Fermion-Parity Anomaly of the Critical Supercurrent in the Quantum Spin-Hall Effect

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The helical edge state of a quantum spin-Hall insulator can carry a supercurrent in equilibrium between two superconducting electrodes (separation L, coherence length \( \xi \)). We calculate the maximum (critical) current \( I_c \) that can flow without dissipation along a single edge, going beyond the short-junction restriction \( L \ll \xi \) of earlier work, and find a dependence on the fermion parity of the ground state when \( L \) becomes larger than \( \xi \). Fermion-parity conservation doubles the critical current in the low-temperature, long-junction limit, while for a short junction \( I_c \) is the same with or without parity constraints. This provides a phase-insensitive, dc signature of the \( 4\pi \)-periodic Josephson effect.

The quantum Hall effect and quantum spin-Hall effect both refer to a two-dimensional semiconductor with an insulating bulk and a conducting edge, and both exhibit a quantized electrical conductance between two metal electrodes. If the electrodes are superconducting, a current can flow in equilibrium, induced by a magnetic flux without any applied voltage. In the quantum Hall effect, the edge states are chiral (propagating in a single direction only) and two opposite edges are needed to carry a supercurrent [1–3]. Graphene is an ideal system to study this interplay and two opposite edges are needed to carry a supercurrent [4–6]. The interplay of the Josephson effect and the quantum spin-Hall effect, in a strong magnetic field [4–6].

Josephson junctions come in two types [10], depending on whether the separation \( L \) of the superconducting electrodes is small or large compared to the coherence length \( \xi = h v / \Delta \), or equivalently, whether the superconducting gap \( \Delta \) is small or large compared to the Thouless energy \( E_T = h v / L \). Existing literature [7–9,11–18] has focused on the short-junction regime \( L \ll \xi \). The supercurrent is then determined entirely by the phase dependence of a small number of Andreev levels in the gap, just one per transverse mode. The phase dependence of the continuous spectrum above the gap can be neglected. As the ratio \( L / \xi \) increases, the Andreev levels proliferate and also the continuous spectrum starts to contribute to the supercurrent. Since \( \sigma \) is switched by changing the occupation of a single level, one might wonder whether a significant parity dependence remains in the long-junction regime.

Remarkably enough, the parity dependence becomes even stronger. While in a short junction the two branches \( I_+ (\phi) = -I_- (\phi) \) differ only in sign, we find that in a long junction they differ both in sign and in magnitude. In particular, the largest current that can flow without dissipation is twice as large for \( I_- \) as it is for \( I_+ \). The difference is illustrated in Fig. 1, in the zero-temperature limit. The basic physics can be explained in simple terms, as we will do first, and then we will present a complete theory for a finite temperature and for an arbitrary ratio \( L / \xi \).

We set the stage by summarizing the findings of Fu and Kane [7] in the short-junction regime. The spectrum of the Bogoliubov–de Gennes Hamiltonian \( H_{BdG} \) is a \( \pm \epsilon \) symmetric combination of a discrete spectrum for \( |\epsilon| < \Delta \) and a continuous spectrum for \( |\epsilon| > \Delta \). Since backscattering along the quantum spin-Hall edge is forbidden by time-reversal symmetry [19], this is a ballistic single-channel Josephson junction. In the limit \( L / \xi \rightarrow 0 \) the discrete spectrum consists of a pair of levels at \( \epsilon_{\pm} = \mp \Delta |\cos (\phi / 2)| \), while the continuous spectrum is \( \phi \) independent [20]. Quite generally, an eigenvalue \( \epsilon (\phi) \) of \( H_{BdG} \) contributes to the supercurrent an amount

\[
I (\phi) = \frac{g e}{h} \frac{d}{d \phi} \epsilon (\phi),
\]

with \( g \) a factor that counts spin and other degeneracies [21].

