Classification of Type A $\mathcal{N}$-fold Supersymmetry

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Abstract

Type A $\mathcal{N}$-fold supersymmetry of one-dimensional quantum mechanics can be constructed by using $\mathfrak{sl}(2)$ generators represented on a finite dimensional functional space. Using this $\mathfrak{sl}(2)$ formalism we show a general method of constructing Type A $\mathcal{N}$-fold supersymmetric models. We also present systematic generation of known models and several new models using this method.
1 Introduction

$\mathcal{N}$-fold supersymmetry is an extension of the ordinary supersymmetry in one-dimensional quantum mechanics and in fact reduces to it for $\mathcal{N} = 1$ [1]-[3]. It has supercharges that are $\mathcal{N}$-th polynomials of momentum. (Similar higher derivative generalizations of supercharges were also investigated in different contexts [7]-[20].) It shares the nonrenormalization theorems of ordinary supersymmetry [21, 22]: Some of the energy eigenvalues can be obtained in a closed form when $\mathcal{N}$-fold supersymmetry is not spontaneously broken, while perturbative parts of some energy eigenvalues are obtained otherwise. This property is related to what has been known as “quasi-solvability” [23]-[25].

Recently we have shown that the Type A variety of $\mathcal{N}$-fold supersymmetry can be constructed using $\mathfrak{sl}(2)$ generators, which are defined on a functional space formed by solvable energy eigenstates [3]. Using this method of construction, we will in this paper classify models of Type A $\mathcal{N}$-fold supersymmetry, some of which are new.

In the following, we first give a brief review of $\mathcal{N}$-fold supersymmetry and its construction by the use of $\mathfrak{sl}(2)$ in Section 2. In Section 3, we first explain the method of model construction and the way the models are classified, using the fact that physical content of the model remains invariant under a linear transformation of a function $h$ that is used to form a basis of the quasi-solvable space. Then in the rest of Section 3 we construct a representative model for each class. Section 4 contains summary and discussions.

2 Type A $\mathcal{N}$-fold supersymmetry and $\mathfrak{sl}(2)$

We will first briefly review the definitions and the fundamental properties of $\mathcal{N}$-fold supersymmetry, its Type A subclass, and the relation with $\mathfrak{sl}(2)$ [3].

An $\mathcal{N}$-fold supersymmetry model in one-dimensional quantum mechanics has a Hamiltonian $H_\mathcal{N}$ of the form;

$$H_\mathcal{N} = H_\mathcal{N}^- \psi \psi^\dagger + H_\mathcal{N}^+ \psi^\dagger \psi,$$

where $\psi$ and $\psi^\dagger$ are fermionic coordinates;

$$\{ \psi, \psi \} = \{ \psi^\dagger, \psi^\dagger \} = 0, \quad \{ \psi, \psi^\dagger \} = 1,$$

and $H_\mathcal{N}^\pm$ are ordinary Hamiltonians for one ordinary (bosonic) coordinate $q$;

$$H_\mathcal{N}^\pm = \frac{1}{2} p^2 + V_\mathcal{N}^\pm(q)$$

with $p = -i d/dq$.

The $\mathcal{N}$-fold supercharges are generically defined as

$$Q_\mathcal{N} = P_\mathcal{N}^\dagger \psi, \quad Q_\mathcal{N}^\dagger = P_\mathcal{N} \psi^\dagger,$$

where $P_\mathcal{N}$ is an $\mathcal{N}$-th order polynomial of $p$:

$$P_\mathcal{N} = w_\mathcal{N}(q) p^\mathcal{N} + w_{\mathcal{N}-1}(q) p^{\mathcal{N}-1} + \cdots + w_1(q) p + w_0(q).$$
These satisfy the following $\mathcal{N}$-fold supersymmetry algebra;

\[ \{ Q_\mathcal{N}, Q_\mathcal{N}^\dagger \} = \{ Q_\mathcal{N}^\dagger, Q_\mathcal{N} \} = 0, \quad (2.6) \]
\[ [ Q_\mathcal{N}, H_\mathcal{N} ] = [ Q_\mathcal{N}^\dagger, H_\mathcal{N} ] = 0. \quad (2.7) \]

In addition, the anticommutator \( \{ Q_\mathcal{N}, Q_\mathcal{N}^\dagger \} \) induces “mother Hamiltonian”, whose relation to the Hamiltonian \( (2.3) \) is discussed in detail in Ref.\[5\]. The nilpotency \( (2.6) \) is trivially satisfied due to the property of the fermionic coordinates \( (2.2) \). And the latter relation \( (2.7) \) leads to

\[ P_\mathcal{N} H_\mathcal{N}^\pm - H_\mathcal{N}^\pm P_\mathcal{N} = 0, \quad (2.8) \]

and its conjugate. The identity \( (2.8) \) induces differential equations for the functions \( V_\mathcal{N}^\pm(q) \) and \( W_\mathcal{N}(q) \) \((n = 0, 1, \cdots, \mathcal{N})\), which cannot be solved in general.

