ADJOINING ROOTS AND RATIONAL POWERS OF GENERATORS IN $PSL(2, \mathbb{R})$ AND DISCRETENESS

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Abstract. Let $G$ be a finitely generated group of isometries of $\mathbb{H}^m$, hyperbolic $m$-space, for some positive integer $m$. The discreteness problem is to determine whether or not $G$ is discrete. Even in the case of a two generator non-elementary subgroup of $\mathbb{H}^2$ (equivalently $PSL(2, \mathbb{R})$) the problem requires an algorithm [5, 6]. If $G$ is discrete, one can ask when adjoining an $n$th root of a generator results in a discrete group.

In this paper we address the issue for pairs of hyperbolic generators in $PSL(2, \mathbb{R})$ with disjoint axes and obtain necessary and sufficient conditions for adjoining roots for the case when the two hyperbolics have a hyperbolic product and are what as known as stopping generators for the Gilman-Maskit algorithm [5]. We give an algorithmic solution in other cases. It applies to all other types of pair of generators that arise in what is known as the intertwining case. The results are geometrically motivated and stated as such, but also can be given computationally using the corresponding matrices.

1. Introduction

Let $G$ be a finitely generated group of isometries of $\mathbb{H}^m$, hyperbolic $m$-space. The discreteness problem is to determine whether or not $G$ is discrete. Even the two generator non-elementary discreteness problem in $\mathbb{H}^2$ (or equivalently $PSL(2, \mathbb{R})$) requires an algorithm. One such algorithm is the Gilman-Maskit algorithm [5], termed the GM algorithm for short and also known as the intertwining algorithm, taken together with the intersecting axes algorithm [6]. The GM algorithm proceeds by considering geometric types of the pairs of generators. If $G$ is discrete, one can ask when adjoining an $n$th root of a generator results in a discrete group.

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Here we answer the discreteness question for adjoining for roots and rational powers of one or both generators in non-elementary two generator discrete subgroups of $PSL(2, \mathbb{R})$ found by the GM algorithm, the intertwining algorithm. When the algorithm is applied to a two generator group, the pair of generators at which discreteness is determined are termed *discrete stopping generators* and the generators correspond to a certain geometric configuration which we review below (see Section 5). The main result of the this paper is discreteness conditions on adjoining roots of a discrete stopping generator. In all other cases, that is the cases of non-stopping generators, discreteness can be determined by running the algorithm using the root as one of the generators.

The problem of adjoining roots has been addressed in [1, 3, 11]. Beardon gave a necessary and sufficient condition for a discrete group generated by a pair of parabolics and Parker obtained results for rational powers of a pair of generators in the case where neither generator was hyperbolic. In [3] discreteness conditions for hyperbolics are given by inequalities that depend upon the cross ratio and multipliers. Since our technique also applies to some of the intertwining cases that Beardon and Parker addressed but are different than their techniques, we include those cases, too.

The organization of this paper is as follows: In sections 2 and 3 notation is fixed and prior results needed are summarized. Results for square roots, arbitrary roots and their powers appear as Theorems 6.1, 7.1, 7.2 and 7.3. Their proofs are given in sections 6, 7, and 7.1. For example, in section 6 we find necessary and sufficient conditions (Theorem 6.1) for a group generated by a pair of hyperbolics discrete stopping generators with hyperbolic product to be discrete and free when a square root of a stopping generators is added. The results are extended to rational powers (Theorems 7.1 and 7.2) in Sections 7 and 7.1. In section 8 these theorems are extended and stated in greater generality as Theorems 8.1, 8.2 and 8.3.

2. Preliminaries: Notation and Terminology

We recall that elements of $Isom(\mathbb{H}^2)$ and $Isom(\mathbb{H}^3)$ are classified by their geometric action or equivalently by their traces when considered as elements of $PSL(2, \mathbb{R})$ or $PSL(2, \mathbb{C})$. In $\mathbb{H}^2$ they are either hyperbolic, parabolic or elliptic and we use $H, P$ and $E$ to denote such an element type. We consider their action using the unit disc model for $\mathbb{H}^2$. A hyperbolic elements fixes two points on the boundary of the unit disc, its ends, and the geodesic interior connecting these two points, its axis. A parabolic fixes one point on the boundary of the
unit disc and an elliptic fixes one point interior to the disc. In $\mathbb{H}^3$ an elliptic element has an axis; in $\mathbb{H}^2$ it is customary to consider the fixed point of an elliptic, its axis and in both $\mathbb{H}^3$ and $\mathbb{H}^2$ to consider the fixed point of the parabolic on the boundary of hyperbolic space its axis. All transformations fix their axes.

