RADU MIRON

THE GEOMETRY OF MYLLER CONFIGURATIONS.
Applications to Theory of Surfaces and Nonholonomic Manifolds

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Dedicated to academicians

Alexandru Myller and Octav Mayer,

two great Romanian mathematicians
MOTTO

It is possible to bring for the great disappeared scientists varied homages. Sometimes, the strong wind of progress erases the trace of their steps. To not forget them means to continue their works, connecting them to the living present.

Octav Mayer
Preface

In 2010, the Mathematical Seminar of the “Alexandru Ioan Cuza” University of Iași comes to its 100th anniversary of prodigious existence.

The establishing of the Mathematical Seminar by Alexandru Myller also marked the beginning of the School of Geometry in Iași, developed in time by prestigious mathematicians of international fame, Octav Mayer, Gheorghe Vrânceanu, Grigore Moisil, Mendel Haimovici, Ilie Popa, Dimitrie Mangeron, Gheorghe Gheorghiev and many others.

Among the first paper works, published by Al. Myller and O. Mayer, those concerning the generalization of the Levi-Civita parallelism must be specified, because of the relevance for the international recognition of the academic School of Iași. Later on, through the effort of two generations of Iași based mathematicians, these led to a body of theories in the area of differential geometry of Euclidian, affine and projective spaces.

At the half-centenary of the Mathematical Seminary, in 1960, the author of the present opuscule synthesized the field’s results and laid it out in the form of a “whole, superb theory”, as mentioned by Al. Myller. In the same time period, the book The geometry of the Myller configurations was published by the Technical Publishing House. It represents, as mentioned by Octav Mayer in the Foreword, “the most precious tribute ever offered to Alexandru Myller”. Nowadays, at the 100th anniversary of the Mathematical Seminary “Alexandru Myller” and 150 years after the enactment, made by Alexandru Ioan Cuza, to set up the University that carries his name, we are going to pay homage to those two historical acts in the Romanian culture, through the publishing, in English, of the
volume *The Geometry of the Myller Configurations*, completed with an ample chapter, containing new results of the Romanian geometers, regarding applications in the studying of non-holonomic manifolds in the Euclidean space. On this occasion, one can better notice the undeniable value of the achievements in the field made by some great Romanian mathematicians, such as Al. Myller, O. Mayer, Gh. Vrânceanu, Gr. Moisil, M. Haimovici, I. Popa, I. Creangă and Gh. Gheorghiev.

The initiative for the re-publishing of the book belongs to the leadership of the “Alexandru Ioan Cuza” University of Iași, to the local branch of the Romanian Academy, as well as NGO Formare Studia Iași. A competent assistance was offered to me, in order to complete this work, by my former students and present collaborators, Professors Mihai Anastasiei and Ioan Bucătaru, to whom I express my utmost gratitude.

Iași, 2010

Acad. Radu Miron
Preface of the book *Geometria configurațiilor Myller* written in 1966 by Octav Mayer, former member of the Romanian Academy (Translated from Romanian)

One can bring to the great scientists, who passed away, various homages. Some times, the strong wind of the progress wipes out the trace of their steps. To not forget them means to continue their work, connecting them to the living present. In this sense, this scientific work is the most precious homage which can be dedicated to Alexandru Myller.

Initiator of a modern education of Mathematics at the University of Iassy, founder of the Geometry School, which is still flourishing today, in the third generation, Alexandru Myller was also a hardworking researcher, well known inside the country and also abroad due to his papers concerning Integral Equations and Differential Geometry.

Our Academy elected him as a member and published his scientific work. The “A. Humboldt” University from Berlin awarded him the title of “doctor Honoris Causa” for “special efforts in creating an independent Romanian Mathematical School”.

Some of Alexandru Myller’s discoveries have penetrated fruitfully the impetuous torrent of ideas, which have changed the Science of Geometry
during the first decades of the century. Among others, it is the case of “Myller configurations”. The reader would be perhaps interested to find out some more details about how he discovered these configurations.

Let us imagine a surface $S$, on which there is drawn a curve $C$ and, along it, let us circumscribe to $S$ a developing surface $\Sigma$; then we apply (develop) the surface $\Sigma$ on a plane $\pi$. The points of the curve $C$, connected to the respective tangent planes, which after are developed overlap each other on the plane $\pi$, are going to represent a curve $C'$ into this plane. In the same way, a series ($d$) of tangent directions to the surface $S$ at the points of the curve $C$ becomes developing, on the plane $\pi$, a series ($d'$) of directions getting out from the points of the curve $C'$. The directions ($d$) are parallel on the surface $S$ if the directions ($d'$) are parallel in the common sense.

Going from this definition of T. Levi-Civita parallelism (valid in the Euclidian space), Alexandru Myller arrived to a more general concept in a sensible process of abstraction. Of course, it was not possible to leave the curve $C$ aside, neither the directions ($d$) whose parallelism was going to be defined in a more general sense. What was left aside was the surface $S$.

For the surface $\Sigma$ was considered, in a natural way, the enveloping of the family of planes constructed from the points of the curve $C$, planes in which are given the directions ($d$). Keeping unchanged the remainder of the definition one gets to what Alexandru Myller called “parallelism into a family of planes”.

A curve $C$, together with a family of planes on its points and a family of given directions (in an arbitrary way) in these planes constitutes what the author called a “Myller configuration”.

It was considered that this new introduced notion had a central place into the classical theory of surfaces, giving the possibility to interpret and link among them many particular facts. It was obvious that this notion can be successfully applied in the Geometry of other spaces different from the Euclidian one. Therefore the foreworded study worthed all the efforts made by the author, a valuable mathematician from the third generation
of the Iași Geometry School.

By recommending this work, we believe that the reader (who needs only basic Differential Geometry) will be attracted by the clear lecture of Radu Miron and also by the beauty of the subject, which can still be developed further on.

Octav Mayer, 1966
A short biography of Al. Myller

ALEXANDRU MYLLER was born in Bucharest in 1879, and died in Iași on the 4th of July 1965. Romanian mathematician. Honorary Member (27 May 1938) and Honorific Member (12 August 1948) of the Romanian Academy. High School and University studies (Faculty of Science) in Bucharest, finished with a bachelor degree in mathematics.

He was a Professor of Mathematics at the Pedagogical Seminar; he sustained his PhD thesis Gewohnliche Differential Gleichungen Höherer Ordnung, in Göttingen, under the scientific supervision of David Hilbert.

He worked as a Professor at the Pedagogical Seminar and the School
of Post and Telegraphy (1907 - 1908), then as a lecturer at the University of Bucharest (1908 - 1910). He is appointed Professor of Analytical Geometry at the University of Iași (1910 - 1947); from 1947 onward - consultant Professor. In 1910, he sets up the Mathematical Seminar at the “Alexandru Ioan Cuza“ University of Iași, which he endows with a library full of didactic works and specialty journals, and which, nowadays, bears his name. Creator of the mathematical school of Iași, through which many reputable Romanian mathematicians have passed, he conveyed to us research works in fields such as integral equations, differential geometry and history of mathematics. He started his scientific activity with papers on the integral equations theory, including extensions of Hilbert’s results, and then he studied integral equations and self-adjoint of even and odd order linear differential equations, being the first mathematician to introduce integral equations with skew symmetric kernel.

He was the first to apply integral equations to solve problems for partial differential equations of hyperbolic type. He was also interested in the differential geometry, discovering a generalization of the notion of parallelism in the Levi-Civita sense, and introducing the notion today known as concurrence in the Myller sense. All these led to “the differential geometry of Myller configurations” (R. Miron, 1960). Al. Myller, Gh. Țîțeica and O. Mayer have created “the differential centro-affine geometry“, which the history of mathematics refers to as “purely Romanian creation!“ He was also concerned with problems from the geometry of curves and surfaces in Euclidian spaces. His research outcomes can be found in the numerous memoirs, papers and studies, published in Iași and abroad: Development of an arbitrary function after Bessel's functions (1909); Parallelism in the Levi-Civita sense in a plane system (1924); The differential centro-affine geometry of the plane curves (1933) etc. - within the volume “Mathematical Writings“ (1959), Academy Publishing House. Alexandru Myller has been an Honorary Member of the Romanian Academy since 1938. Also, he was a Doctor Honoris Causa of the Humboldt University of Berlin.

Excerpt from the volume “Members of the Romanian Academy 1866
- 1999. Dictionary." - Dorina N. Rusu, page 360.
Introduction

In the differential geometry of curves in the Euclidean space $E_3$ one introduces, along a curve $C$, some versor fields, as tangent, principal normal or binormal, as well as some plane fields as osculating, normal or rectifying planes.

More generally, we can consider a versor field $(C, \xi)$ or a plane field $(C, \pi)$. A pair $\{(C, \xi), (C, \pi)\}$ for which $\xi \in \pi$, has been called in 1960 [23] by the present author, a Myller configuration in the space $E_3$, denoted by $\mathcal{M}(C, \xi, \pi)$. When the planes $\pi$ are tangent to $C$ then we have a tangent Myller configuration $\mathcal{M}_t(C, \xi, \pi)$.

Academician Alexandru Myller studied in 1922 the notion of parallelism of $(C, \xi)$ in the plane field $(C, \pi)$ obtaining an interesting generalization of the famous parallelism of Levi-Civita on the curved surfaces. These investigations have been continued by Octav Mayer which introduced new fundamental invariants for $\mathcal{M}(C, \xi, \pi)$. The importance of these studies was underlined by Levi Civita in Addendum to his book *Lezioni di calcolo differenziale assoluto*, 1925.

Now, we try to make a systematic presentation of the geometry of Myller configurations $\mathcal{M}(C, \xi, \pi)$ and $\mathcal{M}_t(C, \xi, \pi)$ with applications to the differential geometry of surfaces and to the geometry of nonholonomic manifolds in the Euclidean space $E_3$.

Indeed, if $C$ is a curve on the surface $S \subset E_3$, $s$ is the natural parameter of curve $C$ and $\xi(s)$ is a tangent versor field to $S$ along $C$ and $\pi(s)$ is tangent planes field to $S$ along $C$, we have a tangent Myller configuration $\mathcal{M}_t(C, \xi, \pi)$ intrinsic associated to the geometric objects $S, C, \xi$. Conse-
quently, the geometry of the field \((\mathcal{C}, \xi)\) on surface \(S\) is the geometry of the associated Myller configurations \(\mathfrak{M}_i(\mathcal{C}, \xi, \pi)\). It is remarkable that the geometric theory of \(\mathfrak{M}_i\) is a particular case of that of general Myller configuration \(\mathfrak{M}(\mathcal{C}, \xi, \pi)\).

For a Myller configuration \(\mathfrak{M}(\mathcal{C}, \xi, \pi)\) we determine a Darboux frame, the fundamental equations and a complete system of invariants \(G, K, T\) called, geodesic curvature, normal curvature and geodesic torsion, respectively, of the versor field \((\mathcal{C}, \xi)\) in Myller configuration \(\mathfrak{M}(\mathcal{C}, \xi, \pi)\). A fundamental theorem, when the functions \((G(s), K(s), T(s))\) are given, can be proven.

The invariant \(G(s)\) was discovered by Al. Myller (and named by him the deviation of parallelism). \(G(s) = 0\) on curve \(C\) characterizes the parallelism of versor field \((\mathcal{C}, \xi)\) in \(\mathfrak{M}, [23], [24], [31], [32], [33]\). The second invariant \(K(s)\) was introduced by O. Mayer (it was called the curvature of parallelism). Third invariant \(T(s)\) was found by E. Bortolotti [3].

In the particular case, when \(\mathfrak{M}\) is a tangent Myller configuration \(\mathfrak{M}_t(\mathcal{C}, \xi, \pi)\) associated to a tangent versor field \((\mathcal{C}, \xi)\) on a surface \(S\), the \(G(s)\) is an intrinsic invariant and \(G(s) = 0\) along \(C\) leads to the Levi Civita parallelism.

In configurations \(\mathfrak{M}_t(\mathcal{C}, \xi, \pi)\) there exists a natural versor field \((\mathcal{C}, \overline{\alpha})\), where \(\overline{\alpha}(s)\) are the tangent versors to curve \(C\). The versor field \((\mathcal{C}, \overline{\alpha})\) has a Darboux frame \(\mathcal{R} = (P(s); \overline{\alpha}, \overline{\nu}, \overline{\nu})\) in \(\mathfrak{M}_t\) where \(\overline{\nu}(s)\) is normal to plane \(\pi(s)\) and \(\overline{\nu}^s(s) = \overline{\nu}(s) \times \overline{\alpha}(s)\). The moving equations of \(\mathcal{R}\) are:

\[
\begin{align*}
\frac{d\overline{\nu}}{ds} &= \overline{\alpha}(s), \quad s \in (s_1, s_2) \\
\frac{d\overline{\alpha}}{ds} &= \kappa_g(s)\overline{\nu} + \kappa_n(s)\overline{\nu}, \\
\frac{d\overline{\nu}^s}{ds} &= -\kappa_g(s)\overline{\alpha} + \tau_g(s)\overline{\nu}, \\
\frac{d\overline{\nu}}{ds} &= -\kappa_n(s)\overline{\alpha} - \tau_g(s)\overline{\nu}.
\end{align*}
\]

The functions \(\kappa_g(s), \kappa_n(s)\) and \(\tau_g(s)\) form a complete system of invariants
of the curve in $M_t$.

A theorem of existence and uniqueness for the versor fields $(C, \overline{\alpha})$ in $M_t(C, \overline{\alpha}, \pi)$, when the invariant $\kappa_g(s)$, $k_n$ and $\tau_g(s)$ are given, is proved. The function $\kappa_g(s)$ is called the geodesic curvature of the curve $C$ in $M_t$; $\kappa_n(s)$ is the normal curvature and $\tau_g(s)$ is the geodesic torsion of $C$ in $M_t$.

The condition $\kappa_g(s) = 0$, $\forall s \in (s_1, s_2)$ characterizes the geodesic (autoparallel lines) in $M_t$; $\kappa_n(s) = 0$, $\forall s \in (s_1, s_2)$ give us the asymptotic lines and $\tau_g(s) = 0$, $\forall s \in (s_1, s_2)$ characterizes the curvature lines $C$ in $M_t$.

One can remark that in the case when $M_t$ is the associated Myller configuration to a curve $C$ on a surface $S$ we obtain the classical theory of curves on surface $S$. It is important to remark that the Mark Krien's formula (2.10.3) leads to the integral formula (2.10.9) of Gauss-Bonnet for surface $S$, studied by R. Miron in the book [62].

Also, if $C$ is a curve of a nonholonomic manifold $E^2_3$ in $E_3$, we uniquely determine a Myller configuration $M_t(C, \overline{\alpha}, \pi)$ in which $(C, \overline{\alpha})$ is the tangent versor field to $C$ and $(C, \pi)$ is the tangent plane field to $E^2_3$ along $C$, [54], [64], [65].

In this case the geometry of $M_t$ is the geometry of curves $C$ in the nonholonomic manifolds $E^2_3$. Some new notions can be introduced as: concurrence in Myller sense of versor fields $(C, \overline{\xi})$ on $E^2_3$, extremal geodesic torsion, the mean torsion and total torsion of $E^2_3$ at a point, a remarkable formula for geodesic torsion and an indicatrix of Bonnet for geodesic torsion, which is not reducible to a pair of equilateral hyperbolas, as in the case of surfaces.

The nonholonomic planes, nonholonomic spheres of Gr. Moisil, can be studied by means of techniques from the geometry of Myller configurations $M_t$.

We finish the present introduction pointing out some important developments of the geometry of Myller configurations:

The author extended the notion of Myller configuration in Riemannian Geometry [24], [25], [26], [27]. Izu Vaisman [76] has studied the
Myller configurations in the symplectic geometry.

Mircea Craioveanu realized a nice theory of Myller configurations in infinit dimensional Riemannian manifolds. Gheorghe Gheorghiev developed the configuration $\mathcal{M}$, [11], [55] in the geometry of versor fields in the Euclidean space and applied it in hydromechanics.

N.N. Mihalieanu studied the Myller configurations in Minkowski spaces [61]. For Myller configurations in a Finsler, Lagrange or Hamilton spaces we refer to the paper [58] and to the books of R. Miron and M. Anastasiei [68], R. Miron, D. Hrimiuc, H. Shimada and S. Sabău [70].

All these investigations underline the usefulness of geometry of Myller configurations in differential geometry and its applications.
Chapter 1

Versor fields. Plane fields in $E_3$

First of all we investigate the geometry of a versor field $(C, \xi)$ in the Euclidean space $E_3$, introducing an invariant frame of Frenet type, the moving equations of this frame, invariants and proving a fundamental theorem. The invariants are called $K_1$-curvature and $K_2$-torsion of $(C, \xi)$. Geometric interpretations for $K_1$ and $K_2$ are pointed out. The parallelism of $(C, \xi)$, concurrence of $(C, \xi)$ and the enveloping of versor field $(C, \xi)$ are studied, too.

A similar study is made for the plane fields $(C, \pi)$ taking into account the normal versor field $(C, \nu)$, with $\nu(s)$ a normal versor to the plane $\pi(s)$.

1.1 Versor fields $(C, \xi)$

In the Euclidian space a versor field $(C, \xi)$ can be analytical represented in an orthonormal frame $\mathcal{R} = (0, \vec{i}_1, \vec{i}_2, \vec{i}_3)$, by

$$(1.1.1) \quad \tau = \tau(s), \quad \xi = \xi(s), \quad s \in (s_1, s_2)$$
where $s$ is the arc length on curve $C$,

$$\overline{r}(s) = \overline{OP}(s) = x(s)\overrightarrow{\mathbf{i}}_1 + y(s)\overrightarrow{\mathbf{i}}_2 + z(s)\overrightarrow{\mathbf{i}}_3$$

and

$$\overline{\xi}(s) = \overline{\xi}(\overline{r}(s)) = \overline{PQ} = \xi^1(s)\overrightarrow{\mathbf{i}}_1 + \xi^2(s)\overrightarrow{\mathbf{i}}_2 + \xi^3(s)\overrightarrow{\mathbf{i}}_3,$$

$$\|\overline{\xi}(s)\|^2 = \langle \overline{\xi}(s), \overline{\xi}(s) \rangle = 1.$$  

All geometric objects considered in this book are assumed to be of class $C^k, k \geq 3$, and sometimes of class $C^\infty$. The pair $(C, \overline{\xi})$ has a geometrical meaning. It follows that the pair $(C, \frac{d\overline{\xi}}{ds})$ has a geometric meaning, too. Therefore, the norm:

$$K_1(s) = \left\| \frac{d\overline{\xi}(s)}{ds} \right\|$$

is an invariant of the field $(C, \overline{\xi})$.

We denote

$$\overline{\xi}_1(s) = \overline{\xi}(s)$$

and let $\overline{\xi}_2(s)$ be the versor of vector $\frac{d\overline{\xi}_1}{ds}$. Thus we can write

$$\frac{d\overline{\xi}_1}{ds} = K_1(s)\overline{\xi}_2(s).$$

Evidently, $\overline{\xi}_2(s)$ is orthogonal to $\overline{\xi}_1(s)$.

It follows that the frame

$$\mathcal{R}_F = (P(s); \overline{\xi}_1, \overline{\xi}_2, \overline{\xi}_3), \quad \overline{\xi}_3(s) = \overline{\xi}_1 \times \overline{\xi}_2$$

is orthonormal, positively oriented and has a geometrical meaning. $\mathcal{R}_F$ is called the Frenet frame of the versor field $(C, \overline{\xi})$.

We have:

**Theorem 1.1.1** The moving equations of the Frenet frame $\mathcal{R}_F$ are:

$$\frac{d\overline{r}}{ds} = a_1(s)\overrightarrow{\mathbf{i}}_1 + a_2(s)\overrightarrow{\mathbf{i}}_2 + a_3(s)\overrightarrow{\mathbf{i}}_3 \quad a_1^2(s) + a_2^2(s) + a_3^2(s) = 1$$
and

\[
\begin{align*}
\frac{d\xi_1}{ds} &= K_1(s)\xi_2, \\
\frac{d\xi_2}{ds} &= -K_1(s)\xi_1(s) + K_2(s)\xi_3, \\
\frac{d\xi_3}{ds} &= -K_2(s)\xi_2(s),
\end{align*}
\]

(1.1.6)

where \( K_1(s) > 0 \). The functions \( K_1(s), K_2(s), a_1(s), a_2(s), a_3(s), s \in (s_1, s_2) \) are invariants of the versor field \((C, \xi)\).

The proof does not present difficulties.

The invariant \( K_1(s) \) is called the curvature of \((C, \xi)\) and has the same geometric interpretation as the curvature of a curve in \( E_3 \). \( K_2(s) \) is called the torsion and has the same geometrical interpretation as the torsion of a curve in \( E_3 \).

The equations (1.1.5), (1.1.6) will be called the fundamental or Frenet equations of the versor field \((C, \xi)\).

In the case \( a_1(s) = 1, a_2(s) = 0, a_3(s) = 0 \) the tangent versor \( \frac{d\tau}{ds} \) is denoted by

\[
(1.1.5') \quad \alpha(s) = \frac{d\tau}{ds}(s)
\]

The equations (1.1.5), (1.1.6) are then the Frenet equations of a curve in the Euclidian space \( E_3 \).

For the versor field \((C, \xi)\) we can formulate a fundamental theorem:

**Theorem 1.1.2** If the functions \( K_1(s) > 0, K_2(s), a_1(s), a_2(s), a_3(s), (a_1^2 + a_2^2 + a_3^2 = 1) \) of class \( C^\infty \) are apriori given, \( s \in [a, b] \) there exists a curve \( C : [a, b] \to E_3 \) parametrized by arclengths and a versor field \( \xi(s), s \in [a, b], \) whose the curvature, torsion and the functions \( a_i(s) \) are \( K_1(s), K_2(s) \) and \( a_i(s) \). Any two such versor fields \((C, \xi)\) differ by a proper Euclidean motion.
For the proof one applies the same technique like in the proof of Theorem 11, p. 45 from [75], [76].

Remark 1.1.1 1. If $K_1(s) = 0$, $s \in (s_1, s_2)$ the versors $\xi_1(s)$ are parallel in $E_3$ along the curve $C$.
2. The versor field $(C, \xi)$ determines a ruled surface $S(C, \overline{\xi})$.
3. The surface $S(C, \overline{\xi})$ is a cylinder iff the invariant $K_1(s)$ vanishes.
4. The surface $S(C, \overline{\xi})$ is with director plane iff $K_2(s) = 0$.
5. The surface $S(C, \overline{\xi})$ is developing iff the invariant $a_3(s)$ vanishes.

If the surfaces $S(C, \overline{\xi})$ is a cone we say that the versors field $(C, \overline{\xi})$ is concurrent.

Theorem 1.1.3 A necessary and sufficient condition for the versor field $(C, \overline{\xi})$, $(K_1(s) \neq 0)$, to be concurrent is the following

\[
\frac{d}{ds} \left( \frac{a_2(s)}{K_1(s)} \right) - a_1(s) = 0, \quad a_3(s) = 0, \quad \forall s \in (s_1, s_2).
\]

1.2 Spherical image of a versor field $(C, \overline{\xi})$

Consider the sphere $\Sigma$ with center a fix point $O \in E_3$ and radius 1.

Definition 1.2.1 The spherical image of the versor field $(C, \overline{\xi})$ is the curve $C^*$ on sphere $\Sigma$ given by $\overline{\xi^*}(s) = \overrightarrow{OP^*}(s) = \overline{\xi}(s), \forall s \in (s_1, s_2)$.

From this definition we have:

\[
(1.2.1) \quad d\overline{\xi^*}(s) = \overline{\xi_2(s)}K_1(s)ds.
\]

Some immediate properties:
1. It follows

\[
(1.2.2) \quad ds^* = K_1(s)ds.
\]

Therefore:
1.3 Plane fields \((C, \pi)\)

The arc length of the curve \(C^*\) is

\[
(1.2.3) \quad s^* = s_0^* + \int_{s_0}^s K_1(\sigma) d\sigma
\]

where \(s_0^*\) is a constant and \([s_0, s] \subset (s_1, s_2)\).

2. The curvature \(K_1(s)\) of \((C, \xi)\) at a point \(P^*(s)\) is expressed by

\[
(1.2.4) \quad K_1(s) = \frac{ds^*}{ds}
\]

3. \(C^*\) is reducible to a point iff \((C, \xi)\) is a parallel versor field in \(E_3\).

4. The tangent line at point \(P^* \in C^*\) is parallel with principal normal line of \((C, \xi)\).

5. Since \(\vec{\xi}_1(s) \times \vec{\xi}_2(s) = \vec{\xi}_3(s)\) it follows that the direction of binormal versor field \(\vec{\xi}_3\) is the direction tangent to \(\Sigma\) orthogonal to tangent line of \(C^*\) at point \(P^*\).

6. The geodesic curvature \(\kappa_g\) of the curve \(C^*\) at a point \(P^*\) verifies the equation

\[
(1.2.5) \quad \kappa_g ds^* = K_2 ds.
\]

7. The versor field \((C, \bar{\xi})\) is of null torsion (i.e. \(K_2(s) = 0\)) iff \(\kappa_g = 0\). In this case \(C^*\) is an arc of a great circle on \(\Sigma\).

1.3 Plane fields \((C, \pi)\)

A plane field \((C, \pi)\) is defined by the versor field \((C, \varpi(s))\) where \(\varpi(s)\) is normal to \(\pi(s)\) in every point \(P(s) \in C\). We assume the \(\pi(s)\) is oriented. Consequently, \(\varpi(s)\) is well determined.

Let \(\mathcal{R}_\pi = (P(s), \varpi_1(s), \varpi_2(s), \varpi_3(s))\) be the Frenet frame, \(\varpi(s) = \varpi_1(s)\), of the versor field \((C, \varpi)\).

It follows that the fundamental equations of the plane field \((C, \pi)\) are:

\[
(1.3.1) \quad \frac{d\varpi}{ds}(s) = b_1(s)\varpi_1 + b_2(s)\varpi_2 + b_3(s)\varpi_3, \quad b_1^2 + b_2^2 + b_3^2 = 1.
\]
\[
\begin{align*}
\frac{d\nu_1}{ds} &= \chi_1(s)\nu_2, \\
\frac{d\nu_2}{ds} &= -\chi_1\nu_1 + \chi_2(s)\nu_3 \\
\frac{d\nu_3}{ds} &= -\chi_2(s)\nu_2.
\end{align*}
\] (1.3.2)

The invariant \(\chi_1(s) = \left\| \frac{d\nu_1}{dr} \right\|\) is called the curvature and \(\chi_2(s)\) is the torsion of the plane field \((C, \pi(s))\).

The following properties hold:

1. The characteristic straight lines of the field of planes \((C, \pi)\) cross through the corresponding point \(P(s) \in C\) iff the invariant \(b_1(s) = 0, \forall s \in (s_1, s_2)\).

2. The planes \(\pi(s)\) are parallel along the curve \(C\) iff the invariant \(\chi_1(s)\) vanishes, \(\forall s \in (s_1, s_2)\).

3. The characteristic lines of the plane field \((C, \pi)\) are parallel iff the invariant \(\chi_2(s) = 0, \forall s \in (s_1, s_2)\).

