Charged particles’ tunnelling from the Kerr-Newman black hole

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In this letter, Parikh-Wilczek tunnelling framework, which treats Hawking radiation as a tunnelling process, is extended, and the emission rate of a charged particle tunnelling from the Kerr-Newman black hole is calculated. The emission spectrum takes the same functional form as that of uncharged particles and consists with an underlying unitary theory but deviates from the pure thermal spectrum. Moreover, our calculation indicates that the emission process is treated as a reversible process in the Parikh-Wilczek tunnelling framework, and the information conservation is a natural result of the first law of black hole thermodynamics.

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I. INTRODUCTION

In 2000, Parikh and Wilczek proposed an approach to calculate the emission rate at which particles tunnel across the event horizon[1]. They treat Hawking radiation as a tunnelling process, and the WKB method is used[2, 3]. In this way a corrected spectrum, which is accurate to a first approximation, is given. Their results are considered to be in agreement with an underlying unitary theory. Following this method, a lot of static or stationary rotating black holes are studied[4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. The same results, that is, Hawking radiation is no longer pure thermal, unitary theory is satisfied and information is conserved, are obtained. In this letter, we extend the Parikh-Wilczek tunnelling framework to calculate the emission rate of a charged particle from the Kerr-Newman black hole. We will treat the charged particle as a de Broglie wave. There are two difficulties to overcome. The first is that in order to do a computation, one need to find the equation of motion of a charged massive tunnelling particle. The second is how to take into account the effect of the electromagnetic field outside the hole. In fact, the first problem is solved in Ref. [13]. In similar manner we can obtain the concrete expression of the equation of motion. About the second problem, we find that in our discussion the Lagrangian density of the electromagnetic field can be expressed by a set of generalized coordinates $A_\mu = (A_t, A_1 A_2 A_3)$. But these coordinates are ignorable coordinates in dragged coordinate system. To eliminate the freedoms corresponding to these coordinates, we modify the Lagrangian function and use the WKB method to get the corrected emission spectrum.

The remainder of the letter is organized as follows. In section 2, we introduce the Painlevé-Kerr-Newman coordinate system, calculate the phase velocity of the de Broglie wave, and therefore obtain the equation of motion of a charged

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particle. In section 3, we modify the Lagrangian function and use Parikh-Wilczek tunnelling framework to get the corrected emission spectrum. Finally, in section 4, we present a short discussion about our result. Throughout the paper, the geometrized units \( G \equiv c \equiv \hbar \equiv 1 \) are used.

II. PHASE VELOCITY AND ELECTROMAGNETIC POTENTIAL

The line element of the Kerr-Newman black hole can be written in the form

\[
\begin{align*}
\text{ds}^2 &= -(1 - \frac{2Mr - Q^2}{\rho^2}) dt^2_k + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left[(r^2 + a^2) \sin^2 \theta + \frac{(2Mr - Q^2)a^2 \sin^4 \theta}{\rho^2}\right] d\varphi^2 \\
&\quad - \frac{2(2Mr - Q^2)a \sin^2 \theta}{\rho^2} dt_k d\varphi,
\end{align*}
\]

where

\[
\rho^2 \equiv r^2 + a^2 \cos^2 \theta,
\]

\[
\Delta \equiv r^2 + a^2 + Q^2 - 2Mr.
\]

The event horizon \( r = r_H \) is given by

\[
r_H = M + \sqrt{M^2 - a^2 - Q^2},
\]

and the 4-dimensional electromagnetic potential is

\[
A_\alpha = -\rho^{-2}Qr[(dt)_a - a \sin^2 \theta (d\varphi)_a].
\]

To calculate the emission rate, we should adopt the dragged coordinate system. The line element in this coordinate system is

\[
\begin{align*}
\text{ds}^2 &= -\frac{\rho^2 \Delta}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta} dt^2_d + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \\
&= \tilde{g}_{00} dt^2_d + g_{11} dr^2 + g_{22} d\theta^2.
\end{align*}
\]

where

\[
\tilde{g}_{00} = -\frac{\rho^2 \Delta}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta},
\]

\[
g_{11} = \frac{\rho^2}{\Delta},
\]

\[
g_{22} = \rho^2.
\]

