CARDINAL INVARIANTS FOR THE $G_\delta$ TOPOLOGY

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Abstract. We prove upper bounds for the spread, the Lindelöf number and the weak Lindelöf number of the $G_\delta$ topology on a topological space and apply a few of our bounds to give a short proof to a recent result of Juhász and van Mill regarding the cardinality of a $\sigma$-countably tight homogeneous compactum.

1. Introduction

All spaces are assumed to be $T_1$. The word compactum indicates a compact Hausdorff space.

Given a topological space $X$ we can consider a finer topology on $X$ by declaring countable intersections of open subsets of $X$ to be a base. The new space is called the $G_\delta$ topology of $X$ and is denoted with $X_\delta$.

There are various papers in the literature investigating what properties of $X$ are preserved when passing to $X_\delta$ and presenting bounds for cardinal invariants on $X_\delta$ in terms of the cardinal invariants of $X$ (see for example [14], [12], [20], [17]). Moreover, results of that kind have found applications to central topics in general topology like the study of covering properties in box products (see, for example, [18]), cardinal invariants for homogeneous compacta (see, for example [2], [6], [7] and [22]) and spaces of continuous functions (See [1]).

Two of the early results on this topic are Juhász’s bound $c(X_\delta) \leq 2^{c(X)}$ for every compact Hausdorff space $X$, where $c(X)$ denotes the cellularity of $X$ and Arhangel’skii’s result that the $G_\delta$ topology on a Lindelöf regular scattered space is Lindelöf. Juhász’s bound is tight in the sense that it’s not possible to prove that $c(X_\delta) \leq c(X)^\omega$ for every compact space $X$ (see [11]) and the scattered property is essential in Arhangel’skii’s result because there are compact Hausdorff spaces whose $G_\delta$-topology even has (weak) Lindelöf number $c^+$ (see [22]).

In this paper we prove various new bounds for cardinal invariants on the $G_\delta$ topology. For example we prove that $s(X_\delta) \leq 2^{s(X)}$ for every

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space $X$, where $s(X)$ is the spread of $X$ (that is, the supremum of the cardinalities of the discrete subsets of $X$). For a regular space we prove that $L(X_\delta) \leq \min \{ psw(X)^d(X), 2^{s(X)} \}$, where $L(X)$ denotes the Lindelöf degree of $X$, $psw(X)$ denotes the point-separating weight of $X$ and $d(X)$ denotes the density of $X$.

Many questions are left open. For example we don’t know whether the inequality $t(X_\delta) \leq 2^{s(X)}$ is true, where $t$ denotes the tightness, even when $X$ is a compact space.

Finally, we exploit a few of our results to give a short proof of a recent result of Juhász and van Mill on the cardinality of homogeneous compacta.

Our notation regarding cardinal functions follows [15]. The remaining undefined notions can be found in [10].

In our proofs we often use elementary submodels of the structure $(H(\mu), \in)$. Dow’s survey [9] is enough to read our paper, and we give a brief informal refresher here. Recall that $H(\mu)$ is the set of all sets whose transitive closure has cardinality smaller than $\mu$. When $\mu$ is regular uncountable, $H(\mu)$ is known to satisfy all axioms of set theory, except the power set axiom. We say, informally, that a formula is satisfied by a set $S$ if it is true when all existential quantifiers are restricted to $S$. A set $M \subset H(\mu)$ is said to be an elementary submodel of $H(\mu)$ (and we write $M \prec H(\mu)$) if a formula with parameters in $M$ is satisfied by $H(\mu)$ if and only if it is satisfied by $M$.

The downward Löwenheim-Skolem theorem guarantees that for every $S \subset H(\mu)$, there is an elementary submodel $M \prec H(\mu)$ such that $|M| \leq |S| \cdot \omega$ and $S \subset M$. This theorem is sufficient for many applications, but it is often useful (especially in cardinal bounds for topological spaces) to have the following closure property. We say that $M$ is $\kappa$-closed if for every $S \subset M$ such that $|S| \leq \kappa$ we have $S \in M$. For large enough regular $\mu$ and for every countable set $S \subset H(\mu)$ there is always a $\kappa$-closed elementary submodel $M \prec H(\mu)$ such that $|M| = 2^\kappa$ and $S \subset M$.

