Relativistic-invariant quantum entanglement between the spins of moving bodies

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The entanglement between spins of a pair of particles may change because the spin and momentum become mixed when viewed by a moving observer [R.M. Gingrich and C. Adami, Phys. Rev. Lett. 89, 270402 (2002)]. In this paper, it is shown that, if the momenta are appropriately entangled, the entanglement between the spins of the Bell states can remain maximal when viewed by any moving observer. Further, we suggest a relativistic-invariant protocol for quantum communication, with which the non-relativistic quantum information theory could be invariantly applied to relativistic situations.

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I. INTRODUCTION

The relativistic thermodynamics has been an intriguing problem for decades [1]. It has been shown that probability distributions can depend on the frames, and thus the entropy and information may change if viewed from different frames [2]. Recently, the effect of Lorentz boosts on quantum states, quantum entanglement and quantum information has drawn particular interests [2, 3, 4, 5, 6]. The relativistic quantum information theory may be invariant to relativistic situations. Though, in this paper, we restrict ourselves to spin-1/2 cases, the generalization to larger spins could be done analogously. Particularly, the generalization to spin-1 massless particles, such as photons [6], may be of special interests since current experiments for quantum communications are mostly based on photons.

II. ENTANGLEMENT BETWEEN THE SPINS, WITH THE PRESENCE OF MOMENTUM ENTANGLEMENT

We start by investigating the bipartite state that, in the momentum representation, has the following form viewed from the rest frame,

\[ \Psi(p, q) = g(p, q) |\psi^-(\theta)\rangle, \]

where \( p \) and \( q \) are the momenta for the first and second particles, respectively (For review of the definition of the momentum eigenstates for massive particles with spin and the transformation under Lorentz boosts, one may refer to Refs. [2, 3, 4, 5]). The spin part of the state is the singlet Bell state

\[ |\psi^-(\theta)\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \]

with

\[ |\uparrow\rangle = |0\rangle \otimes |\uparrow\rangle, \quad |\downarrow\rangle = |\downarrow\rangle \otimes |\uparrow\rangle, \]

\[ |\uparrow\downarrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

The momentum distribution \( g(p, q) \) is normalized according to

\[ \int |g(p, q)|^2 \tilde{p}\tilde{q} = 1, \]

where \( \tilde{p} \) (\( \tilde{q} \)) is the Lorentz-invariant momentum integration measures given by (We use natural units: \( c = 1 \).)
$$\tilde{d}p = \frac{d^3p}{2\sqrt{p^2 + m^2}}$$  \hspace{1cm} (5)$$

Note that there is no entanglement between the spin and the momentum parts of $\Psi(p,q)$. The spins are maximally entangled, while the entanglement between momenta depends on $g(p,q)$. In what follows, we use $p$ to represent the momentum 4-vector as in Eq. (7) unless it is ambiguous.

To an observer in a frame Lorentz transformed by $\Lambda^{-1}$, the state $\Psi(p,q)$ appears to be transformed by $\Lambda \otimes \Lambda$. Therefore the state viewed by this observer appears to be

$$\Psi'(p,q) = U(\Lambda \otimes \Lambda) \Psi(p,q) = [U_{\Lambda^{-1}p} \otimes U_{\Lambda^{-1}q}]\Psi(\Lambda^{-1}p,\Lambda^{-1}q), \hspace{1cm} (6)$$

where $U(\Lambda \otimes \Lambda)$ represents the unitary transformation induced by the Lorentz transformation. Here, for compactness of notation, we define $U_p \equiv D^{(1/2)}(R(\Lambda,p))$ as the spin-1/2 representation of the Wigner rotation $R(\Lambda,p)$ \cite{13}. Because $\Psi'(p,q)$ differs from $\Psi(p,q)$ by only local unitary transformations, the entanglement will not change provided we do not trace out a part of the state. However, in looking at the entanglement between the spins, tracing out over the momentum degrees of freedom is implied. In $\Psi'(p,q)$ the spin and momentum may appear to be entangled, therefore the entanglement between the spins may change when viewed by the Lorentz-transformed observer. By writing $\Psi'(p,q)$ as a density matrix and tracing over the momentum degrees of freedom, the entanglement between the spins (viewed by the Lorentz-transformed observer) could be obtained by calculating the Wootters’ concurrence \cite{14} of the reduced density matrix for spins.

