Equivariant heat invariants of the Laplacian and nonmininmal operators on differential forms

Yong Wang

Abstract

In this paper, we compute the first two equivariant heat kernel coefficients of the Bochner Laplacian on differential forms. The first two equivariant heat kernel coefficients of the Bochner Laplacian with torsion are also given. We also study the equivariant heat kernel coefficients of nonmininmal operators on differential forms and get the equivariant Gilkey-Branson-Fulling formula.

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1 Introduction

In [Do1], Donnelly computed heat kernel coefficients of the Bochner Laplacian with torsion on differential forms. In [Do2], the equivariant heat kernel asymptotics of the Laplacian on functions was established and the first two equivariant heat kernel coefficients were evaluated. Donnelly’s results were used to study the asymptotic expansion of the heat kernel for orbifolds in [DGGW]. On the other hand, in [GBF], Gilkey, Branson and Fulling studied the heat equation asymptotics of nonmininmal operators on differential forms and computed the first two heat kernel coefficients. In [AV], the heat kernel expansion for a general nonmininmal operator on the spaces $C^\infty(\wedge^k)$ and $C^\infty(\wedge^{p,q})$ was studied. In [Gi], Gilkey studied the heat kernel asymptotics of nonmininmal operators for manifolds with boundary. In [PC], the Gilkey-Branson-Fulling formula was generalized to the case of the $h$-Laplacian. The purpose of this paper is to compute the first two equivariant heat kernel coefficients of the Bochner Laplacian and the Bochner Laplacian with torsion on differential forms and give the equivariant version of the Gilkey-Branson-Fulling formula. We firstly compute the first two equivariant heat kernel coefficients of the Bochner Laplacian on differential forms which are given in Section 2. In section 3, we give the first two equivariant heat kernel coefficients of the Bochner Laplacian with torsion on differential forms. In Section 4, we prove the equivariant Gilkey-Branson-Fulling formula.
2 The computation of the equivariant heat kernel coefficients

Let $M$ be a compact oriented Riemannian manifold of dimension $d$ and $T : M \rightarrow M$ be an isometry preserving orientation with fixed point set $\Omega$. $\Omega$ is the disjoint union of closed connected submanifolds $N$ of dimension $n$. Let $TM$ and $T^*M$ be the tangent and cotangent bundles and let $\wedge^p$ be the bundle of exterior $p$-forms over $M$ and $\Delta$ is the Bochner Laplacian associated to the Levi-Civita connection on $\wedge^p$. $T$ induces a map $T^*$ on $\wedge^p$ which commutes with $\Delta$. If $\lambda$ is an eigenvalue of $\Delta$, then $T^*$ maps the $\lambda$ eigenspace into itself. We shall be interested in the sum $\sum_\lambda \text{Tr}(T^*_\lambda)e^{-t\lambda}$, $t > 0$ where $\text{Tr}$ denotes the trace. Using the same discussions of Theorem 4.1 in [Do] (also see [Gi1]), we get

Theorem 1 Let $T : M \rightarrow M$ be an isometry with fixed point set $\Omega$. Then there is an asymptotic expansion

$$\sum_\lambda \text{Tr}(T^*_\lambda)e^{-t\lambda} \sim \sum_{N \in \Omega} \left(4\pi t\right)^{-\frac{d}{2}} \sum_{k=0}^{\infty} t^k \int_N b_k(T,a) d\text{vol}_N(a), \quad t \to 0,$$

(2.1)

where the functions $b_k(T,a)$ depend only upon the germ of $T$ and the Riemannian metric of $M$ near points $a \in N$.

Let $u_i(x,y)$ be the $i$-th term of the asymptotic expansion of the heat kernel of $e^{-t\Delta}$ and $A$ denote the endomorphism induced by $T$ on the fibre of the normal bundle over $a$ and $B = (I - A)^{-1}$. Let $U_N$ be a sufficient small tubular neighborhood of $N$ and $\pi : U_N \rightarrow N$ be the projection. Let $x$ be coordinates for a normal coordinate chart on $\pi^{-1}(a)$ for $a \in N$ and $\varphi = x - T(x)$. Let $d\text{vol}_M(x) = \psi(x) dx \text{vol}_N(a))$. By the Morse lemma, one can find a smooth coordinate change so that (see Appendix in [Do])

