BOUNDS ON CONVEX BODIES IN PAIRWISE INTERSECTING MINKOWSKI ARRANGEMENT OF ORDER $\mu$

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Abstract. A generalization of pairwise intersecting Minkowski arrangement of centrally symmetric convex bodies is the pairwise intersecting Minkowski arrangement of order $\mu$. Here, the homothetic copies of a centrally symmetric convex body are so that none of their interiors intersect the $\mu$-kernel of any other. We give general upper and lower bounds on the cardinality of such arrangements, and study two special cases: For $d$-dimensional translates in classical pairwise intersecting Minkowski arrangement we prove that the sharp upper bound is $3^d$. For $\mu = 1$ the general version yields to another known problem: The Bezdek–Pach Conjecture asserts that the maximum number of pairwise touching positive homothetic copies of a convex body in $\mathbb{R}^d$ is $2^d$. We verify the conjecture on the plane, that is, when $d = 2$. Indeed, we show that the number in question is four for any planar convex body.

1. Introduction

A positive homothetic copy of a convex body (i.e. a compact convex set with non-empty interior) $K$ in Euclidean $d$-space $\mathbb{R}^d$ is a set of the form $\lambda K + t$ where $\lambda > 0$ and $t \in \mathbb{R}^d$. Two sets on the plane are said to touch each other if they intersect but their interiors are disjoint.

Pairwise intersecting homothets of a centrally symmetric convex body in the $d$-dimensional Euclidean space are in pairwise intersecting Minkowski arrangement if none of them contains the center of any other in its interior.

Polyanskii [10] recently proved that such a family of convex bodies has at most $3^{d+1}$ members. This result was improved by Naszódi and Swanepoel [7] showing an upper bound of $2 \cdot 3^d$, which is still not sharp. The conjectured maximum number of elements is $3^d$.

In Section 2 we prove the following upper bound on the cardinality of a family containing translates of a centrally symmetric convex body in pairwise intersecting Minkowski arrangement:

**Theorem 1.** In $\mathbb{R}^d$ a pairwise intersecting Minkowski arrangement consisting of translates of a centrally symmetric convex body $K$ contains at most $3^d$ elements. This bound is sharp, equality holds if and only if $K$ is a $d$-dimensional parallelootope.

We show a construction for arbitrary centrally symmetric convex body that gives a linear lower bound on the cardinality of maximal pairwise intersecting Minkowski arrangements of translates.

**Theorem 2.** For a centrally symmetric convex body $K \subset \mathbb{R}^d$ ($d \geq 2$) a maximal pairwise intersecting Minkowski arrangement consisting of translates of $K$ has at least $2d + 3$ elements.

In Section 3 we introduce some generalizations of the problem based on an idea of Böröczky and Szabó [4]. For $0 \leq \mu \leq 1$ they defined the $\mu$-kernel of a centrally symmetric convex body $K$ as $\mu K$ concentric to $K$.

Using this notion, for homothets of a centrally symmetric convex body we can consider a pairwise intersecting Minkowski arrangement of order $\mu$, where the homothets are pairwise intersecting but none of their interiors intersect the center of any other.

We prove an upper bound on the cardinality of such an arrangement, then, for some special bodies we verify the existence of an exponential lower bound.
Theorem 3. In $\mathbb{R}^d$ a pairwise intersecting Minkowski arrangement of order $\mu$ consisting of translates of a centrally symmetric convex body $K$ contains at most $\left(1 + \frac{2}{1+\mu}\right)^d$ elements.

Theorem 4. For $\mu < \sqrt{2} - 1$ a pairwise intersecting $\mu$-Minkowski arrangement of translates of a euclidean ball can have exponentially many elements.

The generalized notion of Minkowski arrangement provides a connection between the original problem of pairwise intersecting Minkowski arrangements, and the Bezdek–Pach Problem [2], detailed in Section 4.

