CACCIOPPOLI TYPE INEQUALITY FOR NON-NEWTONIAN STOKES SYSTEM AND A LOCAL ENERGY INEQUALITY OF NON-NEWTONIAN NAVIER-STOKES EQUATIONS WITHOUT PRESSURE

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(Communicated by Luis Silvestre)

ABSTRACT. We prove a Caccioppoli type inequality for the solution of a parabolic system related to the nonlinear Stokes problem. Using the method of Caccioppoli type inequality, we also establish the existence of weak solutions satisfying a local energy inequality without pressure for the non-Newtonian Navier-Stokes equations.

1. Introduction. Let $\Omega$ be a domain in $\mathbb{R}^3$ and $T > 0$. We consider a non-stationary and nonlinear variant of the Stokes system in $Q_T = \Omega \times (0, T)$:

$$
\partial_t u - \text{div} \, S(D(u)) + \nabla \pi = f, \quad \text{div} \, u = 0 \quad \text{in} \ Q_T,
$$

where $u$ is the velocity field of the fluid, $\pi$ is the pressure of the fluid, $f$ is the external force, and $S(D(u)) = (S_{ij}(D(u)))_{i,j=1,\ldots,n}$ is the extra stress tensor given as

$$
S_{ij}(D(u)) = \nu(|D(u)|)D_{ij}(u),
$$

$$
D_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad |D(u)| = \left( \sum_{i,j=1}^{3} |D_{ij}(u)|^2 \right)^{\frac{1}{2}}.
$$

Here viscosity $\nu(|D(u)|)$ is a function of $|D(u)|$, given as $\nu(s) = \mu_0 + (\mu_1 + s)^{q-2}$. In this article, we consider the degenerate case, $\mu_0 = \mu_1 = 0$. namely

$$
S_{ij}(D(u)) = |D(u)|^{q-2} D_{ij}(u).
$$

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Let \( z_0 = (x_0, t_0) \in Q_T \) and \( r < \frac{1}{2} \min \{ \sqrt{t_0}, \text{dist}(x_0, \partial \Omega) \} \). For notational convenience, we denote
\[
B_{x_0,r} = \{ x \in \mathbb{R}^n : |x - x_0| < r \}, \quad S_{x_0,r} = \{ x \in \mathbb{R}^n : |x - x_0| = r \},
\]
\[
Q_{z_0,r} = B_{x_0,r} \times (t_0 - r^2, t_0).
\]
For simplicity, we assume that \( f = 0 \) and \( z_0 = (0, 0) \) via translation and we denote \( B_r = B_{x_0,r} \), \( S_r = S_{x_0,r} \) and \( Q_r = Q_{z_0,r} \), unless there is any confusion to be expected. If we consider a similar situation for the heat equation, \( \partial_t v - \Delta v = 0 \), it is well-known that the following Caccippoli inequality holds:
\[
\int_{Q_r} |\nabla v|^2 \, dz \leq \frac{C}{r^2} \int_{Q_{2r}} |v|^2 \, dz.
\]
However, due to non-local effect of the pressure, Caccippoli inequality of the system (1) doesn’t seem to be obvious. Nevertheless, introducing the notion of local pressure and using a suitable decomposition of the pressure in \([27]\), the Caccippoli type inequality of the Navier-Stokes equations was obtained in the interior (see also \([28]\)). The first author in \([17]\) proved independently the Caccippoli inequality for the Stokes system, i.e. for the case \( \nu(s) = \mu_0 \), by testing a suitable divergence free vector field. We remark that the methods of proofs in \([17]\) and \([27]\) are different, although the same Caccippoli inequality can be derived.

Our main motivation is to establish a Caccippoli type inequality of the nonlinear Stokes system (1) as an extension of the result in \([17]\) and also show existence of weak solutions of the non-Newtonian Navier-Stokes equations satisfying local energy inequality without pressure.

First we introduce a notion of strong solution of the system.

**Definition 1.1 (Strong solution).** Let \( 1 < q < \infty \). We say that \( u \) is a strong solution of the system (1) with extra stress tensor (4), with initial data \( u_0 \in L^2_0 \cap W^{1,q} \) and with no slip boundary condition if \( u \) solves the equation (1) in the sense of distributions, and
1. It holds that
\[
u \in L^\infty([0,T];L^q(\Omega)), \quad \nabla u \in L^q([0,T];L^q(\Omega)), \quad \pi \in L^{\frac{2q}{q+2}}([0,T];L^{\frac{2q}{q+2}}(\Omega)), \quad (6)
\]
\[
\nabla u \in L^\infty([0,T];L^q(\Omega)), \quad \partial_t u \in L^\infty([0,T];L^q_0(\Omega)). \quad (7)
\]
2. In case that \( q > 2 \), it holds that
\[
\nabla S(D(u)) \in L^{\frac{2q}{q+2}}([0,T];L^{\frac{2q}{q+2}}(\Omega')) \quad \nabla \pi \in L^{\frac{2q}{q+2}}([0,T];L^{\frac{2q}{q+2}}(\Omega')) \quad (8)
\]
In case that \( 1 < q < 2 \), we have
\[
\nabla^2 u \in L^q([0,T];L^q(\Omega')). \quad (9)
\]
for any bounded subdomain \( \Omega' \Subset \Omega \).

**Remark 1.** Let \( u_0 \in L^2_0(\Omega) \cap W^{1,q}(\Omega) \). In the case that \( \nu(s) = \mu_0 + (\mu_1 + s)^{q-2} \), \( \mu_0 > 0 \) or \( \mu_1 > 0 \), and \( q > 2 \), there are known results regarding existence of the unique solution for the system (1) with no slip boundary data (see e.g. \([3, 4, 13, 14, 22, 24]\) and references therein).

For example, it was shown in \([13, 14]\) that there is unique solution \( u \) of the system (1) with \( \nu(s) = (1 + s^2)^{\frac{2q}{n}} \) such that for any bounded subdomain \( \Omega' \Subset \Omega \subset \mathbb{R}^n \) and for any \( \delta > 0 \)
\[
(1 + |D(u)|^2)^{\frac{2q}{n}} Du \in L^2([\delta,T) \times \Omega'), \quad \pi \in L^2([\delta,T) \times \Omega') \quad 2 < q < \frac{2n}{n-2},
\]
\[ \partial_t u \in L^2([0, T) \times \Omega), \quad u \in L^\infty([0, T); W^{1,q}(\Omega)) \quad 2 < q < \infty. \]

In section 2, we present the proof of the existence of unique strong solution in Definition 1.1 including the case of degenerate power-law type as well as \( q > 1 \) (see Proposition 1). Such results may be known to experts but we cannot find it in the literature.

We are now ready to state our main results. The first result is Caccioppoli type inequality and the main point is that pressure does not appear on the righthand side.

**Theorem 1.2.** (Caccioppoli type inequality) Let \( 1 < q < \infty \), \( z_0 = (x_0, t_0) \in Q_T := \Omega \times (0, T) \) and \( \overline{Q_{2r, \rho}} \subset Q_T \). Suppose that \( u \) is a strong solution of non-Newtonian Stokes system in Definition 1.1. Then, for any \( r > 0 \) with \( 2r < \rho \),

\[
\int_{Q_{2r, r}} |\nabla u|^q \, dz \leq \frac{C}{r^q} \int_{Q_{2r, r}} |u|^q \, dz + \frac{C}{r^2} \int_{Q_{2r, 2r}} |u|^2 \, dz. \tag{10}
\]

The second result is the estimates of the higher regularity regarding spatial derivative of the second order. We treat separately the shear thinning case, i.e. \( 1 < q < 2 \), and shear thickening case, i.e. \( q > 2 \), since structures of two cases are dissimilar, which cause quite different methods of proofs. Next theorem deals with shear thinning case \((1 < q < 2)\).

**Theorem 1.3.** Let \( 1 < q < 2 \), \( z_0 = (x_0, t_0) \in Q_T := \Omega \times (0, T) \) and \( \overline{Q_{2r, \rho}} \subset Q_T \). Suppose that \( u \) is a strong solution of non-Newtonian Stokes system in Definition 1.1. Then, the following estimates hold for any \( r > 0 \) with \( 2r < \rho \):

\[
\sup_{t_0 - r^2 < t < t_0} \int_{B_{2r}, r} |\nabla \times u(t)|^2 \, dx + \frac{1}{r^{2-q}} \int_{Q_{2r, r}} |\nabla D(u)|^q \, dx
\leq \frac{C}{r^2} \int_{Q_{2r, 2r}} |\nabla u|^q \, dz + \frac{C}{r^2} \int_{Q_{2r, 2r}} |\nabla u|^2 \, dz. \tag{11}
\]

For shear-thickening case \((2 < q < \infty)\), we obtain the following:

**Theorem 1.4.** Let \( 2 < q < \infty \), \( z_0 = (x_0, t_0) \in Q_T := \Omega \times (0, T) \) and \( \overline{Q_{2r, \rho}} \subset Q_T \). Suppose that \( u \) is a strong solution of non-Newtonian Stokes system in Definition 1.1. Then, the following estimates hold for any \( r > 0 \) with \( 2r < \rho \):

\[
\sup_{t_0 - r^2 < t < t_0} \int_{B_{2r}, r} |\nabla \times u(t)|^2 \, dx + \frac{1}{r^{2-q}} \int_{Q_{2r, r}} |\nabla S(D)|^{q/2} \, dz
\leq \frac{C}{r^2} \int_{Q_{2r, 2r}} |\nabla u|^q \, dz + \frac{C}{r^2} \int_{Q_{2r, 2r}} |\nabla u|^2 \, dz. \tag{12}
\]

As an application of Caccioppoli type inequality in Theorem 1.2, we can construct a weak solution of non-Newtonian Navier-Stokes equations satisfying a local energy inequality without pressure. To be more precise, we consider the non-Newtonian Navier-Stokes equations

\[ \partial_t u + u \cdot \nabla u - \text{div} S(D(u)) + \nabla \pi = 0, \quad \text{div} u = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{13} \]

with no slip boundary condition, i.e. \( u = 0 \) on \( \partial \Omega \times [0, T] \) and divergence free initial data \( u(\cdot, 0) = u_0 \) in \( L^2(\Omega) \). Next, we recall the notion of weak solution of the non-Newtonian Navier-Stokes equations 13.
Definition 1.5. Let $1 < q < \infty$. We say that $u$ is a weak solution of the system \eqref{13} with extra stress tensor \eqref{4} if
\[ u \in L^\infty(0, T : L^2(\Omega)) \cap L^q(0, T; W^{1,q}_0(\Omega)) \]
and
\[ \int_{Q_T} S(D(u)) : D(\Psi)dz = \int_{Q_T} u \cdot [\partial_t \Psi + (u \cdot \nabla \Psi)]dz + \int_\Omega u_0 \cdot \Psi(0)dx \tag{14} \]
for all $\Psi \in C_0^\infty((0, T) \times \Omega)$ with $\text{div } \Psi = 0$.

