FINITE-DIMENSIONAL IRREDUCIBLE $\Box_q$-MODULES
AND THEIR DRINFEL’D POLYNOMIALS

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Abstract. Let $F$ denote an algebraically closed field with characteristic 0, and let $q$ denote a nonzero scalar in $F$ that is not a root of unity. Let $Z_4$ denote the cyclic group of order 4. Let $\Box_q$ denote the unital associative $F$-algebra defined by generators $\{x_i\}_{i \in Z_4}$ and relations

$$
q x_{i+1}x_i - q^{-1}x_i x_{i+1} = 1,
$$

$$
x_i^3 x_{i+2} - [3]_q x_i x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_i x_{i+2} x_i^3 = 0,
$$

where $[3]_q = (q^3 - q^{-3})/(q - q^{-1})$. There exists an automorphism $\rho$ of $\Box_q$ that sends $x_i \mapsto x_{i+1}$ for $i \in Z_4$. Let $V$ denote a finite-dimensional irreducible $\Box_q$-module of type 1. To $V$ we attach a polynomial called the Drinfel’d polynomial. In our main result, we explain how the following are related:

(i) the Drinfel’d polynomial for the $\Box_q$-module $V$;
(ii) the Drinfel’d polynomial for the $\Box_q$-module $V$ twisted via $\rho$.

Specifically, we show that the roots of (i) are the inverses of the roots of (ii). We discuss how $\Box_q$ is related to the quantum loop algebra $U_q(L(sl_2))$, its positive part $U_q^+$, the $q$-tetrahedron algebra $\boxtimes_q$, and the $q$-geometric tridiagonal pairs.

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1. Introduction

Throughout the paper let $F$ denote an algebraically closed field with characteristic 0. We will discuss a number of algebras over $F$. In [5] B. Hartwig and P. Terwilliger introduced the tetrahedron Lie algebra $\boxtimes$. In [5] Theorem 11.5 they showed that $\boxtimes$ is isomorphic to the three-point $sl_2$ loop algebra. In [3] B. Hartwig classified up to isomorphism the finite-dimensional irreducible $\boxtimes$-modules. For more information about $\boxtimes$ see [19]. Fix $0 \neq q \in F$ such that $q$ is not a root of unity. In [8] T. Ito and P. Terwilliger introduced a quantum analog $\boxtimes_q$ of $\boxtimes$ called the $q$-tetrahedron algebra. In [8] Section 10 they classified up to isomorphism the finite-dimensional irreducible $\boxtimes_q$-modules. For more information about $\boxtimes_q$ see [10,13,16,18]. We now mention a chain of algebra homomorphisms that involves $\boxtimes_q$. Consider the quantum loop algebra $U_q(L(sl_2))$ [2] Section 3.3 and its positive part $U_q^+$ [11] Definition 1.1. By [18] Propositions 4.1, 4.3, there exists a chain of algebra homomorphisms

$$
U_q^+ \longrightarrow U_q(L(sl_2)) \longrightarrow \boxtimes_q.
$$
In the above chain each map is injective. In [7] T. Ito, K. Tanabe and P. Terwilliger introduced the notion of a tridiagonal pair. In [6] T. Ito, K. Nomura and P. Terwilliger classified up to isomorphism the tridiagonal pairs over $\mathbb{F}$. In [8, Theorems 10.3, 10.4] and [11, Theorem 2.7], the finite-dimensional irreducible $\mathbb{E}_q$-modules are linked to a family of tridiagonal pairs said to have $q$-geometric type. For more information about tridiagonal pairs see [1,10,12,20]. The existing literature contains a bijection between any two of the following sets:

(i) the isomorphism classes of $q$-geometric tridiagonal pairs;
(ii) the isomorphism classes of NonNil finite-dimensional irreducible $U_q^+$-modules of type $(1,1)$;
(iii) the isomorphism classes of finite-dimensional irreducible $U_q(L(sl_2))$-modules $V$ of type 1 such that $P_V(1) \neq 0$, where $P_V$ is the Drinfel’d polynomial of $V$;
(iv) the isomorphism classes of finite-dimensional irreducible $\mathbb{E}_q$-modules of type 1;
(v) the polynomials in $\mathbb{F}[z]$ that have constant coefficient 1 and do not vanish at $z = 1$.

The bijection between the sets (i), (ii) is established in [11, Lemma 4.8]. The bijection between the sets (ii), (iii) is established in [11, Theorems 1.6, 1.7] by using the map $U_q^+ \rightarrow U_q(L(sl_2))$ in the chain (1.1). The bijection between the sets (ii), (iv) is established in [8, Theorems 10.3, 10.4] by using the map $U_q^+ \rightarrow \mathbb{E}_q$ in the chain (1.1). The bijection between the sets (iii), (v) is established in [2, p. 261].

In [21] P. Terwilliger introduced a variation of $\mathbb{E}_q$ called $\square_q$. By [18, Propositions 4.1, 4.3] and [21, Proposition 5.5], the chain (1.1) has an extension

$$U_q^+ \rightarrow \square_q \rightarrow U_q(L(sl_2)) \rightarrow \mathbb{E}_q.$$ (1.2)

In the above chain each map is injective.

In the present paper we focus on the finite-dimensional irreducible $\square_q$-modules. We have two main results. Our first main result is about how the isomorphism classes of finite-dimensional irreducible $\square_q$-modules are related to the sets (i)–(v). We identify a class of finite-dimensional irreducible $\square_q$-modules said to have type 1. Using the chain (1.2) we show that the above sets (i)–(v) are in bijection with

(vi) the isomorphism classes of finite-dimensional irreducible $\square_q$-modules of type 1.

Our second main result is about what happens when we twist a finite-dimensional irreducible $\square_q$-module of type 1, via a certain automorphism $\rho$ of $\square_q$ that has order 4.

We next describe our main results in more detail. We begin with some notation and basic concepts. Recall the ring of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4. Let $\mathcal{A}$ denote an $\mathbb{F}$-algebra and let $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ denote an $\mathbb{F}$-algebra homomorphism. Let $V$ denote an $\mathcal{A}$-module. There exists an $\mathcal{A}$-module structure on $V$, called $V$ twisted via $\varphi$, that behaves as follows: for all $a \in \mathcal{A}$ and $v \in V$, the vector $av$ computed in $V$ twisted via $\varphi$ coincides with the vector $\varphi(a)v$ computed in the original $\mathcal{A}$-module $V$. Sometimes we abbreviate $\varphi V$ for $V$ twisted via $\varphi$. Let $z$ denote an indeterminate and let $\mathbb{F}[z]$ denote the $\mathbb{F}$-algebra consisting of the polynomials in $z$ that have all coefficients in $\mathbb{F}$. We now recall the algebra $\square_q$. 
Definition 1.1. (See [21, Definition 5.1].) Let $\square_q$ denote the $F$-algebra with generators $\{x_i\}_{i \in \mathbb{Z}_4}$ and relations

\begin{align*}
qx_i x_{i+1} - q^{-1} x_{i+1} x_i &= 1, \\
x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^3 &= 0,
\end{align*}

where $[3]_q = (q^3 - q^{-3})/(q - q^{-1})$.

Lemma 1.2. There exists an automorphism $\rho$ of $\square_q$ that sends $x_i \mapsto x_{i+1}$ for $i \in \mathbb{Z}_4$. Moreover $\rho^4 = 1$.

Proof. By Definition 1.1.

Lemma 1.3. For nonzero $\alpha \in F$ there exists an automorphism of $\square_q$ that sends $x_0 \mapsto \alpha^{-1} x_0$, $x_1 \mapsto \alpha x_1$, $x_2 \mapsto \alpha^{-1} x_2$, $x_3 \mapsto \alpha x_3$.

Proof. By Definition 1.1.

We have some comments about $\square_q$-modules. Let $V$ denote a finite-dimensional irreducible $\square_q$-module. We will show that each generator $x_i$ of $\square_q$ is semisimple on $V$. Moreover there exist an integer $d \geq 0$ and nonzero $\gamma \in F$ with the following property:

(i) for $x_0$ and $x_2$, the set of distinct eigenvalues on $V$ is $\{\gamma q^{d-2i}\}_{i=0}^d$;

(ii) for $x_1$ and $x_3$, the set of distinct eigenvalues on $V$ is $\{\gamma^{-1} q^{d-2i}\}_{i=0}^d$.

We call $d$ the diameter of $V$, and call $\gamma$ the type of $V$. Observe that the $\square_q$-module $\rho V$ has diameter $d$ and type $\gamma^{-1}$. Next we twist the $\square_q$-module $V$ via the automorphism from Lemma 1.3. The resulting $\square_q$-module has diameter $d$ and type $\gamma \alpha^{-1}$. In particular, if $\alpha = \gamma$ then this $\square_q$-module has type 1. Motivated by the above comments, we focus on the finite-dimensional irreducible $\square_q$-modules of type 1.

Let $V$ denote a finite-dimensional irreducible $\square_q$-module of type 1. We will associate to $V$ a polynomial $P_V \in F[z]$ that has constant coefficient 1. This $P_V$ depends on the isomorphism class of the $\square_q$-module $V$. We call $P_V$ the Drinfel’d polynomial of $V$. We will compare the Drinfel’d polynomials for $V$ and $\rho V$.

The significance of $P_V$ is indicated in the following proposition, which is our first main result.

Proposition 1.4. The map $V \mapsto P_V$ induces a bijection between the following two sets:

(i) the isomorphism classes of finite-dimensional irreducible $\square_q$-modules of type 1;

(ii) the polynomials in $F[z]$ that have constant coefficient 1 and do not vanish at $z = 1$.

Before stating our second main result, we have some comments.

Definition 1.5. Pick any $f \in F[z]$ with constant coefficient 1. Write

$$f = a_0 + a_1 z + \cdots + a_d z^d,$$

$a_0 = 1$, $a_d \neq 0$. 

For $0 \leq i \leq d$ replace $a_i$ by $a_{d-i}/a_d$ to get a polynomial

$$f^\vee = \frac{a_d + a_{d-1}z + \cdots + a_0z^d}{a_d}.$$  

Observe that $f^\vee$ has constant coefficient 1 and $(f^\vee)^\vee = f$. We say that $f, f^\vee$ are partners. Note that

$$f^\vee(z) = \frac{z^df(z^{-1})}{a_d}.$$  

Let $z_1, z_2, \ldots, z_d$ denote the roots of $f$. Then $z_1^{-1}, z_2^{-1}, \ldots, z_d^{-1}$ are the roots of $f^\vee$.

We now state our second main result.

**Theorem 1.6.** Let $V$ denote a finite-dimensional irreducible $\Box_q$-module of type 1. Then the Drinfel’d polynomials for $V$ and $\rho^2 V$ are partners.

**Corollary 1.7.** Let $V$ denote a finite-dimensional irreducible $\Box_q$-module of type 1. Then the $\Box_q$-modules $V$ and $\rho^2 V$ are isomorphic.