In Ref.\[6\] we showed that when a quasi-solvability condition is satisfied an $\mathcal{N}$-fold supersymmetric model can be constructed. In Ref.\[6\] we used this method of construction for the Type A subclass of $\mathcal{N}$-fold supersymmetry, which is defined by the following form of the supercharges \[4\]:

\[ P_\mathcal{N} = \left( \bar{D} + i \frac{\mathcal{N} - 1}{2} E(q) \right) \left( \bar{D} + i \frac{\mathcal{N} - 3}{2} E(q) \right) \cdots \left( \bar{D} - i \frac{\mathcal{N} - 1}{2} E(q) \right), \quad (2.9) \]

where \( \bar{D} \equiv p - i \bar{W}(q) \). We choose the functional subspace where solvable energy eigenstates belong as the space \( \mathcal{V} \) spanned by the bases \( \{1, h(q), h(q)^2, \cdots, h(q)^{\mathcal{N}-1}\} U^{-1} \), where \( h(q) \) is a solution of a differential equation

\[ E(q) = \frac{h(q)''}{h(q)'}, \quad (2.10) \]

(prime denotes derivative with respect to \( q \)) and

\[ U \equiv \exp \left[ \int \left( \bar{W}(q) + \frac{\mathcal{N} - 1}{2} E(q) \right) dq \right]. \quad (2.11) \]

We showed that the Hamiltonian \( H_\mathcal{N}^\pm \) of Type A $\mathcal{N}$-fold supersymmetry is always written as;

\[ H_\mathcal{N}^\pm = U^{-1} \left[ - \sum_{i,j=+,0,-} a_{ij} J^i J^j + \sum_{i=+,0,-} b_i J^i + C \right] U, \quad (2.12) \]

where \( J^i \) are \( sl(2) \) generators on \( \{1, h, h^2, \cdots, h^{\mathcal{N}-1}\} \);

\[ J^+ \equiv h^2 \frac{d}{dh} - (\mathcal{N} - 1) h, \quad J^0 \equiv h \frac{d}{dh} - \frac{\mathcal{N} - 1}{2}, \quad J^- \equiv \frac{d}{dh}, \quad (2.13) \]

and \( a_{ij}, b_i \) and \( C \) are constants. These generators satisfy the algebra,

\[ [J^+, J^-] = -2 J^0, \quad [J^\pm, J^0] = \mp J^\pm, \quad (2.14) \]
and form the following Casimir operator:

\[ \frac{1}{2} (J^+ J^- + J^- J^+) - (J^0)^2 = -\frac{1}{4}(N^2 - 1). \]  

(2.15)

By using this identity, we choose \( a_{++} = 0 \) in the Hamiltonian (2.12) without losing generality. As a result, there are eight independent real parameters \( \{a_{++}, a_{+0}, a_{00}, a_{--}, b_+, b_0, b_-\} \) in the Hamiltonian.

Given the above form of the Hamiltonian, the energy eigenstates are obtained in a trivial manner: One can use \( N \times N \) matrix representations of the generators \( J^i \) and write \( U^{-1} H U \) as an \( N \times N \) matrix and simply diagonalize it.

In Ref.\[6\], we derived the following conditions by requiring that the Hamiltonian \( H_N^{-} \) reduce to the canonical form (2.3);

\[ P_4(h) \equiv a_{++} h^4 + a_{+0} h^3 + a_{00} h^2 + a_{--} h + a_- = \frac{1}{2} (h')^2, \]  

(2.16)

and

\[ P_3(h) \equiv 2(N-2) a_{++} h^3 + \left( \frac{3N-5}{2} a_{+0} + b_+ \right) h^2 \\
+ ((N-2) a_{00} + b_0) h + \frac{N-1}{2} a_{--} + b_- \\
= h' \left( \bar{W} + N - 2 \frac{E}{2} \right) \]  

