We define quaternionic Hermite polynomials by analogy with two families of complex Hermite polynomials. As in the complex case, these polynomials constitute orthogonal families of vectors in ambient quaternionic $L^2$-spaces. Using these polynomials, we then define regular and anti-regular subspaces of these $L^2$-spaces, the associated reproducing kernels and the ensuing quaternionic coherent states.

1. Introduction

Building quantum mechanics on quaternionic Hilbert spaces has been a much studied problem for many years (see, for example, [1] and the many references cited therein). Associated to this problem is that of building appropriate families of coherent states on quaternionic Hilbert spaces. The fact that the analogues of the usual canonical coherent states cannot be built using a group theoretical argument in the case of quaternions, has been elaborated in [2]. On the other hand, analogues of such coherent states in a quaternionic setting have been constructed using other methods in [3] and [19]. In this paper we study the possibility of constructing some analogues of the so-called non-linear coherent states on quaternionic Hilbert spaces, using the recently developed holomorphic function theory for quaternionic variables [7, 11, 12].

Recall that the real Hermite polynomials are defined by
\begin{equation}
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}
\end{equation}
and it is well known that the functions $C_n e^{-x^2/2} H_n(x)$, for some normalization constants $C_n$, are the eigensolutions of the quantum harmonic oscillator [5]. As an immediate extension of the real Hermite polynomials, by replacing the real variable $x$ by a complex number $z$, the complex Hermite polynomials, $H_n(z)$, were studied in [22]. In particular, in [22], it has been shown that these complex Hermite polynomials form an orthonormal basis of a certain Hilbert space of complex functions over $\mathbb{C}$ and this Hilbert space is a reproducing kernel Hilbert space. In [10], using these holomorphic Hermite polynomials, a set of coherent states (CS) have been built, which is then used to study some quantum mechanical issues and a quantization of the non-commutative plane. Apart from $H_n(z)$, another interesting set of Hermite polynomials, $H_{n,m}(z, \overline{z})$, were studied in [13, 16, 17] (see also references therein). In these papers it has been shown that the functions, $e^{-|z|^2/2} H_{n,m}(z, \overline{z})$ are eigensolutions of the Landau problem [5, 13]. Recently,
several interesting features of the $H_{n,m}(z,\bar{z})$ have been studied by fixing either $n = 0$ or $m = 0$. In fact, by so fixing one can recover the holomorphic and anti-holomorphic subspaces of a certain $L^2$-space and the subspaces so obtained can also be identified as the well known Bargmann spaces of holomorphic and anti-holomorphic functions. These are also reproducing kernel Hilbert spaces with reproducing kernels associated to the canonical coherent states. More generally, other reproducing kernel Hilbert spaces have been obtained using subsets of $H_{n,m}(z,\bar{z})$ as bases [5, 8], which also admit coherent states. In [8], these coherent states have been used to implement quantizations of $\mathbb{C}$. In [8, 10] the Hermite polynomials $H_{n,m}(z,\bar{z})$ and $H_n(z)$ and kernels associated with these polynomials have been used to obtain coherent states and the authors have used the CS so obtained to study some quantum phenomena and quantizations. In fact, the procedure used in [8, 10] to build CS was earlier worked out in [15] and later in [4, 20] as generalization of the definition of canonical CS. In this paper we shall use a similar approach to obtain quaternionic coherent states.

For the sake of completeness we briefly revisit the procedure for building generalized and nonlinear coherent states.

Let $\{\phi_m\}_{m=0}^{\infty}$ be an orthonormal basis of an abstract separable Hilbert space $\mathcal{H}$. The well known canonical coherent states are defined by:

$$|z\rangle = e^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} \phi_m \in \mathcal{H},$$

where $z = re^{i\theta} \in \mathbb{C}$, the complex plane.

A possible generalization of the above definition of canonical coherent states, to the so-called nonlinear coherent states, goes as follows: Let $\mathcal{D}$ be an open subset of $\mathbb{C}$. For $z \in \mathcal{D}$ set

$$|z\rangle = \mathcal{N}(|z|)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{\rho(m)}} \phi_m \in \mathcal{H},$$

where $\{\rho(m)\}_{m=0}^{\infty}$ is a positive sequence of real numbers and $\mathcal{N}(|z|)$ is the normalization factor ensuring that $\langle z | z \rangle = 1$. If in addition $\{|z\rangle | z \in \mathcal{D}\}$ satisfy

$$\int_\mathcal{D} |z\langle z | d\mu = I_\mathcal{H},$$

where $d\mu$ is an appropriately chosen measure on $\mathcal{D}$ and $I_\mathcal{H}$ is the identity operator on $\mathcal{H}$, then $\{|z\rangle | z \in \mathcal{D}\}$ is said to be a set of nonlinear coherent states on $\mathcal{D}$.

More generally, (generalized) CS can be constructed as follows: Let $(\Omega, \mu)$ be a measure space and $\mathbb{H}$ a closed subspace of $L^2(\Omega, \mu)$. Let $\{\Phi_m\}_{m=0}^{\dim(\mathbb{H})}$, $\dim(\mathbb{H})$ denoting the dimension of $\mathbb{H}$, be an orthonormal basis of $\mathbb{H}$ satisfying:

$$\sum_{m=0}^{\dim(\mathbb{H})} |\Phi_m(x)|^2 < \infty$$

for all $x \in \Omega$. Let $\mathcal{H}$ be another Hilbert space such that $\dim(\mathbb{H}) = \dim(\mathcal{H})$. Let $\{\phi_m\}_{m=0}^{\dim(\mathcal{H})}$ be an orthonormal basis of $\mathcal{H}$. Define

$$K(x,y) = \sum_{m=0}^{\dim(\mathbb{H})} \Phi_m(x) \Phi_m(y).$$
Then $K(x, y)$ is a reproducing kernel, that is, $K(x, y)$ satisfies

(a) hermiticity, $K(x, y) = \overline{K(y, x)}$ for all $x, y \in \Omega$;
(b) positivity, $K(x, x) \geq 0$ for all $x \in \Omega$;
(c) idempotence, $\int_{\Omega} K(x, y) K(y, z) d\mu(y) = K(x, z),$

and $\mathbb{H}$ is the corresponding reproducing kernel Hilbert space. For $x \in \Omega$, define

\begin{equation}
| x \rangle = K(x, x)^{-\frac{1}{2}} \sum_{m=0}^{\dim(\mathbb{H})} \Phi_m(x) \phi_m \in \mathcal{H}.
\end{equation}

Then,

\begin{equation}
\langle x \mid x \rangle = K(x, x)^{-1} \sum_{m=0}^{\dim(\mathbb{H})} \Phi_m(x) \Phi_m(x) = 1,
\end{equation}

and

$\mathcal{W} : \mathcal{H} \rightarrow \mathbb{H}$ with $\mathcal{W}\phi(x) = K(x, x)^{\frac{1}{2}} \langle x \mid \phi \rangle$

is an isometry. Then, for $\phi, \psi \in \mathcal{H}$ we have

\begin{equation}
\langle \phi \mid \psi \rangle_{\mathcal{H}} = \langle \mathcal{W}\phi \mid \mathcal{W}\psi \rangle_{\mathbb{H}} = \int_{\Omega} \overline{\mathcal{W}\phi(x)} \mathcal{W}\psi(x) d\mu(x)
\end{equation}

\begin{equation}
= \int_{\Omega} \langle \phi \mid x \rangle \langle x \mid \psi \rangle K(x, x) d\mu(x),
\end{equation}

and

\begin{equation}
\int_{\Omega} \langle x \mid x \rangle K(x, x) d\mu(x) = I_{\mathcal{H}},
\end{equation}

where $K(x, x)$ plays the role of a positive weight function. Thus, the set of states $\{ | x \rangle \mid x \in \Omega \}$ forms a set of (generalized) CS.

