DECAY OF ALMOST PERIODIC SOLUTIONS OF ANISOTROPIC DEGENERATE PARABOLIC-HYPERBOLIC EQUATIONS

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Abstract. We prove the well-posedness and decay of Besicovitch almost periodic solutions for nonlinear degenerate anisotropic hyperbolic-parabolic equations. The decay property is proven for the case where the diffusion term is given by a non-degenerate nonlinear $d'' \times d''$ diffusion matrix and the complementary $d'$ components of flux-function form a non-degenerate flux in $\mathbb{R}^{d'}$, with $d' + d'' = d$. For this special case we also prove that the strong trace property at the initial time holds, which allows, in particular, to require the assumption of the initial data only in a weak sense, and gives the continuity in time of the solution with values in $L^1_{\text{loc}}(\mathbb{R}^d)$. So far, for the decay property, we need also to impose that the bounded Besicovitch almost periodic initial function can be approximated in the Besicovitch norm by almost periodic functions whose $\varepsilon$-inclusion intervals $l_\varepsilon$ satisfy $l_\varepsilon/|\log \varepsilon|^{1/2} \to 0$ as $\varepsilon \to 0$. This includes, in particular, generalized limit periodic functions, that is, limits in the Besicovitch norm of purely periodic functions.

1. Introduction

We address the problem of the decay to the mean-value of $L^\infty$ Besicovitch almost periodic solutions to nonlinear degenerate anisotropic hyperbolic-parabolic equations. Consider the Cauchy problem

\begin{align}
\partial_t u + \nabla_x \cdot f(u) &= \nabla_x \cdot (A(u)\nabla_x u), & x \in \mathbb{R}^d, & t > 0, \\
u(0, x) &= u_0, & x \in \mathbb{R}^d,
\end{align}

where $f = (f_1, \cdots, f_d)$, $A(u) = (a_{ij}(u))_{i,j=1}^d$, with $f_i(u), a_{ij}(u) : \mathbb{R} \to \mathbb{R}$ smooth functions. $A(u)$ is a symmetric non-negative matrix and so we may write

\begin{equation}
a_{ij}(u) = \sum_{k=1}^d \sigma_{ik}(u)\sigma_{jk}(u),
\end{equation}

with $\sigma_{ij}(u) : \mathbb{R} \to \mathbb{R}$ smooth functions, that is, $(\sigma_{ij}(u))_{i,j=1}^d$ is the square root of $A(u)$. We assume to begin with that $u_0 \in L^\infty(\mathbb{R}^d)$.

In this paper, we are concerned with the large-time behavior of entropy solutions of (1.1),(1.2) with initial function $u_0$ satisfying

\begin{equation}
u_0 \in L^\infty(\mathbb{R}^d) \cap \text{BAP}(\mathbb{R}^d).\end{equation}

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Here, BAP(\(\mathbb{R}^d\)) denotes the space of the Besicovitch almost periodic functions (with exponent \(p = 1\)), which can be defined as the completion of the space of trigonometric polynomials, i.e., finite sums \(\sum_{\lambda} a_{\lambda} e^{2\pi i x \cdot \alpha} (i = \sqrt{-1} \text{ is the purely imaginary unity})\) under the semi-norm

\[ N_1(g) := \limsup_{R \to \infty} \frac{1}{R^d} \int_{C_R} |g(x)| \, dx, \]

where, for \(R > 0\),

\[ C_R := \{ x \in \mathbb{R}^d : |x|_{\infty} := \max_{i=1, \ldots, d} |x_i| \leq R/2 \}. \]

We observe that the semi-norm \(N_1\) is indeed a norm over the trigonometric polynomials, so the referred completion through it is a well defined Banach space.

Equivalently, the space BAP(\(\mathbb{R}^d\)) is also the completion through \(N_1\) of the space of uniform (or Bohr) almost periodic functions, AP(\(\mathbb{R}^d\)), which is defined as the closure in the sup-norm of the trigonometric polynomials.

We begin by stating the definition of entropy solution for (1.1),(1.2), which is in part motivated by [9]. We use the normal trace property of \(L^2\)-divergence measure fields (see, e.g., [6, 7]).

**Definition 1.1.** An entropy solution for (1.1),(1.2), with \(u_0 \in L^\infty(\mathbb{R}^d)\), is a function \(u(t,x) \in L^\infty((0, \infty) \times \mathbb{R}^d)\) such that

(i) (Regularity) For any \(R > 0\), we have

\[ \sum_{i=1}^d \partial_{x_i} \beta_{ik}(u) \in L^2((0, \infty) \times C_R), \]

for \(k = 1, \ldots, d\), for \(\beta_{ik}(u) = \int_u^\infty \sigma_{ik}(v) \, dv.\)

(ii) (Chain Rule) For any function \(\psi \in C_0(\mathbb{R})\) with \(\psi(u) \geq 0\) and any \(k = 1, \ldots, d\) the following chain rule holds:

\[ \sum_{i=1}^d \partial_{x_i} \beta^\psi_{ik}(u) = \sqrt{\psi(u)} \sum_{i=1}^d \partial_{x_i} \beta_{ik}(u) \in L^2((0, \infty) \times C_R), \]

for \(k = 1, \ldots, d\), for \((\beta^\psi_{ik})' = \sqrt{\psi} \beta'_{ik},\)

for any \(R > 0\).

(iii) (Entropy Inequality) For any convex \(C^2\) function \(\eta : \mathbb{R} \to \mathbb{R}\), and \(q(u) = \eta'(u) f'(u), r_{ij}(u) = \eta'(u) a_{ij}(u),\) we have

\[ \partial_t \eta(u) + \nabla_x \cdot q(u) - \sum_{ij=1}^d \partial_{x_i}^2 r_{ij}(u) \leq -\eta''(u) \sum_{k=1}^d \left( \sum_{i=1}^d \partial_{x_i} \beta_{ik}(u) \right)^2, \]

in the sense of distributions in \((0, \infty) \times \mathbb{R}^d,\) and

\[ \eta(u(t,x))|\{t = 0\} = \eta(u_0(x)), \]

in the sense of the normal trace of the \(L^2\)-divergence measure field

\[ \left( \eta(u), q(u) - \left( \sum_{j=1}^d \partial_{x_j} r_{ij}(u) \right)_{i=1}^d \right). \]
Remark 1.1. We remark that condition (iii) in the Definition 1.1 implies that for all \( k \in \mathbb{R} \) we have
\[
(1.9) \quad \partial_t |u(t, x) - k| + \nabla_x \cdot \text{sgn}(u(t, x) - k)(f(u) - f(k)) - \sum_{i,j=1}^d \partial_{x_i,x_j} \text{sgn}(u(t, x) - k)(A_{ij}(u) - A_{ij}(k)) \leq 0,
\]
where \( A_{ij}(u) = a_{ij}(u) \), in the sense of distributions in \((0, \infty) \times \mathbb{R}^d\).

Remark 1.2. We also remark that \((1.8)\), valid for all \( t > 0 \), implies, for any \( R > 0 \),
\[
(1.10) \quad \lim_{t \to u+} \int_{C_R} |u(t, x) - u_0(x)| \, dx = 0,
\]
as essentially follows from theorem 4.5.1 in [11] (see [15]) which establish es that \((1.8)\) implies
\[
\lim_{t \to u+} \int_{\mathbb{R}^d} \eta(u(t, x))\phi(x) \, dx = \int_{\mathbb{R}^d} \eta(u_0(x))\phi(x) \, dx,
\]
for all \( \phi \in C_0^\infty(\mathbb{R}^d) \), which by a well known convexity argument implies \((1.10)\).

