A QUANTITATIVE SHRINKING TARGET RESULT ON STURMIAN SEQUENCES FOR ROTATIONS

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Abstract. Let \( R_\alpha \) be an irrational rotation of the circle, and code the orbit of any point \( x \) by whether \( R_i \alpha (x) \) belongs to \([0, \alpha)\) or \([\alpha, 1)\) – this produces a Sturmian sequence. A point is undetermined at step \( j \) if its coding up to time \( j \) does not determine its coding at time \( j + 1 \). We prove a pair of results on the asymptotic frequency of a point being undetermined, for full measure sets of \( \alpha \) and \( x \).

1. Introduction.

1.1. Statement of the problem and main results. In this paper we study a shrinking target problem. Let \( \alpha \in [0, 1) \), let \( R_\alpha : [0, 1) \to [0, 1) \) be the rotation \( R_\alpha (x) = x + \alpha \mod 1 \), and let \( \lambda \) denote Lebesgue measure. The following theorem, due to Weyl, is well known.

Theorem 1.1. Let \( \alpha \notin \mathbb{Q} \). Then for any \( x, y \in [0, 1) \), and any \( \epsilon > 0 \),

\[
\lim_{N \to \infty} \frac{\sum_{i=1}^N \chi_{B(y, \epsilon)}(R_i^N x)}{\sum_{i=1}^N \lambda(B(y, \epsilon))} = 1.
\]

That is, the asymptotic for the number of visits of the orbit of \( x \) to the target set \( B(y, \epsilon) \) by step \( N \) is given by the sum of the size of the target over those \( N \) steps.

The statement is written here in a slightly unusual way – the denominator is clearly \( N \epsilon \) (assuming \( \epsilon \leq \frac{1}{2} \) and identifying \([0, 1)\) with \( S^1 \)). But it suggests the following sort of problem. Let \( \{B_i\} \) be a sequence of measurable sets in \([0, 1)\). What can be said about the behavior of \( \sum_{i=1}^N \chi_{B_i}(R_i^N x) \); in particular, is it asymptotic to \( \sum_{i=1}^N \lambda(B_i) \)?

This is, of course, an enormously varied problem. Cases which have generated significant interest are shrinking target problems, in which the \( B_i \) form a decreasing, 2010 Mathematics Subject Classification. Primary: 37E10, 37A05, 37B10.

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This is the set of ‘undetermined’ points at step $c$ (or a portion thereof, if the sequence is finite). Let $\Sigma$ be the set of finite codings of minimal complexity, or with minimal block growth. They were introduced by Hedlund and Morse [6], and have been studied extensively.

For a sequence $(c_0, c_1, \ldots)$ (finite or infinite) of 0’s and 1’s, let $C_{c_0,c_1,\ldots} = \{ x : R_{c_i}^n x \in A_i, \text{ for all } i \}$. If $x \in C_{c_0,c_1,\ldots}$, then $(c_0, c_1, \ldots)$ is a coding for the orbit of $x$ (or a portion thereof, if the sequence is finite). Let $\Sigma$ be the set of finite codings $c_0, c_1, \ldots, c_n$ which actually occur, i.e. for which $C_{c_0,\ldots,c_n} \neq \emptyset$. Let

$$V_j = \{ x : x \in C_{c_0,\ldots,c_j} \text{ and such that } c_0, \ldots, c_j, 0 \text{ and } c_0, \ldots, c_j, 1 \in \Sigma \}.$$ 

This is the set of ‘undetermined’ points at step $j$, that is, points whose coding up to step $j$ does not determine the coding at step $j+1$. The word $c_0, \ldots, c_j$ is also known as a right special word (see, e.g., [5, §2.1.1].)

We want to find asymptotics on how often a point is undetermined; specifically, we will prove

**Theorem A.** For almost all $\alpha$,

$$\lim_{n \to \infty} \frac{\log \sum_{j=1}^n \chi_{V_j}(x)}{\log \sum_{j=1}^n \lambda(V_j)} = 1$$

for almost all $x \in (0,1)$.

As in [2], the full measure condition on $\alpha$ is a diophantine condition involving the continued fraction expansion of $\alpha$. It will be stated explicitly in the proof.

To understand why Theorem A constitutes a shrinking target problem, consider the following. Let $P_j = V_j \cap \bigcup_{k=0}^{\infty} \mathcal{P}$, the partition generated by $\mathcal{P}$ and its first $j$ translates. For $x \in X$, denote by $[[x]]$, the atom of $x$ in $P_j$. The coding $c_0, \ldots, c_j$ determines only the atom $[[R_{c_j}^n x]]$. A point $x$ will belong to $V_j$ if and only if $R_{c_j}^n x$ is in $[[1-\alpha]]$, as the image of this atom under one more rotation contains points in both $A_0$ and $A_1$. We will denote $[[1-\alpha]]$ by $U_j$—these are the shrinking targets which we are trying to hit. Note that $U_j = R_{c_j}^n(V_j)$.

The logarithms in Theorem A indicate a weaker asymptotic result than in [2]. The stronger version is not true:
Theorem B. For almost all \( \alpha \),

\[
\lim_{N \to \infty} \frac{\sum_{j=1}^{N} \lambda_{V_j}(x)}{\sum_{j=1}^{N} \lambda(V_j)}
\]

do not exist for almost every \( x \in [0,1) \).

Thus, Theorem A is in some sense the best one can hope for in this setting, an interesting contrast with the stronger results obtained for targets of the form \( B(y,r_i) \).

1.2. Notation and an outline of the paper. The key tool throughout the paper is the continued fraction expansion of \( \alpha \) and its close relationship to the dynamics of the rotation by \( \alpha \). Throughout, \( \alpha \in (0,1) \) is assumed to be irrational. We write

\[
\alpha = [0; a_1, a_2, a_3, \ldots]
\]

for the continued fraction expansion of \( \alpha \). Note that the elements \( a_i \) of the continued fraction depend on \( \alpha \); we will at times write \( a_i(\alpha) \) to emphasize this dependence. The convergents to \( \alpha \) are the rationals \( p_k/q_k \). The \( k \)th convergent is the best rational approximation to \( \alpha \) with denominator \( \leq q_k \). The \( q_k \) can be computed by the recurrence relation

\[
q_{k+1} = a_{k+1}q_k + q_{k-1}; q_0 = 1, q_1 = a_1.
\]

We will prove Theorem B first, in Section 2. The almost sure existence of elements of the continued fraction expansion which are very large in relation to the preceding elements drives the argument.

Theorem A is proved in Section 3. There we prove a set of looser bounds on \( \sum_{i=1}^{n} a_i \) and \( \sum_{j=1}^{n} \lambda_{V_j}(x) \) which hold for almost all \( \alpha \) and which are sufficient for the statement of Theorem A.

2. Failure of a stronger convergence. We start with the proof of Theorem B. First, we prove the almost-sure existence of very large elements \( a_n \) for the continued fraction expansion. We then use this to show that, for very long stretches of time certain points are undetermined more often than \( \sum_{j=1}^{n} \lambda(V_j) \) predicts.

