Representations of conformal Galilei algebra with integer spin and an application

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Abstract. Conformal Galilei algebra is a class of non-semisimple Lie algebras. Each member of the class is labelled by two parameters \(d\) (positive integer) and \(\ell\) (positive integer or positive half-integer). We investigate the lowest weight representations of the conformal Galilei algebra for \(d = 1\) and any integer \(\ell\). We start with the Verma modules and then extract all the irreducible lowest weight modules. This is done by searching and explicitly constructing singular vectors. As an application of our construction of singular vectors, we obtain the partial differential equations which are symmetric by kinematical transformation generated by the conformal Galilei algebra.

1. Introduction

Our aim in the present work is to study representation theory and its application to a particular class of non-semisimple Lie algebras. The class is called conformal Galilei algebra (CGA) and can be regarded as a nonrelativistic version of conformal algebras [1]. Each member of the class is labelled by two parameters \(d\) (positive integer) and \(\ell\) (positive integer or positive half-integer). CGA has a vast variety of physical applications. However such studies of CGA was limited to lower values of \(\ell\) (see e.g. [10] for references on \(\ell = 1/2\) and \(\ell = 1\) CGA). It is in the last couple of years that more focus has been directed towards CGA with higher values of \(\ell\) [2, 3, 4, 5, 6, 7, 8, 9]. In this work we focus on \(d = 1\) CGA for any integer \(\ell\), and study the lowest weight Verma module in some detail. This shall be done by computing the Kac determinant and by the construction of singular vectors. As an application of our results, we are able to obtain partial differential equations which are symmetric under the kinematical transformations generated by the CGA. The present work is a continuation of our previous works on the lowest (or highest) weight representations of CGA done in Refs. [10, 11] in which \(d = 1\) CGA with half-integer \(\ell\) are studied. We remark a very recent work [12] discussing a similar problem. In Ref. [12] a classification of all finite weight modules over the \(d = 1\) CGA for any \(\ell\) has been done. We would like to be more explicit in the present work. Namely, we start with the Verma modules and show explicitly how to arrive at the irreducible modules. We give the structure of the irreducible modules through a concrete formula.

The plan of this article is as follow: We give the definition of \(d = 1\) CGA in the next section. The Verma modules over the \(d = 1\) CGA are introduced in §3 and their reducibility is studied
by singular vectors. Those singular vectors are also used to obtain symmetric partial differential equations in §4. Finally, a classification of all irreducible lowest weight modules is given in §5.

2. Preliminary: definition of \( d = 1 \) CGA
Throughout this paper we denote the \( d = 1 \) CGA by \( \mathfrak{g}_\ell \). The complex Lie algebra \( \mathfrak{g}_\ell \) for a fixed integer \( \ell \) has the elements [1]:

\[
D, \ H, \ C, \ P_n \ (n = 0, 1, \ldots, 2\ell).
\]

Their nonvanishing commutators are given by

\[
[D, H] = H, \quad [D, C] = -C, \quad [C, H] = 2D, \\
[H, P_n] = -nP_{n-1}, \quad [D, P_n] = (\ell - n)P_n, \quad [C, P_n] = (2\ell - n)P_{n+1}.
\]

One may see from this that \( \langle P_0, P_1, \ldots, P_{2\ell} \rangle \) is an Abelian ideal of \( \mathfrak{g}_\ell \) so that \( \mathfrak{g}_\ell \) is not semisimple. It is known that this Lie algebra has no central extensions [13]. The subalgebra spanned by \( \langle H, D, C \rangle \) is isomorphic to \( \text{so}(2,1) \simeq \text{sl}(2,\mathbb{R}) \simeq \text{su}(1,1) \). The Abelian subalgebra spanned by \( \langle P_n \rangle_{n=0,1,\ldots,2\ell} \) carries the spin \( \ell \) representation of the \( \text{sl}(2,\mathbb{R}) \) subalgebra.