The following theorem is also used often: let $M \prec H(\mu)$ such that $\kappa + 1 \subset M$ and $S \in M$ be such that $|S| \leq \kappa$. Then $S \subset M$.

2. Cardinal invariants for the $G_\delta$ topology

Let’s start by listing the simplest bounds for cardinal functions of the $G_\kappa$ topology. They are probably folklore, and we include them just for the convenience of the reader.

Proposition 1.
(1) \( w(X_\kappa) \leq (w(X))^{\kappa} \).
(2) \( \chi(X_\kappa) \leq (\chi(X))^{\kappa} \).
(3) If \( X \) is regular then \( d(X_\kappa) \leq 2^{d(X)^{\kappa}} \).
(4) If \( X \) is regular, then \( \pi w(X_\kappa) \leq 2^{\pi w(X)^{\kappa}} \).

Proof. The first two items are easy.
As for the third item, recalling that \( w(X) \leq 2^{d(X)} \) for regular spaces, we have that \( d(X_\kappa) \leq w(X_\kappa) \leq w(X)^{\kappa} \leq 2^{d(X)^{\kappa}} \).
To prove the fourth item, recall that \( w(X) \leq (\pi w(X))^{\epsilon(X)} \) for every regular space \( X \). Hence \( \pi w(X_\kappa) \leq w(X_\kappa) \leq w(X)^{\kappa} \leq (\pi w(X))^{\epsilon(X)^{\kappa}} \leq (\pi w(X))^{\pi w(X)^{\kappa}} \leq 2^{\pi w(X)^{\kappa}} \).

Regularity is essential in both the third and the fourth item, as the following example shows.

Example 2. A Hausdorff space \( X \) such that:
\[
\pi w(X_\kappa) \geq d(X_\delta) > 2^{\pi w(X)} \geq 2^{d(X)}
\]
Proof. Let \( X = \beta \omega \), provided with the following topology: every principal ultrafilter is isolated. A basic neighbourhood of a non-principal ultrafilter \( p \) has the form \( \{p\} \cup A \setminus F \), where \( A \in p \) and \( F \) is a finite set. The space \( X \) has a countable \( \pi \)-base, but \( X_\delta \) is a discrete set of cardinality \( 2^\epsilon \).

The following example shows that, unlike in the case of the \( \pi \)-weight, there is no bound on the \( \pi \)-character of the \( G_\delta \)-topology on a regular space of countable \( \pi \)-character.

Example 3. For every cardinal \( \kappa \), there is a hereditarily normal space of countable \( \pi \)-character \( X(\kappa) \) such that \( \pi \chi(X(\kappa)_{\delta}) \geq \kappa \).

Proof. Let \( X(\kappa) \) be the space obtained by taking the sum of a convergent sequence and the one-point compactification of a discrete set of size \( \kappa \) and then collapsing the limit points to a single point \( \infty \). In the resulting space, every point is isolated except for \( \infty \), which nevertheless has a countable \( \pi \)-base. So \( \pi \chi(X(\kappa)) = \omega \). However, \( X(\kappa)_{\delta} \) is homeomorphic to the one-point Lindelöfication of a discrete set of size \( \kappa \). So its \( \pi \)-character is no smaller than \( \kappa \).

One of the early results regarding cardinal invariants for the \( G_\delta \) topology was proved by Juhász in \[14\] and was originally motivated by a problem of Arhangel’skii regarding the weak Lindelöf number of the \( G_\delta \) topology on a compactum. Its proof is an application of the Erdös-Rado theorem from infinite combinatorics.
Theorem 4. (Juhász) Let $X$ be a countably compact regular space. Then $c(X_\delta) \leq 2^{c(X)}$.

Note that regularity is essential in the above theorem as Vaughan \cite{Vaughan} constructed a countably compact Hausdorff space with points $G_\delta$ and cardinality larger than the continuum which is even separable.

We also exploit the Erdős-Rado theorem in our next result. Recall that regularity is essential in the above theorem as Vaughan \cite{Vaughan}.

Theorem 5. Let $X$ be any space and $\kappa$ be a cardinal. Then $s(X_\kappa) \leq 2^{s(X)\cdot\kappa}$.

