Global Weak Solutions to the Equations of Compressible Flow of Nematic Liquid Crystals in Two Dimensions✩

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Abstract

We consider weak solutions to a two-dimensional simplified Ericksen-Leslie system of compressible flow of nematic liquid crystals. An initial-boundary value problem is first studied in a bounded domain. By developing new techniques and estimates to overcome the difficulties induced by the supercritical nonlinearity $|\nabla d|^2 d$ in the equations of angular momentum on the direction field, and adapting the standard three-level approximation scheme and the weak convergence arguments for the compressible Navier-Stokes equations, we establish the global existence of weak solutions under a restriction imposed on the initial energy including the case of small initial energy. Then the Cauchy problem with large initial data is investigated, and we prove the global existence of large weak solutions by using the domain expansion technique and the rigidity theorem, provided that the second component of initial data of the direction field satisfies some geometric angle condition.

Keywords: Liquid crystals, compressible flows, weak solutions, Galerkin method, weak convergence arguments.

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1. Introduction

We consider the existence of weak solutions to an initial-boundary value problem for the following two-dimensional simplified version of the Ericksen-Leslie model in a bounded domain $\Omega \subset \mathbb{R}^2$ which describes the motion of a compressible flow of nematic liquid crystals:

\begin{align}
\partial_t \rho + \text{div}(\rho \mathbf{v}) &= 0, \\
\partial_t (\rho \mathbf{v}) + \text{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P(\rho) &= \mu \Delta \mathbf{v} + (\mu + \lambda) \nabla \text{div} \mathbf{v} - \nu \text{div} \left( \nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I} \right), \\
\partial_t \mathbf{d} + \mathbf{v} \cdot \nabla \mathbf{d} &= \theta (\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}),
\end{align}

where $\rho$ is the density of the nematic liquid crystals, $\mathbf{v}$ the velocity and $P(\rho)$ the pressure, $\mathbf{d} \in \mathbb{S}^1 := \{ \mathbf{d} \in \mathbb{R}^2 \mid |\mathbf{d}| = 1 \}$ represents the macroscopic average of the nematic liquid crystal orientation field. The constants $\mu$, $\lambda$, $\nu$, and $\theta$ denote the shear viscosity, the bulk viscosity, the competition between kinetic energy and potential energy, and the microscopic elastic relation.

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time for the molecular orientation field, respectively. \( \mathbb{I} \) denotes the \( 2 \times 2 \) identical matrix. The term \( \nabla \mathbf{d} \odot \nabla \mathbf{d} \) denotes the \( 2 \times 2 \) matrix whose \((i, j)\)-th entry is given by \( \partial_{x_i} \mathbf{d} \cdot \partial_{x_j} \mathbf{d} \), for \( 1 \leq i, j \leq 2 \), i.e., \( \nabla \mathbf{d} \odot \nabla \mathbf{d} = (\nabla \mathbf{d})^\top \nabla \mathbf{d} \), where \( (\nabla \mathbf{d})^\top \) denotes the transpose of the \( 2 \times 2 \) matrix \( \nabla \mathbf{d} \).

In 1989, Lin [25] first derived a simplified Ericksen-Leslie system modeling liquid crystal flows when the fluid is incompressible and viscous. Subsequently, Lin and Liu [27, 28] established some analysis results on the simplified Ericksen-Leslie system, such as the existence of weak and strong solutions and the partial regularity of suitable solutions, under the assumption that the liquid crystal director field is of varying length by Leslie’s terminology, or variable degree of orientation by Ericksen’s terminology. We refer readers to [5, 9, 11, 21] for details concerning the so-called Ericksen-Leslie system.

When the fluid is allowed to be compressible, the Ericksen-Leslie system becomes more complicated. From the viewpoint of partial differential equations, the system (1.1)–(1.3) is a highly nonlinear hyperbolic-parabolic coupled system, and it is challenging to analyze such a system, in particular, as the density function \( \rho \) may vanish. It should be noted that the system (1.1)–(1.3) includes two important subsystems. When \( \rho \) is constant and \( \mathbf{v} = 0 \), the system (1.1)–(1.3) reduces to the equations for heat flow of harmonic maps into \( S^1 \), on which there have been extensive studies in the past few decades (see, for example, the monograph by Lin and Wang [29] and the references therein). When \( \mathbf{d} \) is constant, the system (1.1), (1.2) becomes the compressible Navier-Stokes equations, which have attracted great interest in the past decades and there have been many important developments (see, e.g., [7, 13, 31, 34] for a survey of recent developments).

Since the supercritical nonlinearity \( |\nabla \mathbf{d}|^2 \mathbf{d} \) causes significant mathematical difficulties, Lin in [25] introduced a Ginzburg-Landau approximation of the simplified Ericksen-Leslie system, i.e., \( |\nabla \mathbf{d}|^2 \mathbf{d} \) in (1.3) is replaced by the Ginzburg-Landau penalty function \( (1 - |\mathbf{d}|^2)/\epsilon \) or by a more general penalty function. Consequently, by establishing some estimates to deal with the direction field and its coupling/interaction with the fluid variables, a number of results on the Navier-Stokes equations can be successfully generalized to such Ginzburg-Landau approximation model. For example, when \( \rho \) is a constant, i.e., the homogeneous incompressible case, Lin and Liu [27] proved the global existence of weak solutions in two and three dimensions. In particular, they also obtained the existence and uniqueness of global classical solutions either in two dimensions or in three dimensions for large fluid viscosity \( \mu \). In addition, the existence of weak solutions to the density-dependent incompressible flow of liquid crystals was proved in [17, 32]. Recently, Wang and Yu [38], and Liu and Qin [33] independently established the global existence of weak solutions to the three-dimensional compressible flow of liquid crystals with the Ginzburg-Landau penalty function.

In the past a few years, progress has also been made on the analysis of the model (1.1)–(1.3) by overcoming the difficulty induced by the supercritical nonlinearity \( |\nabla \mathbf{d}|^2 \mathbf{d} \). For the incompressible case, the existence of weak solutions in two dimensions was established in [26], and the local and global existence of small strong solutions in three dimensions was proved in [12, 23, 30, 37]. For the compressible case, the existence of strong solutions have been investigated extensively. For example, the local existence of strong solutions and a blow-up criterion were obtained in [15, 16], while the existence and uniqueness of global strong solutions to the Cauchy problem in critical Besov spaces were proved in [14] provided that the initial data are close to an equilibrium state, and the global existence of classical solutions to the Cauchy problem was shown in [22] with smooth initial data that has small energy but possibly large oscillations with possible vacuum and constant state as far-field condition. To our best knowledge, however, there are no results available on weak solutions of the multi-dimensional problem (1.1)–(1.3) with large initial data due to the difficulties induced by the compressibility and the supercritical nonlinearity. It
seems that the only global existence of large weak solutions to (1.1)–(1.3) was shown in the one-dimensional case in [3].

In this paper, we will establish the global existence of weak solutions to the two-dimensional problem (1.1)–(1.3) in a domain \( \Omega \subset \mathbb{R}^2 \) with initial conditions:

\[
\rho(x, 0) = \rho_0(x), \quad d(x, 0) = d_0(x), \quad (\rho v)(x, 0) = m_0(x) \quad \text{in} \ \Omega,
\]

and boundary conditions:

\[
\begin{cases}
  d(x, t) = d_0(x), \quad v(x, t) = 0, \quad x \in \partial \Omega, \ t > 0, \text{ if } \Omega \text{ is a bounded domain;} \\
  (\rho, v, d)(x, t) \to (\rho_\infty, 0, e_2) \quad \text{as} \ |x| \to \infty, \quad \text{if } \Omega = \mathbb{R}^2,
\end{cases}
\]

where \( \rho_\infty \geq 0 \) is a constant and \( e_2 = (0, 1) \in \mathbb{R}^2 \) is the unit vector. For simplicity, we call (1.1)–(1.5) the initial-boundary value problem when \( \Omega \in \mathbb{R}^2 \) is a bounded domain, and the Cauchy problem in the case of \( \Omega = \mathbb{R}^2 \).

Before stating our main result, we remark that the constants \( \mu, \lambda, \nu, \text{ and } \theta \) satisfy the physical conditions:

\[
\mu > 0, \quad \lambda + \mu \geq 0, \quad \nu > 0, \quad \theta > 0.
\]

And the pressure \( P(\rho) \) is usually determined through the equations of state, here we focus our study on the case of isentropic flows as in [38] and assume that

\[
P(\rho) = A\rho^\gamma, \quad \text{with } A > 0, \ \gamma > 1.
\]

In addition, for the sake of simplicity, we define

\[
I := I_T := (0, T), \quad Q_T = \Omega \times I,
\]

\[
F(t) := F(\rho, v, d) := \int_\Omega \left( \mu |\nabla v|^2 + (\lambda + \mu) |\text{div} v|^2 + \theta (|\Delta d + |\nabla d|^2|d|)| \right) \text{dx},
\]

and

\[
E(t) := E(\rho, m, d) = \int_\Omega \left( \frac{1}{2} \frac{|m|^2}{\rho} 1_{\{\rho > 0\}} + Q(\rho) + \frac{\nu \theta |\nabla d|^2}{2} \right) \text{dx} \quad \text{with} \quad m = \rho v,
\]

where \( 1_{\{\rho > 0\}} \) denotes the character function, and

\[
Q(s) = \frac{A}{\gamma - 1} \left( s^\gamma - \gamma s^\rho_\infty^{-1} + (\gamma - 1) \rho_\infty^\gamma \right) \quad \text{for some constant } \rho_\infty \geq 0.
\]

Obviously, \( Q(s) \geq 0 \) for any \( s \geq 0 \), and \( E(t) \geq 0 \). Throughout this paper, we use the bold fonts to denote the product spaces, for example,

\[
\mathbf{L}^p(\Omega) := (L^p(\Omega))^2, \quad \mathbf{H}^k(\Omega) := (H^k_0(\Omega))^2 = (W^{k,2}_0(\Omega))^2, \quad \mathbf{H}^k(\Omega) := (W^{k,2}(\Omega))^2;
\]

and the Sobolev space with weak topology is defined as

\[
C^0(\bar{I}, \mathbf{L}^q_{\text{weak}}(\Omega)) := \left\{ f : I \to \mathbf{L}^q(\Omega) \mid \int_\Omega f \cdot g \text{dx} \in C(\bar{I}) \text{ for any } g \in \mathbf{L}^q(\Omega) \right\}.
\]

Our first result on the existence of weak solutions in a bounded domain \( \Omega \) reads as follows.
Theorem 1.1. Let the constant \( \rho_\infty \geq 0, \Omega \) be a bounded domain of class \( C^{2,\alpha} \) with \( \alpha \in (0,1) \), and the initial data \( \rho_0, \mathbf{m}_0, \mathbf{d}_0 \) satisfy the following conditions:

\[
Q(\rho_0) \in L^1(\Omega), \quad \rho_0 \geq 0 \text{ a.e. in } \Omega, \tag{1.9}
\]

\[
\mathbf{m}_0 \in L^{\frac{2+\kappa}{\kappa}}(\Omega), \quad \mathbf{m}_0 1_{\{\rho_0=0\}} = 0 \text{ a.e. in } \Omega, \quad \frac{|\mathbf{m}_0|^2}{\rho_0} 1_{\{\rho_0 > 0\}} \in L^1(\Omega), \tag{1.10}
\]

\[
|\mathbf{d}_0| = 1 \text{ in } \Omega, \quad \mathbf{d}_0(x) \in \mathbf{H}^2(\Omega). \tag{1.11}
\]

Then, there exists a constant \( C_0 \), such that if \( \mathcal{E}_0 := \mathcal{E}(0) \) satisfies

\[
\mathcal{E}_0 < C_0 \nu, \tag{1.12}
\]

the initial-boundary value problem \((1.1)-(1.5)\) has a global weak solution \((\rho, \mathbf{v}, \mathbf{d})\) on \( I = I_T \) for any given \( T > 0 \), with the following properties:

1. Regularity:

\[
0 \leq \rho \text{ a.e. in } Q_T, \quad \rho \in C^0(I, \mathcal{L}^\kappa_{\text{weak}}(\Omega) \cap C^0(I, L^p(\Omega))) \cap L^\gamma+\eta(Q_T), \tag{1.13}
\]

\[
\mathbf{v} \in L^2(I, \mathbf{H}^1_0(\Omega)), \rho \mathbf{v} \in L^\infty(I,L^{\frac{2+\kappa}{\kappa}}(\Omega)) \cap C^0(I, L^{\frac{2+\kappa}{\kappa}}(\Omega)), \tag{1.14}
\]

\[
|\mathbf{d}| = 1 \text{ a.e. in } Q_T, \quad \mathbf{d} \in L^2(I, \mathbf{H}^2(\Omega)) \cap C^0(I, \mathbf{H}^1(\Omega)), \quad \partial_t \mathbf{d} \in L^4(I, L^2(\Omega)), \tag{1.15}
\]

\[
|\mathbf{d}|_{\partial \Omega} = \mathbf{d}_0|_{\partial \Omega} \in C^{0,1}(\partial \Omega) \text{ in the sense of trace for a.e. } t \in I, \tag{1.16}
\]

where \( p \in [1, \gamma] \), and \( \eta \in (0, \gamma-1) \).

2. Equations \((1.1)\) and \((1.2)\) hold in \((\mathcal{D}'(Q_T))^3\), and equation \((1.3)\) holds a.e. in \( Q_T \).

3. Equation \((1.7)\) is satisfied in the sense of renormalized solutions, that is, \( \rho, \mathbf{v} \) satisfy

\[
\partial_t b(\rho) + \text{div} [b(\rho) \mathbf{v}] + [\rho b'(\rho) - b(\rho)] \text{div} \mathbf{v} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^2 \times I),
\]

provided \((\rho, \mathbf{v})\) is prolonged to be zero on \( \mathbb{R}^2 \setminus \Omega \), for any \( b \) satisfying

\[
b \in C^0[0,\infty) \cap C^1(0,\infty), \quad |b'(s)| \leq cs^{-\lambda_0}, \quad s \in (0,1], \quad \lambda_0 < 1, \tag{1.17}
\]

and the growth conditions at infinity:

\[
|b'(s)| \leq c s^{\lambda_1}, \quad s \geq 1, \text{ where } c > 0, \quad 0 < 1 + \lambda_1 < \frac{2\gamma - 1}{2}. \tag{1.18}
\]

4. The following finite and bounded energy inequalities hold:

\[
\frac{d\mathcal{E}(t)}{dt} + \mathcal{F}(t) \leq 0 \quad \text{in } \mathcal{D}'(I), \tag{1.19}
\]

\[
\mathcal{E}(t) + \int_0^t \mathcal{F}(s) ds \leq \mathcal{E}_0 \text{ for a.e. } t \in I. \tag{1.20}
\]

Remark 1.1. It should be noted that for the incompressible case \([4]\) the constant \( C_0 \) in a small energy condition similar to \((1.12)\) depends on the domain \( \Omega \), since the authors in \([4]\) used the following interpolation inequality: Given a bounded domain \( \Omega \subset \mathbb{R}^2 \), there exists an interpolating constant \( c_1(\Omega) \) depending on \( \Omega \), such that

\[
\|\nabla \mathbf{d}\|_{L^1(\Omega)}^2 \leq c_1(\Omega)(\|\nabla \mathbf{d}\|_{L^2(\Omega)}^2 + \|\nabla^2 \mathbf{d}\|_{L^2(\Omega)}^2) \|\nabla \mathbf{d}\|_{L^2(\Omega)}^2 \tag{1.21}
\]

for any \( \nabla \mathbf{d} \in \mathbf{H}^1(\Omega) \) (see \([4\), Lemma 2.4]). In this paper, we use another version of interpolation inequality (see \((2.12)\)), where the interpolating constant takes the value of 2. Thus our constant \( C_0 \) in \((1.12)\) is independent of \( \Omega \). In particular, from our computations (see Proposition 2.1) we can take \( C_0 = 1/4096 \) in \((1.12)\). Of course, this value is not optimal.
Remark 1.2. In the above theorem, we assume that \( d_0 \in H^2(\Omega) \) for simplicity. If this condition is replaced by \( d_0 \in H^1(\Omega) \) and \( d_0 \in C^{2,\beta}(\partial\Omega) \) as in [26], then the above theorem still holds. In addition, the proof of Theorem 1.1 remains basically unchanged if the motion of the fluid is driven by a small bounded external force, i.e., when (1.2) contains an additional term \( \rho f(x, t) \) with \( f \) a small, bounded and measurable function.

We now describe the main idea of the proof of Theorem 1.1. For the Ginzburg-Landau approximation model to (1.1)–(1.3), based on some new estimates to deal with the direction field and its coupling/interaction with the fluid variables, Wang and Yu in [38] adopted a classical three-level approximation scheme which consists of the Faedo-Galerkin approximation, artificial viscosity, artificial pressure, and the celebrated weak continuity of the effective viscous flux to overcome the difficulty of possible large oscillations of the density, and established the existence of weak solutions. These techniques were developed by Lions, Feireisl, et al. for the compressible Navier-Stokes equations in [8, 19, 31], and we refer to the monograph [34] for more details. Compared with the Ginzburg-Landau approximation model in [38], however, the system (1.1)–(1.3) is much more difficult to deal with due to the supercritical nonlinearity \( |\nabla d|^2 d \) in (1.3). Consequently, one can not deduce sufficiently strong estimates on \( d \) from the basic energy inequality (1.20) in the general case, such as \( \nabla^2 d \in L^2(I, L^2(\Omega)) \) as in [38]; and on the other hand, one can not establish the global existence of large solutions to the equation (1.3) for any given \( v \in C^0(I, C_0^0(\Omega)) \) as in the Ginzburg-Landau approximation model. Recently, Ding and Wen [4] obtained the global existence and uniqueness of strong solutions to the two-dimensional density-dependent incompressible model with small initial energy and positive initial density away from zero. In [4], they can deduce \( \nabla^2 d \in L^2(I, L^2(\Omega)) \) from the basic energy inequality in two dimensions under the condition of small initial energy. In fact, they first got \( \nu \| \nabla d \|_{L^2(Q_T)}^2 \leq \mathcal{E}_0 + \nu \theta \| \nabla d \|_{L^2(Q_T)}^4 \) from the basic energy inequality. Then, by interpolation inequality (1.21) and elliptic estimates, they immediately obtain \( \| \nabla^2 d \|_{L^2(Q_T)}^2 \leq C(\mathcal{E}_0, \| \nabla^2 d_0 \|_{L^2(\Omega)}) \) for some constant \( C(\mathcal{E}_0, \| \nabla^2 d_0 \|_{L^2(\Omega)}) \) depending on the arguments, if initial energy is sufficiently small. Motivated by this study, for our compressible case we impose the restriction (1.12) on the initial energy, and consequently deduce the desired energy estimates on \( d \) from the basic energy inequality as in [38]. With these estimates in hand, we also find that the three-level approximation scheme can be applied to the initial-boundary value problem (1.1)–(1.5), if we can construct solutions to the following third approximate problem:

\[
\rho_t + \text{div}(\rho \mathbf{v}) = \varepsilon \Delta \rho,
\]

\[
\int_{\Omega} (\rho \mathbf{v})(t) \cdot \Psi \, dx - \int_{\Omega} m_0 \cdot \Psi \, dx
\]

\[
= \int_0^t \int_{\Omega} \mu \Delta \mathbf{v} + (\mu + \lambda) \nabla \text{div} \mathbf{v} - A \nabla \rho^\gamma - \delta \nabla \rho^\beta - \varepsilon (\nabla \rho \cdot \nabla \mathbf{v})
\]

\[
- \text{div}(\rho \mathbf{v} \otimes \mathbf{v}) - \nu \text{div} \left( \nabla \mathbf{d} \otimes \nabla \mathbf{d} - \frac{|\nabla \mathbf{d}|^2 I}{2} \right) \] \cdot \Psi \, dx \, ds
\]

for all \( t \in I \) and any functions \( \Psi \in X_n \),

\[
\partial_t \mathbf{d} + \mathbf{v} \cdot \nabla \mathbf{d} = \theta (\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}),
\]

where the \( n \)-dimensional Euclidean space \( X_n \) will be introduced in Section 3, \( \varepsilon, \delta, \beta > 0 \) are constants. Fortunately, we can show the local existence of strong solutions \( \mathbf{d} \in C^0(\bar{I}, H^2(\Omega)) \) to the equation (1.23) with smooth initial data for any given \( \mathbf{v} \in \{ \mathbf{v} \in C^0(\bar{I}, H^1(\Omega) \cap L^\infty(\Omega)) \mid \partial_t \mathbf{v} \in \} \)
Given a problem in solution to the whole time interval \( I \), Theorem 1.2. Assume that, for some positive constants \( \rho_\infty \) and \( d_{02} \), \( (\rho_0, m_0) \) satisfies (3.9), (4.10) with \( \rho \in L^\infty(I, L^1(\mathbb{R}^2)) \), \( \rho \in C^0(\bar{I}, L^1(\mathbb{R}^2)) \) and establish some additional auxiliary estimates on \( \partial_t v \), we can apply a fixed point theorem in some bounded subspace of the Banach space \( \{ v \in C([0, T^*_n], X_n) \mid \partial_t v \in L^2((0, T^*_n), H^1(\Omega)) \} \) for some \( T^*_n \in (0, T] \) to obtain the local existence of the third approximate problem. Noting that the regularity of \( \partial_t v \) depends on \( \partial_t \rho \), this is why we need the properties on \( \partial_t \rho \) in Proposition 3.1 on the solvability of (4.22). Then, we can repeat the fixed point argument to extend the local solution to the whole time interval \( I \) by establishing the uniform-in-time energy estimates for the third approximate problem.

