Conformal Models of Magnetohydrodynamic Turbulence

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ABSTRACT

Following the previous work of Ferretti and Yang on the role of magnetic fields in the theory of conformal turbulence, we show that non-unitary minimal model solutions to 2-dimensional magnetohydrodynamics (MHD) obtained by dimensional reduction from 3-dimensions exist under different (and more restrictive) conditions. From a 3-dimensional point of view, these conditions are equivalent to perpendicular flow, in which the magnetic and velocity fields are orthogonal. We also extend the analysis to the finite conductivity case and present some approximate solutions, whose connection to the exact ones of the infinite conductivity case is also discussed.

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1 Introduction

In recent years it has become clear that the proposal by Polyakov [1,2], that turbulent flow in two spatial dimensions may be understood in terms of certain non-unitary conformal field theories (CFT) (at least in the inviscid limit), has provided new insights into the problem of 2-d turbulence in general. A number of authors extended this work, and found minimal model solutions under a variety of different conditions [3-6]. In addition, generalizations to flows in the presence of boundaries have been studied [7,8]. More recently, there have been proposals concerning possible CFT solutions to turbulence even when viscosity is present [9]. There are also results concerning perturbations of the underlying CFT solutions to 2-d turbulence where the perturbations are either 2-dimensional in origin [10] or three-dimensional [11].

Despite the elegance of the CFT approach, one of its least attractive aspects is the appearance of an infinite number of possible solutions (the inviscid case). The existence of an infinite number of solutions is an indication that not all the physically relevant constraints are being imposed. In an effort to understand these issues further, it is useful to study a more complex turbulent system than that of an ordinary fluid in 2-dimensions. To this end, Ferretti and Yang [13] studied CFT solutions to an effectively 2-dimensional theory of ideal (inviscid, infinite conductivity) magnetohydrodynamical turbulence obtained by dimensional reduction from 3 dimensions. Here there is at least a possibility of obtaining more stringent conditions than those found in ordinary fluids. The single Navier-Stokes equation of the simple fluid is replaced by 4 coupled equations, so that the resulting Hopf equations are rather more complicated. However, the authors of [13] considered a self-consistent truncation of these equations, by setting $B = V = 0$, where $B$ and $V$ are scalars interpreted as third components of the magnetic and velocity fields of a 3-dimensional plasma.$^3$ In this limit, the CFT solutions obtained are different from those of ordinary Polyakov turbulence, but as found in [14] it is still apparent that one can easily generate a large (possibly infinite) number of such solutions via non-unitary minimal models. So in this respect, we are no closer to finding a ‘less dense’ set of solutions.

In this paper amongst other things we shall also consider CFT solutions to effective 2-d ideal magnetohydrodynamical turbulence, but with a different self-consistent trun-

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$^3$ This limit defines pure MHD in 2-dimensions.
cation than the one considered in [13]. Our choice is physically equivalent to so-called ‘perpendicular’ flow in 3-dimensions, in which the magnetic and velocity fields are constrained to be at right angles to one another. We find that the corresponding constraints on CFT solutions imply that they should be a subset of those found in usual Polyakov turbulence, and furthermore they are very much more stringent than those discussed previously in the literature. Indeed, in the first 80 or so non-unitary minimal model solutions describing turbulence with either constant enstrophy or energy [3], only 3 of these correspond to solutions of MHD turbulence in the limit mentioned. As well as ideal 2-d MHD, we also consider the situation with finite conductivity $\sigma$. Although this necessarily introduces dissipative effects, we show that it is possible to define a generalized vorticity which satisfies the steady Hopf equations, and upon which one can impose constant flux. The conditions in this case are so restrictive that thus far we have only found approximate solutions after searching through all non-unitary minimal models with $q < 500$.

The structure of the paper is as follows. In section 2, we discuss the equations of 3-d MHD when dimensionally reduced to 2-d, and briefly review the results of [13]. We then analyze CFT solutions under the perpendicular flow conditions mentioned above, when a variety of different flux constraints are imposed, and exhibit a number of exact solutions. In section 3, the Hopf equations of 2-d MHD turbulence with finite conductivity are studied together with flux constraints, and approximate solutions are given. We conclude in section 4 with some comments concerning the relation between the exact and approximate solutions found.
2 Minimal model solutions of 2-D MHD Turbulence

It is well known (see for example [12]) that an approximate description of 3-dimensional plasmas is given by Magnetohydrodynamic (MHD) equations, which combine Maxwell’s equations with those of hydrodynamics under the simplifying assumptions of low frequency, low temperature and infinite conductivity. Under the further assumption of incompressibility, the ideal MHD equations take the form [12]

\[
\begin{align*}
\nabla \cdot \mathbf{v} &= 0 \\
\nabla \cdot \mathbf{B} &= 0 \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\frac{1}{\rho_m} \nabla P + \frac{1}{c\rho_m} \mathbf{J} \times \mathbf{B} \\
\n\nabla \times (\mathbf{v} \times \mathbf{B}) &= \partial_t \mathbf{B} \\
\n\nabla \times \mathbf{B} &= \frac{4\pi}{c} \mathbf{J}
\end{align*}
\]

Here \( \mathbf{v} \) is the velocity field, \( \mathbf{B} \) is the magnetic field, \( \mathbf{J} \) the electric current, \( P \) the pressure, and \( \rho_m \) the mass density of the plasma. We can use the last equation to substitute for \( \mathbf{J} \) in the third (the Navier-Stokes equation). Then after taking the curl of the Navier-Stokes equation to get rid of the pressure term, the MHD equations become:

\[
\begin{align*}
\nabla \cdot \mathbf{v} &= 0 \\
\nabla \cdot \mathbf{B} &= 0 \\
\n\nabla \times (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) &= \frac{1}{4\pi\rho_m} \nabla \times (\mathbf{B} \cdot \nabla \mathbf{B}) \\
(\mathbf{B} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{B} &= \partial_t \mathbf{B}
\end{align*}
\]

