Arbitrary Spin Representations in de Sitter from dS/CFT with Applications to dS Supergravity

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Abstract

We present a simple group representation analysis of massive, and particularly “partially massless”, fields of arbitrary spin in de Sitter spaces of any dimension. The method uses bulk to boundary propagators to relate these fields to Euclidean conformal ones at one dimension lower. These results are then used to revisit an old question: can a consistent de Sitter supergravity be constructed, at least within its intrinsic horizon?
1 Introduction

In addition to (perhaps) being our macroscopic habitat \[1\], the conformally flat, constant curvature, de Sitter (dS) space has long been a seminal laboratory for field theory in curved backgrounds, being the simplest possible generalization of the Minkowski vacuum. Indeed, dS effects on a spin 1/2 field’s propagation were first studied seven decades ago \[2\]. Ever since, systems of all spins and masses have been analyzed in both dS and anti de Sitter (AdS), with obvious connections to string expansions\[1\]. A particularly striking effect, that of partial masslessness or partial gauge invariance, arising first at spin 2 \[4, 5\], was recently discussed by us in some detail \[6, 7, 8, 9, 10\]. (see also \[11, 12\]). The purpose of the present work is twofold: First, we formalize the kinematics underlying these properties in a group theoretic context, using a dS/CFT correspondence between dS\(_{n+1}\) and Euclidean space \(\mathbb{R}^n\). This will exhibit the details of partial masslessness for all spin, in all dimensions \(d = n + 1\), as well as the ensuing propagation properties at these thresholds\[2\]. Secondly, we will use the above systematics to reassess the case for (and against) extending models of cosmological supergravity from their natural AdS arena to dS. Before proceeding to details, we immediately disclaim any global ambitions for dS: we will be working primarily on one patch of the full dS, and there only within its intrinsic event horizon, this being the lamppost under which the physics is (relatively) clear.

In Section 2, we review the description of dS as a coset and the corresponding algebra as that of the Euclidean conformal group in one lower dimension. Section 3 illustrates our method in the simplest case of scalar fields. We will relate fields in dS\(_{n+1}\) with conformal ones on \(\mathbb{R}^n\) by using an intertwiner, essentially establishing a (dS/CFT) correspondence in terms of a proper time propagator. In Section 4, we generalize to bosons of arbitrary spin. We next apply the above results to partially massless higher spin bosons, where discrete ratios of (mass)\(^2\) to cosmological constant imply light cone propagation and (partial) gauge invariances. In Section 6, we treat fermions in the \textit{a priori} hostile dS context. Finally, in Section 7

\[1\] A detailed analysis of partially massless representations has previously been given by solving the Laplace equation on spheres for higher spin harmonics and then analytically continuing these solutions to dS space \[3\]. Although this analysis has the advantage of applying to the entire dS space, it is rather complicated and the simple interpretation of partially massless fields in terms of missing lower helicity states is obscured.

\[2\] Note that the partial masslessness phenomenon occurs in AdS as well. In that case, however, only strictly massless representations are unitary. The representation theory of these fields is much easier to understand, since the algebra can be graded with respect to the generator of the \(SO(2)\) factor of the maximal compact subgroup of the AdS isometry group. Partially massless representations are obtained by searching for null descendants. We thank M. Vassiliev for pointing this out to us.
we revisit the old question of the extent, if any, to which a dS supergravity can be viable, presenting the pros and cons in terms of our framework.

2 de Sitter Space

The dS_{n+1} spacetime is the coset SO(n + 1, 1)/SO(n, 1) and is geometrically described by the one sheeted hyperboloid

\[ Z^M Z_M = -(Z^0)^2 + \sum_{i=1}^{n} (Z^i)^2 + (Z^{n+1})^2 = 1, \]  

(1)

embedded in \( \mathbb{R}^{(n+1,1)} \). The cosmological constant \( \Lambda \) is positive in dS space and usually enters the equation (1) as \( n/\Lambda \) on the right hand side. We will work in units \( \Lambda = n \) throughout, however.

The dS group \( SO(n + 1, 1) \) acts naturally on \( \mathbb{R}^{(n+1,1)} \) with generators

\[ M_{MN} = i(Z_M \partial_N - Z_N \partial_M) \]  

(2)

obeying the Lie algebra

\[ [M_{MN}, M_{RS}] = i\eta_{NR}M_{MS} - i\eta_{NS}M_{MR} + i\eta_{MS}M_{NR} - i\eta_{MR}M_{NS}. \]  

(3)

We will employ a coordinate system in which spatial sections are flat and therefore rewrite this algebra in terms of

\[ P_i \equiv M_{i0} + M_{n+1,i}, \quad D \equiv M_{n+1,0}, \quad K_i \equiv M_{i0} - M_{n+1,i}, \]  

(4)

and \( M_{ij} \) so that

\[ [P_i, P_j] = 0, \quad [K_i, K_j] = 0, \]

\[ [D, P_i] = iP_i, \quad [D, M_{ij}] = 0, \quad [D, K_i] = -iK_i, \]

\[ [M_{ij}, P_k] = i(P_i\delta_{jk} - P_j\delta_{ik}), \quad [M_{ij}, K_k] = i(K_i\delta_{jk} - K_j\delta_{ik}), \]

\[ [P_i, K_j] = -2i\delta_{ij}D + 2iM_{ij}. \]  

