Interference of longitudinal and transversal fragmentations in the Josephson tunneling dynamics of Bose-Einstein condensates

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The dynamics of bosons in Josephson junctions have drawn much attention where the bosons are initially condensed. When interacting bosons tunnel back and forth along the junction, depletion and eventually fragmentation develop. Here, we pose the question how do fragmented bosons tunnel in a bosonic Josephson junction? To this end, we exploit the transverse degree-of-freedom of the junction to encode initial fragmentation to the bosonic cloud. We analyze the survival probability along the junction, fluctuations of particle positions across the junction, and the occupancy of the lowest single-particle states. The dynamics found is rich and includes the speed up of the collapse of density oscillations and slow down of the revival process. It is found that a fully fragmented state significantly accelerates the revival process compared to the conventional Bose-Einstein condensate. To explain the underlying many-body mechanism, we show that the initial fragmentation in the transverse direction interferes with the development of fragmentation in time along the junction. The dynamics of occupation in the first excited single-particle state defines whether interference of fragmentations occurs in the junction. The interference mechanism is a purely many-body effect that does not occur in the mean-field dynamics. All in all, we show that the interference of longitudinal and transversal fragmentations leads to new rules for macroscopic tunneling phenomena of interacting bosons in traps.

I. INTRODUCTION

Atomic Bose-Einstein condensates (BECs) are a unique state of matter and have been used as a flexible platform to explore a wealth of physical phenomena [1–5]. A particular example of interest in the context of the present work is the Josephson effect [6]. Over the years, various exotic features in Josephson junctions of ultracold atoms, such as, Josephson oscillations [7, 8], macroscopic self-trapping [9, 10], collapse and revival sequences [11], matter wave interferometry [12], and squeezing

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have been investigated. Josephson effects have been observed in complex systems, such as, spinor condensates [15], polariton condensates [16], fermionic superfluids [17], and spin-orbit coupled BECs [18].

In many-particle systems, interactions between the particles lead to correlations and their manifestation in a BEC is fragmentation which is a widely-studied phenomenon [19–26]. The concept of fragmentation arises when the reduced one-body density matrix starts to have more than one macroscopic eigenvalue in contrast to the conventional BEC with just one macroscopically occupied state. Hitherto, the development of fragmentation has been explored in the atomic Josephson junction only for fully condensed states [27–31]. In recent years, the dynamics of Josephson junctions in two dimensions has been studied taking into account physics that emerges due to the transversal degree-of-freedom [32–37].

The transversal degree-of-freedom can also enrich the amount of initial correlations in a BEC. In the present context, fragmentation of a bosonic system in a two-dimensional setup can be generated due to the transversal degree-of-freedom alone. This opens up the opportunity to ask and explore how different degrees of fragmentation along the transverse direction would impact the Josephson-junction dynamics in the longitudinal direction. More so, when initially condensed bosons tunnel back and forth (longitudinally) in a Josephson junction, they develop fragmentation. Now, when we let (transversely) fragmented bosons tunnel back and forth in the Josephson junction, would the initial fragmentation impact the longitudinal fragmentation that develops in time in a Josephson junction? So far, to the best of our knowledge, the possible interference of different fragmentations in a two-dimensional BEC has not been discussed. The question is the following, can we combine the concepts of Josephson junction and different fragmentations in a way which would allow one to have different internally correlated states? This intriguing combination cannot be generated in one dimension and so far has not been explored. It turns out that the macroscopic tunneling process becomes more complicated when the different fragmentations interfere. Therefore, in this work, we propose a set-up to study tunneling processes involving both transversal and longitudinal fragmentations, and the possible interference between them in time. As we shall see below, the tunneling dynamics is significantly enriched.

With the emergence of fragmentation as a key perspective of quantum many-boson physics, we examine the tunneling dynamics of a plethora of different initially fragmented states, and observe an intricate paradigm of tunneling dynamics taking into account the interference of fragmentations, to be defined precisely below. We show that the impact of transversal fragmentation on longitudinal fragmentation sets up new rules of the quantum tunneling problem. The rules of tunneling of a
fragmented BEC manifests in quantum mechanical quantities, namely, survival probability, build-up of occupation in the excited fragment, and fluctuations of particles’ positions.

II. THEORETICAL SETUP

The many-body Hamiltonian of \(N\) interacting bosons in two spatial dimensions reads:

\[
\hat{H}(r_1, r_2, \ldots, r_N) = \sum_{j=1}^{N} \left[ \hat{T}(r_j) + \hat{V}(r_j) \right] + \sum_{j<k} \hat{W}(r_j - r_k). \tag{2.1}
\]

Here \(\hat{T}(r_j)\) is the kinetic energy of \(j\)-th boson and \(\hat{V}(r_j)\) represents the trap potential. \(\hat{W}(r_j - r_k)\) is the inter-boson interaction which is modelled as repulsive Gaussian function \([38–40]\) with \(W(r_j - r_k) = \lambda_0 \frac{e^{-(r_j - r_k)^2/2\sigma^2}}{2\pi \sigma^2}\) where \(\sigma = 0.25\sqrt{\pi}\) is the width of the Gaussian function. We take a Gaussian inter-boson interaction because in two spatial dimensions a delta-function potential does not scatter, see, e.g., \([39]\). The robustness of our findings to the range \(\sigma\) is demonstrated in the supplemental material \([41]\). Furthermore, we also demonstrate the robustness of the results to the shape of the inter-bosons interaction, taking dipolar interaction as a case study \([41]\). Here \(\lambda_0\) is the interaction strength which defines the interaction parameter \(\Lambda_0 = \lambda_0(N-1)\). Throughout this work \(r = (x, y)\) and the natural units \(\hbar = m = 1\) are employed. The number of bosons considered in this work is \(N = 10\).

The simplest trap one can imagine for our purpose is a double well of double wells. In order to create the different fragmented ground states, we design the one-body trap potential as \(V(x, y) = \frac{1}{2}(x+2)^2 + V(y)\), where \(V(y) = \frac{1}{2}y^2 + V_L e^{-y^2/8}\), which transforms from single well to double well potential along the transverse direction as \(V_L\) is increased. The form of \(V(x, y)\) manifests that the atoms are initially trapped at the left side of space. For the out-of-equilibrium tunneling dynamics, we quench the trap potential from \(V(x, y)\) to \(V'(x, y)\). The form of \(V'(x, y)\) for the whole range of \(y\) is \(\frac{1}{2}(x+2)^2 + V(y)\) for \(x < -\frac{1}{2}\), \(\frac{1}{2}(x-2)^2 + V(y)\) for \(x > +\frac{1}{2}\), and \(\frac{3}{2}(1-x^2) + V(y)\) for \(|x| \geq \frac{1}{2}\). Here, the ramping up of \(V_L\) leads to a four-well trap, see Fig. 1 for selective barrier heights, \(V_L = 0, 12, \) and 16. The quench of the trap potential described above is analogous to the quench of a standard bosonic Josephson junction, see, e.g. \([27, 36]\). Note that the longitudinal and transversal directions correspond to \(x\)- and \(y\)-directions, respectively.

We make use of the powerful numerical many-body method, the bosonic version of the multiconfigurational time-dependent Hartree method \([25, 27, 42–45]\), also see its multi-layer version for mixtures \([47–51]\). The method incorporates quantum correlations exhaustively to obtain an
in-principle numerically exact ground states of different initial fragmentations, and their out-ofequilibrium dynamics. To obtain the wavefunction, a variationally optimal ansatz which is a linear combination of all permanents generated by distributing the \( N \) bosons over \( M \) time-adaptive orbitals is used. The many-body wavefunction is described as \[ (2.2) \]

\[ |\Psi(t)\rangle = \sum_{\{n\}} C_n(t)|n; t\rangle, \]

where \( C_n(t) \) are the expansion coefficients and \( |n; t\rangle = |n_1, n_2, ..., n_M; t\rangle \). The number of time-dependent permanents \( |n; t\rangle \) is \( \binom{N+M-1}{N} \). We recall that for \( M = 1 \) the many-body ansatz Eq. 2.2 boils down to the time-dependent Gross-Pitaevskii equation [1]. As the number of orbitals is increased, convergence of quantities, analyzed in this work, with \( M \) is obtained. The theoretical method is well documented in the literature [45]. In order to accurately capture the many-body physics, we have performed the many-body computations with \( M = 8 \) time-adaptive orbitals and the convergence is checked with \( M = 10 \) time-adaptive orbitals (see the supplemental material [41]). For the numerical solution we use a grid of \( 128 \times 128 \) points in a box of size \([-10, 10] \times [-10, 10]\) with periodic boundary conditions. Convergence of the results with the number of grid points has been verified using a grid of \( 256 \times 256 \) points and presented in [41].

III. PREPARATION AND PROPERTIES OF THE INITIAL STATE

To study the tunneling dynamics of fragmented BECs, we have to classify their basic properties depending on the degree of fragmentation. In order to characterize the condensed or fragmented state in the initial transversal double well, we present the occupation of the first natural orbital,
$n_1$, as a function of $V_L$. $n_1$ defines whether the ground state is condensed or fragmented in terms of the degree of condensation in the system. The degree of condensation is determined from diagonalization of the reduced one-particle density matrix

$$
\rho(r, r') = N \int dr_2...dr_N \Psi^*(r', r_2, ..., r_N) \Psi(r, r_2, ..., r_N),
$$

where $\rho(r, r') = \rho(r)$ is the density of the bosons. Fig. 2(a) depicts the degree of condensation of the ground state as a function of $V_L$ for two inter-boson interaction strengths $\Lambda_0 = 0.01\pi$ and $10\Lambda_0$. The inter-boson interactions are weak in the sense that at $V_L = 0$ the ground state is more than 99.99% condensed for $\Lambda_0$ and more than 99.9% condensed for $10\Lambda_0$. As expected, the degree of condensation of the ground state decreases with the barrier height, and the ground state becomes 48.55% fragmented for $\Lambda_0$ and 49.85% fragmented for $10\Lambda_0$ when $V_L = 16$. We find that the ground state becomes two-fold fragmented from about $V_L \geq 7$, with the marginally occupied third and fourth orbitals having the occupancy of around $10^{-7}$, and with the remaining four orbitals having occupancy of less than $10^{-7}$. The two-fold fragmented ground state implies that two natural orbitals are macroscopically occupied.

