \partial D^{0} \), one can show that the terms including \( \mathcal{H}_{c_{j}} \) vanishes from (5.11), by Jordan’s lemma.

For analyzing the other terms, the following identities (see [2] or [25]) should be considered

\[
\int_{-t}^{t} \mathcal{F}(t,s)e^{-4ik^{3}(t-s)} ds = 2 \int_{0}^{t} \mathcal{F}(t,2\tau - t)e^{8ik^{3}(\tau-t)} d\tau, \quad (5.22)
\]

where \( \mathcal{F}(t,s) \) is an arbitrary function such that the integral is well defined, and

\[
\int_{\partial D^{0}} k^{2} \int_{0}^{t} \gamma(t)e^{8ik^{3}(t-t')} d\tau dk = \frac{\pi}{12} \gamma(t'), \quad (5.23a)
\]
\[
\int_{\partial D^{0}} k^{m} \int_{0}^{t} \gamma(t)e^{8ik^{3}(t-t')} d\tau dk = \int_{\partial D^{0}} k^{m} \left( \int_{0}^{t} \gamma(t)e^{8ik^{3}(t-t')} d\tau - \frac{1}{8ik^{3}} \gamma(t') \right) dk, \quad (5.23b)
\]

where \( m = 3, 4 \) and \( \gamma(t) \) is a smooth function for \( 0 < \tau < t \). Then by using Jordan’s lemma and by considering the integration by parts, one can show that one can arrive at the limit as \( t' \to t \) in the right-hand side of (5.23b).

By considering the integral term including \( \mathcal{H}_{1j} \) with (5.23b), we find

\[
\int_{\partial D^{0}} \text{diag}\{k^{2}, k^{2}, k\} \mathcal{E}^{-1}(k)\mathcal{H}_{1j}e^{8ik^{3}(t-t')} dk
\]
\[= \int_{\partial D^{0}} \begin{pmatrix}
k^{2} & 0 & 0 \\
0 & k^{2} & 0 \\
0 & 0 & k
\end{pmatrix} \mathcal{E}^{-1}(k) \begin{pmatrix}
\tilde{\mathcal{G}}_{1j}(t,t',k) \\
\tilde{\mathcal{G}}_{1j}(t,t',\alpha k) \\
e^{-2i\alpha^{2}kL}\tilde{\mathcal{G}}_{1j}(t,t',\alpha^{2}k)
\end{pmatrix} dk, \quad (5.24)
\]
where the functions \( \{ \bar{g}_{ij}(t, t', k) \}_{j=1}^{2} \) are given by

\[
\bar{g}_{ij}(t, t', k) = -2e^{-2i\kappa kL} \left\{ k \int_{0}^{t'} \left[ \mathcal{M}_{1,j+1}(t, 2\tau - t) \right. \right. \\
- \frac{1}{2} \left( \bar{f}_{01}(t)\bar{\mathcal{M}}_{2,j+1}(t, 2\tau - t) + \bar{f}_{02}(t)\bar{\mathcal{M}}_{3,j+1}(t, 2\tau - t) \right) \\
- \frac{1}{8ik^2} \left[ \mathcal{M}_{1,j+1}(t, 2\tau' - t) - \left( \bar{f}_{01}(t)\bar{\mathcal{M}}_{2,j+1}(t, 2\tau - t) \\
+ \bar{f}_{02}(t)\bar{\mathcal{M}}_{3,j+1}(t, 2\tau - t) \right) \right] \left\} \right. \\
- 2 \left\{ k^2 \int_{0}^{t'} N_{j+1,1}(t, 2\tau - t)e^{8ik^3(\tau - t')}d\tau - \frac{1}{8ik^2} N_{j+1,1}(t, 2\tau' - t) \right\} \\
- 2e^{-2i\kappa kL} \left\{ k^2 \int_{0}^{t'} \bar{\mathcal{M}}_{1,j+1}(t, 2\tau - t)e^{8ik^3(\tau - t')}d\tau - \frac{1}{8ik^2} \bar{\mathcal{M}}_{1,j+1}(t, 2\tau' - t) \right\}. \tag{5.25}
\]

