Rotating Atomic Traps for Bosons and the Centrifuge Effect

T. Majmudar and A. Widom
Physics Department, Northeastern University, Boston MA 02115

A rigorous time independent Hamiltonian for rotating atomic traps is discussed. The steady states carry a mass current and thereby an angular momentum. It is shown that the rotation positions the atoms away from the rotation axis (after taking both the time and quantum mechanical averages) as in a conventional centrifuge. Some assert that the rotation for Bose condensates cause the atoms to move towards the rotation axis; i.e. act oppositely to fluids in a centrifuge. The opposing physical pictures are reminiscent of the difference between the rotational motion views of Newton and Cassini.

PACS: 05.30.J, 03.75.F, 03.75.F, 42.50.G

I. INTRODUCTION

In recent years there has been considerable interest in the possibility of Bose condensation [1–3] of trapped mesoscopic clusters of atoms [4]. One of the more difficult experimental problems to solve has been the design of effective traps [5–8]. An often used Bose fluid container employs the “time averaged orbiting potential” (TOP) trap [9].

The design of the TOP trap begins with a magnetic bottle formed by the superposition of a uniform magnetic field and a quadrapole magnetic field, i.e. the effective single atom adiabatic potential in the magnetic bottle is given (in cylindrical coordinates) by

\[ U(\rho, \phi, z) = \hbar \gamma G \sqrt{\rho^2 + b^2 + 4z^2 + 2\rho b \cos \phi} \]  

(1)

where \( \gamma \) is the atomic gyromagnetic ratio, \( G \) the quadrapole magnetic field gradient and \( b = (B_0 / G) \), where \( B_0 \) is the magnitude of uniform part of the magnetic field. In the \( z = 0 \) plane, the conical potential is shown in Fig.1.

An experimental problem with the conical potential is that atoms located near the tip of the cone can make a transition onto another adiabatic potential and drift away. There is a leak at the bottom of the cone. In order to plug up the leak, one must move the atoms away from the bottom tip of the cone. In some experiments, a laser beam blasts the atoms away from the tip. In other experiments one simply rotates the potential around an axis as shown in Fig.2.

![FIG. 1. Shown is the conical potential produced by a static magnetic field \( U(\rho, \phi, z = 0) = \hbar \gamma G b U^*(\rho \cos \phi, \rho \sin \phi) \) as defined in Eq.(1).](image)

One may try [10,11] to treat the rotating potential in analogy with the workings of a centrifuge. It might be imagined that the atoms are thrown outwards to large values of \( \rho > b \) as shown schematically in the FIG.2. However, it has been reported in the literature that Bose condensed atoms are thrown inwards to smaller values.
of the radial coordinates $\rho < b$. The observation of such a TOP trap contraction of the cluster has not been direct. Most of the direct observations of the atomic positions take place after the trap potential has been removed. However, the notion of the contraction of the cluster would follow theoretically if one replaced the actual potential in Eq.(1) with the time averaged (over one rotational period) potential [23, 24]

$$\bar{U}(\rho, z) = \left(\frac{\Omega}{2\pi}\right) \int_{-\pi/\Omega}^{\pi/\Omega} U(\rho, \phi + \Omega t, z) dt. \quad (2)$$

Our work is organized as follows: In Sec.II a rigorous time independent Hamiltonian for a rotating potential will be derived [21, 22]. It will be shown that the time independent energy eigenstates carry a mass current $\mathbf{J}$, and thereby an angular momentum

$$\mathbf{L} = \int (\mathbf{r} \times \mathbf{J}) d^3 \mathbf{r} = \sum_{i=1}^{N} (\mathbf{r}_i \times \mathbf{p}_i). \quad (3)$$

The rotating fluid with angular momentum $\mathbf{L}$ is held inside the bottle with a force pointing towards the rotation axis as shown in FIG.2. In order for the force to point towards the rotation axis, the atoms must be positioned so that $\rho > b$. If $\rho < b$, then the bottle walls push the atoms away from the axis. It has been maintained in the literature on top traps that the atoms are positioned atoms away from the axis. It has been maintained in the literature on top traps that the atoms are positioned so that $\rho < b$ which implies a wall force directed away from the rotation axis. In Sec.III, rigorous time averaging theorems will be proved. In the concluding Sec.IV, the experimental importance of the theoretical results will be examined.

