Integrability of Invariant Geodesic Flows on
$n$-Symmetric Spaces

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Abstract
In this paper, by modifying the argument shift method, we prove Liouville integrability of geodesic flows of normal metrics (invariant Einstein metrics) on the Ledger-Obata $n$-symmetric spaces $K^n/\text{diag}(K)$, where $K$ is a semisimple (respectively, simple) compact Lie group.

Keywords: noncommutative and commutative integrability, invariant polynomials, translation of argument, homogeneous spaces, Einstein metrics

MSC: 70H06, 37J35, 53D25

1 Introduction

Invariant Geodesic Flows. We study integrability of $G$-invariant geodesic flows on a class of homogeneous spaces

$$Q = G/H, \quad G = \underbrace{K \times \cdots \times K}_n, \quad H = \text{diag}(K) = \{(g, \ldots, g) \mid g \in K\} \quad (1)$$

where $K$ is a compact connected semisimple Lie group. The homogeneous space $Q$ is diffeomorphic to the direct product $K^{n-1}$, however as a $G$-homogeneous space it is a basic example of a $n$-symmetric Riemannian space, see Ledger and Obata [9].

Let $\mathfrak{g} = \mathfrak{t}^n = \mathfrak{t}_1 \oplus \mathfrak{t}_2 \oplus \cdots \oplus \mathfrak{t}_n$, $\mathfrak{h} = \{(x, \ldots, x) \mid x \in \mathfrak{t}\}$, $\mathfrak{t}$ be the Lie algebras of $G$, $H$ and $K$, respectively ($\mathfrak{t}_i \cong \mathfrak{t}$ is the $i$-th factor). For simplicity, both negative Killing forms on $\mathfrak{g}$ and $\mathfrak{t}$ will be denoted by $\langle \cdot, \cdot \rangle$. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}, \quad \mathfrak{v} = \{x = (x_1, \ldots, x_n) \in \mathfrak{t}^n \mid x_1 + \cdots + x_n = 0\} \quad (2)$$

be the orthogonal decomposition with respect to the Killing form.

The linear subspace $\mathfrak{v}$ can be naturally identified with $T_{\rho(e)}Q$, where $\rho : G \to Q = G/H$ is the canonical projection. Then $G$-invariant metrics on $Q$, via
restrictions to $T_p(c)Q \cong \mathfrak{v}$, are in one-to-one correspondence with $\text{Ad}_H$-invariant scalar products (e.g., see [1])

$$ (\cdot, \cdot)_v = (I(\cdot), \cdot), \quad I : \mathfrak{v} \to \mathfrak{v}, \quad I \circ \text{Ad}_h = I \circ \text{Ad}_h, \quad h \in H. \quad (3) $$

The negative Killing form itself defines normal (or standard) metric $ds^2$. Note that, in the case when $K$ is a simple group, the normal $G$-invariant metric on $Q$ is Einstein (see Wang and Ziller [22]). Besides, Nikonorov proved that, up to the isometry and homothety, the homogeneous space $Q$ for $n = 3$ ($n \geq 4$) admits exactly (respectively, at least) two $G$-invariant Einstein metrics [17].

**General Setting.** Let $\mathcal{F}$ be a collection of functions closed under the Poisson bracket on a Poisson manifold $(M, \{\cdot, \cdot\})$ and let $\Lambda$ be the Poisson bivector related to $\{\cdot, \cdot\}$. Consider the linear space $F_x \subset T_x^*M$ spanned by differentials of functions in $\mathcal{F}$. Suppose that the numbers $\text{dim } F_x$ and $\text{dim ker } \Lambda|_{F_x}$ are constant almost everywhere on $M$ and denote them by $\text{ddim } \mathcal{F}$ and $\text{dind } \mathcal{F}$, respectively (differential dimension and differential index of $\mathcal{F}$). The set $\mathcal{F}$ is called complete if: $\text{ddim } \mathcal{F} + \text{dind } \mathcal{F} = \text{dim } M + \text{corank } \{\cdot, \cdot\}$. It is complete at $x \in M$ if $\text{dim } F_x + \text{dim ker } \Lambda|_{F_x} = \text{dim } M + \text{ker } \Lambda$, i.e., $F_x$ is isotropic: $F^0_x \subset F_x, \quad F^0_x = \{\xi \in T_x^*M \mid \Lambda_x(\xi, F_x) = 0\}$.

The Hamiltonian system $\dot{f} = \{f, h\}$ is completely integrable in the non-commutative sense if it possesses a complete set of first integrals $\mathcal{F}$. Then (under compactness condition) $M$ is almost everywhere foliated by $(\text{dind } \mathcal{F} - \text{corank } \{\cdot, \cdot\})$-dimensional invariant tori. As in the Liouville theorem, the Hamiltonian flow restricted to regular invariant tori is quasi-periodic (see [16, 13]). Mishchenko and Fomenko stated the conjecture that non-commutative integrable systems are integrable in the usual commutative sense by means of integrals that belong to the same functional class as the original non-commutative integrals [13]. In the analytic case, when $\mathcal{F}$ is a finite-dimensional Lie algebra, the conjecture has been proved by Sadetov [20]. The conjecture is also proved in $C^\infty$-smooth case for infinite-dimensional algebras (see [4]).

