Non-direct limits of simple dimension groups with finitely many pure traces

Abstract There exist simple dimension groups which cannot be expressed as a direct limit of simple, or even approximately divisible dimension groups, each with finitely many pure traces, and we can specify its infinite-dimensional Choquet simplex of traces; a more drastic property is noted. On the other hand, a very easy argument shows that if $G$ is a $p$-divisible simple dimension group (for some integer $p > 1$), then it can be expressed as such a direct limit. We also enlarge the class of initial objects for AF (and slightly more general) C*-algebras.

David Handelman

Thinking about properties of traces on dimension groups (see for example, [BeH]), especially simple dimension groups, I realized that it would have been nice to be able to reduce to simple dimension groups with finite pure trace space, or better, to simple dimension groups of finite rank (these automatically have finitely many pure traces), or better still, to simple dimension groups finitely generated as abelian groups.

This suggests three conjectures:

CONJECTURES Every noncyclic simple dimension group is a direct limit (that is, over a directed set, and with positive maps) of simple dimension groups

(a) that are free of finite rank;

(b) that have finite rank;

(c) that have only finitely many pure traces.

Obviously, the truth of conjecture (a) would imply that of (b), which would in turn imply that of (c). In fact, it turns out that (c) is false (so that (a) and (b) are too), via a reasonably well-known example, and not only that, it is false in an extreme way: there is a noncyclic simple dimension group $R$ (in fact, an ordered ring) for which there are no nonzero positive group homomorphisms $\alpha : G \to R$ for any approximately divisible dimension group $G$ with only finitely many pure traces. (All noncyclic simple dimension groups are approximately divisible, so this is a drastic way of showing $R$ cannot be a direct limit of those with only finitely many pure traces.)

This contrasts with the corresponding (dual) question for Choquet simplices: a fundamental result of Lazar & Lindenstrass [LL] is that every metrizable Choquet simplex is an inverse limit (over the positive integers) of finite-dimensional simplices. As pointed out in [G], a quick proof of this is derivable using dimension groups, from the main result of [EHS].

However, there is an easy result that yields many examples that are such direct limits, even for case (b). Case (a) is somewhat problematic, but there is an example later, $\mathbb{Z}[\sqrt{2}][x]$ equipped the strict ordering as functions on $[1/3, 2/3]$, which is a direct limit for case (a) (the pure trace space is a continuum, so the result is not entirely trivial).

All groups (and partially ordered groups) appearing here are torsion-free abelian; free means free as an abelian group. Recall that the rank\(^2\) of a (torsion-free) abelian group $J$ is the dimension over $\mathbb{Q}$ of the rational vector space $J \otimes \mathbb{Q}$. All partially ordered groups appearing here are unperforated.

\(^1\)Supported in part by a Discovery grant from NSERC.

\(^2\)Some authors have very unfortunately defined the rank of a dimension group to be the width of the minimal Bratteli diagram realizing it. This is different from the rank of the underlying group, and should be called the minimal width.
The first section gives fairly easy examples of simple dimension groups that can be expressed as a direct limit of simple dimension groups with finitely many pure traces; this leads to a definition of pro-finite dimensional (for a somewhat larger collection of dimension groups with order unit) and the class consisting of them is denoted $\mathcal{H}$. The second section is devoted to construction of a single example of what we call (strongly) anti-finite dimensional (Anti-FD), a simple dimension group $R$ such that for all $G \in \mathcal{H}$, the only positive group homomorphism $\phi : G \to R$ is zero ($R$ has an even stronger property than this).

The third section shows that Anti-FD simple dimension groups can be constructed to have any Choquet simplex as their trace space (and if the Choquet simplex is metrizable, the dimension group can be chosen to be countable as well). Finally, the fourth section provides a characterization of Anti-FD; it boils down to triviality of the infinitesimal subgroup and an elementary topological constraint, that every finite rank subgroup be discrete with respect to the norm obtained from the affine representation.

Section 6 deal with examples arising from actions of tori on UHF algebras. Sections 7 and 8 show that a small class of the latter are initial objects in the appropriate category, enlarging on work of Elliott and Rørdam [ER].

1 Pro-finite dimensional dimension groups

The following formalizes a well-known construction. Let $G$ be a partially ordered abelian group (not necessarily a dimension group) having an order unit $u$. We define the simplification of $G$, $G^s$, to be the same underlying group $G$, but with positive cone consisting of the set of order units of $G$ (denoted $G^{++}$) together with 0. This puts a generally coarser ordering on the group, although the identity map $G^s \to G$ induces a natural affine homeomorphism $S(G,u) \to S(G^s,u)$ (since $G^+ + G^{++} = G^{++}$, and the order units of $G$ and of $G^s$ are identical). The simplification of an ordered group is simple, and obviously, if $G$ is already simple, it equals its simplification. Although not formally named, this process occurs in many (well, some) papers on dimension groups.

In general, if $G$ is a dimension group, $G^s$ need not be: for any $n > 1$, take $G = \mathbb{Z}^n$ with the usual ordering, so that $G^s$ is $\mathbb{Z}^n$ with the strict ordering, that is, its positive cone consists of 0 together with the set of $n$-tuples, $(k(1), \ldots, k(n))$ such that $k(i) > 0$ for all $i$. This admits discrete traces, so cannot be a simple dimension group $[G]$.

A dimension group $G$ with order unit is approximately divisible if for all pure traces $\tau$, $\tau(G)$ is a dense subgroup of $\mathbb{R}$. This is compatible with other definitions of approximately divisible because of the following result.

THEOREM 1.1 [GH] A dimension group $G$ with order unit is approximately divisible iff its image in $\text{Aff} S(G,u)$ is norm dense for one (hence for all) order unit(s) $u$. All noncyclic simple dimension groups are approximately divisible.

In particular, if $A$ is a unital AF C*-algebra, then $K_0(A)$ is approximately divisible if and only if $A$ has no finite-dimensional representations. This combined with the fact that a dense subgroup of $\text{Aff} K$, equipped with the strict ordering, is a dimension group iff $K$ is a (Choquet) simplex, yields the following.

COROLLARY 1.2 Let $(G, u)$ be a partially ordered abelian group with more than one pure trace. Then its simplification is a dimension group if and only if (a) $S(G, u)$ is a Choquet simplex and (b) the image of $G$ is norm dense in $\text{Aff} S(G, u)$. In particular, if $G$ is an approximately divisible dimension group, then $G^s$ is a dimension group.

Now suppose the partially ordered group $G$ with order unit $u$ can be expressed as a direct limit of unperforated partially ordered groups, in particular, $G = \lim \phi_n : G_n \to G_{n+1}$ (the following remarks also apply when uncountable directed sets are used, however, not only does the notation
become tedious, but uncountable direct limits are not very important in this context), where \( \phi_n \) are order preserving (that is, positive group homomorphisms). Suppose each \( G_n \) has an order unit \( u_n \) such that \( \phi_n(u_n) \) is an order unit of \( G_{n+1} \).

Then \( \phi_n(G_n^{++}) \subseteq G_{n+1}^{++} \). To see this, note that \( x \in G_n^{++} \) iff there exists an integer \( k \) such that \( u_n \leq kx \); hence \( k\phi_n(x) \geq \phi_n(u_n) \), so that \( k\phi_n(x) \) is an order unit in \( G_{n+1} \), and by unperforation, \( \phi_n(x) \) is thus an order unit. Hence, the \( \phi_n \) are positive homomorphisms \( \phi_n : G_n^s \to G_{n+1}^s \) between their simplifications.

The identity mapping sends \( \lim \phi_n : G_n^s \to G_{n+1}^s \) to \( G = \lim \phi_n : G_n \to G_{n+1} \), and is obviously order preserving (since the positive cones in the simplified groups are contained in the positive cones of the originals). If we now assume that \( G \) itself is simple, then this map is an order-isomorphism: if \( g \) is in \( G^+ \setminus \{0\} \), then \( g \in G^{++} \), so there exists a positive integer \( k \) such that \( kg \geq u \); hence there exists \( n \) such that \( g \) is represented by \([h,n]\) and \( u \) is represented by \([v,n]\) with \( v, h \in G_n^+ \) and \( v \in G_n^{++} \), and \( kh \geq v \) in \( G_n \). Hence \( h \) is an order unit of \( G_n \) and \([h,n]\) represents \( g \). Thus \( h \) is positive as an element of \( G_n^s \), so everything that is positive in \( G \) is represented as the image of a positive element in \( \lim \phi_n : G_n^s \to G_{n+1}^s \). We have thus proved,

**Lemma 1.3** Suppose that \( G = \lim \phi_n : G_n \to G_{n+1} \) is a simple partially ordered group, each \( G_n \) is unperforated, and each \( G_n \) contains an order unit \( u_n \) such that \( \phi(u_n) \) is an order unit in \( G_{n+1}^s \). Then \( G \) is also the ordered direct limit of simple unperforated groups, \( \lim \phi_n : G_n^s \to G_{n+1}^s \).

If we begin with a not necessarily simple ordered group \( G = \lim \phi_n : G_n \to G_{n+1} \) but with the same hypotheses on \( G_n \) and \( \phi_n \), then we quickly see that \( G^s = \lim \phi_n : G_n^s \to G_{n+1}^s \) by essentially the same argument.

If \( G \) is a dimension group, we can write \( G = \lim A_n : \mathbb{Z}^{k(n)} \to \mathbb{Z}^{k(n+1)} \). To satisfy the order unit condition, we telescope and delete rows if necessary so that the matrices \( A_n \) have no zero rows. Then \( G^s = \lim A_n : (\mathbb{Z}^{k(n)})^s \to (\mathbb{Z}^{k(n+1)})^s \). This is not very illuminating, since \( (\mathbb{Z}^{k(n)})^s \) is not a dimension group (unless \( k(n) = 1 \)). However, if \( G \) is a \( \mathbb{Z}[1/p] \)-module for some integer \( p > 1 \) (that is, \( G \) is \( p \)-divisible), then \( G \cong G \otimes \mathbb{Z}[1/p] \) (ordered tensor product over \( \mathbb{Z} \)), and thus we can write \( G \cong \lim A_n : \mathbb{Z}[1/p]^{k(n)} \to \mathbb{Z}[1/p]^{k(n+1)} \). Now \( \mathbb{Z}[1/p]^k \) is dense in \( \mathbb{R}^k \), and so its simplification is a dimension group, obviously simple.

A dimension group \( G \) is **pro-finite dimensional** (pro-fd) if it can be realized as a direct limit (over a directed set) of positive group homomorphisms \( \phi_{\alpha,\beta} : G_\alpha \to G_\beta \) where each \( G_\alpha \) is an approximately divisible dimension group with order unit, \( u_\alpha \), such that \( \phi_{\alpha,\beta}(u_\alpha) \) is an order unit in \( G_\beta \), and each \( \partial_{\beta\alpha}(G_\alpha, u_\alpha) \) is finite (finite-dimensional trace space). Necessarily, a pro-FD dimension group has an order unit. Most of the time, specifically when \( G \) is countable, the index set can be taken to be the set of positive integers, in which case the notation simplifies to \( \phi_i : G_i \to G_{i+1} \). The class of pro-FD dimension groups will be denoted \( \mathcal{H} \). Rather surprisingly, not all simple noncyclic dimension groups belong to \( \mathcal{H} \).

We have seen that if \( G \) is simple and in \( \mathcal{H} \), then \( G \) can be represented as a limit of simple dimension groups each with finite-dimensional trace space.

**Corollary 1.4** If \( G \) is a dimension group that is \( p \)-divisible for some integer \( p > 1 \), then \( G^s \) is a direct limit of simple dimension groups each with finitely many pure traces. In particular, every simple \( p \)-divisible dimension group is a direct limit of simple dimension groups, each having only finitely many pure traces, and is thus pro-fd.

A \( p \)-divisible dimension group is automatically approximately divisible. A similar construction occurs if \( G \) can be represented as a limit of the form \( R^{k(n)} \to R^{k(n+1)} \) where \( R \) is a noncyclic ordered subring of the reals (for example, \( R = \mathbb{Z}[\sqrt{2}] \). If \( G \) is simple, \( G \) is again a direct limit of simple dimension groups each with only finitely many pure traces.
COROLLARY 1.5 If $G$ is a simple dimension group that is $p$-divisible for some prime $p$, then $G$ is a limit of simple dimension groups of finite rank.

