THE CRAMÉR CONJECTURE HOLDS WITH A POSITIVE PROBABILITY

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Abstract. We prove that a positive proportion of the intervals of any fixed scalar multiple of \( \log(X) \) in the dyadic interval \([X, 2X]\) contain a prime number. We also show that a positive proportion of the congruence classes modulo \( q \) contain a prime number smaller than any fixed scalar multiple of \( \varphi(q) \log(q) \).

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1. Introduction

1.1. Motivation. Let \( \pi(X) \) be the number of the prime numbers less than \( X \). Then the prime number theorem states that \( \pi(X) \) is asymptotically \( \text{Li}(X) = \int_0^X \frac{dt}{\log t} \approx \frac{X}{\log(X)} \). Hence, on average we have one prime number in an interval of size \( \log(X) \) inside the dyadic interval \([X, 2X]\). In this paper, we study how the prime numbers are distributed in the short intervals with the length \( \lambda \log(X) \) where \( \lambda > 0 \). First, we give the conjectural answer that is predicted by the Cramér model and then the conditional result of Gallagher proving this result by assuming the prime \( k \)-tuple conjecture of Hardy and Littlewood [HL23]. We refer the reader to the nice exposition of Soundararajan [Sou07] for further discussion of this problem and the related results. We cite the following formulation of the Cramér model from [Sou07].

Cramér’s model 1.1. The primes behave like independent random variables \( X(n) \) \((n \geq 3)\) with \( X(n) = 1 \) (the number \( n \) is ‘prime’) with probability \( 1/\log n \), and \( X(n) = 0 \) (the number \( n \) is ‘composite’) with probability \( 1 - 1/\log n \).

Define \( P_k(\lambda, X) \) to be

\[
P_k(\lambda, X) := \frac{1}{X} \# \{X \leq n \leq 2X : \pi(n + \lambda \log(X)) - \pi(n) = k \}.
\]

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It follows from the above Cramér model that; see [Sou07]
\[ \lim_{X \to \infty} P_k(\lambda, X) = e^{-\lambda \frac{e^{\lambda}}{k!}}. \]
In fact Gallagher [Gal76, Theorem 1] proved the above limit holds by assuming the prime \( k \)-tuple conjecture of Hardy and Littlewood. Moreover, by using an upper bound sieve for the \( k \)-tuple problem Gallagher [Gal76, Theorem 2], gives an unconditional exponentially decaying upper bound for \( P_k(\lambda, X) \) in terms of \( k \). In this paper, we prove the following result

**Theorem 1.2.** Let \( \lambda > 0 \) be any positive number and \( P_k(\lambda, X) \) be as above. Then,
\[ \liminf_{X \to \infty} \sum_{k \geq 1} P_k(\lambda, X) > \frac{\lambda}{4\lambda + 1} + o(1) > 0. \]
In other words, with probability at least \( \frac{\lambda}{4\lambda + 1} + o(1) > 0 \), an interval of size \( \lambda \log X \) inside the dyadic interval \([X, 2X]\) contains a prime number.

**Corollary 1.3.** Cramér conjecture holds with a positive probability.

We also show that the \( p \)-adic analogue of the above statement holds that is the Linnik’s conjecture. More precisely, let \( q \) be an integer. We show that a positive proportion of congruence classes modulo \( q \) contain a prime number smaller than \( \lambda \log(q) \phi(q) \) for any \( 0 < \lambda \) without any assumption.

**Theorem 1.4.** Let \( q \) be an integer and \( X \geq \lambda \phi(q) \log(q) \) for some fixed \( \lambda > 0 \). Then a positive proportion, that only depends on \( \lambda \), of congruence classes modulo \( q \) contains a prime number smaller than \( X \). Conversely if a positive number of congruence classes modulo \( q \) contains a prime number less than \( X \) then \( X \gg \phi(q) \log(q) \).

**Corollary 1.5.** Linnik’s conjecture holds with a positive probability.

In [Sar], we generalize our method to the class of the binary quadratic forms of discriminant \( -D \). We show that a positive proportion (independent of \( D \)) of quadratic form of discriminant \( -D \) represent a prime number less than any fixed scalar multiple of \( h(D) \log(D) \) by assuming a Littlewood type zero free region for the Dirichlet L-function \( L(s, \chi_{-D}) \) which holds for almost all \( D \).

1.2. **Further questions.** An interesting question is to give unconditional lower bounds for individual \( k \)
\[ \lim_{X \to \infty} P_k(\lambda, X) > \psi(\lambda, k) > 0, \]
and study the limiting behavior of such lower bounds as \( \lambda \) and \( k \) varies.