(5)

\(^{3}\)Our index conventions are \( M, N = 0, \ldots, n + 1 \), raised and lowered by the Minkowski metric of the \( (n + 1) + 1 \) dimensional embedding space \( \eta_{MN} = \text{diag}(-, +, \ldots, +)_{MN} \); \( i, j = 1, \ldots, n \) (also denoted by vectors), covariant with respect to the \( SO(n) \) subgroup generated by \( M_{ij} \), raised and lowered by the Kronecker delta.
The generators $M_{ij}$ satisfy the $so(n)$ angular momentum Lie algebra, so identifying the remaining generators $P_i$ as momenta, $D$ as dilations and $K_i$ as conformal boosts, we see that the algebra \( (n) \) generates the Euclidean conformal group in \( n \) dimensions. To make the relation between the coset $SO(n+1,1)/SO(n,1)$ and hyperboloid explicit, we begin by introducing the matrix representation of the generators (4).

\[
P_i = i \begin{pmatrix} 0 & \vec{e}_i & 0 \\ \vec{e}_i & 0 & \vec{e}_i^\dagger \\ 0 & -\vec{e}_i & 0 \end{pmatrix}, \quad D = i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad K_i = i \begin{pmatrix} 0 & \vec{e}_i & 0 \\ -\vec{e}_i & 0 & -\vec{e}_i^\dagger \\ 0 & \vec{e}_i & 0 \end{pmatrix}.
\]

(6)

[All entries of the \( n \) dimensional vector \( \vec{e}_i \) vanish save for the \( i^{th} \) slot which is unity.] Now choose a standard vector $X^M = (0, \ldots, 0, 1)$. Since this is the defining representation of $SO(n+1,1)$, the vector $Z^M = (gX)^M$ satisfies the hyperboloid condition (1) for any $g \in SO(n+1,1)$ and in fact the entire dS space can be obtained this way. The stabilizer of $X^M$ is the subgroup $SO(n,1) \subset SO(n+1,1)$, therefore $dS_{n+1} \cong SO(n+1,1)/SO(n,1)$.

Parameterizing coset elements as

\[
g = \exp(-i\vec{P}\cdot\vec{x})\exp(iDt) = \begin{pmatrix} 1 + \frac{1}{2} \vec{x}^2 & \vec{x} & \frac{1}{2} \vec{x}^2 \\ \vec{x} & 1 & \vec{x}^\dagger \\ \frac{1}{2} \vec{x}^\dagger & -\vec{x} & 1 - \frac{1}{2} \vec{x}^2 \end{pmatrix} \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix},
\]

(7)

we find the embedding coordinates to be

\[
Z^M = (gX)^M = \left( \sinh t + \frac{1}{2} e^t \vec{x}^2, \ e^t \vec{x}, \ \cosh t - \frac{1}{2} e^t \vec{x}^2 \right).
\]

(8)

The dS metric then follows:

\[
d s^2 = dZ^M \eta_{MN} dZ^M = -dt^2 + e^{2t} d\vec{x}^2 \equiv dx^\mu g_{\mu\nu} dx^\nu.
\]

(9)

While the parameterization of the coset (7) only spans one half, $Z^{n+1} > Z^0$, of the hyperboloid, the physical region within the intrinsic horizon

\[
0 > -1 + e^{2t} \vec{x}^2 = \xi^\mu \xi_\mu, \quad \xi^\mu = (-1, \vec{x}^i),
\]

(10)

is covered by this coordinate patch.

### 3 Representing Fields in de Sitter: Scalars

We now use the simplest, scalar, fields in $dS_{n+1}$ to illustrate the correspondence with conformal fields on $\mathbb{R}^n$ based on an intertwiner for on-shell dS field represen-
A scalar field in a gravitational background is described by the action

\[ S = \frac{1}{2} \int d^{n+1}x \sqrt{-g} \varphi \left( D_\mu D^\mu - \frac{n-1}{4n} R - m^2 \right) \varphi , \]

and yields a conformally improved scalar at \( m^2 = 0 \). For the constant curvature dS background with metric (9), the scalar curvature \( R = n(n+1) \). The action is then invariant under the isometries of dS\(_{n+1}\) (for any value of the mass \( m \)) whose action on the coordinates \( x^\mu = (t, x^i) \) may be deduced by examining the transformation of the embedding coordinates \( Z^M \) in (8) generated by the matrices (6). This yields symmetry transformations of \( \varphi(x) \) generated by

\[ iP_i = \partial_i , \]
\[ iD = -\frac{d}{dt} + \vec{x} \cdot \vec{\partial} , \]
\[ iM_{ij} = x_i \partial_j - x_j \partial_i , \]
\[ iK_i = -2x_i \left( -\frac{d}{dt} + \vec{x} \cdot \vec{\partial} \right) + \left( -e^{-2t} + \vec{x}^2 \right) \partial_i . \]

We can also read off the \( \frac{1}{2} (n+1)(n+2) \) Killing vectors of dS\(_{n+1}\) from the Lie derivatives above. In particular \( iD = \xi^\mu \partial_\mu \), where \( \xi^\mu = (-1, x^i) \) is the timelike Killing vector within the horizon. Observe that the horizon condition (10) is invariant under the action of \( D \).

