Off-shell \( D = 5, \mathcal{N} = 2 \) Riemann squared supergravity

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Abstract

We construct a new off-shell invariant in \( \mathcal{N} = 2, D = 5 \) supergravity whose leading term is the square of the Riemann tensor. It contains a gravitational Chern–Simons term involving the vector field that belongs to the supergravity multiplet. The action is obtained by mapping the transformation rules of a spin connection with bosonic torsion and a set of curvatures to the fields of the Yang–Mills multiplet with gauge group \( \text{SO}(4,1) \). We also employ the circle reduction of an action that describes locally supersymmetric Yang–Mills theory in six dimensions.

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1. Introduction

It is well known that in string theory the Einstein action of general relativity gets modified by an infinite series of terms of higher order in the Riemann tensor. To obtain the supersymmetric extension of such an infinite series is an open problem. In lower dimensions there exist auxiliary field formulations of Poincaré supergravity which makes it possible to construct supersymmetric actions that contain only the first-order correction to the Einstein action. This has been done for \( \mathcal{N} = 2 \) supersymmetry in \( D = 6 \) dimensions a long time ago [1–3]. The resulting action contains an Einstein plus a Riemann tensor squared term. The construction of [1–3] was based on the observation that the Weyl multiplet underlying Poincaré supergravity has formally the same supersymmetry transformation rules as a Yang–Mills multiplet, for Yang–Mills group \( \text{SO}(5,1) \). Actually, there exist two different Weyl multiplets [4] and this analogy only works for the so-called dilaton Weyl multiplet which contains, amongst others, a dilaton-like scalar field that can be used as the compensating field for dilatations and an antisymmetric tensor gauge field. In this analogy, the Yang–Mills vector \( A_\mu^i \) transforms formally the same as a certain torsionful spin connection \( \omega_{\mu}^{\nu\rho} \) where the torsion is proportional to the 3-form field-strength tensor of the antisymmetric tensor gauge field. The supersymmetry
rules only coincide after fixing the conformal symmetries using the scalar field to fix the dilatations.

Another example of a supersymmetric higher order invariant has been constructed in $D = 5$ dimensions [5]. The leading term of this invariant is a Weyl curvature squared term that is multiplied by a compensating scalar, to make it invariant under dilatations. This compensating scalar belongs to a separate gauge multiplet. An interesting feature of this invariant is that it contains a mixed gauge-gravitational Chern–Simons term

$$A \wedge \text{tr}(R \wedge R), \quad (1.1)$$

where $A$ belongs to the gauge multiplet and $R$ to the supergravity multiplet. This mixed Chern–Simons term plays an important role in discussing higher order corrections to black hole entropy, see e.g. [6, 7], and higher order effects in the AdS/CFT correspondence, see e.g. [8, 9].

In this paper, we show that by applying the techniques of [1–3] to $\mathcal{N} = 2$ supersymmetry in $D = 5$ dimensions we can construct a higher order invariant that differs from the one presented in [5]. The leading term of this new invariant is the Riemann tensor squared. Unlike the invariant of [5] this one is purely gravitational in the sense that the compensating scalar for dilatations, that multiplies the Riemann tensor squared term, belongs to the Weyl multiplet. In analogy to (1.1), the new invariant contains a purely gravitational Chern–Simons term

$$C \wedge \text{tr}(R \wedge R), \quad (1.2)$$

where both $C$ and $R$ belong to the supergravity multiplet. It is natural to expect that this term will also be relevant in exploring the effects of higher derivative corrections in black hole entropy and the AdS/CFT correspondence.

This paper is organized as follows. In section 2, we will record the relevant elements of the so-called dilaton Weyl multiplet and Yang–Mills multiplet with $\mathcal{N} = 2$ supersymmetry in $5D$. In section 3, we shall go over to a convenient basis for the fields which is equivalent to fixing the superconformal symmetries, in the sense that the new fields are invariant under dilatations and $S$-supersymmetry. Section 4 contains the key observation of this paper which states that the transformation rules of a spin connection with bosonic torsion and a set of curvatures in the dilaton Weyl multiplet formally transform in the same manner as the fields of a Yang–Mills multiplet with gauge group $\text{SO}(4, 1)$. The explicit correspondence can be found in equation (4.17). After we dimensionally reduce a locally supersymmetric Yang–Mills action from $6D$ to $5D$ in section 5, we make use of this observation to write down the supersymmetrization of the Riemann squared term in section 7. Further directions and comments are presented in the concluding section.

2. Conformal multiplets

In this section, we will briefly recall some elements of the $\mathcal{N} = 2$ superconformal tensor calculus in five dimensions that will be useful in the construction of the new higher derivative supergravity invariant. More specifically, we will review the relevant Weyl multiplet containing the various gauge fields of the superconformal symmetries and the Yang–Mills multiplet. Most of the results presented in this and in the following section can be found in [10, 11].

2.1. The dilaton Weyl multiplet

There exist two Weyl multiplets in five dimensions known as the standard Weyl multiplet and the dilaton Weyl multiplet. They were constructed in [10] and contain the same number of
gauge fields but differ in their matter field content. The multiplet that is relevant for this paper is the dilaton Weyl multiplet. It consists of the vielbein $e_{\mu}^{a}$, the gravitino $\psi_{\mu}^{a}$, the dilaton gauge field $b_{\mu}$ and the $SU(2)$ gauge field $V^{ij}_{\mu} = V^{ij}_{\mu}$. These gauge fields are supplemented with matter fields to form a multiplet consisting of 32 bosonic and 32 fermionic off-shell degrees of freedom. For the dilaton Weyl multiplet, these matter fields are given by a vector $C_{\mu}$, an antisymmetric tensor $B_{\mu\nu}$, a dilaton field $\sigma$ and a fermion field $\psi^{i}$. The $Q$- and $S$-supersymmetry transformations (with parameters $\epsilon^{i}$, $\eta^{i}$ respectively) are given by

$$
\delta e_{\mu}^{a} = \frac{1}{2} \bar{\epsilon} \gamma^{a} \psi_{\mu},
$$

$$
\delta \psi_{\mu}^{a} = D_{\mu}(\bar{\omega}) \epsilon^{i} + i \gamma^{a} \cdot T \gamma_{\mu} \epsilon^{i} - i \gamma_{\mu} \eta^{i},
$$

$$
\delta V_{\mu}^{ij} = -\frac{3}{2} i \bar{\epsilon} \gamma^{a} \psi_{\mu}^{a} + 4 \bar{\epsilon} \gamma^{a} \chi^{a} + i \bar{\epsilon} \gamma^{a} \cdot T \psi_{\mu}^{a} + \frac{i}{2} \bar{\epsilon} \gamma^{a} \psi_{\mu}^{a},
$$

$$
\delta C_{\mu} = -\frac{i}{4} \bar{\epsilon} \gamma^{a} \psi_{\mu} + \frac{i}{2} \bar{\epsilon} \gamma_{\mu} \psi,
$$

$$
\delta B_{\mu\nu} = \frac{1}{2} \bar{\epsilon} \gamma^{a} \gamma_{\mu\nu} \psi + C_{[\mu} \bar{\epsilon} \gamma_{\nu]} \psi + C_{[\mu} \delta(\epsilon) C_{\nu]}. 
$$

