A note on Aleksandrov type theorem for $k$-convex functions

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Abstract
In this note we show that $k$-convex functions on $\mathbb{R}^n$ are twice differentiable almost everywhere for every positive integer $k > n/2$. This generalizes the classical Aleksandrov’s theorem for convex functions.

1 Introduction

A classical result of Aleksandrov [1] asserts that convex functions in $\mathbb{R}^n$ are twice differentiable a.e., (see also [3], [8] for more modern treatments). It is well known that Sobolev functions $u \in W^{2,p}$, for $p > n/2$ are twice differentiable a.e.. The following weaker notion of convexity known as $k$-convexity was introduced by Trudinger and Wang [12, 13]. Let $\Omega \subset \mathbb{R}^n$ be an open set and $C^2(\Omega)$ be the class of continuously twice differentiable functions on $\Omega$. For $k = 1, 2, \ldots, n$ and a function $u \in C^2(\Omega)$, the $k$-Hessian operator, $F_k$, is defined by

$$F_k[u] := S_k(\lambda(\nabla^2 u)),$$

where $\nabla^2 u = (\partial_{ij} u)$ denotes the Hessian matrix of the second derivatives of $u$, $\lambda(A) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ the vector of eigenvalues of an $n \times n$ matrix.
$A \in \mathbb{M}^{n \times n}$ and $S_k(\lambda)$ is the $k$-th elementary symmetric function on $\mathbb{R}^n$, given by

$$S_k(\lambda) := \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}. \quad (1.2)$$

Alternatively we may write

$$F_k[u] = [\nabla^2 u]_k \quad (1.3)$$

where $[A]_k$ denotes the sum of $k \times k$ principal minors of an $n \times n$ matrix $A$, which may also be called the $k$-trace of $A$. The study of $k$-Hessian operators was initiated by Caffarelli, Nirenberg and Spruck [2] and Ivochkina [6] with further developed by Trudinger and Wang [10, 12, 13, 14, 15].

A function $u \in C^2(\Omega)$ is called $k$-convex in $\Omega$ if $F_j[u] \geq 0$ in $\Omega$ for $j = 1, 2, \ldots, k$; that is, the eigenvalues $\lambda(\nabla^2 u)$ of the Hessian $\nabla^2 u$ of $u$ lie in the closed convex cone given by

$$\Gamma_k := \{ \lambda \in \mathbb{R}^n : S_j(\lambda) \geq 0, \ j = 1, 2, \ldots, k \}. \quad (1.4)$$

(see [2] and [13] for the basic properties of $\Gamma_k$.) We notice that $F_1[u] = \Delta u$, is the Laplacian operator and 1-convex functions are subharmonic. When $k = n$, $F_n[u] = \det(\nabla^2 u)$, the Monge-Ampère operator and $n$-convex functions are convex. To extend the definition of $k$-convexity for non-smooth functions we adopt a viscosity definition as in [13]. An upper semi-continuous function $u : \Omega \to [-\infty, \infty]$ ($u \neq -\infty$ on any connected component of $\Omega$) is called $k$-convex if $F_j[q] \geq 0$, in $\Omega$ for $j = 1, 2, \ldots, k$, for every quadratic polynomial $q$ for which the difference $u - q$ has a finite local maximum in $\Omega$. Henceforth, we shall denote the class of $k$-convex functions in $\Omega$ by $\Phi_k(\Omega)$. When $k = 1$ the above definition is equivalent to the usual definition of subharmonic function, see, for example (Section 3.2, [5]) or (Section 2.4, [7]). Thus $\Phi^1(\Omega)$ is the class of subharmonic functions in $\Omega$. We notice that $\Phi^k(\Omega) \subset \Phi^1(\Omega) \subset L^1_{\text{loc}}(\Omega)$ for $k = 1, 2, \ldots, n$, and a function $u \in \Phi^k(\Omega)$ if and only if it is convex on each component of $\Omega$. Among other results Trudinger and Wang [13] (Lemma 2.2) proved that $u \in \Phi^k(\Omega)$ if and only if

$$\int_\Omega u(x) \left( \sum_{i,j} a^{ij} \partial_{ij} \phi(x) \right) \, dx \geq 0 \quad (1.5)$$

for all smooth compactly supported functions $\phi \geq 0$, and for all constant $n \times n$ symmetric matrices $A = (a^{ij})$ with eigenvalues $\lambda(A) \in \Gamma_k$, where $\Gamma_k$ is dual cone defined by

$$\Gamma_k^* := \{ \lambda \in \mathbb{R}^n : \langle \lambda, \mu \rangle \geq 0 \text{ for all } \mu \in \Gamma_k \}. \quad (1.6)$$
In this note we prove the following Alexsandrov type theorem for \( k \)-convex functions.