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2. Proof of Theorem 1.2

Proof. Let
\[ I_n := \pi(n + \lambda \log(X)) - \pi(n), \]
and
\[ R_X := \{X \leq n \leq 2X : I_n(\lambda, X) \geq 1\}. \]
By Cauchy inequality, we have
\[ R_X \left( \sum_{n=X}^{2X} I_n^2 \right) \geq \left( \sum_{n=X}^{2X} I_n \right)^2. \]
First, we give a lower bound on the right hand side of the above inequality. By a simple double counting formula, we have
\[ \sum_{n=X}^{2X} I_n = [\lambda \log(X)](\pi(2X) - \pi(X)) + O(\log(X)^2) \geq (\lambda + o(1))X. \]
Next, we give a double count formula for \( S := \sum_{n=X}^{2X} I_n \). Let \( H := \{h \in \mathbb{Z} : 0 \leq h \leq \lambda \log(X)\} \). \( S \) gives the number of the pairs of prime numbers \((p_1, p_2)\) such that \( p_1 = n + h_1 \)
\[ p_2 = n + h_2 \]
where \( X < p_1, p_2 < 2X \) and \( h_1, h_2 \in H \). We fix \( h_1 \) and \( h_2 \) and let
\[ A(X, h_1, h_2) := \sum_{n=X}^{2X} \chi_p(n + h_1)\chi_p(n + h_2), \]
where \( \chi_p \) is the characteristic function of the prime numbers. Therefore, we have
\[ \sum_{n=X}^{2X} I_n^2 = \sum_{\{h_1, h_2\} \subseteq H} A(X, h_1, h_2). \]
If \( h_1 = h_2 \), then \( A(X, h_1, h_2) = \pi(2X) - \pi(X) + O(\log(X)) \) and the contribution of the diagonal terms are
\[ \sum_{h \in H} A(X, h, h) = [\lambda \log(X)](\pi(2X) - \pi(X)) + O(\log(X)^2). \]
Next, we give an upper bound on the non-diagonal terms by applying the Selberg upper bound sieve and the result of Gallagher on the asymptotic of the average of the Hardy-Littlewood singular series. By the Selberg upper bound sieve; see [HR74, Theorem 5.7], we have
\[ A(X, h_1, h_2) \leq 4\mathcal{S}(h_1, h_2) \frac{X}{\log(X)} \times \left( 1 + O\left( \frac{\log \log(3X) + \log \log 3|h_1 - h_2|}{\log X} \right) \right), \]
where the constant implied by \( O \) term is absolute and
\[ \mathcal{S}(h_1, h_2) := \prod_p \left( 1 - \frac{\nu_p(h_1 - h_2)}{p} \right)(1 - \frac{1}{p})^{-2}, \]
\[ \nu_p(h) = \begin{cases} 1 & \text{if } p|h \\ 2 & \text{otherwise}. \end{cases} \]
\( S(h_1, h_2) \) is the singular series in the Hardy-Littlewood prime \( k \)-tuples conjecture associated to the set \( \{ h_1, h_2 \} \). By applying the above inequalities and summing over \( \{ h_1, h_2 \} \subset H \), we obtain

\[
\sum_{n=X}^{2X} I_n^2 = \sum_{h_1 \neq h_2 \leq \lambda \log(X)} A(X, h_1, h_2) \leq 4 \frac{X}{\log(X)^2} \sum_{h_1 \neq h_2 \leq \lambda \log(X)} S(h_1, h_2).
\]

(2.5)

By using Gallagher’s result on the average of the Hardy-Littlewood singular series, [Gal76, equation (3)], we have

\[
\sum_{h_1 \neq h_2 \leq \lambda \log(X)} S(h_1, h_2) = (\lambda \log(X))^2 (1 + o(1)).
\]

Therefore, by the above and the equation (2.3), we obtain

\[
\sum_{n=X}^{2X} I_n^2 \leq (4\lambda^2 + \lambda + o(1))X.
\]

By inequality (2.1), (2.2) and the above inequality, we obtain

\[
R_X \geq \left( \frac{\lambda}{4\lambda + 1} + o(1) \right) X.
\]

This completes the proof of our theorem. \( \square \)

3. Proof of Theorem 1.4

Proof. We begin by showing that if a positive proportion of congruence classes modulo \( q \) contain a prime number less than \( X \), then

\[
\varphi(q) \log(q) \ll X.
\]

The number of prime numbers less than \( X \) is asymptotically \( X/\log(X) \). We have

\[
R(X, q) \leq \pi(X) \leq \frac{X}{\log(X)}.
\]

(3.1)

