Circular Sequences and the Diameter of Multipermutohedra

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Abstract

We derive bounds on the number of switches at an arbitrary set of positions in a circular sequence of permutations and relate them to the diameter of Multipermutohedra.

Keywords: Circular sequences, Multipermutohedra

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1 Introduction

A sequence of \( \vec{N} + 1 \) (with \( \vec{N} = \binom{N}{2} \)) permutations, \( \mathcal{P} = \langle \pi_0, \ldots, \pi_N \rangle \), in the symmetric group \( S_N \) is called a circular sequence if \( \pi_0 = 1, \ldots, n \) is the identity permutation, \( \pi_N = n (n-1) \ldots 1 \) is the reverse permutation and \( \pi_{i+1} \) differs from \( \pi_i \) by an adjacent transposition, i.e., \( \pi_{i+1} = \pi_i (j, j+1) \) for some \( j \) such that \( \pi_i(j) < \pi_i(j+1) \). These sequences are used for bounding the number of \( k \)-sets of a point configuration \( X \) in the plane (a \( k \)-set of \( X \) is a subset of size \( k \) separated by a line from the other points in \( X \)). The connection between \( k \)-sets and circular sequences was first established by Goodman and Pollack [5]. For a subset \( K \subseteq \left[ \frac{n-1}{2} \right] \), one can also bound the number of \( k \)-sets, \( k \in K \) over all point configurations \( X \). Such bounds were established for specific choices of \( K \) using this connection (see [1, 6]).

For the sequence \( \mathcal{P} \), defined above, the process of moving from \( \pi_i \) to \( \pi_{i+1} \) is referred to as a switch \( (\pi_i(j), \pi_i(j+1)) \) at position \( j \) (note that the numbers being swapped are \( \pi_i(j) \) and \( \pi_i(j+1) \)). We define \( s_j(\mathcal{P}) \) to be the total number of switches at position \( j \). We count the total number of switches at a given set of positions \( y = \{y_1, \ldots, y_n\} \subseteq [N-1] \), with \( y_1 < y_2 < \cdots < y_n \) which is

\[
s(\mathcal{P}, y) = s_{y_1}(\mathcal{P}) + \cdots + s_{y_n}(\mathcal{P}).
\]

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Our goal is to derive a bound on \( s(\mathcal{P}, y) \) and show how it relates to the diameter of multipermutohedra (which, as we will see later, are essentially “permutohedra on multisets”).

In the next section, we estimate \( s(\mathcal{P}, y) \). In Section 3, we introduce the multipermutohedron and derive bounds on its diameter. In Section 4, we show how the diameter of multipermutohedra relates to a variant of the \( k \)-set problem.

We fix some notations. \([n]\) represents the set \( \{1, 2, \ldots, n\} \). If \( x \) is not an integer, we write \([x]\) instead of \( \lfloor x \rfloor \). For \( S \subseteq [n] \) and \( x \in \mathbb{R}^n \) define \( x(S) := \sum_{i \in S} x_i \) and denote the size of \( S \) by \( |S| \).

A composition of \( n \) is a sequence of positive integers \( \lambda := \langle \lambda_1, \ldots, \lambda_k \rangle \) with \( \sum \lambda_i = n \).

Permutations in \( S_n \) will be represented as words. We call \( e := 1 \, 2 \, \ldots \, N \) the identity permutation and \( \bar{e} := N \, N - 1 \, \ldots \, 1 \) the reverse permutation.

## 2 A Bound on Circular Sequences

With \( \mathcal{P} \) and \( y \) as defined above, we evaluate \( s(\mathcal{P}, y) \) in the following way. Each \( i \in [N] \) starts at position \( i \) in the identity permutation \( e \) and reaches position \( N - i + 1 \) in \( \bar{e} \). Following the notation of [8], we refer to the positions \( y_1, \ldots, y_n \) as gates and define \( c_i \) to be the number of gates that \( i \) must cross in order to reach position \( N - i + 1 \). \( c_i \) is interpreted as the cost in moving \( i \) to position \( N - i + 1 \). As \( c_i = c_{N-i+1} \), the total cost \( c(N) \) is

\[
c(N) := \sum_{i=1}^{N} c_i = 2 \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor} c_i.
\]