II. ROTATIONAL HAMILTONIAN

With the single atom Hamiltonian

$$\tilde{h}_i(t) = -\left(\frac{\hbar^2}{2M}\right) \nabla_i^2 + U(\rho_i, \phi_i + \Omega t, z_i), \quad (4)$$

the Hamiltonian for $N$ atoms in a TOP trap is given by

$$H(t) = \sum_{i=1}^{N} \tilde{h}_i(t) + \sum_{i<j}^{N} u_{ij} \quad (5)$$

wherein the two body potential commutes with the angular momentum operator

$$\mathbf{L} = -i\hbar \sum_{i=1}^{N} \mathbf{r}_i \times \nabla_i. \quad (6)$$

Employing the canonical transformation

$$\mathbf{H} = S^\dagger(t) H(t) S(t) - i\hbar S^\dagger(t) \frac{\partial S(t)}{\partial t} \quad (7)$$

with

$$S(t) = \exp\left(-i\Omega L z t/\hbar\right), \quad (8)$$

one finds a rigorously exact expression for the time independent Hamiltonian corresponding to a TOP angular velocity $\Omega$

$$\mathbf{H} = \sum_{i=1}^{N} h_i + \sum_{i<j}^{N} u_{ij}, \quad (9)$$

where $h_i = \tilde{h}_i(0) - \Omega l_i, \quad l_i = -i\hbar \left(\frac{\partial}{\partial \phi_i}\right). \quad (10)$

In more detail, the time independent Hamiltonian for atoms in a TOP trap has the form

$$\mathbf{H} = \frac{1}{2M} \sum_{i=1}^{N} (\mathbf{p}_i - M\Omega \times \mathbf{r}_i)^2 \quad (11)$$

$$+ \sum_{i<j}^{N} u(\mathbf{r}_i, \mathbf{r}_j), - \frac{M}{2} \sum_{i=1}^{N} |\Omega \times \mathbf{r}_j|^2 \quad (11)$$

where $\Omega$ is the angular velocity (axial) vector along the $z$-axis. The last term on the right hand side of Eq.(11) represents the centrifugal (effective) potential energy.

That the Hamiltonian $\mathbf{H}$ describes the TOP trap atoms in a rotating frame is evident from the velocity operator for the $i^{th}$ atom.

$$\mathbf{v}_i = \left(\frac{i}{\hbar}\right) [\mathbf{H}, \mathbf{r}_i] = \left(\frac{\mathbf{p}_i}{M}\right) - \Omega \times \mathbf{r}_i. \quad (12)$$

Note that two components of the velocity operator do not commute; i.e.

$$[v_{xi}, v_{yj}] = -\left(\frac{2i\hbar\Omega}{M}\right) \delta_{ij}. \quad (13)$$

The acceleration

$$\mathbf{a}_i = \left(\frac{i}{\hbar}\right) [\mathbf{H}, \mathbf{v}_i] \quad (14)$$

is given by

$$M\mathbf{a}_i = \mathbf{f}_i + M (2\mathbf{v}_i \times \mathbf{\Omega} + \mathbf{\Omega} \times (\mathbf{r}_i \times \mathbf{\Omega})). \quad (15)$$

where the internal atomic forces and the confining force obey

$$\mathbf{f}_i = -\nabla_i \sum_{j \neq i}^{N} u(\mathbf{r}_i, \mathbf{r}_j) - \nabla_i U(\mathbf{r}_i). \quad (16)$$

In Eq.(16), the confining potential $U(\mathbf{r})$ is given in Eq.(1). Eq.(15) describes the normal, Coriolis and centrifugal forces in a fully quantum mechanical operator framework.
Summing Eqs.(15) and (16) over all of the atoms confined in the TOP trap, yields

\[ M \sum_{i=1}^{N} a_i + \sum_{i=1}^{N} \nabla_i U(r_i) = \]
\[ M \sum_{i=1}^{N} (2v_i \times \Omega + \Omega \times (r_i \times \Omega)), \quad (17) \]
where the internal forces in the sum cancel due to the equality of action and reaction forces, i.e. momentum conservation.

Thus far, our considerations are rigorously true for the model. The rigorously exact results can be extended to theorems regarding the quantum and also the time averaged properties of the system. The following two theorems are of central importance.

