Abstract
The aim of the present paper is to introduce a new numerical method for solving nonlinear Volterra integro-differential equations involving delay. We apply trapezium rule to the integral involved in the equation. Further, Daftardar-Gejji and Jafari method (DGJ) is employed to solve the implicit equation. Existence-uniqueness theorem is derived for solutions of such equations and the error and convergence analysis of the proposed method is presented. We illustrate efficacy of the newly proposed method by constructing examples.

Keywords: Volterra integro-differential equations, delay, Trapezium rule, Daftardar-Gejji and Jafari method, numerical solution, error, convergence.

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1. Introduction
During the last few decades, Volterra integro-differential equations (VIDEs) are widely used for mathematical modeling of various physical and biological phenomena. In this paper we deal with VIDEs that incorporate memory effect. Such VIDEs are useful in mathematical modeling of hereditary phenomena. The study of Volterra delay-integro-differential equations (VDIDEs) has been an active area of research. It is found that VDIDEs are more effective than standard VIDEs in modeling the real-life phenomena. Hence researchers have developed theoretical and numerical analysis of VDIDEs. For example, Brunner [1] has given a survey of some recent developments in the numerical treatment of VDIDEs. Stability and boundedness of the solutions are discussed by Cemil Tunc [2]. Zhang et al. [3] have discussed general linear methods for solving VDIDEs. Explicit and implicit Runge-Kutta methods for solving neutral Volterra integro-differential equations with delay have been developed by Enright et al. [4]. Readers may refer to [5, 6, 7] for more details in this regard.

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In the present paper we consider VDIDE of the following form:

\[ u'(x) = g(x, u(x)) + \int_{x_0}^{x} K(x, t, u(t - \tau))dt, \quad u(x) = \phi(x) \text{ for } x \in [-\tau, x_0]. \] (1.1)

We employ a decomposition method proposed by Daftardar-Gejji and Jafari \[8\] to generate a new, accurate and fast numerical method for solving nonlinear VDIDEs.

The paper is organized as follows: Preliminaries are given in Section 2. A new numerical method is presented in Section 3. Analysis of this numerical method is given in Section 4. Section 5 deals with different types of illustrative examples. Conclusions are summarized in Section 6.

2. Preliminaries

2.1. Basic Definitions and Notations

In this section, we discuss some basic definitions and results.

**Definition 2.1.** \[9\] Let \( u_i \) denote the approximation to the exact value \( u(x_i) \) obtained by a given method with step-size \( h \). Then

(i) a method is said to be convergent if and only if

\[ \lim_{h \to 0} | u(x_i) - u_i | \to 0, \quad i = 1, 2 \ldots N. \] (2.1)

(ii) a method is said to be of order \( m \) if \( m \) is the largest number for which there exists a finite constant \( C \) such that

\[ | u(x_i) - u_i | \leq Ch^m, \quad i = 1, 2 \ldots N. \] (2.2)

2.2. Daftardar-Gejji and Jafari Method

A new iterative method was introduced by Daftardar-Gejji and Jafari (DGIJ) (2006) \[8\] for solving nonlinear functional equations of the form

\[ u = g + L(u) + N(u), \] (2.3)

where \( g \) is a known function, \( L \) and \( N \) are linear and nonlinear operators respectively. DGIJ method provides a solution to Eq.(2.3) in the form of a series of the form

\[ u = \sum_{i=0}^{\infty} u_i, \] (2.4)

where

\[
\begin{align*}
\quad u_0 &= g, \\
\quad u_{m+1} &= L(u_m) + G_m, \quad m = 0, 1, 2, \ldots, \\
\quad G_m &= N \left( \sum_{j=0}^{m} u_j \right) - N \left( \sum_{j=0}^{m-1} u_j \right), \quad m \geq 1.
\end{align*}
\] (2.5)
The $k$-term approximate solution is given by
\[ u = \sum_{i=0}^{k-1} u_i \] (2.6)
for suitable integer $k$. DGJ method has been employed by many researchers to develop new numerical methods [10, 11, 12] for solving differential equations.

