ON THE MAXIMAL NUMBER OF COPRIME SUBDEGREES IN
FINITE PRIMITIVE PERMUTATION GROUPS

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Abstract. The subdegrees of a transitive permutation group are the orbit
lengths of a point stabilizer. For a finite primitive permutation group which is
not cyclic of prime order, the largest subdegree shares a non-trivial common
factor with each non-trivial subdegree. On the other hand it is possible for
non-trivial subdegrees of primitive groups to be coprime, a famous example
being the rank 5 action of the small Janko group $J_1$ on 266 points which has
subdegrees of lengths 11 and 12. We prove that, for every finite primitive
group, the maximal size of a set of pairwise coprime non-trivial subdegrees is
at most 2.

1. Introduction

In this paper we are concerned with the subdegrees of a finite primitive permuta-
tion group. The set of of a transitive group $G$ is the set of orbit lengths of the
stabilizer $G_\omega$ of a point $\omega$, and we say that a subdegree $d$ of $G$ is non-trivial
if $d \neq 1$. We announced in [10, Theorem 1.7] that a primitive permutation group
could not have as many as three pairwise coprime non-trivial subdegrees. Here we
prove this theorem.

Theorem 1.1. Let $G$ be a finite primitive permutation group. Then the largest
subset of pairwise coprime non-trivial subdegrees of $G$ has cardinality at most 2.

This theorem is related to a classical result on finite primitive groups. In 1935
Marie Weiss [23, Theorem 3] showed that, if $G$ is a finite primitive group which is
not cyclic of prime order, then the largest of the subdegrees has non-trivial divisors
in common with all the other non-trivial subdegrees. It was observed by Peter
Neumann in 1973 [24, Corollary (2), page 93] that Weiss’s theorem implies that a
finite primitive group with $k$ pairwise coprime non-trivial subdegrees has rank at
least $2^k$. Neumann remarked that ‘groups of small rank with non-trivial co-prime
subdegrees appear to be rather rare’, and posed a question of Peter Cameron [23,
Problem 1, page 93] on the existence of a primitive rank 4 group with two coprime
non-trivial subdegrees, that is to say, a group meeting the bound $2^k$ with $k = 2$.
That no such group exists was verified by Cameron himself (see [6, Remark in
Section 1.32]), using the finite simple group classification. The smallest rank for
coprime subdegrees to occur is 5 with the famous example of $J_1$ of degree 266 with

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subdegrees 1, 11, 12, 110 and 132 first studied by Livingstone [21]. Our Theorem 1.1 shows that the parameter $k$ is at most 2. We emphasise that a primitive group may have several pairs of coprime non-trivial subdegrees – examples are given in [10, Example 4.3]. Our result simply prohibits triples of pairwise coprime subdegrees.

We say that a subgroup $L$ of a nonabelian simple group $T$ is pseudo-maximal in $T$ if there exists an almost simple group $A$ with socle $T$ and a maximal subgroup $M$ of $A$ with $T \not\leq M$ and $L = T \cap M$ (see [10, Definition 1.8]). Theorem 1.10 in [10] shows that Theorem 1.1 holds true if the following result on nonabelian simple groups (see [10, Theorem 1.9]) is true.

**Theorem A.** Let $T$ be a transitive nonabelian simple permutation group and assume that the stabilizer of a point is pseudo-maximal in $T$. Then the largest subset of pairwise coprime non-trivial subdegrees of $T$ has cardinality at most 2.

The aim of this paper is to prove Theorem A using the Classification of Finite Simple Groups, thus proving Theorem 1.1. The structure of the paper is straightforward. In Section 2 and 3 we collect some auxiliary results. We prove Theorem A for the alternating groups in Section 4, for the classical groups in Section 5, for the exceptional groups of Lie type in Section 6 and finally for the sporadic simple groups in Section 7.

One of the most efficient methods for analyzing a finite primitive permutation group $G$ is to study the socle $N$ of $G$, that is, the subgroup generated by the minimal normal subgroups of $G$. The O’Nan-Scott theorem describes in detail the embedding of $N$ in $G$ and collects some useful information on the action of $N$. In [24] eight types of primitive groups are defined (depending on the structure and on the action of the socle), namely HA (Holomorphic Abelian), AS (Almost Simple), SD (Simple Diagonal), CD (Compound Diagonal), HS (Holomorphic Simple), HC (Holomorphic Compound), TW (Twisted wreath), PA (Product Action), and it is shown in [17] that every primitive group belongs to exactly one of these types. Combining [10, Theorem 1.3, 1.4] with Theorem 1.1 we have the following corollary determining the maximal number of non-trivial pairwise coprime subdegrees of a primitive group according to its O’Nan-Scott type.

**Corollary 1.2.** Let $G$ be a finite primitive group. If $G$ has two non-trivial coprime subdegrees, then $G$ is of AS, PA or TW type.

Results concerning the subdegrees of a finite permutation group can often give interesting applications in Field Theory (see for example [10, Corollary 1.9]). In fact, Theorem 1.1 has the following surprising application.

**Corollary 1.3.** Let $K = k[\theta]$ be a minimal separable field extension that is not Galois. Let $f(x) \in k[x]$ be the minimal polynomial of $\theta$ over $k$ and write $f(x) = (x - \theta)g_1(x) \cdots g_r(x)$ with $g_i(x) \in K[x]$ irreducible over $K$, for each $i \in \{1, \ldots, r\}$. Then the maximal number of $g_i(x)$ of pairwise coprime degree is 2.

**Proof.** Let $L$ be the normal closure of $K/k$. As $K$ is separable, $L/k$ is a Galois extension. Let $G$ be the Galois group $\text{Gal}(L/k)$ and set $H = \text{Gal}(L/K)$. By the minimality of $K$, the group $G$ acts primitively on the right cosets $G/H$ of $H$ in $G$. The degrees of the $g_i(x)$ are precisely the non-trivial subdegrees of $H$ in the action on $G/H$. Now apply Theorem 1.1. □
2. Embedding results

The main results in this section are Propositions 2.7 and 2.9 which will prove to be important in the proof of Theorem 1.1.

Definition 2.1. If $G$ is a finite group, we let $\mu(G)$ denote the maximal size of a set $\{G_i\}_{i \in I}$ of proper subgroups of $G$ with $|G : G_i|$ and $|G : G_j|$ relatively prime, for each two distinct elements $i$ and $j$ of $I$.

Remark 2.2. The number $\mu(G)$ equals the maximal size of a set $\{M_j\}_{j \in J}$ of maximal subgroups of $G$ with $|G : M_i|$ and $|G : M_j|$ relatively prime, for each two distinct elements $i$ and $j$ in $J$. Clearly, $|J| \leq \mu(G)$. Conversely, let $\{G_i\}_{i \in I}$ be a family of proper subgroups of $G$ with relatively prime index in $G$. Let $M_i$ be a maximal subgroup of $G$ with $A_i \leq M_i$. Since $|G : G_i|$ is coprime to $|G : G_j|$ for $i \neq j$, we have $M_i \neq M_j$ and $|G : M_i|$ is coprime to $|G : M_j|$. Thus $\mu(G) \leq |J|$.

We recall that a finite group $E$ is said to be quasisimple if $E = [E, E]$ and $E/Z(E)$ is a nonabelian simple group, where $Z(E)$ is the centre of the group $E$. Furthermore, we say that the finite group $G$ is a central product of $A$ and $B$, if $A$ and $B$ are non-identity proper subgroups of $G$ with $[A, B] = 1$ and $G = AB$. We recall that a component $E$ of $G$ is a quasisimple subnormal subgroup of $G$. As usual, we denote by $E(G)$ the group generated by the set $\{E_1, \ldots, E_\ell\}$ (possibly empty) of components of $G$, by $F(G)$ the Fitting subgroup of $G$ and by $F^*(G) = F(G)E(G)$ the generalized Fitting subgroup of $G$. If is well-known [2] Chapter 11] that $E(G) = E_1 \cdots E_\ell$ is a central product of $E_1, \ldots, E_\ell$ (here $E(G) = 1$ if $G$ has no components), that $[F^*(G), E(G)] = 1$, that $E(E(G)) = E(G)$ and that $C_G(F^*(G)) \leq F^*(G)$.

Lemma 2.3. Suppose that $K = E_1 \cdots E_\ell$ is a central product of $\ell$ quasisimple groups with $E_i/Z(E_i) \cong E_j/Z(E_j)$, for each $i, j \in \{1, \ldots, \ell\}$. Then $\mu(K) \leq 2$.

Proof. As $E_i$ is quasisimple for each $i \in \{1, \ldots, \ell\}$, we have $K = \langle K, K \rangle$ and $K/Z(K) = T_1 \times \cdots \times T_\ell$ with $T_i = E_iZ(K)/Z(K)$. By hypothesis there is a non-abelian simple group $T$ such that, for each $i \in \{1, \ldots, \ell\}$, we have $T_i \cong T$. We argue by contradiction and we assume that $\mu(K) \geq 3$, that is, $K$ has three proper subgroups $A_1, A_2$ and $A_3$ with $[K : A_1], [K : A_2]$ and $[K : A_3]$ relatively prime. Assume first that $A_1 Z(K) < K$, for each $j \in \{1, 2, 3\}$. So, $A_1 Z(K)/Z(K), A_2 Z(K)/Z(K)$ and $A_3 Z(K)/Z(K)$ are three proper subgroups of $K/Z(K)$ with relatively prime indices, that is, $\mu(K/Z(K)) \geq 3$. Now, from [10] Lemma 5.2 we have $\mu(T^e) \leq 2$, and hence we obtain a contradiction. This shows that $K = A_{j_0} Z(K)$, for some $j_0 \in \{1, 2, 3\}$. Now, we have $K = \langle K, K \rangle = A_{j_0} Z(K), A_{j_0} Z(K) = [A_{j_0}, A_{j_0}] \leq A_{j_0}$, but this contradicts the fact that $A_{j_0}$ is a proper subgroup of $K$. □

Lemma 2.4. Let $G$ be a transitive permutation group on $\Omega$ and let $\omega \in \Omega$. Suppose that $N$ is normal in $G_\omega$ and $N$ fixes a unique point on $\Omega$. Then the maximal size of a subset of pairwise coprime non-trivial subdegrees of $G$ is at most $\mu(N)$.

Proof. Let $\omega_1, \ldots, \omega_r$ be elements of $\Omega \setminus \{\omega\}$ with $|\omega_i^{-1}\omega|_{G_\omega}$ relatively prime to $|\omega_j^{-1}\omega|_{G_\omega}$, for distinct elements $i$ and $j$ in $\{1, \ldots, r\}$. For every $i \in \{1, \ldots, r\}$, set $N_i = N_{\omega_i}$. Since $N$ fixes only the point $\omega$ of $\Omega$, the group $N_i$ is a proper subgroup of $N$. Furthermore, as $N \leq G_\omega$, the index $|N : N_i|$ divides $|\omega_i^{-1}\omega|_{G_\omega}$. From Definition 2.1, we obtain $r \leq \mu(N)$. □
Lemma 2.5. Let $H$ be a finite group such that $C_H(E(H)) = 1$ and $E(H) \cong T^\ell$ for some nonabelian simple group $T$ and for some $\ell \geq 1$. Let $F$ be a subgroup of $H$ with $F = F_1 \cdots F_r$, the central product of $\ell'$ quasisimple groups with $F_i / Z(F_i) \cong T$ and $\ell' \geq \ell$. Then $F = E(H)$.

Proof. Write $E(H) = T_1 \times \cdots \times T_\ell \cong T^\ell$ with $T_i \cong T$ for each $i \in \{1, \ldots, \ell\}$. As $C_H(E(H)) = 1$, the group $H$ is isomorphic to a subgroup of $\text{Aut}(E(H))$. So, replacing $H$ by $\text{Aut}(E(H))$ if necessary, we may assume that $H = \text{Aut}(E(H))$.

Write $E = \text{E}(H)$. We argue by induction on $\ell$. Assume that $\ell = 1$. Since Out($T$) is soluble and $F = [F, F]$, we obtain $F \leq E$ and so $F = E$. Assume that $\ell > 1$. We prove a preliminary claim from which the proof will follow.

Claim 1. If $F$ is a subgroup of $\text{Sym}(m)$, then $m \geq \ell d$, where $d$ is the minimal degree of a faithful permutation representation of $T$. In particular, $m \geq 5\ell'$.

We prove it by induction on $|F|$. Let $A_1, \ldots, A_k$ be the orbits of $F$ on $\{1, \ldots, m\}$ and let $L_j$ be the permutation group induced by $F$ on $A_j$. In particular, $L_j / Z(L_j) \cong T^{\ell_j}$ for some $0 \leq \ell_j \leq \ell'$, and $\ell' \leq \sum_{j=1}^k \ell_j$. If, for each $j \in \{1, \ldots, k\}$, we have $|L_j| < |F|$, then by induction we obtain $|A_j| \geq \ell_j d$. In particular, $m = \sum_{j=1}^k |A_j| \geq \sum_{j=1}^k \ell_j d \geq \ell d$. Therefore, we may assume that $|F| = |L_j|$ for some $j \in \{1, \ldots, \ell\}$, that is, $F$ acts faithfully and transitively on $A_j$. In particular, replacing the set $\{1, \ldots, m\}$ by $A_j$ if necessary, we may assume that $F$ is a transitive subgroup of $\text{Sym}(m)$.

Let $B$ be the system of imprimitivity consisting of the orbits of $Z(F)$. Let $K$ be the kernel of the action of $F$ on $B$ and let $F^B$ be the permutation group induced by $F$ on $B$. Clearly, $Z(F) \leq K$. Assume that $Z(F) < K$. In particular, since $F / Z(F) \cong T^{\ell'}$, there exists $i \in \{1, \ldots, \ell'\}$ with $F_i \leq K$. Let $B$ be a $Z(F)$-orbit and $\lambda \in B$. Since $Z(F)$ is abelian, $Z(F)$ acts regularly on $B$ and hence $F_i = Z(F)(F_i)_{\lambda}$. In particular, $F_i = [F_i, F_i] = [F_i \lambda, (F_i)_{\lambda}] = (F_i)_{\lambda}$ and $F_i$ fixes the point $\lambda$ of $B$. Since $F_i \leq F$, $F$ is transitive and $F_i \leq F_{\lambda}$, we see that $F_i = 1$, a contradiction. Thus $K = Z(F)$ and $F^B \cong T^{\ell'}$. From [11, Theorem 3.1], we have $|B| \geq \ell' d$. Therefore, $m \geq |B| \geq \ell' d$. Finally, since $\text{Sym}(4)$ is soluble, we have $d \geq 5$.

Let $K$ be the kernel of the action by conjugation of $H$ on the set $\{T_1, \ldots, T_\ell\}$ of $\ell$ simple direct factors of $E$. Clearly, $F \cap K$ is a normal subgroup of $F$. Assume that $F \cap K \leq Z(F)$. Then $FK / K \cong F / (F \cap K)$ is isomorphic to a subgroup of $\text{Sym}(\ell)$ and hence, by Claim 1 applied to $F / (F \cap K)$, we obtain $\ell \geq 5\ell'$, a contradiction since by assumption $\ell' \geq \ell$. Thus $F \cap K \nsubseteq Z(F)$. Since $F \cap K \leq F$ and $F / Z(F) \cong T^{\ell'}$, there exists $i \in \{1, \ldots, \ell\}$ with $F_i \leq K$. Relabelling $F_i$ by $F_1$ if necessary, we may assume that $F_1 \leq K$.

