A new result for boundedness in the
quasilinear parabolic-parabolic Keller-Segel
model (with logistic source)

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Abstract

The current paper considers the boundedness of solutions to the following quasilinear Keller-Segel model (with logistic source)

\[
\begin{align*}
\frac{u_t}{u_t} &= \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u \nabla v) + \mu (u - u^2), \quad x \in \Omega, t > 0, \\
\frac{v_t}{v_t} - \Delta v &= u - v, \quad x \in \Omega, t > 0, \\
(D(u) \nabla u - \chi u \cdot \nabla v) \cdot \nu &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 1) \) is a bounded domain with smooth boundary \( \partial \Omega, \chi > 0 \) and \( \mu \geq 0 \). We prove that for nonnegative and suitably smooth initial data \((u_0, v_0)\), if \( D(u) \geq C_D(u + 1)^{m-1} \) for all \( u \geq 0 \) with some \( C_D > 0 \) and some \( m > 2 - \frac{2}{N} \frac{\chi \max\{1, \lambda_0\}}{\chi \max\{1, \lambda_0\} - \mu} \) or \( m = 2 - \frac{2}{N} \) and \( C_D > \frac{C_{GN}(1+\|u_0\|_{L^1(\Omega)})}{3} (2 - \frac{2}{N})^2 \max\{1, \lambda_0\} \chi \), the \((KS)\) possesses a global classical solution which is bounded in \( \Omega \times (0, \infty) \), where \( C_{GN} \) and \( \lambda_0 \) are the constants which are corresponding to the Gagliardo-Nirenberg inequality (see Lemma 2.2) and the maximal Sobolev regularity (see Lemma 2.3). One novelty of this paper is that we use the Maximal Sobolev regularity approach to find a new a-priori estimate.

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\[ \int_{\Omega} u^{x_{\max}(1, \lambda_0)} \chi_{\max} (1, \lambda_0) \mu_0^{-\varepsilon} (x, t) dx \quad \text{(for all } \varepsilon, t > 0 \text{ and } \mu > 0, \text{ see Lemma 3.4)} \]

so that we develop new \( L^p \)-estimate techniques and thereby obtains the boundedness results. To our best knowledge, this seems to be the first rigorous mathematical result which (precisely) gives the relationship between \( m \) and \( \frac{\mu}{\lambda} \) that yields to the boundedness of the solutions. These results significantly improve or extend previous results of several authors.

**Key words:** Boundedness; Chemotaxis; Keller-Segel; Parabolic-parabolic Logistic source; Nonlinear diffusion

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1 Introduction

The motion of cells moving towards the higher concentration of a chemical signal is called chemotaxis. In 1970, a classical mathematical model for chemotaxis was proposed by [11], which is called the classical Keller-Segel model. In fact, let $u, v$ and $\chi > 0$, respectively, denote the cell density, the chemo-attractant and the chemotactic sensitivity. Hillen and Painter ([6]) introduced the following model

$$
\begin{align*}
&u_t = \nabla \cdot (D(u) \nabla u) - \chi \nabla \cdot (u \nabla v), &x \in \Omega, t > 0, \\
v_t = \Delta v - v + u, &x \in \Omega, t > 0,
\end{align*}
$$

(1.1)

which is a generalization of the classical Keller-Segel chemotaxis model, where the diffusion term $D(u)$ is a (nonlinear) nonnegative function which satisfies

$$
D \in C^2([0, \infty))
$$

(1.2)

and

$$
D(u) \geq C_D(u + 1)^{m-1} \text{ for all } u \geq 0
$$

(1.3)

with $m \in \mathbb{R}$ and $C_D > 0$. During the past four decades, the quasilinear Keller-Segel model (1.1) has attracted more and more attention, and also has been constantly modified by various authors to characterize more biological phenomena. The main issue of the investigation was whether the solutions of the models (1.1) are bounded or blow-up (see e.g. Burger et al. [2], Calvez and Carrillo [3], Cieślak et. al. [5, 4], Laurençot and Mizoguchi [12], Winkler et al. [28, 26, 11, 8], Horstmann [7]). In fact, as we all know that $m = 2 - \frac{2}{N}$ has been uniquely detected to be the critical blow-up exponent for (1.1) in higher space dimensions $N \geq 2$. For instance, if $m > 2 - \frac{2}{N}$, then all solutions of (1.1) are global and uniformly bounded [17, 30], whereas if $m < 2 - \frac{2}{N}$, (1.1) possess some solutions which blow up in finite time (see Winkler et. al. [11, 23]). From the above analysis we know that the large exponent $m$ ($> 2 - \frac{2}{N}$) benefits the boundedness of solutions. We should pointed that the idea of [17] relying on the boundedness of $\int_{\Omega} (u(x, t) + |\nabla v|^{\gamma_0})dx$ (with $\gamma_0 < \frac{N}{N-1}$) and the core step is to establish the estimates of the functional

$$
y(t) := \int_{\Omega} u^p(\cdot, t) + \int_{\Omega} |\nabla v(\cdot, t)|^{2q} \text{ for any } p > 1 \text{ and } q > 1, t \geq 0.
$$

(1.4)
However, the method seems not be used to solve the case $m = 2 - \frac{2}{N}$ (see the proof of Lemma 3.3 in [17]), and so, if $m = 2 - \frac{2}{N}$, we should find other method to deal with it.

Apart from the aforementioned system, in order to describe the death and proliferation of cells, a source of logistic type $\mu(u - u^2)$ is included in (1.1). In this paper, we consider the following the quasilinear Keller–Segel system with the logistic source

\begin{align*}
    u_t &= \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u \nabla v) + \mu(u - u^2), \quad x \in \Omega, \ t > 0, \\
    v_t &= \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
    (D(u)\nabla u - \chi u \cdot \nabla v) \cdot \nu &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
    u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}

(1.5)

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial \Omega$. In the last decade, much attention has been devoted to studying the type of model (1.5) and its variations. (see Xiang [31], Tello and Winkler [18], Zheng et al. [17, 48], Zheng et al. [33, 34, 36, 39]). And global existence, boundedness, asymptotic behavior and blow-up of solution were studied in [13, 15, 19, 21, 27, 35, 46, 23, 29, 40, 47, 48, 42, 49]. In fact, if $D(u) \equiv 1$, it is known that arbitrarily small $\mu > 0$ guarantee the global existence and boundedness of solutions for (1.5) when $N = 2$ ([16]), and that appropriately large $\mu$ precludes blow-up in the case $N \geq 3$ ([24]). Thus, the large $\mu$ also benefits the boundedness of solutions. The question how far such systems (1.5) at all are globally solvable when $N \geq 3$ and $\mu > 0$ is small remains completely open. Connected to the above analysis, it is a natural question to ask:

Can we provide an explicit condition involving the exponent $m$ of nonlinear diffusion, the coefficient $\chi$ of chemosensitivity and coefficient $\mu$ of the logistic source to ensure global bounded solutions in the system (1.5)?

This article presents a relationship between the constant $\chi$ of the chemosensitivity as well as the coefficient $\mu$ of logistic source and the diffusion exponent $m$ which implies the boundedness of (1.5). To the best of our knowledge, this is the first result which gives the clear and definite relationship between $m$ and $\frac{\mu}{\chi}$ that yields to the boundedness of the solution. Our main result is the following:

**Theorem 1.1.** Assume that $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\infty}(\bar{\Omega})$ both are nonnegative, $D$ satisfies (1.2)–(1.3). If one of the following cases holds:
\( (i) \ m > 2 - \frac{2}{N} \frac{\chi \max\{1, \lambda_0\}}{\chi \max\{1, \lambda_0\} - \mu_+} \);

\( (ii) \ m = 2 - \frac{2}{N} \) and \( C_D > \frac{C_{GN}(1 + \|u_0\|_{L^1(\bar{\Omega})})(2 - \frac{2}{N})^2 \max\{1, \lambda_0\}}{\chi} \) if \( \mu > 0 \) and \( \mu > 2 - \frac{2}{N} \frac{\chi \max\{1, \lambda_0\}}{\chi \max\{1, \lambda_0\} - \mu_+} \);

then there exists a pair \((u, v)\) which solves (1.5) in the classical sense, where \( C_{GN} \) and \( \lambda_0 \) are the constants which are corresponding to the Gagliardo–Nirenberg inequality (see Lemma 2.2) and the maximal Sobolev regularity (see Lemma 2.3). Moreover, both \( u \) and \( v \) are bounded in \( \Omega \times (0, \infty) \).