In the case where $\{ \Phi_m \}_{m=0}^{\dim(\mathbb{H})}$ is an orthogonal basis of $\mathbb{H}$, one can define $\rho(m) = \| \Phi_m \|^2$; $m = 0, 1, 2, \ldots, \dim(\mathbb{H})$, and obtain an orthonormal basis

\begin{equation}
\left\{ \frac{\Phi_m}{\sqrt{\rho(m)}} \right\}_{m=0}^{\dim(\mathbb{H})}
\end{equation}

of $\mathbb{H}$. Then, setting

\begin{equation}
| x \rangle = K(x, x)^{-\frac{1}{2}} \sum_{m=0}^{\dim(\mathcal{H})} \frac{\Phi_m(x)}{\sqrt{\rho(m)}} \phi_m \in \mathcal{H},
\end{equation}

one obtains the desired result which is analogous to (1.3).

The above discussion motivates the following definition.

**Definition 1.1.** Let $\mathcal{D}$ be an open subset of $\mathbb{C}$. Let

$\Phi_m : \mathcal{D} \rightarrow \mathbb{C}, \quad m = 0, 1, 2, \ldots,$

be a sequence of complex functions. Define

\begin{equation}
| z \rangle = \mathcal{N}(|z|)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\Phi_m(z)}{\sqrt{\rho(m)}} \phi_m \in \mathcal{H}; \quad z \in \mathcal{D},
\end{equation}

where $\mathcal{N}(|z|)$ is a normalization function.
where $N(|z|)$ is a normalization factor and $\{\rho(m)\}_{m=0}^{\infty}$ is a sequence of nonzero positive real numbers. The set of vectors in (1.9) is said to form a set of CS if

(a) $\langle z | z \rangle = 1$ for all $z \in \mathcal{D}$;

(b) the states $\{|z\rangle | z \in \mathcal{D}\}$ satisfy a resolution of the identity:

$$
(1.10) \quad \int_{\mathcal{D}} |z\rangle \langle z | d\mu = I_q,
$$

where $d\mu$ is an appropriately chosen measure and $I_q$ is the identity operator on $\mathcal{H}$.

In [18], we studied what were called quaternionic CS in a complex Hilbert space. These were defined as vector CS, built using a quaternionic variable. However, recently, in [19] we have defined the canonical quaternionic CS by replacing the $z$ in (1.3) by a quaternion $q$ and considered these CS as vectors in a quaternionic Hilbert space (see also [2] for Prelemov type quaternionic CS). Further, we have also defined the quaternionic version of Hermite polynomials by replacing $z$ in $H_{n,m}(z, \bar{z})$ by a quaternion. Since there is more than one way of defining a derivative with respect to a quaternionic variable [7], in [19] the quaternionic derivative was introduced in a formal sense. However, recent developments of quaternionic analysis give a definition of a quaternionic derivative, the so-called Cullen derivative [11, 12], which is more useful for our purposes and we adopt it here.

The novelty of the present paper can be summarized as follows: using the Cullen derivative we define the quaternionic counterparts of the Hermite polynomials $H_n(z)$ and $H_{n,m}(z, \bar{z})$, as vectors in quaternionic Hilbert spaces by replacing $z$ by a quaternion $q$. The two index quaternionic Hermite polynomials $H_{n,m}(q, \bar{q})$ will span a subspace $\mathcal{H}_q$ of a quaternionic $L^2$-space. By fixing $n = 0$ or $m = 0$ in $H_{n,m}(q, \bar{q})$ we shall obtain the regular and anti-regular subspaces of $\mathcal{H}_q$ (as the counterparts of holomorphic and anti-holomorphic subspaces). We shall also prove that the single indexed quaternionic Hermite polynomials $H_n(q)$ and $H_q(\bar{q})$ serve as bases for certain regular and anti-regular subspaces respectively. Apart from these, by defining kernels with $H_n(q)$ and $H_{n,m}(q, \bar{q})$ we will obtain CS resembling (1.9) over quaternionic Hilbert spaces and realize the regular subspace as a reproducing kernel Hilbert space. The functions $e^{-|q|^2/2}H_{n,m}(q, \bar{q})$ are shown to be eigenfunctions of an operator $\mathfrak{L}_H$ with infinite degeneracy as in the Landau problem [19] (see also end of Section II). However, we are unable at this point to give a physical meaning to $\mathfrak{L}_H$. Further investigation of this point, quantization of the type done in [8, 10] and a study of the modular structures along the lines of [5], using quaternionic coherent states, are left for future work.

2. Mathematical prelminaries

In order to make the paper self-contained, we recall a few facts about quaternions which may not be well-known. In particular, we revisit the $2 \times 2$ complex matrix representations of quaternions, quaternionic Hilbert spaces, their duals, the Cullen derivative and the definition of regularity of a function of a quaternionic variable.

2.1. Quaternions. Let $H$ denote the field of quaternions. Its elements are of the form $q = x_0 + x_1i + x_2j + x_3k$ where $x_0, x_1, x_2$ and $x_3$ are real numbers, and $i, j, k$ are imaginary units such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. The quaternionic conjugate of $q$ is defined to be $\bar{q} = x_0 - x_1i - x_2j - x_3k$. We shall find it convenient to use the representation of quaternions by $2 \times 2$ complex matrices:

$$
(2.1) \quad q = x_0\sigma_0 + i\sigma_1 \cdot q,
$$
with \( x_0 \in \mathbb{R}, \ x = (x_1, x_2, x_3) \in \mathbb{R}^3, \ \sigma_0 = 1_2 \), the 2 \times 2 identity matrix, and \( \underline{\sigma} = (\sigma_1, -\sigma_2, \sigma_3) \), where the \( \sigma_i, \ \ell = 1, 2, 3 \) are the usual Pauli matrices. The quaternionic imaginary units are identified as, \( i = \sqrt{-1}\sigma_1, \ j = -\sqrt{-1}\sigma_2, \ k = \sqrt{-1}\sigma_3 \). Thus,

\[
(2.2) \quad q = \begin{pmatrix} x_0 + ix_3 & -x_2 + ix_1 \\ x_2 + ix_1 & x_0 - ix_3 \end{pmatrix} \quad \text{and} \quad \bar{q} = q^\dagger \quad (\text{matrix adjoint}).
\]

Introducing the polar coordinates:

\[
\begin{aligned}
  x_0 &= r \cos \theta, \\
  x_1 &= r \sin \theta \sin \phi \cos \psi, \\
  x_2 &= r \sin \theta \sin \phi \sin \psi, \\
  x_3 &= r \sin \theta \cos \phi,
\end{aligned}
\]

where \( r \in [0, \infty) \), \( \theta, \phi \in [0, \pi] \), and \( \psi \in [0, 2\pi) \), we may write

\[
(2.3) \quad q = A(r) e^{i\theta \sigma(\bar{n})},
\]

where

\[
(2.4) \quad A(r) = r \sigma_0 \quad \text{and} \quad \sigma(\bar{n}) = \begin{pmatrix} \cos \phi & \sin \phi e^{i\psi} \\ \sin \phi e^{-i\psi} & -\cos \phi \end{pmatrix}.
\]

The matrices \( A(r) \) and \( \sigma(\bar{n}) \) satisfy the conditions,

\[
(2.5) \quad A(r) = A(r)^\dagger, \quad \sigma(\bar{n})^2 = \sigma_0, \quad \sigma(\bar{n})^\dagger = \sigma(\bar{n}) \quad \text{and} \quad [A(r), \sigma(\bar{n})] = 0.
\]

Note that \(|\underline{q}|^2 := \bar{q}q = r^2 \sigma_0 = (x_0^2 + x_1^2 + x_2^2 + x_3^2) 1_2 \) defines a real norm on \( H \).

Using the above complex representation for quaternions, we defined a set of vector coherent states (VCS) in [18]. To recall that construction briefly, if \( \{q_m\}_{m=0}^\infty \) is an orthonormal basis of an abstract, complex separable Hilbert space \( \mathcal{H} \) and \( \{\chi^1, \chi^2\} \) is an orthonormal basis of \( \mathbb{C}^2 \), then the VCS take the form,

\[
(2.6) \quad |\underline{q}, j\rangle = \mathcal{N}_1(|\underline{q}|)^{-1/2} \sum_{m=0}^\infty \frac{q^m}{\sqrt{x_m}} \chi^j \otimes \phi_m \in \mathbb{C}^2 \otimes \mathcal{H}, \quad j = 1, 2,
\]

where \( \mathcal{N}_1(|\underline{q}|) \) and \( x_m! = \rho(m) \) can be chosen appropriately. Using [25] we can determine the normalization constant \( \mathcal{N}_1(|\underline{q}|) \), and the resolution of the identity (for details see [18]).