Proposition 2.1. For any \( C \in \mathbb{R} \) and almost every \( \alpha \) there exist infinitely many \( m \) such that

\[
a_m > C \sum_{i=1}^{m-1} a_i.
\]

We need a series of preliminary results to prove this. The following lemma appears in [4, page 60].

Lemma 2.2. For any \( n, b_1, \ldots, b_n \in \mathbb{N} \) we have

\[
\frac{1}{3b_n^2} \leq \frac{\lambda(\{ \alpha : a_1(\alpha) = b_1, \ldots, a_n(\alpha) = b_n \})}{\lambda(\{ \alpha : \text{a_1(\alpha) = b_1, \ldots, a_{n-1}(\alpha) = b_{n-1} \})} < \frac{2}{b_n^2}.
\]

From this it is an easy exercise to deduce:

Corollary 2.3.

\[
\frac{1}{3b_n} \leq \frac{\lambda(\{ \alpha : a_1(\alpha) = b_1, \ldots, a_n(\alpha) \geq b_n \})}{\lambda(\{ \alpha : a_1(\alpha) = b_1, \ldots, a_{n-1}(\alpha) = b_{n-1} \})} < \frac{4}{b_n}.
\]

Let \( W_n = \left\{ \alpha : \sum_{i=1}^{n} a_i(\alpha) < 10n \log n \right\} \).
Lemma 2.4. $\lambda(W_n) > \frac{1}{10n}$ for $n \geq 10$.

Proof. Let $A_n = \{ \alpha : a_i(\alpha) < n^2 \text{ for all } i \leq n \}$. By Corollary 2.3, $\lambda(\{a_i(\alpha) \geq n^2\}) < \frac{4}{n^2}$ for any $i$. Thus, $\lambda(A_n^c) < \frac{4}{n^2}$.

Consider $\sum_{i=1}^{n} \int_{A_n} a_i(\alpha) d\lambda$. We have,

$$\sum_{i=1}^{n} \int_{A_n} a_i(\alpha) d\lambda = \sum_{i=1}^{n} \sum_{j=1}^{n^2} j \cdot \lambda(A_n \cap \{a_i(\alpha) = j\}) < \sum_{i=1}^{n} \sum_{j=1}^{n^2} \frac{j}{j^2},$$

where we have bounded $\lambda(A_n \cap \{a_i(\alpha) = j\}) < \frac{2}{j^2}$ using Lemma 2.2. The double sum is less than or equal to $2n(1 + \log n^2)$ which is bounded above by $5n \log n$, for $n \geq 10$.

Using Markov’s inequality, we have

$$\lambda(W_n^c \cap A_n) \leq \frac{1}{10n \log n} \int_{\alpha \in A_n} \sum_{i=1}^{n} a_i(\alpha) \, d\lambda \leq \frac{1}{10n \log n} \cdot 5n \log n \leq \frac{1}{2}.$$

Since $\lambda(A_n) \geq 1 - \frac{4}{n^2}$, we conclude that $\lambda(W_n \cap A_n) \geq \frac{1}{2} - \frac{4}{n^2}$. Therefore, $\lambda(W_n) > \frac{1}{10n}$ for $n \geq 10$. □

Remark 2.5. The bound in Lemma 2.4 is not optimal, as is easily seen from the proof. We are only concerned to find some bound away from zero.

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. Fix $C > 0$. Corollary 2.3 and the definition of $W_{m-1}$ imply that

$$\lambda(\{ \alpha : a_1(\alpha) = b_1, \ldots, a_m(\alpha) = b_{m-1}, \text{ and } a_m(\alpha) \geq 10C(m-1) \log(m-1) \} \cap W_{m-1}) \geq \lambda(\{ \alpha : a_1(\alpha) = b_1, \ldots, a_{m-1}(\alpha) = b_{m-1} \} \cap W_{m-1}) \cdot \frac{30C(m-1) \log(m-1)}{30C(m-1) \log(m-1)}.$$

From this we have that

$$\lambda(W_{m-1} \cap \{ \alpha : a_m(\alpha) \geq 10C(m-1) \log(m-1) \}) \geq \frac{\lambda(W_{m-1})}{30C(m-1) \log(m-1)}.$$

Let $G_m = W_{m-1} \cap \{ \alpha : a_m \geq 10C(m-1) \log(m-1) \}$. Notice that $\alpha \in G_m$ implies that $a_m(\alpha) > C \sum_{i=1}^{m-1} a_i(\alpha)$. Then, for $m \geq 10$, using Lemma 2.4

$$\lambda(G_m) \geq \frac{\lambda(W_{m-1})}{30C(m-1) \log(m-1)} > \frac{1}{300C(m-1) \log(m-1)}.$$
Using this estimate,
\[
\sum_{m=1}^{\infty} \lambda(G_m) > \sum_{m=10}^{\infty} \frac{1}{300C(m-1) \log(m-1)} = \infty.
\]

To complete the proof we need two lemmas. The first is a well known partial converse to the Borel-Cantelli Lemma for quasi-independent sets (as opposed to independent sets). Its proof is included for completeness.

**Lemma 2.6.** Let \( A_i \) be measurable subsets of a space with probability measure \( \lambda \). If there exists \( C > 0 \) such that \( \lambda(A_i \cap A_j) < C \lambda(A_i) \lambda(A_j) \) and \( \sum_{i=1}^{\infty} \lambda(A_i) = \infty \), then \( \lambda \left( \bigcap_{N=1}^{\infty} \bigcup_{i=N}^{\infty} A_i \right) > \frac{1}{4C} > 0 \).

**Proof.** Let \( B_{N,M} = \bigcup_{i=N}^{M} A_i \). If \( \sum_{i=N}^{M} \lambda(A_i) < \frac{1}{4C} \) then for any \( j \notin [N,M] \) we have that
\[
\lambda(A_j \cap B_{N,M}) \geq \lambda(A_j) - \sum_{i=N}^{M} C \lambda(A_j) \lambda(A_i) > \frac{1}{2} \lambda(A_j).
\]
Because \( \sum \lambda(A_i) = \infty \), the above implies that \( \lambda(B_{N,\infty}) \geq \frac{1}{4C} \) for all \( N \). Because we are in a finite measure space it follows that \( \lambda \left( \bigcap_{N=1}^{\infty} \bigcup_{i=N}^{\infty} A_i \right) = \lim_{N \to \infty} \lambda \left( \bigcup_{i=N}^{\infty} A_i \right) \) and so is at least \( \frac{1}{4C} \).