4. The versor field \((C, \nu_3)\) determines the directions of the characteristic line of \((C, \pi)\).

5. The versor field \((C, \nu_3)\), with \(\chi_2(s) \neq 0\), is concurrent iff:

\[b_1(s) = 0, \ b_3(s) + \frac{d}{dt} \left( \frac{b_2(s)}{\chi_2(s)} \right) = 0.\]

6. The curve \(C\) is on orthogonal trajectory of the generatrices of the ruled surface \(\mathcal{R}(C, \pi_3)\) if \(b_3(s) = 0, \forall s \in (s_1, s_2)\).

7. By means of equations (1.3.1), (1.3.2) we can prove a fundamental theorem for the plane field \((C, \pi)\).
Chapter 2

Myller configurations
\(\mathcal{M}(C, \xi, \pi)\)

The notions of versor field \((C, \xi)\) and the plane field \((C, \pi)\) along to the same curve \(C\) lead to a more general concept named Myller configuration \(\mathcal{M}(C, \xi, \pi)\), in which every versor \(\xi(s)\) belongs to the corresponding plane \(\pi(s)\) at point \(P(s) \in C\). The geometry of \(\mathcal{M} = \mathcal{M}(C, \xi, \pi)\) is much more rich as the geometries of \((C, \xi)\) and \((C, \pi)\) separately taken. For \(\mathcal{M}\) one can define its geometric invariants, a Darboux frame and introduce a new idea of parallelism or concurrence of versor \((C, \xi)\) in \(\mathcal{M}\). The geometry of \(\mathcal{M}\) is totally based on the fundamental equations of \(\mathcal{M}\). The basic idea of this construction belongs to Al. Myller [31], [32], [33], [34] and it was considerable developed by O. Mayer [20], [21], R. Miron [23], [24], [62] (who proposed the name of Myller Configuration and studied its complete system of invariants).

2.1 Fundamental equations of Myller configuration

Definition 2.1.1 A Myller configuration \(\mathcal{M} = \mathcal{M}(C, \xi, \pi)\) in the Euclidean space \(E_3\) is a pair \((C, \xi), (C, \pi)\) of versor field and plane field,
having the property: every $\xi(s)$ belongs to the plane $\pi(s)$.

Let $\overline{\nu}(s)$ be the normal versor to plane $\pi(s)$. Evidently $\overline{\nu}(s)$ is uniquely determined if $\pi(s)$ is an oriented plane for $\forall s \in (s_1, s_2)$.

By means of versors $\overline{\xi}(s), \overline{\nu}(s)$ we can determine the Darboux frame of $\mathfrak{N}$:

\begin{equation}
\mathcal{R}_D = (P(s); \overline{\xi}(s), \overline{\mu}(s), \overline{\nu}(s)),
\end{equation}

where

\begin{equation}
\overline{\mu}(s) = \overline{\nu}(s) \times \overline{\xi}(s).
\end{equation}

$\mathcal{R}_D$ is geometrically associated to $\mathfrak{N}$. It is orthonormal and positively oriented.

Since the versors $\overline{\xi}(s), \overline{\mu}(s), \overline{\nu}(s)$ have a geometric meaning, the same properties have the vectors $\frac{d\xi}{ds}, \frac{d\mu}{ds}$ and $\frac{d\nu}{ds}$.

Therefore, we can prove, without difficulties:

**Theorem 2.1.1** The moving equations of the Darboux frame of $\mathfrak{N}$ are as follows:

\begin{equation}
\frac{d\overline{\nu}}{ds} = \overline{\alpha}(s) = c_1(s)\overline{\xi} + c_2(s)\overline{\mu} + c_3(s)\overline{\nu}; \quad c_1^2 + c_2^2 + c_3^2 = 1
\end{equation}

and

\begin{equation}
\begin{aligned}
\frac{d\overline{\xi}}{ds} &= G(s)\overline{\mu} + K(s)\overline{\nu}, \\
\frac{d\overline{\mu}}{ds} &= -G(s)\overline{\xi} + T(s)\overline{\nu}, \\
\frac{d\overline{\nu}}{ds} &= -K(s)\overline{\xi} - T(s)\overline{\mu}
\end{aligned}
\end{equation}

and $c_1(s), c_2(s), c_3(s); G(s), K(s), T(s)$ are uniquely determined and are invariants.

The previous equations are called the *fundamental equations* of the Myller configurations $\mathfrak{N}$.

Terms:
2.1. Fundamental equations of Myller configuration

\(G(s)\) – is the geodesic curvature, \(K(s)\) – is the normal curvature, \(T(s)\) – is the geodesic torsion of the versor field \((C, \xi)\) in Myller configuration \(\mathcal{M}\).

For \(\mathcal{M}\) a fundamental theorem can be stated:

**Theorem 2.1.2** Let be a priori given \(C^\infty\) functions \(c_1(s), c_2(s), c_3(s), [c_1^2 + c_2^2 + c_3^2 = 1], G(s), K(s), T(s), s \in [a, b]\). Then there is a Myller configuration \(\mathcal{M}(C, \xi, \pi)\) for which \(s\) is the arclength of curve \(C\) and the given functions are its invariants. Two such configurations differ by a proper Euclidean motion.

*Proof.* By means of given functions \(c_1(s), \ldots, G(s), \ldots\) we can write the system of differential equations (2.1.3), (2.1.4). Let the initial conditions \(\tau_0 = \overline{OP}_0, (\xi_0, \mu_0, \nu_0)\), an orthonormal, positively oriented frame in \(E_3\).

From (2.1.4) we find an unique solution \((\xi(s), \mu(s), \nu(s)), s \in [a, b]\) with the property

\[\xi(s_0) = \xi_0, \quad \mu(s_0) = \mu_0, \quad \nu(s_0) = \nu_0,\]

with \(s_0 \in [a, b]\) and \((\xi(s), \mu(s), \nu(s))\) being an orthonormal, positively oriented frame.

Then consider the following solution of (2.1.3)

\[\tau(s) = \tau_0 + \int_{s_0}^{s} [c_1(\sigma)\xi(\sigma) + c_2(\sigma)\mu(\sigma) + c_3(\sigma)\nu(\sigma)]d\sigma,\]

which has the property \(\tau(s_0) = \tau_0\), and \(\left\|\frac{d\tau}{ds}\right\| = 1\).

Thus \(s\) is the arc length on the curve \(\tau = r(s)\).

Now, consider the configuration \(\mathcal{M}(C, \xi, \pi(s)), \pi(s)\) being the plane orthogonal to versor \(\nu(s)\) at point \(P(s)\).

We can prove that \(\mathcal{M}(C, \xi, \pi)\) has as invariants just \(c_1, c_2, c_3, G, K, T\).

The fact that two configurations \(\mathcal{M}\) and \(\mathcal{M}'\), obtained by changing the initial conditions \((\tau_0; \xi_0; \mu_0; \nu_0)\) to \((\tau'_0; \xi'_0; \mu'_0; \nu'_0)\) differ by a proper
Euclidean motions follows from the property that the exists an unique Euclidean motion which apply \((\mathbf{r}_0; \xi_0; \mu_0; \nu_0)\) to \((\mathbf{r}', \xi', \mu', \nu')\).

## 2.2 Geometric interpretations of invariants

The invariants \(c_1(s), c_2(s), c_3(s)\) have simple geometric interpretations:

\[
c_1(s) = \cos \langle \mathbf{r}, \xi \rangle, \quad c_2(s) = \cos \langle \mathbf{r}, \mu \rangle, \quad c_3 = \cos \langle \mathbf{r}, \nu \rangle.
\]

We can find some interpretation of the invariants \(G(s), K(s)\) and \(T(s)\) considering a variation of Darboux frame

\[
R_D(P(s); \xi(s), \mu(s), \nu(s)) \rightarrow R'_D(P' + \Delta s, \xi' + \Delta s, \mu' + \Delta s, \nu' + \Delta s),
\]

obtained by the Taylor expansion

\[
\begin{align*}
\tau(s + \Delta s) &= \tau(s) + \frac{\Delta s}{1!} \frac{d\tau}{ds} + \frac{(\Delta s)^2}{2!} \frac{d^2\tau}{ds^2} + \ldots + \\
&\quad + \frac{(\Delta s)^n}{n!} \left( \frac{d^n\tau}{ds^n} + \bar{\omega}_0(s, \Delta s) \right), \\
\bar{\xi}(s + \Delta s) &= \bar{\xi}(s) + \frac{\Delta s}{1!} \frac{d\xi}{ds} + \frac{(\Delta s)^2}{2!} \frac{d^2\xi}{ds^2} + \ldots + \\
&\quad + \frac{(\Delta s)^n}{n!} \left( \frac{d^n\xi}{ds^n} + \bar{\omega}_1(s, \Delta s) \right), \\
(2.2.1)
\bar{\mu}(s + \Delta s) &= \bar{\mu}(s) + \frac{\Delta s}{1!} \frac{d\mu}{ds} + \frac{(\Delta s)^2}{2!} \frac{d^2\mu}{ds^2} + \ldots + \\
&\quad + \frac{(\Delta s)^n}{n!} \left( \frac{d^n\mu}{ds^n} + \bar{\omega}_2(s, \Delta s) \right), \\
\bar{\nu}(s + \Delta s) &= \bar{\nu}(s) + \frac{\Delta s}{1!} \frac{d\nu}{ds} + \frac{(\Delta s)^2}{2!} \frac{d^2\nu}{ds^2} + \ldots + \\
&\quad + \frac{(\Delta s)^n}{n!} \left( \frac{d^n\nu}{ds^n} + \bar{\omega}_3(s, \Delta s) \right),
\end{align*}
\]
where
\[
\lim_{\Delta s \to 0} \overline{w}_i(s, \Delta s) = 0, \quad (i = 0, 1, 2, 3).
\]

Using the fundamental formulas (2.1.3), (2.1.4) we can write, for \( n = 1 \):
\[
\begin{align*}
\tau(s + \Delta s) &= \tau(s) + \Delta s (c_1 \xi + c_2 \mu + c_3 \nu + \overline{w}_0(s, \Delta s)), \\
\xi(s + \Delta s) &= \xi(s) + \Delta s (G(s) \xi + K(s) \nu + \overline{w}_1(s, \Delta s)), \\
\mu(s + \Delta s) &= \mu(s) + \Delta s (-G(s) \xi + T(s) \nu + \overline{w}_2(s, \Delta s)) \\
\nu(s + \Delta s) &= \nu(s) + \Delta s (-K(s) \xi - T(s) \mu + \overline{w}_3(s, \Delta s)).
\end{align*}
\]
and (2.2.2) being verified.

Let \( \overline{\xi}(s + \Delta s) \) be the orthogonal projection of the versor \( \xi(s + \Delta s) \) on the plane \( \pi(s) \) at point \( P(s) \) and let \( \Delta \psi_1 \) be the oriented angle of the versors \( \xi(s), \overline{\xi}(s + \Delta s) \). Thus, we have

**Theorem 2.2.1** *The invariant \( G(s) \) of versor field \( (C, \xi) \) in Myller configuration \( \mathcal{M}(C, \overline{\xi}, \pi) \) is given by*

\[
G(s) = \lim_{\Delta s \to 0} \frac{\Delta \psi_1}{\Delta s}.
\]

By means of second formula (2.2.3), this Theorem can be proved without difficulties.

Therefore the name of *geodesic curvature* of \( (C, \overline{\xi}) \) in \( \mathcal{M} \) is justified.

Consider the plane \( (P(s); \overline{\xi}(s), \nu(s)) \)-called the normal plan of \( \mathcal{M} \) which contains the versor \( \overline{\xi}(s) \).

Let be the vector \( \overline{\xi}^*(s + \Delta s) \) the orthogonal projection of versor \( \overline{\xi}(s + \Delta s) \) on the normal plan \( (P(s); \overline{\xi}(s), \nu(s)) \). The angle \( \Delta \psi_2 = \angle(\overline{\xi}(s), \overline{\xi}^*(s + \Delta s)) \) is given by the formula

\[
\sin \Delta \psi_2 = \frac{(\overline{\xi}(s), \overline{\xi}(s + \Delta s), \mu(s))}{\|\overline{\xi}^*(s + \Delta s)\|}.
\]
By (2.2.3) we obtain
\[
\sin \Delta \psi_2 = \frac{K(s) + \langle \xi(s), \varpi_1(s, \Delta s), \varpi(s) \rangle}{\|\xi^*(s + \Delta s)\|} \Delta s.
\]

Consequently, we have:

**Theorem 2.2.2** The invariant $K(s)$ has the following geometric interpretation

\[
K(s) = \lim_{\Delta s \to 0} \frac{\Delta \psi_2}{\Delta s}.
\]

Based on the previous result we can call $K(s)$ the normal curvature of $(C, \xi)$ in $\mathfrak{M}$.

A similar interpretation can be done for the invariant $T(s)$.

**Theorem 2.2.3** The function $T(s)$ has the interpretation:

\[
T(s) = \lim_{\Delta s \to 0} \frac{\Delta \psi_3}{\Delta s},
\]

where $\Delta \psi_3$ is the oriented angle between $\varpi(s)$ and $\varpi^*(s + \Delta s)$-which is the orthogonal projection of $\varpi(s + \Delta s)$ on the normal plane $(P(s); \varpi(s), \nu(s))$.

This geometric interpretation allows to give the name geodesic torsion for the invariant $T(s)$.

### 2.3 The calculus of invariants $G, K, T$

The fundamental formulae (2.1.3), (2.1.4) allow to calculate the expressions of second derivatives of the versors of Darboux frame $\mathcal{R}_D$. We have:

\[
\frac{d^2 \tau}{ds^2} = \left( \frac{dc_1}{ds} - Gc_2 - Kc_3 \right) \xi + \left( \frac{dc_2}{ds} - Gc_1 - Tc_3 \right) \varpi + \left( \frac{dc_3}{ds} + Kc_1 + Tc_2 \right) \nu
\]

(2.3.1)
and
\[
\frac{d^2 \xi}{ds^2} = -(G^2 + T^2) \xi + \left( \frac{dG}{ds} - KT \right) \mu + \left( \frac{dK}{ds} + GT \right) \nu,
\]
\[
\frac{d^2 \mu}{ds^2} = - \left( \frac{dG}{ds} + KT \right) \xi - (G^2 + T^2) \mu + \left( \frac{dT}{ds} - GT \right) \nu,
\]
\[
\frac{d^2 \nu}{ds^2} = \left( -\frac{dK}{ds} + GT \right) \xi - \left( \frac{dT}{ds} + GT \right) \mu - (K^2 + T^2) \nu.
\]

These formulae will be useful in the next part of the book.

From the fundamental equations (2.1.3), (2.1.4) we get

**Theorem 2.3.1** The following formulae for invariants \( G(s), K(s) \) and \( T(s) \) hold:

\[
G(s) = \left\langle \bar{\xi}, \frac{d\xi}{ds}, \nu \right\rangle,
\]
\[
K(s) = \left\langle \frac{d\bar{\xi}}{ds}, \nu \right\rangle = - \left\langle \bar{\xi}, \frac{d\nu}{ds} \right\rangle, T(s) = \left\langle \bar{\xi}, \nu, \frac{d\nu}{ds} \right\rangle.
\]

Evidently, these formulae hold in the case when \( s \) is the arclength of the curve \( C \).

### 2.4 Relations between the invariants of the field \((C, \bar{\xi})\) and the invariants of \((C, \xi)\) in \( \mathcal{M}(C, \bar{\xi}, \pi) \)

The versors field \((C, \xi)\) in \( E_3 \) has a Frenet frame \( \mathcal{R}_F = (P(s), \xi_1, \xi_2, \xi_3) \) and a complete system of invariants \((a_1, a_2, a_3; K_1, K_2)\) verifying the equations (1.1.5) and (1.1.6).

The same field \((C, \bar{\xi})\) in Myller configuration \( \mathcal{M}(C, \bar{\xi}, \pi) \) has a Darboux frame \( \mathcal{R}_D = (P(s), \bar{\xi}, \mu, \nu) \) and a complete system of invariants \((c_1, c_2, c_3; G, K, T)\). If we relate \( \mathcal{R}_F \) to \( \mathcal{R}_D \) we obtain
\[
\bar{\xi}_1(s) = \bar{\xi}(s),
\]
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\[\xi_2(s) = \overline{\pi}(s) \sin \varphi + \overline{\nu}(s) \cos \varphi,\]
\[\xi_3(s) = -\mu(s) \cos \varphi + \overline{\nu}(s) \sin \varphi,\]

with $\varphi = \angle(\xi_2, \nu)$ and $\langle \xi_1, \xi_2, \xi_3 \rangle = \langle \xi, \pi, \nu \rangle = 1$.

Then, from (1.1.5) and (2.1.3) we can determine the relations between the two systems of invariants.

From
\[\frac{d\overline{\nu}}{ds}(s) = a_1 \xi_1 + a_2 \xi_2 + a_3 \xi_3 = c_1 \xi + c_2 \pi + c_3 \nu,\]

it follows
\[c_1(s) = a_1(s),\]
\[c_2(s) = a_1(s) \sin \varphi - a_3(s) \cos \varphi,\]
\[c_3(s) = a_2(s) \cos \varphi + a_3(s) \sin \varphi.\]

And, (1.1.6), (2.1.4) we obtain
\[G(s) = K_1(s) \sin \varphi,\]
\[K(s) = K_1(s) \cos \varphi,\]
\[T(s) = K_2(s) + \frac{d\varphi}{ds}.\]

These formulae allow to investigate some important properties of $(C, \xi)$ in $\mathfrak{M}$ when some invariants $G, K, T$ vanish.

2.5 Relations between invariants of normal field $(C, \overline{\nu})$ and invariants $G, K, T$

The plane field $(C, \pi)$ is characterized by the normal versor field $(C, \overline{\nu})$, which has as Frenet frame $\mathcal{R}_F = (P(s); \overline{\nu}_1, \overline{\nu}_2, \overline{\nu}_3)$ with $\overline{\nu}_1 = \overline{\nu}$ and has $(b_1, b_2, b_3, \chi_1, \chi_2)$ as a complete system of invariants. They satisfy the formulae (1.3.1), (1.3.2). But the frame $\mathcal{R}_F$ is related to Darboux frame
\( R_D \) of \((C, \xi)\) in \(\mathcal{M}\) by the formulae

\[
\begin{align*}
\nu_1 &= \nu(s) \\
-\nu_2 &= \sin \sigma \xi + \cos \sigma \mu \\
\nu_3 &= -\cos \sigma \xi + \sin \sigma \mu
\end{align*}
\]

where \( \sigma = \angle(\xi(s), \nu_3(s)) \).

Proceeding as in the previous section we deduce

**Theorem 2.5.1** The following relations hold:

\[
\begin{align*}
c_1 &= -b_2 \sin \sigma + b_3 \cos \sigma \\
c_2 &= b_2 \cos \sigma + b_3 \sin \sigma \\
c_3 &= b_2
\end{align*}
\]

and

\[
\begin{align*}
K &= \chi_1 \sin \sigma \\
T &= \chi_1 \cos \sigma \\
G &= \chi_2 + \frac{d\sigma}{ds}.
\end{align*}
\]

A first consequence of previous formulae is given by

**Theorem 2.5.2** The invariant \( K^2 + T^2 \) depends only on the plane field \((C, \pi)\). We have

\[
K^2 + T^2 = \chi_1.
\]

The proof is immediate, from (2.5.3).

### 2.6 Meusnier’s theorem. Versor fields \((C, \bar{\xi})\) conjugated with tangent versor \((C, \bar{\alpha})\)

Consider the vector field \(\xi^*(s + \Delta s)\), \(|\Delta s| < \varepsilon, \varepsilon > 0\), the orthogonal projection of versor \(\bar{\xi}(s + \Delta s)\) on the normal plane \((P(s); \bar{\xi}(s), \nu(s))\).
Since, up to terms of second order in $\Delta s$, we have

$$\bar{\xi}^{**}(s + \Delta s) = \bar{\xi}(s) + \frac{\Delta s}{1!} K(s)\overline{\nu}(s) + \overline{\theta}(s, \Delta s)\frac{(\Delta s)^2}{2!},$$

for $\Delta s \to 0$ one gets:

$$\frac{d\bar{\xi}^{**}}{ds} = K(s)\overline{\nu}(s). \tag{2.6.1}$$

Assuming $K(s) \neq 0$ we consider the point $P^{**}$ called the center of curvature of the vector field $(C, \bar{\xi}^{**})$, given by

$$\overrightarrow{PP}^{**} = \frac{1}{K(s)}\overline{\nu}(s).$$

On the other hand the field of versors $(C, \bar{\xi})$ have a center of curvature $P_c$ given by $\overrightarrow{PP}_c = \frac{1}{K_1(s)}\bar{\xi}_2$.

The formula (2.4.2), i.e., $K(s) = K_1(s)\cos \varphi$, shows that the orthogonal projection of vector $\overrightarrow{PP}_c$ on the (osculating) plane $(P(s); \bar{\xi}_1, \bar{\xi}_2)$ is the vector $\overrightarrow{PP}_c$.

Indeed, we have

$$\cos \varphi = \frac{1}{K} \tag{2.6.2}$$

As a consequence we obtain a theorem of Meusnier type:

**Theorem 2.6.1** The curvature center of the field $(C, \bar{\xi})$ in $\mathfrak{M}$ is the orthogonal projection on the osculating plane $(M; \bar{\xi}_1, \bar{\xi}_2)$ of the curvature center $P^{**}$.

**Definition 2.6.1** The versor field $(C, \bar{\xi})$ is called conjugated with tangent versor field $(C, \bar{\alpha})$ in the Myller configuration $\mathfrak{M}(C, \bar{\xi}, \pi)$ if the invariant $K(s)$ vanishes.

Some immediate consequences:

1. $(C, \bar{\xi})$ is conjugated with $(C, \bar{\alpha})$ in $\mathfrak{M}$ iff the line $(P; \bar{\xi})$ is parallel in $E_3$ to the characteristic line of the planes $\pi(s)$, $s \in (s_1, s_2)$.
2. $(C, \bar{\xi})$ is conjugated with $(C, \bar{\alpha})$ in $\mathfrak{M}$ iff $|T(s)| = \chi_1(s)$. 
3. \((C, \xi)\) is conjugated with \((C, \alpha)\) in \(M\) iff the osculating planes 
\((P; \xi_1, \xi_2)\) coincide to the planes \(\pi(s)\) of \(M\).

4. \((C, \xi)\) is conjugated with \((C, \xi)\) iff the asymptotic planes of the 
ruled surface \(R(C, \xi)\) coincide with the planes \(\pi(s)\) of \(M\).

### 2.7 Versor field \((C, \overline{\xi})\) with null geodesic torsion

A new relation of conjugation of versor field \((C, \overline{\xi})\) with the tangent versor field \((C, \alpha)\) is obtained in the case \(T(s) = 0\).

**Definition 2.7.1** The versor field \((C, \overline{\xi})\) is called orthogonal conjugated 
with the tangent versor field \((C, \alpha)\) in \(M\) if its geodesic torsion \(T(s) = 0\), 
\(\forall s \in (s_1, s_2)\).

*Some properties*

1. \((C, \overline{\xi})\) is orthogonal conjugated with \((C, \alpha)\) in \(M\) iff \(\pi(s)\) are parallel 
with the characteristic line of planes \(\pi(s)\) along the curve \(C\).

2. \((C, \overline{\xi})\) is orthogonal conjugated with \((C, \alpha)\) in \(M\) if \(|K_1(s)| = \chi_1(s)\), 
along \(C\).

**Theorem 2.7.1** Assuming that the versor field \((C, \overline{\xi})\) in the configuration 
\(M(C, \overline{\xi}, \pi)\) has two of the following three properties then it has the 
third one, too:

a. The osculating planes \((P; \xi_1, \xi_2)\) are parallel in \(E_3\) along \(C\).

b. The osculating planes \((P; \overline{\xi}_1, \overline{\xi}_2)\) have constant angle with the plans 
\(\pi(s)\) on \(C\).

c. The geodesic torsion \(T(s)\) vanishes on \(C\).

The proof is based on the formula \(T(s) = K_2(s) + \frac{d\varphi}{ds} \varphi = \angle(\overline{\xi}_2, \overline{\nu})\).
Consider two Myller configurations $\mathcal{M}(C, \xi, \pi)$ and $\mathcal{M}'(C, \xi', \pi')$ which have in common the versor field $(C, \xi)$. Denote by $\varphi = \angle(\xi_2, \nu)$, $\varphi' = \angle(\xi_2', \nu')$. Then the geodesic torsions of $(C, \xi)$ in $\mathcal{M}$ and $\mathcal{M}'$ are follows:

$$T(s) = K_2(s) + \frac{d\varphi}{ds}, \quad T'(s) = K_2(s) + \frac{d\varphi'}{ds}.$$ 

Evidently, we have $\varphi - \varphi' = \angle(\nu, \nu')$.

By means of these relations we can prove, without difficulties:

**Theorem 2.7.2** If the Myller configurations $\mathcal{M}(C, \xi, \pi)$ and $\mathcal{M}'(C, \xi, \pi')$ have two of the following properties:

a) $(C, \xi)$ has the null geodesic torsion, $T(s) = 0$ in $\mathcal{M}$.

b) $(C, \xi)$ has the null geodesic torsion $T'(s)$ in $\mathcal{M}'$.

c) The angle $\angle(\nu, \nu')$ is constant along $C$, then $\mathcal{M}$ and $\mathcal{M}'$ have the third property.

**Remark 2.7.1** The versor field $(C, \nu_2)$ is orthogonally conjugated with the tangent versors $(C, \pi)$ in the configuration $\mathcal{M}(C, \xi, \pi)$.

### 2.8 The vector field parallel in Myller sense in configurations $\mathcal{M}$

Consider $(C, \nabla)$ a vector field, along the curve $C$. We denote $\nabla(s) = \nabla(\pi(s))$ and say that $(C, \nabla)$ is a vector field in the configuration $\mathcal{M} = \mathcal{M}(C, \xi, \pi)$ if the vector $\nabla(s)$ belongs to plane $\pi(s)$, $\forall s \in (s_1, s_2)$.

**Definition 2.8.1** The vectors field $(C, \nabla)$ in $\mathcal{M}(C, \xi, \pi)$ is parallel in Myller sense if the vector field $\frac{d\nabla}{ds}$ is normal to $\mathcal{M}$, i.e. $\frac{d\nabla}{ds} = \lambda(s)\nu(s)$, $\forall s \in (s_1, s_2)$. 
The parallelism in Myller sense is a direct generalization of Levi-Civita parallelism of tangent vector fields along a curve $C$ of a surface $S$.

It is not difficult to prove that the vector field $\mathbf{V}(s)$ is parallel in Myller sense if the vector field

$$\mathbf{V}'(s + \Delta s) = pr_{\pi(s)}\mathbf{V}(s + \Delta s)$$

is parallel in ordinary sense in $E_3$ up to terms of second order in $\Delta s$.

