Multiple quadrotors carrying a flexible hose: dynamics, differential flatness, control

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Abstract: Using quadrotors UAVs for cooperative payload transportation using cables has been actively gaining interest in the recent years. Understanding the dynamics of these complex multi-agent systems would help towards designing safe and reliable systems. In this work, we study one such multi-agent system comprising of multiple quadrotors transporting a flexible hose. We model the hose as a series of smaller discrete links and derive a generalized coordinate-free dynamics for the same. We show that certain configurations of this under-actuated system are differentially-flat. We linearize the dynamics using variation-based linearization and present a linear time-varying LQR to track desired trajectories. Finally, we present numerical simulations to validate the dynamics, flatness and control.

Keywords: Coordinate-free dynamics, variation based linearization, co-operative control, aerial manipulation, differential-flatness

1. INTRODUCTION

Aerial manipulation has been an active research area for many years now, due to the simplicity of the dynamics and control of multi-rotors. The ubiquity of these aerial vehicles resulted in their use in a wide range of applications. Few such applications include search and rescue [Bernard et al. (2011)] and disaster management, for instance, using UAVs to monitor forest fires [Merino et al. (2012)]. Payload delivery using aerial vehicles [X-Wing (2019), PrimeAir (2019), Palunko et al. (2012)] is another application that has earned much attention in the last few years.

One extension of the payload carrying research is developing multi-rotor vehicles for active fire-fighting [Aerones (2018)] using a tethered hose that carries water and power. This enables carrying a fire hose to heights higher than a typical fire-truck ladder as well as fly longer due to the tethered power supply. Multi-rotors are also used to help string power cables between poles [SkyScopes (2017)], which typically is achieved using manned helicopters. To achieve stable and safe control of these complex systems, it is important to understand the underlying governing principles and dynamics. In this work, we aim to model and control the dynamics of a multiple quadrotor system carrying a flexible cable/hose.

1.1 Related Work

There is a lot of literature on co-operative aerial manipulation, especially towards grasping and transporting payloads using multiple quadrotors [Maza et al. (2009), Mellinger et al. (2013), Jiang and Kumar (2012), Lee and Kim (2017), Michael et al. (2011)]. Trajectory tracking control for point-mass/rigid-body payloads suspended from multiple quadrotors is studied in [Lee et al. (2013), Goodarzi and Lee (2016), Sreenath and Kumar (2013), Wu and Sreenath (2014)]. Similarly, for loads suspended using flexible cables, stabilizing controllers are presented in [Goodarzi et al. (2015), Goodarzi and Lee (2015)] and these systems are shown to be differentially-flat in [Kotaru et al. (2018)]. Tethered aerial vehicles have also been extensively studied in the literature, for instance stabilization of tethered quadrotor and nonlinear-observers for the same are discussed in [Lupashin and D’Andrea (2013), Nicotra et al. (2014), Tognon and Franchi (2015)]. Geometric control of a tethered quadrotor with a flexible tether is presented in [Lee (2015)].
Most of the work discussed in the previous section models the tethers/cables either as rigid-links or as a series of links. Partial differential equations have also been used to model a continuous mass system, such as the aerial refueling cable shown in [Liu et al. (2017)]. However, modeling the aerial cable as a finite-segment lumped mass [Williams and Trivailo (2007), Ro and Kamman (2010)] is quite common in the literature due to the finite dimensionality of the state-space. However, most of these works assume Euler angles in the local frame to represent the attitude of the links. This results in complex equations of motion for the system that are also prone to singularities in case of aggressive motions. Therefore, in this work, we make use of coordinate-free representation that results in singularity-free and compact equations of motion.

1.2 Challenges

Multiple quadrotors carrying a flexible hose has multiple challenges in both modeling the dynamics and also designing a controller. Even though modeling the hose as a finite-segment lumped mass results in a finite-dimensional state space, it would still result in a large number of states depending on the choice of the number of discrete links. In addition, developing a controller is challenging due to the high under-actuation in the system. The swing of the hose. The contributions of this work are as follows,

- We derive a generalized coordinate-free dynamics for multiple quadrotors carrying a flexible hose. These dynamics can be extended to a tethered multiple quadrotor system.
- We show that this system is differentially-flat for certain configurations.
- We present variation-based linearized dynamics and implement a time-varying LQR to track a time-varying desired trajectory.
- Finally, we present numerical simulations to validate the dynamics and control.

To the best of authors knowledge, this is novel configuration of multiple quadrotors with a flexible hose and has not been studied prior to this work.

1.3 Contributions

In this paper, we build upon the work done in the literature to develop the dynamics and control of multiple quadrotors carrying a flexible hose. This work is a step towards developing a system, with multiple quadrotor carrying a water-hose. However, for the purpose of this paper and as a first step, we consider no water flow in the hose. The contributions of this work are as follows,

- We present variation-based linearized dynamics and implement a time-varying LQR to track a time-varying desired trajectory.
- Finally, we present numerical simulations to validate the dynamics and control.

To the best of authors knowledge, this is novel configuration of multiple quadrotors with a flexible hose and has not been studied prior to this work.

1.4 Organization

Rest of the paper is organized as follows. Section 2 explains the system definition, notations and presents the derivation of the dynamics. In Section 3 we show that the system is differentially-flat. In section 4, we present a LQR control on the variation-linearized dynamics. Section 5 presents numerical simulations validating the proposed controller. Finally, some of the limitations in this paper and potential directions to address them are discussed in Section 6. Concluding remarks are in Section 7.

2. DYNAMICS

Consider a flexible hose connected to multiple quadrotor UAVs as shown in Figure 1. In this section, we present the coordinate-free dynamics for this system. We consider the following assumptions before proceeding to derive the dynamics:

A1. No water/water-flow in the hose and thus also no pressure forces;
A2. Hose is modeled as a series of n smaller links connected by spherical joints;
A3. Each link is massless with lumped point-masses at the end with the hose mechanical properties like stiffness and torsional forces ignored.
A4. The quadrotors attach to the hose at their respective center-of-masses.