Proof. Without loss of generality we can assume that $s(X) \leq \kappa$. Suppose by contradiction that there is a discrete set $D \subset X_\kappa$ of cardinality $\geq (2^\kappa)^+$. For every $x \in D$ we can find a $G_\kappa$ set $G_x$ in $X$ such that $G_x \cap D = \{x\}$. Let $\{U^x_\alpha : \alpha < \kappa\}$ be a sequence of open sets such that $G_x = \bigcap\{U^x_\alpha : \alpha < \kappa\}$. Let $\prec$ be a linear ordering on $X$. For every $\alpha, \beta < \kappa$ let $C_{\alpha, \beta} = \{\{x, y\} \in |D|^2 : x \prec y \land x \notin U^y_\alpha \land y \notin U^x_\beta\}$. Then $\{C_{\alpha, \beta} : (\alpha, \beta) \in \kappa^2\}$ is a coloring of $|D|^2$ into $\kappa$ many colors. By the Erdős-Rado theorem we can find a set $T \subset D$ of cardinality $\kappa^+$ and a pair of ordinals $(\gamma, \delta) \in \kappa^2$ such that $|T|^2 \subset C_{\gamma, \delta}$. Note now that $U^x_\gamma \cap U^y_\delta \cap T = \{x\}$ for every $x \in T$. Hence $T$ is a discrete subset of $X$ of cardinality $\kappa^+$, which contradicts $s(X) = \kappa$.

Corollary 6. (Hajnal and Juhász) Let $X$ be a $T_1$ space. Then $|X| \leq 2^{s(X)\cdot\psi(X)}$.

Proof. Set $\kappa = s(X)\cdot\psi(X)$. By the above theorem we have $s(X_\kappa) \leq 2^\kappa$, but since $X_\kappa$ is discrete we must have $|X| \leq 2^\kappa$.

The next example shows that $2^{s(X)\cdot\kappa}$ cannot be replaced with $s(X)^\kappa$ in Theorem 5 even for compact LOTS.

Example 7. There is a compact linearly ordered space $L$ such that $s(L_\delta) > s(L)^\omega$.

Proof. Fleissner constructed in \cite{Fleissner} a compact linearly ordered space $L$ such that $c(L) \leq \mathfrak{c}$ and $L$ has a $\mathfrak{c}^+$-sized subset $S$ consisting of $G_\delta$ points. Since $c(X) = s(X)$ for every linearly ordered space $X$ we must have $s(L) \leq \mathfrak{c}$, but it’s clear that $s(L_\delta) \geq \mathfrak{c}^+$.

Recall that the Lindelöf degree of a topological space $X$ ($L(X)$) is defined as the minimum cardinal $\kappa$ such that for every open cover of $X$ has a $\kappa$-sized subcover.

The weak Lindelöf degree of $X$ ($wL(X)$) is defined as the minimum cardinal $\kappa$ such that, for every open cover $U$ of $X$ there is a $\kappa$-sized subcollection $V \subset U$ such that $X \subset \bigcup V$. 


At the 1970 International Congress of Mathematicians in Nice, France, Arhangel’skii asked whether the weak Lindelöf degree of a compact space with its $G_δ$ topology is always bounded by the continuum. A counterexample has recently been given in [22] but various related bounds for the (weak) Lindelöf number of the $G_δ$ topology have been presented in the literature (see, for example [12], [20], [14] and [7]).

A set $G \subset X$ is called a $G_κ^c$ set if there is a family $\{U_α : α < κ\}$ of open subsets of $X$ such that $G = \bigcap\{U_α : α < κ\} = \bigcap\{\overline{U_α} : α < κ\}$.

Given a space $X$, we denote with $X_κ^c$ the topology generated by the $G_κ^c$ subsets of $X$. Obviously if $X$ is regular, then $X_κ = X_κ^c$.

**Theorem 8.** Let $X$ be any space and $κ$ be a cardinal. Then $L(X_κ^c) \leq 2^{s(X)·κ}$.

**Proof.** Without loss we can assume $s(X) \leq κ$. Fix a cover $F$ of $X$ by $G_κ^c$ sets.

Let $θ$ be a large enough regular cardinal and $M$ be a $κ$-closed elementary submodel of $H(θ)$ such that $X, F ∈ M, 2^κ + 1 \subset M$ and $|M| = 2^κ$.