For any pair of points $r$ and $s$ in $\mathbb{H}^2$, we let $[r, s]$ denote the unique geodesic connecting the points. If $r$ is on the boundary we consider the point $r$ to be an (improper) geodesic following \cite{2}, denote it by $[r, r]$.

A hyperbolic transformation moves points along its axis a fixed distance in the hyperbolic metric, called it translation length toward one end, the attracting fixed point on the boundary and away from the other, the repelling fixed point. An elliptic transformation rotates by an angle $\theta$ about its fixed point where $\theta/2$ is the angle between the two geodesics $L$ and $M$ meeting at the fixed where the elliptic is the product of reflections in $L$ and $M$.

If $X$ is any geodesic, there is an orientation reversing element of order two that fixes $X$ and its ends that is called the half-turn about $X$ and denoted by $H_X$. It is a reflection through $X$\footnote{In $\mathbb{H}^3$ a half-turn about a geodesic is the orientation preserving element of order two fixing the geodesic point-wise. This can be viewed as the product of a reflection through the geodesic in any hyperbolic plane containing the geodesic and a reflection in the plane itself. Since the restriction of a half-turn to $\mathbb{H}^2$ is a reflection, it is customary to use $H_M$ there instead of $R_M$.}.

Any hyperbolic element of $Isom(\mathbb{H})^2$ can be factored in many ways as the product of two half turns about geodesics perpendicular to its axes. Here the two half-turn geodesics intersect the axis half the translation length apart. An elliptic element it is the product of two half-turn geodesics intersecting the axis (a point) and making an angle of $\theta/2$ with each other there. For a parabolic the half-turns geodesics intersect at the fixed point on the boundary.

The discreteness algorithm consists of two independent parts: the intertwining algorithm \cite{5} addresses pairs of hyperbolics with disjoint axes and other types of pairs that follow in this case and the intersecting axes case \cite{6}. The case of hyperbolics with intersecting axes and those that follow from it will be treated elsewhere. The ideas are similar but requires additional and different notation.

Note that we let $Ax_X$ denote the axis of $X$ if $X$ is any transformation but for clarity we sometimes also write for the axes (i) if $X$ is parabolic, the point on the boundary of the unit disc, $pp_X$ or $[p_X, p_X]$ using notation for an improper line as in Fenchel \cite{2} and (ii) if $X$ is elliptic $p_X$ with fixed point interior to the unit disc.
Following [11], we note that an \textit{nth root of a hyperbolic or parabolic} (and thus any rational power) is defined unambiguously. To define an \textit{nth root of an elliptic} we need to consider that it is always conjugate to \( z \mapsto Kz \) considered as an isometry in \( \text{PSL}(2, \mathbb{C}) \) and to take the root there and then conjugate back. Further, a \textit{geometrically primitive} root of an elliptic is an element that corresponds to a minimal rotation in the cyclic group it generates. Thus if \( E \) is a primitive rotation so is \( E^{-1} \). An element is \textit{algebraically primitive} if it generates the entire cyclic group, but here we do not consider such elements to be primitive. The algorithm assumes one can determine whether or not an elliptic is of finite order.

The figures here are schematic. All geodesics are perpendicular to the boundary the unit disc. Blue circles are used to indicate intersections that are perpendicular. Figures for some representative cases are presented, but these are not exhaustive.

3. Preliminaries: the GM algorithm, \( G \), and Hexagons

Assume that \( G = \langle A, B \rangle \) is a non-elementary two generator subgroup of \( \text{PSL}(2, \mathbb{R}) \). The Gilman-Maskit discreteness algorithm considers the intertwining cases. The GM algorithm begins with a pair of hyperbolic generators with disjoint axes and at each step either stop and outputs that the group is discrete or that the group is not discrete, or outputs the next pair of generators to consider. An implementation of the algorithm can begin with any geometric type of pairs of generators that arise in the algorithm. The generators where the algorithm outputs discreteness are termed the \textit{discrete stopping generators}.

Given \( A \) and \( B \), there will be a unique geodesic \( L \), the \textit{core geodesic}, that is a common perpendicular to their axes. We assume that \( L \) is oriented from the axis of \( A \) towards the axis of \( B \).

Further, given \( L \), we can find geodesics \( L_A \) and \( L_B \) such that \( A = H_L H_{L_A} \) and \( B = H_L H_{L_B} \) so that \( A^{-1}B = H_{L_A} H_{L_B} \). We let \( 3G = \langle H_L, H_{L_A}, H_{L_B} \rangle \). We note that \( 3G \) and \( G \) are simultaneously discrete or non-discrete as \( G \) is a subgroup of index 2 in \( 3G \).

The axes and half-turn lines determine a geometric configuration, a hexagon (see [2]). For a given \( 3G \) the hexagon may or may not be convex. (See Figure 1 for examples of a convex and a non-convex hexagon.) The hexagon will have three axis sides and three half-turn sides. In \( \mathbb{H}^2 \) one or more of the axis sides may reduce to a point that is interior or on the boundary, but the half-turn sides will not.

