LINEAR EMBEDDINGS OF GRAPHS AND GRAPH LIMITS

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Abstract. Many real-life networks can be modelled by stochastic processes with a spatial embedding. In such processes, the link probability decreases with distance. Using the theory of graph limits, we show how to recognize graph sequences produced by random graph processes with a linear embedding (a natural embedding into $\mathbb{R}$). We define an operator $\Gamma$ which applies to graph limits, which assumes the value zero precisely for graph limits with a linear embedding. Moreover we introduce a corresponding graph parameter $\Gamma^*$ and show that, for graph sequences which converge to a graph limit under the cut-norm, the $\Gamma^*$-values converge to the $\Gamma$-value of the limit.

1. Introduction

The principal purpose of modelling real-life networks is to be able to extract information about these networks. If one assumes that such networks are the manifestation of an underlying reality, then a useful way to model these networks is to take a latent space approach. In this approach, the formation of the graph is informed by the hidden spatial reality. The graph formation is modelled as a stochastic process, where the probability of a link occurring between two vertices decreases as their metric distance increases.

The spatial reality can be used to represent attributes of the vertices which are inaccessible or unknown, but which are assumed to inform link formation. For example, in a social network, vertices may be considered as members of a social space, where the coordinates represent the interests and background of the users. Given only the graph, such a spatial model allows us to mine the underlying spatial reality. This approach was taken by Hoff et al. in [19]. Recent spatial models for web-based networks, where the vertices clearly exist in an underlying space can be found in [5, 10, 20], while protein-protein interaction networks are modelled as geometric graphs in [27]. A one-dimensional spatial model, the niche model, is proposed in [29] to model food webs.

A fundamental question is to determine whether a given network structure, obtained from a real-life network, is compatible with a spatial model. That is, given a graph, how can we judge whether this graph is likely generated by a spatial model, and if so, what are the dimension and other characteristics of the underlying metric space? This question can be answered if a sequence of graphs (snapshots of the stochastic process) is available. The use of graph sequences to model dynamic network data has been used, for example, in [13, 21].

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In particular, if we have a sequence of graphs of growing size which exhibit similar structures, then a notion of convergence can be formulated. If this notion is translated into the formal definition of a limit, then the limiting object encapsulates the common structure of the graphs in the sequence. The theory of graph limits, which applies to sequences of dense graphs, is developed by Lovász and Szegedy in [24].

In this theory, convergence is based on homomorphism densities, and the limit is a symmetric function. The theory is developed and extended to sequences of random graphs by Borgs et al. in [6, 7, 9] and is explored further by Lovász et al., and others (see for example [8, 11, 26]. See also the recent book [23]). As shown by Diaconis and Janson in [15], the theory of graph limits is closely connected to the probabilistic theory of exchangeable arrays. A different view, where the limit object is referred to as a *kernel*, is provided by Bollobás, Janson and Riordan in [1, 2]. The connection with the results of Borgs et al. and an extension of the theory to sparse graphs are presented in [4].

The approach we take in this paper is based on a paper by Bollobás, Janson and Riordan on monotone graph limits (see [3]). In that paper, a graph parameter Ω which assumes value zero only for threshold graphs is introduced. It is then shown that sequences of graphs for which Ω converges to zero have a limit that is a *monotone* function. Thus, monotone graph limits are seen as generalizations of threshold graphs.

Diaconis, Holmes and Janson also consider the limits of threshold graphs (see [15]). In [16], they consider the limits of interval graphs. Note that the one-dimensional geometric graphs mentioned in our paper are a special class of interval graphs: namely unit interval or proper interval graphs. However, the authors of [16] focus on different properties and generalizations of interval graphs, and their results do not apply to the problems we consider here.

We show that examining the limit of a graph sequence allows us to conclude, whether the sequence has a pseudo-random structure that is similar to the typical structure of graphs produced by a probabilistic process (random graph model) which has a spatial embedding. In this work we only consider random graphs with linear embeddings *i.e.* embeddings in \( \mathbb{R} \). We consider embeddings in \( \mathbb{R} \) in the most general sense: A random graph has a linear embedding if its vertex set is sampled from \( \mathbb{R} \), and for a vertex \( v \), and a constant \( a \in [0,1] \), all the points with link probability to \( v \) that is greater than \( a \) lie in an interval around \( v \).

This paper is organized as follows. In Section 2, we briefly review the results from the theory of graph limits. In Section 3 we give precise definitions for the concepts of spatial embedding and linear embedding for a random graph model, and introduce a new graph parameter \( \Gamma^* \), which characterizes one-dimensional geometric graphs. In Section 4 we introduce its continuous analogue, \( \Gamma \), which applies to symmetric measurable functions. In Section 5 we show that, for any graph \( G \), \( \Gamma^*(G) \) is asymptotically equal to the value of \( \Gamma \) applied to the \( \{0,1\} \)-valued function representing \( G \). Finally, in Section 6 we show that, for a convergent graph sequence \( \{G_n\} \), the sequence \( \{\Gamma^*(G_n)\} \) converges to the infimum of \( \Gamma(w) \) over all functions \( w \) in the equivalence class of functions which forms the limit of \( \{G_n\} \). In particular \( \Gamma^*(G_n) \to 0 \) as \( n \to \infty \) if and only if \( \{G_n\} \) converges to a function \( w \) which has \( \Gamma \)-value arbitrarily small.
2. Preliminaries: graph limits

In this section we summarize the basic definitions and results from the theory of graph limits, insofar as they are relevant to this paper. For more background, the reader is referred to the papers referenced in the introduction. A thorough study of the subject can be found in [23]. In this section, we follow the terminology of [24].

Let $F$ and $G$ be two simple graphs, i.e. graphs without loops or multiple edges. Let $V(F)$ and $V(G)$ be vertex sets of $F$ and $G$ respectively. A map $V(F) \to V(G)$ is called a homomorphism from $F$ to $G$ if it maps adjacent vertices in $F$ to adjacent vertices in $G$. Let $\text{hom}(F,G)$ be the number of homomorphisms of $F$ into $G$. The homomorphism density of $F$ into $G$ is defined as

$$t(F,G) = \frac{\text{hom}(F,G)}{|V(G)|^{|V(F)|}}.$$ 

The homomorphism density can be interpreted as the probability that a random mapping $V(F) \to V(G)$ is a homomorphism.

Let $\{G_n\}$ be a sequence of simple graphs such that $|V(G_n)| \to \infty$. We can define a notion of convergence based on homomorphism densities.

**Definition 2.1.** We say that the sequence $\{G_n\}$ converges if for every simple graph $F$, the sequence $\{t(F,G_n)\}$ converges.

This definition of convergence is non-trivial only for dense graphs, i.e. for graph sequences $\{G_n\}$ with the property that $|E(G_n)| = \Omega(|V(G_n)|^2)$. When $\{G_n\}$ consists of sparse graphs, then for all graphs $F$ with at least one edge, $t(F,G_n) \to 0$.

As shown in [6], the notion of convergence of graph sequences is closely connected to a certain metric space described as follows: Let $W_0$ denote the set of all measurable functions $w : [0,1]^2 \to [0,1]$ which are symmetric, i.e. $w(x,y) = w(y,x)$ for every $x,y \in [0,1]$. The elements of $W_0$ are called graphons. We also denote by $W$ the space of all the bounded symmetric measurable functions from $[0,1]^2$ to $\mathbb{R}$. We can extend the definition of homomorphism densities to $W$ as follows. For each function $w \in W$, let

$$t(F,w) = \int_{[0,1]^k} \prod_{i,j \in E(F)} w(x_i,x_j) dx_1 \ldots dx_k,$$

where $V(F) = \{1,2,\ldots,k\}$.