$$d^2(\varphi + T(x), T(x)) = \sum_{i=1}^{s} y_i^2 = |y|^2.$$

Let $|J(\varphi,y)|$ denote the absolute value of the Jacobian determinant of this change of variables. Let $\Box_y = \sum_{i=1}^{s} \frac{\partial^2}{\partial y_i^2}$ and

$$h_i(x(y)) = |\det B|\text{tr}[T^*u_i(T(x),x)]|J(\varphi,y)|\psi(x),$$

(2.2)

then by the same discussions as in [Do], we have

$$b_k(T,a) = \sum_{j=0}^{k} \frac{1}{j!} \Box_y^j(h_{k-j})(0).$$

(2.3)

Let $\tau(y,x) : \wedge^p_x \rightarrow \wedge^p_y$ be the parallel transport along the geodesic curve from $x$ to $y$. Fix $x$ and suppose $y$ is in some normal coordinate neighborhood $w_j$ of $x$ and
\[ g_{ij} = g(\partial/\partial w_i, \partial/\partial w_j). \] By Chapter 2 in [BGV], we have \( u_0(y, x) = (\det g_{ij})^{-\frac{1}{2}}\tau(y, x). \)

By (2.3),

\[ b_0(T, a) = |\det B| \text{tr}[T^* u_0(T(a), a)] = |\det B| \text{tr}[T^*_a]_{\wedge P}. \]

We write \( \tilde{A} = \left( \begin{array}{cc} I & 0 \\ 0 & A^t \end{array} \right) \) and define

\[ \wedge^p \tilde{A}(\theta_1 \wedge \cdots \wedge \theta_p) = \tilde{A} \theta_1 \wedge \cdots \wedge \tilde{A} \theta_p, \]

then

\[ \text{tr}[\wedge^p \tilde{A}] = \sum_{0 \leq p_1 \leq p} \sum_{n+1 \leq i_1 \leq \cdots \leq i_{p_1} \leq d} \sum_{l \leq p_1 \leq d} \epsilon_{i_1 \cdots i_{p_1}} A_{i_1 l_1} \cdots A_{i_{p_1} l_{p_1}}, \]

where \( \epsilon_{i_1 \cdots i_{p_1}} \) is the generalized Kronecker symbol and

\[ b_0(T, a) = |\det B| \text{tr}[\wedge^p \tilde{A}]. \]

By (2.3),

\[ b_1(T, a) = |\det B| (h_1(0) + \Box_y (\text{tr}[T^* u_0(T(x), x)]|J(x, y)|\psi(x))(0)). \]

Denote \( \tau_0 = \sum_{a,b=1}^d R_{abab} \) and \( \rho_{ab} = \sum_{c=1}^d R_{abcd} \) the scalar curvature and Ricci tensor of \( M \). By Lemma 4.8.7 in [Gi1],

\[ u_1(a, a) = \frac{\tau_0}{6}, \]

so

\[ h_1(0) = \text{tr}[T^*_a u_1(T(a), a)] = \frac{\tau_0}{6} \text{tr}[\wedge^p \tilde{A}], \]

By the Taylor expansions in page 169 in [Do], we know that

\[ \frac{\partial}{\partial y_j}|_{y=0} \left( (\det g_{ij})^{-\frac{1}{4}} |J(\overline{x}, y)|\psi(x) \right) = 0, \]

so

\[ \Box_y (\text{tr}[T^* u_0(T(x), x)]|J(\overline{x}, y)|\psi(x))(0) = \Box_y (\text{tr}[T^* \tau(T(x), x)]|J(\overline{x}, y)|\psi(x))(0) + \text{tr}[\wedge^p \tilde{A}] \Box_y \left( (\det g_{ij})^{-\frac{1}{4}} |J(\overline{x}, y)|\psi(x) \right)(0). \]

In the following, we adopt the convention of summing Greek indices \( 1 \leq \alpha, \beta, \gamma \leq n \) from 1 to \( n \) and Latin indices \( n+1 \leq i, j, k \leq d \) from \( n+1 \) to \( d \). By the computations in [Do,p.170], we know that

\[ \Box_y \left( (\det g_{ij})^{-\frac{1}{4}} |J(\overline{x}, y)|\psi(x) \right)(0) = \frac{1}{6} \rho_{kk} + \frac{1}{3} R_{iksh} B_{ki} B_{hs} + \frac{1}{3} R_{iksh} B_{kt} B_{hi} - R_{kaha} B_{ks} B_{hs}. \]

