Weak Response of Nuclear Matter at low Momentum transfer

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Abstract. A quantitative understanding of the weak nuclear response is a prerequisite for the computer simulations of astrophysical phenomena like supernovae explosions and neutron star cooling. In order to reduce the systematic uncertainties associated with the simulations, a consistent framework, able to take into account dynamical correlation effects, is needed to compute neutrino-nucleon and neutrino-nucleus reaction rates. In this paper we describe the many-body theory of the weak nuclear response at low energy regime. We show how to include both short and long correlations effects in a consistent fashion.

1. Introduction
The description of neutrino interactions at low momentum transfer ($E_\nu \sim 10$ MeV) with nuclei, and nuclear matter in general, is relevant to the study of many different problems, from supernovae explosions [1] to neutron star cooling [2].

The systematic uncertainty associated with computer simulations depends heavily on the values of the neutrino-nucleon and neutrino-nucleus reaction rates used as inputs.

Nuclear many body theory provides a scheme allowing for a consistent treatment of neutrino-nucleus interactions. Within this approach, nuclear dynamics is described by a phenomenological hamiltonian, whose structure is completely determined by the available data on two- and three-nucleon systems, and both short and long range dynamical correlations are taken into account.

Over the past decade, the formalism based on correlated wave functions, originally proposed to describe quantum liquids [3], has been employed to carry out highly accurate calculations of the binding energies of both nuclei and nuclear matter, using either the Monte Carlo method [4, 5, 6] or the cluster expansion formalism and the Fermi Hypernetted Chain integral equations [7, 8, 9].

A different approach, recently proposed in Refs. [10, 11] exploits the correlated wave functions to construct an effective interaction suitable for use in standard perturbation theory. This scheme has been employed to obtain a variety of nuclear matter properties, including the neutrino mean free path [10] and the transport coefficients [11, 12].

In this work we describe the application of the formalism based on correlated wave functions and the effective interaction to the calculation of the weak response of uniform nuclear matter.
2. Many-body theory of the weak nuclear response

2.1. Neutrino-nucleus cross section

The differential cross section of the process

$$\nu_\ell + A \rightarrow \ell + X,$$

in which a neutrino carrying initial four-momentum \( k \equiv (E_\nu, \mathbf{k}) \) interacts with a nuclear target, producing a lepton in a state of four-momentum \( k' \equiv (E_\ell, \mathbf{k}') \), the target final state being undetected, can be written in Born approximation as (see, e.g., Ref. [16])

$$\frac{d\sigma}{d\Omega_\ell dE_\ell} = \frac{G^2}{32\pi^2} \frac{|k'|}{|k|} L_{\mu\nu} W^{\mu\nu},$$

where \( G = G_F \cos \theta_C \), \( G_F \) and \( \theta_C \) being Fermi’s coupling constant and Cabibbo’s angle. The leptonic tensor, that can be written, neglecting all lepton mass, as

$$L_{\mu\nu} = 8 \left[ k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(kk') - i\epsilon_{\mu\nu\alpha\beta}k^\alpha k'^\beta \right],$$

is completely determined by lepton kinematics, whereas the nuclear tensor \( W^{\mu\nu} \) contains all the information on target structure. Its definition involves the initial and final hadronic states \( |0\rangle \) and \( |X\rangle \), carrying four-momenta \( p_0 \) and \( p_X \), respectively, as well as the nuclear weak current operator \( J^{\mu} \):

$$W^{\mu\nu} = \sum_X \langle 0|J^{\mu}|X\rangle\langle X|J^{\nu}|0\rangle \delta^{(4)}(p_0 + q - p_X),$$

where the sum includes all hadronic final states.

In the low momentum transfer regime, we expect the non relativistic approximation to be applicable. Within this approach, the initial and final states can be obtained from nuclear many-body theory (NMBT), while the weak current entering the definition of the hadronic tensor [4] is expanded in powers of \(|q|/m\). At leading order, the resulting response can be written in the simple form

$$S(q, \omega) = \frac{1}{N} \sum_n \langle 0|O_{q}^\dagger|n\rangle\langle n|O_{q}|0\rangle \delta(\omega + E_0 - E_n).$$

where, in the case of charged current interactions, \( O_{q} \) is the operator corresponding to Fermi or Gamow-Teller transitions, whose expressions in the coordinate space are:

$$O_i^F(q) = g_V \delta(r_i - r'_i) \ e^{iqr_i \tau_i^+},$$

$$O_i^{GT}(q) = g_A \delta(r_i - r'_i) \ e^{iqr_i \sigma_i \tau_i^+},$$

where \( r_i \) specifies the position of the \( i \)-th particle.

