Proposed Fermi-surface reservoir-engineering and application to realizing unconventional Fermi superfluids

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We theoretically propose an idea based on reservoir engineering to process the structure of a Fermi edge to split into multiple Fermi edges, so as to be suitable for the state which we want to realize. When one appropriately tunes the chemical-potential difference between two reservoirs being coupled with the system, the system is shown to be in the non-equilibrium steady state with the momentum distribution having a two-edge structure. We argue that these edges play similar roles to two Fermi surfaces, which can be designed to realize exotic quantum many-body states. To demonstrate this, we consider a model driven-dissipative two-component Fermi gas with an attractive interaction as a paradigmatic example and show that it exhibits an unconventional Fermi superfluid. While the superfluid order parameter of this state has the same form as that in the Fulde-Ferrell state discussed in metallic superconductivity under an external magnetic field, the former non-equilibrium pairing state is not accompanied by any spin imbalance. Our proposed reservoir engineering to process the Fermi momentum distribution would provide further possibilities of many-body quantum phenomena beyond the thermal equilibrium case.

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ally produces a two-step structure in $n$ with the model driven-dissipative Fermi gas in Fig. 1(a). This is quite different from our idea in the non-equilibrium case with the ordinary Fermi distribution function, which is proposed to realize this unconventional Fermi superfluid.

Interaction [40], and an artificial field [41–43], have been used to control the Fermi surface. (c) Pairing structure in the ordinary (thermal equilibrium) FF state under an external magnetic field.

has two “Fermi edges” at $p_{F1}$ and $p_{F2}$. This leads to the pairing $|\langle B \rangle\rangle = |p_{F1}, \uparrow\rangle - |p_{F2}, \downarrow\rangle$, in addition to $|\langle A \rangle\rangle$. In a sense, the non-equilibrium FF-like (NEFF) state may be viewed as a mixture of two FF states under external magnetic fields $B$ and $-B$.

We note that possible routes to the FF state in the spin-balanced case has been discussed in the literature, where the shift of single-particle energy induced by external current [38], a size effect [39], an inter-atomic interaction [40], and an artificial field [41–43], have been proposed to realize this unconventional Fermi superfluid. However, these ideas are all in the thermal equilibrium case with the ordinary Fermi distribution function, which is quite different from our idea in the non-equilibrium state.

In what follows, we confirm our scenario by dealing with the model driven-dissipative Fermi gas in Fig. 1(a). We show that our proposed FS reservoir-engineering really produces a two-step structure in $n_{p,\sigma}$. We then show that this processed momentum distribution causes the NEFF phase transition. The latter result proves that two edges imprinted on $n_{p,\sigma}$ work like two Fermi surfaces. Throughout this letter, we set $\hbar = k_B = 1$, and the system volume $V$ is taken to be unity, for simplicity.

The driven-dissipative Fermi gas in Fig. 1(a) is described by the Hamiltonian $H = H_{\text{sys}} + H_{\text{env}} + H_{t}$, consisting of the main system term $H_{\text{sys}}$, the reservoir term $H_{\text{env}}$, as well as the tunneling term $H_{t}$ [44, 45]. The reservoir term has the form,

$$H_{\text{env}} = \sum_{\alpha=L,R} \sum_{p,\sigma} \xi_{p,\sigma}^{\alpha} c_{p,\sigma}^{\alpha†} c_{p,\sigma},$$

where $\alpha = L, R$ denote the left and right reservoirs, and $c_{p,\sigma}^{\alpha}$ is the annihilation operator of a fermion with (pseudo-)spin $\sigma = \uparrow, \downarrow$ in the $\alpha$-reservoir. Each reservoir is assumed to be a free Fermi gas with the kinetic energy $\xi_{p}^{\alpha = L,R} = \varepsilon_{p} - \mu_{\alpha}$, measured from the Fermi chemical potential $\mu_{\alpha}$, where $\varepsilon_{p} = p^2/(2m)$ with $m$ being a particle mass. We also assume that the reservoirs are huge compared to the main system and are always in the ground state at $T_{\text{env}} = 0$. Fermions in these reservoirs thus obey the ordinary Fermi distribution function at $T_{\text{env}} = 0$, $f(\xi_{p}^{\alpha}) = \Theta(-\xi_{p}^{\alpha})$, where $\Theta(x)$ is the step function.