By considering the integral in the left-hand side of (5.19) and by using the equations (5.9), (5.23a), one can find the following equation

\[
\left( \begin{array}{c}
\bar{L}_{1,j+1}(t, 2\tau' - t) + \frac{1}{2} \left[ f_{01}(t)\bar{\mathcal{M}}_{2,j+1}(t, 2\tau' - t) + f_{02}(t)\bar{\mathcal{M}}_{3,j+1}(t, 2\tau' - t) \right] \\
\bar{L}_{j+1,1}(t, 2\tau' - t) - \frac{1}{2} g_{0j}(t)\bar{\mathcal{M}}_{11}(t, 2\tau' - t) + \frac{1}{2} g_{1j}(t)\bar{N}_{11}(t, 2\tau' - t) \\
M_{j+1,1}(t, 2\tau' - t) + \frac{1}{2} g_{0j}(t)\bar{N}_{11}(t, 2\tau' - t) 
\end{array} \right) \\
\frac{6}{\pi} \int_{\partial D^0} \left( \begin{array}{ccc}
k^2 & 0 & 0 \\
0 & k^2 & 0 \\
0 & 0 & k 
\end{array} \right) \mathcal{E}^{-1}(k) \left( \begin{array}{c}
\bar{G}_{ij}(t, t', k) \\
\bar{G}_{ij}(t, t', \alpha k) \\
e^{2i\alpha^2 kL} \bar{G}_{ij}(t, \alpha^2 k) 
\end{array} \right) dk \\
\frac{6}{\pi} \int_{\partial D^0} \left( \begin{array}{ccc}
k^2 & 0 & 0 \\
0 & k^2 & 0 \\
0 & 0 & k 
\end{array} \right) \mathcal{E}^{-1}(k) \left( \begin{array}{c}
\bar{G}_{2j}(t, t', k) \\
\bar{G}_{2j}(t, t', \alpha k) \\
e^{2i\alpha^2 kL} \bar{G}_{2j}(t, \alpha^2 k) 
\end{array} \right) e^{8ik^3(\tau - t')} dk,
\right. \tag{5.26}
\]

Based on the initial conditions (5.5a)-(5.5c) and the above system at \( t' = t \), one can derive the expressions of \( \{ g_{1j}(t) \}_{j=1}^{2} \), \( \{ g_{2j}(t) \}_{j=1}^{2} \) and \( \{ f_{2j}(t) \}_{j=1}^{2} \) as follows

\[
g_{1j}(t) = \frac{1}{2} g_{0j}(t)\bar{N}_{11}(t, t) - \frac{6}{\pi} \int_{\partial D^0} k [\mathcal{E}^{-1}(k)]_3 \left( \begin{array}{c}
\bar{G}_{ij}(t, k) \\
\bar{G}_{ij}(t, \alpha k) \\
e^{2i\alpha^2 kL} \bar{G}_{ij}(t, \alpha^2 k) 
\end{array} \right) dk \\
- \frac{6}{\pi} \int_{\partial D^0} k [\mathcal{E}^{-1}(k)]_3 \left( \begin{array}{c}
\bar{G}_{2j}(t, k) \\
\bar{G}_{2j}(t, \alpha k) \\
e^{2i\alpha^2 kL} \bar{G}_{2j}(t, \alpha^2 k) 
\end{array} \right) e^{8ik^3(\tau - t')} dk, \tag{5.27a}
\]