**Lemma 2.7.** If \( m > n \geq 10 \) then \( \lambda(G_m \cap G_n) \leq 120 \lambda(G_m) \lambda(G_n) \).

**Proof.** By Corollary 2.3, if \( A_1 = \{ \alpha : a_i(\alpha) = b_i \text{ for } 1 \leq i < m \} \), \( A_2 = \{ \alpha : a_i(\alpha) = c_i \text{ for } 1 \leq i < m \} \) are both subsets of \( W_{m-1} \) then
\[
\frac{1}{12} \frac{\lambda(A_2 \cap G_m)}{\lambda(A_2)} \leq \frac{\lambda(A_1 \cap G_m)}{\lambda(A_1)} \leq 12 \frac{\lambda(A_2 \cap G_m)}{\lambda(A_2)}.
\]
Write \( G_n = \bigcup_i A_i \) with each \( A_i \) of the form \( A_i = \{ \alpha : a_\ell(\alpha) = b_\ell, 1 \leq \ell < m \} \). Then \( G_m \cap G_n = \bigcup_i (G_m \cap A_i) \), where we can assume all \( A_i \subset W_{m-1} \). Then, using equation 2,
\[
\lambda(G_m \cap G_m) = \sum_i \lambda(G_m \cap A_i)
\leq \sum_i 12 \frac{\lambda(A^* \cap G_m)}{\lambda(A^*)} \lambda(A_i)
\leq 12 \lambda(G_n) \frac{\lambda(A^* \cap G_m)}{\lambda(A^*)}
\]
for an arbitrary \( A^* \subset W_{m-1} \) of the form above. Since a subcollection of the \( A_i \) form a partition of \( W_{m-1} \), by restricting the above estimate to that subcollection we have \( \lambda(G_m \cap G_n) \leq 12 \frac{1}{\lambda(W_{m-1})} \lambda(G_m) \lambda(G_n) \). The result follows, using Lemma 2.4.

Applying these two lemmas we conclude that there is a positive measure set of \( \alpha \) for which \( a_m(\alpha) \geq C \sum_{i=1}^{m-1} a_i(\alpha) \) infinitely often. If \( \alpha \) is in this set, its image under the Gauss map is as well, so by the ergodicity of that map the set of such \( \alpha \) in fact has full measure.
The following two lemmas on the shrinking targets $U_j$ are also needed to complete our proof of Theorem B. Recall that $U_j = R^j_v(V_j)$ and $V_j = \{x : x \in C_{c_0}, \ldots, c_j} and such that $c_0, \ldots, c_j, 0$ and $c_0, \ldots, c_j, 1 \in \Sigma\}.

These lemmas are proved using the partial fraction expansion of $\alpha$. We will denote by $\langle y \rangle$ the value modulo 1 of a real number $y$ and by $\langle \langle y \rangle \rangle$ the distance from $y$ to the nearest integer.

**Lemma 2.8.** Let

$$r_j = \max\{q_k : q_k \leq j\}$$

$$s_j = \max\{q_k : q_{k+1} \leq j\}$$

$$t_j = \max\{T \in \mathbb{N} : s_j + Tr_j \leq j\}.$$  

Then

$$R_\alpha(U_j) = [\{s_j\alpha\} + t_j\{r_j\alpha\}, \{r_j\alpha\}]$$

or

$$R_\alpha(U_j) = [\{r_j\alpha\}, \{s_j\alpha\} - t_j(1 - \{r_j\alpha\})],$$

and

$$\lambda(U_j) = \lambda(V_j) = \langle \langle r_j\alpha \rangle \rangle - (\langle s_j\alpha \rangle - t_j \langle r_j\alpha \rangle).$$

**Remark 2.9.** Note that if $r_j = q_k$, $s_j = q_{k-1}$ and $t_j < a_{k+1}$.

**Proof.** Note that $\langle \langle r_j \alpha \rangle \rangle$ is smaller than or equal to $\langle \langle i\alpha \rangle \rangle$ for all $i \leq j$.

**Case 1.** $0 < \{r_j\alpha\} < 1/2$. As the convergents alternate in approximating $\alpha$ from above and below, $1/2 < \{s_j\alpha\} < 1$. The only possible improvement in $\{r_j\alpha\}$ as an upper bound for $R_\alpha(U_j)$ would come from finding some $l$ with $\langle \langle l\alpha \rangle \rangle < \langle \langle r_j\alpha \rangle \rangle$. This is not possible for $l \leq j$. Thus the upper endpoint of $R_\alpha(U_j)$ is $\{r_j\alpha\}$ as desired.

The lower bound on $R_\alpha(U_j)$ given by $\{s_j\alpha\}$ can be improved only by adding $\{r_j\alpha\}$ some number of times, as $r_j$ is the only integer $\leq j$ with $\langle \langle r_j\alpha \rangle \rangle < \langle \langle s_j\alpha \rangle \rangle$. The lower endpoint will thus be of the form $y = \{s_j\alpha\} + T\{r_j\alpha\}$ and will be found by taking $T$ as large as possible such that the $s_j + Tr_j$ rotations required to produce this point do not exceed $j$; this number is $t_j$.

We calculate that $\lambda(U_j) = \langle \langle r_j\alpha \rangle \rangle + (1 - \{s_j\alpha\} - t_j\{r_j\alpha\})$ using the fact that in this case $\langle \langle r_j\alpha \rangle \rangle = \{r_j\alpha\}$. Since $\langle \langle s_j\alpha \rangle \rangle = 1 - \{s_j\alpha\}$, this simplifies to the desired result.

**Case 2.** $1/2 < \{r_j\alpha\} < 1$. Then $0 < \{s_j\alpha\} < 1/2$ and the lower endpoint of $R_\alpha(U_j)$ is $\{r_j\alpha\}$. As before, the upper endpoint is of the form $\{s_j\alpha\} - T(1 - \{r_j\alpha\})$. The best such endpoint is found by taking $T$ as large as possible, i.e. equal to $t_j$.

Finally, we calculate again

$$\lambda(U_j) = \langle \langle s_j\alpha \rangle \rangle - t_j(1 - \{r_j\alpha\}) + (1 - \{r_j\alpha\}) = \langle \langle s_j\alpha \rangle \rangle - t_j\langle \langle r_j\alpha \rangle \rangle + \langle \langle r_j\alpha \rangle \rangle.$$

For use in the lemma below as well as later in the paper, we fix some notation. We will adopt interval notation $[m, n)$ to denote intervals of integers; context will make the distinction between these and subsets of the real interval $[0, 1)$ clear.

We let $I_i = [\alpha_i, \alpha_{i+1})$. We let

$$J_i^b = \begin{cases} 
[\alpha_i, \alpha_{i-1} + \alpha_i) & \text{if } b = 1, \\
(\alpha_{i-1} + (b - 1)\alpha_i, \alpha_{i-1} + b\alpha_i) & \text{if } 1 < b \leq \alpha_{i+1}.
\end{cases}$$
Let $\mathcal{J}$ denote the collection of all the $J_b^i$’s. We note that $J_b^i \subset I_i$ and that these intervals are disjoint.

