Strong convergence of the tamed and the semi-tamed Euler schemes for stochastic differential equations with jumps under non-global Lipschitz condition

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Abstract

We consider the explicit numerical approximations of stochastic differential equations (SDEs) driven by Brownian process and Poisson jumps. It is well known that under non-global Lipschitz condition, Euler Explicit method fails to converge strongly to the exact solution of such SDEs without jumps, while implicit Euler method converges but requires much computational efforts. Following the first idea on tamed methods in \cite{2}, we investigate the strong convergence of tamed Euler and semi-tamed methods for stochastic differential driven by Brownian process and Poisson jump, both in compensated and non compensated forms. We proved that under non-global Lipschitz condition and superlinearly growing of the drift term, these schemes converge strongly with the standard one-half order. Numerical simulations to sustain the theoretical results are provided.

\textbf{Keywords:} Stochastic differential equation, Euler schemes, Strong convergence, Poisson process, one-sided Lipschitz.
1. Introduction

In this work, we consider jump-diffusion Itô’s stochastic differential equations (SDEs) of the form:

\[ dX(t) = f(X(t^-))dt + g(X(t^-))dW(t) + h(X(t^-))dN(t), \quad X(0) = X_0, \quad t \in [0, T], \quad T > 0. \quad (1) \]

Here \( W_t \) is a \( m \)-dimensional Brownian motion, \( f : \mathbb{R}^d \to \mathbb{R}^d, d \in \mathbb{N} \) satisfies the one-sided Lipschitz condition and the polynomial growth condition, the functions \( g : \mathbb{R}^d \to \mathbb{R}^{d \times m} \) and \( h : \mathbb{R}^d \to \mathbb{R}^d \) satisfy the globally Lipschitz condition, and \( N_t \) is a one dimensional poisson process with parameter \( \lambda \). Extension to vector-valued jumps with independent entries is straightforward. The one-sided Lipschitz function \( f \) can be decomposed as \( f = u + v \), where the function \( u : \mathbb{R}^d \to \mathbb{R}^d \) is the global Lipschitz continuous part and \( v : \mathbb{R}^d \to \mathbb{R}^d \) is the non-global Lipschitz continuous part. Using this decomposition, we can rewrite the jump-diffusion SDEs (1) in the following equivalent form

\[ dX(t) = (u(X(t^-)) + v(X(t^-))) dt + g(X(t^-))dW(t) + h(X(t^-))dN(t). \quad (2) \]

This decomposition will be used only for semi-tamed schemes. Equations of type (1) arise in a range of scientific, engineering and financial applications [11, 10, 6]. Most of such equations do not have explicit solutions and therefore one requires numerical schemes for their approximations. Their numerical analysis has been studied in [4, 9, 12, 13] with implicit and explicit schemes where strong and weak convergence have been investigated. The implementation of implicit schemes requires significantly more computational effort than the explicit Euler-type approximations as Newton method is usually required to solve nonlinear systems at each time iteration in implicit schemes. The standard explicit method for approximating SDEs of type (1) is the Euler-Maruyama method [13]. Recently it has been proved (see [14, 1]) that the Euler-Maruyama method often fails to converge strongly to the exact solution of nonlinear SDEs of the form (1) without jump term when at least one of the functions \( f \) and \( g \) grows superlinearly. To overcome this drawback of the Euler-Maruyama method, numerical approximation which computational cost is close to that of the Euler-Maruyama method and which converge strongly even in the case the function \( f \) is superlinearly growing was first introduced in [2] and strong convergence was investigated. Further investigations have been performed in
the literature (see for example [15, 16, 8] and references therein), where in [15] the time step $\Delta t$ in [2] is replaced by his power $\Delta t^\alpha$, $\alpha \in (0, 1/2]$ in the denominator of the taming drift term. Recently the work in [15] has been extended for SDEs driven by compensated Levy noise in [3, 17].

In opposite to [3, 17], following the breakthrough idea in [2], we present new numerical schemes by extending their tamed scheme and the corresponding semi-tamed scheme developed in [19] to SDEs (1) driven by Brownian process and Poisson jump. Furthermore, we prove the strong convergence of the corresponding numerical approximations, both in compensated and non compensated forms. The extensions are not straightforward as several technical lemmas are needed. The linear and nonlinear mean-square stabilities of these schemes are provided in the accompanied paper [22].

The paper is organized as follows. Section 2 presents the classical result of existence and uniqueness of the solution $X$ of (1). The compensated and non compensated tamed schemes and semi-tamed scheme are presented in Section 3. The proof of strong convergence of compensated tamed scheme is provided at Section 4, while the proof of strong convergence of semi-tamed scheme and non compensated tamed scheme are provided respectively in Section 5 and Section 6. We end in Section 7 by providing some numerical simulations to sustain our theoretical results.

2. Notations, assumptions and well posedness

Throughout this work, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$. For all $x, y \in \mathbb{R}^d$, we denote by $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_d y_d$, $\|x\| = \langle x, x \rangle^{1/2}$, $\|A\| = \sup_{x \in \mathbb{R}^d, \|x\| \leq 1} \|Ax\|$ for all $A \in \mathbb{R}^{m \times d}$ and $\|X\|_{L^p(\Omega, \mathbb{R})} := (\mathbb{E}\|X\|^p)^{1/p}$, for all $p \in [1, +\infty)$ and all $(\mathcal{F}_t)$-adapted process $X$. We use also the following convention: $\sum_{i=u}^{n} = 0$ for $u > n$. $a \lor b = \max(a, b)$ for all $a, b \in \mathbb{R}$.

We first ensure that SDEs (1) is well posed. The following assumption is needed.

**Assumption 2.1.** We assume that:

(A.1) For all $p > 0$, there exists $M_p > 0$ such that $\mathbb{E}\|X_0\| \leq M_p$, and $f, g, h \in C^1(\mathbb{R}^d)$. 


(A.2) The functions $g$, $h$ and $u$ satisfy the following global Lipschitz condition

$$\|g(x) - g(y)\| \vee \|h(x) - h(y)\| \vee \|u(x) - u(y)\| \leq C \|x - y\| \quad \forall \ x, y \in \mathbb{R}^d.$$ 

(A.3) The function $f$ satisfies the following one-sided Lipschitz condition

$$\langle x - y, f(x) - f(y) \rangle \leq C \|x - y\|^2 \quad \forall \ x, y \in \mathbb{R}^d.$$ 

(A.4) The function $f$ satisfies the following superlinear growth condition

$$\|f(x) - f(y)\| \leq C(1 + \|x\|^c + \|y\|^c) \|x - y\| \quad \forall \ x, y \in \mathbb{R}^d,$$

where $C$ and $c$ are positives constants.

Remark 2.1. Note that from Assumption 2.1, $u$ satisfies the global Lipschitz condition, and $f$ satisfies the one-sided Lipschitz condition and the superlinear growth condition, which implies that the function $v$ satisfies the one-sided Lipschitz condition (A.3) and the superlinear growth condition (A.4) in Assumption 2.1.

Theorem 2.1. Under the conditions (A.1), (A.2) and (A.3) of Assumption 2.1, the SDEs (1) has an unique solution with all bounded moments.

Proof. See [4, Lemma 1].

3. Numerical Schemes and main results

We consider the SDEs (1) in the current non compensated form. Applying the tamed Euler scheme (as in [2]) in the drift term of (1) yields the following scheme that we will call non compensated tamed scheme (NCTS)

$$X_{n+1}^M = X_n^M + \frac{\Delta t f(X_n^M)}{1 + \Delta t \|f(X_n^M)\|} + g(X_n^M) \Delta W_n^M + h(X_n^M) \Delta N_n^M,$$

where $\Delta t = T/M$ is the time step-size, $M \in \mathbb{N}$ is the number of time subdivisions, $\Delta W_n^M = W(t_{n+1}) - W(t_n)$ and $\Delta N_n^M = N(t_{n+1}) - N(t_n)$. Applying the semi-tamed Euler scheme (as in [19]) in the non globally Lipschitz part $v$ of the drift term of (2) yields the following scheme that we will call semi-tamed scheme (STS)

$$Z_{n+1}^M = Z_n^M + u(Z_n^M) \Delta t + \frac{\Delta t v(Z_n^M)}{1 + \Delta t \|v(Z_n^M)\|} + g(Z_n^M) \Delta W_n + h(Z_n^M) \Delta N_n^M.$$
Recall that the compensated poisson process $N(t) := N(t) - \lambda t$ is a martingale and satisfies the following properties

$$\mathbb{E}(N(t+s) - N(t)) = 0 \quad \mathbb{E}|N(t+s) - N(t)|^2 = \lambda s, \quad s, t \geq 0.$$  \hspace{1cm} (5)

We can easily check that $N^2(t) - \lambda t$ is also a martingale, and therefore its quadratic variation is given by $[N, N]_t = \lambda t$.

We can therefore rewrite the jump-diffusion SDEs (1) in the following equivalent form

$$dX(t) = f_{\lambda}(X(t^-))dt + g(X(t^-))dW(t) + h(X(t^-))dN(t),$$  \hspace{1cm} (6)

where $f_{\lambda}(x) = f(x) + \lambda h(x)$. Note that as $f$, the function $f_{\lambda}$ satisfies the one-sided Lipschitz condition (A.3) and the superlinear growth (A.4). Applying the tamed Euler scheme in the drift term of (6) as in [2] yields the following updated scheme for jump SDEs (1) that we will call compensated tamed scheme (CTS)

$$Y_{n+1}^M = Y_n^M + \frac{\Delta t f_{\lambda}(Y_n^M)}{1 + \Delta t \|f_{\lambda}(Y_n^M)\|} + g(Y_n^M)\Delta W_n + h(Y_n^M)\Delta N_n^M,$$  \hspace{1cm} (7)

where $\Delta N_n^M = N(t_{n+1}) - N(t_n)$.

Note that if the equivalent model (2) is putting in the compensated form, and the semi-tamed is applied on the non globally Lipschitz part $v$ of the drift term $f$, we will obtain the same scheme as in (1).

We define the continuous time interpolations of the discrete numerical approximations of (3), (1) and (7) respectively by

$$\bar{X}_t^M = X_n^M + \frac{(t - n\Delta t)f(X_n^M)}{1 + \Delta t \|f(X_n^M)\|} + g(X_n^M)(W_t - W_n) + h(X_n^M)(N_t - N_n),$$  \hspace{1cm} (8)

$$\bar{Z}_t^M = Z_n^M + (t - n\Delta t) \left( u(Z_n^M) + \frac{v(Z_n^M)}{1 + \Delta t \|v(Z_n^M)\|} \right) + g(Z_n^M)(W_t - W_n) + h(Z_n^M)(N_t - N_n),$$  \hspace{1cm} (9)

and

$$\bar{Y}_t^M = Y_n^M + \frac{(t - n\Delta t)f_{\lambda}(Y_n^M)}{1 + \Delta t \|f_{\lambda}(Y_n^M)\|} + g(Y_n^M)(W_t - W_n) + h(Y_n^M)(N_t - N_n),$$  \hspace{1cm} (10)
for all \( t \in [n\Delta t, (n+1)\Delta t), \ n \in \{0, \cdots, M-1\} \).

The main result of this work is given in the following theorem.

**Theorem 3.1.** [Main result]

Let \( X_t \) be the exact solution of (1) and \( \chi^M_t \) the discrete continuous form of the numerical approximations given by (8), (9) and (10) (\( \chi^M_t = X^M_t \) for scheme NCTS, \( \chi^M_t = Z^M_t \) for scheme STS and \( \chi^M_t = \Upsilon^M_t \) for scheme CTS). Under Assumption 2.1, for all \( p \in [1, +\infty) \) there exists a constant \( C_p > 0 \) such that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \| X_t - \chi^M_t \|^p \right]^{1/p} \leq C_p \Delta t^{1/2}, \quad \Delta t = T/M. \tag{11}
\]

### 4. Proof of Theorem 3.1 for \( \chi^M_t = \Upsilon^M_t \)

Before giving the proof of Theorem 3.1 some preparatory results are needed. Here we consider the compensated tamed scheme (CTS) given by (7).

