LOCAL-IN-SPACE BLOW-UP CRITERIA FOR
TWO-COMPONENT NONLINEAR DISPERSEVE WAVE SYSTEM

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Abstract. We investigate the blow-up phenomena for the two-component generalizations of Camassa-Holm equation on the real line. We establish some a local-in-space blow-up criterion for system of coupled equations under certain natural initial profiles. Presented result extends and specifies the earlier blow-up criteria for such type systems.

1. Introduction. In this paper, we consider the following two-component Cauchy problem for the generalized Camassa-Holm equation

\[ u_t - u_{xxt} + 3u_x u - uu_{xxx} - 2uu_{xx}u_x + [g(u)]_x + \rho \rho_x = 0 \] (1)

\[ \rho_t + (\rho u)_x = 0 \] (2)

\[ u(x, 0) = u_0(x) \] (3)

\[ \rho(x, 0) = \rho_0(x) \] (4)

where \( u(t, x) \) denotes the horizontal velocity of the fluid and \( \rho(t, x) \) is a parameter related to the free surface elevation from equilibrium (or scalar density). When \( g(u) = ku \) and \( \rho \equiv 0 \) (1) becomes the Camassa-Holm equation

\[ u_t - u_{xxt} + ku_x + 3uu_x + \rho \rho_x = 2uu_{xx} + uu_{xxx}, \quad t > 0, x \in \mathbb{R} \] (5)

where \( k \) is a dispersive coefficient related to the critical shallow water speed. Two-component model for C-H was first derived in [49] and can be viewed as an important model in the context of shallow water theory [13, 26, 36]. It is worth to note that the papers [13] and [26] incorporate vorticity into the fluid model (physically, vorticity is vital for incorporating the ubiquitous effects of currents and wave-current interactions in fluid motion, also the mathematical analysis of the full-governing equations for water waves with vorticity is of particular interest). In the paper [35] a classification of integrable two-component systems of non-evolutionary partial differential equations that are analogous to the Camassa–Holm equation is carried

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out via the perturbative symmetry approach. In [13] the authors show that the only way for singularities to occur in smooth solutions is through wave breaking, whereby the solution $u$ remains bounded but the spatial derivative becomes unbounded in finite time. The occurrence of the wave-breaking phenomenon is established for suitably defined initial data. Furthermore, in [13] authors also prove the global existence of both small amplitude solutions of (2)-(5), and large amplitude travelling wave solutions with initial data having a suitable rate of decay. An interesting consequence of these investigations is that the solitary wave solutions of must be smooth, thus ruling out the existence of peakon solutions for this particular two-component CH generalisation. In [25] local well-posedness for the two component Camassa-Holm system with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, with $s \geq 2$ have been investigated by applying Kato’s theory [39], also in this provided some precise blow-up scenarios for strong solutions to the system. The local well-posedness is improved by Gui and Liu [29] to the Besov spaces, and authors also showed that the finite time blow-up is determined by either the slope of the first component $u$ or the slope of the second component (also see [13, 25]). The blow-up criterion is made more precise in [41], where the authors showed that the wave-breaking in finite time only depends on the slope of $u$. Properties of the modified two-component Camassa-Holm system also have been considered in many works, e.g. [1, 11, 12, 28, 30, 41, 56, 51].

The details concerning the hydrodynamical relevance of Camassa–Holm equation were mathematically rigorously described by Constantin and Lannes in [15] (earlier Johnson [38] derived (formally) the CH equation from the full governing equations by using asymptotic analysis), where, in addition, authors investigate in what sense model under consideration gives us insight into the wave breaking phenomenon. The equation has bi-Hamiltonian structure [27] and is completely integrable [2, 3, 10, 16, 20]. When $k = 0$, Camassa–Holm equation possesses a solitary waves with discontinuous first derivatives ([9]), which named “peakon” (travelling wave solutions with a corner at their peak). It is worth pointing out that these solutions resemble the Stokes waves of greatest height which arise as extreme travelling wave solutions to the governing equations for water waves in irrotational flow (see the discussion in the papers [17, 18, 19, 42, 43, 44]). The fact that after wave breaking the solution of the Camassa-Holm equation can be continued uniquely as either global conservative [7] or global dissipative solutions has been noticed by Bressan and Constantin [8]. It is also worth mentioning that from the point of view of theory of water waves the fact that solutions that originate from smooth localized initial data can develop singularities only in the form of breaking waves, as proved in the paper [21], is especially interesting. Wave breaking for a large class of initial data has been established in [14, 40, 37]. The infinite propagation speed for the Camassa–Holm equation (and the two-component extensions of it) for $k = 0$ was investigated in [22, 31, 32, 33, 34] (see also [47] for $k \neq 0$). Also, the wide range of problems for CH equation with non-zero dispersion coefficient was considered in [40, 54, 47, 46]. In particular, certain conditions on the initial datum to guarantee that the corresponding solution exists globally or blows up in finite time were established. For $f(u) = ku^2/2$ and $g(u) = (3 - k) u^2/2$, (1) becomes the hyperelastic rod wave equation which was introduced by Dai [23, 24], and describes far-field, finite length, finite amplitude radial deformation waves in cylindrical compressible hyperelastic rods and $u$ represents the radial stretch relative to a pre-stressed state.
The local well-posedness of Cauchy problem for equation (1) has been considered in [52] (also, see [53, 55, 50, 6]). It should be noted that blowup criteria for such type equation systematically involved the computation of some global quantities or other global conditions like antisymmetry assumptions or sign conditions on the associate potential. In this context, it is interesting that in [4] (see also [5, 6]), contrary to previously known blow-up criteria, Brandolese and Cortez suggested a local-in-space blow-up criteria which only involves the values of $u'(x_0)$ and $u_0(x_0)$ in a single point $x_0$ of the real line. In the paper [48] author improved the blow-up criteria which established in the paper [4].