In 1962, Danzer and Grünbaum [4] proved that the maximum cardinality of a family of pairwise touching translates of a convex body $K$ in $\mathbb{R}^d$ is $2^d$, which bound is attained if, and only if, $K$ is an affine image of a cube. Petty [9] showed that every convex body in the plane (resp., in 3-space) has three (resp., four) pairwise touching translates. As an extension of this problem, Bezdek and Pach [2] conjectured in 1988 that the maximum number of pairwise touching positive homothetic copies of a convex body in $\mathbb{R}^d$ is $2^d$. They showed that any such family of homothetic copies has at most $3^d$ elements, and if $K$ is a $d$-dimensional Euclidean ball, then the maximum is equal to $d + 2$. Naszódi [8] improved the first estimate of Bezdek and Pach by proving the upper bound $2^{d+1}$. In [3], Lángi and Naszódi proved (using a result [1] of Bezdek and Brass about one-sided Hadwiger numbers) the upper bound $3 \cdot 2^{d-1}$ in the case when $K$ is centrally symmetric.

In Section 3 of the present paper we show that the conjecture holds on the plane, moreover, every planar convex body has four pairwise touching homothets.

Theorem 5. For any convex body $K$ in $\mathbb{R}^2$, the maximum number of pairwise touching positive homothetic copies of $K$ is exactly four.

For two points $a, b$ in $\mathbb{R}^2$, we denote the closed (resp., open) line segment connecting them by $[a, b]$ (resp., $(a, b)$). We use the standard notations $\text{conv}$, $\partial$ and $\text{int}$ to denote the convex hull, the boundary and the interior of a set in $\mathbb{R}^2$, respectively.

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2. Bounds on pairwise intersecting Minkowski arrangements

Definition 1. Homothets of a centrally symmetric convex body are said to be in pairwise intersecting Minkowski arrangement if they are pairwise intersecting but none of them contains the center of any other in its interior.

The conjecture is that in $\mathbb{R}^d$ a pairwise intersecting Minkowski arrangement consisting of homothets of a centrally symmetric convex body contains at most $3^d$ elements. Here we prove this upper bound for the case, when all the homothets in the arrangement are translates of the given body.

2.1. Proof of Theorem 5

Any centrally symmetric convex body can be considered as the unit ball of a normed space. Thus, it is enough to show that the statement holds for the unit ball of any normed space. It is easy to see that having a Minkowski arrangement is equivalent to the following two conditions on the distances between centers: none of them can be farther than $2$, nor closer than $1$ to any other.

Lemma 2.1. Consider a centrally symmetric convex body $K$ in $\mathbb{R}^d$ and $v_1, v_2, ..., v_n \in \mathbb{R}^d$, so that $1 \leq \|v_i - v_j\|_K \leq \lambda$ for any $i \neq j$. Then $n \leq (\lambda + 1)^d$.

Proof. Because of the assumption, for $i \in \{1, ..., n\}$ different indices the bodies $v_i + \frac{1}{2}K$ are disjoint. Let $Q = \text{conv} \left[ \bigcup_{i=1}^n \left(v_i + \frac{1}{2}K\right) \right]$. Since $\text{diam}_K(Q) \leq \lambda + 1$, using the isodiametric inequality for Minkowski spaces we get that...
\begin{equation}
\frac{n}{2^d} \text{Vol}(K) \leq \text{Vol}(Q) \leq \text{Vol}\left(\frac{\lambda + 1}{2} K\right).
\end{equation}

From this, \( n \leq (\lambda + 1)^d \) follows. \hfill \Box

Applying this lemma for \( \lambda = 2 \), we get that the number of points with pairwise distances between 1 and 2 is at most \( 3^d \), what is equivalent to the statement of Theorem 1.

To reach this bound, (1) has to hold with two equalities. From the following lemma of Groemer [5] we can see that this happens if and only if \( K \) is a \( d \)-dimensional parallelotope.