(2.1)

The ‘soft’ algebra that the dilaton Weyl multiplet realizes is given in [10]. Several definitions, some of which will be needed later, are as follows. Firstly,

$$
D_{\mu}(\bar{\omega}) \epsilon^{i} = \partial_{\mu} \epsilon^{i} + \frac{1}{2} b_{\mu} \epsilon^{i} + \frac{1}{2} \bar{\omega}^{ab} \gamma_{\mu} \epsilon^{i} - V_{\mu}^{ij} \epsilon_{j},
$$

$$
\tilde{D}_{\mu} \sigma = \partial_{\mu} \sigma - b_{\mu} \sigma - \frac{1}{2} \bar{\epsilon} \psi_{\mu} \psi,
$$

$$
\tilde{D}_{\mu} \psi^{i} = (\partial_{\mu} - \frac{1}{2} b_{\mu} + \frac{1}{2} \bar{\omega} \gamma_{\mu} \psi^{i} - V_{\mu}^{ij} \psi_{j} + \frac{1}{2} \gamma \cdot \tilde{G} \psi^{i} + \frac{1}{2} \bar{\epsilon} \gamma \phi_{\mu}^{i}
$$

$$
- \sigma \gamma \cdot T \psi^{i} + \frac{1}{2} \bar{\epsilon} \gamma \phi_{\mu}^{i} \psi^{i} - \sigma \phi_{\mu}^{i}. 
$$

(2.2)

Moreover, in the transformation rules we have used the composite fields

$$
T_{ab} = \frac{1}{2} \delta^{-2}(\sigma \tilde{G}_{ab} + \frac{1}{2} \delta_{ab}^{cd} \tilde{G}^{cd} + \frac{1}{2} \bar{\epsilon} \gamma \phi_{ab} \psi),
$$

$$
\chi^{i} = \frac{1}{2} \delta^{-2} \tilde{G} \psi^{i} + \frac{1}{16} \gamma \cdot T \psi^{i} + \frac{1}{2} \bar{\epsilon} \gamma \phi_{\mu}^{i} \psi^{i}.
$$

(2.3)

In fact, $T_{ab}$, $\chi^{i}$ and a scalar field $D$, which does not arise here, constitute the matter fields of the so-called standard Weyl multiplet [10], and the above expressions are needed to pass from this to the dilaton Weyl multiplet. The expressions for the dependent spin connection $\tilde{\omega}^{ab}_{\mu}$ and the $S$-supersymmetry gauge field $\phi_{\mu}^{i}$ are given by

$$
\tilde{\omega}^{ab}_{\mu} = 2 \bar{\epsilon} \gamma^{a} \tilde{R}_{\mu\nu}^{ab}(Q) - \frac{1}{2} \bar{\epsilon} \gamma^{a} \gamma^{b} \psi_{\mu} \psi,
$$

$$
\phi_{\mu}^{i} = \frac{1}{2} \bar{\epsilon} \gamma^{a} \tilde{R}_{\mu\nu}^{ab}(Q) + \frac{1}{2} \bar{\epsilon} \gamma^{a} \gamma^{b} \psi_{\mu} \psi.
$$

(2.4)

We have the field strengths for $C_{\mu}$, $B_{\mu\nu}$ and $V_{\mu}^{ij}$ defined as

$$
\tilde{G}_{\mu\nu} = 2 \partial_{[\mu} C_{\nu]} + \frac{1}{2} \bar{\epsilon} \gamma^{a} \psi_{[\mu} \psi_{\nu]} \psi,
$$

$$
\tilde{H}_{\mu\nu\rho} = 3 \partial_{[\mu} B_{\nu\rho]} + \frac{1}{2} C_{[\mu} G_{\nu\rho]} - \frac{3}{2} \bar{\epsilon} \gamma^{a} \psi_{[\mu} \psi_{\rho]} + \frac{1}{2} \bar{\epsilon} \gamma^{a} \gamma_{\mu} \psi_{\nu\rho} \psi,
$$

$$
\tilde{R}_{\mu\nu}^{ij}(V) = 2 \partial_{[\mu} V_{\nu]}^{ij} - 2 V_{[\mu}^{ij} V_{\nu]}^{\rho j} - 3 \bar{\epsilon} \gamma^{a} \psi_{[\mu} \psi_{\nu]} - 8 \bar{\epsilon} \gamma^{a} \psi_{[\mu} \psi_{\nu]} \chi_{j} - i \bar{\epsilon} \gamma^{a} \gamma_{\mu\nu} \psi_{i},
$$

and the gravitino curvatures as

$$
\tilde{R}_{\mu\nu}^{ij}(Q) = \tilde{R}_{\mu\nu}^{ij}(Q) - 2 \bar{\epsilon} \gamma^{a} \phi_{\mu}^{i},
$$

(2.8)
We have used here the superconformally covariant derivatives
\[ \widetilde{\mathcal{D}}_{\mu}^i (Q) = 2 \delta_{[\mu} \psi_{\nu]}^i + \frac{1}{2} \delta_{[\mu} \gamma_{ab} \psi_{\nu]}^i + b_{[\mu} \psi_{\nu]}^i - 2 V_{[\mu} \psi_{\nu]}^{i(j)} + 2 i \gamma^i \cdot T \gamma_{[\mu} \psi_{\nu]}^j. \] (2.9)

Note that \( \widetilde{\mathcal{D}}_{\mu \nu} \) is invariant under the bosonic gauge transformations
\[ \delta C_{\mu} = \partial \Lambda_{\mu}, \quad \delta B_{\mu \nu} = 2 \delta_{[\mu} \Lambda_{\nu]} - \frac{1}{2} \Lambda \Gamma_{\mu \nu}, \]
where \( \Gamma_{\mu \nu} = 2 \delta_{[\mu} C_{\nu]} \). Finally, for future reference we give the transformation rules
\[ \delta \bar{\psi}^a = \frac{1}{2} i \bar{\psi}^{a(x)} \gamma^a \psi - \frac{1}{2} i \bar{\psi}^{a(y)} \gamma^a \psi - i \bar{\psi}^{a(x)} \gamma^a \cdot T \gamma^b \psi_{\mu} \]
\[ = \frac{1}{2} i \bar{\psi}^{a(x)} [\widetilde{\mathcal{D}}_{\mu}^{a(x)} (Q) - \frac{1}{2} \bar{\psi}^{a(x)} \gamma_{\nu} \widetilde{\mathcal{D}}^{a(x)} (Q) - 4 \epsilon_{\mu \nu} \bar{\psi}^{a(y)} \chi], \]
\[ \delta \widetilde{\mathcal{D}}_{\mu \nu} = - \frac{1}{2} \bar{\psi}^{a(x)} \gamma_{\nu} \widetilde{\mathcal{D}}^{(a)(x)} (Q) + \frac{1}{2} i \bar{\psi}^{a(y)} \gamma_{\nu} \widetilde{\mathcal{D}}^{(a)(y)} \psi_{\mu} \]
\[ - \frac{1}{2} i \bar{\psi}^{a(x)} \gamma_{\nu} \cdot T \gamma_{\nu} \psi_{\mu} - \frac{1}{2} \bar{\psi}^{a(x)} \gamma_{\nu} \widetilde{\mathcal{D}}^{(a)(x)} \psi_{\mu} - \frac{1}{2} \gamma_{\nu} \bar{\psi}^{a(y)} \psi_{\mu}, \]
\[ \delta \widetilde{\mathcal{D}}_{\mu \nu} = - \frac{1}{2} i \bar{\psi}^{a(y)} \widetilde{\mathcal{D}}_{\mu \nu} (Q) - \frac{1}{2} \bar{\psi}^{a(y)} \gamma_{\nu} \psi_{\mu} - i \bar{\psi}^{a(y)} \gamma_{\nu} \cdot T \gamma_{\nu} \psi_{\mu} + \bar{\psi}^{a(y)} \gamma_{\nu} \psi_{\mu}. \]

This concludes our short review of the dilaton Weyl multiplet.