Motivated by the above-mentioned papers [4, 5, 6, 48] we would like to establish a "local-in-space" blowup criterion for two-component systems (1)−(4). The presented result extend and specify the earlier blow-up criteria for such type the system.

Our paper is organized as follows. In section 2, we recall several useful results which are crucial in the proof of the new blowup result. We give this proof in Section 3.

2. Preliminaries. In this section, we present the local well-posedness result, the precise blow-up scenario of the nonlinear dispersive wave equation and one result from [4] in order to pursue our goal.

Note that, the local well-posedness of the Cauchy problem for (1) was proved in [50, 52] by classical Kato’s approach [39]. For the system (1)−(4) the following theorem holds

**Theorem 2.1.** Assume that $g \in C^\infty(\mathbb{R})$. Let $(u_0,\rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > \frac{5}{2}$. Then there exists $T > 0$, with $T = T(u_0,\rho_0,g)$ and a unique solution to the Cauchy problem (1)-(4) such that $(u,\rho - 1) \in C([0,T^*),H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0,T^*),H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))$. Moreover, the solution depends continuously on the initial data.

Since the proof of this theorem is similar to results from [25, 11, 56], with corresponding changing according to proof of Theorem 3.1 from [50], we omit this proof.

Also, by standard way we obtain the following

**Lemma 2.2.** Let $(u,\rho)$ be the solution of (1)−(4) with initial data $(u_0,\rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > \frac{5}{2}$, and $T$ the maximal time of existence. Then for all $t \in [0,T)$ we have

$$
\int_\mathbb{R} (u^2 + u_x^2 + (\rho - 1)^2)dx = \int_\mathbb{R} (u_0^2 + u_x_0^2 + (\rho_0 - 1)^2)dx.
$$

(6)

Analogical reasoning to those made in [25] for a similar system show that the following result holds

**Theorem 2.3.** Assume that $g \in C^\infty(\mathbb{R})$. Let $(u,\rho)$ be the solution of the system (1)-(4) with initial data $(u_0,\rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > \frac{5}{2}$ and $T^* > 0$. Then $(u,\rho)$ blows up in finite time if and only if

$$
\lim_{t \to T^*} \inf \left\{ \inf_{x \in \mathbb{R}} [u_x(x,t)] \right\} \to -\infty.
$$

We finally recall the following useful lemma which will be used in the sequel.
Lemma 2.4. (See [4]) Let \( f, g_1 \in C^\infty(\mathbb{R}) \) with \( f'' \geq \gamma \geq 0 \) and \( 1_{R_{\pm}} \) denote one of the two indicator functions \( 1_{R^+} \) or \( 1_{R^-} \).

(i) \( \exists \, c \in \mathbb{R} \) such that \( m = g_1(c) = \min_{\mathbb{R}} g_1 \)

- The map \( \varphi: \mathbb{R} \to \mathbb{R} \) given by \( \varphi = \sqrt{(g_1-m)/\gamma} \) is \( K \)-Lipschitz with \( 0 \leq K \leq 1 \).

Then the following estimate holds:

\[
p1_{R_{\pm}} \left( g_1(u) + \frac{f''(u)}{2} u_x^2 \right) \geq \frac{\alpha}{2} (g_1(u) - m) + \frac{m}{2} \tag{7}
\]

where \( p(x) = \frac{1}{2} e^{-|x|} \) and \( \alpha = \frac{1}{4K^2} (\sqrt{1 + 8K^2} - 1) \).

(ii) \( \exists \, c \in \mathbb{R} \) such that \( M = g_1(c) = \max_{\mathbb{R}} g_1 \)

- The map \( \psi: \mathbb{R} \to \mathbb{R} \) given by \( \psi = \sqrt{(M-g_1)/\gamma} \) is \( K \)-Lipschitz with \( 0 \leq K \leq 1/\sqrt{8} \). Then we have:

\[
p1_{R_{\pm}} \left( g_1(u) + \frac{f''(u)}{2} u_x^2 \right) \geq \frac{\alpha}{2} (g_1(u) - M) + \frac{M}{2} \tag{8}
\]

with

\[
\alpha = \frac{1}{4K^2} \left( 1 - \sqrt{1 - 8K^2} \right).
\]

In the case \( g_1 = m = M \) be a constant function (this corresponds to \( K = 0 \)), the right-hand side of the above convolution estimates reads \( p1_{R_{\pm}} \left( g_1(u) + \frac{f''(u)}{2} u_x^2 \right) \geq g/2 \).

3. Main result. We are now ready to formulate and prove our main result.

Theorem 3.1. Let \( (u, \rho) \) be the solution of the system (1) – (4) with initial date \( (u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \), with \( s > 5/2 \). Let \( g \in C^\infty(\mathbb{R}) \). The maximal time \( T^* \) of the corresponding solution \( (u, \rho) \) of the system (1) – (4) in \( C([0,T^*), H^s(\mathbb{R})) \) is \( C^1([0,T^*), H^{s-1}(\mathbb{R})) \times H^{s-2}(\mathbb{R}) \) must be finite, if at least one of the two following conditions (i) or (ii) is fulfilled:

(i) - for function \( g_1(s) = s^2 + g(s) \) \( \exists \, c \in \mathbb{R} \) such that \( m = g_1(c) = \min_{\mathbb{R}} g_1 \)

- The map \( \varphi: \mathbb{R} \to \mathbb{R} \) given by \( \varphi = \sqrt{(g_1-m)} \) is \( K \)-Lipschitz with \( 0 \leq K \leq 1 \).