For every $F ∈ F$ choose open sets $\{U_α(F) : α < κ\}$ witnessing that $F$ is a $G_κ^c$-set. Note that when $F ∈ F$ we can assume that $\{U_α(F) : α < κ\} \subset M$ and hence $\{U_α(F) : α < κ\} \subset M$.

**Claim 1.** $F \cap M$ covers $X \cap M$.

**Proof of Claim 1.** Suppose this is not true and let $p ∈ X \cap M \setminus \bigcup(F \cap M)$. For every $x ∈ X \cap M$ we can find $F_x ∈ F \cap M$ such that $x ∈ F_x$. Moreover, there must be $α(x) < κ$ such that $p \notin U_{α(x)}(F_x)$. Now, $O = \{U_{α(x)}(F_x) : x ∈ X \cap M\}$ is an open cover of $X \cap M$. By Shapirovskii’s Lemma (see [13]) there is a discrete set $D ⊂ X \cap M$ and a subcollection $U ⊂ O$ with $|U| = |D| ≤ κ$ such that $X \cap M ⊂ \overline{D} \cup \bigcup U$. By $κ$ closedness of $M$ we have $D, U ∈ M$ hence $M \models X ⊂ \overline{D} \cup \bigcup U$. Therefore by elementarity $H(θ) \models X \subset \overline{D} \cup \bigcup U$. Since $p \notin \bigcup U$ we must have $p ∈ \overline{D}$.

Let now $F$ be an element of $F$ such that $p ∈ F$. We have $p ∈ \overline{U_α(F)} \cap \overline{D}$ for every $α < κ$ and $\overline{U_α(F)} \cap \overline{D} ∈ M$, by $κ$-closedness of $M$. Define $B = \bigcap\{\overline{U_α(F)} \cap \overline{D} : α < κ\}$. Then $B ∈ M$. Note that we have $H(θ) \models (∃G ∈ F)(B \subset G)$, hence by elementarity $M \models (∃G ∈ F)(B \subset G)$, which implies the existence of $H ∈ F \cap M$ such that $p ∈ B ⊂ H$. But this contradicts the fact that $p \notin \bigcup(F \cap M)$. Hence $F \cap M$ covers $X \cap M$ and the claim is proved.

**Claim 2.** $F \cap M$ covers $X$. 

△
**Proof of Claim 2.** Suppose this is not true and let $p$ be a point of $X \setminus \bigcup (\mathcal{F} \cap M)$. For every $F \in \mathcal{F} \cap M$ we can find $\beta(F) < \kappa$ such that $p \notin U_{\beta(F)}(F)$.

It follows from Claim 1 that the family $\mathcal{V} := \{U_{\beta(F)}(F) : F \in \mathcal{F} \cap M\}$ is an open cover of $X \cap M$. By Shapirovskii’s Lemma we can find a discrete $D \subset X \cap M$ and a family $W \subset \mathcal{V}$ such that $|W| = |D| < \kappa$ and $X \cap M \subset D \cup \bigcup W$. Note that $D, W \in M$, by $\kappa$-closedness of $M$. This implies that $M \models X \subset D \cup \bigcup W$ by elementarity. But this is a contradiction because $p \notin W$, for every $W \in \mathcal{W}$ and since $D \subset X \cap M$ we also have that $p \notin D$.

Since $|M| \leq 2^\kappa$ it follows that $\mathcal{F} \cap M$ is a $2^\kappa$-sized subfamily of $\mathcal{F}$ covering $X$ and hence we are done. △

It’s not possible to replace $X_\delta$ with $X_\delta$ in the above result, as the following example shows.