Now, let $Q = G/H$ be a homogeneous space of a compact Lie group $G$, $\Phi : T^*Q \to \mathfrak{g}^*$ be the momentum mapping of the natural $G$-action on $T^*Q$, $\mathcal{F}_1 = \Phi^*(\mathbb{R}[\mathfrak{g}])$ be the set of Noether’s functions and $\mathcal{F}_2$ be the set of $G$-invariant functions, polynomial in momenta. Both $\mathcal{F}_1$ and $\mathcal{F}_2$ are Lie subalgebras of $(C^\infty(T^*Q), \{\cdot, \cdot\})$, where $\{\cdot, \cdot\}$ is the canonical Poisson bracket. From Noether’s theorem we have $\{\mathcal{F}_1, \mathcal{F}_2\} = 0$. Also $\mathcal{F}_1 + \mathcal{F}_2$ is a complete set of functions on $T^*Q$ (see Bolsinov and Jovanović [3, 5]).

The Hamiltonian function $H_0$ of the normal metric $ds^2$ is a Casimir function within $\mathcal{F}_2$, so it Poisson commute both with $\mathcal{F}_1$ and $\mathcal{F}_2$. Thus the geodesic flow of the normal metric is completely integrable in the non-commutative sense by means of analytic functions, polynomial in momenta.

**Integrable Pairs.** Within the class of Noether’s integrals $\mathcal{F}_1$, for example by using the argument translation method [12], one can always construct a complete commutative subset of function $\mathcal{F}^0 \subset \mathcal{F}_1 \quad (\text{ddim } \mathcal{F}^0 = \frac{1}{2}(\text{ddim } \mathcal{F}_1 + \text{dind } \mathcal{F}_1))$. 


Thus, for the case of the geodesic flow of the normal metric, the Mishchenko-Fomenko conjecture reduces to the construction of a complete commutative subset $F \subset F_2$:
\[
\text{ddim } F = \frac{1}{2} (\text{ddim } F_2 + \text{dind } F_2) .
\]
Indeed, from the completeness of $F_1 + F_2$ it follows that $F^0 + F$ is a complete commutative set on $T^*Q$ (see [2, 3]).

If the required subset $F \subset F_2$ exist, we say that $(G, H)$ is an integrable pair. In [4] the conjecture is stated that all pairs $(G, H)$ are integrable. If $(G, H)$ is a spherical pair, in particular if $G/H$ is a symmetric space, the algebra $F_2$ is already commutative. In this case we need only Noether’s integrals $F_1$ to integrate the geodesic flow (see Mishchenko [11], Brailov [6] and Mikityuk [14]). There are several known classes of integrable pairs (see [3, 5, 15, 8]) but the general problem rest still unsolved. For a related problem on the integrability of geodesic flows on homogeneous spaces of noncompact Lie groups see, e.g., [7, 10].

**Results and Outline of the Paper.** Let $Q = G/H$ be the Ledger-Obata $n$-symmetric space (1). By using the flag of subalgebras
\[
g_1 = \mathfrak{k}_1 \subset g_2 = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \subset \cdots \subset g_n = g = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_n ,
\]
we modify the argument shift method to construct a complete set of polynomials on $g$ with respect to the usual Lie-Poisson bracket (Theorem 2, Section 2). It allows us to find a complete commutative subset of polynomials within $F_2$ (Theorem 3, Corollary 2, Section 2) implying:

**Theorem 1** The geodesic flow of the normal metric on the Ledger-Obata $n$-symmetric space (1) is Liouville integrable by means of analytic integrals, polynomial in momenta.

As a corollary, the complete commutative integrability of the geodesic flows of invariant Einstein metrics constructed by Nikonov [17] (Corollary 3, Section 3) is obtained.

## 2 Liouville Integrability of Geodesic Flows

**H-invariant Euler Equations.** Consider the left trivialization $T^*G \cong g \times G$, where the identification $g^* \cong g$ is given by $(\cdot, \cdot)$. Let $\hat{I} : g \to g$ be a positive definite operator which defines left-invariant metric $ds^2_I$ on $G$.

The left $G$-reduction of the geodesic flow of the metric $ds^2_I$ is described by the Euler equations on $g^* \cong g$:
\[
\dot{x}_i = [x_i, \xi], \quad \xi_i = \nabla x \hat{h}(x_1, \ldots, x_n) = \text{pr}_{\xi_i} \hat{A}(x_1, \ldots, x_n), \quad i = 1, \ldots, n ,
\]
where $\hat{h} = \frac{1}{2}(\hat{A}(x), x)$ is the Hamiltonian, $\hat{A} = \hat{I}^{-1}$ and $\text{pr}_{\xi_i}$ is the projection to $i$-th factor: $\text{pr}_{\xi_i}(x_1, \ldots, x_n) = x_i$.  

