THE CAPILLARITY PROBLEM FOR COMPRESSIBLE LIQUIDS

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ABSTRACT. In this paper we study existence and regularity of solutions to the capillarity problem for compressible liquids in a tube. We introduce an appropriate space of functions of bounded variation, in which the energy functional recently introduced by Robert Finn can be defined. We prove existence of a locally Lipschitz minimiser in this class.

1. INTRODUCTION

Extensive work has been published on the behaviour of capillary (liquid-air or liquid-liquid) interfaces when the liquid is assumed to be incompressible. As an authoritative introduction we refer to [6] by Finn. Two major approaches have been used to obtain existence and regularity results: classical PDE techniques for surfaces of prescribed mean curvature (see, for example, [11, 15, 16]), and the functions of bounded variation and sets of finite perimeter setting for minimising the energy (see, for example, [4, 10, 12, 14, 15]).

Results concerned with compressible liquids are very recent and comparatively few, the model having been introduced by Finn in 2001 [7], see also [8]. Following that paper we consider a capillary tube of cross section \(\Omega \subset \mathbb{R}^n\), which is simply connected and has Lipschitz boundary \(\Sigma := \partial \Omega\). We also assume that it satisfies an interior sphere condition of radius \(R\).

The capillary surface \(S\) is given as a graph of a function \(u\) over the domain \(\Omega\). We assume uniform downwards gravity \(g\) and consider a compressible fluid of density \(\rho\). (In the incompressible fluid case, \(\rho\) is constant.)

One can assume prescribed mass \(M\), but the results in the present paper are for an infinite container.

We consider the energy for a capillary surface to consist of the following components:

Energy of the free surface (surface tension):
\[
E_S = \frac{\sigma}{\rho_0} \int_{\Omega} \Phi(u; p_0) \sqrt{1 + |Du|^2} \, dx;
\]

Potential energy:
\[
W = g \int_{\Omega} \int_{0}^{u} h \Phi(h; p_0) \, dh \, dx;
\]

Wetting energy:
\[
E_\Sigma = -\sigma \int_{\Sigma} \beta \int_{0}^{u} \Phi(h; p_0) \, dh \, ds;
\]

here \(\beta \in L^\infty(\Sigma)\) is the relative adhesion coefficient, satisfying \(|\beta| \leq 1 - a\) with \(a > 0\); \(\sigma\) and \(g\) are the surface tension and gravitational constants; \(\Phi(h; p_0)\) is the density function depending on
height \( h \) and pressure \( p \), which we assume to be given by one of the two models proposed by Finn [1, 7]. In the following, \( p_0 \) and \( \rho_0 \) will denote pressure and density at a reference level \( u \equiv 0 \).

**Mass:** In the case of a mass constraint, a term \( \lambda M \) is added to the energy, where \( \lambda \) is a Lagrange multiplier and the mass is

\[
M = \int_{\Omega} \int_0^u \Phi(h; p_0) \, dh \, dx.
\]

The total energy (and in particular the wetting energy) need not be positive.

A smooth minimizer of the total energy \( E_S + W + E_\Sigma + \lambda M \) will satisfy the Euler-Lagrange equation

\[
\text{div} \frac{Du}{\sqrt{1 + |Du|^2}} = \frac{g\rho_0}{\sigma} u + \frac{D_1 \Phi(u; p_0)}{\Phi(u; p_0)} \frac{1}{\sqrt{1 + |Du|^2}} + \frac{\lambda \rho_0}{\sigma} \text{ on } \Omega,
\]

with boundary condition

\[
\beta = \frac{1}{\rho_0 \sqrt{1 + |Du|^2}} \text{ on } \Sigma,
\]

via standard calculus of variations techniques.

The present paper is based on one of the models proposed by Finn for an isothermal fluid: the density is assumed to be linear in the pressure, from which one obtains that

\[
\Phi(h; p_0) = \rho_0 e^{-\chi gh},
\]

for some positive constant \( \chi \).

We may assume that \( \chi = 1, g = 1, \rho_0 = 1, \sigma = 1 \); other values of these constants correspond to different weightings on the components of the energy (that is, our energy becomes \( \gamma_1 E_S + \gamma_2 W + \gamma_3 E_\Sigma + \gamma_4 \lambda M \) for \( \gamma_i > 0 \)), and a scaling of the domain \( \Omega \).

Then

\[
\Phi(h; p_0) = e^{-h},
\]

and the diverse components of the energy are as follows:

\[
E_S = \int_{\Omega} e^{-u} \sqrt{1 + |Du|^2} \, dx,
\]

\[
W = \int_{\Omega} \int_0^u he^{-h} \, dh \, dx = \int_{\Omega} \left[1 - e^{-u} (1 + u)\right] \, dx,
\]

\[
E_\Sigma = \int_{\Sigma} \beta \int_0^u e^{-h} \, dh \, ds = - \int_{\Sigma} \beta \left(1 - e^{-u}\right) \, ds,
\]

\[
M = \int_{\Omega} \int_0^u e^{-h} \, dh \, dx = \int_{\Omega} (1 - e^{-u}) \, dx.
\]

As we are dealing with the case of an infinite container, we choose \( \lambda = 0 \). Without loss of generality (this will be shown when necessary, in Lemma 3.6) we may set \( \gamma_i \equiv 1 \), and then seek to minimize the energy

\[
J(u) := E_S(u) + W(u) + E_\Sigma(u).
\]

The following results have been recently obtained for the capillarity problem of a compressible fluid.

For slightly compressible fluids Finn [7] introduced the model we are using here. In the case of a tube closed at the bottom he found the necessary condition on the mass for existence of a solution is

\[
M < \rho_0 |\Omega| / \chi g.
\]
For a circular tube, Finn and Luli [9] show that for any boundary contact angle $\gamma$ with $0 \leq \gamma < \pi$ there is at least one symmetric solution of the problem, and that the height of this solution will lie above any prescribed level if $M$ is sufficiently large. If $\gamma \leq \pi/2$, the solution is unique among symmetric solutions with that mass.

Finn and Athanassenas [1] follow the classical PDE approach. They include the situation where on the right hand side of the prescribed mean curvature equation (1.1) the term $\rho_0 g \sigma u$ is replaced by $\rho_0 - \chi_0 |\Omega| > -|\Sigma| \beta$ (here $\beta$ is taken to be constant).

In the present paper we use functions of bounded variation techniques.

