Exhaustion of an interval by iterated Rényi parking

Michael Mackey and Wayne G. Sullivan
August 1, 2016

Abstract

We study a variant of the Rényi parking problem in which car length is repeatedly halved and determine the rate at which the remaining space decays.

1 Introduction

Rényi modeled a long interval randomly jammed with cars of unit length. More precisely, each arriving car is parked at a location chosen according to uniform probability distribution over the set of available parking locations until no parking locations remain. The resulting gap lengths constitute a random variable with range $(0, 1)$. The constant $C_R \approx 0.747598$ is the limit, as the interval length tends to infinity, of the proportion of the interval covered and was analytically determined [5]. This Rényi parking constant is scale invariant, in the sense that it holds for cars of any fixed length. Variations on Rényi’s parking problem consider higher dimensions, a discretised setting, or cars of mixed lengths. Modelling of physical processes, such as random sequential adsorption, frequently refer to the idealised Rényi model (see for example [3, 4, 6] and references therein). Here, we consider another variation. For our purposes it is more convenient to assume we have initially jammed with cars of length 2 leaving us with gaps distributed according to a probability distribution $X_1$ with range $(0, 2)$. In our formulation, the next step is that these $X_1$-distributed intervals are jammed by cars of length 1 to give a new gap distribution $X_2$ with range $(0, 1)$. We then introduce and jam with cars of length $\frac{1}{2}$ to get $X_3$ with range $(0, \frac{1}{2})$, then cars of length $\frac{1}{4}$ and so on. It is clear that our initial intervals are exhausted by this process. Our motivating question is that of determining the rate of exhaustion. We will show that, if $L_n$ denotes the expected length of interval which remains uncovered after $n$ stages, then $L_{n+1}/L_n$ tends to a finite limit $R_1 \approx 0.61$. We will numerically calculate this limit with error bounds and also see that the limiting behaviour is largely independent of the initial probability distribution $X_1$. In particular, $X_1$ need not be the Rényi distribution outlined above. Proof of these facts requires finding a limiting distribution $X$ to the sequence $\{X_n\}$ and for this we need to normalise at each stage. Integral equations governing the evolution of $\{X_n\}$, their corresponding densities $\{f_n\}$, and steady state analogues will be found. We approach the solution to these equations by considering an infinite dimensional eigenvalue problem. Numerical analysis with
error bounds shows that there is a unique eigenvalue of maximum absolute value. Convergence of the iterates of the linear system to the associated eigenspace yields convergence of the sequence of probability densities.

The paper is laid out as follows. In Section 2, we consider the sequence of cumulative probability distributions that arise as a result of the physical process of jamming with cars of fixed length at each stage before halving the car length for the next stage. This yields an integral equation relating the distribution at Stage $n+1$ to that at Stage $n$. We deduce from this the evolution equation in the case of a probability density which naturally leads to an integral equation governing any probability density which is a fixed point of this process. From this, we observe functional properties of any such fixed point which is sufficiently regular.

In Section 3, we use series representations of the iterated sequence of probability densities in order to recast the integral formulation of density evolution as an infinite linear system. The sought-after asymptotic decay rate $R_2$ is seen to be one half of the spectral radius of the infinite matrix and spectral properties of this matrix, $A$, are explored.

Following mention of floating point concerns in Section 4, careful numerical analysis of a similarity transformation of the matrix $A$ is provided in Section 5, based on Gershgorin’s Theorem with error and truncation estimates. Using the same similarity transformation, we deduce a convergence result for iteration of $\ell^1$ vectors under $A$, corresponding to convergence of our probability densities.

Finally, in Section 6, we consider convergence for more general initial distributions without density.

2 The Model

Let $N$ be the large number of gaps in a long interval of length $L$ formed, say, by parking cars of length 2 until jamming occurs. The gap lengths form a random variable, $X_1$, with range $(0, 2)$. The expected number of these gaps which have length greater than 1 is $NP(X_1 > 1)$. Each of these has a car of length 1 placed in it, randomly, by uniform distribution over the set of available locations. This creates $2NP(X_1 > 1)$ new gaps, referred to as Stage 2 gaps, of length in $(0, 1)$ and removes $NP(X_1 > 1)$ Stage 1 gaps of length in $(1, 2)$. The length of one of these $2NP(X_1 > 1)$ Stage 2 gaps is a random variable with range $(0, 1)$ which we call $Y$. The distribution of all $N + NP(X_1 > 1)$ gaps, whose length ranges over $(0, 1)$, will be represented by the random variable $\tilde{X}_2$ which we normalise to $X_2 = 2\tilde{X}_2$ with range $(0, 2)$.

We calculate the probability distribution of $\tilde{X}_2$. Observe that each gap at this point may have existed at the first stage or have been created on introduction of cars at the second stage. We refer to these as Stage 1 gaps and Stage 2 gaps respectively. For $t \in (0, 1),$

$$P(\tilde{X}_2 < t) = P(\text{gap from Stage 1})P(X_1 < t|X_1 < 1) + P(\text{gap from Stage 2})P(Y < t).$$

(2.1)

Clearly,

$$P(\text{gap from Stage 1}) = \frac{N - NP(X_1 > 1)}{N + NP(X_1 > 1)} = \frac{1 - C_1}{1 + C_1}$$
where \( C_1 = P(X_1 > 1) \) and

\[
P(\text{gap from Stage 2}) = \frac{2C_1}{1 + C_1}.
\]

We need \( P(Y < t) \). Consider a Stage 1 gap of length \( \lambda \in (1, 2) \), which has a unit length car randomly placed in it by way of uniform distribution over \((0, \lambda - 1)\), resulting in two new Stage 2 (i.e. \( Y \)-)gaps. The probability that one of these \( Y \)-gaps has length \( < t \), \( P(Y^s < t) \) is 1 if \( \lambda < t + 1 \) and \( \frac{t}{\lambda - 1} \) otherwise. Hence

\[
P(Y < t) = \int_{t=1}^{2} P'(X_1 < \lambda | X_1 > 1)P(Y^s < t)d\lambda
\]

\[
= \int_{1}^{2} \frac{P'(X_1 < \lambda)}{P(X_1 > 1)}P(Y^s < t)d\lambda
\]

\[
= \frac{1}{C_1} \int_{j=1}^{t+1} P'(X_1 < \lambda)d\lambda + \frac{1}{C_1} \int_{j=t+1}^{2} P'(X_1 < \lambda)\frac{t}{\lambda - 1}d\lambda
\]

\[
= \frac{1}{C_1} \left[ P(X_1 < t + 1) - P(X_1 < 1) + \frac{tP(X_1 < \lambda)}{\lambda - 1} \right] + \frac{t^2}{t+1} \int_{t+1}^{2} P(X_1 < \lambda)(\lambda - 1)^2 d\lambda
\]

\[
= \frac{1}{C_1} \left[ t - 1 + C_1 + t \int_{t+1}^{2} \frac{P(X_1 < \lambda)}{\lambda - 1} d\lambda \right].
\]

Substituting into (2.1) we have

\[
P(X_2 < t) = \frac{1 - C_1}{1 + C_1} P(X_1 < t | X_1 < 1) + \frac{2C_1}{1 + C_1} \frac{1}{C_1} \left[ t + C_1 - 1 + t \int_{t+1}^{2} \frac{P(X_1 < \lambda)}{\lambda - 1} d\lambda \right]
\]

\[
= \frac{1}{1 + C_1} P(X_1 < t) + \frac{2}{1 + C_1} \left[ t + C_1 - 1 + t \int_{t+1}^{2} \frac{P(X_1 < \lambda)}{\lambda - 1} d\lambda \right]
\]

Now for \( X_2 = 2X_2 \) and \( t \in (0, 2) \) we have

\[
P(X_2 < t) = P(X_2 < \frac{t}{2})
\]

\[
= \frac{1}{1 + C_1} P(X_1 < \frac{t}{2}) + \frac{2}{1 + C_1} \left[ \frac{t}{2} + C_1 - 1 + \frac{t}{2} \int_{(t+2)/2}^{2} \frac{P(X_1 < \lambda)}{\lambda - 1} d\lambda \right]
\]

\[
= \frac{1}{1 + C_1} \left[ P(X_1 < \frac{t}{2}) + t + 2(C_1 - 1) + t \int_{1+t/2}^{2} \frac{P(X_1 < \lambda)}{\lambda - 1} d\lambda \right].
\]

Letting \( F_s(t) = P(X_s < t) \) and \( C_s = P(X_s > 1) = 1 - F_s(1) \), the map \( S : F_1 \mapsto F_2 \) above can be applied repeatedly to gain a sequence of probability distributions \( \{F_s\} \) on \((0, 2)\), with associated constants \( C_s = P(X_s > 1) \).
We seek a limiting distribution which is a fixed point of this process and are led to consider the evolution equation

\[(1 + C_s)F_{s+1}(x) = F_s(x/2) + x + 2(C_s - 1) + x \int_{(x+2)/2}^{2} \frac{F_s(y)}{(y-1)^2} dy.\]  

(2.2)

Assuming differentiability of $F_1$, we have a corresponding evolution of probability densities $\{f_s\}$,

\[(1 + C_s)f_{s+1}(x) = \frac{1}{2}f_s(x/2) + \int_{1+x/2}^{2} \frac{f_s(y)}{y-1} dy\]

(2.3)

and another differentiation yields the form

\[(1 + C_s)xf_{s+1}'(x) = \frac{x}{4}f_s'(\frac{x}{2}) - f_s(\frac{x + 2}{2})\]

(2.4)

with $f_{s+1}$ fixed by the requirement that $\int_0^2 f_{s+1} = 1$. Note that if $f_s$ is continuously differentiable, then so too is $f_{s+1}$.

Remark that if $F_s \to F$ pointwise, then the limit distribution $F$ will satisfy the steady state version of (2.2), namely

\[(1 + C)F(x) = F(x/2) + x + 2(C - 1) + x \int_{(x+2)/2}^{2} \frac{F(y)}{(y-1)^2} dy\]

(2.5)

where $C = 1 - F(1)$. There are corresponding steady state equations for the limiting probability density $f$ in both integral form,

\[(1 + C)f(x) = \frac{1}{2}f(x/2) + \int_{1+x/2}^{2} \frac{f(y)}{y-1} dy\]

(2.6)

and, assuming it is differentiable, differential form,

\[4(1 + C)xf'(x) = xf'(\frac{x}{2}) - 4f(1 + \frac{x}{2}), \quad (x \in (0,2))\]

(2.7)

where $C = \int_{(1,2)} f$. This is a linear first order ordinary differential equation whose solution is hampered by the presence of two time delays and non-linearity due to the dependence of $C$ on the solution. We will establish uniform convergence of the sequence of probability densities on compact subsets of $(0,2)$ from a general starting point, $f_0$, to a solution of (2.6) and pay particular attention to numerical estimation of $C$.

