Can \(d\)-Density Wave (DDW) Scenario Describe the New Hidden Charge Order in Cuprates?

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In this Letter, I show that the theory of high \(T_c\) superconductivity based on adiabatic switching on of interaction applied to Fermi liquid metal with \(d\)-density wave (DDW) scenario in the pseudogap phase is able to describe the recently observed hidden charge order in several families of Cuprates. In particular, by computing and comparing energies of charge orders of different modulation directions derived from a full microscopic theory with \(d\)-density wave scenario, the axial charge order \(\phi_{X(Y)}\) with wavevector \(Q = (Q_0, 0)((0, Q_0))\) is shown to be unambiguously energetically favorable over the diagonal charge order \(\phi_{X \pm Y}\) with wavevector \(Q = (Q_0, \pm Q_0)\) at least in commensurate limit, to be expected also to hold even to more general incommensurate case, in agreement with experiment. Furthermore, within DDW scenario, the two axial charge orders \(\phi_X\) and \(\phi_Y\) are found to be degenerate and should therefore appear together rather than separately; the charge order is biaxial.

The mechanism of high \(T_c\) superconductivity in Cuprates is one of the biggest unsolved problems in condensed matter physics and remains not fully resolved and understood ever since its discovery in 1986 [1] with hundreds of thousands of papers published on the subject. Most of theories of Cuprates rest on the basic assumption that the parent compound can be described by Mott insulator [2][3], which is inherently strongly correlated electron system which cannot be described by Landau Fermi liquid theory of weakly interacting electrons. A totally opposite point of view sees Cuprate superconductivity just as an instability away from Fermi liquid and should therefore be describable in terms of interacting electrons as interaction is adiabatically turned on, on top of weakly interacting Fermi liquid of metals [4].

Pseudogap phase remains one of the most mysterious parts in the Cuprate phase diagram. It is characterized by a gap in the electron spectral function but without true long range coherence that characterizes superconducting order. Available theories of pseudogap phase propose that it be described by some type of symmetry breaking order. The experimental results on this pseudogap regime that accumulate over the years support this idea to some extent. Related to this, pseudogap phase has also been hypothesized to be a time reversal symmetry-broken state [5] and several theories have been proposed to explain the phenomenon, in the form of loop current order [6] and \(d\)-density wave (DDW) order [7][8][9]. In view of the latter, it has been argued that DDW is a natural description of pseudogap phase where the \(d\)-wave symmetry of the symmetry breaking order in pseudogap phase evolves into \(d\)-wave symmetry of superconducting state (DSC) on lowering the temperatures [10].

After several years of pacificity, the field of high \(T_c\) in Cuprates heats up again with recent discovery of hidden ordered state in the underdoped regime of the phase diagram, within pseudogap phase [11][22]. In short, recent experiments observe what has been hypothesized as hidden charge order in the underdoped regime of several families of Cuprates. This hypothesis was deduced from indirect signatures of such charge order detected with various techniques including scanning tunneling microscopy [11][14], resonant x-ray scattering [13][16][20], nuclear magnetic resonance [21][22], ultrasound study [23], and other methods. Motivated by this experimental discovery, various theoretical proposals have been put forward to explain the presence of such order [24][28].

One of the most intensively studied questions in this regard concerns the character of the charge order, especially the intra and inter unit cell structures of the order parameter. The inter unit cell character concerns the ordering wavevector \(Q\) of the charge order. Different microscopic (or semi microscopic) theories so far lead to varying conclusions. Spin-fermion model [29] leads to axial charge order \(\phi_X, \phi_Y\) with \(Q\) along \(x\) or \(y\) axis respectively [25][26]. In \(t-J\) model, as first discussed in [30] within linear hot spot approximation, the emergent \(SU(2)\) particle-hole symmetry and Hartree-Fock calculation at quadratic level [24][31] put \(d\)-wave SC as the leading instability, followed by diagonal CDW \(\phi_{X \pm Y}\) and then by axial CDW \(\phi_X, \phi_Y\) as the next leading instabilities. This however disagrees with experimental result where the ordering wavevector \(Q\) is found to be axial \(\phi_X, \phi_Y\) [11][12][20].

In this Letter, I adopt the model proposed in [10], argued to be the complete microscopic model of Cuprates, with DDW scenario [8] and show, using variational calculation, that the axial charge order is exclusively energetically favorable. Furthermore, DDW scenario predicts that the axial charge order should be biaxial rather than uniaxial.

