Self-Induced Fractional Fourier Transform and Revivable Higher Order Spatial Solitons in Strongly Nonlocal Nonlinear Media

Daquan Lu, Wei Hu, Yajian Zheng, Yanbin Liang, Longgui Cao, Sheng Lan, and Qi Guo

Laboratory of Photonic Information Technology,
South China Normal University, Guangzhou 510631, China

(Dated: February 2, 2022)

Abstract

The fractional Fourier transform (FRFT) naturally exists in the strongly nonlocal nonlinear (SNN) media and the propagation of optical beams in SNN media can be simply regarded as a self-induced FRFT. Through FRFT technic the evolution of fields in SNN media can be conveniently dealt with and an arbitrary square-integrable input field presents generally as a revivable higher order spatial soliton which reconstructs its profile periodically after every 4 times of Fourier transforms. The self-induced FRFT would illuminate the prospect of the SNN media in new applications such as continuously tunable nonlinearity-induced FRFT devices.

PACS numbers: 42.65.Tg, 42.30.Kq, 42.65.Jx

*Electronic address: huwei@scnu.edu.cn*
Since Fourier suggested the usage of Fourier analysis to solve the heat conduction problem in 1807, the Fourier transform (FT) has been applied widely in many branches of science [1]. In the field of optics, the FT is one of the most important and basic tools in dealing with physical optics and optical information processing [1]. In fact, the term Fourier optics is often used synonymously with optical information process. The fractional Fourier transform (FRFT), which is an extension of the FT [2], has been introduced to optics since 1993 when Mendlovic and Ozaktas find this operator can be optically performed by the quadratic graded-index (GRIN) media [3]. In respect that the FRFT can show the characteristics of the signal changing continuously from the spatial domain to the spectrum domain, it interests optics scientists and engineers and plays an important role in many optics fields [4, 5, 6, 7, 8, 9, 10, 11, 12], such as diffraction [7], transmission [8, 9], imaging [10], information processing [11, 12], etc..

On the other hand, since the first observation of nonlinear optical phenomena by Franken et al in 1961 [13], one year after the invention of the laser, the nonlinear optics has been a rapidly expanded active field and widely influenced other fields. A special branch of the nonlinear optics, the field of optical solitons, have grown enormously in the past decades. In particular, the soliton in strongly nonlocal nonlinear (SNN) media which supports (2+1)-dimension solitons, have attracted extensive interest and been widely investigated in the past few years [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27], since Snyder and Mitchell simplified the nonlocal nonlinear Schrödinger equation (NNLSE) to a linear model (called Snyder-Mitchell mode (SMM) in our papers) in the SNN case and found an exact Gaussian-shaped “accessible soliton” [14]. And the “accessible soliton” has shown its interest in potential applications such as photonic switching and logic gating [27].

The Fourier optics and the nonlinear optics might seem to be independent of each other in that the FT and the FRFT are linear transforms and well known as technic provided to solve problems in linear system. But we note that the SNN media, the propagation equation in which can be mathematically simplified to the linear SMM [14], would provide an opportunity to intersect them with each other.

In this letter, the FRFT induced by the input field itself through the SNN effect is introduced. It is shown that the FRFT naturally exists in the SNN media. The order of the self-induced FRFT in the SNN media can be steered with the input power in addition to the propagation distance \( z \), quite different from the traditional linear (intensity-independent)
FRFT devices such as the lens series and the quadratic GRIN media. Due to the self-induced FRFT, propagation of an arbitrary square-integrable input field presents generally as a revivable higher order spatial soliton (RHOSS) which is similar to the higher order (1+1)D temporal soliton in fiber. The build-block-like solitons, build-block-like breathers, and their interaction, can be regarded as special cases of the RHOSS and can be conveniently investigated with the FRFT technique.

The propagation of beams in nonlocal nonlinear media can be phenomenologically described by the NNLSE: 

\[ 2ikn_0\partial_z \Phi + n_0 \Delta_\perp \Phi + 2k^2 \Delta n \Phi = 0, \]

where \( k \) represents the wave number in the media with the linear part of the refractive index \( n_0 \) when the nonlinear perturbation of refractive index \( \Delta n \) equals zero, \( \Delta n = n_2 \int R(\vec{r} - \vec{r}_a)\Phi(\vec{r})^2 d^2\vec{r}_a \) (\( n_2 \) is the nonlinear index coefficient and \( R \) is the normalized symmetric real spatial response function of the media). In the case of SNN media we need only keep the first two terms of the expansion of \( \Delta n \) and the NNLSE is simplified to the SMM [14]

\[ 2ik\partial_z \Phi + \Delta_\perp \Phi - k^2 \gamma^2 P_0 r^2 \Phi = 0, \tag{1} \]

where \( \gamma \) is a material constant, \( P_0 = \int |\Phi|^2 d^2\vec{r} \) is the input power.