2.1 Rate of Decay of Remaining Space

Consider the question of exhaustion rate of an interval by this iterative process. Let $X$ be the random variable representing gap lengths whose distribution $F$ satisfies (2.5) with $C = 1 - F(1) = P(X > 1)$. If $F$ is differentiable then we let $f = F'$ be the corresponding density function. For a large number $N$ of such $X$-gaps, the total length has expected value $N\mathbb{E}(X)$. Jamming with unit length cars, we can expect to fit $NP(X > 1)$ of same with total length...
NP(X > 1). The uncovered length has reduced to NE(X) − NP(X > 1) and the ratio $R_2$ is $(NE(X) − NP(X > 1))/(NE(X))$ or

$$R_2 = 1 - \frac{P(X > 1)}{E(X)} = 1 - \frac{C}{E(X)}. \quad (2.8)$$

The expected value $E(X) = \int_0^2 tF(dt) = \int_0^2 tf(t)dt = 2 - \int_0^2 F$. On integrating (2.5) and changing order in the resultant double integral we find

$$(1 + C) \int_0^2 F = 2 \int_0^1 F + 2 + 4(C - 1) + \int_{x=0}^2 \int_{y=1+\frac{1}{2}}^2 \frac{xF(y)}{(y-1)^2} dy dx$$

$$= 2 \int_0^1 F + 4C - 2 + \int_{y=1}^2 2F(y)dy$$

$$= 2 \int_0^2 F + 4C - 2.$$

Thus $\int_0^2 F = \frac{2(1-2C)}{1-C}$ and $E(X) = \frac{2C}{1-C}$. Now, from (2.8), we have the rate of exhaustion as

$$R_2 = \frac{1+C}{2}. \quad (2.9)$$

2.2 Properties of a solution

If $f$ is a continuous probability density on $(0, 2)$ satisfying (2.6) then $\lim_{x \to 2} f(x)$ exists and we may take $f(2)$ to be this limit. Then, $f(1) = 2(1 + C)f(2)$ and, as $C$ is a probability, it follows

$$f(1) \leq 4f(2) \leq 2f(1). \quad (2.10)$$

2.1 Proposition Let $f_0$ be a probability density on $(0, 2)$, continuous in a neighbourhood of 0, generating the sequence of probability densities $\{f_s\}$ according to (2.3). There exists $\sigma \in \mathbb{N}$ such that $f_s > 0$ on $(0, 2)$ for all $s > \sigma$. Indeed, for each $s > \sigma$, there exists $\varepsilon_s > 0$ such that $f_s > \varepsilon_s$ on $(0, 2]$.

Proof. By hypothesis, $f_0$ is continuous on $(0, \alpha)$ for some $\alpha > 0$ and so (2.3) implies that $f_1$ is continuous on $(0, 2\alpha)$. After a finite number of iterations we have a continuous probability density on $(0, 2)$. Without loss of generality therefore, we may assume that $f_0$ is continuous, as are subsequent iterations. Choose $\alpha \in (0, 2)$ such that $f_0(\alpha) > 0$. Indeed, by continuity, we may assume $\alpha$ is not a (negative) power of 2, and thus for some $k \in \mathbb{N}$, $\beta = 2^k \alpha \in (1, 2)$. From (2.3), $f_1(2\alpha) \geq \frac{1}{4}f_0(\alpha) > 0$, and by repetition, $f_k(\beta) > 0$. Continuity of $f_k$ implies that the integral $\int_{1+\frac{1}{2}}^2 \frac{f_0(y)}{y} dy > 0$ whenever $1 + \frac{1}{2} \beta < \beta$, that is $x < 2(\beta - 1)$ and therefore $\int_{1+\frac{1}{2}}^2 \frac{f_0(y)}{y} dy > 0$. In particular, there exists $j \in \mathbb{N}$ such that $f_{k+1} > 0$ on $(0, 2^{-j})$ and it follows, as above, that for $k = j + 1$, $f_\sigma > 0$ on $(0, 2)$. Remark also that $f_\sigma > 0$ on $(0, 2)$ implies that $f_{\sigma+1} > 0$ on $(0, 2]$.

Next, we can choose $\delta > 0$ such that $f_\sigma(y) > \frac{1}{2}f_\sigma(1)$ for $y \in (1, 1 + \delta)$. Then $f_{\sigma+1}(x) \geq \frac{1}{4}f_\sigma(2) + \frac{1}{2} \int_{1+\frac{1}{2}}^2 \frac{f_\sigma(y)}{y} dy > \frac{1}{4}f_\sigma(1) \int_{1+\frac{1}{2}}^2 \frac{1}{y} dy \to \infty$ as $x \to 0^+$. That is, lim$_{x \to 0^+} f_{\sigma+1}(x) = \infty$. In particular, there exists $x_0 \in (0, 2)$ such that inf$_{[0, 2]} f_{\sigma+1} = \inf_{[x_0, 2]} f_{\sigma+1} > 0$. □
2.2 Remark The hypothesis in the above result can be weakened to deal with more general probability measures. See Proposition 6.3.

2.3 Proposition Suppose \( f \) is a continuously differentiable probability density on \([0, 2]\) which is a fixed point of \( (2.3) \) (that is, a solution of \( (2.6) \)).

(i) \( f \) is strictly positive.

(ii) If there exists \( M \in \mathbb{R} \) such that \( f' \leq M \) on \([0, \varepsilon)\) for some \( \varepsilon > 0 \) then \( f \) is decreasing.

(iii) If further, \( f \) is twice continuously differentiable and there exists \( N \in \mathbb{R} \) such that \( f'' \geq N \) then \( f \) is convex.

Proof. The first part follows immediately from Proposition 2.1. For the second part, suppose, for sake of contradiction, that \( f'(\xi) \geq 0 \) for some \( \xi \in (0, 2) \). Then, using \( (2.7) \),

\[
0 \leq \xi f'(\frac{\xi}{2}) - 4f(1 + \frac{\xi}{2})
\]

and so \( 0 < 2f(1 + \frac{\xi}{2}) \leq \frac{\xi}{2} f'(\frac{\xi}{2}) \). In particular, \( f'(\frac{\xi}{2}) > 0 \) and we can repeat the argument to gain, for every \( n \in \mathbb{N} \),

\[
0 < 2f(1 + \frac{\xi}{2^n}) \leq \frac{\xi}{2^n} f'(\frac{\xi}{2^n}).
\]

Thus \( \lim \sup_n \frac{\xi}{2^n} f'(\frac{\xi}{2^n}) \geq 2f(1) > 0 \) which is incompatible with \( f' \) being bounded above near 0, proving the claim.

Towards the statement on convexity, we may differentiate \( (2.7) \) to find for all \( x \in (0, 2) \)

\[
K(f'(x) + x f''(x)) = f'(\frac{\xi}{2}) + \frac{x}{2} f''(\frac{\xi}{2}) - 2f'(1 + \frac{x}{2})
\]

\[
= f'(\frac{\xi}{2}) - 4f(1 + \frac{\xi}{2}) + 4f(1 + \frac{x}{2}) - \frac{x}{2} f''(\frac{\xi}{2}) - 2f'(1 + \frac{x}{2})
\]

\[
K x f''(x) = 4f(1 + \frac{\xi}{2}) + \frac{x}{2} f''(\frac{\xi}{2}) - 2f'(1 + \frac{x}{2}),
\]

where \( K = 4(1 + C) \). Suppose \( f''(\xi) < 0 \) some \( \xi \in (0, 2) \). Then the above, together with positivity of \( f \) and \( -f'' \), implies \( \frac{\xi}{2} f''(\frac{\xi}{2}) < K \xi f''(\xi) < 0 \). We can repeat with \( \frac{\xi}{2} \) in place of \( \xi \) to get

\[
\frac{\xi}{4} f''(\frac{\xi}{4}) < K \frac{\xi}{2} f''(\frac{\xi}{2}) < K^2 \xi f''(\xi).
\]

Further repetition yields, \( f''(\frac{\xi}{2^n}) < (2K)^n f''(\xi) \to -\infty \) which does not allow that \( f'' \) is bounded below. \( \square \)

If \( \{f_n\} \) is a sequence of \( C^1 \) functions which converge to \( f \), also \( C^1 \), then it does not follow that \( f'_n \to f' \), even if the convergence is uniform– a common example provided is the sequence \( f_n(x) = \frac{1}{n} \sin(nx) \to 0 \) on \([0, \pi]\), which even has a uniform bound on \( \{f'_n\} \). Nevertheless, an application of the mean value theorem yields the following.

2.4 Lemma Suppose \( \{f_n\} \) is a sequence of \( C^1 \) functions which converge to \( f \), also \( C^1 \), on an interval \( I \). If \( \sup_{n,x \in I} |f'_n(x)| \leq M \) then \( \sup_{x \in I} |f'(x)| \leq M \).
2.5 Corollary Suppose \( \{f_n\} \) is a sequence of \( C^1 \) functions which converge to \( f \), also \( C^1 \), on \( I = (0, L] \). If \( \sup_{n,x \in I} |xf_n'(x)| \leq M \) then \( \sup_{x \in I} |xf'(x)| \leq M \).

Proof. Let \( g_n(x) = xf_n(x) + \int_x^L f_n(y) dy \). For \( x \in (0, L] \), we have uniform convergence of \( f_n \) to \( f \) on the compact interval \([x, L]\) and it follows \( \int_x^L f_n(y) dy \rightarrow \int_x^L f(y) dy \). Then \( g_n(x) \rightarrow g(x) := xf(x) + \int_x^L f(y) dy \) for all \( x \in (0, L] \). Since \( g_n'(x) = xf_n'(x) \) is bounded by hypothesis, and \( g'(x) = xf'(x) \), Lemma 2.4 guarantees that \( \sup_{x \in [0, L]} |xf'(x)| \) has the same bound. \( \square \)

2.6 Proposition Let \( f_0 \) be a continuously differentiable probability density on \((0, 2]\) with \( \sup_{[0,2]} |xf_0'(x)| < \infty \), and let the sequence \( \{f_s\} \) be generated by \(2.3\). If \( f_s \rightarrow f \in C^1((0, 2]) \) pointwise then

(i) \( f \) is a probability density and a solution of \((2.6)\),

(ii) \( f > 0 \),

(iii) \( f \) is decreasing,

(iv) \( \sup_{[0,2]} |tf'(t)| \) is finite,

(v) \( \lim_{t \rightarrow 0} tf'(t) \) exists in \((-2, 0)\).

Proof. \( \)

(iv) Each \( f_s \) is continuously differentiable on \((0, 2]\), and

\[
K_sxf_{s+1}'(x) = xf_s'(x) - 4f_s(1 + \frac{x}{2})
\] (2.11)

where \( K_s = 4(1 + C_s) \geq 4 \).

Since \( f_s \rightarrow f \) uniformly on the compact set \([1, 2]\), \( D := \sup_{x \in [1,2], s \in \mathbb{N}} f_s(x) < \infty \). Letting \( g_s(x) = xf_s'(x) \), equation \(2.11\) reads

\[
g_{s+1}(x) = \frac{2}{K_s}g_s(\frac{x}{2}) - \frac{4}{K_s}f_s(1 + \frac{x}{2})
\]

and

\[
|g_{s+1}(x)| \leq \frac{1}{2}|g_s(\frac{x}{2})| + D.
\]

Let \( G_s = \sup_{[0,2]} |g_s(x)| \). Then \( G_{s+1} \leq \frac{1}{2}G_s + D \) for each \( s \). By hypothesis, \( G_0 \) is finite, and it follows that every \( G_s \) is finite with

\[
G_s \leq \frac{1}{2^s}G_0 + (1 + \frac{1}{2} + \cdots + \frac{1}{2^{s-1}})D \leq G_0 + 2D.
\]

That is, there exists \( M > 0 \), such that for all \( s \in \mathbb{N} \), \( \sup_{[0,2]} |xf_s'(x)| < M \) and it follows from \(2.5\) that \( \sup_{[0,2]} |xf'(x)| \leq M < \infty \).
(i) From the estimates above, \( \int_{a}^{b} |f_s(x)| \, dx \leq |f_{b}^{b} \, dx| \leq M|\log(b/a)| \). Then \( f_s(a) \leq f_s(b) + M|\log(b/a)| \). For each \( s \), we may choose \( b \in [1,2) \) with \( f_s(b) \leq 1 \) so that \( f_s(a) \leq 1 + M\log(2/a) \) for all \( a \in (0,1] \). Then for \( 0 < \delta \leq 1 \),
\[
\int_{0}^{\delta} f_s(x) \, dx \leq \delta(1 + M(1 + \log 2)) + M\delta \log \delta^{-1}.
\]
Hence, as \( \delta \to 0 \), \( \int_{0}^{\delta} f_s \to 0 \), and \( \int_{\delta}^{2} f_s(x) \, dx \to 1 \) uniformly in \( s \).