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The Euler equations are Hamiltonian with respect to the Lie-Poisson bracket (the product of the Lie-Poisson brackets on factors $t_i$):

$$\{f, g\}(x_1, \ldots, x_n) = -\sum_{i=1}^{n} (x_i, [\nabla_{x_i} f, \nabla_{x_i} g]).$$

(7)

The right $H$-action on $T^*G$ is Hamiltonian with momentum mapping, in the left-trivialization, given by

$$\mu(x) = x_1 + \cdots + x_n.$$  

(8)

The geodesic flow is invariant with respect to the right $H$-action if and only if the Hamiltonian $\hat{T}$ is $\text{Ad}_H$-invariant, i.e.,

$$\langle [x, h], \hat{T}(x) \rangle = 0 \iff \text{pr}_\mathfrak{h}[\hat{T}(x), x] = 0 \iff \sum_{i=1}^{n} [\text{pr}_{t_i}, \hat{T}(x), x_i] = 0,$$

where we used

$$\text{pr}_\mathfrak{h}(x_1, \ldots, x_n) = \frac{1}{n}(x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n).$$

(9)

If the Hamiltonian $\hat{h}$ is $\text{Ad}_H$-invariant, then the momentum $\mu$ is preserved by geodesic flow and we can perform the symplectic reduction of the flow to $\mu^{-1}(0)/H \cong T^*Q$. The reduced flow is the geodesic flow of a $G$-invariant submersion metric on $Q$.

Contrary, for a given $\text{Ad}_H$-invariant positive definite operator $I : \mathfrak{v} \to \mathfrak{v}$, let $ds^2$ be a $G$-invariant metric defined by [3]. It can be seen as a submersion metric of an appropriate left $G$-invariant and right $H$-invariant metric $ds^2_k$, simply by taking

$$\hat{I}(x) = s \cdot \text{pr}_\mathfrak{g}(x) + I \text{pr}_\mathfrak{h}(x), \quad x \in \mathfrak{g},$$

where $s > 0$.

**Algebra of $G$-invariant functions on $T^*Q$.** The algebra $\mathcal{F}_2$ of $G$-invariant functions on $T^*Q$, polynomial in momenta, can be identified with $\mathbb{R}[\mathfrak{v}]^H$ ($\text{Ad}_H$-invariant polynomials on $\mathfrak{v}$). Within this identification, the Hamiltonian of the metric $ds_j^2$ is given by $h(x) = \frac{1}{2}\langle Ax, x \rangle$, $A = I^{-1}$, while the Hamiltonian of the normal metric $ds_0^2$ is simply $h_0(x) = \frac{1}{2}\langle x, x \rangle$. Further, the canonical Poisson bracket on $T^*Q$ corresponds to the restriction of the Lie-Poisson bracket (7) to $\mathbb{R}[\mathfrak{v}]^H$ (see Thimm [21]):

$$\{f, g\}_\mathfrak{v}(x) = -\langle x, [\nabla_f(x), \nabla g(x)] \rangle, \quad f, g : \mathfrak{v} \to \mathbb{R}.$$  

(10)

Let $\mathfrak{g}_x$, $\mathfrak{h}_x$ and $\mathfrak{t}_x$, be isotropy algebras of $x$ and $x_i$ in $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{t}$.

Consider the space $\mathfrak{j}_x \subset \mathfrak{v}$ spanned by gradients of all polynomials in $\mathbb{R}[\mathfrak{v}]^H$. For a generic point $x \in \mathfrak{v}$ we have (see [3] [15]):

$$\mathfrak{j}_x = ([x, h]^{-1})^\perp \cap \mathfrak{v} = \{\eta \in \mathfrak{v} | \langle \eta, [x, h] \rangle = 0 \} = \{\eta \in \mathfrak{v} | [x, \eta] \subset \mathfrak{v}\}.$$

$$\mathfrak{j}_x = \{(\xi_1, \ldots, \xi_n) \in \mathfrak{g} | \sum_{i=1}^{n} \xi_i = 0, \quad \sum_{i=1}^{n} [x_i, \xi_i] = 0\}. $$


The Poisson bracket (10) on the algebra \( \mathbb{R}[v]^H \) corresponds to the restriction of the Lie-Poisson bivector
\[
\Lambda_x = \lambda_{x_1} \times \cdots \times \lambda_{x_n} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},
\]
\[
\lambda_{\xi} : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}, \quad \lambda_{\xi}(\eta_1, \eta_2) = -\langle \xi, [\eta_1, \eta_2] \rangle, \quad \xi, \eta_1, \eta_2 \in \mathfrak{k}
\]
to \( j_x \). Denote this restriction by \( \bar{\Lambda} \). Note that the kernel of \( \bar{\Lambda} \) is [5]:
\[
\ker \bar{\Lambda} = \text{pr}_v \ker \Lambda \supset \text{pr}_v \mathfrak{g} = \text{pr}_v(\mathfrak{k}_{x_1}, \ldots, \mathfrak{k}_{x_n}) \subset j_x.
\]
Here, for simplicity, the gradient operator with respect to the restriction of \( \langle \cdot, \cdot \rangle \) to \( v \) is also denoted by \( \nabla \).

Note that (10) is a Poisson bracket within \( \mathbb{R}[v]^H \), while it is an almost-Poisson bracket within the algebra of polynomials on \( v \) (it does not satisfies the Jacobi identity).

We have the following simple basic statement.