Any Lorentz transformation could be written as a rotation followed by a boost \cite{13}, and tracing over the momentum after a rotation will not change the spin concurrence \cite{14}, therefore we can look only at pure boosts. Without loss of generality we may choose boosts in the $z$-direction and write the momentum 4-vector in polar coordinates as

$$p = (E_p,p \cos \varphi_p \sin \theta_p,p \sin \varphi_p \sin \theta_p,p \cos \theta_p), \hspace{1cm} (7)$$

with $E_p = \sqrt{p^2 + m^2}$, $0 \leq \theta_p \leq \pi$ and $0 \leq \varphi_p < 2\pi$. Let $\Lambda = L(\xi)$ be the boost along the $z$-direction (as defined in Ref. \cite{14}), where $\xi$ is the rapidity of the boost and let $\xi = |\xi|$. With Eq. (7), we obtain

$$U_p = \begin{pmatrix} \alpha_p & \beta_p e^{-i\varphi_p} \\ -\beta_p e^{i\varphi_p} & \alpha_p \end{pmatrix}, \hspace{1cm} (8)$$

where

$$\alpha_p = \sqrt{E_p + m} \left(\cosh \frac{\xi}{2} + \frac{p \cos \theta_p}{E_p + m} \sinh \frac{\xi}{2}\right), \hspace{1cm} (9)$$

$$\beta_p = \frac{p \sin \theta_p}{\sqrt{(E_p + m)(E_p + m)}} \sinh \frac{\xi}{2} \left. \right| \hspace{1cm} (10)$$

and $E'_p = E_p \cosh \xi + p \cos \theta_p \sinh \xi$. The similar is for the second particle with momentum $q$. Substituting Eq. (8) into Eq. (6), we obtain the state viewed by the Lorentz-boosted observer as

$$\Psi'(\Lambda p, \Lambda q) = \frac{g(p,q)}{\sqrt{2}} \begin{pmatrix} \alpha_p \beta_q e^{-i\varphi_q} - \alpha_q \beta_p e^{-i\varphi_p} \\ \alpha_p \beta_q + \beta_p \beta_q e^{-i(\varphi_p - \varphi_q)} - \alpha_q \beta_p e^{-i\varphi_p} - \alpha_p \beta_q e^{-i\varphi_q} \end{pmatrix}. \hspace{1cm} (11)$$

At the present stage, we use an “entangled Gaussian” with width $\sigma$ for the momentum distribution, as follows,

$$g(p,q) = \sqrt{\frac{1}{N} \exp \left[-\frac{p^2 + q^2}{4\sigma^2}\right]} \exp \left[-\frac{p^2 + q^2 - 2xp \cdot q}{4\sigma^2(1 - x^2)}\right], \hspace{1cm} (12)$$

where $x \in [0, 1]$ and $N$ is the normalization. In Eq. (12), for a given $\sigma$, $x$ could be reasonably regard as a measure of the entanglement between momenta. When $x = 0$, the momentum part of the state is separable, i.e. the momentum entanglement is zero. However at the limit $x \to 1$, we have

$$\lim_{x \to 1} g(p,q) = \sqrt{\frac{1}{N'} \exp \left[-\frac{p^2}{2\sigma^2}\right]} \delta^3(p - q), \hspace{1cm} (13)$$

where $N'$ is the normalization. Eq. (13) indicates a perfect correlation between the momenta. Note that in Eq. (13) the momenta are not necessarily maximally entangled.