By Theorem 1.1 we derive the following Corollary:

**Corollary 1.1.** Assume that \( u_0 \in C^0(\bar{\Omega}) \) and \( v_0 \in W^{1, \infty}(\bar{\Omega}) \) both are nonnegative, \( D \) satisfies (1.2)–(1.3). If \( \mu > 0 \) and \( \mu > 2 - \frac{2}{N} \frac{\chi \max\{1, \lambda_0\}}{\chi \max\{1, \lambda_0\} - \mu_+} \), then (1.5) possesses a global classical solution \((u, v)\) which is bounded in \( \Omega \times (0, \infty) \).

**Remark 1.1.**

(i) If \( \mu > \frac{(N-2)_+}{N} \chi \max\{1, \lambda_0\} \), then \( 2 - \frac{2}{N} \frac{\chi \max\{1, \lambda_0\}}{\chi \max\{1, \lambda_0\} - \mu_+} < 1 \), then, Theorem 1.1 is improves the result of Zheng et. al. (11).

(ii) Obviously, for any \( \mu > 0 \), then \( 2 - \frac{2}{N} \frac{\chi \max\{1, \lambda_0\}}{\chi \max\{1, \lambda_0\} - \mu_+} < 2 - \frac{2}{N} \), therefore, Corollary 1.1 partly improves the results of [20], [31] and [32], respectively.

(iii) If \( \mu > \frac{(N-2)_+}{N} \chi \max\{1, \lambda_0\} \) and \( D(u) \equiv 1 \), hence, Corollary 1.1 extends the results of Winkler ([24]), who proved the possibility of boundness, in the cases \( \mu > 0 \) is sufficiently large, and with \( \Omega \subset \mathbb{R}^N \) is a convex bounded domains.

(iv) If \( \mu > \chi \max\{1, \lambda_0\} \), then for any \( m \in \mathbb{R} \), then problem (1.5) admits a global classical solution \((u, v)\) which is bounded in \( \Omega \times (0, \infty) \).

(v) As far as we know that this is the first result which gives certainly relationship between \( m \) and \( \frac{\mu}{\chi} \) that yields to boundedness of the solution.

(vi) Theorem 1.1 asserts that, as in the corresponding two-dimensional Keller-Segel system (see Osaki et al. [16]), even arbitrarily small quadratic degradation of cells (for any \( \mu > 0 \)) is sufficient to rule out blow-up and rather ensure boundedness of solutions.

(vii) The idea of the paper can also be solved other type of the models, e.g., chemotaxis-haptotaxis model (with nonlinear chemosensitivity) (see [43]), parabolic-elliptic Keller-Segel model (with logistic source) (see [14]), Keller-Segel–Stokes system with nonlinear diffusion (and logistic source) (see [45]).
(viii) It concludes from Theorem 1.1 that large exponent \( m + \frac{2}{N} \frac{\chi_{\max\{1,\lambda_0\}}}{(\chi_{\max\{1,\lambda_0\}} - \mu_+)} \) benefits the boundedness of solutions.

(ix) From Corollary 1.1, we know that if \( \mu > 0 \) and \( m > 2 - \frac{2}{N} \frac{\chi_{\max\{1,\lambda_0\}}}{(\chi_{\max\{1,\lambda_0\}} - \mu_+)} \), which implies that \( m > 2 - \frac{2}{N} \), therefore, our results improve the result of [14] and [21] provided that the haptotaxis is ignored (\( w \equiv 0 \) in [14] and [21]).

In view of Theorem 1.1 we also conclude the following Corollary:

**Corollary 1.2.** Assume that \( u_0 \in C^0(\bar{\Omega}) \) and \( v_0 \in W^{1,\infty}(\bar{\Omega}) \) both are nonnegative, \( D \) satisfies \((1.2)-(1.3)\). Assume that \( \mu = 0 \). Then if \( m > 2 - \frac{2}{N} \) or \( m = 2 - \frac{2}{N} \) and \( C_D > \frac{C_{GN}(1+\|u_0\|_{L^1(\Omega)})}{3}(2 - \frac{2}{N})^2 \max\{1, \lambda_0\} \chi \), \((1.3)\) admits a global classical solution \((u, v)\) which is bounded in \( \Omega \times (0, \infty) \).

**Remark 1.2.** (i) When \( m > 2 - \frac{2}{N} \), Corollary 1.2 is (partly) coincides with Theorem 0.1 of [17], however, we should pointed that the method in [17] seems to not be used to solve the case \( m = 2 - \frac{2}{N} \).

(ii) To the best of knowledge, this is the first result which solve the case \( m = 2 - \frac{2}{N} \) that yields to the boundedness of solution to problem \((1.5)\).

We sketch here the main ideas and methods used in this article. One novelty of this paper is that we use the Maximal Sobolev regularity approach to prove the existence of bounded solutions. Moreover, by careful analysis, firstly, one can derive new a-priori estimate \( \int_\Omega u^{\gamma_0}(x, t)dx \) (for all \( 1 < \gamma_0 < \frac{\chi_{\max\{1,\lambda_0\}}}{(\chi_{\max\{1,\lambda_0\}} - \mu_+)} \), \( t > 0 \) and \( \mu > 0 \), see Lemma 3.4), then we develop new \( L^p \)-estimate techniques to raise the a priori estimate of solutions from \( L^{\gamma_0}(\Omega) \to L^p(\Omega) \) (for all \( p > 1 \)) (see Lemma 3.5). While if \( \mu = 0 \) and \( m = 2 - \frac{2}{N} \), with the help of the Maximal Sobolev regularity approach, we firstly get the bounded of \( \int_\Omega u^{1+\epsilon}(x, t)dx \) (Lemma 3.6), so that, in light of the Maximal Sobolev regularity approach again, we can obtain the boundedness of \( \int_\Omega u^p(x, t)dx \) (for all \( p > 1 \) and \( t > 0 \), see Lemma 3.7). Finally, in view of the standard semigroup arguments and the Moser iteration method (see e.g. Lemma A.1 of [17]), we can establish the \( L^\infty \) bound of \( u \) (see the proof of Theorem 1.1).
2 Preliminaries

In order to prove the main results, we first state several elementary lemmas which will be needed later. We also present some known results on quasilinear Keller-Segel model (with logistic source).

**Lemma 2.1.** ([37, 38]) Let $s \geq 1$ and $q \geq 1$. Assume that $p > 0$ and $a \in (0, 1)$ satisfy

$$\frac{1}{2} - \frac{p}{N} = (1 - a) \frac{q}{s} + a(\frac{1}{2} - \frac{1}{N}) \quad \text{and} \quad p \leq a.$$ 

Then there exist $c_0, c'_0 > 0$ such that for all $u \in W^{1,2}(\Omega) \cap L^\frac{q}{s}(\Omega)$,

$$\|u\|_{W^{p,2}(\Omega)} \leq c_0 \|\nabla u\|_{L^2(\Omega)}^2 \|u\|_{L^\frac{q}{s}(\Omega)}^{1-a} + c'_0 \|u\|_{L^\frac{q}{s}(\Omega)}.$$

**Lemma 2.2.** ([33]) Let $\theta \in (0, p)$. There exists a positive constant $C_{GN}$ such that for all $u \in W^{1,2}(\Omega) \cap L^\theta(\Omega)$,

$$\|u\|_{L^p(\Omega)} \leq C_{GN}(\|\nabla u\|_{L^2(\Omega)}^2 \|u\|_{L^\theta(\Omega)}^{1-a} + \|u\|_{L^\theta(\Omega)})$$

is valid with $a = \frac{\frac{N}{\theta} - \frac{N}{p}}{1 - \frac{N}{\theta} + \frac{N}{p}} \in (0, 1)$.

