NON-LINEAR GRASSMANNIANS AS COADJOINT ORBITS

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Abstract. For a given manifold $M$ we consider the non-linear Grassmann manifold $\text{Gr}_n(M)$ of $n$-dimensional submanifolds in $M$. A closed $(n+2)$-form on $M$ gives rise to a closed 2-form on $\text{Gr}_n(M)$. If the original form was integral, the 2-form will be the curvature of a principal $S^1$-bundle over $\text{Gr}_n(M)$. Using this $S^1$-bundle one obtains central extensions for certain groups of diffeomorphisms of $M$. We can realize $\text{Gr}_{m-2}(M)$ as coadjoint orbits of the extended group of exact volume preserving diffeomorphisms and the symplectic Grassmannians $\text{SGr}_{2k}(M)$ as coadjoint orbits in the group of Hamiltonian diffeomorphisms. We also generalize the vortex filament equation as a Hamiltonian equation on $\text{Gr}_{m-2}(M)$.

1. Introduction

Let $M$ be a smooth connected closed manifold of dimension $m$. We are interested in the space of closed submanifolds of $M$. More precisely we fix a dimension $n$ and let $\text{Gr}_n(M)$ denote the space of all $n$-dimensional oriented compact boundaryless submanifolds of $M$. This is easily seen to be a Fréchet manifold in a natural way. We consider this as a non-linear analogue of the classical Grassmann manifolds.

Every closed differential form $\alpha$ of degree $n+2$ on $M$ gives rise to a closed 2-form $\tilde{\alpha}$ on $\text{Gr}_n(M)$. If $\alpha$ was integral, our first theorem says that there is a principal $S^1$-bundle $P \to \text{Gr}_n(M)$ with principal connection, whose curvature form is $\tilde{\alpha}$. The group of equivariant connection preserving diffeomorphisms of $P$ then is a central extension of the group of Hamiltonian diffeomorphisms on $\text{Gr}_n(M)$. The latter makes sense, even if $\tilde{\alpha}$ is degenerate. Restricting everything to a connected component of $\text{Gr}_n(M)$, the extension becomes 1-dimensional with fiber $S^1$.

Now the group of diffeomorphisms of $M$ which preserve $\alpha$ acts symplectically on $\text{Gr}_n(M)$. In some cases there are interesting subgroups $G$, which actually act in a Hamiltonian way. In such a situation the pull back of the central extension described above gives a central extension $1 \to S^1 \to \tilde{G} \to G \to 1$.

Let us describe two cases in more detail. Suppose $\alpha$ was an integral volume form and let $\text{Gr}_{m-2}(M)$ denote the space of codimension 2 submanifolds. As mentioned above, the volume form gives a closed 2-form on $\text{Gr}_{m-2}(M)$, which turns out to be (weakly) non-degenerate. Then the group of exact volume preserving diffeomorphisms acts in a Hamiltonian way on $\text{Gr}_{m-2}(M)$. So the pull back gives central extensions $\tilde{G}$ of the group of exact volume preserving diffeomorphisms by $S^1$. This is Ismagilov’s way of constructing these extensions, see [190]. Using the moment map we will then realize the symplectic manifold $\text{Gr}_{m-2}(M)$ as a coadjoint orbit.
of the group $\hat{G}$. We even get Lie group structure on the extensions $\hat{G}$ of the group of exact volume preserving diffeomorphisms.

For the second situation we have in mind we start with a symplectic manifold $(M, \omega)$. Taking $\alpha := \omega^{k+1}$ we get a closed 2–form $\tilde{\alpha}$ on $\text{Gr}_2k(M)$. This form is no longer symplectic. However, when restricted to the open subset $S\text{Gr}_2k(M)$ of symplectic submanifolds it will become non-degenerate. We will refer to $S\text{Gr}_2k(M)$ as a non-linear symplectic Grassmannian. The group of Hamiltonian diffeomorphisms of $M$ now acts in a Hamiltonian way on $S\text{Gr}_2k(M)$. So the procedure above yields central extensions of $\text{Ham}(M, \omega)$ by $S^1$. These extensions are not very interesting, since the associated extensions of Lie algebras turn out to be trivial. However, it permits us to realize the symplectic manifold $S\text{Gr}_2k(M)$ as a coadjoint orbit of $\text{Ham}(M, \omega)$.

For a Riemannian manifold $(M, g)$ the non-linear Grassmannian $\text{Gr}_{m−2}(M)$ of codimension two submanifolds has a canonical almost Kähler structure. The $g$–volume of the submanifold gives a smooth function on $\text{Gr}_{m−2}(M)$ and its Hamiltonian equation generalizes the vortex filament equation.

Finally, let us remark that everything generalizes to non-compact $M$ in a straightforward way. The diffeomorphism groups then have to be replaced by the compactly supported ones.

2. Non-linear Grassmannians

Throughout the whole paper $M$ will be a smooth closed connected $m$–dimensional manifold. Let $\text{Gr}_n(M)$ denote the space of all oriented compact $n$–dimensional not necessarily connected submanifolds without boundary. This is easily seen to be a Fréchet manifold in a natural way, see [KM97]. Note that there is a natural action of the group $\text{Diff}(M)$ on $\text{Gr}_n(M)$. A classical theorem due to R. Thom implies that $\text{Diff}(M)_0$, the connected component in the group of diffeomorphisms, acts transitively on every connected component of $\text{Gr}_n(M)$.

Suppose $N \in \text{Gr}_n(M)$. Then the tangent space of $\text{Gr}_n(M)$ at $N$ can naturally be identified with the space of smooth sections of the normal bundle $TN^\perp := TM|_N/TN$. Any $\alpha \in \Omega^k(M)$ gives rise to $\tilde{\alpha} \in \Omega^{k−n}(\text{Gr}_n(M))$ via:

$$(\tilde{\alpha})_N(Y_1, \ldots, Y_{k−n}) := \int_N i_{Y_1} \cdots i_{Y_n} \alpha.$$ 

Here $N \in \text{Gr}_n(M)$ and $Y_j$ are tangent vectors at $N$, i.e. sections of $TN^\perp$. Then $i_{Y_1} \cdots i_{Y_n} \alpha \in \Omega^n(N)$ does not depend on representatives $Y_j$ and integration is well defined, for $N \in \text{Gr}_n(M)$ comes with an orientation.