In this form the MHD equations display some evident symmetry between the \( \mathbf{v} \) and \( \mathbf{B} \) fields. It is therefore reasonable to expect that \( \mathbf{B} \) will play a similar role to \( \mathbf{v} \) in some generalized MHD conformal turbulence scenario. This observation was exploited by Ferretti and Yang [13]. First they reduced the theory to an effectively two-dimensional one by requiring that all fields be independent of the \( z \)-coordinate:

\[
\partial_3 \mathbf{v} = 0, \quad \partial_3 \mathbf{B} = 0
\]
Equation (3) then implies that the first two components of \( \mathbf{v} \) and \( \mathbf{B} \) can be written as

\[
B_\alpha = \epsilon_{\alpha\beta} \partial_\beta A, \quad v_\alpha = \epsilon_{\alpha\beta} \partial_\beta \psi, \quad \alpha, \beta = 1, 2. \tag{4}
\]

The third components \( B_3 \equiv B \) and \( v_3 \equiv V \), together with the stream function \( \psi \) and magnetic potential \( A \) are all two-dimensional scalars.

If we now define, in analogy with vorticity \( \omega \), the “magnetic vorticity”

\[
\Omega \equiv \epsilon_{\alpha\beta} \partial_\alpha B_\beta = -\partial_\alpha \partial_\alpha A, \tag{5}
\]

and the two-dimensional operator

\[
\mathcal{A} \equiv \epsilon_{\alpha\beta} \partial_\beta A \partial_\alpha \tag{6}
\]

the 2-D MHD equations take the form [13]:

\[
\begin{align*}
\dot{\omega} + \epsilon_{\alpha\beta} \partial_\beta \psi \partial_\alpha \omega &= \frac{1}{4\pi \rho_m} \mathcal{A} \Omega \\
\dot{A} + \epsilon_{\alpha\beta} \partial_\beta \psi \partial_\alpha A &= 0 \\
\dot{V} + \epsilon_{\alpha\beta} \partial_\beta \psi \partial_\alpha V &= \frac{1}{4\pi \rho_m} \mathcal{A} B \\
\dot{B} + \epsilon_{\alpha\beta} \partial_\beta \psi \partial_\alpha B &= \mathcal{A} V
\end{align*} \tag{7}
\]

It is evident from eqn (7) that \( V \) and \( B \) are absent from the equations for \( \dot{\omega} \) and \( \dot{A} \). Hence, one can set \( B \) and \( V \) to zero self-consistently, and study a simplified set of equations.

We therefore begin by examining the meaning and possible solution of the first two equations in (7). Ferretti and Yang [13] pointed out that the additional term on the right hand side of the first equation destroys enstrophy conservation. However, the second equation implies that the analogous quantity

\[
G = \frac{1}{2} \int A^2 \ d^2 x \tag{8}
\]

is now conserved. One can also check that total energy

\[
E = \frac{1}{2} \int (v_\alpha v_\alpha + B_\alpha B_\alpha) \ d^2 x \tag{9}
\]

is a constant of the motion. Proceeding in analogy with ordinary Polyakov (hydrodynamic) turbulence [1,2], the authors of [13] interpreted \( \psi \) and \( A \) as primary fields in
some CFT and proceeded to derive constraints on such solutions by the requirements of steady solutions to the inviscid Hopf equations, i.e.

\[ \dot{A} = 0 \]  

and constancy of $A$-flux on scales $r \sim L$

\[ \langle \dot{A}(r)A(0) \rangle \approx r^0. \]  

$L$ being an infrared (IR) cutoff whose value is typically the size of the largest coherent motion in the system. Beginning with the second equation in (7)

\[ \dot{A} = -\epsilon_{\alpha\beta}\partial_\beta \psi \partial_\alpha A \]  

point-split regularization gives:

\[ \dot{A} \sim |a|^{2(\Delta_\chi - \Delta_\psi - \Delta_A + 1)}[L_{-2}\bar{L}_{-1}^2 - \bar{L}_{-2}L_{-1}^2] \chi, \]  

where $\chi$ is the minimal dimension field in the OPE of $\psi$ with $A$. This fixes the dimension of $\dot{A}$ as $2 + \Delta_\chi$. Thus the Hopf equation for $A$ implies

\[ \Delta_\chi > \Delta_\psi + \Delta_A - 1 \]  

while constancy of $A$-flux gives

\[ \Delta_\chi + \Delta_A = -2 \]  

Next we consider the relevance of the first equation in (7) in the CFT context. The basic idea is that stationary turbulence implies vanishing of $\dot{\omega}$ in the inviscid limit, with $\dot{\omega}$ defined through the Navier-stokes equation for $\omega$, (7). In the point-splitting scheme, this is implemented by the limit $a \to 0$, where $a$ is the usual ultraviolet cutoff, proportional to the viscosity. The first equation can be interpreted as a definition of $\dot{\omega}$:

\[ \dot{\omega} = \epsilon_{\alpha\beta}\partial_\beta \psi \partial_\alpha \partial^2 \psi - \frac{1}{4\pi\rho_m}\epsilon_{\alpha\beta}\partial_\beta A \partial_\alpha \partial^2 A \]  

After performing the regularization, operator product expansions (OPE’s), and differentiations, we obtain

\[ \dot{\omega} = |a|^{2(\Delta_\psi - 2\Delta_A)}\mathcal{L}\phi - |a|^{2(\Delta_\phi - 2\Delta_\psi)}\mathcal{L}\phi = 0, \]  

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where in eqn (17), $\mathcal{L} = [L_2 L_1^2 - \bar{L}_2 L_1^2]$ and $A \times A = [\varphi] + \cdots, \psi \times \psi = [\phi] + \cdots$ where $\varphi$ and $\phi$ are minimal dimension fields. The Hopf equation for $\omega$ thus dictates two Hopf conditions for $\psi$ and $A$ respectively:

$$\Delta \varphi > 2\Delta A, \quad \Delta \phi > 2\Delta \psi$$

These are then the two additional constraints coming from the first equation in (7) that any prospective CFT solution of 2-D MHD must satisfy. This brings the total number of constraints to four (three Hopf conditions and one constant flux constraint).

