Translation and modern interpretation of Laplace’s Théorie Analytique des Probabilités, pages 505-512, 516-520.

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Abstract

The text of Laplace, *Sur l’application du calcul des probabilités à la philosophie naturelle*, (Théorie Analytique des Probabilités. Troisième Édition. Premier Supplément), 1820, is quoted in the context of the Gram-Schmidt algorithm. We provide an English translation of Laplace’s manuscript (originally in French) and interpret the algorithms of Laplace in a contemporary context. The two algorithms given by Laplace computes the mean and the variance of two components of the solution of a linear statistical model. The first algorithm can be interpreted as reverse square-root-free modified Gram-Schmidt by row algorithm on the regression matrix. The second algorithm can be interpreted as the reverse square-root-free Cholesky algorithm.

1 Introduction

This translation work is inspired by the one of Pete Stewart [3] who translates from Latin to English the “Theoria Combinationis Observationum Erroribus Minimis Obnoxiae” of Gauss in the SIAM book “Theory of the Combination of Observations Least Subject to Errors, Part One, Part Two, Supplement.” Stewart translates 101 original pages of Gauss, and he also provides an important contribution (28 pages) to place the work of Gauss in a historical framework. This manuscript is a more modest contribution. I translate thirteen pages and explain the relation of Laplace’s algorithm with our contemporary algorithms. I would like to thanks Pete Stewart to have inspired me by his work. I also would like to thank Åke Björck for giving me my first version of Laplace’s manuscript back in 2004 and Serge Gratton for useful comments on an early draft of the manuscript.

The goal of Laplace is to compute the mass of Jupiter (or Saturn) from a system of normal equations provided by Bouvard and from this same system to compute the distribution of error in the solution assuming a normal distribution of the noise on the observations. The parameters of the noise distribution are not known. Laplace explains how to compute the standard deviation of two variables of a linear statistical model. His algorithm can be interpreted as performing the Cholesky factorization of the normal equations and then compute the two standard deviations from the Cholesky factor. A second method used by Laplace to justify the first is to perform a QR factorization of the regression matrix and compute the standard deviation from the R factor. Laplace was performing the QR factorization through the modified Gram-Schmidt algorithm.

Laplace did not know what a factorization was, nor a matrix. I interpret his result through factorizations but certainly do not claim that Laplace invented all this.

The first method which Laplace introduces consists in successively projecting the system of equations orthogonally to a column of the observation matrix. This action eliminates the associated variable from the updated system. Ultimately, Laplace eliminates all the variables but the one of interest in the linear least squares problem, which eliminates all the columns but one in the observation matrix. Laplace is indeed introducing the main idea behind the Gram-Schmidt algorithm (successive orthogonal projections.) Laplace
gives an example on a 6-by-6 system. Once the observation matrix is reduced to a vector column, Laplace is able to relate the standard deviation of the variable of interest to the norm of this vector column and the norm of the residual vector.

While Laplace could have stopped here and performed the modified Gram-Schmidt algorithm onto the original overdetermined system, he explains how to compute the norm of the projected column of observations of interest directly from the normal equations. He observes that, if he performs a Cholesky factorization of the normal equations, the last coefficient computed will be equal to the norm of the last column orthogonally projected successively to the span of the remaining columns. In the mean time, Laplace observes that this method (Cholesky) provides a way to get the value of the solution from the normal equations. Laplace also generalizes this approach to more than one variable.

Laplace has used the modified Gram-Schmidt algorithm as a tool to derive the Cholesky algorithm on the normal equations. Laplace did not interpret his results with orthogonality, in particular, he did not observe that, after orthogonal projection with respect to the last column, all the remaining projected columns were made orthogonal to that column. The orthogonal projections are interpreted as elimination conserving orthogonality with the residual. Laplace correctly explains and observes the property that all the remaining projected columns, after elimination/projection, are orthogonal to the residual of the least squares problem and that the residual vector is conserved.

Laplace then uses his Cholesky algorithm to solve two 6-by-6 systems of normal equations given to him by the French astronom Bouvard to recompute the mass of Jupiter and Saturn, the originality of the work consists in assessing the reliability of these computations by estimating the standard deviation of the distribution of error in the solution.

In Section 2, I set up the background for Laplace’s work. This background is briefly recalled in Section 1 of Laplace’s manuscript. I chose not to translate this Section directly. It is fairly hard to read indeed and I have preferred to explain it and refer to the equations in it. In Section 3 I provide a translation from French to English of Laplace’s Sur l’application du calcul des probabilités à la philosophie naturelle, (Théorie Analytique des Probabilités. Troisième Édition. Premier Supplément), 1820. I have translatted Sections 2 and 5 which represent pp.505-512 and pp.516-520.

This manuscript of Laplace is quoted in the book of Farebrother [2, Chap.4] and the book of Björck [1, p.61]). In both books, the authors claim that Laplace is using the modified Gram-Schmidt algorithm.

The text is available in French from the Bibliothèque Nationale de France website [1]. There is one typo (in the pages we are translatting). On p.517, the seventh equation should read $-668486\cdot70 = -13208350z + 413134432z' - 151992,0z''' - 34876,7z^{'''}$.

We present some terminology used by Laplace. The overdetermined system of equations is named: les équations générales de condition des éléments. The Linear Least Squares method is named: la méthode la plus avantageuse. The poids (=weight) $P$ of a normal distribution is related to the standard deviation $\sigma$ by

$$P = \left(2\sigma^2\right)^{-1}.$$

## 2 Background

The main goal of this manuscript is to provide a translation of Laplace’s algorithmic contribution. However to put things into context, we start in this section with some background and notations.

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1 See http://gallica.bnf.fr/ark:/12148/bpt6k775950. For Section 2, type 505 in the box Aller Page. For Section 5, type 516 in the box Aller Page.
2.1 Covariance matrix of the regression coefficients of a linear statistical model

Laplace considers the classical linear statistical model

\[ \mathbf{Ax} = \mathbf{b}^* + \mathbf{e}, \quad \mathbf{A} \in \mathbb{R}^{s \times n}, \quad \mathbf{b}^* \in \mathbb{R}^s, \]

where \( \mathbf{e} \) is a vector of random errors having normal distribution and we will denote \( \sigma_e \) the standard deviation of \( \|\mathbf{e}\| \). In statistical language, the matrix \( \mathbf{A} \) is referred to as the regression matrix and the unknown vector \( \mathbf{x} \) is called the vector of regression coefficients. In our context, the matrix \( \mathbf{A} \) is full rank. If \( \mathbf{e} = 0 \) (there is no error in the data), then we denote the solution of the consistent overdetermined linear system of equations as \( \mathbf{x}^* \),

\[ \mathbf{Ax}^* = \mathbf{b}^*. \]

Given a vector of \( s \) observations \( \mathbf{b} \), Laplace considers the linear estimate \( \mathbf{x} \) given by the solution of the linear least squares method. We call \( \mathbf{e}' \) the residual of the linear least squares solution

\[ \mathbf{e}' \equiv \mathbf{Ax} - \mathbf{b}. \]

In Laplace terms, (see, e.g., [p.501] last sentence), by the conditions de la méthode la plus avantageuse, we have

\[ \sum p^{(i)} e^{(i)} = 0, \quad \sum q^{(i)} e^{(i)} = 0, \quad \ldots; \]

where \( p^{(i)} \) is the element (i,1) of \( \mathbf{A} \), \( q^{(i)} \) is the element (i,2) of \( \mathbf{A} \), \ldots, and \( e^{(i)} \) is the element i of \( \mathbf{e}' \). In other words,

\[ \mathbf{e}' = \mathbf{Ax} - \mathbf{b} \perp \text{Span} (\mathbf{A}). \]

Several other definitions for \( \mathbf{x} \), the linear least squares solution, are possible. We give two more equivalent definitions

\[ \mathbf{x} \text{ is such that } \|\mathbf{Ax} - \mathbf{b}\| = \min_{\mathbf{y}} \|\mathbf{Ay} - \mathbf{b}\|, \quad \text{or, equivalently, } \mathbf{x} \text{ is such that } \mathbf{A}^T \mathbf{Ax} = \mathbf{A}^T \mathbf{b}. \]

We define \( \mathbf{u} = (u, u', \ldots) \) the random vector which represents the error between the vector \( \mathbf{x}^* \) and the linear estimator \( \mathbf{x} \). From p.501 to p.504, Laplace derives the formula of the joint distribution of the random variables \( u, u', \ldots \)

We know that the joint distribution of the multivariate centered normal variable \( \mathbf{u} \) is proportional to

\[ \exp \left( -\frac{1}{2} \mathbf{u}^T \mathbf{C}^{-1} \mathbf{u} \right), \]

where \( \mathbf{C} \) is the covariance matrix of \( \mathbf{u} \). In our case we have

\[ \mathbf{C} = \sigma_b^2 (\mathbf{A}^T \mathbf{A})^{-1}, \]

therefore, the joint distribution of \( \mathbf{u} \) is proportional to

\[ \exp \left( -\frac{1}{2\sigma_b^2} \mathbf{u}^T \mathbf{A}^T \mathbf{A} \mathbf{u} \right). \tag{1} \]

In practice one does not know \( \sigma_b^2 \) and we therefore rely on an unbiased estimate, for example

\[ \frac{1}{s - n} \|\mathbf{b} - \mathbf{Ax}\|^2. \]
Now if we approximate $s - n$ with $s$, we obtain that the random vector $u$ follows a multivariate normal distribution with covariance matrix

$$\frac{1}{s} \| \mathbf{b} - \mathbf{Ax} \|_2^2 \left( \mathbf{A}^T \mathbf{A} \right)^{-1} = \frac{1}{s} \| \mathbf{e}' \|_2^2 \left( \mathbf{A}^T \mathbf{A} \right)^{-1}.$$ 

So that the joint distribution of the random variables $u, u', \ldots$ is proportional to

$$\exp \left( -\frac{s}{2\| \mathbf{e}' \|_2^2} \mathbf{u}^T \mathbf{A}^T \mathbf{u} \right).$$

This result is given in term of the variable $v = u / \sqrt{s}$ by Laplace. Laplace states that the joint distribution of the random variables $v, v', \ldots$ is proportional to (see first formula top of p.504)

$$\exp \left( -\frac{1}{2} \sum \left( p^{(i)} v + q^{(i)} v' + \ldots \right)^2 \right).$$

Note that

$$\exp \left( -\frac{s}{2\| \mathbf{e}' \|_2^2} \mathbf{u}^T \mathbf{A}^T \mathbf{u} \right) = \exp \left( -\frac{1}{2} \sum \left( p^{(i)} v + q^{(i)} v' + \ldots \right)^2 \right).$$

Therefore, Laplace’s framework fits our standard linear statistical model framework.

## 2.2 Laplace’s algorithm to compute of the standard deviation of one variable of a linear statistical model

### 2.2.1 Laplace and the modified Gram Schmidt algorithm

The background is now half set. We have a linear statistical model to which we seek a regression vector $\mathbf{x}$ through the linear least squares method and we know that the covariance matrix of the regression vector is given by the matrix $\mathbf{C}$. Laplace wants to compute only the first variable, $z$, of the regression vector, $\mathbf{x}$. He also seeks the standard deviation, $\sigma_z = \sigma_u$, of this variable.

From p.504 to p.505, Laplace explains that the modified Gram-Schmidt process applied to the matrix $\mathbf{A}$ enables him to find the standard deviation of the first variable. Laplace applies the modified Gram-Schmidt in a backward manner, that is, he projects the columns 1 to $n - 1$ orthogonally to the span of the column $n$ and obtains the matrix $\mathbf{A}_1$, then, working from the updated matrix $\mathbf{A}_1$, he projects the columns 1 to $n - 2$ orthogonally to the span of the column $n - 1$, etc.

The reverse modified Gram-Schmidt by row algorithm on the matrix $\mathbf{A}$ is formally given as follows.

Following Laplace, we name the columns of $\mathbf{A}$

$$\mathbf{A} = (\mathbf{p}, \mathbf{q}, \mathbf{r}, \ldots, \mathbf{t}, \mathbf{g}, \mathbf{l}).$$

First step is to project column 1 to $n - 1$ of $\mathbf{A}$ orthogonally to the span of its column $n$, so we define

$$\mathbf{p}_1 = \left( \mathbf{I} - \frac{\mathbf{l}^T}{\| \mathbf{l} \|_2^2} \right) \mathbf{p},$$

$$\mathbf{q}_1 = \left( \mathbf{I} - \frac{\mathbf{l}^T}{\| \mathbf{l} \|_2^2} \right) \mathbf{q},$$

$$\vdots$$

$$\mathbf{t}_1 = \left( \mathbf{I} - \frac{\mathbf{l}^T}{\| \mathbf{l} \|_2^2} \right) \mathbf{t},$$

$$\mathbf{g}_1 = \left( \mathbf{I} - \frac{\mathbf{l}^T}{\| \mathbf{l} \|_2^2} \right) \mathbf{g}.$$
This defines the $s$-by-$(n-1)$ matrix $A_1 = (p_1, q_1, r_1, \ldots t_1, g_1)$.

