Existence of complete Kähler Ricci-flat metrics on crepant resolutions

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Abstract

In this work, we obtain some existence results for complete Ricci-flat Kähler metrics on crepant resolutions of singularities. The method allows us to provide a wider class of examples of complete Ricci-flat Kähler metrics with richer topology at infinity.

1 Introduction

In 1978, Yau [Y] proved the Calabi Conjecture, by showing the existence and uniqueness of Kähler metrics with prescribed Ricci curvature on compact Kähler manifolds.

Although there are natural versions of this conjecture for non-compact manifolds, this case turns out to be much more subtle. A first major advance in this direction was the proof by Tian and Yau [TY1] of a non-compact version of the Calabi Conjecture on quasi-projective manifolds that can be compactified by the addition of a smooth, ample divisor. Shortly after [TY2], the two authors extended their results to the case where the divisor has multiplicity greater than 1 and is allowed to have orbifold-type singularities.

In [S], the asymptotic behavior of the metrics constructed by Tian and Yau was fully described. First, a sequence of explicit Kähler metrics with special approximating properties is constructed. Using those metrics as starting point, we then work out the asymptotic expansion of the metrics given in [TY1].

The aim of this paper is to study the Calabi Conjecture on manifolds which are crepant resolutions of complete Kähler spaces with isolated singularities, and ultimately to construct further examples of Calabi-Yau manifolds with prescribed behavior at infinity.

One of our main results is the following:

Theorem 1.1 Let \((M, \omega)\) be an \(n\)-dimensional, complete Calabi-Yau space with bounded sectional curvature and \(\beta\)-polynomial growth, with singularities contained in a compact subset of \(M\), and such that \(H_{2n-2}(M, \mathbb{C}) = 0\).

If \(M\) admits a crepant resolution \(\pi : \tilde{M} \to M\), then, for \(\varepsilon\) sufficiently small, there exists a Ricci-flat \(\tilde{\omega}\) on \(\tilde{M}\), where

\[ \tilde{\omega} \in \pi^*[\omega] - \varepsilon^2[E], \]
where $[E]$ is the Poincaré Dual of the $(2n - 2)$—homology class of the exceptional divisor $E$.

Although the results in Theorem 1.1 are interesting in themselves, being a natural expansion of the works of Yau [Y], Tian-Yau [TY1] and Joyce [J] on the Calabi Conjecture, one of their virtues is to provide a much wider class of examples of complete Calabi-Yau manifolds. This will be carefully explained in Section 5.

The above mentioned applications have as inspiration the 4-dimensional case of gravitational instantons, connected hyperkähler manifolds, with great relevance in quantum gravity and low-energy supersymmetric solutions of String Theory. Since the only compact examples of gravitational instantons are flat tori and $K3$-surfaces, we should seek for complete, non-compact examples, where compactness is replaced by a suitable condition of decay of the hyperkähler metric a flat metric at infinity.

In [CK1] and [E], the terminology used for the various cases split according to the topology at infinity:

- **ALE** (asymptotically locally flat) manifolds: when the hyperkähler metric decays to the Euclidean metric;
- **ALF** (asymptotically locally flat) manifolds: the metric decays to the flat metric on $\mathbb{R}^3 \times S^1$;
- **ALG**: the metric decays to a product flat metric on $\mathbb{R}^2 \times T^2$;
- **ALH**: the manifold resembles structure the product of $\mathbb{R}$ and one of the six flat orientable 3 manifolds at infinity.

In [Kr1] and [Kr2], Kronheimer classified all ALE hyperkähler manifolds as resolutions of $\mathbb{C}^2/\Gamma$, where $\Gamma$ is either $A_k, D_k$ or $E_j, j = 6, 7, 8$. For the ALF case, $A_k$ and $D_k$ families of ALF metrics have been constructed by Cherkis and Kapustin [CK2] and explicitly by Cherkis-Hitchin [CH]. Using String Theory motivations, they conjecture that those may be the only ALF examples of gravitational instantons.

For the ALG case, it has also been conjectured that the only examples are $D_k$, $k = 0, \ldots, 5$, and $E_j, j = 6, 7, 8$.

Our next result includes both 4-dimensional and higher dimensional analogues of gravitational instantons, but without the hyperkähler constraint. The following theorem provides more specific results about the decay of the metrics in Theorem 1.1 for the special case of manifolds which we call *asymptotically locally flat of order $k$*. A more precise definition is provided in Section 2. In rough terms, a manifold is said to be $ALF_k$ if its structure at infinity resembles $\mathbb{R}^{2n-k} \times T^k$ to some fixed decay rate.

**Theorem 1.2** Let $(M, \omega)$ be a manifold as in Theorem 1.1, which is also asymptotically locally flat of order $k$.

Then, the complete Ricci-flat metric $\tilde{\omega}$ given by Theorem 1.1 satisfies

$$
\tilde{\omega} = \omega + C \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} (\rho^{2+k-2n}) + \partial \bar{\partial} \Psi,
$$

(1)
where $\rho$ is the distance function on $\tilde{M}$, and $\Psi$ is a smooth function on $\tilde{M}$ such that
\[ \nabla^k \Psi = O(\rho^{-\gamma}), \] where $\gamma \in (1 + k - 2n, 2 + k - 2n)$.

