A General Framework for the Security Analysis of Blockchain Protocols *

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Abstract

Blockchain protocols differ in fundamental ways, including the mechanics of selecting users to produce blocks (e.g., proof-of-work vs. proof-of-stake) and the method to establish consensus (e.g., longest chain rules vs. Byzantine fault-tolerant (BFT) inspired protocols). These fundamental differences have hindered “apples-to-apples” comparisons between different categories of blockchain protocols and, in turn, the development of theory to formally discuss their relative merits.

This paper presents a parsimonious abstraction sufficient for capturing and comparing properties of many well-known permissionless blockchain protocols, simultaneously capturing essential properties of both proof-of-work (PoW) and proof-of-stake (PoS) protocols, and of both longest-chain-type and BFT-type protocols. Our framework blackboxes the precise mechanics of the user selection process, allowing us to isolate the properties of the selection process that are significant for protocol design.

We demonstrate the utility of our general framework with several concrete results that delineate which properties are achievable by different types of blockchains under different synchrony assumptions. For example:

- We prove a CAP-type impossibility theorem asserting that liveness with an unknown level of participation (as in typical PoW protocols) rules out security in a partially synchronous setting (as enjoyed by several BFT-type PoS protocols).
- Delving deeper into the partially synchronous setting, we prove that a necessary and sufficient condition for security is the production of “certificates,” meaning stand-alone proofs of block confirmation.
- Restricting to synchronous settings, we prove that typical protocols with a known level of participation (including longest chain-type PoS protocols) can be adapted to provide certificates, but those with an unknown level of participation cannot.
- Finally, we use our framework to articulate a modular two-step approach to blockchain security analysis: (i) prove liveness and security properties for a permissioned version of a protocol; (ii) prove that the method of user selection extends these properties (with high probability) to the original permissionless protocol.

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1 Introduction

The task of a permissionless blockchain protocol is to establish consensus for message ordering over a network of users. This job is made difficult by the fact that, being subject to the laws of physics, the underlying communication network must have latency, i.e. broadcast messages will necessarily take time to travel over the network of users. As a consequence of this latency, malicious users may purposely cause the order in which messages are first seen to vary for different honest users, and some differences in ordering will anyway be an honest consequence of varying propagation times between different nodes of the network [DW13].

Network latency is especially problematic when we work at the level of individual transactions, which may be produced at a rate which is high compared to network latency. For this reason, it is standard practice to collect transactions together into blocks, which can then be produced at a rate which is much lower compared to network latency. Given that we are working in a permissionless setting, the basic question then becomes, “Who should produce the blocks?” In some broad sense all permissionless protocols answer this question in the same way. According to one protocol we might select users with probability depending on their hashing power, while according to another we might select users with probability proportional to their wealth. Users might be selected one at a time, or they might be selected in batches and asked to carry out Byzantine-Fault-Tolerant (BFT) protocols defined in the permissioned setting. In all of these cases, however, we decide upon a method of user selection and then we allow those selected users to act as permissioned users, giving them roles such as block production, or voting on blocks.

The main aim of this paper is to establish a framework for analysing permissionless blockchain protocols that blackboxes the precise mechanics of the user selection process, allowing us to isolate the properties of the selection process that impact the way in which the protocol must be designed, or that influence properties of the resulting protocol (such as security in a range of synchrony settings). In Section 2 we describe a framework of this kind, according to which protocols run relative to a resource pool. This resource pool specifies a balance for each user over the duration of the protocol execution (such as hashrate or stake in the currency), which may be used in determining which users are permitted to make broadcasts updating the state.

1.1 Our Contributions

We demonstrate the utility of our general framework with several concrete results that delineate which properties are achievable by different types of blockchains under different synchrony assumptions. We prove several results that show how basic properties of the user selection process necessarily impact the security guarantees that can be provided by the protocol; these results demonstrate fundamental differences in the security guarantees that can be provided by proof-of-work and proof-of-stake protocols. We will be concerned, in particular, with the distinction between scenarios in which the size of the resource pool is known (e.g. PoS), and scenarios where the size of the resource pool is unknown (e.g. PoW). We shall refer to the former case as the sized setting and the latter as the unsized setting.

A CAP-type impossibility result. By way of introduction, we establish a CAP-type theorem for our framework, which asserts that a protocol cannot be live and secure in the unsized and partially syn-
chronous setting. While CAP-type theorems have already been shown for other blockchain frameworks (see, for example, [GPS19, PS17]), the aim here is to begin the process of using our framework to formally differentiate the security guarantees that can be achieved by protocols using resource pools with fundamentally different properties. In particular, of course, this means differentiating the capabilities of PoW and PoS protocols.

**Theorem 3.1.** No protocol is live and secure in the unsized and partially synchronous setting.

We shall use the term adaptive as shorthand for “live in the unsized setting” and the term partition secure for “secure in the partially synchronous setting.” Theorem 3.1 thus establishes a simple dichotomy for permissionless blockchain protocols. A protocol can be adaptive or it can be partition secure, but not both. It also draws a clean and formal line between longest chain protocols such as Bitcoin and Ethereum [N+08, But18], or PoS implementations such as Snow White [BPS16] on the one hand, and BFT protocols such as Algorand [CM19], Casper FFG [BG17] and PoS implementations of Tendermint [Buc16] or Hotstuff [YMR+19] on the other. While the former group are all adaptive, the latter group are all partition secure. Another interesting conclusion that can be drawn from Theorem 3.1 concerns PoW protocols. PoS protocols are generally best modelled using the sized setting, while PoW protocols are generally best modelled using the unsized setting—the total stake is typically information which is available to a protocol from the beginning of its execution, while the amount of computational power used to provide PoW can vary over time in an unpredictable way. To the extent that PoW protocols must operate in an unsized setting (and guarantee liveness), Theorem 3.1 implies that they cannot be partition secure.

**A necessary and sufficient condition for partition security.** Next we prove a result that, to the best of our knowledge, has no previous analogues in the literature. Protocols such as Algorand and other BFT-type protocols achieve partition security through the production of what we shall call certificates. These are sets of broadcast messages whose very existence suffices to establish block confirmation and which cannot be produced by a (suitably bounded) adversary given the entire duration of the execution of the protocol. Bitcoin does not produce certificates, because the existence of a certain chain does not prove that it is the longest chain – a user will only believe that a certain chain is the longest chain until presented with a longer (possibly incompatible) chain. Algorand does produce certificates, on the other hand, because the very existence of a certain chain, together with appropriate committee signatures for all the blocks in the chain, can suffice to guarantee (beyond a reasonable doubt) that the blocks in that chain are confirmed. We will formally define what it means for a protocol to produce certificates in Section 4.

The production of certificates is also functionally useful, beyond providing security against network partitions. The production of certificates means, for example, that a single untrusted user is able to convince another user of block confirmation (by relaying an appropriate certificate), and this is potentially

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1Liveness and security will be formally defined in Section 2. Roughly, a protocol is live if the number of confirmed blocks can be relied on to increase during extended intervals of time during which message delivery is reliable. A protocol is secure if rollback on confirmed blocks is unlikely.

2It is standard in the distributed computing literature [Lyn96] to consider a variety of synchronous, partially synchronous, or asynchronous settings, in which users may or may not have clocks which are almost synchronised, or run at varying speeds, and where message delivery might be reliable or subject to various forms of failure. These terms, together with the sized and unsized settings, will be formally defined in Section 2. Roughly, in the synchronous setting that we consider, there is an upper bound on message delay, while in the partially synchronous setting there may be unbounded network partitions.
very useful in the context of sharding. If a user wishes to learn the state of a blockchain they were not previously monitoring, then it is no longer necessary to perform an onboarding process in which one samples the opinions of users until such a point that it is likely that at least one of them was ‘honest’ – one simply requests a certificate proving confirmation for a recently timestamped block.\(^3\)

In the spirit of our investigation, which looks to understand to what extent today’s protocols “have to look the way they are” given the security guarantees they achieve (as opposed to there being totally unexplored regions of the protocol design space), a key question is then:

Q1. Are certificates fundamental to partition security, or an artifact of Algorand’s specific implementation? That is, are certificates the only way to achieve security in the partially synchronous setting?

Our next result gives an affirmative response to Q1.

**Theorem 4.1.** If a protocol is partition secure then it produces certificates.

**Certificates in the synchronous setting.** Theorem 4.1 shows that the production of certificates is equivalent to security in the partially synchronous setting. But what about Bitcoin? While Bitcoin does not satisfy the conditions of Theorem 4.1, it clearly has some non-trivial security. The standard formalisation in the literature [GKL18, PSS17, Ren19] is that Bitcoin satisfies non-trivial security guarantees in the synchronous setting, for which there is an upper bound on message delivery time. Even working in the synchronous setting, though, it is clear that Bitcoin does not produce certificates. Again, we are led to ask whether this is a necessary consequence of the paradigm of protocol design:

Q2. Could there be a Bitcoin-like protocol that, at least in the synchronous setting, has as strong a security guarantee in terms of the production of certificates as BFT-type protocols do in the partially synchronous setting?

Again, the crucial distinction is between the sized and unsized settings. The term “non-trivial adversary”, which is used in Theorem 6.1 below, will be defined in Section 6 so as to formalise the idea that the adversary may have at least a certain minimum resource balance throughout the execution. With these basic definitions in place, we can then give a negative answer to Q2.

**Theorem 6.1.** Consider the synchronous and unsized setting. If a protocol is live then, in the presence of a non-trivial adversary, it does not produce certificates.

So, in the unsized setting (in which protocols like Bitcoin operate), useful protocols cannot produce certificates. Following on from our previous discussion regarding the relevance of certificates to sharding, one direct application of this result is that it rules out certain approaches to sharding for PoW protocols. In the sized and synchronous setting, though, it is certainly possible for protocols to produce certificates. It therefore becomes a natural question to ask how far we can push this:

Q3. Does the production of certificates come down purely to properties of the process of user selection? Is it simply a matter of whether one is in the sized or unsized setting?

\(^3\)Such techniques can avoid the need to store block hashes in a sharding “main chain,” and the information withholding attacks that come with those approaches.
Our final theorem gives a form of positive response to Q3. We state an informal version of the theorem below. A formal version will be given in Section 6.

**Theorem 6.2 (Informal Version).** Consider the synchronous and sized setting, and suppose a protocol is of ‘standard form’. Then there exists a ‘recalibration’ of the protocol which produces certificates.

Theorem 6.2 says, in particular, that all ‘standard’ PoS protocols can be tweaked to get the strongest possible security guarantee, since being of ‘standard form’ will entail satisfaction of a number of conditions that are normal for such protocols. Roughly speaking, one protocol will be considered to be a recalibration of another if running the former just involves running the latter for a computable transformation of the input parameters and/or using a different notion of block confirmation. The example of Snow White may be instructive here. At a very high level, Snow White might be seen as a PoS version of Bitcoin. It is a PoS longest chain protocol, and it is not difficult to see that, with the standard notion of confirmation, it does not produce certificates – an adversary can produce chains of blocks which are not confirmed, but which would be considered confirmed in the absence of other blocks which have been broadcast. So whether a block is confirmed depends on the whole set of broadcast messages. On the other hand, it is also not difficult to adjust the notion of confirmation so that Snow White does produce certificates. An example would be to consider a block confirmed when it belongs to a long chain of sufficient density (meaning that it has members corresponding to most possible timeslots) that it could not likely be produced by a (sufficiently bounded) adversary. We will see further examples like this explained in greater depth in Section 6. Theorem 6.2 implies much more generally that PoS protocols can always be modified so as to produce certificates in this way.

The punchline is that whether or not a protocol produces certificates comes down essentially to whether one is working in the sized or unsized setting (e.g. whether the protocol is PoS or PoW). This follows from the following results that we have already described:

(i) According to Theorem 3.1, only protocols which work in the sized setting can be secure in the partially synchronous setting. According to Theorem 4.1, all such protocols produce certificates.

(ii) Theorem 6.1 tells us that, in the synchronous and unsized setting, protocols cannot produce certificates.

(iii) Theorem 6.2 tells us that all standard protocols in the sized and synchronous setting can be recalibrated to produce certificates.

**Reducing permissionless to permissioned: A modular approach to blockchain security analysis.**

Finally, we show how our framework can be used to reduce the description and analysis of permissionless protocols to the permissioned case. This means that describing a permissionless blockchain protocol and proving that it satisfies basic properties, such as liveness and security, is now broken down into two tasks:

(a) Providing such an analysis for a permissioned reduct, i.e. a permissioned form of the protocol that we will describe in Section 7;

(b) Specifying and analysing the properties of an appropriate process of user selection, which selects users according to their balance as specified by the resource pool.
Our work here can be viewed as a common generalization of, for example, the security analyses of Algorand in [CM19] and of Bitcoin in [GKL18].

