Improvements on perturbative oscillation formulas including non-standard neutrino Interactions

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ABSTRACT: We present perturbative oscillation probabilities for electron and muon channels including non-standard interaction (NSI) effects. The perturbation was performed in standard parameters $\frac{\Delta m_{21}^2}{\Delta m_{31}^2}$ and $\sin^2(\theta_{13})$ as in non-standard interaction couplings. Our goal is to match non-standard parameters with the standard ones. This leads to oscillation probabilities with NSI compact and with functional structure similar to the Standard Oscillation (SO) case. Such formalism allows us to recognize degeneracies between standard oscillation parameters and NSI parameters. In such scenario, we also have an educated guess about the origin of the reported behavior of long-baseline experiments degeneracies, which should be due to marginalization on standard oscillation parameters $\delta_{CP}, \theta_{23}$ and NSI parameters.

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1 Introduction

The neutrino mass-mixing formalism that emerged from the last decades of neutrino phenomenology [1–11] is actually known as Standard Neutrino Oscillations (SO), and contains 6 different parameters. Two mass differences ($\Delta m^2_{32}, \Delta m^2_{21}$), one CP phase ($\delta_{\text{CP}}$) and three mixing angles, ($\theta_{12}, \theta_{13}, \theta_{23}$). The current values for these parameters can be found in [12]. As consequence of SO, the difference between the squared neutrino mass eigenvalues must be $\Delta m^2_{32} \approx 2.4 \times 10^{-3}$ eV$^2$ and $\Delta m^2_{21} \approx 7.8 \times 10^{-5}$ eV$^2$. However, in the Standard Model of Particles and Fields (SM) from [13–15] and others, neutrinos are included as massless particles. This suggests that the mechanism responsible to give mass to the neutrino should be other than the Higgs model [16–18]. Henceforth, it is straightforward to search for physics beyond the SM to account to neutrino masses. In this sense, in this work we take into account the so-called Non-Standard Neutrino Interactions (NSI) which was firstly proposed by Wolfenstein [19, 20], and are claimed as the most natural SM extension in the neutrino sector [21–27].

In the standard three neutrino formalism [28], the time evolution of a neutrino flavor state {$|\nu_e, \nu_\mu, \nu_\tau,\rangle$} is given by a Schroedinger-like equation [28]. When NSI are taken into account, the neutrino time evolution equation assumes the form:

$$\frac{d}{dt} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = H \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix},$$

(1.1)
where

\[ H = \Delta \begin{bmatrix} 0 & 0 & 0 \\ 0 & r_\Delta & 0 \\ 0 & 0 & 1 \end{bmatrix} U^\dagger + r_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + r_A \begin{bmatrix} \epsilon_{ee} & \epsilon_{em} & \epsilon_{et} \\ \epsilon_{em}^* & \epsilon_{mu} & \epsilon_{mt} \\ \epsilon_{et}^* & \epsilon_{mt}^* & \epsilon_{rr} \end{bmatrix} \] \tag{1.2}

Here we adopt the parametrization for the mixing matrix \( U = R(\theta_{23}) \Gamma(\delta_{CP}) R(\theta_{13}) \Gamma^\dagger(\delta_{CP}) R(\theta_{12}) \) as given in [29]. Also, we have defined

\[ \Delta = \Delta m^2_{31} / 2E_\nu, \quad r_\Delta = \frac{\Delta m^2_{21}}{\Delta m^2_{31}}, \quad r_A = \frac{A}{\Delta m^2_{31}} \] \tag{1.3}

where \( \Delta m^2_{ji} \equiv m_j^2 - m_i^2 \) is the mass square difference between the two mass eigenstates \( j \) and \( i \). Also, \( A = 2E_\nu V_{CC} \) and \( V_{CC} = \sqrt{2}G_F n_e \) is the matter potential that neutrinos feel while they cross a medium with the electron number density \( n_e = N_A (Z/A) \). \( N_A \) is the number of Avogadro, \( \rho \) is the matter density, and \( (Z/A) \) is the averaged ratio between nucleus charge and mass number in the medium that neutrino crosses. In this work we assume the following values for the mixing angles, \( \sin^2 \theta_{12} = 0.31 \), \( \sin^2 \theta_{13} = 0.023 \), \( \sin^2 \theta_{23} = 0.5 \), and squared mass differences, \( \Delta m^2_{21} = 2.4 \times 10^{-3} \text{ eV}^2 \), \( \Delta m^2_{31} = 7.5 \times 10^{-5} \text{ eV}^2 \) [24] and the CP phase equal to 0. In last term in eq. (1.1) we include the effective NSI parameters, \( \epsilon_{\alpha\beta} \), as effective matter potentials summed up in each element in the neutrino time evolution Hamiltonian and which are related with the coupling constants \( \epsilon_{\alpha\beta}^{f} \) by,

\[ \epsilon_{\alpha\beta} = \sum_f Y_f(x) \epsilon_{\alpha\beta}^{f}, \quad \alpha, \beta = e, \mu, \tau. \] \tag{1.4}

Here, \( Y_f(x) = n_f(x)/n_e(x) \), where \( n_f(x) \) is the number of fermions in the medium and \( \epsilon_{\alpha\beta}^{fV} = \epsilon_{\alpha\beta}^{fL} + \epsilon_{\alpha\beta}^{fR} \) are the coupling parameters in the non-standard interaction effective Lagrangian

\[ L_{\text{NSI}} = \sum_f \epsilon_{\alpha\beta}^{fP} 2\sqrt{2} G_F (\overline{\nu}_\alpha \gamma_\rho P_L \nu_\beta)(\overline{\nu} \gamma^\rho P_X f), \] \tag{1.5}

where \( (P_L, P_R) = (1 - \gamma^5, 1 + \gamma^5) / \sqrt{2}, \) \( P_X = (P_L, P_R) \). At fundamental level, \( \epsilon_{\alpha\beta} \) is related with the couplings of neutrino flavor states with the charged leptons and quarks, and it expresses the ratio between the strength of new interaction over the strength of SM weak coupling. Moreover, at the present there is no evidence for NSI and we have upper bounds for the parameters \( \epsilon_{\alpha\beta} \). A recent analysis of global constraints on NSI parameters were made in Ref. [30]. In addition, recent works on the sensitivity of future experiments was made in [31, 32].

