FreDo: Frequency Domain-based Long-Term Time Series Forecasting

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Abstract

The ability to forecast far into the future is highly beneficial to many applications, including but not limited to climatology, energy consumption, and logistics. However, due to noise or measurement error, it is questionable how far into the future one can reasonably predict. In this paper, we first mathematically show that due to error accumulation, sophisticated models might not outperform baseline models for long-term forecasting. To demonstrate, we show that a non-parametric baseline model based on periodicity can actually achieve comparable performance to a state-of-the-art Transformer-based model on various datasets. We further propose FreDo, a frequency domain-based neural network model that is built on top of the baseline model to enhance its performance and which greatly outperforms the state-of-the-art model. Finally, we validate that the frequency domain is indeed better by comparing univariate models trained in the frequency v.s. time domain.

1 Introduction

Time series forecasting is an interdisciplinary field that has a wide range of applications. Scientists at weather stations research models that help predict tomorrow’s temperature and precipitation. Engineers at manufacturing plants study models that can anticipate malfunctions of machines. Analysts at companies develop models that estimate the price and demand of products in the coming weeks. All of these work in different fields with different kinds of data using different types of models, but all are deeply engaged in time series forecasting.

Most previous work on time series forecasting focuses on short-term forecasting, especially for the immediate \( t + 1 \) term. The terms that are closer to the present are easier to forecast and usually have higher importance in most cases. However, the ability to perform long-term forecasting accurately is desirable. With the exponentially increasing amount of data and computing power, combined with the advancements in deep learning, forecasting longer into the future might eventually be feasible.

However, intuitively, errors accumulate as we forecast farther and farther into the future. Thus, it is questionable how far into the future one can accurately predict. In fact, as shown in Sec. 4, a complex model might not outperform a simple baseline model that always outputs the average value as the horizon increases. Various previous works \([17, 23]\) have ignored the problem of error accumulation and designed sophisticated models based on Transformers \([20]\) for long-term forecasting. In this paper, we show that a baseline model based on periodicity actually achieves comparable results to such Transformer-based models on most datasets. Additionally, we propose FreDo, a frequency domain-based model that is built on top of the baseline model to enhance its performance. Finally, by univariately evaluating models based on the frequency v.s. time domain, we validate that learning in the frequency domain further improves model performance.

Our main contributions are:

- We mathematically derive that, under some general constraints, errors accumulate in forecasting when we forecast farther into the future. We also show that complex models might not outperform a simple baseline model.
Additionally, following the description and setup in [21], we can specify long-term forecasting as with the same sample rate over $T$ time steps. A time series dataset is collected by $N$ sensors at the $x$-th time step. Consequently, the whole dataset is a matrix $X = \{x_1, \ldots, x_t, \ldots, x_T\} \in \mathbb{R}^{T \times N}$, where $x_t = [x_{t,1}, \ldots, x_{t,N}] \in \mathbb{R}^N$ are the recorded signals by all $N$ sensors at the $t$-th time step.

The goal of forecasting is to predict the future horizons $\{x_t, x_{t+1}, \ldots\}$ given the histories $\{x_1, \ldots, x_{t-1}\}$. We use $\{\hat{x}_t, \hat{x}_{t+1}, \ldots\}$ to denote the predictions made by a model. In practice [12, 21, 22, 23], only the $I$ most recent histories $\{x_{t-I}, \ldots, x_{t-1}\}$ are fed into a model as input. This not only establishes fair comparisons between different methods, but also makes the memory usage reasonable while assuming histories before $t - I$ have little influence on the future.

Currently, to the best of our knowledge, there is no precise definition as to what is considered short-term or long-term in time series forecasting. In general, as stated in [23], the purpose of long-term forecasting is to forecast significantly longer into the future than most previous methods can achieve. Additionally, following the description and setup in [21], we can specify long-term forecasting as predicting $O$ future horizons given $I$ histories where $O \geq I$, although the value of $I$ is actually a choice of hyperparameter rather than a part of the problem statement. Combining both statements,
we consider long-term forecasting to be when \( O \geq I \) and both \( I \) and \( O \) are much larger than those achieved in most previous work.