This likely extends to groups satisfying the following property. Say an element $g$ of a torsion-free abelian group is $i$-divisible (i for infinitely) if there exist infinitely many positive integers $n$ such that the equation $g = nx_n$ can be solved (with $x_n$ in $G$). Now say the abelian group $G$ is wi-divisible if every element is a sum of i-divisible elements, that is, the i-divisible elements span the group. It is plausible that $p$-divisible can be weakened to wi-divisible in the statement of the result.

As an example, let $H_n = \bigoplus_{i=1}^{k(n)} U_{i,n}$ where each $U_{i,n}$ is a noncyclic subgroup of $\mathbb{Q}$, and let $(A_n)$ be a sequence of strictly positive matrices, with $A_n$ being of size $k(n+1) \times k(n)$ and having integer entries. We can take $G = \lim A_n : H_n \rightarrow H_{n+1}$. The outcome is a simple dimension group that is wi-divisible; uninterestingly, because the $A_i$ are strictly positive, there is a noncyclic subgroup $U$ of $\mathbb{Q}$ for which we can rewrite the terms as $U^{k(n)}$, so that the group is i-divisible. If we impose the strict, rather than the ordinary direct sum ordering on each $H_n$, then each $H_n$ is now a simple dimension group, and the $A_n$ are still positive homomorphisms between them (this is why we insisted the $A_n$ have no zero rows; if there were any zero rows, some positive elements would be sent to a non-positive elements), and as in the argument above, simplicity of $G$ entails that the limit is just $G$ with its original ordering. Hence $G$ is a limit of simple dimension groups of finite rank, specifically the $H_n$ with the strict ordering.

On the other hand, $G = \mathbb{Z}^{[1/2]} \oplus \mathbb{Z}^{[1/3]}$ with the strict ordering (as a subgroup of $\mathbb{R}^2$) is wi-divisible without being i-divisible.

2 Drastic example

At the opposite extreme to pro-fd, are two properties of dimension groups defined below. An approximately divisible dimension group $(G, u)$ with order unit is anti-finite dimensional (anti-fd) if there are no nonzero positive group homomorphisms $\phi : H \rightarrow G$ for any approximately divisible dimension group $H$ having finite-dimensional pure trace space (equivalently, for any $H \in \mathcal{H}$). This is a much stronger property than merely not being a limit of simple dimension groups with finitely many pure traces. That simple anti-fd dimension groups even exist is somewhat surprising; however, it turns out that for every Choquet simplex $K$, there exists one with trace space $K$ (and if $K$ is metrizable, we can find a countable one). Actually we show there exist lots of approximately divisible dimension groups with an even stronger property.

Let $(G, u)$ be a partially order abelian group with order unit, and form the representation $G \rightarrow \text{Aff}.S(G, u)$, given by $g \mapsto \tilde{g}$, where $\tilde{g}(\tau) = \tau(g)$ as usual. This induces a pseudo-norm on $G$ arising from the supremum norm on $\text{Aff}.S(G, u)$; it can also be characterized purely in terms of the ordering on $G$. If the ordered group is unperforated, then the kernel of the representation consists of the infinitesimals, Inf$G$. We say a group homomorphism $\phi : G \rightarrow H$ between partially ordered groups with order unit (but not necessarily sending order units to order units) is continuous if it is continuous with respect to the pseudo-norms on $G$ and $H$; equivalently, $\phi(\text{Inf}G) \subset \text{Inf}H$ and the induced map $\overline{\phi} : G/\text{Inf}G \rightarrow H/\text{Inf}H$ is continuous with respect to to the norms on the groups induced by their affine representations.

When $G$ has finite-dimensional trace space, every continuous group homomorphism $G \rightarrow H$ is bounded, hence will extend to a norm-continuous map between their completions. A continuous (or bounded, see [G, section 7] for a discussion of bounded group homomorphisms on dimension groups) group homomorphism from a dense subgroup of $\mathbb{R}^n$ to any Banach space automatically extends to a bounded linear function from $\mathbb{R}^n$ to the Banach space (since continuous maps need not send Cauchy sequences to Cauchy sequences, this is not completely obvious; however, if $x_n \rightarrow x/k$ where $x_n$ and $x$ are in the dense subgroup (and $k$ is an integer), then $kx_n \rightarrow x$, so continuity implies $\alpha(kx_n) \rightarrow \alpha(x)$, and this implies extension to a rational subspace, to which the standard method of...
showing continuity implies boundedness for linear transformations applies.) Every positive group homomorphism is automatically continuous, so the following even more drastic property implies anti-finite dimensionality.

We say a dimension group with order unit $R$ is strongly anti-finite dimensional (Anti-FD; we use three upper case letters to distinguish it from anti-fd and Anti-fd, the latter being anti-fd at the beginning of a sentence) if for every $H$ in $\mathcal{H}$, every continuous group homomorphism $\phi : H \to R$ is zero. This condition forces $\text{Inf} R = \{0\}$ for trivial reasons. A few examples of simple anti-fd dimension groups with no infinitesimals that are not Anti-FD are given in section 4.

While characterization of anti-fd dimension groups is a little complicated, the characterization of Anti-FD dimension groups (within the class of approximately divisible dimension groups with order unit) is easy: $\text{Inf} R = \{0\}$ and every finite rank subgroup is discrete (in the norm topology), Corollary 4.5. In this section we show sufficiency, then give more examples (having as trace space, all Choquet simplices) in the next, and finally show necessity in the last section. The most interesting feature is that Anti-FD dimension groups exist. This means that $\mathcal{H}$ is far from exhaustive, even for simple dimension groups.

Now we prepare to provide an example of a simple dimension group that cannot be obtained as a direct limit of simple dimension groups, each with finitely many pure traces, in fact is Anti-FD, and in particular, is a counter-example to (c).

For a pseudo-normed abelian group $H$, let $\psi$ denote the map to its completion; that is, first factor out the elements of norm zero, and complete the resulting normed abelian group. Of course, the completion need not be a vector space (e.g., if the norm is discrete).

**Proposition 2.1** Suppose that $H$ is a pseudo-normed group such that the completion of $\psi(H)$ a finite dimensional real vector space, and $R$ is a normed abelian group with the following properties:

- (0) the completion of $R$ is a Banach space
- (a) every countable subgroup of $R$ is free as an abelian group
- (b) every finitely generated subgroup of $R$ is discrete.

Then every continuous group homomorphism $\phi : H \to R$ is zero.

**Proof.** We immediately reduce to the case that $H$ is a dense subgroup of a finite dimensional real vector space (so we can dispense with $\psi$). Now consider the subgroup $\phi(H)$ of $R$. If $\phi(H)$ is nonzero and has finite rank (as an abelian group), it is countable, as a subgroup of a free group, it is itself free; a free group of finite rank is finitely generated. Hence $\phi(H)$ is discrete. There thus exists a continuous linear functional on the completion of $R$, $\rho$, such that $\rho(\phi(H)) = \mathbb{Z}$.

Then $\nu := \rho \circ \phi : H \to \mathbb{R}$ is a continuous group homomorphism. Since the completion of $H$ actually pseudo-completion, but $\psi$ kills the subgroup consisting of those elements of $H$ with zero norm, so we might as well assume $H$ is dense in $\mathbb{R}^n$ is a finite dimensional vector space, $\nu$ extends to a bounded linear functional on the completion. But its restriction to (the image of) $H$ has discrete range, which is impossible since $H$ is dense in a vector space.

We are thus reduced to the case that $\phi(H)$ has infinite rank. Suppose the (real) vector space dimension of the completion of $H$ is $n$. Every abelian group of infinite rank contains a subgroup of every finite rank. Hence there exists a rank $n + 1$ subgroup $J$ of $\phi(H)$. As a subgroup of $R$, it is free, and thus free on $n + 1$ generators, call them $\phi(h_i)$. By (c), the group they generate is discrete, and thus there exist continuous linear functionals $\alpha_i$ on $B$ (and thus continuous real-valued group homomorphisms on $R$) such that $\alpha_i(\phi(h_j)) = \delta_{ij}$ (Kronecker delta).

Then $\{v_i := \alpha_i \circ \phi\}$ is a collection of continuous additive real-valued group homomorphisms on $H$, each of which extends to a bounded linear functional on the completion. However, $v_i(h_j) = \delta_{ij}$ entails that the set of bounded linear functionals is linearly independent (over the reals of course),
so that the dimension of the dual space of the completion, and thus of the completion itself, exceeds $n$, a contradiction.

Properties (a) and (b) together can be restated as a single property (of normed abelian groups), (c) every finite rank subgroup of $R$ is discrete.

To see the equivalence, note that a finite rank discrete group is free [actually, a result due to Steprans [S], generalizing a result of Lawrence [L], asserts that every discrete normed abelian group is free]; by [Gr, Theorem 137, p 101], a countable group for which every finite rank subgroup is free, is itself free. And of course, if $R$ is a free abelian group, then every subgroup is free.

It is convenient, however, to separate (a) and (b), in view of the closely related examples we obtain.

If $\{x_i\}_{i=1}^n$ is a finite and (real) linearly independent subset of a Banach space, then the abelian group the set generates, $\sum x_i \mathbf{Z} = \oplus x_i \mathbf{Z}$ not only is free (as an abelian group) on $\{x_i\}$ but is discrete: the map $x_i \mapsto e_i$ (where $e_i$ run over the standard basis of $\mathbf{R}^n$) extends to an automatically continuous and real linear isomorphism (with continuous inverse) $\sum x_i \mathbf{R} \to \mathbf{R}^n$ that sends $\sum x_i \mathbf{Z}$ to the usual copy of $\mathbf{Z}^n \subset \mathbf{R}^n$.

**EXAMPLE 2.2** A simple Anti-FD dimension group.

Set $R = \mathbf{Z}[x]$, the ring of polynomials with integer coefficients. Let $I$ be a closed real interval (of nonzero length) that contains no integers. Then a 1925 result due to Chlodovsky [C] (as cited in [F]), says that $R$ is dense in $C(I, \mathbf{R})$ with respect to the supremum norm on $I$. In particular, if we impose the strict ordering on $R$ (that is, $f \in \mathbf{R}^+ \setminus 0$ iff $f$ is strictly positive as a function on $I$), then $R$ is a simple dimension group. We note that $R$ is an ordered ring with 1 as order unit, and its pure trace space consists of the point evaluations at members of $I$, so its pure trace space is an interval.

Moreover, $R = \sum x^i \mathbf{Z}$ expresses $R$ as free on $\{x^i\}$. Finally, every finitely generated subgroup of $R$ is contained in $L = \sum_{i=0}^n x^i \mathbf{Z}$ for some $n$, so to prove the subgroup is discrete, it suffices to show $L$ is discrete. However, $\{x^i\}$ is real linearly independent as a subset of $C(I, \mathbf{R})$ (a polynomial with real coefficients that vanishes on $I$ is zero), and thus $L$ is discrete. So $R$ is a simple dimension group satisfying conditions (0, a, b) of Proposition 2.1, and is thus Anti-FD.

The affine representation of $R$ (using 1 as an order unit) is simply the inclusion $R \subset C(I, \mathbf{R})$ and its pure trace space is naturally homeomorphic to $I$.

If $H$ is an approximately divisible dimension group with order unit, then the image of $H$ is dense in its affine representation. Hence if $H$ is approximately divisible and has only finitely many pure traces, then the natural pseudo-norm from the affine representation has Euclidean space as its completion. The kernel of the affine representation consists of infinitesimals, which is exactly the subset to be factored out in constructing the completion. Any positive homomorphism between partially ordered abelian groups with order unit is automatically continuous with respect to the pseudo-norm topologies on each.

An abelian group homomorphism $\phi : H \to J$ where $H$ is a pseudo-normed abelian group and $J$ is a normed abelian group is weakly continuous if for every continuous group homomorphism $\rho : J \to \mathbf{R}$, the composition $\rho \circ \phi : H \to \mathbf{R}$ is continuous. The proof of Proposition 2.1 only requires that $\phi$ be weakly continuous. However, when the pseudo-completion of $H$ is a finite-dimensional real vector space, weak continuity implies continuity anyway. In the following result, dealing with maps from members of $\mathcal{H}$, we have two ways of proceeding: directly, using weak continuity of the restriction to the image of something with finite dimensional completion, or using that the restriction of a weakly continuous group homomorphism to one of the constituents in the direct limit is automatically continuous.