The geodesics that determine the sides of the hexagon will have subintervals that actually correspond to sides of the hexagon and it
will generally be clear from the context when whether we are talking about a side or the entire geodesic. For any positive integer \( n \), there are geodesics \( L_B^2, L_B^3, L_B^4 \cdots \) such that \( B = L_B^{n-1} L_B^n \) and \( B^n = H_L H_{L_B^n} \). We can also find geodesics \( L_B^{1/2} \) or \( L_B^{1/n} \) with \( B^{1/2} = H_L H_{L_B^{1/2}} \) and \( B^{1/n} = H_L H_{L_B^{1/n}} \). When \( B \) is hyperbolic these geodesics are perpendicular to \( Ax_B \). When \( B \) is parabolic or elliptic, the geodesics pass through the point that is \( Ax_B \).

4. First Result

We begin with the following lemma.

**Lemma 4.1.** Let \( G \) be any group generated by half-turns about three disjoint geodesics, \( L, M, N \). If the half turn geodesics bound a region, that is, no one half-turn geodesic separates the other two, then \( G \) is discrete and free.

If one or more pairs of half-turns intersect, the product of the pair is elliptic or parabolic. If the product is elliptic of finite order and the
angle between the half-turn geodesics of the elliptic is half a primitive angle or parabolic, then the group is discrete providing the half-turn geodesics still bound a region and the transformations are oriented so that the vertex angle hypotheses of the Poincaré Polygon Theorem apply. If the pairs of half-turns only intersect on the boundary, then the group is also free.

Proof. Apply the Poincaré Polygon Theorem [1] or [9]. □

5. Geometric Stopping Generators and Discrete Stopping Configurations

Let H, P and E denote respectively a hyperbolic, parabolic, or elliptic generator.

Note that the stopping configurations which are all hexagons, may look like hyperbolic pentagons, quadrilaterals or triangles because the axes may be points in \( \mathbb{H}^2 \) and also note that when we discuss discrete stopping generators that include elliptic we assume the generator to be geometrically primitive.

We illustrate some, but not all, figures for the discrete stopping cases in Figure 2.

**Theorem 5.1.** For each of the eleven possible ordered stopping configurations the hexagon is convex and satisfies the vertex hypotheses of the Poincaré Polygon theorem in the case of an elliptic generator.

Proof. For each pair ordered pair of generators where the order of the elements is determined by type, there are three types of subcases which are H, P, or E. We list those that are discrete stopping configurations following [5]: page 16 (I-7); page 24, (II-5); page 25 (III-5), page 26 (IV-5); page 29 Theorem; page 30 (VI-6), page 30 (VI-9), they are

1. HxH (i) H: \( Ax_A, L, Ax_B, L_B, Ax_B^{-1}, L_A \)
2. HxP (i) H: \( Ax_A, L, pp_B, L_B, Ax_B^{-1}, L_A \)
3. PxP (i) H: \( pp_A, L, pp_B, L_B, AX_B^{-1}, L_A \); (ii) P: \( pp_A, L, pp_B, L_B, pp_A^{-1}, L_A \)
4. HxE (i) H: \( Ax_A, L, p_B, L_B, Ax_B^{-1}, L_A \) (ii) P: \( Ax_A, L, p_B, L_B, pp_A^{-1}, L_A \)
5. PxE (i) H: \( pp_A, L, p_B, L_B, AX_B^{-1}, L_A \) (ii) P: \( p_A, L, p_B, L_B, pp_A^{-1}, L_A \)
6. ExE (i) H: \( p_A, L, p_B, L_B, AX_B^{-1}, L_A \) (ii) P: \( p_A, L, p_B, L_B, pp_A^{-1}, L_A \)
   (iii) E: \( p_A, L, p_B, L_B, pA^{-1}, L_A \)

Because some half-turn lines reduce to points, for clarity we identify the geometry of the stopping configurations more specifically as follows
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in the next Theorem. The references to $\ref{2}$ and $\ref{1}$ are to be taken modulo a permutation of the order of the generators illustrated in the figures.

**Theorem 5.2.** The discrete stopping configurations are

1. $HxHxH$ The configuration is a convex hexagon as shown in Figure 1.

2. $HxPxH$ The configuration is a pentagon, the convex hexagon of Figure 2 where the $AX_B$ is replaced by a point on the boundary of the unit disc where $L$ and $L_B$ meet. See Figure 2.