Since $K / E \cong \text{Out}(T)^{\ell}$, $\text{Out}(T)$ is soluble and $F_1 = [F_1, F_1]$, we see that $F_1 \leq E$.

For each $j \in \{1, \ldots, \ell\}$, let $\pi_j : E \rightarrow T_j$ be the projection onto the $j$th coordinate of $E$ and let $L_j$ be the kernel of $\pi_j$. Since $F_1 \leq E$ and $L_j \leq E$, we have $F_1 \cap L_j \leq F_1$ and so either $F_1 \cap L_j \leq Z(F_1)$ or $F_1 \leq L_j$. Write $J = \{j \in \{1, \ldots, \ell\} \mid F_1 \leq L_j\}$. If $J = \{1, \ldots, \ell\}$, then $F_1 \leq \cap_{j=1}^\ell L_j = 1$, a contradiction. Thus, relabelling the set $\{1, \ldots, \ell\}$ if necessary, we may assume that $J = \{m+1, \ldots, \ell\}$ for some $m \geq 1$. Fix $j$ in $\{1, \ldots, m\}$. Now, as $F_i \cap L_j \leq Z(F_i)$, we have $|T| \geq \pi_j(F_i) = |F_i : F_i \cap L_j| = |F_1 : Z(F_1)| |Z(F_1) : F_1 \cap L_j| = |T| |Z(F_1) : F_1 \cap L_j|$ and hence $\pi_j(F_1) = T_j$ and $F_1 \cap L_j = Z(F_1)$. Since this argument does not depend on $j \in \{1, \ldots, m\}$, we have $Z(F_1) = F_1 \cap (\cap_{j=1}^m L_j) = F_1 \cap (T_{m+1} \times \cdots \times T_\ell)$. Moreover, since for each
j \in \{1, \ldots, m\} \text{ we have } T_j = \pi_j(F_1), \text{ we see that } F_1 \leq D \times T_{m+1} \times \cdots \times T_{\ell} \text{ where } D \text{ is a diagonal subgroup of } T_1 \times \cdots \times T_m, \text{ that is, } D \text{ is conjugate under an element of } H \text{ to the diagonal subgroup } \{(t, \ldots, t) \mid t \in T\} \text{ of } T_1 \times \cdots \times T_m. \text{ Summing up, this gives } F_1 = D \times \mathbb{Z}(F_1). \text{ As } F_1 = [F_1, F_1], \text{ we have } \mathbb{Z}(F_1) = 1 \text{ and } F_1 = D.

Since } H = \text{Aut}(E), \text{ we have } C_H(F_1) \cong \text{Sym}(m) \times \text{Aut}(T^{\ell-m}). \text{ Let } A \text{ be the normal subgroup of } C_H(F_1) \text{ isomorphic to } \text{Sym}(m) \text{ and let } B \text{ be the normal subgroup of } C_H(F_1) \text{ isomorphic to } \text{Aut}(T^{\ell-m}). \text{ Now the group } F_2 \cdots F_{\ell'} \text{ is contained in } C_H(F_1) = A \times B. \text{ From Claim 1, } A \text{ contains at most } m/5 \text{ of the components } F_2 \cdots F_{\ell'}. \text{ Also, by induction, we have that } B \text{ contains at most } \ell - m \text{ of the components } F_2 \cdots F_{\ell'} \text{ and, if equality is met then } F_2 \cdots F_{\ell'} = T_{m+1} \times \cdots \times T_{\ell}. \text{ Therefore, } \ell' - 1 \leq m/5 + \ell - m. \text{ Since } \ell' \geq \ell, \text{ this gives } \ell' = \ell, m = 1, F_1 = T_1 \text{ and } F_2 \cdots F_{\ell'} = T_2 \times \cdots \times T_{\ell}. \text{ In particular, } F = E.

\textbf{Lemma 2.6.} Let } H \text{ be a finite group and } E = E(H). \text{ Assume that } C_H(E) \text{ is soluble and } E/\mathbb{Z}(E) \cong T^\ell \text{ for some nonabelian simple group } T \text{ and for some } \ell \geq 1. \text{ If } f:E \to H \text{ is an injective homomorphism, then } f(E) = E.

\textbf{Proof.} \text{ We write } E = E(H), Z = \mathbb{Z}(E) \text{ and } \overline{T} = H/C_H(E). \text{ Let } \overline{f} : H \to \overline{T} \text{ be the natural projection. Here we use the “bar” notation, that is, we denote by } \overline{X} \text{ the image under } - \text{ of the subgroup } X \text{ of } H.\n
\text{ In this paragraph we show that } C_{\overline{T}}(E) = 1. \text{ We have } C_{\overline{T}}(E) = C/C_H(E) \text{ for some subgroup } C \text{ of } H. \text{ Since } [C, E] = 1 \text{ and } E \leq H, \text{ we obtain } [C, E] \leq C_H(E) \cap E = \mathbb{Z}(E). \text{ In particular, } [C, E], E = 1 \text{ and } [[E, C], E] = 1. \text{ Now, from the Three Subgroup Lemma, we have } [E, C] = [[E, E], C] = 1. \text{ Thus } C \leq C_H(E) \text{ and } \overline{C} = 1.

Since every component of } \overline{T} \text{ is either contained in } \overline{E} \text{ or commutes with } \overline{E}, \text{ and since } C_{\overline{T}}(E) = 1, \text{ we obtain that } \overline{E} = E(\overline{T}). \text{ Write } F = f(E) \text{ and } \overline{F} = \overline{f}(E). \text{ As } C_H(E) \text{ is soluble and } f \text{ is injective, we have } F \cap C_H(E) \leq \mathbb{Z}(F) \text{ and } \overline{F} \cong F/(F \cap C_H(E)) \text{ is a central product of } \ell \text{ quasisimple groups. Since } E \cong \overline{T}/\mathbb{Z}(\overline{T}) \cong T^\ell, \text{ from Lemma 2.3 we have } \overline{T} = E. \text{ Therefore, } F/C_H(E) = E/C_H(E). \text{ Since } F = [F, F], E = [E, E] \text{ and } C_H(E) \text{ is soluble, we obtain that the last term of the derived series of } F/C_H(E) \text{ (respectively } E/C_H(E) \text{) is } F \text{ (respectively } E), \text{ that is, } F = E. \textbf{ } \Box

\textbf{Proposition 2.7.} Let } G \text{ be a transitive permutation group on } \Omega. \text{ For } \omega \in \Omega, \text{ assume that } C_{G_{\omega}}(E(G_{\omega})) \text{ is soluble, that } E(G_{\omega})/\mathbb{Z}(E(G_{\omega})) \cong T^\ell \text{ for some nonabelian simple group } T \text{ and for some } \ell \geq 1, \text{ and that } G_{\omega} = N_G(E(G_{\omega})). \text{ Then the maximal size of a subset of non-trivial pairwise coprime subdegrees of } G \text{ is at most 2.}

\textbf{Proof.} \text{ Fix } \omega \in \Omega \text{ and write } E = E(G_{\omega}). \text{ Assume that } E \text{ fixes an element } \omega' \text{ of } \Omega. \text{ Let } g \in G \text{ with } \omega^g = \omega'. \text{ Now } E^{\omega^g} \leq G_{\omega} \text{ and so, from Lemma 2.6 applied with } H = G_{\omega}, \text{ we have } E^{\omega^g} = E \text{ and hence } g \in N_G(E) = G_{\omega}. \text{ This yields } \omega' = \omega \text{ and hence } E \text{ fixes a unique point of } \Omega. \text{ Now the proof follows from Lemmas 2.3 and 2.4} \textbf{. } \Box

Let } G = T_1 \times \cdots \times T_{\ell} \text{ be the direct product of nonabelian simple groups. We say that } T_i \text{ has } \text{multiplicity } r \text{ in } G, \text{ if } G \text{ has exactly } r \text{ simple direct factors isomorphic to } T_i.
Lemma 3.1. Let $H$ be a finite group. Assume that each simple direct factor of $\mathbf{E}(H)/\mathbf{Z}(H)$ has multiplicity at most 4 and that $H$ has a unique component $Q$ such that $Q/\mathbf{Z}(Q)$ has largest order among the components of $H$. If $f : Q \to H$ is an injective homomorphism, then $f(Q) = Q$.

Proof. Write $R = f(Q)$ and $\mathbf{E}(H) = E_1 \cdots E_\ell$ with $E_1, \ldots, E_\ell$ the components of $\mathbf{E}(H)$. Set $\mathbf{E}(H)/\mathbf{Z}(\mathbf{E}(H)) = T_1 \times \cdots \times T_\ell$ with $\ell \geq 1$ and with $T_i$ a nonabelian simple group, for each $i \in \{1, \ldots, \ell\}$. Relabelling the index set $\{1, \ldots, \ell\}$ if necessary, we may assume that $E_1 = Q$. The group $H$ acts by conjugation on the set $\{T_1, \ldots, T_\ell\}$ of $\ell$ simple direct factors of $\mathbf{E}(H)/\mathbf{Z}(\mathbf{E}(H))$. The kernel of the action of $H$ on $\{T_1, \ldots, T_\ell\}$ is $K = \cap_{i=1}^{\ell} N_H(T_i)$. Since $T_i$ has multiplicity at most 4 in $\mathbf{E}(H)/\mathbf{Z}(\mathbf{E}(H))$, we see that $H/K$ has orbits of length at most 4 and hence $H/K$ is soluble.

As $R$ is quasisimple, this yields $R \leq K$. As $\text{Out}(T_i)$ is soluble for each $i \in \{1, \ldots, \ell\}$ and since $K/\mathbf{E}(H)$ is isomorphic to a subgroup of $\text{Out}(T_1) \times \cdots \times \text{Out}(T_\ell)$, we obtain that $K/\mathbf{E}(H)$ is soluble. As $R$ is quasisimple, we get $R \leq \mathbf{E}(H)$.

For each $j \in \{1, \ldots, \ell\}$, let $\pi_j : \mathbf{E}(H) \to T_j$ the natural projection onto the $j$th factor $T_j$ of $\mathbf{E}(H)/\mathbf{Z}(\mathbf{E}(H))$ and let $L_j$ be the kernel of $\pi_j$. Since $R \cap L_j \leq R$, we have that either $R \cap L_j \leq \mathbf{Z}(R)$ or $R \leq L_j$. In the former case, $\{T_j\} \geq \pi_j(R) = \{R : R \cap L_j \geq \{R : \mathbf{Z}(R)\} = T_1\}$ and hence $j = 1$ because of the maximality and uniqueness of $T_1$. Therefore this yields $R \leq \cap_{j=2}^{\ell} L_j = E_1\mathbf{Z}(\mathbf{E}(H))$. Since $R = [R, R]$, we obtain $R \leq [E_1\mathbf{Z}(\mathbf{E}(H)), E_1\mathbf{Z}(\mathbf{E}(H))] = E_1$ and hence, since $f$ is injective, $R = E_1$. □

Proposition 2.9. Let $G$ be a transitive permutation group on $\Omega$. For $\omega \in \Omega$, assume that each simple direct factor of $\mathbf{E}(G_\omega)/\mathbf{Z}(\mathbf{E}(G_\omega))$ has multiplicity at most 4, and that $\mathbf{E}(G_\omega)$ has a unique component $Q$ such that $Q/\mathbf{Z}(Q)$ has largest order among the components of $\mathbf{E}(H)$. Suppose that $\mathbf{N}_G(Q) = G_\omega$. Then the maximal size of a subset of non-trivial pairwise coprime subdegrees of $G$ is at most 2.

Proof. If $Q$ fixes the element $\omega'$ of $\Omega$, then there exists $g \in G$ with $\omega' = \omega^g$ and $Q^{g^{-1}} \leq G_\omega$. From Lemma 2.8, we have $Q^{g^{-1}} = Q$ and so $g \in \mathbf{N}_G(Q) = G_\omega$. This yields $\omega' = \omega$ and $Q$ fixes a unique point of $\Omega$. Now the proof follows from Lemmas 2.3 and 2.4. □

The following proposition is taken from [27] Theorem 3.7.

Proposition 2.10. Let $G$ be a transitive permutation group on $\Omega$. For $\omega \in \Omega$, assume that $G_\omega$ contains the normalizer of a Sylow $p$-subgroup of $G$. Then $p$ divides the degree of every non-trivial suborbit of $G$.

Proof. See [27] Theorem 3.7. □

3. Auxiliary Lemmas

We say that a factorization $H = AB$ is coprime if $|H : A|$ is relatively prime to $|H : B|$ and both $A, B$ are proper subgroups of $H$ (see [10] Section 2). Also $H = AB$ is maximal if $A$ and $B$ are maximal subgroups of $H$.

Lemma 3.1. Let $H$ be a finite group, $r$ a prime, and $R$ a normal $r$-subgroup of $H$. Assume that $H/R = E_1 \cdots E_\ell$ is a central product of $\ell$ quasisimple groups with $E_i/\mathbf{Z}(E_i) \cong E_j/\mathbf{Z}(E_j)$, for each $i, j \in \{1, \ldots, \ell\}$. Then $\mu(H) \leq 3$. 

Assume further that \( \mu(H) = 3 \) and let \( A_1, A_2 \) and \( A_3 \) be maximal subgroups of \( H \) with \( |H : A_i| \) relatively prime to \( |H : A_j| \), for distinct \( i \) and \( j \) in \( \{1, 2, 3\} \). Let \( U \) be the normal subgroup of \( H \) with \( U/R = Z(H/R) \). Then relabelling the set \( \{A_1, A_2, A_3\} \) if necessary, \( |H : A_3| \) is divisible by \( r \), \( U \leq A_1, A_2 \) and \( H/U = (A_1/U)(A_2/U) \) is a coprime maximal factorization of \( H/U \).

**Proof.** Suppose that \( \mu(H) \geq 3 \) and let \( A_1, A_2 \) and \( A_3 \) be maximal subgroups of \( H \) with \( |H : A_i| \) relatively prime to \( |H : A_j| \), for distinct \( i \) and \( j \) in \( \{1, 2, 3\} \). Since \( A_i \) is maximal, we obtain that either \( H = A_iR \) or \( R \leq A_i \). In the former case, the index \( |H : A_i| = |R : A_i \cap R| \) is divisible by \( r \). In the latter case, \( A_i/R \) is a maximal subgroup of \( H/R \). From Lemma 2.3, we have \( \mu(H/R) \leq 2 \) and so, in particular, \( \mu(H) = 3 \) (since we are assuming \( \mu(H) \geq 3 \)). We note that this proves the first assertion of the lemma. Since \( \mu(H/R) \leq 2 \), there exists exactly one element \( A_i \) in \( \{A_1, A_2, A_3\} \) with \( R \not\sim A_i \), and there are exactly two elements \( A_j \) and \( A_k \) in \( \{A_1, A_2, A_3\} \) with \( R \leq A_j, A_k \). Thus, replacing \( A_i \) by \( A_3 \) if necessary, we may assume that \( R \leq A_1, A_2 \) and that \( |H : A_3| \) is divisible by \( r \). Since \( A_1/R \) and \( A_2/R \) are maximal subgroups of \( H/R \) and as \( H/R \) is a central product of quasisimple groups, we have that \( U \leq A_1, A_2 \). Hence \( H/U = (A_1/U)(A_2/U) \) is a maximal factorization of the characteristically simple group \( H/U \) with \( gcd(|H : A_1|, |H : A_2|) = 1 \).