The tunneling between these reservoirs and the main system is described by $H_{t} = \sum_{\alpha=L,R} H_{t}^{\alpha}$, where

$$H_{t}^{\alpha} = \sum_{j=1}^{N_{l}} \sum_{p,q,\sigma} \left[ e^{i[p_{j} - q R_{j}^{\alpha}\gamma]} \Lambda_{\alpha} c_{p,\sigma}\Lambda_{\alpha}^{\dagger} c_{p,\sigma} + \text{H.c.} \right].$$

In Eq. (2), particles tunnel between spatial positions $R_{j}^{\alpha}$ in the $\alpha$-reservoir and $r_{j}$ in the main system ($j = 1, \ldots, N_{l} \gg 1$). Although the translational invariance of the main system is broken in Eq. (2), this symmetry property recovers by taking spatial averages over $R_{j}^{\alpha}$ and $r_{j}$ [46, 47]. For simplicity, we set the tunneling matrix elements as $\Lambda_{L} = \Lambda_{R} = \Lambda$. The main system becomes in the non-equilibrium state when $\mu_{L} = \mu + \delta \mu$ and $\mu_{R} = \mu - \delta \mu$ with $\delta \mu \neq 0$.

To grasp effects of $\delta \mu \neq 0$ in more detail, we first consider the simple case when the main system is a two-component free Fermi gas, that is $H_{\text{sys}} = H_{0} = \sum_{p,\sigma} \varepsilon_{p} a_{p,\sigma}^{\dagger} a_{p,\sigma}$ (where $a_{p,\sigma}$ is the annihilation operator of a fermion in the main system). In this case, after a long time has passed since the system was connected to the reservoirs, the main system would reach NESS, where the gain and loss of particles in the main system are balanced. The momentum distribution $n_{p,\sigma} = \langle a_{p,\sigma}^{\dagger} a_{p,\sigma} \rangle$ in NESS can be evaluated by the ordinary Keldysh Green’s function technique [44, 45], which yields

$$n_{p,\sigma} = \frac{1}{2} - \frac{1}{2\pi} \sum_{\xi_{p}^{\pm}} \text{Tan}^{-1} \left( \frac{\xi_{p}^{\pm} + \xi_{\delta \mu}}{2\gamma} \right).$$

Here, $\xi_{p} = \varepsilon_{p} - \mu$, and $\gamma = \pi N_{l} \rho |\Lambda|^{2}$ is the quasi-particle damping rate, where $\rho$ is the single-particle density of...
states in the reservoirs. In obtaining Eq. (3), we have taken the spatial averages over the tunneling positions, and have ignored the $\alpha (= R, L)$ and $\omega$ dependence of $\rho_{p\sigma}$ [10]. In the weak-damping limit ($\gamma \to +0$), the momentum distribution in Eq. (2) has a two-step structure, as $n_{p\sigma} = \frac{1}{2}[\Theta(\xi_p + \delta\mu) + \Theta(\xi_p - \delta\mu)]$. Thus, it has two edges at $p_{F1} = \sqrt{2m(\mu - \delta\mu)}$ and $p_{F2} = \sqrt{2m(\mu + \delta\mu)}$ as schematically shown in Fig. 1(a). Although these edges become obscure as $\gamma$ increases, they remain as far as $\gamma \ll \delta\mu$.

**NEFF superfluid instability**— We next show that the imprinted two edges on $n_{p\sigma}$ play similar roles to Fermi surfaces. For this purpose, we consider the case when the particle-particle scattering vertex, where $\xi$, is usually in cold atom physics [50], we measure the interaction of Cooper pairs with non-zero center of mass momentum. This situation is similar to the FF case in the superfluid instability, when the particle-particle scattering vertex $\xi$ is the damping rate. The main system exhibits the NEFF (BCS) superfluid instability on the solid (dashed) line. As one further increases $\gamma$, the two steps in $n_{p\sigma}$ is completely smeared out and the overall structure becomes similar to the thermal equilibrium case at high temperatures. As a result, the main system is in the normal state, when $\gamma/\mu > 0.056$ in Fig. 2(a).

**Non-equilibrium superfluid phase**— We now enter the superfluid phase below the transition line in Fig. 2(a), by employing the following ansatz for the superfluid order parameter $\Delta(r, t)$:

$$\Delta(r, t) = \Delta_0 e^{-2i\mu_0 t} e^{iQ \cdot r}. \quad (6)$$

Here, $\Delta_0 \equiv U \sum_p (a_p a_{-p})$ is taken to be positive real, without loss of generality. For simplicity, we do not consider the Larkin-Ovchinnikov type solution $\Delta(r, t) = \Delta_0 e^{-2i\mu_0 t} \cos(Q \cdot r)$ [51] in this letter.