#### 4.1. Preparatory results

Throughout this work the following notations will be used with slight modification in the next section

\[
\begin{align*}
\alpha^M_k &:= \mathbb{1}_{\{\|Y^M_k\| \geq 1\}} \left( \frac{Y^M_k}{\|Y^M_k\|} \Delta W^M_k \right), \\
\beta^M_k &:= \mathbb{1}_{\{\|Y^M_k\| \geq 1\}} \left( \frac{Y^M_k}{\|Y^M_k\|} \Delta N^M_k \right), \\
\beta &:= (1 + K + 2C + KTC + TC + T + \|f(0)\| + \|g(0)\| + \|h(0)\|)^4, \\
D^M_n &:= (\beta + \|X_0\|) \exp \left( \frac{3\beta}{2} + \sup_{u \in \{0, \cdots, n\}} \sum_{k=u}^{n-1} \left[ \frac{3\beta}{2} \|\Delta W^M_k\|^2 + \frac{3\beta}{2} \|\Delta N^M_k\| + \alpha^M_k + \beta^M_k \right] \right), \\
\Omega^M_n &:= \left\{ \omega \in \Omega : \sup_{k \in \{0,1, \cdots, n-1\}} \sup_{k \in \{0,1, \cdots, n-1\}} \left( \|\Delta W^M_k(\omega)\| \vee |\Delta N^M_k(\omega)| \right) \leq 1 \right\}.
\end{align*}
\]  

The aims of this section is to update all lemmas used in [2] and provide typically lemmas for Poisson jump.

**Lemma 4.1.** For all positive real numbers \( a \) and \( b \), the following inequality holds

\[ 1 + a + b^2 \leq e^{a+\sqrt{b}}. \]
Proof. For \( a \geq 0 \) fixed, let’s define the function \( l(b) = e^{a+\sqrt{2}b} - 1 - a - b^2 \). It can be easily checked that \( l'(b) = \sqrt{2}e^{a+\sqrt{2}b} - 2b \) and \( l''(b) = 2(e^{a+\sqrt{2}b} - 1) \). Since \( a \) and \( b \) are positive, it follows that \( l''(b) \geq 0 \) for all \( b \geq 0 \). So \( l' \) is a non-decreasing function. Therefore, \( l'(b) \geq l'(0) = \sqrt{2}e^a > 0 \) for all \( b \geq 0 \). This implies that \( l \) is a non-decreasing function. Hence \( l(b) \geq l(0) = e^a - 1 - a \) for all \( b \geq 0 \). Since \( 1 + a \leq e^a \) for all positive number \( a \), it follows that \( l(b) \geq 0 \) for all positive number \( b \), so \( 1 + a + b^2 \leq e^{a+\sqrt{2}b}, \forall b \geq 0 \). Therefore for all \( a \geq 0 \) fixed, \( 1 + a + b^2 \leq e^{a+\sqrt{2}b}, \forall b \geq 0 \). \( \square \)

Following closely [2, Lemma 3.1], we have the following main lemma.

**Lemma 4.2.** The following inequality holds for all \( M \in \mathbb{N} \) and all \( n \in \{0, 1, \cdots, M\} \)

\[
1_{\Omega_n^M} \|Y_n^M\| \leq D_n^M, \tag{13}
\]

where \( D_n^M \) and \( \Omega_n^M \) are given in (12).

**Proof.** As \( \frac{\Delta t}{1 + \Delta t\|f_{\lambda(x)}\|} \leq T \), using Assumption 2.1 on the functions \( g, h \) and \( f_{\lambda} \), following closely [2], the following estimation holds on \( \Omega_{n+1}^M \cap \{\omega \in \Omega : \|Y_n^M(\omega)\| \leq 1\} \), for all \( n \in \{0, 1, \cdots, M-1\} \)

\[
\|Y_{n+1}^M\| \leq \|Y_n^M\| + \frac{\Delta t\|f_{\lambda(Y_n^M)}\|}{1 + \Delta t\|f_{\lambda(Y_n^M)}\|} + \|g(Y_n^M)\|\|\Delta W_n^M\| + \|h(Y_n^M)\|\|\Delta \mathcal{N}_n^M\|
\leq \|Y_n^M\| + T\|f_{\lambda(Y_n^M)} - f_{\lambda}(0)\| + T\|f_{\lambda}(0)\| + \|g(Y_n^M) - g(0)\| + \|g(0)\|
+ \|h(Y_n^M) - h(0)\| + \|h(0)\|
\leq \|Y_n^M\| + TC(K + \|Y_n^M\|c + \|0\|c)\|Y_n^M - 0\| + T\|f_{\lambda}(0)\| + C\|Y_n^M\| + C\|Y_n^M\|
+ \|g(0)\| + \|h(0)\|.
\]

On \( \Omega_{n+1}^M \cap \{\omega \in \Omega : \|Y_n^M(\omega)\| \leq 1\} \), we therefore have

\[
\|Y_{n+1}^M\| \leq 1 + KTC + TC + 2C + T\|f_{\lambda}(0)\| + \|g(0)\| + \|h(0)\| \leq \beta. \tag{14}
\]

Using Cauchy-Schwartz inequality and Hölder inequality, after some simplifications, we have

\[
\|Y_{n+1}^M\|^2 \leq \|Y_n^M\|^2 + 3\Delta t^2\|f_{\lambda(Y_n^M)}\|^2 + 3\|g(Y_n^M)\|^2\|\Delta W_n^M\|^2 + 3\|h(Y_n^M)\|^2\|\Delta \mathcal{N}_n^M\|^2
+ 2\Delta t \langle Y_n^M, f_{\lambda(Y_n^M)} \rangle + 2\langle Y_n^M, g(Y_n^M)\Delta W_n^M \rangle
+ 2\langle Y_n^M, h(Y_n^M)\Delta \mathcal{N}_n^M \rangle. \tag{15}
\]
on $\Omega$, for all $M \in \mathbb{N}$ and all $n \in \{0, 1, \ldots, M - 1\}$.

Using Assumption 2.1 we can easily prove (see [2]) that for all $x \in \mathbb{R}^d$ such that $\|x\| \geq 1$ we have

$$
\begin{align*}
\|g(x)\|^2 &\leq \beta \|x\|^2 \\
\|h(x)\|^2 &\leq \beta \|x\|^2 \\
\langle x, f_\lambda(x) \rangle &\leq \sqrt{\beta} \|x\|^2 \\
\|f_\lambda(x)\|^2 &\leq M \sqrt{\beta} \|x\|^2.
\end{align*}
$$

(16)

Using (16) in (15) yields

$$
\begin{align*}
\|Y_{n+1}^M\|^2 &\leq \|Y_n^M\|^2 + \frac{3T^2 \sqrt{\beta}}{M} \|Y_n^M\|^2 + 3\beta \|Y_n^M\|^2 \|\Delta W_n^M\|^2 + 3\beta \|Y_n^M\|^2 \|\triangle N_n^M\|^2 \\
&+ \frac{2T \sqrt{\beta}}{M} \|Y_n^M\|^2 + 2 \langle Y_n^M, g(Y_n^M) \Delta W_n^M \rangle + 2 \langle Y_n^M, h(Y_n^M) \Delta N_n^M \rangle \\
&\leq \|Y_n^M\|^2 + \left( \frac{3T^2 + 2T}{M} \right) \|Y_n^M\|^2 + 3\beta \|Y_n^M\|^2 \|\Delta W_n^M\|^2 + 3\beta \|Y_n^M\|^2 \|\Delta N_n^M\|^2 \\
&+ 2 \langle Y_n^M, g(Y_n^M) \Delta W_n^M \rangle + 2 \langle Y_n^M, h(Y_n^M) \Delta N_n^M \rangle.
\end{align*}
$$

(17)

Using the fact that $3T^2 + 2T \leq 3 \sqrt{\beta}$, it follows that

$$
\begin{align*}
\|Y_{n+1}^M\|^2 &\leq \|Y_n^M\|^2 + \frac{3\beta}{M} \|Y_n^M\|^2 + 3\beta \|Y_n^M\|^2 \|\Delta W_n^M\|^2 + 3\beta \|Y_n^M\|^2 \|\triangle N_n^M\|^2 \\
&+ 2 \langle Y_n^M, g(Y_n^M) \Delta W_n^M \rangle + 2 \langle Y_n^M, h(Y_n^M) \Delta N_n^M \rangle \\
&= \|Y_n^M\|^2 \left( 1 + \frac{3\beta}{M} + 3\beta \|\Delta W_n^M\|^2 + 3\beta \|\Delta N_n^M\|^2 \right) + 2 \left\langle \frac{Y_n^M}{\|Y_n^M\|}, \frac{g(Y_n^M)}{\|Y_n^M\|} \Delta W_n^M \right\rangle \\
&+ 2 \left\langle \frac{Y_n^M}{\|Y_n^M\|}, \frac{h(Y_n^M)}{\|Y_n^M\|} \Delta N_n^M \right\rangle \\
&= \|Y_n^M\|^2 \left( 1 + \frac{3\beta}{M} + 3\beta \|\Delta W_n^M\|^2 + 3\beta \|\Delta N_n^M\|^2 + 2\alpha_n + 2\beta_n \right).
\end{align*}
$$

(18)

Using Lemma [11] for $a = \frac{3\beta}{M} + 3\beta \|\Delta W_n^M\|^2 + 2\alpha_n + 2\beta_n$ and $b = \sqrt{3\beta} \|\Delta N_n^M\|$ it follows from (18) that :

$$
\|Y_{n+1}^M\|^2 \leq \|Y_n^M\|^2 \exp \left( \frac{3\beta}{M} + 3\beta \|\Delta W_n^M\|^2 + 3\beta \|\Delta N_n^M\|^2 + 2\alpha_n + 2\beta_n \right)
$$

(19)

on $\{w \in \Omega : 1 \leq \|Y_n^M(w)\| \leq M^{1/2c} \}$, for all $M \in \mathbb{N}$ and all $n \in \{0, 1, \ldots, M - 1\}$.

Our proof is concluded by induction exactly as in [2, Lemma 3.1]. Details can be found in [21].
The proofs of the following lemmas can be found in [2, 21].

**Lemma 4.3.** The following inequality holds

\[
\sup_{M \in \mathbb{N}, M \geq 4\beta p T} \mathbb{E} \left[ \exp \left( \beta p \sum_{k=0}^{M-1} \|\Delta W_k^M\|^2 \right) \right] < \infty, \quad \forall p \in [1, \infty).
\]

**Lemma 4.4.** Let \( \alpha_n^M : \Omega \to \mathbb{R}, M \in \mathbb{N}, n \in \{0, 1, \ldots, M\} \) be the process defined in (12).

The following inequality holds

\[
\sup_{z \in \{-1, 1\}} \sup_{M \in \mathbb{N}} \left\| \sup_{n \in \{0, 1, \ldots, M\}} \exp \left( z \sum_{k=0}^{n-1} \alpha_k^M \right) \right\|_{\mathbb{L}^p(\Omega, \mathbb{R})} < \infty,
\]

for all \( p \in [2, +\infty) \).

**Lemma 4.5.** Let \( c \in \mathbb{R} \), the following equality holds

\[
\mathbb{E}[\exp(c \Delta N_n^M)] = \exp \left( \frac{(e^c - 1) \lambda T}{M} \right)
\]

for all \( M \in \mathbb{N} \) and all \( n \in \{0, \cdots, M\} \).