**Lemma 1** The differential dimension and index of \( \mathbb{R}[v]^H \) are given by:
\[
\text{ddim } \mathbb{R}[v]^H = (n - 2) \dim K, \quad \text{dind } \mathbb{R}[v]^H = n \dim K.
\]

**Proof.** The differential dimension of \( \mathbb{R}[v]^H \) is equal to the codimension of a generic orbit \( \text{Ad}_H(x) \) within \( v \), that is \( \text{ddim } \mathbb{R}[v]^H = \dim j_x = \dim v - \dim H + \dim \mathfrak{h}_x \), for a generic \( x \in v \). Since
\[
\dim \mathfrak{h}_x = \dim (\mathfrak{k}_{x_1} \cap \cdots \cap \mathfrak{k}_{x_n}) = 0,
\]
for a generic \( x \in v \), we obtain the first relation in (12). On the other side, from (13) we get
\[
\text{dind } \mathbb{R}[v]^H = \dim \ker \bar{\Lambda}_x = \dim \text{pr}_v \mathfrak{g}_x = \dim \mathfrak{g}_x - \dim \mathfrak{h}_x
\]
\[
= \dim \mathfrak{g}_x = \dim \mathfrak{k}_{x_1} + \cdots + \dim \mathfrak{k}_{x_n} = n \dim K,
\]
for a generic \( x \in v \). □

Let \( \mathbb{R}[\mathfrak{k}]^K \) be the algebra of \( \text{Ad}_K \)-invariant polynomials on \( \mathfrak{k} \). It is generated by \( r = \dim K \)-invariant homogeneous polynomials \( f_1, \ldots, f_r \). The algebra of \( \text{Ad}_G \)-invariant polynomials on \( \mathfrak{g} \) is then generated by \( n \dim K \) polynomials
\[
Z = \{ f_\alpha^i = f_\alpha \circ \text{pr}_{\mathfrak{k}_i} \mid i = 1, \ldots, n, \alpha = 1, \ldots, r \}
\]
and the restrictions of invariants \( Z \) to \( v \) give \( n \dim K \) independent Casimir functions of \( \mathbb{R}[v]^H \).

**Translation of Argument and Flag of Subalgebras.** Mishchenko and Fomenko showed that the set of polynomials induced from the invariants by shifting the argument \( A = \{ f_{\alpha, k}^\alpha(x) \mid k = 1, \ldots, \deg f_\alpha, \alpha = 1, \ldots, r \} \),
\[
f_{\alpha, t}^\alpha(x) = f_\alpha(x + ta) = \sum_{k=0}^{\deg f_\alpha} f_{\alpha, k}^\alpha(x)t^k,
\]

is a complete commutative set on $\mathfrak{k}$:

$$\text{ddim } \mathcal{A} = \frac{1}{2}(\dim \mathcal{K} + \text{rank } \mathcal{K}),$$  \hspace{1cm} (14)

for a generic $a \in \mathfrak{t}$ (see [12], [2], [19]). As was already mentioned, the argument shift method allows us to construct a complete commutative subalgebra in $\mathcal{F}_1$. Now we shall modify the method, by using the flag of subalgebras (5) to use it to construct such a subalgebra within $\mathcal{F}_2$.

Let

$$\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \cdots + \mathcal{B}_{n-1} + \mathcal{Z},$$

$$\mathcal{B}_i = \{ f^\alpha_{i,k}(x) \mid k = 1, \ldots, \text{deg } f^\alpha, \alpha = 1, \ldots, r \},$$

where polynomials $f^\alpha_{i,k}(x)$ are defined by:

$$f^\alpha_{i,k}(x_1, \ldots, x_n) = f^\alpha(x_1 + \cdots + x_i + tx_{i+1}) = \sum_{k=0}^{\text{deg } f^\alpha} f^\alpha_{i,k}(x_1, \ldots, x_n)t^k. \hspace{1cm} (15)$$

**Theorem 2**

(i) The set $\mathcal{B}$ is a commutative set of $\text{Ad}_H$-invariant polynomials on $\mathfrak{g}$.

(ii) The set $\mathcal{B} + \mu^*(\mathfrak{r}[\mathfrak{r}])$ is a complete set of polynomials on $\mathfrak{g}$. In particular, if $\mathcal{A}$ is any complete commutative set on $\mathfrak{t}$, then $\mathcal{B} + \mu^*(\mathcal{A})$ will be a complete commutative set on $\mathfrak{g}$.

**Proof.** Step 1. The polynomial in $\mathcal{B}$ are $\text{Ad}_H$-invariant. Indeed, let $h = (k, \ldots, k) \in H$. Then

$$f^\alpha_{i,t}(\text{Ad}_h(x)) = f^\alpha(\text{Ad}_k(x_1) + \cdots + \text{Ad}_k(x_i) + t \text{Ad}_k(x_{i+1}))$$

$$= f^\alpha(\text{Ad}_k(x_1 + \cdots + x_i + tx_{i+1}))$$

$$= f^\alpha(x_1 + \cdots + x_i + tx_{i+1}) = f^\alpha_{i,t}(x).$$

Step 2. The set $\mathcal{B}$ is involutive. Take polynomials $f^\alpha_{i,t_1}(x)$ and $f^\beta_{j,t_2}(x)$ given by (15).