3. $PxP$ The configuration is a pentagon, the convex hexagon of Figure 2 where the $AX_A$ and the $AX_B$ are replaced respectively by points $p_A$ and $p_B$ on the boundary of the unit disc where one of the following happens: $L_A$ and $L_B$ are disjoint so the figure looks like a quadrilateral or $L_A$ and $L_B$ intersect on the boundary so the figure looks like a triangle with all each vertex on the boundary of the disc.

**Figure 2.** Some discrete stopping configuration $HEH, HHE, HPH, PEP$
(4) HxExE Here the $Ax_B$ is replaced by a point interior to the unit disc where $L$ and $L_B$ meet, and either $L_A$ and $L_B$ are disjoint, so that $A^{-1}B$ is hyperbolic and the figure is a pentagon (see Figure 2) or $L_A$ and $L_B$ intersect on the boundary with $A^{-1}B$ parabolic and the figure is a quadrilateral.

(5) PxExE The configuration is a pentagon, the convex hexagon of where the $Ax_A$ and the $Ax_B$ are replaced respectively by points $p_A$ on the boundary and $p_B$ interior to the unit disc with $L$ the geodesic connecting these two points and where $L$ and $L_A$ meet at $p_A$ and $L$ and $L_B$ at $p_B$. If $L_A$ and $L_B$ are disjoint, the figure is a quadrilateral and if $L_A$ and $L_B$ intersect on the boundary or in the interior, the figure is a triangle (see Figure 2).

(6) ExExE There are three cases: The hexagon reduces to

(i) a quadrilateral with $L_A$ and $L_B$ disjoint when we have ExExH: $p_A, L, p_B, L_B, Ax_A^{-1}B, L_A$. (ii) a triangle with two interior vertices and one on the boundary of the disc when we have ExExP: $p_A, L, p_B, L_B, pp_A^{-1}B, L_A$ (iii) a triangle with all interior vertices when we have ExExExE: $p_A, L, p_B, L_B, p_A^{-1}B, L_A$.

Proof. Follow the GM algorithm through to each discrete stopping case.

Remark 5.1. In the above lists, the stopping generators are given in the order found in the GM algorithm. Later we will see that for our purposes the order does not matter.

Remark 5.2. Thus in what follows we can modify the cyclic order of the stopping generators and consider, for example, HHP and HHE instead of HPH and HEH. That is, the discrete stopping configuration can be rotated, as needed.

Remark 5.3. We note that in all of these cases the convex stopping hexagons lie below (that is, to the right of) $L$ if $L$ is oriented from the axis of $A$ towards the axis of $B$ and the smaller rotation angles of elliptics and parabolics are interior to the hexagon. This assumption allows to ignore consideration of traces of pull-back to $SL(2, \mathbb{R})$ or coherent orientation used in other papers.

We begin with square roots.

6. Adjoining Square Roots

We consider adjoining $B^{1/2}$, the square root of $B$, in the cases above where $B$ is hyperbolic. The results depends upon the location of $L_{B^{1/2}}$
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Figure 3. $A^{-1}B$ hyperbolic and $L_{B/2}$ intersects $Ax_A$, $L_A$

Figure 4. $A^{-1}B$ hyperbolic and $L_{B1/2}$ (iii) intersects $Ax_{A^{-1}B}$ in its interior or (iv) intersects $L_A$ at a vertex

as it enters and exits the hexagon. There are essentially three possibilities, but since the conclusion includes the possibilities of the new group being either free or not free, the results of the theorem are stated using more cases. In Figures 3 and 4 we show some possible locations for $L_{B^{1/2}}$. The hyperbolic law of sines is used to position some of the geodesics.

**Theorem 6.1.** Assume that $(A, B)$ are a pair of hyperbolic discrete stopping generators for $G = \langle A, B \rangle$ with hexagon sides $Ax_A, L, Ax_B, L_B, Ax_{A^{-1}B}, L_A$. Let $L_{B^{1/2}}$ be chosen perpendicular to $Ax_B$ so that $B^{1/2} = H_L H_{L_{B^{1/2}}}$. 

(H) If $A^{-1}B$ also hyperbolic, then $\hat{G} = \langle A, B^{1/2} \rangle$ is discrete and free $\iff$ either
(i) \(L_{B^{1/2}} \cap Ax_A \neq \emptyset\) or (ii) \(L_{B^{1/2}} \cap Ax_{A^{-1}B} \neq \emptyset\) but neither intersection is a vertex of the hexagon interior to \(\mathbb{H}^2\).

\[\textbf{(P)}\] If \(A^{-1}B\) is parabolic, then \(\hat{G} = \langle A, B^{1/2} \rangle\) is discrete and free \iff\ either

(i) \(L_{B^{1/2}} \cap Ax_A \neq \emptyset\) or (ii) \(L_{B^{1/2}} \cap Ax_{A^{-1}B} \neq \emptyset\) but neither intersection is a vertex of the hexagon interior to \(\mathbb{H}^2\).