Both works found an intriguing pattern in the upper values for parameters in the \( |\epsilon_{ee}| \otimes |\epsilon_{et}| \). There are solutions compatible with null NSI parameters, \( |\epsilon_{ee}| \sim |\epsilon_{et}| \sim 0 \) as well solutions that the both NSI parameters are not non-zero and large. In the first case the solutions are a minor perturbation of standard scenario, but in the second case, it is not a small perturbation of the standard oscillations. In ref. [31] it was shown, under certain assumptions, there is a exact symmetry between \( \epsilon_{ee}, \Delta m^2_{32} \) and \( \epsilon_{ee} \rightarrow 2 - \epsilon_{ee}, \Delta m^2_{32} \rightarrow -\Delta m^2_{32} \), that it means there is a fundamental degeneracy in neutrino oscillation probability and then only neutrino scattering experiments can broken this symmetry [33–36].
There are in the literature methods to solve eq. (1.1) based perturbation theory. See for example [24, 25, 37]. The aim of this work is to improve the existing perturbative solutions to extend their range of applicability in the NSI parameters domain. Then we apply the resulting analytic solutions to the long-baseline (LBL) experiments. Then we can extract from the formalism information about the degeneracies analytically.

The paper is organized as follow: In Section 2 we introduce the perturbative methods. Section 3 is dedicated to the formalism of neutrino propagation through quantum perturbation theory of Hamiltonian systems, including NSI. In Section 4 we present the resulting probabilities from the perturbation method, apply it in the long-baseline experiments and compare it with numerical solutions. As discussed in the text, these results can be useful until a long-baseline (LBL) of the order of $\approx 3800$ km depending on the NSI parameters. Furthermore, here we anticipate that our formalism is valid for DUNE, but, as expected, cannot be applied to the atmospheric neutrino case, as Super-Kamiokande [1] and IceCube [11]. In Section 5 we study the degeneracy behavior of neutrino oscillation formulas resulting from perturbation theory. Conclusions are in 6.

2 Perturbative approaches and the neutrino time-evolution

The eq. (1.1) describes a three-neutrino system with the addition of NSI parameters. First we will discuss the standard case, where we have only the three-neutrino system with standard matter effect and latter the case for NSI. In the SO case, exact solutions of eq. (1.1) are possible for vacuum [28] as well as for constant matter case [38, 39]. However, for varying matter potentials, full solutions of Schroedinger equation are only possible numerically. Henceforth, a common approach in literature is to use perturbative methods to found approximate semi-analytical solutions. Such works are based on the perturbative quantum theory for time-dependent Hamiltonians [40]. Even the solution for constant density it is not known easily understandable the physical meaning. A solution given by Ref. [41, 42] \footnote{We should aware that there are mistyping in the Ref. [41] accordingly with Ref. [43]} use the eigenvalues and eigenvectors of the $3 \times 3$ matrix that is not much illuminating. Other analytical solutions involve

- a perturbative approach of full oscillation probability such as (I) Cervera’s expansion [37] (II) Improved $\theta_{13}$ expansion [25],
- a specific rotation that made the problem separable in two $2 \times 2$ systems [44]

In the ref. [26] the perturbative formalism was improved and NSI are included. Because of the non-zero value of $\theta_{13}$ [7, 45], further developments of this formalism were done to extend the theory [25]. Therefore, in this work we adopt the expansion parameter defined as

$$\kappa \simeq \sin^2 \theta_{13} \simeq \frac{\Delta m_{21}^2}{\Delta m_{31}^2} \simeq 0.03, \quad (2.1)$$

which is motivated by recent global data analysis [12] and was firstly applied in [25]. The state of art of perturbative methods applied to neutrino time evolution can be found in
Also, in a recent analysis [47] perturbation theory is used to take into account analytically the different density values that neutrinos feel while they crosses the DUNE LBL.

In next section we will show an improved perturbative approach that results in a larger range of applicability of NSI parameters $\epsilon_{\alpha\beta}$.

3 Perturbation Theory with NSI

Here we address the formalism to solve neutrino time-evolution including NSI. We use perturbation theory through Dyson Series and consider as guidelines that the final expression for the probabilities should obey the following conditions:

i - Include the main features of the exact solution.

ii - Be concise enough to allow direct interpretation and use.

iii - Have the functional form as close as possible to the SO case in presence of matter effects.

iv - Cover most part the allowed phase-space of NSI parameters shown in sensitivity studies for DUNE, [31, 32].

In the case of standard solution for neutrino oscillation (the limit of $\epsilon_{\alpha\beta} \to 0$ in eq. (1.1) we have the matter effect it is invariant under $\theta_{23}$ rotation. When we include the NSI term this invariance it is broken. Nevertheless to keep the approach to have the oscillation probabilities in NSI framework as closed to the standard oscillation probabilities we will used this $\theta_{23}$ rotation as well. Let we start defining a propagation basis,

\[ \\{ |\nu'_\alpha\rangle \} = R_{23}^\dagger \{ |\nu_\alpha\rangle \}, \]

(3.1)

in which in this new basis it is

\[ \tilde{H} = R_{23}^\dagger H R_{23}, \quad \tilde{\epsilon}_{\alpha\beta} = R_{23}^\dagger \epsilon_{\alpha\beta} R_{23}. \]

(3.2)

A good way to achieve the conditions i - iv is to assume that the NSI parameters $\tilde{\epsilon}_{\alpha\beta}$ have the same order of the respective non-vanishing element of the standard mass-mixing formalism. Within this choice for the magnitude of NSI parameters the rotated Hamiltonian can be decomposed in terms of it order of the perturbative parameter $\kappa$ as\(^2\):

\[ \tilde{H} = \sum_n \tilde{H}^{(n)} = \tilde{H}^{(0)} + \tilde{H}^{(1/2)} + \tilde{H}^{(1)} + \tilde{H}^{(3/2)} + \tilde{H}^{(2)}. \]

(3.3)

where we will keep only terms until $\kappa^2$ in the final expression of the Hamiltonian. The most important consequence of our choice of NSI hierarchy is that it assures that each term of

\(^2\)In this work we do not consider terms with $O(\kappa^n)$ where $n > 2$. 

