Quantile hedging on markets with proportional transaction costs

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Abstract

In the paper a problem of risk measures on a discrete-time market model with transaction costs is studied. Strategy effectiveness and shortfall risk is introduced. This paper is a generalization of quantile hedging presented in [4].

Key words: quantile hedging, shortfall risk, transaction costs, risk measures.

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1 Introduction

It is well known that on a classical market without transaction costs the price $x_0$ of a contingent claim $C$ is given as $x_0 = \sup_{Q \in \mathcal{Q}} E^Q[C]$, where $\mathcal{Q}$ is a set of all martingale measures equivalent to the objective measure $P$. This means that if we have an initial endowment $x \geq x_0$ then we can hedge $C$. Thus for $x$ there exists a self-financing strategy $B$ for which the terminal value $X_T^{x,B}$ is not smaller then $C$. If $x < x_0$ then we no longer can hedge $C$. For each strategy $B$ we have $P(X_T^{x,B} < C) > 0$. The investor who wants to hedge $C$ in some way must consider some risk connected with the fact that he is not able to hedge $C$ entirely. There appeared many risk measures introduced for instance by Cvitač and Karatzas [1], Pham [8], Föllmer and Leukert [4] and [5]. Cvitač and Karatzas study the following risk measure: $\inf_{B \in \mathcal{B}} E[(C - X_T^{x,B})^+]$, where $\mathcal{B}$ is a set of all self-financing strategies. Pham introduced $L^p$ hedging in [8] and his risk measure is defined as $\inf_{B \in \mathcal{B}} E[l_p((C - X_T^{x,B})^+)]$, where $l_p(x) = \frac{x^p}{p}$. Another examples of risk measures are provided by Föllmer and Leukert in [4]. They consider so called quantile hedging problem introducing a random variable connected with the strategy $(x, B)$ by defining

$$\varphi_{x,B} = \frac{1_{\{X_T^{x,B} \geq C\}}}{C} + \frac{X_T^{x,B}}{C} 1_{\{X_T^{x,B} < C\}}.$$

This random variable is called ”success function” and its expectation is an effectiveness measure connected with the strategy $(x, B)$. Success function takes its values in the interval $[0, 1]$. If $(x, B)$ is a hedging strategy, then $\varphi_{x,B} = 1$, otherwise $P(\varphi_{x,B} < 1) > 0$ what implies $E[\varphi_{x,B}] < 1$. Their aim is to find the strategy $B$ to maximize $E[\varphi_{x,B}]$ for a given $x$. In the next paper [5]
they also examine another risk measure which is given as \( \inf_{B \in \mathcal{B}} \mathbb{E}[l(((C - X_T^{v,B})^+))] \), where \( l \) is a loss function.

In this paper we study a problem of risk measures on markets with proportional transaction costs. The main idea is based on papers of Föllmer and Leukert on quantile hedging [4] and minimizing shortfall risk [5]. On markets with transaction costs we are given a multi-dimensional contingent claim \( H \), multi-dimensional wealth process \( V_T^{v,B} \) and some cone \( K_T \) which is constructed on a basis of transaction costs. The cone \( K_T \) indicates a partial ordering “\( \succeq_T \)” in \( \mathbb{R}^d \) in the sense that \( x \succeq_T y \iff x - y \in K_T \). We say that strategy \((v, B)\) hedges \( H \) if \( V_T^{v,B} \succeq_T H \). In papers [2], [6] and [7] the authors provide characterization of the set \( \Gamma(H) \subseteq \mathbb{R}^d \) of initial endowments for which exists a hedging strategy \( B \) such that \( V_T^{v,B} \succeq_T H \). The problem arises, what in a sense, is an optimal strategy for an initial endowment \( v \notin \Gamma(H) \). For the terminal wealth \( V_T^{v,B} \) we introduce a set of proportional transfers which is denoted by \( \mathcal{L}(V_T^{v,B}, H) \). Simply speaking, for \( L \in \mathcal{L}(V_T^{v,B}, H) \) we have \( \frac{(V_T^{v,B} - L)_i}{H_i} = \frac{(V_T^{v,B} - L)_j}{H_j} \) for all \( i, j \) where \( V_T^{v,B} \) is a terminal wealth after a proportional transfer \( L \). For this ratio we denote \( \frac{V_T^{v,B}_L}{H} := \frac{(V_T^{v,B} - L)_i}{H_i} \). In section 3 we introduce the “success function” which expectation is an effectiveness measure of the strategy \((v, B)\) by setting

\[
\varphi_{v,B} = 1 \{ V_T^{v,B} \succeq_T H \} + \mathbb{E} \left[ \sup_{L \in \mathcal{L}(V_T^{v,B}, H)} \frac{V_T^{v,B}_L}{H} \right] 1 \{ V_T^{v,B} \succeq_T H \}^c.
\]

We establish some useful properties of the success function. It appears that \( \varphi_{v,B} \in [0, 1] \) and if \( v \in \Gamma(H) \) then \( \varphi_{v,B} = 1 \) for the hedging strategy \( B \), whereas for \( v \notin \Gamma(H) \) we have \( P(V_T^{v,B} < 1) > 0 \) for each strategy \( B \) what implies \( \mathbb{E}[\varphi_{v,B}] < 1 \). Our aim is to find the strategy \( B \) for the initial endowment \( v \) to maximize \( \mathbb{E}[\varphi_{v,B}] \). We consider also another problem. For \( 1 \geq \varepsilon \geq 0 \) we characterize the set \( \Gamma_\varepsilon(H) \subseteq \mathbb{R}^d \) of initial endowments for which exists the strategy \( B \) such that \( \mathbb{E}[\varphi_{v,B}] \geq 1 - \varepsilon \). These are two aspects of quantile hedging which are analogous to problems presented by Föllmer and Leukert.