1.2 Related Work

While a number of excellent papers (see, for example, [GKL18, PSS17, BPS16]) describe frameworks for the analysis of PoW or PoS protocols, to our knowledge the framework presented here is the first to fully formalise a general approach in which users’ protocol interactions are governed by variable resource balances (such as hashrate or stake), which dictate a user’s ability to produce blocks and other messages over the duration of the protocol execution. Defining a framework of this kind allows us to formalise notions such as the sized and unsized settings, that play a crucial role in the statement of Theorems 3.1, 4.1, 6.1 and 6.2. The idea of blackboxing the process of user selection as an oracle (akin to our permitter, as described in Section 2) was explored in [AM+17]. Our paper may be seen as taking the same basic approach, and then fleshing out the details to the point where it becomes possible to prove impossibility results like those presented here. As here, a broad aim of [AM+17] was to understand the relationship between permissionless and permissioned consensus protocols, but the focus of that paper was somewhat different than our objectives in this paper. While our aim is to describe a framework which is as general as possible, and to establish impossibility results which hold for all protocols (and with as few assumptions on the process of user selection as possible), the aim of [AM+17] was to examine specific permissioned consensus protocols, such as Paxos [L+01], and to understand on a deep level how their techniques for establishing consensus connect with and are echoed by Bitcoin.

The CAP theorem is one of the most celebrated theorems in the distributed computing literature. The theorem proved by Gilbert and Lynch [GL02] is a formal version of a conjecture due to Brewer [Bre00], which is made in the context of distributed web services. The theorem establishes an impossibility result: It is impossible for such a distributed service to simultaneously achieve the three desirable properties of consistency, availability, and partition tolerance. For formal definitions of these terms we refer the reader to [GL02] and [Lyn96]. Roughly speaking, ‘security’ in our framework corresponds to ‘atomic consistency’ in the framework in which the CAP Theorem is proved in [GL02], and ‘liveness’ corresponds to ‘availability’. These correspondences are not exact, however. While availability requires a response even during extended periods of asynchrony, our definition of liveness (see Section 2) explicitly rules out the requirement that new confirmed blocks should be produced under such conditions. The key observation in the proof of Theorem 3.1 is that, in the unsized setting, extended periods of asynchrony cannot be distinguished from a waning resource pool. Liveness therefore forces the production of new confirmed blocks during appropriately chosen periods of asynchrony. Liveness and security are thus incompatible in the partially synchronous and unsized setting, while the same is not true in the partially synchronous and sized setting.

CAP-type theorems have been shown for other blockchain frameworks. In [GPS19], for example, the authors developed a useful weakening of the synchronous model, in which honest users need not always be online, but where message delivery and honest participation suffice to establish consensus protocols that can handle more than 1/3 of users being corrupt⁴ – the model considered there is permissioned, at least in the sense that bounds on the adversary concern the number of corrupted users. While honest users may not be online in every round/timeslot, the guarantee is made that, for some \( \chi \in [0, 1] \), more

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⁴Such protocols are not possible in the partially synchronous setting [DLS88].
than a $\chi$ fraction of nodes are not only honest but also online in each round. Precisely which set of honest nodes are online in each round is controlled by the adversary. For this model, the authors prove a CAP-type theorem establishing that consensus is not possible when $\chi < \frac{1}{2} - n$ (with $n$ being the number of users). The framework is different than that presented here, but the fundamental reasoning behind the proof is the same as that for Theorem 3.1, and the same as for the proof of the original CAP Theorem [GL02]. Very significantly (and with a lot more work), the authors then show that consensus is possible when $\chi \geq \frac{1}{2} - n$.

One consequence of Theorem 3.1 is that Bitcoin is not partition secure. This was proved in [PSS17].

1.3 The Structure of this Paper

Section 2 describes the basic framework that we shall use in order to blackbox the process of user selection for permissionless blockchain protocols. The partially synchronous, synchronous, sized and unsized settings will be formally defined in this section. We also define what it means for a protocol to be live, and define our two basic security notions in this section. We prove Theorem 3.1 in Section 3.

In Section 4, we define what it means to produce certificates. We show that the two security notions introduced in Section 2 are actually equivalent in the partially synchronous setting, and we prove Theorem 4.1 for the resulting (single) security notion. In Sections 5 and 6 we then move on to consider the synchronous setting. In Section 5 we show that, while the two security notions defined in Section 2 are not equivalent in the synchronous setting, one can establish weak conditions under which they do satisfy a form of equivalence. We then prove Theorem 6.1 and Theorem 6.2 in Section 6.

In Section 7, we define the notion of a permissioned reduct and show how it can be used to describe modular proofs of liveness and security. In order to demonstrate this point we consider the example of Bitcoin, and show how a modular presentation of the proof of liveness and security from [GKL18] can be described in terms of a permissioned reduct.

Finally, in Section 8, we discuss gaps in the existing analysis, and directions for future research.

2 The Framework

2.1 Predetermined and Undetermined Variables

So that we can define properties such as liveness and security later on, it will be convenient to consider protocols that are specified relative to a finite set of input parameters. For Algorand to run securely, for example, one must first decide how long the protocol is to run for, and then choose committee sizes accordingly. The duration of the execution is therefore required as a parameter of the protocol. Variables that are specified before the execution of the protocol as input parameters, or which take the same value for all executions of the protocol, are referred to as predetermined. Variables (such as the number of users) that are not predetermined, will be referred to as undetermined.
2.2 The Users

Protocols are executed by an undetermined set of pseudonymous users, this set being of undetermined size. Each user controls a set of public keys by which they will be known to other users. The variable $U$ is used to range over users, while $U$ will be used to range over public keys – each user may have many public keys. Amongst all users, there is one who is distinguished as the adversary and who controls an undetermined set of public keys. Users other than the adversary are called honest.

We suppose that each user is a deterministic computing device, which has amongst the actions it can perform calls to certain oracles, as well as certain external functionalities such as the ability to broadcast messages. The protocol specifies an instruction set, which is a program which is run by every user, other than the adversary. The adversary can follow any program of their choosing.

While we might think of the set of users as forming a network over which messages can propagate, we do not make the network explicit in our framework. Users simply have the ability to broadcast messages. Once a message is broadcast by a public key belonging to a given user, it may subsequently be delivered to other users at different stages of the execution.

2.3 Network Failures

The protocol gives instructions to be carried out at each in a sequence of timeslots $t = 0, 1, 2, \ldots$. For each execution of the protocol, the number of timeslots is given as an input parameter, together with a real time corresponding to each timeslot (i.e. the actual time at which those instructions should be carried out). In order that the execution can actually be carried out, the real time between timeslots must therefore suffice to carry out the given instructions. The appropriate length of time between timeslots thus depends on the protocol to be modelled. For many PoS protocols, an appropriate length is slightly more than the network latency, i.e. the time it takes a block to propagate the network. (Thus, each user might carry out many instructions during a single timeslot.) We will see that PoW protocols might be better modelled using very short timeslots.

For a given execution, the sequence of timeslots for which the protocol is to be run is called the duration $D$. At the beginning of each timeslot in the duration, broadcast messages may be delivered to various users. The message state relative to a given user is the set of all broadcast messages which have been delivered to them, and is therefore monotonically increasing over time. In order to be broadcast, a message must be valid, meaning that it must have a certain structure and that certain other conditions, expanded on below, are also satisfied.

For example, if modelling Bitcoin or Snow White, a user’s message state will be the set of (valid) blocks that have been delivered to them. Thus the message state will not, in general, be a single chain of blocks. For Algorand, a user’s message state will be all those messages which have been delivered to them, which are either valid blocks, or else the signed messages of committee members exchanged while reaching consensus on blocks.

It is standard in the distributed computing literature [Lyn96] to consider a variety of synchronous, partially synchronous, or asynchronous settings, in which users may or may not have clocks which are almost synchronised, or run at varying speeds, and where message delivery might be reliable or subject to various forms of failure. For the sake of simplicity, we will suppose here that users’ clocks
are synchronised. We will, though, allow for periods of network failure, during which the adversary is able to control message delivery. In order to formalise this, we will suppose that the duration is divided into intervals that are labelled either synchronous or asynchronous (meaning that each timeslot is either synchronous or asynchronous). We will suppose that, during asynchronous intervals, the adversary is able to interfere with message delivery as they choose, i.e. the adversary can deliver messages when they choose, or can stop messages being delivered at all. During synchronous intervals, we shall suppose that messages are delivered within $\Delta$ many timeslots. So $\Delta$ is fixed (and known to the protocol), and if $t_1, t_2 \in I$ for a synchronous interval $I$, with $t_2 - t_1 \geq \Delta$, then any message broadcast at $t_1$ will be delivered to all other users by $t_2$. It will be convenient to suppose that $\Delta \geq 1$.

We then distinguish two synchronicity settings. In the synchronous setting it is assumed that there are no asynchronous intervals during the duration, while in the partially synchronous setting there may be undetermined asynchronous intervals.

2.4 The Structure of the Blockchain

Amongst all broadcast messages, there is a distinguished set referred to as blocks, and one block which is referred to as the genesis block. Unless it is the genesis block, each block $B$ has a unique parent block $\text{Par}(B)$, which must be uniquely specified within the block message. Each block is produced by a single user, $\text{Miner}(B)$, but may contain other broadcast messages which have been produced by other users. No block can be broadcast by $U := \text{Miner}(B)$ at a point strictly prior to that at which its parent has been delivered to $U$. $\text{Par}(B)$ is defined to be an ancestor of $B$, and all of the ancestors of $\text{Par}(B)$ are also defined to be ancestors of $B$. If $B$ is not the genesis block, then it must have the genesis block as an ancestor. At any point during the duration, the set of broadcast blocks thus forms a tree structure. If $M$ is a set of messages, then we shall say that it is downward closed if it contains the parents of all blocks in $M$. By a leaf of $M$, we shall mean a block in $M$ which is not a parent of any block in $M$. If $M$ is downward closed and contains a single leaf, then we shall say that $M$ is a chain.

2.5 The Resource Pool

Protocols are run relative to a (predetermined or undetermined) resource pool, which in the general case is a function $R: \mathcal{U} \times D \times M \to \mathbb{R}_{\geq 0}$, where $\mathcal{U}$ is the set of public keys, $D$ is the duration and $M$ is the set of all possible sets of messages. So $R$ can be thought of as specifying the resource balance of each public key at each timeslot in the duration, possibly relative to a given message state. For a PoW protocol like Bitcoin, the resource balance of each public key will be their (relevant) computational power at the given timeslot (and hence independent of the message state). For PoS protocols, such as Snow White, Ouroboros [KRDO17] and Algorand, however, the resource balance will be fully determined by ‘on-chain’ information, i.e. information recorded in the message state $M$, and will reflect the user’s currency balance.

By the total resource balance $T$, we mean the sum of the resource balances of all public keys; this is the function $T: D \times M \to \mathbb{R}_{\geq 0}$ defined by $T(t, M) = \sum_U R(U, t, M)$. It should be noted that the resource balance $R$ is a function of the message state $M$. This is consistent with the idea that the resource pool is determined by the message state.

5As described more precisely in Section 2.6, whether the resource pool is predetermined or determined will decide whether we are in the sized or unsized setting.
pool is a variable, meaning that a given protocol may be expected to be live and secure with respect to a range of resource pools.

2.6 The Sized and Unsized Settings

Just as we considered two synchronicity settings earlier, we also consider two resource settings. The basic idea is that in the sized setting, the total resource balance is information which is available to the protocol (and the permitter, as described in Section 2.7), while in the unsized setting it is not. The precise details are as follows.

The unsized setting. For the unsized setting, \( R \) (and hence \( T \)) is undetermined, with the only restrictions being:

1. \( R \) will be a function from \( U \times D \times M \) to \( \mathbb{R}_{\geq 0} \) satisfying the requirement that, at all timeslots in the duration, the total resource balance belongs to a fixed interval \([R_0, R_1]\), where \( R_0 > 0 \) is sufficiently small and \( R_1 > R_0 \) is sufficiently large.\(^6\)

2. There may also be bounds placed on the resource balance of the adversary.

We shall refer to the set of all resource pools satisfying these restrictions as the possible resource pools, and, in Section 2.10, we shall define a protocol to be live if it is live for all possible resource pools.

The sized setting. For the sized setting, the total resource balance \( T \) is a predetermined function \( T : D \times M \rightarrow \mathbb{R}_{\geq 0} \).

The basic idea is that PoS protocols will generally be best modelled using the sized setting, while PoW protocols are best modelled using the unsized setting, since one does not know the total resource balance (e.g., total hashrate in each timeslot) in advance.

2.7 The Permitter Oracle

In order to specify how the resource pool is to be used, we make use of the notion of a permitter oracle. This is the part of the model that blackboxes user selection, since it is the permitter oracle that grants permission to broadcast messages. The permitter oracle need not be implemented explicitly in the blockchain being modelled, and is a mathematical abstraction that allows for the discussion and comparison of blockchains of very different types. It is designed to be as simple as possible, subject to this goal.