Next, we define the vector \( x = (x_1, \ldots, x_n) \) where \( x_j \) represents the distance of \( y_j \) from one of the ends, i.e.

\[
x_j := \min\{y_j, N - y_j\}.
\]

Let \((p_1, \ldots, p_n)\) be the permutation of \((x_1, \ldots, x_n)\) with \( N/2 \geq p_1 \geq \cdots \geq p_n > 0 \). Any number \( j \) such that \( p_{i+1} < j \leq p_i \) has cost \( c_j = i \) (we assume \( p_{n+1} = 0 \)). Hence

\[
c(N) = 2 \sum_{i=1}^{n} i(p_i - p_{i+1}) = 2\rho([n]) = 2x([n]).
\]

We now amortize the cost \( c(N) \) over \( \mathcal{P} \). Each switch \((i, j)\) across a gate is interpreted as contributing +1 or −1 to \( c_i \) according to whether \( i \) moves towards
or away from position $N - i + 1$. A switch $(i, j)$ across a gate is called good if it contributes 1 to both $c_i$ and $c_j$ and bad if it contributes 1 to $c_i$ (or $c_j$) and $-1$ to the other. We observe that every switch across a gate is either good or bad, i.e., must contribute +1 to the cost of at least one of the numbers being moved. This follows from the fact that $i < j$, so if $i$ is in position $> N - i > N - j$, then $j$ moves towards $N - j + 1$.

Hence, good switches contribute 2 to $c_i$ while bad switches contribute 0. Since $c_i = 2x([n])$, the number of good switches is $x([n])$. Thus if $s_b(\mathcal{P}, y)$ is the number of bad switches, then

$$s(\mathcal{P}, y) = x([n]) + s_b(\mathcal{P}, y). \quad (2)$$

Thus, it suffices to estimate $s_b(\mathcal{P}, y)$, in order to bound $s(\mathcal{P}, y)$. Let $l$ be the number of $y_i$ smaller than $N/2$, i.e., $y_l < N/2$ while $y_{l+1} \geq N/2$, and let $r := n - l$. In other words, $l$ and $r$ represent the number of gates to the left and right of $N/2$ respectively. The following result bounds $s(\mathcal{P}, y)$.

**Theorem 1** Let

$$s(y) = \min\{s(\mathcal{P}, y) \mid \mathcal{P} \text{ is a circular sequence}\}$$

be the minimum of the number of switches at a set of positions given by $y$ over all circular sequences of permutations in $\mathcal{S}_N$. Then,

$$x([n]) \leq s(y) \leq x([n]) + lr.$$

**Proof.** From (2), we observe that $s(\mathcal{P}, y) \geq x([n])$ for any circular sequence $\mathcal{P}$. This proves the lower bound for $s(y)$.

To show the upper bound, we construct a circular sequence $\mathcal{P}$ such that the number of bad switches $s_b(\mathcal{P}, y) \leq lr$. We construct the sequence in two phases.

In the first phase, we move the numbers $1, \ldots, r$ to positions $y_r, y_{r-1}, \ldots, y_1$ and the numbers $N, N-1, \ldots, N-l+1$ to positions $y_{r+1}, \ldots, y_n$ in the following way. Starting from $e$, we move 1 to position $y_r$ through a sequence of switches $(1, i), i \leq y_r$. Next, 2 is moved to position $y_{r-1}$ with the switches $(2, i), i \leq y_{r-1}$. Continuing this way, the first $r$ and the last $l$ numbers (in the order $N, N-1, \ldots, N-l+1$) are moved to positions $y_r, y_{r-1}, \ldots, y_1$ and $y_{r+1}, \ldots, y_n$ respectively.

The resulting permutation is

$$r + 1 \ldots r | \ldots r - 1 | \ldots 2 | \ldots 1 | \ldots N | \ldots N - 1 | \ldots N - l + 1 | \ldots N - l$$

where the $|$ indicates the gate positions $y_1, \ldots, y_n$ and the numbers $r+1, \ldots, N-l$ are in the remaining positions in increasing order.
The second phase consists of moving $N - i + 1$ to position $i$ for $i = 1, 2, \ldots, \lfloor N/2 \rfloor$. This is done by starting with $N$ and moving it to the first position by switching it in succession with the numbers $1, \ldots, r$. This brings $1, \ldots, r$ to positions $y_{r+1}, y_r, \ldots, y_2$ and $r + 1$ to position $y_1$ if $y_1 > 1$. Next we move $N - 1$ to the second position. In general, suppose $\gamma$ is a permutation in this sequence with the last $i$ numbers in the first $i$ positions and the first $j$ numbers in the last $j$ positions, and $\gamma(i + 1) = k$, i.e.