**Theorem 1.1.** Let \( k > n/2, n \geq 2 \) and \( u : \mathbb{R}^n \to [-\infty, \infty) \) (\( u \not\equiv -\infty \) on any component of \( \mathbb{R}^n \)), be a \( k \)-convex function. Then \( u \) is twice differentiable almost everywhere. More precisely, we have the Taylor's series expansion for \( \mathcal{L}^n \) a.e.,

\[
| u(y) - u(x) - \langle \nabla u(x), y - x \rangle - \frac{1}{2} \langle \nabla^2 u(x)(y - x), y - x \rangle | = o(|y - x|^2),
\]

as \( y \to x \).

In Section 3 (see, Theorem 3.2.), we also prove that the absolute continuous part of the \( k \)-Hessian measure (see, [12, 13]) \( \mu_k[u] \), associated to a \( k \)-convex function for \( k > n/2 \) is represented by \( F_k[u] \). For the Monge-Ampère measure \( \mu[u] \) associated to a convex function \( u \), such result is obtained in [16].

To conclude this introduction we note that it is equivalent to assume only \( F_k[q] \geq 0 \), in the definition of \( k \)-convexity [13]. Moreover \( \Gamma_k \) may also be characterized as the closure of the positivity set of \( S_k \) containing the positive cone \( \Gamma_n \), [2].

## 2 Notations and preliminary results

Throughout the text we use following standard notations. \( | \cdot | \) and \( \langle \cdot, \cdot \rangle \) will stand for Euclidean norm and inner product in \( \mathbb{R}^n \), and \( B(x, r) \) will denote the open ball in \( \mathbb{R}^n \) of radius \( r \) centered at \( x \). For measurable \( E \subset \mathbb{R}^n \), \( \mathcal{L}^n(E) \) will denote its Lebesgue measure. For a smooth function \( u \), the gradient and Hessian of \( u \) are denoted by \( \nabla u = (\partial_1 u, \cdots, \partial_n u) \) and \( \nabla^2 u = (\partial_{ij} u)_{1 \leq i,j \leq n} \) respectively. For a locally integrable function \( f \), the distributional gradient and Hessian are denoted by \( Df = (D_1 f, \cdots, D_n f) \) and \( D^2 u = (D_{ij} u)_{1 \leq i,j \leq n} \) respectively.

For the convenience of the readers, we cite the following Hölder and gradient estimates for \( k \)-convex functions, and the weak continuity result for \( k \)-Hessian measures, [12, 13].

**Theorem 2.1.** (Theorem 2.7, [13]) For \( k > n/2 \), \( \Phi^k(\Omega) \subset C^{0,\alpha}_{\text{loc}}(\Omega) \) with \( \alpha := 2 - n/k \) and for any \( \Omega' \subset \subset \Omega \), \( u \in \Phi^k(\Omega) \), there exists \( C > 0 \), depending...
only on $n$ and $k$ such that

$$\sup_{x, y \in \Omega'} d_{x,y}^{\alpha} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq C \int_{\Omega'} |u|, \quad (2.1)$$

where $d_x := \text{dist}(x, \partial \Omega ') \text{ and } d_{x,y} := \min\{d_x, d_y\}$.

**Theorem 2.2.** (Theorem 4.1, [13]) For $k = 1, \ldots, n$, and $0 < q < \frac{nk}{n-k}$, the space of $k$-convex functions $\Phi^k(\Omega)$ lie in the local Sobolev space $W^{1,q}_{\text{loc}}(\Omega)$. Moreover, for any $\Omega' \subset \Omega'' \subset \Omega$ and $u \in \Phi^k(\Omega)$ there exists $C > 0$, depending on $n, k, q, \Omega'$ and $\Omega''$, such that

$$\left( \int_{\Omega'} |Du|^q \right)^{1/q} \leq C \int_{\Omega''} |u|. \quad (2.2)$$

