EXTREMAL BETTI NUMBERS OF EDGE IDEALS

TAKAYUKI HIBI, KYOUKO KIMURA AND KAZUNORI MATSUDA

ABSTRACT. Given integers \( r \) and \( b \) with \( 1 \leq b \leq r \), a finite simple connected graph \( G \) for which \( \text{reg}(S/I(G)) = r \) and the number of extremal Betti numbers of \( S/I(G) \) is equal to \( b \) will be constructed.

Let \( S = K[x_1, \ldots, x_n] \) denote the polynomial ring in \( n \) variables over a field \( K \) with each \( \deg x_i = 1 \) and \( I \subset S \) a homogeneous ideal. Let

\[
F_{S/I} : 0 \to \bigoplus_{j \geq 1} S(-a_{p,j})^\beta_{p,p+j}(S/I) \to \cdots \to \bigoplus_{j \geq 1} S(-a_{1,j})^\beta_{1,1+j}(S/I) \to S \to S/I \to 0
\]

be the minimal graded free resolution of \( S/I \) over \( S \), where \( p = \text{proj dim}(S/I) \) is the projective dimension of \( S/I \) and \( \beta_{i,i+j}(S/I) \) is the \( (i,i+j) \)-th graded Betti number of \( S/I \). The (Castelnuovo–Mumford) regularity of \( S/I \) is

\[ \text{reg}(S/I) = \{ j : \beta_{i,i+j}(S/I) \neq 0 \}. \]

A graded Betti number \( \beta_{i,i+j}(S/I) \neq 0 \) is said to be extremal ([2, Definition 4.3.13]) if \( \beta_{k,k+\ell}(S/I) = 0 \) for all pairs \( (k, \ell) \neq (i,j) \) with \( k \geq i \) and \( \ell \geq j \).

Let \( G \) be a finite simple graph (i.e. a graph with no loop and no multiple edge) on the vertex set \( V(G) = \{ x_1, x_2, \ldots, x_n \} \) with \( E(G) \) its edge set. The edge ideal of \( G \) is

\[ I(G) = (x_i x_j : \{ x_i, x_j \} \in E(G)) \subset S = K[V(G)] = K[x_1, x_2, \ldots, x_n]. \]

In the present paper, given integers \( r \) and \( b \) with \( 1 \leq b \leq r \), the existence of a finite simple connected graph \( G \) for which \( \text{reg}(S/I(G)) = r \) and the number of extremal Betti numbers of \( S/I(G) \) is equal to \( b \) will be shown.

Theorem 1. Let \( r, b \) be integers with \( 1 \leq b \leq r \). Then there exists a finite simple connected graph \( G_{r,b} \) with \( \text{reg} K[V(G_{r,b})]/I(G_{r,b}) = r \) such that the number of extremal Betti numbers of \( K[V(G_{r,b})]/I(G_{r,b}) \) is equal to \( b \).

In order to prove Theorem 1 we use the following non-vanishing theorem for Betti numbers [3].

Proposition 2 ([3, Theorem 4.1]). Let \( G \) be a finite simple graph on \( V \) and \( S = K[V] \). Suppose that \( G \) is chordal. Then \( \beta_{i,i+j}(S/I(G)) \neq 0 \) if and only if there exists a subset \( W \) of \( V \) such that the induced subgraph \( G_W \) contains a strongly disjoint set of bouquets of type \( (i,j) \).