Next, we state our second result on the existence of global weak solutions to the Cauchy problem in \( \mathbb{R}^2 \) with large initial data:

**Theorem 1.2.** Assume that, for some positive constants \( \rho_\infty \) and \( d_{02} \), \( (\rho_0, m_0) \) satisfies (3.9), (4.10) with \( \mathbb{R}^2 \) in place of \( \Omega \), and

\[
|d_0| = 1 \text{ a.e. in } \mathbb{R}^2, \quad d_0(x) - e_2 \in H^1(\mathbb{R}^2), \quad d_{02} \geq d_{02},
\]

where \( d_{02} \) denotes the second component of \( d_0 \). Then, the Cauchy problem (1.1)–(1.3) has a global bounded energy weak solution \( (\rho, v, d) \) for any given \( T > 0 \) with the following properties:

1. Regularity:

\[
0 \leq \rho \text{ a.e. in } \mathbb{R}^2 \times I, \quad Q(\rho) \in L^\infty(I, L^1(\mathbb{R}^2)),
\]

\[
\rho \in C^0(\bar{I}, L_{\text{weak}}^1(\Omega')) \cap C^0(\bar{I}, L^p(\Omega')) \cap L^{\gamma+q}(\mathbb{R}^2 \times I),
\]

\[
v \in L^2(I, (D^{1,2}(\mathbb{R}^2))^2) \cap L^2(I, L^{2,loc}(\Omega)), \quad \rho \in C^0(\bar{I}, L^1(\mathbb{R}^2)) \cap L^2(I, L^{q,loc}(\mathbb{R}^2)),
\]

\[
\partial_t d \in L^2(I, L^{2,loc}(\mathbb{R}^2)), \quad |d| = 1 \text{ a.e. in } \mathbb{R}^2 \times I,
\]

\[
d \in C^0(\bar{I}, H^1_{\text{loc}}(\mathbb{R}^2)), \quad \nabla d \in L^2(I, H^1(\mathbb{R}^2)) \cap L^4(\mathbb{R}^2 \times I),
\]

where \( p \in [1, \gamma], \eta \in (0, \gamma - 1), q \geq 1 \), and \( \Omega' \subset \mathbb{R}^2 \) is an arbitrary bounded subdomain.

2. Equations (1.1), (1.2) holds in \( \mathcal{D}'(\mathbb{R}^2 \times I) \), equation (1.3) holds a.e. in \( \mathbb{R}^2 \times I \). Moreover, equation (1.1) is satisfied in the sense of renormalized solutions.

3. The solution satisfies the energy equality (1.20) with \( \mathbb{R}^2 \) in place of \( \Omega \).

**Remark 1.3.** The notation \( D^{1,2}(\mathbb{R}^2) \) in the above theorem denotes the homogeneous Sobolev space on \( \mathbb{R}^2 \), i.e., \( D^{1,2}(\mathbb{R}^2) = C_0^\infty(\mathbb{R}^2, \|v\|_{L^2(\mathbb{R}^2)}) \), where the symbol “overline with norm” means completion with respect to that norm. Please refer to [34, Section 1.3.6] for more basic conclusions concerning the homogeneous Sobolev spaces.

**Remark 1.4.** In Theorem 1.2, the choice of \( e_2 \) is for convenience only. In general, one can choose any reference vector \( e \in S^1 \) and require that the image of \( d \) is contained in a hemisphere around \( e \). Let \( \rho_\infty = 0 \) and suppose that in addition to the assumptions above we have also \( \rho_0 \in L^1(\mathbb{R}^2) \). Then the result above remains valid, moreover there holds \( \int_{B^2} \rho d\mathbf{x} = \int_{B^2} \rho_0 d\mathbf{x} \) in \( \bar{I} \). In addition, in view of the proof of Theorem 1.2 and the following interpolation inequality: there exists a constant \( c \) such that

\[
\|v\|_{L^4(B_R)}^4 \leq c \|v\|_{H^1(B_R)}^2 \|v\|_{L^2(B_R)}^2 \text{ for any } v \in H^1(B_R) \text{ with } R \geq 1,
\]
the above theorem still holds if we impose a small initial energy condition similar to (1.12) to replace the geometric angle condition $d_{02} \geq d_{02}$ in (1.25). Here $B_R \subset \mathbb{R}^2$ is the ball of radius $R$.

**Remark 1.5.** Recently, Lin-Lin-Wang [26] established the existence of large weak solutions for the corresponding incompressible case in a two-dimensional bounded domain. However, it is not clear whether the method of localized small energy can be applied to the compressible model, due to lack of regularity of $\|v\|_{L^\infty(I,L^2(\Omega))}$. Hence it is still an interesting open problem whether a general existence result of large weak solutions holds in bounded domains for the compressible case. In particular, we can not generalize the above result of the Cauchy problem to the bounded domain case, since it is not clear whether the rigidity theorem (see Proposition 6.2) holds in the bounded domain case.

**Remark 1.6.** When we completed this paper, we were informed that Wu and Tan [39] just finished the proof of the existence of global weak solutions to (1.1)–(1.3) in $\mathbb{R}^3$ by using Suen and Hoff’s method [35], if the initial energy is sufficiently small, the coefficients $\mu$ and $\lambda$ satisfy $0 \leq \lambda + \mu < \frac{3 + \sqrt{21}}{6} \mu$, and the initial data $(v_0, d_0)$ satisfies $\|v_0\|_{L^p(\mathbb{R}^3)} + \|d_0\|_{L^p(\mathbb{R}^3)} < \infty$ with $p > 6$. However, it is not clear whether their proof can be applied to (1.1)–(1.3) in a three-dimensional bounded domain.

Here we also describe the main idea of the proof of Theorem 1.2. Recently Lei, Li and Zhang [20] proved the rigidity theorem in $\mathbb{R}^2$ (see Proposition 6.2) and obtained the estimate on $\|\nabla^2 d\|_{L^2(I,L^2(\mathbb{R}^2))}$ from the energy inequality, if the second component of initial data of $d$ satisfies some geometric angle condition. Motivated by this study, we impose the geometric angle condition $d_{02} \geq d_{02}$ in (1.25) in place of (1.12). We first establish the local solvability of the Cauchy problem (1.24) on $d$ (see Proposition 6.1), which can be shown by following the proof of the local solvability of $d$ for the problem (1.24) defined in a bounded domain and using domain expansion technique. Then, using the rigidity theorem, elliptic estimates and interpolation inequalities in $\mathbb{R}^2$, we can follow the proof of global existence of solutions to the third approximate problem (1.22)–(1.24) defined in a bounded domain to establish the global existence of solutions to the approximate problem (1.22)–(1.24) defined in a ball $B_R$ and (1.24) defined in $\mathbb{R}^2$ (see Proposition 6.4). Finally, we adapt the proof in [34, Section 7.11] on the Navier-Stokes equations to prove Theorem 1.2 by using Proposition 6.4 with the mass and momentum equations defined in the bounded invading domain $B_R$ and letting $R \to \infty$.

The rest of paper is organized as follows. In Section 2, we deduce the basic energy equalities from (1.1)–(1.3) and derive more energy estimates on $d$ under the assumption (1.12). In Section 3 we list some preliminary results on solvability of two sub-systems (1.22) and (1.23) in the third approximate problem, while in Sections 4 we establish the unique solvability of the third approximate problem. In Sections 5 exploiting the standard three-level approximation scheme, we complete the proof of Theorem 1.1. Finally we provide the proof of Theorem 1.2 in Section 6.

## 2. Energy estimates

This section is devoted to formally deriving the energy estimates from (1.1)–(1.3) in a bounded domain $\Omega$, which will play a crucial role in the proof of existence. Some of our results have been established in [4, 24] for the incompressible case.
2.1. Energy equalities

We consider a classical solution \((\rho, \mathbf{v}, d)\) of the problem (1.1)–(1.3) with initial and boundary conditions (1.4) and (1.5). First we verify that

\[ |d| \equiv 1 \text{ in } Q_T, \text{ if } |d_0| = 1 \text{ in } \Omega. \]  

(2.1)

Multiplying the \(d\)-system (1.3) by \(d\), we obtain

\[
\frac{1}{2} \partial_t |d|^2 + \frac{1}{2} \mathbf{v} \cdot \nabla |d|^2 = \theta (\Delta d \cdot d + |\nabla d|^2 |d|^2).
\]

Since

\[ \Delta |d|^2 = 2 |\nabla d|^2 + 2 \Delta d \cdot d, \]

it follows that

\[
\partial_t (|d|^2 - 1) - \theta \Delta (|d|^2 - 1) + \mathbf{v} \cdot \nabla (|d|^2 - 1) - 2 \theta |\nabla d|^2 (|d|^2 - 1) = 0. \]  

(2.2)

Multiplying (2.2) by \((|d|^2 - 1)\) and then integrating over \(\Omega\), we use the boundary conditions to get

\[
\frac{d}{dt} \int_{\Omega} (|d|^2 - 1)^2 dx \leq \int_{\Omega} (4 \theta |\nabla d|^2 + \text{div} \mathbf{v})(|d|^2 - 1)^2 dx \leq \|4 \theta |\nabla d|^2 + |\text{div} \mathbf{v}|\|_{L^\infty(\Omega)} \int_{\Omega} (|d|^2 - 1)^2 dx.
\]

We assume that \((\mathbf{v}, d)\) satisfies the following regularity:

\[
\|4 \theta |\nabla d|^2 + |\text{div} \mathbf{v}|\|_{L^1(I, L^\infty(\Omega))} < \infty,
\]

then, using Gronwall’s inequality, we immediately verify (2.1). We shall see that all the couples \((\mathbf{v}_n, d_n)\) in the third approximate solutions constructed in Section 4 satisfy the regularity above.

Similarly, we can verify

\[ d_2 \geq d_0^2 \text{ if } d_0 \geq d_0^2 \text{ for some given constant } d_0^2, \]  

(2.3)

which will play a crucial role in the existence of large weak solutions to the Cauchy problem. Here we have denoted the second component of \(d\) and \(d_0\) by \(d_2\) and \(d_0^2\), respectively.

Next, we give the deduction of (2.3) for the reader’s convenience. Let

\[ \omega = d_2 - d_0^2, \quad \omega^- = \min\{\omega, 0\}, \]

we can deduce from (1.3) that

\[
\partial_t \omega - \theta \Delta \omega = \theta |\nabla d|^2 (\omega + d_0^2) - \mathbf{v} \cdot \nabla \omega. \]  

(2.4)

Multiplying (2.4) by \(\omega^-\) and integrating over \(\Omega\), we get

\[
\frac{1}{2} \frac{d}{dt} \|\omega^-\|^2_{L^2(\mathbb{R}^2)} + \theta \|\nabla \omega^-\|^2_{L^2(\Omega)} \\
= \int_{\Omega} [\theta |\nabla d|^2 (\omega + d_0^2) - \mathbf{v} \cdot \nabla \omega] |\omega^-| dx \\
\leq \left\| \theta |\nabla d|^2 + \frac{1}{2} \text{div} \mathbf{v} \right\|_{L^\infty(\Omega)} \|\omega^-\|^2_{L^2(\Omega)} + d_0^2 \theta \|\nabla d\|_{L^\infty(\Omega)} \int_{\Omega} |\nabla \omega^-| |\omega^-| dx \\
\leq \left( \left\| \theta |\nabla d|^2 + \frac{1}{2} \text{div} \mathbf{v} \right\|_{L^\infty(\Omega)} + \frac{1}{2} d_0^2 \theta \|\nabla d\|_{L^\infty(\Omega)}^2 \right) \|\omega^-\|^2_{L^2(\Omega)} + \frac{\theta}{2} \|\nabla \omega^-\|^2_{L^2(\Omega)},
\]
which yields
\[
\frac{d}{dt} \|\omega^-\|_{L^2(\Omega)}^2 \leq \left( 2 \left\| \theta |\nabla d|^2 + \frac{1}{2} \text{div} v \right\|_{L^\infty(\Omega)} + 2L_2^2 \theta \|\nabla d\|_{L^2(\Omega)}^2 \right) \|\omega^-\|_{L^2(\Omega)}^2.
\]
Hence, by Gronwall’s inequality, one obtains
\[
\|\omega^-(t)\|_{L^2(\Omega)}^2 \leq \|\omega^-(0)\|_{L^2(\Omega)}^2 e^{\int_0^t \left( 2 \left\| \theta |\nabla d|^2 + \frac{1}{2} \text{div} v \right\|_{L^\infty(\Omega)} + 2L_2^2 \theta \|\nabla d\|_{L^2(\Omega)}^2 \right) ds} = 0,
\]
which implies (2.3).

Now we derive the energy equality. With the help of (2.1), we can deduce the basic energy equality. To this end, we multiply equation (1.2) by $v$ and integrate over $\Omega$ to deduce that
\[
\frac{d}{dt} \int_\Omega \left( \frac{1}{2} \rho |v|^2 + Q(\rho) \right) dx + \int (\mu |\nabla v|^2 + (\lambda + \mu) \text{div} v |v|^2) dx = -\nu \int (\nabla d)^T \Delta d \cdot v dx,
\]
where we have used integration by parts, the boundary condition of $v$, the mass equation (1.1) and the equality
\[
\text{div} \left( \nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 \right) = (\nabla d)^T \Delta d := (\partial_i d_j)_{2 \times 2} \Delta d.
\]

On the other hand, multiplying (1.3) by $-(\Delta d + |\nabla d|^2 d)$ and integrating over $\Omega$, we obtain
\[
- \int_\Omega \partial_t d \cdot \Delta d dx - \int_\Omega (v \cdot \nabla d) \cdot \Delta d dx = -\theta \int_\Omega |\Delta d + |\nabla d|^2 d|^2 dx,
\]
where we have used the fact that $|d| = 1$ to get
\[
(\partial_t d + v \cdot \nabla d) \cdot |\nabla d|^2 d = \frac{1}{2} (|\nabla d|^2 \partial_i |d|^2 + |\nabla d|^2 v \cdot \nabla |d|^2) = 0.
\]

Using the boundary condition of $d$, and integrating by parts, we deduce from (2.6) that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla d|^2 dx + \theta \int_\Omega |\Delta d + |\nabla d|^2 d|^2 dx = \int_\Omega (v \cdot \nabla d) \cdot \Delta d dx.
\]

Consequently, (2.5) + (2.7) \times \nu gives the energy equality in the differential form:
\[
\frac{d}{dt} \mathcal{E}(t) + \mathcal{F}(t) = 0,
\]
where the energy $\mathcal{E}(t)$ and dissipation term $\mathcal{F}(t)$ are defined by (1.7) and (1.6). Finally, integrating (2.8) with respect to time over $(0, t)$, we get the energy equality
\[
\mathcal{E}(t) + \int_0^t \mathcal{F}(s) ds = \mathcal{E}_0, \quad t \in I,
\]
where $\mathcal{E}_0 = \mathcal{E}(0)$ denotes the initial energy. Due to the weak lower semicontinuity of norms on the right-hand side of (2.9), for weak solutions one expects an inequality rather than an equality.
2.2. More a priori estimates on $d$ under the condition $(1.12)$

In this subsection, we will deduce more estimates of $d$ under the assumption $(1.12)$, including the case of small initial energy. For simplicity in the deduction, we always use the positive constant $c_1(\Omega)$ to denote various constants depending on $\Omega$.

Multiplying $(1.3)$ by $\Delta d$, integrating the resulting equation over $\Omega$, and using integration by parts, we find that

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla d|^2 dx + \int_\Omega |\Delta d|^2 dx = \theta \int_\Omega |\nabla d|^4 dx + \int_\Omega (v \cdot \nabla d) \cdot \Delta dd. \tag{2.10}
$$

Thus, $(2.5)+(2.10) \times \nu$ gives

$$
\frac{d}{dt} \mathcal{E}(t) + \int_\Omega [\mu|\nabla v|^2 + (\lambda + \mu)|\text{div}v|^2 + \nu|d|^2] \, dx = \nu \int_\Omega |\nabla d|^4 dx. \tag{2.11}
$$

Recalling $\Omega \subset \mathbb{R}^2$, exploiting [36, Lemma 3.3], i.e.,

$$
||v||^4_{L^4(\Omega)} \leq 2||\nabla v||^2_{L^2(\Omega)}||v||^2_{L^2(\Omega)} \text{ for any } v \in H^1_0(\Omega), \tag{2.12}
$$

we can infer that

$$
||\partial_i d_j||^4_{L^4(\Omega)} \leq \left( ||\partial_i d_{0j}||^4_{L^4(\Omega)} + 2^{1/4} ||\partial_i \nabla (d_j - d_{0j})||^2_{L^2(\Omega)} ||\partial_i (d_j - d_{0j})||^2_{L^2(\Omega)} \right)^4
\leq 8 \left( ||\partial_i d_{0j}||^4_{L^4(\Omega)} + 2 ||\partial_i \nabla (d_j - d_{0j})||^2_{L^2(\Omega)} ||\partial_i (d_j - d_{0j})||^2_{L^2(\Omega)} \right)
\leq 32 \left( c_1(\Omega) ||\partial_i d_{0j}||^2_{H^1(\Omega)} ||\partial_i (d_j - d_{0j})||^2_{L^2(\Omega)} \right)
\leq 32 \left( 8 ||\partial_i d_{0j}||^2_{L^2(\Omega)} ||\partial_i (d_j - d_{0j})||^2_{L^2(\Omega)} \right)
\leq 128 \nu \theta \sum_{1 \leq i,j \leq 2} \left( c_1(\Omega) ||\partial_i d_{0j}||^2_{H^1(\Omega)} ||\partial_i (d_j - d_{0j})||^2_{L^2(\Omega)} \right) \tag{2.13}
$$

Consequently, making use of $(2.9)$, $(2.13)$, Hölder’s inequality, and the elliptic estimate

$$
||\nabla^2 d_j||_{L^2(\Omega)} \leq 2 ||\Delta d_j||_{L^2(\Omega)} + 6 ||\nabla^2 d_{0j}||_{L^2(\Omega)} \tag{2.14}
$$

deduced from [10, Collary 9.10], we see that the last term in $(2.11)$ can be estimated as follows:

$$
\nu \int_\Omega |\nabla d|^4 dx \leq 128 \nu \theta \sum_{1 \leq i,j \leq 2} \left( c_1(\Omega) ||\partial_i d_{0j}||^2_{H^1(\Omega)} ||\partial_i (d_j - d_{0j})||^2_{L^2(\Omega)} \right)
\leq 128 \nu \theta \sum_{1 \leq i,j \leq 2} \left( c_1(\Omega) ||\partial_i d_{0j}||^2_{H^1(\Omega)} ||\partial_i (d_j - d_{0j})||^2_{L^2(\Omega)} \right)
\leq 128 \nu \theta \left( c_1(\Omega) + 12 \right) ||d_0||^2_{H^2(\Omega)} \left( ||d_0||^2_{H^2(\Omega)} + \frac{2 \mathcal{E}_0}{\nu} \right) + 1024 \theta \mathcal{E}_0 ||d||^2_{L^2(\Omega)}. \tag{2.15}
$$

Choosing $\mathcal{E}_0 > 0$, such that $\mathcal{E}_0 < \nu/4096$, we have

$$
\frac{d}{dt} \mathcal{E}(t) + \int_\Omega \left( \mu|\nabla v|^2 + (\lambda + \mu)|\text{div}v|^2 + \frac{3 \nu \theta}{4} |\Delta d|^2 \right) \, dx
\leq 128 \nu \theta \left( c_1(\Omega) + 12 \right) ||d_0||^2_{H^2(\Omega)} \left( ||d_0||^2_{H^2(\Omega)} + \frac{2 \mathcal{E}_0}{\nu} \right) := g_1. \tag{2.16}
$$
By (2.14), we obtain
\[ \| \nabla^2 d \|_{L^2(Q_T)}^2 \leq 8 \left( \frac{4(g_1 T + E_0)}{3\nu \theta} + 9\| \nabla^2 d_0 \|_{L^2(\Omega)}^2 \right), \]
where
\[ \| \nabla^2 d \|_{L^2(\Omega)}^2 := \sum_{1 \leq i,j,l \leq 2} \| \partial_i \partial_j d_l \|_{L^2(\Omega)}^2. \]
Moreover, recalling \(|d| = 1\), (2.16) and (2.9), we see that
\[ \int_0^T \int_\Omega |\nabla d|^4 dx dt \leq 2 \left( \sqrt{\frac{E_0}{\nu \theta}} + \sqrt{\frac{4(g_1 T + E_0)}{3\nu \theta}} \right)^2 =: g_2. \quad (2.17) \]

Finally, utilizing (2.9), (2.17), Hölder’s and Poincaré’s inequalities, we get from the equation (1.3) that
\[ \| \partial_t d_j \|_{L^{4/3}(I,L^2(\Omega))} \leq \| v \cdot \nabla d_j \|_{L^{4/3}(I,L^2(\Omega))} + \theta \| \Delta d_j \|_{L^{4/3}(I,L^2(\Omega))} + |\nabla d|^2 d_j \|_{L^4(Q_T)} + \theta T^{1/4} \| \Delta d_j \|_{L^2(Q_T)} \]
\[ \leq c_1(\Omega) \sqrt{\frac{E_0 g_2}{\mu}} + T^{1/4} \sqrt{\frac{E_0 \theta}{\nu}}. \]

In addition, we can also deduce that
\[ \| \partial_t d_j \|_{L^2(I,(H^1(\Omega))^*)} \leq c_1(\Omega) \sqrt{\frac{E_0}{\mu}} + \sqrt{\frac{E_0 \theta}{\nu}}, \]
where \((H^1(\Omega))^*\) denotes the dual space of \(H^1(\Omega)\).