(2.17)

have to be satisfied. From the fact that \( P_4(h) \) is (at most) a fourth order polynomial of \( h \) and by the use of the relation (2.10), we found the following condition on \( E(q) \):

\[ \left( \frac{d}{dq} - 2E \right) \left( \frac{d}{dq} - E \right) \frac{d}{dq} \left( \frac{d}{dq} + E \right) E = 0, \]  

(2.18)

for \( N \geq 2 \). Also, by comparing the coefficients of the highest order terms of \( P_4(h) \) and \( P_3(h) \), we found

\[ \left( \frac{d}{dq} - E \right) \frac{d}{dq} \left( \frac{d}{dq} + E \right) \bar{W} = 0, \]  

(2.19)

for \( N \geq 3 \). When these conditions are met, the potentials \( V_N^\pm(q) \) are given by the following:

\[ V_N^\pm = \frac{1}{2} \bar{W}^2 + \frac{N^2 - 1}{24} \left( E^2 - 2E' \right) \pm \frac{N}{2} \bar{W}'. \]  

(2.20)

It is also possible to derive the conditions (2.18)–(2.20) by a direct calculation of the \( N \)-fold supersymmetry algebra (2.8) \[31\].
3 Model construction

From the results reviewed in the previous section, we see that one method of constructing a Type A $\mathcal{N}$-fold supersymmetry model is to find a solution $E(q)$ of the differential equation (2.18), solve Eq.(2.19) to find a solution $\tilde{W}(q)$, and then obtain the potentials $V^\pm_N(q)$ according to the relation (2.20). We analyzed several models in this manner in Ref.[6].

This method, however, is difficult to carry out, or at least cumbersome. The construction reviewed above provides a rather simple and compact alternative to the above program: Instead of solving the nonlinear differential equation (2.18), we can use its origin, Eq.(2.16), which can be solved as,

$$q = \pm \int \frac{dh}{\sqrt{2P_4(h)}}. \quad (3.1)$$

Once $h(q)$ is obtained by inverting Eq.(3.1), we can obtain $E(q)$ according to Eq.(2.10) and $\tilde{W}(q)$ according to the relation (2.17), or

$$\tilde{W}(q) = \frac{P_3(h)}{h'} - \frac{N-2}{2} E$$

$$= \frac{1}{h'} \left( P_3(h) - \frac{N-2}{2} \frac{d}{dh} P_4(h) \right)$$

$$= \frac{1}{h'} \left( \bar{b}_+ h^2 + b_0 h + \bar{b}_- \right), \quad (3.2)$$

where we have used a relation

$$E(q) = \frac{h''}{h'} = \frac{1}{h'^2} \frac{d}{dq} \left( \frac{1}{2} h'^2 \right) = \frac{1}{h'} \frac{d}{dh} P_4(h), \quad (3.3)$$

and absorbed some of the $a$-coefficients in the redefinition of $b$-coefficients:

$$\bar{b}_+ \equiv b_+ + \frac{1}{2} a_{+0}, \quad \bar{b}_- \equiv b_- + \frac{1}{2} a_{0-}. \quad (3.4)$$

Once this is done, the potentials are simply given by the expression (2.20) from $E(q)$ and $\tilde{W}(q)$. They are given as the following functions of $h$,

$$V^\pm_N = \frac{1}{16P_4(h)} \left( (\mathcal{N}^2 - 1) \left( \frac{d}{dh} P_4(h) \right)^2 + 4w(h)^2 \pm 4\mathcal{N} w(h) \frac{d}{dh} P_4(h) \right)$$

$$- \frac{\mathcal{N}^2 - 1}{12} \frac{d^2}{dh^2} P_4(h) \pm \mathcal{N} \frac{d}{dh} w(h), \quad (3.5)$$

where $w(h) = \bar{b}_+ h^2 + b_0 h + \bar{b}_-$.

Unfortunately, for the most general fourth order polynomial $P_4(h)$, the integral in Eq.(3.1) involves elliptic integrals, and therefore the function $h(q)$ cannot be written down explicitly. For this reason, we will start with the most simple $P_4(h)$ and proceed to more complex ones. In doing so, it is important to classify resulting models, as many of them
are equivalent to each other. This is because of the following: Let us define a new function \( \hat{h} \) by the following;

\[
h = s\hat{h} + t,
\]

where \( s \) and \( t \) are \( q \)-independent constants. Using \( \hat{h} \) in place of \( h \) does not affect the physical content of the model, because \( h \) is introduced to form the basis \( \{1, h, h^2, \ldots, h^{N-1}\} \) that defines the space \( V \) and the new base \( \{\hat{h}, \hat{h}^2, \ldots, \hat{h}^{N-1}\} \) defines the same space of \( V \). This invariance under the linear transformation (3.6) is also evident in the expression \( E(q) \) in Eq. (2.10). A caution, however, is needed for \( P_4(h) \). In order for Eq. (2.16) to be satisfied with \( \hat{h} \), or equivalently, for \( q \) to be invariant (up to the trivial integration constant) in Eq. (3.1), we need to replace as \( P_4(h) \) as follows:

\[
P_4(h) \rightarrow \hat{P}_4(\hat{h}) = \frac{1}{s^2} P_4(h).
\]