As described in Section 2.2, we consider each user to be a computing device with access to certain external oracles and functionalities. At any given timeslot \( t \in D \), a user’s state is entirely specified by the

\(^6\)We consider resource pools with range restricted to the fixed interval \([R_0, R_1]\) because it turns out to be an overly strong condition to require a protocol to be live without any further conditions on the total resource balance, beyond the fact that it is a function to \( \mathbb{R}_{\geq 0} \). We wish to be able to talk about Bitcoin as live in the unsized setting, for example, but liveness will certainly fail if \( R \) is the constant 0 function, or if the total resource balance decreases sufficiently quickly over time.
set of public keys they control, the input parameters, their message state and the set of permissions they have been given by the permitter oracle \( O \). The protocol \( P = (I, O) \) is then a pair, where the instruction set \( I \) is a set of deterministic and efficiently computable instructions, which specifies precisely what actions honest users should carry out at each timeslot, as a function of the timeslot and their state at that timeslot. The instructions of the protocol are therefore a function of the timeslot, the keys controlled by the user, the input parameters, their message state, and the set of permissions they have been given by the permitter.

One of the external functionalities each user has is the ability to broadcast valid messages. Amongst the conditions required for validity is that the public key responsible for the broadcast has been given permission by the permitter oracle \( O \), which is an oracle to which users have access. We thus suppose that users can make ‘requests’ to the permitter, of the form \( (U, M, t', A) \), where \( U \) is a public key under their control, \( M \) is a possible message state, \( t' \) is a timeslot, and where \( A \) is some (possibly empty) extra data. Given a request of this form, the permitter may then respond by giving them permission to broadcast certain messages. The response of the permitter to a request \( (U, M, t', A) \) will be assumed to be a probabilistic function of the protocol’s input parameters, the actual timeslot \( t \), the previous requests made by \( U \), the tuple \( (U, M, t', A) \), and of the user’s resource balance \( R(U, t, M) \).

It should be noted that the roles of the resource pool and the permitter are different in the sense that, while the resource pool is a variable (meaning that a given protocol may be expected to be live and secure with respect to a range of resource pools), the permitter is part of the protocol description (meaning that a protocol is only required to run relative to a specific permitter oracle).

So far we haven’t described any conditions requiring that the behaviour of the permitter must be influenced by the resource pool. Prior to Section 6, the only assumption of this kind that we shall make is stated below.

**No balance, no voice:** No \( U \) will be given permission to broadcast messages in response to a request \( (U, M, t', A) \) for which \( R(U, t', M) = 0 \).

### 2.8 Modelling Simple PoW and PoS Protocols

For concreteness, we next consider how some simple PoS and PoW protocols can be modelled using our framework. In this section, we will give a brief summary. In Section 7 we give a more in-depth description for Bitcoin. We remind the reader that our goal is not to literally model the step-by-step operation of these protocols, but rather to replicate the essential properties of their user selection mechanisms with a suitable choice of a permitter oracle.

First, consider a PoW protocol like Bitcoin. To keep things simple, let us ignore Bitcoin’s adjustable ‘difficulty parameter’ (i.e., how hard the PoW is to produce). To model a simple PoW protocol of this form, we can consider very short timeslots (say 1 second each, or even shorter). The resource level (i.e., hashrate) of a user in a given timeslot is independent of the message state, so we can restrict attention to resource pools \( R: \mathcal{U} \times \mathcal{D} \to \mathbb{R}_{\geq 0} \). We interpret a user request \( (U, M, t', A) \) in a timeslot \( t \)
as all of U’s efforts during timeslot t to extend the message state M\(^8\) – if a user submits more than one request during a timeslot, the permitter ignores all but the first. For example, we can interpret A as a choice and ordering of transactions within a proposed block, along with a choice of predecessor, with the understanding that the user will try as many different nonces as possible during the timeslot. The permitter then gives U permission to broadcast with probability proportional to R(U, t) (so long as A can be legally added to M).\(^9\) A notable feature of this permitter is that permission is granted for the broadcast of specific messages (i.e., a specific choice of A), rather than for a collection of messages meeting certain criteria.

There are various ways in which ‘standard’ PoS selection processes can work. Let us restrict ourselves, just for now and for the purposes of this example, to considering protocols in which the only broadcast messages are blocks, and let us consider a longest chain PoS protocol which works as follows: For each broadcast chain C and for all timeslots in a set T(C), the protocol being modelled selects precisely one public key who is permitted to produce blocks extending C (i.e. blocks whose parent is the unique leaf of C), with the probability each public key is chosen being proportional to their wealth as recorded in C.\(^10\) Then we can consider a permitter which chooses one public key U for each chain C and each timeslot t’ in T(C), each public key U being chosen with probability R(U, C)/T(C). (This is well defined because the total resource pool \(T\) is known to the protocol.) That chosen public key U corresponding to C and t’, is then given permission to broadcast blocks extending C whenever U makes a request (U, M, t’, \(\emptyset\)) for which C is the longest chain in M. A notable feature of this permitter is that the permission it gives is for the broadcast of sets of messages satisfying certain criteria, i.e. when the permitter gives permission it is for any (otherwise valid) block extending a given chain C.

To model a BFT PoS protocol like Algorand, the basic approach will be very similar to that described for the longest chain PoS protocol above, except that certain other signed messages might be now required in M (such as signed votes on blocks) before permission to broadcast is granted, and permission may now be given for the broadcast of messages other than blocks (such as votes on blocks).

### 2.9 The Extended Protocol and the Meaning of Probabilistic Statements

In order to define what it means for a protocol to be secure or live, we first need a notion of confirmation for blocks. This is a function C mapping any message state to a chain that is a subset of that message state, in a manner that depends on the protocol input parameters, including a parameter \(\varepsilon > 0\) called

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\(^8\)The parameter t’ is automatically interpreted as the current timeslot t; the parameter t’ is relevant only for PoS protocols.

\(^9\)I.e., with probability \(R(U, t)/M\), where M is a large constant that depends on an assumed upper bound on hashrate and the timeslot length.

\(^10\)Note that being permitted to broadcast a block is not the same as being instructed by the protocol to broadcast a block, and does not determine how other users will treat the block – there are many contexts in which users might be able to produce valid blocks for which broadcast is not instructed by the protocol (e.g., a block extending a chain other than the longest one). A user may also be permitted to produce two valid blocks whose broadcast constitutes an overt deviation from the protocol, and which might be punished.

\(^11\)Note that in many PoS protocols the relevant balance is actually U’s wealth according to some proper initial segment of C, and that in modelling such protocols one should adjust \(R\) accordingly. As mentioned earlier, it is also standard to insist that U has been recorded as a public key with non-zero stake for a minimum number of timeslots.
the security parameter. The intuition behind $\epsilon$ is that it should upper bound the probability of false confirmation. Given any message state, $C$ returns the set of confirmed blocks.

In Section 2.7, we stipulated that the protocol $P = (I, O)$ is a pair, where the instruction set $I$ is a set of deterministic and efficiently computable instructions specifying precisely what actions an honest user should carry out at each timeslot, and where $O$ is the permitter. In general, however, a protocol might only be considered to run relative to a specific notion of confirmation $C$. We will refer to the triple $(I, O, C)$ as the extended protocol. Often we shall suppress explicit mention of $C$, and assume it to be implicitly attached to a given protocol. We shall talk about a protocol being live, for example, when it is really the extended protocol to which the definition applies. It is important to understand, however, that the notion of confirmation $C$ is separate from $P$, and does not impact the instructions of the protocol. In principle, one can run the same Bitcoin protocol relative to a range of different notions of confirmation. While the set of confirmed blocks might depend on $C$, the instructions of the protocol do not.

An execution of the extended protocol will depend on the message delivery rule: A message delivery rule is a partial function $d$ which maps any tuple $(U, m, t, U')$, such that $U$ is a public key, $m$ is a possible message, $t$ is a timeslot in the duration, and $U'$ is a user, to a timeslot $t' \geq t$. So a fixed message delivery rule determines when each message $m$, broadcast by $U$ at timeslot $t$, will be delivered to each user $U'$.\(^1\) We restrict attention to message delivery rules which are consistent with $\Delta$ (see Section 2.3), the set of synchronous and asynchronous intervals and the adversary, and which satisfy the condition that $d(U, m, t, U') = t$ whenever $U$ is a public key controlled by $U'$.

Each execution of the extended protocol is then entirely determined by:

(I1) The input parameters;
(I2) The set of users and their public keys;
(I3) An index specifying the program executed by the adversary;
(I4) The resource pool (which may or may not be undetermined);
(I5) The message delivery rule;
(I6) The probabilistic responses of the permitter.

With respect to the extended protocol $(I, O, C)$, we call a particular set of choices for (I1)-(I5) a protocol instance. Generally, when we discuss an extended protocol, we shall do so within the context of a setting, which constrains the set of possible protocol instances. The setting might restrict the set of resource pools to those in which the adversary is given a limited resource balance, for example. When we make a probabilistic statement to the effect that a certain condition holds with at most/least a certain probability, this means that the probabilistic bound holds for all protocol instances consistent with the setting.

Where convenient, we may also refer to the pair $(P, C)$ as the extended protocol, where $P = (I, O)$.

\(^1\)Note that a single message delivery rule might result in many different sequences of message deliveries, if different sequences of messages are broadcast. Note also that messages are broadcast by keys but delivered to users.
2.10 Defining Liveness

There are a number of papers that successfully describe liveness and security notions for blockchain protocols, see for example [PSS17, GKL18, KRS18]. Our interest here is in identifying the simplest definitions that suffice to express our later results. Consider a protocol with a notion of confirmation $C$, and let $|C(M)|$ denote the number of blocks in $C(M)$ for any message state $M$. For timeslots $t_1 < t_2$, let $l_1$ be the maximum value $|C(M_1)|$ for any $M_1$ which is a message state of any user at any timeslot $t \leq t_1$, and let $l_2$ be the minimum value $|C(M_2)|$ for any $M_2$ which is a message state of any user at timeslot $t_2$. We say that $[t_1, t_2]$ is a growth interval if $l_2 > l_1$. For any duration $D$, let $|D|$ be the number of timeslots in $D$. For $\ell_{\varepsilon,D}$ which takes values in $\mathbb{N}$ depending on $\varepsilon$ and $D$, let us say that $\ell_{\varepsilon,D}$ is sublinear in $D$ if, for each $\varepsilon > 0$ and each $\alpha \in (0, 1)$, $\ell_{\varepsilon,D} < \alpha |D|$ for all sufficiently large values of $|D|$ (the motivation for considering sublinearity will be described shortly).

**Definition 2.1.** A protocol is **live** if, for every choice of security parameter $\varepsilon > 0$ and duration $D$, there exists $\ell_{\varepsilon,D}$, which is sublinear in $D$, and such that for each pair of timeslots $t_1 < t_2 \in D$ the following holds with probability at least $1 - \varepsilon$: If $t_2 - t_1 \geq \ell_{\varepsilon,D}$ and $[t_1, t_2]$ is entirely synchronous, then $[t_1, t_2]$ is a growth interval.

So, roughly speaking, a protocol is live if the number of confirmed blocks can be relied on to grow over time during synchronous intervals of sufficient length. The reason we require $\ell_{\varepsilon,D}$, to be sublinear in $D$ is so that the number of confirmed blocks likely increases with sufficient increase in synchronous duration. For example, a protocol that confirms a block with probability only $2^{-|D|}$ at each timeslot should not be considered live.

Generally, assertions of liveness and security will be made within the confines of a particular setting, which might limit the resource balance of the adversary or specify a minimum number of active users for each timeslot, for example. Note also, that while Definition 2.1 only refers explicitly to protocols, it is really the extended protocol to which the definition applies. The following stronger notion will also be useful.

**Definition 2.2.** A protocol is **uniformly live** if, for every choice of security parameter $\varepsilon > 0$ and duration $D$, there exists $\ell_{\varepsilon,D}$, which is sublinear in $D$, and such that the following holds with probability at least $1 - \varepsilon$: For all pairs of timeslots $t_1 < t_2 \in D$, if $t_2 - t_1 \geq \ell_{\varepsilon,D}$ and $[t_1, t_2]$ is entirely synchronous, then $[t_1, t_2]$ is a growth interval.

So the difference between being live and uniformly live is that the latter definition requires that, with probability at least $1 - \varepsilon$, all appropriate intervals are growth intervals. The former definition only requires the probabilistic bound to hold for each interval individually. An initial impression might be that it should follow from the Union Bound that Definitions 2.1 and 2.2 are essentially equivalent. This is not so. Firstly, this is because the protocol and notion of confirmation take the security parameter $\varepsilon$ as input. Nevertheless, one might think that if a protocol is live then a ‘recalibration’, which takes some appropriate transformation of the security parameter as input, should necessarily be uniformly live. This does not follow (in part) because there is no guarantee that the resulting $\ell_{\varepsilon,D}$ will be sublinear in $D$ – see Section 5 for a detailed analysis.

**Definition 2.3.** We define a protocol to be **adaptive** if it is live in the unsized setting.
2.11 Defining Security

Roughly speaking, security requires that confirmed blocks normally belong to the same chain. Let us say that two distinct blocks are incompatible if neither is an ancestor of the other, and are compatible otherwise. If \( B \in C(M) \) where \( M \) is the message state of user \( U \) at time \( t \), then we shall say that \( B \) is confirmed for \( U \) at \( t \).