Summing up the above estimates, we conclude that

**Proposition 2.1.** Let \(\Omega\) be a bounded domain. We have the following a priori estimate for the initial-boundary value problem (1.1)–(1.5) with initial data \(|d_0| \equiv 1\):
\[ \sup_{t \in I}(\| \sqrt{\rho} v \|_{L^2(\Omega)} + \| \rho \|_{L^2(\Omega)} + \| \nabla d \|_{L^2(\Omega)}) + \| \nabla v \|_{L^2(I,L^2(\Omega))} \leq c_2(E_0). \quad (2.18) \]

Moreover, if the initial energy satisfies \(E_0 < \nu/4096\), then
\[ \| \nabla^2 d \|_{L^2(Q_T)} + \| \nabla d \|_{L^2(Q_T)} + \| \partial_t d \|_{L^{4/3}(I,L^2(\Omega))} + \| \partial_t d \|_{L^2(I,(H^1(\Omega))^*)} \]
\[ \leq c_3(\|d_0\|_{H^2(\Omega)},E_0). \quad (2.19) \]

Here \(c_2\) and \(c_3\) are positive constants which are, in particular, nondecreasing in their variables. Moreover, \(c_2\) also depends on the given physical parameters \(A, \mu, \nu\) and \(\theta\); and \(c_3\) on \(\nu, \theta\) and the domain \(\Omega\).
3. Strong solvability of sub-systems in the third approximate problem

In order to get a weak solution to the problem (1.1)–(1.5) in a bounded domain $\Omega$, we shall first investigate the existence of solutions to the third approximate problem of the original problem (1.1)–(1.5):

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho \mathbf{v}) &= \varepsilon \Delta \rho, \quad (3.1) \\
\partial_t \mathbf{d} + \mathbf{v} \cdot \nabla \mathbf{d} &= \theta(\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}), \quad (3.2)
\end{align*}
\]

\[
\int_{\Omega} (\rho \mathbf{v})(t) \cdot \Phi \, d\mathbf{x} - \int_{\Omega} m_0 \cdot \Phi \, d\mathbf{x}
= \int_{0}^{t} \int_{\Omega} \left[ \mu \Delta \Phi + (\mu + \lambda) \nabla \text{div} \mathbf{v} - A \nabla \rho \gamma - \delta \nabla \rho^2 - \varepsilon (\nabla \rho \cdot \nabla \mathbf{v}) \right] \cdot \Phi \, d\mathbf{x} ds
\]

for all $t \in I$ and any $\Phi \in \mathbf{X}_n$, with boundary conditions

\[
\nabla \rho \cdot \mathbf{n}|_{\partial \Omega} = 0, \quad \mathbf{v}|_{\partial \Omega} = 0, \quad \mathbf{d}|_{\partial \Omega} = \mathbf{d}_0, \quad (3.4)
\]

and modified initial data

\[
\begin{align*}
\rho(x,0) &= \rho_0 \in W^{1,\infty}(\Omega), \quad 0 < \rho_0 \leq \bar{\rho} < \infty, \quad (3.5) \\
\mathbf{d}(x,0) &= \mathbf{d}_0 \in H^3(\Omega), \quad \mathbf{v}(x,0) = \mathbf{v}_0 \in \mathbf{X}_n, \quad (3.6)
\end{align*}
\]

where $\mathbf{n}$ denotes the outward normal to $\partial \Omega$, and $\varepsilon, \delta, \beta, \rho, \bar{\rho} > 0$ are constants. Here we briefly introduce the finite dimensional space $\mathbf{X}_n$. We know from [34, Section 7.4.3] that there exist countable sets

\[
\{\lambda_i\}_{i=1}^{\infty}, \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \text{ and}
\{\Psi_i\}_{i=1}^{\infty} \subset W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega), \quad 1 \leq p < \infty,
\]

such that

\[-\mu \Delta \Psi_i - (\mu + \lambda) \nabla \text{div} \Psi_i = \lambda_i \Psi_i, \quad i = 1, 2, \cdots,\]

and $\{\Psi_i\}_{i=1}^{\infty}$ is an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in $H^3(\Omega)$ with respect to the scalar product $\int_{\Omega} [\mu \partial_i u \cdot \partial_i v + (\mu + \lambda) \text{div} u \text{div} v] \, d\mathbf{x}$. We define a $n$-dimensional Euclidean space $\mathbf{X}_n$ with scalar product $<\cdot, \cdot>$ by

\[
\mathbf{X}_n = \text{span}\{\Psi_i\}_{i=1}^{n}, \quad <\mathbf{u}, \mathbf{v}> = \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x}, \quad \mathbf{u}, \mathbf{v} \in \mathbf{X}_n,
\]

and denote by $\mathcal{P}_n$ the orthogonal projection of $L^2(\Omega)$ onto $\mathbf{X}_n$.

In the next section, we will show the unique solvability of the third approximate problem based on the idea from [34, Section 7.7] with certain restrictions imposed on the initial approximate energy. Moreover, the unique solution enjoys the energy estimates such as (2.18), (2.19). As a preliminary, this section is devoted to the global solvability of the Neumann problem (3.1) for the density and the local solvability of the non-homogeneous Derichlet problem (3.2) when $v$ is given. Due to the difficulty of the supercritical nonlinearity $|\nabla \mathbf{d}|^2 \mathbf{d}$, we need here higher regularity imposed on the initial data $\mathbf{d}_0$ (see (3.6)) in order to get the local strong solution $\mathbf{d} \in L^2(I_\ast, H^3(\Omega))$ for some time interval $I_\ast$. Thus, using the embedding theorem, $\nabla \mathbf{d} \in L^2(I_\ast, L^\infty(\Omega))$. Such regularity is very important to deduce the pointwise property of $\mathbf{d}$ (see (2.1)).
3.1. The Neumann problem for the density

We consider the following problem:

$$\partial_t \rho + \text{div}(\rho \mathbf{v}) = \varepsilon \Delta \rho,$$

where can be uniquely solved in terms of $\mathbf{v}$ such that $\rho$ satisfies

$$\rho(x, 0) = \rho_0(x) \in W^{1,\infty}(\Omega), \quad 0 < \rho \leq \rho_0(x) \leq \bar{\rho} < \infty,$$

$$\nabla \rho \cdot \mathbf{n}|_{\partial \Omega} = 0,$$

which can be uniquely solved in terms of $\mathbf{v} \in L^\infty(I, W^{1,\infty}_0(\Omega))$. The global well-posedness of the above problem can be read as follows (see, e.g., [34, Proposition 7.39]).

Proposition 3.1. Let $0 < \alpha < 1$, $\Omega$ be a bounded domain of class $C^{2,\alpha}$, and $\rho_0$ satisfy (3.8). Then there exists a unique mapping

$$\mathcal{S}_{\rho_0} : L^\infty(I, W^{1,\infty}_0(\Omega)) \to C^0(\bar{I}, H^1(\Omega)),$$

such that

1. $\mathcal{S}_{\rho_0}(\mathbf{v})$ belongs to the function class

$$\mathcal{R}_T := \{ \rho | \rho \in L^2(I, W^{2,q}(\Omega)) \cap C^0(I, W^{1,q}(\Omega)), \partial_t \rho \in L^2(I, L^q(\Omega)) \}, 1 < q < \infty. \quad (3.10)$$

2. The function $\rho = \mathcal{S}_{\rho_0}(\mathbf{v})$ satisfies (3.7) a.e. in $Q_T$, (3.8) a.e. in $\Omega$ and (3.9) in the sense of traces a.e. in $I$.

3. $\mathcal{S}_{\rho_0}(\mathbf{v})$ is pointwise bounded, i.e.,

$$\rho \leq \int_0^t \| \mathbf{v} \|_{W^{1,\infty}(\Omega)} \, dt \leq \mathcal{S}_{\rho_0}(\mathbf{v})(x, t) \leq \bar{\rho} e^{\int_0^t \| \mathbf{v} \|_{W^{1,\infty}(\Omega)} \, dt}, \quad t \in \bar{I} \text{ for a.e. } x \in \Omega. \quad (3.11)$$

4. If $\| \mathbf{v} \|_{L^\infty(I, W^{1,\infty}(\Omega))} \leq \kappa_v$, then

$$\| \mathcal{S}_{\rho_0}(\mathbf{v}) \|_{L^\infty(I_t, H^1(\Omega))} \leq C_1 \| \rho_0 \|_{H^1(\Omega)} e^{C_2(\kappa_v + \kappa_2)t} \quad (3.12)$$

$$\| \nabla^2 \mathcal{S}_{\rho_0}(\mathbf{v}) \|_{L^2(Q_t)} \leq \frac{C_1}{\varepsilon} \sqrt{t} \| \rho_0 \|_{H^1(\Omega)} \kappa_v e^{C_2(\kappa_v + \kappa_2)t} \quad (3.13)$$

$$\| \partial_t \mathcal{S}_{\rho_0}(\mathbf{v}) \|_{L^2(Q_t)} \leq C \sqrt{t} \| \rho_0 \|_{H^1(\Omega)} \kappa_v e^{C_2(\kappa_v + \kappa_2)t} \quad (3.14)$$

for any $t \in \bar{I}$, where $I_t := (0, t)$, and $Q_t := \Omega \times I_t$. The constant $C_1$ in (3.12)–(3.14) depends at most on $\Omega$, and is independent of $\kappa_v, \varepsilon, T, \rho_0$ and $\mathbf{v}$.

5. $\mathcal{S}_{\rho_0}(\mathbf{v})$ depends continuously on $\mathbf{v}$, i.e.,

$$\| [\mathcal{S}_{\rho_0}(\mathbf{v}_1) - \mathcal{S}_{\rho_0}(\mathbf{v}_2)](t) \|_{L^2(\Omega)} \leq C_2(\kappa_v, \varepsilon, T) t \| \rho_0 \|_{H^1(\Omega)} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{L^\infty(I_t, W^{1,\infty}(\Omega))}, \quad (3.15)$$

$$\| \partial_t [\mathcal{S}_{\rho_0}(\mathbf{v}_1) - \mathcal{S}_{\rho_0}(\mathbf{v}_2)](t) \|_{L^2(Q_t)} \leq C_2(\kappa_v, \varepsilon, T) \sqrt{t} \| \rho_0 \|_{H^1(\Omega)} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{L^\infty(I_t, W^{1,\infty}(\Omega))} \quad (3.16)$$

for any $t \in \bar{I}$, and for any $\| \mathbf{v}_1 \|_{L^\infty(I, W^{1,\infty}(\Omega))} \leq \kappa_v$ and $\| \mathbf{v}_2 \|_{L^\infty(I, W^{1,\infty}(\Omega))} \leq \kappa_v$. The constant $C_2$ is nondecreasing in the first variable and may depend on $\Omega$. 

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PROOF. All the results above have been shown in [31, Proposition 7.39], except (3.16). Hence it suffices to verify (3.16). For simplicity, we always use the positive constant $C_2(\kappa, \varepsilon, T)$ to denote various constants depending on $\kappa$, $\varepsilon$ and $T$.

Let $\rho_1, \rho_2 \in \mathcal{R}_T$ be two solutions of the problem (3.7)–(3.9) with $\mathbf{v}$ replaced by $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Then after a straightforward calculation, we find that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla (\rho_1 - \rho_2)|^2 \, dx + \varepsilon \int_{\Omega} |\Delta (\rho_1 - \rho_2)|^2 \, dx
= \int_{\Omega} \left( \mathbf{v}_2 \cdot \nabla (\rho_1 - \rho_2) + (\rho_1 - \rho_2) \text{div} \mathbf{v}_2 + \rho_1 \text{div} (\mathbf{v}_1 - \mathbf{v}_2) + (\mathbf{v}_1 - \mathbf{v}_2) \cdot \nabla \rho_1 \right) (\Delta \rho_1 - \Delta \rho_2) \, dx.
\] (3.17)

Using Cauchy-Schwarz’s and Poincaré’s inequalities, the right hand side of (3.17) can be bounded from above by

\[
\frac{C_1(\Omega)}{\varepsilon} \left( \|\mathbf{v}_2\|_{W^{1, \infty}(\Omega)}^2 \|\nabla (\rho_2 - \rho_1)\|_{L^2(\Omega)}^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_{W^{1, \infty}(\Omega)}^2 \|\rho_1\|_{H^1(\Omega)}^2 \right) + \frac{\varepsilon}{2} \|\Delta (\rho_1 - \rho_2)\|_{L^2(\Omega)}^2
\]

for some constant $C_1(\Omega)$ depending on $\Omega$, which, together with (3.17), yields

\[
\frac{d}{dt} \int_{\Omega} |\nabla (\rho_1 - \rho_2)|^2 \, dx + \varepsilon \int_{\Omega} |\Delta (\rho_1 - \rho_2)|^2 \, dx
\leq \frac{2C_1(\Omega)}{\varepsilon} \left( \|\mathbf{v}_2\|_{W^{1, \infty}(\Omega)}^2 \|\nabla (\rho_2 - \rho_1)\|_{L^2(\Omega)}^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_{W^{1, \infty}(\Omega)}^2 \|\rho_1\|_{H^1(\Omega)}^2 \right).
\]

Therefore, by Gronwall’s inequality and (3.12),

\[
\|\nabla (\rho_1 - \rho_2)(t)\|_{L^2(\Omega)} \leq C_2(\kappa, \varepsilon, T) \sqrt{t} \|\rho_0\|_{H^1(\Omega)} \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^\infty(I, W^{1, \infty}(\Omega))}. \tag{3.18}
\]

Moreover, we have

\[
\|\Delta (\rho_1 - \rho_2)\|_{L^2(Q_t)} \leq C_2(\kappa, \varepsilon, T) \sqrt{t} \|\rho_0\|_{H^1(\Omega)} \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^\infty(I, W^{1, \infty}(\Omega))}. \tag{3.19}
\]

Consequently, using (3.18), (3.19), we get from (3.7) that

\[
\|\partial_t (\rho_1 - \rho_2)\|_{L^2(Q_t)}
= \|\mathbf{v}_2 \cdot \nabla (\rho_1 - \rho_2) + (\rho_1 - \rho_2) \text{div} \mathbf{v}_2 + \rho_1 \text{div} (\mathbf{v}_1 - \mathbf{v}_2) + (\mathbf{v}_1 - \mathbf{v}_2) \cdot \nabla \rho_1 - \varepsilon \Delta (\rho_1 - \rho_2)\|_{L^2(Q_t)}
\leq C_1(\Omega) \left( \kappa \|\nabla (\rho_1 - \rho_2)\|_{L^2(Q_t)} + \sqrt{t} \|\rho_1\|_{L^\infty(I, H^1(\Omega))} \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^\infty(I, W^{1, \infty}(\Omega))} \right) + \varepsilon \|\Delta (\rho_1 - \rho_2)\|_{L^2(Q_t)}
\leq C_2(\kappa, \varepsilon, T) \sqrt{t} \|\rho_0\|_{H^1(\Omega)} \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^\infty(I, W^{1, \infty}(\Omega))},
\]

which implies (3.16).
3.2. The local solvability of the direction vector

Now we turn to show the local solvability of the direction vector. More precisely, we will show that for
\[ \mathbf{v} \in \mathbb{V} := \{ \mathbf{v} \mid \mathbf{v} \in C^0(I, H^1(\Omega) \cap L^\infty(\Omega)), \ \partial_t \mathbf{v} \in L^2(I, H^1(\Omega)) \}, \]
there exists a \( T_d^* \in (0, T] \), such that the equation
\[ \partial_t \mathbf{d} + \mathbf{v} \cdot \nabla \mathbf{d} = \theta(\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}) \] (3.20)
has a unique strong solution \( \mathbf{d}(x, t) \) on \([0, T_d^*)\) satisfying the initial and boundary conditions
\[ \mathbf{d}(x, 0) = \mathbf{d}_0(x), \quad \partial_t \mathbf{d} \big|_{\partial \Omega} = \mathbf{d}_0(x) \quad \text{for} \ t \in (0, T_d^*). \] (3.21)
(3.22)
This local existence can be shown by modifying the arguments in [4, Section 3]. Here we give its proof, since some arguments in our proof are very different from those in [4, Section 3], and can be applied to the Cauchy problem in Section 6 by using the Ehrling-Nirenberg-Gagliardo interpolation inequality and domain expansion technique. Moreover, we shall show that the solution \( \mathbf{d} \) continuously depends on \( \mathbf{v} \), which will be also used in the proof of the local existence of solutions to the third approximate problem.

3.2.1. Linearized problems

Denote
\[ \begin{align*}
\mathbb{D}^* := & \{ \mathbf{b} \in C^0(I_d^*, H^2(\Omega)) \mid \partial_t \mathbf{b} \in C^0(I_d^*, L^2(\Omega)) \cap L^2(I_d^*, H^1(\Omega)) \}, \\
\mathbb{V}_K := & \{ \mathbf{v} \in \mathbb{V} \mid \| \mathbf{v} \|_V \leq K \}, \\
\mathbb{D}_{\kappa_d}^* := & \{ \mathbf{b} \in \mathbb{D}^* \mid \mathbf{b}(x, 0) = \mathbf{d}_0(x), \ \| \mathbf{b} \|_{\mathbb{D}^*} \leq \kappa_d \},
\end{align*} \]
where \( K \) and \( \kappa_d \) are positive constants,
\[ \| \mathbf{v} \|_V := \left( \| \mathbf{v} \|_{C^0(I, H^1(\Omega))}^2 + \| \mathbf{v} \|_{C^0(I, L^\infty(\Omega))}^2 + \| \partial_t \mathbf{v} \|_{L^2(I, H^1(\Omega))}^2 \right)^{\frac{1}{2}}, \]
(3.23)
\[ \| \mathbf{b} \|_{\mathbb{D}^*} := \left( \| \partial_t \mathbf{b} \|_{L^2(I_d^*, H^1(\Omega))}^2 + \| \partial_t \mathbf{b} \|_{L^2(I_d^*, L^2(\Omega))}^2 + \| \mathbf{b} \|_{L^\infty(I_d^*, H^2(\Omega))}^2 + \| \mathbf{b} \|_{L^2(I_d^*, H^1(\Omega))}^2 \right)^{\frac{1}{2}}, \]
and \( I_d^* := (0, T_d^*) \subset I \). Without loss of generality, we assume \( \kappa_d \geq 1 \).