Second step is to project column 1 to $n-2$ of $A_1$ orthogonally to the span of its column $n-1$, so we define

$$p_2 = \left( I - \frac{g_1 g_1^T}{\|g_1\|^2} \right) p_1,$$

$$q_2 = \left( I - \frac{g_1 g_1^T}{\|g_1\|^2} \right) q_1,$$

$$\vdots$$

$$t_2 = \left( I - \frac{g_1 g_1^T}{\|g_1\|^2} \right) t_1.$$

This defines the $s$-by-$(n-2)$ matrix $A_2 = (p_2, q_2, r_2, \ldots t_2)$.

At the end of step $n-2$, we have computed the $s$-by-2 matrix $A_{n-2}$. The step $n-1$ consists in projecting the first column of $A_{n-2}$ orthogonally to the span of its second column

$$p_{n-1} = \left( I - \frac{q_{n-2} q_{n-2}^T}{\|q_{n-2}\|^2} \right) p_{n-2}.$$

Nowadays we are use to describe the modified Gram-Schmidt the other way around: project orthogonally to column 1, then column 2, etc. In either case, we note that we need to order our variables correctly. With Laplace’s method (reverse modified Gram-Schmidt), we will see that it is crucial to have the variable of interest ordered first. (And ordered last in the case of forward modified Gram-Schmidt.)

Forward modified Gram-Schmidt generates a QR factorization of the matrix $A$, that is, we compute

$$A = QR,$$

where $Q$ is $s$–by–$n$ with orthonormal columns and $R$ is $n$–by–$n$ upper triangular.

(Without loss of generality, we will impose the diagonal elements of $R$ to be positive.) On the other hand, reverse modified Gram-Schmidt generates a QL factorization of the matrix $A$, that is, we compute

$$A = QL,$$

where $Q$ is $s$–by–$n$ with orthonormal columns and $L$ is $n$–by–$n$ lower triangular.

(Without loss of generality, we will impose the diagonal elements of $L$ to be positive.)

We note that Laplace does not generate the matrix $Q$. Laplace applies what we could call the reverse square-root-free modified Gram-Schmidt by row algorithm. If we define

$$D_M = \text{diag} \left( \|p_{n-1}\|^2, \|q_{n-2}\|^2, \|r_{n-3}\|^2, \ldots \|t_2\|^2, \|g_1\|^2, \|l\|^2 \right) = T^T T,$$

then we have

$$T = QD_M^{1/2}.$$

So that we also have the factorization

$$A = T \left( D_M^{-1/2} L \right).$$

The matrix $\left( D_M^{-1/2} L \right)$ is lower triangular with ones on the diagonal.
2.2.2 A standard relation between the standard deviation of the last variable of a statistical model and the QR factorization of the regression matrix

The standard deviation of the variable \( i \) of \( u \) is given by the square-root of the entry \( (i, i) \) of the covariance matrix \( C \). If we are interested in the standard deviation \( \sigma_u \) of the first variable of \( u, u \), we need to be able to compute the entry \( (1, 1) \) of \( (A^TA)^{-1} \). We outline below a standard way to compute this quantity.

Once we have the QL factorization of \( A \), we write

\[
A^T A = L^T Q^T Q L = L^T L
\]

therefore

\[
\sigma_u = \sqrt{\text{entry}(1, 1) \text{ of } C} = \sigma_b \sqrt{\text{entry}(1, 1) \text{ of } (A^T A)^{-1}} = \sigma_b \sqrt{\text{entry}(1, 1) \text{ of } L^{-1} L^{-T}}.
\]

And, so using the fact that the matrix \( L \) is lower triangular, we have

\[
\sigma_u = \frac{1}{\ell_{1,1}} \sigma_b. \tag{5}
\]

We can prove that

\[
\ell_{1,1} = ||p_{n-1}||,
\]

(where \( p_{n-1} \) is the vector obtained at the last step of reverse modified Gram-Schmidt algorithm), so that we obtain that the marginal probability density function of the first variable \( z \) is

\[
\exp\left(-\frac{1}{2\sigma_b^2} ||p_{n-1}||^2 u^2 \right), \tag{6}
\]

and if we use the fact that \( \frac{1}{s} ||e'||^2 \) can be used as an approximation of an unbiased estimate of \( \sigma_b^2 \), we obtain that the marginal probability density function of the first variable \( z \) is

\[
\exp\left(-\frac{1}{2} \frac{s}{||e'||^2} ||p_{n-1}||^2 u^2 \right).
\]

This formula is assessed by Laplace on top p.505. We read:

“\textit{This exponential becomes} \\
\exp(-Pu^2),”
where

\[
P = \frac{\sum p_{n-1}^{(i)2}}{2 \sum \epsilon^{(i)2}}.
\]

\( u \) being the error of the random variable \( z \), \( P \) is what I called the poids (weight) of this value."

The poids is related to the standard deviation \( \sigma \) with \( P = \left(2\sigma^2\right)^{-1} \). The term poids was chosen by Laplace for the following reason (see p.499):

“la probabilité décroit avec rapidité quand il [le poids] augmente, en sorte que le résultat obtenue pèse, si je puis ainsi dire, vers la vérité, d’autant que ce module est plus grand.”

which gives

“the probability quickly decreases with it [le poids] increases, so that the result weights, if I can says so, towards the truth as much as this modulus is larger.”

Other reasons are given in the same paragraph.

### 2.2.3 Laplace’s derivation of the standard deviation of the last variable from the QL factorization of the regression matrix

The overall strategy of Laplace to compute \( \sigma_u \) is well-known nowadays. How did Laplace derive it in the first place? Starting from the fact that the joint density function of \( u, u', \ldots, u^{(n)} \) is proportional to

\[
\exp \left( -\frac{1}{\sigma_b^2} \|Au\|^2 \right),
\]

(see Equation[1], Laplace is interested in computed a function proportional to the marginal probability density function of the first variable \( u, \sigma_u \), that is Laplace wants to compute a function of the variable \( u \) proportional to

\[
\int_{u'=\infty}^{-\infty} \int_{u''=\infty}^{-\infty} \ldots \int_{u^{(n)}=\infty}^{-\infty} \exp \left( -\frac{1}{\sigma_b^2} \|Au\|^2 \right) du'du'' \ldots du^{(n)}
\]

Laplace proposes to proceed by steps. First we will seek a function proportional to the joint density function of \( u, u', \ldots, u^{(n-1)} \); then we will seek a function proportional to the joint density function of \( u, u', \ldots, u^{(n-2)} \), etc. we will eventually end up with a function proportional to the marginal probability density function of the first variable \( u \).

To perform the first step, we therefore need to compute a function of the variables \( u, u', \ldots, u^{(n-1)} \) proportional to

\[
\int_{u^{(n)}=\infty}^{-\infty} \exp \left( -\frac{1}{\sigma_b^2} \|Au\|^2 \right) du^{(n)}.
\]

Laplace observes that, (Pythagorean theorem),

\[
\|Au\|^2 = \| (I - \frac{11^T}{\|1\|^2}) Au \|^2 + \frac{11^T}{\|1\|^2} \|Au\|^2,
\]

\[
\|Au\|^2 = \| (I - \frac{11^T}{\|1\|^2}) Au \|^2 + \frac{11^T}{\|1\|^2} \|Au\|^2,
\]

\[
\|Au\|^2 = \| (I - \frac{11^T}{\|1\|^2}) Au \|^2 + \frac{11^T}{\|1\|^2} \|Au\|^2,
\]
and so

$$
||A\begin{pmatrix} u \\
                 u' \\
                 \vdots \\
                 u^{(n-1)} \\
                 u^{(n)} \end{pmatrix}||^2 = ||A_1\begin{pmatrix} u \\
                               u' \\
                               \vdots \\
                               u^{(n-1)} \\
                               u^{(n)} \end{pmatrix}||^2 + ||\frac{1}{||I||^2} I^T A \begin{pmatrix} u \\
                                                      u' \\
                                                      \vdots \\
                                                      u^{(n-1)} \\
                                                      u^{(n)} \end{pmatrix}||^2,
$$

$$
= ||A_1\begin{pmatrix} u \\
                     u' \\
                     \vdots \\
                     u^{(n-1)} \\
                     u^{(n)} \end{pmatrix}||^2 + ||I||^2 \left(u^{(n)} + \frac{1}{||I||^2} I^T (p, q, \ldots, t, g) \begin{pmatrix} u \\
                                                                                       u' \\
                                                                                       \vdots \\
                                                                                       u^{(n-1)} \\
                                                                                       u^{(n)} \end{pmatrix}\right)^2
$$

The joint density function of $u, u', \ldots, u^{(n)}$ is therefore proportional to

$$
\exp\left(-\frac{1}{\sigma_b^2} ||A_1\begin{pmatrix} u \\
                                           u' \\
                                           \vdots \\
                                           u^{(n-1)} \\
                                           u^{(n)} \end{pmatrix}||^2\right) \cdot \exp\left(-\frac{1}{\sigma_b^2} ||I||^2 \left(u^{(n)} + \frac{1}{||I||^2} I^T (p, q, \ldots, t, g) \begin{pmatrix} u \\
                                                                                u' \\
                                                                                \vdots \\
                                                                                u^{(n-1)} \\
                                                                                u^{(n)} \end{pmatrix}\right)^2\right).
$$

(This latter equation corresponds to the seventh equation on p.504.)

As previously explained, the first step of Laplace’s derivation consists in integrating this last term for $u^{(n)}$ ranging from $-\infty$ to $+\infty$ in order to obtain a function proportional to the joint density function of $u, u', \ldots, u^{(n-1)}$. So let us do this. We write

$$
\int_{u^{(n)}=-\infty}^{+\infty} \exp\left(-\frac{1}{\sigma_b^2} ||A_1\begin{pmatrix} u \\
                                                              u' \\
                                                              \vdots \\
                                                              u^{(n-1)} \\
                                                              u^{(n)} \end{pmatrix}||^2\right) \cdot \exp\left(-\frac{1}{\sigma_b^2} ||I||^2 \left(u^{(n)} + \frac{1}{||I||^2} I^T (p, q, \ldots, t, g) \begin{pmatrix} u \\
                                                                                   u' \\
                                                                                   \vdots \\
                                                                                   u^{(n-1)} \\
                                                                                   u^{(n)} \end{pmatrix}\right)^2\right) du^{(n)}
$$

$$
= \exp\left(-\frac{1}{\sigma_b^2} ||A_1\begin{pmatrix} u \\
                                       u' \\
                                       \vdots \\
                                       u^{(n-1)} \\
                                       u^{(n)} \end{pmatrix}||^2\right) \cdot \int_{u^{(n)}=-\infty}^{+\infty} \exp\left(-\frac{1}{\sigma_b^2} ||I||^2 \left(u^{(n)} + \frac{1}{||I||^2} I^T (p, q, \ldots, t, g) \begin{pmatrix} u \\
                                                                               u' \\
                                                                               \vdots \\
                                                                               u^{(n-1)} \\
                                                                               u^{(n)} \end{pmatrix}\right)^2\right) du^{(n)}.
$$

The second term is of the form

$$
\int_{u^{(n)}=-\infty}^{+\infty} \exp\left(-\mu \left(u^{(n)} + g(u, u', \ldots, u^{(n-1)})\right)^2\right) du^{(n)}.
$$

We note that this term is independent of the variables $u, u', \ldots, u^{(n-1)}$. Therefore we can remove this term from the previous equation and conclude that the joint density function of $u, u', \ldots, u^{(n-1)}$ is proportional to

$$
\exp\left(-\frac{1}{\sigma_b^2} ||A_1\begin{pmatrix} u \\
                                      u' \\
                                      \vdots \\
                                      u^{(n-1)} \\
                                      u^{(n)} \end{pmatrix}||^2\right),
$$

(7)
Continuing the process, we end up with the fact that the marginal probability density function of the first variable $u$ is proportional to
\[
\exp\left(-\frac{1}{\sigma^2_b} \|p_{n-1}\|^2 \cdot u^2 \right).
\]
We recover Equation (6) also given on top p.505.