We now mention the idea behind our proof of Proposition 1.4. Using the map $U_q^+ \to \Box_q$ in the chain (1.2), we will establish a bijection between the isomorphism classes of finite-dimensional irreducible $\Box_q$-modules of type 1 and the isomorphism classes of NonNil finite-dimensional irreducible $U_q^+$-modules of type $(1, 1)$. In [11], the NonNil finite-dimensional irreducible $U_q^+$-modules of type $(1, 1)$ are classified using the Drinfel’d polynomial of $U_q(L(\mathfrak{sl}_2))$. Using these facts we establish the bijection in Proposition 1.4.

We now mention the idea behind our proof of Theorem 1.6. Let $V$ denote a finite-dimensional irreducible $\Box_q$-module of type 1. We will show that for $i \in \mathbb{Z}_4$ the $\Box_q$ generators $x_i, x_{i+2}$ act on $V$ as a $q$-geometric tridiagonal pair (see [11, Definition 2.6]). Using the Drinfel’d polynomials for these tridiagonal pairs (see [12]) together with some results from [2], we prove Theorem 1.6.

Given the earlier literature, Theorem 1.6 and Corollary 1.7 should not be surprising. Concerning Theorem 1.6 we mention one item [15, Lemma 9.11]. This is about the algebra $\mathbb{E}_q$. Let $V$ denote an evaluation module for $\mathbb{E}_q$ with evaluation parameter $t$ (see [15, Section 9]). Let $\rho$ denote the automorphism of $\mathbb{E}_q$ from [8, Lemma 6.3]. It was shown in [15, Lemma 9.11] that the $\mathbb{E}_q$-module $\rho^2 V$ is an evaluation module with evaluation parameter $t^{-1}$. Concerning Corollary 1.7 we mention one item [11, Theorem 2.5]. This is about $q$-geometric tridiagonal pairs. Let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension, and let $A, A^*$ denote a $q$-geometric tridiagonal pair on $V$. It was shown in [11, Theorem 2.5] that the tridiagonal pairs $A, A^*$ and $A^*, A$ are isomorphic.

The paper is organized as follows. Section 2 contains the preliminaries. Section 3 contains some basic facts about $U_q(L(\mathfrak{sl}_2))$. Section 4 contains some basic facts about Drinfel’d polynomials for $U_q(L(\mathfrak{sl}_2))$-modules. Section 5 contains some basic facts about $U_q^+$. In Sections 6–8 we describe the finite-dimensional irreducible $\Box_q$-modules and prove Proposition 1.4. Section 9 contains some basic facts about tridiagonal pairs and their Drinfel’d polynomial. In Section 10 we prove Theorem 1.6. In Section 11 we obtain an analogous result for $\mathbb{E}_q$. 

2. Preliminaries

We now begin our formal argument. Throughout this paper, every algebra without the Lie prefix is meant to be associative and have a 1. Let $A$ denote an $F$-algebra. By an automorphism of $A$ we mean an $F$-algebra isomorphism from $A$ to $A$. Let $B$ denote an $F$-algebra and $\varphi : A \to B$ denote an $F$-algebra homomorphism. For a $B$-module $V$, we pull back the $B$-module structure on $V$ via $\varphi$ and turn $V$ into an $A$-module. We call the $A$-module $V$ the pullback of the $B$-module $V$ via $\varphi$. If $\varphi$ is surjective, then the $B$-module $V$ is irreducible if and only if the $A$-module $V$ is irreducible. We mention a special case in which $A = B$. In this case the following are the same: (i) the pullback of the $B$-module $V$ via $\varphi$; (ii) the $B$-module $V$ twisted via $\varphi$.

Recall the set of natural numbers $N = \{0, 1, 2, \ldots \}$. For the duration of this paragraph fix $d \in N$. Let $V$ denote a vector space over $F$ with dimension $d + 1$. Let $\text{End}(V)$ denote the $F$-algebra consisting of the $F$-linear maps from $V$ to $V$. Let $I \in \text{End}(V)$ denote the identity map. Let $L \in \text{End}(V)$. The map $L$ is said to be nilpotent whenever there exists a positive integer $n$ such that $L^n = 0$. For $\theta \in F$ define

$$V_L(\theta) = \{ v \in V \mid Lv = \theta v \}.$$

Observe that $\theta$ is an eigenvalue of $L$ if and only if $V_L(\theta) \neq 0$; in this case $V_L(\theta)$ is the corresponding eigenspace. We say that $L$ is semisimple whenever $V$ is spanned by the eigenspaces of $L$. Let $\text{Mat}_{d+1}(F)$ denote the $F$-algebra consisting of the $d+1$ by $d+1$ matrices that have all entries in $F$. We index the rows and columns by $0, 1, \ldots, d$. Let $\{v_i\}_{i=0}^d$ denote a basis for $V$. For $A \in \text{End}(V)$ and $M \in \text{Mat}_{d+1}(F)$, we say that $M$ represents $A$ with respect to $\{v_i\}_{i=0}^d$ whenever $Av_j = \sum_{i=0}^d M_{ij}v_i$ for $0 \leq j \leq d$. For $n \in \mathbb{Z}$ define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

For $n \in \mathbb{N}$ define

$$[n]_q! = \prod_{i=1}^n [i]_q.$$

We interpret $[0]_q! = 1$.

3. The quantum loop algebra $U_q(L(\mathfrak{sl}_2))$

In this section, we recall the quantum loop algebra $U_q(L(\mathfrak{sl}_2))$. 
Definition 3.1. (See [2] Section 3.3.) Let $U_q(L(sl_2))$ denote the $F$-algebra with generators $e_i^\pm, K_i^{\pm 1}, i \in \{0, 1\}$ and relations

\[
K_i K_i^{-1} = K_i^{-1} K_i = 1, \\
K_0 K_1 = K_1 K_0 = 1, \\
K_i e_i^\pm K_i^{-1} = q^{\pm 2} e_i^\pm, \\
K_i e_j^\pm K_i^{-1} = q^{\mp 2} e_j^\pm, \quad i \neq j, \\
e_i^+ e_i^- - e_i^- e_i^+ = K_i - K_i^{-1} \\
e^+_0 e^-_0 = e^+_1 e^-_1, \\
(e^+_i)^3 e^+_j - [3]_q (e^+_i)^2 e^+_j + [3]_q e^+_i (e^+_j)^2 - e^+_j (e^+_i)^3 = 0, \quad i \neq j.
\]

We call $e_i^\pm, K_i^{\pm 1}, i \in \{0, 1\}$ the Chevalley generators for $U_q(L(sl_2))$.

In a moment we will discuss some objects $X_{ij}$. The subscripts $i, j$ are meant to be in $\mathbb{Z}_4$. We now recall the equitable presentation for $U_q(L(sl_2))$.

Lemma 3.2. (See [10] Theorem 2.1, [18] Proposition 4.2.) The $F$-algebra $U_q(L(sl_2))$ is isomorphic to the $F$-algebra with generators

\[
X_{01}, \ X_{12}, \ X_{23}, \ X_{30}, \ X_{13}, \ X_{31}
\]

and the following relations:

\[
X_{13} X_{31} = X_{31} X_{13} = 1, \\
\frac{q X_{01} X_{13} - q^{-1} X_{13} X_{01}}{q - q^{-1}} = 1, \quad \frac{q X_{13} X_{30} - q^{-1} X_{30} X_{13}}{q - q^{-1}} = 1, \\
\frac{q X_{23} X_{31} - q^{-1} X_{31} X_{23}}{q - q^{-1}} = 1, \quad \frac{q X_{31} X_{12} - q^{-1} X_{12} X_{31}}{q - q^{-1}} = 1;
\]

\[
\frac{q X_{i,i+1} X_{i+1,i+2} - q^{-1} X_{i+1,i+2} X_{i,i+1}}{q - q^{-1}} = 1 \quad (i \in \mathbb{Z}_4), \\
X_{i,i+1} X_{i+2,i+3} - [3]_q X_{i,i+1}^2 X_{i+2,i+3} + [3]_q X_{i,i+1} X_{i+2,i+3} X_{i,i+1}^2 - X_{i+2,i+3} X_{i,i+1}^3 = 0 \quad (i \in \mathbb{Z}_4).
\]

An isomorphism sends

\[
X_{01} \mapsto K_0 + q (q - q^{-1}) K_0 e^+_0, \quad X_{23} \mapsto K_1 + q (q - q^{-1}) K_1 e^-_1, \\
X_{12} \mapsto K_1 - (q - q^{-1}) e^+_1, \quad X_{30} \mapsto K_0 - (q - q^{-1}) e^-_0, \\
X_{13} \mapsto K_1, \quad X_{31} \mapsto K_0.
\]

The inverse isomorphism sends

\[
e^-_0 \mapsto q^{-1} (q - q^{-1})^{-1} (X_{13} X_{01} - 1), \quad e^-_1 \mapsto q^{-1} (q - q^{-1})^{-1} (X_{31} X_{23} - 1), \\
e^+_1 \mapsto (q - q^{-1})^{-1} (X_{13} - X_{12}), \quad e^+_0 \mapsto (q - q^{-1})^{-1} (X_{31} - X_{30}), \\
K_1 \mapsto X_{13}, \quad K_0 \mapsto X_{31}.
\]

Note 3.3. For notational convenience, we identify the copy of $U_q(L(sl_2))$ given in Definition 3.1 with the copy given in Lemma 3.2 via the isomorphism given in Lemma 3.2.
Lemma 3.5. (See [21, Proposition 5.5], [18, Propositions 4.1, 4.3].) The above homomorphism $\psi$ is injective.

We next recall some facts about finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-modules.

Let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\{s_i\}_{i=0}^d$ denote a finite sequence of positive integers whose sum is the dimension of $V$. By a decomposition of $V$ of shape $\{s_i\}_{i=0}^d$ we mean a sequence $\{U_i\}_{i=0}^d$ of subspaces of $V$ such that $U_i$ has dimension $s_i$ for $0 \leq i \leq d$, and the sum $V = \sum_{i=0}^d U_i$ is direct. We call $d$ the diameter of the decomposition. For $0 \leq i \leq d$ we call $U_i$ the $i$th component of the decomposition. For notational convenience define $U_{-1} = 0$ and $U_{d+1} = 0$. By the inversion of $\{U_i\}_{i=0}^d$ we mean the decomposition $\{U_{d-i}\}_{i=0}^d$.

Lemma 3.6. (See [2, Proposition 3.2].) Let $V$ denote a finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module. Then there exist a unique scalar $\gamma \in \{1, -1\}$ and a unique decomposition $\{U_i\}_{i=0}^d$ of $V$ such that

$$(K_0 - \gamma q^{2i-d} I)U_i = 0, \quad (K_1 - \gamma q^{d-2i} I)U_i = 0$$

for $0 \leq i \leq d$. Moreover, for $0 \leq i \leq d$ we have

$$e_i^0 U_i \subseteq U_{i+1}, \quad e_i^1 U_i \subseteq U_{i+1}, \quad e_i^- U_i \subseteq U_{i-1}, \quad e_i^+ U_i \subseteq U_{i-1}.$$

Definition 3.7. Let $V$ denote a finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module. By the diameter of $V$ we mean the scalar $d$ from Lemma 3.6. By the type of $V$ we mean the scalar $\gamma$ from Lemma 3.6. The sequence $\{U_i\}_{i=0}^d$ is the weight space decomposition of $V$ relative to $K_0$ and $K_1$.