I consider the Laughlin model [10] given by the following Hamiltonian

\[
H = H_0 + \Delta H
\]

\[
H_0 = -t \sum_{\langle ij \rangle} \sum_{\sigma} c_{i,\sigma}^\dagger c_{j,\sigma} + h.c. + t' \sum_{\langle i \rangle} \sum_{\sigma} c_{i,\sigma}^\dagger c_{j,\sigma} + h.c. - \mu \sum_i c_{i,\sigma}^\dagger c_{i,\sigma}
\]
The noninteracting Hamiltonian $H_0$ contains kinetic hopping up to third nearest neighbor and chemical potential terms. The interaction Hamiltonian $\Delta H$ consists of onsite repulsion $U$, antiferromagnetic spin exchange $J$, hopping renormalizing interaction $V_t$, nearest-neighbor repulsion $V_n$, and Cooper pair hopping term $V_c$. I will perform mean field theory on this model Eq. (1) with the DDW scenario and show that axial charge order is exclusively preferred over diagonal charge order, in complete agreement with the experimental observations so far.

The noninteracting kinetic hopping Hamiltonian $H_0$ gives the usual tight binding dispersion $\epsilon_k = -2t(\cos k_x + \cos k_y) + 4t' \cos k_x \cos k_y - \mu$ where $t, t' > 0$ and normally $t' \ll t$. The nearest neighbor hopping produces square Fermi surface while the next nearest neighbor produces the curvature. The location of Fermi surface of free noninteracting fermions determined only by kinetic hopping term is given by $\epsilon_k = -2t(\cos k_x + \cos k_y) + 4t' \cos k_x \cos k_y - \mu = 0$ for the model with nearest neighbor and next nearest neighbor hoppings. The particle-hole scattering process in the density wave channel is mediated most importantly by antiferromagnetic exchange where only fermions near hot spots : points on the Fermi surface connected by ordering density wavevector (the 'nesting wavevector') contribute effectively to the scattering process. The location of hot spots can be determined geometrically as given by the crossing points of the antiferromagnetic reciprocal lattice and Fermi surface. An example of Fermi surface with hot spots is shown in Fig. 1.

Experimentally, charge ordering is found to have periodicity with normally larger than two (around 3 ~ 4) multiple of unit lattice spacing. This corresponds to incommensurate order. Its simplest description proposes that it be modeled by fermion scattering between nearby hot spots (e.g. hot spots 1 and 2 in Fig. 1). The simpler case with commensurate charge order, where $Q_0 = \pi$, can be described by fermion scattering between hot spots on commensurate Fermi surface. Considering the commensurate limit, I decompose the interaction Hamiltonian $\Delta H$ into DDW and DSC channels by deriving reduced Hamiltonians. The overall state is still heavily dominated by $d$-wave symmetry, both in particle-hole and particle-particle channels. I henceforth assume the $d$-density wave and $d$-wave superconducting states to be the dominant instabilities in pseudogap phase of Cuprates. This crucial point is in tune also with the physical argument that the hidden order in the pseudogap phase of Cuprates should necessarily have $d$-wave symmetry because it is a crystal of Cooper pairs and is therefore continuation of the $d$-wave superconducting state at lower temperatures. I will show posteriori that this physical consideration is able to reproduce what has been observed in experiments.

$$H_{DDW} = -g_{DDW} \int_{k,k'} f_k f_{k'} c_{k+Q_0,\sigma}^\dagger c_{k,\sigma} c_{k',\sigma'}^\dagger c_{-k',\downarrow} + Q_0,\sigma'$$

$$H_{DSC} = -g_{DSC} \int_{k,k'} f_k f_{k'} c_{k+Q_0,\sigma}^\dagger c_{k,\sigma} c_{k',\sigma'}^\dagger c_{-k',\downarrow} + Q_0,\sigma'$$
on hidden charge order.

