MAXIMAL LEXICOGRAPHIC SPECTRA AND RANKS FOR STATES WITH FIXED UNIFORM MARGINS

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ABSTRACT. We find the spectrum in maximal lexicographic order for quantum states $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ with margins $\rho_A = \frac{1}{n}I_n$ and $\rho_B = \frac{1}{m}I_m$ and discuss the construction of $\rho_{AB}$. By nonzero rectangular Kronecker coefficients, we give counterexamples for Klyachko’s conjecture which says that a quantum state with maximal lexicographical spectrum has minimal rank among all states with given margins. Moreover, we show that quantum states with the maximal lexicographical spectrum are extreme points.

1. Introduction

The quantum marginal problem is about relations between spectrum of mixed state $\rho_{AB}$ of two (or multi) component system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and that of reduced states $\rho_A$ and $\rho_B$ \cite{4, 8, 12, 13}. As margins of a pure state are isospectral, for $\text{Spec } \rho_A \neq \text{Spec } \rho_B$ state $\rho_{AB}$ can’t be pure. It is interesting to measure the closeness between $\rho_{AB}$ and the pure states. A state $\rho$ is pure if and only if its maximal eigenvalue is equal to one. Hence the maximal eigenvalue may be considered as a measure of purity. On the other hand, a state $\rho$ is pure if and only if its rank equals to one. So pure states can be also characterized by their rank. In \cite{12} Sec.6.4, Klyachko raised the following conjecture:

Conjecture. State $\rho_{AB}$ with maximal lexicographical spectrum has minimal rank among all states with given margins $\rho_A, \rho_B$.

Let $C(\frac{1}{n}I_n, \frac{1}{m}I_m)$ denote the convex set of states with margins $\frac{1}{n}I_n, \frac{1}{m}I_m$, where $I_n, I_m$ are identity matrices of size $n$ and $m$. Since the spectra of $\frac{1}{n}I_n$ and $\frac{1}{m}I_m$ are uniform probability distributions, we call them uniform margins. Motivated by Klyachko’s conjecture, in this paper we study the maximal lexicographic spectrum and ranks of states in $C(\frac{1}{n}I_n, \frac{1}{m}I_m)$. We give counterexamples for Klyachko’s conjecture, and show that there exist states which have the maximal lexicographic spectrum, but they don’t have the minimal rank. Moreover, we discuss how to construct the states in $C(\frac{1}{n}I_n, \frac{1}{m}I_m)$ with prescribed ranks, which generalize the construction in \cite{3}. Our discussion is based on the correspondence between Kronecker coefficients and the spectra of density operators \cite{4, 5, 6, 12}.

The paper is organized as follows. In Section 2 we give the definitions and results used in the paper. In Section 3 we construct the maximal lexicographic spectrum of states in $C(\frac{1}{n}I_n, \frac{1}{m}I_m)$. We provide two classes of counterexamples for Klyachko’s conjecture and show that states with the maximal lexicographical spectrum are extreme points. In Section 4 we give the construction of states with prescribed ranks in $C(\frac{1}{n}I_n, \frac{1}{m}I_m)$.

2010 Mathematics Subject Classification. Primary 20C30; Secondary 15A18.

Key words and phrases. Quantum marginal problem, Maximal lexicographic spectrum, Rank, Rectangular Kronecker coefficients, Generalized discrete Weyl operators.

The research is supported by National Natural Science Foundation of China (Grant No.11626211).


2. Preliminaries

2.1. Partitions and Kronecker coefficients. A partition \( \lambda \) of \( n \in \mathbb{N} \) is a monotonically decreasing sequence \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) of natural numbers such that \( \sum_{i=1}^{k} \lambda_i = n \) and denoted by \( \lambda \vdash n \). The length \( l(\lambda) \) of \( \lambda \) is defined as the number of its nonzero parts and its size as \( |\lambda| := \sum_{i=1}^{k} \lambda_i \). If \( \lambda_1 = \lambda_2 = \cdots = \lambda_k \), we call \( \lambda \) a rectangular partition. The normalization \( \overline{\lambda} := \lambda/n = (\lambda_1/n, \lambda_2/n, \ldots, \lambda_k/n) \) defines a probability distribution on \( \mathbb{N} \).

The Young diagram of a partition \( \lambda \) is a top-aligned and left-aligned array of boxes such that in row \( i \) we have \( \lambda_i \) boxes. If we transpose a Young diagram at the main diagonal we obtain another Young diagram, the corresponding partition is denoted by \( \lambda' \). For \( \ell \in \mathbb{N} \), we let \( \ell \lambda \) stand for the partition arising by multiplying all components of \( \lambda \) by \( \ell \). If \( \mu = (\mu_1, \mu_2, \ldots) \) is another partition, we denote by \( \lambda \cap \mu = (\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2), \ldots) \) which is also a partition.

