1 Introduction

Let $G$ be the Heisenberg group.

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

For a positive integer $c$, let $H_c$ be the subgroup of $G$ obtained when $x, y, cz$ are integers. The Heisenberg manifold $M_c$ is the quotient $G/H_c$. Nonzero Poisson brackets on $M_c$ invariant under left translation by $G$ are parametrized by two real parameters $\mu, \nu$ with $\mu^2 + \nu^2 \neq 0$ [11]. For each positive integer $c$ and real numbers $\mu, \nu$ Rieffel constructed a C*-algebra $A_{\mu,\nu}^{c,\hbar}$ as example of deformation quantization along a Poisson bracket [11]. These algebras have further been studied by [1] [2] [15]. It was also remarked in [11] that it should be possible to construct example of non-commutative geometry as expounded in [5] in these algebras also. It is known [11] that Heisenberg group acts ergodically on $A_{\mu,\hbar}$ and $A_{\mu,\nu}^{c,\hbar}$ accomodates a unique invariant tracial state $\tau$. Using the group action we construct a family of spectral triples. It is shown
that they induce same element in K-homology. We also show that the associated Kasparov module is non-trivial. This has been achieved by constructing explicitly the pairing with a unitary. We also compute the space of forms as described in [3]. Then we characterize torsionless and unitary connections. From that easily follows that a torsionless unitary connection can not exist. For a family of unitary connections we compute Ricci curvature and scalar curvature as introduced in [6]. This family has non-trivial curvature. Following [4] one can construct Quantum Dynamical Semigroups which will be natural candidate for heat semigroup. In a forthcoming version of the paper we wish to describe the construction and dilate the semigroup.

Organization of the paper is as follows. In section 2 after introducing the algebra we compute the GNS space of $\tau$ using a crucial result of Weaver [15]. In the next section following a general principle of construction of spectral triple on a C*-dynamical system with dynamics governed by a Lie group we construct spectral triples and compute the hyper trace [9], [4] associated with the spectral triple. In section 3 we compute the space of forms (Chapter V of [3]). There are not too many instances of this computation in the literature. In section 4 after briefly recalling the notions introduced in [6] we compute the space of $L^2$-forms. Then we characterize torsionless/unitary connections and show a connection can not simultaneously be torsionless and unitary. In the next section for a concrete family of unitary connections we compute Ricci curvature and scalar curvature. In section 7 we show that the spectral triples we consider give rise to same Kasparov element. Then we also show that they have non-trivial Chern character.

2 The Quantum Heisenberg Algebra

Notation: for $x \in \mathbb{R}$, $e(x)$ stands for $e^{2\pi i x}$

**Definition 2.1** For any positive integer $c$ let $S^c$ denote the space of $C^\infty$ functions $\Phi : \mathbb{R} \times \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$ such that

a) $\Phi(x + k, y, p) = e(ckpy)\Phi(x, y, p)$ for all $k \in \mathbb{Z}$

b) for every polynomial $P$ on $\mathbb{Z}$ and every partial differential operator $\tilde{X} = \frac{\partial^{m+n}}{\partial x^m \partial y^n}$ on $\mathbb{R} \times \mathbb{T}$ the function $P(p)(\tilde{X}\Phi)(x, y, p)$ is bounded on $K \times \mathbb{Z}$ for any compact subset $K$ of $\mathbb{R} \times \mathbb{T}$.

For each $h, \mu, \nu \in \mathbb{R}, \mu^2 + \nu^2 \neq 0$, let $A^\infty_h$ denote $S^c$ with product and involu-
tion defined by

$$(\Phi \ast \Psi)(x,y,p) = \sum_q \Phi(x-h(q-p)\mu,y-h(q-p)\nu,q)\Psi(x-h\mu,y-h\nu,p-q)$$ (2.1)

$$\Phi^*(x,y,p) = \bar{\Phi}(x,y,-p)$$ (2.2)

$$\pi : \mathcal{A}_h^{\infty} \to \mathcal{B}(L^2(\mathbb{R} \times T \times \mathbb{Z}))$$ given by

$$(\pi(\Phi)\xi)(x,y,p) = \sum_q \Phi(x-h(q-2p)\mu,y-h(q-2p)\nu,q)\xi(x,y,p-q)$$ (2.3)

gives a faithful representation of the involutive algebra $\mathcal{A}_h^{\infty}$.

$\mathcal{A}_{c,\mu,\nu}^c = \text{norm closure of } \pi(\mathcal{A}_h^{\infty})$ is called the Quantum Heisenberg Manifold.

$N_h = \text{weak closure of } \pi(\mathcal{A}_h^{\infty})$

We will identify $\mathcal{A}_h^{\infty}$ with $\pi(\mathcal{A}_h^{\infty})$ without any mention. Since we are going to work with fixed parameters $c, \mu, \nu, h$ we will drop them altogether and denote $\mathcal{A}_{c,\mu,\nu}^c$ simply by $\mathcal{A}_h$ here the subscript remains merely as a reminiscent of Heisenberg only to distinguish it from a general algebra.

**Action of the heisenberg group:** For $\Phi \in S^c, (r,s,t) \in \mathbb{R}^3 \equiv G,$ (as a topological space)

$$(L_{(r,s,t)}\phi)(x,y,p) = e(p(t+cs(x-r)))\phi(x-r,y-s,p)$$ (2.4)

extends to an ergodic action of the Heisenberg group on $\mathcal{A}_{c,\mu,\nu}^c$.

**The Trace:** $\tau : \mathcal{A}_h^{\infty} \to \mathbb{C}$, given by $\tau(\phi) = \int_0^1 \int_T \phi(x,y,0)dxdy$ extends to a faithful normal tracial state on $N_h$. $\tau$ is invariant under the Heisenberg group action. So, the group action can be lifted to $L^2(\mathcal{A}_h^{\infty})$. We will denote the action at the Hilbert space level by the same symbol.

**Theorem 2.2 (Weaver)** Let $\mathcal{H} = L^2(\mathbb{R} \times T \times \mathbb{Z})$ and $V_f, W_k, X_r$ be the operators defined by

$$(V_f\xi)(x,y,p) = f(x,y)\xi(x,y,p)$$

$$(W_k\xi)(x,y,p) = e(-ck(p^2h\nu + py))\xi(x + k, y, p)$$

$$(X_r\xi)(x,y,p) = \xi(x-2hr\mu, y-2hr\nu, p+r)$$

Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in N_h$ iff $T$ commutes with the operators $V_f, W_k, X_r$ for all $f \in L^\infty(\mathbb{R} \times T), k, r \in \mathbb{Z}$. 

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Lemma 2.3 Let $S^c_{\infty,1} = \{ \psi : \mathbb{R} \times T \times \mathbb{Z} \rightarrow \mathbb{C} | (i) \psi \text{ is measurable,} \\
(ii) \psi_n = \sup_{x \in \mathbb{R}, y \in T} |\psi(x, y, p)| \text{ is an } l_1 \text{ sequence,} \\
(iii) \psi(x + k, y, p) = e(ckyp)\psi(x, y, p) \text{ for all } k \in \mathbb{Z} \}$. 
Then, for $\phi \in S^c_{\infty,1}$ $\pi(\phi)$ defined by the same expression as in (2.3) gives a bounded operator on $L^2(\mathbb{R} \times T \times \mathbb{Z})$.

Proof: Let $\phi' : \mathbb{Z} \rightarrow \mathbb{R}_{+}$ be $\phi'(n) = \sup_{x \in \mathbb{R}, y \in T} |\phi(x, y, n)|$. 
Then $|\pi(\phi(x, y, p)| \leq |\phi' * \xi(x, y,{.})|$, where $* \text{ denotes convolution on } \mathbb{Z}$ and $|\xi(x, y,{.})|$ is the function $p \mapsto |\xi(x, y, p)|$. 
By Young’s inequality $\|\pi(\phi(x, y, .))\|_{l_2} \leq \|\phi' * \xi(x, y,. )\|_{l_2}$
Therefore, $\|\pi(\phi)\| \leq \|\phi\|_{\infty,1}$, where $\|\phi\|_{\infty,1} = \|\phi'\|_{l_1}$.

Remark 2.4 i) product and involution defined by (2.1, 2.2) turns $S^c_{\infty,1}$ into an involutive algebra.
ii) $\phi \mapsto \|\phi\|_{\infty,1}$ is a $*$-algebra norm.

Lemma 2.5 $\pi(S^c_{\infty,1}) \subseteq N_h$.

Proof: Follows from Weaver’s characterization of $N_h$. 

Proposition 2.6 $L^2(\mathcal{A}_h^\infty, \tau)$ is unitarily equivalent with $L^2(\mathbb{T} \times T \times \mathbb{Z}) \cong L^2([0,1] \times [0,1] \times \mathbb{Z})$.