3. Numerical Method

In the present section we construct a new numerical method based on DGJ decomposition to solve Volterra delay-integro-differential equations (VDIDEs) of the following form:
\[ u'(x) = g(x, u(x)) + \int_{x_0}^{x} K(x, t, u(t - \tau)) dt, \quad u(x_0) = u_0, \] (3.1)
\[ u(x) = \phi(x) \text{ for } x \in [-\tau, x_0]. \] (3.2)

Consider a uniform partition of the interval $[-\tau, X]$ with grid points $x_j = jh : j = -M, -M+1, \ldots, -1, 0, 1, \ldots, N$ where $M$ and $N$ are integers such that $N = X/h$ and $M = \tau/h$.

Integrating Eq. (3.1) from $x = x_j$ to $x = x_{j+1} + h$, we get
\[ u(x_{j+1} + h) = u(x_j) + \int_{x_j}^{x_{j+1}} g(x, u(x)) dx + \int_{x_j}^{x_{j+1}} \int_{x_0}^{x} K(x, t, u(t - \tau)) dt dx \] (3.3)

Applying trapezium rule [13] to evaluate integrals on right of Eq. (3.3), we obtain
\[ u(x_{j+1} + h) = u(x_j) + \frac{h}{2} g(x_j, u_j) + \frac{h^2}{4} (K(x_j, x_0, u(x_0 - \tau)) + K(x_j, x_j, u(x_j - \tau))) \]
\[ + K(x_{j+1}, x_0, u(x_0 - \tau))) + \frac{h^2}{2} \left( \sum_{i=1}^{j} K(x_j, x_i, u(x_i - \tau)) \right) \]
\[ + \frac{h^2}{2} \left( \sum_{i=1}^{j} K(x_{j+1}, x_i, u(x_i - \tau)) \right) + \frac{h}{2} g(x_{j+1}, u_{j+1}) + O(h^3) \]
\[ + \frac{h^2}{4} K(x_{j+1}, x_{j+1}, u(x_{j+1} - \tau)) \] (3.4)

If $u_j$ denotes approximation to $u(x_j)$, then approximate solution at $x = x_j$ is given by
\[ u_{j+1} = u_j + \frac{h}{2} g(x_j, u_j) + \frac{h^2}{4} (K(x_j, x_0, u(x_0 - \tau)) + K(x_j, x_j, u(x_j - \tau))) \]
\[ + \frac{h^2}{2} \left( \sum_{i=1}^{j} K(x_j, x_i, u(x_i - \tau)) \right) + \frac{h}{2} g(x_{j+1}, u_{j+1}) \]
\[ + \frac{h^2}{4} K(x_{j+1}, x_{j+1}, u(x_{j+1} - \tau)) \] (3.5)
where the delay term is approximated as given below:

\[
  u(x_j - \tau) = u(jh - Mh) = u((j - M)h) = u(x_{j-M}), j = 0, 1, \ldots, N
\]  

(3.6)

and \( u(x_j) = \phi(x_j), j = -M, -M + 1, \ldots, 0 \). 

(3.7)

Eq. (3.5) is of the form (2.3), where

\[
  u = u_{j+1},
\]

\[
  g = u_j + \frac{h}{2} g(x_j, u_j) + \frac{h^2}{4} (K(x_j, x_0, u(x_M)) + K(x_j, x_j, u(x_M)) + K(x_{j+1}, x_0, u(x_M)))
\]

\[
  + \frac{h^2}{2} \left( \sum_{i=1}^{j-1} K(x_j, x_i, u(x_i-M)) + \sum_{i=1}^{j} K(x_{j+1}, x_i, u(x_i-M)) \right)
\]

\[
  + \frac{h^2}{4} K(x_{j+1}, x_{j+1}, u(x_{j+1-M}))
\]

\[
  N(u) = \frac{h}{g(x_j, u_j+1)}.
\]

Applying DGJ method to obtain 3-term approximate solution of eq. (3.5), we get

\[
  u_{j+1} = u_j + \frac{h}{2} g(x_j, u_j) + \frac{h^2}{4} \left( K(x_j, x_0, u_0-M) + K(x_j, x_j, u_{j-M}) + K(x_{j+1}, x_0, u_{j-M}) \right)
\]

\[
  + \frac{h^2}{2} \left( \sum_{i=1}^{j-1} K(x_j, x_i, u_{i-M}) + \sum_{i=1}^{j} K(x_{j+1}, x_i, u_{i-M}) \right) + \frac{h^2}{4} K(x_{j+1}, x_{j+1}, u(x_{j+1-M}))
\]

\[
  + \frac{h^2}{2} g(x_{j+1}, u_j + \frac{h}{2} g(x_j, u_j) + \frac{h^2}{4} (K(x_j, x_0, u_0-M) + K(x_j, x_j, u_{j-M}) + K(x_{j+1}, x_0, u_{j-M}))
\]

\[
  + \frac{h^2}{2} \left( \sum_{i=1}^{j-1} K(x_j, x_i, u_{i-M}) + \sum_{i=1}^{j} K(x_{j+1}, x_i, u_{i-M}) \right) + \frac{h^2}{4} K(x_{j+1}, x_{j+1}, u(x_{j+1-M})) \right) \right).
\]