The representation (12-15) acting on off-shell fields \( \varphi(x) \) is not reducible, as evidenced by the quadratic Casimir

\[ C_2 \equiv \frac{1}{2} M_{MN}M^{NM} \]
\[ = D^2 + \frac{1}{2} \left( P_i K_i + K_i P_i \right) - \frac{1}{2} M_{ij}M^{ij} \]
\[ = -\frac{d^2}{dt^2} - \frac{n}{d} \frac{d}{dt} + e^{-2t} \vec{\partial}^2 . \]

Incidentally, the quartic Casimir\(^5\) vanishes for the scalar representation,

\[ C_4 \equiv -M_{MN}M^{NO}M_{OP}M^{PM} + 2C_2^2 - (n-2)(n-3)C_2 = 0 . \]
The operator $C_2$ in (16) is precisely the Laplace-Beltrami operator $D_\mu D^\mu$ on $dS_{n+1}$, so we obtain an irreducible representation by solving the field equation

$$\left( D_\mu D^\mu - \frac{1}{4} (n^2 - 1) - m^2 \right) \phi = 0.$$  

(18)

To this end, we expand $\phi$ in spatial Fourier modes

$$\phi(x) = \int d^n x e^{i \vec{k} \cdot \vec{x}} a(\vec{k}, t) + \text{c.c.}, \quad k \equiv |\vec{k}|,$$

(19)

and must now solve

$$\left( -\frac{d^2}{dt^2} - n \frac{d}{dk} - e^{-2t} k^2 - \frac{1}{4} (n^2 - 1) - m^2 \right) a(\vec{k}, t) = 0.$$  

(20)

This is Bessel’s equation in the “conformal time” variable $u = - \exp(-t)$ for the function $u^{-n/2} a(k, t)$. Hence we may express $a(k, t)$ in terms of a Hankel function

$$a(\vec{k}, t) = k^{-\nu} e^{-\frac{1}{2} n t} H_\nu(-ke^{-t}) \, a(\vec{k}) \equiv f_k(t; \nu) a(\vec{k}),$$

(21)

with index

$$\nu = \sqrt{\frac{1}{4} - m^2}.$$  

(22)

Since the Hankel function is just a plane wave (multiplied by a slowly varying time dependent amplitude) for index $\nu = 1/2$, conformally improved scalars ($m^2 = 0$) propagate on the light cone.

On-shell scalar fields are labeled by the arbitrary function $a(\vec{k})$, so we now study the irreducible representation of the $dS$ group acting on these functions induced by the original representation in terms of isometries of $dS_d$. In mathematical terms, we want to study the intertwiner

$$\phi(x) = \int d^n x f_k(t; \nu) e^{i \vec{k} \cdot \vec{x}} a(\vec{k}) + \text{c.c.}$$

(23)

between the isometry representation on functions $\phi(x)$ and the irreducible representation on functions $a(\vec{k})$.

The first step is the intertwining spatial fourier transform which replaces $x^i \rightarrow i \partial / \partial k_i$ and $\partial / \partial \bar{x}^i \rightarrow ik_i$ (maintaining the ordering). Denoting the resulting generators obtained from (12) to (15) this way by tildes, we have the useful identities

$$i \tilde{D} f_k(t; \nu) = \left( \nu - \frac{n}{2} \right) f_k(t; \nu), \quad i \tilde{K}_i f_k(t; \nu) = 0.$$  

(24)

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6The metric $ds^2 = u^{-2}(-da^2 + d\bar{x}^2)$ is then manifestly conformally Minkowskian.
Therefore we get a very simple action on functions \( a(\vec{k}) \) (dropping the irksome tildes)

\[
iP_i = ik_i, \quad (25)
\]

\[
iD = -\vec{k} \cdot \frac{\partial}{\partial \vec{k}} + \nu - \frac{n}{2}, \quad (26)
\]

\[
iM_{ij} = k_i \frac{\partial}{\partial k^j} - k_j \frac{\partial}{\partial k^i}, \quad (27)
\]

\[
iK_i = 2i(\vec{k} \cdot \frac{\partial}{\partial \vec{k}} + \frac{n}{2} - \nu) \frac{\partial}{\partial k^i} - ik_i \frac{\partial}{\partial \vec{k}} \cdot \frac{\partial}{\partial \vec{k}}. \quad (28)
\]

Performing an inverse spatial Fourier transform, \( k_i \rightarrow -i\partial/\partial y^i \) and \( \partial/\partial k_i \rightarrow -iy^i \) intertwines to the usual representation of the Euclidean conformal group acting on scalars in \( n \) dimensions

\[
iP_i = \frac{\partial}{\partial y^i}, \quad (29)
\]

\[
iD = \vec{y} \cdot \frac{\partial}{\partial \vec{y}} + \Delta, \quad (30)
\]

\[
iM_{ij} = y_i \frac{\partial}{\partial y}^j - y_j \frac{\partial}{\partial y^i}, \quad (31)
\]

\[
iK_i = -2y_i(\vec{y} \cdot \frac{\partial}{\partial \vec{y}} + \Delta) + \vec{y}^2 \frac{\partial}{\partial y^i}. \quad (32)
\]