**Lemma 2.2.** Suppose that \( K \subset \mathbb{R}^d \) is a convex body such that for some \( 1 < t \in \mathbb{R} \) the body \( tK \) can be decomposed into translates of \( K \). Then \( K \) is a \( d \)-dimensional parallelotope and \( t \) is an integer. The partition is unique. \hfill \Box

### 2.2. Proof of Theorem 2

First we show a construction of seven bodies in \( \mathbb{R}^2 \), then the lower bound \( 2d + 3 \) for the higher dimensional cases will follow recursively. In \( \mathbb{R}^2 \), consider an affine-regular hexagon inscribed to \( K \) that is symmetric to the center of \( K \). There exist seven translates of this hexagon in Minkowski arrangement, shown in Figure 1, where if a center point is contained in the boundary of another hexagon, then it is a vertex. Translate \( K \) in a way that the center points are the same as the centers of the above hexagons. Now a center of any translate is either not contained in another body, or lies on its boundary. Furthermore, these translates share a common point, so they are pairwise intersecting. This means, that the construction gives a Minkowski arrangement.

![Figure 1](image)

For a centrally symmetric convex body \( K \subset \mathbb{R}^d \) denote by \( M_d(K) \) the maximal number of translates in Minkowski arrangement. It is easy to see, that for any such \( K_1 \in \mathbb{R}^1 \), \( M_1(K_1) = 3 \), and we showed that for \( K_2 \in \mathbb{R}^2 \), \( M_2(K_2) \geq 7 \).

In dimension \( d \geq 3 \) consider the translates of \( K \) in the cartesian coordinate system. Let \( x_1, \ldots, x_d \) denote the coordinate axes and suppose that \( K \) is \( o \)-symmetric. Using the above planar construction, we can take 7 translates of \( K \) in Minkowski arrangement such that their centers lie in the \( x_1x_2 \) plane. Along each coordinate axis \( x_3, \ldots, x_d \) we can add two further translates of \( K \) to the arrangement so that they contain \( o \) on their boundary. This way we get that \( M_{d+1}(K) \geq M_d(K) + 2 \). \hfill \Box
3. Minkowski arrangements of order $\mu$

The notion of pairwise intersecting Minkowski arrangement can be generalized in the following way:

**Definition 2.** For $K$ centrally symmetric convex body and $0 \leq \mu \leq 1$ the translate of $\mu K$ concentric to $K$ is called the $\mu$-kernel of $K$.

**Definition 3.** Homothets of a centrally symmetric convex body are said to be in **pairwise intersecting Minkowski arrangement of order $\mu$** if they are pairwise intersecting but none of their interiors intersects the $\mu$-kernel of any other body.

This generalization provides a connection between the original Minkowski arrangement and the Bezdek–Pach Conjecture, where the homothets of a convex body are in pairwise touching position. In both problems we consider pairwise intersecting Minkowski arrangements of order $\mu$, but in the earlier version $\mu = 0$, while in the latter case $\mu = 1$.

The proof of Theorem 3 goes the same way as of Theorem 1. In this case, the distance between two centers is at least $1 + \mu$ but at most 2. After applying a homothety, this is equivalent to the problem when the distances are between 1 and $\frac{2}{1+\mu}$. Using Lemma 2.1 we get the result. $\square$

The bound in Theorem 3 gives the earlier result 5 for $\mu = 0$ and 2 for the pairwise touching case, when $\mu = 1$.

Notice that the value of $\mu$ plays an important role, that is, it is not enough to know whether $\mu = 0$ or not. For example when $\mu = \frac{1}{3}$ the above theorem in the plane gives 5 as an upper bound, but there exists a sufficiently small positive $\mu$ for which we can find a planar $\mu$-Minkowski arrangement of 6 translates.

For some special convex bodies we can prove exponential lower bound using probabilistic methods.

4. **Proof of the upper bound in Theorem 3**

In this section, we show that the maximum number of pairwise touching positive homothetic copies of a convex body $K \subset \mathbb{R}^2$ is at most four.

Let $K_1, K_2, \ldots, K_n$ be a family of pairwise touching positive homothetic copies of a planar convex body $K$.

If there is a point that belongs to four of the homothets, then we can enlarge (or shrink) each of the four bodies from that point as a center, to obtain four touching translates of $K$.

