Bilinear Bochner–Riesz Square Function and Applications

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Received: 4 August 2022 / Revised: 30 July 2023 / Accepted: 5 September 2023 / Published online: 11 October 2023
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Abstract
In this paper, we introduce Stein’s square function associated with bilinear Bochner–Riesz means and investigate its $L^p$–boundedness properties. Further, we discuss several applications of the square function in the context of bilinear multipliers. In particular, we obtain results for maximal function associated with generalised bilinear Bochner–Riesz means. This extends the results proved in [22]. Another application concerns the $L^p$–estimates for bilinear fractional Schrödinger multipliers. Finally, we improve upon a result of Grafakos, He and Honzik [17] in the context of bilinear radial multipliers and provide a dimension-free sufficient condition on the bilinear multipliers for $L^2 \times L^2 \rightarrow L^1$–boundedness of the associated maximal function. The generalised bilinear spherical maximal function is a particular example of such maximal functions.

Keywords Stein’s square function · Bochner–Riesz means · Bilinear multipliers · Sparse operators · Maximal functions
1 Introduction

1.1 Stein’s Square Function

The square function associated with Bochner–Riesz means was introduced by Stein in [34]. It is commonly referred to as Stein’s square function and is defined by

$$G^\alpha(f)(x) := \left( \int_0^\infty \left| \frac{\partial}{\partial t} B_{\alpha+1}^t f(x) \right|^2 t \, dt \right)^{\frac{1}{2}} = \left( \int_0^\infty |K_{\frac{t}{2}} \ast f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

where $\widehat{B_{\alpha}^t f}(\xi) = \left( 1 - \frac{|\xi|^2}{t} \right)^{\alpha} \hat{f}(\xi)$ is the classical Bochner–Riesz operator with index $\alpha$. Note that the kernel is given by $\widehat{K_{\frac{t}{2}}}(\xi) = 2(\alpha + 1) \frac{|\xi|^2}{t^2} \left( 1 - \frac{|\xi|^2}{t} \right)^{\alpha}$. Here $\hat{f}$ denotes the Fourier transform of $f$ defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx$.

The square function naturally appears in the study of maximal Fourier multiplier operators and plays a crucial role. We refer the reader to [4, 5, 7, 25–28, 34, 36] and references therein for details.

The $L^p$–estimates

$$\|G^\alpha(f)\|_p \lesssim \|f\|_p$$

for the square function $G^\alpha$ have been studied extensively in the literature. The Plancherel theorem yields $L^2(\mathbb{R}^n)$–boundedness of $G^\alpha$ for $\alpha > -\frac{1}{2}$, see [34]. For $p \neq 2$, it is conjectured that the estimate (1) holds for $1 < p < 2$ if, and only if $\alpha > n \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{1}{2}$. Whereas for the range $p > 2$ it is conjectured that the estimate (1) holds if, and only if $\alpha > \alpha(p) - \frac{1}{2}$, where

$$\alpha(p) = \max \left\{ n \left| \frac{1}{p} - \frac{1}{2} \right| - \frac{1}{2}, 0 \right\}.$$

The conjecture for the range $1 < p < 2$ has been settled, i.e. the estimate (1) holds if, and only if $\alpha > n \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{1}{2}$ for $1 < p \leq 2$ and $n \geq 1$. The proof uses the idea of Stein’s analytic interpolation for a family of operators between $L^2$–estimate for $\alpha > -\frac{1}{2}$ and $L^p$–estimate for $\alpha > \frac{n-1}{2}$, see [27, 36] for details. Further, in dimensions $n = 1, 2$, the conjecture has been proved to hold for the range $p > 2$ as well, see [24] and [4] for the case of $n = 1$ and $n = 2$ respectively. However, for $n \geq 3$ and $p > 2$ the sufficient part of the conjecture is not known yet completely. There are many interesting developments in this direction, see [7, 26–28, 32] and references therein for more details. In order to state the recent development on the conjecture, we set some notation.

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For \( n \geq 2 \), define \( p_0(n) = 2 + \frac{12}{4n - 6 - k} \) where \( n \equiv k \mod 3 \), \( k = 0, 1, 2 \). Denote
\[
p_n = \min \left\{ p_0(n), \frac{2(n + 2)}{n} \right\}.
\]

Lee [26] proved the following result.

**Theorem 1.1** [26] For \( n \geq 2 \), the square function \( G^\alpha \) satisfies the estimate (1) for \( p \geq p_n \) and \( \alpha > n \left( \frac{1}{2} - \frac{1}{p} \right) - 1 \).

Motivated by the recent progress on the bilinear Bochner–Riesz problem and a wide scope of applications of Stein’s square function, in this paper we introduce and study the bilinear analogue of the Stein’s square function. Consequently, we discuss several connections of the square function in the context of bilinear multipliers. This allows us to obtain new results for maximal function associated with generalised bilinear Bochner–Riesz means and bilinear fractional Schrödinger operator. Also, we improve upon a result by Grafakos, He and Honzik [17] for radial bilinear multipliers. These results are described in Sects. 1.2, 2.3 and 2.4. Let us first briefly recall some recent developments in the direction of bilinear Bochner–Riesz means.

### 1.2 Bilinear Bochner–Riesz Means and Maximal Function

For Schwartz class functions \( f, g \in \mathcal{S}(\mathbb{R}^n) \), \( n \geq 1 \) and \( \alpha \geq 0 \), the bilinear Bochner–Riesz mean \( B^\alpha_R(f, g) \) is defined by
\[
B^\alpha_R(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( 1 - \frac{|\xi|^2 + |\eta|^2}{R^2} \right)^\alpha \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \, d\xi \, d\eta, \ \ R > 0.
\]

The problem of finding necessary and sufficient conditions on exponents \( p_1, p_2, p \) and the index \( \alpha \) for which the estimate (2) holds is commonly referred to as the bilinear Bochner–Riesz problem. Here we assume that the exponents \( p_1, p_2, p \) satisfy the Hölder relation \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \). Observe that due to the standard dilation argument, it is enough to study the estimate (2) with \( R = 1 \). For \( R = 1 \) we drop the suffix \( R \) from \( B^\alpha_R \) and simply denote it by \( B^\alpha \).

If \( \alpha = 0 \), the operator \( B^0 \) (denoted by \( B \)) is called the bilinear ball multiplier operator. In dimension \( n = 1 \) Grafakos and Li [18] proved the estimate (2) for \( B \) for all \( p_1, p_2, p \) satisfying \( 2 \leq p_1, p_2, p' < \infty \). Here \( p' \) denotes the conjugate index of \( p \) given by \( \frac{1}{p} + \frac{1}{p'} = 1 \). The range \( L = \{(p_1, p_2, p) : 2 \leq p_1, p_2, p' < \infty \} \) is referred to as the local \( L^2 \)-range of exponents. Later, Diestel and Grafakos [11] showed that in dimension \( n \geq 2 \), the operator \( B \) fails to satisfy the estimate (2) provided exactly one of \( p_1, p_2 \) or \( p' \) is less than 2. In [1] Bernicot et al. initiated the study of the operator \( B^\alpha, \alpha > 0 \) and established the estimate (2) under certain conditions on \( p_1, p_2, \alpha \) and
the dimension $n$. In dimension $n = 1$, the results proved in [1] provide an almost complete picture of the Banach triangle, i.e., $1 \leq p_1, p_2, p \leq \infty$. However, in higher dimensions, the results are far from being optimal. Liu and Wang [30] extended some of these results, specifically to the non-Banach triangle (i.e. when $p < 1$) thereby improving the range of $p_1, p_2, p$. Later, Jeong, Lee, and Vargas [21] improved the range of exponents significantly when $p_1, p_2 \geq 2$ and lowered the bounds on $\alpha$ for the estimate (2). They decomposed the operator $\mathcal{B}^\alpha$ into discretized square functions and obtained new results, see [Section 3, [21]] for details. In particular, they proved optimal results for the estimate (2) when $p_1 = p_2 = 2$ and $\alpha > 0$ for all $n \geq 2$.

The maximal function associated with the bilinear Bochner–Riesz means, defined by

$$
\mathcal{B}_R^\alpha(f, g)(x) = \sup_{R > 0} |\mathcal{B}_R^\alpha(f, g)(x)|
$$

plays a key role in addressing the almost everywhere convergence of the bilinear Bochner–Riesz means $\mathcal{B}_R^\alpha(f, g)$ as $R \to \infty$. We refer to Grafakos, He and Honzik [17] and Jeong and Lee [20] for initiating the study of $L^p$–estimates for $\mathcal{B}_n^\alpha$. Recently, Jotsaroop and Shrivastava [22] introduced a different approach to study $L^p$–boundedness of bilinear maximal function $\mathcal{B}^\alpha$. Their approach works uniformly in all dimensions. They recovered the results obtained in [21] for Bochner–Riesz means when $n \geq 2$ and provided new and improved results for the case of dimension $n = 1$ for exponents in the non-Banach triangle. We also refer to [23] for weighted estimates for the bilinear Bochner–Riesz means $\mathcal{B}_n^{\frac{1}{2}}$.

### 1.3 Bilinear Bochner–Riesz Square Function

The bilinear Bochner–Riesz square function of order $\alpha$, denoted by $\mathcal{G}^\alpha$, is defined by

$$
\mathcal{G}^\alpha(f, g)(x) := \left( \int_0^\infty \left( \frac{\partial}{\partial R} \mathcal{B}_R^{\alpha+1}(f, g)(x) \right)^2 R dR \right)^{\frac{1}{2}}.
$$

Note that $\frac{\partial}{\partial R} \mathcal{B}_R^{\alpha+1}(f, g)(x)$ makes sense for $\alpha > -1$ for each $R > 0$. We rewrite the square function $\mathcal{G}^\alpha(f, g)$ in the following way.

$$
\mathcal{G}^\alpha(f, g)(x) = \left( \int_0^\infty |\mathcal{K}^\alpha_R \ast (f \otimes g)(x, x)|^2 \frac{dR}{R} \right)^{\frac{1}{2}},
$$

where we have used the notation

$$
\mathcal{K}^\alpha_R \ast (f \otimes g)(x, x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{K}^\alpha_R(x - y, x - z)f(y)g(z)dydz.
$$
The kernel $K^\alpha_R$ is given by $\hat{K}^\alpha_R(\xi, \eta) = 2(\alpha + 1) \left| \frac{\xi^2 + \eta^2}{R^2} \right|^\alpha_+$. In the spatial variables, the kernel is of the form

$$K^\alpha_R(y_1, y_2) = c_{n+\alpha} R^{2n-2} \Delta \left( \frac{J_{\alpha+n}(2\pi |(Ry_1, Ry_2)|)}{|(Ry_1, Ry_2)|^{\alpha+n}} \right), \quad y_1, y_2 \in \mathbb{R}^n.$$ 

Here $J_{\alpha+n}$ denotes the Bessel function of order $\alpha + n$.

Let us denote

$$G^\alpha_R(f, g)(x) = K^\alpha_R * (f, g)(x).$$

Our main goal in this paper is to investigate the necessary and sufficient conditions on the exponents $p_1, p_2, p$ and $\alpha$ so that the estimate

$$\|G^\alpha(f, g)\|_p \lesssim \|f\|_{p_1} \|g\|_{p_2} \quad (4)$$

holds. Here the notation $A \lesssim B$ in the above means that there is an implicit constant $C > 0$ such that $A \leq CB$. The constant $C$ is independent of essential quantities like functions appearing in the estimate. However, it may depend on parameters $\alpha, n, p_1$ and $p_2$. Sometimes, we will also use the notation $A \lesssim \epsilon B$ to emphasize the dependence of the implied constant on the parameter $\epsilon$. Also, we use calligraphy letters to denote the bilinear operators, whereas the corresponding capital letters are used to denote the operators from the theory of linear operators.

### Organization of the paper

In Sect. 2 we provide statements of results of this paper. The proofs of square function boundedness results, namely Theorems 2.2 and 2.1 are given in Sects. 4 and 5 respectively. We discuss the idea of analytic interpolation for $G^\alpha$ in Sect. 6 and prove Theorem 2.3. The necessary conditions on $\alpha$ are obtained in Sect. 7. Finally, the sparse domination result Theorem 7.2 is proved in Appendix 7.

## 2 Results

### 2.1 $L^p$—estimates for Bilinear Square Function $G^\alpha$

Let $n \geq 2$ and $1 \leq p_1, p_2 \leq \infty$. Let us consider the following notation.
\[
\alpha(p_1) + \alpha(p_2) \quad \text{when } p_n \leq p_1, p_2 \leq \infty; \\
\alpha(p_1) + \left(\frac{1-2p_2^{-1}}{1-2(p_n)^{-1}}\right) \alpha(p_n) \quad \text{when } p_n \leq p_1 \leq \infty \text{ and } 2 \leq p_2 < p_n; \\
\left(\frac{1-2p_1^{-1}}{1-2(p_n)^{-1}}\right) \alpha(p_n) + \alpha(p_2) \quad \text{when } 2 \leq p_1 < p_n \text{ and } p_n \leq p_2 \leq \infty; \\
\left(\frac{2-2p_1^{-1}-2p_1^{-1}}{1-2(p_n)^{-1}}\right) \alpha(p_n) \quad \text{when } 2 \leq p_1, p_2 < p_n.
\]

The following \(L^p\)-boundedness results for \(G^\alpha\) hold.

**Theorem 2.1** Let \(n = 1\) and \(1 < p_1, p_2 < \infty\) be such that \(\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}\). The bilinear Bochner–Riesz square function \(G^\alpha\) maps \(L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})\) into \(L^p(\mathbb{R})\) for each of the following cases.

1. \(p_1, p_2 \geq 2\) and \(\alpha > 0\).
2. \(1 < p_1 < 2, p_2 \geq 2\) and \(\alpha > \frac{1}{p_1} - \frac{1}{2}\).
3. \(1 < p_2 < 2, p_1 \geq 2\) and \(\alpha > \frac{1}{p_2} - \frac{1}{2}\).
4. \(1 < p_1, p_2 < 2\) and \(\alpha > \frac{1}{p} - 1\).

**Theorem 2.2** Let \(n \geq 2\) and \((p_1, p_2, p)\) be such that \(p_1, p_2 \geq 2\) and \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\), then for \(\alpha > \alpha^*(p_1, p_2)\) the bilinear Bochner–Riesz square function \(G^\alpha\) maps \(L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)\) into \(L^p(\mathbb{R}^n)\).

Next, we make use of Stein’s interpolation for an analytic family of bilinear operators (see [Theorem 7.2.9, [16]]) to extend the boundedness of \(G^\alpha\) when either of the exponents \(p_1\) or \(p_2\) is less than 2. This idea requires \(L^p\)-estimates for \(G^\alpha\) when \(\alpha\) is larger than the critical index \(n - \frac{1}{2}\). The \(L^p\)-estimates for \(G^\alpha\) for \(\alpha > n - \frac{1}{2}\) can be easily proved using the arguments from its linear counterpart. Since this part does not require any non-trivial modification in the existing arguments, we skip the details for now and provide them in the Appendix for completeness.

**Theorem 2.3** Let \(1 < p < 2\), then \(G^\alpha\) is bounded from \(L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)\) to \(L^{p/2}(\mathbb{R}^n)\) for \(\alpha > (2n-1)(\frac{1}{p} - \frac{1}{2})\).

**Remark 2.4** Observe that when \(n = 1\) we have that \(\frac{1}{p} - \frac{1}{2} < \frac{2}{p} - 1\) for \(p < 2\). Therefore, we get an improved range of exponents in Theorem 2.3 as compared to case (4) when \(p_1 = p_2\) in Theorem 2.1.

The next result describes the necessary conditions for the \(L^p\)-boundedness of the square function \(G^\alpha\).

**Proposition 2.5** Assume that \(G^\alpha\) is bounded from \(L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)\) to \(L^p(\mathbb{R}^n)\). Then the exponents satisfy the following necessary conditions.
Finally, we show that $L^p$-estimates for $G^\alpha$ can be used to prove new results for various types of bilinear operators.

### 2.2 Generalised Bilinear Bochner–Riesz Means and Maximal Function

Let $\alpha, \lambda > 0$. For Schwartz class functions $f, g \in \mathcal{S}(\mathbb{R}^n), n \geq 1$ consider the generalised bilinear Bochner–Riesz mean $B^\alpha_{\lambda, R}(f, g)$ defined by

$$B^\alpha_{\lambda, R}(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(1 - \frac{|(\xi, \eta)|^\lambda}{R^\lambda}\right)^\alpha \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta, \quad R > 0,$$

Note that when $\lambda = 2$ we have $B^\alpha_{\lambda, R}(f, g)(x) = B^\alpha_{R}(f, g)(x)$. Consider the maximal function

$$B^\alpha_{\lambda, *}(f, g)(x) = \sup_{R > 0} |B^\alpha_{\lambda, R}(f, g)(x)|.$$

The following estimate holds.