Let \( \chi^\lambda, \chi^\mu \) denote the complex irreducible characters of the symmetric group \( S_n \) corresponding to the partitions \( \lambda, \mu \) of \( n \). Their Kronecker product \( \chi^\lambda \otimes \chi^\mu \) is also a character of \( S_n \). The Kronecker coefficient \( g(\lambda, \mu; \nu) \) associated with three partitions \( \lambda, \mu, \nu \) of \( n \) is defined as the multiplicity of \( \chi^\nu \) in \( \chi^\lambda \otimes \chi^\mu \), that is, the coefficient of \( \chi^\nu \) in the expansion

\[
\chi^\lambda \otimes \chi^\mu = \sum_{\nu \in \Phi} c(\lambda, \mu; \nu) \chi^\nu.
\]

In above, all partitions corresponding to the set of nonzero Kronecker coefficients is denoted by

\[
\Phi(\lambda, \mu) = \{ \nu \mid g(\lambda, \mu; \nu) \neq 0 \}.
\]

Kronecker coefficients are only understood in some special cases. It is a difficult open problem to give a combinatorial interpretation of the numbers \( g(\lambda, \mu; \nu) \) \[12\] \[17\].

Let \( \lambda \vdash n, \mu \vdash m, \nu \vdash n+m \) and \( \chi^\lambda \otimes \chi^\mu \) be the outer product of \( \chi^\lambda \) and \( \chi^\mu \). The Littlewood-Richardson coefficient \( c^\nu_{\lambda\mu} \) is the multiplicity of \( \chi^\nu \) in \( \chi^\lambda \otimes \chi^\mu \). There is an efficient algorithm for calculation \( c^\nu_{\lambda\mu} \), known as Littlewood-Richardson rule, see \[11\] \[17\] for details. By the semigroup property of Littlewood-Richardson coefficients (see for example \[5\]), we have that if \( c^\nu_{\lambda\mu} > 0 \) then \( c^\nu_{\ell\lambda\ell\mu} > 0 \) for all \( \ell > 0 \).

2.2. Spectra of quantum states and their orders. Let \( \mathcal{H} \) be a \( d \)-dimensional complex Hilbert space and denote by \( \mathcal{L}(\mathcal{H}) \) the space of linear operators mapping \( \mathcal{H} \) into itself. A positive semidefinite operator \( \rho \in \mathcal{L}(\mathcal{H}) \) is called a density operator if \( Tr(\rho) = 1 \). Denote the set of density operators in \( \mathcal{L}(\mathcal{H}) \) by \( D(\mathcal{H}) \). Density operators are the mathematical formalism to describe the states of quantum objects. Denote the spectrum of \( \rho \) by \( \text{Spec} \rho \), it will always be understood as the vector \( (r_1, \ldots, r_d) \) of eigenvalues of \( \rho \) in decreasing order, that is, \( r_1 \geq \ldots \geq r_d \). The rank of a density operator \( \rho \) is denoted by \( \text{rank} \rho \).

Suppose that \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_d) \) are spectra of two quantum states or partitions of some \( n \in \mathbb{N} \). Recall that \( \lambda \) is less than \( \mu \) in lexicographic order if, for some index \( i \),

\[
\lambda_j = \mu_j \text{ for } j < i \text{ and } \lambda_i < \mu_i,
\]

which is denoted by \( \lambda \leq \mu \). On the other hand, \( \lambda \) is less than \( \mu \) in dominance order (or \( \lambda \) is majorized by \( \mu \)) if

\[
\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \text{ for all } k \geq 1,
\]

which is denoted by \( \lambda \preceq \mu \). It is not hard to see that if \( \lambda \preceq \mu \) then we have \( \lambda \leq \mu \), that is, lexicographic order is a refinement of the dominance order \[16\].
The state of a system composed of particles $A$ and $B$ is described by a density operator on a tensor product of two Hilbert spaces, $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$. The partial trace $\rho_A = tr_B(\rho_{AB}) \in D(\mathcal{H}_A)$ of $\rho_{AB}$ obtained by tracing over $B$ then defines the state of particle $A$. Similarly, $\rho_B = tr_A(\rho_{AB})$ is obtained by tracing out the subsystem $A$. In this way, $\rho_A, \rho_B$ are called marginal states (or margins) of $\rho_{AB}$ [12]. For any two density operators $\rho_A \in D(\mathcal{H}_A)$, $\rho_B \in D(\mathcal{H}_B)$, the set of states in $D(\mathcal{H}_A \otimes \mathcal{H}_B)$ with margins $\rho_A$ and $\rho_B$ is defined as

$$C(\rho_A, \rho_B) := \{\rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)| tr_B(\rho) = \rho_A, \ tr_A(\rho) = \rho_B\}.$$ 

The set of spectra of states in $C(\rho_A, \rho_B)$ is defined as

$$S(\rho_A, \rho_B) := \{S \text{ pec } \rho| \rho \in C(\rho_A, \rho_B)\}.$$ 

It was shown in [4,6,12] that $S(\rho_A, \rho_B)$ is a convex polytope. Hence, Klyachko’s conjecture states that if the spectrum of a state in $C(\rho_A, \rho_B)$ has maximal lexicographic order in $S(\rho_A, \rho_B)$, then it has minimal rank among all other states in $C(\rho_A, \rho_B)$.

2.3. The spectra and nonzero Kronecker coefficients. Given a description of the set of possible triples of spectra $(S \text{ pec } \rho_{AB}, S \text{ pec } \rho_A, S \text{ pec } \rho_B)$ for fixed $d_A = \dim \mathcal{H}_A$ and $d_B = \dim \mathcal{H}_B$ is fundamental in quantum marginal problems.