**Proof.** From the moment generating function of a poisson process \( Y \) with parameter \( \lambda \), we have

\[
\mathbb{E}[\exp(c Y)] = \exp(\lambda(e^c - 1)).
\]

Since \( \Delta N_n \) follows a poisson law with parameter \( \lambda \Delta t \), it follows that

\[
\mathbb{E}[\exp(c \Delta N_n^M)] = \mathbb{E}[\exp(c \Delta N_n^M + c \lambda t)]
\]

\[
= \mathbb{E} \left[ \exp \left( \frac{c \lambda T}{M} \right) \exp(c \Delta N_n^M) \right]
\]

\[
= \exp \left( \frac{c \lambda T}{M} \right) \exp \left( \frac{\lambda T}{M} (e^c - 1) \right)
\]

\[
= \exp \left( \frac{(e^c + c - 1) \lambda T}{M} \right).
\]

\[\square\]

**Lemma 4.6.** The following inequality holds

\[
\mathbb{E} \left[ \exp \left( \sum_{\|x\| \geq 1} \left( x, \frac{h(x)}{\|x\|} \Delta N_n^M \right) \right) \right] \leq \exp \left[ \frac{\lambda \left( e^{p(C + \|h(0)\|)} + p(C + \|h(0)\|) \right)}{M} \right],
\]

for all \( M \in \mathbb{N}, z \in \{-1, 1\}, \) all \( p \in [1, +\infty) \) and all \( n \in \{0, \cdots, M\} \).
Proof. For $x \in \mathbb{R}^d$ such that $\|x\| \neq 0$, we have
\[
\mathbb{E} \left[ \exp \left( p z \left( \frac{x}{\|x\|}, \frac{h(x)}{\|x\|} \Delta N_n^M \right) \right) \right] \leq \mathbb{E} \left[ \exp \left( p z \frac{\|h(x)\|}{\|x\|^2} \Delta N_n^M \right) \right] = \mathbb{E} \left[ \exp \left( p z \frac{\|h(x)\|}{\|x\|} \Delta N_n^M \right) \right].
\]
For all $x \in \mathbb{R}^d$ such that $\|x\| \geq 1$, since $h$ satisfied the global Lipschitz condition $h$, we have
\[
\frac{\|h(x)\|}{\|x\|} \leq \frac{\|h(x) - h(0)\| + \|h(0)\|}{\|x\|} \leq C + \|h(0)\|. \quad (20)
\]
So from inequality (20) and using Lemma 4.5 it follows that
\[
\mathbb{E} \left[ \exp \left( p z 1_{\{|x| \geq 1\}} \left( \frac{x}{\|x\|}, \frac{h(x)}{\|x\|} \Delta N_n^M \right) \right) \right] \leq \mathbb{E} \left[ \exp (p z (C + \|h(0)\|) \Delta N_n^M) \right]
\leq \exp \left( \frac{(e^{p(C+\|h(0)\|)} + p(C + \|h(0)\|) - 1) \lambda T}{M} \right)
\leq \exp \left( \frac{(e^{p(C+\|h(0)\|)} + p(C + \|h(0)\|) \lambda T}{M} \right).
\]

Lemma 4.7. Let $\beta_n^M : \Omega \rightarrow \mathbb{R}$ be the process defined in (12) for all $M \in \mathbb{N}$ and all $n \in \{0, \ldots, M\}$. The following inequality holds
\[
\sup_{z \in \{-1, 1\}} \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \mathbb{E} \left[ \exp \left( z \sum_{k=0}^{n-1} \beta_k^M \right) \right] \leq +\infty, \quad \forall p \in (1, \infty).
\]

Proof. The time discrete stochastic process $z \sum_{k=0}^{n-1} \beta_k^M$ is an $(\mathcal{F}_{nT/M})_{n \in \{0, \ldots, M\}}^+$ martingale for every $z \in \{-1, 1\}$ and every $M \in \mathbb{N}$. So $\exp \left( p z \sum_{k=0}^{n-1} \beta_k^M \right)$ is a positive $(\mathcal{F}_{nT/M})^+$ submartingale for all $M \in \mathbb{N}$, all $z \in \{-1, 1\}$ and all $n \in \{0, \ldots, M\}$. Using Doob’s maximal inequality we have:
\[
\left\| \sup_{n \in \{0, \ldots, M\}} \exp \left( z \sum_{k=0}^{n-1} \beta_k^M \right) \right\|_{L^p(\Omega, \mathbb{R})} \leq \left( \frac{p}{p-1} \right) \left\| \exp \left( z \sum_{k=0}^{M-1} \beta_k^M \right) \right\|_{L^p(\Omega, \mathbb{R})}, \quad (21)
\]
\[
\left\| \exp \left( z \sum_{k=0}^{M-1} \beta_k^M \right) \right\|_{L^p(\Omega, \mathbb{R})}^p = \mathbb{E} \left[ \exp \left( p z \sum_{k=0}^{M-1} \beta_k^M \right) \right] = \mathbb{E} \left[ \exp \left( p z (\sum_{k=0}^{M-2} \beta_k^M + p \beta_{M-1}^M) \right) \right]
= \mathbb{E} \left[ \exp \left( p z \sum_{k=0}^{M-2} \beta_k^M \right) \mathbb{E} \left[ \exp \left( p z \beta_{M-1}^M / \mathcal{F}_{(M-1)T/M} \right) \right] \right].
\]
Using Lemma 4.6 it follows that
\[
\| \exp \left( z \sum_{k=0}^{M-1} \beta_k^M \right) \|_{L^p(\Omega, R)}^p \leq E \left[ \exp \left( p z \sum_{k=0}^{M-2} \beta_k^M \right) \right] \exp \left[ \frac{\left( e^{p(C+\|h(0)\|)} + p(C + \|h(0)\|) \right) \lambda T}{M} \right].
\]

Iterating this last inequality \(M\) times leads to
\[
E \left[ \exp \left( p z \sum_{k=0}^{M-1} \beta_k^M \right) \right] \leq \exp \left[ \lambda T \left( e^{p(C+\|h(0)\|)} + Tp(C + \|h(0)\|) \right) \right],
\]
for all \(M \in \mathbb{N}\), all \(p \in (1, \infty)\) and all \(z \in \{-1, 1\}\).

Combining inequalities (21) and (22) completes the proof of Lemma 4.7. \(\square\)

**Lemma 4.8.** The following inequality holds
\[
\sup_{M \in \mathbb{N}} E \left[ \exp \left( p \beta \sum_{k=0}^{M-1} |\Delta N_k^M| \right) \right] < +\infty,
\]
for all \(p \in [1, +\infty)\).

**Proof.** Using the independence and the stationarity of \(\Delta N_k^M\), along with Lemma 4.5, it follows that
\[
\sup_{M \in \mathbb{N}} E \left[ \exp \left( p \beta \sum_{k=0}^{M-1} |\Delta N_k^M| \right) \right] = \prod_{k=0}^{M-1} E[\exp(p\beta|\Delta N_k^M|)]
\]
\[
= \left( E[\exp(p\beta|\Delta N_k^M|)] \right)^M
\]
\[
= \left( \exp \left[ \frac{(e^{p\beta} + p\beta - 1)\lambda T}{M} \right] \right)^M
\]
\[
= \exp[pT(e^{p\beta} + p\beta - 1)] < +\infty,
\]
for all \(p \in [1, +\infty)\). \(\square\)

Inspired by \cite[Lemma 3.5]{2}, we have the following estimation.

**Lemma 4.9.** [Uniformly bounded moments of the process \(D_n^M\)]

Let \(D_n^M : \Omega \rightarrow [0, \infty), M \in \mathbb{N}, n \in \{0, 1, \ldots, M\}\) be the process defined in (12), then we have
\[
\sup_{M \in \mathbb{N}, M \geq 8\lambda p T} \left\| D_n^M \right\|_{L^p(\Omega, R)} < \infty,
\]
for all \(p \in [1, \infty)\).
Proof. Recall that
\[ D_n^M = (\beta + \| X_0 \|) \exp \left( \frac{3\beta}{2} + \sup_{u \in \{0, \ldots, M\}} \sum_{k=u}^{n-1} \frac{3\beta}{2} \| \Delta W_k^M \|^2 + \frac{3\beta}{2} \| \Delta N_k^M \| + \alpha_k^M + \beta_k^M \right). \]
Using Hölder’s inequality, it follows that
\[
\sup_{M \in \mathbb{N}, M \geq 8\lambda pT} \left\| \sup_{n \in \{0, \ldots, M\}} D_n^M \right\|_{L_p(\Omega, \mathbb{R}^d)} \leq e^{3\beta/2} (\beta + \| X_0 \|_{L_{4p}(\Omega, \mathbb{R})})
\times \sup_{M \in \mathbb{N}, M \geq 8\lambda pT} \left\| \exp \left( \frac{3\beta}{2} \sum_{k=0}^{M-1} \| \Delta W_k^M \|^2 \right) \right\|_{L_{4p}(\Omega, \mathbb{R})}
\times \sup_{M \in \mathbb{N}} \left\| \exp \left( \frac{3\beta}{2} \sum_{k=0}^{M-1} | \Delta N_k^M | \right) \right\|_{L_{4p}(\Omega, \mathbb{R})}
\times \left( \sup_{M \in \mathbb{N}} \left\| \sup_{n \in \{0, \ldots, M\}} \exp \left( \sup_{u \in \{0, \ldots, n\}} \sum_{k=u}^{n-1} \alpha_k^M \right) \right\|_{L_{4p}(\Omega, \mathbb{R})} \right)
\times \left( \sup_{M \in \mathbb{N}} \left\| \sup_{n \in \{0, \ldots, M\}} \exp \left( \sup_{u \in \{0, \ldots, n\}} \sum_{k=u}^{n-1} \beta_k^M \right) \right\|_{L_{4p}(\Omega, \mathbb{R})} \right)
= A_1 \times A_2 \times A_3 \times A_4 \times A_5.
\]
By assumption $A_1$ is bounded. Lemmas 4.3 and 4.8 show that $A_2$ and $A_3$ are bounded. Using again Hölder’s inequality and Lemma 4.4, it follows that
\[ A_4 = \left\| \sup_{n \in \{0, \ldots, M\}} \exp \left( \sup_{u \in \{0, \ldots, n\}} \sum_{k=u}^{n-1} \alpha_k^M \right) \right\|_{L_{4p}(\Omega, \mathbb{R})}, \]
\[ \leq \left\| \sup_{n \in \{0, \ldots, M\}} \exp \left( \sum_{k=0}^{n-1} \alpha_k^M \right) \right\|_{L_{4p}(\Omega, \mathbb{R})} \times \left\| \sup_{u \in \{0, \ldots, M\}} \exp \left( - \sum_{k=0}^{u-1} \alpha_k^M \right) \right\|_{L_{4p}(\Omega, \mathbb{R})} < +\infty, \]
for all $M \in \mathbb{N}$ and all $p \in [1, \infty)$.
Along the same lines as above, we prove that $A_5$ is bounded.
Since each of the terms $A_1, A_2, A_3, A_4$ and $A_5$ is bounded, this complete the proof of Lemma 4.9.

The following lemma is an extension of [2, Lemma 3.6]. Here, we include the jump part.
Lemma 4.10. Let $\Omega^M_M \in \mathcal{F}$, $M \in \mathbb{N}$ be the process defined in \cite{12}. The following holds

$$\sup_{M \in \mathbb{N}} (M^p \mathbb{P}[(\Omega^M_M)^c]) < +\infty,$$

for all $p \in [1, \infty)$.

Proof. Using the subadditivity of the probability measure and the Markov’s inequality, it follows that

\[
\mathbb{P}[(\Omega^M_M)^c] \leq \mathbb{P} \left[ \sup_{n \in \{0, \ldots, M-1\}} D^M_n > M^{1/2c} \right] + M \mathbb{P} [\|W_T/M\| > 1] + M \mathbb{P} [\|N_T\| > 1] \\
\leq \mathbb{P} \left[ \sup_{n \in \{0, \ldots, M-1\}} |D^M_n| > M^{q/2c} \right] + M \mathbb{P} [\|W_T\| > \sqrt{M}] + M \mathbb{P} [\|N_T\| > M] \\
\leq \mathbb{E} \left[ \sup_{n \in \{0, \ldots, M-1\}} |D^M_n|^q \right] M^{-q/2c} + \mathbb{E}[\|W_T\|^q] M^{1-q/2} + \mathbb{E}[\|N_T\|^q] M^{1-q},
\]

for all $q > 1$.

Multiplying both sides of the above inequality by $M^p$ leads to

$$M^p \mathbb{P}[(\Omega^M_M)^c] \leq \mathbb{E} \left[ \sup_{n \in \{0, \ldots, M-1\}} |D^M_n|^q \right] M^{p-q/2c} + \mathbb{E}[\|W_T\|^q] M^{p+1-q/2} + \mathbb{E}[\|N_T\|^q] M^{p+1-q}$$

for all $q > 1$.

For $q > \max\{2pc, 2p+2\}$, we have $M^{p+1-q/2} < 1$, $M^{p-q/2c} < 1$ and $M^{p+1-q} < 1$. It follows for this choice of $q$ that

$$M^p \mathbb{P}[(\Omega^M_M)^c] \leq \mathbb{E} \left[ \sup_{n \in \{0, \ldots, M-1\}} |D^M_n|^p \right] + \mathbb{E}[\|W_T\|^q] + \mathbb{E}[\|N_T\|^q].$$

Using Lemma 4.9 and the fact that $W_T$ and $N_T$ are independents of $M$, it follows that

$$\sup_{M \in \mathbb{N}} (M^p \mathbb{P}[(\Omega^M_M)^c]) < +\infty.$$

The following lemma can be found in \cite{7, Theorem 48 pp 193} or in \cite{20, Theorem 1.1, pp 1].
Lemma 4.11. [Burkholder-Davis-Gundy inequality]

Let $M$ be a martingale with càdlàg paths and let $p \geq 1$ be fixed. Let $M^*_t = \sup_{s \leq t} \|M_s\|$. Then there exist constants $c_p$ and $C_p$ such that

$$c_p \left[ \mathbb{E} \left( \left[ M, M^*_t \right]^{p/2} \right) \right]^{1/p} \leq \mathbb{E}(M^*_t)^{p/2} \leq C_p \left[ \mathbb{E} \left( \left[ M, M_t^* \right]^{p/2} \right) \right]^{1/p},$$

for all $0 \leq t \leq \infty$, where $[M, M]_t$ stand for the quadratic variation of the process $M$.