With $R$ the example of 2.2 (or any other anti-fd simple dimension group), we have the following.
Of course, \( R \) itself has the metric topology from the sup norm on the interval, and this coincides with the metric (not just pseudo-metric) obtained from its affine representation (which would be as a dense subring of \( \text{Aff}(\mathcal{M}^+(I)) = C(I, \mathbb{R}), \mathcal{M}^+ \) being the collection of probability measures on \( I \)).

**Lemma 2.3** Let \( H \in \mathcal{H} \). If \( \phi : H \to R \) is a weakly continuous group homomorphism, then \( \phi = 0 \). This applies automatically if \( \phi \) is an order-preserving group homomorphism.

This contrasts with \( \mathbb{Z}[^{1/2}][\sqrt{]} \) and \( \mathbb{Z}[^{\sqrt{2}}][\sqrt{]} \), each equipped with the strict ordering from restriction to the interval; \( \mathbb{Z}[^{1/2}][\sqrt{]} \) is 2-divisible, so is even a direct limit of simple dimension groups of finite rank. And \( \mathbb{Z}[^{\sqrt{2}}][\sqrt{]} \) is order isomorphic to the ordered tensor product \( R \otimes \mathbb{Z}[^{\sqrt{2}}] \) (where \( \mathbb{Z}[^{\sqrt{2}}] \) is given the total ordering inherited from \( \mathbb{R} \): we can write \( R = \lim A_n : \mathbb{Z}[^{k(n)}] \to \mathbb{Z}[^{k(n+1)}] \) where the \( A_n \) are strictly positive matrices (since \( R \) is simple, this can be arranged by telescoping), and since \( R \) and thus \( R \otimes \mathbb{Z}[^{\sqrt{2}}] \) is simple, the latter is the limit \( A_n \otimes 1 : (\mathbb{Z}[^{\sqrt{2}}][k(n)])^s \to (\mathbb{Z}[^{\sqrt{2}}][k(n+1)])^s \), each direct sum equipped with the strict ordering. Every \( (\mathbb{Z}[^{\sqrt{2}}][k(n)])^s \) is a simple dimension group (from being dense in \( \mathbb{R}[^{k(n)}] \)). In this latter example, the underlying group is free, and it is a direct limit of simple dimension groups that are finitely generated and free. In particular, both \( \mathbb{Z}[^{1/2}][\sqrt{]} \) and \( \mathbb{Z}[^{\sqrt{2}}][\sqrt{]} \) equipped with the strict ordering from some closed interval in \( (0,1) \) are simple dimension groups in \( \mathcal{H} \), but their intersection, \( R = \mathbb{Z}[\sqrt{]}] \) with the strict ordering on the same interval, is a simple dimension group which has no nonzero incoming weakly continuous group homomorphisms from any element of \( \mathcal{H} \).

The ring \( \mathbb{Z}[^{\sqrt{2}}][\sqrt{]} \) with the strict ordering satisfies (a) of Proposition 2.1, but not (b); on the other hand, \( \mathbb{Z}[^{1/2}][\sqrt{]} \) satisfies (b) but not (a). Both satisfy (0).

We actually have uncountably many choices for \( I \) that give rise to nonisomorphic simple dimension groups: for example, the unordered pair consisting of left and right endpoints of the interval, \( I = [a, b] \), is topologically determined from the pure trace space (since they are the only two points with no neighbourhoods homeomorphic to an open interval), and the corresponding value groups, \( \{\mathbb{Z}[a], \mathbb{Z}[b]\} \), viewed as subgroups of the reals, is an order-theoretic invariant of \( \mathbb{Z}[\sqrt{]}] \) with the strict ordering coming from \( I \) (and there are uncountably many different order isomorphism classes of \( \mathbb{Z}[a] \)). In the next section, we will show that arbitrary Choquet simplices can be realized as the trace space of simple Anti-FD dimension groups.

The argument in Proposition 2.1 was divided in two parts; first, dealing with subgroups of finite rank, then with those of infinite rank. Had the following notion (not good enough to be a conjecture) been true, the second argument would have been unnecessary.

*Notion* If \( G \) is a dense subgroup of \( \mathbb{R}^n \), there exists a finite rank subgroup of \( G \) that is also dense. This is plausible but false; this phenomenon accounts for problems in characterizing anti-fd dimension groups, as discussed in section 4.

**Example 2.4** A countable dense subgroup of \( \mathbb{R}^2 \) which contains no dense subgroups of finite rank. This yields a simple dimension group with exactly two pure traces to which there are no positive homomorphisms from any noncyclic simple dimension group of finite rank.

Let \( \{\alpha_0 = 1, \alpha_1, \alpha_2, \ldots\} \) be an infinite set of real numbers that is linearly independent over the rationals. For \( i = 0, 1, 2, \ldots \) set \( x_i = (\alpha_i, 2^{-i}) \in \mathbb{R}^2 \). Let \( G = \sum x_i \mathbb{Z} \subset \mathbb{R}^2 \). By examining the first coordinates, we see that \( G \) is a free abelian group on \( \{x_i\} \), obviously of infinite rank.

We observe that \( 2x_1 - x_0 = (2\alpha_1 - 1, 0) \) and \( 4x_2 - x_0 = (4\alpha_2 - 1, 0) \). Since \( \{2\alpha_1 - 1, 4\alpha_2 - 1\} \) is linearly independent over the rationals, we see that \( \mathbb{R} \oplus 0 \) is contained in the closure of \( G \). Thus for all \( i \), \((0, 2^{-i}) \) is contained in the closure of \( G \), so that \( 0 \oplus \mathbb{R} \) is also contained in the closure, and thus, as the closure is a group, \( \mathbb{R}^2 \) is the closure. So \( G \) is dense in \( \mathbb{R}^2 \).
Now suppose $H$ is a subgroup of finite rank. Since $G$ is free, $H$ must be free, and as its rank is finite, it is finitely generated as an abelian group. Hence there exists $n$ such that $H \subseteq \sum_{i=1}^{n} x_i \mathbb{Z}$. The restriction of the second coordinate functional has values in $2^{-n} \mathbb{Z}$, which is discrete. In particular, $H$ has a trace with discrete range, so cannot be the image of a noncyclic simple dimension group (even dropping the requirement that the image be dense in $G$).

Imposing the strict ordering on $G$; it is the desired simple dimension group; all noncyclic simple dimension groups have dense range in their affine representation.

As a special case, let $\alpha$ be a real transcendental number, and set $\alpha_i = \alpha^i$. The resulting $G$ is simply the ring $\mathbb{Z}[x]$ equipped with the ordering derived from the two ring homomorphisms $x \mapsto \alpha$ and $x \mapsto \frac{1}{2}$; the image is dense in $\mathbb{R}^2$, making $\mathbb{Z}[x]$ into a simple dimension group (and an ordered ring), with these two homomorphisms as the only pure traces (up to normalization at 1). Since $\alpha$ is transcendental, the map $x \mapsto \alpha$ has zero kernel. This example is simply $\mathbb{Z}[x]$ with a much larger positive cone than that of $R$ in Example 2.2.

\section*{3 Other Choquet simplices}

In the construction of the Anti-FD simple dimension group $R = \mathbb{Z}[x]$ with the strict ordering from restriction to $I$, the set $I$ is a closed interval; everything works just as well if we let $I$ be any infinite compact subset of the open interval (since a polynomial is determined by its values on any infinite set). As we can embed a Cantor set in the open interval (for example, truncate the usual Cantor set at both ends), we can even arrange that the pure trace space of $\mathbb{Z}[x]$ equipped with the strict ordering be a Cantor set, hence totally disconnected; the result will still be a simple dimension group satisfying (0), (a), and (b) of Proposition 2.1. However, we can go much farther.

We show that for every (metrizable) Choquet simplex $K$, there exists a (countable) simple dimension group with order unit $(G, u)$ whose trace space is $K$ that satisfies conditions (a) and (b) of the proposition, and therefore has the strong anti-finite dimensional property.

The following is a minor variation on a well-known result.

**Lemma 3.1** Let $B$ be a separable infinite-dimensional real Banach space, and let $Q$ be a countable linearly independent subset. Then there exists a countable dense set $P$ such that $P \cap Q = \emptyset$ and $P \cup Q$ is linearly independent.

**Proof.** Since $B$ is separable, there exists a countable dense set $\{p_i\}_{i=1}^{\infty}$. Since $B$ is complete and not finite-dimensional, its dimension (as a real vector space) is uncountable. Select a basis for $B$ (as a real vector space), $A = \{e_\alpha\}$. Every element of $B$ is uniquely representable in the form $b = \sum \lambda_\alpha e_\alpha$ with $\lambda_\alpha \in \mathbb{R}$ where all but finitely many $\lambda_\alpha$ are zero; define the support of $b$, given by $\text{supp} b = \{e_\alpha \in A \mid \lambda_\alpha \neq 0\}$.

Let $J_0$ be the union of the supports of all the elements of $Q$; this is countable. Let $J_1$ be the union of the supports of all the elements of $\{p_i\}$. Since $J := J_1 \cup J_0$ is countable, we can find in $A \setminus J$, a countable family of pairwise disjoint subsets $V_i$ each of which is itself countably infinite. Index the members of $V_i$ as $V_i = \{v(i, 1), v(i, 2), \ldots \}$, where $v(i, n) \in A \setminus J$. For each pair $(i, n)$, set $P_{i,n} = p_i + 2^{-n}v(i, n)/\|v(i, n)\|$. Then we note that $\lim_{n \to \infty} P_{i,n} = p_i$, so that the countable set $P := \{P_{i,n}\}_{i \times N}$ is dense.

Since the support of every element of $Q$ lies in $J_0$, $P \cap Q = \emptyset$; since for any fixed pair $(i', n')$, the element $v(i', n')$ appears exactly once in the supports of $P_{i,n}$ as $(i, n)$ varies, it easily follows that $P \cup Q$ is linearly independent (using linear independence of $Q$).

We can ask whether (*) for every real infinite-dimensional Banach space, there exists a dense linearly independent subset.

This has been solved affirmatively in [BDHMP; Proposition 3.2] (I am indebted to Ilijas Farrah...
for explaining their argument to me).

The argument in the separable case used the fact that the dimension (as a real vector space) exceeded the cardinality of one of the dense subsets. However, this dimension property fails (when CH holds) for all non-separable Banach spaces whose dimension is $\aleph_1$ (and probably fails, in the presence of GCH, for all non-separable Banach spaces). The argument of the separable case can be adapted if $B$ has a dense subspace of codimension at least as large as its dimension.

A weaker hypothesis that would still be enough for our purposes is the following:

(**) for any real infinite dimensional Banach space, there exists a linearly independent set $\{e_\alpha\}$ whose span (that is, $\sum e_\alpha Z$) is dense.

Whenever this occurs, $\sum e_\alpha Z$ is free (on $\{e_\alpha\}$) and has the property that every finitely generated subgroup is discrete.

COROLLARY 3.2 Let $K$ be a metrizable Choquet simplex. Then there exists a countable dense subgroup $G$ of $\text{Aff} K$ that is free as an abelian group and for which every finitely generated subgroup is discrete. Equipped with the strict ordering, this $G$ is a countable simple dimension group with trace space $K$, with the strong anti-finite dimensional property.

Proof. The Banach space $B = \text{Aff} K$ is separable (since $K$ is metrizable), and thus there exists a countable, dense, real linearly independent subset of $B$, $\{e_n\}_{n=1}^\infty$; we specify $Q = \{1\}$ (consisting of the constant function), so that if we take as order unit, $u = 1$, the affine representation agrees with the inclusion of $G$ in $\text{Aff} K$. Then $G = \sum e_n Z$ is dense; linear independence over $R$ implies that $G$ is free on $\{e_n\}$, and every finitely generated subgroup of $G$ is contained in $\sum_{i=1}^n e_i Z$, which is discrete, and thus the subgroup is itself discrete. The rest is immediate.

These examples have a property that our original $R = \mathbb{Z}[x]$ does not: the group is free on a set which is itself dense.

By the affirmative solution to (*) in [BDHMP], we also obtain immediately the following.

