A note on powers of Boolean spaces with internal semigroups

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Abstract

Boolean spaces with internal semigroups generalize profinite semigroups and are pertinent for the recognition of not-necessarily regular languages. Via recognition, the study of existential quantification in logic on words amounts to the study of certain spans of Boolean spaces with internal semigroups. In turn, these can be understood as the superposition of a span of Boolean spaces and a span of semigroups. In this note, we first study these separately. More precisely, we identify the conditions under which each of these spans gives rise to a morphism into the respective power or Vietoris construction of the corresponding structure. Combining these characterizations, we obtain such a characterization for spans of Boolean spaces with internal semigroups which we use to describe the topo-algebraic counterpart of monadic second-order existential quantification. This is closely related to a part of the earlier work on existential quantification in first-order logic on words by Gehrke, Petrişan and Reggio. The observation that certain morphisms lift contravariantly to the appropriate power structures makes our analysis very simple.

1 Introduction

The interaction of recognition and quantification involves the analysis of spans of topological spaces of the form

\[
\begin{array}{ccc}
\beta f & \beta S \\
\beta T & \overset{\beta f}{\searrow} & \overset{g}{\swarrow} & X
\end{array}
\]

where \(T\) is the set of structures for the logic at hand, \(S\) is the set of all models over these structure (i.e. a set of free variables is fixed and elements of \(S\) consist of a structure from \(T\) equipped with an interpretation of the free variables). In the case of the existential quantifier, the map \(f : S \to T\) is just the map taking a model to the underlying structure, and \(\beta\) is the Stone–Čech compactification (or, equivalently, \(\beta f\) is the Stone dual of the Boolean algebra homomorphism \(f^{-1} : \mathcal{P}(T) \to \mathcal{P}(S)\)). The idea of recognition is that, instead of specifying a subalgebra of \(\mathcal{P}(S)\) corresponding to the formulas with free variables we want to study, \(g\) is a continuous map to a Boolean space dual to the subalgebra.

If \(\mathcal{B}\) is the Boolean algebra of clopens of \(X\), then the Boolean algebra \(g^{-1}[\mathcal{B}] = \{g^{-1}(U) \mid U \in \mathcal{B}\}\) consists of the model classes of the formulas with free variables that we want to study. Also, for \(L = g^{-1}(U) \cap S\), \(L_\exists = f[L]\), is the set of structures satisfying the corresponding (existentially) quantified formula, and we are interested in building a recognizer for the Boolean subalgebra of \(\mathcal{P}(T)\) generated by the sets \(f[g^{-1}(U)]\), for \(U \in \mathcal{B}\).

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In [11], it is shown that the desired recognizer is of the form
\[ h : \beta T \to \mathcal{V}(X) \]  
and is given dually by \( \Diamond V \mapsto f[g^{-1}(V)] \), invoking Vietoris for Boolean spaces as the dual of the monad for modal logic. A categorical approach for a similar construction, also having in mind applications in formal language theory, may be found in [7].

A much simpler analysis would ensue if we can obtain \( h \) by lifting the pair (1) as follows
\[ \beta T \hookrightarrow \mathcal{V}(\beta T) \xrightarrow{(\beta f)^*} \mathcal{V}(\beta S) \xrightarrow{V(g)} \mathcal{V}(X). \]  
Here \( \mathcal{V}(g) \) is the forward image under \( g \), while \( (\beta f)^* \) is the inverse image under \( f \). It is well known that forward image is always continuous and \( \mathcal{V} \) is indeed viewed as a covariant endofunctor with \( \mathcal{V}(g) \) given by forward image. On the other hand, inverse image under a continuous map is not in general continuous on the Vietoris spaces.

Our short and simple analysis here consists in observing that for certain continuous maps (those whose duals have lower adjoints, and for the \( \beta f \)'s in particular) inverse image gives a continuous map. Indeed, we show that all continuous maps of the form (2) come about from a span composing inverse and direct image.

In the setting of recognition, we have to deal with a superposition of semigroup structure and topology. We show that a similar phenomenon is at play for semigroups and that these combine correctly to give a simple description of the topo-algebraic recognizing maps of the form (2) by means of a composition as in (3).

The work in [11] and further papers in this direction for quantification in first-order logic [12, 4] deal with a second complication which is much deeper, and which stems from the combined fact that the set \( S \) of models does not carry an appropriate semigroup structure and that \( f \) is not a homomorphism of semigroups. We also mention that, although they do not refer to logic on words, the corresponding treatment for the fragments of logic defining regular languages may be inferred from [2]. Here, we treat a much simpler case, namely monadic second-order logic, for which \( T \) and \( S \) are semigroups and \( f \) is a length-preserving semigroup homomorphism (i.e., one of those for which inverse images lift homomorphically to the powerset semigroups).