For the considered geometry of the trap, the ground orbital is gerade ($g$-orbital) and the excited orbital is ungerade ($u$-orbital) along the $y$-direction. In other words, the first fragment has $g$-symmetry and the second fragment has $u$-symmetry. As fragmentation develops the $g$- and $u$-orbitals tend to be equally occupied. The topology of the investigation indicates that parity in the $y$-direction is a good quantum number. Therefore, we may call this fragmentation, developed by occupying the $u$-orbital along the $y$-axis, the transversal fragmentation.

**FIG. 2.** Initial conditions. (a) Degree of condensation and (b) transversal position variance, $\frac{1}{N} \Delta^2_y$, as a function of the longitudinal barrier height ($V_L$) for two interaction strengths. The inset in panel (a) shows the density per particle, $\frac{1}{N} \rho(r)$, for the minimal and maximal $V_L$. The number of bosons is $N = 10$. We show here dimensionless quantities.
Now, to analyze the spatial distribution of bosons for different transversely fragmented ground states, we present the many-body transversal position variance,

\[
\frac{1}{N} \Delta_Y^2 = \frac{1}{N} \left[ \langle \Psi(t)|\hat{Y}^2|\Psi(t) \rangle - \langle \Psi(t)|\hat{Y}|\Psi(t) \rangle^2 \right],
\]

in Fig. 2 (b). The motivation is to use this quantity to analyze the dynamics of the initially transversely fragmented condensate as it tunnels back and forth along the Josephson junction, see section V. The many-particle position variance is known to be sensitive to correlations [37, 46].

Initially, \( \frac{1}{N} \Delta_Y^2 \) monotonously grows with \( V_L \) from the initial value 0.5, i.e., the value for the harmonic potential at \( V_L = 0 \), and reaches its maximal value at about \( V_L = 9 \) for \( \Lambda_0 \) and \( V_L = 8 \) for \( 10\Lambda_0 \). Interestingly, due to the appreciable amount of the transversal fragmentation developed, \( \frac{1}{N} \Delta_Y^2 \) decays and tends to saturate when \( V_L \) reaches the value 16. It is worthwhile mentioning that, unlike the many-body result, the mean-field \( \frac{1}{N} \Delta_Y^2 \) monotonously grows from the initial value 0.5 with increasing of the barrier height. Also the mean-field \( \frac{1}{N} \Delta_Y^2 \) practically overlaps for \( \Lambda_0 \) and \( 10\Lambda_0 \), see details in [41]. It is noted that the many-body and mean-field variances in the longitudinal direction, \( \frac{1}{N} \Delta_X^2 \), are essentially the same for all \( V_L \) with value 0.5, and are practically independent of the interaction strength chosen here (and therefore need not be plotted). This suggests a very weak coupling between the \( x \)- and \( y \)-directions of the interacting bosons in the considered two-dimensional double-well trap for the initial conditions for all \( V_L \).

Summarizing, we have prepared initial states of different degrees of fragmentation in the transversal direction depending on the value of \( V_L \). To investigate their out-of-equilibrium Josephson dynamics we propose to quench the trap to a four-well potential where the bosons can tunnel back and forth between the left and right transverse double wells. The main research question can be formulated now more precisely: Will a transversely fragmented condensates develop longitudinal fragmentation in time? And if so, will the two fragmentations interfere with and impact each other?

**IV. OUT-OF-EQUILIBRIUM TUNNELING DYNAMICS OF A FRAGMENTED BEC: EMERGENCE OF INTERFERENCE**

We begin investigating the tunneling dynamics with a basic quantity, namely, the survival probability in the left part of space,

\[
P(t) = \int_{x=-\infty}^{0} \int_{y=-\infty}^{+\infty} d\mathbf{r} \frac{\rho(\mathbf{r};t)}{N},
\]
where $\rho(r; t)$ is the time-dependent density. In the many-body dynamics, one can find that the frequencies of oscillations of $P(t)$ for different $V_L$ are essentially the same, see in Fig. 3, due to the practically same Rabi frequency along the $x$-direction [41]. Note that time is scaled by $t_{\text{Rabi}}$ in what follows where $t_{\text{Rabi}}(= 132.498)$ is the time of a Rabi cycle [41]. The attractive feature lies in the decay rate of the amplitude of the many-body $P(t)$. We find that the amplitude of the many-body $P(t)$ decays in time with a different rate for different $V_L$. This decay occurs due to the development of longitudinal fragmentation in the tunneling process. To distinctly identify the rate of decay of $P(t)$ with different barrier heights, we present it as a function of $V_L$ for a fixed time in Fig. 3 (b). Here, we choose a particular time $t = 20t_{\text{Rabi}}$ when the bosons tunnel back to the left side of space, and is applicable for every even multiple of $t_{\text{Rabi}}$. To compare $P(t)$ with the mean-field dynamics, we also plot the same at the mean-field level. Let us discuss first the mean-field dynamics of $P(t)$. As the trapping potential along the tunneling direction does practically not change with the barrier height, all the bosons continue to tunnel back and forth between the left and right parts of space at the mean-field level, resulting in "no-decay" of the density oscillations, see also [41] for the mean-field $P(t)$. Therefore, in the mean-field description of the dynamics, $P(t)$ is essentially invariant to the internal structure of the junction.

Now, we come back to the details of many-body survival probability. The many-body dynamics of the system can be divided into three regimes as a function of barrier height, i.e., lower barrier heights from $V_L = 0$ to 6, intermediate barrier heights from $V_L = 7$ to 12, and higher barrier heights from $V_L = 13$ to 16. Ramping up of the barrier from $V_L = 0$ to 6, the initial ground state remains condensed but the decay rate monotonously reduces with $V_L$ as the next transversal band gradually becomes closer to the lowest band. Therefore in Fig. 3 (b), $P(t = 20t_{\text{Rabi}})$ hits its maximal value for $V_L = 6$. Once we pass $V_L = 6$, the initial ground state is starting to lose its degree of condensation and it becomes a fragmented state, see Fig. 2(a). Therefore, more bosons are pushed to the $u$-orbital initially (at $t = 0$) while we gradually increase the barrier height from $V_L = 7$. The increased initial occupancy in the $u$-orbital leads to acceleration in the decay rate of $P(t)$ and thus the collapse of density oscillations speeds up. This gradual increase of the decay rate of $P(t)$ holds until the initial connection between the two orbitals ($g$-orbital and $u$-orbital) starts to be weaker for a certain internal structure of the junction. For the considered interaction parameter $\Lambda_0$, this particular barrier height is $V_L = 12$. The speeding up of the collapse of the density oscillations indicates that an intriguing and non-trivial effect occurs between the longitudinal fragmentation developed during the tunneling process and the initial transversal fragmentation. This is a purely many-body effect that cannot be observed in the mean-field survival probability, see for details...
in [41]. For $V_L > 12$, the density collapse slows down again until $V_L = 16$, implying that the two orbitals of the ground state start to behave like two independent states, and the intriguing many-body effect tends to diminish.

By examining Fig. 3 (b), one can find that the decay rate for a fully condensed system, i.e., for $V_L = 0$, is faster than for the fully fragmented system, i.e., for $V_L = 16$. For the fully fragmented system, the initial occupancy is almost equally distributed between the $g$-orbital and $u$-orbital. On top of that there is no connection between the two orbitals due to the combined effect of the inter-boson interaction and barrier height. Thus the two orbitals behave like two independent states tunneling back and forth, and each state consists of almost $N/2$ bosons. For a fixed interaction strength $\lambda_0$, the decay rate of $P(t)$ is faster for a larger number of bosons, which explains the slower decay rate of a fully fragmented system compared to the fully condensed system. See the Appendix for the many-body survival probability of $N = 5$ and $N = 10$ bosons in a two-dimensional double-well.

![FIG. 3. Description of tunneling dynamics using many-body survival probability. (a) Many-body survival probability in the left side of space, $P(t)$, for the barrier heights $V_L = 0$ (black), 7 (magenta), 10 (green), 12 (red), and 16 (blue) as a function of time. For ease of discussion, the least decaying to most decaying of $P(t)$ are for the barrier height $V_L = 7$ (magenta), 10 (green), 16 (blue), 12 (red), and 0 (black), respectively. For the reader, we have separately plotted $P(t)$ for each barrier height in Fig. S4 of the supplemental material. (b) $P(t)$ is plotted at $t = 20t_{\text{Rabi}}$ as a function of the barrier height. The red squares and blue circles show the mean-field and many-body decay rate of $P(t)$, respectively. Unlike the many-body dynamics, the mean-field results do not distinguish the impact of the transverse direction. The inter-boson interaction is $\Lambda_0$ and the number of bosons $N = 10$. See the text for further details. We show here dimensionless quantities.]

Now, to understand the underlying physics of the time-dependent effect discussed above, we analyze further quantities, starting from the occupations of the natural orbitals. Fig. 4 depicts
FIG. 4. Description of tunneling dynamics using the occupation numbers of the first two natural orbitals. Occupations of the first and second natural orbitals, \( \frac{n_1(t)}{N} \) and \( \frac{n_2(t)}{N} \), respectively, for (a) \( V_L = 0 \), (b) \( V_L = 12 \), (c) \( V_L = 13 \), and (d) \( V_L = 16 \). The traps for \( V_L = 0 \), \( V_L = 12 \), and \( V_L = 16 \) are shown in Fig. 1. The inset in panel (b) depicts the magnitude of interference for the barrier heights from \( V_L = 7 \) to \( V_L = 14 \). The interference \( \left[ (n_{u}^{\text{maximal}} - n_{u}^{\text{minimal}})/N \right] \) occurs only for fragmented BEC where \( n_u \) is the occupation of \( u \)-orbital, provided \( n_u^{\text{maximal}} \) is achieved at a later time than \( n_u^{\text{minimal}} \). The inter-boson interaction is \( \Lambda_0 \) and the number of bosons \( N = 10 \). See the text for further details. We show here dimensionless quantities.