PROPOSITION 3.3 Let $K$ be a Choquet simplex. Then there exists a dense subgroup $G$ of $\text{Aff} K$ that is free as an abelian group and for which every finitely generated subgroup is discrete. Equipped with the strict ordering, this $G$ is a simple dimension group with trace space $K$, with the strong anti-finite dimensional property.

4 Characterization of Anti-FD
The distinction between anti-finite dimensionality and its strong form is not large.

LEMMA 4.1 Let $(R, v)$ be an anti-finite dimensional approximately divisible dimension group with order unit such that $\text{Inf} R = \{0\}$. Let $(G, u)$ be an approximately divisible dimension group with order unit such that $\partial_v S(G, u)$ is finite. Then there exists no continuous group homomorphism $\phi : G \to R$ such that $\phi(G)$ contains an order unit of $R$.

Proof. Suppose $\phi(G)$ contains an order unit. We may replace $G$ by its simplification $G^s$; the topology is unchanged as is the set of infinitesimals, but now $G^s$ is a simple dimension group, and thus so is the quotient $G^s/\text{Inf}(G^s)$. Since $\phi(\text{Inf} G) \subset \text{Inf} R$ (from continuity of $\phi$ with respect to the pseudo-norms), $\phi$ induces a continuous map from $G^s/\text{Inf} G^s$ to $R$.

Hence we are reduced to the case that $G$ is a simple dimension group with $\text{Inf} G = 0$; the latter implies we may regard $G$ as a dense subgroup of $\mathbb{R}^n = \text{Aff} S(G, u)$. A continuous group homomorphism from $G$ is automatically bounded, hence sends Cauchy sequences to Cauchy sequences, so that $\phi$ extends to a map from the completion of $G$, $\mathbb{R}^n$, to the completion of $R$, which is $\text{Aff} S(R, v)$; call it $\Phi : \mathbb{R}^n \to \text{Aff} S(R, v)$.
Now $C := \Phi^{-1}(\text{Aff}(S, v)^{++})$ is open; by hypothesis, it is nonempty. Obviously, $C \cap -C = \emptyset$, $C - C = \mathbb{R}^n$ (since $C$, and therefore $C - C$, contains an open ball), and $C + C \subseteq C$. Thus $C' = C \cup \{0\}$ is a proper convex cone in $\mathbb{R}^n$ containing an open ball. Hence we may find a basis for $\mathbb{R}^n, \{a_1, \ldots, a_n\}$ inside $C$. Let $D$ be the convex cone spanned by this basis; then $D$ is a simplicial cone on $\mathbb{R}^n$ (that is, it is obtained from the original ordering by applying an invertible matrix to everything in sight).

Density of $G$ in $\mathbb{R}^n$ implies that if we impose the strict ordering on $\mathbb{R}^n$ given by the basis (that is, if nonzero $x = \sum \lambda_ia_i$ where $\lambda_i \in \mathbb{R}$, then $x$ is in the positive cone iff $\lambda_i > 0$ for all $i$), then $G$ becomes a simple dimension group with respect to this ordering (call it (as an ordered group) $G'$. Since the positive cone is contained in $D$, which in turn is contained in $C'$, it follows that $\phi$ is positive as a group homomorphism from $G'$ to $R$. Since the trace space of $G'$ is finite-dimensional, we reach a contradiction; $\phi$ must be zero, as $R$ is anti-finite dimensional.

Now work toward completing the characterization of strong anti-finite dimensionality.

**Lemma 4.2** Let $\{f_i\}$ be a finite linearly independent subset of a Banach space $B$. Then there exists $K > 0$ (depending only on $\{f_i\}$) such that if $e_i$ are elements of $B$ with $\|e_i - f_i\| \leq K$ for all $i$, then $\{e_i\}$ is linearly independent.

**Proof.** Form the finite-dimensional vector space $\sum f_i\mathbb{R}$; since all norms on finite dimensional vector spaces are equivalent, choosing the $l^1$-norm, there exists $K$ such that for all choices of $\lambda_i \in \mathbb{R}$, we have $\|\sum \lambda_if_i\| > K \sum |\lambda_i|$. If on the other hand, $\sum \lambda_ie_i = 0$, then

$$K \sum |\lambda_i| \geq \sum |\lambda_i||f_i - e_i| \geq \left\| \sum \lambda_i(f_i - e_i) \right\| = \left\| \sum \lambda_if_i \right\| > K \sum |\lambda_i|,$$

a contradiction. 

**Lemma 4.3** Suppose that $H$ is a finite rank subgroup of a Banach space $B$. Then we may decompose $H = K \oplus F$ where $F$ is a discrete group, $\overline{F} = \overline{K} \oplus F$ and $\overline{K}$ is the maximal real subspace of $\overline{H}$.

**Proof.** Since $HQ$ (the rational vector space spanned by $H$ inside $B$) is a finite dimensional rational vector space, $\overline{F}$ is contained in a finite dimensional subspace $V$ of $B$. Since all norms are equivalent on finite dimensional spaces, we can write $\overline{F} = (\sum x_j\mathbb{R}) \oplus (\sum f_i\mathbb{Z})$ where the complete set $\{x_j\} \cup \{f_i\}$ is linearly independent, and the maximal real subspace is the left summand. Necessarily, $\sum f_i\mathbb{Z}$ is discrete.

By density, we may find $e_i \in H$ such that $\|e_i - f_i\|$ is as small as we like; thus with $x_j = e_j$, we obtain that $\{x_j\} \cup \{e_i\}$ is linearly independent; in particular, $\sum e_i\mathbb{Z}$ is discrete. It easily follows that $\overline{F} = (\sum x_j\mathbb{R}) \oplus (\sum e_i\mathbb{Z})$.

Now consider the map $H \subset \overline{F} \to \sum e_i\mathbb{Z}$; this is onto (since each $e_i \in H$ and the target is a free abelian group), and so its kernel $K$ splits, that is, $H = K \oplus (\sum e_i\mathbb{Z})$ where $K \subseteq \sum x_j\mathbb{R}$. Now it is routine to verify that $\overline{H} = \overline{K} \oplus (\sum e_i\mathbb{Z})$, and thus $\overline{K}$ must be the maximal real subspace of $\overline{F}$.

Lawrence [L] has shown that any countable discrete normed abelian group (in particular, any countable discrete subgroup of a Banach space) is free; this was extended by Steprans [S], dropping countable. In any event, a finite rank discrete abelian group is free on a finite set, and this does not require either of these results.

**Corollary 4.4** Let $(R, v)$ be an approximately divisible dimension group with order unit.

(a) If $R$ is strongly anti-finite dimensional, then $\text{Inf}R = 0$ and all finite rank subgroups of $R$ are discrete. In particular, every countable subgroup of $R$ is free.
(b) If \( R \) is anti-finite dimensional and \( \text{Inf} R = \{0\} \), then for every finite rank subgroup \( H \) of \( R \) whose closure is a vector space, \( H \cap R^{++} = \emptyset \).

**Proof.** First we prove that if \( R \) is strongly anti-finite dimensional, then \( \text{Inf} R = 0 \). This is trivial: if \( L = \text{Inf} R \), form the simple dimension group \( H = \mathbb{Q} \oplus L \) with positive cone \( (q,l) > 0 \) if \( q > 0 \). Then \( H \) is a simple noncyclic dimension group, and the map \( (q,l) \mapsto l \in \text{Inf} R \) is automatically pseudo-norm continuous, hence must be zero. So \( L \) is zero.

Let \( H \) be a finite rank subgroup of \( R \) that is not discrete. Taking the closure in the Banach space \( \text{Aff} S(R,v) \), we have a split decomposition \( H = K \oplus F \) where \( F \) is free and discrete and the closure of \( K \) is the maximal real subspace of \( \overline{F} \); if \( H \) is not discrete, then \( K \neq 0 \). Since \( K \) is dense in a finite dimensional vector space, we can equip it with the structure of a noncyclic simple dimension group such that the affine representation of \( K \) is simply the embedding in \( \overline{K} \). Necessarily, the identity map \( K \rightarrow K \subset R \) is continuous, so must be zero, a contradiction to strong anti-finite dimensionality.

If instead, \( R \) is only anti-finite dimensional, we can form \( R^a \) and factor out the infinitesimals; then the hypotheses still apply, so we can reduce to the case that \( \text{Inf} R = 0 \). Now using \( K \) as in the preceding paragraph, the hypotheses of lemma 4.1 above yield a positive map from a simple noncyclic dimension group to \( R \), again a contradiction.

**COROLLARY 4.5** Let \((R,v)\) be an approximately divisible dimension group with order unit. Then \( R \) is Anti-FD if and only if \( \text{Inf} R = 0 \) and every finite rank subgroup of \( R \) is discrete.

**Proof.** Follows from 2.1 and 4.4.

One would like a corresponding characterization of anti-fd, along the lines of, the simple dimension group \( R \) is anti-fd if and only if every finite rank subgroup \( H \) of \( R/\text{Inf} R \) whose closure is a vector space misses \( R^{++} \). However, there is a problem arising from the phenomenon illustrated in Example 2.4, that dense subgroups of \( R^a \) need not contain any dense subgroups of finite rank. Necessary and sufficient, in case \( R \) is simple, is that (after factoring out \( \text{Inf} R \)), if \( H \) is a subgroup of \( R \) whose closure is a finite dimensional real vector space, then \( H \cap R^{++} = \{0\} \), but this is practically tautological.

Finally we can give an example of a countable simple anti-fd with no infinitesimals that is not Anti-FD. Let \( R = \mathbb{Z}[x] + (1 - 2x)\mathbb{Q} \) (a subgroup of \( \mathbb{Q}[x] \)) with the strict ordering from the interval \( [1/3, 2/3] \); this is a countable simple dimension group whose pure trace space can be identified with \( I \). The set \( \{1, 1 - 2x, x^2, x^3, \ldots \} \) is linearly independent, and \( (1 - 2x)\mathbb{Q} \) misses the positive cone. Let \( \phi : G \rightarrow R \) be a positive group homomorphism, where \( G \) has finitely many pure traces.

If \( \phi(G) \subset \mathbb{Z}[x] + n^{-1}(1 - 2x)\mathbb{Z} \subset (1/n)\mathbb{Z}[x] \) for some \( n \), then as the latter is order isomorphic to \( \mathbb{Z}[x] \) with the strict ordering and is thus Anti-FD, and we obtain \( \phi = 0 \).

Now \( \phi(G) \) contains a real basis for its closure, say \( \{p_1, p_2, \ldots, p_n\} \), where each \( p_i \) is a polynomial; this basis is contained in the finite dimensional space \( V := \sum_{j=0}^m x^j \mathbb{R} \) for some \( m \), and \( G \) is contained in this closed subspace. Hence the closure of \( \phi(G) \) is contained in \( V \).

Since \( \phi(G) \) thus consists of polynomials of degree less than or equal \( m \) and is contained in \( R \), that is, \( \phi(G) \subseteq V \cap R \), and the latter is just \( \sum_{i=0}^m x^i \mathbb{Z} + (1 - 2x)\mathbb{Q} \) which is of finite rank. Hence \( \phi(G) \) is of finite rank; the closure of \( \sum_{i=0}^m x^i \mathbb{Z} + (1 - 2x)\mathbb{Q} \) is just \( (1 - 2x)\mathbb{R} + \sum_{i=0}^m x^i \mathbb{Z} \); hence \( \phi(G) \), being dense in a vector space, must map into the real subspace, so \( \phi(G) \subseteq \phi(G) \subseteq (1 - 2x)\mathbb{R} \). But this is impossible, as \( (1 - 2x)\mathbb{R} \) contains no positive elements of \( R \).

This example satisfies condition (b) of Proposition 2.1, but not (a). If instead we take \( R = \mathbb{Z}[x] + \sqrt{2}(1 - 2x)\mathbb{Z} \), the same argument shows that the resulting ordered group is again an anti-
fd simple dimension group that is not Anti-FD, but satisfies condition (a) and not (b). And 
\[ R = \mathbb{Z}[x] + (1 - 2x)\mathbb{Q}[\sqrt{2}] \] is yet another example of a simple anti-fd but not Anti-FD dimension group, this time satisfying neither (a) nor (b).