As far as the conformal dimension of $\dot{\omega}$ is concerned, it is determined in the limit $a \to 0$ by whichever is the lower dimension field, $\phi_m$, between the minimal dimension fields $\phi$ and $\varphi$ in the OPE $\psi \times \psi$ and $A \times A$ respectively:

$$\Delta \dot{\omega} = 2 + \Delta \phi_m$$

The simplest solution found by Ferretti and Yang is the (2,13) non-unitary minimal model, which comprises six primary fields $\psi_{1,1}$ to $\psi_{1,6}$. Interestingly, there is a one-to-one correlation between these and the six fields $I$, $\psi$, $\phi$, $A$, $\chi$ and $\varphi$ respectively. As an immediate physical consequence, one can compute the kinetic and magnetic energy spectra. These are given by

$$E_k \sim k^{1+4\Delta \psi}, \quad E_m \sim k^{1+4\Delta A}. \quad (20)$$

For the (2,13) model, we have $\Delta \psi = -5/13$ and $\Delta A = -12/13$, giving

$$E_k \sim k^{-7/13}, \quad E_m \sim k^{-35/13}. \quad (21)$$

It is not difficult to find more minimal models satisfying the above constraints. Indeed, because the latter are not very stringent, one encounters a proliferation of solutions with a correspondingly wide range of spectra (see Rahimi et al [14]). The situation here is somewhat worse than that of Polyakov turbulence; for example there are five models of the form $(2, 2n+1)$: $(2, 13)$, $(2, 17)$, $(2, 19)$, $(2, 23)$ and $(2, 27)$ with completely different spectra predictions in each case. It seems evident, therefore, that some more stringent constraint is to be sought that would limit the number of possible solutions and make definite predictions regarding the spectra.

As a step towards placing further restrictions on solutions of MHD turbulence we shall consider a special limit of the MHD equations and derive additional constraints on
possible CFT solutions. We set $A = V = 0$ in the equations of motion (7). Note that this implies that the fields $v$ and $B$ are of the form $v = (v^1, v^2, 0)$ and $B = (0, 0, B)$, i.e. the magnetic field is perpendicular to the plane of fluid flow. These so called ‘perpendicular flow’ conditions are familiar in the study of 3-dimensional MHD [15]. In the 2-dimensional case, the resulting equations of motion then take the particularly simple and symmetric form:

$$\partial_t \omega + \epsilon_{\alpha\beta}(\partial_{\beta}\psi)(\partial_{\alpha}\omega) = 0$$

$$\partial_t B + \epsilon_{\alpha\beta}(\partial_{\beta}\psi)(\partial_{\alpha}B) = 0$$

Eqn (22) is just the usual inviscid Navier-Stokes equation of hydrodynamic turbulence whilst eqn (23) is its magnetic analogue. Hence any solution of perpendicular flow MHD has to satisfy the usual Polyakov constraints. We further note that eqn (22) is exactly of the same form as the equation for $A$, with $A$ being replaced by $B$. Hence they have the same solutions. We therefore conclude that, as would perhaps be expected, our solutions will form a subset of both the hydrodynamic set of solutions and magnetic ones mentioned above. This is a rather stringent requirement and indeed we found that solutions are somewhat scarce. We can categorize these in terms of different possible (and mutually exclusive) constant flux constraints: constant enstrophy and constant energy, (the latter allows for energy cascades from small to large scales as envisaged by Kraichnan [16], Leith [17] and Batchelor [18]). This arises from the observation that enstrophy and kinetic energy can now both be conserved (since $B_\alpha = 0$). We also now have a third conserved quantity, the integral of the square of the magnetic field $B$. This allows us to impose a constant $B$-flux constraint as well, leading to the following possibilities:

(a) **Constant enstrophy flux + constant $B$-flux**

$$\Delta_\psi + \Delta_\phi = -3$$

$$\Delta_B + \Delta_\chi = -2$$

(b) **Constant energy flux + constant $B$-flux**

$$\Delta_\psi + \Delta_\phi = -2$$

$$\Delta_B + \Delta_\chi = -2$$
where $\psi \times \psi = [\phi] + \cdots$, $B \times \psi = [\chi] + \cdots$. In either case the Hopf conditions $\dot{\omega} = 0$ and $\dot{B} = 0$ further require that,

$$\Delta_\phi > 2\Delta_\psi \quad (26)$$
$$\Delta_\chi > \Delta_B + \Delta_\psi - 1 \quad (27)$$

We note that the second inequality, eqn (??), implies together with the $B$-flux constraint eqn (25), that $\Delta_\chi > -3/2 + \Delta_\psi/2$.