One may introduce the algebraic anti-involution \( \omega : \mathfrak{g}_\ell \to \mathfrak{g}_\ell \) by

\[
\omega(D) = D, \quad \omega(H) = C, \quad \omega(P_n) = P_{2\ell-n}.
\]

It is not difficult to verify that \( \omega \) satisfies the required relations:

\[
\omega([X,Y]) = [\omega(Y),\omega(X)], \quad \omega^2(X) = X \quad \forall X \in \mathfrak{g}_\ell.
\]

Let us define the degree of the generators based on their commutators with respect to \( D \):

\[
\deg(D) = 0, \quad \deg(H) = 1, \quad \deg(C) = -1, \quad \deg(P_n) = \ell - n.
\]

With respect to the sign of the degree one may define the triangular decomposition of \( \mathfrak{g}_\ell \):

\[
\mathfrak{g}_\ell = \mathfrak{g}_\ell^+ \oplus \mathfrak{g}_\ell^0 \oplus \mathfrak{g}_\ell^-,
\]

where

\[
\mathfrak{g}_\ell^+ = \langle H, \ P_0, \ P_1, \ldots, \ P_{\ell-1} \rangle \\
\mathfrak{g}_\ell^0 = \langle D, \ P_\ell \rangle \\
\mathfrak{g}_\ell^- = \langle C, \ P_{\ell+1}, \ P_{\ell+2}, \ldots, \ P_{2\ell} \rangle
\]

Eqn. (6) is a decomposition of \( \mathfrak{g}_\ell \) as a direct sum of the vector spaces.

3. Lowest weight Verma modules and their irreducibility
In this section we study the lowest weight Verma modules over \( \mathfrak{g}_\ell \). First let us define the Verma modules. We assume the existence of the lowest weight vector \( |\delta,p\rangle \) defined by

\[
D |\delta,p\rangle = \delta |\delta,p\rangle, \quad P_\ell |\delta,p\rangle = p |\delta,p\rangle, \quad X |\delta,p\rangle = 0, \quad \forall X \in \mathfrak{g}_\ell^-.
\]

The Verma module over \( \mathfrak{g}_\ell \) is defined, as usual (see e.g. [14]), as a module induced from \( |\delta,p\rangle : V_\ell^{\delta,p} = U(\mathfrak{g}_\ell^+) |\delta,p\rangle \) where \( U(\mathfrak{g}_\ell^+) \) is the enveloping algebra of \( \mathfrak{g}_\ell^+ \). In the sequel we assume that \( p \neq 0 \). The case of \( p = 0 \) is discussed in [16].
We choose the following basis of $V^{\delta,p}_\ell$:

$$H^k P_{\ell-1}^m P_{\ell-2}^m \cdots P_0^m |\delta,p \rangle.$$  

This is an eigenvector of $D$ with the eigenvalue $\delta + N$ where

$$N = k + \sum_{i=1}^\ell i m_i,$$

and we refer to $N$ as the level as usual. Then $V^{\delta,p}_\ell$ has a grading structure according to the value of $N$:

$$V^{\delta,p}_\ell = \bigoplus_{N=0}^\infty (V^{\delta,p}_\ell)_N, \quad (V^{\delta,p}_\ell)_N = \{ |v \rangle \in V^{\delta,p}_\ell \mid D |v \rangle = (\delta + N) |v \rangle \}.$$ 

One may define an inner product for the vectors in $V^{\delta,p}_\ell$. Let $|x\rangle$ and $|y\rangle$ be any two vectors in $V^{\delta,p}_\ell$:

$$|x\rangle = X |\delta,p \rangle, \quad |y\rangle = Y |\delta,p \rangle, \quad X, Y \in U(g^+\ell).$$

We define the inner product of $|x\rangle$ and $|y\rangle$ by

$$\langle x \mid y \rangle = \langle \delta, p \mid \omega(X)Y |\delta,p \rangle, \quad \langle \delta, p \mid \delta, p \rangle = 1.$$  

Next we define the Kac determinant at level $N$ (see e.g. [15]). Let $|v_1\rangle, |v_2\rangle, \ldots, |v_r\rangle$ be a set of basis of the subspace $(V^{\delta,p}_\ell)_N$. We consider the matrix $(\langle v_i \mid v_j \rangle)$ whose entries are the inner products of the basis of $(V^{\delta,p}_\ell)_N$. The determinant of this matrix is called the Kac determinant at level $N$:

$$\Delta^{(\ell)}_N = \det(\langle v_i \mid v_j \rangle).$$

Our first result is an explicit formula of the Kac determinant which is helpful to know the reducibility of $V^{\delta,p}_\ell$.