This theorem says that the $ALF_k$ behavior at infinity is preserved under our construction. We shall exploit this fact in Section 5.

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2 Background Material

Throughout this paper, we say that $f$ is of order $O(g)$ if there exists a uniform constant $C$ such that $|f| \leq C|g|$ near infinity.

We should start by providing the definition of Asymptotically Locally Flat manifolds. Since the author was not able to find in the literature a precise definition of those objects even in the 4-dimensional case, we set our own, inspired by the definition of Asymptotically Locally Euclidean manifolds in [J].

Definition 2.1 Let $(X, g)$ be an $n$-dimensional non-compact Riemannian manifold. We say that $(X, g)$ is an Asymptotically Locally Flat manifold of order $k$ (or $ALF_k$, for short), if the following conditions hold.

There must exist a compact subset $S \subset X$ and a map (to be referred as an asymptotic coordinate system)
\[ \pi : X \setminus S \to \mathbb{R}^{n-k} \times T^k \] (2)
which is a diffeomorphism between $X \setminus S$ and the set \{(x, y) \in \mathbb{R}^{n-k} \times T^k; r(x) > R\}, for some large $R$, where $r$ is the radius function on $\mathbb{R}^{n-k} \times T^k$.

Furthermore, we require that the push-forward metric $\pi_*(g)$ satisfies
\[ \nabla^m(\pi_*(g) - h) = O(r^{k-n-m}) \quad \text{on} \quad \{(x, y) \in \mathbb{R}^{n-k} \times T^k; r(x) > R\}, \] (3)
where $h$ is the standard flat metric on $\mathbb{R}^{n-k} \times T^k$.

A note about our choice of the decay conditions in the definition above. Let $\omega$ be the Kähler form of the $ALF_k$ Kähler manifold $M$, and let $\omega_0$ be a Ricci-flat Kähler metric on $(\mathbb{R}^{n-k} \times T^k)$ [J]. In every case, $\omega$ must satisfy
\[ \Pi^* \omega = \omega_0 + \partial \bar{\partial} \phi, \] (4)
for some function $\phi$ that satisfies the Monge-Ampère equation
\[ (\omega_0 + \partial \bar{\partial} \phi)^n = \omega^n_0. \] (5)
From (5), we see that $\Delta \phi \to 0$ when $r$ is very large. However, we note that $\Delta (r^{2+k-n}) = 0$ on $\mathbb{R}^{n-k} \times T^k$, so the optimal decay rate that can be expected is $\phi = O(r^{2+k-n})$. Therefore, (4) tells us that $\Pi^* \omega = \omega_0 + O(r^{k-n})$, as claimed.

1If $k \neq 1$, we can simply take the product flat metric. If $k = 1$, consider the Taub-NUT metric of LeBrun [L].
Definition 2.2 Let \((X, g)\) be an ALF\(_k\) manifold. We say that \(\rho : X \to [1, \infty)\) is a distance function on \(X\) if given any asymptotic coordinate system \(\pi : X \setminus S \to \mathbb{R}^{n-k} \times T^k\), we have
\[
\nabla^m(\pi_*(\rho) - r) = O(r^{1+k-n-m}) \quad \text{on} \quad \{(x, y) \in \mathbb{R}^{n-k} \times T^k; r(x) > R\}.
\]

The reader will notice that the condition on Definition 2.2 is independent of the choice of asymptotic coordinate system.

We shall now characterize the necessary regularity and volume growth for our results.

Definition 2.3 Let \((M, g)\) be a Ricci-flat manifold. We say that \(M\) has \(\beta\)-polynomial growth if the volume of the geodesic ball \(B_1(x)\) of radius 1 around a point \(x \in M\) satisfies
\[
\text{vol}(B_1(x)) \geq C^{-1}(1 + \text{dist}_g(x, x_0))^{-\beta},
\]
where \(x_0\) is a fixed point in \(M\), and \(C = C(M)\) is a uniform constant.

Observe that, even though the name might be misleading, we are not bounding above the volume growth of balls on the definition of \(\beta\)-polynomial growth.

Definition 2.4 Let \((M, g)\) be a complete Kähler manifold. We say that \((M, g)\) is of quasi-finite geometry of order \(\ell + \delta\) if there exist \(r > 0, r_1 > r_2 > 0\) such that for any \(x \in M\), there exists a holomorphic map
\[
\Phi_x : U_x \to B_r(x)
\]
such that \(B_{r_2} \subset U_x \subset B_{r_1}\), and \(\Phi_x^*(g)\) is a Kähler metric on \(U_x \subset \mathbb{C}^n\) such that its metric tensor has derivatives up to order \(\ell\) bounded and \(\delta\)-Holder-continuously bounded.

Remark (Tian and Yau, [TY1]): If a complete Kähler manifold \((M, g)\) has its sectional curvature and covariant derivative of the scalar curvature bounded, then \((M, g)\) is of quasi-finite geometry of order \(2 + 1/2\).

3 Proof of Theorem 1.1

In this section, we shall explain the proof of our main theorem.