- \( \exists \, x_0 \in \mathbb{R} \) such that \( \rho_0(x_0) = 0 \) and

\[
u'_0(x_0) < - \frac{1}{2K} \left( \sqrt{1 + 8K^2} - 1 \right) \varphi(u_0(x_0)). \tag{9}\]

(ii) Or, otherwise

- for function \( g_1(s) = s^2 + g(s) \) \( \exists \, c \in \mathbb{R} \) such that \( M = g_1(c) = \max_{\mathbb{R}} g_1 \)

- The map \( \psi: \mathbb{R} \to \mathbb{R} \) given by \( \psi = \sqrt{(M-g_1)} \) is \( K \)-Lipschitz with \( 0 \leq K \leq 1/\sqrt{8} \).

- \( \exists \, x_0 \in \mathbb{R} \) such that \( \rho_0(x_0) = 0 \) and

\[
u'_0(x_0) < - \frac{1}{2K} \left( 1 - \sqrt{1 - 8K^2} \right) \psi(u_0(x_0)). \tag{10}\]

Proof. We introduce the following notation: \( y = u - u_{xx} \) and

\[
E_1(t) = \int_{-\infty}^{\infty} e^{q(x_0,t)} y(\xi,t) d\xi,
\]
where \( q(x_0, t) \) is the solution of the following problem:

\[
q_t = u(q(x, t), t),
\]

\[
q(x, 0) = x.
\]

Notice that the assumptions made on \( u \) imply that \( q \in C^1([0, T^\ast) \times \mathbb{R}, \mathbb{R}) \) is well defined on the whole time interval \([0, T^\ast)\) (see [45, 21, 50, 4]).

Then, differentiating \( E_1(t) \) we get

\[
\frac{dE_1(t)}{dt} = \int_{-\infty}^{q(x_0,t)} e^\xi y_t(\xi, t) \, d\xi + e^{q(x_0,t)} y(q(x_0, t), t) q_t(x_0, t). \tag{12}
\]

Integration by parts yields the following identities

\[
\int_{-\infty}^{q(x_0,t)} e^\xi y_t(\xi, t) \, d\xi = \int_{-\infty}^{q(x_0,t)} e^\xi (-3u_xu + u_{xxx}u + 2u_xu_{xx} - \left[ g(u) \right]_x - \rho \rho_x) \, d\xi =
\]

\[
\int_{-\infty}^{q(x_0,t)} e^\xi (-u_xu + u_{xxx}u + 2u_xu_{xx} - \left[ g(u) + u^2 \right]_x - \rho \rho_x) \, d\xi =
\]

\[
\left[-\frac{1}{2} e^\xi u^2 + e^\xi uu_{xx} + \frac{1}{2} e^\xi u_x^2 - e^\xi g_1(u) - \frac{1}{2} e^\xi \rho^2 \right]_{q(x_0,t)} -_{\infty} +
\]

\[
\frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi u^2 \, d\xi - \frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi uu_{xx} \, d\xi - \frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi u_x^2 \, d\xi + \int_{-\infty}^{q(x_0,t)} e^\xi g_1(u) \, d\xi +
\]

\[
\frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi \rho^2 \, d\xi = \left[-\frac{1}{2} e^\xi u^2 + e^\xi uu_{xx} - \frac{1}{2} e^\xi \rho^2 \right]_{q(x_0,t)} -_{\infty} +
\]

\[
\left[\frac{1}{2} e^\xi u_x^2 - e^\xi g_1(u) - e^\xi uu_x \right]_{q(x_0,t)} -_{\infty} + \frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi u_x^2 \, d\xi +
\]

\[
\frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi uu_x \, d\xi + \frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi u_x^2 \, d\xi - \frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi uu_x \, d\xi + \frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi g_1(u) \, d\xi +
\]

\[
\frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi \rho^2 \, d\xi = \left[\frac{1}{2} e^\xi u^2 - e^\xi uu_x \right]_{q(x_0,t)} -_{\infty} +
\]

\[
\left[-\frac{1}{2} e^\xi u^2 + e^\xi uu_{xx} + \frac{1}{2} e^\xi u_x^2 - e^\xi g_1(u) - \frac{1}{2} e^\xi \rho^2 \right]_{q(x_0,t)} -_{\infty} +
\]

\[
\int_{-\infty}^{q(x_0,t)} \frac{1}{2} e^\xi u^2 \, d\xi - \int_{-\infty}^{q(x_0,t)} \frac{1}{2} e^\xi u^2 \, d\xi + \frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi u_x^2 \, d\xi +
\]
Then, inserting (13) into (12) we have

\[
\begin{align*}
\frac{dE_1(t)}{dt} &= \int_{-\infty}^{q(x_0,t)} e^\xi y_t(\xi, t) \, d\xi + e^{q(x_0,t)} y(q(x_0,t), t) q_t(x_0, t) \geq \\
&\geq \left[ e^\xi u(u - u_x) - e^\xi u_y + \frac{1}{2} e^\xi u_x^2 - e^\xi (g_1(u) - m) - \frac{1}{2} e^\xi \rho^2 \right]_{-\infty}^{q(x_0,t)} + \\
&+ e^{q(t,x_0)} y(q(x_0,t), t) q_t(x_0, t) + \frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi u_x^2 \, d\xi + \\
&+ \frac{q(x_0,t)}{q(x_0,t)} \int_{-\infty}^{q(x_0,t)} e^\xi (g_1(u) - m) \, d\xi + \frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi \rho^2 \, d\xi
\end{align*}
\]