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FIG. 1 (color online). Phase-dependent excitation spectrum of a Josephson junction along a quantum spin-Hall (QSH) edge (left panels) and corresponding zero-temperature supercurrent (right panels). The supercurrent $I_{\phi}$ at $\phi = 0$ is $4\pi$-periodic, with two branches $I_+$ (blue solid), $I_-$ (red solid) distinguished by the ground-state fermion parity and with a parity switch at $\phi = \pm \pi$. The top row shows the short-junction limit of Ref. [7], the bottom row the long-junction limit calculated here. (The jump in $I_+$ at $\phi = 0$ occurs because of the change in slope indicated by the green arrows in the magnified central part of the spectrum.) The $2\pi$-periodic supercurrent $I_{2\pi}$ without parity constraints is also shown (green dashed). The critical current is the same for $I_{4\pi}$ and $I_{2\pi}$ in the short junction, but different by a factor of two in the long junction.

$$I_{\pm}(\phi) = \pm \frac{e\Delta}{2h} \sin(\phi/2), \quad |\phi| < \pi. \quad (2)$$

To discuss the fermion-parity anomaly we assume, for definiteness, that the total number $\mathcal{N}$ of electrons in the system is even. (A different choice amounts to a $2\pi$ phase shift, or equivalently, an interchange of $I_+$ and $I_-\ldots$) The ground-state fermion parity $\sigma$ is even for $\phi = 0$ and switches to odd when $\phi$ crosses $\pi$. Since $\mathcal{N}$ is fixed, this topological phase transition must be accompanied by a switch between an even and odd number of quasiparticle excitations. At zero temperature, only the two levels $\varepsilon_{\pm}$ closest to the Fermi level ($\varepsilon = 0$) play a role, and the parity switch of $\sigma$ means that a quasiparticle is transferred from $\varepsilon_- < 0$ to $\varepsilon_+ > 0$. It cannot relax back from $\varepsilon_-$ to $\varepsilon_+$ at fixed parity of $\mathcal{N}$.

The resulting current-phase relationship can be represented by a switch between $2\pi$-periodic branches $I_{\pm}(\phi)$ (reduced zone scheme), or equivalently as a $4\pi$-periodic function $I_{4\pi}(\phi)$ (extended zone scheme). Both representations are shown in Fig. 1, upper panels. We also include the $2\pi$-periodic current $I_{2\pi}$ that results if the system can relax to its lowest energy state without constraints on the parity of $\mathcal{N}$.

So much for the short-junction limit. An elementary discussion of the long-junction regime (to be made rigorous in just a moment) goes as follows. For $L \gg \xi$ we may assume [22–24] a local linear relation between the current density $I$ and the phase gradient $\phi/L \ll 1/\xi$, of the form $I = \text{const} \times e\nu\phi/L$. The linear increase of $I_-$ is interrupted at $\phi = 0$ by a discontinuity $\Delta I_- = 2e\nu/L$. Half of it results from the jump in the slope of the lowest occupied positive energy level $\varepsilon = (\pi - |\phi|)\hbar\nu/2L$ [green arrows in Fig. 1(e)]. The jump in the slope of the highest occupied negative energy level contributes the other half. In the extended zone scheme, the resulting supercurrent $I_{4\pi}$ is a $4\pi$-periodic sawtooth with a slope $\Delta I_-/4\pi = eE_T/2\pi\hbar$.

The corresponding parity-dependent supercurrents in the reduced zone scheme are

$$I_+ = \frac{eE_T}{2\pi\hbar} \phi, \quad I_- = \frac{eE_T}{2\pi\hbar} (\phi - 2\pi \text{sgn}\phi), \quad |\phi| < \pi. \quad (3)$$

The $4\pi$-periodic supercurrent $I_{4\pi}$ switches from $I_+$ to $I_-$ at $\phi = \pi$, while $I_{2\pi}$ remains in the branch $I_+$ by compensating the switch in ground-state fermion parity $\sigma$ by a switch in the parity of the electron number $\mathcal{N}$. These are the curves plotted in Fig. 1 (lower panels).