We assume \( \Phi(\vec{r}, z) = a(\vec{r}) \exp(-i\beta z) \) to seek the stationary solutions of Eq. (1). Substituting this expression into Eq. (1) gives

\[ 2\beta ka = k^2 \gamma^2 P_0 r^2 a - \Delta_\perp a, \]

the eigen solutions of which are Hermite Gaussian soliton family \( a^{(p)} = HG_{m,n}^{(p)} = c^{(p)} H_m\left(\frac{x}{w_c}\right)H_n\left(\frac{y}{w_c}\right)e^{-\frac{x^2}{2w_c^2}} \) (\( m + n = p \)) in cartesian coordinate [26], Laguerre Gaussian soliton family

\[ a^{(p)} = \begin{cases} 
LG_{l,q}^{c(\ell)} = c^{(p)} \left(\frac{x}{w_c}\right)^l L_q^l \left(\frac{x}{w_c}\right) \cos(l\theta) e^{-\frac{x^2}{2w_c^2}} \\
LG_{l,q}^{s(\ell)} = c^{(p)} \left(\frac{x}{w_c}\right)^l L_q^l \left(\frac{x}{w_c}\right) \sin(l\theta) e^{-\frac{x^2}{2w_c^2}} 
\end{cases} \] (2q + l = p) \tag{3}

in circular cylindrical coordinate [26], or Ince Gaussian soliton family in elliptical coordinate [23, 24]. In Eqs. (2) and (3), \( p = 0, 1, 2... \) is the order of the solution with which the soliton eigenvalue can be straightforwardly obtained:

\[ \beta^{(p)} = (p + 1)\beta_0, \tag{4} \]

\( c^{(p)} \) is the normalized coefficient ensures \( \int |a^{(p)}|^2 d^2\vec{r} = P_0, \) \( w_c = (k^2\gamma^2 P_0)^{-1/4} \) is a generalized beam width, \( H_m(x) \) and \( L_q^l (x) \) respectively represents the Hermite and the Laguerre polynomials, \( \beta_0 = \sqrt{P_0\gamma} = 1/kw_c^2. \)
Mathematically, the profile of the eigen soliton solution, \( a^{(p)} \), is simultaneously the eigen function of the FRFT \([2, 3, 4]\):

\[
\hat{F}_\alpha \{g(\vec{r}_1)\}(\vec{r}_2) = \frac{\exp[i(\alpha - \frac{\pi}{2})]}{2\pi w_c^2 \sin \alpha} \exp \left[ \frac{i\vec{r}_2^2}{2w_c^2 \tan \alpha} \right] \int \exp \left[ \frac{i\vec{r}_1^2}{2w_c^2 \tan \alpha} - \frac{i\vec{r}_1 \cdot \vec{r}_2}{w_c^2 \sin \alpha} \right] g(\vec{r}_1) d^2 \vec{r}_1,
\]

i.e., \( a^{(p)} \) is the eigen solution of the equation \( \hat{F}_\alpha \{a^{(p)}(\vec{r}_1)\} = a^{(p)}(\vec{r}_2) e^{-i\alpha} \) with the eigen value \( e^{-i\alpha} \), where the order of the FRFT is

\[\alpha = \beta_0 z = \sqrt{P_0 \gamma z}. \quad (5)\]

Therefore the field of the eigen soliton at \( z \) is connected with that at the entrance plane through the FRFT: \( \Phi^{(p)}(\vec{r}_2, z) = \hat{F}_\alpha \{\Phi^{(p)}(\vec{r}_1, 0)\} \times \exp(-i\alpha) \).

An arbitrary square-integrable input field can be expressed as a linear superposition of an arbitrary one of the three families of eigen soliton solutions: \( \Phi(\vec{r}_1, 0) = \sum_{p=0}^{\infty} c_p \Phi^{(p)}(\vec{r}_1, 0) \).