Then, in section 6, we introduce a shortfall risk in quantile hedging. Shortfall is defined as

\[
s(V_T^{v,B}) = \begin{cases} 0 & \text{on the set } \{ V_T^{v,B} \succeq_T H \} \\ \left(1 - \mathbb{E} \left[ \sup_{L \in \mathcal{L}(V_T^{v,B}, H)} \frac{V_T^{v,B}_L}{H} \right] \right) & \text{on the set } \{ V_T^{v,B} \not\succeq_T H \}^c. \end{cases}
\]

Shortfall is a \([0, 1]\)-valued random variable which is equal to 0 if \( V_T^{v,B} \succeq_T H \) and it is strictly positive if \( V_T^{v,B} \not\succeq_T H \). It describes the part of the contingent claim which is not hedged by the strategy \((v, B)\). We study the problem of minimizing shortfall risk given as \( \mathbb{E}[u(s(V_T^{v,B}))] \), where \( u : [0, 1] \rightarrow \mathbb{R} \) is a loss function. We accept here the assumption, that the investor considers only the percentage of the contingent claim which is not hedged as a loss, not the value of this part. As before, we study two problems. Firstly, in section 6, we characterize the strategy \( B \) which minimizes shortfall risk. Secondly, in section 7, we characterize the set \( \Gamma_\alpha(H) \) of initial endowments for which there exists the strategy \( B \) such that \( \mathbb{E}[u(s(V_T^{v,B}))] \leq \alpha \) for a given number \( \alpha \geq 0 \).

In section 8 we show how Föllmer’s and Leukert’s theory can be obtained under zero transaction costs. Since condition \( \text{EF} \) imposed in [6] is not satisfied, we use results shown in [2].
2 Market with proportional transaction costs

In this section we present some results obtained by Kabanov, Rásonyi, Stricker in papers [6] and [7] which deal with conditions for the absence of arbitrage under friction. We particularly need a hedging theorem providing description of the set of initial endowments which allow to hedge the contingent claim.

Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1,...,T}, P) \) be a probability space equipped with a complete, discrete-time filtration. We assume that \( \mathcal{F}_0 \) is a trivial \( \sigma \)-field and that \( \mathcal{F}_T = \mathcal{F} \). On \( \Omega \) we are given a strictly positive \( \mathbb{R}^d \)-valued, adapted process \( S_t \) which describes the prices of \( d \) traded securities. We can assume that, for instance the first component is a price of a bond, but it is not necessary for further consideration. Proportional transaction costs are given as the process \( \Lambda_t = (\lambda_t^{ij})_{i,j=1,2,...,d} \) with values in the set \( M^d_+ \) of matrices with non-negative, adapted entries and zero diagonal. If we want to increase the \( j \)-th stock account by the amount \( L_t^{ij} \geq 0 \) at time \( t \), then we have to transfer an amount \((1 + \lambda_t^{ij})L_t^{ij} \) from the \( i \)-th account. The quantity \( \lambda_t^{ij}L_t^{ij} \) is lost because of occurring transaction costs. Given an initial endowment \( v \in \mathbb{R}^d \) we invest in stocks at each time \( t = 0, 1, ..., T \). The agent’s position at time \( t \) can be described either by vector \( V_t \) of stock units or by vector \( V_t \) of values invested in each stock. The relation between these quantities is: \( V_t^i = V_t^i S_t^i \). Operator “\( \hat{\cdot} \)” will be used also for any random vector \( Z \) and \( \hat{Z} \) stands for \((\frac{Z^1}{S^1}, ..., \frac{Z^d}{S^d})\). A self-financing portfolio is defined by its increments as follows

\[
\Delta V_t^i = \hat{V}_{t-1}^i \cdot \Delta S_t^i + \Delta B_t^i \quad i = 1, ..., d, \quad t = 0, 1, ..., T,
\]

with convention for initial values \( V_{-1}^i = v^i, \ S_{-1} = S_0, \ L_{-1}^{ij} = 0 \) for \( i,j = 1,2, ..., d \) and where

\[
\Delta B_t^i := \sum_{j=1}^d \Delta L_t^{ij} - \sum_{j=1}^d (1 + \lambda_t^{ij}) \Delta L_t^{ji}.
\]

Here we denote \( \Delta Y_t = Y_t - Y_{t-1} \) for each process \( Y \). The adapted, increasing and non-negative process \( L_{ij} \) represents the net cumulative transfers from the position \( i \) to the position \( j \) under transaction costs. The increment \( \Delta V_t^i \) of value on \( i \)-th stock account consists of two parts: the increment \( \hat{V}_{t-1}^i \Delta S_t^i \) due to the price movements and the increment \( \Delta B_t^i \) caused by agent’s action at time \( t \). Since the pair \((v, B)\) determines the wealth process \( V_t^{v,B} \), we will treat it as a trading strategy.

In the sequel we will use the following notation: \( L^0(A, \mathcal{F}_t) \), where \( A \subseteq \mathbb{R}^d \) is a set of \( \mathcal{F}_t \)-measurable random variables which take values in the set \( A \). \( L^0(M^d_+, \mathcal{F}_t) \) stands for matrices which entries are non-negative and \( \mathcal{F}_t \) measurable random variables. Let

\[
M_t(\omega) := \left\{ x \in \mathbb{R}^d : \exists L \in L^0(M^d_+, \mathcal{F}_t) \text{ such that} \right\}
\]

be a set of position which can be converted into zero by a non-negative transfer. This set is a polyhedral cone. Let \( K_t := \mathbb{R}^d_+ + M_t \) and \( F_t := K_t \cap (-K_t) \). The set \( K_t \), which is called the solvency region, is a polyhedral cone. It is formed by vectors which can be transformed into a vector with only non-negative components by a positive transfer, thus by adding a vector from \(-M_t\). \( F_t \) represents positions which can be converted into zero and vice versa. \( F_t \) is a linear space.

We shall say that a strategy \((0, B)\) is a weak arbitrage opportunity at time \( t \) if \( V_t^{0,B} \in K_t \).
and \( P(V_t^{0,B} \in K_t \setminus F_t) > 0 \). There is an absence of a weak arbitrage opportunity if there does not exist arbitrage opportunity at any time. The absence of a weak arbitrage opportunity (strict no-arbitrage property) can be expressed in geometric terms:

\[
\text{NA}^s : \quad R_t \cap L^0(K_t, F_t) \subseteq L^0(F_t, F_t) \quad \text{for } t = 0, 1, ..., T,
\]

where \( R_t := \left\{ V_t^{0,B} : B \in B; \ B\text{-set of all strategies} \right\} \).

The set \( R_t \) describes wealth at time \( t \) which can be obtained starting with the zero initial endowment.

Let us define an efficient friction condition.

\[
\text{EF} : \quad \text{The cones } K_t(\omega) \text{ are proper, i.e. } F_t(\omega) = \{0\} \text{ for each } (\omega, t).
\]

Under \( \text{EF} \) the condition \( \text{NA}^s \) can be rewritten as \( R_t \cap L^0(K_t, F_t) = \{0\} \) for \( t = 0, 1, ..., T \). Under \( \text{EF} \) there are some equivalent conditions to \( \text{NA}^s \). For more details see [6].