This is precisely the action of the conformal group acting on scalars (quasi-primaries) of conformal weight

\[
\Delta = \frac{n}{2} + \sqrt{\frac{1}{4} - m^2}. \quad (33)
\]

The intertwiner is given by

\[
\varphi(x^\mu) = \int d^n k e^{i\vec{k} \cdot \vec{x}} f_k(t; \nu) \int d^n y e^{-i\vec{k} \cdot \vec{y}} \Phi(\vec{y}) + \text{c.c.}. \quad (34)
\]

The field \( \Phi(\vec{y}) \) is a quasi-primary of weight \( \Delta \). The \( \vec{k} \) integral can be performed by employing the integral representation of the Hankel function

\[
k^{-\nu} H_\nu(uk) = -\frac{i}{\pi} e^{-\frac{1}{2}iu\pi} \int_0^\infty \frac{d\tau}{\tau^{\nu+1}} \exp \left[ \frac{1}{2} iu \left( \tau + \frac{k^2}{\tau} \right) \right]. \quad (35)
\]

We find

\[
\varphi(x^\mu) = \int d^n y \Delta(\vec{x} - \vec{y}, t) \Phi(\vec{y}). \quad (36)
\]
Here
\[ \Delta(\vec{x} - \vec{y}, t) = \mathcal{N} \int_{0}^{\infty} d\tau \tau^{n-\frac{s}{2}+1} \exp \left[ -\frac{i\tau}{2} \left( -u^2 + (\vec{x} - \vec{y})^2 \right) \right]. \] (37)

is the Schwinger proper time parametrization of the bulk to boundary propagator from conformal time \( u = 0 \) to \( u = -\exp(-t) \) (up to some overall normalization \( \mathcal{N} \)). Therefore the intertwiner exactly realizes the dS/CFT correspondence of [14].

We end this section by noting that the quadratic Casimir for the conformal representation is
\[ C_2 = -\Delta(\Delta - n) = \frac{1}{4} (n^2 - 1) + m^2, \] (38)

which agrees with the one imposed by the field equation (20). The quartic Casimir invariant vanishes also in the conformal representation, as it should for scalars.

4 Representing Arbitrary Spin Bosons

A massive bosonic spin \( s \) field in dS space can be described by a completely symmetric tensor \( \varphi_{\mu_1...\mu_s} \) subject to the field equation and constraints
\[ \left( D_{\mu} D^{\mu} - 2n + 4 + (n-5)s + s^2 - m^2 \right) \varphi_{\mu_1...\mu_s} = 0 = D_{\rho} \varphi_{\rho\mu_2...\mu_s} = \varphi_{\rho\mu_3...\mu_s}. \] (39)

For generic values of the mass \( m \), \( \varphi_{\mu_1...\mu_s} \) describes the
\[ \mu(n, s) = \frac{(n + 2s - 2)(n + s - 3)!}{s!(n - 2)!} \] (40)
degrees of freedom of a spin \( s \) symmetric field in \( n + 1 \) dimensions. The mass parameter is defined so that the theory is strictly massless for \( m^2 = 0 \) with a

\footnote{We concentrate on completely symmetric higher spin representations. See [15, 16] for a discussion of higher spins for other tensor symmetries. It is also worth noting that to obtain the simple field equation and constraints in (39) from an action for spins \( s \geq 5/2 \), it is necessary to introduce auxiliary fields. They play, however, no rôle in the representation theoretic analysis given here.}

\footnote{For scalars, there is no gauge invariance, but one often chooses \( m^2 \) such that vanishing mass yields a conformally improved scalar in general backgrounds as we did in Section 3. It is a peculiarity of four dimensions that the choice of mass parameter required for gauge invariance of spins \( s \geq 1 \) also yields the conformally improved scalar when continued to \( s = 0 \).}
gauge invariance\(^9\)
\[
\varphi_{\mu_1...\mu_s} = \partial_{(\mu_1} \xi_{\mu_2...\mu_s)} ,
\]
(subject to \(\xi_{\rho_3...\mu_s} = 0\)). The degree of freedom count is then
\[
\mu(n, s) - \mu(n, s - 1) = \frac{(n + 2s - 3)(n + s - 4)!}{s!(n - 3)!} .
\]

Actions may be written down for these free theories, both massive and massless; see [8, 11] for details. The dS isometries act on off-shell fields \(\varphi_{\mu_1...\mu_s}\) as

\[
iP_i \varphi_{\mu_1...\mu_s} = \partial_i \varphi_{\mu_1...\mu_s} ,
\]
\[
iD \varphi_{\mu_1...\mu_s} = \left( -\frac{d}{dt} + \vec{x} \cdot \vec{\partial} \right) \varphi_{\mu_1...\mu_s} + s \delta_{(\mu_1} \varphi_{\mu_2...\mu_s)} j \, ,
\]
\[
iM_{ij} \varphi_{\mu_1...\mu_s} = (x_i \partial_j - x_j \partial_i) \varphi_{\mu_1...\mu_s} - s \varphi_{(\mu_1} \delta_{\mu_2...\mu_s)} j + s \varphi_{(\mu_1} \delta_{\mu_2...\mu_s)} i \, ,
\]
\[
iK_i \varphi_{\mu_1...\mu_s} = \left[ -2x_i \left( -\frac{d}{dt} + \vec{x} \cdot \vec{\partial} \right) + \left( -e^{-2t} + \vec{x}^2 \right) \partial_i \right] \varphi_{\mu_1...\mu_s} + 2s \delta_{(\mu_1} \varphi_{\mu_2...\mu_s)} j x_j + 2s \delta_{(\mu_1} \varphi_{\mu_2...\mu_s)} i - 2s \delta_{(\mu_1} \varphi_{\mu_2...\mu_s)} j x_j + 2s \delta_{(\mu_1} \varphi_{\mu_2...\mu_s)} i .
\]