**Proposition 1** The Kac determinants $\Delta^{(\ell)}_N$ at level $N$ of $g_\ell$ are given as follows (up to overall sign):

(i) $\Delta^{(1)}_N = \left( \prod_{m=0}^N m! \right)^2 (2p)^{N(N+1)},$

(ii) $\Delta^{(\ell)}_N = (\ell + 1)^2 p^2 \delta N_1, \quad (\ell \geq 2).$  

The proof is found in [16]. (i) shows that if $\ell = 1$ then the Verma module $V^{\delta,p}_1$ has no singular vectors so that it is irreducible. (ii) shows that if $\ell \geq 2$ then there exist no singular vectors in the level $N = 1$ subspace of $V^{\delta,p}_\ell$. As shown in Proposition 3 and 4 there exist singular vectors in the subspace of other levels. Hence we have shown the following:

**Proposition 2**

(i) $V^{\delta,p}_1$ is irreducible if $\ell = 1$.

(ii) $V^{\delta,p}_\ell$ is reducible if $\ell \geq 2$.

**Proposition 3** Following are the singular vectors in $V^{\delta,p}_\ell$ for $\ell \geq 2$:

$$\langle S^{(2n)}k |\delta,p \rangle, \quad \langle S^{(2n+1)}k |\delta,p \rangle, \quad k = 1, 2, \ldots$$
where $n$ takes the value of a positive integer. The maximal value of $n$ is determined by $\ell$ and $N$ in such a way that $S^{(2n)}$ and $S^{(2n+1)}$ given below are well-defined.

$$
S^{(2n)} = \ell \cdot p_{\ell-2n} + \sum_{j=1}^{n-1} a_j p_{\ell-2n+j} p_{\ell-j} + \frac{(-1)^n \ell! (\ell + 2n)!}{((\ell + n)!)^2} P_{\ell-n}^2,
$$

$$
S^{(2n+1)} = p^2 \ell \cdot p_{\ell-2n-1} - \frac{\ell + 2n + 1}{(n + \frac{1}{2})(\ell + 1)} \cdot p_{\ell-1} S^{(2n)} - \frac{n - \frac{1}{2} \ell + 2n + 1}{n + \frac{1}{2} \ell + 1} \cdot p_{\ell-1} p_{\ell-2n} - \sum_{j=1}^{n-1} b_j p_{\ell-2n+j} p_{\ell-j-1},
$$

where

$$
a_j = (-1)^j \frac{\ell! (\ell + 2n)!}{(\ell + j)! (\ell + 2n - j)!}, \quad b_j = (-1)^j \frac{n - \frac{1}{2} - j}{n + \frac{1}{2}} \frac{p \ell! (\ell + 2n + 1)!}{(\ell + j + 1)! (\ell + 2n - j)!}.
$$

**Proposition 4** Following are the singular vectors in $V_{\ell}^{\delta,p}$ for $\ell \geq 2$.

$$
(\tau^{(2n+1)})^k |\delta, p\rangle, \quad k = 1, 2, \ldots
$$

where $n$ takes the value of a positive integer. The maximal value of $n$ is determined by $\ell$ and $N$ in such a way that $\tau^{(2n+1)}$ given below is well-defined.

$$
\tau^{(2n+1)} = p_{\ell-1} ((\ell + 2) P_{\ell-1}^2 - 2p (\ell + 1) P_{\ell-2})^n + \sum_{j=0}^{n-1} c_j P_{\ell-1}^{2(n-j-1)} P_{\ell-2}^j + \sum_{j=1}^{n} d_j P_{\ell-1}^{2n-2j+1} P_{\ell-2}^j,
$$

where

$$
c_j = \binom{n - 1}{j} \left( -\frac{2p(\ell + 1)}{\ell + 2} \right)^j c_0, \quad d_1 = -p(\ell + 1) (\ell + 2)^{n-1}.
$$

Other coefficients $c_0$, $d_j$ ($j \leq 1 \leq n$) are determined by the relations

$$(j + 1) (\ell + 2) d_{j+1} + (2n - 2j + 1) p (\ell + 1) d_j + (\ell + 3) c_{j-1} + \binom{n}{j} (-2)^j (p(\ell + 1))^{j+1} (\ell + 2)^{n-j} = 0, \quad (j = 1, 2, \ldots, n)
$$

with $d_{n+1} = 0$.

Proposition 3, 4 are proved by verifying that the vectors (14) and (17) satisfies the definition of singular vectors (see [16]).