The most important result for this paper is a description of the set of initial endowments which allow to hedge the contingent claim. Let us start with the fact that the cone \( K_t \) generates a partial ordering \( "\succeq_t" \) on \( \mathbb{R}^d \) in the sense that \( x \succeq_t y \iff x - y \in K_t \). Contingent claim \( H \) is an \( \mathbb{R}^d \) valued random variable and the set

\[
\Gamma(H) := \{ v \in \mathbb{R}^d : \text{there exists a strategy } B \text{ such that } V_t^{v,B} \succeq_t H \}\,
\]

stands for all hedging initial endowments. For simplicity we assume that \( H \succeq_t c \mathbf{1} \) for some \( c \in \mathbb{R} \). The next theorem presented in [7] provides description of the set \( \Gamma(H) \).

**Theorem 2.1** Assume that \( \text{EF} \) and \( \text{NA}^s \) are satisfied. Then

\[
\Gamma(H) = \left\{ v \in \mathbb{R}^d : \hat{Z}_0 v \geq \mathbf{E} \hat{Z}_T H \quad \forall Z \in \mathcal{Z} \right\}
\]

where \( \mathcal{Z} \) is the set of bounded martingales such that \( \hat{Z}_t \in L^0(K_t^*, F_t) \) for \( t = 0, 1, ..., T \) and where \( K_t^* \) denotes the dual cone to the cone \( K_t \).

From now on we assume that conditions \( \text{EF} \) and \( \text{NA}^s \) are satisfied.

**3 Strategy effectiveness**

In this section we introduce a success function \( \varphi_{v,B} \) for the strategy \( (v, B) \) and establish its properties. Its expectation under \( P \) is in fact some kind of risk measure, but more adequate risk measure will be defined in section 6. This one we accept rather as an effectiveness measure.

We will consider only admissible strategies and from now on we assume that \( H \succeq_t 0 \) almost everywhere.

**Definition 3.1** Strategy \( (v, B) \) is admissible if \( V_t^{v,B} \succeq_t 0 \).

Let \( (v, B) \) be an admissible strategy. Our aim is to describe its effectiveness regarding the contingent claim \( H \). Divide \( \Omega \) into two parts: \( \{ V_t^{v,B} \succeq_t H \} \) and \( \{ V_t^{v,B} \succeq_t H \}^c \). On the set \( \{ V_t^{v,B} \succeq_t H \} \) we put \( \varphi_{v,B} = 1 \). The next part of this section is to define \( \varphi_{v,B} \) on the set \( \{ V_t^{v,B} \succeq_t H \}^c \) and examine its basic properties.
For the terminal wealth $V_T^{v,B}$ and transfer $L \in L^0(M^d_+, \mathcal{F}_T)$ we will consider $V_T^{v,B}$ after transfer $L$ under transaction costs at time $T$ given by

$$(V_T^{v,B} | L)^i = (V_T^{v,B})^i + \sum_{j=1}^d L_{ji} - \sum_{j=1}^d (1 + \lambda_T^j)L_{ji}.$$ 

In the set of all transfers $L^0(M^d_+, \mathcal{F}_T)$ we distinguish a subclass of proportional transfers.

**Definition 3.2** Assume that for an admissible strategy $(v, B)$ holds $V_T^{v,B} \not\geq_T H$. Transfer $L \in L^0(M^d_+, \mathcal{F}_T)$ is a proportional transfer if there exists $c_L \in L^0(\mathbb{R}, \mathcal{F}_T)$ such that 

$$V_T^{v,B} | L = c_L \cdot H.$$ 

$\mathcal{L}(V_T^{v,B}, H)$ stands for the class of all proportional transfers and for $L \in \mathcal{L}(V_T^{v,B}, H)$ we denote 

$$\frac{V_T^{v,B}}{H} \mid L := c_L.$$ 

**Remark 3.3** $\mathcal{L}(V_T^{v,B}, H)$ is not empty since $(v, B)$ is admissible. This means that there exists $L_0 \in L^0(M^d_+, \mathcal{F}_T)$ for which $V_T^{v,B} | L_0 = 0$, thus $c_{L0} = \frac{V_T^{v,B} | L_0}{H} = 0$.

The meaning of the class of proportional transfers is to achieve the same ”rate of hedge” on each stock account. We want to make this rate as high as possible. Thus on the set $\{V_T^{v,B} \geq_T H\}^c$ we define $\varphi_{v,B}$ as 

$$\varphi_{v,B} = 1_{\{V_T^{v,B} \geq_T H\}} + \text{ess sup}_{L \in \mathcal{L}(V_T^{v,B}, H)} \frac{V_T^{v,B} | L}{H} 1_{\{V_T^{v,B} \geq_T H\}}^c.$$ 

**Lemma 3.4** Assume that $V_T^{v,B} \not\geq_T H$. There exists an optimal transfer $\hat{L} \in \mathcal{L}(V_T^{v,B}, H)$ such that 

$$\text{ess sup}_{L \in \mathcal{L}(V_T^{v,B}, H)} \frac{V_T^{v,B} | L}{H} = \frac{V_T^{v,B} | \hat{L}}{H}.$$ 

**Proof**

Let us consider two geometrical objects which depend on $\omega$: the translated polyhedron cone $V_T^{v,B} + (-M_T)$ with its boundary $\partial(V_T^{v,B} + (-M_T))$ and the line spanned by the vector $H$. 

$V_T^{v,B} + (-M_T)$ is generated by $m$ measurable vectors $\xi_1, \xi_2, ..., \xi_m$, where $d \leq m \leq d(d-1)$ and can be represented as an intersection of $l$ half-spaces for some $l$. The $i$-th half-space is spanned by $d-1$ generators $\xi_{i1}, \xi_{i2}, ..., \xi_{i,l-1}$ from the set $\xi_1 \xi_2, ..., \xi_m$. Putting $g_i = \xi_{i1} \times \xi_{i2} \times ... \times \xi_{i,l-1}$ where $\times$ denotes the cross product, we obtain a measurable vector which is orthogonal to each vector from the set $\xi_{i1}, \xi_{i2}, ..., \xi_{i,l-1}$. Thus the $i$-th half-space has the following representation:

$$\left\{ x \in \mathbb{R}^d : (x - V_T^{v,B}) \cdot g_i \geq 0 \right\},$$

and the boundary of the cone can be represented as:

$$x \in \partial(V_T^{v,B} + (-M_T)) \iff \begin{cases} (x - V_T^{v,B}) \cdot g_i \geq 0 & \forall i = 1, 2, ..., l  \\ (x - V_T^{v,B}) \cdot g_i = 0 & \text{for some } i = 1, 2, ..., l. \end{cases}$$
On the other hand the line spanned by the vector $H$ can be represented as

$$x \in \text{span}\{H\} \iff x \cdot h_i = 0 \quad \forall i = 1, 2, \ldots, d - 1,$$

where \(\{H, h_1, h_2, \ldots, h_{d-1}\}\) is a basis in \(\mathbb{R}^d\), each vector \(h_i\) is measurable and \(H \perp h_i\) for all \(i = 1, 2, \ldots, d - 1\). Such basis can be obtained by taking the set \(\{H, H + e_1, H + e_2, \ldots, H + e_d\}\), where \(\{e_1, e_2, \ldots, e_d\}\) is a standard basis in \(\mathbb{R}^d\), choosing a subset of \(d\) linear independent vectors containing \(H\) and then orthogonalizing it starting with the vector \(H\).