A simple graph $G$, with vertex set $V(G) = \{1,2,\ldots,n\}$ and adjacency matrix $A$, can be represented by a function $w_G \in W_0$, which takes values in $\{0,1\}$. Split the interval $[0,1]$ into $n$ equal intervals $I_1, I_2, \ldots, I_n$. Now for $(x,y) \in I_i \times I_j$, let

$$w_G(x,y) = \begin{cases} A_{i,j} & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}.$$ 

Our definition of $w_G$ differs slightly from that given in [24] since we give the diagonal blocks $I_i \times I_i$ value one, not zero. The advantage of this choice becomes apparent when we discuss “diagonally increasing” functions. It is a convenience and is not essential for the results.

Note that a graph can be represented by many different functions $w_G$. Each labelling of the vertices of $G$ results in a permutation of the rows and columns of
the adjacency matrix, and leads to a trivially different function. Since a graph represents an entire isomorphism class, we need to introduce an equivalent notion for functions in \( \mathcal{W} \). Recall that a map \( \phi : [0,1] \to [0,1] \) is measure-preserving if for every measurable set \( X \subseteq [0,1] \), the pre-image \( \phi^{-1}(X) \) is measurable with the same measure as \( X \). Let \( \Phi \) be the set of all invertible maps \( \phi : [0,1] \to [0,1] \) such that both \( \phi \) and its inverse are measure-preserving. Any \( \phi \in \Phi \) acts on a function \( w \in \mathcal{W} \) by transforming it into a function \( w^\phi \), where \( w^\phi(x,y) = w(\phi(x), \phi(y)) \).

The notion of the convergence of a graph sequence can be better understood if \( \mathcal{W} \) is equipped with a distance derived from the cut-norm, introduced in \cite{17} and defined as follows: For all \( w \in \mathcal{W} \),

\[
\|w\|_\square = \sup_{S,T \subset [0,1]} \left| \int_{S \times T} w(x,y) \, dx \, dy \right|,
\]

where \( S \) and \( T \) are measurable subsets of \([0,1]\). We then define the cut-distance of two functions \( w_1 \) and \( w_2 \) in \( \mathcal{W} \) by

\[
\delta_{\square}(w_1, w_2) = \inf_{\phi \in \Phi} \|w_1 - w_2^\phi\|_\square = \inf_{\phi \in \Phi} \sup_{S,T \subset [0,1]} \left| \int_{S \times T} (w_1 - w_2^\phi) \right|.
\]

This yields the definition of the cut-distance of two (unlabelled) graphs \( G \) and \( G' \), defined as

\[
\delta_{\square}(G, G') = \delta_{\square}(w_G, w_{G'}).
\]

The choice of term “distance” rather than “metric” is due to the fact that \( \delta_{\square}(G, G') \) can be zero for different graphs \( G \) and \( G' \), for example when \( G' \) is the \( k \)-fold blow-up of \( G \) (see \cite{13} for more details).

It is shown in Theorem 3.8 of \cite{6} that a graph sequence \( \{G_n\} \) converges whenever the corresponding sequence of functions \( w_{G_n} \) is \( \delta_{\square} \)-Cauchy. Moreover, to a convergent graph sequence \( \{G_n\} \), one assigns a “limit object” represented by a function \( w \in \mathcal{W}_0 \) (not necessarily integer-valued, or corresponding to a graph). More precisely, for every convergent sequence \( \{G_n\} \), there exists \( w \) in \( \mathcal{W}_0 \) such that the homomorphism densities \( t(F, G_n) \) converge to the homomorphism densities \( t(F, w) \) for every finite simple graph \( F \). If this is the case, we say \( \{G_n\} \) converges to \( w \), and write \( G_n \to w \). Such a function \( w \) encodes the common structure of the graphs of the sequence. For more details, see \cite{24}. In this paper, we use the following characterization of convergent graph sequences which is given in \cite{13}.

**Theorem 2.2.** \cite{6} A sequence \( \{G_n\} \) converges to a function \( w \) in \( \mathcal{W}_0 \) if and only if \( \delta_{\square}(w_{G_n}, w) \to 0 \). Furthermore, if this is the case, and \( \|V(G_n)\| \to \infty \), then there is a way to label the vertices of the graphs \( G_n \) such that \( \|w_{G_n} - w\|_\square \to 0 \).

The limit object of a convergent graph sequence is unique up to measure-preserving transformations. Namely \( w \) and \( w' \) are limits of a convergent graph sequence \( \{G_n\} \) if and only if \( w^\phi = w'^\psi \) almost everywhere for some measure-preserving maps \( \phi, \psi : [0,1] \to [0,1] \) (or equivalently whenever \( \delta_{\square}(w, w') = 0 \)). Note that cut-distance does not define a metric on \( \mathcal{W} \), as two different functions can have \( \delta_{\square} \)-distance zero. We say two functions \( w', w \in \mathcal{W}_0 \) are equivalent, and we write \( w' \approx w \), if \( \delta_{\square}(w', w) = 0 \). Identifying equivalent functions \( w \) and \( w' \) in \( \mathcal{W} \), we consider the cut-distance as a metric on the quotient space \( \mathcal{W}/\approx \), denoted by \( \overline{\mathcal{W}} \).
Similarly, we define the set \( \mathcal{W}_0 \) of unlabelled graphons. It was shown in [24] that \( \mathcal{W}_0 \) is in fact a compact metric space.

Finally, given any function \( w \in \mathcal{W}_0 \) and integer \( n \), we define the random graph \( G(n, w) \) to be the probability space of graphs on vertex set \( \{1, 2, \ldots, n\} \) obtained through the following stochastic process: Each vertex \( j \) receives a value \( x_j \), drawn independently and uniformly at random from \([0, 1]\). For each pair \( i < j \), independently, vertices \( i \) and \( j \) are then linked with conditional probability \( w(x_i, x_j) \). In [24], it is shown that, asymptotically almost surely, for any finite graph \( F \), the homorphism density \( \ell(F, G) \) for a graph \( G \) produced by \( G(n, w) \) is arbitrarily close to \( \ell(F, w) \). Thus, a graph sequence \( \{G_n\} \), where for each \( n \), \( G_n \) is produced by \( G(n, w) \), almost surely converges to \( w \).

3. Linear Embeddings and the Parameter \( \Gamma^* \)

In this section, we will define a graph parameter \( \Gamma^* \) which is zero precisely when the graph is a one-dimensional geometric graph, and thus has a natural linear embedding. In subsequent sections we will then introduce a related parameter \( \Gamma \) which applies to functions in \( \mathcal{W}_0 \). Using graph limits, we will show a close relationship between the two parameters, especially when applied to convergent graph sequences.

First, we need precise definitions of the concepts discussed in the introduction. Following the convention, see for example [22], we use both random graph and random graph model to denote a discrete probability space where the sample space is the set of all graphs on a given vertex set. The notation \( u \sim v \) signifies “\( u \) is adjacent to \( v \)”. The link probability for a given pair of vertices \( u, v \) is the probability of the event \( u \sim v \).