**Theorem 2.3.** [Theorem 1.1, [13]] For any $u \in \Phi^k(\Omega)$, there exists a Borel measure $\mu_k[u]$ in $\Omega$ such that

(i) $\mu_k[u](V) = \int_V F_k[u](x) \, dx$ for any Borel set $V \subset \Omega$, if $u \in C^2(\Omega)$ and

(ii) if $(u_m)_{m \geq 1}$ is a sequence in $\Phi^k(\Omega)$ converges in $L^1_{\text{loc}}(\Omega)$ to a function $u \in \Phi^k(\Omega)$, the sequence of Borel measures $(\mu_k[u_m])_{m \geq 1}$ converges weakly to $\mu_k[u]$.

Let us recall the definition of the dual cones, [11]

$$\Gamma^*_k := \{ \lambda \in \mathbb{R}^n : \langle \lambda, \mu \rangle \geq 0 \text{ for all } \mu \in \Gamma_k \},$$

which are also closed convex cones in $\mathbb{R}^n$. We notice that $\Gamma^*_j \subset \Gamma^*_k$ for $j \leq k$ with $\Gamma^*_n = \Gamma_n = \{ \lambda \in \mathbb{R}^n : \lambda_i \geq 0, \ j = 1, 2, \ldots, n \}$, $\Gamma^*_1$ is the ray given by

$$\Gamma^*_1 = \{ t(1, \cdots, 1) : t \geq 0 \},$$

and $\Gamma^*_2$ has the following interesting characterization,

$$\Gamma^*_2 = \left\{ \lambda \in \Gamma_n : |\lambda|^2 \leq \frac{1}{n-1} \left( \sum_{i=1}^n \lambda_i \right)^2 \right\}. \quad (2.3)$$

We use this explicit representation of $\Gamma^*_2$ to establish that the distributional derivatives $D_{ij} u$ of the $k$-convex function $u$ are signed Borel measures for $k \geq 2$, (see also [13]).
Theorem 2.4. Let $2 \leq k \leq n$ and $u : \mathbb{R}^n \to [-\infty, \infty)$, be a $k$-convex function. Then there exist signed Borel measures $\mu^{ij} = \mu^{ji}$ such that
\[
\int_{\mathbb{R}^n} u(x) \partial_{ij} \phi(x) \, dx = \int_{\mathbb{R}^n} \phi(x) \, d\mu^{ij}(x) \quad \text{for } i, j = 1, 2, \ldots, n, \quad (2.4)
\]
for all $\phi \in C^\infty_c(\mathbb{R}^n)$.