2.2. The Yang–Mills multiplet

The off-shell non-Abelian \( D = 5, N = 2 \) vector multiplet consists of \( 8 n + 8 n \) bosonic and fermionic degrees of freedom (where \( n \) denotes the dimension of the gauge group). Denoting the Yang–Mills index by \( I = 1, \ldots, n \), the bosonic field content consists of vector fields \( A_{\mu}^I \), scalar fields \( \rho^I \) and auxiliary fields \( Y^{ij} = Y^{(i)j} \), that are \( SU(2) \)-triplets. The fermion fields are given by \( SU(2) \)-doublets \( \chi^I \).

The \( Q \)- and \( S \)-transformations of the vector multiplet, in the background of the dilaton Weyl multiplet, are given by [11]
\[ \delta A_{\mu}^I = - \frac{1}{2} i [i \rho^J \bar{\psi}^J + \frac{1}{2} \bar{\psi}^{a(x)} \chi^I, \]
\[ \delta Y^{ij} = - \frac{1}{2} i [\bar{\psi}^{a(x)} \gamma^{ij} + \frac{1}{2} i \bar{\psi}^{a(y)} \sim^{ij} + \frac{1}{2} \bar{\psi}^{a(y)} \gamma^{ij} - \frac{1}{2} \bar{\psi}^{a(y)} f^{jk} \rho^{k} \chi^{ij}], \]
\[ \delta \chi^I = - \frac{1}{2} \gamma^I \cdot \sim^J \chi^{J} - \frac{1}{2} \bar{\psi}^{a(y)} \gamma^I \chi^{J} + \frac{1}{2} \bar{\psi}^{a(y)} \gamma^I \chi^{J}, \]
\[ \delta \rho^J = \frac{1}{2} i \bar{\psi}^{a(x)} \chi^I. \]

We have used here the superconformally covariant derivatives
\[ \mathcal{D}_{\mu}^I = (\delta_{\mu} - b_{\mu}) \rho^J + g f^{jk} A_{\mu}^J \rho^K - \frac{1}{2} i \bar{\psi}^{a(x)} \gamma^I. \]
\[ \mathcal{D}_{\mu}^{IJ} = (\delta_{\mu} - b_{\mu} + \frac{1}{2} \delta_{\mu} \gamma_{ab} \gamma_{ab} \gamma^I \gamma^K - V^{Ij} \gamma^J - g f^{jk} A_{\mu}^J \gamma^K + \frac{1}{2} Y^{Ij} \gamma^J + Y^{IJ} \gamma^J - \rho^J \gamma^I \cdot T \gamma^J - \rho^J \gamma^I), \]
and the supercovariant Yang–Mills curvature
\[ \mathcal{F}^{IJ}_{\mu \nu} = 2 \delta_{[\mu} A^{I}_{\nu]}^J + g f^{jk} A_{\mu}^J A_{\nu}^K - \bar{\psi}^{a(x)} \gamma_{\nu} \gamma^I \gamma^J + \frac{1}{2} i \rho^J \bar{\psi}^{a(x)} \gamma_{\nu} \gamma^I. \]

3. Change of basis

In what follows, it turns out to be convenient to change to a basis, denoted by tilded fields, in which all fields are dilatation and S-supersymmetry invariant. In terms of the original fields the tilded fields are given by
\[ \bar{e}^\mu = e^\mu, \]
\[ \bar{e}^\mu = e^\mu, \]
\[ \bar{V}_{\mu}^{ij} = V_{\mu}^{ij} - \frac{1}{2} i \omega^{-1} \bar{\psi}^{a(x)} \gamma^{ij} + \frac{1}{2} \sigma^{-2} \bar{\psi}^{a(x)} \gamma^{ij}, \]
\[ \bar{C}_{\mu} = C_{\mu}, \]
\[ \bar{\psi}^{a(x)} = \sigma^{a(x)} \bar{\psi}^{a(x)}. \]
Dropping the tildes for convenience in notation, we find that the supersymmetry transformation rules in the new basis are given by

\[ \delta \epsilon_{\mu}' = \frac{1}{2} \bar{\psi}^{\alpha}' \sigma_{\alpha} \psi_{\mu}, \]
\[ \delta \psi_{\mu}' = D_{\nu} (\bar{\omega}_{-}) \epsilon' - \frac{1}{2} \bar{\psi}^{i} \gamma^{\nu} \epsilon', \]
\[ \delta V_{\mu}^{ij} = \frac{1}{2} \bar{\psi}^{[i} \gamma^{\nu} \psi_{\mu]}^{j} = - \frac{1}{6} \bar{\psi}^{(i} \gamma \cdot \hat{H} \psi_{\mu)^j} = - \frac{1}{4} i \bar{\psi}^{(i} \gamma \cdot \hat{G} \psi_{\mu)^j}. \]  

(3.2)

\[ \delta \epsilon_{\mu} = - \frac{1}{2} i \bar{\psi} \psi_{\mu}, \]
\[ \delta B_{\mu
u} = \frac{1}{2} \bar{\psi} \gamma_{[\mu} \psi_{\nu]} + C_{[\mu} \delta (\epsilon) C_{\nu]}, \]

where we have used the torsionful spin connection

\[ \bar{\omega}_{(i}^{ab} = \bar{\omega}_{(i}^{ab} \pm \hat{H}_{(i}^{ab}, \]
\[ \bar{\omega}_{(i}^{ab} = 2 \epsilon^{[a} \delta_{\mu}^{b]} - \epsilon^{[a} \epsilon^{b]} e_{\nu} \delta_{\mu}^{\epsilon} + \frac{1}{2} \bar{\psi}^{[a} \gamma^{b]} \psi_{\mu} + \frac{1}{2} \bar{\psi}^{a} \gamma_{\mu} \psi_{b} \]  

(3.3)

and the supercovariant curvatures

\[ \hat{\psi}_{\mu} = 2 D_{(\mu} (\bar{\omega}_{-}) \psi_{\nu)} + i y^{r} \hat{G}_{(\mu} \psi_{\nu)}, \]
\[ \hat{G}_{\mu
u} = 2 \partial_{(\mu} C_{\nu)} + \frac{1}{2} \bar{\psi} \gamma^{(\mu} \psi_{\nu)} + C_{(\mu} \hat{G}_{\nu)} \]
\[ \hat{H}_{\mu
u} = 3 \partial_{(\mu} B_{\nu)} - \frac{1}{2} \bar{\psi} \gamma_{(\mu} \psi_{\nu)} + \frac{1}{2} C_{(\mu} G_{\nu)}. \]  