If there is a point that belongs to three of the homothets, then we can enlarge (or shrink) each of the four homothets.

If there is a point that belongs to three of the homothets and $K$ has at least four members, then we will show that this point also belongs to a fourth body.

**Proposition 4.1.** Let $n \geq 4$ and $K_1, K_2, K_3, K_4$ pairwise touching positive homothets of the convex body $K \subset \mathbb{R}^2$. If $K_1 \cap K_2 \cap K_3 \neq \emptyset$, then $K_1 \cap K_2 \cap K_3 \cap K_4 \neq \emptyset$.

**Proof.** Let $p_{123} \in K_1 \cap K_2 \cap K_3$. For $i \in \{1, 2, 3\}$ and a line $e$ through $p_{123}$ for which $e \cap K_i = \emptyset$ denote by $H^e_i$ the halfplane bounded by $e$ that contains $K_i$. Let $C_i = \bigcap_{p_{123} \in e \cap K_i = \emptyset} H^e_i$ the smallest cone with vertex $p_{123}$ containing $K_i$.

Now we show that $\text{int} C_i \cap \text{int} C_j = \emptyset$ for any $i \neq j$, $i, j \in \{1, 2, 3\}$.

Suppose that for a pair $i \neq j$ there exists $c \in (\text{int} C_i \cap \text{int} C_j)$. Then the line $p_{123}c$ intersects the interior of both $K_i$ and $K_j$ because $C_i$ and $C_j$ are the smallest cones containing $K_i$ and $K_j$ respectively. Hence due to the convexity of the bodies, $K_i$ overlaps $K_j$, which is a contradiction.

Suppose that $K_1 \cap K_2 \cap K_3 \cap K_4 = \emptyset$. Then $p_{123} \notin K_4$, thus there exists a supporting line $L_4$ of $K_4$ that does not go through $p_{123}$ and separates $K_4$ from $p_{123}$. $K_4$ touches $K_1$, $K_2$ and $K_3$, hence each of these three bodies has a point in both of the closed halfplanes bounded by $L_4$. From this it follows that $L_4$ intersects...
the cones $C_1$, $C_2$ and $C_3$. For every $i \in \{1, 2, 3\}$, $L_4 \cap C_i$ is a connected subset of $L_4$, thus there is a middle one of them. Without loss of generality we can assume that this one is $K_1$. Let $v_1$ be the vector for which $p_{123} = x_1 + v_1$. The image of $p_{123}$ by the homothety that maps $K_1$ to $K_4$ is the point $x_4 + \frac{x_3}{x_1} \cdot v_1 \in L_4$. The same homothety maps $C_1$ to the cone $C_1' := C_1 + \left( x_4 - x_1 + v_1 \cdot \left( \frac{x_3}{x_1} - 1 \right) \right)$. (Figure 2) As $K_1 \subset C_1$ and the bodies are positive homothets, $K_4 \subset C_1'$ follows. At least one pair of the bounding lines of $C_1$ and $C_1'$ are different, thus due to the fact that $\text{int} C_1 \cap \text{int} C_j = \emptyset$ for any $i \neq j$, $i, j \in \{1, 2, 3\}$ $C_1'$ is disjoint to at least one of the cones $C_2$ and $C_3$. But in this case $K_4$ cannot touch the body lying in this cone, which is a contradiction.

Thus, it is enough to consider the case when no point belongs to three of the homothets.

**Proposition 4.2.** Let $K_1$, $K_2$, ..., $K_n \subset \mathbb{R}^2$ be pairwise touching convex bodies, such that no three share a common point. Then $n \leq 4$.

**Proof.** For each $i \in \{1, ..., n\}$ choose an interior point $p_i \in K_i$. The bodies are pairwise touching, so we can draw a curve between any two of the chosen points $p_i$, $p_j$ so that it goes in $K_i \cup K_j$. Since no three of the bodies share a common point, these curves intersect only in the interior of the bodies. It is easy to see that we can eliminate these intersections with a perturbation. This way we draw the complete graph of $n$ vertices on the plane, from which $n \leq 4$ follows immediately.