\[
g_{2j}(t) = 2g_{0j}(t) \left[ g_{0j}(t) + g_{0j}(t) \right] - g_{0j}(t)\bar{N}_{11}(t, t) + \frac{1}{2} g_{1j}(t)\bar{N}_{11}(t, t) \\
- \frac{12}{\pi} \int_{\partial D^0} k^2 [\mathcal{E}^{-1}(k)]_2 \left( \begin{array}{c}
\bar{G}_{ij}(t, k) \\
\bar{G}_{ij}(t, \alpha k) \\
e^{2i\alpha^2 kL} \bar{G}_{ij}(t, \alpha^2 k) 
\end{array} \right) dk.
\]
From theorem (5.1) of the GLM representation, we have
\[
G_{ij}(t, k) = e^{2i\alpha^2kL}G_{ij}(t, \alpha^2k) \quad \forall j = 1, 2, 3
\]
and
\[
f_{ij}(t) = f_{0j}(t) \left[ f_{01}(t) + f_{02}(t) \right] + f_{01}(t)\overline{M}_{2,j+1}(t, t) + f_{02}(t)\overline{M}_{3,j+1}(t, t)
\]
Next, we should express the functions \(G_{ij}(t, k)\) and \(f_{ij}(t)\) in terms of the spectral functions \(\Phi_{ij}(t, k)\) and \(\phi_{ij}(t, k)\). In fact, we can write the functions \(G_{ij}(t, k)\) as the following form
\[
G_{ij}(t, k) = e^{-2ikL} \left[ \Phi_{ij}(t, k) \right] = e^{-2ikL} \left[ \Phi_{ij}(t, k) \right] = e^{-2ikL} \left[ \Phi_{ij}(t, k) \right]
\]
where \([E^{-1}(k)]_{j, j} \neq 0\) imply the \(j\)-th row of the inverse matrix of \(E(k)\), and \(\{\tilde{G}_{ij}(t, k)\}_{j=1}^{2}\) are given by
\[
\tilde{G}_{ij}(t, k) = -2e^{-2ikL} \left\{ k \int_{0}^{t} \left[ \overline{N}_{1,j+1}(t, 2\tau - t) - \frac{1}{2} (f_{01}(t)\overline{N}_{2,j+1}(t, 2\tau - t) + f_{02}(t)\overline{N}_{3,j+1}(t, 2\tau - t)) \right] e^{8ik(\tau - t)} d\tau \\
- \frac{1}{8ik} \left[ f_{01}(t) - \frac{1}{2} (f_{01}(t)\overline{N}_{2,j+1}(t, 2\tau - t) + f_{02}(t)\overline{N}_{3,j+1}(t, 2\tau - t)) \right] \right\}
\]
Next, we should express the functions \(\{\tilde{G}_{ij}(t, k)\}_{j=1}^{2}\) in terms of the spectral functions \(\Phi_{ij}(t, k)\) and \(\phi_{ij}(t, k)\). In fact, we can write the functions \(G_{ij}(t, k)\) as the following form
\[
G_{21}(t, k) = -e^{-2ikL} \left[ \Phi_{21}(t, k) = -e^{-2ikL} \left[ \Phi_{21}(t, k) \right] = -e^{-2ikL} \left[ \Phi_{21}(t, k) \right]
\]
From theorem (5.1) of the GLM representation, we have
\[
\Phi_{11}(t, k) = 1 + \int_{-t}^{t} \left[ L_{11}(t, s) + kM_{11}(t, s) + k^2N_{11}(t, s) \right] e^{-4ik(t-s)} ds
\]
which yields
\[
3k^2 \int_{-t}^{t} N_{11}(t, s)e^{-4ik(t-s)} ds = \Phi_{11}(t, k) + \alpha\Phi_{11}(t, k) + \alpha^2\Phi_{11}(t, k)
\]
By integrating the above system along the contour \( \partial D^0 \) and by considering (5.22) and (5.23a), we can express \( N_{11}(t,t) \) and \( \tilde{M}_{11}(t,t) \) in terms of \( \Phi_{11}(t,k) \), \( \Phi_{11}(t,\alpha_k) \) and \( \Phi_{11}(t,\alpha^2_k) \):

\[
N_{11}(t,t) = \frac{2}{\pi} \int_{\partial D^0} \left[ \Phi_{11}(t,k) + \alpha \Phi_{11}(t,\alpha_k) + \alpha^2 \Phi_{11}(t,\alpha^2_k) \right] dk,
\]