2010 Mathematics Subject Classification. 05E40, 13H10.

Key words and phrases. edge ideal, Castelnuovo–Mumford regularity, extremal Betti number.
Recall fundamental materials on graph theory to understand Proposition 2. Let $G$ be a finite simple graph on the vertex set $V$ and $W$ a subset of $V$. The induced subgraph of $G$ on $W$ is the subgraph $G_W$ of $G$ with the vertex set $V(G_W) = W$ and with the edge set $E(G_W) = \{\{x_i, x_j\} \in E(G) : x_i, x_j \in W\}$. A finite simple graph $G$ is chordal if every cycle in $G$ of length $> 3$ has a chord. A subset $M$ of $E(G)$ is a matching of $G$ if, for any $e, e' \in M$ with $e \neq e'$, one has $e \cap e' = \emptyset$. A matching $M$ is called an induced matching of $G$ if, for any $e, e' \in M$ with $e \neq e'$, there is no $e'' \in E(G)$ satisfying both of $e \cap e'' \neq \emptyset$ and $e' \cap e'' \neq \emptyset$. The induced matching number $\text{ind-match}(G)$ of $G$ is the maximum size of induced matching of $G$. A complete bipartite graph of type $(1, d)$ is called a bouquet. Let $\mathcal{B} = \{B_1, \ldots, B_s\}$ be a set of bouquets, where $B_i$ is a subgraph of $G$, and set $V(\mathcal{B}) = V(B_1) \cup \cdots \cup V(B_s)$. We say that $\mathcal{B}$ is a strongly disjoint set of bouquets of $G$ if $V(B_k) \cap V(B_\ell) = \emptyset$ for all $k \neq \ell$ and if, for each $1 \leq k \leq s$, there exists $e_k \in E(B_k)$ such that $\{e_1, \ldots, e_s\}$ forms an induced matching of $G$. When $\mathcal{B}$ is a strongly disjoint set of bouquets, we define the type of $\mathcal{B}$ as $|V(\mathcal{B})| - s, s)$. Finally, we say that $G$ contains a strongly disjoint set of bouquets of type $(i, j)$ if there exists a strongly disjoint set of bouquets $\mathcal{B}$ of $G$ whose type is $(i, j)$ and which satisfies $V(\mathcal{B}) = V(G)$.

We now turn to a proof of Theorem 1.

(First Step) Let $1 = b \leq r$ and $G_{r,1}$ the graph consisting of $r$ paths $P_3$ with the common vertex $z$, that is,

$$V(G_{r,1}) := \{x_1, \ldots, x_r\} \cup \{y_1, \ldots, y_r\} \cup \{z\},$$

$$E(G_{r,1}) := \{\{z, y_i\} : i = 1, \ldots, r\} \cup \{\{x_i, y_i\} : i = 1, \ldots, r\}.$$

Note that the graph $G_{r,1}$ is a tree, in particular a chordal graph. By virtue of Proposition 2, one has $\beta_{r+1, (r+1)+r}(S/I(G_{r,1}))$ is the only extremal Betti number of $S/I(G_{r,1})$ and $\text{reg}(S/I(G_{r,1})) = r$, as required.

(Second Step) Let $r$ and $b$ be integers with $2 \leq b \leq r$. We then introduce the finite simple connected graphs $G_{r,j}$ for $j = 2, \ldots, b$ constructed as follows. Starting with the star triangle consisting of $r$ triangles with the common vertex $z$, we define $G_{r,2}$ by adding a new vertex $w_1$ joining with all vertices of a triangle, say, $z, x_1, y_1$, to it. Thus

$$V(G_{r,2}) := \{x_1, \ldots, x_r\} \cup \{y_1, \ldots, y_r\} \cup \{z\} \cup \{w_1\},$$

$$E(G_{r,2}) := \{\{z, x_i\} : i = 1, \ldots, r\} \cup \{\{z, y_i\} : i = 1, \ldots, r\} \cup \{\{x_i, y_i\} : i = 1, \ldots, r\}
\cup \{\{w_1, x_1\}, \{w_1, y_1\}, \{w_1, z\}\}.$$
to \(G_{r,j}\). In other words,
\[
V(G_{r,j+1}) := \{x_1, \ldots, x_r\} \cup \{y_1, \ldots, y_r\} \cup \{z\} \cup \{w_1, \ldots, w_j\},
\]
\[
E(G_{r,j+1}) := E(G_{r,j}) \cup \{w_j, z\} \cup \{w_j, w_i\} : i = 1, \ldots, j - 1
\]
\[
\cup \{w_j, x_i \} : i = 1, \ldots, j \cup \{w_j, y_i \} : i = 1, \ldots, j \}.
\]

The following lemma completes Second Step of our proof.