4. **Differential equations symmetric under the group generated by $g_\ell$**

The singular vectors obtained in the previous section can be used to derive partial differential equations having particular symmetries. The symmetries are generated by $g_\ell$, i.e., the symmetry group is the exponentiation of $g_\ell$, and the partial differential equations are invariant under the change of independent variables, i.e., the kinematical symmetries, caused by the group. This can be done by applying the method developed for real semisimple Lie groups in [17]. In this section,
before presenting our results, we give a brief review of the method with suitable modification for the present case.

The basic idea of [17] is to realize the Verma modules in a space of $C^\infty$-class functions. Then the generators of $\mathfrak{g}_\ell$ will be differential operators and the singular vector will be converted to differential equations.

Let $G$ be a complex semisimple Lie group and $\mathfrak{g}$ its Lie algebra. The Lie algebra $\mathfrak{g}$ has the triangular decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$. The corresponding decomposition of $G$ is denoted by $G = G_+ G_0 G_-$. Consider the space of $C^\infty$-class functions on $G$ having the property called right covariance:

$$C_\Lambda = \{ f \in C^\infty(G) \mid f(gxg^-) = e^{\Lambda(H)}f(g) \} ,$$

(20)

where $\Lambda \in \mathfrak{g}^*$ (algebra dual to $\mathfrak{g}_\ell$), $g \in G$, $H \in \mathfrak{g}^0$, $x = e^H \in G_0$, $g^- \in G_-$. Because of the right covariance, the functions of $C_\Lambda$ are actually function on $G/B$ with $B = G_0 G_-$, or on $G_+$. We keep using the same notations for the restricted representation space of functions on $G_+$. Then the Lie algebra $\mathfrak{g}_\ell$ is realized by the standard left action of $\mathfrak{g}_\ell$ on $C_\Lambda$:

$$\pi_L(X)f(g) = \frac{d}{dt} f(e^{-tX}g) \bigg|_{t=0} , \quad X \in \mathfrak{g}, \; g \in G. \quad (21)$$

We introduce the right action of $\mathfrak{g}$ on $C_\Lambda$ by the standard formula:

$$\pi_R(X)f(g) = \frac{d}{dt} f(g e^{tX}) \bigg|_{t=0} , \quad X \in \mathfrak{g}, \; g \in G. \quad (22)$$

Using the right covariance we can show that the function $f \in C_\Lambda$ has the properties of lowest weight vector:

$$\pi_R(H)f(g) = \Lambda(H)f(g), \quad \pi_R(X)f(g) = 0, \quad H \in \mathfrak{g}^0, \; X \in \mathfrak{g}^-.$$  

(23)

This allows us to realize the Verma module $V^\Lambda \simeq U(\mathfrak{g}^+) v_0$ with the lowest weight vector $v_0$ in terms of the function in $C_\Lambda$ and differential operators $\pi_R(X)$, $X \in \mathfrak{g}^+$. Now suppose that the Verma module $V^\Lambda$ has a singular vector. The general structure of a singular vector is

$$v_s = \mathcal{P}(X_1, X_2, \ldots, X_s)v_0, \quad X_k \in \mathfrak{g}^+.$$  

(24)

where $\mathcal{P}$ denotes a homogeneous polynomial in its variables. It is shown in [17] that if the operator $\pi_R(\mathcal{P})$ has a nontrivial kernel

$$\pi_R(\mathcal{P})\psi = 0,$$  

(25)

for some function $\psi$ on $G_+$, then the differential equation (25) has the kinematical symmetry generated by $\mathfrak{g}_\ell$.

Now let us apply this scheme to our case. We parametrize an element of $G_+$ as $g = \exp(tH) \exp \left( \sum_{n=0}^{\ell-1} x_n P_n \right)$. Then the right action of $\mathfrak{g}_\ell^+$ yields

$$\pi_R(P_n) = \frac{\partial}{\partial x_n}, \quad \pi_R(H) = \frac{\partial}{\partial t} + \sum_{j=1}^{\ell-1} j x_j \frac{\partial}{\partial x_{j-1}}.$$  

(26)

From (25), Proposition 3 and Proposition 4 we obtain the following hierarchies of partial differential equations.
The corresponding equations for \( n \)

The following equations have the kinematical symmetry generated by \( g_\ell \).