From this equation, Laplace deduce that, to compute the standard deviation of $u$, he needs to compute $\|p_{n-1}\|^2$. While it is clear that he (or Bouvard) can work on $A$ and perform the modified Gram Schmidt algorithm, Laplace finds it easier to work on the normal equations. Quoting Laplace:

"Mais il est plus simple d’appliquer le procédé dont nous venons de faire usage aux équations finales qui déterminent les éléments, pour les réduire à une seule, ce qui donne une méthode facile de résoudre ces équations."

which means

"But it is easier to apply the method we have just used to the final equations which define the variables, in order to reduce it to a single, which gives a convenient way of solving these equations."

Therefore the next question that needs to be answered is: how can we compute $\|p_{n-1}\|^2$ from $A^T A$ without accessing $A$? This question is the matter of Section 2 of Laplace’s treatise from p.505 to p.512. An example of application of the technique is proposed in the Section 5 of the same manuscript from p.516 to p.520. We provide a translation of these two parts in the next section.

If we consider the QL factorization of the matrix $A$ given by
\[
A = \begin{pmatrix} Q_1 & q \end{pmatrix} \begin{pmatrix} L_1 & z^T \alpha \end{pmatrix},
\]
$C_1$, the covariance matrix of joint normal distribution of the variables $u, u_1, \ldots, u_{n-1}$, is
\[
C_1 = \sigma^2_b (A^T A)^{-1} = \sigma^2_b (L_1^T L_1)^{-1}.
\]
We can derive this relation from Laplace’s analysis for example. Another way to derive, this result is to remember that the covariance matrix $C_1$ of the joint normal distribution of the variable $u, u_1, \ldots, u_{n-1}$ is the $(n-1)$-by-$(n-1)$ block of the covariance matrix $C$ of the joint normal distribution of the variable $u, u_1, \ldots, u_{n-1}, u_n$. So if we write
\[
\sigma^2_b (A^T A)^{-1} = \sigma^2_b \left( \begin{pmatrix} L_1 & z \end{pmatrix} \begin{pmatrix} L_1 & z \end{pmatrix}^T \alpha \right)^{-1}
\]
\[
= \sigma^2_b \left( L_1^T L_1 + zz^T \frac{zz^T}{\alpha z} \right)^{-1}
\]
\[
= \sigma^2_b \left( \begin{pmatrix} L_1^T L_1 \end{pmatrix}^{-1} - \frac{1}{\alpha} z \frac{L_1^T L_1}{\alpha}^{-1} - \frac{1}{\alpha} z \frac{L_1^T L_1}{\alpha}^{-1} \right),
\]
we see that the the $(n-1)$-by-$(n-1)$ block is $\sigma^2_b (L_1^T L_1)^{-1} = C_1$. This is two ways to explain a standard result.
3 Translation

We now present a translation of Laplace’s text. We proceed by couple of pages. First page gives the French version. Second page gives the translatted version. We recall that the notation $S$ stands for $\sum$. 
2. Reprenons l’équation générale de condition, et, pour plus de simplicité, bornons-la aux six éléments \( z, z', z'', z''' \), \( z''', z'''' \), \( z'''', z''''', z'''''' \); elle devient alors

\[
\varepsilon^{(i)} = p^{(i)} z + q^{(i)} z' + r^{(i)} z'' + t^{(i)} z''' + \lambda^{(i)} z'' + \lambda^{(i)} z''' - \alpha^{(i)}.
\]

(1)

En la multipliant par \( \lambda^{(i)} \) et réunissant tous les produits semblables, on aura

\[
S \lambda^{(i)} \varepsilon^{(i)} = z S \lambda^{(i)} p^{(i)} + z' S \lambda^{(i)} q^{(i)} + \ldots - S \lambda^{(i)} \alpha^{(i)},
\]

le signe intégral \( S \) s’étendant à toutes les valeurs de \( i \), depuis \( i = 0 \) jusqu’à \( i = s - 1 \), \( s \) étant le nombre des observations employées. Par les conditions de la méthode la plus avantageuse, on a \( S \lambda^{(i)} \varepsilon^{(i)} = 0 \); l’équation précédente donnera donc

\[
z^{(i)} = -z'' S \lambda^{(i)} r^{(i)} - z''' S \lambda^{(i)} r^{(i)} - \alpha^{(i)}
\]

Si l’on substitue cette valeur dans l’équation (1) et si l’on fait

\[
\gamma^{(i)}_1 = \gamma^{(i)} - \lambda^{(i)} \frac{S \lambda^{(i)} \gamma^{(i)}}{S \lambda^{(i)}},
\]

\[
r^{(i)}_1 = r^{(i)} - \lambda^{(i)} \frac{S \lambda^{(i)} r^{(i)}}{S \lambda^{(i)}},
\]

\[
q^{(i)}_1 = q^{(i)} - \lambda^{(i)} \frac{S \lambda^{(i)} q^{(i)}}{S \lambda^{(i)}},
\]

\[
p^{(i)}_1 = p^{(i)} - \lambda^{(i)} \frac{S \lambda^{(i)} p^{(i)}}{S \lambda^{(i)}},
\]

\[
\alpha^{(i)}_1 = \alpha^{(i)} - \lambda^{(i)} \frac{S \lambda^{(i)} \alpha^{(i)}}{S \lambda^{(i)}},
\]

on aura

\[
\varepsilon^{(i)} = p^{(i)} z + q^{(i)} z' + r^{(i)} z'' + t^{(i)} z''' + \gamma^{(i)}_1 z'' - \alpha^{(i)};
\]

(2)

par ce moyen, l’élément \( z'' \) a disparu des équations de condition que représente l’équation [2]. En multipliant cette équation par \( \gamma^{(i)}_1 \) et réunissant tous les produits semblables, en observant ensuite que l’on a

\[
S \gamma^{(i)}_1 \varepsilon^{(i)} = 0
\]

en vertu des équations

\[
0 = S \lambda^{(i)} \varepsilon^{(i)} ; \quad 0 = S \gamma^{(i)}_1 \varepsilon^{(i)}
\]

que donnent les conditions de la méthode la plus avantageuse, on aura

\[
0 = z S \gamma^{(i)}_1 p^{(i)} + z' S \gamma^{(i)}_1 q^{(i)} + z'' S \gamma^{(i)}_1 r^{(i)} + z''' S \gamma^{(i)}_1 t^{(i)} + z'''' S \gamma^{(i)}_1 \alpha^{(i)} - S \gamma^{(i)}_1 \alpha^{(i)};
\]

d’où l’on tire

\[
z^{(i)} = -z'' S \gamma^{(i)}_1 t^{(i)} - z''' S \gamma^{(i)}_1 r^{(i)} - z'' S \gamma^{(i)}_1 q^{(i)} - z' S \gamma^{(i)}_1 p^{(i)} + S \gamma^{(i)}_1 \alpha^{(i)}.
\]
2. We consider again the overdetermined system of equations, and, for the sake of simplicity, we restrict it to the six elements \( z, z', z'', z'^', z^v, z^v' \); it then becomes
\[
\varepsilon^{(i)} = p^{(i)} z + q^{(i)} z' + t^{(i)} z'' + t^{(i)} z''' + q^{(i)} z'^v + \lambda^{(i)} z^v - \alpha^{(i)}.
\] (1)

Multiplying by \( \lambda^{(i)} \) and grouping all similar products, we have
\[
\mathcal{S} \lambda^{(i)} \varepsilon^{(i)} = z \mathcal{S} \lambda^{(i)} p^{(i)} + z' \mathcal{S} \lambda^{(i)} q^{(i)} + \ldots - \mathcal{S} \lambda^{(i)} \alpha^{(i)},
\]
the integral sign \( \mathcal{S} \) ranging for all the values of \( i \), from \( i = 0 \) to \( i = s - 1 \), \( s \) being the number of observations. By the conditions of *la méthode la plus avantageuse*, we have \( \mathcal{S} \lambda^{(i)} \varepsilon^{(i)} = 0 \); the former equation consequently gives
\[
\varepsilon^v = -z'' \frac{\mathcal{S} \lambda^{(i)} t^{(i)}}{\mathcal{S} \lambda^{(i)} 2} - z''' \frac{\mathcal{S} \lambda^{(i)} r^{(i)}}{\mathcal{S} \lambda^{(i)} 2} - z'''' \frac{\mathcal{S} \lambda^{(i)} q^{(i)}}{\mathcal{S} \lambda^{(i)} 2} - z'''' \frac{\mathcal{S} \lambda^{(i)} p^{(i)}}{\mathcal{S} \lambda^{(i)} 2} + \frac{\mathcal{S} \lambda^{(i)} \alpha^{(i)}}{\mathcal{S} \lambda^{(i)} 2}.
\]
If we replace this value in Equation (1) and if we perform
\[
\begin{align*}
\gamma_1^{(i)} &= \gamma^{(i)} - \lambda^{(i)} \frac{\mathcal{S} \lambda^{(i)} t^{(i)}}{\mathcal{S} \lambda^{(i)} 2}, \\
t_1^{(i)} &= t^{(i)} - \lambda^{(i)} \frac{\mathcal{S} \lambda^{(i)} t^{(i)}}{\mathcal{S} \lambda^{(i)} 2}, \\
r_1^{(i)} &= r^{(i)} - \lambda^{(i)} \frac{\mathcal{S} \lambda^{(i)} r^{(i)}}{\mathcal{S} \lambda^{(i)} 2}, \\
q_1^{(i)} &= q^{(i)} - \lambda^{(i)} \frac{\mathcal{S} \lambda^{(i)} q^{(i)}}{\mathcal{S} \lambda^{(i)} 2}, \\
p_1^{(i)} &= p^{(i)} - \lambda^{(i)} \frac{\mathcal{S} \lambda^{(i)} p^{(i)}}{\mathcal{S} \lambda^{(i)} 2}, \\
\alpha_1^{(i)} &= \alpha^{(i)} - \lambda^{(i)} \frac{\mathcal{S} \lambda^{(i)} \alpha^{(i)}}{\mathcal{S} \lambda^{(i)} 2},
\end{align*}
\]
we have
\[
\varepsilon^{(i)} = p_1^{(i)} z + q_1^{(i)} z' + t_1^{(i)} z'' + t_1^{(i)} z''' + q_1^{(i)} z'^v + \gamma_1^{(i)} z^v - \alpha_1^{(i)};
\] (2)
by this technique, the element \( z^v \) has disappeared from the system of equations represented by Equation (2). Multiplying this equation by \( \gamma_1^{(i)} \), grouping all similar products, and observing that we have
\[
\mathcal{S} \gamma_1^{(i)} \varepsilon^{(i)} = 0
\]
from the equations
\[
0 = \mathcal{S} \lambda^{(i)} \varepsilon^{(i)}, \quad 0 = \mathcal{S} \gamma^{(i)} \varepsilon^{(i)}
\]
given by the conditions of *la méthode la plus avantageuse*, we have
\[
0 = z \mathcal{S} \gamma_1^{(i)} p_1^{(i)} + z' \mathcal{S} \gamma_1^{(i)} q_1^{(i)} + z'' \mathcal{S} \gamma_1^{(i)} r_1^{(i)} + z''' \mathcal{S} \gamma_1^{(i)} t_1^{(i)} + z'''' \mathcal{S} \gamma_1^{(i)} q_1^{(i)} + z'''' \mathcal{S} \gamma_1^{(i)} p_1^{(i)} - \mathcal{S} \gamma_1^{(i)} \alpha_1^{(i)};
\]
from which we draw
\[
\varepsilon^v = -z'' \frac{\mathcal{S} \gamma_1^{(i)} t_1^{(i)}}{\mathcal{S} \gamma_1^{(i)} 2} - z''' \frac{\mathcal{S} \gamma_1^{(i)} r_1^{(i)}}{\mathcal{S} \gamma_1^{(i)} 2} - z'''' \frac{\mathcal{S} \gamma_1^{(i)} q_1^{(i)}}{\mathcal{S} \gamma_1^{(i)} 2} - z'''' \frac{\mathcal{S} \gamma_1^{(i)} p_1^{(i)}}{\mathcal{S} \gamma_1^{(i)} 2} + \frac{\mathcal{S} \gamma_1^{(i)} \alpha_1^{(i)}}{\mathcal{S} \gamma_1^{(i)} 2}.
\]
Si l’on substitue cette valeur dans l’équation (2) et si l’on fait
\[
\begin{align*}
t_2^{(i)} &= t_1^{(i)} - \gamma_1^{(i)} \frac{Y_1^{(i)} t_1^{(i)}}{Y_1^{(i)2}}, \\
r_2^{(i)} &= r_1^{(i)} - \gamma_1^{(i)} \frac{Y_1^{(i)} r_1^{(i)}}{Y_1^{(i)2}}, \\
q_2^{(i)} &= q_1^{(i)} - \gamma_1^{(i)} \frac{Y_1^{(i)} q_1^{(i)}}{Y_1^{(i)2}}, \\
p_2^{(i)} &= p_1^{(i)} - \gamma_1^{(i)} \frac{Y_1^{(i)} p_1^{(i)}}{Y_1^{(i)2}}, \\
\alpha_2^{(i)} &= \alpha_1^{(i)} - \gamma_1^{(i)} \frac{Y_1^{(i)} \alpha_1^{(i)}}{Y_1^{(i)2}},
\end{align*}
\]
on aura
\[
\varepsilon^{(i)} = p_2^{(i)} z + q_2^{(i)} z' + r_2^{(i)} z'' + t_2^{(i)} z''' - \alpha_2^{(i)}.
\] (3)