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\( \tilde{H} \) assumes a block-diagonal form, in which all the terms in \( \tilde{H}^{(0)} \) are of order \( \kappa^{(0)} \), all the terms in \( \tilde{H}^{(1/2)} \) are of order \( \kappa^{(1/2)} \), and so on:

\[
\tilde{H}^{(0)} = \Delta \begin{pmatrix} r_A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \Delta r_A \begin{pmatrix} \tilde{\epsilon}_{ee} & 0 & 0 \\ 0 & \tilde{\epsilon}_{\mu\mu} & 0 \\ 0 & 0 & \tilde{\epsilon}_{\tau\tau} \end{pmatrix} ,
\]

\[
\tilde{H}^{(1/2)} = \Delta \begin{pmatrix} 0 & 0 & s_{13} e^{-i\delta} \\ 0 & 0 & 0 \\ s_{13} e^{i\delta} & 0 & 0 \end{pmatrix} + \Delta r_A \begin{pmatrix} 0 & 0 & \tilde{\epsilon}_{\tau\tau} \\ 0 & 0 & 0 \\ 0 & \tilde{\epsilon}_{\tau e} & 0 \end{pmatrix} ,
\]

\[
\tilde{H}^{(1)} = \Delta \begin{pmatrix} r \Delta s_{12}^2 + s_{13}^2 & r \Delta c_{12} s_{12} & 0 \\ r \Delta c_{12} s_{12} & r \Delta c_{12}^2 & 0 \\ 0 & 0 & -s_{13}^2 \end{pmatrix} + \Delta r_A \begin{pmatrix} 0 & \tilde{\epsilon}_{\mu e} & 0 \\ \tilde{\epsilon}_{e\mu} & 0 & \tilde{\epsilon}_{\mu\tau} \\ 0 & \tilde{\epsilon}_{\tau\mu} & 0 \end{pmatrix} ,
\]

\[
\tilde{H}^{(3/2)} = -\Delta \begin{pmatrix} 0 & 0 & (r \Delta s_{12}^2 + \frac{1}{2} s_{13}^2) s_{13} c_{12} s_{13} e^{-i\delta} \\ (r \Delta s_{12}^2 + \frac{1}{2} s_{13}^2) s_{13} c_{12} s_{13} e^{i\delta} & r \Delta s_{12} c_{12} s_{13} e^{-i\delta} & 0 \\ 0 & 0 & 0 \end{pmatrix} ,
\]

\[
\tilde{H}^{(2)} = -\Delta r \Delta \begin{pmatrix} s_{12}^2 s_{13}^2 & \frac{1}{2} c_{12} s_{12} s_{13} & 0 \\ \frac{1}{2} c_{12} s_{12} s_{13} & 0 & 0 \\ 0 & 0 & -s_{13}^2 \end{pmatrix} .
\]

As it can be seen from eqs. (3.4-3.8), our option for the relative strength of NSI with respect to SO led to all the terms \( \tilde{H}^{(n)} \) to be block-diagonal. The only exception is in eq. (3.6), where was included the parameter \( \tilde{\epsilon}_{\mu\tau} \approx \kappa \), which correspondent element in SO perturbation theory appear only at order \( \kappa^{3/2} \). We have verified that the block-diagonal structure of each term in \( \tilde{H} \) is responsible for the compactness of the resulting oscillation formulas. This is the case of eq.(11) from [25], where NSI are disregarded. However, in eq. (33) of the same reference, when NSI are included it is done considering \( \tilde{\epsilon}_{\alpha\beta} \approx \kappa \), for all \( \alpha, \beta = e, \mu, \tau \), breaking the block-diagonal structure.

Our option for the NSI strenght keeps the block-diagonal structure of eqs. (3.4-3.8). We have verified that this feature turns easier the calculus of oscillation probabilities, which also have a simpler form. It also has the advantage that the perturbation theory is now applicable for a larger intensity of NSI. The main goal is that the parameters \( \tilde{\epsilon}_{ee}, \tilde{\epsilon}_{\mu\mu}, \tilde{\epsilon}_{\tau\tau} \) are now into the non-perturbed Hamiltonian \( \tilde{H}_0 \), as given in eq. (3.4), and can assume values of order of \( \kappa^0 \). As pointed previously, in the [25] \( \tilde{\epsilon}_{\alpha\beta} \approx \kappa \), for all \( \alpha, \beta = e, \mu, \tau \). Henceforth, our choice will increase the range of applicability of NSI. The maximum that NSI parameters can assume in our formalism are summarized in Table 1.

The neutrino time-evolution Hamiltonian, \( \tilde{H} \), as given in eqs. (3.3-3.8), can now be evolved in time using the perturbation theory. In the Appendix (A) we give details about the
This Work & [25, 27]  \\[\tilde{\epsilon}_{ee} \sim \kappa^0 & \tilde{\epsilon}_{ee} \sim \kappa \\
\tilde{\epsilon}_{\mu\mu} \sim \kappa^0 & \tilde{\epsilon}_{\mu\mu} \sim \kappa \\
\tilde{\epsilon}_{e\tau} \sim \kappa & \tilde{\epsilon}_{e\tau} \sim \kappa \\
\tilde{\epsilon}_{e\mu} \sim \kappa & \tilde{\epsilon}_{e\mu} \sim \kappa \\
\tilde{\epsilon}_{\mu\tau} \sim \kappa & \tilde{\epsilon}_{\mu\tau} \sim \kappa
table 1. Summary of maxima values that NSI parameters can assume in our formalism and in refs. [25, 27]. $\kappa^0$ means that the parameter is in include in the initial Hamiltonian, $\tilde{H}_0$ given in eq. (3.4)

formalism adopted. The desired probabilities can be written as $P_{\nu_\alpha \nu_\beta} = |S_{\beta\alpha}|^2$, where the $S$ matrix is given from eq. (A.9). As we have verified, the oscillation probabilities obtained from such method can be factorized in terms of the order $n$ of the parameter of the expansion, $\kappa^n$. Considering al contributions in which $n \leq 2$, the neutrino flavor oscillation probability assumes the factorized form

$$P(\nu_\alpha \rightarrow \nu_\beta) = \sum_{n<2} P^{(n)}_{\nu_\alpha \nu_\beta} = P^{(0)}_{\nu_\alpha \nu_\beta} + P^{(1)}_{\nu_\alpha \nu_\beta} + P^{(3/2)}_{\nu_\alpha \nu_\beta} + P^{(2)}_{\nu_\alpha \nu_\beta}.$$ (3.9)

In the next sections we show our results for the oscillation probabilities within this procedure.