Given a convex region \( S \subseteq \mathbb{R}^k \) equipped with a metric \( d \) derived from one of the \( L_p \) norms, we define a symmetric function \( f : S \times S \rightarrow [0, 1] \) to be a spatial link function if for every \( a \in [0, 1] \) and for every \( x \in S \), the region \( R_a(x) = \{ y : f(x, y) \geq a \} \) is a convex set containing \( x \). Thus, if we move a point \( y \) away from a given point \( x \) along a ray starting at \( x \), then \( f(x, y) \) decreases as the distance from \( x \) increases. This does not mean that \( f(x, y) \) is decreasing in the distance \( d(x, y) \), however. For example, if \( S = [0, 1] \), one can define a spatial link function \( f \) as follows. For a given \( x \), \( f(x, y) = 2(y - x) \) for \( y > x \), and \( f(x, y) = \frac{1}{2}(x - y) \) for \( y < x \). Thus, \( f(x, x + 0.1) = 0.2 > 0.1 = f(x, x - 0.2) \), but \( d(x, x + 0.1) < d(x, x - 0.2) \).

Let \( k \) be a positive integer, and \( S \) be a convex region in \( \mathbb{R}^k \). Let \( d \) denote a metric derived from one of the \( L_p \) norms on \( S \). Fix \( n \in \mathbb{N} \). For a spatial link function \( f \) and a probability measure \( \mu \) on \( S \), we define a spatial random graph \( SG(S, d, f, \mu, n) \) to be a random graph with vertex set \( \{1, 2, \ldots, n\} \) formed according to the following process. Each vertex \( j \) receives a value \( x_j \), drawn from \( S \) according to the probability distribution given by \( \mu \). For each pair \( i < j \), independently, vertices \( i \) and \( j \) are then linked with a conditional probability which equals \( f(x_i, x_j) \).

**Definition 3.1.** A random graph on the vertex set \( \{1, 2, \ldots, n\} \) has a spatial embedding into a given metric space \((S, d)\) if there exist a probability distribution \( \mu \) and a link probability function \( f \) so that the random graph corresponds to the spatial random graph \( SG(S, d, f, \mu, n) \) (i.e. gives the same probability distribution on the sample space of all graphs with vertex set \( \{1, 2, \ldots, n\} \)). A linear embedding is a spatial embedding into \((\mathbb{R}, |\cdot|)\).
The notion of spatial embedding can be seen as a “fuzzy” version of a random geometric graph. A graph $G$ is called a geometric graph on a bounded region $S \subseteq \mathbb{R}^k$ with metric $d$ if there exists an embedding $\pi$ of the vertices of $G$ in $S$, and a threshold value $r > 0$, such that for every two vertices $u$ and $v$ of $G$, $u$ is adjacent to $v$ if and only if $d(\pi(u), \pi(v)) \leq r$. Geometric graphs have been studied extensively; see for example [12, 14, 28]. The random geometric graph $RG(S, n, r)$ is the geometric graph which results if the embeddings of the vertices are chosen randomly from $S$. Random geometric graphs clearly have a spatial embedding. Link probabilities in this case can only be 1 or 0. Precisely, the spatial link function $f$ is given by $f(x, y) = 1$ if $d(x, y) \leq r$, and $f(x, y) = 0$ otherwise. For all $a \in [0, 1]$, $R_a(x)$ equals the closed ball around $x$ of radius $r$, so clearly $f$ is a spatial link function. In this paper, we restrict ourselves to geometric graphs on the one-dimensional space $(\mathbb{R}, |\cdot|)$ and will refer to these as one-dimensional geometric graphs.

We introduce first a graph parameter $\Gamma^*$, which characterizes geometric graphs in $(\mathbb{R}, |\cdot|)$. One-dimensional geometric graphs are also known as unit interval graphs. The correspondence becomes clear if we associate each vertex $u$ of a one-dimensional geometric graph with the interval $[\pi(u) - \frac{r}{2}, \pi(u) + \frac{r}{2}]$, where $\pi$ is the geometric embedding. (We can always assume, without loss of generality, that $r = 1$.) Now vertices $u$ and $v$ are adjacent if and only if the associated intervals overlap.

It is well known that unit interval graphs are characterized by the consecutive 1s property of the vertex-clique matrix (see [18]). Restating this property, it follows that a graph $G$ is one-dimensional geometric if and only if there exists an ordering $\prec$ on the vertex set of $G$ such that

\[ \forall v, z, w \in V(G), \ v \prec z \prec w \ \text{and} \ v \sim w \Rightarrow z \sim v \ \text{and} \ z \sim w. \tag{6} \]

To be self-contained, we present a direct proof below.

**Proposition 3.2.** A graph $G$ is a one-dimensional geometric graph if and only if there exists an ordering $\prec$ on $V(G)$ that satisfies (6).

**Proof.** The forward direction is clear. To prove the converse, we proceed by induction. Suppose that for every graph $G$ with $k < n$ vertices, if $V(G)$ satisfies (6) for an ordering $\prec$, then there exists a linear embedding $\pi$ of vertices of $G$, with the additional conditions that $\pi$ is injective, and that the distance between adjacent vertices is strictly less than one. Also, we assume that the embedding respects the ordering $\prec$, so $u \prec v$ implies that $\pi(u) < \pi(v)$.

Suppose that $G$ is a graph with $n$ vertices, and there exists an ordering $\prec$ on vertices of $G$ which satisfies (6).

Let $\{v_1, \ldots, v_n\}$ be the vertices of $G$ labeled such that $v_i \prec v_j$ whenever $i < j$. The ordering $\prec$ restricted to $V(G) \setminus \{v_n\}$ satisfies Condition (6) for $G - v_n$. Thus, by the induction hypothesis, $G - v_n$ has a linear embedding $\pi$ of $V(G) \setminus \{v_n\}$ into the real line which satisfies the additional conditions. Suppose that $m$ is the smallest index such that $v_m$ is adjacent to $v_n$. Let $\ell = \max\{\pi(v_{n-1}), \pi(v_{m-1}) + 1\}$, and consider the interval $(\ell, \pi(v_m) + 1)$. By the induction hypothesis, $\pi(v_{m-1}) < \pi(v_m)$, and, since $v_m$ and $v_n$ are adjacent, so are $v_m$ and $v_{n-1}$, and thus $\pi(v_{n-1}) < \pi(v_m) + 1$. This implies that $\ell < \pi(v_m) + 1$, and thus the interval is non-empty. Moreover, every point in the interval has distance greater than one to all embeddings of non-neighbours of $v_n$, and distance less than one to all embeddings of neighbours of $v_n$. Therefore, choosing $\pi(v_n)$ in this interval results in a linear embedding of $V(G)$ with the desired properties, and we are done. \qed
Using Condition (3), we define a parameter $\Gamma^*$ on graphs which identifies the one-dimensional geometric graphs. Let $G$ be a graph with a linear order $\prec$ on its vertices. For every $v \in V(G)$, we define the down-set $D(v)$ and the up-set $U(v)$ of $v$ as follows:

$$D(v) = \{ x \in V(G) : x \prec v \} \text{ and } U(v) = \{ x \in V(G) : v \prec x \}.$$ 

For every vertex $v$, the collection of all the neighbours of $v$ is denoted by $N(v)$.

