Development and Control of Rotary Inverse Pendulum by LQR with Integral Action

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Abstract. This research a basic well-known hardware, is introduced. The inverse pendulum is the focused in this research. In order to prove the control algorithm LQR with integral action. The dynamic model is obtained by Euler-Lagrange equation. Motor actuator is included in the model in order to get the correct model. The model is linearize by Taylor series expansion. The model parameters are identified by software simulation. The inversed pendulum prototype is manufactured and assembled. The Linear Quadratic Regulator with Integral (LQR+I) controller is presented and applied to control the inverse pendulum. The control simulation and experiment results are present in the performance of this controller.

Keyword: Rotary Inverse Pendulum, Conventional LQR, LQR with Integral action

1. Introduction
Inverse Pendulum is the basic tool to prove the designing of control algorithm. By simple hardware and mechanism, the inverse pendulum is the good example in control system, because of dynamic model of inverse pendulum is non-linear model but it not complicate. The model type of inverse pendulum can be divide into two types. The first is the convention inverse pendulum or linear inverse pendulum [1] (LIP). The limitation of LIP is the boundary of the movement, it depend on the length of the guide of the hardware. In order to eliminate this limit hardware problem, the rotary inverse pendulum [2] (RIP) can increase working space. By this technique, the experiment results can be collected more. By simply of structure and dynamic model, many control algorithms are applied to prove the concept. Neural network is the one of non-linear algorithm. [3] are applied to control the inverse pendulum by present ability to adjust any parameter in the system. The linear controller also applied to control the inverse pendulum. [4] applied LQR controller to manipulate pendulum upright position. To increase the performance of the LQR, [5] put the fuzzy controller to adjust the controller gain of the LQR. The controller gain can change depend on error of the system. This paper present the conventional LQR with integral action. The performance of LQR with integral is evaluated in this paper.

2. Hardware Design
The rotary inverse pendulum is designed by increasing working space. The rotary inverse pendulum compose of two parts; the first part is moving part and the second part is stationary part. The rotary inverse pendulum model is presented in figure 1.
The moving part presents the pendulum rod and cart. To measure angle of the pendulum rod, rotary encoder is directly attached to the shaft which links to the pendulum rod. There is encoder electronic device on the moving part, then the cable of the encoder is the obstacle of full rotation of the moving cart. To solve the problem of the cable, Slip ring is the equipment which attached to divide moving part and stationary part. The moving part can be fully rotating. The RTD1024 is rotary encoder with resolution 1024 pulse per revolution (2.84 pulse per degree). And the significant power to rotate the pendulum is 100 watts of DC motor. Those power transmits through pulley and belt with ratio (18:7).

The 32 Bit of micro-controller is applied to compute encoder data and control rule. The electronic diagrams of the inverse pendulum shown in figure 2. And the structure of the inverse pendulum is built from aluminium and steel as shown in figure 3.
3. Dynamic Model

The dynamic model of inverse pendulum is split into two parts: the first part is inverse pendulum and another part is pendulum arm. Both of them are the moving part. The 2D model is presented in figure 4.

The Euler-Lagrange equation (1) is solved the dynamic model of the inverse pendulum. Friction in the system is small and negligible.

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = f$$

When

- $L$ is the difference between total kinetic energy and potential energy
- $q$ is the state variable
- $f$ is the external input

Then the Lagrange equation is
\[ L = \frac{1}{2} J_{\text{hub}} \dot{\theta}_1^2 + \frac{1}{2} m_1 \left( l_1 \left( \dot{\theta}_1 + l_1 \cos \theta_2 \right)^2 + \frac{1}{2} m_2 \left( -l_2 \sin \theta_2 \right)^2 + \frac{1}{2} J_{\text{pendulum}} \dot{\theta}_2^2 - m_2 g l_1 \cos \theta_2 \right) \]  

From the equation (1), the non-linear model is rearranged to equation (3)

\[ M(q) \ddot{q} + C(q, \dot{q}) + G(q) + D = F \]  

when

\[
M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}
\]

\[ M_{11} = J_{\text{hub}} + l_1^2 m_2 \]
\[ M_{12} = l_1 l_2 m_2 \cos \theta_2 \]
\[ M_{21} = M_{12} \]
\[ M_{22} = J_{\text{pendulum}} + l_2^2 m_2 \]

\[ C_1 = -l_1 l_2 m_2 \sin \theta_2 \dot{\theta}_2^2 + R_{\text{vol}} \dot{\theta}_2 + \frac{K_r^2 K_v^2}{G_{\text{ratio}} R_{\text{vol}}} \theta_1 \]
\[ C_2 = 0 \]
\[ G_1 = 0 \]
\[ G_2 = -l_1 m_2 g \sin \theta_2 \]
\[ F_1 = \frac{K_r K_v}{G_{\text{ratio}} R_{\text{vol}}} \]
\[ F_2 = 0 \]