I obtain for the coupling constants in the reduced Hamiltonians Eq. (2) from Eq. (1) the following result.

\[ g_{DDW} = 6J + 8V_n + 8V_c, g_{DSC} = 6J - 8V_n \] (3)

We see that the onsite \( U \) term does not contribute to either DDW or DSC channel. The antiferromagnetic spin exchange coupling \( J \) contributes in both channels and reflects its dominant effect in dictating the energetics of the system and thus justifies my previous consideration regarding its role in determining the location of hot spots. The nearest neighbor repulsion with coupling constant \( V_n \) manifests its effect in both \( d \)-wave channels. The \( s \)-wave Cooper pair hopping term with coupling \( V_c \) in Eq. (1) contributes to the DDW channel. From Eq. (2), the DDW and DSC states are energetically favorable and may exist only if \( g_{DDW}, g_{DSC} > 0 \). From Eq. (3), we see that DDW is always energetically favorable and is bound to exist whereas DSC requires \( J > J_c = 4V_n/3 \) to exist. It was found that these microscopic parameters do not depend on doping level and in particular, fit to experiments suggests \( V_c = 0 \) and allows setting \( V_n = 0 \) [10], whereas the \( J, V_c \) are positive. These altogether secure the existence of DDW and DSC in the appropriate regime of Cuprates’ phase diagram.

Performing Hubbard-Stratonovich transformation, I write

\[ Z = \int \mathcal{D}c^\dagger \int \mathcal{D}c e^{-S[c,\bar{c}]} \]

\[ = \int \mathcal{D}c^\dagger \int \mathcal{D}c \int \mathcal{D}\phi \int \mathcal{D}\epsilon \epsilon e^{-S[c,\phi,\Delta]} \]

where, using finite temperature \( T = 1/\beta \) (imaginary) time-independent Euclidean space-time formalism, I have

\[ S[c,\phi,\Delta] = \int_0^\beta d\tau \mathcal{H}[c,\phi,\Delta] \]

\[ = \int_0^\beta d\tau \sum_\sigma \int_k \epsilon_k c_k^\dagger c_k + \frac{\vert\phi\vert^2}{g_{DDW}} + \frac{\vert\Delta\vert^2}{g_{DSC}} + \phi^* \sum_\sigma \int_k f_k c_k^\dagger c_{k+\sigma} + h.c. + \Delta^* \int_k f_k c_k^\dagger c_{k-\sigma} + h.c. \]

where the DDW and DSC order parameter fields are given by

\[ \phi = g_{DDW} \langle c_i^\dagger c_j \rangle_{DDW} = g_{DDW} \int_k \langle c_k^\dagger c_{k+Q,\sigma} c_k \rangle \] (6)

\[ \Delta = g_{DSC} \langle c_i^\dagger c_{i,-\sigma} \rangle_{DSC} = g_{DSC} \int_k \langle c_k^\dagger c_{k-\sigma} \rangle \] (7)

It is to be noted that both the DDW and DSC orders considered in this theory are bond order type in real space; that is, they represent order parameters defined on Cu-O bond rather than at the Cu site [8][13]. The (commensurate) DDW state breaks translational symmetry by one lattice spacing into two lattice spacings (i.e. doubles the unit cell) and breaks time reversal symmetry since it is represented by arrow of current on the bond (which corresponds to complex value for DDW order parameter \( \phi \)) and 4-fold rotational symmetry [13][25]. DSC state is also defined on the bond; it breaks 4-fold rotational \((x vs. y)\) symmetry but still preserves translational symmetry by one lattice spacing and \(d_{x^2-y^2}\) the time reversal symmetry.

I write the resulting mean-field Hamiltonian from the action Eq. (3) in terms of basis \( 4 \times 1 \) spinor \( \Psi_k = (c_k^\dagger, c_{k+Q,\sigma}^\dagger, c_{-k,\sigma}, c_{-(k+Q,\sigma)}^\dagger) \), from which I obtain the following Hamiltonian matrix

\[ H_{\text{red}} = \begin{pmatrix} \epsilon_k & \phi f_k & \Delta^* f_k & 0 \\ \phi^* f_k & \epsilon_k + Q & 0 & \Delta^* f_{k+Q} \\ \Delta f_k & 0 & -\epsilon_{-k} & -\phi f_{k+Q} \\ 0 & \Delta f_{k+Q} & -\phi^* f_{k+Q} & -\epsilon_{-(k+Q)} \end{pmatrix} \]

(8)

contributing to the full action Eq. (1) term of the form \( \int_k \Psi_k^\dagger H_{\text{red}} \Psi_k \). Despite the complicated expressions of those eigenvalues, in principle however, the Hamiltonian matrix in Eq. (8) can be diagonalized [36] that leads to new expression for the action Eq. (5) that can be written as

\[ S[c,\phi,\Delta] = \int_0^\beta d\tau \left[ \frac{\vert\phi\vert^2}{g_{DDW}} + \frac{\vert\Delta\vert^2}{g_{DSC}} + \int_k \sum_{i=1}^4 E_k^i d_k^i d_k^i \right] \]