Let $\zeta$ denote the infinitesimal $\text{Diff}(M)$–action on $\text{Gr}_n(M)$, that is for every vector field $X \in \mathfrak{x}(M)$ on $M$ we have a fundamental vector field $\zeta X$ on $\text{Gr}_n(M)$. One easily verifies the following

**Lemma 1.** For every $X \in \mathfrak{x}(M)$, $N \in \text{Gr}_n(M)$, $k \in \mathbb{N}$, $\alpha \in \Omega^k(M)$ and every $\varphi \in \text{Diff}(M)$ we have:

(i) $\zeta_X(N) = X|_N$.
(ii) $d\alpha = d\tilde{\alpha}$.
(iii) $i_{\zeta_X} \tilde{\alpha} = \tilde{i}_X \alpha$.
(iv) $L_{\zeta_X} \tilde{\alpha} = \tilde{L}_X \alpha$.
(v) $\varphi^* \tilde{\alpha} = \varphi^* \alpha$. 


Suppose $\alpha$ is a closed $k$–form on $M$. Then we get a closed 2–form $\alpha$ on $\text{Gr}_k - 2(M)$. Our first theorem states that if $[\alpha] \in H^k(M; \mathbb{R})$ is integral then $\alpha$ will be the curvature form of a principal $S^1$–bundle over $\text{Gr}_k - 2(M)$.

**Theorem 1.** Let $M$ be a closed connected manifold and let $\alpha$ be a closed $k$–form representing an integral cohomology class of $M$. Then there exist a principal $S^1$–bundle $\mathcal{P} \to \text{Gr}_k - 2(M)$ and a principal connection $\eta \in \Omega^1(\mathcal{P})$ whose curvature form is $\alpha$.

**Proof.** Note that it suffices to prove this for one representative of $[\alpha] \in H^k(M; \mathbb{R})$. For if $\eta$ is a principal connection with curvature $\tilde{\alpha}$ then $\eta + \pi^*\tilde{\beta}$ is a principal connection with curvature form $\alpha + d\beta$.

Pick a smooth triangulation $\Delta^v$ of $M$ and let $\Delta^{m-k}$ denote its $(m-k)$–skeleton. Choose an open neighborhood $U$ of $\Delta^{m-k}$ which deformation retracts onto $\Delta^{m-k}$. Moreover set $A := M \setminus U$ and $V := A^0 = M \setminus \overline{U}$. One easily checks the following properties:

(i) $H^j(A; \mathbb{Z}) = 0$ for all $j \geq k$. We will actually only use $H^k(A; \mathbb{Z}) = 0$.

(ii) For all compact $K \subseteq M \setminus \Delta^{m-k}$ there exists $g \in \text{Diff}(M)_0$ with $g(K) \subseteq V$.

If moreover $K^1 \subseteq V$ compact, then $g$ and the diffeotopy connecting it to the identity can be chosen to fix the points in $K^1$.

Using (i) and considering

$$
\begin{array}{ccc}
H^k(M, A; \mathbb{Z}) & \longrightarrow & H^k(M; \mathbb{R}) \\
\uparrow & & \uparrow \\
H^k(M, A; \mathbb{Z}) & \longrightarrow & H^k(M; \mathbb{Z}) & \longrightarrow & H^k(A; \mathbb{Z})
\end{array}
$$

we see that $[\alpha] \in H^k(M; \mathbb{R})$ has a representative which vanishes on $V$ and which represents an integral class in $H^k(M, A; \mathbb{R})$, i.e. lies in the image of $H^k(M, A; \mathbb{Z}) \to H^k(M, A; \mathbb{R})$. Since it suffices to construct the bundle and the connection for some representative we may assume from now on

(iii) $\alpha$ vanishes on $V$.

(iv) $\alpha$ represents an integral class in $H^k(M, A; \mathbb{R})$.

**Lemma 2.** Suppose $L$ is a compact manifold of dimension $l < k$, which might have a boundary (even corners) and suppose $f : L \to M$ smooth. Then there exists $g \in \text{Diff}(M)_0$, such that $g(f(L)) \subseteq V$. Moreover if $f(\partial L) \subseteq V$ then $g$ and the diffeotopy connecting it with the identity can be chosen to fix the points in $f(\partial L)$.

**Proof of Lemma 2.** A well known transversality argument shows that there exists $g_1 \in \text{Diff}(M)_0$ with $g_1 \circ f$ transversal to $\Delta^{m-k}$. Since $l < k$ we thus must have $g_1(f(L)) \cap \Delta^{m-k} = \emptyset$. From (ii) we get $g_2 \in \text{Diff}(M)_0$ with $g_2(g_1(f(L))) \subseteq V$. The second part is proved similarly.

Let us continue with the proof of Theorem 1. Let $I = [0, 1]$ denote the unit interval. For $\varphi \in C^\infty(I, \text{Diff}(M))$ with $\varphi_0 = \text{id}$ we define $U_\varphi := \{ N \in \text{Gr}_{k-2}(M) : \varphi_1(N) \subseteq V \}$. Lemma 2 implies that $U_\varphi$ is an open covering of $\text{Gr}_{k-2}(M)$. Moreover we set

$$
\lambda_\varphi := -\int_0^1 \varphi_1^* \alpha dt \in \Omega^k(M).
$$
A one line computation shows $d\lambda_x = \alpha - \varphi^1 \alpha$. Because of (11) we particularly have $d\lambda_x = \alpha$ on $\varphi^1(V)$ and thus $d\tilde{\lambda}_x = \tilde{\alpha}$ on $U_x$. The $\tilde{\lambda}_x$ will be the connection forms, we are going to define the transition cocycle defining the $S^1$–bundle.

Fix $\varphi, \psi \in C^{\infty}(I, \text{Diff}(M))$ with $\varphi_0 = \psi_0 = \text{id}$ and consider homotopies $\Phi \in C^{\infty}(I \times I, \text{Diff}(M))$ with $\Phi_{0,t} = \varphi_t$, $\Phi_{1,t} = \psi_t$ and $\Phi_{s,0} = \text{id}$ for all $s \in I$. For such a $\Phi$ we set

$$U_\Phi := \{ N \in \text{Gr}_{k-2}(M) : \Phi_{s,1}(N) \subseteq V \text{ for all } s \in I \}.$$  

Clearly $U_\Phi$ are open subsets of $U_\varphi \cap U_\psi$. Lemma 2 shows that for every $N \in U_\varphi \cap U_\psi$ there exists a homotopy $\Phi$ with ends $\varphi$ and $\psi$, such that $N \in U_\Phi$. In other words $U_\Phi$ constitute an open covering of $U_\varphi \cap U_\psi$, as $\Phi$ varies with fixed ends $\varphi$ and $\psi$. For such a $\Phi$ we define

$$\tau_\Phi := \int_0^1 \int_0^1 \Phi_{s,t}^* \delta \Phi(\partial_s)i_{\delta \Phi(\partial_t)} \alpha \, ds \, dt \in \Omega^{k-2}(M).$$