Consider \((M, g)\), a (singular) Kähler manifold with zero Ricci curvature, and denote by \(\rho\) its distance function. Let \(\pi : \tilde{M} \to M\) be a crepant resolution and let \(S \subset M\) be a compact set that contains the singular set of \(M\).

Consider the positive semi-definite \((1, 1)\)-form \(\Pi^*(\omega_g)\) on \(\tilde{M}\), where \(\omega_g\) is the singular metric on \(M\). This metric is positive definite on the directions which are transverse to the exceptional divisor \(E\). Thanks to our assumption, the homology group \(H_{n-2}(\tilde{M}, \mathbb{C})\) must be generated by the class of the exceptional divisor \([E]\). By Poincaré Duality, \(H_{2n-2}(\tilde{M}, \mathbb{C}) = H^2_c(\tilde{M}, \mathbb{C})\). Hence, all any closed \((1,1)\)-form on \(\tilde{M}\) is cohomologous to a compactly supported 2-form.

Next, let us define \(\omega = \Pi^*(\omega_g) - \varepsilon^2 \omega_E\), a complete metric on \(\tilde{M}\). In order to prove the existence of a Ricci-flat Kähler metric on \(\tilde{M}\), we shall first prove the following result.
Theorem 3.1 Let $(X, g)$ be a Kähler manifold with bounded geometry of order $2+1/2$ and of $\beta$-polynomial growth. Let $f$ be a compactly supported function satisfying the integrability condition $\int_X (e^f - 1)w^n = 0$. Then, there exists a solution $u$ to the equation

\[
\begin{cases}
\omega + \frac{\sqrt{-1}}{2\pi} u = e^f w^n, \\
\omega + \frac{\sqrt{-1}}{2\pi} u > 0 \quad \text{on } X.
\end{cases}
\]

The first step in the proof of Theorem 3.1 is to show that (6) is solvable.

The proof of this result will follow closely the ideas on [TY1], with some refinements provided by [S]. The idea of the proof consists of obtaining the function $u$ as the uniform limit of solutions $u_\varepsilon$, as $\varepsilon$ goes to zero, of perturbed Monge-Ampère equations

\[
\begin{cases}
\omega + \frac{\sqrt{-1}}{2\pi} u_\varepsilon = e^{f+\varepsilon u_\varepsilon} w^n, \\
\omega + \frac{\sqrt{-1}}{2\pi} u_\varepsilon > 0 \quad \text{on } X.
\end{cases}
\]

The existence of those $u_\varepsilon$, for every $\varepsilon > 0$, is guaranteed by [CY1], since we are assuming that our manifold $(X, g)$ has bounded geometry.

The remainder of this section will be dedicated to the proof of uniform $C^{2,1/2}$-estimates for $u_\varepsilon$, since standard elliptic theory will guarantee that we can take the limit when $\varepsilon$ approaches zero and obtain a solution $u$ to (6).

The very first step is to check that the integrability condition is satisfied for each $u_\varepsilon$.

Lemma 3.1 Let $(X, g)$ be as in Theorem 3.1, and let $u_\varepsilon$ be a solution to (7). Then

$$
\int_M (e^{f+\varepsilon u_\varepsilon} - 1)\omega^n = 0
$$

Proof: In order to be able to perform integration by parts, we consider a cut-off function $\Phi : \mathbb{R} \to [0, 1]$ such that $\Phi(t) = 1$ for $t \leq 1$, and $\Phi(t) = 0$ for $t \geq 2$, and $|\Phi'| \leq 2$, and $|\Phi''| \leq 2$.

Let $\rho(x)$ be the distance (for the metric $g$) from $x$ to a fixed point $x_0 \in X$. Set $\Phi_R(x) = \Phi(\frac{\rho(x)}{R})$.

Multiplying both sides of (7) by $\Phi_R(x)$, integrating by parts, and using the fact that all metrics $\omega + \frac{\sqrt{-1}}{2\pi} u_\varepsilon$ are equivalent to $\omega$, it follows that

$$
\int_M \Phi_R(x)(e^{f+\varepsilon u_\varepsilon} - 1)\omega^n \leq \frac{C}{R} \int_M |\nabla u_\varepsilon|\omega^n,
$$

where the derivative is taken with respect to the metric $g$. Hence, it suffices to show that $\int_M |\nabla u_\varepsilon|\omega^n$ is bounded. Holder’s inequality gives us

$$
\int_M |\nabla u_\varepsilon|\omega^n \leq \left( \int_M (1 + \rho(x))^{2q}|\nabla u_\varepsilon|^2\omega^n \right)^{1/2} \left( \int_M (1 + \rho(x))^{-2q}\omega^n \right)^{1/2},
$$

where $q$ is chosen such that the last integral on the right-hand side is finite.
Therefore, it only remains to prove that

$$\int_M (1 + \rho(x))^{2q} |\nabla u_\varepsilon|^2 \omega^n < \infty. \quad (8)$$