First note that, in order for the norm of the vector \(|\underline{q}, j\rangle\) to be finite, we must have,

\[
(2.7) \quad \langle q, j | \ q, j \rangle = \mathcal{N}_1(|\underline{q}|)^{-1} \sum_{m=0}^\infty \frac{r^{2m} x_m}{x_m!} < \infty.
\]

Thus, if \( \lim_{m \to \infty} x_m = x \), we need to restrict \( r \) to \( 0 \leq r < L = \sqrt{x} \) for the convergence of the above series. In this case, define

\[
(2.8) \quad \mathcal{D} = \{(r, \theta, \phi, \psi) \mid 0 \leq r < L, \ 0 \leq \phi \leq \pi, \ 0 \leq \theta, \psi < 2\pi\},
\]

The resolution of the identity is then given with respect to a measure of the type

\[
(2.9) \quad d\zeta(r, \theta, \phi, \psi) = d\tau(r) d\theta d\Omega(\phi, \psi), \quad \text{with} \quad d\Omega(\phi, \psi) = \frac{1}{4\pi} \sin \phi d\phi d\psi.
\]

2.2. Quaternionic Hilbert spaces. In this subsection we define left and right quaternionic Hilbert spaces. For details we refer the reader to [11]. We also define the Hilbert space of square integrable functions on quaternions based on [23].
2.2.1. Right Quaternionic Hilbert Space. Let $V_H^R$ be a linear vector space under right multiplication by quaternionic scalars (again $H$ standing for the field of quaternions). For $f, g, h \in V_H^R$ and $q \in H$, the inner product

$$\langle \cdot | \cdot \rangle : V_H^R \times V_H^R \rightarrow H$$

satisfies the following properties

(i) $\langle f | g \rangle = \langle g | f \rangle$

(ii) $\|f\|^2 = \langle f | f \rangle > 0$ unless $f = 0$, a real norm

(iii) $\langle f | g + h \rangle = \langle f | g \rangle + \langle f | h \rangle$

(iv) $\langle f | gq \rangle = \langle f | g \rangle q$

(v) $\langle fq | g \rangle = \overline{q} \langle f | g \rangle$

where $\overline{q}$ stands for the quaternionic conjugate. We assume that the space $V_H^R$ is complete under the norm given above. Then, together with $\langle \cdot | \cdot \rangle$ this defines a right quaternionic Hilbert space, which we shall assume to be separable. Quaternionic Hilbert spaces share most of the standard properties of complex Hilbert spaces. In particular, the Cauchy-Schwartz inequality holds on quaternionic Hilbert spaces as well as the Riesz representation theorem for their duals (see below). Thus, the Dirac bra-ket notation can be adapted to quaternionic Hilbert spaces:

$$|fq\rangle = |f\rangle q, \quad \langle qf| = \overline{q} \langle f|,$$

for a right quaternionic Hilbert space, with $|f\rangle$ denoting the vector $f$ and $\langle f|$ its dual vector.

2.2.2. Left Quaternionic Hilbert Space. Let $V_H^L$ be a linear vector space under left multiplication by quaternionic scalars. For $f, g, h \in V_H^L$ and $q \in H$, the inner product

$$\langle \cdot | \cdot \rangle : V_H^L \times V_H^L \rightarrow H$$

satisfies the following properties

(i) $\langle f | g \rangle = \langle g | f \rangle$

(ii) $\|f\|^2 = \langle f | f \rangle > 0$ unless $f = 0$, a real norm

(iii) $\langle f | g + h \rangle = \langle f | g \rangle + \langle f | h \rangle$

(iv) $\langle qf | g \rangle = q \langle f | g \rangle$

(v) $\langle f | qg \rangle = \langle f | g \rangle \overline{q}$

Again, we shall assume that the space $V_H^L$ together with $\langle \cdot | \cdot \rangle$ is a separable Hilbert space. Also,

$$(2.10) \quad |qf\rangle = |f\rangle \overline{q}, \quad \langle qf| = q \langle f|.$$

Note that, because of our convention for inner products, for a left quaternionic Hilbert space, the bra vector $\langle f|$ is to be identified with the vector itself, while the ket vector $|f\rangle$ is to be identified with its dual. (There is a natural left multiplication by quaternionic scalars on the dual of a right quaternionic Hilbert space and a similar right multiplication on the dual of a left quaternionic Hilbert space.)

Separable quaternionic Hilbert spaces admit countable orthonormal bases. Let $V_H^L$ be a left quaternionic Hilbert space and let $\{e_\nu\}_{\nu=0}^N$ ($N$ could be finite or infinite) be an orthonormal basis for it. Then, $\langle e_\nu | e_\mu \rangle = \delta_{\nu\mu}$ and any vector $f \in V_H^L$ has the expansion $f = \sum_\nu f_\nu e_\nu$, with $f_\nu = \langle f | e_\nu \rangle \in H$. Using such a basis, it is possible to introduce a
multiplication from the right on \( V_H^L \) by elements of \( H \). Indeed, for \( f \in V_H^L \) and \( q \in H \) we define,
\[
(f q) = \sum_{\nu} (f, q)e_{\nu}.
\]

The field of quaternions \( H \) itself can be turned into a left quaternionic Hilbert space by defining the inner product \( \langle q \mid q' \rangle = qq'^* = q^*q' \) or into a right quaternionic Hilbert space with \( \langle q \mid q' \rangle = q'q = \overline{qq'} \).

2.2.3. Quaternionic Hilbert Spaces of Square Integrable Functions. Let \( (X, \mu) \) be a measure space and \( H \) the field of quaternions, then
\[
L^2_H(X, \mu) = \left\{ f : X \rightarrow H \mid \int_X |f(x)|^2 d\mu(x) < \infty \right\}
\]
is a left quaternionic Hilbert space, with the (left) scalar product
\[
\langle f \mid g \rangle = \int_X f(x)\overline{g(x)}d\mu(x),
\]
where \( \overline{g(x)} \) is the quaternionic conjugate of \( g(x) \), and (left) scalar multiplication \( af, a \in H \), \( (af)(q) = af(q) \) (see [23] for details). Similarly, one could define a right quaternionic Hilbert space of square integrable functions.

2.3. Dual spaces. In order to obtain a quaternionic version of the Riesz representation theorem, we need to recall a few facts about the dual space of a quaternionic Hilbert space. We follow [21] in order to do this. Let \( \mathcal{H}_{ld} \) be the left dual space of the left quaternionic Hilbert space \( V_H^L \). That is,
\[
\mathcal{H}_{ld} = \{ h : V_H^L \rightarrow H \mid h \text{ is bounded and left linear} \}
\]
with the usual norm \( \|h\| = \sup\{|h(x)| \mid \|x\| = 1, \ x \in V_H^L\} \). It is known that \( \mathcal{H}_{ld} \) is a real vector space. Moreover, \( \mathcal{H}_{ld} \) can be transformed into a quaternionic Hilbert space. Indeed, as noted above, \( V_H^L \) also admits a right multiplication by quaternionic scalars.

Using this fact, for any functional \( h \in \mathcal{H}_{ld} \) and any \( \lambda \in H \) define
\[
(\lambda h)(x) = h(x\lambda), \quad (h\lambda)(x) = h(x)\lambda; \ x \in V_H^L.
\]
Then \( \mathcal{H}_{ld} \) becomes a two-sided quaternionic Banach space, with the scalar multiplication so defined.