**Definition 2.4** (Security). A protocol is **secure** if the following holds for every choice of security parameter \( \varepsilon > 0 \), for every \( U_1, U_2 \) and for all timeslots \( t_1, t_2 \) in the duration: With probability \( > 1 - \varepsilon \), all blocks which are confirmed for \( U_1 \) at \( t_1 \) are compatible with all those which are confirmed for \( U_2 \) at \( t_2 \).

The following stronger notion will also be useful.

**Definition 2.5** (Uniform Security). A protocol is **uniformly secure** if the following holds for every choice of security parameter \( \varepsilon > 0 \): With probability \( > 1 - \varepsilon \), there do not exist incompatible blocks \( B_1, B_2 \), timeslots \( t_1, t_2 \) and \( U_1, U_2 \) such that \( B_i \) is confirmed for \( U_i \) at \( t_i \) for \( i \in \{1, 2\} \).

So the difference between security and uniform security is that the latter requires the probability of even a single disagreement to be bounded, while the former only bounds the probability of disagreement for each pair of users at each timeslot pair. Just as for liveness and uniform liveness, it does not follow from the Union Bound that security is essentially equivalent to uniform security. In Section 5 we will perform a detailed analysis of the relationship between these notions.

**Definition 2.6.** A protocol is **partition secure** if it is secure in the partially synchronous setting. A protocol is **uniformly partition secure** if it is uniformly secure in the partially synchronous setting.

Note that BFT protocols such as Algorand are normally designed to be partition secure.

3 The Impossibility of being Adaptive and Partition Secure

Now that the framework and all required definitions are in place, we can formally prove Theorem 3.1.

**Theorem 3.1.** No protocol is both adaptive and partition secure.

As stated previously, this theorem can be seen as an analogue of the CAP Theorem [GL02] from distributed computing for our framework. Now that we have formally defined adaptivity, security, and liveness, it may be useful to say a little more about the relationship to the CAP Theorem. While the CAP Theorem asserts that (under the threat of unbounded network partitions), no protocol can be both available and consistent, it is possible for BFT protocols such as Algorand to be both live and secure in the partially synchronous setting. This is possible because liveness is a fundamentally weaker property than availability: Liveness does not require new confirmed blocks to be produced during extended periods of asynchrony. For example, Algorand is live, even though block production may stop during network partitions. The key idea behind the proof of Theorem 3.1 is that, in the unsized (and partially synchronous) setting, this fundamental difference disappears, with network partitions indistinguishable from waning resource pools. Liveness then forces the existence of growth intervals during network partitions. In the unsized and partially synchronous setting, security and liveness thus become incompatible, just as consistency and availability are incompatible according to the CAP Theorem.
Proof. (of Theorem 3.1) The idea behind the proof can be summed up as follows. We consider executions of the protocol in which there are at least two users, both of which are honest, and who control public keys $U_0$ and $U_1$ respectively. Suppose that, in an execution of the protocol in the unsized and partially synchronous setting, $U_0$ and $U_1$ both have the same constant and non-zero resource balance, and that all other users have resource balance zero throughout the duration. According to the assumption ‘no balance, no vote’, this means that $U_0$ and $U_1$ will be the only public keys which are able to broadcast messages. For as long as the adversary is able to prevent messages broadcast by each $U_i$ from being delivered to $U_{1-i}$ ($i \in \{0, 1\}$), the execution will be indistinguishable, as far as $U_i$ is concerned, from one in which only $U_i$ has the same constant and non-zero resource balance. The fact that the protocol is live means that, with high probability, $U_0$ and $U_1$ will see confirmed blocks within a bounded period of time. The confirmed blocks for $U_0$ will be incompatible with those for $U_1$, so long as these confirmed blocks appear before any point at which a message broadcast by $U_i$ has been delivered to $U_{1-i}$ for some $i \in \{0, 1\}$. This contradicts security for the protocol in the partially synchronous setting.

To describe the argument in more detail, let $U_0$ and $U_1$ be public keys controlled by different honest users. For a duration $D$ which is sufficiently long, we consider three different resource pools:

- $\mathcal{R}_0$: We let $\mathcal{R}_0$ assign the constant value $R > 0$ to both $U_0$ and $U_1$ over the entire duration, while all other users are assigned the constant value 0.
- $\mathcal{R}_1$: We let $\mathcal{R}_1$ assign the constant value $R$ to $U_0$ over the entire duration, while all other users are assigned the constant value 0.
- $\mathcal{R}_2$: We let $\mathcal{R}_2$ assign the constant value $R$ to $U_1$ over the entire duration, while all other users are assigned the constant value 0.

We consider three different instances of the protocol with the same parameters, for the unsized setting in which the resource pool is an undetermined variable:

- $\mathcal{I}_{n_0}$: Here $\mathcal{R} := \mathcal{R}_0$. All timeslots are asynchronous and the adversary prevents the delivery of messages broadcast by $U_i$ to the user controlling $U_{1-i}$, for $i \in \{0, 1\}$.
- $\mathcal{I}_{n_1}$: Here $\mathcal{R} := \mathcal{R}_1$, and we work in the synchronous setting (or in the partially synchronous setting, but without interference by the adversary).
- $\mathcal{I}_{n_2}$: Here $\mathcal{R} := \mathcal{R}_2$, and we work in the synchronous setting.

According to the assumption of ‘no balance, no voice’, it follows that only $U_0$ and $U_1$ will be able to broadcast messages in any of these three instances. Our framework stipulates that the instructions of the protocol for a given user at a given timeslot must be a deterministic function of the protocol parameters, the timeslot, the keys controlled by the user, their message state and the set of permissions they have been given by the permitter (see Section 2.7). It also stipulates that the response of the permitter to a request $(U, M, t', A)$ is a probabilistic function of the protocol parameters, the actual timeslot $t$, previous requests made by $U$, the request $(U, M, t', A)$, and the user’s resource level $\mathcal{R}(U, t', M)$. It therefore follows by induction on timeslots that, because the resource pool is undetermined:

$(†)$ For each $i \in \{0, 1\}$, and for all timeslots in $\mathcal{I}_{n_0}$, the probability distribution on the state of the user controlling $U_i$ is identical to the corresponding distribution at the same timeslot in $\mathcal{I}_{n_{1+i}}$.

If the protocol is adaptive, then it follows from Definition 2.1 that we can find a timeslot $t_0$ satisfying the following condition: In both $\mathcal{I}_{n_{1+i}} (i \in \{0, 1\})$, it holds with probability $> \frac{3}{4}$ that there is at least
one block which is confirmed for $U_i$ at $t_0$. By (†) it then holds for $\mathbb{I}_{0}$, and for each $i \in \{0, 1\}$, that with probability $> 3/4$ there is at least one block which is confirmed for $U_i$ at $t_0$. We stipulated in Section 2.4 that no block $B$ can be broadcast by $U := \text{ Miner}(B)$ at a point strictly prior to that at which its parent has been delivered to $U$. It follows that in $\mathbb{I}_{0}$ all blocks which are confirmed for $U_i$ at $t_0$ must be incompatible with all blocks which are confirmed for $U_{1-i}$. The definition of security therefore fails to hold for timeslot $t_0$, and with respect to the security parameter $1/2$. □

4 Certificates in the Partially Synchronous Setting

The rough idea is that ‘certificates’ should be proofs of confirmation. Towards formalising this idea, let us first consider a version which is too weak.

**Definition 4.1.** If $B \in \mathbb{C}(M)$ then we shall refer to $M$ as a **subjective certificate** for $B$.

We will say that a set of messages $M$ is broadcast if every member is broadcast, and that $M$ is broadcast by timeslot $t$ if every member of $M$ is broadcast at a timeslot $\leq t$ (different members potentially being broadcast at different timeslots). If $M$ is a subjective certificate for $B$, then there might exist $M' \supset M$ for which $B \notin \mathbb{C}(M')$. So the fact that $M$ is broadcast does not constitute proof that $C$ is confirmed with respect to any user (unless you are that user and you also know that $M$ is your entire message state). When do we get harder forms of proof than subjective certificates? Definition 4.2 below gives a natural and very simple way of formalising this.

**Definition 4.2.** We say that a protocol with a notion of confirmation $C$ **produces certificates** if the following holds with probability $> 1 - \varepsilon$ when the protocol is run with security parameter $\varepsilon$: There do not exist incompatible blocks $B_1, B_2$, a timeslot $t$ and $M_1, M_2$ which are broadcast by $t$, such that $B_i \in \mathbb{C}(M_i)$ for $i \in \{1, 2\}$.

It is important to stress that, in the definition above, the $M_i$ are not necessarily the message states of any user, but are rather arbitrary subsets of the set of all broadcast messages. The basic idea is that, if a protocol produces certificates, then subjective certificates constitute proof of confirmation. Algorand is an example of a protocol which produces certificates: The protocol is designed so that it is unlikely that two incompatible blocks will be produced at any point in the duration together with appropriate committee signatures verifying confirmation for each.

Our next aim is to show that, in the partially synchronous setting, producing certificates is equivalent to security. In fact, producing certificates is clearly at least as strong as uniform security, so it suffices to show that if a protocol is secure then it must produce certificates.

**Theorem 4.1.** If a protocol is partition secure then it produces certificates.

**Proof.** Towards a contradiction, suppose that the protocol with notion of confirmation $C$ is secure in the partially synchronous setting, but that there exists a protocol instance $\mathbb{I}_{1}$ with security parameter $\varepsilon$, such that the following holds with probability $\geq \varepsilon$: There exist incompatible blocks $B_1, B_2$, a timeslot $t$ and $M_1, M_2$ which are broadcast by $t$, such that $B_i \in \mathbb{C}(M_i)$ for $i \in \{1, 2\}$. This means that the following

13See Section 2.9 for the definition of a protocol instance.
holds with probability $\geq \varepsilon$ for $t_{\text{last}}$, which is the last timeslot in the duration: There exist incompatible blocks $B_1, B_2$ and $M_1, M_2$ which are broadcast by $t_{\text{last}}$, such that $B_i \in \mathcal{C}(M_i)$ for $i \in \{1, 2\}$.

Consider the protocol instance $\text{In}_2$ which has the same parameters as $\text{In}_1$, the same adversary, and the same set of users who are active at the same set of timeslots with the same set of public keys, except that now there are two extra users $U_1$ and $U_2$, who are active at all timeslots with keys $U_1$ and $U_2$ respectively. Suppose that the resource pool for $\text{In}_2$ is the same as that for $\text{In}_1$ when restricted to public keys other than $U_1$ and $U_2$, and that $U_1$ and $U_2$ have zero resource balance throughout the duration. Suppose further that the message delivery rule for $\text{In}_2$ is the same as that for $\text{In}_1$ when restricted to tuples $(U, m, t, U')$ such that $U \not\in \{U_1, U_2\}$ and $U' \not\in \{U_1, U_2\}$, but that now all timeslots are asynchronous (we can suppose that the adversary is such that this message delivery rule is still consistent with the adversary, even when all timeslots are asynchronous). Our framework stipulates that the instructions of the protocol for a given user at a given timeslot must be a deterministic function of the protocol parameters, the timeslot, the keys controlled by the user, their message state and the set of permissions they have been given by the permitter (see Section 2.7). It also stipulates that the response of the permitter to a request $(U, M, t', A)$ is a probabilistic function of the input parameters, the actual timeslot $t$, previous requests made by $U$, the request $(U, M, t', A)$, and the user’s resource level $R(U, t', M)$. From the assumption ‘No balance, no voice’, it therefore follows by induction on timeslots that the probability distribution on the set of broadcast messages is the same at each timeslot for $\text{In}_2$ as for $\text{In}_1$, independent of which deliveries the adversary chooses to make to $U_1$ and $U_2$. It therefore holds for the protocol instance $\text{In}_2$ that with probability $\geq \varepsilon$ there exist incompatible blocks $B_1, B_2$, and $M_1, M_2$ which are broadcast by $t_{\text{last}}$, such that $B_i \in \mathcal{C}(M_i)$ for $i \in \{1, 2\}$. Suppose that the adversary withholds all messages from $U_1$ and $U_2$ until $t_{\text{last}}$, and then delivers $M_1$ and $M_2$ (if they exist) to $U_1$ and $U_2$ respectively. This suffices to demonstrate that the definition of security is violated with respect to $t_{\text{last}}, \varepsilon, U_1$ and $U_2$. \hfill $\Box$

**Corollary 4.3.** Security and uniform security are equivalent in the partially synchronous setting.

**Proof.** This follows from Theorem 4.1 and the fact that producing certificates clearly implies uniform security. \hfill $\Box$

## 5 Defining Recalibrations

Theorem 4.1 presents a rather tidy picture for the partially synchronous setting. It is not difficult to see, however, that security and uniform security will not be equivalent in the synchronous setting. To see this, we can consider the example of Bitcoin. Suppose that we operate in the standard way for Bitcoin, and use a notion of confirmation $\mathcal{C}$ that depends only on the security parameter $\varepsilon$, and not on the duration $\mathcal{D}$. In this case, the protocol is secure in the synchronous setting, but will not be uniformly secure in a setting where the adversary controls a non-zero amount of mining power: If a fixed number of blocks are required for confirmation then, given enough time, the adversary will eventually complete a double spend. It is also clear, however, how to ‘recalibrate’ the protocol to deal with different durations – to make the protocol uniformly secure, the number of blocks required for confirmation should be a function of both $\varepsilon$ and $\mathcal{D}$.

