Title:
Cohen-Macaulay \( r \)-partite graphs with minimal clique cover

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ABSTRACT. In this paper, we give some necessary conditions for an $r$-partite graph such that the edge ring of the graph is Cohen-Macaulay. It is proved that if there exists a cover of an $r$-partite Cohen-Macaulay graph by disjoint cliques of size $r$, then such a cover is unique.

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1. Introduction

Mainly, after using the notion of simplicial complexes and its algebraic interpretation by Stanley in 1970s to prove the upper bound conjecture for number of simplicial spheres [10], this notion has been one of the main streams of research in commutative algebra. In this stream, characterization and classification of Cohen-Macaulay simplicial complexes have been extensively studied in last decades. It is known that the Cohen-Macaulay property of a simplicial complex and complement of its comparability graph coincide [8]. Therefore, to characterize all simplicial complexes which are Cohen-Macaulay, it is enough to characterize all graphs with this property [10].
To examine special classes of graphs, Estrada and Villarreal in [3] found some necessary conditions for bipartite graphs to be Cohen-Macaulay. Finally, Herzog and Hibi in [5] presented a combinatorial characterization for bipartite graphs equivalent to the Cohen-Macaulay property of these graphs. This purely combinatorial method can not be generalized for $r$-partite graphs in general. Because, as shown in Example 2.3, the Cohen-Macaulay property may depend on characteristics of the base field. In this paper, we consider $r$-partite graphs with a minimal clique cover and find a necessary condition for Cohen-Macaulay property of these graphs. More precisely, we prove that in a Cohen-Macaulay $r$-partite graph with a minimal clique cover, there is a vertex of degree $r - 1$ and the cover is unique.

2. Preliminaries

A simple graph is an undirected graph with no loop or multiple edge. A finite graph is denoted by $G = (V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. Let $|V(G)| = n$. We use $[n] = \{1, 2, \ldots, n\}$ as vertices of $G$. The complementary graph of $G$ is the graph $\overline{G}$ on $[n]$ whose edge set $E(\overline{G})$ consists of those edges $\{i, j\}$ which are not in $E(G)$. An independent set of vertices is a set of pairwise nonadjacent vertices. An $r$-partite graph is a graph whose set of vertices can be partitioned into $r$ disjoint subsets such that each set is independent. A subset $A \subset [n]$ is called a minimal vertex cover of $G$ if (i) each edge of $G$ is incident with at least one vertex in $A$, and (ii) there is no proper subset of $A$ with property (i). It is easy to check that any minimal vertex cover of a graph is the complement set of a maximal independent set of the graph. A graph $G$ is called unmixed (well-covered) if any two minimal vertex covers of $G$ have the same cardinality. A clique in a graph is a set of pairwise adjacent vertices, and by an $r$-clique we mean a clique of size $r$. An $r$-matching in $G$ is a set of pairwise disjoint $r$-cliques in $G$ and a perfect $r$-matching is an $r$-matching which covers all vertices of $G$.

Let $\omega(G)$ denote the maximum size of cliques in $G$, which is called the clique number of $G$. Let $f : V(G) \to [k]$ be a map such that if $v_1$ is adjacent to $v_2$ then $f(v_1) \neq f(v_2)$. If such a map exists, we say that $G$ is colorable by $k$ colors. The smallest such $k$ is called the chromatic number of the graph and is denoted by $\chi(G)$. A graph $G$ is called perfect if $\omega(H) = \chi(H)$ for each induced subgraph $H$ of $G$. The class
of perfect graphs plays an important role in graph theory and most of computations in this class can be done by fast algorithms. L. Lovász in [9] has proved that a graph is perfect if and only if its complement is perfect. M. Chudnovsky et al in [2] have proved that a necessary and sufficient condition for a graph $G$ to be perfect is that $G$ does not have an odd hole (a cycle of odd length greater than 3) or an odd anti-hole (complement of an odd hole) as induced subgraph.

Let $G$ be a graph on $[n]$. Let $S = K[x_1, \ldots, x_n]$, the polynomial ring over a field $K$. The edge ideal $I(G)$ of $G$ is defined to be the ideal of $S$ generated by all square-free monomials $x_ix_j$ provided that $i$ is adjacent to $j$ in $G$. The quotient ring $R(G) = S/I(G)$ is called the edge ring of $G$.