Then, inserting (13) into (12) we have

\[
\begin{align*}
\frac{dE_1(t)}{dt} &= \int_{-\infty}^{q(x_0,t)} e^\xi y_t(\xi, t) \, d\xi + e^{q(x_0,t)} y(q(x_0,t), t) q_t(x_0, t) \geq \\
&\geq \left[ e^\xi u(u - u_x) - e^\xi u_y + \frac{1}{2} e^\xi u_x^2 - e^\xi (g_1(u) - m) - \frac{1}{2} e^\xi \rho^2 \right]_{-\infty}^{q(x_0,t)} + \\
&+ e^{q(t,x_0)} y(q(x_0,t), t) q_t(x_0, t) + \frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi u_x^2 \, d\xi + \\
&+ \frac{q(x_0,t)}{q(x_0,t)} \int_{-\infty}^{q(x_0,t)} e^\xi (g_1(u) - m) \, d\xi + \frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi \rho^2 \, d\xi
\end{align*}
\]

Bearing in the mind equality (11) we obtain that

\[
\begin{align*}
\frac{dE_1(t)}{dt} &\geq \left[ e^\xi u(u - u_x) + \frac{1}{2} e^\xi u_x^2 - e^\xi (g_1(u) - m) - \frac{1}{2} e^\xi \rho^2 \right]_{-\infty}^{q(x_0,t)} + \\
&+ \frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi u_x^2 \, d\xi + \frac{q(x_0,t)}{q(x_0,t)} \int_{-\infty}^{q(x_0,t)} e^\xi (g_1(u) - m) \, d\xi + \frac{1}{2} \int_{-\infty}^{q(x_0,t)} e^\xi \rho^2 \, d\xi
\end{align*}
\]

Multiplying the inequality (14) by \( e^{-q(x_0,t)} \) we get

\[
\begin{align*}
\frac{dE_1(t)}{dt} &\geq \left[ u(u - u_x) + \frac{1}{2} u_x^2 - (g_1(u) - m) \right] (q(x_0,t), t) + \\
&+ \frac{1}{2} e^{-q(x_0,t)} \int_{-\infty}^{q(x_0,t)} e^\xi u_x^2 \, d\xi + e^{-q(x_0,t)} \int_{-\infty}^{q(x_0,t)} e^\xi (g_1(u) - m) \, d\xi - \\
&- \frac{1}{2} e^{-q(x_0,t)} \int_{-\infty}^{q(x_0,t)} e^\xi \rho^2 \, d\xi
\end{align*}
\]
Adding the expression \(-q(t, x, t) e^{-q(x, t)} E_1(t)\) to the right-hand side and left-hand side of the latter inequality, by virtue of equality (11) we have

\[
-q(t, x, t) e^{-q(x, t)} E_1(t) + e^{-q(x, t)} \frac{dE_1(t)}{dt} \geq -q(t, x, t) e^{-q(x, t)} E_1(t) +
\]

\[
\left[ u(u - u_x) + \frac{1}{2} u_x^2 - (g_1(u) - m) - \frac{1}{2} \rho^2 \right] (q(x, t), t) + \frac{1}{2} e^{-q(x, t)} \int_{-\infty}^{q(x, t)} e^\xi u_x^2 d\xi +
\]

\[
e^{-q(x, t)} \int_{-\infty}^{q(x, t)} e^\xi (g_1(u) - m) d\xi + \frac{1}{2} e^{-q(x, t)} \int_{-\infty}^{q(x, t)} e^\xi \rho^2 d\xi
\]

Besides, it is easy to see that

\[
E_1(t) = \int_{-\infty}^{q(x, t)} e^\xi y(\xi, t) d\xi = e^{q(x, t)} (u(q(x, t), t) - u_x(q(x, t), t))
\]

Therefore,

\[
\frac{d}{dt} \left( e^{-q(x, t)} E_1(t) \right) \geq \frac{1}{2} u_x^2 (q(x, t), t) -
\]

\[
(g_1(u(q(x, t), t)) - m) - \frac{1}{2} \rho^2 \int_{-\infty}^{q(x, t)} e^\xi u_x^2 d\xi +\]

\[
e^{-q(x, t)} \int_{-\infty}^{q(x, t)} e^\xi (g_1(u) - m) d\xi + \frac{1}{2} e^{-q(x, t)} \int_{-\infty}^{q(x, t)} e^\xi \rho^2 d\xi
\]