It turns out that the admissible spectral triples correspond to nonzero Kronecker coefficients. It was shown in [12] (see also [4,6]) that for a density operator $\rho_{AB}$ with the rational spectral triple $(S \text{ pec } \rho_A, S \text{ pec } \rho_B, S \text{ pec } \rho_{AB}) = (r_A, r_B, r_{AB})$ there is an integer $m > 0$ such that $g(mr_A, mr_B, mr_{AB}) \neq 0$. Conversely, suppose that $\lambda, \mu, \nu \vdash n$ are partitions with lengths $l(\lambda) \leq m, l(\mu) \leq n, l(\nu) \leq nm$. In [6] the authors showed that if $g(\lambda, \mu; \nu) \neq 0$ then there exists a density operator $\rho_{AB}$ on $\mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^m \otimes \mathbb{C}^n$ with spectra $S \text{ pec } \rho_A = \lambda, S \text{ pec } \rho_B = \mu, S \text{ pec } \rho_{AB} = \nu$. Hence the length of $\nu$ is the rank of $\rho_{AB}$.

3. The maximal lexicographic spectrum of $C(\frac{1}{n}I_n, \frac{1}{m}I_m)$ and counterexamples for Klyachko’s conjecture

In this section, through the correspondence between the spectra and nonzero Kronecker coefficients we will find the maximal lexicographic spectrum for states in $C(\frac{1}{n}I_n, \frac{1}{m}I_m)$ and give two classes of counterexamples for Klyachko’s conjecture. Moreover, we discuss their extremity. The following proposition is well-known (see e.g. [10]).

Proposition 3.1 (Transposition property). Suppose that $\lambda, \mu, \nu \vdash n$. Then we have $g(\lambda, \mu; \nu) = g(\lambda, \mu'; \nu')$.

By the discussion in Section 6.4 of [12] we have the following proposition which is also well-known. It gives a lower bound for ranks of states in $\rho \in C(\rho_A, \rho_B)$. In many cases the lower bound is best, see Remark [3,4].

Proposition 3.2. Let $\rho \in C(\rho_A, \rho_B)$ and denote $k_A = \text{rank } \rho_A$, $k_B = \text{rank } \rho_B$. Suppose that $k_A \leq k_B$. Then $\lceil \frac{k_A}{k_B} \rceil \leq \text{rank } \rho \leq k_A k_B$.

Proposition 3.3. Suppose that $\lambda = S \text{ pec } \rho_A, \mu = S \text{ pec } \rho_B$ are rational spectra and $n_{AB}$ the minimal positive integer such that $\lambda = n_{AB}\lambda$ and $\mu = n_{AB}\mu$ are partitions. Then $S(\rho_A, \rho_B)$ is the closure of $\frac{1}{n_{AB}}\Phi(\ell \lambda, \ell \mu)$ for $\ell \geq 1$, that is,

$$S(\rho_A, \rho_B) = \bigcup_{\ell=1}^{\infty} \frac{1}{n_{AB}}\Phi(\ell \lambda, \ell \mu).$$
Proof. Let $QS(\rho_A, \rho_B)$ be the set of rational spectra in $S(\rho_A, \rho_B)$. It suffices to show that

$$QS(\rho_A, \rho_B) = \bigcup_{\ell=1}^{\infty} \frac{1}{\ell n_{AB}} \Phi(\ell \lambda, \ell \mu).$$

(3.1)

By Theorem 2.3 of [6], for any $\tilde{\nu} \in QS(\rho_A, \rho_B)$ there exists an integer $m$ such that $m\tilde{\nu}$, $m\bar{\lambda}$ and $m\bar{\mu}$ are partitions and

$$g(m\bar{\lambda}, m\bar{\mu}; m\tilde{\nu}) \neq 0.$$  

(3.2)

Since $\bar{\lambda}$, $\bar{\mu}$ consist of rational numbers, let

$$\bar{\lambda} = (a_1/b_1, a_2/b_2, \ldots, a_s/b_s)$$

and

$$\bar{\mu} = (c_1/d_1, c_2/d_2, \ldots, c_t/d_t)$$

where $a_i$ and $b_i$ $(i = 1, 2, \ldots, s)$, $c_j$ and $d_j$ $(j = 1, 2, \ldots, t)$ are integers and relatively prime. Then we have that $n_{AB}$ (resp. $m$) is the least common multiple (resp. the common multiple) of $\{b_1, b_2, \ldots, b_s, d_1, d_2, \ldots, d_t\}$. Hence we have $n_{AB} | m$. Let $k = n_{AB} | m$, then (3.2) is equivalent to

$$\tilde{\nu} \in \frac{1}{kn_{AB}} \Phi(k\bar{\lambda}, k\bar{\mu}).$$

Thus we have that $QS(\rho_A, \rho_B) \subseteq \bigcup_{\ell=1}^{\infty} \frac{1}{\ell n_{AB}} \Phi(\ell \lambda, \ell \mu)$, and therefore (3.1) holds by Theorem 3.2 of [6].

□

Given $\lambda$ and $\mu$ partitions such that $\mu_i \leq \lambda_i$ for all $i \geq 1$, we write $\mu \in \lambda$ (or $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for some $i$). In [19] (see also [7]), the author introduced a construction, which can be used to obtain the maximal component, in the lexicographic order in $\chi^\lambda \otimes \chi^\mu$. The construction is as follows.