There exists exactly one positive point \(\hat{V}\) of intersection \(\partial(\mathbb{V}_{v,B}^T + (-M_T))\) with \(\text{span}\{H\}\). Since it is a solution of linear system with measurable coefficients

$$\begin{cases}
(x - \mathbb{V}_{v,B}^T \cdot g_i \geq 0 & \forall i = 1, 2, \ldots, l \\
(x - \mathbb{V}_{v,B}^T \cdot g_i = 0 & \text{for some } i = 1, 2, \ldots, l \\
x \cdot h_i = 0 & \forall i = 1, 2, \ldots, d - 1,
\end{cases}$$

it is a measurable random vector. Hence also measurable is \(\hat{c}\), where \(\hat{V} = \hat{c}H\).

Each transfer is represented by adding to \(\mathbb{V}_{v,B}^T\) some vector from the cone \((-M_T)\). As \(\hat{L}\) we get the transfer represented by \(\hat{V} - \mathbb{V}_{v,B}^T\). From construction of \(\hat{V}\) we conclude that for any other proportional transfer such that \(\mathbb{V}_{v,B}^T |_{\hat{L}} = \hat{c}H\) we have \(\hat{c} \leq \hat{c}\). As a consequence we obtain

$$\hat{c} = \text{ess sup} \frac{\mathbb{V}_{v,B}^T |_{\hat{L}}}{H} = \frac{\mathbb{V}_{v,B}^T |_{\hat{L}}}{H}. \quad \square$$

**Remark 3.5** The success function fulfils

$$0 \leq \varphi_{v,B} 1_{\mathbb{V}_{v,B}^T \geq T H}^c < 1$$

**Proof**

$$\varphi_{v,B} 1_{\mathbb{V}_{v,B}^T \geq T H}^c \geq 0$$

since \((v, B)\) is admissible. If \(\varphi_{v,B} 1_{\mathbb{V}_{v,B}^T \geq T H}^c \geq 1\) then \(1_{\mathbb{V}_{v,B}^T \geq T H}^c \mathbb{V}_{v,B}^T |_{\hat{L}} / H \geq 1\). This implies \(\mathbb{V}_{v,B}^T |_{\hat{L}} \geq H\) on the set \(\mathbb{V}_{v,B}^T \geq T H\) but this means that \(\mathbb{V}_{v,B}^T \geq T H\) which is a contradiction. \(\square\)

To summarize, the success function \(\varphi_{v,B}\) is equal to 1 if \(\mathbb{V}_{v,B}^T \geq T H\) and strictly smaller then 1 if \(\mathbb{V}_{v,B}^T \not\geq T H\).

In the next part of the paper we will work with the set

$$\mathcal{R} := \{\varphi : 0 \leq \varphi \leq 1; \varphi \text{ is } \mathcal{F}_T \text{ measurable}\}$$

of \(\mathcal{F}_T\) measurable functions which takes values in \([0, 1]\).

We start with two useful properties of the success function.

**Lemma 3.6** Assume that \((v, B)\) is an admissible strategy. Then \(v \in \Gamma(H \varphi_{v,B})\).
Proof
In view of lemma 3.4 we have
\[ H \varphi _{v,B} = H \mathbf{1}_{\{ V^{v,B}_T \succeq _T H \}} + H \text{ ess sup } \frac{V^{v,B}_T}{H} \mathbf{1}_{\{ V^{v,B}_T \succeq _T H \}^c} \]
\[ = H \mathbf{1}_{\{ V^{v,B}_T \succeq _T H \}} + V^{v,B}_T |_{\hat{L}} \mathbf{1}_{\{ V^{v,B}_T \succeq _T H \}^c} \]
where \( \hat{L} \) is an optimal proportional transfer. Thus we have \( v \in \Gamma (H \varphi _{v,B}) \).
\[ \square \]

Lemma 3.7 Assume that \((v, B)\) is a hedging strategy for a modified contingent claim \( H \varphi \) for some function \( \varphi \in \mathcal{R} \). Then \( \varphi _{v,B} \geq \varphi \).

Proof
Since \( V^{v,B}_T \succeq _T H \varphi \), there exists transfer \( M \in L^0(M_x^d, \mathcal{F}_T) \) such that \( V^{v,B}_T |_{M} - H \varphi \geq 0 \). Let \( N \in \mathcal{L}(V^{v,B}_T |_{M} - H \varphi, H) \) be any proportional transfer on the set \( \{ V^{v,B}_T \succeq _T H \}^c \) such that \( \mathbf{1}_{\{ V^{v,B}_T \succeq _T H \}^c} |_{N} = \gamma \) for some \( \gamma \geq 0 \). Let us consider the terminal wealth \( V^{v,B}_T \) on the set \( \{ V^{v,B}_T \succeq _T H \}^c \) after transfer \( K \) described as follows: first change \( V^{v,B}_T \) by the transfer \( M \) and then change \( V^{v,B}_T |_{M} - H \varphi \) by transfer \( N \). The terminal wealth \( V^{v,B}_T \) after transfer \( K \) is thus given as
\[ V^{v,B}_T |_{K} = H \varphi + (V^{v,B}_T |_{M} - H \varphi) |_{N}. \]
It is clear that \( K \in \mathcal{L}(V^{v,B}_T, H) \) since
\[ V^{v,B}_T |_{K} = H \varphi + H \gamma = (\varphi + \gamma) H. \]
This leads to the following inequalities
\[ \varphi _{v,B} = \mathbf{1}_{\{ V^{v,B}_T \succeq _T H \}} + \text{ ess sup } \frac{V^{v,B}_T}{H} \mathbf{1}_{\{ V^{v,B}_T \succeq _T H \}^c} \]
\[ \geq \mathbf{1}_{\{ V^{v,B}_T \succeq _T H \}} + \frac{V^{v,B}_T |_{K}}{H} \mathbf{1}_{\{ V^{v,B}_T \succeq _T H \}^c} \]
\[ = \mathbf{1}_{\{ V^{v,B}_T \succeq _T H \}} + (\varphi + \gamma) \mathbf{1}_{\{ V^{v,B}_T \succeq _T H \}^c} \]
\[ \geq \varphi. \]
\[ \square \]
4 Quantile hedging - effectiveness maximization