(9)

where \( E_k^i, i = 1, 2, 3, 4 \) are the four eigenvalues of Hamiltonian matrix Eq. (5) and \( d_k^i, d_k^i \) are the Bogoliubov quasiparticle operators. Integrating out the fermions and considering (imaginary) time-independent field theory (where the Hamiltonian only has no explicit dependence on imaginary time so that \( \int_0^\beta d\tau H = \beta H \)), I obtain

\[ H[\phi,\Delta] = \frac{\vert\phi\vert^2}{g_{DDW}} + \frac{\vert\Delta\vert^2}{g_{DSC}} - T \int_k \ln \left[ \sum_{i=1}^4 e^{-\frac{E_k^i}{T}} \right] \]

(10)

where the functions \( \alpha, \beta_1, \beta_2 \) are given in the Supplementary Material. I am going to treat this effective Hamiltonian as the free energy (per unit cell) that I will minimize variationally [38]. Common approximation employed in studying Cuprates with hot spots on Fermi surface is that in analyzing the fermion scattering process, only contributions from fermions near hot spots are considered and
phases in terms of their phase boundaries, but rather the phase diagram of Cuprates involving DDW and DSC way down to low temperatures to coexist with relevant range of temperatures $T$ to $75K$ regime in the commensurate limit. The units of both axes are in eV. The solid lines are guide to the eye.

The fermion kinetic energy dispersion $\epsilon_k$ is expanded to linear (or quadratic) order around the hot spots. In evaluating the energy Eq. (10) however, rather than considering such expansion in deriving the effective action for the order parameters, I use the full expression for energy dispersion $\epsilon_k$ but integrate over momenta within appropriate regime in Fourier space.

I compute the energy given in Eq. (10) and minimize it as function of the variational mean field variables $|\Delta|$ and $|\phi|$ at given microscopic parameters $(t, t', t'', \mu, U, J, V_n, V_v, V_c)$ as function of temperature in the low temperature ($T \lesssim 250K$) regime. I consider both axial $\phi_X, \phi_Y$ and diagonal $\phi_{X+Y}, \phi_{X-Y}$ charge orders (ordering wavevector $Q = (Q_0, 0), (0, Q_0)$ and $Q = (Q_0, \pm Q_0)$ respectively) in the commensurate limit $Q_0 = \pi$ and compare their optimized variational energies from Eq. (10), which gives the total energy of the system with DDW and (or) DSC. This calculation is not intended to derive the phase diagram of Cuprates involving DDW and DSC phases in terms of their phase boundaries, but rather aimed at determining the relative energetics of the two types of charge order. Besides, the charge order is believed to exist below a critical temperature $T_{CD}$ all the way down to low temperatures to coexist with $d$-wave superconducting state that sets in at $T_c$ within certain range of hole doping levels. The result is presented in Fig. 3.

In obtaining Fig. 3 the microscopic parameters are set to $t = 1.0, t' = 0.3, \mu = 0.0, V_n = 0, U = 0.767t, J = 0.75t, V_v = 0.87t, V_v = 0$. I show the result for relevant range of temperatures $T$ where DDW and (or) DSC exist. We can see that the two components of axial charge order with $(Q_0, 0)$ and $(0, Q_0)$ are degenerate. So are the diagonal $(Q_0, Q_0)$ and $(Q_0, -Q_0)$ charge orders. This originates microscopically from the $x$ vs. $y$ symmetry of square lattice in the model Eq. (11). This is further supported by the finding that the effective Hamiltonian Eq. (10) is invariant under $C_4$ rotations which implies exact degeneracy. This degeneracy suggests that the charge order should be biaxial in character rather than uniaxial; it contains both $\phi_X \equiv (Q_0, 0)$ and $\phi_Y \equiv (0, Q_0)$ components or both $\phi_X \equiv (Q_0, Q_0)$ and $\phi_{X+Y} \equiv (Q_0, -Q_0)$ components rather than each of them separately. This is a crucial prediction of this work.