The aim of this section is formalise the idea of recalibration and to show that, if a protocol is secure, then (under fairly weak conditions) a recalibration will be uniformly secure. The basic idea is very
simple – one runs the initial (unrecalibrated) protocol for smaller values of $\varepsilon$ as the duration increases, but one has to be careful that the resulting $\ell_{\varepsilon, D}$ is sublinear in $D$.

**Definition 5.1.** We shall say $(P_2, C_2)$ is a recalibration of the extended protocol $(P_1, C_1)$ if running $P_2$ given certain input parameters means running $P_1$ for a computable transformation of those parameters, and then terminating after $|D|$ many steps are complete.

So if running $P_2$ with security parameter $\varepsilon$ and for $n$ many timeslots means running $P_1$ with input parameters that specify a security parameter $\varepsilon/10$ and that specify a duration consisting of $2n$ many timeslots, and then terminating after $n$ many timeslots have been completed, then we might describe $P_2$ as a recalibration of $P_1$.\(^{14}\) Note also, that we allow the recalibration to use a different notion of confirmation.

In the following, we shall say that $\ell_{\varepsilon, D}$ is independent of $D$ if $\ell_{\varepsilon, D} = \ell_{\varepsilon, D'}$ for all $\varepsilon > 0$ and all $D, D'$. When $\ell_{\varepsilon, D}$ is independent of $D$, we shall often write $\ell_{\varepsilon}$ for $\ell_{\varepsilon, D}$.

**Definition 5.2.** In the bounded user setting we assume that there is a finite upper bound on the number of users, which holds for all protocol instances.\(^{15}\)

**Proposition 5.3.** Consider the synchronous and bounded user setting. Suppose $P$ satisfies liveness with respect to $\ell_{\varepsilon, D}$, that $\ell_{\varepsilon, D}$ is independent of $D$, and that for each $\alpha > 0$, $\ell_{\varepsilon} < \alpha \varepsilon^{-1}$ for all sufficiently small $\varepsilon > 0$. If $P$ is secure, there exists a recalibration of $P$ that is uniformly live and uniformly secure.

The conditions on $\ell_{\varepsilon, D}$ in the statement of Proposition 5.3 can reasonably be regarded as weak, because existing protocols which are not already uniformly secure will normally satisfy the conditions that:

- $(\dagger_a)$ $\ell_{\varepsilon, D}$ is independent of $D$, and;
- $(\dagger_b)$ For some constant $c$ and any $\varepsilon \in (0, 1)$, we have $\ell_{\varepsilon} < c \ln \frac{1}{\varepsilon}$.

The example of Bitcoin might be useful for the purposes of illustration here. Bitcoin is secure in the synchronous setting, and the number of blocks required for confirmation is normally considered to be independent of the duration. The number of blocks required for confirmation does depend on how sure one needs to be that an adversary cannot double spend in any given time interval, but it’s also true that an adversary’s chance of double spending in a given time interval decreases exponentially in the number of blocks required for confirmation as well. So Bitcoin is an example of a protocol satisfying $(\dagger_a)$ and $(\dagger_b)$ above.

**Proof of Proposition 5.3.** It is useful to consider a security notion that is intermediate between security and uniform security. For the purposes of the following definition, we shall say that a block is confirmed at timeslot $t$ if there exists at least one user for whom that is the case.

**Definition 5.4** (Timeslot Security). A protocol is **timeslot secure** if the following holds for every choice of security parameter $\varepsilon > 0$, and for all timeslots $t_1, t_2$ in the duration: With probability $> 1 - \varepsilon$, all blocks which are confirmed at $t_1$ are compatible with all blocks which are confirmed at $t_2$.

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\(^{14}\) The reader might wonder why one should specify a duration of $2n$ timeslots and then terminate after $n$ many. This is because the instructions of the first $n$ timesteps can depend on the intended duration. In Algorand, committee sizes will depend on the intended duration, for example.

\(^{15}\) Note that the requirement here is that the number of users is bounded, rather than the number of public keys.
So the difference between timeslot security and uniform security is that the latter requires the probability of even a single disagreement to be bounded, while the former only bounds the probability of disagreement for each pair of timeslots. Similarly, the difference between security and timeslot security is that, for each pair of timeslots, the latter requires the probability of even a single disagreement to be bounded, while the former only bounds the probability of disagreement for each pair of users at that timeslot pair.

Now suppose $P$ is live and secure, and that the conditions of Proposition 5.3 hold. Then it follows directly from the Union Bound that, if the number of users is bounded, then some recalibration of $P$ is live and timeslot secure and satisfies the conditions of Proposition 5.3. Since a recalibration of a recalibration of $P$ is a recalibration of $P$, our main task is therefore to show that, if $P$ is live and timeslot secure and the conditions of Proposition 5.3 hold, then there exists a recalibration of $P$ that is uniformly live and uniformly secure.

So suppose $(P, C)$ is live and timeslot secure, and that the conditions of Proposition 5.3 hold. Suppose we are given $e_0$ and $D_0$ as input parameters to our recalibration $(P', C')$. We wish to find an appropriate security parameter $e_1$ and a duration $D_1 \geq D_0$ to give as inputs to $P$ and $C$, so that uniform security is satisfied with respect to $e_0$ and $D_0$ if we run $P$ with inputs $e_1$ and $D_1$ and then terminate after $|D_0|$ many timeslots. The difficulty is to ensure that $\ell_{e_1}$ remains sublinear in $D_0$. To achieve this, let $n := |D_0|$, set $e_1 := e_0/2n$ and choose $|D_1| > n + \ell_{e_1}$, so that $D_0$ is the first $n$ timeslots in $D_1$. This defines the recalibration. It remains to establish uniform liveness and uniform security.

For uniform liveness we must have that, for each $\alpha \in (0, 1)$, $\ell_{e_1} < \alpha n$ for all sufficiently large values of $n$—if this condition holds then it follows from the Union Bound that our recalibration will satisfy uniform liveness (and the required sublinearity in $|D_0|$) with respect to $\ell_{e_1}': = \ell_{e_1}$. The condition holds since we are given that for each $\alpha > 0$, $\ell_{e_1} < \alpha e_0^{-1}$ for all sufficiently small $e > 0$. Suppose given $\alpha > 0$, and put $\alpha' := \alpha e_0/2$. Then we have that, for all sufficiently large $n$:

$$\ell_{e_1} < \alpha'(e_0/2n)^{-1} = \alpha n.$$  

Next we must show that the conditions for uniform security are satisfied. Suppose $P$ is given inputs $e_1$ and $D_1$ and is actually run for $|D_1|$ many timeslots. We aim to show that, with probability $1 - e_0$, there do not exist incompatible blocks $B_1, B_2$, timeslots $t_1, t_2 \in D_0$ and $U_1, U_2$ such that $B_1$ is confirmed for $U_i$ at $t_i$ for $i \in \{1, 2\}$. Let $t_{\text{last}}$ be the last timeslot of the duration $D_1$ and define $r^* := t_{\text{last}} - \ell_{e_1}$. The basic idea is that the two following conditions hold with high probability: (i) $[r^*, t_{\text{last}}]$ is a growth interval, and (b) There does not exist $t_1 \in D_0$, users $U_1, U_2$ and incompatible blocks $B_1, B_2$, such that $B_1$ is confirmed for $U_1$ at $t_1$ and $B_2$ is confirmed for $U_2$ at $t_{\text{last}}$. When both these conditions hold, and since $r^* > n$, this suffices to show that no incompatible and confirmed blocks exist during the duration $D_0$. Now let us see that in more detail.

By the choice of $D_1$, $r^* > n$. It follows from the definition of liveness that $(\hat{\tau}_1)$ below fails to hold with probability $\leq e_1$:

$(\hat{\tau}_1)$ $[r^*, t_{\text{last}}]$ is a growth interval.

Note that, so long as $(\hat{\tau}_1)$ holds, every user has more confirmed blocks at $t_{\text{last}}$ than any user does at any timeslot in $D_0$. It also follows from the Union Bound, and the definition of liveness and timeslot security, that $(\hat{\tau}_2)$ below fails to hold with probability $\leq n e_1 = e_0/2$:
There does not exist $t_1 \in D_0$, users $U_1, U_2$ and incompatible blocks $B_1, B_2$, such that $B_1$ is confirmed for $U_1$ at $t_1$ and $B_2$ is confirmed for $U_2$ at $t_{\text{last}}$.

Now note that:

(a) If $(\dagger_1)$ and $(\dagger_2)$ both hold, then there do not exist incompatible blocks $B_1, B_2$, timeslots $t_1, t_2 \in D_0$ and $U_1, U_2$ such that $B_i$ is confirmed for $U_i$ at $t_i$ for $i \in \{1, 2\}$.

(b) With probability $> 1 - \epsilon_1 - \epsilon_0/2 \geq 1 - \epsilon_0$, $(\dagger_1)$ and $(\dagger_2)$ both hold.

So uniform security is satisfied with respect to $\epsilon_0$ and $D_0$, as required. □

**Definition 5.5.** We say $P$ has **standard functionality** if it is uniformly live and uniformly secure. We say that a recalibration of $P$ is **faithful** if it has standard functionality when $P$ does.

Proposition 5.3 justifies concentrating on protocols which have standard functionality where it is convenient to do so, since protocols which are live and secure will have recalibrations with standard functionality, so long as the rather weak conditions of Proposition 5.3 are satisfied. Again, when we talk about the security and liveness of a protocol, it is really the extended protocol that we are referring to.

### 6 Certificates in the Synchronous Setting

#### 6.1 The Synchronous and Unsized Setting

As outlined in the introduction, one aim of this paper is to give a positive answer to Q3, by showing that whether a protocol produces certificates comes down essentially to properties of the user selection process: In the unsized setting protocols cannot produce certificates, while in the sized setting, recalibrated protocols will automatically produce certificates, at least if they are of ‘standard form’. For the partially synchronous setting, the results of Sections 3 and 4 already bear this out – the sized setting is required for security and all secure protocols must produce certificates. The following theorem now deals with the unsized and synchronous setting. Recall that, in the unsized setting, the total resource balance belongs to a fixed interval $[R_0, R_1]$. We say that the protocol operates ‘in the presence of a non-trivial adversary’ if the setting allows that the adversary may have resource balance at least $R_0$ throughout the duration.

**Theorem 6.1.** Consider the synchronous and unsized setting. If a protocol is live then, in the presence of a non-trivial adversary, it does not produce certificates.

**Proof.** The basic idea is that the adversary with resource balance at least $R_0$ can ‘simulate’ their own execution of the protocol, in which only they have non-zero resource balance, while honest users carry out an execution in which the adversary does not participate. Simulating their own execution means that the adversary carries out the protocol as usual, while ignoring messages broadcast by honest users, but does not initially broadcast messages when given permission to do so. Liveness (together with the fact that the resource pool is undetermined) guarantees that, with high probability, both the actual and simulated executions produce blocks which look confirmed from their own perspective. These blocks will be incompatible with each other, and (once the adversary finally broadcasts the messages that they
have been given permission for), these blocks will all have subjective certificates which are subsets of
the set of broadcast messages. This suffices to show that the protocol does not produce certificates.

More precisely, we consider two instances of the protocol $\mathcal{I}_0$ and $\mathcal{I}_1$ in the synchronous and unsized
setting, which have the same parameters, including the same small security parameter $\epsilon$, the same long
duration $D$, the same set of users and the same message delivery rule, but which differ as follows:

- In $\mathcal{I}_0$ a set of users $\mathcal{U}_0$ control public keys in a set $\mathcal{U}_0$, which are the only public keys that do
  not have zero resource balance throughout the duration. The total resource balance $T$ has a fixed
  value, $R_0$ say.

- In $\mathcal{I}_1$ it is the adversary who controls the public keys in $\mathcal{U}_0$, and those keys have the same
  resource balance throughout the duration as they do in $\mathcal{I}_0$. Now, however, another set of honest
  users $\mathcal{U}_1$ control public keys in a set $\mathcal{U}_1$ (disjoint from $\mathcal{U}_0$), and the public keys in $\mathcal{U}_1$ have total
  resource balance $R \geq R_0$ throughout the duration, i.e. the resource balances of these keys always
  add to $R$.

In $\mathcal{I}_1$, we suppose that the adversary simulates the users in $\mathcal{U}_0$ for $\mathcal{I}_0$, which means that the adversary
carries out the instructions for those users, with the two following exceptions. Until a certain timeslot $t^*$, to be
detailed subsequently, they:

(a) Ignore all messages broadcast by honest users, and;

(b) Do not actually broadcast messages when permitted, but consider them delivered to the simulated
users in $\mathcal{U}_0$ as per the message delivery rule.