4 Results for perturbation theory until $O(\kappa^2)$

At this point we present our results for the neutrino oscillation probabilities obtained from the perturbation theory described in Sec. 3. Please remember that expansions considering $s_{13}$ and $\tilde{\epsilon}_{e\tau}$ in different orders of $\kappa$ have already been considered [25–27, 37]. In this work we perform modifications in such perturbation theory that result in an increasing in the allowed domain for the parameters $\tilde{\epsilon}_{\alpha\beta}$. Also, as pointed in previous section, it is interesting that our resulting formulas present the functional form as close as possible to the SO case in presence of matter effects. For this purpose we define:

$$\Sigma = |\Sigma| e^{i\phi_{\Sigma}} \equiv s_{13} e^{-i\delta_{\text{CP}}} + r_A \tilde{\epsilon}_{e\tau},$$
$$\Omega = |\Omega| e^{i\phi_{\Omega}} \equiv r_{A} c_{13} s_{12} + r_A \tilde{\epsilon}_{e\mu},$$
$$\Lambda \equiv \frac{1}{r_{A}} + \tilde{\epsilon}_{\mu\tau} - \tilde{\epsilon}_{\mu\mu},$$
$$\Gamma \equiv (1 + \tilde{\epsilon}_{ee} - \tilde{\epsilon}_{\mu\mu})$$
$$\eta \equiv \Lambda - \Gamma.$$ (4.1)

In our notation $\tilde{\epsilon}_{\alpha\beta} = |\tilde{\epsilon}_{\alpha\beta}| e^{i\phi_{\alpha\beta}}$ for all $\alpha, \beta = e, \mu, \tau$.

As pointed in Section 3, the desired neutrino oscillation probability $P_{\nu_\alpha \nu_\beta}$ is obtained from the $S$ matrix, $P_{\nu_\alpha \nu_\beta} = |S_{\beta\alpha}|^2$. Also, they can be organized by the order of the
parameter of expansion, \( P_{\nu_\alpha \nu_\beta} = \sum_{n \leq 2} P_{\nu_\alpha \nu_\beta}^{(n)} \), and arranged within a compact form which has the same functional structure than SO case. Explicitly, for the muon to electron-neutrino oscillation case, using the eqs. (4.1), each order of the oscillation probability is given by:

\[
P_{\nu_\mu \nu_e}^{(1)} = \frac{\lvert \Sigma \rvert^2 s_{23}^2 \sin^2 \left( \frac{\Delta x}{2} r_A \eta \right)}{r_A^2 \eta^2}
\]

\[
P_{\nu_\mu \nu_e}^{(3/2)} = \frac{8c_{23}s_{23} \lvert \Sigma \rvert \sin \left( \frac{\Delta x}{2} r_A \eta \right) \sin \left( \frac{\Delta x}{2} r_A \eta \right) \cos \left( \frac{\Delta x}{2} r_A \Lambda - \phi_\Sigma + \phi_\Omega \right)}{r_A^2 \Gamma \eta}
\]

\[
P_{\nu_\mu \nu_e}^{(2)} = \frac{4c_{23}^2 \lvert \Omega \rvert^2 \sin^2 \left( \frac{\Delta x}{2} r_A \Gamma \right) + 2 \lvert \Sigma \rvert^2 s_{23} \left( 2 \lvert \Sigma \rvert^2 - \frac{r_A^2 s_{12}^2 + 2 s_{13}^2}{r_A^2 \eta^2} \right) (\Delta x) \sin (r_A \eta \Delta x)}{r_A^2 \eta^4}
\]

\[
- 4s_{23} \left( \frac{4 \lvert \Sigma \rvert^4}{r_A^4 \eta^4} - 2 \lvert \Sigma \rvert^2 \left( r_A s_{12}^2 + 2 s_{13}^2 \right) \sin \left( \frac{\Delta x}{2} r_A \eta \right) \sin \left( \frac{\Delta x}{2} r_A \eta \right) \right)
\]

\[
\times \left( \frac{\sin \left( \phi_\mu + \frac{\Delta x}{2} r_A \eta \right)}{r_A^2 \eta \Gamma} \sin \left( \phi_\mu + \frac{\Delta x}{2} r_A \eta \right) + \frac{\sin \left( \phi_\tau + \frac{\Delta x}{2} r_A \eta \right)}{r_A^2 \eta^2 \Lambda} \right)
\]

\[
\text{The same approach within perturbation theory can be applied to calculate muon neutrino survival probability. In Appendix B we show our results for } P_{\nu_\mu \nu_e}, \text{ considering terms until } \kappa^n \text{ where } n \leq 3/2.
\]

The formulas given in eqs. (4.2-4.4) can be directly applied for the DUNE case, where the LBL is of the order of \( L = 1300 \text{km} \) and where the matter density value is assumed to be \( \rho = 2.8 \text{ g/cm}^3 \). In Figure (1), upper panel, we compare such predictions with our numerical solutions, which are obtained using the Runge-Kutta method [48]. The results obtained using the ref. [25] for the same two sets of NSI parameters are also shown. All the predictions from perturbation theory return the right values for the phase of oscillation. However, for this intensity of NSI parameters, the predictions from ref. [25] overestimate the oscillation amplitude. Furthermore, for both sets of NSI parameters, in all the energy interval we clearly see the convergence of our results from perturbation theory. For \((\epsilon_{ee}, \epsilon_{e\tau}, \epsilon_{\tau\tau}) = (0.6, 0.15, 0.03)\) even our result for \( P_{\nu_\mu \nu_e}^{(1)} \) does agree with the exact solution within few percent error. When \((\epsilon_{ee}, \epsilon_{e\tau}, \epsilon_{\tau\tau}) = (0.6, 0.3, 0.3)\) is used, we must use \( P_{\nu_\mu \nu_e}^{(1)} + P_{\nu_\mu \nu_e}^{(3/2)} + P_{\nu_\mu \nu_e}^{(2)} \) to reach such precision level. In the energy interval of \( E_\nu \leq 0.5 \text{ GeV} \), while \( P_{\nu_\mu \nu_e}^{(1)} \) returns a to small amplitude, \( P_{\nu_\mu \nu_e}^{(1)} + P_{\nu_\mu \nu_e}^{(3/2)} \) predicts negative non-physical values. Moreover, our formula for \( P_{\nu_\mu \nu_e}^{(1)} + P_{\nu_\mu \nu_e}^{(3/2)} + P_{\nu_\mu \nu_e}^{(2)} \) is in accordance with the numerical solution in few percent in all the energy domain. In resume, the perturbation theory we develop led to an increment on the intensity of NSI parameters in which the model is valid when compared with ref. [25]. The limit of applicability depends on the combination of
Figure 1. Upper Panel: Comparison between perturbation theory and numerical solutions for neutrino oscillation probability for the DUNE case, where the matter density value is assumed to be $\rho = 2.8 \text{ g/cm}^3$ and the distance traveled is $L = 1300 \text{ km}$. The non-zero NSI parameters are indicated in the plots. Solid black line is our numerical (exact) solution to the neutrino time-evolution. The purple dotted-line refers to the results from [25] until $n = 2$, and the other colored lines refers to our results for the perturbative calculation of $P_{\nu_{\mu}\nu_e}$, considering the maximum value of $n$ as indicated in the plot. Middle Panel: The same for a baseline of $L = 3800 \text{ km}$ and $\rho = 2.8 \text{ g/cm}^3$. Bottom Panel: The same for $\rho = 3.6 \text{ g/cm}^3$.

