When is a Locally Convex Space Eberlein–Grothendieck?

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Abstract. The weak topology of a locally convex space (lcs) $E$ is denoted by $w$. In this paper we undertake a systematic study of those lcs $E$ such that $(E, w)$ is (linearly) Eberlein–Grothendieck (see Definitions 1.2 and 3.1). The following results obtained in our paper play a key role: for every barrelled lcs $E$, the space $(E, w)$ is Eberlein–Grothendieck (linearly Eberlein–Grothendieck) if and only if $E$ is metrizable ($E$ is normable, respectively). The main applications concern to the space of continuous real-valued functions on a Tychonoff space $X$ endowed with the compact-open topology $C_k(X)$. We prove that $(C_k(X), w)$ is Eberlein–Grothendieck (linearly Eberlein–Grothendieck) if and only if $X$ is hemicompact ($X$ is compact, respectively). Besides this, we show that the class of $E$ for which $(E, w)$ is linearly Eberlein–Grothendieck preserves linear continuous quotients. Various illustrating examples are provided.

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1. Introduction

All topological spaces in the paper are assumed to be Tychonoff and all vector spaces are over the field of real numbers $\mathbb{R}$. By $C_k(X)$ and $C_p(X)$ we mean the space $C(X)$ of real-valued continuous functions defined on a Tychonoff space $X$ equipped with the compact-open and pointwise convergence topology, respectively. For a locally convex space (lcs) $E$ we denote by $w$ the weak topology.
topology \( w = \sigma(E, E^*) \) of \( E \); similarly, the weak*-topology \( \sigma(E^*, E) \) of the dual \( E^* \) is denoted by \( w^* \).

The classic Eberlein–Šmulian theorem states the equivalence of compactness, countable compactness and sequential compactness for subsets \( A \subseteq (E, w) \), where \( E \) is any Banach space. Remarkably, one can derive a proof of the Eberlein–Šmulian theorem from the analogous results about these versions of compactness in the function spaces \( C_p(X) \).

Recall that a subset \( A \) is countably compact in \( Y \) if every countable sequence in \( A \) has an accumulation point in \( Y \). The following significant statement which is intentionally formulated below in a simplified form is due to A. Grothendieck.

**Theorem 1.1** [1, p. 107]. Let \( X \) be a compact space. The closure of any countably compact set \( A \subseteq C_p(X) \) is compact.

For further numerous developments of the Eberlein–Grothendieck theorem, we refer to [1] and references therein. Motivated by these facts A. V. Arkhangel’skii introduced the following definition.

**Definition 1.2** [1, p. 95]. A topological space \( Y \) is called an Eberlein–Grothendieck space, if there is a homeomorphic embedding of \( Y \) into the space \( C_p(K) \) for some compact space \( K \).

Remark here only that every metrizable space is Eberlein–Grothendieck [1, Theorem IV.1.25]. Also, every Eberlein–Grothendieck topological space \( Y \) has countable tightness, i.e. whenever a point \( x \) is in the closure of a set \( A \subseteq Y \), then there is a countable \( B \subseteq A \) such that \( x \) is in the closure of \( B \). Some questions about Eberlein–Grothendieck topological spaces were considered in [2].

In this paper we undertake a systematic study of those lcs \( E \) such that \((E, w)\) is Eberlein–Grothendieck. To the best of our knowledge this class of lcs has not been investigated before in such generality. Since every finite-dimensional lcs trivially is isomorphic to a space \( C_p(K) \) with a finite \( K \), we consider only infinite-dimensional lcs.

First, in Sect. 2 we show that the class of lcs \( E \) for which \((E, w)\) is Eberlein–Grothendieck is closed under the operations of taking (1) linear subspaces; (2) linear continuous quotients; (3) countable products, but it does not preserve countable inductive limits. Further, we observe that if \( E \) is an Eberlein–Grothendieck lcs, then also \((E, w)\) is Eberlein–Grothendieck. Example 4.1 shows that the converse direction in general does not hold. Thus, for every metrizable lcs \( E \), the space \((E, w)\) is Eberlein–Grothendieck (Corollary 2.3).

Note that for \( C_p(X) \) the weak and the original topologies coincide. A topological space represented as a countable union of compact subsets is called \( \sigma \)-compact. The following two fundamental results on \( C_p(X) \) are due to O. Okunev [12].
Theorem 1.3 [12]. If an lcs $H$ is a continuous open image of a subspace of $C_p(K)$ for some compact space $K$, then the dual lcs $(H^*, w^*)$ is $\sigma$-compact.

Corollary 1.4 [12]. An lcs $C_p(X)$ is Eberlein–Grothendieck if and only if $X$ is $\sigma$-compact.

Since $C_p(X)$ is metrizable if and only if $X$ is countable, there are plenty of nonmetrizable lcs $E$ such that $(E, w)$ is Eberlein–Grothendieck. Despite this, we demonstrate that $(E, w)$ is Eberlein–Grothendieck if and only if $E$ is metrizable for several classes lcs $E$ important for applications.

The following result obtained in our paper plays a key role: for every barrelled lcs $E$, if the space $(E, w)$ is Eberlein–Grothendieck then $E$ must be metrizable (Theorem 2.7). The main application of Theorem 2.7 concerns to the space $C_k(X)$. Theorem 2.11 provides an exhaustive characterization: $(C_k(X), w)$ is Eberlein–Grothendieck if and only if $X$ is hemicompact. Also, for any LF-space $E$, $(E, w)$ is Eberlein–Grothendieck if and only if $E$ is metrizable.

