A SIMILARITY INVARIANT OF A CLASS OF \( n \)-NORMAL OPERATORS IN TERMS OF \( K \)-THEORY

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Abstract. In this paper, we prove an analogue of the Jordan canonical form theorem for a class of \( n \)-normal operators on complex separable Hilbert spaces in terms of von Neumann’s reduction theory. This is a continuation of our study of bounded linear operators, the commutants of which contain bounded maximal abelian set of idempotents. Furthermore, we give a complete similarity invariant for this class of operators by \( K \)-theory for Banach algebras.

1. Introduction

In this paper the authors continue the study on generalizing the Jordan canonical form theorem for bounded linear operators on separable Hilbert spaces, which was initiated in [9] and carried on in [12]. Throughout this article, we only discuss Hilbert spaces which are complex and separable. Denote by \( \mathcal{L}(\mathcal{H}) \) the set of bounded linear operators on a Hilbert space \( \mathcal{H} \). An idempotent \( P \) on \( \mathcal{H} \) is an operator in \( \mathcal{L}(\mathcal{H}) \) such that \( P^2 = P \). A projection \( Q \) in \( \mathcal{L}(\mathcal{H}) \) is an idempotent such that \( Q = Q^* \). An operator \( A \) in \( \mathcal{L}(\mathcal{H}) \) is said to be irreducible if its commutant \( \{A\}' \triangleq \{B \in \mathcal{L}(\mathcal{H}) : AB = BA\} \) contains no projections other than 0 and the identity operator \( I \) on \( \mathcal{H} \), introduced by P. Halmos in [7]. (The separability assumption is necessary because on a nonseparable Hilbert space every operator is reducible.) An operator \( A \) in \( \mathcal{L}(\mathcal{H}) \) is said to be strongly irreducible if \( XAX^{-1} \) is irreducible for every invertible operator \( X \) in \( \mathcal{L}(\mathcal{H}) \), introduced by F. Gilfeather in [6]. This shows that the commutant of a strongly irreducible operator contains no idempotents other than 0 and \( I \). We observe that strong irreducibility stays invariant up to similar equivalence while irreducibility is an invariant up to unitary equivalence. For an operator \( A \) in \( \mathcal{L}(\mathcal{H}) \), a nonzero idempotent \( P \) in \( \{A\}' \) is said to be minimal if every idempotent \( Q \) in \( \{A\}' \cap \{P\}' \) satisfies \( QP = P \) or \( QP = 0 \). For a minimal idempotent \( P \) in \( \{A\}' \), the restriction \( A|_{\text{ran}P} \) is strongly irreducible on \( \text{ran}P \). For \( n \) in \( \mathbb{N} \cup \{\infty\} \), we write \( \mathcal{H}^{(n)} \) for the orthogonal direct sum of \( n \) copies of \( \mathcal{H} \), where we denote by \( \mathbb{N} \) the set of positive integers. For an operator \( T \) in \( \mathcal{L}(\mathcal{H}) \) and \( n \) in \( \mathbb{N} \cup \{\infty\} \), the orthogonal direct sum of \( T \) with itself \( n \) times is denoted by \( T^{(n)} \). Let \( \mathcal{F} \) be a subset of \( \mathcal{L}(\mathcal{H}) \). Then we write \( \mathcal{F}^{(n)} \) for \( \{T^{(n)} \in \mathcal{L}(\mathcal{H}^{(n)}) : T \in \mathcal{F}\} \) and \( \mathcal{F}' \) for the commutant of \( \mathcal{F} \).

On a finite dimensional Hilbert space \( \mathcal{H} \), the Jordan canonical form theorem shows that every operator \( B \) in \( \mathcal{L}(\mathcal{H}) \) can be uniquely written as a (Banach) direct sum of Jordan blocks up to similarity. An important observation is that for any two

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bounded maximal abelian sets of idempotents $\mathcal{D}$ and $\mathcal{P}$ in the commutant $\{B\}'$, there exists an invertible operator $X$ in $\{B\}'$ such that

$$X \mathcal{D} X^{-1} = \mathcal{P}. \quad (1.1)$$

Thus, we obtain $K_0(\{B\}') \cong \mathbb{Z}^{(k)}$ and $K_1(\{B\}') \cong 0$ by a routine computation, where we let $k$ denote the number of minimal idempotents in $\mathcal{P}$. Furthermore, the ordered $K_0$ groups can be viewed as a complete similarity-invariant in the following way. Let $B_1$ and $B_2$ be in $\mathcal{L}(\mathcal{H})$ such that

$$\theta_1(K_0(\{B_1\}')) = \theta_2(K_0(\{B_1 \oplus B_2\}')) = \mathbb{Z}^{(k)}, \quad (1.2)$$

and $\theta_1([I_{\{B_1\}'},]) = n_1 e_1 + \cdots + n_k e_k$ and $\theta_2([I_{\{B_1 \oplus B_2\}'},]) = 2n_1 e_1 + \cdots + 2n_k e_k$, where $\theta_1$ and $\theta_2$ are group isomorphisms essentially induced by the standard traces of matrices and $\{e_i\}_{i=1}^k$ are the generators of the semigroup $N^{(k)}$ of $\mathbb{Z}^{(k)}$ and $I_{\{B_1\}'}$, is the unit of $\{B_1\}'$, then $B_1$ is similar to $B_2$. The reader is referred to Chapter 2 of [8] for the details skipped above.

In our first attempt to prove an analogue of the Jordan canonical form theorem in [9], we observe that minimal idempotents in $\{A\}'$, for $A \in \mathcal{L}(\mathcal{H})$, play an important role in the construction of the Jordan canonical form of $A$. However, for a single self-adjoint generator $N$ of a diffuse masa, the commutant $\{N\}'$ contains no minimal idempotents. This fact shows us that, on considering a generalization of the Jordan canonical form theorem, direct sums of Jordan blocks need to be replaced by direct integrals of strongly irreducible operators with regular Borel measures to represent certain operators in $\mathcal{L}(\mathcal{H})$.

We briefly introduce some concepts in the von Neumann’s reduction theory that will be employed in this paper. For the most part, we follow [2, 11]. Once and for all, let $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots \subset \mathcal{H}_\infty$ be a sequence of Hilbert spaces with $\mathcal{H}_n$ having dimension $n$ and $\mathcal{H}_\infty$ spanned by the remaining $\mathcal{H}_n$’s. Let $\mu$ be (the completion of) a finite positive regular Borel measure supported on a compact subset $\Lambda$ of $\mathbb{R}$. (We realize this by virtue of [11, Theorem 7.12].) And let $\{\Lambda_\infty\} \cup \{\Lambda_n\}_{n=1}^{\infty}$ be a Borel partition of $\Lambda$. Then we form the associated direct integral Hilbert space

$$\mathcal{H} = \int_\Lambda^\oplus \mathcal{H}(\lambda) d\mu(\lambda) \quad (1.3)$$

which consists of all (equivalence classes of) measurable functions $f$ and $g$ from $\Lambda$ into $\mathcal{H}_\infty$ such that

1. $f(\lambda) \in \mathcal{H}(\lambda) \equiv \mathcal{H}_n$ for $\lambda \in \Lambda_n$;
2. $\|f\|^2 \equiv \int_\Lambda \|f(\lambda)\|^2 d\mu(\lambda) < \infty$;
3. $(f, g) \equiv \int_\Lambda (f(\lambda), g(\lambda)) d\mu(\lambda)$.