More importantly, we clearly see that the axial charge order consistently has lower energy than the diagonal charge order and therefore axial charge order should be observed experimentally rather than diagonal order. This is the result in the commensurate limit. The treatment of general incommensurate case is very delicate. However, considering the absolute energetic competitiveness of axial charge order over diagonal one over large temperature range demonstrated in Fig. 2 it is expected that as the microscopic parameters are tuned adiabatically to give the more realistic incommensurate case, this result still holds very firmly. I have checked this prediction by considering cases slightly away from commensurability by choosing chemical potential $\mu$ in the range $-0.5 \lesssim \mu \lesssim 0$ with all other parameters the same as those used in Fig. 2 and find that axial charge order still has decisively lower energy than diagonal one. Real situation in experiments finds periodicity of $3 \sim 4$ lattice spacings $\equiv \Delta_3$, corresponding to $1.75 \lesssim Q_0 \lesssim 2.10$ and can be reproduced by chemical potential in the range $-0.5 \lesssim \mu \lesssim -0.3$ with all other parameters as used above. This realistic situation is thus well covered and this suggests that the energetic advantage of axial charge order concluded above from the commensurate limit should remain valid even to realistic situation. This final result comes in remarkable agreement with experiments, as they indeed consistently observe axial charge order rather than diagonal one [11, 12, 20]. This is the main result of this Letter.

—Discussion: We observe that the energetic preference for axial charge order over diagonal charge order arises naturally in the mean field theory of Laughlin model Eq. (10) with DDW order scenario. While DDW order is assumed to begin with, no special emergent symmetry is assumed in this work. The assumption of $d$-wave symmetry itself is directly responsible for the energetic advantage of axial charge order over the diagonal one. The full effective Hamiltonian Eq. (10) can be expanded in terms of order parameters $\phi$ and $\Delta$. The expansion automatically includes higher than quadratic (cubic, quartic, . . .) order terms which describe the interplay between charge order and superconductivity. The energetic calculations presented above apparently suggests that the delicate interplay unambiguously singles out and prefers axial charge order over diagonal charge order. The interplay between DDW charge order and superconductivity is embodied naturally in the effective action Eq. (10) obtained upon integrating out the fermions including the feedback effect of superconductivity to charge order, which has been argued to suppress diagonal charge order and favors axial charge order [40]. The result Eq. (10) contains terms which represent the feedback effect of superconductivity dynamics at all higher (than quadratic)
orders in order parameters.

Experiments have not been able to conclusively decide on the uniaxiality versus biaxiality of the charge order. Some theoretical works suggest uniaxial charge order, while BCS-Eliashberg type calculation on semi microscopic spin-fermion model. My result from mean der \[26\] from BCS-Eliashberg type calculation on semi orders in order parameters.

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[32] Rather than considering the full phase diagram of Cuprates, I focus on the two types of order directly relevant to the discussion of charge order and its interplay with superconductivity in Cuprates in the form of DDW and DSC phases.

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[35] It is however important to note that the (commensurate) DDW order parameter will eventually involve only the imaginary part of the full bond order parameter \(\langle c_{i\sigma}^\dagger c_{j\sigma}\rangle\): \(\phi\) is to be taken as purely imaginary with the phase fixed to \(\pi/2\) [8]. The real part of \(\langle c_{i\sigma}^\dagger c_{j\sigma}\rangle\) contributes to correction of kinetic energy \(\epsilon_k\), which cannot be treated as order parameter [10] and is small in magnitude as it has much smaller effective coupling constant. This small correction to ground state energy coming from interactions will not change the qualitative conclusion on the relative energetic of different types of charge orders to be considered soon, as it is independent of \(Q\).

[36] We may diagonalize this action using both Bogoliubov-Valatin transformation by defining new fermionic quasi-particle creation and annihilation operators \(d_{k\alpha}^\dagger, d_{k\beta}\):

\[
\begin{align*}
\langle d_{k\alpha}^\dagger \rangle &= \sin \theta_{ck+Q\uparrow} + \cos \theta_{ck\downarrow} \\
\langle d_{k\beta} \rangle &= \sin \theta_{ck+Q\downarrow} + \cos \theta_{ck\uparrow}
\end{align*}
\]

[37] The full explicit expression of the energy eigenvalues \(E_k\) of the Hamiltonian given in Eq. \[33\] are very complicated but they can be written as \(E_k^{\pm,1} = \alpha \pm \beta_1, E_k^{\pm,2} = -\alpha \pm \beta_2\) [33].