The occupation of the first, \( \frac{n_1(t)}{N} \), and second, \( \frac{n_2(t)}{N} \), natural orbitals for \( V_L = 0, 12, 13, \) and \( 16 \). To demonstrate how the occupation of the various orbitals affects the overall dynamics, we have included the symmetries of the two maximally occupied orbitals for all the barrier heights from \( V_L = 0 \) to 16 during the dynamical evolution in Table I. It is found that \( \frac{n_1(t)}{N} \) monotonously decreases, remains as \( g \)-orbital (see also section III for the initial orbitals), and eventually tends to saturate for all the initial states. This holds true whether the system is initially fully condensed, partially fragmented, or fully fragmented, see Table I. Remarkably, the dynamical occupation of the second natural orbital shows an intriguing feature. We observe that when the initial state is fully condensed at \( V_L = 0 \) (or fully fragmented at \( V_L = 16 \)), \( \frac{n_2(t)}{N} \) monotonously increases (or
decreases). For the fully condensed system, the second natural orbital is an excited $g$-orbital and for the fragmented system it is a $u$-orbital, see Table I. Interestingly, in Figs. 4 (b) and (c), we find that, initially, the $u$-orbital loses its occupation and thereafter the occupation builds up in time. This transition, from decreasing of occupation to building up of occupation in the $u$-orbital represents the non-trivial effect which we call the interference of the longitudinal and transversal fragmentations, taking place for intermediate barrier heights from $V_L = 7$ to $V_L = 14$. Clearly, the interference of fragmentations occurs when all the following three conditions are satisfied chronologically, (i) initially occupied $u$-orbital, (ii) loss of occupation in the $u$-orbital, and (iii) build-up of occupation in the $u$-orbital. Moreover, we observe that the $u$-orbital loses its occupation until and unless there is a swapping of orders between higher natural orbitals, for details see [41].

For completeness, we have tabulated the symmetry of the first two maximally occupied orbitals during the time evolution, for all barrier heights from $V_L = 0$ to 16, in Table I. By inspecting Table I, one can find that the second maximally occupied orbital is changing its symmetry during the time evolution for the intermediate barrier heights $V_L = 7, 8, 9, 10, 11,$ and $14$. Among these intermediate heights, only for $V_L = 11$ the nature of the second natural orbital swaps between $u$-orbital and excited $u$-orbital. While for other intermediate barrier heights, the swapping takes place between the $u$-orbital and excited $g$-orbital, i.e., the symmetry changes. This swappings of orbitals are found by scrutinizing the occupancy of the orbitals with time, for details we refer to [41]. Moreover, for $V_L = 11$, it is found that the nature of the second natural orbital is swapping multiple times between $u$-orbital and excited $u$-orbital. This swapping of orbitals showcases the intriguing dynamics of a fragmented BEC undergoing tunneling in the junction.

Now the magnitude of the interference between the longitudinal and transversal fragmentations is calculated from the difference between the maximal occupancy of the $u$-orbital after building up of occupation and its minimal occupancy when the transition from loss to build up of occupation takes place, see the inset in Fig. 4 (b). Therefore, for fragmented BEC the magnitude of interference is $\frac{n_{u\text{maximal}}^{u} - n_{u\text{minimal}}^{u}}{N}$, where $n_u$ is the occupation of $u$-orbital, provided $n_{u\text{maximal}}^{u}$ is achieved at a later time than $n_{u\text{minimal}}^{u}$. This analysis identifies that the maximal interference of the longitudinal and transversal fragmentations takes place at $V_L = 12$ for $\Lambda_0$.

V. ANALYSIS OF THE INTERFERENCE OF FRAGMENTATIONS

Now, in order to show the consequences of the interference of the longitudinal and transversal fragmentations on a quantum mechanical observable, we present the time evolution of the
many-body $\frac{1}{N}\Delta_{Y}^{2}(t)$ in Fig. 5 (a). As the inter-boson interaction is weak and since for $V_{L} = 0$ the initial state is fully condensed, $\frac{1}{N}\Delta_{Y}^{2}(t)$ is practically frozen in time due to no interference between the longitudinal and transversal fragmentations. At $V_{L} = 7$, when the ground state is initially barely depleted (around 0.01%, see section III), $\frac{1}{N}\Delta_{Y}^{2}(t)$ shows small oscillations during the tunneling process originating from small interference of fragmentations. Increasing of the initial fragmentation in the ground state, say, at $V_{L} = 10$, $\frac{1}{N}\Delta_{Y}^{2}(t)$ shows oscillatory nature with constant amplitude and frequency of oscillations, due to a stronger interference. Interestingly, for $V_{L} = 12$, we find that $\frac{1}{N}\Delta_{Y}^{2}(t)$ grows as time progresses. This growing nature of $\frac{1}{N}\Delta_{Y}^{2}(t)$ signifies strong coupling of the $x$- and $y$-directions and it emerges from the strong interference between the longitudinal and transversal fragmentations. For $V_{L} > 12$, $\frac{1}{N}\Delta_{Y}^{2}(t)$ slowly tends toward essentially frozen dynamical behavior which exhibits decoupling of the fragmentations. Eventually, at $V_{L} = 16$, we observe essentially constant dynamical behavior due to the practically null interference of the longitudinal and transversal fragmentations. Therefore, we conclude that the detailed investigation of the fluctuations in particle positions across the junction, being a sensitive probe of correlations, encodes the interference of fragmentations. All in all, we find that the oscillations of the many-body $\frac{1}{N}\Delta_{Y}^{2}(t)$ depend significantly on the barrier height. Compared to the many-body dynamics, the mean-field $\frac{1}{N}\Delta_{Y}^{2}(t)$ monotonously increases, showing almost frozen dynamics with fluctuation of around $10^{-3}$, for all barrier heights (see details in [41]).

Let us define a quantity to which we call the normalized occupation, $\eta(t) = n_{1}(t)/n_{1}(0)$. We find that $\eta(t)$ would help us to compare the decrease in the first occupation number of systems

![Fig. 5. Interference of longitudinal and transversal fragmentations. (a) Time-dependent transversal position variance per particle, $\frac{1}{N}\Delta_{Y}^{2}(t)$, and (b) normalized occupation of the $g$-orbital, $\eta(t) = n_{1}(t)/n_{1}(0)$. The inter-boson interaction is $\Lambda_{0}$ and the number of bosons $N = 10$. Color codes are explained in panel (b). We show here dimensionless quantities.](image-url)
with different degrees of initial fragmentation. \( \eta(t) \) also exhibits that the observed interference of fragmentations varies in time during the evolution. One can analyze the time-dependent nature of the interference of fragmentations by analyzing the real-time decay of the normalized occupation \( \eta(t) \) of the \( g \)-orbital, see Fig. 5 (b). As one increases the barrier height from \( V_L = 0 \) to \( V_L = 6 \), the rate of decay of \( \eta(t) \) decreases. Further increase of the barrier height, until \( V_L = 12 \), the rate of decay of \( \eta(t) \) increases and, for \( V_L > 12 \), it follows the previous trend of decreasing. We examine the dynamics of \( \eta(t) \) in time for different \( V_L \). If we look into the details of \( \eta(t) \) for \( V_L = 12 \), we notice that it crosses the corresponding \( \eta(t) \)'s found at \( V_L = 0, 13, \) and 16 as time progresses, and that it exhibits the maximal rate of loss of the degree of condensation. The behavior of \( \eta(t) \) suggests that the interference between the longitudinal and transversal fragmentations is indeed time dependent. By analyzing Fig. 5 (b), we find the following rule of tunneling: the interference between the longitudinal and transversal fragmentations speeds up the rate of loss of the degree of condensation \( \eta(t) \) in the system.

VI. IMPACT ON THE REVIVAL PROCESS

So far, we have dealt with the dynamics of various quantities of a rather weakly interacting system consists of \( N = 10 \) bosons. Now, to see the revival in reasonable time, we investigate the same problem with stronger interaction, \( 10\Lambda_0 \). As discussed in Section III, for \( V_L = 0 \), the initial system \((t = 0)\) is about 99.99\% condensed for the weaker interaction \((\Lambda_0)\), and for the stronger interaction \((10\Lambda_0)\), it is about 99.9\% which is still rather condensed. These values give us a sense of how much stronger is the interaction \( 10\Lambda_0 \) compared to \( \Lambda_0 \).

Here we would like to inspect what is the impact of interference of fragmentations, if any, on the revival process [11]. Fig. 6 presents the time evolution of \( P(t) \) and \( \eta(t) \) for the interaction parameter \( 10\Lambda_0 \). Here, \( P(t) \) exhibits substantial richer physics of the tunneling dynamics for different fragmented initial states. First of all, the density never tunnels 100\% back for all barrier heights, as the longitudinal fragmentation develops at the very beginning of the tunneling process due to the stronger interaction. In addition to the density collapse, also found for the weaker interaction \( \Lambda_0 \), here we observe a different rate of revival of the density oscillations. We attribute this effect to the interference of the initial transversal fragmentation and the developed longitudinal fragmentation in the tunneling process. We find that when the barrier height gradually increases from \( V_L = 0 \) to \( V_L = 10 \), the rate of revival of the density oscillations decreases, see the black and red curves of Fig. 6(a). Further increase of the barrier height, i.e., when \( V_L > 10 \), the revival of
the density oscillations increases and reaches its maximal value, when the ground state is initially fully fragmented.

FIG. 6. Tunneling dynamics for stronger interaction. (a) Time-dependent survival probability in the left side of space, $P(t)$, and (b) normalized occupation of the ground orbital, $\eta(t) = n_1(t)/n_1(0)$. The inter-boson interaction is $10\Lambda_0$ and the number of bosons $N = 10$. Color codes are explained in panel (b). We show here dimensionless quantities.

The different rates of the revival process can be explained by the dynamics of $\eta(t)$, shown in Fig. 6 (b). The general feature is that all the ground states lose their degree of condensation with an oscillatory background as time progresses. The maximal decay rate of $\eta(t)$, found at $V_L = 10$, is the result of the strongest interference of the longitudinal and transversal fragmentations, see [41]. Here we also find, as seen in the case of weaker interaction, that the rate of loss of degree of condensation of a fully fragmented BEC is slower compared to the fully condensed state. Moreover, the rate of revival decreases when one moves from $V_L = 0$ to $V_L = 10$. With further increase of $V_L$, $\eta(t)$ increases and becomes maximal for $V_L = 16$.