**Theorem 2.1** (Riesz representation theorem). For any functional \( h \in \mathcal{H}_{ld} \), exactly as in the real and complex cases,
\[
h(x) = \langle x \mid y \rangle, \quad x \in V_H^L
\]
for a vector \( y \in V_H^L \), and then \( \|h\| = \|y\| \).

Let \( \{e_\nu \mid \nu \in \Lambda \} \) be a fixed orthonormal basis of \( V_H^L \) and define \( J : \mathcal{H}_{ld} \rightarrow V_H^L \) by the canonical mapping defined by the relation \( \langle x \mid y \rangle = \langle x \mid Jh \rangle \); \( x \in V_H^L \).

Then it can be shown that \( J \) is additive, isometric and bijective. Now define
\[
K : V_H^L \rightarrow V_H^L, \quad K(x) = \sum_{\nu \in \Lambda} \langle x \mid e_\nu \rangle e_\nu.
\]
Clearly, \( K \) is additive. (Note, \( x = \sum_{\nu \in \Lambda} \langle x \mid e_\nu \rangle e_\nu \).
Theorem 2.2. The left dual space $\mathcal{S}_{ld}$ of $V^L_H$, is also a two-sided quaternionic Hilbert space, if we introduce the inner product in $\mathcal{S}_{ld}$ by
\begin{equation}
\langle h|k \rangle = \langle KJh|KJk \rangle; \ h, k \in \mathcal{S}_{ld}.
\end{equation}
The inner product (2.14) is consistent with the norm of $\mathcal{S}_{ld}$.

Further, if we define $U : \mathcal{S}_{ld} \rightarrow V^L_H$ by $U = KJ$, then $U$ is a two-linear bijective map, and thereby $\mathcal{S}_{ld}$ is isomorphic to $V^L_H$.

2.4. Cullen regular functions. There have been a number of different suggestions in the literature on how the notion of holomorphy could be extended to functions of a quaternionic variable. We mention two here, of which we shall adopt the second definition. For a brief history of quaternionic holomorphy we refer the reader to [7]. The first definition of quaternionic holomorphy is given via the Cauchy-Fueter equations, which attempts to mimic the Cauchy-Riemann equations in a straightforward way.

Definition 2.3. (Cauchy-Fueter equations) [2,11] Let $f : H \rightarrow H$ be a quaternion valued function of a quaternionic variable. We say that $f$ is left-regular if it satisfies the Cauchy-Fueter equation
\begin{equation}
\frac{\partial f}{\partial q} = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} + k \frac{\partial f}{\partial x_3} = 0,
\end{equation}
and that $f$ is right-regular if it satisfies the other Cauchy-Fueter equation
\begin{equation}
\frac{\partial f}{\partial q} = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1} i + \frac{\partial f}{\partial x_2} j + \frac{\partial f}{\partial x_3} k = 0.
\end{equation}

Using this definition a theory of regular functions has been developed as a well struc-tured theory. However, the unpleasant feature of this definition is that under the Cauchy-Fueter equations the function $f(q) = q$ is not regular, and thereby none of the monomials or polynomials are regular. There have been several attempts to overcome this feature. The most promising, and recent attempt has appeared in [11] (see also [12]) where the following definition is offered. Let $S = \{x = x_1 + x_2j + x_3j \mid x_1, x_2, x_3 \in \mathbb{R}, x_1^2 + x_2^2 + x_3^2 = 1\}$.

Definition 2.4. (Slice-regular functions [11]) Let $\Omega$ be a domain in $H$. A real differentiable (i.e., with respect to $x_0$ and the $x_i$, $i = 1, 2, 3$) function $f : \Omega \rightarrow H$ is said to be slice left regular if, for every quaternion $I \in S$, the restriction of $f$ to the complex line $L_I = \mathbb{R} + II \mathbb{R}$ passing through the origin, and containing 1 and $I$, has continuous partial derivatives (with respect to $x$ and $y$, every element in $L_I$ being uniquely expressible as $x + yI$) and satisfies
\begin{equation}
\overline{D}_1 f(x + yI) := \frac{1}{2} \left( \frac{\partial f_1(x + yI)}{\partial x} + i \frac{\partial f_1(x + yI)}{\partial y} \right) = 0.
\end{equation}
Similarly, it is said to be slice right regular if
\begin{equation}
\overline{D}_1 f(x + yI) := \frac{1}{2} \left( \frac{\partial f_1(x + yI)}{\partial x} + \frac{\partial f_1(x + yI)}{\partial y} I \right) = 0.
\end{equation}

With this definition all monomials of the form $aq^n$, $a \in H$, $n \in \mathbb{N}$, are slice right regular while those of the form $q^n a$, $a \in H$, $n \in \mathbb{N}$, are slice left regular. Since regularity respects addition, all polynomials of the form $f(q) = \sum_{t=0}^n a_t q^t$, with $a_t \in H$, are slice right regular and similarly polynomials of the form $f(q) = \sum_{t=0}^n q^t a_t$, are slice left regular. Further, an analog of Abel's theorem guarantees convergence of appropriate infinite power series.
Proposition 2.5. [11] For any non-real quaternion \( q \in H - \mathbb{R} \), there exist, and are unique, \( x, y \in \mathbb{R} \) with \( y > 0 \), and \( I \in \mathbb{S} \) such that \( q = x + yI \).

Henceforth we shall refer to slice right (left) regular functions simply as right (left) regular functions. We now define the Cullen derivative of regular functions.

Definition 2.6. (Cullen derivative [11, 12]) Let \( \Omega \) be a domain in \( H \), and let \( f : \Omega \rightarrow H \) be a left regular function. The Cullen derivative of \( f \), \( \partial_c f \), is defined as

\[
\partial_c f(q) = \begin{cases}
\frac{1}{2} \left( \frac{\partial f(x + Iy)}{\partial x} - I \frac{\partial f(x + Iy)}{\partial y} \right) & \text{if } y \neq 0 \\
\frac{\partial f}{\partial x}(x) & \text{if } q = x \text{ is real}
\end{cases}
\]

Similarly, for a right regular function \( f \) its Cullen derivative is defined as

\[
\partial_c f(q) = \begin{cases}
\frac{1}{2} \left( \frac{\partial f(x + Iy)}{\partial x} - \frac{\partial f(x + Iy)}{\partial y} I \right) & \text{if } y \neq 0 \\
\frac{\partial f}{\partial x}(x) & \text{if } q = x \text{ is real}
\end{cases}
\]

Under the above definition the Cullen derivative of a regular function is regular and with \( a_n \in H \) we have, for example, for a right regular power series,

\[
(2.17) \quad \partial_c \left( \sum_{n=0}^{\infty} a_n q^n \right) = \sum_{n=0}^{\infty} na_n q^{n-1}.
\]

The following theorem gives the quaternionic version of holomorphy via a Taylor series. Let \( B(0, R) \) be an open ball in \( H \), of radius \( R \) and centered at 0.

Theorem 2.7. [11] A function \( f : B(0, R) \rightarrow H \) is right, respectively left, regular if and only if it has a series expansion of the form

\[
f(q) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0) q^n, \quad \text{respectively,} \quad f(q) = \sum_{n=0}^{\infty} q^n \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0),
\]

converging on \( B(0, R) \).

3. Complex Hermite polynomials

For the construction and analysis of the quaternionic Hermite polynomials, \( H_n(q) \) and \( H_{n,m}(q, \bar{q}) \), it would be useful to first review some facts about their complex counterparts, specially since the results we obtain here in the quaternionic case are parallel to those in the complex case. For a detailed discussion of the complex Hermite polynomials \( H_n(z) \) and \( H_{n,m}(z, \bar{z}) \), we refer the reader to [5, 10, 13, 16, 17, 22].