**Theorem 2.6** Let $n \geq 2$ and $\alpha > \alpha_*(p_1, p_2) + 1/2$. Then the maximal function $B^\alpha_{\lambda, *}(f, g)$ maps $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ where $p_1, p_2 \geq 2$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

We will invoke $L^p-$boundedness results for the square function from Theorems 2.1 and 2.2 to get $L^p-$estimates for the maximal function $B^\alpha_{\lambda, *}(f, g)(x)$. Note that this generalises the results proved in [21, 22] to the setting of generalised bilinear Bochner–Riesz means. We would like to emphasise here that the methods used in [21, 22] explicitly use the fact that $\lambda = 2$ and do not apply to the case of $\lambda \neq 2$ directly. Therefore, the results obtained in Theorem 2.6 are new for $\lambda \neq 2$. This is possible due to the use of the square function. However, we conjecture that the range of $L^p-$boundedness of $B^\alpha_{\lambda, *}$ for $\lambda \neq 2$ in theorem 2.6 above should be similar to the standard maximal bilinear Bochner Riesz means, i.e. the case of $\lambda = 2$.

### 2.3 Bilinear Fractional Schrödinger Multiplier

The fractional Schrödinger equation is defined as

$$\frac{\partial}{\partial s} u(s, x) = (-\Delta)^{\beta} u(s, x) \quad \text{when } s > 0, \quad u(0, x) = f(x).$$
The solution to this equation with initial data \( f \) is of the form

\[
e^{is(-\Delta)^\beta} f(x) = \int_{\mathbb{R}^n} m_\beta(s|\xi|^2) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi, \quad s > 0,
\]

where \( m_\beta(u) = e^{i|u|^\beta}, u \in \mathbb{R}, \beta > 0 \) and it is called the fractional Schrödinger multiplier. The problem of finding optimal \( \gamma \) for \( \beta = 1 \) such that \( e^{-is\Delta} f \to f \) a.e. as \( s \to 0^+ \) when \( (I + (-\Delta)^{\gamma/2}) f \in L^2(\mathbb{R}^n) \) has been resolved recently in [12, 13]. When \( \gamma > \frac{n}{2(n+1)} \) and \( (I + (-\Delta)^{\gamma/2}) f \in L^2(\mathbb{R}^n) \) it is known that \( \lim_{s \to 0^+} e^{-is\Delta} f = f \) a.e. This result is sharp except at the end-point \( \gamma = \frac{n}{2(n+1)} \) for \( n \geq 2 \), see [12, 13]. When \( n = 1 \) it was proved that \( e^{-is\Delta} f \to f \) a.e. as \( s \to 0^+ \) if and only if \( \gamma \geq \frac{1}{4} \), see [3, 9].

We consider the bilinear fractional Schrödinger operator defined by

\[
T_{m_\beta,s}(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m_\beta(s^2|\xi, \eta|^2) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta. \tag{5}
\]

Note that \( T_{m_\beta,s}(f, g)(x) \) is the restriction of the solution of the linear fractional Schrödinger equation in \( \mathbb{R}^{2n} \) with initial data given by product type function, i.e. of the form \( (f \otimes g)(y, z) = f(y)g(z) \). We are concerned with the problem of convergence of \( T_{m_\beta,s}(f, g)(\cdot) \) a.e. to \( f(\cdot)g(\cdot) \) when \( (I + (-\Delta)^{\gamma/2}) f, (I + (-\Delta)^{\gamma/2}) g \in L^2(\mathbb{R}^n) \) for some \( \gamma \). In order to address this problem, we establish the following result for the associated maximal function.

**Theorem 2.7** Let \( T_{m_\beta,s}(f, g) \) be as defined in (5). If \( \beta (\alpha_*(p_1, p_2) + 1) < \gamma \) then

\[
\| \sup_{0<s<1} |T_{m_\beta,s}(f, g)| \|_p \lesssim \|(I - \Delta)^\gamma f \|_{p_1} \|(I - \Delta)^\gamma g \|_{p_2}. \tag{6}
\]

Consequently, we get that \( T_{m_\beta,s}(f, g)(x) \to f(x)g(x) \) as \( s \to 0^+ \) for a.e. \( x \) whenever the right hand side of (6) is finite.

**Remark 2.8** Since \( \alpha_*(2, 2) = 0 \), Theorem 2.7 implies that \( T_{m_\beta,s}(f, g)(x) \to f(x)g(x) \) a.e. for any \( \gamma > \beta \) provided \( (I - \Delta)^\gamma f, (I - \Delta)^\gamma g \in L^2(\mathbb{R}^n) \). Note that when \( \beta = 1 \) we have \( m_1 \left( |(\xi, \eta)|^2 \right) = e^{i|\xi|^2} e^{i|\eta|^2} \). In this case, we can directly use the result from the linear theory to prove the estimate (6) for \( \gamma > \frac{n}{2(n+1)} \). When \( \beta \neq 1 \), the results of the type (6) are new in the bilinear setting and Theorem 2.7 provides us with a range of \( \beta \) for which the results hold. However, the problem of finding an optimal regularity for \( f \) and \( g \) for which (6) holds needs to be investigated further.

### 2.4 General Bilinear Radial Multipliers

Let \( m : \mathbb{R}^{2n} \to \mathbb{C} \) be a bounded measurable function. Let \( T_{m,s}, s > 0 \) be the corresponding bilinear multiplier operator defined as

\[
T_{m,s}(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m(s\xi, s\eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.
\]
In the study of bilinear multiplier operators, the lack of Plancherel theorem argument poses a big difficulty. In [17], Grafakos, Honzik and He obtained some sufficient conditions on $m$ so that the corresponding bilinear maximal function $T_m^*(f, g) := \sup_{s>0} |T_{m,s}(f, g)|$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$. To be precise, they proved that if $m \in C^\infty(\mathbb{R}^{2n})$ and satisfies

$$|\partial^\beta m(\xi, \eta)| \leq C_\beta |(\xi, \eta)|^{-\alpha}, \quad \forall |\beta| \leq [\frac{n}{2}] + 2,$$

where $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$ and $\alpha > \frac{n}{2} + 1$, then $T_m^*$ is a bounded operator from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$. Note that the above condition is dependent on the dimension. Here we provide an improved sufficient condition for the case of bilinear radial multipliers. In doing so, we make use of the bilinear Bochner–Riesz square function.

Let $m_0 : [0, \infty) \to \mathbb{C}$ be a bounded measurable function and $m(\xi, \eta) = m_0(|(\xi, \eta)|^2)$ be a radial function on $\mathbb{R}^{2n}$. Let $T_m^*(f, g)$ denote the bilinear maximal function associated with $m(\xi, \eta) = m_0(|(\xi, \eta)|^2)$ defined as above. Observe that if $m_0$ is a smooth function on $[0, \infty)$, it is easy to see that $m(\xi, \eta) := m_0(|(\xi, \eta)|^2)$ is also a smooth function on $\mathbb{R}^{2n}$. Let $\varphi, \phi : (0, \infty) \to \mathbb{C}$ be compactly supported smooth functions such that $\sum_{j \geq 1} \varphi(2^{-j} x) + \phi(x) \equiv 1$ on $[0, \infty)$ and $\text{supp}(\varphi) \subset [1/2, 2]$ and $\text{supp}(\phi) \subset [0, 3/2]$. We establish the following result concerning sufficient condition on $m_0$ so that $T_m^*$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$.

**Theorem 2.9** Let $m_0 : [0, \infty) \to \mathbb{C}$ be a smooth function on $[0, \infty)$. Let

$$m_j(t) := m_0(t)\varphi\left(2^{-j} t\right), \quad t \geq 0$$

and there exists $\epsilon > 0$ and $\beta > 1$ such that

$$\|m_j\|_{L^2_\beta} \leq C 2^{-j\epsilon}$$

for all $j \geq 1$ with $C$ independent of $j$. Then, the operator $T_m^*$ extends as a bounded operator from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$.

Here we have used the notation $\|m\|_{L^2_\beta}^2 := \int_0^\infty |t^{\beta+1} (d/dt)^\beta \frac{m(t)}{t}|^2 t^{-1} dt$.

Indeed, the idea of the proof of Theorem 2.9 yields the following result in terms of the derivatives of $m_0$.

**Theorem 2.10** Let $m_0 : [0, \infty) \to \mathbb{C}$ be a smooth function such that

$$|m_0^k(t)| \lesssim t^{-k-\epsilon} \quad \text{as} \quad t \to \infty$$

for some $\epsilon > 0$ and $k = 0, 1, 2$. Here $m_0^k$ denotes the $k$th derivative of $m_0$. Then $T_m^*$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$.
Theorem 2.9 can be applied to deduce \( L^p \)-estimates for the generalized bilinear spherical maximal function, which is defined as follows.

For \( n \geq 1 \) define the generalized bilinear spherical means by

\[
S_{\omega_{\mu},s}(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \omega_{\mu}(s^2|\xi, \eta|^2) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta,
\]

where

\[
\omega_{\mu}(t) = 2^{\mu+n-1} \frac{\Gamma(\mu + n + 1)}{|t|^{\frac{1}{2}(\mu+n)}} J_{\mu+n}(\frac{|t|^{\frac{1}{2}}}{2})
\]

for \( \mu \in \mathbb{C} \) such that \( \mu \neq -n - 1, -n - 2, \ldots -n - k, \ldots \). Here \( J_{\mu} \) denotes the Bessel function of the first kind of order \( \mu \).

The generalised bilinear spherical maximal function is defined by

\[
S_{\omega_{\mu}}^*(f, g)(x) = \sup_{s > 0} |S_{\omega_{\mu},s}(f, g)(x)|.
\]

Note that when \( \beta = -1 \) the operator \( S_{\omega_{\mu}}^* \) is the bilinear spherical maximal function

\[
S_{-1}^*(f, g)(x) = M_S(f, g)(x) = \sup_{s > 0} \left| \int_{S^{2n-1}} f(x - sy)g(x - sz) d\sigma(y, z) \right|,
\]

where \( d\sigma \) is the normalised Lebesgue measure on the unit sphere \( S^{2n-1} \subset \mathbb{R}^{2n} \). We refer to [8, 20] for results on the bilinear spherical maximal function.

As a consequence of Theorem 2.9 we get the following result.

**Theorem 2.11** For \( \mu > -n + \frac{1}{2} \), we have

\[
\|S_{\omega_{\mu}}^*(f, g)\|_1 \lesssim \|f\|_2 \|g\|_2
\]

Observe that the theorem above also includes the bilinear spherical maximal function for \( n \geq 2 \). We will skip the proof of the theorem above. It may be completed using the asymptotic expansion of the Bessel functions along with Theorem 2.9.

### 3 Proofs of Theorems 2.6, 2.7, 2.9 and 2.10

The methods developed in [20] and [22] for studying the boundedness of the maximal bilinear Bochner–Riesz means (i.e. when \( \lambda = 2 \)) do not apply directly to deduce the \( L^p \)-estimates for \( B_{\lambda, \ast}^{\alpha} \) for \( \lambda \neq 2 \). Therefore, the role of the bilinear square function is crucial here. We will establish a pointwise relation between the maximal function \( B_{\lambda, \ast}^{\alpha} \) and square function \( G^{\alpha} \). In particular, we show that

\[
B_{\lambda, \ast}^{\alpha}(f, g)(x) \lesssim_{\lambda, \alpha, \beta} G^{\beta - 1}(f, g)(x) + Mf(x)Mg(x) \quad \text{for a.e.} \quad x \in \mathbb{R}^n,
\]
where \( \alpha - \beta + 1/2 > 0, \beta > 1/2, \lambda > 0 \) and \( M \) denotes the classical Hardy-Littlewood maximal operator. Observe that invoking Theorem 2.2 regarding the boundedness of bilinear square function, the inequality above yields the desired estimate for \( B^{\lambda, \ast} \) in Theorem 2.6. In order to prove the inequality (7), we require the Riemann-Liouville formula (see [5] for details).

**Lemma 3.1** [5] Let \( h \in L^2(\mathbb{R}) \) and for \( \beta \geq 0 \) let \( (d/dt)^{\beta} h(\nu) = (-2\pi i \nu)^{\beta} \hat{h}(\nu) \). Suppose that \( \text{supp}(h) \subseteq (-\infty, a] \) and \((d/dt)^{\beta} h \in L^2(\mathbb{R}) \) for \( \beta > 1/2 \). Then \( \text{supp}((d/dt)^{\beta} h) \subseteq (-\infty, a] \) and

\[
h(x) = c_{\beta} \int_{x}^{\infty} (t - x)^{\beta - 1} (d/dt)^{\beta} h(t) dt, \quad \text{for a.e. } x.
\]

(8)

Let \( m \in C_c^\infty((0, \infty)) \). Applying Lemma 3.1 to \( \tilde{m}(u) = \frac{m(u)}{u} \) we can write

\[
m(u) = c_{\beta} \int_{0}^{\infty} \left( 1 - \frac{u}{t} \right)^{\beta - 1} \frac{u}{t} (d/dt)^{\beta} \tilde{m}(t) dt,
\]

where \( \beta > 1/2 \). Write \( u = |(\xi, \eta)|^2 \) in the formula above to get

\[
m(|(\xi, \eta)|^2) = c_{\beta} \int_{0}^{\infty} \left( 1 - \frac{|(\xi, \eta)|^2}{t} \right)^{\beta - 1} \frac{|(\xi, \eta)|^2}{t} (d/dt)^{\beta} \tilde{m}(t) dt.
\]

(9)

Consider the bilinear multiplier operator given by

\[
T_{m, s}(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m(s^2 |(\xi, \eta)|^2) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.
\]

Observe that using the formula (9) we can rewrite \( T_{m, s}(f, g) \) as

\[
T_{m, s}(f, g)(x) = c_{\beta} \int_{0}^{\infty} \mathcal{K}_{s}^{\beta - 1} * (f, g)(x) t^{\beta} (d/dt)^{\beta} \tilde{m}(t) dt.
\]

(10)

Cauchy-Schwarz inequality in the above yields

\[
\sup_{s > 0} |T_{m, s}(f, g)(x)| \leq c_{\beta} \|m\|_{L^2_{\beta}} \mathcal{G}^{\beta - 1}(f, g)(x),
\]

(11)

where \( \|m\|_{L^2_{\beta}} = \int_{0}^{\infty} |t^{\beta + 1} (d/dt)^{\beta} \tilde{m}(t)|^2 t^{-1} dt \) and \((d/dt)^{\beta} \tilde{m}(t) \) is a distributional derivative of \( \tilde{m} \) of order \( \beta \).

We make use of the above analysis to prove Theorem 2.6.
Proof of theorem 2.6

For convenience, let us work with $2\lambda$ in place of $\lambda$. Let $m_j(|(\xi, \eta)|^2) = (2^j (1 - |(\xi, \eta)|^{2\lambda}))^\alpha \psi(2^j (1 - |(\xi, \eta)|^{2\lambda}))$, where $\psi$ is a smooth compactly supported function on $[\frac{1}{2}, 2]$. We can write

$$B_{2\lambda, R}(f, g)(x) = B_{0, R}(f, g)(x) + \sum_{j \geq 1} 2^{-j\alpha} \int_{\mathbb{R}^{2n}} m_j(R^{-2} |(\xi, \eta)|^2) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta,$$

where

$$B_{0, R}(f, g)(x) = \int_{\mathbb{R}^{2n}} \left(1 - \frac{|(\xi, \eta)|^{2\lambda}}{R^{2\lambda}}\right)^\alpha \varphi(|(\xi, \eta)|^{2\lambda}) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta$$

and

$$\varphi(x) + \sum_{j \geq 1} \psi(2^j (1 - x)) \equiv 1 \text{ on } [0, 1).$$

Let $B_j(f, g)(x) = \sup_{R > 0} \left|\int_{\mathbb{R}^{2n}} m_j(R^{-2} |(\xi, \eta)|^2) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta\right|$. For $j \geq 1$ we claim that

$$B_j(f, g)(x) \lesssim 2^{j\beta - j/2} g^\beta(f, g)(x) \text{ for any } \beta > 1/2.$$

This follows from the inequality (11) along with the estimate $\|m_j\|_{L^2_\beta} \simeq 2^{j\beta - j/2}$. For, observe that when $\beta = 0$ we have that $\|m_j\|_{L^2_\beta}^2 = \int_0^\infty |\psi(2j(1 - t^{2\lambda})))|^2 t^{-1} dt \simeq 2^{-j}$. Further, when $\beta = 1$ we can easily verify that $\|m_j\|_{L^2_1} \simeq 2^j$. Therefore, interpolating between $\beta = 0$ and $\beta = 1$ we get the estimate for $0 < \beta < 1$. In fact, the same argument yields the desired bound for any $\beta > 0$.