Remark 2. The existence of weak solutions for $q \geq \frac{3n+2}{n+2}$ had first appeared in \cite{18, 19, 20}, which is unique for $q > \frac{2n+2}{n+2}$ (see also \cite{21}). The existence of measure-valued solutions was shown for $q > \frac{2n}{n+2}$ in \cite{22, 24}. Later, the existence of weak solution had been investigated for $q > \frac{2n}{n+2}$ in \cite{6, 12, 22, 23, 24}. Those results hold for both the space-periodic problem and the Dirichlet problem.

For the spatially periodic domains, the strong solution had been obtained globally in time for $q \geq \frac{11}{5}$ if $n = 3$, and for $q > 1$ if $n = 2$ in \cite{6, 22}: the local-in-time strong solution for $q > \frac{3}{2}$ in \cite{22}, for $2 > q > \frac{3}{2}$ in \cite{11}, and for $2 > q > 1$ in \cite{7} when $n = 3$.

When the case $\mu_1 > 0$ had been considered, regularity for the Dirichlet problem has also been well studied: the global-in-time strong solution was obtained for $q \geq \frac{5}{2}$ if $n = 3$ and $q \geq 2$ if $n = 2$ in \cite{24}, and the regularity result was extended to the case $q \geq \frac{11}{5}$ in \cite{12, 13} when $n = 3$: the local-in-time strong solution was obtained for $q > 1$ in \cite{11, 8}. The uniqueness of weak solution is known for $q > \frac{5}{2}$ in \cite{18} and for $q > \frac{11}{5}$ in \cite{9}. Regularity of weak solution for the degenerate case is considered in \cite{2, 25}.

As mentioned earlier, we can show the existence of weak solutions satisfying a certain type of local energy inequality, where pressure term does not appear. To be more precise, our last main result reads as follows:

Theorem 1.6. Let $q \geq 9/5$ and $N(x)$ be the fundamental solution of $-\Delta$ in $\mathbb{R}^3$. Then, there exists a weak solution $u$ of the non-Newtonian Navier-Stokes equations \eqref{13} satisfying the following local energy inequality: For almost every $\tau \in (0, T)$
\[
\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \psi(x, \tau)dx + \mu_0 \int_0^\tau \int_{\Omega} \phi^2 |\nabla u|^2 \psi dx dz + \int_0^\tau \int_{\Omega} \phi^2 |D(u)|^q \psi dx d\tau \\
\leq -\int_0^\tau \int_{\Omega} S(D(u)) \cdot (\nabla \phi^2 \otimes u + D(\nabla \phi \cdot v + \phi w)) \psi dx dz + \int_0^\tau \int_{\Omega} (u \cdot \nabla \phi^2) \frac{|u|^2}{2} \psi dx d\tau \\
+ \int_0^\tau \int_{\Omega} (u \otimes u) : \nabla (\nabla \phi \cdot v + \phi w) \psi dx dz + \int_0^\tau \int_{\Omega} \frac{1}{2} |\nabla u|^2 \psi' dx d\tau, \tag{15}
\]
where $0 \leq \phi(x) \in C_0^\infty(\Omega), 0 \leq \psi(t) \in C_0^\infty([0, T),)\),
\[
v := N \ast (\phi \nabla \cdot u) = \int_{\Omega} N(x - y) \frac{\phi(y) \nabla y \times u(y, t)}{dy}
\tag{16}
\]
and
\[
w := \nabla [N \ast (\phi \nabla \cdot u)] - \nabla \times [N \ast (\phi \nabla \times u)].
\]

Remark 3. In Theorem 1.6 if $\mu_0 > 0$, then the assumption $q \geq 9/5$ is not required. The condition $u \in L^\infty(0, T : L^2(\Omega)) \cap L^q(0, T; W^{1,q}_0(\Omega))$ in Definition 1.5 implies $u \in L^\frac{4}{q}(\Omega \times (0, T))$. From the right hand side of \eqref{15}, $u \in L^3_{loc}(Q_T)$ is necessary, and this holds for $q \geq \frac{9}{5}$. 
Next corollary is direct consequence of the local energy inequality

**Corollary 1.** The weak solution $u$ constructed in Theorem 1.6 satisfies

$$
sup_{-r^2 < t < 0} \int_{\mathbb{R}^3} |\nabla v(\cdot, t)|^2 dx + \int_{Q_r} |\nabla u|^q dz
\leq \frac{c}{r} \left( \int_{Q_{2r}} |u|^q dz \right)^\frac{1}{q} \left( \int_{Q_{2r}} |u|^p dz \right)^\frac{1}{p} + \frac{c}{r} \int_{Q_{2r}} |u|^2 dz + \frac{c}{r} \int_{Q_{2r}} |u|^3 dz, \tag{17}
$$

where $v$ is given in (16).

**Remark 4.** We recall that the non-Newtonian Navier-Stokes equations (13) satisfy the following scaling invariance:

$$u_\lambda(x, t) = \lambda^{\frac{4}{3-q}} u(\lambda x, \lambda^{\frac{2}{3-q}} t), \quad p_\lambda(x, t) = \lambda^{\frac{2(4-q)}{3-q}} p(\lambda x, \lambda^{\frac{2}{3-q}} t).$$

We also note that $v$ in (16) hold $v_\lambda(x, t) = \lambda^{\frac{2(4-q)}{3-q}} v(\lambda x, \lambda^{\frac{2}{3-q}} t)$. Therefore, the local energy inequality (17) holds for $u_\lambda$, $p_\lambda$ and $v_\lambda$ with replacement of $Q_r$ by $Q_r = B_r \times (-r^{\frac{2}{3-q}}, 0)$. From the above scaling invariance, since $r^{\frac{2}{3-q}} \int_{Q_r} |\nabla u|^q dz$ is scaling invariant, we conjecture that in case that $2 < q < \frac{11}{7}$, the size of a possible singular set is of $\frac{11-5q}{3-q}$ dimensional Hausdorff measure zero, and we leave it as an open question.

**Remark 5.** For the magnetohydrodynamic equations, the local energy estimate similar to (15) was recently constructed in [10] by using the Caccioppoli type inequality of the Stokes system proved in [17]. One of our motivation is to extend such estimate to the non-Newtonian fluid flow. The inequality (15) contains the non-local effect of pressure, which appears as a different form $v$ defined in (16). The advantage of the inequality (15) is that the non-local effect does not present itself on the righthand side but it is included on the left hand side, which yields the Caccioppoli type inequality (17). The price to pay is, however, lack of control of the local $L^\infty_q L^2_r$-norm in the inequality, although it can be estimated in terms of the right side together with itself in a bigger parabolic cylinder (see Corollary 2). On the other hand, as mentioned earlier, another type of local energy inequality was established in [27] and [28], which contains a part of pressure on the right hand of the inequality (see [28] Definition 3.1]), but such pressure can be controlled in terms of velocity fields similarly as in right hand side of (15) and it leads to the Caccioppoli type inequality (see [28] Lemma 4.1).

This paper is organized as follows. In section 2 we recall some useful lemmas and existence of strong solution is shown. The section 3 and section 4 are devoted to providing the proofs of Theorem 1.2 Theorem 1.6

2. Preliminaries. In this section we introduce the notations and present preparatory results that are useful to our analysis. We start with the notations. Let $\Omega$ be a domain with smooth boundary in $\mathbb{R}^3$. For $1 \leq q \leq \infty$, we denote by $W^{k,q}(\Omega)$ the usual Sobolev spaces, namely $W^{k,q}(\Omega) = \{ f \in L^q(\Omega) : D^\alpha f \in L^q(\Omega), 0 \leq |\alpha| \leq k \}$. The set of $q$-th power Lebesgue integrable functions on $\Omega$ is denoted by $L^q(\Omega)$ and $L^q_{\text{loc}}(\Omega)$ indicates the set of locally $q$-th power Lebesgue integrable functions defined on $\Omega$. The letter $c$ is used to represent a generic constant, which may change from line to line, and $c(\ast, \cdots, \ast)$ is considered a positive constant depending on $\ast, \cdots, \ast$.

Next we present a local version of Sobolev inequality and Korn’s inequality.
Lemma 2.1 (Local Sobolev’s inequality). Let $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^3)$ with $1 < p < 3$, $q = \frac{3p}{3-p}$ and $x_0 \in \mathbb{R}^3$. Then, for any $r,s$ with $0 < r < s < \infty$

$$\|u\|_{L^q(B_{r},r)} \leq c \left( \left\| \nabla u \right\|_{L^{p}(B_{r},r)} + \frac{1}{s-r} \left\| u \right\|_{L^{p}(B_{s},s)} \right).$$

(18)

Lemma 2.2 (Local Korn’s inequality). If $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^3)$ with $1 < p < \infty$ and $x_0 \in \mathbb{R}^3$, then there exists $r_0 > 0$ such that for any $r,s$ with $0 < r < s < r_0$,

$$\|\nabla u\|_{L^{p}(B_{r},r)} \leq c \left( \left\| D(u) \right\|_{L^{p}(B_{r},r)} + \frac{1}{s-r} \left\| u \right\|_{L^{p}(B_{s},s)} \right).$$

(19)

We omit the details of the above lemmas since proofs are rather straightforward via the Sobolev embedding theorem or Korn’s inequality for $v = u\phi$, where $\phi \in C_{0}^{\infty}(B_{r})$ is a standard cut-off function satisfying $\phi = 1$ in $B_{r}$ and $|\nabla \phi| \leq \frac{c}{r}$.

Next, we recall the following lemma proved in [15], which is parabolic analog to a result of elliptic cases (see e.g. [16]).