Proof. Let $k \geq 2$ and $u \in \Phi^k(\mathbb{R}^n)$. Since $\Phi^k(\mathbb{R}^n) \subset \Phi^2(\mathbb{R}^n)$ for $k \geq 2$, it is enough to prove the theorem for $k = 2$. Let $u$ be a 2-convex function in $\mathbb{R}^n$. For $A \in S^{n \times n}$, the space of $n \times n$ symmetric matrices, define the distribution $T_A : C^2_c(\mathbb{R}^n) \to \mathbb{R}$, by
\[
T_A(\phi) := \int_{\mathbb{R}^n} u(x) \sum_{i,j} a^{ij} \partial_{ij} \phi(x) \, dx
\]
By (1.5), $T_A(\phi) \geq 0$ for $A \in S^{n \times n}$ with eigenvalues $\lambda(A) \in \Gamma^*_2$, and $\phi \geq 0$. Therefore, by Riesz representation (see, for example Theorem 2.14 in [9] or Theorem 1, Section 1.8 in [3]), there exist a Borel measure $\mu^A$ in $\mathbb{R}^n$, such that
\[
T_A(\phi) = \int_{\mathbb{R}^n} \phi(\sum_{i,j} a^{ij} D_{ij} u) \, dx = \int_{\mathbb{R}^n} \phi(x) \, d\mu^A, \quad (2.5)
\]
for all $\phi \in C^2_c(\mathbb{R}^n)$ and all $n \times n$ symmetric matrices $A$ with $\lambda(A) \in \Gamma^*_2$. In order to prove the second order distributional derivatives $D_{ij} u$ of $u$ to be signed Borel measures, we need to make special choices for the matrix $A$. By taking $A = I_n$, the identity matrix, $\lambda(A) \in \Gamma^*_1 \subset \Gamma^*_2$, we obtain a Borel measure $\mu^I_n$ such that
\[
\int_{\mathbb{R}^n} \phi \sum_{i=1}^n D_{ii} u \, dx = \int_{\mathbb{R}^n} \phi \, d\mu^I_n, \quad (2.6)
\]
for all $\phi \in C^2_c(\mathbb{R}^n)$. Therefore, the trace of the distributional Hessian $D^2 u$, is a Borel measure. For each $i = 1, \ldots, n$, let $A_i$ be the diagonal matrix with all entries 1 but the $i$-th diagonal entry being 0. Then by the characterization of $\Gamma^*_2$ in (2.3), it follows that $\lambda(A_i) \in \Gamma^*_2$. Hence there exist a Borel measure $\mu^i$ in $\mathbb{R}^n$ such that
\[
\int_{\mathbb{R}^n} \phi \sum_{j \neq i} D_{jj} u \, dx = \int_{\mathbb{R}^n} \phi \, d\mu^i, \quad (2.7)
\]
for all $\phi \in C^2_c(\mathbb{R}^n)$. From (2.6) and (2.7) it follows that, the diagonal entries $D_{ii} u = \mu^{I_n}_n - \mu^i := \mu^{ii}$ are signed Borel measure and
\[
\int_{\mathbb{R}^n} u \partial_{ij} \phi \, dx = \int_{\mathbb{R}^n} \phi \, d\mu^{ii}, \quad (2.8)
\]
for all $\phi \in C^2_c(\mathbb{R}^n)$. Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis in $\mathbb{R}^n$ and for $a, b \in \mathbb{R}^n$, $a \otimes b := (a^t b')$, denotes the $n \times n$ rank-one matrix. For $0 < t < 1$ and $i \neq j$, let us define $A_{ij} := I_n + t[e_i \otimes e_j + e_j \otimes e_i]$. By a straightforward calculation, it is easy to see that the vector of eigenvalues $(A_{ij}) = (1-t, 1+t, 1, \ldots, 1) \in \Gamma_2^n$, for $0 < t < (n/2(n-1))^{1/2}$. Note that for this choice of $A_{ij}$

$$\sum_{k,l=1}^{n} a^{kl} \partial_{kl} \phi = \sum_{k=1}^{n} \partial_{kk} \phi + 2t \partial_{ij} \phi.$$ 

Thus for $i \neq j$, (2.5) and (2.6) yields

$$\int_{\mathbb{R}^n} u \partial_{ij} \phi \, dx = \frac{1}{2t} \left[ \int_{\mathbb{R}^n} u \sum_{k,l=1}^{n} a^{kl} \partial_{kl} \phi \, dx - \int_{\mathbb{R}^n} u \sum_{k=1}^{n} \partial_{kk} \phi \, dx \right]$$

$$= \frac{1}{2t} \left[ \int_{\mathbb{R}^n} \phi \, d\mu^{A_{ij}} - \int_{\mathbb{R}^n} \phi \, d\mu^{I_n} \right]$$

$$= \int_{\mathbb{R}^n} \phi \, d\mu^{ij}, \quad (2.9)$$

where

$$\mu^{ij} := \frac{1}{2t} (\mu^{A_{ij}} - \mu^{I_n}) = \frac{1}{2t} \left( \mu^{A_{ij}} - \sum_{k=1}^{n} \mu^{kk} \right).$$

Therefore $D_{ij} u = \mu^{ij}$, are signed Borel measures and satisfies the identity (2.4).

A function $f \in L^1_{loc}(\mathbb{R}^n)$ is said to have **locally bounded variation** in $\mathbb{R}^n$ if for each bounded open subset $\Omega'$ of $\mathbb{R}^n$,

$$\sup \left\{ \int_{\Omega'} f \, \text{div} \phi \, dx : \phi \in C^1_c(\Omega'; \mathbb{R}^n), \, |\phi(x)| \leq 1 \text{ for all } x \in \Omega' \right\} < \infty.$$ 

We use the notation $BV_{loc}(\mathbb{R}^n)$, to denote the space of such functions. For the theory of functions of bounded variation readers are referred to [4, 17, 3].