4 Error Accumulation in Long-term Forecasting

Errors or noises in time series are almost unavoidable. In terms of forecasting, we have the following data-generating process (DGP):

\[
x_t = f(x_{t-1}, x_{t-2}, \ldots) + e_t,
\]

where \( f \) can be any function and \( e_t \) is the error with mean 0 and variance \( \sigma^2 \). Here, \( e_t \) at different time steps are assumed to be mutually independent, though in the real world this might not be the case [17]. Furthermore, imagine that we already know the exact parameters of the function \( f \) in Eq. (1), which is nearly impossible in practice.

Usually, the scale of \( e_t \) is smaller than \( f(x_{t-1}, x_{t-2}, \ldots) \) so the prediction (under DGP) is useful. However, errors accumulate over time. Hence, conceptually, the errors of model predictions increase as we try to forecast farther and farther into the future. Eventually, the error is so large that it makes the model useless even if we know the DGP.

To formally show error accumulation, we assume that the DGP is a \( p \)-th order autoregressive (AR) model:

\[
x_t = c + \sum_{i=1}^{p} \theta_i x_{t-p} + e_t = c + \left( \sum_{i=1}^{p} \theta_i L^{i-1} \right) x_{t-p} + e_t := c + \phi(L; \theta) x_{t-1} + e_t,
\]

where \( \theta_1, \ldots, \theta_p \) are the parameters of the DGP, \( c \) is a constant, \( L \) is the lag operator, and \( \phi(L; \theta) = \sum_{i=1}^{p} \theta_i L^{i-1} \) is a polynomial of lag operators. Now, after observing \( p \) data points \( x_0, \ldots, x_{p-1} \) as input and assuming \( x_t = 0, \forall t < 0 \), we want to forecast \( x_p, x_{p+1}, \ldots \) using the DGP. Notice that \( \text{Var}[x_t] = \text{Var}[e_t] = 0, \forall t < p \) because \( x_0, \ldots, x_{p-1} \) are observed values, but \( \text{Var}[e_t] = \sigma^2, \forall t \geq p \).

To illustrate, for the first two forecast steps, we have

\[
x_p = c + \phi(L; \theta) x_{p-1} + e_p \quad \text{and} \quad x_{p+1} = c + \phi(L; \theta) x_{p} + e_{p+1} = c + \phi(L; \theta)(c + \phi(L; \theta) x_{p-1} + e_p) + e_{p+1} = (1 + \phi(L; \theta))c + \phi^2(L; \theta) x_{p-1} + \phi(L; \theta)e_p + e_{p+1}.
\]

Thus, we have

\[
\text{Var}[x_p] = \text{Var}[e_p] = \sigma^2 \quad \text{and} \quad \text{Var}[x_{p+1}] = \text{Var}[\phi(L; \theta) e_p + e_{p+1}] = \text{Var}\left[ \sum_{i=1}^{p} \theta_i e_{p+1-i} + e_{p+1} \right] = (\theta_1^2 + 1) \sigma^2.
\]

We can see that the variance increases from time step \( p \) to \( p+1 \), which means that the best mean-squared error (MSE) achievable by any model increases strictly monotonically. Next, we want to show that this is true for any \( t \geq p \). Following Eq. (2), we can always replace the \( x \) terms on the right-hand-side with greater lag terms, for example from Eq. (4) to Eq. (5). Eventually, similar to Eq. (6), we have

\[
x_{p+k} = c \sum_{i=0}^{k} \phi^i (L; \theta) + \phi^{k+1}(L; \theta) x_{p-1} + \sum_{i=0}^{k} \phi^i e_{p+k-i}, \forall k \geq 0.
\]

Then, its variance is \( \text{Var}[x_{p+k}] = \text{Var}\left[ \sum_{i=0}^{k} \phi^i e_{p+k-i} \right] \), because the constant term and the \( x_{p-1} \) term have zero variance. If we compare the variance of \( x_{p+k} \) and \( x_{p+k+1} \):

\[
\text{Var}[x_{p+k}] = \text{Var}\left[ \sum_{i=0}^{k} \phi^i e_{p+k-i} \right] = \text{Var}[e_{p+k} + \phi(L; \theta)e_{p+k-1} + \phi^2(L; \theta)e_{p+k-2} + \ldots],
\]

\[
\text{Var}[x_{p+k+1}] = \text{Var}\left[ \sum_{i=0}^{k+1} \phi^i e_{p+k+1-i} \right] = \text{Var}[e_{p+k+1} + \phi(L; \theta)e_{p+k} + \phi^2(L; \theta)e_{p+k-1} + \ldots],
\]