The set $\Gamma(H)$ is a set of all initial endowments which allow to hedge the contingent claim $H$. If $v \in \Gamma(H)$ then there exists a strategy $B \in B$ such that $V_T^{v,B} \succeq_T H$. Suppose that we are given an initial capital $v_0$, such that $v_0 \notin \Gamma(H)$. A natural question arises: what is an optimal strategy for $v_0$? As the optimality criteria we accept an expectation of the success function under measure $P$. If for two admissible strategies $(v, B)$ and $(\bar{v}, \bar{B})$ holds $E[\varphi_{v,B}] \geq E[\varphi_{\bar{v},\bar{B}}]$ then strategy $(v, B)$ is at least as effective as $(\bar{v}, \bar{B})$. If $(v, B)$ is at least as effective as any other admissible strategy, then it is called optimal. The problem of finding optimal strategy for $v_0$ is a first aspect of quantile hedging problem and we formally formulate it as follows:

For a fixed initial endowment $v_0 \in \Gamma(0)$ such that $v_0 \notin \Gamma(H)$ find an admissible strategy $(v, B)$, where $v_0 \succeq v$, such that $E[\varphi_{v,B}] \to \text{max}$. To describe optimal strategy, we start with the following theorem.

Theorem 4.1 There exists a function $\hat{\varphi} \in R$ which is a solution of the problem

$$E[\varphi] \to \text{max}$$

$v_0 \in \Gamma(H\varphi)$.

Proof

Let us denote $R_0 := \{ \varphi \in R : v_0 \in \Gamma(H\varphi) \}$, $R_0 \neq \emptyset$ since $0 \in R_0$. Let $\varphi_n \in R_0$ be a sequence of elements such that $E[\varphi_n] \to \sup_{\varphi \in R_0} E[\varphi]$. Since $\{\varphi_n\}$ is a sequence of elements from a hull in $L^\infty(\Omega)$, there exists a subsequence $\varphi_{nk}$ which converges to $\hat{\varphi}$ in a weak * topology. One can prove that $\hat{\varphi}$ belongs to $R$. We will show that $v_0 \in \Gamma(H\hat{\varphi})$. Each element of the sequence $\{\varphi_n\}$ satisfies $\hat{Z}_0v_0 \geq E[\hat{Z}_TH\varphi_n]$ $\forall Z \in Z$, and $\hat{\varphi}$ as a weak limit satisfies

$$\forall Z \in Z \quad \hat{Z}_0v_0 \geq E[\hat{Z}_TH\varphi_{nk}] \xrightarrow{k} E[\hat{Z}_TH\hat{\varphi}].$$

Thus $v_0 \in \Gamma(H\hat{\varphi})$. $\square$

The next theorem provides the solution of our problem.

Theorem 4.2 Let $\hat{\varphi}$ be a function from theorem 4.1, and the strategy $(v_0, B)$ be a hedging strategy for the modified contingent claim $H\hat{\varphi}$. Then $(v_0, B)$ is an optimal strategy. Furthermore, $\hat{\varphi} = \varphi_{v_0,B}$.

Proof

$(v_0, B)$ is admissible since $V_T^{v_0,B} \succeq_T H\hat{\varphi} \succeq_T 0$.

Let $(\bar{v}, \bar{B})$ be any admissible strategy such that $v_0 \succeq \bar{v}$. Then by lemma 3.6 we have: $\bar{v} \in \Gamma(H\varphi_{\bar{v},\bar{B}})$ and this implies that $v_0 \in \Gamma(H\varphi_{\bar{v},\bar{B}})$. From theorem 4.1 we have

$$E[\varphi_{\bar{v},\bar{B}}] \leq E[\hat{\varphi}]. \tag{4.2.1}$$

Now, let us consider the strategy $(v_0, B)$. Since $V_T^{v_0,B} \succeq_T H\hat{\varphi}$, by lemma 3.7 we have:

$$\varphi_{v_0,B} \geq \hat{\varphi}. \tag{4.2.2}$$

By virtue of (4.2.1) and (4.2.2) we have $\varphi_{v_0,B} = \hat{\varphi}$. Hence $(v_0, B)$ is optimal. $\square$
5 Quantile hedging - sets with a fixed level of effectiveness

Assume, that we are given a number \( \varepsilon \in [0, 1] \). We want to characterize strategies which effectiveness is not smaller than \( 1 - \varepsilon \). This is the second aspect of quantile hedging and in fact our task is to characterize the set \( \Gamma_\varepsilon(H) \) which is given as

\[
\Gamma_\varepsilon(H) = \left\{ v \in \mathbb{R}^d : \text{there exists an admissible strategy } B \text{ such that } E[\varphi_{v,B}] \geq 1 - \varepsilon \right\}.
\]

It is clear that \( \Gamma_{\varepsilon_1}(H) \subseteq \Gamma_{\varepsilon_2}(H) \) if \( \varepsilon_1 \leq \varepsilon_2 \). Hence set \( \Gamma_\varepsilon(H) \) contains the set \( \Gamma(H) = \Gamma_0(H) \), for any \( \varepsilon \in [0, 1] \) but it can contain more elements as the initial capitals which allow to hedge \( H \) with some loss of effectiveness.

Let us set

\[
\mathcal{M} := \{ \varphi \in \mathcal{R} : E[\varphi] \geq 1 - \varepsilon \}.
\]

The next theorem provides a description of the set \( \Gamma_\varepsilon(H) \).