En continuant ainsi, on parviendra à une équation de la forme
\[
\varepsilon^{(i)} = p_S^{(i)} z - \alpha_S^{(i)}.
\] (4)
Il résulte du n°20 du Livre II que, si la valeur de $z$ est déterminée par cette équation et que $u$ soit l’erreur de cette valeur, la probabilité de cette erreur est
\[
\sqrt{\frac{s P_S^{(i)2}}{2s \varepsilon^{(i)2} \pi}} e^{-\frac{s P_S^{(i)2}}{2s \varepsilon^{(i)2} u^2}},
\]

$s \varepsilon^{(i)2}$ étant la somme des carrés des restes des équations de condition, lorsqu’on y a substitué les éléments déterminés par la méthode la plus avantageuse. Le poids $P$ de cette erreur est donc égal à $s P_S^{(i)2}$.

Il s’agit maintenant de déterminer $s P_S^{(i)2}$. Pour cela, on multipliera respectivement chacune des équations de condition représentées par l’équation (1), d’abord par le coefficient du premier élément, et l’on prendra la somme de ces produits; ensuite par le coefficient du second élément, et l’on prendra la somme de ces produits, et ainsi du reste. On aura, en observant que par les conditions de la méthode la plus avantageuse $S p^{(i)} \varepsilon^{(i)} = 0, S q^{(i)} \varepsilon^{(i)} = 0, \ldots$, les six équations suivantes :

\[
\begin{align*}
p \alpha &= p^{(2)} z + p q z' + p r z'' + p \gamma z''' + p \lambda z^v + p \alpha z^w, \\
q \alpha &= q^{(2)} z' + q r z'' + q \gamma z''' + q \lambda z^v + q \alpha z^w, \\
r \alpha &= r^{(2)} z'' + r q z''' + r \gamma z^v + r \lambda z^w, \\
\gamma \alpha &= \gamma^{(2)} z''' + \gamma q z^v + \gamma r z^w, \\
\lambda \alpha &= \lambda^{(2)} z^v + \lambda q z^w + \lambda r z^w,
\end{align*}
\]

\(A\)

où l’on doit observer que nous supposons
\[
p^{(2)} = S p^{(i)2}, \quad p q = S p^{(i)} q^{(i)}, \quad q^{(2)} = S q^{(i)2}, \quad q r = S q^{(i)} r^{(i)}, \ldots
\]
If we replace this value in Equation (2) and if we perform
\[ t_2^{(i)} = t_1^{(i)} - \gamma_1^{(i)} \frac{S_1^{(i)} t_1^{(i)}}{S_1^{(i)2}}, \]
\[ r_2^{(i)} = r_1^{(i)} - \gamma_1^{(i)} \frac{S_1^{(i)} r_1^{(i)}}{S_1^{(i)2}}, \]
\[ q_2^{(i)} = q_1^{(i)} - \gamma_1^{(i)} \frac{S_1^{(i)} q_1^{(i)}}{S_1^{(i)2}}, \]
\[ p_2^{(i)} = p_1^{(i)} - \gamma_1^{(i)} \frac{S_1^{(i)} p_1^{(i)}}{S_1^{(i)2}}, \]
\[ \alpha_2^{(i)} = \alpha_1^{(i)} - \gamma_1^{(i)} \frac{S_1^{(i)} \alpha_1^{(i)}}{S_1^{(i)2}}, \]
we have
\[ \epsilon^{(i)} = p_2^{(i)} z + q_2^{(i)} z' + r_2^{(i)} z'' + t_2^{(i)} z''' - \alpha_2^{(i)}. \] (3)

Continuing in a similar manner, we end up with an equation of the form
\[ \epsilon^{(i)} = p^*_2 z - \alpha^*_2. \] (4)

From n°20 of Livre II, we know that, if the value of \( z \) is determined by this equation and if \( u \) is the error of the value, the probability of this error will be
\[ \sqrt{\frac{sS_p^{(i)2}}{2S\epsilon^{(i)2} \pi}} e^{-\frac{sS_p^{(i)2}}{2S\epsilon^{(i)2} \pi} u^2}, \]
where \( S\epsilon^{(i)2} \) is the sum of the squares of the residuals of the equations of condition, after we replaced the elements determined by la méthode la plus avantageuse. Le poids \( P \) of this error is then equal to \( \frac{sS_p^{(i)2}}{2S\epsilon^{(i)2} \pi} \).

Our next task is to determine \( S_p^{(i)2} \). For this, we multiply each of these equations represented by Equation (1), first by the coefficient of the first element, and we take the sum of these products; then by the coefficient of the second element, and we take the sum of these products, and so on for the remaining. We have, by observing that the conditions of la méthode la plus avantageuse \( Sp^{(i)} \epsilon^{(i)} = 0, Sq^{(i)} \epsilon^{(i)} = 0, \ldots \), the six following equations:

\[
\begin{align*}
\overline{p\alpha} &= p^{(2)} z + \overline{pq} z' + \overline{pr} z'' + \overline{pt} z''' + \overline{p\gamma} z^iv + \overline{p\lambda} z^v, \\
\overline{q\alpha} &= \overline{pq} z + \overline{q^{(2)}} z' + \overline{qr} z'' + \overline{qt} z''' + \overline{q\gamma} z^iv + \overline{q\lambda} z^v, \\
\overline{r\alpha} &= \overline{rp} z + \overline{rq} z' + \overline{r^{(2)}} z'' + \overline{rt} z''' + \overline{r\gamma} z^iv + \overline{r\lambda} z^v, \\
\overline{t\alpha} &= \overline{tp} z + \overline{tq} z' + \overline{t^{(2)}} z'' + \overline{tt} z''' + \overline{t\gamma} z^iv + \overline{t\lambda} z^v, \\
\overline{\gamma\alpha} &= \overline{\gamma p} z + \overline{\gamma q} z' + \overline{\gamma r} z'' + \overline{\gamma t} z''' + \overline{\gamma^{(2)}} z^iv + \overline{\gamma\lambda} z^v, \\
\overline{\lambda\alpha} &= \overline{\lambda p} z + \overline{\lambda q} z' + \overline{\lambda r} z'' + \overline{\lambda t} z''' + \overline{\lambda\gamma} z^iv + \overline{\lambda\lambda} z^v,
\end{align*}
\]

(A)

where we have defined
\[ p^{(2)} = Sp^{(i)2}, \quad \overline{pq} = Sp^{(i)} q^{(i)}, \quad q^{(2)} = Sq^{(i)2}, \quad \overline{qr} = Sq^{(i)} r^{(i)}, \ldots \]
Si l’on multiplie pareillement les équations de condition représentées par l’équation (2) respectivement par les coefficients de $z$ et que l’on ajoute ces produits, ensuite par les coefficients de $z'$ en ajoutant encore ces produits, et ainsi de suite, on aura le système suivant d’équations, en observant que $Sp_1^{(i)} e^{(i)} = 0$, $Sq_1^{(i)} e^{(i)} = 0$, ..., par les conditions de la méthode la plus avantageuse,

$$
\begin{align*}
\begin{cases}
    p_1 \alpha_1 &= p_2^{(2)} z + p_1 q_1^{(2)} z' + p_1 r_1^{(2)} z'' + p_1 s_1^{(2)} z''' + p_1 t_1^{(2)} z^iv, \\
    q_1 \alpha_1 &= p_1 q_1^{(2)} z + q_1 q_1^{(2)} z' + q_1 r_1^{(2)} z'' + q_1 s_1^{(2)} z''' + q_1 t_1^{(2)} z^iv, \\
    r_1 \alpha_1 &= p_1 r_1^{(2)} z + q_1 r_1^{(2)} z' + r_1 r_1^{(2)} z'' + r_1 s_1^{(2)} z''' + r_1 t_1^{(2)} z^iv, \\
    t_1 \alpha_1 &= p_1 t_1 r_1^{(2)} z + q_1 t_1 r_1^{(2)} z' + t_1 r_1^{(2)} z'' + t_1 s_1^{(2)} z''' + t_1 t_1^{(2)} z^iv, \\
    y_1 \alpha_1 &= p_1 y_1^{(2)} z + q_1 y_1^{(2)} z' + r_1 y_1^{(2)} z'' + s_1 y_1^{(2)} z''' + t_1 y_1^{(2)} z^iv,
\end{cases}
\end{align*}
$$

(B)

où l’on doit observer que

$$
\begin{align*}
    p_1 q_1^{(2)} = Sp_1^{(i)} q_1^{(i)}, \quad p_2^{(2)} = Sp_1^{(i)} q_1^{(i)}, \quad \ldots
\end{align*}
$$

En substituant, au lieu de $p_1^{(i)}$, $q_1^{(i)}$, ..., leurs valeurs précédentes, on a

$$
\begin{align*}
    p_1 q_1^{(2)} = Sp_1^{(i)} q_1^{(i)} - \frac{Sp_1^{(i)} \lambda^{(i)} p_1^{(i)} S\lambda^{(i)} q_1^{(i)}}{S\lambda^{(i)}},
\end{align*}
$$

ou

$$
\begin{align*}
    p_1 q_1^{(2)} &= p_1 q_1^{(2)} - \frac{\lambda p \lambda q_1^{(2)}}{\lambda^{(2)}},
\end{align*}
$$

on a pareillement

$$
\begin{align*}
    p_1^{(2)} &= p_1^{(2)} - \frac{\lambda p \lambda q_1^{(2)}}{\lambda^{(2)}}, \\
    q_1^{(2)} &= q_1^{(2)} - \frac{\lambda q_1^{(2)} \lambda p \lambda q_1^{(2)}}{\lambda^{(2)}}, \\
    r_1^{(2)} &= r_1^{(2)} - \frac{\lambda q_1^{(2)} \lambda p \lambda q_1^{(2)}}{\lambda^{(2)}}, \\
    t_1^{(2)} &= t_1^{(2)} - \frac{\lambda q_1^{(2)} \lambda p \lambda q_1^{(2)}}{\lambda^{(2)}}, \\
    y_1^{(2)} &= y_1^{(2)} - \frac{\lambda q_1^{(2)} \lambda p \lambda q_1^{(2)}}{\lambda^{(2)}},
\end{align*}
$$

Ainsi les coefficients du système des équations (B) se déduisent facilement des coefficients du système des équations (A).