we can see that the coefficients for the error terms are the same for the first \( k \) terms, but the variance of \( x_{p+k+1} \) has one additional \( k + 1 \)-th term. That is, if we expand all lag polynomials, group coefficients by different error terms, and sum the variances, we can conclude that \( \text{Var}[x_{p+k+1}] > \text{Var}[x_{p+k}] \). This means that the best achievable MSE increases strictly monotonically no matter what model is used, even for non-recurrent, multi-horizon prediction models.
To design a strong yet simple baseline model, we make a very simple but important observation: the best we can do is to average the cycles observed in the input and predict the averaged subseries temporally to the appropriate output horizon. Notice that there is no trainable parameter in the process and there are no interactions between series. Conceptually, AverageTile averages the cycles observed in the input and predicts the same averaged cycle for any future time step. The process is illustrated in Fig. 2.

The above derivation applies to AR models. For general nonlinear function $f$, the best we can do is to use first-order Taylor expansion, so the same applies. If the time series is not stationary, the variance can increase to infinity, which renders any model, including the DGP, useless in the long-term. In the case of weakly stationary time series, it is assumed that both $\mathbb{E}[x_t]$ and $\text{Var}[x_t]$ are fixed values as $t \to \infty$. Thus, there is a bound on the variance, which is exactly $\text{Var}[x_t]$. In other words, the MSE starts from $\sigma^2$ at the first step and approaches $\text{Var}[x_t]$ strictly monotonically as we forecast farther into the future. Although the variance is bounded, the optimal model prediction is still $\mathbb{E}[x_t]$. This, again, means that any model is not better than just predicting the average of observed data points.

Finally, we want to point out that Eq. (1) means that the value of $x_t$ is neither dependent on other exogenous variables nor on the timestamp $t$. If this is not the case, then it is possible that the error does not accumulate at all. For instance, if the DGP is instead $x_t = f(t) + e_t$, then there is no error accumulation. In the real world, the data is probably a mix of $f(x_{t-1}, \ldots)$ and $f(t)$, so the error accumulation might not increase monotonically but will still trend upward over the long-term.

Notice that the model is still useful if the forecast horizon is short. There is no clear cutoff with respect to horizon as to whether a model is useful vs. useless, especially for real-world datasets. However, we emphasize that in many cases a sophisticated model might not outperform a simple baseline model (e.g., always predicting the average value) when the forecast horizon is long enough. Indeed, in the next section, we propose a baseline model, which astonishingly achieves similar performance to the state-of-the-art Transformer-based Autoformer model [21] on most datasets considered.

5 The AverageTile Baseline Model

To design a strong yet simple baseline model, we make a very simple but important observation: there are strong periodicities in most time series datasets. In Fig. 1 we plot part of a univariate series in both time and frequency domain from four common benchmark datasets. Out of the four datasets, only Exchange rate has no meaningful periodicity. In fact, Exchange rate is the only one without clear periodicity out of the seven datasets we benchmark. Thus, we assume that a time series dataset has at least one (often dominant) periodicity of $P$. Series-wise. Finally, we tile the averaged subseries until the appropriate output horizon.

Based on this observation, we propose a baseline model, AverageTile, that has zero trainable parameters. In AverageTile, we need to first determine $P$. Then, given the input histories $\{x_{t-I}, \ldots, x_{t-1}\}$ with length $I = rP$, $r \in \mathbb{Z}^+$, the future prediction output for horizon $o$, $o \in \{0, \ldots, O-1\}$ is

$$
\hat{x}_{t+o} = \frac{\sum_{i=1}^{r} x_{t+(o \mod P)-iP}}{r}.
$$

In other words, AverageTile cuts the input histories into $r$ non-overlapping subseries with length $P$, then averages them series-wise, and finally tiles the averaged subseries temporally to the appropriate output horizon. Notice that there is no trainable parameter in the process and there are no interactions between series. Conceptually, AverageTile averages the cycles observed in the input and predicts the same averaged cycle for any future time step. The process is illustrated in Fig. 2.