(5.32a)

\[
\tilde{M}_{11}(t,t) = [\phi_{01}^2(t) + \phi_{02}^2(t)] + \frac{2}{\pi} \int_{\partial D^0} k \left[ \Phi_{11}(t,k) + \alpha^2 \Phi_{11}(t,\alpha_k) + \alpha \Phi_{11}(t,\alpha^2_k) \right] dk.
\]

(5.32b)

Similarly, one can find

\[
N_{22}(t,t) = \frac{2}{\pi} \int_{\partial D^0} \left[ \phi_{22}(t,k) + \alpha \phi_{22}(t,\alpha_k) + \alpha^2 \phi_{22}(t,\alpha^2_k) \right] dk,
\]

(5.33a)

\[
N_{32}(t,t) = \frac{2}{\pi} \int_{\partial D^0} \left[ \phi_{32}(t,k) + \alpha \phi_{32}(t,\alpha_k) + \alpha^2 \phi_{32}(t,\alpha^2_k) \right] dk,
\]

(5.33b)

and

\[
\tilde{M}_{22}(t,t) = -f_{01}^2(t) + \frac{2}{\pi} \int_{\partial D^0} k \left[ \phi_{22}(t,k) + \alpha^2 \phi_{22}(t,\alpha_k) + \alpha \phi_{22}(t,\alpha^2_k) \right] dk,
\]

(5.34a)

\[
\tilde{M}_{32}(t,t) = -f_{01}(t)f_{02}(t) + \frac{2}{\pi} \int_{\partial D^0} k \left[ \phi_{32}(t,k) + \alpha^2 \phi_{32}(t,\alpha_k) + \alpha \phi_{32}(t,\alpha^2_k) \right] dk.
\]

(5.34b)

By considering the following equation

\[
\phi_{12}(t,k) = \int_{-t}^{t} \left[ \mathcal{L}_{12}(t,s) + k \mathcal{M}_{12}(t,s) + k^2 \mathcal{N}_{12}(t,s) \right] e^{4ikt^3(t-s)} ds,
\]

(5.35)

one finds

\[
3k^2 \int_{-t}^{t} \mathcal{N}_{12}(t,s) e^{4ikt^3(t-s)} ds = \phi_{12}(t,k) + \alpha \phi_{12}(t,\alpha_k) + \alpha^2 \phi_{12}(t,\alpha^2_k),
\]

(5.36a)

\[
3k \int_{-t}^{t} \mathcal{M}_{12}(t,s) e^{4ikt^3(t-s)} ds = \phi_{12}(t,k) + \alpha^2 \phi_{12}(t,\alpha_k) + \alpha \phi_{12}(t,\alpha^2_k),
\]

(5.36b)

which yield the following results

\[
k^2 \int_{0}^{t} \mathcal{N}_{12}(t,2\tau - t) e^{8ikt^3(t-\tau)} d\tau = \frac{1}{6} \left[ \phi_{12}(t,k) + \alpha \phi_{12}(t,\alpha_k) + \alpha^2 \phi_{12}(t,\alpha^2_k) \right],
\]

(5.37a)

\[
k \int_{0}^{t} \mathcal{M}_{12}(t,\tau) e^{8ikt^3(\tau-\tau)} d\tau = \frac{1}{6} \left[ \phi_{12}(t,k) + \alpha^2 \phi_{12}(t,\alpha_k) + \alpha \phi_{12}(t,\alpha^2_k) \right].
\]

(5.37b)

By recalling that

\[
\tilde{M}_{12}(t,s) = \mathcal{M}_{12}(t,s) + \frac{1}{2} \left[ \tilde{f}_{01}(t) \mathcal{N}_{22}(t,s) + \tilde{f}_{02}(t) \mathcal{N}_{32}(t,s) \right].
\]