Since

\[
\left(\frac{\partial}{\partial t_d}\right)^a = \left(\frac{\partial}{\partial t_k}\right)^a + \Omega \left(\frac{\partial}{\partial \varphi}\right)^a,
\]

\[
\text{ds}^2 = -(1 - \frac{2Mr - Q^2}{\rho^2}) dt^2_k + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left[(r^2 + a^2) \sin^2 \theta + \frac{(2Mr - Q^2)a^2 \sin^4 \theta}{\rho^2}\right] d\varphi^2 \\
&\quad - \frac{2(2Mr - Q^2)a \sin^2 \theta}{\rho^2} dt_k d\varphi,
\]
we can easily obtain the components of the electromagnetic potential in the dragged coordinate system

\[ A'_0 = A_a \left( \frac{\partial}{\partial t_d} \right)^a = -\rho^{-2}Qr[1 - a\Omega \sin^2 \theta], \quad A'_1 = A'_2 = 0, \quad (11) \]

where \( \Omega = -g_{03}/g_{33} \) is the dragged angular velocity. As discussed in Ref. [12], there is a coordinate singularity in the metric (6) at the radius of the event horizon. To calculate the emission rate, we should introduce the Painlevé-Kerr-Newman coordinate system.

The line element in the Painlevé-Kerr-Newman coordinate system is given in Ref. [12]. Namely,

\[
d s^2 = \hat{g}_{00} dt^2 + 2\sqrt{\hat{g}_{00}(1 - g_{11})} dt d\rho + dr^2 + \hat{g}_{00} G(r, \theta)^2 + g_{22} d\theta^2 + 2\sqrt{\hat{g}_{00}(1 - g_{11})} G(r, \theta) d\rho d\theta, \quad (12)\]

which is obtained from (6) by the coordinate transformation

\[ dt_k = dt + F(r, \theta) dr + G(r, \theta) d\theta, \quad (13) \]

where \( F(r, \theta) \) satisfies

\[ g_{11} + \hat{g}_{00} F(r, \theta)^2 = 1, \quad (14) \]

and \( G(r, \theta) \) is decided by

\[ G(r, \theta) = \int \frac{\partial F(r, \theta)}{\partial \theta} dr + C(\theta), \quad (15) \]

here, \( C(\theta) \) is an arbitrary analytic function of \( \theta \).

The components of the electromagnetic potential in the Painlevé-Kerr-Newman coordinate system is

\[ A_0 = -\rho^{-2}Qr[1 - a\Omega \sin^2 \theta], \quad A_1 = A_2 = 0. \quad (16) \]

From (16) we obtain the electromagnetic potential on the event horizon

\[ A_0|_{r_H} = -V_0 = -\frac{Qr_H}{r_H^2 + a^2}, \quad A_1 = A_2 = 0. \quad (17) \]

Similar to Ref. [14], we treat the charged particle as a de Broglie wave and we can easily obtain the expression of \( \dot{r} \). Namely,

\[ \dot{r} = v_p = -\frac{1}{2} \hat{g}_{00} = \Delta \frac{\rho^2}{2} \sqrt{\frac{\rho^2}{(\rho^2 - \Delta)[(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta]}}. \quad (18) \]

Note that to calculate the emission rate correctly, we should take into account the self-gravitation of the tunnelling particle with energy \( \omega \) and electric charge \( q \). That is, we should replace \( M \) and \( Q \) with \( M - \omega \) and \( Q - q \) in (12) and (18), respectively.