Let $R$ be a commutative ring with an identity. The depth of $R$, denoted by depth$(R)$, is the largest integer $r$ such that there is a sequence $f_1, \ldots, f_r$ of elements of $R$ such that $f_i$ is not a zero-divisor in $R/(f_1, \ldots, f_{i-1})$ for all $1 \leq i \leq r$, and $(f_1, \ldots, f_r) \neq R$. Such a sequence is called a regular sequence. The depth is an important invariant of a ring. It is bounded by another important invariant, the Krull dimension, the length of the longest chain of prime ideals in the ring. A ring $R$ is called Cohen-Macaulay if depth$(R) = \dim(R)$. A graph $G$ is called Cohen-Macaulay if the ring $R(G)$ is Cohen-Macaulay.

**Theorem 2.1.** [11, Proposition 6.1.21] If $G$ is a Cohen-Macaulay graph, then $G$ is unmixed.

A simplicial complex $\Delta$ on $n$ vertices is a collection of subsets of $[n]$ such that the following conditions hold:
(i) $\{i\} \in \Delta$ for each $i \in [n]$,
(ii) if $E \in \Delta$ and $F \subseteq E$ then $F \in \Delta$.

An element of $\Delta$ is called a face and a maximal face with respect to inclusion is called a facet. The set of all facets of $\Delta$ is denoted by $\mathcal{F}(\Delta)$. The dimension of a face $F \in \Delta$ is defined to be $|F| - 1$ and dimension of $\Delta$ is the maximum of dimensions of its faces. A simplicial complex is called pure if all of its facets have the same dimension. For more details on simplicial complexes see [10].

The clique complex of a finite graph $G$ on $[n]$ is the simplicial complex $\Delta(G)$ on $[n]$ whose faces are cliques of $G$. Let $\Delta$ be a simplicial complex on $[n]$. We say that $\Delta$ is shellable if its facets can be ordered as $F_1, F_2, \ldots, F_m$ such that for all $j \geq 2$ the subcomplex $(F_1, \ldots, F_{j-1}) \cap F_j$ is pure of dimension $\dim F_j - 1$. An order of the facets satisfying this condition is called a shelling order. To say that $F_1, F_2, \ldots, F_m$ is a shelling order.
order of $\Delta$ is equivalent to say that for all $i$, $2 \leq i \leq m$ and all $j < i$, there exists $l \in F_i \setminus F_j$ and $k < i$ such that $F_i \setminus F_k = \{l\}$. A graph $G$ is called shellable if $\Delta(G)$ is a shellable simplicial Complex.

Let $\Delta$ be a simplicial complex on $[n]$ and $I_{\Delta}$ be the ideal of $S = K[x_1, \ldots, x_n]$ generated by all square-free monomials $x_{i_1} \cdots x_{i_t}$, provided that $\{i_1, \ldots, i_t\}$ is not a face of $\Delta$. The ring $S/I_{\Delta}$ is called the Stanley-Reisner ring of $\Delta$. A simplicial complex is called Cohen-Macaulay if its Stanley-Reisner ring is Cohen-Macaulay.

**Theorem 2.2.** [6, Theorem 8.2.6] If $\Delta$ is a pure and shellable simplicial complex, then $\Delta$ is Cohen-Macaulay.

Estrada and Villarreal in [3] have proved that for a bipartite graph $G$ the Cohen-Macaulay property and pure shellability are equivalent. This is not true in general for $r$-partite graphs when $r > 2$ (Example 2.3).

Also in bipartite graphs, Cohen-Macaulay property does not depend on characteristic of the ground field. But again, this is not true in general as shown in the following example.

**Example 2.3.** Let $G$ be the graph in Figure 1. Then, $R(G)$ is Cohen-Macaulay when the characteristic of the ground field $K$ is zero but it is not Cohen-Macaulay in characteristic 2. Therefore the graph $G$ is not shellable ([7]).
3. The Cohen-Macaulay property and uniqueness of perfect \(r\)-matching

M. Estrada and R. H. Villarreal in [3] have proved that if \(G\) is a Cohen-Macaulay bipartite graph and has at least one vertex of positive degree, then there is a vertex \(v\) such that \(\text{deg}(v) = 1\). By \(\text{deg}(v)\) we mean the number of vertices adjacent to \(v\). J. Herzog and T. Hibi in [5] have proved that a bipartite graph \(G\) with parts \(V\) and \(W\) is Cohen-Macaulay if and only if, \(|V| = |W|\) and there is an order on the vertices of \(V\) and \(W\) as \(v_1, \ldots, v_n\) and \(w_1, \ldots, w_n\) respectively, such that:

1) \(v_i \sim w_i\) for \(i = 1, \ldots, n\),
2) if \(v_i \sim w_j\), then \(i \leq j\),
3) for each \(1 \leq i < j < k \leq n\) if \(v_i \sim w_j\) and \(v_j \sim w_k\), then \(v_i \sim w_k\).