Given a finite group \( G \), we say that the normal subgroup \( N \) of \( G \) is the last term of the derived series of \( G \) if \( G/N \) is soluble and \( N = [N, N] \).

**Lemma 3.2.** Let \( T \) be a transitive permutation group on \( \Omega \), let \( \omega \) be in \( \Omega \) and let \( N \) be the last term of the derived series of \( T_\omega \). If \( T_\omega = N_T(N) \), then \( N \) fixes only the point \( \omega \).

**Proof.** Suppose that \( N \) fixes \( \omega' \) and write \( \omega' = \omega^g \), for some \( g \in T \). Set \( K = N^{\omega^{-1}} \).

Since \( K \leq T_\omega \), \( T_\omega/N \) is soluble and \( NK/N \) is isomorphic to \( K/(K \cap N) \), we see that \( K/(K \cap N) \) is soluble. Since \( K = [K, K] \), we obtain that \( K = N \cap K \) and so \( N = N^{\omega^{-1}} \). This shows that \( g \in N_T(N) = T_\omega \). So \( \omega' = \omega \) and \( N \) fixes only the point \( \omega \) of \( \Omega \).

**Remark 3.3.** Let \( T \) be a nonabelian simple permutation group on a set \( \Omega \) and let \( T_\omega \) be pseudo-maximal in \( T \), with \( \omega \in \Omega \). So, there exists an almost simple group \( A \) with socle \( T \) and a maximal subgroup \( M \) of \( A \) such that \( T \not\sim M \) and \( T_\omega = T \cap M \). Let \( N \) be a characteristic subgroup of \( T_\omega \). Then \( M = N_A(N) \), because \( M \) is maximal in \( A \) and \( T_\omega \). Hence, \( T_\omega = T \cap N_A(N) = N_T(N) \). Furthermore, as \( T_\omega = N_T(T_\omega) \), we obtain that \( \omega \) is the only fixed point of \( T_\omega \) in \( \Omega \).

We will use these two facts repeatedly in the following.

4. **Alternating groups**

**Proof of Theorem A for the alternating groups.** A subgroup \( H \) of \( \text{Sym}(n) \) is either intransitive, imprimitive or primitive in its action on \( \{1, \ldots, n\} \). In the proof of this theorem we consider these three cases separately.

Let \( T = \text{Alt}(n) \), for some \( n \geq 5 \). Fix \( \omega \in \Omega \) and write \( H = T_\omega \). Assume that \( H \) is intransitive in the natural action of \( T \) of degree \( n \). Then \( H \cong (\text{Sym}(k) \times \text{Sym}(n-k)) \cap T \), for some \( k \) with \( 1 \leq k < n/2 \). (Note that for \( n \) even, \( (\text{Sym}(n/2)^{\mathcal{B}} \times \text{Sym}(n/2)) \cap T \) is not pseudo-maximal in \( T \).) In particular, the action of \( T \) on \( \Omega \) is permutation equivalent to the action of \( \text{Alt}(n) \) on the \( k \)-subsets of \( \{1, \ldots, n\} \).
Suppose that \( n - k \geq 5 \). Let \( N \) be the minimal normal subgroup of \( H \) isomorphic to \( \text{Alt}(n - k) \). Clearly, \( N \) is simple and fixes a unique \( k \)-subset of \( \{1, \ldots, n\} \). So, by Lemmas 2.3 and 2.4 the group \( T \) has at most 2 non-trivial coprime subdegrees on \( \Omega \). Now, suppose that \( n - k \leq 4 \). If \( k \leq 2 \), then the rank of \( T \) is at most 3 and the assertion follows immediately. If \( k \geq 3 \), then \((n, k) = (7, 3)\) and by direct inspection we see that \( T \) has no pair of non-trivial coprime subdegrees.

Assume next that \( H \) is imprimitive in the natural action of \( T \) of degree \( n \). Then \( H \cong (\text{Sym}(k) \wr \text{Sym}(n/k)) \cap T \), for some divisor \( k \) of \( n \) with \( 1 < k < n \). In particular, the action of \( T \) on \( \Omega \) is permutation equivalent to the action of \( \text{Alt}(n) \) on the set \( \mathcal{P} \) of partitions of \( \{1, \ldots, n\} \) into \( n/k \) parts all of size \( k \). Suppose that \( k \geq 5 \). Let \( N \) be the socle of \( H \). Clearly, \( N \cong \text{Alt}(k)^{n/k} \) and \( N \) fixes a unique element of \( \mathcal{P} \). So, Lemmas 2.3 and 2.4 yield that \( T \) has at most 2 non-trivial pairwise coprime subdegrees on \( \Omega \). It remains to consider the case that \( k \in \{2, 3, 4\} \). Let \( N \) be the normal subgroup of \( H \) isomorphic to \( \text{Sym}(k)^{n/k} \cap T \). Clearly, \( N \) fixes a unique element of \( \mathcal{P} \). Furthermore, since \( N \) is a \( \{2, 3\} \)-group, we have \( \mu(N) \leq 2 \). Therefore Lemma 2.4 yields that \( T \) has at most 2 non-trivial coprime subdegrees.

It remains to consider the case that \( H \) is a primitive subgroup of \( T \) in the natural action of degree \( n \). Let \( N \) be the socle of \( H \). Suppose that \( N \cong S^\ell \) for some nonabelian simple group \( S \) and \( \ell \geq 1 \). Clearly, \( N = E(H) \) and \( C_H(N) = 1 \), and \( N_T(N) = H \) because \( H \) is pseudo-maximal in \( T \). In particular, from Proposition 2.7 we see that \( T \) has at most 2 non-trivial pairwise coprime subdegrees. Finally assume that \( N \) is an elementary abelian \( p \)-group. In the rest of the proof, we identify \( \text{AGL}(d, p) \) with its image under the natural affine permutation representation. So \( H = \text{AGL}(d, p) \cap T \) and hence \( H \) is isomorphic to a subgroup of index 2 in \( \text{AGL}(d, p) \) if \( p \) is odd, and \( H = \text{AGL}(d, p) \) if \( p = 2 \). (Note that the affine general linear group \( \text{AGL}(d, p) \) is a subgroup of \( \text{Alt}(p^d) \) only for \( p = 2 \).) If \( d = 1 \), then by Proposition 2.10 every subdegree of \( T \) is divisible by \( p \). Assume now that \( d > 1 \). Suppose that \( T \) has two coprime subdegrees \( n_1 = |\omega_1^H| \) and \( n_2 = |\omega_2^H| \). We show that either \( n_1 \) or \( n_2 \) is divisible by \( p \), from which it follows that \( T \) has at most 2 non-trivial coprime subdegrees. We argue by contradiction and we assume that \( n_1 \) and \( n_2 \) are not divisible by \( p \). In particular, each of \( H_{\omega_1} \) and \( H_{\omega_2} \) contains a Sylow \( p \)-subgroup of \( H \). Therefore, from \([25, \text{ Theorem 1}]\) we have that \( H_{\omega_1} = (N \rtimes P_1) \cap T \) and \( H_{\omega_2} = (N \rtimes P_2) \cap T \) with \( P_1 \) and \( P_2 \) maximal parabolic subgroups of \( \text{GL}(d, p) \), that is,

\[
P_i \cong \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in \text{GL}(d_i, p), \text{GL}(d - d_i, p), B \in \text{Mat}(d_i \times (d - d_i), p) \right\}
\]

where \( 1 \leq d_i \leq d - 1 \), for \( i = 1, 2 \). For each \( i \in \{1, 2\} \), we have \( N \leq H_{\omega_i} \), and so, from the modular law, we obtain \( H_{\omega_i} = N \rtimes (P_i \cap T) \). Therefore

\[
n_i = |H : H_{\omega_i}| = |\text{GL}(d, p) \cap T : P_i \cap T| = |\text{GL}(d, p) : P_i|,
\]

for \( i = 1, 2 \). Since \( n_1 \) and \( n_2 \) are coprime, \( \text{GL}(d, p) = P_1 P_2 \), leading to a factorization of \( \text{PGL}(d, p) \) by two maximal parabolics. No such factorization exists, see for example \( [18, \text{ Table 1}] \). \( \square \)

5. Classical groups

In this section we prove Theorem 1.1 when the simple group \( T \) is a classical group. We use Aschbacher’s theorem, which subdivides the maximal subgroups
of the almost simple groups with socle $T$ in nine classes $C_1, \ldots, C_8$ and $S$. In particular, in what follows we use the notation, the treatment and the terminology in [3] Chapter 3 and 4).

We start with a preliminary proposition which will prove to be helpful in the proof of the main result of this section. First we set some notation and some terminology.

**Notation 5.1.** Let $V$ be an $n$-dimensional vector space over a field $\mathbb{F}_q$ of size $q$ and let $V_1 \oplus \cdots \oplus V_t$ be a direct sum decomposition of $V$ into $t$ subspaces. Let $H$ be a subgroup $\text{GL}(V)$ leaving invariant each summand of this decomposition, that is, $V_i^h = V_i$ for all $h \in H$ and for all $i \in \{1, \ldots, t\}$. Let $H_i$ be the linear group induced by $H$ in its action on $V_i$. Note that $H_i$ centralizes $V_j$ (that is, $H_i$ acts trivially on $V_j$) for each $j \neq i$. We assume that, for each $i \in \{1, \ldots, t\}$, the subspace $V_i$ is an irreducible $H_i$-module. Fix $i$ and $j$ two distinct elements of $\{1, \ldots, t\}$. We suppose that for each $a_i \in H_i$, there exists $a_j \in H_j$ with $a_i a_j \in H$. (In particular, the element $a_i a_j$ of $H$ acts trivially on $V_k$, for each $k \neq i, j$.) Finally, we assume that for each $i$, the group $H_i$ contains an element fixing no non-zero vector of $V_i$.

**Proposition 5.2.** Let $V$ and $V_1, \ldots, V_t$ be as in Notation 5.1. If $t \geq 3$, then $V_1 \oplus \cdots \oplus V_t$ is the unique decomposition of $V$ as a direct sum of irreducible $H$-submodules of $V$.

**Proof.** Assume that $t \geq 3$. Let $U$ be an irreducible $H$-submodule of $V$. We show that $U = V_i$, for some $i \in \{1, \ldots, t\}$, from which the proof follows. Let $u$ be a non-zero element of $U$ and write $u = u_1 + \cdots + u_t$ with $u_i \in V_i$. Since $u \neq 0$, relabelling the direct summands, if necessary, we may assume that $u_1 \neq 0$. Using Notation 5.1, choose $a_1 \in H_1$ fixing no non-zero element of $V_1$. From Notation 5.1 we see that there exists $a_2 \in H_2$ with $a = a_1 a_2 \in H$. Now, as $a$ centralizes $u_3, \ldots, u_t$, we obtain $u - u^a = (u_1 - u_1^a) + (u_2 - u_2^a) \in U \cap (V_1 \oplus V_2)$. So, replacing $u$ by $u - u^a$ if necessary, we may assume that $u = u_1 + u_2 \in V_1 \oplus V_2$ and that $u_1 \neq 0$.

Since $t \geq 3$, from Notation 5.1 we see that there exists $a_3 \in H_1$ with $b = a_1 a_3 \in H$. Then $u - u^b = u_1 - u_1^a \in U \cap V_1$. So, replacing $u$ by $u - u^b$ if necessary, we may assume that $u \in V_1$. Since $H$ acts irreducibly on $U$, we obtain $U = \langle u^h \mid h \in H \rangle \leq V_1$. As $V_1$ is an irreducible $H$-module, we have $U = V_1$. □

We observe that Proposition 5.2 does not hold for $t = 2$. Consider, for instance, the group

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{F}_q \setminus \{0\} \right\}$$

| Group   | Soluble case | Isomorphisms                        |
|---------|--------------|-------------------------------------|
| $\text{PSL}_n(q)$ | $n = 1$ or $(n,q) = (2,2),(2,3)$ | $\text{PSL}_2(q) = \text{SL}_2(q)$ |
| $\text{PSp}_n(q)$ | $(n,q) = (2,2),(2,3)$ | $\text{PSL}_2(q) = \text{SL}_2(q)$ |
| $\text{PSU}_n(q)$ | $n = 1$ or $(n,q) = (2,2),(2,3),(3,2)$ | $\text{PSL}_2(q) = \text{SL}_2(q)$ |
| $\text{PO}_n(q)$ | $n = 1$ or $(n,q) = (3,3)$ | $\text{PSL}_2(q) = \text{SL}_2(q), \text{PS}^+_{3}(q) = \text{PSp}(q)$ |
| $\text{PO}^+_n(q)$ | $n = 2$ or $(n,q) = (4,2),(4,3)$ | $\text{PSL}_2(q) \times \text{PSL}_2(q)$ |
| $\text{PO}_n^-(q)$ | $n = 2$ | $\text{PSL}_2(q)$ |
| $\text{PO}_4(q)$ | | $\text{PSL}_4(q)$ |
| $\text{PO}_6(q)$ | | $\text{PSL}_6(q)^2$ |
| $\text{PSL}_6(q)$ | | $\text{PSL}_6(q)$ |

**Table 1.** Some information on simple classical groups

[...](...continued)
Proof. Let $\omega$ be a unique point of $\Omega$ (and 4). Proposition 5.4. \omega shows that for some $\varepsilon \in \{0, +, -\}$ respectively. Write $T = G/\mathbb{Z}(G)$ and assume that $T$ is a nonabelian simple group. Assume that $T$ is a transitive permutation group on $\Omega$ with pseudo-maximal point stabilizer $T_\omega$.

Let $A$ be an almost simple group with socle $T$ and $M$ be a maximal subgroup of $A$ with $T \not\subseteq M$ and with $T_\omega = M \cap T$. Suppose that $M$ lies in the Aschbacher class $C_2$, that is, $M$ is the stabilizer in $A$ of a direct sum decomposition $V_1 \oplus \cdots \oplus V_{n/m}$ of $V$. So, $M$ is of type $\text{GL}(m)(q)$ wr $\text{Sym}(n/m)$ if $T = \text{PSL}(n)(q)$, of type $\text{GU}(m)(q)$ wr $\text{Sym}(n/m)$ if $T = \text{PSU}(n)(q)$, of type $\text{Sp}(m)(q)$ wr $\text{Sym}(n/m)$ if $T = \text{PSp}(n)(q)$, and of type $\text{O}^+_m(q)$ wr $\text{Sym}(n/m)$ if $T = \text{PO}^+_m(q)$ (see [14] Chapter 3 and 4.2 for details and terminology).

Proposition 5.4. Let $T, \Omega$ and $M$ be as in Notation 5.3. If $n/m \geq 3$ and if $(T, M) \neq (\text{PSL}_n(2), \text{GL}_2(q) \wr \text{Sym}(n))$, $(\text{PO}^+_n(3), \text{O}^+_2(2) \wr \text{Sym}(n/2))$ or $(\text{PO}^+_n(3), \text{O}^+_2(3) \wr \text{Sym}(n/2))$, then the kernel of the $T_\omega$-action on the $V_i$ fixes a unique point of $\Omega$.