**Lemma 2.3.** ([?]) Suppose that $\gamma \in (1, +\infty)$ and $g \in L^\gamma((0,T); L^\gamma(\Omega))$. Consider the following evolution equation

$$\begin{align*}
v_t - \Delta v + v &= g, \quad (x, t) \in \Omega \times (0, T), \\
\frac{\partial v}{\partial \nu} &= 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
v(x, 0) &= v_0(x), \quad (x, t) \in \Omega.
\end{align*}$$

For each $v_0 \in W^{2,\gamma}(\Omega)$ such that $\frac{\partial v_0}{\partial \nu} = 0$ and any $g \in L^\gamma((0,T); L^\gamma(\Omega))$, there exists a unique solution $v \in W^{1,\gamma}((0,T); L^\gamma(\Omega)) \cap L^\gamma((0,T); W^{2,\gamma}(\Omega))$. In addition, if $s_0 \in [0, T)$, $v(\cdot, s_0) \in W^{2,\gamma}(\Omega)(\gamma > N)$ with $\frac{\partial v(\cdot, s_0)}{\partial \nu} = 0$, then there exists a positive constant $\lambda_0 := \lambda_0(\Omega, \gamma, N)$ such that

$$\int_{s_0}^{T} e^{\gamma s} \|v(\cdot, t)\|_{W^{2,\gamma}(\Omega)}^\gamma ds \leq \lambda_0 \left( \int_{s_0}^{T} e^{\gamma s} \|g(\cdot, s)\|_{L^\gamma(\Omega)}^\gamma ds + e^{\gamma s_0}(\|v_0(\cdot, s_0)\|_{W^{2,\gamma}(\Omega)}) \right).$$
The first lemma concerns the local solvability of problems (1.5), which can be proved by a straightforward adaption of the corresponding procedures in Lemma 3.1 of [1] (see also Lemma 1.1 of [21] and [20, 22, 33]) to our current setting:

**Lemma 2.4.** Suppose that \( \Omega \subset \mathbb{R}^N(N \geq 1) \) is a bounded domain with smooth boundary, \( D \) satisfies (1.2)–(1.3). Then for nonnegative triple \((u_0, v_0) \in C(\bar{\Omega}) \times W^{1,\infty}(\bar{\Omega})\), problem (1.5) has a unique local-in-time non-negative classical functions

\[
\begin{cases}
    u \in C^0(\bar{\Omega} \times [0,T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\text{max}})), \\
    v \in C^0(\bar{\Omega} \times [0,T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\text{max}})),
\end{cases}
\]

where \( T_{\text{max}} \) denotes the maximal existence time. Moreover, if \( T_{\text{max}} < +\infty \), then

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as} \quad t \nearrow T_{\text{max}}
\]

is fulfilled.

According to the above existence theory, for any \( s \in (0,T_{\text{max}}) \), \((u(\cdot, s), v(\cdot, s)) \in C^2(\bar{\Omega})\), so that, without loss of generality, we can assume that there exists a constant \( K \) such that

\[
\|u_0\|_{C^2(\Omega)} \leq K \quad \text{and} \quad \|v_0\|_{C^2(\Omega)} \leq K.
\]

The following properties of solutions of (1.5) are well known.

**Lemma 2.5.** Assume that \( \mu > 0 \). There exists a positive constant \( K_0 \) such that the solution \((u, v)\) of (1.5) satisfies

\[
\|u(\cdot, t)\|_{L^1(\Omega)} \leq K_0 \quad \text{for all} \quad t \in (0,T_{\text{max}})
\]

and

\[
\int_t^{t+\tau} \int_\Omega u^2 \leq K_0 \quad \text{for all} \quad t \in (0,T_{\text{max}} - \tau),
\]

where

\[
\tau := \min\{1, \frac{1}{6} T_{\text{max}}\}.
\]

**Lemma 2.6.** Let \( \mu = 0 \), then the solution \((u, v)\) of (1.5) satisfies

\[
\|u(\cdot, t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \text{for all} \quad t \in (0,T_{\text{max}}).
\]


3 A priori estimates and the proof of the main results

This section is devoted to prove Theorem 1.1 by preparing a series of lemmas in this section. To this end, we first prove three important lemmas which are similar to Lemma 3.4 of [34] (see also [41]).

**Lemma 3.1.** Let
\[ H(y) = y(l_0 - 1) + \frac{1}{\lambda_0 + 1} \left( \frac{\lambda_0}{l_0} \right)^{l_0 - 1} (l_0 - 1) \chi \lambda_0 \] for \( y > 0 \). For any fixed \( l_0 \geq 1, \chi, \lambda_0 > 0 \), Then
\[ \min_{y > 0} H(y) = (l_0 - 1)^{\frac{1}{\lambda_0 + 1}} \chi. \]

Proof. It is easy to verify that
\[ H'(y) = (l_0 - 1) \chi \left[ 1 - \lambda_0 \left( \frac{l_0}{y(l_0 + 1)} \right)^{l_0 + 1} \right]. \]

Let \( H'(y) = 0 \), we have
\[ y = \lambda_0^\frac{1}{\lambda_0 + 1} \frac{l_0}{l_0 + 1}. \]

Direct computation shows that \( \lim_{y \to 0^+} H(y) = +\infty \) and \( \lim_{y \to +\infty} H(y) = +\infty \), so that,
\[ \min_{y > 0} H(y) = H\left( \lambda_0^\frac{1}{\lambda_0 + 1} \frac{l_0}{l_0 + 1} \right) = (l_0 - 1)^{\frac{1}{\lambda_0 + 1}} \chi. \]

\[ \square \]

**Lemma 3.2.** Let
\[ B_1 = \frac{1}{p + 1} \left[ \frac{p + 1}{p} \right]^{-p} \left( \frac{p - 1}{p} \right)^{p + 1} \] (3.1)

and \( \tilde{H}(y) = y + B_1 y^{-p} \chi^{p + 1} \lambda_0 \) for \( y > 0 \). For any fixed \( p \geq 1, \chi, \lambda_0 > 0 \), Then
\[ \min_{y > 0} \tilde{H}(y) = \frac{(p - 1)}{p} \lambda_0^{\frac{1}{p + 1}} \chi. \]

Proof. A straightforward computation shows that
\[ \tilde{H}'(y) = 1 - B_1 p \lambda_0 \left( \frac{\chi}{y} \right)^{p + 1}. \]

Let \( \tilde{H}'(y) = 0 \), we have
\[ y = (B_1 \lambda_0 p)^{\frac{1}{p + 1}} \chi. \]
On the other hand, by \( \lim_{y \to 0^+} \tilde{H}(y) = +\infty \) and \( \lim_{y \to +\infty} \tilde{H}(y) = +\infty \), we have
\[
\min_{y > 0} \tilde{H}(y) = \tilde{H}[ (B_1 \lambda_0 p) \frac{1}{p+1} \chi ] = (B_1 \lambda_0 p)^{\frac{1}{p+1}} (p^{\frac{1}{p+1}} + p^{-\frac{1}{p+1}}) \chi = \frac{(p-1)}{p} \lambda_0^{\frac{1}{p+1}} \chi.
\]