**Definition 3.3.** Let $A \subseteq V(G)$, and $\prec$ be a linear order of the vertex set of $G$. We define,

$$\Gamma^*(G, \prec, A) = \frac{1}{|V(G)|^3} \sum_{u \prec v} \left[ |N(v) \cap A \cap D(u)| - |N(u) \cap A \cap D(u)| \right]_+ + \frac{1}{|V(G)|^3} \sum_{u \prec v} \left[ |N(u) \cap A \cap U(v)| - |N(v) \cap A \cap U(v)| \right]_+,$$

where

$$[x]_+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}.$$ 

We also define

$$\Gamma^*(G, \prec) = \max_{A \subseteq V(G)} \Gamma^*(G, \prec, A),$$

and

$$\Gamma^*(G) = \min_{\prec} \Gamma^*(G, \prec),$$

where the minimum is taken over all the linear orderings of $V(G)$.

**Proposition 3.4.** A graph $G$ is one-dimensional geometric if and only if $\Gamma^*(G) = 0$.

**Proof.** Let $G$ be a one-dimensional geometric graph, and $A$ be an arbitrary subset of $V(G)$. Let $\prec$ be a linear ordering that satisfies Condition (3). Fix an arbitrary pair of vertices $u \prec v$ of $G$. By Condition (3), if $z$ belongs to $N(v) \cap A \cap D(u)$ then $z$ is adjacent to $u$ as well. Thus $|N(v) \cap A \cap D(u)| \leq |N(u) \cap A \cap D(u)|$. Similarly, $|N(u) \cap A \cap U(v)| \leq |N(v) \cap A \cap U(v)|$, which implies that $\Gamma^*(G, \prec) = 0$. Thus $\Gamma^*(G) = 0$.

Conversely, let $G$ be a graph such that $\Gamma^*(G) = 0$. Let $\prec$ be the linear order of $V(G)$ such that $\Gamma^*(G, \prec) = 0$. Let $u \prec v$ be an arbitrary pair of adjacent vertices of $G$, and take $z$ so that $u \prec z \prec v$. Since $\Gamma^*(G, \prec, A) = 0$ for all $A \subseteq V(G)$, choosing $A = \{ v \}$ gives that $1 = |N(u) \cap \{ v \} \cap U(z)| \leq |N(z) \cap \{ v \} \cap U(z)|$. This implies that $z$ is adjacent to $v$. Similarly, one can show that $z$ is adjacent to $u$. Thus Condition (3) is satisfied for $G$, and $G$ is a geometric graph. \qed

Next, we extend Condition (3) to functions in $W_0$. The generalization is obtained by considering functions representing graphs, as introduced in the previous section. Let $G$ be a one-dimensional geometric graph with a linear ordering $\prec$ of its vertices that satisfies Condition (3). Let $w_G$ be the function in $W_0$ that represents $G$ with respect to the labelling of $V(G)$ obtained from the linear ordering $\prec$. It follows that $w_G(x,z) = 1$ and $x \leq y \leq z$ imply that $w_G(x,y) = 1$ and $w_G(y,z) = 1$. We generalize this property as follows:

**Definition 3.5.** A function $w \in W$ is diagonally increasing if for every $x, y, z \in [0,1]$, we have
A function $w$ in $W$ is diagonally increasing almost everywhere if there exists a diagonally increasing function $w'$ which is equal to $w$ almost everywhere.

Combining definitions 3.1 and 3.3 it is clear that a symmetric function $w$ is a spatial link function on $[0,1]$ if and only if $w$ is diagonally increasing. In the following remark, we show that a $w$-random graph has a “reasonable” linear embedding whenever $w$ is equivalent to a diagonally increasing function.

Remark. Note that the random graphs $G(n,w)$ and $G(n,w')$ are the same, i.e. they are identical as probability distributions, if $w \approx w'$. To see this, let $Pr_w(F)$ denote the probability assigned to a simple graph $F$ on vertex set $\{1,2,\ldots,n\}$ in $G(n,w)$. Clearly, $Pr_w(F) = \int \prod_{i \neq j} w(x_i, x_j) \prod_{k \neq l} (1 - w(x_k, x_l)) = \sum_{F'} (-1)^{|e(F')| - |e(F)|} t(F', w)$, where the sum is taken over all graphs $F'$ on vertex set $\{1,2,\ldots,n\}$ which contain $F$ as their subgraph. Our claim clearly follows from Corollary 3.10 of [6], which we state below:

For two graphons $w$ and $w'$ we have $\delta_C(w,w') = 0$ if and only if $t(F, w) = t(F, w')$ for every simple graph $F$.

Thus, if $w$ is equivalent to a diagonally increasing function, then for any integer $n > 1$, the random graph $G(n,w)$ has a linear embedding.

The converse is also true, under certain conditions. Namely, suppose $G(n,w)$ has a linear embedding $SG([0,1], |\cdot|, f, \mu, n)$. Also suppose that $\mu$ is a continuous probability distribution (i.e. absolutely continuous with respect to Haar measure), that assigns nonzero measures to open intervals in $[0,1]$. Let $F$ be the cumulative distribution function of $\mu$ on $[0,1]$. Then, if $x$ is sampled uniformly from $[0,1]$, $F(x)$ is sampled according to $\mu$. Let $w'(x,y) = f(F(x), F(y))$, where $f$ is the spatial link function. An argument similar to our previous discussion implies that for every simple graph $H$, the densities $t(H, w)$ and $t(H, w')$ are the same. Thus, $\delta_C(w,w') = 0$. Moreover, $w'$ is diagonally increasing, since $F$ is increasing and $f$ is a spatial link function. Therefore $w$ is equivalent to a diagonally increasing function.

Clearly, a graph is a one-dimensional geometric graph if and only if it has a function representative in $W_0$ which is diagonally increasing. (Remember that we assume the function representative to have all blocks on the diagonal equal to 1.) Indeed, the function representative will be the function $w_G$ where the vertices are ordered according to a linear ordering that satisfies Condition [6]. More important is the connection between diagonally increasing functions and linear embeddings, which follows in the next section.

4. THE PARAMETER $\Gamma$ ON $W$

Next, we introduce a parameter $\Gamma$ which generalizes the graph parameter $\Gamma^*$ to functions in $W$. We will see that $\Gamma$ identifies the diagonally increasing functions.
**Definition 4.1.** Let \( A \) denote the collection of all measurable subsets of \([0, 1]\). Let \( w \) be a function in \( W \), and \( A \in A \). We define

\[
\Gamma(w, A) = \int \int_{y < z} \left[ \int_{x \in A \cap [0, y]} (w(x, z) - w(x, y)) \, dx \right]_+ dy dz + \int \int_{y < z} \left[ \int_{x \in A \cap [z, 1]} (w(x, y) - w(x, z)) \, dx \right]_+ dy dz.
\]

Moreover, \( \Gamma(w) \) is defined as

\[
\Gamma(w) = \sup_{A \in A} \Gamma(w, A),
\]

where the supremum is taken over all the measurable subsets of \([0, 1]\).

It follows directly from the definitions that any function \( w \in W \) which is almost everywhere diagonally increasing has \( \Gamma(w) = 0 \). The converse also holds, as is stated in the following proposition.

**Proposition 4.2.** Let \( w \) be a function in \( W \). The function \( w \) is diagonally increasing almost everywhere if and only if \( \Gamma(w) = 0 \).