The element in $\mathcal{H}$ represented by the function $\lambda \to f(\lambda)$ is denoted by $\int_\Lambda f(\lambda) d\mu(\lambda)$. An operator $A$ in $\mathcal{L}(\mathcal{H})$ is said to be decomposable if there exists a strongly $\mu$-measurable operator-valued function $A(\cdot)$ defined on $\Lambda$ such that $A(\lambda)$ is an operator in $\mathcal{L}(\mathcal{H}(\lambda))$ and $(Af)(\lambda) = A(\lambda)f(\lambda)$, for all $f \in \mathcal{H}$. We write $A \equiv \int_\Lambda A(\lambda) d\mu(\lambda)$ for the equivalence class corresponding to $A(\cdot)$. If $A(\lambda)$ is a scalar multiple of the identity on $\mathcal{H}(\lambda)$ for almost all $\lambda$, then $A$ is said to be diagonal. The collection of all diagonal operators is said to be the diagonal algebra of $\Lambda$. It is an abelian von Neumann algebra. A decomposable operator $A$ in $\mathcal{L}(\mathcal{H})$ is essentially a direct sum of $n$-normal operators with respect to $n$. Let $\Lambda = \Lambda_n$ and $A$ in $\mathcal{L}(\mathcal{H})$ be decomposable, then $A$ is $n$-normal. An operator $A$ in $\mathcal{L}(\mathcal{H})$ is said to be $n$-normal,
if there exists a unitary operator $U$ from $\mathcal{H}$ to $(L^2(\nu))^{(n)}$ such that
\[ UAU^* = \begin{pmatrix} M_{f_{11}} & \cdots & M_{f_{1n}} \\ \vdots & \ddots & \vdots \\ M_{f_{n1}} & \cdots & M_{f_{nn}} \end{pmatrix}_{n \times n} \begin{pmatrix} L^2(\nu) \\ \vdots \\ L^2(\nu) \end{pmatrix} \tag{1.4} \]
where $\nu$ is a finite positive regular Borel measure supported on a compact subset $\Gamma$ of $\mathbb{C}$ and $M_{f_{ij}}$ is a Multiplication operator for $f_{ij}$ in $L^\infty(\nu)$ and $1 \leq i, j \leq n$. In the sense of direct integral decomposition, the operator $UAU^*$ is in the form
\[ UAU^* = \int_\Gamma \begin{pmatrix} f_{11}(\lambda) & \cdots & f_{1n}(\lambda) \\ \vdots & \ddots & \vdots \\ f_{n1}(\lambda) & \cdots & f_{nn}(\lambda) \end{pmatrix} d\nu(\lambda). \tag{1.5} \]
Furthermore, by virtue of ([1], Corollary 2), for every $n$-normal operator $A$ on $(L^2(\nu))^{(n)}$ and positive integer $n < \infty$, there exists an $n$-normal unitary operator $U$ on $(L^2(\nu))^{(n)}$ such that $UAU^*$ is an upper triangular $n$-normal operator, i.e.
\[ UAU^* = \begin{pmatrix} M_{f_{11}} & M_{f_{12}} & \cdots & M_{f_{1n}} \\ 0 & M_{f_{22}} & \cdots & M_{f_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{f_{nn}} \end{pmatrix}_{n \times n} \begin{pmatrix} L^2(\nu) \\ \vdots \\ L^2(\nu) \end{pmatrix} \tag{1.6} \]

The following two basic results will be used in the sequel:

(1) An operator acting on a direct integral of Hilbert spaces is decomposable if and only if it commutes with the corresponding diagonal algebra ([11], p. 22).

(2) Every abelian von Neumann algebra is (unitarily equivalent to) an essentially unique diagonal algebra ([11], p. 19).

By the above observation of the Jordan canonical form theorem, our first question is whether the commutant $\{A\}'$ contains a bounded maximal abelian set of idempotents for every operator $A$ in $\mathcal{L}(\mathcal{H})$. In [9], we gave a negative answer by constructing two operators $A$ and $B$ in the forms
\[ A = \bigoplus_{i=1}^\infty A_i, \quad A_i = \begin{pmatrix} \frac{1}{\pi} & 1 \\ 0 & -\frac{1}{\pi^2} \end{pmatrix} \in M_2(\mathbb{C}), \quad B = \begin{pmatrix} N_\mu & I \\ 0 & -\frac{1}{\pi}N_\mu \end{pmatrix} \in \mathcal{L}(L^2(\mu)) \tag{1.7} \]
where the multiplication operator $N_\mu$ is defined on $L^2(\mu)$ by $N_\mu f = z \cdot f$ for each $f$ in $L^2(\mu)$ and $\mu$ is a finite regular Borel measure supported on a compact subset of $\mathbb{C}$. (And it is well known that the normal operator $N_\mu$ is star-cyclic.) Furthermore, we proved the following theorem:

**Theorem 1.1** ([9], Theorem 1.2). An operator $A$ in $\mathcal{L}(\mathcal{H})$ is similar to a direct integral of strongly irreducible operators if and only if its commutant $\{A\}'$ contains a bounded maximal abelian set of idempotents.

That an operator $A$ is similar to a direct integral of strongly irreducible operators denoted by $XAX^{-1} = \int A B(\lambda) d\mu(\lambda)$ for some invertible operator $X$ in $\mathcal{L}(\mathcal{H})$ means that the Hilbert space $\mathcal{H} = \int A \mathcal{H}(\lambda) d\mu(\lambda)$ is in the sense of (1.3) and the operator $XAX^{-1}$ is decomposable with respect to the corresponding diagonal
algebra such that the integrand $B(\cdot)$ is a bounded strongly $\mu$-measurable operator-valued function defined on $\Lambda$ and $B(\lambda)$ is strongly irreducible on the corresponding fibre space $\mathcal{H}(\lambda)$ for almost every $\lambda$ in $\Lambda$. For related concepts and results about von Neumann’s reduction theory, the reader is referred to [2, 3, 4, 10, 11].

Since then, we have paid more attention to the subset $\mathcal{I}$ of $\mathcal{L}(\mathcal{H})$, where the set $\mathcal{I}$ consists of the operators $A$ in $\mathcal{L}(\mathcal{H})$ such that every $\{A\}'$ contains a bounded maximal abelian set of idempotents. We also found that for an operator $A$ in $\mathcal{L}(\mathcal{H})$, the commutant $\{A\}'$ may both contain a bounded maximal abelian set of idempotents and an unbounded maximal abelian set of idempotents.

Inspired by (1.1), our second question is whether the equality (1.1) holds in the commutant $\{A\}'$ for each operator $A$ in $\mathcal{I}$. In [12], we gave a negative answer by constructing an operator $C$ in the form

$$
C = \begin{pmatrix}
N_\mu^{(\infty)} & I \\
0 & N_\mu^{(\infty)}
\end{pmatrix}
(L^2(\mu))^{(\infty)}.
$$

We denote by $\mathcal{I}_U$ the subset of $\mathcal{I}$ such that for every operator $A$ in $\mathcal{I}_U$, the equality (1.1) holds for any two bounded maximal abelian sets of idempotents in the commutant $\{A\}'$. Compared with the Jordan canonical form theorem, we define and say that the strongly irreducible decomposition of every operator $A$ in $\mathcal{I}_U$ is unique up to similarity. Therefore, our third question is what the structure of an operator $A$ in $\mathcal{I}_U$ is. In [12], inspired by ([1], Corollary 2), the author mainly proved that an $n$-normal operator $A$ in $\mathcal{L}(\mathcal{H})$ unitarily equivalent to the following form is in $\mathcal{I}_U$:

$$
A = \begin{pmatrix}
N_\mu & M_{f_{i2}} & \ldots & M_{f_{in}} \\
0 & N_\mu & \ldots & M_{f_{2n}} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & N_\mu
\end{pmatrix}_{n \times n},
$$

where $m, n < \infty$, $\mu$ and $N_\mu$ are as in (1.7), $f_{ij}$ is in $L^\infty(\mu)$ for $1 \leq i, j \leq n$ and the inequality $f_{i,i+1}(\lambda) \neq 0$ holds for almost every $\lambda$ in the support of $\mu$ and $1 \leq i < n - 1$.

In the present paper, we have two motivations. One is to generalize the main result of [12]. Precisely, we prove the operator

$$
A = \bigoplus_{n=1}^{\infty} \bigoplus_{m=1}^{\infty} A_{nm}^{(m)}
$$

in $\mathcal{L}(\mathcal{H})$ is in $\mathcal{I}_U$, where $A_{nm} = 0$ holds for all but finitely many $m$ and $n$ in $\mathbb{N}$, $A_{nm}$ is unitarily equivalent to the form

$$
\begin{pmatrix}
N_{\nu_{nm}} & M_{f_{nm;12}} & \ldots & M_{f_{nm;1n}} \\
0 & N_{\nu_{nm}} & \ldots & M_{f_{nm;2n}} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & N_{\nu_{nm}}
\end{pmatrix}_{n \times n},
$$

the measures $\nu_{nm_1}$ and $\nu_{nm_2}$ are mutually singular compactly supported finite positive regular Borel for $m_1 \neq m_2$, the function $f_{nm;ij}$ is in $L^\infty(\nu_{nm})$ for $1 \leq i, j \leq n$ such that the inequality

$$
f_{nm; i, i+1}(\lambda) \neq 0
$$

(1.12)
holds for almost every \( \lambda \) in the support of \( \nu_{mn} \) for \( 1 \leq i \leq n - 1 \).

Since \( \{ A_{nm_1} \oplus A_{nm_2} \} = \{ A_{nm_1} \}' + \{ A_{nm_2} \}' \), for the sake of simplicity and without loss of generality, the above object can be fulfilled by proving that

\[
A = A_{n_1}^{(m_1)} \oplus A_{n_2}^{(m_2)} \oplus A_{n_3}^{(m_3)}
\]

in \( \mathcal{L(\mathcal{H})} \) is in \( \mathcal{U} \), where \( A_{n_k} \) is in the form

\[
A_{n_k} = \begin{pmatrix}
N_{\mu} & M_{f_{12,k}} & \cdots & M_{f_{1n_k,k}} \\
0 & N_{\mu} & \cdots & M_{f_{2n_k,k}} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & N_{\mu}
\end{pmatrix}
\]

for \( k = 1, 2, 3, n_1 > n_2 > n_3 \), the measure \( \mu \) is as in (1.7), the function \( f_{ij,k} \) is in \( L^\infty(\mu) \) for \( 1 \leq i, j \leq n_k \) such that the inequality

\[
f_{i,i+1,k}(\lambda) \neq 0
\]

holds for almost every \( \lambda \) in the support of \( \mu \) for \( 1 \leq i \leq n_k - 1, 1 \leq k \leq 3 \). The condition (1.12) (or (1.15)) is necessary and sufficient for an operator in the form of (1.11) (or (1.14)) to be strongly irreducible almost everywhere on the support of the corresponding measure in the sense of direct integral decomposition as in (1.5), which was proved in ([12], Lemma 3.1).