This will be proved by showing that the inequality holds for $(u_\varepsilon)_+$ and $(u_\varepsilon)_-$. So, we shall now assume that $u = (u_\varepsilon)_+$. Multiplying (7) by $(1 + \rho(x))^{2p} \Phi_R^2(x)u$, we obtain

$$\int_X (1 + \rho(x))^{2p} \Phi_R^2(x) u (e^{f + \varepsilon u} - 1) \omega^n = \frac{-1}{2\pi} \int_X (1 + \rho(x))^{2p} \Phi_R^2(x) u \partial \overline{\partial} u_\varepsilon \wedge (\omega^{n-1} + \cdots \omega_\varepsilon^{n-1}).$$

After some manipulations involving integration by parts, the following inequality can be derived:

$$\int_X (1 + \rho(x))^{2p} \Phi_R^2(x) u e^{f} (e^{\varepsilon u} - 1) \omega^n + \int_X |\nabla((1 + \rho(x))^{p} u)\Phi_R^2(x)\omega^n \leq C \left\{ \int_X (1 + \rho(x))^{2p} \Phi_R^2(x) u e^{f} - 1 |\omega^n + \int_X (1 + \rho(x))^{2p-2} \Phi_R^2(x) u^2 \omega^n + \right.$$  

$$+ \int_X \frac{|\Phi'|{(\rho/R)}}{R^2} (1 + \rho(x))^{2p} u \omega^n \right\}. \quad (9)$$

Notice that the first term on the left-hand side of (9) is bounded below by

$$C \int_X (1 + \rho(x))^{2p} \Phi_R^2(x) u^2 e^{f} \omega^n \geq C \int_X (1 + \rho(x))^{2p} \Phi_R^2(x) u^2 \omega^n,$$

since $e^{\varepsilon u} - 1 \geq \varepsilon u$, and $f$ has compact support.

Therefore, since the first integral on the right-hand side of (9) is finite, and $\frac{|\Phi'|{(\rho/R)}}{R^2} < C\Phi_{2R}^2$, we have

$$\int_X (1 + \rho(x))^{2p} \Phi_R^2(x) u^2 \omega^n + \int_X |\nabla((1 + \rho(x))^{p} u)\Phi_R^2(x)\omega^n \leq$$

$$\leq C \left\{ 1 + \int_X (1 + \rho(x))^{2p-2} \Phi_{2R}^2(x) u^2 \omega^n \right\}. \quad (10)$$

Now, the result of Cheng and Yau [CY1] ensures that, for each $\varepsilon > 0$, the solution $u_\varepsilon$ is bounded. So, there exists an $N$ (that depends on the volume growth of $\omega$) verifying

$$\int_X (1 + \rho(x))^{-N} u^2 \omega^n < \infty.$$

Therefore, by inductively using (10) in order to get $p$ to be negative enough, and of course letting $R \to \infty$, we obtain the desired statement.

This completes the proof of the proposition. \qed

The higher order estimates are shown in [Y].
Theorem 3.2 (Yau, [TY]) There are constants $C_3$ and $C_4$, independent of $\varepsilon$, such that
\[ 0 \leq n + \Delta u_\varepsilon \leq C_3 e^{u_\varepsilon - \inf_X u_\varepsilon}. \]

Furthermore, there are a priori bounds for $|\nabla^3 u_\varepsilon|$ in terms of $\sup_X |u_\varepsilon|, |\Delta u_\varepsilon|, |f|, |\nabla f|, |\nabla^2 f|.$

Finally, the last ingredient on the proof of uniform bounds for $u_\varepsilon$ is essentially proved in [TY1]:

Proposition 3.1 There exists a constant $C$ (independent of $\varepsilon$) such that $\sup_X |u_\varepsilon| \leq C$.

The key observation about this proposition is that we can still use the weighted Sobolev inequality derived in [TY1], since our starting manifold $(X, g)$ is Ricci-flat, with the right regularity and volume growth.

Proof: The reader should be warned about the constants $C$, that will denote, unless otherwise stated, various different constants which are all independent of $\varepsilon$.

In the course of the proof, we shall make use of uniform bound on the Green’s Kernel on balls, as follows. Applying the maximum principle for (7), we can see that the maximum and the minimum of $u_\varepsilon$ are attained inside the (compact) support of $f$. Let $x_{\max}$ (resp. $x_{\min}$) be points where the maximum (resp. minimum) of $u_\varepsilon$ are achieved.

The expression
\[ \int_M e^f (e^{u_\varepsilon} - 1) \omega^n = \int_M (e^{f+\varepsilon u_\varepsilon} - 1) \omega^n + \int_M (e^f - 1) \omega^n = 0 \]
implies that $u_\varepsilon(x_{\max}) > 0$ and $u_\varepsilon(x_{\min}) < 0$.

Let us assume that the weighted average
\[ \text{Ave}_\rho(u_\varepsilon) = \frac{\int_M (1 + \rho(x))^{-N} u_\varepsilon(x) \omega^n}{\int_M (1 + \rho(x))^{-N} \omega^n} \]
of $u_\varepsilon$ with respect to the weight $(1 + \rho(x))^{-N}$ ($N$ chosen accordingly to the volume growth of $\omega$) is non-negative. The proof in the other case is analogous.

To complete the proof of this proposition, the following result is needed.