Before we give the proof, we introduce some notations which will be used later. Let \( w \in W_0 \), and \( A \) and \( B \) be measurable subsets of \([0, 1]\). We define \( \hat{w}(A, B) \) to be the average of \( w \) on \( A \times B \), i.e.

\[
\hat{w}(A, B) = \frac{1}{\mu(A) \mu(B)} \int_{A \times B} w(x, y) \, dx dy,
\]

where \( \mu \) is the Lebesgue measure on \([0, 1]\). Let \( n \) be a positive integer. For each \( 0 \leq i \leq n - 1 \), let \( I_i = \left[ \frac{i}{n}, \frac{i+1}{n} \right] \). We define the symmetric functions \( w_n, w_n^+, \) and \( w_n^- \) on \([0, 1]^2\) as follows.

\[
w_{n,i,j}^+ = \hat{w}(I_i, I_j) \text{ for } 0 \leq i, j \leq n - 1,
\]

\[
w_n(x, y) = w_{n,i,j}^+ \text{ if } (x, y) \in I_i \times I_j,
\]

\[
w_n^-(x, y) = \begin{cases} w_{n,i-1,j+1}^+ & \text{if } (x, y) \in I_i \times I_j \text{ & } 1 \leq i \leq j \leq n - 2 \\ 0 & \text{if } (x, y) \in I_0 \times I_j \\ 0 & \text{if } (x, y) \in I_i \times I_{n-1} \end{cases}
\]

\[
w_n^+(x, y) = \begin{cases} w_{n+1,j-1}^+ & \text{if } (x, y) \in I_i \times I_j \text{ & } i \leq j - 2 \\ 1 & \text{if } (x, y) \in I_i \times I_i \\ 1 & \text{if } (x, y) \in I_i \times I_{i+1} \end{cases}
\]

Let \( A \) and \( B \) be subsets of \([0, 1]\). We say \( A \leq B \) if every \( a \) in \( A \) is smaller than or equal to every \( b \) in \( B \).

We now give the proof of Proposition 4.2. This proof is inspired by the proof of Lemma 4.6 of [3]. However, we include the proof to make the paper self-contained.

**Proof of Proposition 4.2.** Clearly, if \( w \) is diagonally increasing almost everywhere then \( \Gamma(w) = 0 \). We now prove the other direction. First, let us assume that \( w \) is a function in \( W_0 \) with \( \Gamma(w) = 0 \). Let \( A, B, \) and \( C \) be measurable subsets of \([0, 1]\) such that \( C \leq A \leq B \). Since \( \Gamma(w) = 0 \), for almost every \( y \in A \) and almost every \( z \in B \),

\[
\int_{x \in C} w(x, z) \, dx \leq \int_{x \in C} w(x, y) \, dx.
\]
Taking repeated integrals of both sides of Equation (7) over \( A \) and then \( B \) and then dividing by \( \mu(A) \), we conclude that

\[
(8) \quad \int_{C \times B} w(x, z)dz \leq \frac{\mu(B)}{\mu(A)} \int_{C \times A} w(x, y)dy.
\]

Similarly, one can show that for subsets \( A, B, \) and \( C \) of \([0, 1] \) with \( A \leq B \leq C \), we have

\[
(9) \quad \int_{A \times C} w(x, y)dydx \leq \frac{\mu(A)}{\mu(B)} \int_{B \times C} w(x, z)dzdx.
\]

Applying the above inequalities to the sets \( I_i \), we have that for every \((x, y) \in [0, 1]^2\), \( w_n^-(x, y) \leq w_n(x, y) \leq w_n^+(x, y) \). Now let \( A \) and \( B \) be measurable subsets of \([0, 1] \). From Equations (8) and (9) it follows that, if \( 0 \leq i \leq j - 2 \leq n - 3 \), then

\[
\int_{(A \cap I_i) \times (B \cap I_j)} w(x, y)dydx \leq \frac{\mu(B \cap I_j)}{\mu(I_{j-1})} \int_{(A \cap I_i) \times I_{j-1}} w(x, y)dydx
\]

\[
\leq \frac{\mu(A \cap I_i)\mu(B \cap I_j)}{\mu(I_{i+1})\mu(I_{j-1})} \int_{I_{i+1} \times I_{j-1}} w(x, y)dydx
\]

\[
= \mu(A \cap I_i)\mu(B \cap I_j)w_{n+1,i,j-1}^+.
\]

Thus,

\[
(10) \quad \int_{(A \cap I_i) \times (B \cap I_j)} w(x, y)dydx \leq \int_{(A \cap I_i) \times (B \cap I_j)} w_n^+(x, y)dydx.
\]

By definition of \( w_n^+ \), similar inequalities hold trivially for the cases where \( i = j - 1 \) or \( i = j \). Finally, using the fact that \( w \) is symmetric, we conclude that (10) holds for every \( i \) and \( j \). Therefore,

\[
\int_{A \times B} w(x, y)dydx = \sum_{i,j=0}^{n-1} \int_{(A \cap I_i) \times (B \cap I_j)} w(x, y)dydx
\]

\[
\leq \sum_{i,j=0}^{n-1} \int_{(A \cap I_i) \times (B \cap I_j)} w_n^+(x, y)dydx
\]

\[
= \int_{A \times B} w_n^+(x, y)dydx.
\]

Moreover, since measurable subsets of \([0, 1]^2\) can be approximated in measure by finite unions of disjoint rectangles, we get

\[
\int_E w(x, y)dydx \leq \int_E w_n^+(x, y)dydx,
\]

for every measurable subset \( E \) of \([0, 1]^2\). Thus, \( w \leq w_n^+ \) (and similarly \( w_n^- \leq w \)) almost everywhere in \([0, 1]^2\). Therefore,

\[
\|w - w_n\|_1 \leq \|w_n^+ - w_n^-\|_1 = \int_{[0,1]^2} (w_n^+ - w_n^-)(x, y)dydx.
\]

By the definitions of \( w_n^+ \) and \( w_n^- \), we have \( \int_{I_i \times I_j} w_n^-(x, y)dydx = \int_{I_{i-2} \times I_{j+2}} w_n^+(x, y)dydx \) for every pair \( i, j \) satisfying \( 2 \leq i \leq j - 1 \leq n - 4 \). Moreover, \( w_n^+, w_n^- \in \mathcal{W}_0 \). Thus,

\[
\|w - w_n\|_1 \leq \frac{8}{n}.
\]
Using the Borel-Cantelli lemma, we conclude that the sequence \( \{w_{2n}\}_{n \in \mathbb{N}} \) converges to \( w \) almost everywhere in \([0, 1]^2\), i.e. \( \psi := \limsup_{n \to \infty} w_{2n} = w \) almost everywhere.

Finally, by Equations (3) and (4), each \( w_n \) is a diagonally increasing function. Therefore, \( \psi \) is diagonally increasing as well. This proves the converse for the case where \( w \in \mathcal{W}_0 \).

Now let \( w \) be an element of \( \mathcal{W} \) such that \( \Gamma(w) = 0 \). Define the new symmetric function \( w' \) to be \( w' = \frac{w-a}{b-a} \), where \( a \) (respectively \( b \)) is a lower bound (respectively upper bound) for \( w \). Then \( w' \in \mathcal{W}_0 \) and \( \Gamma(w') = 0 \). Therefore, by the previous part of the proof, we have that \( w' \) is diagonally increasing almost everywhere. Hence, \( w \) is diagonally increasing almost everywhere as well. \( \square \)

5. Parameters \( \Gamma^* \) and \( \Gamma \) asymptotically agree on graphs

A graph \( G \) can be represented as a function \( w_G \in \mathcal{W}_0 \), but it is not necessarily true that \( \Gamma^*(G) = \Gamma(w_G) \), even when the representation \( w_G \) is obtained by using the ordering of the vertices that achieves \( \Gamma^*(G) \). This is due to the fact that a set \( A \) which determines the value of \( \Gamma(w) \) does not have to be consistent with the partition of \([0, 1] \) into \( n \) equal-sized parts on which \( w_G \) is defined. However, we show that \( \Gamma^*(G) \) and \( \Gamma(w_G) \), computed using the same ordering of the vertices, are asymptotically equal. This result follows as a corollary from the following theorem.