Let $\lambda, \mu$ be partitions of $n$, together with two strictly decreasing sequences of partitions

$$\lambda = \lambda(1) \supset \cdots \supset \lambda(r) \supset \lambda(r+1) = \emptyset,$$

$$\mu = \mu(1) \supset \cdots \supset \mu(r) \supset \mu(r+1) = \emptyset,$$

(3.3)

such that

$$c_{\lambda(i) \cap \mu(i)} \neq 0$$

and

$$c_{\lambda(i+1) \cap \mu(i+1)} \neq 0,$$

for all $1 \leq i \leq r$. Set

$$v_i = |\lambda(i) \cap \mu(i)|$$

for all $1 \leq i \leq r$. Then $v = (v_1, \ldots, v_r)$ is a partition of $n$. Any $v$ obtained in this way is called a partition of strip type derived from $(\lambda, \mu)$ [19]. For example, if we let $\lambda = (2^5)$, $\mu = (5^2)$ and $v = (4, 4, 1, 1)$, then $v$ is a partition of strip type derived from $(\lambda, \mu)$. The corresponding sequences of partitions are

$$(2^5) \supset (3^2) \supset (2) \supset (1) \supset \emptyset,$$

$$(5^2) \supset (3^2) \supset (1^2) \supset (1) \supset \emptyset.$$  

Clausen and Meier showed that the maximal component $\chi^v$ of $\chi^\lambda \otimes \chi^\mu$ in the lexicographic order corresponds to a derived partition of strip type [7].

Observe that $\lambda \cap \mu \not\subseteq \lambda$. In the Young diagram of $\lambda$ we let $\lambda \setminus \lambda \cap \mu$ denote boxes which belong to $\lambda$ but not $\lambda \cap \mu$ (similarly for $\mu \setminus \lambda \cap \mu$). It is called skew diagram in [19] (see also [11]). $\lambda \setminus \lambda \cap \mu$ may correspond to a partition. For example, if we let $\lambda = (2^5)$ and $\mu = (5^2)$, then $\lambda \cap \mu = (2, 2)$ and $\lambda \setminus \lambda \cap \mu = (2^3)$ which is also a partition.

**Proposition 3.4.** Suppose that $\lambda$, $\mu \vdash n$ are two rectangular partitions. Then there is exactly one partition of strip type derived from $(\lambda, \mu)$ which has the maximal lexicographic order in $\Phi(\lambda, \mu)$. 

Proof. Since λ and μ are two rectangular partitions, by Lemma 3.3 of [1] there exists only one pair of partitions λ(2), μ(2) such that \( c_{\lambda(2),\mu(2)}^{(i)} \neq 0 \) and \( c_{\mu(2),\lambda(2)}^{(i)} \neq 0 \) where λ(2) = λ \( \cap \mu \) and μ(2) = μ \( \cap \lambda \) are also rectangular partitions. Similarly, if we continue the construction described in (3.3), for each 1 ≤ i ≤ r there exists only one pair of rectangular partitions \( \lambda(i+1), \mu(i+1) \) such that \( c_{\lambda(i+1),\mu(i+1)}^{(i)} \neq 0 \) and \( c_{\mu(i+1),\lambda(i+1)}^{(i)} \neq 0 \). Hence, there is exactly one partition of strip type derived from \( (\lambda, \mu) \) denoted by \( \nu \). By Theorem 3.5 of [19] we have that \( \nu \in \Phi(\lambda, \mu) \). Since the maximal component of \( \chi^\lambda \otimes \chi^\mu \) in the lexicographic order corresponds to a derived partition of strip type, by uniqueness we have that \( \nu \) has the maximal lexicographic order in \( \Phi(\lambda, \mu) \) [7, 19].

The following proposition can be obtained from (6.10) of [12] which gives us the value of Kronecker coefficient in Proposition 3.4.

Proposition 3.5. Suppose that \( \lambda, \mu \vdash n \) are two rectangular partitions. Let \( \nu \) be the partition of strip type derived from \( (\lambda, \mu) \). Then \( g(\lambda, \mu; \nu) = 1 \).

In the following, we let \( \text{lcm}(n, m) \) denote the least common multiple of \( n \) and \( m \).

Theorem 3.6. For \( n \leq m \), let \( \lambda = (a^n), \mu = (b^m) \vdash k \) be two rectangular partitions, where \( k = \text{lcm}(n, m) \), \( a = n \mid k \) and \( b = m \mid k \). Suppose that \( \nu \) is the partition of strip type derived from \( (\lambda, \mu) \). Then the maximal lexicographic spectrum for states in \( C(\frac{1}{n}I_n, \frac{1}{m}I_m) \) is \( \nu/k \).