These are not the only possibilities. Cateau et al [6] also discuss the so-called Saffman (or discontinuity of vorticity) constraint, which has been proposed as an alternative to the constant enstrophy or constant energy conditions. In this case the conformal dimensions $\Delta_\psi$ and $\Delta_\phi$ satisfy

$$5\Delta_\psi + \Delta_\phi + 9 = 0 \quad (28)$$

We surveyed Polyakov models in the range $q \leq 500$ for constant enstrophy (44 models) and for constant energy (41 models) as listed in the paper by Lowe [3]. We also considered the Saffman solutions given in Cateau et al [6] (9 models).

### Constant enstrophy solutions

We only found one solution in this category in the range considered, the $(21,166)$ model. This is a large model with 1650 primary fields! The identifications are as follows: $\psi \equiv \psi_{4,31}, \Delta_\psi \approx -1.496$, $\phi \equiv \psi_{7,55}, \Delta_\phi \approx -1.504$ and $B \equiv \psi_{7,61}, \Delta_B \approx -0.492$, $\chi \equiv \psi_{10,79}, \Delta_\chi \approx -1.508$. Here $\chi$ is actually the minimal dimension field in the model.

### Constant energy solutions

In this case we note an interesting possibility, namely, if we take a solution of the constant energy constraint equation and identify $B$ with $\phi$ and $\chi$ with $\psi$ then we obtain a simultaneous solution of the second constraint equation. With this identification the Hopf condition $\Delta_\chi > \Delta_B + \Delta_\phi - 1$ implies that $\Delta_\phi < 1$, which is automatically satisfied. However we still have to fulfill the requirement that $\chi$ (or $\psi$) is the minimal dimension field in the OPE of $B$ with $\psi$ (ie $\phi$ with $\psi$). This is a non-trivial condition, and
Table 1: Primary fields and their identifications in (2, 17).

| Field | $\psi_{1,1}$ | $\psi_{1,2}$ | $\psi_{1,3}$ | $\psi_{1,4}$ | $\psi_{1,5}$ | $\psi_{1,6}$ | $\psi_{1,7}$ | $\psi_{1,8}$ |
|-------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Interpretation | $I$ | $B$ | $\varphi$ | $?$ | $\psi$ | $\chi$ | $\phi$ |
| $\Delta \approx$ | 0 | $-0.41$ | $-0.76$ | $-1.05$ | $-1.29$ | $-1.47$ | $-1.59$ | $-1.65$ |

it seriously limits the possible solutions. Amongst all of the constant energy minimal model solutions ($p, q$) with $q < 500$ (see Lowe [3]) only (10, 59) and (59, 344) satisfy this. In the first case we have $\psi \equiv \psi_{1,6}$, $\phi \equiv \psi_{1,5}$ and in the second, $\psi \equiv \psi_{6,135}$, $\phi \equiv \psi_{5,29}$. In both cases we have $B \equiv \phi$, $\chi \equiv \psi$, where $\psi$ (or $\chi$) is the lowest dimension operator in the model. We also note that in both models $\psi$ and $\phi$ have roughly the same dimension, each being very close to 1. Indeed the OPE constraints tend to indicate that this is a generic feature of any similar solutions. No other solutions were found in the range investigated.

**Saffman solutions**

Here things are somewhat easier. If we consider CFT solutions where the Saffman conditions (eqn. (27)) are imposed on the enstrophy, the very first model (2, 17) provides a solution to the MHD case. This is a small model with only 8 fields. The field content and identifications are given in Table 1. Here $\varphi$ is the minimal field in the OPE $B \times B$. Another solution was the (5, 46) model, with $\psi \equiv \psi_{1,13}$, $\phi \equiv \psi_{1,9}$ and $B \equiv \psi_{1,17}$, $\chi \equiv \psi_{1,9}$. Thus we have found that two of the 9 models listed in [6] are also solutions to the corresponding MHD case. Compared to the previous cases of constant energy enstrophy (where we only found 3 models out of a list of about 80), conformal solutions to MHD turbulence with Saffman conditions imposed are easier to find.

**Energy spectra**

One can easily work out the immediate physical consequence of these solutions, namely the energy spectra. Here kinetic energy $1/2 \int (v_\alpha v_\alpha) \, d^2x$ and magnetic energy $1/2 \int B^2 \, d^2x$ are each conserved independently. We define their respective spectra
Table 2: All exact minimal model solutions of perpendicular flow turbulent MHD and their spectra, with $q < 500$

We summarize our results in Table 2, which gives the dimensions of all the relevant fields as well as the spectra predictions of each model. We note in particular the approximate Kolmogorov-like kinetic energy spectrum of the constant energy models. This is due to the dimension of $\psi$ being close to 1 in these solutions, as remarked earlier. It is also of interest that the spectra of the other models are roughly consistent with each other.

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**Approximate solutions**

As a matter of interest, let us mention some approximate solutions of conformal turbulence. They might be relevant if one cannot easily find an exact solution, as in the
finite conductivity case to be discussed below. Falkovich et al [4] listed a number of minimal models with a pair of fields satisfying the dimensional constraints of constant enstrophy but not the minimal field requirement of the OPE. We consider the example of the (13, 107) model for the purpose of illustration. In this model, \( \psi \equiv \psi_{5,39} \) and \( \phi \equiv \psi_{4,34} \) with \( \Delta_\psi + \Delta_\phi = -3 \). However, \( \phi \) is not quite the minimal field in the OPE \( \psi \times \psi \), but has dimension very close to it. Let us denote the true minimal field by \( \varphi \). Thus,

\[
\psi \times \psi \sim [\varphi] + \cdots, \quad \Delta_\varphi > 2\Delta_\psi.
\]

In the \( a \to 0 \) limit, \( \dot{\omega} \) becomes a level 2 field in \([\varphi]\) and therefore acquires dimension \( 2 + \Delta_\varphi \). The enstrophy flux is then given by

\[
< \dot{\omega} \omega > \sim L^\Delta,
\]

where

\[
\Delta = \Delta_\varphi + \Delta_\omega = (2 + \Delta_\varphi) + (1 + \Delta_\psi) = \Delta_\varphi - \Delta_\phi,
\]

using \( \Delta_\psi + \Delta_\phi = -3 \) in the last line of eqn (34). Hence, if \( \Delta \) is sufficiently small, we can obtain a very good approximation to constant enstrophy flux.