Next, we claim that

$$\sup_{R > 0} |B_{0, R}(f, g)(x)| \lesssim Mf(x) M^g(x) \text{ for every } \lambda > 0.$$

Note that $\varphi \equiv 1$ in $[0, \delta]$ for some $\delta > 0$ and supp($\varphi$) $\subset [0, 1/2]$. Let $\rho$ be a smooth function on $\mathbb{R}^{2n}$ supported on $\{(\xi, \eta) : 1/2 \leq |(\xi, \eta)| \leq 2\}$ such that $\sum_{j \geq 0} \rho(2^j (\xi, \eta)) \equiv 1$ on $\{(\xi, \eta) : 0 < |(\xi, \eta)| \leq 1\}$. Consider the kernel

$$K(y, z) = \int_{\mathbb{R}^{2n}} \left(1 - |(\xi, \eta)|^{2\lambda}\right)^\alpha \varphi(|(\xi, \eta)|^{2\lambda}) e^{2\pi i (y \cdot \xi - z \cdot \eta)} d\xi d\eta.$$

Let $\rho \in C^\infty([1/2, 1])$ and write

$$\tilde{K}(y, z) := \sum_{j \geq 0} \int_{\mathbb{R}^{2n}} \left(1 - |(\xi, \eta)|^{2\lambda}\right)^\alpha - 1 \rho(2^j (\xi, \eta)) \varphi(|(\xi, \eta)|^{2\lambda}) e^{2\pi i (y \cdot \xi - z \cdot \eta)} d\xi d\eta$$

$$= \sum_{j \geq 0} \tilde{K}_j(y, z).$$
We need to estimate \( \tilde{K}_j \) for large \( j \). When \( j \) is large we have that \( \rho(2^j (\xi, \eta)) \varphi((\xi, \eta))^{2^k} = \rho(2^j (\xi, \eta)) \).

Using power series expansion of \((1 - r)^\alpha\), \(0 \leq r \leq \delta \leq 1 \) around \( r = 0 \) we can write

\[
(1 - |(\xi, \eta)|^{2^j})^{\alpha} - 1 = \sum_{k \geq 1} (-1)^k k! (\alpha - 1) \ldots (\alpha - k + 1)|(\xi, \eta)|^{2^k}.
\]

It is easy to verify that the integration by parts argument gives us

\[
|\tilde{K}_j(y, z)| \lesssim \frac{2^{-j\lambda} 2^{-2nj}}{(1 + 2^{-j}|(y, z)|)^{2n+1}} \lesssim 2^{-j\lambda} \frac{2^{-nj}}{(1 + 2^{-j}|y|)^{n+\frac{1}{2}}} \frac{2^{-nj}}{(1 + 2^{-j}|z|)^{n+\frac{1}{2}}}.
\]

Finally, from the definition \( K(y, z) - \tilde{K}(y, z) = \int_{\mathbb{R}^{2n}} \varphi(|(\xi, \eta)|^{2^k}) e^{2\pi i (y,z) \cdot (\xi, \eta)} d\xi d\eta \).

Since \( \varphi(|(\xi, \eta)|^{2^k}) = 1 \) near the origin, using integration by parts again we get that

\[
|K(y, z) - \tilde{K}(y, z)| \lesssim_{\lambda} (1 + |(y, z)|)^{-2n-1} \lesssim_{\lambda} (1 + |y|)^{-n-\frac{1}{2}} (1 + |z|)^{-n-\frac{1}{2}} \text{ for any } \lambda > 0.
\]

This completes the proof. \( \square \)

**Proof of theorem 2.7**

Let \( \varphi \in C_0^{\infty}(\mathbb{R}) \) be an even function such that \( \varphi \equiv 1 \) in a neighbourhood of the origin. Write \( m_\beta(u) = m_{\beta,1}(u) + m_{\beta,2}(u) \), where \( m_{\beta,1}(u) = m_{\beta}(u) \varphi(u) \). Note that we can realise these as functions on \( \mathbb{R}^{2n} \) by putting \( u = |(\xi, \eta)|^2 \) for \( \xi, \eta \in \mathbb{R}^n \). The 2n—dimensional Fourier transform is given by

\[
\hat{m}_{\beta,1}(x) = \int_{\mathbb{R}^{2n}} m_{\beta,1}((\xi, \eta))^2 e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.
\]

We will show the following estimate for \( \hat{m}_{\beta,1} \).

\[
|\hat{m}_{\beta,1}(x)| \lesssim_{\beta} |x|^{-(2n+1)} \text{ for } |x| \text{ large and } 0 < \beta. \tag{12}
\]

Consider

\[
\hat{m}_{\beta,1}(x) = \int_{\mathbb{R}^d} (e^{i|\xi|^2 \beta} - 1) \varphi(|\xi|^2) e^{2\pi i x \cdot \xi} d\xi + \int_{\mathbb{R}^d} \varphi(|\xi|^2) e^{2\pi i x \cdot \xi} d\xi.
\]

It is easy to verify that

\[
\left| \int_{\mathbb{R}^d} \varphi(|\xi|^2) e^{2\pi i x \cdot \xi} d\xi \right| \lesssim (1 + |x|)^{-2n - \beta'} \text{ for any } \beta' > 0.
\]
Next, for the remaining part, we perform a partition of unity argument and consider the integrals

\[ I_j(x) = \int_{\mathbb{R}^d} (e^{i|x|^2\beta} - 1)\phi(2^j \xi) e^{2\pi i x \cdot \xi} d\xi \]

where \( \phi \) is a smooth function supported in \( 1/4 \leq |\xi| \leq 4 \) and \( \sum_{j \geq 0} \phi(2^j \xi) = 1 \) on \( 0 < |\xi| \leq 2 \). Note that for \( j \) large enough, we know that \( \varphi(|\xi|^2)\phi(2^j \xi) = \phi(2^j \xi) \).

Therefore, the desired estimate for \( \hat{m}_{\beta,1}(x) \) follows from suitable estimate on \( I_j(x) \) for \( j \) large.

Applying a change of variable argument we get that

\[ I_j(x) = 2^{-jd} \int_{\mathbb{R}^d} (e^{i(2^{-j}|\xi|)^2\beta} - 1)\phi(\xi) e^{2\pi i 2^{-j}x \cdot \xi} d\xi \]

Note that for \( j \) large, \( 2^{-2j\beta}|\xi|^{2\beta} \) is very small on the support of \( \phi \) which in turn implies that \( (e^{i(2^{-j}|\xi|)^2\beta} - 1) \) is very small. Write \( e^{i(2^{-j}|\xi|)^2\beta} = \sum_{k=0}^{\infty} \frac{1}{k!} (2^{-j}|\xi|)^{2k\beta} \) and use integration by parts argument to get that

\[ \left| \int_{\mathbb{R}^d} (2^{-j}|\xi|)^{2k\beta}\phi(\xi) e^{2\pi i 2^{-j}x \cdot \xi} d\xi \right| \leq c(k, d, \beta) 2^{-2j\beta} 2^{-jd} (1 + 2^{-j}|x|)^{-d-1}, \]

where the constant \( c(k, d, \beta) \) is at most a polynomial in \( k \) of a fixed degree for all \( k \geq 1 \). This estimate gives us

\[ |I_j(x)| \lesssim 2^{-2j\beta} 2^{-jd} \left( 1 + 2^{-j}|x| \right)^{-d-1}. \]

Summing over \( j \) implies that

\[ \sup_{s > 0} |T_{m_{\beta,1,s}}(f, g)(x)| \lesssim Mf(x) Mg(x). \]

Next, we show that \( \|m_{\beta,2}(\cdot)/\cdot^\gamma\|_{L^2_\alpha} < \infty \) for \( \gamma > \beta \alpha \). Let \( \Phi \) be a smooth function supported in \([1, 2]\) such that

\[ \sum_{j \geq 0} \Phi(2^{-j}u) \equiv 1, \quad \text{on } |u| \geq 1. \]

Observe that it suffices to obtain the required estimate on the \( L^2_\alpha \) norm of \( m_{\beta,2}^j(u) = m_{\beta,2}(u)\phi(2^{-j}u) \) for large \( j \). Consider

\[ \|m^j_{\beta,2}\|_{L^2_\alpha}^2 = \int_0^\infty u^{2\alpha+1} \left| \frac{d}{du} \left( \frac{d}{\gamma+1} \Phi(2^{-j}u) \right) \right|^2 du \]
\[= 2^{-2j\gamma} \int_0^\infty u^{2\alpha+1} \left| \left( \frac{d}{du} \right)^\alpha \left( e^{i2j^\beta |\cdot|^\beta \Phi(\cdot)} \right)(u) \right|^2 du,\]

where \(\tilde{\Phi}(u) = \frac{\phi(u)}{|u|^\gamma + 1}\).

Note that \(\tilde{\Phi}\) is a smooth function supported in \([1, 2]\). We know that the support of \((\frac{d}{du})^\alpha \left( e^{i2j^\beta |\cdot|^\beta \Phi(\cdot)} \right)(u)\) is contained in \((-\infty, 2]\) when \(\alpha > \frac{1}{2}\). Therefore, it is enough to estimate

\[J_\alpha = \int_{\mathbb{R}} \left| \left( \frac{d}{du} \right)^\alpha \left( e^{i2j^\beta |\cdot|^\beta \Phi(\cdot)} \right)(u) \right|^2 du.\]

Interpolation between integral values of \(\alpha\) gives us that \(J_\alpha \lesssim 2^{2j\alpha\beta}\). Combining this with the estimates above we get

\[\int_0^\infty u^{2\alpha+1} \left| \left( \frac{d}{du} \right)^\alpha \left( e^{i2j^\beta |\cdot|^\beta \Phi(\cdot)} \right)(u) \right|^2 du \lesssim 2^{2j(\alpha\beta - \gamma)}. \quad (13)\]

This yields that \(\|m_{\beta,2}(\cdot)(\cdot)^\gamma\|_{L^2_\alpha} < \infty\) when \(\gamma > \beta\alpha\).

Next, note that when \(\gamma\) is an integer, using binomial expansion we can write

\[(-\triangle)^\gamma (f \otimes g)(x, y) = \sum_{0 \leq \mu_1 + \mu_2 \leq \gamma} c(\gamma, \mu_1, \mu_2)(-\triangle)^{\mu_1} f(x)(-\triangle)^{\mu_2} g(y).\]

Using the estimate (13) for \(\gamma_0 = m\) and \(\gamma_1 = m + 1, m \geq 0\), we get

\[
\sup_{0 < s < 1} \left| T_{m_{\beta,s}}^j (f, g)(x) \right| \lesssim 2^j(\alpha\beta - \gamma_k) \sum_{0 \leq \mu_1 + \mu_2 \leq \gamma_k} c(\gamma_k, \mu_1, \mu_2) \left( G_{\alpha-1}^{\alpha-1} ((-\triangle)^{\mu_1} f, (-\triangle)^{\mu_2} g)(x) \right)
\]

where \(k = 0, 1\) and \(m \geq 0\).

Invoking Theorem 2.2 we get that

\[\sup_{0 < s < 1} \| T_{m_{\beta,s}}^j (f, g) \|_p \lesssim 2^{j(\alpha\beta - \gamma_k)} \| (I - \triangle)^\gamma_k f \|_{p_1} \| (I - \triangle)^\gamma_k g \|_{p_2}\]

for \(k = 0, 1\) and \(\alpha > \alpha_*(p_1, p_2) + 1\).

An interpolation argument (see Theorem 6.4.5 on page 152 and Theorem 4.4.1 on page 96 in [2]) yields

\[\sup_{0 < s < 1} \| T_{m_{\beta,s}}^j (f, g) \|_p \lesssim 2^{j(\alpha\beta - \gamma)} \| (I - \triangle)^\gamma f \|_{p_1} \| (I - \triangle)^\gamma g \|_{p_2}\]

for any \(m < \gamma < m + 1, m \geq 0\) and \(j \geq 1\).
When $\gamma > \alpha \beta$ summing over $j$ gives us
\[
\| \sup_{0 < s < 1} |T_{m_j,s}(f, g)|_p \| < \|(I - \Delta)^{\gamma} f\|_p \| (I - \Delta)^{\gamma} g\|_p,
\]
where $\alpha > \alpha^*(p_1, p_2) + 1$.

**Proof of Theorem 2.9**

Recall that we need to prove the required estimates for bilinear maximal functions associated with the following operators
\[
T^s_j(f, g)(x) = \int_{\mathbb{R}^{2n}} m_j(s^2|\xi, \eta|^2) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \quad \text{for } j \geq 1 \text{ and }
\]
\[
T^s_0(f, g)(x) = \int_{\mathbb{R}^{2n}} m_0(s^2|\xi, \eta|^2) \phi(s^2|\xi, \eta|^2) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.
\]

First, observe that $m_0(|(\xi, \eta)|^2) \phi(|(\xi, \eta)|^2)$ is a compactly supported smooth function on $\mathbb{R}^{2n}$. Therefore, the corresponding maximal function $\sup_{s > 0} |T^s_0(f, g)|$ can be dominated by $Mf(x) Mg(x)$ pointwise a.e. and hence the desired $L^p$—estimate follows.

Next, we will show that $\sup_{s > 0} |T^s_j(f, g)|$ extends to a bounded operator on $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ and its norm is bounded by $\|m_j\|_{L^2_{\beta}}$ for each $j \geq 1$. From the inequality (11) we see that
\[
\sup_{s > 0} |T^s_j(f, g)(x)| \lesssim \|m_j\|_{L^2_{\beta}} \mathcal{G}^{\beta - 1}(f, g)(x) \quad \text{for a.e. } x.
\]

Using the boundedness of bilinear Stein’s square function (see Theorem 2.2), we know that $\mathcal{G}^{\beta - 1}(f, g)$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ for any $\beta > 1$. The given criteria on $m_j$ allow us to sum the R.H.S. in the inequality above. This completes the proof.

**Proof of Theorem 2.10**

The proof follows along the same lines as the theorem above. Again it is easy to check that $T^s_0$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$. For the remaining part, using (11) and applying it for $\beta = 2$ gives us the result. We skip the details here.

**4 Proof of Theorem 2.2**

We decompose the bilinear Bochner–Riesz square function along the same lines as carried out in [Section 3, [22]]. This involves decomposing the bilinear Bochner–
Riesz multiplier \((1 - \frac{|\xi|^2 + |\eta|^2}{R^2})_+^\alpha\) along the \(\xi\) and \(\eta\)–axes separately. We proceed as follows.

Let \(\psi \in C_0^\infty[\frac{1}{2}, 2]\) and \(\psi_0 \in C_0^\infty[-\frac{3}{4}, \frac{3}{4}]\) be such that

\[
\sum_{j \geq 2} \psi(2^j(1 - t)) + \psi_0(t) = 1 \text{ for all } t \in [0, 1).
\]

This allows us to write

\[
\hat{K}^\alpha_R(\xi, \eta) = \frac{|\xi|^2 + |\eta|^2}{R^2} \left(1 - \frac{|\xi|^2 + |\eta|^2}{R^2}\right)_+^\alpha + \sum_{j \geq 2} \mathfrak{m}^\alpha_{j,R}(\xi, \eta) + \mathfrak{m}^\alpha_{0,R}(\xi, \eta),
\]

where for \(j \geq 2\),

\[
\mathfrak{m}^\alpha_{j,R}(\xi, \eta) = \psi(2^j \left(1 - \frac{|\xi|^2}{R^2}\right)) \frac{|\xi|^2 + |\eta|^2}{R^2} \left(1 - \frac{|\xi|^2}{R^2}\right)_+^\alpha \left(1 - \frac{|\eta|^2}{R^2} \left(1 - \frac{|\xi|^2}{R^2}\right)^{-1}\right)_+^\alpha
\]

and

\[
\mathfrak{m}^\alpha_{0,R}(\xi, \eta) = \psi_0 \left(\frac{|\xi|^2}{R^2}\right) \frac{|\xi|^2 + |\eta|^2}{R^2} \left(1 - \frac{|\eta|^2}{R^2}\right)_+^\alpha.
\]

This yields the following decomposition of the operator

\[
g^\alpha_R(f, g)(x) = \sum_{j \geq 2} g^\alpha_{j,R}(f, g)(x) + g^\alpha_{0,R}(f, g)(x),
\]

where

\[
g^\alpha_{j,R}(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathfrak{m}^\alpha_{j,R}(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta, \quad j \geq 2
\]

and

\[
g^\alpha_{0,R}(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathfrak{m}^\alpha_{0,R}(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.
\]

Consequently, we get that

\[
G^\alpha(f, g)(x) \leq G^\alpha_0(f, g)(x) + \sum_{j \geq 2} G^\alpha_{j}(f, g)(x) \quad (14)
\]

where

\[
G^\alpha_{j}(f, g)(x) = \left(\int_0^\infty |g^\alpha_{j,R}(f, g)(x)|^2 \frac{dR}{R}\right)^{\frac{1}{2}}, \quad j = 0, 2, 3, \ldots
\]
Therefore, our job of proving $L^p(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ boundedness of $G^\alpha$ is reduced to obtaining the same for new square functions $G^\alpha_j$ with $\|G^\alpha_j\|_{L^p \times L^{p_2} \rightarrow L^p} \leq C_j$ such that $\sum_{j \geq 2} C_j < \infty$. We will address the problem of boundedness of $G^\alpha_0$ and $G^\alpha_j$, $j \geq 2$ separately.