\[\textbf{(E)}\] If \(A^{-1}B\) is elliptic, it is primitive since it is a stopping generator. Then \(\hat{G} = \langle A, B^{1/2} \rangle\) is discrete \iff\ either

(i) \(L_{B^{1/2}} \cap Ax_A \neq \emptyset\) but the intersection is not at the vertex \(Ax_A \cap L_A\) or

(ii) \(L_{B^{1/2}} \cap L_A \neq \emptyset\) with \(H_L A H_L B^{1/2}\) primitive elliptic

In all other cases, the group is not free and one applies the algorithm to the case where \(A^{-1}B\) is elliptic to determine discreteness.

Proof. We consider the \(HHH\) case and work first with the ordered hyperbolic generators \(A, B, A^{-1}B\).

Since the hexagon is convex and \(L_{B^{1/2}}\) is perpendicular to the Axis of \(B\) intersecting it along the interior side of the axis, it must also either intersects \(Ax_A, L_A\) or \(Ax_{A^{-1}B}\). If it intersects \(L_A\), then \(H_L A H_L B^{1/2}\) is elliptic so \(\hat{G}\) is not free except and not discrete except possibly when the elliptic is primitive. If \(L_{B^{1/2}}\) it intersects \(Ax_A\) but not at a vertex of the hexagon, then the lower region, the region of the hexagon below \(L_{B^{1/2}}\) is part of the hexagon for \(G_2 = \langle A^{-1}B^{1/2}, B^{1/2} \rangle\). This is the hexagon with sides \(Ax_{A^{-1}B^{1/2}}, L_A, Ax_{A^{-1}B}, L_B, Ax_B, L_{B^{1/2}}\). Thus \(G_2\) is discrete and free because the half-turn lines bound a region. If \(L_{B^{1/2}}\) intersects \(Ax_{A^{-1}B}\), then the region of the hexagon above \(L_{B^{1/2}}\) is a hexagon with half-turn lines \(L, L_{B^{1/2}}, L_A\) with axis sides along \(Ax_A, Ax_{A^{-1}B^{1/2}}, Ax_B\). Thus the group \(G_2 = \langle A, B^{1/2} \rangle\) is discrete and free. Of course, \(G_2\) is the same group as \(\hat{G}\).

The analysis of the cases for \(A^{-1}B\) elliptic or parabolic are similar and thus omitted after noting that in the case that \(A^{-1}B\) is primitive elliptic.

\[\square\]

7. Adjoining \(n\)th Roots

The ideas used in adjoining \(n\)th roots are the same as in adjoining square roots except that there are more cases to consider depending upon where \(L_B^{1/n}\) intersects the hexagon and which of its powers intersect an interior side of the hexagon and which interior side, which
intersect vertices and which do not intersect vertices. For ease of exposition we refer to a vertex as a side since it is a degenerate side. Write \( v_{A \cap L_A} \) and \( v_{L_A \cap A^{-1}B} \) for the vertices.

For any stopping configuration we have a hexagon corresponding to transformations \( A, B \) and \( A^{-1}B \) and half-turn sides \( L, L_A, L_B \). We consider \( n \)th roots of \( B \). The segment of \( L_{B^{1/n}} \) that passes through the hexagon can exit along \( L_A, Ax_A \) or \( Ax_A^{-1}B \) (It cannot cross \( L \) or \( Ax_X \) or \( L_B \)). If it crosses \( L_A \) then the group has an elliptic element and one goes to the elliptic case where \( A^{-1}B \) is elliptic and then applies the algorithm appropriately.

While we know that \( L_{B^{1/n}} \) intersects \( Ax_A \) in its interior, there are five choices for where each of the other \( L \)-lines described below exits the hexagon: interior to \( Ax_A \), interior to \( L_A \), interior to \( Ax_A^{-1}B \), at \( v_{A \cap L_A} \) or at \( v_{L_A \cap Ax_A^{-1}B} \). The cases we need to consider involve two integers \( r \) and \( s \) with \( 1 < s < r < n \). We assume that going counter clockwise from \( L_{B^{1/n}} \), one encounters next \( L_{B^{1/n}}^{(s)} \) and then \( L_{B^{1/n}}^{(s+1)} \). We term these integers splitting integers if they determine a jump in the side of the hexagon that these \( L \)-lines intersect. That is, if \( L_{B^{1/n}}^{(s)} \) and \( L_{B^{1/n}}^{(s+1)} \) do not intersect the same side.

