Twistor-like superstrings with $D = 3, 4, 6$ target-superspace and $N = (1,0), (2,0), (4,0)$ world-sheet supersymmetry

F. Delduc $^*$, E. Ivanov $^{**}$ and E. Sokatchev $^{***}$

Abstract

We construct a manifestly $N = (4,0)$ world-sheet supersymmetric twistor-like formulation of the $D = 6$ Green-Schwarz superstring, using the principle of double (target-space and world-sheet) Grassmann analyticity. The superstring action contains two Lagrange multiplier terms and a Wess-Zumino term. They are written down in the analytic subspace of the world-sheet harmonic $N = (4,0)$ superspace, the target manifold being too an analytic subspace of the harmonic $D = 6$, $N = 1$ superspace. The kappa symmetry of the $D = 6$ superstring is identified with a Kac-Moody extension of the world-sheet $N = (4,0)$ superconformal symmetry. It can be enlarged to include the whole world-sheet reparametrization group if one introduces the appropriate gauge Beltrami superfield into the action. To illustrate the basic features of the new $D = 6$ superstring construction, we first give some details about the simpler (already known) twistor-like formulations of $D = 3$, $N = (1,0)$ and $D = 4$, $N = (2,0)$ superstrings.

$^*$ Laboratoire de Physique Théorique, Unité associée au CNRS URA 1436, Ecole Normale Supérieure, 46 allée d’Italie, 69364 Lyon Cedex 07, France.

$^{**}$ Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna near Moscow, Russia.

$^{***}$ Physikalisches Institut, Universität Bonn, Nussallee 12, 5300 Bonn 1, Germany.

† On leave from the Institute for Nuclear Research and Nuclear Energy, Sofia, Bulgaria.
1 Introduction

Recently, essential progress has been made in understanding the nature of the mysterious kappa symmetry, which is the basic ingredient of any self-consistent covariant super $p$-brane theory [1 - 10].

The first decisive step in this direction has been made by Sorokin, Tkach, Volkov and Zheltukhin (STVZ) [1, 2] in the case of superparticle ($p = 0$). Replacing the superparticle momentum by a new twistor variable (a commuting spinor), these authors reformulated the massless $D = 3$, 4 superparticle actions in a manifestly world-line and target-space supersymmetric way. In this new formulation, the local kappa symmetry inherent to the original form of the action becomes part of the diffeomorphism group of the world-line $N = 1$ and $N = 2$ superspace, respectively. More precisely, it is identified with the odd sector of the group of local world-line superconformal transformations and so it is realized as an off-shell symmetry of the action. The latter is naturally written down in terms of the world-line superfields.

These results have been further extended in [3 - 8]. Delduc and Sokatchev [3] have put the $D = 3$, 4 STVZ actions in an arbitrary curved background and have given a similar description for the massless $D = 6$ superparticle. In the case $D = 4$ the key role of the combined complex Grassmann analyticity (chirality) in the world-line and target superspaces has been revealed. A new crucial feature of the $D = 6$ action is that it relies on the concept of double harmonic analyticity which generalizes the complex analyticity of the $D = 4$ case: The target superspace is the analytic harmonic $D = 6$ superspace [11] and its coordinates are superfields living in an analytic subspace of the world-line harmonic $N = 4$ superspace. Ivanov and Kapustnikov [4] have developed a twistor-type formulation for the massive $D = 2$ superparticle, where the corresponding kappa symmetry could once again be identified with local superconformal symmetry in the $N = 1$ world-line superspace. Based on this formulation, they have also found a very simple and efficient algorithm for constructing higher-order supersymmetric and kappa invariant corrections to the minimal superparticle action. In [4] this result has been extended to the case of the massive $D = 3$ superparticle. Also, it has been observed [5, 6] that the twistor-like superfield actions of the massive $D = 2$, $D = 3$ superparticles follow via a dimensional reduction from those of the massless $D = 3$, $D = 4$ superparticles and the harmonic superspace action for the massive $D = 5$ superparticle has been derived in this way from the $D = 6$ action of ref. [4]. A group-theoretical analysis of the STVZ construction has been undertaken by Galperin, Howe and Stelle [7]. Howe and Townsend [8] interpreted the world-line superfield STVZ actions as those of supersymmetric Chern-Simons mechanics.

Though being very interesting and suggestive in their own right, the twistor formulation of the superparticle and the new interpretation of the relevant kappa symmetry should be considered as preparatory stages before approaching the cases of super $p$-branes with $p \geq 1$ and, first of all, superstrings ($p = 1$) [12]. The first steps in generalizing this approach to superstrings have been made in ref. [2, 8]. Berkovits [9] proposed a twistor-like action for the $N = 1$ Green-Schwarz (GS) superstring which has the same form for all classically allowed superstring dimensions. However, this action possesses only one manifest off-shell world-line supersymmetry, while the kappa symmetry of the GS action is known to involve 1, 2, 4 and 8 Grassmann parameters for target spaces of dimension
3, 4, 6 and 10, respectively. Thus, the formulation of [3] could only fully explain kappa symmetry for three target-space dimensions. The complete twistor action for the $D = 4$ GS superstring, with manifest $N = (2,0)$ world-sheet supersymmetry, has been found in [4] and [4]. Like Berkovits’ action, it is written as a pure Wess-Zumino term in the world-sheet superspace, however the latter is now identified with the complex chiral subspace of the world-sheet $N = (2,0)$ superspace. Kappa symmetry turns out to be directly related to a restricted class of diffeomorphisms of the world-sheet superspace, namely to $N = (2,0)$ superconformal transformations with the parameters having an arbitrary dependence on the remaining (inert under supersymmetry) world-line bosonic coordinate. The guiding principle in constructing the twistor description of the $D = 4$ superstring, just as in the case of the $D = 4$ superparticle [3], was that of double Grassmann analyticity. The coordinates of the target chiral $D = 4$ $N = 1$ superspace were defined as chiral superfields with respect to world-sheet $N = (2,0)$ supersymmetry. In [3] it has been suggested that this principle could be applied in twistor-like formulations of other super $p$-branes with kappa invariance.

The main result of the present article (Section 4) is a twistor formulation of the $D = 6$ $N = 1$ superstring based on a combination of the ideas underlying the twistor descriptions of the $D = 6$ $N = 1$ superparticle and the $D = 4$ $N = 1$ superstring. It is given in terms of the coordinates of the analytic harmonic $D = 6$ target superspace, which are in turn world-sheet $N = (4,0)$ harmonic analytic superfields. The action consists of a Wess-Zumino term and two Lagrange multiplier terms. The latter imply on-shell constraints on the superfields, which restrict their harmonic dependence. We analyse the component content of the theory and show that it is equivalent to the GS superstring in six dimensions. The superstring kappa symmetry is identified with the odd sector of the $N = (4,0)$ superconformal group (gauged, as in the case of $D = 4$ superstring, by the supersymmetry-inert world-sheet coordinate). We show that by introducing a gauge $N = (4,0)$ Beltrami supermultiplet, the action can be made invariant under an extended world-sheet superdiffeomorphism group including arbitrary bosonic world-sheet reparametrizations.