Further, let $J_b^i = (q_{i-1} + q_i, q_{i-1} + 2q_i)$ for all $i$, whether $a_{i+1} \geq 2$ or not. If $a_{i+1} = 1$, $J_b^i \subset I_{i+1}$ and it equals $J_l^{i+1}$, but we note that in any case $\{J_b^i\}_{i \in \mathbb{N}}$ consists of pairwise disjoint intervals.

**Lemma 2.10.** For any $J \in \mathcal{J}$, and for all $l \in J$, the sets $V_l = R_{\alpha}^{-l} U_l$ are pairwise disjoint.

**Proof.** Fix $J_b^i \in \mathcal{J}$. For $l \in J_b^i$, Lemma 2.8 tells us that $R_{\alpha} U_l$ is the interval containing 0 bounded by $R_{\alpha}^q (0)$ and $R_{\alpha}^{q_l+1(b-1)q_i} (0)$.

Suppose that $l > k \in J_b^i$. Then $U_l = U_k =: U$, and $R_{\alpha}^{-l} U \cap R_{\alpha}^{-k} U \neq \emptyset$ if and only if $R_{\alpha} U \cap R_{\alpha}^{l-k} (R_{\alpha} U) \neq \emptyset$. For such an intersection to occur, $R_{\alpha}^{l-k}$ of some endpoint of $R_{\alpha} U$ must lie in $R_{\alpha} U$.

We examine the two cases: $b = 1$ and $b > 1$.

If $b = 1$, $J_b^i = [q_i, q_{i-1} + q_i)$, $1 < l - k < q_{i-1}$, and the endpoints of $R_{\alpha} U$ are $R_{\alpha}^q (0)$ and $R_{\alpha}^{q_i+1(b-1)q_i} (0)$. The first time after $q_{i-1}$ that the orbit of 0 hits $U$ is $q_{i-1} + q_i$. But $(l - k) + q_{i-1} < q_{i-1} + q_i$ and $(l - k) + q_i < q_{i-1} + q_i$, so neither endpoint of $R_{\alpha} U$ will return to $R_{\alpha} U$ under $R_{\alpha}^{l-k}$, proving the desired disjointness.

If $b > 1$, $J_b^i = [q_i + (b-1)q_i, q_{i-1} + bq_i)$, $1 < l - k < q_i$, and the endpoints of $R_{\alpha} U$ are $R_{\alpha}^q (0)$ and $R_{\alpha}^{q_i+1(b-1)q_i} (0)$. The first time after $q_{i-1} + (b-1)q_i$ that the orbit of 0 hits $U$ is $q_{i-1} + bq_i$. But $(l - k) + q_i < q_{i-1} + bq_i$ since $l - k < q_i$ and $b \geq 2$ and $(l - k) + (q_{i-1} + (b-1)q_i) < q_{i-1} + bq_i$ since $l - k < q_i$, so neither endpoint of $R_{\alpha} U$ will return to $R_{\alpha} U$ under $R_{\alpha}^{l-k}$, again proving disjointness. \(\square\)

**Corollary 2.11.** For all $m$,

$$\sum_{j=1}^{q_m-1} \lambda(V_j) < \sum_{i=1}^{m} a_i.$$  

**Proof.** In $\mathcal{J}$, there are $a_i$ intervals $J_b^{i-1}$ contained in $I_{i-1} = [q_{i-1}, q_i)$. By Lemma 2.10, $\sum_{j \in J_b^{i-1}} \lambda(V_j) < 1$ since the $V_l$ are disjoint over these indices. Therefore, $\sum_{j=q_{i-1}}^{q_i-1} \lambda(V_j) < a_i$ and the result follows. \(\square\)

The following technical tool, a consequence of equidistribution of points under the rotation $R_{\alpha}$ and regularity of measures will be used in the proof of Theorem B:

**Lemma 2.12.** Let $A \subset [0, 1]$ have positive measure and fix $\delta > 0$. Suppose we have families $\{X_m\}_{m \in \mathbb{N}}$ and $\{Y_m\}_{m \in \mathbb{N}}$ of subsets of $[0, 1]$ such that

- $\lambda(X_m), \lambda(Y_m) > \delta > 0$ for all $m$,
- For each $m$, $X_m = \bigcup_{k=1}^{K_m} R_{\alpha}^k (U_m)$ and $Y_m = \bigcup_{k=1}^{K_m} R_{\alpha}^k (V_m)$ where $U_m$ and $V_m$ are intervals and $K_m \to \infty$ as $m \to \infty$.

Then, for any sufficiently large $m$, there exists a pair of points $x^* \in X_m \cap A$ and $y^* \in Y_m \cap A$ with $|x^* - y^*| < \delta$.

**Proof.** Choose a positive $\epsilon$ satisfying $\epsilon < \frac{(0.99 \lambda(A) \delta)}{2 + (0.99 \delta)}$. This choice guarantees that $(\frac{1}{2}) (0.99) (\lambda(A) - \epsilon) > \epsilon$. Since $A$ has finite measure, there is a finite, disjoint union of open intervals $B = \bigcup_{i=1}^{M} I_i$ such that $\lambda(A \Delta B) < \epsilon$. By the equidistribution of points under $R_{\alpha}$ and the fact that $K_m \to \infty$, we may pick $M > 0$ so large that for all $m > M$,

$$\lambda(I_i \cap X_m) > 0.99 \lambda(I_i) \delta$$
\[
\lambda(I_i \cap Y_m) > .99\lambda(I_i)\delta
\]
for all \(i = 1, \ldots, n\), using our lower bound on the measures of \(X_m\) and \(Y_m\). Further pick \(M\) so large that for \(m > M\), the maximum separation between two adjacent points in \(\{R^{k_0}_{n}k\}_{k=1}^{k_m}\) is < \(\delta\).

Consider the intervals forming \(X_m\) and \(Y_m\) which are contained in \(I_i\). For each interval \(U\) which is a connected component of \(X_m\), let \(V_U\) be its nearest neighbor to the right among the connected components of \(Y_m\). (Such a neighbor exists for all but possibly the last such \(U\) contained in \(I_i\). We may choose \(M\) so large that the number of \(X_m\) intervals in \(I_i\) is very large, making this exceptional subinterval’s contribution to the argument below negligible.) Note that \(\max_{x \in V_U, y \in V_U} |x - y| < \delta\).