Lemma 2.3. Let $f,f_{1}, \ldots, f_{l}$ be nonnegative functions in $L^{1}_{\text{loc}}(Q_{p})$ and $\alpha_{1}, \ldots, \alpha_{l}$ given nonnegative numbers. There exists $\epsilon_{0}$, depending on $\alpha_{1}, \ldots, \alpha_{l}$, such that for any $\epsilon$ with $\epsilon \leq \epsilon_{0}$ and for any $2r < \rho$ if

$$\int_{Q_{r}} f dz \leq \epsilon \int_{Q_{2r}} f dz + \epsilon \sum_{j=1}^{l} \left( r^{-\alpha_{j}} \int_{Q_{2r}} f_{j} dz \right)$$

for some constant $c$, then there exists a constant $c > 0$, independent of $\epsilon$ such that following inequality holds:

$$\int_{Q_{r}} f dz \leq c \sum_{j=1}^{l} \left( r^{-\alpha_{j}} \int_{Q_{2r}} f_{j} dz \right).$$

Next, we prove an auxiliary lemma, which we will use often in proofs of Theorems. Let $2r < \rho$ and $\phi_{r} = \phi_{r}(x)$ indicate a standard cut-off function satisfying

$$\phi_{r} = 1 \text{ in } B_{r}, \quad \phi_{r} = 0 \text{ in } B_{\rho}\setminus B_{r}, \quad |\nabla \phi_{r}| \leq \frac{c}{\rho}, \quad |\nabla^{2} \phi_{r}| \leq \frac{c}{\rho^{2}}.$$  

(20)

We denote by $N(x) = \frac{1}{3|x|^{3}}$ the fundamental solution of $-\Delta$ in $\mathbb{R}^{3}$ and for a given $u \in W^{1,q}_{\text{loc}}(\mathbb{R}^{3})$ with $1 < q < \infty$ we define $v$ as follows:

$$v(x) := \int_{\mathbb{R}^{3}} N(x-y) \left[ \phi_{r}(y) \nabla u(y) \right] dy.$$  

(21)

Next lemma shows some estimates of Sobolev norm for $v$.

Lemma 2.4. Suppose $u \in W^{1,q}_{\text{loc}}(\mathbb{R}^{3})$ with $1 < q < \infty$ and $v$ is given as in (21). Then $v \in W^{2,q}_{\text{loc}}(\mathbb{R}^{3})$ and $v$ satisfies following estimates: For any $\rho > 0$

$$\left\| v \right\|_{L^{q}(B_{\rho})} \leq c \left\| u \right\|_{L^{q}(B_{\rho})},$$

(22)

$$\left\| \nabla v \right\|_{L^{q}(B_{\rho})} \leq c \left\| u \right\|_{L^{q}(B_{\rho})},$$

(23)

$$\left\| \nabla^{2} v \right\|_{L^{q}(\mathbb{R}^{3})} \leq c \left\| \nabla u \right\|_{L^{q}(B_{\rho})}.$$  

(24)

Proof. We note that $v = \nabla \times (N \ast (\phi_{r}u)) + N \ast ((\nabla \phi_{r}) \times u) := v_{1} + v_{2}$. We observe that if $|x| \leq \rho$ and $|y| \leq \rho$, then $|x-y| \leq 2\rho$, and so $|v_{2}|_{|x| \leq \rho} \leq \|N\chi_{B_{2\rho}}\ast((\nabla \phi) \times u\chi_{B_{\rho}})\|_{L^{q}(B_{\rho})}$. Applying Young’s theorem to $v_{2}$, we have

$$\left\| v_{2} \right\|_{L^{q}(B_{\rho})} \leq c \|N\|_{L^{1}(B_{2\rho})}\|(\nabla \phi) \times u\|_{L^{q}(B_{\rho})} \leq c \|u\|_{L^{q}(B_{\rho})},$$

(24)
where we used that \( \|N\|_{L^1(\Omega)} = \int_{|z| \leq \rho} |z|^{-1}dz = \rho^2 \) and \([20]\). Likewise, \[ \|v_1\|_{L^3(\Omega)} \leq c\|
abla N\|_{L^3(\Omega)}\|\phi u\|_{L^3(\Omega)} \leq c\|\phi\|_{L^3(\Omega)}. \]

Via similar computations, we can see that \[ \|\nabla v_2\|_{L^3(\Omega)} \leq c\|
abla N\|_{L^3(\Omega)}\|v\times u\|_{L^3(\Omega)} \leq c\|u\|_{L^3(\Omega)}. \]

On the other hand, by Calderon-Zygmund estimates, \[ \|\nabla v_1\|_{L^3(\Omega)} \leq c\|\phi u\|_{L^3(\Omega)} \leq c\|u\|_{L^3(\Omega)}. \]

Summing up the above estimate, we obtain \([22]\) and \([23]\). Due to standard estimates of singular integral, the estimate \([24]\) is immediate, and thus we omit the details. \(\square\)

**Remark 6.** We remark that \(B_p\) in the left hand side of \([22]\) and \([23]\) could be replaced by \(\mathbb{R}^3\), in case that \(q > 3\) and \(q > 3/2\), respectively. Indeed, if \(q > 3/2\), due to Calderon-Zygmund estimates and Sobolev embedding, we have
\[
\|\nabla v\|_{L^3(\Omega)} \leq \|\nabla v_1\|_{L^3(\Omega)} + \|\nabla v_2\|_{L^3(\Omega)} \leq C\|u\|_{L^3(\Omega)} + C\|\nabla v_2\|_{L^{\frac{3q}{q+3}}(\Omega)} \leq C\|u\|_{L^3(\Omega)} + C\|\phi\|_{L^3(\Omega)}\|u\|_{L^3(\Omega)} \leq C\|u\|_{L^3(\Omega)}.
\]

Using the above estimate, when \(q > 3\), it is immediate via Sobolev embedding that \[ \|v\|_{L^3(\Omega)} \leq C\|\nabla v\|_{L^{\frac{3q}{q+3}}(\Omega)} \leq C\|u\|_{L^3(\Omega)}, \]
where we used that \(3q/(3+q) > 3/2\).

Next, we prove the existence of strong solutions in Definition \(1.1\).

**Proposition 1.** Let \(\Omega \subset \mathbb{R}^3\) be an open domain with smooth boundary, unless \(\partial\Omega\) is empty. Then, there exists a unique strong solution in Definition \(1.1\).

**Proof.** Without loss of generality, we assume that the initial data \(u_0\) is regular enough. Before providing the proof, we begin with the solutions \(\{(u^\epsilon, \pi^\epsilon) | \epsilon > 0\}\), which solves the following approximate system:
\[
\begin{align*}
\partial_t u^\epsilon - \text{div}S^\epsilon(D(u^\epsilon)) + \nabla \pi^\epsilon &= 0, & \text{div} u^\epsilon &= 0 & \text{in } Q_T, \\
u^\epsilon |_{t=0} &= u_0, & u^\epsilon |_{\partial\Omega} &= 0,
\end{align*}
\]

where \(S^\epsilon(s) = \mu_\epsilon(s)s, \mu_\epsilon(s) = \epsilon + (\epsilon^2 + |s|^2)^\frac{3-2}{2}\).

As in \([13]\) and \([14]\) (see also \([3, 4]\)), it can be shown that unique solution \(u^\epsilon\) can be constructed in the class \(u^\epsilon \in L^\infty(0, T; W^{1, \max(2, q)}(\Omega)), \partial_t u^\epsilon \in L^2(0, T; L^2(\Omega)), u^\epsilon \in L^2(0, T; W^{2, 2}_{\text{loc}}(\Omega)), \nabla S^\epsilon \in L^{\min(2, \frac{3}{\pi+4})}(0, T; L^{\min(2, \frac{3}{\pi+4})}(\Omega)), \nabla \pi^\epsilon \in L^{\min(2, \frac{3}{\pi+4})}(0, T; L^{\min(2, \frac{3}{\pi+4})}(\Omega)), \)

Below, we derive some uniform estimates independent of \(\epsilon\). For simplicity, denote \(D^\epsilon = D(u^\epsilon)\).

By standard argument, that is, by multiplying \([25]\) with \(u^\epsilon\) and \(D_t u^\epsilon\), respectively, and by using the integration by parts, we then have
\[
\begin{align*}
\frac{d}{dt} \frac{1}{2} \int_\Omega |u^\epsilon|^2dx + \int_\Omega \left((\epsilon^2 + |D^\epsilon|^2)^\frac{3-2}{2} |D^\epsilon|^2 + \epsilon|\nabla u^\epsilon|^2\right) dx &\leq 0, \\
\frac{d}{dt} \int_\Omega \left(\frac{1}{q}(\epsilon^2 + |D^\epsilon|^2)^\frac{3-2}{2} + \frac{1}{2}\epsilon|\nabla u^\epsilon|^2\right) dx + \int_\Omega |\partial_t u^\epsilon|^2dx &\leq 0.
\end{align*}
\]
Integrating in time, we obtain
\[
\sup_{0 < t < T} \int_\Omega |u^\epsilon(t)|^2 \, dx + \int_0^T \int_\Omega \left( (\epsilon^2 + |D\epsilon|^2)^{\frac{q-2}{2}} |D^\epsilon|^2 + \epsilon |\nabla u^\epsilon|^2 \right) \, dx \, dt \leq c \int |u_0|^2 \, dx,
\]
\[
\sup_{0 < t < T} \int_\Omega \left( (\epsilon^2 + |D\epsilon|^2)^{\frac{q}{2}} + \epsilon |\nabla u^\epsilon(t)|^2 \right) \, dx + \int_0^T \int_\Omega |\partial_t u^\epsilon|^2 \, dx \, dt
\leq c \int \left( (\epsilon + |D(u_0)|^2)^{\frac{q}{2}} + \epsilon |\nabla u_0|^2 \right) \, dx.
\]
Due to Biot-Savart law and Korn’s inequality, the above estimates imply that
\[
\begin{aligned}
&u^\epsilon \in L^\infty(0, T; L^2(\Omega)), \quad D_t u^\epsilon \in L^2(0, T; L^2(\Omega)), \\
&\nabla u^\epsilon \in L^q((0, T) \times \Omega) \cap L^\infty(0, T; L^q(\Omega)) \quad \text{uniformly in } \epsilon.
\end{aligned}
\]
Via Bogovski’s formula, this again implies that \( \pi^\epsilon \in L^{\min\{2, \frac{q+1}{q-1}\}}((0, T) \times (\Omega')) \) uniformly in \( \epsilon \).