Considering all of the possibilities gives:

**Theorem 7.1.** If \( A, B \) are hyperbolic discrete stopping generators with \( A^{-1}B \) hyperbolic, parabolic or elliptic, consider the stopping configuration along with \( L_{B^{1/n}} \) and \( L_{B^{r/n}} \) for \( r \) an integer with \( 1 \leq r \leq n \). Let \( s \) be an integer with \( 1 \leq s \leq r \) so that \( r \) and \( s \) are splitting integers.

**CASE I: \( L_{B^{r/n}} \) does not intersect any vertex of the hexagon.**

**H** Assume that \( A^{-1}B \) also hyperbolic. Then \( \hat{G} = \langle A, B^{1/n} \rangle \) is discrete and free \( \iff \) either

1. \( L_{B^{1/n}} \cap Ax_A^{-1}B \neq \emptyset \) or
2. \( L_{B^{(s/n)}} \cap Ax_A \neq \emptyset \) and \( L_{B^{(s+1)/n}} \cap Ax_A^{-1}B \neq \emptyset \), for some integer \( s \leq r \).

If \( L_{B^{(s/n)}} \cap L_A \neq \emptyset \) for some integer \( 1 \leq s \leq n \), then \( \langle A, B^{1/n} \rangle \) is not free, it may be discrete if \( A^{-1}B^{s/n} \) is primitive, otherwise one must go to an appropriate elliptic case of the algorithm to determine discreteness.

**P** Assume that \( A^{-1}B \) is parabolic so that its axis is a boundary vertex. Then \( \hat{G} = \langle A, B^{1/n} \rangle \) is discrete and free \( \iff \)

\[ L_{B^{(s/n)}} \cap Ax_A \neq \emptyset \ \forall s, \ 1 \leq s \leq r. \]

If \( L_{B^{(s/n)}} \cap L_A \neq \emptyset \) for some integer \( 1 \leq s \leq n \) but the intersection is not at the point \( Ax_A^{-1}B \), then \( \hat{G} = \langle A, B^{1/n} \rangle \) is not free. It is discrete if
$A^{-1}B^{s/n}$ is primitive. Otherwise one must go to an appropriate elliptic case of the algorithm to determine discreteness.

**E** Assume $A^{-1}B$ is elliptic so that its axis is an interior point. Then $\hat{G} = \langle A, B^{1/n} \rangle$ is discrete if $L_{B^{-1}} \cap Ax_A = \emptyset \quad \forall s$ and $L_{B^{-1}} \cap Ax_{A^{-1}B} = \emptyset$ or if $L_{B^{-1}} \cap Ax_{A^{-1}B} = Ax_A \cap L_A$ for some $s$ and $A^{-1}B^{1/n}$ is primitive elliptic.

For all other cases, the elliptic case of algorithm must be applied to determine discreteness.

**CASE II:** $L_{B^{r/n}}$ intersects a vertex, either $v_{L_A \cap Ax_A}$ or $v_{L_A \cap Ax_{A^{-1}B}}$

If either of these intersections are on the boundary of $\mathbb{H}^2$, the group is discrete and free. Intersections at interior vertices will give elliptic elements and the group will be discrete if the rotation of the elliptic is primitive. Otherwise apply the GM algorithm for elliptic elements.

It follows immediately that

**Theorem 7.2.** If $s/n$ is a rational number with $s/n > 1$ and $s = w + r$, we note that $\langle A, B^w \rangle$ is discrete whenever $\langle A, B \rangle$ is and we can apply the Theorem 7.1 then to $\langle A, Y^{r/n} \rangle$ where $Y = B^w$.

**7.1. B parabolic or elliptic.** In the case that the stopping generator $B$ is parabolic or elliptic, $L_{B^{1/n}}$ will have a segment that begins at $Ax_B$ which in this case is a point and passes through the interior of the stopping hexagon and the options for exiting the hexagon are unchanged. Thus we can conclude

**Theorem 7.3.** If $B$ is parabolic, the conclusions of Theorem 7.1 still apply, as do those of Theorem 7.2.

If $B$ is primitive elliptic, the conclusions of Theorem 7.1 with $\hat{G}$ discrete, but not free. The conclusions of Theorem 7.2 also apply again with $\hat{G}$ discrete, but not free.

**8. General Formulation Theorems**

In this section we state the results above in greater generality. Assume that $(X, Y)$ are discrete stopping generators. This means that the hexagon is convex and that the angles at the any elliptic vertices are half of a primitive elliptic angle and the direction of rotation of parabolics and elliptics is interior to the hexagon.

The hexagon sides are $Ax_X$, $Ax_Y$ and $AX_{X^{-1}Y}$ and the half-turn geodesic sides as $L$, $L_X$, and $L_Y$. All of the half-turn sides are subintervals of proper geodesics. The axis sides may be single points. In the case
of an elliptic element its axis is a point interior to $\mathbb{H}^2$ and in the case of a parabolic element its axis is a point on the boundary of $\mathbb{H}^2$.