In Section 2, we introduce $\text{BV}$, the space of functions of bounded variation. After a transformation of $u$, the weighted surface area term is well defined in $\text{BV}$. Transforming the remaining components of the energy gives us a new energy, $J_1$.

In Section 3, we prove height estimates. In Lemma 3.1 and Lemma 3.5 we have two Stampacchia type results needed in our case.

In Section 4, we show that the energy functional is bounded from below, and that a minimizing sequence for the energy functional is uniformly bounded in the $\text{BV}$-norm. Existence then follows via the standard compactness theorem and by the lower semicontinuity of the functional.

Finally, in Section 5 we show that there exists a locally Lipschitz minimiser.

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2. The Energy in the Isothermal Case

As in [13] the $\text{BV}$-seminorms are:

$$\int_{\Omega} \sqrt{1 + |Du|^2} = \sup \left\{ \int_{\Omega} g_{n+1} + u \text{div}_n g \, dx : g_i \in C_0^1(\Omega) \forall i = 1, \ldots, n+1, \sum_{i=1}^{n+1} g_i^2 \leq 1 \right\}.$$

and

$$\int_{\Omega} |Du| = \sup \left\{ \int_{\Omega} u \text{div}_n g \, dx : g_i \in C_0^1(\Omega) \forall i = 1, \ldots, n, \sum_{i=1}^{n} g_i^2 \leq 1 \right\},$$

where $\text{div}_n$ is the divergence of the first $n$ components, $\text{div}_n g = \sum_{i=1}^{n} D_i g_i$.

One then defines the spaces $\text{BV}(\Omega) := \{ u \in L^1(\Omega) : \int_{\Omega} \sqrt{1 + |Du|^2} < \infty \}$, and $\text{BV}^+(\Omega) := \{ u \in \text{BV}(\Omega) : u \geq 0 \text{ almost everywhere in } \Omega \}.$
In the case that \( u \in C^1(\Omega) \), the surface energy term \( E_S = \int_{\Omega} e^{-u} \sqrt{1 + |Du|^2} \, dx \) may be simplified by writing \( v = e^{-u} \). Then we can rewrite it as \( \int_{\Omega} \sqrt{v^2 + |Dv|^2} \, dx \), which bears close resemblance to the integral investigated by Bemelmans and Dierkes [2], which was \( \int_{\Omega} \sqrt{v + |Dv|^2/4} \, dx \); see also [3].

The focus of our investigation now shifts to \( v \), rather than \( u \) itself.

Define

\[
\int_{\Omega} \sqrt{v^2 + |Dv|^2} := \sup \left\{ \int_{\Omega} v \left( g_{n+1} + \text{div}_n g \right) \, dx : g_i \in C^1(\Omega) \, \forall i = 1, \ldots, n+1, \sum_{i=1}^{n+1} g_i^2 \leq 1 \right\}.
\]

**Lemma 2.1.** If \( v \) is smooth, \( \int_{\Omega} \sqrt{v^2 + |Dv|^2} = \int_{\Omega} \sqrt{v^2 + |Dv|^2} \, dx \).

**Proof.** We consider the test function \( g^\varepsilon = \chi_{\varepsilon} \left[ v^2 + |Dv|^2 \right]^{-1/2} (-Dv, v) \), where \( \chi_{\varepsilon} \) is a sequence of \( C_0^\infty(\Omega) \) functions with \( \chi_{\varepsilon} \leq 1 \), converging to \( \chi_\Omega \), the characteristic function of \( \Omega \), in \( L^1 \). Then

\[
\int_{\Omega} \sqrt{v^2 + |Dv|^2} \geq \int_{\Omega} v(g_{n+1}^\varepsilon + \text{div}_n g^\varepsilon) \, dx \\
= \int_{\Omega} g_{n+1}^\varepsilon v - Dv \cdot g^\varepsilon \, dx \\
= \int_{\Omega} \chi_{\varepsilon} \sqrt{v^2 + |Dv|^2} \, dx \\
\rightarrow \int_{\Omega} \sqrt{v^2 + |Dv|^2} \, dx
\]
as \( \varepsilon \to 0 \). The other direction is similar. \( \square \)

We note the following fact:

**Lemma 2.2.** The quantity \( \int_{\Omega} \sqrt{v^2 + |Dv|^2} \) is finite exactly when \( v \) is in \( BV(\Omega) \).

**Proof.** Suppose that \( v \) is in \( BV(\Omega) \). Then

\[
\int_{\Omega} \sqrt{v^2 + |Dv|^2} = \sup \left\{ \int_{\Omega} v \left( g_{n+1} + \text{div}_n g \right) \, dx : g_i \in C^1(\Omega), \sum_{i=1}^{n+1} g_i^2 \leq 1 \right\} \\
\leq \sup \left\{ \int_{\Omega} (|v| + 1) g_{n+1} + v \text{div}_n g \, dx : g_i \in C^1(\Omega), \sum_{i=1}^{n+1} g_i^2 \leq 1 \right\} \\
\leq \sup \left\{ \int_{\Omega} g_{n+1} + v \text{div}_n g \, dx : g_i \in C^1(\Omega), \sum_{i=1}^{n+1} g_i^2 \leq 1 \right\} \\
+ \sup \left\{ \int_{\Omega} g_{n+1} |v| \, dx : g_{n+1} \in C^1(\Omega), \, g_{n+1}^2 \leq 1 \right\} \\
= \int_{\Omega} \sqrt{1 + |Dv|^2} + \|v\|_{L^1(\Omega)} \\
< \infty.
\]

On the other hand, if \( \int_{\Omega} \sqrt{v^2 + |Dv|^2} < \infty \), then \( v \in L^1(\Omega) \), since if not, we can take \( g_i = 0 \) for \( i < n+1 \) and \( g_{n+1} = \chi_{\varepsilon} \) (where \( \chi_{\varepsilon} \) is as in Lemma 2.1) so that

\[
\int_{\Omega} \sqrt{v^2 + |Dv|^2} \geq \int_{\Omega} |v| \chi_{\varepsilon} \, dx \rightarrow \infty
\]
as \( \varepsilon \to 0 \), contradicting our assumption. Finally we can check

\[
\int_{\Omega} \sqrt{1 + |Dv|^2} = \sup \left\{ \int_{\Omega} g_{n+1} + v \, \text{div}_n g + v g_{n+1} - v g_{n+1} \, dx : g_i \in C^1_0(\Omega), \sum_{i=1}^{n+1} g_i^2 \leq 1 \right\}
\]

\[
\leq \sup \left\{ \int_{\Omega} v \, \text{div}_n g + v g_{n+1} \, dx : g_i \in C^1_0(\Omega), \sum_{i=1}^{n+1} g_i^2 \leq 1 \right\}
\]

\[
+ \sup \left\{ \int_{\Omega} g_{n+1}(1 - v) \, dx : g_{n+1} \in C^1_0(\Omega), g_{n+1}^2 \leq 1 \right\}
\]

\[
\leq \int_{\Omega} \sqrt{v^2 + |Dv|^2} + \|v\|_{L^1(\Omega)} + |\Omega| \leq \infty.
\]