Lemma 3.3. Let
\[
C_D > \frac{C_{GN}(1 + \|u_0\|_{L^1(\Omega)})}{3} (2 - \frac{2}{N})^2 \max\{1, \lambda_0\} \chi
\] (3.2)
and
\[
h(p) := \frac{4C_D}{C_{GN}(1 + \|u_0\|_{L^1(\Omega)})} - \frac{(1 - \frac{2}{N} + p)^2}{p} \max\{1, \lambda_0\} \chi,
\]
where \( p \geq 1, C_D, C_{GN}, \lambda_0 \) and \( \chi \) are positive constants. Then there exists a positive constant \( p_0 > 1 \) such that
\[
h(p_0) > 0. \tag{3.3}
\]

Proof. Due to (3.2), \( h(1) > \frac{3C_D}{C_{GN}(1 + \|u_0\|_{L^1(\Omega)})} - (2 - \frac{2}{N})^2 \max\{1, \lambda_0\} \chi > 0 \). Next, by basic calculation, we derive that for any \( p \geq 1 \), \( h'(p) = \frac{(1 - \frac{2}{N} + p)(p + \frac{2}{N} - 1)}{p^2} > 0 \). Therefore, from the continuity of \( h \), there exists a positive constant \( p_0 > 1 \) such that (3.3) holds. \( \square \)

With the Lemma 3.1 in hand, in view of the Maximal Sobolev regularity approach, we derive the following new a-priori estimate \( \int_{\Omega} u^{\gamma_0}(x, t) dx \) (for all \( 1 < \gamma_0 < \frac{\chi \max\{1, \lambda_0\}}{(\chi \max\{1, \lambda_0\} - \mu)^+} \), \( t > 0 \) and \( \mu > 0 \)), which plays a critical role in obtaining the main results.

Lemma 3.4. Let \((u, v)\) be a solution to (1.5) on \((0, T_{max})\) and \( \mu > 0 \). Then for all \( 1 < p < \frac{\chi \max\{1, \lambda_0\}}{(\chi \max\{1, \lambda_0\} - \mu)^+} \), there exists a positive constant \( C := C(p, |\Omega|, \mu, \lambda_0, \chi) \) such that
\[
\int_{\Omega} u^p(x, t) \leq C \quad \text{for all} \quad t \in (0, T_{max}). \tag{3.4}
\]

Proof. Multiplying the first equation of (1.5) by \( u^{\mu-1} \), integrating over \( \Omega \) and using (1.3), we get
\[
\frac{1}{l_0} \frac{d}{dt} \|u\|_{L^6(\Omega)}^6 + C_D(l_0 - 1) \int_{\Omega} u^{m+l_0-3} |\nabla u|^2 \leq -\chi \int_{\Omega} \nabla \cdot (u \nabla v) u^{\mu-1} + \int_{\Omega} u^{l_0-1}(\mu u - \mu u^2) \quad \text{for all} \quad t \in (0, T_{max}), \tag{3.5}
\]
which implies that,
\[
\frac{1}{l_0} \frac{d}{dt} \|u\|_{L^0(\Omega)}^l \leq -\frac{l_0 + 1}{l_0} \int u^0 - \chi \int \nabla \cdot (u \nabla v)u^{l-1} + \int \left( \frac{l_0 + 1}{l_0} u^l + u^{l-1}(\mu u - \mu u^2) \right) \text{ for all } t \in (0, T_{max}).
\] (3.6)

Next, for any positive constant \(\delta_1 > 0\), we derive from the Young inequality that
\[
\int \left( \frac{l_0 + 1}{l_0} u^l + u^{l-1}(\mu u - \mu u^2) \right) = \frac{l_0 + 1}{l_0} \int u^l + \mu \int u^l - \mu \int u^{l+1}
\]
\[
\leq (\delta_1 - \mu) \int u^{l+1} + C_1(\delta_1, l_0) \text{ for all } t \in (0, T_{max}),
\] (3.7)

where
\[
C_1(\delta_1, l_0) = \frac{1}{l_0 + 1} \left( \delta_1 \frac{l_0 + 1}{l_0} \right)^{\frac{l-1}{l}} \left( \frac{l_0 + 1}{l_0} + \mu \right)^{l+1} |\Omega|.
\]

Now, integrating by parts to the first term on the right hand side of (3.5) and using the Young inequality, we obtain
\[
-\chi \int \nabla \cdot (u \nabla v)u^{l-1}
\]
\[
= (l_0 - 1) \chi \int u^{l-1} \nabla u \cdot \nabla v
\]
\[
= -\frac{l_0 - 1}{l_0} \chi \int u^l \Delta v
\]
\[
\leq \frac{l_0 - 1}{l_0} \chi \int u^l |\Delta v| \text{ for all } t \in (0, T_{max}).
\] (3.8)

Now, let
\[
\kappa_0 := (A_1 l_0) \frac{1}{\chi} l_0 + 1 \chi,
\] (3.9)

where
\[
A_1 = \frac{1}{l_0 + 1} \left[ \frac{l_0 + 1}{l_0} \right]^{-\frac{l-1}{l}} \left( \frac{l_0 - 1}{l_0} \chi \right)^{l+1}.
\] (3.10)

Additionally, by applying (3.8) and the Young inequality, we observe
\[
-\chi \int \nabla \cdot (u \nabla v)u^{l-1}
\]
\[
\leq \kappa_0 \int u^{l+1} + \frac{1}{l_0 + 1} \left[ \kappa_0 \frac{l_0 + 1}{l_0} \right]^{-\frac{l-1}{l}} \left( \frac{l_0 - 1}{l_0} \chi \right)^{l+1} \int |\Delta v|^{l+1}
\]
\[
= \kappa_0 \int u^{l+1} + A_1 \kappa_0^{l-1} \chi^{l+1} \int |\Delta v|^{l+1} \text{ for all } t \in (0, T_{max}).
\] (3.11)
Substitute (3.7) and (3.11) into (3.6), we derive that

\[
\frac{1}{l_0} \frac{d}{dt} \|u\|_{L^0_t(\Omega)} \leq \left( \delta_1 + \kappa_0 - \mu \right) \int_\Omega u^{l_0+1} - \frac{l_0 + 1}{l_0} \int_\Omega u^{l_0} + A_1 \kappa_0^{-l_0} \chi^{l_0+1} \int_\Omega |\Delta v|^{l_0+1} + C_1(\delta_1, l_0) \quad \text{for all } t \in (0, T_{\text{max}}).
\]

For any \( t \in (0, T_{\text{max}}) \), applying the variation-of-constants formula to the above inequality, we show that

\[
\frac{1}{l_0} \|u(\cdot, t)\|_{L^0_t(\Omega)} \leq \frac{1}{l_0} e^{-(l_0+1)t} \|u_0(\cdot)\|_{L^0_t(\Omega)} + (\delta_1 + \kappa_0 - \mu) \int_0^t e^{-(l_0+1)(t-s)} \int_\Omega u^{l_0+1} + A_1 \kappa_0^{-l_0} \chi^{l_0+1} \int_0^t e^{-(l_0+1)(t-s)} \int_\Omega |\Delta v|^{l_0+1} + C_1(\delta_1, l_0) \int_0^t e^{-(l_0+1)(t-s)} \int_\Omega u^{l_0+1} + A_1 \kappa_0^{-l_0} \chi^{l_0+1} \int_0^t e^{-(l_0+1)(t-s)} \int_\Omega |\Delta v|^{l_0+1} + C_2(l_0, \delta_1),
\]

where

\[
C_2 := C_2(l_0, \delta_1) = \frac{1}{l_0} \|u_0(\cdot)\|_{L^0_t(\Omega)} + C_1(\delta_1, l_0) \int_0^t e^{-(l_0+1)(t-s)} ds.
\]