Now we compute \( \text{tr}[T^* \tau(T(x), x)] \). Let \( E = (E_1, \cdots, E_d) \) be an oriented orthonormal frame field in a neighborhood of \( a \) such that for \( x \in N, E_1(x), \cdots, E_n(x) \) are tangent to \( N \) while the vector fields \( E_{n+1}(x), \cdots, E_d(x) \) are normal to \( N \) and \( E \) is parallel.
along geodesics normal to $N$ and $dT$ is expressed as a matrix-valued function $A(x)$ as $dTE(x) = E(Tx)A(x)$. Then $A(a) = \begin{pmatrix} J & 0 \\ 0 & A \end{pmatrix}$. Let $x = \exp_a \left( \sum_{i=n+1}^d x_i E_i(a) \right)$.

By Lemma 3.1 in [LYZ], $A(x) = A(a)$. We consider $\Lambda^p(T^*M)$ as the associated bundle $SO(T^*M) \times_{\mu} \Lambda^p(R^n)$ where the action $\mu$ is defined by (2.4). Let $\sigma = (E_1^* \cdots E_d^*)$ be the local section of $SO(T^*M)$ on $V$. Then a local section of $\Lambda^p(T^*M)$ on $V$ can be expressed as $[(\sigma, f)]$ where $f : V \to \Lambda^p(R^n)$ be a smooth function. Let

$$T^*[(\sigma(Tx), c)] = [(\sigma(x), T^*(c)]$$

where $T^*(x) : V \to$ End$(R^n)$. Let

$$\tau(Tx, x)[(\sigma(x), c)] = [(\sigma(Tx), \tau^*(x)]$$

Then $T^*(x) = \Lambda^p \tilde{A}$ is a constant matrix-valued function. As in [LYZ, p.575], define the oriented frame field $ETx$ over the patch $V$ by requiring that $ETx(Tx) = E(Tx)$ and that $ETx$ be parallel along geodesics through $Tx$ and a map $\Phi : V \to so(d)$ by $ETx = E(x)e^{\Phi(x)}$. Then $E^{\ast}(x) = E^*(x)e^{-\Phi(x)}$, that is

$$\tau(Tx, x)\sigma(x) = \sigma(Tx)e^{-\Phi(x)}$$

so $\tau^*(x) = \Lambda^p e^{-\Phi(x)}$. By Lemma 3.3 in [LYZ], $\Phi = \begin{pmatrix} 0 & 0 \\ 0 & \Psi \end{pmatrix}$ and

$$\Psi_{ij}(x) = -\frac{1}{2} A_{rl} x_l x_s R_{rsij}(a) + O(|x|^3),$$

By $x_i = B_{ij} \overline{x}_j$ and $\overline{x}_j = y_j + O(|y|^3)$, so

$$\Psi_{ij}(x) = -\frac{1}{2} A_{rl} B_{lk} B_{sq} R_{rsij}(a) y_k y_q + O(|x|^3).$$

$$e^{-\Phi(x)} = 1 - \Phi(x) + O(|x|^3).$$

$$\text{tr}[T^*\tau(T(x), x)] = \text{tr}[(\Lambda^p(W))] = \text{tr}[(\Lambda^p(\tilde{A}(1 - \Phi(x))))] + O(|x|^3),$$

$$C := \Box y(\text{tr}[(\Lambda^p(W))]|y = 0 = \sum_{\delta = 1}^{d-n} \sum_{1 \leq i_1 < \cdots < i_p \leq d} \sum_{l_1, \cdots, l_p} \varepsilon_{l_1, \cdots, l_p} \sum_{s=1}^p W_{1i_1} \cdots W_{is-1i_{s-1}}$$

$$\cdot (\frac{\partial^2}{\partial y_0^2} W_{i_{s+1} i_{s+1}}) \cdots W_{i_p i_p} |y = 0$$

$$= \sum_{\delta = 1}^{d-n} \sum_{1 \leq l_1 < \cdots < l_p \leq d} \sum_{s=1}^p \varepsilon_{l_1, \cdots, l_p} A_{il_1} \cdots A_{is-1l_{s-1}} A_{i_{s+1} l_{s+1}} \cdots A_{i_p l_p} A_{rl} B_{lk} B_{sq} R_{u\delta} A_{mlu} R_{vm\delta}$$