In the following section, we present the notation used to describe the system.

2.1 Notation

Dynamics for the model are defined using geometric-representation of the states. Each link is a spherical-joint and is represented using a unit-vector \( q \in \mathbb{S}^2 := \{x \in \mathbb{R}^3 \mid \|x\| = 1\} \). The position of one end of the cable is given in \( \mathbb{R}^3 \) and finally, the rotation matrix \( R \in SO(3) := \{R \in \mathbb{R}^{3 \times 3} | R^T R = 1, det(R) = +1\} \) is used to represent the attitude of the quadrotor.

Let the hose be discretized into \( n \) links with the cable joints indexed as \( S = \{0, 1, \ldots, n\} \) as shown in Figure 1. The position of one (starting) end of the hose is given as \( x_0 \in \mathbb{R}^3 \) in the world-frame. The position of the link joints/point-masses is represented by \( x_i \in \mathbb{R}^3 \), where the link attitude between \( x_{i-1} \) and \( x_i \) is given by \( q_i \in \mathbb{S}^2 \) and length of this link-segment is \( l_i \), i.e., \( x_i = x_{i-1} + l_i q_i \). Also, \( m_i \) is the mass of the lumped point-mass for link \( i \). Let the set \( I \subseteq S \) be the set of indices where the cable is attached to the quadrotor and \( n_I = |I| \) is the number of quadrotors.

For the \( j^{th} \) quadrotor, \( R_j \in SO(3) \) is the attitude, \( m_{Qj}, J_j \) is its mass and inertia matrix (in body-frame) and \( f_j \in \mathbb{R}, M_j \in \mathbb{R}^3 \) are the corresponding thrust and moment for all \( j \in I \). Finally, the configuration space of this system is given as \( Q := \mathbb{R}^3 \times (\mathbb{S}^2)^n \times (SO(3))^n \).

Table 1 lists the various symbols used in this paper.

2.2 Derivation

The kinematic relation between the different link positions is given using link attitudes as,

\[
x_i = x_0 + \sum_{k=1}^{i} l_k q_k, \quad \forall i \in S \setminus \{0\},
\]

(1)

and the corresponding velocities and accelerations are related as,

\[
v_i = v_0 + \sum_{k=1}^{i} l_k \dot{q}_k, \quad a_i = a_0 + \sum_{k=1}^{i} l_k \ddot{q}_k.
\]

(2)

Potential energy \( U : TQ \rightarrow \mathbb{R} \) of the system, due to hose and quadrotors’ mass is computed as shown below,

\[
U = \sum_{i \in \mathbb{S}} m_i x_i \cdot \hat{e}_3,
\]

(3)

where \( m_i = m_i + m_{Qj} \) is the net-mass at index \( i \) and \( 1_i := 1_I(i) = \begin{cases} 1, & \text{if } i \in I \\ 0, & \text{else} \end{cases} \) is an indicator function for the set \( I \).
Equations of motion for multiple quadrotors carrying a flexible hose

\[ \dot{x}_0 = v_0, \quad \dot{\omega}_i = \omega_i \times q_i, \]  

\[
\begin{bmatrix}
M_{00}I_3 & -q_1^x M_{01} & -q_2^x M_{02} & \ldots & -q_n^x M_{0n} \\
-M_{10}q_1^x & -M_{11}I_3 & M_{12}q_2^x & \ldots & M_{1n}q_n^x \\
-M_{20}q_2^x & M_{21}q_1^x & -M_{22}I_3 & \ldots & M_{2n}q_n^x \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-M_{n0}q_n^x & M_{n1}q_1^x & M_{n2}q_2^x & \ldots & -M_{nn}I_3 \\
\end{bmatrix} = \begin{bmatrix}
\sum_{i=1}^n M_0 \|\omega_i\|^2 q_i + \sum_{k=0}^n a_k \\
-\sum_{k=1}^n (M_{1k} \|\omega_k\|^2 q_k^x) - l_1 q_1^x \sum_{k=1}^n u_k \\
-\sum_{k=1}^n (M_{2k} \|\omega_k\|^2 q_k^y) - l_2 q_2^x \sum_{k=2}^n u_k \\
\vdots \\
-\sum_{k=1}^n (M_{nk} \|\omega_k\|^2 q_k^x) - l_n q_n^x u_n \\
\end{bmatrix},
\]

(5)

\[ R_i = R_j \Omega_j^x, \quad J_f \Omega_j = M_j - \Omega_j - J_i \Omega_j, \]

(6)

\[ \forall i \in S \setminus \{0\}, \quad j \in \mathcal{I}, \quad u_i = (-m_i g e_3 + f_i R_i e_3 l_i). \]

Kinetic energy \( \mathcal{T} : TQ \to \mathbb{R} \) is similarly given as,

\[ \mathcal{T} = \sum_{i \in \mathcal{I}} \frac{1}{2} m_i \langle v_i, v_i \rangle + \sum_{j \in \mathcal{J}} \frac{1}{2} \Omega_j \langle J_f R_j, e_3 \rangle, \]

(7)

where \( \Omega_j \) is the angular velocity of the quadrotor \( j \) in its body-frame. Dynamics are derived using the Lagrangian method, where Lagrangian \( \mathcal{L} : TQ \to \mathbb{R} \) is given as,

\[ \mathcal{L} = \mathcal{T} - \mathcal{U}. \]

We derive the equations of motion using the Langrange-d’Alembert principle of least action, given below,

\[ \delta \int_{t_0}^{t_1} \mathcal{L} dt + \int_{t_0}^{t_1} \delta \mathcal{W} \, dt = 0, \]