According to the linearity of Eq. (1) and the FRFT, the propagation in the SNN media is presented as the FRFT on the input field

\[\Phi(\vec{r}_2, z) = \hat{F}_\alpha \{\Phi(\vec{r}_1, 0)\} \times \exp(-i\alpha). \quad (6)\]

In the special case \( \alpha = \pi/2 \), the propagated field at \( z \) is deduced to the well-known FT of the input field, reads

\[\Phi(\vec{r}_2, z) = \frac{-i}{2\pi w_c^2} \int \Phi(\vec{r}_1, 0) \exp \left[ -\frac{i\vec{r}_1 \cdot \vec{r}_2}{w_c^2} \right] d^2 \vec{r}_1. \quad (7)\]

Equations (5)-(7) indicate that the SNN media naturally performs the FRFT and FT on the input field. The physical origination of this effect is as follows: When a beam is input into a SNN media, it would induce a quadratic GRIN channel in the medium through the nonlocality. The propagation in the channel then performs the FRFT and FT, just as in the traditional quadratic GRIN media[3]. Because the grads of the refractive index distribution can be steered with the power, the order of the self-induced FRFT in the SNN media can be steered with the input power in addition to the propagation distance \( z \), quite different from the traditional linear FRFT devices.

The simulation based on the NNLSE shows that: the self-induced FRFT becomes more and more distinct with the increase of the nonlocality or the input power (Fig. (c)). In fact, the increase of the power is tantamount to the increase of the nonlocality, in that it decreases \( w_c \) and the average beam width in propagation. On the other hand, the smoother
FIG. 1: (Color online) Propagation dynamics of the super Gaussian field \( \exp \left[ -\left( \frac{x}{\sqrt{2} w_1} \right)^8 \right] \) in SNN media with Gaussian response function \( R(r) = \frac{1}{2\pi w_R^2} \exp \left[ -\frac{r^2}{2w_R^2} \right] \) [24, 25, 26], based on numerical simulation of the NNLSE. (a) Evolution of the profile in propagation. (b), (c) Intensity distribution at \( \beta_0 z = \pi/2 \) and \( \pi \) respectively. The nonlocality degree \( \Gamma = w_B/w_R = 1/10 \), \( w_B \) is the second-order moment width of the beam at the entrance plane, \( P_1 \) is the critical power for the eigen soliton with generalized width \( w_1 \).

the shape of the input field is, the less the nonlocality is required to support the FRFT, because it contains less higher-order eigen soliton solutions in the superposition. Generally, the profile of the eigen soliton holds when \( \Gamma < 1/10 \). When the order of a constituent eigen soliton in the superposition is high enough so that \( \Gamma > 1/10 \), the profile would be distorted in propagation. Therefore the SNN media act as a low-pass spatial frequency filter.

Based on the self-induced FRFT, it is convenient to investigate the propagation in SNN media from the angle of Fourier optics. We can predict the field would present a periodical evolution with the period \( \Delta z = 2\pi/\sqrt{P_0 \gamma} \) (corresponding to \( \Delta \alpha = 2\pi \)) (Figs. 1, 2). As a result of cascade FT, the pattern evolves to the inversion of the input pattern at \( z = (2n + 1)\pi/\sqrt{P_0 \gamma} \) and revive to the input pattern at \( z = 2n\pi/\sqrt{P_0 \gamma} \) (Fig. 2) (we call those cross sections the revived planes or imaging planes). At \( z = (n + 1/2)\pi/\sqrt{P_0 \gamma} \) (we call these cross sections the Fourier planes), the patterns are the FT spectrum of the input field or the inversion. From Eq. (7) the spatial frequency \( \vec{k}_r = r_2/w_c^2 \), thus at the Fourier planes the
FIG. 2: Propagation dynamics of Gaussian beams \( \exp[-(x/\sqrt{2}w_1)^2] \) truncated by triangular (rows 1 and 3) and square (rows 2 and 4) super Gaussian diaphragm in SNN media with Gaussian response function based on numerical simulation of the NNLSE. \( P_0 = 4P_1, \Gamma = 1/20 \).

beam width \( w(z) \propto 1/\sqrt{P_0} \), which obeys the scaling rule of FT (Fig. 1(b)). Because of the periodic revivable evolution, which is similar to the higher order temporal soliton in fiber, we call this type of propagation the RHOSS (The RHOSS should be distinguished from the traditionally mentioned “higher order spatial soliton” which refers to the stationarily propagated multipole spatial soliton). There is an interesting difference between the higher order temporal soliton in fiber and the RHOSS: the existence of the higher order temporal soliton in fiber requires much more power than that required for the fundamental soliton, whereas to support the RHOSS, the power can be lower than the critical power of the stationarily propagated fundamental soliton.