Lemma 3.8. Let $V$ denote a finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module of type 1. Let $\{U_i\}_{i=0}^d$ denote the weight space decomposition of $V$ from Lemma 3.6. Then $U_0$ and $U_d$ have dimension 1.

Proof. By [11, Lemma 3.12].

We now give a detailed description of the finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-modules of type 1 and diameter 1.

Lemma 3.9. (See [2, Section 4].) There exists a family of finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-modules

$$(3.5) \quad V(1, a), \quad 0 \neq a \in \mathbb{F}$$

with the following property: $V(1, a)$ has a basis $v_0, v_1$ with respect to which the Chevalley generators are represented by the following matrices:

$$e_0^+ : \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}, \quad e_0^- : \begin{bmatrix} 0 & a^{-1} \\ 0 & 0 \end{bmatrix}, \quad K_0 : \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix},$$

$$e_1^+ : \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_1^- : \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad K_1 : \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}.$$
Moreover $V(1,a)$ has type 1 and diameter 1. Every finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module of type 1 and diameter 1 is isomorphic to exactly one of the modules $\mathcal{Y}_n$.

**Definition 3.10.** Referring to Lemma 3.9 we call $a$ the evaluation parameter of $V(1,a)$.

We now describe how the $U_q(L(\mathfrak{sl}_2))$-modules (3.5) look from the equitable point of view.

**Lemma 3.11.** Recall the basis $v_0, v_1$ of $V(1,a)$ from Lemma 3.9. With respect to this basis the equitable generators are represented by the following matrices:

$$X_{01} : \begin{bmatrix} q^{-1} & (q - q^{-1})a^{-1} \\ 0 & q \end{bmatrix}, \quad X_{23} : \begin{bmatrix} q & 0 \\ q^{-1} - q & q^{-1} \end{bmatrix},$$

$$X_{12} : \begin{bmatrix} q & q^{-1} - q \\ 0 & q^{-1} \end{bmatrix}, \quad X_{30} : \begin{bmatrix} q^{-1} & 0 \\ (q^{-1} - q)a & q \end{bmatrix},$$

$$X_{13} : \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}, \quad X_{31} : \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix}.$$

**Proof.** By Lemma 3.9.

**Lemma 3.12.** For $0 \neq a \in \mathbb{F}$, the following equations hold on the $U_q(L(\mathfrak{sl}_2))$-module $V(1,a)$ from Lemma 3.9

\begin{align*}
(3.6) \quad & \text{tr}(X_{01}X_{23}) = 2 + (q - q^{-1})^2a^{-1}, \\
(3.7) \quad & \text{tr}(X_{12}X_{30}) = 2 + (q - q^{-1})^2a,
\end{align*}

where tr denotes trace.

**Proof.** Use Lemma 3.11.

**Lemma 3.13.** For $0 \neq a \in \mathbb{F}$, let $v_0, v_1$ denote the basis of $V(1,a)$ from Lemma 3.9. Let $U_0, U_1$ denote the weight space decomposition of $V(1,a)$. Then $U_i$ is spanned by $v_i$ for $i = 0, 1$.

**Proof.** Use the matrices that represent $K_0$ and $K_1$ in Lemma 3.9.

## 4. The Drinfel’d Polynomial of a $U_q(L(\mathfrak{sl}_2))$-Module

In this section we recall the Drinfel’d polynomial of a finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module of type 1.

**Definition 4.1.** Let $V$ denote a finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module of type 1 and diameter $d$. Let $\{U_i\}_{i=0}^d$ denote the weight space decomposition of $V$ from Lemma 3.9. Pick $j \in \mathbb{N}$. By Lemmas 3.6 and 3.8 we see that $U_0$ is an eigenspace for $(e_1^+)^j(e_0^+)j^2$; let $\sigma_j$ denote the corresponding eigenvalue. Observe that $\sigma_0 = 1$, and $\sigma_j = 0$ if $j > d$.

**Definition 4.2.** (See [2] Section 3.5.) Let $V$ denote a finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module of type 1 and diameter $d$. We define a polynomial $P_V \in \mathbb{F}[z]$ by

$$P_V = \sum_{i=0}^d \frac{(-1)^i \sigma_i z^i}{([i]_q)_{z^2}},$$

where the scalars $\sigma_i$ are from Definition 4.1. Observe that $P_V$ has constant coefficient 1. We call $P_V$ the Drinfel’d polynomial of $V$. 

The polynomial $P_V$ has the following property.

**Theorem 4.3.** (See [2, p. 261].) The map $V \mapsto P_V$ induces a bijection between the following two sets:

(i) the isomorphism classes of finite-dimensional irreducible $U_q(L(s\mathfrak{l}_2))$-modules of type 1;

(ii) the polynomials in $\mathbb{F}[z]$ that have constant coefficient 1.

We have been discussing the Drinfel’d polynomial $P_V$. We now mention a related polynomial $Q_V$.

**Definition 4.4.** Let $V$ denote a finite-dimensional irreducible $U_q(L(s\mathfrak{l}_2))$-module of type 1 and diameter $d$. Let $\{U_i\}_{i=0}^d$ denote the weight space decomposition of $V$ from Lemma 3.9. Pick $j \in \mathbb{N}$. By Lemmas 3.6 and 3.8 we see that $U_0$ is an eigenspace for $(e_0)^j(e_1)^j$; let $\mu_j$ denote the corresponding eigenvalue. Observe $\mu_0 = 1$, and $\mu_j = 0$ if $j > d$.

**Definition 4.5.** (See [2, Section 3.5].) Let $V$ denote a finite-dimensional irreducible $U_q(L(s\mathfrak{l}_2))$-module of type 1 and diameter $d$. We define a polynomials $Q_V \in \mathbb{F}[z]$ by

$$Q_V = \sum_{i=0}^d \frac{(-1)^i \mu_i z^i}{(|i|_q)^2},$$

where the scalars $\mu_i$ are from Definition 4.4. Observe that $Q_V$ have constant coefficient 1.

**Lemma 4.6.** (See [2, p. 269].) The polynomials $P_V$ and $Q_V$ are partners in the sense of Definition 4.3.

**Note 4.7.** The polynomials $P, Q$ in [2, Section 3.5] are related to our polynomials $P_V, Q_V$ as follows: $P_V(z) = P(q^{-1}z)/P(0)$ and $Q_V(z) = Q(q^{-1}z)/Q(0)$. This is explained in [2, p. 268, 269].

We now compute $P_V$ and $Q_V$ for the case in which $V$ has type 1 and diameter 1.

**Lemma 4.8.** Pick $0 \neq a \in \mathbb{F}$. Let $V$ denote the $U_q(L(s\mathfrak{l}_2))$-module $V(1, a)$ from Lemma 3.9. Then $P_V(z) = 1 - az$ and $Q_V(z) = 1 - a^{-1}z$.

**Proof.** Using the matrices that represent $e_0^+, e_1^+$ in Lemma 3.9 along with Definition 4.4 we obtain $\sigma_1 = a$. Therefore $P_V(z) = 1 - az$ in view of Definition 4.3. Similarly using Lemma 3.9 and Definition 4.4 we obtain $\mu_1 = a^{-1}$. Therefore $Q_V(z) = 1 - a^{-1}z$ in view of Definition 4.5.

5. The positive part of $U_q(L(s\mathfrak{l}_2))$

In this section we recall a certain algebra $U_q^+$ called the positive part of $U_q(L(s\mathfrak{l}_2))$.

**Definition 5.1.** (See [11, Definition 1.1].) Let $U_q^+$ denote the $\mathbb{F}$-algebra with generators $x, y$ and relations

$$x^3y - [3]_q x^2yx + [3]_q xyx^2 - yx^3 = 0,$$

$$y^3x - [3]_q y^2xy + [3]_q yxy^2 - xy^3 = 0.$$

We call $x, y$ the standard generators for $U_q^+$. 

Remark 5.2. (See [17, Corollary 3.2.6].) There exists an injective \(\mathbb{F}\)-algebra homomorphism from \(U_q^+\) to \(U_q(L(\mathfrak{sl}_2))\) that sends \(x \mapsto e_0^+\) and \(y \mapsto e_1^+\). Consequently we call \(U_q^+\) the positive part of \(U_q(L(\mathfrak{sl}_2))\).

Comparing the relations in Definition 1.1 with the relations in Definition 5.1 we obtain an \(\mathbb{F}\)-algebra homomorphism \(\kappa : U_q^+ \to \Box_q\) that sends \(x \mapsto x_0\) and \(y \mapsto x_2\).

Lemma 5.3. (See [21, Proposition 5.5].) The above homomorphism \(\kappa\) is injective.

Recall the map \(\psi\) from above Lemma 3.5. Consider the composition (5.1) \(\psi \circ \kappa : U_q^+ \xrightarrow{\kappa} \Box_q \xrightarrow{\psi} U_q(L(\mathfrak{sl}_2)).\) Note that the map \(\psi \circ \kappa\) is different from the map in Remark 5.2. Let \(V\) denote a finite-dimensional irreducible \(U_q(L(\mathfrak{sl}_2))\)-module of type 1. Consider the pullback of \(V\) via \(\psi \circ \kappa\). We now describe the resulting \(U_q^+\)-module \(V\).

Lemma 5.4. (See [11, Proposition 12.1].) The above \(U_q^+\)-module \(V\) is irreducible if and only if \(P_V(1) \neq 0\).

Definition 5.5. (See [11, Definition 1.3].) Let \(V\) denote a finite-dimensional \(U_q^+\)-module. This module is called NonNil whenever the standard generators \(x, y\) are not nilpotent on \(V\).

Lemma 5.6. (See [11, Corollary 2.8].) Let \(V\) denote a NonNil finite-dimensional irreducible \(U_q^+\)-module. Then the standard generators \(x, y\) are semisimple on \(V\). Moreover there exist \(d \in \mathbb{N}\) and nonzero scalars \(\gamma, \gamma' \in \mathbb{F}\) such that the set of distinct eigenvalues of \(x\) (resp. \(y\)) on \(V\) is \(\{\gamma q^{d-2i}\}_{i=0}^d\) (resp. \(\{\gamma' q^{d-2i}\}_{i=0}^d\)).

Definition 5.7. (See [11, Definition 2.9].) Let \(V\) denote a NonNil finite-dimensional irreducible \(U_q^+\)-module. By the diameter of \(V\) we mean the scalar \(d\) from Lemma 5.6. By the type of \(V\) we mean the ordered pair \((\gamma, \gamma')\) from Lemma 5.6.

We will use the following notation.

Definition 5.8. Let \(A, B\) denote \(\mathbb{F}\)-algebras and \(\varphi : A \to B\) denote an \(\mathbb{F}\)-algebra homomorphism. Let \(\varphi^*\) denote the map that sends each \(B\)-module to its pullback via \(\varphi\).

Referring to (5.1) and Definition 5.8 note that (5.2) \((\psi \circ \kappa)^* = \kappa^* \circ \psi^*\).

In the next result we describe \((\psi \circ \kappa)^*\). In Section 8 we describe the maps \(\kappa^*, \psi^*\).