**Boundedness of $G^\alpha_j$, $j \geq 2$:**

We further decompose the bilinear operator $g^\alpha_{j,R}$. Let $\varphi_R(\xi) = \left(1 - \frac{|\xi|^2}{R^2}\right)_+$. Now using an identity from Stein and Weiss [35], page 278, we have the following relation.

\[
\left(1 - \frac{|\eta|^2}{R^2}\varphi_R(\xi)\right)^\alpha = c_{\delta,\beta} R^{-2\alpha} \varphi_R(\xi)^{-\alpha} \int_0^R \left(R^2 \varphi_R(\xi) - t^2\right)^{\beta-1} t^{2\delta+1} \left(1 - \frac{|\eta|^2}{t^2}\right)^\delta + dt,
\]

where $\beta > \frac{1}{2}$, $\delta > -\frac{1}{2}$ and $\beta + \delta = \alpha$. Moreover, the constant $c_{\delta,\beta}$ is given by

\[
c_{\delta,\beta} = \frac{2 \Gamma(\delta + \beta + 1)}{\Gamma(\delta + 1) \Gamma(\beta)},
\]

where $\Gamma$ denotes the Gamma function.

The identity (15) yields the following decomposition

\[
g^\alpha_{j,R}(f, g)(x) = c_{\delta,\beta} R^{-2\alpha} \int_0^{R_j} \left(B_{j,\beta}^R f(x) B_{\delta}^\beta g(x) + A_{j,\beta}^R f(x) A_{\delta}^\beta g(x)\right) t^{2\delta+1} dt,
\]

where $R_j = R \sqrt{2-j+1}$. Further, the operators on the right-hand side of (17) are defined by

\[
B_{j,\beta}^R f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) \psi \left(2^j \left(1 - \frac{|\xi|^2}{R^2}\right)\right) \left|\xi|^2 \varphi_R(\xi) - t^2\right)^{\beta-1} e^{2\pi i x \cdot \xi} d\xi,
\]

\[
A_{j,\beta}^R f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) \psi \left(2^j \left(1 - \frac{|\xi|^2}{R^2}\right)\right) \left(R^2 \varphi_R(\xi) - t^2\right)^{\beta-1} e^{2\pi i x \cdot \xi} d\xi
\]

and

\[
A_{\delta}^\beta g(x) = \int_{\mathbb{R}^n} \hat{g}(\eta) \left|\eta|^2 \right. \left(1 - \frac{|\eta|^2}{t^2}\right)^\delta e^{2\pi i x \cdot \eta} d\eta.
\]
Note that $B^\delta_t$ in (17) denotes the classical Bochner–Riesz means of index $\delta$. Consider the equation (17) and write it as

$$g_\alpha^j, R(f, g)(x) = I_{j, R} + II_{j, R},$$

where

$$I_{j, R} = c_\alpha R^{-2\alpha} \int_0^{R_j} B_{j, \beta}^{R, t} f(x) B_i^\delta t(x) t^{2\delta+1} dt,$$

and

$$II_{j, R} = c_\alpha R^{-2\alpha} \int_0^{R_j} A_{j, \beta}^{R, t} f(x) A_i^\delta t(x) t^{2\delta+1} dt.$$

First, we estimate the square function corresponding to the term $II_{j, R}$. Apply Cauchy-Schwarz inequality to get that

$$|II_{j, R}| \lesssim R^{-2\alpha} \left( \int_0^{R_j} |A_{j, \beta}^{R, t} f(\cdot) t^{2\delta+1}|^2 dt \right)^{1/2} \left( \int_0^{R_j} |A_i^\delta t(x)|^2 dt \right)^{1/2}.$$

Making a change of variable $t \to Rt$ in the integral $\left( \int_0^{R_j} |A_{j, \beta}^{R, t} f(\cdot) t^{2\delta+1}|^2 dt \right)^{1/2}$ we get that

$$\int_0^{R_j} |A_{j, \beta}^{R, t} f(x) t^{2\delta+1}|^2 dt = R^{4\alpha-1} \int_0^{\sqrt{2}-j+1} |\tilde{S}_{j, \beta}^{R, t} f(x) t^{2\delta+1}|^2 dt,$$

where

$$\tilde{S}_{j, \beta}^{R, t} f(x) = \int_{\mathbb{R}^n} \psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \right) \left( 1 - \frac{|\xi|^2}{R^2} - t^2 \right)^{\beta-1} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

We also make the change $t \to Rt$ in the other integral involving the term $A_i^\delta t g$ and apply Hölder’s inequality with $p_1$ and $p_2$ to get that

$$\left( \int_0^{\infty} |II_{j, R}|^2 \frac{dR}{R} \right)^{1/2} \lesssim \sup_{R > 0} \left( \int_0^{\sqrt{2}-j+1} |\tilde{S}_{j, \beta}^{R, t} f(\cdot) t^{2\delta+1}|^2 dt \right)^{1/2} \left( \int_0^{\infty} \left( \int_0^{\sqrt{2}-j+1} |A_i^\delta t g(x)|^2 dt \right) \frac{dR}{R} \right)^{1/2}.$$

We know that the term involving maximal function in the estimate above satisfies the following $L^p$–estimates, see [22, Theorem 5.1].

$$\left( \sup_{R > 0} \left( \int_0^{\sqrt{2}-j+1} |\tilde{S}_{j, \beta}^{R, t} f(\cdot) t^{2\delta+1}|^2 dt \right)^{1/2} \right)_{p_1} \lesssim 2^{j(\alpha(p_1)+\frac{1}{2}-\beta+\epsilon)} \|f\|_{p_1},$$
for $p_1 \geq p_n$ or $p_1 = 2$ when $n \geq 2$ and $\beta > \alpha(p_1) + \frac{1}{2}$.

The remaining term is dealt with by using $L^p$-estimates for the Bochner–Riesz square function in the following way.

\[
\left\| \left( \int_0^\infty \left( \int_0^{\sqrt{2-j-1}} |A_{R,t}^\delta g(x)|^2 \, dR \right) \frac{dR}{R} \right)^{\frac{1}{2}} \right\|_{L^p} = \left\| \left( \int_0^\infty \left( \int_0^{\sqrt{2-j-1}} |A_{R,t}^\delta g(x)|^2 \, dR \right) \frac{dR}{R} \right)^{\frac{1}{2}} \right\|_{L^p} \leq 2^{\frac{n}{4}(-j+1)} \left\| \left( \int_0^\infty |A_{R}^\delta g(x)|^2 \, dR \right)^{\frac{1}{2}} \right\|_{L^p} \leq 2^{\frac{n}{4}(-j+1)} \|g\|_{L^p},
\]

where $p_2 \geq p_n$ or $p_2 = 2$ and $\delta > \alpha(p_2) - \frac{1}{2}$.

Therefore, for $p_1, p_2 \geq p_n$ or $p_1, p_2 = 2$, we get

\[
\left\| \left( \int_0^\infty |I_{j,R}|^2 \, dR \right)^{\frac{1}{2}} \right\| \lesssim 2^{j(\alpha(p_1) - \beta - 1+\epsilon)} \|f\|_{L^p} \|g\|_{L^p},
\]

when $\beta > \alpha(p_1) + \frac{1}{2}$, $\delta > \alpha(p_2) - \frac{1}{2}$ and the exponent of $2^j$ is negative.

Next, we need to deal with the bilinear square function associated with the term $I_{j,R}$. We begin in a similar fashion as in the previous case of the term $II_{j,R}$. Making use of the change of variable $t \rightarrow R_t$ and Hölder inequality, we arrive at the following.

\[
\left\| \left( \int_0^\infty |I_{j,R}|^2 \, dR \right)^{\frac{1}{2}} \right\| \lesssim 2^{-\frac{1}{4}} \left\| \left( \int_0^\infty |S_{j,R}^R f(\cdot) t^{2+1}|^2 \, dt \right) \, dR \right\| \left\| \sup_{R>0} \left( \frac{1}{R_J} \int_0^{R_J} |B_{R}^\delta g(x)|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^p},
\]

where

\[
S_{j,R}^R f(x) = \int_{R^n} \psi \left( 2j \left( 1 - \frac{\xi^2}{R^2} \right) \right) \frac{\xi^2}{R^2} \left( 1 - \frac{|\xi|^2}{R^2} - t^2 \right)^{-\frac{1}{2}} \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi.
\]
Invoking the $L^p −$estimates for the maximal operator $f \mapsto \sup_{R > 0} \left( \frac{1}{R} \int_0^R |B^\delta_t f(\cdot)|^2 dt \right)^{1/2}$ from [22, Lemma 4.4], for $\delta > \alpha(p) - \frac{1}{2}$ we get that

$$\left\| \sup_{R > 0} \left( \frac{1}{R} \int_0^R |B^\delta_t g(x)|^2 dt \right)^{1/2} \right\|_{L^p} \lesssim \|g\|_{L^p},$$

where $p_2 \geq p_n$ or $p_2 = 2$. For the other term using Minkowski integral inequality, we can write

$$\left\| \left[ \int_0^\infty \left( \int_0^{\tau_j} \left| S^{\mathcal{R},t}_{j,\beta} f(\cdot) t^{2\delta+1} \right|^2 dt \right) \frac{dR}{R} \right]^{1/2} \right\|_{L^p} \leq \left( \int_0^{\tau_j} \left[ \int_{\mathbb{R}^n} \left( \int_0^\infty \left| S^{\mathcal{R},t}_{j,\beta} f(\cdot) \right|^2 \frac{dR}{R} \right)^{\frac{p_1}{2}} dx \right]^{\frac{2}{p_1}} t^{4\delta+2} dt \right)^{1/2}. \quad (20)$$

Here $\tau_j = \sqrt{2} - j + 1$. Therefore, we need to prove the desired estimates for the square function corresponding to operator $S^{\mathcal{R},t}_{j,\beta}$.

Let $\psi \in C^N([1/2, 2]), n \geq 1$ and $0 < \nu < \frac{1}{16}$. Consider the following operators

$$B^\psi_{v,t} f(x) = \int_{\mathbb{R}^n} \psi \left( v^{-1} \left( 1 - \frac{|\xi|^2}{t^2} \right) \right) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

The following result is proved in [22]; also see Jeong, Lee and Vargas [21, Lemma 2.6] for a similar result.

**Lemma 4.1** [22] Let $n \geq 2, 0 < \nu < \frac{1}{16}$ and $\epsilon > 0$. Then for $p \geq p_n$ or $p = 2$, there exists $N \geq 1$ such that for all $\psi \in C^N([1/2, 2])$ the following holds

$$\left\| \left( \int_0^\infty |B^{\psi}_{v,t} f(\cdot)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim_{\epsilon, N} \nu^{\left( \frac{1}{2} - \alpha(p) \right) \nu - \epsilon} \|f\|_{L^p(\mathbb{R}^n)},$$

where the implicit constant depends on $\epsilon$ and $N$.

We will make use of Lemma 4.1 to prove the following estimate for the square function associated with the operators $S^{\mathcal{R},t}_{j,\beta}$.

**Theorem 4.2** Let $n \geq 2$. When $\beta > \alpha(p) + \frac{1}{2}$, then the following estimate holds

$$\left\| \left( \int_0^\infty |S^{\mathcal{R},t}_{j,\beta} f(\cdot)|^2 \frac{dR}{R} \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim 2^{j(\alpha(p) - \beta - \frac{1}{4} + \epsilon)} \|f\|_{L^p(\mathbb{R}^n)}$$

where the implicit constant is independent of $t \in [0, \sqrt{2} - j + 1]$ when $p \geq p_n$ or $p = 2$. 
Firstly, assuming Theorem 4.2, we get

\[ \left\| \int_0^\infty \left( \int_0^{\tau_j} |S_{j,\beta} f(\cdot) \tau_j^{\delta + 1}|^2 dt \right) dR \right\|_{p_1}^{1/2} \lesssim 2^{j(\alpha(p_1) - \beta - \frac{1}{2} + \epsilon)} \|f\|_{p_1} \left( \int_0^{\tau_j} \tau_j^{\delta + 1} dt \right)^{1/2} \]

\[ \lesssim 2^{j(\alpha(p_1) - \beta - \delta - 1 + \epsilon)} \|f\|_{p_1}. \]

Therefore, for \( \beta > \alpha(p_1) + \frac{1}{2}, \delta > \alpha(p_2) - \frac{1}{2} \), we have

\[ \left\| \int_0^\infty |II_{j,R}|^2 dR \right\|_{p}^{1/2} \lesssim 2^{j(\alpha(p_1) - \beta - \delta - 1 + \epsilon)} \|f\|_{p_1} \|g\|_{p_2}, \]

where \( p_1, p_2 \geq p_n \) or \( p_1, p_2 = 2 \) and the exponent of \( 2^j \) is negative.

**Proof of Theorem (21)** Write \( \beta = \gamma + 1 \) in \( S_{j,\beta} f \) and note that when \( t^2 \in [0, 2^{-j-1}) \) we can rewrite the function

\[ \psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \right) \left( 1 - \frac{|\xi|^2}{R^2} - t^2 \right)^{\gamma} \frac{|\xi|^2}{R^2} + \rho \sum_{k \geq 0} \frac{\Gamma(\rho + k)}{\Gamma(\rho)k!} \left( \frac{2^j t^2}{2^j (1 - \frac{|\xi|^2}{R^2})} \right)^k. \]

Let \( 0 < \epsilon_0 \ll 1 \) be a fixed number and split the interval \([0, 2^{-j+1}]\) into subintervals \([0, 2^{-j-1-\epsilon_0}]\) and \([2^{-j-1-\epsilon_0}, 2^{-j+1}]\). We will deal with the operators corresponding to each subinterval separately.

**Case I: When \( t \in [0, \sqrt{2^{-j-1-\epsilon_0}}] \)**

First, consider the range \(-1 < \gamma < 0\). Write \( \gamma = -\rho \), where \( \rho > 0 \). Note that using Taylor’s expansion, we can write

\[ \left( 1 - \frac{|\xi|^2}{R^2} \right)^{-\rho} \left( 1 - \frac{t^2}{1 - \frac{|\xi|^2}{R^2}} \right)^{-\rho} = 2^j \rho \left( 1 - \frac{|\xi|^2}{R^2} \right)^{-\rho} \]

\[ \sum_{k \geq 0} \frac{\Gamma(\rho + k)}{\Gamma(\rho)k!} \left( \frac{2^j t^2}{2^j (1 - \frac{|\xi|^2}{R^2})} \right)^k \]

\[ -2^{j(\rho-1)} \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \right)^{-\rho+1} \sum_{k \geq 0} \frac{\Gamma(\rho + k)}{\Gamma(\rho)k!} \left( \frac{2^j t^2}{2^j (1 - \frac{|\xi|^2}{R^2})} \right)^k. \]
Since \( \frac{t^2}{1-|\frac{\gamma}{t}|^2} \leq 2^{-j-1-\epsilon_0} 2^{j+1} < 1 \), the series in the estimate above converges. Thus, we get that

\[
S_{j, y+1}^{R, t} f = 2^{j \rho} \sum_{k \geq 0} \frac{\Gamma(\rho + k)}{\Gamma(\rho) k!} \left( 2^j t^2 \right)^k B_{2^{-j}, R}^{\psi^k} f - 2^{j(\rho - 1)} \sum_{k \geq 0} \frac{\Gamma(\rho + k)}{\Gamma(\rho) k!} \left( 2^j t^2 \right)^k B_{2^{-j}, R}^{\psi^k-1} f,
\]

where \( \psi^k(x) := x^{-k-\rho} \psi(x) \).

Observe that \( \psi^k \in C_0^\infty([\frac{1}{2}, 2]) \) and \( \frac{d}{dx} \psi^k(x) = -(k + \rho)x^{-k-\rho-1} \psi(x) + x^{-k-\rho} \frac{d}{dx} \psi(x) = (k + \rho)(k + \rho + 1)x^{-k-\rho-2} \psi(x) - 2(k + \rho)x^{-k-\rho-1} \frac{d}{dx} \psi(x) + x^{-k-\rho} \frac{d^2}{dx^2} \psi(x) \). We can see that

\[
\sup_{x \in [\frac{1}{2}, 2]} \left| \frac{d^2}{dx^2} \psi^k(x) \right| \lesssim C(\rho) 2^{k+2}(1 + k)^2.
\]

Similarly, computing higher order derivatives, we get \( \psi^k \) satisfies the estimate

\[
\sup_{x \in [\frac{1}{2}, 2], 0 \leq l \leq N} \left| \frac{d^l \psi^k}{dx^l} \right| \leq C(\rho) 2^{N+k}(1 + k)^N
\]

for \( N \geq 20n \).