**Theorem 8.1.** Assume that $(X, Y)$ are discrete stopping generators so that the hexagon with sides $Ax, X^{-1}Y, L, L^{-1}Y, L^{-1}X, LX$ is a convex stopping hexagon. Let $\hat{G} = \langle X^{1/n}, Y \rangle$ where $n$ is a positive integer and $X$ is any type of transformation, $H, E$ or $P$. Let $L_{X^{1/n}}$ be the geodesic with $X^{1/n} = L \circ L_{X^{1/n}}$.

There are three possibilities:

1. $L_{X^{1/n}} \cap Ax \neq \emptyset$
2. $L_{X^{1/n}} \cap Ax^{-1}Y \neq \emptyset$
3. $L_{X^{1/n}} \cap L \neq \emptyset$.

We have

**No vertex intersections:** Assume that none of these intersections are at vertices of the hexagon, then

$$\hat{G} = \langle X^{1/n}, Y \rangle$$

is discrete $\iff$

- item 1 or 2 occurs or
- item 3 occurs with $H_X H_Y$ a primitive rotation.

**Vertex Intersections:** An intersection that occurs at a vertex will be either at $Ax \cap L$ or $Ax^{-1}Y \cap L$. The group $\hat{G}$ is discrete

- if the vertex is interior and the rotation there is primitive.

The group $\hat{G}$ is free

- in cases 1 or 2 or
- if 3 occurs with the intersection point on the boundary of $\mathbb{H}^2$.

**Proof.** This follows directly from applying Theorems 6.1, 7.1 and 7.2 but allowing the order of the generators to be cyclically permuted.

We have

**Theorem 8.2.** If $X, Y$ are hyperbolic discrete stopping generators with $X^{-1}Y$ hyperbolic, parabolic or elliptic, consider the stopping configuration along with $L_{X^{1/n}}$ and $L_{X^{r/n}}$ for $r$ an integer with $1 \leq r \leq n$. Let $s$ be an integer with $1 \leq s \leq r$.

**Case 1:** Assume first that $L_{X^{r/n}}$ does not intersect any vertex of the hexagon.

**IH:** Assume that $X^{-1}Y$ also hyperbolic.
Then $\hat{G} = \langle X^{1/n}, Y \rangle$ is discrete and free $\iff$ either

1. $L_{X^{1/n}} \cap Ax_{X^{-1}Y} \neq \emptyset$ or
2. $L_{X^{(s)/n}} \cap Ax_{Y} \neq \emptyset$ and $L_{X^{(s+1)/n}} \cap Ax_{X^{-1}Y} \neq \emptyset$, for some integer $s \leq r$.

If $L_{X^{s/n}} \cap L_{Y} \neq \emptyset$ for some integer $1 \leq s \leq n$, then $\langle X^{-1}, Y \rangle$ is not free.

It may be discrete if $X^{s/n}Y^{-1}$ is primitive, otherwise one must go to an appropriate elliptic case of the algorithm to determine discreteness.

**IP:** Assume that $X^{-1}Y$ is parabolic so that its axis is a boundary vertex.

Then $\hat{G} = \langle X^{1/n}, Y \rangle$ is discrete and free $\iff$

$$L_{X^{(s)/n}} \cap Ax_{Y} \neq \emptyset \ \forall s, 1 \leq s \leq r.$$ 

If $L_{X^{s/n}} \cap L_{Y} \neq \emptyset$ for some integer $1 \leq s \leq n$ but the intersection is not at the point $Ax_{X^{-1}Y}$, then $\hat{G} = \langle X^{1/n}, Y \rangle$ is not free.

It is discrete if $X^{s/n}Y^{-1}$ is primitive.

Otherwise one must go to an appropriate elliptic case of the algorithm to determine discreteness.

**IE:** Assume $X^{-1}Y$ is elliptic so that its axis is an interior point.

Then $\hat{G} = \langle X^{1/n}, Y \rangle$ is discrete if

$$L_{X^{(s)/n}} \cap Ax_{Y} \neq \emptyset \ \forall s \text{ but } L_{X^{s/n}} \cap L_{Y} = \emptyset \text{ unless the intersection is at } Ax_{Y} \cap L_{Y} \text{ and } H_{X^{s/n}}H_{L_{Y}} \text{ is primitive elliptic.}$$

For all other cases, the elliptic case of algorithm must be applied to determine discreteness.

**II:** If $L_{X^{r/n}}$ intersects a vertex, it must be at $L_{Y} \cap Ax_{Y}$ or $L_{Y} \cap Ax_{X^{-1}Y}$. If either of these intersections are on the boundary of $\mathbb{H}^2$, the group is discrete and free. Intersections at interior vertices will give elliptic elements and the group will be discrete if the rotation of the elliptic is primitive. Otherwise apply the elliptic cases of the algorithm.