Proof. It is equivalent to show that \( \nu/k \) is maximal in the lexicographic order in \( S(\frac{1}{n}I_n, \frac{1}{m}I_m) \). For \( k = \text{lcm}(n, m) \), it is the minimal integer such that both \( a = k/n \) and \( b = k/m \) are integers. For \( \lambda = (a^n), \mu = (b^m) \vdash k \), by Proposition 3.3 we have that

\[
S(\frac{1}{n}I_n, \frac{1}{m}I_m) = \bigcup_{\ell=1}^{\infty} \frac{1}{\ell k} \Phi(\ell \lambda, \ell \mu).
\]

Let \( \nu^{(1)} = \nu \) and \( \nu^{(2)} \) be the partition of strip type derived from \( (\ell \lambda, \ell \mu) \) for all \( \ell \geq 1 \). By (3.3), let \( \nu^{(1)}(1) = (v_1^{(1)}, v_2^{(1)}, \ldots, v_r^{(1)}) \) be derived from the following two strictly decreasing sequences of partitions

\[
\lambda = \lambda(1) \supset \cdots \supset \lambda(r) \supset \lambda(r+1) = \emptyset, \\
\mu = \mu(1) \supset \cdots \supset \mu(r) \supset \mu(r+1) = \emptyset,
\]

(3.4)

such that

\[
c_{\lambda(i)\cap\mu(i),\lambda(i+1)}^{(i)} \neq 0 \quad \text{and} \quad c_{\mu(i)\cap\lambda(i),\mu(i+1)}^{(i)} \neq 0,
\]

(3.5)

and

\[
\nu^{(1)}_i = |\lambda(i) \cap \mu(i)|
\]

for \( 1 \leq i \leq r \). Then by (3.4), (3.5) and the semigroup property of Littlewood-Richardson coefficients, for all \( \ell \geq 1 \) we have

\[
\ell \lambda = \ell \lambda(1) \supset \cdots \supset \ell \lambda(r) \supset \ell \lambda(r+1) = \emptyset, \\
\ell \mu = \ell \mu(1) \supset \cdots \supset \ell \mu(r) \supset \ell \mu(r+1) = \emptyset,
\]

(3.6)

and

\[
c_{\ell \lambda(i)\cap\ell \mu(i),\ell \lambda(i+1)}^{(i)} \neq 0 \quad \text{and} \quad c_{\ell \mu(i)\cap\ell \lambda(i),\ell \mu(i+1)}^{(i)} \neq 0.
\]

(3.7)

By (3.6) and (3.7) we have that \( \nu^{(\ell)} (\ell \geq 1) \) are derived from \( (\ell \lambda, \ell \mu) \) where \( \ell \lambda, \ell \mu \) are still rectangular partitions. Moreover, we have

\[
\nu^{(\ell)} = (v_1^{(\ell)}, v_2^{(\ell)}, \ldots, v_r^{(\ell)}), \quad \text{where} \quad v_i^{(\ell)} = |\ell \lambda(i) \cap \ell \mu(i)| = \ell v_i^{(1)}
\]
for \( i = 1, 2, \ldots, r \). That is,
\[
y^{(l)} = \ell y^{(1)}. \tag{3.8}
\]

By Proposition 3.4 we have that \( y^{(l)} \) has maximal lexicographic order in \( \Phi(\lambda, \mu) \) for each \( \ell \geq 1 \). Thus \( y^{(l)}/\ell k \) has maximal lexicographic order in \( \frac{1}{\ell k} \Phi(\lambda, \ell \mu) \) for each \( \ell \geq 1 \). By (3.3) we have that all their normalizations \( y^{(l)}/\ell k \) are equal to \( y^{(1)}/k \). Hence \( y^{(1)}/k = y/k \) has the maximal lexicographic order in \( \bigcup_{\ell = 1}^{\infty} \frac{1}{\ell k} \Phi(\lambda, \ell \mu) \). Thus, by the density of rational spectra, we have that \( y/k \) has the maximal lexicographic order in \( \mathcal{S}(\frac{1}{m} I_n, \frac{1}{n} I_m) \).

3.1. Two classes of counterexamples for Klyachko’s conjecture. In the following two examples, we will show that there exist states \( \rho \in C(\frac{1}{m} I_n, \frac{1}{n} I_m) \) which have the maximal lexicographic spectrum, but they do not have the minimal rank.

Example 3.7. Let \( m \geq 3 \) be odd and write it as \( 2k + 1 \) for an integer \( k \). Suppose that \( \lambda = (m, m) \) and \( \mu = (2^m) \). Then it is not hard to see that the partition of strip type derived from \( (\lambda, \mu) \) is
\[
\nu = (4^k, 1, 1). \tag{3.9}
\]
Since \( \text{lcm}(2, m) = 2m \), by Theorem 3.6 we have \( \nu/2m \) has the maximal lexicographic order in \( \mathcal{S}(\frac{1}{m} I_2, \frac{1}{m} I_m) \). Thus, states with maximal lexicographic spectrum in \( C(\frac{1}{m} I_2, \frac{1}{m} I_m) \) have rank \( k + 2 \).

On the other hand, let \( \gamma = (4^{k-1}, 3, 3) \). By Theorem 1.6 of [18] and Proposition 3.1 we have that \( g(\lambda, \mu; \gamma) = g(\lambda, \mu; \gamma') = 1 \). Hence, \( \gamma = (4^{k-1}, 3, 3) \in \Phi(\lambda, \mu) \). Hence, there exist states in \( C(\frac{1}{2} I_2, \frac{1}{m} I_m) \) with rank \( k + 1 \). Moreover, by Proposition 3.2 we have that \( k + 1 \) is the minimal rank for \( C(\frac{1}{m} I_2, \frac{1}{m} I_m) \).

In the following example, we will find that their differences can be large.