2.2. Description of the initial and final states

Understanding the properties of matter at densities comparable to the central density of atomic nuclei is made difficult by both the complexity of the interactions and the approximations implied in any theoretical description of quantum mechanical many-particle systems.

The main problem associated with the use of the nuclear potential models in a many-body calculation lies in the strong repulsive core of the NN force, which cannot be handled within standard perturbation theory.
Within NMBT, a nuclear system is seen as a collection of point-like protons and neutrons whose dynamics are described by the Hamiltonian

\[ H = \sum_i t(i) + \sum_{j>i} v(ij) + \ldots , \]  

(7)

where \( t(i) \) and \( v(ij) \) denote the kinetic energy operator and the bare NN potential, respectively, while the ellipses refer to the presence of additional many-body interactions.

Carrying out perturbation theory in the basis provided by the eigenstates of the noninteracting system requires a renormalization of the NN potential. This is the foundation of the widely employed approach developed by Brückner, Bethe and Goldstone, in which \( v(ij) \) is replaced by the well-behaved G-matrix, describing NN scattering in the nuclear medium (see, e.g. Ref.[13]). Alternatively, the many-body Schrödinger equation, with the Hamiltonian of Eq.(7), can be solved using either the variational method or stochastic techniques. These approaches have been successfully applied to the study of both light nuclei [4] and uniform neutron and nuclear matter [7, 3, 6, 17].

Our work has been carried out using the scheme of correlated basis function (CBF) theory in which nonperturbative effects due to the short-range repulsion are embodied in the basis functions.

The correlated states of nuclear matter are obtained from the Fermi gas (FG) states \( |n_{\text{FG}}\rangle \) through the transformation \[ |n\rangle = \frac{F|n_{\text{FG}}\rangle}{\langle n_{\text{FG}}|F^{\dagger}F|n_{\text{FG}}\rangle^{1/2}}. \]  

(8)

In the above equation, \( |n_{\text{FG}}\rangle \) is a determinant of single particle states describing \( N \) noninteracting nucleons. The operator \( F \), embodying the correlation structure induced by the NN interaction, is written in the form

\[ F(1, \ldots, N) = S \prod_{j>i=1}^{N} f_{ij}, \]  

(9)

where \( S \) is the symmetrization operator which takes care of the fact that, in general,

\[ [f_{ij}, f_{ik}] \neq 0. \]  

(10)

The structure of the two-body correlation functions \( f_{ij} \) must reflect the complexity of the NN potential. The shape of the radial functions \( f^{n}(r_{ij}) \) is determined through functional minimization of the expectation value of the nuclear Hamiltonian in the correlated ground state

\[ E_{0}^{\text{V}} = \langle 0|H|0 \rangle. \]  

(11)

As an example, Fig.1 shows the radial dependence of the potentials and correlation functions acting in the spin-isospin channels \( S = 0 \) and \( T = 0 \) and \( S = 0 \) and \( T = 1 \).

It has to be pointed out that the correlation operator of Eq.(9) is defined such that, if any subset of the particles, say \( i_{1}, \ldots i_{p} \), is removed far from the remaining \( i_{p+1}, \ldots i_{N} \), it factorizes according to

\[ F(1, \ldots, N) \rightarrow F_{p}(i_{1}, \ldots i_{p})F_{N-p}(i_{p+1}, \ldots i_{N}). \]  

(12)

The above property is the basis of the cluster expansion formalism, that allows one to write the matrix element of a many-body operator between correlated states as a sum, whose terms correspond to contributions arising from isolated subsystems (clusters) involving an increasing number of particles.
3. Correlation effects in the transition matrix elements

Using correlated states implies severe difficulties in the explicit calculation of the weak matrix element. In the FG model, the nuclear response is non vanishing only when the final nuclear state differs from the initial state for the presence of a particle excited outside the Fermi sea and a hole in the Fermi sea. In the presence of correlations, which can induce virtual nucleon-nucleon scattering processes leading to excitation of nucleons to states outside the Fermi sea, more complex scenarios must also be considered. For example, if the initial state has a two particle-two hole component, the final state can be a three particle-three hole state or, if the probe interacts with an excited nucleon, a two particle-two hole state.