According to properties of the FRFT, there are three special cases of the RHOSS (Fig. 3):

1) **Build-block-like soliton.** As shown in Eqs. (4), during propagation, every degenerate eigen soliton with the order \( p \) has the same propagation-induced phase \( \beta(z) \) (or in other words, has the same FRFT eigen value \( \exp(-ip\alpha) \)). Therefore, when i) the input field is the linear superposition of the eigen soliton solutions with the same order \( p \), beam center, and generalized width \( w_1 \); and ii) the power is the critical power \( P_1 = 1/(k^2w_1^4\gamma^2) \) which supports
the eigen solitons with the generalized width $w_1$, the field would propagate stationarily, i.e., the soliton occurs (rows 1-2 in Fig. 3). Because the field of this type can be freely composed of the eigen soliton solutions with the same order, we call it build-block-like soliton.

2) Build-block-like breather. When all are the same as the build-block-like soliton except the input power $P_0$ deviate from the critical power $P_1$, the FRFT keeps the shape of the input field and periodically varies the width with the period $\Delta \alpha = \pi$. Subsequently the evolution is presented as breather with the period $\Delta z = \pi/\sqrt{P_0 \gamma}$ (row 3 in Fig. 3), and the change of the build-block-like breather’s width is the same as the prediction for the Gaussian breather in Ref. [14]. At $z = (n + 1/2)\pi/\sqrt{P_0 \gamma}$, the FRFT is deduced to the FT and $w(z) \propto 1/\sqrt{P_0 w_1}$.

3) Multi soliton interaction. In this case, the traditional technic might require much effort in mathematical treatment because orbits of the interacting solitons are influenced by each other. But by introducing the FRFT, the evolution of the orbits is presented simply as a shift in the FRFT reads $\Phi(\vec{r}_2, z) = \hat{F}_\alpha \{\Phi(\vec{r}_1 - \vec{r}_0, 0)\} \exp[-i\alpha]$, where $\vec{r}_0$ represents the

![FIG. 3: (Color online) Propagation dynamics of build-block-like soliton (rows 1-2), build-block-like breather (row 3), and multisoliton interaction (row 4) in SNN media with Gaussian response function based on numerical simulation of the NNLSE. $\Gamma = 1/10$, $P_0 = 2P_1$ for row 3 and $P_0 = P_1$ for others. The input fields are respectively ($HG^{(8)}_{4,4} - HG^{(8)}_{0,8}/10$) (row 1), ($HC^{(8)}_{0,8} + HG^{(8)}_{8,0}$) (row 2), ($LC^{(s,8)}_{8,0} + iLG^{(s,8)}_{8,0} - HG^{(8)}_{0,8}/125$) (row 3) and [$HG^{(3)}_{0,3}(x+3w_1, y+3w_1) + HG^{(4)}_{1,0}(x-3w_1, y-3w_1)$](row 4). The general width of all constituent beams are $w_1$ at the entrance plane.](image)
initial deviation of the beam center of the interacting soliton from the mass center. Under the vertical incidence condition (row 4 in Fig. 3), the solitons intersect each other at $z = (n + 1/2)\pi/\sqrt{P_0}\gamma$, evolve to the inversion $\Phi(\vec{r}_0 - \vec{r}, 0)$ at $z = (2n + 1)\pi/\sqrt{P_0}\gamma$, and recur to the input field $\Phi(\vec{r} - \vec{r}_0, 0)$ at $z = 2n\pi/\sqrt{P_0}\gamma$.

To verify the prediction about the self-induced FRFT (or in other words, the RHOSS), we carried out the experiment in a columned lead glass. The experimental setup is illustrated in Fig. 4. The beam from a Verdi laser is focused by the collimation lens. A diaphragm is placed at the focus and the real image is produced at the entrance plane of the lead glass by the confocal lenses pair. When the input power is adjusted, the intensity distribution at the entrance and exit plane are monitored by imaging the beams onto a CCD camera. The lead glass is 59.8 mm in length and 15.1 mm in diameter. The beam at the entrance of lead glass is 85 µm × 80 µm in size (for square diaphragm) or 106 µm in diameter of circumscribed circle.
The experimental results are shown in Fig. 5(a)-(l): by changing the input power, the $w_c$ changes and the patterns similar to the FRFT spectrums with the orders $\alpha = 0$, $\pi/4$, $\pi/2$, $3\pi/4$, and $\pi$ in Fig. 2 are recorded. In case of the triangle (square) diaphragm, the output pattern recur to the input pattern when $P_0 = 551$ mW (Fig. 5(b)) ($P_0 = 590$ mW (Fig. 5(h))) and evolves to the inversion when $P_0 = 823$ mW (Fig. 5(f)) ($P_0 = 810$ mW (Fig. 5(l))). Because the higher-frequencies are filtrated in propagation, the output patterns are smoother than the input ones. In Fig. 5(z), the variation of the FRFT order $\alpha$ with the square of the input power, i.e. $\sqrt{P_0}$, is illustrated. The linear fit shows that the FRFT order $\alpha$ is directly proportional to $\sqrt{P_0}$, as predicted in Eq. (5).