$$= \sum_{\delta = 1}^{d-n} \sum_{1 \leq p_1 \leq p_1 \leq n} \sum_{1 \leq i_1 < \cdots < i_{p_1} \leq d} \sum_{s=1}^p \varepsilon_{l_1, \cdots, l_p} A_{i_1 l_1} \cdots A_{i_{p_1} l_{p_1}} A_{i_{p_1 + 1} l_{p_1 + 1}} \cdots A_{i_p l_p} A_{rl} B_{lk} B_{sq} R_{u\delta} A_{mlu} R_{vm\delta}$$
\[ A_{i_1+1} \cdots A_{i_p+1} A_{r_1} B_{t_1} B_{v_1} A_{m_1} R_{v_1 m_1} \]  

(2.17)

So by (2.5), (2.7), (2.11), (2.12) and (2.17), we get

\[
b_1(T, a) = |\det B| \left\{ C + \text{tr}[\wedge^p A] \left( \frac{T_0}{6} + \frac{1}{6} \rho_{kk} \right) + \frac{1}{3} R_{iksh} B_{ki} B_{hs} + \frac{1}{3} R_{ikth} B_{kt} B_{hi} - R_{k\alpha h\alpha} B_{ks} B_{hs} \right\}.
\]

(2.18)

**Theorem 2** The coefficient \( b_k(T, a) \) of Theorem 1 is of the form \( b_k(T, a) = |\det B| b'_k(T, a) \) where \( b'_k(T, a) \) is an invariant polynomial in the components of \( A, B \) and the curvature tensor \( R \) and its covariant derivative at \( a \). In particular, \( b_0(T, a), b_1(T, a) \) are determined by (2.6) and (2.18).

**Remark** Theorem 2 can be used to evaluate the heat kernel coefficients of the Laplacian on forms on orbifolds as in [DGGW].

### 3 The computation of the equivariant heat kernel coefficients of the Bochner Laplacian with torsion

The Levi-Civita connection \( \nabla \) is the unique torsion zero connection on \( TM \) which preserves the metric. More generally, let \( \mathcal{T} : TM \times TM \rightarrow TM \) be a skew-symmetric linear map, i.e. \( \mathcal{T}(X, Y) = -\mathcal{T}(Y, X) \). Then there is exactly one metric-preserving connection \( \nabla = \nabla + \mathcal{Q} \) on \( TM \) with torsion tensor \( \mathcal{T} \). Let \( Q(X, Y, Z) = g(Q(X, Y), Z); \mathcal{T}(X, Y, Z) = g(\mathcal{T}(X, Y), Z) \) and \( Q_{ijk} = Q(E_i, E_j, E_k); \mathcal{T}_{ijk} = \mathcal{T}(E_i, E_j, E_k), \) then

\[
Q_{ijk} = \frac{1}{2} (\mathcal{T}_{ijk} + \mathcal{T}_{kij} + \mathcal{T}_{kji}).
\]

(3.1)

Let \( \overline{\nabla}^* = \nabla^* + Q^* \) be the dual connection of \( \overline{\nabla} \) on \( T^* M \) and \( \overline{Q}_{ijk} = g(Q^*(E_i)E_j^*, E_k^*), \) then \( \overline{Q}_{ijk} = -Q_{ijk}. \overline{\nabla}^* \) induces a connection on \( \wedge^p T^* M \). We still denote it by \( \overline{\nabla}^* \).

Let the Bochner Laplacian with torsion be

\[
\overline{\Delta} = -\sum_{i=1}^{d} \left( \nabla_{E_i}^* \nabla_{E_i} - \nabla_{E_i}^* \nabla_{E_i}^* \right).
\]

Let \( \overline{\pi}(y, x) : \wedge^p y \rightarrow \wedge^p x \) be the parallel transport along the geodesic curve from \( x \) to \( y \) associated to the connection \( \overline{\nabla}^* \). Define the oriented frame field \( E^{x,*} \) over the patch \( V \) by requiring that \( \overline{E}^{x,*}(x) = E^*(x) \) and that \( \overline{E}^{x,*} \) be parallel along geodesics through \( x \) associated to the connection \( \overline{\nabla}^* \). Let \( z \) be the normal coordinates for the center \( x \). Let \( E^{x,*}(z) = E^{x,*}(z) \overline{\nu}(x, z); \overline{\nu}(x, Tx) = \overline{\nu}(x). \)