It is known that for a normed space $E$ there exists even a canonical linear embedding $T : (E, w) \rightarrow C_p(K)$, where $K$ is the closed unit ball in $(E^*, w^*)$. Being motivated by this simple fact we shall say that for an lcs $E$ the space $(E, w)$ is a linearly Eberlein–Grothendieck space if $(E, w)$ can be linearly embedded into $C_p(K)$ for some compact space $K$.

In Sect. 3 we observe that for every $E$, an lcs $(E, w)$ is linearly Eberlein–Grothendieck if and only if the dual lcs $(E^*, w^*)$ is compactly generated, i.e. $(E^*, w^*)$ admits a compact subset $K$ whose linear span covers $E^*$ (Theorem 3.2). Relying on a hereditary property of compactly generated lcs we prove that the class of lcs $E$ for which $(E, w)$ is linearly Eberlein–Grothendieck is closed under the operation of taking linear continuous quotients (Theorem 3.5).

We develop further the technique of Theorem 2.7 and show that for every barrelled lcs $E$, if the space $(E, w)$ is linearly Eberlein–Grothendieck then $E$ must be normable (Theorem 3.8). As an application, completing our study of the spaces $C_k(X)$ we prove that $(C_k(X), w)$ is linearly Eberlein–Grothendieck if and only if $X$ is compact (Theorem 3.11).

In the last Sect. 4 we provide several illustrating examples. For instance, both the space of test functions $\mathcal{D}(\Omega)$ and the space of distributions $\mathcal{D}'(\Omega)$ endowed with the weak topologies are not Eberlein–Grothendieck (Example 4.5).

Definitions of several classes of locally convex spaces in use are reminded explicitly in the text. All standard notions can be found in the books [7], [13].

2. Eberlein–Grothendieck lcs

The following characterization will be applied repeatedly in the sequel.

Proposition 2.1 For every $E$, an lcs $(E, w)$ is Eberlein–Grothendieck if and only if the dual lcs $(E^*, w^*)$ is $\sigma$-compact.
Proof \( H = (E, w) \) is Eberlein–Grothendieck \( \implies (E^*, w^*) \) is \( \sigma \)-compact is an immediate consequence of Theorem 1.3. The converse implication follows from the well-known fact that \( (E, w) \) embeds into \( C_p(L) \) by a linear homeomorphism, where \( L \) is the space \( (E^*, w^*) \). Indeed, the required embedding is defined by the formula

\[
\xi : x \mapsto \xi_x, \quad \xi_x(x^*) = x^*(x), \quad x \in E, x^* \in E^*.
\]

Since \( L \) is \( \sigma \)-compact we can apply Corollary 1.4.

As a straightforward application of Theorem 1.3 and Proposition 2.1 we have

**Corollary 2.2** If \( E \) is an Eberlein–Grothendieck lcs, then also \((E, w)\) is Eberlein–Grothendieck.

Later, in Example 4.1 we show that the converse direction in Corollary 2.2 in general does not hold. Since every metrizable topological space is Eberlein–Grothendieck we immediately obtain

**Corollary 2.3** For every metrizable lcs \( E \), the space \((E, w)\) is Eberlein–Grothendieck.

Another more informative argument implying Corollary 2.3 will be presented later in the proof of Proposition 3.12. The class of lcs \( E \) with Eberlein–Grothendieck \((E, w)\) is invariant under certain basic topological operations.

**Proposition 2.4**

(a) Let \((E, w)\) be an Eberlein–Grothendieck lcs. If there is a linear continuous quotient mapping \( \pi \) from \( E \) onto an lcs \( F \), then \((F, w)\) is also Eberlein–Grothendieck.

(b) Let \((E, w)\) be an Eberlein–Grothendieck lcs. Then \((F, w)\) is Eberlein–Grothendieck for every linear subspace \( F \subset E \).

(c) Let \((E_n, w)\) be an Eberlein–Grothendieck lcs, where \( n \in \omega \). Then the countable product \( E = \prod_{n \in \omega} E_n \) also has the property that \((E, w)\) is Eberlein–Grothendieck.

**Proof**

(a) We define the dual mapping \( \pi^* : (F^*, w^*) \to (E^*, w^*) \) by the formula:

\[
\pi^*(\phi) = \phi \circ \pi \text{ for every } \phi \in F^*.
\]

Since \( \pi \) is a continuous and open surjection, the dual mapping \( \pi^* \) establishes a linear homeomorphism between \((F^*, w^*)\) and a closed subspace of \((E^*, w^*)\) [11, Theorems 8.12.1, 8.12.3]. The latter space is \( \sigma \)-compact, by Proposition 2.1, hence \((F^*, w^*)\) is also \( \sigma \)-compact. Finally, \((F, w)\) is Eberlein–Grothendieck, again by Proposition 2.1.