For $\mathcal{I}_0$ (so long as the duration is sufficiently long), liveness guarantees the existence of a timeslot $t_0$
for which the following holds with probability $> 1 - \epsilon$:

$(\nabla_0)$ At $t_0$ there exists a set of broadcast messages $M_0$ and a block $B_0$ such that $B_0 \in C(M_0)$.

For $\mathcal{I}_1$, liveness guarantees the existence of a timeslot $t_1$ for which the following holds with probability
$> 1 - \epsilon$:

$(\nabla_1)$ At $t_1$ there exists a set of broadcast messages $M_1$ and a block $B_1$ such that $B_1 \in C(M_1)$.

Choose $t^* > t_0, t_1$. Our framework stipulates that the instructions of the protocol for a given user at a
given timeslot must be a deterministic function of the input parameters, the timeslot, the keys controlled
by the user, their message state and the set of permissions they have been given by the permitter (see
Section 2.7). It also stipulates that the response of the permitter to a request $(U, M, t', A)$ is a probabilistic
function of the protocol input parameters, the actual timeslot $t$, previous requests made by $U$, the request
$(U, M, t', A)$, and the resource level $R(U, t', M)$. Since we are working in the unsized setting, $\mathcal{I}_1$ and $\mathcal{I}_0$
have the same input parameters. It therefore follows by induction on timeslots $t \leq t^*$, that the following
is true at all points until the end of timeslot $t$:

$(\nabla_2)$ The probability distribution for $\mathcal{I}_0$ on the set of permissions given by the permitter is identical
to the probability distribution for $\mathcal{I}_1$ on the set of permissions given by the permitter to the
adversary.

Now suppose that at timeslot $t^*$ the adversary broadcasts all messages for which they have been given
permission by the permitter. Note that, according to the assumptions of Section 2.4, any block $B_0$
broadcast by the adversary at $t^*$ will be incompatible with any block $B_1$ that has been broadcast by any
honest user up to that point. Combining \((\nabla_0), (\nabla_1)\) and \((\nabla_2)\), we see that (so long as \(\varepsilon\) is sufficiently small that \(\varepsilon < 1 - 2\varepsilon\)) the following holds with probability \(> \varepsilon\) for \(t'\) and \(\mathbb{I}_1\): There exist incompatible blocks \(B_0, B_1, M_0, M_1\) which are broadcast by the end of \(t'\), such that \(B_i \in C(M_i)\) for \(i \in \{0, 1\}\). This suffices to show that the protocol does not produce certificates.

\[\square\]

### 6.2 The Synchronous and Sized Setting

**The Example of Sized Bitcoin.** Our aim in this subsection is to show that, if we work in the synchronous and sized setting, and if a protocol is of ‘standard form’, then a recalibration will produce certificates. To make this precise, however, it will be necessary to recognise the potentially time dependent nature of proofs of confirmation. To explain this idea, it is instructive to consider the example of Bitcoin in the sized setting: The protocol is Bitcoin, but now we are told in advance precisely how the hash rate capability of the network varies over time, as well as bounds on the hash rate of the adversary.\(^{16}\) To make things concrete, let us suppose that the total hash rate is fixed over time, and that the adversary has 10% of the hash rate at all times. Suppose that, during the first couple of hours of running the protocol, the difficulty setting is such that the network as a whole (with the adversary acting honestly) will produce an expected one block every 10 minutes. Suppose further that, after a couple of hours, we see a block \(B\) which belongs to a chain \(C\), in which it is followed by 10 blocks. In this case, the constraints we have been given mean that it is very unlikely that \(B\) does not belong to the longest chain. So, at that timeslot, \(C\) might be considered a proof of confirmation for \(B\), i.e. the existence of the chain \(C\) can be taken as proof that \(B\) is confirmed. The nature of this proof is time dependent, however. The same set of blocks a large number of timeslots later would not constitute proof of confirmation.

If we now consider a PoS version of the example above, modified to work for Snow White rather than Bitcoin, then the proof produced will not be time dependent. This is because PoS protocols are timed, i.e. when permission is given to broadcast \(m\) in response to a request \((U, t, M, A)\), other users are able to determine \(t\) from \(m\). This is another fundamental distinction between PoW and PoS protocols.

**Definition 6.1.** We shall refer to the setting as being **timed** if both of the following hold:

- For each broadcast message \(m\), there exists a unique timeslot \(t_m\) such that permission to broadcast \(m\) was given in response to some request \((U, t_m, M, A)\).
- \(t_m\) is computable from \(m\).

In order to prove that (recalibrated) protocols in the sized setting produce certificates, we shall have to make the assumption that we are also working in the timed setting.

**Protocols in Standard Form.** The basic intuition behind the production of certificates in the sized setting can be seen from the example of “Sized Bitcoin” above. Once a block is confirmed, honest users will work ‘above’ this block. So long as those honest users possess a majority of the total resource balance, and so long as the permitter reflects this fact in the permissions it gives, then those honest users will broadcast a set of messages which suffices (by its existence rather than the fact that it is

\[\text{\footnotesize\(^{16}\)}\text{Normally we think of PoW protocols as operating in the unsized setting, precisely because such guarantees on the hash rate are not realistic.}\]
the full message state of any user) to give proof of confirmation. This proof of confirmation might be temporary, but it will not be temporary in the timed setting.

This intuitive argument, however, assumes that the protocol satisfies certain standard properties. As alluded to above, there is an assumption that the set of messages broadcast by a group of users will reflect their resource balances and that the adversary will have a minority resource balance. There is also an assumption that broadcast messages will (in some sense) point to a particular position on the blockchain. So we will have to formalise these ideas, and the results we prove will only hold modulo the assumption that these standard properties are satisfied.

First, let us formalise the idea that messages always point to a position on the blockchain.

**Definition 6.2.** We shall say that a protocol is in standard form if it satisfies all of the following:

1. **The protocol has standard functionality** (see Definition 5.5).
2. **Every broadcast message is ‘attached’ to a specific block** (blocks being attached to themselves).
3. **While B is confirmed for U, the instruction set Σ will only instruct U to broadcast messages which are attached to B or descendants of B.**

**Reflecting the Resource Pool.** Now let us try to describe how the permitter might reflect the resource pool. We’ll need a simple way to say that one set of users consistently has a higher resource balance than another.

**Definition 6.3.** For Θ > 1, we say a set of public keys \( U_1 \) dominates another set \( U_2 \), denoted \( U_1 >_\Theta U_2 \), if the following holds for all sets of broadcast messages \( M \) and all timeslots \( t \):

\[
\sum_{m \in U_1} R(U, t, M) > \Theta \cdot \sum_{m \in U_2} R(U, t, M).
\]

Next, we will need to formalise the idea that, if one set of keys dominates another, then they will be able to broadcast discernibly different sets of messages. Recall that, in the timed setting, each message \( m \) corresponds to a timeslot \( t_m \), which can be determined from \( m \). We shall write \( M[t_1, t_2] \) to denote the set \( \{ m \mid m \in M, \ t_m \in [t_1, t_2] \} \). We will say that the set of keys \( U_0 \) is directed to broadcast \( M \) if, for every \( m \in M \), there is some member of \( U_0 \) that is given permission to broadcast \( m \) and is directed to broadcast \( m \) by the protocol. We will say that \( U_0 \) is able to broadcast \( M \) if, for every \( m \in M \), there is some member of \( U_0 \) that is given permission to broadcast \( m \). We define \( M^* := \{ M \mid M \text{ is finite} \} \). We let \( T \) be the set of functions \( T : D \times M \to R_{\geq 0} \) (so that the total resource balance \( T \in T \)). We say that a set of keys \( U_0 \) has total resource balance \( T : D \times M \to R_{\geq 0} \) if \( T(t, M) = \sum_{m \in U_0} R(U, t, M) \). In the definition below, we shall say \( r \) is sublinear in \( |D| \) if, for each \( \Theta, \varepsilon, T, \) and for every \( \alpha \in (0, 1) \), it holds that \( r(\Theta, \varepsilon, T, |D|) < \alpha |D| \) for all sufficiently large \(|D|\).

**Definition 6.4.** We say that \((I, 0, C)\) reflects the resource pool if there exist computable finite valued functions \( r : R_{>1} \times R_{>0} \times T \times N \to N \) and \( X : R_{>1} \times R_{>0} \times T \times N \to 2^M \), such that:

1. \( r \) is sublinear in \(|D|\).
2. If \( U_1 \cup U_2 \) has total resource balance \( T \), and if \( U_1 >_\Theta U_2 \), then, when \( P \) is run with security parameter \( \varepsilon \) and for \(|D| \) many timeslots, the following holds with probability \( > 1 - \varepsilon \):

   For all intervals of timeslots \([t_1, t_2]\) with \( t_2 - t_1 \geq r(\Theta, \varepsilon, T, |D|) \), there exists some element of
with respect to $X(\Theta, \epsilon, T, |\mathcal{D}|) \cap M[t_1, t_2]$ which $\mathcal{U}_1$ is directed to broadcast, while $\mathcal{U}_2$ is not able to broadcast any element of $X(\Theta, \epsilon, T, |\mathcal{D}|) \cap M[t_1, t_2]$.

So in Definition 6.4, $r$ specifies a number of timeslots. Then $X$ specifies certain sets of messages $M$ such that, if $\mathcal{U}_1 \supseteq \mathcal{U}_2$ and $\mathcal{U}_1 \cup \mathcal{U}_2$ has total resource balance $T$, then $\mathcal{U}_1$ can be expected to broadcast one of these sets $M$ in any interval of sufficient length (i.e. the length specified by $r$). To make this interesting, we also have that $\mathcal{U}_2$ can be expected not to make such broadcasts. To see why this is a natural and reasonable condition to assume, it is instructive to consider the example of Sized Bitcoin. Suppose that in some execution the honest users always have at least 60% of the mining power. Then, over any long period of time $r$, we can be fairly sure that honest users will get to make at least 50% of the expected number of block broadcasts, while the adversary is unlikely to be able to make such broadcasts if $r$ is large enough. In fact, the exponentially fast convergence for the law of large numbers guaranteed by bounds like Hoeffding’s inequality, means $r$ only needs to grow with $\ln 1/p$, where $p$ is the probability of error (i.e. the probability these conditions on the block broadcasts don’t hold in a given interval). It’s therefore easy to see that Sized Bitcoin would reflect the resource pool if it could be implemented in a timed setting. Similar arguments can be made for all well known PoS protocols, and these are implemented in the timed setting.

**Definition 6.5.** In the bounded adversary setting it is assumed that:

(i) $\mathcal{U}_1 \supset \mathcal{U}_2$ for some predetermined input parameter $\Theta > 1$, where $\mathcal{U}_1$ is the set of keys controlled by honest users, and $\mathcal{U}_2$ is the the set of keys controlled by the adversary.

(ii) $(I, 0, C)$ reflects the resource pool.

Finally, we can now formalise the idea that under standard conditions, standard protocols in the sized setting produce certificates.

**Theorem 6.2.** Consider the timed, bounded adversary and sized setting. If $\mathcal{P}$ is in standard form, then there exists a faithful recalibration that produces certificates.

**Proof.** To define our recalibration $(\mathcal{P}', C')$, suppose we are given parameters $\epsilon, T, \Theta$ and $\mathcal{D}$. We need to specify a value $\epsilon'$ to give as input to $\mathcal{P}$ (we will leave other parameters unchanged), and we must also define $C'$. Then we need to show that the new extended protocol is uniformly live and produces certificates.

We define $\epsilon' := \epsilon/4$. Towards defining $C'$, suppose that $\mathcal{P}$ satisfies uniform liveness with respect to $t_{\epsilon', \mathcal{D}}$. We divide the duration into intervals of length $t_{\epsilon', \mathcal{D}}$, by defining $t_i := i \cdot (t_{\epsilon', \mathcal{D}} + r(\Theta, \epsilon', T, |\mathcal{D}|))$. From the definition of uniform liveness we have the following.

($\S_1$) With probability $> 1 - \epsilon/4$ it holds that, for all $i$ with $t_i \leq |\mathcal{D}|$, all users have at least $i$ many confirmed blocks by the end of timeslot $t_i$.

Now suppose $(\mathcal{P}, C)$ satisfies Definition 6.4 with respect to $r$ and $X$. For each $i > 0$, define $t_i' := t_i + r(\Theta, \epsilon', T, |\mathcal{D}|)$. Let $I_i$ be the interval $[t_i, t_i']$, and write $M[I_i]$ to denote $M[t_i, t_i']$. Let $\mathcal{U}_1$ be the set of keys controlled by honest users, and let $\mathcal{U}_2$ be the the set of keys controlled by the adversary. According to Definition 6.4, we can then conclude that:

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17Note that this is not the same as 50% of blocks in the longest chain.
(§2) It holds with probability $> 1 – \varepsilon/4$ that, whenever $I_i$ is contained in the duration, there exists some element of $X(\Theta, \varepsilon', \mathcal{T}, |\mathcal{D}|) \cap M[I_i]$ which $\mathcal{U}_1$ is directed to broadcast, while $\mathcal{U}_2$ is not able to broadcast any element of this set.