The other motivation is to prove a complete similarity invariant of an operator \( A \) as in (1.13) by \( K \)-theory for Banach algebras. This similarity invariant is different from the necessary and sufficient conditions for two \( n \)-normal operators similar to each other proved by D. Deckard and C. Pearcy in ([5], Theorem 1).

Precisely, we prove the following theorems in this paper. In \( K \)-theory for Banach algebras, by \( V(\{A\}') \) we denote the semigroup \( \cup_{n=1}^\infty P_n(\{A\}')/\sim \), where \( P_n(\{A\}') \) is the set of idempotents in \( M_n(\{A\}') \) and by “\( \sim \)” we denote that similarity relation in the corresponding algebra. By \( K_0(\{A\}') \) we denote the Grothendieck group generated by \( V(\{A\}') \), which is well known as the \( K_0 \)-group of \( \{A\}' \).

**Theorem 1.2.** Let \( A \in \mathcal{L(\mathcal{H})} \) be assumed as in (1.13). Then the following statements hold:

(a) The strongly irreducible decomposition of \( A \) is unique up to similarity;
(b) \( K_0(\{A\}') \cong \{ f(\lambda) \in \mathbb{Z}(3) : f \text{ is bounded and Borel on } \sigma(A) \} \).

When we deal with a finite direct sum of operators as in (1.10), inspired by ([3], Chapter 9, Theorem 10.16) we obtain a generalization of the above theorem in the following form.

**Theorem 1.3.** Let \( A \in \mathcal{L(\mathcal{H})} \) be assumed as in (1.10). Then the following statements hold:

(a) The strongly irreducible decomposition of \( A \) is unique up to similarity;
(b) There exists a bounded \( N \)-valued simple function \( r_A \) on \( \sigma(A) \) such that

\[
K_0(\{A\}') \cong \{ f(\lambda) \in \mathbb{Z}^{(r_A(\lambda))} : f \text{ is bounded and Borel on } \sigma(A) \}.
\]

For operators as in (1.13), we characterize the similarity with \( K \)-theory for Banach algebras as follows.
Theorem 1.4. Let \( A = A_{n_1}^{(m_1)} \oplus A_{n_2}^{(m_2)} \oplus A_{n_3}^{(m_3)} \) and \( B = B_{l_1}^{(k_1)} \oplus B_{l_2}^{(k_2)} \oplus B_{l_3}^{(k_3)} \) be as in (1.13) and every entry of \( A \) and \( B \) is in \( L^\infty(\mu) \) as in (1.14). Then \( A \) and \( B \) are similar if and only if there exists a group isomorphism \( \theta \) such that the following statements hold:

1. \( \theta(K_0([A \oplus B]')) = \{ f(\lambda) \in \mathbb{Z}^{(3)} : f \text{ is bounded and Borel on } \sigma(A) \} \);
2. \( \theta([I_{A \oplus B}]) = 2m_1e_1 + 2m_2e_2 + 2m_3e_3 \),

where \( \{e_i(\lambda)\}_{i=1}^{3} \) are the generators of the semigroup \( \mathbb{N}^{(3)} \) of \( \mathbb{Z}^{(3)} \) for every \( \lambda \) in \( \sigma(A) \) and \( I_{A \oplus B}' \) is the unit of \( \{ A \oplus B \}' \).

By a more complicated computation, we obtain a generalization of the above theorem as follows.

Theorem 1.5. Let \( A = \sum_{i=1}^{s} A_{n_i}^{(m_i)} \) and \( B = \sum_{j=1}^{t} B_{l_j}^{(k_j)} \) be in the sense of (1.13), and every entry of \( A_{n_i} \) and \( B_{l_j} \) is in \( L^\infty(\mu) \) as in (1.14), for \( 1 \leq i \leq s < \infty \) and \( 1 \leq j \leq t < \infty \), where \( n_i \neq n_j \) for \( i \neq j \), and \( m_i, n_i, k_j, l_j \) are in \( \mathbb{N} \) for every \( i \) and \( j \). Then \( A \) and \( B \) are similar if and only if there exists a group isomorphism \( \theta \) such that the following statements hold:

1. \( \theta(K_0([A \oplus B]')) = \{ f(\lambda) \in \mathbb{Z}^{(s)} : f \text{ is bounded and Borel on } \sigma(A) \} \);
2. \( \theta([I_{A \oplus B}]) = 2m_1e_1 + 2m_2e_2 + \cdots + 2m_se_s \),

where \( \{e_i(\lambda)\}_{i=1}^{s} \) are the generators of the semigroup \( \mathbb{N}^{(s)} \) of \( \mathbb{Z}^{(s)} \) for every \( \lambda \) in \( \sigma(A) \) and \( I_{A \oplus B}' \) is the unit of \( \{ A \oplus B \}' \).

Let the support of every spectral measure \( \nu_{nm} \) in the sense of (1.10) and (1.11) be a single point in \( \mathbb{C} \), then Theorem 1.3 shows that the strongly irreducible decomposition of every matrix \( A \) in \( M_n(\mathbb{C}) \) is unique up to similarity, and Theorem 1.4 characterizes a necessary and sufficient condition that two matrices are similar. This is identified with the Jordan canonical form theorem.

This paper is organized as follows. In section 2, we prove Theorem 1.2 and Theorem 1.4. In section 3, we develop a method of decomposing an upper triangular \( n \)-normal operator \( A \) of the following form with respect to the multiplicity function of the (1,1) entry:

\[
A = \begin{pmatrix}
M_{f_{11}} & M_{f_{12}} & \cdots & M_{f_{1m}} \\
0 & M_{f_{22}} & \cdots & M_{f_{2m}} \\
& \ddots & \ddots & \vdots \\
0 & 0 & \cdots & M_f
\end{pmatrix}
\in L^2(\mu) \\
\in L^2(\mu) \\
\in L^2(\mu) \\
\in L^2(\mu)
\]

where \( n < \infty, \mu \) are as in (1.7), \( f \) and \( f_{ij} \) are in \( L^\infty(\mu) \) for \( 1 \leq i, j \leq n \) and the inequality \( f_{i,i+1}(\lambda) \neq 0 \) holds for almost every \( \lambda \) in the support of \( \mu \) and \( 1 \leq i \leq n - 1 \).

2. Proofs

For an \( n \)-normal operator \( A \) in the form as in (1.6), an application of ([12], Lemma 3.1) shows that for a fixed \( \lambda \) in the support of \( \nu \), the operator \( A(\lambda) \) is strongly irreducible if and only if \( f_{ii}(\lambda) = f_{in}(\lambda) \) and \( f_{i,i+1}(\lambda) \neq 0 \) hold for \( 1 \leq i \leq n - 1 \). Therefore for an \( n \)-normal operator \( A \) in the form as in (1.14) and (1.15), \( A(\lambda) \) is strongly irreducible for almost every \( \lambda \) in the support of \( \mu \) in the sense of (1.5). We need to mention that the multiplication operators \( M_{f_{i,i+1,k}} \) may not
be invertible in general. This makes the computation become more complicated. However, the commutant \( \{ A_n \} \)' is a subalgebra of \( \{ N_\mu \} \)' by (12, Lemma 3.2) for an operator \( A_n \) in the form

\[
A_n = \begin{pmatrix}
N_\mu & M_{f12} & \cdots & M_{f1n} \\
0 & N_\mu & \cdots & M_{f2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & N_\mu \\
\end{pmatrix}_{n \times n},
\]

(2.1)

where the measure \( \mu \) is as in (1.7), the function \( f_{ij} \) is in \( L^\infty(\mu) \) for \( 1 \leq i, j \leq n \) such that the inequality

\[
f_{i,i+1}(\lambda) \neq 0
\]

holds for almost every \( \lambda \) in the support of \( \mu \) for \( 1 \leq i \leq n - 1 \). Precisely, by (12, Lemma 3.2), every operator \( X \) in \( \{ A_n \} \)' is in the form

\[
X = \begin{pmatrix}
M_\psi & M_{\psi_{12}} & M_{\psi_{13}} & \cdots & M_{\psi_{1n}} \\
0 & M_\psi & M_{\psi_{23}} & \cdots & M_{\psi_{2n}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & M_\psi \\
\end{pmatrix}_{n \times n},
\]