Remark 1.3. Take \( \eta(u) = \frac{1}{2}u^2 \) in \((1.7)\) and as test function \( \phi_R(x)\chi_\nu(t) \), with \( \phi_R \in C_0^\infty(\mathbb{R}^d) \), \( 0 \leq \phi_R(x) \leq 1 \), for all \( x \in \mathbb{R}^d \), \( \phi_R(x) = 1 \), for \( |x| \leq R \), \( \phi_R(x) = 0 \), for \( |x| \geq R + 1 \), and \( \|D^\alpha \phi_R\|_\infty \leq C \), \( |\alpha| \leq 2 \), for some \( C > 0 \) independent of \( R \), and \( \chi_\nu(t) = \theta(t-t_0) - \theta(t-t_1) \), with
\[
\theta_\nu(t) = \int_0^t \delta_\nu(s) \, ds = \int_0^\nu \sigma(s) \, ds, \quad \delta_\nu(s) = \nu \sigma(\nu s),
\]
with \( \sigma \in C_0^\infty(\mathbb{R}) \), \( \text{supp } \sigma \subset [0, 1] \), \( \sigma \geq 0 \), \( \int_\mathbb{R} \sigma(s) \, ds = 1 \). Then, sending \( \nu \to \infty \) we deduce that for some constant \( C > 0 \), independent of \( R \), we have, for all \( t > 0 \),
\[
(1.11) \quad \int_0^t \int_{C_R} \sum_{k=1}^d \left( \sum_{i=1}^d \partial_{x_i} \beta_{ik}(u) \right)^2 \, dx \, dt \leq C(R + 1)^d + C \nu(R + 1)^{d-1}.
\]
In particular, for any \( t > 0 \),
\[
(1.12) \quad \limsup_{R \to \infty} R^{-d} \int_0^t \int_{C_R} \sum_{k=1}^d \left( \sum_{i=1}^d \partial_{x_i} \beta_{ik}(u) \right)^2 \, dx \, dt \leq C.
\]

For any \( g \in \text{BAP}(\mathbb{R}^d) \), its mean value \( M(g) \), defined by
\[
M(g) := \lim_{R \to \infty} R^{-d} \int_{C_R} g(x) \, dx,
\]
exists (see, e.g., [2]). The mean value \( M(g) \) is also denoted by \( \mathcal{F}_{\mathbb{R}^d} g \). Also, the Bohr-Fourier coefficients of \( g \in \text{BAP}(\mathbb{R}^d) \)
\[
a_\lambda = M(ge^{-2\pi i \lambda \cdot x}),
\]
are well defined and we have that the spectrum of \( g \), defined by
\[
\text{Sp}(g) := \{ \lambda \in \mathbb{R}^d : a_\lambda \neq 0 \},
\]
is at most countable (see, e.g., [22]). We denote by $\text{Gr}(g)$ the smallest additive subgroup of $\mathbb{R}^d$ containing $\text{Sp}(g)$ (cf. [22], where $\text{Gr}(g)$ was introduced and denoted by $M(g)$).

The first result of this paper is the following.

**Theorem 1.1.** For any $u_0 \in L^\infty(\mathbb{R}^d)$, there exists a unique weak entropy solution $u(t,x)$ of (1.1),(1.2). Moreover, if $u_0$ satisfies (1.4), then

$$u \in L^\infty((0,\infty), \text{BAP}(\mathbb{R}^d)) \bigcap L^\infty(\mathbb{R}^{d+1}_+),$$

and $\text{Gr}(u(t,\cdot)) \subset \text{Gr}(u_0)$, for a.e. $t > 0$.

A particular case of (1.1) is the following

$$\partial_t u + \nabla_x \cdot f(u) = \nabla_{x'}(B(u)\nabla_{x''}u), \quad x \in \mathbb{R}^d, \quad t > 0,$$

where $B(u) = (b_{ij}(u))_{i,j=1}^d$, and $1 \leq d' < d$, so $B(u)$ is a symmetric non-negative $d'' \times d''$-matrix, $d'' = d - d'$, and $\nabla_{x''} := (\partial_{x_{d'+1}}, \ldots, \partial_{x_d})$. Also, we assume the non-degeneracy condition:

For any $(\tau,\kappa') \in \mathbb{R}^{d+1}$, with $\tau^2 + \kappa'^2 = 1$, and $\kappa'' \in \mathbb{R}^{d''}$, with $|\kappa''| = 1$, denoting $\pi_{d'}(f(u)) = (f_1(u), \ldots, f_{d'}(u))$,

$$L^1\{\xi \in \mathbb{R} : |\xi| \leq \|u_0\|_\infty, \tau + \pi_{d'}(f(\xi)) \cdot \kappa' = 0\} = 0,$$

$$L^1\{\xi \in \mathbb{R} : |\xi| \leq \|u_0\|_\infty, \kappa'' B(\xi) \kappa'' = 0\} = 0.$$

Although (1.14) is a particular case of (1.1), under the non-degeneracy conditions (1.15) and (1.16) we may relax (1.8) in Definition 1.1 to

$$u(t,x)|\{t = 0\} = u_0(x),$$

in the sense of the normal trace of the $L^2$ divergence-measure field

$$(u, f(u) - (0, \ldots, 0, \sum_{j=d'+1}^d \partial_{x_j} B_{ij}(u)\big|_{i=d'+1}^d)),$$

We call $u(t,x) \in L^\infty((0,\infty) \times \mathbb{R}^d)$ a weak entropy solution of (1.14),(1.2) if it satisfies all the corresponding conditions of Definition 1.1 except that instead of (1.8), we now impose the weaker (1.17).

The second result of this paper concerns weak entropy solutions of (1.14),(1.2).

**Theorem 1.2.** Let $u$ be weak entropy solution of (1.14),(1.2). Then,

$$u \in C([0,\infty), L^1_{\text{loc}}(\mathbb{R}^d)).$$

In particular, for any $R > 0$,

$$\lim_{t \to 0^+} \int_{|x| < R} |u(t,x) - u_0(x)| \, dx = 0.$$

Moreover, if $u_0$ satisfies (1.4), then

$$u \in L^\infty([0,\infty), \text{BAP}(\mathbb{R}^d)) \bigcap L^\infty(\mathbb{R}^{d+1}_+),$$

and

$$\text{Gr}(u(t,\cdot)) \subset \text{Gr}(u_0),$$

for a.e. $t > 0$. 
From Theorem 1.2, we deduce that weak entropy solutions of (1.14), (1.2) are indeed entropy solutions of (1.14), (1.2) in the sense of Definition 1.1, so that Theorem 1.1 applies to them. As we will see in Section 3, the proof of Theorem 1.2 amounts to show the validity of the strong trace property for the solution of (1.14), (1.2).

Finally, we establish the following decay property as the third result of this paper. We remark that, in particular, the hypotheses on the initial function are clearly satisfied by the generalized limit periodic functions, that is, limits in the Besicovitch norm induced by $N_1$ of purely periodic functions.

**Theorem 1.3.** Assume, in addition to (1.4), that $u_0$ can be approximated in the Besicovitch norm induced by $N_1$ by a sequence of almost periodic functions $u_0\nu$ which, for each $\varepsilon$, possess $\varepsilon$-inclusion intervals, $l_\nu\varepsilon$, satisfying $l_\nu\varepsilon/|\log \varepsilon|^{1/2} \to 0$, as $\varepsilon \to 0$. Then, the entropy solution of (1.14), (1.2) satisfies

$$
(1.22) \quad \lim_{t \to +\infty} M(|u(t,\cdot) - M(u_0)|) = 0.
$$

There is a large literature related with degenerate parabolic equations, being the first important contribution by Vol’pert and Hudjaev in [27]. Uniqueness for the homogeneous Dirichlet problem, for the isotropic case, was only achieved many years later by Carrillo in [3], using an extension of Kruzhkov’s doubling of variables method [18]. The result in [3] was extended to non-homogeneous Dirichlet data by Mascia, Porretta and Terracina in [20]. An $L^1$ theory for the Cauchy problem for anisotropic degenerate parabolic equations was established by Chen and Perthame [9], based on the kinetic formulation (see [23]), and later also obtained using Kruzhkov’s approach in [1, 8] (see also, [17], [13] and the references therein). Decay of almost periodic solutions for general nonlinear systems of conservation laws of parabolic and hyperbolic types was first addressed in [14], as an extension of the ideas put forth in [4]. Only recently the problem of the decay of almost periodic solutions was retaken, specifically for scalar conservation laws, by Panov in [22], where some elegant ideas were introduced to successfully extend the result in [14] in that specific case.

We first give a brief account on the way Theorem 1.1 is proven. The part of existence and uniqueness are by now well known and for most of that we just refer to [8], which deals with the case of initial function in $L^1(\mathbb{R}^d)$. Nevertheless, (1.12) is new and of great interest in the case of initial functions in $L^\infty(\mathbb{R}^d)$. For the invariance of the class of $L^\infty$ Besicovitch almost periodic functions with exponent $p = 1$, we use the elegant method of reduction to the periodic case introduced by Panov in [22].

Concerning Theorem 1.2, the first part, including (1.18), (1.19) and (1.20), which improves the regularity given in Theorem 1.1, is consequence of the strong trace property enjoyed by (1.14) as is shown here.