Proof: For $\phi \in S^c_{\infty,1}$, $\Gamma \phi : \mathbb{R} \times T \times \mathbb{Z} \rightarrow \mathbb{C}$ is given by

$$\Gamma \phi(x, y, p) = \begin{cases} e(-cxy)\phi(x, y, p) & \text{for } y < 1 \\
\phi(x, y, p) & \text{for } y = 1 \end{cases}.$$ 

Then $\Gamma \phi(x + k, y, p) = \Gamma \phi(x, y, p)$. So, $\Gamma \phi$ is a map from $\mathbb{T} \times T \times \mathbb{Z}$ to $\mathbb{C}$.

$$\tau(\phi^* * \phi) = \int_0^1 \int_T \sum_q |\phi(x - h\mu, y - h\nu, -q)|^2 dxdy = \int_0^1 \int_T \sum_q |\phi(x, y, q)|^2 dxdy.$$

Since, $|\phi(x + k, y, p)| = |\phi(x, y, p)|$ for all $x \in \mathbb{R}, y \in T, k, p \in \mathbb{Z}$. 
Therefore $\tau(\phi^* * \phi) = \|\Gamma \phi\|^2$, i.e., $\Gamma : L^2(\mathcal{A}_h^\infty, \tau) \rightarrow L^2(\mathbb{T} \times T \times \mathbb{Z})$ is an isometry. To see $\Gamma$ is an unitary observe,
(i) $N_h \subseteq L^2(\mathcal{A}_h^\infty, \tau)$, since $\tau$ is normal.
(ii) \( \phi_{m,n,k} = \begin{cases} e(cxyp)e(mx + ny)\delta_{kp}, & \text{for } 0 \leq y \leq 1 \\ \delta_{kp}e(mx) & \text{for } y = 1 \end{cases} \)

is an element of \( S_{\infty,\infty,1}^c \subseteq N_h \)

(iii) \( \{ \Gamma \phi_{m,n,k} \}_{m,n,k \in \mathbb{Z}} \) is an orthonormal basis in \( L^2(\mathbb{T}^2 \times \mathbb{Z}) \). \( \square \)

**Remark 2.7** \( \phi \mapsto \phi|_{[0,1] \times \mathbb{T} \times \mathbb{Z}} \) gives an unitary isomorphism.

**Corollary 2.8** Let \( M_{yp} \) be the multiplication operator on \( \mathcal{H} = L^2(\mathbb{T} \times \mathbb{T} \times \mathbb{Z}) \).

If we consider \( \mathcal{A}_h^\infty \) as a subalgebra of \( \mathcal{B}(\mathcal{H}) \) by the left regular representation then, \( [M_{yp}, \mathcal{A}_h^\infty] \subseteq \mathcal{B}(\mathcal{H}) \).

**Proof:** Note for \( \phi \in \mathcal{A}_h^\infty \), \( (M_{yp}\phi)(x,y,p) = yp\phi(x,y,p) \) gives an element in \( S_{\infty,\infty,1}^c \), hence a bounded operator.

\[
[M_{yp}, \phi] \psi(x,y,p) = \sum_q (yp - (y - hq\nu)(p - q))\phi(x - h(q - p)\mu, y - h(q - p)\nu, q) \times \psi(x - hq\mu, y - hq\nu, p - q) = \sum_q q(y - h(q - p)\nu)\phi(x - h(q - p)\mu, y - h(q - p)\nu, q) \times \psi(x - hq\mu, y - hq\nu, p - q) = (M_{yp}(\phi \star \psi))(x,y,p)
\]

for \( \psi \in \mathcal{A}_h^\infty \). This completes the proof. \( \square \)

### 3 A class of spectral triples

Let \( (\mathcal{A}, G, \alpha) \) be a \( C^* \) dynamical system with \( G \) an \( n \) dimensional Lie group, and \( \tau \) a \( G \)-invariant trace on \( \mathcal{A} \). Let \( \mathcal{A}_h^\infty \) be the space of smooth vectors, \( \mathcal{K} = L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^N \) where \( N = 2^{[n/2]} \). Fix any basis \( X_1, X_2, \ldots, X_n \) of \( L(G) \) the Lie algebra of \( G \). Since \( G \) acts as a strongly continuous unitary group on \( \mathcal{H} = L^2(\mathcal{A}, \tau) \) we can form selfadjoint operators \( d_{X_i} \) on \( \mathcal{H} \). \( D : \mathcal{K} \to \mathcal{K} \) is given by \( D = \sum_i d_{X_i} \otimes \gamma_i \), where \( \gamma_1, \ldots, \gamma_n \) are selfadjoint matrices in \( M_N(\mathbb{C}) \) such that \( \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij} \) along with \( \mathcal{A}_h^\infty \) and \( \mathcal{K} \) should produce a spectral triple. For such a \( D , [D, \mathcal{A}_h^\infty] \subseteq \mathcal{A}_h^\infty \otimes M_N(\mathbb{C}) \).
Proposition 3.1 For the quantum Heisenberg manifold, if we identify the Lie algebra of Heisenberg group with the Lie algebra of upper triangular matrices, then $D$ as described becomes a selfadjoint operator with compact resolvent for the following choice:

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & c\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\alpha \in \mathbb{R}$ is greater than one.

Proof: Let $\mathcal{D}(D) = \{ f \in L^2([0, 1] \times [0, 1] \times \mathbb{Z}) | f(x, 0, p) = f(x, 1, p), f(1, y, p) = e(cpy)f(0, y, p), \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \in L^2 \} \otimes \mathbb{C}^2$

$D(f \otimes u) = \sum_{j=1}^3 id_j(f) \otimes \sigma_j(u)$, where

$$id_1(f) = -i\frac{\partial f}{\partial x}$$
$$id_2(f) = -2\pi cpxf(x, y, p) - i\frac{\partial f}{\partial y}$$
$$id_3(f) = -2\pi p\alpha f(x, y, p)$$

and $\sigma_j$'s are the spin matrices.

Let $\eta : L^2([0, 1] \times [0, 1] \times \mathbb{Z}) \to L^2([0, 1] \times [0, 1] \times \mathbb{Z})$ be the unitary given by

$$\eta(f)(x, y, p) = \begin{cases} e(-cxy)p f(x, y, p) & \text{for } y < 1 \\ f(x, y, p) & \text{for } y = 1 \end{cases}$$

Let $\mathcal{D}(D') = (\eta \otimes I_2)\mathcal{D}(D)$, and $D' = (\eta \times I_2)D(\eta \otimes I_2)^{-1}$.

Then $\mathcal{D}(D') = \{ f \in L^2([0, 1] \times [0, 1] \times \mathbb{Z}) | f(x, 0, p) = f(x, 1, p), f(0, y, p) = f(1, y, p), \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, pf \in L^2 \} \otimes \mathbb{C}^2$ and

$D'(f \otimes u) = \sum_{j=1}^3 id'_j(f) \otimes \sigma_j(u)$ where,

$$id'_1(f)(x, y, p) = -2\pi icypf(x, y, p) - \frac{\partial f}{\partial x}(x, y, p)$$
$$id'_2(f)(x, y, p) = -\frac{\partial f}{\partial y}(x, y, p)$$
$$id'_3(f)(x, y, p) = 2\pi p\alpha f(x, y, p)$$
Note, on \( D(D') \), \( D' = T + S \) where, \( \mathcal{D}(T) = \mathcal{D}(D') \subseteq \mathcal{D}(S) \)

\[
T = -i \frac{\partial}{\partial x} \otimes \sigma_1 - i \frac{\partial}{\partial y} \otimes \sigma_2 - 2\pi c \alpha M_\rho \otimes \sigma_3
\]

\( S = 2\pi c M_\rho \otimes \sigma_1 \) are selfadjoint operators on their respective domains. Also note \( T \) has compact resolvents. Our conclusion follows from the Rellich lemma since \( S \) is relatively bounded with respect to \( T \) with relative bound less than \( \frac{1}{\alpha} < 1 \).

\[ \blacksquare \]

**Theorem 3.2** Let \( \mathcal{H} = L^2(A_{\hbar}^\infty, \tau) \otimes \mathbb{C}^2 \cdot \). \( A_{\hbar}^\infty \) with its diagonal action becomes a subalgebra of \( \mathcal{B}(\mathcal{H}) \). \( (A_{\hbar}^\infty, \mathcal{H}, D) \) is an odd spectral triple of dimension 3.

**Proof**: \( (A_{\hbar}^\infty, \mathcal{H}, D) \) is a spectral triple follows from the previous proposition and the remark preceding that. We only have to show \( |D|^{-3} \in \mathcal{L}^{1, \infty} \), the ideal of Dixmier traceable operators. For that observe:

(i) Since \( T \) is the dirac operator on \( \mathbb{T}^3 \), \( \mu_n(T^{-1}|_{\ker T}) = O(1/n^{1/3}) \), \( \mu_n \) stands for the \( n \)th singular value.

(ii) \( S \) is relatively bounded with relative bound less than \( \frac{1}{\alpha} < 1 \), hence \( \|S(T + i)^{-1}\| \leq \frac{1}{\alpha} \) and \( \|(1 + S(T + i)^{-1})^{-1}\| \leq \frac{\alpha}{\alpha - 1} \)

(iii) \( \mu_n(AB) \leq \mu_n(A)\|B\| \), for bounded operators \( A, B \).