(8)

where \( \delta \mathcal{W} \) is the infinitesimal work done by the external forces. \( \delta \mathcal{W} \) can be computed as,

\[ \delta \mathcal{W} = \sum_{j \in \mathcal{J}} \left( (W_{1,j}, M_j) + (W_{2,j}, f_j R_j e_3) \right), \]

(9)

\[ W_{1,j} = R_j \delta R, \]

(10)

\[ W_{2,j} = \delta x_j = \delta x_0 + \sum_{k=1}^n l_k \delta q_k, \]

(11)

are variational vector fields [Goodarzi et al. (2015)] corresponding to quadrotor attitudes and positions. The infinitesimal variations on \( q \) and \( R \) are expressed as,

\[ \delta q = \xi^x \xi = -q^x \xi, \xi \in \mathbb{R}^3 \text{ s.t. } \xi \cdot q = 0, \]

\[ \delta q = -q^y \xi - q^x \xi, \]

\[ \delta R = R\eta^y, \delta \Omega^y = (\Omega^y \eta)^x + \eta^x, \eta \in \mathbb{R}^3, \]

with the constraints \( q \cdot \dot{q} = 0 \) and \( q \cdot \omega = 0, \omega \) is the angular velocity of \( q \), s.t. \( \dot{q} = \omega \times q \). The cross-map is defined as \((\cdot)^x : \mathbb{R}^3 \to \mathfrak{so}(3)\) s.t. \( x^x y = x \times y, \forall x, y \in \mathbb{R}^3 \). Similarly, variations on the link positions are given as,

\[ \delta x_i = \delta x_0 + \sum_{k=1}^i l_k \delta q_k = \delta x_0 - \sum_{k=1}^i l_k q_k^x \xi_k, \]

(12)

\[ \delta v_i = \delta v_0 + \sum_{k=1}^i l_k \delta q_k = \delta v_0 - \sum_{k=1}^i l_k (q_k^y \xi_k + q_k^x \xi_k), \]

(13)

Finally, we obtain the equations of motion for the system by solving (8). See Appendix A for the detailed derivation. Equations of motion for the multiple quadrotors carrying a flexible hose are given in (4)-(6). Note the mass-matrix \( M_{\{q_i\}} \) is a function of link attitudes \( \{q_i\} = \{q_1, q_2, \ldots, q_n\} \) and we use the following notation similar to [Goodarzi et al. (2014)]

\[ M_{00} = \sum_{k=0}^n m_k, M_{0i} = \sum_{k=0}^n m_k, \]

\[ M_{0i} = M_{0i}, M_{ij} = \sum_{k=\max(i,j)}^n m_k l_j. \]

(14)

Remark: 1. In (5), note the use of \( f_i, R_i \) for \( i \notin \mathcal{I} \), (since \( i \notin \mathcal{I} \) implies no quadrotor is attached at index \( i \) and thus cannot have \( f_i \) and \( R_i \)). However, this notation is used for convenience, since \( i \notin \mathcal{I} \Rightarrow \mathbf{1} = 0 \) and thus \( f_i R_i e_3 l_i = 1 \), there by ensuring the right inputs to the system.

Remark: 2. Degrees of freedom for the multiple quadrotors carrying a flexible hose is \( DOF = 3(nq+1)+2n \) when \( 2n \) corresponds to the link attitudes \( DOF, 3nq \) the rotational DOF of the quadrotors and \( 3 \) for the initial position \( x_0 \). Similarly, the degrees of actuation are \( DOA = 4nq \) corresponding to the 4 inputs for each quadrotor. Thus, the degrees of under-actuation are \( DOA-A = 2n + 3 - nq \). For a typical setup we have \( n \gg nq \), i.e., system is highly under-actuated.

Remark: 3. For a tethered system, we can assume \( x_0 = 0 \) \( \forall t \), i.e., the system is tethered to origin of the inertial frame, and we can derive the dynamics as earlier. Equations of motion for this system would be same as (4)-(6), without the equation corresponding to \( \dot{v}_0 \).

3. DIFFERENTIAL FLATNESS

In the previous section, we derived the dynamics for multiple quadrotors carrying a flexible hose. The system is under-actuated and thus the control of the system is challenging. In this section, we show that the system is differentially-flat. A nonlinear system is differentially-flat if a set of outputs of the system (equal to the number of inputs) and their derivatives can be used to determine the states and inputs without integration.

Definition: 1. Differentially-Flat System, [Murray et al. (1995)]: A system \( x = f(x, u), x \in \mathbb{R}^n, u \in \mathbb{R}^m \), is differentially flat if there exists flat outputs \( y \in \mathbb{R}^m \) of the form \( y = y(x, u, u, \ldots, u^{(P)}) \) such that the states and the inputs can be expressed as \( x = x(y, \dot{y}, \ldots, y^{(q)}), u = u(y, \dot{y}, \ldots, y^{(q)}) \), where \( p, q \) are positive finite integers.
Differential-flatness is used in planning the system trajectories, where the flat outputs are used to plan in lower-dimension space and the corresponding desired states and inputs are computed using differential flatness.

A quadrotor is a differentially-flat system with the quadrotor center-of-mass and yaw as the flat outputs [Mellinger and Kumar (2011)]. A quadrotor with cable suspended load (with the cable modeled as a massless link) is also shown to be differentially-flat with load position and quadrotor yaw as the flat outputs [Sreenath et al. (2013)]. Similarly, a quadrotor with flexible cable suspended load, with the cable modeled as series of smaller links is shown to be differentially-flat [Kotaru et al. (2018)]. Again, here
In the next section, we present the linearized dynamics about any desired time-varying trajectory and use an LQR to track desired trajectories.