(b) It suffices to note that \((F, w)\) is a subspace of \((E, w)\) provided \( F \subset E \) [11, Theorem 8.12.2].
(c) If \( E = \prod_{n \in \omega} E_n \) then \((E, w) = \prod_{n \in \omega} (E_n, w) \) (see [7, Proposition 8.8.7]). Since \((E_n, w)\) homeomorphically embeds into \(C_p(K_n)\) for some compact space \(K_n\) for each \(n \in \omega\), we have that \((E, w)\) homeomorphically embeds into \(C_p(X)\), where \(X = \bigoplus_{n \in \omega} K_n\). It suffices to recall that the latter space \(C_p(X)\) is Eberlein–Grothendieck because \(X\) is \(\sigma\)-compact. The proof is complete. \(\Box\)

Remark 2.5 Inductive limit of the countable sequence of lcs \( \{E_n : n \in \omega\} \), where each \((E_n, w)\) is Eberlein–Grothendieck, does not have to satisfy the same property. Denote by \(\varphi\) the \(\aleph_0\)-dimensional vector space endowed with the finest locally convex topology. The space \(\varphi\) can be identified with the strict inductive limit of the sequence of Euclidean spaces \(\mathbb{R}^n\) [16]. If \(E = \varphi\) then the dual lcs \((E^*, w^*)\) is isomorphic to the countable product \(\mathbb{R}^\omega\) which clearly is not \(\sigma\)-compact. Therefore, \(\varphi\) endowed with the weak topology is not Eberlein–Grothendieck, by Proposition 2.1. Alternatively, according to Corollary 2.10, \(\varphi\) endowed with the weak topology is not Eberlein–Grothendieck because \(\varphi\) is a strict LB-space.

For the reader’s convenience we recall definitions of several classes of locally convex spaces which will be used. A Fréchet space is a completely metrizable locally convex space. An lcs that is a locally convex inductive limit of a countable inductive system of Fréchet spaces (Banach spaces) is called an \((LF)\)-space \((\text{or } (LB)\)-space). Recall that a subset \(A\) in a locally convex space \(E\) is bounded, it every neighborhood of zero \(U\) in \(E\) absorbs \(A\), i.e. there exists a scalar \(\gamma > 0\) such that \(A \subset \gamma U\). Clearly, the closure and the absolute convex hull of a bounded set are again bounded. The strong topology \(\beta(E^*, E)\) on the dual \(E^*\) is the topology of uniform convergence on bounded subsets of \(E\).

An absorbing balanced convex closed set in an lcs \(E\) is a barrel. Every lcs \(E\) has a base of neighborhoods of 0 consisting of barrels, and \(E\) is called barrelled if each barrel is a neighborhood of 0.

A subset \(B \subset E\) is called bornivorous if it absorbs all bounded subsets of \(E\). A quasibarrelled space is an lcs for which every bornivorous barrel is a neighborhood of 0. An lcs \(E\) is quasibarrelled if and only if every \(\beta(E^*, E)\)-bounded set \(B\) in \(E^*\) is equicontinuous, see [7, 11.2]. The following well-known fact, which will be used below, can be found for example in [16, Note (p. 1066)].

**Proposition 2.6** A locally convex space \(E\) is barrelled if and only if \(E\) is quasibarrelled and has the Banach-Mackey property, i.e. every \(w^*\)-bounded set in \(E^*\) is \(\beta(E^*, E)\)-bounded.

The next result plays a key role in our paper.

**Theorem 2.7** Let \(E\) be a barrelled locally convex space. If \((E, w)\) is an Eberlein–Grothendieck space, then \(E\) is metrizable.
Proof By Proposition 2.1 the weak* dual \((E^*, w^*)\) of \(E\) is \(\sigma\)-compact. Let \(\{K_n : n \in \omega\}\) be an increasing sequence of \(w^*\)-compact sets covering \(E^*\). Denote by \(B_n\) the \(w^*\)-closure of the absolutely convex hull of \(K_n\). Then \(B_n\) is \(w^*\)-compact and absolutely convex, see [13, Observation 4.1.5]. Evidently, each set \(B_n\) is \(w^*\)-bounded and closed. Applying the above Proposition 2.6 we derive that every set \(B_n\) is bounded in the strong topology \(\beta(E^*, E)\).

Set \(A_n = 2^n B_n\) for all \(n \in \omega\). It suffices to prove that \(\{A_n : n \in \omega\}\) is a fundamental sequence of bounded sets in \((E^*, \beta(E^*, E))\), i.e. every \(\beta(E^*, E)\)-bounded set \(A \subset E^*\) is contained in some \(A_m\).

Indeed, assume that we have already proved this claim. Then the strong bidual \((E^{**, \beta(E^{**, E}))}\) (being the strong dual of \((E^*, \beta(E^*, E))\)) would be metrizable. This follows from the definition of the topology \(\beta(E^{**, E})\) as the topology of the uniform convergence on \(\beta(E^*, E)\)-bounded sets and an assumption that the latter topology admits a fundamental sequence of bounded sets. Consider \(E\) canonically embedded as a linear subspace of the bidual \(E^{**}\). Since \(E\) is barreled, the topology \(\beta(E^{**, E})\) restricted to \(E\) coincides with the original topology of \(E\) by applying [7, Proposition 11.2.]. We conclude that \((E^*, \beta(E^*, E))\) is metrizable implies that \(E\) is metrizable as well.

Therefore, we need only to show that \(\{A_n : n \in \omega\}\) forms a fundamental sequence of bounded sets in \((E^*, \beta(E^*, E))\). Let \(A\) be any absolutely convex \(\beta(E^*, E)\)-bounded set. Since \(E\) is barreled, the set \(A\) is equicontinuous. Hence, there exists a neighborhood of zero \(U \subset E\) such that \(A \subset U^o\), where \(U^o = \{x^* \in E^* : |x^*(x)| \leq 1\text{ for each } x \in U\}\) is the polar of \(U\) in \(E^*\). By the Alaoglu-Bourbaki theorem [7, Theorem 8.5.2] the absolutely convex set \(U^o\) is \(w^*\)-compact, so the closure \(\overline{B}\) of \(A\) in \((E^*, w^*)\) is compact and absolutely convex. By [13, Corollary 3.2.5] the linear span \(E_{\overline{B}}^*\) endowed with the Minkowski norm topology \(\tau_B\) is a Banach space and also \(w^*|_{E_{\overline{B}}^*} \leq \tau_B\), see [13, Proposition 3.2.2].