5 Consequences
By direct translation from well-known results (too well-known to bother referring to), we have some consequences for unital AF C*-algebras and minimal actions on Cantor sets. Pro-fd simple dimension groups are \( K_0 \) of AF algebras which are direct limits of simple AF algebras each with finitely many pure traces. On the other hand, if \( A \) is an AF algebra whose \( K_0 \) group is anti-fd, then \( A \) contains no simple infinite dimensional AF-subalgebra with finitely many pure traces (more generally, the simple subalgebra need not be AF, but its \( K_0 \) group should be dense in a finite dimensional vector space).

If \((X,T)\) is a minimal action on a Cantor set, and its dimension group (ordered Čech cohomology) is anti-fd, then \((X,T)\) is not even orbit equivalent to a minimal system with a non-trivial map to a minimal action on a Cantor set that has only finitely many ergodic measures.

6 More examples
Very often, a vaguely ring-like structure on dimension groups is enough to guarantee the condition that every finite rank subgroup is discrete.

Let \( A = \mathbb{Z}[x,x^{-1}] \) equipped with the coordinatewise ordering. For a \( f \in A \), define \( c(f) \), the content of \( f \), as the greatest common divisor of the nonzero coefficients of \( f \). Gauss’ Lemma implies that \( c(fg) = c(f)c(g) \) for \( f,g \in A \). Let \( (P_i) \) be a sequence of elements of \( A^+ \), and form \( R = \lim \times P_i : A \to A \) as a partially ordered \( A \)-module. This is obviously a dimension group.

The order-theoretic properties of such dimension groups, beginning instead with \( A = \mathbb{R}[x,x^{-1}] \) are studied in detail in [BH]. The determination of the positive cone and the pure trace (there called state) space is independent of the choice of coefficient rings (\( \mathbb{Z} \) or \( \mathbb{R} \)), although of course the topological and underlying group structures are dependent on it. When we quote a result from [BH], it will apply to the pure trace space or to strong positivity.

Define \( Q_j = \prod_{i=1}^j P_i \). For \( f \in A \), define \( \log f = \{ k \in \mathbb{Z} \mid (f,x^k) \neq 0 \} \), where \( (f,x^k) \) denotes the coefficient of \( x^k \) in \( f \).

Now let \( R \) be the order ideal of \( R \) generated by the constant function 1; since an order ideal of a dimension group is again a dimension group, \( R \) is a dimension group. We can identify \( R \) with the following subgroup of \( \mathbb{Z}[x^{-1},p_i^{-1}] \),

\[ R \equiv R(p_i) = \left\{ \frac{f}{Q_n} \mid n \in \mathbb{Z}^+; \ \log f \subseteq \log Q_n \right\} \]

Now assume that \( c(p_i) = 1 \) for almost all \( i \). We may delete finitely many \( p_i \) and so assume that \( c(p_i) = 1 \) for all \( i \). We now show that every finite rank subgroup of \( R \) is discrete.

Form \( G_n = \sum_{i \in \log Q_n} \left( \frac{a_i}{Q_n} \right) \mathbb{Z} \); this is a finitely generated subgroup of \( R \).

(i) If \( a \in R \) and \( ma \in G_n \) for some nonzero integer \( m \), then \( a \in G_n \). Write \( a = \frac{f}{Q_r} \) for some \( f \in A \) and \( r \geq 1 \) such that \( \log f \subseteq \log Q_r \). Then there exists integers \( t_i \) such that \( mf/Q_r = \sum_{i \in \log Q_n} t_i x^i/Q_n \). Let \( g \) be the numerator of the right side, so that \( \log g \subseteq \log Q_n \). Set \( s = \max \{ r, n \} \), and multiply the equation \( mf/Q_r = g/Q_n \) by \( Q_s \). Since \( Q_s/Q_n \) and \( Q_s/Q_r \) are products of \( p_i \), each has content one. Thus \( c(g) = c(mf) = mc(f) \). Thus \( m \) divides \( c(g) \), so that \( h = g/m \) is in \( A \). Thus \( a = \frac{f}{Q_r} = \frac{h}{Q_n} \in G_n \).

(ii) \( G_n \) is discrete. It suffices to show \( \{ x^i/Q_n \} i \in \log Q_n \) is linearly independent over the reals. If \( \{ \alpha_i \} \) are real numbers such that \( \sum \alpha_i x^i/Q_n = 0 \), on multiplying by \( Q_n \), we obtain \( \sum \alpha_i x^i = 0 \), forcing all \( \alpha_i = 0 \).
(iii) Any finite rank subgroup of R is a subgroup of some $G_n$. If S is a finite rank subgroup, we may find a finite subset $s_i$ of S such that $S' = \sum(s_i, Z)$ has rank equalling that of S. There exists $n$ such that all $s_i \in G_n$. If $s \in S$, there exists a nonzero integer $m$ such that $ms \in S' \subset G_n$. By (i), $s \in G_n$. Hence $S \subseteq G_n$.

(iv) Every finite rank subgroup of R is discrete. Follows from (ii) and (iii).

Thus we have proved most of the following bifurcation result.

**Proposition 6.1** Let $p_i$ be a family of elements of $A^+$, and set $R = R(p_i)$. Then all finite rank subgroups of R are discrete if and only if $c(p_i) = 1$ for almost all i. If $c(p_i) > 1$ for infinitely many i, then $R \in \mathcal{H}$.

**Proof.** If $c(p_i) = 1$ for almost all i, then we have just shown all finite rank subgroups are discrete. Otherwise, $c(p_i) > 1$ for infinitely many i. By telescoping, we may assume that $d_i := c(p_i) > 1$ for all i. Then we can write $p_i = d_iP_i$ for some $P_i \in A^+$, and it easily follows that if $U = \lim \times d_i : Z \to Z$, then $R(p_i) \cong U \otimes R(P_i)$ as ordered abelian groups, hence we can write $R(p_i)$ as a limit of dimension groups of the form $U^n(i)$, hence $R(p_i) \in \mathcal{H}$. 

We can also often tell when the ordered groups $R(p_i)$ have dense range in their affine representation (we normally use 1 as the order unit for the representation).

Density of the image of $R(p_i)$ in its affine representation with respect to the order unit 1 (hence as this is a dimension group, with respect to any order unit) is equivalent to the nonexistence of discrete pure traces. The kernel of a pure discrete trace of a dimension group is a maximal order ideal (with quotient $Z$), by [GH, xxx]. Of course, we assume that all $p_i$ have at least two nonzero terms.

There are two obvious maximal order ideals in $R(p_i)$, namely, the kernels of the point evaluations at 0, that is, $\tau_0 : f/Q_n \mapsto \lim_{t \to 0} f(t)/Q_n(t)$ and $\tau_\infty : f/Q_n \mapsto \lim_{t \to \infty} f(t)/Q_n(t)$. It is easy to evaluate the range of these two traces, in terms of leading and terminal coefficients. By multiplying each $p_i$ by a suitable power of $x$ (and converting the corresponding $P_n$), we may assume $0 = \min \log p_i$ for all i, and let $d_i = \max \log p_i$, so $D_n := \max \log P_n = \sum_{i \le n} d_i$. Then $\tau_0(f/P_n) = (f, x^0)/(P_n, x^0)$ and $\tau_\infty(f/P_n) = (f, x^{D(n)})/(P_n, x^{D(n)})$.

It easily follows that $\tau_0(G) = \lim \times (p_i, x^0) : Z \to Z$ and $\tau_\infty(G) = \lim \times (p_i, x^{d_i}) : Z \to Z$. Hence necessary and sufficient for both these traces to have dense image is that

\[
\text{for infinitely many } i, \quad (p_i, x^0) > 1 \\
\text{for infinitely many } i, \quad (p_i, x^{d_i}) > 1.
\]

(Recall that we have altered the $p_i$ so the smallest exponent with nonzero coefficient is the constant term.)

Hence, whenever $\ker \tau_0$ and $\ker \tau_\infty$ are the only maximal order ideals of $R(p_i)$, these conditions are also sufficient for density of the image of $R(p_i)$ in its affine representation, and thus for approximate divisibility.

There are lots of situations in which these are the only maximal order ideals. We first make an observation: if $\log p_i = \log q_i$, then there is an obvious order isomorphism between the lattices of order ideals of $R(p_i)$ and those of $R(q_i)$. So to determine when they are the only maximal ideals, we can replace all the nonzero coefficients of all the $p_i$ by 1. When we apply this process to $p_i$, the result will be denoted $p_i'$. Although the pure trace space changes when we go from $R(p_i)$ to $R(p_i')$, the lattice of order ideals does not.

If $(p_i)$ is strongly positive (see [BH]); this is independent of the choice of coefficients, then the pure traces are precisely the point evaluations (including $\tau_0$ and $\tau_\infty$); the converse does not hold. Since maximal order ideals are automatically contained in kernels of pure traces (not necessarily
discrete), it follows that if $(p_i)$ is strongly positive, then ker $\tau_0$ and ker $\tau_\infty$ are the only maximal order ideals. Hence, if there exists a strongly positive sequence $(q_i)$ such that Log $p_i = \text{Log} q_i$, then $R(p_i)$ has only the two maximal order ideals.

If $p$ is a polynomial written in the form $p = \sum (p_i x^i)$, we call an exponent $i$ (or the corresponding monomial) an _isolani_ if $(p, x^{i-1})$ and $(p, x^{i+1})$ are both zero. Next we note that if infinitely many $p_i$ have no leading or terminal isolani, then the sequence $(p'_i)$ (obtained by replacing each nonzero coefficient in $p_i$ by 1) is strongly positive (an easy consequence of the Superposition Lemma of [BH]). Hence if infinitely many $p_i$ have no isolani, then $R(p_i)$ has dense range in its affine representation if and only if (†) holds.

The pure trace space is similarly just the set of point evaluations when infinitely many $p_i$ are equal to each other (and projectively faithful, that is Log $p_i - \text{Log} p_i$ generates all of $\mathbb{Z}$). Hence if infinitely many Log $p_i$ are equal to each other and projectively faithful, then $R(p_i)$ has only the two maximal order ideals, and so † is necessary and sufficient for density of $R(p_i)$ in its affine representation.

If there is a bound on the number of isolani, more generally, if there is an integer $N$ such that infinitely many $p_i$ have at most $N$ isolani and their diameter (distance from the smallest isolani to the largest) is bounded, then I think the pure trace space of $R(p'_i)$ again is the set of point evaluations. The thesis by Alan Kelm [K] contains numerous results on strong positivity and related ideas, improving some of those in [BH].

On the other hand, there exist lacunary choices for $(p_i)$ for which there are lots of maximal order ideals, and even lots of discrete traces (the former depends only on (Log $p_i$), the latter depends on the actual coefficients). For example, let $p'_i = 1 + x^{3^i}$; these aren’t projectively faithful, but we can also take $p'_i = 1 + x^{3^i} + x^{5^i}$. Obtaining conditions on the coefficients in lacunary examples so that density holds is not trivial.

Anyway, we have at least some results.

**PROPOSITION 6.2** Suppose that $p_i \in A^+$ such that $|\text{Log } p_i| \geq 2$ for all $i$ and (†) holds. Then sufficient for $R(p_i)$ to have dense image in its affine representation is either (a) for infinitely many $i$, $p_i$ has no leading or terminal isolani.

(b) there exists an infinite subset $S$ of $\mathbb{N}$ such that for all $i \in S$, Log $p_i$ are equal to each other.

If there is a bound on the degrees of the (nontrivial) polynomials $p_i$, then (b) applies. If there are no gaps, then there are no isolanis either (gaps can even persist, e.g., if $p'_i = (1 + x)(1 + x^{3^i})$, then $(p'_i)$ is strongly positive because of the factor $(1 + x)$ but all finite products will have gaps), and so the proposition applies to such sequences as well.

**COROLLARY 6.3.** Suppose that $p_i \in A^+$ such that Log $p_i \geq 2$ for all $i$. If all of the following hold, then $R(p_i)$ is Anti-FD.

(i) for infinitely many $n$, the coefficient of the smallest degree term in $p_n$ exceeds 1;
(ii) for infinitely many $n$, the coefficient of the largest degree term in $p_n$ exceeds 1;
(iii) for almost all $n$, $c(p_n) = 1$
(iv) either (a) or (b) of Proposition 6.2.

(Conditions (i) and (ii) together are equivalent to (†) for $R(p_i)$.) Now we can take the simplifications of such $R$, so obtaining simple dimension groups that are Anti-FD.