The maximal supercurrent is reached near $\phi = 2\pi$ for $I_{4\pi}$ (with parity constraint) and near $\phi = \pi$ for $I_{2\pi}$ (without parity constraint). There is a factor of two difference in magnitude of these critical currents in a long junction,

$$I_{4\pi,c} = eE_T/\hbar, \quad I_{2\pi,c} = eE_T/2\hbar. \quad (4)$$

In contrast, for a short junction both are the same (equal to $e\Delta/2\hbar$).

To determine the crossover from the short-junction limit (2) to the long-junction limit (3), including the temperature dependence, we adapt the scattering theory of
Josephson effect [25] to include the fermion parity constraints. Input is the scattering matrix $s_0$ of electrons in the normal region and the Andreev reflection matrix $r_A$ at the normal-superconductor interfaces. These take a particularly simple $2 \times 2$ form at the quantum spin-Hall edge, but our general formulas are applicable also to multichannel topological superconductors.

The parity-dependent partition function is [12–14,26]

$$Z_\pm = \frac{1}{2} \left( \prod_{\varepsilon > 0} e^{\beta \varepsilon / 2} \right) \left( \prod_{\varepsilon > 0} (1 + e^{-\beta \varepsilon}) \pm \prod_{\varepsilon > 0} (1 - e^{-\beta \varepsilon}) \right)$$

$$= \frac{1}{2} Z_0 \left[ 1 \pm \prod_{\varepsilon > 0} \tanh (\beta \varepsilon / 2) \right],$$

(5)

with $\beta = 1/k_B T$ and $Z_0 = \prod_{\varepsilon > 0} 2 \cosh (\beta \varepsilon / 2)$ the partition function without parity constraints. From the expression for $Z_\pm$ one can see that the $\pm$ selects terms that contain an even $(\pm)$ or an odd $(-\pm)$ number of quasiparticle excitation factors $e^{-\beta \varepsilon}$, as is dictated by the ground-state fermion parity. The partition function $Z$ gives the free energy $F$ and hence the supercurrent $I$ [27],

$$I_\pm = \frac{2 e}{\hbar} \frac{dF_\pm}{d\phi}, \quad F_\pm = -\beta^{-1} \ln Z_\pm,$$

(6)

$$I_{2\pi} = I_0 = \frac{2 e}{\hbar} \frac{dF_0}{d\phi}, \quad F_0 = -\beta^{-1} \ln Z_0.$$  

(7)

The density of states $\rho(\varepsilon)$ contains both the discrete spectrum for $|\varepsilon| < \Delta$ (a sum of delta functions at the Andreev levels) and the continuous spectrum for $|\varepsilon| > \Delta$, including also a contribution $\rho_S$ from the superconducting electrodes. Scattering theory gives the expression [25]

$$\rho(\varepsilon) = \text{Im} \frac{d}{d\varepsilon} \nu(\varepsilon + i0^+) + \rho_S(\varepsilon),$$

(8)

$$\nu(\varepsilon) = -\pi^{-1} \ln \text{Det} X(\varepsilon), \quad X = (1 - M)^{-1/2},$$

(9)

$$M(\varepsilon) = r_A^*(\varepsilon) s_0^0(\varepsilon) r_A(\varepsilon) s_0(\varepsilon).$$

(10)

The factor $M^{-1/2}$ in the definition of $X$, as well as the term $\rho_S$, give a $\phi$-independent additive contribution to $F_0$ without any effect on $I_0$, but we need to retain these terms here because they do enter into the parity constraint for $I_\pm$.