Theorem 5.9. (See [11, Theorems 1.6, 1.7].) The map \((\psi \circ \kappa)^*\) gives a bijection between the following two sets:

(i) the isomorphism classes of finite-dimensional irreducible \(U_q(L(\mathfrak{sl}_2))\)-modules \(V\) of type 1 such that \(P_V(1) \neq 0\);

(ii) the isomorphism classes of NonNil finite-dimensional irreducible \(U_q^+\)-modules of type \((1, 1)\).

Let \(V\) denote a NonNil finite-dimensional irreducible \(U_q^+\)-module of type \((1, 1)\). Via Theorem 5.9 the vector space \(V\) becomes a finite-dimensional irreducible \(U_q(L(\mathfrak{sl}_2))\)-module of type 1 such that \(P_V(1) \neq 0\).
Definition 5.10. Let $V$ denote a NonNil finite-dimensional irreducible $U_q^+$-module of type $(1, 1)$. Let $P_V$ (resp. $Q_V$) denote the polynomial from Definition 4.2 (resp. Definition 4.5) associated with the $U_q(L(sl_2))$-module $V$ from Theorem 5.9.

We now give a detailed description of the NonNil finite-dimensional irreducible $U_q^+$-modules of type $(1, 1)$ and diameter 1. Pick $0 \neq a \in \mathbb{F}$. Via Theorem 5.9 the $U_q(L(sl_2))$-module $V(1, a)$ from Lemma 5.9 becomes a NonNil $U_q^+$-module.

Lemma 5.11. The NonNil $U_q^+$-modules

\begin{equation}
V(1, a) \quad a \in \mathbb{F} \setminus \{0, 1\}
\end{equation}

are irreducible. With respect to the basis $v_0, v_1$ from Lemma 3.11, the standard generators $x, y$ are represented by the following matrices:

\begin{equation}
x : \begin{bmatrix}
q^{-1} & (q - q^{-1})a^{-1} \\
0 & q
\end{bmatrix}, \quad y : \begin{bmatrix}
q & 0 \\
q^{-1} & q^{-1}
\end{bmatrix}.
\end{equation}

Moreover $V(1, a)$ has type $(1, 1)$ and diameter 1. Every NonNil finite-dimensional irreducible $U_q^+$-module of type $(1, 1)$ and diameter 1 is isomorphic to exactly one of the modules (5.3).

Proof. Pick $0 \neq a \in \mathbb{F}$. By Lemma 4.8 and Lemma 5.4 the $U_q^+$-module $V(1, a)$ is irreducible if and only if $a \neq 1$. The matrices in (5.4) are obtained from Lemma 3.11 and Theorem 5.9. The remaining assertions follow by Lemma 5.9 and Theorem 5.9. \hfill \square

Lemma 5.12. Pick $a \in \mathbb{F} \setminus \{0, 1\}$. Let $V$ denote the $U_q^+$-module $V(1, a)$ from Lemma 5.11. Then $P_V(z) = 1 - az$ and $Q_V(z) = 1 - a^{-1}z$.

Proof. By Lemma 4.8 and Definition 5.10. \hfill \square

6. The type of a finite-dimensional irreducible $\Box_q$-module

In Theorem 5.9, we described the map $(\psi \circ \kappa)^\sharp$. We will describe the maps $\kappa^\sharp, \psi^\sharp$ in Section 9. The main result for $\kappa^\sharp$ (resp. $\psi^\sharp$) is Theorem 5.1 (resp. Theorem 5.3). In Sections 7, 8 we obtain the results needed to prove the above theorems.

Let $V$ denote a finite-dimensional irreducible $\Box_q$-module. Our next goal is to show that each generator $x_i$ of $\Box_q$ is semisimple on $V$, and we find its eigenvalues.

We begin with some results concerning the relation (1.3).

Lemma 6.1. (See [8] Lemma 11.2.) Let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $C, D \in \text{End}(V)$. Then for all nonzero $\theta \in \mathbb{F}$ the following are equivalent:

(i) The expression $qCD - q^{-1}DC - (q - q^{-1})I$ vanishes on $V_C(\theta)$;
(ii) $(D - \theta^{-1}I)V_C(\theta) \subseteq V_C(q^{-2}\theta)$.

Lemma 6.2. (See [8] Lemma 11.3.) Let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $C, D \in \text{End}(V)$. Then for all nonzero $\theta \in \mathbb{F}$ the following are equivalent:

(i) The expression $qCD - q^{-1}DC - (q - q^{-1})I$ vanishes on $V_D(\theta)$;
(ii) $(C - \theta^{-1}I)V_D(\theta) \subseteq V_D(q^2\theta)$. 

Lemma 6.3. (See [8] Lemma 11.4.) Let $V$ denote a vector space over $F$ with finite positive dimension. Let $C, D \in \text{End}(V)$ such that
\[
\frac{qCD - q^{-1}DC}{q - q^{-1}} = I.
\]
Then for all nonzero $\theta \in F$,
\[
\sum_{n=0}^{\infty} V_C(q^{-2n}\theta) = \sum_{n=0}^{\infty} V_D(q^{2n}\theta^{-1}).
\]

Lemma 6.4. (See [8] Lemma 11.5.) With the notation and assumptions of Lemma 6.3
\[
\dim V_C(\theta) = \dim V_D(\theta^{-1}).
\]

Lemma 6.5. Let $V$ denote a finite-dimensional irreducible $\Box_q$-module. Then for all nonzero $\theta \in F$ we have
\[
\dim V_{x_0}(\theta) = \dim V_{x_1}(\theta^{-1}) = \dim V_{x_2}(\theta) = \dim V_{x_3}(\theta^{-1}).
\]
Proof. By (6.1) we have
\[
 x_0 \to x_1 \to x_2 \to x_3
\]
where $r \to s$ means
\[
\frac{qrs - q^{-1}sr}{q - q^{-1}} = 1.
\]
Applying Lemma 6.4 to each arrow in (6.1) we obtain the result. \(
\)

We mention a result concerning the relation (1.4).

Lemma 6.6. (See [8] Lemma 11.1.) Let $V$ denote a finite-dimensional vector space over $F$. Let $C, D \in \text{End}(V)$. Then for all nonzero $\theta \in F$ the following are equivalent:
(i) The expression $C^3D - [3]qC^2DC + [3]qCDC^2 - DC^3$ vanishes on $V_C(\theta)$;
(ii) $DV_C(\theta) \subseteq V_C(q^2\theta) + V_C(\theta) + V_C(q^{-2}\theta)$.

Lemma 6.7. Let $V$ denote a finite-dimensional irreducible $\Box_q$-module. Then each generator $x_i$ of $\Box_q$ is semisimple on $V$. Moreover there exist $d \in \mathbb{N}$ and nonzero $\gamma \in F$ with the following property:
(i) for $x_0$ and $x_2$, the set of distinct eigenvalues on $V$ is $\{\gamma q^{d-2i} \}_{i=0}^{d};$
(ii) for $x_1$ and $x_3$, the set of distinct eigenvalues on $V$ is $\{\gamma^{-1} q^{d-2i} \}_{i=0}^{d}.$

Proof. Since $F$ is algebraically closed and $V$ has finite positive dimension, there exists $c \in F$ such that $V_{x_0}(c) \neq 0$. By [22] Proposition 5.2, the action of $x_0$ is invertible on $V$. Therefore 0 is not an eigenvalue of $x_0$ on $V$, so $c \neq 0$. By this and since $q$ is not a root of 1, the scalars $c, q^2c, q^4c, \ldots$ are mutually distinct. Consequently these scalars can not all be eigenvalues for $x_0$ on $V$. Therefore there exists a nonzero $\tau \in F$ such that $V_{x_0}(\tau) \neq 0$ and $V_{x_0}(q^2\tau) = 0$. Moreover there exists $d \in \mathbb{N}$ such that $V_{x_0}(q^{-2n}\tau)$ is nonzero for $0 \leq n \leq d$ and zero for $n = d + 1$. We will show that
\[
V_{x_0}(\tau) + V_{x_0}(q^{-2}\tau) + \cdots + V_{x_0}(q^{-2d}\tau)
\]
is equal to $V$. Our strategy is to show that the subspace (6.2) is a $\Box_q$-submodule of $V$. By construction (6.2) is invariant under $x_0$. By Lemma 6.1 and since
Let \( q \) denote a vector space over \( F \) with finite positive dimension. Let \( \{s_i\}_{i=0}^{d} \) denote a sequence of positive integers whose sum is the dimension of \( V \). By a flag on \( V \) of shape \( \{s_i\}_{i=0}^{d} \), we mean a nested sequence \( U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d \) of subspaces of \( V \) such that the dimension of \( U_i \) is \( s_0 + s_1 + \cdots + s_i \) for \( 0 \leq i \leq d \). Observe that \( U_d = V \). We call \( U_i \) the \( i \)th component of the flag. We call \( d \) the diameter of the flag.

The following construction yields a flag on \( V \). Let \( \{V_i\}_{i=0}^{d} \) denote a decomposition of \( V \) with shape \( \{s_i\}_{i=0}^{d} \). Define

\[
U_i = V_0 + V_1 + \cdots + V_i \quad (0 \leq i \leq d).
\]

Then the sequence \( U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d \) is a flag on \( V \) of shape \( \{s_i\}_{i=0}^{d} \). We say this flag is induced by the decomposition \( \{V_i\}_{i=0}^{d} \).

We now recall what it means for two flags to be opposite. Suppose we are given two flags on \( V \) with the same diameter: \( U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d \) and \( U'_0 \subseteq U'_1 \subseteq \cdots \subseteq U'_d \). We say these flags are opposite whenever there exists a decomposition
The decomposition \( \{ V_i \}_{i=0}^d \) of \( V \) that induces \( U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d \) and its inversion \( \{ V_{d-i} \}_{i=0}^d \) induces \( U_0' \subseteq U_1' \subseteq \cdots \subseteq U_d' \). In this case
\[
U_i \cap U_j' = 0 \quad (0 \leq i, j \leq d, \ i + j < d)
\]
and
\[
V_n = U_n \cap U_{d-n}' \quad (0 \leq n \leq d).
\]
The decomposition \( \{ V_i \}_{i=0}^d \) is uniquely determined by the given flags.

**Theorem 7.1.** Let \( V \) denote a finite-dimensional irreducible \( \Box_q \)-module of type 1. Then there exists a collection of flags on \( V \), denoted \([h], h \in \mathbb{Z}_4\), that have the following property: for each generator \( x_i \) of \( \Box_q \) the decomposition \([i, i+1]\) of \( V \) induces \([i]\) and its inversion \([i+1, i]\) induces \([i+1]\).

**Proof.** For all \( h \in \mathbb{Z}_4 \) let \([h]\) denote the flag on \( V \) induced by the decomposition \([h, h+1]\). By Lemma \( \ref{lem:flags} \) (with \( C = x_{h-1} \) and \( D = x_h \)) the flag on \( V \) induced by \([h, h+1]\) is equal to \([h]\). The result follows. \( \square \)

**Theorem 7.2.** Let \( V \) denote a finite-dimensional irreducible \( \Box_q \)-module of type 1. Then for \( i \in \mathbb{Z}_4 \) the flags \([i],[i+1]\) are opposite.