In Darboux frame, $\mathbf{V}(s)$ can be represented by its coordinate as follows:

$$\mathbf{V}(s) = V^1(s)\xi(s) + V^2(s)\pi(s).$$

(2.8.1)

By virtue of fundamental equations (2.1.4) we find:

$$\frac{d\mathbf{V}}{ds} = \left(\frac{dV^1}{ds} - GV^2\right)\xi + \left(\frac{dV^2}{ds} + GV^1\right)\pi + (KV^1 + TV^2)\nu.$$ 

(2.8.2)

Taking into account the Definition 2.8.1, one proves:

Theorem 2.8.1 The vector field $\mathbf{V}(s)$, (2.8.1) is parallel in Myller sense in configuration $\mathfrak{M}(C, \xi, \pi)$ iff coordinates $V^1(s), V^2(s)$ are solutions of the system of differential equations:

$$\frac{dV^1}{ds} - GV^2 = 0, \quad \frac{dV^2}{ds} + GV^1 = 0.$$ 

(2.8.3)

In particular, for $\mathbf{V}(s) = \xi(s)$, we obtain

Theorem 2.8.2 The versor field $\xi(s)$ is parallel in Myller sense in $\mathfrak{M}(C, \xi, \pi)$ iff the geodesic curvature $G(s)$ of $(C, \xi)$ in $\mathfrak{M}$ vanishes.

This is a reason that Al. Myller says that $G$ is the deviation of parallelism [31]. Later we will see that $G(s)$ is an intrinsic invariant in the geometry of surfaces in $E_3$.

By means of (2.8.3) we have
Chapter 2. Myller configurations \( \mathfrak{M}(C, \xi, \pi) \)

**Theorem 2.8.3** There exists an unique vector field \( \mathbf{V}(s) \), \( s \in (s_1', s_2') \subset (s_1, s_2) \) parallel in Myller sense in the configuration \( \mathfrak{M}(C, \xi, \pi) \) which satisfy the initial condition \( \mathbf{V}(s_0) = \mathbf{V}_0 \), \( s_0 \in (s_1', s_2') \) and \( \langle \mathbf{V}_0, \nu(s_0) \rangle = 0 \).

Evidently, theorem of existence and uniqueness of solutions of system (2.8.3), is applied in this case.

In particular, if \( G(s) = \text{constant} \), then the general solutions of (2.8.3) can be obtained by algebric operations.

An important property of parallelism in Myller sense is expressed in the next theorem.

**Theorem 2.8.4** The Myller parallelism of vectors in \( \mathfrak{M} \) preserves the lengths and angles of vectors.

Proof. If \( \frac{d\mathbf{V}}{ds} = \lambda(s)\nu \), then \( \frac{d}{ds} \langle \mathbf{V}, \mathbf{V} \rangle = 0 \). Also, \( \frac{d\mathbf{V}'}{ds} = \lambda(s)\nu \), \( \frac{d\mathbf{U}}{ds} = \lambda'(s)\nu \), then \( \frac{d}{ds} \langle \mathbf{V}(s), \mathbf{U}'(s) \rangle = 0 \)

### 2.9 Adjoint point, adjoint curve and concurrence in Myller sense

The notions of adjoint point, adjoint curve and concurrence in Myller sense in a configuration \( \mathfrak{M} \) have been introduced and studied by O. Mayer [20] and Gh. Gheorghiev [11], [55]. They applied these notions, to the theory of surfaces, nonholomorphic manifolds and in the geometry of versor fields in Euclidean space \( E_3 \).

In the present book we introduce these notions in a different way.

Consider the vector field

\[
\xi^*(s + \Delta s) = \text{pr}_\pi(s)\xi(s + \Delta s).
\]

Taking into account the formula (2.2.1)' we can write up to terms of second order in \( \Delta s \):

\[
(2.9.1) \quad \xi^*(s + \Delta s) = \xi(s) + \Delta s(G\mu(s) + \omega^*(s, \Delta s)).
\]
with
\[ \overline{\omega}^*(s, \Delta s) \to 0, (\Delta s \to 0). \]

Let \( C' \) be the orthogonal projection of the curve \( C \) on the plane \( \pi(s) \). A neighbor point \( P'(s + \Delta s) \) is projected on plane \( \pi(s) \) in the point \( P^*(s + \Delta s) \) given by

\[
\begin{align*}
\overline{r}^*(s + \Delta s) &= \overline{r}(s) + \Delta s(c_1 \overline{\xi} + c_2 \overline{\mu} + \overline{\omega}_0(s, \Delta s)), \\
\overline{\omega}_0(s, \Delta s) &\to 0, (\Delta s \to 0).
\end{align*}
\]

**Definition 2.9.1** The adjoint point of the point \( P(s) \) with respect to \( \overline{\xi}(s) \) in \( \mathfrak{M} \) is the characteristic point \( P_a \) on the line \( (P, \overline{\xi}) \) of the plane ruled surface \( R(C^*, \overline{\xi}^*) \).

One proves that the position vector \( \overline{R}(s) \) of adjoint point \( P_a \) for \( G \neq 0 \), is as follows:

\[
\overline{R}(s) = \overline{r}(s) - \frac{c_2}{G} \overline{\xi}(s).
\]

The vector field \( (C^*, \overline{\xi}^*) \) from (2.9.3) is called geodesic field. A result established by O. Mayer [20] holds:

**Theorem 2.9.1** If the versor field \( (C, \overline{\xi}) \) is enveloping in space \( E_3 \), then the adjoint point \( P_a \) of the point \( P(s) \) in \( \mathfrak{M} \) is the contact point of the line \( (P, \overline{\xi}) \) with the cuspidale line.

**Definition 2.9.2** The geometric locus of the adjoint points corresponding to the versor field \( (C, \overline{\xi}) \) in \( \mathfrak{M} \) is the adjoint curve \( C_a \) of the curve \( C \) in \( \mathfrak{M} \).

The adjoint curve \( C_a \) has the vector equations (2.9.3) for \( \forall s \in (s_1, s_2) \).

Now, we can introduce

**Definition 2.9.3** The versor field \( (C, \overline{\xi}) \) is concurrent in Myller sense in \( \mathfrak{M}(C, \overline{\xi}, \pi) \) if, at every point \( P(s) \in C \) the geodesic vector field \( (C^*, \overline{\xi}^*) \) is concurrent.
For $G(s) \neq 0$, we have

**Theorem 2.9.2** The versor field $(C, \xi)$ is concurrent in Myller sense in $\mathcal{M}$ iff the following equation hold

$$\frac{d}{ds} \left( \frac{c_2}{G} \right) = c_1.$$  

(2.9.4)

For the proof see Section 1.1, Chapter 1, Theorem 1.1.3.

### 2.10 Spherical image of a configuration $\mathcal{M}$

In the Section 2, Chapter 1 we defined the spherical image of a versor field $(C, \xi)$. Applying this idea to the normal vectors field $(C, \nu)$ to a Myller configuration $\mathcal{M}(C, \xi, \pi)$ we define the notion of spherical image $C^*$ of $\mathcal{M}$ as being

$$\nu^*(s) = O\hat{P}(s) = \nu(s)$$  

(2.10.1)

Thus, the relations between the curvature $\chi_1$ and torsion $\chi_2$ of $(C, \nu)$ and geodesic curvature $\kappa_g$ of $C^*$ at a point $P^* \in C^*$ and arclength $s^*$ are as follows

$$\chi_1 = \frac{ds^*}{ds}, \quad \chi_2 ds = \kappa_g ds^*.$$  

The properties enumerated in Section 2, ch 1, can be obtained for the spheric image $C^*$ of configuration $\mathcal{M}$.

Consider the versor field $\overrightarrow{P_1^*} = \xi^*(s) = \xi(s)$ and the angle $\theta = \angle(\nu_2, \xi)$. It follows

$$K = -\frac{ds^*}{ds} \cos \theta, \quad (\text{for} \chi_2 \neq 0).$$  

(2.10.2)

Thus, for $\theta = \pm \frac{\pi}{2}$ one gets $K = 0$, which leads to a new interpretation of the fact that $\xi(s)$ is conjugated to $\alpha(s)$ in Myller configurations.

Analogous one can obtain $T = -\frac{ds^*}{ds} \cos \tilde{\theta}, \quad \tilde{\theta} = \angle(\nu_2, \nu^*)$ (with $\nu^*(s) = \nu(s)$, applied at the point $P^*(s) \in C^*$). For $\tilde{\theta} = \pm \frac{\pi}{2}$ it fol-
lows that $\xi(s)$ are orthogonally conjugated with $\pi(s)$.

The problem is to see if the Gauss-Bonet formula can be extended to Myller configurations $\mathfrak{M}(C, \xi, \pi)$. In the case ($\alpha(s) \in \pi(s)$, i.e. $\pi(s) \perp \pi(s)$) such a problem was suggested by Thomson [18] and it has been solved by Mark Krein in 1926, [18].

Here, we study this problem in the general case of Myller configuration, when $\langle \pi(s), \pi(s) \rangle \neq 0$.

First of all we prove

\textbf{Lemma 2.10.1} Assume that we have:

1. $\mathfrak{M}(C, \xi, \pi)$ a Myller configuration of class $C^3$, in which $C$ is a closed curve, having $s$ as arclength.
2. The spherical image $C^*$ of $\mathfrak{M}$ determines on the support sphere $\Sigma$ a simply connected domain of area $\omega$.

In this hypothesis we have the formula

$\omega = 2\pi - \int_C \left( \pi \cdot \frac{d\pi}{ds} \cdot \frac{d^2\pi}{ds^2} \right) / \left\| \frac{d\pi}{ds} \right\|^2 ds.$ (2.10.3)

Proof. Let $\Sigma$ be the unitary sphere of center $O \in E_3$ and a simply connected domain $D$, delimited by $C^*$ on $\Sigma$. Assume that $D$ remains to left with respect to an observer looking in the sense of versor $\pi(s)$, when he is going along $C^*$ in the positive sense.

Thus, we can take the following representation of $\Sigma$:

$x^1 = \cos \varphi \sin \theta$
$x^2 = \sin \varphi \sin \theta$
$x^3 = \cos \varphi, \ \varphi \in [0, 2\pi), \ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$ (2.10.4)

The curve $C^*$ can be given by

$\varphi = \varphi(s), \ \theta = \theta(s), \ s \in [0, s_1]$ (2.10.5)

with $\varphi(s), \theta(s)$ of class $C^3$ and $C^*$ being closed: $\varphi(0) = \varphi(s_1), \ \theta(0) =$
\( \theta(s_1) \).

The area \( \omega \) of the domain \( D \) is

\[
\omega = \int_{0}^{s_1} \int_{0}^{\theta(s_1)} \sin \theta d\theta = \int_{0}^{s_1} (1 - \cos \theta) \frac{d\varphi}{ds} ds = \\
= 2\pi - \int_{0}^{s_1} \cos \theta \frac{d\varphi}{ds} ds.
\]  

Noticing that the versor \( \nu^* = \nu(s) \) has the coordinate (2.10.4) a straightforward calculus leads to

\[
\left\langle \nu, \frac{d\nu}{ds}, \frac{d^2\nu}{ds^2} \right\rangle / \left\| \frac{d\nu}{ds} \right\|^2 = \cos \theta \frac{d\varphi}{ds} + \frac{d}{ds} \arctg \frac{\sin \theta \frac{d\varphi}{ds}}{\frac{d\theta}{ds}}.
\]  

Denoting by \( \psi \) the angle between the meridian \( \varphi = \varphi_0 \) and curve \( C^* \), oriented with respect to the versor \( \nu \) we have

\[
\operatorname{tg} \psi = \left( \sin \theta \frac{d\varphi}{ds} \right) / \frac{d\theta}{ds}.
\]  

The previous formulae lead to

\[
\left\langle \nu, \frac{d\nu}{ds}, \frac{d^2\nu}{ds^2} \right\rangle / \left\| \frac{d\nu}{ds} \right\|^2 = \cos \theta \frac{d\varphi}{ds} + \frac{d\psi}{ds}.
\]  

But in our conditions of regularity \( \int_{C} \frac{d\psi}{ds} ds = 0 \). Thus (2.10.7), (2.10.8) implies the formula (2.10.3)

It is not difficult to see that the formula (2.10.3) can be generalized in the case when the curve \( C \) of the configuration \( \mathcal{M} \) has a finite number of angular points. The second member of the formula (2.10.3) will be additive modified with the total of variations of angle \( \psi \) at the angular points corresponding to the curve \( C \).

Now, one can prove the generalization of Mark Krein formula.

**Theorem 2.10.1** Assume that we have

1. \( \mathcal{M}(C, \xi, \pi) \) a Myller configuration of class \( C^3 \) (i.e. \( C \) is of class
$C^3$ and $\xi(s)$, $\overline{\nu}(s)$ are the class $C^2$ in which $C$ is a closed curve, having $s$ as natural parameter.

2. The spherical image $C^*$ of $\mathfrak{M}$ determine on the support sphere $\Sigma$ a simply connected domain of area $\omega$.

3. $\sigma$ the oriented angle between the versors $\overline{\nu}_3(s)$ and $\overline{\xi}(s)$. In these conditions the following formula hold:

\[
\omega = 2\pi - \int_C G(s) ds + \int_C d\sigma.
\]

Proof. The first two conditions allow to apply the Lemma 2.10.1. The fundamental equations (1.3.1), (1.3.2) of $(C, \overline{\nu})$ give us for $\overline{\nu} = \overline{\nu}_1$:

\[
\frac{d\overline{\nu}_1}{ds} = \chi_1 \overline{\nu}_2, \quad \frac{d^2\overline{\nu}_1}{ds^2} = \frac{d\chi_1}{ds} \overline{\nu}_2 + \chi_1(-\chi_1 \overline{\nu}_1 + \chi_2 \overline{\nu}_3).
\]

So,

\[
\left\langle \frac{\overline{\nu}_1}{ds}, \frac{d^2\overline{\nu}_1}{ds^2} \right\rangle = \chi_1^2 \chi_2.
\]

Thus, the formula (2.10.3), leads to the following formula

\[
\omega = 2\pi - \int_C \chi_2(s) ds.
\]

But, we have $G(s) = \chi_2(s) + \frac{d\sigma}{ds}$, $G(s)$ being the geodesic curvature of $(C, \overline{\xi})$ in $\mathfrak{M}$. Then the last formula is exactly (2.10.9).

If $G = 0$ for $\mathfrak{M}$, then we have $\omega = 2\pi$.

Indeed $G(s) = 0$ along the curve $C$ imply $\omega = 2\pi + 2k\pi$, $k \in \mathbb{N}$. But we have $0 \leq \omega < 4\pi$, so $k = 0$.

A particular case of Theorem 2.10.1 is the famous result of Jacobi:

The area $\omega$ of the domain $D$ determined on the sphere $\Sigma$ by the closed curve $C^*$-spherical image of the principal normals of a closed curve $C$ in $E_3$, assuming $D$ a simply connected domain, is a half of area of sphere $\Sigma$.

In this case we consider the Myller configuration $\mathfrak{M}(C, \overline{\alpha}, \pi)$, $\pi(s)$
being the rectifying planes of $C$. 
Chapter 3

Tangent Myller configurations $\mathcal{M}_t$

The theory of Myller configurations $\mathcal{M}(C, \xi, \pi)$ presented in the Chapter 2 has an important particular case when the tangent versor fields $\overline{\alpha}(s)$, $\forall s \in (s_1, s_2)$ belong to the corresponding planes $\pi(s)$. These Myller configurations will be denoted by $\mathcal{M}_t = \mathcal{M}_t(C, \xi, \pi)$ and named **tangent Myller configuration**.

The geometry of $\mathcal{M}_t$ is much more rich than the geometrical theory of $\mathcal{M}$ because in $\mathcal{M}_t$ the tangent field has some special properties. So, $(C, \overline{\alpha})$ in $\mathcal{M}_t$ has only three invariants $\kappa_g, \kappa_n$ and $\tau_g$ called **geodesic curvature**, **normal curvature** and **geodesic torsion**, respectively, of the curve $C$ in $\mathcal{M}_t$.

The curves $C$ with $\kappa_g = 0$ are **geodesic lines** of $\mathcal{M}_t$; the curves $C$ with the property $\kappa_n = 0$ are the **asymptotic lines** of $\mathcal{M}_t$ and the curve $C$ for which $\tau_g = 0$ are the **curvature lines** for $\mathcal{M}_t$. The mentioned invariants have some geometric interpretations as the geodesic curvature, normal curvature and geodesic torsion of a curve $C$ on a surfaces $S$ in the Euclidean space $E_3$. 
3.1 The fundamental equations of $\mathfrak{M}_t$

Consider a tangent Myller configuration $\mathfrak{M}_t = (C, \xi, \pi)$. Thus we have

\begin{equation}
\langle \alpha(s), \nu(s) \rangle = 0, \forall s \in (s_1, s_2).
\end{equation}

The Darboux frame $\mathcal{R}_D$ is $\mathcal{R}_D = (P(s); \xi(s), \pi(s), \nu(s))$ with $\pi(s) = \nu(s) \times \xi(s)$.

The fundamental equations of $\mathfrak{M}_t$ are obtained by the fundamental equations (2.1.3), (2.1.4), Chapter 2 of a general Myller configuration $\mathfrak{M}$ for which the invariant $c_3(s)$ vanishes.

**Theorem 3.1.1** The fundamental equations of the tangent Myller configuration $\mathfrak{M}_t(C, \xi, \pi)$ are given by the following system of differential equations:

\begin{equation}
\frac{d\pi}{ds} = c_1(s)\xi(s) + c_2(s)\pi(s), \quad (c_1^2 + c_2^2 = 1),
\end{equation}

\begin{equation}
\begin{aligned}
\frac{d\xi}{ds} &= G(s)\pi(s) + K(s)\nu(s) \\
\frac{d\pi}{ds} &= -G(s)\xi(s) + T(s)\nu(s) \\
\frac{d\nu}{ds} &= -K(s)\xi(s) - T(s)\pi(s).
\end{aligned}
\end{equation}

The invariants $c_1, c_2, G, K, T$ have the same geometric interpretations and the same denomination as in Chapter 2. So, $G(s)$ is the geodesic curvature of the field $(C, \xi)$ in $\mathfrak{M}_t$, $K(s)$ is the normal curvature and $T(s)$ is the geodesic torsion of $(C, \xi)$ in $\mathfrak{M}_t$.

The cases when some invariants $G, K, T$ vanish can be investigate exactly as in the Chapter 2.

In this respect, denoting $\varphi = \angle (\xi_2(s), \nu(s))$ and using the Frenet formulae of the versor field $(C, \xi)$ we obtain the formulae

\begin{equation}
G = K_1 \sin \varphi, \quad K = K_1 \cos \varphi, \quad T = K_2 + \frac{d\varphi}{ds}.
\end{equation}
In §5, Chapter 2 we get the relations between the invariants of \((C, \xi)\) in \(\mathcal{M}_t\) and the invariants of normal versor field \((C, v)\), (Theorem 2.5.1, Ch 2.) For \(\sigma = \angle(\xi(s), \nu_3(s))\) we have

\[
K = \chi_1 \sin \sigma, \quad T = \chi_1 \cos \sigma, \quad G = \chi_2 + \frac{d\sigma}{ds}.
\]

(3.1.5)

Others results concerning \(\mathcal{M}_t\) can be deduced from those of \(\mathcal{M}\).

For instance

**Theorem 3.1.2** (Mark Krein) Assuming that we have:

1. \(\mathcal{M}_t(C, \xi, \pi)\) a tangent Myller configuration of class \(C^3\) in which \(C\) is a closed curve, having \(s\) as natural parameter.

2. The spherical image \(C^*\) of \(\mathcal{M}_t\) determines on the support sphere \(\Sigma\) a simply connected domain of area \(\omega\).

3. \(\sigma = \angle(\xi, \nu_3)\).

In these conditions the following Mark-Krein’s formula holds:

\[
\omega = 2\pi - \int_C G(s) ds + \int_C d\sigma.
\]

(3.1.6)

### 3.2 The invariants of the curve \(C\) in \(\mathcal{M}_t\)

A smooth curve \(C\) having \(s\) as arclength determines the tangent versor field \((C, \bar{\alpha})\) with \(\bar{\alpha}(s) = \frac{d\gamma}{ds}\). Consequently we can consider a particular tangent Myller configuration \(\mathcal{M}_t(C, \bar{\alpha}, \pi)\) defined only by curve \(C\) and tangent planes \(\pi(s)\).

In this case the geometry of Myller configurations \(\mathcal{M}_t(C, \bar{\alpha}, \pi)\) is called the geometry of curve \(C\) in \(\mathcal{M}_t\). The Darboux frame of curve \(C\) in \(\mathcal{M}_t\) is \(\mathcal{R} = (P(s); \bar{\alpha}(s), \bar{\nu}^*(s), \bar{v}(s))\), \(\bar{\nu}^*(s) = \bar{\nu}(s) \times \bar{\alpha}(s)\). It will be called the Darboux frame of the curve \(C\) in \(\mathcal{M}_t\).

**Theorem 3.2.1** The fundamental equations of the curve \(C\) in the Myller configuration \(\mathcal{M}_t(C, \bar{\alpha}, \pi)\) are given by the following system of differential
Chapter 3. Tangent Myller configurations $\mathfrak{M}_t$

equations:

(3.2.1) \[ \frac{d\tau}{ds} = \overline{\alpha}(s), \]

\[ \frac{d\overline{\alpha}}{ds} = \kappa_g(s)\overline{\mu}^*(s) + \kappa_n(s)\overline{\nu}(s), \]

(3.2.2) \[ \frac{d\overline{\mu}^*}{ds} = -\kappa_g(s)\overline{\alpha}(s) + \tau_g(s)\overline{\nu}(s), \]

\[ \frac{d\overline{\nu}}{ds} = -\kappa_n(s)\overline{\alpha}(s) - \tau_g(s)\overline{\mu}^*(s). \]

Of course (3.2.1), (3.2.2) are the moving equations of the Darboux frame $\mathcal{R}_D$ of the curve $C$.

The invariants $\kappa_g$, $\kappa_n$ and $\tau_g$ are called: the geodesic curvature, normal curvature and geodesic torsion of the curve $C$ in $\mathfrak{M}_t$.

Of course, we can prove a fundamental theorem for the geometry of curves $C$ in $\mathfrak{M}_t$:

**Theorem 3.2.2** A priori given $C^\infty$-functions $\kappa_g(s)$, $\kappa_n(s)$, $\tau_g(s)$, $s \in [a,b]$, there exists a Myller configuration $\mathfrak{M}_t(C, \overline{\alpha}, \pi)$ for which $s$ is arclength on the curve $C$ and given functions are its invariants. Two such configurations differ by a proper Euclidean motion.

The proof is the same as proof of Theorem 2.1.2, Chapter 2.

**Remark 3.2.1** Let $C$ be a smooth curve immersed in a $C^\infty$ surface $S$ in $E_3$. Then the tangent plans $\pi$ to $S$ along $C$ uniquely determines a tangent Myller configuration $\mathfrak{M}_t(C, \overline{\alpha}, \pi)$. Its Darboux frame $\mathcal{R}_D$ and the invariants $k_g$, $\kappa_n$, $\tau_g$ of $C$ in $\mathfrak{M}_t$ are just the Darboux frame and geodesic curvature, normal curvature and geodesic torsion of curve $C$ on the surface $S$.

Let $\mathcal{R}_F = (P(s); \overline{\alpha}_1(s), \overline{\alpha}_2(s), \overline{\alpha}_3(s))$, be the Frenet frame of curve $C$ with $\overline{\alpha}_1(s) = \overline{\alpha}(s)$.

The Frenet formulae hold:

(3.2.3) \[ \frac{d\overline{\alpha}_1}{ds} = \overline{\alpha}(s) \]
3.3. Geodesic, asymptotic and curvature lines in $\mathcal{M}_t(C, \pi, \pi)$

The notion of parallelism of a versor field $(C, \alpha, \pi)$ in $\mathcal{M}_t$ along the curve $C$, investigated in the Section 8, Chapter 2 for the general case, can be

\[
\begin{align*}
\frac{d\alpha_1}{ds} &= \kappa(s)\alpha_2(s); \\
\frac{d\alpha_2}{ds} &= -\kappa(s)\alpha_1(s) + \tau(s)\alpha_3(s) \\
\frac{d\alpha_3}{ds} &= -\tau(s)\alpha_2(s)
\end{align*}
\]

(3.2.4)

where $\kappa(s)$ is the curvature of $C$ and $\tau(s)$ is the torsion of $C$.

The relations between the invariants $\kappa_g, \kappa_n, \tau_g$ and $\kappa, \tau$ can be obtained like in Section 4, Chapter 2.

**Theorem 3.2.3** The following formulae hold good

\[
\kappa_g(s) = \kappa \sin \varphi^*, \quad \kappa_n(s) = \kappa \cos \varphi^*, \quad \tau_g(s) = \tau + \frac{d\varphi^*}{ds},
\]

(3.2.5) with $\varphi^* = \angle(\alpha_2(s), \nu(s))$.

In the case when we consider the relations between the invariants $\kappa_g, \kappa_n, \tau_g$ and the invariants $\chi_1, \chi_2$ of the normal versor field $(C, \nu)$ we have from the formulae (3.1.5):

\[
\kappa_n = \chi_1 \sin \sigma, \quad \tau_g = \chi_1 \cos \sigma, \quad \kappa_g = \chi_2 + \frac{d\sigma}{ds},
\]

(3.2.6) where $\sigma = \angle(\alpha(s), \nu_3(s))$.

It is clear that the second formula (3.2.5) gives us a theorem of Meusnier type, and for $\varphi^* = 0$ or $\varphi^* = \pm \pi$ we have $\kappa_g = 0, \kappa_n = \pm \kappa, \tau_g = \tau$.

For $\sigma = 0$, or $\sigma = \pm \pi$, from (3.2.6) we obtain $\kappa_n = 0, \tau_g = \pm \chi_1, \kappa_g = \chi_2$.

### 3.3 Geodesic, asymptotic and curvature lines in $\mathcal{M}_t(C, \alpha, \pi)$

The notion of parallelism of a versor field $(C, \alpha, \pi)$ in $\mathcal{M}_t$ along the curve $C$, investigated in the Section 8, Chapter 2 for the general case, can be
applied now for the particular case of tangent versor field \((C, \overline{\alpha})\). It is defined by the condition \(\kappa_g(s) = 0, \forall s \in (s_1, s_2)\). The curve \(C\) with this property is called \textit{geodesic line} for \(\mathbb{M}_t\) (or \textit{autoparallel curve}).

The following properties hold:

1. \textit{The curve} \(C\) \textit{is a geodesic in the configuration} \(\mathbb{M}_t\) \textit{iff at every point} \(P(s)\) \textit{of} \(C\) \textit{the osculating plane of} \(C\) \textit{is normal to} \(\mathbb{M}_t\).
2. \(C\) \textit{is geodesic in} \(\mathbb{M}_t\) \textit{iff the equality} \(|\kappa_n| = \kappa\) \textit{holds along} \(C\).
3. \textit{If} \(C\) \textit{is a straight line in} \(\mathbb{M}_t\) \textit{then} \(C\) \textit{is a geodesic of} \(\mathbb{M}_t\).