### III. EMISSION RATE

When we investigate a charged particle’s tunnelling, the effect of the electromagnetic field should be taken into account. That is, the matter-gravity system consists of the black hole and the electromagnetic field outside the hole. We write the Lagrangian function of the matter-gravity system as

\[ L = L_m + L_e, \quad (19) \]
where $L_C = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ is the Lagrangian function of the electromagnetic field corresponding to the generalized coordinates $A_\mu = (A_t, \ 0, \ 0)$ in the Painlevé-Kerr-Newman coordinate system. When a charged particle tunnels out, the system transit from one state to another. But from the expression of $L_C$ we find that $A_\mu = (A_t, \ 0, \ 0)$ is an ignorable coordinate. Moreover, in the dragged coordinate system, the coordinate $\varphi$ does not appear in the line element expressions \[^{[12]}\] and \[^{[12]}\]. That is to say, $\varphi$ is also an ignorable coordinate in the Lagrangian function $L$. To eliminate these two freedoms completely, the action for the classically forbidden trajectory should be written as

$$S = \int_{t_f}^{t_i} (L - P_{A_t} \dot{A_t} - P_\varphi \dot{\varphi}) dt,$$

which is related to the emission rate of the tunnelling particle by

$$\Gamma \sim e^{-2 \text{Im} S}.$$  

Therefore, the imaginary part of the action is

$$\text{Im} S = \text{Im} \left\{ \int_{r_i}^{r_f} \left[ P_r - \frac{P_{A_t} \dot{A_t}}{r} - \frac{P_\varphi \dot{\varphi}}{r} \right] dr \right\}$$

$$= \text{Im} \left\{ \int_{r_i}^{r_f} \left[ \int_{(0,0,0)}^{(P_r,P_{A_t},P_\varphi)} dP_r - \frac{\dot{A_t}}{r} dP_{A_t} - \frac{\dot{\varphi}}{r} dP_{\varphi} \right] dr \right\},$$

where $P_{A_t}$ and $P_\varphi$ are the canonical momentums conjugate to $A_t$ and $\varphi$, respectively. If we treat the black hole as a rotating sphere and consider the particle self-gravitation, we have

$$\dot{\varphi} = \Omega'_H,$$

and

$$J' = (M - \omega')a = P'_\varphi,$$

where $\Omega'_H$ is the dragged angular velocity of the event horizon. The imaginary part of the action can be rewritten as

$$\text{Im} S = \text{Im} \left\{ \int_{r_i}^{r_f} \left[ \int_{(0,0,J)}^{(P_r,P_{A_t},J-\omega a)} dP_r - \frac{\dot{A_t}}{r} dP_{A_t} - \frac{\Omega'_H}{r} dJ' \right] dr \right\}.$$  

We now eliminate the momentum in favor of energy by using Hamilton’s equations

$$\dot{r} = \frac{dH}{dP_r} \big|_{(r;A_t,P_{A_t},\varphi,P_\varphi)} = \frac{d(M - \omega')}{dP_r} = \frac{dM'}{dP_r},$$

$$\dot{A_t} = \frac{dH}{dP_{A_t}} \big|_{(A_t;\varphi,P_{A_t},P_\varphi)} = \frac{V_0' dQ'}{dP_{A_t}} = \frac{(Q - q')r_H}{r_H^2 + a^2} \cdot \frac{d(Q - q')}{dP_{A_t}}.$$  

Note that to derive \[^{[27]}\] we have treated the black hole as a charged conducting sphere.\[^{[18]}\]

Based on similar discussion to \[^{[1]}\] \[^{[2]}\], it follows directly that a charged particle tunnelling across the event horizon sees the effective metric of Eq. \[^{[12]}\], although with the replacements $M \rightarrow M - \omega'$ and $Q \rightarrow Q - q'$. The same substitutions in Eq. \[^{[18]}\] yield the desired expression of $\dot{r}$ as a function of $\omega'$ and $q'$. Thus, we can rewrite \[^{[25]}\] in the following explicit manner