In summary, the self-induced FRFT in SNN media made the nonlinear optics and the Fourier optics intersect with each other. The introducing of the FRFT technic would release one from complicated mathematical calculation in propagation problems such as soliton solutions in SNN media. The RHOSS, including the build-block-like solitons, build-block-like breathers, and their interaction, would greatly enrich the nonlocal soliton family. The fact that the order of the self-induced FRFT in SNN media is related not only to the propagation distance but also to the input power, quite different from that in the traditional linear devices, would illuminate the prospect of new applications of the SNN media such as developing power-controlled continuously tunable FRFT devices.

This research was supported by the National Natural Science Foundation of China (Nos. 10674050 and 10804033), the Program for Innovative Research Team of the Higher Education in Guangdong (No. 06CXTD005), and Specialized Research Fund for the Doctoral Program of Higher Education (No. 20060574006).

[1] R.N. Bracewell, The Fourier Transform and its Applications (McGraw-Hill, New York, 2000), 3rd ed.
[2] V. Namias, J. Inst. Math. Appl. 25, 241 (1980).
[3] D. Mendlovic and H.M. Ozaktas, J. Opt. Soc. Am. A 10, 1875 (1993).
[4] M.A. Bandres and J.C. Gutiérrez-Vega, Opt. Lett. 30, 540 (2005).
[5] Y.J. Cai and F. Wang, Opt. Lett. 31, 2278 (2006).
[6] A. Shahin, H.M. Ozaktas and D. Mendlovic, Opt. Commun. 120, 134 (1995).
[7] P. Pellat-Finet, Opt. Lett. 19, 1388 (1994).
[8] H.M. Ozaktas and D. Mendlovic, J. Opt. Soc. Am. A 12, 743 (1995).
[9] A.W. Lohmann, J. Opt. Soc. Am. A 10, 2181 (1993).
[10] L.M. Bernardo and O.D.D. Soares, J. Opt. Soc. Am. A 11, 2622 (1994).
[11] M.A. Kutay and H.M. Ozaktas, J. Opt. Soc. Am. A 15, 825 (1998).
[12] J. Hahn, H. Kim, and B. Lee, Opt. Exp. 14, 11103 (2006).
[13] P.A. Franken, A.E. Hill, C.W. Peters, and G. Weinreich, Phys. Rev. Lett. 7, 118 (1961).
[14] A.W. Snyder and D.J. Mitchell, Science 276, 1538 (1997).
[15] C. Conti, M. Peccianti, and G. Assanto, Phys. Rev. Lett. 92, 113902 (2004).
[16] C. Rotschild, O. Cohen, O. Manela, M. Segev, and T. Carmon, Phys. Rev. Lett. 95, 213904 (2005).
[17] A. Dreischuh, D.N. Neshev, D.E. Petersen, O. Bang, and W. Krolikowski, Phys. Rev. Lett. 96, 043901 (2006).
[18] A.I. Yakimenko, V.M. Lashkin, and O.O. Prikhodko, Phys. Rev. E 73, 066605 (2006).
[19] C. Rotschild, M. Segev, Z.Y. Xu, Y.V. Kartashov, L. Torner, and O. Cohen, Opt. Lett. 31, 3312 (2006).
[20] A.V. Mamaev, A.A. Zozulya, V.K. Mezentsev, D.Z. Anderson, and M. Saffman, Phys. Rev. A 56, R1110 (1997).
[21] N.I. Nikolov, D. Neshev, W. Królikowski, O. Bang, J.J. Rasmussen, and P.L. Christiansen, Opt. Lett. 29, 286 (2004).
[22] W. Królikowski, M. Saffman, B. Luther-Davies, and C. Denz, Phys. Rev. Lett. 80, 3240 (1998).
[23] D.M. Deng, and Q. Guo, Opt. Lett. 32, 3206 (2007).
[24] S. Lopez-Aguayo, and J.C. Gutiérrez-Vega, Opt. Exp. 15, 18326 (2007).
[25] D. Briedis, D.E. Petersen, D. Edmundson, W. Krolikowski, and O. Bang, Opt. Express 13, 435 (2005).
[26] D. Buccoliero, A.S. Desyatnikov, W. Krolikowski, and Y.S. Kivshar, Phys. Rev. Lett. 98, 053901(2007).
[27] M. Peccianti, C. Conti, G. Assanto, A.D. Luca, and C. Umeton, Appl. Phys. Lett. 81, 3335 (2002).