Since $P$ is uniformly secure, we also know that:

(§3) With probability $> 1 – \varepsilon/4$, there do not exist incompatible blocks $B_1, B_2$, timeslots $t_1, t_2$ and $U_1, U_2$ such that $B_i$ is confirmed for $U_i$ at $t_i$ for $i \in \{1, 2\}$.

So now define $X' (\Theta, \varepsilon', \mathcal{T}, |\mathcal{D}|)$ to be all those $M$ in $X(\Theta, \varepsilon', \mathcal{T}, |\mathcal{D}|)$ for which there exists $i$ such that all of the following hold:

(i) $I_i \subseteq \mathcal{D}$.

(ii) $M \in M[I_i]$, and;

(iii) For some chain $C$ of length $i$ with leaf $B$, all messages in $M$ are attached $B$ or its descendants.

Now if $M \in X' (\Theta, \varepsilon', \mathcal{T}, |\mathcal{D}|)$, then let $i_M$ be the (unique) $i$ such that (i)–(iii) hold for $i$ and $M$, let $C$ be as specified in (iii) for $i_M$, and define $C'(M) := C$. We also define $C'(\emptyset) = \emptyset$. This function $C'$ is almost the notion of confirmation that we want for our recalibration, but the problem is that it is only defined for very specific values of $M$. We will use $C'$ to help us define $C'$ that is defined for all possible $M$.

Combining (§1), (§2) and (§3), and the definition of $X^*$, it follows that with probability $> 1 – \varepsilon$ both of the following hold:

1. If $M, M' \in X'(\Theta, \varepsilon', \mathcal{T}, |\mathcal{D}|)$ are both broadcast, then all blocks in $C'(M)$ are compatible with all those in $C'(M')$.

2. For every $i > 0$, there exists $M \in X'(\Theta, \varepsilon', \mathcal{T}, |\mathcal{D}|)$ which is broadcast and such that $i_M = i$.

In order to define $C'$ for our recalibration, we can then proceed as follows. Given arbitrary $M$, choose $M' \subseteq M$ such that $M' \in X'(\Theta, \varepsilon', \mathcal{T}, |\mathcal{D}|)$ and $i_{M'}$ is maximal, or if there exists no $M'$ satisfying these conditions then define $M' := \emptyset$. We define $C'(M) := C'(M')$. It follows from (1) and (2) above that $(P', C')$ produces certificates and satisfies uniform liveness with respect to $\ell'_{\varepsilon, \mathcal{D}} := \ell_{\varepsilon, \mathcal{D}} + 2r(\Theta, \varepsilon', \mathcal{T}, |\mathcal{D}|)$. □

7 Reducing to the Permissioned Case

An advantage of our framework is the opportunity it provides for modular analysis, by breaking the description and evaluation of a permissionless blockchain protocol down into two tasks:

(a) Providing such an analysis for a permissioned reduct, i.e. a probability free and (in a sense to be clarified) permissioned form of the protocol that we will describe in this section;

(b) Specifying and analysing the properties of an appropriate choice of permitter.

In this section we will define permissioned reducts and show how they can be used to formally describe this modular approach. The reader may note that for some BFT-inspired PoS protocols such as Algorand [CM19], this modular approach is already the natural approach taken – to define Algorand one considers a certain BFT protocol in the permissioned setting, and then considers an appropriate process of user
selection, giving selected users the task of carrying out the BFT protocol. The keen reader may also note that, even where it is not immediately so obvious what such a modular approach might mean (as is the case, for example, with Bitcoin), existing proofs of liveness and security do often end up taking something like a modular approach of this form (even if it is not quite so explicit as for our account here). It is a benefit of formalising this breakdown through our framework, however, that this reduction to the permissioned case can be seen to hold much more generally. We hope that describing this general reduction also helps clarify the relationship between permissionless and permissioned protocols.

After we have defined permissioned reducts, we will consider the example of Bitcoin. Taking the proof of liveness and security given in [GKL18], we outline how it can be presented in an entirely modular fashion, in which one first deals with a permissioned reduct for Bitcoin, and then proves that liveness and security carry over to the permissionless setting.

7.1 Permissioned Reducts

Roughly, we consider a framework which is the same as that in Section 2, except that the permitter and resource pool are replaced by a deterministic set of rules specifying precisely which keys are allowed to broadcast and when. In order to make these ideas precise, however, we need a few new definitions.

In Section 2 we specified that the response of the permitter to a request \((U, M, t', A)\) is a probabilistic function of the request and: (i) The protocol’s input parameters; (ii) The actual timeslot \(t\); (iii) The previous requests made by \(U\); (iv) The resource balance \(R(U, t', M)\). This response could be permission to broadcast any and all messages in a certain set, \(M'\) say. Now consider a pair \((U, D)\), where \(U\) is a set of keys. A permission table \(T\) for the pair is a (deterministic) function that takes any request \((U, M, t', A)\) together with values for (i)-(iv) above as input (assuming \(U \in U\) and \(t, t' \in D\)), and outputs a value \(M'\) (which could be empty). We will say that the permission table is consistent with \(O\), if whenever it outputs a value \(M'\), there is always non-zero chance that the permitter \(O\) would give permission to broadcast messages in \(M'\) in response to those same inputs.

So a permission table is a simple object, which can be thought of as a probability free version of the permitter. The permission table operates in a permissioned setting, in the sense that the set of keys is given as an input before the permission table decides who will be given permission to broadcast and when.

One final note before we define permissioned reducts: We will consider only protocols that are live and secure, and can therefore assume that \(\ell_{e,D}\) has been chosen so as to satisfy Definition 2.1. A permissioned reduct will be defined relative to this choice of \(\ell_{e,D}\).

7.1.1 Defining Permissioned Reducts for the Protocol \(P = (I, O)\).

For a given instance of the protocol, let \(U_0\) be the set of all keys controlled by honest users, let \(U_1\) be the set of all keys controlled by the adversary. We consider a framework which is the same as that described in Section 2, except that, for each execution of the protocol, permission to broadcast is now determined by a permission table \(T\) for \((U_0 \cup U_1, D)\) – we shall say that the execution happens relative to \(T\). So now the permission table is one of the variables determining the instance, and when a user makes a request \((U, M, t', A)\) at timeslot \(t\), it is the permission table which dictates the resulting permission to broadcast.
When the execution happens relative to a permission table, we will refer to it as a *permissioned execution* – we will expand on the relationship between permissioned executions and the traditional permissioned setting later.

To define a permissioned reduct $PR = (I, T)$ for $P = (I, 0)$, we replace the permitter $O$ with a function $T$, which maps each tuple $(U_0, U_1, \varepsilon, D)$, to a set $T$ of permission tables satisfying the conditions that:

(i) All $T \in T$ are consistent with $O$, and;

(ii) The following probability free versions of liveness and security are satisfied for all possible executions of the protocol consistent with the setting, so long as $U_0$ is the set of keys controlled by honest users, $U_1$ is the set of keys controlled by the adversary, and so long as the execution is relative to some permission table for $(U_0 \cup U_1, D)$ in $T$.

**Absolute Liveness.** For each pair of timeslots $t_1 < t_2 \in D$, if $t_2 - t_1 \geq \ell_{e,D}$ and $[t_1, t_2]$ is entirely synchronous, then $[t_1, t_2]$ is a growth interval.

**Absolute Security.** For every $U_1, U_2$ and for all timeslots $t_1, t_2$ in the duration, all blocks which are confirmed for $U_1$ at $t_1$ are compatible with all those which are confirmed for $U_2$ at $t_2$.

For each choice of $\ell_{e,D}$ there may therefore be many permissioned reducts. It is clear, however, that for each $\ell_{e,D}$ there exists a single permissioned reduct which is *maximal*, in the sense that $T$ maps each tuple $(U_0, U_1, \varepsilon, D)$ to the set of *all* $T$ for which absolute liveness and absolute security are satisfied, and which are consistent with $O$.

The reader might find it useful to think of the permissioned reduct as determining a set of ‘good executions’, while simultaneously establishing which permissions to broadcast will result in such executions.

While the definition above is quite simple, it may also initially appear a little abstract. An example will make things clear.

### 7.1.2 Defining a Permissioned Reduct for Bitcoin

**The model.** Before defining the permissioned reduct, we need to decide how we will model Bitcoin as a protocol $P = (I, 0)$. In Section 2.8 we described a simple approach to modelling Bitcoin. Here we take the same simple approach, but it will be necessary to flesh out the details of the instruction set somewhat. We will consider a notion of confirmation that makes Bitcoin uniformly live and uniformly secure in the synchronous setting.

For the sake of concreteness, we fix timeslots of one second each. At each timeslot $t$, the *selected chain* for each user $U$ is the longest chain in their message state – where there is a tie between longest chains, the selected chain is that which was delivered first, with ties between deliveries at the same timeslot ordered by least hash.\(^\text{18}\) At timeslot $t$, each public key $U$ controlled by $U$ is instructed to put in a single request $(U, M, t, A)$, where $M$ is $U$’s message state, and where $A$ specifies a block (without

\(^{18}\)Note that, according to our framework, users do not choose which messages are delivered to them. Of course, the adversary may choose to ignore certain messages, but we shall suppose that the selected chain for any user is defined (as above) as a function of their sequence of message deliveries. So the adversary does not choose their selected chain, but may choose to produce blocks that do not extend it.
PoW) extending the selected chain. In response to this request, the permitter may give permission to broadcast the block. If given permission to broadcast, the instructions dictate that the block be delivered to all users by the blockon the selected chain. In response to this request, the permitter may give permission to broadcast the block. If given permission to broadcast, the instructions dictate that the block be immediately understood. We work in the synchronous and unsized setting, so that \( R \) is undetermined, with the only restrictions being that:

- \( R \) is assumed to be a function from \( U \times D \) to \( \mathbb{R}_{\geq 0} \) satisfying the requirement that, at all timeslots in the duration, the total resource balance belongs to a fixed interval \([R_0, R_1]\), where \( R_1 \geq R_0 > 0 \).
- For some fixed \( \mu > 0 \), it holds at all timeslots that the adversary controls a proportion at most \( 0.5 - \mu \) of the total resource balance.\(^{19}\) As in [GKL18], bounds have to be placed on \( \mu \) that only allow it to be close to 0 when the ratio between \( \Delta \) and the rate at which blocks are produced is sufficiently small. We shall not worry about achieving optimal values for \( \mu \) – see [Ren19] for an excellent account that considers values for \( \mu \) in more detail.

We suppose we are given two functions \( N_{e,D} \) and \( \ell_{e,D} \) such that:

- \( N_{e,D} \) defines the notion of confirmation \( C \). More precisely, a user considers block \( B \) confirmed once it belongs to their selected chain, and is followed by at least \( N_{e,D} \) many blocks in that chain.
- When the protocol \( P = (I, 0) \) is run with security parameter \( \varepsilon \), it holds with probability at least \( 1 - \varepsilon/2 \) that: \((\dagger_0)\) For every interval of timeslots of length at least \( \ell_{e,D}^{\varepsilon} \), at least one honest user is given permission to broadcast a block in that interval.

That appropriate values can be chosen for \( N_{e,D} \) and \( \ell_{e,D}^{\varepsilon} \) has to be shown later as part of the probabilistic analysis of the behaviour of the permitter, once we have dealt with the permissioned reduct. It is worth immediately understanding the relevance of \( \ell_{e,D}^{\varepsilon} \), however. This value is chosen so as to ensure that it holds with probability at least \( 1 - \varepsilon/2 \) that, once all users have a non-empty set of confirmed blocks, every interval \([t_1, t_2]\) of length at least

\[
\Delta^* := \ell_{e,D}^{\varepsilon} + 2\Delta
\]

will be a growth interval. This follows since the longest chain \( C \) in any user’s message state at \( t_1 \) will be delivered to all users by \( t_1 + \Delta \) (to be in a message state, a message has to be broadcast). Then, if \((\dagger_0)\) holds, some honest user will broadcast a chain strictly longer than \( C \) by \( t_1 + \Delta + \ell_{e,D}^{\varepsilon} \), which will be delivered to all users by \( t_1 + \ell_{e,D}^{\varepsilon} + 2\Delta \). Given this analysis, it’s tempting to define \( \ell_{e,D} \) to be \( \Delta^* \). The difficulty, however, is that we need to allow some initial time for users to build up at least \( N_{e,D} \) many blocks in their longest chain, so that they begin to have confirmed blocks. For this reason we define:

\[
\ell_{e,D} := (N_{e,D} + 1) \cdot \Delta^*.
\]

**The permissioned reduct.** We have already specified the instruction set, so it suffices to determine an appropriate set \( T \) of permission tables for each tuple \((U_0, U_1, e, D)\). For a given execution (whether permissioned or relative to the permitter), we let \( \mathcal{E} \) be the set of all pairs \((U, t)\) such that \( U \) is given permission to broadcast a block \( B \) at \( t \) – we will refer to \( B \) as the block corresponding to \((U, t)\). To

\(^{19}\)For the sake of notational convenience, we have assumed that \( \mu \) is fixed as part of the setting here, and so does not need to be specified as another input parameter to the protocol (meaning, for example, that we consider \( \ell_{e,D} \) rather than \( \ell_{e,D,\mu} \)). It is not difficult, however, to see how our presentation could be modified to deal with varying \( \mu \). The same is true for other values such as \( R_0, R_1 \) and \( \Delta \), which we consider for presentational convenience to be fixed, but which could alternatively be specified as input parameters.
establish absolute security, we will use a version of the approach in [GKL18], simplified since we only need to define and analyse the permissioned reduct at this stage. In particular, this means comparing the number of blocks that the adversary is permitted to broadcast with the number of blocks that honest users are permitted to broadcast in isolated intervals: We define \((U, t) \in E\) to be isolated if \(U\) is honest and there do not exist any \((U', t') \in E\) such that \(U'\) is honest and \(t' \in [t - \Delta, t + \Delta]\). If \((U, t)\) is isolated, then we may also refer to the corresponding block as isolated. If \((U, t) \in E\) then we may refer to the pair and the corresponding block as honest/dishonest when \(U\) is honest/dishonest. The point about isolated blocks is the following (where the height of a block is its number of ancestors):

\[ (\dagger_1) \]
If \(B\) is isolated, then any broadcast block \(B' \neq B\) of the same height must be dishonest.