[38] In the calculations to be presented, the order parameters are taken to be complex-valued in general to accommodate DDW as well as DSC. With \(d_{k\downarrow,\beta}\) \(d\)-wave SC state assumed, \(\Delta\) eventually takes real value. Furthermore, as noted earlier [33], since \(\phi\) is purely imaginary, then only the magnitudes of \(\Delta\) and \(\phi\) are varied and the phases drop out as they are fixed to 0 and \(\pi/2\) respectively.

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In this Supplementary Material, I give several important notes on the calculations, proof of exact degeneracy of the two orthogonal components of charge order, and the detailed expression for the functions $\alpha, \beta_1, \beta_2$ appearing in the effective Hamiltonian Eq. (10) in the main text.

In the calculations presented in the main text, I considered the most general character of the mean field order parameters; both $\phi$ and $\Delta$ are allowed to be complex valued in general even though both of them are assumed to take $d$-wave form. If we assume the $\Delta(k) \sim \Delta_0 f_k$ to describe $d$-wave superconductivity with $d_{x^2-y^2}$ symmetry without complex mixing with other $d$-wave symmetries, then $\Delta_0$ will be real-valued automatically.

The full expression for the coefficients $\alpha, \beta_1, \beta_2$ appearing in the effective Hamiltonian shown in the main text are given as follows.

$$\alpha = \frac{1}{2\sqrt{6}} [-4a_2 + \frac{2\times 2^{1/3}(a_2^3 + 12a_0)}{(2a_2^2 + 27a_1^2 - 72a_2a_0 + f(a_0, a_1, a_2))^{1/3}} + 2^{2/3}(2a_2^3 + 27a_1^2 - 72a_2a_0 + f(a_0, a_1, a_2))^{1/3}]^{1/2}$$

$$\beta_1 = \frac{1}{2\sqrt{6}} [-8a_2 - \frac{2\times 2^{1/3}(12a_0 + a_2^3)}{(27a_1^2 - 72a_2a_0 + 2a_2^3 + f(a_0, a_1, a_2))^{1/3}} - 2^{2/3}(27a_2^3 - 72a_0a_2 + 2a_2^3 + f(a_0, a_1, a_2))^{1/3}$$

$$\beta_2 = \frac{1}{2\sqrt{6}} [-8a_2 - \frac{2\times 2^{1/3}(12a_0 + a_2^3)}{(27a_1^2 - 72a_2a_0 + 2a_2^3 + f(a_0, a_1, a_2))^{1/3}} - 2^{2/3}(27a_2^3 - 72a_0a_2 + 2a_2^3 + f(a_0, a_1, a_2))^{1/3}$$

$$+ \frac{12\sqrt{6}a_1}{\sqrt{(-4a_2 + \frac{2\times 2^{1/3}(12a_0 + a_2^3)}{(27a_1^2 - 72a_2a_0 + 2a_2^3 + f(a_0, a_1, a_2))^{1/3}} + 2^{2/3}(27a_2^3 - 72a_0a_2 + 2a_2^3 + f(a_0, a_1, a_2))^{1/3})^{1/2}}$$

where

$$f(a_0, a_1, a_2) = \sqrt{-4(a_2^3 + 12a_0)^3 + (2a_2^3 + 27a_1^2 - 72a_2a_0)^2}$$

and the coefficients $a_0, a_1, a_2$ are given by

$$a_0 = (|\Delta|^2 f_k^2 + \epsilon_k^2)(|\Delta|^2 f_{k+Q}^2 + \epsilon_k^2 + Q) + (|\phi|^2 f_k^2 - \epsilon_k f_k + Q)(|\phi|^2 f_{k+Q}^2 - \epsilon_k f_{k+Q}) + 2|\Delta|^2 f_k^2 f_{k+Q}^2 |\phi|^2 - \epsilon_k^2 f_{k+Q}^2$$

$$a_1 = |\phi|^2 (\epsilon_k + f_{k+Q}^2 (f_k^2 + f_{k+Q}^2 - f_k^2))$$

$$a_2 = -[(\epsilon_k^2 + f_{k+Q}^2) + (|\Delta|^2 + |\phi|^2)(f_k^2 + f_{k+Q}^2)]$$