For the (13, 107) model, we have

\[
\psi \times \psi = \psi_{5,39} \times \psi_{5,39}
\]

\[
= \sum_{i=1}^{9} \sum_{j=1}^{77} \psi_{i,j}
\]

\[
= [\psi_{5,41}] + \cdots
\]

(Here the summations take place in increments of 2). Hence \( \varphi = \psi_{5,41} \), with \( \Delta_\varphi \approx -1.587 \). Moreover \( \Delta_\phi = \Delta_{4,34} \approx -1.553 \) so \( \Delta \approx -0.03 \).

One could also scan such solutions for approximate solutions of the further \( B \)-flux constraint

\[
\Delta_B + \Delta_\chi = -2, \quad B \times \psi \sim [\chi] + \cdots
\]
The (13,107) model again furnishes a good example, with \( B = \psi_{5,35}, \Delta_B \approx -1.562 \) and \( \chi = \psi_{4,32}, \Delta_\chi \approx -0.438 \). Checking the OPE \( B \times \psi \), we find

\[
B \times \psi = \psi_{5,35} \times \psi_{5,39} = \sum_{i=1}^{9} \sum_{j=1}^{73} \psi_{i,j} = [\psi_{5,41}] + \cdots
\]  
\tag{37}

(Again the summations are in step of 2). Thus again \( B \times \psi \sim [\varphi] \).

Repeating the above argument, we obtain for the \( B \)-flux

\[
< \dot{BB} > \sim L^{\Delta'}
\]  
\tag{38}

where

\[
\Delta' = \Delta_\dot{B} + \Delta_B = (2 + \Delta_{\varphi}) + \Delta_B
\]  
\tag{39}

Then, using \( (2 + \Delta_B) = -\Delta_\chi \), we obtain

\[
\Delta' = \Delta_{\varphi} - \Delta_\chi \approx -0.04
\]  
\tag{40}

In a similar spirit we can look for approximate solutions of the \( B \)-flux constraint among the exact solutions of Polyakov turbulence. We found a few such close solutions within the range of models investigated. In the constant enstrophy category the first is the model (25,234) with \( \psi \equiv \psi_{9,79}, \phi \equiv \psi_{11,103}, \ B \equiv \psi_{11,95}, \ \chi \equiv \psi_{9,83} \) and \( \Delta' \approx -0.04 \).

There were more such approximate solutions for constant energy, with the first example being the low-lying model (8,47) with \( \psi \equiv \psi_{3,17}, \phi \equiv \psi_{3,18}, \ B \equiv \phi, \ \chi \equiv \psi \) and \( \Delta' \approx -0.02 \). For the Saffman condition we have the solution (6,53) with \( \psi \equiv \psi_{1,12}, \phi \equiv \psi_{1,9}, \ B \equiv \psi_{1,16}, \ \chi \equiv \psi_{1,8} \) and \( \Delta' \approx -0.02 \).
3 MHD with finite conductivity

In this case the full set of the modified MHD equations is as follows [12]:

\[ \nabla \cdot \mathbf{v} = 0 \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \cdot \mathbf{J} = 0 \]
\[ \nabla \cdot \mathbf{E} = \rho_c \]
\[ \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho_m} \nabla P + \frac{\rho_c}{\rho_m} \mathbf{E} + \frac{1}{c \rho_m} \mathbf{J} \times \mathbf{B} \]  
\[ \mathbf{J} = \sigma (\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B}) \]
\[ \nabla \times \mathbf{E} = -\frac{1}{c} \partial_t \mathbf{B} \]
\[ \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} \]

Now we have an additional \( \mathbf{E} \) field, and new parameters \( \sigma \) (conductivity) and \( \rho_c \) (electric charge density). After substituting for \( \mathbf{J} \) as before and again taking the curl of the Navier-Stokes equation to eliminate the pressure term, we obtain

\[ \nabla \cdot \mathbf{v} = 0 \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \cdot \mathbf{E} = \rho_c \]
\[ \nabla \times (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = \frac{\rho_c}{\rho_m} \nabla \times \mathbf{E} + \frac{1}{c \rho_m} \nabla \times (\mathbf{B} \cdot \nabla \mathbf{B}) \]  
\[ \nabla \times \mathbf{E} = -\frac{1}{c} \partial_t \mathbf{B} \]
\[ \nabla \times \mathbf{B} = \frac{4\pi \sigma}{c} (\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B}) \]

We note that these equations correctly reduce to the set of eqns (1) and (2) in the infinite conductivity limit (where \( \sigma \to \infty \) and \( \rho_c \to 0 \)).