For convenience of the reader in Sections 2, 3 we recall the already known examples. In Section 2 we review the simplest case of a twistor-like superstring, the one with a $D = 3$ target superspace and $N = (1,0)$ world-sheet supersymmetry. There one can already see the rôle played by the twistor variables and the relationship between kappa symmetry and world-sheet supersymmetry. We study the symmetries of the action and we show how to promote the right conformal invariance to full right diffeomorphisms by introducing a Beltrami gauge field. In Section 3 we present a modification of the $D = 4$ $N = (2,0)$ superstring of ref. [3], in which the Wess-Zumino term is given as a full superspace integral and the on-shell condition on the superfields is included in the action with a Lagrange multiplier.

\footnote{An $N = (2,0)$ world-sheet supersymmetric twistor-like formulation of the $D = 10$ superstring, with two of the eight kappa supersymmetries traded for world-sheet superconformal ones, has been independently and simultaneously proposed by Tonin [10].}
2 D=3 superstring with N=(1,0) world-sheet supersymmetry

In this section we review the formulation of twistor-like superstring theories with N = (1,0) world-sheet supersymmetry. Although this can be done for the target-space dimensions $D = 3, 4, 6, 10$, we shall restrict ourselves to the case $D = 3$. The reason is that with just one world-sheet supersymmetry one can fully explain the kappa symmetry of a three-dimensional superstring only.

2.1 Superspace action

The action we shall consider has $N = (1,0)$ supersymmetry, so we introduce a Grassmann coordinate $\eta$ besides the light-cone world-sheet coordinates $\xi^+ , \xi^-$. The heterotic nature of $N = (1,0)$ supersymmetry means that we supersymmetrize only one of the world-sheet directions, e.g., $\xi^-$. The corresponding covariant spinor derivative on the world sheet is $D = \partial_\eta + i\eta \partial (-), D^2 = i\partial (-)$. The target superspace coordinates $X^\mu (x, \eta), \Theta^\alpha (x, \eta)$ are defined as world-sheet superfields. $\Theta^\alpha$ is a $D = 3$ Majorana spinor. In the following, heavy use will be made of the relation\textsuperscript{2}

\begin{equation}
(\gamma^\mu)_{\alpha(\beta(\gamma_\mu)_{\delta\epsilon})} = 0 .
\end{equation}

The action describing the $D = 3$ superstring is

\begin{equation}
S = \int d^2 \xi d\eta \left[ \Pi^\mu (+) D\Theta^\mu \Theta - i P_\mu (DX^\mu - i D\Theta^\mu \Theta) \right],
\end{equation}

where

\begin{equation}
\Pi^\mu (+) = \partial (+) X^\mu - i \partial (+) \Theta^\mu \Theta ,
\end{equation}

and $P_\mu$ is a Lagrange multiplier superfield. This action is invariant under global space-time supersymmetry transformations,

\begin{equation}
\delta \Theta^\alpha = \epsilon^\alpha , \quad \delta X^\mu = i \Theta^\mu \epsilon .
\end{equation}

Indeed, up to total derivatives and making use of the identity (1), one can write down the variation of the first term $S_1$ of this action as follows

\begin{equation}
\delta S_1 = \int d^2 \xi d\eta (DX^\mu - i D\Theta^\mu \Theta) \partial (+) \Theta^\mu \epsilon .
\end{equation}

Clearly, this can be absorbed into a variation of the Lagrange multiplier $P_\mu$ (the constraint introduced by $P_\mu$ is invariant in its own right).

The action (2) is also invariant under a restricted class of left superdiffeomorphisms:

\begin{equation}
\delta \xi^(-) = \Lambda (-) - \frac{1}{2} \eta D\Lambda (-), \quad \delta \eta = -\frac{i}{2} D\Lambda (-).
\end{equation}

\textsuperscript{2}This identity is valid in $D = 3, 4, 6, 10$. It is crucial for the consistency of the Green-Schwarz superstring theories. At the same time, it is in the basis of the twistor-like approach.
(the world-sheet coordinate $\xi^{(+)}$ is not affected by (1)). They do not change the form of the spinor covariant derivative:
\[
\delta D = -\frac{1}{2} \partial (-) \Lambda (-) \ D .
\]
This transformation law is reminiscent of a superconformal transformation. However, the $\xi^{(+)}$-dependence of the superfield parameter $\Lambda (-)(\xi^{(+)}, \xi^{(-)}, \eta)$ is not restricted and so these transformations actually constitute a Kac-Moody extension of the $N = (1,0)$ superconformal group. Their active form on any scalar superfield $\phi$ is
\[
\delta \phi = \Lambda (-) \partial (-) \phi - \frac{i}{2} D \Lambda (-) D \phi .
\]

(5)

Taking for the time being all fields to transform as scalars, one finds, up to total derivatives and using again eq. (1):
\[
\delta S = -i \int d^2 \xi d\eta \ (DX^\mu - i D \Theta^\mu \Theta) \left[ D \left( \partial (+) \Lambda (-) D \Theta^\mu \Theta \right) - \frac{1}{2} \partial (+) D \Lambda (-) D \Theta^\mu \Theta \right] .
\]
It is thus clear that, like in the previous case, one can choose an additional variation of the Lagrange multiplier so that the action (2) will be invariant.

Finally, the action (2) is invariant under right conformal transformations:
\[
\delta \xi^{(+)} = \Lambda^{(+)\partial (+)}(\xi^{(+)}) ,
\]
the superfields $X^\mu$ and $\Theta$ being scalars, and the Lagrange multiplier $P_\mu$ a density. The fact that this is not a full diffeomorphism invariance is an indication that one of the two Virasoro constraints does not follow from the action. This conformal invariance can be promoted to a full right diffeomorphism invariance, $\Lambda^{(+)} = \Lambda^{(+)\partial (+)}(\xi^{(+)}, \xi^{(-)}, \eta)$ by introducing a new field which will at the same time generate the missing constraint. To clarify this procedure, one could consider a simplified example. Take a bosonic string with the following action:
\[
S = \int d^2 \xi \left( \partial (+) X^\mu \partial (-) X_\mu + \nu \partial (-) X^\mu \partial (-) X_\mu \right) .
\]

(6)