If we set

\[
  M_1 = u_j + \frac{h}{2} g(x_j, u_j) + \frac{h^2}{4} \left( K(x_j, x_0, u_0-M) + K(x_j, x_j, u_{j-M}) + K(x_{j+1}, x_0, u_{j-M}) \right)
\]

\[
  + \frac{h^2}{2} \left( \sum_{i=1}^{j-1} K(x_j, x_i, u_{i-M}) + \sum_{i=1}^{j} K(x_{j+1}, x_i, u_{i-M}) \right) + \frac{h^2}{4} K(x_{j+1}, x_{j+1}, u(x_{j+1-M}))
\]

\[
  M_2 = M_1 + \frac{h}{2} g(x_{j+1}, M_1),
\]

then equation (3.8) becomes

\[
  u_{j+1} = M_1 + \frac{h}{2} g(x_{j+1}, M_2),
\]

(3.8)

which is our new numerical method (NNM) for solving VDIDEs of the form (3.1)-(3.2).
4. Analysis of Numerical Method

4.1. Existence and Uniqueness Theorem

The following result is generalization of Theorem (1) in [7].

**Theorem 1.** Consider the Volterra integro-differential equation

\[ u'(x) = g(x, u(x)) + \int_{x_0}^{x} K(x, t, u(t - \tau)) dt, \]
\[ u(x) = \phi(x) \text{ for } x \in [-\tau, x_0]. \]

Assume that \( g \) and \( K \) are continuous and satisfy Lipschitz condition

\[ \| g(x, u_1) - g(x, u_2) \| \leq L_1 \| u_1 - u_2 \| \]
\[ \| K(x, t, u_1(t - \tau)) - K(x, t, u_2(t - \tau)) \| \leq L_2 \| u_1(t - \tau) - u_2(t - \tau) \| \]

for every \( x - x_0 \leq a, \| t - x_0 \| \leq a, \| u_1 \| < \infty, \| u_2 \| < \infty \) and \( a > 0 \). Then the IVP (4.1) has unique solution.

**Proof.** Integrating eq. (4.1) and using initial condition, we get

\[ u(x) = u_0 + \int_{x_0}^{x} g(z, u(z)) dz + \int_{x_0}^{x} \int_{z}^{x} K(z, t, u(t - \tau)) dt dz \]
\[ = u_0 + \int_{x_0}^{x} \left( g(z, u(z)) + \int_{z}^{x} K(z, t, u(t - \tau)) dt \right) dz \]
\[ = u_0 + \int_{x_0}^{x} G(z, u(z), u(z - \tau)) dz, \]

where \( G(z, u(z), u(z - \tau)) = g(z, u(z)) + \int_{x_0}^{z} K(z, t, u(t - \tau)) dt \). We have the following observations:

1. \( u_0 \) is continuous because it is constant,
2. Kernel \( G \) is continuous for \( 0 \leq x \leq a \), because \( g \) and \( K \) are continuous in the same domain,
3. \( G \) satisfies Lipschitz condition:

\[ \| G(x, u_1(x), u_1(x - \tau)) - G(x, u_2(x), u_2(x - \tau)) \| = \| g(x, u_1(x)) + \int_{x_0}^{x} K(x, t, u_1(t - \tau)) dt - g(x, u_2(x)) - \int_{x_0}^{x} K(x, t, u_2(t - \tau)) dt \| \]
\[ \leq \| g(x, u_1(x)) - g(x, u_2(x)) \| + \| \int_{x_0}^{x} K(x, t, u_1(t - \tau)) dt + \int_{x_0}^{x} K(x, t, u_2(t - \tau)) dt \| \]
\[ \leq L_1 \| u_1 - u_2 \| + aL_2 \| u_1(x - \tau) - u_2(x - \tau) \| \quad (\because \| x - x_0 \| \leq a) \]
Thus all the conditions of Theorem (1) in [7] are satisfied. Hence, The IVP (4.1) has a unique solution.