If a pair \((x^*, y^*)\) as desired does not exist, then for each pair \((U, V_U)\), at least one of \(U, V_U\) contains no points in \(A\). Therefore, \(\lambda(I_i \setminus A) > \frac{1}{2}(.99)\lambda(I_i)\delta\). Thus,

\[
\lambda(B \setminus A) > \sum_{i=1}^{n} \frac{1}{2}(.99)\lambda(I_i)\delta \\
= \frac{1}{2}(.99)\lambda(B)\delta \\
> \frac{1}{2}(.99)(\lambda(A) - \epsilon)\delta > \epsilon
\]
by our choice of \(\epsilon\). But this contradicts our choice of \(B\), proving the lemma. \(\square\)

To simplify notation a bit, we set for all integers \(m\):

\[
f_m(x) := \frac{\sum_{q_m-1}^{q_m-1} \chi_{V_j}(x)}{\sum_{j=1}^{q_m-1} \lambda(V_j)}.
\]

Where it exists, we set

\[
f(x) := \lim_{N \to \infty} \frac{\sum_{j=1}^{N} \chi_{V_j}(x)}{\sum_{j=1}^{N} \lambda(V_j)}.
\]

Note that, where it exists, \(\lim_{m \to \infty} f_m(x) = f(x)\) and \(f\) is measurable. In addition, by Fatou’s Lemma, \(f\) will be integrable over the set where it is defined, since \(\int_{[0,1]} f_m d\lambda = 1\) for all \(m\). Therefore we can assume \(f\) takes only finite values.

We are now ready to prove Theorem B.

**Proof of Theorem B.** Fix \(C > 0\) and apply Proposition 2.1 to find a full measure set of \(\alpha\) satisfying equation 1 for infinitely many \(m\). Fix any such \(m\).

For all \(b \in [2, a_m]\), let

\[
W_b = \bigcup_{j \in J_m^{m-1}} V_j.
\]

Note that by Lemma 2.10, this is a disjoint union, and using Lemma 2.8,

\[
\lambda(W_b) = q_m - 1 \left[ \langle q_m - 2 \alpha \rangle - (b - 2)\langle q_m - 1 \alpha \rangle \right].
\]

In addition, if \(x \in W_b\), then it will belong to exactly one \(V_j\) with \(j \in J_m^{m-1}\) for all \(b' \leq b\), and because \(V_{j+q_k} \subset V_j\) for \(q_k \leq j \leq q_{k+1} - q_k\),

\[
\sum_{j=q_m-1}^{q_{m-1}} \chi_{V_j}(x) \geq b \quad \text{for all} \quad x \in W_b.
\]
Choose any $\rho \in (1/8, 1/4)$ in such a way that $\rho a_m \in \mathbb{N}$ (possible since $a_m$ is very large), and let $X_m = W_{\rho a_m}$. We then estimate the measure of $X_m$ below using standard results on the convergents:

$$\lambda(X_m) = q_{m-1} \left[ \langle \langle q_{m-2} \alpha \rangle \rangle - (\rho a_m - 1) \langle \langle q_{m-1} \alpha \rangle \rangle \right]$$

$$\geq q_{m-1} \left[ \frac{1}{q_{m-1} + q_{m-2}} - \frac{\rho a_m}{q_m} \right]$$

$$\geq \frac{1}{2} - \frac{a_m q_{m-1}}{q_m}$$

$$\geq \frac{1}{2} - \rho$$

$$\geq \frac{1}{4}.$$

Second, choose $\sigma \in (1/16, 1/8)$ so that $\sigma a_m \in \mathbb{N}$ and is $\geq 2$. Let $Y_m = W_1 \setminus W_{\sigma a_m}$. Then, as any $y \in Y_m$ will not belong to $V_j$ when $j \in J^m_b$ for $b \geq \sigma a_m$,

$$\sum_{j=q_{m-1}}^{q_m-1} \chi_{V_j}(y) \leq \sigma a_m - 1 \text{ for all } y \in Y.$$

Using Corollary 2.11,

$$\sum_{j=1}^{q_m-1} \chi_{V_j}(y) \leq \sigma a_m - 1 + \sum_{i=1}^{m-1} a_i \text{ for all } y \in Y.$$

We can also estimate the measure of this set (recalling that $a_m$ is very large):

$$\lambda(Y_m) = q_{m-1} \left[ (\sigma a_m - 1) \langle \langle q_{m-1} \alpha \rangle \rangle \right]$$

$$\geq q_{m-1} (\sigma a_m - 1) \frac{1}{q_m + q_{m-1}}$$

$$\geq \frac{1}{2} \sigma \frac{a_m q_{m-1}}{2q_m}$$

$$= \frac{1}{4} \sigma \frac{a_m q_{m-1}}{a_m q_{m-1} + q_{m-2}}$$

$$\geq \frac{1}{4} \sigma \frac{a_m q_{m-1}}{(a_m + 1) q_{m-1}}$$

$$\geq \frac{1}{4} \sigma \frac{1}{2} \geq \frac{1}{128}.$$

Estimates here are certainly not precise; the key point is that $X_m$ and $Y_m$ have a positive lower bound on their measures which is independent of $m$. Let $\delta = \frac{1}{128}$.

Using the results above, for all $x \in X_m$ and $y \in Y_m$,

$$\sum_{j=1}^{q_m-1} \chi_{V_j}(x) - \sum_{j=1}^{q_m-1} \chi_{V_j}(y) \geq \rho a_m - \sigma a_m + 1 - \sum_{i=1}^{m-1} a_i$$

$$\geq (\rho - \sigma - 1/C)a_m.$$

Finally, using Corollary 2.11,

$$|f_m(x) - f_m(y)| = \frac{\left| \sum_{j=1}^{q_m-1} \chi_{V_j}(x) - \sum_{j=1}^{q_m-1} \chi_{V_j}(y) \right|}{\sum_{j=1}^{q_m-1} \lambda(V_j)} \geq \frac{(\rho - \sigma - 1/C)a_m}{\sum_{i=1}^{m-1} a_i}$$
As in the proof of Lemma 2.4, set 
\[
\rho = \frac{\sigma - 1/C}{1 + 1/C}.
\]

By choosing \( C \) sufficiently large, and since \( \rho > \sigma \), we have \(|f_m(x) - f_m(y)| \geq D > 0\) for all \( m \) such that equation 1 holds and all \( x \in X_m, y \in Y_m \).

Let \( \lambda = \{x : f(x) \text{ exists}\} \). Towards a contradiction, assume \( \lambda(Z) > 0 \). Fix \( \epsilon < D/3 \) and \( \lambda(Z) \).

Since \( f \) is measurable, by Luzin’s Theorem there is a compact set \( G \subset Z \) with \( \lambda(G) > \lambda(Z) - \epsilon \) over which \( f \) is (uniformly) continuous. Let \( \delta > 0 \) be such that \(|x - y| < \delta\) and \( x, y \in G \) imply \(|f(x) - f(y)| < \epsilon\).

Let \( Z = \{x \in Z : \text{ for all } n \geq N, \frac{\sum_{j=1}^{n} \chi_{V_j}(x)}{\sum_{j=1}^{n} \lambda(V_j)} \text{ is within } \epsilon \text{ of } f(x)\} \).

Under our assumption \( \lambda(Z_N) \to \lambda(Z) \) as \( N \to \infty \). Pick \( N_0 \) so large that \( \lambda(Z_{N_0}) > \lambda(Z) - \epsilon \) and, therefore, \( \lambda(G \cap Z_{N_0}) > \lambda(Z) - 2\epsilon > 0 \) by the choice of \( \epsilon \).