(3.4)

(3.5)

(3.6)

(3.7)

For the purposes of the following section, we also define the supercovariant curvature

\[ \hat{V}_{\mu}^{ij} = 2 \partial_{(\mu} V_{\nu)}^{ij} - 2 V_{(\mu}^{li} V_{\nu)}^{j} - \bar{\psi}_{(i}^{l} \gamma^{r} \psi_{j)}^{r} + \frac{1}{2} \bar{\psi}^{l} \gamma_{\mu} \psi_{j}^{r} + \frac{1}{2} i \bar{\psi}^{l} \gamma_{\mu} \psi_{j} \]

(3.8)

The redefinitions (3.1) are in fact equivalent to fixing the dilatation, special conformal transformations and S-supersymmetry by imposing the gauge conditions

\[ \sigma = 1, \quad b_{\mu} = 0, \quad \psi^{i} = 0. \]  

(3.9)

The first of these conditions fixes the dilatation symmetry, the second fixes the special conformal transformations, while the last condition fixes the S-supersymmetries. We stress that the gauge fixing performed here is merely a way to describe a field redefinition and will not be used to obtain an off-shell Poincaré supergravity theory. In fact, for that purpose a new compensating multiplet, which has been taken to be a linear multiplet in [11], is needed.

In order for the condition \( \psi^{i} = 0 \) to be invariant under supersymmetry, one has to modify the supersymmetry rules by adding a compensating S-supersymmetry transformations with parameter

\[ \eta^{i} = \left( - \gamma \cdot T + \frac{1}{2} \gamma \cdot \hat{G} \right) \epsilon^{i}. \]  

(3.10)

The second gauge condition in (3.9), in turn, leads to the compensating conformal boost transformations with parameter

\[ \Lambda_{\epsilon}^{\mu} = - \frac{1}{2} i \bar{\psi} \phi_{\mu} - \frac{1}{2} i \bar{\eta} \psi_{\mu} + \bar{\epsilon} \gamma_{\mu} \chi. \]  

(3.11)

with \( \eta \) as given in (3.10).

Similarly, we change basis for the Yang–Mills multiplet by defining the dilatation and S-supersymmetry invariant tilded fields as

\[ \bar{\psi}^{i} = \bar{\psi}^{i}, \]
\[ \bar{V}_{(i}^{j} = \sigma^{-1} \gamma^{j} \psi^{i} + \frac{1}{2} i \sigma^{-1} \bar{V}_{(i}^{j} \psi^{k} \psi^{l)}, \]
\[ \bar{C}_{(i}^{j} = \sigma^{-3/2} \gamma^{j} \psi^{i} + \frac{1}{2} i \sigma^{-3/2} \bar{C}_{(i}^{j} \psi^{k} \psi^{l)}, \]
\[ \bar{H}_{(i}^{j} = \sigma^{-3} \gamma^{j} \psi^{i} + \frac{1}{2} i \sigma^{-3} \bar{H}_{(i}^{j} \psi^{k} \psi^{l)}. \]  

(3.12)
Again, dropping the tildes for convenience in notation, we find the supersymmetry transformation rules

\[ \delta A_\mu^I = -\frac{1}{2} i \rho^j \bar{\psi}_\mu + \frac{1}{2} i \bar{\psi} \gamma_\mu \lambda^I, \]

\[ \delta Y^{ij} = -\frac{1}{2} \epsilon^{ij} \tilde{D} \lambda^{ij} - \frac{1}{2} \bar{\psi} \gamma^{ij} \bar{\psi} \tilde{D} \lambda^{ij} - \frac{1}{2} i g \epsilon^{ij} f_{jK} \rho^I \lambda^{jIK}, \]

\[ \delta \lambda^d = -\frac{1}{2} (\gamma \cdot \tilde{F} - \rho^j \gamma \cdot \tilde{G}) \epsilon^j - \frac{1}{2} i \bar{\psi} \rho^I \epsilon^i - Y^{ij} \epsilon_j, \]

\[ \delta \rho^I = \frac{1}{2} i \bar{\psi} \lambda^I, \]

where \( \tilde{F}^{ij} \) and \( \tilde{D}_\mu \rho^I \) are as defined in (2.17) and (2.15), respectively, and

\[ \tilde{D}_\mu \lambda^{ij} = (\partial_\mu + \frac{1}{2} \tilde{D}_\mu \gamma_{ab}) \lambda^{ij} - V_a^{ij} \lambda^j + g f_{jK} A_k^I \lambda^{jIK} \]

\[ + \frac{1}{2} (\gamma \cdot \tilde{F} - \rho^j \gamma \cdot \tilde{G}) \psi^j + \frac{1}{2} i \bar{\psi} \rho^I \psi^I + Y^{ij} \psi^I, \]

(3.13)

With these results at hand, we are ready to make a connection between the dilaton Weyl multiplet and Yang–Mills multiplet transformation rules.

4. The Weyl multiplet as a Yang–Mills multiplet

In this section, we will show that the following multiplet of fields

\[ (\omega^{ab}_{\mu+}, -\bar{V}^{ab}_{ij}, \bar{G}_{ab}), \]

(4.1)

defined in (3.3), (3.5), (3.8) and (3.6), respectively, transforms as a Yang–Mills multiplet

\[ (A^I_\mu, \chi^I, Y^{ij}_I, \rho^I), \]

where the antisymmetric index pair \( ab \) plays the role of the Yang–Mills index \( I \), for Yang–Mills group \( SO(4, 1) \). In the above, the definition of the gravitino curvature that follows from (3.5) is given by

\[ \tilde{\psi}_{ab} = 2 D_{[a}(\omega_{[a}, \omega_{b]} \psi_{b]} + i \gamma^\mu \bar{G}_{[a} \psi_{b]}, \]

(4.3)

where it is important to note that in \( D_{[a}(\omega_{[a}, \omega_{b]} \psi_{b]} \), the connection \( \omega \) rotates the Lorentz vector index, while the connection \( \tilde{\omega} \) rotates the Lorentz spinor index.