To show the local existence of solutions, we will construct a sequence of approximate solutions to (3.20)–(3.22) and use the technique of iteration based on the set \( \mathbb{D}_{\kappa_d}^* \). First we linearize the original system (3.20) for \( \mathbf{b} \in \mathbb{D}_{\kappa_d}^* \) as follows:
\[ \partial_t \mathbf{d} - \theta \Delta \mathbf{d} = \theta |\nabla \mathbf{b}|^2 \mathbf{b} - \mathbf{v} \cdot \nabla \mathbf{b} \] (3.24)
with a given function \( \mathbf{v} \in \mathbb{V}_K \), and initial and boundary conditions
\[ \mathbf{d} \big|_{t=0} = \mathbf{d}_0, \ \mathbf{x} \in \Omega, \] (3.25)
\[ \mathbf{d} \big|_{\partial \Omega} = \mathbf{d}_0 \quad \text{in} \ I_d^*. \] (3.26)
Since \( F(\mathbf{b}) := |\nabla \mathbf{b}|^2 \mathbf{b} - \mathbf{v} \cdot \nabla \mathbf{b} \in L^2(I_d^*, H^1(\Omega)) \) and \( \partial_t F(\mathbf{b}) \in L^2(I_d^*, (H^1(\Omega))^*) \), we can apply a standard method, such as a semidiscrete Galerkin method in [5], to prove that the initial-boundary value problem (3.21)–(3.26) has a unique solution
\[ \begin{align*}
\mathbf{d} & \in C^0(I_d^*, H^2(\Omega) \cap L^2(I_d^*, H^3(\Omega))), \\
\partial_t \mathbf{d} & \in L^2(I_d^*, (H^1(\Omega))^*)), \\
\partial_t \mathbf{d} & \in C^0(I_d^*, L^2(\Omega) \cap L^2(I_d^*, H^1(\Omega))).
\end{align*} \]
Here we have used \((H^1(\Omega))^*\) to denote the dual space of \(H^1(\Omega)\). Therefore, we can get a solution \(d^1\) to \([3.24]\) with \(b\) replaced by some given \(d^0 \in D^*_\kappa\). Assuming \(d^{k-1} \in D^*_\kappa\) for \(k \geq 1\), we can construct an approximate solution \(d^k\) satisfying
\[
\partial_t d^k - \theta \Delta d^k = \theta |\nabla d^{k-1}|^2 d^{k-1} - \nu \cdot \nabla d^{k-1}
\]  
with initial and boundary conditions:
\[
d^k|_{t=0} = d_0, \; x \in \Omega, \\
d^k|_{\partial \Omega} = d_0, \; t \in I_d^*.
\]

3.2.2. Uniform estimates

Next we derive the uniform estimates:
\[
\|d^k\|_{D^*} \leq \kappa_d
\]  
(3.28)
for some constant \(\kappa_d\) depending on \(d_0\) and the given constant \(K\). We mention that in the following estimates the letter \(c \geq 1\) will denote various positive constants independent of \(\kappa_d, K, \nu, d_0\) and \(\Omega\); the letter \(C_1(\Omega)\) will denote various positive constants depending on \(\Omega\), and the letter \(C(\ldots)\) various positive constants depending on its variables (it may depend on \(\Omega\) sometimes, however we omit them for simplicity), and is nondecreasing in its variables. Of course, they may also depend on the fixed value \(\theta\).

First, we can deduce from \([3.27]\) and integration by parts that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |d^k|^2 dx + \theta \int_\Omega |\nabla d^k|^2 dx \\
= \theta \int_\Omega |\nabla d^{k-1}|^2 d^{k-1} \cdot d^k dx - \int_\Omega \nu \cdot \nabla d^{k-1} \cdot d^k dx + \theta \int_{\partial \Omega} n \cdot \nabla d_0 \cdot d_0 dx
\]
and
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla d^k|^2 dx + \theta \int_\Omega |\Delta d^k|^2 dx \\
= -\theta \int_\Omega |\nabla d^{k-1}|^2 d^{k-1} \cdot \Delta d^k dx + \int_\Omega \nu \cdot \nabla d^{k-1} \cdot \Delta d^k dx,
\]
which, together with Cauchy-Schwarz’s and Hölder’s inequalities, imply that
\[
\frac{1}{2} \frac{d}{dt} \|d^k\|^2_{H^1(\Omega)} \leq \left( \theta |\nabla d^{k-1}|^2_{L^4(\Omega)} \|d^{k-1}\|_{L^4(\Omega)} + \|\nu\|^2_{L^\infty(\Omega)} \|\nabla d^{k-1}\|^2_{L^4(\Omega)} \right) + \|d^k\|^2_{L^2(\Omega)} + \theta \|\nabla d_0\|^2_{L^2(\partial \Omega)} \|d_0\|_{L^2(\partial \Omega)}.
\]
Hence, using the embeddings \(H^2(\Omega) \hookrightarrow L^\infty(\Omega)\) for \(d\), \(H^1(\Omega) \hookrightarrow L^4(\Omega)\) for \((\nabla d, \nu)\), and trace theorem, we have
\[
\frac{d}{dt} \|d^k\|^2_{H^1(\Omega)} \leq C(K, \|d_0\|_{H^2(\Omega)})(\kappa_d^6 + \kappa_d^2 K^2) + 2\|d^k\|^2_{H^1(\Omega)},
\]  
(3.29)
which, together with Gronwall’s inequality, implies
\[
\|d^k(t)\|^2_{H^1(\Omega)} \leq \left( \|d_0\|^2_{H^1(\Omega)} + C(K, \|d_0\|_{H^2(\Omega)})(\kappa_d^6 + \kappa_d^2 K^2) t \right) e^{2t}
\]
for any \( t \in I_d^* \). Taking \( T_1^* = \kappa_d^{-6} \leq 1 \) and letting \( T_2^* \leq T_1^* \), we conclude

\[
\|d^k\|_{H^2(\Omega)}^2 \leq C(K, \|d_0\|_{H^2(\Omega)}).
\] (3.30)

Next we derive bounds on \( \partial_t d \). It follows from (3.27) and integration by parts that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\partial_t d^k|^2 dx + \theta \int_{\Omega} |\nabla \partial_t d^k|^2 dx
\]

\[
= -2\theta \left( \int_{\Omega} (\Delta d^k - \partial_t d^{k-1}) \partial_t d^k dx - \int_{\Omega} \nabla \partial_t d^k \cdot \nabla \partial_t d^k dx + \theta \int_{\Omega} |\nabla d^k| |\partial_t d^k| dx \right) + \theta \int_{\Omega} |\nabla d^k| |\partial_t d^k| dx
\]

\[
= -\int_{\Omega} \nabla \partial_t d^k \cdot \nabla \partial_t d^k dx - \int_{\Omega} \partial_t \nabla \partial_t d^k \cdot \partial_t d^k dx.
\] (3.31)

Noting that \( (\partial_t d, \partial_t v, v) = 0 \) on \( \partial \Omega \), applying Cauchy-Schwarz’s and Hölder’s inequalities, and \( (2,12) \), the right hand side of (3.31) can be bounded from above by

\[
c \left( \|\Delta d^{k-1}\|_{L^2(\Omega)}^2 \|d^{k-1}\|_{L^\infty(\Omega)}^2 + \|\nabla d^{k-1}\|_{L^4(\Omega)}^4 \|\partial_t d^{k-1}\|_{L^2(\Omega)}^2 \|\nabla \partial_t d^{k-1}\|_{L^2(\Omega)}^2 \right)
\]

\[
+ \|d^{k-1}\|_{L^2(\Omega)}^2 \|\nabla d^{k-1}\|_{L^4(\Omega)}^2 \|\partial_t d^{k-1}\|_{L^2(\Omega)} \|\nabla \partial_t d^{k-1}\|_{L^2(\Omega)}
\]

\[
+ \|v\|_{L^2(\Omega)}^2 \|\nabla \partial_t d^{k-1}\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}^2 \|\nabla \partial_t v\|_{L^2(\Omega)} \|\partial_t d^k\|_{L^2(\Omega)}^2
\]

\[
+ \left( \|\nabla \partial_t d^{k-1}\|_{L^2(\Omega)} + \|\partial_t v\|_{L^2(\Omega)} \|\nabla \partial_t v\|_{L^2(\Omega)} \|\partial_t d^k\|_{L^2(\Omega)}^2 \right) + \frac{\theta}{2} \|\nabla \partial_t d^k\|_{L^2(\Omega)}^2.
\]

Using \( H^2(\Omega) \hookrightarrow L^\infty(\Omega) \) and \( H^1(\Omega) \hookrightarrow L^4(\Omega) \) for \( d \) and \( \nabla d \) respectively, the above term can be further bounded from above by

\[
C_1(\Omega) \left( (\kappa_5^5 + K^2)(\|\nabla \partial_t d^{k-1}\|_{L^2(\Omega)} + \|\nabla \partial_t d^{k-1}\|_{L^2(\Omega)}^2) + \kappa_d^2 \right.
\]

\[
+ \left( \|\nabla \partial_t d^{k-1}\|_{L^2(\Omega)} + \|\partial_t v\|_{L^2(\Omega)} \|\nabla \partial_t v\|_{L^2(\Omega)} \|\partial_t d^k\|_{L^2(\Omega)}^2 \right) + \frac{\theta}{2} \|\nabla \partial_t d^k\|_{L^2(\Omega)}^2.
\] (3.32)

Hence, we have

\[
\frac{d}{dt} \|\partial_t d^k\|_{L^2(\Omega)}^2 + \theta \|\nabla \partial_t d^k\|_{L^2(\Omega)}^2
\]

\[
\leq C_1(\Omega) \left( (\kappa_5^5 + K^2)(\|\nabla \partial_t d^{k-1}\|_{L^2(\Omega)} + \|\nabla \partial_t d^{k-1}\|_{L^2(\Omega)}^2) + \kappa_d^2 \right.
\]

\[
+ \left( \|\nabla \partial_t d^{k-1}\|_{L^2(\Omega)} + \|\partial_t v\|_{L^2(\Omega)} \|\nabla \partial_t v\|_{L^2(\Omega)} \|\partial_t d^k\|_{L^2(\Omega)}^2 \right).
\] (3.33)

Consequently, using Gronwall’s and Hölder’s inequalities, we get

\[
\|\partial_t d^k(t)\|_{L^2(\Omega)}^2 \leq \left( \|\partial_t d_0\|_{L^2(\Omega)}^2 + C_1(\Omega)(\kappa_d^5 + K^2)(t + \kappa_d \sqrt{t} + \sqrt{\kappa_d^5}) \right) e^{C_1(\Omega)(\kappa_d \sqrt{t} + K^2)}
\]

for any \( t \in I_d^* \). Now, taking \( T_3^* = \kappa_d^{-12} \leq 1 \), letting \( T_2^* \leq T_3^* \), and noting that

\[
\|\partial_t d(0)\|_{L^2(\Omega)} = \|\theta(\Delta d_0 + |\nabla d_0|^2 d_0) - v_0 \cdot \nabla d_0\|_{L^2(\Omega)},
\]

we conclude

\[
\sup_{t \in I_d^*} \|\partial_t d^k\|_{L^2(\Omega)} \leq C(K, \|d_0\|_{H^2(\Omega)}).
\] (3.34)
Moreover, using (3.27), (3.33), (3.34), Cauchy’s inequality, and the elliptic theory, we obtain the following estimates,

\[ \| \partial_t d^k \|_{L^2(I_d^*, H^1(\Omega))} \leq C(K, \| d_0 \|_{H^1(\Omega)}), \]

(3.35)

\[ \| \nabla^2 d^k \|_{L^2(\Omega)}^2 \leq C \left( \| \nabla^2 d_0 \|_{L^2(\Omega)}^2 + \| \partial_t d^k \|_{L^2(\Omega)}^2 \right) + \| \nabla \cdot \nabla d^{k-1} \|_{L^2(\Omega)}^2 + \| \nabla \nabla d^{k-1} \|_{L^2(\Omega)}^2 \]

(3.36)

\[ \leq C(K, \| d_0 \|_{H^2(\Omega)}^2) + c \left( \| \nabla d^{k-1} \|_{L^2(\Omega)}^2 + c \| \nabla \nabla d^{k-1} \|_{L^2(\Omega)}^2 \right), \]

and

\[ \| \nabla^3 d^k \|_{L^2(I_d^*, L^2(\Omega))} \leq C(K, \| d_0 \|_{H^2(\Omega)}^2) + \| \nabla^3 d_0 \|_{L^2(\Omega)}^2 + c \| \nabla (\nabla d^{k-1}) \|_{L^2(I_d^*, L^2(\Omega))}^2 \]

(3.37)

where

\[ \| \nabla^3 d^k \|_{L^2(I_d^*, L^2(\Omega))}^2 := \sum_{1 \leq i, j, l, m \leq 2} \| \partial_i \partial_j \partial_m d^k \|_{L^2(I_d^*, L^2(\Omega))}^2. \]

To complete the derivation of (3.28), it suffices to deduce the uniform estimates of the last two terms in (3.36) and (3.37) as follows.

Since \( d \in \mathbb{D}^s \), in view of [34, Lemmas 1.65, 1.66], we have

\[ d(t) - d_0 = \int_0^t \partial_t d(\tau) d\tau \text{ for any } t \in I_d^*, \]

which implies that

\[ \| d(t) - d_0 \|_{H^1(\Omega)} \leq \sqrt{t} \| \partial_t d \|_{L^2(I_d^*, H^1(\Omega))}. \]

(3.38)

Using (3.38), we obtain

\[ \| \nabla \cdot \nabla d^{k-1} \|_{L^2(\Omega)}^2 \leq C(v) \| \nabla d_0 \|_{L^2(\Omega)}^2 \left( \| \nabla d^{k-1} - \nabla d_0^{k-1} \|_{L^2(\Omega)}^2 \right) + \| \nabla d_0^{k-1} \|_{L^2(\Omega)}^2 \]

\[ \leq C(K^2(t \kappa_d^2 + \| d_0 \|_{H^1(\Omega)}^2). \]

Recalling that \( 0 \leq t \leq T_d^* \leq \kappa_d^{-12} \), we have

\[ \| \nabla \cdot \nabla d^{k-1} \|_{L^2(\Omega)}^2 \leq C(K, \| d_0 \|_{H^1(\Omega)}^2). \]

(3.39)

To estimates the last term in (3.38) and the the last two terms in (3.37), we shall introduce the following well-known Ehrling-Nirenberg-Gagliardo interpolation inequality:

**Lemma 3.1.** Let \( \Omega \) be a domain in \( \mathbb{R}^N \) satisfying the cone condition. If \( mp > N \), let \( 0 < q < \infty \); if \( mp < N \), let \( 0 < q \leq p^* = Np/(N - mp) \). Then there exists a constant \( c_1 \) depending on \( m \), \( N \), \( p \), \( q \) and the dimension of the cone \( \mathbf{C} \), such that for all \( v \in W^{m,p}(\Omega) \),

\[ \| v \|_{L^q(\Omega)} \leq c_1 \| v \|_{W^{m,p}(\Omega)}^\alpha \| v \|_{L^p(\Omega)}^{1-\alpha}, \]

(3.40)

where \( \alpha = (N/mp) - (N/mq) \).

**Proof.** The proof can be found in [1, Chapter 5].
In particular, we can deduce from (3.40) that
\[
\|v\|_{L^\infty(\Omega)}^2 \leq c_1(\Omega)\|v\|_{H^2(\Omega)}\|v\|_{L^2(\Omega)} \tag{3.41}
\]
\[
\|v\|_{L^4(\Omega)}^2 \leq c_1(\Omega)\|v\|_{H^1(\Omega)}\|v\|_{L^2(\Omega)} \tag{3.42}
\]
for some constants \(c_1(\Omega)\) depending on \(\Omega\).

Making use of (3.38), (3.41) and (3.42), it is easy to see that
\[
\|\nabla d^{k-1} d^{k-1}(t)\|^2_{L^2(\Omega)} \leq c\|\nabla d^{k-1}\|^2_{L^2(\Omega)}\|d^{k-1}\|^2_{L^2(\Omega)}
\leq c\left(\|\nabla d^{k-1}\|^4_{L^4(\Omega)}\|d^{k-1}\|^2_{L^\infty(\Omega)} + \|\nabla d^5_0\|^2_{L^4(\Omega)}\|d^5_0\|^2_{L^\infty(\Omega)} + \|\nabla d^{k-1} - \nabla d^5_0\|^2_{L^4(\Omega)}\|d^5_0\|^2_{L^\infty(\Omega)}\right)
\leq C_1(\Omega)\left(\kappa^5\|d^{k-1} - d_0\|_{H^1(\Omega)} + \|d_0\|_{H^2(\Omega)}\right)
\leq C_1(\Omega)\left(\kappa^5\|\partial_t d\|_{L^2(I_d^*, L^2(\Omega))}\sqrt{t} + \|d_0\|_{H^2(\Omega)}\right)
\]
for any \(t \in I_d^*\). Since \(T_d^* \leq \kappa_d^{-12}\), we have
\[
\|\nabla d^{k-1} d^{k-1}\|_{L^2(\Omega)} \leq C_1(\Omega) \left(1 + \|d_0\|_{H^2(\Omega)}\right). \tag{3.43}
\]

Using Hölder’s inequality, (3.41) and (3.42), we infer that
\[
\|\nabla (v \cdot \nabla d^{k-1})\|^2_{L^2(I_d^*, L^2(\Omega))} + c\|\nabla (\|\nabla d^{k-1}\|^2 d^{k-1})\|^2_{L^2(I_d^*, L^2(\Omega))}
\leq c\left(\|\nabla v\|^2_{L^\infty(I_d^*, L^2(\Omega))}\|\nabla d^{k-1}\|^2_{L^2(I_d^*, L^\infty(\Omega))} + \|v\|^2_{L^\infty(Q_r)}\|\nabla^2 d^{k-1}\|^2_{L^2(I_d^*, L^2(\Omega))} + \int_{I_d^*} \|\nabla d^{k-1}\|^2_{L^\infty(\Omega)}\|\nabla d^{k-1}\|^4_{L^2(\Omega)} \, ds\right)
\leq C_1(\Omega)\left(K^2\|\nabla d^{k-1}\|^2_{L^\infty(I_d^*, L^2(\Omega))}\|d^{k-1}\|^2_{L^1(I_d^*, H^3(\Omega))} + K^2\|\nabla d^{k-1}\|^2_{L^\infty(I_d^*, L^2(\Omega))} T_d^* + \|d^{k-1}\|^5_{L^\infty(I_d^*, H^2(\Omega))}\|d^{k-1}\|^4_{L^1(I_d^*, H^3(\Omega))}\right)
\leq C_1(\Omega)\left(K^2\kappa^2 \sqrt{T_d^*} + K^2\kappa^2 T_d^* + \kappa^6 \right),
\]
which, together with the fact \(T_d^* \leq \kappa_d^{-12} \leq 1\), yields
\[
\|\nabla (v \cdot \nabla d^{k-1})\|^2_{L^2(I_d^*, L^2(\Omega))} + c\|\nabla (\|\nabla d^{k-1}\|^2 d^{k-1})\|^2_{L^2(I_d^*, L^2(\Omega))} \leq C_1(\Omega) (1 + K^2). \tag{3.44}
\]

Finally, putting (3.30), (3.31), (3.34), (3.43) and (3.44) together, we then obtain that
\[
\|d^k\|_{W^*} \leq C(K, \|d_0\|_{H^2(\Omega)}, \|\nabla^3 d_0\|_{L^2(\Omega)}).
\]
Choosing \(\kappa_d \geq \max\{C(K, \|d_0\|_{H^2(\Omega)}, \|\nabla^3 d_0\|_{L^2(\Omega)}, \|d_0\|_{H^2(\Omega), 1}\}\), we get (3.28).