Next, we again perform dimensional reduction, additionally constraining the \( \mathbf{E} \) field such that

\[ \partial_3 \mathbf{E} = 0 \]  
and denoting the two-dimensional scalar field \( E_3 \) by \( E \). We further note that even though the two-dimensional quantity \( \partial_\alpha E_\alpha \) is non-vanishing, any vector in 2-dimensions
may be decomposed into curl-free and divergence-free parts

\[ E_\alpha = \epsilon_{\alpha\beta} \partial_\beta C + \partial_\alpha D. \quad (44) \]

After some algebraic manipulation one finally arrives at the following set of equations for the finite-conductivity MHD equations in 2-D:

\[
\begin{align*}
\dot{\omega} + \epsilon_{\alpha\beta} \partial_\beta \psi \partial_\alpha \omega &= \frac{1}{4\pi \rho_m} A \Omega - \frac{\rho_e}{\rho_m} \left( \frac{c}{4\pi \sigma} \partial_\alpha \partial_\alpha B - \frac{1}{c} [\epsilon_{\alpha\beta} \partial_\beta \psi \partial_\alpha B - AV] \right) \\
\dot{A} + \epsilon_{\alpha\beta} \partial_\beta \psi \partial_\alpha A &= -\frac{c^2}{4\pi \sigma} \Omega + \text{constant.} \\
\dot{V} + \epsilon_{\alpha\beta} \partial_\beta \psi \partial_\alpha V &= \frac{1}{4\pi \rho_m} AB + \frac{\rho_e}{\rho_m} \left( \frac{c}{4\pi \sigma} \Omega - \frac{1}{c} A \psi \right) \\
\dot{B} + \epsilon_{\alpha\beta} \partial_\beta \psi \partial_\alpha B &= AV + \frac{c^2}{4\pi \sigma} \partial_\alpha \partial_\alpha B \\
\partial_\alpha \partial_\alpha D &= \rho_c
\end{align*}
\]

Again, we find that the infinite conductivity limit is consistent with the previous ideal MHD equations. One should note that, interestingly, the field \( C \) drops out completely while \( D \) decouples from the dynamics. Thus, we still only have four independent dynamical variables \( \omega, A, B \) and \( V \). However, the presence of \( \sigma \) has a non-trivial effect on the equations. In particular, it is no longer obviously consistent to set \( B = V = 0 \), as we shall now discuss.

**Case (I):** \( B = V = 0 \)

In this case the above equations reduce to

\[
\begin{align*}
\dot{\omega} + \epsilon_{\alpha\beta} \partial_\beta \psi \partial_\alpha \omega &= \frac{1}{4\pi \rho_m} A \Omega \\
\dot{A} &= \left( \frac{c^2}{4\pi \sigma} \partial_\alpha \partial_\alpha A + A \psi \right) + \text{constant.} \\
\frac{c^2}{4\pi \sigma} \partial_\alpha \partial_\alpha A + A \psi &= 0
\end{align*}
\]

The last equation in (46) is a constraint which forces \( A \) to have a trivial time-dependence (\( \dot{A} = \lambda \), where \( \lambda \) is a constant). Hence, defining \( A' = A - \lambda t \), we have

\[ A' = 0 \quad (47) \]
In other words the magnetic field is static. Clearly quantities of the form \( \int d^2x A^p \) and \( \int d^2x (B_\alpha B_\alpha)^p \), for any positive integer \( p \), will all be conserved. The existence of a constraint on \( A \) (eqn (46)) however, proves fatal to the conformal approach. In field theory, point-splitting tells us that \( A\psi \) is a level 2 field in the conformal family \([\chi]\) of some minimal dimension field \( \chi \). On the other hand, the other term \( \partial_\alpha \partial_\alpha A \) is in \([A]\) at level 1. Also we note that while \( A \) is an antisymmetric operator, \( \partial_\alpha \partial_\alpha \) is symmetric. Thus parity considerations also rule out a CFT solution to this problem. We interpret this as a limitation of the conformal approach to turbulence rather than as a fact of underlying physical significance.

**Case (II): \( A = V = 0 \)**

Next we consider the limit \( A = V = 0 \), as in the infinite conductivity case. The \( V \) and \( A \) equations are then automatically satisfied if we choose the constant to be zero in eqns. (46). Thus unlike case (I) we do not obtain constraint equations, which as we have seen are obstacles to CFT solutions. The remaining equations of motion become

\[
\dot{\omega} + \epsilon_{\alpha\beta} \partial_\beta \psi \partial_\alpha \omega = -\frac{\rho_c}{\rho_m} \left( \frac{c}{4\pi\sigma} \partial_\alpha \partial_\alpha B - \frac{1}{c} \epsilon_{\alpha\beta} \partial_\beta \psi \partial_\alpha B \right)
\]

(48)

\[
\dot{B} + \epsilon_{\alpha\beta} \partial_\beta \psi \partial_\alpha B = \frac{c^2}{4\pi\sigma} \partial_\alpha \partial_\alpha B
\]

(49)

The immediate observation here is that we now have non-trivial finite effects due to the \( \partial_\alpha \partial_\alpha B \) terms in both equations. Therefore \( \dot{\omega} \) and \( \dot{B} \) do not vanish in the inviscid limit \( a \to 0 \). One can also check that in this case there are no ‘classical’ quadratic conserved quantities as in the infinite \( \sigma \) case. Indeed, generically the presence of finite \( \sigma \) induces decay in \( B \) and \( \omega \) as discussed earlier. At first sight this might seem fatal for conformal turbulence. However, further thought reveals an interesting feature which we now describe.