Proof. Let $\overline{H}$ be the subgroup of $G$ leaving invariant each direct summand $V_i$ of $V$, let $H$ be the projection of $\overline{H}$ in $T_\omega$ and, for each $i \in \{1, \ldots, n/m\}$, let $\overline{H}_i$ be the matrix group induced by $H$ in its action on $V_i$. In particular, $H$ is the kernel of the $T_\omega$-action on the $V_i$. Furthermore, we have $\overline{H}_i = \text{GL}(V_i), \text{GU}(V_i), \text{Sp}(V_i)$ and $\text{O}^2(V_i)$ respectively. Note that $H \trianglelefteq T_\omega$. From [14] Chapter 2, we see that $\overline{H}_i$ acts irreducibly on $V_i$, except when $\overline{H}_i \cong \text{GL}_1(q)$ and $q = 2$, or $\overline{H}_i \cong \text{O}^+_2(q)$ and $q = 2, 3$. Furthermore, for each distinct $i$ and $j$, and for each $a_i \in \overline{H}_i$, there exists $a_j \in \overline{H}_j$ with $a_i a_j \in \overline{H}_i$. Finally, for each $i$, if $\overline{H}_i \not\cong \text{GL}_2(1)$, we see with a direct inspection that $\overline{H}_i$ contains an element fixing no non-zero vector of $V_i$. This shows that for $\overline{H}_i \not\cong \text{GL}_2(1)$, $\text{O}^+_2(2)$ and $\text{O}^+_2(3)$ we are in the position to apply Proposition 5.2.

From Proposition 5.2 the group $\overline{H}$ fixes a unique direct sum decomposition of $V$ in $n/m$ vector spaces of dimension $m$. Assume that $H$ fixes $\omega'$ and write $\omega' = \omega^g$, for some $g \in T$. Let $\overline{g} \in G$ be an element projecting to $g$ in $T$. Now, $\overline{g}$ stabilizes the direct sum decomposition $V_1^{\overline{g}} \oplus \cdots \oplus V_{n/m}^{\overline{g}}$. From Proposition 5.2 we obtain that $\overline{g}$ stabilizes the direct sum decomposition $V_1 \oplus \cdots \oplus V_{n/m}$. So from the maximality of $M$ in $A$, we have that $g \in M \cap T = T_\omega$ and $\omega' = \omega$. \hfill \Box

Proof of Theorem A for the classical groups. By the results in Section 3 we may assume that $T$ is one of: $\text{PSL}_n(q)$ for $n \geq 2$ with $(n, q) \neq (4, 2)$ and, if $n = 2$, then $q \geq 7$ and $q \neq 9$; $\text{PSU}_n(q)$ with $n \geq 3$ and $(n, q) \neq (3, 2)$; $\text{PSp}_n(q)$ with $n \geq 4$ and $(n, q) \neq (4, 2)$; $\text{PO}^+_n(q)$ with $n \geq 7$ and $nq$ odd; or $\text{PO}^+_n(q)$ with $n \geq 8$ and $n$ even. Write $q = p^f$ for some prime $p$ and some $f \geq 1$. We assume that $T$ is transitive on $\Omega$ and that, for $\omega \in \Omega$, $T_\omega = T \cap M$, where $M$ is a maximal subgroup of some almost simple group $A$ with socle $T$, and $T \not\subseteq M$. of scalar matrices acting on the 2-dimensional vector space $F_q^2$. If $V_1 = (1, 0)F_q$ and $V_2 = (0, 1)F_q$, then $V = V_1 \oplus V_2$ is a direct decomposition that satisfies Notation 5.1 with $t = 2$ (here the group induced by $H$ on $V_i$ is the multiplicative group of the field $F_q$ acting by multiplication). Clearly, every pair of 1-dimensional subspaces of $V$ forms an $H$-invariant decomposition, and hence Proposition 5.2 does not hold for $t = 2$.

Notation 5.3. Let $V$ be an $n$-dimensional vector space over a field $F_q$ of size $q$. We let $G$ be a subgroup of $\text{GL}(V)$ and we suppose that $G = \text{SL}(V)$, or that $V$ is endowed with a non-degenerate Hermitian, symplectic or quadratic form and $G = \text{SU}(V), \text{Sp}(V) or \Omega^\varepsilon(V)$ (with $\varepsilon \in \{0, +, -\}$) respectively. Write $T = G/\mathbb{Z}(G)$ and assume that $T$ is a nonabelian simple group. Assume that $T$ is a transitive permutation group on $\Omega$ with pseudo-maximal point stabilizer $T_\omega$.
In order to avoid a long list of exceptions in some general arguments that we use later in the proof, we first deal with the case \( T = \text{PSL}_2(q) \) and we use Dickson’s classification of the subgroup lattice of \( T \) (see [26, Section 3.6, Theorem 6.25, 6.26]).

As above \( q \geq 7 \) and \( q \neq 9 \). If \( T_\omega \cong \text{Sym}(3), \text{Alt}(4), \text{Sym}(4) \) or \( \text{Alt}(5) \) (that is, \( T_\omega \) is as in [26, Theorem 6.25 (c)]), then by direct inspection we see that \( \mu(T_\omega) \leq 2 \) and the result follows from Lemma 2.4 (applied with \( N = T_\omega \)). Assume that \( M \) contains the stabilizer of a subfield of \( \mathbb{F}_q \), that is, \( T_\omega = M \cap T \cong \text{PSL}(2, r) \) or \( \text{PGL}(2, r) \) for \( r = p^s \) with \( s \) dividing \( f \) (that is, \( T_\omega \) as in [26, Theorem 6.25 (d)]). If \( r \neq 2 \) or 3, then from Proposition 2.7, each set of pairwise coprime non-trivial subdegrees of \( T \) has size at most two. If \( r = 2 \) or 3, then we have already dealt with these cases as \( \text{PSL}_2(2) = \text{PGL}_2(2) \cong \text{Sym}(3), \text{PSL}_3(3) \cong \text{Alt}(4) \) and \( \text{PGL}_3(3) = \text{Sym}(4) \).

Assume that \( M \) contains a parabolic subgroup, that is, \( T_\omega \) is a Borel subgroup of \( T \) (here \( T_\omega \) is as in [26, Theorem 6.25 (a)]). In particular, the action of \( T \) on \( \Omega \) is permutation equivalent to the action of \( T \) on the projective line. Therefore \( T \) is 2-transitive and has only one non-trivial subdegree, namely \( q \).

Assume that \( M \) contains the normalizer of a maximal torus of \( T \), that is, \( T_\omega \) is a dihedral group of order \( 2(q \pm 1)/\gcd(2, q - 1) \) (here \( T_\omega \) is as in [26, Theorem 6.25 (b)]). If \( T_\omega \) is a 2-group, then every non-trivial subdegree of \( T \) is even. Suppose that \( T_\omega \) is not a 2-group and let \( r \) be a prime with \( r \mid |T_\omega| \) and \( r \neq 2 \). Let \( R \) be a Sylow \( r \)-subgroup of \( T_\omega \). From the description of the subgroup lattice of \( T \) in [26, Theorem 6.25, 6.26], we see that \( R \) is a Sylow \( r \)-subgroup of \( T \) and \( N_T(R) \leq T_\omega \). In particular, from Proposition 2.10, every non-trivial subdegree of \( T \) is divisible by \( r \). This concludes the analysis for \( \text{PSL}_2(q) \).

Now, to avoid a few more small exceptions in the general arguments below, we consider separately the cases where \( T = \text{PSL}_3(3), \text{PSL}_3(4), \text{PSL}_4(3), \text{PSU}_3(3), \text{PSU}_4(3) \) and \( \text{PSp}_4(3) \). In each of these groups, we see with a direct inspection with \textit{magma} or with [8] that the theorem holds true. Finally, for the remaining cases we use Aschbacher’s theorem and in particular we use extensively Tables 3.5A–F in [14].

**Case** \( M \in C_1 \): \( M \) is the stabilizer of totally singular or non-singular subspaces.

We first consider the case that \( M \) is of type \( P_m \), that is, \( M \) is a maximal parabolic subgroup of \( A \). In particular, \( M \) and, hence also \( T_\omega \), contain the normalizer of a Sylow \( p \)-subgroup of \( T \). It follows from Proposition 2.10 that every non-trivial subdegree of \( T \) is divisible by \( p \).

Now suppose that \( M \) is of type \( \text{GL}_m(q) \oplus \text{GL}_{n-m}(q) \) if \( T = \text{PSL}_n(q) \), of type \( \text{GU}_m(q) \perp \text{GU}_{n-m}(q) \) if \( T = \text{PSU}_n(q) \), of type \( \text{Sp}_m(q) \perp \text{Sp}_{n-m}(q) \) if \( T = \text{PSp}_n(q) \), of type \( \text{O}_m(q) \perp \text{O}_{n-m}(q) \) if \( T = \text{PO}_m(q) \), of type \( \text{O}^\varepsilon_m(q) \perp \text{O}^\varepsilon_{n-m}(q) \) if \( T = \text{PO}^\varepsilon_m(q) \), and of type \( \text{O}^{e_m}(q) \perp \text{O}^{e_m}_{n-m}(q) \) if \( T = \text{PO}^{e_m}(q) \). Note that, from [14, Table 3.5A–F], we take \( m < n - m \) (except for \( T = \text{PO}_m(q) \) and possibly for \( T = \text{PO}^{e_m}(q) \)). Moreover, if \( T = \text{PO}_m(q) \) and \( n = 2m \), then \( m \) is even and \( M \) is of type \( \text{O}^{e_m}_m(q) \perp \text{O}^{e_m}_m(q) \) with \( \text{PO}^{e_m}_m(q) \neq \text{PO}^{e_m}_m(q) \) (see [14, Proposition 4.1.6]). With a direct inspection in each of these cases and using Table 11 we see that either \( 1 \): each simple direct factor of \( E(T_\omega)/\mathbb{Z}(E(T_\omega)) \) has multiplicity at most two and there exists a unique factor having size strictly bigger than the others, or \( 2 \): \( E(T_\omega)/\mathbb{Z}(E(T_\omega)) \) is the direct product of pairwise isomorphic simple groups, or \( 3 \): \( T_\omega \) is soluble. Indeed, \( 3 \) arises if and only if \( T = \text{PSL}_n(q) \) and \( (n, m, q) = (3, 1, 2), (3, 1, 3) \), or \( T = \text{PSU}_n(q) \) and \( (n, m, q) = (3, 1, 3), (4, 1, 2) \), or \( T = \text{PO}_n(q) \) and \( (n, m, q, \varepsilon) = (7, 3, 3, +) \).
each of these cases, we see from [14, Proposition 4.1.4, 4.1.6] that $T_\omega$ is a $(2,3)$-group. So $\mu(T_\omega) \leq 2$ and the result follows from Lemma 2.4. Moreover, if (i) or (ii) holds, then from [14, Proposition 4.1.3–4, 4.1.6] $\mathcal{C}_{T_\omega}(E(T_\omega))$ is soluble and hence the theorem follows from Proposition 2.9 or 2.7 respectively.

Now suppose that $T = \text{PSL}_n(q)$ and that $M$ is of type $P_{m,n-m}$. From [14, Proposition 4.1.22], we see that $M$ contains a parabolic subgroup (not necessarily maximal) of $T$. Therefore $T_\omega$ contains a Borel subgroup of $T$ and so, $T_\omega$ contains the normalizer of a Sylow $p$-subgroup of $T$. Now from Proposition 2.10 every non-trivial subdegree of $T$ is divisible by $p$.

It remains to consider the case that $T = \text{PO}_n^\pm(q)$ and $M$ is of type $\text{Sp}_{n/2}(q)$ with $q$ even. From [14, Proposition 4.1.7], we see that $\mathcal{C}_{T_\omega}(E(T_\omega))$ is soluble and that $E(T_\omega)/Z(E(T_\omega)) \cong \text{PSp}_{n/2}(q)$ is simple. Therefore each set of pairwise coprime non-trivial subdegrees of $T$ has size at most two, by Proposition 2.7.

**Case** $M \in C_2$: $M$ is the stabilizer of a direct sum decomposition.

We first consider the case that $M$ is of type $\text{GL}_{n/2}(q^2).2$ if $T = \text{PSU}_n(q)$, of type $\text{GL}_{n/2}(q).2$ or $\text{O}_{n/2}(q)^2$ (with $n/2 \geq 5$ odd) if $T = \text{PO}_n^+(q)$, and of type $\text{O}_{n/2}(q)^2$ (with $n/2 \geq 5$ odd) if $T = \text{PO}_n^-(q)$. From [14, Proposition 4.2.4–5, 4.2.7, 4.2.16], we see that $\mathcal{C}_{T_\omega}(E(T_\omega))$ is soluble, and that either $(E(T_\omega)/Z(E(T_\omega))) \cong S^T$ for some nonabelian simple group $S$ (here $\ell = 1$ or 2) or $T_\omega$ is soluble. In the former case, from Proposition 2.7 each set of pairwise coprime non-trivial subdegrees of $T$ has size at most two. The latter case occurs only for $T = \text{PSp}_{14}(3)$, which we excluded from this analysis.

In the rest of the proof of this case we use the detailed information on the Sylow normalizers of the Lie type groups in [22, Section 5]. Given a connected reductive algebraic group $G$ defined over a finite field $\mathbb{F}_q$ and $F: G \to G$ the corresponding Frobenius endomorphism, we adopt the terminology in [22] for the Sylow $\Phi_r$-tori of $G$ and we refer to as Sylow $\Phi_r(q)$-tori their subgroups of fixed points (under $F$) in the finite Lie type group $G = G^F$. Furthermore, we deal with each family of classical groups separately. In fact, although the arguments are very similar in every case, there are some slight differences that can be presented neatly only by dealing with one family at a time.

**The groups** $T = \text{PSL}_n(q)$. Assume that $M$ is of type $\text{GL}_m(q)$ wr Sym$(n/m)$ with $m \geq 1$. Let $\mathbb{F}_q^n = V_1 \oplus \cdots \oplus V_{n/m}$ be the direct sum decomposition preserved by $T_\omega$ and let $H$ be the normal subgroup of $T_\omega$ fixing every direct summand $V_i$, for $i \in \{1, \ldots, n/m\}$. If $m \geq 3$, or $m = 2$ and $q \geq 4$, we see from [14, Proposition 4.2.9] that $\mathcal{C}_T(T_\omega)$ is soluble and that $E(T_\omega)/Z(E(T_\omega))$ is isomorphic to a direct product of $m \geq 5$ pairwise isomorphic nonabelian simple groups. So from Proposition 2.7 each set of pairwise coprime non-trivial subdegrees of $T$ has size at most two. This leaves the cases $m = 1$, and $(m,q) = (2,2)$ and $(2,3)$.