This note is organized as follows. In Section 2, we set up the notation and define the main concepts used later. In Section 3 we study power constructions both for compact Hausdorff and Boolean spaces and for semigroups. Finally, in Section 4 we show that forward images under length-preserving homomorphisms of languages recognized by a Boolean space with an internal semigroup are precisely those that are recognized by its Vietoris. This generalizes a known result for regular languages [17, 19]. Interpreting the result in the context of logic on words leads to a topo-algebraic description of second-order existential quantification.

2 Preliminaries

The reader is assumed to have some acquaintance with Stone duality and with semigroups and the theory of formal languages, including logic on words. Nevertheless, we briefly recall the main concepts and results that we will need. For further reading on topology, we refer to [21], on Stone duality to [8], on semigroup and formal language theory to [1], and on logic on words to [20].

**Stone duality.** Stone duality establishes a correspondence between Boolean algebras and Boolean spaces. On the level of objects it goes as follows. Given a Boolean space \( X \), the set of its clopen

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1 A Boolean space is a compact Hausdorff topological space with a basis of clopen sets.
(i.e., both open and closed) subsets, $\text{Clop}(X)$, forms a Boolean algebra (actually, this is true for every topological space). Conversely, if $B$ is a Boolean algebra, then the set of ultrafilters $\mathfrak{b}$ of $B$, $X_B$, is a Boolean space when equipped with the topology generated by the sets of the form

$$\hat{b} = \{ x \in X_B \mid b \in x \},$$

for $b \in B$. We have isomorphisms $B \cong \text{Clop}(X_B)$ and $X \cong X_{\text{Clop}(X)}$. For the morphisms, we have the following correspondence. If $f : X \to Y$ is a continuous function between Boolean spaces, then taking preimages of clopens defines a homomorphism $f^{-1} : \text{Clop}(Y) \to \text{Clop}(X)$ of Boolean algebras, and if $\alpha : B \to C$ is a homomorphism of Boolean algebras, for every ultrafilter $y \in X_C$, the set $\{ b \in B \mid \alpha(b) \in y \}$ is an ultrafilter of $B$, and this correspondence yields a continuous function $X_C \to X_B$.

Dual spaces of powerset Boolean algebras will play a role in the sequel. For any set $S$, the Stone dual of $\mathcal{P}(S)$ is the Stone-Čech compactification of $S$, which we will denote by $\beta S$. The set $S$ embeds densely in $\beta S$ via the map $s \mapsto \{ x \in \beta S \mid s \in x \}$. We say that a subset $L \subseteq S$ is set-theoretically recognized by a Boolean space $X$ provided there is a continuous map $f : \beta S \to X$ and a clopen subset $C \subseteq X$ such that $L = f^{-1}(C) \cap S$. Any function of sets $f : S \to T$ yields a homomorphism of Boolean algebras $f^{-1} : \mathcal{P}(T) \to \mathcal{P}(S)$ and the corresponding dual map will be denoted $\beta f : \beta S \to \beta T$.

**Formal languages and recognition.** An alphabet is a finite set of symbols $A$, also called letters, a (non-empty) word over $A$ is an element of the free semigroup $A^+$ and a (formal) language $L$ is a set of words over some alphabet. The Boolean algebra of all languages over $A$, $\mathcal{P}(A^+)$, is naturally equipped with a biaction of $A^+$ given by

$$u^{-1}Lv^{-1} = \{ w \in A^+ \mid uwv \in L \},$$

for every $u, v \in A^+$ and $L \subseteq A^+$. Languages of the form $[i]$ are called quotients of $L$.

We say that a language $L$ is recognized by a semigroup $S$ provided there is a homomorphism $f : A^+ \to S$ and a subset $Q \subseteq S$ satisfying $L = f^{-1}(Q)$, or equivalently, provided $L = f^{-1}(f[L])$. Notice that the set of all languages recognized by a given homomorphism forms a Boolean algebra closed under quotients. Given a homomorphism $h : B^+ \to A^+$ between finitely generated free semigroups and a language $L \subseteq A^+$ recognized by $S$, we have that $h^{-1}(L)$ is also recognized by $S$. The homomorphism $h$ is said to be length-preserving, or an $lp$-morphism, if it maps letters to letters. Languages recognized by finite semigroups are called regular.

To handle non-regular languages topo-algebraically, one needs to consider richer structures, which we introduce below. Notice that an attempt to make the notion of recognition as uniform as possible was first made in [3] and followed up by [6]. But in both cases the main concern is to capture several computational models within the same framework, and not going beyond regularity.

**Boolean spaces with internal semigroups.** This concept, for monoids, was introduced in [11], based on ideas of [10]. We give a short introduction. For more details see [12] or [13].