neutrino energy, baseline and matter density. For the DUNE case, we verified that such limits can be pushed even further for $E_{\nu} \leq 10 \text{ GeV}$.

In order to determine how good are the oscillation formulas obtained from perturbation theory, in the middle and bottom panel of Figure (1) we compare the predictions from eqs. (4.2-4.4) with the results from [25] in function of neutrino energy, for the case where neutrinos travel a distance of $L = 3800 \text{ km}$. We also present our numerical results for the exact solution. The NSI parameters and medium density values are indicated in the panel. As we can see, the predictions from [25] and our results do agree with the numerical solution with respect to the positions of maxima and minima of oscillations. Moreover, in all panel, we can see the convergence of our results from perturbation theory to the numerical solution. For the middle-left panel, where $(\epsilon_{ee} = 0.6, \epsilon_{e\tau} = 0.15, \epsilon_{\tau\tau} = 0.03)$, the agreement of our prediction for $P_{\nu_{\mu}\nu_e}^{(1)} + P_{\nu_{\mu}\nu_e}^{(3/2)} + P_{\nu_{\mu}\nu_e}^{(2)}$ with the exact solution for the
entire energy domain shown in the plot is $\geq 90\%$. As it can be noticed, for this values of NSI parameters, the modifications we made in the perturbation theory gives better results for the amplitude of oscillation than ref. [25]. Also, as the energy increases, the NSI terms which depend on $r_A = 2\sqrt{2} G_F N_e E / \Delta m_{31}^2$ increase too. This would lead to the NSI terms to be greater than the order of perturbation in which they were included in our model, and hence, the worst agreement with a numerical solution is obtained. This behavior can be verified in all plots of Figure (1). In the lower-left panel the density is increased from $2.8 \text{ g/cm}^3$ to $3.6 \text{ g/cm}^3$. The same few percent levels of difference between our perturbative results $P_{\nu_\mu \nu_e}^{(1)} + P_{\nu_\mu \nu_e}^{(3/2)} + P_{\nu_\mu \nu_e}^{(2)}$ and the numerical solution was found. In the middle and bottom right panel of Figure (1) we present the same results for the case of $(\epsilon_{ee} = 0.6, \epsilon_{e\tau} = 0.3, \epsilon_{\tau\tau} = 0.3)$. Again, for $E_\nu > 4 \text{ GeV}$, our results from perturbation theory gets worst for both medium densities. This difference reaches a factor $\approx 8$ for $E_\nu = 15 \text{ GeV}$. As expected, with this intensity of NSI parameters and for $\rho = 2.8 \text{ g/cm}^3$, the predictions from the ref. [25] are no longer valid, since it returns non-physical values to the oscillation probability. If we increase the density to $\rho = 3.6 \text{ g/cm}^3$, for the same values of NSI parameters, also our formula summed up until $\sum_{n\leq 2} P_{\nu_\mu \nu_e}^{(2)}$ starts to return non-physical values, as can be seen in down-left panel. This case configures an upper limit of applicability of our formalism.

One advantage of perturbation formulas is make possible analytic description of NSI effects in the LBL experiments, like the DUNE. In the left panel of Figure 2 we show our results for $P_{\nu_\mu \nu_e}^{(1)} + P_{\nu_\mu \nu_e}^{(3/2)} + P_{\nu_\mu \nu_e}^{(2)}$ for three different values of $\epsilon_{e\tau}$, all of them positive, and all the other NSI parameters set to zero. In the right panel of Figure 2 we show the same probability for three cases where the only non-vanishing NSI parameter is $\epsilon_{ee}$, all of them negative. In these two scenarios where only one NSI parameter is non-zero, NSI should be easily noticed by the DUNE experiment, since its signature is the enhancement (attenuation) of oscillation amplitude. For $E_\nu > 1.2 \text{ GeV}$, the increase of oscillation amplitude due to $\epsilon_{e\tau} > 0$ in left panel of figure 2 as well as the decrease of oscillation amplitude due to $\epsilon_{ee} < 0$ in the right panel of the same figure can be described using only $P_{\nu_\mu \nu_e}^{(1)}$. From eqs. (4.1, 4.2), one can see that $\epsilon_{e\tau} > 0$ ($\epsilon_{ee} < 0$) implies in larger $\Sigma$ ($\eta$) respectively. At $E_\nu \leq 1.2 \text{ GeV}$, for the same NSI parameters, there is an complete inversion of its effect over the resulting oscillation pattern: $\epsilon_{e\tau} > 0$ ($\epsilon_{ee} < 0$) decreases (increases) the amplitude of oscillation. Such behavior is described by the relative signs present in the $P_{\nu_\mu \nu_e}^{(2)}$ formula, as given in eq. (4.4).

Moreover, the inclusion of only one non-zero NSI parameter was the case considered in many previous analyses of neutrino experiments. [49]. However, as can be seen, $\epsilon_{e\tau} > 0$ has the exact opposite effect of $\epsilon_{ee} < 0$ in all the energy domain shown in the picture. Hence, the combined effect of more than one NSI parameter greater then zero can be the canceling the NSI signature. To illustrate that, in right panel of Figure 3 we show two cases where $\epsilon_{ee}$ and $\epsilon_{e\tau}$ are relatively large and with opposite sign. As it is clearly seen the net result is the almost completely canceling of NSI effect. In such scenario, unfortunately, NSI and SO cases are completely degenerated. Moreover, from eqs. (4.1-4.4) we clearly identify that our resulting oscillation probabilities are intrinsically degenerated. Furthermore, in Section 5 we make an analysis of degeneracies within perturbation theory formalism.
Figure 2. Examples of how NSI affects neutrino oscillations for the DUNE baseline using the eqs. (4.2), (4.3), (4.4). $\delta_{CP}$ and all NSI parameters other than the specified in the plots are set to zero. Left Panel: Three different values of $\epsilon_{e\tau} > 0$. Right Panel: Three different values of $\epsilon_{ee} < 0$. All the plots are calculated until order 2 in the perturbation theory.