3.1. The polynomials \( H_n(z) \). Let \( z = x + iy \in \mathbb{C}, \ 0 < s < 1 \),

\[
d\nu(z) = d\nu(x, y) = \exp\left[-(1-s)x^2 - \left(\frac{1}{s} - 1\right)y^2\right] \, dx \, dy
\]

and

\[
b_n(s) = \frac{\pi^{s/2}}{1-s} \left( \frac{2 + s}{1-s} \right)^n n!.
\]
Define
\[ H_n(z) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m (2z)^{m-2}}{m!(n-2m)!} = n! \sum_{m=0}^{[n/2]} C_{nm} z^{n-2m}. \]
([x] denoting the integer part of x) and observe that \( H_n(x + iy) = H_n(x - iy) \). The \( H_n \) are just the usual Hermite polynomials, written in terms of a complex argument. From [10] and [22] we have
\[ \int_{\mathbb{R}^2} H_n(x + iy)H_m(x + iy)d\nu(x, y) = b_n(s) \delta_{nm}. \]
Define
\[ h_{n,s}(z) = b_n(s)^{-\frac{1}{2}} e^{-z^2/2} H_n(z) \]
and the Hilbert space of entire functions
\[ \mathcal{X}_s = \left\{ f \mid \int_{\mathbb{R}^2} |f(x + iy)|^2 \exp(sx^2 - \frac{1}{s}y^2)dx dy < \infty \right\}. \]
It has been shown in [22] that \( \{ h_{n,s}(z) \}_{n=0}^\infty \) is an orthonormal basis of \( \mathcal{X}_s \) and \( \mathcal{X}_s \) is a reproducing kernel Hilbert space with the kernel
\[ \mathcal{R}_s(z, w) = \sum_{n} b_{n,s}(z)\overline{b_{n,s}(w)} = \frac{1 - s^2}{2\pi s} \exp \left[ -\frac{1 + s^2}{4s} (z^2 + \overline{w}^2) + \frac{1 - s^2}{2s} zw \right], \]
where \( z, w \in \mathbb{C} \). The expression for the kernel can be reduced to (see [10])
\[ \mathcal{R}_s(z, \overline{z}) = \frac{1 - s^2}{2\pi s} \exp\left[ -sx^2 + \frac{1}{s}y^2 \right], \quad z = x + iy. \]
If we take
\[ h_{n,s}(z) = b_n(s)^{-\frac{1}{2}} H_n(z) \]
then for \( z, w \in \mathbb{C} \),
\[ K_s(z, w) = \sum_{n} h_{n,s}(z)\overline{h_{n,s}(w)} = \frac{1 - s^2}{2\pi s} \exp \left[ -\frac{(s - 1)^2}{4s} (z^2 + \overline{w}^2) + \frac{1 - s^2}{2s} zw \right]. \]
Further by replacing \( w \) by \( z \) we get
\[ K_s(z, \overline{z}) = \frac{1 - s^2}{2\pi s} \exp \left[ -\frac{s - 1}{2} x^2 + \frac{s^2 - 3s + 2}{2s} y^2 \right], \quad z = x + iy. \]
The kernel (3.9) is also a reproducing kernel. In fact it is easily seen that the corresponding reproducing kernel Hilbert space, \( \mathcal{H}_{\text{hol}} \), which is again a Hilbert space of analytic functions, and for which the vectors (3.8) form an orthonormal basis, is a subspace of the Hilbert space \( L^2(\mathbb{C}, d\mu_s) \), where
\[ d\mu_s(x, y) = e^{-[(1-s)x^2 + (4-1)y^2]} \ dx \ dy. \]
Similarly, the polynomials \( h_{n,s}(\overline{z}) \) would span a reproducing kernel Hilbert space \( \mathcal{H}_{\text{a-hol}} \subset L^2(\mathbb{C}, d\mu_s) \) of anti-analytic functions, with reproducing kernel \( K_s(z, \overline{w}) \).
3.2. The polynomials \( H_{n,m}(z,\overline{z}) \). A second class of complex Hermite polynomials have been studied in [17]. These are defined, for positive integers \( n, m \), as

\[
H_{n,m}(z,\overline{z}) = (-2)^{n+m}\exp(z\overline{z}/2) \left( \frac{\partial}{\partial z} \right)^n \left( \frac{\partial}{\partial \overline{z}} \right)^m \exp(-z\overline{z}/2).
\]

The generating function for this polynomial is

\[
\exp[(a\overline{z} + \overline{a}z - a\overline{a})/2] = \sum_{n,m} \frac{(a/2)^n(\overline{a}/2)^m}{n!m!} H_{n,m}(z,\overline{z}).
\]

When \( m \geq n \) and \( n \geq 0 \), the complex Hermite polynomials can also be written as

\[
H_{n,m}(z,\overline{z}) = \frac{m!}{(m-n)!}z^{m-n}(-2)^{n+1}F_1(-n, m - n + 1, z\overline{z}/2).
\]

These polynomials satisfy the orthogonality relation

\[
\int_{-\infty}^{\infty} \exp(-z\overline{z}/2)H_{n,m}(z,\overline{z})H_{\nu,\mu}(z,\overline{z}) \, dx \, dy = 2\pi \delta_{n\nu}\delta_{m\mu}n!m!2^{n+m}.
\]

Now let

\[
h_{n,m}(z,\overline{z}) = (-1)^{n+m}\exp(|z|^2) \left( \frac{\partial}{\partial z} \right)^n \left( \frac{\partial}{\partial \overline{z}} \right)^m \exp(-|z|^2) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} z^i\overline{z}^j.
\]

From (3.14) (see also [8, 13]), the polynomials \( h_{n,m}(z,\overline{z}) \) form a complete orthogonal system in the Hilbert space \( L^2(\mathbb{C}, e^{-|z|^2}d\lambda) \), where \( d\lambda = \frac{1}{\pi}d^2z \) is the Lebesque measure on \( \mathbb{C} \). For proof of completeness see [13]. Also from (3.14) we have

\[
\int_{\mathbb{C}} e^{-|z|^2}h_{n,m}(z,\overline{z})h_{n,m}(z,\overline{z}) \, d\lambda = n!m!.
\]

and thereby

\[
\|h_{n,m}\|_{L^2} = \sqrt{n!m!}.
\]

The operator

\[
\mathcal{L} = -\frac{1}{4} \left\{ 4 \frac{\partial^2}{\partial z\partial \overline{z}} + 2 \left( z \frac{\partial}{\partial z} - \overline{z} \frac{\partial}{\partial \overline{z}} \right) - |z|^2 \right\}
\]

is the Hamiltonian of a nonrelativistic quantum particle moving on the plane under the action of a constant external magnetic field applied perpendicularly to the plane. The functions \( e^{-|z|^2/2}h_{n,m}(z,\overline{z}) \) are the eigenfunctions of \( \mathcal{L} \) with eigenvalues \( n + 1/2 \), each eigenvalue being infinitely degenerate. The degeneracy is given by \( m = 0, 1, 2, \cdots \). For details see [13, 16].

4. Quaternionic Hermite Polynomials

In this section we define the quaternionic Hermite polynomials \( H_n(q) \) and \( H_{n,m}(q,\overline{q}) \), by analogy with the complex polynomials introduced above. We shall identify the set of all polynomials \( H_n(q) \) as an orthogonal set in a \( L^2 \)-space and similarly for the \( H_{n,m}(q,\overline{q}) \). We also obtain reproducing kernels using the polynomials \( H_n(q) \) and \( H_{n,m}(q,\overline{q}) \) and the corresponding reproducing kernel Hilbert spaces. Finally we look at an operator \( \mathcal{L}_H \) which can be considered as the quaternionic version of the Landau operator (3.17).
It is well-known (see, e.g., [18]) that any \( q \in H \), in the \( 2 \times 2 \) matrix representation, can be written as

\[
(4.1) \quad q = u_q Z u_q^\dagger,
\]

where

\[
u_q = \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \begin{pmatrix} \cos \phi & i \sin \phi \\ i \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \in SU(2),
\]

and \( Z = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \), \( z \in \mathbb{C} \). Since \( SU(2) \) is a compact group, let \( d\omega(u_q) \) be the normalized Haar measure on it. From the decomposition (4.1), \( d\mu(q) := e^{-|z|^2} d\lambda d\omega(u_q) \) is a measure on \( H \) and \( L^2_H(H, e^{-|z|^2} d\lambda d\omega(u_q)) \) can be considered to be a left quaternionic Hilbert space, with an inner product defined as in (2.12).