Lemma 4.1 gives us

\[
\left\| \left( \int_0^\infty |B_{2^{-j}, R}^{\psi^k} f|^2 \frac{d R}{R} \right)^{\frac{1}{2}} \right\|_p \lesssim_{N, p} 2^{N+k}(1 + k)^N 2^{j(\rho - 1) + \frac{l}{2}} \| f \|_p.
\]

Consequently, we get that

\[
\left\| \left( \int_0^\infty |S_{j, y+1}^{R, t} f (\cdot)|^2 \frac{d R}{R} \right)^{\frac{1}{2}} \right\|_p \lesssim 2^{j \rho} \sum_{k \geq 0} \frac{\Gamma(\rho + k)}{\Gamma(\rho) k!} \left( 2^j t^2 \right)^k \left( \int_0^\infty |B_{2^{-j}, R}^{\psi^k} f|^2 \frac{d R}{R} \right)^{\frac{1}{2}}_p
\]

\[
\lesssim 2^{j(\rho + \alpha)(p) - 1} \| f \|_p \sum_{k \geq 0} \frac{\Gamma(\rho + k)}{\Gamma(\rho) k!} \left( 2^j t^2 \right)^k 2^{N+k}(1 + k)^N.
\]

Since \( \frac{\Gamma(\rho + k)}{\Gamma(\rho) k!} \approx k^\rho \) and \( t^2 \leq 2^{-j-1-\epsilon_0} \), the series in the expression above converges.

This completes the proof of Theorem 4.2 for the range \(-1 < \gamma < 0\). When \( \gamma \geq 0 \), the proof follows in a similar manner as in Theorem 5.1 in [22]. The details are left to the reader.
Case II: When $t \in [\sqrt{2^{-j-1-\epsilon_0}}, \sqrt{2^{-j+1}}]$ 

Note that in this case $t \approx 2^{-j/2}$. We rewrite

$$
\psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \right) \left( 1 - \frac{|\xi|^2}{s^2 R^2} - t^2 \right)^{\gamma} \frac{|\xi|^2}{R^2} = (1 - t^2)^{\gamma+1} \psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \right) \\
\left( 1 - \frac{|\xi|^2}{R^2 (1 - t^2)} \right)^\gamma \frac{|\xi|^2}{R^2 (1 - t^2)}.
$$

Let $1 - t^2 = s^2$ and consider the operator $S_{j,\gamma+1}^{\ast R,s}$.

$$
S_{j,\gamma+1}^{\ast R,s} f(x) = \int_{\mathbb{R}^n} \psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \right) \left( 1 - \frac{|\xi|^2}{s^2 R^2} \right)^\gamma \frac{|\xi|^2}{s^2 R^2} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi,
$$

where $s \in [\sqrt{1 - 2^{-j+1}}, \sqrt{1 - 2^{-j-1-\epsilon_0}}]$.

Note that we need to establish the $L^p$-estimates for the square function

$$
\left( \int_0^\infty |S_{j,\gamma+1}^{\ast R,s} f(\cdot)|^2 \frac{dR}{R} \right)^{\frac{1}{2}}.
$$

Let $\Lambda > 100$ be a large number. By considering the cases $j \geq \Lambda$ and $2 \leq j < \Lambda$ separately, we have the following results.

**Proposition 4.3** For $j \geq \Lambda$, $0 < \epsilon < 1$ and $\gamma > \alpha(p) - \frac{1}{2}$ the estimate

$$
\sup_{s \in [s_1, s_2]} \left\| \left( \int_0^\infty |S_{j,\gamma+1}^{\ast R,s} f(\cdot)|^2 \frac{dR}{R} \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \epsilon 2^{-j\gamma} 2^{j(\alpha(p)-\frac{1}{2})} 2^{\epsilon j} \|f\|_{L^p(\mathbb{R}^n)},
$$

holds when $p \geq p_n$ or $p = 2$ where $s_1 = \sqrt{1 - 2^{-j+1}}$ and $s_2 = \sqrt{1 - 2^{-j-1-\epsilon_0}}$.

**Proposition 4.4** When $\gamma > \alpha(p) - \frac{1}{2}$ and $0 < \epsilon < 1$ we have

$$
\sup_{s \in [s_1, s_2]} \left\| \left( \int_0^\infty |S_{j,\gamma+1}^{\ast R,s} f(\cdot)|^2 \frac{dR}{R} \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \leq C(\Lambda, \gamma, \epsilon) \|f\|_{L^p(\mathbb{R}^n)}
$$

for all $2 \leq j \leq \Lambda$, $p \geq p_n$ or $p = 2$.

**Proof of Proposition 4.4**

Note that

$$
S_{j,\gamma+1}^{\ast R,s} f(x) = \int_{\mathbb{R}^n} \psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \right) \left( 1 - \frac{|\xi|^2}{s^2 R^2} \right)^\gamma \frac{|\xi|^2}{s^2 R^2} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \\
= B_{2^{-j} R}^\psi A_{Rs}^{\gamma} f(x),
$$

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where
\[ B_{2^{-j}, R}^\psi (f)(x) = \int_{\mathbb{R}^n} \psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \right) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \]
and
\[ A_{R,s}^\gamma (f)(x) = \int_{\mathbb{R}^n} \left( 1 - \frac{|\xi|^2}{R^2 s^2} \right)^\gamma \frac{|\xi|^2}{R^2 s^2} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi. \]

Since \( \psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \right) \) is a compactly supported smooth function, we can use integration by parts to show that
\[ \sup_{R > 0} |B_{2^{-j}, R}^\psi f(x)| \leq 2^{jn} Mf(x). \]

This implies that
\[ \left\| \left( \int_0^\infty |B_{2^{-j}, R/s}^\gamma A_{R,s} f(\cdot)|^2 \frac{dR}{R} \right)^{\frac{1}{2}} \right\|_p \lesssim 2^{jn} \left\| \left( \int_0^\infty |MA_{R,s}^\gamma f(\cdot)|^2 \frac{dR}{R} \right)^{\frac{1}{2}} \right\|_p. \]

The \( L^p \) boundedness of the vector-valued extension of the Hardy-Littlewood Maximal function \( M \) (see [10, 14]) in the inequality above gives us
\[ \left\| \left( \int_0^\infty |S_{j, R/s} f(\cdot)|^2 \frac{dR}{R} \right)^{\frac{1}{2}} \right\|_p \lesssim 2^{jn} \left\| \left( \int_0^\infty |A_{R,s} f(\cdot)|^2 \frac{dR}{R} \right)^{\frac{1}{2}} \right\|_p. \]

Finally, using \( L^p \) boundedness of Bochner–Riesz square function when \( \gamma > \alpha(p) - \frac{1}{2} \) for \( p \geq p_n \) and \( p = 2 \) we get the desired result. \( \square \)

**Proof of Proposition 4.3**

In order to prove our estimates, we exploit the method from [22, Proposition 5.4]. For the convenience of the reader and an easy reference, we use the same notation as in [22].

Let \( R = 1 \) and write
\[ \left( 1 - \frac{|\xi|^2}{s^2} \right)^\gamma \frac{|\xi|^2}{s^2} = \sum_{k \geq 2} 2^{-k\gamma} \tilde{\psi} \left( 2^k \left( 1 - \frac{|\xi|^2}{s^2} \right) \right) \frac{|\xi|^2}{s^2} + \tilde{\psi}_0 \left( \frac{|\xi|^2}{s^2} \right) \frac{|\xi|^2}{s^2}, \]
where \( \tilde{\psi} \in C_0^\infty ([1/2, 2]) \) and \( \tilde{\psi}_0 \in C_0^\infty ([0, 3/4]) \).

Observe that the corresponding expression for \( R \neq 1 \) can be written by replacing \( \xi \) with \( \xi / R \) in the equation above.
First, observe that the product \( \psi \left( 2^j \left( 1 - |\xi|^2 \right) \right) \hat{\psi} \left( 2^k \left( 1 - \frac{|\xi|^2}{s^2} \right) \right) \) vanishes if \( k < j - 2 \) and \( s \in \left[ \sqrt{1 - 2^{-j+1}}, \sqrt{1 - 2^{-j-1}} \right] \). Therefore, we can write

\[
\psi \left( 2^j \left( 1 - |\xi|^2 \right) \right) \left( 1 - \frac{|\xi|^2}{s^2} \right)^\gamma \frac{|\xi|^2}{s^2} = \sum_{k \geq j-2} 2^{-k} \psi \left( 2^k \left( 1 - \frac{|\xi|^2}{s^2} \right) \right) \psi \left( 2^j \left( 1 - |\xi|^2 \right) \right).
\]

Further, since \( \frac{|\xi|^2}{s^2} \approx 1 \) on the support of \( \hat{\psi} \left( 2^k \left( 1 - \frac{|\xi|^2}{s^2} \right) \right) \) for all \( k \geq \Lambda - 2 \), the term \( \frac{|\xi|^2}{s^2} \) can be ignored. Therefore, it is enough to work with the multiplier

\[
\psi \left( 2^j \left( 1 - |\xi|^2 \right) \right) \left( 1 - \frac{|\xi|^2}{s^2} \right)^\gamma = \sum_{k \geq j-2} 2^{-k} \psi \left( 2^k \left( 1 - \frac{|\xi|^2}{s^2} \right) \right) \psi \left( 2^j \left( 1 - |\xi|^2 \right) \right).
\]

We need to further decompose each piece \( \hat{\psi} \left( 2^k \left( 1 - \frac{|\xi|^2}{s^2} \right) \right) \psi \left( 2^j \left( 1 - |\xi|^2 \right) \right) \), \( k \geq j - 2 \). We refer to [22, Equation (29)] for this decomposition and in order to avoid repetition and new notations, we directly import it from there. It goes as follows.

For \( 0 < \epsilon < 1 \) let \( v = \frac{2}{s} \frac{2^{-k} \epsilon}{2^{-k} \epsilon m} \). Write \( 2^k (2^{-k+1} + 2^{-k}(1+\mu)) = m \). Then we have

\[
S_{j, \gamma+1} f(x) = \sum_{k \geq j-2} 2^{-k} \sum_{0 \leq m \leq v^{-1}+1} \sum_{q=0}^{N-1} \left( -\frac{1}{q} \right) (2\pi i) 2^{-k} e^{q \frac{dx}{x}} d\hat{\psi} \left( 2^k \left( 1 - \frac{|\xi|^2}{s^2} \right) \right) U_{R, R(k, m)} f(x) (x)
\]

\[
+ \sum_{k \geq j-2} 2^{-k} \sum_{m \leq v^{-1}+1} \int_{\mathbb{R}} \hat{\psi}(\mu) e^{2\pi i \frac{dx}{x} \mu} d\mu \int_{\mathbb{R}^n} \varphi_{\epsilon, \mu} \left( 2^{(1+\epsilon)k} \xi_{k, m} \right) \psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \right) \hat{f}(\xi) e^{2\pi i x \xi} d\xi = 21.
\]

where

\[
U_{R, R(k, m)} f(x) = \int_{\mathbb{R}^n} \varphi_q \left( 2^{(1+\epsilon)k} \xi_{k, m} \right) \psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \right) \hat{f}(\xi) e^{2\pi i x \xi} d\xi,
\]

and

\[
P_{R, R(k, m)} f(x) = \int_{\mathbb{R}^n} \varphi \left( 2^{(1+\epsilon)k} \xi_{k, m} \right) \psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \right) R_N \left( 2^{k} \xi_{k, m} \mu \right) \hat{f}(\xi) e^{2\pi i x \xi} d\xi.
\]

In the expression (21) we have used the notation \( \varphi_q(x) = x^q \varphi(x) \) where \( \varphi \in C_c^\infty([-1, 1]) \) is such that \( \sum_{m \in \mathbb{Z}} \varphi(x - m) = 1 \) on \( \mathbb{R} \). Let \( L = \sup_{0 \leq l \leq N, x \in [-1, 1]} \left| \frac{d^l \varphi}{dx^l} \right| \).
The remainder term \( r_N \) satisfies the estimate
\[
\left| \frac{d^q r_N}{ds^q}(s) \right| \leq s^{N-q}.
\]

Observe that in order to prove Proposition 4.3 it suffices to obtain required estimates for square functions associated with \( U_{Rs,R}^{\varphi_q,j}(k, m) \) and \( P_{Rs,R}^{l,k}(\mu, m) \). Since \( \varphi_q \) behaves the same way for all \( 0 \leq q \leq N-1 \), it is enough to describe proof for the operator \( U_{Rs,R}^{\varphi_q,j}(k, m) \) with \( q = 0 \). Also, note that \( \varphi_q \) is a smooth function supported in \([-1, 1]\) and it satisfies the estimate
\[
\sup_{0 \leq l \leq N, x \in [-1,1]} \left| \frac{d^l \varphi_q}{dx^l} \right| \leq LNq!,
\]
where \( L \) is the constant defined as above.

Let \( R = 1 \) (for \( R \neq 1 \) replace \( \xi \) by \( \xi / R \)) and decompose \( \psi(2^j (1 - |\xi|^2)) \) into smooth functions having their supports in the annulus \([\xi : |\xi|^2 \in [a + s^2 2^{-(1+\epsilon)k}(m-1), a + s^2 2^{-(1+\epsilon)k}(m+1)]], 0 \leq m \leq [v^{-1}] + 1\), where \( a = s^2(1-2^{-k+1}) \). Consider the following operators
\[
V_{Rs}^{\varphi_q,j}(k, m) f(x) = \int_{\mathbb{R}^n} \varphi_l \left( 2^{(1+\epsilon)k} \xi_{k,m} \right) \hat{f}(\xi)e^{2\pi i x \cdot \xi} d\xi,
\]
where \( \varphi_l(x) = x^l \varphi(x) \) and
\[
Q_{N,R,s}^k(\mu) f(x) = \int_{\mathbb{R}^n} \varphi \left( 2^{(1+\epsilon)k} \xi_{k,m} \right) r_N \left( 2^j s^2 \xi_{k,m} \mu \right) \hat{f}(\xi)e^{2\pi i x \cdot \xi} d\xi.
\]

Equation (30) in [22] allows us to write
\[
U_{Rs,R}^{\varphi_q,j}(k, m) f(x) = \sum_{l \geq 0} \frac{1}{l!} s^{2l} 2^{-k(1+\epsilon)l} 2^{-j/2} d^l \psi(2^j (1 - a - s^2 2^{-(1+\epsilon)k} m)) V_{Rs}^{\varphi_q,j}(k, m) f(x)
\]
\[
+ X_{N,R,s}^k f(x),
\]
where \( \psi \in C^{N+1}([1/2, 2]) \) and
\[
X_{N,R,s}^k f(x) = \int_{\mathbb{R}} \hat{\psi}(\mu) e^{2\pi i (2^j (1-a-s^2 2^{-(1+\epsilon)k} m) \mu}) Q_{N,R,s}^k(\mu) f(x) d\mu.
\]

In [22, Lemma 5.5] authors proved \( L^p \) –estimates for a maximal function variant corresponding to the operator \( V_{Rs}^{\varphi_q,j}(k, m) \). With suitable modifications in their proof, we get the following estimate for the square function
\[
\left\| \left( \int_0^\infty |V_{Rs}^{\varphi_q,j}(k, m) f(\cdot)|^2 \frac{dR}{R} \right)^{1/2} \right\|_p \leq C(p) l! 2^{(1+\epsilon)(\alpha(p)-\frac{1}{2})k} \|f\|_p.
\]
where $p$ and $\alpha(p)$ satisfy the assumption of Proposition 4.3, i.e., $p \geq p_n$ and $\gamma > \alpha(p) - \frac{1}{2}$. Moreover, the implicit constant in the inequality above is uniform with respect to $s \in [s_1, s_2]$. We skip the details.

Next, for the square function corresponding to $X_{N,R,s}^k$, we have the following estimate.

**Lemma 4.5** For $1 < p < \infty$, we have

$$\sup_{s \in [s_1, s_2]} \left\| \left( \int_0^\infty \left| X_{N,R,s}^k f \right|^2 \frac{d R}{R} \right)^{1/2} \right\|_p \lesssim 2^{((1+\epsilon)\alpha(p)-\frac{1}{2})k} \| f \|_p.$$  

**Proof** Recall the definition of $X_{N,R,s}^k$ from equation (23) and note that it is enough to show that

$$\sup_{s \in [s_1, s_2]} \left\| \left( \int_0^\infty \left| Q_{N,R,s}^k f \right|^2 \frac{d R}{R} \right)^{1/2} \right\|_p \lesssim (1 + |\mu|)N 2^{-j\epsilon N} 2^{-d(1+\epsilon)N} 2^{(1+\epsilon)k(n+1)} \| f \|_p,$$  

(25)

where $k = j + d$.