$$
\gamma = N N - 1 \ldots N - i + 1 k \ldots k - 1| \ldots k - 2| \ldots j + 1| \ldots N - i j j - 1 \ldots 1.
$$

Here the numbers $j + 1, \ldots, k - 1$ are at the gate positions and the numbers $k, k + 1, \ldots, N - i - 1$ are in the remaining positions in increasing order. Switching $N - i$ in succession with $j + 1, j + 2, \ldots, k$ moves $N - i$ to position $i + 1$, and $j + 1$ to position $N - j$ resulting in the permutation

$$
N N - 1 \ldots N - i k + 1 \ldots k| \ldots k - 1| \ldots j + 2| \ldots N - 1 j + 1 j \ldots 1.
$$

Repeating this operation, we finally obtain the permutation $\hat{e}$.

We observe that all switches in the second phase are good; for a switch $(i, j)$ across a gate, the numbers $1, \ldots, (i - 1)$ are already to the right of $i$. So $i$ is at position $< N - i + 1$. The same argument holds for $j$.

In the first phase, suppose $l \leq r$. Then the number of bad switches are at most $l(r - l)$ for moving the numbers $1, \ldots, r - l$ to positions $y_r, y_{r-1}, \ldots, y_{l+1}$; $(l - 1) + (l - 2) + \cdots + 0 = l(l - 1)/2$ for moving the numbers $r - l + 1, \ldots, r$ to positions $y_l, y_{l-1}, \ldots, y_1$ and $l + (l - 1) + \cdots + 1 = l(l + 1)/2$ for moving the numbers $N, N - 1, \ldots, N - l + 1$ to positions $y_{r+1}, \ldots, y_n$ adding up to a total of at most $lr$ bad switches. The case $l > r$ is handled similarly. This proves the result.

\begin{example}
Let $N = 8$ and $y = \{1, 4, 6, 7\}$. By \cite{1}, $x = (1, 4, 2, 1)$. Since $y_1 < N/2$ and $y_2 \geq N/2$, $l = 1$ and $r = n - l = 3$. By Theorem \cite{1}

$$
8 = \sum x_i \leq s(y) \leq 8 + lr = 11
$$

To construct a circular sequence $\mathcal{P}$ with $s(\mathcal{P}, y) = 11$, we start with the identity permutation and move the numbers $1, 2, 8$ to positions $y_3, y_2, y_4$ leading to $34526187$. Next we move $8$ to position $1$ by switching it in succession with the numbers $1, 2, 3$ giving $84536217$. Next $7$ is moved to the second place. We summarize by showing some of the permutations in this sequence.

$$
\begin{align*}
12345678 & \rightarrow 34526187 & \rightarrow 34526187 \\
8452617 & \rightarrow 87546321 & \rightarrow 87546321
\end{align*}
$$

where the numbers above the arrows indicate the number of switches required to move to the next permutation, the ‘|’ show the positions where the switches are counted and the numbers being switched are underlined.
\end{example}
We see that the lower bound for \( s(y) \) is attained when all the gates are to the left (or right) of the middle. In this case, \( r = 0 \) (resp. \( l = 0 \)) and the two bounds for \( s(y) \) coincide. The upper bound is attained when \( y = [N - 1] \). In this case, \( s(y) = \binom{N}{2} \), i.e. we count switches at all positions.

### 3 Circular Sequences and Multipermutohedra

The multipermutohedron is a generalization of the permutohedron \( P_N \) which is the convex hull of all permutations of the point \((1, 2, \ldots, N) \in \mathbb{R}^N\). A multipermutohedron is defined by taking the convex hull of all permutations of a multiset, that is, a “set” with repeated elements.