**Theorem 5.1.** Let \( n \in \mathbb{N} \). Let \( w \in \mathcal{W}_0 \) be a function which is measurable with respect to the product algebra \( A_n^* \times A_n^* \), where the algebra \( A_n^* \) is generated by the intervals \( \{I_i : 0 \leq i \leq n-1\} \). Then

\[
\Gamma(w) = \sup_{A \in A_n^*} \Gamma(w, A) = \max_{A \in A_n^*} \Gamma(w, A) + O\left(\frac{1}{n}\right).
\]

**Proof.** Let \( n \in \mathbb{N} \) and \( w \in \mathcal{W}_0 \) be as above. Note that \( w \) is constant on the rectangles \( I_i \times I_j \), since it is measurable with respect to the product algebra \( A_n^* \times A_n^* \). For each \( i, j \in \{0, \ldots, n-1\} \), let \( w(x, y) = a_{ij} \) whenever \( (x, y) \in I_i \times I_j \). Fix \( A \in A \), and let \( \beta_k = \mu(A \cap I_k) \) for every \( 0 \leq k \leq n-1 \). The expression for \( \Gamma(w, A) \) as given in Definition (1) can now be simplified.

Consider \( y < z \) so that \( y \in I_i \) and \( z \in I_j \). If \( i = j \), then for all \( x, w(x, z) = w(x, y) \), so \( \left[ \int_{x \in A \cap [0, y]} (w(x, z) - w(x, y)) \, dx \right]_+ = 0 \). If \( 0 \leq i < j \leq n-1 \), then

\[
\begin{align*}
\left[ \int_{x \in A \cap [0, y]} (w(x, z) - w(x, y)) \, dx \right]_+ & = \sum_{k=0}^{i-1} \int_{A \cap I_k} (a_{kj} - a_{ki}) \, dx + \int_{A \cap I_i \cap [0, y]} (a_{ij} - a_{ii}) \, dx_+ \\
& = \sum_{k=0}^{i-1} \mu(A \cap I_k)(a_{kj} - a_{ki}) + \mu(A \cap I_i \cap [0, y])(a_{ij} - a_{ii})_+ \\
& \leq \left( \left[ \sum_{k=0}^{i-1} \beta_k(a_{kj} - a_{ki}) \right]_+ + \frac{2}{n} \right).
\end{align*}
\]

In the last step, we use the inequality \( |x + y|_+ \leq |x|_+ + |y|_+ \), and the fact that \( w \) is bounded by 1, so \( |\mu(A \cap I_i \cap [0, y])(a_{ij} - a_{ii})| \) is at most \( \frac{2}{n} \).
Similarly, we have that
\[
\left[ \int_{x \in A \cap [z,1]} (w(x,y) - w(x,z)) \, dx \right]_+ \\
= \left[ \sum_{k=j+1}^{n-1} \mu(A \cap I_k)(a_{ki} - a_{kj}) + \mu(A \cap I_j \cap [z,1])(a_{ji} - a_{jj}) \right]_+ \\
\leq \left( \left[ \sum_{k=j+1}^{n-1} \beta_k(a_{ki} - a_{kj}) \right]_+ + \frac{2}{n} \right).
\]

Using this, we can bound \( \Gamma(w,A) \):
\[
\Gamma(w, A) \leq \sum_{0 \leq i < j \leq n-1} \int_{y \in I_i} \int_{z \in I_j} \left( \left[ \sum_{k=0}^{i-1} \beta_k(a_{kj} - a_{ki}) \right]_+ + \frac{2}{n} \right) \, dy \, dz \\
+ \sum_{0 \leq i < j \leq n-1} \int_{y \in I_i} \int_{z \in I_j} \left( \left[ \sum_{k=j+1}^{n-1} \beta_k(a_{ki} - a_{kj}) \right]_+ + \frac{2}{n} \right) \, dy \, dz \\
= \sum_{0 \leq i < j \leq n-1} \frac{1}{n^2} \left[ \sum_{k=0}^{i-1} \beta_k(a_{kj} - a_{ki}) \right]_+ + \frac{n-1}{n^2} \\
+ \sum_{0 \leq i < j \leq n-1} \frac{1}{n^2} \left[ \sum_{k=j+1}^{n-1} \beta_k(a_{ki} - a_{kj}) \right]_+ + \frac{n-1}{n^2}.
\]

Now define,
\[
g_w(A) = g_w(\beta_0, \ldots, \beta_{n-1}) \\
= \sum_{0 \leq i < j \leq n-1} \frac{1}{n^2} \left( \left[ \sum_{k=0}^{i-1} \beta_k(a_{kj} - a_{ki}) \right]_+ + \left[ \sum_{k=j+1}^{n-1} \beta_k(a_{ki} - a_{kj}) \right]_+ \right).
\]

Thus,
\[
(11) \quad \Gamma(w, A) \leq g_w(A) + \frac{2(n-1)}{n^2} \leq g_w(A) + \frac{2}{n}.
\]

Similarly, one can use the inequality \([x + y]_+ \geq [x]_+ - |y|\) to show that
\[
(12) \quad \Gamma(w, A) \geq g_w(A) - \frac{2}{n}.
\]

Since \( x \mapsto [x]_+ \) is a convex function, \( g_w \) is the sum of convex functions, and therefore is itself also convex. Moreover, since \( \beta_k \in [0, \frac{1}{n}] \), the function \( g_w \) achieves its maximum when each of the coefficients \( \beta_k \) is either 0 or \( \frac{1}{n} \). Since \( \beta_k = \mu(A \cap I_k) \), this implies that the maximum is achieved when, for each \( k \), either \( A \) contains \( I_k \), or is disjoint from \( I_k \). Hence, \( \sup_{A \in \mathcal{A}} g_w(A) = \max_{A \in \mathcal{A}_n^+} g_w(A) \).
Let \( A' \in \mathcal{A}_n^* \) be such that \( \max_{A \in \mathcal{A}_n^*} g_w(A) = g_w(A') \). Then, by Equation (11) and (12) we have,

\[
\sup_{A \in \mathcal{A}} \Gamma(w, A) \leq \sup_{A \in \mathcal{A}} g_w(A) + \frac{2}{n} \leq \Gamma(w, A') + \frac{4}{n}
\]

On the other hand, we clearly have

\[
\max_{A \in \mathcal{A}_n^*} \Gamma(w, A) \leq \sup_{A \in \mathcal{A}} \Gamma(w, A),
\]

completing the proof.

\[\square\]

Corollary 5.2. Let \( G \) be a graph with \( n \) vertices, and \( w_G \) be the function in \( W_0 \) that represents \( G \) with respect to a linear ordering \( \prec \) of the vertices of \( G \). Then

\[
\Gamma^*(G, \prec) = \Gamma(w_G) + \mathcal{O}(\frac{1}{n}).
\]

Proof. Let \( A \in \mathcal{A}_n^* \), and define \( \tilde{A} = \{0 \leq i \leq n-1 : I_i \subseteq A \} \). From the proof of Theorem 5.1 it is easy to observe that \( \Gamma^*(G, \prec, \tilde{A}) = g_{w_G}(A) \), and \( g_{w_G}(A) - \frac{2}{n} \leq \Gamma(w_G, A) \leq g_{w_G}(A) + \frac{2}{n} \). Thus,

\[
\max_{A \in \mathcal{A}_n^*} \Gamma^*(G, \prec, \tilde{A}) - \frac{2}{n} \leq \max_{A \in \mathcal{A}_n^*} \Gamma(w_G, A) \leq \max_{A \in \mathcal{A}_n^*} \Gamma^*(G, \prec, \tilde{A}) + \frac{2}{n},
\]

Using Theorem 5.1 we conclude that \( |\Gamma^*(G, \prec) - \Gamma(w_G)| \leq \frac{6}{n} \), and we are done. \[\square\]

6. Continuity of the parameter \( \Gamma \).

Our final result is that the parameter \( \Gamma \) is continuous under the cut-norm. In particular, let \( \{G_n\} \) be a graph sequence converging to a limit \( w \in W_0 \), and let \( \prec_n \) be appropriate labellings of vertices of \( G_n \) such that the functions in \( W_0 \) representing \( G_n \) with respect to \( \prec_n \) converge to \( w \) in cut-norm. Then \( \Gamma^*(G_n, \prec_n) \) converges to \( \Gamma(w) \).