Define

$$M_{s,\mu}(\xi) = 2^{j\epsilon N} 2^{d(1+\epsilon)N} \varphi \left( 2^{(1+\epsilon)k} \xi_{k,m}^s \right) R_N \left( 2^{j} s^2 \xi_{k,m}^s \mu \right)$$

and write $M_{R,s,\mu}(\xi) = M_{s,\mu}(\xi / R)$. Note that

$$Q_{N,R,s}^k(\mu) f = 2^{-j\epsilon N} 2^{-d(1+\epsilon)N} f * (\mathcal{F}^{-1} M_{R,s,\mu}).$$

Using the definition of $\xi_{k,m}^s$ and $\varphi$, one can verify that support of $M_{R,s,\mu}(\xi)$ is contained in the set $\{ \xi : |\xi| / R \approx 1 \}$.

We have the following estimate, see proof of [Lemma 5.6, [22]] for details.

$$\left| \frac{\partial^\beta}{\partial \xi^\beta} M_{s,\mu}(\xi) \right| \leq c L (1 + |\mu|)^N 2^{(1+\epsilon)k|\beta|}$$

for all $|\beta| \leq N$. Also, observe that the supremum over $s$ is outside the integral in (25) we can use Plancherel theorem to prove it when $p = 2$.

Next, using integration by parts and the pointwise estimate on $\frac{\partial^\mu}{\partial \xi^\mu} M_{s,\mu}(\xi)$ as above, we can show that

$$\left| \sum_{i=1}^n \frac{\partial}{\partial x_i} \mathcal{F}^{-1} M_{R,s,\mu}(x) \right| \leq c LR^{n+1} (1 + |\mu|)^N (1 + 2^{-(1+\epsilon)k} |R_x|)^{-N'},$$  

(26)

where $N' \leq N$. 

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Choose $N'$ large enough in the inequality (26) to show that $\mathcal{F}^{-1} M_{R,s,\mu}$ verifies the conditions of vector-valued Calderón-Zygmund kernel. More precisely, for $x \neq 0$, we have

$$
\sup_{s \in [s_1, s_2]} \left( \int_0^\infty \left| \sum_{l=1}^n \frac{\partial}{\partial x_i} \mathcal{F}^{-1} M_{R,s,\mu}(x) \right|^2 \frac{dR}{R} \right)^{\frac{1}{2}} \lesssim L(1 + |\mu|)^N (2^{-1(1+\epsilon)k}|x|)^{-n-1}.
$$

Using the vector-valued Calderón-Zygmund theory we first show that $\int_0^\infty |Q_{N,R,s}^k f|^{2} \frac{dR}{R} \right)^{\frac{1}{2}}$ is weak type $(1, 1)$ which implies the result for $1 < p < 2$. The estimate for $2 < p < \infty$ is obtained by duality. This yields the desired estimate (25).

From the decomposition of $U^\varphi_{R,s,R} (k, m) f$ (see (22) observe that the estimate (24) and Lemma 4.5 yield the desired estimate for operator $U^\varphi_{R,s,R} (k, m) f$ as claimed in Proposition 4.3. This takes care of the first part in the expression (21).

Next, we need to deal with the operators $P^j_{R,s,R} (\mu, m)$ to complete the proof of Proposition 4.3. This part follows using more or less the same arguments as used in proving Lemma 4.5. With $m \leq [\nu^{-1}] + 1$ we decompose $P^j_{R,s,R} (\mu, m)$ as follows.

$$
P^j_{R,s,R} (\mu, m) f(x) = \sum_{l=0}^{N-1} \frac{1}{l!} \int_\mathbb{R} \int_\mathbb{R^n} \hat{\psi}(\mu') s^R_{N} (\mu, \mu') \varphi(2^{j+1+\epsilon}k R^{s}_{k,m}) \hat{f}(\xi) e^{2\pi i x \cdot \xi} e^{2\pi i 2^{j+1-2^{-(1+\epsilon)k}m}\mu'} d\mu' d\xi,
$$

where $\psi_j (k, m, l) = \frac{d^l \psi}{dx^l} (2^j (1 - a - s^2 2^{-1(1+\epsilon)k}m))$, $s^R_{N} (\mu, \mu') = r_N (2^{k} \xi_{k,m}^R \mu) r_N (2^{j} R^{s}_{k,m} \mu')$ and for $0 \leq l \leq N - 1$ the operator $D^j_{R,s,R} (\mu, m)$ is defined by

$$
D^j_{R,s,R} (\mu, m) f(x) = \int_\mathbb{R^n} \varphi_l (2^{j+1+\epsilon}k R_{s,k,m}^R) r_N (2^{k} \xi_{k,m} R_{s,k,m}^R) e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi.
$$

The square function corresponding to each of these operators can be dealt with in a similar way as we did for $X^k_{N,R,s}$ in Lemma 4.5 with the bounds being uniform in $s \in [s_1, s_2]$. We leave the details here. This completes the proof of Proposition 4.3.

Bounding of $G^\alpha_0$ : This part is dealt with by making minor modifications to the arguments used in the previous case. We need to carry out a similar decomposition with respect to the other variable $\eta$ in order to deal with the square function $G^\alpha_0$. We leave the details to avoid repetition as this part can be completed following the steps in the previous section along with the corresponding ideas from [22].

Remark 4.6 The proof of Theorems 2.1 and 2.2 can be simplified to a great extent for the particular case of $p_1 = p_2 = 2$ as compared to other values of exponents. The details are left to the reader.
5 Proof of Theorem 2.1

We follow the proof of Theorem 2.2 in Sect. 4 and arrive at equation (18). Then we estimate the square functions corresponding to terms $I_{j,R}$ and $II_{j,R}$. Consider the second part $II_{j,R}$ as we did in the proof of Theorem 2.2 in Sect. 4. Invoking [22, Theorem 6.1] we know that

$$\| \sup_{R > 0} \left( \int_0^{\sqrt{2^{-j+1}}} \left| \tilde{S}_{j,\beta}^{R,t} f(\cdot) t^{2\delta+1} \right|^2 dt \right)^{\frac{1}{2}} \|_{p_1} \lesssim 2^j \left( \frac{1}{2} - \tilde{\alpha} \right) \| f \|_{p_1},$$

for $\beta > \frac{1}{2}$, when $2 \leq p_1 < \infty$ and for $\beta > \frac{1}{p_1}$, when $1 < p_1 < 2$. Here $\tilde{\alpha} = \min\{\alpha, \delta + \frac{1}{2}\}$.

The remaining term is estimated using the boundedness of Bochner–Riesz square function [24, 36]. We have

$$\left\| \left( \int_0^{\infty} \left( \int_0^{\sqrt{2^{-j+1}}} |A_R^\delta g(x)|^2 dt \right) \frac{dR}{R} \right)^{\frac{1}{2}} \right\|_{p_2} \lesssim 2^{\frac{\delta}{2}(-j+1)} \| g \|_{p_2},$$

for $\delta > -\frac{1}{2}$ when $2 \leq p_2 < \infty$ and for $\delta > \frac{1}{p_2} - 1$ when $1 < p_2 < 2$.

From these estimates, we get that

$$\left\| \left( \int_0^{\infty} |II_{j,R}|^2 \frac{dR}{R} \right)^{\frac{1}{2}} \right\|_p \lesssim 2^j \left( \frac{1}{2} - \tilde{\alpha} \right) + \frac{\delta}{2}(-j+1) \| f \|_{p_2} \| g \|_{p_2},$$

holds for the claimed range of $p_1, p_2, p$ and $\alpha$ in Theorem 2.1.

Next, consider

$$\left\| \left( \int_0^{\infty} |I_{j,R}|^2 \frac{dR}{R} \right)^{\frac{1}{2}} \right\|_p \leq 2^{-\frac{1}{4}} \left\| \left( \int_0^{\infty} \left( \int_0^{\sqrt{2^{-j+1}}} |S_{j,\beta}^{R,t} f(\cdot) t^{2\delta+1} \right|^2 dt \right) \frac{dR}{R} \right)^{\frac{1}{2}} \left\|_{p_1} \sup_{R > 0} \left( \frac{1}{R} \int_0^{R} |B_R^\delta g(x)|^2 dt \right)^{\frac{1}{2}} \right\|_{p_2}.$$

Recall that for the maximal function term in the above, we have

$$\left\| \sup_{R > 0} \left( \frac{1}{R} \int_0^{R} |B_R^\delta g(x)|^2 dt \right)^{\frac{1}{2}} \right\|_{p_2} \lesssim \| g \|_{p_2},$$

for $\delta > -\frac{1}{2}$ when $2 \leq p_2 < \infty$ and for $\delta > \frac{1}{p_2} - 1$ when $1 < p_2 < 2$. 

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If we have the following estimate
\[
\left\| \int_0^\infty \left( \int_0^\infty |S_{j,\beta}^R f(\cdot) t^{2\delta+1}|^2 dt \right) \frac{dR}{R} \right\|_{p_1}^{\frac{1}{2}} \lesssim 2^{-(j-1)(\delta+\frac{1}{2})} \|f\|_{p_1} \tag{28}
\]
when \( \beta > \frac{1}{2} \) for \( 2 \leq p_1 < \infty \) and \( \beta > \frac{1}{p_1} \) for \( 1 < p_1 < 2 \). Then, the following holds
\[
\left\| \left( \int_0^\infty |I_{j,R}|^2 \frac{dR}{R} \right)^{\frac{1}{2}} \right\|_{p} \lesssim 2^{-(j-1)(\delta+\frac{1}{2})} \|f\|_{p_1} \|g\|_{p_2},
\]
for the range of \( p_1, p_2, p \) and \( \alpha \) as in Theorem 2.1.

In order to prove the above estimate (28), we rewrite the multiplier in the following way.
\[
\psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \left( 1 - \frac{|\xi|^2}{R^2} - t^2 \right) \right)^{\beta-1} \frac{|\xi|^2}{R^2} + (1 - t^2)^\beta \psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \left( 1 - \frac{|\xi|^2}{R^2} - t^2 \right) \right)^{\beta-1} \frac{|\xi|^2}{R^2(1 - t^2)}.
\]

Denote
\[
S'_{j,\beta}^R = \int_{\mathbb{R}} \psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \left( 1 - \frac{|\xi|^2}{s^2 R^2} \right) \right)^{\beta-1} \frac{|\xi|^2}{s^2 R^2} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.
\]

Therefore, we get that
\[
\int_0^{\sqrt{2^{-j+1}}} |S_{j,\beta}^R f(\cdot) t^{2\delta+1}|^2 dt = \int_0^{\sqrt{2^{-j+1}}} |S'_{j,\beta} f(\cdot)|^2 t^{4\delta+2} (1 - t^2)^{2\beta} dt
\]
\[
\lesssim 2^{-j(2\delta+1)} \int_0^{\sqrt{2^{-j+1}}} |S'_{j,\beta} f(\cdot)|^2 dt.
\]

Note that
\[
S'_{j,\beta} f(x) = B_{2^{-j},R}^\psi B_{R^s}^{\beta-1} f(x),
\]

where
\[
B_{2^{-j},R}^\psi f(x) = \int_{\mathbb{R}} \psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \right) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi,
\]
and
\[
B_R^\beta(f)(x) = \int_{\mathbb{R}} \left( 1 - \frac{|\xi|^2}{R^2 s^2} \right)^\beta \left( \frac{1}{R^2 s^2} f(\xi) \right) e^{2\pi i x \cdot \xi} d\xi.
\]

Since \( \psi \left( 2^j \left( 1 - \frac{|\xi|^2}{R^2} \right) \right) \) is a compactly supported smooth function, an integration by parts argument gives us that \( |B_{2^{-j}, R}^\psi f(x)| \lesssim Mf(x) \). Consequently, we get that
\[
\left\| \int_0^{\sqrt{2^{-j}+1}} \int_0^\infty |B_{2^{-j}, R}^\psi B_{R(1-t^2)}^{\beta-1} f(\cdot)|^2 \frac{dR}{R} dt \right\|_{L^p}^{1/2} \lesssim \left\| \int_0^{\sqrt{2^{-j}+1}} \int_0^\infty |MB_{R(1-t^2)}^{\beta-1} f(\cdot)|^2 \frac{dR}{R} dt \right\|_{L^p}^{1/2}.
\]

From here the desired result follows using a change of variables \( R(1 - t^2) \to R \), vector-valued boundedness of the Hardy-Littlewood maximal operator \( M \) and finally the boundedness of Bochner–Riesz square function.

This takes care of \( G_{\alpha}^j \) for \( j \geq 2 \). We carry out the same analysis for the square function \( G_{\alpha}^j \) as in the proof of Theorem 2.2 and use the one-dimensional results for Bochner–Riesz square function to conclude the proof of Theorem 2.1. Since this part does not require any modification in the proof given in the previous section, we leave the details to the reader. \( \square \)

6 Analytic Interpolation for \( G^\alpha \) and Proof of Theorem 2.3

In this section, we obtain new results for \( G^\alpha \) by using multilinear complex interpolation. In order to do so we will require \( L^p \) estimates for \( G^\alpha \) for \( \alpha > n - \frac{1}{2} \), which is the subject of Theorem 7.2, see Appendix for a proof. We assume it for a moment and complete the proof of Theorem 2.3.

Let \( z = \alpha + i \tau \in \mathbb{C} \) and consider the linearised operator with complex order
\[
T_b^z(f, g)(x) = \int_0^\infty \int_{\mathbb{R}^n \times \mathbb{R}^n} \left( 1 - \frac{|\xi|^2 + |\eta|^2}{R^2} \right)^z \left( \frac{|\xi|^2 + |\eta|^2}{R^2} \right) f(\xi) g(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta b(x, R) \frac{dR}{R},
\]
where \( b(x, R) \in L^2((0, \infty), dR/R) \) with \( \int_0^\infty |b(x, R)|^2 \frac{dR}{R} \leq 1 \).

We need to track down the bounds for the operator norm \( \|T_b^z\|_{L^p \times L^p \to L^p} \) in terms of the imaginary part of \( z \) in Theorems 2.2 and 7.2. This is possible as we have the precise formula for constants in terms of the parameter \( z \) in the decomposition (17) of the bilinear multiplier symbol \( \left( 1 - \frac{|\xi|^2 + |\eta|^2}{R^2} \right)^z \), which is given by
\[ c_z = C_{\beta, \delta + \iota \tau} = \frac{\Gamma(\beta + \delta + 1 + \iota \tau)}{\Gamma(\beta) \Gamma(\delta + 1 + \iota \tau)}, \]

where \( \alpha = \beta + \delta \) with \( \beta > \frac{1}{2} \) and \( \delta > -\frac{1}{2} \), see equation (16) and [35, page 279] for details. Using estimates of gamma function from [[15], page no. 422–423] we have

\[ |\Gamma(\beta + \delta + 1 + \iota \tau)| \leq |\Gamma(\beta + \delta + 1)| \quad \text{and} \quad \frac{1}{|\Gamma(\delta + 1 + \iota \tau)|} \leq \frac{e^{C(\delta)|\tau|^2}}{|\Gamma(\delta + 1)|}, \]

where \( C(\delta) = \max\{(1 + \delta)^{-2}, (1 + \delta)^{-1}\} \).

Therefore, \( |c_z| \) increases atmost by a constant multiple of \( e^{C|\tau|^2} \) where \( C \) is a fixed constant.

Further, we need to employ Calderón-Zygmund method, in which we need to use estimates on the kernel of \( G^\alpha \). Again for this part, we have the following estimates of Bessel functions, for \( Re(z) > -\frac{1}{2} \) and \( |x| \geq 1 \)

\[ |J_z(|x|)| \leq C_{Re(z)} e^{a_1 |Im(z)|^2 |x|^{-\frac{1}{2}}} \]

where \( a_1 = \max\{(Re(z) + \frac{1}{2})^{-2}, (Re(z) + \frac{1}{2})^{-1} + \pi/2\} \) and for \( Re(z) > -\frac{1}{2} \) and \( 0 < |x| \leq 1 \)

\[ |J_z(|x|)| \leq C'_{Re(z)} e^{a_2 |Im(z)|^2 |x|^{Re(z)}} \]

where \( a_2 = \max\{(Re(z) + \frac{1}{2})^{-2}, (Re(z) + \frac{1}{2})^{-1}\} \). Note that, for the kernel estimates, we have constants with growth at most \( e^{C_2|\tau|^2} \).

With this discussion on bounds for various constants as above, an inspection of proof of Theorem 2.2 yields that \( \|G^z\|_{L^2 \times L^2 \rightarrow L^1} \) increases by a constant multiple of \( e^{C|\tau|^2} \), where \( z = \alpha + \iota \tau \) with \( \alpha > 0 \). Consequently, the bilinear Calderón-Zygmund theory (see [19] for details) yields that \( \|G^z\|_{L^{p_1} \times L^{p_2} \rightarrow L^{p, \infty}} \) increases at most as \( e^{C|\tau|^2} \) for some \( c > 0 \) where \( 1 \leq p_1, p_2 < \infty \) and \( \alpha > n - \frac{1}{2} \). Finally, the real interpolation between these two estimates gives us similar bounds for the quantity \( \|G^z\|_{L^{p_1} \times L^{p_2} \rightarrow L^{p}} \) for all \( 1 < p_1, p_2 < \infty \) and \( \alpha > n - \frac{1}{2} \). These estimates ensure the admissible growth while we try to use the bilinear analytic interpolation theorem (see Theorem 7.2.9, [16]) for \( G^z \).