In the absence of parity constraints, Ref. [28] gives the free energy

$$F_0 = -\beta^{-1} \sum_{p=0}^\infty \ln \text{Det} X(i\omega_p),$$

(11)

as a sum over fermionic Matsubara frequencies $\omega_p = (2p + 1)\pi/\beta$. A similar calculation [29] gives the parity dependence in the form

$$F_\sigma = F_0 - \beta^{-1} \ln \frac{1}{2} \left[ 1 + \sigma e^{\beta s} \sqrt{\text{Det} X(0)} \right] \times \exp \left( \sum_{p=1}^{\infty} (-1)^p \ln \text{Det} X(i\Omega_p / 2) \right),$$

(12)

$$\sigma = \text{sgn}[\text{Pf}(r_A s_0 - s_0^0 r_A^*)/(\text{Det} s_0)^{-1/2}]_{x=0}.$$  

(13)

with bosonic Matsubara frequencies $\Omega_p = 2p \pi / \beta$. The ground-state fermion parity $\sigma$ is given in terms of the Pfaffian of the antisymmetrized scattering matrix, evaluated at the Fermi energy. The sign ambiguity in the square root is resolved by fixing $\sigma = 1$ at $\phi = 0$.

Equation (12) contains a contribution from the superconducting electrodes,

$$J_S = \int_\Delta^\infty d\varepsilon \rho_S(\varepsilon) \ln \tanh (\beta \varepsilon / 2),$$

(14)

which only plays a role at temperatures $T \geq \Delta / k_B$. The factor $e^{\beta s}$ can therefore be replaced by unity in the long-junction regime, when $k_B T \leq E_T \ll \Delta$.

We now specify these general formulas for the quantum spin-Hall edge, with the Hamiltonian [30]

$$H_{\text{BdG}} = \left( \begin{array}{cc} v p \sigma_x + U(x) & \Delta^*(x) \sigma_y \\ \Delta(x) \sigma_y & v p \sigma_z - U(x) \end{array} \right).$$

(15)

The edge runs along the $x$ axis, $p = -i \hbar \partial_x$ is the momentum operator, and the electrostatic potential is $U(x)$ (measured relative to the Fermi level). The pair potential $\Delta(x)$ vanishes in the normal region $|x| < L/2$. In the two superconducting regions we set $\Delta(x) = \Delta e^{\pm i \phi / 2}$, with a step at $x = \pm L/2$. This so-called “rigid boundary condition” is justified for a single channel coupled to a bulk superconducting reservoir [10].

A mode-matching calculation gives the scattering matrices

$$s_0 = \left( \begin{array}{cc} 0 & e^{i\chi} \\ e^{i\chi} & 0 \end{array} \right), \quad \chi(\varepsilon) = \chi_0 + \varepsilon / E_T,$$

(16)

$$r_A = \left( \begin{array}{cc} \alpha e^{i\phi / 2} & 0 \\ 0 & -\alpha e^{-i\phi / 2} \end{array} \right), \quad \alpha(\varepsilon) = \sqrt{1 - \frac{\varepsilon^2}{\Delta^2} + i \frac{\varepsilon}{\Delta}},$$

$$\text{Det} X(\varepsilon) = 2 \cos \phi + \alpha^2 e^{2i\chi / E_T} + \alpha^2 e^{-2i\chi / E_T}.$$  

(17)

We discuss the various terms in these expressions. The electron scattering matrix $s_0$ is purely off diagonal, because of the absence of backscattering along the quantum spin-Hall edge. The transmission phase $\chi$ depends linearly on energy because of the linear dispersion. Electrostatic potential fluctuations contribute only to the energy-independent offset $\chi_0 = -(\hbar v)^{-1} \int_0^\chi U dx$, which drops out in Eq. (9). The Andreev reflection matrix $r_A$ (from electron to hole) is unitary below the gap. Above the gap there is also propagation into the superconductor, so $r_A$ is subunitary. The same expression (16) for $r_A$ applies at all energies, evaluated at $\varepsilon + i0^+$ to avoid the branch cut of the square root.
Putting all pieces together [29], we obtain the parity-dependent supercurrent for arbitrary ratio $\Delta/E_T$. In the short-junction limit $\Delta/E_T \to 0$ we recover the known result (2), when the energy dependence of the scattering matrix and the phase sensitivity of the continuous spectrum can both be ignored. In the opposite long-junction limit $\Delta/E_T \to \infty$ we find