\( \square \)

**Corollary 2.3.** If, in addition to the above conditions, \( v \) is in \( BV(\Omega) \), and \( v_k \) is a mollification of \( v \), then

\[
\int_{\Omega} \sqrt{v_k^2 + |Dv_k|^2} \to \int_{\Omega} \sqrt{v^2 + |Dv|^2}.
\]

This may be proved in the same manner as Lemma A1 of [10].

Under the transformation \( v = e^{-u} \), the wetting energy is \( E_\Sigma = -\int_\Sigma \beta[1 - e^{-u}] \, ds = -\int_\Sigma \beta[1 - v] \, ds \), where we consider \( v|_\Sigma \) as a trace of \( v \). As in [13, Theorem 2.10] if \( \Sigma \) is Lipschitz, each function in \( BV(\Omega) \) has a trace in \( L^1(\Sigma) \). Furthermore, if \( \Sigma \) also satisfies an interior sphere condition with radius \( R \), then the following estimate holds (see [10], Remark 2):

\[
(2.1) \quad \int_\Sigma |v| \, ds \leq \int_{\Omega} |Dv| + c_R \int_{\Omega} |v| \, dx,
\]

where \( c_R \) depends on \( n, R, \) and \( \Sigma \).

The integrand of \( W \), the potential energy term, becomes \( \int_1^v \ln h \, dh \) and so the complete energy, in the isothermal case, is

\[
J_1(v) = \int_{\Omega} \sqrt{v^2 + |Dv|^2} + \int_{\Omega} \int_1^v \ln h \, dh \, dx - \int_\Sigma \beta[1 - v] \, ds
\]

Here we are reminded of the energy studied by Claus Gerhardt in [10], which was

\[
\int_{\Omega} \sqrt{1 + |Dv|^2} + \int_{\Omega} \int_0^v H(x, h) \, dh \, dx - \int_\Sigma \beta v \, ds
\]

for \( \beta \in L^\infty(\Sigma) \) and \( H \) satisfying the conditions (a) \( \frac{\partial H}{\partial h} > 0 \), and (b) \( H(x, h_0) \geq (1 + c) H(x, h_0) \leq -(1 + c) \) for some \( h_0 \geq 0 \) and a given \( c \). The conditions imply that for large values of \( |v| \), the potential energy term (the one involving \( H \)) is strictly positive and increasing at least linearly in \( |v| \). The current case is an improvement on this situation: a strictly positive potential energy, which increases like \( v \ln v \) for large values of \( v \).
3. Height bounds

In this section, we assume that $v$ minimises $J_1$ in $BV(\Omega)$ and seek height bounds. Note that a bound from above on $v$ would correspond to a bound from below on $u$, while a strictly positive bound from below on $v$ corresponds to a bound from above on $u$.

At the end of this section we show an easier way to find one-sided estimates in the cases of $\beta$ being either positive (for which we show $v$ bounded from above) or negative (for which we show $v$ bounded from below).

3.1. Height bounds from above on $v$. To estimate a minimiser $v$ from above, we follow an approach similar to [10, 15] leading to a Stampacchia iteration [18]. We use the following variant of the original Stampacchia lemma:

**Lemma 3.1.** Suppose $B(t)$, non-negative and non-increasing in $t$, satisfies

$$(h - k)B(h) \leq Ck[B(k)]^\gamma,$$

for all $h, k$ such that $0 < k_0 \leq k < h$, for some constants $C, k_0 > 0$ and $\gamma > 1$. Then $B(K) = 0$ for some sufficiently large $K$ dependent on $C, \gamma, k_0$ and $B(k_0)$.

We will use the following in the proof of the above lemma:

**Lemma 3.2.** For all $\alpha > 1$ and $d > -\alpha$, the sequence

$$s_m = \left(1 + \frac{d}{\alpha^m}\right)\left(1 + \frac{d}{\alpha^{m-1}}\right)\ldots\left(1 + \frac{d}{\alpha}\right),$$

converges to a non-zero limit.

**Proof.** We examine the sequence $\{\ln s_n\}$, writing each term as the partial sum $\sum_{j=1}^{m} \ln \left(1 + \frac{d}{\alpha^j}\right)$, and using the ratio test for the convergence of series:

$$\lim_{j \to \infty} \left[\ln \left(1 + \frac{d}{\alpha^{j+1}}\right)\right] \left[\ln \left(1 + \frac{d}{\alpha^j}\right)\right]^{-1} = \lim_{j \to \infty} \left[\frac{\partial}{\partial j} \ln \left(1 + \frac{d}{\alpha^{j+1}}\right)\right] \left[\frac{\partial}{\partial j} \ln \left(1 + \frac{d}{\alpha^j}\right)\right]^{-1}$$

$$= \lim_{j \to \infty} \left[\frac{d \ln \alpha}{\alpha^{j+1} + d}\right] \left[\frac{d \ln \alpha}{\alpha^j + d}\right]^{-1}$$

$$= \frac{1}{\alpha}.$$ 

As this series converges to some limit $L$, $\{s_m\}$ converges to $e^L > 0$. \hfill \Box

**Proof of Lemma 3.1.** We begin by defining the sequence $k_m := k_0s_m$, where $s_m$ is as in the preceding lemma, with $\alpha = 2$ and $d = C[B(k_0)]^{\gamma-1}2^{\gamma/(\gamma-1)} > 0$ (we assume here that $B(k_0) \neq 0$, otherwise the lemma is trivially true). Note that as $k_{m+1} - k_m = k_0d2^{-(m+1)} > 0$, $\{k_m\}$ is positive and increasing, and, by the above result, converges to some limit $K$.