Example 3.8. Let \( \lambda = ((n+1)^n), \mu = (n^{n+1}) \) and \( \nu \) be the partition of strip type derived from \( (\lambda, \mu) \). Then we have that \( \nu = (\nu_1, \nu_2, \ldots, \nu_{n+1}) \) where \( \nu_1 = n^2, \nu_2 = \cdots = \nu_{n+1} = 1 \). Then by Theorem 3.6 the maximal lexicographic spectrum of states in \( C(\frac{1}{n} I_n, \frac{1}{n+1} I_{n+1}) \) is
\[
\nu = \frac{n(n+1)}{n+1}.
\]
Hence the rank of those states are \( n + 1 \).

On the other hand, by Proposition 6.9 of [10], we have that \( g(\lambda, \mu; \gamma) = 1 \), where
\[
\gamma = \left(\frac{n(n+1)}{2}, \frac{n(n+1)}{2}\right)
\]
is a two row partition. Hence, there exist states with rank \( 2 \) in \( C(\frac{1}{n} I_n, \frac{1}{n+1} I_{n+1}) \). By Proposition 3.2 we have that the minimal rank of states in \( C(\frac{1}{n} I_n, \frac{1}{n+1} I_{n+1}) \) is \( 2 \).

3.2. On the extremity of states with maximal lexicographic spectrum. Let \( H \) be a Hermitian matrix. Denote the diagonal entries of \( H \) by \( D(H) \) which are arranged decreasingly. The well-known Schur Theorem states that \( D(H) \preceq \text{Spec } H \) [9, 15]. When \( D(H) = \text{Spec } H \), by Corollary 4.3.34 and Theorem 4.3.45 in [9] we have the following proposition.

Proposition 3.9. Let \( H \) be a Hermitian matrix. Then \( D(H) = \text{Spec } H \) if and only if \( H \) is diagonal.
Theorem 3.10. Let $\mathcal{D}(\mathcal{H})$ be the set of density operators on $\mathcal{H}$. Suppose that $\mathcal{C}(\mathcal{H}) \subseteq \mathcal{D}(\mathcal{H})$ is a convex subset. If $\rho \in \mathcal{C}(\mathcal{H})$ has the maximal lexicographic spectrum among all other states, then $\rho$ is an extreme point of $\mathcal{C}(\mathcal{H})$.

Proof. Suppose that there exist $\sigma, \tau \in \mathcal{C}(\mathcal{H})$ such that

$$\rho = p_1 \sigma + p_2 \tau,$$

where $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$. 

Denote $\lambda = Spec \rho$, $\mu = Spec \sigma$, $\eta = Spec \tau$. In the following, we don’t distinguish between the spectrum and the diagonal matrix with diagonal entries consist of it. Let $U$ be the unitary matrix such that $U^* \lambda U = \rho$. Then we have

$$\lambda = p_1 U^* \sigma U + p_2 U^* \tau.$$

Let $D(U^* \sigma U), D(U^* \tau)$ be the diagonal of $U^* \sigma U$ and $U^* \tau$. Then we have

$$\lambda = p_1 D(U^* \sigma U) + p_2 D(U^* \tau).$$

By Schur’s Theorem we have that $D(U^* \sigma U) \leq \mu$ and $D(U^* \tau) \leq \eta$. Moreover, since lexicographic order is a refinement of dominance order, we have $D(U^* \sigma U) \leq \mu$ and $D(U^* \tau) \leq \eta$. Since $\lambda$ has the maximal lexicographic order, we have

$$\lambda = p_1 D(U^* \sigma U) + p_2 D(U^* \tau) \leq p_1 \mu + p_2 \eta$$

$$\leq p_1 \lambda + p_2 \lambda$$

$$= \lambda.$$

Hence we have

$$\lambda = p_1 D(U^* \sigma U) + p_2 D(U^* \tau) = p_1 \mu + p_2 \eta.$$  \hspace{1cm} (3.10)

Since $D(U^* \sigma U) \leq \mu \leq \lambda$ and $D(U^* \tau) \leq \eta \leq \lambda$, by (3.10) we should have that

$$\lambda = \mu = D(U^* \sigma U) \quad \text{and} \quad \lambda = \eta = D(U^* \tau).$$

Hence by Proposition 3.9 we have

$$\lambda = \mu = U^* \sigma U \quad \text{and} \quad \lambda = \eta = U^* \tau,$$

which is equivalent to $\rho = \sigma = \tau$. \hfill \Box

Since $\mathcal{C}(\rho_A, \rho_B) \subseteq \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is convex, we have the following corollary.

Corollary 3.11. If $\rho \in \mathcal{C}(\rho_A, \rho_B)$ has maximal lexicographic spectrum, then it is an extreme point.

4. Ranks of states in $\mathcal{C}(\frac{1}{n} I_n, \frac{1}{m} I_m)$

For $0 < n \leq m$, write $m = np + r$ where $p \geq 1$ and $0 \leq r \leq n - 1$. In this section, we construct states with prescribed ranks in $\mathcal{C}(\frac{1}{n} I_n, \frac{1}{m} I_m)$ which generalizes the construction in [3].

Suppose that $n \leq m$. Let $|0\rangle, \ldots, |m-1\rangle$ denote the standard orthonormal basis of $\mathbb{C}^m$.