Applying (i),(ii),(iii) to \( (D' + i)^{-1} = (T + i)^{-1}(1 + S(T + i)^{-1})^{-1} \) we get the desired conclusion for \( D' \) and hence for \( D \). \[ \blacksquare \]

**Corollary 3.3** Let \( T, S, D, D' \) be as in the proof of proposition (3.1). \( A = (\eta \otimes I_2)^{-1}T(\eta \otimes I_2) \) Then \( (A_{\hbar}^\infty, \mathcal{H}, A) \) is an odd spectral triple of dimension 3

**Proof**: We only have to show \( [A, A_{\hbar}^\infty] \subseteq \mathcal{B}(\mathcal{H}) \).

Let \( B = (\eta \otimes I_2)^{-1}S(\eta \otimes I_2) \). Then since \( \eta \otimes I_2 \) commutes with \( S \), \( B = S \).

By corollary (2.8), \( [B, A_{\hbar}^\infty] \subseteq \mathcal{B}(\mathcal{H}) \). Now the previous theorem along with \( D = A + B \) completes the proof. \[ \blacksquare \]

**Remark 3.4** One can similarly show \( (A_{\hbar}^\infty, \mathcal{H}, A_t) \) forms an odd spectral triple of dimension 3, for \( t \in [0, 1] \). Here \( A_t \) stands for \( A_t = A + tB \).

**Remark 3.5** \( D, A \) constructed above depends on \( \alpha \).
Proposition 3.6  The positive linear functional on \( \mathcal{A}_h \otimes M_2(\mathbb{C}) \) given by \( \int : a \mapsto tr_\omega a |D|^{-3} \) is nothing but \( \frac{1}{2} (tr_\omega |D|^{-3}) \tau \otimes tr \).

Proof:

\[
D^2 = - \begin{pmatrix}
(d_1^2 + d_2^2 + (d_3 + \frac{1}{2\alpha})^2 - \frac{1}{4\alpha^2}) & 0 \\
0 & (d_1^2 + d_2^2 + (d_3 - \frac{1}{2\alpha})^2 - \frac{1}{4\alpha^2})
\end{pmatrix}
\]

It is easily seen that

(i) compactness of resolvents of \( D^2 \) implies that for \( X_1, X_2 \)
(ii) eigenvalues of \( X_1, X_2 \) have similar asymptotic behaviour.

Therefore \( X_1^{-3/2}, X_2^{-3/2} \in L^{4,\infty} \) and \( tr_\omega a X_1^{-3/2} = tr_\omega a X_2^{-3/2} \) for any \( a \in \mathcal{B}(L^2(\mathcal{A}_h)) \)

Consider the unitary group on \( \mathcal{H} \cong L^2([0,1] \times \mathbb{T} \times \mathbb{Z}) \otimes \mathbb{C}^2 \) given by

\[
U_t(x \otimes y \otimes e_p \otimes z) = e(pt)(x \otimes y \otimes e_p \otimes z).
\]

Then \( U_tD = DU_t \) and

\[
\int A = tr_\omega U_tAU_t^* |D|^{-3} = tr_\omega \int_0^1 U_tAU_t^* |D|^{-3} dt = \int (A)_0
\]

Here

\[
A = \begin{pmatrix}
\psi_{11} & \psi_{12} \\
\psi_{21} & \psi_{22}
\end{pmatrix} \mapsto (A)_0 = \begin{pmatrix}
(\psi_{11})_0 & (\psi_{12})_0 \\
(\psi_{21})_0 & (\psi_{22})_0
\end{pmatrix}
\]

is the CPmap explicitly given for \( \psi \in S^c \) by \((\psi)_0(x, y, p) = \delta_{p0}\psi(x, y, p)\)

Since \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) commutes with \( |D|^{-3} \), we get

\[
\int A = tr_\omega (a_{11})_0 X_1^{-3/2} + tr_\omega (a_{22})_0 X_2^{-3/2} = tr_\omega ((a_{11})_0 + (a_{22})_0) X_1^{-3/2}
\]

Consider the homomorphism \( \Phi : C(\mathbb{T}^2) \to \mathcal{A}_h \) given by

\( \Phi(f)(x, y, p) = \delta_{p0} f(x, y) \). Now by riesz representation theorem for

\[ \int \circ (\Phi \otimes I_2) : C(\mathbb{T}^2) \to \mathbb{C}, \]

we get a measure \( \lambda \) on \( \mathbb{T}^2 \) such that

\( tr_\omega 2(\psi)_0 X_1^{-3/2} = \int (\psi)_0(x, y, 0) d\lambda \) implying

\[
\int A = \frac{1}{2} \int ((a_{11})_0 + (a_{22})_0) d\lambda
\]

(3.1)
In the next lemma we show $\lambda$ is proportional to Lebesgue measure. That will prove $\int \propto \tau \otimes \text{tr}$, and the proportionality constant is obtained by evaluating both sides on $I$.

**Lemma 3.7** If $\{1, \hbar \mu, \hbar \nu\}$ is rationally independent then $\lambda$ as obtained in the previous proposition is proportional to Lebesgue measure.

**Proof:** It is known [9] [4] that for a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with $|D|^{-p} \in \mathcal{L}^{(1,\infty)}$ for some $p$, $a \mapsto \text{tr}_\omega a|D|^{-p}$ is a trace on the algebra. This along with (3.1) gives

$$\int (\phi \ast \psi)(x, y, 0)d\lambda(x, y) = \int (\psi \ast \phi)(x, y, 0)d\lambda(x, y), \forall \phi, \psi \in S^c$$

Taking $\phi(x, y, p) = e(c[x]y)p(x - [x])g(y)\delta_{1p}$ where $g : T \to \mathbb{C}$, $f : [0, 1] \to \mathbb{C}$ are smooth functions with $\text{supp}(f) \subseteq [\epsilon, 1 - \epsilon]$ for some $\epsilon > 0$ and $\psi = \phi^*$ we get from (3)

$$\int |\phi(x + \hbar \mu, y + \hbar \nu, 1)|^2d\lambda(x, y) = \int |\phi \circ \gamma(x + \hbar \mu, y + \hbar \nu, 1)|^2\lambda(x, y)$$

where $\gamma : T^2 \to T^2$ is given by $\gamma(x, y) = (x - 2\hbar \mu, y - 2\hbar \nu)$. The hypothesis of linear independence of $(1, \hbar \mu, \hbar \nu)$ over the rationals implies that $\gamma$-orbits are dense. This along with (3) proves the lemma. 

**Remark 3.8** In the rest of the paper $\int$ will denote $\frac{1}{2} \tau \otimes \text{tr}$.

### 4 Space of forms

**Lemma 4.1** Let $\mathcal{A}$ be a dense subalgebra of a unital $C^*$-algebra $\bar{\mathcal{A}}$ closed under holomorphic function calculus, then $\mathcal{A}$ is simple provided $\bar{\mathcal{A}}$ is so.

**Proof:** Let $J \subseteq \mathcal{A}$ be an ideal. Then $\bar{J} = \bar{\mathcal{A}}$, since $\bar{\mathcal{A}}$ is simple. There exists $x \in J$ such that $\|x - I\| < 1$. Then $x^{-1} \in \bar{\mathcal{A}}$, hence in $\mathcal{A}$ because $\mathcal{A}$ is closed under holomorphic function calculus. Therefore $1 = xx^{-1} \in J$. 

**Remark 4.2** $\mathcal{A}_h^\infty$ is simple because $\mathcal{A}_h$ is so.
Definition 4.3 (Connes) Let \((\mathcal{A}, \mathcal{H}, D)\) be a spectral triple. \(\Omega^k(\mathcal{A}) = \{ \sum_{i=1}^n a_i^0 \delta a_1 \cdots \delta a_k | n \in \mathbb{N}, a_i^j \in \mathcal{A} \} \Omega^\bullet(\mathcal{A}) = \bigoplus_0^\infty \Omega^k(\mathcal{A})\) is the unital graded algebra of universal forms. Here \(\delta\) is an abstract linear operator with \(\delta^2 = 0, \delta(ab) = \delta(a)b + a\delta(b)\). \(\Omega^\bullet(\mathcal{A})\) becomes a *algebra under the involution \((\delta a)^* = -\delta(a^*)\forall a \in \mathcal{A}\). Let \(\pi : \Omega^\bullet(\mathcal{A}) \to \mathcal{B}(\mathcal{H})\) be the *-representation given by \(\pi(a) = A, \pi(\delta a) = [D, a]\) Let \(J_k = \ker \pi |_{\Omega^k(\mathcal{A})}\) The unital graded differential *-algebra of differential forms \(\Omega^\bullet_D(\mathcal{A})\) is defined by

\[
\Omega^\bullet_D(\mathcal{A}) = \bigoplus_0^\infty \Omega^k_D(\mathcal{A}), \Omega^k_D(\mathcal{A}) = \Omega^k(\mathcal{A})/(J_k + \delta J_{k-1}) \cong \pi(\Omega^k(\mathcal{A}))/\pi(\delta J_{k-1})
\]

Notation: (i) Let \(\phi \in S^c, [D, \phi] = \sum \delta_i(\phi) \otimes \sigma_i\) where \(\delta_i(\phi) = id_j(\phi)\) (see proof of proposition 3.1 for \(d_i\)) but looked upon as derivation on \(\mathcal{A}^\infty\).