**Case III:** If $X$ is parabolic, the conclusion of 8.1 still apply.

**Case IV:** If $X$ is primitive elliptic, the conclusion of 8.1 with $\hat{G}$ discrete but not free.

**Proof.** Applying Theorems 7.1, 7.3 and 8.1 to the permutation $(X, Y, X^{-1}Y)$ of the triple $(X, Y, X^{-1}Y)$.

$\square$
If \( s, n, w \) and \( r \) are positive integers with \( s/n > 1 \) and \( S = wn + r \), then \( G = \langle X^w, Y \rangle \) is discrete whenever \( \langle X, Y \rangle \) is. Applying the above we have immediately

**Theorem 8.3.** [Powers of Roots] Let \( s, n, w \) and \( r \) be positive integers with \( s/n > 1 \) and \( s = wn + r \). The group \( \langle X^w, Y \rangle \) is discrete whenever \( \langle Z, Y \rangle \) is where \( Z = X^{r/n} \) and the discreteness of \( \langle Z, Y \rangle \) can be determined by Theorem 8.2.

### 9. Miscellaneous Remarks

**Remark 9.1. Roots of Non-stopping generators** A generator \( X \) for a rank two discrete free group \( G = \langle A, B \rangle \) is a primitive generator if there exists an element \( Y \) such that \( G = \langle X, Y \rangle \) \([10]\). The pair \( (X, Y) \) is called a a primitive pair.

Given a primitive pair, if \( G \) is discrete and free, there is a sequence of integers, known as the F-sequence or the Fibonacci sequence, \( [n_1, ..., n_t] \) such that the sequence stops at a pair \( (C, D) \) of discrete stopping generators after applying appropriate Nielsen transformations determined by the \( n_i \) starting with the pair \( (X, Y) \). Using the reverse F sequence, one can write \( X \) and \( Y \) as words in the stopping generators and thus obtain \( \langle X, Y \rangle \) as words in \( \langle C, D \rangle \). One can apply the GM algorithm to \( \langle X^{r/n}, Y \rangle \) to see whether the group is discrete or not. Starting with \( X \) and \( Y \) written as words in \( C \) and \( D \) will often shorten the implementation of the algorithm. Alternately, if it is known that \( \langle X, Y \rangle \) is discrete and free, one can apply the GM algorithm to find its stopping generators \( (C, D) \) and then write \( \langle X, Y^{r/n} \rangle \) as words in \( \langle C, D \rangle \) before running the algorithm.

**Remark 9.2.** If \( G \) contains elliptic elements, there is an extended F-sequence \([8]\). It contains extra terms that correspond geometrically to replacing an elliptic element by its primitive power and the extra integer is that power. The same idea applies.

**Remark 9.3. Matrix calculations** Using Fenchel’s theory of matrices and extending it as necessary allows one to turn these geometric algorithms into purely computational matrix procedures. We use, for example, we some of the following results from \([2]\). (i) If \( f \in SL(2, \mathbb{C}) \) is a matrix determining a transformation \( f \), then \( f - f^{-1} \) is a line matrix. It corresponds to a half-turn about a geodesic whose ends are the fixed points of the line matrix. The geodesic is, of course, the axis of \( f \). (ii) If \( f \) and \( g \) are transformations with distinct axes and with line matrices \( L_f \) and \( L_g \). The axes of \( f \) and \( g \) are perpendicular if the trace of \( L_g L_f = 0 \). This holds even if \( Ax_f \) and or \( Ax_g \) are improper lines.
(iii) The trace of $fg$ tells us the angle of intersection (see also [1]). The computational matrix theory is developed in full detail in [4].

**Remark 9.4.** The question has been raised as to whether this translates to an algebraic treatment using the Purzitsky-Rosenberger trace minimizing algorithm [12, 13, 14]. The trace minimizing method is to replace $(A, L_{B^1/n})$ when $Tr A \geq Tr L_{B^1/n}$ by one of the ordered pairs $(L_{B^1/n}, AL_{B^1/n})$, $(L_{B^1/n}, AL_{B^1/n}^{-1})$, $(AL_{B^1/n}, L_{B^1/n})$ or $(AL_{B^1/n}^{-1}, L_{B^1/n})$ depending upon the sizes of the traces. Thus it seems that one would have to start the algorithm with $\langle A, L_{B^1/n} \rangle$ and that even if the pair $(A, B)$ were the algebraic stopping generators, computations would have to be carried out to reflect the intersection properties or the algorithmic steps.

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ADJOINING ROOTS AND RATIONAL POWERS OF GENERATORS IN $PSL(2, \mathbb{R})$ AND DISCRETENESS

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