Since \( f_s \to f \) uniformly on \([1,2] \), we have convergence of the integrals \( C_s = \int_{1}^{2} f_s \) to \( C = \int_{1}^{2} f \), and of \( \int_{1+\delta/2}^{2} \frac{f_s(y)}{y} \, dy \to \int_{1+\delta/2}^{2} \frac{f(y)}{y} \, dy \) for every \( x > 0 \). Thus we take limits in \( (2.3) \) to conclude \( f \) satisfies \( (2.6) \). It is clear that \( f \) is non-negative valued and on considering compact sub-intervals \([\delta,2]\), we also have that \( \int_{0}^{s} f = \lim_{\delta \to 0} \int_{\delta}^{2} f = \lim_{\delta \to 0} \int_{0}^{s} f = 1 \).

(ii) follows from (i) and Proposition 2.3.

(vi) (iii) Letting \( L = \lim \sup_{x \to 0^+} xf'(x) \), equation \( (2.7) \) implies \( 4(1 + C)L = 2L - 4f(1) \). From (iv), we find \( L \) is finite and from (ii) \( f(1) > 0 \) so \( L(4C + 2) = -4f(1) < 0 \) and \( L = \frac{4f(1)}{4C + 2} < 0 \). Notice the same quantity arises on taking the limit infimum (which is also finite from (iv)), and thus \( \lim_{x \to 0^+} xf'(x) = L = -\frac{4f(1)}{4C + 2} \in (-\infty,0) \). This implies that \( f' \) is negative in a neighborhood of 0 and so from Proposition 2.3 (i). \( f \) is decreasing on \((0,2]\). Moreover, since \( f \) is decreasing and \( (2.10) \) \( f(2) \geq f(1)/4, 1 = \int_{0}^{2} f \geq f(1) + f(2) \geq \frac{5}{4}f(1) \). Hence \( f(1) \leq 4/5 \) and \( L = -\frac{4f(1)}{4C + 2} \geq -\frac{8}{5} \).

Later (Corollary 5.9), we prove the existence of a probability density \( f^* \) on \((0,2]\) of the form \( f^*(t) = \ell^* \log(t) + \sum_{k=0}^{\infty} a_k^*(t - 1)^k \) which is a solution of \( (2.6) \). The coefficient vector \((\ell^*, a_0^*, a_1^*, \ldots) \) is in \( \ell_1^* \) for \( 1 \leq \rho \leq 3 \) (see Definition 3.3). This means \( f^* \) is a \( C^\infty \) function on \((0,2]\). Moreover, \( f^* \) is reached by iterating \( (2.6) \) from any initial probability density \( f_0 = \ell \log(t) + \sum_{k=0}^{\infty} a_k(t - 1)^k \) with \( (\ell, (a_k)) \in \ell_1 \), for example, from \( f_0(t) \equiv \frac{1}{2} \). Consequently, \( f^* \) satisfies the conclusions of Proposition 2.6. In particular, we have from Proposition 2.6 (v) that \( \ell^* = \lim_{t \to 0} tf''(t) < 0 \) which, in turn, guarantees that \( f'' \) has a finite infimum over \((0,2]\), so by Proposition 2.3 (iii), \( f^* \) is also convex.

3 Linearisation

We write successive iterations of an initial probability density \( f_0 \) on \((0,2]\) as
\[
f_s(t) = \ell_s \log t + \sum_{k=0}^{\infty} a_{k,s}(t - 1)^k
\]
with the desired limit function written as
\[
f(t) = \ell \log t + \sum_{k=0}^{\infty} a_k(t - 1)^k. \tag{3.1}
\]
We will identify \( f \) with the sequence of coefficients \((\ell_s, a_0, a_1, s, \ldots)\).
Substituting the series representation of \( f \) into (2.6) we have

\[
\int \frac{f(\tau)}{\tau - 1} d\tau = \ell \int \frac{\log(1 + (\tau - 1))}{\tau - 1} d\tau + \sum_{k \geq 1} a_k (\tau - 1)^{k-1} d\tau + \int \frac{a_0}{\tau - 1} d\tau
\]

\[
= \ell \int \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} (\tau - 1)^{k-1} d\tau + a_0 \log(\tau - 1) + \sum_{k \geq 1} a_k (\tau - 1)^k
\]

and

\[
\int_{1+\frac{t}{2}}^{2} \frac{f(\tau)}{\tau - 1} d\tau = \ell \left( \sum_{k \geq 1} \frac{(-1)^{k+1} k}{k^2 2^k} - \sum_{k \geq 1} \frac{(-1)^{k+1} k^2}{k^2 2^k} \right)
\]

\[
- a_0 \log \frac{t}{2} + \sum_{k \geq 1} \frac{a_k}{k} - \sum_{k \geq 1} \frac{a_k k^2}{k^2 2^k}.
\]

Since \( \sum_{k \geq 1} \frac{(-1)^{k+1} k}{k^2} = \pi^2 \) and \( t^k = (1 + (t - 1))^k = \sum_{j=0}^{k} \frac{k}{j} (t - 1)^j \) we gain

\[
\int_{1+\frac{t}{2}}^{2} \frac{f(\tau)}{\tau - 1} d\tau = \ell \left( \frac{\pi^2}{12} + \sum_{k \geq 1} \frac{(-1)^k}{2^k k^2} \sum_{j=0}^{k} \frac{k}{j} (t - 1)^j \right) - a_0 \log \frac{t}{2} + \sum_{k \geq 1} \frac{a_k}{k} - \sum_{k \geq 1} \frac{a_k k^2}{k^2 2^k} \sum_{j=0}^{k} \frac{k}{j} (t - 1)^j.
\]

We also find

\[
\frac{1}{2} f\left(\frac{t}{2}\right) = \ell \left( \frac{\pi^2}{12} + \sum_{k \geq 0} \frac{(-1)^k}{2^k k^2} \sum_{j=0}^{k} \frac{k}{j} (t - 1)^j \right)
\]

and thus (2.6) becomes

\[
(1 + C) \left( \ell \log t + \sum_{k \geq 0} a_k (t - 1)^k \right) = \left( \ell - a_0 \right) \log \frac{t}{2} + \sum_{k \geq 0} \frac{a_k}{2^k + 1} \sum_{j=0}^{k} \frac{k}{j} (t - 1)^j (-1)^{k-j}
\]

\[
+ \ell \left( \frac{\pi^2}{12} + \sum_{k \geq 0} \frac{(-1)^k}{2^k k^2} \sum_{j=0}^{k} \frac{k}{j} (t - 1)^j \right)
\]

\[
+ \sum_{k \geq 1} \frac{a_k}{k} - \sum_{k \geq 1} \frac{a_k k^2}{k^2 2^k} \sum_{j=0}^{k} \frac{k}{j} (t - 1)^j.
\]

Equating the coefficients of \( \log t \) gives \((1 + C)\ell = \ell - a_0\) while the constant terms yield

\[
(1 + C) a_0 = \left( \ell - a_0 \right) (-\log 2) + \sum_{k \geq 0} (-1)^k \frac{a_k}{2^k + 1} + \ell \left( \frac{\pi^2}{12} + \sum_{k \geq 1} \frac{(-1)^k}{k^2 2^k} + \sum_{k \geq 1} \frac{a_k}{k} (1 - 2^{-k}) \right).
\]
The sum $\sum_{k \geq 1} \frac{(-1)^k}{k^{2k^2}}$ equals $\text{Li}_2(\frac{3}{2})$, approximately $-0.4484$, where $\text{Li}_2$ denotes the dilogarithm. Finally, the coefficient of $(t - 1)^r$, $r \geq 1$, gives

$$(1 + C)a_r = \sum_{k \geq r} \frac{a_k}{2k+1} \binom{k}{r} (-1)^{k-r} + \theta \sum_{k \geq r} \binom{k}{r} \frac{(-1)^k}{2k^{2r}} - \sum_{k \geq r} \binom{k}{r} \frac{a_k}{k^{2r}}.$$ 

The sum $\sum_{k=r}^{\infty} \frac{(-1)^k(k^r)}{k^{2r}}$ is numerically slow to estimate but can be simplified.

### 3.1 Lemma

Let $x \in [0, \frac{1}{2})$. Then

$$\sum_{k=r}^{\infty} \frac{(-1)^k(k^r)}{k^{2r}} \left( \frac{x}{1-x} \right)^k = \frac{(-1)^r}{r} \sum_{k=r}^{\infty} \frac{1}{k^r} x^k.$$ 

In particular, taking $x = \frac{1}{3}$,

$$\sum_{k=r}^{\infty} \frac{(-1)^k(k^r)}{k^{2r} 2^k} = \frac{(-1)^r}{r} \sum_{k=r}^{\infty} \frac{1}{k^r} 3^k.$$ 

**Proof.** First, we use the expansion $\log(1 - x) = -\sum_{k=1}^{\infty} \frac{1}{k} x^k$ to write

$$\sum_{k=1}^{\infty} \frac{1}{k} x^k = \log \frac{1}{1-x} = \log(1 - \frac{-x}{1-x}) = -\sum_{k=1}^{\infty} \frac{(-1)^k(k^r)}{k^r} \left( \frac{x}{1-x} \right)^k.$$ 

This gives the desired identity when $r = 1$ and we proceed by induction; assuming the statement is true for $r$ we have,

$$\sum_{k=r}^{\infty} \frac{x^k}{k} = (-1)^r r \sum_{k=r}^{\infty} \frac{(-1)^k(k^r)}{k^{2r}} \left( \frac{x}{1-x} \right)^k.$$ 

Divide across by $x^r$, differentiate, and multiply by $x^{r+1}$ to gain

$$\sum_{k=r+1}^{\infty} \frac{x^k}{k} - \sum_{k=r+1}^{\infty} \frac{x^k}{k} = (-1)^r \sum_{k=r}^{\infty} \frac{(-1)^k(k^r)}{k^{2r}} \left( \frac{x^{k+1}}{(1-x)^{k+1}} + \frac{(k-r)x^k}{(1-x)^k} \right)$$

which gives

$$\frac{x^{r+1}}{1-x} - r \sum_{k=r+1}^{\infty} \frac{x^k}{k} = (-1)^r \sum_{k-1=r-1}^{\infty} (-1)^k(k-1) \left( \frac{x}{1-x} \right)^{k+1} + (-1)^r r \sum_{k=r+1}^{\infty} \frac{(-1)^k(k^r)(k-r)}{k^{2r}} \left( \frac{x}{1-x} \right)^k.$$
The first term on the right hand side can be simplified using $\frac{1}{(1-t)^{s+1}} = \sum_{k=s}^{\infty} \binom{k}{s} t^{k-s}$ leading to

$$
\frac{x^{r+1}}{1-x} - r \sum_{k=r+1}^{\infty} \frac{x^k}{k} = \left( \frac{x}{1-x} \right)^{r+1} + \frac{1}{k^2} \sum_{k=r+1}^{\infty} (-1)^k \binom{k}{r+1} \left( \frac{x}{1-x} \right)^k.
$$

Removing the common term and dividing by $-r$ gives the statement for $r+1$ as required.