Note: \([\delta_1, \delta_3] = [\delta_2, \delta_3] = 0, [\delta_1, \delta_2] = \delta_3\)

(ii) \(\phi_{m,n}(x, y, p) = e(mx + ny)\delta_{j=0}\)

Lemma 4.4 Let \(\mathcal{A}\) be a unital simple algebra, \(M \subseteq \bigoplus_{\text{ntimes}} \mathcal{A}\) a \(\mathcal{A}\-\mathcal{A}\) bimodule. Suppose \(\exists a_{ij}, 1 \leq n, 1 \leq j \leq i\) such that

(i) \(a_{ii} \neq 0\), (ii) \(b_i = (a_{i1}, \ldots, a_{ii}, 0, \ldots, 0) \in M\)

Then \(M \cong \bigoplus_{\text{ntimes}} \mathcal{A}\) as an \(\mathcal{A}\-\mathcal{A}\) bimodule.

Proof: By induction on \(n\),

For \(n = 1\), \(0 \neq M\) is an ideal in \(\mathcal{A}\), hence \(M = \mathcal{A}\).

Let \(\pi : M \to \mathcal{A}\) be \(\pi(a_1, \ldots, a_n) = a_n\).

Then by hypothesis \(\pi(M)\) is a nontrivial ideal in \(\mathcal{A}\) hence equals \(\mathcal{A}\). So, we have a split short exact sequence

\[0 \to \ker(\pi) \to M \to \mathcal{A} \to 0\]

Therefore \(M = \ker(\pi) \oplus \text{Im}\pi = \ker(\pi) \oplus \mathcal{A} = \bigoplus_{\text{ntimes}} \mathcal{A}\). In the last equality we have used induction hypothesis for \(\ker(\pi)\).

\[\square\]

Proposition 4.5

(i) \(\Omega^1_D(\mathcal{A}^\infty_h) = \{ \sum_{i} a_i \otimes \sigma_i | a_i \in \mathcal{A}^\infty_h, \sigma_i\text{‘}s \text{ are spin matrices} \}\)

\[= \mathcal{A}^\infty_h \oplus \mathcal{A}^\infty_h \oplus \mathcal{A}^\infty_h\]

(ii) \(\pi(\Omega^k(\mathcal{A}^\infty_h)) = \mathcal{A}^\infty_h \otimes M_2(\mathbb{C}) = \mathcal{A}^\infty_h \oplus \mathcal{A}^\infty_h \oplus \mathcal{A}^\infty_h \oplus \mathcal{A}^\infty_h\)
Proof: $\Omega^1_D(A^\infty_\hbar) = \pi(\Omega^1(A^\infty_\hbar)) \subseteq \text{R.H.S.}$

Let $\phi_{m,n}(x,y,p) = \delta_{p0}e(mx + ny)$ and $\phi \in S^c$ be such that $\phi(x,y,p) = \delta_{p1}\phi(x,y,p)$. Then applying the previous lemma to $[D, \phi_{01}], [D, \phi_{10}], [D, \phi] \in \pi(\Omega^1(A))$ we get the result.

(ii)(i) along with $\Omega^k(A^\infty_\hbar) = \Omega^1(A^\infty_\hbar) \otimes A^\infty_\hbar \ldots \otimes A^\infty_\hbar \Omega^1(A^\infty_\hbar)$ proves the result.

Proposition 4.6

(i) $\pi(\delta J_1) = A^\infty_\hbar$
(ii) $\Omega^2_J(A^\infty_\hbar) = A^\infty_\hbar \oplus A^\infty_\hbar \oplus A^\infty_\hbar$

Proof: (i) Let $\omega = \sum a_i \delta(b_i) \in J_1$. Then $\pi(\omega) = \sum a_i \delta(b_i) \sigma_j = 0$ gives $\sum a_i \delta_j(b_i) = 0, \forall j$

\[
\pi(\delta \omega) = \sum_i \left( \sum_j \delta_j(a_i) \sigma_j \right) \left( \sum_k \delta_k(b_i) \sigma_k \right)
= \sum_i \left( \sum_j \delta_j(a_i) \delta_j(b_i) \right) \otimes I_2 + \sum_i \left( \sum_{j<k} \left( \delta_j(a_i) \delta_k(b_i) - \delta_k(a_i) \delta_j(b_i) \right) \sigma_j \sigma_k \right)
\]
(4.1)

\[
\sum_i [\delta_j, \delta_k](a_i b_i) = \sum_i \delta_j(\delta_k(a_i) b_i) - \delta_k(\delta_j(a_i) b_i) \quad \text{[Since } \sum a_i \delta_j(b_i) = 0, \forall j \]
= \sum_i [\delta_j, \delta_k](a_i b_i) + \sum_i (\delta_k(a_i) \delta_j(b_i) - \delta_j(a_i) \delta_k(b_i))
\]
(4.2)

Also note

\[
\sum_i [\delta_j, \delta_k](a_i b_i) = \sum_i [\delta_j, \delta_k](a_i b_i) + \sum_i a_i [\delta_j, \delta_k](b_i)
= \sum_i \sum [\delta_j, \delta_k](a_i) b_i
\]
(4.3)

Comparing rhs of (4.2, 4.3) we see second term on the rhs of (4.1) vanishes proving $\pi(\delta J_1) \subseteq A^\infty_\hbar$.

To see actually equality holds note, 

$\omega = 2\phi_{02} \delta(\phi_{01}) - \phi_{01} \delta(\phi_{02}) \in J_1 \pi(\delta \omega) = 2\phi_{03} \otimes I_2 \neq 0$. An application of lemma 4.4 proves (i).
(ii) Let $\phi \in S^c$ be such that $\phi(x, y, p) = \delta_1 p \phi(x, y, p)$

$\omega_1 = \delta(\phi_{1,0})\delta(\phi_{0,1}) \omega_2 = \delta(\phi_{1,1})\delta(\phi) \omega_1 = \delta(\phi_{0,1})\delta(\phi)$ Now lemma \ref{lem:4.4} together with (i) implies the result.

Lemma 4.7 $\pi(\delta J_2) = \{\sum a_j \otimes \sigma_j | a_j \in A_\infty^\infty\} = A_\infty^\infty \oplus A_\infty^\infty \oplus A_\infty^\infty$

Proof: Let $\omega = \sum a_i \delta(b_i)\delta(c_i) \in J_2$

$$0 = \pi(\omega) = \sum a_i(\sum_j \delta_j(b_i)\sigma_j)(\sum_j \delta_k(c_i)\sigma_k)$$

$$= a_i\delta_j(b_i)\delta_j(c_i) + \sum_{j<k} a_i(\delta_j(b_i)\delta_k(c_i) - \delta_k(b_i)\delta_j(c_i))\sigma_j\sigma_k$$

Comparing the coefficients of the various spin matrices we get

$$\sum a_i\delta_j(b_i)\delta_j(c_i) = 0 \quad (4.4)$$

$$\sum a_i(\delta_j(b_i)\delta_k(c_i) - \delta_k(b_i)\delta_j(c_i)) = 0, \forall j \neq k \quad (4.5)$$

from (4.4),

$$0 = \sum \delta_1(a_i)(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i))$$

$$+ \sum \delta_1(a_i)(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i))$$

Therefore,

$$\sum \delta_1(a_i)(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i)) = -\sum a_i\delta_1(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i)) \quad (4.7)$$

Similarly we get two more equalities. Let $A = \text{be coefficient of } I_2 \text{ in } \pi(\delta \omega)$.

Then

$$\sqrt{-1}A = \sum \delta_1(a_i)(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i))$$

$$+ \sum \delta_2(a_i)(\delta_3(b_i)\delta_1(c_i) - \delta_1(b_i)\delta_3(c_i))$$

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Here second equality follows from (4.7) and the last equality follows from (4.5) since \( \delta_j \)'s form a lie algebra. This shows

\[
\pi(\delta J_2) \subseteq \{ \sum_{j=1}^{3} a_j \sigma_j | a_j \in A^\infty_n \} \cong A^\infty_n \oplus A^\infty_n \oplus A^\infty_n
\]

Let \( \phi \in S^c \) be such that \( \phi(x, y, p) = \delta_{1\mu} \phi(x, y, p) \). Then,

\[
\begin{align*}
\omega_1 &= 2\phi_{0,2}\delta(\phi_{0,1})\delta(\phi_{0,1}) - \phi_{0,1}\delta(\phi_{0,2})\delta(\phi_{0,1}) \in J_2 \\
\omega_2 &= 2\phi_{2,0}\delta(\phi_{1,0})\delta(\phi_{1,0}) - \phi_{1,0}\delta(\phi_{2,0})\delta(\phi_{1,0}) \in J_2 \\
\omega_3 &= \phi_{0,2}\delta(\phi_{0,1})\delta(\phi) - \phi_{0,1}\delta(\phi_{0,2})\delta(\phi) \in J_2
\end{align*}
\]

satisfies,

\[
\begin{align*}
\pi(\delta \omega_1) &= 2\phi_{0,4}\sigma_2 \\
\pi(\delta \omega_2) &= 2\phi_{4,0}\sigma_1 \\
\pi(\delta \omega_3) &= 2\phi_{0,3}\delta(\phi)\sigma_1 + 2\phi_{0,3}\delta(\phi)\sigma_2 + 2\phi_{0,3}\delta(\phi)\sigma_3
\end{align*}
\]

Therefore by Lemma 4.4 we get equality in (4). \( \square \).