This action has the same type of invariances as (2), namely full left diffeomorphisms and right conformal invariance. Note that the Lagrange multiplier $\nu$ is the gauge field for left diffeomorphisms. At the same time, it generates the Virasoro constraint $\partial (-) X^\mu \partial (-) X_\mu = 0$. To restore the right diffeomorphisms, one makes an arbitrary change of coordinate from $\xi^{(+)}$ to a new coordinate $\xi^{(+)} = \xi^{(+)}(\xi^{(+)}, \xi^{(-)})$. The Jacobian of this transformation may be reabsorbed into a rescaling of the Lagrange multiplier $\nu$. The only change is thus in the derivative
\[
\partial (-) = \mathcal{D}(-) = \partial (-) + \mu(\xi) \partial (+) , \quad \mu(\xi) = \tilde{\partial}(-) \xi^{(+)} .
\]
The new field $\mu$ is the gauge field for the right diffeomorphisms. Further, the variation with respect to $\mu$ of the action (2), with $\partial (-)$ replaced by $\mathcal{D}(-)$, produces the second Virasoro constraint
\[
\partial (+) X^\mu \partial (+) X_\mu + 2 \nu \partial (+) X^\mu \mathcal{D}(-) X_\mu = 0 .
\]

(7)
Effectively, the fields $\mu$ and $\nu$ are two of the components of the world-sheet metric. The third one is not present because it corresponds to a Weyl rescaling. In an analogous way, we can covariantize the derivatives $D, \partial(-)$ in the superstring action (2) as follows:

$$D \rightarrow D = D + \chi \partial(+)$$

$$\partial(-) \rightarrow \partial(-) = -iD^2,$$  \hspace{1cm} (8)

where the superfield $\chi$ transforms under right diffeomorphisms:

$$\delta_{\text{right}}\chi = -\Lambda^{(+)}\partial(+)\chi + \mathcal{D}\Lambda^{(+)} \quad \Lambda^{(+)} = \Lambda^{(+)}(\xi^+, \xi^-, \eta).$$  \hspace{1cm} (9)

The action (2) with these replacements made is still invariant under global space-time supersymmetry, as well as under left superdiffeomorphisms (4) provided the derivatives in the transformation law (3) are replaced by covariant derivatives, the transformation law for the Lagrange multiplier $P_\mu$ is suitably modified and $\chi$ is inert under (4) \hspace{1cm} (10)

Note that the so covariantized left superdiffeomorphisms, being rewritten in a passive form, involve an induced field-dependent transformation of the coordinate $\xi^{(+)}$. In components this leads to a non-standard form of the world-sheet reparametrizations. Besides, these transformations close with a field-dependent bracket parameter. By making use of the invariance under $\xi^{(+)}$ diffeomorphisms one may “subtract” this unwanted induced shift of $\xi^{(+)}$ from the left diffeomorphisms and restore the original Lie bracket structure. As a result of this redefinition, the variation $\delta_{\text{left}}\chi$ becomes

$$\tilde{\delta}_{\text{left}}\chi = \tilde{\Lambda}^{(+)}\partial(+)\chi - \mathcal{D}\tilde{\Lambda}^{(+)} , \quad \tilde{\Lambda}^{(+)} = i(\Lambda^{(-)}\mathcal{D}\chi + \frac{1}{2}\mathcal{D}\Lambda^{(-)}\chi).$$  \hspace{1cm} (11)

We point out that $\delta_{\text{left}}$ and $\tilde{\delta}_{\text{left}}$ coincide modulo a $\xi^{(+)}$ diffeomorphism transformation.

In fact, the superfield $\chi$ is a pure gauge. In particular, using the freedom in the parameter $\Lambda^{(+)}$, one can fix a Wess-Zumino gauge (and we shall assume it in what follows), where the only surviving component of $\chi$ is the Beltrami parameter $\mu(\xi) = -iD\chi|_{\eta=0}$. Note that this does not restrict the bosonic part of the world-sheet reparametrization.

We would like to point out that there is an alternative approach, in which one could have started with the complete formalism of $N = (2, 0)$ supergravity on the world sheet (see, for example, [10]). There one introduces the full set of zweibeins and connections. However, most of them simply drop out from the twistor-like action (3). As we have just seen, the only gauge superfield really needed for maintaining the gauge symmetries (local supersymmetry and diffeomorphisms) is the Beltrami superfield.

2.2 Component action. World-sheet conformal supersymmetry versus kappa symmetry

We shall denote the physical components of the superfields $X, \Theta$ and $P$ by:

$$x^\mu(\xi) = X^\mu|_{\eta=0}, \quad \theta^\alpha(\xi) = \Theta^\alpha|_{\eta=0}, \quad \lambda(\xi) = D\Theta^\alpha|_{\eta=0}, \quad p_\mu(\xi) = P_\mu|_{\eta=0}$$

These transformation laws of $\chi$ are compatible with the Lie bracket structure: $[\delta_{\text{left}}, \delta_{\text{right}}] \sim \delta_{\text{right}}$.
(we discard some purely auxiliary fields which are expressed in terms of the physical ones on shell). Further, we introduce the notation:

\[ \pi_\mu^{(+)} = \partial^{(+)} x^\mu - i \partial^{(+)} \theta \gamma^\mu \theta, \]

\[ \hat{\pi}_\mu^{(-)} = D^{(-)} x^\mu - i D^{(-)} \theta \gamma^\mu \theta \equiv \pi_\mu^{(-)} + \mu \pi_\mu^{(+)}, \]

with \( D^{(-)} = \partial^{(-)} + \mu \partial^{(+)} \). Then it is not hard to obtain the component form of the action (2):

\[ S = \int d^2 \xi \left( \pi_\mu^{(+)} \lambda \gamma^\mu \lambda + i \pi_\mu^{(+) \mu} D^{(-)} \theta \gamma^\mu \theta - i \lambda \gamma^\mu \lambda \partial^{(+)} \theta \gamma^\mu \theta + p_\mu \left( \hat{\pi}_\mu^{(-)} - \lambda \gamma^\mu \lambda \right) \right). \]  

(12)

The left superdiffeomorphisms (5) (covariantized with respect to \( \xi^{(+)} \) diffeomorphisms) contain, in particular, local left supersymmetry with parameter \( \rho = -\frac{i}{2} D \Lambda^{(-)} |_{\eta=0} \). It acts on the above fields in the following way:

\[ \delta x^\mu = i \rho \lambda \gamma^\mu \theta, \quad \delta \theta^\alpha = \rho \lambda^\alpha, \]

\[ \delta \lambda^\alpha = i \rho D^{(-)} \theta, \quad \delta p_\mu = i \partial^{(+)} (\rho \lambda \gamma^\mu \theta). \]  

(13)

The field \( p_\mu \) is a Lagrange multiplier for the constraint

\[ \hat{\pi}_\mu^{(-)} = \lambda \gamma^\mu \lambda. \]  

(14)

With the help of the identity (1) this implies that \( \hat{\pi}_\mu^{(-)} \) is a light-like vector:

\[ \hat{\pi}_\mu^{(-)} \hat{\pi}_\mu^{(-)} = 0. \]  