Using the Maurer–Cartan equation for the left logarithmic derivative, cf. [KM97],

$$[\delta \Phi(\partial_s), \delta \Phi(\partial_t)] = \frac{\partial}{\partial s} \delta \Phi(\partial_s) - \frac{\partial}{\partial t} \delta \Phi(\partial_t),$$

an easy computation yields

$$d\tau_\Phi = \lambda_\psi - \lambda_\varphi + \int_0^1 \Phi_{s,1}^* i_{\delta \Phi(\partial_s)} \alpha \, ds.$$  

Particularly $\tilde{\lambda}_\psi - \tilde{\lambda}_\varphi = d\tilde{\tau}_\Phi$ on $U_\Phi$. Note that $\tilde{\tau}_\Phi$ is a function on $\text{Gr}_{k-2}(M)$ and obviously $\tilde{\tau}_\Phi(N) = \int_{I \times I \times N} \Phi_N^* \alpha$ with $\Phi_N(s,t,x) = \Phi_{s,t}(x)$, $x \in N$. If $\Psi$ is another homotopy with ends $\varphi$ and $\psi$ and $N \in U_\varphi \cap U_\Psi$ then

$$\tilde{\tau}_\Psi(N) - \tilde{\tau}_\Phi(N) \in \mathbb{Z}.$$  

Indeed, $\tilde{\Psi}_N - \tilde{\Phi}_N$ represents a class in $H_k(M,A;\mathbb{Z})$ and $\tilde{\tau}_\Psi(N) - \tilde{\tau}_\Phi(N)$ is the pairing of this class with $[\alpha] \in H^k(M,A;\mathbb{R})$. From (17) we see that the result must be integral as well.

So when considered as functions $U_\Phi \to S^1 := \mathbb{R}/\mathbb{Z}$ the $\tilde{\tau}_\Phi$ fit together and define well defined smooth $f_{\varphi,\psi} : U_\varphi \cap U_\psi \to S^1$ satisfying $df_{\varphi,\psi} = \tilde{\lambda}_\psi - \tilde{\lambda}_\varphi$. A similar argument shows that they satisfy the cocycle condition $f_{\varphi,\psi} + f_{\psi,\rho} - f_{\varphi,\rho} = 0$ as functions $U_\varphi \cap U_\psi \cap U_\rho \to S^1$, where $S^1$ is written additively.

Now define $P$ to be the principal $S^1$–bundle one obtains when gluing $U_\varphi \times S^1$ with the help of $f_{\varphi,\psi}$. On $U_\varphi \times S^1$ we define $\eta_\varphi := \lambda_\varphi + d\theta$, where $d\theta$ denotes the standard volume form on $S^1$. These locally defined $\eta_\varphi$ define a global principal connection $\eta \in \Omega^1(P)$, for we have $df_{\varphi,\psi} = \tilde{\lambda}_\psi - \tilde{\lambda}_\varphi$ on $U_\varphi \cap U_\psi$. Since $d\tilde{\lambda}_\varphi = \tilde{\alpha}$ on $U_\varphi$ its curvature form is $\tilde{\alpha}$. This finishes the proof of the theorem.

**Example 1.** Let us consider the case $k = 2$. So $\alpha$ is a closed integral 2–form and $\text{Gr}_{k-2}(M)$ is the space of oriented points in $M$. Let $\mathcal{M}$ denote the connected component of $\text{Gr}_{k-2}(M)$ where the submanifolds consist of a single positively oriented point. Certainly $\mathcal{M} = M$ and $\tilde{\alpha} = \alpha$. So in this case the restriction of the bundle $\mathcal{P} \to \text{Gr}_{k-2}(M)$ to $\mathcal{M}$ gives the classical circle bundle with connection corresponding to the closed integral 2–form $\alpha$.

**Remark 1.** A Theorem of R. Thom implies that the action of $\text{Diff}(M)_0$ on connected components of $\text{Gr}_n(M)$ is transitive. Hence connected components of $\text{Gr}_n(M)$ are
homogeneous spaces of $\text{Diff}(M)_0$. Actually Thom’s theorem shows that $\text{Diff}(M)_0$ acts transitively on connected components of $\text{Emb}(N,M)$, the space of smooth embeddings of $N$ in $M$. So connected components of $\text{Emb}(N,M)$ are homogeneous spaces of $\text{Diff}(M)_0$ too. Below we will see that similar statements hold for the group of volume preserving diffeomorphisms.

Moreover the connected components of $\text{Emb}(N,M)$ are principal bundles over corresponding connected components of $\text{Gr}_n(M)$, see [KM97]. The structure group is the group of orientation preserving diffeomorphisms of $N$.

3. Universal construction

Suppose we have a principal $S^1$–bundle $\pi : (\mathcal{P}, \eta) \to (\mathcal{M}, \Omega)$ with connection $\eta$ and curvature $\Omega$. We assume $\mathcal{M}$ connected but it may very well be infinite dimensional. We associate Kostant’s exact sequence of groups, see [K70]:

$$1 \to S^1 \to \text{Aut}(\mathcal{P}, \eta) \to \text{Ham}(\mathcal{M}, \Omega) \to 1.$$ 

Here $\text{Aut}(\mathcal{P}, \eta)$ is the connected component of the group of equivariant connection preserving diffeomorphisms of $\mathcal{P}$ and $\text{Ham}(\mathcal{M}, \Omega)$ is the group of Hamiltonian diffeomorphisms of $\mathcal{M}$. The latter can either be described as the connected component of holonomy preserving diffeomorphisms, or as the kernel of a flux homomorphism [NV].

The group $\text{Aut}(\mathcal{P}, \eta)$ acts on $\mathcal{M}$ in a Hamiltonian way with equivariant moment map

$$\hat{\mu} : \mathcal{M} \to \text{aut}(\mathcal{P}, \eta)^*, \quad \hat{\mu}(x)(\xi) = -(i\xi\eta)(\pi^{-1}(x)).$$

This moment map is universal in the following sense: Whenever we have a Hamiltonian action of a Lie group $G$ on $\mathcal{M}$, we can pull back Kostant’s extension and get a 1–dimensional central group extension:

$$\begin{array}{ccc}
S^1 & \longrightarrow & \text{Aut}(\mathcal{P}, \eta) \\
\downarrow & & \downarrow \\
S^1 & \longrightarrow & \tilde{G} \\
\downarrow & & \downarrow \\
G & \longrightarrow & G
\end{array}$$