Lemma 3.2 If $\text{Ave}_\rho(u_\varepsilon) \geq 0$, then there exists a constant $C$ independent of $\varepsilon$ such that
\[ u_\varepsilon(x_{\min}) \geq -C \quad \text{and} \quad \sup_M (u_\varepsilon(x) - \text{Ave}_\rho(u_\varepsilon)) \leq C. \]

Let us complete the proof of Proposition 3.1 assuming the Lemma 3.2 and return to its proof later.

Consider a small convex geodesic ball $B_r(x_{\min})$, and a cut-off function $\eta$ on it, which is identically 1 on $B_{r/2}(x_{\min})$, and zero outside $B_{3r/2}(x_{\min})$. 

7
Yau’s higher order estimates \[ Y \] give that
\[
\Delta u_\varepsilon + n \leq Ce^{C(u_\varepsilon - \inf_M u_\varepsilon)}.
\]

Set \( \psi = (u_\varepsilon - \inf_M u_\varepsilon - 1)_- \). Clearly, \( 0 \leq \psi \leq 1 \), \( \psi(x_{\min}) = 1 \), and \( \psi \neq 0 \Rightarrow u_\varepsilon - \inf_M u_\varepsilon < 1 \).

Now, multiply (11) by \( \eta^2(x)\psi(x)G(x_{\min}, x) \), where \( G(x, y) \) is the Green’s kernel for the Dirichlet problem on the ball \( B_r(x_{\min}) \), so as to obtain
\[
- \int_M \Delta \psi \eta^2 \psi G(x_{\min}, x)\omega^n \leq C \int_M \psi \eta^2 \psi G(x_{\min}, x)\omega^n.
\]

Integrating by parts, it follows
\[
\int_M \eta^2 \nabla \psi |\nabla|^2 G(x_{\min}, x)\omega^n + \int_M \nabla (\eta^2) \psi \nabla G(x_{\min}, x)\omega^n + \int_M \eta^2 \nabla \psi \nabla G\omega^n \leq C \int_M \eta^2 \psi G(x_{\min}, x)\omega^n.
\]

Note that
\[
\int_M \eta^2 \nabla \psi \nabla G\omega^n = \frac{1}{2} \int_M \nabla (\eta^2) \psi \nabla G\omega^n - \int_M \nabla (\eta^2) \psi \nabla G\omega^n,
\]
and Holder’s inequality yields
\[
\int_M \nabla (\eta^2) \psi \nabla G(x_{\min}, x)\omega^n \leq \int_M |\nabla \eta| |\nabla G(x_{\min}, x)\omega^n| + \int_M \eta^2 |\nabla \psi| G(x_{\min}, x)\omega^n.
\]

Therefore, rearranging the terms in (12) and recalling that \( \text{supp}(\eta) \subset B_r(x_{\min}) \), we obtain
\[
\frac{1}{2} \int_{B_r(x_{\min})} \nabla (\eta^2) \psi \nabla G\omega^n + \frac{1}{2} \int_{B_r(x_{\min})} |\nabla \psi|^2 \eta^2 G(x_{\min}, x)\omega^n \leq \]
\[
\leq C \left\{ \int_{B_r(x_{\min})} \psi(x) G(x_{\min}, x)\omega^n + \int_{B_r(x_{\min})} \psi^2 |\nabla \eta| G(x_{\min}, x)\omega^n + \int_{B_r(x_{\min})} \psi^2 |\nabla \eta|^2 G\omega^n \right\}.
\]

As mentioned before, we want to use the fact that \( G(x_{\min}, x) \) and \( \nabla G(x_{\min}, x) \) are bounded independently of \( \varepsilon \). We have
\[
1 = \psi^2(x_{\min}) = \int_{B_r(x_{\min})} \Delta G(x_{\min}, x)\psi^2\omega^n
\]
\[
\leq C \left[ \int_{B_r(x_{\min})} G(x_{\min}, x)\omega^n + \int_{B_r(x_{\min})} |\psi|^2\omega^n \right]
\]
\[
\leq C \left( \int_{B_r(x_{\min})} \psi^{2n-2} \right) \left( \int_{B_r(x_{\min})} G^{\frac{2n-1}{2n+1}} \right)^{\frac{2n-1}{2n+1}} + \int_{B_r(x_{\min})} |\psi|^2\omega^n
\]
Therefore, \( \text{vol}(\text{supp}(\psi) \cap B_r(x_{\text{min}})) \geq 1 \). Hence,

\[
\int_M (1 + \rho(x))^{-N} u_\varepsilon \omega^n \leq \int_{M \setminus \text{supp}(\psi) \cap B_r(x_{\text{min}})} (1 + \rho(x))^{-N} u_\varepsilon \omega^n + C.
\]

However,

\[
\int_{M \setminus \text{supp}(\psi) \cap B_r(x_{\text{min}})} (1 + \rho(x))^{-N} u_\varepsilon \omega^n \leq \sup_M u_\varepsilon \int_M (1 + \rho(x))^{-N} \omega^n - \sup_M u_\varepsilon \int_{\text{supp}(\psi) \cap B_r(x_{\text{min}})} (1 + \rho(x))^{-N} \omega^n,
\]

and \( \int_{\text{supp}(\psi) \cap B_r(x_{\text{min}})} (1 + \rho(x))^{-N} \omega^n \geq C(2 + \rho(x_{\text{min}}))^{-N} \), since the volume of the region of integration is bounded below.