**Theorem 5.1** The set \( \Gamma_\varepsilon(H) \) admits the following representation

\[
\Gamma_\varepsilon(H) = \bigcup_{\varphi \in \mathcal{M}} \Gamma(H\varphi).
\]

**Proof**

\( \subseteq \)

Let \( v \in \Gamma_\varepsilon(H) \). Then there exists \( B \in \mathcal{B} \) such that \( V_T^{v,B} \succeq_T 0 \) and \( E[\varphi_{v,B}] \geq 1 - \varepsilon \). Thus \( \varphi_{v,B} \in \mathcal{M} \) and

\[
\Gamma(H\varphi_{v,B}) \subseteq \bigcup_{\varphi \in \mathcal{M}} \Gamma(H\varphi).
\]

\( \supseteq \)

Let \( v \in \bigcup_{\varphi \in \mathcal{M}} \Gamma(H\varphi) \). Then there exists \( \varphi \in \mathcal{M} \) such that \( v \in \Gamma(H\varphi) \). Let us consider the strategy \((v, B)\) which hedges the modified contingent claim \( H\varphi \). Then by lemma 3.7 we have

\[
V_T^{v,B} \succeq_T H\varphi \implies \varphi_{v,B} \geq \varphi,
\]

and as a consequence \( E[\varphi_{v,B}] \geq E[\varphi] \geq 1 - \varepsilon \). Finally, we have \( v \in \Gamma_\varepsilon(H) \). \( \square \)

6 Risk measure in quantile hedging - minimizing shortfall risk

On markets without transaction costs shortfall is defined as \( (C - X_T^{v,B})^+ \), where \( a^+ = \max\{a, 0\} \).

In this section we introduce a shortfall connected with the strategy \((v, B)\) under transaction costs. To this end we use the set of proportional transfers. Shortfall risk is introduced as an expectation of a loss function of shortfall. Our aim is to minimize shortfall risk for a fixed initial capital over all admissible strategies.

In section 3 we introduced a random variable

\[
\text{ess sup}_{L \in \mathcal{L}(V_T^{v,B}, H)} \frac{V_T^{v,B}}{H}
\]

defined on the set \( \{V_T^{v,B} \succeq_T H\}^c \).

It describes the part of the contingent claim which is successfully hedged. As shortfall we accept the remaining part: \( 1 - \text{ess sup}_{L \in \mathcal{L}(V_T^{v,B}, H)} \frac{V_T^{v,B}}{H} \). Let us start with formal definition.
Similarly to previous sections we formulate the first aspect of risk measure problem as:

\[
\begin{aligned}
\text{observe a sequence } \tilde{\phi} \\
\text{and it is strictly positive if } V_{T}^{v,B} \geq_{T} H \\
\text{for two admissible strategies.}
\end{aligned}
\]

Let us denote \( V_{T}^{v,B} \) be a sequence of elements such that

\[
\begin{aligned}
E_{\mathcal{L}(V_{T}^{v,B},H)} \left( \begin{array}{c}
\text{on the set } \left\{ V_{T}^{v,B} \geq_{T} H \right\} \\
\text{on the set } \left\{ V_{T}^{v,B} \geq_{T} H \right\}^{c}.
\end{array} \right)
\end{aligned}
\]

**Proof**

Shortfall can be expressed in terms of the success function. We have

\[
1 - \varphi_{v,B} = 0 \mathbf{1}_{\left\{ V_{T}^{v,B} \geq_{T} H \right\}} + \left( 1 - \text{ess sup}_{L \in \mathcal{L}(V_{T}^{v,B},H)} \frac{V_{T}^{v,B}}{H} \right) \mathbf{1}_{\left\{ V_{T}^{v,B} \geq_{T} H \right\}}^{c} = s(V_{T}^{v,B}).
\]

Shortfall is a random variable which takes values in the interval \([0, 1]\). It is equal to 0 if \( V_{T}^{v,B} \geq_{T} H \) and it is strictly positive if \( V_{T}^{v,B} \not\geq_{T} H \).

Let \( u : [0, 1] \rightarrow \mathbb{R} \) be a continuous, non-decreasing function such that \( u(0) = 0 \) and \( u(1) < \infty \). We regard such function as a loss function. Basing on a loss function we define the shortfall risk of an admissible strategy as \( E[u(s(V_{T}^{v,B}))] \). It is clear that if \( v \in \Gamma(H) \) then shortfall risk is equal to 0 for the hedging strategy, otherwise it is positive. If for two admissible strategies \( (v, B) \) and \( (\tilde{v}, \tilde{B}) \) holds \( E[u(s(V_{T}^{v,B}))] \leq E[u(s(V_{T}^{\tilde{v},\tilde{B}}))] \) then we regard the strategy \( (v, B) \) as not as risky as \( (\tilde{v}, \tilde{B}) \). If the shortfall risk of the strategy \( (v, B) \) is not greater than any other, then \( (v, B) \) is called optimal or risk-minimizing.

Similarly to previous sections we formulate the first aspect of risk measure problem as:

For a fixed initial endowment \( v_{0} \in \Gamma(0) \) such that \( v_{0} \not\in \Gamma(H) \) find an admissible strategy \((v, B)\), where \( v_{0} \geq_{0} v \), such that \( E[u(s(V_{T}^{v,B}))] \rightarrow \min \).

We start with the auxiliary lemma proved in [3].

**Lemma 6.3** Let \( X_{1}, X_{2}, \ldots \) be a sequence of \([0, \infty)\) random variables. There exists a sequence \( X_{n} \in \text{conv}\{X_{n}, X_{n+1}, \ldots\} \) such that \( X_{n} \) converges almost surely to a \([0, \infty)\) valued random variable \( \tilde{X} \).

To describe optimal strategy we start with the following theorem.

**Theorem 6.4** There exists a function \( \tilde{\varphi} \in \mathcal{R} \) which is a solution of the problem

\[
E[u(1 - \varphi)] \rightarrow \min
\]

\( v_{0} \in \Gamma(H\varphi) \).

**Proof**

Let us denote \( \mathcal{R}_{0} := \{ \varphi \in \mathcal{R} : v_{0} \in \Gamma(H\varphi) \} \). \( \mathcal{R}_{0} \neq \emptyset \) since \( 0 \in \mathcal{R}_{0} \). Let \( \varphi_{n} \in \mathcal{R}_{0} \) be a sequence of elements such that \( E[u(1 - \varphi_{n})] \rightarrow \inf_{\varphi \in \mathcal{R}_{0}} E[u(1 - \varphi)] \). In view of lemma 6.3 there exists a sequence \( \tilde{\varphi}_{n} \in \text{conv}\{\varphi_{n}, \varphi_{n+1}, \ldots\} \) which converges almost surely to \( \tilde{\varphi} \in \mathcal{R} \). Since \( u(1 - \tilde{\varphi}_{n}) \leq u(1) \leq \infty \), by dominated convergence theorem we obtain

\[
E[u(1 - \tilde{\varphi})] = \lim_{n \rightarrow \infty} E[u(1 - \varphi_{n})] = \inf_{\varphi \in \mathcal{R}_{0}} E[u(1 - \varphi)].
\]
From Fatou’s lemma we have
\[ E[\tilde{Z}_T H \tilde{\varphi}] = E[\lim_n \tilde{Z}_T H \tilde{\varphi}_n] \leq \liminf_n E[\tilde{Z}_T H \tilde{\varphi}_n] \leq \tilde{Z}_0 v_0 \quad \forall Z \in \mathcal{Z}. \]

Hence \( v_0 \in \Gamma(H \tilde{\varphi}). \) \( \square \)

The next theorem provides a description of the risk-minimizing strategy for \( v_0 \).