We define the generalized discrete Weyl operators $X, Z_n \in \mathcal{L}(\mathbb{C}^m)$ by

$$X|i\rangle = |i+1\rangle; \quad Z_n|i\rangle = \omega^n |i\rangle,$$

where $\omega^n = 1, i = 0, 1, \ldots, m - 1$ and the addition is modulo $m$. If $n = m$, these are called the discrete Weyl operators [3].

For $n \leq m$, the maximal entangled state of $\mathbb{C}^n \otimes \mathbb{C}^m$ are defined by

$$|\psi_{(n)}\rangle := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i\rangle|i\rangle.$$
Theorem 4.2. Suppose that \( n \leq m \). Then \( \rho \in C(\frac{1}{n} I_n, \frac{1}{m} I_m) \) with rank \( k \).

Proof. Suppose that \( \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \lambda_{i,j} = 1 \) and \( \lambda_{i,j} \geq 0 \) for \( i = 0, 1, \ldots, m-1 \) and \( j = 0, 1, \ldots, n-1 \). Let

\[
\Lambda = \begin{pmatrix}
\lambda_{0,0} & \lambda_{0,1} & \cdots & \lambda_{0,n-1} \\
\lambda_{1,0} & \lambda_{1,1} & \cdots & \lambda_{1,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{m-1,0} & \lambda_{m-1,1} & \cdots & \lambda_{m-1,n-1}
\end{pmatrix}.
\]

(4.1)

Denote the row vector of \( \Lambda \) by \( \text{row}(\Lambda) = (\mu_0, \mu_1, \ldots, \mu_{m-1}) \) where \( \mu_i = \sum_{j=0}^{n-1} \lambda_{i,j} \) for \( i = 0, 1, \ldots, m-1 \).

For \( |\psi_{ij}\rangle \) discussed in Proposition 4.1, let

\[
\rho = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \lambda_{i,j} |\psi_{ij}\rangle \langle \psi_{ij}|.
\]

Then \( \rho \) is a state of the system \( \mathbb{C}^n \otimes \mathbb{C}^m \). Next, we find conditions when \( \rho \in C(\frac{1}{n} I_n, \frac{1}{m} I_m) \) with rank \( k \) for \( m \leq k \leq mn \).

For \( |\psi_{ij}\rangle \), we have

\[
\text{tr}_B(\langle \psi_{ij}| \psi_{ij}\rangle) = \text{tr}_B \left( \frac{1}{n} \sum_{s,s'=0}^{n-1} |s\rangle \langle s'| \otimes X^i Z^j_s \langle s'| \otimes Z^{-i}_s X^{-j} \right)
\]

\[
= \frac{1}{n} I_n.
\]

Hence we have

\[
\text{tr}_B(\rho) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \lambda_{i,j} \text{tr}_B(\langle \psi_{ij}| \psi_{ij}\rangle) = \frac{1}{n} I_n.
\]

(4.2)
Since \( Z^j_n(s) = \omega^j(s) \), we have
\[
Z^j_n(s)\langle s| Z^{-j}_n = \omega^j \omega^{-j}(s)\langle s| = |s\rangle\langle s|.
\]

Then we have
\[
tr_A(\langle \psi_{ij}\rangle \langle \psi_{ij}\rangle) = tr_A \left( \frac{1}{n} \sum_{i,j=0}^{n-1} |s\rangle\langle s'| \otimes X^i Z^j_n(s) s'(s') Z^{-j}_n X^{-i} \right)
\]
\[= \frac{1}{n} \sum_{i,j=0}^{n-1} X^i Z^j_n(s) s'(s') Z^{-j}_n X^{-i}
\]
\[= \frac{1}{n} X^i \left( \sum_{j=0}^{n-1} |s\rangle\langle s'| \right) X^{-i}.
\]

Let \( P_{ij} = tr_A(\langle \psi_{ij}\rangle \langle \psi_{ij}\rangle) \). Then for \( 0 \leq i \leq m-1 \) we have \( P_{i,0} = P_{i,1} = \cdots = P_{i,n-1} \) which denoted by \( P_i \). Hence we have
\[
tr_A(\rho) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \lambda_{ij} tr_A(\langle \psi_{ij}\rangle \langle \psi_{ij}\rangle) = \left( \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \lambda_{ij} P_{ij} \right)
\]
\[= \sum_{i=0}^{m-1} P_i \left( \sum_{j=0}^{n-1} \lambda_{ij} \right)
\]
\[= \sum_{i=0}^{m-1} \mu_i P_i,
\]
where \( \mu_i = \sum_{j=0}^{n-1} \lambda_{ij} \) is the sum of row \( i \) of \( \Lambda \). Note that \( \sum_{i=0}^{m-1} P_i = I_m \). If we let \( \mu_i = \frac{1}{m} \) for \( i = 0, 1, \ldots, m-1 \), then we have
\[
tr_A(\rho) = \sum_{i=0}^{m-1} \mu_i P_i = \frac{1}{m} \sum_{i=0}^{m-1} P_i = \frac{1}{m} I_m.
\] (4.3)

By (4.2) and (4.3) we have that if \( row(\Lambda) = (\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}) \), then \( \rho \in C(\frac{1}{n} I_n, \frac{1}{m} I_m) \) where \( \Lambda \) is defined in (4.1). Since the number of nonzero entries of \( \Lambda \) is the rank of \( \rho \), for \( m \leq k \leq mn \) it is not hard to find \( k \) nonzero entries such that \( row(\Lambda) = (\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}) \). \( \square \)

**Theorem 4.3.** Suppose that \( n \leq m \) and \( nm \). Then for each \( \frac{m}{n} \leq k \leq m \) there exist states \( \rho \in C(\frac{1}{n} I_n, \frac{1}{m} I_m) \) with rank \( \rho = k \).