\textbf{Asymptotics}

The curve \(C\) is called \textit{asymptotic} in \(\mathbb{M}_t\) if \(\kappa_n = 0, \forall s \in (s_1, s_2)\). An asymptotic \(C\) is called an \textit{asymptotic line}, too. The following properties can be proved without difficulties:

1. \(C\) \textit{is asymptotic line in} \(\mathbb{M}_t\) \textit{iff at every point} \(P(s)\) \textit{of} \(C\) \textit{the osculating plane of} \(C\) \textit{coincides to the plane} \(\pi(s)\).
2. \(C\) \textit{is a straight line then} \(C\) \textit{is asymptotic in} \(\mathbb{M}_t\).
3. \(C\) \textit{is asymptotic in} \(\mathbb{M}_t\) \textit{iff along} \(C\), \(|\kappa_g| = \kappa\).
4. \(C\) \textit{is asymptotic in} \(\mathbb{M}_t\), \textit{then} \(\tau_g = \tau\) \textit{along} \(C\).
5. \(C\) \textit{is asymptotic line in} \(\mathbb{M}_t\) \textit{iff} \(C\) \textit{is asymptotic line in} \(\mathbb{M}_t\).

Therefore we may say that the asymptotic line in \(\mathbb{M}_t\) are the auto-conjugated lines.

\textbf{Curvature lines}

The curve \(C\) is called the \textit{curvature line} in the configurations \(\mathbb{M}_t(C, \overline{\xi}, \pi)\) if the ruled surface \(\mathcal{R}(C, \overline{\nu})\) is a developing surface.

One knows that \(\mathcal{R}(C, \overline{\nu})\) is a developing surface iff the following equation holds:

\[
(\overline{\alpha}(s), \overline{\nu}(s), \frac{d\overline{\nu}}{ds}(s)) = 0, \quad \forall s \in (s_1, s_2).
\]

Taking into account the fundamental equations of \(\mathbb{M}_t\) one gets:

1. \(C\) \textit{is a curvature line in} \(\mathbb{M}_t\) \textit{iff its geodesic torsion} \(\tau(s) = 0\), \(\forall s\).
2. \(C\) \textit{is a curvature line in} \(\mathbb{M}_t\) \textit{iff the versors field} \((C, \overline{\nu}')\) \textit{are conjugated to the tangent} \(\overline{\alpha}(s)\) \textit{in} \(\mathbb{M}_t\).
Theorem 3.3.1 If the curve $C$ of the tangent Myller configuration $\mathcal{M}_t(C, \alpha, \pi)$ satisfies two from the following three conditions

a) $C$ is a plane curve.

b) $C$ is a curvature line in $\mathcal{M}_t$.

c) The angle $\varphi^*(s) = \angle(\overline{\nu}_2(s), \overline{\nu}(s))$ is constant,

then the curve $C$ verifies the third condition, too.

For proof, one can apply the Theorem 3.2.3, Chapter 3.

More general, let us consider two tangent Myller configurations $\mathcal{M}_t(C, \alpha, \pi)$ and $\mathcal{M}_t^*(C, \alpha^*, \pi^*)$ and $\varphi^{**}(s) = \angle(\overline{\nu}(s), \overline{\nu}^*(s))$.

We can prove without difficulties:

Theorem 3.3.2 Assuming satisfied two from the following three conditions

1. $C$ is curvature line in $\mathcal{M}_t$.

2. $C$ is curvature line in $\mathcal{M}_t^*$.

3. The angle $\varphi^{**}(s)$ is constant for $s \in (s_1, s_2)$. Then, the third condition is also verified.

3.4 Mark Krein’s formula

In the Section 10, Chapter 2, we have defined the spherical image of a general Myller configuration $\mathcal{M}$. Of course the definition applies for tangent configurations $\mathcal{M}_t$, too.

But now will appear some new special properties.

If in the formulae from Section 10, Chapter 2 we take $\xi(s) = \pi(s), \forall s$, the invariants $G, K, T$ reduce to geodesic curvature, normal curvature and geodesic torsion, respectively, of the curve $C$ in $\mathcal{M}_t$.

Such that we obtain the formulae:

\[ \kappa_n = -\frac{ds^*}{ds} \cos \theta_1, \quad \theta_1 = \angle(\overline{\nu}_2, \overline{\alpha}) \]
\[ \tau_g = -\frac{ds^*}{ds} \cos \theta_2, \quad \theta_2 = \angle(\nu_2, \mu^*) \] (3.4.2)

which have as consequences:

1. \( C \) is asymptotic in \( \mathcal{M}_t \) iff \( \theta_1 = \pm \frac{\pi}{2} \).
2. \( C \) is curvature line in \( \mathcal{M}_t \) iff \( \theta_2 = \pm \frac{\pi}{2} \).

The following Mark Krein’s theorem holds:

**Theorem 3.4.1** Assume that we have

1. A Myller configuration \( \mathcal{M}_t(C, \xi, \pi) \) of class \( C^k \), \( k \geq 3 \) where \( s \) is the natural parameter on the curve \( C \) and \( C \) is a closed curve.

2. The spherical image \( C^* \) of \( \mathcal{M}_t \) determine on the support sphere \( \Sigma \) a simply connected domain of area \( \omega \).

3. \( \sigma = \angle(\nu_3, \alpha) \).

In these conditions Mark Krein’s formula holds:

\[ \omega = 2\pi - \int_C \kappa_g(s) ds + \int_C d\sigma. \] (3.4.3)

Indeed the formula (2.10.9) from Theorem 2.10.1, Chapter 2 is equivalent to (3.4.3).

The remarks from the end of Section 10, Chapter 2 are valid, too.

### 3.5 Relations between the invariants \( G, K, T \) of the versor field \( (C, \xi) \) in \( \mathcal{M}_t(C, \xi, \pi) \) and the invariants of tangent versor field \( (C, \alpha) \) in \( \mathcal{M}_t \)

Let \( \mathcal{M}_t(C, \xi, \pi) \) be a tangent Myller configuration, \( \mathcal{R}_D = (P(s), \xi(s), \nu(s), \nu(s)) \) its Darboux frame and the tangent Myller configuration \( \mathcal{M}_t'(C, \alpha, \pi) \) determined by the plane field \( (C, \pi(s)) \), with \( \alpha = \frac{d\nu}{ds} \) tangent versors field
3.5. Relations between the invariants \( G, K, T \) of the versor field \((C, \xi)\) in \( \mathcal{M}_t(C, \xi, \pi) \) and the invariants of tangent versor field \((C, \alpha)\) in \( \mathcal{M}_t \) to the oriented curve \( C \). The Darboux frame \( \mathcal{R}_D = (P(s) ; \overline{\pi}(s), \overline{\tau}(s), \overline{\nu}) \) and oriented angle \( \lambda = \angle (\overline{\pi}, \overline{\xi}) \) allow to determine \( \mathcal{R}_D \) by means of formulas:

\[
\begin{align*}
\overline{\xi} &= \overline{\pi} \cos \lambda + \overline{\tau} \sin \lambda \\
\overline{\pi} &= -\overline{\pi} \sin \lambda + \overline{\tau} \cos \lambda \\
\overline{\nu} &= \overline{\nu}.
\end{align*}
\]

(3.5.1)

The moving equations (3.1.2), (3.1.3) and (3.2.1), (3.2.3) of \( \mathcal{R}_D \) and \( \mathcal{R}'_D \) lead to the following relations between invariants \( \kappa_g, \kappa_n, \tau_g \) of the curve \( C \) in \( \mathcal{M}_t \) and the invariants \( G, K, T \) of versor field \((C, \xi)\) in \( \mathcal{M}_t \):

\[
\begin{align*}
K &= \kappa_n \cos \lambda + \tau_g \sin \lambda \\
T &= -\kappa_n \sin \lambda + \tau_g \cos \lambda \\
G &= \kappa_g + \frac{d\lambda}{ds}.
\end{align*}
\]

(3.5.2)

The two first formulae (3.5.2) imply

\[
K^2 + T^2 = \kappa_n^2 + \tau_g^2 = \left( \frac{ds^*}{ds} \right)^2.
\]

(3.5.3)

The formula (3.5.2) has some important consequences:

**Theorem 3.5.1**

1. The curve \( C \) has \( \kappa_g(s) \) as geodesic curvature on the developing surface \( E \) generated by planes \( \pi(s) \), \( s \in (s_1, s_2) \).

2. \( G \) is an intrinsec invariant of the developing surface \( E \).

**Proof.**

1. Since the planes \( \pi(s) \) pass through tangent line \((P(s), \overline{\pi}(s))\) to curve \( C \), the developing surface \( E \), enveloping by planes \( \pi(s) \) passes through curve \( C \). So, \( \kappa_g \) is geodesic curvature of \( C \subset E \) at point \( P(s) \).

2. The invariants \( \kappa_g(s) \) and \( \lambda(s) \) are the intrinsec invariants of surface \( E \). It follows that the invariant \( G \) has the same property.

Now, we investigate an extension of Bonnet result.
Theorem 3.5.2  Supposing that the versor field \((C, \xi)\) in tangent Myller configuration \(\mathfrak{M}_t(C, \xi, \pi)\) has two from the following three properties:

1. \(\xi(s)\) is a parallel versor field in \(\mathfrak{M}_t\),
2. The angle \(\lambda = \angle(\alpha, \xi)\) is constant,
3. The curve \(C\) is geodesic in \(\mathfrak{M}_t\),

then it has the third property, too.

The proof is based on the last formula (3.5.2). Also, it is not difficult to prove:

Theorem 3.5.3  The normal curvature and geodesic torsion of the versor field \((C, \mu^*)\) in tangent Myller configuration \(\mathfrak{M}_t(C, \xi, \pi)\), respectively, are the geodesic torsion and normal curvature with opposite sign of the curve \(C\) in \(\mathfrak{M}_t\).

The same formulae (3.5.2) for \(K(s) = 0\), \(s \in (s_1, s_2)\) and \(\tau_g(s) \neq 0\) imply:

\[
\tan \lambda = \frac{-\kappa_n(s)}{\tau_g(s)}.
\]

(3.5.4)

In the theory of surfaces immersed in \(E^3\) this formula was independently established by E. Bortolotti [3] and Al. Myller [31-34].

Definition 3.5.1  A curve \(C\) is called a geodesic helix in tangent configuration \(\mathfrak{M}_t(C, \xi, \pi)\) if the angle \(\lambda(s) = \angle(\alpha, \mu)\) is constant \((\neq 0, \pi)\).

The following two results can be proved without difficulties.

Theorem 3.5.4  If \(C\) is a geodesic helix in \(\mathfrak{M}_t(C, \xi, \pi)\), then

\[
\frac{\kappa_n}{\tau_g} = \pm \frac{\kappa}{\tau}.
\]

Another consequence of the previous theory is given by:
3.5. Relations between the invariants $G, K, T$ of the versor field $(C, \xi)$ in $M_t(C, \xi, \pi)$ and the invariants of tangent versor field $(C, \pi)$ in $M_t$.

**Theorem 3.5.5** If $C$ is geodesic in $M_t(C, \xi, \pi)$, it is a geodesic helix for $M_t$ iff $C$ is a cylinder helix in the space $E_3$.

We can make analogous consideration, taking $\tau = 0$ in the formula (3.5.2).

We stop here the theory of Myller configurations $M_t(C, \xi, \pi)$ in Euclidean space $E_3$. Of course, it can be extended for Myller configurations $M(C, \xi, \pi)$ with $\pi$ a $k$-plane in the space $E^n, n \geq 3, k < n$ or in more general spaces, as Riemann spaces, [26], [27].
Chapter 4

Applications of theory of Myller configuration $\mathcal{M}_t$ in the geometry of surfaces in $E_3$

A first application of the theory of Myller configurations $\mathcal{M}(C, \xi, \pi)$ in the Euclidean space $E_3$ can be realized to the geometry of surfaces $S$ embedded in $E_3$. We obtain a more clear study of curves $C \subset S$, a natural definition of Levi-Civita parallelism of vector field $\nabla$, tangent to $S$ along $C$, as well as the notion of concurrence in Myller sense of vector field, tangent to $S$ along $C$. Some new concepts, as those of mean torsion of $S$, the total torsion of $S$, the Bonnet indicatrix of geodesic torsion and its relation with the Dupin indicatrix of normal curvatures are introduced. A new property of Bacaloglu-Sophie-Germain curvature is proved, too. Namely, it is expressed in terms of the total torsion of the surface $S$. 
4.1 The fundamental forms of surfaces in \(E_3\)

Let \(S\) be a smooth surface embedded in the Euclidean space \(E_3\). Since we use the classical theory of surfaces in \(E_3\) [see, for instance: Mike Spivak, Diff. Geom. vol. 1, Bealkley, 1979], consider the analytical representation of \(S\), of class \(C^k\), \((k \geq 3,\ \text{or} \ k = \infty)\):

\[
\mathbf{r} = \mathbf{r}(u,v), \quad (u,v) \in D.
\]

\((4.1.1)\)

\(D\) being a simply connected domain in plane of variables \((u,v)\), and the following condition being verified:

\[
\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0} \quad \text{for} \ \forall (u,v) \in D.
\]

\((4.1.2)\)

Of course we adopt the vectorial notations

\[
\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}, \ \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}.
\]

Denote by \(C\) a smooth curve on \(S\), given by the parametric equations

\[
u = u(t), \ v = v(t), \ t \in (t_1, t_2).
\]

\((4.1.3)\)

In this chapter all geometric objects or mappings are considered of \(C^\infty\)-class, up to contrary hypothesis.

In space \(E_3\), the curve \(C\) is represented by

\[
\mathbf{r} = \mathbf{r}(u(t), v(t)), \ t \in (t_1, t_2).
\]

\((4.1.4)\)

Thus, the following vector field

\[
d\mathbf{r} = \mathbf{r}_u du + \mathbf{r}_v dv
\]

\((4.1.5)\)

is tangent to \(C\) at the points \(P(t) = P(\mathbf{r}(u(t)), v(t)) \in C\). The vectors \(\mathbf{r}_u\)
and \( \overline{r}_v \) are tangent to the parametric lines and \( d\overline{r} \) is tangent vector to \( S \) at point \( P(t) \).

From (4.1.2) it follows that the scalar function

\[
\Delta = \|\overline{r}_u \times \overline{r}_v\|^2 \quad (4.1.6)
\]

is different from zero on \( D \).

The unit vector \( \overline{\nu} \)

\[
\overline{\nu} = \frac{\overline{r}_u \times \overline{r}_v}{\sqrt{\Delta}} \quad (4.1.7)
\]

is normal to surface \( S \) at every point \( P(t) \).

The tangent plane \( \pi \) at \( P \) to \( S \), has the equation

\[
\langle \overline{R} - \overline{r}(u,v), \overline{\nu} \rangle = 0. \quad (4.1.8)
\]

Assuming that \( S \) is orientable, it follows that \( \overline{\nu} \) is uniquely determined and the tangent plane \( \pi \) is oriented (by means of versor \( \overline{\nu} \)), too.

The first fundamental form \( \phi \) of surface \( S \) is defined by

\[
\phi = \langle d\overline{r}, d\overline{r} \rangle, \quad \forall (u,v) \in D \quad (4.1.9)
\]

or by \( \phi(du, dv) = \langle d\overline{r}(u,v), d\overline{r}(u,v) \rangle \), \( \forall (u,v) \in D \).

Taking into account the equality (4.1.5) it follows that \( \phi(du, dv) \) is a quadratic form:

\[
\phi(du, dv) = Edu^2 + 2Fudvdv + Gdv^2. \quad (4.1.10)
\]

The coefficients of \( \phi \) are the functions, defined on \( D \):

\[
E(u,v) = \langle \overline{r}_u, \overline{r}_u \rangle, \quad F(u,v) = \langle \overline{r}_u, \overline{r}_v \rangle, \quad G(u,v) = \langle \overline{r}_v, \overline{r}_v \rangle. \quad (4.1.11)
\]

But we have the discriminant \( \Delta \) of \( \phi \):

\[
\Delta = EG - F^2. \quad (4.1.12)
\]
(4.1.13) \[ E > 0, \ G > 0, \ \Delta > 0. \]

Consequence: the first fundamental form \( \varphi \) of \( S \) is positively defined.

Since the vector \( d\vec{r} \) does not depend on parametrization of \( S \), it follows that \( \varphi \) has a geometrical meaning with respect to a change of coordinates in \( E_3 \) and with respect to a change of parameters \((u, v)\) on \( S \).

Thus \( ds \), given by:

(4.1.14) \[ ds^2 = \varphi(du, dv) = Edu^2 + 2Fdu dv + Gdv^2 \]

is called the element of arclength of the surface \( S \).

The arclength of a curve \( C \) an \( S \) is expressed by

(4.1.15) \[ s = \int_{t_0}^{t} \left\{ E \left( \frac{du}{d\sigma} \right)^2 + 2F \frac{du}{d\sigma} \frac{dv}{d\sigma} + G \left( \frac{dv}{d\sigma} \right)^2 \right\}^{1/2} d\sigma. \]

The function \( s = s(t), \ t \in [a, b] \subset (t_1, t_2), \ t_0 \in [a, b] \) is a diffeomorphism from the interval \([a, b] \rightarrow [0, s(t)]\). The inverse function \( t = t(s) \), determines a new parametrization of curve \( C: \vec{r} = \vec{r}(u(s), v(s)) \). The tangent vector \( \frac{d\vec{r}}{ds} \) is a versor:

(4.1.16) \[ \alpha = \frac{d\vec{r}}{ds} = \vec{r}_u \frac{du}{ds} + \vec{r}_v \frac{dv}{ds}. \]

For two tangent versors \( \alpha = \frac{d\vec{r}}{ds}, \alpha_1 = \frac{\delta \vec{r}}{\delta s} \) at point \( P \in S \) the angle \( \angle(\alpha, \alpha_1) \) is expressed by

(4.1.17) \[ \cos \angle(\alpha, \alpha_1) = \frac{Edu\delta u + F(du\delta v + dv \delta u) + Gdv\delta v}{\sqrt{Edu^2 + 2Fdu dv + Gdv^2} \sqrt{E\delta u^2 + 2F\delta u \delta v + G\delta v^2}}. \]

The second fundamental form \( \psi \) of \( S \) at point \( P \in S \) is defined by

(4.1.18) \[ \psi = \langle \vec{v}, d^2 \vec{r} \rangle = -\langle d\vec{r}, d\vec{v} \rangle \ \forall (u, v) \in D. \]

Also, we adopt the notation \( \psi(du, dv) = -\langle d\vec{r}(u, v), d\vec{v}(u, v) \rangle. \psi \) is a
quadratic form:

\[(4.1.19)\quad \psi(du, dv) = Ldu^2 + 2Mdu dv + Ndv^2,\]

having the coefficients functions of \((u, v) \in D:\)

\[(4.1.20)\quad L(u, v) = \frac{1}{\sqrt{\Delta}} \langle \bar{r}_u, \bar{r}_u, \bar{r}_{uu} \rangle, \quad M(u, v) = \frac{1}{\sqrt{\Delta}} \langle \bar{r}_u, \bar{r}_v, \bar{r}_{uv} \rangle, \quad N(u, v) = \frac{1}{\sqrt{\Delta}} < \bar{r}_u, \bar{r}_v, \bar{r}_{vv} > .\]

Of course from \(\psi = -\langle d\bar{r}, d\bar{\nu} \rangle\) it follows that \(\psi\) has geometric meaning.

The two fundamental forms \(\phi\) and \(\psi\) are enough to determine a surface \(S\) in Euclidean space \(E_3\) under proper Euclidean motions. This property results by the integration of the Gauss-Weingarten formulae and by their differential consequences given by Gauss-Codazzi equations.

### 4.2 Gauss and Weingarten formulae

Consider the moving frame

\[\mathcal{R} = (P(\bar{r}); \bar{r}_u, \bar{r}_v, \bar{v})\]

on the smooth surface \(S\) (i.e. \(S\) is of class \(C^\infty\)).

The moving equations of \(R\) are given by the Gauss and Weingarten formulae.

The Gauss formulae are as follows:

\[(4.2.1)\quad\begin{align*}
\bar{r}_{uu} &= \begin{cases} 1 \\ 11 \end{cases} \bar{r}_u + \begin{cases} 2 \\ 11 \end{cases} \bar{r}_v + L\bar{r}, \\
\bar{r}_{uv} &= \begin{cases} 1 \\ 12 \end{cases} \bar{r}_u + \begin{cases} 2 \\ 12 \end{cases} \bar{r}_v + M\bar{v}, \\
\bar{r}_{vv} &= \begin{cases} 1 \\ 22 \end{cases} \bar{r}_u + \begin{cases} 2 \\ 22 \end{cases} \bar{r}_v + N\bar{v}.\end{align*}\]
The coefficients \( \{ i_j^k \} \), \((i, j, k = 1, 2; u = u^1, v = u^2)\) are called the Christoffel symbols of the first fundamental form \( \phi \):

\[
\begin{align*}
\{ 1_{11} \} &= -\frac{1}{\sqrt{D}} \langle \nabla, \nabla_{u} \nabla_{uu} \rangle, \\
\{ 1_{21} \} &= -\frac{1}{\sqrt{\Delta}} \langle \nabla, \nabla_{v} \nabla_{uv} \rangle, \\
\{ 1_{22} \} &= -\frac{1}{\sqrt{\Delta}} \langle \nabla, \nabla_{v} \nabla_{vv} \rangle, \\
\{ 2_{11} \} &= -\frac{1}{\sqrt{\Delta}} \langle \nabla, \nabla_{u} \nabla_{uu} \rangle, \\
\{ 2_{21} \} &= -\frac{1}{\sqrt{\Delta}} \langle \nabla, \nabla_{v} \nabla_{uv} \rangle, \\
\{ 2_{22} \} &= -\frac{1}{\sqrt{\Delta}} \langle \nabla, \nabla_{v} \nabla_{vv} \rangle.
\end{align*}
\]

The Weingarten formulae are given by:

\[
\begin{align*}
\frac{\partial \nabla}{\partial u} &= \frac{1}{\sqrt{\Delta}} \{(FM - GL)\nabla_{u} + (FL - EM)\nabla_{v}\} \\
\frac{\partial \nabla}{\partial v} &= \frac{1}{\sqrt{\Delta}} \{(FN - GM)\nabla_{u} + (FM - EN)\nabla_{v}\}.
\end{align*}
\]

Of course, the equations (4.2.1) and (4.2.3) express the variation of the moving frame \( R \) on surface \( S \).

Using the relations \( \nabla_{uu} \nabla_{v} = \nabla_{uv} \nabla_{u} = \nabla_{vv} \nabla_{u} \) and \( \frac{\partial^2 \nabla}{\partial u \partial v} = \frac{\partial^2 \nabla}{\partial v \partial u} \) applied to (4.2.1), (4.2.3) one deduces the so called fundamental equations of surface \( S \), known as the Gauss-Codazzi equations.

A fundamental theorem can be proved, when the first and second fundamental from \( \phi \) and \( \psi \) are given and Gauss-Codazzi equations are verified (Spivak [75]).
4.3 The tangent Myller configuration $\mathcal{M}_t(C, \bar{\xi}, \pi)$ associated to a tangent versor field $(C, \bar{\xi})$ on a surface $S$

Assuming that $C$ is a curve on surface $S$ in $E_3$ and $(C, \bar{\xi})$ is a versor field tangent to $S$ along $C$, there is an uniquely determined tangent Myller configuration $\mathcal{M}_t = \mathcal{M}_t(C, \bar{\xi}, \pi)$ for which $(C, \pi)$ is tangent field planes to $S$ along $C$.

The invariants $(c_1, c_2, G, K, T)$ of $(C, \bar{\xi})$ in $\mathcal{M}_t(C, \bar{\xi}, \pi)$ will be called the invariants of tangent versor field $(C, \bar{\xi})$ on surface $S$.

Evidently, these invariants have the same values on every smooth surface $S'$ which contains the curve $C$ and is tangent to $S$.

Using the theory of $\mathcal{M}_t$ from Chapter 3, for $(C, \bar{\xi})$ tangent to $S$ along $C$ we determine:

1. The Darboux frame

\begin{equation}
R_D = (P(s); \bar{\xi}(s), \overline{\mu}(s), \overline{\nu}(s)),
\end{equation}

where $s$ is natural parameter on $C$ and

\begin{equation}
\overline{\nu} = \frac{1}{\sqrt{\Delta}} \overline{r}_u \times \overline{r}_v, \quad \overline{\mu} = \overline{\nu} \times \bar{\xi}.
\end{equation}

Of course $R_D$ is orthonormal and positively oriented.

The invariants $(c_1, c_2, G, K, M)$ of $(C, \bar{\xi})$ on $S$ are given by (3.1.2), (3.1.3) Chapter 3. So we have the moving equations of $R$.

**Theorem 4.3.1** The moving equations of the Darboux frame of tangent versor field $(C, \bar{\xi})$ to $S$ are given by

\begin{equation}
\frac{d\overline{\nu}}{ds} = \overline{\alpha}(s) = c_1(s)\bar{\xi} + c_2(s)\overline{\mu}, \quad (c_1^2 + c_2^2 = 1)
\end{equation}

and
\[
\frac{d\xi}{ds} = G(s)\overline{\mu}(s) + K(s)\overline{\nu}(s),
\]
\[
\frac{d\overline{\mu}}{ds} = -G(s)\xi(s) + T(s)\overline{\nu}(s),
\]
\[
\frac{d\overline{\nu}}{ds} = -K(s)\xi - T(s)\overline{\mu}(s),
\]
\[
(4.3.4)
\]
where \(c_1(s), c_2(s), G(s), K(s), T(s)\) are invariants with respect to the changes of coordinates on \(E_3\), with respect of transformation of local coordinates on \(S\), \((u,v) \rightarrow (\tilde{u}, \tilde{v})\) and with respect to transformations of natural parameter \(s \rightarrow s_0 + s\).

A fundamental theorem can be proved exactly as in Chapter 2.

**Theorem 4.3.2** Let be \(c_1(s), c_2(s), [c_1^2 + c_2^2 = 1]\), \(G(s), K(s), T(s)\), \(s \in [a,b]\), a priori given smooth functions. Then there exists a tangent Myller configuration \(\mathfrak{M}_t(C, \xi, \pi)\) for which \(s\) is the arclength of curve \(C\) and the given functions are its invariants. \(\mathfrak{M}_t\) is determined up to a proper Euclidean motion. The given functions are invariants of the versor field \((C, \xi)\) for any smooth surface \(S\), which contains the curve \(C\) and is tangent planes \(\pi(s), s \in [a,b]\).

The geometric interpretations of the invariants \(G(s), K(s)\) and \(T(s)\) are those mentioned in the Section 2, Chapter 3.