4. CONTROL

Having presented differential-flatness in the previous section, we proceed to present control to track desired-trajectories generated using the flat-outputs in this section. As presented in the Remark 2, the given system is highly underactuated and thus controlling the system is challenging. In this section, we present a way to control the system by linearizing the dynamics in (4)-(6) about a given desired time-varying trajectory \( x_{d}(t), v_{d}(t), q_{d}(t), \omega_{d}(t), R_{d}(t), \Omega_{d}(t) \), \( \forall i \in S, j \in I \) and then implementing a linear controller.

4.1 Variation Based Linearization

In this sub-section, we present the coordinate-free linear dynamics, obtained through variation based linearization of the nonlinear dynamics in (4)-(6). We use the variation linearization techniques described in [Wu and Sreenath, 2015] to obtain the linear dynamics. The error state of the linear-dynamics is given as,

\[ \delta x = [\delta x, \xi_1, \ldots, \xi_n, \delta \omega, \ldots, \delta \omega_n, \eta_{j_1}, \ldots, \delta \Omega_{j_1}, \ldots, \delta \Omega_{jn_Q}]^T, \]  

(15)

and the corresponding inputs as,

\[ \delta u = [\delta f_{j_1}, \delta f_{j_2}, \ldots, \delta f_{j_{n_Q}}, \delta M_{j_1}^T, \delta M_{j_2}^T, \ldots, \delta M_{jn_Q}^T]^T, \]  

(16)

where \( j_1, j_2, \ldots, j_{n_Q} \) are elements of \( I \) arranged in increasing order. The individual elements of the error state are computed as,

\[ \delta x = x - x_d, \quad \delta v = v - v_d, \]

\[ \xi_i = q_{id}^{x_i} \quad \delta \omega_n = \omega_n - (q_n^{x_n})^2 \omega_{id}, \]

\[ \eta_j = \frac{1}{2} (R_{id}^T R_j - R_{jd}^T R_j^T) \delta \Omega_j = \Omega_j - R_{jd}^T R_j \Omega_{jd}. \]

Finally, the linearized dynamics (See Appendix C for detailed derivation of the linearized dynamics) about a time-varying desired trajectory are given below,

\[ \delta x = A \delta x + B \delta u, \]  

(17)

\[ \delta C \delta x = 0. \]  

(18)

The linear dynamics matrices \( A, B \) are,

\[ A = \begin{bmatrix} 0_{3n, 3} & 0_{3n, 3} & I_{3n, 3} & 0_{3n, 3q} & 0_{3n, q} \\ 0_{3q, 3} & 0_{3q, 3} & 0_{3q, 3} & 0_{3q, 3q} & 0_{3q, q} \\ \alpha & \alpha & \beta & \gamma & I_{3n, 3q} \\ 0_{3n, 3q} & 0_{3n, 3q} & 0_{3n, 3q} & 0_{3n, 3n} & \gamma \\ 0_{3n, q} & 0_{3n, q} & 0_{3n, q} & 0_{3n, 3} & \nu \end{bmatrix} M_{(n_Q, q)}^{-1}, \]  

(19)

with,

\[ F = \begin{bmatrix} 0_{3n, 3} & 0_{3n, 3} & \omega_e \end{bmatrix}, \quad \alpha = 0_{3n, 3} \quad \beta = \text{bdia}(I_2 - q_n^{x_n} q_{id}), \quad \gamma = \text{bdia}(-R_{jd}^T, -R_{jd}, \ldots, -R_{sq}^T), \quad \nu = \text{bdia}(I_{\nu} - \Sigma^{x}_e - \Sigma^{x}_n), \]

and

\[ B = \begin{bmatrix} \Omega_{(n+1), 3q} + Q_{10} \alpha \\ \Omega_{3n, 3n} + Q_{20} \end{bmatrix}, \]  

with \( \mu = \text{bdia}(J_{j_1}, \ldots, J_{j_{n_Q}}) \).

The rest of the elements are described below,

\[ c_1 = b \text{dia}(g_1 q_{d1}, \ldots, g_{3} q_{d3}), \quad c_2 = \text{bdia}(-w_{1}, \ldots, -w_{n}). \]

Next, the constraint matrix \( C \) is defined as,

\[ c = \begin{bmatrix} O_{(n, 3)} & c_1 & O_{(n, 3n)} & O_{(n, 0)}, \quad O_{(n, 3)} & c_2 & O_{(n, 3n)} & O_{(n, 0)}, \quad O_{(n, 3)} & c_3 & O_{(n, 3n)} & O_{(n, 0)} \end{bmatrix}, \]  

(21)

with \( c_1 = b \text{dia}(q_{d1}, q_{d2}, \ldots, q_{d3}), \quad c_2 = b \text{dia}(-w_{1}, \ldots, -w_{n}). \)

The above equation is solved offline and stored in a table for online computation. Note that the explicit time dependence of \( P, A, B \) is dropped for convenience. Finally, the feedback gain for the control input is computed as,

\[ u(t) = -K x(t). \]  

(23)

Since the gains are computed backwards in time, the computed input result in a stable control for the constrained linear-system. The net control-input to the nonlinear system can be computed as,

\[ u(t) = u_d(t) + \delta u. \]  

(24)
In the next section, we present few numerical simulations with the finite-horizon LQR performing tracking control on the full nonlinear-dynamics.