One finds out that the many-body $P(t)$ and the time-evolution of $\eta(t)$ for stronger interaction exhibit a new rule for the revival dynamics of fragmented BECs: the interference of the longitudinal and transversal fragmentations opposes the revival process. Moreover, we observe that a fully fragmented state, without coupling between the longitudinal and transversal fragmentations, speeds up the process of revival compared to the conventional BEC. As discussed above, the fully fragmented BEC can be identified as two independent BECs with essentially $N/2$ bosons in each fragment. Hence, the observation of revival dynamics of fully condensed and fully fragmented systems implies that, for a fixed inter-boson interaction strength $\lambda_0$, the revival process takes place at a slower rate for a BEC with a larger number of bosons. To connect further the revival dynamics of a fully condensed system and a fully fragmented system, we present the survival probability of
$N = 5$ and $N = 10$ bosons for the two dimensional double-well in the Appendix.

VII. CONCLUDING REMARKS

In conclusion, we explore the Josephson dynamics of involved bosonic objects which undergo a rich pathway from condensation to fragmentation in a transversal double-well trap. We would like to bring out that such intricate ground states shows new rules while tunneling. We have demonstrated the physics behind the tunneling dynamics of the fragmented BEC by emphasizing three limiting cases: the first, when the ground state is initially fully condensed, the second, when the interference of the longitudinal and transversal fragmentations is maximal, and the third, when the ground state is initially fully fragmented with no coupling between the fragmentations in the tunneling process. The interference of fragmentations in the tunneling process occurs when the following three criteria are satisfied chronologically, (a) an initial occupation in the second fragment (the ungerade orbital along the $y$-direction), (b) the initial decrease of occupation of the second fragment in time, and (c) the subsequent build up of occupation in the second fragment in time. The magnitude of the interference is defined by the difference between the maximal and minimal respective occupations of the second fragment.

We have established that the interference of the fragmentations constitutes a new mechanism of the tunneling process by analyzing the survival probability, details of fragmentation dynamics, and transversal position variance in the junction. We find and explain how the interference of fragmentations governs the collapse and revival of the density oscillations and formulate general rules for macroscopic tunneling: (i) the interference between the longitudinal and transversal fragmentations speeds up the loss of degree of condensation in the junction, (ii) there is an optimal geometry that maximizes the interference of fragmentations while tunneling, (iii) for a fixed inter-boson interaction strength $\lambda_0$, the loss of degree of condensation occurs at a slower rate but the revival process takes place at a faster rate for a smaller number of bosons in a BEC, and (iv) the interference of fragmentations delays the revival process.

As the four-well set-up considered here is a minimal substructure of a two-dimensional optical lattice, tunneling of different quantum phases, such as, superfluid, Mott insulator [52], and fermionized Tonks–Girardeau gas [53, 54], can be studied by tuning the barrier height and the strength of inter-particle interaction. Moreover, dipolar bosonic crystal orders and the dynamics of bosons in two-dimensional optical lattices, including the impact of competition between longitudinal and transversal fragmentations, have the future scope to be investigated. The many-body physics pre-
sented here could be relevant to the community working on atomtronics [55] and metrology [56]. Further investigations are warranted.

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APPENDIX

The appendix describes the tunneling dynamics in terms of the survival probability in the left side of space, \( P(t) \), in a regular double-well potential in two spatial dimensions, i.e., for \( V_L = 0 \). The interaction strength is \( \lambda_0 = 0.0111 \) which corresponds to the interaction parameter \( \Lambda_0 = 0.1 \) for \( N = 10 \) bosons. The results are shown for \( N = 5 \) and \( N = 10 \) bosons, see Fig. 7. The results depict the collapse of density oscillations and the revival dynamics. It is clearly seen that, for a fixed inter-boson interaction strength \( \lambda_0 \), the decay rate of \( P(t) \) is faster for \( N = 10 \) while the revival time is faster for \( N = 5 \). These results would help one to demonstrate and discuss the tunneling dynamics of a fully condensed system and a fully fragmented system presented in the main text.

[1] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).
[2] L. S. Cederbaum, A. I. Streltsov, Y. B. Band, and O. E. Alon, Phys. Rev. Lett. 98, 110405 (2007).
[3] O. E. Alon, A. I. Streltsov, and L. S. Cederbaum, Phys. Rev. Lett. 95, 030405 (2005).
[4] A. Bhowmik, P. K. Mondal, S. Majumder, and B. Deb, Phys. Rev. A 93, 063852 (2016).
[5] M. Schmidt, L. Lassablière, G. Quéméner, and T. Langen, Phys. Rev. Research 4, 013235 March (2022).
[6] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. 80, 885 (2008).
[7] A. Burchinati, C. Fort, and M. Modugno, Phys. Rev. A 95, 023627 (2017).
[8] S. Levy, E. Lahoud, I. Shomroni, and J. Steinhauer, Nature (London) 449, 579 (2007).
[9] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, Phys. Rev. Lett. 79, 4950 (1997).
FIG. 7. Survival probability in the left side of space, $P(t)$, for the barrier height $V_L = 0$, which describes a regular double-well potential in two dimensions. The inter-boson interaction is the same as described in the main text. The interaction strength is $\lambda_0 = 0.0111$ which corresponds to the interaction parameter $\Lambda_0 = 0.1$ for $N = 10$ bosons. The results obtained are for the numbers of bosons $N = 5$ and $N = 10$. We show here dimensionless quantities.

[10] M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani, M. K. Oberthaler, Phys. Rev. Lett. 95, 010402 (2005).
[11] G. J. Milburn, J. Corney, E. M. Wright, D. F. Walls, Phys. Rev. A 55, 4318 (1997).
[12] T. Schumm, S. Hofferberth, L. M. Andersson, S. Wildermuth, S. Groth, I. Bar-Joseph, J. Schmiedmayer, and P. Krüger, Nature Physics 1, 57 (2005).
[13] C. Orzel, A. K. Tuchman, M. L. Fenselau, M. Yasuda, and M. A. Kasevich, Science 291, 2386 (2001).
[14] Q. Wu, L. Mancino, M. Carlesso, M. A. Ciampini, L. Magrini, N. Kiesel, and M. Paternostro, PRX Quantum 3, 010322 (2022).
[15] T. Zibold, E. Nicklas, C. Gross, and M. K. Oberthaler, Phys. Rev. Lett. 105, 204101 (2010).
[16] M. Abbarchi, A. Amo, V. G. Sala, D. D. Solnyshkov, H. Flayac, L. Ferrier, I. Sagnes, E. Gañopin, A. Lemaitre, G. Malpuech, and J. Bloch, Nature Physics 9, 275 (2013).
[17] G. Valtolina, A. Burchianti, A. Amico, E. Neri, K. Xhani, J. A. Seman, A. Trombettoni, A. Smerzi, M. Zaccanti, M. Inguscio, and G. Roati, Science 350, 1505 (2015).
[18] J. Hou, X. -W. Luo, K. Sun, T. Bersano, V. Gokhroo, S. Mossman, P. Engels, and C. Zhang, Phys. Rev. Lett. 120, 120401 (2018).
[19] P. Nozières and D. Saint James, J. Phys. 43, 1133 (1982).
[20] R. W. Spekkens and J. E. Sipe, Phys. Rev. A 59, 3868 (1999).
[21] E. J. Mueller, T. -L. Ho, M. Ueda, and G. Baym Phys. Rev. A 74, 033612 (2006).
[22] P. Bader and U. R. Fischer, Phys. Rev. Lett. 103, 060402 (2009).
[23] Q. Zhou and X. Cui, Phys. Rev. Lett. 110, 140407 (2013).
[24] M. -K. Kang and U. R. Fischer, Phys. Rev. Lett. 113, 140404 (2014).
[25] A. U. J. Lode and C. Bruder, Phys. Rev. Lett. 118, 013603 (2017).
[26] B. Chatterjee, C. Lévêque, J. Schmiedmayer, and A. U. J. Lode, Phys. Rev. Lett. 125, 093602 (2020).
[27] K. Sakmann, A. I. Streltsov, O. E. Alon, and L. S. Cederbaum, Phys. Rev. Lett. 103, 220601 (2009).
[28] J. Vargas, M. Nuske, R. Eichberger, C. Hippler, L. Mathey, and A. Hemmerich, Phys. Rev. Lett. 126, 200402 (2021).
[29] J. Erdmann, S. I. Mistakidis, and P. Schmelcher, Phys. Rev. A 98, 053614 (2018).
[30] F. Theel, K. Keiler, S. I. Mistakidis, and P. Schmelcher, New J. Phys. 22, 023027 (2020).
[31] C. Sias, A. Zenesini, H. Lignier, S. Wimberger, D. Ciampini, O. Morsch, and E. Arimondo, Phys. Rev. Lett. 98, 120403 (2007).
[32] O. Fialko, A. S. Bradley, and J. Brand, Phys. Rev. Lett. 108, 015301 (2012).
[33] G. Spagnolli, G. Semeghini, L. Masi, G. Ferioli, A. Trenkwalder, S. Coop, M. Landini, L. Pezzè, G. Modugno, M. Inguscio, A. Smerzi, and M. Fattori, Phys. Rev. Lett. 118, 230403 (2017).
[34] A. Burchianti, F. Scazza, A. Amico, G. Valtolina, J. A. Seman, C. Fort, M. Zaccanti, M. Inguscio, and G. Roati, Phys. Rev. Lett. 120, 025302 (2018).
[35] K. Xhani, E. Neri, L. Galantucci, F. Scazza, A. Burchianti, K.-L. Lee, C. F. Barenghi, A. Trombettoni, M. Inguscio, M. Zaccanti, G. Roati, and N. P. Proukakis, Phys. Rev. Lett. 124, 045301 (2020).
[36] A. Bhownik, S. K. Haldar, and O. E. Alon, Sci. Rep. 10, 21476 (2020).
[37] A. Bhownik and O. E. Alon, Sci. Rep. 12, 627 (2022).
[38] J. Christensson, C. Forssén, S. Åberg, S. M. Reimann, Phys. Rev. A 79, 012707 (2009).
[39] R. A. Doganov, S. Klaiman, O. E. Alon, A. I. Streltsov, and L. S. Cederbaum, Phys. Rev. A 87, 033631 (2013).
[40] U. R. Fischer, A. U. J. Lode, and B. Chatterjee, Phys. Rev. A 91, 063621 (2015).
[41] Supplemental materials of this work.
[42] A. I. Streltsov, O. E. Alon, and L. S. Cederbaum Phys. Rev. Lett. 99, 030402 (2007).
[43] O. E. Alon, A. I. Streltsov, and L. S. Cederbaum, Phys. Rev. A 77, 033613 (2008).
[44] J. H. V. Nguyen, M. C. Tsatsos, D. Luo, A. U. J. Lode, G. D. Telles, V. S. Bagnato, and R. G. Hulet, Phys. Rev. X 9, 011052 (2019).
[45] A. U. J. Lode, C. Lévêque, L. B. Madsen, A. I. Streltsov, and O. E. Alon, Rev. Mod. Phys. 92, 011001 (2020).
[46] S. Klaiman and O. E. Alon, Phys. Rev. A 91, 063613 (2015).
[47] S. Krönke, L. Cao, O. Vendrell, and P. Schmelcher, New J. Phys. 15, 063018 (2013).
[48] L. Cao, S. Krönke, O. Vendrell, and P. Schmelcher, J. Chem. Phys. 139, 134103 (2013).
[49] J. Chen, J. M. Schurer, and P. Schmelcher, Phys. Rev. Lett. 121, 043401 (2018).
[50] J. M. Schurer, A. Negretti, and P. Schmelcher, Phys. Rev. Lett. 119, 063001 (2017).
[51] S. I. Mistakidis, G. C. Katsimiga, G. M. Koutentakis, Th. Busch, and P. Schmelcher, Phys. Rev. Lett. 122, 183001 (2019).
[52] M. Capello, F. Becca, M. Fabrizio, and S. Sorella, Phys. Rev. Lett. 99, 056402 (2007).
[53] V. Dunjko, V. Lorent, and M. Olshanii, Phys. Rev. Lett. 86, 5413 (2001).
[54] B. Paredes, A. Widera, V. Murg, O. Mandel, S. Fölling, I. Cirac, G. V. Shlyapnikov, T. W. Hänsch, and I. Bloch, Nature (London) 429, 277 (2004).