From (4.1), for any \( i, j \in \mathbb{N} \) we have

\[
q^i = u_q \begin{pmatrix} z^i & 0 \\ 0 & \bar{z} \end{pmatrix} u_q^\dagger \quad \text{and} \quad \overline{q}^j = u_q \begin{pmatrix} \overline{z}^j & 0 \\ 0 & \overline{z} \end{pmatrix} u_q^\dagger
\]

and thereby,

\[
(4.2) \quad q^i \overline{q}^j = u_q \begin{pmatrix} z^i \overline{z}^j & 0 \\ 0 & \overline{z} \end{pmatrix} u_q^\dagger.
\]

4.1. The quaternionic Hermite polynomials \( H_n(q) \). Let \( d\eta(q) = d\mu_s(z) d\omega(u_q) \), with \( d\mu_s \) as in (3.11). Since

\[
q^{n-2m} = u_q \begin{pmatrix} z^{n-2m} & 0 \\ 0 & \overline{z}^{n-2m} \end{pmatrix} u_q^\dagger
\]

and the \( C_{nm} \) in (3.3) are real numbers, we have

\[
C_{nm} q^{n-2m} = u_q \begin{pmatrix} C_{nm} z^{n-2m} & 0 \\ 0 & C_{nm} \overline{z}^{n-2m} \end{pmatrix} u_q^\dagger
\]

and thereby

\[
n! \sum_{m=0}^{[n/2]} C_{nm} q^{n-2m} = u_q \left( n! \sum_{m=0}^{[n/2]} C_{nm} z^{n-2m} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) u_q^\dagger
\]

That is

\[
H_n(q) = u_q \begin{pmatrix} H_n(z) & 0 \\ 0 & H_n(\overline{z}) \end{pmatrix} u_q^\dagger.
\]

Observe that

\[
H_n(q) = H_n(\overline{q}) = u_q \begin{pmatrix} H_n(z) & 0 \\ 0 & H_n(\overline{z}) \end{pmatrix} u_q^\dagger.
\]

Similarly, from (3.3) we easily see that

\[
H(q) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m (2q)^{n-2m}}{m!(n-2m)!} ,
\]

which means that they satisfy the same recursion relations,

\[
(4.4) \quad q H_n(q) = \frac{1}{2} H_{n+1}(q) + n H_{n-1}(q),
\]
as the real Hermite polynomials $H_n(x)$, or also
\[ H_{n+1}(q) = 2qH_n(q) - H'_n(q), \]
the prime denoting the (Cullen) derivative.

Let
\[ L^2_H(H, d\eta(q)) = \left\{ f : H \to H \mid \int_H f(q)\overline{f(q)}d\eta(q) < \infty \right\}. \]
This is a left quaternionic Hilbert space with a scalar product as in (2.12).

**Theorem 4.1.** The set \( \{H_n(q) \mid n \in \mathbb{N}\} \) is an orthogonal set in \( L^2_H(H, d\eta(q)) \).

**Proof.** Consider
\[ \int_H H_n(q)\overline{H_m(q)}d\eta(q) = \int_{SU(2)} u_q \left( \int_{\mathbb{R}^2} H_n(z)H_m(\overline{z})d\nu(x, y) \right) u_q^\dagger d\omega(u_q) \]
\[ = \int_{SU(2)} u_q \left( b_n(s) n! \delta_{mn} 0 \right) u_q^\dagger d\omega(u_q) \]
\[ = \int_{SU(2)} u_q u_q^\dagger d\omega(u_q) b_n(s) n! \delta_{mn} \]
\[ = b_n(s) n! \delta_{mn}. \]

Redefine
\[ H^s_n(q) = b_n(s)^{-\frac{1}{2}}H_n(q), \]
then \( H^s_n(q) \in L^2_H(H, d\eta(q)) \) and
\[ \int_H H^s_n(q)\overline{H^s_m(q)}d\eta(q) = \delta_{nm}. \]

Define the kernel
\[ K_s(q_1, q_2) = \sum_{n=0}^{\infty} H^s_n(q_1)\overline{H^s_n(q_2)}, \]
then from (3.9) \( K_s(q_1, q_2) \) is a reproducing kernel and the corresponding reproducing kernel Hilbert space is
\[ A_s = \text{span}\{H^s_n(q) \mid n \in \mathbb{N}\}, \]
the span and its closure being taken under left multiplication by quaternionic constants. Note that the above reproducing kernel is the quaternionic equivalent of the kernel (3.9) and the Hilbert space \( A_s \) the equivalent of \( \mathcal{K}_{hol}^s \), generated by that kernel.

**Lemma 4.2.** \( K_s(q, \overline{q}) = K_s(z, \overline{z})\mathbb{1}_2 \).

**Proof.** Since
\[ H_n(q) = u_q \left( \begin{array}{cc} H_n(z) & 0 \\ 0 & H_n(\overline{z}) \end{array} \right) u_q^\dagger, \]
we have
\[ H_n(q)\overline{H_n(q)} = u_q \left( \begin{array}{cc} H_n(z)\overline{H_n(z)} & 0 \\ 0 & H_n(z)\overline{H_n(z)} \end{array} \right) u_q^\dagger. \]
Since $b_n(s)$ are real numbers,
\[
\begin{align*}
  b_n(s)^{-1} H_n(q) \overline{H_n(q)} &= u_q \left( b_n(s)^{-1} H_n(z) \overline{H_n(z)} \begin{pmatrix} 0 & 0 \\ b_n(s)^{-1} H_n(z) & 0 \end{pmatrix} \right) u_q^* \\
  &= u_q \left( h_{n,s}(z) \overline{h_{n,s}(z)} \begin{pmatrix} 0 & 0 \\ h_{n,s}(z) & 0 \end{pmatrix} \right) u_q^*.
\end{align*}
\]

Therefore
\[
\begin{align*}
  \sum_n b_n(s)^{-1} H_n(q) \overline{H_n(q)} &= u_q \left( \sum_n h_{n,s}(z) \overline{h_{n,s}(z)} \begin{pmatrix} 0 & 0 \\ h_{n,s}(z) & 0 \end{pmatrix} \right) u_q^* \\
  &= u(q) \left( K_s(z,z) \begin{pmatrix} 0 & 0 \\ K_s(z,z) & 0 \end{pmatrix} \right) u_q^* \\
  &= K_s(z,\overline{z}) \mathbb{I}_2.
\end{align*}
\]

That is,
\[
(4.10) \quad K_s(q,\overline{q}) = K_s(z,\overline{z}) \mathbb{I}_2.
\]

4.2. The quaternionic Hermite polynomials $H_{n,m}(q,\overline{q})$. The exponential series
\[
e^q = \sum_{n=0}^{\infty} \frac{q^n}{n!}, \quad q \in H,
\]
converges absolutely, and uniformly on compact subsets with respect to the norm of $H$ [9] (p 204). Thereby
\[
e^{-|q|^2} = e^{-q\overline{q}} = \sum_{n=0}^{\infty} (-1)^n \frac{(q\overline{q})^n}{n!}
\]
converges uniformly. Further $q$ and $\overline{q}$ commute and real numbers commute with quaternions. Therefore, as in the complex case, as an extension of complex hermite polynomials, using Definition $2.6$ we get
\[
h_{n,m}(q,\overline{q}) = (-1)^{n+m} e^{q^2} \frac{\partial^{n+m}}{\partial q^n \partial \overline{q}^m} e^{-|q|^2} = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} q^i \overline{q}^j
\]
(4.11)
\[
= n!m! \sum_{j=0}^{\min\{n,m\}} \frac{(q)^{n-j} (\overline{q})^{m-j}}{(n-j)! (m-j)!}.
\]
(4.12)

where the $a_{ij}$ are real coefficients and the derivatives should be understood in the Cullen sense. Further from (4.2) we have
\[
a_{ij} q^i \overline{q}^j = u_q \left( \begin{pmatrix} a_{ij} z^i \overline{z}^j & 0 \\ 0 & a_{ij} \overline{z}^i z^j \end{pmatrix} \right) u_q^*.
\]