This is a Lie group extension, even if Kostant’s extension is only a group extension in this infinite dimensional setting, see [NV]. Moreover the pull back $\tilde{\mu} : \mathcal{M} \to \tilde{g}^*$ of $\hat{\mu}$ is a smooth equivariant moment map for the $\tilde{G}$–action on $\mathcal{M}$. Consider the corresponding central extension of Lie algebras:

$$\begin{array}{ccc}
\mathbb{R} & \longrightarrow & \text{aut}(\mathcal{P}, \eta) \\
\downarrow & & \downarrow \\
\mathbb{R} & \longrightarrow & \hat{\mathfrak{g}} \\
\downarrow & & \downarrow p \\
\mathfrak{g} & \longrightarrow & \mathfrak{g}
\end{array}$$

Proposition 1. In the situation above suppose moreover that $\mathcal{G}$ acts transitively on $\mathcal{M}$ and admits an injective but not necessarily equivariant moment map $\mu : \mathcal{M} \to \mathfrak{g}^*$. Then the equivariant moment map $\tilde{\mu} : \mathcal{M} \to \tilde{\mathfrak{g}}^*$ is one-to-one onto a coadjoint orbit of $\tilde{\mathcal{G}}$. Moreover it pulls back the Kirillov–Kostant–Souriau symplectic form to $\Omega$.

Proof. Note first, that $p^* \circ \mu : \mathcal{M} \to \tilde{\mathfrak{g}}^*$ is an injective but not necessarily equivariant moment map for the $\tilde{G}$–action on $\mathcal{M}$. Since $\mathcal{M}$ is connected, two moment maps differ by a constant in $\tilde{\mathfrak{g}}^*$. Thus every moment map for the $\tilde{G}$–action on $\mathcal{M}$ is
injective, particularly $\hat{\mu}$. Next, $\hat{G}$ acts transitively, for $G$ does. So the equivariance of $\hat{\mu}$ implies that $\hat{\mu}$ is onto a single coadjoint orbit. A straight forward calculation shows that the pull back of the Kirillov–Kostant–Souriau symplectic form is $\Omega$. \hfill \square

Till the end of the section we will denote all the left $G$–actions by a dot. Suppose we have a not necessarily equivariant moment map $\mu : \mathcal{M} \rightarrow \mathfrak{g}^*$. Let $\hat{h} : \hat{g} \rightarrow C^\infty(\mathcal{M}, \mathbb{R})$ denote the dual map, that is $h_X(x) = \mu(x)(X)$, for $x \in \mathcal{M}$ and $X \in \mathfrak{g}$. The universal property of the pull back implies that there is a unique section $\sigma : \mathcal{M} \rightarrow \hat{g}$ with $i_{\sigma(X)}\eta = -\pi^*h_X$. Conversely every section is obtained in this way. So we have a one-to-one correspondence of not necessarily equivariant moment maps $\mu : \mathcal{M} \rightarrow \mathfrak{g}^*$ and sections of $p : \hat{g} \rightarrow g$. Every such choice gives a linear isomorphism

$$\mathbb{R} \oplus \mathfrak{g} \rightarrow \hat{g}, \quad (a, X) \mapsto a + \sigma(X). \quad \quad (1)$$

Via (1) the equivariant moment map $\hat{\mu} : \mathcal{M} \rightarrow \mathfrak{g}^*$ we constructed above is

$$\hat{\mu} : \mathcal{M} \rightarrow (\mathbb{R} \oplus \mathfrak{g})^* = \mathbb{R}^* \oplus \mathfrak{g}^*, \quad \hat{\mu} = (-1^*, \mu).$$

Here $1^*$ is the dual base to 1 considered as base of $\mathbb{R}$. Equivalently $\hat{\mu}(x)(a, X) = \mu(x)(X) - a$, for $x \in \mathcal{M}$, $X \in \mathfrak{g}$ and $a \in \mathbb{R}$.

Define $\kappa : G \rightarrow \mathfrak{g}^*$ by $-\kappa(g^{-1})(X) = g \cdot \sigma(X) - \sigma(g \cdot X)$. So $\kappa$ is the failure of $\sigma$ to be $G$–equivariant. Then via (1) the adjoint action is

$$g \cdot (a, X) = (a - \kappa(g^{-1})(X), g \cdot X), \quad \text{for } g \in G.$$ 

The function $\kappa$ satisfies $\kappa(g_1g_2) = \kappa(g_1) + g_1 \cdot \kappa(g_2)$, hence it is a 1–cocycle (derivation) on $G$ with values in $\mathfrak{g}^*$. For $g \in G$ and $X \in \mathfrak{g}$ the function $h_{g^{-1}}X - g \cdot h_X$ is locally constant, hence constant since $\mathcal{M}$ is connected. So we get a function $G \rightarrow \mathfrak{g}^*$ which measures the failure of the moment map to be $G$–equivariant. One readily checks $-\kappa(g^{-1})(X) = h_{g^{-1}}X - g \cdot h_X$, equivalently $\kappa(g) = \mu(g \cdot x_0) - g \cdot \mu(x_0)$, for every $x_0 \in \mathcal{M}$. So the section $\sigma$ is $G$–equivariant iff the corresponding moment map is $G$–equivariant.

Via (1) we can express the Lie bracket as

$$[(a, X), (b, Y)] = (c(X, Y), [X, Y]),$$

where $c \in \Lambda^2\mathfrak{g}^*$ is the cocycle $c(X, Y) = [\sigma(X), \sigma(Y)] - \sigma([X, Y])$. Note that $c$ also is a measure for the failure of $\sigma$ to be $g$–equivariant. By choosing different sections $\sigma$ we obtain all 2–cocycles $c$ in one cohomology class, but different sections could define the same 2–cocycle. Moreover the differential of $\kappa : G \rightarrow \mathfrak{g}^*$ at the identity satisfies $(T_e\kappa \cdot X)(Y) = c(X, Y)$. Since we had $-\kappa(g^{-1})(X) = h_{g^{-1}}X - g \cdot h_X$, we get

$$c(X, Y) = h_{[X,Y]} + L_{\xi_X}h_Y = h_{[X,Y]} + \{h_X, h_Y\} = h_{[X,Y]} - \Omega(\xi_X, \xi_Y). \quad \quad (2)$$

The unexpected signs of the second summands stem from the convention for the Lie derivative of functions, which is an infinitesimal right action and quite confusing. Thus $c$ also is a measure for the failure of the moment map to be $g$–equivariant. Particularly the moment map $\mu$ is $g$–equivariant iff the corresponding section $\sigma$ is $g$–equivariant. Finally for every point $x_0 \in \mathcal{M}$ we have $c(X, Y) = h_{[X,Y]}(x_0) - \Omega(\xi_X, \xi_Y)(x_0)$. So we see that $c(X, Y) = -\Omega(\xi_X, \xi_Y)(x_0)$ is a cocycle describing the extension $0 \rightarrow \mathbb{R} \rightarrow \hat{g} \rightarrow g \rightarrow 0$ and corresponds to moment maps satisfying $\mu(x_0) = 0$. 