By integrating over the momenta, we obtain the reduced density matrix, viewed by the Lorentz-boosted observer, as

$$\rho = \int \Psi'(p,q) \Psi'(p,q)^\dagger \tilde{d}p \tilde{d}q. \hspace{1cm} (14)$$

The entanglement between the spins viewed by the Lorentz-boosted observer is obtained by calculating the Wootters’ concurrence \cite{14}, denoted as $C(\rho)$. The change in the Lorentz-transformed concurrence $C(\rho)$ depends on $\sigma/m$, $x$, and $\xi$. Fig. \cite{14} shows the concurrence as a function of rapidity $\xi$, for different values of $\sigma/m$ and $x$. Similar to Ref. \cite{14}, the decrease from the maximum value ($C(\rho) = 1$ for Bell states) documents the boost-induced decoherence of the spin entanglement \cite{14}. However, it is interesting to see that for fixed $\sigma/m$ and $\xi$, the concurrence decreases less for non-zero $x$. Further, it is surprising that at the limit $x \to 1$, the concurrence does not decrease, no matter what $\sigma/m$ and $\xi$ are. Indeed, at the limit $x \to 1$, not only the concurrence but also the reduced density matrix for spins are independent of $\sigma/m$ and $\xi$.

One possible explanation for that the concurrence decreases less with the presence of momentum entanglement is as follows. Boosting the state, we move some of the spin entanglement to the momentum \cite{14}, however
the momentum entanglement appears to be moved to spins simultaneously. The transfer of momentum entanglement to spins hence compensates the decrease of spin entanglement, and the Lorentz-transformed concurrence decreases less. When the momenta of the two particles are perfectly correlated, even though may be not maximally entangled, the transfer of entanglement from momenta to spins happens to fully compensate the decrease of spin entanglement, so the entanglement of the reduced spin state remains maximal when viewed by any Lorentz-boosted observer. Particularly, for the singlet Bell states all have invariant reduced density matrices for spins viewed from any frame Lorentz boosted along the z-axis.

where $\alpha_p^2 + \beta_p^2 = 1$ due to the unitarity of $U_p$. For the singlet Bell state shown in Eq. (1) with momentum distribution given in Eq. (12), the reduced density matrix remains the same as in the rest frame when viewed by any Lorentz-boosted observer. Thus the entanglement between the spins remains maximal if viewed from any Lorentz-transformed frame. Indeed, the following four “Bell” states all have invariant reduced density matrices for spins viewed from any frame Lorentz boosted along the z-axis.

$$\Phi_f^+ = \sqrt{f(p)} \delta(p - q) \delta_{p, q} \delta_{p, q} \sigma^{2} |\phi^+\rangle,$$  \hspace{1cm} (17a)

$$\Phi_f^- = \sqrt{f(p)} \delta(p - q) \delta_{p, q} \sigma^{2} |\phi^-\rangle,$$  \hspace{1cm} (17b)

$$\Psi_f^+ = \sqrt{f(p)} \delta(p - q) \delta_{p, q} \sigma^{3} |\psi^+\rangle,$$  \hspace{1cm} (17c)

$$\Psi_f^- = \sqrt{f(p)} \delta(p - q) \sigma^{3} |\psi^-\rangle.$$  \hspace{1cm} (17d)

Here we define $\delta_{x,y} = \delta((x - y) \mod 2\pi)$ for compactness of notation. In Eqs. (17), $f(p)$ could be any distribution as long as the state is normalized. $|\phi^\pm\rangle = (|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle)/\sqrt{2}$ and $|\psi^\pm\rangle = (|\downarrow\uparrow\rangle \pm |\uparrow\downarrow\rangle)/\sqrt{2}$ are the conventional Bell states. Further, the states in Eqs. (17), together with those differing by only rotations, constitute a set of states of which the entanglement between the spins remains maximal when viewed from any Lorentz-transformed frame. This invariance of spin entanglement leads to possible applications to the relativistic quantum information processing. Here we shall note that, in Eqs. (17) as well as in the remaining part of this paper, the $\delta$-functions should be regard as limits of analytical functions under certain conditions, such as Eq. (18) is the limit of Eq. (12) at $x \to 1$. The only restriction on $f(p)$ is that the states in Eqs. (17) could be normalized.