Assume next that $m = 1$. From the structure and of the order of $T_\omega$ we see that $T_\omega$ is the normalizer of a Sylow $\Phi_1(q)$-torus $S_1$ of $T$, that is, $T_\omega = \mathcal{N}_T(S_1)$. Recall that $n \geq 3$. Let $r$ be the largest prime dividing $q - 1$. Now, if $q > 3$, or if $r = 2$ and $q \equiv 1 \mod 4$, then from [22, Theorems 5.14 and 5.19] we obtain that $\mathcal{N}_T(S_1)$ contains the normalizer of a Sylow $r$-subgroup of $T$. In this case every non-trivial subdegree of $T$ is divisible by $r$ by Proposition 2.10. It remains to consider the case that either $q = 2$, or 2 and 3 are the only primes dividing $q - 1$. Assume that $q = 2$. If $n \leq 4$, then $T_\omega$ is a $(2,3)$-group, so $\mu(T_\omega) \leq 2$ and the result follows from Lemma 2.4. Suppose that $n \geq 5$ and let $N$ be the last term of the derived
series of \( T_\omega \). From Lemma 3.2 the group \( N \) fixes a unique point of \( \Omega \), and the result follows from Lemmas 2.3 and 2.4. So, we may now assume that \( q \neq 2 \). If 3 divides \( q - 1 \), then from [22] Theorems 5.14] we obtain that either \( \mathbb{N}_{T}(S_1) \) contains the normalizer of a Sylow 3-subgroup of \( T \) or \( n = 3 \). In the former case, every non-trivial subdegree of \( T \) is divisible by 3 from Proposition 2.10. In the latter case, as \( q - 1 \) is only divisible by the primes 2 and 3, we have that \( T_\omega \) is a \( \{2, 3\} \)-group and the result follows from Lemma 2.4. Therefore, it remains to deal with the case that 2 is the only prime dividing \( q - 1 \) and \( q \equiv 3 \mod 4 \), that is, \( q = 3 \). We do this in the following paragraph.

Assume \((m, q) = (1, 3), (2, 2) \) or \((2, 3) \). If \( n/m \leq 4 \), then \( T_\omega \) is a \( \{2, 3\} \)-group and the result follows from Lemma 2.4. Suppose that \( n/m \geq 5 \). From Proposition 5.4 the kernel \( H \) of the \( T_\omega \)-action on the direct summands \( V_i \) of \( V \) fixes a unique point of \( \Omega \). In each case \( H \) is a \( \{2, 3\} \)-group and hence \( \mu(H) \leq 2 \) and the result follows from Lemma 2.4. The analysis for the remaining classical groups is similar.

The groups \( T = \text{PSU}_n(q) \). Assume that \( M \) is of type \( \text{GU}_m(q) \wr \text{Sym}(n/m) \) with \( m \geq 1 \). Let \( \mathbb{F}_q^n = V_1 \oplus \cdots \oplus V_{n/m} \) be the direct sum decomposition preserved by \( T_\omega \) and let \( H \) be the normal subgroup of \( T_\omega \) fixing every direct summand \( V_i \), for \( i \in \{1, \ldots, n/m\} \). If \( m \geq 4 \), or if \( m = 3 \) and \( q \geq 3 \), or if \( m = 2 \) and \( q \geq 4 \), we see from [14] Proposition 4.2.9] that \( C_T(\text{E}(T_\omega)) \) is soluble and that \( \text{E}(T_\omega)/\text{Z}(\text{E}(T_\omega)) \) is isomorphic to the direct product of pairwise isomorphic nonabelian simple groups. So from Proposition 2.7 \( T \) has at most 2 non-trivial coprime subdegrees. We now consider the remaining cases, namely, \( m = 1 \) and \((m, q) = (2, 2), (2, 3) \) and \((3, 2) \).

Assume that \( m = 1 \). Now the order of \( \text{GU}_1(q) \) is divisible by \( q+1 \) and so \( T_\omega \) is the normalizer of a Sylow \( \Phi_2(q) \)-torus \( S_2 \) of \( T \). Set \( r = 2 \) if \( q \equiv 3 \mod 4 \), or choose the largest prime \( r > 2 \) dividing \( q+1 \) if \( q \equiv 3 \mod 4 \). From [22] Theorem 5.14, 5.19], we have that either \( T_\omega \) contains the normalizer of a Sylow \( r \)-subgroup of \( T \) (and hence every non-trivial subdegree of \( T \) is divisible by \( r \) from Proposition 2.10] or \( n = 3 \) and \( r = 3 \). In the latter case, by our choice of \( r \), the only primes dividing \( q+1 \) are 2 and 3. Since \( n = 3 \), we obtain that \( T_\omega \) is a \( \{2, 3\} \)-group and by Lemma 2.4 \( T \) has at most 2 non-trivial coprime subdegrees.

Assume that \((m, q) = (2, 2), (2, 3) \) or \((3, 2) \). If \( n/m \leq 4 \), then \( T_\omega \) is a \( \{2, 3\} \)-group and the result follows from Lemma 2.4. Suppose that \( n/m \geq 5 \). From Proposition 5.4 the group \( H \) fixes a unique point of \( \Omega \). As \( H \) is a \( \{2, 3\} \)-group, we obtain \( \mu(H) \leq 2 \) and the result follows from Lemma 2.4.

The groups \( T = \text{PSp}_n(q) \). Assume that \( M \) is of type \( \text{Sp}_m(q) \wr \text{Sym}(n/m) \) with \( m \geq 2 \) even. Let \( \mathbb{F}_q^n = V_1 \oplus \cdots \oplus V_{n/m} \) be the direct sum decomposition preserved by \( T_\omega \) and let \( H \) be the normal subgroup of \( T_\omega \) fixing every direct summand \( V_i \), for \( i \in \{1, \ldots, n/m\} \). If \( m \geq 4 \), or if \( m = 2 \) and \( q \geq 4 \), we see from [14] Proposition 4.2.10] that \( C_T(\text{E}(T_\omega)) \) is soluble and that \( \text{E}(T_\omega)/\text{Z}(\text{E}(T_\omega)) \) is isomorphic to a direct product of pairwise isomorphic nonabelian simple groups. So from Proposition 2.7 \( T \) has at most 2 non-trivial coprime subdegrees. We now consider the remaining cases.

Assume that \((m, q) = (2, 2) \) or \((2, 3) \). If \( n/m \leq 4 \), then \( T_\omega \) is a \( \{2, 3\} \)-group and hence the result follows from Lemma 2.4. Suppose that \( n/m \geq 5 \). From Proposition 5.4 the group \( H \) fixes a unique point of \( \Omega \). As \( H \) is a \( \{2, 3\} \)-group, we obtain \( \mu(H) \leq 2 \) and the result follows from Lemma 2.4.

The groups \( T = \text{PO}_n(q) \) (\( n \) odd). Assume that \( M \) is of type \( \text{O}_m(q) \wr \text{Sym}(n/m) \) with \( m \geq 1 \) (where \( q = p \geq 3 \) if \( m = 1 \)). Let \( \mathbb{F}_q^n = V_1 \oplus \cdots \oplus V_{n/m} \) be the direct sum decomposition preserved by \( T_\omega \) and let \( H \) be the normal subgroup of \( T_\omega \) fixing every direct summand \( V_i \), for \( i \in \{1, \ldots, n/m\} \). If \( m \geq 4 \), or if \( m = 2 \) and \( q \geq 4 \), we see from [14] Proposition 4.2.10] that \( C_T(\text{E}(T_\omega)) \) is soluble and that \( \text{E}(T_\omega)/\text{Z}(\text{E}(T_\omega)) \) is isomorphic to a direct product of pairwise isomorphic nonabelian simple groups. So from Proposition 2.7 \( T \) has at most 2 non-trivial coprime subdegrees. We now consider the remaining cases.

Assume that \((m, q) = (2, 2) \) or \((2, 3) \). If \( n/m \leq 4 \), then \( T_\omega \) is a \( \{2, 3\} \)-group and hence the result follows from Lemma 2.4. Suppose that \( n/m \geq 5 \). From Proposition 5.4 the group \( H \) fixes a unique point of \( \Omega \). As \( H \) is a \( \{2, 3\} \)-group, we obtain \( \mu(H) \leq 2 \) and the result follows from Lemma 2.4.
decomposition preserved by $T_\omega$ and let $H$ be the normal subgroup of $T_\omega$ fixing every direct summand $V_i$, for $i \in \{1, \ldots, n/m\}$. If $m \geq 5$, or if $m = 3$ and $q \neq 3$, we see from [14, Proposition 4.2.12] that $C_T(E(T_\omega))$ is soluble and that $E(T_\omega)/Z(E(T_\omega))$ is isomorphic to a direct product of pairwise isomorphic nonabelian simple groups. So from Proposition 2.7, $T$ has at most 2 non-trivial coprime subdegrees. We now consider the remaining cases.

Assume $m = 1$ or $(m, q) = (3, 3)$. Note that $O_1(q)$ has order 2 and is generated by $-1$. If $n/m \leq 4$, then $T_\omega$ is a $\{2, 3\}$-group and hence the result follows from Lemma 2.3. Suppose that $n/m \geq 5$. From Proposition 6.4, the group $H$ fixes a unique point of $\Omega$. As $H$ is a $\{2, 3\}$-group, we obtain $\mu(H) \leq 2$ and the result follows from Lemma 2.3.

**The Groups** $T = PO_n^+(q)$ ($n$ even). Assume that $M$ is of type $O_m^\varepsilon(q) \wr \text{Sym}(n/m)$ with $\varepsilon \in \{\circ, +, -\}$ (where $e^{n/m} = +$ if $m$ is even) and with $q = p \geq 3$ if $m = 1$. Let $F_q = V_1 \oplus \cdots \oplus V_{n/m}$ be the direct sum decomposition preserved by $T_\omega$ and let $H$ be the normal subgroup of $T_\omega$ fixing every direct summand $V_i$, for $i \in \{1, \ldots, n/m\}$. If $m \geq 5$, or if $m = 4$ and $q \geq 4$, or if $m = 4$ and $\varepsilon = -$, or if $m = 3$ and $q \neq 3$, we see from [14, Proposition 4.2.11, 4.2.14] that $C_T(E(T_\omega))$ is soluble and that $E(T_\omega)/Z(E(T_\omega))$ is isomorphic to a direct product of pairwise isomorphic nonabelian simple groups. So from Proposition 2.7, $T$ has at most 2 non-trivial coprime subdegrees. We now consider the remaining cases.

Assume $m = 1$, or $(m, q, \varepsilon) = (3, 3, \circ)$, $(4, 2, +)$ or $(4, 3, +)$ (recall from [14, Table 4.2A] that if $m = 1$ then $q = p \geq 3$). In each of these cases, $H$ is a $\{2, 3\}$-group. If $n/m \leq 4$, then $T_\omega$ is a $\{2, 3\}$-group and hence the result follows from Lemma 2.3. Suppose that $n/m \geq 5$. From Proposition 6.4, the group $H$ fixes a unique point of $\Omega$. $\mu(H) \leq 2$ and the result follows from Lemma 2.3.

Assume $m = 2$. Note that if $\varepsilon = -$, then $n/2$ is even because $e^{n/2} = +$. Now, $O_2^\circ(q)$ is a dihedral group of order $2(q - 1)$, and $O_2^\circ(q)$ is a dihedral group of order $2(q + 1)$. The largest power of the polynomial $x - 1$ dividing the generic order of $PO_n^\circ(q)$ is $n/2$. Similarly, if $n/2$ is even, the largest power of the polynomial $x + 1$ dividing the generic order of $PO_n^\circ(q)$ is $n/2$. Therefore, considering the structure of $T_\omega$ and its order, we obtain that $T_\omega$ is the normalizer of a $\Phi_1(q)$-torus $S_1$ of $T$ if $\varepsilon = +$ and is the normalizer of a $\Phi_2(q)$-torus $S_2$ of $T$ if $\varepsilon = -$. Assume first that $\varepsilon = -$. Set $r = 2$ if $q \equiv 3 \mod 4$, or choose a prime $r$ dividing $q + 1$ and coprime to $q - 1$ if $q \neq 3 \mod 4$. From [22, Theorem 5.14, 5.19], $T_\omega$ contains the normalizer of a Sylow $r$-subgroup of $T$. In this case every non-trivial subdegree of $T$ is divisible by $r$ by Proposition 2.10. Assume now that $\varepsilon = +$. Then each $V_i$ is a hyperbolic plane for its stabilizer $M_i \cong O_2^\circ(q)$ in $M$. As any hyperbolic plane contains exactly two isotropic lines, then $M$ is the stabilizer in $A$ of a decomposition of $V$ in 1-dimensional spaces. So we are back to the case $m = 1$, which has already been considered.

**The Groups** $T = O_n^\varepsilon(q)$ ($n$ even). Assume that $M$ is of type $O_m^\varepsilon(q) \wr \text{Sym}(n/m)$ with $\varepsilon \in \{\circ, -\}$ and with $q = p \geq 3$ if $m = 1$. Let $F_q = V_1 \oplus \cdots \oplus V_{n/m}$ be the direct sum decomposition preserved by $T_\omega$ and let $H$ be the normal subgroup of $T_\omega$ fixing every direct summand $V_i$, for $i \in \{1, \ldots, n/m\}$. If $m \geq 4$, or if $m = 3$ and $q \neq 3$, we see from [14, Proposition 4.2.11, 4.2.14] that $C_T(E(T_\omega))$ is soluble and that $E(T_\omega)/Z(E(T_\omega))$ is isomorphic to a direct product of pairwise isomorphic nonabelian simple groups. So from Proposition 2.7, $T$ has at most 2 non-trivial coprime subdegrees.
Assume that \( m = 1 \) or \( (m, q) = (3, 3) \) (recall that if \( m = 1 \) then \( q = p \geq 3 \)). In each of these cases, \( H \) is a \( \{2, 3\} \)-group. If \( n/m \leq 4 \), then \( T_\omega \) is a \( \{2, 3\} \)-group and hence the result follows from Lemma 2.4. Suppose that \( n/m \geq 5 \). From Proposition 5.4, the group \( H \) fixes a unique point of \( \Omega \), \( \mu(H) \leq 2 \) and the result follows from Lemma 2.4.

Assume that \( m = 2 \). Note that from [14 Table 3.5F], \( n/2 \) is odd. Now, \( O_2^n(q) \) has order divisible by \( q + 1 \). Since \( n/2 \) is odd, the largest power of the polynomial \( x + 1 \) dividing the generic order of \( P\Omega_2^n(q) \) is \( n/2 \). Therefore, considering the structure of \( T_\omega \) and its order, we obtain that \( T_\omega \) is the normalizer of a \( \Phi_2(q) \)-torus of \( T \). Set \( r = 2 \) if \( q \equiv 3 \mod 4 \), or choose a prime \( r \) dividing \( q + 1 \) and coprime to \( q - 1 \) if \( q \not\equiv 3 \mod 4 \). From Theorem 5.14, 5.19, \( T_\omega \) contains the normalizer of a Sylow \( r \)-subgroup of \( T \). So every non-trivial subdegree of \( T \) is divisible by \( r \) from Proposition 2.4.

Case \( M \in C_3: M \) is the stabilizer of a structure on \( V \) as an \( n/r \)-dimensional space over an extension field of \( F \) of prime index \( r \).

From [14 Tables 3.5A–F], we see that \( M \) is of type \( GL_m(q^r) \) if \( T = PSL_n(q) \), of type \( GU_m(q^r) \) if \( T = PSU_n(q) \), of type \( Sp_m(q^r) \) or \( GU_{n/2}(q) \) (with \( q \) odd) if \( T = PSp_n(q) \), of type \( O_{n/r}(q^r) \) (with \( n/r \geq 3 \)) if \( T = P\Omega_{n/2}(q) \), of type \( GU_{n/2}(q) \), \( O_{n/r}(q^r) \) (with \( n/r, \geq 4 \)), or \( O_{n/2}(q^r) \) (with \( n/2 \) odd) if \( T = P\Omega_n(q) \), and of type \( GU_n(q) \), \( O_n(q) \) (with \( n/r \geq 4 \)), or \( O_{n/2}(q^2) \) (with \( n/2 \) odd) if \( T = P\Omega_n(q) \).