The sequence of \(\tau_B\)-closed and absolutely convex sets \(\{A_n \cap E_{\overline{B}}^* : n \in \omega\}\) covers the Banach space \((E_{\overline{B}}^*, \tau_B)\). By the Baire Category Theorem there exist \(k \in \omega, x \in A_k \cap E_{\overline{B}}^*\) and \(\epsilon > 0\) such that \(x + \epsilon B \subset A_k \cap E_{\overline{B}}^*\). Then we have that

\[
2\epsilon B \subset (x + \epsilon B) - (x + \epsilon B) \subset 2A_k.
\]

Hence, \(A \subset B \subset A_m\) for some \(m \in \omega\). This conclusion completes the proof. \(\square\)

Remark 2.8 Note that the proof of Theorem 2.7 uses only the properties that \(E\) is quasibarrelled and each set \(B_n\) is \(\beta(E^*, E)\)-bounded. On the other hand, if \(E\) is metrizable (hence quasibarrelled) with a countable base \(\{U_n : n \in \omega\}\) of absolutely convex neighborhoods of zero in \(E\), then the polars \(U_n^o\) in \(E^*\) form a cover of \((E^*, w^*)\) consisting of absolutely convex \(w^*\)-compact sets (which are also \(\beta(E^*, E)\)-bounded). Now, if \(E^*\) is covered by another sequence of absolutely convex \(w^*\)-compact sets \(\{B_n : n \in \omega\}\), then each set \(B_n\) is also
\( \beta(E^*, E) \)-bounded. This follows from the proof of Theorem 2.7, applying again the Baire Category Theorem. Indeed, one shows that each set \( B_n \) is contained in some \( 2^p U_p \), which implies that \( B_n \) is \( \beta(E^*, E) \)-bounded. However, it is not clear if the assumption on \( B_n \) to be absolutely convex can be removed.

Metrizable barrelled spaces have attracted much attention of the researchers. Several concrete examples of metrizable (not complete) barrelled spaces can be found in [15] and the books [5], [13]. Since every \( (LF) \)-space is barrelled (see [13, Proposition 4.2.6] or [7, Proposition 11.3.1 (c)]) and no strict \( (LB) \)-space is metrizable [13, Proposition 8.5.18] we infer immediately

**Corollary 2.9** Let \( E \) be an \( (LF) \)-space. Then \((E, w)\) is Eberlein–Grothendieck if and only if \( E \) is metrizable.

**Corollary 2.10** Let \( E \) be a strict \( (LB) \)-space. Then \((E, w)\) is not Eberlein–Grothendieck.

Now we present the main application of Theorem 2.7. A space \( X \) is said to be hemicompact if there is a sequence \( \{K_n : n \in \omega\} \) of compact subsets of \( X \) with the following property: if \( K \subset X \) is compact then \( K \subset K_n \) for some \( n \in \omega \).

**Theorem 2.11** For any Tychonoff space \( X \) the following are equivalent

(a) \( (C_k(X), w) \) is Eberlein–Grothendieck.
(b) \( C_k(X) \) is metrizable.
(c) \( X \) is hemicompact.

**Proof** The equivalence (b) \( \iff \) (c) is very well known (see, for instance [11, Theorem 5.8.5]). (b) \( \implies \) (a) is true for every metrizable lcs. In order to prove the converse implication we need to do some preparatory work.

For every \( x \in X \) consider the evaluation mapping \( \delta_x \), which is defined by the formula

\[
\delta_x(f) = f(x) \text{ for every } f \in C(X).
\]

It is well known that \( \delta_x \in C_p(X)^* \subset C_k(X)^* \) for every \( x \in X \).

We will use the description of the elements from the dual space \( C_k(X)^* \). Below \( \nu X \) denotes the Hewitt realcompactification of a Tychonoff space \( X \). We identify the linear spaces \( C(X) \) and \( C(\nu X) \). The essential of the following statement is well known and can be found in [17]. Some relevant information can be found also in [20]. The measures considered in the paper are real-valued. The support \( \text{supp}(\mu) \) of a measure \( \mu \) on \( X \) is defined as the set of all \( x \in X \) such that every neighborhood \( U \) of \( x \) satisfies \( |\mu|(U) > 0 \).

**Theorem 2.12** [17, Theorem III.3.3] Let \( \phi \) be a continuous linear functional defined on \( C_k(X) \). Then there exists a unique Radon measure \( \mu \) on \( \nu X \) with the compact support \( \text{supp}(\mu) \subset X \) which fulfills the following properties

1. \( \phi(f) = \int f d\mu \) for every \( f \in C(X) \).
(2) If \( f \in C(X) \) is such that \( f = 0 \) on \( \text{supp}(\mu) \), then \( \phi(f) = 0 \).

(3) For any open subset \( U \) of \( \nu X \) which satisfies the condition \( U \cap \text{supp}(\mu) \neq \emptyset \), there exists some \( f \in C(X) \) such that \( f = 0 \) on \( \nu X \setminus U \) and \( \phi(f) = 1 \).

**Proposition 2.13** The subspace \( \{ \delta_x : x \in X \} \subset (C_k(X)^*, w^*) \) is homeomorphic to \( X \) and is closed in \( (C_k(X)^*, w^*) \).

**Proof** The first claim follows from the fact that the space \( \{ \delta_x : x \in X \} \) considered as a subspace of the double function space \( C_p(C_p(X)) \) is homeomorphic to \( X \) (see [1, Corollary 0.0.5]).