As for Theorem 1.3, namely, the decay property (1.22), it is obtained essentially using ideas in [14]. Unfortunately we cannot use the reduction to the periodic case for getting the decay of the solution, as in [22]. In particular, we cannot apply the result on the decay of periodic entropy solutions for nonlinear anisotropic degenerate parabolic-hyperbolic equations of Chen and Perthame in [10]. The reason is that we miss here the necessary non-degeneracy condition for the equation in higher space dimensions corresponding to the uplifting to the periodic context.
This paper is organized as follows. After this Introduction, in Section 2, the proof of Theorem 1.1 is given, split in a number of auxiliary results, starting with the important in its own Proposition 2.1, followed by three lemmas. In Section 3, we prove Theorem 1.2, which establishes the strong trace property at the initial time and the continuity in time of the solution with values in $L^1_{\text{loc}}(\mathbb{R}^d)$. Finally, in Section 4, we prove Theorem 1.3, namely, the decay property.

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1 through a number of auxiliary results and results that establish parts of the its statement.

We begin with a proposition which is central in the whole strategy of reducing to the periodic case as devised in [22]. We will need the following technical lemma of [22], to which we refer for the proof.

**Lemma 2.1.** Suppose that $u(x,y) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^m)$,

$$E = \{ x \in \mathbb{R}^n : (x,y) \text{ is a Lebesgue point of } u(x,y) \text{ for a.e. } y \in \mathbb{R}^m \}.$$

Then $E$ is a set of full measure and $x \in E$ is a common Lebesgue point of the functions $I(x) = \int_{\mathbb{R}^m} u(x,y) \rho(y) \, dy$, for all $\rho \in L^1(\mathbb{R}^m)$.

**Proposition 2.1.** (mean $L^1$-contraction).

Let $u(t,x), v(t,x) \in L^\infty(\mathbb{R}^{d+1}_+) \times \mathbb{R}$ be two entropy solutions of (1.1), (1.2), with initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$. Then for a.e. $0 < t_0 < t_1$

$$N_1(u(t_1, \cdot) - v(t_1, \cdot)) \leq N_1(u(t_0, \cdot) - v(t_0, \cdot)),$$

and also for a.e. $t > 0$,

$$N_1(u(t, \cdot) - v(t, \cdot)) \leq N_1(u_0 - v_0),$$

**Proof:** We follow closely with the due adaptations the proof of proposition 1.3 in [22]. We first recall that by using the doubling of variables method of Kruzkov [18], as adapted by Carrillo [3] to the isotropic degenerate parabolic case and [1] to the anisotropic one, we obtain

$$|u - v|_t + \nabla \cdot \text{sgn}(u-v)(f(u) - f(v)) \leq \sum_{i,j=1}^d \partial^2_{x,xx_j} \text{sgn}(u-v)(A_{ij}(u) - A_{ij}(v))$$

in the sense of distributions in $\mathbb{R}^{d+1}_+$. As usual, we define a sequence approximating the indicator function of the interval $(t_0, t_1)$, by setting for $\nu \in \mathbb{N}$,

$$\delta_\nu(s) = \nu \sigma(\nu s), \quad \theta_\nu(t) = \int_0^t \delta_\nu(s) \, ds = \int_0^{\nu t} \sigma(s) \, ds,$$

where $\sigma \in C_0^\infty(\mathbb{R})$, supp $\rho \subset [0,1]$, $\sigma \geq 0$, $\int_{\mathbb{R}} \sigma(s) \, ds = 1$. We see that $\delta_\nu(s)$ converges to the Dirac measure in the sense of distributions in $\mathbb{R}$ while $\theta_\nu(t)$ converges everywhere to the Heaviside function. For $t_1 > t_0 > 0$, if $\chi_\nu(t) = \theta_\nu(t - t_0) - \theta_\nu(t - t_1)$, then $\chi_\nu \in C_0^\infty(\mathbb{R}_+) \times \mathbb{R}$, $0 \leq \chi_\nu \leq 1$, and the sequence $\chi_\nu(t)$ converges everywhere, as $\nu \to \infty$, to the indicator function of the interval $(t_0, t_1)$. Let us take $g \in C_0^\infty(\mathbb{R}^d)$, satisfying $0 \leq g \leq 1$, $g(y) \equiv 1$ in the cube $C_1$, $g(y) \equiv 0$ outside
the cube $C_k$, with $k > 1$. We apply (1.8) to the test function $\varphi = R^{-d} \chi_{\nu}(t)g(x/R)$, for $R > 0$. We then get

\begin{equation}
(2.4) \quad \int_0^\infty (R^{-d} \int_{\mathbb{R}^d} |u(t, x) - v(t, x)| g(x/R) \, dx) \left( \delta_{\nu}(t - t_0) - \delta_{\nu}(t - t_1) \right) \, dt \\
+ R^{-d-1} \int_{\mathbb{R}^{d+1}} \text{sgn}(u - v)(f(u) - f(v)) \cdot \nabla_y g(x/R) \chi_{\nu}(t) \, dx \, dt \\
- R^{-d-1} \sum_{i,j=1}^{d} \int_{\mathbb{R}^{d+1}} \text{sgn}(u - v) \partial_{x_i}(A_{ij}(u) - A_{ij}(v)) \partial_{x_j} g(x/R) \chi_{\nu}(t) \, dx \, dt \geq 0.
\end{equation}

Define

$$F = \{ t > 0 : (t, x) \text{ is a Lebesgue point of } |u(t, x) - v(t, x)| \text{ for a.e. } x \in \mathbb{R}^d \}.$$ 

As a consequence of Fubini’s theorem, $F$ is a set of full Lebesgue measure and by Lemma 2.1 each $t \in F$ is a Lebesgue point of the functions

$$I_R(t) = R^{-d} \int_{\mathbb{R}^d} |u(t, x) - v(t, x)| g(x/R) \, dx,$$

for all $R > 0$ and all $g \in C_0(\mathbb{R})$. Now we assume $t_0, t_1 \in F$ and take the limit as $\nu \to \infty$ in (1.9), to get

\begin{equation}
(2.5) \quad I_R(t_1) \leq I_R(t_0) + R^{-d-1} \int_{(t_0, t_1) \times \mathbb{R}^d} \text{sgn}(u - v)\left(f(u) - f(v)\right) \cdot \nabla_y g(x/R) \, dx \, dt \\
- R^{-d-1} \sum_{i,j=1}^{d} \int_{(t_0, t_1) \times \mathbb{R}^d} \text{sgn}(u - v) \partial_{x_i}(A_{ij}(u) - A_{ij}(v)) \partial_{x_j} g(x/R) \, dx \, dt.
\end{equation}

Now, we have

\begin{equation}
(2.6) \quad R^{-d-1} \left| \int_{(t_0, t_1) \times \mathbb{R}^d} \text{sgn}(u - v)\left(f(u) - f(v)\right) \cdot \nabla_y g(x/R) \, dx \, dt \right| \\
\leq R^{-1} \|f(u) - f(v)\|_{\infty} \int_{(t_0, t_1) \times \mathbb{R}^d} |\nabla_y g(y)| \, dy \, dt \to 0, \text{ as } R \to \infty.
\end{equation}
Also, we have

\[
R^{-d-1} \left| \sum_{i,j=1}^{d} \int_{R^d} \frac{\partial}{\partial x_i} \left( A_{ij}(u) - A_{ij}(v) \right) \partial_x g(x/R) \chi_\nu(t) \, dx \right|
\]

\[
\leq R^{-d-1} \left| \int_{R^d} \frac{\partial}{\partial x_k} \left( \sum_{i=1}^{d} \partial_{x_i} \beta_{jk}(u) \right) \partial_x g(x/R) \chi_\nu(t) \, dx \right|
\]

\[
+ R^{-d-1} \left| \int_{R^d} \frac{\partial}{\partial x_k} \left( \sum_{i=1}^{d} \partial_{x_i} \beta_{jk}(v) \right) \partial_x g(x/R) \chi_\nu(t) \, dx \right|
\]

\[
\leq CR^{-1} \sum_{k=1}^{d} \left( R^{-d} \int_{(t_0,t_1) \times C_{kR}} \left( \sum_{i=1}^{d} \partial_{x_i} \beta_{jk}(u) \right)^2 + \left( \sum_{i=1}^{d} \partial_{x_i} \beta_{jk}(v) \right)^2 \, dx \right)^{1/2}
\]

\[
\times \int_{(t_0,t_1) \times R^d} |\nabla y g(y)|^2 \, dy \, dt \right)^{1/2}
\]

\[
\rightarrow 0 \quad \text{as} \ R \rightarrow \infty,
\]

where we have used (1.12). On the other hand, we have

\[
N_1(u(t, \cdot) - v(t, \cdot)) \leq \limsup_{R \to \infty} I_R(t) \leq k^d N_1(u(t, \cdot) - v(t, \cdot)),
\]

so taking the limit as \( R \to \infty \) in (2.5), for \( t_0, t_1 \in F, t_0 < t_1 \), we get

\[
N_1(u(t_1, \cdot) - v(t_1, \cdot)) \leq k^d N_1(u(t_0, \cdot) - v(t_0, \cdot)),
\]

and since \( k > 1 \) is arbitrary we can make \( k \to 1^+ \) to get the desired result. Finally, for \( t_0 = 0 \), we use (1.10) to send \( t_0 \to 0^+ \) in (2.5) and proceed exactly as we have just done.