(3.2)
Nextly, we compute the Taylor expansion of $\Psi(x)$. Let $Q^*E^x(z) = E^x(z)A^z(z)$ and $L(x) = A^z(0)$, then
\begin{equation}
Q^*E^x(x) = E^x(x)L(x), \quad L(E_k)_{ij} = Q_{kji}.
\end{equation}

Let $\gamma$ be the geodesic curve from $x$ to $z$. By $\nabla^L_\gamma E^x = 0$, $\nabla^L E^x = 0$ and (3.2), we have
\begin{equation}
\frac{d}{dt} \Psi(x, \gamma(t)) + A^x(\dot{\gamma}(t))\Psi(x, \gamma(t)) = 0.
\end{equation}

By $\gamma(t) = tz; \dot{\gamma} = \sum_{j=1}^d z_j \frac{\partial}{\partial z_j},$
\begin{equation}
\sum_{j=1}^d \frac{\partial \Psi}{\partial z_j}(x, tz)z_j + \sum_{j=1}^d z_j A^x(\frac{\partial}{\partial z_j})(0)\Psi(x, tz) = 0.
\end{equation}

By $\Psi(x, 0) = id$ and setting $t = 0$, we get
\begin{equation}
\sum_{j=1}^d \frac{\partial \Psi}{\partial z_j}(x, 0)z_j + \sum_{j=1}^d z_j A^x(\frac{\partial}{\partial z_j})(0) = 0.
\end{equation}

Taking the derivative about $t$ and setting $t = 0$, we get
\begin{equation}
\sum_{i,j=1}^d \frac{\partial^2 \Psi}{\partial z_i \partial z_j}(x, 0)z_i z_j + \sum_{i,j=1}^d z_i z_j A^x (\frac{\partial}{\partial z_j})(0) \frac{\partial \Psi}{\partial z_i}(x, 0) + \sum_{i,j=1}^d z_i z_j \frac{\partial A^x (\frac{\partial}{\partial z_j})(0)}{\partial z_i}(0) = 0.
\end{equation}

By (3.6) and (3.7), we have
\begin{equation}
\Psi(x, z) = Id - \sum_{j=1}^d z_j A^x (\frac{\partial}{\partial z_j})(0) + \frac{1}{2} \sum_{i,j=1}^d A^x (\frac{\partial}{\partial z_j})(0) A^x (\frac{\partial}{\partial z_i})(0) z_i z_j

- \frac{1}{2} \sum_{i,j=1}^d z_i z_j \frac{\partial A^x (\frac{\partial}{\partial z_j})(0)}{\partial z_i}(0) + O(|z|^3).
\end{equation}

Let $T x = \exp_x(\sum_{i=1}^d u_i E_i(x))$ and $x = (a, c)$ be the orthogonal coordinates in [LYZ,p.574], then by the proposition in [Yu,p.84], we have
\begin{equation}
u_i = O(|c|^3), \quad 1 \leq i \leq n, \quad u_i = c_{i-n} - c_{i-n} + O(|c|^3), \quad n + 1 \leq i \leq d,
\end{equation}
where $c = (c_1, \ldots, c_{d-n})A$. Then
\begin{equation}
\Psi(x) = \Psi(x, u) = Id - \sum_{j=n+1}^d u_j A^x (\frac{\partial}{\partial z_j})(0) + \frac{1}{2} \sum_{i,j=n+1}^d A^x (\frac{\partial}{\partial z_j})(0) A^x (\frac{\partial}{\partial z_i})(0) u_i u_j