5. NUMERICAL SIMULATIONS

In this section, we present numerical results to validate the dynamics and control discussed in the earlier sections. We present numerical simulations for tracking control for a desired setpoint and circular trajectory.\(^2\)

5.1 Setpoint Tracking

(i). Two Quadrotor system: Following parameters are considered for the simulations, 
\(n = 10, \ n_Q = 2, \ \mathcal{I} = \{0, 10\}, \ m_i = 0.0909kg, \ l_i = 0.1m,\)
\(mQ_1 = 0.85kg, \ J_0 = diag([0.0557, 0.0557, 0.0150])kgm^2\)
and the setpoint is given as,
\[
x_{0d} = [0, 0, 0]^T, \ x_{nd} = [0.6, 0.0, 0.0]^T,
\]
with the cable hanging between these two points. Degrees of freedom and under-actuation for this setup are \#DOF = 29, \#DOU = 21 respectively. The linear dynamics \(A, B, C\) are computed about this setpoint \(x_n\). Here, we compare two different controllers, (i) the finite-horizon LQR discussed in the previous-section and (ii) position-controllers on the two quadrotors with feed-forward forces due to the cable at steady state. We start with some initial error in the cable orientation and the resulting error plots are shown in Figure 4. As seen in the Figure, errors for cable position \(x_0\), cable attitudes and quadrotors’ attitude converge to origin. Attitude errors for the hose links is defined as the configuration error on \(S^2\),
\[
\Psi_{x} = 1 - q_{x}^Tq_{x},
\]
and similar quadrotor attitude error is defined as,
\[
\Psi_{R} = 0.5|\tau(I - R_{0d}^TR_{x})|.
\]
For the position control with feed-forward forces, even though the quadrotor attitudes are zero, the initial error in cable orientation results in oscillations in the cable. These oscillations are not accounted for in the control and can be seen in Figure 4.

(ii). Three Quadrotor system: Setpoint tracking for cable suspended from three-quadrotors is presented here. Parameters for the system are as follows,
\(n = 10, \ n_Q = 3, \ \mathcal{I} = \{0, 5, 10\}, \ m_i = 0.0909kg, \ l_i = 0.2m, \)
and \#DOF = 32, \#DOU = 20. Various tracking errors for the system are presented in Figure 5 and snapshots for the system are shown in Figure 6.

5.2 Trajectory Tracking

In this section, we show that the presented controller tracks a desired time-varying trajectory with initial errors. We use the following system parameters,
\(n = 5, \ n_Q = 2, \ \mathcal{I} = \{0, 5\}, \ m_i = 0.1667kg, \ l_i = 0.25m, \)
and the rest same as those given in Section 5.1. We consider the following flat output trajectory,
\[
x_0 = \begin{bmatrix} a_x(1 - \cos(2f_1\pi t)) \\ a_y \sin(2f_2\pi t) \\ a_z \cos(2f_3\pi t) \end{bmatrix}, \ T_{1} = \begin{bmatrix} 2.74 \\ 0.0 \\ -3.27 \end{bmatrix}, \ \psi_0 = \psi_5 = 0,
\]
\(f_1 = \frac{1}{4}, f_2 = \frac{1}{5}, f_3 = \frac{1}{7}, a_x = 2, a_y = 2.5, a_z = 1.5.\)
Rest of the states and inputs can be computed using differential-flatness. We use the linearized-dynamics and the finite-horizon LQR presented in the previous sections to achieve the tracking control. Following weights are used for the LQR,
\(Q_x = 0.5I_6, \ Q_0 = 0.5I_{2i}, \ Q_R = I_{3nQ}, \ Q_\Omega = 0.75I_{3nQ}, \)
\(Q = bdiag(Q_x, Q_0, Q_R, \Omega_0), \)
\(R = 0.21I_{4nQ}, \ P_T = 0.01I_{4n_2}, \)
where \(n_x\) is \(6 + 6n + 6nQ\). Figure 7 shows snapshots of the system at different instants along the trajectory. The proposed controller tracks the desired trajectory (shown in red) when started with an initial error.

6. RESULTS AND DISCUSSION

Having presented numerical results to validate our controller, we now present some discussion on limitations and future work.

Limitations

Though increasing discretization helps better represent the dynamics of an hose system, it also increases the computation-complexity. To better study the effect of discretization we ran multiple simulations with different discretizations for a fixed cable length and mass. We used only control on the quadrotor-positions with feed-forward control on the whole system (lqr) and feed-forward control on the quadrotor-position (ff). \(\Psi_q\) is the hose link attitude errors as defined in (25) and \(\Psi_R\) is the quadrotor attitude configuration error defined in (26).

\(\)
While differential-flatness can be used to plan trajectories in the flat-output space and compute desired states and inputs, this computation requires computing $q_i$ and its derivatives from tension $T_i$ and its derivatives, i.e. $q_i = \frac{T_i}{\|T_i\|}$. The complexity of this computation increases for higher-derivatives.

In addition, as listed in the Section 2, we don’t consider the mechanical properties of the hose when deriving the dynamics. Thus, the dynamics derived and the subsequent presented control might not completely capture the system fully and might lead to instability in cases when hose properties are important, such as when water flows in the hose.

**Future Work**

As part of future work, we would like to address some of the limitations listed in the previous sections, such as, (i) number of discretizations, (ii) number of derivatives to computed, and (iii) water flow in the hose. We would like to study the current system along with all the mechanical properties of the cable and develop controllers for such systems. In addition, to implement the control we require state estimation of the cable which as shown is modeled as $(S^2)^n$. Towards this end we hypothesize [Kotaru and Sreenath (2019)] can be extended to estimate the cable state. A possible method to improve the computation time would be to use limited cable states like mid-position of the cable etc., to develop a controller.

7. CONCLUSION

In this work we have studied the multiple quadrotors carrying a flexible hose system. We modeled the flexible-hose as a series of smaller discrete-links with lumped mass and derived the coordinate-free dynamics using Langrange-d’Alembert’s principle. We also showed that the given system is differentially-flat, as long as the end of the hose is connected to a quadrotor. Variation-based linearized dynamics were derived about time-varying desired trajectory. We showed tracking control for the system using finite-horizon LQR for the linear dynamics and validated this through numerical simulations with up-to 10 discretizations of the hose. Finally, we discussed some of the limitations due to the assumptions and directions for future work.