Now, let us derive uniform estimates of the higher derivatives in space variables. Let \( \Omega' \) be any open bounded domain with \( \Omega' \subseteq \Omega \). We denote by \( \phi \) a smooth cut-off function with
\[
\phi = 1 \quad \text{in } \Omega', \quad \phi = 0 \quad \text{in } \Omega \setminus \tilde{\Omega},
\]
where \( \tilde{\Omega} \) is a bounded domain with \( \Omega \subseteq \tilde{\Omega} \subseteq \Omega \) such that \( \text{dist}(\Omega', \tilde{\Omega}) > 0 \) and \( \text{dist}(\Omega, \Omega') > 0 \). Testing \( \nabla \times (\phi^2 \nabla \times u^\epsilon) \) to \([1]\) and integrating it by parts, we obtain
\[
\begin{aligned}
\frac{d}{dt} \int_\Omega \phi^2 |\nabla \times u^\epsilon|^2 \, dx &+ \int_\Omega \left( \phi^2 \nabla \left( (\epsilon^2 + |D\epsilon|^2)^{\frac{q-2}{2}} D^\epsilon \right) : \nabla D^\epsilon + \epsilon \phi^2 |\nabla D^\epsilon|^2 \right) \, dx \\
&= \int_\Omega (\nabla \times \left( (\epsilon^2 + |D\epsilon|^2)^{\frac{q-2}{2}} D^\epsilon \right)) \nabla \phi^2 \times \nabla \times u^\epsilon \, dx \\
&- \int_\Omega (D_x \left( (\epsilon^2 + |D\epsilon|^2)^{\frac{q-2}{2}} D^\epsilon \right)) (\nabla \phi^2 \cdot \nabla u^\epsilon) \, dx \\
&+ \int_\Omega (D_{x_k} \left( (\epsilon^2 + |D\epsilon|^2)^{\frac{q-2}{2}} D^\epsilon \right)) (D_{x_k} \phi^2) (D_{x_k} u^\epsilon) \, dx \\
&+ \epsilon \int_\Omega \Delta u^\epsilon \nabla \phi^2 \times \nabla \times u^\epsilon - (D_x, D^\epsilon_{ij}) (\nabla \phi^2 \cdot \nabla u^\epsilon) \, dx \\
&+ \int_\Omega (D_{x_k} D^\epsilon_{ij}) (D_{x_k} \phi^2) (D_{x_k} u^\epsilon) \, dx.
\end{aligned}
\]
(29)

• (The case \( q > 2 \)) We note that
\[
\left( \phi \left| \nabla \left( (\epsilon^2 + |D\epsilon|^2)^{\frac{q-2}{2}} D^\epsilon \right) \right| \right)^{\frac{q}{q-1}} \leq c \left( \phi (\epsilon^2 + |D\epsilon|^2)^{\frac{q-2}{2}} |\nabla D^\epsilon| \right)^{\frac{q}{q-1}}
\leq \left( \phi^2 (\epsilon^2 + |D\epsilon|^2)^{\frac{q-2}{2}} |\nabla D^\epsilon|^2 \right)^{\frac{q}{q-1}} \| D^\epsilon \|_{L^q} \| \nabla D^\epsilon \|_{L^2}
\leq c \phi^2 (\epsilon^2 + |D\epsilon|^2)^{\frac{q-2}{2}} |\nabla D^\epsilon|^2 + c |D\epsilon|^q.
\]
On the other hand, applying Hölder’s inequalities and Young’s inequalities, the right hand side in \([29]\) is dominated by
\[
c \| \phi \nabla \left( (\epsilon^2 + |D\epsilon|^2)^{\frac{q-2}{2}} D^\epsilon \right) \|_{L^\frac{q}{q-1}} \| \nabla u^\epsilon \|_{L^q} + c \epsilon \| \phi \nabla^2 u^\epsilon \|_{L^2} \| \nabla u^\epsilon \|_{L^2}
\leq \delta \| \phi \nabla \left( (\epsilon^2 + |D\epsilon|^2)^{\frac{q-2}{2}} D^\epsilon \right) \|_{L^\frac{q}{q-1}} + \delta \epsilon \| \phi \nabla^2 u^\epsilon \|_{L^2}^2 + \frac{c}{\delta q - 1} \| \nabla u^\epsilon \|_{L^q}^q + \frac{c \epsilon}{\delta} \| \nabla u^\epsilon \|_{L^2}^2.
\]
where \( \delta \) is a constant. Taking \( \delta \) small enough, \((29)\) reduces to the inequality
\[
\frac{d}{dt} \int \phi^2 |\nabla \times u^\epsilon|^2 dx + \int_\Omega \left( (\phi |\nabla (\epsilon^2 + |D^\epsilon|^2)^{\frac{2}{q-2}} D^\epsilon) \right)^{\frac{q}{q-1}} + c\phi^2 |\nabla^2 u^\epsilon|^2 dx
\leq c \int_\Omega |\nabla u^\epsilon|^q dx + ce \int_\Omega |\nabla u^\epsilon|^2 dx.
\]
(31)

This implies, after integration in time, that
\[
\sup_{0 \leq t \leq T} \int_\Omega \phi^2 |\nabla \times u^\epsilon|^2 dx + c \int_0^T \int_\Omega \left( (\phi |\nabla (\epsilon^2 + |D^\epsilon|^2)^{\frac{2}{q-2}} D^\epsilon) \right)^{\frac{q}{q-1}} + c\phi^2 |\nabla^2 u^\epsilon|^2 dx
\leq c \int_0^T \int_\Omega |\nabla u^\epsilon|^q dx + ce \int_0^T \int_\Omega |\nabla u^\epsilon|^2 dx + c \int_\Omega \phi^2 |\nabla \times u_0|^2.
\]

From Biot-Savart law and Korn's inequality, the above estimate implies that uniformly in \( \epsilon \)
\[
\nabla u^\epsilon \in L^\infty(0, T; L^2(\Omega')), \quad \nabla(|D^\epsilon|^{q-2} D^\epsilon) \in L^{\frac{4}{q-2}}((0, T) \times \Omega').
\]

In addition, since \( D_t u^\epsilon \in L^2(0, T; L^2(\Omega)) \) and \( \nabla (\epsilon^2 + |D^\epsilon|^2)^{\frac{2}{q-2}} D^\epsilon \in L^{\frac{4}{q-2}}((0, T) \times \Omega') \) uniformly in \( \epsilon \), it follows from the equations that \( \nabla \pi^\epsilon \in L^{\frac{4}{q-2}}((0, T) \times \Omega') \), independent of \( \epsilon \). Since \( \Omega \) is arbitrary open bounded domain, there exists a subsequence \( \{\epsilon_k\}_{k=1}^\infty \) so that as \( k \to \infty, \epsilon_k \to 0 \) and \( \{u^{\epsilon_k}\}_{k=1}^\infty \), which converges to \( \tilde{u} \) and satisfies the regularity assumption in Definition \([11]\).

Moreover, using Aubin-Lions theorem \( u^{\epsilon_k} \) converges strongly to \( \tilde{u} \) in \( L^r_{\text{loc}}, r < \frac{5q}{4q-2} \), and using Minty theorem \( D(u^{\epsilon_k}) \) converges strongly to \( D(\tilde{u}) \) in \( L^q_{\text{loc}} \) (see \([14]\) and \([26]\) for details). This again implies that \( \tilde{u} \) solves the equation \((1)\) in the sense of distributions. Since uniqueness of the strong solution is straightforward due to the monotonicity property of \( S(D) : D \), we skip its details.

- (The case \( 1 < q < 2 \)) We note that
\[
(\phi |\nabla D^\epsilon|)^q \leq \left( \phi^2 (\epsilon^2 + |D^\epsilon|^2)^\frac{2}{q-2} |\nabla D^\epsilon|^2 \right)^{\frac{q}{2}} \left( (\epsilon^2 + |D^\epsilon|^2)^\frac{2}{q-2} \right)^{\frac{q}{2}}
\leq c\phi^2 (\epsilon^2 + |D^\epsilon|^2)^\frac{2}{q-2} |\nabla D^\epsilon|^2 + c(\epsilon^2 + |D^\epsilon|^2)^\frac{q}{2}.
\]

Via integration by parts, the first three integrands of the right hand side in \((29)\) becomes
\[
\int_\Omega \text{div} \left( (\epsilon^2 + |D^\epsilon|^2)^\frac{2}{q-2} D^\epsilon \right) \nabla \phi^2 \times \nabla \times u^\epsilon - \int_\Omega D_{ijx} \left( (\epsilon^2 + |D^\epsilon|^2)^\frac{2}{q-2} D^\epsilon_{ij} \right) (\nabla \phi^2 \cdot \nabla u^\epsilon_{ij})
\]
\[
+ \int_\Omega D_{xx}(\epsilon^2 + |D^\epsilon|^2)^\frac{2}{q-2} D^\epsilon_{ij} (D_{xx}, \phi^2)(D_{xx} u^\epsilon_{ij})
\]
\[
= -\int_\Omega (\epsilon^2 + |D^\epsilon|^2)^\frac{2}{q-2} D^\epsilon_{ij} (\nabla \phi^2 \times \nabla \times u^\epsilon_{ij}) - \int_\Omega (\epsilon^2 + |D^\epsilon|^2)^\frac{2}{q-2} D^\epsilon_{ij} D_{xx} \left( \nabla \phi^2 \cdot \nabla u^\epsilon_{ij} \right)
\]
\[
+ \int_\Omega (\epsilon^2 + |D^\epsilon|^2)^\frac{2}{q-2} D^\epsilon_{ij} D_{xx} \left( D_{xx}, \rho^2 \right)(D_{xx} u^\epsilon_{ij})
\]

Hence, applying Hölder’s inequalities and Young’s inequalities, the right hand side of \((29)\) is dominated by
\[
c\|\phi \nabla^2 u^\epsilon\|_{L^q} \left( \int_\Omega (\epsilon^2 + |D^\epsilon|^2)^\frac{q}{2} \right)^\frac{q-1}{q} + c\|\nabla u^\epsilon\|_{L^q} \left( \int_\Omega (\epsilon^2 + |D^\epsilon|^2)^\frac{q}{2} \right)^\frac{q-1}{q}
\]
\[
+ ce\|\phi \nabla^2 u^\epsilon\|_{L^q} \|\nabla u^\epsilon\|_{L^q}
\]
Taking \( \delta \) small enough, (29) reduces to the inequality
\[
\frac{d}{dt} \int_\Omega \phi^2 |\nabla \times \mathbf{u}'|^2 dx + \int_\Omega ((\phi|\nabla D')^q + \epsilon \phi^2 |\nabla \mathbf{u}'|^2) dx \\
\leq c(1 + \frac{1}{\delta^{\frac{q}{p-2}}}) (\|\nabla \mathbf{u}'\|_{L^q}^q + \epsilon^q) + c\epsilon \int_\Omega |\nabla \mathbf{u}'|^2 dx,
\]
whieh yields by integrating in time that
\[
\sup_{0 < t < T} \int_\Omega \phi^2 |\nabla \times \mathbf{u}'|^2 dx + \int_0^T \int_\Omega ((\phi|\nabla D')^q + \epsilon \phi^2 |\nabla \mathbf{u}'|^2) dx dt \\
\leq c(1 + \frac{1}{\delta^{\frac{q}{p-2}}}) \left( \int_0^T \int_\Omega |\nabla \mathbf{u}'|^q dx dt + \epsilon^q T \right) \\
+ c\epsilon \int_0^T \int_\Omega |\nabla \mathbf{u}'|^2 dx dt + c \int_\Omega \phi^2 |\nabla \times \mathbf{u}_0'|^2 dx.
\]
From Biot-Sawart law and Korn’s inequality, (33) implies uniformly in \( \epsilon \) that
\[
\nabla \mathbf{u}' \in L^\infty(0, T; L^q(\Omega)), \quad \nabla^2 \mathbf{u}' \in L^3((0, T) \times \Omega').
\]
As in the case \( q > 2 \), we can find a subsequence \( \{\epsilon_k\}_{k=1}^\infty \) so that as \( k \to \infty, \epsilon_k \to 0 \) and \( \{u'^k\}_{k=1}^\infty \) which converges to some \( \tilde{u} \) and satisfies the regularity assumption in Definition 1.1.