Then, using (7) (precisely, inequality (3.5) in [4]) for \(f(s) = \frac{1}{2} s^2\) and \(g_1(s) = s^2 + g(s)\) we obtain that

\[
\frac{d}{dt} \left( e^{-q(x, t)} E_1(t) \right) \geq \frac{1}{2} u_x^2 (q(x, t), t) - \left( 1 - \frac{1}{4K} \left( \sqrt{1 + 8K^2} - 1 \right) \right) \times
\]

\[
(g_1(u(q(x, t), t)) - m) - \frac{1}{2} \rho^2 \int_{-\infty}^{q(x, t)} e^\xi u_x^2 d\xi +\]

\[
\int_{-\infty}^{q(x, t)} e^\xi (g_1(u(q(x, t), t)) - m) d\xi -
\]

\[
\frac{1}{2} \rho^2 \int_{-\infty}^{q(x, t)} e^\xi \rho^2 d\xi
\]

Obviously,

\[
\frac{1}{2K} \left( \sqrt{1 + 8K^2} - 1 \right) = \sqrt{\frac{2}{K^2} \left( K^2 - \frac{1}{4} \left( \sqrt{1 + 8K^2} - 1 \right) \right)}
\]

Thus,

\[
\frac{d}{dt} \left( e^{-q(x, t)} E_1(t) \right) = \frac{d}{dt} (u(q(x, t), t) - u_x(q(x, t), t)) \geq
\]
Further, according to (16) and (17) we have:

\[
\frac{1}{2} \left( u_x^2 (q (x, t), t) - \left( \frac{1}{2K} \left( \sqrt{1+8K^2} - 1 \right) \varphi (u (q (x, t), t)) \right)^2 \right) - \frac{1}{2} \varphi^2 |q (x, t)| - \frac{1}{2} e^{-q (x, t)} q (x, t) + \frac{1}{2} e^{-2q (x, t)} \int_{q (x, t)} e^\xi \rho^2 d\xi \]

(15)

Next, we would like to obtain the similar inequality for the following integral:

\[
E_2 (t) = \int_{q (x, t)}^\infty e^{-\xi} y_t (\xi, t) d\xi.
\]

Indeed, by differentiating we get:

\[
\frac{dE_2 (t)}{dt} = \int_{q (x, t)}^\infty e^{-\xi} y_t (\xi, t) d\xi - e^{-q (x, t)} y (q (x, t), t) q_t (x, t).
\]

(16)

Integration by parts yields the following identities

\[
\int_{q (x, t)}^\infty e^{-\xi} y_t (\xi, t) d\xi = \int_{q (x, t)}^\infty e^{-\xi} \left( - \left[ \frac{1}{2} u^2 \right] x + [uu_{xx}]_x - [g_1 (u)]_x + \left[ \frac{1}{2} u_x^2 \right] x - \left[ \frac{1}{2} \rho^2 \right] x \right) d\xi =
\]

\[
\left[ - \frac{1}{2} e^{-\xi} u^2 + e^{-\xi} uu_{xx} + \frac{1}{2} e^{-\xi} u_x^2 - e^{-\xi} g_1 (u) - \frac{1}{2} e^{-\xi} \rho^2 \right]_{q (x, t)}^\infty - \frac{1}{2} \int_{q (x, t)}^\infty e^{-\xi} u_x^2 d\xi + \int_{q (x, t)}^\infty e^{-\xi} uu_{xx} d\xi + \int_{q (x, t)}^\infty e^{-\xi} u_x^2 d\xi =
\]

\[
\left[ \frac{1}{2} e^{-\xi} u^2 - e^{-\xi} g_1 (u) + e^{-\xi} uu_x - \frac{1}{2} e^{-\xi} u_x^2 + e^{-\xi} uu_{xx} + e^{-\xi} \frac{1}{2} u_x^2 - \frac{1}{2} e^{-\xi} \rho^2 \right]_{q (x, t)}^\infty - \frac{1}{2} \int_{q (x, t)}^\infty e^{-\xi} u_x^2 d\xi - \int_{q (x, t)}^\infty e^{-\xi} g_1 (u) d\xi - \frac{1}{2} \int_{q (x, t)}^\infty e^{-\xi} \rho^2 d\xi =
\]

\[
\left[ e^{-\xi} u (u + u_x) - e^{-\xi} uu_y - e^{-\xi} g_1 (u) + \frac{1}{2} e^{-q (x, t)} u_x^2 - \frac{1}{2} e^{-\xi} \rho^2 \right]_{q (x, t)}^\infty - \frac{1}{2} \int_{q (x, t)}^\infty e^{-\xi} u_x^2 d\xi - \int_{q (x, t)}^\infty e^{-\xi} g_1 (u) d\xi - \frac{1}{2} \int_{q (x, t)}^\infty e^{-\xi} \rho^2 d\xi
\]

(17)

Further, according to (16) and (17) we have:

\[
- \frac{dE_2 (t)}{dt} = - \int_{q (x, t)}^\infty e^{-\xi} y_t (\xi, t) d\xi + e^{-q (x, t)} y (q (x, t), t) q_t (x, t) \geq
\]
Meanwhile, integration by parts gives

\[ e^{-q(x_0, t)} y (q (x_0, t) , t) q_t (x_0, t) + \]

\[ \int_{q(x_0, t)}^\infty e^{-\xi} (g_1 (u) - m) d\xi + \frac{1}{2} e^{-q(x_0, t)} \rho^2 |_{q(x_0, t)}^\infty + \frac{1}{2} \int_{q(x_0, t)}^\infty e^{-\xi} \rho^2 d\xi \]

Applying (11) we have

\[ -e^{q(x_0, t)} E_2 (t) = \int_{q(x_0, t)}^\infty e^{-\xi} y (\xi, t) d\xi = e^{-q(x_0, t)} (u (q (x_0, t) , t) + u_x (q (x_0, t) , t)) \]