Therefore,

\[
\int_M (1 + \rho(x))^{-N} u_\varepsilon \omega^n \leq C + \sup_M u_\varepsilon \left[ \int_M (1 + \rho(x))^{-N} \omega^n - C(2 + \rho(x_{\text{min}}))^{-N} \right],
\]

which implies, together with the Lemma 3.2 that \( \sup_m u_\varepsilon \leq C \). This completes the proof of Theorem 3.1.

**Proof of Lemma 3.2:** Let us write \( \psi = (u_\varepsilon - \text{Ave}_\rho(u_\varepsilon))_+ \), so as to have \( \psi(e^{\varepsilon u_\varepsilon} - 1) \geq 0 \) on \( M \). We have

\[
- \int_M |\nabla (\psi^{2+1})|^2 \omega^n = - \left( \frac{q + 1}{2} \right)^2 \int_M \psi^{q-1} u_\varepsilon (e^{f+\varepsilon u_\varepsilon} - 1) \omega^n.
\]

Thanks to (7), the last estimate yields

\[
\int_M |\nabla (\psi^{2+1})| \omega^n \leq \left( \frac{q + 1}{2} \right)^2 \int_M |\psi|^q e^f - 1 \omega^n.
\]

Our manifold satisfies the \( \beta \)-polynomial growth condition and is assumed to have bounded sectional curvature, therefore the weighted Sobolev inequalities developed in [TY1] still hold. It then follows that

\[
\int_M (1 + \rho(x))^{-N} \psi^{2+1} - \text{Ave}_\rho(\psi^{2+1}) \psi^{2+1} \omega^n \leq C \int_M (1 + \rho(x))^{-N} |\psi|^q \omega^n, \tag{17}
\]

where the last constant depends on \( q \), but not on \( \varepsilon \).

Since the weighted volume \( \int_M (1 + \rho(x))^{-N} \omega^n \) is bounded (for \( N \) large enough), we can apply H"older’s inequality again, to obtain

\[
||1 + |\psi||_{q+1} \frac{2n+1}{2n+4} \leq C(q, n)||1 + |\psi||_{q+1},
\]

where \( ||f||_p \) stands for the weighted \( L^p \)-norm \( ||f||_p = (\int_M (1 + \rho)^{-N} f^p)^{\frac{1}{p}} \).

Set \( q_0 = 2 \frac{2n+1}{2n+4} \), and \( q_{j+1} = q_j \frac{2n+1}{2n+4} \). Inductively,

\[
||1 + |\psi||_{q_{j+1}} \leq C(q, n)||1 + |\psi||_{q_0} \leq C \int_M (1 + \rho)^{-N} |\psi| \omega^n \leq C \left( \int_M (1 + \rho)^{-N} |\psi|^2 \omega^n \right)^{\frac{1}{2}} \left( \int_M (1 + \rho)^{-N} \omega^n \right)^{\frac{1}{2}}.
\]
Note that the first term on the right-hand side of this inequality is bounded, as shown in the proof of Lemma 3.1. Therefore, by letting $j \to \infty$,
\[
\lim_{j \to \infty} ||\psi||_{q_j} = \sup_M (u_0 \epsilon - \text{Ave}_\rho(u_\epsilon))_+ \leq C,
\]
which finishes the proof of the Lemma 3.2.

With the uniform bounds, we are able to consider a subsequence $u_\epsilon \to u$, where $u$ is a solution to (6). We note that the solution $u$ will be bounded, and $\int_X |\nabla u|^2 \omega^n < \infty$.

To see this, we multiply (7) by $u_\epsilon$ and after integrating by parts once, we conclude
\[
\int_X |\nabla u_\epsilon|^2 \omega^n < C \int_X |u_\epsilon| e^{\epsilon u_\epsilon} - 1 |\omega^n|,
\]
which clearly proves the claim.

4 Asymptotics

In this section, we shall give the proof of Theorem 1.2. The main idea is borrowed from [5], since we can take advantage of $f$ being compactly supported. This theorem implies that the ALF$_k$-type of the manifold is preserved under our construction.

We begin by showing the first order decay of $u$, which is a result that actually is independent of the topological type (at infinity) of our Ricci-flat manifold. It only requires the same assumptions as for Theorem 1.1.

Lemma 4.1 Let $(M, g)$ as in Theorem 1.1 and let $u$ be the solution to (6) constructed in Theorem 1.1. Then $u(x)$ converges uniformly to zero as $\rho(x) \to \infty$, where $\rho(x)$ denotes the distance (using W.R.T. the metric $g$) from a fixed point on $M$ to $x$.

Proof: We shall use the fact that the solution $u$ was obtained as the uniform limit of solutions $u_\epsilon$ to (7), and prove uniform bounds on $u_\epsilon$ by using the maximum principle.

Let us define the Monge-Ampère operator
\[
M(\phi) = \log \left( \frac{(\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi)^n}{\omega^n} \right).
\]

Observe that for $\beta \in (2, n-1)$, we have that $M(C \rho^{-\beta}(x)) = C\beta \rho^{-\beta-2}(x) + O(\rho^{-\beta-3}(x))$. Also, for each solution $u_\epsilon$, we have that $M(u_\epsilon) = f + \epsilon u_\epsilon$.