\[\square\]

**Lemma 2.2. (Uniqueness)** The problem (1.1),(1.2) has at most one entropy solution.

**Proof:** The proof follows through standard arguments (cf., e.g., [27]). So, let \( u, v \in L^\infty(\mathbb{R}^{d+1} \times [0,T]) \) be two weak entropy solutions. As in Proposition 2.1, by using the doubling of variables method of Kruzhkov [18], as adapted by Carrillo [3] to the isotropic degenerate parabolic case and [1] to the anisotropic one, we obtain

\[
\int_{\mathbb{R}^{d+1}} \left\{ |u - v| \phi_t + \text{sgn}(u - v) (f(u) - f(v)) \cdot \nabla \phi \right. 
\]

\[
+ \sum_{i,j=1}^{d} \text{sgn}(u - v) (A_{ij}(u) - A_{ij}(v)) \partial^2_{x_i x_j} \phi \right\} \, dx \, dt \geq 0,
\]

for all \( 0 \leq \phi \in C^\infty(\mathbb{R}^{d+1}) \). We take \( \phi(t,x) = \rho(x) \chi_\nu(t) \), where \( \rho(x) = e^{-\sqrt{1+x^2}} \) and \( \chi_\nu \) is as in the proof of Proposition 2.1. We observe that

\[
\sum_{i=1}^{d} |\partial_x \rho(x)| + \sum_{i,j=1}^{d} |\partial^2_{x_i x_j} \rho(x)| \leq C \rho(x),
\]
for some constant $C > 0$ depending only on $d$. Hence, making $\nu \to 0$, we arrive at

$$\int_{\mathbb{R}^d} |u(t_1, x) - v(t_1, x)|\rho(x) \, dx \leq \int_{\mathbb{R}^d} |u(t_0, x) - v(t_0, x)|\rho(x) \, dx$$

$$+ \hat{C} \int_{t_0}^{t_1} \int_{\mathbb{R}^d} |u(s, x) - v(s, x)|\rho(x) \, dx \, dt,$$

for a.e. $0 < t_0 < t_1$, for some $\hat{C} > 0$ depending only on $\mathbf{f}, A$ and the dimension $d$. Therefore, using Gronwall and (1.10), we conclude

$$(2.9) \quad \int_{\mathbb{R}^d} |u(t, x) - v(t, x)|\rho(x) \, dx \leq e^{\hat{C}t} \int_{\mathbb{R}^d} |u_0(x) - v_0(x)|\rho(x) \, dx,$$

which gives the desired result.

Observe that in the same way we got (2.9) from (2.8), we may get

$$(2.10) \quad \int_{\mathbb{R}^d} (u(t, x) - v(t, x))_+\rho(x) \, dx \leq e^{\hat{C}t} \int_{\mathbb{R}^d} (u_0(x) - v_0(x))_+\rho(x) \, dx,$$

from

$$(2.11) \quad \int_{\mathbb{R}^{d+1}_+} \left\{ (u - v)_+ \phi_1 + \text{sgn}(u - v)_+ (\mathbf{f}(u) - \mathbf{f}(v)) \cdot \nabla \phi \right.$$

$$+ \sum_{i,j=1}^d \text{sgn}(u - v)_+ (A_{ij}(u) - A_{ij}(v)) \partial_{x_i x_j} \phi \} \, dx \, dt \geq 0,$$

where $(u - v)_+ = \max\{0, u - v\}$ and $\text{sgn}(u - v)_+ = H(u - v)$ where $H(s)$ is the Heaviside function. Taking $v = k$, with $k > \|u_0\|_{\infty}$, and then reversing the roles of $u$ and $v$, making $u = k$ and $v = u$, with $k < -\|u_0\|_{\infty}$, we deduce that

$$(2.12) \quad |u(t, x)| \leq \|u_0\|_{\infty}, \quad \text{for a.e. } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

**Lemma 2.3. (Existence)** There exists an entropy solution to the problem (1.1),(1.2).

**Proof:** We consider the problem (1.1),(1.2) with initial function $u_{0,R}(x) = u_0(x)\chi_{R}(x)$, where $B_R = B(0, R)$ is the open ball with radius $R$ centered at the origin. By the existence theorem in [9], which holds for initial data in $L^1(\mathbb{R}^d)$, we obtain an entropy solution $u_R(t, x)$ of (1.1),(1.2)$_R$. Now, using (2.9), we see that, for a.e. $t > 0$,

$$(2.13) \quad \int_{\mathbb{R}^d} |u_R(t, x) - u_{0,R}(x)|\rho(x) \, dx \leq e^{\hat{C}t} \int_{\mathbb{R}^d} |u_{0,R}(x) - u_{0,0,R}(x)|\rho(x) \, dx$$

$$\to 0, \quad \text{as } R, \tilde{R} \to \infty.$$

Therefore, $u_R(t, x)$ converges in $L^1_{\text{loc}}((0, \infty) \times \mathbb{R}^d)$ to a function $u(t, x)$, which satisfies the bound in (2.12) since it holds for all $u_R$. It is now easy to deduce from the fact that the $u_R$’s satisfy all conditions of Definition 1.1 that $u(t, x)$ also satisfies
all those conditions. We just observe that for the verification of (1.7) from the fact that the $u_R$’s satisfy (1.7), we use the uniform boundedness in $L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ of
\[
\sum_{k=1}^{d} \left( \sum_{i=1}^{d} \partial_{x_i} \beta_{ik}(u_R) \right)^2
\]
and Fatou’s Lemma. Also, (1.8) is proved by including the initial function in (1.7), with $u(t,x)$ replaced by $u_R(t,x)$, tested against any function in $C^\infty_0(\mathbb{R}^{d+1})$, and taking the limit as $R \to \infty$, to conclude that (1.8) also holds.

In the next lemma, we prove that the solution operator for (1.1),(1.2) take bounded Besicovitch almost periodic functions into bounded Besicovitch almost periodic functions and that $\text{Gr}(u(t,\cdot)) \subset \text{Gr}(u_0(\cdot))$.

**Lemma 2.4.** Let $u(t,x)$ be the entropy solution of (1.1),(1.2) with $u_0$ satisfying (1.4). Let $G_0 = \text{Gr}(u_0)$. Then, $u(t,x) \in L^\infty([0,\infty), \text{BAP}(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}_+^{d+1})$ and $\text{Sp}(u(t,\cdot)) \subset G_0$, for a.e. $t > 0$.