Let \( m \) be chosen so large that the following hold:

- \( q_m > N_0 \),
- \( a_m \) satisfies condition 1, and
- \( \{X_m\}_{m \in \mathbb{N}} \) and \( \{Y_m\}_{m \in \mathbb{N}} \) satisfy Lemma 2.12 with \( A = G \cap Z_{N_0} \).

Then we may take \( x^* \in X_m \cap G \cap Z_{N_0} \) and \( y^* \in Y_m \cap G \cap Z_{N_0} \) with \(|x^* - y^*| < \delta\).

As \( x^*, y^* \in Z_{N_0} \) and \( q_m > N_0 \), \(|f_m(x^*) - f(x^*)| < \epsilon\) and \(|f_m(y^*) - f(y^*)| < \epsilon\). As both points are in \( G \) and \(|x^* - y^*| < \delta\), \(|f(x^*) - f(y^*)| < \epsilon\).

We conclude that \(|f_m(x^*) - f_m(y^*)| < 3\epsilon < D\). But this contradicts our result above on the minimum difference between the values of \( f_m \) at points in \( X_m \) and \( Y_m \) when \( a_m \) satisfies 1.

Therefore there is a set of full measure where the \( f_m \) do not converge, completing the proof.

3. Proof of Theorem A. Towards Theorem A, we claim the following set of inequalities:

There exists a positive constant \( C_1 \) such that for almost every \( \alpha \) and \( x \in [0, 1) \),

\[
C_1 n(\log n)^3 > n \sum_{i=1}^{n} a_i(\alpha) \geq \sum_{j=1}^{q_n-1} \chi_{V_j}(x) > \frac{1}{4}(n - 2).
\]  

(3)

The middle inequality follows from almost the same proof as Corollary 2.11. We prove the other two inequalities in the following sequence of Lemmas. Lemma 3.1 specifies the full measure set of \( \alpha \) for which we prove Theorem A.

**Lemma 3.1.** There exists a positive constant \( C_1 \) such that for almost every \( \alpha \), \( C_1 n(\log n)^3 > \sum_{i=1}^{n} a_i(\alpha) \) for all sufficiently large \( n \).

**Remark 3.2.** Note that how large \( n \) must be for the given bound to hold does depend on \( n \).

**Proof.** As in the proof of Lemma 2.4, set \( A_n = \{\alpha : a_i(\alpha) < n^2 \text{ for all } i \leq n\} \). As before, \( \int_{A_n} n \sum_{i=1}^{n} a_i(\alpha) d\lambda(\alpha) \leq 5n \log n \) (for \( n > 7 \)). Note also that \( \lambda\text{-a.e. } \alpha \) belongs
to $A_n$ for all but finitely many $n$. It follows from Markov’s inequality that
\[
\lambda \left( \{ \alpha \in A_n : \sum_{i=1}^{n} a_i(\alpha) > 10n(\log n)^{2.1} \} \right) \leq \frac{1}{10n(\log n)^{2.1}} \int_{A_n} \sum_{i=1}^{n} a_i(\alpha) d\lambda \\
\leq \frac{1}{2} \left( \frac{1}{\log n} \right)^{1.1}.
\]

Since almost every $\alpha$ belongs to $A_n$ for all but finitely many $n$, almost every $\alpha$ belongs to $A_{10^k}$ for all but finitely many $k$. Then
\[
\lambda \left( \left\{ \alpha \in A_{10^k} : \sum_{i=1}^{10^k} a_i(\alpha) > 10^{k+1}(\log 10^k)^{2.1} \right\} \right) \leq \left( \frac{1}{\log 10^k} \right)^{1.1}.
\]

These measures form a summable sequence, so for a.e. $\alpha$,
\[
\sum_{i=1}^{10^k} a_i(\alpha) \leq 10^{k+1}(\log 10^k)^{2.1} \quad \text{for all but finitely many } k.
\]

This implies the Lemma for the subsequence $n = 10^k$ because for large enough $k$, we have $10^k(\log 10^k)^3 > 10^{k+1}(\log 10^{k+1})^{2.1}$.

In general, given $\alpha$, suppose that $\sum_{i=1}^{10^k} a_i(\alpha) < 10^k(\log 10^k)^3$ for all $k \geq k^*$. Then for any $n \geq 10^k$, if $10^k \leq n \leq 10^{k+1}$, then
\[
\sum_{i=1}^{n} a_i(\alpha) \leq 10^{k+1}(\log 10^{k+1})^{3} \leq 10n(\log 10n)^3
\]
and the result holds after an appropriate choice of $C_1$. \hfill \Box

We will give a lower bound on $\sum_{j=1}^{q_n} \chi_{J_j}(x)$ by bounding below the sum over the $J_j$. As we noted above, $\{J_j\}_{i \in \mathbb{N}}$ is a disjoint set of intervals. Let
\[
h_i(x) = \sum_{j \in J_i^j} \chi_{J_j}(x).
\]

**Lemma 3.3.** For all $i$,
\[
\int_{[0,1]} h_i(x) d\lambda > 1/2.
\]

**Proof.** As per Lemma 2.10, over $j \in J_i^j$, the $V_j$ are disjoint, so $h_i(x) \in \{0,1\}$. The length of the interval $J_j$ is $q_i$, and for $j \in J_i^j$,
\[
\lambda(V_j) = \langle \langle q_{i-1} \alpha \rangle \rangle,
\]
using the description of $R_\alpha(U_j)$ provided by Lemma 2.8. By Theorem 13 in [4], $\langle \langle q_{i-1} \alpha \rangle \rangle > \frac{1}{q_i - q_{i-1} + q_i}$. We may then bound the integral from below by
\[
\int_{[0,1]} h_i(x) d\lambda > \frac{q_i}{q_i + q_{i-1}} > \frac{q_i}{2q_i} = \frac{1}{2}.
\]

\hfill \Box

The following sequence of results prove that the random variables $h_i(x)$ are (approximately) independent.

**Lemma 3.4.** Let $[c, d] \subset [0, 1]$. Let $f_{[c, d]}(i, b) = \# \{ [c, d] \cap \cup_{j \in J_i} R_\alpha^{-1}(0) \}$. Then
\[
\lambda ([c, d]) |J_b| - 2 \leq f_{[c, d]}(i, b) \leq \lambda ([c, d]) |J_b| + 2.
\]
with multiplicative errors yields

Furthermore, for any \( V \), as the sets \( V \) is hit by the left endpoints of the \( V \) intervals \( J \) contains exactly one point of \( R_a^{-l}(0) \) with \( 1 \leq l \leq q_m \). Recall that \( |J_b^l| = q_m \) where \( m = i - 1 \) if \( b = 1 \) and \( m = i \) if \( b > 1 \). Therefore, if \( a = \min J_b^l \), each \( I_j := R_a^{-a-1}(\frac{j}{q_m}, \frac{j+1}{q_m}) \) contains exactly one point of \( R_a^{-l}(0) \) for \( l \in J_b^l \).