Moreover, sing Aubin-Lions theorem \( u'^k \) converges strongly to \( \tilde{u} \) in \( L^r_{loc}, r < \frac{5q}{3} \), and using Minty theorem \( D(u'^k) \) converges strongly to \( D(\tilde{u}) \) in \( L^q_{loc} \) (see [17] and [20] for details). Therefore, it is straightforward that \( \tilde{u} \) solves the equation (1) in the sense of distributions. Uniqueness of the strong solution satisfying Definition 1.1 is rather standard.

3. Non-Newtonian Stokes system. In this section, we present the proofs of Theorem 1.2, Theorem 1.3 and Theorem 1.4 (see [17] for Newtonian case).

3.1. Proof of Theorem 1.2. This subsection is devoted to giving the proof of Theorem 1.2.

Proof of Theorem 1.2. According to Lemma 2.4, \( \partial_t \mathbf{u} \in L^{\frac{3q}{3q-1}}(0, T; W^{-1, \frac{3q}{3q-1}}(\Omega)) \) and \( \mathbf{u} \in L^\infty(0, T; L^q(\Omega)) \cap L^q(0, T; W^{1,q}(\Omega)) \) imply that \( \partial_t \mathbf{v} \in L^{\frac{3q}{3q-1}}(0, T; L^{\frac{3q}{3q-1}}(\mathbb{R}^3)) \) and \( \mathbf{v} \in L^\infty(0, T; W^{1,2}(\mathbb{R}^3)) \) imply that \( \mathbf{v} \in L^q(0, T; W^{2,q}(\mathbb{R}^3)) \). Fix \( z_0 = (x_0, t_0) \) in \( Q_T \) and let \( r < \frac{1}{2} \min\{\sqrt{r_0}, \text{dist}(x_0, \partial \Omega)\} \). We denote \( B_r = B_{r,x_0} \) and \( Q_r = Q_{r}(z_0) \). Let \( \psi_r \) be given in (30) and \( \psi_r(t) \) a smooth cut off function in time variable such that
\[
\psi_r(t) = 1 \text{ if } t \geq -r^2, \quad \psi_r = 0 \text{ if } t \leq -4r^2, \quad |\psi_r'| \leq \frac{c}{r^2}.
\]
Testing \( \psi_r \nabla \times (\phi_r \mathbf{v}) \) to (1), we have
\[
0 = \int_{-4r^2}^{r^2} \int_{B_{2r}^c} \partial_t \mathbf{u} \cdot \nabla \times (\phi_r \mathbf{v}) \psi_r dz + \int_{-4r^2}^{r^2} \int_{B_{2r}^c} S(D(\mathbf{u} : D(\nabla \times (\phi_r \mathbf{v}))\psi_r dz.
\]
We first show that
\[
\int_{-4r^2}^{r} \langle \partial_t \mathbf{u} , \nabla \times (\phi_r \mathbf{v}) \rangle \psi_r dt = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \mathbf{v}|^2(\tau) d\tau + \frac{1}{2} \int_{-4r^2}^{r} \int_{\mathbb{R}^3} |\nabla \mathbf{v}|^2 \psi_r' dz
\]
a.e. \( \tau \in (0, -r^2) \). Indeed, we note that
\[
\langle \partial_t \mathbf{u} , \nabla \times (\phi_r \mathbf{v}) \rangle = \langle \nabla \times \partial_t \mathbf{u}, \phi_r \mathbf{v} \rangle = \langle \phi_r \nabla \times \mathbf{u}, \mathbf{v} \rangle.
\]
\[ \Delta \phi_r v, v > = \nabla_\phi_r \nabla_v \nabla v = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla_v|^2 \, dx. \]

Integrating the above with \( \psi_r \) in time variable, we obtain \[ 36 \]. Next we consider the second term in \[ 35 \]. We note first that
\begin{align*}
\nabla \times (\phi_r v) &= \nabla \phi_r \times v + \phi_r \nabla \times [N * \nabla \times (\phi_r u)] - \phi_r \nabla \times [N * (\nabla \phi_r \times u)] \\
&= \nabla \phi_r \times v + \phi_r^2 u + \phi_r \nabla [N * (\nabla \phi_r \cdot u)] - \phi_r \nabla \times [N * (\nabla \phi_r \times u)]. \tag{37}
\end{align*}

Therefore, we obtain
\begin{align*}
\int_{B_{2r}} S(D(u)) : D(\nabla \times (\phi_r v)) \, dx &= \int_{B_{2r}} \phi_r^2 |D|^q + S(D(u)) : (\nabla \phi_r \otimes u) \\
&+ S(D(u)) : D \left( \nabla \phi_r \times v + S(D(u)) : D (\phi_r \nabla [N * (\nabla \phi_r \cdot u)]) \\
&- S(D(u)) : D (\phi_r \nabla \times [N * (\nabla \phi_r \times u)]) \right) \, dx.
\end{align*}

Summing up above estimates, we obtain that
\begin{align*}
\frac{1}{2} \sup_{-r^2 < \tau < 0} \int_{\mathbb{R}^3} |\nabla v|^2 \, dx + \int_{Q_{2r}} \psi_r \phi_r^2 |D|^q \, dz \\
&\leq \left| \int_{-4r^2}^0 \int_{\mathbb{R}^3} |\nabla v|^2 \psi_r' \, dz \right| + \int_{Q_{2r}} \psi_r S(D(u)) : (\nabla \phi_r \otimes u) \, dz \\
&+ \int_{Q_{2r}} \psi_r S(D(u)) : D (\nabla \phi_r \times v) \, dz \\
&+ \int_{Q_{2r}} \psi_r S(D(u)) : D (\phi_r \nabla [N * (\nabla \phi_r \cdot u)]) \, dz \\
&+ \int_{Q_{2r}} \psi_r S(D(u)) : D (\phi_r \nabla \times [N * (\nabla \phi_r \times u)]) \, dz \\
&\quad := I_0 + I_1 + I_2 + I_3 + I_4. \tag{38}
\end{align*}

It is direct via Hölder’s and Young’s inequalities that
\begin{align*}
I_1 &= \int_{Q_{2r}} S(D(u)) : (\nabla \phi_r \otimes u) \leq \epsilon \int_{Q_{2r}} |D(u)|^q + \frac{c}{r^q} \int_{Q_{2r}} |u|^q. \tag{39}
\end{align*}

By Hölder’s inequality, we estimate \( I_2 \) as follows:
\begin{align*}
I_2 &\leq \frac{c}{r^2} \int_{Q_{2r}} |D(u)|^{q-1} ||u||_{L^1(Q_{2r})} \, dz + \frac{c}{r} \int_{Q_{2r}} |D(u)|^{q-1} ||\nabla v||_{L^1(Q_{2r})} \\
&\leq \frac{c}{r^2} ||D(u)||_{L^{q/(q-1)}(Q_{2r})} ||u||_{L^q(Q_{2r})} + \frac{c}{r} ||D(u)||_{L^{q/(q-1)}(Q_{2r})} ||\nabla v||_{L^q(Q_{2r})} \\
&\leq \frac{c}{r} ||D(u)||_{L^{q/(q-1)}(Q_{2r})} ||u||_{L^q(Q_{2r})} \leq \epsilon ||D(u)||_{L^q(Q_{2r})}^q + \frac{c}{r^q} ||u||_{L^q(Q_{2r})}^q. \tag{40}
\end{align*}

On the other hand, as in Lemma \[ 24 \], we can observe that
\begin{align*}
||D (\phi_r \nabla [N * (\nabla \phi_r \cdot u)])||_{L^q(B_{2r})} + ||D (\phi_r \nabla \times [N * (\nabla \phi_r \times u)])||_{L^q(B_{2r})} \\
&\leq \frac{c}{r} ||u||_{L^q(B_{2r})}. \tag{41}
\end{align*}

Hence, the terms \( I_3 \) and \( I_4 \) are controlled as follows:
\begin{align*}
I_3 + I_4 &\leq \frac{c}{r} ||D(u)||_{L^{q/(q-1)}(Q_{2r})} ||u||_{L^q(Q_{2r})} \leq \epsilon ||D(u)||_{L^q(Q_{2r})}^q + \frac{c}{r^q} ||u||_{L^q(Q_{2r})}^q. \tag{42}
\end{align*}

Via Lemma \[ 24 \] we also have
\begin{align*}
I_0 &\leq \frac{c}{r^2} \int_{-4r^2}^{-r^2} \int_{B_{2r}} |u|^2 \, dz. \tag{43}
\end{align*}
Summarizing all estimates above, we obtain
\[
\sup_{-r^2 < r < 0} \int_{\mathbb{R}^3} |\nabla v|^2 + \int_{Q_r} |D(u)|^q \leq \epsilon \int_{Q_{2r}} |D(u)|^q + \frac{c}{r^q} \int_{Q_{2r}} |u|^q + \frac{c}{r^2} \int_{Q_{2r}} |u|^2.
\]
Due to Lemma 2.3 we obtain
\[
\sup_{-r^2 < r < 0} \int_{\mathbb{R}^3} |\nabla v|^2 + \int_{Q_r} |D(u)|^q \leq \frac{c}{r^q} \int_{Q_{2r}} |u|^q + \frac{c}{r^2} \int_{Q_{2r}} |u|^2.
\] (44)
Using Korn’s inequality, we again obtain
\[
\int_{Q_{\frac{3}{2}}} |\nabla u|^q \leq \frac{c}{r^q} \int_{Q_{2r}} |u|^q + \frac{c}{r^2} \int_{Q_{2r}} |u|^2.
\]
We complete the proof. \(\square\)

As mentioned earlier in Introduction, the Caccioppoli type inequality (10) doesn’t control the local \(L^\infty L^2\)–norm of \(u\) by right hand side of (10). Next corollary, however, shows that \(L^\infty L^2\)–norm of a somehow can be estimated in terms of righthand side of (10) as well as itself in a bigger cylinder.