Now, by Lemma 2.3, we have

\[
A_1 \kappa_0^{-l_0} \chi^{l_0+1} \int_0^t e^{-(l_0+1)(t-s)} \int_\Omega |\Delta v|^{l_0+1} = A_1 \kappa_0^{-l_0} \chi^{l_0+1} e^{-(l_0+1)t} \int_0^t e^{(l_0+1)s} \int_\Omega |\Delta v|^{l_0+1} \leq A_1 \kappa_0^{-l_0} \chi^{l_0+1} e^{-(l_0+1)t} \lambda_0 \int_0^t \int_\Omega e^{(l_0+1)s} u^{l_0+1} + (\|v_0(\cdot)\|_{L^0_t(\Omega)} + \|\Delta v_0(\cdot)\|_{L^0_t(\Omega)})
\]

for all \( t \in (0, T_{\text{max}}) \). By substituting (3.13) into (3.12), using (3.9) and Lemma 3.1 we get

\[
\frac{1}{l_0} \|u(\cdot, t)\|_{L^0_t(\Omega)} \leq (\delta_1 + \kappa_0 + A_1 \kappa_0^{-l_0} \chi^{l_0+1} \lambda_0 - \mu) \int_0^t e^{-(l_0+1)(t-s)} \int_\Omega u^{l_0+1} + A_1 \kappa_0^{-l_0} \chi^{l_0+1} e^{-(l_0+1)t} \lambda_0 (\|v_0(\cdot)\|_{L^0_t(\Omega)} + \|\Delta v_0(\cdot)\|_{L^0_t(\Omega)}) + C_2(l_0, \delta_1)
\]

\[
= (\delta_1 + \left( \frac{l_0 - 1}{l_0} \right) \lambda_0 \chi - \mu) \int_0^t e^{-(l_0+1)(t-s)} \int_\Omega u^{l_0+1} + A_1 \kappa_0^{-l_0} \chi^{l_0+1} e^{-(l_0+1)t} \lambda_0 (\|v_0(\cdot)\|_{L^0_t(\Omega)} + \|\Delta v_0(\cdot)\|_{L^0_t(\Omega)}) + C_2(l_0, \delta_1)
\]

\[
\leq (\delta_1 + \left( \frac{l_0 - 1}{l_0} \right) \max \{1, \lambda_0\} \chi - \mu) \int_0^t e^{-(l_0+1)(t-s)} \int_\Omega u^{l_0+1} + A_1 \kappa_0^{-l_0} \chi^{l_0+1} e^{-(l_0+1)t} \lambda_0 (\|v_0(\cdot)\|_{L^0_t(\Omega)} + \|\Delta v_0(\cdot)\|_{L^0_t(\Omega)}) + C_2(l_0, \delta_1).
\]
For any \( \varepsilon > 0 \), we choose \( l_0 = \frac{\chi \max\{1, \lambda_0\}}{\chi \max\{1, \lambda_0\} - \mu} - \varepsilon \). Then
\[
\frac{(l_0 - 1)}{l_0} \max\{1, \lambda_0\} \chi < \mu,
\]
thus, pick \( \delta_1 = \frac{1}{2} \varepsilon \) such that
\[
0 < \delta_1 < \mu - \frac{(l_0 - 1)}{l_0} \lambda_0 \varepsilon.
\]
Then in light of (3.14), we derive that there exists a positive constant \( C_3 \) such that
\[
\int_\Omega u^{l_0}(x, t) dx \leq C_3 \quad \text{for all } t \in (0, T_{\text{max}}).
\]
Then upon combined with the arbitrariness of \( \varepsilon \) and the Young inequality, we can derive (3.15). The proof Lemma 3.4 is complete.

We proceed to establish the main step towards our boundedness proof.

**Lemma 3.5.** Assume that \( \mu > 0 \). If
\[
m > 2 - \frac{2 \chi \max\{1, \lambda_0\}}{N (\chi \max\{1, \lambda_0\} - \mu)},
\]
then for \( p > \max\{N+1, N(m+1)\} \), there exists a positive constant \( C = (p, |\Omega|, \mu, \lambda_0, \chi, m, C_D) \) such that the solution of (1.5) from Lemma 2.4 satisfies
\[
\int_\Omega u^p(x, t) dx \leq C \quad \text{for all } t \in (0, T_{\text{max}}).
\]

**Proof.** For any \( p > \max\{N + 1, N(m + 1), l_0 - 1, 1, 1 - m + \frac{N-2}{N} l_0\} \), we multiply the first equation of (1.5) by \( u^{p-1} \) and integrate the resulting equation to discover
\[
\frac{1}{p} \frac{d}{dt} \|u\|_{L^p(\Omega)}^p + C_D (p - 1) \int_\Omega (u + 1)^{m+p-3} |\nabla u|^2 \\
\leq -\chi \int_\Omega u^{p-1} \nabla \cdot (u \nabla v) + \int_\Omega u^{p-1} (\mu u - \mu u^2) \\
= \chi (p - 1) \int_\Omega u^{p-1} \nabla u \cdot \nabla v + \int_\Omega u^{p-1} (\mu u - \mu u^2) \\
\leq \chi (p - 1) \int_\Omega u^{p-1} \nabla u \cdot \nabla v + \mu \int_\Omega u^p \quad \text{for all } t \in (0, T_{\text{max}}),
\]
which leads to
\[
\frac{1}{p} \frac{d}{dt} \|u\|_{L^p(\Omega)}^p + C_D (p - 1) \int_\Omega u^{m+p-3} |\nabla u|^2 \\
\leq \frac{p + 1}{p} \int_\Omega u^p + \chi (p - 1) \int_\Omega u^{p-1} \nabla u \cdot \nabla v + \left(\frac{p + 1}{p} + \mu\right) \int_\Omega u^p
\]
(3.19)
for all $t \in (0,T_{\text{max}})$.

Here, the Young inequality guarantees that

$$
\left( \frac{p+1}{p} + \mu \right) \int_{\Omega} u^p \leq \int_{\Omega} u^{p+1} + C_1(p),
$$

(3.20)

where

$$
C_1(p) = \frac{1}{p+1} \left( \frac{p+1}{p} \right)^{-p} \left( \frac{p+1}{p} + \mu \right)^{p+1} |\Omega|.
$$

Once more integrating by parts, we also find that

$$
(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v = \chi(p-1) \int_{\Omega} \nabla u^p \cdot \nabla v \leq \chi(p-1) \int_{\Omega} u^p |\nabla v|.
$$

(3.21)

Here we use the Young inequality to estimate the integrals on the right of (3.36) according to

$$
\chi \int_{\Omega} u^p |\nabla v| \leq \int_{\Omega} u^{p+1} + \frac{1}{p+1} \left[ \frac{p+1}{p} \right]^{-p} \left( \chi \right)^{p+1} \int_{\Omega} |\nabla v|^{p+1}
$$

(3.22)

where

$$
A_1 := \frac{1}{p+1} \left[ \frac{p+1}{p} \right]^{-p} \left( \chi \right)^{p+1}.
$$

While (3.19), (3.20) and (3.22) imply that

$$
\frac{1}{p} \frac{d}{dt} \|u\|_{L^p(\Omega)}^p + \frac{4C_D(p-1)}{(m+p-1)^2} \|\nabla u^\frac{m+p-1}{2}\|_{L^2(\Omega)}^2 \\
\leq 2 \int_{\Omega} u^{p+1} - \frac{p+1}{p} \int_{\Omega} u^p \\
+ A_1 \int_{\Omega} |\nabla v|^{p+1} + C_1(p) \quad \text{for all } t \in (0,T_{\text{max}}).
$$

(3.23)
Employing the variation-of-constants formula to (3.23), we obtain

\[
\frac{1}{p} \|u(\cdot, t)\|_{L^p(\Omega)}^p \leq \frac{1}{p} e^{-(p+1)t} \|u_0(\cdot)\|_{L^p(\Omega)}^p + \frac{4C_D(p - 1)}{(m + p - 1)^2} \int_0^t e^{-(p+1)(t-s)} \|\nabla u^{\frac{m+p-1}{2}}\|_{L^2(\Omega)}^2 ds \\
+ 2 \int_0^t e^{-(p+1)(t-s)} \int_\Omega u^{p+1} ds \\
+ A_1 \int_0^t e^{-(p+1)(t-s)} \int_\Omega |\Delta v|^{p+1} dx ds + C_1(p) \int_0^t e^{-(p+1)(t-s)} ds
\]

for all \( t \in (0, T_{\text{max}}) \), where

\[
C_2 := C_2(p) = \frac{1}{p} e^{-(p+1)t} \|u_0\|_{L^p(\Omega)}^p + C_1(p) \int_0^t e^{-(p+1)(t-s)} ds.
\]