Meanwhile, integration by part gives

\[ E_2 (t) = \int_{q(x_0, t)}^\infty e^{-\xi} y (\xi, t) d\xi = e^{-q(x_0, t)} (u (q (x_0, t) , t) + u_x (q (x_0, t) , t)) \]

Next, adding the expression \( q_t (x_0, t) e^{q(x_0, t)} E_2 (t) \) to the right-hand side and left-hand side of the inequality (18), by virtue of equality (11) and (7) we have

\[ -\frac{d}{dt} (e^{q(x_0, t)} E_2 (t)) \geq \frac{1}{2} u_x^2 (q (x_0, t) , t) - (g_1 (u (q (x_0, t) , t)) - m) + \]

\[ \frac{1}{2} e^{q(x_0, t)} \int_{q(x_0, t)}^\infty e^{-\xi} u_x^2 d\xi + e^{q(x_0, t)} \int_{q(x_0, t)}^\infty e^{-\xi} (g_1 (u) - m) d\xi - \]

\[ \frac{1}{2} \rho^2 |_{-\infty}^{q(x_0, t)} + \frac{1}{2} e^{-q(x_0, t)} \int_{-\infty}^{q(x_0, t)} e^{\xi} \rho^2 d\xi \geq \frac{1}{2} u_x^2 (q (x_0, t) , t) - \]

\[ \left( 1 - \frac{1}{4K^2} \left( \sqrt{1 + 8K^2} - 1 \right) \right) (g_1 (u (q (x_0, t) , t)) - m) + \]

\[ \frac{1}{2} \rho^2 |_{q(x_0, t)}^\infty + \frac{1}{2} e^{q(x_0, t)} \int_{q(x_0, t)}^\infty e^{-\xi} \rho^2 d\xi \]

In virtue of (2) and (11) we have

\[ \frac{d}{dt} \rho (q (x_0, t) , t) = \rho_t (q (x_0, t) , t) + \rho_x (q (x_0, t) , t) q'_t (x_0, t) = \]

\[ \rho_t (q (x_0, t) , t) + \rho_x (q (x_0, t) , t) u (q (x_0, \tau) , \tau) = -\rho (q (x_0, t) , t) u_x (q (x_0, \tau) , \tau) . \]

Therefore,

\[ \rho (q (x_0, t) , t) = \rho_0 (x_0) e^{\int_{-u_x(q(x_0,\tau),\tau)}^{q(x_0,\tau)} d\tau} \quad \text{and since} \quad \rho_0 (x_0) = 0 \quad \text{we obtain that} \quad \rho (q (x_0, t) , t) = 0 \quad \text{for every} \quad t. \]
Thus,
\[
\frac{d}{dt} (-u(q(x_0,t),t) - u_x(q(x_0,t),t)) \geq \\
\frac{1}{2} \left( u_x^2(q(x_0,t),t) - \left( \frac{1}{2K} \left( \sqrt{1 + 8K^2} - 1 \right) \varphi(u(q(x_0,t),t)) \right)^2 \right)
\]

Besides, from (15) we obtain that
\[
\frac{d}{dt} (u(q(x_0,t),t) - u_x(q(x_0,t),t)) \geq \\
\frac{1}{2} \left( u_x^2(q(x_0,t),t) - \left( \frac{1}{2K} \left( \sqrt{1 + 8K^2} - 1 \right) \varphi(u(q(x_0,t),t)) \right)^2 \right)
\]

Therefore, the functions $R_1(t)$ and $R_2(t)$ defined by $R_1(t) = u(q(x_0,t),t) - u_x(q(x_0,t),t)$ and $R_2(t) = -u(q(x_0,t),t) - u_x(q(x_0,t),t)$ are both monotone increasing functions as long as function
\[
N_1(t) = -u_x(q(x_0,t),t) - \frac{1}{2K} \left( \sqrt{1 + 8K^2} - 1 \right) \varphi(u_0(x_0)) > 0.
\]

Note that, by condition (9) of Theorem 3.1
\[
N_1(0) = -u_0'(x_0) - \frac{1}{2K} \left( \sqrt{1 + 8K^2} - 1 \right) \varphi(u_0(x_0)) > 0.
\]

By continuity of $N_1(t)$, we conclude that $R_1(t)$ and $R_2(t)$ are increasing functions on some intervals $[0, T_1)$. This means that $R_1(t_2) \geq R_1(t_1)$ and $R_2(t_2) \geq R_2(t_1)$ for $t_1 < t_2 < T_1$, or, equivalently,
\[
(-u_x(q(x_0,t_2),t_2)) - (-u_x(q(x_0,t_1),t_1)) \geq -(u(q(x_0,t_2),t_2) - u(q(x_0,t_1),t_1))
\]
\[
(-u_x(q(x_0,t_2),t_2)) - (-u_x(q(x_0,t_1),t_1)) \geq u(q(x_0,t_2),t_2) - u(q(x_0,t_1),t_1)
\]

Hence,
\[
(-u_x(q(x_0,t_2),t_2)) - (-u_x(q(x_0,t_1),t_1)) \geq \|u(q(x_0,t_2),t_2) - u(q(x_0,t_1),t_1)\|
\]