**Proof.** Let \( p = nm \). Suppose that \( \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \tau_{i,j} = 1 \) and \( \tau_{i,j} \geq 0 \) for \( i = 0, 1, \ldots, p-1 \) and \( j = 0, 1, \ldots, n-1 \). Let
\[
T = \begin{pmatrix}
\tau_{0,0} & \tau_{0,1} & \cdots & \tau_{0,n-1} \\
\tau_{1,0} & \tau_{1,1} & \cdots & \tau_{1,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\tau_{p-1,0} & \tau_{p-1,1} & \cdots & \tau_{p-1,n-1}
\end{pmatrix}.
\] (4.4)

Denote the row vector of \( T \) by \( row(T) = (v_0, v_1, \ldots, v_{p-1}) \) where \( v_i = \sum_{j=0}^{n-1} \tau_{i,j} \) for \( i = 0, 1, \ldots, p-1 \).
Thus the maximal lexicographic spectrum for states in $\mathbb{C}$ attainable. Combined with Theorem 4.2, if $n$ with ranks from 2 to $T$ where $k$ is defined in (4.4). Since the number of nonzero entries of $\rho$, for $p \leq k \leq m$ it is not hard to find $k$ nonzero entries such that $\text{rank}(\rho) = \frac{m}{p}$.

Remark 4.4. From Theorem 4.3 we can see that the lower bound of Proposition 3.2 is attainable. Combined with Theorem 4.2 if $\frac{n}{m}$ we have that for each $\frac{n}{m} \leq k \leq m$ there exists $\rho \in C(\frac{1}{n}I_n, \frac{1}{m}I_m)$ where $T$ is defined in (4.4). Since the number of nonzero entries of $T$ is the rank of $\rho$, for $p \leq k \leq m$ it is not hard to find $k$ nonzero entries such that $\text{rank} = (\frac{1}{n}, \frac{1}{p}, \ldots, \frac{1}{p})$.

For $n \leq m$, by the discussion in Example 4.3 we have that if $m = n + 1$ then there exist states in $C(\frac{1}{n}I_n, \frac{1}{m}I_m)$ with rank $\frac{n+1}{2} \leq 2$. Thus, the lower bound of Proposition 3.2 is also attainable. By Theorem 2.2 of [13] we have that there exist states in $C(\frac{1}{n}I_n, \frac{1}{m}I_m)$ with ranks from 2 to $mn$. When $n \leq m$ it is interesting to give the construction of states with ranks from $\frac{mn}{n}$ to $mn$. Recently, in [2] the authors discussed the construction of locally maximally entangled state of multipart quantum systems. By their results, we can decide whether there exist states with spectra $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ in $C(\frac{1}{m}I_m, \frac{1}{n}I_n)$ where $1 \leq k \leq mn$. When $k = 2$, they gave an explicit construction of such states.

Suppose that $\frac{nm}{n}$ and $p = \frac{m}{n}$. Then in Theorem 3.3 we have $k = m, a = p$ and $b = 1$. If $\nu$ is the partition of strip type derived from $\lambda = (p^n)$ and $\mu = (1^m)$, then we have $\nu = (n^p)$. Thus the maximal lexicographic spectrum for states in $C(\frac{1}{m}I_m, \frac{1}{n}I_n)$ is $\nu/m = (\frac{1}{p}, \frac{1}{p}, \ldots, \frac{1}{p})$. So the rank of these states is $p$. Comparing with Proposition 4.2 when $\frac{nm}{n}$ we can see that the rank of states with maximal lexicographic spectrum is minimal in $C(\frac{1}{m}I_m, \frac{1}{n}I_n)$. Thus if
If \( n \mid m \) then Klyachko’s conjecture is true for states in \( C(\frac{1}{n} I_n, \frac{1}{m} I_m) \). In Theorem 4.3 we give a construction of such states.

The geometric complexity theory program is an approach to separate algebraic complexity classes. Rectangular Kronecker coefficients play an important role in geometric complexity theory \([3, 10]\). For example, it can be used to prove the lower bounds of determinantal complexity. By the construction in Theorem 4.2 and 4.3 and the proof of main results in \([3]\), we can get nonzero stretched Kronecker coefficients for a pair of different rectangular partitions. For example, just as Theorem 1 in \([3]\) we have the following corollary.

**Corollary 4.5.** Suppose that \( n \mid m \), \( p = \frac{m}{n} \) and \( a \in \mathbb{N} \). Let \( \lambda = (a^m) \) and \( \mu = ((pa)^n) \). For each partition \( \nu \vdash ma \) if there exists a \( p \times n \) nonnegative matrix \( A \) with constant row sum \( na \) such that its nonzero entries consist of all parts of \( \nu \), then there exists a stretching factor \( k \in \mathbb{N} \) such that \( g(k \lambda, k \mu; k \nu) \neq 0 \).

**Acknowledgments**

We would like to thank the referee for many helpful comments and suggestions.

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