From [14 Section 4.3], the group \( C_{T_\omega}(E(T_\omega)) \) is soluble. Furthermore, in each of the cases, considering the restrictions on \( n, q \) and \( r \) that we have given above, we see from Table 1 that either \( T_\omega \) is soluble or \( E(T_\omega)/Z(E(T_\omega)) \cong S^\ell \) for some nonabelian simple group \( S \) (where \( \ell = 1 \), or \( \ell = 2 \) if \( T = P\Omega_n(q) \) and \( M \) is of type \( O_3^3(q^r) \)). In the latter case, the theorem follows from Proposition 2.4.

Assume now that \( T_\omega \) is soluble. Since we are excluding \( T = PSp_4(3) \), with a direct inspection we see that \( T = PSL_r(q) \) or \( PSU_r(q) \), and in particular that \( r \geq 3 \).

From [14 Proposition 4.3.6], the group \( T_\omega \) is isomorphic to \( Z_a \times Z_r \) with \( a = (q^r - \varepsilon)/(q - \varepsilon) \gcd(q - \varepsilon, r) \) (here \( \varepsilon = 1 \) if \( T = PSL_r(q) \) and \( \varepsilon = -1 \) if \( T = PSU_r(q) \)). In particular, \( T_\omega \) is the normalizer of a \( \Phi_1(q) \)-torus of \( T \) if \( T = PSL_r(q) \) and is the normalizer of a \( \Phi_2(q) \)-torus of \( T \) if \( T = PSU_r(q) \). From Zsigmondy’s theorem, we see that there exists a prime \( s \) dividing \( q^r - \varepsilon \) and coprime to \( q^r - \varepsilon \) for every \( i \in \{1, \ldots, r - 1\} \) (note that \( r \geq 3 \) is prime and that we are excluding \( PU_3(2) \) since it is soluble). Clearly, \( s \geq 3 \). Moreover if \( 3 \) divides \( q^r - \varepsilon \), then \( q - \varepsilon \equiv 0 \) (mod 3) if \( r = -1 \), and \( q^r - \varepsilon \equiv 0 \) (mod 3) if \( r = +1 \); since \( r \geq 3 \), this implies that \( s \neq 3 \). Thus \( s \geq 3 \).

From Theorem 5.14, we obtain that \( T_\omega \) contains the normalizer of a Sylow \( s \)-subgroup of \( T \), and hence every non-trivial subdegree of \( T \) is divisible by \( r \) by Proposition 2.4.

Case \( M \in C_4: M \) is the stabilizer of a tensor product decomposition.

From [14 Section 3.5], we get that \( M \) is of type \( GL_m(q) \otimes GL_{n/m}(q) \) if \( T = PSL_n(q) \) (with \( n \neq m^2 \)), of type \( GU_m(q) \otimes GU_{n/m}(q) \) if \( T = PSU_n(q) \) (with \( n \neq m^2 \)), of type \( Sp_m(q) \otimes O_{n/m}^+(q) \) if \( T = PSp_n(q) \), of type \( O_m(q) \otimes O_{n/m}(q) \) (with \( n \neq m^2 \)) if \( T = P\Omega_{n/2}(q) \), of type \( Sp_m(q) \otimes Sp_{n/m}(q) \) (with \( n \neq m^2 \)) or \( O_{n/m}^+(q) \otimes O_{n/m}^+(q) \) if \( T = P\Omega_n(q) \), and of type \( O_{n/m}(q) \otimes O_{n/m}^+(q) \) if \( T = P\Omega_n(q) \). Note that if \( T = PSp_n(q) \), then \( q \) is odd (see [14 Table 3.5C]). With a direct inspection (using [14 Proposition 4.4.10–12, 4.4.14, 4.4.17, 4.4.18]) we see that \( C_{T_\omega}(E(T_\omega)) \) is soluble. We claim that either \((i) : T_\omega \) is soluble, or \((ii) : E(T_\omega)/Z(E(T_\omega)) \) is
a direct product of (at least one) isomorphic simple groups, or \((iii)\) : each simple direct factor of \(E(T_n)/Z(E(T_n))\) has multiplicity one, or \((iv)\) : if \(T = PSU_4(q)\) and \(M\) is of type \(PSp_m(q) \otimes SL_2(q) \otimes SL_2(q)\), or \(T = PO^-_{4m}(q)\) and \(M\) is of type \(O^+_{m}(q) \otimes SL_2(q) \otimes SL_2(q)\).

As usual, \((i)\) occurs only for small values of \(n\) and \(q\). (Recall that \(q\) is odd if \(T = PSp_n(q)\), and \(m,n/m \geq 4\) if \(E(T_n)/Z(E(T_n))\) of type \(PSp_m(q) \otimes SL_2(q) \otimes SL_2(q)\) for \(n = 6, q = 2\) and \(T = PSp_n(q)\), when \(n = 6, q = 3\) and \(T = PSp_n(q)\) (here \(M\) is of type \(Sp_2(3) \otimes O_5(3)\) or \(Sp_2(3) \otimes O^+_5(3)\)), and when \(n = 12, q = 3\) and \(T = PO^-_{4m}(q)\) (here \(M\) is of type \(O_3(3) \otimes O^-_7(3)\)). In each of these cases, from \([14], Proposition 4.4.10, 4.4.14, 4.4.17\] we see that \(M\) is a \(\{2,3\}\)-group. Then \(\mu(T_n) \leq 2\) and the theorem follows from Lemma 2.4.

Now we consider \((ii)\). Again this occurs in a small list of cases, typically when one of the two central factors of the type of \(M\) is soluble. Namely, \((ii)\) arises for \(T = PSL_n(q)\) when \(m = 2, q = 2; 3\); for \(T = PSU_n(q)\) when \((m, q) = (2, 2), (2, 3), (3, 2); \) for \(T = PSp_n(q)\) when \(m = 2\) and \(q = 3\), or \(n = 3m\) and \(q = 3\), or \(n = 4m\) and \(q = 3\), or \(n = 20\) (here \(M\) is of type \(Sp_2(3) \otimes O_5(3)\)), or \(n = 6\) (here \(M\) is of type \(Sp_2(3) \otimes O_5(3)\)), or \(n = 8\) (here \(M\) is of type \(Sp_2(3) \otimes O^+_5(3)\) for \(T = PSp_n(q)\) when \(m = 3, q = 3\) (recall that \(n \neq m^2\)); for \(T = PO^-_{4m}(q)\) when \(m = 2 \) and \(q = 2, 3\), or \(m = 3 \) and \(q = 3\) (here \(M\) is of type \(O_3(3) \otimes O^-_{n/3}(q)\)), or \(m = 4 \) and \(q = 3\) (here \(M\) is of type \(O_4(3) \otimes O^-_{n/4}(q)\)); for \(T = PO^-_{4m}(q)\) when \(m = 3\) and \(q = 3\). In each of these cases, from Proposition 2.9 the group \(T\) has at most two non-trivial coprime subdegrees.

Finally, with a direct inspection on the type of \(M\), we see that if \((i), (ii)\) and \((iii)\) do not hold, then both central factors of \(M\) are insoluble and one of the two is the central product of smaller quasisimple groups. From Table 1 this happens only when \(O^+_7(q)\) is one of the central factors of \(M\). Now from [14], Table 3.5A – F, we see that either \(T = PSp_{4m}(q)\) or \(T = PO^+_{4m}(q)\) and our claim is proved. In \((iv)\) we may use Proposition 2.9 to conclude that \(T\) has at most two non-trivial coprime subdegrees.

**Case** \(M \in C_3\): \(M\) is the stabilizer of a subfield of \(F_q\) of prime index \(r\).

From [14] Section 3.5, \(M\) is of type \(GL_{n}(q^{1/r})\) if \(T = PSL_n(q)\), of type \(GU_n(q^{1/r})\), \(O^+_n(q)\) or \(Sp_n(q)\) if \(T = PSU_n(q)\), of type \(Sp_n(q^{1/r})\) if \(T = PSp_n(q)\), of type \(O_n(q^{1/r})\) if \(T = PSp_n(q)\), of type \(O^+_n(q^{1/r})\) or \(O^-_n(q^{-1})\) if \(T = PO^-_{4m}(q)\), and of type \(O^-_n(q^{1/r})\) if \(T = PO^-_{4m}(q)\). Since we are excluding the cases \(T = PSU_3(3)\) and \(PSU_4(3)\), the group \(E(T_n)/Z(E(T_n))\) is either simple, or a direct product of two isomorphic simple groups (which occurs when \(T = PSU_4(q)\) and \(M\) is of type \(O^+_7(q)\)), or \(T = PSU_3(2^r)\). In the third case, we see from [14] Proposition 4.5.3 (II), that \(M\) is a \(\{2,3\}\)-group, and then \(\mu(T_n) \leq 2\) and the result follows from Lemma 2.4.

In the remaining cases, from [14] Proposition 4.5.3–6, 4.5.8, 4.5.10, we see that \(C_{T_n}(E(T_n))\) is soluble and so, from Proposition 2.7 \(T\) has at most two non-trivial coprime subdegrees.

**Case** \(M \in C_6\): \(M\) is the normalizer of an extraspecial \(r\)-group in an absolutely irreducible representation.

From [14] Section 3.5, the group \(M\) is of type \(r^{2m} Sp_{2m}(r)\) if \(T = PSL_n(q)\) or \(T = PSU_n(q)\) (with \(n = r^m\)), of type \(2^{1+2m} O^-_{2m}(2)\) if \(T = PSp_n(q)\) (with \(n = 2^m\), and
of type $2^{1+2m} \Omega_{2m}^+(2)$ if $T = P\Omega^+_m(q)$ (with $n = 2m$). From [14] Proposition 4.6.5–6, 4.6.8–9, we see that $C_{T_\omega}(E(T_\omega))$ is soluble. Furthermore, since we are excluding the group $PSL_2(q)$ (which we studied in the first part of the proof), from Table 1 we have that $T_\omega$ is soluble if and only if $T = PSL_3(q)$, $PSU_3(q)$. (Recall that $n \geq 4$ if $T = PSp_n(q)$ and $n \geq 8$ if $T = P\Omega^+_n(q)$.) If $T = PSL_3(q)$ or $PSU_3(q)$, then with a direct inspection of the structure of $M$ described in [14] Proposition 4.6.5–6, we see that $M$ is a $\{2, 3\}$-group and so $\mu(T_\omega) \leq 2$. Hence from Lemma 2.4 the group $T$ has at most two non-trivial coprime subdegrees.

It remains to consider the case that $M$ is insoluble. Let $N$ be the last term of the derived series of $M$. Since $M/T_\omega$ is soluble, we have $N \leq T_\omega$. Furthermore, from the group structure of $M$, the group $N$ contains a characteristic $r$-subgroup $R$ with $N/R \cong Sp_{2m}(r)$ if $T = PSL_n(q)$, $PSU_n(q)$, with $N/R \cong (O^*_{2m}(2))'$ if $T = PSp_n(q)$, and with $N/R \cong (O^*_{2m}(2))'$ if $T = P\Omega^+_n(q)$.

From Lemma 3.2, the group $N$ fixes only the point $\omega$ of $\Omega$. We show that $\mu(N) \leq 2$, from which the theorem follows (in this case) from Lemma 2.4. From Lemma 3.1 we have $\mu(N) \leq 3$. If $\mu(N) \leq 2$, then the result follows from Lemma 2.4. Suppose that $\mu(N) = 3$ and let $A_1, A_2, A_3$ be three maximal subgroups of $N$ having pairwise relatively prime indices in $N$. Let $U$ be the normal subgroup of $N$ with $R \leq U$ and with $N/U$ simple (that is, $U/R = Z(N/R)$). From Lemma 3.1 relabelling the $A_i$ if necessary, we have that $r$ divides $|N : A_2|$ and that $N = (A_1/U)(A_2/U)$ is a maximal factorization of the simple group $N/U$ with $\gcd(|N : A_1|, |N : A_2|) = 1$. Therefore $(N/U, A_1/U, A_2/U)$ is one of the triples in [10] Table 1. Suppose that $T = PSL_n(q)$ or $PSU_n(q)$, that is, $N/U \cong PSp_{2m}(r)$. From [10] Table 1, we see that $r$ divides $|N : A_1|$ or $|N : A_2|$, contradicting the fact that $|N : A_1|$ is coprime with $|N : A_1|$ and with $|N : A_2|$. Now suppose that $T = PSp_n(q)$, that is, $N/U \cong P\Omega^+_n(2)$. (Recall that $r = 2$.) From [10] Table 1 we see that $P\Omega^+_n(2) = PSL_2(4)$ and $P\Omega^+_n(2) = PSU_4(2)$ are the only orthogonal groups $P\Omega^+_n(2)$ admitting a coprime factorization. Furthermore, 2 divides $|N : A_1|$ or $|N : A_2|$, a contradiction. Finally suppose that $T = P\Omega^+_n(q)$, that is, $N/U \cong P\Omega^+_n(2)$. From [10] Table 1, we see that 2 divides $|N : A_1|$ or $|N : A_2|$, again a contradiction because $r = 2$.

Case $M \in C_\omega$: $M$ is the stabilizer of a homogeneous tensor decomposition of $V$.

From [14] Section 3.5, we have that the group $M$ is of type $G_\ell_n(q) \wr \text{Sym}(t)$ if $T = PSL_n(q)$ (with $m \geq 3$), of type $G_\ell_n(q) \wr \text{Sym}(t)$ if $T = PSL_n(q)$ (with $m \geq 3$ and $(m, q) \neq (3, 2)$), of type $Sp_m(q) \wr \text{Sym}(t)$ if $T = PSp_n(q)$ (with $q$ odd, $m \geq 2$ and $(m, q) \neq (2, 3)$), of type $O_n(m) \wr \text{Sym}(n/m)$ if $T = P\Omega^+_n(q)$ (with $m \geq 3$ and $(m, q) \neq (3, 3)$), of type $Sp_m(q) \wr \text{Sym}(t)$ (with $m \geq 2$ and $(m, q) \neq (2, 2), (2, 3)$), or $O_n(m) \wr \text{Sym}(t)$ (with $q$ odd, and $m \geq 6$ if $\varepsilon = +$ and $m \geq 4$ if $\varepsilon = -$) if $T = P\Omega^+_n(q)$. In particular, from Table 1 we see that $E(T_\omega)/Z(E(T_\omega))$ is a direct product of isomorphic simple groups. Furthermore, from [14] Proposition 4.7.3–5, 4.7.6–8 we see that $C_{T_\omega}(E(T_\omega))$ is soluble. Now as usual from Proposition 2.7 we obtain that $T$ has at most two non-trivial coprime subdegrees.

Case $M \in C_\ell$: $M$ is a classical subgroup.

From [14] Section 3.5, we see that the group $M$ is of type $Sp_n(q)$, $O_n^+(q)$ or $SU_n(q/2)$ if $T = PSL_n(q)$, and of type $O_n^+(q)$ if $T = PSp_n(q)$ and $q$ is even.

Since we are excluding the cases $T = PSL_3(3)$, $PSL_3(4)$, $PSL_4(2)$, $PSL_4(3)$ and $PSp_4(2)$, the group $E(T_\omega)/Z(E(T_\omega))$ is either simple or a direct product of two isomorphic simple groups (in fact, the latter case occurs when $T = PSL_4(q)$ or
PSp$_4(q)$ and $M$ is of type $Q^+_4(q)$. From [14, Proposition 4.8.3-6], we see that $C_T(E(T_\omega))$ is soluble and so, from Proposition 2.7, $T$ has at most two non-trivial coprime subdegrees on $\Omega$.