As for \( \tilde{W}(q) \), in doing the replacement (3.6) we can transform the coefficients so that \( \tilde{W}(q) \) remains invariant. This way, the potentials remain invariant under the transformation (3.6) and (3.7).

It is also important to note that on the right hand side of Eq. (3.1), even a complex integration constant is allowed, since it still is the solution of Eq. (2.16). Therefore, we can freely perform the following complex translation of \( q \) in \( h(q) \), \( E(q) \), \( \tilde{W}(q) \) and \( V^\pm_N(q) \);

\[
q \rightarrow q + q_0,
\]

where \( q_0 \) is an arbitrary complex constant. We will also use this when necessary.

In the following construction of model, we will classify the resulting models according to the distribution of zero points (and their orders) of \( P_4(h) \). And then we will use the transformation (3.6) and (3.7) to choose representatives of each class by moving the zero points of \( P_4(h) \) to convenient locations; typically \( h = 0 \) and \( |h| = 1 \), with the zero point \( h = 0 \) having the highest order, as much as we can. (In doing so, we will avoid distinguishing \( \hat{h} \) from \( h \) and \( \hat{P}_4 \) from \( P_4 \) unless it is necessary.) We will denote the resulting models by the set of the zero points of \( P_4(h) \) with the order of each zero point as a superindex (if more that one); for example \( \{0^2, 1\} \) denotes a model who has a zero point of order 2 at \( h = 0 \) and a zero point of order 1 at \( h = 1 \). Also, in the following we will choose the sign on the right hand side of Eq. (3.1), by using reflection of \( q \), so that the resulting expression is free from extra minus signs.

### 3.1 Case (0): Constant \( P_4(h) \); \( \{\phi\} \)

The most simple model is given by a constant \( P_4(h) \). For \( P_4(h) = a_{--} \), we obtain the following:

\[
h(q) = \sqrt{2a_{--}} q,
\]

which induces,

\[
E(q) = 0,
\]

\[
\tilde{W}(q) = \sqrt{2a_{--}} \bar{b}_+ q^2 + b_0 q + \frac{\bar{b}_-}{\sqrt{2a_{--}}},
\]
3.2 Case (1): Linear $P_4(h) : \{0\}$

Applying the transformation (3.7), we can get rid of the parameter $a_{--}$ (and still not lose the generality). We choose to do so with $s = \sqrt{2a_{--}}$, so that we have $P_4(h) = 1/2$, which is equivalent to setting $a_{--} = 1/2$ to start with. This way, we obtain the following:

$$h(q) = q,$$
$$E(q) = 0,$$
$$\tilde{W}(q) = \tilde{b}_+ q^2 + b_0 q + \tilde{b}_-,$$
$$V_{\tilde{N}}(q) = \frac{\tilde{b}_+^2}{2} q^4 + b_+ b_0 q^3 + \left( \tilde{b}_+ \tilde{b}_- + \frac{1}{2} b_0^2 \right) q^2 + (b_0 \tilde{b}_- + N\tilde{b}_+) q + \frac{1}{2} \left( \tilde{b}_-^2 \pm N b_0 \right).$$  

With suitable definition of parameters, the above potentials become asymmetric double-well potentials, whose $N$-fold supersymmetry is spontaneously broken by nonperturbative effects. The solvable energy eigenvalues represent only the perturbative parts (by suitable definition of the coupling constant). Its nonperturbative aspects and the asymptotic behaviors of the perturbation series were studied in Refs.[1, 2] and resulted in the discovery of $N$-fold supersymmetry [1, 2].