The calculations above govern the relationship between the coefficients of steady state solution $f$, but the corresponding equations relating the coefficients of $f_{s+1}$ to those of $f_s$ amount to only a notational change and are summarised in the following infinite system.

\[
(1+C_s) \begin{bmatrix}
\ell_{s+1} \\
a_0_{s+1} \\
a_{1,s+1} \\
\vdots 
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} & -1 & 0 & 0 & 0 & \ldots \\
0.027 & 1.193 & \cdots & \frac{1}{k} + \frac{(-1)^k}{k^{s+1}} - \frac{1}{k^2^{s+1}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{(-1)^k}{k} \sum_{j=r}^{\infty} \frac{1}{j^{3k}} & 0 & \left( \frac{k}{r} \right) \left( \frac{(-1)^k}{2k^{s+1}} - \frac{1}{k^{2k}} \right) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots 
\end{bmatrix} \begin{bmatrix}
\ell_s \\
a_0_s \\
a_{1,s} \\
\vdots 
\end{bmatrix}
\]

where $k, r \geq 1$ and $\binom{k}{r} = 0$ for $k < r$. The constants 0.027 and 1.193 are just numerical placeholders for the true values $\frac{\pi^2}{12} + \text{Li}_2(\frac{3}{2}) - \frac{\log^2 2}{2}$ and $\frac{1}{2} + \log(2)$ respectively. We have used $r$ and $k$ above to index rows and columns respectively, but these indices are given relative to the sub-matrices depicted. This matrix of coefficients will hereafter be denoted by $A$ and its $(n+1) \times (n+1)$ principal submatrix by $A_n$. For example, $A_4$ is given by

\[
\begin{bmatrix}
0.5 & -1.0 & 0.0 & 0.0 & 0.0 \\
0.02747926 & 1.1931472 & 0.25000000 & 0.50000000 & 0.22916667 \\
-0.40546510 & 0.0 & -0.25000000 & -0.50000000 & 0.062500000 \\
0.036065890 & 0.0 & 0.0 & 0.0 & -0.31250000 \\
-0.0055254100 & 0.0 & 0.0 & 0.0 & 0.020833333
\end{bmatrix}
\]

Where necessary, $A_n$ may be padded by zeroes so that expressions such as $A - A_n$ and $A_n - A_m$ are well-defined.

### 3.1 Spectrum

Let us make some observations on $A$. As an operator on $\ell^\infty$, $A$ is not bounded since the absolute row sum of the second row majorises the harmonic series. $A$ is however bounded on $\ell_1$, with $A_1 \simeq 2.193$, as provided by the absolute second column sum. Indeed, for $k \geq 1$, the absolute sum of column $k$ is convergent to $\frac{1}{2}$ and bounded above by 1, while the absolute sum of the only infinite column is bounded by 1.08, as provided by the first of the following estimates.
3.2 Lemma

\[ (i) \quad \sum_{r=1}^{\infty} \frac{1}{r} \sum_{j=1}^{\infty} \frac{1}{j^3} < 0.55, \tag{3.3} \]

\[ (ii) \quad \sum_{r=m}^{\infty} \frac{1}{r} \sum_{j=1}^{\infty} \frac{1}{j^3} < \frac{9}{4(m^2m)}. \tag{3.4} \]

Proof. As \( \sum_{j=1}^{\infty} \frac{1}{j^3} \), the first double sum is bounded by \( \frac{3}{2} \sum_{r=1}^{\infty} \frac{1}{r^23^r} = \frac{3}{2} \text{Li}_2(2/3) \approx 0.5493. \) The second double sum is similarly bounded by \( \frac{3}{2} \sum_{r=m}^{\infty} \frac{1}{m^23^r} = \frac{9}{4(m^2m)}. \)

While \( A \) is bounded on \( \ell_1 \), it is apparent that \( A_n - A_1 \to \frac{1}{2} \), that is, \( A \) is not approximated by its finite rank truncations. For this reason, we will consider \( A \) acting on the scaled \( \ell^1 \) space, \( \ell^1_\rho. \)

3.3 Definition For \( \rho \geq 1, \ell^1_\rho = \{(x_n)_n : (x_n)_{n1,\rho} = \sum \rho^n|x_n| < \infty\}. \)

If \( D_\rho \) is the diagonal matrix \( \text{diag}(1, \rho, \rho^2, \ldots) \) then \( A^\rho := D_\rho AD_\rho^{-1} \), that is, \( (A^\rho)_{i,j} = (A)_{i,j} \rho^{|i-j|}. \) The norm of \( A \) as a bounded linear operator on \( \ell^1_\rho \) is given by \( A_{\ell^1_\rho} = A^\rho. \)

In particular, \( A \in \mathcal{L}(\ell^1_\rho) \) for \( 1 \leq \rho \leq 3 \) since the absolute column sums of \( A^\rho \) converge to zero and the first column sum is finite for \( \rho \leq 3 \). Moreover, \( A^\rho - A^\rho_{n1} = A - A_n \ell^1_\rho \to 0 \) which implies that \( A \) is approximated by its finite rank truncations on \( \ell^1_\rho \) for \( 1 < \rho \leq 3 \) and so is a compact operator there. Thus, its spectrum on \( \ell^1_\rho, 1 < \rho \leq 3 \) consists only of non-zero eigenvalues with a possible cluster point at zero. Moreover, we have convergence of the spectrum of \( A_n \) to that of \( A \) (see for example \cite{Xi} XI-9.5). Of particular interest is the eigenvalue corresponding to the spectral radius (we will see in due course that there is only one such, and it is positive) and the corresponding left and right eigenvectors. Our immediate objective is to estimate these to within acceptable tolerances.

3.2 Similarity Transforms

The first step is to find a similarity transform whose action on \( A \) yields a matrix for which the application of Gershgorin’s Theorem gives a good bound for the maximum eigenvalue. This is done by choosing approximate row and column eigenvectors for \( A_m. \) We express these here as \( \hat{w} \) and \( \hat{v}. \) Let the standard unit column and row vectors be denoted, respectively, by \( e_i \) and \( \hat{e}_i, \) \( i = 0, \ldots, m. \) For a similarity transform we require that \( \hat{w}v = 1 \) and the \( j^{th} \) entry of \( v \) satisfies \( v_j = \hat{e}_jv = 1. \) The identity operator \( I \) and the operator \( f^{(j)} \) which has the projection \( e_j\hat{e}_j \) removed are given by

\[ I := \sum_{i=0}^{m} e_i\hat{e}_i, \quad f^{(j)} := \sum_{i\neq j} e_i\hat{e}_i. \tag{3.5} \]
We use the similar notation $v^{(i)}$ and $\hat{w}^{(i)}$ to denote the vectors with the $j^{th}$ entry set to 0: $\hat{w}^{(i)} = \hat{w} - v_j, I^{(i)} v = v^{(i)}$. For fixed $j$, $0 \leq j \leq m$, we define two matrices

$$W := e_j \hat{w} - v_j e_j + I = e_j \hat{w} - v^{(j)} e_j + I^{(j)}, \quad V := v(e_j - \hat{w}^{(j)}) + I^{(j)}. \quad (3.6)$$

3.4 Lemma For fixed $j$, let $e_j v = 1$ and $\hat{w} v = 1$. Then $W, V$ as given by (3.6) satisfy $V W = W V = I$. If, in addition, $A v = \lambda v$ and $\hat{w} A = \lambda \hat{w}$, then the entries of $W A V$ are given by

$$\left\{ \begin{array}{ll}
(W A V)_{ii} = \lambda & \text{if } i \neq j; \\
(W A V)_{ij} = (W A V)_{ji} = 0 & \text{otherwise.} 
\end{array} \right. \quad (3.7)$$

Proof. $(e_j \hat{w}) V = e_j (e_j - \hat{w}^{(j)}) + e_j \hat{w}^{(j)}$, $(-v_j e_j) V = (-v_j) v(e_j - \hat{w}^{(j)}) + I^{(j)}$. Also $I V = v(e_j - \hat{w}^{(j)}) + I^{(j)}$. Collecting terms using $v_j = 1$ we find $W V = e_j e_j + I^{(j)} = I$. A similar argument shows $V W = I$.

For the second part, $A V = \lambda v e_j + I^{(j)}$. Then using $\hat{w} v = e_j v = 1$, $$(e_j \hat{w} - v^j e_j + I) \lambda v e_j = (e_j - v + v)(e_j - \hat{w}^{(j)}) = \lambda (e_j e_j - e_j \hat{w}^{(j)}). \quad (3.8)$$

$$(e_j \hat{w} - v^j e_j + I^{(j)} A I^{(j)}) \lambda v e_j = \lambda (e_j \hat{w}^{(j)} - v^{(j)} e_j A I^{(j)}) + I^{(j)} A I^{(j)}. \quad (3.9)$$

$$W A V = \lambda (e_j e_j) + I^{(j)} A I^{(j)} - v^{(j)} e_j A I^{(j)}. \quad (3.10)$$

Similarity transforms as above are used to give bounds on the maximum eigenvalue. In one case we estimate $\hat{w}$ and $v$ numerically. For a similarity transform we need to show that the crucial requirements of $\hat{w} v = 1$ and $e_j v = 1$ are satisfied. The $j^{th}$ entry in $v$ can be set to 1, but a numerical computation yielding $\hat{w} v = 1$ does not necessarily mean $\hat{w} v = 1$. However, when the numerical computation is done with bounds, one can show that there are values close to those given so that this sum of products yields exactly 1.

3.5 Lemma Let

$$1 - \alpha \leq \sum_{0}^{m} v_i w_i \leq 1 + \alpha, \quad (3.11)$$

where $0 < \alpha < 1/2$. With $\lambda := \alpha/(1 - \alpha)$ there exist $\{w^*_i : i = 0, \ldots, m\}$ so that

$$\sum_{0}^{m} v_i w^*_i = 1, \quad |w_i - w^*_i| \leq \lambda |w_i|, \quad i = 0, \ldots, m. \quad (3.12)$$

Proof. Let $\sum_{0}^{m} v_i w_i$ denote the sum including exactly the terms with $v_i w_i > 0$. Then $\sum_{0}^{m} v_i w_i \geq 1 - \alpha$. Define $s_i := (1 - \lambda) w_i, t_i := (1 + \lambda) w_i$ for those $i$’s with $v_i w_i > 0; s_i = t_i = w_i$ otherwise. Now

$$\sum_{0}^{m} \lambda v_i w_i \geq \alpha \implies \sum_{0}^{m} v_i s_i \leq \sum_{0}^{m} v_i w_i - \alpha \leq 1 + \alpha - \alpha = 1. \quad (3.13)$$

Also

$$\sum_{0}^{m} v_i t_i \geq \sum_{0}^{m} v_i w_i + \alpha \implies \sum_{0}^{m} v_i t_i \geq 1 - \alpha + \alpha = 1. \quad (3.14)$$

This implies there exist $\{w^*_i : |s_i w_i| \leq |w^*_i| \leq |t_i w_i|, i = 0, \ldots, m\}$ so that the sum is unity. \(\square\)
4 Numerical Approximation

The behaviour of \( f(t) = \ell \log(x) + \sum_{i=1}^{\infty} a_i (t - 1)^k \) under the parking and scaling procedure corresponds to the action of the matrix \( A \) on the coefficients \( \{\ell, a_0, a_1, \ldots\} \). We study this behaviour using the largest eigenvalue of \( A \) together with the right and left eigenvalue pair \((3.6)\). We work with finite dimensional approximates \( A_m \) of \( A \), and many of the entries of these are also approximate. Numerical techniques are used to get estimates of the principal eigenvalue and the corresponding eigenvectors. We also need error bounds on our approximations to use in connection with Gershgorin's Theorem to give bounds for the principal eigenvalue of \( A_m \).

A significant part of the proof depends on numerical calculation using floating point arithmetic. The nonzero real number \( x \) has the binary floating point representation given in terms of a sign bit, an exponent \( k \) so that \( 2^{k-1} \leq |x| < 2^k \) and a \( b \)-bit integer significand \( m \): \( 2^{-k}|x| \) is approximated by \( m2^b \), \( 2^{b-1} \leq m < 2^b \). The sign of \( x \) is stored in the sign bit of the floating point representation. The specific floating point format specifies \( b \) and the interval \([k_{\min}, k_{\max}]\) of allowed values of \( k \). Numbers with absolute value greater than \( 2^{k_{\max}} \) cannot be represented in this format. Zero has a special representation; numbers with absolute value less than \( 2^{k_{\min}} \) are often converted to zero. A nonzero calculation which yields a result with absolute value less than \( 2^{k_{\min}} \) is called exponential underflow; a calculation exceeding the maximum exponent is called exponential overflow. Problems with exponential overflow and underflow do not arise in the calculations used here, but the limited accuracy of floating point arithmetic does.