\[
\begin{align*}
\left(p \frac{\partial}{\partial x_{\ell-2n}} + \sum_{j=1}^{n-1} a_j \frac{\partial^2}{\partial x_{\ell-2n-j} \partial x_{\ell-j}} + \frac{(-1)^n \ell!(\ell+2n)!}{2(\ell+n)!^2} \frac{\partial^2}{\partial x_{\ell-n}^2} \right)^k \psi(x) &= 0, \\
\left(p^2 \frac{\partial}{\partial x_{\ell-2n-1}} - \ell + 2n + 1 \frac{\partial}{(n+\frac{1}{2})(\ell+1)} \frac{\partial^2}{\partial x_{\ell-1} \partial x_{\ell-n-1}} + \frac{(-1)^n \ell!(\ell+2n)!}{2(\ell+n)!^2} \frac{\partial^2}{\partial x_{\ell-n}^2} \right)^k \psi(x) &= 0,
\end{align*}
\]

\[
\begin{align*}
\left(\frac{\partial}{\partial x_{\ell-1}} \left((\ell+2) \frac{\partial^2}{\partial x_{\ell-1}^2} - 2p(\ell+1) \frac{\partial}{\partial x_{\ell-2}}\right)^n + \sum_{j=0}^{n-1} c_j \left(\frac{\partial}{\partial x_{\ell-1}}\right)^{2n-j-1} \left(\frac{\partial}{\partial x_{\ell-2}}\right)^j \frac{\partial}{\partial x_{\ell-3}} + \sum_{j=1}^{n} d_j \left(\frac{\partial}{\partial x_{\ell-1}}\right)^{n-2j+1} \left(\frac{\partial}{\partial x_{\ell-2}}\right)^j \frac{\partial}{\partial x_{\ell-3}} \right)^k \psi(x) &= 0.
\end{align*}
\]

We have obtained highly nontrivial differential equations. To have a close look at the equations, we give examples of the hierarchies of equations for \( n = 1, 2 \). For \( n = 1 \) the equations (27)-(29) read as follows:

\[
\begin{align*}
\left(p \frac{\partial}{\partial x_{\ell-2}} - \frac{\ell+2}{2(\ell+1)} \frac{\partial^2}{\partial x_{\ell-1}^2} \right)^k \psi(x) &= 0, \\
\left(p^2 \frac{\partial}{\partial x_{\ell-3}} - \frac{\ell+3}{\ell+1} p \frac{\partial^2}{\partial x_{\ell-2} \partial x_{\ell-1}} + \frac{(\ell+2)(\ell+3)}{3(\ell+1)^2} \frac{\partial^3}{\partial x_{\ell-1}^3} \right)^k \psi(x) &= 0,
\end{align*}
\]

\[
\begin{align*}
\left((\ell+2) \frac{\partial^3}{\partial x_{\ell-1}^3} - 3(\ell+1) p \frac{\partial^2}{\partial x_{\ell-1} \partial x_{\ell-2}} + \frac{3(\ell+1)^2}{\ell+3} p^2 \frac{\partial}{\partial x_{\ell-3}} \right)^k \psi(x) &= 0.
\end{align*}
\]

The corresponding equations for \( n = 2 \) are given by

\[
\begin{align*}
\left(p \frac{\partial}{\partial x_{\ell-4}} - \frac{\ell+4}{\ell+1} \frac{\partial^2}{\partial x_{\ell-3} \partial x_{\ell-1}} + \frac{(\ell+3)(\ell+4)}{2(\ell+1)(\ell+2)} \frac{\partial^2}{\partial x_{\ell-2}^2} \right)^k \psi(x) &= 0, \\
\left(p^2 \frac{\partial}{\partial x_{\ell-5}} - \frac{\ell+5}{\ell+1} p \frac{\partial^2}{\partial x_{\ell-4} \partial x_{\ell-1}} + \frac{(\ell+4)(\ell+5)}{5(\ell+1)(\ell+2)} \frac{\partial^2}{\partial x_{\ell-2}^2} \\
+ \frac{2(\ell+4)(\ell+5)}{5(\ell+1)^2} \frac{\partial^3}{\partial x_{\ell-3} \partial x_{\ell-1}^2} - \frac{3(\ell+1)^2(\ell+5)}{5(\ell+1)^2(\ell+2)} \frac{\partial^3}{\partial x_{\ell-2} \partial x_{\ell-1}} \right)^k \psi(x) &= 0,
\end{align*}
\]

\[
\begin{align*}
\left((\ell+2)^2 \frac{\partial^5}{\partial x_{\ell-1}^5} - 5(\ell+1)(\ell+2) p \frac{\partial^4}{\partial x_{\ell-2} \partial x_{\ell-1}^3} + \frac{3(\ell+1)^2(\ell+2)}{\ell+3} p^2 \frac{\partial^3}{\partial x_{\ell-1}^3} \partial x_{\ell-3} \right) \\
+ 6(\ell+1)^2 p^2 \frac{\partial^3}{\partial x_{\ell-1} \partial x_{\ell-2}^2} - \frac{6(\ell+1)^3}{\ell+3} p^3 \frac{\partial^2}{\partial x_{\ell-2} \partial x_{\ell-3}} \right)^k \psi(x) &= 0.
\end{align*}
\]
5. Irreducible lowest weight modules over \( g_\ell \)