The proof of the following lemma can be found in [2, Lemma 3.7] or [21].

Lemma 4.12. Let $k \in \mathbb{N}$ and let $Z : [0, T] \times \Omega \rightarrow \mathbb{R}^{k \times m}$ be a predictable stochastic process satisfying $\mathbb{P} \left[ \int_0^T \|Z_s\|^2 ds < +\infty \right] = 1$. Then we have the following inequality

$$\left\| \sup_{s \in [0,t]} \left\| \int_0^s Z_u dW_u \right\| \right\|_{L^p(\Omega, \mathbb{R})} \leq C_p \left( \int_0^t \sum_{i=1}^m \|Z_s \hat{e}_i \|^2_{L^p(\Omega, \mathbb{R}^k)} ds \right)^{1/2},$$

for all $t \in [0, T]$ and all $p \in [1, \infty)$, where $(\hat{e}_1, \cdots, \hat{e}_m)$ is the canonical basis of $\mathbb{R}^m$.

The following lemma can be found in [2, Lemma 3.8, pp 16] or [21].

Lemma 4.13. Let $k \in \mathbb{N}$ and let $Z^l_t \in \Omega \rightarrow \mathbb{R}^{k \times m}$, $l \in \{0, 1, \cdots, M - 1\}$, $M \in \mathbb{N}$ be a family of mappings such that $Z^l_t$ is $\mathcal{F}_{IT/M} / \mathcal{B}(\mathbb{R}^{k \times m})$-measurable for all $l \in \{0, 1, \cdots, M - 1\}$ and $M \in \mathbb{N}$. Then the following inequality holds:

$$\left\| \sup_{j \in \{0,1, \cdots, n\}} \left\| \sum_{i=0}^{j-1} Z^l_{t} \Delta W^M_{t} \right\| \right\|_{L^p(\Omega, \mathbb{R})} \leq C_p \left( \sum_{i=0}^{n-1} \sum_{i=1}^{m} \|Z^l_{t} \Delta W^M_{t} \hat{e}_i \|^2_{L^p(\Omega, \mathbb{R}^k)} \frac{T}{M} \right)^{1/2},$$

for all $p \geq 1$.

Lemma 4.14. Let $k \in \mathbb{N}$ and $Z : [0, T] \times \Omega \rightarrow \mathbb{R}^k$ be a predictable stochastic process satisfying $\mathbb{P} \left[ \int_0^T \|Z_s\|^2 ds < +\infty \right] = 1$. Then the following inequality holds:

$$\left\| \sup_{s \in [0,t]} \left\| \int_0^s Z_u d\mathcal{N}_u \right\| \right\|_{L^p(\Omega, \mathbb{R})} \leq C_p \left( \int_0^T \|Z_s\|^2_{L^p(\Omega, \mathbb{R}^k)} ds \right)^{1/2},$$

for all $t \in [0, T]$ and all $p \in [1, +\infty)$.

Proof. Since $\overline{\mathcal{N}}$ is a martingale with càdlàg paths satisfying $d[\overline{\mathcal{N}}, \overline{\mathcal{N}}]_s = \lambda ds$, it follows from the property of the quadratic variation (see [6, (8.21), pp 219]) that

$$\left[ \int_0^t Z_u d\overline{\mathcal{N}}_s, \int_0^t Z_s d\overline{\mathcal{N}}_s \right] = \int_0^t \|Z_s\|^2 \lambda ds. \quad (23)$$
Applying Lemma 4.11 for $M_t = \sup_{0 \leq t \leq T} \int_0^t Z_s dN_s$ and using (23) leads to:

\[
\left[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| \int_0^t Z_s dN_s \right\|^p \right] \right]^{1/p} \leq C_p \left[ \mathbb{E} \left( \int_0^T \| Z_s \|^2 ds \right)^{p/2} \right]^{1/p}, \tag{24}
\]

where $C_p$ is a positive constant depending on $p$ and $\lambda$.

Using the definition of $\| X \|_{L^p(\Omega, \mathbb{R}^d)}$ for any random variable $X$, it follows that

\[
\left\| \sup_{s \in [0,T]} \left\| \int_0^s Z_u dN_u \right\| \right\|_{L^p(\Omega, \mathbb{R})} \leq C_p \left( \int_0^T \| Z_s \|^2 ds \right)^{1/2}, \tag{25}
\]

Using Minkowski’s inequality in its integral form yields

\[
\left\| \sup_{s \in [0,T]} \left\| \int_0^s Z_u dN_u \right\| \right\|_{L^p(\Omega, \mathbb{R})} \leq C_p \left( \int_0^T \left\| \int_{[lT/M, (l+1)T/M]} Z_j \Delta N_{l,M} \right\|_{L^p(\Omega, \mathbb{R})} ds \right)^{1/2}.
\]

This completes the proof of the lemma. \qed

**Lemma 4.15.** Let $k \in \mathbb{N}$, $M \in \mathbb{N}$ and $Z_i^M : \Omega \rightarrow \mathbb{R}^k, l \in \{0, 1, \cdots, M - 1\}$ be a family of mappings such that $Z_i^M$ is $\mathcal{F}_{IT/M}/\mathcal{B}(\mathbb{R}^k)$-measurable for all $l \in \{0, 1, \cdots, M - 1\}$, then $\forall n \in \{0, 1, \cdots, M\}$ the following inequality holds

\[
\left\| \sup_{j \in \{0, 1, \cdots, n\}} \left\| \sum_{l=0}^{j-1} Z_i^M \Delta N_{l,M} \right\|_{L^p(\Omega, \mathbb{R})} \right\|_{L^p(\Omega, \mathbb{R})} \leq C_p \left( \sum_{j=0}^{n-1} \| Z_j^M \|_{L^2(\Omega, \mathbb{R}^k)}^2 \frac{T}{M} \right)^{1/2},
\]

for all $p \geq 1$, where $C_p$ is a positive constant independent of $M$.

**Proof.** Let's define $Z_i^M : [0, T] \times \Omega \rightarrow \mathbb{R}^k$ such that $Z_i^M := Z_i^M$ for all $s \in \left[ \frac{IT}{M}, \frac{(l+1)T}{M} \right)$, $l \in \{0, 1, \cdots, M - 1\}$. 


Using the definition of stochastic integral and Lemma 4.14 it follows that

\[
\left\| \sup_{j \in \{0,1,\ldots,n\}} \sum_{l=0}^{j-1} Z^M_l \Delta N^M_l \right\|_{L^p(\Omega, \mathbb{R})} = \left\| \sup_{j \in \{0,1,\ldots,n\}} \int_0^{jT/M} Z^M_u \, dN^M_u \right\|_{L^p(\Omega, \mathbb{R})} \\
\leq \left\| \sup_{s \in [0,nT/M]} \int_0^s Z^M_u \, dN^M_u \right\|_{L^p(\Omega, \mathbb{R}^k)} \\
\leq C_p \left( \int_0^{nT/M} \left\| Z^M_u \right\|_{L^p(\Omega, \mathbb{R}^k)}^2 \, ds \right)^{1/2} \\
= C_p \left( \sum_{j=0}^{n-1} \left\| Z^M_j \right\|_{L^p(\Omega, \mathbb{R}^k)}^2 \frac{T}{M} \right)^{1/2}.
\]

This completes the proof of the lemma. \qed

**Lemma 4.16.** Let \( Y^M_n : \Omega \rightarrow \mathbb{R}^d \) be defined by (7) for \( n \in \{0, \ldots, M\} \) and all \( M \in \mathbb{N} \). The following inequality holds

\[
\sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \mathbb{E} \left[ \left\| Y^M_n \right\|^p \right] < +\infty
\]

for all \( p \in [1, \infty) \).

**Proof.** Let’s first represent the numerical approximation \( Y^M_n \) in the following appropriate form

\[
Y^M_n = Y^M_{n-1} + \frac{\Delta t f_\lambda(Y^M_{n-1})}{1 + \Delta t \| f_\lambda(Y^M_{n-1}) \|} + g(Y^M_{n-1}) \Delta W^M_{n-1} + h(Y^M_{n-1}) \Delta N^M_{n-1}
\]

\[
= X_0 + \sum_{k=0}^{n-1} \frac{\Delta t f_\lambda(Y^M_k)}{1 + \Delta t \| f_\lambda(Y^M_k) \|} + \sum_{k=0}^{n-1} g(Y^M_k) \Delta W^M_k + \sum_{k=0}^{n-1} h(Y^M_k) \Delta N^M_k
\]

\[
= X_0 + \sum_{k=0}^{n-1} g(0) \Delta W^M_k + \sum_{k=0}^{n-1} h(0) \Delta N^M_k + \sum_{k=0}^{n-1} \frac{\Delta t f_\lambda(Y^M_{n-1})}{1 + \Delta t \| f_\lambda(Y^M_{n-1}) \|} \\
+ \sum_{k=0}^{n-1} (g(Y^M_k) - g(0)) \Delta W^M_k + \sum_{k=0}^{n-1} (h(Y^M_k) - h(0)) \Delta N^M_k,
\]

for all \( M \in \mathbb{N} \) and all \( n \in \{0, \ldots, M\} \).

Using the inequality

\[
\left\| \frac{\Delta t f_\lambda(Y^M_k)}{1 + \Delta t \| f_\lambda(Y^M_k) \|} \right\|_{L^p(\Omega, \mathbb{R}^d)} < 1
\]
Using Lemma 4.13 and Lemma 4.15, it follows that:

\[
\|Y_n^M\|_{L^p(\Omega,\mathbb{R}^d)} \leq \|X_0\|_{L^p(\Omega,\mathbb{R}^d)} + \left\| \sum_{k=0}^{n-1} g(0) \Delta W^M_k \right\|_{L^p(\Omega,\mathbb{R}^d)} + \left\| \sum_{k=0}^{n-1} h(0) \Delta N^M_k \right\|_{L^p(\Omega,\mathbb{R}^d)} + M + \left\| \sum_{k=0}^{n-1} (g(Y_k^M) - g(0)) \Delta W^M_k \right\|_{L^p(\Omega,\mathbb{R}^d)} + \left\| \sum_{k=0}^{n-1} (h(Y_k^M) - h(0)) \Delta N^M_k \right\|_{L^p(\Omega,\mathbb{R}^d)}.
\]

Using Lemma 4.13 and Lemma 4.15 it follows that:

\[
\|Y_n^M\|_{L^p(\Omega,\mathbb{R}^d)} \leq \|X_0\|_{L^p(\Omega,\mathbb{R}^d)} + C_p \left( \sum_{k=0}^{n-1} m \|g_i(0)\|^2 \frac{T}{M} \right)^{1/2} + C_p \left( \sum_{k=0}^{n-1} \|h(0)\|^2 \frac{T}{M} \right)^{1/2} + M + C_p \left( \sum_{k=0}^{n-1} \|\left(g_i(Y_k^M) - g_i(0)\right)\Delta W^M_k\|^2_{L^p(\Omega,\mathbb{R}^d)} \frac{T}{M} \right)^{1/2} \leq \|X_0\|_{L^p(\Omega,\mathbb{R}^d)} + C_p \left( \sum_{k=0}^{n-1} m \|g_i(0)\|^2 \frac{T}{M} \right)^{1/2} + C_p \left( \sum_{k=0}^{n-1} \|h(0)\|^2 \frac{T}{M} \right)^{1/2} + M + C_p \left( \sum_{k=0}^{n-1} \|h(Y_k^M) - h(0)\|^2_{L^p(\Omega,\mathbb{R}^d)} \frac{T}{M} \right)^{1/2}.
\]

From \(\|g_i(0)\|^2 \leq \|g_i(0)\|^2\) and the global Lipschitz condition satisfied by \(g\) and \(h\), we obtain

\[
\|g_i(Y_k^M) - g_i(0)\|_{L^p(\Omega,\mathbb{R}^d)} \leq C \|Y_k^M\|_{L^p(\Omega,\mathbb{R}^d)}
\]

\[
\|h(Y_k^M) - h(0)\|_{L^p(\Omega,\mathbb{R}^d)} \leq C \|Y_k^M\|_{L^p(\Omega,\mathbb{R}^d)}.
\]

So using (26), we obtain

\[
\|Y_n^M\|_{L^p(\Omega,\mathbb{R}^d)} \leq \|X_0\|_{L^p(\Omega,\mathbb{R}^d)} + C_p \sqrt{Tm} \|g(0)\| + C_p \sqrt{T} \|h(0)\| + M + C_p \left( \frac{Tm}{M} \sum_{k=0}^{n-1} \|Y_k^M\|^2_{L^p(\Omega,\mathbb{R}^d)} \right)^{1/2} + C_p \left( \frac{T}{M} \sum_{k=0}^{n-1} \|Y_k^M\|^2_{L^p(\Omega,\mathbb{R}^d)} \right)^{1/2}.
\]