(2.3)

and in special, every idempotent \( E \) in \( \{ A_n \} \)' is in the form \( E = M_\chi_{\Delta}^{(n)} \) for some characteristic function \( \chi_{\Delta} \) in \( L^\infty(\mu) \), where \( \Delta \) is a Borel subset in the support of \( \mu \). Let \( \mathcal{E}_n \) denote the set of idempotents in \( \{ A_n \} \)' Then \( \mathcal{E}_n \) is the only maximal abelian set of idempotents in \( \{ A_n \} \)' and obviously, the set \( \mathcal{E}_n \) is bounded. We observe that the bounded set of idempotents

\[
\mathcal{E} \triangleq (\mathcal{E}_{n_1} \oplus \cdots \oplus \mathcal{E}_{n_1}) \oplus (\mathcal{E}_{n_2} \oplus \cdots \oplus \mathcal{E}_{n_2}) \oplus (\mathcal{E}_{n_3} \oplus \cdots \oplus \mathcal{E}_{n_3})
\]

(2.4)

\( (m_1 \text{ copies of } \mathcal{E}_{n_1}, m_2 \text{ copies of } \mathcal{E}_{n_2}, \text{ and } m_3 \text{ copies of } \mathcal{E}_{n_3} \) is maximal abelian in the commutant of \( A = A_{n_1}^{(m_1)} \oplus A_{n_2}^{(m_2)} \oplus A_{n_3}^{(m_3)} \) as mentioned from (1.13) to (1.15). In the rest of this article, we define \( \mathcal{E} \) to be the standard bounded maximal abelian set of idempotents in \( \{ A \}' \) where \( A \) is defined as in (1.13). The following two preliminary lemmas are needed to prove Theorem 1.2.

**Lemma 2.1.** Let \( A_{n_1} \) and \( A_{n_2} \) \( (n_1 > n_2) \) be assumed as in (1.14). Then the following statements hold:

1. the equality \( A_{n_1} X = X A_{n_2} \) yields that \( X = (X_1^T, 0_{n_2 \times (n_1-n_2)})^T \), where \( X_1 \) is an upper triangular \( n_2 \)-by-\( n_2 \) operator-valued matrix such that every entry of \( X_1 \) is in \( \{ N_\mu \} \)' and the transpose of \( X_1 \) is denoted by \( X_1^T \);
2. the equality \( A_{n_2} Y = Y A_{n_1} \) yields that \( Y = (0_{n_2 \times (n_1-n_2)}, Y_1) \), where \( Y_1 \) is an upper triangular \( n_1 \)-by-\( n_2 \) operator-valued matrix such that every entry of \( Y_1 \) is in \( \{ N_\mu \}' \).
Proof: If $A_{n_1} = A_{n_2}$, then this lemma is identified with ([12], Lemma 3). For the sake of simplicity, let operators $A_{n_1}$ and $A_{n_2}$ be in the form

$$A_{n_2} = \begin{pmatrix} N_\mu & M_{f_{12}} & M_{f_{13}} & \cdots & M_{f_{1n_2}} \\ 0 & N_\mu & M_{f_{23}} & \cdots & M_{f_{2n_2}} \\ 0 & 0 & N_\mu & \cdots & M_{f_{3n_2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N_\mu \end{pmatrix} L^2(\mu)_{n_2 \times n_2}$$

and

$$A_{n_1} = \begin{pmatrix} N_\mu & M_{g_{12}} & M_{g_{13}} & \cdots & M_{g_{1n_1}} \\ 0 & N_\mu & M_{g_{23}} & \cdots & M_{g_{2n_1}} \\ 0 & 0 & N_\mu & \cdots & M_{g_{3n_1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N_\mu \end{pmatrix} L^2(\mu)_{n_1 \times n_1}$$

Let $E_\mu(\cdot)$ be the spectral measure for $N_\mu$. For a Borel subset $\Delta$ of $\sigma(N_\mu)$ such that $E_\mu(\Delta)$ is a nontrivial projection in $\{N_\mu\}$, we write $P_1 = E_\mu(\Delta)$ and $P_2 = E_\mu(\sigma(N_\mu) \setminus \Delta)$, meanwhile write $\mu_1$ for $\mu|\Delta$ and $\mu_2$ for $\mu|\sigma(N_\mu) \setminus \Delta$. Hence the operators $A_{n_1}$, $A_{n_2}$ and $X$ can be expressed in the form

$$A_{n_1} = \begin{pmatrix} A_{n_1,1} & 0 \\ 0 & A_{n_1,2} \end{pmatrix} \text{ran} P_1^{(n_1)}, \quad A_{n_2} = \begin{pmatrix} A_{n_2,1} & 0 \\ 0 & A_{n_2,2} \end{pmatrix} \text{ran} P_2^{(n_2)},$$

and $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$

where

$$A_{n_1,i} = \begin{pmatrix} N_{\mu_i} & M_{g_{12,i}} & M_{g_{13,i}} & \cdots & M_{g_{1n_1,i}} \\ 0 & N_{\mu_i} & M_{g_{23,i}} & \cdots & M_{g_{2n_1,i}} \\ 0 & 0 & N_{\mu_i} & \cdots & M_{g_{3n_1,i}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N_{\mu_i} \end{pmatrix} \text{ran} P_i^{(n_1)}, \quad i = 1, 2$$

and

$$A_{n_2,i} = \begin{pmatrix} N_{\mu_i} & M_{f_{12,i}} & M_{f_{13,i}} & \cdots & M_{f_{1n_2,i}} \\ 0 & N_{\mu_i} & M_{f_{23,i}} & \cdots & M_{f_{2n_2,i}} \\ 0 & 0 & N_{\mu_i} & \cdots & M_{f_{3n_2,i}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N_{\mu_i} \end{pmatrix} \text{ran} P_i^{(n_2)}, \quad i = 1, 2.$$
The equality $A_{n_1}X = XA_{n_2}$ yields $A_{n_1,i}X_{12} = X_{12}A_{n_2,j}$. And this equality can be expressed in the form

$$
\begin{pmatrix}
N_{\mu_1} & M_{g_{12,1}} & \cdots & M_{g_{1n_1,1}} \\
0 & N_{\mu_1} & \cdots & M_{g_{2n_1,1}} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & N_{\mu_1}
\end{pmatrix}
\begin{pmatrix}
X_{12,11} & X_{12,12} & \cdots & X_{12,1n_2} \\
X_{12,21} & X_{12,22} & \cdots & X_{12,2n_2} \\
\vdots & \vdots & \ddots & \vdots \\
X_{12,n_1,1} & X_{12,n_1,2} & \cdots & X_{12,n_1,n_2}
\end{pmatrix}
= 
\begin{pmatrix}
N_{\mu_2} & M_{f_{12,2}} & \cdots & M_{f_{1n_2,2}} \\
0 & N_{\mu_2} & \cdots & M_{f_{2n_2,2}} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & N_{\mu_2}
\end{pmatrix}
$$

(2.10)

Since the measures $\mu_1$ and $\mu_2$ are mutually singular, the equality $N_{\mu_1}X_{12,n_1,1} = X_{12,n_1,1}N_{\mu_2}$ yields that $X_{12,n_1,1} = 0$. Thus the equality $N_{\mu_1}X_{12,n_1,2} = X_{12,n_1,2}N_{\mu_2}$ yields that $X_{12,n_1,2} = 0$. By this method, we obtain that every entry in the $n_1$-th row of $X_{12}$ is zero. The same result holds for the the $(n_1 - 1)$-th row of $X_{12}$. By induction, we obtain that $X_{12} = 0$. By a similar discussion, we have that $X_{21} = 0$.

This means that the equality $P_i^{(n_1)}X = XP_i^{(n_2)}$ holds for every Borel subset $\Delta$ of $\sigma(N\mu)$. Therefore the operator $X$ can be expressed in the form

$$
X = 
\begin{pmatrix}
M_{h_{11}} & M_{h_{12}} & M_{h_{13}} & \cdots & M_{h_{1n_2}} \\
M_{h_{21}} & M_{h_{22}} & M_{h_{23}} & \cdots & M_{h_{2n_2}} \\
M_{h_{31}} & M_{h_{32}} & M_{h_{33}} & \cdots & M_{h_{3n_2}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
M_{h_{n_1,1}} & M_{h_{n_1,2}} & M_{h_{n_1,3}} & \cdots & M_{h_{n_1,n_2}}
\end{pmatrix}_{n_1 \times n_2},
$$

(2.11)

where $h_{ij}$ is in $L^\infty(\mu)$, $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$. By the assumption, we have that $d_{i,i+1}(\lambda) \neq 0$ and $g_{j,j+1}(\lambda) \neq 0$ for $1 \leq i \leq n_2 - 1$, $1 \leq j \leq n_1 - 1$, and almost every $\lambda$ in $\sigma(N\mu)$. The equality $A_{n_1}X = XA_{n_2}$ yields that

$$
N_{\mu}M_{h_{n_1-1,i}} + M_{g_{n_1-1,n_1}}M_{h_{n_1,1}} = M_{h_{n_1-1,i}}N_{\mu}. 
$$

(2.12)

This equality yields that $M_{h_{n_1,1}} = 0$. Thus the equality

$$
N_{\mu}M_{h_{n_1-2,1}} + M_{g_{n_1-2,n_1-1}}M_{h_{n_1-1,1}} = M_{h_{n_1-2,1}}N_{\mu}
$$

(2.13)

yields that $M_{h_{n_1-1,1}} = 0$. By computation, we obtain that $M_{h_{j,1}} = 0$ for $2 \leq j \leq n_1$.