**Corollary 4.8** \( \Omega^2_D(A^\infty_n) = A^\infty_n \)

**Proof:** Immediate from the previous lemma and proposition 3.5(ii). \( \square \)

**Lemma 4.9** (i) \( \Omega^4_D(A^\infty_n) = 0 \)

(ii) \( \Omega^k_D(A^\infty_n) = 0, \forall k > 4 \)
Proof: (i) It suffices to show \( \pi(\delta J_3) = A_h^\infty \oplus A_h^\infty \oplus A_h^\infty \oplus A_h^\infty \).

For that note,

\[
\begin{align*}
\omega_1 &= 2\phi_{0,2}\delta(\phi_{0,1})\delta(\phi_{0,1}) - \phi_{0,1}\delta(\phi_{0,2})\delta(\phi_{0,1}) \in J_3 \\
\omega_2 &= 2\phi_{0,2}\delta(\phi_{0,1})\delta(\phi_{0,1}) - \phi_{0,1}\delta(\phi_{0,2})\delta(\phi_{0,1}) \in J_3 \\
\omega_3 &= 2\phi_{0,2}\delta(\phi_{0,1})\delta(\phi_{0,1}) - \phi_{0,1}\delta(\phi_{0,2})\delta(\phi_{0,1}) \in J_3 \\
\omega_4 &= 2\phi_{0,2}\delta(\phi_{0,1})\delta(\phi_{0,1}) - \phi_{0,1}\delta(\phi_{0,2})\delta(\phi_{0,1}) \in J_3
\end{align*}
\]

satisfies

\[
\begin{align*}
\pi(\delta \omega_1) &= 2\phi_{0,5} \otimes I_2 \\
\pi(\delta \omega_2) &= 2\phi_{1,4}\sigma_2\sigma_1 \\
\pi(\delta \omega_3) &= 2\phi_{0,4}\delta_2(\phi) \otimes I_2 + 2\phi_{0,4}\delta_1(\phi)\sigma_2\sigma_1 + 2\phi_{0,4}\delta_3(\phi)\sigma_2\sigma_3 \\
\pi(\delta \omega_4) &= 2\phi_{1,3}\delta_1(\phi)I_2 + 2\phi_{1,3}\delta_2(\phi)\sigma_1\sigma_2 + 2\phi_{1,3}\delta_3(\phi)\sigma_1\sigma_3
\end{align*}
\]

Now an application of Lemma 4.4 completes the proof.

(ii) The same argument as in (i) does the job with the following choice,

\[\omega'_i = \omega_i \delta(\phi_{0,1})\ldots\delta(\phi_{0,1}) \quad i = 1, 4 \quad \Box\]

5 Connections:– torsinless/unitary

Definition 5.1 [6] (i)\( \int \) determines a semi-definite sesquilinear form on \( \Omega^\bullet(A_h^\infty) \) by setting

\[
(\omega, \eta) = \int \pi(\omega)\pi(\eta)^*d\omega, \eta \in \Omega^\bullet(A_h^\infty)
\]

(ii) Let

\[
K_k = \{ \omega \in \Omega^k(A_h^\infty)(\omega, \omega) = 0 \}, \quad K = \bigoplus_{k=0}^\infty K_k
\]

\( K, K + \delta K \) are two sided *-ideals, the later is closed under differential.

\[
\tilde{\Omega}^\bullet(A_h^\infty) = \bigoplus_{k=0}^\infty \tilde{\Omega}^k(A_h^\infty), \quad \Omega^k(A_h^\infty) = \Omega^k(A_h^\infty)/K_k
\]

(iii)\( \tilde{H}^k \) denotes the Hilbert space completion of \( \tilde{\Omega}^k(A_h^\infty) \) with respect to the scalar product. \( \tilde{H}^\bullet = \bigoplus_{k=0}^\infty \tilde{H}^k \), \( \tilde{H}^k \) is to be interpreted as the space of square-integrable k-forms.
(iv) The algebra multiplication of $\Omega^\bullet(A_h^\infty)$ descends to a linear map
$m : \tilde{\Omega}^\bullet(A_h^\infty) \otimes_{A_h^\infty} \tilde{\Omega}^\bullet(A_h^\infty) \to \tilde{\Omega}^\bullet(A_h^\infty)$.

(v) The unital graded differential *-algebra of square-integrable differential forms is defined by
\[
\tilde{\Omega}_D^\bullet(A_h^\infty) = \oplus_{k=0}^\infty \tilde{\Omega}_D^k(A_h^\infty), \tilde{\Omega}_D^k(A_h^\infty) = \tilde{\Omega}^k(A_h^\infty)/K_k + \delta K_{k-1}
\]

(vi) $\delta : \Omega^{\bullet+1}(A_h^\infty) \to \Omega^{\bullet+1}(A_h^\infty)$ descends to a linear map
$\delta : \tilde{\Omega}_D^\bullet(A_h^\infty) \to \tilde{\Omega}_D^{\bullet+1}(A_h^\infty)$

(vii) A connection $\nabla$ on a finitely generated projective $A_h^\infty$ module $E$ is a $C$ linear map
\[
\nabla : \tilde{\Omega}_D^\bullet(A_h^\infty) \otimes E \to \tilde{\Omega}_D^{\bullet+1}(A_h^\infty) \otimes E
\]
such that $\nabla(\omega s) = \delta(\omega)s + (-1)^k\omega\nabla(s)$ for all $\omega \in \tilde{\Omega}_D^\bullet(A_h^\infty)$ and all $s \in \tilde{\Omega}_D^\bullet(A_h^\infty) \otimes E$

(viii) The curvature of a connection $\nabla$ on $E$ is given by
\[
R(\nabla) = -\nabla^2 : E \to \tilde{\Omega}_D^{k}(A_h^\infty) \otimes_{A_h^\infty} E
\]

Remark 5.2 $\omega \in \tilde{\Omega}^k(A_h^\infty)$ determines two operators
$m_L(\omega), m_R(\omega) : \tilde{\Omega}^n(A_h^\infty) \to \tilde{\Omega}^{n+k}(A_h^\infty)$ given by $m_L(\omega)(\eta) = m(\omega \otimes \eta)$,
$m_R(\omega)(\eta) = m(\eta \otimes \omega)$. These operators extend to bounded linear operators
$m_L(\omega), m_R(\omega) : \mathcal{H}^\infty \to \mathcal{H}^{n+k}$ for all $n$.

Proposition 5.3 (i) $\tilde{\Omega}^k(A_h^\infty) = A_h^\infty \otimes M_2(\mathbb{C}) \cong A_h^\infty \oplus A_h^\infty \oplus A_h^\infty \oplus A_h^\infty$

(ii) $\tilde{\Omega}_D^k(A_h^\infty) \cong \Omega^k(A_h^\infty)$

(iii) $\mathcal{H}^k = L^2(A_h^\infty, \tau) \otimes \mathbb{C}^4$

Proof: (i) By the faithfulness of the linear functional $A \mapsto \int A$ defined on $\pi(\Omega^\bullet(A_h^\infty)) = A_h^\infty \otimes M_2(\mathbb{C})$ we get $J_k = K_k$.

hence $\tilde{\Omega}^k(A_h^\infty) = \Omega^k(A_h^\infty)/\ker(\pi) \cong \pi(\Omega^k(A_h^\infty)) = A_h^\infty \otimes M_2(\mathbb{C})$

(ii) Follows from (i) and proposition 5.3

(iii) In (i) we have already seen $J_k = K_k$. That gives the result.

Remark 5.4 Since $\tilde{\Omega}_D^1(A_h^\infty)$ is free with 3 generators, we can and will identify $\tilde{\Omega}_D^1(A_h^\infty) \otimes_{A_h^\infty} \tilde{\Omega}_D^1(A_h^\infty)$ with $A_h^\infty \otimes M_3(\mathbb{C})$ and a connection $\nabla$ is specified by its value on the generators.
Definition 5.5 A connection $\nabla : \tilde{\Omega}^1_D(A^\infty_\hbar) \to \tilde{\Omega}^1_D(A^\infty_\hbar) \otimes A^\infty_\hbar \tilde{\Omega}^1_D(A^\infty_\hbar)$ is called torsionless if $T(\nabla) = \delta - m \circ \nabla : \tilde{\Omega}^1_D(A^\infty_\hbar) \to \tilde{\Omega}^2_D(A^\infty_\hbar)$ vanishes.