**Theorem 2.5.** Let $n \geq 2$, $k > n/2$ and $u : \mathbb{R}^n \to [-\infty, \infty)$, be a $k$-convex function. Then $u$ is differentiable a.e. $\mathcal{L}^n$ and $\frac{\partial u}{\partial x_i} \in BV_{loc}(\mathbb{R}^n)$, for all $i = 1, \ldots, n$.

**Proof.** Observe that for $k > n/2$, we can take $n < q < \frac{nk}{n-k}$ and by the gradient estimate (2.2), we conclude that $k$-convex functions are differentiable.
Let \( \Omega' \subseteq \mathbb{R}^n \), \( \phi = (\phi^1, \ldots, \phi^n) \in C^1_c(\Omega'; \mathbb{R}^n) \) such that \( |\phi(x)| \leq 1 \) for \( x \in \Omega' \). Then by integration by parts and the identity (2.4), we have for \( i = 1, \ldots, n \),

\[
\int_{\Omega'} \frac{\partial u}{\partial x_i} \text{div} \phi \, dx = -\sum_{j=1}^{n} \int_{\Omega'} u \frac{\partial^2 \phi^j}{\partial x_i \partial x_j} \, dx
=-\sum_{j=1}^{n} \int_{\Omega'} \phi^j \, d\mu^{ij}
\leq \sum_{j=1}^{n} |\mu^{ij}|(\Omega') < \infty ,
\]

where \( |\mu^{ij}| \) is the total variation of the Radon measure \( \mu^{ij} \). This proves the theorem. \( \square \)

### 3 Twice differentiability

Let \( u \) be a \( k \)-convex function, \( k \geq 2 \), then by the Theorem 2.4, we have \( D^2 u = (\mu^{ij})_{i,j} \), where \( \mu^{ij} \) are Radon measures. By Lebesgue’s Decomposition Theorem, we may write

\[
\mu^{ij} = \mu^{ij}_{ac} + \mu^{ij}_s \quad \text{for} \ i, j = 1, \ldots, n ,
\]

where \( \mu^{ij}_{ac} \) is absolutely continuous with respect to \( \mathcal{L}^n \) and \( \mu^{ij}_s \) is supported on a set with Lebesgue measure zero. Let \( u_{ij} \) be the density of the absolutely continuous part, i.e., \( d\mu^{ij}_{ac} = u_{ij} \, dx \), \( u_{ij} \in L^1_{\text{loc}}(\mathbb{R}^n) \). Set \( u_{ij} := \frac{\partial^2 u}{\partial x_i \partial x_j} \),

\[
\nabla^2 u := \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{i,j} = (u_{ij})_{i,j} \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^{n \times n}) \quad \text{and} \quad [D^2 u]_s := (\mu^{ij}_s)_{i,j} .
\]

Thus the vector valued Radon measure \( D^2 u \) can be decomposed as \( D^2 u = [D^2 u]_{ac} + [D^2 u]_s \), where \( d[D^2 u]_{ac} = \nabla^2 u \, dx \). Now we are in a position to prove the theorem 1.1. To carry out the proof, we use a similar approach to Evans and Gariepy, see, Section 6.4, in [3].

**Proof of Theorem 1.1.** Let \( n \geq 2 \) and \( u \) be a \( k \)-convex function on \( \mathbb{R}^n \), \( k > n/2 \). Then by Theorem 2.4, and Theorem 2.5, we have for \( \mathcal{L}^n \) a.e. \( x \)

\[
\lim_{r \to 0} \int_{B(x,r)} |\nabla u(y) - \nabla u(x)| \, dy = 0 , \quad (3.1)
\]

\[
\lim_{r \to 0} \int_{B(x,r)} |\nabla^2 u(y) - \nabla^2 u(x)| \, dy = 0 \quad (3.2)
\]
and
\[
\lim_{r \to 0} \frac{|[D^2u]_s|(B(x, r))}{r^n} = 0.
\]
(3.3)

where \( \int_E f \, dx \) we denote the mean value \((\mathcal{L}^n(E))^{-1} \int_E f \, dx \). Fix a point \( x \) for which (3.1)-(3.3) holds. Without loss generality we may assume \( x = 0 \).