(15)

In fact, eq. (15) is one of the two Virasoro constraints for the superstring (the second Virasoro constraint can be obtained by varying the Beltrami parameter \( \mu \) in (12), see below). Here we see the main idea of the twistor approach in action: A light-like vector is represented as a pair of commuting spinors (twistor variables). Further, we note that the twistor variable \( \lambda^\alpha \) appears in (12) only in the combination \( \lambda \gamma^\mu \lambda \), so (14) may be used to eliminate it from the action, which then becomes:

\[ S = \int d^2 \xi \left( \pi_\mu^{(+)} \hat{\pi}_\mu^{(+)} + i \pi_\mu^{(+ \mu)} \partial^{(-)} \theta \gamma^\mu \theta - i \hat{\pi}_\mu^{(-)} \partial^{(+)} \theta \gamma^\mu \theta \right). \]  

(16)

This action, accompanied by the constraint (13), is just the action of the GS superstring (12). As a consequence of the elimination of the twistor-like variable \( \lambda^\alpha \), the action (16) has lost the local left supersymmetry (13) of the action (12). Instead, it has a new local symmetry,

\[ \delta x^\mu = i \kappa \gamma^\rho \gamma^\mu \theta \hat{\pi}_\mu^{(-)}, \quad \delta \theta^\alpha = (\kappa \gamma^\mu)^\alpha \hat{\pi}_\mu^{(-)}, \]  

(17)

which is just the familiar kappa symmetry of the GS superstring. Actually, the transformations (17) can be recast in the form of local supersymmetry (13), if one defines \( \rho = \lambda^\alpha \kappa_\alpha \), and then uses the on-shell condition (14), as well as the Fierz identity for the \( D = 3 \) gamma matrices. This procedure shows that kappa symmetry is equivalent to world-sheet supersymmetry only on shell (and hence the on-shell closure of the algebra of
kappa symmetry). Further, we recall the well-known fact that because of the presence of
the light-like vector $\hat{\pi}_{(-)}$ in (17) only half of $\kappa^\alpha$ are true gauge parameters (they are used
to gauge away half of $\Theta^\alpha$). In the allowed dimensions $D = 3, 4, 6, 10$ that half of $\kappa^\alpha$
can be matched by $N = (1, 0), (2, 0), (4, 0), (8, 0)$ world-sheet supersymmetries. In this
section we consider $N = (1, 0)$, so it was natural to associate it with the $D = 3$ super-
string. As a matter of fact, the whole discussion above applies to any of the dimensions
$D = 3, 4, 6, 10$, except for the relationship between local world-sheet supersymmetry
and kappa symmetry.

Let us explain in more detail how the second Virasoro constraint follows from the action
(12). After a redefinition of the Lagrange multiplier as
$$p^\mu = \tilde{p}^\mu - i \partial(+)\theta \gamma^\mu \theta + \pi^\mu(+),$$
the action takes the form
$$S = \int d^2 \xi \left( \pi_{(+)\mu} \hat{\pi}_{(-)}^\mu + i \pi_{(+)\mu} \partial(D(-)\theta \gamma^\mu \theta - i \hat{\pi}_{(-)\mu} \partial(+)\theta \gamma^\mu \theta + \tilde{p}_\mu (\hat{\pi}_{(-)}^\mu - \lambda \gamma^\mu \lambda) \right).$$
Varying (18) with respect to $\tilde{p}_\mu$ leads to the already known twistor constraint (14), whose
corollary is eq. (15), while varying with respect to $\mu$ gives
$$\langle \tilde{p}_\mu + \pi_{(+)\mu} \rangle \pi^\mu(+) = 0.$$ 
Let us now look at the equation of motion for $\lambda^\alpha$
$$\tilde{p}_\mu (\gamma^\mu \lambda)_\alpha = 0.$$  
It is well known [1, 2, 6, 7] that for $D = 3$ (as well as for $D = 4, 6, 10$) this equation has
the general solution
$$\tilde{p}^\mu = c \lambda \gamma^\mu \lambda,$$
where $c$ is an arbitrary scalar field on the world sheet. Comparing (20) with (14) we find
$$\tilde{p}^\mu = c \hat{\pi}_{(-)}^\mu.$$ 
Further, substituting this into into eq. (19) we see that (19) is just the second Virasoro
constraint (7), provided one identifies $c = 2 \nu$.

We conclude this section by the remark that the first term in the action (2) is in fact
a typical Wess-Zumino term. This becomes clear after rewriting the action in the form:
$$S = \int d^2 \xi d\eta \left[ (D\Theta^\alpha \partial(+)X^\mu - D X^\mu \partial(+)\Theta^\alpha)(\gamma^\mu \Theta)_\alpha - iP_\mu (DX^\mu - iD\Theta \gamma^\mu \Theta) \right],$$
obtained by a redefinition of the Lagrange multiplier $P_\mu$. Indeed, the first term now looks
like a Wess-Zumino term $\int \partial Z^M \partial Z^N B_{NM}(Z)$, where the only non-vanishing component
of the two-form is $B_{\mu\alpha} = (\gamma^\mu \Theta)_\alpha$.  

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4The presence of a WZ term is a characteristic feature of the GS superstring [13].
3 D=4 superstring with N=(2,0) world-sheet supersymmetry

The formulation of the $D=4$ $N=(2,0)$ superstring presented in this section is equivalent to the one of ref. [4], the main difference is that the WZ term is given as a full world-sheet superspace integral, and not as a chiral one, as in [4]. This allows us to introduce the on-shell constraints on the superfields $X$ and $\Theta$ in the action via a Lagrange multiplier, so the new action involves only unconstrained objects. At the end of this section we show how the formulation of [4] can be obtained from the present one.

We begin by introducing an $N=(2,0)$ world-sheet superspace with coordinates $\xi^{(+),(-)}, \eta, \bar{\eta}$. As before, supersymmetry acts only in the direction of the coordinate $\xi^{(-)}$. Further, following the principle of double Grassmann analyticity formulated in the Introduction, we introduce the coordinates of the left and right chiral bases in $D=4$ superspace, $X^\mu_L$, $\Theta^\alpha$ and $X^\mu_R = \bar{X}^\mu_L$, $\bar{\Theta}^{\dot{\alpha}} = \bar{\Theta}^{\tau}$, as left and right-handed chiral world-sheet superfields, respectively:

$$\bar{D}X^\mu_L = \bar{D}\Theta^\alpha = 0 \ , \ D\bar{X}^\mu_R = D\bar{\Theta}^{\dot{\alpha}} = 0 \ , \ (22)$$

where

$$D = \partial/\partial \eta + i\bar{\eta}\partial/\partial \xi^{(-)} \ , \ \bar{D} = -\partial/\partial \bar{\eta} - i\eta\partial/\partial \xi^{(+)} .$$

The usual, real coordinate $X^\mu$ is identified with the real part of the chiral ones,

$$X^\mu = \frac{1}{2}(X^\mu_L + X^\mu_R) ,$$

after restricting the imaginary part to be

$$\frac{i}{2}(X^\mu_L - X^\mu_R) = -\Theta^\sigma\bar{\Theta} . \ (23)$$

The constraint (23) is in fact an equation of motion (see [4] for the analogous case of the $D=4$ $N=2$ superparticle), so we are going to introduce it in the action with a Lagrange multiplier (cf. (2)).