**Example 9.** There are $T_1$ spaces $X$ of countable spread where $\ell(X_\delta)$ can be arbitrarily large.

**Proof.** Let $\kappa$ be a cardinal of uncountable cofinality and $\mu = cf(\kappa)$. Define a topology on $X = \kappa$ by declaring sets of the form $[0, \alpha] \setminus F$ to be a base, where $\alpha$ is an ordinal less than $\kappa$ and $F$ is a finite set. It is easy to see that $s(X) = \omega$. Moreover $\{[0, \alpha] : \alpha < \kappa\}$ is an open cover of $X$ without subcovers of cardinality less than $\mu$ and hence $\ell(X_\delta) \geq \mu$. □

However, for regular spaces, the $G_\delta$ modification and the $G_\delta^c$ modification coincide, so we obtain the following result:

**Theorem 10.** Let $X$ be a regular space. Then $\ell(X_\kappa) \leq 2^{s(X) \cdot \kappa}$.

Recall that the tightness of a point $x$ in the space $X$ ($t(x, X)$) is defined as the minimum cardinal $\kappa$ such that for every subset $A$ of $X$ with $x \in \overline{A} \setminus A$ there is a subset $B \subset A$ such that $|B| \leq \kappa$ and $x \in \overline{B}$. The tightness of the space $X$ is then defined as $t(X) = \sup \{t(x, X) : x \in X\}$. A space of countable tightness is also called countably tight.

**Theorem 11.** Let $X$ be a countably compact space with a dense set of points of countable character. Then $w\ell(X_\kappa) \leq 2^{t(X) \cdot w\ell_c(X) \cdot \kappa}$.

**Proof.** Without loss of generality we can assume that $w\ell_c(X) \cdot t(X) \leq \kappa$. Fix a cover $\mathcal{F}$ of $X$ by $G_\kappa$ sets

Let $\theta$ be a large enough regular cardinal and $M$ be a $\kappa$-closed elementary submodel of $H(\theta)$ such that $X, \mathcal{F} \in M$ and $|M| = 2^\kappa$. □
Proof of Claim 1. Let \( F \subseteq X \setminus M \) and use \( t(X) \leq \kappa \) to fix a \( \kappa \)-sized set \( A \subseteq X \setminus M \) such that \( x \notin A \). Note \( A \subseteq M \). Let \( F \subseteq \mathcal{F} \) be such that \( x \in F \) and let \( \{ U_\alpha : \alpha < \kappa \} \) be a sequence of open sets witnessing that \( F \) is a \( G^*_\kappa \) set.

Note that the set \( B = \bigcap \{ A \cap U_\alpha : \alpha < \kappa \} \) is in \( M \) and \( x \in B \subseteq F \). Now \( H(\theta) \mid (\exists F \subseteq \mathcal{F}) (B \subseteq F) \). Hence \( M \mid (\exists F \subseteq \mathcal{F}) (B \subseteq F) \). Therefore we can find \( G \subseteq \mathcal{F} \cap M \) such that \( x \in B \subseteq G \), which is what we wanted. \( \triangle \)

Claim 2. \( \mathcal{F} \cap M \) has dense union in \( X \).

Proof of Claim 2. Suppose not and let \( p \in X \setminus \bigcup \mathcal{F} \cap M \) be a point of countable character. Fix a local base \( \{ V_n : n < \omega \} \) at \( p \).

For every \( x \in X \setminus M \) pick \( F_x \subseteq \mathcal{F} \cap M \) such that \( x \in F_x \) and let \( \{ V_\alpha^x : \alpha < \kappa \} \subseteq M \) be a sequence of open sets witnessing that \( F_x \) is a \( G^*_\kappa \) set. Since \( p \notin F_x \), there must be \( \alpha < \kappa \) such that \( p \notin V_\alpha^x \). Hence there must be \( n_x < \omega \) such that \( V_{n_x} \cap V_\alpha^x = \emptyset \). let \( U_n = \bigcup \{ V_\alpha^x : n_x = n \} \). Then \( \{ U_n : n < \omega \} \) is a countable open cover of the countably compact space \( X \setminus M \). So there is \( k < \omega \) such that \( \{ U_n : n < k \} \) covers \( X \setminus M \). Let now \( \mathcal{U} = \{ U_\alpha^x : n_x < k \} \). Then \( \mathcal{U} \) covers \( X \setminus M \), hence \( wL^c(X) \leq \kappa \) implies the existence of \( \mathcal{V} \subseteq [\mathcal{U}]^\kappa \) such that \( X \cap M \subseteq \bigcup \mathcal{V} \). But that implies \( M \mid X \subseteq \bigcup \mathcal{V} \) and hence \( H(\theta) \mid X \subseteq \bigcup \mathcal{V} \), which contradicts \( V_k \cap (\bigcup \mathcal{V}) = \emptyset \). \( \triangle \)

Corollary 12. (Alas, [3]) Let \( X \) be a countably compact \( T_2 \) space with a dense set of points of countable character. Then \( |X| \leq 2^{|wL^c(X) \cap t(X)|} \).