Les équations de condition représentées par l’équation (3) donneront semblablement le système suivant d’équations

$$
\begin{align*}
\begin{cases}
    p_2 \alpha_2 &= p_2^{(2)} z + p_2 q_2^{(2)} z' + p_2 r_2^{(2)} z'' + p_2 s_2^{(2)} z''' + p_2 t_2^{(2)} z^iv, \\
    q_2 \alpha_2 &= p_2 q_2^{(2)} z + q_2 q_2^{(2)} z' + q_2 r_2^{(2)} z'' + q_2 s_2^{(2)} z''' + q_2 t_2^{(2)} z^iv, \\
    r_2 \alpha_2 &= p_2 r_2^{(2)} z + q_2 r_2^{(2)} z' + r_2 r_2^{(2)} z'' + r_2 s_2^{(2)} z''' + r_2 t_2^{(2)} z^iv, \\
    t_2 \alpha_2 &= p_2 t_2 r_2^{(2)} z + q_2 t_2 r_2^{(2)} z' + t_2 r_2^{(2)} z'' + t_2 s_2^{(2)} z''' + t_2 t_2^{(2)} z^iv,
\end{cases}
\end{align*}
$$

(C)

et l’on a

$$
\begin{align*}
    p_2^{(2)} &= p_1^{(2)} - \frac{\lambda p_2^{(2)} \lambda q_1^{(2)}}{\lambda^{(2)}}, \\
    p_2 q_2^{(2)} &= p_1 q_1^{(2)} - \frac{\lambda p_1^{(2)} \lambda q_2^{(2)}}{\lambda^{(2)}}, \\
    p_2 r_2^{(2)} &= p_1 r_1^{(2)} - \frac{\lambda p_1^{(2)} \lambda r_1^{(2)}}{\lambda^{(2)}}, \\
    p_2 t_2^{(2)} &= p_1 t_1^{(2)} - \frac{\lambda p_1^{(2)} \lambda t_1^{(2)}}{\lambda^{(2)}},
\end{align*}
$$

15
If we multiply similarly the equations represented by Equation (2) respectively by the coefficients of $z$ and we add these products, then by the coefficient of $z'$ adding again these products, and so on, we have the following system of equations, by noting that $Sp_{i}^{(i)}e^{(i)} = 0$, $Sq_{i}^{(i)}e^{(i)} = 0$, ..., from the conditions of la méthode la plus avantageuse,

$$
\begin{align*}
\begin{cases}
p_{1}α_1 &= p_{1}^{(2)} z + \frac{p_{1}q_{1}}{\lambda_1} z' + \frac{p_{1}r_{1}}{\lambda_1} z'' + \frac{p_{1}t_{1}}{\lambda_1} z''' + \frac{p_{1}γ_{1}}{\lambda_1} z^{iv}, \\
q_{1}α_1 &= \frac{p_{1}q_{1}}{\lambda_1} z + q_{1}^{(2)} z' + \frac{q_{1}r_{1}}{\gamma_1} z'' + \frac{q_{1}t_{1}}{\gamma_1} z''' + \frac{q_{1}γ_{1}}{\gamma_1} z^{iv}, \\
r_{1}α_1 &= \frac{p_{1}r_{1}}{\lambda_1} z + \frac{q_{1}r_{1}}{\gamma_1} z' + r_{1}^{(2)} z'' + \frac{q_{1}r_{1}}{\gamma_1} z''' + \frac{r_{1}γ_{1}}{γ_1} z^{iv}, \\
t_{1}α_1 &= \frac{p_{1}t_{1}}{\lambda_1} z + \frac{q_{1}t_{1}}{\gamma_1} z' + t_{1}^{(2)} z''' + \frac{t_{1}γ_{1}}{γ_1} z^{iv}, \\
γ_{1}α_1 &= \frac{p_{1}γ_{1}}{\lambda_1} z + \frac{q_{1}γ_{1}}{\gamma_1} z' + γ_{1}^{(2)} z^{iv},
\end{cases}
\end{align*}
$$

where we have defined

$$p_{1}q_{1} = S_{p_{1}}^{(i)}q_{1}^{(i)}, \quad p_{2} = S_{p_{1}}^{(i)}.$$ Substituting $p_{1}^{(i)},q_{1}^{(i)},...$ with their previous values, we have

$$p_{1}q_{1} = \frac{Sp_{1}(i)q_{1}(i) - \frac{S\lambda(i)p_{1}(i)S\lambda(i)q_{1}(i)}{S\lambda(i)^{2}}}{\lambda(i)^{2}}$$

or

$$p_{1}q_{1} = p_{1}q - \frac{\lambda p \lambda q}{\lambda(2)};$$

we have similarly

$$p_{2}^{(2)} = p_{2}^{(2)} - \frac{\lambda_{2}^{2}}{\lambda(2)},$$

$$q_{2}^{(2)} = q_{2}^{(2)} - \frac{\lambda_{2}^{2}}{\lambda(2)},$$

$$p_{1}r_{1} = p_{1}r - \frac{\lambda p \lambda r}{\lambda(2)},$$

$$p_{1}t_{1} = p_{1}t - \frac{\lambda p \lambda t}{\lambda(2)},$$

and so on.

Doing so, the coefficients of the system of equations (B) are easily computable from the coefficients of the system of equations (A).

The equations represented by Equation (3) similarly give the following system of equations

$$
\begin{align*}
\begin{cases}
p_{2}α_2 &= p_{2}^{(2)} z + \frac{p_{2}q_{2}}{\lambda_2} z' + \frac{p_{2}r_{2}}{\lambda_2} z'' + \frac{p_{2}t_{2}}{\lambda_2} z''' + \frac{p_{2}γ_{2}}{\lambda_2} z^{iv}, \\
q_{2}α_2 &= \frac{p_{2}q_{2}}{\lambda_2} z + q_{2}^{(2)} z' + \frac{q_{2}r_{2}}{γ_2} z'' + \frac{q_{2}t_{2}}{γ_2} z''' + \frac{q_{2}γ_{2}}{γ_2} z^{iv}, \\
r_{2}α_2 &= \frac{p_{2}r_{2}}{\lambda_2} z + \frac{q_{2}r_{2}}{γ_2} z' + r_{2}^{(2)} z'' + \frac{q_{2}r_{2}}{γ_2} z''' + \frac{r_{2}γ_{2}}{γ_2} z^{iv}, \\
t_{2}α_2 &= \frac{p_{2}t_{2}}{\lambda_2} z + \frac{q_{2}t_{2}}{γ_2} z' + t_{2}^{(2)} z''' + \frac{t_{2}γ_{2}}{γ_2} z^{iv}, \\
γ_{2}α_2 &= \frac{p_{2}γ_{2}}{\lambda_2} z + \frac{q_{2}γ_{2}}{γ_2} z' + γ_{2}^{(2)} z^{iv},
\end{cases}
\end{align*}
$$

and we have

$$p_{2}^{(2)} = p_{1}^{(2)} - \frac{\lambda p^{(2)}}{\lambda(2)},$$

$$p_{2}q_{2} = \frac{p_{1}q_{1}}{γ_2} - \frac{\lambda p^{(2)}q_{1}}{γ_2}$$

$$p_{2}r_{2} = \frac{p_{1}r_{1}}{γ_2} - \frac{\lambda p^{(2)}r_{1}}{γ_2},$$

$$p_{2}t_{2} = \frac{p_{1}t_{1}}{γ_2} - \frac{\lambda p^{(2)}t_{1}}{γ_2},$$

and so on.
On a pareillement le système d’équations

\[
\begin{align*}
\overline{p_3\alpha_3} &= p_3^{(2)} z + p_3 q_3^{(2)} z' + p_3 r_3^{(2)} z'', \\
q_3\alpha_3 &= q_3^{(2)} z + q_3 q_3^{(2)} z' + q_3 r_3^{(2)} z'', \\
r_3\alpha_3 &= r_3^{(2)} z + q_3 r_3^{(2)} z' + r_3^{(2)} z'' \quad \text{(D)}
\end{align*}
\]

en faisant

\[
\begin{align*}
p_3^{(2)} &= p_2^{(2)} - \frac{p_4^3}{q_4}, \\
p_3 q_3^{(2)} &= p_2 q_2 - \frac{p_4 q_4^3}{q_4}, \\
p_3 r_3^{(2)} &= p_2 r_2 - \frac{p_4 r_4^3}{q_4},
\end{align*}
\]

on aura encore

\[
\begin{align*}
\overline{p_4\alpha_4} &= p_4^{(2)} z + p_4 q_4^{(2)} z', \\
q_4\alpha_4 &= q_4^{(2)} z + q_4 q_4^{(2)} z', \\
r_4\alpha_4 &= r_4^{(2)} z + q_4 r_4^{(2)} z', \quad \text{(E)}
\end{align*}
\]

en faisant

\[
\begin{align*}
P_4^{(2)} &= P_3^{(2)} - \frac{P_4^3}{Q_4}, \\
P_4 q_4^{(2)} &= P_3 q_3^{(2)} - \frac{P_4 q_4^3}{Q_4}, \\
P_4 r_4^{(2)} &= P_3 r_3^{(2)} - \frac{P_4 r_4^3}{Q_4},
\end{align*}
\]

Enfin on aura

\[
\overline{p_5\alpha_5} = p_5^{(2)} z, \quad \text{(F)}
\]

en faisant

\[
p_5^{(2)} = P_4^{(2)} - \frac{P_4 q_4^2}{Q_4}, \quad p_5 \alpha_5 = P_4 \alpha_4 - \frac{P_4 q_4^{2} Q_4 \alpha_4}{Q_4},
\]

\(p_5^{(2)}\) est la valeur \(S p_5^{(i)2}\), et le poids \(P\) sera

\[
\frac{S p_5^{(2)}}{2 S e^{(i)2}}.
\]

On voit par la suite des valeurs \(p^{(2)}, p_1^{(2)}, p_2^{(2)}, \ldots\) qu’elles vont en diminuant sans cesse, et qu’ainsi, pour le même nombre d’observations, le poids \(P\) diminue quand le nombre des éléments augmente.

Si l’on considère la suite des équations qui déterminent \(p_5\alpha_5\), on voit que cette fonction, développée suivant les coefficients du système des équations \(A\), est de la forme

\[
\overline{p\alpha} + \overline{M q\alpha} + \overline{N r\alpha} + \ldots,
\]

le coefficient de \(\overline{p\alpha}\) étant l’unité. Il suit de là que si l’on résout les équations \(A\), en y laissant \(\overline{p\alpha}, q\alpha, r\alpha, \ldots\) comme indéterminées, \(\frac{1}{p_5^{(2)}}\) sera, en vertu de l’équation \(F\), le coefficient de \(\overline{p\alpha}\) dans l’expression de \(z\). Pareillement, \(\frac{1}{q_5^{(2)}}\) sera le coefficient de \(q\alpha\) dans l’expression de \(z'\); \(\frac{1}{r_5^{(2)}}\) sera le coefficient de \(r\alpha\) dans l’expression de \(z''\); et ainsi de suite du reste; ce qui donne un moyen de simple d’obtenir \(p_5^{(2)}, q_5^{(2)}, \ldots\); mais il est plus simple encore de les déterminer ainsi.
We similarly have the system of equations

\[
\begin{aligned}
\frac{p_3 \alpha_3}{q_3 \alpha_3} &= p_3 z + \frac{p_3 q_3}{q_3 \alpha_3} z' + \frac{p_3 r_3}{r_3 \alpha_3} z'', \\
\frac{q_3 \alpha_3}{r_3 \alpha_3} &= \frac{q_3}{r_3} z + \frac{q_3}{r_3} z',
\end{aligned}
\]

by doing

\[
\begin{aligned}
p_3 &= p_2 - \frac{p_2 q_3}{q_3^2}, \\
p_3 q_3 &= p_2 q_2 - \frac{p_2 q_3}{q_3}, \\
p_3 r_3 &= p_2 r_2 - \frac{p_2 q_3}{r_3},
\end{aligned}
\]

we also have

\[
\begin{aligned}
p_4 &= p_2 - \frac{p_2 q_3}{q_3}, \\
p_4 q_4 &= p_2 q_2 - \frac{p_2 q_3}{q_3}, \\
p_4 r_4 &= p_2 r_2 - \frac{p_2 q_3}{r_3},
\end{aligned}
\]

Finally we have

\[
p_5 = p_2 z,
\]

by doing

\[
\begin{aligned}
p_5 &= p_4 - \frac{p_4 q_4}{q_4}, \\
p_5 q_4 &= p_4 q_4 - \frac{p_4 q_4}{q_4}, \\
p_5 r_4 &= p_4 r_4 - \frac{p_4 q_4}{r_4},
\end{aligned}
\]

p_5 is the value \(S p_5^{(1)}\), and le poids \(P\) is

\[
\frac{S p_5^{(2)}}{2 \varepsilon \eta^{(1)}},
\]

We see from the sequence of values \(p_1^{(2)}, p_2^{(2)}, p_3^{(2)}, \ldots\) that they always go diminishing, and so, for the same number of observations, le poids \(P\) decreases when the number of elements increases.