The second claim is based on the description of the elements of \( C_k(X)^* \) outlined in Theorem 2.12 above. Let \( \phi \in C_k(X)^* \) be defined by a measure \( \mu \) on \( \nu X \) with \( \text{supp}(\mu) \subset X \). Assume that \( \text{supp}(\mu) \) contains at least two different points \( x_1 \) and \( x_2 \). Then according to Theorem 2.12 there are two continuous functions \( f_1, f_2 \in C(X) \) such that \( \text{supp}(f_1) \cap \text{supp}(f_2) = \emptyset \) and \( \phi(f_1) = \phi(f_2) = 1 \).

Define an open neighborhood of \( \phi \) in \( (C_k(X)^*, w^*) \) as follows:

\[
V = \{ \zeta \in C_k(X)^* : \zeta(f_1) > 0, \zeta(f_2) > 0 \}.
\]

Then no functional \( \delta_x \) belongs to \( V \) since no \( x \in X \) belongs both to \( \text{supp}(f_1) \) and \( \text{supp}(f_2) \). Hence \( \phi \) is not in the closure of \( \{ \delta_x : x \in X \} \). \( \square \)

Now we finish the proof of (a) \( \implies \) (b). First, by Proposition 2.1, \( (C_k(X)^*, w^*) \) is \( \sigma \)-compact. Second, by Proposition 2.13, \( X \) itself is \( \sigma \)-compact. Third, every \( \sigma \)-compact space is Lindelöf, in particular, \( X \) is a \( \mu \)-space, i.e. every closed bounded subset of \( X \) is compact. Hence, \( C_k(X) \) is barrelled by the classic Nachbin-Shirota theorem (see [13, Theorem 10.1.20] or [7, Theorem 11.7.5]). It remains to apply Theorem 2.7 and complete the proof. \( \square \)

### 3. Linearly Eberlein–Grothendieck lcs

**Definition 3.1** A locally convex space \( H \) is called *linearly Eberlein–Grothendieck* if \( H \) is linearly isomorphic to a subspace of the space \( C_p(K) \) for some compact space \( K \).

It is easily seen that if an lcs \( H \) satisfies Definition 3.1 above, then the topology of \( H \) coincides with its weak topology, because this is true for every \( C_p(K) \). Recall also that \( (E, w) \) is linearly Eberlein–Grothendieck for every normed space \( E \). Hence any infinite-dimensional normed space \( E \) provides an example of an lcs which is not linearly Eberlein–Grothendieck while \( (E, w) \) is linearly Eberlein–Grothendieck.

In this section we study lcs \( E \) such that \( (E, w) \) are linearly Eberlein–Grothendieck. Our first goal here is to find a counterpart of Proposition 2.1 for this new stronger notion.

The following classic concept of topological algebra appears to be relevant to the subject. A topological group \( G \) which is algebraically generated by one
of its compact subsets is called \textit{compactly generated}. By analogy, a topological vector space \( L \) is called compactly generated if \( L \) has a compact basis \( K \), meaning that the linear span of \( K \) is equal to \( L \). Evidently, a Banach space is compactly generated if and only if it is finite-dimensional. Nevertheless, the free locally convex space \( L(X) \), as well as \( L_p(X) = (L(X), w) \), over a compact space \( X \) are compactly generated. We will show shortly that compactly generated lcs play an important role in our study.

\textbf{Theorem 3.2} For every \( E \), an lcs \( (E, w) \) is linearly Eberlein–Grothendieck if and only if the dual lcs \( (E^*, w^*) \) is compactly generated.

\textit{Proof} Assume first that there is an isomorphic embedding \( \pi: (E, w) \rightarrow C_p(K) \), where \( K \) is some compact space. The dual of \( C_p(K) \) is the space \( L_p(K) \). Then the adjoint mapping \( \pi^*: L_p(K) \rightarrow (E^*, w^*) \) is onto. Therefore, the compact space \( \pi^*(K) \) linearly generates \( (E^*, w^*) \).

Opposite direction. Assume that a compact subset \( B \) linearly generates \( (E^*, w^*) \). In the proof of Proposition 2.1 we used a mapping of evaluation which always linearly embeds \( (E, w) \) into \( C_p(L) \), where \( L \) is the whole space \( (E^*, w^*) \). Since \( B \) linearly generates \( L \), here it is enough to take \( C_p(B) \) as a target space. Formally, the linear embedding \( T: (E, w) \rightarrow C_p(B) \) is defined as follows:

\[
(Tx)(x^*) = x^*(x), \quad x \in E, x^* \in B.
\]

Below we explain why this embedding is a 1-to-1 mapping. Take two different \( x \) and \( y \) in \( E \). If we assume that \( b(x) = b(y) \) for every \( b \in B \), then \( x^*(x) = x^*(y) \) for every \( x^* \in L \), which is of course false. So, the linear isomorphic embedding \( T \) witnesses that \( (E, w) \) is a linearly Eberlein–Grothendieck space. \( \square \)

A closed subgroup of an arbitrary compactly generated abelian group in general does not have to be compactly generated. Also, the discrete free group \( F_2 \) generated by two elements has a subgroup that is not finitely generated. However, closed subgroups of a compactly generated locally compact abelian group are compactly generated (for the details see [14]).

Does every closed linear subspace of a compactly generated lcs remain compactly generated? Surprisingly, we were unable to find this question formulated and answered explicitly in any published source dealing with topological vector spaces. By this reason we include its simple affirmative solution.