$$I_{4\pi} = I_0 - \frac{2e}{h \beta} \frac{d}{d\phi} \ln \left[ \frac{1}{2} + \cos(\phi/2) e^{S-\pi/2E_T} \right]. \tag{18}$$

$$S = \sum_{p=1}^{\infty} (-1)^p \ln \left( 1 + 2 e^{-\Omega_p/E_T} \cos \phi + e^{-2\Omega_p/E_T} \right), \tag{19}$$

$$I_{2\pi} = I_0 - \frac{2e}{h \beta} \sin \phi \sum_{p=0}^{\infty} \left[ \cos \phi + \cosh(2\omega_p/E_T) \right]^{-1}. \tag{20}$$

The plot of the results in Fig. 2 shows that the crossover from a sine to a sawtooth shape occurs early: already for $L = \xi$, this would apply even if the effective superconducting gap is well below the bulk value of Nb. The corresponding Thouless energy is $E_T/k_B = 1.5 \text{ K}$, so at $T = 100 \text{ mK}$ one should be close to the low-temperature limit.

In the ongoing search for the $4\pi$-periodic Josephson effect the first results have been reported [34] for the ac effect (fractional Shapiro steps [9,15–18]). A dc measurement of the current-flux ($I-\Phi$, $\phi = 2e\Phi/h$) relationship, for times large compared to the time $\tau_{qp} \approx \mu\text{s}$ for unpaired quasiparticles to tunnel into the system [35], will measure the $2\pi$ periodic $I_{2\pi}$ rather than $I_{4\pi}$. Such a phase-sensitive measurement (Fig. 2, upper inset) would produce the critical current $I_{2\pi,c}$ without any signature of the parity anomaly. In contrast, a phase-insensitive measurement of the critical current through the current-voltage ($I-V$) characteristic (lower inset) will produce $I_{4\pi,c}$ even on time scales $\gg \tau_{qp}$, because the phase of a resistively shunted (overdamped) circuit can adjust to a change in $N$ on time scales much smaller than $\tau_{qp}$. A change in the parity of $N$ will be compensated by a $2\pi$ phase shift, without a change in critical current [29]. In a short junction, $I_{2\pi,c}$ and $I_{4\pi,c}$ are the same, so this does not help, but in a long junction they differ by up to a factor of two.

In conclusion, we have presented a theory for the $4\pi$-periodic Josephson effect on large scales compared to the superconducting coherence length. A multitude of subgap states, as well as a continuum of states above the gap, contribute to the supercurrent for $L \gg \xi$, but still the parity anomaly responsible for the $4\pi$ periodicity persists. In fact, we have found that in a long junction the anomaly manifests itself also in a phase-insensitive way, through a doubling of the critical current. This opens up new possibilities for the detection of this topological effect at the quantum spin-Hall edge [31–33], and possibly also in semiconductor nanowires [34,36–41].

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It is a pervasive misunderstanding that the factor $g = 2$ in the relation (1) between supercurrent and Andreev levels accounts for the charge $2e$ of the Cooper pairs, rather than counting the spin degeneracy. Because of this misunderstanding, the results for the supercurrent carried by Majorana zero modes (which have $g = 1$) should be divided by two in Ref. [9] and many follow-up papers. The origin of the misunderstanding is discussed in more detail by N. M. Chchelkatchev and Yu. V. Nazarov, Phys. Rev. Lett. 90, 0512610 (2003).

The factor of two in the relation (6) between supercurrent and free energy accounts for the Cooper pair charge, with all degeneracies included in the partition function. Please see Ref. [21] to avoid any misunderstanding.

See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.110.017003 for details of the calculations.