If we consider the sequence of equations which determine \(p_5 \alpha_5\), we see that this function, developed according to the coefficients of the system of equations (A), is of the form

\[
\overline{p \alpha} + M q \alpha + N r \alpha + \ldots,
\]

the coefficient of \(p \alpha\) being the unity. It follows from there that if we solve the equations (A), by leaving \(p \alpha\), \(q \alpha\), \(r \alpha\), \ldots as unknowns, \(\frac{1}{p_5^{(2)}}\) is, due to Equation (F), the coefficient of \(p \alpha\) in the expression of \(z\). Similarly, \(\frac{1}{q_5^{(2)}}\) is the coefficient of \(q \alpha\) in the expression of \(z'\); \(\frac{1}{r_5^{(2)}}\) is the coefficient of \(r \alpha\) in the expression of \(z''\); and so on for the others; this gives a simple mean to obtain \(p_5^{(2)}, q_5^{(2)}, \ldots\); but it is even simpler to compute them as follows.
D’abord l’équation (F) donne la valeur de \( p_5^{(2)} \) et de \( z \). Si dans le système des équations (E) on élimine \( z \) au lieu de \( z' \), on aura une seule équation en \( z' \), de la forme

\[
q_5 \alpha_5 = q_5^{(2)} z';
\]

en faisant

\[
q_5^{(2)} = q_4^{(2)} - \frac{p_4 q_4^2}{p_4^{(2)}}, \quad q_5 \alpha_5 = \frac{p_4 q_4 \alpha_4}{p_4^{(2)}}.
\]

Si dans le système des équations (D) on élimine \( z \) au lieu de \( z'' \), pour ne conserver à la fin du calcul que \( z'' \), on aura \( r_5^{(2)} \) en changeant dans la suite des équations qui, à partir de ce système, déterminent \( p_5^{(2)} \), la lettre \( p \) dans la lettre \( r \), et réciproquement. On aura ainsi

\[
\begin{align*}
    r_4^{(2)} &= r_3^{(2)} - \frac{p_3 r_3^2}{p_3^{(2)}}, \\
    r_4 q_4 &= r_3 q_3 - \frac{p_3 q_3 r_3}{p_3^{(2)}}, \\
    q_4^{(2)} &= q_3^{(2)} - \frac{p_3 q_3^2}{p_3^{(2)}}, \\
    r_5^{(2)} &= r_4^{(2)} - \frac{p_4 r_4^2}{q_4^{(2)}}.
\end{align*}
\]

Pour avoir \( t_5^{(2)} \), on partira du système (C), en changeant, dans la suite des valeurs de \( p_3^{(2)} \), \( p_3 q_3 \), \( r_3^{(2)} \), \( q_3 r_3 \), \( \ldots \), la lettre \( p \) dans la lettre \( t \), et réciproquement.

On aura pareillement la valeur de \( \gamma_5^{(2)} \), en partant du système des équations (B) et changeant dans la suite des valeurs de \( p_2^{(2)} \), \( p_3^{(2)} \), \( \ldots \), la lettre \( p \) dans la lettre \( \gamma \), et réciproquement.

Enfin, on aura la valeur de \( \lambda_5^{(2)} \) en changeant, dans la suite des valeurs de \( p_1^{(2)} \), \( p_2^{(2)} \), \( \ldots \), la lettre \( p \) dans la lettre \( \lambda \), et réciproquement.
Firstly Equation (F) gives the value of $p_2^{(2)}$ and of $z$. If in the system of equations (E) we eliminate $z$ instead of $z'$, we have a single equation in $z'$, of the form

$$q_5 \alpha_5 = q_5^{(2)} z';$$

by doing

$$q_5^{(2)} = q_4^{(2)} - \frac{p_4 q_4^2}{p_4^{(2)}}, \quad q_5 \alpha_5 = \alpha_4^{(2)} - \frac{p_4 q_4 p_4 \alpha_4}{p_4^{(2)}}.$$

If in the system of equations (D) we eliminate $z$ instead of $z''$, in order to only keep at the end of the computation $z''$, we have $r_5^{(2)}$ by changing in the sequence of equations which, starting from this system, determine $p_5^{(2)}$, the letter $p$ by the letter $r$, and reciprocally. We then have

$$r_4^{(2)} = r_3^{(2)} - \frac{p_3 r_3^2}{p_3^{(2)}},$$

$$r_4 q_4 = r_3 q_3 - \frac{p_3 r_3 q_3}{p_3^{(2)}},$$

$$q_4^{(2)} = q_3^{(2)} - \frac{p_3 q_3^2}{p_3^{(2)}},$$

$$r_5^{(2)} = r_4^{(2)} - \frac{p_4 r_4 q_4}{q_4^{(2)}}.$$

In order to have $t_5^{(2)}$, we start from the system (C), by changing, in the sequence of values of $p_3^{(2)}$, $p_3 q_3$, ..., $r_3^{(2)}$, $q_3 r_3$, ..., the letter $p$ by the letter $r$, and reciprocally.

We similarly have the value of $\gamma_5^{(2)}$, starting from the system of equations (B) and changing in the sequence of values of $p_2^{(2)}$, $p_3^{(2)}$, ..., the letter $p$ by the letter $\gamma$, and reciprocally.

Finally, we have the value of $\lambda_5^{(2)}$ by changing, in the sequence of values of $p_1^{(2)}$, $p_2^{(2)}$, ..., the letter $p$ by the letter $\lambda$, and reciprocally.
5. Appliquons maintenant cette méthode à un exemple. Pour cela, j’ai profité de l’immense travail que Bouvard a fait terminer sur les mouvements de Jupiter et de Saturne, dont il a construit des Tables très précises. Il a fait usage de toutes les oppositions observées par Bradley et par les astronomes qui l’ont suivi : il les a discutées de nouveau et avec le plus grand soin, ce qui lui a donné 126 équations de condition pour le mouvement de Jupiter en longitude et 129 équations pour le mouvement de Saturne. Dans ces dernières équations, Bouvard a fait entrer la masse d’Uranus comme indéterminée. Voici les équations finales qu’il a conclues par la méthode la plus avantageuse :

\[
\begin{align*}
7212^\prime, 600 &= 795938z - 12729398z' + 6788,2z'' - 1959,0z''' + 696,13z^{iv} + 2602z^v, \\
-738297^\prime, 800 &= -12729398z + 424865729z' - 153106,5z'' - 39749,1z''' - 5459z^{iv} + 5722z^v, \\
237^\prime, 782 &= 6788,2z - 153106,5z' + 71,8720z'' - 3,2252z''' + 1,2484z^{iv} + 1,3371z^v, \\
-40^\prime, 335 &= -1959,0z - 39749,1z' - 3,2252z'' + 57,1911z''' + 3,6213z^{iv} + 1,1128z^v, \\
-343^\prime, 455 &= 696,13z - 5459z' + 1,2484z'' + 3,6213z''' + 21,543z^{iv} + 46,310z^v, \\
-1002^\prime, 900 &= 2602z + 5722z' + 1,3371z'' + 1,1128z''' + 46,310z^{iv} + 129z^v.
\end{align*}
\]

Dans ces équations, la masse d’Uranus est supposée \( \frac{1+z''}{15956} \); la masse de Jupiter est supposée \( \frac{1+z'}{10677,09} \); \( z'' \) est le produit de l’équation du centre par la correction du périhélie employé d’abord par Bouvard; \( z''' \) est la correction de l’équation du centre; \( z^{iv} \) est la correction séculaire du mouvement de Jupiter; \( z^v \) est la correction de l’époque de la longitude au commencement de 1750. La seconde du degré décimal est prise pour unité.

Au moyen des équations précédentes renfermées dans le système (A), j’ai conclu les suivantes, renfermées dans le système (B) :

\[
\begin{align*}
27441^\prime, 68 &= 743454z - 12844814z' + 6761,23z'' - 1981,45z''' - 237,97z^{iv}, \\
-693812^\prime, 58 &= -12844814z + 424611920z' - 153165,81z'' - 39798,46z''' - 7513,15z^{iv}, \\
248^\prime, 1772 &= 6761,23z - 153165,81z' + 71,8581z'' - 3,2367z''' + 0,7684z^{iv}, \\
-31^\prime, 6836 &= -1981,45z - 39798,46z' - 3,2367z'' + 57,1815z''' + 3,2218z^{iv}, \\
16^\prime, 5783 &= -237,97z - 7513,15z' + 0,7684z'' + 3,2218z''' + 4,9181z^v.
\end{align*}
\]

De ces équations, j’ai tiré les quatre suivantes, renfermées dans le système (C) :

\[
\begin{align*}
28243^\prime, 85 &= 731939,5z - 13208350z' + 6798,41z'' - 1825,56z'''', \\
-668486^\prime, 70 &= -13208350z + 413134432z' - 151992,0z'' - 34876,7z'''', \\
245^\prime, 5870 &= 6798,41z - 151992,0z' + 71,7381z'' - 3,7401z''', \\
42^\prime, 5434 &= -1825,56z - 34876,7z' - 3,7401z''' + 55,0710z'''';
\end{align*}
\]

ces dernières équations donnent les suivantes, renfermées dans le système (D) :

\[
\begin{align*}
26833^\prime, 55 &= 671414,7z - 14364541z' + 6674,43z''', \\
-695430^\prime, 0 &= -14364541z + 391046861z' - 154360,6z''', \\
242^\prime, 6977 &= 6674,43z - 154360,6z' + 71,4841z'''.
\end{align*}
\]

Enfin j’ai conclu de là les deux équations, renfermées dans le système (E) :

\[
4172^\prime, 95 = 48442z + 48020z' , \quad -171455^\prime, 2 = 48020z + 57725227z'.
\]
5. We now apply this method to an example. For this, I have benefited from the immense work that Bouvard has just finished on the movements of Jupiter and Saturn, from which he has constructed extremely accurate Tables. He has used all the observations from Bradley and from the astronomers that have followed him: he has discussed them again and with the greatest care, which has given him 126 equations for the movement of Jupiter in longitude and 129 equations for the movement of Saturn. In these equations, Bouvard has introduced the mass of Uranus as unknown. Here are the final equations that he has obtained by *la méthode la plus avantageuse*:

\[
\begin{align*}
7212''.600 &= 795938z - 12729398z' + 6788.2z'' + 1959.0z''' + 696.1z'''' + 2602z'', \\
-738297''.800 &= -12729398z + 424865729z' - 153106.5z'' - 39749.1z''' - 5459z'''' + 5722z'', \\
237''.782 &= 6788.2z - 153106.5z'' + 71.8720z''' - 3.2252z'''' + 1.2484z'''' + 1.3371z, \\
-40''.335 &= -1959.0z - 39749.1z' - 3.2252z'' - 57.1911z''' + 3.6213z'''' + 1.1128z'', \\
-343''.455 &= 696.13z - 5459z' + 1.2484z'' + 3.6213z''' + 21.543z'''' + 46.310z'', \\
-1002''.900 &= 2602z + 5722z' + 1.3371z'' + 1.1128z'''' + 46.310z'''' + 129z''.
\end{align*}
\]

In these equations, the mass of Uranus is supposed to be \( \frac{1+z}{1057.09} \); the mass of Jupiter is supposed to be \( \frac{1+z'}{1959} \); \( z'' \) is the product if the equation of the center by the correction of the periapsis firstly employed by Bouvard; \( z''' \) is the correction of the equation of the center; \( z'''' \) is the secular correction of the mean movement; \( z \) is the correction of the epoch of the longitude beginning in 1750. The second of the decimal degree is taken for unit.