\textbf{Proposition 3.3} Every closed linear subspace of a compactly generated lcs is also compactly generated.

\textit{Proof} Assume that an lcs \( L \) has a compact basis \( K \) and \( H \) is a closed linear subspace of \( L \). Define

\[
K_n = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in K, \lambda_i \in [-1, 1] \right\} \subset L.
\]
Then each $K_n$ is a compact set because it is a continuous image of the product $[-1,1]^n \times K^n$. Denote by $H_n = K_n \cap H$. Each $H_n$ is a compact subset of $H$ since $H$ is closed in $L$, and $\bigcup_{n \in \mathbb{N}} H_n = H$ since $\bigcup_{n \in \mathbb{N}} K_n = L$. Finally, we define

$$B = \bigcup_{n \in \mathbb{N}} \frac{1}{n^2} H_n \subset H.$$ 

Clearly, $0 \in B$ and $\text{span}(B) = H$ because $H_n \subset \text{span}(B)$ for every $n \in \mathbb{N}$. It remains to show that $B$ is compact. Let $U$ be any absolutely convex neighborhood of zero in $L$. Every compact set is bounded, therefore, there is a natural number $n_0$ such that $K_n \subset nU$ for every $n > n_0$, or, equivalently, $\frac{1}{n} x \in U$ for every $x \in K, n > n_0$. Hence $\frac{1}{n^2} H_n \subset U$ for every $n > n_0$. Indeed, every element of the set $\frac{1}{n^2} K_n$ is of the form $\sum_{i=1}^{n} \frac{\lambda_i}{n} y_i$, where each $\lambda_i \in [-1,1], y_i \in U$ and $U$ is an absolutely convex set. We get that for every open cover of $B$ the neighborhood of zero covers all points of $B$ except perhaps for finitely many compact sets $\frac{1}{n^2} H_n$. Finite union of compact sets is covered by finitely many open sets and the proof is complete. □

Remark 3.4 In general, Proposition 3.3 is not true for non-locally convex spaces. Consider the free topological vector space $V(X)$, where $X$ is the closed unit interval $[0,1]$. By construction, $V(X)$ is generated by the compact set $X = [0,1]$. It is known that the free topological vector space $V(Y)$ is isomorphic as a topological vector space to a closed vector subspace of $V([0,1])$, where $Y$ is the free union of countably many copies of the segment $[0,1]$ (see [9, Corollary 2.6]). The space $V(Y)$ is not compactly generated because every compact subset $K$ of $V(Y)$ is contained in a proper subspace $V(C) \subset V(Y)$ generated by a compact subset $C \subset Y$. Similarly, the free abelian group $A([0,1])$ contains an isomorphic closed copy of $A(Y)$ which is not compactly generated (see [10]). This phenomenon cannot occur with the free locally convex space $L([0,1])$, by the results of [10]. Note that the latter fact now is an immediate consequence of our Proposition 3.3.

It would be interesting to know whether the property established in Proposition 3.3 characterizes compactly generated locally convex spaces among all compactly generated topological vector spaces.

Theorem 3.5 Let $(E,w)$ be a linearly Eberlein–Grothendieck lcs. If there is a linear continuous quotient mapping $\pi$ from $E$ onto an lcs $F$, then $(F,w)$ is also linearly Eberlein–Grothendieck.

Proof As in the proof of Proposition 2.4, we define the dual mapping $\pi^* : (F^*,w^*) \to (E^*,w^*)$ by the formula:

$$\pi^*(\phi) = \phi \circ \pi \text{ for every } \phi \in F^*.$$ 

Since $\pi$ is a continuous and open surjection, the dual mapping $\pi^*$ establishes a linear homeomorphism between $(F^*,w^*)$ and a closed linear subspace of $(E^*,w^*)$. The space $(E^*,w^*)$ is compactly generated, by Theorem 3.2, hence
$(F^*, w^*)$ is also compactly generated, by Proposition 3.3. Finally, $(F, w)$ is linearly Eberlein–Grothendieck, again in view of Proposition 3.3. □

The proof of the following statement is very similar to the proof of Proposition 2.4 and is left to the reader.

**Proposition 3.6**

(a) Let $(E, w)$ be a linearly Eberlein–Grothendieck lcs. Then $(F, w)$ is linearly Eberlein–Grothendieck for every linear subspace $F \subset E$.

(b) Let $(E, w)$ be a Eberlein–Grothendieck lcs, where $k = 1, 2, \ldots, n$. Then the finite product $E = \prod_{k=1}^{n} E_k$ also has the property that $(E, w)$ is linearly Eberlein–Grothendieck.

**Proposition 3.7** Let $E_k$ be lcs for all $k \in \omega$. Then for the countable product $E = \prod_{k \in \omega} E_k$ the space $(E, w)$ is not linearly Eberlein–Grothendieck.

**Proof** Countable product $E$ contains an isomorphic copy of $\mathbb{R}^\omega$. The weak topology on $\mathbb{R}^\omega$ coincides with the original product topology. Hence, by Proposition 3.6 (a), it suffices to observe that $\mathbb{R}^\omega$ is not linearly Eberlein–Grothendieck.