We propose the following action for the $D=4$ $N=(2,0)$ superstring:

$$S = \frac{1}{2} \int d^2 \xi d\eta d\bar{\eta} \left[ (\partial_{(+)}X^\mu_L + \partial_{(+)}X^\mu_R - i\partial_{(+)}\Theta^\sigma\bar{\Theta} + i\Theta^\sigma\partial_{(+)}\bar{\Theta})\Theta^{\tau}\bar{\Theta} \right. \right.$$

$$+ P^\mu \left( \frac{i}{2}(X^\mu_L - X^\mu_R) + \Theta^\sigma\bar{\Theta} \right) \] . \ (24)$$

Comparing this action with that from [4] (see (29)), we see that the main difference is in the first (WZ) term in (24). Note also that the Lagrange multiplier term in (24) has exactly the same form as the action for the $D=4$ $N=2$ superparticle of ref. [3].

The action (24) has several symmetries.

Firstly, it has global target-space supersymmetry. In the left-handed chiral basis of $D=4$ superspace it is realized as follows:

$$\delta X^\mu_L = 2i\Theta^\sigma\epsilon^\sigma \ , \ \delta \Theta^\alpha = \epsilon^\alpha . \ (25)$$
and similarly for the right-handed basis. The invariance of the Lagrange multiplier term in (24) is obvious (for the moment, we do not vary $P^\mu$). To check the invariance of the WZ term one has to use the chirality conditions (22), the Fierz identity for the matrices $\sigma^\mu$ and to ascribe the following transformation law to the Lagrange multiplier:

$$\delta P^\mu = 2i(\partial_{(+)}\Theta\sigma^\mu \bar{\epsilon} - \epsilon\sigma^\mu\partial_{(+)}\bar{\Theta}) .$$

Secondly, the action (24) is invariant under restricted left superdiffeomorphisms:

$$\delta \xi^{(-)} = \Lambda^{(-)} + \frac{1}{2} \bar{\eta} \dot{D}\Lambda^{(-)} + \frac{1}{2} D\Lambda^{(-)} \eta , \quad \delta \eta = \frac{i}{2} \dot{D}\Lambda^{(-)} , \quad \delta \bar{\eta} = -\frac{i}{2} D\Lambda^{(-)} ,$$

where $\Lambda^{(-)}(\xi^{(+)}, \xi^{(-)}, \eta, \bar{\eta}) = \bar{\Lambda}^{(-)}$. These transformations leave the volume of the real world-sheet superspace invariant,

$$\delta(d^2\xi d\eta d\bar{\eta}) = 0 ,$$

and transform the left and right-handed coordinates of the target superspace as scalars. Consequently, one finds

$$\delta(\partial_{(+)}X^\mu_L) = -\frac{i}{2}(\dot{D}\partial_{(+)}\Lambda)DX^\mu_L - (\partial_{(+)}\Lambda)\partial_{(-)}X^\mu_L ,$$

etc. Using all this, as well as the on-shell relation (24) (in other words, this means finding appropriate compensating transformations of the Lagrange multiplier), one can show that the WZ term in (24) is invariant up to total derivatives. The Lagrange multiplier term is manifestly invariant too. Like in the case $D = 3$, the transformations (26) constitute an $N = (2,0)$ world-sheet superconformal group with the parameters local the supersymmetry-inert coordinate $\xi^{(+)}$. Their relation to kappa supersymmetry is precisely the same as in the case of $D = 3$ superparticle (details can be found in [4]). Note that (26) leave invariant the chiral subspace $(\xi_L^{(-)} = \xi^{(-)} - i\bar{\eta}\eta, \eta)$ and can be regarded as a particular class of general diffeomorphisms of the latter,

$$\delta \xi_L^{(-)} = \tilde{\Lambda}^{(-)}(\xi_L^{(-)}, \eta, \xi^{(+)}) , \quad \delta \eta = \omega(\xi_L^{(-)}, \eta, \xi^{(+)}) ,$$

which preserve the flat definition of the world-sheet chiral subspace

$$Im \xi_L^{(-)} = -i\eta\bar{\eta} .$$

Finally, the action (24) is obviously invariant under right conformal reparametrizations of $\xi^{(+)}$. In the end of this section we shall sketch how these can be promoted to general ones.

Here we shall not investigate the component content of the action (24). Instead, we shall show how it can be reduced to the chiral action of ref. [4], where the issue of components has been discussed. Firstly, we impose the equation of motion (23). Thus the Lagrange multiplier term in (24) drops out, but the fields become constrained. Secondly, we convert one of the Grassmann integrations, e.g., $d\bar{\eta}$ into a spinor derivative, $\dot{D}$. Using
the relation $\bar{D}X^\mu_R = 2i\Theta \sigma^\mu \bar{D}\Theta$ following from (23) and integrating $\partial_{(+)}$ by parts, we obtain the chiral form of the WZ term from [4]:

$$S_{WZ} = -\int d\xi^{(+)} d\xi^{(-)} d\eta (\partial_{(+)} X^\mu + i \Theta \sigma^\mu \partial_{(+)} \bar{\Theta} - i \partial_{(+)} (\Theta \sigma^\mu \bar{\Theta}) \Theta \sigma^\mu \bar{D} \bar{\Theta}) .$$

(29)

Note that the chirality of the integrand in (29) and the reality of the action are not manifest, but they follow from the on-shell relation (23).

The WZ term (29) has the standard geometric form $\int \partial Z^M \partial Z^N B_{MN}$, where the two-form is represented by $B_{\mu \dot{\alpha}} = (\Theta \sigma^\mu)_{\dot{\alpha}}$. In contrast, what we find in the real form (24) of the WZ term is not the two-form itself, but its chiral-basis prepotential $\Theta \sigma^\mu \bar{\Theta}, B_{\mu \dot{\alpha}} = -\bar{D}_{\dot{\alpha}} (\Theta \sigma^\mu \bar{\Theta})$.