And the linear model is solved from equation (3) by Taylor series expansion. The linear model of the inverse pendulum is shown in equation (4) and rearranged to state space form.

\[ \dot{x} = Ax + Bu \]  

or:

\[
\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = M^{-1} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -B_{\text{vol}} & 0 & 0 \\ 0 & 0 & -\frac{K_r^2 K_v^2}{G_{\text{ratio}} R_{\text{vol}}} & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ M \text{ is the symmetrical positive definite matrix of the robot inertia and evaluated at the upright position.} \]

4. Control Algorithm

This section is presented the implemented control algorithm. By adding integral term to conventional LQR to improve the performance.

4.1 Conventional LQR Controller

The conventional LQR controller is the reference controller refer to [5] that optimally determines the gains by compromising the state and control-input cost. The LQR cost function \( (J) \) is expressed by

\[ J = \int_0^\infty \left( x^T Q x + u^T R u \right) dt \]  

The \( Q \) and \( R \) matrices are the state and control weighting matrices. The control signal follows equation (5)
The optimal gain matrix is solved by the algebraic Riccati equation (6-7).

\[ A^T P + PA + Q - PBR^{-1}B^T P = 0 \]
\[ K = R^{-1}B^T P \]

The metrics \( Q \) and \( R \) are the weighting metrics which define depend on significant state.

### 4.2 LQR with Integral Action Controller

The LQR with Integral (LQR+I) is improvement of controller form the conventional LQR. The LQR+I increase integral state to control the system by modify state of the LQR as show in equation (9)

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} \dot{x}_r \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} z \end{bmatrix} + A\dot{x} + Bu \\
\dot{e} &= \begin{bmatrix} \dot{e} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} z \end{bmatrix} + A\dot{e} + Bu
\end{align*}
\]

when
- \( x \) is error state
- \( z \) is \( \int \text{edt} \)
- \( x \) is the reference input

And the control signal of the LQR+I is

\[
u = -Kx = -K \begin{bmatrix} x_r \\ z_r \end{bmatrix} = -K\begin{bmatrix} x_r \\ z_r \end{bmatrix} = -K(x - x_r) - K\left(\int (x - x_r)\right)dt
\]

Increasing state of the integral term improves steady error of the system. This term can eliminate error and overshoot of the system caused by incomplete dynamic model, friction in the system. The block diagrams of the LQR+I shows in figure 5.

This technique can produce good dynamic and static performance.

### 5. Simulation Results

In this section, the balancing performance of inverse pendulum are simulated and compared with convention LQR and LQR+I. The simulation results are simulated by MATLAB/Simulink. The initial condition of simulation is set to 5 degree of angle. The simulation results are presented in figure 6 and 7.
6. Experimental result

After simulate the response results, the inverse pendulum system is test by the real hardware. The control rule of the LQR is implemented by velocity control and convert it to discrete control. The discrete control rule is implement to microcontroller as shown in equation (11).

\[
\begin{bmatrix}
1 & 0 \\
K & 0
\end{bmatrix}
\begin{bmatrix}
e_{\phi}[n] - e_{\phi}[n-1] \\
e_{\theta}[n] - e_{\theta}[n-1]
\end{bmatrix}
+ \begin{bmatrix}
\frac{(e_{\phi}[n] - 2e_{\phi}[n-1] + e_{\phi}[n-2])}{\Delta t} \\
\frac{(e_{\theta}[n] - 2e_{\theta}[n-1] + e_{\theta}[n-2])}{\Delta t}
\end{bmatrix}
+ \begin{bmatrix}
e_{\phi}[n]\Delta t \\
e_{\theta}[n]\Delta t
\end{bmatrix}
\]

(11)

The experimental results are presented in inverse pendulum angle and arm link angle as shown in figure 8 and 9.
7. Conclusion

The inverse pendulum is simulated and tested by LQR and LQR+I controller. Both control algorithms are compared. The performance of the LQR+I can reduce overshoot and eliminates steady state error. By this algorithm, the performance is improved but high energy consume also.
Figure 10 The experimental result of inverse pendulum of LQR+I

Figure 11 The power to control inverse pendulum of LQR+I

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