If $B \subseteq \mathcal{P}(A^+)$ is a Boolean algebra of languages, then dually, we have a continuous quotient $q : \beta(A^+) \to X_B$. If the Boolean algebra $B$ is closed under quotients, then $q[A^+]$ has a natural semigroup structure, which is inherited from the duals of the homomorphisms $u^{-1}(\cup)v^{-1} : B \to B$. Moreover, the restriction of $q$ to $A^+$ induces a homomorphism $A^+ \to q[A^+]$ onto a dense subset of $X_B$. The pair $(q[A^+], X_B)$ encodes the essential information about the Boolean algebra closed.

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*An ultrafilter is a proper maximal nonempty upset $x \subseteq B$ closed under binary meets.*
under quotients $B$. This is the reasoning motivating the definition of a Boolean space with an internal semigroup.

A Boolean space with an internal semigroup, or BiS for short, is a pair $(S, X)$ where $S$ is a semigroup densely contained in a Boolean space $X$, and such that the natural actions of $S$ on itself extend to continuous endomorphisms of $X$, that is, for each $s \in S$, there are continuous functions $\lambda_s, \rho_s : X \to X$ such that the following diagrams commute:

$$
\begin{array}{ccc}
S & \xrightarrow{\cdot s} & X \\
\downarrow & & \downarrow \\
S & \xrightarrow{s} & X \\
\end{array}
\begin{array}{ccc}
S & \xrightarrow{s \cdot (\cdot)} & X \\
\downarrow & & \downarrow \\
S & \xrightarrow{\cdot (\cdot) s} & X \\
\end{array}
$$

The prime example of a BiS is the pair $(A^+, \beta(A^+))$, which corresponds to taking $B = \mathcal{P}(A^+)$. A morphism of BiSs, $f : (S, X) \to (T, Y)$, is a continuous map $f : X \to Y$ so that $f$ restricts to a semigroup homomorphism $f \upharpoonright : S \to T$.

A language $L \subseteq A^+$ is recognized by the BiS $(S, X)$ if there exists a morphism $f : (A^+, \beta(A^+)) \to (S, X)$ and a clopen subset $C \subseteq X$ such that $L = f^{-1}(C) \cap A^+$. Notice that each semigroup homomorphism $A^+ \to S$ completely determines a morphism of BiSs $(A^+, \beta(A^+)) \to (S, X)$. It is not hard to verify that the set of all languages recognized by a BiS via a fixed homomorphism is a Boolean algebra closed under quotients.

Logic on words. As the name suggests, logic on words is meant to express properties of words. We consider two kinds of variables: first-order and second-order variables. Intuitively, first-order variables provide information about positions in a word, while the second-order variables stand for sets of positions. First-order variables are denoted by $x, x_1, x_2, \ldots$, and the second-order ones by $X, X_1, X_2, \ldots$. There are the following three types of atomic formulas:

- if $R \subseteq \mathbb{N}^k$, then $R(x_1, \ldots, x_k)$ is a (uniform) $k$-ary numerical predicate (expressing that “the tuple of positions $(x_1, \ldots, x_k)$ belongs to $R$”);

- if $a \in A$, then $a(x)$ is a letter predicate (expressing that in the “position $x$ there is an $a$”);

- if $x$ is a first-order variable and $X$ is a second-order variable, then $X(x)$ is an atomic formula (expressing that “$x$ belongs to $X$”).

Then, Boolean combinations of formulas are formulas and if $\phi$ is a formula, then both $\exists x \phi$ and $\forall X \phi$ are formulas. It is well known that every monadic second-order sentence is equivalent, over words, to a formula of the form $\exists X_1 \cdots \exists X_N \phi(X_1, \ldots, X_N)$, for some first-order formula $\phi$ with free variables in $\{X_1, \ldots, X_N\}$. To interpret formulas with $N$ second-order free variables, one usually considers words over the extended alphabet $A \times 2^N$. For instance, if $N = 1$, then the word $(a, 1)(b, 0)(b, 1)(b, 0)(a, 1) \in (A \times 2)^+$ encodes the word $w = abbba$ with the only second-order variable interpreted in the set of odd positions of $w$. Thus, every formula $\phi = \phi(X_1, \ldots, X_N)$ with free variables in $\{X_1, \ldots, X_N\}$ defines a language $L_\phi(X_1, \ldots, X_N) \subseteq (A \times 2^N)^+$, and the language over $A^+$ definable by $\exists X_1 \cdots \exists X_N \phi(X_1, \ldots, X_N)$ consists of all the words $w$ for which there is an interpretation of the free variables satisfying $\phi$. In other words, $T = A^+$ is the set of all structures for logic on words in $A$, $S = (A \times 2^N)^+$ is the set of all MSO models in $N$ free variables on words in $A$, and the $\lambda$-morphism $\pi_N : (A \times 2^N)^+ \to A^+$, given by the projection $A \times 2^N \to A$, gives rise to existential quantification in the sense that $L_{\exists X_1 \cdots \exists X_N \phi(X_1, \ldots, X_N)} = \pi_N[L_{\phi(X_1, \ldots, X_N)}]$. More generally, given a language $L \subseteq (A \times 2^N)^+$, we denote $L_{\exists N} = \pi_N[L]$.  