We use \([x]\) to denote the floating point representation of \( x \), which we assume exists in the format specified above. \(|[x]|\) is selected so that \(||[x]| - |x|\|\) is minimal. If \( 2^{k-1} \leq |x| \leq 2^k \), hence \( 2^{b-1} \leq 2^{b-k}|x| \leq 2^b \), there exists an integer \( m, 2^{b-1} \leq m \leq 2^b \), so that

\[
|2^{b-k}|x| - m| \leq 1/2 \Rightarrow |2^{-k}|x| - m2^{-b}| \leq 2^{b-1} \Rightarrow \frac{|[x] - |x||}{|x|} \leq 2^{-b}, \tag{4.1}
\]

since \( 2^{-k}|x| \geq 1/2 \). A special case is when the \( m = 2^b \), which will yield the exponent \( k + 1 \) and significand \( 2^{b-1} \) with the relative error is close to \( 2^{b-1} \). Thus the \( b \)-bit significand floating point representation of \( x \) has a relative deviation of at most \( 2^{-b}|x| \) and there may be two distinct floating point representations of \( x \) satisfying this relation.

Let \([x],[y]\) be floating point representations and \( z := [x] \cdot [y] \) be the exact result one of the operations \(+, \times \) or \( \div \). Assume \( z \) is defined, nonzero and within the range of the floating point format. The value \( z \) is not necessarily expressible using \( b \)-significand bits, but one can select \([z]\) so that \(|z - [z]|/|z| \leq 2^{-b}\). Well designed floating point routines, e.g. \([2]\), have this property. This requires an intermediate result with greater than \( b \)-bit accuracy. In the absence of exponent underflow, the result zero arises only if at least one of the terms in \( x \cdot y \) is zero or the operation is additive with summands equal in magnitude and opposite in sign: if \( x \cdot y \) yields zero, no rounding is needed.

In numerical computations used in the proofs, Intel extended floating point operations with \( b = 64 \) is used. For basic calculation round to nearest as discussed above is employed. However we also use other rounding settings to get bounds for the calculation. The setting round up yields a value greater than or equal to \( x \cdot y \); round down yields a value less than or equal to \( x \cdot y \). We use a simplified interval arithmetic with intervals of the form \( x \pm x_\varepsilon \) and \( y \pm y_\varepsilon \), and compute \( z, z_r \), for the operation \( x \cdot y \), where \( \cdot \) is one of \(+, \times, \div \). We use rounding settings...
to compute $z_r$ from $(x, x_r)$ and $(y, y_r)$ so that if $x - x_r \leq s \leq x + x_r$ and $y - y_r \leq t \leq y + y_r$, then $z - z_r \leq s \cdot t \leq z + z_r$. The base value of $x \cdot y$ is computed using round to nearest (in the case of equidistant nearest values, the even significand is chosen). We then compute the bound $z_r$ using round up and round down.

The use of interval arithmetic also allows us to bound the maxima and minima of an arithmetic formula over an interval by dividing the interval into subintervals and searching, using interval arithmetic on the subintervals.

5 Numerical Bounds for the Largest Eigenvalue

The discussion below uses Gershgorin’s Theorem and similarity transforms involving numerical estimates. It is not sufficient to compute the transform: we need good bounds on the calculations. Thus the computations are done with interval arithmetic. One step involves computing with bounds the transform of the $8 \times 8$ matrix $A_7$ which transforms the log term $\ell$ and the polynomial coefficients $a_0, \ldots, a_6$. This is used together with theoretical bounds involving $A_m, m > 7,$ to get an interval containing the maximum eigenvalue. We obtain bounds for the right and left eigenvectors for $m > 7$. Standard methods are available for solving linear equations and eigenvalue problems, but usually these do not provide error estimates. We have combined standard methods with routines for simplified interval arithmetic to supply the needed bounds.

Our program is written for the Free Pascal (http://www.freepascal.org) compiler for the Intel x86 or x86-64 processor. Our source code is available at [http://maths.ucd.ie/renyi](http://maths.ucd.ie/renyi). This file also contains an estimate of $R$ to approximately 1000 digits.

The finite dimensional approximation $A_m, m > 4,$ has a single eigenvalue $\lambda_m$ of maximum magnitude; other points of the spectrum have absolute value less than 0.5. The value of $\lambda_m$ for $m > 15$ changes by less than $10^{-10}$. Computed values of $\lambda_m$ strongly suggest convergence. Gershgorin’s Theorem provides a direct proof.

Theorem 5.1 Let $A = \{a_{ij} : i, j = 0, \ldots, m\}$ be a matrix with complex entries. Each eigenvalue of $A$ is contained in one of the discs

$$\{z \in \mathbb{C} : |z - a_{ij}| \leq \sum_{i \neq j} |a_{ij}| \}, \ j = 0, \ldots, m. \tag{5.1}$$

If the union of the discs about $j_1, \ldots, j_k$ has an empty intersection with all the remaining discs, this union contains exactly $k$ eigenvalues, counting multiplicity.

See [7] for details.

In the discussion below, the term disc means Gershgorin disc. Two matrices related by a similarity transform have the same eigenvalues and the eigenvectors are related by similarity. We use the above theorem in connection with some similarity transforms to deduce bounds. The similarity transforms we use are based on the right and left eigenvectors associated with the maximum eigenvalue of $A_m$ for various $m$. For simplicity of notation the subscript $m$ is often omitted, when it can be inferred from context. We write $v_\lambda$ and $w_\lambda$ for these eigenvectors. We initially require $(v_\lambda)_0 = (w_\lambda)_0 = 1$, but for use with (3.6), $(v_\lambda)_0 = 1$, but $w_\lambda$ is scaled so that $\hat{w}_\lambda v_\lambda = 1$. Define $\hat{A}$ with

$$\hat{A}_{ij} = A_{ij}, \ i, j = 1, 2, 3, \ldots, \tag{5.2}$$
\( \hat{A} \) is the matrix \( A \) with row and column 0 omitted. We denote the associated truncated eigenvectors with the 0 entries removed by \( \hat{v}_\lambda \) and \( \hat{w}_\lambda \). The eigenvalue equations \( Av_\lambda = \lambda v_\lambda, \hat{w}_\lambda A = \lambda \hat{w}_\lambda \) yield the linear equations

\[
(\lambda - \hat{A})\hat{v}_\lambda = \hat{A}_{*0}; \quad \hat{w}_\lambda (\lambda - \hat{A}) = \hat{A}_{0*},
\]

(5.3)

where \( \hat{A}_{*0} \) and \( \hat{A}_{0*} \) denote the column and row of \( A \) indexed by 0 with the entries indexed by zero removed. We use the \( \ell^1 \) norm on column vectors and the \( \ell^\infty \) norm on row vectors:

\[
v = \sum_i |v_i|; \quad w_\infty = \max_i \{|w_i|\};
\]

(5.4)

\[
v_\lambda = \hat{v}_\lambda + 1; \quad w_{\lambda\infty} = \max \{1, \hat{w}_{\lambda\infty}\}.
\]

(5.5)

The associated matrix norm, \( \hat{A} \), is the maximum \( \ell^1 \) norm of the columns of \( \hat{A} \). Note that \( \hat{A} \) is upper triangular. The absolute sum of column \( k + 1 \) is bounded by

\[
\frac{1}{k} + 2^{k-1}(1 + \frac{1}{k})\sum_{j=0}^{k} \binom{k}{j} \leq 1
\]

(5.6)

for \( k \geq 3 \). Checking the columns for \( k < 3 \) shows that the supremum of the absolute column sums of \( \hat{A} \) occurs at the first column: \( \hat{A} = (A)_{1,1} = 0.5 + \log 2 \). For \( \lambda > (A)_{1,1} \), which is the case for the maximum eigenvalue, we have

\[
(\lambda - \hat{A})^{-1} = \sum_{0}^{\infty} \lambda^{-k-1}(A)^k; \quad (\lambda - \hat{A})^{-1} \leq \frac{1}{\lambda - A_{1,1}}.
\]

(5.7)

In the case of \( A_m \), \( \hat{A} \) is indexed by \( 1 \ldots m \times 1 \ldots m \) and \( \lambda = \lambda_m \), the maximum eigenvalue of \( A_m \). Because \( \lambda - \hat{A} \) is non-singular, \( Av_\lambda = \lambda v_\lambda \) with \( v_0 = 0 \) implies \( v = 0 \). Thus, if \( Av_\lambda = \lambda v_\lambda \) with \( v_\lambda \neq 0 \), we can scale \( v_\lambda \) to make \( (v_\lambda)_0 = 1 \). The inverse of \( (\lambda - \hat{A}) \) can be computed by the power series above or by using the upper triangular property of \( \hat{A} \). One can use back substitution to solve for \( \hat{v}_\lambda \). The upper triangular forward substitution for \( \hat{w}_\lambda \) depends on \( \lambda \) but the evaluation of \( (\hat{w}_\lambda)_k \) depends only on \( \lambda \) and \( A_{i,j} : i, j \leq k \).

5.2 Lemma Let \( a_i = i \) for \( i > 1 \), \( a_1 = 0 \), \( a_0 = 1 \). Then \((-1)^{1+a_i}(\hat{v}_\lambda)_i > 0 \) for each integer \( i \geq 0 \).

Proof. Define the matrix \( B \) by

\[
B_{ij} := (-1)^{a_i-a_j}A_{ij}.
\]

(5.8)

Then all the entries \( B_{ij} \) for \( i \neq 1 \) except \( B_{2,2} \) are nonnegative. Define \( \hat{B} \) as \( \hat{A} \) in (5.2) except that row and column 1 are removed. Then all entries of \( \hat{B} \) are nonnegative except the term coming from \( B_{2,2} = -1/4 \). Let \( Bz = \lambda z \) with \( z_1 = 1 \). Let \( z \) be \( z \) with the 1 entry removed. \( \hat{B}_{*1} \) has a single nonzero entry \( \hat{B}_{0,1} = 1 \). As in (5.6) one can show that \( \hat{B} < 1 \), then

\[
\hat{z} = \sum_{0}^{\infty} (\lambda + 1)^{-k-1}(\hat{B} + I)^k(\hat{B}_{*1}),
\]

(5.9)
so the entries of \( \mathbf{z} \) are strictly positive including \( z_1 = 1 \). We have \( v_0 = 1 \) by definition so \( v_j = (-1)^{j-1} z_j / z_0 \) with \( z_0 = 1 \) from the similarity transform relating \( \mathbf{A} \) and \( \mathbf{B} \), which uses the self-inverse diagonal matrix with diagonal entries \((-1)^{j-1}\).

\[ \Box \]

5.3 Lemma For \( m \geq 7 \), if \( \lambda_m \) satisfies \( 1.232 \leq \lambda_m \leq 1.234 \), then for \( \mathbf{A}_m v_{\lambda_m} = \lambda_m v_{\lambda_m} \) with \( v_0 = 1 \)

\[ 0.23 \times \sum_{i=0}^{m} |v_i| < 0.70, \quad -0.76 < v_i < -0.72. \]  \( (5.10) \)