So, we get

\[
G(s) = \lim_{\Delta s \to 0} \frac{\Delta \psi_1}{\Delta s}, \text{ with } \Delta \psi_1 = \angle(\xi(s), \text{pr}_{\pi(s)}\xi(s + \Delta s)),
\]

\[
K(s) = \lim_{\Delta s \to 0} \frac{\Delta \psi_2}{\Delta s}, \text{ with } \Delta \psi_2 = \angle(\overline{\mu}, \text{pr}_{(P,\pi,\pi)}\xi(s + \Delta s)),
\]

\[
T(s) = \lim_{\Delta s \to 0} \frac{\Delta \psi_3}{\Delta s}, \text{ with } \Delta \psi_3 = \angle(\overline{\mu}, \text{pr}_{(P,\overline{\mu},\overline{\mu})}\xi(s + \Delta s)).
\]

For this reason \(G(s)\) is called the geodesic curvature of \((C, \xi)\) on \(S\); \(K(s)\) is called the normal curvature of \((C, \xi)\) on \(S\) and \(T(s)\) is named the geodesic torsion of the tangent versor field \((C, \xi)\) on \(S\).
The calculus of invariants $G, K, T$ on a surface is exactly the same as it has been done in Chapter 2, (2.3.3), (2.3.4):

$$G(s) = \langle \xi, \frac{d\xi}{ds}, \nu \rangle, \quad K(s) = \langle \frac{d\xi}{ds}, \nu \rangle = -\langle \xi, \frac{d\nu}{ds} \rangle,$$

(4.3.5) \hspace{1cm} T(s) = \langle \xi, \nu, \frac{d\nu}{ds} \rangle.

The analytical expressions of invariants $G, K, T$ on $S$ are given in next section.

### 4.4 The calculus of invariants $G, K, T$ on a surface

The tangent versor $\alpha = \frac{d\tau}{ds}$ to the curve $C$ in $S$ is

(4.4.1) \hspace{1cm} \alpha(s) = \frac{d\tau}{ds} = \tau_u \frac{du}{ds} + \tau_v \frac{dv}{ds}.

$\alpha(s)$ is the versor of the tangent vector $d\tau$, from (4.4)

(4.4.2) \hspace{1cm} d\tau = \tau_u du + \tau_v dv.

The coordinate of a vector field $(C, \nabla)$ tangent to surface $S$ along the curve $C \subset S$, with respect to the moving frame $\mathcal{R} = (P; \tau_u, \tau_v, \nu)$ are the $C^\infty$ functions $V^1(s), V^2(s)$:

$$\nabla(s) = V^1(s)\tau_u + V^2(s)\tau_v.$$

The square of length of vector $\nabla(s)$ is:

$$\langle \nabla(s), \nabla(s) \rangle = E(V^1)^2 + 2FV^1V^2 + G(V^2)^2$$

and the scalar product of two tangent vectors $\nabla(s)$ and $\overline{U}(s) = U^1(s)\tau_u +$
$U^2(s)\tau_s$ is as follows

$$\langle U, V \rangle = EU^1V^1 + F(U^1V^2 + V^1U^2) + GU^2V^2.$$ 

Let $C$ and $C'$ two smooth curves on $S$ having $P(s)$ as common point. Thus the tangent vectors $d\bar{\tau}, \delta\bar{\tau}$ at a point $P$ to $C$, respectively to $C'$ are

$$d\bar{\tau} = \tau_u du + \tau_v dv,$$

$$\delta\bar{\tau} = \tau_u \delta u + \tau_v \delta v.$$ 

They correspond to tangent directions $(du, dv)$, $(\delta u, \delta v)$ on surface $S$.

Evidently, $\bar{\alpha}(s) = \frac{d\bar{\tau}}{ds}$ and $\bar{\xi}(s) = \frac{\delta\bar{\tau}}{\delta s}$ are the tangent versors to curves $C$ and $C'$, respectively at point $P(s)$, and $(C, \bar{\alpha})$, $(C, \bar{\xi})$ are the tangent versor fields along the curve $C$.

The Darboux frame $\mathcal{R}_D = (P(s), \bar{\xi}(s), \bar{\mu}(s), \bar{\nu}(s))$ has the versors $\bar{\xi}(s), \bar{\mu}(s), \bar{\nu}(s)$ given by

$$\bar{\xi}(s) = \tau_u \frac{\delta u}{\delta s} + \tau_v \frac{\delta v}{\delta s},$$

(4.4.3) $$\bar{\mu}(s) = \frac{1}{\sqrt{\Delta}} \left[ \left( E\tau_v - F\tau_u \right) \frac{\delta u}{\delta s} + \left( F\tau_v - G\tau_u \right) \frac{\delta v}{\delta s} \right],$$

$$\bar{\nu}(s) = \frac{1}{\Delta} (\tau_u \times \tau_v).$$

Taking into account (4.3.3) and (4.4.3) it follows:

(4.4.4) $$c_1 = \langle \bar{\alpha}, \bar{\xi} \rangle = \frac{Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v}{\sqrt{\phi(du, dv)}\sqrt{\phi(\delta u, \delta v)}},$$

$$c_2 = \langle \bar{\alpha}, \bar{\mu} \rangle = \frac{\sqrt{\Delta}(du\delta v - dv\delta u)}{\sqrt{\phi(du, dv)}\sqrt{\phi(\delta u, \delta v)}}$$

where

$$\phi(du, dv) = \langle d\bar{\tau}, d\bar{\tau} \rangle, \quad \phi(\delta u, \delta v) = \langle \delta\bar{\tau}, \delta\bar{\tau} \rangle.$$ 

In order to calculate the invariants $G, K, T$ we need to determine the
vectors $\frac{d\xi}{ds}, \frac{d\nu}{ds}$.

By means of (4.4.3) we obtain

$$\frac{d\xi}{ds} = \frac{d}{ds} \left( \frac{\delta \nu}{\delta s} \right) = \tau_{uv} \frac{du}{ds} \frac{\delta u}{\delta s} + \tau_{uv} \left( \frac{du}{ds} \frac{\delta v}{\delta s} + \frac{\delta u}{\delta s} \frac{dv}{ds} \right)$$

(4.4.5)

$$+ \tau_{vv} \frac{dv}{ds} \frac{\delta v}{\delta s} + \tau_{u} \frac{d}{ds} \left( \frac{\delta u}{\delta s} \right) + \tau_{v} \frac{d}{ds} \left( \frac{\delta v}{\delta s} \right).$$

The scalar mixt $\langle \tau_u, \tau_v, \nu \rangle$ is equal to $\sqrt{\Delta}$. It results:

$$\langle \xi, \frac{d\xi}{ds}, \nu \rangle = \sqrt{\Delta} \left[ \frac{d}{ds} \left( \frac{\delta \nu}{\delta s} \right) \frac{du}{ds} - \frac{d}{ds} \left( \frac{\delta u}{\delta s} \right) \frac{dv}{ds} \right]$$

$$+ \langle \tau_u, \tau_{uu}, \nu \rangle \frac{du}{ds} \left( \frac{\delta u}{\delta s} \right)^2 +$$

(4.4.6)

$$+ \langle \tau_u, \tau_{uv}, \nu \rangle \left( \frac{du}{ds} \frac{\delta v}{\delta s} + \frac{dv}{ds} \frac{\delta u}{\delta s} \right) \frac{\delta u}{\delta s} +$$

$$+ \langle \tau_v, \tau_{vv}, \nu \rangle \frac{dv}{ds} \frac{\delta v}{\delta s} \frac{\delta u}{\delta s} +$$

$$+ \langle \tau_v, \tau_{uv}, \nu \rangle \frac{dv}{ds} \left( \frac{\delta v}{\delta s} \right)^2.$$

Taking into account (4.2.1), (4.2.2), (4.3.5) and (4.4.5) we have

**Proposition 4.4.1** The geodesic curvature of the tangent versor field $(C, \xi)$ on $S$, is expressed as follows:

$$G(\delta, d) = \sqrt{\Delta} \left\{ \frac{d}{ds} \left( \frac{\delta \nu}{\delta s} \right) \frac{\delta u}{\delta s} - \frac{d}{ds} \left( \frac{\delta u}{\delta s} \right) \frac{\delta v}{\delta s} \right\}$$

(4.4.7)

$$+ \left\{ \left\{ \frac{2}{11} \right\} \frac{du}{ds} + \left\{ \frac{2}{12} \right\} \frac{dv}{ds} \right\} \left( \frac{\delta u}{\delta s} \right)^2$$

$$- \left\{ \left\{ \frac{1}{11} \right\} - \left\{ \frac{2}{12} \right\} \right\} \frac{du}{ds} - \left\{ \left\{ \frac{2}{22} \right\} - \left\{ \frac{1}{12} \right\} \right\} \frac{dv}{ds} \frac{\delta u}{\delta s} \frac{\delta v}{\delta s}$$

$$- \left\{ \left\{ \frac{1}{21} \right\} \frac{du}{ds} + \left\{ \frac{2}{22} \right\} \frac{dv}{ds} \right\} \left( \frac{\delta v}{\delta s} \right)^2.$$
Remark that the Christoffel symbols are expressed only by means of the coefficients of the first fundamental form $\phi$ of surface $S$ and their derivatives. It follows a very important result obtained by Al. Myller [34]:

**Theorem 4.4.1** The geodesic curvature $G$ of a tangent versor field $(C, \xi)$ on a surface $S$ is an intrinsic invariant of $S$.

The invariant $G$ was named by Al. Myller the deviation of parallelism of the tangent field $(C, \xi)$ on surface $S$.

The expression (4.4.7) of $G$ can be simplified by introducing the following notations

\[
\frac{D}{ds} \left( \frac{\delta u^i}{\delta s} \right) = \frac{d}{ds} \left( \frac{\delta u^i}{\delta s} \right) + \sum_{j,k=1}^{2} \left\{ \frac{i}{jk} \right\} \frac{\delta u^j}{\delta s} \frac{du^k}{ds} \quad (u = u^1; v = u^2; i = 1, 2).
\]

The operator (4.4.8) is the classical operator of covariant derivative with respect to Levi-Civita connection.

Denoting by

\[
G^{ij}(\delta, d) = \sqrt{\Delta} \left\{ \frac{D}{ds} \left( \frac{\delta u^i}{\delta s} \right) \frac{\delta u^j}{\delta s} - \frac{D}{ds} \left( \frac{\delta u^j}{\delta s} \right) \frac{\delta u^i}{\delta s} \right\}, (i, j = 1, 2)
\]

and remarking that $G^{ij}(\delta, d) = -G^{ji}(\delta, d)$ we have $G^{11} = G^{22} = 0$.

**Proposition 4.4.2** The following formula holds:

\[
G(\delta, d) = G^{12}(\delta, d).
\]

Indeed, the previous formula is a consequence of (4.4.7), (4.4.8) and (4.4.9).

**Corollary 4.4.1** The parallelism of tangent versor field $(C, \xi)$ to $S$ along the curve $C$ is characterized by the differential equation

\[
G^{ij}(\delta, d) = 0 \quad (i, j = 1, 2).
\]
In the following we introduce the notations

\[(4.4.12) \quad G(s) = G(\delta, d), \quad K(s) = K(\delta, d), \quad T(s) = T(\delta, d), \]

since \(\xi(s) = \tau_u \frac{\delta u}{\delta s} + \tau_v \frac{\delta v}{\delta s}, \quad \alpha(s) = \tau_u \frac{du}{ds} + \tau_v \frac{dv}{ds}.\)

The second formula (4.3.5), by means of (4.4.5) gives us the expression of normal curvature \(K(\delta, d)\) in the form

\[(4.4.13) \quad K(\delta, d) = \frac{L du \delta u + M (du \delta v + dv \delta u) + N dv \delta v}{\sqrt{E du^2 + 2F du dv + G dv^2} \sqrt{E \delta u^2 + 2F \delta u \delta v + G \delta v^2}}.\]

It follows

\[(4.4.14) \quad K(\delta, d) = K(d, \delta)\]

**Remark 4.4.1** The invariant \(K(\delta, d)\) can be written as follows

\[K(\delta, d) = \frac{\psi(\delta, d)}{\sqrt{\phi(d, d) \sqrt{\phi(\delta, \delta)}}}\]

where \(\psi(\delta, d)\) is the polar form of the second fundamental form \(\psi(d, d)\) of surface \(S\).

\(K(\delta, d) = 0\) gives us the property of conjugation of \((C, \xi)\) with \((C, \alpha)\).

The calculus of invariant \(T(\delta, d)\) can be made by means of Weingarten formulae (4.2.3).

Since we have

\[(4.4.15) \quad \frac{d\tau}{ds} = \frac{1}{\Delta} \left\{ \left[ (FM - GL) \tau_u + (FL - EM) \tau_v \right] \frac{du}{ds} \right. \]

\[\left. + \left[ (FN - GM) \tau_u + (FM - EN) \tau_v \right] \frac{dv}{ds} \right\},\]

we deduce

\[(4.4.16) \quad T(\delta, d) = \frac{1}{\sqrt{\Delta}} \left\{ \left[ (FM - GL) \frac{du}{ds} + (FN - GM) \frac{dv}{ds} \right] \frac{\delta v}{\delta s} \right. \]

\[\left. - \left[ (FL - EM) \frac{du}{ds} + (FM - EN) \frac{dv}{ds} \right] \frac{\delta u}{\delta s} \right\}.\]
For simplicity we write $T(\delta, d)$ from previous formula in the following form

\[(4.4.17)\ T(\delta, d) = \frac{1}{\sqrt{\Delta} \sqrt{\phi(d, d) / \phi(\delta, \delta)}} \begin{vmatrix} E\delta u + F\delta v & F\delta u + G\delta v \\ Ldu + Mdv & Mdu + Ndv \end{vmatrix}.\]

Remember the expression of mean curvature $H$ and total curvature $K_t$ of surface $S$:

\[(4.4.18)\ H = \frac{EN - 2FM + GN}{2(EG - F^2)}, \quad K_t = \frac{LN - M^2}{EG - F^2}.\]

It is not difficult to prove, by means of (4.4.16), the following formula

\[(4.4.19)\ T(\delta, d) - T(d, \delta) = 2\sqrt{\Delta} H \left( \frac{\delta u}{\delta s} \frac{dv}{ds} - \frac{\delta v}{\delta s} \frac{du}{ds} \right).\]

It has the following nice consequence:

**Theorem 4.4.2** The geodesic torsion $T(\delta, d)$ is symmetric with respect of directions $\delta$ and $d$, if and only if $S$ is a minimal surface.

Finally, $(C, \overline{\xi})$ is orthogonally conjugated with $(C, \overline{\pi})$ if and only if $T(\delta, d) = 0$.

**Remark 4.4.2** The invariant $K(\delta, d)$ was discovered by O. Mayer [20] and the invariant $T(\delta, d)$ was found by E. Bertoltti [3].

## 4.5 The Levi-Civita parallelism of vectors tangent to $S$

The notion of Myller parallelism of vector field $(C, \overline{V})$ tangent to $S$ along curve $C$, in the associated Myller configuration $\mathfrak{M}_C(C, \overline{\xi}, \pi)$ is exactly the Levi-Civita parallelism of tangent vector field $(C, \overline{V})$ to $S$ along the curve $C$. Indeed, taking into account that the tangent versor field $(C, \overline{\xi})$ has

\[(4.5.1)\ \overline{\xi}(s) = \xi^1(s)\overline{\pi}_u + \xi^2(s)\overline{\pi}_v\]
and expression of the operator $\frac{D}{ds}$ is

$$\frac{D\xi^i}{ds} = \frac{d\xi^i}{ds} + \sum_{j,k=1}^{2} \xi^j \left\{ \frac{d\xi^k}{ds} \right\} \right) \quad (i = 1, 2).$$

(4.5.2)

we have

$$G^{ij} = \frac{D\xi^i}{ds} \xi^j - \frac{D\xi^j}{ds} \xi^i, \quad (i = 1, 2).$$

Thus the parallelism of versor $(C, \xi)$ along $C$ on $S$ is expressed by $G^{ij} = 0$. But these equations are equivalent to

$$\frac{D(\lambda(s)\xi^i(s))}{ds} = 0 \quad (i = 1, 2), \quad (\lambda(s) \neq 0).$$

(4.5.3)

This is the definition of Levi-Civita parallelism of vectors $\nabla(s) = \lambda(s)\xi(s)$, $(\lambda(s) = \|V\|)$ tangent to $S$ along $C$. Writing $\nabla = V^1(s)\bar{r}_u + V^2(s)\bar{r}_v$ and putting

$$\frac{DV^i}{ds} = \frac{dV^i}{ds} + \sum_{j,k=1}^{2} V^j \left\{ \frac{dV^k}{ds} \right\},$$

(4.5.4)

the Levi-Civita parallelism is expressed by

$$\frac{DV^i}{ds} = 0.$$

(4.5.5)

In the case $\nabla(s) = \xi(s)$ is a versor field, parallelism of $(C, \xi)$ in the associated Myller configurations $\mathfrak{M}_t(C, \xi, \pi)$ is called the parallelism of directions tangent to $S$ along $C$. We can express the condition of parallelism by means of invariants of the field $(C, \xi)$, as follows:

1. $(C, \xi)$ is parallel in the Levi-Civita sense on $S$ along $C$ iff $G(\delta, d) = 0$.

2. $(C, \xi)$ is parallel in the Levi-Civita sense on $S$ along $C$ iff the versor $\xi(s')$, $\{ |s' - s| < \varepsilon, \varepsilon > 0, s' \in (s_1, s_2) \}$ is parallel in the space $E_3$ with the normal plan $(P(s); \xi(s), \nabla(s))$.

3. A necessary and sufficient condition for the versor field $(C, \xi)$ to be
parallel on $S$ along $C$ in Levi-Civita sense is that developing on a plane the ruled surface $E$ generated by tangent planes field $(C, \pi)$ along $C$ to $E$- the directions $(C, \xi)$ after developing to be parallel in Euclidean sense.

4. The directions $(C, \xi)$ are parallel in the Levi-Civita sense on $S$ along $C$ iff the versor field $(C, \xi_2)$ is normal to $S$.

5. $(C, \xi)$ is parallel on $S$ along $C$, iff $|K| = K_1$.

6. If $(C, \xi)$ is parallel on $S$ along $C$, then $|K| = K_1, T = K_2$.

7. If the ruled surface $R(C, \xi)$ is not developable, then $(C, \xi)$ is parallel in the Levi-Civita sense iff $C$ is the striction line of surface $R(C, \xi)$.

Taking into account the system of differential equations (4.5.5) in the given initial conditions, we have assured the existence and uniqueness of the Levi-Civita parallel versor fields on $S$ along $C$.

Other results presented in Section 9, Chapter 2 can be particularized here without difficulties.

A first application. The Tchebishev nets on $S$ are defined as a net of $S$ for which the tangent lines to a family of curves of net are parallel on $S$ along to every curve of another family of curves of net and conversely.

A Bianchi result

**Theorem 4.5.1** In order that the net parameter $(u = u_0, v = v_0)$ on $S$ to be a Tchebishev is necessary and sufficient to have the following conditions:

\[
\begin{bmatrix} 1 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix} = 0.
\]

Indeed, we have $G(\delta, d) = G(d, \delta) = 0$ if (4.5.6) holds.

But (4.5.6) is equivalent to the equations $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0$. So, with respect to a Tchebishev parametrization of $S$ its arclength element $ds^2 = \phi(d, d)$ is given by

\[
ds^2 = E(u)du^2 + 2F(u, v)dudv + G(v)dv^2.
\]

We finish this paragraph remarking that the parallelism of vectors $(C, \overline{V})$ on $S$ or the concurrence of vectors $(C, \overline{V})$ on $S$ can be studied
using the corresponding notions in configurations $\mathcal{M}_t$ described in the
Chapter 3.

4.6 The geometry of curves on a surface

The geometric theory of curves $C$ embedded in a surface $S$ can be derived
from the geometry of tangent Myller configuration $\mathcal{M}_t(C, \alpha, \pi)$ in which $\alpha(s)$ is tangent versor to $C$ at point $P(s) \in C$ and $\pi(s)$ is tangent plane
to $S$ at point $P$ for any $s \in (s_1, s_2)$.

Since $\mathcal{M}_t(c, \alpha, \pi)$ is geometrically associated to the curve $C$ on $S$ we
can define its Darboux frame $R_D$, determine the moving equations of $R_D$
and its invariants, as belonging to curve $C$ on surface $S$.

Applying the results established in Chapter 3, first of all we have

**Theorem 4.6.1** For a smooth curve $C$ embedded in a surface $S$, there
exists a Darboux frame $R_D = (P(s); \alpha(s), \pi^*(s), \pi(s))$ and a system of
invariants $\kappa_g(s), \kappa_n(s)$ and $\tau_g(s)$, satisfying the following moving equa-
tions

\[
\frac{d\pi}{ds} = \alpha(s), \quad \forall s \in (s_1, s_2)
\]

and

\[
\begin{align*}
\frac{d\alpha}{ds} &= \kappa_g(s)\pi^* + \kappa_n(s)\pi, \\
\frac{d\pi^*}{ds} &= -\kappa_g(s)\alpha + \tau_g(s)\pi, \\
\frac{d\pi}{ds} &= -\kappa_n(s)\alpha - \tau_g(s)\pi^*, \quad \forall s \in (s_1, s_2). 
\end{align*}
\]

The functions $\kappa_g(s), \kappa_n(s), \tau_g(s)$ are called the *geodesic curvature*, the
*normal curvature* and *geodesic torsion* of curve $C$ at point $P(s)$ on surface
$S$, respectively.

Exactly as in Chapter 3, a fundamental theorem can be enounced and
proved.

The invariants $\kappa_g, \kappa_n$ and $\tau_g$ have the same values along $C$ on any
smooth surface $S'$ which passes through curve $C$ and is tangent to surface $S$ along $C$.

The geometric interpretations of these invariants and the cases of curves $C$ for which $\kappa_g(s) = 0$, or $\kappa_n(s) = 0$ or $\tau_g(s) = 0$ can be studied as in Chapter 3.

The curves $C$ on $S$ for which $\kappa_g(s) = 0$ are called (as in Chapter 3) geodesics (or *autoparallel curves*) of $S$. If $C$ has the property $\kappa_n(s) = 0$, $\forall s \in (s_1, s_2)$, it is *asymptotic curve* of $S$ and the curve $C$ for which $\tau_g(s) = 0$, $\forall s \in (s_1, s_2)$ is the *curvature line* of $S$.

The expressions of these invariants can be obtained from those of the invariants $G(\delta, d)$, $K(\delta, d)$ and $T(\delta, d)$ for $\xi(s) = \overline{\mathcal{r}}(s)$ given in Chapter 4:

\begin{align}
\kappa_g &= G(d, d), \quad \kappa_n(s) = K(d, d), \quad \tau_g(s) = T(d, d).
\end{align}

So, we have

\begin{align}
\kappa_g &= \sqrt{\Delta} \left\{ \frac{D}{ds} \left( \frac{du^i}{ds} \right) \frac{du^j}{ds} - \frac{D}{ds} \left( \frac{du^j}{ds} \right) \frac{du^i}{ds} \right\}, \quad (i, j = 1, 2) \\
\kappa_n &= \frac{Ldu^2 + 2Mduv + Ndv^2}{Edv^2 + 2Fduv + Gdv^2}, \\
\tau_g &= \frac{1}{\sqrt{\Delta}} \begin{vmatrix} Ldu + Mdv & Mdu + Ndv \\ Edu + Fdv & Fdu + Gdv \end{vmatrix}.
\end{align}

Evidently, $\kappa_g$ is an intrinsic invariant of surface $S$ and $\kappa_n$ is the ratio of fundamental forms $\psi$ and $\Phi$.

The Mark Krein formula (3.4.3), Chapter 3 gives now the Gauss-Bonnet formula for a surface $S$. 

4.7 The formulae of O. Mayer and E. Bortolotti

The geometers O. Mayer and E. Bortolotti gave some new forms of the invariants $K(\delta, d)$ and $T(\delta, d)$ which generalize the Euler or Bonnet formulae from the geometry of surfaces in Euclidean space $E_3$.

Let $S$ be a smooth surface in $E_3$ having the parametrization given by curvature lines. Thus the coefficients $F$ and $M$ of the first fundamental and the second fundamental form vanish.

We denote by $\theta = \angle \left( \pi, \frac{r_u}{\sqrt{E}} \right)$ and obtain

\[
\cos \theta = \frac{\sqrt{E} du}{\sqrt{E du^2 + G dv^2}}, \quad \sin \theta = \frac{\sqrt{G} dv}{\sqrt{E du^2 + G dv^2}}.
\]  

(4.7.1)

The principal curvature are expressed by

\[
\frac{1}{R_1} = \frac{L}{E}, \quad \frac{1}{R_2} = \frac{N}{G}.
\]

(4.7.2)

The mean curvature and total curvature are

\[
H = \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) = \frac{1}{2} \left( \frac{L}{E} + \frac{N}{G} \right),
\]

\[
K_t = \frac{1}{R_1 R_2} = \frac{LN}{EG}.
\]

(4.7.3)

Consider a tangent versor field $(C, \xi)$, $\xi = \tau_u \frac{\delta u}{\delta s} + \tau_v \frac{\delta v}{\delta s}$ and let be $\sigma = \angle \left( \xi, \frac{r_u}{\sqrt{E}} \right)$. One gets

\[
\cos \sigma = \frac{\sqrt{E} \delta u}{\sqrt{E \delta u^2 + G \delta v^2}}, \quad \sin \sigma = \frac{\sqrt{G} \delta v}{\sqrt{E \delta u^2 + G \delta v^2}}.
\]

(4.7.4)

Consequently, the expressions (4.4.13) and (4.4.17) of the normal
The curvature and geodesic torsion of \((C, \xi)\) on \(S\) are as follows

\[
K(\delta, d) = \frac{\cos \sigma \cos \theta}{R_1} + \frac{\sin \sigma \sin \theta}{R_2},
\]

\[
T(\delta, d) = \frac{\cos \sigma \sin \theta}{R_2} - \frac{\sin \sigma \cos \theta}{R}.
\]

The first formula was established by O. Mayer [20] and second formula was given by E. Bortolotti [3].

In the case \(\xi(s) = \pi(s)\) these formulae reduce to the known Euler and Bonnet formulas, respectively:

\[
\kappa_n = \frac{\cos^2 \theta}{R_1} + \frac{\sin^2 \theta}{R_2},
\]

\[
\tau_g = \frac{1}{2} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \sin 2\theta.
\]

For \(\theta = 0\) or \(\theta = \frac{\pi}{2}\), \(\kappa_n\) is equal to \(\frac{1}{R_1}\) and \(\frac{1}{R_2}\) respectively. For \(\theta = \pm \frac{\pi}{4}\), \(\tau_g\) takes the extremal values.

\[
\frac{1}{T_1} = \frac{1}{2} \left( \frac{1}{R_2} - \frac{1}{R_1} \right), \frac{1}{T_2} = -\frac{1}{2} \left( \frac{1}{R_2} - \frac{1}{R_1} \right).
\]

Thus

\[
T_m = \frac{1}{2} \left( \frac{1}{T_1} + \frac{1}{T_2} \right), \ T_i = \frac{1}{T_1} \frac{1}{T_2}
\]

are the mean torsion of \(S\) at point \(P \in S\) and the total torsion of \(S\) at point \(P \in S\).

For surfaces \(S\), \(T_m\) and \(T_i\) have the following properties:

\[
T_m = 0, \ T_i = -\frac{1}{4} \left( \frac{1}{R_1} - \frac{1}{R_2} \right)^2.
\]

**Remark 4.7.1** 1. As we will see in the next chapter the nonholomorphic manifolds in \(E_3\) have a nonvanishing mean torsion \(T_m\).