We now prove that $B(k_m) \leq B(k_0)2^{\mu m}$ for $\mu = (1 - \gamma)^{-1} < 0$, by induction.

The base step, for $m = 1$, is as follows: by assumption (3.1),

$$(k_1 - k_0)B(k_1) \leq Ck_0B(k_0)^\gamma,$$
and so using $k_1 - k_0 = k_0 d/2$ we find that
\[ B(k_1) \leq C \frac{2^{1-\mu}}{d} B(k_0)^{\gamma - 1} B(k_0)^{2\mu} \]
\[ = 2^{1-\mu - \gamma/(\gamma-1)} B(k_0)^{2\mu} \]
\[ = B(k_0)^{2\mu}. \]

Now we make the inductive assumption that $B(k_m) \leq B(k_0)^{2\mu m}$. We use this and condition (3.1) to estimate
\[ B(k_{m+1}) \leq C \frac{2^{m+1}}{d} B(k_m)^{\gamma} \]
\[ \leq C \frac{2^{m+1}}{d} [B(k_0)^{2\mu m}]^{\gamma} \]
\[ = 2^{m+1-\gamma/(\gamma-1)} B(k_0)^{1-\gamma} [B(k_0)^{2\mu m}]^{\gamma} \]
\[ \leq B(k_0)^{2\mu (m+1)}. \]

Finally, the monotonicity of $B$ implies that $B(K) \leq \lim_{m \to \infty} B(k_m) \leq \lim_{m \to \infty} B(k_0)^{2\mu m} = 0$.

**Theorem 3.3.** Let $v$ minimise $J_1$ in $BV^+(\Omega)$, where $J_1$ is given by (2.2). Assume in addition $\partial \Omega$ to be Lipschitz and to satisfy an interior sphere condition. Then $v$ is bounded above.

**Proof.** We set $A(k) = \{ x \in \Omega : v(x) > k \}$ for $k > k_0$, $k_0$ to be chosen later, the goal being to show that the non-increasing $|A(k)|$ vanishes for some large $k$. We also write $w := \min(v, k)$. As $v$ minimises $J_1$, we have $J_1(v) \leq J_1(w)$ for all eligible $w$, which after rearranging gives
\[ 0 \geq \left\{ \int_{A(k)} \sqrt{v^2 + |Dv|^2} - \int_{A(k)} k \, dx \right\} + \int_{A(k)} \int_k^v \ln h \, dh \, dx + \int_\Sigma \beta(v-w) \, ds. \tag{3.2} \]

Here, we make use of the fact that $w \in BV(\Omega)$, that $Dw = Dw$ in $\Omega \setminus A(k)$ and $Dw = 0$ in $A(k)$.

We estimate the boundary term using (2.1):
\[ \left| \int_\Sigma \beta(v-w) \, ds \right| \leq (1-a) \left[ \int_\Omega |D(v-w)| + c_R \int_\Omega |v-w| \, dx \right]. \]

One can easily show that $\int_\Omega |Dv| \leq \int_\Omega \sqrt{v^2 + |Dv|^2}$, so that (3.2) gives
\[ k |A(k)| \geq a \int_{A(k)} |D(v-k)| + (\ln k - (1-a)c_R) \int_{A(k)} |v-k| \, dx. \tag{3.3} \]

For BV functions on $C^{0,1}$ domains, $\Omega$, we have the Sobolev inequality
\[ \left[ \int_\Omega |f|^{\frac{n}{n-1}} \, dx \right]^{\frac{n-1}{n}} \leq c_2 \left[ \int_\Omega |Df| + \int_\Omega |f| \, dx \right]. \]

To see this, we first note that the inequality holds true for $W^{1,1}(\Omega)$ functions, since the space $W^{1,1}(\Omega)$ is continuously embedded in $L^{n/(n-1)}(\Omega)$ for $n > 1$ (see Theorem 7.26 in [11]). The results extends to $f \in BV(\Omega)$, after approximating $f$ by smooth functions as in Theorem 1.17 of [13], and then following the steps of the proof of Theorem 1.28 of [13].
We rearrange this inequality as
\[
a \int_{\Omega} |Df| \geq \frac{a}{c_{\Omega}} \left[ \int_{\Omega} |f|^{\frac{n}{n-1}} \, dx \right]^{\frac{n-1}{n}} - a \int_{\Omega} |f| \, dx.
\]
Using the above estimate with \( f = v - w \), (3.3) becomes
\[
k|A(k)| \geq \frac{a}{c_{\Omega}} \left[ \int_{\Omega} |v - w|^{\frac{n}{n-1}} \, dx \right]^{\frac{n-1}{n}} + (\ln k - (1 - a)c_R - a) \int_{A(k)} |v - w| \, dx.
\]
By choosing \( k_0 \) (the lower bound on \( k \)) large enough, we can ensure that \((\ln k - (1 - a)c_R - a)\) is positive, and drop this term altogether. The Hölder inequality \([13, \text{Theorem 1.28}]\) gives
\[
\|f\|_{L_{n-1}^n(A(k))} \geq \|f\|_{L^1(A(k))}|A(k)|^{-1/n},
\]
and so for all \( h, k \) with \( h > k \geq k_0 \), we have
\[
\frac{a}{c_{\Omega}} (h - k)|A(h)| \leq \frac{a}{c_{\Omega}} \int_{A(k)} (v - k) \, dx \leq k|A(k)|^{1 + \frac{1}{n}}.
\]
Now we can apply Lemma 3.1 and conclude that for sufficiently large \( K \), \(|A(K)| = 0\). \(\Box\)

3.2. **Height bounds on \( v \) from below.** We start by remarking that bounds from below on \( v \) — or from above on \( u \), where the surface is given by \( \text{graph } u \) — are not essential for the existence proof. We will see in the next section that the energy is bounded from below irrespective of such an estimate, and that subsequent results, leading to existence, also hold. However, they are important for the correspondance between \( u \) and \( v \), and for the regularity results.

**Theorem 3.4.** Suppose that \( v \) minimises \( J_1 \) in \( BV^+(\Omega) \). Then there exists a bound from below on \( v \), \( 0 < c \leq v \), \( L^1 \)-almost everywhere.

The proof is similar to that of Theorem 3.3, but we use a slightly stronger Stampacchia-type result.