[55] L. Amico, D. Anderson, M. Boshier, J.-P. Brantut, L.-C. Kwek, A. Minguzzi, and W. von Klitzing, Rev. Mod. Phys. 94, 041001 (2022).

[56] J.-G. Baak and U. R. Fischer, Phys. Rev. Lett. 132, 240803 (2024).
The symmetry of the first two maximally occupied orbitals are highlighted for the barrier heights $V_L = 0$ to 16. Here $t = 0$ represents the initial condition and $t > 0$ represents the out-of-equilibrium dynamics. For all barrier heights the first orbital remains as the g-orbital throughout the dynamical evolution (third column). On the other hand, the symmetry of the second orbital is different depending on the barrier height and time of evolution (fourth column). For $V_L = 0$ to 6, the system is fully condensed and thus only one orbital, i.e., g-orbital, is practically occupied at $t = 0$, see also Fig. 2(a). For $V_L \geq 7$, two orbitals are occupied at $t = 0$ and they are g-orbital and u-orbital. Only for $V_L = 12, 13, 15,$ and 16, the u-orbital remains as the second orbital throughout the dynamics. While for the other barrier heights, the second orbital switches its nature between u-orbital and excited g-orbital or excited u-orbital. As an example, for $V_L = 7$, the initial ($t = 0$) second orbital is u-orbital, then in the dynamics ($0 < t \leq 0.20t_{Rabi}$) the second orbital remains as u-orbital, and subsequently the second orbital becomes excited g-orbital for the time window $0.20t_{Rabi} < t \leq 30t_{Rabi}$. The number of bosons is $N = 10$ and the interaction parameter $\Lambda_0$. For details see [41].

| Barrier height | time | 1st orbital | 2nd orbital |
|---------------|------|-------------|-------------|
| $V_L = 0$ to 6 | $t = 0$ | g-orbital | – | g-orbital | excited g-orbital |
| $V_L = 7$ | $t = 0$ | g-orbital | u-orbital | g-orbital | u-orbital | g-orbital | excited g-orbital |
| $V_L = 8$ | $0 < t \leq 0.81t_{Rabi}$ | g-orbital | u-orbital | g-orbital | u-orbital | g-orbital | excited g-orbital |
| $V_L = 9$ | $0 < t \leq 2.11t_{Rabi}$ | g-orbital | u-orbital | g-orbital | u-orbital | g-orbital | excited g-orbital |
| $V_L = 10$ | $0 < t \leq 0.76t_{Rabi}$ | g-orbital | u-orbital | g-orbital | u-orbital | g-orbital | excited g-orbital |
| $V_L = 11$ | $0 < t \leq 16.79t_{Rabi}$ | g-orbital | u-orbital | g-orbital | u-orbital | g-orbital | excited u-orbital | g-orbital | excited u-orbital |
| $V_L = 12$ and 13 | $t = 0$ | g-orbital | u-orbital | g-orbital | u-orbital |
| $V_L = 14$ | $0 < t \leq 25.70t_{Rabi}$ | g-orbital | u-orbital | g-orbital | u-orbital | excited g-orbital |
| $V_L = 15$ and 16 | $t = 0$ | g-orbital | u-orbital | g-orbital | u-orbital | g-orbital | u-orbital |
SUPPLEMENTAL MATERIAL: INTERFERENCE OF LONGITUDINAL AND TRANSVERSAL FRAGMENTATIONS IN THE JOSEPHSON TUNNELING DYNAMICS OF BOSE-EINSTEIN CONDENSATES

In this supplemental material, we promote the main text with further details. Here, we start with a brief mathematical description on the many-particle variance discussed in the main text. Next, we present the initial-state mean-field position variance as a function of barrier height. Moving forward, to represent and compare Josephson dynamics of different fragmented states, we require a fixed time scale. As a useful information, we present the longitudinal and transversal Rabi frequencies in the four-well setup, considered in the main text, as a function of $V_L$. Further, as a basic analysis tool of the Josephson dynamics presented here, we provide the mean-field results of the dynamics of the survival probability, and the longitudinal and transversal position variances. Moreover, in order to go deeper into the understanding of the interference in different fragmentation channels, we demonstrate the details of the fragmentation processes in terms of the occupancy of the higher natural orbitals and their connection with many-body longitudinal and transversal position variances. Next, we discuss the robustness of our results to the width of the inter-boson interaction potential. We also demonstrate the robustness of our results to the shape of inter-boson interaction, employing the popular dipole interaction as a case study. Finally, we present the convergences of our results.
VIII. MANY-PARTICLE POSITION VARIANCE

The main text contains the results of the many-particle position variance which exhibits the impact of interference of the longitudinal and transversal fragmentations developed during the tunneling process. Here we present the mathematical formula of the many-particle position variance. The time-dependent variance per particle of an operator, \( \hat{A} \), is determined by the combination of the expectation values of \( \hat{A} \) and \( \hat{A}^2 \). The expectation value of \( \hat{A} = \sum_{j=1}^{N} \hat{a}(r_j) \) depends only on one-body operator. On the other hand, \( \hat{A}^2 = \sum_{j=1}^{N} \hat{a}^2(r_j) + \sum_{j<k} 2\hat{a}(r_j)\hat{a}(r_k) \) consists of one-body and two-body operators. Consequently, the variance is expressed as

\[
\frac{1}{N} \Delta_{\hat{A}}^2(t) = \frac{1}{N} \left[ \langle \Psi(t)|\hat{A}^2|\Psi(t)\rangle - \langle \Psi(t)|\hat{A}|\Psi(t)\rangle^2 \right] 
= \frac{1}{N} \left\{ \sum_{j} n_j(t) \int \! dr \phi_j^*(r; t)\hat{a}^2(r)\phi_j(r; t) - \left[ \sum_{j} n_j(t) \int \! dr \phi_j^*(r; t)\hat{a}(r)\phi_j(r; t) \right]^2 \right\} 
+ \sum_{jpkq} \rho_{jpkq}(t) \left[ \int \! dr \phi_j^*(r; t)\hat{a}(r)\phi_k(r; t) \right] \left[ \int \! dr \phi_p^*(r; t)\hat{a}(r)\phi_q(r; t) \right] \right\},
\]

(S1)

where \{\phi_j(r; t)\} are the natural orbitals, \{n_j(t)\} the natural occupations, and \( \rho_{jpkq}(t) \) are the elements of the reduced two-particle density matrix,

\[
\rho(r_1, r_2, r_1', r_2'; t) = \sum_{jpkq} \rho_{jpkq}(t) \phi_j^*(r_1'; t)\phi_p^*(r_2'; t)\phi_k(r_1; t)\phi_q(r_2; t).
\]

(S2)

For one-body operators which are local in position space, the variance described in Eq S1 reduces to

\[
\frac{1}{N} \Delta_{\hat{A}}^2(t) = \int \! dr \frac{\rho(r; t)}{N} \hat{a}^2(r) - N \left[ \int \! \frac{\rho(r; t)}{N} \hat{a}(r) \right]^2 + \int \! dr_1 dr_2 \frac{\rho^{(2)}(r_1, r_2, r_1, r_2; t)}{N} \hat{a}(r_1)\hat{a}(r_2).
\]

(S3)

IX. MEAN-FIELD TRANSVERSAL POSITION VARIANCE OF THE INITIAL STATE

In the main text, we demonstrate the initial-state many-body transversal position variance as a function of the barrier height. Here, we investigate how the initial mean-field \( \frac{1}{N} \Delta_{\hat{Y}}^2 \) behaves with \( V_L \), see in Fig. S1. We observe that the mean-field \( \frac{1}{N} \Delta_{\hat{Y}}^2 \) monotonously increases with the barrier height which is unlike to the corresponding many-body result. Moreover, the mean-field \( \frac{1}{N} \Delta_{\hat{Y}}^2 \) is essentially independent of the interaction strength.
FIG. S1. Mean-field transversal position variance, $\frac{1}{N} \Delta_Y^2$, of the initial state as a function of the longitudinal barrier height ($V_L$) for two interaction strengths. The number of bosons is $N = 10$. We show here dimensionless quantities.