Therefore
\[
\begin{align*}
  \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} q^i \overline{q}^j &= u_q \left( \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} z^i \overline{z}^j \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) u_q^* \\
  &= u_q \left( \begin{pmatrix} h_{n,m}(z,\overline{z}) & 0 \\ 0 & h_{m,n}(z,\overline{z}) \end{pmatrix} \right) u_q^*.
\end{align*}
\]

Therefore from (4.11) we get
\[
(4.13) \quad h_{n,m}(q,\overline{q}) = u_q \left( \begin{pmatrix} h_{n,m}(z,\overline{z}) & 0 \\ 0 & h_{m,n}(z,\overline{z}) \end{pmatrix} \right) u_q^*.
\]
We can see from (3.15) that $h_{n,m}(q,\overline{q})$ is obtained by substituting $z$ by $q$ in the definition (3.15) of $h_{n,m}(z,\overline{z})$. Thereby, using the fact that $h_{n,m}(z,\overline{z}) \in L^2(\mathbb{C},e^{-|z|^2}d\lambda)$ and $SU(2)$ is a compact group, we have $h_{n,m}(q,\overline{q}) \in L^2_H(H,e^{-|z|^2}d\lambda d\omega(u_q))$.

**Theorem 4.3.** The set $\{h_{n,m}(q,\overline{q}) \mid m,n \in \mathbb{N}\}$ is an orthogonal set in $L^2_H(H,e^{-|z|^2}d\lambda d\omega(u_q))$.

**Proof.** See [19].

**Theorem 4.4.** $\|h_{m,n}(q,\overline{q})\|_{L^2_H} = \sqrt{n!}m!$ in $L^2_H(H,e^{-|z|^2}d\lambda d\omega(u_q))$.

**Proof.** From (3.16) and (4.13) we have

\[
\int_H h_{n,m}(q,\overline{q})h_{n,m}(q,\overline{q})e^{-|z|^2}d\lambda d\omega(u_q)
= \int_{SU(2)} \int_{\mathbb{C}} u_q \left( \begin{array}{c} h_{n,m}(z,\overline{z})h_{n,m}(z,\overline{z}) \\ 0 \\ h_{n,m}(z,\overline{z})h_{n,m}(z,\overline{z}) \\ 0 \end{array} \right) u_q^\dagger e^{-|z|^2}d\lambda d\omega(u_q)
= \int_{SU(2)} u_q \left( \begin{array}{c} \int_{\mathbb{C}} h_{n,m}(z,\overline{z})h_{n,m}(z,\overline{z})e^{-|z|^2}d\lambda \\ 0 \\ \int_{\mathbb{C}} h_{n,m}(z,\overline{z})h_{n,m}(z,\overline{z})e^{-|z|^2}d\lambda \end{array} \right) u_q^\dagger d\omega(u_q)
= n!m! \int_{SU(2)} u_q u_q^\dagger d\omega(u_q)
= n!m! \|2\).

From [13] [16], (3.17) and (4.13) it is clear that the functions $e^{-|q|^2/2}h_{n,m}(q,\overline{q})$ are the eigenfunctions of the operator

\[\Lambda_H = u_q \left( \begin{array}{cc} L & 0 \\ 0 & \overline{L} \end{array} \right) u_q^\dagger\]

with spectrum $n + \frac{1}{2}$, each level being infinitely degenerate ($m = 0, 1, 2, 3, \ldots$). Even though this operator can be considered as the quaternionic version of the complex Landau operator, we do not have at the moment a proper physical understanding of it.

5. Coherent states

In this section we define CS over quaternionic Hilbert spaces and in particular CS arising from quaternionic Hilbert spaces of (slice) regular functions. As examples we build CS using the quaternionic Hermite polynomials $H_n(q)$ and $H_{n,m}(q,\overline{q})$.

5.1. The general construction. Coherent states may be built on quaternionic Hilbert spaces, in more or less the same way as was outlined in Section 1 for coherent states on complex Hilbert spaces. Indeed, let $V^L_H$ be a left quaternionic Hilbert space whose dimension could be finite or countably infinite and let $\phi_m, \ m = 0, 1, 2, \ldots$, be an orthonormal basis of this space. Let $X$ be a locally compact space and $\mu$ a (Radon) measure on it. Consider a set of functions $\Phi_m : X \to H, \ m = 0, 1, 2, \ldots$, of the same cardinality as the dimension of $V^L_H$, and which satisfy the two conditions,

1. $0 < \mathcal{N}(x) := \sum_m |\Phi_m(x)|^2 < \infty$, for all $x \in X$.
2. $\int_X \Phi_m(x)\overline{\Phi_n(x)} \ d\mu(x) = \delta_{mn}$, for all $m$ and $n$. 


A family of coherent states \( \{ \eta_x \mid x \in X \} \subset V_H^L \) can now be defined to be the vectors,
\[
(5.1) \quad \eta_x = \mathcal{N}(x)^{-\frac{1}{2}} \sum_m \Phi_m(x) \phi_m.
\]

By construction, these coherent states are seen to be normalized, i.e., \( \| \eta_x \|^2 = 1 \), for all \( x \in X \), and to satisfy the resolution of the identity,
\[
\int_X \langle f \mid \eta_x \rangle \langle \eta_x \mid g \rangle \mathcal{N}(x) d\mu(x) = \langle f \mid g \rangle, \quad f, g \in V_H^L.
\]
Moreover, taking \( L^2_H(X, d\mu) \) to be a left quaternionic Hilbert space, the map
\[
(5.2) \quad W : V_H^L \rightarrow L^2_H(X, d\mu), \quad \text{with} \quad Wf(x) = \mathcal{N}(x)^{\frac{1}{2}} \langle f \mid \eta_x \rangle_{V_H^L}
\]
is a linear isometry onto a closed subspace
\[
\mathcal{H}_K := WV_H^L \subset L^2_H(X, d\mu).
\]
The subspace \( \mathcal{H}_K \) is a reproducing kernel Hilbert space, with reproducing kernel
\[
(5.3) \quad K : X \times X \rightarrow H, \quad K(y, x) = \left[ \mathcal{N}(y) \mathcal{N}(x) \right]^{\frac{1}{2}} \langle \eta_y \mid \eta_x \rangle = \sum_m \Phi_m(y) \overline{\Phi_m(x)}.
\]
Thus, if \( F \in \mathcal{H}_K \), so that \( F(x) = \mathcal{N}(x)^{\frac{1}{2}} \langle f \mid \eta_x \rangle \), for some \( f \in V_K^L \), then
\[
\int_X F(y) K(x, y) \, d\mu(y) = F(x),
\]
for all \( x \in X \). This also means that for each \( x \in X \), the evaluation map, \( E_x : \mathcal{H}_K \rightarrow H \), with \( E_x(F) = F(x) \), is continuous and
\[
(5.4) \quad |F(x)| \leq \mathcal{N}(x)^{\frac{1}{2}} \| f \|_{V_K^L} = \mathcal{N}(x)^{\frac{1}{2}} \| F \|_{\mathcal{H}_K}.
\]

All these results are familiar from the theory of coherent states on complex Hilbert spaces. Thus, entirely analogous results hold on quaternionic Hilbert spaces.

Suppose, in particular, that \( X = \mathcal{D} \) (some open ball in \( H \), centered at the origin) and that the functions \( \Phi_m, \quad m = 0, 1, 2, \ldots, \) are elements in \( L^2_H(\mathcal{D}, d\mu) \), which are regular functions, whose Taylor expansions (around the origin) have real coefficients (for example, they could be normalized polynomials, of degree \( m \) in the quaternionic variable \( \mathbf{q} \), with real coefficients). Then, for any \( f \in V_K^L \) the transformed function \( Wf \in L^2_H(\mathcal{D}, d\mu) \) is a series in the conjugate variable \( \overline{\mathbf{q}} \) i.e., it has the form
\[
Wf(\overline{\mathbf{q}}) = \sum_m \alpha_m \mathbf{q}^m, \quad \alpha_m \in H,
\]
the sum converging in the \( L^2 \)-norm. We now show that the above series is in fact a right anti-regular function (i.e., regular in the variable \( \overline{\mathbf{q}} \)). For this we need the following lemma, which is an adaptation of Lemma 2.5 in [11].