### 3.2.3. Taking limits

In order to take limits in (3.27), we shall further show that \(\{d^k\}_{k=1}^\infty\) is a Cauchy sequence. To this end, we define
\[
\tilde{d}^{k+1} = d^{k+1} - d^k
\]
which satisfies
\[ \partial_t \tilde{d}^{k+1} - \theta \Delta \tilde{d}^{k+1} = \theta [\nabla \tilde{d}^k : (\nabla \tilde{d}^k + \nabla \tilde{d}^{k-1})] \tilde{d}^k + \theta |\nabla \tilde{d}^{k-1}|^2 \tilde{d}^k - v \cdot \nabla \tilde{d}^k \]  
(3.45)
with initial and boundary conditions:
\[ \tilde{d}^{k+1}|_{t=0} = 0, \quad x \in \Omega, \]
\[ \tilde{d}^{k+1}|_{\partial \Omega} = 0, \quad t \in I_t^*. \]

Using Cauchy-Schwarz’s and Hölder’s inequalities, (3.41) and (3.42) again, it follows from (3.45) and the boundary condition that
\[ \frac{d}{dt} \| \tilde{d}^{k+1} \|_{H^1(\Omega)}^2 + 2 \left( \| \nabla \tilde{d}^{k+1} \|_{L^2(\Omega)}^2 + \| \Delta \tilde{d}^{k+1} \|_{L^2(\Omega)}^2 \right) \leq c \left( \| \nabla \tilde{d}^k \|_{L^2(\Omega)}^2 \| \nabla (\tilde{d}^k + d^{k-1}) \|_{L^\infty(\Omega)} \| \tilde{d}^k \|_{L^2(\Omega)} \right) + \| \tilde{d}^{k+1} \|_{H^1(\Omega)}^2 \]
(3.46)
\[ \leq C_1(\Omega) \left( K^2 + \kappa_d^2 (\| d^k \|_{H^1(\Omega)} + \| d^{k-1} \|_{H^2(\Omega)}) \right) + \| \tilde{d}^{k+1} \|_{H^1(\Omega)}^2 + \| d^{k-1} \|_{H^2(\Omega)} \]

Then, using Gronwall’s and Hölder inequalities, we have
\[ \frac{d}{dt} \| \tilde{d}^{k+1} \|_{H^1(\Omega)}^2 \leq C_1(\Omega) \left( K^2 t + \kappa_d^3 \int_0^t \left( \| d^k \|_{H^1(\Omega)} + \| d^{k-1} \|_{H^2(\Omega)} \right) ds \right) \sup_{0 \leq s \leq t} \| \tilde{d}^k \|_{H^1(\Omega)}^2 \]
\[ \leq C(K, \kappa_d)(t + \sqrt{t}) \sup_{0 \leq s \leq t} \| \tilde{d}^k \|_{H^1(\Omega)}^2. \]

Taking
\[ T_3^* = \min\{ T_2^*, C^{-2}(K, \kappa_d)/(64e^2), 1/2 \} \]
and letting \( T_3^* \leq T_3^* \), we have
\[ \sup_{t \in I_t^*} \| \tilde{d}^{k+1} \|_{H^1(\Omega)}^2 \leq \frac{1}{4} \sup_{t \in I_t^*} \| \tilde{d}^k \|_{H^1(\Omega)}^2. \]
(3.47)
Furthermore, from (3.46), (3.47) and the elliptic estimates we get
\[ \| \tilde{d}^{k+1} \|_{L^2(I_t^*, H^2(\Omega))}^2 \leq \frac{1}{4} \sup_{t \in I_t^*} \| \tilde{d}^k \|_{H^1(\Omega)}^2. \]

By iteration, we have
\[ \sup_{t \in I_t^*} \| \tilde{d}^{k+1} \|_{H^1(\Omega)}^2 + \| \tilde{d}^{k+1} \|_{L^2(I_t^*, H^2(\Omega))}^2 \leq \frac{1}{2^{k-1}} \sup_{t \in I_t^*} \| \tilde{d}^k \|_{H^1(\Omega)}^2 \]
for any \( k \geq 2. \)

In particular,
\[ \sum_{k=2}^{\infty} \left( \| d^k \|_{L^\infty(I_t^*, H^1(\Omega))} + \| \tilde{d}^k \|_{L^2(I_t^*, H^2(\Omega))} \right) < \infty, \]
which implies \( \{ d_k \}_{k=1}^\infty \) is a Cauchy sequence in \( L^\infty(I_t^*, H^1(\Omega)) \cap L^2(I_t^*, H^2(\Omega)). \) Therefore,
\[ d^k \to d \text{ strongly in } L^\infty(I_t^*, H^1(\Omega)) \cap L^2(I_t^*, H^2(\Omega)). \]
(3.48)

As a consequence, \( d \) is accurately a solution to problem (3.20)–(3.22) satisfying the regularity \( \| d \|_D \leq \kappa_d \), by virtue of (3.28) and (3.48), the lower semi-continuity, the elliptic estimate and the compactness theorem with time (see [4, Lemma 2.5]).
3.2.4. Continuous dependence

Finally we show that the solution $d$ continuously depends on $v$. Let $T^*_d \in (0, T^*_3]$, $v_1, v_2 \in V_K$, $d_1$ and $d_2$ be two solutions of the initial-boundary value problem (3.20)–(3.22), corresponding to $v = v_1$ and $v = v_2$ respectively. Moreover, two solutions satisfy
\[
\|d_1\|_{D^*} + \|d_2\|_{D^*} < \infty.
\]

Multiplying the difference of the equations for $d_1 - d_2$ by $d_1 - d_2$ and integrating by parts, we obtain
\[
\frac{d}{dt} \int_\Omega (d_1 - d_2)^2 dx + 2\theta \int_\Omega |\nabla (d_1 - d_2)|^2 dx = 2\theta \int_\Omega |\nabla d_1|^2 |d_1 - d_2|^2 dx + \int_\Omega (\nabla (d_1 - d_2) : \nabla (d_1 + d_2)) d_2 \cdot (d_1 - d_2) dx
\]
\[
= 2\theta \int_\Omega (v_1 - v_2) \cdot \nabla d_1 \cdot (d_1 - d_2) dx - 2 \int_\Omega v_2 \cdot \nabla (d_1 - d_2) \cdot (d_1 - d_2) dx.
\]

Similarly to that in the derivation of (3.46), we use (3.41) and (3.42) to see that the right-hand side of the above identity can be bounded from above by
\[
C(\|d_1\|_{D^*}, \|d_2\|_{D^*}, K) \left( 1 + \|d_1\|_{H^3(\Omega)} + \|d_2\|_{H^3(\Omega)} \right) \times \left( \|d_1 - d_2\|^2_{L^2(\Omega)} + \|v_1 - v_2\|_{L^2(\Omega)} \|d_1 - d_2\|_{L^2(\Omega)} + \theta \|\nabla (d_1 - d_2)\|^2_{L^2(\Omega)} \right),
\]

Hence,
\[
\frac{d}{dt} \|d_1 - d_2\|^2_{L^2(\Omega)} + \theta \|\nabla (d_1 - d_2)\|^2_{L^2(\Omega)} \leq C(\|d_1\|_{D^*}, \|d_2\|_{D^*}, K) \left( 1 + \|d_1\|_{H^3(\Omega)} + \|d_2\|_{H^3(\Omega)} \right) \times \left( \|d_1 - d_2\|^2_{L^2(\Omega)} + \|v_1 - v_2\|_{L^2(\Omega)} \|d_1 - d_2\|_{L^2(\Omega)} \right),
\]
in particular,
\[
\frac{d}{dt} \|d_1 - d_2\|_{L^2(\Omega)} \leq C(\|d_1\|_{D^*}, \|d_2\|_{D^*}, K) \left( 1 + \|d_1\|_{H^3(\Omega)} + \|d_2\|_{H^3(\Omega)} \right) \times \left( \|d_1 - d_2\|_{L^2(\Omega)} + \|v_1 - v_2\|_{L^2(\Omega)} \right).
\]

Thus, applying Gronwall’s and Cauchy-Schwarz’s inequalities, we conclude
\[
(\|d_1 - d_2\|_{L^2(\Omega)})(t) \leq (t + \sqrt{t})C(\|d_1\|_{D^*}, \|d_2\|_{D^*}, K, T^*_d))\|v_1 - v_2\|_{L^\infty(I, L^2(\Omega))} \tag{3.49}
\]
for any $t \in I^*_d$. Moreover,
\[
\|\nabla (d_1 - d_2)\|_{L^2(Q_t)} \leq (t + \sqrt{t})C(\|d_1\|_{D^*}, \|d_2\|_{D^*}, K, T^*_d))\|v_1 - v_2\|_{L^\infty(I, L^2(\Omega))}.
\]

Obviously, the uniqueness of local solutions in the function class $D^*$ follows from (3.49) immediately. In particular, $\|d_1\|_{D^*}, \|d_2\|_{D^*} \leq \kappa_d$. As the end of this subsection, we summarize our previous results on the local existence of $d$.

**Proposition 3.2.** Let $K > 0$, $0 < \alpha < 1$, $\Omega$ be a bounded domain of class $C^{2,\alpha}$, $v \in V_K$, and $d_0 \in H^3(\Omega)$. Then there exist a finite time
\[
T^*_d := h_1(K, \|d_0\|_{H^3(\Omega)}, \|\nabla^3 d_0\|_{L^2(\Omega)}) \in (0, \min\{1, T\}),
\]
and a corresponding unique mapping

\[ \mathcal{D}_{d_0}^K : \mathbb{D}_K^* (\text{with } T_d^K \text{ in place of } T) \to C^0(I_d^K, \mathcal{H}^2(\Omega)), \]

where \( h_1 \) is nonincreasing in its first two variables and \( Q_{T_d^K} := \Omega \times I_d^K := \Omega \times (0, T_d^K) \), such that

1. \( \mathcal{D}_{d_0}^K(\mathbf{v}) \) belongs to the following function class

\[ \mathcal{R}_{T_d^K} := \{ \mathbf{d} \mid \mathbf{d} \in L^2(I_d^K, \mathcal{H}^3(\Omega)), \partial_{\alpha \beta}^2 \mathbf{d} \in L^2(I_d^K, (\mathcal{H}^1(\Omega))^*), \partial_t \mathbf{d} \in C^0(I_d^K, \mathcal{L}^2(\Omega)) \cap L^2(I_d^K, \mathcal{H}^1_0(\Omega)) \}, \tag{3.50} \]

2. \( \mathbf{d} = \mathcal{D}_{d_0}^K(\mathbf{v}) \) satisfies \((3.21)\) a.e. in \( Q_{T_d^K} \), \((3.21)\) in \( \Omega \) and \((3.22)\) in \( I_d^K \). Moreover, \( \mathbf{d}|_{t=0} = \mathbf{d}_0 \).

3. \( \mathcal{D}_{d_0}^K(\mathbf{v}) \) enjoys the following estimate:

\[ \| \mathcal{D}_{d_0}^K(\mathbf{v}) \|_{\mathbb{D}_d^*} \leq C(K, \| \mathbf{d}_0 \|_{\mathcal{H}^2(\Omega)}, \| \nabla^3 \mathbf{d}_0 \|_{\mathcal{L}^2(\Omega)}), \tag{3.51} \]

where \( C \) is nondecreasing in its variables, and \( T_d^K \) in the definition of \( \mathbb{D}_d^* \) should be replaced by \( T_d^K \). Moreover (in view of \((2.1)\) and \((2.3)\)),

\[ |\mathbf{d}(t)| \equiv 1 \text{ for any } t, \text{ if } |\mathbf{d}_0| \equiv 1; \]

\[ d_2(t) \geq d_{02} \text{ for any } t, \text{ if } d_{02} \geq d_{02}. \]

4. \( \mathcal{D}_{d_0}^K(\mathbf{v}) \) continuously depends on \( \mathbf{v} \) in the following sense:

\[ \| [\mathcal{D}_{d_0}^K(\mathbf{v}_1) - \mathcal{D}_{d_0}^K(\mathbf{v}_2)](t) \|_{\mathcal{L}^2(\Omega)} + \| \nabla (\mathcal{D}_{d_0}^K(\mathbf{v}_1) - \mathcal{D}_{d_0}^K(\mathbf{v}_2)) \|_{\mathcal{L}^2(\Omega)} \leq \sqrt{\varepsilon} C(K, \| \mathbf{d}_0 \|_{\mathcal{H}^2(\Omega)}, \| \nabla^3 \mathbf{d}_0 \|_{\mathcal{L}^2(\Omega)}), \tag{3.52} \]

for any \( t \in I_d^K \), where \( C \) is nondecreasing in its variables.

**Remark 3.1.** We can further show that \( \mathcal{D}_{d_0}^K(\mathbf{v}) \) continuously depends on \( \mathbf{v} \) as the form of \((3.16)\). However, the estimate \((3.52)\) is sufficient to prove the local existence of solutions to the third approximate problem. It should be noted that the constants \( h_1 \) and \( C \) above depend on the domain \( \Omega \). However, in Section 4 we will see that the above result in bounded domains can be generalized to the Cauchy problem.

### 4. Unique solvability of the third approximate problem

In this section, we establish the global existence of a unique solution to the third approximate problem \((3.1)-(3.6)\). We first use the iteration technique and fixed point theorem to show the local existence. For this purpose, with the help of Proposition \((3.1)\) and \((3.2)\) we shall introduce the operator form of the approximate momentum equations \((3.3)\) to construct a contractive mapping.

#### 4.1. Operator form

Given

\[ \rho \in C^0(I_d^K, L^1(\Omega)), \quad \partial_t \rho \in L^1(\Omega_T), \quad \inf_{(x, t) \in \Omega_T} \rho(x, t) \geq \rho > 0, \tag{4.1} \]

we define, for all \( t \in I \)

\[ \mathcal{M}_{\rho(t)} : X_n \to X_n \]
It is easy to observe that $M$ will be repeatedly used in the estimates below. First one has

$$
\|L^{\infty}\|_{\mathcal{L}(X_n, X_n)} \leq c(n) \int_\Omega \rho(t) dx, \quad t \in \bar{I}.
$$

(4.3)

It is easy to observe that $\mathcal{M}_\rho(t)$ exists for all $t \in \bar{I}$ and

$$
\|\mathcal{M}_\rho(t)\|_{\mathcal{L}(X_n, X_n)} \leq \frac{1}{\rho^2},
$$

(4.4)

where $\mathcal{L}(X_n, X_n)$ denotes the set of all continuous linear operators mapping $X_n$ to $X_n$.

By virtue of (4.3) and (4.4), we have

$$
\|\mathcal{M}_\rho(t)\|_{\mathcal{L}(X_n, X_n)} \leq \frac{c(n)}{\rho^2} \|\rho_1(t)\|_{L^2(\Omega)}, \quad t \in \bar{I}.
$$

(4.5)

For the difference, the following inequality

$$
\|\mathcal{M}_{\rho_1(t)} - \mathcal{M}_{\rho_2(t)}\|_{\mathcal{L}(X_n, X_n)} \leq c(n) \|\rho_2 - \rho_1\|_{L^2(\Omega)}, \quad t \in \bar{I},
$$

(4.6)

holds. Due to the identity $\mathcal{M}_{\rho_2(t)} - \mathcal{M}_{\rho_1(t)} = \mathcal{M}_{\rho_1(t)}(\mathcal{M}_{\rho_1(t)} - \mathcal{M}_{\rho_2(t)})\mathcal{M}_{\rho_1(t)}$, we find that

$$
\|\mathcal{M}_{\rho_2(t)} - \mathcal{M}_{\rho_1(t)}\|_{\mathcal{L}(X_n, X_n)} \leq \frac{c(n)}{\rho^2} \|\rho_2 - \rho_1\|_{L^2(\Omega)}, \quad t \in \bar{I}
$$

(4.7)

for $\rho_1(t), \rho_2(t)$ satisfying (4.1).

Next, we shall look for $T_n^* \subset (0, T_d^{\bar{K}}]$ and

$$
v \in A := \{v \in C(T_n^*, X_n) \mid \partial_t v \in L^2(T_n^*, X_n)\}, \quad T_n^* := (0, T_n^* \subset (0, T_d^{\bar{K}})
$$

with $\|v\|_{C(T_n^*, H^2(\Omega))} + \|\partial_t v\|_{L^2(T_n^*, H^1(\Omega))} \leq \bar{K}$ for some $\bar{K}$, satisfying

$$
\int_\Omega (\rho v)(t) \cdot \Psi dx - \int_\Omega m_0 \cdot \Psi dx
$$

$$
= \int_0^t \int_\Omega \left[ \mu \Delta v + (\mu + \lambda) \nabla \text{div} v - A \nabla \rho - \delta \nabla \rho^2 - \varepsilon (\nabla \rho \cdot \nabla v)
$$

$$
- \text{div}(\rho v \otimes v) - \nu \text{div} \left( \nabla d \otimes \nabla d - \frac{|\nabla d|^2}{2} \right) \right] \cdot \Psi dx dt
$$

(4.8)

for all $t \in [0, T_n]$ and any $\Psi \in X_n$, where $\rho(t) = [\mathcal{L}_{\rho_0}(v)](t)$ is the solution of the problem (3.7)–(3.9) constructed in Proposition 3.1 and $d(t) = \mathcal{D}^{K}_{d_0}(v)(t)$ is the solution of the problem (3.20)–(3.22).
Moreover, one has constructed in Proposition 3.2. By the regularity of \((\rho, d)\) in Propositions 3.1, 3.2 and the operator \(M_{\rho(t)}\), the equations (1.8) can be rephrased as

\[
v(t) = M_{[\mathcal{I}_{\rho_0}(v)](t)}^{-1} \left( (\mathcal{P} m_0 + \int_0^t \mathcal{P} [\mathcal{N}(\mathcal{I}_{\rho_0}(v), v, \mathcal{D}_{d_0}(v))] ds \right)
\]

with \(m_0 = (\rho v)(0)\), where \(\mathcal{P} := \mathcal{P}_n\) is the orthogonal projection of \(L^2(\Omega)\) to \(X_n\), and

\[
\mathcal{N}(\rho, v, d) = \mu \Delta v + (\mu + \lambda) \nabla \text{div} v - A \nabla \rho^\gamma - \delta \nabla \rho^\beta - \varepsilon (\nabla \rho \cdot \nabla v) - \text{div}(\rho v \otimes v) - \nu \text{div} \left( \nabla d \otimes \nabla d - \frac{\|\nabla d\|^2_2}{2} \right).
\]

Moreover, one has

\[
\partial_t v(t) = M_{[\mathcal{I}_{\rho_0}(v)](t)}^{-1} M_{\mathcal{D}[\mathcal{I}_{\rho_0}(v)](t)} M_{[\mathcal{I}_{\rho_0}(v)](t)}^{-1} \left( \mathcal{P} m_0 + \int_0^t [\mathcal{P} \mathcal{N}(\mathcal{I}_{\rho_0}(v), v, \mathcal{D}_{d_0}(v))] (s) ds \right)
\]

\[
+ M_{[\mathcal{I}_{\rho_0}(v)](t)}^{-1} \int_0^t \left[ \mathcal{P} \mathcal{N}(\mathcal{I}_{\rho_0}(v), v, \mathcal{D}_{d_0}(v)) \right] (t) ds.
\]

4.2. Auxiliary estimates

We shall derive some auxiliary estimates on \((\rho = \mathcal{I}_{\rho_0}(v), v, d = \mathcal{D}_{d_0}(v_n))\) and \((\rho_k = \mathcal{I}_{\rho_0}(v_k), v_k, d = \mathcal{D}_{d_0}(v_k))\), \(k = 1, 2\), where \(v\) and \(v_k\) belong to the class \(A_n\) and

\[
\|v\|_{C(I^*_n, X_n)} + \|\partial_t v\|_{L^2(I^*_n, X_n)} \leq K, \quad \|v_k\|_{C(I^*_n, X_n)} + \|\partial_t v_k\|_{L^2(I^*_n, X_n)} \leq K,
\]

with \(K\) being a positive constant. By the equivalence of norms in (4.2), we have

\[
\left( \|v\|^2_{C(I^*_n, H^1(\Omega))} + \|v\|^2_{C(I^*_n, L^\infty(\Omega))} + \|\partial_t v\|^2_{L^2(I^*_n, H^1(\Omega))} \right)^{\frac{1}{2}} \leq c(n) K =: \tilde{K}.
\]

We denote \((\mathcal{I}(v), \mathcal{D}(v)) = (\mathcal{I}_{\rho_0}(v), \mathcal{D}_{d_0}(v))\) for simplicity. From (4.10) and (4.2) we get

\[
\|\mathcal{P} \mathcal{N}(\rho, v, d)\|_{X_n} \leq c(n) \left( \|v\|_{X_n} + \|\rho\|_{L^\infty(\Omega)} (\|v\|_{X_n} + \|v\|_{X_n}^2) + \|\rho\|^2_{L^\infty(\Omega)} + \|\nabla d\|^2_{L^2(\Omega)} \right).
\]

From (3.11), (3.23), (3.51) and (4.12), it follows that

\[
\|\mathcal{P} \mathcal{N}(\mathcal{I}(v), v, \mathcal{D}(v))(t)\|_{X_n} \leq h_2(K, \rho, \|d_0\|_{H^2(\Omega)}, \|\nabla d_0\|_{L^2(\Omega)}, T, n), \quad t \in I^*_n,
\]

where the constant \(h_2\) is nondecreasing in its first three variables.

Employing (4.10) and the formula \(F(z_1) - F(z_2) = \int_{z_1}^{z_2} F'(s) ds\) with \(F(s) = s^\gamma\) and \(F(s) = s^\beta\),
we obtain
\[
< \mathcal{N}(\rho_1, v_1, d_1) - \mathcal{N}(\rho_2, v_2, d_2), \Phi >
= \int_{\Omega}\left[\mu \Delta(v_1 - v_2) + (\mu + \lambda)\nabla \text{div}(v_1 - v_2)\right] \cdot \Phi \, dx
+ \int_{\Omega}[(\rho_1 - \rho_2)u^i_1 u^i_1 + \rho_2(u^i_1 - u^i_2)u^i_1 + \rho_2 u^j_2(u^j_1 - u^j_2)] \partial_j \Phi \, dx
+ \varepsilon \int_{\Omega}\rho_2[\Delta(v_1 - v_2) \cdot \Phi + \partial_j(u^i_1 - u^i_2)\partial_j \Phi] \, dx + \varepsilon \int_{\Omega}(\rho_1 - \rho_2)(\Delta v_1 \cdot \Phi + \partial_j u^i_1 \partial_j \Phi) \, dx
+ \nu \int_{\Omega}\partial_i d^k_2(\partial_j d^k_1 - \partial_j d^k_2)\partial_j \Phi \, dx + \nu \int_{\Omega}\partial_j d^k_1(\partial_i d^k_1 - \partial_i d^k_2) \partial_i \Phi \, dx
+ \frac{\nu}{2} \int_{\Omega}(\partial_j d^k_1 + \partial_j d^k_2)(\partial_j d^k_1 - \partial_j d^k_2) \partial_i \Phi \, dx + \int_{\Omega}\left[\int_{\rho_2}^{\rho_1}(\gamma A s^{\gamma - 1} + \delta \beta s^{\beta - 1}) \, ds\right] \text{div} \Phi \, dx
\] (4.14)
for any \( \Phi \in W^{1,\infty}_0(\Omega) \). Making use of (3.11), (3.5), (4.2), (4.14), and the elementary properties of the projection \( \mathcal{P} := \mathcal{P}_n \) (see [34, Exercise 7.33]), we get
\[
\left\| \left[ \mathcal{N}(\rho_1, v_1, d_1) - \mathcal{N}(\rho_2, v_2, d_2) \right](t) \right\|_{X_n}
\leq h_2(K, \bar{\rho}, \|d_0\|_{H^1(\Omega)}, \|\nabla^3 d_0\|_{L^2(\Omega)}, T, n) \left\{ \left\| (v_1 - v_2)(t) \right\|_{X_n} + \left\| (\rho_1 - \rho_2)(t) \right\|_{L^1(\Omega)} + \left\| \nabla (d_1 - d_2)(t) \right\|_{L^2(\Omega)} \right\}, \quad t \in I^*_n, \tag{4.15}
\]
where \( h_2 \) is again nondecreasing in its first three variables.