(5.38)

From the equation,

\[
\Phi_{21}(t,k) = \int_{-t}^{t} \left[ \mathcal{L}_{21}(t,s) + k \mathcal{M}_{21}(t,s) + k^2 \mathcal{N}_{21}(t,s) \right] e^{-4ikt^3(t-s)} ds,
\]

(5.39)
we find
\[ 3k^2 \int_{-t}^{t} N_{21}(t, s)e^{-4ik^3(t-s)}ds = \Phi_{21}(t, k) + \alpha\Phi_{21}(t, \alpha k) + \alpha^2\Phi_{21}(t, \alpha^2 k), \] (5.40)
which shows that
\[ k^2 \int_{0}^{t} N_{21}(t, 2\tau - t)e^{8ik^3(\tau - t)}d\tau = \frac{1}{6} \left[ \Phi_{21}(t, k) + \alpha\Phi_{21}(t, \alpha k) + \alpha^2\Phi_{21}(t, \alpha^2 k) \right]. \] (5.41)

Following the same way, we can derive the similar results
\[ \mathcal{N}_{23}(t, t) = \frac{2}{\pi} \int_{\partial D^o} \left[ \phi_{23}(t, k) + \alpha\phi_{23}(t, \alpha k) + \alpha^2\phi_{23}(t, \alpha^2 k) \right] dk, \] (5.42a)
\[ \mathcal{N}_{33}(t, t) = \frac{2}{\pi} \int_{\partial D^o} \left[ \phi_{33}(t, k) + \alpha\phi_{33}(t, \alpha k) + \alpha^2\phi_{33}(t, \alpha^2 k) \right] dk, \] (5.42b)
and
\[ \tilde{\mathcal{N}}_{23}(t, t) = -f_{01}f_{02}(t) + \frac{2}{\pi} \int_{\partial D^o} k \left[ \phi_{23}(t, k) + \alpha^2\phi_{23}(t, \alpha^2 k) \right] dk, \] (5.43a)
\[ \tilde{\mathcal{N}}_{33}(t, t) = -f_{02}^2(t) + \frac{2}{\pi} \int_{\partial D^o} k \left[ \phi_{33}(t, k) + \alpha^2\phi_{33}(t, \alpha^2 k) \right] dk. \] (5.43b)

By considering the equation
\[ \overline{\phi_{13}(t, k)} = \int_{-t}^{t} [\mathbf{L}_{13}(t, s) + k\mathbf{M}_{13}(t, s) + k^2\mathbf{N}_{13}(t, s)] e^{4ik^3(t-s)}ds, \] (5.44)
one finds
\[ 3k^2 \int_{-t}^{t} \mathbf{N}_{13}(t, s)e^{4ik^3(t-s)}ds = \phi_{13}(t, k) + \alpha\phi_{13}(t, \alpha k) + \alpha^2\phi_{13}(t, \alpha^2 k), \] (5.45a)
\[ 3k \int_{-t}^{t} \mathcal{N}_{13}(t, s)e^{4ik^3(t-s)}ds = \phi_{13}(t, k) + \alpha^2\phi_{13}(t, \alpha^2 k), \] (5.45b)
which yield
\[ k^2 \int_{0}^{t} \mathbf{N}_{13}(t, 2\tau - t)e^{8ik^3(\tau - t)}d\tau = \frac{1}{6} \left[ \phi_{13}(t, k) + \alpha\phi_{13}(t, \alpha k) + \alpha^2\phi_{13}(t, \alpha^2 k) \right], \] (5.46a)
\[ k \int_{0}^{t} \mathcal{N}_{13}(t, s)e^{8ik^3(\tau - t)}d\tau = \frac{1}{6} \left[ \phi_{13}(t, k) + \alpha^2\phi_{13}(t, \alpha^2 k) \right]. \] (5.46b)