In the thermal equilibrium state, $\Delta_0$ and $Q$ of the FF superfluid order parameter can conveniently be determined from the minimization conditions for the free energy in terms of these quantities. However, this approach is not applicable to NESS, so that we take the following strategy: We derive the NESS gap equation from the self-consistent condition for $\Delta_0 = \sum_p (a_p a_{-p})$. Within the framework of the non-equilibrium Hartree-Fock-Bogoliubov approximation (NEHFB) [60, 48], this condition gives

$$1 = U \frac{1}{4\pi} \sum_p \frac{1}{E_{p\sigma}} \sum_{\eta, \zeta = \pm} \frac{\tan^{-1} \left( \frac{E_{p\sigma}}{2\gamma} \right)}{E_{p\sigma}^{\eta, \zeta} - \xi_{p\sigma}^{\eta, \zeta}}, \quad (7)$$

where $E_{p\sigma}^{\eta, \zeta} = [\xi_{p\sigma}^{\eta, \zeta} / 2 + \xi_{p\sigma}^{\eta, \zeta} / 2] / 2$, and $\xi_{p\sigma}^{\eta, \zeta} = \xi_{p\sigma}^{\eta, \zeta} + \xi_{p\sigma}^{\eta, \zeta} / 2 + \delta\mu$.

Figure 2(a) shows the superfluid phase transition line in the $\gamma = \mu$ plane, determined from the pole condition $\chi(Q, \nu = 2\mu) = 0$. The region above this line is in the normal state, where the superfluid order is destroyed by strong incoherent pumping and decay of particles by the two reservoirs. When $\gamma/\mu < 0.04$, the pole condition is satisfied at $Q \neq 0$ (see Fig. 2(b1)), indicating that the superfluid instability is associated with the Bose condensation of Cooper pairs with non-zero center of mass momentum. This situation is similar to the FF case in the thermal equilibrium state [33, 34]. Thus, the two edges in $n_{p\sigma}$ is found to really work like two Fermi surfaces with different sizes $p_{F1}$ and $p_{F2}$.

Because the damping $\gamma$ makes the two-step structure in $n_{p\sigma}$ obscure, the superfluid instability of this non-equilibrium FF-like state (NEFF) ($Q \neq 0$) changes to the BCS-type phase transition with $Q = 0$, when $\gamma/\mu \gtrsim 0.04$ (see Fig. 2(b2)). As one further increases $\gamma$, the two steps in $n_{p\sigma}$ is completely smeared out and the overall structure becomes similar to the thermal equilibrium case at high temperatures. As a result, the main system is in the normal state, when $\gamma/\mu > 0.056$ in Fig. 2(a).

**FIG. 2**: (a) Calculated chemical potential difference $\delta\mu$ at the superfluid phase transition in the model driven-dissipative Fermi gas in Fig. 1(a). We take $T_{\text{env}} = 0$ and $(ppa_s)^{-1} = 1$. $\gamma$ is the damping rate. The main system exhibits the NEFF (BCS) superfluid instability on the solid (dashed) line. (b) Inverse particle-particle scattering vertex $\chi(Q, \nu = 2\mu)^{-1}$, as a function of $Q$. Upper (lower) panel shows the result along the path (b1) (path (b2)) in (a).
The second equation to determine \((\Delta_0, Q)\) is obtained from the net current \(J_{\text{net}}\) in the main system. In NEHFB, \(J_{\text{net}} = \sum_{\sigma = \uparrow, \downarrow, \sum_p} \langle p + Q/2 \rangle \eta, \zeta \rangle \). where

\[
n_{p, \sigma} = \frac{1}{2} - \frac{1}{4\pi} \sum_{\eta, \zeta = \pm} \frac{E_{p, \eta}}{\frac{p, \eta}{\gamma}} \left[ \eta + \frac{\xi_{p, \eta}}{\xi_{p, Q} E_{p, \eta}} \right].
\]

According to the Bloch’s theorem \(53, 54\), \(J_{\text{net}}\) must vanish in the thermal equilibrium state. However, this theorem does not work out of equilibrium. Indeed, one may consider the current-carrying superfluid state driven by external/boundary conditions \(55, 56\). In this paper, however, to simplify our discussions, we restrict our discussions to the case with \(J_{\text{net}} = 0\). That is, we solve the coupled gap equation \(7\) with the vanishing-current condition, \(J_{\text{net}} = 0\), to self-consistently determine \((\Delta_0, Q)\).

It can be shown that Eq. \(7\) in the limit \(\gamma \to 0\) has the same form as the FF gap equation in the thermal equilibrium state \(53, 54\). This means that the two edges at \(p_r1\) and \(p_r2\) in \(n_{p, \sigma}\) work like large and small Fermi surfaces as in the FF case under an external magnetic field. On the other hand, when we further set \(Q = 0\), Eq. \(7\) has the same form as the ordinary BCS gap equation in the presence of an external magnetic field.