4. Codimension two Grassmannians

Let $M$ be a closed $m$–dimensional manifold with integral volume form $\nu$, that is $\int_M \nu \in \mathbb{Z}$. From Theorem 1 we get a principal $S^1$–bundle $P \to \text{Gr}_{m-2}(M)$ and a principal connection $\eta$ whose curvature form is $\Omega := \tilde{\nu}$. Recall that $\Omega_N(Y_1, Y_2) = \int_N Y_1 Y_2 \nu$ for tangent vectors $Y_1$ and $Y_2$ at $N$, i.e. sections of $TN^\perp$. Note that $\Omega$ is symplectic, i.e. (weakly) non-degenerate. The action of the group of volume preserving diffeomorphisms $\text{Diff}(M, \nu)$ on $\text{Gr}_{m-2}(M)$ preserves the symplectic form $\Omega$. In dimension $m = 3$ the symplectic form $\Omega$ is known as the Marsden–Weinstein symplectic form on the space of unparameterized oriented links, see [MWS83].

Let $\text{Ham}(M, \nu)$ denote the group of exact volume preserving diffeomorphisms with Lie algebra

$$\text{ham}(M, \nu) = \{ X \in \mathfrak{X}(M) : i_X \nu \text{ exact differential form} \}.$$ 

The action of $\text{Ham}(M, \nu)$ on $\text{Gr}_{m-2}(M)$ is Hamiltonian. Indeed, this follows from $[\text{ham}(M, \nu), \text{ham}(M, \nu)] = \text{ham}(M, \nu)$, see [L74], the fact that $\text{ham}(M, \nu)$ acts symplectically and the fact that the Lie bracket of two symplectic vector fields (on $\text{Gr}_{m-2}(M)$) will be Hamiltonian. In our special situation we do not actually need this general argument, for we have the following

Lemma 3. Let $M$ be a connected component of $\text{Gr}_{m-2}(M)$ and choose $N_0 \in M$. Then

$$\mu : M \to \text{ham}(M, \nu)^*, \quad \mu(N)(X) = \int_N X - \int_{N_0} X, \quad \text{where } i_X \nu = d\alpha$$

is a well defined and injective moment map for the $\text{Ham}(M, \nu)$–action on $M$. Particularly $\text{Ham}(M, \nu)$ acts in a Hamiltonian way on $M$.

Proof. The definition is meaningful since $\mu(N)(X) = \int_T i_X \nu$ for any bordism $\tau$ in $M$ with boundary $N - N_0$, and this expression does not depend on the choice of $\tau$, for $i_X \nu$ is exact.

The fundamental vector field of $X \in \text{ham}(M, \nu)$ is $\zeta_X(N) = X|_N$. To show that $\mu$ is a moment map, we verify that the function $h(N) := \mu(N)(X)$ is a Hamiltonian function for the vector field $\zeta_X$. Indeed, up to a constant, $h$ equals $\tilde{\alpha}$, and thus

$$dh = d\tilde{\alpha} = d\alpha = i_{\zeta_X} \nu = i_{\zeta_X} \nu = i_{\zeta_X} \Omega.$$ 

The injectivity of this moment map is easily seen choosing $\alpha$ with appropriate support. \qed

Proposition 2. The action of $\text{Ham}(M, \nu)$ on connected components of $\text{Gr}_n(M)$ is transitive, provided $m - n \geq 2$.

Proof. We will show more. Namely we will prove that $\text{Ham}(M, \nu)$ acts transitively on every connected component of $\text{Emb}(N, M)$, the space of embeddings of $N$ in $M$.

First we show that the action of $\text{Ham}(M, \nu)$ on $\text{Emb}(N, M)$ is infinitesimal transitive, i.e. every vector field along a closed submanifold $N$ in $M$ of codimension at least two, can be extended to an exact divergence free vector field on $M$. We start with an arbitrary extension $Y \in \mathfrak{X}(M)$ of the given vector field $X \in \Gamma(TM|_N)$. By the relative Poincaré lemma for the $m$–form $\beta = L_Y \nu$, there exists an $(m-1)$–form $\lambda$ on a tubular neighborhood $U$ of $N$ in $M$, such that $d\lambda = \beta$ on $U$ and $\lambda|_N = 0$. The relation $i_Z \nu = \lambda$ defines a vector field $Z \in \mathfrak{X}(U)$ with properties: $Z|_N = 0$ and $L_Z \nu = \beta$. Then $Y - Z$ is a divergence free vector field on $U$ extending $X$. Since
If $m - 1 > n$ we have $H^{m-1}(U) = 0$, in particular $Y - Z$ is an exact divergence free vector field. It can be extended to an exact divergence free vector field $\tilde{X} \in \mathfrak{X}(M)$ with $\tilde{X}|_N = X$.

Next we show that every isotopy of $N$ in $M$ extends to an exact volume preserving diffeotopy of $M$. Indeed, an isotopy $h_t : N \to M$ determines a smooth family of vector fields $X_t$ on $M$ along $N_t = h_t(N) \subset M$. By the infinitesimal transitivity we can extend each $X_t$ to an exact divergence free vector field $\tilde{X}_t$. Looking closer at the construction above, we see that the extension can be chosen smoothly depending on $t$. The diffeotopy $\phi_t$ determined by $\tilde{X}_t$ is exact volume preserving and extends the isotopy $h_t$, i.e. $h_t = \phi_t \circ h_0$. So $\text{Ham}(M, \nu)$ acts transitively on connected components of $\text{Emb}(N, M)$.

**Remark 2.** The connected components of $\text{Gr}_n(M)$ and $\text{Emb}(N, M)$ can be written as homogeneous spaces of $\text{Ham}(M, \mu)$. In the first case the isotropy group is the subgroup of exact volume preserving diffeomorphisms leaving $N \in \text{Gr}_n(M)$ invariant, in the second case it is the subgroup of exact volume preserving diffeomorphisms fixing $N$ pointwise.

Proposition 1, Lemma 3, Proposition 2 and Theorem 1 prove the following

**Theorem 2.** Let $M$ be a closed $m$–dimensional manifold with integral volume form $\nu$ and let $\mathcal{M}$ be a connected component of $\text{Gr}_{m-2}(M)$ equipped with the symplectic form $\Omega = \tilde{\nu}$. Then there exists a central extension of $\text{Ham}(M, \nu)$ by $S^1$ such that $\mathcal{M}$ is a coadjoint orbit of this extension. Particularly this coadjoint orbit is prequantizable.