In the following we will consider only the dominant transition, between the correlated ground state and a correlated one particle-one hole (ph) state. The corresponding weak matrix element can be written

\[ M_{ph} = \frac{\langle ph | F^\dagger O | 0 \rangle}{\langle ph | F^\dagger F | ph \rangle^{\frac{1}{2}} \langle 0 | F^\dagger F | 0 \rangle^{\frac{1}{2}}} , \]  

(13)

where \( F \) is the correlation operator defined in Eq.(9). Here the kets \(|0\rangle\) and \(|ph\rangle\) correspond to the ground and one particle-one hole Fermi Gas states, respectively, and \( O = \sum_i O_i \), \( O_i \) being the Fermi or the Gamow-Teller transition operator (see Eqs.(6)).

Note the the \( g_{ij} = f_{ij} - 1 \) is short ranged, and therefore its matrix elements are small. At two-body level, the cluster expansion of Eq.(13) yields \[ \langle ph | F^\dagger OF | 0 \rangle \approx \langle ph | (1 + \sum_{j > i} g_{ij}) O (1 + \sum_{j > i} g_{ij}) | 0 \rangle . \]  

(14)

The above equation suggests the definition of an effective operator \( O_{12}^{\text{eff}} \), acting on Fermi Gas states. From

\[ \langle ph | F^\dagger OF | 0 \rangle = \langle ph | O_{12}^{\text{eff}} | 0 \rangle . \]  

(15)
it follows that, at the two-body cluster level (compare to Eq. (14))

\[
\frac{1}{N} O_{12}^{\text{eff}} = O_1 + \frac{N-1}{2} \{O_1 + O_2, g_{12}\} + \frac{N-1}{2} [g_{12}(O_1 + O_2)g_{12}],
\]

(16)

where \(\{A, B\} = AB + BA\) and \(N\) denotes the number of particles. Note that the \(O^{\text{eff}}\) is a two-body operator, as it includes screening effects arising from nucleon-nucleon correlations.

We have carried out the calculation of \(M_{ph}\) on a cubic lattice, for a discrete set of \(N_h\) states \(h_i\), satisfying the conditions \(h_i < k_f\) and \(h_i + q > k_f\), \(k_f\) being the Fermi momentum. Figure 2 shows a comparison between the correlated and the Fermi Gas matrix element for a Fermi transition. It has to be noticed that the effects of correlations is enhanced in the calculation of the response, see Eq. (5), whose definition involves the square of the correlated matrix element.

**Figure 2.** Fermi transition matrix element at \(|q| = 0.3 \text{ fm}^{-1}\) as a function of the magnitude of hole momentum \(|h|\). The dashed horizontal line corresponds to the result of the FG model.

### 4. The effective interaction

At lowest order of CBF, the effective interaction \(V_{\text{eff}}\) is defined by

\[
\langle H \rangle = \langle 0_{FG}|F^\dagger HF|0_{FG} \rangle = \langle 0_{FG}|T_0 + V_{\text{eff}}|0_{FG} \rangle.
\]

(17)

As the above equation suggests, the approach based on the effective interaction allows one to obtain any nuclear matter observables using perturbation theory in the FG basis. However, as discussed in the previous Section, the calculation of the hamiltonian expectation value in the correlated ground state, needed to extract \(V_{\text{eff}}\) from Eq. (17), involves severe difficulties. In this work we follow the procedure developed in Refs. [14, 10], whose authors derived the expectation value of the effective interaction by carrying out a cluster expansion of the rhs of Eq. (17), and keeping only the two-body cluster contribution. The resulting expression can be written

\[
\langle 0_{FG}|V_{\text{eff}}|0_{FG} \rangle = \sum_{i<j} \langle ij|v_{\text{eff}}(12)|ij \rangle_a = \sum_{i<j} \langle ij|f_{12} \left[ -\frac{1}{m} (\nabla^2 f_{12}) + v(12)f_{12} \right] |ij \rangle_a,
\]

(18)

where the laplacian operates on the relative coordinate and the suffix \(a\) denotes that the ket \(|ij\rangle_a\) is anti-symmetrized.

In the left panel of Fig. 3 the central components of the effective interaction obtained from the \(v_8'\) potential of Ref. [15] at equilibrium density is compared to the corresponding component...
5. Correlation effects on the response.