It is tedious but not difficult to solve the field equations in the frame (9) and then construct the higher spin bulk to boundary propagator

\[
\varphi_{\mu_1...\mu_s} = \int d^9 y \Delta (\vec{x} - \vec{y}, t) \varphi_1...\varphi_s V_{\mu_1...\mu_s} .
\]

The higher spin boundary fields \(V_{i_1...i_s}\) are totally symmetric and traceless in \(n\) dimensions and transform under the conformal algebra as

\[
iP_i V_{(s)} = \partial_i V_{(s)} ,
\]
\[
iD_i V_{(s)} = (y^i \partial_i + \Delta_s) V_{(s)} ,
\]
\[
iM_{ij} V_{(s)} = (y_i \partial_j - y_j \partial_i) - \mathcal{S}_{ij} V_{(s)} ,
\]
\[
iK_i V_{(s)} = (-2iy_i D + i\vec{y}^2 P_i) V_{(s)} + y_j \mathcal{S}_{ij} V_{(s)} .
\]

Here \(V_{(s)}\) is shorthand for \(V_{i_1...i_s}\) and the intrinsic spin operator \(\mathcal{S}_{ij}\) acts as

\[
\mathcal{S}_{ij} V_{i_1...i_s} = \sum_{\alpha=1}^{s} V_{i_1...i_{\alpha-1}i_{\alpha+1}...i_s} \delta_{ji\alpha} - (i \leftrightarrow j) .
\]

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\(^9\)For example, the massive spin 1 field equation \(G_{\mu} = (D^2 - n - m^2) \varphi_{\mu} - D_\mu D \varphi\) obeys a Bianchi identity \(D G = 0\) at \(m^2 = 0\), as is easily verified using \([D_\mu, D_\nu] \varphi_{\nu} = 2g_{\rho\nu} \varphi_{\nu}\) in dS.
The weight $\Delta_s$ is computed by examining the quadratic Casimir of this representation
\[ C_2 = -\Delta_s(\Delta_s - n) - s(s + n - 2). \tag{53} \]
For higher spins, the quadratic Casimir and Laplacian are no longer equal; instead, a simple computation reveals (see \[17\] [18]) that
\[ D_\mu D^\mu \varphi_{\mu_1...\mu_s} = \left( C_2 + s(s + n - 1) \right) \varphi_{\mu_1...\mu_s}. \tag{54} \]
Comparing (53), (54) and (39) relates the mass parameter $m^2$ and weight $\Delta_s$,
\[ m^2 = -\Delta_s(\Delta_s - n) + (s - 2)(s - 2 + n). \tag{55} \]
Our next task is to find the values of $m^2$ making fields massless or partially massless.

## 5 Partially Massless Bosons

We have assembled all the relevant machinery to provide a very simple description of partially massless bosons in $\text{dS}_{n+1}$ in terms of representations of the $n$ dimensional conformal group\(^\text{10}\).

We recall that the physical polarizations of a strictly massless field satisfy
\[ \partial^i V_{(s-1)} \equiv \partial^i V_{i_1i_2...i_s} = 0. \tag{56} \]
thanks to the gauge invariance (41) which projects out all but the maximal helicity $s$ excitations\(^\text{11}\). A field obeying (56) has the correct degree of freedom count, as given in (42), for a strictly massless field.

For partially massless fields, gauge invariances of the form
\[ \delta \varphi_{\mu_1...\mu_s} = D_{(\mu_1...\mu_t}\varphi_{\mu_{t+1}...\mu_s)} + \cdots \tag{57} \]
imply that the requirement (56) is relaxed and replaced by
\[ \partial^{i_1} \cdots \partial^{i_t} V_{i_1...i_s} = 0, \quad (t \leq s). \tag{58} \]
We call such a field “partially massless of depth $t$”. This amounts to projecting out all helicities save $(s, \ldots, t+1)$ and gives $\mu(n, s) - \mu(n, s-t)$ degrees of freedom.

\(^{10}\)These representations have been studied in a conformal field theory context in \[19\] [20] [21] [22].