X. LONGITUDINAL AND TRANSVERSAL RABI PERIODS FOR THE FOUR-WELL SETUP

The tunneling of bosons, demonstrated in the main text, occurs for various barrier heights, $V_L$. Therefore, to compare our results, a natural choice of time scale is required. Fig. S2 provides the Rabi periods (in logarithmic scale) as a function of barrier height along the longitudinal and transversal directions, $t_{Rabi} = \frac{2\pi}{E_X - E_0}$ and $t'_{Rabi} = \frac{2\pi}{E_Y - E_0}$, respectively, where $E_0$ is the energy of the ground state, and $E_X$ and $E_Y$ represent the energies of the first excited states along the $x$- and $y$-directions, respectively. Here, the Rabi periods are calculated by diagonalizing the single-particle Hamiltonian using the discrete variable representation method. The trapping potential described in the main text, with the increasing barrier height, changes from a double-well to a four-well potential. As the one-body Hamiltonian is separable, it is noticed that $t_{Rabi}$ does not change with the barrier height with value 132.498 whereas $t'_{Rabi}$ monotonously grows. $t'_{Rabi}$ crosses $t_{Rabi}$ when $E_Y = E_X$ at $V_L \approx 8$. As we are interested in the tunneling dynamics along the longitudinal direction, we set the time-scale of the dynamics as $t/t_{Rabi}$.
XI. NO INTERFERENCES BETWEEN THE LONGITUDINAL AND TRANSVERSAL DEGREES-OF-FREEDOM WITHIN THE MEAN-FIELD DYNAMICS

We have seen in the main text that the amplitude of the many-body survival probability, $P(t)$, decays for all barrier heights due to development of the longitudinal fragmentation. Here we show the dynamics of $P(t)$ as if we would have investigated the tunneling phenomenon under mean-field theory. Fig. S3 depicts the survival probability in the left side of space for the inter-particle interactions $\Lambda_0$ and $10\Lambda_0$. Here, in the mean-field dynamics, we observe that $100\%$ of the particles tunnel back and forth between the left and right parts of space with practically the same frequency of oscillations as a function of the barrier height. Therefore, Fig. S3 exhibits that the mean-field survival probability essentially does not depend on the shape of the initial density structure of the ground states, barrier height, and the considered inter-bosons interaction strengths. This is in sharp distinction from the many-body dynamics. In other words, the mean-field dynamics show no interference between the transversal and longitudinal degrees-of-freedom. Hence, the simplicity of the mean-field dynamics may be used as a reference to define the interference of fragmentations at the many-body level of theory.

FIG. S2. The Rabi periods (in logarithmic scale) along the longitudinal and transversal directions are computed as a function of barrier height, $V_L$, for the four-well trapping potential in which the dynamics of the main text takes place. We show here dimensionless quantities.
FIG. S3. Time-dependent mean-field survival probability in the left side of space, \( P(t) \), for the interaction strengths (a) \( \Lambda_0 \) and (b) \( 10\Lambda_0 \). See the text for further discussion. We show here dimensionless quantities.

XII. MANY-BODY SURVIVAL PROBABILITY

The many-body survival probability in the left side of space are presented in Fig. S4 for the barrier heights \( V_L = 1, 7, 10, 12, \) and \( 16 \). In the main text, we presented the many-body survival probability for the mentioned barrier heights in a single plot to compare the decay rate of the survival probability and density collapse with respect to the barrier height, see Fig. 3(a) of the main text. Here, we separately plot the many-body survival probability for each barrier height to get a better visibility for the reader.

XIII. DETAILS OF THE MANY-BODY FRAGMENTATIONS

The main text shows that the occupation of the first natural orbital decreases with time, which implies the growing occupations of the higher natural orbitals. Now, we examine and compare the microscopic mechanism of how the higher natural orbitals, \( \frac{n_{j=2,3,4}(t)}{N} \), become populated as a function of \( V_L \). Figs. S5 and S6 depict the occupations of the most dominant higher natural orbitals, i.e., the second, third, and fourth natural orbitals for the interaction strengths \( \Lambda_0 \) and \( 10\Lambda_0 \), respectively.

Let us start with the discussion of Fig. S5. If one gradually moves from \( V_L = 1 \) to 6, the initial state continuously deforms yet maintaining its coherency. The deformation of the initial ground state delays the process of losing the coherence in the dynamics. Therefore, from \( V_L = 0 \) to \( V_L = 6 \), although all the higher natural orbitals become occupied with time, the rate of occupancy of the second natural orbital is faster for \( V_L = 0 \) compared to \( V_L = 6 \). From \( V_L = 0 \) to 6, the
FIG. S4. Time-dependent many-body survival probability in the left side of space, $P(t)$, for the interaction strength $\Lambda_0$ and number of bosons $N = 10$. The barrier heights are (a) $V_L = 0$, (b) $V_L = 7$, (c) $V_L = 10$, (d) $V_L = 12$, and (e) $V_L = 16$. For comparison in a single plot, we refer Fig. 3(a) of the main text. See the text for further discussion. We show here dimensionless quantities.

second, third, and fourth natural orbitals are excited $g$-orbital, $u$-orbital, and excited $u$-orbital, respectively, discussed in the main text. The order of populations of the orbitals is preserved throughout the dynamics until $V_L = 6$, i.e., excited $g$-orbital $> u$-orbital $> \text{excited } u$-orbital.

Remarkably, at the beginning of the dynamics found for $V_L \geq 7$, the second natural orbital loses its coherence along with the first natural orbital. Here, the second, third, and fourth natural
orbitals are $u$-orbital, excited $g$-orbital, and excited $u$-orbital, respectively. The loss of coherence of the $u$-orbital occurs due to its sufficient initial occupation at $V_L \geq 7$ and it mimics the trend of the $g$-orbital. This loss of coherence of the $u$-orbital is, of course, a purely many-body phenomenon and it occurs only for the fragmented ground state (also discussed in the main text). If the ground state is initially more fragmented, depending on the barrier height, the $u$-orbital follows the pattern of loss of coherence for longer times in the process of tunneling. The $u$-orbital loses its coherence until the moment in time when there is a swapping of orders of populations between two higher natural orbitals. At $V_L = 9$ and 10, we observe that the $u$-orbital and excited $g$-orbital exchange their orders at $t = 0.76$ and 2.11, respectively, and afterwards, the $u$-orbital builds up coherence, see the insets of Figs. S5 (c) and (d). This build up of coherence in the $u$-orbital defines the interference of the longitudinal and transversal fragmentations, see the main text.

At $V_L = 11$, the microscopic mechanism of the fragmentation becomes even richer. Here we find that the build up of coherence in the $u$-orbital is accompanied by the swapping of orders of populations of excited $g$-orbital and excited $u$-orbital at $t = 4.56$. Further, around $t = 17$, the $u$-orbital and excited $u$-orbital exchange their orders three times and afterwards, the $u$-orbital builds up coherence. At $V_L = 12$, the build up of coherence in the $u$-orbital occurs when the excited $g$-orbital and excited $u$-orbital swap their orders. Also, similar to the fragmentation mechanism found at $V_L = 9$ and 10, the build up of coherence in the $u$-orbital is accompanied by a swapping of the orders of the $u$-orbital and excited $g$-orbital.

At $V_L = 16$, as the initial state is essentially fully fragmented and there is no coupling between the longitudinal and transversal fragmentations, the $u$-orbital retains the trend of loss of coherence, also see Fig. 2(d) of the main text. Accordingly, the populations of the excited $g$-orbital and excited $u$-orbital monotonously grow. All in all, the interference of the longitudinal and transversal fragmentations occurs when (i) the $u$-orbital has sufficient initial occupancy, (ii) it loses its coherency in the initial dynamical evolution, and then (iii) it builds up its coherency. We find that the interference is maximal at $V_L = 12$ for the considered inter-boson interaction $\Lambda_0$.

Now, we discuss the time-evolution of $\frac{\sum_{j=2,3,4} N_j(t)}{N}$ for the stronger inter-boson interaction $10\Lambda_0$, see Fig. S6, and find how it leads to a qualitatively richer microscopic mechanism of interference of fragmentations found for the weaker interaction $\Lambda_0$. We observed for the weaker interaction that, whenever the $u$-orbital breaks its natural trend from loss of coherence to build up of coherence, the $u$-orbital and excited $g$-orbital or the excited $g$-orbital and excited $u$-orbital interchange their orders. Here, for $10\Lambda_0$, we find that the orders of the orbitals interchange multiple times, for example see for $V_L = 10$ and $V_L = 12$, see Fig. S6. Although at the smaller barrier height, say at
$V_L = 9$, the u-orbital and excited g-orbital exchange their orders only once, but for $V_L = 10$, the u-orbital breaks its trend twice from loss of coherence to build up of coherence, at $t = 0.66t_{Rabi}$ and $3.60t_{Rabi}$. For the former time, the excited g-orbital and excited u-orbital, and for the latter time, the u-orbital and excited u-orbital interchange their orders. Remarkably, at $V_L = 12$, the u-orbital and excited g-orbital interchange their orders twelve times between $t = 3t_{Rabi}$ and $6t_{Rabi}$. Therefore, for strong interaction, we observe that the u-orbital breaks its trend of loss of coherence to build up of coherence multiple times due to the interchange of order by one of three possibilities: either the u-orbital and excited g-orbital, or the excited g-orbital and excited u-orbital, or the u-orbital and excited u-orbital. Here we find that the transition from loss of coherence to build up of coherence or vice versa is always accompanied by the interchange of orders of higher natural orbitals until the revival takes place. In the two extreme barrier heights considered, at $V_L = 1$, the u-orbital shows transition from build up of coherence to loss of coherence and, at $V_L = 16$, from loss of coherence to build up of coherence, due to the pure effect of the revival process.