Let \( b_1 < b_2 < \cdots < b_n \) be \( n \) distinct numbers \( (n > 1) \) and consider the multiset with \( k_i \) copies of \( b_i \) for \( i = 1, \ldots, n \), i.e.

\[
M := \{b_1^{k_1}, b_2^{k_2}, \ldots, b_n^{k_n}\} = \{a_1, \ldots, a_N\} \quad k_i > 0, \ i = 1, \ldots, n \quad (3)
\]

with \( a_1 \leq a_2 \leq \cdots \leq a_N \) and \( N = \sum_{i=1}^{n} k_i \). Let \( P(M) \subset \mathbb{R}^N \) be the polytope formed by taking the convex hull of all permutations of the point \((a_1, \ldots, a_N) \in \mathbb{R}^N\). We call \( P(M) \) the multipermutohedron on the multiset \( M \).

Multipermutohedra were first studied by Schoute [7], who constructed them all from the simplex by a sequence of simple operations of expansion and contraction. Independent treatments of multipermutohedra also appear in [2, 8]. The proofs for the assertions that follow can be found in [2].

The inequality description of \( P(M) \) is a direct generalization of the description for the permutohedron.

\[
P(M) = \{ x \in \mathbb{R}^N \mid x([N]) = \sum_{i=1}^{N} a_i, \ x(S) \geq \sum_{i=1}^{\vert S \vert} a_i, \text{ for all } S \subset [N] \}.
\]

Thus if \( M = \{1, 2, 2, 3\} \), then \( P(M) \subset \mathbb{R}^4 \) is given by

\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 & = 8 \\
1 \leq x_i & \leq 3 \quad i = 1, \ldots, 4 \\
x_i + x_j & \geq 3 \quad 1 \leq i < j \leq 4.
\end{align*}
\]

Figure [1] shows \( P(\{1,2,2,3\}) \). It has 14 facets corresponding to these 14 inequalities, which correspond, in turn, to the 14 ordered partitions of the set \( \{1, 2, 3, 4\} \) into two parts.

Faces of \( P(M) \) are products of lower dimensional multipermutohedra and correspond to ordered partitions of \([N]\). As in the permutohedron, the face lattice of the multipermutohedron does not depend on the numbers being permuted but only on their multiplicities. Henceforth we assume that, in \([3]\), \( b_i = i \)
and rewrite $M$ as

$$M = \{1^{k_1}, \ldots, n^{k_n}\}. \quad (4)$$

Given $\sigma \in S_N$, the vertex $a_\sigma$ has 1’s in positions $\sigma(1), \ldots, \sigma(k_1)$, 2’s in positions $\sigma(k_1 + 1), \ldots, \sigma(k_1 + k_2)$ and so on, i.e.

$$a_\sigma := (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(N)}). \quad (5)$$

Note that this correspondence is not one-to-one; several permutations will correspond to the same vertex of $P(M)$.

Adjacency of vertices on $P(M)$ is obtained by switching two components whose values are consecutive. The relation to permutations is as follows. For $j = 1, \ldots, n$, define

$$y_j := k_1 + \cdots + k_j \quad (6)$$

and let $y_0 = 0$. We define blocks of consecutive integers $S_1, \ldots, S_n$ by

$$S_j := [y_j] \setminus [y_{j-1}].$$

Given $\sigma, \pi \in S_N$, the vertices $a_\sigma$ and $a_\pi$ are adjacent if $a_\pi = a_{\sigma(i,j)}$ for some transposition $(i, j)$ with $i$ and $j$ in successive blocks. In other words, $\pi$ is obtained from $\sigma$ by switching $\sigma(i)$ and $\sigma(j)$ for some $i$ and $j$ in successive blocks and possibly permuting the values of $\sigma(i,j)$ within each block.

**Example 3** Let $M = \{1, 2, 2, 3\}$ as earlier. Let $e = 1 2 3 4$ be the identity permutation. This corresponds to the vertex $a_e = (1, 2, 2, 3)$ of $P(M)$. Multiplying $e$ by transpositions $(i, j)$ with $i, j$ in successive blocks leads to the permutations $2 1 3 4, 3 2 1 4, 1 4 3 2$ and $1 2 4 3$. These in turn correspond to the vertices $(2, 1, 2, 3), (2, 2, 1, 3), (1, 3, 2, 2)$ and $(1, 2, 3, 2)$ that are adjacent to $(1, 2, 2, 3)$ as shown in Figure 1. \[\square\]
For $\sigma, \pi \in S_N$, we denote by $d(\sigma, \pi)$, the shortest distance between the vertices $a_{\sigma}$ and $a_{\pi}$ in $P(M)$. For convenience, we also denote the vertex $a_{\sigma}$ in $P(M)$ by $\sigma$.