### 3.2 Case (1): Linear $P_4(h) : \{0\}$

The function $P_4(h) = a_{0-} h + a_{--}$ leads to the following when the integration constant is appropriately chosen:

$$h(q) = \frac{a_{0-}}{2} q^2 + \sqrt{2a_{--}} q,$$

which results in:

$$E(q) = \frac{a_{0-}}{a_{0-} q + \sqrt{2a_{--}}},$$

$$\tilde{W}(q) = \frac{1}{4(a_{0-} q + \sqrt{2a_{--}})} \left[ a_{0-} \tilde{b}_+ q^4 + 4\sqrt{2a_{--}} a_{-} \tilde{b}_+ q^3 + (8 a_{-} \tilde{b}_+ + 2 a_{0-} b_0) q^2 + 4 \sqrt{2a_{--}} b_0 q + 4 \tilde{b}_- \right].$$

If we set $a_{0-} = 0$, we reproduce Case (0) studied above. For $a_{0-} \neq 0$, the above lead to potentials that are rational polynomials of $q$. In order to obtain a simple representative model, we apply the translation ($t$) to move the zero point of $P_4(h)$ to $h = 0$ and use the scaling ($s$) to set $P_4(h) = 2h$. This leads to the following:

$$h(q) = q^2,$$
$$E(q) = \frac{1}{q},$$

$$\tilde{W}(q) = \frac{1}{2} \left[ \tilde{b}_+ q^2 + b_0 q + \frac{\tilde{b}_-}{q} \right].$$
The potentials are given by the following;

\[ V_{N}^{\pm}(q) = \frac{b_2^2}{8} q^6 + \frac{b_+b_0}{4} q^4 + \frac{1}{8} \left( b_0^2 + 2b_+ b_- \pm 6N b_+ \right) q^2 + \frac{1}{8} \left( \bar{b}_-^2 + N^2 - 1 \mp 2N\bar{b}_- \right) \frac{1}{q^2} + \frac{1}{4} b_0 \left( \bar{b}_- \pm N \right) . \]  

(3.22)

This model contains the sextic potentials studied in Refs. [5, 27, 28] as special cases.

### 3.3 Case (2): Quadratic \( P_4(h) \)

The general quadratic function \( P_4(h) = a_{00} h^2 + a_{0-} h + a_{-} \) leads to the following;

\[ h(q) = \frac{a_{0-}}{2a_{00}} \left( \cosh(\alpha q) - 1 \right) + \sqrt{\frac{a_{-}}{a_{00}}} \sinh(\alpha q), \]  

(3.23)

\[ E(q) = \alpha \frac{a_{0-} \cosh(\alpha q) + 2\sqrt{a_{-} a_{00}} \sinh(\alpha q)}{a_{0-} \sinh(\alpha q) + 2\sqrt{a_{-} a_{00}} \cosh(\alpha q)}, \]  

(3.24)

where \( \alpha \equiv \sqrt{2a_{00}} \). Reproduction to Case (1) for \( a_{00} = 0 \) is evident in these expressions.

Models in this category are classified to two cases; (2a) one zero point of order 2, and (2b) two different zero points. We will choose the zero points for each case as; (2a) \( \{0^2\} \) and (2b) \( \{\pm 1\} \). There is also a notable variation to (2b), namely Case (2b') with zero points at \( \{\pm i\} \), which will be also studied.

**Case (2a): \( \{0^2\} \)**

In this case, we choose \( P_4(h) = a_{00} h^2 \). Note that since \( a_{00} \) is invariant under the transformation \( \{8.7\} \), it cannot be set to a particular value without losing generality. We obtain the following:

\[ h(q) = e^{\alpha q}, \]  

(3.25)

\[ E(q) = \alpha, \]  

(3.26)

\[ \tilde{W}(q) = \frac{1}{\alpha} \left[ \bar{b}_+ e^{\alpha q} + b_0 + \bar{b}_- e^{-\alpha q} \right], \]  

(3.27)

\[ V_{N}^{\pm}(q) = \frac{b_2^2}{4a_{00}} e^{2\alpha q} + \frac{b_+}{2} \left( \frac{b_0}{a_{00}} \pm N \right) e^{\alpha q} + \frac{b_-}{2} \left( \frac{b_0}{a_{00}} \mp N \right) e^{-\alpha q} + \frac{b_2^2}{4a_{00}} e^{-2\alpha q} + \frac{b_0^2}{4a_{00}} + \frac{b_+ b_-}{2a_{00}} + \frac{a_{00}}{12} \left( N^2 - 1 \right). \]  

(3.28)

These induces exponential potentials for \( a_{00} > 0 \) and periodic potentials for \( a_{00} < 0 \), whose special cases were studied in Refs. [3, 26].