**Theorem I:** For a stationary state density matrix \( \rho \) obeying \( i\hbar \dot{\rho} = [\mathcal{H}, \rho] = 0 \), the mean value of the time rate of change of a bounded quantity \( Q \) vanishes i.e.

\[ \langle \dot{Q} \rangle = 0. \quad (18) \]

**Proof:** Employing \( \dot{Q} = (i/\hbar) [\mathcal{H}, Q] \) and the cyclic invariance of the trace

\[ \langle \dot{Q} \rangle = Tr(\rho \dot{Q}) = Tr(\dot{\rho}Q) = 0. \quad (19) \]

To apply this theorem for atoms in a rotating trap, let us consider mean acceleration of one atom via Eqs.(17); i.e.

\[ \langle (a + 2\Omega \times v + \Omega \times (\Omega \times r)) \rangle = - \langle \nabla U(r) \rangle / M. \quad (20) \]

In a stationary state, Eq.(18), \( a \geq 0 \) and \( v = 0 \). Thus we arrive at

**Theorem II:** In any rotational stationary state, the (mean) mass times the centripetal acceleration of an atom is equal to the (mean) force exerted on the atom by the confining potential

\[ M \langle (\Omega \times (\Omega \times r)) \rangle = - \langle \nabla U(r) \rangle. \quad (21) \]

Let us now return to FIG.2. Clearly the (mean) centripetal acceleration \( \Omega \times (\Omega \times r) \) points towards the rotation axis. Thus, the mean force due to the wall potential \( - \langle \nabla U(r) \rangle \) also points towards the rotational axis. This proves, beyond any doubt, that the atoms must be positioned at a distances \( \rho > b \) as in FIG.2. It is not possible in a stationary state to have the atoms on the average at a distances closer than \( b \).

The above theorems can be further extended to nonstationary states if time averaging techniques are employed. If \( < Q(t) > \) is a quantum mean value of a physical quantity at time \( t \), then the time average of that mean value is defined by

\[ \overline{Q} = \lim_{\tau \to \infty} \left( \frac{1}{\tau} \right) \int_{t_0 -(\tau/2)}^{t_0 +(\tau/2)} <Q(t)> dt. \quad (22) \]

It is a simple matter to prove the following

**Theorem III:** If \( < Q(t) > \) is a bounded function of time, then

\[ \overline{Q} = 0. \quad (23) \]

**Proof:**

\[ \overline{Q} = \lim_{\tau \to \infty} \left( \frac{1}{\tau} \right) \int_{t_0 -(\tau/2)}^{t_0 +(\tau/2)} \frac{dQ(t)}{dt} dt = \]
\[ \lim_{\tau \to \infty} \left( \frac{<Q(t_0 + (\tau/2)> - <Q(t_0 -(\tau/2)>)}{\tau} \right) = 0. \quad (24) \]

Finally, proceeding as before we prove the central result of this work.

**Theorem IV:** In any state with finite quantum and time averages, the mass times the mean centripetal acceleration of an atom is equal to the mean force exerted on the atom by the confining potential:

\[ M\Omega \times (\Omega \times r) = -\nabla U. \quad (25) \]

**Proof:** Apply Eq.(24), in the form \( \Pi = 0 \) and \( \nabla = 0 \), to Eq.(20).

For a classical centrifuge with rotating walls, our central Eq.(25) can be employed to prove the usual result that particles are thrown outwards by the rotation. What we have quite rigorously proved is that Eq.(25) is also true for identical Bosons with quantum mechanical effects fully taken into account.

**III. CONCLUSIONS**

The reported experimental studies which maintain that the rotating TOP trap pulls the particles inward (opposite to the centrifuge effect) appears more than just a little puzzling to us. As far as we know there have been no direct observations of particles localized on the rotation axis of a TOP trap. (i) In some cases the atoms have been observed flying outwards after the TOP trap has been removed. (ii) In an in situ measurement, the absorption of light by atoms in the trap was from an incident beam directed normal to the rotation axis. Again, the notion of atoms clustered on the rotation axis is (at best) only indirectly inferred from experimental data. (iii) In in situ measurements, the probe pulse of light is synchronized to the angular velocity of rotation. If the fluid in the trap responded only to the time averaged potential, then synchronization should have no experimental consequences. Thus the notion of employing a static harmonic oscillator potential to model atoms in a dynamic TOP trap may be unreliable.
Most of the central theorems proved above for rotating quantum mechanical systems, have been previously derived for classical fluids rotating in steady state. When the fluids are in a state of rotational flow, they tend to form ellipsoidal figures of equilibrium. The history of mathematical studies of the stability of such ellipsoidal figures has been reviewed in detail by Chandrasekhar. The averaging procedure for classical rotating fluids relies heavily on virial moments of the mass and velocity distributions. Some of the final results obtained by these classical virial methods are identical to those we have achieved by time and quantum mechanical averaging procedures. The classical rotational high angular velocity centrifuge effects retain their validity even in the quantum domain.