However, by condition of Theorem 3.1
\[
|\varphi(u(q(x_0,t_2),t_2)) - \varphi(u(q(x_0,t_1),t_1))| \leq K |u(q(x_0,t_2),t_2) - u(q(x_0,t_1),t_1)|,
\]

where $K \leq 1$. Thus, we arrive at
\[
\frac{1}{K} |\varphi(u(q(x_0,t_2),t_2)) - \varphi(u(q(x_0,t_1),t_1))| \geq \\
\frac{1}{2K} \left( \sqrt{1 + 8K^2} - 1 \right) |\varphi(u(q(x_0,t_2),t_2)) - \varphi(u(q(x_0,t_1),t_1))|.
\]

That means that
\[
N_1(t_2) \geq N_1(t_1) > 0. \tag{20}
\]

Also, we obtain that the function $N_2(t)$ defined by
\[
N_2(t) = (-u_x(q(x_0,t),t)) + \frac{1}{2K} \left( \sqrt{1 + 8K^2} - 1 \right) \varphi(u(q(x_0,t),t))
\]
is a monotone increasing function on interval $[0, T_1)$. Thus, the functions $N_1(t)$, $N_2(t)$ and $N_1(t)N_2(t)$ are increasing function on interval $[0, T_1)$. Besides, in virtue of (20) we have $N_1(T_1) > 0$. Hence, using the same arguments as above, we
obtain that \( R_1(t), R_2(t), N_1(t) \) and \( N_2(t) \) are a monotone increasing function for all \( t > T_1 \). Since \( N_1(t) N_2(t) \geq N_1(0) N_2(0) > 0 \), we get
\[
\frac{d}{dt} (u(x,0,t), t) - u_x(q(x,0,t), t)) \geq (-u_x(q(x,0,t), t))^2 - \left( \frac{1}{2K} \left( \frac{1}{\sqrt{1+8K^2}} - 1 \right) \varphi(u(q(x,0,t), t)) \right)^2 \geq (-u_x(x,0,0))^2 - \left( \frac{1}{2K} \left( \frac{1}{\sqrt{1+8K^2}} - 1 \right) \varphi(u(x,0)) \right)^2 = l > 0. \quad (21)
\]

Similarly,
\[
\frac{d}{dt} (-u(q(x,0,t), t) - u_x(q(x,0,t), t)) \geq l \quad (22)
\]

Summing up inequalities (21) and (22) we obtain
\[
\frac{d}{dt} (-u_x(q(x,0,t), t)) \geq l
\]

Hence,
\[-u_x(q(x,0,t), t) \geq -u_x(x,0,0) + lt
\]

However, from (6) we see that \( ||u||_{L\infty} \leq \frac{K}{\sqrt{1 + 8K^2}} \leq \frac{1}{\sqrt{2}} \int_{\mathbb{R}} (u_0^2 + u_0^2 + (\rho_0 - 1)^2) dx \) (see [21]). This implies that \( u(q(x,0,t), t) \) (and therefore \( \varphi(u(q(x,0,t), t)) \)) is bounded. So, there exist some sufficiently large \( t_0 \) such that
\[-\frac{1}{2} u_x(q(x,0,t), t) \geq \left( \frac{1}{\sqrt{1+8K^2}} - 1 \right) |u(q(x,0,t), t) - c|
\]

Thus, from (21) and (22) we have
\[
\frac{d}{dt} (u(q(x,0,t), t) - u_x(q(x,0,t), t)) \geq \frac{1}{4} (-u_x(q(x,0,t), t))^2
\]

and
\[
\frac{d}{dt} (-u(q(x,0,t), t) - u_x(q(x,0,t), t)) \geq \frac{1}{4} (-u_x(q(x,0,t), t))^2
\]

for \( t > t_0 \).

Hence,
\[
\frac{d}{dt} (-u_x(q(x,0,t), t)) \geq \frac{1}{4} (-u_x(q(x,0,t), t))^2
\]

Therefore,
\[
-\frac{d}{dt} (-u_x(q(x,0,t), t))^{-1} \geq \frac{1}{4}.
\]

Integrating over \((t_0, \tau + t_0)\) gives
\[-(-u_x(q(x,0,\tau + t_0), \tau + t_0))^{-1} + (-u_x(q(x,0,t_0), t_0))^{-1} \geq \frac{1}{4} \tau
\]

So,
\[-(-u_x(q(x,0,\tau + t_0), \tau + t_0))^{-1} - \frac{1}{4} \tau \geq (-u_x(q(x,0,\tau + t_0), \tau + t_0))^{-1}
\]

and
\[-u_x(q(x,0,\tau + t_0), \tau + t_0) \geq \frac{1}{4} \left( -u_x(q(x,0,t_0), t_0))^{-1} \right) - \frac{1}{4} \tau.
\]

Thus, \(-u_x(q(x,0,\tau + t_0), \tau + t_0) \rightarrow \infty\) as \( \tau \rightarrow 4 \left( -u_x(q(x,0,t_0), t_0))^{-1} \right) \). Therefore, the conclusion (i) of Theorem 3.1 follows.
The necessary changes to deal with the conditions of Part (ii) of the theorem are slight. In this case, instead of (15) and (19) we have (by using (8))

\[
\frac{d}{dt}(u(q(x_0,t), t) - u_x(q(x_0,t), t)) \geq \frac{1}{2}\left(u^2_{x}\big|_{q(t,x_0)} - \left(\frac{1}{2K}\left(1 - \sqrt{1 - 8K^2}\right)\psi(u(q(x_0,t), t))\right)^2\right).
\]

and

\[
\frac{d}{dt}(-u(q(x_0,t), t) - u_x(q(x_0,t), t)) \geq \frac{1}{2}\left(u^2_{x}\big|_{q(t,x_0)} - \left(\frac{1}{2K}\left(1 - \sqrt{1 - 8K^2}\right)\psi(u(q(x_0,t), t))\right)^2\right).
\]