**Corollary 2.** Let \(1 < q < \infty\) and \(u\) be a strong solution of non-Newtonian Stokes system in Definition 1.1. Then, for any \(r > 0\) with \(2r < \rho\)
\[
\sup_{-r^2 < r < 0} \int_{B_r} |u|^2 \leq c \left(\frac{r}{\rho}\right)^3 \sup_{-\rho^2 < r < 0} \int_{B_{\rho}} |u|^2 + \frac{c}{\rho^3} \int_{Q_{\rho}} |u|^q + \frac{c}{\rho^2} \int_{Q_{\rho}} |u|^2.
\] (45)

**Proof.** Let \(v\) be given in (21). We observe first that \(u - \nabla \times v\) is harmonic in \(B_r\). By the mean value property of the harmonic functions, we have
\[
\int_{B_r} |u - \nabla \times v|^2 dx \leq c \left(\frac{r}{\rho}\right)^3 \int_{B_{\rho}} |u - \nabla \times v|^2 dx.
\]
By triangle inequality and the above inequality, we have
\[
\int_{B_r} |u|^2 \leq \int_{B_r} |u - \nabla \times v|^2 + \int_{B_r} |\nabla \times v|^2 \leq c \left(\frac{r}{\rho}\right)^3 \int_{B_{\rho}} |u - \nabla \times v|^2 + \int_{B_r} |\nabla \times v|^2
\]
\[
\leq c \left(\frac{r}{\rho}\right)^3 \int_{B_{\rho}} |u|^2 + c \int_{B_r} |\nabla \times v|^2 \leq c \left(\frac{r}{\rho}\right)^3 \int_{B_{\rho}} |u|^2 + c \int_{\mathbb{R}^3} |\nabla v|^2.
\] (46)
From (46), we have
\[
\sup_{-r^2 < r < 0} \int_{B_r} |u(t)|^2 \leq c \left(\frac{r}{\rho}\right)^3 \sup_{-\rho^2 < r < 0} \int_{B_{\rho}} |u(t)|^2 + \int_{\mathbb{R}^3} |\nabla v(t)|^2.
\] (47)
From (44) in the proof of Theorem 1.2 we also have the inequality
\[
\sup_{-r^2 < r < 0} \int_{\mathbb{R}^3} |\nabla v(t)|^2 + \int_{Q_r} |D(u)|^q \leq \frac{c}{\rho^3} \int_{Q_{\rho}} |u|^q + \frac{c}{\rho^2} \int_{Q_{\rho}} |u|^2, \quad r < \frac{\rho}{2}.
\] (48)
Combining (47) and (48), we obtain (45). This completes the proof. \(\square\)
3.2. Proofs of Theorem 1.3 and Theorem 1.4

This subsection is devoted to proving Theorem 1.3 and Theorem 1.4. Before providing the proof, we begin to recognize that $u$ is the limit of the solution of the approximate system, which is considered in the proof of Proposition 1.

\[ \frac{\partial u^e}{\partial t} - \text{div} S^e(D(u^e)) + \nabla \pi^e = 0, \quad \text{div} u^e = 0 \quad \text{in } Q_T, \]

where

\[ S^e(s) = \mu(s)s, \quad \mu(s) = \epsilon + (\epsilon^2 + |s|^2)^{\frac{\alpha - 2}{2}}. \]

As we have already mentioned it in the proof of Proposition 1, following similar procedures in [13] and [14], it can be shown the existence of unique solution $u^e$ satisfying that

\[ u^e \in L^\infty(0, T; W^{1, q}(\Omega)), \quad D_t u^e \in L^2(0, T; L^2(\Omega)), \quad u^e \in L^2(0, T; W^{2, 2}(\Omega)), \]

\[ \nabla S^e \in L^2(0, T; L^2(\Omega)), \quad \nabla \pi^e \in L^2(0, T; L^2(\Omega)). \]

Furthermore, there is a subsequence $\{u^{e_k}\}_{k=1}^\infty$ which converges to some $u$, satisfying Definition 1.1. More precisely,

\[ u^{e_k} \to u \quad \text{strongly in } L^r_{\text{loc}}, \quad r < \frac{5q}{3}, \]

\[ D(u^{e_k}) \to D(u) \quad \text{strongly in } L^q_{\text{loc}}, \]

\[ \epsilon_k \nabla u^{e_k}, \epsilon_k \nabla^2 u^{e_k} \to 0 \quad \text{strongly in } L^q_{\text{loc}}, \]

\[ \nabla \left( (\epsilon_k^2 + |D^{e_k}|^2)^{\frac{q-2}{2}} D^{e_k} \right) \to \nabla |D(u)|^{q-2} D(u) \quad \text{weakly in } L^\infty_{\text{loc}}, \quad \text{if } q > 2 \]

\[ \nabla D(u^{e_k}) \to \nabla D(u) \quad \text{weakly in } L^q_{\text{loc}}, \quad \text{if } 1 < q < 2. \]

Now we will show that $u$ satisfies the estimates in Theorem 1.3 and Theorem 1.4. As before, fix $z_0 = (x_0, t_0) \in Q_T$ and $r < \frac{1}{4} \min \{ \sqrt{t_0}, \text{dist} (x_0, \partial \Omega) \}$. We denote $B_r = B_r(x_0)$ and $Q_r = Q_r(z_0)$, and we write $\phi_r$ and $\psi_r$ given in (20) as $\phi$ and $\psi$, respectively. For the simplicity, we also denote $u^k = u^{e_k}, D^k = D(u^{e_k})$ and $S^k = S^{e_k}(D(u^{e_k}))$.

Taking inner product by $\nabla \times (\phi^2 \nabla \times u^k)$ to the equation (1) and integrating over $\Omega$, due to the orthogonality of $\nabla$ and $\nabla \times$, we have the identity that

\[ 0 = \int_0^T \int_{B_{2r}} (\partial_t u^k \cdot \text{div} S^k + \nabla p^k) \cdot \nabla \times (\phi^2 \nabla \times u^k) \]

\[ = \int_0^T \int_{B_{2r}} \partial_t u^k \cdot \nabla \times (\phi^2 \nabla \times u^k) \]

\[ - \int_{B_{2r}} \int_{-4r^2}^0 \text{div} S^k \cdot [\nabla \times (\phi^2 \nabla \times u^k)]. \]

We note that

\[ \int_\Omega \partial_t u^k \cdot \nabla \times (\phi^2 \nabla \times u^k) dx = \frac{1}{2} \int_\Omega \partial_t (\phi^2 \nabla \times u^k)^2 dx. \]

Since $\nabla \times (\phi^2 \nabla \times u^k) = \nabla \phi^2 \times \nabla \times u^k + \phi^2 \nabla \times \nabla \times u^k$ and $\nabla \times \nabla \times u^k = -\Delta u^k$, we have

\[ \int_{B_{2r}} \text{div} S^k \cdot [\nabla \times (\phi^2 \nabla \times u^k)] dx \]

\[ = -\int_{B_{2r}} \phi^2 \nabla S^k \cdot \nabla D_{ij} + \partial_x (S_{ij} \partial_x \phi^2) \partial_x u^k - \partial_x (S_{ij} \partial_x \phi) \partial_x u^k \]

\[ + \partial_x (S_{ij} \partial_x \phi) \partial_x u^k \]

\[ = -\int_{B_{2r}} \phi^2 \nabla S^k \cdot \nabla D_{ij} + \partial_x (S_{ij} \partial_x \phi^2) \partial_x u^k - \partial_x (S_{ij} \partial_x \phi) \partial_x u^k. \]
+ \text{div} S^k \cdot (\nabla \phi^2 \times \nabla \times u^k) \right) dx.

Hence, due to (55) with a temporal test function $\psi$, we obtain
\[
\frac{1}{2} \sup_{-r^2 < \tau < 0} \int_{B_{2r}} \phi^2 |\nabla \times u^k|^2(\tau) dx + \int_{Q_{2r}} \psi \phi^2 \nabla S^k_{ij} \cdot \nabla D_{ij} dz
\leq \int_{Q_{2r}} \left[ \psi \partial_x \left( S^k_{ij} \partial_x \phi^2 \right) \partial_x u_j - \psi \partial_x \left( S^k_{ij} \partial_x \phi^2 \right) \partial_x u_j^k \right.
\left. - \psi \text{div} S^k \cdot (\nabla \phi^2 \times \nabla \times u^k) \right] dz + \frac{1}{2} \int_{Q_{2r}} \psi' \phi^2 |\nabla \times u^k|^2(\tau) dz. \tag{56}
\]

- (Case 1 < $q < 2$) We observe that
\[
\nabla S^k_{ij} \cdot \nabla D_{ij} \geq (q - 1)(\epsilon^2_k + |D|^2)^{\frac{q - 2}{2}} |\nabla D|^2 + \epsilon_k |\nabla D|^2. \tag{57}
\]

Due to $|\nabla D|^q = \left( (\epsilon^2_k + |D|^2)^{\frac{q - 2}{2}} |\nabla D|^2 \right)^{\frac{q - 2}{q - 1}}$, we have
\[
\frac{1}{r^2} \| \phi \nabla D \|^q_{L^1(\Omega_{2r})} \leq C \left( \int_{Q_{2r}} (\epsilon^2_k + |D|^2)^{\frac{q - 2}{2}} |\nabla D|^2 \right)^{\frac{q}{2}} \left( \frac{1}{r^2} \int_{Q_{2r}} (\epsilon^2_k + |D|^2)^{\frac{q}{2}} \right)^{\frac{q - 2}{q}}.
\]

Integrating by parts, the first term of the right hand side of (58) becomes
\[
\int_{Q_{2r}} \left[ \psi \partial_x \left( S^k_{ij} \partial_x \phi^2 \right) \partial_x u_j - \psi \partial_x \left( S^k_{ij} \partial_x \phi^2 \right) \partial_x u_j^k \right.
\left. - \psi \text{div} S^k \cdot (\nabla \phi^2 \times \nabla \times u^k) \right] dz
\]
\[
= - \int_{Q_{2r}} \left[ \psi S^k_{ij} \partial_x \phi^2 \partial_x \partial_x u_j - \psi S^k_{ij} \partial_x \phi^2 \partial_x \partial_x u_j^k + \psi S^k : \nabla (\nabla \phi^2 \times \nabla \times u^k) \right] dz.
\]