Next, we calculate the transformation rules of \( \omega^{ab}_{\mu+} \) and \( \bar{G}_{ab} \). In the new basis, we find

\[ \delta \omega^{ab}_{\mu+} = -\frac{1}{2} i \bar{\psi} \gamma^\mu \omega^{ab}_{\mu+} - \frac{1}{2} i \bar{\psi} \gamma^\mu \psi^{ab}_{\mu+} - \frac{1}{2} i \bar{\psi} \gamma^\mu \tilde{H}_{cab} - \frac{1}{2} i \bar{\psi} \psi_{\mu+} \tilde{G}_{ab}, \]

\[ \delta \tilde{H}_{cb} = -\frac{1}{2} i \bar{\psi} \gamma_{b} \psi_{\mu+} - \frac{1}{2} i \bar{\psi} \gamma_{b} \psi_{\mu+} + \frac{1}{2} i \bar{\psi} \gamma_{b} \psi_{\mu+} \tilde{H}_{cab}, \]

\[ \delta \tilde{G}_{ab} = -\frac{1}{2} i \bar{\psi} \psi_{ab}, \]

(4.4)

From the first two equations, it readily follows that

\[ \delta \omega^{ab}_{\mu+} = -\frac{1}{2} i \bar{G}^{ab} \psi_{\mu+} - \frac{1}{2} i \bar{\psi} \psi_{\mu+} \psi^{ab} \psi_{\mu+}. \]

(4.6)

Comparing these results with the transformation rules of the Yang–Mills multiplet, one sees that they indeed agree upon making the identification

\[ \omega^{ab}_{\mu+} \leftrightarrow A^I_\mu, \quad \psi^{ab}_{ij} \leftrightarrow -\chi^I, \quad \tilde{G}^{ab} \leftrightarrow \rho^I. \]

(4.7)

Next, we compute the transformation rule for \( \tilde{\psi}_{ab} \). We find

\[ \delta \tilde{\psi}_{ab} = \frac{1}{2} \tilde{R}_{abcd}(\omega_{[a} \gamma_{b]} \gamma^{cd} \epsilon^i - \tilde{V}^{ij}_{ab} \psi_j - i \tilde{D}_{[a}(\omega_{[a} \tilde{G}_{b]} \gamma^{cd} \epsilon^i + \frac{1}{2} \tilde{G}_{cd} \tilde{G}_{[a]} \gamma^{cd} \epsilon^i, \]

(4.8)

where \( \tilde{R}_{abcd}(\omega_{[a} \tilde{G}_{b]} \gamma^{cd} \epsilon^i - \tilde{V}^{ij}_{ab} \psi_j - i \tilde{D}_{[a}(\omega_{[a} \tilde{G}_{b]} \gamma^{cd} \epsilon^i + \frac{1}{2} \tilde{G}_{cd} \tilde{G}_{[a]} \gamma^{cd} \epsilon^i, \]

where \( \tilde{R}_{abcd}(\omega_{[a} \gamma_{b]} \gamma^{cd} \epsilon^i - \tilde{V}^{ij}_{ab} \psi_j - i \tilde{D}_{[a}(\omega_{[a} \tilde{G}_{b]} \gamma^{cd} \epsilon^i + \frac{1}{2} \tilde{G}_{cd} \tilde{G}_{[a]} \gamma^{cd} \epsilon^i, \]

(4.8)

where \( \tilde{R}_{abcd}(\omega_{[a} \gamma_{b]} \gamma^{cd} \epsilon^i - \tilde{V}^{ij}_{ab} \psi_j - i \tilde{D}_{[a}(\omega_{[a} \tilde{G}_{b]} \gamma^{cd} \epsilon^i + \frac{1}{2} \tilde{G}_{cd} \tilde{G}_{[a]} \gamma^{cd} \epsilon^i, \]

(4.8)

where \( \tilde{R}_{abcd}(\omega_{[a} \gamma_{b]} \gamma^{cd} \epsilon^i - \tilde{V}^{ij}_{ab} \psi_j - i \tilde{D}_{[a}(\omega_{[a} \tilde{G}_{b]} \gamma^{cd} \epsilon^i + \frac{1}{2} \tilde{G}_{cd} \tilde{G}_{[a]} \gamma^{cd} \epsilon^i, \]

(4.8)
Finally, we calculate the transformation rule of \( \hat{\omega}_- \). This follows from (4.3). Next, using the Bianchi identity for \( \hat{R} \),

\[
\hat{D}_{[a}(\hat{\omega})\hat{H}_{bcd]} = \frac{1}{2} \hat{G}_{[ab}[\hat{G}_{cd}].
\]

one finds that

\[
\hat{R}_{abcd}(\hat{\omega}_-) = \hat{R}_{c[ab} (\hat{\omega}_+) - (\hat{G}_{ab}\hat{G}_{cd} + 2\hat{G}_{a[c}\hat{G}_{d]b}).
\]

(4.10)

If one furthermore uses the Bianchi identity \( \hat{D}_{[a}(\hat{\omega})\hat{G}_{c]d]} = 0 \), one finds the final result

\[
\delta \hat{\psi}^i_{ab} = \frac{1}{2} \gamma^{cd} \hat{R}_{c[ab} (\hat{\omega}_+) \epsilon^{e} - \hat{\psi}^{i}_{ab} \epsilon^{e} + \frac{i}{2} \gamma^{cd} \hat{D}_{a}(\hat{\omega}_+) \hat{G}_{cd} \epsilon^{e} - \frac{1}{2} \hat{G}_{ab} \gamma \cdot \hat{G} \epsilon^{e},
\]

(4.11)

where in \( \hat{D}_{a}(\hat{\omega}_+) \hat{G}_{ab} \), the connection \( \hat{\omega}_+ \) rotates both of the indices \( a \) and \( b \). Upon using the identifications (4.7), one sees that this transformation rule indeed assumes the form of \( \delta \lambda_i^{[a} \), see (3.13), if one makes the extra identification

\[
\hat{\psi}^{i}_{ab} \leftrightarrow -Y^{i[a}.
\]

(4.12)

Finally, we calculate the transformation rule of \( \hat{V}^{i[a} \) in a similar way. We find

\[
\delta \hat{V}^{i[a}_b = \hat{\epsilon}^{i[a} \gamma^{d} \hat{D}_{a}(\hat{\omega}_+) \hat{\psi}_{b]c} = -\frac{1}{6} \hat{\epsilon}^{i[a} \gamma \cdot \hat{H} \hat{\psi}^{i]}_b - \frac{1}{4} \hat{\epsilon}^{i[a} \gamma \cdot \hat{G} \hat{\psi}^{i]}_b,
\]

(4.13)

where in \( \hat{D}_{a}(\hat{\omega}_+) \hat{\psi}_{b]c} \) the connection \( \hat{\omega}_+ \) acts on the index \( b \) while the connection \( \hat{\omega}_- \) acts on the index \( c \) and the spinor index. Upon using the Bianchi identity

\[
\hat{D}_{a}(\hat{\omega}_+) \hat{\psi}_{b]c} = -\frac{1}{2} \gamma^{d} \left( 2\hat{G}_{d[ab} \hat{\psi}^{i]}_c + \hat{G}_{d[bc} \hat{\psi}^{i]}_{a} \right),
\]

(4.14)

we then find

\[
\delta \hat{V}^{i[a}_b = -\frac{1}{2} \hat{\epsilon}^{i[a} \gamma \cdot \hat{H} \hat{\psi}^{i]}_b - \frac{1}{6} \hat{\epsilon}^{i[a} \gamma \cdot \hat{G} \hat{\psi}^{i]}_b - i \hat{\epsilon}^{i[a} \left( \hat{G}^{d[a}_{c} \hat{\psi}^{i]}_b \right),
\]

(4.15)

where in \( \hat{D}_{a}(\hat{\omega}_+) \hat{\psi}_{b]c} \) the connection \( \hat{\omega}_+ \) rotates the spinor index, while the connection \( \hat{\omega}_- \) rotates the Lorentz vector indices. Expression (4.15) can equivalently be written as

\[
\delta \hat{V}^{i[a}_b (V) = -\frac{1}{2} \hat{\epsilon}^{i[a} \gamma \cdot \hat{H} \hat{\psi}^{i]}_b - \frac{1}{3} \hat{\epsilon}^{i[a} \gamma \cdot \hat{G} \hat{\psi}^{i]}_b - i \hat{\epsilon}^{i[a} \left( \hat{G}^{d[a}_{c} \hat{\psi}^{i]}_b \right),
\]

(4.16)

where in \( \hat{D}_{a}(\hat{\omega}_+) \hat{\psi}_{ab} \) the connection \( \hat{\omega}_+ \) acts on the spinor index, while \( \hat{\omega}_- \) acts on both of the indices \( a \) and \( b \). This result indeed agrees with the corresponding Yang–Mills transformation rule, upon using identifications (4.7) and (4.12).