Now, we determine at which barrier height the interference of longitudinal and transversal fragmentations is maximal for the inter-boson interaction $10\Lambda_0$, see Fig. S7. The interference is calculated from the difference between maximal occupancy of the u-orbital after building up of coherence (but before the revival process) and its minimal occupancy. We find that at the intermediate barrier heights, from $V_L = 7$ to $V_L = 12$, the interference of longitudinal and transversal fragmentations take place and, for $10\Lambda_0$, the maximal interference occurs at $V_L = 10$. Therefore, with the increase of inter-boson interaction, the maximal interference appears at the lower barrier height, compare to Fig. 2(c) of the main text.
FIG. S5. Details and mechanisms of the fragmentations (weak interaction). Time evolution of the occupation numbers per particle of the higher natural orbitals, \( \frac{n_j}{N} \), for different barrier heights. The inter-boson interaction is \( \Lambda_0 \) and the number of bosons \( N = 10 \). The insets of panels (a) to (g) magnify the same plots. In the insets of panels (b) to (g), we also present swapping of orders of the orbitals. This swapping of orbitals happens when the \( u \)-orbital shows a transition from loss of coherence to build up of coherence. To guide the eye, we mark the time of swapping of orders of the orbitals with a blue circle. Color codes are explained in panels (a) and (b). We show here dimensionless quantities.
FIG. S6. Details and mechanisms of the fragmentations (stronger interaction). Time evolution of the occupation numbers per particle of the higher natural orbitals, $n_{j=2,3,4}(t)/N$, for different barrier heights. The inter-boson interaction is $10\Lambda_0$ and the number of bosons $N = 10$. The insets of panels (a) to (c) magnify the same plots. In the inset of panel (b), we present swapping of orders of the orbitals. This swapping of orders happens when the $u$-orbital shows a transition from loss of coherence to build up of coherence. To guide the eye, we mark the time of swapping of orbitals with a blue circle. Color codes are explained in panel (a). We show here dimensionless quantities.
FIG. S7. Interference of longitudinal and transversal fragmentations. Interference of fragmentations requires three conditions to be satisfied in time, and they are (a) initially fragmented ground state, (b) loss of coherence in the $u$-orbital during initial dynamics, and then (c) build up of coherence in the $u$-orbital. The magnitude of interference is defined by the difference between maximal occupancy of the $u$-orbital after building up of coherence and its minimal occupancy. For the barrier height $V_L < 7$, the interference slowly decreases and eventually becomes zero when the $u$-orbital has essentially no initial occupancy. The inter-boson interaction is $10\Lambda_0$ and the number of bosons $N = 10$. We show here dimensionless quantities.
XIV. DYNAMICS OF LONGITUDINAL AND TRANSVERSAL POSITION VARIANCES

A. Mean-field dynamics

Here we start our discussion with the position variance and use it to analyze: (i) how ground states of different shapes tunnel when the fragmentation is not considered in the system and (ii) the Josephson tunneling dynamics in the mean-field limit. Fig. S8 records the mean-field longitudinal and transversal position variances, $\frac{1}{N}\Delta_{X}^{2}(t)$ and $\frac{1}{N}\Delta_{Y}^{2}(t)$, respectively, for the inter-particle interaction $\Lambda_{0}$. Fig. S8 (a) shows that the mean-field dynamics of $\frac{1}{N}\Delta_{X}^{2}(t)$ is practically independent of the barrier height, $V_{L}$. Whereas the mean-field $\frac{1}{N}\Delta_{X}^{2}(t)$ essentially does not depend on the barrier height for the time durations considered here, $\frac{1}{N}\Delta_{Y}^{2}(t)$ monotonously increases with $V_{L}$ having almost a frozen dynamical behavior (fluctuation around $10^{-3}$). This monotonous increase of the base value of $\frac{1}{N}\Delta_{Y}^{2}(t)$ originates due to the initial deformation in the ground state, see Fig. S1. For the increased interaction strength $10\Lambda_{0}$, we find that $\frac{1}{N}\Delta_{X}^{2}(t)$ and $\frac{1}{N}\Delta_{Y}^{2}(t)$ show practically the same dynamical behavior as compared to their corresponding results found for $\Lambda_{0}$. Therefore, it suggests that the regime of the inter-boson interaction strength considered in this work does not have any qualitative impact on the mean-field dynamics of the longitudinal and transversal position variances.

![Figure S8](image-url)

**FIG. S8.** Time-dependent many-particle position variances per particle along the longitudinal and transversal directions, $\frac{1}{N}\Delta_{X}^{2}(t)$ and $\frac{1}{N}\Delta_{Y}^{2}(t)$, respectively, for different barrier heights. The inter-boson interaction is $\Lambda_{0}$. The results are obtained by applying the mean-field theory. The results for $10\Lambda_{0}$ essentially overlap those for $\Lambda_{0}$ and are not plotted. Color codes are explained in the left panel. We show here dimensionless quantities.
B. Many-body dynamics

As the main text demonstrates the dynamics of the many-body transversal position variance for the interaction strength $\Lambda_0$, here we discuss the impact of the interference of longitudinal and transversal fragmentations on the many-body dynamics of the longitudinal position variance, $\frac{1}{N}\Delta_X^2(t)$, see Fig. S9. As a general feature, we find that the many-body $\frac{1}{N}\Delta_X^2(t)$ oscillates with a growing amplitude and the rate of growth varies depending on $V_L$. In comparison with the corresponding mean-field dynamics, the maximal deviation of the many-body $\frac{1}{N}\Delta_X^2(t)$ occurs at $V_L = 1$ and the minimal at $V_L = 12$. We find that the growth of the many-body $\frac{1}{N}\Delta_X^2(t)$ maintains its order from $V_L = 1$ to $V_L = 10$ which is consistent with the many-body $P(t)$ and the dynamics of the normalized occupation number, $\eta(t)$, discussed in the main text. For example, if we consider the barrier height until $V_L = 10$, the many-body $\frac{1}{N}\Delta_X^2(t)$ shows maximal deviation at $V_L = 7$ and minimal at $V_L = 10$ in comparison with the respective many-body dynamics at $V_L = 1$, which is consistent with $P(t)$ and the normalized occupation number. By examining Fig. S9, we notice that the saturation value of the many-body $\frac{1}{N}\Delta_X^2(t)$ is almost double for the fully condensed state (at $V_L = 1$) compared to the corresponding saturation value for the fully fragmented state (at $V_L = 16$). Moreover, the saturation value of the many-body $\frac{1}{N}\Delta_X^2(t)$ is minimal when the interference of the longitudinal and transversal fragmentations is maximal. Therefore, by analyzing the many-body $\frac{1}{N}\Delta_X^2(t)$ and $\frac{1}{N}\Delta_Y^2(t)$ (in the main text), we observe that the interference of the longitudinal and transversal fragmentations hinders attaining a higher saturation value for $\frac{1}{N}\Delta_X^2(t)$ and helps to increase the amplitude of oscillations of $\frac{1}{N}\Delta_Y^2(t)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure_s9}
\caption{Time-dependent many-body longitudinal position variance per particle, $\frac{1}{N}\Delta_X^2(t)$, for different barrier heights. The inter-boson interaction is $\Lambda_0$ and the number of bosons $N = 10$. See the text for further discussion. We show here dimensionless quantities.}
\end{figure}
Now we would demonstrate the many-body longitudinal and transversal position variances for the interaction strength $10\Lambda_0$, see Fig. S10 (a) and (b), respectively, and provide a further connection with the respective many-body survival probability, $P(t)$, and normalized loss of coherence, $\eta(t)$, discussed in the main text. From Fig. S10 (a), we find that the many-body dynamics of $\frac{1}{N}\Delta_X^2(t)$ gradually increases in an oscillatory manner due to the development of longitudinal fragmentation and eventually it saturates until the revival takes place, see corresponding survival probability plot in Fig.4(a) of main text. The saturation value of $\frac{1}{N}\Delta_X^2(t)$ is maximal for $V_L = 1$, when the initial state is fully condensed, and minimal for $V_L = 10$ when the interference of the longitudinal and transversal fragmentations is maximal. The signature of revival shown in the many-body dynamics of $\frac{1}{N}\Delta_X^2(t)$ complements the time-evolution of the many-body $P(t)$ and normalized loss of coherence, $\eta(t)$, see Fig. 4 of main text. It is found that the revival in the many-body dynamics of $\frac{1}{N}\Delta_X^2(t)$ takes place faster when the system is initially fully fragmented and there is essentially no interference between the longitudinal and transversal fragmentations in the process of tunneling, i.e., at $V_L = 16$. Moreover, the process of revival is delayed when the maximal interference of fragmentations occurs.

FIG. S10. Time-dependent many-body position variances per particle along the longitudinal and transversal directions, $\frac{1}{N}\Delta_X^2(t)$ and $\frac{1}{N}\Delta_Y^2(t)$, respectively, for different barrier heights. The inter-boson interaction is $10\Lambda_0$ and the number of bosons $N = 10$. See the text for further discussion. We show here dimensionless quantities.

Fig. S10 (b) records the many-body dynamics of $\frac{1}{N}\Delta_Y^2(t)$. For the lowest barrier height $V_L = 1$, when the initial ground state is fully condensed, the dynamics of $\frac{1}{N}\Delta_Y^2(t)$ is found to be practically frozen. As the interference of the longitudinal and transversal fragmentations takes place, we observe oscillatory nature in the dynamics of $\frac{1}{N}\Delta_Y^2(t)$, see for $V_L = 10$. Compared to the weak inter-boson interaction, here we find that the period of oscillations of $\frac{1}{N}\Delta_Y^2(t)$ changes with time.
due to the combined effect of transversal many-body dynamics and breathing-motion frequencies. The maximal fluctuations of $\frac{1}{N}\Delta^2_Y(t)$ with a growing amplitude is observed for $V_L = 10$ which goes hand in hand with the maximal coupling of the longitudinal and transversal fragmentations. As we increase the barrier height $V_L > 10$, the interference of the longitudinal and transversal fragmentations gradually becomes smaller and the oscillations of $\frac{1}{N}\Delta^2_Y(t)$ slowly decay. Eventually, at $V_L = 16$, we observe essentially constant dynamical behavior due to the practically null interference of the longitudinal and transversal fragmentations. Therefore, we come to the conclusion that the detailed investigation of the many-body dynamics of $\frac{1}{N}\Delta^2_Y(t)$, being a sensitive probe of correlations, encodes the interference of longitudinal and transversal fragmentations.

XV. ROBUSTNESS OF THE RESULTS TO THE WIDTH OF THE INTER-BOSON INTERACTION POTENTIAL

In the main text, we have made a detailed investigation on the physics of interference of the longitudinal and transversal fragmentations in the tunneling process in two spatial dimensions. For our study, we have used for the inter-boson interaction a Gaussian model potential of finite width, $\sigma = 0.25\sqrt{\pi}$. In order to demonstrate the robustness of our results, we recomputed all the properties discussed in this work for two additional smaller widths, $\sigma = 0.25$ and $\sigma = 0.25/\sqrt{\pi}$. Note that we purposely do not use the delta-function to model the inter-boson interaction as the delta-function in two-dimensions does not scatter.

Here, we begin with the discussion of static properties, i.e., the loss of coherence and transversal position variance as a function of the longitudinal barrier height for the widths of the Gaussian model potential $\sigma = 0.25\sqrt{\pi}$, 0.25, and 0.25/√π, see Fig. S11. We find that the qualitative physics of the static properties, presented in this work, does not depend on $\sigma$. Moreover, the difference between the results, found for $\sigma = 0.25$ and 0.25/√π, is smaller compared to the respective difference obtained for $\sigma = 0.25\sqrt{\pi}$ and 0.25. This suggests that decreasing of the width, $\sigma$, slowly reduces the quantitative difference of a particular quantity, at least for the properties examined here.