At least \( \lambda([c, d])|J_b^l| - 2 \) of the \( I_j \) above are completely contained in \([c, d]\), and at most \( \lambda([c, d])|J_b^l| + 2 \) of them intersect \([c, d]\). The result then follows. \( \square \)

**Proposition 3.5.** Fix \( k \). For all \( i \) such that \( q_i > k \) and any \( 1 \leq b \leq a_{i+1} \),

\[
\left( \frac{\lambda(V_k)|J_b^l| - 3}{\lambda(V_k)|J_b^l|} \right) \lambda(V_k) \lambda \left( \bigcup_{l \in J_b^l} V_l \right)
\]

\[
\leq \lambda \left( V_k \cap \bigcup_{l \in J_b^l} V_l \right)
\]

\[
\leq \left( \frac{\lambda(V_k)|J_b^l| + 3}{\lambda(V_k)|J_b^l|} \right) \lambda(V_k) \lambda \left( \bigcup_{l \in J_b^l} V_l \right).
\]

**Proof.** Fix \( k \). Let \( i \) be so large that \( q_i > k \). By the previous lemma, the interval \( V_k \) is hit by the left endpoints of the \( V_i \) between \( \lambda(V_k)|J_b^l| - 2 \) and \( \lambda(V_k)|J_b^l| + 2 \) times. As the sets \( V_i \) are disjoint and of the same measure over \( l \in J_b^l \), this easily yields

\[
(\lambda(V_k)|J_b^l| - 3) \lambda(V_i) \leq \lambda \left( V_k \cap \bigcup_{l \in J_b^l} V_l \right) \leq (\lambda(V_k)|J_b^l| + 3) \lambda(V_i) \quad \text{for any } l^* \in J_b^l.
\]

Furthermore, for any \( l^* \in J_b^l, |J_b^l| \lambda(V_i) = \lambda(\bigcup_{l \in J_b^l} V_i). \) Translating to an inequality with multiplicative errors yields

\[
\left( \frac{\lambda(V_k)|J_b^l| - 3}{\lambda(V_k)|J_b^l|} \right) \lambda(V_k) \lambda \left( \bigcup_{l \in J_b^l} V_l \right)
\]

\[
\leq \lambda \left( V_k \cap \bigcup_{l \in J_b^l} V_l \right)
\]

\[
\leq \left( \frac{\lambda(V_k)|J_b^l| + 3}{\lambda(V_k)|J_b^l|} \right) \lambda(V_k) \lambda \left( \bigcup_{l \in J_b^l} V_l \right).
\]

\( \square \)

Proposition 3.5 asserts near independence of the events \( V_k \) and \( \bigcup_{l \in J_b^l} V_l \). Using it for all \( k \in J_b^l \), where \( j < i \) (which guarantees \( k < q_i \)) we get the following corollary. It relates to calculating the correlation between a point being undetermined in the intervals \( J_b^l \) and \( J_b^l \).

**Corollary 3.6.** For any \( k \in J_b^l \), and \( J_b^l, J_b^l \) disjoint, \( j > i \),

\[
\left( \frac{\lambda(V_k)|J_b^l| - 3}{\lambda(V_k)|J_b^l|} \right) \lambda \left( \bigcup_{k \in J_b^l} V_k \right) \lambda \left( \bigcup_{l \in J_b^l} V_l \right)
\]

\[
\leq \lambda \left( \bigcup_{k \in J_b^l} V_k \cap \bigcup_{l \in J_b^l} V_l \right)
\]
Proof. This follows from summing Proposition 3.5’s inequalities over the disjoint sets \( V_k \) for \( k \in J_q^i \). (The desire to compute this sum explains our preference for the formulation in terms of multiplicative bounds above.) \( \square \)

**Proposition 3.7.** For \( j > i \)

\[
\left(1 - \frac{3q_{i+1}}{q_j}\right) \int h_id\lambda \int h_jd\lambda \leq \int h_i h_j d\lambda \leq \left(1 + \frac{3q_{i+1}}{q_j}\right) \int h_id\lambda \int h_jd\lambda.
\]

Proof. First,

\[
\int h_i(x)h_j(x)d\lambda = \int \left(\sum_{l \in J_2} \chi_{V_l}(x)\right) \left(\sum_{l \in J_2} \chi_{V_l}(x)\right) d\lambda.
\]

As over \( J_2^i \) and over \( J_2^j \) the sets \( V_l \) are disjoint, the integrand of the above has value 0 or 1 according to whether \( x \in \left(\bigcup_{l \in J_2^i} V_l\right) \cap \left(\bigcup_{l \in J_2^j} V_l\right) \). Thus,

\[
\int h_i h_j d\lambda = \lambda \left(\bigcup_{l \in J_2^i} V_l \cap \bigcup_{l \in J_2^j} V_l\right).
\]

By Corollary 3.6, for \( l \in J_2^j \) we get

\[
\left(\frac{\lambda(V_l)|J_2^i|}{\lambda(V_l)|J_2^j|} - 3\right) \lambda \left(\bigcup_{l \in J_2^i} V_l\right) \lambda \left(\bigcup_{l \in J_2^j} V_l\right)
\]

\[
\leq \lambda \left(\bigcup_{l \in J_2^i} V_l \cap \bigcup_{l \in J_2^j} V_l\right)
\]

\[
\leq \left(\frac{\lambda(V_l)|J_2^i| + 3}{\lambda(V_l)|J_2^j|}\right) \lambda \left(\bigcup_{l \in J_2^i} V_l\right) \lambda \left(\bigcup_{l \in J_2^j} V_l\right).
\]

To assess the value of the terms \( \left(1 \pm \frac{3}{\lambda(V_l)|J_2^q|}\right) \) consider an arbitrary \( l \in J_2^j \). As \( U_l = R_{a_l} V_l \), using the description of \( R_{a_l} U_l \) given by Proposition 2.8 and [4, Theorem 13], \( \lambda(V_l) > \langle \langle q_l, a_l \rangle \rangle > \frac{1}{q_{l+1} + q_l} \geq \frac{1}{q_{l+1}} \). From its description, \( |J_2^q| = q_j \). Using these two bounds, \( \frac{3}{\lambda(V_l)|J_2^q|} < \frac{3q_{i+1}}{q_j} \).

Returning to our inequalities for \( \int h_i h_j \), as the \( V_l \) are disjoint over \( J_2^i \) or \( J_2^j \) we can translate back into integrals as so:

\[
\left(1 - \frac{3q_{i+1}}{q_j}\right) \int \sum_{l \in J_2^i} \chi_{V_l}(x) d\lambda \int \sum_{l \in J_2^j} \chi_{V_l}(x) d\lambda
\]

\[
\leq \int h_i h_j d\lambda \leq \left(1 + \frac{3q_{i+1}}{q_j}\right) \int \sum_{l \in J_2^i} \chi_{V_l}(x) d\lambda \int \sum_{l \in J_2^j} \chi_{V_l}(x) d\lambda.
\]

These are the desired bounds on \( \int h_i h_j d\lambda \). \( \square \)
The independence result we want is the following.