R. Zaare-Nahandi in [12] has proved that a well-covered bipartite graph \(G\) is Cohen-Macaulay if and only if there is a unique perfect 2-matching in \(G\).

Let \(\alpha(G)\) denote the maximum cardinality of independent sets of vertices of \(G\). Let \(\mathcal{G}\) be the class of graphs such that for each \(G \in \mathcal{G}\) there are \(k = \alpha(G)\) cliques in \(G\) covering all its vertices. For each \(G \in \mathcal{G}\) and cliques \(Q_1, \ldots, Q_k\) such that \(V(Q_1) \cup \cdots \cup V(Q_k) = V(G)\), we may take \(Q'_1 = Q_1\) and for \(i = 2, \ldots, k\), \(Q'_i\) the induced subgraph on the vertices \(V(Q_i) \setminus (V(Q_1) \cup \cdots \cup V(Q_{i-1}))\). Then \(Q'_1, \ldots, Q'_k\) are \(k\) disjoint cliques covering all vertices of \(G\). We call such a set of cliques, a basic clique cover of the graph \(G\). Therefore any graph in the class \(\mathcal{G}\) has a basic clique cover.

**Proposition 3.1.** Let \(G\) be an \(r\)-partite, unmixed and perfect graph such that all maximal cliques are of size \(r\). Then \(G\) is in the class \(\mathcal{G}\).

**Proof.** Let \(V_1, \ldots, V_r\) be parts of \(G\). By [13], \(|V_1| = |V_2| = \cdots = |V_r| = \alpha(G)|\). Also by [9], the complement graph \(\overline{G}\) is perfect. On the other hand, \(V_i\) is a clique of maximal size in \(\overline{G}\) for each \(1 \leq i \leq r\). Therefore, \(\chi(G) = \omega(G) = \alpha(G)|\). This implies that \(\overline{G}\) is \(\alpha(G)\)-partite. Therefore there are \(\alpha(G)\) disjoint maximal cliques in \(G\) covering all vertices. \(\square\)

The converse of the above proposition is not true as the following example shows.

**Example 3.2.** Let \(G\) be the graph in Figure 2. Then \(G\) is a graph in class \(\mathcal{G}\) which is 4-partite, unmixed and all maximal cliques are of size
4. But the induced subgraph on \{A, B, C, D, E\} is a cycle of length 5 and therefore, by [2], the graph $G$ is not perfect.

An easy computation by Singular [4] shows that the dimension and the depth of the edge ring of $G$ are both 4 and therefore, $G$ is Cohen-Macaulay.

Let $H$ be a graph and $v$ be a vertex of $H$. Let $N(v)$ be the set of all vertices of $H$ adjacent to $v$.

**Theorem 3.3.** [11, Proposition 6.2.4] *If $H$ is Cohen-Macaulay and $v$ is a vertex of $H$, then $H \setminus (v, N(v))$ is Cohen-Macaulay.*

**Theorem 3.4.** [13] *Let $G$ be an $r$-partite unmixed graph such that all maximal cliques are of size $r$. Then all parts have the same cardinality and there is a perfect 2-matching between each two parts.*

Now, we present the main theorem of this paper which is a generalization of [3, Theorem 2.4].

**Theorem 3.5.** *Let $G$ be an $r$-partite graph in the class $\mathcal{G}$ such that each maximal clique is of size $r$. If $G$ is Cohen-Macaulay then there is a vertex of degree $r - 1$ in $G$.*

**Proof.** By Theorem 3.4 all parts have the same cardinality. So there is a positive integer $n$ such that $|V| = rn$. Assume that for all vertices $v$ in $G$ we have $\text{deg}(v) \geq r$. Let $Q_i = \{x_{1i}, x_{2i}, \ldots, x_{ri}\}$ for $i = 1, \ldots, n$ are cliques in a basic clique cover of $G$. Without loss of generality, assume that $v_{11}$ be a vertex of the minimal degree. If $\text{deg}(v_{11}) = (r - 1)n$ then
$G = K_{n,n,...,n}$ is a complete $r$–partite graph. Thus $G$ is not Cohen-Macaulay by [1, Exercise 5.1.26] and we get a contradiction. Therefore, $r \leq \deg(v_{11}) \leq (r - 1)n - 1$.