Lemma 5.1. If \( f \) is a right regular function on \( B = B(0, R) \), then for every \( I \in \mathbb{S} \) and every \( J \in \mathbb{S} \), perpendicular to \( I \), there exist two holomorphic functions \( F, G : B \cap L_I \rightarrow L_I \) such that the restriction of \( f \) to \( L_I \) can be split as the sum
\[
(5.5) \quad f_I(z) = F(z) + JG(z), \quad z = x + yI \in L_I.
\]

Note that if \( f \) were to be an anti-regular function, a similar splitting into two anti-holomorphic functions would hold.
Hence each
function is anti-holomorphic, Lemma 5.1 implies that $f$ is also anti-regular, proving the theorem. □

Henceforth we shall denote the space $\mathcal{H}_K$ by $\mathcal{H}_{a-reg}$. In an entirely analogous manner we can also define a reproducing kernel Hilbert space $\mathcal{H}_{a-reg}$ of regular functions, starting from coherent states built out of the polynomials $H_m(\bar{q})$ instead of $H_m(q)$. Both these spaces are contained as subspaces of $L^2(H, d\mu)$.

5.2. Two examples. We now construct coherent states following the above procedure and using the Hermite polynomials introduced in Section 4. From (4.9) and Lemma 4.2 it is clear that the functions $H_n(\bar{q})$ in (4.6) satisfy the conditions 1. and 2. stated at the beginning of Section 5.1. Thus, we have the result:

**Theorem 5.3.** Let $\{\phi_n\}_{n=0}^{\infty}$ be an orthonormal basis of the left quaternionic Hilbert space $V^L_H$. For $q \in H$, $0 < s < 1$, $K_s(q, \bar{q})$ as in (4.10) and $H_n^s(q)$ as in (4.10), the set of vectors

$$\eta_{q,s} = K_s(q, \bar{q})^{-\frac{1}{2}} \sum_{n=0}^{\infty} H_n^s(q)\phi_n \in V^L_H$$

forms a set of coherent states.

By Theorem 5.3 the reproducing kernel Hilbert space associated to this family of CS is a space of right anti-regular functions. Similarly, had we constructed these CS using the functions $H_n^s(\bar{q})$, the corresponding reproducing kernel Hilbert space would have consisted of right regular functions, and in fact would have been the space $A_s$ in (4.9).

Next, for each fixed $n$, let $B_n = \{h_{n,m}(q, \bar{q}) \mid m \in \mathbb{N}_0\}$ and let $A_n(H)$ be the closed linear span, under left multiplication by quaternionic scalars, of the vectors in $B_n$. Then $B_n$ is a basis of $A_n(H)$ and $\bigoplus_{n=0}^{\infty} A_n(H)$ is a left quaternionic Hilbert space, which is a closed subspace of $L^2_H(H, e^{-|z|^2}d\lambda d\omega(u_q))$. Note that unlike in the case of the complex polynomials, discussed in Section 5.2 where the vectors $h_{n,m}$ spanned the entire space $L^2(\mathbb{C}, e^{-|z|^2}d\lambda)$, here $\bigoplus_{n=0}^{\infty} A_n(H)$ is only a proper subspace of $L^2_H(H, e^{-|z|^2}d\lambda d\omega(u_q))$. 
Further

\begin{equation}
K_n(q_1, q_2) = \sum_{m=0}^{\infty} \frac{1}{n^m m!} h_{n,m}(q_1) h_{n,m}(q_2)
\end{equation}

is a reproducing kernel of the Hilbert subspace \( A_n(H) \). The convergence of the above sum easily follows from the convergence of the analogous sum in the complex case, i.e., with \( q \) replaced by the complex variable \( z \) [1]. In particular we have \( K_0(q, q) = e^{|q|^2} \).

Assume that \( \{\phi_m\}_{m=0}^{\infty} \) is an orthonormal basis of \( V_H^L \). For \( q \in H \), define

\begin{equation}
\eta_{q,n} := K_n(q, q)^{-1/2} \sum_{m=0}^{\infty} \frac{h_{n,m}(q, q)}{\sqrt{n^m m!}} \phi_m \in V_H^L.
\end{equation}

The vectors \( \{\eta_{q,n} \mid q \in H\} \) are then a family of coherent states, for each \( n \). In particular, the vectors

\[ \eta_{q,0} = e^{-\frac{|q|^2}{2}} \sum_{m=0}^{\infty} \frac{q^m}{\sqrt{m!}} \phi_m \in V_H^L, \]

are the so-called canonical quaternionic coherent states. The corresponding reproducing kernel, which is easily computed using (5.3), is seen to be

\[ K_0(q_1, q_2) = \sum_{n=0}^{\infty} \frac{q_1^n q_2^n}{n!}, \]

the quaternionic analogue of the well-known complex Bargmann kernel. (Note that, since \( q_1 \) and \( q_2 \) do not necessarily commute, we cannot write this as \( \exp[q_1 q_2] \)).

Again, the reproducing kernel Hilbert space associated to these canonical quaternionic CS is a space of right anti-regular functions. It is of course a subspace of the bigger space \( L_H^2(H, e^{-|z|^2} d\lambda d\omega(u_q)) \). This is analogous to the space of anti-analytic functions generated by the canonical CS [12] on a complex Hilbert space. Similarly, we could have constructed the conjugate family of canonical quaternionic CS and the reproducing kernel space would have consisted of right regular functions, again as in the complex case. Thus, we get the two spaces (completion under left multiplication by quaternionic scalars is implied),

\[ \mathcal{H}_{reg} = \text{span} \{h_{0,m}(q, q) \mid m \in \mathbb{N}\} \]

and

\[ \mathcal{H}_{a-reg} = \text{span} \{h_{n,0}(q, q) \mid n \in \mathbb{N}\}, \]

of right regular and right anti-regular functions, respectively. The first Hilbert space is the quaternionic analogue of the Bargmann space of analytic functions, with

\[ h_{0,m}(q, q) = \frac{q^m}{\sqrt{m!}}, \quad m = 0, 1, 2, \ldots. \]

It is interesting to note that if we define the formal annihilation and creation operators on this space by

\[ a h_{0,m} = \sqrt{m} h_{0,m-1}, \quad a^\dagger h_{0,m} = \sqrt{m} + 1 h_{0,m+1}, \]

then these have realizations by the (Cullen) derivative w.r.t. \( q \) and multiplication by \( q \), respectively (again in complete analogy with the complex case).

Note that since the \( h_{n,m}(q, q) \) are mutually orthogonal elements in the ambient space \( L_H^2(H, e^{-|z|^2} d\lambda d\omega(u_q)) \), the elements of \( \mathcal{H}_{reg} \) and \( \mathcal{H}_{a-reg} \) are mutually orthogonal except for the one-dimensional subspace generated by the vector \( h_{0,0}(q, q) = 1 \) which is common to both spaces.
6. Conclusion

Using the notion of the Cullen derivative and the related notions of regular and anti-regular functions of quaternionic variables, we have obtained a wide ranging generalization of certain physically interesting classes of coherent states to quaternionic Hilbert spaces. The analysis shows, among others, that all the so-called non-linear coherent states, which can be realized on Hilbert spaces of analytic or anti-analytic functions, have quaternionic generalizations. In the process we have also obtained fairly straightforward generalizations of two different types of complex orthogonal Hermite polynomials to analogous polynomials in a quaternionic variable, again satisfying similar orthogonality and recursion relations. It would be interesting to explore other families of orthogonal polynomials in the same vein. In recent years, formulations of quantum mechanics on quaternionic Hilbert spaces have been proposed to address some of the conceptual problems associated to particle interactions at very short distances (see, for example, [1]). In view of the importance of coherent states in usual quantum mechanics, it is expected that they would also be of importance in quaternionic quantum mechanics.

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