From the knowledge about superconductivity, this coexistence means that the two-step structure in \(n_{p, \sigma}\) does not promote the formation of Cooper-pairs (C) and (D) in Fig. 1(b), but rather suppresses the pair formation around \(p_r = \sqrt{2m \mu}\).

**Phase diagram of driven-dissipative Fermi gas**—Figure 3(a) shows the steady-state phase diagram of a model driven-dissipative Fermi gas. We have confirmed that all the states appearing in this figure are (meta-)stable in the sense that the time evolution of a small deviation from each state always decays. (For more details, see Ref. \(52\).) As expected from Fig. 2(a), NEFF \((\Delta_0 > 0, Q = 0)\) appears in the region (B), where the chemical potential difference \(\delta \mu > 0\) is large enough to produce a clear two-step structure in \(n_{p, \sigma}\) but the damping \(\gamma\) is not strong enough to smear out this structure.

We note that the BCS-type superfluid \((\Delta_0 > 0, Q = 0)\) is also stable in the region (B). This so-called bistability is a characteristic non-equilibrium phenomenon and has been observed in various systems \(57, 59\). This is quite different from the thermal equilibrium case, where the ground state is uniquely identified as the state with the lowest free energy. In the region (C), the bistability of the BCS state and the normal state occurs.

In the bistability regions (B) and (C), which state is realized would depend on how to reach these regions. When one varies \(\delta \mu\) adiabatically, one expects the appearance of the hysteresis shown in Fig. 3(b): As \(\delta \mu\) increases from \(\delta \mu = 0\), the BCS-type state \((\Delta_0 > 0, Q = 0)\) would be maintained both in the regions (B) and (C). As one decreases \(\delta \mu\) from the region (D), on the other hand, the phase transition from the normal state to NEFF \((\Delta_0 > 0, Q \neq 0)\) would occur at the boundary between (B) and (C).

To see the difference between the BCS-type and NEFF states in Fig. 3(a), we compare in Figs. 4(a) and (b) their pair amplitudes, both of which are commonly given by

\[
\langle a_{p, \sigma} \rangle = -\frac{1}{4\pi} \sum_{\eta, \zeta = \pm} \tan^{-1} \left( \frac{E_{p, \eta}}{2\gamma} \right) \frac{\Delta_0}{E_{p, \eta}}.
\]

While the pair amplitude is isotropic in the BCS-type state (see Fig. 4(a)), Fig. 4(b) shows that it vanishes around the “equator” of the Fermi sphere in NEFF, when \(Q\) points to the \(p_r\) direction.

We emphasize that this structure of the NEFF pair amplitude is also different from the FF case in the thermal equilibrium state shown in Fig. 4(c): The vanishing region (which is also referred to the blocking region in the superconductivity literature) spreads over the lower hemisphere. Noting that NEFF may be viewed as a mixture of two FF states with \(Q \neq 0\) and \(Q = 0\), Fig. 4(d), we find that their blocking regions give the vanishing pair amplitude around the equator in Fig. 4(b).

We briefly note that the difference of pair amplitude \(\langle a_{p, \sigma} \rangle\) between NEFF and FF does not reflect the order parameter \(\Delta_0 = U \sum_p \langle a_{p, \sigma} \rangle\) in the weak damping limit \(\gamma \to 0\). In this limit, thus, they follow the same gap equation.

So far, we have only discussed stable solutions of the non-equilibrium gap equation \(7\) under the vanishing current condition \(J_{\text{net}} = 0\). Here, we briefly comment on unstable solutions: One is a gapless uniform superfluid state \((Q = 0)\), being accompanied by the Cooper pairs (C) and (D) in Fig. 1(b). The other is another FF-like state where \(Q\) is nonzero but smaller than the stable NEFF state. We note that similar unstable solutions are known as the Sarma(-Liu-Wilczek) state \(60, 61\), as well as the saddle point \(64\) state in spin- and mass-imbalanced Fermi gases, respectively.
Since many-fermion systems are sensitive to their FSs, many-fermion phenomena associated with FSs.

Summary and future work—We have proposed an idea to process the Fermi momentum distribution $n_{p,\sigma}$, by using reservoirs with different chemical potentials. We considered a driven-dissipative two-component Fermi gas, to show that the pumping and decay of fermions by two reservoirs can imprint a two-edge structure on the non-equilibrium steady state. We clarified the similarity and difference between this unconventional pairing state and the FF state known in superconductivity under an external magnetic field.

Our proposed FS-engineering can be applied to a variety of situations. In (optical) lattice systems, the combination of the band structure and our technique may trigger unconventional ordered phases, such as nonequilibrium spin- and charge-density wave-like states. Since many-fermion systems are sensitive to their FSs, the proposed FS-engineering would contribute to further exploration for unknown many-body quantum phenomena associated with FSs.