Proposition 5.6 A connection is torsionless iff its values on the generators $\sigma_1, \sigma_2, \sigma_3$ are given by

$$\nabla(\sigma_1) = \begin{pmatrix} \Box & a & b \\ a & \Box & c \\ b & c & \Box \end{pmatrix}, \nabla(\sigma_2) = \begin{pmatrix} \Box & d & e \\ d & \Box & f \\ e & f & \Box \end{pmatrix}, \nabla(\sigma_3) = \begin{pmatrix} \Box & p - 1 & q \\ p & \Box & r \\ q & r & \Box \end{pmatrix}.$$

Proof:

$$\delta(\sum_{i,j} a_i \delta_j(b_i) \sigma_j) = -\sqrt{-1}(\sum_i(\delta_1(a_i)\delta_2(b_i) - \delta_2(a_i)\delta_1(b_i))\sigma_3$$

$$+ \sum_i(\delta_2(a_i)\delta_3(b_i) - \delta_3(a_i)\delta_2(b_i))\sigma_1$$

$$+ \sum_i(\delta_3(a_i)\delta_1(b_i) - \delta_1(a_i)\delta_3(b_i))\sigma_2)$$

$$m \circ \nabla(\sum_{i,j} a_i \delta_j(b_i) \sigma_j) = m(\sum_{i,j} \delta(a_i \delta_j(b_i)) \otimes \sigma_j) + \sum_{i,j} a_i \delta_j(b_i) m \circ \nabla(\sigma_j)$$

$$= m(\sum_{i,j,k} \delta_k(a_i \delta_j(b_i)) \sigma_k \otimes \sigma_j) + \sum_{i,j} a_i \delta_j(b_i) m \circ \nabla(\sigma_j)$$

Torsion of $\nabla$ vanishes iff $(\delta - m \circ \nabla)(\sum_{i,j} a_i \delta_j(b_i) \sigma_j) \equiv 0$, or equivalently ,

$$\sum_i (\delta_j(a_i) \delta_k(b_i) - \delta_k(a_i) \delta_j(b_i)) = \sum_i (\delta_j(a_i \delta_k(b_i)) - \delta_k(a_i \delta_j(b_i)))$$

$$+ \sum_{i,l} a_i \delta_l(b_i)(m \circ \nabla(\sigma_l))_n$$

whenever $j \neq k$ and $n$ satisfies $\sigma_j \sigma_k \sigma_n = \sqrt{-1}$. This happens iff

$$0 = \sum_i a_i [\delta_j, \delta_k](b_i) + \sum_{i,l} a_i \delta_l(b_i)(m \circ \nabla(\sigma_l))_n$$

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Using the Lie algebra relations between the $\delta_j$’s we get equivalence of the above system of equations with

$$0 = \sum_i a_i \delta_3(b_i) + \sum_{i,l} a_i \delta_l(b_i)(m \circ \nabla(\sigma_l))_3$$

$$0 = \sum_{i,l} a_i \delta_l(b_i)(m \circ \nabla(\sigma_l))_2$$

$$0 = \sum_{i,l} a_i \delta_l(b_i)(m \circ \nabla(\sigma_l))_1$$

whenever $j \neq k$ and $n$ satisfies $\sigma_j \sigma_k \sigma_n = \sqrt{-1}$.

Taking $b_i = \phi_{0,1}, a_i = 1$ we get $\delta_1(b_i) = \delta_3(b_i) = 0, \delta_2(b_i) = b_i$. Putting these values in the above relations we get $(m \circ \nabla(\sigma_2))_j = 0$ for $j = 1, 2, 3$.

Similarly taking $b_i = \phi_{1,0}, a_i = 1$ we get $(m \circ \nabla(\sigma_1))_j = 0$ for $j = 1, 2, 3$.

Substituting these values in the above equations we get,

$$(m \circ \nabla(\sigma_3))_1 = (m \circ \nabla(\sigma_3))_2 = \sum_i a_i \delta_3(b_i)(1 + (m \circ \nabla(\sigma_3))_3) = 0$$

Note, $J = \{\sum a_i \delta_3(b_i) | n \in \mathbb{N}, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{A}_h^\infty\}$ is a nontrivial ideal in $\mathcal{A}_h^\infty$ hence equals $\mathcal{A}_h^\infty$. Therefore $(m \circ \nabla(\sigma_3))_3 = -1$. Now the result follows from the anticommutation relation between the spin matrices. \(\square\)

**Definition 5.7** A connection on a Finitely generated projective $\mathcal{A}_h^\infty$ module $\mathcal{E}$, endowed with an $\mathcal{A}_h^\infty$ valued inner product $\langle \cdot, \cdot \rangle$ is called unitary if

$$\delta(s, t) = \langle \nabla s, t \rangle - \langle s, \nabla t \rangle, \forall s, t \in \mathcal{E}$$

Where the right hand side of this equation is defined by

$$\langle \omega \otimes s, t \rangle = \omega(s, t), \langle s, \eta \otimes t \rangle = \langle s, t \rangle\eta^*$$

**Proposition 5.8** A connection $\nabla$ on $\bar{\Omega}_D^1(\mathcal{A}_h^\infty)$ is unitary iff its values on the generators $\sigma_1, \sigma_2, \sigma_3$ are given by

$$\nabla(\sigma_1) = \begin{pmatrix} x & y & z \\ y & u & p \\ z & v & q \end{pmatrix}, \nabla(\sigma_2) = \begin{pmatrix} y & u & v \\ u & r & s \\ p & s & f \end{pmatrix}, \nabla(\sigma_3) = \begin{pmatrix} z & p & q \\ v & s & f \\ q & f & g \end{pmatrix}$$

where $x, y, z, p, q, r, s, u, v, f, g \in \mathcal{A}_h^\infty$ are selfadjoint elements.
\textbf{Proof:} Taking \( s = a_i \sigma_i, t = b_j \sigma_j \) in the defining condition of a unitary connection we get
\[
\delta(\delta_{ij}a_i b_j^*) = a_i(\langle \nabla(\sigma_i), \sigma_j \rangle - \langle \sigma_i, \nabla(\sigma_j) \rangle)b_j^* + \delta_{ij}(\delta(a_i)b_j^* - a_i(\delta(b_j))^*)
\]
(5.1)
implying \( \langle \nabla(\sigma_i), \sigma_j \rangle = \langle \sigma_i, \nabla(\sigma_j) \rangle \)
That is jth row of \( \nabla(\sigma_i) \) is the star of the ith column of \( \nabla(\sigma_j) \). This completes the proof. \( \Box \)

\textbf{Corollary 5.9} A connection \( \nabla \) can not simultaneously be torsionless and unitary.

\textbf{Proof:} If possible let \( \nabla \) be one such. Let \( v, p \) be as in proposition 3.16 and \( c \) be as in proposition 3.14. Then \( v = c = p \) and also \( v - p = -1 \). This leads to a contradiction. \( \Box \)

\section{Connections with Positive Scalar Curvature}

\textbf{Definition 6.1} (Theorem 2.9 of [6]) There is a sesquilinear map \( \langle \cdot, \cdot \rangle_D : \tilde{\Omega}_D^k(A^\infty_h) \otimes \tilde{\Omega}_D^k(A^\infty_h) \to A^\infty_h \) satisfying \( \langle x, \langle \omega, \eta \rangle_D \rangle = \int x\eta\omega^* \),
for all \( x \in A_h \)

In the following proposition we identify \( \tilde{\Omega}^k(A^\infty_h) \) with \( A^\infty_h \otimes M_2(\mathbb{C}) \)

\textbf{Proposition 6.2} \( \langle \omega, \eta \rangle_D = \frac{1}{2}(I \otimes tr)(\omega\eta^*) \)

\textbf{Proof:} Let \( \omega = \omega_0 \otimes I_2 + \sum_{i=1}^3 \omega_i \otimes \sigma_i, \eta = \eta_0 \otimes I_2 + \sum_{i=1}^3 \eta_i \otimes \sigma_i \)
Then \( \frac{1}{2}(I \otimes tr)(\omega\eta^*) = \sum_{i=0}^3 \omega_i \eta_i^* \)
\( (x, \sum_{i=0}^3 \omega_i \eta_i^*) = \tau(x\eta_0 \omega_0^*) = (x, \langle \omega, \eta \rangle_D) \) for all \( x \in A_h \). This completes the proof since \( A_h \) is dense in \( \mathcal{H}_0^0 \). \( \Box \).