**Proof:** The proof follows by the elegant method of reduction to the periodic case introduced by Panov in [22], more specifically theorems 2.1 and 2.2 in [22]. Here we limit ourselves to indicate the few adaptations that need to be made. The method begins by considering the case where the initial function $u_0$ is given by a trigonometric polynomial,
\[
(2.14) \quad u_0(x) = \sum_{\lambda \in \Lambda} a_{\lambda} e^{2\pi i \lambda \cdot x},
\]
where $\Lambda = \text{Sp}(u_0) \subset \mathbb{R}^d$ is a finite set. Since $u_0$ is real we have that $-\Lambda = \Lambda$ and $a_{-\lambda} = \bar{a}_{\lambda}$, where as usual $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$. The first observation is that we may find a basis for $G_0$, $\{\lambda_1, \ldots, \lambda_m\}$, so that any $\lambda \in G_0$ can be uniquely written as $\lambda = \lambda(\bar{k}) = \sum_{j=1}^{m} k_j \lambda_j$, $\bar{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m$, and the vectors $\lambda_j$ are linearly independent over $\mathbb{Z}$ and so also over $\mathbb{Q}$. Let $J = \{k \in \mathbb{Z}^m : \lambda(\bar{k}) \in \Lambda\}$. Then
\[
(2.15) \quad u_0(x) = \sum_{k \in J} a_{\bar{k}} e^{2\pi i \sum_{j=1}^{m} k_j \lambda_j \cdot x}, \quad a_{\bar{k}} := a_{\lambda(\bar{k})}.
\]
We then have $u_0(x) = v_0(y(x))$, where
\[
(2.16) \quad v_0(y) = \sum_{k \in J} a_k e^{2\pi i \bar{k} \cdot y}
\]
is a periodic function, $v_0(y + e_i) = v_0(y)$, $i = 1, \ldots, m$, $e_i$ the elements of the canonical basis of $\mathbb{R}^m$, and
\[
y(x) = (y_1, \ldots, y_m), \quad y_j = \lambda_j \cdot x = \sum_{k=1}^{d} \lambda_j k x_k, \quad \lambda_j = (\lambda_{j1}, \ldots, \lambda_{jd}).
\]
We then consider the nonlinear degenerate parabolic-hyperbolic equation
\[
(2.17) \quad v_t + \nabla_y \cdot \tilde{f}(v) = (\mathcal{B}\nabla_y) \cdot (A(v)(\mathcal{B}\nabla_y)v), \quad v = v(t,y), \quad t > 0, \quad y \in \mathbb{R}^m,
\]
with $\tilde{f} = (f_1, \ldots, f_m)$ and
\[
\tilde{f}_j(v) = \lambda_j \cdot f(v) = \sum_{k=1}^{d} \lambda_j k f_k(v), \quad j = 1, \ldots, m, \quad \mathcal{B} = \frac{\partial y}{\partial x}^\top.
\]
and
\[ B\nabla_y = \frac{\partial y}{\partial x}^T \nabla_y = \left( \sum_{j=1}^{m} \lambda_{j1} \partial_{y_j}, \cdots, \sum_{j=1}^{m} \lambda_{jd} \partial_{y_j} \right). \]

We consider the Cauchy problem for (2.17) with initial data
\[ (2.18) \quad v(0, y) = v_0(y). \]

Existence and uniqueness of the entropy solution \( v(t, y) \in L^\infty(\mathbb{R}^{m+1}_+) \) of (2.17), (2.18) follow from the analogs of Lemmas 2.3 and 2.2 for (2.17), (2.18), and it is easy to see that \( v(t, y) \) is also spatially periodic, namely, \( v(t, y + e_i) = v(t, y), \) for all \( y \in \mathbb{R}^m, \)
\( t > 0, \) where \( e_j, j = 1, \cdots, m, \) is the canonical basis of \( \mathbb{R}^m. \) The following assertion corresponds to theorem 2.1 of [22] and its proof follows by the same lines as the proof of that result, so we just refer to [22] for the proof.

\textit{Assertion #1.} For a.e. \( z \in \mathbb{R}^m \) the function \( u(t, x) = v(t, z + y(x)) \) is an entropy solution of (1.1), (1.2) with initial data \( v_0(z + y(x)). \)

The next step is another observation in [22] that it follows from Birkhoff individual ergodic theorem [12] that, for any \( w \in L^1(\Pi^m), \) where \( \Pi^m := \mathbb{R}^m / \mathbb{Z}^m, \) for almost all \( z \in \Pi^m, \) we have
\[ (2.19) \quad \int_{\mathbb{R}^m} w(z + y(x)) \, dx = \int_\Pi w(y) \, dy. \]

Moreover, if \( w \in C(\Pi^m), \) then (2.19) holds for all \( z \in \Pi^m \) and \( w^z(x) := w(z + y(x)) \)
is a (Bohr) almost periodic function for each \( z \in \Pi^m. \)

The next main assertion corresponds to the first part of theorem 2.2 of [22], that is, it does not include the part about the decay of the entropy solution, and again its proof follows exactly as the one of the referred theorem and we refer to [22] for the proof. Also, in the present case we can no longer assert the continuity of the solution in \( t \) taking values in \( \text{BAP}(\mathbb{R}^d), \) which is essentially based on the continuity of the periodic solution of the hyperbolic problem corresponding to (2.17), (2.18), which in general is not known for the degenerate parabolic-hyperbolic equation (2.17). As we will see in the next section such continuity holds in the special case of the degenerate parabolic-hyperbolic equation (1.14), under the non-degeneracy conditions (1.15) and (1.16). We leave the claim about the decay of the weak entropy solution to be addressed in a subsequent statement by itself.

\textit{Assertion #2.} Let \( u(t, x) \) be a weak entropy solution of (1.1), (1.2), assume that the initial function \( u_0(x) \) is a trigonometrical polynomial with \( G_0 = \text{Gr}(u_0). \) Then
\[ u \in L^\infty([0, \infty), \text{BAP}(\mathbb{R}^d)) \bigcap L^\infty(\mathbb{R}^{d+1}_+) \]
and \( \text{Sp}(u(t, \cdot)) \subset G_0 \) for a.e. \( t > 0. \)

We just observe that Assertion #2 is proved (cf. [22]) by using Assertion #1 and showing, for a suitable sequence \( z_l \) converging to 0 as \( l \to \infty, \) belonging to the set of full measure of \( z \in \mathbb{R}^m \) given by Assertion #1, for each fixed \( t \) in a set of full measure in \( \mathbb{R}^+, \) the convergence of the entropy solutions \( u^{z_l}(t, \cdot) = v(t, z_l + y(x)) \)
in \( \text{BAP}(\mathbb{R}^d), \) as \( z_l \to 0, \) uniformly with respect to \( t, \) and using that for each \( z_l \)
\[ u^{z_l} \in L^\infty([0, \infty), \text{BAP}(\mathbb{R}^d)) \bigcap L^\infty(\mathbb{R}^{d+1}_+) \]
and \( \text{Sp}(u^{z_l}(t, \cdot)) \subset G_0 \) for a.e. \( t > 0. \)

Now, let us consider the general case where \( u_0 \in \text{BAP}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \) Let \( u(t, x) \) be the entropy solution of (1.1), (1.2) obtained above. Following [22], let \( \text{Gr}(u_0) \)
be the minimal additive subgroup of $\mathbb{R}^d$ containing $\text{Sp}(u_0)$. We then consider a sequence $u_{0l}$ of trigonometrical polynomials such that $u_{0l} \to u_0$ as $l \to \infty$, in $\text{BAP}(\mathbb{R}^d)$ and $\text{Sp}(u_{0l}) \subset \text{Gr}(u_0)$, which may be obtained from the Bochner-Fejér trigonometrical polynomials (see [2], p.105). We denote by $u_t(t, x)$ the weak entropy solution of (1.1), (1.2) with initial function $u_{0l}(x)$. By Proposition 2.1, there exists a set $F \subset \mathbb{R}_+$ of full measure such that, for all $t \in F$ and for every $l \in \mathbb{N}$, we have

$$N_l(u(t, \cdot) - u_{0l}(t, \cdot)) \leq N_l(u_{0l} - u_0) \to 0, \quad \text{as } l \to \infty.$$  

Since $u_{0l}$ has finite spectrum, by Assertion #2 we see that $u_t(t, x) \in L^{\infty}([0, \infty), \text{BAP}(\mathbb{R}^d))$ and $\text{Sp}(u_{0l}(t, \cdot)) \subset \text{Gr}(u_0)$, for all $t \in F$, for all $l \in \mathbb{N}$. Therefore, $u \in L^{\infty}([0, \infty), \text{BAP}(\mathbb{R}^d))$. Moreover, we easily see that $\text{Sp}(u(t, \cdot)) \subset \text{Gr}(u_0)$, for a.e. $t > 0$.

$$\square$$

3. PROOF OF THEOREM 1.2

In this section we prove the first part of Theorem 1.2, namely (1.18), (1.19) and (1.20). This amounts to proving the strong trace property for the weak entropy solution of (1.14), (1.2) at all hyperplane $t = t_0$, for all $t_0 \geq 0$. Indeed, by the Gauss-Green Theorem (see, e.g., [6], [7]), applied to the (divergence-free) $L^2$-divergence-measure field $(u, f(u) - \nabla_x b(u))$, we easily deduce that the limits $\lim_{t \to t_0^\pm} u(t, x)$ exist in the weak star topology of $L^\infty(\mathbb{R}^d)$, for $t_0 > 0$, and just the limit for $t_0^+$ when $t_0 = 0$. By the same result, for $t_0 > 0$, using the fact that the referred field is divergence-free, we easily deduce that the limits for $t_0^+$ and $t_0^-$ must coincide. We also refer to Theorem 4.5.1 of [11] whose proof establishes the continuity of $u(t, \cdot)$ from $(0, \infty)$ into $L^1_{\text{loc}}(\mathbb{R}^d)$ except for a countable set of $t \in (0, \infty)$. As observed in [11], the continuity at $t_0$ would follow if the entropy inequality included the initial time, which is not the case here, where we consider the weak initial prescription (1.17).