![Figure 1: Periodicities observed in time series datasets. Among four datasets shown here, only Exchange has no meaningful periodicity.](image1)

![Figure 2: The computational steps of Average Tile. First, given the dataset, we determine a single meaningful periodicity $P$. Then, given input histories with length $I$ where $I = rP$, $r \in \mathbb{Z}^+$, we cut the input histories into $r$ non-overlapping subseries. We then overlap the $r$ subseries and average them series-wise. Finally, we tile the averaged subseries until the appropriate output horizon.](image2)
DFT and extract where the loss and gradients are calculated. In summary, the model is trained in the frequency domain after taking the “DFT and extract” operations, where

\[
\zeta = [\Re(\zeta_0^{raw}), \ldots, \Re(\zeta_{I-1}^{raw}), \Im(\zeta_1^{raw}), \ldots, \Im(\zeta_{I-1}^{raw})] \in \mathbb{R}^I, \text{ if } I \text{ is even,}
\]

\[
\zeta = [\Re(\zeta_0^{raw}), \ldots, \Re(\zeta_{I-2}^{raw}), \Im(\zeta_1^{raw}), \ldots, \Im(\zeta_{I-2}^{raw})] \in \mathbb{R}^I, \text{ if } I \text{ is odd,}
\]

where \(\Re\) and \(\Im\) denote the real and imaginary part of a complex number. Notice that in Eq. (9) and Eq. (10), the \(\zeta\) vector still has the same length as the input \(x\). We refer these two operations as “DFT and extract.” As for inverse DFT, we can insert the entries in \(\zeta\) into the corresponding positions in \(\zeta_{raw}\) and thus obtain the inverse DFT. We refer to the inverse operations as “insert and inverse DFT.”

After taking the “DFT and extract” operations, \(\zeta\) is fed into a linear layer that projects the dimension from \(I\) to \(O\). Then, it is fed into several layers of Mixer modules sequentially, where each Mixer module is two linear layers with \(O\) hidden dimensions and a ReLU activation function in between. Finally, we apply the “insert and inverse DFT” so the vector is transformed back to time domain where the loss and gradients are calculated. In summary, the model is trained in the frequency domain but with real values so we do not have to deal with complex-valued training. In fact, our experimental results show that training in real domain yields better results in most cases as shown in Appendix B.
Table 1: Periodicities of datasets and the chosen $P$ in two settings. In the “Autoformer setting,” the input length $I$ is 36 for ILI and 96 for other datasets. Under this constraint, we cannot choose $P$ larger than $I$, thus we use $P = 1$ in Weather, Solar, and ILI, and $P = 24$ in Electricity and Traffic. In the case of “one-cycle setting,” we choose $I = P$ so one cycle of data points is observed.

| Dataset   | ETm2 | Electricity | Exchange | Traffic | Weather | Solar | ILI |
|-----------|------|-------------|----------|---------|---------|-------|-----|
| Sample interval | 15 min. | 1 hour | 1 day | 1 hour | 10 min. | 10 min. | 1 week |
| Periodicities | daily | daily, weekly | none | daily, weekly | daily | daily | yearly |
| Periodicities in $P$ | 96 | 24, 168 | none | 24, 168 | 144 | 144 | 52 |
| $P$ in “Autoformer setting” | 96 | 24 | 1 | 24 | 1 | 1 | 1 |
| $P$ in “one-cycle setting” | 96 | 168 | 1 | 168 | 144 | 144 | 52 |

We have tried to directly learn the mapping from the input $\{x_{t-I}, \ldots, x_{t-1}\}$ to the output $\{x_t, \ldots, x_{t+O-1}\}$, but the results are quite poor. Instead, we decide to build on top of the already good baseline AverageTile. Thus, we take in the prediction made by AverageTile, apply “DFT and extract,” and refine it in frequency domain by adding outputs of the Mixer modules, as shown in Fig. 3. The refined prediction is then the output of the model, which also goes through the “insert and inverse DFT” phase to obtain the final time domain output. Essentially, we add learnable parameters to the baseline AverageTile model to further enhance its performance.

Compared to state-of-the-art methods such as Informer [23] and Autoformer [21] which are highly sophisticated, FreDo is not only much simpler and smaller, but also outperforms by a huge margin.

7 Experimental Results

We extensively perform several experiments on seven public datasets to demonstrate the superiority of our models. First, we show that the baseline AverageTile model achieves comparable performances against Autoformer. Next, we compare FreDo against Autoformer under two settings, and see that it significantly outperforms. Finally, we show that learning in the frequency domain is indeed better by comparing the univariate performance of models in both frequency and time domain.