5.1. Short range correlations.
We have calculated the nuclear response for a Fermi transition (Eqs. 5 and 6). The inclusion of interaction leads to sizable modifications of the FG response. Correlation effects in the transition matrix elements, taken into account through the use of the effective operator, produce a quenching of \( \sim 15\% \). This feature is apparent in Fig. 4 where the FG response, represented by the green line, is compared to that obtained using the effective operator in the calculation of \( M_{ph} \) and the FG single particle spectrum, represented by the red line. An even larger modification is produced by interaction effects on the single particle energies. Replacing the FG single particle energies with the HF energies (Eq. 19) leads to a sizable broadening of \( \omega \) region corresponding to non-vanishing response as shown by the black line in Fig. 4.

5.2. Long range correlations: the Tamm-Dancoff Approximation (TDA)
In the previous section we have discussed the nuclear response in the correlated Hartree-Fock (HF) approximation, in which the bare Fermi transition operator is replaced by the effective operator of Eq. 15 and the final state is assumed to be a one particle-one hole state.

It is important to realize that the FG one particle-one hole states, while being eigenstates of the HF hamiltonian, defined as

\[
H_{HF} = \sum_i e_i ,
\]  

(20)
with $c_i$ given by Eq. (19), are not eigenstates of the full nuclear Hamiltonian. As a consequence, there is a residual interaction $V_{res}$, which can be identified with the effective interaction defined in Section 4 that can induce transitions between different one particle-one hole states, as long as their total momentum $q$, spin, and isospin are conserved.

In order to include the effects of these transitions, we use the TDA, which amounts to expanding the final state in the basis of one particle-one hole states according to

$$|f⟩ = |q, TSM⟩ = \sum_i c_i^{TSM}|h_i, p_i = h_i + q, TSM⟩,$$

where $S$ and $T$ denote the total spin and isospin of the particle hole pair and $M$ is the spin projection.

At fixed $q$, the excitation energy of the state (21), $\omega_f$, as well as the coefficients $c_i^{TSM}$, are obtained solving the eigenvalue equation

$$H|f⟩ = (H_{HF} + V_{res})|f⟩ = \left(\sum_i e_i + \sum_{ij} v_{eff}(ij)\right)|f⟩ = (E_0 + \omega_f)|f⟩,$$

where $E_0$ is the ground state energy. In the TDA, the response can be written as

$$S(q, \omega) = \sum_{TM, S} \sum_n \left| \sum_i (c_n^{TSM})_i(h_i, p_i, TSM|O_{eff}(q)|0⟩) \right|^2 \delta(\omega - \omega_n^{TSM}).$$

We have performed the diagonalization of the Hamiltonian operator over a basis of $N_h \sim 3000$ one particle-one hole states for any spin-isospin channel. The appearance of an eigenvalue lying well outside the particle hole continuum, corresponding to a collective excitation, reminiscent to the plasmon mode of the electron gas, is clearly visible in Fig. 5, where the calculation of the response for a Fermi transition within TDA is compared with the response of the correlated HF approximation. The TDA response exhibits a sharp isolated peak corresponding to the collective mode, lying $\sim 4$ MeV above the upper limit of the particle hole continuum.

6. Conclusions
We have carried out calculations of the charged current weak response of nuclear matter in the low momentum transfer regime. The quantitative understanding of this quantity is required in
Figure 5. Nuclear matter response in the TD approximation at $|q| = 0.3$ fm$^{-1}$ (solid line). The dashed lines show the results of the correlated HF approximation.

many problems of physics, ranging from simulations of supernovæ explosions to neutron star cooling.

The calculation has been performed using a many-body approach based on a realistic nuclear Hamiltonian, including two- and three-nucleon interactions, yielding a good description of the properties of both the two-nucleon systems and uniform nuclear matter. The response associated with Fermi transitions at low momentum transfer has been calculated from an effective interaction, derived using the formalism of correlated basis functions and the cluster expansion technique. Our work improves upon existing effective interaction models in that it includes the effects of many-nucleon forces, which become sizable, at high density. The responses calculated within the correlated HF approximation show that the inclusion of short range correlations leads to a significant quenching of the transition matrix elements and shifts the strength towards larger values of the energy transfer, $\omega$, for all values of $|q|$. At low momentum transfer, long range correlations have been taken into account within the TDA. The excitation of the coherent state can be clearly seen in our results at $|q| = 0.3$ fm$^{-1}$.

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