The last part of the proof proceeds in the same manner, by applying (10). Theorem is proved. \(\square\)

From the proof of Theorem 3.1 it is easy to see that the following Corollaries hold

**Corollary 1.** Let \(g \in C^\infty(\mathbb{R})\), \(\rho \equiv 0\) and \(u \in C([0, \infty), H^s(\mathbb{R}))\) be global smooth solution of the problem (1), (3) (with \(s > 5/2\)). Also suppose that at least one of the two following conditions (i) or (ii) is fulfilled:

(i) - for function \(g_1(s) = s^2 + g(s) \exists c \in \mathbb{R}\) such that \(m = g_1(c) = \min_{\mathbb{R}} g_1\)

- The map \(\varphi: \mathbb{R} \to \mathbb{R}\) given by \(\varphi = \sqrt{(g_1 - m)}\) is \(K\)-Lipschitz with \(0 \leq K \leq 1\)

(ii) Or, otherwise

- for function \(g_1(s) = s^2 + g(s) \exists c \in \mathbb{R}\) such that \(M = \max_{\mathbb{R}} g_1\)

- The map \(\psi: \mathbb{R} \to \mathbb{R}\) given by \(\psi = \sqrt{(M - g_1)}\) is \(K\)-Lipschitz with \(0 \leq K \leq 1/\sqrt{8}\).

Then, for all \(t \geq 0\), the two following (i') and (i'') conclusions holds:

(i') under condition (i)

\[
u_x(x,t) > -\frac{1}{2K}\left(\sqrt{1 + 8K^2} - 1\right)\varphi(u(x,t)) \geq -\frac{1}{2}\left(\sqrt{1 + 8K^2} - 1\right)|u(x,t) - c|; \tag{23}\]

(ii') Or, under condition (ii)

\[
u_x(x,t) > -\frac{1}{2K}\left(1 - \sqrt{1 - 8K^2}\right)\psi(u(x,t)) \geq -\frac{1}{2}\left(1 - \sqrt{1 - 8K^2}\right)|u(x,t) - c|. \tag{24}\]

**Corollary 2.** Under the conditions of Corollary 1

(i') let \(c < 0\), then \(u(x,t) - c \geq 0\) for all \(x \in \mathbb{R}\)

(i'') let \(c > 0\), then \(u(x,t) - c \leq 0\) for all \(x \in \mathbb{R}\).

**Proof.** If \([a,b]\) is an interval where \(u(x,t) - c > 0\) for all \(x \in [a,b]\) and fixed \(t\), then by (23) (or (24)) we have

\[-\gamma \int_a^b e^{\gamma x} (u(x,t) - c) \, dx \leq \int_a^b e^{\gamma x} u_x(x,t) \, dx =\]
\[
e^{-\gamma_b} (u(b,t) - c) - e^{-\gamma_a} (u(a,t) - c) - \gamma \int_{a}^{b} e^{-\gamma_x} (u(x,t) - c) \, dx
\]

where \(\gamma = \frac{1}{2} \left( \sqrt{1 + 8K^2} - 1 \right)\) (in the case (i) of Corollary 1) or \(\frac{1}{2} \left( 1 - \sqrt{1 - 8K^2} \right)\) (in the case (ii) of Corollary 1). From here we obtain that

\[
0 \leq e^{-\gamma_a} (u(a,t) - c) \leq e^{-\gamma_b} (u(b,t) - c).
\]

Therefore, \(u(b,t) > c\) for all \(b > a\). Suppose that \(u(x,t) < c\) in some point of the left neighbourhood of the point \((a,t)\), for example, \(u(p,t) < c\) for some \(p < a\).

Then,

\[
\gamma \int_{s}^{p} e^{-\gamma_x} (u(x,t) - c) \, dx \leq \int_{s}^{p} e^{-\gamma_x} u_x(x,t) \, dx =
\]

\[
e^{-\gamma_p} (u(p,t) - c) - e^{-\gamma_s} (u(s,t) - c) + \gamma \int_{s}^{p} e^{-\gamma_x} (u(x,t) - c) \, dx
\]

Consequently,

\[
e^{-\gamma_s} (u(s,t) - c) \leq e^{-\gamma_p} (u(p,t) - c) < 0
\]

However, \(c < 0\) and \(u(s,t) \to 0\) as \(s \to -\infty\). Hence, left-hand side of the latter inequality should nonnegative for some \(s < p\), while \((u(s,t) - c) < 0\). Thus, we obtain a contradiction which show that \(u(x,t) - c > 0\) for all \(x \in \mathbb{R}\).

By similar way we can prove the validity of the statement \((i'')\). Corollary 2 is thus established.

**Remark 1.** Also note that Theorem 3.1 holds for the system with the generalized hyperelastic-rod wave equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} - u_{xxt} + 3u_x u - c (u_{xxx} u + 2u_x u_{xx}) + [g(u)]_x + \rho \rho_x &= 0 \\
\frac{\partial \rho}{\partial t} + (cu \rho)_x &= 0 \\
u(x,0) &= u_0(x) \\
\rho(x,0) &= \rho_0(x)
\end{align*}
\]

where \(0 < c < 3\).

The proof proceeds in the same manner with the necessary slight changes.

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