With these former equations corresponding to the system (A), I have concluded the followings, corresponding to the system (B):

\[
\begin{align*}
27441''.68 &= 743454z - 12844814z' + 6761.23z'' - 1981.45z''' - 237.97z'''' , \\
-693812''.58 &= -12844814z + 424611920z' - 153165.81z'' - 39798.46z''' - 7513.15z'''' , \\
248''.1772 &= 6761.23z - 153165.81z' + 71.8581z'' - 3.2367z'''' + 0.7684z'''' , \\
-31''.6836 &= -1981.45z - 39798.46z' - 3.2367z'' + 57.1815z''' + 3.2218z'''' , \\
16''.5783 &= -237.97z - 7513.15z' + 0.7684z'' + 3.2218z'''' + 4.9181z'''' .
\end{align*}
\]

From these equations, I have drawn the four followings, corresponding to the system (C):

\[
\begin{align*}
28243''.85 &= 731939.5z - 13208350z' + 6798.41z'' - 1825.56z'''' , \\
-668486''.70 &= -13208350z + 413134432z' - 151992.0z'' - 34876.7z'''' , \\
245''.5870 &= 6798.41z - 151992.0z' + 71.7381z'' - 3.7401z'''' , \\
42''.5434 &= -1825.56z - 34876.7z' - 3.7401z'' + 55.0710z'''' .
\end{align*}
\]

these latest equations give us the followings, corresponding to the system (D):

\[
\begin{align*}
26833''.55 &= 671414.7z - 14364541z' + 6674.43z'' , \\
-695430''.0 &= -14364541z + 391046861z' - 154360.6z'' , \\
242''.6977 &= 6674.43z - 154360.6z' + 71.4841z'' .
\end{align*}
\]

Finally I have concluded from there the two equations, corresponding to the system (E):

\[
\begin{align*}
4172''.95 &= 48442z + 48020z' , \\
-171455''.2 &= 48020z + 57725227z'.
\end{align*}
\]
Je m’arrête à ce système, parce qu’il est facile d’en conclure les valeurs du poids $P$ relatives aux deux éléments $z$ et $z’$ que je désirais particulièrement de connaître. Les formules du n° 3 donnent, pour $z$,

$$P = \frac{s}{2Se(i)^2} \left[ 48442 - \frac{(48020)^2}{57725227} \right]$$

et, pour $z’$,

$$P = \frac{s}{2Se(i)^2} \left[ 57725227 - \frac{(48020)^2}{48442} \right].$$

Le nombre $s$ des observations est ici 129 et Bouvard a trouvé

$$Se(i)^2 = 31096;$$
on a donc, pour $z$,

$$\log P = 2,0013595;$$
et, pour $z’$,

$$\log P = 5,0778624.$$

Les équations précédentes donnent

$$z’ = -0,00305,$$
$$z = 0,08916.$$

La masse de Jupiter est $\frac{1}{1067,09} (1 + z’).$ En substituant pour $z’$ sa valeur précédente, cette masse devient $\frac{1}{1070,35}.$ La masse du Soleil est prise pour unité. La probabilité que l’erreur de $z’$ est comprise dans les limites $\pm U$ est, par le n° 1,

$$\frac{\sqrt{P}}{\sqrt{\pi}} \int du e^{-Pu^2},$$

l’intégrale étant prise depuis $u = -U$ jusqu’à $u = U$. On trouve ainsi la probabilité que la masse de Jupiter est comprise dans les limites

$$\frac{1}{1070,35} \pm \frac{1}{100 1067,09},$$
egale à $\frac{1000000}{1000001}$; en sorte qu’il y a un million à très peu à parier contre un que la valeur $\frac{1}{1070,35}$ n’est pas en erreur d’un centième de sa valeur; ou, ce qui revient à fort peu près au même, qu’après un siècle de nouvelles observations, ajoutées aux précédentes et dicitées de la même manière, le nouveau résultat ne différera pas du précédent d’un centième de sa valeur.

Newton avait trouvé, par les observations de Pound, sur les élongations des satellites de Jupiter, la masse de cette planète égale à la $1067^{e}$ partie de celle du Soleil, ce qui diffère très peu du résultat de Bouvard.

La masse d’Uranus est $\frac{1+z}{17907}$. En substituant pour $z$ sa valeur précédente, cette masse devient $\frac{1}{17907}$. La probabilité que cette valeur est comprise dans les limites

$$\frac{1}{17907} \pm \frac{1}{4 19504}$$
est égale à $\frac{2508}{2509}$, et la probabilité que cette masse est comprise dans les limites

$$\frac{1}{17907} \pm \frac{1}{5 19504}$$
est égale à $\frac{215.6}{216.6}$. 

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I stop with this system, because it is easy to conclude from it the values of the poids $P$ corresponding to the two elements $z$ and $z'$ which I particularly wish to know. The formula from n° 3 give, for $z$,

$$P = \frac{s}{2S\epsilon^2} \left[ 48442 - \frac{(48020)^2}{57725227} \right]$$

and, for $z'$,

$$P = \frac{s}{2S\epsilon^2} \left[ 57725227 - \frac{(48020)^2}{48442} \right].$$

The number $s$ of observations is here 129 and Bouvard has found $S\epsilon^2 = 31096$; we then have, for $z$,

$$\log P = 2.0013595;$$

and, for $z'$,

$$\log P = 5.0778624.$$

The former equations give

$$z' = -0.00305, \quad z = 0.08916.$$

The mass of Jupiter is $\frac{1}{1067.09} (1 + z')$. Replacing $z'$ by its former value, this mass becomes $\frac{1}{1070.35}$. The mass of the Sun is taken as unity. The probability that the error in $z'$ is between the limit $\pm U$ is, from n° 1,

$$\frac{\sqrt{P}}{\sqrt{\pi}} \int du \ e^{-\mu^2},$$

the integral being taken from $u = -U$ to $u = U$. We then find that the probability for the mass of Jupiter to be between the limits

$$\frac{1}{1070.35} \pm \frac{1}{100 1067.09},$$

is equal to $\frac{2508}{2509}$; so that there is one million to very few to bet against one that the value $\frac{1}{1070.35}$ is not in error of one hundredth of its value; or, which is more or less the same, that after one century of new observations, added to the former and discussed in the same manner, the new result does not differ from the former of more than one hundredth of its value.

Newton had found, from the observations of Pound, on the elongations of Jupiter’s satellites, the mass of this planet equal to the $1067^{th}$ part of the Sun, which differs very few from the result of Bouvard.

The mass of Uranus is $\frac{1+z}{19503}$. Replacing for $z$ its former value, this mass becomes $\frac{1}{17907}$. The probability that this value is between the limits

$$\frac{1}{17907} \pm \frac{1}{4 19504},$$

is equal to $\frac{2508}{2509}$, and the probability that this mass is between the limits

$$\frac{1}{17907} \pm \frac{1}{5 19504},$$

is equal to $\frac{215.6}{216.6}$. 
Les perturbations qu’Uranus produit dans le mouvement de Saturne étant peu considérables, on ne doit pas encore attendre des observations de ce mouvement une grande précision dans la valeur de sa masse. Mais, après un siècle de nouvelles observations, ajoutées aux précédentes et discutées de la même manière, la valeur de $P$ augmentera de manière à donner cette masse avec une grande probabilité que sa valeur sera contenue dans d’étroites limites; ce qui sera de beaucoup préférable à l’emploi des élongations des satellites d’Uranus, à cause de la difficulté d’observer ces élongations.

Bouvard, en appliquant la méthode précédente aux 126 équations de condition que lui ont données les observations de Jupiter et en supposant la masse de Saturne égale à $\frac{1}{3534,08}$, a trouvé

$$z = 0,00620$$

et

$$\log P = 4,8856829.$$  

Ces valeurs donnent la masse de Saturne égale à $\frac{1}{3512,3}$, et la probabilité que cette masse est comprise dans les limites

$$\frac{1}{3512,3} \pm \frac{1}{100} \frac{1}{3534,08}$$

est égale à $\frac{11327}{11328}$.

Newton avait trouvé par les observations de Pound sur la plus grande élongation du quatrième satellite de Saturne, la masse de cette planète égale à $\frac{1}{3012}$, ce qui surpasse d’un sixième le résultat précédent. Il y a des millions de milliards à parier contre un que celui de Newton est en erreur, et l’on n’en sera point surpris si l’on considère la difficulté d’observer les plus grandes élongations des satellites de Saturne. La facilité d’observer celles des satellites de Jupiter a rendu, comme on l’a vu, beaucoup plus exacte la valeur que Newton a conclue des observations de Pound.
The perturbations that Uranus induces in the movement of Saturn being negligible, we should not expect a great accuracy in the value of the mass from these observations of the movement. But, after a century of new observations, added to the previous and discussed in the same manner, the value of $P$ increases so that the mass is given with a large probability that its value is contained within tight bounds; which is a lot better than using the elongations of the Uranus’ satellites, because these elongations are difficult to observe.

Bouvard, applying the former method to 126 equations given from the observations of Jupiter and assuming that the mass of Saturn is equal to $\frac{1 + z}{3534.08}$, has found

$$z = 0.00620$$

and

$$\log P = 4.8856829.$$  

These values give the mass of Saturn equal to $\frac{1}{3512.3}$, and the probability that this mass is between the limits

$$\frac{1}{3512.3} \pm \frac{1}{100} \frac{1}{3534.08}$$

is equal to $\frac{11327}{11328}$.

Newton has found, from Pound’s observations on the largest elongations of Saturn’s fourth satellite, that the mass of this planet is equal to $\frac{1}{3012}$, which overestimates from than one sixth the former result. There are millions of billions to bet against one that the one of Newton is in error, and we should not be surprised considering the difficulty to observe the greatest elongations of Saturn’s satellites. The easiness to observe the ones from Jupiter’s satellites has given, as we have seen, a much more exact value than the one concluded by Newton from Pound’s observations.
4 Comments

Although Laplace presents his algorithm for two variables, in Sections 4.1 and 4.2 we will assume that he is only seeking one variable and its standard deviation. This makes explanations easier. Then in Section 4.3 we generalize to two variables.

4.1 Laplace’s algorithm as a factorization algorithm

The procedure used by Laplace is a variant of the Cholesky factorization of the normal equations. We do not claim that Laplace interprets his algorithm as a factorization. We state that, in Matrix Computation term, we can interpret Laplace’s algorithm as a factorization.

In Matlab notation, if we initialize \( M \) with the lower part of \( A^T A \) in input, his algorithm writes

\[
\text{for } k=n:-1:2, \\
M(1:k-1,1:k-1) = M(1:k-1,1:k-1) - M(1:k-1,k)*M(1:k-1,k)' / M(k,k); \\
\text{end}
\]

The operation is a symmetric rank-1 update and Laplace only updates the lower part of the matrix \( M \) at each step. After this operation, one obtains a reverse square-root-free Cholesky factorization of the form:

\[
(A^T A) = (D_M^{-1} \cdot M)^T M,
\]

where \( D_M \) is the matrix corresponding to the diagonal of \( M \). This is a reverse Cholesky factorization because it is a upper triangular matrix times a lower triangular matrix as opposed to being a lower triangular matrix times a upper triangular matrix. It is square root free because the left factor, \((D_M^{-1} \cdot M)^T\), has a unit diagonal and the right factor, \( M \), has a non-unit diagonal. reverse Cholesky (with square roots) gives \( D_M^{1/2} \) to each factor so that the factorization writes

\[
(A^T A) = (D_M^{-1/2} \cdot M)^T (D_M^{-1/2} \cdot M) = L^T L.
\]

Cholesky factorization reverse square-root-free Cholesky factorization

In Matlab notation, after Laplace’s algorithm, we could compute the solution with a backward and a forward solve.

\[
z = \text{diag}(\text{diag}(M))' \backslash \alpha; \\
z = M \backslash z;
\]

where the first line is the backward solve and the second line is the forward solve.

In Laplace’s algorithm, the backward solve is done on the fly as

\[
z = \alpha; \\
\text{for } k=n:-1:2, \\
z(1:k-1) = z(1:k-1) - M(1:k-1,k)*z(k) / M(k,k); \\
\text{end}
\]
On the fly means that this loop is inserted in the loop of the factorization. Since Laplace only wishes the first variable, the forward solve reduces to
\[ z(1) = z(1) / M(1,1); \]

Another way to understand Laplace’s factorization is to follow Trefethen and Bau’s explanations of LU[4]. Laplace is applying a sequence of unit upper triangular matrices, \( U_i \), to \( A^T A \) in order to reduce \( A^T A \) to lower triangular form. We obtain

\[
\begin{pmatrix}
  1 & 1 & 1 \\
  1 & 1 & 1 \\
  1^* & 1^* & 1^*
\end{pmatrix}
\begin{pmatrix}
  A^T A
\end{pmatrix} =
\begin{pmatrix}
  M
\end{pmatrix}
\]

And with a few “strokes of luck”, we can prove that
\[
(U_{n-1}, \ldots, U_2, U_1)^{-1} = (D^{-1}_M \cdot M)^T
\]

so that we finally recover our factorization: \((A^T A) = (D^{-1}_M \cdot M)^T \cdot M.\)

If we follow this interpretation then, the backward solve is the application of the sequence of unit upper triangular matrices to the right-hand side itself.

A third way to understand Laplace’s factorization is to follow Laplace’s explanations. Laplace is implicitly computing the L factor of the QL factorization of \( A \) from the normal equations \( A^T A \).

4.2 Laplace’s algorithm to compute the standard deviation of a variable

We have seen in Section 2.2.2 that we can derive quite easily the standard deviation of the first variable of a statistical model from the QL factorization of the regression matrix. The relation is given in Equation (5).

Since the L factor obtained by the QL factorization is also the reverse Cholesky factor, Equation (5) explains how to compute the standard deviation of the first variable with reverse Cholesky. We think this is the best way to understand Laplace’s algorithm with our contemporary tools.