Hence (56) reduces to the inequality
\[
\frac{1}{2} \sup_{-r^2 < \tau < 0} \int_{B_{2r}} \phi^2 |\nabla \times u^k|^2(\tau) dx + \int_{Q_{2r}} \psi \phi^2 \left[ \frac{1}{r^2} |\nabla D|^q + \epsilon_k |\nabla D|^2 \right] dz
\leq \frac{c}{r^2} \int_{Q_{2r}} (\epsilon^2_k + |D|^2)^{\frac{q}{2}} + \frac{1}{2} \int_{Q_{2r}} \psi' \phi^2 |\nabla \times u^k|^2(\tau) dz
\]
\[
- \int_{Q_{2r}} \left[ \psi S^k_{ij} \partial_x \phi^2 \partial_x \partial_x u_j - \psi S^k_{ij} \partial_x \phi^2 \partial_x \partial_x u_j^k + \psi S^k : \nabla (\nabla \phi^2 \times \nabla \times u^k) \right] dz. \tag{59}
\]

Passing to the limit, (59) reduces to the inequality
\[
\frac{1}{2} \sup_{-r^2 < \tau < 0} \int_{B_{2r}} \phi^2 |\nabla \times u|^2(\tau) dx + \frac{1}{r^2} \int_{Q_{2r}} \psi \phi^2 |\nabla D|^q dz
\leq \frac{c}{r^2} \int_{Q_{2r}} |D|^q dz + \frac{1}{2} \int_{Q_{2r}} \psi' \phi^2 |\nabla \times u|^2(\tau) dz
\]
\[
- \int_{Q_{2r}} \left[ \psi S^k_{ij} \partial_x \phi^2 \partial_x \partial_x u_j - \psi S^k_{ij} \partial_x \phi^2 \partial_x \partial_x u_j^k + \psi S^k : \nabla (\nabla \phi^2 \times \nabla \times u) \right] dz
\]= \frac{c}{r^2} \int_{Q_{2r}} |D|^q dz + I_1 + I_2. \tag{60}
\]
Here we use (50), (51), (52), (54) and lower semi-continuity on the left hand side. First, we note that
\[ I_1 \leq C \int_{Q_{2r}} \phi^2 |\nabla \times u|^2 \psi'(s) dz \leq C \int_{Q_{2r}} |\nabla u|^2 dz = \frac{C}{r^2} \|\nabla u\|_{L^2(Q_{2r})}^2. \]
On the other hand, Hölder's inequality yields
\[ I_2 \leq \frac{C}{r^2} \left( \int_{Q_{2r}} |D|^q dz \right)^{\frac{q-1}{q}} \|\phi \nabla^2 u\|_{L^s(Q_{2r})} + \frac{C_4}{r^2} \|\nabla u\|_{L^2(Q_{2r})}^q. \]
Choosing \( \delta > 0 \) small enough and summing up all the estimates, we obtain
\[ \sup_{-r^2 < t < 0} \int_{B_r(t)} |\nabla \times u(t)|^2 dx + \frac{1}{r^2} \int_{Q_r} |\nabla D|^q dz \leq \frac{C}{r^2} \|\nabla u\|_{L^2(Q_{2r})}^q + \frac{C}{r^2} \|\nabla u\|_{L^2(Q_{2r})}^2. \]
This completes the proof of the case 1 \( q < 2 \).

- (Case \( q > 2 \)) Recalling (57) and using \( |\nabla S^k| \leq c(\epsilon^2 + |D|^2)^{\frac{q-2}{2}} |\nabla D| \), we note that
\[ \frac{1}{r^{q-2}} \|\nabla S^\kappa\|_{L^{\frac{q}{q-1}}(Q_{2r})} \]
\[ \leq \left( \int_{Q_{2r}} (\epsilon^2 + |D|^2)^{\frac{q-2}{2}} |\nabla D|^2 \phi^2 \right)^{\frac{q-1}{q}} \left( \frac{1}{r^2} \int_{Q_{2r}} (\epsilon^2 + |D|^2)^{\frac{q}{2}} \right)^{\frac{q-1}{q}} \]
\[ \leq \int_{Q_{2r}} (\epsilon^2 + |D|^2)^{\frac{q-2}{2}} |\nabla D|^2 \phi^2 dz + \frac{c}{r^2} \int_{Q_{2r}} (\epsilon^2 + |D|^2)^{\frac{q}{2}}. \]
Hence (56) reduces to the inequality
\[ \frac{1}{2} \sup_{-r^2 < \tau < 0} \int_{B_{2r}(\tau_0)} \phi^2 |\nabla \times u^k(\tau)|^2 + \int_{Q_{2r}} \psi \phi^2 \left[ \frac{1}{r^{q-2}} |\nabla S^\kappa|_{L^{\frac{q}{q-1}}} + \epsilon_k |\nabla D|^2 \right] \]
\[ \leq \frac{c}{r^2} \int_{Q_{2r}} (\epsilon^2 + |D|^2)^{\frac{q}{2}} \]
\[ + \int_{Q_{2r}} \left[ \psi \partial_{x_1} \left( S^k_{ij} \partial_{x_i} \phi^2 \right) \partial_{x_j} u^k + \psi \partial_{x_{k-1}} \left( S^k_{ij} \partial_{x_i} \phi^2 \right) \partial_{x_j} u^k + \psi \partial_{x_{k-1}}(\nabla \phi^2 \times \nabla \times u^k) \right] \]
\[ + \frac{1}{2} \int_{Q_{2r}} \psi \phi^2 |\nabla \times u^k|^2(\tau). \]
Passing to the limit, (63) reduces to the inequality
\[ \frac{1}{2} \sup_{-r^2 < \tau < 0} \int_{B_{2r}(\tau_0)} \phi^2 |\nabla \times u|^2(\tau) + \int_{Q_{2r}} \psi \phi^2 \left[ \frac{1}{r^{q-2}} |\nabla S|_{L^{\frac{q}{q-1}}} \right] \]
\[ \leq \frac{c}{r^2} \int_{Q_{2r}} |D|^q \]
\[ + \int_{Q_{2r}} \left[ \psi \partial_{x_1} \left( S^k_{ij} \partial_{x_i} \phi^2 \right) \partial_{x_j} u^k - \psi \partial_{x_{k-1}} \left( S^k_{ij} \partial_{x_i} \phi^2 \right) \partial_{x_j} u^k - \psi \div S \cdot (\nabla \phi^2 \times \nabla \times u) \right] \]
\[ + \frac{1}{2} \int_{Q_{2r}} \psi' \phi^2 |\nabla \times u|^2(\tau) dz = \frac{c}{r^2} \int_{Q_{2r}} |D|^q + I_1 + I_2. \]
As in the case $1 < q < 2$, we use (50), (51), (52), and lower semi-continuity on the left hand side. We observe that

\[
I_2 \leq C \int_{Q_{2r}} \phi^2 |\nabla \times u|^2 |\psi'(s)| \, dz \leq \frac{C}{r^2} \int_{Q_{2r}} |\nabla u|^2 \, dz = \frac{C}{r^2} \|\nabla u\|_{L^2(\Omega)}^2.
\]

Via Hölder’s inequality, we get

\[
I_1 \leq \frac{C}{r^2} \|\phi \nabla S\|_{L^{r^2} Q_{2r}} \|\nabla u\|_{L^q Q_{2r}} + \frac{C}{r^2} \|\nabla u\|_{L^q Q_{2r}}^q \leq \frac{\delta}{r^2} \|\phi \nabla S\|_{L^{r^2} Q_{2r}}^\frac{r}{q} + \frac{C}{r^2} \int_{Q_{2r}} |\nabla u|^q.
\]

Combining all the estimates, we have the inequality

\[
\sup_{-r^2 < t < 0} \int_{B_r} |\nabla \times u(t)|^2 \, dx + \int_{Q_{2r}} |Dk|^{q-2} |\nabla D|^2 \phi^2 \, dz \leq \frac{\delta}{r^2} \|\phi \nabla S\|_{L^{r^2} Q_{2r}}^\frac{r}{q} + \frac{C}{r^2} \int_{Q_{2r}} |\nabla u|^q + \frac{C}{r^2} \int_{Q_{2r}} |\nabla u|^2 \, dz.
\]

Choosing $\delta > 0$ in (65) small enough and summing up above estimates, we obtain

\[
\sup_{-r^2 < t < 0} \int_{B_r} |\nabla \times u(t)|^2 \, dx + \int_{Q_{2r}} |Dk|^{q-2} |\nabla D|^2 \phi^2 \, dz \leq C \int_{Q_{2r}} |\nabla u|^q \, dz + \frac{C}{r^2} \int_{Q_{2r}} |\nabla u|^2 \, dz.
\]

This completes the proof of the case $q > 2$, and thus we deduce Theorem 1.3 and Theorem 1.6.

4. Non-Newtonian Navier-Stokes equations. In this section, we give the proof of Theorem 1.6.

Proof of Theorem 1.6. Let $\Phi : [0, \infty) \to [0, 1]$ be a smooth and non-increasing function satisfying

\[
\Phi = 1 \text{ on } [0, 1], \quad \Phi = 0 \text{ on } [2, \infty), \quad |\Phi'| \leq 2 \text{ on } [1, 2].
\]

For each positive integer $m$, we set $\Phi_m(\tau) = \Phi(\frac{\tau}{m})$ for $\tau \in [0, \infty)$. We recall the following approximate system introduced in [26].

\[
\partial_t u_m + \nabla \cdot (u_m \otimes u_m) + \nabla \cdot \Phi_m(|u_m|) - \div S(|Du_m|) + \nabla p_m = \nabla \cdot q_m, \quad \div u_m = 0 \quad \text{in } Q_T
\]

with initial data $u_m \big|_{t=0} = u_0$. Existence of weak solution $u_m$ is known (see [26] Theorem 3.1) and it was also shown (possibly subsequence but, for simplicity, use the same index) independent of $\epsilon$, that

\[
\|u_m\|_{L^\infty(0, T; L^2(\Omega))} + \|\nabla u_m\|_{L^q(Q_T)} + \|S(|Du_m|)\|_{L^\frac{q}{q-1}(Q_T)} < c.
\]