Using the inequality \((a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2\), it follows that:

\[
\|Y_n^M\|^2_{L^p(\Omega,\mathbb{R}^d)} \leq 3 \left( \|X_0\|_{L^p(\Omega,\mathbb{R}^d)} + C_p \sqrt{Tm} \|g(0)\| + C_p \sqrt{T} \|h(0)\| + M \right)^2 + \frac{3T(C_p \sqrt{m} + C_p)^2}{M} \sum_{k=0}^{n-1} \|Y_k^M\|^2_{L^p(\Omega,\mathbb{R}^d)}.
\]
for all \( p \in [1, \infty) \). Using the fact that \( \frac{3T(C_p\sqrt{m} + C_p)^2}{M} < 3T(C_p\sqrt{m} + C_p)^2 \) we obtain the following estimation
\[
\|Y_n^M\|_{L^p(\Omega,\mathbb{R}^d)}^2 \leq 3 \left( \|X_0\|_{L^p(\Omega,\mathbb{R}^d)} + C_p\sqrt{Tm}\|g(0)\| + C_p\sqrt{T}\|h(0)\| + M \right)^2 + 3T(C_p\sqrt{m} + C_p)^2 \sum_{k=0}^{n-1} \|Y_k^M\|_{L^p(\Omega,\mathbb{R}^d)}^2.
\]

Applying Gronwall lemma to (27) leads to
\[
\|Y_n^M\|_{L^p(\Omega,\mathbb{R}^d)}^2 \leq e^{3T(C_p\sqrt{m}+C_p)^2} \left( \|X_0\|_{L^p(\Omega,\mathbb{R}^d)} + C_p\sqrt{Tm}\|g(0)\| + C_p\sqrt{T}\|h(0)\| + M \right)^2.
\]

Taking the square root and the supremum in the both sides of (28) leads to
\[
\sup_{n \in \{0,\ldots,M\}} \|Y_n^M\|_{L^p(\Omega,\mathbb{R}^d)} \leq 2e^{3T(C_p\sqrt{m}+C_p)^2} \left( \|X_0\|_{L^p(\Omega,\mathbb{R}^d)} + C_p\sqrt{Tm}\|g(0)\| + C_p\sqrt{T}\|h(0)\| + M \right).
\]

Unfortunately, (29) is not enough to conclude the proof of the lemma due to the term \( M \) in the right hand side. Using the fact that \( (\Omega_n^M)_n \) is a decreasing sequence and by using Hölder’s inequality, we obtain :
\[
\sup_{M \in \mathbb{N}} \sup_{n \in \{0,\ldots,M\}} \left( \|1_{(\Omega_n^M)^c} Y_n^M\|_{L^p(\Omega,\mathbb{R}^d)} \right) \leq \sup_{M \in \mathbb{N}} \sup_{n \in \{0,\ldots,M\}} \left( \|1_{(\Omega_n^M)^c}\|_{L^{2p}(\Omega,\mathbb{R})} \|Y_n^M\|_{L^{2p}(\Omega,\mathbb{R}^d)} \right)
\]
\[
\leq \left( \sup_{M \in \mathbb{N}} \sup_{n \in \{0,\ldots,M\}} \left( M \|1_{(\Omega_n^M)^c}\|_{L^{2p}(\Omega,\mathbb{R})} \right) \right) \times \left( \sup_{M \in \mathbb{N}} \sup_{n \in \{0,\ldots,M\}} \left( M^{-1}\|Y_n^M\|_{L^{2p}(\Omega,\mathbb{R}^d)} \right) \right).
\]

Using inequality (29) yields
\[
\left( \sup_{M \in \mathbb{N}} \sup_{n \in \{0,\ldots,M\}} \left( M^{-1}\|Y_n^M\|_{L^{2p}(\Omega,\mathbb{R}^d)} \right) \right) \leq 2e^{3(C_p\sqrt{m}+C_p)^2} \left( \frac{\|X_0\|_{L^{2p}(\Omega,\mathbb{R}^d)} + C_p\sqrt{Tm}\|g(0)\| + C_p\sqrt{T}\|h(0)\| + 1}{M} \right)
\]
\[
\leq 2e^{3(C_p\sqrt{m}+C_p)^2} \left( \frac{\|X_0\|_{L^{2p}(\Omega,\mathbb{R}^d)} + C_p\sqrt{Tm}\|g(0)\| + C_p\sqrt{T}\|h(0)\| + 1}{M} \right) < +\infty,
\]

for all \( p \geq 1 \). From the relation
\[
\|1_{(\Omega_n^M)^c}\|_{L^{2p}(\Omega,\mathbb{R})} = \mathbb{E} \left[ 1_{(\Omega_n^M)^c} \right]^{1/2p} = \mathbb{P} \left[ (\Omega_n^M)^c \right]^{1/2p}
\]
it follows using Lemma 4.10 that
\[ \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left( M \left\| \mathbb{1}_{(\Omega_n^M)^c} \right\|_{L^p(\Omega, \mathbb{R})} \right) = \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left( M^{2p} \mathbb{P} \left[ (\Omega_n^M)^c \right] \right)^{1/2p} < +\infty, \] (32) for all \( p \geq 1. \)

So plugging (31) and (32) in (30) leads to
\[ \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| \mathbb{1}_{(\Omega_n^M)^c} Y_n^M \right\|_{L^p(\Omega, \mathbb{R})} < +\infty. \] (33)

Furthermore, we have
\[ \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| Y_n^M \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| \mathbb{1}_{(\Omega_n^M)} Y_n^M \right\|_{L^p(\Omega, \mathbb{R}^d)} + \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| \mathbb{1}_{(\Omega_n^M)^c} Y_n^M \right\|_{L^p(\Omega, \mathbb{R}^d)}. \] (34)

From (33), the second term of inequality (34) is bounded, while using Lemma 4.2 and Lemma 4.9 we have
\[ \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| \mathbb{1}_{(\Omega_n^M)} Y_n^M \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| D_n^M \right\|_{L^p(\Omega, \mathbb{R})} < +\infty. \] (35)

Finally plugging (33) and (35) in (34) leads to
\[ \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| Y_n^M \right\|_{L^p(\Omega, \mathbb{R}^d)} < +\infty. \]

\[ \square \]

**Lemma 4.17.** Let \( Y_n^M \) be defined by (7) for all \( M \in \mathbb{N} \) and all \( n \in \{0, 1, \ldots, M\} \), then we have
\[ \sup_{M \in \mathbb{N}} \sup_{n \in \{0, 1, \ldots, M\}} \left( \mathbb{E} \left[ \| f_\lambda(Y_n^M) \|^p \right] \vee \mathbb{E} \left[ \| g(Y_n^M) \|^p \right] \vee \mathbb{E} \left[ \| h(Y_n^M) \|^p \right] \right) < +\infty, \]
for all \( p \in [1, \infty) \).

**Proof.** As \( f_\lambda \) satisfies the polynomial growth condition, for all \( x \in \mathbb{R}^d \) we have
\[ \| f_\lambda(x) \| \leq C(K + \| x \|^c) \| x \| + \| f_\lambda(0) \| = CK \| x \| + C \| x \|^{c+1} + \| f_\lambda(0) \|. \]

\[ \bullet \text{ If } \| x \| \leq 1, \text{ then } CK \| x \| \leq CK, \text{ hence} \]
\[ \| f_\lambda(x) \| \leq CK + C \| x \|^{c+1} + \| f_\lambda(0) \| \]
\[ \leq KC + KC \| x \|^{c+1} + C + C \| x \|^{c+1} + \| f_\lambda(0) \| + \| f_\lambda(0) \| \| x \|^{c+1} \]
\[ = (KC + C + \| f_\lambda(0) \|) (1 + \| x \|^{c+1}). \] (36)
Lemma 4.18. Let $\mathbf{Y}^M_t$ be the time continuous approximation given by (10). For all $p \in [1, \infty)$, there exists a constant $C_p$ such that the following inequalities hold

\begin{align*}
\sup_{t \in [0,T]} \left\| \mathbf{Y}^M_t - \mathbf{Y}^M_{[t]} \right\|_{L^p(\Omega, \mathbb{R}^d)} & \leq C_p \Delta t^{1/2}, \\
\sup_{M \in \mathbb{N}} \sup_{t \in [0,T]} \left\| \mathbf{Y}^M_t \right\|_{L^p(\Omega, \mathbb{R}^d)} & < \infty, \\
\sup_{t \in [0,T]} \left\| f_\lambda(\mathbf{Y}^M_t) - f_\lambda(\mathbf{Y}^M_{[t]}) \right\|_{L^p(\Omega, \mathbb{R}^d)} & \leq C_p \Delta t^{1/2}.
\end{align*}

In the sequel, for all $s \in [0, T]$ we denote by $\lfloor s \rfloor$ the greatest grid point less than $s$.

Lemma 4.18. Let $\mathbf{Y}^M_t$ be the time continuous approximation given by (10). For all $p \in [1, \infty)$, there exists a constant $C_p$ such that the following inequalities hold

\begin{align*}
\sup_{t \in [0,T]} \left\| \mathbf{Y}^M_t - \mathbf{Y}^M_{[t]} \right\|_{L^p(\Omega, \mathbb{R}^d)} & \leq C_p \Delta t^{1/2}, \\
\sup_{M \in \mathbb{N}} \sup_{t \in [0,T]} \left\| \mathbf{Y}^M_t \right\|_{L^p(\Omega, \mathbb{R}^d)} & < \infty, \\
\sup_{t \in [0,T]} \left\| f_\lambda(\mathbf{Y}^M_t) - f_\lambda(\mathbf{Y}^M_{[t]}) \right\|_{L^p(\Omega, \mathbb{R}^d)} & \leq C_p \Delta t^{1/2}.
\end{align*}

In the sequel, for all $s \in [0, T]$ we denote by $\lfloor s \rfloor$ the greatest grid point less than $s$. In other hand, using the local Lipschitz condition satisfied by $g$ and $h$, it follows that

\begin{align*}
\sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| f_\lambda(Y^M_n) \right\|_{L^p(\Omega, \mathbb{R}^d)} & \leq (KC + C + \|f_\lambda(0)\|) \left( 1 + \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| Y^M_n \right\|_{L^p(\Omega, \mathbb{R}^d)}^{c+1} \right) \\
& \leq +\infty,
\end{align*}

for all $p \in [1, \infty)$. Using inequality (38) and Lemma 4.16, it follows that

\begin{align*}
\sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| f_\lambda(Y^M_n) \right\|_{L^p(\Omega, \mathbb{R}^d)} & \leq (KC + C + \|f_\lambda(0)\|) \left( 1 + \sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| Y^M_n \right\|_{L^p(\Omega, \mathbb{R}^d)}^{c+1} \right) \\
& \leq +\infty,
\end{align*}

for all $p \in [1, \infty)$. Using the same argument as for $f_\lambda$ the following holds

\begin{align*}
\sup_{M \in \mathbb{N}} \sup_{n \in \{0, \ldots, M\}} \left\| h(Y^M_n) \right\|_{L^p(\Omega, \mathbb{R}^d)} & < +\infty,
\end{align*}

for all $p \in [1, +\infty)$. This complete the proof of Lemma 4.17. \hfill \square
Proof. Using Lemma 4.14, Lemma 4.12 and the time continuous approximation (10), it follows that

\[ \sup_{t \in [0,T]} \left\| \bar{Y}^M_t - \bar{Y}^M_{\lfloor t \rfloor} \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq \frac{T}{M} \left( \sup_{t \in [0,T]} \left\| \frac{f_\lambda(Y^M_{\lfloor t \rfloor})}{1 + \Delta t} \right\|_{L^p(\Omega, \mathbb{R}^d)} + \sup_{t \in [0,T]} \left\| \int_{\lfloor t \rfloor}^{\lfloor t \rfloor + \Delta t} g(Y^M_{\lfloor t \rfloor})dW_s \right\|_{L^p(\Omega, \mathbb{R}^d)} \right) + \sup_{t \in [0,T]} \left\| \int_{\lfloor t \rfloor}^{\lfloor t \rfloor + \Delta t} h(Y^M_{\lfloor t \rfloor})d\bar{N}_s \right\|_{L^p(\Omega, \mathbb{R}^d)} \]

\[ \leq \frac{T}{\sqrt{M}} \left( \sup_{n \in \{0, \ldots, M\}} \left\| f_\lambda(Y^M_n) \right\|_{L^p(\Omega, \mathbb{R}^d)} + \frac{T}{M} \sum_{i=1}^{m} \left\| \int_{\lfloor t \rfloor}^{\lfloor t \rfloor + \Delta t} g_i(Y^M_s)ds \right\|_{L^p(\Omega, \mathbb{R}^d)} \right)^{1/2} \]

\[ + \sup_{t \in [0,T]} \left( \frac{TC_p}{\sqrt{M}} \left\| h(Y^M_s) \right\|_{L^p(\Omega, \mathbb{R}^d)} \right)^{1/2} \]

\[ \leq \frac{T}{\sqrt{M}} \left( \sup_{n \in \{0, \ldots, M\}} \left\| f_\lambda(Y^M_n) \right\|_{L^p(\Omega, \mathbb{R}^d)} + \frac{\sqrt{M}}{\sqrt{T}} \left( \sup_{i \in \{1, \ldots, m\}} \sup_{n \in \{0, \ldots, M\}} \left\| g_i(Y^M_n) \right\|_{L^p(\Omega, \mathbb{R}^d)} \right) \right) \]

\[ + \frac{C_p\sqrt{T}}{\sqrt{M}} \left( \sup_{n \in \{0, \ldots, M\}} \left\| h(Y^M_n) \right\|_{L^p(\Omega, \mathbb{R}^d)} \right), \]

for all \( M \in \mathbb{N} \).