We rewrite (1.7) for the present case. For any convex $C^2$ function $\eta : \mathbb{R} \to \mathbb{R}$, and $q_i(u) = \eta'(u)f_i(u)$, $r_{ij}(u) = \eta'(u)b_{ij}(u)$, $i,j = d'+1, \ldots, d$, we have

$$\partial_t \eta(u) + \nabla_x q_i(u) - \sum_{i,j=d'+1}^d \partial_{x,x}^2 r_{ij}(u) - \sum_{k=d'+1}^d \left( \sum_{i=1}^d \partial_{x,i} \beta_{ik}(u) \right)^2,$$

in the sense of distributions in $(0, \infty) \times \mathbb{R}^d$, where $\beta_{ik}(u) = \int^u \sigma_{ik}(v) dv$ and $\Sigma(u) = (\sigma_{ij}(u))_{i,j=1}^d$ satisfies $B(u) = \Sigma(u)^2$.

We will use the kinetic formulation for (1.1) (cf. [9]). So, we introduce the kinetic function $\chi$ on $\mathbb{R}^2$:

$$\chi(\xi; u) = \begin{cases} 
1 & \text{for } 0 < \xi < u, \\
-1 & \text{for } u < \xi < 0, \\
0 & \text{otherwise.}
\end{cases}$$

The following representation holds for any $S \in C^1(\mathbb{R})$,

$$S(u) = \int_{\mathbb{R}} S'(\xi) \chi(\xi; u) d\xi.$$
which yields the following kinetic equation equivalent to (3.1):

\begin{equation}
\partial_t \chi(\xi; u) + a(\xi) \cdot \nabla_x \chi(\xi; u) - \sum_{i,j=d'+1}^{d} b_{ij}(\xi) \partial_{x_i,x_j}^2 \chi(\xi; u) = \partial_t (m + n)(t, x, \xi)
\end{equation}

in the sense of distributions in \((0, \infty) \times \mathbb{R}^{d+1}\).

In (3.3), \(m(t, x, \xi), n(t, x, \xi)\) are non-negative measures satisfying, for all \(R, T > 0\),

\begin{equation}
\int_{C_{R,T}} (m + n)(t, x, \xi) \, dx \, dt \leq \mu_{R,T}(\xi) \in L_0^\infty(\mathbb{R}),
\end{equation}

where \(C_{R,T} = (0, T) \times C_R\), and by \(L_0^\infty\) we mean \(L^\infty\) with compact support, and

\begin{equation}
n(t, x, \xi) = \delta(\xi - u(t, x)) \sum_{k=d'+1}^{d} \left( \sum_{i=d'+1}^{d} \partial_{x_i} \sigma_{ik}(u(t, x)) \right)^2.
\end{equation}

Also, taking \(\eta(u) = \frac{1}{2} u^2\) in (1.7), we see that

\begin{equation}
\int_{\mathbb{R}} \int_{C_{R,T}} (m + n)(t, x, \xi) \, dx \, dt \, d\xi \leq C(R, T),
\end{equation}

for all \(R, T > 0\), for some constant \(C(R, T) > 0\) depending only on \(R, T\) and \(\|u_0\|_\infty\).

Equation (3.1) implies that for any convex entropy \(\eta\), the vector field \(F = (\eta(u), q(u) - (\sum_{i=1}^{d} \partial_x r_{ij}(u)))_{j=1}^{d'} \in D\mathcal{M}^2(C_{R,T})\), where

\[
\hat{r}_{ij}(u) = \begin{cases} 
0, & \text{for } 1 \leq i \leq d' \text{ or } 1 \leq j \leq d', \\
\tau_{ij}(u), & \text{for } d'+1 \leq i, j \leq d
\end{cases}
\]

and for any \(R > 0\), \(T > 0\), that is, it is an \(L^2\) divergence-measure field on \(C_{R,T}\).

By theorems 3.1 and 3.2 in [15], or essentially also from lemma 1.3.3 in [11], the normal trace of the \(D\mathcal{M}^2\)-field \(F\) at the hyperplane \(t = t_* \in (0, T)\), from above, that is, as a part of the boundary of \(C_{R,T} \cap \{ t > t_* \}\), as well as from below, that is, as part of the boundary of \(C_{R,T} \cap \{ t < t_* \}\), is simply given by

\[
\langle F \cdot \nu, \phi \rangle_{t=t_* \pm} = \int_{\mathbb{R}^d} \eta(u(t_*, x)) \phi(x) \, dx,
\]

for a.e. \(t_* > 0\), for any \(\phi \in C^1_c(\mathbb{R}^d)\), where \(\langle F \cdot \nu, \cdot \rangle_{t=t_* \pm}\) denotes the normal trace at \(\{ t = t_* \}\) from above and \(\langle F \cdot \nu, \cdot \rangle_{t=t_* \mp}\) the one from below. Also, from theorem 3.2 in [15] or also essentially from lemma 1.3.3 in [11], we deduce that, for any \(t_0 > 0\),

\begin{equation}
\langle F \cdot \nu, \phi \rangle_{t=t_0 \pm} = \text{ess lim}_{t \to t_0 \pm} \int_{\mathbb{R}^d} \eta(u(t, x)) \phi(x) \, dx,
\end{equation}

for any \(\phi \in C^1_c(\mathbb{R}^d)\), and for \(t_0 = 0\) we have, similarly,

\begin{equation}
\langle F \cdot \nu, \phi \rangle_{t=0 \pm} = \text{ess lim}_{t \to 0 \pm} \int_{\mathbb{R}^d} \eta(u(t, x)) \phi(x) \, dx.
\end{equation}

Now, using (3.1) and the representation (3.2) for an arbitrary convex \(\eta\), we deduce that, for \(f(t, x, \xi) = \chi(\xi; u(t, x))\), there exists the limit

\begin{equation}
\lim_{t \to t_0 \pm} f(t, \cdot, \cdot) = f^*(\cdot, \cdot),
\end{equation}

where \(f^*(\cdot, \cdot)\) is the convex envelope of \(f(t, \cdot, \cdot)\).
in the weak star topology of \( L^\infty(C_R \times (-L, L)) \), for any \( R > 0 \), and any \( L > 0 \) satisfying \( \|u\|_{L^\infty(\mathbb{R}^d \times (-L, L))} \leq L \). Similarly, we have

\[
\lim_{t \to t_0^-} f(t, \cdot, \cdot) = f^\tau(\cdot, \cdot),
\]

in the weak star topology of \( L^\infty(C_R \times (-L, L)) \). We observe that for \( \eta(u) = u \), for all \( t_0 > 0 \),

\[
\text{ess} \lim_{t \to t_0} \int_{\mathbb{R}^d} u(t, x)\phi(x) \, dx = \text{ess} \lim_{t \to t_0} \int_{\mathbb{R}^d} u(t, x)\phi(x) \, dx,
\]

for all \( \phi \in C^1_c(\mathbb{R}^d) \), as a consequence of (3.7), (3.8) and the Gauss-Green formula

\[
\text{ess} lim_{t \to 0} \int_{\mathbb{R}^d} u(t, x)\phi(x) \, dx = \lim_{t \to 0} \int_{\mathbb{R}^d} u(t, x)\phi(x) \, dx,
\]

as in [6, 7] (essentially also from lemma 1.3.3 in [11]). Therefore, if the existence of strong trace of \( u(t, x) \) at \( t = t_0 \) can be proved, both from above and below, these strong traces must coincide. Since the proof of the strong trace property from below is totally analogous to that for the strong trace from above, it will suffice to investigate the latter.

Following the method in [26], in order to prove that the limits in (3.9) and (3.10) can be taken as the strong convergence in \( L^1(C_{R,T} \times (-L, L)) \), it suffices to prove that \( f^\tau(\cdot, \cdot) \) is a \( \chi \)-function, which is proved by using localization method introduced in [26]. For simplicity we just consider the case \( t_0 = 0 \).