Figure 3. Two different sets of NSI parameters which leaves oscillation probability completely degenerated with the SO case in DUNE baseline. We emphasize that the minus sign in $\epsilon_{e\tau}$ is equivalent to the phase $\phi_{e\tau} = \pi$.

5 Study of Degeneracies

Here, we analyze the degenerate behavior of neutrino oscillations in the presence of NSI in the light of perturbative formalism developed in Section 3. In general, degenerate solutions for the neutrino time-evolution can be due to:

i Only SO parameters
ii Only NSI parameters

iii Combinations of SO and NSI parameters

The case i is the most common in the literature [49]. Here we focus the attention mainly to the cases ii and iii. When NSI is included in the neutrino time evolution, a degenerated region emerge in the allowed phase-space for both SO and NSI parameters from studies of sensitivities in DUNE experiment. DUNE sensitivities to NSI parameters are calculated in [32]. It is clear from that work that DUNE will have the sensitivity to NSI parameters of order $\epsilon_{\alpha\beta} \approx 0.1$. A similar picture is found in [31]. In special, the allowed region for the parameters $\epsilon_{ee}$ and $\epsilon_{e\tau}$ in these both references shows the puzzling behavior, where one of these NSI parameters can be big if the other is also big.

It is natural to think that the origin of the degenerate allowed region relies on the neutrino time evolution mechanism. Henceforth, this degeneracy should be manifested in the neutrino oscillation probabilities. In [31, 32], the degenerate behavior is analytically determined for a specific set of NSI parameters. These analysis motivated us to ask if the perturbation theory is able to explain (or at least mimic) the degenerate comportment. To answer such question we will use oscillation formulas within perturbation theory in two different ways:

- Find analytic expressions for the degeneracies from eqs. (4.1-4.4),
- Perform a graphic comparison between perturbative solution in the $\epsilon_{ee} \otimes \epsilon_{e\tau}$ plane, and the results from the literature.

In what follows, we apply the perturbation theory to determine analytically the origin of the degeneracies in neutrino oscillation probabilities. As the general rule, within perturbation theory, the degeneracy behavior depends on the order in which NSI are included. For analytic simplification purposes, initially one could consider only $P_{\nu_1 \nu_e}^{(1)}$. As the eqs. (4.1-4.4), where NSI are already included, present the same functional structure than SO formulas, an obvious degenerate situation occurs when these equations coincide with the SO case, which means the phase and the amplitude of neutrino oscillation with NSI are identical to standard case. Also, as the relation between propagation and flavor basis is given by eq. (3.1), it must be taken into account the possibility of $\theta_{23}$ to vary within its 3$\sigma$ allowed region. When the value of $\theta_{23}$ is not the best fit value from [50] we indicate it as $\theta'_{23}$. Henceforth we define the degeneracy condition considering all the NSI parameters (and associated phases) as

$$\left[ P_{\nu_1 \nu_e}^{(1)}(\tilde{\epsilon}_{\alpha\beta}, \theta'_{23}) \right]^{(\text{NSI})} - \left[ P_{\nu_1 \nu_e}^{(1)}(\tilde{\epsilon}_{\alpha\beta} = 0, \theta_{23}) \right]^{(\text{SO})} = 0. \quad (5.1)$$

In the simpler case, where $\theta'_{23} = \theta_{23}$, then the eq. (5.1) is satisfied if the following two conditions are satisfied:

$$\Lambda - \Gamma = \frac{(1 - r_A)}{r_A},$$

$$\Sigma = s_{13}^{2} e^{-i\delta}, \quad (5.2)$$
which are respectively verified if
\[ \tilde{\epsilon}_{ee} = \tilde{\epsilon}_{\tau\tau} \quad \leftrightarrow \quad \epsilon_{ee} = -s_{23}(s_{23}\epsilon_{\mu\mu} + c_{23}\epsilon_{\mu\tau}) + c_{23}(-s_{23}\epsilon_{\mu\tau} + c_{23}\epsilon_{\tau\tau}), \]
\[ \tilde{\epsilon}_{e\tau} = 0 \quad \leftrightarrow \quad \epsilon_{e\mu} = c_{23} \frac{s_{23}}{s_{23}} \epsilon_{\tau\tau}. \] (5.3)

If in eqs. (5.2-5.3) the first condition is obeyed, then the phase of oscillation in \( P_{\nu\mu\nu}^{(1)} \) (NSI) will be identical to the phase in SO case. It also implies that the denominator in eq (4.2) is identical to the SO case. However, to the amplitude of oscillation be the same in SO and NSI cases, it is also necessary that the second condition be satisfied.

If the rotation angle in eq. (3.1) is varied within its 3σ allowed region, \( \theta'_{23} \neq \theta_{23} \), then eq. (5.3) implies in
\[ \frac{r_A^2}{s_{13}} |\tilde{\epsilon}_{e\tau}|^2 + 2 \frac{r_A}{s_{13}} |\tilde{\epsilon}_{e\tau}| \cos(\zeta) + 1 = \left( s_{23} \frac{s_{23}'}{s_{23}} \right)^2 \frac{\sin^2(\zeta)}{\sin^2(y')}, \] (5.4)
where the phase \( \zeta \) is given by
\[ \zeta = \delta_{cp} + \phi_{e\tau}, \] (5.5)
and where the functions \( \text{sinc}(x) \) are defined as \( \text{sinc}(x) \equiv \frac{\sin(x)}{x} \). Also,
\[ y = z + \frac{\Delta r_A}{2}(\tilde{\epsilon}_{\tau\tau} - \tilde{\epsilon}_{ee}), \] (5.6)
where \( z \) contains only SO terms
\[ z = \frac{\Delta}{2} x(r_A + 1). \] (5.7)