7.1 Datasets

The descriptions of the seven datasets are as follows: (1) ETTm2 [23] is collected from electricity transformer temperatures every 15 minutes from July 2016 to July 2018; (2) Electricity [12] records hourly electricity consumption from 2012 to 2014; (3) Exchange [12] includes the daily exchange rate of eight currencies from 1990 to 2016; (4) Traffic represents hourly road occupancy rates on the San Francisco Bay Area freeways from 2015 to 2016; (5) Weather records 21 meteorological indicators every 10 minutes from a weather station in Germany; (6) Solar measures solar power production in 2006 from photovoltaic power plants in Alabama; and (7) ILI counts the weekly ratio of recorded influenza-like illness (ILI) patients among all patients from 2002 to 2021.

We follow the setup in Autoformer [21] and split all datasets into training/validation/testing chronologically by 60%/20%/20% for the ETT and by 70%/10%/20% for other datasets. We also normalize the data according to the mean and variance of the training set for each series independently. Lastly, we calculate the mean-squared error (MSE) and mean-absolute error (MAE) on the test set to compare different models.

1https://www.bgc-jena.mpg.de/wetter
2https://gis.cdc.gov/grasp/fluview/fluportaldashboard.html

Figure 4: The error curves of AverageTile, Autoformer, TimeDo, and FreDo under three datasets (ETTm2, Traffic, Electricity) when output length is 720 with the “one-cycle setting.” The error curve measures the errors for the next 720 time steps $\{\hat{x}_t, \ldots, \hat{x}_{t+719}\}$ averaged over all possible $t$. 
Table 2: Results under the “Autoformer setting.” The input lengths are 36 for ILI and 96 for other datasets. Four output lengths are tested for each dataset. They are 24, 36, 48, 60 for ILI dataset and 96, 192, 336, 720 for others. Best performance among all models is in boldface and the second best is underlined. The results of Autoformer are referenced directly from their paper [17] except Solar, where we run their released code 5 times and report the average. Under the AverageTile column, numbers with asterisk indicate that the baseline model outperforms the Autoformer.

| Dataset     | Autoformer       | AverageTile | TimeDo   | FreDo   |
|-------------|------------------|-------------|----------|---------|
|             | Metric           | MSE ± MAE   | MSE ± MAE| MSE ± MAE|
|             |                  | MAE ± std   | MAE ± std| MAE ± std|
| ETTm2       | 96               | 255 ± .339  | .263 ± .301* | .184 ± .018  | .281 ± .020  | .169 ± .011  | .268 ± .007 |
|             | 192              | 281 ± .340  | .321 ± .337* | .216 ± .003  | .309 ± .003  | .200 ± .006  | .295 ± .004 |
|             | 336              | 339 ± .372  | .376 ± .370* | .257 ± .003  | .340 ± .003  | .237 ± .006  | .323 ± .003 |
|             | 720              | 422 ± .419  | .471 ± .422  | .321 ± .003  | .384 ± .003  | .297 ± .003  | .366 ± .001 |
|             | 96               | 613 ± .388  | .831 ± .434  | .526 ± .004  | .333 ± .005  | .512 ± .005  | .332 ± .007 |
|             | 192              | 616 ± .382  | .762 ± .413  | .514 ± .003  | .315 ± .006  | .492 ± .007  | .315 ± .005 |
|             | 336              | 622 ± .337  | .768 ± .414  | .524 ± .006  | .312 ± .002  | .500 ± .005  | .312 ± .004 |
|             | 720              | 660 ± .406  | .806 ± .429  | .564 ± .002  | .323 ± .004  | .538 ± .006  | .323 ± .007 |
|             | 96               | 456 ± .380  | .531 ± .377  | .336 ± .003  | .361 ± .003  | .331 ± .013  | .347 ± .020 |
|             | 192              | 496 ± .427  | .911 ± .734  | .215 ± .004  | .270 ± .009  | .216 ± .003  | .276 ± .003 |
|             | 336              | 549 ± .342  | .764 ± .364  | .224 ± .003  | .273 ± .006  | .211 ± .008  | .264 ± .002 |
|             | 720              | 610 ± .342  | .915 ± .734  | .229 ± .003  | .278 ± .009  | .237 ± .004  | .286 ± .005 |
| Solar       | 96               | 408 ± .421  | .911 ± .734  | .215 ± .004  | .270 ± .009  | .216 ± .003  | .276 ± .003 |
|             | 192              | 763 ± .649  | .925 ± .741  | .228 ± .003  | .281 ± .002  | .239 ± .006  | .288 ± .003 |
|             | 336              | 1.102 ± 0.914 | .915 ± .734  | .229 ± .003  | .278 ± .009  | .237 ± .004  | .286 ± .005 |
|             | 720              | 678 ± .603  | .879 ± .715  | .229 ± .003  | .274 ± .004  | .230 ± .006  | .279 ± .007 |
| ILI         | 24               | 3.482 ± 1.287 | 6.643 ± 1.953 | 2.944 ± .0875 | 1.129 ± .0256 | 3.286 ± .0592 | 1.307 ± .0160 |
|             | 36               | 3.103 ± 1.148 | 6.267 ± 1.889 | 3.329 ± .0682 | 1.210 ± .0312 | 3.252 ± .0505 | 1.279 ± .0148 |
|             | 48               | 2.669 ± 1.085 | 5.444 ± 1.693 | 3.438 ± .0917 | 1.231 ± .0179 | 3.298 ± .0209 | 1.275 ± .0054 |
|             | 60               | 2.770 ± 1.125 | 4.798 ± 1.593 | 3.645 ± .0561 | 1.278 ± .0098 | 4.480 ± .0110 | 1.303 ± .0036 |