We also show here a longer version of the proof:

Using the Jordan curve theorem, it is easy to show that the complement of $K_1 \cup K_2 \cup K_3$ in $\mathbb{R}^2$ has two connected components, one bounded and one unbounded. We call the closure of the bounded component the internal region surrounded by $K_1, K_2, K_3$.

**Proposition 4.3.** Let $K_1$, $K_2$, $K_3$, $K_4 \subset \mathbb{R}^2$ be pairwise touching convex bodies, such that no three share a common point. Then one of them lies in the internal region surrounded by the other three.

First observe that $n \leq 4$ follows from Proposition 4.3 Indeed, we may assume that $K_4$ is in the internal region $I$ surrounded by $K_1, K_2, K_3$. Suppose that $n \geq 5$. Since $K_5$ touches $K_4$, it must also lie in $I$. On the other hand, $K_5$ touches $K_1, K_2, K_3$ at points that do not belong to $K_4$. Now, $(\text{bd} I) \setminus K_4$ is the union of three open arcs, and $K_4$ must have a point on at least two of these arcs to touch $K_1, K_2, K_3$. However, then the interior of $K_5$ intersects the interior of at least one set from $K_1, ..., K_4$, a contradiction.

**Proof of Proposition 4.3.** For each $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, choose a point $p_{ij} \in K_i \cap K_j$. Due to our assumptions, these are pairwise distinct points. It follows that

$$ (p_{ik}, p_{il}) \cap (p_{jm}, p_{jn}) = \emptyset \quad \text{for any} \quad i, j, k, l \in \{1, 2, 3, 4\} \quad \text{with} \quad i \neq j. $$

Note that no three of the six touching points are collinear, since no three homothets of $K$ share a point in common.

We examine the convex hull $C = \text{conv}(p_{12}, \ldots, p_{15})$ of the six touching points.

Assume that $C$ is a hexagon. Consider the points $p_{12}$, $p_{13}$, $p_{14}$. We can select two of them, say $p_{12}$ and $p_{14}$ such that they are second neighbors in the natural cyclic order of the vertices of $C$, thus $p_{23}$, $p_{24}$ or $p_{34}$ is a common neighbour of them (Figure 4). Consider the segment connecting this common neighbour and another point of $p_{23}$, $p_{24}$ and $p_{34}$ (Figure 4). Then $(p_{12}, p_{14})$ and this segment intersect, contradicting (2).

Next, assume that $C$ is a pentagon. Without loss of generality we may assume that $p_{12}$ is the point which is not a vertex of $C$. Every vertex of $C$ has a 3 or a 4 in its indices. Thus, $p_{34}$ has a second neighbor in the cyclic ordering of the vertices of $C$, which has 3 or 4 in its indices. We consider the case when it is $p_{23}$ (see Figure 5), the other cases can be handled in the same manner. We connect $p_{12}$ with each of the two neighbors of $p_{34}$. One of these two line segments intersects $(p_{34}, p_{23})$ contradicting (2).

Next, assume that $C$ is a quadrilateral. Without loss of generality we may assume that $p_{12}$ is one of the two points that are not vertices of $C$. It is sufficient to examine those two cases, when the other such point
is $p_{13}$ or $p_{34}$, any other arrangement can be obtained from these by changing the roles (Figure 6).

If $p_{13}$ is not a vertex of the convex hull: The cyclic order of the vertices of $C$ is $p_{23}, p_{34}, p_{14}, p_{24}$, otherwise the diagonals of the quadrilateral would contradict (2). Then the segment $(p_{24}, p_{34})$ and the polygon $(p_{23}, p_{12}, p_{14})$ connect two opposite vertices of the quadrilateral, therefore they intersect each other. Again, contradicting (2).

If $p_{34}$ is not a vertex of the convex hull then the cyclic order of the vertices of $C$ is $p_{23}, p_{14}, p_{12}, p_{24}$, and a very similar argument to the previous case yields the desired contradiction.