For example, if $p_i = 2x + 3$ for all $i$, all the conditions hold, and the corresponding $R$ (which also happens to be a ring, resembling the GICAR dimension group, but with values $\mathbb{Z}[1/3]$ and

* based on Nimzovich’s term for an isolated d-pawn in chess.
identical, and does not require the ordered ring structure. The condition that was obtained in the case of a partially ordered ring with $u = 1$; however, the proof is identical, and does not require the ordered ring structure. The condition that $\hat{G}$ contain $\hat{u}R$ in its closure is equivalent to the stronger property of approximate divisibility when $G$ is a dimension group, but not for more general partially ordered groups.

**7 Initial object preliminaries**

The proof of the following is based on the proof of [H, Proposition 1.2, last few paragraphs]. It was obtained en passant in the case of a partially ordered ring with $u = 1$; however, the proof is identical, and does not require the ordered ring structure. The condition that $\hat{G}$ contain $\hat{u}R$ in its closure is equivalent to the stronger property of approximate divisibility when $G$ is a dimension group, but not for more general partially ordered groups.

**LEMMA 7.1** Suppose $(G, u)$ is a partially ordered unperforated abelian group with order unit that the closure of $\hat{G}$ contains $\hat{u}R$. Let $p$ and $q$ be relatively prime positive integers; then for all $\epsilon > 0$ for all $r \in (0, 1]$, there exist order units $v$ and $w$ such that $u = pv + qw$ and $\|\hat{v} - r1/p\| < \epsilon/p$ and $\|\hat{w} - (1-r)1/q\| < \epsilon/q$.

**Proof.** By interchanging $p$ and $q$ if necessary, we may assume that $r \in [1/2, 1]$. There exists a positive integer $pk \equiv 1 \mod q$, so that we can find integer $t$ such that $pk = qt + 1$. Now form the subgroup $kuZ + qG$ of $G$; since this contains $qG$, its image in the affine representation (with respect to $u$) is dense. Hence there exists $z \in G$ such that $1 \cdot \max \{r - \epsilon, 1/2\}/p < (ku - qz) < 1 \cdot r/p < 1/p$. Set $v = ku - qz$, so $\hat{v} \gg 0$, and thus $v$ is an order unit. Now $u - pv = u - pkv + pqz$, so $u - pv = (1 - pk)v + qz = q(tu + z)$. Set $w = tu + z$; then $u = pv + qw$; moreover, $\hat{w} = u - pv/q = q^{-1}(1 - \hat{v}) \gg 0$, the latter since $\hat{v} < p^{-1}1$. Hence $w$ is an order unit.

It also follows that $\|\hat{w} - 1 \cdot (1-r)/q\| < \epsilon/q$ (just use the triangle inequality applied to $\|q\hat{w} - (1-r)\| = \|1 - p\hat{w} - (1-r)\|$). \hfill \bullet

The following was proved by Perera and Rørdam [PR], in the context of homomorphisms from finite dimensional algebras to real rank zero C*-algebras (it applies to AF algebras in particular). Here and in the preceding, we do not require $G$ to be a dimension group, merely to have dense image in its affine representation. If $G$ is a dimension group, then dense image is equivalent to all pure traces not being discrete.

**COROLLARY 7.2** Let $(G, u)$ be a partially ordered abelian group with order unit such that the image of $G$ in its affine representation is dense. Suppose that $\{p_i\}$ is a finite set of positive integers such that $\gcd\{p_i\} = 1$. Given an order unit $U$ of $G$, there exist order units $v_i$ such that $U = \sum p_i v_i$.

**Proof.** Let $q = \gcd\{p_2, \ldots, p_n\}$, so that $p_1$ and $q$ are relatively prime. Given an order unit $u$, by the preceding there exist order units $v_1$ and $w$ such that $u = p_1 v_1 + qw$. Now $qG$ has dense image in whatever affine representation $G$ has dense image. Since $\gcd\{p_i/q\} = 1$, so by induction applied to $qG$, $qw$ can be expressed as as $\sum_{i \geq 2} p_i q^{-1} z_i$ where $z_i$ are order units in $qG$. Then $z_i = q v_i$ for some $v_i$ in $G$, necessarily order units, and $u = \sum p_i v_i$, completing the induction. \hfill \bullet

In particular, if $\gcd\{p_i\} = 1$, then $\prod_{i=1}^n M_{p_i} \mathbb{C}$ is an initial object for approximately divisible AF algebras. The paper by Perera and Rørdam [PR] proves more; real rank zero and no representations which hits the compacts is sufficient to get this.

There is a peculiarity, noted previously, in the characterization of Anti-FD partially ordered
groups. Another approach emphasizes the peculiarity.

Let \((H, v)\) be an unperforated partially ordered abelian group with order unit \(v\), and suppose that \(\text{Inf} H = \{0\}\). For \(m\) a positive integer, we say \((H, v)\) (or just \(H\)) is Anti-FD\((m)\) if all subgroups \(H_0\) of rank \(m\) or less are discrete (with respect to the norm topology induced by the order unit \(v\)); that is, \(H_0\) is a discrete subgroup of \(\text{Aff} \, S(H, v)\) in the sup-norm).

By the characterization above (Corollary 4.5), \((H, v)\) is Anti-FD\((m)\) for all \(m\) iff \((H, v)\) is Anti-FD. Examples include the critical groups of \([H2]\): a subgroup \(G\) of \(R^m\) is critical if it is dense and of rank \(k + 1\); equipped with the strict ordering inherited from the vector space, these are simple dimension groups with \(m\) pure traces, and every subgroup of rank \(m\) or less is discrete. Hence Anti-FD\((m)\) simple dimension groups which are not Anti-FD\((m+1)\) exist for every \(m \in \mathbb{N}\).

Anti-FD\((m)\) partially ordered groups have the expected (relatively weak) property dealing with continuous homomorphisms.

**PROPOSITION 7.3** Suppose \((G, u)\) is an approximately divisible unperforated partially ordered abelian group, and \(m\) is an integer such that \(\text{rank} \, G/\text{Inf} \, G \leq m\). If \((H, v)\) is Anti-FD\((m)\) and \(\text{Inf} H = \{0\}\), then any continuous group homomorphism \(\phi : G \to H\) is zero.

We require a few elementary lemmas. The norm (or pseudo-norm) on a group \((G, u)\) is determined by the order unit \(u\) (changing the order unit changes the norm to an equivalent one, but even for simple dimension groups, may actually change the Choquet simplex into one that is not affinely homeomorphic to the original!).

**LEMMA 7.4** Let \((G, u)\) be an unperforated approximately divisible group, and \(n\) a positive integer. Let \(S_n = \{g \in G^+ \mid \|g\| < 1/n\}\). Then \(S_n\) generates \(G\) as an abelian group.

**Proof.** It suffices to show \(G^+\) is contained in the group generated by \(S_n\). Select an order unit \(w\), and a relatively prime pair of positive integers, \(p\) and \(q\). By Lemma 7.1 above, there exist order units \(y\) and \(z\) in \(G\) such that \(w = py + qz\). Since \(w, y, z\) are all in the positive cone, we have \(\|y\| \leq \|w\|/p\) and \(\|z\| \leq \|w\|/q\). If we choose \(p, q > n\|w\|\), we have \(y, z \in S_n\), and we are done. \(\bullet\)

**LEMMA 7.5** Let \((G, u)\) and \((H, v)\) be unperforated partially ordered groups with order unit, and let \(\phi : G \to H\) be a positive group homomorphism (not necessarily sending order units to order units). Then \(\|\phi(g)\| \leq \|g\| \cdot \|\phi(u)\|\) (that is, \(\|\phi\| \leq \|\phi(u)\|\)).

**Proof.** Suppose \(g \in G\) and \(\|g\| = \delta > 0\). If \(p\) and \(q\) are positive integers such that \(\delta < p/q\), then \(\|g\| < p/q\) entails \(-q \leq -p\|g\| < q\), so by unperforation, \(\|\phi(g)\| < (p/q)\|\phi(u)\|\). Applying \(\phi\), we obtain \(-q\|\phi(u)\| < p\|\phi(g)\| < q\|\phi(u)\|\), and since \(\phi(u)\) is positive, we deduce \(\|\phi(g)\| \leq (p/q)\|\phi(u)\|\). Since this is true for all \(p/q > \delta\), the result follows. \(\bullet\)

**Proof.** (Proposition 7.3). By definition, a continuous group homomorphism maps \(\text{Inf} \, G\) to \(\text{Inf} \, H = \{0\}\), so without loss of generality, we may factor out \(\text{Inf} \, G\), and thus assume that \(G\) itself has rank \(m\) or less. Then \(H_0 := \phi(G)\) has rank at most \(m\), and so is discrete in the affine representation of \((H, v)\). There thus exists \(\epsilon > 0\) such that \(\|\hat{h}\| < \epsilon\) for \(h \in H_0\), then \(\hat{h} = 0\), and thus \(h = 0\).

From the continuity of \(\phi\) at \(0\), given \(\epsilon\), there exists a positive integer \(n\) such that for all \(g \in G\), \(\|g\| < 1/n\) entails \(\|\phi(g)\| < \epsilon\). Thus if we define, as in 7.4, \(S_n = \{g \in G \mid \|g\| < 1/n\}, \) then \(\phi(S_n) = \{0\}\), and since \(S_n\) generates \(G\) as an abelian group, \(\phi(G) = 0\). \(\bullet\)

Now we see the peculiarity. If we let \(m \to \infty\), we only obtain that there are no nontrivial homomorphisms from finite rank simple dimension groups, whereas we know that we can replace finite rank by finite pure trace space, and we also know that not all finite pure trace space dimension groups can be written as a direct limit of finite rank approximately divisible groups.
8 Initial objects

Let \((H, v)\) be a partially ordered unperforated abelian group with order unit \(u\). We say that \((H, v)\) is an initial object if for all approximately divisible partially ordered abelian groups with order unit, \((G, u)\), there exists an order-preserving group homomorphism \((H, v) \to (G, u)\). Normally, \(H\) will be a dimension group \((G \text{ need not be, but in cases of interest, it will be})\). This is designed to be compatible with initial objects in the class of unital C*-algebras (typically AF), via their ordered \(K_0\)-groups. Obviously if \(v = 2x\) for some \(x \in H\) (and necessarily in \(H^{++}\) as a consequence of unperforation), then \((H, v)\) cannot be an initial object, although it is still possible for \((H, x)\) to be one.

Initial objects in the category of dimension groups (which can be defined in many different, and equally plausible ways) that are approximately divisible are automatically at least anti-fd (by [ER], the pure trace space of an approximately divisible initial object is infinite). Some of the \(R(p_i)\) are initial objects; if \(p_i = 1 + x\) for all \(i\), this was shown in [ER]. We show by completely different methods that if \(p_i = a_i + b_ix\) with \(a_i, b_i\) positive integers exceeding 1 such that \(\gcd(a_i, b_i) = 1\) and \(\sum_i \min \{a_i, b_i\} / (a_i + b_i) < \infty\), then \(R(p_i)\) is an initial object. Their pure trace spaces are all the one-point compactification of \(\mathbb{Z}^+\). All \(R(p_i)\) are the ordered Grothendieck group of the fixed point algebras of product type actions of a circle on UHF C*-algebras.

If \(g, h \in G\) and \(k\) is a positive integer, then we use either \(g \equiv h \mod k\) or \(g \equiv h \mod kG\) to denote \(h - k \in kG\).

**Lemma 8.1** Let \((G, u)\) be a partially ordered abelian group which has dense image in \(\text{Aff} S(G, u)\). Then for all integers \(k\), all \(\epsilon > 0\), all \(g \in G\) there exists \(h \in G\) such that \(h \equiv g \mod k\) and \(\|h\| < \epsilon\).

**Proof.** Since the image of \(G\) is dense, so is that of \(kG\) and therefore that of \(-g + kG\). Hence \(-g + kG\) contains elements of arbitrarily small norm.

We will show the following. Let \(\{u_i\}_{i=0}^{N-1}\) be a set of order units such that for some \(\delta\), there exists a positive rational \(c\) such that \(\|u_i - c\| < \delta\) for all \(i\) (in this case, \(c\) represents the constant function with value \(c\)). Suppose that \(1 < a < b\) are positive and relatively prime integers. Then there exist elements \(\{v_i\}_{i=0}^{N}\) of \(G\) such that

\[
\left\| \hat{v}_i - \frac{c}{a + b} \right\| < \frac{\delta}{b - a} \quad \text{for} \quad i = 0, 1, 2, \ldots, N
\]

\[
u_i = bv_i + av_{i+1} \quad \text{for} \quad i = 0, 1, 2, \ldots, N - 1.
\]

A consequence is that if \(\delta < c(b - a)/(a + b)\), then the \(v_i\) are all order units, hence in \(G^+\).