- \frac{1}{2} \sum_{i,j=n+1}^d u_i u_j \frac{\partial A^x (\frac{\partial}{\partial z_j})(0)}{\partial z_i}(0) + O(|u|^3).
\end{equation}
We know that
\[ A^x(\frac{\partial}{\partial x_j})(0) = A^x(0)(E_j(x)) = L(x)(E_j(x)), \] (3.11)
\[ \frac{\partial A^{(a,0)}}{\partial z_j}(0) = \frac{\partial A^{(a,0)}}{\partial z_{i-n}}(0) = \frac{\partial L(E_j)}{\partial c_{i-n}}(0), \ n + 1 \leq i, j \leq d, \] (3.12)
\[ L(E_j) = L(E_j)(a, 0) + \sum_{i=n+1}^{d} \frac{\partial L(E_j)}{\partial c_{i-n}}(a, 0)c_{i-n} + O(|c|^2), \] (3.13)
so
\[ \Psi(x) = Id - \sum_{j=n+1}^{d} u_j L(E_j)(a, 0) - \sum_{i,j=n+1}^{d} \frac{\partial L(E_j)}{\partial c_{i-n}}(a, 0)c_{i-n}u_j \]
\[ + \frac{1}{2} \sum_{i,j=n+1}^{d} L(E_i)(a, 0)L(E_j)(a, 0)u_iu_j - \frac{1}{2} \sum_{i,j=n+1}^{d} \frac{\partial L(E_i)}{\partial c_{i-n}}(a, 0)u_iu_j + O(|c|^3). \] (3.14)
By
\[ u_i = -y_{i-n} + O(|y|^3), \ c_{i-n} = B_{ij}y_{j-n} + O(|y|^3), \ n + 1 \leq i, j \leq d, \] (3.15)
we have
\[ \Psi(x) = Id + \sum_{j=n+1}^{d} y_{j-n} L(E_j)(a, 0) + \sum_{i=n+1}^{d} \frac{\partial L(E_j)}{\partial c_{i-n}}(a, 0)y_{i-n}B_{ik}y_{k-n} \]
\[ + \frac{1}{2} \sum_{i,j=n+1}^{d} L(E_i)(a, 0)L(E_j)(a, 0)y_{i-n}y_{j-n} - \frac{1}{2} \sum_{i,j=n+1}^{d} \frac{\partial L(E_i)}{\partial c_{i-n}}(a, 0)y_{i-n}y_{j-n} + O(|y|^3). \] (3.16)
By (3.2),
\[ \overline{E}^{x,*}(T^x) = E^* (T^x)e^{-\Phi(x)\Psi(x)}. \] (3.17)
Let
\[ \overline{\pi}(T^x, x) = \Pi(T^x)e^{-\Phi(x)\Psi(x)} = [(\sigma(T^x), \overline{\pi}(x)c)]. \] (3.18)
Let \( \overline{b}_l(T, a) \) denote the equivariant heat kernel coefficients of the Bochner Laplacian with torsion. Similar to the discussions in Section 2, \( \overline{b}_0(T, a) = b_0(T, a) \). Similar to (2.18), we have
\[ \overline{b}_1(T, a) = |\text{det}B| \left\{ \overline{\mathcal{C}} + \text{tr}[\wedge^p \overline{A}][\frac{\overline{\tau}_0}{6} + \frac{1}{6}\rho_{kk} \]
\[ + \frac{1}{3}R_{iksh}B_{ki}B_{hs} + \frac{1}{3}R_{ikth}B_{ki}B_{hs} - R_{kaka}B_{kh}B_{hs} \right\}. \] (3.19)
where
\[ \overline{\mathcal{C}} = \Box_y(\text{tr}[T^x\overline{\pi}(T^x, x)])|_{y=0} = \text{tr}[\wedge^p (\overline{A}e^{-\Phi(x)\Psi(x)})] := \text{tr}[\wedge^p \overline{\mathcal{W}}]. \] (3.20)
Theorem 3 The coefficient $\overline{b}_k(T,a)$ is of the form $\overline{b}_k(T,a) = |\det B|\overline{f}_k(T,a)$ where $\overline{f}_k(T,a)$ is an invariant polynomial in the components of $A$, $B$ and the curvature tensor $R$ and the torsion tensor $T$ and its covariant derivative at $a$. In particular, $\overline{b}_1(T,a)$ are determined by (3.24) and (3.19).
In the following, we define another Bochner’s Laplacian with torsion and compute its equivariant heat invariants. Let

$$\hat{\Delta} = -\sum_{i=1}^{d} (\nabla_{E_i} \nabla_{E_i}^* - \nabla_{E_i}^* \nabla_{E_i}) = \Delta + Q_{iij} E_j + F.$$ 