Finally, assume that $C$ is a triangle. The vertices of the triangle have 6 indices in total, so by the pigeon hole principle, some index occurs in at least two of the vertices. Thus, we will assume that $p_{12}$ and $p_{13}$ are vertices of $C$. Examine which point of the remaining four can be the third vertex of $C$. It cannot be $p_{14}$, as in this case the other three touching points would be in the interior of $K_1$, which is a contradiction. If the third vertex was $p_{24}$ then one of the three line segments $(p_{34}, p_{12}), (p_{34}, p_{13}), (p_{34}, p_{24})$ would intersect one of the three line segments $(p_{14}, p_{12}), (p_{14}, p_{13}), (p_{14}, p_{24})$ contradicting (2) (see Figure 7). The same reasoning shows that the third vertex of $C$ cannot be $p_{34}$. Thus the third vertex is $p_{23}$. This is indeed possible, and in this case $K_4$ is in the internal region surrounded by $K_1, K_2, K_3$, finishing the proof of Proposition 4.3.

**Figure 2.**

**Remark 4.4.** The following proposition is also true and can be proved the same way considering further cases:

If $K_1, K_2, K_3, K_4 \subset \mathbb{R}^2$ are pairwise touching convex bodies and $K_1 \cap K_2 \cap K_3 \cap K_4 = \emptyset$, then one of them lies in the internal region surrounded by the other three.

5. **Proof of the lower bound in Theorem 3**

In this section, we show that for any planar convex body $K$, there are four pairwise touching homothets of $K$.

Consider two distinct parallel support lines of $K$ that each touch $K$ at one point: $x_1$ and $x_2$. The existence of such pair of lines follows from Theorem 2.2.11. of [12] (Theorem 2.2.9. of [11]), but may also be proved as an exercise.
Let $K_1 = K$ and $K_2 = K + x_2 - x_1$. Let $f$ be the line through the single point of contact, $x_2$, of $K_1$ and $K_2$ parallel to $x_2 - x_1$. On both sides of $f$, there is a translate of $K$ that touches both $K_1$ and $K_2$. Indeed, if we push $K$ around $K_1$ so that it always touches $K_1$ then, by continuity, such two positions will be found.

If on both sides we can find such translates of $K$ that also contains $x_2$ then $x_2$ is a common point of four translates of $K$ and we are done. Thus we assume that at least one of these translates does not contain $x_2$. We call this translate $K_3$. 
Now, $K_1, K_2, K_3$ are pairwise touching translates of $K$ that do not share a common point. It follows that they surround a bounded region $R$ with non-empty interior. Consider the largest homothet $K_4$ of $K$.
contained in \( R \). To finish the proof, we claim that \( K_4 \) touches \( K_1, K_2 \) and \( K_3 \). Indeed, assume \( K_4 \) touches only two of them, say \( K_1 \) and \( K_2 \). Consider a line that separates \( K_4 \) and \( K_1 \), and another line that separates \( K_4 \) and \( K_2 \). Let \( u_1 \) and \( u_2 \) be the unit normal vectors of these two lines respectively, pointing away from \( K_4 \). Clearly, if the origin is not in \( \text{conv}(u_1, u_2) \) then \( K_1 \) can be moved a little inside \( R \) so that it does not touch either \( K_1 \), \( K_2 \) or \( K_3 \). Then, we may enlarge \( K_4 \) slightly within \( R \) contradicting the maximality of \( K_4 \). Thus \( o \in \text{conv}(u_1, u_2) \), that is \( u_1 = -u_2 \). However in this case, \( K_1 \) and \( K_2 \) are strictly separated, which is a contradiction.

6. A TOPOLOGICAL NOTE

In this section we formulate a more general version of Proposition 4.3. This generalization is not needed for the proof of Theorem 5, but may of some interest.

An arc in the plane is the image of an injective continuous map of the \([0, 1]\) interval into the plane. A Jordan curve in the plane is the image of an injective continuous map of the circle into the plane. We will call the closed bounded region bounded by a Jordan curve a Jordan region.