We break this into three steps. We form \(G_0 = G \otimes \mathbb{Q}\), elements of which we regard as formal fractions, \(g/q\) where \(g \in G\) and \(q \in \mathbb{Q}^{++}\); it has the obvious positive cone. Since it is a vector space over the rationals, solutions of linear systems (with integer coefficients) are relatively easy to deal with.

We define the \(N \times (N + 1)\) matrix \(A\) (with rows indexed 0, 1, 2, \ldots, \(N - 1\) and columns indexed 0, 1, 2, \ldots, \(N\)) via

\[
A_{ij} = \begin{cases} 
  b & \text{if } i = j \\
  a & \text{if } j = i + 1 \\
  0 & \text{else.}
\end{cases}
\]

The equations in the second line above can be compressed into the simple \(AV = U\), where \(V = (v_i)^T\) and \(U = (u_i)^T\).
For each \( j = 0, 1, 2, \ldots, N \), define \( T_j = u_0 - ab^{-1}u_1 + \cdots + (ab^{-1})^ju_j \), that is, \( T_j = \sum_{i=0}^{j}(a/b)^iu_i \in G_0 \).

**LEMMA 8.2** All solutions \( X = (x_i)^T \in G_0^{N+1} \) (indexed beginning at zero) to \( AX = U \) are of the form given by

\[
x_i = \left( \frac{-b}{a} \right)^i \left( x_0 - \frac{T_{i-1}}{b} \right) \quad i = 1, 2, \ldots, N.
\]

**Proof.** We first describe all the solutions to the homogeneous equation \( AX = 0 \). If \( Z = (z_i)^T \) is a solution, then \( b z_0 + a z_1 = 0 \) implies \( z_1 \) is a rational multiple of \( z_0 \), and by induction, every \( z_i \) is a rational multiple of \( z_0 \). Hence we may write \( Z = (1, q_1, q_2, \ldots, q_n)^T z_0 \) where all the \( q_s \) are rational numbers. We easily see from the equations that \( q_i = (-b/a)q_{i-1} \), and thus \( z_i = (-b/a)^i z_0 \). Let \( w = (1, -b/a, b^2/a^2, \ldots, (-b/a)^N)^T \in Q^{N+1} \). We have just shown that all solutions to \( AZ = 0 \) are of the form \( Z = wz_0 \).

Next, define the vector \( S = (S_i)^T \) where \( S_i = (-b/a)^i - 1 T_{i-1}/a \) for \( 1 \leq i \leq N \) and \( S_0 = 0 \). We claim that \( AS = U \), so \( S \) is a particular solution. The top coordinate of \( AS \) is \( aT_0/a = T_0 = u_0 \). We simply calculate the rest; for \( 1 \leq j \leq N - 1 \),

\[
(AS)_j = bS_j + aS_{j+1} = \left( \frac{-b}{a} \right)^j \left( \frac{b}{a} T_{j-1} + \frac{-ba}{a^2} T_j \right) = \left( \frac{-b}{a} \right)^j (T_j - T_{j-1}) = \left( \frac{-b}{a} \right)^j \left( \frac{-a}{b} \right)^j u_j = u_j.
\]

Now let \( X \) be any solution in \( G_0^{N+1} \) to \( AX = U \). Then \( A(X - S) = 0 \), so there exists \( z_0 \in G_0 \) such that \( X = S + wz_0 \), which is precisely the desired statement. (Note that \( x_0 = z_0 \).) \( \bullet \)

**LEMMA 8.3** Suppose \( u_i \) in \( G_0 \) satisfy \( \|\hat{u}_i - c\| < \delta \) for some positive constant \( c \) and some \( \delta > 0 \). Then there exists a solution \( X = (x_i) \in G_0^{N+1} \) to \( AX = U \) such that \( \|\hat{x}_i - c/(a+b)\| < \delta/(b-a) \) for all \( i = 0, 1, \ldots, N \).

**Proof.** Set \( Y = c1/(a+b) \) (the column of constants). Then \( AY = c1 \), which is close (in the infinity norm) to \( U \). Let \( M \) be the upper triangular \( N \times N \) matrix (with coordinates indexed from 0 to \( N - 1 \)) with 1 directly above the diagonal and zeros elsewhere. Set \( B = b1 + aM \) (so it constitutes the first \( N \) columns of \( A \)). Then \( B^{-1} \) (with entries in \( Q \)) is \( b^{-1} \left( 1 + \sum_{i=1}^{N-1} (-a/b)^i M^i \right) \), and we calculate \( B^{-1}A \) has the identity as its first \( N \) columns, and has \( a((-a/b)^{N-1-i})^T \) as the last column (recalling the indexing begins at \( i = 0 \)).

The equation \( AX = U \) is equivalent to \( B^{-1}AX = B^{-1}U \). Writing \( X = xX_0^T \) (where \( x \) is the final coordinate of \( X \)) and \( B^{-1}U = (r_0, r_1, \ldots, r_{N-1})^T \), we have \( X_0 + ax((-a/b)^{N-1-i})^T = B^{-1}U \). We are free to vary \( x \), and the value of \( x \) uniquely determines \( X \). Set \( x = -cu/(a+b) \) (recall that the affine representation of \( G \) is with respect to the order unit \( u \), so \( \hat{x} = -c/(a+b) \), the constant function.

The rest of the coordinates of \( X = (x_i) \) \( (x_N = -u/(a+b)) \) are given by

\[
x_i = \frac{1}{b} \left( u_i - \frac{a}{b} u_{i+1} + \left( \frac{a}{b} \right)^2 u_{i+2} + \cdots + \left( \frac{a}{b} \right)^{N-1-i} u_{N-1} \right) = \frac{acu}{a+b} \left( \frac{-a}{b} \right)^{N-1-i}.
\]
Now we approximate $\tilde{x}_i$. 

$$\| \tilde{x}_i - c \left( 1 - \frac{a}{b} + (\frac{a}{b})^2 + \cdots + (\frac{a}{b})^{N-1-i} \right) \| - \frac{ac}{a+b} \left( \frac{-a}{b} \right)^{N-1-i} < \frac{a\delta}{b} \left( 1 + \frac{a}{b} + \frac{a^2}{b^2} + \cdots + \frac{(a/b)^{N-1-i}}{b} \right)$$

$$= \frac{a\delta}{b} \cdot \frac{1 - (a/b)^{N-i}}{b-a}$$

$$< \frac{a\delta}{b}.$$ 

Finally, 

$$\left( 1 - \frac{a}{b} + (\frac{a}{b})^2 + \cdots + (\frac{a}{b})^{N-1-i} \right) \frac{1}{b} - \frac{a}{a+b} \left( \frac{a}{b} \right)^{N-1-i} = 1/(a+b).$$

**Lemma 8.4** If $X = (x_i)^T$ is a solution in $G_0^{N+1}$ to $AX = U$, then for all $\epsilon$, there exists a solution $V = (v_i)$ in $G^{N+1}$ to $AV = U$ such that $\|\tilde{x}_i - \tilde{v}_i\| < \epsilon$.

**Proof.** First, we may find $y \in G$ such that $\|\tilde{x}_i - \tilde{y}_i\| < \epsilon/a$. Now we perturb $y_0$ by an element of the form $h = \sum_i = Nh_i a^i$ where $\|h_i\| < a^{-i-2}$ so that $v_i := (-b/a)^i(y + h - T_{i-1}/b) \in G$ (and $v_0 = y + h$).

Set $y_i = (-b/a)^i(y - T_{i-1}/b) \in G_0$. Multiply by $b^i$, so we obtain 

$$a^iy_i = (-b)^iy - (-b)^{i-1}T_{i-1}.$$

The term $t_{i-1} := (-b)^{i-1}T_{i-1} = (-b)^{i-1}u_0 + (-b)^{i-2}au_1 + (-b)^{i-3}a^2u_2 + \cdots + a^{i-1}u_{i-1}$ belongs to $G$. We want to find $h \in G$ with various properties such that $(-b)^i(y - h) \equiv -t_{i-1}$ mod $a^i$ for $i = 1, 2, \ldots, N$. We write $h$ formally as $\sum_{i=0}^{N} a^i h_i$, and find the $h_i$ inductively.

If $i = 1$, the equation reduces to $-bh_0 \equiv u_0 - by$ mod $a$; since gcd $\{a, b\} = 1$, this has a solution in $G$, and we may choose a solution $h_0 \in G$ such that $\|h_0\| < \epsilon/a^2$. Now suppose $h_0, \ldots, h_{k-1}$ have been defined with $\|h_i\| < a^{-2i+2}$ for $i = 0, 1, 2, \ldots, k-1$ and if $h^{(j)} = \sum_{i=0}^{k} a^i h_i_j$, then $(-b)^i(y - h^{(i-1)}) \equiv -t_{i-1}$ mod $a^i$ for all $i \leq k$. We want to find $h_k$ such that $(-b)^{k+1}(y - h^{(k)} - a^k h_{k+1}) \equiv (-b)^k T_k$ mod $a^{k+1}$.

We have $T_k = T_{k-1} + (-b/a)^k u_k$, so $(-b)^k T_k = a^k u_k + (-b)^{-1} k T_{k-1}$. Also, the induction assumption implies $(-b)^k (y - h^{(k-1)}) \equiv (-b)^k(-1)^{k-1}T_k$ mod $a^k$; thus $x = (-b)^k(y - h^{(k-1)}) + (b)(-b)^{(k-1)} T_{k-1}/a^k$ belongs to $G$. The equation reduces thus to $a^k(-b)(x - (-b)^k h_k) \equiv 0$ mod $a^{k+1}$. This is equivalent to $(-b)(x - (-b)^k h_k) \equiv 0$ mod 2. Since gcd $\{a, b\} = 1$, we can solve this with arbitrarily small norm. This completes the induction.

If we assume $b < a$ (and still have gcd $\{a, b\} = 1$) instead of $a < b$, then we index the $u_i$ in reverse order; then the corresponding result (with $[b-a]$ replacing $b-a$) applies in this case.

**Theorem 8.5** Let $p_n = b_n + a_n x$ with $1 < a_n, b_n$, $\sum \min \{a_n, b_n\}/(a_n + b_n) < \infty$, and gcd $\{a_n, b_n\} = 1$ for all $n = 1, 2, \ldots$. Form $R(p_i) \equiv R = \{ f/\Pi_{i=1}^{n} p_i \mid \Log f \subseteq \Log \Pi_{i=1}^{n} p_i \}$, the order ideal generated by 1 in $\lim x^A : A \rightarrow A$. Then $R$ is an initial object in the class of approximately divisible dimension groups.

**Proof.** Let $d = \prod_{i=1}^{\infty} |a_i - b_i|/(a_n + b_n)$; by hypothesis, $d > 0$. Let $(G, u)$ be an approximately divisible dimension group with order unit. Begin with $u^0 = u$. By 8.3, we can find order units $u_i^0, u_i^1$ of $G$ such that $\|\tilde{u}_i^1 - 1/(a_1 + b_1)\| < d/|a_1 - b_1|$ and $u^0 = a_1 u_i^0 + b_1 u_i^1$. Since $d < |a_1 - b_1|/|a_1 + b_1|$, the $\tilde{u}_i$ are strictly positive, and thus $u_i^1$ are order units, and the process may be continued. Now we proceed by induction.
After the $N - 1$st iteration, we can find $\{u_j(N)\}_{j=0}^N$ such that $u_j^{N-1} = a_n u_j^N + b_n u_{i+1}^N$ for $i = 0, 1, 2, \ldots, N - 1$ and for all $j = 0, 1, \ldots, N$,

$$\left\| \hat{u}_j^N - \frac{1}{\prod_{t=1}^N (a_t + b_t)} \right\| < \frac{d}{\prod_{t=1}^N \left| a_t - b_t \right|}.$$ Since $d < \prod_{t=1}^N \left| a_t - b_t \right|/(a_t + b_t)$, it follows that $\hat{u}_j^N$ are strictly positive, hence $u_j^N$ are order units, and we may continue to iterate this process.