Assume that a positive proportion of congruence classes contain a prime number less than \( X \). Then

\[
c\varphi(q) < R(X, q),
\]

for some positive constant \( 0 < c \) that is independent of \( q \). Then by the above inequality and inequality (3.1), we obtain

\[
c' \varphi(q) \log(q) \leq X,
\]

for some \( c' > 0 \) that only depends on \( c \). In this section, we show that the inverse of the above necessary condition holds which means the above bound is optimal. Namely, if

\[
X \geq c\varphi(q) \log(q),
\]

for some \( 0 < c \) then

\[
R(X, q) > c' \varphi(q),
\]

for some \( 0 < c' \leq 1 \) that only depends on \( 0 < c \). Assume that

\[
(3.2) \quad \varphi(q) \ll X/\log(X).
\]
Let $\pi(X, a, q)$ be the number of prime numbers $p < X$ such that $p \equiv a \mod q$. We proceed by applying the Cauchy inequality and obtain

$$R(X, q)(\sum_{a \mod q} \pi(X, a, q)^2) \geq \left(\sum_{a \mod q} \pi(X, a, q)\right)^2.$$  

Note that $\sum_{a \mod q} \pi(X, a, q) = \pi(X) \approx X/\log(X)$, is the number of prime numbers less than $X$. Hence

$$(3.3) \quad R(X, q)(\sum_{a \mod q} \pi(X, a, q)^2) \geq X^2/\log(X)^2.$$  

Next, we give a double count formula for $\sum_{a \mod q} \pi(X, a, q)^2$. Note that this sum counts the pairs of prime numbers $(p_1, p_2)$ such that $p_1, p_2 < X$ and $p_1 \equiv p_2 \mod q$. This counting problem is reduced to counting the integral solutions to the following additive problem in prime numbers $(x, y)$ and some integer $t$

$$(3.4) \quad x = y + tq,$$  

such that $0 < x, y < X$. Since $0 < x, y < X$, then $|tq| < X$ and hence $|t| < X/q$. Let $A(X, tq)$ denote the number of prime solutions $(x, y)$ to equation (3.4) such that $0 < x, y < X$. Then

$$(3.5) \quad \sum_{|t| < X/q} A(X, tq) = \sum_{a \mod q} \pi(X, a, q)^2.$$  

If $t = 0$ then it corresponds to the diagonal elements $p_1 = p_2$ and we obtain

$$A(X, 0) = \pi(X) \approx X/\log(X).$$  

If $t \neq 0$ then by Selberg upper bound sieve; see [HR74, Theorem 5.7], we have

$$A(X, tq) \leq 4\mathcal{S}(tq) \frac{X}{\log(X)^2} \times \left(1 + O\left(\frac{\log \log(3X) + \log \log 3|tq|}{\log X}\right)\right),$$  

where the constant implied by $O$ term is absolute and

$$\mathcal{S}(tq) := \prod_p \left(1 - \frac{\nu_p(tq)}{p}\right)(1 - \frac{1}{p})^{-2}$$  

$$\nu_p(tq) = \begin{cases} 1 & \text{if } p\mid tq \\ 2 & \text{otherwise} \end{cases}.$$  

(3.6) $\mathcal{S}(tq)$ is the singular series in the Hardy-Littlewood prime $k$-tuples conjecture associated to the set $\{0, tq\}$. By applying the above inequalities and summing over $0 \leq t \leq X/q$, we obtain

$$\sum_{|t| < X/q} A(X, tq) \leq X/\log(X) + 4\frac{X}{\log(X)^2} \sum_{|t| < X/q} \mathcal{S}(tq).$$  

It follows that

$$\mathcal{S}(tq) \leq \frac{q}{\varphi(q)} \mathcal{S}(t) \prod_{p|q} \left(1 + \frac{1}{p(p-2)}\right)$$  

$$\ll \frac{q}{\varphi(q)} \mathcal{S}(t).$$
Therefore,

\[ \sum_{|t|<X/q} A(X, tq) \ll X/\log(X) + \frac{q}{\varphi(q) \log(X)^2} \sum_{|t|<X/q} \Theta(t). \]

By applying the result of Gallagher on the average size of the Hardy-Littlewood singular series \[\text{Gal76}\], we obtain

\[ (3.7) \quad \sum_{a \mod q} \pi(X, a, q)^2 \ll X/\log(X) + X^2/(\varphi(q) \log(X)^2). \]

By inequalities \((3.7)\) and \((3.8)\), we obtain

\[ X^2/\log(X)^2 \ll R(X, q)(X^2/(\varphi(q) \log(X)^2) + X/\log(X)). \]

By our assumption in \((3.2)\)

\[ \varphi(q) \leq X/\log(X). \]

Hence, \(X/\log(X) \leq X^2/(\varphi(q) \log(X)^2)\) and we obtain

\[ (3.8) \quad \varphi(q) \ll R(X, d). \]

This concludes our theorem.

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