**Lemma 3.5.** Suppose \( B(t) \), non-negative and non-increasing in \( t \), satisfies
\[
(h - k)B(h) \leq C h [B(k)]^\gamma,
\]
for all \( h, k \) such that \( 0 < k_0 \leq k < h \), for some constants \( C, k_0 > 0 \) and \( \gamma > 1 \). If
\[
CB(k_0)^{\gamma - 1} < 1,
\]
then there exists a \( K < \infty \) such that \( B(K) = 0 \).

**Proof.** From (3.5), we may choose \( d > 0 \) and \( \alpha > 1 \) such that \( CB(k_0)^{\gamma - 1} \gamma ^{-1} \leq d < \alpha \).

Next, define the sequence \( k_m := k_0/s_m = k_{m-1}(1 - d/\alpha m)^{-1} \), where \( s_m \) is as in Lemma 3.2 (but note the change of sign on \( d \)); here \( \{k_m\} \) is positive, increasing, and by Lemma 3.2, convergent to some \( K \).

We now prove that \( B(k_m) \leq B(k_0) \alpha^{-1} \) by induction.

The base step, for \( m = 1 \), is as follows: by the definition of the sequence, we have \( k_1 = k_0(1 - d/\alpha)^{-1} \), and so our assumption (3.4) gives
\[
(k_1 - k_0)B(k_1) \leq Ck_1 B(k_0)^\gamma,
\]
which leads to

\[ B(k_1) \leq \frac{C \alpha}{d} B(k_0)^\gamma \leq B(k_0) \alpha^{-\frac{m}{\gamma-1}}. \]

Now we make the inductive assumption that \( B(k_m) \leq B(k_0) \alpha^{-\frac{m}{\gamma-1}} \), and show that this then holds for \( k_{m+1} \): we use (3.4) to estimate

\[
B(k_{m+1}) \leq C \frac{k_{m+1}}{k_{m+1} - k_m} B(k_m)^\gamma \\
\leq C \frac{\alpha^{m+1}}{d} B(k_0)^{\gamma - 1} \alpha^{-\frac{m}{\gamma-1} + \frac{m+1}{\gamma-1}} [B(k_0) \alpha^{-\frac{m}{\gamma-1}}] \\
\leq B(k_0) \alpha^{-\frac{m}{\gamma-1}}.
\]

Finally, the monotonicity of \( B \) implies that \( B(K) \leq \lim_{m \to \infty} B(k_m) \leq \lim_{m \to \infty} B(k_0) \alpha^{-\frac{m}{\gamma-1}} = 0. \) \( \Box \)

We will need to show that the measure of the set where \( v \) is small is small enough to satisfy (3.5). This is the only place in this paper where it is not immediately clear that rescaling the constants \( \gamma_i \) to 1 does not result in a loss of generality. Consequently, we include the arbitrary weightings in \( J_1 \) in the following step.

\textbf{Lemma 3.6.} Let \( v \) minimise \( J_1 = \gamma_1 E_S + \gamma_2 W + \gamma_3 E_\Sigma \) in \( BV^+(\Omega) \), and set \( B(k) := \{ x \in \Omega : v(x) < 1/k \} \). Then for all \( \eta > 0 \) we can find a \( k \) such that \( |B(k)| \leq \eta \).

\textit{Proof.} Define the comparison function \( w := \max \{ v, 1/k \} \in BV^+(\Omega) \) for any \( k \geq k_0, \) \( k_0 \) to be chosen later. Note that \( 0 \leq w - v \leq 1/k \). Since \( v \) minimises \( J_1 \), we have \( J_1(v) \leq J_1(w) \). We use \( \int \sqrt{u^2 + |Du|^2} \geq \int |u| dx \) for \( u \in BV^+(\Omega) \) to estimate

\[
0 \geq J_1(v) - J_1(w) \\
= \gamma_1 \int_{B(k)} \sqrt{v^2 + |Dv|^2} - \gamma_1 \int_{B(k)} \frac{1}{k} dx - \gamma_3 \int_{\Sigma} \beta(w - v) ds \\
+ \gamma_2 \int_{B(k)} v \ln v - v - \frac{1}{k} \ln \frac{1}{k} + \frac{1}{k} dx \\
\geq \int_{B(2k)} -|\gamma_1 - \gamma_2| \left| v - \frac{1}{k} \right| + \gamma_2 \left( v \ln v - \frac{1}{k} \ln \frac{1}{k} \right) dx \\
+ \int_{B(k) \setminus B(2k)} -|\gamma_1 - \gamma_2| \left| v - \frac{1}{k} \right| + \gamma_2 \left( v \ln v - \frac{1}{k} \ln \frac{1}{k} \right) dx \\
- \gamma_3 (1-a)|\Sigma| \frac{1}{k}
\]
— now choose $k_0$ large enough so that $x \ln x$ is decreasing for $0 < x \leq \frac{1}{k_0}$ —

$$\geq \int_{B(2k)} -|\gamma_1 - \gamma_2| \frac{1}{k} + \gamma_2 \left( \frac{1}{2k} \ln \frac{1}{2k} - \frac{1}{k} \ln \frac{1}{k} \right) \, dx + \int_{B(k) \setminus B(2k)} -|\gamma_1 - \gamma_2| \frac{1}{2k} \, dx$$

\[
\begin{align*}
&- \gamma_3(1 - a)|\Sigma| \frac{1}{k} \\
&\geq -|\gamma_1 - \gamma_2| \frac{2}{k} |\Omega| - \gamma_3(1 - a)|\Sigma| \frac{1}{k} + \gamma_2 \int_{B(2k)} \left( -\frac{1}{2k} \right) (\ln \xi + 1) \, dx \\
&\quad \text{for some } \xi \in \left( \frac{1}{2k}, \frac{1}{k} \right)
\end{align*}
\]

Rearranging, and choosing $k_0$ large enough that $\ln \frac{1}{2k_0} < -1$, we find that

$$|B(2k)| \leq \frac{4|\gamma_1 - \gamma_2||\Omega| + \gamma_3(1 - a)|\Sigma|}{\gamma_2 (\ln 2k - 1)} < \eta$$

for sufficiently large $k > k_0$. 