**Remark 3.** Recall that the central extension in Theorem 2 is the pull-back of Kostant’s extension by the Hamiltonian action of $\text{Ham}(M, \nu)$ on $\mathcal{M}$. Choose an element $N_0$ in $\mathcal{M}$. The moment map $\mu$ from Lemma 3 vanishes at $N_0$, so by 2 the corresponding Lie algebra cocycle on $\text{ham}(M, \nu)$ is $c_{N_0} = -\Omega(\zeta_\nu(N_0)) = -\int_{N_0} i\zeta_i X \nu$. The failure of the moment map $\mu$ to be equivariant is $\kappa(\mu)(X) = \int_{\phi(N_0)} \alpha - \int_{N_0} \alpha$, with $i\zeta_i \nu = d\alpha$.

**Remark 4.** A result of Roger [95] says that the second Lie algebra cohomology group of $\text{ham}(M, \nu)$ is isomorphic to $H_{m-2}(M; \mathbb{R})$, the 2–cocycle on $\text{ham}(M, \nu)$ defined by the $(m - 2)$–cycle $\sigma$ on $M$ being $c_\sigma(X, Y) = \int_{\sigma} i\zeta_i X \nu$. Every homology class $\sigma$ in $H_{m-2}(M; \mathbb{Z})$ has a representative which is a closed submanifold of codimension 2 in $M$. The representative $N_0$ can be taken to be the zero set of a section transversal to the zero section in a rank two vector bundle with Euler class the Poincaré dual of $\sigma$. It follows that all 1–dimensional central extensions of $\text{ham}(M, \nu)$ corresponding to $\sigma \in H_{m-2}(M; \mathbb{Z})$ can be integrated to group extensions. The original construction is due to Ismagilov [96]. However, we even get the Lie group structure on the extensions by using a result in [NV].

**Remark 5.** Suppose $\mathcal{M}$ and $\mathcal{M}'$ are two connected components of $\text{Gr}_{m-2}(M)$ corresponding to homologous submanifolds of $M$. Choose $N_0 \in \mathcal{M}$ and $N'_0 \in \mathcal{M}'$. Since $N_0$ and $N'_0$ are homologous we can choose a smooth $(m - 1)$–chain $B_0$ in $M$ with $\partial B_0 = N'_0 - N_0$. Denote $g := \text{ham}(M, \nu)$, $G := \text{Ham}(M, \nu)$ and define

$$\lambda_0 \in g^*, \quad \lambda_0(X) := -\int_{B_0} iX \nu.$$
This does not depend on the choice of $B_0$, for $i_X \nu$ is exact. Via Lemma $N_0$ and $N'_0$ give rise to moment maps $\mu_0 : M \to g^*$ and $\mu'_0 : M' \to g^*$ with corresponding cocycles $c_0$ and $c'_0$ and $\kappa_0 : G \to g^*$ and $\kappa'_0 : G \to g^*$, respectively, see Remark $c_0(X, Y) - c_0(X, Y) = \lambda_0(-[X, Y])$ and $\kappa'_0(\varphi) - \kappa_0(\varphi) = \lambda_0 - \varphi \cdot \lambda_0$ (3)

An easy calculation shows

\begin{align*}
X \cdot Y & = \mathcal{L}_X Y - [X, Y] \\
R &\text{ action, whereas on Lie groups the Lie bracket } [X, Y] \text{ is an infinitesimal left action.}
\end{align*}

Moreover the moment maps give rise to Lie algebra isomorphisms $\mathbb{R} \oplus_{c_0} \mathfrak{g} \to \mathfrak{g}$ and $\mathbb{R} \oplus c'_0 \mathfrak{g} \to \mathfrak{g}'$. Using these identifications and $\lambda_0$ from above we can define a mapping

$$\Phi_0 : \mathfrak{g} \simeq \mathbb{R} \oplus c_0 \mathfrak{g} \to \mathbb{R} \oplus c'_0 \mathfrak{g} \simeq \mathfrak{g}'$$

by $\Phi_0(a, X) = (a + \lambda_0(X), X)$. This is an isomorphism of Lie algebras and $G$–equivariant for we have (3). Particularly the Lie algebra extensions $0 \to \mathbb{R} \to \mathfrak{g} \to g \to 0$ and $0 \to \mathbb{R} \to \mathfrak{g}' \to g' \to 0$ are isomorphic, as expected.

When defining $\Phi_0 : \mathfrak{g} \to \mathfrak{g}'$ we made two choices, namely $N_0$ and $N'_0$. We claim that $\Phi_0$ is independent of them. Indeed, suppose $N_1 \in \mathcal{M}$ and $N'_1 \in \mathcal{M}'$, choose $B_{01}$ and $B'_{01}$ such that $\partial B_{01} = N_1 - N_0$ and $\partial B'_{01} = N'_1 - N'_0$ and define $\rho_0 \in \mathfrak{g}^*$ by $\rho_0(X) = -\int_{B_{01}} i_X \nu$ and $\rho_0 \in \mathfrak{g}^*$ by $\rho'_0(X) = -\int_{B'_{01}} i_X \nu$, respectively. Again this does not depend on the choice of $B_{01}$ or $B'_{01}$. Moreover choose $B_1$, such that $\partial B_1 = N'_1 - N_1$ and define $\lambda_1(X) := -\int_{B_1} i_X \nu$. One easily checks that the composition

$$\mathbb{R} \oplus_{c_0} \mathfrak{g} \to \mathfrak{g} \to \mathbb{R} \oplus_{c_1} \mathfrak{g} \simeq \mathfrak{g}'$$

and similarly for $c'_0, c'_1, \mu'_0, \mu'_1$ and $\rho'_{01}$. Thus $\Phi_0 = \Phi_1$ is equivalent to $\rho_0 + \lambda_1 = \lambda_0 + \rho'_{01}$ which is equivalent to

$$-\int_{B_{01}} i_X \nu - \int_{B_1} i_X \nu = -\int_{B_0} i_X \nu - \int_{B'_{01}} i_X \nu,$$

but this follows since $i_X \nu$ is exact and the integral is over a cycle.

Summarizing we have seen that whenever the components $\mathcal{M}$ and $\mathcal{M}'$ consist of homologous submanifolds, there is a canonic $G$–equivariant isomorphism of Lie algebras $\Phi : \mathfrak{g} \to \mathfrak{g}'$. Particularly the coadjoint orbits of $\mathcal{G}$ and $\mathcal{G}'$ coincide. We are not aware of a more intrinsic definition of $\Phi$, and we don’t know if the corresponding group extensions are isomorphic in this situation.