Finally, we outline very briefly how the $\xi^{(+)}$ conformal invariance of the $D = 4$ action (24) can be extended to full diffeomorphisms. Like in the case $D = 3$, this can be done by introducing into the action (24) an $N = (2,0)$ Beltrami superfield. A new point compared to $D = 3$ is that the right superdiffeomorphism group should preserve the notion of worldsheet chirality. The natural way to satisfy this requirement is to apply the approach of ref. [14] to $N = 1$ $D = 4$ supergravity in superspace (for applications to $N = 2$ worldsheet geometry in the context of $N = 2$ chiral bosons see also ref. [15]). One changes the coordinate $\xi^{(+)}$ to a complex one $\xi^{(+), L} = \xi^{(+)} + i \chi^{(+)} (\xi^{(-), L}, \xi^{(+), R}, \bar{\eta}, \eta)$ and replaces the conformal shifts of $\xi^{(+)}$ by chiral diffeomorphisms of $\xi^{(+), L}$,

$$\delta \xi^{(+), L} = \Lambda^{(+)} (\xi^{(+), L}, \xi^{(-), L}, \eta) .$$

Here $\Lambda^{(+)}$ is an arbitrary complex function of its arguments and, as before, $\xi^{(-), L} = \xi^{(-)} - i \bar{\eta} \eta$. The real superfield $\chi^{(+)}$ accomodates the $N = (2,0)$ Beltrami gauge multiplet. In the Wess-Zumino gauge it reduces to $\chi^{(+)} = -\bar{\eta} \eta \mu (\xi)$ where $\mu$ is the Beltrami parameter, the same as in the case $D = 3$. The covariantized form of the left diffeomorphisms can be obtained by replacing $\xi^{(+)}$ by $\xi^{(+), L}$ in the transformation laws (27) and requiring that the modified transformations still preserve the flat relation (28). The resulting transformation laws are nonlinear and nonpolynomial in $\chi^{(+)}$ but they radically simplify in the Wess-Zumino gauge. The only place in the action (24) where the new Beltrami superfield $\chi^{(+)}$ appears is in the coordinates of the chiral superfields $X_L, X_R, \Theta$ and $\bar{\Theta}$. The details of the construction are out of the scope of our presentation here. We only note that the action considered in [4] can be viewed as a particular gauge-fixed form of the Beltrami-covariantized action, with the whole of $\chi^{(+)}$ gauged to zero.

4 D=6 superstring with N=(4,0) world-sheet supersymmetry

4.1 Superspace action and symmetries

The world-sheet super-coordinates of the $N = (4,0)$ superstring will be denoted by $\xi^{(+)}$, $\xi^{(-)}, \eta^i, \bar{\eta}_i$ ($i$ is an $SU(2)$ doublet index). To those we add the harmonic coordinates\footnote{The Lorentz weights ($\pm$) should not be confused with the $U(1)$ charges $\pm$.}
$u^\pm_i$ of the sphere $S^2$ (see [11]). They are used to project the supersymmetric covariant derivatives:

$$\{D_i, D^i\} = i\delta^i_j \partial_{(-)} \rightarrow D^\pm = u^\pm_i D^i, \quad D^\pm = u^\pm_i D^i.$$ 

The usual target superspace of the $D = 6$ superstring has coordinates $X^{\alpha\beta} = -X^{\beta\alpha} \equiv (\gamma^\mu)_{\alpha\beta} X^\mu$ and $\Theta^{\alpha a}$. Here $\alpha, \beta$ are $SU(4)^*$ spinor indices and $\Theta$ satisfies the pseudo-Majorana condition $\Theta^{\alpha a} = \epsilon_{ij} C^a_{\alpha} \Theta^{\bar{\alpha} a}$.

As we saw in the preceding section, the $D = 4$ superstring is formulated in terms of the coordinates of the chiral subspaces of the target superspace. At the same time, they are taken as chiral superfields with respect to world-sheet supersymmetry. This is actually the principle of double Grassmann analyticity mentioned in the Introduction. In the case $D = 6$ the notion corresponding to $D = 4$ chirality is that of $SU(2)$ harmonic Grassmann analyticity [11]. Following this idea we choose to formulate the $D = 6$ superstring in terms of the coordinates $X^{\alpha\beta}(\xi^+, \eta^+, u)$, $\Theta^{\alpha a}(\xi^+, \eta^+, u)$ (where $\Theta^{\alpha a} = C^a_{\alpha} \Theta^{\alpha a}$), defined as analytic harmonic superfields:

$$D^+ X^{\alpha\beta} = \bar{D}^+ X^{\alpha\beta} = 0, \quad D^+ \Theta^{\alpha a} = \bar{D}^+ \Theta^{\alpha a} = 0.$$ 

Such superfields can be considered as unconstrained objects in the analytic subspace $\xi^{ (+)} , \xi^{ (-)} = \xi^{ (-)} + i\eta^i \bar{\eta}^j u^+_i u^-_j , \eta^+ = u^+_i \eta^i , \bar{\eta}^+ = u^-_i \bar{\eta}^i$. Later on, after imposing the harmonic equations of motion, the usual coordinate $\theta^{\alpha a}$ will reappear as the first term of the harmonic expansion of $\Theta^{\alpha a}$.

We shall consider the following action for the $D = 6$ $N = (4,0)$ superstring:

$$S = \int d^2 \xi [du] d^2 \eta^+ \left[ \partial_{(+)} X^{\alpha\beta} \Theta^{\alpha \gamma} \Theta^{\beta \delta} \epsilon_{\alpha\beta\gamma\delta} + P_{\alpha \beta} (D^{++} X^{\alpha\beta} - i \Theta^{\alpha a} \Theta^{\beta a} + Q^a_{\alpha} D^{++} \Theta^{\alpha a}) \right]. \tag{30}$$

Here $D^{++} = u^{+i} \partial / \partial u^{-i} + i\eta^i \bar{\eta}^j \partial_{(-)}$ is the analytic basis form of the harmonic covariant derivative. The superfields $P_{\alpha \beta}$ and $Q^a_{\alpha}$ are analytic Lagrange multipliers restricting the $u$-dependence of the on-shell fields. The last two terms in (30) are exactly the same as in the case of the $D = 6$ $N = 4$ superparticle (see [3]).