**Theorem 6.5** Let \( \tilde{\varphi} \) be a function from theorem 6.4 and the strategy \((v_0, B)\) be a hedging strategy for the modified contingent claim \( H \tilde{\varphi} \). Then \((v_0, B)\) is an optimal strategy. Furthermore, \( \tilde{\varphi} = \varphi_{v_0, B} \).

**Proof**

\((v_0, B)\) is admissible since \( V_{v_0, B} \succeq_T H \tilde{\varphi} \succeq_T 0 \).

Let \((\tilde{v}, \tilde{B})\) be any admissible strategy such that \( v_0 \succeq_0 \tilde{v} \). Then by lemma 3.6 we have \( \tilde{v} \in \Gamma(H \varphi_{	ilde{v}, \tilde{B}}) \) what implies \( v_0 \in \Gamma(H \varphi_{v_0, B}) \). From remark 6.2 and theorem 6.4 we obtain

\[ E[u(s(V_{v_0, B}^\tilde{B}))] = E[u(1 - \varphi_{\tilde{v}, \tilde{B}})] \geq E[u(1 - \varphi)]. \]  \hspace{1cm} (6.5.3)

Now let us consider the strategy \((v_0, B)\). Since \( V_{v_0, B} \succeq_T H \tilde{\varphi} \), by lemma 3.7 we have:

\[ \varphi_{v_0, B} \geq \tilde{\varphi}. \]  \hspace{1cm} (6.5.4)

Taking (6.5.3) and (6.5.4) into account we have \( \varphi_{v_0, B} = \tilde{\varphi} \), thus \( E[u(s(V_{v_0, B}^\tilde{B}))] = E[u(H - H \tilde{\varphi})] \)

and this proves that \((v_0, B)\) is optimal. \( \square \)

### 7 Risk measure in quantile hedging - sets with a fixed level of shortfall risk

Assume, that we are given a number \( \alpha \geq 0 \). We want to characterize strategies for which shortfall risk is not larger than \( \alpha \). This is the second aspect of risk measure problem in quantile hedging. Our task is to provide a description of the set \( \Gamma^u_\alpha(H) \) given as

\[ \Gamma^u_\alpha(H) := \left\{ v \in \mathbb{R}^d : \text{there exists an admissible strategy } B \right\} \quad \text{such that } E[u(s(V_{v, B}^\tilde{B}))] \leq \alpha. \]

It is clear that \( \Gamma^u_{\alpha_1}(H) \subseteq \Gamma^u_{\alpha_2}(H) \) if \( \alpha_1 \leq \alpha_2 \). Since for the hedging strategy \((v, B)\) holds \( E[u(s(V_{v, B}^\tilde{B}))] = 0 \), we conclude that set \( \Gamma^u_\alpha(H) \) contains the set \( \Gamma(H) = \Gamma^u_0(H) \) for any \( \alpha \geq 0 \).

Let us set

\[ \mathcal{N} := \{ \varphi \in \mathcal{R} : E[u(1 - \varphi)] \leq \alpha \} \]

The next theorem provides a description of the set \( \Gamma^u_\alpha(H) \).

**Theorem 7.1** The set \( \Gamma^u_\alpha(H) \) admits the following representation

\[ \Gamma^u_\alpha(H) = \bigcup_{\varphi \in \mathcal{N}} \Gamma(H \varphi). \]
Proof
This proof is similar to the proof of theorem 5.1.
Let \( v \in \Gamma_0^u(H) \). Then there exists a strategy \( B \in \mathcal{B} \) such that \( V^{v,B}_T \succeq_T 0 \) and \( \mathbb{E}[u(s(V^{v,B}_T))] = \mathbb{E}[u(1 - \varphi_{v,B})] \leq \alpha \). Thus \( \varphi_{v,B} \in \mathcal{N} \) and by lemma 3.6 we have \( v \in \Gamma(H\varphi_{v,B}) \subseteq \bigcup_{\varphi \in \mathcal{N}} \Gamma(H\varphi) \).

Let \( v \in \bigcup_{\varphi \in \mathcal{N}} \Gamma(H\varphi) \). Then there exists \( \varphi \in \mathcal{N} \) such that \( v \in \Gamma(H\varphi) \). Let us consider the strategy \((v, B)\) which hedges the modified contingent claim \( H\varphi \). Then by lemma 3.7 we have \( V^{v,B}_T \succeq_T H\varphi \implies \varphi_{v,B} \geq \varphi \) and this implies \( \mathbb{E}[u(s(V^{v,B}_T))] = \mathbb{E}[u(1 - \varphi_{v,B})] \leq \mathbb{E}[u(1 - \varphi)] \leq \alpha \).

In effect we have \( v \in \Gamma_0^u(H) \). \(\square\)

8 Quantile hedging under zero transaction costs

In this section we show how the theory of Föllmer and Leukert can be obtained. All previous sections required the EF condition which of course is not satisfied under zero transaction costs.

We will base on results obtained by Delbaen, Kabanov, Valkeila [2] which are less general than results used so far, but the condition EF is not required there. First, we give a short description of these results, then recall two aspects of quantile hedging studied by Föllmer and Leukert and then show how their theory can be obtained under zero transaction costs.

In cited paper we assume that transaction costs are constant in time, given by a matrix \( \Lambda \). Contingent claim is bounded from below in the sense of partial ordering determined by the cone \( K := M + \mathbb{R}_d^+ \) for some \( c \in \mathbb{R} \). \( K \) is independent on \( t \) and \( \omega \). We denote by \( Q \) the set of probability measures \( Q \sim P \) such that \( S_t \) follows a local martingale in respect to \( Q \). We shall need EMM condition.