**Proof.** We invoke Theorem \( \ref{thm:opposite} \). The flags \([i],[i+1]\) are opposite since the decomposition \([i, i+1]\) induces \([i]\) and its inversion \([i+1, i]\) induces \([i+1]\). \( \square \)

**Note 7.3.** Theorem \( \ref{thm:opposite} \) can be strengthened as follows: the flags \([i],[i]\) are mutually opposite. This follows from Theorem \( \ref{thm:opposite} \) and \( \ref{thm:opposite} \). We don’t discuss the details since we do not need this fact.

**Theorem 7.4.** Let \( V \) denote a finite-dimensional irreducible \( \Box_q \)-module of type 1 and diameter \( d \). Pick a generator \( x_i \) of \( \Box_q \) and consider the corresponding decomposition \([i, i+1]\) of \( V \) from Definition \( \ref{def:decomposition} \). For \( 0 \leq n \leq d \) the \( n \)th component of \([i, i+1]\) is the intersection of the following two sets:
\[
\begin{align*}
(i) \quad & \text{component } n \text{ of the flag } [i]; \\
(ii) \quad & \text{component } d-n \text{ of the flag } [i+1];
\end{align*}
\]

**Proof.** By Theorem \( \ref{thm:opposite} \) and \( \ref{thm:opposite} \). \( \square \)

8. The maps \( \kappa^\sharp, \psi^\sharp \)

In Theorem \( \ref{thm:kappa} \) we described the map \( (\psi \circ \kappa)^\sharp \). In this section we describe the maps \( \kappa^\sharp, \psi^\sharp \). We first show that \( \kappa^\sharp \) gives a bijection between the isomorphism classes of finite-dimensional irreducible \( \Box_q \)-modules of type 1 and the isomorphism classes of NonNil finite-dimensional irreducible \( U_q^{\pm 1} \)-modules of type \((1, 1)\). Then we show that the map \( \psi^\sharp \) gives a bijection between the isomorphism classes of finite-dimensional irreducible \( U_q(L(\mathfrak{sl}_2)) \)-modules \( V \) of type 1 such that \( P_V(1) \neq 0 \) and the isomorphism classes of finite-dimensional irreducible \( \Box_q \)-modules of type 1. After establishing these results, we define the Drinfel’d polynomial of a finite-dimensional irreducible \( \Box_q \)-module of type 1 and prove Proposition \( \ref{prop:kappa} \).

**Proposition 8.1.** Let \( V \) denote a finite-dimensional irreducible \( \Box_q \)-module of type 1. Let \( W \) denote a nonzero subspace of \( V \) such that \( x_0 W \subseteq W \) and \( x_2 W \subseteq W \). Then \( W = V \).
Proof. Without loss we may assume that \( W \) is irreducible as a module for \( x_0, x_2 \). Let \( \{V_i\}_{i=0}^{d} \) denote the decomposition \([0, 1]\) of \( V \) and let \( \{V'_{i}\}_{i=0}^{d} \) denote the the decomposition \([2, 3]\) of \( V \). Recall that \( x_0 \) (resp. \( x_2 \)) is semisimple on \( V \) with eigenspaces \( \{V_i\}_{i=0}^{d} \) (resp. \( \{V'_{i}\}_{i=0}^{d} \)). By this and since \( W \) is invariant under each of \( x_0, x_2 \) we find

\[
W = \sum_{n=0}^{d} W \cap V_n, \quad W = \sum_{n=0}^{d} W \cap V'_n.
\]

Define

\[
(8.1) \quad m = \min\{n \mid 0 \leq n \leq d, \quad W \cap V_n \neq \emptyset\},
\]

\[
(8.2) \quad m' = \min\{n \mid 0 \leq n \leq d, \quad W \cap V'_n \neq \emptyset\}.
\]

We claim that \( m = m' \). To prove this claim, we first show that \( m \leq m' \). Suppose \( m > m' \). By (8.2) and the equation on the left in (8.1), the space \( W \) is contained in component \( d - m \) of the flag \([1]\). By construction \( W \) has nonzero intersection with component \( m' \) of the flag \([2]\). By Theorem 7.2 (with \( i = 1 \) and \( m > m' \), the component \( d - m \) of \([1]\) has zero intersection with component \( m' \) of \([2]\), which is a contradiction. Therefore \( m \leq m' \). Applying the above argument to \( \partial^2 V \) we obtain \( m' \leq m \). We have shown \( m \leq m' \leq m \). Therefore the claim is proved. The claim implies that the component \( d - m \) of the flag \([1]\) contains \( W \), the component \( d - m \) of the flag \([3]\) contains \( W \), the component \( m \) of the flag \([0]\) has nonzero intersection with \( W \), and the component \( m \) of the flag \([2]\) has nonzero intersection with \( W \). We can now easily show \( W = V \). Since the \( \square_q \)-module \( V \) is irreducible, it suffices to show that \( W \) is invariant under each generator \( x_i \) of \( \square_q \). By construction \( W \) is invariant under \( x_0 \) and \( x_2 \). Let \( x_r \) denote one of \( x_1, x_3 \). We show \( x_r W \subseteq W \). Let \( W' \) denote the span of the set of vectors in \( W \) that are eigenvectors for \( x_r \). By construction \( W' \subseteq W \) and \( x_r W' \subseteq W' \). We show \( W' = W \). To this end we show that \( W' \) is nonzero and invariant under each of \( x_0, x_2 \). We now show \( W' \neq 0 \). By Theorem 7.4 the intersection of the component \( d - m \) of the flag \([1]\) and the component \( m \) of the flag \([2]\) is equal to component \( d - m \) of the decomposition \([1, 2]\) which is an eigenspace for \( x_1 \). By this and the comment after the preliminary claim we have \( W' \neq 0 \) for \( x_r = x_1 \). Similarly we show \( W' \neq 0 \) for \( x_r = x_3 \). We now show \( x_0 W' \subseteq W' \). To this end we pick \( v \in W' \) and show \( x_0 v \in W' \). Without loss we may assume that \( v \) is a nonzero and invariant under each of \( x_0, x_2 \). By Theorem 7.4 the vector \( x_0 v \) is contained in an eigenspace \( x_r \), so \( (x_0 - \theta^{-1} I) v \in W' \). By this and since \( v \in W' \) we have \( x_0 v \in W' \). We have shown \( x_0 W' \subseteq W' \). Similarly we show \( x_2 W' \subseteq W' \). So far we have shown that \( W' \) is nonzero and invariant under each of \( x_0, x_2 \). By the irreducibility of \( W \) we have \( W' = W \), so \( x_r W \subseteq W \). It follows that \( W \) is \( \square_q \)-invariant. The space \( V \) is irreducible as a \( \square_q \)-module so \( W = V \).

Recall the map \( \kappa \) from Lemma 5.3. Let \( V \) denote a finite-dimensional irreducible \( \square_q \)-module of type 1. Pulling back the \( \square_q \)-module structure on \( V \) via \( \kappa \), we turn \( V \) into a \( U_q^+ \)-module.

Lemma 8.2. The above \( U_q^+ \)-module \( V \) is NonNil and irreducible of type \((1, 1)\).
Proof. The $U^+_q$-module $V$ is irreducible by Proposition 8.1. By Lemma 6.7 and the construction, for each generator $x, y$ of $U^+_q$ the action on $V$ is semisimple with eigenvalues $\{q^{d-2n} \mid 0 \leq n \leq d\}$. Therefore the $U^+_q$-module $V$ is NonNil and type $(1, 1)$. The result follows. □

Lemma 8.3. Let $V$ denote a NonNil finite-dimensional irreducible $U^+_q$-module of type $(1, 1)$. Then there exists a unique $\Box_q$-module structure on $V$ such that the standard generators $x$ and $y$ act as $x_0$ and $x_2$ respectively. This $\Box_q$-module is irreducible of type 1.

Proof. By [8, Theorem 10.4], there exists a $\Box_q$-module $V$ such that the standard generators $x$ and $y$ act as $x_0$ and $x_2$ respectively. By this and since the $U^+_q$-module $V$ is irreducible, the $\Box_q$-module $V$ is irreducible. By Lemma 6.7 and since the $U^+_q$-module $V$ is of type (1, 1), the $\Box_q$-module $V$ is of type 1. It suffices to show this $\Box_q$-module structure on $V$ is unique. For each generator $x_i$ of $\Box_q$ the action on $V$ is determined by the decomposition $[i, i+1]$. By Theorem 7.4 the decomposition $[i, i+1]$ is determined by the flags $[i]$ and $[i+1]$. Therefore our $\Box_q$-module structure on $V$ is determined by the flags $[h], h \in \mathbb{Z}_4$. By construction the flags $[0]$ and $[1]$ are determined by the decomposition $[0, 1]$ and hence by the action of $x$ on $V$. Similarly the flags $[2]$ and $[3]$ are determined by the decomposition $[2, 3]$ and hence by the action of $y$ on $V$. Therefore the given $\Box_q$-module structure on $V$ is determined by the action of $x$ and $y$ on $V$, so this $\Box_q$-module structure is unique. The result follows. □

Theorem 8.4. The map $\kappa$ gives a bijection between the following two sets:

(i) the isomorphism classes of finite-dimensional irreducible $\Box_q$-modules of type 1;

(ii) the isomorphism classes of NonNil finite-dimensional irreducible $U^+_q$-modules of type $(1, 1)$.

Proof. By Lemmas 8.2 and 8.3. □

Recall the map $\psi$ from above Lemma 3.5.

Theorem 8.5. The map $\psi$ gives a bijection between the following two sets:

(i) the isomorphism classes of finite-dimensional irreducible $U_q(L(sl_2))$-modules $V$ of type 1 such that $P_V(1) \neq 0$;

(ii) the isomorphism classes of finite-dimensional irreducible $\Box_q$-modules of type 1.

Proof. The result follows from 5.2 along with Theorems 5.9 and 8.4. □

Let $V$ denote a finite-dimensional irreducible $\Box_q$-module of type 1. Via Theorem 8.5 the vector space $V$ becomes a finite-dimensional irreducible $U_q(L(sl_2))$-module of type 1 such that $P_V(1) \neq 0$. Via Theorem 8.4 the vector space $V$ becomes a NonNil finite-dimensional irreducible $U^+_q$-module of type $(1, 1)$.

Definition 8.6. Let $V$ denote a finite-dimensional irreducible $\Box_q$-module of type 1. Let $P_V$ (resp. $Q_V$) denote the polynomial from Definition 4.2 (resp. Definition 4.5) associated with the $U_q(L(sl_2))$-module $V$ from Theorem 8.5. Observe that the polynomial $P_V$ (resp. $Q_V$) is equal to the polynomial from Definition 6.10 associated with the $U^+_q$-module $V$ from Theorem 8.4.
Proof of Proposition 1.4. By Theorems 4.3 and 8.5.

We now give a detailed description of the finite-dimensional irreducible $\mathbb{C}_q$-modules of type 1 and diameter 1. Pick $0 \neq a \in \mathbb{F}$. Via Theorem 8.5 the $U_q(L(\mathfrak{sl}_2))$-module $V(1,a)$ from Lemma 3.9 becomes a $\mathbb{C}_q$-module.