\(^{11}\)Helicity in dimensions $n > 3$ can be defined in terms of $\vec{P}^{-2}(\epsilon^{i_1...i_n}P_{i_1}M_{i_2i_3})^2$. Strictly speaking, in the discussion above, we should refer to the absolute value of the helicity.
The subalgebra of translations, dilations and rotations leaves the condition (58) invariant. However, conformal boosts do not, unless one tunes the conformal weights $\Delta_s$ appropriately. To obtain these tunings we study

$$\partial^i \cdots \partial^i K_i V_{i_1 \ldots i_s} = 0,$$

for depth $t$ partially massless polarizations $V_{(s)}$ subject to (58). It is a simple combinatorics problem to compute the (unique) value of $\Delta_s$ as a function of the depth $t$ such that the condition holds. We state the result below. The main idea is conveyed by the simplest non-trivial example, spin 2.

For a spin 2 field $V_{ij} - V_{ji} = 0 = V_{ii}$, the conformal boost acts as

$$iK_i V_{jk} = i(-2y_i\mathcal{D} + \vec{y}^2 P_j)V_{jk} + 4y_{(j}V_{k)i} - \delta_{i(j} y_{k)i}.$$

The field $V_{ij}$ is strictly massless whenever

$$\partial^i V_{ij} = 0,$$

so we test whether this condition is respected by conformal boosts by computing

$$\partial^k K_i V_{jk} = 2i(\Delta_s - n) V_{ij}.$$  

Here we have relied on the divergence constraint (61). Hence we find the strictly massless tuning

$$\Delta_s = n.$$ 

Using this relation as well as $s = 2$ in (55) gives $m^2 = 0$, the correct tuning for a strictly massless spin 2 boson.

To study partially massless spin 2, we replace the single divergence condition (61) by the double divergence one

$$\partial^i \partial^j V_{ij} = 0.$$ 

Now we must compute

$$\partial^i \partial^k K_i V_{jk} = 4i(\Delta_s - n + 1) \partial_j V_{ij}$$

where we used (64) but not (61). Therefore we obtain the partially massless spin 2 tuning

$$\Delta_s = n - 1.$$ 

It is also clear that the partially massless representation is irreducible. One might have thought it to be a direct sum of spin 2 and spin 1 strictly massless representations. However, since the tunings (63) and (66) differ, the strictly massless spin 2 field components satisfying (61) mix with the remaining ones when $\Delta_s \neq n$. 

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We calculate the tunings for arbitrary spin in the same way by imposing (58) and computing
\[ \partial^{i_1} \ldots \partial^{i_t} K_i V_{i_1 \ldots i_s} = 2it(\Delta_s - n - s + t + 1) \partial^{i_2} \ldots \partial^{i_t} V_{i_2 \ldots i_s}. \] (67)

The tunings are therefore
\[ \Delta_s = n + s - t - 1. \] (68)

Inserting the tuning condition (68) in the dS mass–conformal weight relation (55) yields the mass conditions for depth \( t \) partial masslessness
\[ m^2 = (t - 1)(2s - 3 + n - t). \] (69)

Firstly note that for depth \( t = 1 \), i.e. strictly massless fields, the mass parameter \( m^2 = 0 \). Also when \( n = 3 \), the result agrees with the one conjectured in [6] on the basis of requiring light cone propagation for partially massless fields.

### 6 Fermion Representations

There is no fundamental difficulty in adapting the above bosonic manipulations to fermions, only tedium. Rather than performing the computation we note that a massive spin \( s \equiv \sigma + 1/2 \) fermionic field satisfies
\[ (\slashed{D} + m)\psi_{\mu_1 \ldots \mu_s} = 0 = D_\nu \psi_{\mu_2 \ldots \mu_s} = \gamma_\nu \psi_{\mu_2 \ldots \mu_s}. \] (70)

Here \( \psi_{\mu_1 \ldots \mu_s} \) is a completely symmetric tensor-spinor. With the above choice of the parameter \( m \), strict masslessness occurs at\(^{12}\)
\[ m^2 = -\left(s + \frac{n}{2} - 2\right)^2. \] (71)

Accounting for this “offset”, we find partially massless tunings for fermions at
\[ m^2 = -\left(s - t + \frac{n}{2} - 1\right)^2, \] (72)

in precise agreement with the result of [6] for dS4. Recalling that the cosmological constant is reinstated by multiplying the right hand side of (72) by \( \Lambda/n \), we note (as observed in [6]), that fermionic tunings are satisfied for real values of \( m \) only for \( \Lambda < 0 \), i.e. in AdS space. The parameter \( m \), (not its square) appears in the

\(^{12}\)Again, this criterion applies to masslessness imposed by a gauge invariance and is valid for spins \( s \geq 3/2 \). In four dimensions, coincidentally, it yields \( m^2 = 0 \) for \( s = 1/2 \), the same value required for a conformally improved spinor.
action, for these theories. For the choice of square root $m = +i (s - t + n/2 - 1)$, the action for partially (and strictly) massless dS fermionic fields no longer obeys a reality condition, but is invariant under a formal gauge invariance. This property is the root cause of the difficulty defining dS supergravities, the topic of the next section.

7 de Sitter Supergravity Revisited

Our considerations thus far have led us to the following picture (illustrated in Figure 1) of particles in (A)dS backgrounds. Partially massless fields, of either statistics, are always unitary in dS, while in AdS only strictly massless ones are. From Figure 1 this behavior is understood by turning on cosmological constants of either sign and following their effects on the signs of lower helicity state norms. A sequence of unitary partially massless fields is only encountered when starting from Minkowski space ($\Lambda = 0$) and first turning on a positive cosmological constant. The bad news, however, is that partially massless fermions require tunings with negative $m^2$ as already noted in cosmological supergravity [23, 24]. This led to the rejection of (see especially [25]) dS supergravity as a consistent local QFT, a rejection bolstered later by the difficulties in defining string theory on dS backgrounds (see for example [26]).