A direct argument is the following: $C_p(K)$ is covered by a sequence of closed bounded sets, but $\mathbb{R}^\omega$ cannot be covered by a sequence of closed bounded sets, in view of the Baire Category Theorem. An alternative argument: $\mathbb{R}^\omega$ is not linearly Eberlein–Grothendieck by Corollary 3.9 below, because $\mathbb{R}^\omega$ is a particular example of a Fréchet space. □

**Theorem 3.8** Let $E$ be a barrelled lcs such that $(E, w)$ is linearly Eberlein–Grothendieck. Then $E$ is normable.

**Proof** By our Theorem 3.2 there exists a $w^*$-compact set $K$ in $E^*$ such that the linear span of $K$ coincides with $E^*$. Similarly to the proof of Theorem 2.7, let $A$ be the $w^*$-closure of the absolutely convex hull of $K$. Then $A \subset E^*$ is a $w^*$-compact and absolutely convex set. It follows that $E^* = \bigcup \{nA : n \in \omega \}$. Also, since $E$ is barrelled, the set $A$ is equicontinuous. Let $U$ be the polar of $A$ in $E$, i.e. $U = \{x \in E : |x^*(x)| \leq 1 \text{ for each } x^* \in A\}$. By the barrelledness of $E$ the absolutely convex closed absorbing set $U$ is a neighborhood of zero in $E$. We claim that $U$ is also a bounded set in $E$. Indeed, assume on the contrary that $U$ is not bounded. Then there exists a continuous linear functional $x^* \in E^*$ such that $\sup \{|x^*(x)| : x \in U\} = \infty$. However, there exists $n \in \omega$ with $x^* \in nA$. This means that $\frac{1}{n} x^* \in A$, therefore $|x^*(x)| \leq n$ for each $x \in U$. The obtaining contradiction says that $U$ is a bounded neighborhood of zero in $E$, hence, $E$ is normable by the Kolmogorov’s theorem, see [7, Proposition 6.9.4]. □

Below, as an easy application of Theorem 3.8, we characterize Fréchet and $(LF)$-spaces $E$ such that $(E, w)$ are linearly Eberlein–Grothendieck spaces.

**Corollary 3.9** Let $E$ be a Fréchet space. Then $(E, w)$ is linearly Eberlein–Grothendieck if and only if the metric of $E$ can be generated by a complete norm, i.e. $E$ is isomorphic to a Banach space.
Corollary 3.10 Let $E$ be an (LF)-space. Then $(E, w)$ is linearly Eberlein–Grothendieck if and only if $E$ is normable. Remark that a normed (LF)-space does not have to be complete (see [13, Section 8.7]).

We apply Theorem 3.8 to the space $C_k(X)$. A Tychonoff space $X$ is called pseudocompact if every continuous function $f : X \to \mathbb{R}$ is bounded.

Theorem 3.11 For any Tychonoff space $X$ the following conditions are equivalent:

(i) $X$ is compact;

(ii) $C_p(X)$ is linearly Eberlein–Grothendieck;

(iii) $(C_k(X), w)$ is linearly Eberlein–Grothendieck.

Proof We show only non-trivial implications (ii) $\implies$ (i); (iii) $\implies$ (i). Assuming (ii) we know that $X$ must be $\sigma$-compact, by Corollary 1.4. On the other hand, $X$ must be pseudocompact because otherwise $C_p(X)$ would contain an isomorphic copy of $\mathbb{R}^\omega$ [19], which is not linearly Eberlein–Grothendieck. Every $\sigma$-compact and pseudocompact space is compact and the proof of (i) is finished.

Similarly, assuming (iii) we know that $X$ must be $\sigma$-compact, by Theorem 2.11. Hence, $C_k(X)$ is barreled, then, by Theorem 3.8, $C_k(X)$ is normable, which implies that $X$ is a compact space. $\square$

Proposition 3.12 If $E$ is a metrizable lcs, then $(E, w)$ embeds linearly into $C_p(K, \mathbb{R}^\omega) \cong C_p(X)$, where $K$ is a compact space and $X$ is the free sum of countably many copies of $K$.

Proof Let $\{(U_n) : n \in \omega\}$ be a decreasing base of absolutely convex and closed neighborhoods of zero in $E$. For each $n \in \omega$ define the polar set

$$U_n^o = \{x^* \in E^* : |x^*(x)| \leq 1 \text{ for each } x \in U_n\}.$$

Then each $U_n^o$ is a $w^*$-compact set in $E^*$ by the classic Alaoglu-Bourbaki theorem, and $E^* = \bigcup_{n \in \omega} U_n^o$. Hence $K = \prod_{n \in \omega} U_n^o$ is a compact set in the space $\prod_{n \in \omega} (E_n^*)^w$, where $E_n^* = E^*$ for each $n \in \omega$. Define a linear mapping

$$\xi : x \mapsto \xi_x \in C_p(K, \mathbb{R}^\omega), x \in E,$

by the formula

$$\xi_x(x^*) = (x_n^*(x)), \text{ where } x^* = (x_n^*) \in \prod_{n \in \omega} U_n^o.$$
In order to show that the inverse to $\xi$ is continuous, assume that $\xi x_\gamma \rightarrow 0$. We need to show that $x_\gamma \rightarrow 0$ in $(E, w)$. Take any $x^* \in E^*$. As above, there exists $n \in \omega$ such that $y^* = (0, ..., 0, x^*, 0, ... ) \in U_n^* \times \prod_{m \neq n} U_m^*$. By assumption we have that $\xi x_\gamma (y^*) \rightarrow 0$. This implies that $\xi x_\gamma (y^*) = (0, ..., x^*(x_\gamma), 0, ... ) \rightarrow 0$, hence $x^*(x_\gamma) \rightarrow 0$.

The last claim that $C_p(K, \mathbb{R}^\omega)$ is canonically isomorphic with $C_p(X)$ is well known (see [1, Propositions 0.3.3 and 0.3.4]).