Thanks to (4.7), (3.15), (3.11), and Hölder’s inequality, we obtain
\[
\left\| \mathcal{M}^{-1}_{[\mathcal{N}]}(t) - \mathcal{M}^{-1}_{[\mathcal{N}]}(v_1, v_2) \right\|_{L^2(\Omega, X_n)} \leq h_3(K, \|\rho_0\|_{H^1(\Omega)}, \bar{\rho}, T, n) \left\| v_1 - v_2 \right\|_{C^0(I^*_n, X_n)}, \tag{4.16}
\]
for any \( v_1, v_2 \in C^0(I^*_n, X_n) \) and \( t \in I^*_n \). Here the constant \( h_3 \) is nondecreasing in its first two variables, and is nonincreasing in its third variable. Since
\[
\mathcal{M}^{-1}_{[\mathcal{N}]}(t) \int_0^t [\mathcal{P} N(\mathcal{N}(v), v, \mathcal{D}(v))] \, ds - \mathcal{M}^{-1}_{[\mathcal{N}]}(v_1, v_2) \int_0^t [\mathcal{P} N(\mathcal{N}(v_1, v_2))] \, ds,
\]
we make use of (3.11), (4.4), (4.7) and (4.13) to find that
\[
\mathcal{M}^{-1}_{[\mathcal{N}]}(t) \int_0^t [\mathcal{P} N(\mathcal{N}(v), v, \mathcal{D}(v))] \, ds \in C^0(I^*_n, X_n).
\]
And notice that
\[
\left\| \mathcal{M}^{-1}_{[\mathcal{N}]} \int_0^t [\mathcal{P} N(\mathcal{N}(v), v, d)] \, ds \right\|_{C(I^*_n, X_n)} \leq h_4(K, \bar{\rho}, \|d_0\|_{H^2(\Omega)}, \bar{\rho}, \|\nabla^3 d_0\|_{L^2(\Omega)}, T, n), \tag{4.17}
\]
for any \( t \in I^*_n \), where \( h_4 \) is nondecreasing in its first three variables and nonincreasing in its fourth variable. Similarly, \( \mathcal{M}^{-1}_{[\mathcal{N}]}(t)(\mathcal{P} m_0) \in C^0(I^*_n, X_n) \) and
\[
\left\| \mathcal{M}^{-1}_{[\mathcal{N}]}(t)(\mathcal{P} m_0) \right\|_{C(I^*_n, X_n)} \leq \rho_*^{-1} e^{Kt} \left\| \mathcal{P} m_0 \right\|_{X_n}, \quad t \in I^*_n. \tag{4.18}
\]
Finally, we denote
\[
\mathbf{u} = \mathcal{M}_{\mathcal{I}(v)}^{-1} \mathcal{M}_{\mathcal{I}(v)} \mathcal{M}_{\mathcal{I}(v)}^{-1} \left( \mathcal{P} \mathbf{m}_0 + \int_0^t [\mathcal{P} \mathcal{N}(\mathcal{I}(v), \mathcal{I}(v))] \right) ds \]
(4.19)

Thus, we can use (3.11), (4.4), (4.5) and (4.13) to evaluate (4.19) and obtain that
\[
\|\mathbf{u}\|_{X_n} \leq h_4(K, \hat{\rho}, \|\mathbf{d}_0\|_{H^2(\Omega)}, \hat{\rho}), \|\nabla^3 \mathbf{d}_0\|_{L^2(\Omega), T, n} \leq (\|\partial_t \mathcal{I}(v)\|_{L^1(\Omega)}(1 + \|\mathcal{P} \mathbf{m}_0\|_{X_n}) + 1),
\]
which, together with (3.14) and Hölder’s inequality, yields
\[
\|\mathbf{u}\|_{L^2(I_t, X_n)} \leq h_5(K, \hat{\rho}, \|\rho_0\|_{H^1(\Omega)}, \|\mathbf{d}_0\|_{H^2(\Omega)}, \hat{\rho}, \|\nabla^3 \mathbf{d}_0\|_{L^2(\Omega), T, n}) \leq (1 + \|\mathcal{P} \mathbf{m}_0\|_{X_n}), \quad (4.20)
\]
where \(h_5\) is nondecreasing in its first four variables and nonincreasing in its fifth variable.

4.3. Local existence

Now we are in a position to prove existence of a local solution by applying a fixed point theorem. To this end, we assume that
\[
5 \max \left\{ \frac{\|\mathcal{P} \mathbf{m}_0\|_{X_n}}{\rho}, \|\mathbf{v}(0)\|_{X_n} \right\} < K, \quad (4.21)
\]
and take
\[
T_0 := T_0(\hat{\rho}, \|\rho_0\|_{H^1(\Omega)}, \|\mathbf{d}_0\|_{H^2(\Omega)}, \hat{\rho}, K, T, n) = \min \left\{ \frac{K}{4 T_1}, \frac{\ln(5/4)}{K}, \frac{K^2}{16 h_5^2}, \frac{25}{16 \rho_0^2 h_5^2}, \frac{T_0^2}{\hat{\rho}^2} \right\}, \quad (4.22)
\]
so that \(T_0\) is nonincreasing in its first three variables and nodecreasing in the fourth variable.

With this choice we have
\[
h_4 T_0 \leq \frac{K}{4}, \quad \hat{\rho}^{-1} e^{K T_0}\|\mathcal{P} \mathbf{m}_0\|_{X_n} < \frac{K}{4}, \quad h_5 \sqrt{T_0} (1 + \|\mathcal{P} \mathbf{m}_0\|_{X_n}) < \frac{K}{2}.
\]

Therefore, by virtue of (4.17), (4.18), (4.20) and (4.21), the mapping
\[
\mathcal{T} : \mathbb{A} \rightarrow \mathbb{A},
\mathcal{T}(\mathbf{w}) := \mathcal{M}_{\mathcal{I}(w)}^{-1} \left\{ \mathcal{P} \mathbf{m}_0 + \int_0^t [\mathcal{P} \mathcal{N}(\mathcal{I}(w), \mathcal{I}(\mathbf{w}))] \right\} ds, \quad (4.23)
\]
maps
\[
B_{K, \tau_0} = \left\{ \mathbf{w} \in \mathbb{A} \mid \|\mathbf{w}\|_{C(I_{\tau_0}, X_n)} + \|\partial_t \mathbf{w}\|_{L^2(I_{\tau_0}, X_n)} \leq K \right\}
\]
into itself for any \(0 < \tau_0 < T_0\), where we can take \(X_n = H^1_0(\Omega)\).

In the next step, we prove that \(\mathcal{T}\) is contractive. Keeping in mind that
\[
\mathcal{M}_{\mathcal{I}(w_1)}^{-1}(\mathbf{w}_1) - \mathcal{M}_{\mathcal{I}(w_2)}^{-1}(\mathbf{w}_2) = (\mathcal{M}_{\mathcal{I}(w_1)}^{-1} - \mathcal{M}_{\mathcal{I}(w_2)}^{-1})(\mathbf{w}_1) + \mathcal{M}_{\mathcal{I}(w_2)}^{-1}(\mathbf{w}_1 - \mathbf{w}_2),
\]
we get from (4.23) that
\[
\mathcal{T}(\mathbf{w}_1) - \mathcal{T}(\mathbf{w}_2)
= (\mathcal{M}_{\mathcal{I}(w_1)}^{-1} - \mathcal{M}_{\mathcal{I}(w_2)}^{-1}) \left( \mathcal{P} \mathbf{m}_0 + \int_0^t [\mathcal{P} \mathcal{N}(\mathcal{I}(\mathbf{w}_1), \mathcal{I}(\mathbf{w}_1))] \right) ds
+ \mathcal{M}_{\mathcal{I}(w_2)}^{-1} \int_0^t (\mathcal{P} \mathcal{N}(\mathcal{I}(\mathbf{w}_1), \mathcal{I}(\mathbf{w}_1))) - (\mathcal{P} \mathcal{N}(\mathcal{I}(\mathbf{w}_2), \mathcal{I}(\mathbf{w}_2))) (s) ds. \quad (4.24)
\]
Recalling that \( \underline{\rho} \leq \rho(0) \leq \bar{\rho} \), one has
\[
\| \mathcal{P} m_0 \|_{\mathcal{X}_n} \leq c(n) \bar{\rho} \| v(0) \|_{\mathcal{X}_n} \leq c(n) \bar{\rho} K.
\] (4.25)

We apply (4.13), (4.16) and (4.25) to bound the first term in (4.24), and use (3.11), (3.15), (3.52), (4.4), (4.14) and Hölder’s inequality to majorize the second term in (4.24), and obtain
\[
\| T(w_1) - T(w_2) \|_{\mathcal{X}_n}(t)
\leq h_6(\bar{\rho}, \| \rho_0 \|_{H^1(\Omega)}, \| d_0 \|_{H^2(\Omega)}, \| \nabla^3 d_0 \|_{L^2(\Omega)}, K, T, n)\sqrt{t} \| w_1 - w_2 \|_{C^0(\bar{t}, \mathcal{X}_n)},
\] (4.26)
for \( t \in [0, T_0] \), \( w_1, w_2 \in B_{K, T_0} \), where \( h_6 \) is nondecreasing in its first three variables and nonincreasing in the fourth variable.

Similar to (4.24), we also have the identity
\[
\partial_t T(w_1) - \partial_t T(w_2)
= (\mathcal{M}^{-1}_{\mathcal{J}(w_1)} - \mathcal{M}^{-1}_{\mathcal{J}(w_2)}).\mathcal{M}_{\partial_t \mathcal{J}(w_1)}.\mathcal{M}^{-1}_{\mathcal{J}(w_1)} \left\{ \mathcal{P} m_0 + \int_0^t \left[ \mathcal{P} N(\mathcal{J}(w_1), w_1, D(w_1)) \right](s) ds \right\}
+ \mathcal{M}^{-1}_{\mathcal{J}(w_2)}(\mathcal{M}_{\partial_t \mathcal{J}(w_1)} - \mathcal{M}_{\partial_t \mathcal{J}(w_2)}).\mathcal{M}^{-1}_{\mathcal{J}(w_1)} \left\{ \mathcal{P} m_0 + \int_0^t \left[ \mathcal{P} N(\mathcal{J}(w_1), w_1, D(w_1)) \right](s) ds \right\}
+ \mathcal{M}^{-1}_{\mathcal{J}(w_2)}(\mathcal{M}_{\partial_t \mathcal{J}(w_2)} - \mathcal{M}_{\partial_t \mathcal{J}(w_2)}).\mathcal{M}^{-1}_{\mathcal{J}(w_2)} \left\{ \int_0^t \left[ \mathcal{P} N(\mathcal{J}(w_1), w_1, D(w_1)) - \mathcal{P} N(\mathcal{J}(w_2), w_2, D(w_2)) \right](s) ds \right\}.
\]
Now, employing (3.11), (3.52), (4.3), (4.4), (4.6), (4.13), (4.14), (4.15), (4.16) and (4.25) to control the six terms on the right-hand side of the equality above, we deduce that
\[
\| \partial_t T(w_1) - \partial_t T(w_2) \|_{\mathcal{X}_n}(t)
\leq h_6(\bar{\rho}, \| \rho_0 \|_{H^1(\Omega)}, \| d_0 \|_{H^2(\Omega)}, \| \nabla^3 d_0 \|_{L^2(\Omega)}, K, T, n) \left( \| \partial_t [\mathcal{J}(w_1) - \mathcal{J}(w_2)](t) \|_{L^1(\Omega)} \right)
+ (\| \partial_t \mathcal{J}(w_1)(t) \|_{L^1(\Omega)} + 1) \| w_1 - w_2 \|_{C^0(\bar{t}, \mathcal{X}_n)} + \| \nabla(d_1 - d_2)(t) \|_{L^2(\Omega)}),
\] (4.27)
for a.e. \( t \in [0, T_0] \) and for any \( w_1, w_2 \in B_{K, T_0} \), where \( h_6 \) is nondecreasing in its three variables and nonincreasing in the fourth variable again.

Consequently, substituting (3.14), (3.16), and (3.52) into (4.27), we find that
\[
\| \partial_t T(w_1) - \partial_t T(w_2) \|_{L^2(\bar{t}, \mathcal{X}_n)}
\leq h_6(\bar{\rho}, \| \rho_0 \|_{H^1(\Omega)}, \| d_0 \|_{H^2(\Omega)}, \| \nabla^3 d_0 \|_{L^2(\Omega)}, K, T, n) \sqrt{t} \| w_1 - w_2 \|_{C^0(\bar{t}, \mathcal{X}_n)}.
\] (4.28)

Adding (4.28) to (4.26), we finally get
\[
\| T(w_1) - T(w_2) \|_{\mathcal{X}_n}(t) + \| \partial_t T(w_1) - \partial_t T(w_2) \|_{L^2(\bar{t}, \mathcal{X}_n)}
\leq \tilde{h}_6(\bar{\rho}, \| \rho_0 \|_{H^1(\Omega)}, \| d_0 \|_{H^2(\Omega)}, \| \nabla^3 d_0 \|_{L^2(\Omega)}, K, T, n) t
\left( \| w_1 - w_2 \|_{C^0(\bar{t}, \mathcal{X}_n)} + \| \partial_t (w_1 - w_2) \|_{L^2(\bar{t}, \mathcal{X}_n)} \right)
\]
where \( \tilde{h}_6 \) is nondecreasing in its first three variables and nonincreasing in the fourth variable.
for any \( t \in [0, T_0] \).

If we take 
\[
T^*_n = \min \left\{ T_0, \frac{1}{2h_0} \right\},
\]
then \( \mathcal{T} \) maps \( B_{K,T^*_n} \subset C(\bar{I}T^*_n, X_n) \) into itself and is contractive. Therefore, it possesses in \( B_{K,T^*_n} \) a unique fixed point \( v \) which satisfies (4.38). Thus, we have a solution \((\rho = \mathcal{P}(v), v, \mathcal{P}(v))\) which is defined in \( Q_{T^*_n} \) and satisfies the initial-boundary value problem (3.1)-(3.6) for each given \( n \). Moreover, we see that \( T^*_n \) has the form

\[
0 < T^*_n = h(\bar{\rho}, \|\rho_0\|_{H^1(\Omega)}, \|d_0\|_{H^2(\Omega)}, \|\nabla^3 d_0\|_{L^2(\Omega)}, K, T, n) \leq T_0,
\]
where \( h \) is nonincreasing in its first three variables and nondecreasing in the fourth variable. This means that we can find a unique maximal solution \((\rho_n, v_n, d_n)\) defined in \([0, T_n) \times \Omega\) for each given \( n \), where \( T_n \leq T \).

### 4.4. Global existence

In order to show the maximal time \( T_n = T \) for any \( n \), it suffices to derive uniform bounds for \( \rho_n, v_n, d_n \) and \( \mathcal{P}_n(\rho_n v_n) \). However, we need to impose an additional (smallness) condition in the initial approximate energy to get the uniform bound of \( \|d_n\|_{L^\infty([0,T_n]H^2(\Omega))} \). We remark that \( T_n \) in (4.29) depends on \( \|\nabla^3 d_0\|_{L^2(\Omega)} \), since we have used the fact \( d^{K-1} = d_0 \) on \( \partial \Omega \) and the elliptic estimate to obtain (3.37), and thus we need an auxiliary term \( \|\nabla^3 d_0\|_{L^2(\Omega)} \). Such auxiliary term will not change, if only the initial data of \( d(x, t) \) changes and the boundary value of \( d(x, t) \) does not change. This is why we do not estimate the uniform bound of \( \|\nabla^3 d_n\|_{L^\infty([0,T_n]L^2(\Omega))} \).

For simplicity of notations, we denote

\[
(\rho, v, d, \mathcal{P}m) := (\rho_n, v_n, d_n, \mathcal{P}_n(\rho_n v_n)).
\]

We mention that in the following estimates the letter \( G(\ldots) \) will denote various positive constants depending on its variables.

First, we derive energy estimates similar to Proposition 3.1. Differentiating (4.8) with respect to \( t \), integrating by parts, employing (3.1) and the regularity of \((\rho, v)\), we obtain

\[
\frac{d}{dt} E_\delta(t) + \int_{\Omega} \left( \mu |\nabla v|^2 + (\lambda + \mu) |\text{div} v|^2 + \varepsilon\delta\beta\rho^{\beta - 2} |\nabla \rho|^2 \right) \, dx \leq \nabla \delta \cdot \nabla \rho \]

where

\[
E_\delta(t) = \int_{\Omega} \left( \frac{1}{2} |m|^2 - 1_{\{\rho > 0\}} + Q(\rho) + \frac{\delta}{\beta - 1} \rho^\beta \right) \, dx.
\]

Noting that \( d \) belongs to the function class (3.50) with \( T_n \) in place of \( T^*_d \), \( d \) satisfies (2.7), and one further has

\[
\frac{d}{dt} E_\delta(t) + \int_{\Omega} \left( \mu |\nabla v|^2 + (\lambda + \mu) |\text{div} v|^2 + \nu \theta(\delta |\Delta d| + |\nabla d|^2 d|^2) + \varepsilon\delta\beta\rho^{\beta - 2} |\nabla \rho|^2 \right) \, dx \leq 0,
\]

where

\[
E_\delta(t) := E_\delta(\rho, m, d) := \int_{\Omega} \left( \frac{1}{2} |m|^2 - 1_{\{\rho > 0\}} + \frac{A}{\gamma - 1} \rho^\gamma + \frac{\delta}{\beta - 1} \rho^\beta + \frac{\nu |\nabla d|^2}{2} \right) \, dx
\]

(4.30)
with \( m = \rho v \). Integrating (4.30) over \((0, t)\), we get

\[
\mathcal{E}(t) + \int_0^t \int_\Omega \left( \mu |\nabla v|^2 + (\lambda + \mu) |\text{div} v|^2 + \nu \theta (|\Delta d| + |\nabla d|^2)^2 \right) \, dx \, ds
\leq \mathcal{E}_0(t) := \mathcal{E}_0(\rho_0, m_0, d_0) < \mathcal{E}_0,
\]

where \( \mathcal{E}_0 \) is a given positive constant. In particular,

\begin{align}
\| \sqrt{\rho} v \|_{L^\infty(t, L^2(\Omega))} &\leq 2 \mathcal{E}_0, \\
\| \nabla d \|_{L^\infty(t, L^2(\Omega))} &\leq \mathcal{E}_0/\nu, \\
\| v \|_{L^2(t, H^1(\Omega))} &\leq c(\Omega) \mathcal{E}_0/\mu.
\end{align}

With the help of (4.31)–(4.33), we can deduce more uniform bounds on \((\rho, v)\). Using (3.11) and (4.33), thanks to the norm of equivalence on \( X_n \) (see (4.2)), we find that

\[
G_1(\rho_0, \mathcal{E}_0, T, n) \leq \rho \leq G_2(\bar{\rho}, \mathcal{E}_0, T, n),
\]

from which, (4.31) and (4.2), it follows that

\[
\| v \|_{C^0(t, X_n)} \leq G(\bar{\rho}, \mathcal{E}_0, T, n)
\]

and

\[
\| \mathcal{P}(\rho v) \|_{C^0(t, X_n)} \leq G(\bar{\rho}) \| \rho v \|_{C^0(t, L^2(\Omega))} \leq G(\bar{\rho}, \mathcal{E}_0, T, n).
\]

Applying (4.35) to (3.12), one gets

\[
\| \rho \|_{C^0(t, L^1(\Omega))} \leq G(\bar{\rho}) \| \rho_0 \|_{L^1(\Omega)}, \mathcal{E}_0, T, n).
\]

Utilizing (4.32), (4.34)–(4.36), arguing similarly to that for (4.20), we obtain from (4.11) that

\[
\| \partial_t v \|_{L^2(t, X_n)} \leq G(\bar{\rho}) \| \rho_0 \|_{H^1(\Omega)}, \| d_0 \|_{H^2(\Omega)}, \mathcal{E}_0, T, n).
\]

Hence, we have shown the uniform boundedness of \( \rho, \bar{\rho}, \| \rho \|_{C^0(t, X_n)}, \| \mathcal{P}(\rho v) \|_{C^0(t, X_n)}, \| v \|_{C^0(t, X_n)} \) and \( \| \partial_t v \|_{L^2(t, X_n)} \). It remains to show the uniform boundedness of \( \| d \|_{L^\infty(t, H^2(\Omega))}, \mathcal{E}_0 \), which can be obtained by following the spirit of proof in Section 3.2.2. For the reader’s convenience, we give the proof in the following.