To prove \((\dagger_1)\), suppose that \((U, t)\) is isolated, and that the corresponding block \(B\) is of height \(k\). Towards a contradiction, suppose also that \(B' \neq B\) is of height \(k\) and is honest. Then \(B'\) cannot be broadcast strictly prior to \(t - \Delta\), otherwise \(B'\) (and all ancestors) would be delivered to the user who controls \(U\) before \(t\), and \(B\) would not be broadcast by \(U\) at \(t\). Also, \(B'\) cannot be broadcast during \([t - \Delta, t + \Delta]\) because \(B\) is isolated. Then we also know that \(B'\) cannot be broadcast after \(t + \Delta\) because it is honest, and \(B\) and all ancestors will have been delivered to the honest producer of \(B\) by \(t + \Delta\). This gives the required contradiction.

For any interval \([t_1, t_2]\), we define:

- \(X_{t_1, t_2}\) to be the blocks corresponding to pairs \((U, t) \in E\) such that \(t \in [t_1, t_2]\);
- \(Y_{t_1, t_2}\) to be the blocks corresponding to isolated pairs \((U, t) \in E\) such that \(t \in [t_1 + \Delta, t_2 - \Delta]\);
- \(Z_{t_1, t_2}\) to be the blocks corresponding to dishonest pairs \((U, t) \in E\) such that \(t \in [t_1, t_2]\).

To define the permissioned reduct, we let \(T\) be the set of all permission tables \(T\) such that the following holds for every execution relative to \(T\) consistent with the setting:

\[ (\circ_0) \]
\(T\) is consistent with \(0\) and outputs \(M' = 0\) whenever given input \((U, M, t', A)\) at timeslot \(t\), for which \(t' \neq t\), or when a previous request has been submitted with respect to \(U\) at \(t\).

\[ (\circ_1) \]
For each interval \([t_1, t_2]\) of length \(\geq l'_{\epsilon, \Delta^*}\), there exists at least one honest \((U, t) \in E\) with \(t \in [t_1, t_2]\).

\[ (\circ_2) \]
For any interval \([t_1, t_2]\), if \(|X_{t_1, t_2}| \geq N_{\epsilon, \Delta}\) then \(|Y_{t_1, t_2}| > |Z_{t_1, t_2}|\).

Roughly, \((\circ_0)\) just says that the permission table is like \(0\), in the sense that it only listens to one request per key at each timeslot, and then only gives permission to broadcast at most a single block at a time (we will specify \(0\) precisely in Section 7.2). We will show that \((\circ_1)\) suffices to ensure absolute liveness, and we will show that \((\circ_2)\) suffices to ensure absolute security.

**Establishing absolute liveness.** In order to prove that \(T\) is a permissioned reduct, we must show that it ensures absolute liveness and absolute security. To deal with absolute liveness, it is useful to consider a weakened version of growth intervals. Let \(\|M\|\) denote the length of the longest chain in \(M\). For timeslots \(t_1 < t_2\), let \(l_1\) be the maximum value \(\|M_t\|\) for any \(M_t\) which is a message state of any user at any timeslot \(t \leq t_1\), and let \(l_2\) be the minimum value \(\|M_t\|\) for any \(M_t\) which is a message state of any user at timeslot \(t_2\). We say that \([t_1, t_2]\) is a weak growth interval if \(l_2 > l_1\). In order to show that absolute liveness is satisfied, we will prove that: \((\dagger_2)\) So long as \(T \in T\), any interval of length \(\Delta^*\) is a weak growth interval. This suffices to show absolute liveness, since then, letting \(t'\) be the first timeslot at which all users have a non-empty set of confirmed blocks:
• Every interval \([t_1, t_2]\) with \(t_1 \geq t^*\) and of length at least \(\Delta^*\) is a growth interval.

• It follows from the definition \(\ell_{e,D} := (N_{e,D} + 1) \cdot \Delta^*\) and from (\(\dagger 2\)) (applied to the first \(N_{e,D} \cdot \Delta^*\) timeslots) that \((t^* \text{ exists and})\) every interval of length \(\ell_{e,D}\) contains an interval \([t_1, t_2]\) with \(t_1 \geq t^*\) and of length at least \(\Delta^*\).

That (\(\dagger 2\)) holds follows from precisely the same argument that we described previously in this subsection, when explaining the relevance of \(\ell_{e,D}^*\). Consider an interval \([t_1, t_2]\) of length at least \(\Delta^*\). The longest chain \(C\) in any user’s message state at \(t_1\) will be delivered to all users by \(t_1 + \Delta\). Then it follows from (\(\diamondsuit 1\)) that some honest user will broadcast a chain properly extending \(C\) by \(t_1 + \Delta + \ell_{e,D}^*\) which will be delivered to all users by \(t_1 + \ell_{e,D}^* + 2\Delta = t_1 + \Delta^*\).

**Establishing absolute security.** For any chain \(C\), let \(C^*\) be \(C\) with the \(N_{e,D}\) many blocks of greatest height removed. Towards a contradiction, suppose now that absolute security does not hold, so that there exist \(U_1, U_2, C_1, C_2\) and \(t_1, t_2\), such that \(C_1\) is the selected chain for \(U_1\) at \(t_1\), \(C_2\) is the selected chain for \(U_2\) at \(t_2\), but \(C_1^*\) and \(C_2^*\) are incompatible, i.e. neither is an initial segment of the other. Without loss of generality, we may suppose that these values are chosen so that \(|t_1 - t_2|\) is minimal, and so that \((t_1, t_2)\) is the lexicographically least possible choice subject to \(|t_1 - t_2|\) being minimal. Let \(B_0\) be the honest block of greatest height in \(C_1 \cap C_2\). Let \(t_0\) be the timeslot at which \(B_0\) is given permission for broadcast (or if \(B_0\) is the genesis block, let \(t_0 = 0\)). Since the length of the selected chain for each user is monotonically increasing over time, our choice of \(t_1\) and \(t_2\) leaves two possibilities. Either \(t_1 = t_2\), or else \(t_2 = t_1 + 1\).

Consider first the possibility that \(t_1 = t_2\), and without loss of generality suppose \(|C_2| \geq |C_1|\). It follows from (\(\diamondsuit 2\)) that \(|Y_{t_0,t_1}| > |Z_{t_0,t_1}|\). To obtain a contradiction, it therefore suffices to show that each element of \(Y_{t_0,t_1}\) can be paired with a distinct element of \(Z_{t_0,t_1}\). To see this, note that none of the blocks in \(Y_{t_0,t_1}\) can be of height greater than the leaf of \(C_1\), because this would contradict the fact that \(C_1\) is the selected chain for \(U_1\) at \(t_1\). The claim therefore follows from the existence of \(C_1\) and \(C_2\) at \(t_1\), and from (\(\dagger 1\)).

Consider next the possibility that \(t_2 = t_1 + 1\). To obtain a contradiction, it suffices to show that each element of \(Y_{t_0,t_2}\) can be paired with a distinct element of \(Z_{t_0,t_2}\). Now we divide into two subcases. Suppose first that \(|C_2| \leq |C_1|\). In this case, no element of \(Y_{t_0,t_2}\) can be of height greater than the leaf of \(C_2\), since this would contradict the fact that \(C_2\) is the selected chain for \(U_2\) at \(t_2\). As before, the claim therefore follows from (\(\dagger 1\)). Finally, consider the case that \(|C_2| > |C_1|\). In this case, no element of \(Y_{t_0,t_2}\) can be of height greater than the leaf of \(C_1\), since such blocks must be broadcast at a timeslot \(\leq t_1\), and this would contradict the fact that \((t_1, t_2)\) was specified as the lexicographically least possible choice subject to \(|t_1 - t_2|\) being minimal. Once again, the claim therefore follows from (\(\dagger 1\)).

### 7.2 Specifying the Permitter

For a given execution of the protocol, we say that a set of responses from the permitter are consistent with \(T\) if the permission table gives the same responses as the permitter to all sets of inputs given to the permitter during the course of the execution.

Given a permissioned reduct \(PR = (I, T)\), we have two tasks with respect to the permitter \(0\):

1. We must specify how the permitter is defined, i.e. we must specify the distribution on the response of the permitter given each possible set of inputs.20

20 Of course, there is some flexibility in terms of the order in which definitions are given here. One might also specify the
2. We must show that, when the protocol is run with security parameter $\varepsilon$, it holds with probability $> 1 - \varepsilon$ that the permitter produces a set of responses that are consistent with some $T \in T(U_0, U_1, \varepsilon, D)$.

If we can achieve (2) above, then this suffices to establish uniform liveness and uniform security for the protocol.

For the example of Bitcoin, we have already specified the permitter and the permissioned reduct. So it remains to show that when the protocol is run with security parameter $\varepsilon$, it holds with probability $> 1 - \varepsilon$ that the permitter produces a set of responses that are consistent with some $T \in T(U_0, U_1, \varepsilon, D)$. This now follows essentially as in [GKL18] (Theorem 31, which establishes that executions are ‘typical’ with high probability) from standard concentration bounds. We refer the reader to that paper for the details.

The relationship to the permissioned setting. In the traditional permissioned setting, one can think of the set of keys/users as being given as an input parameter. Who is allowed to broadcast and when is then determined as a function of this input. More generally, the instructions to be carried out might depend on the set of users as input, but it is instructive for us now to consider permissioned protocols in which this input is used only to determine permissions to broadcast. The permissioned reduct, as we have defined it here, is then a set of permissioned protocols. To deal with the permissionless setting, one defines a permitter and then shows that, with high probability, it produces responses consistent with one of the permissioned protocols in that set.

8 Final Discussion

There are many ways in which the framework we have presented could be expanded to more accurately model consensus protocols in the permissionless setting. Here we just point out some of the principal weaknesses.

User incentives. It is generally recognised in the blockchain community that the classical division into honest and dishonest users is overly simplistic – while the relevance of user incentives is not specific to the permissionless context, it is nevertheless something which tends to be (rightly) prioritised in that context. In the real world, users are not simply honest or otherwise, and a game theoretic analysis can give much more robust performance guarantees than reliance on the honesty of certain users. For the sake of simplicity, we have omitted any such game theoretic analysis here, but we regard it as an important task to properly integrate our framework with an appropriate treatment of user incentives.

Realising the permitter. Part of the strength of the framework we have presented is that it blackboxes the precise mechanics of the user selection process, allowing us to isolate the properties of the selection process which are significant in the sense that they impact the way in which the protocol must be designed, or influence properties of the resulting protocol. This allows us to prove general impossibility results. In order to fully specify a usable protocol, however, it remains true that one has to actually implement (an approximation to) the permitter oracle. It would be interesting to extend the framework we have presented here, so as to also formalise the process of permitter implementation. Specifying and analysing the full protocol would then consist of three tasks:

- protocol first, and then the permissioned reduct second.
(a) Specifying a permissioned reduct;

(b) Specifying and analysing an appropriate permitter oracle;

(c) Proving that one can implement an appropriate approximation to the permitter.

Again, an advantage of this approach might be in the opportunity it provides for modular analysis – many protocols will use essentially the same method of permitter implementation.

Other performance measures. For the sake of simplicity, we have concentrated here on the notions of liveness and security. It is standard in the literature to consider also certain notions of ‘chain quality’ (see [GKL18], for example), which require that honest users produce at least a certain proportion of confirmed blocks. This suffices to ensure that transactions will actually be included in the set of confirmed blocks within a certain bounded time (with high probability). Such considerations are clearly important for a rigorous analysis of protocol performance.

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