By the equality $A_{n_1}X = XA_{n_2}$, we have

$$
N_{\mu}M_{h_{n_1-1,2}} + M_{g_{n_1-1,n_1}}M_{h_{n_1,2}} = M_{h_{n_1-1,2}}N_{\mu}.
$$

(2.14)

This yields that $M_{h_{n_1,2}} = 0$. Thus the equality

$$
N_{\mu}M_{h_{n_1-2,2}} + M_{g_{n_1-2,n_1-1}}M_{h_{n_1-1,2}} = M_{h_{n_1-2,2}}N_{\mu}
$$

(2.15)

yields that $M_{h_{n_1-1,2}} = 0$. By computation, we obtain that $M_{h_{j,2}} = 0$ for $3 \leq j \leq n_1$.

By induction, we have $M_{h_{j,i}} = 0$ for $i < j$. The proof of the first assertion is finished.
In the proof of the second assertion, by a similar computation, we obtain that $Y$ is an $n_2$-by-$n_1$ operator-valued matrix as in (2.11). Therefore, we apply the equality $A_{n_2}Y = YA_{n_1}$ to obtain that

$$N_\mu M_{h_2} = M_{h_1} M_{g_1} + N_{h_2}. \quad (2.16)$$

This equality yields that $M_{h_2} = 0$. Thus the equality

$$N_\mu M_{h_2} = M_{h_1} M_{g_1} + M_{h_2} N_{h_2}, \quad (2.17)$$

yields that $M_{h_2} = 0$. By computation, we obtain that $M_{h_2} = 0$ for $1 \leq j \leq n_1 - 1$.

By the equality $A_{n_2}Y = YA_{n_1}$, we have

$$N_\mu M_{h_2} = M_{h_1} M_{g_1} + M_{h_2} N_{h_2}. \quad (2.18)$$

This yields that $M_{h_2} = 0$. Thus the equality

$$N_\mu M_{h_2} = M_{h_1} M_{g_1} + M_{h_2} N_{h_2}, \quad (2.19)$$

yields that $M_{h_2} = 0$. By computation, we obtain that $M_{h_2} = 0$ for $1 \leq j \leq n_1 - 2$. By induction, we have $M_{h_2} = 0$ for $j \leq n_1 - n_2 + i - 1$. Therefore, the proof of the second assertion is finished.

A fact we need to mention is that if $n_1 = n_2$, then $X$ is an $n_1$-by-$n_1$ upper triangular operator-valued matrix such that every entry of $X$ is in $\{N_{\mu}\}$ and the entries of $X$ have further relations with others. \qed

**Lemma 2.2.** For an operator $A$ defined from (1.13) to (1.15) and every idempotent $P$ in $\{A\}$, there exists an invertible operator $X$ in $\{A\}$ such that $XPX^{-1}$ is in $\mathcal{E}$ (defined as in (2.4)).

**Proof.** As defined from (1.13) to (1.15), we have $A = A^{(m_1)}_{n_1} \oplus A^{(m_2)}_{n_2} \oplus A^{(m_3)}_{n_3}$ for positive integers $n_1 > n_2 > n_3$.

Let $B$ be an operator in $\{A\}$. Then $B$ can be expressed in the form

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix},$$

where

$$B_{ij} = \begin{pmatrix} B_{ij;11} & \cdots & B_{ij;1m_j} \\ \vdots & \ddots & \vdots \\ B_{ij;m_i,1} & \cdots & B_{ij;m_i,m_j} \end{pmatrix}_{m_i \times m_j},$$

and $B_{ij;st}$ is in the set $\{X$ is bounded linear : $A_{nt}X = XA_{nt} \}$, for $1 \leq i, j \leq 3$.

For $B$ in $\{A\}$, there exists a unitary operator $U$ which is a composition of finitely many row-switching transformations such that $C = UBU^*$ is in the form

$$C = \begin{pmatrix} C_{11} & \cdots & C_{1n_1} \\ \vdots & \ddots & \vdots \\ C_{n_1,1} & \cdots & C_{n_1,n_1} \end{pmatrix},$$

(2.22)
where $C_{lk}$ consists of the $(l, k)$ entries of each $B_{ij;st}$, and the relative positions of these entries stay invariant in $C_{lk}$. Notice that $C_{lk}$ is not square for $l \neq k$, and $C_{11}$, $C_{n_3+1,n_3+1}$ and $C_{n_2+1,n_2+1}$ are not of the same size. By Lemma 2.1, we have that $C_{ij} = 0$ for $i > j$.

For $1 \leq i \leq n_3$, the block entry $C_{ii}$ is in the form

$$C_{ii} = \begin{pmatrix} C_{ii;11} & C_{ii;12} & C_{ii;13} \\ 0 & C_{ii;22} & C_{ii;23} \\ 0 & 0 & C_{ii;33} \end{pmatrix},$$

where

$$C_{ii;kl} = \begin{pmatrix} b_{kl;11}^{ii} & \cdots & b_{kl;1m_i}^{ii} \\ \vdots & \ddots & \vdots \\ b_{kl;m_i1}^{ii} & \cdots & b_{kl;m_im_i}^{ii} \end{pmatrix}_{m_i \times m_i},$$

and the operator $b_{kl;st}^{ii}$ is the $(i, i)$ entry of the block $B_{kl;st}$, for $1 \leq k, l \leq 3$, and $1 \leq s \leq m_k$, and $1 \leq t \leq m_l$.

For $n_3 < i \leq n_2$, the block entry $C_{ii}$ is in the form

$$\begin{pmatrix} b_{11;11}^{ii} & \cdots & b_{11;1m_1}^{ii} & b_{12;11}^{ii} & \cdots & b_{12;1m_2}^{ii} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{11;m_11}^{ii} & \cdots & b_{11;m_1m_1}^{ii} & b_{12;m_11}^{ii} & \cdots & b_{12;m_1m_2}^{ii} \\ b_{12;11}^{ii} & \cdots & b_{12;m_11}^{ii} & b_{22;11}^{ii} & \cdots & b_{22;1m_2}^{ii} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{m_2 \times m_1} & \cdots & b_{22;m_21}^{ii} & \cdots & b_{22;m_2m_2}^{ii} \end{pmatrix},$$

and for $n_2 < j \leq n_1$ the block entry $C_{jj}$ is in the form

$$\begin{pmatrix} b_{11;11}^{jj} & \cdots & b_{11;1m_1}^{jj} \\ \vdots & \ddots & \vdots \\ b_{11;m_11}^{jj} & \cdots & b_{11;m_1m_1}^{jj} \end{pmatrix},$$

where the operator $b_{kl;st}^{jj}$ is the $(i, i)$ entry of the block $B_{kl;st}$, for $1 \leq k, l \leq 2$, and $1 \leq s \leq m_k$, and $1 \leq t \leq m_l$, and the operator $b_{11;st}^{jj}$ is the $(j, j)$ entry of the block $B_{11;st}$, for $1 \leq s \leq m_1$, and $1 \leq t \leq m_1$.

Let $C'_{ii}$ be the block diagonal matrix in which the diagonal blocks are the same as in $C_{ii}$. For example, the operator $C'_{11}$ is in the form

$$C'_{11} = \begin{pmatrix} C_{11;11} & 0 & 0 \\ 0 & C_{11;22} & 0 \\ 0 & 0 & C_{11;33} \end{pmatrix}.$$
is in the commutant \( \{UAU^*\}' \). Let \( \sigma_{\{UAU^*\}'}(C - C') \) denote the spectrum of \( C - C' \) in the unital Banach algebra \( \{UAU^*\}' \). Then for every operator \( D \) in the commutant \( \{UAU^*\}' \), we obtain the following equality

\[
\sigma_{\{UAU^*\}'}(D(C - C')) = \sigma_{\{UAU^*\}'}((C - C')D) = \{0\}.
\]

Therefore, the operator \( C - C' \) is in the Jacobson radical of \( \{UAU^*\}' \) denoted by \( \text{Rad}(\{UAU^*\}') \).

Let \( C \) be an idempotent in \( \{UAU^*\}' \). Then \( C' \) is also an idempotent in \( \{UAU^*\}' \). Notice that \( 2C' - I \) is invertible in \( \{UAU^*\}' \). Then the equality

\[
(2C' - I)(C + C' - I) = I + (2C' - I)(C - C')
\]

yields that the operator \( C + C' - I \) is invertible in \( \{UAU^*\}' \), since \( C - C' \) is in \( \text{Rad}(\{UAU^*\}') \). Therefore, we obtain the equality \( (C + C' - I)C = C'(C + C' - I) \) which means that the operators \( C \) and \( C' \) are similar in \( \{UAU^*\}' \).