7.2 Settings

To exhibit the strong performance of AverageTile and the even stronger performance of FreDo, we compare our models to the state-of-the-art Autoformer model [21]. We do not compare against other models such as Informer [23], Reformer [11], or LogTrans [13] because Autoformer has shown to greatly outperform those models. Additionally, to demonstrate the advantage of frequency domain learning, we also show the results of TimeDo, the time-domain version of FreDo. That is, without applying the “DFT and extract,” we directly take the input along with the baseline prediction and feed it into the model. The output is also used directly without the “insert and inverse DFT” step. The number of parameters of FreDo and TimeDo are thus exactly the same.

To compare against Autoformer fairly but to also show how low an error can be achieved using our models, we devise two settings. In both settings, all setups except the input length \( I \) follow exactly the Autoformer [17] paper and have been double checked with their released code. The first setting is the same as in the Autoformer paper [21], that is, \( I = 36 \) for ILI and \( I = 96 \) for other datasets, in order to compare fairly against the results reported in their paper. However, since the \( P \) for some datasets are longer, we further experiment with one cycle of input (i.e., \( I = P \)) to showcase the lowest error rate our models can achieve. We call the first setting the “Autoformer setting,” and name the second the “one-cycle setting.” The summary of both settings can be found in Tab. 1. In addition, detailed hyperparameters of our models can be found in Appendix A.

7.3 Comparing AverageTile to Autoformer

The results under two settings are shown in Tab. 2 and Tab. 3. For both settings, first notice that the baseline model consistently outperforms on the Exchange dataset, which is the only dataset without
Table 3: Results under the “one-cycle setting.” The input lengths are set to the largest $P$ as shown in Tab. 1, so exactly one-cycle of data is given as input. Best performance among all models is in boldface and the second best is underlined. The results of Autoformer are collected by running their released code by 5 times. Under the AverageTile column, numbers with asterisk indicate that the baseline model outperforms the Autoformer. Lower MSE or MAE is better.