**Proof of Theorem 3.4.** Let $B(k)$ be defined as above. Set $w := \max(v, 1/k)$, for some $k \geq k_0$. Again, as $v$ minimizes $\mathcal{J}_1$, then $\mathcal{J}_1(v) \leq \mathcal{J}_1(w)$. Proceeding exactly as in the proof of Theorem 3.3, we obtain

$$0 \geq a \int_{\Omega} |D(v - w)| - \frac{1}{k} |B(k)| - [(1 - a)c_1 + \ln(1/k)] \int_{\Omega} |v - w| \, dx$$

$$\geq \frac{a}{c_{\Omega}} \|v - w\|_{L^\frac{n}{n-1}(\Omega)} - \frac{1}{k} |B(k)| - [(1 - a)c_1 + \ln(1/k) + a] \int_{\Omega} |v - w| \, dx,$$

and if we choose $k_0$ large, so that $\ln k \geq c_1(1 - a) + a$, then the final term above is positive. We drop it and apply the Hölder inequality to the $L^{\frac{n}{n-1}}$ term, leaving us with $(1/k)|B(k)| \geq (a/c_{\Omega}) \|v - w\|_{L^1(B(k))} |B(k)|^{-1/n}$, and so for each $h > k \geq k_0$ we have

$$(h - k)|B(h)| \leq Ch |B(k)|^{1 + \frac{1}{n}}.$$

Lemma 3.6 implies that we can find $k_0$ large enough that $CB(k_0)^{\frac{1}{n}} < 1$. We can then apply the Stampacchia-type Lemma 3.5 to conclude that $|B(K)| = 0$ for large $K$, and so $v \geq \frac{1}{K}$ almost everywhere. 

**3.3. Height estimates in the cases $\beta \leq 0$ and $\beta \geq 0$.** Height estimates are easier to obtain in case $\beta$ is either non-positive or non-negative.

We begin by observing a height bound for $v$ in the surface energy term. This closely follows Lemma 5 of [2], and may be proved in the same way.

**Lemma 3.7.** Let $v \in BV^+(\Omega)$ and suppose that $A(k) = \{x \in \Omega : v(x) > k\}$ has positive measure. Then $w = \min(v, k) \in BV^+(\Omega)$ and for almost all $k$,

$$\int_{\Omega} \sqrt{w^2 + |Dw|^2} < \int_{\Omega} \sqrt{v^2 + |Dv|^2}.$$
Theorem 3.8. Suppose that $v \in BV^+(\Omega)$ minimises $J_1$, and that $\beta \geq 0$. Then $v$ is bounded from above.

Proof. Set $w = \min(v, k)$. Suppose that $A(k)$ is of positive measure for some $k \geq 1$. We may choose $k$ so that Lemma 3.7 gives us

$$\int_\Omega \sqrt{w^2 + |Dw|^2} - \int_\Omega \sqrt{v^2 + |Dv|^2} < 0.$$ 

We note that

$$\int_\Omega \int_1^w \ln h \, dh \, dx - \int_\Omega \int_1^v \ln h \, dh \, dx = \int_\Omega \int_v^w \ln h \, dh \, dx$$

$$= \int_{\Omega \cap \{x : v(x) \geq k\}} \int_k^v \ln h \, dh \, dx$$

$$\leq 0.$$

Finally,

$$- \int_\Sigma \beta [1 - w] \, ds + \int_\Sigma \beta [1 - v] \, ds = \int_\Sigma \beta [w - v] \, ds \leq 0$$

if $\beta \geq 0$. Together, these inequalities give $J_1(w) - J_1(v) < 0$, contradicting that $v$ was a minimum. It follows that $|A(k)|$ cannot be positive, and so $v \leq k$. \hfill $\Box$

Lemma 3.9. Suppose that $v$ minimizes $J_1$, and $\beta \leq 0$. Then $v \geq e^{-1}$.

Proof. Set $w = \max(v, \varepsilon)$, and write $B(\varepsilon) = \{x \in \Omega : v(x) < \varepsilon\}$. Then

$$J_1(v) - J_1(w) = \int_{B(\varepsilon)} \sqrt{v^2 + |Dv|^2} - \int_{B(\varepsilon)} \sqrt{w^2 + |Dw|^2} - \int_\Omega \int_v^{\max(v, \varepsilon)} \ln h \, dh \, dx$$

$$- \int_\Sigma \beta [w - v] \, ds$$

$$= \int_{B(\varepsilon)} \sqrt{v^2 + |Dv|^2} - \int_{B(\varepsilon)} \varepsilon \, dx - \int_{B(\varepsilon)} \int_{B(\varepsilon)} \ln h \, dh \, dx - \int_\Sigma \beta [\max(v, \varepsilon) - v] \, ds$$

$$\geq \int_{B(\varepsilon)} \sqrt{v^2 + |Dv|^2} - \int_{B(\varepsilon)} (\varepsilon - v + v) \, dx + (- \ln \varepsilon) \int_{B(\varepsilon)} (\varepsilon - v) \, dx$$

$$= \int_{B(\varepsilon)} \sqrt{v^2 + |Dv|^2} - \int_{B(\varepsilon)} v \, dx + (- \ln \varepsilon - 1) \int_{B(\varepsilon)} (\varepsilon - v) \, dx$$

$$> 0$$

for all $\varepsilon < e^{-1}$, if $\int_{B(\varepsilon)} |\varepsilon - v| \neq 0$. However, this would contradict our assumption that $v$ is minimal for $J_1$, so we conclude that $|B(\varepsilon)| = 0$ for small enough $\varepsilon$. \hfill $\Box$

4. Existence of a Minimiser

Lemma 4.1 (Lower bounds for the energy). If $v \in BV^+(\Omega)$, then $J_1(v) \geq C(n, R, a, |\Omega|)$, where $C$ is not necessarily positive.
Proof. As before, we can incorporate the wetting energy into the surface tension term using (2.1), so that
\[ J_1(v) \geq \int_{\Omega} \sqrt{v^2 + |Dv|^2} + \int_{\Omega} v \ln h \, dh \, dx - (1 - a)c_R \int_{\Omega} |1 - v| \, dx - (1 - a) \int_{\Omega} |Dv| \]
\[ \geq a \int_{\Omega} |Dv| + \int_{\Omega} f(v) \, dx, \]
where \( f(v) := v(\ln v - 1) + 1 - c_R(1 - a)|1 - v| \) is bounded below by a constant dependent on \( c_R \) and \( a \). The result follows. \( \square \)

We define a minimising sequence for \( J_1 \) as a sequence \( v_j \in BV^+(\Omega) \) with
\[ \lim_{j \to \infty} J_1(v_j) = \inf_{w \in BV^+(\Omega)} J_1(w) := m. \]

Lemma 4.2. A minimising sequence for \( J_1 \) is uniformly bounded in the BV-norm.