Next, it suffices to show that the \((1, 1)\) block of \( C_{11}' \) denoted by \( C_{11;11} \) is similar to an element of the standard bounded maximal abelian set of idempotents in \( M_{m_1}(L^\infty(\mu)) \).

We assert that for every positive integer \( k \), there exists a positive integer \( l_k \) such that for every idempotent \( P \) in \( \mathcal{L}(\mathcal{H}) \) satisfying \( \|P\| \leq k \), there exists an invertible operator \( X \) in \( \mathcal{L}(\mathcal{H}) \) satisfying \( \|X\| \leq l_k \) and \( \|X^{-1}\| \leq l_k \) such that \( XPX^{-1} \) is in the standard bounded maximal abelian set of idempotents of \( \{UAU^*\}' \).

By \((1), \text{Theorem 1}\), we obtain that the Borel map \( \phi_{l_k} : \pi_3(\mathcal{J}_k) \to \pi_3(\mathcal{J}_k) \) is bounded. Therefore the equivalent class of

\[
\phi_{1|1|c_{11;11}|l_k} \circ C_{11;11}(1)
\]

is the invertible operator \( X_{11;11} \) we need in \( M_{m_1}(L^\infty(\mu)) \). In the same way, we obtain the invertible operators \( X_{11;22} \) and \( X_{11;33} \) for \( C_{11;22} \) and \( C_{11;33} \) respectively. Notice that the diagonal entries of \( D_{i_s^t} \) are the same for \( 1 \leq i \leq 3 \) and \( 1 \leq s, t \leq m_i \). Construct an invertible operator \( X \) in the commutant \( \{UAU^*\}' \) with \( X_{11;ii} \) for \( 1 \leq i \leq 3 \) such that \( XC'X^{-1} \) is in the standard bounded maximal abelian set of idempotents of \( \{UAU^*\}' \).

\begin{lemma}
Let \( \mathcal{P} \) be a bounded maximal abelian set of idempotents in the commutant \( \{A\}' \), where \( A \) is defined from \((1.13)\) to \((1.15)\). Then there exists a finite subset \( \mathcal{P}_0 \) of \( \mathcal{P} \) such that the equality

\[
\mathcal{P}_0(\lambda) = \mathcal{P}(\lambda)
\]

holds almost everywhere on \( \sigma(N_\mu) \).
\end{lemma}
Proof: The motivation of this lemma is to find a Borel measurable skeleton of \( \mathcal{P} \).

By Lemma 2.2, for an idempotent \( P \) in \( \mathcal{P} \), there exists a unitary operator \( U \) such that the operator \( C = UPU^* \) is in the form of (2.22), and \( C \) is similar to \( C' \) in \( \{ UAU^* \}' \), where \( C' \) is in the form of (2.28).

Let \( E_i \) be a projection in \( \{ UAU^* \}' \), which is as in the form of (2.28)

\[
E_i = \begin{pmatrix}
E_{i;1} & 0 & \cdots & 0 \\
0 & E_{i;2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & E_{i;n_i}
\end{pmatrix}
\]

for \( i = 1, 2, 3 \), (2.35)

where, as in the form of (2.27) we write \( E_{i;1} \) as a 3-by-3 block matrix, the \((i, i)\) block of \( E_{i;1} \) is the identity of \( M_{n_i}(L^\infty(\mu)) \) and other blocks are 0, compared with \( C'_{11} \) in (2.27). Thus the projections \( E_{i;2}, \ldots, E_{i;n_i} \) can be fixed corresponding to \( E_{i;1} \). Therefore we have the equality \( E_iCE_i = E_iC'E_i \). Define a \( \mu \)-measurable function \( r_i \) in the form

\[
r_i(P)(\lambda) \triangleq \frac{1}{n_i} \text{Tr}_{n_i m_i} (E_i UPU^* E_i(\lambda)), \text{ for almost every } \lambda \in \sigma(N_\mu),
\]

(2.36)

where \( \text{Tr}_{n_i m_i} \) denotes the standard trace on \( M_{n_i m_i}(\mathbb{C}) \).

We assert that there exists an idempotent \( P \) in \( \mathcal{P} \) such that the inequality

\[
0 < r_1(P)(\lambda) < m_1
\]

(2.37)

holds almost everywhere on \( \sigma(N_\mu) \).

If \( r_1(P)(\lambda) = 0 \) or \( r_1(P)(\lambda) = m_1 \) holds almost everywhere on \( \sigma(N_\mu) \) for every \( P \) in \( \mathcal{P} \), then \( \mathcal{P} \) is not bounded maximal abelian. Therefore, there exists a subset \( \Gamma_1 \) of \( \sigma(N_\mu) \) with \( \mu(\Gamma_1) > 0 \) and an idempotent \( P_1 \) in \( \mathcal{P} \) such that \( 0 < r_1(P_1)(\lambda) < m_1 \) holds almost everywhere on \( \Gamma_1 \). In the same way, we have a subset \( \Gamma_2 \) of \( \sigma(N_\mu) \backslash \Gamma_1 \) with \( \mu(\Gamma_2) > 0 \) and an idempotent \( P_2 \) in \( \mathcal{P} \) such that \( 0 < r_1(P_2)(\lambda) < m_1 \) holds almost everywhere on \( \Gamma_2 \). By Zorn lemma, there are sequences \( \{ P_i \}_{i=1}^{\infty} \) in \( \mathcal{P} \) and \( \{ \Gamma_i \}_{i=1}^{\infty} \) with \( \mu(\Gamma_i) > 0 \) for every \( i \) and \( \bigcup_{i=1}^{\infty} (\Gamma_i) = \sigma(N_\mu) \) such that \( 0 < r_1(P_i)(\lambda) < m_1 \) holds almost everywhere on \( \Gamma_i \). Denote by \( P \) the sum of the restrictions of \( P_i \) on \( \Gamma_i \). Therefore, we obtain the above assertion.

Next, we assert that there exists an idempotent \( P \) in \( \mathcal{P} \) such that the equality

\[
r_1(P)(\lambda) = 1
\]

(2.38)

holds almost everywhere on \( \sigma(N_\mu) \).

If \( P \) is described as in the first assertion, then \( \sigma(N_\mu) \) can be divided into at most \( m_1 - 1 \) pairwise disjoint Borel subsets \( \{ \Gamma_i \}_{i=1}^{m_1-1} \) corresponding to \( r_1(P) \) such that the equality \( r_1(P)(\lambda) = i \) holds almost everywhere on \( \Gamma_i \). Assume that \( \mu(\Gamma_{m_1-1}) > 0 \).

By a similar proof of the first assertion, there exists an idempotent \( P_1 \) in \( \mathcal{P} \) such that the inequality \( 0 < r_1(P_1)(\lambda) - m_1 \) holds almost everywhere on \( \Gamma_{m_1-1} \). Let \( Q_1 \) denote the sum of the restriction of \( P_1 \) on \( \Gamma_{m_1-1} \) and the restriction of \( P \) on \( \sigma(N_\mu) \backslash \Gamma_{m_1-1} \). Redivide \( \sigma(N_\mu) \) into at most \( m_1 - 2 \) pairwise disjoint Borel subsets \( \{ \Gamma_i \}_{i=1}^{m_1-2} \) corresponding to \( r(Q_1) \) as above. Assume that \( \mu(\Gamma_{m_1-2}) > 0 \). There exists an idempotent \( P_2 \) in \( \mathcal{P} \) such that the inequality \( 0 < r_1(P_2)(\lambda) < m_1 - 2 \) holds almost everywhere on \( \Gamma_{m_1-2} \). Construct \( Q_2 \) with \( P_2 \) and \( Q_1 \) as above. After at most \( m_1 - 2 \) steps, we obtain an idempotent in \( \mathcal{P} \) as required in the second assertion.
Finally, we assert that there are \( m_1 \) idempotents \( \{P_i\}_{i=1}^{m_1} \) in \( \mathcal{P} \) such that the equality
\[
r_1(P_i)(\lambda) = 1
\] (2.39)
holds almost everywhere on \( \sigma(N_\mu) \), and \( P_i P_j = 0 \) for \( i \neq j \).

By the second assertion, we obtain \( P_1 \) in \( \mathcal{P} \) such that \( r_1(P_1)(\lambda) = 1 \) holds almost everywhere on \( \sigma(N_\mu) \). Then we obtain \( P_2 \) in \( (I - P_1)\mathcal{P} \) such that \( r_1(P_2)(\lambda) = 1 \) holds almost everywhere on \( \sigma(N_\mu) \) by applying the first two assertions. Take these idempotents one by one and we prove the third assertion.