We have shown that the Verma modules over \( g_\ell \) are reducible in many cases (Proposition 2). It is known that the Verma module is, in a sense, the largest lowest weight module. That is, one can derive all irreducible lowest weight modules starting from the Verma module \( V_\ell^{\delta,p} \). Here we give a classification of all irreducible lowest weight modules over \( g_\ell \):

**Theorem 1** All irreducible lowest weight modules over \( g_\ell \) are listed as follows:

(i) the Verma module \( V_\ell^{\delta,p} \) for \( \ell = 1 \).

(ii) the quotient module \( V_\ell^{(i)} \) for \( \ell \geq 2 \). This module is infinite dimensional with the basis vectors \( H^k P_{\ell-1}^m | v_0^{(i)} \rangle \) where \( k, m \) are nonnegative integers. See Lemma 2 for the definition of the lowest weight vector \( | v_0^{(i)} \rangle \).

Theorem 1 coincides with the results in [12] and the statement is stronger than our previous result [16] where the complete classification is not given. Theorem 1 (i) is already mentioned in Proposition 2. So we give an outline of the proof of (ii) below. The proof of Lemma 1, 2 is found in [16].

We consider the quotient module \( V_\ell^{\delta,p}/\mathcal{I}^{(2)} \) where \( \mathcal{I}^{(2)} \) is the largest \( g_\ell \)-submodule of \( V_\ell^{\delta,p} \). Since there is no singular vectors in the \( N = 1 \) subspace \( (V_\ell^{\delta,p})_1 \), \( \mathcal{I}^{(2)} \) will be induced by the singular vector in the \( N = 2 \) subspace.

**Lemma 1** There exists precisely one (up to overall constant) singular vector in the \( N = 2 \) subspace \( (V_\ell^{\delta,p})_2 \). This singular vector is given by

\[
| v_0^{(2)} \rangle = (2p(\ell + 1)P_{\ell-2} - (\ell + 2)P_{\ell-1}^2) | \delta, p \rangle. \tag{36}
\]

Define \( \mathcal{I}^{(2)} = U(g_\ell) | v_0^{(2)} \rangle \), then \( \mathcal{I}^{(2)} \) is the largest \( g_\ell \)-submodule in \( V_\ell^{\delta,p} \). Let \( | u_0^{(2)} \rangle \) be the lowest weight vector in \( V_\ell^{\delta,p}/\mathcal{I}^{(2)} \). Then

\[
D | u_0^{(2)} \rangle = \delta | u_0^{(2)} \rangle, \quad P_\ell | u_0^{(2)} \rangle = p | u_0^{(2)} \rangle, \quad X | u_0^{(2)} \rangle = 0, \quad X \in g_\ell, \quad P_{\ell-2} | u_0^{(2)} \rangle = \frac{\ell + 2}{2p(\ell + 1)} P_{\ell-1}^2 | u_0^{(2)} \rangle. \tag{37}
\]

It follows that the basis of \( V_\ell^{\delta,p}/\mathcal{I}^{(2)} \) is given by

\[
H^k P_{\ell-1}^{m_1} P_{\ell-3}^{m_3} \cdots P_{\ell-1}^{m_\ell} | u_0^{(2)} \rangle. \tag{38}
\]

This is an eigenvector of \( D \) with the eigenvalue \( \delta + k + m_1 + \sum_{i=3}^\ell i m_i \). We define the level \( N^{(2)} \) in the quotient space \( V_\ell^{\delta,p}/\mathcal{I}^{(2)} \) by \( N^{(2)} = k + m_1 + \sum_{i=3}^\ell i m_i \).