One crucial observation that can be made is that the $d$-wave form factors $f_k = \cos k_x - \cos k_y, f_{k+Q} = \cos(k_x + \theta)$, and $f_k^2 + f_{k+Q}^2 = \cos^2 k_x - \sin^2 k_y$.
\[ Q_x - \cos(k_y + Q_y) \] appear only at quadratic order rather than linear order and \( f_k^2 \) is 4-fold rotationally invariant. The \( \epsilon_k \) itself is also invariant under \( C_4 \) rotations. This implies that the full effective Hamiltonian Eq. (10) in the main text, upon symmetric integration over momenta, is invariant under 4-fold rotation by \( \pi/2 \) which brings \( k_x \rightarrow k_y, Q_x \rightarrow Q_y \) and \( k_y \rightarrow -k_x, Q_y \rightarrow -Q_x \). This implies that the \( x \) and \( y \) axial charge orders with \( \langle Q_0, 0 \rangle \) and \( \langle 0, Q_0 \rangle \) should be degenerate, as has been verified numerically in the main text on the energetic calculation.

As a check, for the special case where \( a_1 = 0 \) (the coefficient of linear term in characteristic quartic polynomial equation vanishes) which according to Eq. (13) occurs for the situations where \( \epsilon_{k+Q} = -\epsilon_k \) or \( f_{k+Q} = \pm f_k \), the four eigenvalues should satisfy \( \sum_{i\neq j\neq k} E_i(k)E_j(k)E_k(k) = 0 \). Eventually, the eigenvalues take the form \( E^{1,2}(k) = \alpha \pm \beta, E^{3,4} = -\alpha \pm \beta \) and do satisfy the triplet product condition. Indeed, as can be verified from the expression given for \( \beta_1 \) and \( \beta_2 \) shown above, I obtain \( \beta_1 = \beta_2 = \beta \). In such case, the effective Hamiltonian Eq. (10) simplifies to

\[
H[\phi, \Delta] = \frac{|\phi|^2}{g_{DDW}} + \frac{|\Delta|^2}{g_{DSC}} - T \int_k [\ln(4 \cosh \frac{\alpha}{T} \cosh \frac{\beta}{T})]
\]

where

\[
\beta = \frac{1}{2\sqrt{3}} \left[ -8a_2 - \frac{2 \times 2^{1/3}(12a_0 + a_2^2)}{(-72a_0a_2 + 2a_4^2 + \sqrt{-4(12a_0 + a_2^2)^3 + 72a_0a_2 + 2a_4^2})} \right]^{1/3}
\]

\[
-2^{2/3}(-72a_0a_2 + 2a_4^2 + \sqrt{-4(12a_0 + a_2^2)^3 + 72a_0a_2 + 2a_4^2})^{1/3} \right]^{1/2}
\]

The integration over momenta is supposed to be performed over the area of occupied states in the Brillouin zone within the Fermi surface. It however has to be done carefully with the following considerations especially with regard to the integration range of momenta. First, any scattering process, whether it is in particle-hole (density wave) channel or particle-particle (pairing) channel mostly involves only fermions near the Fermi surface. The reason is the scattering involves quasiparticle excitations; fermionic excitations out of the ‘Fermi sea’; the region of occupied states. Therefore, the closer the momenta near the Fermi surface, the lower the excitation energy and the easier it is to excite the fermions out of the Fermi sea. Second, only momenta near hot spots contribute to the particle-hole (density wave) scattering process because only those momenta can be connected by nesting wave vector that characterizes the density wave order. On the other hand, for particle-particle (pairing) scattering process, any momenta on and near the Fermi surface may participate because for any such wavevector \( \mathbf{k} \), there always exists the corresponding \( -\mathbf{k} \) with which Cooper pair is formed. This whole consideration is equivalent to taking hot spot approximation and performing the momentum integration over momenta with finite UV cutoff \( \Lambda \) around the hot spot when integrating out fermions to get effective action for order parameter. Physically, only fermions around hot spot are scattered by the nesting wavevector. The UV momentum cutoff gives the distance away from such hot spot momentum and should therefore be relatively small with respect to the characteristic wavevector of the Brillouin zone \( \pi \); \( |\mathbf{k} - \mathbf{k}_{ho}| \ll \pi \). A reasonable choice \( |\mathbf{k} - \mathbf{k}_{ho}| \leq 0.05\pi \) has been used in the calculation but the physical result remains qualitatively the same independent of the precise magnitude of the cutoff as long as this latter condition is satisfied.