This action has the three symmetries we already encountered in the preceding Sections. These are:

1) Global space-time supersymmetry:

$$\delta X^{\alpha\beta} = i (\epsilon^{++} \Theta^{\alpha \beta} - \epsilon^{--} \Theta^{\alpha a}), \quad \delta \Theta^{\alpha a} = \epsilon^{\alpha a}$$

with $\epsilon^{\pm a} = u^\pm_i \epsilon^i$, and $\epsilon^i$ is a $SU(2)$-Majorana spinor parameter. Up to total derivatives, the variation of the action (30) is (for the moment, $P_{\alpha \beta}$ and $Q^a_{\alpha}$ are not varied):

$$\delta S = \int d^2 \xi [du] d^2 \eta^+ \left\{ 2(D^{++} X^{\alpha\beta} - i \Theta^{\alpha a} \Theta^{\beta a}) \epsilon^{--} \partial_{(+)} \Theta^{\alpha a} \right. + 2i P_{\alpha \beta} \epsilon^{\alpha a} D^{++} \Theta^{\alpha a} \}

- 2\partial_{(+)} X^{\alpha\beta} \epsilon^{--} D^{++} \Theta^{\alpha a} \epsilon_{\alpha\beta\gamma\delta} + 2i P_{\alpha \beta} \epsilon^{--} D^{++} \Theta^{\alpha a} \right\}, \tag{31}$$

\footnote{The same approach proved successful in the case of the $D = 6$ superparticle [3].}
which may be compensated by the following variations of the Lagrange multipliers:

$$\delta P_{\alpha\beta} = -2\epsilon_{\alpha\beta\gamma\delta} \epsilon^{-\gamma} \partial^{(+)\delta} \Theta^{\gamma\delta}, \quad \delta Q^-_{\alpha} = -2\epsilon_{\alpha\beta\gamma\delta} \partial^{(+)\delta} X^{\beta\gamma\delta} + 2iP_{\alpha\beta}\epsilon^{-\beta}.$$  

2) Left superdiffeomorphisms. These may be written as active transformations on the fields as follows:

$$\delta X^{\alpha\beta} = iD^+ \tilde{D}^+ (\Lambda^{(-)} D^{(-)} X^{\alpha\beta}) , \quad \delta \Theta^{\gamma\alpha} = iD^+ \tilde{D}^+ (\Lambda^{(-)} D^{(-)} \Theta^{\gamma\alpha}) ,$$  

(32)

where \(\Lambda^{(-)}\) is a \(u\)-independent \((D^{++} \Lambda^{(-)} = 0)\) superfield satisfying the constraint

$$D^+ \tilde{D}^+ \Lambda^{(-)} = 0 .$$  

(33)

Notice that the \(\xi^{(+)\delta}\)-dependence of \(\Lambda^{(-)}\) is not restricted just as in the cases considered previously. The components of \(\Lambda^{(-)}\) are easily shown to be the parameters of left diffeomorphisms, local \(N = 4\) left supersymmetry and local \(SU(2)\) transformations (see [3, 4]). Assuming for the time being that the Lagrange multipliers transform according to the standard law (32), the variation of the action (30) under these transformations is given, up to total harmonic derivatives, by

$$\delta S = i \int d^2 \xi [du] d^4 \eta \partial^{(+)\delta} \Lambda^{(-)} D^{(-)} X^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta\gamma\delta} .$$  

(34)

One has to work a little in order to show that this expression can be compensated by choosing the full transformations of the Lagrange multipliers to be as follows:

$$\delta P_{\alpha\beta} = iD^+ \tilde{D}^+ \left[ \Lambda^{(-)} D^{(-)} P_{\alpha\beta} - \epsilon_{\alpha\beta\gamma\delta} \partial^{(+)\delta} \Lambda^{(-)} D^{(-)} (\Theta^{\gamma\delta} D^{(-)} \Theta^{\gamma\delta}) \right]$$

$$\delta Q^-_{\alpha} = iD^+ \tilde{D}^+ \left[ \Lambda^{(-)} D^{(-)} Q^-_{\alpha} + \epsilon_{\alpha\beta\gamma\delta} \partial^{(+)\delta} \Lambda^{(-)} ((D^{(-)})^2 X^{\beta\gamma\delta} \Theta^{\gamma\delta} - \frac{2}{3} \Theta^{\gamma\delta} \Theta^{\gamma\delta} - \frac{2}{9} (D^{(-)})^3 (\Theta^{\gamma\delta} \Theta^{\gamma\delta})) \right] .$$  

(35)

3) Right conformal invariance. The action is invariant under right conformal transformations with parameter \(\Lambda^{(+)\delta}(\xi^{(+)\delta})\) provided the fields \(X\) and \(\Theta\) transform as scalars and the Lagrange multipliers transform as densities. Like in the cases \(D = 3\) and \(D = 4\), this invariance can be promoted to a right superdiffeomorphism one, this time by changing \(\Lambda^{(+)\delta}(\xi^{(+)\delta})\) to a general analytic superfield, if one introduces an analytic einbein \(\chi^{++(+)\delta}\) and replaces the harmonic derivative \(D^{++}\) by a covariant one:

$$D^{++} \rightarrow \mathcal{D}^{++} = D^{++} + \chi^{++(+)\delta} \partial^{(+)\delta}$$

with \(\delta \chi^{++(+)\delta} = -D^{++} \Lambda^{(+)\delta} + \Lambda^{(+)\delta} \partial^{(+)\delta}\chi^{++(+)\delta}\). The action (30) with this replacement made is still globally space-time supersymmetric, and invariant under left superdiffeomorphisms, provided the harmonic derivative \(D^{--}\) in (32) is replaced by a covariant one:

$$D^{--} \rightarrow \mathcal{D}^{--} = D^{--} + \chi^{--(+)\delta} \partial^{(+)\delta} .$$

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\\(7\\)Their passive form acting on the coordinates of the world-sheet superspace can be found in [3]. There a detailed discussion of the \(N = 4\) world-sheet superconformal group is given (see also [4]).
with:
\[\{D^{++}, D^{--}\} = D^0 \Rightarrow D^{++} \chi^{-(+)} - D^{--} \chi^{++(+) = 0.}\]

This equation determines \(\chi^{-(+)}\) as a function of \(\chi^{++(+)}\). The parameter \(\Lambda(-)\) in \(32\), \(33\) is now covariantly \(u\)-independent:
\[D^{++} \Lambda(-) = D^{--} \Lambda(-) = 0.\]

In the Wess-Zumino gauge, the only surviving component of \(\chi^{++(+)\text{ is}}
\[\mu(x) = i \int [du] D^{-} D^{-} \chi^{++(+)_{\eta=0}},\]

which transforms under right diffeomorphisms as
\[\delta \mu(x) = -\partial(-) \Lambda^{+(+)}(x) - \mu(x) \partial(+) \Lambda^{(+)}(x) - \Lambda^{+(+)}(x) \partial(+) \mu(x).\]

4.2 Component action

In order to find out the component content of the superfield action \(30\) one has first to get rid of the harmonic dependence of the superfields in it. This is achieved using the harmonic constraints introduced in the action with the Lagrange multipliers \(P\) and \(Q\) (see \(3\) for the details). Eliminating those infinite sets of auxiliary fields and using the Wess-Zumino gauge for \(\chi^{++(+)\text{ just discussed, we are left with the following component fields:}}\)
\[x^{\alpha\beta}(\xi) = \int [du] X^{\alpha\beta}_{\eta=0}, \quad \theta^{i\alpha}(\xi) = 2 \int [du] u^{-i} \Theta^{+}_{\eta=0},\]
\[\chi^{\alpha} = \int [du] D^{+} \Theta^{+}_{\eta=0}, \quad \bar{\chi}^{\alpha} = \int [du] D^{-} \Theta^{-}_{\eta=0},\]
\[\sigma_{\alpha\beta} = \int [du] P_{\alpha\beta|\eta=0}.\]