The diameter of $P(M)$ (denoted $\text{diam}(P(M))$) is the largest of the distances between pairs of vertices of $P(M)$.

**Proposition 4** The vertex farthest from $\sigma \in P(M)$ is $\bar{\sigma} = \sigma(N) \sigma(N-1) \ldots \sigma(1)$. The diameter of $P(M)$ is $d(e, \bar{e})$.

**Proof.** We first show that the vertex farthest from $e$ in $P(M)$ is $\bar{e}$. This will follow if we can show that for any permutation $\pi$ with $a_{\pi} \neq a_e$, switching a pair of numbers $\pi(i) < \pi(j)$ with $i < j$ in successive blocks yields a permutation $\pi' = \pi(i,j)$ with $d(e, \pi') \geq d(e, \pi)$.

Suppose $d(e, \pi) > d(e, \pi')$. Let $i' = \pi(i)$ and $j' = \pi(j)$. Since $\pi$ and $\pi'$ are adjacent vertices in $P(M)$, $d(e, \pi) = d(e, \pi') + 1$. Consider the shortest path from $e$ to $\pi$ through $\pi'$. Let $\gamma, \alpha_1$ be successive vertices on this path with $\alpha_1^{-1}(j') < \alpha_1^{-1}(i')$ in successive blocks while $\gamma^{-1}(j')$ and $\gamma^{-1}(i')$ are either in the same block or in successive blocks. Hence, in going from $\gamma$ to $\alpha_1$, we have moved $j'$ to the left of $i'$. Let the vertices on the path from $\gamma$ be $\beta_1, \ldots, \beta_{p-1} = \pi', \alpha_p = \pi$.

We construct a shorter path from $\gamma$ to $\pi$, namely $\gamma, \beta_1, \ldots, \beta_{p-1} = \pi$ where for each $m$, $\beta_m = (i', j')\alpha_m$. We see that for each $m > 0$, $\beta_{m+1}$ differs from $\beta_m$ by a transposition across adjacent blocks showing that they are adjacent. Since $\beta_1$ switches $i'$ and $j'$ in $\alpha_1$, we see that either $\beta_1 = \gamma$ or $\beta_1$ and $\gamma$ are adjacent (this case subsumes the possibility of $\beta_1 = \alpha_1$). In any case we see that this path is shorter than our chosen path by at least 1. This contradicts our hypothesis that $d(e, \pi) > d(e, \pi')$ proving our claim.

Given a path from $e$ to $\pi$ in $P(M)$, multiplying the permutations in this path by $\sigma$ yields a path from $\sigma e$ to $\sigma \pi$ of the same length. Hence the vertex farthest from $\sigma$ is $\sigma \bar{e} = \bar{\sigma}$ and $d(\sigma, \bar{\sigma}) = d(e, \bar{e})$ proving the result. \hfill \square

We now relate the diameter of $P(M)$ to circular sequences. Let the multi-set $M$ and $y = (y_1, \ldots, y_{n-1})$ be given by $[7]$ and $[6]$ respectively. From the criterion for adjacency of vertices on $P(M)$, we observe that each circular sequence $P$ corresponds to a path in $P(M)$ from $e$ to $\bar{e}$ of length $s(P,y)$. From Proposition 4 it follows that

$$\text{diam}(P(M)) = \min\{s(P,y) | P \text{ is a circular sequence}\}.$$  

Theorem 1 automatically translates to bounds on diameter of $P(M)$.