Now, due to Lemma 2.3 and the second equation of (1.5) and using the Hölder inequality, we have

\[
A_1 \int_0^t e^{-(p+1)(t-s)} \int_\Omega |\Delta v|^{p+1} ds = A_1 e^{-(p+1)t} \int_0^t e^{(p+1)s} \int_\Omega |\Delta v|^{p+1} ds \\
\leq A_1 e^{-(p+1)t} \lambda_0 \left[ \int_0^t \int_\Omega e^{(p+1)s} u^{p+1} ds + \|v_0\|_{W^{2,p+1}}^{p+1} \right] \\
\leq A_1 e^{-(p+1)t} \lambda_0 \int_0^t e^{(p+1)s} u^{p+1} ds + C_3
\]

for all \( t \in (0, T_{\text{max}}) \), where \( C_3 = A_1 e^{-(p+1)t} \lambda_0 \|v_0\|_{W^{2,p+1}}^{p+1} \). Inserting (3.25) into (3.24), we conclude that

\[
\frac{1}{p} \|u(\cdot, t)\|_{L^p(\Omega)}^p \leq (2 + A_1 \lambda_0) \int_0^t e^{-(p+1)(t-s)} \int_\Omega u^{p+1} ds - \frac{4C_D(p - 1)}{(m + p - 1)^2} \int_0^t e^{-(p+1)(t-s)} \|\nabla u^{\frac{m+p-1}{2}}\|_{L^2(\Omega)}^2 ds + C_2 + C_3.
\]

for all \( t \in (0, T_{\text{max}}) \). Let \( l_0 = \frac{\chi_{\max\{1, \lambda_0\}}}{(\chi_{\max\{1, \lambda_0\}} - \mu_+)} - \varepsilon \), where \( \varepsilon = \frac{1}{3} \frac{N}{2} (m - 2 + \frac{2}{N} \chi_{\max\{1, \lambda_0\}} - \mu_+) \).

On the other hand, since \( m > 2 - \frac{2}{N} \frac{\chi_{\max\{1, \lambda_0\}}}{(\chi_{\max\{1, \lambda_0\}} - \mu_+)} \), yields to \( p + 1 < m + p - 1 + \frac{2}{N} \), so that in particular, according to by the Gagliardo–Nirenberg inequality and (3.15), one can
get there exist positive constants $C_4$ and $C_5$ such that
\[
(2 + A_1 \lambda_0) \int_\Omega u^{p+1} = (2 + A_1 \lambda_0) \|u\|_{L^{m+p+1}}^{(m+p+1)} + \|u\|_{L^{m+p+1}}^{(m+p+1)} \leq C_4 \left( \|\nabla u\|_{L^2(\Omega)}^{(m+p+1)} \right)^{(m+p+1)} \leq C_5 \left( \|\nabla u\|_{L^2(\Omega)}^{(m+p+1)} \right)^{(m+p+1)}.
\]

In view of $m > 2 - \frac{2}{N} \chi_{\max\{1, \lambda_0\}}$, by some basic calculation, we derive that
\[
\frac{N(p + 1) - Nl_0}{(2 - N)l_0 + N(m + p - 1)} < 1,
\]
so that, with the help of the Young inequality, we derive that for any $\delta_1 > 0$,
\[
(2 + A_1 \lambda_0) \int_\Omega u^{p+1} \leq \delta_1 \|\nabla u\|_{L^2(\Omega)}^{(m+p+1)} + C_6.
\]

In combination with (3.26) and (3.28) and choosing $\delta_1$ appropriately small, this shows that
\[
\int_\Omega u^p(x, t)dx \leq C_7 \text{ for all } t \in (0, T_{max}),
\]
which together with the Hölder inequality implies the result. The proof of Lemma 3.3 is completed.

**Lemma 3.6.** Assume that $\mu = 0$. If
\[
m > 2 - \frac{2}{N}
\]
or
\[
m = 2 - \frac{2}{N} \text{ and } C_D > \frac{C_{GN}(1 + \|u_0\|_{L^1(\Omega)})}{3} (2 - \frac{2}{N})^2 \chi_{\max\{1, \lambda_0\}} \chi.
\]
then there exists a positive constant $p_0 > 1$ such that the solution of (1.5) from Lemma 2.4 satisfies
\[
\int_\Omega u^{p_0}(x, t)dx \leq C \text{ for all } t \in (0, T_{max}).
\]

**Proof.** Case $m = 2 - \frac{2}{N}$ and $C_D > \frac{C_{GN}(1 + \|u_0\|_{L^1(\Omega)})}{3} (2 - \frac{2}{N})^2 \chi_{\max\{1, \lambda_0\}}$. Firstly, let $p > 1$. Multiplying the first equation of (1.5) by $u^{p-1}$ and using $\mu = 0$, we derive that
\[
\frac{1}{p} \frac{d}{dt} \|u\|_{L^p(\Omega)}^{p} + (p - 1) \int_\Omega D(u)(u + 1)^{p-\frac{2}{N}-1} |\nabla u|^2 = \chi(p - 1) \int_\Omega u^{p-1} \nabla v \cdot \nabla v \text{ for all } t \in (0, T_{max}).
\]
which combined with (1.3) yields to
\[
\frac{1}{p} \frac{d}{dt} \|u\|^p_{L^p(\Omega)} + C_D(p-1) \int_{\Omega} u^{p-\frac{2}{N}-1} |\nabla u|^2 \\
\leq -\frac{p+1}{p} \int_{\Omega} u^p + \chi(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v + \frac{p+1}{p} \int_{\Omega} u^p
\]
for all \( t \in (0, T_{\text{max}}) \).

Here, according to the Young inequality, it reads that
\[
\frac{p+1}{p} \int_{\Omega} u^p \leq \varepsilon_1 \int_{\Omega} u^{p+1} + C_1(\varepsilon_1, p),
\]
where
\[
C_1(\varepsilon_1, p) = \frac{1}{p+1} \left( \varepsilon_1 \frac{p+1}{p} \right)^{-p} \left( \frac{p+1}{p} \right)^{p+1} |\Omega|.
\]

Once more integrating by parts, we also find that
\[
(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v \leq \chi \frac{(p-1)}{p} \int_{\Omega} u^p |\nabla v|.
\]

On the right of (3.36) we use the Young inequality to find
\[
\chi \frac{(p-1)}{p} \int_{\Omega} u^p |\nabla v| \\
\leq \varepsilon_2 \int_{\Omega} u^{p+1} + \frac{1}{p+1} \left[ \varepsilon_2 \frac{p+1}{p} \right]^{-p} \left( \frac{p+1}{p} \right)^{p+1} \int_{\Omega} |\nabla v|^{p+1} \\
= \varepsilon_2 \int_{\Omega} u^{p+1} + A_1 \int_{\Omega} |\nabla v|^{p+1},
\]
where \( \varepsilon_2 = (B_1 \lambda_0 p)^{\frac{1}{p+1}} \chi \),
\[
A_1 := \frac{1}{p+1} \left[ \varepsilon_2 \frac{p+1}{p} \right]^{-p} \left( \frac{p+1}{p} \right)^{p+1}
\]
and \( B_1 \) is the same as (3.1). Hence (3.34), (3.35) and (3.37) results in
\[
\frac{1}{p} \frac{d}{dt} \|u\|^p_{L^p(\Omega)} + \frac{4C_D(p-1)}{(1-\frac{2}{N}+p)^2} \|\nabla u^{\frac{1-p}{2}}\|_{L^2(\Omega)}^2 \\
\leq (\varepsilon_1 + \varepsilon_2) \int_{\Omega} u^{p+1} - \frac{p+1}{p} \int_{\Omega} u^p \\
+ A_1 \int_{\Omega} |\nabla v|^{p+1} + C_1(\varepsilon_1, p)
\]
for all \( t \in (0, T_{\text{max}}) \).
On the other hand, by the Gagliardo–Nirenberg inequality and (2.4), one can get there exists a positive constant $C_{GN}$ such that