Let $\hat{\nabla} = \nabla - \frac{1}{2p} (Q(E_i) E_i)^*$ be a connection on $\wedge T^p M$ associated to a connection $\hat{\nabla} = \nabla - \frac{1}{2p} (Q(E_i) E_i)^*$ on $T^* M$. Then $\hat{\Delta}$ is a generalized Laplacian associated to the connection $\hat{\nabla}$. In this case, $L(x) = -\frac{1}{2p} (Q(E_i) E_i)^*$ and $L(E_j) = -\frac{1}{2p} Q_{ij}$. By Proposition 3.3, 3.5 and 4.3 [Do1], we have

$$\hat{u}_1(x, x) = \frac{70}{6} - \frac{1}{2} T_{kjk,j} - \frac{1}{4} T_{kjk} T_{ljl}, \quad \hat{b}_1(T, a) = |\det B| \left\{ \hat{C} + \text{tr}[\hat{\nabla}^p \hat{A}](\frac{70}{6} - \frac{1}{2} T_{kjk,j} - \frac{1}{4} T_{kjk} T_{ljl} + \frac{1}{6} \rho_{kk} + \frac{1}{3} R_{iksh} B_{ki} B_{hs} + \frac{1}{3} R_{ikth} B_{ki} B_{hi} - R_{kalpha} B_{ks} B_{hs}) \right\}. \quad (3.25)$$

$$\frac{\partial W_{lm_1lm_2}}{\partial y_1}|_{y=0} = \frac{1}{2p} A_{lm_1lm_2} Q_{iij} + \frac{1}{2p} Q_{iij}.$$ 

$$\frac{\partial^2 W_{lm_1lm_2}}{\partial y_1^2}|_{y=0} = A_{\epsilon l_1lm_2} A_{rl_1B_{ij} B_{kj} A_{lm_1} R_{vrum} m_3} + A_{lm_1lm_2} \left[ -\frac{1}{p} \sum_{j=n+1}^{d} Q_{iij+n} E_{j} \right] + \frac{1}{4p^2} Q_{iij+n} Q_{kkn} + \frac{1}{2p} Q_{iij}.$$ 

So

$$\hat{C} = \sum_{\delta=1}^{d-n} \sum_{1 \leq l_1 < \ldots < l_p \leq d} \sum_{l_1, \ldots, l_p} A_{l_1} \cdots A_{l_p} Q_{iij+n} Q_{kkn} + \sum_{1 \leq m_1 \leq p} A_{l_1} \cdots A_{l_p} \left[ A_{r1} B_{ij} B_{kj} A_{lm_1} R_{vrum} m_3 \right] + \sum_{1 \leq m_3 \leq p} A_{l_1} \cdots A_{l_p} \left[ A_{r1} B_{ij} B_{kj} A_{m_1} R_{vrum} m_3 \right] + \sum_{j=n+1}^{d} Q_{iij+n} E_{j}.$$ 

Theorem 4: The coefficient $\hat{b}_k(T, a)$ is of the form $\hat{b}_k(T, a) = |\det B| \hat{b}_k(T, a)$ where $\hat{b}_k(T, a)$ is an invariant polynomial in the components of $A$, $B$ and the curvature tensor $T$ and the torsion tensor $\hat{T}$ and its covariant derivative at $a$. In particular, $\hat{b}_1(T, a)$ are determined by (3.26) and (3.27).
4 The equivariant Gilkey-Branson-Fulling formula

Since $T$ is a preserving orientation isometry, then $T^*$ commutes with $d$, $\delta$ and $\triangle$, so $T^*$ preserves the Hodge decomposition. Let

\[ \Delta_{d}^{(p)} = \Delta^{(p)}|_{\text{Im}d}; \Delta_{\delta}^{(p)} = \Delta^{(p)}|_{\text{Im}\delta} \]

and for each $t > 0$

\[ f_T(t, d^{(p)}) = \text{Tr}(T^* e^{-t\Delta_{d}^{(p)}}); \quad f_T(t, \delta^{(p)}) = \text{Tr}(T^* e^{-t\Delta_{\delta}^{(p)}}), \]

\[ f_T(t, \triangle^{(p)}) = \text{Tr}(T^* e^{-t\Delta^{(p)}}); \quad \beta_{T,p} = \text{Tr}(T^*|_{\ker(\triangle^{(p)})}) \]

Then we have via the Hodge decomposition theorem:

\[ f_T(t, \triangle^{(p)}) = \beta_{T,p} + f_T(t, d^{(p)}) + f_T(t, \delta^{(p)}). \] (4.1)