Corollary 13. Let \( X \) be a regular countably compact space with a dense set of points of countable character. Then \( wL(X) \leq 2^{wL_c(X) \cap t(X) \cdot \kappa} \).

Corollary 14. Let \( X \) be a normal countably compact space with a dense set of points of countable character. Then \( wL(X) \leq 2^{wL(X) \cap t(X) \cdot \kappa} \).

In a similar way we can prove the following theorem:

Theorem 15. Let \( X \) be a space with a dense set of isolated points. Then \( wL(X) \leq 2^{wL_c(X) \cap t(X) \cdot \kappa} \).

Question 1. Is it true that \( wL(X) \leq 2^{wL_c(X) \cap t(X) \cdot \kappa} \) for any Hausdorff space \( X \)?
We call a cover $\mathcal{U}$ of a space $X$, strongly point-separating if $\bigcap\{\overline{U} : U \in \mathcal{U} \land x \in U\} = \{x\}$.

We define $\text{psw}_{s}(X)$ to be the least cardinal $\kappa$ such that $X$ admits a strongly point-separating open cover of order $\kappa$. Obviously $\text{psw}_{s}(X) = \text{psw}(X)$ for every regular space $X$.

**Theorem 16.** Let $X$ be a $T_2$ space. Then $L(X,\kappa) \leq \text{psw}_{s}(X)^{L(X),\kappa}$.

**Proof.** Let $\lambda = \text{psw}_{s}(X)$ and fix a strongly point-separating open cover $\mathcal{U}$ of $X$ of order $\leq \lambda$. We can assume $L(X) \leq \kappa$. Let $\mathcal{F}$ be a $G_\kappa$ cover of $X$. Since $L(X) \leq \kappa$ we can assume that $\mathcal{F}$ is made up of $\kappa$-sized intersections of elements of $\mathcal{U}$. Let $M$ be a $\kappa$-closed elementary submodel of $H(\theta)$ such that $\lambda^\kappa \subseteq M$, $X, \mathcal{U}, \mathcal{F} \in M$ and $|M| = \lambda^\kappa$.

**Claim 1.** $\mathcal{F} \cap M$ covers $\overline{X \cap M}$.

**Proof of Claim 1.** Let $p \in \overline{X \cap M}$. Let $F \in \mathcal{F}$ be such that $p \in F$. Let $\{U_\alpha : \alpha < \kappa\} \subseteq \mathcal{U}$ be a family of open sets such that $F = \bigcap \{U_\alpha : \alpha < \kappa\}$. Let $x_\alpha$ be any point in $U_\alpha \cap M$. Note that for every $\alpha < \kappa$ we have that $\{U \in \mathcal{U} : x_\alpha \in U\}$ is an element of $M$ of cardinality $\lambda$. Therefore $\{U \in \mathcal{U} : x_\alpha \in U\} \subseteq M$ and hence $U_\alpha \in M$, for every $\alpha < \kappa$. By $\kappa$-closedness of $M$ we have $F = \bigcap \{U_\alpha : \alpha < \kappa\} \subseteq M$, as we wanted. $\triangle$

**Claim 2.** $\mathcal{F} \cap M$ actually covers $X$.

**Proof of Claim 2.** Suppose that is not true and let $p \in X \setminus \bigcup (\mathcal{F} \cap M)$. For every $x \in \overline{X \cap M}$, let $F_x \in \mathcal{F} \cap M$ such that $x \in F_x$ and let $\{U_\alpha^x : \alpha < \kappa\} \subseteq M$ be a sequence of open sets such that $\bigcap \{U_\alpha^x : \alpha < \kappa\} = F_x$. We again have that $\{U_\alpha^x : \alpha < \kappa\} \subseteq M$ and hence, for every $x \in \overline{X \cap M}$ we can find an open neighbourhood $U_x \in M$ of $x$ such that $p \notin U_x$. The family $\mathcal{V} := \{U_x : x \in \overline{X \cap M}\}$ is an open cover of the space $\overline{X \cap M}$, which has Lindelöf number at most $\kappa$ and hence we can find $\mathcal{C} \subseteq [\mathcal{V}]^\kappa$ such that $X \cap M \subseteq \overline{X \cap M} \subseteq \bigcup \mathcal{C}$. By $\kappa$-closedness of $M$ we have $\mathcal{C} \subseteq M$ and hence the previous formula implies $M \models X \subseteq \bigcup \mathcal{C}$. By elementarity we get that $H(\theta) \models X \subseteq \bigcup \mathcal{C}$, which contradicts the fact that $p \notin \bigcup \mathcal{C}$. $\triangle$