$$
\int_{\Omega} u^{p+1} \leq \left\| u \frac{1}{\frac{1}{2} + \frac{1}{p}} \right\|_{L^{\frac{2(p+1)}{1 - \frac{2}{p}}}(\Omega)}^{2(p+1)} \leq C_{GN}(\left\| \nabla u \right\|_{L^{2}(\Omega)}^{1 - \frac{2}{p} + \frac{1}{p}} + \left\| u \right\|_{L^{\frac{2(p+1)}{1 - \frac{2}{p}}}(\Omega)}^{1 - \frac{2}{p} + \frac{1}{p}}) \leq C_{GN}(1 + \left\| u_0 \right\|_{L^{1}(\Omega)})(\left\| \nabla u \right\|_{L^{2}(\Omega)}^{2} + 1),
$$

(3.39)

where $C_{GN}$ is the same as Lemma 2.2. In combination with (3.38) and (3.39), this shows that

$$
\frac{1}{p} \frac{d}{dt} \| u \|_{L^{p}(\Omega)}^{p} \leq (\varepsilon_1 + \varepsilon_2 - \frac{4C_D(p-1)}{(1 - \frac{2}{N} + p)^2} \frac{1}{C_{GN}(1 + \left\| u_0 \right\|_{L^{1}(\Omega)})}) \int_{\Omega} u^{p+1} - \frac{p+1}{p} \int_{\Omega} u^{p} + A_1 \int_{\Omega} |\Delta v|^{p+1} + C_2(\varepsilon_1, p) \text{ for all } t \in (0, T_{\text{max}}),
$$

(3.40)

where

$$
C_2(\varepsilon_1, p) = C_1(\varepsilon_1, p) + \frac{4C_D(p-1)}{(1 - \frac{2}{N} + p)^2}.
$$

Employing the variation-of-constants formula to (3.40), we obtain

$$
\frac{1}{p} \left\| u(\cdot, t) \right\|_{L^{p}(\Omega)}^{p} \leq \frac{1}{p} e^{-(p+1)t} \| u_0 \|_{L^{p}(\Omega)}^{p} + (\varepsilon_1 + \varepsilon_2 - \frac{4C_D(p-1)}{(1 - \frac{2}{N} + p)^2} \frac{1}{C_{GN}(1 + \left\| u_0 \right\|_{L^{1}(\Omega)})}) \int_{0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} u^{p+1} ds + A_1 \int_{0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} |\Delta v|^{p+1} dx ds + C_2(\varepsilon_1, p) \int_{0}^{t} e^{-(p+1)(t-s)} ds \leq (\varepsilon_1 + \varepsilon_2 - \frac{4C_D(p-1)}{(1 - \frac{2}{N} + p)^2} \frac{1}{C_{GN}(1 + \left\| u_0 \right\|_{L^{1}(\Omega)})}) \int_{0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} u^{p+1} ds + A_1 \int_{0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} |\Delta v|^{p+1} dx ds + C_3(\varepsilon_1, p)
$$

(3.41)

with

$$
C_3 := C_3(\varepsilon_1, p) = \frac{1}{p} e^{-(p+1)t} \| u_0 \|_{L^{p}(\Omega)}^{p} + C_2(\varepsilon_1, p) \int_{0}^{t} e^{-(p+1)(t-s)} ds.
$$

Now, due to Lemma 2.3 and the second equation of (1.3) and using the Hölder inequality,
we have
\[
A_1 \int_0^t e^{-(p+1)(t-s)} \int_\Omega |\Delta u|^{p+1} ds = A_1 e^{-(p+1)t} \int_0^t e^{(p+1)s} \int_\Omega |\Delta u|^{p+1} ds \leq A_1 e^{-(p+1) t} \lambda_0 \left[ \int_0^t \int_\Omega e^{(p+1)s} u^{p+1} ds + \|v_0\|_{W^{2,p+1}}^{p+1} \right]
\]
for all \( t \in (0, T_{\max}) \), where \( C_4 = A_1 e^{-(p+1) t} \lambda_0 \|v_0\|_{W^{2,p+1}}^{p+1} \). Recalling (3.41), applying Lemma 3.1 and the Young inequality, we derive that
\[
\frac{1}{p} \|u(\cdot, t)\|_{L^p(\Omega)}^p \leq (\varepsilon_1 + \varepsilon_2 - 4C_D(p-1) \frac{1}{(1-\frac{2}{N}+p)^2 C_GN(1+\|u_0\|_{L^1(\Omega)})} ) \int_0^t e^{-(p+1)(t-s)} \int_\Omega u^{p+1} ds + A_1 \lambda_0 \int_0^t e^{-(p+1)(t-s)} \int_\Omega u^{p+1} ds + C_5 \leq (\varepsilon_1 + \varepsilon_2 + A_1 \lambda_0 - 4C_D(p-1) \frac{1}{(1-\frac{2}{N}+p)^2 C_GN(1+\|u_0\|_{L^1(\Omega)})} ) \int_0^t e^{-(p+1)(t-s)} \int_\Omega u^{p+1} ds + C_5 \]
with \( C_5 = C_4 + C_3(\varepsilon_1, p) \). Observing that
\[
\varepsilon_2 + A_1 \lambda_0 = (B_1 \lambda_0 p)^\frac{1}{p+1} \chi + \left[(B_1 \lambda_0 p)^\frac{1}{p+1} \chi\right]^{-p} \frac{1}{p+1} \left[p+1\right]^{-p} \left(\frac{p-1}{p}\right)^{p+1},
\]
so that, with the help of Lemma 3.2 we derive that
\[
\varepsilon_2 + A_1 \lambda_0 = \frac{(p-1)}{p} \lambda_0^{\frac{1}{p+1}} \chi \leq \frac{(p-1)}{p} \max\{1, \lambda_0\} \chi;
\]
thus, by (3.31), we can choose \( \varepsilon_1 \) small enough in (3.43), using the Hölder inequality, we derive that there exits a positive constant \( p_0 > 1 \) such that
\[
\int_\Omega u^{p_0}(x,t) dx \leq C_6 \quad \text{for all} \quad t \in (0, T_{\max}).
\]
Case \( m > 2 - \frac{2}{N} \) can be proved very similarly, therefore, we omit it. The proof of Lemma 3.6 is completed.

Lemma 3.7. Suppose that the conditions of Lemma 3.6 hold. Then for any \( p > 1 \), there exists a positive constant \( C := C(p, |\Omega|, C_D, \lambda_0, m, \chi) \) such that
\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \text{for all} \quad t \in (0, T_{\max}).
\]
Proof. Firstly, let $p > \max\{N + 1, N(m + 1), p_0 - 1, 1, 1 - m + \frac{N - 2}{N}\}$, where $p_0 > 1$ is the same as Lemma 3.6. Testing the first equation of (1.5) against $u^{p-1}$, using $\mu = 0$ and the Young inequality yields

$$
\frac{1}{p} \frac{d}{dt} \|u\|^p_{L^p(\Omega)} + C_D(p - 1) \int_{\Omega} (u + 1)^{m+p-3} \|
abla u\|^2 + \frac{p + 1}{p} \int_{\Omega} u^p \\
\leq \chi \int_{\Omega} u^{p-1} \nabla \cdot (u \nabla v) + \frac{p + 1}{p} \int_{\Omega} u^p \\
= \chi(p - 1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v + \frac{p + 1}{p} \int_{\Omega} u^p \\
\leq \chi \frac{p - 1}{p} \int_{\Omega} u^p |\Delta v| + \frac{p + 1}{p} \int_{\Omega} u^p \\
\leq 2 \int_{\Omega} u^{p+1} + C_1 \int_{\Omega} |\Delta v|^{p+1} + C_2 \text{ for all } t \in (0, T_{max}),
$$