| Dataset | Metric | Autoformer | AverageTile | TimeDo | FreDo |
|---------|--------|------------|-------------|--------|-------|
|         |        | MSE ± std  | MAE ± std   | MSE ± std  | MAE ± std   | MSE ± std  | MAE ± std   |
| ETTm2   | 96     | .255 ± .0187 | .327 ± .0101 | .263 ± .300* | .199 ± .0192 | .288 ± .0086 | .178 ± .0117 | .270 ± .0023 |
|         | 192    | .293 ± .0083 | .346 ± .0062 | .321 ± .337* | .244 ± .0351 | .322 ± .0167 | .223 ± .0281 | .304 ± .0105 |
|         | 336    | .346 ± .0087 | .378 ± .0032 | .376 ± .370* | .293 ± .0435 | .356 ± .0190 | .270 ± .0404 | .337 ± .0172 |
|         | 720    | .447 ± .0270 | .433 ± .0138 | .471 ± .421* | .385 ± .0842 | .417 ± .0436 | .345 ± .0558 | .390 ± .0295 |
| Solar   | 96     | .196 ± .0023 | .311 ± .0027 | .212 ± .279* | .160 ± .0002 | .250 ± .0002 | .148 ± .0004 | .240 ± .0001 |
|         | 192    | .211 ± .0064 | .324 ± .0053 | .216 ± .283* | .173 ± .0003 | .261 ± .0003 | .160 ± .0005 | .251 ± .0002 |
|         | 336    | .236 ± .0257 | .342 ± .0142 | .228* ± .295* | .187 ± .0002 | .277 ± .0002 | .173 ± .0004 | .266 ± .0004 |
|         | 720    | .267 ± .0237 | .365 ± .0128 | .267 ± .325 | .219 ± .0013 | .307 ± .0008 | .206 ± .0006 | .295 ± .0006 |
| Traffic | 96     | .085 ± .0010 | .203 ± .0015 | .081* ± .196* | .080 ± .0006 | .198 ± .0012 | .078 ± .0004 | .195 ± .0005 |
|         | 192    | .185 ± .0043 | .309 ± .0038 | .167* ± .289* | .164 ± .0031 | .295 ± .0026 | .154 ± .0022 | .285 ± .0017 |
|         | 336    | .341 ± .0102 | .425 ± .0062 | .306* ± .398* | .317 ± .0128 | .417 ± .0074 | .257 ± .0024 | .378 ± .0023 |
|         | 720    | 1.012 ± 0.0234 | .775 ± .0113 | .810* ± .676* | .559 ± .0426 | .576 ± .0196 | .490 ± .0294 | .541 ± .0302 |
| Exchange | 96    | .630 ± .0207 | .392 ± .0119 | .620* ± .296* | .456 ± .0010 | .303 ± .0005 | .445 ± .0010 | .299 ± .0004 |
|         | 192    | .625 ± .0246 | .387 ± .0218 | .624* ± .297* | .457 ± .0017 | .302 ± .0003 | .446 ± .0010 | .299 ± .0006 |
|         | 336    | .619 ± .0138 | .382 ± .0118 | .632 ± .302* | .461 ± .0024 | .300 ± .0006 | .454 ± .0012 | .299 ± .0004 |
|         | 720    | .644 ± .0108 | .395 ± .0080 | .660 ± .319* | .494 ± .0032 | .311 ± .0017 | .474 ± .0010 | .309 ± .0006 |
| Electricity | 96 | .248 ± .0063 | .327 ± .0076 | .317 ± .288* | .196 ± .0010 | .235 ± .0008 | .194 ± .0009 | .227 ± .0006 |
|         | 192    | .339 ± .0121 | .396 ± .0104 | .343 ± .305* | .234 ± .0004 | .270 ± .0009 | .230 ± .0112 | .262 ± .0008 |
|         | 336    | .354 ± .0108 | .393 ± .0094 | .383 ± .331* | .282 ± .0015 | .308 ± .0012 | .276 ± .0008 | .298 ± .0006 |
|         | 720    | .468 ± .0178 | .469 ± .0143 | .443* ± .370* | .353 ± .0023 | .360 ± .0033 | .346 ± .0008 | .348 ± .0016 |
| IIL     | 96     | .466 ± .0487 | .467 ± .0201 | .290* ± .224* | .175 ± .0033 | .234 ± .0033 | .176 ± .0008 | .234 ± .0015 |
|         | 192    | .761 ± .1948 | .618 ± .0895 | .327* ± .243* | .192 ± .0027 | .250 ± .0023 | .193 ± .0112 | .248 ± .0008 |
|         | 336    | .820 ± .0940 | .690 ± .0369 | .372* ± .266* | .199 ± .0018 | .257 ± .0017 | .202 ± .0018 | .255 ± .0019 |
|         | 720    | .834 ± .1498 | .653 ± .0576 | .376* ± .270* | .206 ± .0029 | .261 ± .0017 | .207 ± .0016 | .260 ± .0009 |

clear periodicity. Since $P = 1$ for Exchange dataset, this implies that taking the average of the last $I$ observations as output already outperforms Autoformer. This further confirms the fact that a sophisticated model can actually underperform a baseline model in long-term forecasting. Other than the Exchange dataset, AverageTile also outperforms occasionally on other datasets.

To understand why AverageTile performs comparably against Autoformer, we look at the error curves as shown in Fig. 4. The error curves of AverageTile are clear demonstrations of error accumulation in long-term forecasting. As described in Sec. 4, a good model should have an increasing error curve. The observed error curves suggest that AverageTile is a reasonable, good baseline model. However, the error curves of Autoformer indicate that it has room for improvement, because the left-end of the error curve is inverted and the right-end of the curve explodes, especially on the Traffic and Electricity datasets. In fact, its errors are higher than AverageTile for the first 100+ time steps.