Now the assignment $x^j/\prod_{i \leq n} p_i \mapsto u_j^N$ (for $0 \leq j \leq n$) extends to a well-defined, positive homomorphism $R \to G$, sending 1 to $u$.

We can replace approximately divisible dimension groups by partially ordered abelian groups with dense image in their affine function space, since the constructions never use interpolation.

The sequence $(p_n = b_n + a_n x)$ is either strongly positive (and more), in which case the pure trace space is the two-point compactification of $\mathbb{R}^+$, or not strongly positive, in which case the pure trace space the one-point compactification of $\mathbb{Z}^+$. Strong positivity is equivalent (this is a very special case of a result in [BH]) in this case to $\sum \min \{a_n, b_n\} / (a_n + b_n) = \infty$, precisely the negation of the hypothesis in 8.5. When the sum converges (as in the last result), the product $\prod_{i=1}^N (a_n + b_n)$ converges to an entire function, and the Maclaurin series coefficients determine the pure traces ([BH] again); the pure trace space is thus the one-point compactification of $\mathbb{Z}^+$, and by 6.2, the condition that $a_n, b_n > 1$ more than implies that $R$ is approximately divisible.

Here is an amusing class of initial objects, easily proved to be so, and easily determined when they have dense image in their affine representation. Moreover, they also provide examples of $R(p_i)$ for which checking merely at the two end evaluations is not sufficient to determine density; in fact, the collection of maximal order ideals is indexed by a Cantor space (in a natural way; in fact, the indexing is topological, if we use the usual topology on maximal order ideals), and we have to check density of the image at every one of the extremal traces.

Let $p_i$ be a family of polynomials in $A^+$ such that $|\log p_i| \geq 2$ for all $i$. We say the sequence (or the corresponding dimension group) $(p_i)$ is non-interactive if whenever $x^m \in \prod_{i=1}^N (p_i)$ (for some finite product of distinct $p_i$), there exist unique $w_i \in \log p_i$ such that $m = \sum w_i$. The simplest example arises when $p_i = a_i + b_i x^i$ for $a_i, b_i \in \mathbb{N}$, but there are lots of others. The term comes from ergodic theory, actions of $\mathbb{Z}$, analysed with respect to measure-theoretic equivalence.

**Proposition 8.6** Let $(p_i)$ be a non-interactive sequence of polynomials in $A^+$ such that $c(p_i) = 1$ for almost all $i$. Then $R(p_i)$ is an initial object in the category of approximately divisible unital partially ordered unperforated abelian groups. The range of every pure trace of $R(p_i)$ is a subgroup of the rationals. Moreover, $(R(p_i), 1)$ has dense image in its affine representation if and only if for infinitely many $i$, none of the coefficients appearing $p_i$ is 1 (that is, $(p_i, x^j) \neq 1$ for all $j$).

**Proof.** Define $P_n = \prod_{i \leq n} p_i$. Non-interactive means the Bratteli diagram for $R(p_i)$ (and also for $S(p_i)$, from which the former can be obtained by beginning at a single point in the top level) is non-interactive, meaning if $j \neq k \in \log P_n$, then $j + \log p_{n+1} \cap k + \log p_{n+1} = \emptyset$. Since the set of entries of $\log p_{n+1}$ has greatest common divisor 1, for each vertex of the diagram, corresponding to $x^i/P_n$ with $i \in \log P_n$, we can solve the equations $U = \sum (p_{n+1}, x^i) u_j$ in an arbitrary approximately divisible etc group $G$ (no matter what the order unit $U$ is) at each index, independently of the other indices on the same level.

If we use the notation $u_{i,n}$ to denote the order unit corresponding to $i \in \log P_n$, we have found order units $v_{i+t,n+1}$ order units with $t \in \log p_{n+1}$ such that $u_{i,n} = \sum (p_{n+1}, x^i) v_{i+t,n+1}$. Then for $s \in \log p_{n+1}$ we define $(u_{s,n+1})$ to be $v_{i+(s-i),n+1}$ where $i$ is the unique element of $\log P_n$.
such that \( s \in i + \log p_{n+1} \) (so necessarily \( s - i \in \log p_{n+1} \)). Then the map \( x^i/P_n \mapsto v_{i,n} \) (for \( i \in \log P_n \)) yields a well-defined, positive group homomorphism \((R(p_i),1) \mapsto (G,u_0)\). Hence \( R(p_i) \) is an initial object.

Next, we show that the advertised condition is precisely what is needed to guarantee that \( R(p_i) \) has no (pure) discrete traces. It is practically tautological that the the pure trace space can be identified with the path space of the reduced Bratteli diagram (obtained by collapsing all multiple edges to single edges), which is a Cantor set. Along each path, the corresponding trace divides the value by the coefficient. Hence if all the \( p_i \) admitted 1 as a coefficient, there would be a path corresponding to repeated division by 1, hence the image of the trace would be \( \mathbb{Z} \). Conversely, if every path ran into infinitely many levels where all the nonzero coefficients of \( p_n \) are not one, we would obtain division by infinitely many integers exceeding 1, so that the range of the trace would be a noncyclic subgroup of the rationals. In that case, no extreme trace is discrete, hence \( R(p_i) \) has dense range in its affine representation.

So if we set \( p_i = a_i + b_i x^{2^i} \), then \( (p_i) \) is non-interactive, and \( R(p_i) \) is an initial object. It will have dense range in its affine representation if and only if for infinitely many \( i \), both \( a_i, b_i > 1 \) hold. This is a stronger condition than the condition considered previously, which amounts to (in this case), \( a_i > 1 \) for infinitely many \( i \) and \( b_i > 1 \) for infinitely many \( j \). These two conditions merely guarantee that the \( \tau_0 \) and \( \tau_\infty \) are not discrete. These traces correspond to the extreme left path and the extreme right path in the Bratteli diagram, respectively. The case that \( a_i = 2 \) and \( b_i = 3 \) yields the ordered \( K_0 \)-group of the AF-algebra given as the infinite tensor product, \( \otimes (M_2 \oplus M_3) \), considered in [ER].

There is an easy generalization of the sequence of non-interactive polynomials, which also yields a family of initial objects.

Let \( X \) be an infinite tree, and let \( X^0 \) be the Bratteli diagram obtained from \( X \) by labelling the edges with positive integers. We sometimes identify \( X \) with its path space. Let \( G \) be the resulting direct limit. The pure traces are given by paths in \( X \) (not \( X^0 \)) as follows. Let \( X_n \) be the set of vertices at the \( n \)-th level, so \( X_0 = \{ x_0 \} \) consists of the initial point, and \( X = \cup X_n \). Let \( p := (x_0, x_1, \ldots) \) be a path in \( X \) (where \( x_i \in X_i \)), and suppose the multiplicity of the edge \( x_i \to x_{i+1} \) is \( m(i) \). Let \( f : X_n \to \mathbb{Z} \): the trace \( \tau_p \) is given by the map \( [f,n] \mapsto f(x_n)/m(0)m(1)\cdots m(n-1) \). It is easy to check that this is well-defined and a pure trace.

Associated to each vertex \( x \in X \) is its vector of multiplicities, \( v(x) \); this is the list of multiplicities on the edges emanating from \( x \).

Moreover, \( \ker \tau_p \) is a maximal order ideal (the order ideals of \( C(X,\mathbb{Z}) \) are in bijection with the order ideals of \( G \)), and thus \( \tau_p \) is pure. Moreover, \( \{\tau_p\}_{p \in X} \) is a compact set of pure traces (routine) homeomorphic to the path space, \( X \). To check that this is the pure trace space of \( G \), we note that the embedding \( \mathbb{Z}^{X_n} \to \mathbb{Z}^{X_{n+1}} \) (of which \( G \) is the direct limit) is an order embedding: \([f,n] \geq 0 \) iff \( f \geq 0 \). But if \( f \) is nonnegative along every path, then its values at each of the points in \( X_n \) are nonnegative. Hence \( \{\tau_p\}_{p \in X} \) is a compact set of pure traces that determines the ordering, and thus its closed convex hull is the normalized trace space of \( X \). It follows that \( \{\tau_p\}_{p \in X} \) is the pure trace space of \( G \).

Now \( G \) is an archimedean (in the strong sense) subgroup of \( C(X,\mathbb{R}) \), and density is equivalent to every \( \tau_p \) having dense range. Sufficient for this is that there exist infinitely many \( n \) such that for every \( x \in X_n \), \( 1 \notin v(x) \).

It is a direct consequence of 6.2 that \( G \) is an initial object if \( c(v(x)) = 1 \) for all \( x \in X \). Thus we have the following.

**Proposition 8.7** Let \( X \) be an infinite tree with root \( x_0 \), and \( X_0 \) the corresponding Bratteli diagram obtained by attaching positive-integer-valued weights to each vertex,
and let \((H, u := \chi_{\{x_0\}})\) be the resulting dimension group with order unit.

(i) Sufficient for \((H, u)\) to be an initial object is that for all \(x \in X\), \(c(v(x)) = 1\).

(ii) Sufficient for \(H\) to be approximately divisible is that for infinitely many \(n\), for all \(x \in X_n\), no entry of \(v(x)\) is 1.

(iii) The pure trace space of \((H, u)\) is naturally homeomorphic to the path space of \(X\).

A reasonable question is whether all Anti-FD dimension groups are initial objects. If we extend the definition of initial object to require a one to one map (as was proved for the Pascal’s triangle example in [ER]), then such initial objects must be countable and free (as abelian groups). So this stronger property excludes the non-Anti-FD but anti-fd example \(\mathbb{Z}[x] + (2x - 1)\mathbb{Q}\) discussed at the end of section four.

References

[BDHMP] T Bartoszynski, M Dzamonja, L Halbeisen, E Martinov, A Plichko, On bases in Banach spaces, Studia Mathematica, 170 (2005) 147–171.

[BH] BM Baker & DE Handelman, Positive polynomials and time dependent integer-valued random variables, Canadian J Math 44 (1992) 341.

[BeH] S Bezuglyi & D Handelman, Measures on Cantor set: the good, the ugly, the bad, Trans Amer Math Soc (to appear).

[C] I Chlodovsky, Une remarque sur la représentation des fonctions continues par des polynômes coefficients entiers, Mat Sb 32 (1925) 472–475.

[EHS] EG Effros, David Handelman, & Chao-Liang Shen, Dimension groups and their affine representations, Amer J Math 102 (1980) 385–407.

[ER] GA Elliott & M Rørdam, Perturbation of Hausdorff moment sequences and an application to the theory of \(C^*\)-Algebras of real rank zero, Operator Algebras Abel Symposia, Volume 1 (2006) 97–115.

[F] Le Baron O Ferguson, Approximation by polynomials with integral coefficients, Amer Math Soc, Rhode Island, 1980.

[G] KR Goodearl, Partially ordered abelian groups with interpolation, Mathematical Surveys and Monographs, 20, American Mathematical Society, Providence RI, 1986.

[GH] KR Goodearl & David Handelman, Metric completions of partially ordered abelian groups, Indiana Univ J Math 29 (1980) 861–895.

[Gr] P Griffith, Infinite abelian group theory, University of Chicago press, 1970.

[H] D Handelman, Iterated multiplication of characters of compact connected Lie groups, J of Algebra 173 (1995) 67–96.

[H2] D Handelman, Free rank \(n+1\) dense subgroups of \(\mathbb{R}^n\) and their endomorphisms, J Funct Anal 46 (1982), no. 1, 1–27.

[K] A Kelm, Strong positivity results for polynomials of bounded degree, PhD thesis, University of Ottawa (1993).

[L] J Lawrence, Countable abelian groups with a discrete norm are free, Proc Amer Math Soc 90 (1984) 352–354.

[LL] AJ Lazar and J Lindenstrauss, Banach spaces whose duals are \(L^1\) spaces and their representing matrices, Acta Math 126 (1971) 165193.
[PR] F Perera and M Rørdam, *AF-embeddings into C*-algebras of real rank zero*, J Funct Anal 217 (2004) 142–170.

[S] J Steprans, *A characterization of free abelian groups*, Proc Amer Math Soc 93 (1985) 347–349.

Mathematics Dept, University of Ottawa, Ottawa K1N 6N5 ON, Canada; dehsg@uottawa.ca