Then following similar calculations as in the proof of Theorem 1, Section 6.4 in [3], we obtain,
\[
\int_{B(r)} \left| u(y) - u(0) - \langle \nabla u(0), y \rangle - \frac{1}{2} \langle \nabla^2 u(0)y, y \rangle \right| \, dy = o(r^2),
\]
as \( r \to 0 \). In order to establish
\[
\sup_{B(r/2)} \left| u(y) - u(0) - \langle \nabla u(0), y \rangle - \frac{1}{2} \langle \nabla^2 u(0)y, y \rangle \right| = o(r^2) \text{ as } r \to 0,
\]
we need the following lemma.

**Lemma 3.1.** Let \( h(y) := u(y) - u(0) - \langle \nabla u(0), y \rangle - \frac{1}{2} \langle \nabla^2 u(0)y, y \rangle \). Then there exists a constant \( C > 0 \) depending only on \( n, k \) and \( |\nabla^2 u(0)| \), such that for any \( 0 < r < 1 \)
\[
\sup_{y, z \in B(r)} \frac{|h(y) - h(z)|}{|y - z|^\alpha} \leq \frac{C}{r^\alpha} \int_{B(2r)} |h(y)| \, dy + Cr^{2-\alpha},
\]
(3.6)

where \( \alpha := (2 - n/k) \).

**Proof.** Let \( \Lambda := |\nabla^2 u(0)| \) and define \( g(y) := h(y) + \frac{\Lambda}{2} |y|^2 \). Since \( \frac{\Lambda}{2} |y|^2 - u(0) - \langle \nabla u(0), y \rangle - \frac{1}{2} \langle \nabla^2 u(0)y, y \rangle \) is convex and sum of two \( k \)-convex functions are \( k \)-convex (follows from (1.4)), we conclude that \( g \) is \( k \)-convex. Applying the Hölder estimate in (2.1) for \( g \) with \( \Omega' = B(2r) \), there exists \( C := C(n, k) > 0 \), such that
\[
\sup_{y, z \in B(r)} \frac{|g(y) - g(z)|}{|y - z|^\alpha} = \text{dist}(B(r), \partial B(2r))^{n+\alpha} \sup_{y, z \in B(r)} \frac{|g(y) - g(z)|}{|y - z|^\alpha} \leq \sup_{y, z \in B(2r)} \frac{|g(y) - g(z)|}{|y - z|^\alpha} \leq C \int_{B(2r)} |g(y)| \, dy \leq C \int_{B(2r)} |h(y)| \, dy + Cr^{n+2},
\]
(3.7)
where $d_{y,z} := \min\{\text{dist}(y, \partial B(2r)), \text{dist}(z, \partial B(2r))\}$. Therefore the estimate (3.6) for $h$ follows from the estimate (3.7) and the definition of $g$.  

Proof of Theorem 1.1. (ctd.) To prove (3.5), take $0 < \epsilon, \delta < 1$, such that $\delta^{1/n} \leq 1/2$. Then there exists $r_0$ depending on $\epsilon$ and $\delta$, sufficiently small, such that, for $0 < r < r_0$

\[
\mathcal{L}^n \{ z \in B(r) : |h(z)| \geq \epsilon r^2 \} \leq \frac{1}{\epsilon r^2} \int_{B(r)} |h(z)| \, dz = o(r^n) \text{ by (3.4)} < \delta \mathcal{L}^n(B(r)) \quad (3.8)
\]

Set $\sigma := \delta^{1/n} r$. Then for each $y \in B(r/2)$ there exists $z \in B(r)$ such that

| $|h(z)| \leq \epsilon r^2 \quad \text{and} \quad |y - z| \leq \sigma$.|

Hence for each $y \in B(r/2)$, we obtain by (3.4) and (3.6),

\[
|h(y)| \leq |h(z)| + |h(y) - h(z)| \\
\leq \epsilon r^2 + C|y - z|^{\alpha} \left( \frac{1}{r^\alpha} \int_{B(2r)} |h(y)| \, dy + r^{2-\alpha} \right) \\
\leq \epsilon r^2 + C\delta^{\alpha/n} r^{\alpha} \left( \frac{1}{r^\alpha} \int_{B(2r)} |h(y)| \, dy + r^{2-\alpha} \right) \\
\leq \epsilon r^2 + C\delta^{\alpha/n} \left( \int_{B(2r)} |h(y)| \, dy + r^2 \right) \\
= r^2 \left( \epsilon + C\delta^{\alpha/n} \right) + o(r^2) \quad \text{as} \quad r \to 0
\]