2. \(T_i\) from (4.7.10) gives us the Bacaloglu curvature of surfaces [35].
Consider in plane $\pi(s)$ the cartesian orthonormal frame $(P(s); \vec{i}_1, \vec{i}_2)$, $i_1 = \frac{T^u}{\sqrt{E}}$, $i_2 = \frac{T^v}{\sqrt{G}}$ and the point $Q \in \pi(s)$ with the coordinates $(x, y)$, given by
\[ \overrightarrow{PQ} = x\vec{i}_1 + y\vec{i}_2, \quad \overrightarrow{PQ} = |\kappa_n|^{-1}\overline{\alpha}. \]
But $\overline{\alpha} = \cos \theta_1 + \sin \theta_2$. So we have the coordinates $(x, y)$ of point $Q$:
\[ (4.7.11) \quad x = |\kappa_n|^{-1}\cos \theta; \quad y = |\kappa_n|^{-1}\sin \theta. \]

The locus of the points $Q$, when $\theta$ is variable in interval $(0, 2\pi)$ is obtained eliminating the variable $\theta$ between the formulae (4.7.7) and (4.7.11). One obtains a pair of conics:
\[ (4.7.12) \quad \frac{x^2}{R_1} + \frac{y^2}{R_2} = \pm 1 \]
called the *Dupin indicatrix* of normal curvatures. This indicatrix is important in the local study of surfaces $S$ in Euclidean space.

Analogously we can introduce the Bonnet indicatrix. Consider in the plane $\pi(s)$ tangent to $S$ at point $P(s)$ the frame $(P(s), \vec{i}_1, \vec{i}_2)$ and the point $Q'$ given by
\[ \overrightarrow{PQ'} = |\tau_g|^{-1}\overline{\alpha} = x\vec{i}_1 + y\vec{i}_2. \]
Then, the locus of the points $Q'$, when $\theta$ verifies (4.7.7) and $x = |\tau_g|^{-1}\cos \theta$, $y = |\tau_g|^{-1}\sin \theta$, defines the Bonnet indicatrix of geodesic torsions:
\[ (4.7.13) \quad \left( \frac{1}{R_2} - \frac{1}{R_1} \right) xy = \pm 1 \]
which, in general, is formed by a pair of conjugated equilateral hyperbolae.

Of course, the relations between the indicatrix of Dupin and Bonnet can be studied without difficulties.

Following the same way we can introduce the indicatrix of the invariants $K(\delta, d)$ and $T(\delta, d)$.

So, consider the angles $\theta = \angle(\overline{\alpha}, \vec{i}_1)$, $\sigma = \angle(\overline{\xi}, \vec{i}_1)$ and $U \in \pi(s)$. The
point $U$ has the coordinates $x, y$ with respect to frame $(P(s); \tilde{t}_1, \tilde{t}_2)$ given by
\begin{equation}
(4.7.14) \quad x = |K(\delta, d)|^{-1} \cos \sigma, \quad y = |K(\delta, d)|^{-1} \sin \sigma.
\end{equation}

The locus of points $U$ is obtained from (4.7.5) and (4.7.14):
\begin{equation}
(4.7.15) \quad \frac{x \cos \theta}{R_1} + \frac{y \sin \theta}{R_2} = \pm 1.
\end{equation}

Therefore (4.7.15) is the indicatrix of the normal curvature $K(\delta, d)$ of versor field $(C, \xi)$. It is a pair of parallel straight lines.

Similarly, for
\begin{equation}
\begin{aligned}
x &= |T(\delta, d)|^{-1} \cos \sigma, \\
y &= |T(\delta, d)|^{-1} \sin \sigma
\end{aligned}
\end{equation}

and the formula (4.7.6) we determine the indicatrix of geodesic torsion $T_g(\delta, d)$ of the versor field $(C, \xi)$:
\begin{equation}
\frac{x \sin \theta}{R_2} - \frac{y \cos \theta}{R_1} = \pm 1.
\end{equation}

It is a pair of parallel straight lines.

Finally, we can prove without difficulties the following formula:
\begin{equation}
(4.7.16) \quad \kappa_n(\theta) \kappa_n(\sigma) + \tau_g(\theta) \tau_g(\sigma) = 2HK(\sigma, \theta) \cos(\sigma - \theta) - K_t \cos 2(\sigma - \theta),
\end{equation}

where $\kappa_n(\theta) = \kappa_n(d, d)$, $\kappa_n(\sigma) = \kappa_n(\delta, \delta)$, $K(\sigma, \theta) = K(d, \delta)$.

For $\sigma = \theta$, (i.e. $\tilde{\xi} = \tilde{\alpha}$) the previous formulas leads to a known Baltrami-Enneper formula:
\begin{equation}
(4.7.17) \quad \kappa_n^2 + \tau_g^2 - 2HK_n + K_t = 0,
\end{equation}

for every point $P(s) \in S$.

Along the asymptotic curves we have $\kappa_n = 0$, and the previous equa-
tions give us the Enneper formula

\[ \tau_g^2 + K_t = 0, \ (\kappa_n = 0). \]

Along to the curvature lines, \( \tau_g = 0 \) and one obtains from (4.7.17)

\[ \kappa_n^2 - 2H\kappa_n + K_t = 0, \ (\tau_g = 0). \]

The considerations made in Chapter 4 of the present book allow to affirm that the applications of the theory of Myller configurations in the geometry of surfaces in Euclidean space \( E_3 \) are interesting. Of course, the notion of Myller configuration can be extended to the geometry of nonholonomic manifolds in \( E_3 \) which will be studied in next chapter. It can be applied to the theory of versor fields in \( E_3 \) which has numerous applications to Mechanics, Hydrodynamics (see the papers by Gh. Gheorghiev and collaborators).

Moreover, the Myller configurations can be defined and investigated in Riemannian spaces and applied to the geometry of submanifolds of these spaces, [24], [25]. They can be studied in the Finsler, Lagrange or Hamilton spaces [58], [68], [70].
Chapter 5

Applications of theory of Myller configurations in the geometry of nonholonomic manifolds from $E_3$

The second efficient applications of the theory of Myller configurations can be done in the geometry of nonholonomic manifolds $E_3^2$ in $E_3$. Some important results, obtained in the geometry of manifolds $E_3^2$ by Issaly l’Abbe, D. Sintzov [40], Gh. Vrânceanu [44], [45], Gr. Moisil [72], M. Haimovici [14], [15] Gh. Gheorghiev [10], [11], I. Popa [12], G. Th. Gheorghiu [13], R. Miron [30], [64], [65], I. Creangă [4], [5], [6], A. Dobrescu [7], [8] and I. Vaisman [41], [42], can be unitary presented by means of associated Myller configuration to a curve embedded in $E_3^2$. Now, a number of new concepts appears, the mean torsion, the total torsion, concurrence of tangent vector field. The new formulae, as Mayer, Bortolotti-formulas, Dupin and Bonnet indicatrices etc. will be studied, too.
5.1 Moving frame in Euclidean spaces $E_3$

Let $R = (P; I_1, I_2, I_3)$ be an orthonormed positively oriented frame in $E_3$. The application $P \in E_3 \rightarrow R = (P; I_1, I_2, I_3)$ of class $C^k$, $k \geq 3$ is a moving frame of class $C^k$ in $E_3$. If $\vec{r} = \overrightarrow{OP} = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$ is the vector of position of point $P$, and $P$ is the application point of the versors $I_1, I_2, I_3$, thus the moving equation of $R$ can be expressed in the form (see Spivak, vol. I [76] and Biujghens [2]):

\begin{equation}
(5.1.1) \quad d\vec{r} = \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3,
\end{equation}

where $\omega_i (i = 1, 2, 3)$ are independent 1 forms of class $C^{k-1}$ on $E_3$, and

\begin{equation}
(5.1.2) \quad dI_i = \sum_{j=1}^{3} \omega_{ij} I_j, \quad (i = 1, 2, 3),
\end{equation}

with $\omega_{ij}, (i, j = 1, 2, 3)$ are the rotation Ricci coefficients of the frame $R$. They are 1-forms of class $C^{k-1}$, satisfying the skewsymmetric conditions

\begin{equation}
(5.1.3) \quad \omega_{ij} + \omega_{ji} = 0, \quad (i, j = 1, 2, 3).
\end{equation}

In the following, it is convenient to write the equations (5.1.2) in the form

\begin{align}
(5.1.2)' & \quad dI_1 = rI_2 - qI_3 \\
& \quad dI_2 = pI_3 - rI_1 \\
& \quad dI_3 = qI_1 - pI_2.
\end{align}

Thus, *thus structure equations* of the moving frame $R$ can be obtained by exterior differentiating the equations (5.1.1) and (5.1.2)' modulo the same system of equations (5.1.1), (5.1.2)'.

One obtains, without difficulties

**Theorem 5.1.1** The structure equations of the moving frame $R$ are

\begin{align}
& \quad d\omega_1 = r \wedge \omega_2 - q \wedge \omega_3, \quad dp = r \wedge q,
\end{align}
5.1. Moving frame in Euclidean spaces $E_3$

$$d\omega_2 = p \wedge \omega_3 - r \wedge \omega_1, \quad dq = p \wedge r,$$
$$d\omega_3 = q \wedge \omega_1 - p \wedge \omega_2, \quad dr = q \wedge p. \tag{5.1.4}$$

In the vol. II of the book of Spivak [76], it is proved the theorem of existence and uniqueness of the moving frames:

**Theorem 5.1.2** Let $(p,q,r)$ be 1-forms of class $C^{k-1}, (k \geq 3)$ on $E_3$ which satisfy the second group of structure equation (5.1.4), thus:

1°. In a neighborhood of point $O \in E_3$ there is a triple of vector fields $(I_1, I_2, I_3)$, solutions of equations (5.1.2)*, which satisfy initial conditions $(I_1(0), I_2(0), I_3(0))$-positively oriented, orthonormed triple in $E_3$.

2°. In a neighborhood of point $O \in E_3$ there exists a moving frame $R = (P; I_1, I_2, I_3)$ orthonormed positively oriented for which $\overrightarrow{OP}$ is given by (5.1.1), where $\omega_i(i = 1, 2, 3)$ satisfy the first group of equations (5.1.4). $(I_1, I_2, I_3)$ are given in 1° and the initial conditions are verified: $(P_0 = \overrightarrow{OO}; I_1(0), I_2(0), I_3(0))$-the orthonormed frame at point $O \in E_3$.

Remarking that the 1-forms $\omega_1, \omega_2, \omega_3$ are independent we can express 1-forms $p, q, r$ with respect to $\omega_1, \omega_2, \omega_3$ in the following form:

$$p = p_1 \omega_1 + p_2 \omega_2 + p_3 \omega_3, \quad q = q_1 \omega_1 + q_2 \omega_2 + q_3 \omega_3,$$
$$r = r_1 \omega_1 + r_2 \omega_2 + r_3 \omega_3. \tag{5.1.5}$$

The coefficients $p_i, q_i, r_i$ are function of class $C^{k-1}$ on $E_3$.

Of course we can write the structure equations (5.1.4) in terms of these coefficients. Also, we can state the Theorem 5.1.2 by means of coefficients of 1-forms $p, q, r$.

The 1-forms $p, q, r$ determine the rotation vector $\Omega$ of the moving frame:

$$\Omega = pI_1 + qI_2 + rI_3. \tag{5.1.6}$$

Its orthogonal projection on the plane $(P; I_1, I_2)$ is

$$\overline{\theta} = pI_1 + qI_2. \tag{5.1.7}$$
Chapter 5. Applications of theory of Myller configurations in the geometry of nonholonomic manifolds from $E_3$

Let $C$ be a curve of class $C^k$, $k \geq 3$ in $E_3$, given by

$$\tau = \tau(s), \ s \in (s_1, s_2),$$

(5.1.8)

where $s$ is natural parameter. If the origin $P$ of moving frame describes the curve $C$ we have $R = (P; I_1(s), I_2(s), I_3(s))$, where $P(s) = P(\tau(s))$, $I_j(s) = I_j(\tau(s))$, $(j = 1, 2, 3)$.

So, the tangent versor $\frac{d\tau}{ds}$ to curve $C$ is:

$$\frac{d\tau}{ds} = \frac{\omega_1(s)}{ds}I_1 + \frac{\omega_2(s)}{ds}I_2 + \frac{\omega_3(s)}{ds}I_3$$

(5.1.9)

and $ds^2 = \langle d\tau, d\tau \rangle$ is:

$$ds^2 = (\omega_1(s))^2 + (\omega_2(s))^2 + (\omega_3(s))^2,$$

(5.1.10)

where $\omega_i(s)$ are 1-forms $\omega_i$ restricted to $C$.

The restrictions $p(s), q(s), r(s)$ of the 1-forms $p, q, r$ to $C$ give us:

$$\frac{p(s)}{ds} = p_1(s)\frac{\omega_1(s)}{ds} + p_2(s)\frac{\omega_2(s)}{ds} + p_3(s)\frac{\omega_3(s)}{ds},$$

$$\frac{q(s)}{ds} = q_1(s)\frac{\omega_1(s)}{ds} + q_2(s)\frac{\omega_2(s)}{ds} + q_3(s)\frac{\omega_3(s)}{ds},$$

(5.1.11)

$$\frac{r(s)}{ds} = r_1(s)\frac{\omega_1(s)}{ds} + r_2(s)\frac{\omega_2(s)}{ds} + r_3(s)\frac{\omega_3(s)}{ds}.$$

5.2 Nonholonomic manifolds $E_3^2$

**Definition 5.2.1** A nonholonomic manifold $E_3^2$ on a domain $D \subset E_3$ is a nonintegrable distribution $\mathcal{D}$ of dimension 2, of class $C^{k-1}$, $k \geq 3$.

One can consider $\mathcal{D}$ as a plane field $\pi(P)$, $P \in D$, the application $P \to \pi(P)$ being of class $C^{k-1}$.

Also $\mathcal{D}$ can be presented as the plane field $\pi(P)$ orthogonal to a versor field $\pi(P)$ normal to the plane $\pi(P)$, $\forall P \in D$. 

5.2. Nonholonomic manifolds $E^2_3$

Assuming that $\mathbf{v}(P)$ is the versor of vector

$$\overrightarrow{V}(P) = X(x, y, z)e_1 + Y(x, y, z)e_2 + Z(x, y, z)e_3$$

and $\overrightarrow{OP} = xe_1 + ye_2 + ze_3$ thus the nonholonomic manifold $E^2_3$ is given by the Pfaff equation:

(5.2.1) \[ \omega = X(x, y, z)dx + Y(x, y, z)dy + Z(x, y, z)dz = 0 \]

which is nonintegrable, i.e.

(5.2.2) \[ \omega \wedge d\omega \neq 0. \]

Consider a moving frame $\mathcal{R} = (P; I_1, I_2, I_3)$ in the space $E_3$ and the nonholonomic manifold $E^2_3$, on a domain $D$ orthogonal to the versors field $I_3$. It is given by the Pfaff equations

(5.2.3) \[ \omega_3 = \langle I_3, d\mathbf{v} \rangle = 0, \quad \text{on } D. \]

By means of (5.1.4) we obtain

(5.2.4) \[ \omega_3 \wedge d\omega_3 = -(p_1 + q_2)\omega_1 \wedge \omega_2 \wedge \omega_3. \]

So, the Pfaff equation $\omega_3 = 0$ is not integrable if and only if we have

(5.2.5) \[ p_1 + q_2 \neq 0, \quad \text{on } D. \]

It is very known that:

In the case $p_1 + q_2 = 0$ on the domain $D$, there are two functions $h(x, y, z) \neq 0$ and $f(x, y, z)$ on $D$ with the property

(5.2.6) \[ h\omega_3 = df. \]
Thus the equation $h\omega_3 = 0$, have a general solution

\[(5.2.7)\quad f(x, y, z) = c, \quad (c = \text{const.}), \quad P(x, y, z) \in D.\]

The manifold $E^2_3$, in this case is formed by a family of surfaces, given by (5.2.7).

For simplicity we assume that the class of the manifold $E^2_3$ is $C^\infty$ and the same property have the geometric object fields or mappings which will be used in the following parts of this chapter.

A smooth curve $C$ embedded in the nonholonomic manifolds $E^2_3$, has the tangent vector $\alpha = \frac{d\tau}{ds}$ given by (5.1.9) and $\omega_3 = 0$.

This is

\[(5.2.8)\quad \alpha(s) = \frac{d\tau}{ds} = \frac{\omega_1(s)}{ds} I_1 + \frac{\omega_2(s)}{ds} I_2, \quad \forall P(\tau(s)) \in D.\]

The square of arclength element, $ds$ is

\[(5.2.9)\quad ds^2 = (\omega_1(s))^2 + (\omega_2(s))^2.\]

And the arclength of curve $C$ on the interval $[a, b] \subset (s_1, s_2)$ is

\[(5.2.10)\quad s = \int_a^b \sqrt{(\omega_1(\sigma))^2 + (\omega_2(\sigma))^2} d\sigma.\]

A tangent versor field $\tilde{\xi}(s)$ at the same point $P(s)$ to another curve $C'$, $P \in C'$ and $P \in C$, can be given in the same form (5.2.8):

\[(5.2.11)\quad \tilde{\xi}(s) = \frac{\delta\tau}{\delta s} = \frac{\tilde{\omega}_1(s)}{\delta s} I_1 + \frac{\tilde{\omega}_2(s)}{\delta s} I_2,\]

with $\delta s^2 = (\tilde{\omega}_1(s))^2 + (\tilde{\omega}_2(s))^2$.

Thus, $\tilde{\xi}(s)$ is a versor field tangent to the nonholonomic manifold $E^2_3$ along to the curve $C$.

To any tangent vector field $(C, \tilde{\xi})$ to $E^2_3$ along the curve $C$ belonging to $E^2_3$ we uniquely associated the tangent Myller configuration $\mathfrak{M}_t(C, \tilde{\xi}, I_3)$. 

5.3 The invariants $G, R, T$ of a tangent versor field $(C, \xi)$ in $E_3^2$

We say that: *the geometry of the associated tangent Myller configuration $\mathcal{M}_t(C, \xi, I_3)$ is the geometry of versor field $(C, \xi)$ on $E_3^2$. In particular the geometry of configuration $\mathcal{M}_t(C, \mu, I_3)$ is the geometry of curve $C$ in the nonholonomic manifold $E_3^2$.*

The Darboux frame of $(C, \xi)$ on $E_3^2$ is $R_D = (P, \bar{\xi}, \bar{\mu}, I_3)$, with $\bar{\mu} = I_3 \times \bar{\xi}$, i.e.:

$$
\bar{\mu} = \frac{\tilde{\omega}_1(s)}{\delta s} I_2 - \frac{\tilde{\omega}_2(s)}{\delta s} I_1.
$$

(5.2.12)

In Darboux frame $R_D$, the tangent versor $\bar{\sigma} = \frac{d\sigma}{ds}$ to $C$ can be expressed in the following form:

$$
\bar{\sigma}(s) = \cos \lambda(s) \bar{\xi}(s) + \sin \lambda(s) \bar{\mu}(s).
$$

(5.2.13)

Taking into account the relations (5.2.8), (5.2.11) and (5.2.13), one gets:

$$
\frac{\omega_1}{\delta s} = \cos \lambda \frac{\tilde{\omega}_1}{\delta s} - \sin \lambda \frac{\tilde{\omega}_2}{\delta s},
$$

$$
\frac{\omega_2}{\delta s} = \sin \lambda \frac{\tilde{\omega}_1}{\delta s} + \cos \lambda \frac{\tilde{\omega}_2}{\delta s}.
$$

(5.2.14)

For $\lambda(s) = 0$, we have $\bar{\sigma}(s) = \bar{\xi}(s)$.

### 5.3 The invariants $G, R, T$ of a tangent versor field $(C, \bar{\xi})$ in $E_3^2$

The invariants $G, K, T$ of $(C, \bar{\xi})$ satisfy the fundamental equations

$$
\frac{d\bar{\xi}}{ds} = G\bar{\mu} + KI_3, \quad \frac{d\bar{\mu}}{ds} = -G\bar{\xi} + TI_3, \quad \frac{dI_3}{ds} = -KI_3. 
$$

(5.3.1)

Consequently,

$$
G(\delta, d) = \left\langle \xi, \frac{d\bar{\xi}}{ds}, I_3 \right\rangle.
$$
(5.3.2) \[ K(\delta, d) = \left\langle \frac{d\xi}{ds}, I_3 \right\rangle = -\left\langle \xi, \frac{dI_3}{ds} \right\rangle \]
\[ T(\delta, d) = \left\langle \xi, I_3, \frac{dI_3}{ds} \right\rangle. \]

The proofs and the geometrical meanings are similar to those from Chapter 3.

By means of expression (5.2.11) of versors \( \tilde{\xi}(s) \) we deduce:
\[
\frac{d\tilde{\xi}}{ds} = \left[ \frac{d}{ds} \left( \frac{\tilde{\omega}_1}{\delta s} \right) - \frac{r}{\delta s} \frac{\tilde{\omega}_2}{\delta s} \right] I_1 + \left[ \frac{d}{ds} \left( \frac{\tilde{\omega}_2}{\delta s} \right) + \frac{r}{\delta s} \frac{\tilde{\omega}_1}{\delta s} \right] I_2 +
\]
\[ + \left[ \frac{p}{\delta s} \frac{\tilde{\omega}_2}{\delta s} - \frac{q}{\delta s} \frac{\tilde{\omega}_1}{\delta s} \right] I_3, \]  
(5.3.3)

and
\[
\frac{dI_3}{ds} = \left( q_1 \frac{\omega_1}{ds} + q_2 \frac{\omega_2}{ds} \right) I_1 - \left( p_1 \frac{\omega_1}{ds} + p_2 \frac{\omega_2}{ds} \right) I_2. \]  
(5.3.4)

The formulae (5.3.2) lead to the following expressions for geodesic curvature \( G(\delta, d) \), normal curvature \( K(\delta, d) \) and geodesic torsion \( T(\delta, d) \) of the tangent versor field \( (C, \tilde{\xi}) \) on the nonholonomic manifold \( E_3^2 \):

(5.3.5) \[ G(\delta, d) = \frac{\tilde{\omega}_1}{\delta s} \left[ \frac{d}{ds} \left( \frac{\tilde{\omega}_2}{\delta s} \right) + \frac{r}{\delta s} \frac{\tilde{\omega}_1}{\delta s} \right] - \frac{\tilde{\omega}_2}{\delta s} \left[ \frac{d}{ds} \left( \frac{\tilde{\omega}_1}{\delta s} - \frac{r}{\delta s} \frac{\tilde{\omega}_2}{\delta s} \right) \right], \]

(5.3.6) \[ K(\delta, d) = \frac{p\tilde{\omega}_2 - q\tilde{\omega}_1}{\delta s \cdot ds} \]

(5.3.7) \[ T(\delta, d) = \frac{p\tilde{\omega}_1 + q\tilde{\omega}_2}{\delta s ds}. \]

All these formulae take very simple forms if we consider the angles \( \alpha = \angle(\bar{\alpha}, I_1) \), \( \beta = \angle(\bar{\xi}, I_1) \) since we have
\[
\frac{\omega_1}{ds} = \cos \alpha, \quad \frac{\omega_2}{ds} = \sin \alpha,
\]
\[
\frac{\tilde{\omega}_1}{\delta s} = \cos \beta, \quad \frac{\tilde{\omega}_2}{\delta s} = \sin \beta. \]  
(5.3.8)
5.3. The invariants $G, R, T$ of a tangent versor field $(C, \xi)$ in $E^2_3$

With notations

\[(5.3.9) \quad G(\delta, d) = G(\beta, \alpha); K(\delta, d) = K(\beta, \alpha), T(\delta, d) = T(\beta, \alpha)\]

the formulae (5.3.2) can be written:

\[(5.3.10) \quad G(\beta, \alpha) = \frac{d\beta}{ds} + r_1 \cos \alpha + r_2 \sin \alpha,\]

\[(5.3.11) \quad K(\beta, \alpha) = (p_1 \cos \alpha + p_2 \sin \alpha) \sin \beta - (q_1 \cos \alpha + q_2 \sin \alpha) \cos \beta,\]

\[(5.3.12) \quad T(\beta, \alpha) = (p_1 \cos \alpha + p_2 \sin \alpha) \cos \beta + (q_1 \cos \alpha + q_2 \sin \alpha) \sin \beta.\]

Immediate consequences:

Setting

\[(5.3.13) \quad T_m = p_1 + q_2,\]

(called the mean torsion of $E^2_3$ at point $P \in E^2_3$),

\[(5.3.14) \quad H = p_2 - q_1\]

(called the mean curvature of $E^2_3$ at point $P \in E^2_3$), from (5.3.11) and (5.3.12) we have:

\[(5.3.15) \quad K(\beta, \alpha) - K(\alpha, \beta) = T_m \cos(\alpha - \beta),\]

\[(5.3.15) \quad T(\beta, \alpha) - T(\alpha, \beta) = H \cos(\alpha - \beta).\]

**Theorem 5.3.1** The following properties hold:

1. Assuming $\beta - \alpha \neq \pm \frac{\pi}{2}$, the normal curvature $K(\beta, \alpha)$ is symmetric with respect to the versor field $(C, \xi), (C, \alpha)$ at every point $P \in E^2_3$, if and only if the mean torsion $T_m$ of $E^2_3$ vanishes ($E^2_3$ is integrable).

2. For $\beta - \alpha \neq \pm \frac{\pi}{2}$, the geodesic torsion $T(\beta, \alpha)$ is symmetric with respect to the versor fields $(C, \xi), (C, \alpha)$, if and only if the nonholonomic manifold $E^2_3$ has null mean curvature ($E^2_3$ is minimal).
5.4 Parallelism, conjugation and orthonormal conjugation

The notion of conjugation of versor field $(C, \xi)$ with tangent field $(C, \alpha)$ on the nonholonomic manifold $E^3_2$ is that studied for the associated Myller configuration $\mathfrak{M}_t$, so that $(C, \xi)$ are conjugated with $(C, \alpha)$ on $E^3_2$ if and only if $K(\delta, d) = 0$ i.e.

\[(p_1 \omega_1 + p_2 \omega_2)\tilde{\omega}_2 - (q_1 \omega_1 + q_2 \omega_2)\tilde{\omega}_1 = 0\]

or, by means of (5.3.11):

\[(p_1 \cos \alpha + p_2 \sin \alpha) \sin \beta - (q_1 \cos \alpha + q_2 \sin \alpha) \cos \beta = 0.\]

All propositions established in section 3.3, Chapter 3, can be applied. The notion of orthogonal conjugation of $(C, \xi)$ with $(C, \alpha)$ is given by $T(\delta, d) = 0$ or by

\[(5.4.3) \quad (p_1 \omega_1 + p_2 \omega_2)\tilde{\omega}_1 + (q_1 \omega_1 + q_2 \omega_2)\tilde{\omega}_2 = 0\]

or, by means of (5.3.11), it is characterized by

\[(5.4.4) \quad (p_1 \cos \alpha + p_2 \sin \alpha) \cos \beta + (q_1 \cos \alpha + q_2 \sin \alpha) \sin \beta = 0.\]

The relation of conjugation is symmetric iff $E^2_3$ is integrable ($T_m = 0$) and that of orthogonal conjugation is symmetric iff $E^2_3$ is minimal (i.e. $H = 0$).