4.3 Laplace’s algorithm to compute the standard deviation of two variables

Laplace indeed is interested in the standard deviation of the two first variables. In this case, he stops his reduction process at the 2-by-2 matrix corresponding to
\[
A_{n-2}^T A_{n-2} = \begin{pmatrix}
  p_n^T p_{n-2} & p_n^T q_{n-2} \\
  q_n^T p_{n-2} & q_n^T q_{n-2}
\end{pmatrix} = \begin{pmatrix}
  48442 & 48020 \\
  48020 & 57725227
\end{pmatrix}.
\]

The covariance matrix of \( u \) and \( u' \) is then given by
\[
\sigma_n^2 (A_{n-2}^T A_{n-2})^{-1} = \sigma_b^2 \left( \begin{pmatrix}
  p_n^T p_{n-2} & p_n^T q_{n-2} \\
  q_n^T p_{n-2} & q_n^T q_{n-2}
\end{pmatrix} \right)^{-1} = \sigma_b^2 \begin{pmatrix}
  48442 & 48020 \\
  48020 & 57725227
\end{pmatrix}^{-1}
\]

\[
= \sigma_b^2 \left( \frac{1}{48442 \cdot 57725227 - 48020^2} \begin{pmatrix}
  57725227 & -48020 \\
  -48020 & 48442
\end{pmatrix} \right).
\]
If we use the fact that $\frac{1}{s-n} \|e'\|^2$ is an unbiased estimate of $\sigma^2_z$ and approximate the term $s-n$ with $s$, we get that the standard deviation of $z$, the first variable, and the standard deviation of $z'$, the second variable, are equal to

$$
\sigma^2_z = \frac{1}{s} \|e'\|^2 \frac{57725227}{48442 \times 57725227 - 48020^2} \quad \text{and} \quad \sigma^2_{z'} = \frac{1}{s} \|e'\|^2 \frac{48442}{48442 \times 57725227 - 48020^2}.
$$

Laplace gives these two formulae for the *poids*, $P$, and not for the standard deviation. Using the relation $P = (2\sigma^2)^{-1}$ we find the equation of Laplace for $z$ (top of p. 24 of this document)

$$
P = \frac{s}{2s e'(0)^2} \left[ 48442 - \frac{(48020)^2}{57725227} \right]
$$

and, for $z'$,

$$
P = \frac{s}{2s e'(0)^2} \left[ 57725227 - \frac{(48020)^2}{48442} \right].
$$

### 5 Numerical example

There are two problem sets for Laplace to apply his algorithm. The first one computes the mass of Jupiter, the second one computes the mass of Saturn. Laplace uses observations from Bouvard where $(s = 126, n = 6)$ for Jupiter and $(s = 129, n = 6)$ for Saturn. Actually Laplace only uses from Bouvard the 6–by–6 normal equations and the norm of the residual of the least squares problem, $e'$. On the 6 unknowns (in the $x$ vector), Laplace only seeks one, the second variable $z'$. The mass of Jupiter in term of the mass of the Sun is given by $z'$ and the formula:

$$
\text{mass of Jupiter} = \frac{1 + z'}{1067.09}.
$$

It turns out that the first variable, $z$, represents the mass of Uranus through the formula

$$
\text{mass of Uranus} = \frac{1 + z}{19504}.
$$

Same approach holds for Saturn, so Laplace will indeed compute and report the mass of three planets in his manuscript.

Note that at this time, Bouvard knew that he did not understand the behavior of Uranus. He conjectured that another planet should exist to explain the anomaly in the observed behavior of Uranus. The mass of Uranus is introduced as the auxiliary variable $z$ to try to cure the problem. Laplace correctly predicts that the computed mass for Uranus is not reliable. For the anecdote, the missing planet was Neptun and was found by Johann Gottfried Galle three years after the death of Bouvard.

The number of operations performed by Bouvard is quite remarkable. For the computation of the mass of Jupiter, Laplace accredited Bouvard for the computation of the normal equations ($A^T A$) and of the residual norm ($e' = Ax - b$), this makes about $sn^2 + 2sn$ operations. For this numerical example, Laplace performed the Cholesky factorization which is about $n^3/3$. This represents 6,048 operations for Bouvard and a mere 72 for Laplace! For the computation of the mass of Saturn, the comparison is even worse since Bouvard performed all the operations and reported the results to Laplace. We note that this means that Laplace has explained his algorithm to Bouvart.

The computation of Laplace proved to be quite exact. In Table 1 we compare them with the current NASA values. We see that the values for Jupiter and Saturn in Laplace are quite close from the NASA ones. The value for Uranus is quite far as can be expected from its large variance. (Laplace would have said its small *poids.*) We also note that the NASA values are within Laplace’s bounds for Saturn and Uranus.
Table 1: Fraction of the mass of the Sun. The computed values from Bouvard are given in bold and the bound from Laplace in parenthesis. Laplace proved that his value for the mass of Uranus was not reliable (He was right.) The interval of confidence for Uranus and Saturn from Laplace are correct (i.e. the NASA values are in these intervals).

The value for Jupiter is not within Laplace’s bound which means that the noise in the observations was not normal.

In Table 2, we perform the computation of Laplace again using 64-bit arithmetic and we report the incorrect digits in his computation. It is interesting to see that Laplace conserves a fix number of significant digits along the computation. We can therefore say that Laplace was computing in floating-point arithmetic.

We note that, while the condition number of $A^T A$ is fairly large (above $10^8$), we can equilibrate the matrix $A^T A$ with a diagonal scaling $S$ equal to the inverse of the square-root of the diagonal elements. In this case, the scaled normal equations matrix, $S(A^T A)S$, has ones on its diagonal and its condition number of 104. So, up to a diagonal scaling, the system that Laplace is considered is well-conditioned.

We can check the value given by Laplace for the variance of the variable $z$ and $z'$. On the one hand, Laplace gives the poids of $z$ as $\log P = 2.0013595$ so we obtain that the standard deviation of $z$ is given by $1/\sqrt{2}/\sqrt{10^{(2.0013595)}}$ that is $\sigma_z = 0.0706$. On the other hand we can use the standard formula $\sigma_z = \sigma_b \sqrt{\text{entry}(1,1) \text{ of } (A^T A)^{-1}}$. In Matlab, this gives $\text{sqrt}(31096/129)*\text{sqrt(invATA}(1,1))$ we obtain $\sigma_z = 0.0707$. For the variable $z'$ Laplace gives its poids as 5.0778624, therefore $\sigma'_z = 0.002044343$. Directly from the normal equations, we would have found $\sigma_z = 0.002044348$.

Laplace interprets his result by giving an interval with a confidence level. For example, once, Laplace has computed $z' = 0.08916$, the variable such that the mass of Jupiter is $1 + z'10^6709$, and its associted poids $(P = 10^{2.0778624})$, Laplace uses the fact that

$$\sqrt{\frac{P}{\pi}} \int_{-1/100}^{1/100} du e^{-\frac{P}{2} u^2} \approx \frac{1000000}{1000001}$$

to claim that there is one chance out of one million for the computed value of $z'$ to be between $-1/100$ and $1/100$ of its exact value. This means that there is one chance out of one million for the mass of Jupiter to be between $1/1081$ and $1/1059$ the one of the Sun.

In the same manner, once, Laplace has computed $z = -0.00305$, the variable such that the mass of Uranus is $\frac{1+z}{17907}$, and its associted poids $(P = 10^{2.0013595})$, Laplace uses the fact that

$$\sqrt{\frac{P}{\pi}} \int_{-1/4}^{1/4} du e^{-\frac{P}{2} u^2} \approx \frac{2508}{2509}$$

to claim that there is one chance out of 2,509 for the computed value $z$ to be between $-1/4$ and $1/4$ of its exact value. This means that there is one chance out of 2,509 for the mass of Uranus to be between $1/23,241$ and $1/14,564$ the one of the Sun.
| STEP A:                                                                 |       |       |       |       |       |         |         |         |         |       |
|------------------------------------------------------------------------|-------|-------|-------|-------|-------|---------|---------|---------|---------|-------|
| 795938                                                                | −12729398 | 6788.2 | −1959.0 | 696.13 | 2602  | 7212.600 | −738297.800 |       |         |       |
| 424865729                                                              | −153106.5 | −39749.1 | −5459  | 5722  |       |         | 237.782 |         | −40.335 |       |
| 71.8720                                                                | −3.2252 | 1.2484 | 1.3371 |       | 129   | 1002.900 |         |         |         |       |
| 57.1911                                                                | 3.6213 | 1.1128 |         |       |       |         |         |         |         |       |
| 21.5432                                                                | 46.310 |         |         |       |       |         |         |         |         |       |
| 424865729                                                              | −153106.5 | −39749.1 | −5459  | 5722  |       |         | 237.782 |         | −40.335 |       |
| 71.8720                                                                | −3.2252 | 1.2484 | 1.3371 |       | 129   | 1002.900 |         |         |         |       |
| 57.1911                                                                | 3.6213 | 1.1128 |         |       |       |         |         |         |         |       |
| 21.5432                                                                | 46.310 |         |         |       |       |         |         |         |         |       |
| STEP B:                                                                 |       |       |       |       |       |         |         |         |         |       |
| 743454                                                                | −12844814 | 6761.23 | −1981.45 | −237.97 |       | 27441.68 | −693812.58 | 248.1772 | −31.6836 |       |
| 424611920                                                              | −153165.81 | −39798.46 | −7513.15 | 4.918 | 16.5783 |         |         |         |         |       |
| 71.8581                                                                | −3.2367 | 0.7684 |         |       |       |         |         |         |         |       |
| 57.1815                                                                | 3.2218 |         |         |       |       |         |         |         |         |       |
| 424611920                                                              | −153165.81 | −39798.46 | −7513.15 | 4.918 | 16.5783 |         |         |         |         |       |
| 71.8581                                                                | −3.2367 | 0.7684 |         |       |       |         |         |         |         |       |
| 57.1815                                                                | 3.2218 |         |         |       |       |         |         |         |         |       |
| STEP C:                                                                 |       |       |       |       |       |         |         |         |         |       |
| 731939.5                                                               | −13208350 | 6798.41 | −1825.56 | 28243.85 |       |         |         |         |         |       |
| 413134432                                                              | −151992.0 | −34876.7 | −668486.70 | 245.5870 | 42.5434 |         |         |         |         |       |
| 71.7381                                                                | −3.7401 | 55.0710 |         |       |       |         |         |         |         |       |
| 413134201                                                              | −151991.9 | −34876.6 | −668486.18 | 245.5870 | 42.5441 |         |         |         |         |       |
| 71.7380                                                                | −3.7401 | 55.0709 |         |       |       |         |         |         |         |       |
| STEP D:                                                                 |       |       |       |       |       |         |         |         |         |       |
| 671414.7                                                               | −14364541 | 6674.43 | 26833.55 |       |       |         |         |         |         |       |
| 391046861                                                              | −154360.6 | 242.6977 | −695430.0 |       |       |         |         |         |         |       |
| 71.4841                                                                |       |       |         |       |       |         |         |         |         |       |
| 671423.6                                                               | −14364485 | 6674.43 | 26833.57 |       |       |         |         |         |         |       |
| 391046869                                                              | −154360.6 | 242.6977 | −695429.6 |       |       |         |         |         |         |       |
| 71.4841                                                                |       |       |         |       |       |         |         |         |         |       |
| STEP E:                                                                 |       |       |       |       |       |         |         |         |         |       |
| 48442                                                                  | 48020 | 4172.95 |       |       |       |         |         |         |         |       |
| 57725227                                                              | −171455.2 |       |       |       |       |         |         |         |         |       |
| 48227                                                                  | 48021 | 4173.00 |       |       |       |         |         |         |         |       |
| 57725258                                                              | −171355.9 |       |       |       |       |         |         |         |         |       |
| STEP F:                                                                 |       |       |       |       |       |         |         |         |         |       |
| $z_0 = −0.00305$                                                        |       |       |       |       |       |         |         |         |         |       |
| $z_1 = 0.08916$                                                         |       |       |       |       |       |         |         |         |         |       |
| $z_0 = −0.00304$                                                        |       |       |       |       |       |         |         |         |         |       |
| $z_1 = 0.08916$                                                         |       |       |       |       |       |         |         |         |         |       |
| Table 2: Laplace computations. Comparison with “exact” computation. (“exact” means 64-bit arithmetic.) First line is Laplace’s value. Second line is the value computed with 64-bit arithmetic and Laplace’s algorithm. |       |       |       |       |       |         |         |         |         |       |
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