$$\text{Im} S = \text{Im} \left[ \int_{r_i}^{r_f} \left[ \frac{2\sqrt{(\rho^2 - \Delta')}[(\rho^2 + a^2)^2 - \Delta' a^2 \sin^2 \theta]}{\Delta' \sqrt{\rho^2}} \right] (dM' - \frac{Q' r_H'}{r_H'^2 + a^2} dQ' - \Omega'_H dJ') dr \right].$$
where
\[ \Delta' = r^2 + a^2 + Q'^2 - 2M'r = (r - r'_+)(r - r'_-), \]
\[ r'_\pm = (M - \omega') \pm \sqrt{(M - \omega')^2 - a^2 - (Q - q')^2}, \]
\[ r_i = M + \sqrt{M^2 - a^2 - Q^2}, \]
\[ r_f = M - \omega + \sqrt{(M - \omega)^2 - a^2 - (Q - q)^2}. \]
We see that \( r = r'_+ = (M - \omega') + \sqrt{(M - \omega')^2 - a^2 - (Q - q')^2} \) is a pole. The integral can be evaluated by deforming the contour around the pole, so as to ensure that positive energy solution decay in time. Doing the \( r \) integral first we obtain
\[ \text{Im} S = -\frac{1}{2} \int_{(M,Q)} \frac{4\pi(M'^2 + M'\sqrt{M'^2 - a^2 - Q'^2} - \frac{1}{2}Q'^2)}{\sqrt{M'^2 - a^2 - Q'^2}} (dM' - \frac{Q'\omega_H}{r_{BH}^2 + a^2}dQ' - \Omega_H' dJ') dr. \]
Finishing the integration we get
\[ \text{Im} S = \pi[M^2 - (M - \omega)^2 + M\sqrt{M^2 - a^2 - Q^2} - (M - \omega)\sqrt{(M - \omega)^2 - a^2 - (Q - q)^2} - \frac{1}{2}(Q^2 - (Q - q)^2)] = -\frac{1}{2}\Delta S_{BH}. \]
In fact, if we bear in mind that
\[ T' = \frac{\sqrt{M'^2 - a^2 - Q'^2}}{4\pi(M'^2 + M'\sqrt{M'^2 - a^2 - Q'^2} - \frac{1}{2}Q'^2)}, \]
we easily get
\[ \frac{1}{T'}(dM' - V_0'dQ' - \Omega_H'dJ') = dS'. \]
That is, \( T' \) is a natural result of the first law of black hole thermodynamics.

The tunnelling rate is therefore
\[ \Gamma \sim \exp[-2 \text{Im} S] = e^{\Delta S_{BH}}. \]
Obviously, the emission spectrum \( T' \) deviates from the pure thermal spectrum but consists with an underlying unitary theory and takes the same functional form as that of uncharged massless particles.

**IV. CONCLUSION AND DISCUSSION**

With the Parikh-Wilczek tunnelling framework, We calculated the emission rate of a charged particle tunnelling from the Kerr-Newman black hole. The result supports Parikh-Wilczek’s conclusion, that is, the corrected spectrum
is not perfect thermal but the information is conserved during the emission process. Our calculation also indicates that the emission process satisfies the first law of black hole thermodynamics. In fact, the first law of black hole thermodynamics is a combination of the energy conservation law, \( dM - V_0 dQ - \Omega_H dJ = dQ_h \) (where \( Q_h \) denotes heat quantity), and the second law of thermodynamics, \( dS = \frac{dQ_h}{T} \). The equation of energy conservation is suitable for any process (reversible or irreversible process). But the equation \( dS = \frac{dQ_h}{T} \) is only true for a reversible process. For an irreversible process, \( dS > \frac{dQ_h}{T} \). That is, Parikh-Wilczek tunnelling framework has treated the emission process as an irreversible process. In this treatment, the black hole and the outside approach an thermal equilibrium. There is an entropy flux \( dS = \frac{dQ_h}{T} \) between the black hole and the outside. As the black hole radiate, the entropy of the black hole decreases, but the total entropy of the black hole and the outside is constant, and the information is conserved. Therefore, the information conservation is a natural result of the reversible process and the first law of black hole thermodynamics.

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