4
3 Some power constructions

Compact Hausdorff and Boolean spaces. The power of a compact Hausdorff spaces is the so-called Vietoris space. Vietoris is a covariant endofunctor on compact Hausdorff spaces which restricts to the category of Boolean spaces. At the level of objects it assigns to a space \(X\) the set of all its closed subsets, denoted \(\mathcal{V}(X)\) and called the Vietoris space of \(X\), equipped with the topology generated by the sets of the form

\[
\diamond U = \{ C \in \mathcal{V}(X) \mid C \cap U \neq \emptyset \} \quad \text{and} \quad \Box U = \{ C \in \mathcal{V}(X) \mid C \subseteq U \},
\]

where \(U \subseteq X\) ranges over all open subsets of \(X\). In the case where \(X\) is a Boolean space, taking \(\diamond U\) and \(\Box U\) for \(U\) clopen gives a subbasis of clopen subsets for \(\mathcal{V}(X)\). For more details see \[15\]. Note that, since \(X\) is Hausdorff, each singleton is closed and thus the map \(i_X : X \to \mathcal{V}(X)\) sending each \(x \in X\) to \(\{x\}\) is well defined. Further note that, for any open \(U \subseteq X\), we have

\[
\diamond U \cap \text{im}(i_X) = i_X[U] \quad \text{and} \quad \Box U \cap \text{im}(i_X) = i_X[U]
\]

so that \(X\) is homeomorphically embedded in \(\mathcal{V}(X)\) via the map \(i_X\). When the space \(X\) is clear from the context, we denote this embedding simply by \(i\).

On morphisms, the Vietoris functor acts as follows: if \(g : Z \to X\) is a continuous function, then so is

\[
\mathcal{V}(g) : \mathcal{V}(Z) \to \mathcal{V}(X), \quad C \mapsto g[C].
\]

Indeed, routine computations show that, for an open subset \(U \subseteq X\), we have

\[
\mathcal{V}(g)^{-1}(\diamond U) = \diamond (g^{-1}(U)) \quad \text{and} \quad \mathcal{V}(g)^{-1}(\Box U) = \Box (g^{-1}(U)).
\]

On the other hand, given a continuous map \(f : Z \to Y\), taking preimages also defines a function between the corresponding Vietoris spaces, but in a contravariant way. A natural question is then under which conditions the function \(f^* := f^{-1} : \mathcal{V}(Y) \to \mathcal{V}(Z)\) is continuous.

Proposition 3.1. Let \(f : Z \to Y\) be a continuous function between compact Hausdorff spaces. Then, the following are equivalent:

(a) \(f^* : \mathcal{V}(Y) \to \mathcal{V}(Z)\) is continuous,

(b) \(f^* \circ i : Y \to \mathcal{V}(Z)\) is continuous,

(c) \(f\) is open.

Proof. Since \(Y\) is homeomorphically embedded in \(Y\) via the map \(i\), it is clear that \((a)\) implies \((b)\). The remainder of the proposition is essentially a consequence of the fact that, for any set map \(f\), the forward image under \(f\) is lower adjoint to the inverse image under \(f\). That is,

\[
\forall S \subseteq Y, T \subseteq Z \quad ( f[T] \subseteq S \iff T \subseteq f^{-1}(S) ).
\]

Using this, we see first of all that, for compact Hausdorff spaces, the map \(f^*\) is always continuous with respect to the ‘box-part’ of the topology. That is, for any open \(U \subseteq Z\) and closed \(K \subseteq Y\), we have

\[
K \in (f^*)^{-1}(\Box U) \iff f^{-1}(K) \subseteq U \iff U^c \subseteq (f^{-1}(K))^c = f^{-1}(K^c) \iff f[U^c] \subseteq K^c \iff K \subseteq (f[U^c])^c.
\]
Now, since \( f \) is continuous, \( Y \) is compact, and \( Z \) is Hausdorff, it follows that \( f \) is a closed mapping and thus \( (f[U^c])^c \) is open. That is, we have shown that

\[
(f^* \restriction \Diamond U) = \Diamond (f[U^c]).
\]

Similarly, we have

\[
K \in (f^* \restriction \Diamond U) \iff f^{-1}(K) \cap U \neq \emptyset \iff U \subseteq (f^{-1}(K))^c = f^{-1}(K^c) \iff f[U] \subseteq K^c \iff K \cap f[U] = \emptyset.
\]

Now, if \( f \) is open, then the above calculation shows \( (f^* \restriction \Diamond U) = \Diamond f[U] \) and thus that \((a)\) and \((b)\) hold. Conversely, if \( f \circ i \) is continuous, then, for any open \( U \subseteq Z \), the set

\[
(f \circ i)^{-1} (\Diamond U) = \{ y \in Y \mid \{ y \} \cap f[U] \neq \emptyset \} = f[U]
\]

must be open.