Furthermore, it was proven that if $q > 8/5$, there exists a limit function $u$ such that (possibly subsequence) (see [26] Theorem 1.3)

\[
\nabla u_m \rightharpoonup \nabla u \quad \text{weakly in } L^q(Q_T),
\]

\[
S(|Du_m|) \rightharpoonup S(|Du|) \quad \text{weakly in } L^{\frac{q}{q-1}}(Q_T),
\]

\[
u_m \rightharpoonup u \quad \text{strongly in } L^r(0, T; L^q(\Omega)), \quad r < \frac{5q}{3},
\]

and $u$ is the weak solution of the following non-Newtonian Navier-Stokes equations:

\[
\partial_t u + \nabla \cdot (u \otimes u) - \div S(|D(u)|) + \nabla p = 0, \quad \div u = 0 \quad \text{in } Q_T
\]
From (73)-(75), we have the identity
\[
\int
\]
Now, it remains to consider the second term concerning convection: Using the proof of Theorem 1.2 and we set an auxiliary vector field \( v_m \) as
\[
v_m(x, t) := \int_{\mathbb{R}^3} N(x - y) (\phi(y) \nabla_y u_m(y, t)) \, dy. \tag{71}
\]
According to Lemma 2.4 we note that
\[
u_m \in L^\infty(0, T; L^2(\Omega)) \cap L^q(0, T; W^{1,q}(\Omega)) \quad \text{and} \quad \partial_t u_m \in L^\frac{2}{n-1}(0, T; W^{-1,\frac{n}{n-1}}(\Omega)),
\]
which imply that
\[
v_m \in L^\infty(0, T; W^{1,2}(\mathbb{R}^3)) \cap L^q(0, T; W^{2,q}(\mathbb{R}^3)) \quad \text{and} \quad \partial_t v_m \in L^\frac{2}{n-1}(0, T; L^\frac{2}{n-1}(\mathbb{R}^3)).
\]
Testing \( \psi \nabla \times (\phi v_m) \) to (67), for any \( 0 < \tau < T \) we have the identity that
\[
0 = \int_0^\tau \int_\Omega \partial_t u_m \cdot \nabla \times (\phi v_m) \psi \, dz + \int_0^\tau \int_\Omega \nabla \cdot (u_m \otimes u_m \Phi_m(|u_m|)) \cdot \nabla \times (\phi v_m) \psi \, dz
\]
\[
+ \int_0^\tau \int_\Omega S(D(u_m)) : D(\nabla \times (\phi v_m)) \psi \, dz. \tag{72}
\]
As in the case of Stokes system, the first term and the third term in (72) can be computed as follows:
\[
\int_0^\tau \int_\Omega \frac{\partial u_m}{\partial t} \cdot \nabla \times (\phi v_m) \psi = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla v_m|^2 \psi)(\tau) \, dx - \frac{1}{2} \int_0^\tau \int_{\mathbb{R}^3} |\nabla v_m|^2 \psi', \tag{73}
\]
and
\[
= \int_0^\tau \int_\Omega S(D(u_m)) : D(\nabla \times (\phi v_m)) \psi \, dz
\]
\[
= \int_0^\tau \int_\Omega \phi^2 |D(u_m)|^q \psi + S(D(u_m)) : \left( \nabla (\phi^2 \otimes u_m) + \nabla \phi \otimes v_m + \phi \nabla [N * (\nabla \phi \otimes u_m)] - \phi \nabla \times [N * (\nabla \phi \times u_m)] \right) \psi \, dz. \tag{74}
\]
Now, it remains to consider the second term concerning convection: Using the identity (57),
\[
\int_0^\tau \int_\Omega \nabla \cdot (u_m \otimes u_m \Phi_m(|u_m|)) \cdot \nabla \times (\phi v_m) \psi \, dz
\]
\[
= \int_0^\tau \int_\Omega u_m \otimes u_m \Phi_m(|u_m|) : \nabla (\phi^2 u_m) \psi - u_m \otimes u_m \Phi_m(|u_m|) : \nabla \left( \nabla \phi \otimes v_m - \phi \nabla [N * (\nabla \phi \cdot u_m)] + \phi \nabla \times [N * (\nabla \phi \times u_m)] \right) \psi \, dz. \tag{75}
\]
From (73)-(75), we have the identity
\[
\frac{1}{2} \int_{\mathbb{R}^3} (|\nabla v_m|^2 \psi)(\tau) \, dx + \int_0^\tau \int_\Omega \phi^2 |D(u_m)|^q \psi
\]
\[
= \int_0^\tau \int_\Omega -S(D(u_m)) : (\nabla \phi^2 \otimes u_m) \psi - S(D(u_m)) : D \left( \nabla \phi \otimes v_m \right.
\]
\[
- \phi \nabla [N * (\nabla \phi \cdot u_m)] + \phi \nabla \times [N * (\nabla \phi \times u_m)] \right) \psi \, dz
\]
\[
+ \int_0^\tau \int_\Omega u_m \otimes u_m \Phi_m(|u_m|) : \nabla (\phi^2 u_m) \psi \, dz + \int_0^\tau \int_\Omega (u_m \otimes u_m \Phi_m(|u_m|) :
\]
Next, we treat the term \( m \sigma \). We note that for any \( |m| \leq \infty \)
Therefore, it is straightforward that \( \Phi(|u_m|) \to 1 \) in the set of support of \( \phi \) as \( m \to \infty \).
Again, using strong convergence (69), we observe that
\[
v_m \to v := \int_{\mathbb{R}^n} N(x - y) [\phi(y)\nabla \times u(y, t)] dy
\]
strongly in \( L^q((0, T) : W^{1,q}_{\text{loc}}(\Omega)) \)
\[
\nabla[N * (\nabla \phi \cdot u_m)] - \nabla \times [N * (\nabla \phi \times u_m)]
\]
converges strongly to
\[
w := \nabla[N * (\nabla \phi \cdot u)] - \nabla \times [N * (\nabla \phi \times u)]\text{ in } L^q((0, T) : W^{1,q}_{\text{loc}}(\Omega)).
\]
Summing up, using (68) and (69), we obtain that as \( m \to \infty \)
\[
I_1^n \to \int_0^T \int_\Omega \left[ -S(D(u)) : (\nabla \phi^2 \otimes u)\psi - S(D(u)) : D(\nabla \phi \times v + \phi w)\psi \right],
\]
\[
I_2^n \to \int_0^T \int_\Omega (u \otimes u) : \nabla(\nabla \phi \times v + \phi w)\psi
\]
\[
I_3^n \to \int_0^T \int_\Omega (u \otimes u) : \nabla(\nabla \phi \times v + \phi w)\psi
\]
\[
I_4^n \to \frac{1}{2} \int_0^T \int_\Omega |\nabla \phi'| |\psi| dz,
\]
where \( v \) and \( w \) are defined in (78) and (79), respectively.
It remains to estimate \( I_2^n \). We rewrite \( I_2^n \) as follows:
\[
I_2^n := \int_0^T \int_\Omega u_{m,j} u_{m,i} \Phi_m(|u_m|) \phi \frac{\partial u_{m,i}}{\partial x_j} \psi
\]
\[
+ \int_0^T \int_\Omega u_{m,j} u_{m,i} \Phi_m(|u_m|) \frac{\partial \phi^2}{\partial x_j} u_{m,i} \psi \equiv I_{21}^n + I_{22}^n.
\]
Similarly as before, we can see that
\[
I_{22}^n \to \int_0^T \int_\Omega (u \cdot \nabla \phi^2)|u|^2 |\psi| dz.
\]
Next, we treat the term \( I_{21}^n \). We introduce \( \tilde{\Phi}_m \) and \( \hat{\Phi}_m \), which are defined by
\[
\tilde{\Phi}_m(\sigma) := \Phi_m(\sqrt{\sigma}), \quad \hat{\Phi}_m(\sigma) = \int_0^\sigma \tilde{\Phi}_m(\mu) d\mu.
\]
We note that for any \( \sigma \in [0, \infty) \)
\[
\tilde{\Phi}_m(\sigma) \leq c \sigma^2, \quad \lim_{m \to \infty} \hat{\Phi}_m(\sigma) = \sigma.
\]
We then see that
\[
I_{21}^n = \int_0^T \int_\Omega u_{m,j} u_{m,i} \Phi_m(|u_m|) \phi \frac{\partial u_{m,i}}{\partial x_j} \psi = \frac{1}{2} \int_0^T \int_\Omega u_{m,j} \tilde{\Phi}_m(|u_m|^2) \phi \frac{\partial |u_m|^2}{\partial x_j} \psi
\]
\[
= \frac{1}{2} \int_0^T \int_\Omega u_{m,j} \tilde{\Phi}_m(|u_m|^2) \phi \frac{\partial |u_m|^2}{\partial x_j} \psi.
\]
With the aid of (69) and (86), we obtain
\[
\lim_{m \to \infty} I_{m}^{21} = - \lim_{m \to \infty} \int_{0}^{\tau} \int_{\Omega} (u_{m} \cdot \nabla \phi^{2}) \frac{|u_{m}|^2}{2} \psi \, dz = - \int_{0}^{\tau} \int_{\Omega} (u \cdot \nabla \phi^{2}) \frac{|u|^2}{2} \psi \, dz. \quad (85)
\]

Adding together (83) and (85), we can see that as \( m \to \infty \)
\[
I_{m}^{2} \longrightarrow \int_{0}^{\tau} \int_{\Omega} (u \cdot \nabla \phi^{2}) \frac{|u|^2}{2} \psi \, dz. \quad (86)
\]

Summing up (80), (81), (82) and (86), we obtain the inequality (15). This completes
the proof.

Next corollary is an easy consequence of the above local energy inequality.

Proof of Corollary 1. Instead of \( Q_{T} = \mathbb{R}^{3} \times (0, T) \), let us consider \( Q_{\rho, 2r} \), \( \rho < 2r < T \). Let
\[ v = N * (\phi \nabla \times u) \] for the cut-off function \( \phi(x) \) which is 1 on \( B_{r} \) and zero on \( B_{2r}^{c} \), and \( \psi(t) \) which is 1 on \( [-r^{2}, 0] \) and zero on \( (-\infty, 4r^{2}] \). Then the estimate (17) is
the combination of the local energy inequality (15) and Lemma 2.4. More precisely,
we have the inequality
\[
\sup_{-r^{2} < t < 0} \int_{\mathbb{R}^{3}} |\nabla v(t)|^2 \, dx + \int_{Q_{r}} |\nabla u|^q \, dz \leq c \left( \int_{Q_{2r}} |\nabla u|^q \, dz \right)^{\frac{q-1}{q}} \left( \int_{Q_{2r}} |u|^q \, dz \right)^{\frac{1}{q}} \, dz
\]
\[ + \frac{c}{r^{2}} \int_{Q_{2r}} |u|^2 \, dz + \frac{1}{r} \int_{Q_{2r}} |u|^3 \, dz. \]

Its verification is straightforward, and thus we omit the details.

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Received June 2016; revised April 2017.

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