This continuity means that if a graph sequence \( \{G_n\} \) converges to a diagonally increasing function then \( \{\Gamma^*(G_n)\} \) goes to zero. Namely, if a graph sequence converges to a graph limit \( w \), then the graphs in the sequence have approximately the same homomorphism densities as the \( w \)-random graph \( G(n, w) \) for large \( n \). Moreover, the \( w \)-random graph has a linear embedding. Therefore, the graphs in the sequence are structurally similar to those produced by a random graph process with a linear embedding, and thus their vertices have a natural interpretation as points in \( \mathbb{R} \). Moreover, the optimal order for \( \Gamma^* \) gives the order in which points should be placed in \( \mathbb{R} \). This prompts us to conjecture that \( \Gamma^* \) is a parameter that measures “embeddability” in \( \mathbb{R} \).

We first present the following lemmas.

Lemma 6.1. Let \( w : [0,1]^2 \to [-2,2] \) be a measurable function. Then \( \|w\chi\|_{\Box} \leq 2\sqrt{\|w\|_{\Box}} \), where

\[
\chi(x, y) = \begin{cases} 
1 & x \leq y \\
0 & \text{otherwise}
\end{cases}.
\]
Proof. Let $\Omega = \{(x, y) : 0 \leq x \leq y \leq 1\}$ denote the subset of points above the diagonal in $[0, 1]^2$. Define $k = \lceil \frac{1}{\sqrt{\|w\|_\Box}} \rceil$, which is a positive integer. Now, we can decompose $\Omega$ into $k - 1$ rectangles and $k$ triangles as shown in Figure 6. Precisely, the $i$-th rectangle has width $\frac{1}{k}$ and ranges from $y = \frac{i}{k}$ to $y = 1$, and each triangle has base and height equal to $\frac{1}{k}$. By the definition of cut-norm, the integral of $w$ over each of the rectangles is at most $\|w\|_\Box$, in absolute value. Also, each of the triangles has measure $\frac{1}{2k^2}$, and there are $k$ triangles in total. Since $|w|$ is bounded by 2, the integral of $w$ over the triangles is at most $\frac{1}{2k^2}(2)(k) = \frac{1}{k}$, in absolute value. Therefore, we have

$$\int_0^1 \int_0^1 w(x, y) \, dx \, dy \leq \frac{1}{k} + (k - 1)\|w\|_\Box \leq \sqrt{\|w\|_\Box} + \left(\frac{1}{\sqrt{\|w\|_\Box}}\right)\|w\|_\Box = 2\sqrt{\|w\|_\Box}. \quad (15)$$

For arbitrary subsets $A$ and $B$ of $[0, 1]$, let $\chi_{A \times B}$ denote the characteristic function of the subset $A \times B$ of $[0, 1]^2$. Applying (15) to $w\chi_{A \times B}$ instead of $w$, we get $|\int_0^1 \int_0^1 w\chi_{A \times B}(x, y) \, dx \, dy| \leq 2\sqrt{\|w\chi_{A \times B}\|_\Box} \leq 2\sqrt{\|w\|_\Box}$, which proves that $\|w\|_\Box \leq 2\sqrt{\|w\|_\Box}$. \hfill \Box

**Lemma 6.2.** Let $w_1$ and $w_2$ be elements of $\mathcal{W}_0$. Then $|\Gamma(w_1) - \Gamma(w_2)| \leq 2\|w_1 - w_2\|_\Box + 4\sqrt{\|w_1 - w_2\|_\Box}$. 

**Proof.** Let

$$\Gamma_1(w, A) = \int \int_{y < z} \left[ \int_{x \in A \cap [0, y]} (w(x, z) - w(x, y)) \, dx \right]_+ \, dy \, dz,$$

$$\Gamma_2(w, A) = \int \int_{y < z} \left[ \int_{x \in A \cap [z, 1]} (w(x, y) - w(x, z)) \, dx \right]_+ \, dy \, dz,$$
so $\Gamma(w, A) = \Gamma_1(w, A) + \Gamma_2(w, A)$. Fix a measurable set $A \in \mathcal{A}$. Using again the inequality $|x + y|_+ \leq |x|_+ + |y|_+$, we obtain that

$$\Gamma_1(w_1, A) = \int \int_{y < z} \left[ \int_{x \in A \cap [0,y]} (w_1(x, z) - w_1(x, y)) \, dx \right] \, dy \, dz + \\ \leq \int \int_{y < z} \left[ \int_{x \in A \cap [0,y]} (w_1(x, z) - w_2(x, z)) \, dx \right] \, dy \, dz + \\ + \int \int_{y < z} \left[ \int_{x \in A \cap [0,y]} (w_2(x, z) - w_2(x, y)) \, dx \right] \, dy \, dz + \\ + \int \int_{y < z} \left[ \int_{x \in A \cap [0,y]} (w_2(x, y) - w_1(x, y)) \, dx \right] \, dy \, dz.$$

Recall that a function on $[0, 1]$ attains a value at least as large as the average of the function at some point. Therefore there exists $y_0, z_0 \in [0, 1]$ such that

$$\Gamma_1(w_1, A) \leq \int_{y < z} \left[ \int_{x \in A \cap [0,y_0]} (w_1(x, z) - w_2(x, z)) \, dx \right] \, dy \, dz + \\ + \Gamma_1(w_2, A) + \\ + \int_{y < z_0} \left[ \int_{x \in A \cap [0,y]} (w_2(x, y) - w_1(x, y)) \, dx \right] \, dy + \\ = \int_{x \in T_1} \int_{x \in A \cap [0,y_0]} (w_1(x, z) - w_2(x, z)) \, dx \, dz + \\ + \Gamma_1(w_2, A) + \\ + \int_{y \in T_2} \int_{x \in A \cap [0,y]} (w_2(x, y) - w_1(x, y)) \, dx \, dy,$$

where $T_1$ and $T_2$ are the appropriate sets of points which make the associated expressions positive. From the definition of the cut-norm, it then follows that

$$\Gamma_1(w_1, A) - \Gamma_1(w_2, A) \leq \|w_1 - w_2\|_\square + \|(w_1 - w_2) \chi\|_\square.$$

Similarly, by switching $w_1$ and $w_2$, we get $\Gamma_1(w_2, A) - \Gamma_1(w_1, A) \leq \|w_1 - w_2\|_\square + \|(w_1 - w_2) \chi\|_\square$, which implies that

$$|\Gamma_1(w_1, A) - \Gamma_1(w_2, A)| \leq \|w_1 - w_2\|_\square + \|(w_1 - w_2) \chi\|_\square$$
holds for every subset $A$. Moreover, one can prove the analogous result for $\Gamma_2$. Thus, $|\Gamma_1(w_1, A) - \Gamma_2(w_2, A)| \leq |\Gamma_1(w_1, A) - \Gamma_1(w_2, A)| + |\Gamma_2(w_1, A) - \Gamma_2(w_2, A)| \leq 2\|w_1 - w_2\|_\square + 2\|(w_1 - w_2) \chi\|_\square$.