Now we study the case of tangent versor field $(C, \xi)$ parallel along $C$, in $E^3_2$. Applying the theory of parallelism in $\mathfrak{M}_t(C, \xi, \pi)$, from Chapter 3 we obtain:

**Theorem 5.4.1** The versors $(C, \xi)$, tangent to the manifold $E^2_3$ along the curve $C$ is parallel in the Levi-Civita sense if and only if the following
system of equations holds

\[
\frac{d}{ds} \left( \frac{\tilde{\omega}_1}{\delta s} \right) + \frac{r}{ds \delta s} \frac{\tilde{\omega}_2}{\delta s} - \frac{d}{ds} \left( \frac{\tilde{\omega}_2}{\delta s} \right) - \frac{r}{ds \delta s} \frac{\tilde{\omega}_1}{\delta s} = 0.
\]

But, the previous system is equivalent to the system:

\[
\frac{d}{ds} \left( \frac{\tilde{\omega}_1}{\delta s} \right) - \frac{r}{ds \delta s} \frac{\tilde{\omega}_2}{\delta s} = h(s) \frac{\tilde{\omega}_1}{\delta s},
\]

\[
\frac{d}{ds} \left( \frac{\tilde{\omega}_2}{\delta s} \right) + \frac{r}{ds \delta s} \frac{\tilde{\omega}_1}{\delta s} = h(s) \frac{\tilde{\omega}_2}{\delta s}, \quad \forall h(s) \neq 0.
\]

Since, the tangent vector field \( h(s)\tilde{\xi}(s) \) has the same direction with the versor \( \tilde{\xi}(s) \), from (5.2.11) we obtain \( G(\tilde{\delta},d) = h^2 G(\delta,d) \), \( \tilde{\delta} \) being the direction of tangent vector \( h(s)\tilde{\xi} \). Therefore, the equations \( G(\delta,d) = 0 \) is invariant with respect to the applications \( \tilde{\xi}(s) \to h(s)\tilde{\xi}(s) \). Thus the equations (5.4.3) or (5.4.4) characterize the parallelism of directions \((C,h(s)\tilde{\xi}(s))\) tangent to \( E^2_3 \) along \( C \).

All properties of the parallelism of versors \((C,\xi)\) studied for the configuration \( \mathcal{M}_t \) in sections 2.8, Chapter 2 are applied here.

Theorem of existence and uniqueness of parallel of tangent versors \((C,\tilde{\xi})\) on \( E^2_3 \) can be formulated exactly as for this notion in \( \mathcal{M}_t \).

But a such kind of theorem can be obtained by means of the following equation, given by (5.3.10):

\[
G(\beta,\alpha) \equiv \frac{d\beta}{ds} + r_1(s) \cos \alpha + r_2(s) \sin \alpha = 0.
\]

The parallelism of vector field \((C,\overline{V})\) tangents to \( E^2_3 \) along the curve \( C \) can be studied using the form

\[
\overline{V}(s) = ||\overline{V}(s)||\tilde{\xi}(s)
\]

where \( \overline{\xi} \) is the versor of \( \overline{V}(s) \).

Also, we can start from the definition of Levi-Civita parallelism of
tangent vector field \((C, \nabla)\), expressed by the property

\[
\frac{d\nabla}{ds} = h(s)I_3.
\]  
(5.4.8)

Setting

\[
\nabla(s) = V^1(s)I_1 + V^2(s)I_2
\]  
(5.4.9)

and using (5.4.8) we obtain the system of differential equations

\[
\frac{dV^1}{ds} - r\frac{ds}{ds}V^2 = 0, \quad \frac{dV^2}{ds} + r\frac{ds}{ds}V^1 = 0.
\]  
(5.4.10)

All properties of parallelism in Levi-Civita sense of tangent vectors \((C, \nabla)\) studied in section 2.8, Chapter 2 for Myller configuration are valid. For instance

**Theorem 5.4.2** The Levi-Civita parallelism of tangent vectors \((C, \nabla)\) in the manifold \(E^2_3\) preserves the lengths and angle of vectors.

The concurrence in Myller sense of tangent vector fields \((C, \xi)\) is characterized by Theorem 2.9.2 Chapter 2, by the following equations

\[
\frac{d}{ds} \left( \frac{c_2}{G} \right) = c_1,
\]  
(5.4.11)

where \(c_1 = \langle \alpha, \xi \rangle\), \(c_2 = \langle \alpha, \mu \rangle\), \(G = G(\delta, d) \neq 0\).

Consequence: the concurrence of tangent versor field \((C, \overline{\xi})\) in \(E^2_3\) for \(G \neq 0\) is characterized by

\[
\frac{d}{ds} \left[ \frac{\tilde{\omega}_1\omega_2 - \tilde{\omega}_2\omega_1}{\delta s \cdot ds} \cdot \frac{1}{G} \right] = \frac{\tilde{\omega}_1\omega_1 + \tilde{\omega}_2\omega_2}{\delta sds}
\]  
(5.4.12)

or by:

\[
\frac{d}{ds} \left[ \frac{1}{G} \sin(\alpha - \beta) \right] = \cos(\alpha - \beta).
\]  
(5.4.13)

The properties of concurrence in Myller sense can be taken from section 2.8, Chapter 2.
5.5 Theory of curves in $E^2_3$

To a curve $C$ in the nonholonomic manifold $E^2_3$ one can uniquely associate a Myller configuration $\mathfrak{M}_t = \mathfrak{M}_t(C, \alpha, \pi)$ where $\pi(s)$ is the orthogonal to the normal versor $I_3(s)$ of $E^2_3$ at every point $P \in C$.

Thus the geometry of curves in $E^2_3$ is the geometry of associated Myller configurations $\mathfrak{M}_t$. It is obtained as a particular case taking $\xi(s) = \pi(s)$ from the previous sections of this chapter.

The Darboux frame of the curve $C$, in $E^2_3$ is given by

$$\mathcal{R}_D = \{P(s); \overline{\alpha}(s), \overline{\mu}^*(s), I_3(s)\}, \quad \overline{\mu}^* = I_3 \times \overline{\alpha}$$

and the fundamental equations of $C$ in $E^2_3$ are as follows:

$$\frac{d\overline{\alpha}}{ds} = \overline{\alpha}(s)$$

and

$$\frac{d\overline{\mu}^*}{ds} = \kappa_g(s)\overline{\mu}^* + \kappa_n(s)I_3,$$

$$\frac{dI_3}{ds} = -\kappa_n(s)\overline{\alpha} - \tau_g(s)\overline{\mu}^*.$$

The invariant $\kappa_g(s)$ is the geodesic curvature of $C$ at point $P \in C$, $\kappa_n(s)$ is the normal curvature of curve $C$ at $P \in C$ and $\tau_g(s)$ it geodesic torsion of $C$ at $P \in C$ in $E^2_3$.

The geometrical interpretations of these invariants are exactly those described in the section 3.2, Chapter 3. Also, a fundamental theorem can be enounced as in section 3.2, Theorem 3.2.2, Chapter 3.

The calculus of $\kappa_g$, $\kappa_n$ and $\tau_g$ is obtained by the formulae (5.3.2), for $\xi = \overline{\alpha}$. 


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We have:

\begin{align}
\kappa_g &= \left\langle \frac{d\alpha}{ds}, I_3 \right\rangle, \\
\kappa_n &= \left\langle \frac{d\alpha}{ds}, I_3 \right\rangle = -\left\langle \alpha, \frac{dI_3}{ds} \right\rangle, \\
\tau_g &= \left\langle \alpha, I_3, \frac{dI_3}{ds} \right\rangle.
\end{align}

By using the expressions (3.1.4), Ch. 3 we obtain

\begin{align}
\kappa_g &= \kappa \sin \varphi^*, \\
\kappa_n &= \kappa \cos \varphi^*, \\
\tau_g &= \tau + \frac{d\varphi^*}{ds}
\end{align}

with $\varphi^* = \angle(\bar{\alpha}_2, I_3)$, $\bar{\alpha}_2$ being the versor of principal normal of curve $C$ at point $P$.

A theorem of Meusnier can be formulated as in section 3.1, Chapter 3.

The line $C$ is geodesic (or autoparallel) line of the nonholonomic manifold $E_3^2$ if $\kappa_g(s) = 0$, $\forall s \in (s_1, s_2)$.

**Theorem 5.5.1** Along a geodesic $C$ of the nonholonomic manifold $E_3^2$ the normal curvature $\kappa_n$ is equal to $\pm \kappa$ and geodesic torsion $\tau_g$ is equal to the torsion $\tau$ of $C$.

The differential equations of geodesics are as follows

\begin{align}
\kappa_g &\equiv \frac{\omega_1}{ds} \left[ \frac{d}{ds} \left( \frac{\omega_2}{ds} \right) + r_1 \right] - \frac{\omega_3}{ds} \left[ \frac{d}{ds} \left( \frac{\omega_1}{ds} \right) - r_2 \right] = 0.
\end{align}

This equations are equivalent to the system of differential equations

\begin{align}
\frac{d}{ds} \left( \frac{\omega_1}{ds} \right) - r_2(s) &= h(s) \frac{\omega_1}{ds}, \\
\frac{d}{ds} \left( \frac{\omega_2}{ds} \right) + r_1(s) &= h(s) \frac{\omega_2}{ds},
\end{align}

where $h(s) \neq 0$ is an arbitrary smooth function.

If we consider $\sigma = \angle(I_3, \bar{\nu}_3)$, where $\bar{\nu}_3$ the versor of binormal is of the
field \((C, I_3)\) and \(\chi_2\) is the torsion of \((C, I_3)\), then the \(\kappa_g(s)\), the geodesic curvature of the field \((C, \pi)\) is obtained by the formulas of Gh. Gheorghiev:

\[
\kappa_g = \frac{d\sigma}{ds} + \chi_2.
\]

(5.5.12)

It follows that: the geodesic lines of the nonholonomic manifold \(E_3^2\) are characterized by the differential equations:

\[
\frac{d\sigma}{ds} + \chi_2(s) = 0.
\]

(5.5.13)

By means of this equations we can prove a theorem of existence of uniqueness of geodesic on the nonholonomic manifold \(E_3^2\), when \(\sigma_0 = \angle(I_3(s_0), \pi_3(s_0))\) is a priori given.

### 5.6 The fundamental forms of \(E_3^2\)

The first, second and third fundamental forms \(\phi, \psi\) and \(\chi\) of the nonholonomic manifold \(E_3^2\) are defined as in the theory of surfaces in the Euclidean space \(E_3\).

The tangent vector \(d\pi\) to a curve \(C\) at point \(P \in C\) in \(E_3^2\) is given by:

\[
d\pi = \omega_1(s)I_1 + \omega_2(s)I_2, \; \omega_3 = 0.
\]

(5.6.1)

The first fundamental form of \(E_3^2\) at point \(P \in E_3^2\) is defined by:

\[
\phi = \langle d\pi, d\pi \rangle = (\omega_1(s))^2 + (\omega_2(s))^2, \; \omega_3 = 0.
\]

(5.6.2)

Clearly, \(\phi\) has geometric meaning and it is a quadratic positive defined form.

The arclength of \(C\) is determined by the formula (5.2.10) and the arclength element \(ds\) is given by:

\[
ds^2 = \phi = (\omega^1)^2 + (\omega^2)^2, \; \omega_3 = 0.
\]

(5.6.3)
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The angle of two tangent vectors \(\vec{d\tau}\) and \(\vec{d\tau} = \vec{\omega}_1 I_1 + \vec{\omega}_2 I_2\) is expressed by

\[
\cos \theta = \frac{\langle \vec{d\tau}, \vec{d\tau} \rangle}{\delta sds} = \frac{\vec{\omega}_1 \omega_1 + \vec{\omega}_2 \omega_2}{\sqrt{(\omega_1)^2 + (\omega_2)^2 \sqrt{\omega_1^2 + \omega_2^2}}}.
\]

The second fundamental form \(\psi\) of \(E_3^2\) at point \(P\) is defined by

\[
(5.6.5) \psi = -(d\tau, dI_3) = p\omega_2 - q\omega_1 = p_2\omega_2^2 + (p_1 - q_2)\omega_1\omega_2 - q_1\omega_1^2.
\]

The form \(\psi\) has a geometric meaning. It is not symmetric.

The third fundamental form \(\chi\) of \(E_3^2\) at point \(P\) is defined by

\[
(5.6.6) \quad \chi = \langle d\tau, \vec{\theta} \rangle, \quad \omega_3 = 0,
\]

where \(\vec{\theta}\) is given by (5.1.7).

As a consequence, we have

\[
(5.6.7) \quad \chi = p\omega_1 + q\omega_2 = p_1\omega_1^2 + (p_2 + q_1)\omega_1\omega_2 + q_2\omega_2^2
\]

\(\chi\) has a geometrical meaning and it is not symmetric.

One can introduce a fourth fundamental form of \(E_3^2\), by

\[
\Theta = \langle dI_3, dI_3 \rangle = p^2 + q^2 (mod \omega_3).
\]

But \(\Theta\) linearly depends by the forms \(\phi, \psi, \chi\). Indeed, we have

\[
\Theta = M\psi - K_t\phi - T_m\chi,
\]

where \(M\) is mean curvature, \(T_m\) is mean torsion and \(K_t\) is total curvature of \(E_3^2\) at point \(P\). The expression of \(K_t\) is

\[
(5.6.8) \quad K_t = p_1q_2 - p_2q_1.
\]

The formulae (5.5.6) and (5.5.7) and the fundamental forms \(\phi, \psi, \chi\) allow to express the normal curvature and geodesic torsion of a curve \(C\).
5.7. Asymptotic lines. Curvature lines

at a point $P \in C$ as follows

\[
\kappa_n = \frac{\psi}{\phi} = \frac{p\omega_2 - q\omega_1}{\omega_1^2 + \omega_2^2} = \frac{p_2\omega_2^2 + (p_1 - q_2)\omega_1\omega_2 - q_1\omega_1^2}{\omega_1^2 + \omega_2^2},
\]

and

\[
\tau_g = \frac{\chi}{\phi} = \frac{p\omega_1 + q\omega_2}{\omega_1^2 + \omega_2^2} = \frac{p_1\omega_1^2 + (p_2 + q_1)\omega_1\omega_2 + q_2\omega_2^2}{\omega_1^2 + \omega_2^2}.
\]

The cases when $\kappa_n = 0$ and $\tau_g = 0$ are important. They will be investigated in the next section.

5.7 Asymptotic lines. Curvature lines

An asymptotic tangent direction $d\vec{r}$ to $E^2_3$ at a point $P \in E^2_3$ is defined by the property $\kappa_n(s) = 0$. A line $C$ in $E^2_3$ for which the tangent directions $d\vec{r}$ are asymptotic directions is called an *asymptotic line* of the nonholonomic manifold $E^2_3$.

The asymptotic directions are characterized by the following equation of degree 2.

\[
p_2\omega_2^2 + (p_1 - q_2)\omega_1\omega_2 - q_1\omega_1^2 = 0.
\]

The realisant of this equation is the invariant

\[
K_g = K_t - \frac{1}{4}T_m^2,
\]

called the *gaussian curvature* of $E^2_3$ at point $P \in E^2_3$.

We have

**Theorem 5.7.1** At every point $P \in E^2_3$ there are two asymptotic directions:

- real if $K_g < 0$
- imaginary if $K_g > 0$
- coincident if $K_g = 0$

The point $P \in E^2_3$ is called *planar* if the asymptotic directions of $E^2_3$ at $P$ are nondetermined.
A planar point is characterized by the equations

\begin{equation}
\begin{align*}
p_1 - q_2 &= 0, \\
p_2 &= q_1 = 0.
\end{align*}
\end{equation}

The point \( P \in E^2_3 \) is called \textit{elliptic} if the asymptotic direction of \( E^2_3 \) at \( P \) are imaginary and \( P \) is a \textit{hyperbolic} point if the asymptotic directions of \( E^2_3 \) at \( P \) are real.

Of course, if \( P \) is a hyperbolic point of \( E^2_3 \) then, exists two asymptotic line through the point \( P \), tangent to the asymptotic directions, solutions of the equations (5.7.1).

The curvature direction \( d\tau \) at a point \( P \in E^2_3 \) is defined by the property \( \tau_\gamma(s) = 0 \). A line \( C \) in \( E^2_3 \) for which the tangent directions \( d\tau \) are the curvature directions is called a \textit{curvature line} of the nonholonomic manifold \( E^2_3 \).

The curvature directions are characterized by the following second order equations.

\begin{equation}
\begin{align*}
p_1 \omega_1^2 + (p_2 + q_1)\omega_1\omega_2 + q_2\omega_2^2 &= 0.
\end{align*}
\end{equation}

The realisant of this equations is

\begin{equation}
T_t = K_t - \frac{1}{4}M^2.
\end{equation}

We have

**Theorem 5.7.2** At every point \( P \in E^2_3 \) there are two curvature directions

- real, if \( T_t < 0 \)
- imaginary, if \( T_t > 0 \)
- coincident, if \( T_t = 0 \)

The curvature lines on \( E^2_3 \) are obtained by integrating the equations (5.7.1) in the case \( T_t \leq 0 \).

**Remark 5.7.1** In the case of surfaces \( (T_m = 0) \), the curvature lines coincides with the lines of extremal normal curvature.
5.8 The extremal values of \( \kappa_n \). Euler formula. Dupin indicatrix

The extremal values at a point \( P \in E^2_3 \), of the normal curvature \( \kappa_n = \frac{\psi}{\phi} \) when \( (\omega_1, \omega_2) \) are variables are given by

\[
\frac{\partial \psi}{\partial \omega_i} - \kappa_n \frac{\partial \phi}{\partial \omega_i} = 0, \quad (i = 1, 2)
\]

which are equivalent to the equations

\[
\frac{\partial \psi}{\partial \omega_i} - \kappa_n \frac{\partial \phi}{\partial \omega_i} = 0, \quad (i = 1, 2).
\]

(5.8.1)

Taking into account the form (5.6.5) of \( \psi \) and (5.6.3) of \( \phi \), the system (5.8.1) can be written:

\[
-(q_1 + \kappa_n)\omega_1 + \frac{1}{2}(p_1 - q_2)\omega_2 = 0 \tag{5.8.2}
\]

\[
\frac{1}{2}(p_1 - q_1)\omega_1 + (p_2 - \kappa_n)\omega_2 = 0 \tag{5.8.3}
\]

It is a homogeneous and linear system in \( (\omega_1, \omega_2) \)-which gives the directions \( (\omega_1, \omega_2) \) of extremal values of normal curvature.

But, the previous system has solutions if and only if the following equations hold

\[
\begin{vmatrix}
-(q_1 + \kappa_n) & \frac{1}{2}(p_1 - q_2) \\
\frac{1}{2}(p_1 - q_2) & (p_2 - \kappa_n)
\end{vmatrix} = 0
\]

equivalent to the following equations of second order in \( \kappa_n \):

\[
\kappa_n^2 - (p_2 - q_1)\kappa_n + p_1q_2 - p_2q_1 - \frac{1}{4}(p_1 + q_2)^2 = 0. \tag{5.8.4}
\]

This equation has two real solutions \( \frac{1}{R_1} \) and \( \frac{1}{R_2} \) because its realisant is

\[
\Delta = (p_2 + q_1)^2 + (p_1 - q_2)^2. \tag{5.8.5}
\]

We have \( \Delta \geq 0 \). Therefore the solutions \( \frac{1}{R_1}, \frac{1}{R_2} \) are real, different or
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equal, $\frac{1}{R_1}, \frac{1}{R_2}$ are the extremal values of normal curvature $\kappa_n$.

Let us denote
\begin{equation}
H = \frac{1}{R_1} + \frac{1}{R_2} = p_2 - q_1
\end{equation}
called the mean curvature of the nonholonomic manifold $E_3^2$ at point $P$, and
\begin{equation}
K_g = \frac{1}{R_1 R_2} = p_1 q_2 - p_2 q_1 - \frac{1}{4}(p_1 + q_2).
\end{equation}
called the Gaussian curvature of $E_3^2$ at point $P$.

Substituting $\frac{1}{R_1}, \frac{1}{R_2}$ in the equations (5.8.2) we have the directions of extremal values of the normal curvature-called the principal directions of $E_3^2$ at point $P \in E_3^2$. These directions are obtained from (5.8.2) for $\kappa_n = \frac{1}{R_1}, \kappa_n = \frac{1}{R_2}$. Thus, one obtains the following equations which determine the principal directions:
\begin{equation}
(p_1 - q_2)\omega_2^2 + 2(p_2 + q_1)\omega_1 \omega_2 - (p_1 - q_2)\omega_1^2 = 0.
\end{equation}
Let $(\omega_1, \omega_2), (\tilde{\omega}_1, \tilde{\omega}_2)$ the solution of (5.8.8). Then $d\tau = \omega_1 I_1 + \omega_2 I_2$ and $\delta\tau = \tilde{\omega}_1 I_1 + \tilde{\omega}_2 I_2$ are the principal directions on $E_3^2$ at point $P$.

The principal directions $d\tau, \delta\tau$ of $E_3^2$ in every point $P$ are real and orthogonal.

Indeed, because $\Delta > 0$, and if $\frac{1}{R_1} \neq \frac{1}{R_2}$ we have $\langle \delta\tau, d\tau \rangle = 0$.

The curves on $E_3^2$ tangent to $\delta\tau$ and $d\tau$ are called the lines of extremal normal curvature. So, we have

**Theorem 5.8.1.** At every point $P \in E_3^2$ there are two real and orthogonal lines of extremal normal curvature.

Assuming that the frame $\mathcal{R} = (P; I_1, I_2, I_3)$ has the vectors $I_1, I_2$ in the principal directions $\delta\tau, d\tau$ respectively, thus the equation (5.8.8) implies
\begin{equation}
p_1 - q_2 = 0
\end{equation}
and the extremal values of normal curvature \( \kappa_n \) are given by

\[
\kappa_n^2 - (p_2 - q_1)\kappa_n - p_2q_1 = 0. \tag{5.8.10}
\]

We have

\[
\frac{1}{R_1} = p_2, \quad \frac{1}{R_2} = -q_1. \tag{5.8.11}
\]

Denoting by \( \alpha \) the angle between the versor \( I_1 \) and the versor \( \bar{\xi} \) of an arbitrary direction at \( P \), tangent to \( E_3^2 \) we can write the normal curvature \( \kappa_n \) from (5.6.8) in the form

\[
\kappa_n = \frac{1}{R_1} \cos^2 \alpha + \frac{1}{R_2} \sin^2 \alpha, \quad \alpha = \angle(\xi, I_1). \tag{5.8.12}
\]

The formula (5.8.12) is called the \textit{Euler formula} for normal curvatures on the nonholonomic manifold \( E_3^2 \).

Consider the tangent vector \( \overrightarrow{PQ} = |\kappa_n|^{-1/2}\bar{\xi} \), i.e.

\[
\overrightarrow{PQ} = |\kappa_n|^{-1/2} (\cos \alpha I_1 + \sin \alpha I_2).
\]

The cartesian coordinate \((x, y)\) of the point \( Q \), with respect to the frame \((P; I_1, I_2)\) are given by

\[
x = |\kappa_n|^{-1/2} \cos \alpha, \quad y = |\kappa_n|^{-1/2} \sin \alpha. \tag{5.8.13}
\]

Thus, eliminating the parameter \( \alpha \) from (5.8.12) and (5.8.13) are gets the geometric locus of the point \( Q \) (in the tangent plan \((P; I_1, I_2)\)):

\[
\frac{x^2}{R_1} + \frac{y^2}{R_2} = \pm 1 \tag{5.8.14}
\]

called the \textit{Dupin’s indicatrix} of the normal curvature of \( E_3^2 \) at point \( P \).

It is formed by a pair of conics, having the axis in the principal directions \( \delta \tau, d \tau \) and the invariants: \( H \)-the mean curvature and \( K_g \) - the
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The Dupin indicatrix is formed by two ellipses, one real and another imaginary, if $K_g > 0$. It is formed by a pair of conjugate hyperbolae if $K_g < 0$, whose asymptotic lines are tangent to the asymptotic lines of $E_3^2$ at point $P$. It follows that the asymptotic directions of $E_3^2$ at $P$ are symmetric with respect to the principal directions of $E_3^2$ at $P$.

In the case $K_g = 0$, the Dupin indicatrix of $E_3^2$ at $P$ is formed by a pair of parallel straight lines - one real and another imaginary.

5.9 The extremal values of $\tau_g$. Bonnet formula. Bonnet indicatrix

The extremal values of the geodesic torsion $\tau_g = \frac{\chi}{\phi}$ at the point $P$, on $E_3^2$ can be obtained following a similar way as in the previous paragraph.

Such that, the extremal values of $\tau_g$ are given by the system of equations

$$\frac{\partial \chi}{\partial \omega_i} - \tau_g \frac{\partial \phi}{\partial \omega_i} = 0, (i = 1, 2).$$

Or, taking into account the expressions of $\phi = (\omega_1)^2 + (\omega_2)^2$ and $\chi = p\omega_1 + q\omega_2 = (p_1\omega_1 + p_2\omega_2)\omega_1 + (q_1\omega_1 + q_2\omega_2)\omega_2$, we have:

\begin{align*}
(p_1 - \tau_g)\omega_1 + \frac{1}{2}(p_2 + q_1)\omega_2 &= 0 \\
\frac{1}{2}(p_2 + q_1)\omega_1 + (q_2 - \tau_g)\omega_2 &= 0.
\end{align*}

But, these imply

\begin{align*}
\begin{vmatrix}
  p_1 - \tau_g & \frac{1}{2}(p_2 + q_1) \\
  \frac{1}{2}(p_2 + q_1) & q_2 - \tau_g
\end{vmatrix}
&= 0
\end{align*}
or expanded:

\[
\tau_g^2 - (p_1 + q_2)\tau_g + (p_1 q_2 - p_2 q_1 - \frac{1}{4}(p_2 - q_1)^2) = 0.
\]

The solutions \(\frac{1}{T_1}, \frac{1}{T_2}\) of this equations are the extremal values of geodesic torsion \(\tau_g\). They are real. The realisant of (5.9.3) is

\[
\Delta_1 = (p_1 - q_2)^2 + (p_2 + q_1)^2
\]

Therefore \(\frac{1}{T_1}, \frac{1}{T_2}\) are real, distinct or coincident.

We denote

\[
T_m = \frac{1}{T_1} + \frac{1}{T_2} = p_1 + q_2,
\]

the mean torsion of \(E_3^2\) at point \(P\). And

\[
T_t = \frac{1}{T_1 T_2} = p_1 q_2 - p_2 q_1 - \frac{1}{4}(p_2 - q_1)^2 = K_g - \frac{1}{4}H^2
\]

is called the total torsion of \(E_3^2\) at \(P\).

But, as we know the condition of integrability of the Pfaf equation \(\omega_3 = 0\) is

\[
d\omega_3 = 0, \mod\omega_3,
\]

which is equivalent to \(T_m = p_1 + q_2 = 0\).