By choosing $\delta$ such that, $C\delta^{\alpha/n} = \epsilon$, we have for sufficiently small $\epsilon > 0$ and $0 < r < r_0$,

\[
\sup_{B(r/2)} |h(y)| \leq 2\epsilon r^2 + o(r^2).
\]

Hence

\[
\sup_{B(r/2)} \left| u(y) - u(0) - \langle \nabla u(0), y \rangle - \frac{1}{2} \langle \nabla^2 u(0)y, y \rangle \right| \, dy = o(r^2) \quad \text{as} \quad r \to 0.
\]

This proves (1.7) for $x = 0$ and hence $u$ is twice differentiable at $x = 0$. Therefore $u$ is twice differentiable at every $x$ and satisfies (1.7), for which (3.1)-(3.3) holds. This proves the theorem.  

Let $u$ be a $k$-convex function and $\mu_k[u]$ be the associated $k$-Hessian measure. Then $\mu_k[u]$ can be decomposed as the sum of a regular part $\mu_k^{ac}[u]$ and
a singular part $\mu^s_k[u]$. As an application of the Theorem 1.1, we prove the following theorem.

**Theorem 3.2.** Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in \Phi^k(\Omega)$, $k > n/2$. Then the absolute continuous part of $\mu_k[u]$ is represented by the $k$-Hessian operator $F_k[u]$. That is

$$
\mu^ac_k[u] = F_k[u] \, dx.
$$

**Proof.** Let $u$ be a $k$-convex function, $k > n/2$ and $u_\epsilon$ be the mollification of $u$. Then by (1.5) and properties of mollification (see, for example Theorem 1, Section 4.2 in [3]) it follows that $u_\epsilon \in \Phi^k(\Omega) \cap C^\infty(\Omega)$. Since $u$ is twice differentiable a.e. (by Theorem 1.1) and $u \in W^{2,1}_{loc}(\Omega)$ (by Theorem 2.5), we conclude that $\nabla^2 u_\epsilon \to \nabla^2 u$ in $L^1_{loc}$. Let $\mu_k[u_\epsilon]$ and $\mu_k[u]$ are the Hessian measures associated to the functions $u_\epsilon$ and $u$ respectively. Then by the weak continuity Theorem 2.3 (Theorem 1.1, [13]), $\mu_k[u_\epsilon]$ converges to $\mu_k[u]$ in measure and $\mu_k[u_\epsilon] = F_k[u_\epsilon] \, dx$. It follows that for any compact set $E \subset \Omega$,

$$
\mu_k[u](E) \geq \limsup_{\epsilon \to 0} \mu_k[u_\epsilon](E) = \limsup_{\epsilon \to 0} \int_E F_k[u_\epsilon].
$$

Since $F_k[u_\epsilon] \geq 0$ and $F_k[u_\epsilon](x) \to F_k[u](x)$ a.e., by Fatou’s lemma, for every relatively compact measurable subset $E$ of $\Omega$, we have

$$
\int_E F_k[u] \leq \liminf_{\epsilon \to 0} \int_E F_k[u_\epsilon].
$$

Therefore by Theorem 3.1, [13], it follows that $F_k[u] \in L^1_{loc}(\Omega)$. Let $\mu_k[u] = \mu^ac_k[u] + \mu^s_k[u]$, where $\mu^ac_k[u] = h \, dx$, $h \in L^1_{loc}(\Omega)$ and $\mu^s_k[u]$ is the singular part supported on a set of Lebesgue measure zero. We would like to prove that $h(x) = F_k[u](x) \mathcal{L}^n$ a.e. $x$. By taking $E := \overline{B}(x, r)$, from (3.10) and (3.11), we obtain

$$
\int_{\overline{B}(x, r)} F_k[u] \, dy \leq \frac{\mu_k[u](\overline{B}(x, r))}{\mathcal{L}^n(B(x, r))} = \int_{\overline{B}(x, r)} h \, dy + \frac{\mu^s_k[u](\overline{B}(x, r))}{\mathcal{L}^n(B(x, r))}. \quad (3.12)
$$