Lemma 8.7. The $\mathbb{C}_q$-modules

\begin{equation}
V(1,a) \quad a \in \mathbb{F}\backslash\{0,1\}
\end{equation}

are irreducible. With respect to the basis $v_0, v_1$ from Lemma 3.9 the $\mathbb{C}_q$ generators $\{x_i\}_{i \in \mathbb{Z}_4}$ are represented by the following matrices:

\begin{align}
  x_0 & : \begin{bmatrix} q^{-1} & (q^{-1}a^{-1}) \\ 0 & q \end{bmatrix}, \\
  x_1 & : \begin{bmatrix} q & q^{-1} - q \\ 0 & q^{-1} \end{bmatrix}, \\
  x_2 & : \begin{bmatrix} q & 0 \\ q^{-1} & q^{-1} \end{bmatrix}, \\
  x_3 & : \begin{bmatrix} q^{-1} & 0 \\ (q^{-1} - q)a & q \end{bmatrix}.
\end{align}

Moreover $V(1,a)$ has type 1 and diameter 1. Every finite-dimensional irreducible $\mathbb{C}_q$-module of type 1 and diameter 1 is isomorphic to exactly one of the modules (8.3).

Proof. Pick $0 \neq a \in \mathbb{F}$. By Lemma 5.11 and Theorem 8.4 the $\mathbb{C}_q$-module $V(1,a)$ is irreducible if and only if $a \neq 1$. The matrices in (8.4) are obtained from Lemma 3.11 and Theorem 8.5. The remaining assertions follow by Lemma 3.9 and Theorem 8.5.

Lemma 8.8. Pick $a \in \mathbb{F}\backslash\{0,1\}$. Let $V$ denote the $\mathbb{C}_q$-module $V(1,a)$ from Lemma 8.7. Then $P_V(z) = 1 - az$ and $Q_V(z) = 1 - a^{-1}z$.

Proof. By Lemma 4.8 and Definition 8.6.

9. Tridiagonal pairs

Our next general goal is to prove Theorem 1.6. To do this, it is convenient to bring in the notion of a tridiagonal pair. In this section we recall the definition of a tridiagonal pair and review its basic properties. We are mainly interested in a family of tridiagonal pairs said to have $q$-geometric type. Let $V$ denote a finite-dimensional irreducible $\mathbb{C}_q$-module of type 1. Near the end of this section we show that for $i \in \mathbb{Z}_4$ the $\mathbb{C}_q$ generators $x_i, x_{i+2}$ act on $V$ as a $q$-geometric tridiagonal pair.

From now until the end of Theorem 9.11 let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension.

Definition 9.1. (See [7, Definition 1.1].) By a tridiagonal pair on $V$ we mean an ordered pair $A, A^*$ of elements in $\text{End}(V)$ that satisfy (i)–(iv) below:

(i) Each of $A, A^*$ is semisimple.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that

\begin{equation}
A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),
\end{equation}

where $V_{-1} = 0, V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of $A^*$ such that

\begin{equation}
A V^*_i \subseteq V^*_{i-1} + V^*_i + V^*_{i+1} \quad (0 \leq i \leq \delta),
\end{equation}

where $V^*_{-1} = 0, V^*_{\delta+1} = 0$.
(iv) There does not exist a subspace $W$ of $V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

We say the tridiagonal pair $A, A^*$ is over $F$.

**Note 9.2.** According to a common notational convention, $A^*$ denotes the conjugate transpose of $A$. We are not using this convention. In a tridiagonal pair $A, A^*$ the linear transformations $A$ and $A^*$ are arbitrary subject to (i)–(iv) above.

We recall a few basic facts about tridiagonal pairs. Let $A, A^*$ denote a tridiagonal pair on $V$ and let $d, \delta$ be as in Definition 9.1. By Lemma 4.5 we have $d = \delta$; we call this common value the diameter of $A, A^*$. By Corollary 5.7, for $0 \leq i \leq d$ the subspaces $V_i, V^*_i$ have the same dimension; we denote this common dimension by $\rho_i$. By the construction $\rho_i \neq 0$. By Corollary 5.7, 6.6, the sequence \( \{\rho_i\}_{i=0}^d \) is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq d/2$. We call the sequence $\{\rho_i\}_{i=0}^d$ the shape of $A, A^*$. The pair $A, A^*$ is said to be sharp whenever $\rho_0 = 1$. By Theorem 1.3 and since $F$ is algebraically closed, the tridiagonal pair $A, A^*$ is sharp. An ordering of the eigenvalues of $A$ is called standard whenever it satisfies (9.1). We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenvalues of $A$. Then the ordering $\{V_{d-i}\}_{i=0}^d$ is standard and no further ordering is standard. An ordering of the eigenvalues of $A$ is called standard whenever the corresponding ordering of the eigenvalues of $A$ is standard. Similar comments apply to $A^*$. Let $\{V_i\}_{i=0}^d$ (resp. $\{V^*_i\}_{i=0}^d$) denote a standard ordering of the eigenvalues of $A$ (resp. $A^*$). For $0 \leq i \leq d$ let $\theta_i$ (resp. $\theta^*_i$) denote the eigenvalue of $A$ (resp. $A^*$) that corresponds to the eigenspace $V_i$ (resp. $V^*_i$). By construction the orderings $\{\theta_i\}_{i=0}^d$ and $\{\theta^*_i\}_{i=0}^d$ are standard. By Theorem 4.6, for $0 \leq i \leq d$ the subspace $V^*_i$ is invariant under

$$ (A^* - \theta^*_i I)(A^* - \theta^*_i I) \ldots (A^* - \theta^*_i I)(A - \theta_i I)(A - \theta_i I) \ldots (A - \theta_0 I). $$

Let $\zeta_i$ denote the corresponding eigenvalue. Note that $\zeta_0 = 1$. We call the sequence $\{\zeta_i\}_{i=0}^d$ the split sequence of $A, A^*$ with respect to the ordering $\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d$.

In the tridiagonal pairs over $F$ are classified up to isomorphism. In this classification there are several cases; the "most general" case is called type I. We will recall the type I case shortly.

**Lemma 9.3.** (See Theorem 11.1.) Let $A, A^*$ denote a tridiagonal pair on $V$. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) denote a standard ordering of the eigenvalues of $A$ (resp. $A^*$). Then the expressions

$$ (9.2) \quad \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} $$

are equal and independent of $i$ for $2 \leq i \leq d - 1$.

**Definition 9.4.** (See Definition 4.3.) Let $A, A^*$ denote a tridiagonal pair on $V$. We associate with $A, A^*$ a scalar $\beta \in F$ as follows. If the diameter $d \geq 3$ let $\beta + 1$ denote the common value of (9.2). If $d \leq 2$ let $\beta$ denote any scalar in $F$. We call $\beta$ a base of $A, A^*$. By construction, for $d \geq 3$ the tridiagonal pairs $A, A^*$ and $A^*, A$ have the same base. For $d \leq 2$, we always chose the bases such that $A, A^*$ and $A^*, A$ have the same base.

**Definition 9.5.** Let $A, A^*$ denote a tridiagonal pair on $V$. We say that the base of $A, A^*$ has type I whenever this base is not equal to 2 or $-2$. 

Lemma 9.6. (See [7] Theorem 11.2.) Let $A, A^*$ denote a tridiagonal pair on $V$ that has diameter $d$. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) denote a standard ordering of the eigenvalues of $A$ (resp. $A^*$). Let $\beta$ denote a base of $A, A^*$ and assume that $\beta$ has type I. Fix a nonzero $t \in \mathbb{F}$ such that $t^2 + t^{-2} = \beta$. Then there exists a sequence of scalars $a, b, c, a^*, b^*, c^*$ such that
\[
\theta_i = a + b t^{2i-d} + c t^{d-2i},
\]
\[
\theta_i^* = a^* + b^* t^{2i-d} + c^* t^{d-2i}
\]
for $0 \leq i \leq d$. This sequence is uniquely determined by $t$ provided $d \geq 2$.

With reference to Lemma 9.6 we have $t^4 \neq 1$ since $t^2 + t^{-2} = \beta$ and $\beta \neq \pm 2$. More generally, we have the following result.

Lemma 9.7. (See [7] Theorem 11.2.) With reference to Lemma 9.6 we have $t^{2i} \neq 1$ for $1 \leq i \leq d$.

We now recall from [12] the Drinfel’d polynomial for a tridiagonal pair. We will restrict our attention to the case of type I. Until the end of Lemma 9.9 assume that $d \geq 2$. With reference to Lemma 9.6 for $1 \leq i \leq d$ define $p_i \in \mathbb{F}[z]$ by
\[
p_i = (t^i - t^{-i})^2 (b b^* t^{2i-2d} + c c^* t^{2d-2i} - z).
\]
Define $P_{A, A^*} \in \mathbb{F}[z]$ by
\[
P_{A, A^*} = e \sum_{i=0}^d \zeta_i p_{i+1} p_{i+2} \cdots p_d,
\]
where $e = (-1)^d |\{d_i\}|^{-2} (t - t^{-1})^{-2d}$ and $\{\zeta_i\}_{i=0}^d$ is the split sequence of $A, A^*$ with respect to the ordering $\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d$. By [12] Theorem 12.12, the polynomial $P_{A, A^*}$ is independent of the choice of the standard orderings for the eigenvalues of $A$ and $A^*$. We call $P_{A, A^*}$ the Drinfel’d polynomial for the tridiagonal pair $A, A^*$.

Note 9.8. Our above definition of the Drinfel’d polynomial for the tridiagonal pair $A, A^*$ is slightly different from the one in [12] Definition 9.3]. The Drinfel’d polynomial for $A, A^*$ in [12] Definition 9.3] is equal to the right-hand side of (9.3) except that the factor $e$ is missing.

Lemma 9.9. (See [12] Theorem 12.12.) Let $A, A^*$ denote a tridiagonal pair on $V$ that has diameter at least 2 and a base with type I. Then $P_{A, A^*} = P_{A^*, A}$.

Definition 9.10. (See [11] Definition 2.6.) Let $d \in \mathbb{N}$ and let $A, A^*$ denote a tridiagonal pair on $V$ that has diameter $d$. Fix a nonzero $t \in \mathbb{F}$ such that $t^4 \neq 1$. Then $A, A^*$ is called $t$-geometric whenever the sequence $\{t^{d-2i}\}_{i=0}^d$ is a standard ordering of the eigenvalues of $A$ and a standard ordering of the eigenvalues of $A^*$.

Let $A, A^*$ denote a $t$-geometric tridiagonal pair on $V$. Define $\beta = t^2 + t^{-2}$. By Definitions 9.4 and 9.10, the scalar $\beta$ is a base of $A, A^*$. Observe that $\beta$ has type I.

Theorem 9.11. (See [11] Lemma 4.8.) For $A, A^* \in \text{End}(V)$ the following are equivalent:

(i) the pair $A, A^*$ acts on $V$ as a $q$-geometric tridiagonal pair;
(ii) the vector space $V$ is a NonNil irreducible $U_q^+$-module of type $(1, 1)$ on which the standard generators $x, y$ act as $A, A^*$ respectively.
For the duration of this paragraph assume that \( d \geq 2 \). Let \( V \) denote a NonNil finite-dimensional irreducible \( U_q^+ \)-module of type \((1,1)\) and diameter \( d \). By Theorem \( 9.11 \) the pair \( x, y \) acts on \( V \) as a \( q \)-geometric tridiagonal pair. The polynomials \( P_V, Q_V \) and \( P_{x,y} \) are associated with \( V \). By Lemma \( 4.4 \) and Definition \( 5.10 \) the polynomials \( P_V, Q_V \) are partners. Therefore \( z^dQ_V(z^{-1}) \) is a scalar multiple of \( P_V \).