We write for \( x \in \mathbb{R}^d \), \( x = (x', x'') \), where \( x' \in \mathbb{R}^{d'}, \ x'' \in \mathbb{R}^{d''} \). Fixing, \( x_0 \in \mathbb{R}^d \), we consider the sequence

\[
f_\varepsilon (\underline{x}, \underline{x}, \xi) := f(\varepsilon x_0 + \Lambda(\varepsilon)\underline{x}, \xi),
\]

where \( \Lambda(\varepsilon)\underline{x} := (\varepsilon x'_0, \varepsilon^{1/2} x'') \). So, \( f_\varepsilon \) satisfies

\[
\partial_t f_\varepsilon + a(\xi)' \cdot \nabla_x' f_\varepsilon + \varepsilon^{1/2} a(\xi)'' \cdot \nabla_x'' f_\varepsilon - \sum_{i,j = d'+1}^d b_{ij}(\xi) \partial^2_{\xi_i\xi_j} f_\varepsilon = \partial_t (m_\varepsilon + n_\varepsilon),
\]

where \( a(\xi)' = (\pi^{d'}(a(\xi)), 0, \ldots, 0) \), \( a(\xi)'' = a(\xi) - a(\xi)' \), and \( m_\varepsilon \in \mathcal{M}_{1\text{loc}}^+(0, \infty) \times \mathbb{R}^{d} \times \mathbb{R} \) is defined, for every \( 0 \leq R^0_i < R^1_i < R^2_i, \ i = 1, \ldots, d, \ L_1 < L_2 \), by

\[
(m_\varepsilon + n_\varepsilon) \left( \prod_{0 \leq i \leq d} [R^1_i, R^2_i] \times [L_1, L_2] \right)
\]

\[
= \frac{1}{\varepsilon^{d+2d''}} (m + n) \left( \varepsilon R^0_1, \varepsilon R^0_2 \right) \times (x_0 + \Lambda(\varepsilon) \prod_{1 \leq i \leq d} [R^1_i, R^2_i]) \times [L_1, L_2],
\]

where \( \Lambda(\varepsilon): \mathbb{R}^d \to \mathbb{R}^d \) is defined by \( \Lambda(\varepsilon)\underline{x} := (\varepsilon x'_0, \varepsilon^{1/2} x'') \).

Following [26], as in [16], we have there exists a sequence \( \varepsilon_n \) converging to 0 and a set \( \mathcal{E} \subset \mathbb{R}^d \), with \( L^d(\mathbb{R}^d \setminus \mathcal{E}) = 0 \), such that for all \( x_0 \in \mathcal{E} \)

\[
\lim_{\varepsilon \to 0} (m_\varepsilon + n_\varepsilon) = 0,
\]

in the weak topology of \( \mathcal{M}_{1\text{loc}}^+(0, \infty) \times \mathbb{R}^d \times \mathbb{R} \).

We now observe that

\[
f_\varepsilon (0, \underline{x}, \xi) = f^\tau (x_0 + \Lambda(\varepsilon)\underline{x}, \xi).
\]
Again following [26], as in [16], we have that there exists a subsequence still denoted \( \varepsilon_n \) and a subset \( \mathcal{E}' \) of \( \mathbb{R}^d \) such that for every \( x_0 \in \mathcal{E}' \) and for every \( R > 0 \),

\[
\lim_{\varepsilon_n \to 0} \int_{-L}^{L} \int_{(-R,R)^d} \left| f^*(x_0, \xi) - f^*(x_0 + \Lambda(\varepsilon_n)x, \xi) \right| \, dx \, d\xi = 0. \tag{3.15}
\]

Now, we claim that there exists a sequence \( \varepsilon_n \) which goes to 0 and a \( \chi \)-function \( f_\infty \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d \times (-L, L)) \) such that \( f_{\varepsilon_n} \) converges strongly to \( f_\infty \) in \( L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d \times (-L, L)) \) and

\[
\partial_y f_\infty + a(\xi) \cdot \nabla_y f_\infty - \sum_{i,j=1}^{d+1} b_{ij}(\xi) \partial_{y,\xi}^2 f_\infty = 0. \tag{3.16}
\]

The proof of the claim is very similar to that of proposition 3 of [26], and lemma 3.1 in [16], and relies on a particular case of the version of averaging lemma in [24] (see also [25]). Here, we need the following variation of the standard averaging lemma.

**Lemma 3.1.** Let \( N, N', N'' \) be positive integers with \( N = N' + N'' \), \( f_n(y, \xi) \) be a bounded sequence in \( L^2(\mathbb{R}^N \times \mathbb{R}) \cap L^1(\mathbb{R}^N \times \mathbb{R}) \), \( g_n^i \in L^2(\mathbb{R}^N \times \mathbb{R}) \) be such that \( g_n^i \to g^i \) strongly in \( L^2(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^{N+1}) \), \( i = 1, 2 \), and for \( y \in \mathbb{R}^N \) write \( y = (y', y'') \), \( y' \in \mathbb{R}^{N'} \), \( y'' \in \mathbb{R}^{N''} \). Assume

\[
\alpha(\cdot)' \cdot \nabla_{y'} f_n + \alpha(\cdot)'' \cdot \nabla_{y''} f_n - \sum_{i,j=1}^{d+1} \beta_{ij}(\xi) \partial^2_{y_i y_j} f_n = \partial_\xi \nabla_{y',\xi} g^1_n + \nabla_{y'',\xi} g^2_n,
\]

where \( \alpha(\cdot)' \in C^2(\mathbb{R}; \mathbb{R}^{N'}) \), \( \alpha(\cdot)'' \in C^2(\mathbb{R}; \mathbb{R}^{N''}) \), \( \beta \in C^2(\mathbb{R}) \) satisfy

\[
\mathcal{L}^1 \{ \xi \in \mathbb{R} : \alpha(\xi) \cdot \xi' = 0 \} = 0, \quad \text{for every } \xi' \in \mathbb{R}^{N'}, \quad \text{with } |\xi'| = 1,
\]

where \( \mathcal{L}^1 \) is the Lebesgue measure on \( \mathbb{R} \), and also

\[
\mathcal{L}^1 \{ \xi \in \mathbb{R} : \beta(\xi) = 0 \} = 0.
\]

Then, for any \( \phi \in C_c^\infty(\mathbb{R}) \), the average \( \bar{u}_n(\phi) = \int_\mathbb{R} \phi(\xi) f_n(y, \xi) \, d\xi \) is relatively compact in \( L^2(\mathbb{R}^N) \).

The application of Lemma 3.1 to the problem at hand is made, as in [26], by multiplying (3.12) by \( \phi_1(\xi - \xi_0) \phi_2(\xi) \) where \( \phi_1 \in C_0^\infty((1/2R), 2R) \times (1, 2R)^d \), \( \phi_2 \in C_0^\infty((-2L, 2L), \mathbb{R}) \), both taking values in \([0,1]\), with \( \phi_1(\xi - \xi_0) = 1 \), for \( \xi_0 \in (1/R, R) \times (-2L, 2L) \), \( \phi_2(\xi) = 1 \), for \( \xi \in (-L, L) \). We then consider the equation obtained for \( \phi_1 \partial_\xi f_\varepsilon \), which is easily seen to satisfy the hypotheses of Lemma 2.1, we refer to [26] for the details.

The final step of the proof is to proof that for every \( x_0 \in \mathcal{E}' \),

\[
f_\infty(0, x, \xi) = f^*(x_0, \xi), \tag{3.20}
\]

for a.e. \( (x, \xi) \in \mathbb{R}^d \times (-L, L) \), which the result corresponding to proposition 4 of [26]. The proof is the same as the one of the referred proposition, and consists in proving that, for any \( \phi \in C_0^\infty(\mathbb{R}^d \times (-L, L)) \), the sequence

\[
h_n(t) := \int_{-L}^{L} \int_{\mathbb{R}^d} (f_\varepsilon(t, x, \xi) - f_\infty(t, x, \xi)) \phi(x, \xi) \, dx \, d\xi,
\]

converges to 0 in \( BV((0, 1)) \), which is done exactly as in [26].
Finally, from (3.16) and (3.20), we easily conclude that
\[ f_\infty(t, x, \xi) = f^*(x_0, \xi), \]
for almost all \((t, x, \xi) \in \mathbb{R}^{d+1} \times (-L, L)\), which is constant with respect to \((t, x)\). Hence, since \(f_\infty\) is a \(\chi\)-function for almost all \((t, x)\), we conclude that \(f^*(x_0, \cdot)\) is a \(\chi\)-function, as was to be proved. The proof of the strong trace property at any hyperplane \(\{t = t_0\}\), \(t_0 > 0\), both from above and from below, follows exactly as just done for \(t_0 = 0\), from above. This establishes the strong trace property at the initial time and the continuity in time with values in \(L^1_{\text{loc}}(\mathbb{R}^d)\).

Finally, since we have already proved the strong assumption of the initial data, it follows that the weak entropy solution of (1.14), (1.2) is actually an entropy solution in the sense of Definition 1.1. In particular, (1.20) and (1.21) follow from Theorem 1.1.