First, we rewrite the energy inequality (4.30) in the following form as in (2.11):

\[
\mathcal{E}(t) + \int_0^t \int_\Omega \left[ \mu |\nabla v|^2 + (\lambda + \mu) |\text{div} v|^2 + \nu \theta |\Delta d|^2 + \varepsilon \delta \beta \rho^{-2} |\nabla \rho|^2 \right] \, dx \, ds \leq \mathcal{E}_0 + \nu \theta \| \nabla d \|_{L^4(\Omega)}^4.
\]

Let \( \mathcal{E}_0 = \nu/4094 \). Arguing in the same manner as in the derivation of (2.19), we deduce that

\[
\| \nabla^2 d \|_{L^2(Q_t)} + \| \nabla d \|_{L^4(Q_t)} + \| \partial_t d \|_{L^{4/3}(t, L^2(\Omega))} \leq G(\| d_0 \|_{H^2(\Omega)}, \mathcal{E}_0).
\]

Now we proceed to derive uniform bounds of higher derivatives on \( d \). Differentiating (3.2) with respect to \( t \), multiplying the resulting equations by \( \partial_t d \) in \( L^2(\Omega) \), recalling \( |d| = 1 \), we integrate by parts to infer that

\[
\frac{d}{dt} \int_\Omega |\partial_t d|^2 \, dx
= 2 \int_\Omega \partial_t d \cdot (\theta \Delta d + \theta |\nabla d|^2 \partial_t d + 2\theta (\nabla d \cdot \nabla \partial_t d) d - \partial_t v \cdot \nabla d - v \cdot \nabla \partial_t d) \, dx
\leq -\theta \| \nabla \partial_t d \|_{L^2(\Omega)}^2 + (10\theta + 1) \| \partial_t d \|_{L^2(\Omega)}^2 + 2 \| \partial_t v \|_{L^2(\Omega)}^2 + \frac{1}{\theta} \| v \|_{L^\infty(Q_t)}^2 \| \partial_t d \|_{L^2(\Omega)}^2,
\]
where the second term on the right-hand side of (4.40) can be bounded as follows, using (2.12), Hölder’s and the triangle inequalities.

\[(10\theta + 1)||\partial_t d||^2_{L^2(\Omega)} \leq (10\theta + 1)||\partial_t d||^2_{L^4(\Omega)}||\nabla d||^2_{L^4(\Omega)}\]

\[\leq (10\theta + 1)\sum_{i=1}^2 ||\partial_t d_i||^2_{L^4(\Omega)}||\nabla d||^2_{L^4(\Omega)}\]

\[\leq (10\theta + 1)\sum_{i=1}^2 \sqrt{2}||\partial_t d_i||_{L^2(\Omega)}||\nabla \partial_t d_i||_{L^2(\Omega)}||\nabla d||^2_{L^4(\Omega)}\]

\[\leq \frac{\theta}{2}||\nabla \partial_t d||^2_{L^2(\Omega)} + \frac{(10\theta + 1)^2}{\theta}||\partial_t d||^2_{L^2(\Omega)}||\nabla d||^4_{L^4(\Omega)}\].

Inserting (4.41) into (4.40), we conclude that

\[\frac{d}{dt}||\partial_t d||^2_{L^2(\Omega)} + \frac{\theta}{2}||\nabla \partial_t d||^2_{L^2(\Omega)} \leq \left(\frac{(10\theta + 1)^2}{\theta}||\nabla d||^2_{L^4(\Omega)} + \frac{1}{\theta}||v||^2_{L^\infty(Q_T)}\right) ||\partial_t d||^2_{L^2(\Omega)} + 2||\partial_t v||^2_{L^2(\Omega)}\]

Thus, applying Gronwall’s inequality, we have

\[||\partial_t d||^2_{L^2(\Omega)} \leq \left(||\partial_t d(0)||^2_{L^2(\Omega)} + 2||\partial_t v||^2_{L^2(Q_T)}\right) e^{\frac{(10\theta + 1)^2}{\theta}||\nabla d||^2_{L^4(Q_T)} + \frac{1}{\theta}||v||^2_{L^\infty(Q_T)}}\]

Noting that

\[||\partial_t d(0)||_{L^2(\Omega)} = ||\theta(\Delta d_0 + |\nabla d_0|^2 d_0) - v_0 \cdot \nabla d_0||_{L^2(\Omega)},\]

we use (4.35), (4.38) and (4.39) to arrive at

\[||\partial_t d||^2_{L^\infty(I_n, L^2(\Omega))} + ||\nabla \partial_t d||^2_{L^2(I_n, L^2(\Omega))} \leq G(\rho, \bar{\rho}, ||\rho_0||_{H^1(\Omega)}, ||d_0||_{H^2(\Omega)}, \bar{\varepsilon}_0, T, n)\]

Recalling that |d| \equiv 1, it follows from (3.2) that

\[\theta^2 \int |\Delta d|^2 dx \leq 3\theta^2 \int |\nabla d|^4 dx + 3 \int |\partial_t d|^2 dx + 3 \int |v|^2 |\nabla d|^2 dx.\]

Similarly to the derivation of (2.15), the first term on the right-hand side of (4.43) can be estimated as follows.

\[3\theta^2 \int |\nabla d|^4 dx \leq 384\theta^2(c(\Omega) + 12)^3||d_0||^2_{H^2(\Omega)} \left(||d_0||^2_{H^2(\Omega)} + \frac{2\varepsilon_0}{\nu}\right) + \frac{3072\theta^2\varepsilon_0}{\nu}||\Delta d||^2_{L^2(\Omega)}.\]

Putting (4.43) and (4.44) together, using (4.35), (4.32) and (3.2), we get

\[\frac{\theta^2}{4} \int |\Delta d|^2 dx \leq (c(\Omega) + 12)^3||d_0||^2_{H^2(\Omega)} \left(||d_0||^2_{H^2(\Omega)} + \frac{2\varepsilon_0}{\nu}\right) + 3||\partial_t d||^2_{L^2(\Omega)} + 6||v||^2_{L^\infty(\Omega)}||\nabla d||^2_{L^2(\Omega)}\]

\[\leq G(\rho, \bar{\rho}, ||\rho_0||_{H^1(\Omega)}, ||d_0||_{H^2(\Omega)}, \bar{\varepsilon}_0, T, n).\]
Proposition 4.1. Let

\[
\delta > 0, \ \beta > 0, \ \varepsilon > 0, \ \text{and} \ 0 < \rho \leq \bar{\rho} < \infty.
\]

Assume that \( \Omega \) is a bounded \( C^{2,\alpha} \)-domain (\( \alpha \in (0,1) \)), and the initial data \((\rho_0, m_0, d_0)\) satisfies

\[
\mathcal{E}_\delta(\rho_0, m_0, d_0) < \mathcal{E}_{\delta,0} := \nu/4094,
\]

\[
0 < \rho \leq \rho_0 \leq \bar{\rho}, \ \rho_0 \in W^{1,\infty}(\Omega), \ \|d_0\| = 1, \ (4.49)
\]

\[
v_0 \in X_n, \ d_0 \in H^\delta(\Omega).
\]

Then there exists a unique triple \((\rho_n, v_n, d_n)\) with the following properties:

1. Regularity.

\[
\begin{cases}
\rho_n \text{ satisfies the regularity as in Proposition 3.1 with } T \text{ in place of } T^K_d, \\
v_n \in C^0(I, X_n), \ \partial_t v_n \in L^2(I, X_n), \ \nabla \rho_n \in L^2(I, E_{d,0}^p(\Omega)), \ \rho_n v_n \in C^0(I, E_{d}^0(\Omega)),
\end{cases}
\]

\[
d_n \text{ satisfies the regularity as in Proposition 3.2 with } T \text{ in place of } T^K_d.
\]

2. \((\rho_n, v_n, d_n)\) solves (3.1) and (3.2) a.e. in \( Q_T \), and satisfies (3.3) and \((\rho_n, v_n, d_n)|_{t=0} = (\rho_0, v_0, d_0)\).

3. Finite and bounded energy inequalities:

\[
\frac{d}{dt} \mathcal{E}_\delta^n(t) + \mathcal{F}^n(t) + \int_{\Omega} \varepsilon \delta \rho_n^{-2} |\nabla \rho_n|^2(t) dx \leq 0 \ \text{in } D'(I),
\]

and

\[
\mathcal{E}_\delta^n(t) + \int_0^t \left( \mathcal{F}^n(s) + \int_{\Omega} \varepsilon \delta \rho_n^{-2} |\nabla \rho_n|^2(s) dx \right) ds \leq \mathcal{E}_\delta(\rho_0, m_0, d_0) \ \text{a.e. in } I,
\]

where \( \mathcal{F}^n(t) := \mathcal{F}(\rho_n, v_n, d_n) \) and \( \mathcal{E}_\delta^n(t) := \mathcal{E}_\delta(\rho_n, m_n, d_n) \) with \( m_n = \rho_n v_n \).

4. Additional uniform estimates.

\[
|d_n| \equiv 1 \ \text{in } \tilde{Q}_T,
\]

\[
\|\nabla^2 d_n\|_{L^2(\Omega)} + \|\nabla d_n\|_{L^4(\Omega)} + \|\partial_t d_n\|_{L^4(\Omega)} \leq G(\|d_0\|_{H^\delta(\Omega)}),
\]

\[
\sqrt{\varepsilon} \|\nabla \rho_n\|_{L^2(\Omega)} \leq G(\delta),
\]

\[
\|\rho_n\|_{\frac{4\delta}{3} (\tilde{Q}_T)} \leq G(\varepsilon, \delta).
\]
(see [34, Section 7.7.5.2] for the proof of (4.50) and (4.51)), where $G$ is a positive constant which is independent of $n$ and nondecreasing in its arguments. Moreover, if $\varepsilon$ is not explicitly written in the argument of $G$, then $G$ is independent of $\varepsilon$ as well.

**Remark 4.1.** Here we have used the notation $E^p_0(\Omega)$ concerning the spaces of vector fields with summable divergence introduced in [34, Section 3.2]. For the reader’s convenience, we give the detailed definition. We set

$$E^{q,p}(\Omega) = \{ g \in (L^q(\Omega))^2 : \text{div}g \in L^p(\Omega) \} \hookrightarrow L^q(\Omega)$$

with the norm

$$\|g\|_{E^{q,p}} = \|g\|_{L^q} + \|\text{div}g\|_{L^p(\Omega)}.$$

For the sake of simplicity, $E^{p,p}(\Omega)$ is denoted by $E^p(\Omega).$ Then we define

$$E^{q,p}_0(\Omega) = \overline{D(\Omega)}^{E^{q,p}}$$

and $E^p_0 = \overline{D(\Omega)}^{E^p}.$

The notation $E^{q,p}_0(\Omega)$ will be used in Proposition 5.1.

**5. Proof of Theorem 1.1**

Once we have established Proposition 4.1, we can obtain Theorem 1.1 by the standard three-level approximation scheme and the method of weak convergence as in [8, 31] for the compressible Naiver-Stokes equations. These arguments have also been successfully used to establish the existence of weak solutions to other models from fluid dynamics. Here we briefly describe how to prove Theorem 1.1 and state the existence of solutions to the first and second approximate problems for the reader’s convenience, we refer to [38] or [8, 34] for the details of the limit process.

In view of the proof in [38], it suffices to analyze the convergence of the supercritical nonlinearity $|\nabla d|^2d$. In fact, using the uniform bounds in (4.54) and (4.55) on $d_n$, applying the Arzelà-Ascoli theorem and Aubin-Lions lemma, and taking subsequences if necessary, we deduce that

$$d_n \rightarrow d \text{ strongly in } C^0(I, L^2(\Omega)) \cap L^p(I, H^1(\Omega)) \cap L^r(I, W^{1,p}(\Omega))$$

for any $p \geq 1$ and $r \in [1, 2)$, which implies that

$$|\nabla d_n|^2d_n \rightarrow |\nabla d|^2d \text{ weakly in } L^2(Q_T).$$

Moreover, using Vitali's convergence theorem, and taking subsequences if necessary, we obtain

$$|\nabla d_n|^2d_n \rightarrow |\nabla d|^2d \text{ strongly in } L^r(Q_T) \text{ for any } r \in [1, 2).$$

In addition, we also have the regularity $d \in L^2(I, H^2(\Omega))$ and $\partial_t d \in L^2(I, (H^1(\Omega))^*)$. In view of [34, Proposition 7.31], we get consequently

$$d \in C^0(I, H^1(\Omega)).$$

Putting (5.2) and (5.4) into the proof of [38, Proposition 4.1], we immediately obtain a weak solution of the second approximate problem which is the weak limit $(\rho_\varepsilon, v_\varepsilon, d_\varepsilon)$ of the solution sequence $(\rho_n, v_n, d_n)$ as $n \rightarrow \infty$ constructed in Proposition 4.1. Moreover, referring to the conclusions in [34, Proposition 7.31], the existence of solutions to the second approximate problem reads as follows.
Proposition 5.1. Let $\delta > 0$, $\beta \geq \max\{\gamma, 8\}$, $\varepsilon > 0$, and $0 < \rho \leq \bar{\rho} < \infty$. Assume that $\Omega$ is a bounded $C^{2,\alpha}$-domain ($\alpha \in (0, 1)$), and the initial data $(\rho_0, m_0, d_0)$ satisfies (4.48), (4.49) and
\[
m_0 \in L^2(\Omega), \ d_0 \in H^2(\Omega).
\]
Then there exists a unique triple $(\rho_\varepsilon, v_\varepsilon, d_\varepsilon)$ with the following properties:

1. Regularity.
\[
\begin{cases}
\rho_\varepsilon \geq 0 \ a.e. \ in \ Q_T, \ &\rho_\varepsilon \in C^0(I, L^p_{weak}(\Omega)) \cap C^0(I, L^p(\Omega)), \ 1 \leq p < \beta, \\
\rho_\varepsilon^\beta \in L^2(I, H^1(\Omega)), \ &\partial_t \rho_\varepsilon \in L^{5/2+3/4\beta}(Q_T), \ \nabla^2 \rho_\varepsilon \in L^{5/2+3/4\beta}(\Omega), \\
\nabla \rho_\varepsilon, \ &\rho_\varepsilon v_\varepsilon \in L^2(I, E)_{\varepsilon\theta}^{(10\beta+6, 5/2+3/4\beta)}(\Omega), \ \int \rho_\varepsilon dx = \int \rho_0 dx, \\
m_\varepsilon := \rho_\varepsilon v_\varepsilon \in C^0(I, L^{\beta+1}_{weak}(\Omega)), \ &d_\varepsilon \in C^0(I, H^1(\Omega)), \ d|_{\partial \Omega} = d_0 \in C^{0,1}(\partial \Omega).
\end{cases}
\]

2. Weak-strong solutions.
\[
\begin{align*}
\partial_t \rho_\varepsilon + \text{div}(\rho_\varepsilon v_\varepsilon) - \varepsilon \Delta \rho_\varepsilon &= 0 \ in \ D'(Q_T), \\
\partial_t (\rho_\varepsilon v_\varepsilon) + \partial_j (\rho_\varepsilon v_\varepsilon v_j) - \mu \Delta v_\varepsilon - (\mu + \lambda) \text{div} v_\varepsilon + \nabla A \rho_\varepsilon + \delta \nabla \rho_\varepsilon^\beta + \nu \text{div} \left( \nabla d_\varepsilon \otimes \nabla v_\varepsilon - \frac{|\nabla v_\varepsilon|^2}{2} \right) + \varepsilon \nabla \rho_\varepsilon \cdot \nabla v_\varepsilon &= 0 \ in \ (D'(Q_T))^2, \\
\partial_t d_\varepsilon + v_\varepsilon \cdot \nabla d_\varepsilon &= \theta(\Delta d_\varepsilon + |\nabla d_\varepsilon|^2 d_\varepsilon), \ a.e. \ in \ Q_T.
\end{align*}
\]
Moreover, $(\rho_\varepsilon, v_\varepsilon, d_\varepsilon)(x, 0) = (\rho_0, v_0, d_0)$. (3) $(\rho_\varepsilon, v_\varepsilon, d_\varepsilon)$ satisfies the finite and bounded energy inequalities as in (4.53) and (4.53), and the uniform estimates (4.54) and (4.55).

4. Additional uniform estimates.
\[
\begin{align*}
\|\rho_\varepsilon v_\varepsilon\|_{L^\infty(I, L^{2\beta+3}(\Omega))} + \|\rho_\varepsilon v_\varepsilon\|_{L^2(I, L^{5/2+3/4\beta}(\Omega))} + \|\rho_\varepsilon v_\varepsilon\|_{L^{10\beta+6}(Q_T)} \\
+ \|\rho_\varepsilon|v_\varepsilon|^2\|_{L^2(I, L^{(6/5+3/\beta)}(\Omega))} + \|\rho_\varepsilon\|_{L^{\beta+1}(Q_T)} + \varepsilon \|\nabla \rho_\varepsilon\|_{L^{10\beta+6}(Q_T)} \\
+ \sqrt{\varepsilon} \|\nabla \rho_\varepsilon\|_{L^2(Q_T)} + \varepsilon \|\nabla \rho_\varepsilon \cdot \nabla v_\varepsilon\|_{L^{5/2+3}(Q_T)} \leq G(\delta),
\end{align*}
\]
where $G$ is a positive constant independent of $\varepsilon$.

Here we explain how the assumptions (4.50) in Proposition 4.1 become (5.5) in Proposition 5.1. First, we have Proposition 5.1 for fixed initial data satisfying (4.50). If they satisfy the assumption (5.5) only, then we can approximate $(\rho_0, m_0, d_0)$ by a sequence $(\rho_{0m}, m_{0m}, d_{0m})$ which satisfies (4.50), (4.48), and
\[
\mathcal{E}_\delta(\rho_{0m}, m_{0m}, d_{0m}) \to \mathcal{E}_\delta(\rho_0, m_0, d_0) \text{ as } m \to +\infty;
\]
moreover, $d_{0m} \to d_0$ strongly in $H^2(\Omega) \cap C^{0,1}(\bar{\Omega})$. Then it is sufficient to take $v_{0m} = P_n(m_{0m}/\rho_0) \in X_n$. Obviously, Proposition 5.1 holds with these new initial data, and those uniform estimates in Proposition 5.1 are independent of $m$. Therefore, repeating the limit process as $m \to \infty$, we obtain Proposition 5.1 as well.

Now, with Proposition 5.1 in hand, following the proof of [38, Proposition 5.1] or [34, Proposition 7.27], we can obtain a weak solution $(\rho_\delta, v_\delta, d_\delta)$ of the first approximate problem as the weak limit of the sequence $(\rho_\varepsilon, v_\varepsilon, d_\varepsilon)$ as $\varepsilon \to 0$ constructed in Proposition 5.1. Thus, we have the following existence result.
Proposition 5.2. Let \( \delta > 0 \), \( \beta \geq \max\{\gamma, 8\} \). Assume that \( \Omega \) is a bounded \( C^{2,\alpha} \)-domain (\( \alpha \in (0, 1) \)), and \((\rho_0, m_0, d_0)\) satisfies (1.9)–(1.11) with \( \beta \) in place of \( \gamma \), and (4.48). Then there exists a unique triple \((\rho_\delta, v_\delta, d_\delta)\) with the following properties:

1. Regularity.

\[
\begin{cases}
\rho_\delta \in C^0(\bar{I}, L^3_{\text{weak}}(\Omega)) \cap C^0(\bar{I}, L^p(\Omega)) \cap L^{\beta+1}(\mathbb{R}^3 \times I), \ 1 \leq p < \beta, \\
\rho_\delta \geq 0 \ \text{a.e. in } Q_T, \quad \rho_\delta = 0, \quad v_\delta = 0 \ \text{in } (\mathbb{R}^3 \setminus \Omega) \times I, \\
\rho_\delta |v_\delta|^2 \in L^2(I, L^{\frac{6\beta}{\beta+3}}(\mathbb{R}^2)) \cap L^1(I, L^{\frac{3\beta}{\beta+3}}(\mathbb{R}^2)), \\
m_\delta := \rho_\delta v_\delta \in L^2(I, L^{\frac{6\beta}{\beta+3}}(\mathbb{R}^2)) \cap C^0(\bar{I}, L^{\frac{3\beta}{\beta+3}}(\Omega)), \\
d_\delta \in C^0(I, H^1(\Omega)), \quad d_\delta|_{\partial \Omega} = d_0.
\end{cases}
\]

2. \((\rho_\delta, v_\delta, d_\delta)\) solves (5.6)–(5.8) with \( \varepsilon = 0 \), and also satisfies the finite and bounded energy inequalities as in (4.52) and (4.53) with \( \varepsilon = 0 \). Moreover, \((\rho_\delta, v_\delta, d_\delta)|_{t=0} = (\rho_0, v_0, d_0)\).