Let $\nabla f^\alpha = \nabla f^\alpha|_{x_1 + \cdots + x_i + t_1 x_{i+1}}$ and $\nabla f^\beta = \nabla f^\beta|_{x_1 + \cdots + x_j + t_2 x_{j+1}}$. Then $[\nabla f^\alpha, x_1 + \cdots + x_i + t_1 x_{i+1}] = 0$, $[\nabla f^\beta, x_1 + \cdots + x_j + t_2 x_{j+1}] = 0$ and

$$\nabla f^\alpha_{i,t_1}(x) = \left(\nabla f^\alpha, \ldots, \nabla f^\alpha, t_1 \nabla f^\alpha, 0, \ldots, 0\right), \hspace{1cm} (16)$$

$$\nabla f^\beta_{j,t_2}(x) = \left(\nabla f^\beta, \ldots, \nabla f^\beta, t_2 \nabla f^\beta, 0, \ldots, 0\right). \hspace{1cm} (17)$$

First, consider the case $i < j$. We have

$$\{ f^\alpha_{i,t_1}, f^\beta_{j,t_2} \}(x) = -\langle x_1, \nabla f^\alpha, \nabla f^\beta \rangle - \cdots - \langle x_i, \nabla f^\alpha, \nabla f^\beta \rangle$$

$$-t_1 \langle x_{i+1}, \nabla f^\alpha, \nabla f^\beta \rangle$$

$$= -\langle x_1 + \cdots + x_i + t_1 x_{i+1}, \nabla f^\alpha, \nabla f^\beta \rangle$$

$$= \langle \nabla f^\alpha, x_1 + \cdots + x_i + t_1 x_{i+1}, \nabla f^\beta \rangle = 0.$$
Now, let $i = j$. Then we have

\[
\{f^\alpha_{i,t_1}, f^\beta_{i,t_2}\}(x) = -\langle x_1 + \cdots + x_i, [\nabla f^\alpha, \nabla f^\beta] \rangle \\
- t_1 t_2 (x_{i+1}, [\nabla f^\alpha, \nabla f^\beta]) \\
= -\langle x_1 + \cdots + x_i + t_1 x_{i+1}, [\nabla f^\alpha, \nabla f^\beta] \rangle \\
+ t_1 (x_{i+1}, [\nabla f^\alpha, \nabla f^\beta]) \\
- t_1 (x_1 + \cdots + x_i + t_2 x_{i+1}, [\nabla f^\alpha, \nabla f^\beta]) \\
+ t_1 (x_1 + \cdots + x_i, [\nabla f^\alpha, \nabla f^\beta]) \\
= t_1 (x_1 + \cdots + x_i + x_{i+1}, [\nabla f^\alpha, \nabla f^\beta]).
\]

In the same way:

\[
\{f^\alpha_{i,t_1}, f^\beta_{i,t_2}\}(x) = t_2 (x_1 + \cdots + x_i + x_{i+1}, [\nabla f^\alpha, \nabla f^\beta]).
\]

Therefore $\{f^\alpha_{i,t_1}, f^\beta_{i,t_2}\} = 0$ for $t_1 \neq t_2$ and taking the limit $t_1 \to t_2$, we get

\[
\{f^\alpha_{i,t_1}, f^\beta_{i,t_2}\} = 0
\]

for all $t_1, t_2$. It follows that $\{B, B\} = 0$. Item (i) is proved.

Step 3. For a generic $x_1, \ldots, x_i$, due to the Mishchenko-Fomenko shifting of argument method, the set of polynomials $B_i$, considered as polynomials in variable $x_{i+1}$, form a complete set on $\mathfrak{k}_{i+1}$ with respect to the corresponding Lie-Poisson bracket. Therefore

\[
\dim \text{pr}_{i+1} B_{i,x} = \frac{1}{2} (\dim K + \text{rank } K), \quad B_{i,x} = \text{span} \{\nabla f^\alpha_{i,k}(x)\}, \quad \text{(18)}
\]

for a generic $x \in \mathfrak{g}$.

Let $B_x$ be the linear space spanned by gradients of polynomial in $B$ at $x \in \mathfrak{g}$. From (18) we get

\[
\dim B_x \geq \text{rank } K + \dim \text{pr}_{i+1} B_{i,x} + \cdots + \dim \text{pr}_{n} B_{n-1,x} \\
\quad \geq \text{rank } K + \frac{n-1}{2} (\dim K + \text{rank } K), \quad \text{(19)}
\]

where we used that $B$ contains invariants in variable $x_1$. Thus $\text{ddim } B \geq \frac{1}{2}((n-1) \dim K + (n+1) \text{ rank } K)$.

Step 4. Fix a generic $(n \dim K - n \text{ rank } K)$-dimensional adjoint orbit

\[
\mathcal{O} = \text{Ad}_G(x_1, \ldots, x_n) = \mathcal{O}_1(x_1) \times \cdots \times \mathcal{O}_n(x_n), \quad \mathcal{O}_i(x_i) = \text{Ad}_K(x_i),
\]

such that (13) holds. This means that the action of $H = \text{diag}(K)$ is locally free.

The orbit $\mathcal{O}$ with the Konstant-Kirillov symplectic form $\omega$ is a symplectic leaf in $(\mathfrak{g}, \{\cdot,\cdot\})$. The $\text{Ad}_H$-action, restricted to $\mathcal{O}$, is Hamiltonian with the momentum mapping (9) (e.g., see [13]).
The algebra of $H$-invariant and Noether’s functions $C^\infty_H(O) + \mu^*(C^\infty(\mathfrak{k}))$ is a complete algebra on $(O, \omega)$ and

\[
\begin{align*}
\operatorname{ddim} C^\infty_H(O) &= (n-1) \dim K - n \rank K, \\
\operatorname{ddim} \mu^*(C^\infty(\mathfrak{k})) &= \dim K \\
\operatorname{ddim} (C^\infty_H(O) + \mu^*(C^\infty(\mathfrak{k}))) &= n \dim K - (n+1) \rank K \\
\operatorname{dind} C^\infty_H(O) &= \operatorname{dind} \mu^*(C^\infty(\mathfrak{k})) = \rank K \\
\operatorname{dind} (C^\infty_H(O) + \mu^*(C^\infty(\mathfrak{k}))) &= \rank K
\end{align*}
\]  