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Appendix A. DYNAMICS DERIVATION

In this section, we present the detailed derivation of the equations of motion, (4)-(6), for the given system. Starting with the principle of least action in (8) and substituting for Lagrangian and virtual work from (3),(7) and (9), we have,

\[
\delta \int \left( \sum_{i \in \mathcal{S}} \frac{1}{2} \bar{m}_i (v_i, v_i) - \bar{m}_i g e_3 \cdot x_i + \sum_{j \in \mathcal{I}} \frac{1}{2} (\Omega_j, J \Omega_j) \right) dt.
\]

\[
\int \sum_{j \in \mathcal{I}} \left( W_{ij} + f_{j} R_{j} e_3 \right) dt. \tag{A.1}
\]

Separating and solving the rotational components, we have the following rotational dynamics

\[
J_i \ddot{\Omega}_i = M_i - \Omega_i \times J_i \Omega_i, \quad \forall j \in \mathcal{I}
\]

Taking variation on rest of the equation results in,

\[
\int \sum_{i \in \mathcal{S}} \left[ \bar{m}_i (\delta v_i, v_i) + \delta \bar{m}_i g e_3 + f_i R_i e_3 \right] dt = 0.
\]

Expanding the summation,

\[
\int \left( \bar{m}_i (\delta v_i, v_i) + \delta \bar{m}_i g e_3 + f_i R_i e_3 \right) dt = 0 \tag{A.2}
\]

Replacing the variations \( \delta v_i, \delta x_i \) with their expansions (12), (13), we get,

\[
\int \left( \bar{m}_i (\delta v_i, v_i) + \delta \bar{m}_i g e_3 + f_i R_i e_3 \right) dt = 0 \tag{A.3}
\]

Using the following simplifications in (A.3),

\[
- l_k (q_k^x \dot{q}_k^x + q_k^x \dot{q}_k^x \xi_k), \quad v_k = l_k (q_k^x v) \cdot \xi_k + l_k (q_k^x v) \cdot \xi_k,
\]

and regrouping the respective variations would result in,

\[
\int \left( \bar{m}_i (\delta v_i, v_i) + \delta \bar{m}_i g e_3 + f_i R_i e_3 \right) dt = 0 \tag{A.4}
\]

Integration by parts on the respective variation sets results in

\[
\int \left( \bar{m}_i (\delta v_i, v_i) + \delta \bar{m}_i g e_3 + f_i R_i e_3 \right) dt = 0 \tag{A.5}
\]

and finally,

\[
\int \left( \left( \delta x_0 \cdot \left( \sum_{k=0}^n - \bar{m}_k \dot{v}_k + u_k \right) + \xi_1 \cdot l_1 q_1^x \left( \sum_{k=1}^n - \bar{m}_k \dot{v}_k + u_k \right) \right) dt = 0. \tag{A.6}
\]

By principle of least action the above integral is valid \( \forall \delta x_0, \xi_1, t \) and thus, to ensure the above equation to be zero for all time we have,

\[
\left( - \bar{m}_0 v_0 + \bar{m}_1 v_1 + \ldots + \bar{m}_n v_n \right) + \sum_{k=0}^n u_k = 0
\]

\[
q_1 \times \left( - l_1 q_1^x \left( \sum_{k=1}^n - \bar{m}_k \dot{v}_k + l_1 q_1^x \sum_{k=1}^n u_k \right) \right) = 0, \tag{A.7}
\]

Expanding the \( \dot{v}_i \) we have,

\[
\left( \bar{m}_0 v_0 + \bar{m}_1 v_1 + \ldots + \bar{m}_n v_n \right) + \sum_{k=0}^n u_k = 0,
\]

\[
l_1 (q_1^x)^2 \left( \bar{m}_1 (v_0 + \sum_{k=1}^n l_k q_k) \right) = l_1 (q_1^x)^2 \sum_{k=0}^n u_k,
\]

\[
l_n (q_n^x)^2 \left( \bar{m}_n (v_0 + \sum_{k=1}^n l_k q_k) \right) = l_n (q_n^x)^2 \sum_{k=0}^n u_k.
\]

Simplifying the above equations using (14) and the following relations,

\[
\dot{q} = \omega \times q + \ddot{q},
\]

\[
\ddot{q} = \omega \times \ddot{q} + \dddot{q} = A \omega \times \dddot{q} - ||\omega||^2 \dddot{q}
\]

\[
(q^x)^2 \dddot{q} = q \times (q \times \dddot{q}) = (q \times \dddot{q}) q - (q \times q) \dddot{q} = -(q \times \dddot{q}) q - \dddot{q},
\]

would result in (5).

Appendix B. DIFFERENTIAL FLATNESS

In this section, we present the proof for differential-flatness stated in Lemma 1

**Proof.** Illustration of the differential-flatness is shown in Figure 2. For the purpose of proving differential-flatness we redefine the dynamics of the system using tensions in the cable links as given below,

\[
\bar{m}_0 \ddot{x}_0 = T_1 - \bar{m}_0 g e_3, \tag{B.1}
\]

\[
\bar{m}_y \ddot{x}_0 = T_{a+1} - T_a - \bar{m}_a g e_3, \tag{B.2}
\]

\[
\bar{m}_b \ddot{x}_b = T_{b+1} - T_b - \bar{m}_b g e_3 + f_b R e_3, \tag{B.3}
\]

\[
\bar{m}_y \ddot{x}_y = -T_n - \bar{m}_n g e_3 + f_n R e_3, \tag{B.4}
\]

where \( \forall a \in S \setminus \{0, I\} \), i.e., all the points except the starting point of the cable and those connected to the quadractors and \( \forall b \in I \setminus \{n\} \) and the quadrator attitude dynamics are as given in (6). Also note \( n \in I \), i.e., end the cable is attached to the quadrator. Number of inputs in the system are 4nQ corresponding to the thrust and moment of the quadrators. Number of flat outputs are 3 (for position \( x_0 \)) and 3(nQ - 1) (for each tension \( T_{k+1} \forall k \in I \setminus \{n\} \) and \( q_0 \) for each quadrator yaw \( \psi_j \) = 4nQ.