Let \( K_1, K_2, K_3 \) be three pairwise touching Jordan regions whose pairwise intersections are non-empty arcs (which may be degenerate, that is a single point). Using the Jordan curve theorem, it is easy to show that the complement of \( K_1 \cup K_2 \cup K_3 \) in the plane has two connected components, one bounded and one unbounded. We call the closure of the bounded component the internal region surrounded by \( K_1, K_2, K_3 \), and the closure of the unbounded component the external region.

**Proposition 6.1.** Let \( K_1, K_2, K_3, K_4 \) be four pairwise touching Jordan regions whose pairwise intersections are non-empty arcs (which may be degenerate, that is a single point). Then one of them lies in the internal region surrounded by the other three.

An outline of the proof of Proposition 6.1 We will call the image of the non-negative reals under an injective mapping into the plane an unbounded path if it is an unbounded subset of the plane. The image of 0 is the starting point of the unbounded path.

Assume that \( K_1 \) is not in the internal region surrounded by \( K_2, K_3, K_4 \). Then there is a point \( p_1 \) on the boundary of \( K_1 \) that does not belong to either of the other three sets, and from which there is an unbounded path, \( \gamma_1 \) which is disjoint from the other three sets. Similarly, if \( K_2 \) is not in the internal region surrounded by the other three, then there is a point \( p_2 \) on the boundary of \( K_2 \) that does not belong to either of the other three sets, and from which there is an unbounded path, \( \gamma_2 \) which is disjoint from the other three sets. And the same holds for \( K_3 \) yielding \( p_3 \) and \( \gamma_3 \).

We may assume that \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are pairwise disjoint. Now, \( \gamma_1 \cup \gamma_2 \cup \gamma_3 \) partition the external region of \( K_1, K_2, K_3 \) into three parts. And \( K_4 \) is in one of these three parts. However, each part only intersects two of the sets \( K_1, K_2, K_3 \), which is a contradiction. \( \square \)

**References**

1. Károly Bezdek and Peter Brass, *On \( k^+ \)-neighbour packings and one-sided Hadwiger configurations*, Beiträge Algebra Geom. **44** (2003), no. 2, 493–498. MR MR2017050 (2004i:52017)
2. Károly Bezdek and Robert. Connelly, *Intersection points*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **31** (1988), 115–127 (1989). MR MR1003637 (90i:52014)
3. Károly Böröczky and László Szabó, *Minkowski arrangements of spheres*, Monatshefte für Mathematik **141** (2004), no. 1, 11–19.
4. L. Danzer and B. Grünbaum, *Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V. L. Klee*, Math. Z. **79** (1962), 95–99. MR MR0138040 (25 #1488)
5. Helmut Groemer, *Auszässzungen für die anzahl der konvexen körper, die einen konvexen körper berühren*, Monatshefte für Mathematik **65** (1961), 74–81.
6. Zsolt Lángi and Márton Naszódi, *On the Bezdek-Pach conjecture for centrally symmetric convex bodies*, Canad. Math. Bull. **52** (2009), no. 3, 407–415. MR 2547807 (2010j:52068)
7. Márton Naszódi and Konrad J. Swanepoel, *Arrangements of homothets of a convex body ii*, 2017.
8. Márton Naszódi, *On a conjecture of Károly Bezek and János Pach*, Period. Math. Hungar. **53** (2006), no. 1-2, 227–230. MR MR2286473
9. C. M. Petty, *Equilateral sets in Minkowski spaces*, Proc. Amer. Math. Soc. **29** (1971), 369–374. MR MR0275294 (43 #1051)
10. Alexandr Polyanskii, *Pairwise intersecting homothets of a convex body*, Electronic Notes in Discrete Mathematics 61 (2017), 1003–1009.

11. Rolf Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993. MR 1216521 (94d:52007)

12. ibid., *Convex bodies: the Brunn-Minkowski theory*, expanded ed., Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014. MR 3155183

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