Proof. We can assume that \( J_1(v_j) \leq m + 1 \) for \( j \) large enough. As in the previous lemma, where we defined \( f \), we then have
\[ m + 1 \geq J_1(v_j) \geq a \int_{\Omega} |Dv_j| + \int_{\Omega} f(v_j) \, dx, \]
so the uniform bound follows from the lower bound on \( f \):
\[ \int_{\Omega} |Dv_j| \leq \frac{1}{a} \left( m + 1 - |\Omega| \inf_{h \in \mathbb{R}^+} f(h) \right). \]
Also, since there exist positive constants \( \alpha_1, \alpha_2 \) such that \( f(t) \geq \alpha_1 t - \alpha_2 \), we have the uniform \( L^1 \) bound
\[ \|v_j\|_{L^1(\Omega)} \leq \frac{1}{\alpha_1} \left( \int_{\Omega} [f(v_j) + \alpha_2] \, dx \right) \leq \frac{1}{\alpha_1} (m + 1 + \alpha_2|\Omega|). \]
\( \square \)

Lemma 4.3 (Lower semicontinuity of \( J_1 \)). A sequence \( v_k \in BV^+(\Omega) \) with \( v_k \to v \) in \( L^1(\Omega) \) satisfies
\[ J_1(v) \leq \liminf_{k \to \infty} J_1(v_k). \]

Proof. We show the surface energy term is lower semicontinuous. For any admissible \( g \), we have
\[ \int_{\Omega} v (g_{n+1} + \text{div}_n g) \, dx = \lim_{k \to \infty} \int_{\Omega} v_k (g_{n+1} + \text{div}_n g) \, dx \]
\[ = \liminf_{k \to \infty} \int_{\Omega} v_k (g_{n+1} + \text{div}_n g) \, dx \]
\[ \leq \liminf_{k \to \infty} \int_{\Omega} \sqrt{v_k^2 + |Dv_k|^2}. \]
Lower semicontinuity follows by taking the supremum over all admissible \( g \).
Continuity of the remaining terms of \( J_1 \) follows as in [10, Appendix II]. \( \square \)

Combining all of the above results we have:

Theorem 4.4 (Existence of a minimiser). There exists a function \( v \in BV^+(\Omega) \), such that
\[ J_1(v) = \inf_{w \in BV^+(\Omega)} J_1(w). \]
**Proof.** Let \( \{v_j\} \) be the minimising sequence of Lemma 4.2, with \( \|v_j\|_{BV(\Omega)} \leq C \). By the standard compactness theorem (for example [13, Theorem 1.19]) there exists a subsequence \( v_{j'} \to v \) in \( L^1(\Omega) \).

Since the BV-norm is lower semicontinuous, \( v \) is also in \( BV^+(\Omega) \), and \( J_1(v) \geq \inf_{w \in BV^+(\Omega)} J_1(w) = m \). Lower semicontinuity of \( J_1 \), as in Lemma 4.3, gives \( J_1(v) \leq \lim \inf J_1(v_{j'}) = m \), completing the proof. \( \square \)

### 5. Regularity

In this section we show that a minimiser \( v \in BV^+(\Omega) \) of \( J_1 \) is locally Lipschitz in \( \Omega \) following a procedure similar to [10].

In a subsequent paper, we discuss boundary regularity. If one has boundary regularity, the methods of [1] can be used to derive higher regularity in smooth domains.

**Theorem 5.1.** Let \( v \) be a minimiser of \( J_1 \) in \( BV^+(\Omega) \). Then \( v \) is locally Lipschitz in \( \Omega \).

**Proof.** We mollify \( v \) over the whole of \( \Omega \). The mollification \( v_\varepsilon \) is in \( C^\infty(\Omega) \), and shares the height bounds derived for \( v \) in Section 3 (that is, bounded above and bounded from below away from zero). Furthermore, since \( v \in BV(\Omega) \),

\[
v_\varepsilon \to v \text{ in } L^1(\Omega) \text{ and } \int_\Omega |Dv_\varepsilon| \to \int_\Omega |Dv|.
\]

Corollary 2.3 for the surface energy and standard convergence results for the remaining energy terms then imply that

\[
J_1(v_\varepsilon) \to J_1(v).
\]

Let \( B \subset \Omega \) be any ball of sufficiently small radius \( \rho \), and consider the following two related Dirichlet problems:

\[
\begin{align*}
\text{div} \frac{Dw_\varepsilon}{\sqrt{w_\varepsilon^2 + |Dw_\varepsilon|^2}} &= \frac{w_\varepsilon}{\sqrt{w_\varepsilon^2 + |Dw_\varepsilon|^2}} + \ln w_\varepsilon \text{ in } B, \\
w_\varepsilon &= v_\varepsilon \text{ on } \partial B;
\end{align*}
\]

and

\[
\begin{align*}
\text{div} \frac{Du_\varepsilon}{\sqrt{1 + |Du_\varepsilon|^2}} &= \sigma \left(-\frac{1}{\sqrt{1 + |Du_\varepsilon|^2}} + u_\varepsilon\right) \text{ in } B, \\
u_\varepsilon &= -\sigma \ln(v_\varepsilon) \text{ on } \partial B.
\end{align*}
\]

The second expression is in fact a family of problems, indexed by \( \sigma \in [0, 1] \). This family is of mean curvature type. Note that for smooth \( w_\varepsilon \) and \( u_\varepsilon \), (5.2) is equivalent to (5.3) for \( \sigma = 1 \), with the correspondence \( w_\varepsilon = e^{-u_\varepsilon} \).

Our next step is to solve (5.3) for \( \sigma = 1 \) using the continuity method. We apply [11, Theorem 13.8]. A prerequisite for this is to show that a smooth solution \( u^\sigma \) of (5.3), for any \( \sigma \in [0, 1] \), has height and gradient bounds independent of \( \sigma \).

The height bound may be found in [17]; however, the geometric nature of our problem admits a shorter proof which we present as the following lemma.
Lemma 5.2. Let $u^\sigma$ be a smooth solution to (5.3) corresponding to a $\sigma \in [0, 1]$. Then

$$\sup_B |u^\sigma| < M_1,$$

where $M_1$ depends only on $\sup_\Omega |\ln v_\varepsilon|$.