Let us assume that we have some CFT in which we identify \( \psi \) and \( B \) with suitable primary fields such that the OPE’s \( \psi \times \psi \) and \( \psi \times B \) each have a positive defect of dimensions as before. However, in this case we do not insist on any other condition at this stage. Of course any previous solution of the additional constraints of the infinite conductivity case would automatically be a suitable candidate. Having fixed \( \psi \) and \( B \) one can then use eqns (48) and (49) as a definition of \( \dot{\omega} \) and \( \dot{B} \) respectively, ie

\[
\dot{\omega} \sim [\phi]_2 + [\chi]_2 + \frac{1}{\sigma}[B]_1
\]

(50)
\[ \dot{B} \sim [\chi]_2 + \frac{1}{\sigma} [B]_1 \]  

(51)

where the subscripts on the r.h.s. of eqns. (50) and (51) indicate the level of the corresponding secondary conformal fields. Now by the assumption of positive defect of dimensions, (eqns (26)), \([\phi]_2\) and \([\chi]_2\) both vanish by construction. Thus we obtain

\[ \dot{\omega} = \frac{c \rho_c}{4\pi \sigma \rho_m} \partial_\alpha \partial_\alpha B \]  

(52)

\[ \dot{B} = -\frac{c^2}{4\pi \sigma} \partial_\alpha \partial_\alpha B \]  

(53)

Now we notice that if we define \(W \equiv \omega + \frac{\rho_c}{\rho_m} B\) then

\[ \dot{W} = 0 \]  

(54)

So we discover that although correlation functions of \(\omega\) or \(B\) cannot simultaneously satisfy Hopf equations describing steady flow, correlations involving \(W\) only do satisfy Hopf-like equations at the u.v. (short distance) level that are steady:

\[ <\dot{W}(x_1)W(x_2)\cdots> + <W(x_1)\dot{W}(x_2)\cdots> + \cdots = 0. \]  

(55)

In other words, there is still a remnant of ‘steady’ turbulence present, even though the effect of finite conductivity on the N-S equation is similar to that of finite viscosity in that it destroys the turbulence. Here, because we have a second equation for \(B\) which has a similar finite conductivity dependence, we can still salvage steady turbulence in the variable \(W\). We mention at this point that as well as the \(\sigma\) dependent terms, \([\chi]_2\) also drops out of the equation of motion of \(W\). Thus in principle it is only necessary to require \([\phi]_2\) vanishes to obtain steady turbulence in \(W\). However, solutions under these conditions can never make sense in the \(\sigma \to \infty\) limit. Furthermore, even though we cannot expect a CFT solution to the steady Hopf equation for \(B\), we still have to make sense of eqn (54), which means requiring \(\Delta_\chi > \Delta_\psi + \Delta_B - 1\) (we do not consider here the situation where \([\chi]_2\) is a null field.) Hence in what follows we will impose this condition.

**Flux constraints**

Since \(\dot{B}\) is non-zero, there are short-distance violations of constant \(B\)-flux. Using

\[ \dot{B} \sim [\chi]_2 + \frac{1}{\sigma} [B]_1 \]

\[ = \lim_{a \to 0} \text{const.} \ (a\bar{a})^{\Delta_\chi-\Delta_\psi-\Delta_B+1} \mathcal{L}_\chi - \frac{c^2}{4\pi \sigma} \partial^2 B \]  

(56)
we can evaluate the correlator \( < \dot{B}(r)B(0) > \). Taking \( \Delta_\chi > \Delta_\psi + \Delta_B - 1 \) we find for scales \( r << L \)

\[
< \dot{B}(r)B(0) > \sim -\frac{c^2}{4\pi\sigma} < \partial^2B(r)B(0) > \\
\sim -\frac{c^2}{4\pi\sigma} |r|^{-2(1+2\Delta_B)}
\]

(57)

Clearly, finite \( \sigma \) violates constant \( B \)-flux (as expected), even on scales \( r << L \). We note that, depending on whether \( \Delta_B < -1/2 \) or \( > -1/2 \), this decay of \( B \)-flux will occur quicker at longer or shorter lengthscales. Similar arguments apply to enstrophy.

We next consider \( W \)-flux. The relevant correlator is \( < W(r)\dot{W}(0) > \). From the definition of \( W \) we can evaluate this correlator as

\[
< W(r)\dot{W}(0) > = \langle \omega(r)\dot{\omega}(0) \rangle + \frac{\rho c}{c \rho_m} < \omega(r)\dot{B}(0) > + \frac{\rho c}{c \rho_m} < B(r)\dot{\omega}(0) > \\
+ \left(\frac{\rho c}{c \rho_m}\right)^2 < B(r)\dot{B}(0) >
\]

(58)

There are no u.v. \( (r << L) \) contributions to the r.h.s. of eqn (58) since \( \dot{W} = 0 \) on these scales. However as usual there could be infrared contributions on the lengthscales \( r \sim L \). Evaluating the various terms we find a number of cancellations occur leading to

\[
< W(r)\dot{W}(0) > \sim -L^{-(2+\Delta_\psi+1+\Delta_\phi)} - \frac{\rho c}{c \rho_m} L^{-(\Delta_B+\Delta_\phi+2)}
\]

(59)

At this stage we allow ourselves to speculate on the possibility of imposing constant \( W \)-flux. Such a condition would require that

\[
\Delta_\phi + \Delta_\psi + 3 = 0 \\
\Delta_\phi + \Delta_B + 2 = 0
\]

(60)

The first condition in eqn (60) is the familiar constant enstrophy constraint. It follows naturally by demanding constancy of \( W \)-flux as \( \int W^2d^2x \) is some generalized enstrophy \((W \) is the generalized vorticity). The second is similar to the constant \( B \)-flux condition of the \( \sigma \)-infinite case, with \( \chi \) being the minimal field \( \phi \) of \( \psi \times \psi \), although we should emphasize that here \( B \) is not necessarily constrained by the additional requirement \( B \times \psi \sim [\chi] \). Of course we also need to satisfy the two Hopf conditions corresponding to the requirement that \( [\chi]_2 \) and \( [\phi]_2 \) vanish in the inviscid limit.