**Lemma 3.** Let \(2 \leq b \leq r\) and \(S = K[V(G_{r,b})]\). Then

1. \(\text{reg}(S/I(G_{r,b})) = r\);
2. \(\text{proj dim}(S/I(G_{r,b})) = 2r + b - 1\);
3. the extremal Betti numbers of \(S/I(G_{r,b})\) are

\[
\beta_{r+b+i-1,(r+b+i-1)+(r-i+1)}(S/I(G_{r,b})), \quad i = 1, 2, \ldots, b-1,
\]

and
\[
\beta_{2r+b-1,(2r+b-1)+1}(S/I(G_{r,b})).
\]

**Proof.** We start on proving that \(G_{r,b}\) is a chordal graph.

Let \(j\) be an integer with \(1 \leq j \leq b - 1\). We prove that \(G_{r,j+1}\) is chordal under the hypothesis that \(G_{r,j}\) is chordal. Here, for abuse notation, we use \(G_{r,1}\) to denote the star triangle consisting of \(r\) triangles, obviously it is a chordal graph. Suppose that there exists a cycle \(C\) of \(G_{r,j+1}\) of length \(> 3\) with no chord. Since \(G_{r,j}\) is chordal, it follows that \(C\) must contain \(w_j\). If \(C\) contains a vertex which does not belong to \(N_{G_{r,j+1}}(w_j)\), where 
\[
N_{G_{r,j+1}}(w_j) = N_{G_{r,j+1}}(w_j) \cup \{w_j\} \text{ and } N_{G_{r,j+1}}(w_j) \text{ is the neighbourhood of } w_j \text{ in } G_{r,j+1}.
\]
then \(C\) must be a triangle \(x_{j+k}y_k\) for some \(k \geq j + 1\). This is a contradiction. Thus all vertices of \(C\) must belong to \(N_{G_{r,j+1}}(w_j)\). Since \(C\) contains \(w_j\), this contradicts the hypothesis that \(C\) does not have a chord.

1. Since \(G_{r,b}\) is chordal, the regularity of \(S/I(G_{r,b})\) is equal to the induced matching number \(\text{ind-match}(G_{r,b})\) of \(G_{r,b}\) ([1 Corollary 6.9]). Since \(\{x_1, y_1\}, \ldots, \{x_r, y_r\}\) form an induced matching, one has \(\text{ind-match}(G_{r,b}) = r\). Conversely, let \(\mathcal{M}\) be an induced matching of \(G_{r,b}\) which is different from the above one. If one of edges in \(\mathcal{M}\) contains \(z\), then \(|\mathcal{M}| = 1\) because 
\[
\text{ind-match}(G_{r,b}) = V(G_{r,b}.
\]
Otherwise, one of edges in \(\mathcal{M}\) contains \(w_j\) for some \(j\). Then the number of edges of \(\mathcal{M}\) contained in \(G_{N[w_j]}\) is 1. The other edges of \(\mathcal{M}\) must be contained in \(\{x_{j+1}, y_{j+1}\}, \ldots, \{x_r, y_r\}\). It then follows that \(|\mathcal{M}| \leq r - j + 1 \leq r\). Hence \(\text{ind-match}(G_{r,b}) = r\), as required.

2. Since \(G_{r,b}\) is chordal, we can apply Proposition [2]. One has \(|V(G_{r,b})| = 2r + b\) and the bipartition \(\{z\} \cup (V(G_{r,b}) \setminus \{z\})\) of the vertex set defines a bouquet of type \((2r+b-1,1)\). Hence \(\beta_{2r+b-1,(2r+b-1)+1}(S/I(G_{r,b})) \neq 0\) and \(\text{proj dim}(S/I(G_{r,b})) = 2r + b - 1\), as desired.