Finally, Newton derived the oblate spheroid shape for the rotating earth employing what is presently very well known as a centrifuge effect. The equatorial circle has a slightly larger diameter than distance between the north and south pole due to the daily rotation of the earth. Four generations of the Cassini family argued that the earth was a prolate spheroid with the equatorial circle having a slightly smaller diameter than distance between the north and south pole. The Cassini family argued (against Newton) that the earth’s mass would be drawn towards the rotational axis of the earth, not unlike what has been claimed for atoms in a rotating TOP trap. The considerations of the Cassini family were shown to be incorrect. While the settlement of the shape of the earth required long times and large human and economic expenses of sizable expeditions spreading from France to Lapland to Peru, it is to be hoped that direct observations of fluid shapes in TOP traps can be carried out more expeditiously.

[1] Bose Einstein Condensation ed. A. Griffin, D.W. Snoke, S. Stringari, Cambridge University Press, Cambridge (1995).
[2] H.S.T. Stoof Journal of Low Temperature Physics 114, 11 (1999).
[3] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, Reviews of Modern Physics 71, 463 (1999).
[4] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman and E.A. Cornell, Science, 269, 198 (1995).
[5] C.C. Bradeley, C.A. Sackett and R.G. Hulet, Phys. Rev. Lett. 78, 985 (1997).
[6] C.C. Bradeley, C.A. Sackett, J.J. Tollet, and R.G. Hulet, Phys. Rev. Lett., 75, 1687 (1995).
[7] K.B. Davis, M.O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, it Phys. Rev. Lett. 75, 3969 (1995).
[8] J. Stenger, D.M. Stamper-Kurn, M.R. Andrews, A.P. Chikkatur, S. Inouye, H.J. Miesner, and W. Ketterle, Journal of Low Temp. Phys. 113, 167 (1998).
[9] W. Petrich, M.H. Anderson, J.R. Ensher, and E.A. Cornell, Phys. Rev. Lett., 74, 3352 (1995).
[10] A.B. Kuklov, N. Chencinski, A.M. Levine, M. Sreiber, and J.L. Birman, Phys. Rev. , A55, 488 (1997).
[11] V.G. Minogin, J.A. Richmond, and G.I. Opat, Phys. Rev., A58, 3138 (1998).
[12] J.R. Ensher, D.S. Jin, M.R. Matthews, C. Wieman, and E.A. Cornell, Phys. Rev. Lett 78, 3801 (1997).
[13] S. Giorgini, L.P. Pitaevskii, S. Stringari, Phys. Rev. Lett. 78, 3987 (1997).
[14] D.S. Rokhsar Phys. Rev. Lett. 79, 2164 (1997).
[15] S. Giorgini, L.P. Pitaevskii, S. Stringari, Journal of Low Temp. Phys. 109, 309 (1997).
[16] M. Edwards and K. Burnett, Phys. Rev. A51, 1382 (1995).
[17] G. Baym and C. Pethick, Phys. Rev. Lett. 76, 6 (1996).
[18] F. Dalfovo and S. Stringari, Phys. Rev. A53, 2477 (1996).
[19] M. Edwards, R.J. Dodc, C.W. Clark, P.A. Ruprecht, and K. Burnett, Phys. Rev. A53, R1950 (1996).
[20] M.J. Holland and J. Cooper, Phys. Rev. A53, 154 (1996).
[21] A. Widom Phys. Rev. 168, 150 (1968).
[22] J. Wu and A. Widom arXiv:cond-mat/9806094.
[23] B.P. Anderson, and M.A. Kasevitch, Phys. Rev. A 59, R938 (1999).
[24] S. Chandrasekhar, Ellipsoidal Figures of Equilibrium, Dover, Mineola N.Y. (1987).
[25] I. Newton, Principia Mathematica, Book III, Proposition XVIII.
[26] N. Grossman, The Sheer Joy of Celestial Mechanics, Birkhäuser, Boston (1996).