Let $N(v_{11}) = \{v_{21}, \ldots, v_{2l_2}, v_{31}, \ldots, v_{3l_3}, \ldots, v_{r1}, \ldots, v_{rl_r}\}$. We have $\deg(v_{11}) = l_2 + \cdots + l_r$. Without loss of generality, we may assume that $l_2 \leq l_i$ for $i = 3, \ldots, r$. Set $G' = G \setminus (\{v_{11}\}, N(v_{11}))$. The graph $G'$ is Cohen-Macaulay by Theorem 3.3. If $l_2 = 1$, then, there exists $3 \leq i \leq r$ such that $l_i \geq 2$. The sets

$$\{v_{12}, \ldots, v_{1n}, v_{22}, \ldots, v_{2n}, v_{3(3+1)}, \ldots, v_{3n}, \ldots, (v_{i(i+1)}, \ldots, v_{in}), \ldots, v_{r(i+1)}, \ldots, v_{rn}\}$$

and

$$\{v_{12}, \ldots, v_{1n}, v_{3(3+1)}, \ldots, v_{3n}, \ldots, v_{i(i+1)}, \ldots, v_{in}, \ldots, v_{r(i+1)}, \ldots, v_{rn}\}$$

are two minimal vertex covers for $G'$ and their cardinalities are not equal. Here, by $(v_{i(i+1)}, \ldots, v_{in})$ we mean the vertices $v_{i(i+1)}, \ldots, v_{in}$ are removed from the set. This contradicts to Cohen-Macaulay property of $G'$. Therefore, $l_2 \geq 2$. We claim that

$$\deg(v_{1i}) = l_2 + l_3 + \cdots + l_r = \deg(v_{11}), \quad i = 1, \ldots, l_2.$$  

It is enough to show that $\deg(v_{12}) = l_2 + l_3 + \cdots + l_r$ and analogous argument proves the claim. If $\deg(v_{12}) > l_2 + l_3 + \cdots + l_r$, then there is a $j_t$, $l_t + 1 \leq j_t \leq n$ for some $2 \leq t \leq r$, such that $v_{12} \sim v_{1j_t}$. Without loss of generality we assume that $t = 2$.

If there is $j_2$, $l_2 + 1 \leq j_2 \leq n$, such that $v_{12} \sim v_{2j_2}$ then there is a minimal vertex cover for $G'$ containing the set

$$\{v_{12}, v_{1(l_2+1)}, \ldots, v_{1n}, v_{3(3+1)}, \ldots, v_{3n}, \ldots, v_{r(l_r+1)}, \ldots, v_{rn}\}.$$  

On the other hand, $\{v_{2(2+1)}, \ldots, v_{2n}, \ldots, v_{r(l_r+1)}\} \cup v_{r_{1}}, \ldots, v_{rl_r}\}$ is a minimal vertex cover of $G'$. By $l_2 \geq 2$ and Theorem 2.1, this contradicts the Cohen-Macaulay property of $G'$. Therefore $\deg(v_{12}) = l_2 + l_3 + \cdots + l_r$. Thus, for all $1 \leq i \leq l_2$ we have $N(v_{1i}) = \{v_{21}, \ldots, v_{2l_2}, v_{31}, \ldots, v_{3l_3}, \ldots, v_{r1}, \ldots, v_{rl_r}\}$. Consider the graph $H = G \setminus \{v_{2(2+1)}, \ldots, v_{2n}, \ldots, v_{r(l_r+1)}, \ldots, v_{rn}\} \cup N(v_{2(2+1)}) \cup \cdots \cup N(v_{2n}) \cup \cdots \cup N(v_{r(l+1)}) \cup \cdots \cup N(v_{rn})$. By Theorem 3.3, $H$ is Cohen-Macaulay but the complement of $H$ is not connected. This is a contradiction by [1, Exercise 5.1.26].

Theorem 3.5 implies that the perfect $r$-matching in a Cohen-Macaulay $r$-partite graph is unique.
Corollary 3.6. Let \( G \) be an \( r \)-partite graph in the class \( \mathcal{G} \) such that all maximal cliques are of size \( r \). If \( G \) is Cohen-Macaulay, then there is a unique perfect \( r \)-matching in \( G \).

Proof. Since \( G \) is in the class \( \mathcal{G} \), there is a perfect \( r \)-matching in \( G \). By Theorem 3.5, there is a vertex \( v \in V(G) \) of degree \( r - 1 \). Therefore, the \( r \)-clique in the \( r \)-matching which contains \( v \), must be in all perfect \( r \)-matchings of \( G \). The graph \( G \backslash \{v, N(v)\} \) is again an \( r \)-partite graph in the class \( \mathcal{G} \) which is Cohen-Macaulay by Theorem 3.3. Continuing this process, we find that the chosen perfect \( r \)-matching is the unique perfect \( r \)-matching in \( G \). \( \square \)

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