We recall that
\[ \tilde{\mathcal{M}}_{13}(t, s) = \mathcal{M}_{13}(t, s) + \frac{1}{2} \left[ \tilde{f}_{01}(t)\mathbf{L}_{23}(t, s) + \tilde{f}_{02}(t)\mathbf{L}_{33}(t, s) \right]. \] (5.47)

From the equation,
\[ \Phi_{31}(t, k) = \int_{-t}^{t} [L_{31}(t, s) + kM_{31}(t, s) + k^2N_{31}(t, s)] e^{-4ik^3(t-s)}ds, \] (5.48)
we find
\[ 3k^2 \int_{-t}^{t} N_{31}(t, s)e^{-4ik^3(t-s)}ds = \Phi_{31}(t, k) + \alpha\Phi_{31}(t, \alpha k) + \alpha^2\Phi_{31}(t, \alpha^2 k), \] (5.49)
which shows that
\[ k^2 \int_0^t N_{31}(t, 2\tau - t) e^{8ik^3(\tau - t)} d\tau = \frac{1}{6} \left[ \Phi_{31}(t, k) + \alpha\Phi_{31}(t, \alpha k) + \alpha^2\Phi_{31}(t, \alpha^2 k) \right]. \]

(5.50)

Based on the above analysis, we have
\[ \tilde{G}_{ij} = -e^{-2ikL} \left\{ \frac{1}{3} \left[ \phi_{1,j+1}(t, k) + \alpha^2\phi_{1,j+1}(t, \alpha k) + \alpha\phi_{1,j+1}(t, \alpha^2 k) \right] \right. \]
\[ - \frac{1}{4ik^2} \left[ \tilde{f}_{11}(t) - \frac{1}{\pi} \tilde{f}_0(t) \int_{\partial D^o} \left( \phi_{2,j+1}(t, k) + \alpha\phi_{2,j+1}(t, \alpha k) + \alpha^2\phi_{2,j+1}(t, \alpha^2 k) \right) dk \right] \]
\[ - \frac{1}{\pi} \tilde{f}_0(t) \int_{\partial D^o} \left( \phi_{3,j+1}(t, k) + \alpha\phi_{3,j+1}(t, \alpha k) + \alpha^2\phi_{3,j+1}(t, \alpha^2 k) \right) dk \}
\[ - \left\{ \frac{1}{3} \left[ \Phi_{j+1,1}(t, k) + \alpha\Phi_{j+1,1}(t, \alpha k) + \alpha^2\Phi_{j+1,1}(t, \alpha^2 k) \right] - \frac{1}{4ik} \tilde{g}_0(t) \right\} \]
\[ - e^{-2ikL} \left\{ \frac{1}{3} \left[ \phi_{1,j+1}(t, k) + \alpha\phi_{1,j+1}(t, \alpha k) + \alpha^2\phi_{1,j+1}(t, \alpha^2 k) \right] \right. \]
\[ \left. \left. - \frac{1}{2ik} \tilde{f}_0(t) \right\} \right\}, \quad j = 1, 2. \]

(5.51)

By using previous symbol hypothesis and by considering the equations (5.29a), (5.29b), (5.32a), (5.32b), (5.33a), (5.33b), (5.34a), (5.34b), (5.42a), (5.42b), (5.43a), (5.43b) and (5.51) in (5.27a)-(5.27c), we can derive the expression of \( \{g_{1j}(t)\}_{j=1}^2 \), \( \{g_{2j}(t)\}_{j=1}^2 \) and \( \{f_{2j}(t)\}_{j=1}^2 \) in terms of \( \{\Phi_{ij}(t, k)\}_{i=1}^3 \) and \( \{\phi_{ij}(t, k)\}_{i=1}^3 \) as (5.27a)-(5.27c). Based on these results and the GLM representation (see Theorem 5.1), one can derive a system of nonlinear ODEs for \( \{\Phi_{ij}(t, k)\}_{i=1}^3 \) and \( \{\phi_{ij}(t, k)\}_{i=1}^3 \).

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