For a fixed $\delta > 0$, set $C_1 = \frac{C'}{\delta}$, where $C'_1 = \sup_{\rho(x) \geq \delta^{-1}} (|u_\epsilon| + 1)$, and set $C_2 = -C_1$. We point out that, due to Proposition 3.1, we can choose $C_1$ independently of $\epsilon$.

Hence, for all $x$ such that $\{\rho^{-1}(x) = \delta\}$, we have
\[
C_1 \rho^{-1}(x) > |u_\epsilon(x)| \quad \text{and} \quad C_2 \rho^{-1}(x) > |u_\epsilon(x)|.
\]
Furthermore, in the region \( \{ x \in M; \rho^{-1}(x) = \delta \} \), for \( \delta \) chosen to be sufficiently small, we have

\[
M(C_1\rho^{-1}(x)) < f + \varepsilon u_\varepsilon < M(C_2\rho^{-1}(x)).
\]

Finally, we observe that the solution \( u_\varepsilon \) (for each \( \varepsilon \)) converges uniformly to zero at infinity. Hence, we can apply the maximum principle to the operator \( M(\cdot) \) so as to conclude that there exists a constant \( C \) such that

\[
-C\rho^{-1}(x) \leq u_\varepsilon(x) \leq C\rho^{-1}(x).
\]

We complete the proof of this lemma by observing that thanks to the uniform estimates in \( S \), Proposition 5.2, we can take the constant \( C \) to be independent of \( \varepsilon \). Then, passing to the limit when \( \varepsilon \to 0 \), the claim follows. \( \square \)

**Proposition 4.1** Let \( (M, g) \) as in Theorem 1.1 and assume that \( M \) is an ALF\(_k\)-manifold. Let \( u \) be the solution of \( 6 \) constructed in Theorem 1.1.

Then \( u \) has order of decay of \( O(\rho^{2+k-2n}(x)) \), where \( n \) is the complex dimension of the manifold \( M \) and \( f \in C^\infty(M, \mathbb{R}) \) has compact support.

We emphasize that this estimate is the best that can be expected, in view of the decay of harmonic functions vanishing at infinity on complete ALF\(_k\) manifolds.

**Proof:** We shall use once more the maximum principle for the Monge-Ampère operator \( M(\phi) \), but our choice of barrier shall be carried out in a slightly more careful way.

Observe that the linearization of \( M(\cdot) \) at \( \phi = 0 \) equals the laplacian \( \Delta_g \) of the metric \( g \), so most of the discussion is done around \( \Delta(\phi) \).

The barrier function that will be used here is given by \( \psi = C\rho^{2+k-2n}(-\log \rho)^\ell \), where the constants \( C \) and \( k \) will be carefully chosen in what follows.

After some computations, we obtain

\[
M(C\rho^{2+k-2n}(-\log \rho)^\ell) = C(2\ell + 1 + k - 2n)\rho^{k-2n}(-\log \rho)^{\ell-1}(1 + o(1)).
\]

Next, fix \( \delta > 0 \) and define \( C_1 = \frac{C_1}{\delta} \), where \( C_1' = \sup(|u| + 1) \), where the supremum is taken on the set \( \{ x \in M; \rho^{2+k-2n}(-\log \rho)^\ell(x) \leq \delta \} \). Also, set \( C_2 = -C_1 \). Clearly,

\[
C_1\rho^{2+k-2n}(-\log \rho)^\ell(x) > u(x) \quad \text{and} \quad C_2\rho^{2+k-2n}(-\log \rho)^\ell(x) < u(x).
\]

on the set where \( \rho^{2+k-2n}(-\log \rho)^\ell(x) = \delta \).

Now, we fix \( \ell \) (the exponent of the log term) to be small enough so that \( 2\ell + 1 + k - 2n < 0 \). Since \( f \) has compact support\(^2\), there exists some small \( \delta > 0 \) such that

\[
M(u) = f \geq C_1(2\ell + 1 + k - 2n)\rho^{-2n}(-\log \rho)^\ell(x) (1 + o(1)) \tag{18}
\]

\[
= M(C_1\rho^{2+k-2n}(-\log \rho)^\ell(x)) \tag{19}
\]

\(^2\)In fact, all we need from \( f \) is that it decays strictly faster than \( \rho^{-2n} \).
and analogously for the upper bound with \( C_2 \).

By Lemma 4.1, we know that \( u(x) \) converges uniformly to zero as \( \rho(x) \to \infty \).