\(\text{EMM} : \quad Q \neq \emptyset.\)

Let \( D \) be the set of martingales \( Z \) with \( \hat{Z} \) taking values in \( K^* \) and bounded \( \hat{Z}_T \). Under EMM condition we have the following description of the set of hedging endowments:

\[
\Gamma(H) = \bigcap_{Z \in D} \{ v \in \mathbb{R}^d : \hat{Z}_0 v \geq \mathbb{E}\hat{Z}_TH \}.
\]

It is left as an exercise to check that under this new description of \( \Gamma(H) \) theorems 4.1, 4.2, 5.1, which solve our problems remain true.

Now take a look on a classical market model without transaction costs. Under no-arbitrage condition the price of a scalar contingent claim \( C \) is given as \( \sup_{Q \in Q} \mathbb{E}^Q[C] \). In the quantile hedging problem studied by Föllmer and Leukert we consider only admissible strategies \((x, B)\) for which the wealth process \( X^{x,B}_t \geq 0 \) for all \( t = 0, 1, ..., T \). The authors use as an effectiveness measure the success function defined as

\[
\varphi_{x,B} = 1_{\{X^x,B \geq C\}} + \frac{X^x_B}{C} 1_{\{X^x,B < C\}}.
\]
The first problem
Let \( x_0 \) be a fixed initial endowment. We search for such admissible strategy \((x, B)\), where \( x \leq x_0 \), to maximize \( E[\varphi_{x, B}] \). We write this as
\[
E[\varphi_{x, B}] \rightarrow \max_{x \leq x_0}.
\]

The second problem
Let \( \varepsilon \) be a fixed number in \([0, 1]\). We search for such admissible strategy \((x, B)\) which effectiveness is not smaller than \( 1 - \varepsilon \) in order to minimize the initial capital. We write this problem as
\[
E[\varphi_{x, B}] \geq 1 - \varepsilon \quad x \rightarrow \min.
\]

To show that these problems can be obtained under zero transaction costs we have to find scalar equivalents of multi-dimensional objects on our market. Let \( Y \in \mathbb{R}^d \) describes how our wealth is allocated in stock positions on the market with transaction costs. Now choose the \( i \)-th stock account to transfer capitals from all others on it. Then the wealth of \( Y \) in the \( i \)-th stock is :
\[
Y(i) := \sum_{j=0}^{d} (1 - \lambda_{ji}) Y^j.
\]

Usually \( Y(i) \neq Y(j) \) for \( i \neq j \), but under zero transaction costs we have \( Y(i) = Y(j) = \sum_{i=1}^{d} Y^i \). Thus we accept the following scalar equivalents: for the initial endowment \( v \) we take \( x_v := \sum v^i \), for the wealth process \( V_{t}^{v, B} \) we take \( X_t^{x_v, B} := \sum (V_t^{v, B})^i \), for the contingent claim \( H \) we take \( C_H := \sum H^i \).

Now we show that problems of quantile hedging under zero transaction costs are the same as formulated by the authors for scalar equivalents.

First, note that
\[
\frac{V_T^{v, B}}{H} \mid L = \sum_{i=0}^{d} \frac{(V_T^{v, B})_i}{H^i} \quad \forall L \in \mathcal{L}(V_T, H). \tag{8.0.5}
\]

For each \( L \in \mathcal{L}(V_T, H) \) we have \( \sum \frac{V_T^{v, B}}{H} H^i = \sum \frac{V_T^{v, B}}{H} H^i \) and \( \sum \frac{V_T^{v, B}}{H} \mid L H^i = \sum (V_T^{v, B} \mid L)^i \), so
\[
\frac{V_T^{v, B}}{H} \mid L = \sum \frac{(V_T^{v, B} \mid L)^i}{H^i}.
\]
Since the costs are equal to zero, thus \( \sum (V_T^{v, B} \mid L)^i = \sum (V_T^{v, B})^i \) and (8.0.5) holds.

Since relation “\( \geq \)” becomes a linear ordering “\( \geq \)” for the sums of components, we get the
equality of the success functions.

\[ \varphi_{v,B} = 1_{\{v_T^B \geq H\}} + \text{ess sup}_{L \in \mathcal{L}(V_v^B,H)} \frac{V_T^B}{H} 1_{\{v_T^B \geq H\}^c} \]

\[ = 1_{\{\sum(v_T^B) \geq \sum H\}} + \sum_{i} (V_T^B)_i \frac{\sum H_i}{\sum H} - 1_{\{\sum(v_T^B) < \sum H\}} \]

\[ = 1_{\{X_{x,v,B}^T \geq C_H\}} + \frac{X_{x,v,B}^T}{C_H} 1_{\{X_{x,v,B}^T < C_H\}} \]

\[ = \varphi_{x,v,B} \]

One can check that the set of the hedging endowments is of the form 
\[ \Gamma(H) = \{v \in \mathbb{R}^d : \sum v^i \geq \sup_{Q \in \mathcal{Q}} E^Q[\sum H^i]\}. \]
Then our problem of maximizing effectiveness

\[ E[\varphi_{v,B}] \longrightarrow \max \]

\[ v \leq v_0 \notin \Gamma(H) \]

becomes

\[ E[\varphi_{x,B}] \longrightarrow \max \]

\[ x \leq x_{v_0} < \sup_{Q \in \mathcal{Q}} E^Q[C_H], \]

what is the first problem considered by Föllmer and Leukert.

Our second problem is to determine the set \( \Gamma_{\epsilon}(H) \). First denote that if for \( v, \tilde{v} \in \mathbb{R}^d \) holds \( \sum \tilde{v}^i \geq \sum v^i \) and \( v \in \Gamma_{\epsilon}(H) \) then \( \tilde{v} \in \Gamma_{\epsilon}(H) \). For \( \gamma_v := \sum v^i \) define \( \gamma := \inf_{v \in \Gamma_{\epsilon}(H)} \gamma_v \). If for \( v \in \mathbb{R}^d \) holds \( \sum v^i \geq \gamma \) then \( v \in \Gamma_{\epsilon}(H) \) and if \( \sum v^i < \gamma \) then \( v \notin \Gamma_{\epsilon}(H) \). Thus the set \( \Gamma_{\epsilon}(H) \) is of the form \( \Gamma_{\epsilon}(H) = \{v \in \mathbb{R}^d : \sum v^i \geq \gamma\} \). The problem reduces to finding the number \( \gamma \) which is the cost minimizing capital searched by Föllmer and Leukert.

Remark 8.1 Föllmer and Leukert considered admissible strategies for which \( X_{x,v,B}^T \geq 0 \) for each \( t = 0,1, ..., T \). We only require \( X_{x,v,B}^T \geq 0 \), what is a generalization.

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