Furthermore, eq. (5.4) reveals a quadratic relation between \( |\tilde{\epsilon}_{ee}|, |\tilde{\epsilon}_{e\tau}|, \) and \( |\tilde{\epsilon}_{\tau\tau}| \), which is given in terms of the ratio of two functions \( \text{sinc}(x) \). Hence, given \( \gamma = \tilde{\epsilon}_{ee} - \tilde{\epsilon}_{\tau\tau} \), the ratio on the right side of eq. (5.4) can be approximated by Taylor series around \( r_A \gamma < 1 \):
\[ \frac{\sin^2(y')}{\sin^2(y')} = \left( \frac{B_1}{B_1 + B_2 r_A \gamma} \right)^2, \] (5.8)
where \( B_1 \) and \( B_2 \) are given as,
\[ B_1 = \text{sinc} \left( \frac{\Delta x}{2}(1 - r_A) \right), \]
\[ B_2 = \left( \frac{\Delta x}{2} \right) \frac{\cos(\frac{\Delta x}{2}(1 - r_A)) - \text{sinc} \left( \frac{\Delta x}{2}(1 - r_A) \right)}{\left( \frac{\Delta x}{2}(1 - r_A) \right)}. \] (5.9)

Notice that \( B_1 \) and \( B_2 \) do not depend on \( \gamma \). When we insert eqs. (5.8), (5.9) into eq. (5.4) it assumes the form:
\[ \frac{r_A^2}{s_{13}} |\tilde{\epsilon}_{e\tau}|^2 + 2 \frac{r_A}{s_{13}} |\tilde{\epsilon}_{e\tau}| \cos(\zeta) + 1 = \left( s_{23} \frac{s_{23}'}{s_{23}} \right)^2 \left( \frac{B_1}{B_1 + B_2 r_A \gamma} \right)^2. \] (5.10)

In Figure (4), we show the predictions from eq. (5.10), for DUNE case, applied to the \( \epsilon_{ee} \otimes \epsilon_{e\tau} \) plane, assuming all other NSI parameters to be zero. The solid lines represent
the plots for $s'_{23} = s_{23}$ while the dashed lines represent the region limited by $s'_{23}$ at $3\sigma$ allowed values and the respective phases indicated. Notice, that in each panel, for the black solid line, the values of $\zeta$ used in the region $\epsilon_{ee} > 0$ are different from the values used in $\epsilon_{ee} < 0$, as it is explicit in the plots. We superpose those results with the $1\sigma$, $2\sigma$, and $3\sigma$ allowed regions for the NSI parameters $\epsilon_{ee}$ and $\epsilon_{e\tau}$ obtained in the sensitivity calculation for the DUNE experiment [31], where to obtain the allowed region in the plane $\epsilon_{ee} \otimes \epsilon_{e\tau}$ a marginalization over parameters $\Delta m^2_{31}$, $\theta_{23}$, $\delta_{CP}$ and the complex phase of $\phi_{e\tau}$ was performed. As we can see, the solution from perturbation theory at $n \leq 1$ shows the same degenerate behavior than the one reported in [31] if we set different values of $\zeta$ and $s'_{23}$ in the regions $\epsilon_{ee} > 0$ and $\epsilon_{ee} < 0$. Within this scenario, using perturbative theory, a possible hypothesis to explain the degenerated behavior is that degenerated regions should be caused by the marginalization over the parameters $\theta_{23}$ and $\zeta$. In such process, as these parameters are free to vary, the value that leads to a solution including NSI which is degenerated with the solution due only to SO at one point of the plane $\epsilon_{ee} \otimes \epsilon_{e\tau}$ should be different from the best value for another point.

![Figure 4](image)

**Figure 4.** Comparison of our results (lines) from perturbation theory with the degenerated allowed region in the plane $\epsilon_{ee} \otimes \epsilon_{e\tau}$ (shaded region) reported in ref. [31]. In this region, color darkness refers to the $(1\sigma, 2\sigma, 3\sigma)$ regions the authors found in their sensitivity study of the DUNE experiment. Our results are from equation (5.10). Different line colors refers to the different values of CP phase indicated in the plot. Explicitly, in the $\epsilon_{ee} \leq 0$ region we use $\zeta = \pi/3$ and for $\epsilon_{ee} > 0$ we use $\zeta = 4\pi/3$. Solid lines are generated using the best fit point for $\theta_{23}$, $s^2_{23} = 0.441$, and in dashed lines we use the $3\sigma$ values $s^2_{23} = 0.385 \rightarrow 0.635$. Both values are from [50].

## 6 Conclusions

We study the effects of inclusion of NSI in the perturbative approach to neutrino oscillations in the matter through Dyson Series. We have modify the perturbation Hamiltonian to allow NSI parameters to be as large as possible, keeping the compromise that the calculus procedure and, more important, the resulting formulas to be small and practical to handle. Our formalism covers the most part of the phase-space allowed in DUNE sensitivities calculations as [31, 32]. Our goal was to led the NSI parameters $\tilde{\epsilon}_{\alpha\beta}$ to have the same order of the respective non-vanishing element of the case where only the standard mass-mixing
formalism is considered, as can be directly verified for the case of \( P_{\nu_\mu \nu_e} \) in eqs. 3.4-3.8. The resulting formulas were applied to the case of constant matter potential and for LBL of the order of 1–3 thousand kilometers and a density of \( \rho = 2.8 \) and \( \rho = 3.6 \, \text{g/cm}^3 \). Comparing our results with the exact solutions, as shown in Figure 1 we see that, for the baseline of \( \approx 4000 \, \text{km} \), \( 1.0 \leq E_\nu \leq 10 \, \text{GeV} \) and \( \rho = 3.6 \, \text{g/cm}^3 \), the upper limit of applicability of our formalism is \( (\epsilon_{ee}, \epsilon_{e\tau}, \epsilon_{\tau\tau}) \leq (0.6, 0.3, 0.3) \). Such limits are depending on the distance traveled by neutrinos, their energy and medium density.

We also study the dependences of the degeneracies in \( P_{\nu_\mu \nu_e}^{(1)} \) with respect to the NSI parameters. The formalism we develop led to an educated guess about the origin of degeneracies presented in [31, 32]. Even when only terms until first order in perturbation theory are taken into account, for specific values of NSI parameters our formalism is able to analytic mimic such degeneracy behavior as can be seen in Figure 4 and associated text. As was pointed, such degeneracies should be due to the combined marginalization over SO and SNI parameters, like \( \theta_{23} \) and the phase \( \zeta \).