**Case $M/T_\omega$ is soluble.**

Since $M/T_\omega$ is soluble, we have $E(M) = E(T_\omega)$. From the definition of the class $S$ in [14, Chapter 1], we have that $E(M)$ is a nonabelian simple group and $C_{Aut(T)}(E(M)) = 1$. Thus $E(T_\omega)$ is simple and $C_T(E(T_\omega)) = 1$. In particular, from Proposition 2.7, $T$ has at most two non-trivial coprime subdegrees on $\Omega$. The proof of Conjecture A' for finite classical groups is now complete.

### 6. Exceptional groups of Lie type

**Proof of Theorem A for the exceptional groups of Lie type.** Write $q = p^f$ for some prime $p$ and some $f \geq 1$. The group $T$ is one of the following exceptional simple groups: $F_4(q)$, $G_2(q)$ (with $q > 2$), $E_6(q)$, $E_7(q)$, $E_8(q)$, $^{2}B_2(q)$ (with $p = 2$ and $f = 2f' + 1$, where $f' \geq 1$), $^{3}D_4(q)$, $^{2}G_2(q)$ (with $p = 3$ and $f = 2f' + 1$, where $f' \geq 1$), $^{2}F_4(q)$ (with $p = 2$ and $f \geq 2$) and $^{2}E_6(q)$. The group $^{2}F_4(2)$ is not simple and the Tits group $^{2}F_4(2)'$ will be considered in Section 7 together with the sporadic simple groups.

For the proof of this result we use [19]. Liebeck and Seitz [19, Theorem 2] give a reduction theorem to describe the maximal subgroups of the finite exceptional groups (and their automorphism groups) similar to the well-known result of Aschbacher [4] for the finite classical groups. They show that $M$ is either in one of five well specified families listed in [19, Theorem 2 (a)-(e)] or is contained in the automorphism group of a finite simple group. In the latter case, as $M/T_\omega$ is soluble, the group $F^*(T_\omega)$ is simple and the theorem follows from Remark 3.3 and Proposition 2.7. This shows that in the rest of this proof we may assume that $M$ is in one of the five families described in [19, Theorem 2 (a)-(e)].

**The group $M$ is as in [19, Theorem 2 (a)].**

In this case, $M = N_A(D)$, where $D$ is either a parabolic subgroup of $T$ or $D$ is given in [20, Theorem, Table 5.1 and 5.2]. In the former case, $T_\omega$ contains a parabolic subgroup of $T$ and hence a Borel subgroup of $T$. In particular, $T_\omega$ contains the normalizer of a Sylow $p$-subgroup of $T$ and the theorem follows from Proposition 2.10.

Assume that $D$ is as in [20, Table 5.1]. Now the structure of $T_\omega$ is described in the second column of [20, Table 5.1]. With a direct inspection we see that in each case $C_{T_\omega}(E(T_\omega))$ is soluble and either (i) $E(T_\omega)/Z(E(T_\omega))$ is the direct product of pairwise isomorphic simple groups, or (ii) $E(T_\omega)/Z(E(T_\omega))$ is the direct product of simple groups having multiplicity at most 3 and with a unique factor of largest order, or (iii) $T = G_2(3)$ and $T_\omega$ is of type $2.(L_2(3) \times L_2(3)).2$, or (iv) $T = E_7(3)$ and $T_\omega$ is of type $2^3.(L_2(3))^7.2^4.L_3(2)$, or (v) $T = E_8(3)$ and $T_\omega$ is of type $2^5.(L_2(3))^8.2^4.AGL_3(2)$ (here we are using the notation in [20, Table 5.1]). In particular, in (i) and (ii) the theorem follows from Proposition 2.7 and 2.9 respectively. In (iii), we see that $T_\omega$ is a $\{2,3\}$-group, $\mu(T_\omega) \leq 2$ and the result follows from Lemma 2.4. Now assume that $T$ and $T_\omega$ are as in (iv) or (v). Then $T_\omega$ contains a Sylow 2-subgroup of $T$. As the Sylow 2-subgroups of $T = E_7(3)$ and $T = E_8(3)$ are self-normalizing (see [16, Theorem 6] or [15, Corollary]), then we are done by Proposition 2.10.
Assume that $D$ is as in \cite[Table 5.2]{20}. Suppose that $T$ is not a Suzuki group or a Ree group, that is, $T$ is not $2 \cdot B_2(q)$, $2 \cdot F_4(q)$ or $2 \cdot G_2(q)$. Then with a direct inspection on the order of $T$ and on \cite[Table 5.2]{20}, we see that $T_\omega$ is the normalizer of a Sylow $\Phi_e(q)$-torus of $T$, for some $e$. For instance, in the last row of \cite[Table 5.2]{20}, we have that $T = E_8(q)$ and $T_\omega = T \cap N_A(D)$ where $D$ is a torus of $T$ of order $(q^2 - q + 1)^4$. In particular, since $\Phi_6(q) = q^2 - q + 1$ and since 4 is the largest power of the polynomial $x^2 - x + 1$ dividing the generic order of $E_8$, we obtain that $D$ is a $\Phi_6(q)$-torus of $E_8(q)$. Suppose that $q^e - 1$ has a primitive prime divisor $r$ with $r \geq 3$. It follows from \cite[Theorem 5.14]{22} that either $T_\omega$ contains the normalizer of a Sylow $r$-subgroup of $T$, or $T = G_2(q)$, $r = 3$ and $q \equiv 2, 4, 5$, or $7 \mod 9$. In the former case, every non-trivial subdegree of $T$ is divisible by $r$, by Proposition 2.10. For the latter case, we note that in \cite[Table 5.2]{20} we have $q = 3^f$ if $T = G_2(q)$. Hence 3 does not divide $q^e - 1$ and the latter case does not arise. It remains to consider the case that either $q^e - 1$ has no primitive prime divisors, or 2 is the only primitive prime divisor of $q^e - 1$. Clearly, this happens if and only if $e = 2$ and $q + 1$ is a power of 2, or $(e, q) = (6, 2)$, or $e = 1$ and $q - 1$ is a power of 2. Suppose that $e = 1$ and $q \equiv 1 \mod 4$, or $e = 2$ and $q \equiv 3 \mod 4$. It follows from \cite[Theorem 5.19]{22} that $T_\omega$ contains the normalizer of a Sylow 2-subgroup of $T$ and hence, from Proposition 2.10, every non-trivial subdegree of $T$ is divisible by 2. Therefore, it remains to consider the case that $(e, q) = (6, 2)$ or $(1, 3)$. Suppose $(e, q) = (6, 2)$. A direct inspection in \cite[Table 5.2]{20} shows that if $D$ is a $\Phi_6(q)$-torus of $T$, then $q > 3$ (see the “Condition” column in \cite[Table 5.2]{20}). Suppose that $(e, q) = (6, 2)$. Again a direct inspection in \cite[Table 5.2]{20} shows that if $D$ is a $\Phi_6(q)$-torus of $T$ (that is, $D$ has order a power of $q^2 - q + 1$), then $q = 2$ is permitted only if $T = 3 \cdot D_4(q)$ (see the “Condition” column in \cite[Table 5.2]{20}). Now, if $T = 3 \cdot D_4(q)$, $q = 2$ and $T_\omega$ is the normalizer of a $\Phi_6(q)$-torus of $T$, then from \cite[Table 5.2]{20} we see that $T_\omega$ is a $\{2, 3\}$-group and the theorem follows from Lemma 2.4.

Suppose that $T$ is a Suzuki group or a Ree group. Malle in \cite[Section 8]{22} investigates the Sylow normalizers of $T$. We use the notation and the terminology from \cite[Section 8]{22}. Then with a direct inspection of the order of $T$ and of \cite[Table 5.2]{20}, we see that $T_\omega$ is the normalizer of a Sylow $\Phi_e^{(r')}q$-torus of $T$, for a suitable prime $r$ different from the defining characteristic of $T$. It follows from \cite[Theorem 8.4]{22} that either (i) : $T_\omega$ contains the normalizer of a Sylow $r$-subgroup of $T$, or (ii) : $T = 2 \cdot G_2(3^{2f+1})$, $r = 2$ and $D$ is the torus of size $q + 1$, or (iii) : $T = 2 \cdot F_4(2^{2f+1})$, $r = 3$, $D$ is the torus of size $(q + 1)^2$ and $2^{2f+1} \equiv 2, 5 \mod 9$. In (i), every non-trivial subdegree of $T$ is divisible by $r$, by Proposition 2.10. Suppose that (ii) holds. We may assume that 2 is the only prime dividing $q + 1$ (otherwise we may apply \cite[Theorem 8.4]{22} to a prime $r' \neq 2$ dividing $q + 1$ and we obtain that $T_\omega$ contains the normalizer of a Sylow $r'$-subgroup). Now, as $q = 3^{2f+1}$, we have that $q + 1$ is a power of 2 only if $f = 0$, that is, $T = 2 \cdot G_2(3)$ (which we excluded from our analysis). Finally assume that (iii) holds. Here we have $\Phi_e^{(r')}q = q + 1$. Also, again arguing as in (ii) we may assume that $q + 1$ is a power of 3. Now, \cite[Table 5.1]{20} shows that $T_\omega$ is a $\{2, 3\}$-group and the result follows from Lemma 2.4.

The group $M$ is as in \cite[Theorem 2 (b)]{19}. We have $M = N_A(E)$, where $E$ is the elementary abelian $r$-group given in \cite[Theorem 1 (II)]{17} (here $r \neq p$). We have $T_\omega = N_T(E)$. The pair $(T, E)$ and the structure of $C_T(E)$ and of $N_T(E)$ are as in \cite[Table 1]{17}. We have nine rows to
consider. If \((T, E)\) is in the 5th, 8th or 9th row of \([7] \text{ Table 1}\), then \(E(T_\omega)\) is simple, \(C_{T_\omega}(E(T_\omega))\) is soluble and the result follows from Proposition \(2.7\). Assume that \((T, E)\) is in the 2nd row of \([7] \text{ Table 1}\), that is, \(T = 2 G_2(3)'\). As \(T \cong \text{PSL}_2(8)\), the proof in this case was given in Section \(5\).

Finally, suppose that \((T, E)\) is one of the remaining cases: 1st, 3rd, 4th, 6th or 7th row of \([7] \text{ Table 1}\). With a direct inspection we see that \(T_\omega\) contains a normal \(r\)-subgroup \(R\) with \(T_\omega/R\) a simple group (note that \(\text{SL}_3(2), \text{SL}_3(3), \text{SL}_5(2)\) and \(\text{SL}_3(5)\) are simple). We claim that \(\mu(T_\omega) \leq 2\), from which the theorem follows from Lemma \(2.4\). We argue by contradiction and we assume that \(\mu(T_\omega) \geq 3\) and let \(\{A_1, A_2, A_3\}\) be three maximal subgroups of \(T_\omega\) having pairwise relatively prime index in \(T_\omega\). From Lemma \(5.1\), relabelling the \(A_i\) if necessary, \(r\) divides \([T_\omega : A_i]\) and \(T_\omega/R = (A_1/R)(A_2/R)\) is a maximal coprime factorization of \(T_\omega/R\) (here note that \(\mathbb{Z}(T_\omega/R) = 1\) because \(T_\omega/R\) is simple). Therefore \((T_\omega/R, A_1/R, A_2/R)\) is one of the triples in \([10] \text{ Table 1}\). A direct inspection of \(T_\omega/R, r\) and of the maximal coprime factorizations of \(T_\omega/U\) in \([10] \text{ Table 1}\), shows that \(r\) divides either \([T_\omega : A_1]\) or \([T_\omega : A_2]\), a contradiction.

**The Group \(M\) is as in \([19] \text{ Theorem 2 (c)}\).**

Here \(M\) is the centralizer of a graph, field, or graph-field automorphism of \(T\) of prime order \(r\) (see \([12] \text{ Definition 2.5.13}\) for a definition of these terms). In this case, the structure of \(M\) is described in \([12] \text{ Section 4.4}\). Here we use the notation in \([12]\). Write \(T = \Delta G(q)\), where \(\Delta\) is the Lie type of \(T\), \(q\) is the characteristic and \(d = 1, 2, 3\). We first consider the case that \(M\) is the centralizer of a field automorphism \(x\). Recall that \(2 B_2(2) \cong 5 : 4\). From \([12] \text{ Proposition 4.9.1}\), we have that \(E(M)/\mathbb{Z}(E(M)) \cong \Delta G(q^{1/r})\). Since \(\Delta \in \{E, F, G, B, D\}\), we obtain that \(E(M)/\mathbb{Z}(E(M))\) and hence \(E(T_\omega)/\mathbb{Z}(E(T_\omega))\) is simple except for \(T = 2 B_2(2')\). Furthermore, from \([12] \text{ Chapter 4}\), the group \(C_{T_\omega}(E(T_\omega))\) is soluble. Therefore, if \(T \neq 2 B_2(2')\), the result follows from Proposition \(2.7\). If \(T = 2 B_2(2')\), then \(M \cong (5 : 4) \times r\), \(T_\omega \cong 5 : 4\), \(T_\omega\) is a \(\{2, 5\}\)-group and the result follows from Lemma \(2.4\).

Assume that \(x\) is a graph-field automorphism. Recall that from \([12] \text{ Definition 2.5.13}\), we have \(T = G_2(q), F_4(q)\) or \(E_6(q)\). From \([12] \text{ Proposition 4.9.1}\), we have \(d = 1\), \(r = 2, 3\) and \(E(M)/\mathbb{Z}(E(M)) \cong \Delta G(q^{1/r})\). In particular, \(E(M)/\mathbb{Z}(E(M))\) is simple. Furthermore, from \([12] \text{ Chapter 4}\), the group \(C_{T_\omega}(E(T_\omega))\) is soluble and so the result follows from Proposition \(2.7\).

If remains to study the case that \(x\) is a graph automorphism. Recall that from \([12] \text{ Definition 2.5.13 (b), (d)}\) the groups \(2 B_2(q), 2 F_4(q), 2 G_2(q), F_4(q)\) and \(G_2(q)\) do not admit graph automorphisms. In particular, \(T = E_6(q), 2 E_6(q)\) or \(3 D_4(q)\). We consider separately \(T = 3 D_4(2)\) and we use \(8\). With a direct inspection on the maximal subgroups of \(T\), we see that either \(T_\omega\) contains the normalizer of a Sylow subgroup of \(T\) (and hence the theorem follows from Proposition \(2.10\), or \(E(T_\omega)\) is simple and \(C_{T_\omega}(E(T_\omega))\) is soluble (and hence the theorem follows from Proposition \(2.7\), or \(T_\omega\) is a \(\{2, 3\}\)-group (and hence the theorem follows from Lemma \(2.4\)). Now we continue the proof for the remaining groups. Note that from \([12] \text{ Sections 4.5, 4.7 and 4.9}\) the group \(C_{T_\omega}(E(T_\omega))\) is always soluble. From \([12] \text{ Proposition 4.9.2 (b)}\), we see that for \(T = E_6(q)\) or \(2 E_6(q)\) we have \(E(T_\omega)/\mathbb{Z}(E(T_\omega)) \cong F_4(q)\) if \(p = r = 2\), and for \(T = D_4(q)\) or \(3 D_4(q)\) we have \(E(T_\omega)/\mathbb{Z}(E(T_\omega)) \cong G_2(q)\) if \(p = 3\). Moreover, from \([12] \text{ Tables 4.5.1 and 4.7.3A}\), we see that for \(T = E_6(q)\) or \(2 E_6(q)\) we have \(E(T_\omega)/\mathbb{Z}(E(T_\omega)) \cong F_4(q)\) or \(C_4(q)\) if \(p \neq 2\).
(depending on the conjugacy class of \(x\)), for \(T = D_4(q)\) we have \(E(T_\omega)/Z(E(T_\omega)) \cong G_2(q)\) if \(p \neq 3\), and for \(3 \cdot D_4(q)\) we have \(E(T_\omega)/Z(E(T_\omega)) \cong \text{PSL}_3(q)\) or \(\text{PSU}_3(q)\) (depending whether \(q \equiv 1 \mod 3\) or \(q \equiv -1 \mod 3\) respectively). In particular, in each of these cases (as we are excluding \(3 \cdot D_4(2)\)) we may use Proposition 2.7 and the theorem follows.