In particular we have the following:

**Corollary 3.2.** Let \( X, Y \) and \( Z \) be compact Hausdorff spaces and \( f : Z \to Y \) and \( g : Z \to X \) be continuous functions. If \( f \) is an open map, then

\[
h = \mathcal{V}(g) \circ f^* \circ i : Y \to \mathcal{V}(X), \quad y \mapsto h(y) = g[f^{-1}(\{ y \})]
\]

is continuous. Moreover, for every open subset \( U \subseteq X \), the equality \( h^{-1}(\Diamond U) = f[g^{-1}(U)] \) holds.

**Proof.** The fact that \( h \) is continuous follows immediately from Proposition \( 3.1 \). Moreover, for an open subset \( U \subseteq X \), we have:

\[
h^{-1}(\Diamond U) = (f^* \circ i)^{-1}(\mathcal{V}(g)^{-1}(\Diamond U)) = (f^* \circ i)^{-1}(\Diamond g^{-1}(U)) = f[g^{-1}(U)].
\]

In the next proposition we show that every continuous map \( Y \to \mathcal{V}(X) \) arises from a composition as in Corollary \( 3.2 \).

**Proposition 3.3.** Let \( X \) and \( Y \) be compact Hausdorff spaces and \( h : Y \to \mathcal{V}(X) \) be a continuous function. Then, the subspace

\[
Z = \{ (y, x) \in Y \times X \mid x \in h(y) \}
\]

of \( Y \times X \) is compact and Hausdorff. In particular, \( h \) is of the form \( \mathcal{V}(g) \circ f^* \circ i \), where \( f \) and \( g \) are the restrictions to \( Z \) of the projections to \( Y \) and \( X \), respectively.

**Proof.** The space \( Z \) is compact Hausdorff if \( Z \) is a closed subspace of \( Y \times X \). We show that \( Z^c \) is open. Let \( (y, x) \notin Z \). Then, \( x \notin h(y) \) and since \( h(y) \subseteq X \) is closed and \( X \) is compact and Hausdorff (and thus, regular), there exist disjoint open subsets \( U, V \subseteq X \) so that \( x \in U \) and \( h(y) \subseteq V \). Since \( h \) is continuous, it follows that \( h^{-1}(\Diamond V) \times U \) is an open neighborhood of \( (y, x) \) contained in \( Z^c \).

We finish this section with a characterization of the open maps between Boolean spaces. Recall that, in general, a lower adjoint of a map \( \alpha : P \to Q \) of posets is a map \( \alpha_* : Q \to P \) satisfying

\[
\forall p \in P, \ q \in Q \quad ( \alpha_* (q) \leq p \iff q \leq \alpha(p) )
\]

**Proposition 3.4.** Let \( B \) and \( C \) be Boolean algebras, with duals \( Y \) and \( Z \), respectively. Then, a continuous function \( f : Z \to Y \) is open if and only if the dual map \( \alpha : B \to C \) has a lower adjoint.
Lemma 3.7. We do have the next result. A map between the corresponding powersets, which in general is not a homomorphism. However, they are equivalent:

\[ f[\hat{c}] = \bigcap \{ \hat{b} \mid b \in B, f[\hat{c}] \subseteq \hat{b} \}. \]

Thus, \( f \) is open exactly when the set

\[ \{ \hat{b} \mid b \in B, f[\hat{c}] \subseteq \hat{b} \} \] (7)

has a minimum for inclusion. On the other hand, the fact that \( \alpha \) and \( f \) are dual to each other, translates to the fact that, for every \( b \in B \) and \( c \in C \), we have

\[ f[\hat{c}] \subseteq \hat{b} \iff \hat{c} \subseteq f^{-1}(\hat{b}) = \alpha(\hat{b}) \iff c \leq \alpha(b). \]

Therefore, (7) has a minimum if and only if \( \{ b \in B \mid c \leq \alpha(b) \} \) does. But this amounts to saying that \( \alpha \) admits a lower adjoint. \( \square \)

In particular, since every function \( f : S \to T \) between sets \( S \) and \( T \) induces a complete homomorphism \( f^{-1} : \mathcal{P}(T) \to \mathcal{P}(S) \) between complete Boolean algebras, thus having a lower adjoint, we have the following:

**Corollary 3.5.** For every map of sets \( f : S \to T \), the map \( \beta f : \beta S \to \beta T \) is open.

**Remark 3.6.** We remark that, so far, we proved that, given a set map \( f : S \to T \) and a continuous function \( g : \beta T \to X \) the Boolean algebra generated by the subsets of the form \( f[g^{-1}(U)] \), for \( U \in \text{Clop}(X) \), is precisely \( h^{-1}[\text{Clop}(\mathcal{V}(X))] = \{ h^{-1}(V) \mid V \in \text{Clop}(\mathcal{V}(X)) \} \), where \( h = \mathcal{V}(g) \circ f^* \circ i \).