\textbf{Notation}:- Let \( \omega \in \Omega^1_D(A^\infty_h) \). Since \( K + \delta K \) is an ideal in \( \Omega^*_D(A^\infty_h) \)
m : \( \tilde{\Omega}^*(A^\infty_h) \otimes_{A^\infty_h} \tilde{\Omega}^*(A^\infty_h) \to \tilde{\Omega}^*(A^\infty_h) \) induces two maps denoted by the same symbol
m : \( \tilde{\Omega}^*_D(A^\infty_h) \otimes_{A^\infty_h} \tilde{\Omega}^*_D(A^\infty_h) \to \tilde{\Omega}^{*+1}(A^\infty_h) \)
m : \( \tilde{\Omega}^*_D(A^\infty_h) \otimes_{A^\infty_h} \tilde{\Omega}^1_D(A^\infty_h) \to \tilde{\Omega}^{*+1}(A^\infty_h) \). These in turn induce bounded maps.
Since $\tilde{\Omega}_D^1(\mathcal{A}_h^\infty)$ is free, curvature of a connection $\nabla$, $R(\nabla) = -\nabla^2 : \tilde{\Omega}_D^1(\mathcal{A}_h^\infty) \to \tilde{\Omega}_D^2(\mathcal{A}_h^\infty)$ is given by a $3 \times 3$ matrix $((R_{ij}))$ with entries in $\tilde{\Omega}_D^2(\mathcal{A}_h^\infty)$. Let $P_{SK_1} : \mathcal{H}^2 \to \mathcal{H}^1$ be the projection onto closure of $\pi(\delta K_1) \subseteq \tilde{\Omega}_D^2(\mathcal{A}_h^\infty)$, and $R_{ij}^+ = (I - P_{SK_1})(R_{ij})$. Let $e_1, e_2, e_3$ be the canonical basis of $\tilde{\Omega}_D^1(\mathcal{A}_h^\infty)$. If we denote by $\text{Ric}_j = \sum_i m_L(e_i)^{ad}(R_{ij}^+) \in \mathcal{H}^1$ then Ricci curvature of $\nabla$ is given by

$$\text{Ric}(\nabla) = \sum_j \text{Ric}_j \otimes e_j \in \mathcal{H}^1 \otimes_{\mathcal{A}_h^\infty} \tilde{\Omega}_D^1(\mathcal{A}_h^\infty)$$

Here superscript $\text{ad}$ stands for Hilbert space adjoint. Finally the scalar curvature $r(\nabla)$ of $\nabla$ is given by

$$r(\nabla) = \sum_i m_R(e_i)^{ad}(\text{Ric}_i) \in \mathcal{H}^0$$

**Proposition 6.3** Let $f, g : \mathbb{T} \to \mathbb{R}$ be smooth maps. Henceforth we visualize $f, g$ as elements of $S^* \infty$ in the following way, $f(x, y, p) = \delta_{0p}f(x)$, $g(x, y, p) = \delta_{0p}g(y)$. Similar considerations will be applied for $f', g'$. Let $\nabla$ be the connection given by $\nabla(\sigma_1) = f'\delta(g)\sigma_1 + g'\delta(f)\sigma_2$, $\nabla(\sigma_2) = g'\delta(f)\sigma_1$, $\nabla(\sigma_3) = 0$, then $r(\nabla)$ is $-2f'^2, g'^2$.

**Proof:** By direct computation one gets,

$$\nabla^2(\sigma_1) = -R_{11}\sigma_1 - R_{12}\sigma_2, \nabla^2(\sigma_2) = -R_{21}\sigma_1, \nabla^2(\sigma_3) = 0$$

where

$$R_{11} = f''g\sigma_3, R_{12} = \sqrt{-1}(f'^2g'^2 - g''f')\sigma_3, R_{21} = -\sqrt{-1}(g''f' + f'^2g'^2)\sigma_3,$$

other $R_{ij}$’s are zero.

Then

$$\text{Ric}_1 = -f''g\sigma_2 - (g''f' + f'^2g'^2)\sigma_1, \text{Ric}_2 = (g''f' - f'^2g'^2)\sigma_2$$

implying the desired conclusion $r(\nabla) = -2f'^2g'^2.$ \hfill \Box
Remark 6.4  (i) All these notions of Ricci curvature, scalar curvature were introduced by [8]. To the best of our knowledge it is the first infinite dimensional example where one can have connections with nontrivial scalar curvature.

(ii) Note the choice of the spectral triple depend on a parameter $\alpha$. For the connections we have considered scalar curvature does not depend on the parameter $\alpha$.

7  nontriviality of the chern character associated with the spectral triples

The spectral triple we constructed depends on a real parameter $\alpha$. In this section we show that the Kasparov module associated with the spectral triple $\mathcal{E}$ $\mathcal{B}$ are homotopic. We also argue that they give non-trivial elements in $K^1(A_\hbar)$ by explicitly computing pairing with some unitary in the algebra representing elements of $K^1(A_\hbar)$.

Lemma 7.1  Let $A$ be a selfadjoint operator with a bounded inverse and $B$ a symmetric operator with $D(A) \subseteq D(B)$ on some Hilbert space $\mathcal{H}$. Also suppose that $\|Bu\| \leq a\|Au\|, \forall u \in D(A)$. Then $|A|^{-p}B|A|^{-(1-p)} \in \mathcal{B}(\mathcal{H})$ and $\||A|^{-p}B|A|^{-(1-p)}\| \leq a$.

Proof: Clearly $\|Bu\| \leq a\|A|u\|, \forall u \in D(A)$ implying $\|B|A|^{-1}\| \leq a$. For $u, v \in D(A)$

$$\||A|^{-1}Bu\| = \sup_{v \in D(A), \|v\| \leq 1} |\langle |A|^{-1}Bu, v \rangle|$$

$$= \sup_{v \in D(A), \|v\| \leq 1} |\langle u, B|A|^{-1}v \rangle| \leq a\|u\|$$

Therefore $|A|^{-1}B \in \mathcal{B}(\mathcal{H})$, $\||A|^{-1}B\| \leq a$.

Let $\mathcal{H}_p$ be the Hilbert space completion of $\cap D(A^n)$ with respect to $\|u\|_p = \|A^p u\|$. Let $B_1 : \mathcal{H}_1 \to \mathcal{H}_0, B_0 : \mathcal{H}_0 \to \mathcal{H}_{-1}$ be the maps given by $B_i(u) = B(u)$ for $u \in \cap D(A^n)$. Then $\|B_1\|, \|B_1\| \leq a$. By Calderon-Zygmund interpolation theorem [12] we get maps $B_p : \mathcal{H}_p \to \mathcal{H}_{-(1-p)}$ for $0 \leq p \leq 1$ with $\|B_p\| \leq a$. On $\cap D(A^n), B_p$ agrees with $|A|^{-p}B|A|^{-(1-p)}$ proving the lemma. □
Lemma 7.2 Let $A, B$ be as above with $a < 1$. Let $A_t = A + tB, t \in [0, 1]$. Then $t \mapsto \tan^{-1}(A_t)$ is a norm continuous function.

Proof: Let $C = |A|^{-1/2}B|A|^{-1/2}$, then by the previous lemma $\|C\| \leq a$. For $\lambda \in i\mathbb{R}, ||A|(A - \lambda)^{-1}| \leq 1$.

\[
A_t - \lambda = (A - \lambda) + t|A|^{1/2}C|A|^{1/2} \\
= |A|^{1/2}((A - \lambda)|A|^{-1} + tC)|A|^{1/2} \\
= |A|^{1/2}(1 + tC(A - \lambda)^{-1}|A|)(A - \lambda)|A|^{-1}|A|^{1/2}
\]

Now note $\|tC(A - \lambda)^{-1}|A|| \leq a < 1$ for $0 \leq t \leq 1$. Therefore

\[
(A_t - \lambda)^{-1} = |A|^{-1/2}|A|(A - \lambda)^{-1}(1 + tC(A - \lambda)^{-1}|A|)^{-1}|A|^{-1/2}
\]

So, if we denote by $R_t(\lambda) = (A_t - \lambda)^{-1}$ and $F(\lambda) = |A|(A - \lambda)^{-1}$ then the above equality becomes,

\[
R_t(\lambda) = |A|^{-1/2}|A|R_0(\lambda)(1 + tC|A|F(\lambda))^{-1}|A|^{-1/2} \\
= R_0(\lambda) + |A|^{-1/2}F(\lambda)\sum_{n=1}^{\infty}(-tCF(\lambda))^n|A|^{-1/2}
\]

(7.1)

Let $\lambda \in \mathbb{R}, t, s \in [0, 1], u, v \in D(A)$. Observe

(i)

\[
\| \sum_{n=1}^{\infty}(-tCF(i\lambda))^n|A|^{-1/2}u - \sum_{n=1}^{\infty}(-sCF(i\lambda))^n|A|^{-1/2}u \| \\
\leq \sum_{n=0}^{\infty}||(t^{n+1} - s^{n+1})CF(i\lambda)||^n\|C\|\|F(i\lambda)|A|^{-1/2}u\| \\
\leq \sum_{n=0}^{\infty}|(t^{n+1} - s^{n+1})a^n|\|F(i\lambda)|A|^{-1/2}u\| \\
\leq |(t - s)|\sum_{n=0}^{\infty}n + 1a^{n+1}\|F(i\lambda)|A|^{-1/2}u\| \\
\leq |(t - s)|\frac{a}{(1 - a)^2}\|F(i\lambda)|A|^{-1/2}u\|
\]
\( (i) \)
\[
\int_0^\infty \| F(i\lambda)|A|^{-1/2}u \|^2 d\lambda \leq \int_0^\infty \langle (A^2 + \lambda^2)^{-1}u, |A|u \rangle d\lambda
\]
\[= \frac{1}{2} \iint_0^\infty \langle (A^2 + \xi)^{-1}u, |A|u \rangle \frac{d\xi}{\xi}
\]
\[= \frac{1}{2} \pi \langle A^{2-1/2}u, |A|u \rangle = \frac{\pi}{2} \| u \|^2
\]