Using inequality (43) and Lemma 4.17, it follows that

\[ \left[ \sup_{t \in [0,T]} \left\| \bar{Y}^M_t - \bar{Y}^M_{\lfloor t \rfloor} \right\|_{L^p(\Omega, \mathbb{R}^d)} \right] < C_p \Delta t^{1/2}, \] (44)

for all \( p \in [1, \infty) \).

Using the inequalities (11), \( \|a\| \leq \|a - b\| + \|b\| \) for all \( a, b \in \mathbb{R}^d \) and Lemma 4.16 it follows that

\[ \sup_{t \in [0,T]} \left\| \bar{Y}^M_t \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq \left[ \sup_{t \in [0,T]} \left\| \bar{Y}^M_t - \bar{Y}^M_{\lfloor t \rfloor} \right\|_{L^p(\Omega, \mathbb{R}^d)} \right] + \sup_{t \in [0,T]} \left\| \bar{Y}^M_{\lfloor t \rfloor} \right\|_{L^p(\Omega, \mathbb{R}^d)} \]

\[ \leq \frac{C_p}{M^{1/2}} + \sup_{t \in [0,T]} \left\| \bar{Y}^M_{\lfloor t \rfloor} \right\|_{L^p(\Omega, \mathbb{R}^d)} \]

\[ < C_p T^{1/2} + \sup_{t \in [0,T]} \left\| \bar{Y}^M_{\lfloor t \rfloor} \right\|_{L^p(\Omega, \mathbb{R}^d)} \]

\[ < \infty, \]

for all \( p \in [1, +\infty) \) and all \( M \in \mathbb{N} \). Further, using the polynomial growth condition

\[ \|f_\lambda(x) - f_\lambda(y)\| \leq C(K + \|x\|^c + \|y\|^c)\|x - y\|, \]

where
for all \(x, y \in \mathbb{R}^d\), it follows using Hölder inequality that

\[
\sup_{t \in [0, T]} \left\| f_\lambda(Y_t^M) - f_\lambda(Y_{[t]}^M) \right\|_{L^p(\Omega, \mathbb{R}^d)} \leq C \left( K + 2 \sup_{t \in [0, T]} \left\| Y_t^M \right\|_{L^2_p(\Omega, \mathbb{R}^d)} \right) \times \left( \sup_{t \in [0, T]} \left\| Y_t^M - Y_{[t]}^M \right\|_{L^2_p(\Omega, \mathbb{R}^d)} \right).
\] (45)

Using (45) and (40), the following inequality holds

\[
\left[ \sup_{t \in [0, T]} \left\| f_\lambda(Y_t^M) - f_\lambda(Y_{[t]}^M) \right\|_{L^p(\Omega, \mathbb{R}^d)} \right] < C_p \Delta t^{1/2},
\] (46)

for all \(p \in [1, \infty)\).

Now we are ready to give the proof of Theorem 3.1.

4.2. Main part of the proof for \(\chi_M^t = Y_t^M\)

Recall that for \(s \in [0, T]\), \([s]\) denote the greatest grid point less than \(s\). The time continuous solution (10) can be written into its integral form as bellow

\[
Y_s^M = X_0 + \int_0^s \frac{f_\lambda(Y_{[u]}^M)}{1 + \Delta t \left\| f_\lambda(Y_{[u]}^M) \right\|} \, du + \int_0^s g(Y_u^M) \, dW_u + \int_0^s h(Y_u^M) \, d\mathcal{N}_u,
\] (47)

for all \(s \in [0, T]\) almost surely and all \(M \in \mathbb{N}\).

Let’s estimate first the quantity \(\left\| X_s - Y_s^M \right\|^2\), where \(X_s\) is the exact solution of (11).

\[
X_s - Y_s = \int_0^s \left( f_\lambda(X_u) - \frac{f_\lambda(Y_{[u]}^M)}{1 + \Delta t \left\| f_\lambda(Y_{[u]}^M) \right\|} \right) \, du + \int_0^s \left( g(X_u) - g(Y_{[u]}^M) \right) \, dW_u + \int_0^s \left( h(X_u) - h(Y_{[u]}^M) \right) \, d\mathcal{N}_u.
\]

Using the relation \(d\mathcal{N}_u = d\mathcal{N}_u - \lambda du\), it follows that

\[
X_s - Y_s = \int_0^s \left[ \left( f_\lambda(X_u) - \frac{f_\lambda(Y_{[u]}^M)}{1 + \Delta t \left\| f_\lambda(Y_{[u]}^M) \right\|} \right) - \lambda \left( h(X_u) - h(Y_{[u]}^M) \right) \right] \, du + \int_0^s \left( g(X_u) - g(Y_{[u]}^M) \right) \, dW_u + \int_0^s \left( h(X_u) - h(Y_{[u]}^M) \right) \, d\mathcal{N}_u.
\]
The function \( k : \mathbb{R}^d \to \mathbb{R} \), \( x \mapsto \|x\|^2 \) is twice differentiable. Applying Itô's formula for jumps process \([18, \text{pp. 6-9}]\) to the process \( X_s - \overline{Y}_s^M \) leads to

\[
\|X_s - \overline{Y}_s^M\|^2 = 2 \int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(X_u) - \frac{f_\lambda(\overline{Y}_{[u]}^M)}{1 + \Delta t \|f_\lambda(\overline{Y}_{[u]}^M)\|} \right\rangle \, du
- 2 \lambda \int_0^s \left\langle X_u - \overline{Y}_u^M, h(X_u) - h(\overline{Y}_{[u]}^M) \right\rangle \, du + \sum_{i=1}^m \int_0^s \|g_i(X_u) - g_i(\overline{Y}_{[u]}^M)\|^2 \, du
+ 2 \sum_{i=1}^m \int_0^s \left\langle X_u - \overline{Y}_u^M, g_i(X_u) - g_i(\overline{Y}_{[u]}^M) \right\rangle \, dW^i_u
+ \int_0^s \left[ \|X_u - \overline{Y}_u^M + h(X_u) - h(\overline{Y}_{[u]}^M)\|^2 - \|X_u - \overline{Y}_u^M\|^2 \right] \, dN_u.
\]

Using again the relation \( dN_u = d\overline{N}_u + \lambda du \) leads to

\[
\|X_s - \overline{Y}_s^M\|^2 = 2 \int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(X_u) - \frac{f_\lambda(\overline{Y}_{[u]}^M)}{1 + \Delta t \|f_\lambda(\overline{Y}_{[u]}^M)\|} \right\rangle \, du
- 2 \lambda \int_0^s \left\langle X_u - \overline{Y}_u^M, h(X_u) - h(\overline{Y}_{[u]}^M) \right\rangle \, du + \sum_{i=1}^m \int_0^s \|g_i(X_u) - g_i(\overline{Y}_{[u]}^M)\|^2 \, du
+ 2 \sum_{i=1}^m \int_0^s \left\langle X_u - \overline{Y}_u^M, g_i(X_u) - g_i(\overline{Y}_{[u]}^M) \right\rangle \, dW^i_u
+ \int_0^s \left[ \|X_u - \overline{Y}_u^M + h(X_u) - h(\overline{Y}_{[u]}^M)\|^2 - \|X_u - \overline{Y}_u^M\|^2 \right] \, d\overline{N}_u
+ \lambda \int_0^s \left[ \|X_u - \overline{Y}_u^M + h(X_u) - h(\overline{Y}_{[u]}^M)\|^2 - \|X_u - \overline{Y}_u^M\|^2 \right] \, du
= A_1 + A_2 + A_3 + A_4 + A_5 + A_6. \tag{48}
\]

In the next step, we give some useful estimations of \( A_1, A_2, A_3 \) and \( A_6 \).

\[
A_1 := 2 \int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(X_u) - \frac{f_\lambda(\overline{Y}_{[u]}^M)}{1 + \Delta t \|f_\lambda(\overline{Y}_{[u]}^M)\|} \right\rangle \, du
= 2 \int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(X_u) - f_\lambda(\overline{Y}_u^M) \right\rangle \, du
+ 2 \int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(\overline{Y}_u^M) - \frac{f_\lambda(\overline{Y}_{[u]}^M)}{1 + \Delta t \|f_\lambda(\overline{Y}_{[u]}^M)\|} \right\rangle \, du,
= A_{11} + A_{12}.
\]

Using the one-sided Lipschitz condition satisfied by \( f_\lambda \) leads to

\[
A_{11} := 2 \int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(X_u) - f_\lambda(\overline{Y}_u^M) \right\rangle \, du
\leq 2C \int_0^s \|X_u - \overline{Y}_u^M\|^2 \, du. \tag{49}
\]
Moreover, using the inequality $2\langle a, b \rangle \leq 2\|a\|\|b\| \leq \|a\|^2 + \|b\|^2$ for all $a, b \in \mathbb{R}^d$ leads to

\[
A_{12} = 2 \int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(\overline{Y}_u^M) - \frac{f_\lambda(\overline{Y}_u^M)}{1 + \Delta t}\right\rangle du \\
= 2 \int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(\overline{Y}_u^M) - f_\lambda(\overline{Y}_{[u]}^M)\right\rangle ds \\
+ 2\Delta t \int_0^s \left\langle X_u - \overline{Y}_u^M, \frac{f_\lambda(\overline{Y}_{[u]}^M)\|f_\lambda(\overline{Y}_{[u]}^M)\|}{1 + \Delta t}\right\rangle du \\
\leq \int_0^s \|X_u - \overline{Y}_u^M\|^2 du + \int_0^s \|f_\lambda(\overline{Y}_u^M) - f_\lambda(\overline{Y}_{[u]}^M)\|^2 du \\
+ \int_0^s \|X_u - \overline{Y}_u^M\|^2 du + \frac{T^2}{M^2} \int_0^s \|f_\lambda(\overline{Y}_{[u]}^M)\|^4 du \\
\leq 2 \int_0^s \|X_u - \overline{Y}_u^M\|^2 du + \int_0^s \|f_\lambda(\overline{Y}_u^M) - f_\lambda(\overline{Y}_{[u]}^M)\|^2 du \\
+ \frac{T^2}{M^2} \int_0^s \|f_\lambda(\overline{Y}_{[u]}^M)\|^4 du. 
\tag{50}
\]

Combining (49) and (50) gives the following estimation of $A_1$

\[
A_1 \leq (2C + 2) \int_0^s \|X_u - \overline{Y}_u^M\|^2 du + \int_0^s \|f_\lambda(\overline{Y}_u^M) - f_\lambda(\overline{Y}_{[u]}^M)\|^2 du \\
+ \frac{T^2}{M^2} \int_0^s \|f_\lambda(\overline{Y}_{[u]}^M)\|^4 du. 
\tag{51}
\]

Using again the inequality $2\langle a, b \rangle \leq 2\|a\|\|b\| \leq \|a\|^2 + \|b\|^2$ for all $a, b \in \mathbb{R}^d$ and the global Lipschitz condition satisfied by $h$ leads to

\[
A_2 := -2\lambda \int_0^s \left\langle X_u - \overline{Y}_u^M, h(X_u) - h(\overline{Y}_{[u]}^M)\right\rangle du \\
= -2\lambda \int_0^s \left\langle X_u - \overline{Y}_u^M, h(X_u) - h(\overline{Y}_u^M)\right\rangle du - 2\lambda \int_0^s \left\langle X_u - \overline{Y}_u^M, h(\overline{Y}_u^M) - h(\overline{Y}_{[u]}^M)\right\rangle du \\
\leq (2\lambda + \lambda C^2) \int_0^s \|X_u - \overline{Y}_u^M\|^2 du + \lambda C^2 \int_0^s \|\overline{Y}_u^M - \overline{Y}_{[u]}^M\|^2 du. 
\tag{52}
\]