Similar to the nonequivariant case, we have

\[ f_T(t, d^{(p)}) = f_T(t, \delta^{(p-1)}). \] (4.2)

By (4.1) and (4.2), we have

\[ f_T(t, \delta^{(p)}) = \sum_{j \leq p} (-1)^{p-j} [f_T(t, \triangle^{(j)}) - \beta_{T,j}]. \]

On the other hand,

\[ f_T(t, d^{(p)}) = f_T(t, \delta^{(p-1)}) = \sum_{j \leq p-1} (-1)^{p-j} [f_T(t, \triangle^{(j)}) - \beta_{T,j}]. \]

Let $D^{(p)} = a^2 d \delta + b^2 \delta d$ acting on $\wedge^p$ where $a \neq 0, b \neq 0$. If we make the same computations for operator $D^{(p)}$ we obtain

\[ f_T(t, D^{(p)}) = \text{Tr}(T^* e^{-tD^{(p)}}) = \beta_{T,p} + f_T(a^2 t, d^{(p)}) + f_T(b^2 t, \delta^{(p)}) \]

\[ = f_T(b^2 t, \triangle^{(p)}) + \sum_{j < p} (-1)^{p-j} [f_T(b^2 t, \triangle^{(j)}) - f_T(a^2 t, \triangle^{(j)})]. \] (4.3)

By Lemma 1.8.2 in [Gi1], we have

\[ f_T(t, D^{(p)}) = \sum_{N \in \Omega} (4\pi t)^{-\frac{N}{2}} a^{-N} \sum_{k=0}^{\infty} t^k b_{N,k}^T(D^{(p)}) \]

\[ f_T(b^2 t, \triangle^{(j)}) = \sum_{N \in \Omega} (4\pi t)^{-\frac{N}{2}} b^{-N} \sum_{k=0}^{\infty} t^k b_{N,k}^T(\triangle^{(j)}) b^{2k} \]

\[ f_T(a^2 t, \triangle^{(j)}) = \sum_{N \in \Omega} (4\pi t)^{-\frac{N}{2}} a^{-N} \sum_{k=0}^{\infty} t^k b_{N,k}^T(\triangle^{(j)}) a^{2k} \]
By equating coefficients of $t^l$ in the asymptotic expansion in (4.3), we get

**Theorem 5**

$$\sum_{N \in \Omega} (4\pi)^{-\frac{nN}{2}} b_{N,l}^T (D^{(p)}) = b^{2l} \sum_{N \in \Omega} (4\pi)^{-\frac{nN}{2}} b_{N,l+\frac{nN}{2}} (\Delta^{(p)}) + \sum_{j<p} (-1)^{p-j} (b^{2l} - a^{2l}) \sum_{N \in \Omega} (4\pi)^{-\frac{nN}{2}} b_{N,l+\frac{nN}{2}} (\Delta^{(j)}).$$

(4.4)

**Remark 1.** If $n_N = \text{constant}$ and taking $l = -\frac{n_N}{2}$, then we have

$$\sum_{N \in \Omega} b_{N,0}^T (D^{(p)}) = b^{-n} \sum_{N \in \Omega} b_{N,0}^T (\Delta^{(p)}) + \sum_{j<p} (-1)^{p-j} (b^{-n} - a^{-n}) \sum_{N \in \Omega} b_{N,0}^T (\Delta^{(j)}).$$

(4.5)

If $n_N = \text{constant}$ and taking $l = -\frac{n_N}{2} + 1$, then we have

$$\sum_{N \in \Omega} b_{N,1}^T (D^{(p)}) = b^{-n+2} \sum_{N \in \Omega} b_{N,1}^T (\Delta^{(p)}) + \sum_{j<p} (-1)^{p-j} (b^{-n+2} - a^{-n+2}) \sum_{N \in \Omega} b_{N,1}^T (\Delta^{(j)}).$$

(4.6)

**Remark 2.** Let $M$ be a Kähler manifold and $T$ preserve the orientation and the canonical almost complex structure, then $T^*$ commutes with $\partial, \bar{\partial}, \partial^*, \bar{\partial}^*$. Similar to Theorem 3, we can get the equivariant version of the expression of heat kernel coefficients of nonminimal operators on Kähler manifolds in [AV].

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School of Mathematics and Statistics, Northeast Normal University, Changchun, Jilin 130024, China;

E-mail: wangy581@nenu.edu.cn