Therefore, we can apply the maximum principle to the operator \( M \) and conclude that there exists a constant \( C \) such that

\[
|u(x)| \geq C\rho^{2-2n}(-\log \rho)^k(x).
\]

This completes the proof of the proposition.

\[\square\]

**Proposition 4.2** Let \( u \) be a solution to (6). Then, there exists \( C = C(k) \) such that, for all \( x \in M \),

\[
|\nabla^k u|_g(x) \leq C\rho^{\gamma-k}(x).
\]  

(20)

**Proof:** Since \( f \) has compact support, the statement follows from observing that the leading behavior of the derivatives of \( u \) will depend solely on the term \( C\rho^{2+k-2n}(x) \) appearing on the expansion of the solution \( u \) given by Proposition 4.1.

\[\square\]

## 5 Application: construction of complete non-flat Ricci-flat manifolds

### 5.1 Asymptotically Locally Flat metrics on \( \mathbb{C}^n \), \( n > 2 \)

In \([L]\), LeBrun observed that \( \mathbb{C}^2 \) admits a Ricci-flat Kähler metric which is not flat. The so-called *Taub-NUT metric* of Hawking \([H]\) can be given explicitly on \( S^3 \times \mathbb{R}^+ \) in the form

\[
g_T = \frac{\rho + 1}{4\rho} d\rho^2 + \rho(1 + \rho)[\sigma_1^2 + \sigma_2^2] + \frac{\rho}{\rho + 1} \sigma_3^2,
\]

where \( \{\sigma_1, \sigma_2, \sigma_3\} \) is a left-invariant coframe for \( S^3 \) and \( \rho \in \mathbb{R}^+ \).

Note that the volume of a large ball of geodesic radius \( R \) about the origin is given by

\[
\text{Vol}_{g_T}(B_R(0)) = O(R^3),
\]

so this 4-manifold is an \( ALF_1 \)-manifold\(^3\), hence it has non-zero Riemann curvature tensor.

We should also point out that the metric \( g_t \) is in fact Hyperkähler, since it has holonomy \( SU(2) \).

For our example, we shall consider a copy of \( \mathbb{C}^2 \), endowed with the Taub-NUT metric \( g_T \), and take the product with an Euclidean \( (\mathbb{C}^{n-2}, g_e) \). Clearly, the product metric \( g \otimes g_e \) is a trivial example of a non-flat complete Ricci-flat metric on \( \mathbb{C}^n \), but that is certainly not our main focus of interest.

The group \( \mathbb{Z}^2 \) acts in both components \( \mathbb{C}^2 \) and \( \mathbb{C}^{n-2} \). Let \( \gamma_1 \) and \( \gamma_2 \) be the generators of the action in each factor. Consider the action on \( \mathbb{C}^n \) by the group (of order 2) \( \Gamma \)

\(^3\text{In the literature, an } ALF_1 \text{ 4-manifold is called Asymptotically Locally Flat, or ALF for short.}\)
generated by the pair \((\gamma_1, \gamma_2)\). The quotient \(\mathbb{C}^n/\Gamma\) is not a product of quotients, as \(\Gamma\) has order 2.

Now, consider the manifold \(M = (\mathbb{C}^n, g_T \otimes g_e)/(\Gamma \times G)\). According to our construction, \(M\) will be a singular, complete Ricci-flat manifold, with a quotient singularity at the origin.

Let \(\pi : \tilde{M} \to M\) be a crepant resolution of \(M\). We shall point out here that the work of Sardo Infirri \([SI]\) implies the existence of a crepant resolution for this quotient, since it is a toric variety.

Then, Theorem \([1.1]\) can be applied to provide the existence of a complete, Ricci-flat metric \(\tilde{g}\) on \(\tilde{M}\). In view of Theorem \([1.2]\) this metric will also be a ALF\(_1\) metric, since the ALF\(_k\)-type is preserved under our construction.

This construction could be generalized, in order to provide higher-dimensional examples of ALF\(_k\)-manifolds. Simply start with lower dimensional Ricci-flat pieces with interesting asymptotic behavior (or even product of tori with copies of \((\mathbb{C}^2, g_T)\)), quotient them out by a “mixing action”, and consider a crepant resolution of the quotient. The obvious obstruction to this construction is to guarantee the existence of a crepant resolution to the quotient manifold described above.

### 5.2 4-dimensional Ricci-flat ALG manifolds

In the literature, ALG instantons correspond to 4-dimensional hyperkähler manifolds which are metrically asymptotic to the flat product \(\mathbb{R}^2 \times T^2\).

We want to provide a simple construction of Ricci-flat ALG manifolds.

In this case, we shall start with the standard flat metric on \(\mathbb{C} \times T^2\), or even more interestingly, consider the complement of a singular fiber of a generic elliptic fibration, endowed with the Ricci-flat metric described by Gross and Wilson in \([GW]\).

In analogy with the example described in the previous subsection, write \(\Gamma \subset U(1)\) for a finite subgroup of isometries of \(\mathbb{C}, z\), and let \(G\) be a finite group of rotations of \(T^2, (\theta_1, \theta_2)\). Let \(A : \mathbb{C} \to T^2\) be a \(G\)-invariant map.

The group \(\Gamma \times G\) acts on \(\mathbb{C} \times T^2\) by

\[
(z, \theta) \mapsto (\gamma(z), T_{A(z)}^{-1} \circ g \circ T_{A(z)}(w)),
\]

where \(\gamma \in \Gamma\), \(g \in G\), and \(T_{A(z)}\) denotes the translation on \(\mathbb{C}^{n-2}\) by the element \(A(z)\).

Consider the Ricci-flat quotient of \(\mathbb{C} \times T^2\) by this action. By Theorems \([1.1]\) and \([1.2]\), a crepant resolution of this quotient will admit a ALF\(_2\) (or ALG) Ricci-flat metric.

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