(see Theorem 2.1 and Remark 2.1 in [4], where we used that a generic Ad$K$-orbit in $\mu(O)$ is regular and that the Ad$H$-action is locally free at a generic point $x \in O$). In particular, a commutative set $C \subset C^\infty_H(O)$ is a complete subset if

\[
\operatorname{ddim} C = \frac{1}{2} (\operatorname{ddim} C^\infty_H(O) + \operatorname{dind} C^\infty_H(O)) = \frac{n-1}{2} (\dim K - \rank K). \tag{20}
\]

Let $C = \{f|_O \mid f \in \mathcal{B}\}$. The invariants $Z$ restricted to $O$ are constants, so we have

\[
\operatorname{ddim} C = \operatorname{ddim} \mathcal{B} - n \rank K \geq \frac{1}{2} ((n-1) \dim K - (n-1) \rank K). \tag{21}
\]

From (20), we get that $C$ is a complete commutative subset of $C^\infty_H(O)$. In particular, inequalities in (19) and (21) are equalities.

Since the set of $\mathcal{B} + \mu^* (\mathbb{R}[\mathfrak{k}])$ is a complete set restricted to a generic symplectic leaf $(O, \omega)$, it is a complete set on $(\mathfrak{g}, \{\cdot, \cdot\})$. This completeness the proof. □

By using Theorem 2 we obtain the following integrable model. Consider a left-invariant metric on $G$ defined by the Hamiltonian function

\[
\hat{h}_{s,t} = \sum_{i=1}^{n-1} \frac{1}{2} \left(s_i^2(x_1 + \cdots + x_i) + t_i x_{i+1}, s_i(x_1 + \cdots + x_i) + t_i x_{i+1}, \right), \tag{22}
\]

where parameters $s_i, t_i$ are chosen such that $\hat{h}$ is a positive definite Hamiltonian of the left-invariant metric.

**Corollary 1** *The Euler equations on $\mathfrak{g}$ determined with Hamiltonian (22)*

\[
\begin{align*}
\dot{x}_1 &= [x_1, \sum_{i=1}^{n-1} \left(s_i^2(x_1 + \cdots + x_i) + t_i s_i x_{i+1}\right)], \\
\dot{x}_k &= [x_k, s_{k-1} t_{k-1} (x_1 + \cdots + x_{k-1}) + \sum_{i=k}^{n-1} \left(s_i^2(x_1 + \cdots + x_i) + t_i s_i x_{i+1}\right)], \\
\dot{x}_n &= [x_n, s_{n-1} t_{n-1} (x_1 + \cdots + x_{n-1})], \quad k = 2, \ldots, n-1
\end{align*}
\]

are completely integrable.
Lemma 2 3 If f and g are Ad_H-invariant polynomials on g and \{f, g\} = 0, then \{f|_v, g|_v\}_v = 0, where \{\cdot, \cdot\}_v is the bracket given by 10.

Let \mathcal{F} be the set of polynomials, obtained by restriction of polynomials in \mathcal{B} to \mathfrak{v}.

Theorem 3 The set \mathcal{F} is a complete commutative subset of \mathbb{R}[\mathfrak{v}]^H.

Proof. According to Theorem 2 and Lemma 2, the set \mathcal{F} is commutative. Further, from (4) and Lemma 1, it is complete if and only if

\[ \text{ddim } \mathcal{F} = \frac{1}{2}((n - 2) \dim K + n \rank K). \quad (23) \]

Since

\[ \text{dim } \mathcal{F}_x = \dim \mathcal{B}_x - \dim (\mathcal{B}_x \cap \mathfrak{h}). \quad (24) \]

The relation (18) is satisfied for a generic \( x \in \mathfrak{v} \) and \( i < n - 1 \), while for \( i = n - 1 \) it does not hold. Indeed, from \( x_1 + \cdots + x_n = 0 \), we get that \( f_{n-1,i}^\alpha(x) = f^\alpha((1-t)x_n) = (1-t)^\deg f^\alpha(x_n) \). Thus

\[ \dim \mathcal{B}_x \geq \rank K + \dim \pr_{t_{n-1}} B_{1,x} + \cdots + \dim \pr_{t_1} B_{n-2,x} + \rank K \]

\[ = 2 \rank K + \frac{n - 2}{2} (\dim K + \rank K), \quad (25) \]

for a generic \( x \in \mathfrak{v} \).

On the other hand, it is obvious that

\[ \dim (\mathcal{B}_x \cap \mathfrak{h}) \leq \dim (\text{span}\{\nabla f^\alpha(x_n) | \alpha = 1, \ldots, \rank K\}) = \rank K. \quad (26) \]

Combining (25), (24) and (26) we get

\[ \dim \mathcal{F}_x \geq \frac{1}{2}((n - 2) \dim K + n \rank K), \quad (27) \]

for a generic \( x \in \mathfrak{v} \). According to (23) we have ddim \( \mathcal{F} \leq \frac{1}{2}((n - 2) \dim K + n \rank K) \), i.e., the relation (27) is an equality. \( \square \)

Let \( h_{s,t} \) be the restriction of the Hamiltonian (22) to \( \mathfrak{v} \) and \( ds^2_{s,t} \) be the corresponding \( G \)-invariant submersion metric on \( Q = G/H \).