To present the robustness of the width, $\sigma$, at the many-body dynamics, we select the dynamics of the ground state at $V_L = 12$ when the interference of the longitudinal and transversal fragmentations is maximal. Fig. S12 depicts the many-body dynamics of loss of coherence, $\frac{n_1(t)}{N}$, occupations of the second and third natural orbitals, $\frac{n_2(t)}{N}$ and $\frac{n_3(t)}{N}$, respectively, longitudinal position variance, $\frac{1}{N}\Delta^2_X(t)$, and transversal position variance, $\frac{1}{N}\Delta^2_Y(t)$, at $V_L = 12$ for the widths $\sigma = 0.25\sqrt{\pi}$,
FIG. S11. (a) Loss of coherence and (b) transversal position variance, \( \frac{1}{N} \Delta_x^2 (t) \), for the three different widths of the inter-boson interaction potential, i.e., \( \sigma = 0.25\sqrt{\pi} \), 0.25, and 0.25/\( \sqrt{\pi} \), as a function of the longitudinal barrier height \( V_L \). The inter-boson interaction is \( \Lambda_0 \) and the number of bosons \( N = 10 \). We show here dimensionless quantities.

0.25, and 0.25/\( \sqrt{\pi} \). First, it is found that \( \frac{n_1(t)}{N} \) is the \( g \)-orbital throughout the tunneling process for all \( \sigma \). Also, we observe that the \( g \)-orbital loses coherence comparatively quicker (slower) for \( \sigma = 0.25/\sqrt{\pi} \) (\( \sigma = 0.25\sqrt{\pi} \)) which implies faster (slower) development of fragmentation as shown in Fig. S12 (a). The unchanged qualitative feature of the occupancy of the \( g \)-orbital suggests its robustness with the width of the inter-boson interaction.

Fig. S12 (b) presents the occupation of the second and third natural orbitals. For \( \sigma = 0.25\sqrt{\pi} \), discussed in the main text, the second and third natural orbitals remain as \( u \)-orbital and excited \( g \)-orbital, and the build up of coherence in the \( u \)-orbital manifests the interference of longitudinal and transversal fragmentations. Here, also for the widths \( \sigma = 0.25 \) and 0.25/\( \sqrt{\pi} \), we observe a build up of coherence in the \( u \)-orbital with the faster rate for the width \( \sigma = 0.25/\sqrt{\pi} \). It is found that the qualitative features of occupations of the second and third natural orbitals are the same for all \( \sigma \) until a swapping between the orders of \( u \)-orbital and excited \( g \)-orbital happens for \( \sigma = 0.25 \) and 0.25/\( \sqrt{\pi} \). This swapping of orders of the orbitals would also happen for \( \sigma = 0.25\sqrt{\pi} \) if one would compute the dynamics for a longer time than 30 Rabi Cycles. Therefore, Fig. S12 (b) exhibits the robustness of the interference of longitudinal and transversal fragmentations to the width of the inter-boson interaction potential.

As the variances are sensitive probe of correlations, we find here that the developed fragmentation in the system only quantitatively influences the many-body \( \frac{1}{N} \Delta_X^2 (t) \) and \( \frac{1}{N} \Delta_Y^2 (t) \) for different \( \sigma \). Also, Figs. S12 (c) and (d) attribute that decreasing the width \( \sigma \) of the inter-boson interaction potential gradually diminishes the quantitative difference between the dynamics of a given quantity
FIG. S12. Dependence of the many-body dynamics of the initially-prepared ground state at the barrier height $V_L = 12$ on the three different widths of the inter-boson interaction potential, i.e., $\sigma = 0.25\sqrt{\pi}$, $\sigma = 0.25$, and $\sigma = 0.25/\sqrt{\pi}$. The dynamics is represented by the (a) loss of coherence, (b) occupation of the second and third natural orbitals, (c) longitudinal position variance, and (d) transversal position variance. The inter-boson interaction is $\Lambda_0$ and the number of bosons $N = 10$. In panel (b), the solid and dashed lines represent the occupation of the second, $n_2(t)/N$, and third, $n_3(t)/N$, natural orbitals, respectively. We show here dimensionless quantities.

for the studies made here. All in all, we observe that the width of the inter-boson interaction potential does not qualitatively impact the many-body physics of the interference of fragmentations we focused on and discussed in this work.

Further, we have computed all the quantities for each of the barrier heights for the inter-boson interaction strength $10\Lambda_0$ with the two additional smaller widths, $\sigma = 0.25$ and $0.25/\sqrt{\pi}$. We have checked and verified that the qualitative physics of the dynamics of the different fragmented states is independent of $\sigma$ for the stronger interaction $10\Lambda_0$ as well, and thus they are not shown graphically.

Finally, in the mean-field dynamics, we notice that the time-evolution of all the quantities
obtained for $\sigma = 0.25$ and $\sigma = 0.25/\sqrt{\pi}$ fall on top of the respective results computed for $\sigma = 0.25\sqrt{\pi}$ (not shown). Therefore, the width of the inter-boson interaction does essentially not influence the mean-field dynamics presented in this work, which is interesting for itself.

**XVI. ROBUSTNESS OF THE RESULTS TO THE FORM OF THE BOSON-BOSON INTERACTION: THE CASE OF DIPOLAR INTERACTION**

This section demonstrates the robustness of the interference of longitudinal and transversal fragmentations to the form of the boson-boson interaction. In the main text and throughout the supplemental material, the boson-boson interaction is chosen as a Gaussian form. Here, we investigate the interference of fragmentations for a different shape of the interaction, say, the dipolar interaction which can be written mathematically as, $W(r_j - r_k) = \frac{\lambda_0}{|r_j - r_k|^3 + \Delta^3}$, where $\Delta^3 = 0.07$ is the threshold of the long-range interaction [3]. Such long-range interaction is relevant for atomic clouds made of Cr, Dy, or Er. Here also, $\lambda_0$ is the interaction strength which is related to the interaction parameter $\Lambda_0 = \lambda_0(N - 1)$. Now, we consider the same number of bosons and interaction strength as described in Figure 2(c) and 2(d) of the manuscript. Fig. S13 presents the occupation of the first and second natural orbitals for the barrier heights $V_L = 12$ and $V_L = 16$. We observe that the dynamics of $\frac{n_1(t)}{N}$ and $\frac{n_2(t)}{N}$ are qualitatively the same as described in the main text. For intermediate height of $V_L = 12$, here also we find loss of coherence followed by build up of coherence in the second natural orbital. This investigation exhibits the robustness of the interference of fragmentations to the form of the inter-boson interaction.

![Figure S13](image)

**FIG. S13.** Dynamics for dipolar interaction. Occupations of the first and second natural orbitals, $\frac{n_1(t)}{N}$ and $\frac{n_2(t)}{N}$, respectively, for (a) $V_L = 12$ and (b) $V_L = 16$. The inter-boson interaction is $\Lambda_0$ and the number of bosons $N = 10$. Robustness of the interference of fragmentations to the form of the inter-boson interaction is found. We show here dimensionless quantities.
XVII. DETAILS OF CONVERGENCES OF QUANTITIES IN THE TUNNELING DYNAMICS

The multiconfigurational time-dependent Hartree for bosons (MCTDHB) method is used in the present work to compute the ground (initial) state which eventually becomes fragmented depending on the barrier height $V_L$. To compute the ground state and its subsequent real-time propagation, the many-body Hamiltonian is represented by $128^2$ exponential discrete-variable-representation grid points in a box of size $[-10, 10) \times [-10, 10)$. In the main text, we calculated the many-body quantities with $M = 8$ time-adaptive orbitals. Here we provide the convergences of the quantities discussed in this work, i.e., we show that the time-dependent many-boson wavefunction built from $M = 8$ time-adaptive orbitals leads to numerically converged results. We have verified the convergences of the quantities with $M = 10$ time-adaptive orbitals using $128^2$ exponential discrete-variable-representation grid points and also with $M = 8$ time-adaptive orbitals using increased grid density of $256^2$ exponential discrete-variable-representation grid points for all the barrier heights and the two interaction strengths $\Lambda_0$ and $10\Lambda_0$. In order to demonstrate the convergences, we select the barrier height $V_L = 13$, the maximal barrier height for which the interference of fragmentations is appreciable, for the interaction strength $\Lambda_0$.

Fig. S14 represents the occupations of the first, second, third, and fourth natural orbitals, longitudinal position variance, $\frac{1}{N}\Delta^2_\hat{X}(t)$, and transversal position variance, $\frac{1}{N}\Delta^2_\hat{Y}(t)$, for $V_L = 13$. The overlapping curves for all quantities with increasing grid density and number of orbitals signify that the results presented in this work are very well converged for $M = 8$ time-adaptive orbitals with $128^2$ exponential discrete-variable-representation grid points in a box of size $[-10, 10) \times [-10, 10)$. 
FIG. S14. Convergences of the (a) occupations of the first and second natural orbitals, (b) occupations of the third and fourth natural orbitals, (c) many-body longitudinal position variance, and (d) many-body transversal position variance at the barrier height $V_L = 13$ with the number of orbitals and grid points. In panel (a) [(b)], the solid lines represent the first [third] natural orbital and the dashed lines depict the second [fourth] natural orbital. The inter-boson interaction is $\Lambda_0$ and the number of bosons $N = 10$. Convergences are verified using $M = 10$ time-adaptive orbitals with $128 \times 128$ grid points and $M = 8$ time-adaptive orbitals with $256 \times 256$ grid points. We show here dimensionless quantities.
[1] O. E. Alon, Analysis of a Trapped Bose-Einstein Condensate in Terms of Position, Momentum, and Angular-Momentum Variance. Symmetry 11, 1344 (2019).

[2] A. U. J. Lode, C. Lévéque, L. B. Madsen, A. I. Streltsov, and O. E. Alon, Colloquium: Multiconfigurational time-dependent Hartree approaches for indistinguishable particles. Rev. Mod. Phys. 92, 011001 (2020).

[3] S. Dutta, M. C. Tsatsos, S. Basu, and A. U. J. Lode, Management of the correlations of Ultracold Bosons in triple wells. New Journal of Physics 21, 053044 (2019).