Proof. We consider \( \mathbf{A}_m \) for some \( m \geq 7 \). Let \( \mathbf{z} \) be a row vector with \( |(\mathbf{z})_i| \leq 1 \). Then

\[ \mathbf{z} \mathbf{v} = (\mathbf{z}_0)(\mathbf{v})_0 + \mathbf{z}_0 \mathbf{v} = \mathbf{v} \sum_{i=0}^{m} \left( \mathbf{z} \left( \lambda - \mathbf{A} \right)^{-1} \mathbf{A} \right) \mathbf{v} \]

With \( (\mathbf{A} \mathbf{v}_0)_i = \mathbf{A}_1 \) for \( 1 \leq j \leq 7 \) and \( (\mathbf{A} \mathbf{v}_0)_i = 0 \) for \( i > 7 \), we have

\[ \left| \left( \mathbf{z} \left( \lambda - \mathbf{A} \right)^{-1} \mathbf{A} \right) \mathbf{v} - \left( \mathbf{z} \left( \lambda - \mathbf{A} \right)^{-1} \mathbf{A} \right) \mathbf{v} \right| \leq \left| \left( \mathbf{z} \left( \lambda - \mathbf{A} \right)^{-1} \mathbf{A} \right) \mathbf{v} \right| \]

First we select \( \mathbf{z} \) with \( \mathbf{z}_0 = 1, \mathbf{z}_1 = -1 \) and for \( i > 1 \) \((\mathbf{z})_i = (-1)^{i+1}\) so that \( |(\mathbf{z})_i| = 1 \) and \((\mathbf{z})_i \mathbf{v}_0 > 0 \) because the signs of \((\mathbf{z})_i \mathbf{v}_0\) alternate. We compute using interval arithmetic that 2.032 and 2.072 are bounds for the minimum and maximum of \( \left( \mathbf{z} \left( \lambda - \mathbf{A} \right)^{-1} \mathbf{A} \right) \mathbf{v} \) over \( 1.232 \leq \lambda \leq 1.234 \). Next from \( 3.4 \)

\[ \left| \left( \mathbf{z} \left( \lambda - \mathbf{A} \right)^{-1} \mathbf{A} \right) \mathbf{v} \right| \leq \frac{\mathbf{A} \mathbf{v}_0 - \mathbf{A} \mathbf{v}_0}{2.132 - 0.5 - \log(2)} \leq \frac{2.25 \times 3^{-7}}{49 \times 0.03885} \leq 0.00055, \]

from which follow the first pair of inequalities in \( 5.10 \). For the second pair of inequalities we set \( \mathbf{z}_i = 0 \) for \( i \neq 1, \mathbf{z}_1 = 1 \). Then as above

\[ \left| \left( \mathbf{z} \left( \lambda - \mathbf{A} \right)^{-1} \mathbf{A} \right) \mathbf{v} \right| \leq \frac{\mathbf{A} \mathbf{v}_0 - \mathbf{A} \mathbf{v}_0}{2.132 - 0.5 - \log(2)} \leq \frac{2.25 \times 3^{-7}}{49 \times 0.03885} \leq 0.00055. \]

We compute using interval arithmetic the values \(-0.711\) and \(-0.751\) as upper and lower bounds for \( \left( \mathbf{z} \left( \lambda - \mathbf{A} \right)^{-1} \mathbf{A} \right) \mathbf{v} \) over \( 1.232 \leq \lambda \leq 1.234 \). Following this, \( 5.14 \) shows that the second pair of inequalities obtain.

\[ \Box \]

5.4 Lemma For \( m \geq 7 \), if \( \lambda_m \) satisfies \( 1.232 \leq \lambda_m \leq 1.234 \), then for \( \mathbf{A}_m v_{\lambda_m} = \lambda_m v_{\lambda_m} \) with \( v_0 = 1 \) and \( \mathbf{w}_m \mathbf{A}_m = \lambda_m \mathbf{w}_{\lambda_m} \) with \( w_0 = 1 \)

\[ 0.19 \times \sum_{i=0}^{m} w_i v_i < 0.71, \quad 0.14 \leq \frac{\max_{0 \leq i \leq m} \{ |w_i| \}}{\sum_{i=0}^{m} w_i v_i} \leq 0.35. \]

The matrices \( \mathbf{W}, \mathbf{V} \) defined by \( 3.6 \) satisfy

\[ \mathbf{W} \leq 1 + \mathbf{w}_\infty \leq 2.35, \quad \mathbf{V} \leq 1 + \mathbf{w}_\infty \mathbf{w}_\infty \leq 3.81. \]
Proof. We have

$$\sum_{i=0}^{m} w_i v_i = 1 \times 1 + \hat{w}_\lambda (\lambda - \hat{A})^{-1} \hat{A}_{0\hat{w}}.$$  \hfill (5.17)

$$|\sum_{i=0}^{m} w_i v_i - 1 - \hat{w}_\lambda (\lambda - \hat{A})^{-1} \hat{A}_{0\hat{w}}| \leq 1 \times (\lambda - \hat{A})^{-12} \hat{A}_{0\hat{w}} - \hat{A}_{0\hat{w}},$$ \hfill (5.18)

since in the $\ell^\infty$ norm $\hat{A}_{0\hat{w}} = 1$ and $\hat{w}_\lambda = \hat{A}_{0\hat{w}} (\lambda - \hat{A})^{-1}$. We use interval arithmetic to find

$$19.3 < \sum_{i=0}^{7} w_i v_i < 21.3.$$ \hfill (5.19)

Then $2.25 \times 3^{-7}/(49(1.232 - 0.5 - \log(2))^2) < 0.014$ gives a bound for the terms greater than 7. For the bound on $w$, note that with $(w_j)_i = 1$ forward substitution in (5.3) yields $(w_j)_1 = -1/(\lambda - A_{1,1})$ and (5.7) then shows that $w_j = |(w_j)_1|$. With 19.2 and 21.4 lower and upper bounds for $\sum w_i v_i$ and $1/(1.234 - 0.5 - \log 2)$, 1/(1.234 - 0.5 - \log 2) lower and upper bounds for $\max_{0 \leq i \leq n} |(w_i)|$, which occurs at $i = 1$, we have the last pair of inequalities of (5.15).

Now $W = e_j \hat{w} + v(0) e_j + f(0)$ with $j = 0$. The norm of column 0 of $W$ is $\nu - 1 + 1/\sum w_i v_i < 2.08$. The column with greatest absolute sum is column 1, so we get the scaled $|w_1|$ plus 1. Then (5.15) implies $W \leq 2.35$. We have $V = v(e_j - \hat{w}(0)) + f(0)$ again with $j = 0$. The maximum column absolute sum occurs at $i = 1$. It is bounded by $\nu w_1 + 1$. \hfill $\square$

5.5 Proposition For $m \geq 7$, $A_m$ has a single real eigenvalue in the interval $1.232891 \pm 0.0007$. All remaining eigenvalues have absolute values less than 1.002.

Proof. First we consider $A_7$. We compute approximate right eigenvector $v$ and left eigenvector $\hat{w}$ for $\lambda \approx 1.232891$ with $w_0 = 1$ and $\hat{w} v \approx 1$. First we compute with interval arithmetic that $|\sum w_i v_i - 1| < 7 \times 2^{-64}$, so we can apply Lemma 3.5 to conclude that there exists $w^*$ so that $\sum w_i v_i^* = 1$ and $|w_i - w_i^*| \leq \epsilon |w_i|$ with $\epsilon = 8 \times 2^{-64}$. We define $W$ and $V$ by (3.0). Using interval arithmetic we deduce a disc about 1.232891 with radius less than $53 \times 2^{-64}$. The absolute sum of each of the remaining columns is bounded by $1 + 46 \times 2^{-64}$, which implies that if the complex number $z$ is in any discs except the one about 1.232891, then $|z| \leq 1.001$. Thus there is a single eigenvalue with magnitude greater than 1.001. This must be real; otherwise we would have a conjugate pair of eigenvalues of magnitude greater than 1.001.

Next we consider the case of $A_m$, $m > 7$. Extend $W$ and $V$ to $A_m$ as follows using $h = 7$:

$$\hat{W}_{ij} = \begin{cases} W_{ij} & \text{if } 0 \leq i, j < h; \\ 0 & \text{if } i, j > h, i \neq j; \\ 8 & \text{if } i = j > h; \end{cases} \hat{V}_{ij} = \begin{cases} V_{ij} & \text{if } 0 \leq i, j < h; \\ 0 & \text{if } i \neq j. \end{cases}$$ \hfill (5.20)

We have the similarity transform $\hat{W} A \hat{V}$. We need bounds for the discs about $(\hat{W} A \hat{V})_{ii}$ for $i > 7$ and bounds for the contributions of the terms $(\hat{W} A \hat{V})_{ij}$ to the discs about $(\hat{W} A \hat{V})_{ii}$ for $i \leq 7$. For $j > 7$

$$\sum_{i=0}^{m} |(\hat{W} A \hat{V})_{ij}| \leq \sum_{i=0}^{6} \frac{W}{8} |A_{ij}| + \sum_{i=7}^{j} |A_{ij}| < \frac{2.35}{8} (0.5 + \frac{2}{6}) + 0.5 + \frac{1}{6} \leq 1.$$ \hfill (5.21)
For \( j \leq 7 \leq i, \, A_{ij} \neq 0 \) only if \( j = 0 \). Thus for \( j \leq 7 \),

\[
\sum_{i=7}^{m} |(\tilde{W}A\tilde{V})_{ij}| \leq 8v \sum_{i=7}^{\infty} A_{i0} < 8 * 3.81 * 2.25 * 3^{-7}/7^2 < 0.00064, \tag{5.22}
\]

using (3.4) and (5.16). For \( \tilde{W}A\tilde{V} \) the disc about 1.23289 has radius less than 0.00065; any \( z \) belonging to the remaining discs has \( |z| < 1.001 + 0.00065 \). \( \square \)

5.6 Proposition Let \((\lambda_n, v_n, w_n)\) denote the maximum eigenvalue and corresponding right and left eigenvectors of \( A_n \), where the \((v_n)_1 = (w_n)_1 = 1\). Then the sequence \(\{\lambda_n\}\) converges, while \(\{v_n\}\) converges in \(\ell^1\), \(1 \leq \rho \leq 3\) and \(\{w_n\}\) converges in \(\ell^\infty\).