Hence by letting $\epsilon \to 0$, we obtain

$$
F_k[u](x) \leq h(x) \mathcal{L}^n \text{ a.e. } x. \quad (3.13)
$$

To prove the reverse inequality, let us recall that $h$ is the density of the absolute continuous part of the measure $\mu_k[u]$, that is for $\mathcal{L}^n$ a.e. $x$

$$
h(x) = \lim_{r \to 0} \frac{\mu^ac_k[u](\overline{B}(x, r))}{\mathcal{L}^n(B(x, r))} = \lim_{r \to 0} \frac{\mu_k[u](\overline{B}(x, r))}{\mathcal{L}^n(B(x, r))}. \quad (3.14)
$$
Since $\mu^\epsilon_k[u]$ is supported on a set of Lebesgue measure zero,

$$\mu^\epsilon_k[u](\partial B(x, r)) = 0, \quad \mathcal{L}^1 \text{ a.e. } r > 0.$$ 

Therefore by the weak continuity of $\mu_k[u]$ (see, for example Theorem 1, Section 1.9 [3]), we conclude that

$$\lim_{\epsilon \to 0} \mu_k[u]_e(B(x, r)) = \mu_k[u](B(x, r)), \quad \mathcal{L}^1 \text{ a.e. } r > 0. \quad (3.15)$$

Let $\delta > 0$, then for $\epsilon < \epsilon' = \epsilon(\delta)$ and for $\mathcal{L}^1 \text{ a.e. } r > 0$, $\mathcal{L}^n \text{ a.e. } x$

$$h(x) \leq \lim_{r \to 0} \frac{(1 + \delta)\mu_k[u]_e(B(x, r))}{\mathcal{L}^n(B(x, r))}$$

$$= (1 + \delta) \lim_{r \to 0} \int_{B(x, r)} F_k[u](y) dy$$

$$= (1 + \delta) F_k[u](x) \quad (3.16)$$

By letting $\epsilon \to 0$ and finally $\delta \to 0$, we obtain

$$h(x) \leq F_k[u](x), \quad \mathcal{L}^n \text{ a.e. } x.$$ 

This proves the theorem. \qed

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**References**

[1] A.D. Alexandrov, Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it. *Leningrad State University Annals [Uchenye Zapiski] Math. Ser.* 6, (1939), 3–35 (Russian).

[2] L. Caffarelli, L. Nirenberg and J. Spruck, Dirichlet problem for nonlinear second order elliptic equations III, Functions of the eigenvalues of the Hessian. *Acta Math.* 155 (1985), 261-301.

[3] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions.* Studies in Advanced Mathematics, CRC Press, Boca Raton, Florida, 1992.
[4] E. Giusti, *Minimal surfaces and functions of bounded variation*. Birkhäuser Boston Inc, Boston, 1984.

[5] L. Hörmander, *Notions of Convexity*. Birkhäuser Boston Inc., Boston, 1994.

[6] N. Ivochkina, Solution of the Dirichlet problems for some equations of Monge-Ampère type. *Math Sb.* 128 (1985), 403–415.

[7] M. Klimek, *Pluripotential Theory*. Oxford University Press, New York, 1991.

[8] N. V. Krylov, *Nonlinear Elliptic and Parabolic equations of second order*. Reidel Pub, Co., Dordrecht, 1987.

[9] W. Rudin, *Real and Complex Analysis*. Third edition. McGraw-Hill Book Co., New York, 1987.

[10] N. S. Trudinger, Weak solutions of Hessian equations. *Comm. Partial Differential Equations* 22 (1997), 1251–1261.

[11] N. S. Trudinger, New maximum principles for linear elliptic equations. *Preprint*.

[12] N. S. Trudinger and X. J. Wang, Hessian measures I. *Topol. Methods Nonlinear Anal.* 10 (1997), 225–239.

[13] N. S. Trudinger and X. J. Wang, Hessian measures. II. *Ann. of Math.* 150 (1999), 579–604.

[14] N. S. Trudinger and X. J. Wang, Hessian measures. III. *J. Funct. Anal.* 193 (2002), 1–23.

[15] N. S. Trudinger and X. J. Wang, On the weak continuity of elliptic operators and applications to potential theory. *Amer. J. Math.* 124 (2002), 369–410.

[16] N. S. Trudinger and X. J. Wang, The Affine Plateau problem. *preprint*. Available online at http://arxiv.org/abs/math.DG/0405541

[17] W. P. Ziemer, *Weakly differentiable functions*. Springer, New York, 1989.