One can show that there exists no singular vector in \( N^{(2)} = 1, 2 \) subspaces of \( V_\ell^{\delta,p}/\mathcal{I}^{(2)} \) and precisely one singular vector exists in \( N^{(2)} = 3 \) subspace. Using the singular vector in \( N^{(2)} = 3 \) subspace one can obtain the largest \( g_\ell \)-submodule \( \mathcal{I}^{(3)} \) in \( V_\ell^{\delta,p}/\mathcal{I}^{(2)} \). Then we consider the quotient module \( V_\ell^{\delta,p}/\mathcal{I}^{(2)}/\mathcal{I}^{(3)} \) and search singular vectors in the quotient module. In fact, one can repeat this process until we arrive at \( V_\ell^{\delta,p}/\mathcal{I}^{(2)}/\cdots/\mathcal{I}^{(\ell)} \). This is assured by the next lemma:
Lemma 2 Suppose that we have arrived at the quotient space $V^{(\lambda)}_\ell := V^{\delta,p}/I^{(2)}/\cdots/I^{(\lambda)}$, $(2 \leq \lambda \leq \ell)$. Namely, we have the quotient space with the basis

$$H^k P_{\ell-1}^{m_1} P_{\ell-\lambda-1}^{m_{\lambda+1}} \cdots P_0^{m_r} \left| u^{(\lambda)}_0 \right>, \quad (39)$$

where $\left| u^{(\lambda)}_0 \right>$ is the lowest weight vector in $V^{(\lambda)}_\ell$, defined by

$$D \left| u^{(\lambda)}_0 \right> = \delta \left| u^{(\lambda)}_0 \right>, \quad P_\ell \left| u^{(\lambda)}_0 \right> = p \left| u^{(\lambda)}_0 \right>, \quad X \left| u^{(\lambda)}_0 \right> = 0, \quad X \in g^-_\ell,$$

$$P_{\ell-a} \left| u^{(\lambda)}_0 \right> = \frac{p (\ell + a)}{a!} \frac{1}{(p(\ell + 1))^a} P_\ell^{\delta,p} u^{(\lambda)}_0, \quad (a = 2, 3, \ldots, \lambda). \quad (40)$$

Then

(i) $V^{(\lambda)}_\ell$ is the graded vector space:

$$V^{(\lambda)}_\ell = \bigoplus_{N^{(\lambda)} = 0}^{\infty} (V^{(\lambda)}_\ell)_{N^{(\lambda)}}, \quad (V^{(\lambda)}_\ell)_{N^{(\lambda)}} = \{ |v\rangle \in V^{(\lambda)}_\ell \mid D |v\rangle = (\delta + N^{(\lambda)}) |v\rangle \},$$

$$N^{(\lambda)} = k + m_1 + \sum_{i=\lambda+1}^{\ell} i m_i. \quad (41)$$

(ii) The subspace $(V^{(\lambda)}_\ell)_{N^{(\lambda)}}$ has a nonvanishing Kac determinant if $1 \leq N^{(\lambda)} \leq \lambda$. The Kac determinant is given by (up to a sign factor)

$$\Delta_{N^{(\lambda)}} = (p(\ell + 1))^{N^{(\lambda)}(N^{(\lambda)}+1)} \prod_{k=0}^{N^{(\lambda)}} k! (N^{(\lambda)} - k)!. \quad (42)$$

This implies that there exists no singular vectors in the level $N^{(\lambda)}$ subspaces if $1 \leq N^{(\lambda)} \leq \lambda$.

(iii) If $2 \leq \lambda \leq \ell - 1$, then there exists precisely one (up to overall constant) singular vector in the $N^{(\lambda)} = \lambda + 1$ subspace of $V^{(\lambda)}_\ell$. The singular vector is given by

$$\left| v^{(\lambda+1)}_\lambda \right> = \left( (\lambda + 1)! p^{(\ell + 1)\lambda} P_{\ell-\lambda-1} - (\ell + 2)(\ell + 3) \cdots (\ell + \lambda + 1) P^{\lambda+1}_\ell \right) u^{(\lambda)}_0. \quad (43)$$

Now we consider the module $V^{(\ell)}_\ell = V^{\delta,p}/I^{(2)}/\cdots/I^{(\ell)}$. The basis of this space is $H^k P_{\ell-1}^{m_1} \left| u^{(\ell)}_0 \right>$ where $\left| u^{(\ell)}_0 \right>$ is the lowest weight vector defined by (40) with $\lambda = \ell$. It is not difficult to verify that there exists no singular vector in $V^{(\ell)}_\ell$ so that $V^{(\ell)}_\ell$ is irreducible.