By the above three assertions, we obtain \( m_1 + m_2 + m_3 \) idempotents \( \{P_{ij}\}_{i=1,j=1}^{3,m_1} \) in \( \mathcal{P} \) such that the equality
\[
r_i(P_{ij})(\lambda) = 1
\] (2.40)
holds almost everywhere on \( \sigma(N_\mu) \), and \( (P_{ij})(P_{ik}) = 0 \) for \( i \neq k \) or \( j \neq l \). Construct \( \mathcal{P}_0 \) in the form
\[
\mathcal{P}_0 \triangleq \{ \sum_{i=1}^{3} \sum_{j=1}^{m_i} \alpha_{ij}(P_{ij}) : \alpha_{ij} \in \{0,1\} \}.
\] (2.41)
Then the equality \( \mathcal{P}_0(\lambda) = \mathcal{P}(\lambda) \) holds almost everywhere on \( \sigma(N_\mu) \).

**Proof of Theorem 1.2.** Let \( \mathcal{P} \) be a bounded maximal abelian set of idempotents in \( \{A\}' \). By Lemma 2.3, there exist \( m_1 + m_2 + m_3 \) idempotents \( \{P_{ij}\}_{i=1,j=1}^{3,m_1} \) in \( \mathcal{P} \) such that the equality \( r_i(P_{ij})(\lambda) = 1 \) holds almost everywhere on \( \sigma(N_\mu) \), and \( P_{ij}P_{ik} = 0 \) for \( i \neq k \) or \( j \neq l \). By Lemma 2.2, there exists an invertible operator \( X_{1:1} \) in \( \{A\}' \) such that \( X_{1:1}P_{1:1}X_{1:1}^{-1} \) is in the standard bounded maximal abelian set of idempotents \( \mathcal{E} \) in \( \{A\}' \). Precisely, the idempotent \( X_{1:1}P_{1:1}X_{1:1}^{-1} \) is in the form
\[
X_{1:1}P_{1:1}X_{1:1}^{-1} = (I \oplus 0^{(m_1-1)}) \oplus (0^{(m_2)}) \oplus (0^{(m_3)}),
\] (2.42)
where \( I \) is the identity operator in \( M_{m_1}(L^\infty(\mu)) \). In a similar way, there exists an invertible operator \( X_{2:1} \) in \( \{A\}' \) such that \( (X_{2:1}X_{1:1})P_{1:1}(X_{2:1}X_{1:1})^{-1} \) and \( (X_{2:1}X_{1:1})P_{2:1}(X_{2:1}X_{1:1})^{-1} \) are both in the standard bounded maximal abelian set of idempotents in \( \{A\}' \). The invertible operator \( X_{2:1} \) is in the form
\[
X_{2:1} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},
\] (2.43)
where \( I \) is the identity operator in \( M_{m_1}(L^\infty(\mu)) \). Furthermore, there exist \( m_1 + m_2 + m_3 - 3 \) invertible operators \( \{X_{ij}\}_{i=1,j=1}^{3,m_1} \) in \( \{A\}' \) such that \( X(P_{ij})X^{-1} \) is in the standard bounded maximal abelian set of idempotents in \( \{A\}' \) for every \( i \) and \( j \), where let \( X \) denote the product
\[
X = X_{m_1-3:1} \cdots X_{1:3}X_{m_2-1:2} \cdots X_{1:2}X_{m_1-1:1} \cdots X_{1:1}.
\] (2.44)
Then we obtain that the set \( X \mathcal{P} X^{-1} \) is the standard bounded maximal abelian set of idempotents in the commutant \( \{A\}' \). Therefore, the strongly irreducible decomposition of \( A \) is unique up to similarity.

Next, we compute the \( K \) groups of \( \{A\}' \). We denote by \( \mathcal{J} \) a closed two-sided ideal of \( \{A\}' \) such that for every operator \( B \) in \( \mathcal{J} \), every entry in the main diagonal of \( B_{ij;st} \) is 0 for \( 1 \leq i \leq 3 \) and \( 1 \leq s, t \leq m_i \), where \( B \) and \( B_{ij;st} \) are as in the form of (2.20) and (2.21). By \( \mathcal{B} \) we denote a subalgebra of \( \{A\}' \) such that for every operator \( B \) in \( \mathcal{B} \), every entry of \( B_{ij;st} \) is 0 except ones in the main diagonal of
$B_{ii,st}$, for $1 \leq i,j \leq 3$ and $1 \leq s,t \leq m_i$. By observation, we obtain the following split short exact sequence:

$$0 \rightarrow \mathcal{J} \xrightarrow{i} \{A\}' \xrightarrow{\pi} \mathcal{B} \rightarrow 0$$

where we denote by $i$ and $\alpha$ the inclusion maps and by $\pi$ the map such that for every operator $B$ in $\{A\}'$, every entry of $\pi(B)_{ii,st}$ is 0 except ones in the main diagonal of $B_{ii,st}$, staying invariant with respect to $\pi$, for $1 \leq i,j \leq 3$ and $1 \leq s,t \leq m_i$. Essentially, $\pi$ is the quotient map. Furthermore, we obtain

$$\mathcal{B} \cong M_{m_1}(L^\infty(\mu)) \oplus M_{m_2}(L^\infty(\mu)) \oplus M_{m_3}(L^\infty(\mu)).$$

By Lemma 2.2, we have $K_0(\pi)$ is an isomorphism. Therefore,

$$K_0(\{A\}') \cong K_0(\mathcal{B})$$

and by a routine computation, we obtain

$$K_0(\{A\}') \cong \{ f : \sigma(N_\mu) \rightarrow \mathbb{Z}^{(3)}, f \text{ is bounded Borel} \}.$$  

For a generalized case, we need to combine the proofs as above with respect to different regular Borel measures which are pairwise mutually singular. Since the spectrum of $A_{nm}$ (as in (1.10) and (1.11)) equals $\sigma(N_{\nu_{nm}})$, we construct a normal operator

$$N = \bigoplus_{n=1}^{\infty} \bigoplus_{m=1}^{\infty} N_{\nu_{nm}},$$

where $N_{\nu_{nm}} = 0$ holds for all but finitely many $n$ and $m$ in $\mathbb{N}$, corresponding to the assumption from (1.10) to (1.12). We observe that for $i \neq j$, the scalar-valued spectral measures $\nu_{nm_i}$ and $\nu_{nm_j}$ are mutually singular, but the scalar-valued spectral measures $\nu_{n,m}$ and $\nu_{n,m}$ may not be mutually singular. By [3], IX, Theorem 10.16, the normal operator $N$ can be expressed in a direct sum of finitely many normal operators with pairwise mutually singular scalar-valued spectral measures. Actually, this is a finer decomposition than the one in (2.49). With this expression and the above proof for a special case, we obtain the proof of Theorem 1.2 and Theorem 1.3. 

By Theorem 1.2, we can compute the $K_0$ group of $\{A\}'$, if the strongly irreducible decomposition of $A$ is unique up to similarity. Next, we investigate the uniqueness of the strongly irreducible decomposition of $A$ up to similarity by the $K_0$ group of $\{A\}'$. Let operators $A$ and $B$ be as in the form of (1.14) and (1.15):

$$A = \begin{pmatrix}
N_\mu & M_{f_{12}} & \cdots & M_{f_{1n}} \\
0 & N_\mu & \cdots & M_{f_{2n}} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & N_\mu
\end{pmatrix}_{n \times n},$$

and

$$B = \begin{pmatrix}
N_\mu & M_{g_{12}} & \cdots & M_{g_{1n}} \\
0 & N_\mu & \cdots & M_{g_{2n}} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & N_\mu
\end{pmatrix}_{n \times n}.$$
where $M_{f,j}$ and $M_{g,j}$ is in $\{N_{\mu}^{(n)}\}^\prime$, for $1 \leq i < j \leq n$. Then we have the following lemma.

**Lemma 2.4.** The operators $A^{(m)}$ and $B^{(m)}$ are similar in $M_{nm_1}(L^\infty(\mu))$ $(m_1 \geq m_2)$ if and only if there exists an isomorphism $\theta$ such that

$$
\begin{align*}
(1) & \quad K_0(\{T\}) \equiv \{ f : \sigma(N_\mu) \to \mathbb{Z}, f \text{ is bounded Borel} \}, \\
(2) & \quad \theta([I_{\{T\}'}]) = 2m_1e,
\end{align*}
$$

where $T = A^{(m_1)} \oplus B^{(m_2)}$ and $e(\lambda)$ is the generator of the semigroup $\mathbb{N}$ of $\mathbb{Z}$ for almost every $\lambda$ in $\sigma(N_\mu)$.

**Proof.** If the operators $A^{(m_1)}$ and $B^{(m_2)}$ are similar in $M_{nm_1}(L^\infty(\mu))$, then we obtain $K_0(\{T\})'$ as required by the proof of Theorem 1.2.