\( (ii) \)
\[
\int_0^\infty \| F(i\lambda)|A|^{-1/2}u \|^2 d\lambda \leq \frac{1}{2} \iint_0^\infty \langle (A^2 + \xi)^{-1}u, |A|u \rangle \frac{d\xi}{\sqrt{\xi}}
\]
\[= \frac{1}{2} \pi \langle A^{2-1/2}u, |A|u \rangle = \frac{\pi}{2} \| u \|^2
\]

\( (iii) \) Using (7.1), (i), (ii) we get
\[
\int_0^\infty |\langle (R_t(i\lambda) - R_s(i\lambda))u, v \rangle| d\lambda
\]
\[\leq \int_0^\infty |(t-s)| \frac{a}{(1-a)^2} \| F(i\lambda)|A|^{-1/2}u \| \| F(-i\lambda)|A|^{-1/2}v \| d\lambda
\]
\[\leq |(t-s)| \frac{a}{(1-a)^2} \left( \int_0^\infty \| F(i\lambda)|A|^{-1/2}u \|^2 d\lambda \right)^{1/2} \left( \int_0^\infty \| F(-i\lambda)|A|^{-1/2}v \|^2 d\lambda \right)^{1/2}
\]
\[\leq |(t-s)| \frac{a}{(1-a)^2} \frac{\pi}{2} \| u \| \| v \|
\]
This shows \( \lim_{s \to t} \| \int_0^\infty (R_t(i\lambda) - R_s(i\lambda))d\lambda \| = 0. \) Similarly one can show \( \lim_{s \to t} \| \int_0^\infty (R_t(-i\lambda) - R_s(-i\lambda))d\lambda \| = 0. \) Now the result follows once we observe \( \tan^{-1} A_t = \int_0^\infty (R_t(i\lambda) + R_t(-i\lambda))d\lambda. \)
\( \square \)

**Lemma 7.3** Let \( A, B \) be as above except now we do not require \( A \) to be invertible. Instead we assume \( A \) to have discrete spectrum. Then there exists \( \kappa \geq 0 \) such that \( t \mapsto \tan^{-1}(A_t + \kappa) \) is norm continuous.

**Proof:** Without loss of generality we can assume 0 is an eigenvalue of \( A \). Otherwise we are done by the previous lemma. Choose \( 2 \leq n \in \mathbb{N} \) such that \( b = a \frac{n}{n-1} < 1 \) Choose \( \kappa > 0 \) such that
(i) smallest positive eigenvalue of \( A \) is greater than \( \kappa \).
(ii) if \( \beta \) is the biggest negative eigenvalue then \( \beta < nk \).

Let \( \tilde{A} = A + \kappa, \tilde{A}_t = \tilde{A} + tB. \) Then by choice of \( \kappa \)
(i) \( \tilde{A} \) is an invertible selfadjoint operator.
(ii) \( \| B\tilde{A}^{-1} \| \leq a\| A(\kappa)^{-1} \| \leq a \frac{n}{n-1} < 1 \)
That is $B$ is relatively bounded with respect to $\tilde{A}$ with relative bound $b < 1$. Now an application of the previous result to the pair $\tilde{A}, B$ does the job. □

Combining these two we get

**Proposition 7.4** Let $A, B$ be operators on the Hilbert space $\mathcal{H}$ such that

(i) $A$ is selfadjoint with compact resolvent.
(ii) $B$ is symmetric with $D(A) \subseteq D(B)$, and relatively bounded with respect to $A$ with relative bound less than 1.

Then there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{x \to \infty} f(x) = 1, \lim_{x \to -\infty} f(x) = -1$ such that $t \mapsto f(A + tB)$ is norm continuous.

**Proof:** If $A$ is invertible then by lemma (7.2) $f(x) = \frac{2}{\pi} \tan^{-1}(x)$ serves the purpose. In the other case by lemma (7.3) $f(x) = \frac{2}{\pi} \tan^{-1}(x + \kappa)$ does the job. □

Let the Hilbert space $\mathcal{H}$ and the operators $A, B, D$ be as in corollary (3.3).

**Corollary 7.5** The Kasparov module associated with $(A_\infty^\alpha, \mathcal{H}, D)$ is operatorially homotopic with $(A_\infty^\alpha, \mathcal{H}, A)$

**Proof:** Let $A_t = A + tB$ for $t \in [0, 1]$. Then $D = A_1, A = A_0$. As remarked earlier $(A_\infty^\alpha, \mathcal{H}, A_t)$ are spectral triples. Let $f$ be the function obtained from the previous proposition for the pair $A, B$. Then $((A_h, \mathcal{H}, f(A_t)))_{t \in [0, 1]}$ gives the desired homotopy.

As remarked earlier the operator $A$ depends on a real parameter $\alpha > 1$. Now we will make that explicit and denote $A$ by $A^{(\alpha)}$.

**Proposition 7.6** The Kasparov modules associated with $(A_\infty^\alpha, \mathcal{H}, A^{(\alpha)})$ are operatorially homotopic for $\alpha > 1$

**Proof:** By proposition (2.4), $\mathcal{H} = L^2(\mathbb{T} \times \mathbb{T} \times \mathbb{Z}) \otimes \mathbb{C}^2$. Let $B$ be the operator $-2\pi c M_p \otimes \sigma_3$. Here $p$ denotes the $\mathbb{Z}$ variable in the $L^2$ space. Then $B$ is selfadjoint with $D(A^{(\alpha)}) \subseteq D(B)$. Also $B$ is relatively bounded with respect to $A^{(\alpha)}$ with relative bound less than $\frac{1}{\alpha} < 1$. Let $A^{(\alpha)}_t = A^{(\alpha)} + tB$ for $t \in [0, 1]$. Then $A^{(\alpha)}_t = A^{(\alpha+t)}$. Let $f$ be the function obtained from proposition 5.4 for the pair $A^{(\alpha)}, B$. Then from the norm continuity of $t \mapsto f(A^{(\alpha+t)})$ we see the Kasparov modules $((A_\infty^\alpha, \mathcal{H}, A^{(\alpha+t)}))_{t \in [0, 1]}$ are homotopic. Since $\alpha$ is arbitrary this completes the proof. □
Remark 7.7 Proposition (7.6) and corollary (7.5) together imply the Kasparov module associates with the spectral triple \((\mathcal{A}_h^{\infty}, \mathcal{H}, D)\) is independent of \(\alpha\).

In the next proposition we show \((\mathcal{A}_h^{\infty}, \mathcal{H}, D)\) has non trivial chern character.

**Proposition 7.8** The Kasparov module associated with \((\mathcal{A}_h^{\infty}, \mathcal{H}, D)\) gives a nontrivial element in \(K^1(\mathcal{A}_h)\)

**Proof:** By corollary \(7.5\) \((\mathcal{A}_h^{\infty}, \mathcal{H}, D)\) and \((\mathcal{A}_h^{\infty}, \mathcal{H}, A)\) give rise to same element \([([\mathcal{A}_h^{\infty}, \mathcal{H}, A])] \in K^1(\mathcal{A}_h)\). Let \(\phi \in \mathcal{A}_h^{\infty}\) be the unitary whose symbol in \(S^c\) is given by \(\phi(x, y, p) = \delta_{0p}e^{2\pi iy}\). This gives an element \([\phi] \in K^1(\mathcal{A}_h)\). It suffices to show \(\langle [\phi], [([\mathcal{A}_h^{\infty}, \mathcal{H}, A])] \rangle \neq 0\) where \(\langle \cdot, \cdot \rangle: K^1(\mathcal{A}_h) \times K^1(\mathcal{A}_h) \to \mathbb{Z}\) denotes the pairing coming from Kasparov product. \(\phi\) acts on \(L^2(\mathcal{A}_h) \otimes \mathbb{C}^2 \cong L^2([0, 1] \times \mathbb{T} \times \mathbb{Z}) \otimes \mathbb{C}^2\) as a composition of two commuting unitaries \(U_1 = M_{e(y)} \otimes I_2, U_2 = M_{e(p\nu \hbar)} \otimes I_2\). Then note \(U_2\) commutes with \(A\). Let \(E\) be the projection \(E = I(A \geq 0)\). \(U_2\) also commutes with \(E\). Now by proposition 2 (page 289 of [5]) \(EU_1U_2E\) is a Fredholm operator and \(\langle [\phi], [([\mathcal{A}_h^{\infty}, \mathcal{H}, A])] \rangle = \text{Index}(EU_1U_2E) = \text{Index}(EU_1E)\), last equality holds because \(U_2\) commutes with \(E\). Now \(\text{Index}(EU_1E) \neq 0\) because this is the index pairing of the Dirac operator on \(\mathbb{T}^3\) with the unitary \(U_1\). \(\square\)

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