Summarizing, we find the correspondence

\[
\begin{pmatrix}
A^I_{\mu} \\
Y^{ij} \\
\lambda^I_j \\
\rho \mu
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
\hat{\omega}^{ab}_{\mu} \\
\hat{\psi}^{i}_{ab} \\
\hat{\psi}^{i}_{ab} \\
\hat{G}_{ab}
\end{pmatrix}.
\]

(4.17)

This concludes our discussion of the analogy between the dilaton Weyl multiplet and the Yang–Mills multiplet. After constructing the coupling of the Yang–Mills multiplet to the Weyl multiplet by means of dimensional reduction from 6D in the following section, we shall use the correspondence (4.17) to obtain the Riemann squared invariant in the subsequent section.
5. Yang–Mills coupled to Weyl in 6D and dimensional reduction to 5D

In this section, we will verify that the 5D local supersymmetry transformations of the Weyl multiplet and Yang–Mills multiplet given above follow precisely from a suitable circle reduction and truncation of the known counterparts in 6D [4]. Next, we will reduce the action in 6D that describes the coupling of a Yang–Mills multiplet to the Weyl multiplet [4] down to 5D. Using these results, we shall then construct in section 6 the new supersymmetric Riemann tensor squared invariant by making use of the analogy between the non-Abelian vector multiplet and the dilaton Weyl multiplet, derived in the previous section.

5.1. The 6D action for Yang–Mills coupled to Weyl

The 6D Weyl multiplet with the superconformal symmetries gauge-fixed, or equivalently with suitable field redefinitions amounting to the same thing, consists of the vielbein $E_M^A$, gravitino $\Psi_i^A$, the $SU(2)$-valued vector fields $V^i_{MN}$ and the 2-form potential $B_{MN}$. Their supersymmetry transformations are given by [2]

$$E^A_M \delta E^A_M = \frac{1}{2} \hat{\xi} \Gamma^A \Psi_B,$$

$$\delta \Psi_i = D_A (\hat{\partial}) \Psi_i + \frac{1}{8} \hat{R}^{AB} \Gamma_{BC} \Gamma_{DE} - (E_A^M \delta E^M_B) \Psi_B,$$

$$\delta V^i_{A} = \varepsilon^{ijkl} (\Gamma^j \Psi_{AB} - \frac{1}{2} \hat{R}^{ijkl} \Gamma^j \Psi_{AB} - (E_A^M \delta E^M_B) V^j_B),$$

$$\delta B_{AB} = -\varepsilon \Gamma_{[A} \Psi_B + 2 \left( E_A^M \delta E^M_B \right) B_{[BC]},$$

(5.1)

where

$$\hat{\Psi}^i_{AB} = 2D_{[A} (\hat{\partial}) \Psi^i_{B]} + \frac{1}{2} \hat{R}^{CD} (\hat{\Gamma}^{CD} \Psi^i_B + T_{AB} C \Psi_C,$$

$$\hat{\Gamma}_{ABC} = 3\delta_{[A} \delta_{BC]} + \frac{1}{2} \hat{\Psi}_{[A} \Gamma_B \Psi_{C]} - 3T_{[AB} \delta_{CD]} C_{CD},$$

$$T_{AB} C = E_A^M E_B^N C \left( \delta_M E_N C - \delta_N E_M C \right),$$

(5.2)

and

$$\hat{\omega}^{i}_{AB} = 2E^A_M E^B_C (\partial_M E^N_{[A} \partial_B E^{N] C} - E^{MA} E^{NB} \partial_{[A} E^{BC]} + \frac{1}{4} \hat{\Psi}_{C} \Gamma^{[A} \Psi_{B]} + \frac{1}{2} \hat{\Psi}_{C} \Gamma^{-A} \Psi_{B]} + \frac{1}{4} \hat{\Psi}^{-A} \Gamma_{C} \Psi_{B]} \hat{\Gamma}_{ABC}^{-B} \Psi^{i}_{D}.$$}

(5.3)

Turning to the 6D Yang–Mills multiplet, it consists of a vector $W_M$, a spinor $\Omega^i$, and a triplet of auxiliary fields $Y_{ij}$. The superconformal gauge fixed Yang–Mills supermultiplet transformations take the form [2]

$$\delta W_A = -\varepsilon \Gamma_A \Omega - (E_A^M \delta E^M_B) W_B,$$

$$\delta \Omega^i = \frac{1}{2} \Gamma_{[A} \hat{\partial}_{BC]} \epsilon^i + \frac{1}{2} \gamma^{ij} \epsilon_j,$$

$$\delta \gamma^{ij} = -\varepsilon \Gamma_{1} \hat{\Gamma}_{ABC} \Omega^{ij} - \frac{1}{2} \varepsilon \Gamma^{A} \hat{\Gamma}^{B} \hat{\Gamma}^{C} \Omega^{ij} \hat{\Gamma}_{ABC},$$

where $W_A = W_A^1 T_i$, and similarly for the other members of the Yang–Mills multiplet, where $T_i$ are the generators of the Yang–Mills gauge group, and

$$\hat{\Gamma}_{AB} = 2\partial_M W_B + g[W_A, W_B] + 2\hat{\Psi}_{[A} \Gamma_B \Omega + T_{AB} I W_C.$$

(5.5)

$$\hat{\Gamma}_{1} \hat{\partial}_{1} \hat{\Gamma}_{ABC} \Omega^{ij} - \frac{1}{2} \varepsilon \Gamma^{A} \hat{\Gamma}^{B} \hat{\Gamma}^{C} \Omega^{ij} \hat{\Gamma}_{ABC},$$

(5.6)

Finally, the locally supersymmetric Lagrangian in 6D that describes the couplings of the superconformal gauge fixed Yang–Mills–Weyl multiplet is given by [4]

$$E^{-1} \mathcal{L}_6 = -\frac{1}{4} F_{AB} \hat{\partial}_{CD} F^{CD} + \frac{1}{2} \hat{\Gamma}^{A} \hat{\Gamma}^{B} \hat{\Gamma}^{C} \Omega^{ij} \hat{\partial}_{CD} \hat{\partial}_{[A} \hat{\partial}_{B]} \hat{\partial}_{ij} - \frac{1}{16} \varepsilon \Gamma^{ABCDEF} \hat{\Gamma}_{ABCDEF} \hat{\Gamma}^{[A} \hat{\partial}_{1} \hat{\partial}_{B]} \hat{\partial}_{CD} \hat{\partial}_{[A} \hat{\partial}_{B]} \hat{\partial}_{CD} \hat{\partial}_{ij},$$