Again these turn out to be very stringent requirements for any prospective solution of finite \( \sigma \) MHD. The first constraint tells us that we are looking for a subset of the
constant enstrophy solutions of Polyakov turbulence, which also contains any field $B$ with dimension given by

$$\Delta_B = -2 - \Delta_\phi \quad (61)$$

We have checked all the constant enstrophy minimal models given in Lowe [3] without any success. One can however find approximate solutions. We list the first six of them in Table 3, although we have found others in our search through all minimal model solutions of turbulence with $q < 500$ which we have not listed. Indeed, it turns out to be easier to find approximate solutions in the case of finite $\sigma$ than in the infinite one, because as mentioned above $B$ is not necessarily restricted to satisfy the OPE $B \times \psi \sim \chi$. In Table 3, $\Delta \equiv \Delta_\phi + \Delta_B + 2$ is a measure of the deviation from exact enstrophy flux constancy.

| Model | $\chi$ | $\phi$ | $\psi$ | $B$ | $\Delta$ |
|-------|--------|--------|--------|-----|---------|
| (11, 87) | $\Delta_{5,39} \approx -1.499$ | $\Delta_{3,23} \approx -1.492$ | $\Delta_{2,16} \approx -1.508$ | $\Delta_{4,26} \approx -0.505$ | 0.003 |
| (11, 91) | $\Delta_{3,25} \approx -1.597$ | $\Delta_{3,25} \approx -1.597$ | $\Delta_{2,14} \approx -1.403$ | $\Delta_{4,26} \approx -0.409$ | -0.007 |
| (14, 111) | $\Delta_{5,39} \approx -1.501$ | $\Delta_{1,7} \approx -1.486$ | $\Delta_{1,8} \approx -1.514$ | $\Delta_{5,34} \approx -0.510$ | 0.004 |
| (14, 115) | $\Delta_{4,34} \approx -1.544$ | $\Delta_{1,9} \approx -1.565$ | $\Delta_{1,6} \approx -1.435$ | $\Delta_{4,39} \approx -0.436$ | -0.0008 |
| (16, 135) | $\Delta_{7,60} \approx -1.613$ | $\Delta_{7,59} \approx -1.639$ | $\Delta_{7,56} \approx -1.361$ | $\Delta_{1,15} \approx -0.363$ | -0.002 |
| (21, 166) | $\Delta_{10,79} \approx -1.508$ | $\Delta_{7,55} \approx -1.504$ | $\Delta_{4,31} \approx -1.496$ | $\Delta_{7,61} \approx -0.492$ | 0.003 |

Table 3: Some approximate solutions of finite conductivity MHD.

Of particular noteworthiness is the last model (21,166). This is also an exact solution of the infinite conductivity case as we found earlier. Moreover, surprisingly, the $B$-field here is precisely the same as before ($\psi_{7,61}$), which is a consequence of $\Delta_\chi$ and $\Delta_\phi$ being very close together. In this case the deviation $\Delta$ is given by

$$\Delta = \Delta_B + \Delta_\phi + 2$$

$$= (\Delta_B + \Delta_\chi + 2) + (\Delta_\phi - \Delta_\chi)$$

$$= \Delta_\phi - \Delta_\chi \quad (62)$$
4 Conclusion

First let us make a few comments concerning the connection between the approximate solutions discussed in the previous section, and the exact solutions of the infinite $\sigma$ MHD turbulence with constant enstrophy obtained in section 2. Clearly the limit $\sigma \to \infty$ is non-trivial in that the second condition in eqn (60) is replaced by the $B$-flux condition of eqn (24), and in addition, $B$ is now restricted by the OPE $B \times \psi \sim \chi$. All the approximate solutions in Table 3 except (21,166) fail to satisfy either or both of these conditions, so they remain only approximate solutions when $\sigma$ is finite. Remarkably, as we have already mentioned, (21,166) goes over to an exact solution in the limit $\sigma \to \infty$. It is unfortunate that we can only verify this interesting property for the single exact solution found in the range of models considered. Certainly such a property provides further motivation for finding further exact solutions to the constant enstrophy, $\sigma$ infinite model of magnetohydrodynamical turbulence considered in this paper. Since it would seem very unlikely that the same minimal model could describe these two limits in such a way by pure coincidence, we are left with the open question of whether this might relate to some generic feature of physical interest.

As regards exact solutions for $\sigma$ finite, we can only speculate at this point. At first sight, it may appear that there are two possible outcomes as $\sigma \to \infty$, either such solutions remain exact solutions or they do not. If they remain exact, the simultaneous solutions of eqns (24) and (60) implies that $\Delta \phi = \Delta \chi$ i.e. $\phi$ is identified with $\chi$. Hence $\chi$ (or $\phi$) would have to be the minimal dimension field in both the OPE $B \times \psi$ and $\psi \times \psi$, which is a very restrictive condition. If such solutions do not remain exact, it would be interesting to see if they are at least approximate, because if this is the case then we have the picture that exact solutions with constant enstrophy and $\sigma$ infinite are approximate solutions when $\sigma$ is finite and vice versa.

In conclusion, we have shown that by requiring that CFT describe the special case of perpendicular flow 3-d MHD dimensionally reduced to 2-d, we have been led to impose an extra constraint and thus drastically cut down on the number of possible minimal models that can describe this theory. We showed that the perpendicular $B$-field produced a “hydromagnetic turbulence” effect superimposed upon the usual hydrodynamic turbulence in some minimally interacting manner. Extending the analysis to the finite conductivity case we discovered that the Hopf equation was still satisfied by
some generalized vorticity $W$, producing some mixed $\omega$ and $B$ turbulence phenomenon. There are no analogous conserved quantities but the prospect of $W$-flux constancy is not thereby ruled out. Some interesting instances of minimal models offering approximate realizations of the latter were identified and the issue of the relationship with the infinite conductivity limit was raised.

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