**Case (2b): \( \{\pm 1\} \)**

With \( P_4(h) = a_{00}(h^2 - 1) \), we obtain the following:

\[ h(q) = \cosh(\alpha q), \]  

(3.29)
3.4 Case (3): Cubic \(P_4(h)\)

In case \(P_4(h)\) is a general cubic polynomial we can no longer write down an algebraic expression of \(h(q)\), as the integral in Eq. (3.3) yields expressions involving elliptic functions. On the other hand, for a case when \(P_4(h) = a_{00} h^2 + a_{0} h^3\), we find that we can write an algebraic expression of \(h(q)\). In this case, we obtain:

\[
h(q) = \frac{2a_{00}}{a_{0}} \cosh(\alpha q) - \frac{1}{a_{0} \cosh(\alpha q) + 1}.
\] (3.37)
In the above, the integration constant in Eq. (3.1) is chosen so that the expression of $h(q)$ is most simple. In order to obtain $h(q)$ of Eq. (3.25) of Case (2a), one needs to do a translation on $q$;

$$q \to q - \sqrt{\frac{2}{a_{00}}} \arctanh \sqrt{1 + \frac{a_{+0}}{a_{00}}},$$

(3.38)

first and then take $a_{+0} = 0$.

From the fact that we have the closed expression (3.37) for $P_4(h) = a_{+0} h^3 + a_{00} h^2$, we see that calculable cases are limited to the cases that have a zero point of at least order two. In the following, we choose the representative models as ones with zero points being (3a) $\{0^3\}$ and (3b) $\{0^2, 1\}$.

**Case (3a): $\{0^3\}$**

In this case, we use only the translation to move the zero point to $h = 0$. Using the remaining scaling degree of freedom, we can choose the coefficient of $h^3$ to any value. We choose $P_4(h) = 2 h^3$ to obtain the following:

$$h(q) = \frac{1}{q^2},$$

(3.39)

$$E(q) = -\frac{3}{q},$$

(3.40)

$$\tilde{W}(q) = -\frac{1}{2} \left( \tilde{b}_- q^3 + b_0 q + \frac{\tilde{b}_+}{q} \right),$$

(3.41)

$$V_{\pm N}^\pm(q) = \frac{\tilde{b}_+^2}{8} q^6 + \frac{b_0 \tilde{b}_-}{4} q^4 + \frac{1}{8} \left( \tilde{b}_+^2 + 2 \tilde{b}_+ \tilde{b}_- \mp 6 N \tilde{b}_- \right) q^2 + \frac{1}{4} b_0 \left( \tilde{b}_+ \mp N \right).$$

(3.42)

**Case (3b): $\{0^2, 1\}$**

In case we have one zero point of order 2, we use the translation on $h$ to move it to $h = 0$ and use scaling on $h$ to move the other zero point to $h = 1$. In this manner we obtain $P_4(h) = -a_{00} h^2 (h - 1)$ with $a_{00}$ remaining as an arbitrary parameter. In this case, we find the following using $\beta \equiv \sqrt{a_{00}/2}$:

$$h(q) = \frac{1}{\cosh^2(\beta q)},$$

(3.43)

$$E(q) = -2 \beta \frac{\cosh(2\beta q) - 2}{\sinh(2\beta q)},$$

(3.44)

$$\tilde{W}(q) = -\frac{1}{2\beta \sinh(\beta q)} \left[ \frac{\tilde{b}_+}{\cosh(\beta q)} + b_0 \cosh(\beta q) + \tilde{b}_- \cosh(\beta q) \right],$$

(3.45)

$$V_{\pm N}^\pm(q) = \frac{1}{\cosh^2(\beta q) \sinh^2(\beta q)} \left[ \frac{\tilde{b}_+^2}{4a_{00}} \cosh^8(\beta q) + \tilde{b}_- \left( \frac{b_0}{2a_{00}} \mp \frac{N}{2} \right) \cosh^6(\beta q) \right].$$
3.5 Case (4): Quartic $P_4(h)$

As is already evident in Case (3), we cannot do the integration in Eq.(3.1) for the most general case. This case, however, is rather important and interesting, since no models with $a_{++} \neq 0$ were listed in Ref.[23] or were constructed as an $\mathcal{N}$-fold supersymmetric model in the past. The latter is partly because of the fact that in our previous construction of $\mathcal{N}$-fold supersymmetric models [5], reduction on $\mathcal{N}$ was used with implicit assumption, which were satisfied only for $a_{++} = 0$, as was explained in detail in Ref.[6].