Corollary 2 The geodesic flow of the metric \( ds^2_{s,t} \) is completely integrable. The complete commutative set of analytic functions, polynomial in momenta is

\[ \{\tau(f_{i,k}^\alpha|_{\mathfrak{v}}), \tau(f_{i}^\alpha|_{\mathfrak{v}}), \Phi^*(f_{i,k}^\alpha|_{\mathfrak{v}}) | i = 1, \ldots, n - 1, k = 1, \ldots, \deg f^\alpha, \alpha = 1, \ldots, r\}. \]
Here $\tau$ denotes the bijection $\mathbb{R}[v]^H \to \mathcal{F}_2$, $\Phi : T^*Q \to \mathfrak{g}^*$ is the momentum mapping of the canonical $G$-action,

$$f^\alpha(x_i + t a_i) = \sum_{k=0}^{\deg f^\alpha} f_{a_i}^{\alpha, k}(x_1, \ldots, x_n) t^k$$

and $a_i \in \mathfrak{k}$, $i = 1, \ldots, n$ are in generic position. That is, $\{f_{a_i}^{\alpha, k}\}$ is a complete commutative set on $\mathfrak{g}$ induced from the invariants by the argument translation with $a = (a_1, \ldots, a_n)$.

### Gaudin Type Systems on $G = K_n$.

Consider the Hamiltonian

$$\hat{h}_a(x) = \frac{1}{2} \left\langle \frac{1}{a_1} x_1 + \cdots + \frac{1}{a_n} x_n, \frac{1}{a_1} x_1 + \cdots + \frac{1}{a_n} x_n \right\rangle.$$  

The corresponding Euler equations on $\mathfrak{g} = \mathfrak{k}^n$ are

$$\dot{x}_i = \sum_{j=1}^{n} \frac{1}{a_i a_j} [x_i, x_j], \quad i = 1, \ldots, n. \quad (28)$$

Following [18], we refer to system (28) as a Gaudin type system on $\mathfrak{k}^n$ (the Gaudin system is originally defined for $\mathfrak{k} = \mathfrak{su}(2)$). The system is $H$-invariant, so the momentum mapping (8) is conserved along the flow. By using the pencil of compatible Poisson brackets (e.g., see Bolsinov [2]) defined by the Lie-Poisson bivector (11) and the bivector

$$\hat{\Lambda}_x = a_1 \lambda_{x_1} \times \cdots \times a_n \lambda_{x_n} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},$$

Panasyuk proved the integrability of equations (28) restricted to admissible adjoint orbits $\mathcal{O} = \text{Ad}_G(x_1, \ldots, x_n)$ for a generic value of parameters $a = (a_1, \ldots, a_n)$ [18]. The complete algebra of integrals is $\mathcal{P} + \mu^* (\mathbb{R}[\mathfrak{k}])$, where

$$\mathcal{P} = \left\{ \left. f \left( \frac{x_1}{t_1 + a_1 t_2} + \cdots + \frac{x_n}{t_1 + a_n t_2} \right) \right| t_1, t_2 \in \mathbb{R}, t_1^2 + t_2^2 \neq 0, f \in \mathbb{R}[\mathfrak{k}] \right\}.$$

The set $\mathcal{P}$ is commutative. It could be proved that the set of polynomials, obtained by the restriction of polynomials in $\mathcal{P}$ to $\mathfrak{v}$ is a complete commutative subset of $\mathbb{R}[v]^H$.

### 3 Einstein Metrics

Recall that the Riemannian manifold $(Q, g)$ is called Einstein if the Ricci curvature $\text{Ric}(g)$ satisfies the equation $\text{Ric}(g) = C \cdot g$, for some constant $C$ [11].

From now on we assume that $K$ is a simple Lie group. The normal $G$-invariant metric $ds_0^2$ on (1) is Einstein (see Proposition 5.5, [22]). Up to the isometry and homothety, the homogeneous space $Q$ for $n = 3$ ($n \geq 4$) admits
exactly (respectively, at least) two $G$-invariant Einstein metrics that we shall describe below. Let $(\cdot, \cdot)_v$ be an ad$_{h}$-invariant scalar product on $v$. We can diagonalize $(\cdot, \cdot)_b$ and $(\cdot, \cdot)|_b$ simultaneously (see [17]). Namely, let $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n$ be an unit vector and $\xi \in \mathfrak{k}$ be a linear subspace defined by

$$\mathfrak{k}_{\nu} = \{ (\nu_1 \xi, \ldots, \nu_n \xi) | \xi \in \mathfrak{k} \}. \quad (29)$$

There exist $n-1$ orthogonal ad$_{h}$-invariant irreducible submodules $v_1, \ldots, v_{n-1} \subset v$ and $n-1$ positive numbers $s_1, \ldots, s_{n-1}$ such that

$$v = v_1 \oplus v_2 \oplus \cdots \oplus v_{n-1}, \quad v_1 = \mathfrak{k}_{\nu_1}, \ldots, v_{n-1} = \mathfrak{k}_{\nu_{n-1}}$$

and

$$(\cdot, \cdot)_v = s_1(\cdot, \cdot)|_{v_1} \oplus s_2(\cdot, \cdot)|_{v_2} \oplus \cdots \oplus s_{n-1}(\cdot, \cdot)|_{v_{n-1}}, \quad (30)$$

where $\nu^1, \ldots, \nu^{n-1}$ is the orthonormal base of the hyperplane orthogonal to $(1, \ldots, 1) \in \mathbb{R}^n$. The diagonalization is unique if all $s_i$ are different.