Finally, suppose $\mathcal{M}$ is a component of $\text{Gr}_{m-2}(M)$ which consists of $0$–homologous submanifolds. Then

$$\mu : \mathcal{M} \to g^*, \quad \mu(N)(X) := \int_N \alpha, \quad \text{with } d\alpha = i_X \nu$$

is a $G$–equivariant moment map. So we get a canonic $G$–equivariant isomorphism of Lie algebras $\mathfrak{g} \simeq \mathbb{R} \oplus \mathfrak{g}$. Moreover $\mathcal{M}$ is a coadjoint orbit of $G$, canonically.

5. Generalized vortex filament equation

For a Riemannian metric $g$ on $M$ with induced volume form $\nu(g) = \nu$, we identify the normal bundle $TN^\perp$ with the Riemannian orthonormal bundle $TN^{\perp\prime}$.
and denote by $g$ the induced metric on it. We endow the symplectic manifold $(\text{Gr}_{m-2}(M), \Omega)$ with a Riemannian metric

$$\tilde{g}(Y_1, Y_2) = \int_N g(Y_1, Y_2) \nu(g|_{N}) \quad \text{for} \quad Y_1, Y_2 \in \Gamma(TN^{\perp}).$$

For $N \in \text{Gr}_{m-2}(M)$, the vector bundle $TN^{\perp}$ is oriented, 2–dimensional and has a metric, so we can define a fiber wise complex structure $J$ on $TN^{\perp}$ by rotation with +90 degrees. It induces an almost complex structure $\tilde{J}$ on $\text{Gr}_{m-2}(M)$ which is compatible with $\Omega$ and $\tilde{g}$, that is $\Omega(Y_1, Y_2) = \tilde{g}(\tilde{J}Y_1, Y_2)$.

The $g$–volume of the submanifold gives a smooth function on $\text{Gr}_{m-2}(M)$$

$$h : \text{Gr}_{m-2}(M) \to \mathbb{R}, \quad h(N) = \int_N \nu(g|_{N}). \quad (4)$$

**Lemma 4.** For the $\tilde{g}$–gradient of $h$ we have $(\text{grad } h)(N) = -\text{tr } II_N$, where $II_N \in \Gamma(S^2T^*N \otimes TN^{\perp})$ denotes the second fundamental form of the submanifold $N$.

**Proof.** For $Y \in \Gamma(TN^{\perp})$ we have

$$dh(Y) = \frac{1}{2} \int_N \text{tr}(L_Y g) \nu(g|_{N})$$

$$= \int_N \text{tr}(\nabla Y) \nu(g|_{N})$$

$$= -\int_N \text{tr} g(II_N, Y) \nu(g|_{N})$$

$$= -\tilde{g}(\text{tr } II_N, Y).$$

Since $\tilde{g}$ is weakly non-degenerated we conclude $\text{grad } h = -\text{tr } II$. \hfill \Box

Since $\tilde{J}$, $\Omega$ and $\tilde{g}$ are compatible, the Hamiltonian vector field of $h$ is $X_h = \tilde{J}(\text{grad } h) = J\text{tr } II$ and this proves the following

**Proposition 3.** The Hamiltonian equation for the Hamiltonian function $(4)$ is

$$\frac{\partial}{\partial t} N_t = J\text{tr } II(t).$$

In dimension $m = 3$ this equation is known as the vorticity filament equation, see [MW83].

**Remark 6.** Let $N$ be a closed oriented manifold of dimension $m-2$. The expression $J\text{tr } II$ can also be considered as a vector field on $\text{Emb}(N, M)$. Suppose $t_t$ is a curve of embeddings solving

$$\frac{\partial}{\partial t} t_t = J\text{tr } II(t_t)$$

and let $f$ be an orientation preserving diffeomorphism of $N$. Then $t_t \circ f$ will again be a solution of $(5)$. The geometric interpretation of this fact is the following. When restricting to suitable connected components, the space of embeddings becomes a principal $\text{Diff}(N)$–bundle over the non-linear Grassmannian. Here $\text{Diff}(N)$ denotes the group of orientation preserving diffeomorphisms. Using the Riemannian metric we can write down a connection of this bundle, known as a mechanical connection. For an embedding $\iota : N \to M$, the vertical tangent space is the space of vector fields along $\iota$ tangent to $\iota(N)$. So the space of vector fields along $\iota$ having values in the Riemannian orthogonal complement of $\iota(N)$ is a complement to the vertical
tangent space. This complements define a connection, which is obviously a principal connection. Regarding the expression $J\text{tr} II$ as a vector field on the space of embeddings, just means considering the horizontal lift of $J\text{tr} II$. Since the connection is principal, parallel transport will be $\text{Diff}(N)$–equivariant. This translates to $\iota_t \circ f$ is a solution of \([5]\) iff $\iota_t$ was.

**Remark 7.** Let $\iota_t$ be a curve of embeddings in $M$. Then we get a curve of Riemannian metrics $\iota_t^* g$ on $N$. This gives rise to a curve of volume forms $\nu(\iota_t^* g)$ on $N$. If $\iota_t$ is a solution of \([5]\) this curve will be constant. Indeed, for every horizontal $\iota_t$ one shows $\frac{\partial}{\partial t} \nu(\iota_t^* g) = -g(\text{tr} II, \frac{\partial}{\partial t} \iota_t)$ as in the proof of Lemma 4. If $\iota_t$ solves \([5]\) this implies $\frac{\partial}{\partial t} \nu(\iota_t^* g) = -g(\text{tr} II, \frac{\partial}{\partial t} \iota_t) = g(J \frac{\partial}{\partial t} \iota_t, \frac{\partial}{\partial t} \iota_t) = 0$. In the case of oriented knots in a 3–dimensional $M$, this implies that a solution of \([5]\), parameterized by arc length at time $t_0$, will have the same property for every time $t$.

6. **Symplectic Grassmannians**

Suppose $(M, \omega)$ is a closed connected symplectic manifold. Let $\text{SGr}_{2k}(M) \subseteq \text{Gr}_{2k}(M)$ denote the open subset of oriented submanifolds which are symplectic. We don’t assume the elements in $\text{SGr}_{2k}(M)$ to be oriented by their symplectic form. Note that $\text{SGr}_{2k}(M)$ is invariant under the action of the group of symplectic diffeomorphisms $\text{Diff}(M, \omega)$. Set $\alpha := \omega^{k+1}$. Then $\Omega := \alpha$ is a closed 2–form on $\text{Gr}_{2k}(M)$. Note that $\text{SGr}_{2k}(M) \subseteq \text{Gr}_{2k}(M)$ is an open subset on which $\Omega$ is (weakly) non-degenerate, hence a symplectic manifold. Indeed, for an almost complex structure $J$ on $M$ tamed by $\omega$ and $Y \in \Gamma(TN^{2k})$ we have $\Omega_N(Y, J^k Y) = (k + 1) \int_N \omega(Y, J^k Y) \omega^k$, vanishing iff $Y = 0$.