Furthermore, as consequence of our perturbative expansion we can clearly identify the degeneracies between the usual oscillation paradigm and the NSI oscillation probability, which means that we cannot distinguish the two oscillation scenarios. We identify two cases: when it have a degeneracy (i) involving only NSI parameters and (ii) involving a combination of the NSI parameters and the standard oscillation parameters. In the first case it is a irreducible degeneracy and it cannot be removed by any neutrino oscillation experiment. In the second, a very precise measurement of standard mixing parameters can lift the degeneracies. Using our perturbative formulae we can reproduced the degeneracy between \( \tilde{\epsilon}_{ee} \) and \( \tilde{\epsilon}_{e\tau} \) as shown in Figure 4. Moreover, hopefully, such formalism should allow a more comprehensive understanding of the role played by NSI in LBL experiments, like the DUNE.

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The S matrix is responsible for the neutrino-state time evolution,
\[ |\nu_\alpha(x)\rangle = S_{\alpha\beta}(t)|\nu_\beta(0)\rangle, \]  
where the operator \( S_I(t) \) is given by,
\[ S(t) = \mathcal{T} \exp \left\{ -i \int_0^t H(t')dt' \right\}. \]
In this case, the oscillation probability in then
\[ P_{\nu_\mu\to\nu_e} = P_{\nu_\mu\nu_e} = |S_{e\mu}|^2 \]
Here \( \mathcal{T} \) means temporal ordering. From eq. (3.3) we define the potential \( \tilde{V} = \tilde{H}' - \tilde{H}_0 \), which in the interaction picture assumes the form,
\[ \tilde{V}_I = e^{i\tilde{H}_0x}\tilde{V}e^{-i\tilde{H}_0x}, \]
where \( \tilde{V} \) is the time-dependent potential in the Schroedinger picture. The time-evolution operator in the interaction picture can be defined as [40] \( \Omega(x) \),
\[ \Omega(x) = e^{i\tilde{H}_0x}\tilde{S}(x), \]
which must obey the operator time evolution equation:
\[ i\frac{d}{dt}\Omega(x) = \tilde{V}_I\Omega(x), \]
which is subjected to the initial condition $\Omega(x = 0) = 1$. The eq. A.6 plus the initial condition implies in the integral equation:

$$\Omega(x) = 1 - i \int_0^x \tilde{V}_I(x') \Omega(x') dx'. \quad (A.7)$$

The recursive substitution of $\Omega(x)$ into eq.(A.7) leads to the Dyson Series, which gives the solution for Eq.(A.6) as:

$$\Omega(x) = 1 + (-i) \int_0^x \tilde{V}_I(x') dx' + (-i)^2 \int_0^x \tilde{V}_I(x') dx' \int_0^{x'} \tilde{V}_I(x'') dx'' + (-i)^3 \int_0^x \tilde{V}_I(x') dx' \int_0^{x'} \tilde{V}_I(x'') dx'' \int_0^{x''} \tilde{V}_I(x''') dx''' + O(\epsilon^2). \quad (A.8)$$

Once $\Omega(x)$ is determinate from eq.(A.8), in the propagation basis $\tilde{S}(x) = e^{-i\tilde{H}_0} \Omega(x)$. The $S$ matrix can then be written in the flavor basis as

$$S = U_{33} \tilde{S} U_{23} \dagger. \quad (A.9)$$

The $\tilde{S}_{\alpha\beta}$ elements are explicitly calculated in [51].

## B Muon neutrino survival probability

Here we show our results for muon neutrino survival probability obtained from the same procedure explained in Section 3. We consider terms until $n \leq 3/2$. The resulting formulas show the same functional structure of SO formalism, and the NSI features are incorporated in the ($\Sigma$, $\Omega$, $\Lambda$, $\Gamma$) quantities,

$$P^{(0)}_{\nu_{\mu} \nu_{\mu}} = 1 - 4c_{23}^2 s_{23}^2 \sin^2 \left( \frac{\Delta x}{2} r_A \Lambda \right), \quad (B.1)$$

$$P^{(1)}_{\nu_{\mu} \nu_{\mu}} = -4|\Sigma|^2 s_{23}^4 \sin^2 \left( \frac{\Delta x}{2} r_A (\Gamma - \Lambda) \right) \frac{r_A^2 (\Gamma - \Lambda)^2}{r_A^2 (\Gamma - \Lambda)^2}$$

$$+ 2c_{23}^2 s_{23}^2 \left( c_{12}^2 r_A + \frac{|\Sigma|^2}{r_A (\Gamma - \Lambda)} + s_{13}^2 \right) \sin(r_A \Lambda \Delta x)(\Delta x)$$

$$+ 2|\Sigma|^2 s_{23}^2 c_{23}^2 \left[ \cos(r_A \Gamma \Delta x) - \cos(r_A \Lambda \Delta x) \right] \frac{r_A^2 (\Gamma - \Lambda)^2}{r_A^2 (\Gamma - \Lambda)^2}$$

$$- 8|\bar{\epsilon}_{\mu\tau}| c_{23} s_{23} \left( c_{23}^2 - s_{23}^2 \right) \cos(\bar{\phi}_{\mu\tau}) \sin^2 \left( \frac{\Delta x}{2} r_A \Lambda \right), \quad (B.2)$$

$\Lambda$
$p_{\nu \mu}^{(3/2)} = \frac{8r_{\Delta}c_{12}s_{12}c_{23}s_{23}s_{13}}{r_{A}\Lambda} \left( c_{23}^{2} - s_{23}^{2} \right) \cos(\delta) \sin^{2}\left( \frac{\Delta x}{r_{A}\Lambda} \right) + 4|\Omega|\Sigma|c_{23}s_{23}\cos(\phi_{\Sigma} - \phi_{\Omega}) \left( c_{23}^{2}\cos(r_{A}\Gamma\Delta x) - c_{23}^{2} + s_{23}^{2}\cos(r_{A}(\Gamma - \Lambda)\Delta x) \right) \right) - \frac{4|\Omega|\Sigma|c_{23}s_{23}\cos(\phi_{\Sigma} - \phi_{\Omega}) \left( c_{23}^{2}\cos(r_{A}\Lambda\Delta x) + 1 \right) \right) + \frac{4|\Omega|\Sigma|c_{23}s_{23}\cos(\phi_{\Sigma} - \phi_{\Omega}) \left( s_{23}^{2}\cos(r_{A}\Lambda\Delta x) \right) \right)}{r_{A}^{2}\Gamma\Lambda}. \quad (B.3)$