(5.7)

where $F_{\mu}^{i,j}$ is the ordinary Yang–Mills field strength and

$$F_{\mu}^{i} = D_{\mu} \Omega^i + \frac{1}{2} \hat{\Gamma}_{ABC} \Gamma_{[BC} \Omega^i + \frac{1}{2} \gamma^{ij} \epsilon_j - g[W_A, \Omega^i].$$

(5.8)
5.2. Dimensional reduction to 5D

We begin by making the ansatz\(^3\)

\[ E_M = \begin{pmatrix} e_{\mu}^a & -C_{\mu} \\ 0 & 1 \end{pmatrix}, \quad E_A = \begin{pmatrix} e_a^{\mu} & e_a^{\mu}C_{\mu} \\ 0 & 1 \end{pmatrix}, \]

\[ B_{ab} = -2e_{\nu}^{e_{\nu}^a}b_{\mu \nu}, \quad B_{55} = -e_{\nu}^{e_{\nu}^a}C_{\mu}, \quad \gamma^{ij} = -2e_{\nu}^{e_{\nu}^a}V_{ij}, \quad \gamma^5 = 0, \]

\[ \Psi_{\mu} = e_{\nu}^{e_{\nu}^a}\psi_{\mu}, \quad \Psi_5 = 0, \quad \epsilon = \epsilon, \]

\[ W_{\mu} = e_{\nu}^{e_{\nu}^a}A_{\mu}, \quad W_5 = \rho, \quad \gamma_{ij} = -\gamma_{ij}, \quad \Omega = -\frac{1}{2}\lambda. \quad (5.9) \]

We also let

\[ \Gamma_{\alpha} = i\gamma_{\alpha}\gamma_5, \quad \bar{\epsilon} = i\bar{\epsilon}\gamma_5, \quad \epsilon^{\alpha\beta\gamma\delta} = \epsilon^{\alpha\beta\gamma\delta}. \quad (5.10) \]

The second expression in (5.10) applies to all Dirac conjugated spinors. Note also that we have identified the Kaluza–Klein vector originating from the 6D metric with \(B_{55}\), and have set to zero \(\gamma^5\) and \(\Psi_5\). This amounts to a consistent truncation of a single off-shell vector multiplet in 5D.

It is now a straightforward exercise to show that the above ansatz yields precisely the local supersymmetry transformations (3.2) and (3.13). In doing so, and in reducing the Lagrangian (5.7) to 5D, it is useful to note the relations

\[ \tilde{F}_{ab} = \tilde{F}_{ab} - \rho\tilde{G}_{ab}, \quad \tilde{F}_{ab} = \tilde{G}_{ab}, \]

\[ \tilde{F}_{ab} = -2\tilde{H}_{ab}, \quad \tilde{F}_{ab} = -\tilde{G}_{ab}, \]

\[ \tilde{G}_{ab} = \tilde{G}_{ab}, \quad \tilde{G}_{ab} = \frac{1}{2}\tilde{G}_{ab}, \quad \tilde{G}_{ab} = -\frac{1}{2}\tilde{G}_{ab}, \]

where \(\tilde{F}_{\mu\nu}, \tilde{G}_{\mu
u}, \tilde{H}_{\mu
u\rho}\) and \(\tilde{G}_{ab}\) are defined in (2.17), (3.6), (3.7) and (3.4), respectively.

It is also useful to record the results

\[ D_a\Omega = -\frac{1}{4}D_a(\tilde{G})\lambda + \frac{1}{8}\tilde{G}_{ab}\gamma^b\lambda, \]

\[ D_a\Omega = -\frac{1}{16}\tilde{G}_{ab}\gamma^b\lambda. \quad (5.12) \]

Armed with these results, it is straightforward to reduce the Lagrangian (5.7) to 5D. The result is as follows:

\[ e^{-1}\mathcal{L}_5 = -\frac{1}{4}(F_{\mu\nu} - \rho^lG_{\mu\nu})(F_{\mu\nu} - \rho^lG_{\mu\nu}) - \frac{1}{2}D_{\mu}\rho^lD^\mu\rho^l - \frac{1}{2}\bar{\psi}\lambda^l + \gamma_ijkl\]

\[ + \frac{1}{16}\epsilon^{\mu\nu\rho\sigma\lambda}(F_{\mu\nu} - \rho^lG_{\mu\nu})(F_{\mu\nu} - \rho^lG_{\mu\nu})C_{\lambda} + 8B_{\rho\sigma}D_{\lambda}\rho^l \]

\[ - \frac{1}{2}\bar{\lambda}^l\gamma^\rho\gamma^\mu\psi_{\rho}(D_{\mu}\rho^l - \frac{1}{2}\bar{\psi}\rho^l\lambda^l) \]

\[ - \frac{1}{4}(F_{\mu\nu} - \rho^lG_{\mu
u} - \frac{1}{2}\bar{\psi}_{\mu\lambda}\gamma^\nu\lambda^l)\lambda^l\gamma^\rho\gamma^\mu\psi_{\rho} \]

\[ - \frac{1}{2}\tilde{H}_{\mu\nu\lambda\rho\sigma\lambda'}\gamma^\mu\nu\rho\sigma\lambda'\lambda'^l - \frac{1}{8}\tilde{G}_{\mu
u\rho\sigma\lambda}^l\gamma^\mu\nu\rho\sigma\lambda'. \quad (5.13) \]

This Lagrangian will be our starting point for constructing the supersymmetric Riemann tensor squared action in the following section.

6. The Riemann squared invariant

To obtain the Riemann squared invariant, we make the substitutions (4.17) in the Lagrangian (5.13). Thus, we find the main result of this paper given by