3. Since \(|V(G_{r,b})| = 2r + b = (r + b + i - 1) + (r - i + 1)\), if

\[
\beta_{r+b+i-1,(r+b+i-1)+(r-i+1)}(S/I(G_{r,b})) \neq 0,
\]
then this Betti number is extremal. We have already known that 
\[ \beta_{2r+b-1,(2r+b-1)+1}(S/I(G_{r,b})) \neq 0 \]
by (2). We prove 
\[ \beta_{r+b+i-1,2r+b}(S/I(G_{r,b})) = \beta_{r+b+i-1,(r+b+i-1)+(r-i)+1}(S/I(G_{r,b})) \neq 0 \]
for \( i = 1, 2, \ldots, b-1 \).

For \( i = 1, 2, \ldots, b-1 \), consider the following set \( \mathcal{B}_i = \{B_1^{(i)}, \ldots, B_{r-i}^{(i)}\} \) of bouquets (we describe a bouquet by its vertex bipartition):
\[
B_1^{(i)} : \{w_i\} \sqcup N_{G_{r,b}}(w_i) \\
\quad = \{w_i\} \sqcup \{\{z, x_1, \ldots, x_i, y_1, \ldots, y_i, w_1, \ldots, w_{b-1}\} \setminus \{w_i\}\}, \\
B_{k+1}^{(i)} : \{x_{i+k}\} \sqcup \{y_{i+k}\}, \quad k = 1, 2, \ldots, r-i.
\]

Since 
\[ \{w_i, x_i\} \in E(B_1^{(i)}), \{x_{i+1}, y_{i+1}\} \in E(B_2^{(i)}), \ldots, \{x_r, y_r\} \in E(B_{r-i}^{(i)}) \]
form an induced matching of \( G_{r,b} \), it follows that \( \mathcal{B}_i \) is a strongly disjoint set of bouquets contained in \( G_{r,b} \). Proposition 2 says \( \beta_{r+b+i-1,2r+b}(S/I(G_{r,b})) \neq 0 \).

To complete the proof, our work is to show that
\[ \beta_{r+2b-2+i,(r+2b-2+i)+j}(S/I(G_{r,b})) = 0 \]
for \( i = 1, 2, \ldots, r-b \) and \( j = 2, 3, \ldots, r-b-i+2 \).

We proceed the proof by induction on \( r-b \geq 0 \). When \( r-b = 0 \), there is nothing to prove.

Assume \( r-b > 0 \). We show that there is no set of bouquets which guarantees the non-vanishing of these Betti numbers in meaning of Proposition 2. On the contrary, suppose that there exists a strongly disjoint set of bouquets \( \mathcal{B} \) of \( G_{r,b} \) contained in \( (G_{r,b})_{V(\mathcal{B})} \) of type \( (r+2b-2+i, j) \), where \( 1 \leq i \leq r-b \) and \( 2 \leq j \leq r-b-i+2 \). Let \( \mathcal{B} = \{B_1, \ldots, B_j\} \) and assume that \( e_1, \ldots, e_j \) form an induced matching of \( G_{r,b} \) with each \( e_t \in E(B_t) \).

(Step 1) Assume that \( \{x_k, y_k\} \notin \{e_1, \ldots, e_j\} \) for some \( k \) with \( b \leq k \leq r \). In this case, \( x_k, y_k \notin V(\mathcal{B}) \) because \( j \geq 2 \). Then \( \mathcal{B} \) can be regarded as a strongly disjoint set of bouquets of \( G_{r-1,b} \) of
\[ \text{type} (r+2b-2+i, j) = \text{type} ((r-1)+2b-2+(i+1), j). \]

Since 
\[ \beta_{r-1+2b-2+\alpha,(r-1+2b-2+\alpha)+\beta}(K[V(G_{r-1,b})]/I(G_{r-1,b})) = 0, \]
\[ \alpha = 1, 2, \ldots, r-1-b; \quad \beta = 2, 3, \ldots, (r-1) - b - \alpha + 2 \]
by inductive hypothesis, the possible pairs \((i, j)\) can be
\[ \bullet \ (i, r-b-i+2); \quad (i, r-b-i+1) \quad \text{for} \ 1 \leq i \leq r-b-2; \]
\[ \bullet \ i = r-b-1, \ \text{then} \ j = 2, 3; \]
• \( i = r - b \), then \( j = 2 \).