**Corollary 17.** Let $X$ be a regular space. Then $L(X,\kappa) \leq \text{psw}(X)^{L(X),\kappa}$.

**Question 2.** Is $t(X_\delta) \leq 2^{t(X)}$ true for every (compact) $T_2$ space $X$?

3. AN APPLICATION TO HOMOGENEOUS COMPACTA

**Definition 18.** Let $X$ be a topological space. A set $S \subseteq X$ is called subseparable if there is a countable set $C \subseteq X$ such that $S \subseteq \overline{C}$.
Since \( w(X) \leq 2^{d(X)} \) for every regular space \( X \) and the weight is hereditary every subseparable subspace of a regular topological space has weight at most continuum.

**Lemma 19.** (Juhász and van Mill, [16]) Let \( X \) be a \( \sigma \)-countably tight homogeneous compactum. Then \( X \) contains a non-empty subseparable \( G_\delta \)-subset and has a point of countable \( \pi \)-character.

**Corollary 20.** Every \( \sigma \)-countably tight homogeneous compactum has character at most continuum.

**Proof.** Let \( x \in X \) be any point. By homogeneity we can find a subseparable \( G_\delta \) set \( G \) containing \( x \). Then \( w(G) \leq 2^\omega \). So we can fix a continuum-sized family \( \mathcal{U} \) of open neighbourhoods of \( x \) such that \( G \cap \bigcap \mathcal{U}_x = \{x\} \). Let \( \{U_n : n < \omega\} \) be a countable family of open sets such that \( G = \bigcap\{U_n : n < \omega\} \). Then \( \mathcal{V} = \{U_n : n < \omega\} \cup \mathcal{U} \) is a continuum sized family of open subsets of \( X \) such that \( \bigcap \mathcal{V} = \{x\} \). Since \( X \) is compact, this implies that \( \chi(x, X) \leq 2^\omega \). \( \square \)

**Theorem 21.** Let \( X \) be a homogeneous compactum which is the union of countably many countably tight dense subspaces. Then \( L(X_\delta) \leq 2^\omega \).

**Proof.** Let \( \{X_n : n < \omega\} \), be a countable family of countably tight subspaces covering \( X \). Let \( \mathcal{U} \) be a \( G_\delta \)-cover of \( X \). Without loss we can assume that for every \( U \in \mathcal{U} \) there are open sets \( \{O_n(U) : n < \omega\} \) such that \( O_{n+1}(U) \subseteq O_n(U) \), for every \( n < \omega \) and \( U = \bigcap\{O_n(U) : n < \omega\} \).

Let \( \theta \) be a large enough regular cardinal and let \( M \) be an \( \omega \)-closed elementary submodel of \( H(\theta) \) such that \( |M| = 2^\omega \) and \( M \) contains everything we need.

**Claim.** \( \mathcal{U} \cap M \) covers \( \overline{X \cap M} \).

**Proof of Claim.** Let \( x \in \overline{X \cap M} \).

We claim that \( x \in X_n \cap M \), for every \( n < \omega \). Indeed, fix \( n < \omega \) and let \( V \) be a neighbourhood of \( x \). Pick \( y \in V \cap (X \cap M) \). Then \( y \) has a local base \( \mathcal{U}_y \subseteq M \) having cardinality continuum. By the assumptions on \( M \), \( \mathcal{U}_y \subseteq M \). Since \( X_n \) is dense in \( X \), \( M \) reflects this and therefore for every \( U \in \tau \cap M \) we have \( U \cap X_n \cap M \neq \emptyset \). Hence for every \( U \in \mathcal{U}_y \) we have \( U \cap X_n \cap M \neq \emptyset \). It turns out that \( V \cap X_n \cap M \neq \emptyset \), for every open neighbourhood \( V \) of \( x \), as we wanted.