Since $\Gamma(w_i) = \sup_A \Gamma(w_i, A)$ for $i = 1, 2$, it follows that

$$|\Gamma(w_1) - \Gamma(w_2)| \leq 2\|w_1 - w_2\|_\square + 2\|(w_1 - w_2) \chi\|_\square.$$

This fact, together with Lemma [31], finishes the proof.

We are now ready to prove our continuity result. In order to study the limit of the sequence $\{\Gamma^\ast(G_n)\}$, we need to define the following parameter, which is a
such that it satisfies assumption, Proof. By the definition of \( \tilde{\Gamma} \), for each positive integer \( u \) implies that \( \limsup_{n} \)

By Lemma 6.2 we have, that there exist maps \( \psi_{n} \) as

Thus, for every \( \tilde{\Gamma} \) on the space \( W_{0} \) with the cut-distance \( \delta \).

**Theorem 6.4.** Let \( w \in W_{0} \) be the limit of a \( \delta_{\square} \)-convergent sequence \( \{ w_{n} \}_{n \in \mathbb{N}} \) of functions in \( W_{0} \). Then \( \{ \tilde{\Gamma}(w_{n}) \}_{n \in \mathbb{N}} \) converges to \( \tilde{\Gamma}(w) \) as \( n \to \infty \).

**Proof.** By the definition of \( \tilde{\Gamma} \), for each positive integer \( m \) there exists an element \( u_{m} \in W_{0} \) such that \( \delta_{\square}(w, u_{m}) = 0 \) and \( |\Gamma(u_{m}) - \tilde{\Gamma}(w)| \leq \frac{1}{m} \). Fix such a sequence of graphons \( \{ u_{m} \}_{m=1}^{\infty} \).

Fix \( m \in \mathbb{N} \). Then

as \( n \) goes to infinity. By the definition of cut-distance, this convergence implies that there exist maps \( \psi_{n} \in \Phi \) such that

By Lemma 6.2 we have,

Thus, for every \( m \in \mathbb{N} \),

which implies that \( \limsup_{n \in \mathbb{N}} \tilde{\Gamma}(w_{n}) \leq \tilde{\Gamma}(w) \).

To prove the other inequality, let \( \gamma := \liminf_{n \in \mathbb{N}} \tilde{\Gamma}(w_{n}) \), and recall that, by assumption, \( \delta_{\square}(w_{n}, w) \to 0 \) as \( n \to \infty \). Fix \( 0 < \epsilon < 1 \), and let \( n \in \mathbb{N} \) be chosen such that it satisfies

In addition, let \( w'_{n} \in W_{0} \) be such that \( \delta_{\square}(w'_{n}, w_{n}) = 0 \) and \( |\Gamma(w'_{n}) - \tilde{\Gamma}(w_{n})| < \epsilon/3 \). By definition of the \( \delta_{\square} \)-distance, there exists \( \phi \in \Phi \) such that \( \| w'_{n} - w^{\phi} \|_{\square} < \frac{\epsilon^{2}}{18\epsilon} \) and thus, by Lemma 6.2

Therefore, for every \( 0 < \epsilon < 1 \), \( \tilde{\Gamma}(w) \leq \Gamma(w^{\phi}) \leq \liminf_{n \in \mathbb{N}} \tilde{\Gamma}(w_{n}) + \epsilon \). Combining this with the lower bound, we get

which implies that \( \lim_{n \to \infty} \tilde{\Gamma}(w_{n}) = \tilde{\Gamma}(w) \).

**Corollary 6.5.** Let \( w \in W_{0} \) be the limit of a convergent sequence \( \{ G_{n} \}_{n \in \mathbb{N}} \) of graphs with \( |V(G_{n})| \to \infty \). Then \( \{ \Gamma^{*}(G_{n}) \}_{n \in \mathbb{N}} \) converges to \( \tilde{\Gamma}(w) \) as \( n \to \infty \).
Proof. For each \( n \in \mathbb{N} \), let \( w'_{G_n} \) be the step function representing \( G_n \) with respect to an ordering \( \prec'_{n} \) that is optimal for \( \Gamma^* \). Thus, by Corollary 5.2,

\[
\inf_{n \in \mathbb{N}} \Gamma^*(G_n) = \inf_{n \in \mathbb{N}} \Gamma^*(G_n, \prec'_{n}) = \inf_{n \in \mathbb{N}} \Gamma(w'_{G_n}) \geq \inf_{n \in \mathbb{N}} \tilde{\Gamma}(w'_{G_n}).
\]

Clearly the sequence \( \{w'_{G_n}\} \) converges to \( w \) with respect to \( \delta \)-distance. Thus by Theorem 6.4,

\[
\inf_{n \in \mathbb{N}} \Gamma^*(G_n) \geq \tilde{\Gamma}(w).
\]

On the other hand, let \( u \in \mathcal{W}_0 \) be an element equivalent to \( w \) such that \( \tilde{\Gamma}(w) + \epsilon \geq \Gamma(u) \). Since the sequence \( \{G_n\} \) converges to \( u \), there is a labelling of vertices of graphs \( G_n \), corresponding to an ordering \( \prec_{n} \), for which \( \|w_{G_n} - w\|_\square \to 0 \). Thus by Lemma 5.2 we have \( \Gamma(w_{G_n}) \to \Gamma(w) \). Therefore by another application of Corollary 5.2 we have,

\[
\epsilon + \tilde{\Gamma}(w) \geq \Gamma(u) = \lim_{n \to \infty} \Gamma(w_{G_n}) = \lim_{n \to \infty} \Gamma^*(G_n, \prec_{n}) \geq \sup_{n \in \mathbb{N}} \Gamma^*(G_n).
\]

\( \square \)

In particular, if a convergent graph sequence \( \{G_n\} \) with limit \( w \) has the property that \( \{\Gamma^*(G_n)\} \) converges to zero, the above theorem states that \( \tilde{\Gamma}(w) = 0 \). This implies that there exist functions \( u \) with \( \Gamma(u) \) arbitrarily small so that the graphs \( \{G_n\} \) have similar structure, in terms of homomorphism densities, as the random graph \( G(n, u) \). We would like to conclude that the graphs \( \{G_n\} \) are consistent with having been formed by a random process with a linear embedding. However, it does not follow from our results that any function \( u \) with \( \Gamma(u) \) small is “close” to a diagonally increasing function. We conjecture that, in fact, if \( \Gamma(w) \) is small, then there exists a diagonally function \( u \) which is close to \( w \) in box distance.

**Conjecture 6.6.** There exists a strictly increasing function \( f \) which approaches zero as \( x \to 0 \) such that:

For every \( w \in \mathcal{W}_0 \), there exists \( u \in \mathcal{W}_0 \) with \( \Gamma(u) = 0 \) and \( \|w - u\|_\square \leq f(\Gamma(w)) \).

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