4. Illustrating Examples

Example 4.1 Let $X$ be any uncountable scattered compact space. By $C(X)$ here we mean a Banach space with the sup-norm. We define an lcs $E$ to be $C(X)$ endowed with the coarser vector topology $\tau$ of uniform convergence on the countable compacts subsets of $X$. We claim that the spaces $(E, w)$ and $(C(X), w)$ coincide. Indeed, every linear continuous functional $\xi$ on $C(X)$ for a scattered compact space $X$ is defined by an atomic measure (which has a countable and compact support) on $X$ (see [18, Corollary 19.7.7]). This means that $\xi$ remains continuous being considered as a linear functional on $E$. So, $E^* = C(X)^* = \ell_1(X)$. Since $(E, w)$ and $(C(X), w)$ coincide and $(C(X), w)$ obviously is linearly Eberlein–Grothendieck, we conclude that $(E, w)$ is also linearly Eberlein–Grothendieck. However, $E$ with the original topology $\tau$ does not have to be Eberlein–Grothendieck. Take any uncountable scattered compact space $X$ with the following properties: 1) There is an uncountable subset $Y \subset X$ consisting of isolated points; 2) Every separable closed subset of $X$ is at most countable. For instance, let $X$ be the compact ordered space of ordinals $[0, \omega_1]$, or let $X$ be the one-point compactification of an uncountable discrete set $Y$. For each $y \in Y$ define a function $f_y \in C(X)$ as follows: $f_y(x) = 1$ if $x = y$ and $f_y(x) = 0$ otherwise. One can easily verify that the constant zero function belongs to the closure of $\{ f_y : y \in Y \}$ in $E$ but zero does not belong to the closure of $\{ f_y : y \in A \}$ in $E$ for every countable $A \subset Y$. It appears that $E$ does not have countable tightness, hence $E$ cannot be homeomorphically embedded into $C_p(K)$ for any compact space $K$.

Example 4.2 Let $X$ be the compact ordered space of ordinals $[0, \omega_1]$ and $Y$ be its subspace $[0, \omega_1)$. Then $X$ is compact, while $Y$ is not compact and every continuous real-valued function defined on $Y$ extends continuously to $X$. These very well-known facts show that the linear mapping of restriction $\pi$ maps $C_p(X)$ onto $C_p(Y)$, and $(C_p(Y), w)$ is not linearly Eberlein–Grothendieck despite that $(C_p(X), w)$ is linearly Eberlein–Grothendieck, by Theorem 3.11. We conclude that Theorem 3.5 is not valid in general if the mapping $\pi$ is not open.

Example 4.3 An lcs $X$ is defined to have a *neighborhood* $\omega^\omega$-*base* at zero if there exists a neighborhood base $\{ U_\alpha : \alpha \in \omega^\omega \}$ at zero such that $U_\beta \subset U_\alpha$ for all elements $\alpha \leq \beta$ in $\omega^\omega$. Every metrizable lcs has an $\omega^\omega$-base at zero and
the properties of lcs possessing an $\omega^\omega$-base at zero resemble the properties of metrizable lcs \[8\]. However, Corollary 2.3 cannot be generalized to $\omega^\omega$-based lcs. Let $E$ be the free locally convex space $L(X)$. It has been known that $L(X)$ has a $\omega^\omega$-base at zero for every metrizable compact space $X$ (see \[6\]). But $L(X)$ equipped with its weak topology, $L_p(X)$, is an Eberlein–Grothendieck space if and only if $X$ is finite, by \[12, Corollary 1\]. In particular, for the countable discrete space $X = \omega$ we have that $L(X) \cong \varphi$, and $(L(X)^*, w^*) \cong (\varphi^*, w^*) \cong \mathbb{R}^\omega$.

**Example 4.4** Let $E$ be the lcs $l_\infty = \{(x_n) \in \mathbb{R}^\mathbb{N} : \sup_n |x_n| < \infty\}$ equipped with the topology of pointwise convergence. If we consider the canonical linear mapping $\pi$ of restriction from $C_p(\beta\mathbb{N})$ into $\mathbb{R}^\mathbb{N}$, then the image of $\pi$ is exactly $E$. Since the mapping $\pi$ is not quotient, by this way we cannot decide whether $E$ is linearly Eberlein–Grothendieck. However, it has been proved that the space $C_p(\beta\mathbb{N})$ admits (another) linear continuous and quotient mapping onto $E$ \[3, Theorem 1\]. Hence $E$ is linearly Eberlein–Grothendieck by Theorem 3.5.

Alternatively, we can argue as follows. Look at the set $K = \{\frac{1}{n}\delta_n : n \in \mathbb{N}\} \cup \{0\} \subset E^*$. Then $K$ is compact in $(E^*, w^*)$ and $\text{span}(K) = E^*$. Therefore, $E$ is linearly Eberlein–Grothendieck by Theorem 3.2.

**Example 4.5** If $\Omega \subset \mathbb{R}^n$ is an open set, then the space of test functions $\mathcal{D}(\Omega)$ is a Montel space, i.e. a barrelled lcs in which every closed and bounded subset is compact. As usual, $\mathcal{D}'(\Omega)$ denotes its strong dual, the space of distributions. $\mathcal{D}'(\Omega)$ is also a Montel lcs, hence barrelled. Neither $\mathcal{D}(\Omega)$ nor $\mathcal{D}'(\Omega)$ is metrizable, they are not even sequential (see \[4\]). Applying Theorem 2.7 we conclude that $(E, w)$ is not Eberlein–Grothendieck for $E$ both $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$. Also, by Corollary 2.2, both $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ considered with the original topologies are not Eberlein–Grothendieck.

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