Making use of these dynamics we prove the flatness as follows.

\[
(A.5)
\]
(i) Given, \( x_0 \) is a flat-output and therefore we have the cable start position and its derivatives as shown, 
\[
\{x_0, \dot{x}_0, x_1, x_1(2n+1), \ldots, x_0(2n+1)\}.
\]
(ii) Taking derivatives of (B.1) and making use of (B.5) we have tension vector \( T_1 \) in the first link and its derivatives, 
\[
\{T_1, \dot{T}_1, \ddot{T}_1, x_1^{(2n+1)} \ldots, x_0^{(2n+1)}\}.
\]
(iii) Attitude of the first link is then determined from the tension vector in (B.6) as 
\[
\dot{q}_1 = T_1 \mid T_1 \mid
\]
and its higher derivatives, 
\[
\{\dot{q}_1, \ldots, \dot{q}_1^{(2n+1)}\},
\]
are computed by taking derivatives of (B.7) and (B.8).
(iv) Position and its derivatives of the next link point-mass \( \mathbf{m}_1 \) is computed using (1), 
\[
\{x_1, \dot{x}_1, x_1, \ldots, x_1^{(2n+1)}\}.
\]
(v) Repeating the steps (ii)-(iv), we can compute the link attitudes, tensions and the positions iteratively till \( x_n \).
(vi) Using (B.3) and the fact that \( T_{n+1} \) is a flat-output (note \( b \in \sum \{n\} \)) we can compute the thrust in the quadrotor \( f_b \).
(vii) From \( x_b, f_bR_{e3} \) and their derivatives, the quadrotor attitude, angular velocity \( R_e, \dot{R}_e \) and moment \( \mathbf{J} \) can be computed as shown in [Mellinger and Kumar (2011)].
(viii) Rest of the states and inputs for the multiple quadrotors carrying a flexible hose segments can be iteratively determined as described above.

Proof for Corollary 2 is given below, Proof. For tethered system we have \( x_0 \equiv 0 \) and \( T_1 \) is known since it is a flat-output, i.e., steps (i)-(ii) (see (B.5)-(B.6)). Rest of the proof follows form Lemma 1.

Appendix C. VARIATION-BASED LINEARIZED DYNAMICS

Taking variations with respect to desired states for various states is as follows,
\[
\delta q_i = \xi_i q_{id} - q_{id} \xi_i, \quad \delta \| \omega_i \|^2 = \delta (\omega_i \cdot \omega_i) = 2 \omega_i \delta \omega_i, \quad \delta R_{ij} = R_{ij} \delta \xi_j.
\]
Taking variation on the first row of \((5)\),
\[
\delta \left( M_{0d} \dot{q}_i \right) = \sum_{i=1}^{n} M_{0i} \omega_i \delta \xi_i \quad \text{(C.1)}
\]
\[
\delta \left( \dot{M}_{0d} \dot{q}_i \right) = \sum_{i=1}^{n} \dot{M}_{0i} \omega_i \delta \xi_i \quad \text{(C.2)}
\]
\[
\sum_{i=1}^{n} M_{0i} \dot{\omega}_i \delta \xi_i = \sum_{i=1}^{n} \dot{M}_{0i} \omega_i \delta \xi_i \quad \text{(C.3)}
\]
\[
\sum_{i=1}^{n} \dot{M}_{0i} \dot{\omega}_i \delta \xi_i = - \sum_{i=1}^{n} M_{0i} \omega_i \dot{\omega}_i \delta \xi_i \quad \text{(C.4)}
\]
\[
\sum_{i=1}^{n} \dot{M}_{0i} \dot{\omega}_i \delta \xi_i = - \sum_{i=1}^{n} M_{0i} \omega_i \dot{\omega}_i \delta \xi_i \quad \text{(C.5)}
\]
\[
\sum_{i=1}^{n} \dot{M}_{0i} \dot{\omega}_i \delta \xi_i = - \sum_{i=1}^{n} M_{0i} \omega_i \dot{\omega}_i \delta \xi_i \quad \text{(C.6)}
\]
\[
M_{0i} \left( \dot{\delta q}_i + q_{id} \delta \xi_i \right) + M_{0i} \dot{\omega}_i \delta \xi_i \quad \text{(C.7)}
\]
\[
\sum_{i=1}^{n} \dot{M}_{0i} \dot{\omega}_i \delta \xi_i = - \sum_{i=1}^{n} M_{0i} \omega_i \dot{\omega}_i \delta \xi_i \quad \text{(C.8)}
\]
\[
\sum_{i=1}^{n} \dot{M}_{0i} \dot{\omega}_i \delta \xi_i = - \sum_{i=1}^{n} M_{0i} \omega_i \dot{\omega}_i \delta \xi_i \quad \text{(C.9)}
\]
\[
\sum_{i=1}^{n} \dot{M}_{0i} \dot{\omega}_i \delta \xi_i = - \sum_{i=1}^{n} M_{0i} \omega_i \dot{\omega}_i \delta \xi_i \quad \text{(C.10)}
\]
From (C.5) & (C.10), we have,
\[
\begin{bmatrix}
M_{00}I_3 & -q_{1d}^xM_{01} & -q_{2d}^xM_{02} & \ldots & -q_{nd}^xM_{0n} \\
M_{10}q_{1d}^x & M_{11}I_3 & -M_{12}q_{1d}^xq_{2d}^x & \ldots & -M_{1n}q_{1d}^xq_{nd}^x \\
M_{20}q_{2d}^x & -M_{21}q_{2d}^xq_{1d}^x & M_{22}I_3 & \ldots & -M_{2n}q_{2d}^xq_{nd}^x \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
M_{n0}q_{nd}^x & -M_{n1}q_{nd}^xq_{1d}^x & -M_{n2}q_{nd}^xq_{2d}^x & \ldots & M_{nn}I_3 \\
\end{bmatrix}
\]