**Lemma 9.12.** Assume that \( d \geq 2 \). Let \( V \) denote a NonNil finite-dimensional irreducible \( U_q^+ \)-module of type \((1,1)\) and diameter \( d \). Then
\[
P_{x,y}(z) = z^dQ_V(z^{-1}).
\]

*Proof.* By Lemma \( 9.10 \) it suffices to show that \( P_{y,x}(z) = z^dQ_V(z^{-1}) \). By Theorem \( 8.4 \) the vector space \( V \) becomes a \( \square_q \)-module on which \( x = x_0 \) and \( y = x_2 \). This \( \square_q \)-module \( V \) is irreducible of type 1. Let \( \{W_i\}_{i=0}^d \) (resp. \( \{W_i^*\}_{i=0}^d \) denote the decomposition \([2,3]\) (resp. \([1,0]\)) of the \( \square_q \)-module \( V \). Observe that \( \{W_i\}_{i=0}^d \) (resp. \( \{W_i^*\}_{i=0}^d \) is a standard ordering of the eigenspaces of \( y \) (resp. \( x \)). With respect to this ordering, we have \( a = a' = 0, b = c' = 0, c = b' = 1 \) in view of Lemma \( 9.9 \). By Theorem \( 8.5 \) the \( \square_q \)-module \( V \) becomes a \( U_q(L(\mathfrak{sl}_2)) \)-module on which \( x_0 = X_{01} \) and \( x_2 = X_{23} \). This \( U_q(L(\mathfrak{sl}_2)) \)-module \( V \) is irreducible of type 1 and \( P_V(1) \neq 0 \). By Definition \( 8.3 \) and \( 8.4 \) we have
\[
(e_0^\dagger)(e_1^\dagger)^i = (q - q^{-1})^{-2i}(X_{01} - X_{31})^i(X_{23} - X_{13})^i \quad (i \in \mathbb{N}).
\]

Let \( \{U_i\}_{i=0}^d \) denote the weight space decomposition of the \( U_q(L(\mathfrak{sl}_2)) \)-module \( V \) from Lemma \( 8.4 \). Observe that \( U_0 = W_0^* \). By Definition \( 4.3 \) the left-hand side of \( 9.4 \) acts on \( W_0^* \) as \( \mu I \). By \( 8.2 \), \( 8.4 \) and Lemma \( 8.6 \) we have \( (X_{01} - X_{31})U_i \subseteq U_{i-1} \) and \( (X_{23} - X_{13})U_i \subseteq U_{i+1} \) for \( 0 \leq i \leq d \). By \( 8.4 \) and Lemma \( 8.6 \) along with the fact that \( x = X_{01} \) and \( y = X_{23} \) on \( V \), we have \( X_{01} - X_{31} = x - q^{2i-d}I \) and \( X_{23} - X_{13} = y - q^{d-2i}I \). This \( U_q(L(\mathfrak{sl}_2)) \)-module \( V \) is irreducible of type 1 and \( P_V(1) \neq 0 \). By these comments, the following holds on \( W_0^* \) for \( 0 \leq i \leq d \):
\[
(X_{01} - X_{31})^i(X_{23} - X_{13})^i = (x - q^{2-d}I)(x - q^{4-d}I) \ldots (x - q^{2i-d}I)
\]
\[
(y - q^{d-2i+2}I) \ldots (y - q^{-2}I)(y - q^dI).
\]

Let \( \{\zeta_i\}_{i=0}^d \) denote the split sequence for the tridiagonal pair \( y, x \) with respect to the orderings \( \{W_i\}_{i=0}^d, \{W_i^*\}_{i=0}^d \). By the definition of split sequence above Lemma \( 9.3 \) the right-hand side of \( 9.5 \) acts on \( W_0^* \) as \( \zeta I \). By the above comments, we have \( \zeta_i = (q - q^{-1})^{2i} \mu_i \). By this and Definition \( 4.5 \) along with \( 9.3 \), we routinely obtain \( P_{y,x}(z) = z^dQ_V(z^{-1}) \). The result follows.

Let \( V \) denote a finite-dimensional irreducible \( \square_q \)-module of type 1. Via Theorem \( 8.4 \) the vector space \( V \) becomes a NonNil finite-dimensional irreducible \( U_q^+ \)-module of type \((1,1)\).

**Lemma 9.13.** Assume that \( d \geq 2 \). Let \( V \) denote a finite-dimensional irreducible \( \square_q \)-module of type 1 and diameter \( d \). Then
\[
P_{x_0,x_2}(z) = z^dQ_V(z^{-1}).
\]

*Proof.* Apply Lemma \( 9.12 \) with \( x = x_0 \) and \( y = x_2 \). \( \square \)

**Lemma 9.14.** Let \( V \) denote a finite-dimensional irreducible \( \square_q \)-module of type 1. Then for \( i \in \mathbb{Z}_4 \) the \( \square_q \) generators \( x_i, x_{i+2} \) act on \( V \) as a \( q \)-geometric tridiagonal pair.
Proof. By Theorems 8.4 and 9.11 the pair \(x_0, x_2\) acts on \(V\) as a \(q\)-geometric tridiagonal pair. Applying this to \(\rho V\), \(\rho^2 V\), \(\rho^3 V\) we obtain the remaining assertions.

10. The proof of Theorem 1.6

Let \(V\) denote a finite-dimensional irreducible \(\square_q\)-module of type 1. In the previous section we showed that for \(i \in \mathbb{Z}_4\) the \(\square_q\) generators \(x_i, x_{i+2}\) act on \(V\) as a \(q\)-geometric tridiagonal pair. In this section we will describe the relations among the polynomials

\[
P_{x_i,x_{i+2}}, \quad P_{\rho^i V}, \quad Q_{\rho^i V} \quad (i \in \mathbb{Z}_4).
\]

Using these results, we prove Theorem 1.6.

Lemma 10.1. Assume that \(d \geq 2\). Let \(V\) denote a finite-dimensional irreducible \(\square_q\)-module of type 1 and diameter \(d\). Then

\[
P_{x_1,x_3}(z) = z^d P_{V}(z^{-1}).
\]

Proof. By Lemma 9.10 it suffices to show that \(P_{x_3,x_1}(z) = z^d P_{V}(z^{-1})\). Let \(\{W_i\}_{i=0}^d\) (resp. \(\{W^*_i\}_{i=0}^d\)) denote the decomposition \([0,3]\) (resp. \([1,2]\)) of the \(\square_q\)-module \(V\). Observe that \(\{W_i\}_{i=0}^d\) (resp. \(\{W^*_i\}_{i=0}^d\)) is a standard ordering of the eigenspaces of \(x_3\) (resp. \(x_1\)). With respect to this ordering, we have \(a = a^* = 0\), \(b = b^* = 1\), \(c = c^* = 1\), \(d = 0\) in view of Lemma 9.6. By Theorem 8.5 the \(\square_q\)-module \(V\) becomes a \(U_q(L(\mathfrak{sl}_2))\)-module on which \(x_3 = X_{30}\) and \(x_1 = X_{12}\). This \(U_q(L(\mathfrak{sl}_2))\)-module \(V\) is irreducible of type 1 and \(P_{V}(1) \neq 0\). By Definition 3.1 and \(3.3\) we have

\[
(10.1) \quad (e_0^+)^i(e_0^-)^i = (q - q^{-1})^{-2i}(X_{12} - X_{13})^i(X_{30} - X_{31})^i \quad (i \in \mathbb{N}).
\]

Let \(\{U_i\}_{i=0}^d\) denote the weight space decomposition of the \(U_q(L(\mathfrak{sl}_2))\)-module \(V\) from Lemma 3.6. Observe that \(U_0 = W_0^*\). By Definition 4.1 the right-hand side of \(10.1\) acts on \(W_0^*\) as \(\sigma]\). By \(3.3\), \(3.4\) and Lemma 5.6 we have \((X_{12} - X_{13})U_1 \subseteq U_{i-1}\) and \((X_{30} - X_{31})U_{i} \subseteq U_{i+1}\) for \(0 \leq i \leq d\). By \(4.2\) and Lemma 5.6 along with the fact that \(x_3 = X_{30}\) and \(x_1 = X_{12}\) on \(V\), we have \(X_{12} - X_{13} = x_1 - q^{-2i-1}I \) and \(X_{30} - X_{31} = x_3 - q^{2(d-i)}I \) on \(U_i\) for \(0 \leq i \leq d\). By these comments, the following holds on \(W_0^*\) for \(0 \leq i \leq d\):

\[
(10.2) \quad (X_{12} - X_{13})^i(X_{30} - X_{31})^i = (x_1 - q^{d-2i})^i(x_1 - q^{d-i} I) \ldots (x_1 - q^{d-i} I)
\]

\[
(x_3 - q^{2i-2-d}I) \ldots (x_3 - q^{2-d}I)(x_3 - q^{-d}I).
\]

Let \(\{\zeta_i\}_{i=0}^d\) denote the split sequence for the tridiagonal pair \(x_3, x_1\) with respect to the orderings \(\{W_i\}_{i=0}^d, \{W^*_i\}_{i=0}^d\). By the definition of split sequence above Lemma 9.3 the right-hand side of \(10.2\) acts on \(W_0^*\) as \(\zeta_i\). The by above comments, we have \(\zeta_i = (q - q^{-1})^2 I\). By this and Definition 4.2 along with \(4.3\), we routinely obtain \(P_{x_3,x_1}(z) = z^d P_{V}(z^{-1})\). The result follows.

Proposition 10.2. Assume that \(d \geq 2\). Let \(V\) denote a finite-dimensional irreducible \(\square_q\)-module of type 1 and diameter \(d\). Then the following six polynomials coincide:

\[
P_{x_0,x_2}(z), \quad z^d P_{\rho V}(z^{-1}), \quad z^d Q_{\rho V}(z^{-1}),
\]

\[
P_{x_2,x_0}(z), \quad z^d P_{\rho^3 V}(z^{-1}), \quad z^d Q_{\rho^3 V}(z^{-1}).
\]
Moreover the following six polynomials coincide:

\[ P_{x_1,x_2}(z), \quad z^d P_V(z^{-1}), \quad z^d Q_V(z^{-1}), \]
\[ P_{x_3,x_1}(z), \quad z^d P_{\rho V}(z^{-1}), \quad z^d Q_{\rho V}(z^{-1}). \]

Proof. Apply Lemmas 9.13 and 10.1 to \( \rho V \) for \( i \in \mathbb{Z}_4 \) and use Lemma 9.9 \( \square \)

**Corollary 10.3.** Let \( V \) denote a finite-dimensional irreducible \( \square_q \)-module of type 1. Then \( P_{\rho V} = Q_V \).