Let us first present the reasons for rejection in terms of the present analysis, followed by such mitigating circumstances as we can muster for keeping the possibility in play. For concreteness, we deal primarily here with $N = 1$ supergravity in four dimensions.

- Although (formally) locally supersymmetric actions exist, the mass parameter appearing in the term $m\sqrt{-g} \psi_\mu \gamma^{\mu\nu} \psi_\nu$ must be pure imaginary for light-cone propagation. Therefore, the action of dS supergravity does not obey a reality condition.

- The associated local supersymmetry transformations $\delta \psi_\mu = (D_\mu + \frac{1}{2} m \gamma_\mu) \epsilon$, being complex, cannot preserve Majorana condition on fields. For $N = 2$ supersymmetry, a symplectic reality condition is possible, but the locally supersymmetric action is either still complex or the Maxwell field’s kinetic term has tachyonic sign [25].

- Another way to see that there can be no real supercharges is that the $N = 1$ dS superalgebra ought be the $d = 5$, $N = 1$ superlorentz algebra, but there are no Majorana spinors in $d = 5$ Minkowski space.
Figure 1: Cosmological phase diagrams for partially massless Bose and Fermi fields depicting strictly massless and maximal depth partially massless tunings. All other lines with partially massless gauge invariances appear between these.
• Because $dS_4$ has topology $S^3 \times \mathbb{R}$, gauge charges (being surface integrals) vanish since there are no spatial boundaries $[26]$. This argument is of a different nature from the previous ones as it involves the global considerations which we have chosen to ignore.

• Finally, even for the $N = 2$ case, where a $dS$ superalgebra exists, there are no positive mass, unitary representations $[25]$. Unlike the praiseworthy $AdS$ algebra, our maximal compact subalgebra has no $SO(2)$ factor whose generator could define a positive mass Hamiltonian. Again this is a global issue.

Let us now present the arguments in favor of $dS$ supergravity:

• While the tuned $dS$ supergravity mass

\[ m^2 = -\Lambda/3 . \tag{73} \]

is indeed negative, there are precedents for consistent theories with negative $m^2$, for example scalars in AdS for which a range of such values can be tolerated $[27, 28]$ essentially because the lowest eigenvalue of the Laplacian has a positive offset there. Figure 1 shows that fermions mirror this behavior in $dS$.

• Despite an imaginary mass-like term in the action, at least the free field representations are unitary. For unitarity of spin $3/2$ representations, the relevant quantity is $m^2 - 3\Lambda$, not $m^2$ alone. In addition the linearized equations of motion for physical spin 2, helicity $\pm 2$ and its proposed spin $3/2$, helicity $\pm 3/2$ superpartner degrees of freedom are identical

\[
\begin{align*}
\text{Spin 2:} & \quad \left( -u^2 \frac{d^2}{du^2} - u \frac{d}{du} + u^2 \vec{\partial}^2 + \frac{3}{4} \sqrt{\Lambda} \right) \epsilon_{\pm 2} = 0 , \tag{74} \\
\text{Spin 3/2:} & \quad \left( -u^2 \frac{d^2}{du^2} - u \frac{d}{du} + u^2 \vec{\partial}^2 + \frac{3}{4} \sqrt{\Lambda} \right) \epsilon_{\pm 3/2} = 0 . \tag{75}
\end{align*}
\]

Here we have employed the frame $ds^2 = u^{-2}(-du^2 + d\vec{x}^2)$; a detailed derivation may be found in $[6]$.

• In $dS$ space positivity of energy is possible only for localized excitations within the horizon $[10]$. Only this region of $dS$ possesses a timelike Killing vector. $dS$ gravity is therefore stable against local excitations within the physical region $[29]^{13}$. Observations of supernovae suggest that we inhabit an

\[ ^{13}\text{The same conclusion applies also to partially massless $dS$ fields, see [7].} \]
asymptotically dS universe \[ \Pi \]. Although dS quantum gravity is problematic, nobody would reject cosmological Einstein gravity as an effective description of local physics within the physical region. This looser criterion is all we should require of a sensible dS supergravity.

• As stated, any local supersymmetry is purely formal in the absence of \( d = 4 + 1 \) Majorana spinors (even the fermionic field equation above is not consistent with a reality condition on \( \psi_\mu \)). In any case, a bona fide \( N = 1 \) dS superalgebra with Majorana super charge, would imply a global positive energy theorem from \( \{ Q, Q \} \sim H \), which is already ruled out at the level of dS gravity. Instead, we could envisage relaxing the Majorana condition and allowing only a formal local supersymmetry. A direct Hamiltonian constraint analysis still shows that only helicities \( \pm 3/2 \) propagate. Also there are other examples where the Majorana condition must be dropped, but nonetheless a formal supersymmetry yields the desired Ward identities: continuation to Euclidean space is a familiar case \[ 30, 31, 32, 33, 34, 35 \].