where

\begin{align}
& a_i = M_{0i}\left(\dot{\omega}_{id}^x - \|\omega_{id}\|^2I_3\right)q_{id}^x \\
& b_i = M_{0i}(2q_{id}^x\dot{\omega}_{id}), \quad i \in \{1, \ldots, n\} \\
& c_{ij} = \begin{cases} 
M_{0i}q_{id}^x - \sum_{j=1, j \neq i}^n M_{ij}\left(\dot{\omega}_{jd}^x\dot{\omega}_{jd}^x + \|\omega_{jd}\|^2q_{jd}^x\right) - l_i\left(\sum_{k=1}^n u_k^x\right)(-q_{id}^x) & i = j \\
M_{ij}\left(\dot{\omega}_{jd}^x - \|\omega_{jd}\|^2I\right)q_{jd}^x & i \neq j
\end{cases} \\
& d_{ij} = \begin{cases} O & i = j \\
M_{ij}\left[2q_{jd}^xq_{jd}^x\dot{\omega}_{jd}\right] & i \neq j
\end{cases} \\
& e_j = -f_{jd}R_{jd}^te_3^x, \quad j \in \mathcal{I} \\
& f_{ij} = \begin{cases} 1 & \text{if } j \notin \mathcal{I} \\
-(l_iq_i^x)f_{jd}R_{jd}^te_3^x & \text{if } j \in \mathcal{I}, j \geq i \\
O_{3,3} & \text{if } j \in \mathcal{I}, j < i
\end{cases} \\
& g_j = R_{jd}e_3 \quad (C.18) \\
& h_{ij} = \begin{cases} 1 & \text{if } j \notin \mathcal{I} \\
l_iq_i^xR_{jd}e_3, & \text{if } j \in \mathcal{I}, j \geq i \\
O_{3,1} & \text{if } j \in \mathcal{I}, j < i
\end{cases}
\end{align}
Variation Linearized Dynamics

\[
\delta x = [\delta x, \xi_1, \ldots, \xi_n, \delta v, \delta \omega_1, \ldots, \delta \omega_n, \eta_{j1}, \ldots, \eta_{jnQ}, \delta \Omega_{j1}, \ldots, \delta \Omega_{jnQ}]^T
\]
\[
\delta u = [f_{j1}, f_{j2}, \ldots, f_{jnQ}, M_{j1}^T, M_{j2}^T, \ldots, M_{jnQ}]^T
\]
\[
\delta x = A\delta x + B\delta u
\]

where,

\[
A = \begin{bmatrix}
0_{3,3} & 0_{3,3n} & I_{3,3} & 0_{3,3nQ} & 0_{3,3nQ} \\
0_{3n,3} & 0_{3n,3} & \alpha & 0_{3n,3n} & 0_{3n,3nQ} \\
0_{3nQ,3nQ} & 0_{3nQ,3nQ} & \beta & 0_{3nQ,3nQ} & 0_{3nQ,3nQ} \\
0_{3nQ,3nQ} & 0_{3nQ,3nQ} & 0_{3nQ,3nQ} & \gamma & I_{3nQ,3nQ} \\
0_{3nQ,3nQ} & 0_{3nQ,3nQ} & 0_{3nQ,3nQ} & 0_{3nQ,3nQ} & \nu
\end{bmatrix},
\]
\[
\alpha = \text{blkdiag}\{q_{0d}q_{1d}^T\omega_{nd}^X, q_{2d}q_{2d}^T\omega_{nd}^X, \ldots, q_{nd}q_{nd}^T\omega_{nd}^X\}
\]
\[
\beta = \text{blkdiag}\{I_3-q_{0d}q_{1d}^T, I_3-q_{2d}q_{2d}^T, \ldots, I_3-q_{nd}q_{nd}^T\}
\]
\[
\gamma = \text{blkdiag}\{-\Omega_{j1}^{X1d}, \ldots, -\Omega_{j2d}^{X1d}, \ldots, -\Omega_{jnd}^{X1d}\}
\]
\[
\nu = \text{blkdiag}\{\nu_{j1}^{-1}, \ldots, \nu_{j2d}^{-1}, \ldots, \nu_{jnd}^{-1}\}
\]
\[
B = \begin{bmatrix}
O_{3(n+1),4nQ} \\
M^{-1}G \\
O_{3nQ,4nQ} \\
O_{3nQ,4nQ} \mu
\end{bmatrix}, \mu = \text{blkdiag}\{J_{j1}^{-1}, \ldots, J_{jnQ}^{-1}\}
\]

and

\[
M = \begin{bmatrix}
M_{00}I_3 & -q_{0d}^X M_{01} & \ldots & -q_{nd}^X M_{0n} \\
-M_{01}q_{1d}^X & -M_{11}I_3 & \ldots & -M_{1n}q_{1d}^X \\
\vdots & \vdots & \ddots & \vdots \\
-M_{n0}q_{nd}^X & -M_{n1}q_{1d}^X & \ldots & -M_{nn}I_3
\end{bmatrix}
\]
\[
F = \begin{bmatrix}
O_{3,3} & [a] & O_{3,3} & [b] & \ldots & O_{3nQ,3nQ} \\
O_{3n,3} & [c] & O_{3n,3} & [d] & \ldots & O_{3nQ,3nQ}
\end{bmatrix}
\]
\[
G = \begin{bmatrix}
[g_{i,j}] \\
[p_{i,j}]
\end{bmatrix} O_{(3(n+1),3nQ)}
\]