Using the inequalities $\|g(x) - g(y)\| \leq \|g(x) - g(y)\|$ and $\|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2$ for all
\(a, b \in \mathbb{R}^d\) and the global Lipschitz condition we have

\[
A_3 := \sum_{i=1}^{m} \int_0^s \|g_i(X_u) - g_i(\bar{Y}^M_{[u]})\|^2 du \\
\leq m \int_0^s \|g(X_u) - g(\bar{Y}^M_{[u]})\|^2 du \\
\leq m \int_0^s \|g(X_u) - g(\bar{Y}^M_{u}) + g(\bar{Y}^M_{u}) - g(\bar{Y}^M_{[u]})\|^2 du \\
\leq 2m \int_0^s \|g(X_u) - g(\bar{Y}^M_{u})\|^2 du + 2m \int_0^s \|g(\bar{Y}^M_{u}) - g(\bar{Y}^M_{[u]})\|^2 du \\
\leq 2mc^2 \int_0^s \|X_u - \bar{Y}^M_{u}\|^2 du + 2mc^2 \int_0^s \|\bar{Y}^M_{u} - \bar{Y}^M_{[u]}\|^2 du. \quad (53)
\]

Using once again inequality \(\|a + b\|^2 \leq 2\|a\|^2 + \|b\|^2\) for all \(a, b \in \mathbb{R}^d\) we obtain the following estimation of \(A_6\)

\[
A_6 := \lambda \int_0^s \left[ \|X_u - \bar{Y}^M_{u} + h(\bar{Y}^M_{u}) - h(\bar{Y}^M_{[u]})\|^2 - \|X_u - \bar{Y}^M_{u}\|^2 \right] du \\
\leq \lambda \int_0^s \|X_u - \bar{Y}^M_{u}\|^2 du + 2\lambda \int_0^s \|h(X_u) - h(\bar{Y}^M_{[u]})\|^2 du \\
\leq \lambda \int_0^s \|X_u - \bar{Y}^M_{u}\|^2 du + 4\lambda \int_0^s \|h(X_u) - h(\bar{Y}^M_{u})\|^2 du \\
+ 4\lambda \int_0^s \|h(\bar{Y}^M_{u}) - h(\bar{Y}^M_{[u]})\|^2 du \\
\leq (\lambda + 4\lambda c^2) \int_0^s \|X_u - \bar{Y}^M_{u}\|^2 du + 4\lambda c^2 \int_0^s \|\bar{Y}^M_{u} - \bar{Y}^M_{[u]}\|^2 du. \quad (54)
\]

Inserting (51), (52), (53) and (54) in (48) we obtain

\[
\|X_s - \bar{Y}^M_s\|^2 \leq (2C + 2 + 2mc^2 + 3\lambda + 5\lambda c^2) \int_0^s \|X_u - \bar{Y}^M_u\|^2 du \\
+ (2mc^2 + 5\lambda c^2) \int_0^s \|\bar{Y}^M_u - \bar{Y}^M_{[u]}\|^2 du \\
+ \int_0^s \|f_\lambda(\bar{Y}^M_u) - f_\lambda(\bar{Y}^M_{[u]})\|^2 du + \frac{T^2}{M^2} \int_0^s \|f_\lambda(\bar{Y}^M_{[u]})\|^4 du \\
+ 2 \sum_{i=1}^{m} \int_0^s \left\langle X_u - \bar{Y}^M_u, g_i(X_u) - g_i(\bar{Y}^M_{[u]}) \right\rangle dW^i_u \\
+ \int_0^s \left[ \|X_u - \bar{Y}^M_u + h(X_u) - h(\bar{Y}^M_{[u]})\|^2 - \|X_u - \bar{Y}^M_u\|^2 \right] d\bar{V}_u.
\]
Taking the supremum in both sides of the previous inequality leads to

\[
\sup_{s \in [0,t]} \left\| X_s - \overline{Y}_s^M \right\|^2 \leq (2C + 2 + 2mC^2 + 3\lambda + 5\lambda C^2) \int_0^t \left\| X_u - \overline{Y}_u^M \right\|^2 du
+ (2mC^2 + 5\lambda C^2) \int_0^t \left\| \overline{Y}_u^M - \overline{Y}_{[u]}^M \right\|^2 du
+ \int_0^t \| f_a(\overline{Y}_u^M) - f_b(\overline{Y}_{[u]}^M) \|^2 du + \frac{T^2}{M^2} \int_0^t \| f_a(\overline{Y}_{[u]}^M) \|^4 du
+ 2 \sup_{s \in [0,t]} \left| \sum_{i=1}^m \int_0^s \left< X_u - \overline{Y}_u^M, g_i(X_u) - g_i(\overline{Y}_{[u]}^M) \right> dW_u \right|
+ \sup_{s \in [0,t]} \left| \int_0^s \left[ \left\| X_u - \overline{Y}_u^M + h(X_u) - h(\overline{Y}_{[u]}^M) \right\|^2 \right] d\overline{N}_u \right|
+ \sup_{s \in [0,t]} \left| \int_0^s \left\| X_u - \overline{Y}_u^M \right\|^2 d\overline{N}_u \right|.
\]

(55)

Using Lemma 4.12 we have the following estimation for all \( p \geq 2 \)

\[
B_1 := \left\| 2 \sup_{s \in [0,t]} \left| \sum_{i=1}^m \int_0^s \left< X_u - \overline{Y}_u^M, g_i(X_u) - g_i(\overline{Y}_{[u]}^M) \right> dW_u \right| \right\|_{L^{p/2}(\Omega,\mathbb{R})}
\leq C_p \left( \int_0^t \left\| \sum_{i=1}^m \left< X_u - \overline{Y}_u^M, g_i(X_u) - g_i(\overline{Y}_{[u]}^M) \right> \right\|^2_{L^{p/2}(\Omega,\mathbb{R})} ds \right)^{1/2}.
\]

Moreover, using the inequalities \( ab \leq \frac{a^2}{2} + \frac{b^2}{2} \) and \( (a + b)^2 \leq 2a^2 + 2b^2 \) for all \( a, b \in \mathbb{R} \), we have the following estimations for all \( p \geq 2 \)

\[
B_1 \leq C_p \left( \int_0^t \left\| \sum_{i=1}^m \left< X_u - \overline{Y}_u^M, g_i(X_u) - g_i(\overline{Y}_{[u]}^M) \right> \right\|^2_{L^{p/2}(\Omega,\mathbb{R})} du \right)^{1/2}
\leq C_p \left( \int_0^t \left\| X_u - \overline{Y}_u^M \right\|^2_{L^p(\Omega,\mathbb{R})} \left\| g_i(X_u) - g_i(\overline{Y}_{[u]}^M) \right\|^2_{L^p(\Omega,\mathbb{R},dt)} du \right)^{1/2}
\leq \frac{C_p}{\sqrt{2}} \left( \sup_{s \in [0,t]} \left\| X_s - \overline{Y}_s^M \right\|^2_{L^p(\Omega,\mathbb{R},dt)} \right)^{1/2} \left( 2C^2m \int_0^t \left\| X_s - \overline{Y}_s^M \right\|^2_{L^p(\Omega,\mathbb{R},dt)} ds \right)^{1/2}
\leq \frac{1}{4} \sup_{s \in [0,t]} \left\| X_s - \overline{Y}_s^M \right\|^2_{L^p(\Omega,\mathbb{R},dt)} + C_p^2m \int_0^t \left\| X_s - \overline{Y}_s^M \right\|^2_{L^p(\Omega,\mathbb{R},dt)} ds
\leq \frac{1}{4} \sup_{s \in [0,t]} \left\| X_s - \overline{Y}_s^M \right\|^2_{L^p(\Omega,\mathbb{R},dt)} + 2C_p^2m \int_0^t \left\| X_s - \overline{Y}_s^M \right\|^2_{L^p(\Omega,\mathbb{R},dt)} ds
+ 2C_p^2m \int_0^t \left\| \overline{Y}_s^M - \overline{Y}_{[s]}^M \right\|^2_{L^p(\Omega,\mathbb{R},dt)} ds.
\]

(56)
Using Lemma \ref{lem:ineq} and the inequality \((a + b)^4 \leq 16a^4 + 16b^4\), it follows that

\[
B_2 := \left\| \sup_{s \in [0,t]} \left( \int_0^s \| X_u - \overline{Y}_u^M + h(X_u) - h(\overline{Y}_{[u]}^M) \|^2 d\overline{N}_u \right) \right\|_{L^{p/2}(\Omega,\mathbb{R}^d)} \leq C_p \left( \int_0^t \| X_u - \overline{Y}_u^M + h(X_u) - h(\overline{Y}_{[u]}^M) \|^2_{L^{p/2}(\Omega,\mathbb{R}^d)} du \right)^{1/2}
\]

\[
\leq C_p \left( \int_0^t 16 \| X_u - \overline{Y}_u^M \|^4_{L^{p/2}(\Omega,\mathbb{R}^d)} + 16 \| h(X_u) - h(\overline{Y}_{[u]}^M) \|^4_{L^{p/2}(\Omega,\mathbb{R}^d)} du \right)^{1/2},
\]

for all \(p \geq 2\).

Using the inequality \(\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}\) for \(a, b \in \mathbb{R}^+\), it follows that

\[
B_2 \leq 2C_p \left( \int_0^t \| X_u - \overline{Y}_u^M \|^4_{L^{p/2}(\Omega,\mathbb{R}^d)} du \right)^{1/2} + 2C_p \left( \int_0^t \| h(X_u) - h(\overline{Y}_{[u]}^M) \|^4_{L^{p/2}(\Omega,\mathbb{R}^d)} du \right)^{1/2}
\]

\[
= B_{21} + B_{22}.
\]

Using Hölder’s inequality, it follows that

\[
B_{21} := 2C_p \left( \int_0^t \| X_u - \overline{Y}_u^M \|^4_{L^{p/2}(\Omega,\mathbb{R}^d)} du \right)^{1/2}
\]

\[
\leq 2C_p \left( \int_0^t \| X_u - \overline{Y}_u^M \|^2_{L^{p}(\Omega,\mathbb{R}^d)} \| X_u - \overline{Y}_u^M \|^2_{L^{p}(\Omega,\mathbb{R}^d)} du \right)^{1/2}
\]

\[
\leq \frac{1}{4} \sup_{u \in [0,t]} \| X_u - \overline{Y}_u^M \|^2_{L^{p}(\Omega,\mathbb{R}^d)} 8C_p \left( \int_0^t \| X_u - \overline{Y}_u^M \|^2_{L^{p}(\Omega,\mathbb{R}^d)} du \right)^{1/2}.
\]

Using the inequality \(2ab \leq a^2 + b^2\) for \(a, b \in \mathbb{R}\) leads to

\[
B_{21} \leq \frac{1}{16} \sup_{u \in [0,t]} \| X_u - \overline{Y}_u^M \|^2_{L^{p}(\Omega,\mathbb{R}^d)} + 16C_p^2 \int_0^t \| X_u - \overline{Y}_u^M \|^2_{L^{p}(\Omega,\mathbb{R}^d)} du.
\]

Using the inequalities \((a + b)^4 \leq 4a^4 + 4b^4\) and \(\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}\) for \(a, b \in \mathbb{R}^+\), we obtain

\[
B_{22} := 2C_p \left( \int_0^t \| h(X_u) - h(\overline{Y}_{[u]}^M) \|^4_{L^{p/2}(\Omega,\mathbb{R}^d)} du \right)^{1/2}
\]

\[
\leq 2C_p \left( \int_0^t \left[ 4 \| h(X_u) - h(\overline{Y}_{[u]}^M) \|^4_{L^{p/2}(\Omega,\mathbb{R}^d)} + 4 \| h(\overline{Y}_u^M) - h(\overline{Y}_{[u]}^M) \|^4_{L^{p/2}(\Omega,\mathbb{R}^d)} \right] du \right)^{1/2}
\]

\[
\leq 4C_p \left( \int_0^t \| h(X_u) - h(\overline{Y}_u^M) \|^4_{L^{p/2}(\Omega,\mathbb{R}^d)} du \right)^{1/2} + 4C_p \left( \int_0^t \| h(\overline{Y}_u^M) - h(\overline{Y}_{[u]}^M) \|^4_{L^{p/2}(\Omega,\mathbb{R}^d)} du \right)^{1/2}.
\]