**The group \(M\) is as in [19] Theorem 2 (d)].**

In this case, \(T = E_8(q)\), \(p > 5\) and \(F^*(M) = \text{Alt}(5) \times \text{Alt}(6)\) or \(\text{Alt}(5) \times \text{PSL}_2(q)\). Since \(M/T_\omega\) is soluble, we have \(F^*(M) = F^*(T_\omega)\) and hence the theorem follows from Proposition 2.9.

**The group \(M\) is as in [19] Theorem 2 (e)].**

In this case, \(F^*(M) = F^*(T_\omega)\) is described in detail in [19] Table III. With a direct inspection, we see that either \(F^*(T_\omega)\) is the direct product of two nonabelian simple groups, or \(T = E_8(q)\) and \(F^*(T_\omega) = E(T_\omega) \cong \text{PSL}_2(q) \times G_2(q) \times G_2(q)\) with \(p > 2\) and \(q > 3\). In the former case, the theorem follows from Proposition 2.7 (if the two simple groups are isomorphic) or Proposition 2.9 (if the two simple groups are non-isomorphic).

Suppose that \(T = E_8(q)\) and write \(N = F^*(T_\omega) \cong \text{PSL}_2(q) \times G_2(q) \times G_2(q)\). Since \(T_\omega/N\) is soluble, the group \(N\) is clearly the last term of the derived series of \(T_\omega\). From Lemma 3.2 \(N\) fixes only the point \(\omega\) of \(\Omega\). We claim that \(\mu(N) \leq 2\).

Conjecture A’ will follow from this claim and Lemma 2.4.

It remains to prove that \(\mu(N) \leq 2\). Write \(N = S_1 \times S_2 \times S_3\) with \(S_1 \cong S_2 \cong G_2(q)\) and \(S_3 \cong \text{PSL}_2(q)\). We see from [10] Table 1 that \(\mu(\text{PSL}_2(q)) \leq 2\) and that, for any two maximal subgroups \(M_1\) and \(M_2\) of \(G_2(q)\), the indices \(|G_2(q) : M_1|\) and \(|G_2(q) : M_2|\) are divisible by a non-trivial common factor. Suppose now that \(A_1, A_2, A_3\) are maximal subgroups of \(N\) of pairwise coprime indices. If \(A_3\), say, projects onto each of the three simple direct factors \(\{S_1, S_2, S_3\}\) of \(N\), then \(A_3 = D \times S_3\) where \(D\) is a diagonal subgroup of \(S_1 \times S_2 \cong G_2(q) \times G_2(q)\). As \(|\text{PSL}_2(q)|\) divides \(|G_2(q)|\), this implies that \(|N : A_3|\) is divisible by the orders of each of the simple direct factors, a contradiction. If \(A_2\) and \(A_3\), say, do not project onto \(S_1\), then by maximality we have \(A_2 = B \times S_2 \times S_3\) and \(A_3 = C \times S_2 \times S_3\), where \(B\) and \(C\) are proper subgroups of \(S_1\). Since \(G_2(q)\) has no two maximal subgroups of pairwise coprime index, we obtain a contradiction. A similar argument applies for \(S_2\). Hence, since \(\mu(\text{PSL}_2(q)) = 2\), relabelling the index set \(\{1, 2, 3\}\) if necessary, we have that either \(A_1 = B_1 \times S_2 \times S_3\), \(A_2 = S_1 \times B_2 \times S_3\) and \(A_3 = S_1 \times S_2 \times B_3\) (where, for each \(i\), \(B_i\) is a maximal subgroup of \(S_i\)), or \(A_1 = B_1 \times S_2 \times S_3\), \(A_2 = S_1 \times S_2 \times B_3\) and \(A_3 = S_1 \times S_2 \times B_3\) (where \(B_1\) is a maximal subgroup of \(S_1\) and \(B_3, B_3'\) are maximal subgroups of \(S_3\)). In the former case, as \(G_2(q)\) has no two maximal subgroups of pairwise coprime index, \(|N : A_1| = |S_1 : B_1|\) and \(|N : A_2| = |S_2 : B_2|\) are divisible by non-trivial common factor, a contradiction. We now consider the latter case. If the characteristic \(p\) divides \(|N : A_3|\), then \(|N : A_1|\) and \(|N : A_2|\) are coprime to \(q\) implying that \(B_3, B_3'\) are parabolic subgroups of \(S_3 \cong \text{PSL}_2(q)\), a contradiction. Hence \(p\) is coprime with \(|N : A_1|\) and \(B_1\) is a maximal parabolic subgroup of \(S_1 \cong G_2(q)\). So \(|N : A_3| = (q^6 - 1)/(q - 1)\) (which is divisible by \(q + 1\)). However \(|N : A_2| = |S_2 : B_3|\) and \(|N : A_3| = |S_3 : B_3'|\) are coprime indices of maximal subgroups of \(\text{PSL}_2(q)\), and by [10] Table 7, one of these indices is the index of a parabolic, and hence equal to \(q + 1\), a contradiction. Thus \(\mu(N) = 2\).
7. Sporadic groups

Proof of Theorem A for the sporadic groups. The group $T$ is one of the 27 sporadic simple groups (note that we did not consider the simple group $2^f F_4(2)'$ in Section 6). Fix $\omega \in \Omega$. Since $T_\omega$ is pseudo-maximal in $T$, there exists an almost simple group $A$ with socle $T$ and a maximal subgroup $M$ of $A$ with $T \nmid M$ and $T_\omega = T \cap M$. If $T$ is not the Fisher-Griess Monster, in the proof of this result we may use the complete list of the maximal subgroups of $A$ available in [11,8]; in particular, the tuple $(A,T,M,T_\omega)$ is in [11,8]. If $T$ is the Monster, then Out$(T) = 1$, $T$ has 43 known conjugacy classes of maximal subgroups and, by [5], an unknown maximal subgroup of $T$ is almost simple. In particular, if $T$ is the Monster and $T_\omega$ is conjugate to one of these unknown maximal subgroups, then by Proposition 2.7 we have that $T$ has at most two non-trivial coprime subdegrees. This shows that in the rest of this proof we can simply use the information on the subgroup lattice of the sporadic groups in [11,8], including the Monster. We use the notation in [8].

In order to avoid a long list of cases to consider, we have checked with magma that this theorem holds true for $|\Omega| \leq 2000$ by a direct inspection (all primitive permutation groups of degree at most 2000 are in the PrimitiveGroups database). From the “Specification Structure” column in the list of maximal subgroups of $A$ in [8], it can be readily checked whether Proposition 2.4 or 2.9 applies, in this case the theorem immediately follows. Moreover, from the “Specification Order” column, it is immediate to see whether $T_\omega$ is a $\{p,q\}$-group, from which the theorem follows from Lemma 2.4. Furthermore, from the “Specification Abstract” column, sometimes it can be easily inferred whether $T_\omega$ contains the normalizer of a Sylow $p$-subgroup of $T$, for some prime $p$, so the theorem follows from Proposition 2.10 in this case. (For instance, if $T = J_1$ and $T_\omega \cong 7 : 6$, then we see that $T_\omega$ is the normalizer of a cyclic group of order 7. Since a Sylow 7-subgroup of $T$ has order 7, we obtain that $T_\omega$ contains the normalizer of a Sylow 7-subgroup. For later reference we give another example. If $T = McL$ and $T_\omega = 5^{1+2}_+ : 3 : 8$, we see that $T_\omega$ contains a Sylow 5-subgroup $P$ of $T$ and that $P \trianglelefteq T$. As $T_\omega$ is a maximal subgroup of $T$, we obtain $T_\omega = N_T(P)$.) Now the proof is a case-by-case analysis on the tuples $(A,T,M,T_\omega)$ which do not meet any of the conditions described in this paragraph: Table 2 gives all possible such pairs $(T,T_\omega)$.

Let $(T,T_\omega)$ be one of the pairs in Table 2 and let $N$ be the last term of the derived series of $T_\omega$ (as we defined in Section 6). This means that $N \trianglelefteq T_\omega$, $T_\omega/N$ is soluble and $N = [N,N]$. Note that $N > 1$. From Lemma 3.2 the group $N$ fixes only the point $\omega$ of $\Omega$.

Assume that $(T,T_\omega)$ is not one of the following nine pairs.

$$(J_4, 2^{3+12} (S_5 \times L_3(2))), \quad (Fi'_{24}, 2^{3+12} (L_3(2) \times A_6)), \quad (M, 2^{3+6+12+18} (3S_6 \times L_3(2))),$$
$$(B, [2^{35}] (S_5 \times L_3(2))), \quad (McL, 2^4 : A_7), \quad (Fi_{23}, 2^{6+8} (A_7 \times S_5)),$$
$$(CO_2, 3^{1+4} : 2^{1+4} S_5), \quad (B, 3^{1+8} : 2^{1+6} U_4(2).2), \quad (B, 5^{1+4} : 2^{1+4} A_5, 4).$$

With a direct inspection, we see that $N$ contains a normal $p$-subgroup $P$ such that $N/P$ is either a quasisimple group, or isomorphic to $A_5 \times A_5$ and $p = 2$ (here $T$ is the Harada-Norton group $HN$). We show that $\mu(N) \leq 2$, from which it follows by Lemma 2.4 that $T$ has at most two non-trivial coprime subdegrees. We argue by contradiction and we assume that $\mu(N) \geq 3$ and we let $A_1, A_2, A_3$ be three distinct maximal subgroups of $N$ with pairwise coprime index in $N$. Let $U/P$ be the centre
of \( N/P \). From Lemma 3.1, \( p \) divides \(|N : A_3|\) and \( N/U = (A_1/U)(A_2/U) \) is a maximal coprime factorization of \( N/U \). Suppose first that \( N/U = A_1 \times A_3 \). Since in every coprime factorization of \( A_1 \times A_3 \), one of the two maximal subgroups has even index and as \( p = 2 \), we obtain a contradiction. Suppose now that \( N/U \) is simple. Therefore \( (N/U, A_1/U, A_2/U) \) is in [10] Table 1. A direct inspection on \( N/U \), on \( p \), and on the maximal coprime factorizations of \( N/U \) in [10] Table 1 shows that \( p \) divides either \(|N : A_1|\) or \(|N : A_2|\), a contradiction.

It remains to consider the case that \((T, T_o)\) is one of the nine pairs that we excluded above. Suppose that \((T, T_o)\) is one of the first six pairs (those in the first two rows). It is immediate, comparing the order of \( T_o \) with the order of \( T \), to check that \( T_o \) contains only the points \( \{1\} \) of \( N \). By Proposition 2.10, every non-trivial subdegree of \( T \) is divisible by 2 or 3. Now it is divisible by 3. Hence, by Proposition 2.10, \( T_o \) contains the normalizer of a Sylow 2-subgroup of \( T \) and, hence, from Proposition 2.10, every non-trivial subdegree of \( T \) is even.

Assume that \( T = B \) and \( T_o = 5_+^{1+4} : 2_+^{1+4}.A_5.4 \). From [28] Section 3 and Table III, we see that \( T_o \) contains the normalizer of a Sylow 5-subgroup of \( T \) and hence, by Proposition 2.10, every non-trivial subdegree of \( T \) is divisible by 5.

Assume that \( T = B \) and \( T_o = 3_+^{1+8} : 2_+^{1+6}.U_4(2).2 \). Write \( N = [T_o, T_o] \). From Lemma 3.2, \( N \) fixes only the point \( \omega \) of \( \Omega \). Next we show that \( \mu(N) = 2 \), from which the theorem follows from Lemma 2.4. We argue by contradiction and we assume that \( \mu(N) \geq 3 \) and we let \( A_1, A_2 \) and \( A_3 \) be three maximal subgroups of \( N \) with pairwise coprime index in \( N \). Let \( U \) be the normal subgroup of \( N \) with \( N/U \cong U_4(2) \). Note that if \( U \nsubseteq A_i \), then \( N = A_i/U \) and \(|N : A_i| = |U : (U \cap A_i)|\) is divisible by 2 or 3. Since \( U \) is a \( \{2,3\} \)-group, there exist at most two elements of \( \{A_1, A_2, A_3\} \) containing \( U \). Moreover, if \( U \leq A_i \), then \( A_i/U \) is a maximal subgroup of the simple group \( N/U \). As \( \mu(U_4(2)) \leq 2 \) from Lemma 2.7, there exist at most two elements of \( \{A_1, A_2, A_3\} \) containing \( U \). Therefore, relabelling the set \( \{A_1, A_2, A_3\} \) if necessary, we have two cases to consider: (i) \( U \leq A_1, A_2 \) and \( U \nsubseteq A_3 \), or (ii) \( U \leq A_1 \) and \( U \nsubseteq A_2, A_3 \). In (i), \( N/U = (A_1/U)(A_2/U) \) is a maximal factorization of \( N/U \) with two subgroups having coprime index. With a direct inspection on the subgroup lattice of \( U_4(2) \) (or from [10] Table 1), we see that (replacing \( A_1 \) by \( A_2 \) if necessary) \(|N : A_1| = 27 \) and \(|N : A_2| = 40 \). Since \(|N : A_3| = 2^{2+3^2} \) for some \( \alpha \) and \( \beta \), we obtain a contradiction. In (ii), replacing \( A_2 \) by \( A_3 \) if necessary, we may assume that \(|N : A_2| \) is divisible by 2 and \(|N : A_3| \) is divisible by 3. Now \( A_1/U \) is a maximal subgroup of \( N/U \). With a direct inspection on the subgroup lattice of \( U_4(2) \) we see that \(|N : A_1| \in \{27, 36, 40, 45\} \). Since each of these numbers is divisible by 2 or by 3, we obtain a contradiction.

It remains to consider the case that \( T = C_02 \) and \( T_o = 3_+^{1+4} : 2_+^{1+4}.S_5 \). Comparing the order of \( T \) with the order of \( T_o \) we see that \( T_o \) contains a Sylow 3-subgroup \( S \) of \( T \). From [28] Section 2 and Table I, we see that \(|N_T(S)| = 32|S| \). The generators of \( T_o \) are available in [1]. Now by a computation in magma we check that \(|N_T(S)| = 32|S| \) and hence \( T_o \) contains the normalizer of a Sylow 3-subgroup of \( T \). By Proposition 2.10, every non-trivial subdegree of \( T \) is divisible by 3. \( \square \)

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Table 2. Pseudo-maximal subgroups of $T$ relevant to the proof of Theorem A for the sporadic simple groups