Let us introduce a notation similar to the one used in section 2:
\[\pi^{\alpha\beta}_{(\pm)} = \partial_{(\pm)} x^{\alpha\beta} + \frac{i}{2} \partial_{(\pm)} \theta^{i\alpha} \theta^{i\beta} - \frac{i}{2} \partial_{(\pm)} \theta^{ij} \theta_{i\alpha} \theta_{j\beta}\]
\[\dot{\pi}^{\alpha\beta}_{(-)} = D_{(-)} x^{\alpha\beta} + \frac{i}{2} D_{(-)} \theta^{i\alpha} \theta^{i\beta} - \frac{i}{2} D_{(-)} \theta^{ij} \theta_{i\alpha} \theta_{j\beta} \equiv \pi^{\alpha\beta}_{(-)} + \mu \pi^{\alpha\beta}_{(+)} .\]

Here \(D_{(-)} = -i\{[D^{-}, D^{+}], D^{+}\}\). Then the component form of the action \(30\) is given by:
\[S = \int d^2 \xi \left[ \epsilon_{\alpha\beta\gamma\delta} \left( 2 \pi^{\alpha\beta}_{(+)} \bar{\lambda}^{\gamma} \lambda^{\delta} + 2 i \partial_{(+)} \theta^{i\alpha} \bar{\theta}_{i\beta} \bar{\lambda}^{\gamma} \lambda^{\delta} \right) \right.\]
\[+ \left. i D_{(-)} \theta^{i\alpha} \bar{\theta}_{i\beta} \pi^{\gamma\delta}_{(+)} - i \sigma_{\alpha\beta} \left( \pi^{\alpha\beta}_{(-)} + 2 \bar{\lambda}^{\alpha} \lambda^{\beta} \right) \right].\] (36)

In components, the left local supersymmetry transformations contained in \(32\) read:
\[\delta \theta^{i\alpha} = \rho^{i} \bar{\lambda}^{\alpha} + \bar{\rho}^{\alpha} \lambda^{i},\]
\[\delta x^{\alpha\beta} = -\frac{i}{2} \rho^{i} (\bar{\lambda}^{\alpha} \theta^{i\beta} - \bar{\theta}^{i\alpha} \lambda^{\beta}) - \frac{i}{2} \bar{\rho}^{i} (\lambda^{\alpha} \theta^{i\beta} - \theta^{i\alpha} \lambda^{\beta}),\] (37)
\[\delta \lambda^{\alpha} = i \rho^{i} D_{(-)} \theta^{i\alpha}, \quad \delta \bar{\lambda}^{\alpha} = -i \bar{\rho}^{i} D_{(-)} \theta^{i\alpha},\]
\[\delta \sigma_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} \partial_{(+) \gamma} (\rho^{i} \bar{\lambda}^{\delta} \theta^{i\beta} + \bar{\rho}^{i} \lambda^{\delta} \theta^{i\beta}).\]
In the action (36) the field $\sigma_{\alpha\beta}$ is a Lagrange multiplier for the constraint:

$$\bar{\lambda}^\alpha \lambda^\beta - \bar{\lambda}^\beta \lambda^\alpha = -\hat{\pi}^{\alpha\beta}_{(-)} ,$$

which has as a consequence the Virasoro constraint

$$(\hat{\pi}_{(-)})^2 = \epsilon_{\alpha\beta\gamma\delta} \hat{\pi}^{\alpha\beta}_{(-)} \hat{\pi}^{\gamma\delta}_{(-)} = 0$$

(39)

(the second Virasoro constraint can be obtained in precisely the same manner as in the case $D = 3$, see section 2). Further, only the product $\bar{\lambda}\lambda$ appears in the action, so we can use the constraint (38) to rewrite the action as follows:

$$S = \int d^2 \xi \epsilon_{\alpha\beta\gamma\delta} \left[ -\hat{\pi}^{\alpha\beta}_{(+)} \hat{\pi}^{\gamma\delta}_{(-)} - i \partial_{(+)} \theta^i \theta^j \hat{\pi}^{\gamma\delta}_{(-)} + i D_{(-)} \theta^i \theta^j \hat{\pi}^{\gamma\delta}_{(+)} \right] .$$

(40)

This, together with the constraint (39), is just the action for the GS superstring in six dimensions.

Once again, the procedure of elimination of the twistor variables $\bar{\lambda}, \lambda$ breaks the left local supersymmetry, but some memory of it is kept, which is just kappa symmetry. It emerges after the replacement $\rho^i = \lambda^\alpha \kappa^i_\alpha$, $\bar{\rho}^i = -\bar{\lambda}^\alpha \kappa^i_\alpha$ in eqs. (37) and the elimination of the product $\bar{\lambda}\lambda$ from the resulting transformations with the help of the on-shell constraint (38). In this way one obtains

$$\delta \theta^i = \frac{1}{2} \kappa^i_\beta \hat{\pi}^{\alpha\beta}_{(-)} , \quad \delta x^\alpha = -\frac{i}{2} (\delta \theta^i \theta^j - \delta \theta^j \theta^i) .$$

These can be recognized as the kappa symmetry transformations of the GS superstring. We stress again that this identification is only possible on shell, where the constraint (38) is valid.

5 Conclusions

The obvious question which did not find its answer in the present paper, is how to approach the most interesting case of the $D = 10$ superstring with $N = (8,0)$ world-sheet supersymmetry. The problem is that the notion of complex structure and the associated Grassmann analyticity, heavily used in this paper, do not have a natural extension to the case $D = 10, N = (8,0)$. However, there exists an alternative approach, which is based on the properties of the eight-sphere realized as a coset space of the $D = 10$ lorentz group, and does not make use of any complex structures (see [17] for the case of the superparticle and a forthcoming publication for the superstring).

Amongst possible immediate developments of the results presented here let us mention coupling the above superstring actions to target-space background supergravity and super-Yang-Mills, as well as introducing additional world-sheet superfields in order to describe the internal degrees of freedom of the heterotic superstrings.

Finally, let us point out that in the present paper we dealt with an $n = 1$ target superspace. In the context of a twistor-like formulation this naturally goes together with an world-sheet supersymmetry of the heterotic $(N,0)$ type. Analogous formulations for
the $n = 2$ GS superstrings are expected to be combined with world-sheet supersymmetries of the $(N, N)$ type and thus to involve two independent sets of twistor variables. The latter may be used to replace both vectors $\pi^\mu_\pm$, $\pi^\nu_\pm$ and thus to simultaneously solve both Virasoro constraints of the superstring, along the lines of refs. [2, 3, 11].

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