**Theorem 2.** Assume that 
\[ g \in C[I \times \mathbb{R}^n, \mathbb{R}^n], \ K \in C[I \times I \times \mathbb{R}^n, \mathbb{R}^n] \text{ and } \int_s^x \left| K(t, s, u(s)) \right| \, dt \leq N, \text{ for } x_0 \leq s \leq x \leq x_0 + a, \ u \in \Omega = \{ \phi \in C[I, \mathbb{R}^n] : \phi(x_0) = x_0 \text{ and } |\phi(x) - u_0| \leq b \}. \text{ Then IVP (4.1) possesses at least one solution.} \]

**Proof.** One can prove this result in a similar manner as given in [14].

4.2. Error Analysis

**Theorem 3.** Let \( g \) and \( K \) satisfy Lipschitz condition in second and third variables with Lipschitz constants \( L_1 \) and \( L_2 \) respectively and \( g \) is bounded by \( M, M > 0 \). Then the numerical method (3.8) is of third order.

**Proof.** Suppose \( u_{j+1} \) is an approximation to \( u(x_{j+1}) \). By using Eq.(3.4) and (3.8), we obtain
\[
|u(x_{j+1}) - u_{j+1}| = |\frac{h^2}{2}g(x_{j+1}, u_{j+1}) - \frac{h^2}{2}g(x_{j+1}, M_2)| + O(h^3)
\]
\[
\leq \frac{h^2}{2} L_1 |u_{j+1} - M_2| + O(h^3).
\]

Using Eq.(3.8) and (3.8), we get
\[
|u(x_{j+1}) - u_{j+1}| \leq \frac{h^2}{4} L_1 \left| g(x_{j+1}, M_2) - g(x_{j+1}, M_1) \right| + O(h^3)
\]
\[
\leq \frac{h^2}{4} L_1 \left| M_2 - M_1 \right| + O(h^3)
\]
\[
\leq \frac{h^2}{4} L_1 \left( \frac{h}{2} \left| g(x_{j+1}, M_1) \right| \right) + O(h^3)
\]
\[
\leq h^3 \left( \frac{L_1}{8} \right) + O(h^3) \leq h^3 \left( \frac{L_1}{8} M \right) + O(h^3)
\]
\[
\leq C h^3,
\]
where \( C \) is some constant. \( \Rightarrow \) The numerical method (3.8) is of third order.

**Corollary 4.** The numerical method (3.8) is convergent.

**Proof.** By Theorem (3) and definition (2.1), the numerical method (3.8) is convergent.

5. Illustrative examples

**Example 5.1.** Consider the linear VIDE with delay
\[
u' (x) = e^{-1}(1 - e^x) + u(x) + \int_0^x u(t - 1)\, dt, \quad u(0) = 1, \quad 0 \leq x \leq 1;
\]
\[
u(x) = e^x, \quad x < 0.
\]
The exact solution is $u(x) = e^x$.

| $x$ | Ab. error for $h = 0.01$ | Ab. error for $h = 0.02$ | Ab. error for $h = 0.1$ |
|-----|--------------------------|--------------------------|--------------------------|
|     | CPU time 0.0523884 Sec. | CPU time 0.0139188 Sec. | CPU time 0.0021009 Sec. |
| 0.1 | $1.98103 \times 10^{-5}$ | $7.88502 \times 10^{-5}$ | $1.89658 \times 10^{-3}$ |
| 0.2 | $2.15173 \times 10^{-5}$ | $8.5647 \times 10^{-5}$ | $2.06066 \times 10^{-3}$ |
| 0.3 | $2.34488 \times 10^{-5}$ | $9.33384 \times 10^{-5}$ | $2.24651 \times 10^{-3}$ |
| 0.4 | $2.5633 \times 10^{-5}$ | $1.02037 \times 10^{-4}$ | $2.45688 \times 10^{-3}$ |
| 0.5 | $2.81019 \times 10^{-5}$ | $1.11871 \times 10^{-4}$ | $2.69489 \times 10^{-3}$ |
| 0.6 | $3.08911 \times 10^{-5}$ | $1.22981 \times 10^{-4}$ | $2.96401 \times 10^{-3}$ |
| 0.7 | $3.40406 \times 10^{-5}$ | $1.35527 \times 10^{-4}$ | $3.26816 \times 10^{-3}$ |
| 0.8 | $3.75955 \times 10^{-5}$ | $1.4969 \times 10^{-4}$ | $3.61174 \times 10^{-3}$ |
| 0.9 | $4.16061 \times 10^{-5}$ | $1.65669 \times 10^{-4}$ | $3.99966 \times 10^{-3}$ |
| 1   | $4.6129 \times 10^{-5}$ | $1.83692 \times 10^{-4}$ | $4.43746 \times 10^{-3}$ |