III. RELATIVISTIC-INvariant PROTOCOL FOR QUANTUM INFORMATION PROCESSING

An application of possible interests of the above results is to suggest a relativistic-invariant protocol for quantum communication. Conventionally using the spin of a single spin-1/2 particle to represent a qubit may not be appropriate in relativity theory, because the reduced density matrix for its spin is generally not covariant under Lorentz transformations. If and only if for momentum eigenstates (plane waves), the reduced density matrix for the spin of a single particle could be covariant under Lorentz transformations, but momentum eigenstates are not localized and difficult for feasible applications. However, two spin-1/2 particles that are appropriately entangled, such as in Eqs. (17) without being momentum eigenstates, could indeed have reduced density matrix for spins to be invariant under Lorentz transformation. Such invariance provides us the possibility to feasibly represent a single qubit using two appropriately entangled spin-1/2 particles, in a Lorentz-invariant manner.
into account that in many practical situations of communication, one may need to maintain the particles along desired directions, here we focus on the idea case where the momenta of the pair of particles have deterministic directions, and assume that the two particles are moving along the same deterministic direction. We may also choose the boost $\Lambda$ to be along the $z$-axis, and the momenta to lie in the $x$-$z$ plane, i.e. $\theta_p \equiv \theta_q \equiv \theta$ and $\varphi_p \equiv \varphi_q \equiv 0$, without loss of generality. In this protocol we use the momentum distribution that has the following form in the rest frame,

$$\tilde{g}(p, q) = \sqrt{f(p)} \delta(p-q) \delta_{\theta,\theta} \delta_{\varphi,\varphi} \delta_{\varphi_p,\varphi_q} \delta_{\varphi_q,\varphi},$$

with $f(p)$ being arbitrary as long as $\tilde{g}(p, q)$ is normalized as in Eq. (18). Because Eq. (18) is a simultaneous instance of the momentum distributions of the states in both Eq. (17a) and Eq. (17d), both $\tilde{g}(p, q) |\psi^+\rangle$ and $\tilde{g}(p, q) |\psi^-\rangle$ have invariant reduced density matrices for spins when viewed from any Lorentz-boosted frames. This enables us to use these two states as the orthonormal bases, namely $|\hat{0}\rangle$ and $|\hat{1}\rangle$, of a qubit, as follows.

$$|\hat{0}\rangle \sim \tilde{g}(p, q) |\phi^+\rangle,$$

$$|\hat{1}\rangle \sim \tilde{g}(p, q) |\psi^-\rangle.$$  

Eqs. (19) could be regarded as a representation of a “Lorentz-invariant” qubit, in the sense that we look only at the spin part of the state. The representation of “Lorentz-invariant” multiple qubits could be obtained straightforward. Note that in multi-qubit states, the momentum distributions of individual qubits are not necessarily the same. We can further find operator acting upon a single qubit, in terms of the “Lorentz-invariant” bases, as

$$\tilde{\Omega} = \sum_{\sigma, \tau = 0, 1} \lambda_{\sigma\tau} |\tilde{\sigma}\rangle \langle \tilde{\tau}|.$$  

The operators acting upon multiple qubits can be obtained analogously. We refer the operators as in Eq. (20) to be “Lorentz-invariant” in the sense that, if we look only at spins, the action of the operator on the state $a|\hat{0}\rangle + b|\hat{1}\rangle$ ($\forall a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$) remains the same when viewed in any Lorentz-boosted frame. Within the set of these “Lorentz-invariant” qubits and operators, the entropy, entanglement and measurement results all have invariant meanings, despite that for a single quantum spin and some other situations these quantities may have no invariant meanings in different frames [3][4]. Therefore it is guaranteed that, using such states and operators, the non-relativistic quantum information theory could be invariantly applied to relativistic situation.

IV. CONCLUSION

As observed in Ref. [4], because Lorentz boosts entangle the spin and momentum degrees of freedom, entanglement between the spins may change if viewed from a moving frame. Especially, maximally entangled spin states will most likely decohere due to the mixing with momentum degrees of freedom, depending on the initial momentum wave function [4]. In this paper, we investigate the quantum entanglement between the spins of a pair of spin-1/2 massive particles in moving frames, for the case that the momenta of the particles are entangled. We show that if the momenta of the pair are appropriately entangled, the entanglement between the spins of the Bell states remains maximal when viewed from any Lorentz-transformed frame. Further, we suggest a relativistic-invariant protocol for quantum communication, with which the non-relativistic quantum information theory could be invariantly applied to relativistic situations. Though the investigations are based on spin-1/2 particles, we believe the similar results for larger spins could be obtained analogously. Especially, we hope our work would help to find a relativistic-invariant protocol for quantum information processing based on photons, i.e. the case of massless spin-1 particles.

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