4. Proof of Theorem 1.3

In this section we prove the decay property for the (weak) entropy solution of (1.14), (1.2). The decay property follows using ideas in [14]. We recall that the space of Stepanoff almost periodic functions (with exponent \(p = 1\)) in \(\mathbb{R}^d\), \(\text{SAP}(\mathbb{R}^d)\), is defined as the completion of the trigonometric polynomials with respect to the norm
\[ \|f\|_S := \sup_{x \in \mathbb{R}^d} \int_{C_{x}(x)} |f(y)| \, dy = \sup_{x \in \mathbb{R}^d} \int_{C_1} |f(y + x)| \, dy, \]
where
\[ C_R(x) := \{y \in \mathbb{R}^d : |y - x|_\infty := \max_{i=1,\ldots,d} |y_i - x_i| \leq R/2\}. \]

Another characterization of the Stepanoff almost periodic function (S-a.p., for short) is obtained introducing the concept of \(\varepsilon\)-period of a function \(f\), that is a number \(\tau\) satisfying
\[ (4.1) \quad \|f(\cdot + \tau) - f(\cdot)\|_S \leq \varepsilon. \]

Let \(E_S\{\varepsilon, f\}\) denote the set of such numbers. If the set \(E_S\{\varepsilon, f\}\) is relatively dense for all positive values of \(\varepsilon\), then the function \(f\) is S-a.p. (see, e.g., [2]). By the set \(E_S\{\varepsilon, f\}\) being relatively dense it is meant that there exists a length \(l_\varepsilon\), called \(\varepsilon\)-inclusion interval, such that for any \(x \in \mathbb{R}^d\), \(C_{l_\varepsilon}(x)\) contains an element of \(E_S\{\varepsilon, f\}\). Now, as a consequence of the fact that \(\text{Gr}(u(t, x)) \subset \text{Gr}(u_0)\), we have the following lemma which is of interest in its own.

**Lemma 4.1.** If \(u_0\) is a trigonometric polynomial, then the entropy solution of (1.14), (1.2), \(u(t, x)\), is S-a.p. for all \(t > 0\), and, for any \(\varepsilon > 0\), \(u(t, x)\) possesses an \(\varepsilon\)-inclusion interval, \(l_\varepsilon(t)\), satisfying \(l_\varepsilon(t) = l_\varepsilon^{(c, t)}(0)\), where \(l_\varepsilon^{(c, t)}(0)\) is an \(\varepsilon'\)-inclusion interval of \(u_0(x)\), and \(\varepsilon'(\varepsilon, t) = \varepsilon|\log \varepsilon|^{-1}e^{-Ct}\), for certain \(C > 0\).

**Proof:** Clearly, \(u_0\), being a trigonometric polynomial, is S-a.p. The fact that \(u(t, x)\) is S-a.p. for all \(t > 0\) follows from (2.9), with \(v(t, x) = u(t, x + \tau)\) and...
\( \rho(x - x_0) \) instead of \( \rho(x) \), from which we deduce

\[
\int_{C_1(x_0)} \int_{C_R(x_0)} |u(t, x + \tau) - u(t, x)| \, dx \leq c(t) \int_{C_1(x)} |u_0(x) - u_0(x + \tau)| \rho(x - x_0) \, dx
\]

\[
+ c(t)O\left(\frac{1}{R}\right) \leq c(R, t) \sup_{x \in \mathbb{R}^d} \int_{C_1(x)} |u_0(y + \tau) - u_0(y)| \, dy + c(t)O\left(\frac{1}{R}\right),
\]

where \( c(t) = \tilde{c} t, \tilde{c} > 0 \) only depending on \( r, c(R, t) \) is a positive constant depending only on \( R, t, \) and \( O\left(\frac{1}{R}\right) \) goes to zero when \( R \to \infty \) uniformly with respect to \( x_0 \). So, choosing \( R \) large enough so that \( c(t)O(1/R) \lesssim \varepsilon/2 \) and then taking any \( \tau \in E_S\{\varepsilon/(2c(R, t)), u_0\} \), we get that \( \tau \in E_S\{\varepsilon, u(t, \cdot)\} \), and so \( u(t, \cdot) \) is S-a.p. A technical computation on the terms on the right-hand side of (4.2), using \( p \) to estimate \( R \) as a function of \( \varepsilon/(2c(t)) \), and then getting an expression for \( c(R, t) \), gives the estimate \( l_\varepsilon(t) = l_{\varepsilon}(\varepsilon, t)(0) \), with \( \varepsilon' = \varepsilon/| \varepsilon | \log e^{-C t} \), for certain \( C > 0 \), as desired.

Now we can use Lemma 4.1 to prove the decay property (1.22). Clearly, from Proposition 2.1, it suffices to consider the case where the initial function \( u_0 \) is an almost periodic function whose \( \varepsilon \)-inclusion intervals \( l_\varepsilon \) satisfy \( l_\varepsilon / \log e \varepsilon \to 0 \), as \( \varepsilon \to 0 \). From Lemma 4.1 we see that the \( \varepsilon \)-inclusion interval of the solution \( u(t, x) \) satisfies \( l_\varepsilon(t)/t^{1/2} \to 0 \). Let us then consider the scaling sequence \( u^T(t, x) := u(Tt, T^d x', \sqrt{T} x'') \), and define \( \varepsilon' = x'/t, \varepsilon'' = x''/\sqrt{T} \). So, \( u^T \) is a uniformly bounded sequence of weak entropy solutions of (1.1), (1.2), with initial functions \( u^T_0(x) := u_0(Tx', \sqrt{T} x'') \). Using the Averaging Lemma 3.1, we deduce that \( u^T \) is relatively compact in \( L^1_{\text{loc}}(\mathbb{R}^{d+1}_{+}) \) and the initial functions clearly weakly converge to \( \tilde{u}_0 = M(u_0) \). By passing to a subsequence, which we still denote by \( u^T(t, x) \), we have that \( u^T \to \tilde{u} \) as \( T \to \infty \), in \( L^1_{\text{loc}}(\mathbb{R}^{d+1}) \), for some \( \tilde{u} \in L^\infty(\mathbb{R}^{d+1}) \). We see also that \( \tilde{u} \) satisfies (1.5), (1.6), (1.7) and (1.17), all of which are easy to be verified, and we observe by (1.17) that \( \tilde{u}(0, x) = M(u_0) \). Now, in view of Theorem 1.2, by the uniqueness Lemma 2.2, we conclude that \( \tilde{u}(t, x) = M(u_0) \), that is \( u^T \to M(u_0) \), in \( L^1_{\text{loc}}(\mathbb{R}^{d+1}) \). This, in particular, implies

\[
0 = \lim_{T \to \infty} \frac{1}{T} \int_0^1 \int_{|x'| \leq c', |x''| \leq c''} |u(Tt, T^d x', \sqrt{T} x'') - M(u_0)| \, dx' \, dx'' \, dt
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1}{T+x' \sqrt{T} x''} \int_{|x'| \leq c', |x''| \leq c''} |u(t, x', x'') - M(u_0)| \, dx' \, dx'' \, dt
\]

\[
\geq \frac{1}{2d^2} \lim_{T \to \infty} \frac{1}{T} \int_{T/2}^T \int_{|\xi'| \leq c', |\xi''| \leq c''} |u(t, \xi', \xi'' \sqrt{T}) - M(u_0)| \, d\xi' \, d\xi'' \, dt,
\]

which implies

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{|\xi'| \leq c', |\xi''| \leq c''} |u(t, \xi', \xi'' \sqrt{T}) - M(u_0)| \, d\xi' \, d\xi'' \, dt = 0,
\]

as is easily seen. Now, invoking Lemma 4.1, we can then make a computation similar to that in p.51 of [14] in order to get that, there are constants \( c_1, c_2 > 0 \).
depending only on the dimension, such that, given any \( \varepsilon > 0 \),

\[
\int_{|\xi'| \leq c', |\xi''| \leq c''} |u(t, \xi', \xi'' \sqrt{t}) - M(u_0)| \, d\xi' \, d\xi'' \geq c_1 M(|u(t, \cdot) - M(u_0)|) - c_2 \varepsilon.
\]

Therefore, by (4.3), we deduce

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T M(|u(t, \cdot) - M(u_0)|) \, dt = 0.
\]

Now, by Proposition 2.1, we conclude

\[
\lim_{t \to \infty} M(|u(t, \cdot) - M(u_0)|) \, dt = 0,
\]

which is the desired result.

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