**Semigroups.** We start by recalling that, given a semigroup \( S \), the set \( \mathcal{P}(S) \) of its subsets is equipped with a semigroup structure given by pointwise multiplication:

\[ Q_1 \cdot Q_2 = \{ s_1s_2 \mid s_1 \in Q_1, s_2 \in Q_2 \}, \]

for every subsets \( Q_1, Q_2 \subseteq S \). In particular, there is an embedding of semigroups \( i_S : S \hookrightarrow \mathcal{P}(S) \) given by \( i_S(s) = \{ s \} \). When \( S \) is clear from the context, we just write \( i \). Notice that \( \mathcal{P}(S) \) also admits a monoid structure, with the neutral element being the empty set, but we are only concerned with the semigroup structure of \( \mathcal{P}(S) \). Taking powers defines an endofunctor on semigroups. Indeed, for a homomorphism \( g : S \to T \), taking forward images defines a homomorphism

\[ \mathcal{P}(g) : \mathcal{P}(S) \to \mathcal{P}(T), \quad Q \mapsto g[Q]. \]

Of course, the set \( \mathcal{P}_{\text{fin}}(S) \) consisting of the finite subsets of \( S \) forms a subsemigroup of \( \mathcal{P}(S) \), and for a homomorphism \( g : S \to T \), \( \mathcal{P}(g) \) restricts to a homomorphism \( \mathcal{P}_{\text{fin}}(g) : \mathcal{P}_{\text{fin}}(S) \to \mathcal{P}_{\text{fin}}(T) \).

This observation will be useful in Section 5.

On the other hand, if \( f : S \to T \) is a semigroup homomorphism, then taking preimages defines a map between the corresponding powersets, which in general is not a homomorphism. However, we do have the next result.

**Lemma 3.7.** Let \( f : B^+ \to A^+ \) be a homomorphism between free semigroups. Then, the following are equivalent:

(a) \( f^* : \mathcal{P}(A^+) \to \mathcal{P}(B^+) \) is a homomorphism,
(b) \( f^* \circ i : A^+ \to \mathcal{P}(B^+) \) is a homomorphism,

(c) \( f \) is an lp-morphism.

Proof. The equivalence between [a] and [b] is trivial. Suppose that \( f \) is length-preserving. Then, given \( w_1, w_2 \in A^+ \) and \( u \in B^+ \), we have that \( u \in f^* \circ i(w_1w_2) \) if and only if \( f(u) = w_1w_2 \). Since \( f \) is length-preserving, this happens if and only if \( u \) admits a factorization \( u = u_1u_2 \) satisfying \( f(u_1) = w_1 \) and \( f(u_2) = w_2 \), that is, \( u \in f^* \circ i(w_1) \cdot f^* \circ i(w_2) \). This proves that \( f^* \circ i \) is a homomorphism. Conversely, if \( f \) is not length-preserving, then there exists a letter \( b \in B \) so that \( f(b) \) may be written as \( aw \) for some \( a \in A \) and \( w \in A^+ \). Then, \( b \) belongs to \( f^* \circ i(aw) \) but not to \( f^* \circ i(a) \cdot f^* \circ i(w) \) and so, \( f^* \circ i \) is not a homomorphism.

Remark 3.8. Notice that for an lp-morphism \( f : B^+ \to A^+ \) and a word \( w \in A^+ \), the set \( f^{-1}(w) \) is finite. Thus, by Lemma 3.7, every such \( f \) defines a homomorphism of semigroups \( f^* : A^+ \to \mathcal{P}_{\text{fin}}(B^+) \).

The following is a particular case of a well known result in semigroup theory (see e.g. [16, Chapter XVI, Proposition 1.1]).

Proposition 3.9. Let \( h : T \to \mathcal{P}(S) \) be a homomorphism. Then, the set \( R = \{(t, s) \mid t \in T, s \in h(t)\} \) is a subsemigroup of \( T \times S \). In particular, \( h \) is of the form \( \mathcal{P}(g) \circ f^* \circ i \), where \( f \) and \( g \) are the restrictions to \( R \) of the projections to \( S \) and \( T \), respectively.