Denote \( V = \{z \in \mathbb{C} : 0 < Re(z) < 1\} \). For a fixed \( \delta > 0 \) and \( z \in \overline{V} \) consider the family of bilinear operators \( U^z(f, g) = T_b^{(n-\frac{1}{2})z+\delta} (f, g) \), where \( f \) and \( g \) are Schwartz class functions. Consider

\[ |U^z(f, g)(x)| = \left| \int_0^\infty \int_{\mathbb{R}^n} \left(1 - \frac{|\xi|^2 + |\eta|^2}{R^2}\right)^{(n-\frac{1}{2})z+\delta} \frac{|\xi|^2 + |\eta|^2}{R^2} \hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta b(x, R) \frac{dR}{R} \right| \]
\[
\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} \left( \int_0^\infty \left( 1 - \frac{|\xi|^2 + |\eta|^2}{R^2} \right)^{(n-\frac{1}{2}z+\delta)} \left( \frac{|\xi|^2 + |\eta|^2}{R^2} \right)^2 \frac{dR}{R} \right)^{\frac{1}{2}} d\hat{f}(\xi) \hat{g}(\eta). 
\]

Here in the above, we have used the Cauchy-Schwarz inequality in the \( R \) variable and the fact that \( \|b\|_{L^2} \leq 1 \). Making a change of variable \( \frac{\sqrt{|\xi|^2 + |\eta|^2}}{R} \rightarrow s \), we get

\[
\int_0^\infty \left( 1 - \frac{|\xi|^2 + |\eta|^2}{R^2} \right)^{(n-\frac{1}{2}z+\delta)} \left( \frac{|\xi|^2 + |\eta|^2}{R^2} \right)^2 \frac{dR}{R} = \int_0^1 (1 - s^2)^{Re((n-1)z+2\delta)} s^3 ds.
\]

Since \( Re((n-\frac{1}{2})z+\delta) > 0 \), the integral in the above expression is finite. Therefore, we get that

\[ |U^z(f, g)(x)| \lesssim \| \hat{f} \|_{L^1} \| \hat{g} \|_{L^1}. \]

From this estimate we get that for the family \( \{U^z\} \) and \( 0 \leq a < \pi \), we have

\[ \sup_{z \in \overline{V}} \frac{\log |(U^z(f, g)(x)|}{e^{a|Im(z)|}} < \infty. \]

From this estimate it is clear that the map \( z \rightarrow U^z(f, g) \) is analytic in \( V \), continuous and bounded on the closure \( \overline{V} \) and the family \( \{U^z\}_{z \in \overline{V}} \) is of admissible growth, see [page 513, [16]].

Note that when \( Re(z) = 0 \), the operator \( U^z \) maps \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n) \) with norm bounded by \( K_0(0 + i\tau) = C_{n,\delta} e^{C|\tau|^2} \) for some \( C, C_{n,\delta} > 0 \). When \( Re(z) = 1 \), we have that \( U^z \) maps \( L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \rightarrow L^{p}(\mathbb{R}^n) \) with a bound \( K_1(1 + i\tau) = C_{n,\delta} e^{C|\tau|^2} \) for some \( c > 0 \).

Given a triplet \((q_1, q_2, q)\) of exponents satisfying the Hölder relation we choose another triplet \((p_1, p_2, p)\) satisfying the Hölder relation such that \( \frac{1}{q_i} = \frac{1-\theta}{2} + \frac{\theta}{p_i}, i = 1, 2 \) where \( 0 \leq \theta \leq 1 \). Now applying the bilinear analytic interpolation [16, Theorem 7.2.9] we get that \( T^{\alpha(q_1, q_2)}_b \) is bounded from \( L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n) \) where \( 0 < \alpha(q_1, q_2) < n - \frac{1}{2} \) and it is determined by the relation \( (n-\frac{1}{2})\theta + \delta = \alpha(q_1, q_2) \) for arbitrarily small \( \delta > 0 \). This in turn implies that \( G^{\alpha(q_1, q_2)} \) is bounded from \( L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n) \).

We would like to remark that the discussion above gives us new results for \( G^\alpha \) with \( \alpha \) below the critical index when either of the exponents \( q_1 \) or \( q_2 \) is less than 2. In particular, if we consider the case of \( 1 < q_1 = q_2 < 2 \), we can write down the explicit range of the index \( \alpha \). Write \( q_1 = q_2 = q \). Note that we can choose \( p_1 = p_2 \) arbitrarily close to 1, say \( 1 + \epsilon \) where \( \epsilon \) is very small, then from the convex combination relation we get that \( \theta = \frac{1+\epsilon}{1-\epsilon} (\frac{2}{q} - 1) \). Hence we get that the range of \( \alpha \) is given by \( \alpha = \alpha(q_1, q_2) > (2n-1)(\frac{1}{q} - \frac{1}{2}) \). This proves Theorem 2.3. \qed
7 Necessary Conditions: Proof of Proposition 2.5

The following result will be required in our analysis.

Lemma 7.1 Let \( \alpha, p_1, p_2, p \) be given parameters for which the bilinear square function \( G^\alpha \) maps \( L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \). Then \( G^{\alpha+1} \) is also bounded from \( L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) with norm bounded by a constant multiple of \( \| G^\alpha \|_{L^{p_1} \times L^{p_2} \to L^p} \).

Proof For \( \rho > \alpha + \frac{1}{2} \) we have the following identity for the Bochner–Riesz multiplier symbol from [34, page 105]

\[
(1 - |\xi|^2)^\rho = C_{\rho, \alpha} \int_0^1 (1 - t^2)^{\rho - \alpha - 1} t^{2\alpha + 1} \left( 1 - \frac{|\xi|^2}{t^2} \right)^\alpha dt.
\]

Using this identity for \( \rho = \alpha + 1 > \alpha + \frac{1}{2} \) we get that

\[
K^{\alpha+1}_R * (f, g)(x) = C_\alpha \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\xi|^2 + |\eta|^2}{R^2} \left( 1 - \frac{|\xi|^2 + |\eta|^2}{R^2} \right)^{\alpha+1} \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta
\]

\[
= C_\alpha \int_0^1 t^{2\alpha + 1} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\xi|^2 + |\eta|^2}{R^2} \left( 1 - \frac{|\xi|^2 + |\eta|^2}{t^2 R^2} \right)^\alpha \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta dt
\]

\[
= C_\alpha \int_0^1 t^{2\alpha + 3} K^{\alpha}_R * (f, g)(x) dt.
\]

This relation along with elementary arguments, gives us the pointwise relation.

\[
G^{\alpha+1}(f, g)(x) \lesssim G^\alpha(f, g)(x).
\]

\[\Box\]

Proof of Proposition 2.5 part (1)

Let \( \psi \) be a function such that \( \hat{\psi}(\xi) \equiv 1 \) for \( |\xi| \leq 2 \). Recall that \( K^{\alpha}_R(y) = c_\alpha R^{2n} \left( \frac{J_{\alpha+n}(2\pi R|y|)}{(2\pi R|y|)^{\alpha+n}} - \frac{J_{\alpha+n+1}(2\pi R|y|)}{(2\pi R|y|)^{\alpha+n+1}} \right) , y \in \mathbb{R}^{2n} \). Using the asymptotic properties of the Bessel functions (see [15, page 432]), for large \( |(x, x)| \) we get that

\[
K^{\alpha}_R * (\psi, \psi)(x) = c_1 \frac{\cos \left( 2\pi R |(x, x)| + \frac{\pi}{2} (\alpha + n + \frac{1}{2}) \right)}{|(x, x)|^{\alpha+n+\frac{3}{2}}} + O \left( \frac{1}{|(x, x)|^{\alpha+n+\frac{3}{2}}} \right).
\]

Let \( M \in \mathbb{N} \) and define \( A_M(R) = \{ x \in \mathbb{R}^n : |x| - \frac{M - \frac{\pi}{2} (\alpha + n + \frac{1}{2})}{\sqrt{2} R} \leq \frac{\delta}{2 \pi R} \} \) for \( \delta \ll 1/4 \). From the boundedness of \( G^\alpha \) we have
\[
\left( \int_1^2 \left| \mathcal{K}^\alpha_R \ast (\psi, \psi)(x) \right|^2 dR \right)^{\frac{1}{2}} \lesssim \| \psi \|_{p_1} \| \psi \|_{p_2} < \infty.
\]

This implies that

\[
\limsup_{M \to \infty} \int_1^2 \int_{x \in A_M(R)} |x|^{-p(\alpha+n+\frac{1}{2})} dx \ dR < \infty. \tag{29}
\]

Since \( R \) is localised between \([1, 2]\), the condition (29) implies that \( \alpha > n \left( \frac{1}{p} - 1 \right) - \frac{1}{2} \).

Next, we need to show that we must have \( \alpha > -\frac{1}{2} \). Consider

\[
\int_1^2 |\mathcal{B}^\alpha_R(f, g)(x)|^2 dR
= \int_1^2 \left| \int_{\mathbb{R}^n \times \mathbb{R}^n} \left( 1 - \frac{|\xi|^2 + |\eta|^2}{R^2} \right)^\alpha \hat{f}(\xi) \hat{g}(\eta)e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \right|^2 dR
\]

Adding and subtracting terms \( \left( 1 - \frac{|\xi|^2 + |\eta|^2}{R^2} \right)^{\alpha+j} \) from \( j = 1 \) to \( N \) with \( \alpha + N > n - \frac{1}{2} \) we get that

\[
\left( \int_1^2 |\mathcal{B}^\alpha_R(f, g)(x)|^2 dR \right)^{\frac{1}{2}} \leq \sum_{j=0}^{N-1} G^{\alpha+j} (f, g)(x) + \left( \int_1^2 |\mathcal{B}^{\alpha+N}_R(f, g)(x)|^2 dR \right)^{\frac{1}{2}} \tag{30}
\]

Lemma 7.1 and the estimates on \( \mathcal{K}^\alpha_R \) for \( \alpha > n - \frac{1}{2} \) yield that \( (f, g) \to \left( \int_1^2 |\mathcal{B}^\alpha_R(f, g)(x)|^2 dR \right)^{\frac{1}{2}} \) is bounded for the triplet \((p_1, p_2, p)\) under consideration.

Using linearization argument as earlier we see that for any \( b \in L^2([1, 2), dR) \) the bilinear operator

\[
T_b^\alpha(f, g)(x) = \int_1^2 b(R) \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \left( 1 - \frac{|\xi|^2 + |\eta|^2}{R^2} \right)^\alpha \hat{f}(\xi) \hat{g}(\eta)e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \right) dR
\]

is bounded from \( L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \) into \( L^p(\mathbb{R}^n) \). Note that if \( \hat{f} \) and \( \hat{g} \) are compactly supported smooth functions, then \( T_b^\alpha(f, g) \) makes sense for \( \alpha \geq -\frac{1}{2} \).

Let \( u, v \in \mathbb{S}^{n-1} \) and \( \rho > 0 \). Using translation and dilation arguments for bilinear multiplier operators, we see that the operator

\[
T_{b, B_{u,v,\rho}}^\alpha(f, g)(x) = \int_1^2 b(R) \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \left( 1 - \frac{|\xi - \rho u|^2 + |\eta - \rho v|^2}{2R^2 \rho^2} \right)^\alpha \hat{f}(\xi) \hat{g}(\eta)e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \right) dR.
\]
is bounded form $L^{p_1}([\mathbb{R}^n]) \times L^{p_2}([\mathbb{R}^n])$ into $L^p([\mathbb{R}^n])$ with operator norm uniformly bounded in $u, v \in S^{n-1}$ and $\rho > 0$.

Note that the multiplier $\left(1 - \frac{|\xi - \rho u|^2 + |\eta - \rho v|^2}{2R^2 \rho^2}\right)\alpha$ converges to $(1 - \frac{1}{R^2})\alpha$ as $\rho \to \infty$ for any fixed $\xi, \eta$. The boundedness of $T_{a,b,c,d}^{\alpha}((f, g)$ is bounded on the triplet $(p_1, p_2, p)$ and Fatou’s lemma imply that

$$\left| \int_1^2 \left(1 - \frac{1}{R^2}\right)^\alpha b(R) dR \right| \lesssim \|b\|_2,$$

for all $b \in L^2([1, 2])$. The Riesz representation theorem for $L^2([1, 2])$ implies that $\alpha > -\frac{1}{2}$. This completes the proof of part (1) of Proposition 2.5.

**Proof of Proposition 2.5 part (2)**

We already have that $\alpha > -\frac{1}{2}$. Therefore, we need to prove the remaining two conditions. Consider the linearised local square function given by

$$L_b^\alpha(f, g)(x) = \int_1^2 b(R) \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \left(1 - \frac{|\xi|^2 + |\eta|^2}{R^2}\right)^\alpha \frac{|\xi|^2 + |\eta|^2}{R^2} f(\xi)g(\eta)e^{2\pi i (\xi \cdot \eta)} \frac{d\xi d\eta}{R^4} \right) dR,$$

where $b \in L^2([1, 2])$.

For $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{R}^n$ write $\eta = (\eta', \eta_n)$, where $\eta' = (\eta_1, \eta_2, \ldots, \eta_{n-1})$. Let $\epsilon > 0$ be a small number and let $\psi_\epsilon(\xi, \eta) = \phi_1(\xi/\epsilon)\phi_2(\eta'/\epsilon)\phi_3((1 - \eta_n)/\epsilon)$ where $\phi_1, \phi_2$ and $\phi_3$ are smooth functions supported in the ball $B(0, 1)$.

Consider the bilinear operator

$$\tilde{L}_b^\alpha(f, g)(x) = \int_1^2 b(R) \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \psi_\epsilon(\xi, \eta) \left(1 - \frac{|\xi|^2 + |\eta|^2}{R^2}\right)^\alpha \frac{|\xi|^2 + |\eta|^2}{R^2} f(\xi)g(\eta)e^{2\pi i (\xi \cdot \eta)} \frac{d\xi d\eta}{R^4} \right) dR.$$

Observe that by clubbing $\phi_1(\xi/\epsilon)$ with $\hat{f}(\xi)$ and $\phi_2(\eta'/\epsilon)\phi_3((1 - \eta_n)/\epsilon)$ with $\hat{g}(\eta)$, the $L^p$—boundedness of $L_b^\alpha$ implies the corresponding $L^p$—boundedness of the operator $\tilde{L}_b^\alpha$. Denote

$$\kappa_{\psi, R}^\alpha(y, z) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \psi_\epsilon(\xi, \eta) \left(1 - \frac{|\xi|^2 + |\eta|^2}{R^2}\right)^\alpha \frac{|\xi|^2 + |\eta|^2}{R^2} e^{-2\pi i (y \cdot \xi + z \cdot \eta)} d\xi d\eta.$$

Let $\phi$ be a smooth and compactly supported function with $\phi = 1$ on $B(0, 2)$ and observe that

$$\left| \int_{\mathbb{R}^n} \tilde{L}_b^\alpha(f, g)(x) \phi(x) dx \right| = \left| \int_1^2 \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \kappa_{\psi, R}^\alpha(y, z) f(y)g(z) dy dz \right) b(R) dR \right| \lesssim \|b\|_2 \|f\|_{p_1} \|g\|_{p_2}.$$

(31)
Consider a narrow cone \( C = \{(y, z) \in \mathbb{R}^n \times \mathbb{R}^n : \sqrt{|y|^2 + |z'|^2} \leq \epsilon_0 z_n\} \), where \( \epsilon_0 < \epsilon \). Using the method of stationary phase we know that for \( w = (y, z) \in C \) the kernel \( K_{\psi, R}^\alpha \) satisfies the following estimate

\[
K_{\psi, R}^\alpha(w) = R^{2n} e^{i2\pi R|w|} |Rw|^{-\frac{2n+1}{2} - \alpha} + O(|Rw|^{-\frac{2n+1}{2} - \alpha - 1}).
\]