Let \( k < \omega \) be such that \( x \in X_k \). Using the fact that \( X_k \) has countable tightness we can choose a countable set \( C_k \subseteq X_k \cap M \) such that \( x \in \overline{C_k} \). Note that, since \( M \) is countably closed, every subset of \( C \) is an element of \( M \). Since \( \mathcal{U} \) covers \( X \) there is \( U \in \mathcal{U} \) such that \( x \in U \). Note that \( x \in \overline{O_i(U)} \cap C \), for every \( i < \omega \). Let \( B = \bigcap\{O_i(U) \cap C : i < \omega\} \) and note that \( B \in M \). We have \( H(\theta) \models (\exists U \in \mathcal{U})(B \subseteq U) \). Since every
free variable in the previous formula belongs to $M$, by elementarity we have $M \models (\exists U \in \mathcal{U})(B \subset U)$ and hence there is $U \in \mathcal{U} \cap M$ such that $x \in B \subset U$, which finishes the proof of the Claim. \hfill \triangle$

Let us now prove that $\mathcal{U} \cap M$ actually covers $X$, which will finish the proof.

Suppose this is not the case and let $p \in X \setminus \bigcup(\mathcal{U} \cap M)$. By the claim, for every $x \in X \cap M$ we can pick a $U_x \in \mathcal{U} \cap M$ containing $x$. Then we can choose $m < \omega$ such that $p \notin O_m(U_x) \in M$. This means that we can cover $X \cap M$ by an open family $\mathcal{V} \subset M$ such that $p \notin \bigcup \mathcal{V}$. By compactness we can then take a finite subfamily $\mathcal{F}$ of $\mathcal{U}$ such that $X \cap M \subset \bigcup \mathcal{F}$. Since $\mathcal{F} \in M$ this is equivalent to $M \models X \subset \bigcup \mathcal{F}$, which implies, by elementarity, $H(\theta) \models X \subset \bigcup \mathcal{F}$, and that is a contradiction because $p \in H(\theta) \setminus \bigcup \mathcal{F}$. □

Lemma 22. Let $X$ be a compact homogeneous space which is the union of finitely many countably tight subspaces. Then $L(X_\delta) \leq 2^{\omega}$.

Proof. Let $\mathcal{F}$ be a finite cover of $X$ by countably tight subspaces. We can find a non-empty open subset $V$ of $X$ such that $V \cap F$ is dense in $V$, whenever $V \cap F \neq \emptyset$ and $F \in \mathcal{F}$. Applying the argument proving Lemma 21 to $V$ we obtain that $L(V_\delta) \leq 2^{\aleph_0}$. Using the homogeneity of $X$ we can find an open cover $\mathcal{V}$ of $X$ such that $L(\mathcal{V}_\delta) \leq 2^{\aleph_0}$, for every $V \in \mathcal{V}$. Choosing a finite subcover of $\mathcal{V}$ we see that $L(X_\delta) \leq 2^{\omega}$.

The following lemma was noted independently by de la Vega and Ridderbos (see [21] for the proof of a much more general statement).

Lemma 23. Let $X$ be a homogeneous space. Then $|X| \leq d(X)^{\pi_X(X)}$.

Theorem 24. (Juhász and van Mill) Let $X$ be a compact homogeneous space which is the union of countably many dense countably tight subspaces or of finitely many countably tight subspaces. Then $|X| \leq 2^{\omega}$.

Proof. Use homogeneity to fix, for every $x \in X$, a subseparable $G_\delta$ set $G_x$ containing $x$. We have $w(G_x) \leq 2^{\omega}$. Note that $\mathcal{U} = \{G_x : x \in X\}$ is a $G_\delta$ cover of $X$, so there is $C \in [X]^{2^{\omega}}$ such that $X \subset \bigcup\{G_x : x \in C\}$. For every $x \in C$, we can fix a continuum-sized $D_x \subset G_x$, dense in $G_x$. Then $D = \bigcup\{D_x : x \in C\}$ is a dense subset of $X$ having cardinality at most continuum, proving that $d(X) \leq 2^{\omega}$. Using the above lemmas we obtain that $|X| \leq 2^{\omega}$. □

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