Furthermore, although AverageTile does not outperform in every case, we emphasize that it is a much simpler model than Autoformer in many aspects. Specifically, AverageTile has no learnable parameter, no training, and is univariate without any complex, hand-crafted architecture. In fact, if we search over the hyperparameter $r = \frac{P}{I}$ and choose the best $r$ for each dataset, AverageTile can achieve even better performance. This extra experiment is detailed in Appendix C. It is much more difficult to search hyperparameters for sophisticated models like Autoformer because it is more computationally intensive and the model size increases with longer input length.

### 7.4 Performance of FreDo

From both Tab. 2 and Tab. 3, we can clearly see the superiority of FreDo. Compared to Autoformer, under both settings, FreDo outperforms by a large margin, with at least 20% improvement in most
cases and up to 50% improvement in some cases. Furthermore, the standard deviation is also smaller in most cases. This is expected because the size of Autoformer is much larger, as tabulated in Appendix D, so there is higher chance of overfitting and thus higher variance. Lastly, FreDo uses the same set of parameters for all series in a dataset (which is also the case for AverageTile), so this also reduces overfitting and forces the model to learn more general patterns across all the dataset. In short, FreDo is a more effective model than Autoformer because it achieves both significantly lower bias and lower variance. This is again validated by the error curves in Fig. 4, where FreDo demonstrates slower rate of error accumulation.

TimeDo also outperforms Autoformer significantly. Comparing FreDo with TimeDo, we see that FreDo achieves lower errors in most cases. However, the improvement is statistically insignificant in many cases and even underperforms on the Solar dataset. Thus, we further perform a detailed experiment to prove that the frequency domain-based model is indeed better.

7.5 Frequency vs. Time Domain

Recall that both TimeDo and FreDo are univariate models; we are able to run on multivariate datasets because we use the same set of model parameters for each series (each variable or feature). Thus, the multivariate results in Tab. 2 and Tab. 3 do not provide a complete picture of model capability, because the same univariate model is trying to minimize the errors on all series simultaneously. Instead, we can train the models, both TimeDo and FreDo, with each univariate series independently, to assess full capability of these models. For example, there are 321 series (variables or features) in the Electricity dataset; thus, we can train 321 different TimeDo and FreDo models, one for each series. Then, to compare the performances, we compare the aggregated error over all 321 models and also perform paired t-tests on the 321 models. We note that the model sizes of TimeDo and FreDo are exactly the same given the same input and output lengths. The results of all seven datasets are shown in Tab. 4. From the table, we can confidently conclude that learning in the frequency domain improves model performance with statistical significance.

8 Conclusion and Limitations

In this paper, we mathematically show the problem of error accumulation which can make sophisticated time series models not useful for long-term forecasting. We propose AverageTile, a simple baseline model, that can achieve comparable performance to the state of the art Transformer-based model. Building on the capability of AverageTile, we further propose FreDo, that learns in the frequency domain and greatly outperforms the state-of-the-art. Finally, we evaluate FreDo and TimeDo univariately to validate that learning in the frequency domain is better. The main limitation of FreDo is that it is designed for long-term forecasting so it does not perform better in the short-term. Also, it is a univariate model so it does not consider interactions between series. Thus, future research can explore multivariate frequency domain models that can also achieve better performance in the short-term, and further explore and understand the advantages of frequency domain learning.
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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper’s contributions and scope? [Yes] See last paragraph in Sec. 1.
   (b) Did you describe the limitations of your work? [Yes] See the last few sentences in Sec. 8.
   (c) Did you discuss any potential negative societal impacts of your work? [No] It is not obvious how a better forecasting model can be used in a negative way.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]

2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] In Sec. 4.
   (b) Did you include complete proofs of all theoretical results? [Yes] Not a very complex proof so all results are included.

3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] The (anonymized) URL is in the Abstract. We also submit the code in the supplementary material.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] In Sec. 7.1, Sec. 7.2, and Appendix A.
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes] In Sec. 7 either in the form of standard deviation or p-value.
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [No] Experiments do not require workstation GPUs.

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] We use public datasets and we cite them in Sec. 7.1
   (b) Did you mention the license of the assets? [No] We are unable to find the licenses, but we do cite the papers or URLs.
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   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [No] Our data do not contain identifiable or offensive content.

5. If you used crowdsourcing or conducted research with human subjects...
(a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]

(b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]

(c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]