**Theorem 5.9.1** *The nonholonomic manifold \(E_3^2\) of mean torsion \(T_m\) equal to zero is a family of surfaces in \(E_3\).*

The directions \(\delta\mathbf{r} = \tilde{\omega}_1 I_1 + \tilde{\omega}_2 I_2\) and \(d\mathbf{r} = \omega_1 I_1 + \omega_2 I_2\) of extremal geodesic torsion, corresponding to the extremal values \(\frac{1}{T_1}\) and \(\frac{1}{T_2}\) of \(\tau_g\) are given by the equation obtained from (5.9.2) by eliminating \(\tau_g\) i.e.

\[
(p_2 + q_1)\omega_1^2 - (p_1 - q_2)\omega_1\omega_2 - (p_2 + q_1)\omega_2^2 = 0.
\]

Thus \(\delta\mathbf{r}\) and \(d\mathbf{r}\) are real and orthogonal at every point \(P \in E_3^2\) for
which \( \frac{1}{T_1} \neq \frac{1}{T_2} \). \( \delta \tau \) and \( d \tau \) are called the direction of the extremal geodesic torsion at point \( P \in E_3^2 \).

**Theorem 5.9.2** At every point \( P \in E_3^2 \), where \( \frac{1}{T_1} \neq \frac{1}{T_2} \) there exist two real and orthogonal directions of extremal geodesic torsion.

Now, assuming that, at point \( P \), the frame \( R = (P; I_1, I_2, I_3) \) has the vectors \( I_1 \) and \( I_2 \) in the directions \( \delta \tau \), \( d \tau \) of the extremal geodesic torsion, respectively, from (5.9.7) we deduce

\[
(5.9.8) \quad p_2 + q_1 = 0.
\]

Thus, the equation (5.9.3) takes the form

\[
(5.9.9) \quad \tau_g^2 - (p_1 + q_2) \tau_g + p_1 q_2 = 0.
\]

Its solutions are

\[
(5.9.10) \quad \frac{1}{T_1} = p_1, \quad \frac{1}{T_2} = q_2
\]

and, setting \( \beta = \angle(\xi, I_1) \), from \( \tau_g = \frac{\chi}{\phi} \) one obtains [65] the so called Bonnet formula giving \( \tau_g \) for the nonholonomic manifold \( E_3^2 \):

\[
(5.9.11) \quad \tau_g = \frac{\cos^2 \beta}{T_1} + \frac{\sin^2 \beta}{T_2}, \quad \beta = \angle(\xi, I_1).
\]

If \( T_m = 0 \), (5.9.11) reduce to the very known formula from theory of surfaces:

\[
(5.9.12) \quad \tau_g = \frac{1}{T_1} \cos 2\beta.
\]

By means of the formula (5.9.9) we can determine an indicatrix of geodesic torsions.

In the tangent plane at point \( P \) we take the cartesian frame \( (P; I_1, I_2) \), \( I_1 \) having the direction of extremal geodesic torsion \( \delta \tau \) corresponding to \( \frac{1}{T_1} \) and \( I_2 \) having the same direction with \( d \tau \) corresponding to \( \frac{1}{T_2} \).
Consider the point \( Q' \in \pi(s) \) given by

\[
\overrightarrow{PQ'} = |\tau_g|^{-1}\xi = |\tau_g|^{-1}(\cos \beta I_1 + \sin \beta I_2)
\]

with \( \beta = \angle(\xi, I_1) \). The cartesian coordinate \((x, y)\) at point \( Q' \): \( \overrightarrow{PQ'} = xI_1 + yI_2 \), by means of (5.9.13) give us

\[
x = |\tau_g|^{-1}\cos \beta, \quad y = |\tau_g|^{-1}\sin \beta.
\]

Eliminating the parameter \( \beta \) from (5.9.14) and (5.9.11) one obtains the locus of point \( Q' \):

\[
\frac{x^2}{T_1} + \frac{y^2}{T_2} = \pm 1,
\]

called the *indicatrix of Bonnet* for the geodesic torsions at point \( P \in E^2_3 \).

It consists of two conics having the axis the tangents in the directions of extremal values of geodesic torsion. The invariants of these conics are \( T_m \)-the mean torsion and \( T_t \)-the total torsion of \( E^2_3 \) at point \( P \).

The Bonnet indicatrix is formed by two ellipses, one real and another imaginary, if and only if the total torsion \( T_t \) is positive. It is composed by two conjugated hyperbolas if \( T_t < 0 \). In this case the curvature lines of \( E^2_3 \) at \( P \) are tangent to the asymptotics of these conjugated hyperbolas. Finally, if \( T_t = 0 \), the Bonnet indicatrix (5.9.15) is formed by two pair of parallel straightlines, one being real and another imaginary.

**Remark 5.9.1** 1. If \( T_t < 0 \), the directions of asymptotics of the hyperbolas (5.9.15) are symmetric to the directions of the extremal geodesic torsion.

2. The direction of extremal geodesic torsion are the bisectrices of the principal directions.

3. In the case of surfaces, \( T_m = 0 \), the Bonnet indicatrix is formed by two equilateral conjugates hyperbolas.

Assuming that the Dupin indicatrix is formed by two conjugate hyperbolas and the Bonnet indicatrix is formed by two conjugate hyperbolas, we present here only one hyperbolas from every of this indicatrices as
follows (Fig. 1):
5.10. The circle of normal curvatures and geodesic torsions

Consider a curve $C \subset E^2_3$ and its tangent versor $\overline{\alpha}$ at point $P \in C$. Denoting by $\alpha = \angle(\overline{\alpha}, I_1)$ and using the formulae (5.3.8) we obtain

\[
\kappa_n = p_2 \sin^2 \alpha + (p_1 - q_2) \sin \alpha \cos \alpha - q_1 \cos^2 \alpha
\]
\[
\tau_g = p_1 \cos^2 \alpha + (p_2 + q_1) \sin \alpha \cos \alpha + q_2 \sin^2 \alpha.
\]  

(5.10.1)

Eliminating the parameter $\alpha$ from (5.10.1) we deduce

\[
\kappa_n^2 + \tau_g^2 - H\kappa_n - T_m\tau_g - K_g = 0
\]  

(5.10.2)

where $H = p_2 - q_1$ is the mean curvature of $E^2_3$ at point $P$, $T_m = p_1 + q_2$ is the mean torsion of $E^2_3$ at $P$ and $K_g = p_1 q_2 - p_2 q_1$ is the Gaussian
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curvature of $E_3^2$ at point $P$. Therefore we have

**Theorem 5.10.1** 1. The normal curvature $\kappa_n$ and the geodesic torsion $\tau_g$ at a point $P \in E_3^2$ satisfy the equations (5.10.2).

2. In the plan of variable $(\kappa_n, \tau_g)$ the equation (5.10.2) represents a circle with the center \( \left( \frac{H}{2}, \frac{T_m}{2} \right) \) and of the radius given by

\[
R^2 = \frac{1}{4}(H^2 + T_m^2) - K_g.
\]

(5.10.3)

Evidently, this circle is real if we have

\[
K_g < \frac{1}{4}(H^2 + T_m^2).
\]

It is a complex circle if $K_g > \frac{1}{4}(H^2 + T_m^2)$. This circle, for the nonholonomic manifolds $E_3^2$ was introduced by the author [62]. It has been studied by Izu Vaisman in [41], [42].

In the case of surfaces, $T_m = 0$ and (5.10.3) gives us

\[
R^2 = \frac{1}{4}H^2 - K_g = \frac{1}{4} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^2 - \frac{1}{R_1 R_2} = \frac{1}{4} \left( \frac{1}{R_1} - \frac{1}{R_2} \right)^2.
\]

So, for surfaces the circle of normal curvatures and geodesic torsions is real.

If $T_m = 0$, the total torsion is of the form

\[
T_t = -\frac{1}{4} \left( \frac{1}{R_1} - \frac{1}{R_2} \right)^2.
\]

(5.10.4)

This formula was established by Alexandru Myller [35]. He proved that $T_t$ is just “the curvature of Sophy Germain and Emanuel Bacaloglu” ([35]).
5.11 The nonholonomic plane and sphere

The notions of nonholonomic plane and nonholonomic sphere have been defined by Gr. Moisil who expressed the Pfaff equations, provided the existence of these manifolds, and R. Miron [64] studied them. Gh. Vrânceanu, in the book [44] has studied the notion of nonholonomic quadrics.

The Pfaff equation of the nonholonomic plane $\Pi^2_3$ given by Gr. Moisil [30], [71] is as follows

$$\omega \equiv (q_0 z - r_0 y + a)dx + (r_0 x - p_0 z + b)dy + (p_0 y - q_0 x + c)dz = 0,$$

(5.11.1)

where $p_0, q_0, r_0, a, b, c$ are real constants verifying the nonholonomy condition: $ap_0 + bq_0 + cr_0 \neq 0$.

While, the nonholonomic sphere $\Sigma^2_3$ has the equation given by Gr. Moisil [30]:

$$\omega \equiv [2x(ax + by + cz) - a(x^2 + y^2 + z^2) + \mu x + q_0 z - r_0 y + h]dx$$

$$+ [2y(ax + by + cz) - b(x^2 + y^2 + z^2) + \mu y + r_0 x - p_0 z + k]dy$$

$$+ [2z(ax + by + cz) - c(x^2 + y^2 + z^2) + \mu z + p_0 y - q_0 x + l]dz.$$

(5.11.2)

$\mu, p_0, q_0, r_0, a, b, c, h, k, l$ being constants which verify the condition $\omega \wedge d\omega \neq 0$.

In this section we investigate the geometrical properties of these special manifolds.

1. The nonholonomic plane $\Pi^2_3$

The manifold $\Pi^2_3$ is defined by the property $\psi \equiv 0$, $\psi$ being the second fundamental form (5.6.3) of $E^2_3$.

From the formula (5.6.3) is follows

$$p_2 = q_1 = p_1 - q_2 = 0.$$
The nonholonomic plane \( \Pi^2_3 \) has the invariants \( H, T_m, \ldots \) given by

\[(5.11.4) \quad H = 0, T_m \neq 0, T_t = \frac{1}{4} T_m^2, K_t = T_t, K_g = 0.\]

Conversely, (5.11.4) imply (5.11.3).

Therefore, the properties (5.11.4), characterize the nonholonomic plane \( \Pi^2_3 \).

Thus, the following theorems hold:

**Theorem 5.11.1** 1. The following line on \( \Pi^2_3 \) are nondetermined:
- The lines tangent to the principal directions.
- The lines tangent to the directions of extremal geodesic torsion.
- The asymptotic lines.

Theorem 5.11.2 Also, we have
- The lines of curvature coincide with the minimal lines, \( \langle \delta r, \delta r \rangle = 0 \).

The normal curvature \( \kappa_n \equiv 0 \).
- The geodesic of \( \Pi^2_3 \) are straight lines.

Consequences: the nonholonomic manifold \( \Pi^2_3 \) is totally geodesic. Conversely, a total geodesic manifold \( \Pi^2_3 \) is a nonholonomic plane.

If \( T_m = 0 \), \( \Pi^2_3 \) is a family of ordinary planes from \( E^2_3 \).

2. The nonholonomic sphere, \( \Sigma^2_3 \).

A nonholonomic manifold \( E^2_3 \) is a nonholonomic sphere \( \Sigma^2_3 \) if the second fundamental from \( \psi \) is proportional to the first fundamental form \( \Phi \).

By means of (5.6.3) it follows that \( \Sigma^2_3 \) is characterized by the following conditions

\[(5.11.5) \quad p_1 - q_2 = 0, \; p_2 + q_1 = 0.\]

It follows that we have

\[(5.11.6) \quad \psi = \frac{1}{2} H\phi, \; \chi = \frac{1}{2} T_m\phi, \; \Theta^2 = K_t\phi.\]

Thus, we can say
Proposition 5.11.1 1. The normal curvature $\kappa_n$ at a point $P \in \Sigma_3^2$ is the same ($= \frac{1}{2} H$) in all tangent direction at point $P$.

2. The geodesic torsion $\tau_g$ at point $P \in \Sigma_3^2$ is the same in all tangent direction at $P$.

Proposition 5.11.2 1. The principal directions at point $P \in E_3^2$ are nondetermined.

2. The directions of extremal geodesic torsion at $P$ are nondetermined, too.

Proposition 5.11.3 At every point $P \in \Sigma_3^2$ the following relations hold:

\[ (5.11.7) \quad K_g = \frac{1}{4} H^2, \quad T_t = \frac{1}{4} T_m^2, \quad K_t = \frac{1}{4} (H^2 + T_m^2), \]

Conversely, if the first two relations (5.11.7) are verified at any point $P \in E_3^2$, then $E_3^2$ is a nonholonomic sphere.

Indeed, (5.11.7) are the consequence of (5.11.5). Conversely, from $K_g = \frac{1}{4} H^2, \quad T_t = \frac{1}{4} T_m^2$ we deduce (5.11.5).

Proposition 5.11.4 The nonholonomic sphere $\Sigma_3^2$ has the properties

\[ (5.11.8) \quad K_g > 0, T_t > 0, K_t > 0. \]

Proposition 5.11.5 1. If $H \neq 0$, the asymptotic lines of $\Sigma_3^2$ are imaginary.

2. The curvature lines of $\Sigma_3^2$, $(T_m \neq 0)$ are imaginary.

The geodesic of $\Sigma_3^2$ are given by the equation (5.5.7) or by the equation (5.5.11).

Theorem 5.11.3 The geodesics of nonholonomic sphere $\Sigma_3^2$ cannot be the plane curves.
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Proof. By means of theorem 5.5.1, along a geodesic $C$ we have $\kappa_n = |\kappa|$, $\tau_g = \tau$, $\kappa$ and $\tau$ being curvature and torsion of $C$. From (11.6) we get

$$\kappa_n = \frac{1}{2} H, \quad \tau_g = \frac{1}{2} T_m.$$  

So, the torsion $\tau$ of $C$ is different from zero. It can not be a plane curve.

We stop here the theory of nonholonomic manifolds $E_3^2$ in the Euclidean space $E_3$, remarking that the nonholonomic quadric was investigated by G. Vrânceanu and A. Dobrescu [8], [44], [45].
References

PART A (1966)

[1] Bianchi, L., *Lezioni di geometria differenziale*, Terza ed. vol. I, partea I, 1927

[2] Biusghens, S.S., *Geometria vectornovo polea*, Izv. Acad. Nauk. (10), 1946, p. 73

[3] Bortolotti, E., *Su do una generalizzazione della teoria delle curve e sui sistemi conjugati di una V₂ in Vₙ*, Rend. R. Ist. Lombardo Sci. e Lett. vol LVIII, f. XI, 1925

[4] Creangă, I., *Asupra corespondențelor punctuale între varietăți neolnome din spațiul euclidian cu trei dimensiuni*, Analele Șt. Univ. Iași, T. II, 1956.

[5] Creangă, I., *Asupra transformării unui cuplu de varietăți neolnome prin corespondențe punctuale*, Analele Șt. Univ. Iași, T. VII, 1961

[6] Creangă, I., *Asupra rețelelor Peterson în corespondențe punctuale între varietăți neolnome*, Analele Șt. Univ. Iași, T. III, 1957.

[7] Dobrescu, A., *Geometrie diferențială*, București, Editura didactică și pedagogică, 1963.

[8] Dobrescu, A., *Asupra suprafețelor neolnome*, Lucrările consf. de geometrie, Timișoara, 1955
[9] Gheorghiev, Gh., s.a., Lectii de geometrie, Bucuresti, Ed. didactică și pedagogică, 1965.

[10] Gheorghiev, Gh., Despre geometria intrinsecă a unui câmp de vectori, Studii și Cerc. St. Mat. Iași, 1951

[11] Gheorghiev, Gh., Câteva probleme geometrice legate de un câmp de vectori unitari, Bul. St. Acad. R.P.R., T. VI, 1955

[12] Gheorghiev, Gh., Popa I., Couples des variétés non holonomes de l’espace à trois dimensions (II), C.R. Paris, T. 253, 1961

[13] Gheorghiu, Gh. Th., Sur les variétés non holonomes de l’espace S₃, Revue de Math. pures et appl. Buc. T. II, 1957

[14] Haimovici, M., Sur la géométrie intrinsèque des surfaces non holonomes, Rev. Univ. Tucuman, S.A. Mat. vol. I, 1940 (Argentina).

[15] Haimovici, M., Sur la géométrie intrinsèque des variétés non holonomes, J. de Math. pures et appl., Paris, 51, 1946

[16] Kaul, R.N., On the magnitude of the derived vector of the unit normal a hypersurface, Tensor, t., 1957 (Japonia).

[17] Kovan’tov, N.I., Prostranstvenaia indicatrisa geodezicescaia crucenii triortogonalinii sistemî negolonominh poverhnosti, Dokladi Acad. Nauk. T. XCVII, Nr. 5, 1954.

[18] Krein, M., Üüber den Satz von ”curvatura integra”, Bull. de la Soc. Physico Math. de Kazan, 3-e s. T, 3, 1926.

[19] Intzinger, R., Zur Differentialgeometrie Pfaffscher Mannigfaltigkeiten, Monatshefte für ath. u. Phys. 45.

[20] Mayer, O., Une interprétation géométrique de la seconde forme quadratique d’une surface, C.R. Paris, 1924, t. 178 p. 1954.
[21] Mayer, O., *Étude sur les réseaux de M. Myller*, Ann. St. de l’Univ. de Jassy, T. XIV, 1926.

[22] Mihăilescu, T., *La courbure extérieure des hypersurfaces non holonomes*, Bull. Math. de la Soc. Sci. Math. P.R. 1 (49), 1957.

[23] Miron, R., *Geometria unor configuraţii Myller*, Analele Șt. Univ. Iași, T. VI, f.3, 1960.

[24] Miron, R., *Configuraţii Myller \( \mathfrak{M}(C, \xi^i, T^{n-1}) \) în spaţii Riemann \( V_n \). Aplicaţii la studiul hipersuprafetelor din \( V_n \)*, Studii și Cerc. Șt. Mat. Iași, T. XII, f. 1, 1962.

[25] Miron, R., *Configuraţii Myller \( \mathfrak{M}(C, \xi^i, T^m) \) în spaţii Riemann \( V_n \). Aplicaţii la studiul varietăţilor \( V_m \) din \( V_n \)*, Studii și Cerc. Șt. Mat. Iași, T. XIII, f.1, 1962.

[26] Miron, R., *Les configurations de Myller \( \mathfrak{M}(C, \xi^i, T^{n-1}) \) dans les espaces de Riemann \( V_n \) (I)*, Tensor Nr. 3, 1962, vol. 12 (Japonia).

[27] Miron, R., *Les configurations de Myller \( \mathfrak{M}(C, \xi^i, T^m) \) dans les espaces de Riemann \( V_n \)*, Tensor, Nr. 1, 1964, vol. V (Japonia).

[28] Miron, R., *Curbura geodezică mixtă, curbura normală mixtă și torsiunea geodezică mixtă în teoria suprafetelor*, Gazeta Mat. Fiz. S.A. nr. 5, 1961.

[29] Miron, R., *Asupra sistemului complet de invariăţiuni ai unei familii de direcții tangente unei suprafete*, Gazeta Matematică, S.A., nr. 1-2, 1964.

[30] Moisil, C. Gr., *Sur les propriétés affines et conformes des variétés non holonomes*, Bull. Fac. de St. din Cernăuți, vol. IV, 1930.

[31] Myller, A.I., *Quelques propriétés des surfaces réglées en liaison avec la théorie du parallélisme de Levi-Civita*, C.R. Paris, 1922, T. 174m p. 997.
[32] Myller, A., *Surfaces réglées remarquables passant par une courbe donnée*, C.R. Paris 1922, T. 175, p. 937.

[33] Myller, A.I., *Les systèmes de corubes sur une surface et le parallélisme de Levi-Civita*, C.R. Paris, 1923, T.176, p. 433.

[34] Myller, A.I., *Le parallélisme au sens de Levi-Civita dans un système de plans*, Ann. Sci. Univ. Jassy, T. XIII, f.1-2, 1924.

[35] Myller, A.I. *Scritti matematici*, Buc. Ed. Acad. R.P.R, 1959.

[36] Nagata, Y., *Remarks on the normal curvature of a vector fields*, Tensor, 8, 1958.

[37] Pan, T.K., *Normal curvature of a vector field*, Amer. J. of Math. 74, 1952.

[38] Pauc, Chr., *Extension aux variétés on holonomes $V_n^{n-1}$ de de quelques propertés des surfaces et des $V_3^2$*, Atti dei Lincei, 1938.

[39] Popa, I., Gheorghiev, Gh., *Sur une variété uni-dimensionnelle non holonome de A. Myller*, An. St. Univ. Iași, 4, f.1, 1960.

[40] Sințov, D., *Zur Krümmungstheorie der Integalkurven der Pfaffschen Gleichung*, Math. Annalen. B. 101, 1929.

[41] Vaisman, I., *Despre evolutele varietăților neolonome din $S_3$ euclidian*, Analele Univ. Iași, T.V, f.1, 1959

[42] Vaisman, I., *Varietăți neolonome riglate din $S_3$ euclidian*, St. și Cerc. Șt. T. XI, 1960, Iași.

[43] Vranceanu, G., *Lecții de geometrie diferențială*, vol. I, II, Ed. Acad. R.P.R. București, 1951, 1952.

[44] Vranceanu, G., *Geometrie analitică, proiectivă și diferențială*, Editura didactică și pedagogică, București, 1961
[45] Vrăinceanu, G., *Les espaces non holonomes et leurs applications mécaniques*, Mémorial des Sci. Math. f 76, 1936

[46] Yano, K et Petrescu, St., *Sur les espaces métriques non holonomes complementaires*, Disq. Math. et Phys, 1, f.2, 1940

**PART B (2010)**

[47] Anastasiei M. and M. Crăşmăreanu, *Lecţii de geometrie (Curbe şi suprafeţe)*, Editura TehnoPress, Iaşi, România, 2005.

[48] S. S. Biusghens, *Geometria statienarnovo idealnoi neslimaernoi jid-kosti*, Isvestia Akademii Naun, 12, 1948 pg. 471-512.

[49] W. Boskoff, *Hyperbolic Geometry and Barbilian spaces*, Hadronic Press, 1996.

[50] I. Bucătaru and R. Miron, *Finsler-Lagrange Geometry. Applications to Dynamical Systems*, Ed. Acad. Romane, Bucureşti, 2007.

[51] M. Craioveanu, *Topologia şi geometria varietăţilor Banach* (in Romanian), Ph.D. Thesis, Iaşi, 1968.

[52] O. Constantinescu, M. Crăşmăreanu and M.-I. Munteanu, *Elemente de geometrie superioară*, Ed. Matrix-Rom, Bucureşti, 2007.

[53] O. Constantinescu, *Spaţii Lagrange. Generalizări şi Aplicaţii*, Teza de doctorat (Îndrumător cStiinţific Radu Miron), Iaşi, 2005.

[54] Gh. Gheorghiev, *On Dini nets* (Romanian), Acad. Repub. Pop. Române. Bal. 'Sti. Ser, Mat. Fiz. Chim. 2(1950), 17-20.

[55] A. Haimovici, *Grupuri de transformări. Introducere elementară*, Ed. Didactică şi Pedagogică, 1963.

[56] M. Haimovici, *Opera Matematica*, vol. I, II. Ed. Acad. Romane, București, 1998, 2009.
[57] Q. A. Khu, *Geometria configuratiilor Myller din spatii Finsler*, teza de doctorat (Îndrumător Ştiinţific R. Miron), Iaşi, 1977.

[58] T. Levi-Civita, *Rendiconti del Circolo di Palermo*, t. XLII, 1917, p. 173.

[59] T. Levi-Civita, Lezioni di calcolo differentiale assoluto, Zanichelli, 1925

[60] N. Mihăileanu, *Configuraţii Myller în spaţii Minkowski*, Analele Universităţii Bucureşti, Math.-Mec., 16(1967), 117-119.

[61] R. Miron, *Geometria Configuraţiilor Myller*, Editura Tehnică, Bucureşti, 1966.

[62] R. Miron, *Despre torsiunea totală a unei suprafeţe*, Gazeta Matematică S. A. Nr. 8, 1955.

[63] R. Miron, *Câteva probleme din geometria unui câmp de vectorii unitari*, St. şi Cerc. Şt. Acad. Filiala Iaşi, anul VI, 1955, Nr. 1-2, pg. 173-183.

[64] R. Miron, *Asupra sferei neolonome şi planului neolonom*, An. Şt. Univ. “Al. I. Cuza” din Iaşi, Seria Nouă, Sect. 1, Tom. I, 1955, fasc. 1-2, pg. 43-53.

[65] R. Miron, *Despre reducerea la forma canonice a grupei intrinsec al unui spaţiu neolonom*, Buletin Ştiinşific, Secţia de St. Matem. şi Fiz. T. VIII, nr. 3, 1956, pg. 631-645.

[66] R. Miron, *Problema geometrizării sistemelor mecanice neolonome*, Teza de doctorat (Îndrumător Ştiinţific acad. M. Haimovici), St. şi Cerc. Şt. Acad. Filiala Iaşi, an VII, 1956, 7.1, pg. 15-50.

[67] R. Miron and M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publishers, FTPH nr. 59, 1994.
[68] R. Miron and D. Brânzei, *Backgrounds of Arithmetic and Geometry*, World Scientific Publishing, S. Pure Math. vol. 23, 1995.

[69] R. Miron, D. Hrimiuc, H. Shimada and S. Sabău, *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Acad. Publishers, nr. 118, 2001.

[70] R. Miron, I. Pop, *Topologie Algebrică. Omologie, Omotopie, Spaţii de acoperire*, Ed. Academiei 1974.

[71] Gr. C. Moisil, *Opera matematică, vol.I, II, III*, Editura Academiei, Bucuresti, 1976, 1980, 1992.

[72] Ghe. Mureşescu, *Contribuţii la studiul geometriei diferenţiale a spaţiilor fibre cu grup structural proiectiv*, teza de doctorat, Iaşi, 1969.

[73] I. Pop, *Varietăţi diferenţiabile. Culegere de probleme*, Ed. Univ. “Al. I. Cuza” Iaşi, 1975.

[74] L. Răileanu and R. Miron, *Geometrie diferenţială, vol I, II*, Ed. Univ. “Al. I. Cuza” Iaşi, 1987.

[75] M. Spivak, *A comprehensive introduction to differential geometry. vol. I*, Brandeis. Univ., Waltham, Mass., 1970

[76] I. Vaisman, *Simplectic Geometry and secondary characteristic classes*, Progress in Mathematics, 72, Birkhauser Verlag, Basel, 1994
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Bibliography
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Now, to 100th anniversary of Mathematical Seminar “Alexandru Myller”, in English version, the book is published by Publishing House of Romanian Academy in a new title and is completed with an ample chapter concerning the applications of Myller configurations to the nonholonomic manifolds.

The book contents: the notion of Myller configurations, Darboux frame, fundamental formulae and fundamental theorem of existence of Myller configurations. The complete system of invariants of Myller configurations allow to introduce the notions of Myller parallelism and concurrence as well as a famous Kneut’s formula. By way it is obtained an important generalization of Levi-Civita parallelism. The applications to theory of surfaces and nonholonomic manifolds end the book.