**Corollary 5** Let $M$ be defined by $[4]$ with $k_i > 0$ and let $y = (y_1, \ldots, y_{n-1})$ and $x = (x_1, \ldots, x_{n-1})$ be specified by $[8]$ and $[7]$. Then the diameter of $P(M)$ is bounded by

$$x([n-1]) \leq \text{diam}(P(M)) \leq x([n-1]) + lr$$
Example 6 Consider the multiset $M = \{1, 2^3, 3^2, 4, 5\}$. The vectors $y$ and $x$ given by (6) and (1) are $y = (1, 4, 6, 7)$ and $x = (1, 4, 2, 1)$. Since $y_1 < N/2$ and $y_2 \geq N/2$, $l = 1$ and $r = n - 1 - l = 3$. By Corollary 5

$$8 = \sum x_i \leq \text{diam} (P(M)) \leq 8 + lr = 11.$$ 

A circular sequence $P$ with $s(P, y) = 11$ was constructed in Example 2 and this translates to a path of length 11 between $e$ and $\bar{e}$ in $P(M)$. □

It’s easy to see that the upper bound in Corollary 5 is attained for the permutohedron $P_N$ which has a diameter of $\binom{N}{2}$. Also, for small $n$, it’s possible to derive an explicit expression for $\text{diam}(P(M))$.

Proposition 7 Let the multiset $M$ be given by the composition $(k_1, \ldots, k_n)$ of $N$. If $n = 2$ then $\text{diam} (P(M)) = \min\{k_1, k_3\}$. When $n = 3$,

$$\text{diam} (P(M)) = \begin{cases} x_1 + x_2 & \text{if } k_1 \neq k_3; \\ x_1 + x_2 + 1 = 2k_1 + 1 & \text{if } k_1 = k_3. \end{cases}$$

Proof. For $n = 2$, the lower and upper bounds for the diameter in Corollary 5 are the same and the result follows. If $n = 3$ and $k_1 \neq k_3$ then it is easy to show by a careful choice of switches that the lower bound for the diameter is attainable. If $k_1 = k_3$, then by Corollary 5 the diameter is either $2k_1$ or $2k_1 + 1$. The lower bound is not attained because, for any circular sequence, the switches at positions $k_1$ and $N - k_1$ cannot all involve a number less than $k_1$ and a number greater than $N - k_1$, i.e., there must be at least one bad switch. This proves the result. □

We observe that since the diameter of $P(M)$ is obtained by counting the number of switches at certain positions in a circular sequence in $\mathcal{S}_N$, its value is at most the total number of switches i.e. $\binom{N}{2}$. It would be interesting to determine, in some form, all the integers in the set $[\binom{N}{2}]$ that are the diameters of multipermutohedra $P(M) \subset \mathbb{R}^N$ for some $M$ given by (4).

4 Multipermutohedra and $k$-sets

We now relate the diameter of the multipermutohedron to arrangements of points on a plane. Let $\mathcal{X} \subset \mathbb{R}^2$ be a configuration of $N$ points in general position on a plane, i.e., no three points of $\mathcal{X}$ lie on a line. For $k \leq \lfloor N/2 \rfloor$, we define a left (resp. right) $k$-set to be a set of $k$ points of $\mathcal{X}$ that lie on the left (resp. right) of a line (with respect to a directed reference line, say the X-axis).
For a line parallel to the X-axis, we take the left of the line to be the open half-space above the line. A k-set of $\mathcal{X}$ is either a left k-set or a right k-set. Let $f_l(k, \mathcal{X}), f_r(k, \mathcal{X})$ and $f(k, \mathcal{X})$ denote the number of left k-sets, right k-sets and k-sets of $\mathcal{X}$ respectively.

For subsets $L, R \subseteq [(N - 1)/2]$, we count the number of sets that appear as a left k-set for $k \in L$ or a right k-set for $k \in R$. We define

$$f(L, R, \mathcal{X}) := \sum_{k \in L \cap R} f(k, \mathcal{X}) + \sum_{k \in L \setminus R} f_l(k, \mathcal{X}) + \sum_{k \in R \setminus L} f_r(k, \mathcal{X}). \quad (7)$$

When $L = R \subseteq [(N - 1)/2]$, we are counting the number of k-sets for $k \in L$ and we write $f(K, \mathcal{X})$ for $f(K, K, \mathcal{X})$. Our objective is to derive bounds for $f(L, R, \mathcal{X})$ in terms of the diameters of certain multipermutohedra.