3. For any \( b \) satisfying (1.17) and (1.18), the function \( b(\rho_\delta) \) is in \( C^0(\bar{I}, L^{\frac{2\gamma+1}{\gamma+1}}(\Omega)) \cap C^0(\bar{I}, L^p(\Omega)) \), \( 1 \leq p < \frac{\beta}{\alpha + 1} \). Moreover,

\[
\partial_t b(\rho_\delta) + \text{div}[b(\rho_\delta)v] + [\rho_\delta b'(\rho_\delta) - b(\rho_\delta)] \text{div}v = 0 \ \text{in } \mathcal{D}'(\mathbb{R}^2 \times I).
\]

For any \( b_k \) \((k > 0)\) defined by

\[
b_k(s) = \begin{cases} b(s) & \text{if } s \in [0, k), \\ b(k) & \text{if } s \in [k, \infty), \end{cases} \quad \text{with } \quad (b_k)'_+(s) = \begin{cases} b'(s) & \text{if } s \in [0, k), \\ 0 & \text{if } s \in [k, \infty), \end{cases}
\]

we have \( \partial_t b_k(\rho_\delta) + \text{div}[b_k(\rho_\delta)v] + [\rho_\delta b_k'(\rho_\delta) - b_k(\rho_\delta)] \text{div}v = 0 \ \text{in } \mathcal{D}'(\mathbb{R}^2 \times I) \).

4. \((\rho_\delta, v_\delta, d_\delta)\) satisfies the uniform estimates (4.54) and (4.55). Moreover,

\[
\begin{align*}
& \|\rho_\delta v_\delta\|_{L^\infty(\bar{I}, L^{\frac{2\gamma+1}{\gamma+1}}(\Omega))} + \|\rho_\delta v_\delta\|_{L^2(\bar{I}, L^{\frac{6\beta}{\beta+3}}(\Omega))} + \|\rho_\delta |v_\delta|^2\|_{L^2(\bar{I}, L^{\frac{3\beta}{\beta+3}}(\Omega))} \\
& \quad + \|\rho_\delta |v_\delta|^2\|_{L^2(\bar{I}, L^{\frac{6\beta}{\beta+3}}(\Omega))} + \|\rho_\delta\|_{L^{\gamma+\theta}(Q_T)} + \beta \pi^\theta \|\rho_\delta\|_{L^{\beta+\theta}(Q_T)} \leq G(\theta)
\end{align*}
\]

for any constant \( \theta \in (0, \gamma - 1) \) (the uniform estimate on \( \|\rho_\delta\|_{L^{\gamma+\theta}(Q_T)} \) can be found in [18] for the two-dimensional case). Here \( G(\theta) \) is a positive constant independent of \( \delta \).

Remark 5.1. We assume that the initial density \( \rho_0 \) satisfies (4.49) in Proposition 5.1. However, the condition (4.49) on \( \rho_0 \) can be relaxed by the condition “\( \rho_0 \in L^2(\Omega) \) and \( \rho_0 \geq 0 \)” in Proposition 5.2. This process can be dealt by the approximation method (similarly as in (5.9)), please refer to [34, Section 7.10.7] for the detailed proof.

Finally, with the help of Proposition 5.2, we can follow the proof of [38, Theorem 2.1] or [34, Theorem 7.7] to obtain a weak solution \((\rho, v, d)\) of the original problem (1.1)–(1.4) which is the weak limit as \( \delta \to 0 \) of the weak solution sequence \((\rho_\delta, v_\delta, d_\delta)\) constructed in Proposition 5.2. This completes the proof of Theorem 1.1.

6. Global existence of large solutions to the Cauchy problem

In this section, we will briefly describe how to prove Theorem 1.2 on the Cauchy problem by modifying the proof of Theorem 1.1 and applying the domain expansion technique.
6.1. **Local solvability of the Cauchy problem on the direction vector**

First we establish the local existence of solutions to the following Cauchy problem on the direction vector:

\[ \partial_t d + v \cdot \nabla d = \theta(\Delta d + |\nabla d|^2(d + e_2)) \quad \text{in } \mathbb{R}^2 \times I \]

with initial data

\[ d(x, 0) := d_0 \in H^2(\mathbb{R}^2), \quad |d_0(x) + e_2| = 1, \quad d_{02}(x) + 1 \geq d_{02} > 0 \]

for some given constant \( d_{02} \), where \( d_{02} \) denotes the second component of \( d_0 \), and the known function

\[ v \in \tilde{V} := \{ v \in C^0([I, H^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)] \mid \partial_t v \in L^2(I, H^1(\mathbb{R}^2)) \} \].

The local existence can be shown by following the proof of Proposition 3.2 and applying the domain extension technique. Next, we briefly describe the proof.

By virtue of (6.2), we have a sequence of functions \( \{d_n^0(x)\}_{n \geq 1} \subset H^2(\mathbb{R}^2) \), such that

\[ |d_n^0(x) + e_2| = 1, \quad d_{n2}(x) + 1 \geq d_{02}/2, \quad d_n^0 \to d_0 \text{ in } H^2(\mathbb{R}^2) \]

and \( \text{supp } d_n^0 \subset B_{R_n} := \{ x \in \mathbb{R}^2 \mid |x| < R_n \} \) for some \( R_n \geq 1 \). In view of Proposition 3.2, there exists a unique local strong solution \( d \) to the following Dirichlet problem:

\[ \partial_t d + v \cdot \nabla d = \theta(\Delta d + |\nabla d|^2d) \quad \text{in } B_{R_n} \times I, \]

with initial and boundary conditions

\[ d(x, 0) = d_n^0 \in H^2(B_{R_n}), \quad d(x, t)\vert_{\partial B_{R_n}} = 0 \quad \text{for any } t \in I. \]

Moreover \( d_2 \geq c(d_{02}) \). However, the constants \( h_1 \) and \( C \) in Proposition 3.2 depend on the bounded domain \( \Omega = B_{R_n} \), since we have used the interpolation inequalities (3.41) and (3.42) (of course, the embedding theorems \( H^2(\Omega) \hookrightarrow L^\infty(\Omega) \) and \( H^1(\Omega) \hookrightarrow L^4(\Omega) \) used in (3.32) and (3.29) can be replaced by such two interpolation inequalities, respectively), where the interpolation constants depend on the bounded domain \( \Omega = B_{R_n} \). Fortunately, we have the following results:

\[ \|v\|_{L^\infty(B_{R_n})}^2 \leq c \|v\|_{H^2(B_{R_n})} \|v\|_{L^2(B_{R_n})} \quad \text{and} \quad \|v\|_{L^4(B_{R_n})}^2 \leq c \|v\|_{H^1(B_{R_n})}^2 \|v\|_{L^2(B_{R_n})} \]

for some constant \( c \) independent of \( R_n \geq 1 \), which can be deduced from (3.41) and (3.42) with \( B_1 \) in place of \( \Omega \) by scaling the spatial variables. In addition, the elliptic estimate used in (3.37) is replaced by the following special elliptic estimate: if \( v \in H^1_0(B_{R_n}), f \in H^1(B_{R_n}) \) and \( \Delta v = f \) in \( D'(B_{R_n}) \), then

\[ \|\nabla^3 v\|_{L^2(B_{R_n})}^2 \leq c \|f\|_{H^1(B_{R_n})} \quad \text{for some constant } c \text{ independent of } R_n \geq 1 \]

which can be deduced from the standard elliptic estimate on the domain \( B_1 \) by scaling the spatial variables. With these facts, repeating the proof of Proposition 3.2 with a trivial modification, we can obtain Proposition 3.2 with \( \Omega = B_{R_n} \). Moreover, the constants \( h_1 \) and \( C \) can be independent of \( B_{R_n} \) and \( \|\nabla^3 d_0\|_{L^2(B_{R_n})} \). Then, using the domain expansion technique (see [2]), we can obtain a unique strong solution \( d \) of the Cauchy problem (6.1), (6.2) as the weak limit of the sequence \( d_{R_n} \) with initial data \( d_{R_n} \vert_{t=0} = d_n^0 \) as \( R_n \to \infty \). Thus, we have

**Proposition 6.1.** Proposition 3.2 holds with \( \mathbb{R}^2 \) in place of \( \Omega \), where \( d = D_{d_0}^K(v) \) solves the Cauchy problem (6.1), (6.2), the constants \( h_1 \) and \( C \) only depend on \( K \) and \( \|d_0\|_{H^2(\mathbb{R}^2)} \), and

\[
d_2(x, t) + 1 \geq d_{02}/2 \quad \text{for any } (x, t) \in \mathbb{R}^2 \times I_d^K, \quad \text{if } d_{02}(x) + 1 \geq d_{02}.
\]
6.2. Global solvability of the approximate problem

Once we have established Proposition 6.1, we can see, in view of the proof in Section 4, that there exists a unique local solution to the following approximate problem to the original problem (1.1)–(1.5) with $\Omega = \mathbb{R}^2$:

$$\int_{B_R} (\rho v)(t) \cdot \Psi dx - \int_{B_R} m_0 \cdot \Psi dx$$

$$= \int_0^t \int_{B_R} \left[ \mu \Delta v + (\mu + \lambda) \nabla \div v - A \nabla \rho^\gamma - \delta \nabla \rho^\beta - \varepsilon (\nabla \rho \cdot \nabla v) - \div (\rho v \otimes v) \right]$$

$$- \nu \div \left( \nabla d \otimes \nabla d - \frac{|\nabla d|^2 d}{2} \right) \cdot \Psi dx ds \quad \text{for all } t \in I \text{ and any } \Psi \in \mathbb{X}_n,$$

$$\partial_t \rho + \div (\rho v) = \varepsilon \Delta \rho \quad \text{in } B_R \times I,$$

$$\partial_t d + v \cdot \nabla d = \theta (\Delta d + |\nabla d|^2 d) \quad \text{in } \mathbb{R}^2 \times I,$$

with boundary conditions

$$\nabla \rho \cdot n|_{\partial B_R} = 0, \ v|_{\partial B_R} = 0, \ d(x, t) - e_2 \to 0 \text{ as } |x| \to +\infty$$

and modified initial conditions

$$\rho(x, 0) = \rho_0 \in W^{1,\infty}(B_R), \quad 0 < \rho \leq \rho_0 \leq \bar{\rho} < \infty,$$

$$v(x, 0) = v_0 \in \mathbb{X}_n, \quad d(x, 0) - e_2 \in H^3(\mathbb{R}^2),$$

where $(v, \partial_t v) \equiv 0$ in $(\mathbb{R}^2 \setminus B_R) \times I$ in (6.10). Here we have used the fact that $v \in L^2(I, H^1_0(B_R))$ is equivalent to $v \in L^2(I, H^1(\mathbb{R}^2))$ and $v = 0$ in $(\mathbb{R}^2 \setminus B_R) \times I$. Hence one has $v \in C^0(I, H^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ and $\partial_t v \in L^2(I, H^1(\mathbb{R}^2))$ by zero extension, if $v \in C(\bar{I}, \mathbb{X}_n)$ and $\partial_t v \in L^2(I, \mathbb{X}_n)$ with $\Omega = B_R$.

Similar to the arguments in Section 4.4 we shall derive the uniform-in-time energy estimates in order to extend the local solution globally in time. This can be done by modifying the derivation of the uniform estimates in Section 4.4. The only difference in arguments arises from the energy estimates for $d$ in the whole space, recalling that we have shown the uniform estimates on $\|\Delta d\|_{L^2(I, L^2(\Omega))}$ and $\|\Delta d\|_{L^\infty(I, L^2(\Omega))}$ in (4.39) and (4.46) in a bounded domain. In the case of $\Omega = \mathbb{R}^2$, however, if $d_2$ satisfies some geometric angle condition, then we can use the rigidity theorem to deduce the uniform estimates on $\|\Delta d\|_{L^2(I, L^2(\mathbb{R}^2))}$ and $\|\Delta d\|_{L^\infty(I, L^2(\mathbb{R}^2))}$. The rigidity theorem, which was recently established in [20], reads as follows.

**Proposition 6.2.** Let $d_2 > 0$, $c_0 > 0$. Then there exists a positive constant $\varpi_0 = \varpi(d_2, c_0)$, such that the following holds.

*If $d := (d_1, d_2) : \mathbb{R}^2 \to S^1$, $\nabla d \in H^1(\mathbb{R}^2)$ with $\|\nabla d\|_{L^4(\mathbb{R}^2)} \leq c_0$ and $d_2 \geq d_2$, then

$$\|\nabla d\|_{L^4(\mathbb{R}^2)} \leq (1 - \varpi_0)\|\Delta d\|_{L^4(\mathbb{R}^2)}.$$*

Consequently, for such a map the associated harmonic energy is coercive, i.e.,

$$\|\Delta d + |\nabla d|^2 d\|_{L^2(\mathbb{R}^2)} \geq \frac{\varpi_0}{2} (\|\Delta d\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla d\|_{L^4(\mathbb{R}^2)}^4).$$

(6.14)

On the other hand, we have used the fact $d \equiv 1$ and the boundedness of domain $\Omega$ to get the uniform estimate $\|d\|_{L^\infty(I, L^2(\Omega))}$ in (4.46). For the case of the whole space, we need the uniform
estimate \( \| \mathbf{d} - \mathbf{e}_2 \|_{L^\infty(I,L^2(\mathbb{R}^2))} \), which can be deduced from \( \mathbf{d} \equiv 1 \) and (6.1). More precisely, we have

\[
\frac{1}{2} \frac{d}{dt} \| \mathbf{d} - \mathbf{e}_2 \|_{L^2(\mathbb{R}^2)}^2 + \theta \| \nabla \mathbf{d} \|_{L^2(\mathbb{R}^2)}^2 = \theta \int_{\mathbb{R}^2} \left[ \| \nabla \mathbf{d} \|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \| \mathbf{d} - \mathbf{e}_2 \|_{\text{div}(\mathbb{R}^2)}^2 \right] dx
\]

\[
\leq \left( 2\theta + \frac{\theta^2}{4} \right) \| \nabla \mathbf{d} \|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \| \mathbf{d} - \mathbf{e}_2 \|_{L^2(\mathbb{R}^2)}^2,
\]

which, together with Gronwall’s inequality, yields

\[
\| \mathbf{d} - \mathbf{e}_2 \|_{L^2(\mathbb{R}^2)}^2 \leq \| \mathbf{d}_0 - \mathbf{e}_2 \|_{L^2(\mathbb{R}^2)}^2 + \left( 4\theta^2 T + \frac{\theta^2}{2} \| \nabla \mathbf{d} \|_{L^2(I,L^2(\mathbb{R}^2))} \right) e^{\theta} \| \nabla \mathbf{d} \|_{L^\infty(I,L^2(\mathbb{R}^2))},
\]

(6.15)

Consequently, plugging (6.14), (6.15) and Proposition 6.1 with \( d_{02} > 0 \) into the deduction of the global estimates in Section 4.4, we immediately establish the following global existence of a unique solution \((\rho_n, \mathbf{v}_n, \mathbf{d}_n)\) to the approximate problem (6.8)–(6.13):

**Proposition 6.3.** Let \( \Omega = B_R \) with \( R > 0 \), \( d_{02} \) be a positive constant, \((\beta, \varepsilon, \rho, \bar{\rho})\) satisfy (4.47) and \( \delta \in (0, 1] \). Assume \( \rho_0 \) satisfies (4.47), \( \mathbf{v}_0 \in X_n \) and

\[
d_{02} \geq d_{02}, \quad |\mathbf{d}_0(x)| = 1, \quad \mathbf{d}_0(x) - \mathbf{e}_2 \in H^2(\mathbb{R}^2).
\]

(6.16)

Then there exists a unique triple \((\rho_n, \mathbf{v}_n, \mathbf{d}_n)\) defined on \( \mathbb{R}^2 \times I \) with the following properties:

1. \((\rho_n, \mathbf{v}_n)\) satisfies the regularity (4.51), and \( \mathbf{d}_n \) satisfies

\[
\mathbf{d}_n - \mathbf{e}_2 \in C^0(\bar{I}, H^2(\mathbb{R}^2)) \cap L^2(\mathbb{I}, H^3(\mathbb{R}^2)), \quad \partial_t \mathbf{d}_n \in C^0(\bar{I}, L^2(\mathbb{R}^2)) \cap L^2(\mathbb{I}, H^1(\mathbb{R}^2)),
\]

2. \((\rho_n, \mathbf{v}_n, \mathbf{d}_n)\) solves (6.8) a.e. in \( Q_T \) and (6.10) a.e. in \( \mathbb{R}^2 \times I \), and satisfies (6.8).

3. \((\rho_n, \mathbf{v}_n, \mathbf{d}_n)\) satisfies the finite and bounded energy inequalities as in Proposition 4.1 with \( \mathbb{R}^2 \) in place of \( \Omega \), where \((\rho_n, \mathbf{v}_n, \nabla \rho_n) = 0 \) in \((\mathbb{R}^2 \setminus \Omega) \times I \). Moreover, we have the uniform estimates (4.56), (4.57), and

\[
|\mathbf{d}_n| \equiv 1 \text{ in } \mathbb{R}^2 \times I,
\]

\[
\| \nabla^2 \mathbf{d}_n \|_{L^2(\mathbb{R}^2 \times I)} + \| \nabla \mathbf{d}_n \|_{L^4(\mathbb{R}^2 \times I)} \leq G(d_{02}, \sup_{\delta \in (0, 1]} \mathcal{E}_0(\rho_0, q_0, d_0)).
\]

As a consequence of Proposition 6.3, similarly to Section 5, we can use the standard three-level approximation scheme and the method of weak convergence based on Proposition 6.3 to establish the existence of weak solutions to the following problem:

\[
\partial_t \rho + \text{div}(\rho \mathbf{v}) = 0 \quad \text{in } B_R \times I,
\]

\[
\partial_t (\rho \mathbf{v}) + \text{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P(\rho) = \mu \Delta \mathbf{v} + (\mu + \lambda) \nabla \text{div} \mathbf{v} - \nu \text{div} \left( \nabla \mathbf{d} \otimes \nabla \mathbf{d} - \frac{1}{2} |\nabla \mathbf{d}|^2 \right) \quad \text{in } B_R \times I,
\]

\[
\partial_t \mathbf{d} + \mathbf{v} \cdot \nabla \mathbf{d} = \theta(\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}) \quad \text{in } \mathbb{R}^2 \times I.
\]

(6.17)

(6.18)

(6.19)

More precisely, we have the following conclusion:
Proposition 6.4. Let the initial data \((\rho_0, m_0)\) satisfy (1.9) and (1.10) with \(B_R\) in place of \(\Omega\), and \(d_0\) satisfy
\[
d_{02} \geq d_{02}, \quad |d_0| = 1 \quad \text{a.e. in} \quad \mathbb{R}^2, \quad d_0(x) - e_2 \in H^1(\mathbb{R}^2). \tag{6.20}
\]
Then the initial-boundary value problem (6.17)–(6.19) has a global weak solution \((\rho, v, d)\) defined on \(\mathbb{R}^2 \times I\) for any given \(T > 0\), such that

1. \((\rho, v)\) satisfies the regularity (1.13) and (1.14) with \(R^2\) in place of \(\Omega\), and \(d\) satisfies the regularity (1.26). Moreover, \((\rho, v) = 0\) in \((\mathbb{R}^2 \setminus B_R) \times I\).

2. Equations (6.17), (6.18) hold in \(D'(B_R \times I)\), the equation (6.19) holds a.e. in \(\mathbb{R}^2 \times I\), the equation (6.17) is satisfied in the sense of renormalized solutions, and the solution satisfies the energy equality (1.20) with \(R^2\) in place of \(\Omega\).

3. Additional estimate:
\[
\int_0^T \int_{\mathbb{R}^2} \frac{\theta \omega_0}{2} (|\Delta d|^2 + |\nabla d|^4) dx ds \leq E_0,
\]
where the constant \(\omega_0\) depends on \(d_{02}, \nu\) and
\[
E_0 = \int_{B_R} \left( \frac{1}{2} \frac{|m_0|^2}{\rho_0} 1_{\{\rho_0 > 0\}} + Q(\rho_0) \right) dx + \int_{\mathbb{R}^2} \frac{\nu |\nabla d_0|^2}{2} dx.
\]

Remark 6.1. To replace (6.16) by (6.20), we have used the fact that there exists a sequence of approximate functions \(\{d^0_m\}_{m=1}^\infty \subset H^1(\mathbb{R}^2) + e_2\), such that
\[
|d^0_m| = 1, \quad d^0_m - d_0 \to 0 \quad \text{in} \quad H^1(\mathbb{R}^2), \quad \text{and} \quad d^0_{m2} \geq d_{02}/2.
\]

Finally, we can follow the arguments in [34, Section 7.11] for the compressible Navier-Stokes equations to prove Theorem 1.2 by using Proposition 6.4 on the bounded invading domain \(B_R\) and letting \(R \to \infty\), where one should use the embedding theorem
\[
H^1(\Omega') \hookrightarrow L^p(\Omega') \quad \text{for any} \quad p \geq 1 \quad \text{and any bounded Lipschiz domain} \quad \Omega' \subset \mathbb{R}^2,
\]
to replace \(L^6(\mathbb{R}^3) \hookrightarrow D^{1,2}(\mathbb{R}^3)\) in [34, Section 7.11] in the treatment of the velocity \(v\). Here we omit the proof, since the additional limit process on \(d\) is trivial. We mention that, in the limit process on \(d\), we also have the weak convergence as in (5.2), and the strong convergence as in (5.1) and (5.3) with any bounded space \(\Omega' \subset \mathbb{R}^2\).

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