Let $\text{Ham}(M, \omega)$ denote the Lie group of Hamiltonian diffeomorphisms with Lie algebra $\text{ham}(M, \omega)$ of Hamiltonian vector fields on $M$. The action of $\text{Ham}(M, \omega)$ on $\text{SGr}_{2k}(M)$ is Hamiltonian. Indeed we already know that the action is symplectic and since $[\text{ham}(M, \omega), \text{ham}(M, \omega)] = \text{ham}(M, \omega)$, see \cite{C70}, the action must be Hamiltonian. In our special situation one does not have to use this general argument, for one can write down Hamilton functions.

**Lemma 5.** The mapping

$$\mu : \text{SGr}_{2k}(M) \to \text{ham}(M, \omega)^*, \quad \mu(N)(X) := (k + 1) \int_N f \omega^k,$$

is an injective equivariant moment map for the $\text{Ham}(M, \omega)$–action on $\text{SGr}_{2k}(M)$. Here $f$ is the unique Hamilton function of $X$ with zero integral. Particularly the action is Hamiltonian.

**Proof.** First we show that $h(N) := (k + 1) \int_N f \omega^k$ is a Hamiltonian function for the fundamental vector field $\zeta_X$ of $X \in \text{ham}(M, \omega)$. Note that $h = (k + 1) f \omega^k$ and $(k + 1) df \wedge \omega^k = i_X \omega^{k+1}$. Thus

$$dh = (k + 1) df \omega^k = (k + 1) d(f \omega^k) = i_X \omega^{k+1} = i_{\zeta_X} \omega^{k+1} = i_{\zeta_X} \Omega.$$  

So $\mu$ is a moment map. The injectivity is obvious. Finally for every $\varphi \in \text{Ham}(M, \omega)$ we have

$$\mu(\varphi(N))(X) = (k + 1) \int_{\varphi(N)} f \omega^k = (k + 1) \int_{N} (\varphi^* f) \omega^k = \mu(N)(\varphi^* X).$$

and thus $\mu$ is equivariant. \qed
Proposition 4. The group $\text{Ham}(M, \omega)$ acts transitively on every connected component of $\text{SGr}_{2k}(M)$.

Proof. We first show that the action is infinitesimal transitive. So suppose $N \in \text{SGr}_{2k}(M)$ and let $X$ be a tangent vector at $N$, i.e. a section of the normal bundle $TN^\perp$. Since $N$ is a symplectic submanifold we can identify the normal bundle with the $\omega$-orthogonal complement $TN^\perp_\omega$ of $TN$. So we may assume that $X$ is a section of $TN^\perp_\omega$. Consider $i_X \omega$ as a function, say $\lambda$, on the total space $E$ of $TN^\perp_\omega$ which happens to be linear along the fibers. One easily shows that $d\lambda = i_X \omega$ along $N \subseteq E$. Considering $E$ as a tubular neighborhood of $N$ one easily gets a function $\lambda'$ on $M$ such that $d\lambda' = i_X \omega$ along $N \subseteq M$. So the Hamiltonian vector field to $\lambda'$ will be an extension of $X$. Thus $\text{Ham}(M, \omega)$ acts infinitesimally transitive on $\text{SGr}_{2k}(M)$.

Suppose $N_t$ is a curve in $\text{SGr}_{2k}(M)$ and set $X_t := \frac{\partial}{\partial t}N_t$, a section of $TN^\perp_t$. For every fixed time $t$ the section $X_t$ can be extended to a vector field in $\text{ham}(M, \omega)$ as shown above. Moreover it is clear that this extension can be chosen smoothly with respect to the parameter $t$. Now the flow of this extension clearly gives a curve in $\text{Ham}(M, \omega)$ transporting, say, $N_0$ to $N_1$. □

Proposition 4, Lemma 5, Proposition 3 and Theorem 3 prove the following

Theorem 3. Let $(M, \omega)$ be a symplectic manifold, such that $[\omega]^{k+1} \in H^{2k+2}(M; \mathbb{R})$ is integral and let $M$ denote a connected component of $\text{SGr}_{2k}(M)$ endowed with the symplectic form $\Omega = \omega^{k+1}$. Then $M$ is a coadjoint orbit of $\text{Ham}(M, \omega)$. Particularly this coadjoint orbit is prequantizable.

Remark 8. Since we have an equivariant moment map the Lie algebra extension $0 \to \mathfrak{e} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$ from section 3 with $\mathfrak{g} = \text{ham}(M, \omega)$ is trivial. This is the reason why $\text{SGr}_{2k}(M)$ can be considered as coadjoint orbit of $\text{Ham}(M, \omega)$ rather than as coadjoint orbit of a central extension. However, the group extension

$$1 \to S^1 \to \tilde{G} \to G \to 1$$

(6)

from section 3 with $G = \text{Ham}(M, \omega)$ may very well be non-trivial as the following example shows.

Example 2. Let $M = S^2$, $\omega$ the standard symplectic form of mass 1 and let $k = 0$. Then $M = S^2$ is a connected component of $\text{SGr}_{2k}(M)$ and $\Omega = \omega$, cf. Example 3.

The bundle $\mathcal{P} \to \mathcal{M}$ is the Hopf fibration $S^3 \to S^2$ and $\eta$ is the standard contact structure on $S^3$. Since $M = \mathcal{M}$, the group extension (6) is trivial iff Kostant’s extension

$$1 \to S^1 \to \text{Aut}(S^3, \eta) \to \text{Ham}(S^2, \omega) \to 1$$

is trivial. The equivariant moment map from Lemma 5 provides a Lie algebra homomorphism $\sigma : \text{ham}(S^2, \omega) \to \text{aut}(S^3, \eta)$, right inverse to the projection. So Kostant’s group extension is trivial iff this Lie algebra homomorphism integrates to a group homomorphism. However this is not the case. Indeed, the loop in $\text{Ham}(S^2, \omega)$ given by rotation around an axis does not integrate to a closed curve in $\text{Aut}(S^3, \eta)$. To see this, note first that the Hamilton function generating the rotation vanishes along the equator, for it has zero integral. So $\sigma$ maps the Hamilton vector field to an element of $\text{aut}(S^3, \eta)$ which is horizontal over the equator of $S^3$. So integrating our loop of rotation gives a curve in $\text{Aut}(S^3, \eta)$ whose flow lines over
the equator of $S^2$ are horizontal. Such a flow line has holonomy $1/2$, for this is the total curvature of a hemisphere. Thus it is not closed.

Alternatively one can use the fact that the Hamilton function generating the rotation has values $\pm 1/2$ at the poles.

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