Since the minimum (resp. maximum) of $f(L, R, \mathcal{X})$ is not affected by a slight perturbation of the points of $\mathcal{X}$, we assume that no two points of $\mathcal{X}$ lie on a line parallel to the X-axis. Then, for each $k \leq (N - 1)/2$, there are two k-sets that are both left k- and right k-sets. Hence

$$f_l(k, \mathcal{X}) + f_r(k, \mathcal{X}) = f(k, \mathcal{X}) + 2. \quad (8)$$

Following the approach of Goodman and Pollack [5], we associate with $\mathcal{X}$ a circular sequence of permutations in the following way. Project the points of $\mathcal{X}$ and we write $f_l$.

This gives a lower bound for $f(k, \mathcal{X})$. The projection of the points becomes perpendicular to a line joining a pair of points of $\mathcal{X}$.

If $s$ are both left $k$-sets while the number of right $k$-sets.

Hence by (8), the number of left $k$-sets is

$$f_l(k, \mathcal{X}) = s_k + s_{N-k}. \quad (9)$$

The subsets $L$ and $R$ define a composition $\langle L, R \rangle = (k_1, \ldots, k_n)$ of $N$ where $n = |L| + |R| + 1$ and the elements of $L$ are the partial sums $k_1 + k_2 + \cdots + k_j$ that are at most $(N - 1)/2$ while the elements of $R$ are the partial sums $k_n + k_{n-1} + \cdots + k_j$ that are at most $(N - 1)/2$. Let $P(\langle L, R \rangle)$ be the multipermutohedron defined by this composition. Then the sequence $P(\mathcal{X})$ describes a path in $P(\langle L, R \rangle)$ from $e$ to $\bar{e}$ of length

$$\sum_{k \in L} s_k + \sum_{k \in R} s_{N-k} \text{ which must be at least its diameter.}$$

This gives a lower bound for $f(L, R, \mathcal{X})$. 9
To bound $f(L, R, X)$ from above, we consider the subsets $\bar{L} := \left\lceil \frac{N}{2} \right\rceil \setminus L$ and $\bar{R} := \left\lceil \frac{(N - 1)}{2} \right\rceil \setminus R$. As before, these define a composition $\langle \bar{L}, \bar{R} \rangle$ of $N$ and a multipermutohedron $P(\langle \bar{L}, \bar{R} \rangle)$. Since $\sum_{k=1}^{N-1} s_k = \binom{N}{2}$, we can rewrite (9) as

$$f(L, R, X) = \binom{N}{2} - \left( \sum_{k \in L} s_k + \sum_{k \in R} s_{N-k} \right) + |L \cup R| - |L \cap R|.$$ 

This gives the upper bound for $f(L, R, X)$ and ties the diameter of $P(M)$ to $k$-sets in the following way.

**Theorem 8** Let $X$ be a configuration of $N$ points in $\mathbb{R}^2$ in general position and let $L, R \subseteq \left\lceil \frac{(N - 1)}{2} \right\rceil$. Then the function $f(L, R, X) - |L \cup R| + |L \cap R|$ is bounded from above and below by the diameter of multipermutohedra $P(\langle L, R \rangle)$ and $P(\langle \bar{L}, \bar{R} \rangle)$, that is,

$$\text{diam} \left( P(\langle L, R \rangle) \right) \leq f(L, R, X) - |L \cup R| + |L \cap R| \leq \binom{N}{2} - \text{diam} \left( P(\langle \bar{L}, \bar{R} \rangle) \right).$$

In particular,

$$\text{diam} \left( P(\langle K, K \rangle) \right) \leq f(K, X) \leq \binom{N}{2} - \text{diam} \left( P(\langle \bar{K}, \bar{K} \rangle) \right)$$

thus bounding the number of $k$-sets for $k \in K \subseteq \left\lceil \frac{(N - 1)}{2} \right\rceil$ by diameters of multipermutohedra. 

The problem of $k$-sets and its relation to circular sequences has been extensively studied (see [1, 4, 5, 6]). In [6], $k$-sets are used to derive a lower bound on the number of convex quadrilaterals in a set of $n$ points in the plane.

A lower bound for the number of $k$-sets of a point configuration in general position follows from Proposition 7 and the above theorem. As observed in [6, Example 8], this lower bound of $2k + 1$ can actually be achieved by a configuration of $N$ points ($N \geq 2k + 1$) which consist of a regular $(2k + 1)$-gon with the remaining points situated close to the center of the gon. Finding the upper bound for the number of $k$-sets seems to be a harder problem. Some estimates for this bound are given in [1, 3].

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