where $C_1 = \frac{1}{p+1} \left[ \frac{p+1}{p} \right]^{-p} \chi(p - 1)^{p+1}$ and $C_2 = \frac{1}{p+1} \left( \frac{p+1}{p} \right)^{-p} \chi(p - 1)^{p+1} |\Omega|$. Employing the variation-of-constants formula to (3.46), we obtain

$$
\frac{1}{p} \|u(\cdot, t)\|^p_{L^p(\Omega)} \\
\leq \frac{1}{p} e^{-(p+1)t} \|u_0(\cdot)\|^p_{L^p(\Omega)} - \frac{4C_D(p - 1)}{(m + p - 1)^2} \int_0^t e^{-(p+1)(t-s)} \|\nabla u^{m+p-1}\|^2_{L^2(\Omega)} ds \\
+ 2 \int_0^t e^{-(p+1)(t-s)} \int_{\Omega} u^{p+1} ds \\
+ C_1 \int_0^t e^{-(p+1)(t-s)} \int_{\Omega} |\Delta v|^{p+1} dx ds + C_2(p) \int_0^t e^{-(p+1)(t-s)} ds \\
\leq 2 \int_0^t e^{-(p+1)(t-s)} \int_{\Omega} u^{p+1} ds - \frac{4C_D(p - 1)}{(m + p - 1)^2} \int_0^t e^{-(p+1)(t-s)} \|\nabla u^{m+p-1}\|^2_{L^2(\Omega)} ds \\
+ C_3 \int_0^t e^{-(p+1)(t-s)} \int_{\Omega} |\Delta v|^{p+1} dx ds + C_3(p)
$$

with

$$
C_3 := C_3(p) = \frac{1}{p} e^{-(p+1)t} \|u_0(\cdot)\|^p_{L^p(\Omega)} + C_2(p) \int_0^t e^{-(p+1)(t-s)} ds.
$$

Now, we use Lemma 2.3 the second equation of (1.5) and the Hölder inequality to find

$$
C_1 \int_0^t e^{-(p+1)(t-s)} \int_{\Omega} |\Delta v|^{p+1} ds \\
= C_1 e^{-(p+1)t} \int_0^t e^{(p+1)s} \int_{\Omega} |\Delta v|^{p+1} ds \\
\leq C_1 e^{-(p+1)t} \lambda_0 \left[ \int_0^t \int_{\Omega} e^{(p+1)s} u^{p+1} ds + \|v_0\|_{W^{2,p+1}}^{p+1} \right] \\
\leq C_1 e^{-(p+1)t} \lambda_0 \int_0^t e^{(p+1)s} u^{p+1} ds + C_4
$$

(3.48)
for all \( t \in (0,T_{\max}) \), where \( C_4 = C_1 e^{-(p+1)t} \lambda_0 \| v_0 \|_{W^{2,p+1}}^{p+1} \). Hence (3.47) and (3.48) results in

\[
\frac{1}{p} \| u(\cdot,t) \|_{L^p(\Omega)}^p \leq (2 + C_1 \lambda_0) \frac{\int_0^t e^{-(p+1)(t-s)} \int_\Omega u^{p+1} ds}{\int_0^t e^{-(p+1)(t-s)} \| \nabla u \|_{L^2(\Omega)}^2 ds + C_2 + C_3},
\]

(3.49)

for all \( t \in (0,T_{\max}) \). Therefore, observe that \( m \geq 2 - \frac{2}{N} \) and \( p_0 > 1 \) yields to \( p + 1 < m + p - 1 + \frac{2}{N} p_0 \), so that, in view of the Gagliardo–Nirenberg inequality, (3.32) and using the Young inequality, one can get there exist positive constants \( C_4, C_5 \) and \( C_6 \) such that for any \( \delta_1 > 0 \)

\[
(2 + C_1 \lambda_0) \int_\Omega u^{p+1} \leq (2 + C_1 \lambda_0) \| u \|_{L^\infty(\Omega)}^{m+p-1} \| u \|_{L^\infty(\Omega)} \| \nabla u \|_{L^2(\Omega)}^2 + \| u \|_{L^\infty(\Omega)}^{m+p-1} \| u \|_{L^\infty(\Omega)} \| \nabla u \|_{L^2(\Omega)}^2 + C_6,
\]

where we have used that \( \frac{N(p+1) - Np_0}{(2-N)p_0 + N(m+p-1)} < 1 \) by \( m \geq 2 - \frac{2}{N} \) and \( p_0 > 1 \). Inserting (3.50) into (3.49), choosing \( \delta_1 \) appropriately small and using the Hölder inequality, we can get (3.45).

Now, in light of Lemmas 3.5, 3.7 and 2.4, we shall prove the global boundedness of solutions for (1.5), by using the well-known Moser-Alikakos iteration and the standard semigroup arguments. To this end, we first prove the following Lemma:

**Lemma 3.8.** Suppose that the conditions of Theorem 1.1 hold. Let \( T \in (0,T_{\max}) \) and \( (u,v) \) be the solution of (1.5). Then there exists a constant \( C > 0 \) independent of \( T \) such that the component \( v \) of \( (u,v) \) satisfies

\[
\| \nabla v(\cdot,t) \|_{L^\infty(\Omega)} \leq C \quad \text{for all} \quad t \in (0,T).
\]

(3.51)

**Proof.** Firstly, due to Lemmas 3.5, 3.6 we derive that there exist positive constants \( p_0 > N \) and \( C_1 \) such that

\[
\| u(\cdot,t) \|_{L^{p_0}(\Omega)} \leq C_1 \quad \text{for all} \quad t \in (0,T_{\max}).
\]

21
Next, for any $t \in (0, T)$, in view of (2.3), recalling well-known smoothing properties of the Neumann heat semigroup, we find $C_2$ and $C_3 > 0$ such that

$$
\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \\
\leq C_2 \int_0^t (t-s)^{-\frac{\alpha}{N} - \frac{N}{p_0} e^{-\gamma(t-s)}} \|u(\cdot, s)\|_{L^{p_0}(\Omega)} ds + C_2 t^{-\alpha} \|v_0(\cdot, \cdot)\|_{L^\infty(\Omega)} \\
\leq C_2 \int_0^{\infty} \sigma^{-\frac{\alpha}{N}} \frac{N}{p_0} e^{-\gamma \sigma} d\sigma + C_2 t^{-\alpha} K \\
\leq C_3 \text{ for all } t \in (0, T).
$$

(3.52)

\[\square\]

**Lemma 3.9.** Assume that $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\infty}(\bar{\Omega})$ both are nonnegative. If $m > 2 - \frac{2}{N} \chi_{\max\{1, \lambda_0\}}$ or $m = 2 - \frac{2}{N}$ and $C_D > \frac{C_{GN}(1+\|u_0\|_{L^1(\Omega)})}{3(2 - \frac{2}{N})^2 \max\{1, \lambda_0\}\chi}$, then there exists $C > 0$ such that for every $T \in (0, T_{\max})$

$$
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T).
$$

(3.53)

**Proof.** With the regularity properties from Lemmas 3.5–3.7, 3.8 and at hand, one can readily derive (3.53) by means of the well-known Moser-Alikakos iteration (see e.g. Lemma A.1 of [17]) applied to the first equation in (1.5).

With the above estimate as hand (see Lemma 3.9), now we can immediately pass to our main result.

**The proof of Theorem 1.1**

The statement of global classical solvability and boundedness is a straightforward consequence of Lemma 3.9 and and the extendibility criterion provided by Lemma 2.4. Hence the classical solution $(u, v)$ of (1.5) is global in time and bounded.

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