We find that for $P_4(h) = a_{++}h^4 + a_{+0}h^3 + a_{00}h^2$ we can obtain expression of $h(q)$ explicitly as follows;

$$h(q) = \frac{4a_{00}^{3/2} e^{\alpha q}}{a_{00} - 2a_{+0} \sqrt{a_{00}} e^{\alpha q} + (a_{+0}^2 - 4a_{++}a_{00}) e^{2\alpha q}}.$$  (3.47)

Therefore, we see that closed algebraic expressions can be written down when at least one zero point is of order 2 or higher. Such cases can be classified as follows; (4a) one zero point of order four, with representative model having $\{0^4\}$, (4b) one zero point of order 3 and another different zero point, with $\{0^3,1\}$, (4c) two zero points, both of them being of order 2, with $\{\pm1^2\}$, (4c') variation of (4c) with $\{\pm i^2\}$, (4d) one zero point of order 2 and two different zero points, whose representative model is, for example, $\{0^2,1,\sigma(\neq 1)\}$.

Since Case (4d) results in expressions that are not much simpler than Eq.(3.47), we will skip it and study its special cases of (4d1) $\{0^2,\pm1\}$, and (4d1') $\{0^2,\pm i\}$, which are related to each other by complex scaling on $h$.

**Case (4a): $\{0^4\}$**

In this case, we choose $P_4(h) = h^4/2$, since only the translation is used to move the zero point of order 4 to $h = 0$ and we have the freedom to choose the scaling on $h$. This choice leads to the following:

$$h(q) = \frac{1}{q},$$  \hspace{1cm} (3.48)

$$E(q) = -\frac{2}{q},$$  \hspace{1cm} (3.49)
\[ \widetilde{W}(q) = -\tilde{b}_- q^2 - b_0 q - \tilde{b}_+, \quad (3.50) \]

\[ V_N^\pm(q) = \frac{\tilde{b}_+^2}{2} q^4 + b_0 \tilde{b}_- q^3 + \frac{1}{2} \left( b_0^2 + 2\tilde{b}_+ \tilde{b}_- \right) q^2 + (\tilde{b}_+ b_0 + N\tilde{b}_-) q + \frac{1}{2} \left( \tilde{b}_+^2 + N b_0 \right). \quad (3.51) \]

**Case (4b): \{0^3, 1\}**

For \( P_4(h) = a_{++} h^3 (h - 1) \), we find the following:

\[ h(q) = -\frac{2}{a_{++} q^2 - 2}, \quad (3.52) \]

\[ E(q) = -\frac{3a_{++} q^2 + 2}{q(a_{++} q^2 - 2)}, \quad (3.53) \]

\[ \widetilde{W}(q) = \frac{a_{++} \tilde{b}_-}{4} q^3 - \left( \tilde{b}_- + \frac{b_0}{2} \right) q + \frac{\tilde{b}_+ + b_0 + \tilde{b}_-}{a_{++}} q. \quad (3.54) \]

\[ V_N^\pm(q) = \frac{a_{++}^2 \tilde{b}_+^2}{32} q^6 - \frac{a_{++} \tilde{b}_-(2\tilde{b}_- + b_0)}{8} q^4 + \frac{1}{8} \left( \tilde{b}_- (2\tilde{b}_+ + 6b_0 + 6\tilde{b}_-) + b_0^2 \pm 3Na_{++} \tilde{b}_- \right) q^2 \]

\[ + \left( \frac{(\tilde{b}_+ + b_0 + \tilde{b}_-)^2}{2a_{++}^2} + \frac{N^2 - 1}{8} + \frac{N}{2a_{++}} (\tilde{b}_+ + b_0 + \tilde{b}_-) \right) \frac{1}{q^2} \]

\[ - \frac{(2\tilde{b}_- + b_0)(2(\tilde{b}_+ + b_0 + \tilde{b}_-) \pm Na_{++})}{4a_{++}}. \quad (3.55) \]

**Case (4c): \{\pm 1^2\}**

For \( P_4(h) = a_{++} (h^2 - 1)^2 \), we find the following with \( \gamma \equiv \sqrt{2a_{++}} \):

\[ h(q) = \frac{1}{\tanh(\gamma q)}, \quad (3.56) \]

\[ E(q) = -\frac{2\gamma}{\tanh(\gamma q)}, \quad (3.57) \]

\[ \widetilde{W}(q) = -\frac{1}{\gamma} \sinh^2(\gamma q) \left[ \frac{\tilde{b}_+}{\tanh^2(\gamma q)} + \frac{b_0}{\tanh(\gamma q)} + \tilde{b}_- \right]. \quad (3.58) \]

The expression of the potentials \( V_N^\pm(q) \) is rather long and not particularly informative and therefore we will not list it here.