Now, let $ds^2_{p,q}$ be a $G$-invariant metric defined by the scalar product (30), where

$$\nu^j = \frac{1}{\sqrt{j^2 + j}} (1, \ldots, 1, -j, 0, \ldots, 0), \quad j = 1, \ldots, n-1,$$

$$s_1 = \cdots = s_{n-2} = 1/p, \quad s_{n-1} = 1/q. \quad (31)$$

It is Einstein for $p = n^{1/(n-1)}$, $p^{n-2}q = 1$. Moreover, for $n = 3$, up to isometry and homothety, this is the only $G$-invariant Einstein metric different from the normal one $p = q = 1$ (see [17]).

Together with the scalar product (30), (31) it is natural to consider its extension to an Ad$_{h}$-invariant scalar product on $g$

$$(\cdot, \cdot)_g = \frac{1}{s}(\cdot, \cdot)|_b \oplus \frac{1}{p}(\cdot, \cdot)|_{v_1} \oplus \cdots \oplus \frac{1}{p}(\cdot, \cdot)|_{v_{n-2}} \oplus \frac{1}{q}(\cdot, \cdot)|_{v_{n-1}} \quad (32)$$

and the corresponding left-invariant metric $ds^2_{p,q,s}$ on $G$. Then $ds^2_{p,q}$ can be seen as a submersion metric, induced by $ds^2_{p,q,s}$.

The Hamiltonian of the metric $ds^2_{p,q,s}$, in the left-trivialization, read

$$\hat{h} = \frac{s}{2}(pr_b x, pr_b x) + \frac{p}{2}(x - pr_{v_{n-1}} x, x - pr_{v_{n-1}} x) + \frac{q}{2}(pr_{v_{n-1}} x, pr_{v_{n-1}} x). \quad (33)$$

Note that the orthogonal projection to (29) with respect to the Killing form is given by

$$pr_{\mathfrak{k}_\nu}(x_1, \ldots, x_n) = (\nu_1(x_1 + \cdots + \nu_n x_n), \ldots, \nu_n(x_1 + \cdots + \nu_n x_n)). \quad (34)$$

By using (31) and (34), we easily get:
Lemma 3 \begin{align*}
\hat{h} &= \frac{p}{2} \sum_{k=1}^{n-1} \langle x_k, x_k \rangle + \frac{1}{2} \left( \frac{q n}{n-1} - \frac{p}{n-1} \right) \langle x_n, x_n \rangle \\
&\quad + \frac{1}{2} \left( \frac{s}{n} - \frac{p}{n-1} + \frac{q}{n^2 - n} \right) \langle \mu, \mu \rangle + \left( \frac{p}{n-1} - \frac{q}{n-1} \right) \langle \mu, x_n \rangle
\end{align*}

has the form

where $\mu$ is the momentum mapping \begin{align*}
\mu
\end{align*}

The Euler equations with Hamiltonian \begin{align*}

\text{are}
\end{align*}

\begin{equation}
\begin{aligned}
\dot{x}_k &= [x_k, u(x_1 + \cdots + x_{n-1}) + vx_n], \quad k = 1, \ldots, n-1, \\
\dot{x}_n &= [x_n, v(x_1 + \cdots + x_{n-1})],
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\text{where } u &= s/n - p/(n-1) + q/(n^2 - n) \text{ and } v = s/n - q/n. \text{ In particular, the set } B \text{ is a set of integral of the system (36). Whence, the functions } F \text{ commute with the Hamiltonian } h = \hat{h}|_v \text{ of the metric } ds_{p,q}^2. \text{ Applying Corollary 2 we obtain:}
\end{aligned}
\end{equation}

Corollary 3 \begin{align*}
The geodesic flow of the $G$-invariant Nikonorov’s Einstein metric on \begin{align*}
\text{on } (1)
\end{align*}

is completely commutatively integrable by means of analytic integrals, polynomial in momenta.
\end{align*}

Note that, restricted to the invariant subspace $v = \mu^{-1}(0)$, the equations \begin{align*}
\text{take the form}
\end{align*}

\begin{equation}
\begin{aligned}
\dot{x}_k &= [x_k, (v - u)x_n], \quad k = 1, \ldots, n-1, \\
\dot{x}_n &= 0, \\
x_1 + \cdots + x_n &= 0
\end{aligned}
\end{equation}

\begin{equation}
\end{equation}

\begin{equation}
\end{equation}

The generic solution of (37) is given by

\begin{equation}
\begin{aligned}
x_k(t) &= \text{Ad}_{\exp(t\xi)} x_k^0, \quad x_k^0 = x_k(0), \\
\xi &= (v - u)(x_1^0 + \cdots + x_{n-1}^0), \quad k = 1, \ldots, n-1,
\end{aligned}
\end{equation}

where $\exp : \mathfrak{k} \to K$ is the exponential mapping.

Acknowledgments.

I am greatly thankful to Alexey Bolsinov on useful discussions. This research was supported by the Serbian Ministry of Science Project 144014 Geometry and Topology of Manifolds and Integrable Dynamical Systems.
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