Fig.1: Comparison of solutions $u(x)$ of (5.1) for $h = 0.01$.  
Fig.2: Comparison of solutions $u(x)$ of (5.1) for $h = 0.02$.  
Fig.3: Comparison of solutions $u(x)$ of (5.1) for $h = 0.1$.  

We compare our solution with exact solution for different values of $h$ in Figs. 1-3 and the absolute errors, CPU time in Table 1. It is observed that NNM solution coincides with exact solution and the proposed method is time efficient.

**Example 5.2.** Consider the non-linear VIDE with delay

$$u'(x) = -e^x \sinh x + u(x) + \int_0^x u^2(t - 1)dt, \quad u(0) = 1, \quad 0 \leq x \leq 1,$$

$$u(x) = e^{x+1}, \quad x < 0.$$

The exact solution is $u(x) = e^{x+1}$.

| $x$   | Ab. error for $h = 0.01$ | Ab. error for $h = 0.02$ | Ab. error for $h = 0.1$ | CPU time 0.0551474 Sec. | CPU time 0.0158238 Sec. | CPU time 0.001088 Sec. |
|-------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 0.1   | $5.39924 \times 10^{-5}$ | $2.1491 \times 10^{-4}$ | $5.17093 \times 10^{-3}$ | 0.0551474 Sec.           | 0.0158238 Sec.           | 0.001088 Sec.           |
| 0.2   | $5.91385 \times 10^{-5}$ | $2.35415 \times 10^{-4}$ | $5.66983 \times 10^{-3}$ |                         |                         |                         |
| 0.3   | $6.54017 \times 10^{-5}$ | $2.60385 \times 10^{-4}$ | $6.27996 \times 10^{-3}$ |                         |                         |                         |
| 0.4   | $7.30438 \times 10^{-5}$ | $2.90867 \times 10^{-4}$ | $7.02773 \times 10^{-3}$ |                         |                         |                         |
| 0.5   | $8.2388 \times 10^{-5}$  | $3.28155 \times 10^{-4}$ | $7.94573 \times 10^{-3}$ |                         |                         |                         |
| 0.6   | $9.38929 \times 10^{-5}$ | $3.78444 \times 10^{-4}$ | $9.07421 \times 10^{-3}$ |                         |                         |                         |
| 0.7   | $1.0787 \times 10^{-4}$  | $4.29901 \times 10^{-4}$ | $1.04628 \times 10^{-2}$ |                         |                         |                         |
| 0.8   | $1.25104 \times 10^{-4}$ | $4.98748 \times 10^{-4}$ | $1.21727 \times 10^{-2}$ |                         |                         |                         |
| 0.9   | $1.4628 \times 10^{-4}$  | $5.83369 \times 10^{-4}$ | $1.42792 \times 10^{-2}$ |                         |                         |                         |
| 1     | $1.72316 \times 10^{-4}$ | $6.87436 \times 10^{-4}$ | $1.68753 \times 10^{-2}$ |                         |                         |                         |

Fig. 4: Comparison of solutions $u(x)$ of (5.2) for $h = 0.01$.

Fig. 5: Comparison of solutions $u(x)$ of (5.2) for $h = 0.02$.  

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We compare our solution with exact solution for different values of $h$ in Fig.4-6. and the absolute errors, CPU time in Table 2. It is observed that NNM solution coincides with exact solution and the proposed method is time efficient.

6. Conclusion

A new numerical method is developed for solving nonlinear Volterra delay integro-differential equations (VDIDEs) of the form

$$u'(x) = g(x, u(x)) + \int_{x_0}^{x} K(x, t, u(t - \tau))dt, \quad u(x) = \phi(x) \text{ for } x \in [-\tau, x_0].$$

(6.1)

Existence-uniqueness theorem is derived for solution of VDIDEs and error and convergence analysis of the proposed method is presented. Efficiency of the proposed method is illustrated with various examples and it is observed that they are in very good agreement with the exact solutions.

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