The summary in favor of dS supergravity is then that the free field limit is unitary with time evolution governed by the Hermitean generator \( D = iM_{40} \) subject to a positive energy condition—within the physical region inside the Killing horizon. A formal supersymmetry, similar to that remaining when Euclideanizing supersymmetric Minkowski models, suffices to show that amplitudes obey supersymmetric Ward identities.

\section{Conclusions}

Our work has consisted of two parts, one unambiguously correct, the other more speculative, or better, open-ended. The first, demonstrable, one was devoted to a simple formulation of generic spin massive models in arbitrary dimensional dS, one that was particularly relevant to hierarchies of partial massive higher spins. Use of a dS/CFT correspondence between the fields and their (boundary) Euclidean limits in one lower dimension was an important ingredient in the process, and our conclusions established and generalized to arbitrary dimensions our original conjectures in this respect. In particular we have provided an explicit realization of the suggestion of \[ 12 \], that an AdS/CFT correspondence for partially massless fields should be considered in dS, rather than AdS where these fields are not unitary. Of course the most pressing question now is whether this new perspective yields any insight on the much harder problem of interactions for partially massless fields.

Our second aim was to review the arguments, pro and con, concerning existence, albeit in a restrictive sense, of N=1 dS supergravity. The arguments against
are well known, revolving about the need for an imaginary spin 3/2 mass parameter and correspondingly imaginary term in its action, all in addition to the generic problems inherited from the intrinsic horizon of dS. But imaginarity in turn implies that the local SUSY is purely formal and the supercharges are not Hermitean. The arguments in favor accept, but try to turn, these manifest difficulties into harmless ones. Whatever their force, they do have on their side the fact that we may well be living inside the horizon of some asymptotically dS world, one in which supersymmetric physics should have a place.

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References

[1] S.M. Carroll, Living Rev. Rel. 4 (2001) 1, astro-ph/0004075,
[2] P.A.M. Dirac, Ann. Math. 36 (1936) 657,
[3] A. Higuchi, J. Math. Phys. 28 (1987) 1553,
[4] S. Deser and R.I. Nepomechie, Phys. Lett. B132 (1983) 321,
[5] S. Deser and R.I. Nepomechie, Ann. Phys. 154 (1984) 396,
[6] S. Deser and A. Waldron, Phys. Lett. B513 (2001) 137, hep-th/0105181,
[7] S. Deser and A. Waldron, Phys. Lett. B508 (2001) 347, hep-th/0103255,
[8] S. Deser and A. Waldron, Nucl. Phys. B607 (2001) 577, hep-th/0103198,
[9] S. Deser and A. Waldron, Phys. Rev. Lett. 87 (2001) 031601, hep-th/0102166,
[10] S. Deser and A. Waldron, Nucl. Phys. B631 (2002) 369, hep-th/0112182,
[11] Y.M. Zinoviev, (2001), hep-th/0108192,
[12] L. Dolan, C.R. Nappi and E. Witten, JHEP 10 (2001) 016, hep-th/0109096,
[13] V.K. Dobrev, Nucl. Phys. B553 (1999) 559, hep-th/9812194,
[14] A. Strominger, JHEP 10 (2001) 034, hep-th/0106113,
[15] L. Brink, R.R. Metsaev and M.A. Vasiliev, Nucl. Phys. B586 (2000) 183, hep-th/0005136,
[16] Y.M. Zinoviev, (2002), hep-th/0211233,
[17] K. Pilch and A.N. Schellekens, J. Math. Phys. 25 (1984) 3455,
[18] B. de Wit, (2002), hep-th/0212245,
[19] V. Dobrev et al., Lecture Notes in Physics 63, Berlin 1977.
[20] V.K. Dobrev et al., Phys. Rev. D13 (1976) 887,
[21] V.K. Dobrev and V.B. Petkova, Rept. Math. Phys. 13 (1978) 233,
[22] I. Todorov, M. Mintchev and V. Petkova, Pisa, Italy: Sc. Norm. Sup. (1978).
[23] P.K. Townsend, Phys. Rev. D15 (1977) 2802,
[24] D.Z. Freedman and A. Das, Nucl. Phys. B120 (1977) 221,
[25] K. Pilch, P. van Nieuwenhuizen and M.F. Sohnius, Commun. Math. Phys. 98 (1985) 105,
[26] E. Witten, (2001), hep-th/0106109,
[27] P. Breitenlohner and D.Z. Freedman, Phys. Lett. B115 (1982) 197,
[28] P. Breitenlohner and D.Z. Freedman, Ann. Phys. 144 (1982) 249,
[29] L.F. Abbott and S. Deser, Nucl. Phys. B195 (1982) 76,
[30] H. Nicolai, Nucl. Phys. B140 (1978) 294,
[31] H. Nicolai, Nucl. Phys. B156 (1979) 157,
[32] H. Nicolai, Nucl. Phys. B156 (1979) 177,
[33] H. Nicolai, Phys. Lett. B89 (1980) 341,
[34] P. van Nieuwenhuizen and A. Waldron, Phys. Lett. B389 (1996) 29, hep-th/9608174,
[35] A. Waldron, Phys. Lett. B433 (1998) 369, hep-th/9702057,