Proof. For \( A_m, \, m > 7 \) and \( n > m \), define \( W_m \) and \( V_m \) using \( \zeta w \) for \( w \) in (3.6) with \( \zeta = (vw)^{-1} \). By (5.7) and the upper triangular form of \( \tilde{A} \) with \( \tilde{A}_{00} = 1 \),

\[
w_\infty \leq 1/(\lambda_m - A_{11}), \, w_1 = 1/(\lambda_m - A_{11}), \, \zeta w_\infty = (\lambda_m - A_{11})^{-1}(vw)^{-1}. \tag{5.23}
\]

From (5.10) have \( v_0 = 1, -0.76 < v_1 < -0.70, \sum_i |v_i| < 2.08, \) so \( |v_i| \leq 1 \) for \( i = 0, \ldots, m \). Then for \( n > m \) define \( \tilde{V} \) and \( \tilde{W} \) using (5.20) with \( h = m \). First we consider \((\tilde{W}A_n\tilde{V})_{ij}\) for \( 0 \leq i, j \leq m \). These entries are given by (5.10). For \( 0 \leq i, j \leq m \), \( i \times j = 0 \) the values are zero except the \((0,0)\) entry is \( \lambda_m \). For \( 1 \leq i \leq m, \, 1 \leq j \leq m \),

\[
(\tilde{W}A_n\tilde{V})_{ij} = (A_n)_{ij} - v_i (A_{0j}). \tag{5.24}
\]

Note \((A_{0j}) = 0 \) for \( j > 1 \), so the entries equal \((A_n)_{ij}\) when \( 1 \leq i \leq m \) and \( 2 \leq j \leq m \). The argument for (5.6) and estimates of the form (5.22) show that \( 2 \leq j \Rightarrow \sum_{i=0}^{n} (\tilde{W}A_n\tilde{V})_{ij} < 1.002. \tag{5.25} \)

Only the column indexed by \( j = 1 \) is different.

\[
(\tilde{W}A_n\tilde{V})_{0,1} = 0, \, (\tilde{W}A_n\tilde{V})_{1,1} = 0.5 + \log 2 + v_1, (\tilde{W}A_n\tilde{V})_{i,1} = v_i, \, 1 < i \leq m. \tag{5.26}
\]

Since \( \sum_{i=0}^{m} v_i < 2.08 \) and \( -0.76 < v_1 < -0.70 \),

\[
\sum_{i=0}^{m} |(\tilde{W}A_n\tilde{V})_{1,1}| = \sum_{i=0}^{m} v_i - 1 + 0.5 \log 2 - 2v_1 < 0.874. \tag{5.27}
\]

The argument of (5.22) implies \( \sum_{i=0}^{n} |(\tilde{W}A_n\tilde{V})_{1,1}| < 0.875. \) For \( j \leq m \),

\[
\sum_{i=m+1}^{n} |(\tilde{W}A_n\tilde{V})_{ij}| \leq 8v \frac{2.25 \times 3^{-m}}{m^2}. \tag{5.28}
\]

The above is a bound for the radius of the disc about \( \lambda_m \). For \( n > m \), \( \lambda_n \) is inside this disc. Thus shows that for \( n, k > m \) using (5.16),

\[
|\lambda_n - \lambda_k| \leq 16v \frac{2.25 \times 3^{-m}}{m^2}, \tag{5.29}
\]
hence \( \{\lambda_\eta\} \) is a Cauchy sequence. Then (5.3) and (5.7) show that \( \mathbf{v}_\lambda \) and \( \mathbf{w}_\lambda \) converge as \( \lambda_\eta \) converges, hence \( \{\mathbf{v}_\lambda\} \) converges in \( \ell^1 \) and \( \{\mathbf{w}_\lambda\} \) converges in \( \ell^\infty \). To see that \( \{\mathbf{v}_\lambda\} \) converges in \( \ell^1_\rho \), note that the transformed matrix entries \( (\mathbf{A})_{jk}p^{-k} \) in \( \mathbf{A} \) are not greater than those in \( \mathbf{A} \) for \( \rho \geq 1 \) because \( \mathbf{A} \) is upper triangular. In (5.3) it follows from (3.4) that the transform of \( \mathbf{A} \) is in \( \ell^1_\rho \) for \( 1 \leq \rho \leq 3 \), so \( \mathbf{v}_\lambda \) converges in \( \ell^1_\rho \).

\[ \square \]

5.7 Remark Proposition 5.5 implies the limit \( \lambda = \lim \lambda_\eta \) lies in the interval \( 1.232891 \pm 0.0007 \).

Theorem 5.8 Let \( \lambda, \mathbf{w}_\lambda \) and \( \mathbf{v}_\lambda \) be the limits as above with \( \mathbf{w} \mathbf{v} = \zeta \). Then if \( \mathbf{w} \in \ell^1 \) satisfies \( \mathbf{w} \mathbf{u} \neq 0 \), then

\[ \lim_{n \to \infty} \lambda^{-n} \mathbf{A}^n \mathbf{u} = (\mathbf{w} \mathbf{u}) \mathbf{v} / \zeta. \]  

(5.30)

Also if \( \mathbf{w} \mathbf{u} \neq 0 \),

\[ \lim_{n \to \infty} \mathbf{A}^n \mathbf{u} = (\mathbf{w} \mathbf{u}) \mathbf{v} / (\mathbf{w} \mathbf{v}) \]  

(5.31)

Proof. Use (3.6) to define \( \mathbf{W} \) and \( \mathbf{V} \) on \( \ell^1 \), where one uses \( \mathbf{w} / \zeta \) in the expressions for \( \mathbf{V} \) and \( \mathbf{W} \) to correspond to the \( \mathbf{w} \) used here. Write \( \mathbf{u} = \mathbf{v}(\mathbf{w} / \zeta) + \mathbf{u} - \mathbf{v}(\mathbf{w} / \zeta) \) and define \( \mathbf{z} \) and \( \mathbf{B} = (B_{ij}) \) by

\[ \mathbf{z} = \mathbf{W} \mathbf{u} - \frac{1}{\zeta} (\mathbf{w} \mathbf{u}) \mathbf{e}_0, \quad B = \begin{cases} (\mathbf{W} \mathbf{A} \mathbf{V})_{ij} & \text{if } ij \neq 0; \\ 0 & \text{if } i = 0 \text{ or } j = 0. \end{cases} \]

(5.32)

Then \( \mathbf{W} \mathbf{u} = (\mathbf{w} / \zeta) \mathbf{e}_0 + \mathbf{z} \) with \( \mathbf{e}_0 \mathbf{z} = 0 \), so

\[ (\mathbf{W} \mathbf{A} \mathbf{V})^n \mathbf{W} \mathbf{u} = \lambda^n \frac{1}{\zeta} (\mathbf{w} \mathbf{u}) \mathbf{e}_0 + \mathbf{B}^n \mathbf{z}. \]

(5.33)

The estimates in Proposition 5.6 show that the absolute sum of each column of \( \mathbf{B} \) is not greater than 1, i.e., \( \mathbf{B} \leq 1 \). Then

\[ \lambda^{-n} \mathbf{A}^n \mathbf{u} = \mathbf{V} \lambda^{-n} (\mathbf{W} \mathbf{A} \mathbf{V})^n \mathbf{W} \mathbf{u} = (\mathbf{w} / \zeta) \mathbf{V} \mathbf{e}_0 + \mathbf{V} \lambda^{-n} \mathbf{B}^n \mathbf{z}. \]

(5.34)

Now \( \mathbf{B} \leq 1 \) and \( \mathbf{V} \mathbf{e}_0 = \mathbf{v} \) so

\[ \lambda^{-n} \mathbf{A}^n \mathbf{u} - \frac{1}{\zeta} (\mathbf{w} \mathbf{u}) \mathbf{v} \leq \lambda^{-n} \mathbf{V} \mathbf{B}^n \mathbf{z} \leq \lambda^{-n} \mathbf{V} (\mathbf{W} + \frac{\mathbf{w}}{\zeta}) \mathbf{u}. \]

(5.35)

For (5.31) \( \lim \lambda^{-n} \mathbf{A}^n \mathbf{u} = \lim \lambda^{-n} \mathbf{A}^n \mathbf{u} \); one takes the limit of the ratios.

\[ \square \]

In this section we expressed the transformation of \( \ell \log(t) + \sum_{0}^{\infty} a_k(t-1)^k \) as the matrix operation \( \mathbf{A} \). Now we consider the reverse process. Let \( (\ell, a_0, a_1, \ldots) \in \ell^1 \) and define

\[ g(t) = \ell^* \log(t) + \sum_{k=0}^{\infty} a_k^*(t-1)^k. \]

(5.36)
Starting from any probability density of the form \( f \), the hypotheses of Proposition 2.6. Here we show that the appropriately scaled function corresponding to the eigenvector \( \mathbf{v} \) satisfies the hypotheses of Proposition 2.6.

**5.9 Corollary** There exists a probability density \( f^* \) on \([0, 2]\) of the form

\[
f^*(t) = \ell^* \log(t) + \sum_{k=0}^{\infty} a_k^*(t-1)^k
\]

so that \( f^* \) satisfies (2.6) with \( 1 + C = \lambda \). The coefficients \((\ell^*, a_0^*, a_1^*, \ldots)\) are in \( \ell_1^1 \) for \( 1 \leq \rho \leq 3 \).

Starting from any probability density of the form

\[
f_0(t) = \ell \log(t) + \sum_{k=0}^{\infty} a_k (t-1)^k
\]

with \( f_0(t) \) given by (2.3) for each \( 0 < y < 2 \) we have

\[
\lim_{s \to \infty} \sup_{y \leq t \leq 2} |f_0(t) - f^*(t)| = 0.
\]

**Proof.** First let \( f_0 = 0.5 \) on \([0, 2]\) corresponding to \( u \) with \( \ell = 0 \), \( a_0 = 0.5 \) and \( a_k = 0 \) otherwise. We know that \( A^u f_0 \geq 0 \) and \( \lim_n \lambda^{-n} A^n u = (\mathbf{wu})/\zeta \). Define \( g(t) \) by

\[
g(t) = -v_0 \log(t) - \sum_{k=0}^{\infty} v_{k+1} (t-1)^k.
\]

Then \( A g = \lambda g \). In the representation as functions

\[
\lim_{n \to \infty} \lambda^{-n} A^n f = -\frac{w_1}{2\zeta} g,
\]

because convergence of the coefficients implies uniform convergence on compact intervals of \([0, 2]\). From (5.15) and \( w_1 = -w \) we have \( -w_1 = 1/((\zeta(\lambda - 0.5 - \log(2))) \geq 1.14 \), so the limit of \( \lambda^{-n} A^n f \) is \( c g(t) \) with \( c \geq 0.57 \), hence \( g(t) \geq 0 \). The \( \lim_n I_0 \) is the multiple of \( g \) with \( \int_0^2 g \, dt = 1 \):

\[
f^*(t) = g(t)/\int_0^2 g \, dt.
\]

Now for a general initial probability density of the given form, \( A^k I_0 \) will be strictly positive for some \( k > 0 \) \( I_0 = \tilde{I}_0 + \varepsilon \). Then \( \lambda^{-k} A^k(\tilde{I}_0 + \varepsilon) \geq \lambda^{-k} A^k \varepsilon \). Then the limit \( \lim_{n \to \infty} \lambda^{-n} A^n I_0 \) cannot be the zero function. By taking ratios we find that \( \lim_{s \to \infty} f_s = f^* \). \( \square \)

**5.10 Remark** By virtue of (2.9), we now have an estimate of the rate of decay of remaining space in the interval with error estimate, \( R_2 = 0.616445 \pm 0.0035 \).
6 The Scaled Rényi Parking Transform

We have shown in Corollary 5.9 that a probability density of the form (3.1) with coefficient vector \( \{a_k\} \subseteq \ell^1 \) converges under the iteration to a density \( f^* \). Here we consider more general distributions on \([0, 2]\). In this section, we express the iteration on non-negative measures. This iteration has \( \delta_0 \) as a fixed point, but, in the case of probability measures on \([0, 2]\), is equivalent to that for cumulative distribution functions given in (2.2). In particular, we show that the point measure \( \delta_x \) for \( x \in [0, 2] \) converges to \( f^* dx \).

6.1 Definition \( C[0, 2] \) denotes the continuous functions on \([0, 2]\); \( \mathcal{M} \) denotes the bounded Borel measures on \([0, 2]\); \( \mathcal{M}^+ \) denotes the non-negative measures in \( \mathcal{M} \); \( \mathcal{M}^*_+ \) denotes the measures \( \mu \in \mathcal{M}^+ \) for which \( \mu[0, 2] = 1 \). The integral of \( g \in C[0, 2] \) with respect to \( \mu \in \mathcal{M}^+ \) is denoted by

\[
\langle g, \mu \rangle = \int_0^2 g(x) \mu(dx) \equiv \int g \, d\mu.
\]

(6.1)

For \( \mu, \nu \in \mathcal{M} \) we say that \( \mu \leq \nu \) if \( \nu - \mu \in \mathcal{M}^+ \).