Using the global Lipschitz condition, leads to

\[
B_{22} \leq 4C_p \left( \int_0^t \| X_u - \overline{Y}_u^M \|^4_{L^{p/2}(\Omega,\mathbb{R}^d)} du \right)^{1/2} + 4C_p \left( \int_0^t \| \overline{Y}_u^M - \overline{Y}_{[u]}^M \|^4_{L^{p/2}(\Omega,\mathbb{R}^d)} du \right)^{1/2}.
\]
Using the same estimations as for $B_{21}$, it follows that:

$$B_{22} \leq \frac{1}{16} \sup_{s \in [0,t]} \|X_s - \overline{Y}^M_s\|^2_{L^p(\Omega,\mathbb{R}^d)} + 64C_p \int_0^t \|X_u - \overline{Y}^M_u\|^2_{L^p(\Omega,\mathbb{R}^d)} du$$

$$+ \frac{1}{4} \sup_{s \in [0,t]} \|\overline{Y}^M_s - \overline{Y}^M_{[s]}\|^2_{L^p(\Omega,\mathbb{R}^d)} + 64C_p \int_0^t \|\overline{Y}^M_u - \overline{Y}^M_{[u]}\|^2_{L^p(\Omega,\mathbb{R}^d)} du.$$  

Taking the supremum under the integrand in the last term of the above inequality and using the fact that $C_p$ is a generic constant leads to

$$B_{22} \leq \frac{1}{16} \sup_{s \in [0,t]} \|X_s - \overline{Y}^M_s\|^2_{L^p(\Omega,\mathbb{R}^d)} + 64C_p \int_0^t \|X_u - \overline{Y}^M_u\|^2_{L^p(\Omega,\mathbb{R}^d)} du$$

$$+ C_p \sup_{s \in [0,t]} \|\overline{Y}^M_s - \overline{Y}^M_{[s]}\|^2_{L^p(\Omega,\mathbb{R}^d)}.$$  

(59)

Inserting (58) and (59) into (57) gives

$$B_2 \leq \frac{1}{8} \sup_{s \in [0,t]} \|X_s - \overline{Y}^M_s\|^2_{L^p(\Omega,\mathbb{R}^d)} + C_p \int_0^t \|X_u - \overline{Y}^M_u\|^2_{L^p(\Omega,\mathbb{R}^d)} du$$

$$+ C_p \sup_{s \in [0,t]} \|\overline{Y}^M_s - \overline{Y}^M_{[s]}\|^2_{L^p(\Omega,\mathbb{R}^d)}.$$  

(60)

Using again Lemma 4.14 leads to

$$B_3 := \left\| \sup_{u \in [0,t]} \left( \int_0^u \|X_u - \overline{Y}^M_u\|^2 d\overline{N}_u \right)^{1/2} \right\|_{L^p/2(\Omega,\mathbb{R}^d)}$$

$$\leq C_p \left( \int_0^t \|X_u - \overline{Y}^M_u\|^4_{L^p/2(\Omega,\mathbb{R}^d)} du \right)^{1/2}.$$  

Using the same argument as for $B_{21}$, we obtain

$$B_3 \leq \frac{1}{8} \sup_{u \in [0,t]} \|X_u - \overline{Y}^M_u\|^2_{L^p(\Omega,\mathbb{R}^d)} + C_p \int_0^t \|X_u - \overline{Y}^M_u\|^2_{L^p(\Omega,\mathbb{R}^d)} du.$$  

(61)

Taking the $L^p$ norm in both side of (55), inserting inequalities (56), (60), (61) and using Minkowski’s inequality in its integral form leads to

$$\left\| \sup_{s \in [0,t]} \|X_s - \overline{Y}^M_s\|_{L^p(\Omega,\mathbb{R})}^2 \right\|_{L^p/2(\Omega,\mathbb{R})} = \left\| \sup_{s \in [0,t]} \|X_s - \overline{Y}^M_s\|_{L^p(\Omega,\mathbb{R})}^2 \right\|_{L^p/2(\Omega,\mathbb{R})}.$$
The previous inequality can be rewritten in the following appropriate form:

\[
\left\| \sup_{s \in [0,t]} \| X_s - \bar{Y}_s^M \| \right\|_{L^p(\Omega, \mathcal{F})}^2 \\
\leq C_p \int_0^t \| X_s - \bar{Y}_s^M \|_{L^p(\Omega, \mathcal{F})}^2 ds + C_p \int_0^t \| \bar{Y}_s^M - \bar{Y}_{[s]}^M \|_{L^p(\Omega, \mathcal{F})}^2 ds \\
+ \int_0^t \| f_\lambda(X_s) - f_\lambda(\bar{Y}_{[s]}^M) \|_{L^p(\Omega, \mathcal{F})}^2 ds + C_p \sup_{u \in [0,t]} \| \bar{Y}_u^M - \bar{Y}_{[u]}^M \|_{L^p(\Omega, \mathcal{F})}^2 \\
+ \frac{T^2}{M^2} \int_0^t \| f_\lambda(\bar{Y}_{[s]}^M) \|_{L^p(\Omega, \mathcal{F})}^4 ds + 2C^2 \int_0^t \| \bar{Y}_s^M - \bar{Y}_{[s]}^M \|_{L^p(\Omega, \mathcal{F})}^2 ds.
\]

Applying Gronwall’s lemma to the previous inequality leads to:

\[
\frac{1}{2} \left\| \sup_{s \in [0,t]} \| X_s - \bar{Y}_s^M \| \right\|_{L^p(\Omega, \mathcal{F})}^2 \\
\leq C_p e^{C_p \left( \int_0^T \| f_\lambda(\bar{Y}_s^M) - f_\lambda(\bar{Y}_{[s]}^M) \|_{L^p(\Omega, \mathcal{F})}^2 ds + C_p \sup_{u \in [0,t]} \| \bar{Y}_u^M - \bar{Y}_{[u]}^M \|_{L^p(\Omega, \mathcal{F})}^2 \\
+ \frac{T^2}{M^2} \int_0^T \| f_\lambda(\bar{Y}_{[s]}^M) \|_{L^p(\Omega, \mathcal{F})}^4 ds + C_p \int_0^T \| \bar{Y}_s^M - \bar{Y}_{[s]}^M \|_{L^p(\Omega, \mathcal{F})}^2 ds \right)
\]
From the inequality \( \sqrt{a+b+c} \leq \sqrt{a} + \sqrt{b} + \sqrt{c} \) for all \( a, b, c \in \mathbb{R}^+ \), it follows that

\[
\frac{1}{2} \left\| \sup_{s \in [0,t]} \|X_s - \overline{Y}_s\|^2 \right\|_{L^p(\Omega, \mathbb{R})} \leq C_p e^{C_p} \left( \sup_{t \in [0,T]} \|f_{\lambda}(\overline{Y}^M) - \overline{Y}^M\|_{L^p(\Omega, \mathbb{R}^d)} + C_p \sup_{t \in [0,T]} \|\overline{Y}^M - \overline{Y}^M\|_{L^p(\Omega, \mathbb{R}^d)} \right) + \frac{T}{M} \left( \sup_{n \in \{0, \ldots, M\}} \|f_{\lambda}(Y^M_n)\|^2_{L^p(\Omega, \mathbb{R}^d)} \right) + C_p \sup_{t \in [0,T]} \|\overline{Y}^M - \overline{Y}^M\|_{L^p(\Omega, \mathbb{R}^d)},
\]

(62)

for all \( p \in [2, \infty) \).

Using Lemma 4.17 and Lemma 4.18, it follows from (63) that

\[
\left\| \sup_{t \in [0,T]} \|X_t - \overline{Y}^M_t\|^2 \right\|_{L^p(\Omega, \mathbb{R})} = \left( E \left[ \sup_{t \in [0,T]} \|X_t - \overline{Y}^M_t\|^p \right] \right)^{1/p} \leq C_p \Delta t^{1/2},
\]

(64)

for all \( p \in [2, \infty) \) and all \( M \in \mathbb{N} \). Using Hölder’s inequality, one can prove that (64) holds for \( p \in [1, 2] \). The proof of the theorem is complete.

5. Proof of Theorem 3.1 for STS scheme \( (\chi_t^M = \overline{Z}_t^M) \)

After replacing the increment of the poisson process \( \Delta N^M_n \) by its compensated form \( \Delta \overline{N}^M_n \) in STS (4), we obtain an equivalent scheme similar with the compensated tamed scheme (CTS). Therefore, the proof of the strong convergence of the STS follows exactly the one of compensated tamed scheme (CTS) (7) in Section 4. Here we should make the following changes for our semi-tamed scheme

\[
\alpha_k^M := 1_{\{\|Z^M_k\| \geq 1\}} \left( \frac{Z^M_k + u(\overline{Z}^M_k) \Delta t}{\|Z^M_k\|}, \frac{g(Z^M_k)}{\|Z^M_k\|} \Delta W^M_k \right),
\]

\[
\beta_k^M := 1_{\{\|Z^M_k\| \geq 1\}} \left( \frac{Z^M_k + u(\overline{Z}^M_k) \Delta t}{\|Z^M_k\|}, \frac{h(Z^M_k)}{\|Z^M_k\|} \Delta \overline{N}^M_k \right),
\]

where \( u_\lambda = u + \lambda h \). The function \( v \) which is one-sided Lipschitz (see Remark 2.1) should replace the function \( f_\lambda \) in the proof of the compensated tamed scheme (CTS). It follows from the proof in Section 4 that there exists a constant \( C_p > 0 \) such that

\[
\left( E \left[ \sup_{t \in [0,T]} \|X_t - \overline{Z}_t^M\|^p \right] \right)^{1/p} \leq C_p \Delta t^{1/2},
\]

(65)
for all $p \in [1, \infty)$. Details can also be found in \[21\].

6. Proof of Theorem 3.1 for NCTS scheme $(X_t^M = \overline{X}_t^M)$

Using the relation $\Delta \overline{X}_n^M = \Delta X_n^M - \lambda \Delta t$, the continuous interpolation of (8) can be expressed in the following form

$$\overline{X}_t^M = X_n^M + \lambda(t - n\Delta t)h(X_n^M) + \frac{(t - n\Delta t)f(X_n^M)}{1 + \Delta t||f(X_n^M)||} + g(X_n^M)(W_t - W_{n\Delta t}) + h(X_n^M)(\overline{N}_t - \overline{N}_{n\Delta t}),$$

for all $t \in [n\Delta t, (n + 1)\Delta t]$.

The numerical solution of the non compensated tamed scheme (NCTS) (3) is also equivalent to

$$X_{n+1}^M = X_n^M + \frac{\Delta t f(X_n^M)}{1 + \Delta t||f(X_n^M)||} + g(X_n^M)\Delta W_n^M + h(X_n^M)\Delta N_n^M$$

$$= X_n^M + \lambda h(X_n^M)\Delta t + \frac{\Delta t f(X_n^M)}{1 + \Delta t||f(X_n^M)||} + g(X_n^M)\Delta W_n^M$$

$$+ h(X_n^M)\Delta \overline{N}_n^M. \tag{66}$$

The functions $\lambda h$ and $f$ in the numerical solution of the scheme NCTS given by (3) (or (66)) satisfy respectively the same conditions as the $u_\lambda$ and $v$ in the numerical solution of the STS given by (4). Hence, it follows from the proof in Section 5 that there exists a constant $C_p > 0$ such that

$$\left( \mathbb{E} \left[ \sup_{t \in [0, T]} \| X_t - \overline{X}_t^M \|^p \right] \right)^{1/p} \leq C_p \Delta t^{1/2}, \tag{67}$$

for all $p \in [1, \infty)$.

7. Numerical simulations

In this section, we present some numerical experiments to illustrate our theoretical strong convergence. We consider the stochastic differential equation

$$dX_t = (-4X_t - X_t^3)dt + X_tdW_t + X_t dN_t, \ t \in [0, T], \tag{68}$$
with initial condition $X_0 = 1$. $N_t$ is the scalar Poisson process with parameter $\lambda = 1$. Here $u(x) = -4x$, $v(x) = -x^3$, $g(x) = h(x) = x$. It is obvious to check that $u, v, g$ and $h$ satisfy Assumption 2.1. Indeed $\langle x - y, f(x) - f(x) \rangle \leq c(x - y)^2$ for all $c \geq 0$. As you can observe in Figure 1 all schemes have for strong convergence order 0.5, which confirm the theoretical result in Theorem 3.1.

Figure 1: Strong convergence of the compensated tamed scheme (CTS), the non compensated tamed scheme (NCTS) and the semi-tamed scheme (STS). The initial solution is $X_0 = 1$ and the parameter of the scalar Poisson $\lambda = 1$ and $T = 1$. 
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