Proof. Let \( d \) denote the diameter of the \( \square_q \)-module \( V \). First assume that \( d = 0 \), then \( P_{\rho V} = 1 = Q_V \) by Definitions 11.1 and 8.6. Next assume that \( d = 1 \).

By construction, the \( \square_q \)-module \( \rho V \) is irreducible of type 1 and diameter 1. Via Theorem 11.1, the \( \square_q \)-module \( V \) (resp. \( \rho V \)) becomes a finite-dimensional irreducible \( U_q(L(sl_2)) \)-module of type 1 and diameter 1. Let \( a \) (resp. \( b \)) denote the evaluation parameter of the \( U_q(L(sl_2)) \)-module \( V \) (resp. \( \rho V \)). By Lemma 11.1, we have \( P_{\rho V}(z) = 1 - b z \) and \( Q_V(z) = 1 - a^{-1} z \). To show that \( P_{\rho V} = Q_V \), it suffices to show that \( b = a^{-1} \). By construction, the action of \( X_{01} \) (resp. \( X_{23} \)) on \( \rho V \) is equal to the action of \( X_{12} \) (resp. \( X_{30} \)) on \( V \). Therefore \( \text{tr}(X_{01}X_{23}) \) on \( \rho V \) is equal to \( \text{tr}(X_{12}X_{30}) \) on \( V \). By this, \( 3.6 = 5.7 \) and \( q^2 \neq 1 \) we have \( b = a^{-1} \). Therefore \( P_{\rho V} = Q_V \). Next assume that \( d \geq 2 \). Then \( P_{\rho V} = Q_V \) by Proposition 10.2. We have shown that \( P_{\rho V} = Q_V \) for every value of \( d \). The result follows. \( \square \)

**Proof of Theorem 1.6.** Combine Lemma 1.6 and Corollary 10.3 \( \square \)

**Proof of Corollary 1.7.** Apply Theorem 1.6 twice to \( V \), and use Proposition 1.4 \( \square \)

11. The \( q \)-tetrahedron algebra \( \mathbb{E}_q \)

Recall the result Theorem 1.6 about the algebra \( \square_q \). In this section we obtain an analogous result for \( \mathbb{E}_q \).

**Definition 11.1.** (See [8] Definition 6.1.) Let \( \mathbb{E}_q \) denote the \( \mathbb{F} \)-algebra with generators

\[ \{ x_{ij} \mid i, j \in \mathbb{Z}_4, j - i = 1 \text{ or } j - i = 2 \} \]

and the following relations:

(i) For \( i, j \in \mathbb{Z}_4 \) and \( j - i = 2 \),
\[ x_{ij} x_{ji} = 1. \]

(ii) For \( h, i, j \in \mathbb{Z}_4 \) such that the pair \( (i - h, j - i) \) is one of \( (1,1), (1,2), (2,1), \)
\[ \frac{q x_{hi} x_{ij} - q^{-1} x_{ij} x_{hi}}{q - q^{-1}} = 1. \]

(iii) For \( h, i, j, k \in \mathbb{Z}_4 \) such that \( i - h = j - i = k - j = 1 \),
\[ x_{hi}^3 x_{jk} - [3]_q x_{hi}^2 x_{jk} x_{hi} + [3]_q x_{hi} x_{jk}^2 x_{hi} - x_{jk} x_{hi}^3 = 0. \]

We call \( \mathbb{E}_q \) the \( q \)-tetrahedron algebra.

We will use the following automorphism of \( \mathbb{E}_q \).

**Lemma 11.2.** (See [8] Lemma 6.3.) There exists an automorphism \( \rho \) of \( \mathbb{E}_q \) that sends each generator \( x_{ij} \) to \( x_{i+1,j+1} \). Moreover \( \rho^4 = 1 \).
Note 11.3. We use the same symbol $\rho$ for an automorphism of $\Box_q$ (in Lemma 11.2) and $\otimes_q$ (in Lemma 11.2). As we proceed, it should be clear from the context which algebra is being discussed.

Comparing the relations in Lemma 3.2 with the relations in Definition 11.1, we obtain an $F$-algebra homomorphism $\eta : U_q(L(sl_2)) \to \otimes_q$ that sends $X_{13} \mapsto x_{13}$, $X_{31} \mapsto x_{31}$ and $X_{i,i+1} \mapsto x_{i,i+1}$ for $i \in \mathbb{Z}_4$.

Lemma 11.4. (See [18, Propositions 4.3].) The above homomorphism $\eta$ is injective.

We next recall some facts about finite-dimensional irreducible $\otimes_q$-modules.

Lemma 11.5. (See [8, Theorem 12.3].) Let $V$ denote a finite-dimensional irreducible $\otimes_q$-module. Then each generator $x_{ij}$ of $\otimes_q$ is semisimple on $V$. Moreover there exist $d \in \mathbb{N}$ and $\gamma \in \{1, -1\}$ such that for each generator $x_{ij}$ the set of distinct eigenvalues of $x_{ij}$ on $V$ is $\{\gamma q^{d-2i}d_i\}_{i=0}^d$.

Definition 11.6. (See [8, Definition 12.4].) Let $V$ denote a finite-dimensional irreducible $\otimes_q$-module. By the diameter of $V$ we mean the scalar $d$ from Lemma 11.5. By the type of $V$ we mean the scalar $\gamma$ from Lemma 11.5.

Recall the map $\kappa$ from above Lemma 5.3 and the map $\psi$ from above Lemma 3.5. Consider the composition $\eta \circ \psi \circ \kappa : U_q^+ \overset{\kappa}{\longrightarrow} \Box_q \overset{\psi}{\longrightarrow} U_q(L(sl_2)) \overset{\eta}{\longrightarrow} \otimes_q$.

Theorem 11.7. (See [8, Theorem 10.3, 10.4].) The map $(\eta \circ \psi \circ \kappa)^\sharp$ gives a bijection between the following two sets:

(i) the isomorphism classes of finite-dimensional irreducible $\otimes_q$-modules of type 1;

(ii) the isomorphism classes of NonNil finite-dimensional irreducible $U_q^+$-modules of type $(1, 1)$.

Proof. Observe that $(\eta \circ \psi \circ \kappa)^\sharp = (\psi \circ \kappa)^\sharp \circ \eta^\sharp$. The result follows from this along with Theorems 5.9, 11.7.

Theorem 11.8. The map $\eta^\sharp$ gives a bijection between the following two sets:

(i) the isomorphism classes of finite-dimensional irreducible $\otimes_q$-modules of type 1;

(ii) the isomorphism classes of finite-dimensional irreducible $U_q(L(sl_2))$-modules $V$ of type 1 such that $P_V(1) \neq 0$.

Proof. Observe that $(\eta \circ \psi \circ \kappa)^\sharp = (\psi \circ \kappa)^\sharp \circ \eta^\sharp$. The result follows from this along with Theorems 5.5, 11.8.

Theorem 11.9. The map $(\eta \circ \psi)^\sharp$ gives a bijection between the following two sets:

(i) the isomorphism classes of finite-dimensional irreducible $\otimes_q$-modules of type 1;

(ii) the isomorphism classes of finite-dimensional irreducible $\Box_q$-modules of type 1.

Proof. Observe that $(\eta \circ \psi)^\sharp = \psi^\sharp \circ \eta^\sharp$. The result follows from this along with Theorems 5.3, 11.8.

Let $V$ denote a finite-dimensional irreducible $\otimes_q$-module of type 1. Via Theorem 11.8 the vector space $V$ becomes a finite-dimensional irreducible $U_q(L(sl_2))$-module of type 1 such that $P_V(1) \neq 0$. Via Theorem 11.7 the vector space $V$ becomes a
NonNil finite-dimensional irreducible $U_q^+$-module of type $(1, 1)$. Via Theorem 11.9, the vector space $V$ becomes a finite-dimensional irreducible $\square_q$-module of type 1.

**Definition 11.10.** Let $V$ denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1. Let $P_V$ denote the polynomial from Definition 4.2 associated with the $U_q(L(\mathfrak{sl}_2))$-module $V$ from Theorem 11.8. Observe that $P_V$ is equal to the polynomial from Definition 5.10 associated with the $U_q^+$-module $V$ from Theorem 11.7 and the polynomial from Definition 8.6 associated with the $\square_q$-module $V$ from Theorem 11.9.

**Proposition 11.11.** The map $V \mapsto P_V$ induces a bijection between the following two sets:

(i) the isomorphism classes of finite-dimensional irreducible $\boxtimes_q$-modules of type 1;

(ii) the polynomials in $\mathbb{F}[z]$ that have constant coefficient 1 and do not vanish at $z = 1$.

**Proof.** By Theorems 4.3 and 11.8.

Combining Theorems 8.4, 8.5, 9.11, 11.8, 11.11, we obtain a bijection between any two of the following sets:

(i) the isomorphism classes of $q$-geometric tridiagonal pairs;

(ii) the isomorphism classes of NonNil finite-dimensional irreducible $U_q^+$-modules of type $(1, 1)$;

(iii) the isomorphism classes of finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-modules $V$ of type 1 such that $P_V(1) \neq 0$;

(iv) the isomorphism classes of finite-dimensional irreducible $\boxtimes_q$-modules of type 1;

(v) the polynomials in $\mathbb{F}[z]$ that have constant coefficient 1 and do not vanish at $z = 1$;

(vi) the isomorphism classes of finite-dimensional irreducible $\square_q$-modules of type 1.

Let $V$ denote a finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module of type 1 and $P_V(1) \neq 0$. Via the above bijections, we view $V$ as an $A$-module, where $A$ is any of $U_q^+, \square_q, U_q(L(\mathfrak{sl}_2)), \boxtimes_q$. We call $P_V$ the Drinfel’d polynomial for the $A$-module $V$.

The map $\eta \circ \psi$ is an injective $\mathbb{F}$-algebra homomorphism from $\square_q$ to $\boxtimes_q$. We have the automorphism $\rho$ for $\square_q$ and $\boxtimes_q$. We now explain how these maps are related.

**Lemma 11.12.** The following diagram commutes:

\[
\begin{array}{ccc}
\square_q & \xrightarrow{\eta \circ \psi} & \boxtimes_q \\
\downarrow\rho & & \downarrow\rho \\
\square_q & \xrightarrow{\eta \circ \psi} & \boxtimes_q
\end{array}
\]

In the above diagram the map $\psi$ is from above Lemma 3.3, the map $\eta$ is from above Lemma 11.4, the map $\rho$ on the left is from Lemma 1.3 and the map $\rho$ on the right is from Lemma 11.2.

**Proof.** Use the definitions of the maps in question.
Theorem 11.13. Let $V$ denote a finite-dimensional irreducible $\mathbb{Q}_q$-module of type 1. Then the Drinfel’d polynomials for $V$ and $\rho V$ are partners in the sense of Definition 1.5.

Proof. By Lemmas 11.9, 11.12 and Theorem 1.6. \hfill $\Box$

Corollary 11.14. Let $V$ denote a finite-dimensional irreducible $\mathbb{Q}_q$-module of type 1. Then the $\mathbb{Q}_q$-modules $V$ and $\rho^2 V$ are isomorphic.

Proof. Apply Theorem 11.13 twice to $V$, and use Proposition 11.11. \hfill $\Box$

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