One may wonder what happens if we take a quotient by $g^-_\ell$-submodule which is not largest. It can be shown that this does not produce any new irreducible modules. We explain this by taking the Verma module $V^{(\ell)}_\ell$ as an example. First let us note the following fact:

Lemma 3 All singular vectors in $V^{(\ell)}_\ell$ are independent of $H$.

It is an easy task to prove this lemma. We assume that the singular vector is $k$th order in $H$, then demand that the vector is annihilated by $P_{\ell+1}$. It results that all the terms which is $k$th order in $H$ must disappear from the singular vector. This reduces the order in $H$ by one. Repeating the same argument we prove the lemma.

Next let $|w^N\rangle$ be a singular vector in level $N$ ($2 < N \leq \ell$) subspace of $V^{\delta,p}_\ell$. 

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Lemma 4  In the quotient space $V^{(N-1)}_{\ell}$

(i) $|w^N\rangle$ is reduced to the zero vector if $|w^N\rangle$ is independent of $P_{\ell-N}$
(ii) $|w^N\rangle$ is reduced to singular vector given in (43) if $|w^N\rangle$ depends on $P_{\ell-N}$

This lemma is proved as follows. From Lemma 3, $|w^N\rangle$ is a linear combination of $P_{\ell-1}^{m_1}P_{\ell-2}^{m_2}\cdots P_{\ell-N}^{m_N}|\delta,p\rangle$ so that $|w^N\rangle$ is annihilated by any $P_n \in \mathfrak{g}_\ell^-$. The condition $C|w^N\rangle = 0$ requires some relations among the coefficients of the linear combination. In the quotient space $V^{(N-1)}_{\ell}$ we have additional relations (40). Substituting (40) into $|w^N\rangle$ and using the relation from $C|w^N\rangle = 0$ one may verify that the lemma is true.

Now consider the quotient module $V^{(N-1)}_{\ell}/W$ where $W = U(\mathfrak{g}_\ell^+) |w^N\rangle$. The singular vectors in the level $M$ ($2 \leq M \leq N - 1$) subspaces of $V^{(N-1)}_{\ell}$ are also singular vectors in $V^{(N-1)}_{\ell}/W$. Thus one can consider the quotient module $V^{(N-1)}_{\ell}/W/\mathcal{I}^{(2)}/\cdots/\mathcal{I}^{(N-1)}$. In the case of Lemma 4 (i) the contribution from $W$ disappears, while in the case of Lemma 4 (ii) taking the quotient by $W$ has the same effect as taking the quotient by $\mathcal{I}^{(N)}$. Therefore we arrive at the result.

Lemma 5

(i) $V^{(N-1)}_{\ell}/W/\mathcal{I}^{(2)}/\cdots/\mathcal{I}^{(N-1)} \simeq V^{(N-1)}_{\ell}$ if $|w^N\rangle$ is independent of $P_{\ell-N}$
(ii) $V^{(N-1)}_{\ell}/W/\mathcal{I}^{(2)}/\cdots/\mathcal{I}^{(N-1)} \simeq V^{(N)}_{\ell}$ if $|w^N\rangle$ depends on $P_{\ell-N}$

Lemma 5 means that taking the quotient by $W$ does not produce any new irreducible module.

Finally, let $|z^N\rangle$ be be a singular vector in level $N$ (> $\ell$) subspace of $V^{\delta,p}_{\ell}$. Then by the arguments similar to Lemma 4, 5 one may come to the next lemma:

Lemma 6  In the quotient space $V^{(\ell)}_{\ell}$, any singular vector $|z^N\rangle$ is reduced to the zero vector. Hence $V^{\delta,p}_{\ell}/Z/\mathcal{I}^{(2)}/\cdots/\mathcal{I}^{(\ell)} \simeq V^{(\ell)}_{\ell}$ where $Z = U(\mathfrak{g}_\ell^+) |z^N\rangle$.

Thus we have proved that taking a quotient of $V^{\delta,p}_{\ell}$ by the $\mathfrak{g}_\ell$-submodule which is not the largest does not produce any new irreducible modules. Though we have discussed the case of the Verma module $V^{\delta,p}_{\ell}$, the same arguments hold for the quotient module $V^{(\lambda)}_{\ell}$ and, the quotient by the $\mathfrak{g}_\ell$-submodule does not produce any new irreducible module if the submodule is not the largest.

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