\(\text{Due to the sign in the relation between } B_{\mu\nu} \text{ and } B_{\nu\mu}, \text{ the torsionful spin connection } \tilde{\omega}_- \text{ in [1] corresponds to } \tilde{\omega}_+ \text{ here.}\)
\[ e^{-1} \mathcal{L}(R^2) = -\frac{1}{2} [R_{\mu\nu\lambda\rho}(\tilde{\omega}^+) - G_{\mu\nu} \tilde{G}_{\lambda\rho}] [R_{\rho\mu\nu\lambda}(\tilde{\omega}^+) - G^{\mu\nu} \tilde{G}_{\lambda\rho}] \\
= -\frac{1}{4} D_\mu(\tilde{\omega}^+) \tilde{G}^{ab} D^\mu(\tilde{\omega}^+) \tilde{G}_{ab} + V_{\mu\nu} i\bar{\psi}^{\mu\nu} i \bar{\psi}^{ab} + \frac{1}{2} \bar{\psi}^{ab} D(\tilde{\omega}^+) \tilde{\psi}_{ab} \\
+ \frac{1}{16} e^{i\mu\nu\rho\sigma} (R_{\mu\nu\rho\sigma}(\tilde{\omega}^+) - G_{\mu\nu} \tilde{G}_{\rho\sigma})(D_\lambda(\tilde{\omega}^+) \tilde{G}^{\lambda\rho\sigma}) C_\lambda \\
+ \frac{1}{2} e^{i\mu\nu\rho\sigma} (R_{\mu\nu\rho\sigma}(\tilde{\omega}^+) - G_{\mu\nu} \tilde{G}_{\rho\sigma})(D_\lambda(\tilde{\omega}^+) \tilde{G}^{\lambda\rho\sigma}) B_{\rho\sigma} \\
- \frac{1}{2} \bar{\psi}^{ab} \psi^{\mu\nu} \bar{\psi}^{ab} \tilde{D}_\mu(\tilde{\omega}^+) \tilde{G}_{ab} + \frac{1}{2} (R_{\mu\nu\rho\sigma}(\tilde{\omega}^+) - G_{\mu\nu} \tilde{G}_{\rho\sigma}) \bar{\psi}^{\mu\nu} \psi^{ab} \tilde{\psi}_{ab} \\
- \frac{1}{2} \bar{\psi}^{ab} \psi_{\mu\nu} \bar{\psi}^{ab} \tilde{D}_\mu(\tilde{\omega}^+) \tilde{G}_{ab} \tilde{\psi}_{ab} + \frac{1}{2} \bar{\psi}^{ab} \psi^{\mu\nu} \psi^{ab} \tilde{\psi}_{ab} \tilde{\psi}_{ab}. \]

(6.1)

The action of this Lagrangian is invariant under the off-shell \( N = 2, D = 5 \) supersymmetry transformations given in (3.2). The use of the Lorentz vector indices is motivated by the substitution rule (4.17). As a consequence, in \( D_\mu(\tilde{\omega}^+) \tilde{G}_{ab} \) the spin connection \( \tilde{\omega}^+ \) rotates the indices \( a \) and \( b \), while in \( D_\mu(\tilde{\omega}^+) \tilde{\psi}_{ab} \) the connection \( \tilde{\omega}^+ \) rotates the spinor index, and the connection \( \tilde{\omega}^+ \) rotates the indices \( a \) and \( b \).

The purely bosonic part of the Lagrangian takes the form

\[ e^{-1} \mathcal{L}(R^2)_{\text{bosonic}} = -\frac{1}{2} [R_{\mu\nu\lambda\rho}(\omega^+) - G_{\mu\nu} G_{\lambda\rho}] [R_{\rho\mu\nu\lambda}(\omega^+) - G^{\mu\nu} G^{\lambda\rho}] \\
= -\frac{1}{2} D_\mu(\omega^+) G^{ab} D^\mu(\omega^+) G_{ab} + V_{\mu\nu} i\bar{\psi}^{\mu\nu} i \bar{\psi}^{ab} + \frac{1}{2} \bar{\psi}^{ab} D(\omega^+) \tilde{\psi}_{ab} \\
+ \frac{1}{16} e^{i\mu\nu\rho\sigma} (R_{\mu\nu\rho\sigma}(\omega^+) - G_{\mu\nu} G_{\rho\sigma})(D_\lambda(\omega^+) G^{\lambda\rho\sigma}) C_\lambda \\
+ \frac{1}{2} e^{i\mu\nu\rho\sigma} (R_{\mu\nu\rho\sigma}(\omega^+) - G_{\mu\nu} G_{\rho\sigma})(D_\lambda(\omega^+) G^{\lambda\rho\sigma}) B_{\rho\sigma} \]

(6.2)

It is possible to extend the above result by adding the Hilbert–Einstein term, as well as the Weyl squared invariant of [5]. To do so, one first performs the inverse of the field redefinitions (3.1) making the fields \( \sigma \) and \( \psi^i \) explicit. Next, the Weyl squared invariant of [5], prior to any conformal symmetry gauge fixing, can be added to our action. The standard Weyl multiplet used in that action can be converted to the dilaton Weyl multiplet by using the map that exists between these two multiplets, see equations (2.3) and (2.3). Finally, a superconformal version of the Einstein action, using the linear multiplet as a compensating multiplet, can be added to these two actions. A conformal gauge fixing at the very end then leads to the desired result.

Alternatively, instead of giving the Einstein action a superconformal treatment, one can also extend the off-shell Poincaré supergravity constructed in [4, 12] to 5D in a manner described in this work.

We note that in the off-shell Poincaré supergravity theory the vector fields \( V_\mu \) are auxiliary. However, with the addition of our Riemann squared invariant, these fields acquire kinetic terms and become dynamical. Such dynamical auxiliary fields should be treated with care in a string theory approximation.

7. Conclusions

In this paper, we have constructed a new \( D = 5, N = 2 \) supersymmetric Riemann tensor squared action. A noteworthy feature of this action is that it contains the purely gravitational Chern–Simons action (1.2). This is in contrast to the higher order invariant constructed in [5] whose leading term is the Weyl tensor squared and which contains the mixed Chern–Simons term (1.1). The latter invariant plays an important role in higher order considerations in black hole entropy calculations and the AdS/CFT correspondence. We expect our newly constructed invariant to play a similar role too. In particular, it would be interesting to see whether our new action may lead to higher order corrections to the entropy formula of black holes in a similar

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4 We thank Bernard de Wit for a clarifying discussion on this point.
way as the higher order invariant of [5] did [17]. In this context, it is of interest to note that a similar torsionful Riem\(^2\) invariant in three dimensions did not lead to any correction of the central charge of the corresponding boundary conformal field theory due to a curvature with parallelizing torsion [13]. For general R\(^2\)-invariants, however, one does expect corrections, see e.g. [18].

Our construction was based on the methods developed some time ago in the context of \(D = 6\) dimensions [1–3]. Both in \(D = 6\) and \(D = 5\) dimensions use is made of the observation that the underlying Weyl multiplet contains a dilaton scalar field which acts as the compensating field for dilatations. For this reason, this multiplet was nominated the dilaton Weyl multiplet. It turns out in a special basis where all fields are inert under dilatations and \(S\)-supersymmetry that the dilaton Weyl multiplet transforms precisely as a Yang–Mills multiplet whose action can easily be constructed. This is what makes the construction of the \(D = 5\) Riemann tensor squared invariant feasible.

The maximally symmetric vacuum solutions and the resulting spectrum corresponding to the Riem\(^2\) action (6.2) remain to be investigated. The field redefinitions (3.1) make the model invariant under the superconformal transformations (2.1), similar to a Brans–Dicke-type realization of a conformal Einstein action. Therefore, the conformal symmetries may be viewed as a ‘fake’ symmetry in a sense. Consequently, whether the formulation of the theory in this setup can lead to the possibility of discarding potentially ghostly states in a manner proposed in [14], and investigated further in [15, 16], remains to be studied.

Finally, following [19] the compactification of the new \(R + R^2\) invariant over \(S^2\) to \(D = 3\) dimensions is expected to yield, after truncation, a supersymmetric version of topological massive gravity. It would be interesting to explicitly perform this reduction.

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