The value of \( \langle g, \mu \rangle \) for all \( g \in C[0, 2] \) determines \( \mu \in \mathcal{M}^+ \) uniquely. For bounded Borel functions \( f, g : [0, 2] \to \mathbb{R} \), we have the measures \( f(x) \, dx \leq g(x) \, dx \) if and only if \( f \leq g \, \text{a.e.} \)

We define the operator \( U : \mathcal{M}^+ \to \mathcal{M}^+ \) by

\[
\langle g, U \mu \rangle = \int_0^1 g(2x) \mu(dx).
\]

(6.2)

Thus \( U \mu \) takes the restriction of \( \mu \) to \([0, 1]\) and scales it to an element of \( \mathcal{M}^+ \) using the operator \( x \mapsto 2x \).

For each \( x \in (1, 2] \) we define the measure \( \nu_x \in \mathcal{M}^+ \) by

\[
\langle g, \nu_x \rangle = \frac{1}{x-1} \int_0^{2/(x-1)} g(y) \, I_{[0, 2/(x-1)]}(y) \, dy = \frac{1}{x-1} \int_0^{2/(x-1)} g(y) \, dy,
\]

(6.3)

so \( \nu_x \) is Lebesgue measure on \([0, 2/(x-1)]\) scaled by \( 1/(x-1) \). Note \( \nu_x[0, 2] = 2 \). Here we use the indicator function

\[
I_{[0, 2/(x-1)]}(y) = I_{y \leq 2/(x-1)} = I_{y \geq 1+y/2} = \begin{cases} 1 & \text{if } 0 \leq y \leq 2(x-1) \leq 2; \\ 0 & \text{otherwise}. \end{cases}
\]

(6.4)

Given \( \mu \in \mathcal{M}^+ \), \( g \in C[0, 2] \) we define \( V \mu \) by

\[
\langle g, V \mu \rangle := \int_1^2 \langle g, \nu_x \rangle \mu(dx) = \int_1^2 \left( \int_0^{2/(x-1)} \frac{g(y)}{x-1} \, dy \right) \mu(dx).
\]

(6.5)

Note that the integral is on \((1, 2]\); the value \( \mu\{1\} \) is not included. The treatment of \( \mu\{1\} \) is significant. The measure concentrated at \( 1, \delta_1 \), has \( V \delta_1 = \delta_2 \). If we included the point \( 1 \) in the definition of \( V \), then the image should be \( 2\delta_0 \). We define

\[
T = U + V
\]

(6.6)

as the basic unscaled Rényi transformation. The following is elementary.
6.2 Proposition If $\mu \leq \nu$ then $U\mu \leq U\nu$, $V\mu \leq V\nu$ and $T\mu \leq T\nu$.

The point 1 is a discontinuity in $T$: for $x \leq 1$, $T\delta_x$ is $\delta_x$; for $x > 1$, $T\delta_x = \nu_x$. Note that $\delta_0$ is a fixed point of $T$.

We define the renormalised transform for $\mu \in M^+, \mu \neq 0$,

$$\hat{T}_\mu = \frac{1}{(T\mu)(0, 2]} T\mu,$$

(6.7)

so that $\hat{T}_\mu \in M_1^+$. The transformations, $T, U, V$ are linear. The transformation $\hat{T}$, called the area scaled Rényi transformation, is nonlinear, but $\hat{T}$ maps $M_1^+$ into itself. Note that $\hat{T}^\mu = \hat{T}(T^{\mu-1}\mu)$, so one can work with $T$ and normalise in the last step. For the probability density $f^*$ from Corollary 5.9 with $C^* = \int_1^2 f^*(x)dx$, we define

$$\hat{T}_\mu = \frac{1}{1 + C^*} T\mu.$$

(6.8)

This just a different normalization of $T$ which leaves the measure $f^*dx$ fixed. Notice $\hat{T}f^* = f^*dx$ as well. Also $\mu \leq \nu \Rightarrow \hat{T}_\mu \leq \hat{T}_\nu$, but this need not hold for $T$.

Now we consider the case when $\mu \in M^+$ has a density $\mu(dx) = f(x)dx$. For $g \in C[0, 2]$

$$\langle g, U\mu \rangle = \int_0^1 g(2y)f(y)dy = \int_0^2 g(x)f(x/2) dx/2,$$

(6.9)

so $U$ acts on $f$ by $f \mapsto \frac{1}{2}f(x/2)$ for $x \in [0, 2]$. Next consider $V$.

$$\langle g, V\mu \rangle = \int_1^2 \int_0^{2(x-1)} g(y) \frac{dy}{x-1} f(x)dx = \int_1^2 \int_0^{2(1-(x-1))} g(y) \frac{dy}{x-1} f(x)dy.$$

(6.10)

Changing the order of integration we get

$$\langle g, V\mu \rangle = \int_0^2 \int_1^{1+y/2} g(y) \frac{dy}{x-1} f(x) dx dy = \int_0^2 g(y) \int_1^{1+y/2} \frac{f(x)}{x-1} dx dy.$$

(6.11)

Thus the action of $V$ on the density $f$ is given by

$$f \mapsto \int_{1+y/2}^2 \frac{f(y)}{y-1} dy$$

(6.12)

Comparing the above with (2.3). The following shows that for any $\mu \in M^+$ with $\mu(0, 2] > 0$, $T^\mu - \varepsilon dx \in M^+$ for some $n \in N$ and $\varepsilon > 0$.

6.3 Proposition Let $\mu \in M^+$ satisfy $\mu(0, 2] > 0$. Then there exists $n, \varepsilon > 0$ and $\nu \in M^+$ so that $T^n\mu = \nu + \varepsilon dx$ and $\alpha > 0$ so that $T^{\alpha+1}\mu \geq \alpha f^*dx$. 
Proof. Since \( \mu(0, 2) > 0 \), we can iterate \( T \) as necessary to get \( (T^j \mu)(1, 2) > 0 \). Given \( T^j \mu(1, 2) > 0 \), there exists \( k \) so that \( (T^j \mu)[1 + 2^{-k}, 2] > 0 \). \( V T^j \mu \) integrated over \([1 + 2^{-k}, 2]\) is an integral of \( \nu_x \) over \([1 + 2^{-k}, 2]\). The minimal density of \( \nu_x \) for \( 1 + 2^{-k} \leq x \leq 2 \) occurs at \( 1 \) and equals \( 1 \). Thus \( T^{j+1} \mu \geq (T^j \mu)[1 + 2^{-k}, 2]I_{(0, 2)}dx \). We then have \( UT_i^{j+k} \mu \geq \epsilon I_{[0, 2]}dx \) with \( \epsilon = (T^j \mu)[1 + 2^{-k}, 2] \). Note

\[
VT_i^{j+k} \mu \geq \int_{1+2^{-k}/2}^{2} \frac{\epsilon dy}{y-1} dx = \epsilon \log \frac{2}{x},
\]

and \( f^* \) is a linear combination of \( \log x \) and a bounded function, so there exists \( \alpha > 0 \) so that

\[
T_i^{j+k+1} \mu \geq \epsilon(1 + \log \frac{2}{x})dx \geq \alpha f^* dx.
\]

\( \Box \)

Theorem 6.4 Let \( f(x) = D \log(x) + g(x) \) be a probability density on \((0, 2)\) where \( D \) is a constant and \( g(x) \) is continuous on \([0, 2]\). Then \( \lim_{n \to \infty} T^n f dx = f^* dx \).

Proof. The continuity of \( f \) on \((0, 2)\) implies via \([23]\) the continuity of \( \hat{T}^n f \). Since \( g \) is continuous, for \( \epsilon > 0 \) there exists a polynomial \( q_\epsilon(x) \) so that

\[
\sup_{0 < x \leq 2} |f(x) - D \log(x) - q_\epsilon(x)| \leq \epsilon
\]

because the polynomials are dense in the set of continuous functions on \([0, 2]\). We have

\[
\hat{T}^n(D \log(x) + q_\epsilon(x) - \epsilon) \leq \hat{T}^n f(x) \leq \hat{T}^n(D \log(x) + q_\epsilon(x) + \epsilon).
\]

Corollary 5.9 shows that \( \{\hat{T}^n(D \log(x) + q_\epsilon(x))\} \) converges to \( \beta_\epsilon f^* \) uniformly on compact subsets of \((0, 2)\), where \( \beta_\epsilon > 0 \) is a constant, and \( \lim_{n \to \infty} \hat{T}^n I = w f^* \) for \( w = -w_1 \). For \( 0 < y < 2 \) there exists \( n_{\epsilon, y} \) such that \( j \geq n \) implies

\[
\sup_{y \leq x \leq 2} |\hat{T}^j(D \log(x) + q_\epsilon(x)) - \beta_\epsilon f^*(x)| < \epsilon, \quad \sup_{y \leq x \leq 2} |(\hat{T}^j)1(x) - w f^*(x)| < 1.
\]

Then for \( j > n_{\epsilon, y} \) and \( y \leq x \leq 2 \)

\[
|\hat{T}^j f(x) - \beta_\epsilon f^*(x)| \leq \epsilon \left(1 + |(\hat{T}^j)1(x) - w f^*(x)| + w f^*(y)\right) \leq \epsilon (2 + w f^*(y)),
\]

since \( f^*(y) \geq f^*(x) \) for \( y \leq x \leq 2 \). Thus for \( j, k > n_{\epsilon, y} \) and \( y \leq x \leq 2 \),

\[
|\hat{T}^j f(x) - \hat{T}^k f(x)| \leq \epsilon (2 + w f^*(y)),
\]

so the sequence of functions \( \{\hat{T}^n f(x)\} \) is uniformly Cauchy on each compact subset of \((0, 2)\), which implies convergence. It follows from \([6.18]\) this limit must be of the form \( \beta f^* \), so we write \( \lim_n \hat{T}^n(D \log(x) + g(x)) = \beta^* f^* \). By Proposition 6.3 \( \beta^* > 0 \). Convergence of \( \{\hat{T}^n f\} \) to \( \beta^* f^* \) then implies convergence of \( \{\hat{T}^n \delta_x\} \) to \( f^* \).

\( \Box \)

6.5 Corollary For \( 0 < x \leq 2 \) the sequence \( \{\hat{T}^n \delta_x\} \) converges to \( f^* dx \).

Proof. There exists an integer \( j \geq 0 \) so that \( 1 < 2^j x \leq 2 \). Let \( y = 2^j x - 1 \) so that \( 0 < y < 1 \). Then \( T^j \delta_x = \delta_{y+1} \) and \( T^{j+i} \delta_x = \frac{1}{y^i} \delta_{[0, 2y]}/dx \). There exists an integer \( k \geq 0 \) so that \( 1 < 2^k y \leq 2 \). Let \( z = k y - 1 \). Then

\[
T^{j+k+1} \delta_x = \frac{1}{1+z} (1 + 2 I_{[0 < x \leq 2z]}(x) \log \frac{2z}{x}) dx
\]

is of the form \( D \log(x) + g(x) \) with \( D \) constant and \( g \) continuous.
References

[1] Dunford, N., and Schwartz, J. T. Linear operators. Part II. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988.

[2] IEEE Working Group for Floating-Point Arithmetic. Standard for Floating-Point Arithmetic 754-2008. Tech. rep., IEEE, https://standards.ieee.org/findstds/standard/754-2008.html, 2008.

[3] Mackey, D., Kelly, E., Nooney, R., and McDonagh, C. Modelling random antibody adsorption and immunoassay activity. Mathematical Biosciences and Engineering 13, 6 (2016).

[4] Penrose, M. D. Random parking, sequential adsorption, and the jamming limit. Comm. Math. Phys. 218, 1 (2001), 153–176.

[5] Rényi, A. On a one-dimensional problem concerning random space filling. Magyar Tud. Akad. Mat. Kutató Int. Közl. 3, no 1/2 (1958), 109–127.

[6] Solomon, H., and Weiner, H. A review of the packing problem. Comm. Statist. A—Theory Methods 15, 9 (1986), 2571–2607.

[7] Wilkinson, J. H. The algebraic eigenvalue problem. Clarendon Press, Oxford, 1965.