Similarly to Case (2b'), one could move the zero points of this case to \( h = \pm i \). This results in \( h(q) = \tan(\gamma q) \), \( E(q) = 2\gamma \tan(\gamma q) \).

**Case (4d1): \{0^2, \pm 1\}**

For \( P_4(h) = -a_{00} h^2 (h^2 - 1) \), we find the following:

\[ h(q) = \frac{1}{\cosh(\alpha q)} \quad (3.59) \]
4. Summary and Discussions

\begin{equation}
E(q) = \frac{\alpha}{\sinh(\alpha q) \cosh(\alpha q)} \left[ 2 - \cosh^2(\alpha q) \right].
\end{equation}

\begin{equation}
\tilde{W}(q) = -\frac{1}{\alpha \sinh(\alpha q)} \left[ \tilde{b}_+ + b_0 \cosh(\alpha q) + \tilde{b}_- \cosh^2(\alpha q) \right].
\end{equation}

\begin{equation}
V^\pm_N(q) = \frac{1}{\sinh^2(\alpha q)} \left[ \frac{\bar{b}^2}{4a_{00}} \cosh^4(\alpha q) + \frac{\bar{b}_-}{2} \left( \frac{b_0}{a_{00}} \mp N \right) \cosh^3(\alpha q) \right.
\end{equation}
\begin{equation}
+ \left( \frac{b_0^2 + 2\bar{b}_+ \bar{b}_-}{4a_{00}} + \frac{N^2 - 1}{12} a_{00} \right) \cosh^2(\alpha q) + \left( \frac{\bar{b}_+ b_0}{2a_{00}} + \frac{N}{2} (\bar{b}_+ + 2\bar{b}_-) \right) \cosh(\alpha q) \nonumber
\end{equation}
\begin{equation}
+ \left( \frac{\bar{b}_-^2}{4a_{00}} + \frac{N^2 - 1}{6} a_{00} \pm \frac{N}{2} b_0 \right) \right].
\end{equation}

**Case (4d1'): \{0^2, \pm i\}**

For \( P_4(h) = a_{00} h^2 (h^2 + 1) \), we find the following:

\begin{equation}
h(q) = \frac{1}{\sinh(\alpha q)},
\end{equation}

\begin{equation}
E(q) = \frac{\alpha}{\sinh(\alpha q) \cosh(\alpha q)} \left[ -2 - \sinh^2(\alpha q) \right].
\end{equation}

As noted above, this case is obtained from Case (4d1) by \( h \to ih \) and therefore the rest of the expressions are not listed here.

## 4 Summary and Discussions

In this paper, we have classified the Type A \( N \)-fold supersymmetry models according to the zero point structure of \( P_4(h) \), which is at most a fourth order polynomial of \( h \). We have found that the potentials of the model can be written down explicitly for the most general distribution of the zero points only when the function \( P_4(h) \) is at most second order polynomial of \( h \). In such cases, we have classified the resulting models and have written down the potentials for representative cases of each classes. When the function \( P_4(h) \) is a third or forth order polynomial of \( h \), we have found that the potentials can be written down explicitly only when at least one zero point is of order 2 or higher. For those cases, we have written down relevant expressions for each representative cases. Along the way, we have found several new models not studied previously.

It should be stressed that all these models are connected to each other in the eight-parameter space \( \{a_{++}, a_{+0}, a_{00}, a_{0-}, a_{--}, b_+, b_0, b_-\} \). Evidently, many of them are worth studying in detail, because of the quasi-solvability, as is clear in the expression of the Hamiltonian (2.12). We also note that as far as the resulting potentials are concerned, some of the cases listed in this paper are equivalent. For example, potentials \( V^\pm_N(q) \) of Case (0) and \( V^\pm_N(q) \) of Case (4a) are equivalent to each other when coefficients \( \tilde{b}_+ \) and \( \tilde{b}_- \) are interchanged. Similarly, Cases (1), (3a) and (4b), Cases (2a) and (4c), and, Cases (2b), (3b) and (4d1) have equivalent potentials. They, however, are listed here as separate...
cases, since they all differ in the form of supercharges. This equivalence/difference issue may be worth studying further.

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Note added in proof

After the completion of this work, we came across an interesting paper, A. González-López, N. Kamran, and P. J. Olver, Commun. Math. Phys. 153 (1993) 117, where $GL(2,\mathbb{R})$ symmetry in quasi-solvable systems is discussed. The equivalence of the potentials mentioned in the summary may be explained naturally by this symmetry.

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