**Proposition 3.8.** There exist constants $C, b > 0$ such that
\[
\left| \int_{[0,1)} h_i(x)h_j(x)d\lambda - \int_{[0,1)} h_i(x)d\lambda \int_{[0,1)} h_j(x)d\lambda \right| < Ce^{-b|i-j|}.
\]

**Proof.** We may assume $j > i$. Using Proposition 3.7, we need to show that the expression
\[
\frac{3q_i+1}{q_j} \int h_i d\lambda \int h_j d\lambda
\]
decays exponentially in $|i-j|$. A clear upper bound on each of $\int h_i d\lambda, \int h_j d\lambda$ is 1. As $q_{k+2} > q_k, \frac{n+1}{q_j}$ decays exponentially in $|i-j|$, as desired. $\square$

We can apply this approximate independence to prove the remaining inequality in equation 3. Let $\tilde{h}_i(x) = h_i(x) - \int h_i(x)d\lambda$, and note that $\tilde{h}_i(x) \in (-1,1)$. Let $\tilde{s}_n(x) = \sum_{i=1}^n \tilde{h}_i(x)$.

**Proposition 3.9.** For almost every $x \in S^1$, for sufficiently large $n$,
\[
\sum_{j=1}^{q_i-1} \chi_{V_j}(x) > \frac{1}{4}(n-2).
\]

**Proof.** First, for all $x \in [0,1), \sum_{j=1}^{q_i-1} \chi_{V_j}(x) \geq \sum_{i=1}^{n-2} h_i(x)$ as $j \in J_i^j$ implies $j < q_i+2$.

Consider $\sum_{i=1}^{n-2} \int h_i(x)d\lambda$. By Lemma 3.3 this is bounded below by $\frac{1}{4}(n-2)$; it is bounded above by $n$ as $h_i$ takes only 1 or 0 as a value. Applying Chebyshev's inequality to $\tilde{s}_n$ yields (for any $\epsilon > 0$)
\[
\lambda(\{x : |\tilde{s}_{n-2}(x)| > \epsilon(n-2)\}) < \frac{\int \tilde{s}_{n-2}^2(x)d\lambda}{\epsilon^2(n-2)^2} < \frac{\sum_{i=1}^{n-2} \int \tilde{h}_i^2(x)d\lambda + 2 \sum_{i<j} \int \tilde{h}_i(x)\tilde{h}_j(x)d\lambda}{\epsilon^2(n-2)^2} \leq \frac{D}{\epsilon^2(n-2)^2}.
\]

For the last inequality we have used the facts that $\tilde{h}_i(x) \in (-1,1)$ and therefore $\sum_{i=1}^{n-2} \int \tilde{h}_i^2(x)d\lambda < n-2$, and that for some positive constant $D$, $2 \sum_{i<j} \int \tilde{h}_i\tilde{h}_j d\lambda < (D-1)(n-2)$ by Proposition 3.8.

We restrict our attention to the subsequence of times $\{(n-2)^2\}$, obtaining
\[
\lambda(\{x : |\tilde{s}_{(n-2)^2}(x)| > \epsilon(n-2)^2\}) < \frac{D}{\epsilon^2(n-2)^2}.
\]

Summing the term on the right-hand side of the above inequality over all $n$ yields a convergent series so by the Borel-Cantelli Lemma, for almost every $x \in [0,1)$,
\[
\frac{\tilde{s}_{(n-2)^2}(x)}{(n-2)^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

Consider now the intervals $\{(n-2)^2, (n-1)^2\}$. As $\tilde{h}_i(x) \in (-1,1)$, for $k \in [(n-2)^2, (n-1)^2)$,
\[
|\tilde{s}_{(n-2)^2}(x) - \tilde{s}_k(x)| < 2(n-2) + 1
\]
\[ \left| \hat{s}_k(x) \right| < \left| \hat{s}_{(n-2)^2}(x) \right| + \frac{2(n-2) + 1}{k} \leq \frac{\left| \hat{s}_{(n-2)^2}(x) \right| + 2(n-2) + 1}{(n-2)^2} \to 0 \]

as \( k \to \infty \).

We have now that for almost all \( x \),
\[ \sum_{i=1}^{n-2} h_i(x) - \int h_i(x) d\lambda \to 0. \]

As \( \sum_{i=1}^{n-2} \int h_i(x) d\lambda \in \left(\frac{1}{2}(n-2), (n-2)\right) \), for sufficiently large \( n \), \( \sum_{i=1}^{n-2} h_i(x) > \frac{1}{4}(n-2) \) as desired.

We now prove a similar series of inequalities for \( \sum_{j=1}^{q_n} \lambda(V_j) \), namely:
\[ C_1n(\log n)^3 > \sum_{i=1}^{n} a_i(\alpha) > \sum_{j=1}^{q_n-1} \lambda(V_j) > \frac{1}{2}(n-2). \quad (4) \]

The left-most inequality is Lemma 3.1 and the next is Corollary 2.11. It remains only to prove:

**Lemma 3.10.** For all \( \alpha \),
\[ \sum_{j=1}^{q_n-1} \lambda(V_j) > \frac{1}{2}(n-2). \]

**Proof.** This follows easily from Lemma 3.3 after noting that
\[ \sum_{j=1}^{q_n-1} \lambda(V_j) > \sum_{j=1}^{n-2} \sum_{j \in J} \lambda(V_j) = \sum_{i=1}^{n-2} \int h_i(x) d\lambda. \]

The inequalities collected above enable us to prove the main theorem:

**Proof of Theorem A.** Consider the full measure set of \( \alpha \) satisfying Lemma 3.1. For a given \( \alpha \), suppose \( n \in [q_m, q_{m+1}) \) is large enough for the bound in Lemma 3.1 to hold. Then we have the following for almost every \( x \):
\[ \frac{1}{4}(m-2) < \sum_{j=1}^{q_m-1} \chi_{V_j}(x) \leq \sum_{j=1}^{n} \chi_{V_j}(x) \leq \sum_{j=1}^{q_{m+1}-1} \chi_{V_j}(x) < C_1(m+1)(\log(m+1))^3, \]
and
\[ \frac{1}{2}(m-2) < \sum_{j=1}^{q_m-1} \lambda(V_j) \leq \sum_{j=1}^{n} \lambda(V_j) \leq \sum_{j=1}^{q_{m+1}-1} \lambda(V_j) < C_1(m+1)(\log(m+1))^3. \]

Taking logs and forming the relevant quotient, we see that the \( \log(m-2) \) and \( \log(m+1) \) terms dominate the \( \log(\text{constant}) \) and \( \log(\log(-)) \) terms. As \( \frac{\log(m)}{\log(m+1)} \) and \( \frac{\log(m+1)}{\log(m-2)} \to 1 \), the result follows.

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