In each of the three cases, \( i + j \) is equal to either \( r - b + 1 \) or \( r - b + 2 \). Hence \( |V(\mathcal{B})| = (r + 2b - 2 + i) + j \) is equal to either \( 2r + b - 1 \) or \( 2r + b \), which contradicts \( x_k, y_k \notin V(\mathcal{B}) \).

(Step 2) Assume that \( \{x_k, y_k\} \in \{e_1, \ldots, e_j\} \) for all \( k = b, \ldots, r \). Then \( j \geq r - b + 1 \). Since \( j \leq r - b - i + 2 \) and \( i \geq 1 \), it follows that \( j = r - b + 1 \) and \( i = 1 \). Hence

\[ V(\mathcal{B}) \subset \{z\} \cup \{x_b, \ldots, x_r\} \cup \{y_b, \ldots, y_r\} \]

This contradicts the fact that type of \( \mathcal{B} \) is \( (r + 2b - 2 + 1, r - b + 1) \). \( \square \)

Example 4. Let \( b = 3 \) and \( r = 5 \). The graph \( G_{5,3} \) is the following:

```
Example 6. Let \( 1 = b \leq r < p \) and \( G_{p,r,1} \) be the connected graph for which

\[ V(G_{p,r,1}) = \{x_1, \ldots, x_{p-1}\} \cup \{y_1, \ldots, y_r\} \cup \{z\}, \]
\[ E(G_{p,r,1}) = \{\{z,y_1\} : i = 1, \ldots, r\} \cup \{\{x_i, y_1\} : i = 1, \ldots, r - 1\} \]
\[ \cup \{\{x_j, y_r\} : j = r, \ldots, p - 1\}. \]```
Note that $G_{r+1,r,1} = G_{r,1}$, appears in First Step of the proof of Theorem 1. Then Proposition 2 says that $\beta_{p,p+r}(S/I(G_{p,r,1}))$ is the only extremal Betti number of $S/I(G_{p,r,1})$. In particular,

- $\text{reg}(S/I(G_{r+1,r,1})) = r$;
- $\text{projdim}(S/I(G_{r+1,r,1})) = p$;
- the number of extremal Betti numbers of $K[V(G_{p,r,1})]/I(G_{p,r,1})$ is equal to $1 = b$.

Acknowledgment. The authors were partially supported by JSPS KAKENHI 26220701, 15K17507 and 17K14165.

REFERENCES

[1] H. T. Hà and A. Van Tuyl, Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers, J. Algebraic Combin. 27 (2008), 215–245.

[2] J. Herzog and T. Hibi, “Monomial ideals,” Graduate Texts in Mathematics 260, Springer, London, 2010.

[3] K. Kimura, Non-vanishingness of Betti numbers of edge ideals, in: “Harmony of Gröbner Bases and the Modern Industrial Society” (T. Hibi, Ed.), World Sci. Publ., Hackensack, NJ. 2012, pp. 153–168.

Takayuki Hibi, Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Suita, Osaka 565-0871, Japan

E-mail address: hibi@math.sci.osaka-u.ac.jp

Kyouko Kimura, Department of Mathematics, Faculty of Science, Shizuoka University, 836 Ohya, Suruga-ku, Shizuoka 422-8529, Japan

E-mail address: kimura.kyoko.a@shizuoka.ac.jp

Kazunori Matsuda, Kitami Institute of Technology, Kitami, Hokkaido 090-8507, Japan

E-mail address: kaz-matsuda@mail.kitami-it.ac.jp