See [33, Lemma 2.3.3] for details. Let \( M \gg \epsilon_0^{-100} \) be a large number. Consider the sets \( A_M = \{y : \epsilon_0/(10)M^{\frac{1}{2}} \leq |y| < (\epsilon_0/5)M^{\frac{1}{2}}\} \) and \( B_M = \{z : (\epsilon_0/10)M \leq |z| \leq (\epsilon_0/5)M, |z'| \leq (\epsilon_0/10)|z_n|\} \). Note that \( A_M \times B_M \subset C \). Let \( f(y) = \chi_{A_M}(y) \) and \( g(z) = \chi_{B_M}(z)e^{-i \pi |z|} \). Thus,

\[
\left|\int_1^2 \int_{\mathbb{R}^n \times \mathbb{R}^n} K_{\psi, R}^\alpha(y, z) f(y) g(z) dy dz b(R) dR \right| \\
\geq \left|\int_1^2 \int_{A_M \times B_M} R^{2n} e^{i2\pi R|w| - |z|} |Rw|^{-\frac{2n+1}{2} - \alpha} dy dz b(R) dR \right| \\
= \left|\int_{A_M \times B_M} e^{-i \pi |z|} |w|^{-\frac{2n+1}{2} - \alpha} \left(\int_1^2 R^{2n} e^{i2\pi R|w|} R^{-\frac{2n+1}{2} - \alpha} b(R) dR \right) dy dz \right|
\]

Choose \( b \) such that \( b(R) R^{-\frac{2n+1}{2} - \alpha} R^{2n} = (R-1)^{-\delta} \) where \( 0 < \delta < \frac{1}{2} \), then a change of variables argument gives us that

\[
\int_1^2 R^{2n} e^{i2\pi R|w|} R^{-\frac{2n+1}{2} - \alpha} b(R) dR = e^{i2\pi |w|} \int_0^1 e^{i2\pi R|w|} R^{-\delta} dR.
\]

Now we use the asymptotic estimates on \( \int_0^1 e^{i2\pi R|w|} R^{-\delta} dR \). First, split the integral into two parts, one where \( 2\pi R|w| < \leq \frac{1}{4} \) and the other one being its complement, then we use integration by parts argument on the part where \( 2\pi R|w| \) is large. This gives us that

\[
\int_1^2 R^{2n} e^{i2\pi R|w|} R^{-\frac{2n+1}{2} - \alpha} b(R) dR = ce^{2\pi i |w|} |w|^{-1+\delta} + O(|w|^{-1})
\]

for \( |w| \to \infty \). This estimate yields

\[
\left|\int_{A_M \times B_M} e^{-i \pi |z|} |w|^{-\frac{2n+1}{2} - \alpha} \left(\int_1^2 R^{2n} e^{i2\pi R|w|} R^{-\frac{2n+1}{2} - \alpha} b(R) dR \right) dy dz \right| \\
\geq \left|\int_{A_M \times B_M} e^{i2\pi (|w| - |z|)} |w|^{-\frac{2n+1}{2} - \alpha} |w|^{-1+\delta} dw \right|
\]

As shown in [21, Proposition 4.6] we can verify that on \( A_M \times B_M \) the term \(|w| - |z|\) is small. This implies that

\[
\left|\int_{A_M \times B_M} e^{i2\pi (|w| - |z|)} |w|^{-\frac{2n+1}{2} - \alpha} |w|^{-1+\delta} dw \right| \gtrsim M^{n/2-\alpha-3/2+\delta}
\]

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This estimate along with (31) gives us
\[ M^{-\alpha + \frac{n}{2} - 3/2 + \delta} \lesssim M^{\frac{n}{2p_1}} M^{\frac{n}{2p_2}}. \]
Since \( M \) is arbitrarily large and \( \delta \) can be as close to \( \frac{1}{2} \) as we need, we get that \( \alpha > \frac{n}{2} - \frac{n}{2p_1} - \frac{n}{2p_2} - 1 \). Using the symmetry between \( \xi \) and \( \eta \) we get the desired result.

Appendix: Sparse domination

We refer the reader to [29] for details about the sparse domination principle.

Sparse family of dyadic cubes: Consider a dyadic lattice of cubes in \( \mathbb{R}^n \). Then a family of dyadic cubes \( S \) is said to \( \eta \)--sparse, with \( 0 < \eta < 1 \) if for every cube \( Q \in S \), there exists a measurable subset \( E_Q \subset Q \) such that \( |E_Q| \geq (1 - \eta)|Q| \) and \( \{E_Q\}_{Q \in S} \) are pairwise disjoint.

For a locally integrable function \( f \) and a finite cube \( Q \) we use the notation \( \langle f \rangle_Q \) to denote the average of \( f \) over \( Q \) given by
\[ \langle f \rangle_Q = \frac{1}{|Q|} \int_Q |f(x)| \, dx, \]
where \( |Q| \) stands for the measure of the cube \( Q \). We will always assume that cubes have their sides parallel to coordinate axes. Also, for \( \lambda > 0 \) we use the notation \( \lambda Q \) to denote the concentric cube with \( Q \) such that \( |\lambda Q| = \lambda^n |Q| \). For a cube \( Q \subset \mathbb{R}^n \) the notation \( Q^2 \) stands for the Cartesian product \( Q \times Q \).

Bilinear sparse operator: Let \( S \) be a \( \eta \)--sparse family with \( 0 < \eta < 1 \). Then for compactly supported bounded functions \( f \) and \( g \) the bilinear sparse operator associated with the family \( S \) is defined by
\[ S(f, g)(x) = \sum_{Q \in S} \langle f \rangle_Q \langle g \rangle_Q \chi_Q(x). \]

We have the following pointwise sparse domination result for the square function.

**Theorem 7.2** Let \( \alpha > n - \frac{1}{2} \). Then for compactly supported functions \( f \) and \( g \) defined on \( \mathbb{R}^n \) there exists \( \nu \)-sparse families \( \{S_k\}_{k=1}^{3^n} \) such that
\[ G^\alpha(f, g)(x) \lesssim \sum_{k=1}^{3^n} S_k(f, g)(x), \]
where \( S_k \) denotes bilinear sparse operator defined as above and \( 0 < \nu < 1 \) is a constant depending only on \( n \).

**Remark 7.3** The proof of Theorem 7.2 is based on the sparse domination principle for vector-valued Calderón-Zygmund operators. The proof follows without much difficulty using standard arguments. For completion, we give the details here. Also, note...
that sparse domination gives us weighted estimates for the operator. However, we do not discuss weighted estimates in this paper as the main theme of the paper is to establish unweighted estimates for $G^\alpha$. We leave the details of weighted consequences of Theorem 7.2 to the reader, see [29] for details.

Weak-type estimate at $(1, 1, \frac{1}{2})$ for $\alpha > n - \frac{1}{2}$: Note that we can view the square function as a vector-valued bilinear operator in the following way.

$$G^\alpha(f, g)(x) = \left\| \frac{g^\alpha_R(f, g)(x)}{\sqrt{R}} \right\|_{L^2}.$$

Here the norm $\| \cdot \|_{L^2}$ is taken with respect to $R$ over the interval $(0, \infty)$. Since the kernel $K^\alpha_R$ is a radial function, we write (with a little abuse of notation)

$$K^\alpha_R(x, y) = K^\alpha_R(r),$$

where $r = |(x, y)|, (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Let $\alpha = n - \frac{1}{2} + \delta, \delta > 0$. Sunouchi [36] proved the following estimate

$$\left\| \frac{K^\alpha_R(r + s)}{\sqrt{R}} - \frac{K^\alpha_R(r)}{\sqrt{R}} \right\| \lesssim \min \left\{ R^{-\frac{1}{2} - \delta} r^{-(2n + \delta)}, |s| R^{\frac{1}{2} - \delta} r^{-(2n + \delta)} \right\}. \quad (32)$$

Consider

$$\int_{0 < 2s < r} \left\{ \int_0^\infty \left| \frac{K^\alpha_R(r + s)}{\sqrt{R}} - \frac{K^\alpha_R(r)}{\sqrt{R}} \right|^2 dR \right\}^{\frac{1}{2}} r^{2n-1} dr \leq C \int_{0 < 2s < r} \left\{ \int_0^{1/s} \left( \frac{1}{R^{\frac{1}{2} + \delta} r^{(2n + \delta)}} \right)^2 \frac{dR}{r^{2n + \delta}} \right\}^{\frac{1}{2}} r^{2n-1} dr < \infty$$

Thus, we see that the kernel of $G^\alpha$ verifies the regularity condition for bilinear Calderón-Zygmund operators, see [19, 29] for details on bilinear Calderón-Zygmund theory. Since we already proved (see Theorem 2.2) that $G^\alpha$ is bounded at $(2, 2, 1)$, invoking the bilinear Calderón-Zygmund theory from Grafakos and Torres [19] we get that $G^\alpha$ maps $L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n)$ to $L^{\frac{1}{2} \cdot \infty}(\mathbb{R}^n)$.

**Proof of Theorem 7.2**

Let $h$ be a measurable function defined on $\mathbb{R}^n$ and $E \subseteq \mathbb{R}^n$ be a measurable set. Define

$$\omega(h, E) = \sup_{x \in E} h(x) - \inf_{x \in E} h(x).$$

For $0 < \eta < 1$ and a cube $Q$ set

$$\omega_\eta(h, Q) = \min\{\omega(h, E) : E \subset Q \text{ with } |E| \geq (1 - \eta)|Q|\}.$$

We use the following result from Lerner and Nazarov [29] for sparse domination of $G^\alpha$. 

$$\geq \text{Birkhäuser}$$
Theorem 7.4 [29] Let \( f, g, h \) be functions such that

1. For every \( \epsilon > 0 \) it holds
   \[ |\{x \in [-N, N]^n : |h(x)| > \epsilon\}| = o(N^n) \text{ as } N \to \infty \]

2. For any dyadic cube \( Q \) and \( 0 < \eta \leq 2^{-n-2} \) there exists a \( \delta > 0 \) such that
   \[ \omega_\eta(h, Q) \leq C_\eta \sum_{k=0}^{\infty} 2^{-\delta k} \left( \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f| \right) \left( \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |g| \right). \]

Then there exists \( \nu \)-sparse families \( \{S_k\}_{k=1}^{3^n} \) such that for compactly supported bounded functions \( f \) and \( g \) we have

\[ |h(x)| \lesssim \sum_{k=1}^{3^n} S_k(f, g)(x). \]

Here \( \nu \) is a constant depending only on \( n \).

We will show that the result above holds for the square function under consideration.

Note that the condition (1) of Theorem 7.4 holds for \( h = G^\alpha(f, g) \) as it is weak-type at \((1, 1, \frac{1}{2})\). Therefore, we need to verify the second condition. For convenience write \( T_R(f, g)(x) = \frac{g^\alpha_R(f, g)(x)}{\sqrt{R}} \). Given a cube \( Q \subseteq \mathbb{R}^n \) denote \( Q^2 = Q \times Q \). Let \( Q \) be a dyadic cube and set \( Q_k = (2^{k+1}Q)^2 \setminus (2^k Q)^2 \), for \( k \in \mathbb{N} \). Let \( 0 < \lambda \leq 2^{-n-2} \) and points \( x, x' \in Q \). Then

\[
\left\| T_R(f, g)(x) \right\|_{L^2} - \left\| T_R(f, g)(x') \right\|_{L^2} \leq \left\| T_R((f, g)\chi_{(2^{k_n}Q)^2})(x) - T_R((f, g)\chi_{(2^{k_n}Q)^2})(x') \right\|_{L^2} \\
+ \sum_{k \geq k_n} \left\| T_R((f, g)\chi_{Q_k})(x) - T_R((f, g)\chi_{Q_k})(x') \right\|_{L^2} \\
\leq I_1 + I_2
\]

where

\[ I_1 = \left\| T_R((f, g)\chi_{(2^{k_n}Q)^2})(x) \right\|_{L^2} + \left\| T_R((f, g)\chi_{(2^{k_n}Q)^2})(x') \right\|_{L^2} \]

and

\[ I_2 = \sum_{k \geq k_n} \int_{Q_k} \left\| K_R\alpha^\prime \left( |(x, x) - (y_1, y_2)| \right) - K_R\alpha^\prime \left( |(x', x') - (y_1, y_2)| \right) \right\|_{L^2} |f(y_1)||g(y_2)|dy_1dy_2. \]

Note that \( k_n \in \mathbb{N} \) is a dimensional constant that will be chosen suitably at a later stage.
Let us first estimate the term $I_2$. Set $r = |(x', x') - (y_1, y_2)|$, since $x, x' \in Q$ and $(y_1, y_2) \in Q_k$, we have that $|(x, x) - (y_1, y_2)| = r + s$ where $s \in (-\sqrt{2}|x - x'|, \sqrt{2}|x - x'|)$. Therefore,

$$\|\mathcal{K}_R^\alpha(\cdot - (x_1, y_1)) - \mathcal{K}_R^\alpha(\cdot - (x_2, y_2))\|_{L^2}^2 \leq \|\mathcal{K}_R^\alpha(r + s) - \mathcal{K}_R^\alpha(r)\|_{L^2}^2.$$ 

We can estimate this quantity using the bounds for $\mathcal{K}_R^\alpha$ in (32). Note that in order to use (32) we need to make sure that $r > 2s$, which is possible since $x, x' \in Q$ and $(y_1, y_2) \in Q_k$ with a choice of $k \geq k_n$. Consider

$$\|\mathcal{K}_R^\alpha(r + s) - \mathcal{K}_R^\alpha(r)\|_{L^2}^2$$

$$= \int_0^{|x|^{-1}} |\mathcal{K}_R^\alpha(r + s) - \mathcal{K}_R^\alpha(r)|^2 dR + \int^\infty_{|x|^{-1}} |\mathcal{K}_R^\alpha(r + s) - \mathcal{K}_R^\alpha(r)|^2 dR$$

$$\lesssim \int_0^{|x|^{-1}} |s|^2 2^{1-2\delta} r^{-(4n+2\delta)} dR + \int^\infty_{|x|^{-1}} R^{-(1+\delta)} r^{-(4n+2\delta)} dR$$

$$\approx \frac{|s|^{2\delta}}{r^{4n+2\delta}}.$$ 

Let $l(Q)$ denote the side length of the cube $Q$. Since $r = |(x', x') - (y_1, y_2)| \approx 2^k l(Q)$ and $s \leq \sqrt{2}|x - x'| \lesssim l(Q)$ we get that

$$\frac{|s|^{2\delta}}{r^{4n+2\delta}} \lesssim \frac{l(Q)^{2\delta}}{2^{2k(2n+\delta)} l(Q)^{4n+2\delta}} = \left(\frac{1}{2^{2k(2n+\delta)} |Q|^2}\right)^2.$$ 

Therefore, we have

$$I_2 \lesssim \sum_{k \geq k_n} \int_{Q^{2^k+1} Q \setminus (2^k Q)^2} \frac{1}{2^{2k(2n+\delta)} |Q|^2} |f(y_1)||g(y_2)| dy_1 dy_2$$

$$\lesssim \sum_{k \geq k_n} 2^{-k\delta} \left(\frac{1}{2^{k+1} Q} \int_{2^k Q} |f(y)| dy\right) \left(\frac{1}{2^{k+1} Q} \int_{2^k Q} |g(y)| dy\right)$$

Next, we estimate the quantity $I_1$. This follows from the weak-type boundedness of $G^\alpha$ at $(1, 1, \frac{1}{2})$. For $\beta > 0$ consider the set

$$E^\beta = \{x \in Q : \|T_R((f, g) \chi_{2^{kn} Q})^2(x)\|_{L^2} > \beta\}.$$ 

Using the weak-type boundedness of $G^\alpha$ at $(1, 1, \frac{1}{2})$ we get

$$|E^\beta| \leq \frac{\|G^\alpha\|_{L^1 \times L^1 \rightarrow L^{\frac{1}{2}}, \infty}}{\beta} \left(\int_{2^{kn} Q} |f(y)| dy\right) \left(\int_{2^{kn} Q} |g(y)| dy\right)^{\frac{1}{2}}$$
We can choose $\beta = \|G^{\alpha}\|_{L^1 \times L^1 \rightarrow L^{1/2} \times L^{1/2} \rightarrow L^1} \lambda^{-2} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f(y)|dy \right)^{1/2} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |g(y)|dy \right)^{1/2}$. This implies that $|E^*| \leq \lambda |Q|$. Take $E = Q \setminus E^*$ and observe that for $x \in E$ we have

$$\|T_R((f,g)\chi_{(2^k Q)^2})(x)\|_{L^2(0,\infty)} \lesssim_{\lambda} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f(y)|dy \right)^{1/2} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |g(y)|dy \right)^{1/2}.$$

Moreover, we have that

$$|E| \geq |Q| - |E^*| \geq (1 - \lambda) |Q|.$$ 

Putting these estimates together, we get that with our choice of $E$ for every $x, x' \in E$ the following holds.

$$\|T_R((f,g)(x))\|_{L^2} - \|T_R((f,g)(x'))\|_{L^2} \lesssim C_{\lambda} \sum_{k=0}^{\infty} 2^{-k\delta} \left( \frac{1}{|2^{k+1} Q|} \int_{2^{k+1} Q} |f(y)|dy \right)^{1/2} \left( \frac{1}{|2^{k+1} Q|} \int_{2^{k+1} Q} |g(y)|dy \right)^{1/2}.$$

This proves condition (2) of Theorem 7.4 and hence the proof of Theorem 7.2 is done.

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