4 The power construction for BiSs and MSO

The power construction for BiSs. Combining the power constructions of Section 3 provides a power construction that applies to BiSs:

Definition 4.1 ([12, Theorem III.1]). Let \( (S, X) \) be a BiS. We define the Vietoris of \( (S, X) \) to be the BiS
\[
\mathcal{V}(S, X) = (\mathcal{P}_{\text{fin}}(S), \mathcal{V}(X))
\]
equipped with the actions
\[
\lambda_Q : \mathcal{V}(X) \to \mathcal{V}(X), \quad C \mapsto \bigcup_{s \in Q} \lambda_s[C] \quad \text{and} \quad \rho_Q : \mathcal{V}(X) \to \mathcal{V}(X), \quad C \mapsto \bigcup_{s \in Q} \rho_s[C],
\]
for each \( Q \in \mathcal{P}_{\text{fin}}(S) \).

We remark that, although the fact that \( \mathcal{V}(S, X) \) is a BiS only appears explicitly in [12], this is implicitly present already in [11].

We will say that a morphism \( h : (B^+, \beta(B^+)) \to (A^+, \beta(A^+)) \) of BiSs is length-preserving provided its restriction to \( B^+ \) is length-preserving. Using Corollary 3.2 and Lemma 3.7, and taking into account Remark 3.8, we obtain:

Proposition 4.2. Let \( A \) and \( B \) be alphabets and \( (S, X) \) a BiS. Then, for every span
\[
\begin{array}{c}
(B^+, \beta(B^+)) \\
\downarrow f \\
(A^+, \beta(A^+)) \\
\downarrow g \\
(S, X)
\end{array}
\]

with \( f \) length preserving, the map \( h = \mathcal{V}(g) \circ f^* \circ i \) is a morphism of \( \text{BiSs} \).

**Proposition 4.3.** Let \((S, X)\) and \((T, Y)\) be \( \text{BiSs} \) and \( h : (T, Y) \to \mathcal{V}(S, X) \) a morphism. Then, there is a \( \text{BiS} \) \((R, Z)\) and morphisms \( f : (R, Z) \to (T, Y) \) and \( g : (R, Z) \to (S, X) \) so that \( h = \mathcal{V}(g) \circ f^* \circ i \).

**Proof.** We take \( Z = \{(y, x) \mid y \in Y, x \in h(y)\} \) and \( R = \{(t, s) \mid t \in T, s \in h(s)\} \). By Propositions 3.3 and 3.9 we already know that \( Z \) and \( R \) are, respectively, a Boolean space and a semigroup that do the job. Thus, it remains to show that \((R, Z)\) is a \( \text{BiS} \). Since the pair \((T \times S, Y \times X)\) has a \( \text{BiS} \) structure, we only need to prove that \( R \) is dense in \( Z \). Let \( V \subseteq Y \) and \( U \subseteq X \) be open subsets and \((y, x) \in (V \times U) \cap Z \). We need to show that \((V \times U) \cap R\) is nonempty. Since \( h \) is continuous, \( h^{-1}(\diamond U) \cap V \) is an open subset of \( Y \), and it is nonempty as it contains \((y, x)\). Since \( T \) is dense in \( Y \), there exists an element \( t \in h^{-1}(\diamond U) \cap V \cap T \). In particular, \( h(t) \cap U \neq \emptyset \). Since \( h \) restricts to a semigroup homomorphism \( T \to \mathcal{P}_{\text{fin}}(S) \), this yields the existence of \( s \in h(t) \cap U \cap S \) as required.

As a consequence we obtain the desired result on recognition.

**Corollary 4.4.** Let \((S, X)\) be a \( \text{BiS} \). Then, a language \( L \subseteq A^{+} \) is recognized by \( \mathcal{V}(S, X) \) if and only if it is a Boolean combination of forward images under \( \ell p \)-morphisms of languages recognized by \((S, X)\).

**Proof.** The backwards implication is a trivial consequence of Remark 3.6 and Proposition 3.3. Conversely, let \( h : (A^{+}, \beta(A^{+})) \to \mathcal{V}(S, X) \) be a morphism recognizing \( L \subseteq A^{+} \). Again by Proposition 3.3 there exists a \( \text{BiS} \) \((R, Z)\) and morphisms \( f : (R, Z) \to (A^{+}, \beta(A^{+})) \) and \( g : (R, Z) \to (S, X) \) so that \( h = \mathcal{V}(g) \circ f^* \circ i \). The only thing to notice is that \( R = \{(u, s) \mid u \in A^{+}, s \in h(u)\} \) is the subsemigroup of \( A^{+} \times S \) generated by the finite alphabet \( B = \{(a, s) \mid a \in A, s \in h(a)\} \). Therefore, we have a unique morphism of \( \text{BiSs} \) \( \pi : (B^{+}, \beta(B^{+})) \to (R, Z) \) mapping \( b \in B \) to \( b \in R \), and this morphism is such that \( f \circ \pi \) is length-preserving. The intended conclusion follows then from Remark 3.6.