Proof. We suppose that $u^\sigma$ achieves a positive interior maximum, $u^\sigma(\tilde{x}) = \tilde{M}$ at some point $\tilde{x} \in B$. If $\tilde{M} > 1$, then the mean curvature $H(u^\sigma) = \text{div} \frac{Du^\sigma}{\sqrt{1 + |Du^\sigma|^2}}$ at $\tilde{x}$ must be strictly positive. But a point of positive mean curvature cannot correspond to an interior maximum, contradicting the assumption $\tilde{M} > 1$. We conclude that $u^\sigma \leq \min\{1, \sup_{\partial B} |\sigma \ln v_\varepsilon|\} \leq \min\{1, \sup_{\partial B} |\ln v_\varepsilon|\}.$

A similar argument shows that $u^\sigma$ has no negative internal minimum, so $u^\sigma \geq -\sup_{\partial B} |\ln v_\varepsilon|$. □

Continuing the proof of Theorem 5.1: We find that the gradient bound

$$\sup_B |Du^\sigma| \leq M_2$$

is an application of standard results. Firstly, an interior gradient bound can either be derived by applying a maximum principle to the elliptic equation satisfied by the gradient; or by using [16, Theorem 4], which gives

$$\sup_{B'} |Du^\sigma| \leq M_3$$

where $B' \subset B$ and $M_3$ is dependent on $\text{dist}(B', \partial B)$, $n$ and $\sup |u^\sigma|$.

Secondly, a boundary gradient estimate

$$\sup_{\partial B} |Du^\sigma| \leq M_4$$

results from [11, Corollary 14.5] with the structure condition (14.33). Here $M_4$ is dependent on $|\ln v_\varepsilon|_{C^2(\partial B)}$, $n$, $\sup |u^\sigma|$, and $\rho$. Together these two gradient estimates give us (5.5).

The conditions for the continuity method being satisfied, the problem (5.3), with $\sigma = 1$, has a $C^{2,\alpha}(B)$ solution which we call $u_\varepsilon$. It has height and gradient bounds (5.4) and (5.5). It is also unique: the proof is similar to that of Theorem 2.2 in [1], adjusted to Dirichlet boundary data.

We set $w_\varepsilon = e^{-u_\varepsilon}$. This is a $C^{2,\alpha}(B)$ solution of (5.2) with height bound $e^{-M_1} \leq w_\varepsilon \leq e^{M_1}$ and gradient bound $|Dw_\varepsilon| \leq M_2 e^{M_1}$.

Note that (5.2) is the Euler-Lagrange equation for the energy

$$\mathcal{J}_2(w) := \int_B \sqrt{w^2 + |Dw|^2} + \int_B \int_1^w \ln h \, dh \, dx,$$

and so $w_\varepsilon$ is a critical point of $\mathcal{J}_2$ in the class of $H^{1,2}(B)$ functions with boundary data $v_\varepsilon$. Furthermore, as the integrand of $\mathcal{J}_2$ is convex in $(w, Dw)$, $w_\varepsilon$ is also a minimiser in this class (see, for example, the remark in Section 8.2.3 of [5]) and hence in the smaller set $C^{2,\alpha}(B)$.

In particular, if we compare $w_\varepsilon$ to $v_\varepsilon$, we have

$$\int_B \sqrt{w_\varepsilon^2 + |Dw_\varepsilon|^2} + \int_B \int_1^{w_\varepsilon} \ln h \, dh \, dx \leq \int_B \sqrt{v_\varepsilon^2 + |Dv_\varepsilon|^2} + \int_B \int_1^{v_\varepsilon} \ln h \, dh \, dx.$$
Now let $\tilde{v}_\varepsilon$ be defined by

$$
\tilde{v}_\varepsilon = \begin{cases} 
  w_\varepsilon & \text{in } B \\
  v_\varepsilon & \text{in } \Omega \setminus B.
\end{cases}
$$

Using (5.7) for the region $B$ where $\tilde{v}_\varepsilon$ may be different to $v_\varepsilon$, we see that $J_1(\tilde{v}_\varepsilon) \leq J_1(v_\varepsilon)$.

Now we will show that $\tilde{v}_\varepsilon$ converges to a BV($\Omega$) function which is locally Lipschitz.

Uniform $L^1(\Omega \setminus B)$ bounds are given by the height bounds for $v$ in Section 3. Uniform $L^1(B)$ bounds are given by $\sup_B |w_\varepsilon| \leq e^{M_1}$ where $M_1$ is the constant in (5.4); $M_1$ also depends on the height bounds for $v$.

As a consequence of (5.1), we may assume that $J_1(v_\varepsilon) \leq J_1(v) + 1$. Then $J_1(\tilde{v}_\varepsilon) \leq J_1(v) + 1$, and so

$$
\int_{\Omega} \sqrt{\tilde{v}_\varepsilon^2 + |D\tilde{v}_\varepsilon|^2} \leq J_1(v) + 1 - \int_{\Omega} \int_{\tilde{v}_\varepsilon} \ln h dh dx + \int_{\Sigma} \beta (1 - \tilde{v}_\varepsilon) ds
\leq J_1(v) + 1 + |\Omega| \sup \left( h \ln h - h + 1 \right) + |\Sigma| \sup \left( \beta (1 + \sup |v|) \right)
$$

which is bounded above, independently of $\varepsilon$. Uniform BV bounds follow as in Lemma 2.2. Therefore a subsequence of $\tilde{v}_\varepsilon$ converges to $v_0 \in BV(\Omega)$, and $v_0$ is Lipschitz in $B'$ with bounds given by (5.6).

Lower semicontinuity of the functional now gives

$$
J_1(v_0) \leq \liminf J_1(\tilde{v}_\varepsilon) \leq \liminf J_1(v_\varepsilon) = J_1(v)
$$

but as $v$ was assumed to minimise $J_1$ these must all be equal. We conclude that there exists a minimiser of $J_1$ that is locally Lipschitz on interior sets. \qed

Reconsidering the problem of a capillary surface $S = \text{graph } u$ that minimises the original energy functional $J$ given in the introduction, we conclude that the found $v$ corresponds to a minimiser in the class $\{u : e^{-w} \in BV^+(\Omega)\}$. This solution is given by $u = -\ln v$, and is locally Lipschitz on interior sets.

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