We remark that this result takes care of the first stage of set-theoretic recognition in the first-order setting [11, 4]. The complication in the first-order setting stems from the fact that the first-order models do not form a semigroup. Here we treat the considerable easier case of monadic second-order quantifiers.

**Monadic second-order existential quantification.** We address the following questions: Given a \( \text{BiS} \) \((S, X)\), which \( \text{BiS} \) recognizes the Boolean algebra generated by the languages of the form \( L_{\exists N} \) where \( L \) is recognized by \((S, X)\)? Does it recognize much more? Unlike in the first-order case where an iterative construction is absolutely needed, we will see that, for second-order quantification, taking once the power of \((S, X)\) is enough to recognize every language \( L_{\exists N} \) for every \( N \in \mathbb{N} \), where \( L \) is recognized by \((S, X)\). In fact, since the projection \( \pi_{N} : (A \times 2^{N})^{+} \to A^{+} \) modeling the quantifier \( \exists_{N} \) is length-preserving, this essentially follows from the results above. As already mentioned, the corresponding problem for first-order quantifiers was considered in [11, 4] and it is much more delicate, as the universe of models of formulas with free first-order variables does not admit a semigroup structure.

**Proposition 4.5.** Let \((S, X)\) be a \( \text{BiS} \). If \((S, X) \) recognizes the language \( L \subseteq (A \times 2^{N})^{+} \), then \( \mathcal{V}(S, X) \) recognizes the language \( L_{\exists N} \subseteq A^{+} \). Conversely, if \( K \subseteq A^{+} \) if recognized by \( \mathcal{V}(S, X) \), then there exists a positive integer \( N \), an alphabet \( A' \subseteq A \) and a language \( L \subseteq (A' \times 2^{N})^{+} \) recognized by \((S, X)\) such that \( K = L_{\exists N} \).
there exists an onto homomorphism \( \pi \) big enough \( N \) language \( f \) where \( \text{FO} \) aperiodic semigroups. On the other hand, the languages recognizable by an aperiodic semigroup are semigroup if and only if it is obtained by second-order quantification of a language recognized by \( (S, X) \). In studying fragments of logic and not a single formula, we can always express the property “every letter belongs to \( B \)”. Conversely, let \( K \subseteq A^+ \) be a language recognized by \( \mathcal{V}(S, X) \). By the proof of Corollary \([14]\), there exists a finite alphabet \( B \subseteq A \times S \) such that \( K = f[L] \) for a language \( L \subseteq B^+ \) recognized by \( (S, X) \), where \( f : B \to A \) is the restriction of the projection \( A \times S \to A \). Take \( A' = f[B] \) and choose a big enough \( N \) so that each of the sets \( B \cap (\{a\} \times S) \), with \( a \in A' \), has at most \( N \) elements. Then, there exists an onto homomorphism \( \pi : (A' \times 2^N)^+ \to B^+ \) satisfying \( \pi_N = f \circ \pi \), and so, there is a language \( L' = \pi^{-1}(L) \subseteq (A' \times 2^N)^+ \) recognized by \( (S, X) \) and such that \( K = L'_{\exists N} \).

Remark 4.6. Observe that, if \( B' \subseteq B \) is an inclusion of alphabets, then a language \( L \subseteq (B')^+ \) can always be seen as a language over \( B \) via the inclusion \( (B')^+ \subseteq B^+ \). Nevertheless, the fact that \( L \) is recognized by \( (S, X) \) as a language over \( B' \) does not necessarily implies that it is recognized by \( (S, X) \) as a language over \( B \). On the other hand, the BiS \( (S \times 2, X \times \{0, 1\}) \), where \( 2 \) denotes the two-element semilattice, does recognize \( L \) as a language over \( B \). Since one is usually interested in studying fragments of logic and not a single formula, we can always express the property “every letter belongs to \( B'' \)”, and so, this is not really a constraint.

Notice that, Proposition \([15]\) implies that a language is recognized by the power of an aperiodic semigroup if and only if it is obtained by second-order quantification of a language recognized by aperiodic semigroups. On the other hand, the languages recognizable by an aperiodic semigroup are precisely those definable in \( \text{FO}(<) \) \([13, 14]\), and in turn, the second-order existential quantification of those yields \( \text{MSO}(<) \), a fragment of logic defining precisely the regular languages \([5, 9]\) (i.e., those recognized by a finite semigroup). We may thus derive the following:

**Corollary 4.7.** Every finite semigroup divides a power of an aperiodic one.

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