On the other hand, we suppose that the relations in (2.52) hold. Let $P$ and $Q$ be idempotents in $\{A^{(m_1)}\}'$ and $\{B^{(m_2)}\}'$ respectively such that the equalities

$$
r_{\{A^{(m_1)}\}'}(P)(\lambda) = 1 \quad \text{and} \quad r_{\{B^{(m_2)}\}'}(Q)(\lambda) = 1
$$

hold for almost every $\lambda$ in $\sigma(N_\mu)$. If $P \oplus 0$ and $0 \oplus Q$ are not similar in $\{T\}'$, then we obtain $\theta([P \oplus 0]) = e = \theta([0 \oplus Q])$. Thus $\theta$ is not an isomorphism which contradicts the assumption in (2.52). Therefore $P \oplus 0$ and $0 \oplus Q$ are similar in $\{T\}'$. We can choose projections $E$ and $F$ similar to $P \oplus 0$ in $\{T\}'$ such that $T|_{\text{ran } E} = A$ and $T|_{\text{ran } F} = B$. Thus $A \oplus 0$ is similar to $0 \oplus B$ in $\{T\}'$. The equality $\theta([I_{\{T\}'}]) = 2m_1e_1$ yields that $m_1 + m_2 = 2m_1$. Hence $m_1 = m_2$ and $A^{(m_1)}$ is similar to $B^{(m_2)}$. \hfill $\square$

**Proof of Theorem 1.4.** If the operator $A = \oplus_{i=1}^3 A^{(m_i)}$ is similar to $B = \oplus_{j=1}^3 B^{(k_j)}$, then we can obtain an isomorphism $\theta$ and the $K_0$ group $K_0(\{T\}')$ as required in the theorem by a routine computation.

To show the converse, suppose that there exists an isomorphism $\theta$ such that

(a) $\theta : K_0(\{T\}') \to \{ f : \sigma(N_\mu) \to \mathbb{Z}^{(3)}, f \text{ is bounded Borel} \}$

(b) $\theta([I_{\{T\}'}]) = 2m_1e_1 + 2m_2e_2 + 2m_3e_3$.

In the commutant $\{T\}'$, there exist 3 projections $\{E_i\}_{i=1}^3$ and 3 projections $\{F_j\}_{j=1}^3$ such that

(1) $T|_{\text{ran } E_i} = A_{n_i}$ and $T|_{\text{ran } F_j} = B_{l_j}$;

(2) $E_iE_j = F_iF_j = 0$ and $E_iF_j = 0$ for $i \neq j$;

(3) the equalities $r_i(E_i)(\lambda) = 1$ and $r_j(F_j)(\lambda) = 1$ hold for almost every $\lambda$ in $\sigma(N_\mu)$ and $1 \leq i, j \leq 3$.

The equivalence classes $\{E_i\}_{i=1}^3$ can be considered as the generating set of $K_0(\{T\}')$. If $E_i$ is not similar to $E_i$ in $\{T\}'$ for some $i$, then for $K_0(\{T\}')$, there exists a $\lambda$ in the $\sigma(N_\mu)$ such that the set $\{E_i(\lambda)\}_{i=1}^3 \cup \{F_j(\lambda)\}$ generates $\mathbb{Z}^{(3)}$, which is a contradiction since $\lambda$ cannot be removed from $\sigma(N_\mu)$. Therefore, $E_i$ is similar to $E_i$ in $\{T\}'$ for $1 \leq i \leq 3$. The coefficient of $e_i$ in $\theta([I_{\{T\}'}])$ is $m_i + k_i = 2m_i$ for $1 \leq i \leq 3$. Therefore the equality $m_i = k_i$ holds for $1 \leq i \leq 3$. Thus we obtain that the operator $A$ is similar to $B$. \hfill $\square$

3. Appendix

In this part, we show the relation between an operator $A$ as in (1.16) and $B^{(m)}$ such that $B$ is in the form of (2.51) and $m$ is a positive integer. The motivation
is to obtain a decomposition of an operator $A$ as in (1.16) with respect to the main diagonal entries. Suppose that $A$ is an operator in the form

$$A = \begin{pmatrix} M_f & M_{f_{12}} & \cdots & M_{f_{1m}} \\ 0 & M_f & \cdots & M_{f_{2m}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_f \end{pmatrix}_{n \times n} \begin{pmatrix} L^2(\mu) \\ L^2(\mu) \\ \vdots \\ L^2(\mu) \end{pmatrix}, \tag{3.1}$$

and there exists a unitary operator $V$ such that $VM_fV^* = N_{\nu}^{(m)}$ where $f$ and $f_{ij}$ are in $L^\infty(\mu)$ and $\nu = \mu \circ f^{-1}$. Then we have the following proposition.

**Proposition 3.1.** There is a unitary operator $W$ such that $WM_fW^* = N_{\nu}^{(m)}$ and

$$W^{(n)} A(W^*)^{(n)} = \bigoplus_{k=1}^{m} \begin{pmatrix} N_{\nu} & M_{f_{k,12}} & M_{f_{k,13}} & \cdots & M_{f_{k,1m}} \\ 0 & N_{\nu} & M_{f_{k,23}} & \cdots & M_{f_{k,2m}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N_{\nu} \end{pmatrix}_{n \times n}. \tag{3.2}$$

**Proof.** The multiplication operators $M_f$ and $M_{f_{ij}}$ are in $\{N_{\nu}\}'$ for $i < j$. By the assumption in (3.1), there exists a unitary operator $V$ such that $VM_fV^* = N_{\nu}^{(m)}$. Then we obtain the equality

$$N_{\nu}^{(m)}(VM_{f_{ij}}V^*) = (VM_{f_{ij}}V^*)N_{\nu}^{(m)}. \tag{3.3}$$

Therefore, the operator $VM_{f_{ij}}V^*$ can be expressed in the form

$$VM_{f_{ij}}V^* = \begin{pmatrix} M_{\phi_{ij,11}} & \cdots & M_{\phi_{ij,1m}} \\ \vdots & \ddots & \vdots \\ M_{\phi_{ij,jm}} & \cdots & M_{\phi_{ij,m,m}} \end{pmatrix}_{m \times m}, \tag{3.4}$$

where $\phi_{ij,s:t}$ is in $L^\infty(\nu)$ and $M_{\phi_{ij,s:t}}$ is in $\{N_{\nu}\}'$ for $1 \leq s, t \leq m$. The motivation is to find a unitary operator such that every $VM_{f_{ij}}V^*$ is unitarily equivalent to a diagonal operator in the commutant $\{N_{\nu}^{(m)}\}'$. We observe that

$$\{VN_{\nu}V^*\}'' = \{VN_{\nu}V^*\}' \subseteq \{N_{\nu}^{(m)}\}'. \tag{3.5}$$

Let $\mathcal{E}$ be the set of projections in $\{VN_{\nu}V^*\}'$. Then $\mathcal{E}$ is a maximal abelian set of projections in $\{N_{\nu}^{(m)}\}'$. By (II), Proposition 4.1, we obtain that $\mathcal{E}$ is a bounded maximal abelian set of idempotents in $\{N_{\nu}^{(m)}\}'$. As an application of Lemma 2.3, there exist $m$ projections $\{E_i\}_{i=1}^{m}$ in $\mathcal{E}$ such that $E_iE_j = 0$ for $i \neq j$ and the equality $\text{rank}(E_i(\lambda)) = 1$ holds for almost every $\lambda$ in the support of $\nu$. Then by (II),Corollary 2) and a similar proof of Theorem 1.2, we obtain a unitary operator $V_1$ in $\{N_{\nu}^{(m)}\}'$ such that every projection in $V_1\mathcal{E}V_1^*$ is diagonal. Therefore $(V_1V)^{(n)}$ is as required. \hfill \Box

By Proposition 3.1, we observe that $B^{(m)}$ for $B$ as in (2.51) is a special form of (3.2). When we consider a similar result as Theorem 1.2, the following example
makes the calculation appear to be more complicated. Let the operators $X$ and $Y$ be in the form
\[
X = \begin{pmatrix} N_\mu & I \\ 0 & N_\mu \end{pmatrix}, \quad Y = \begin{pmatrix} N_\mu & 0 \\ 0 & N_\mu \end{pmatrix},
\]
(3.6)

Then $X$ and $Y$ are not similar in $M_2(L^\infty(\mu))$, where the regular Borel measure $\mu$ is supported on the interval $[-1, 1]$. By Lemma 2.1, if $Z$ is a bounded linear operator such that $XZ = ZY$, then $Z$ is in the form
\[
Z = \begin{pmatrix} M_{f_1} & M_{f_{12}} \\ 0 & M_{f_2} \end{pmatrix},
\]
(3.7)

where every entry of $Z$ is in $\{N_\mu\}$. And $M_{f_2} = M_{f_1}N_\mu$. Therefore, the operator $Z$ is not invertible. In (5), $\{E\}$ where every entry of $T$ makes the calculation appear to be more complicated. Let the operators $\{E\}$ be in the form
\[
\text{Proposition 3.2. Let } A \text{ be an operator as in (2.50)}
\]
\[
A^{(\infty)} = \begin{pmatrix} N^{(\infty)}_\mu & M^{(\infty)}_{f_{12}} & M^{(\infty)}_{f_{13}} & \cdots & M^{(\infty)}_{f_{1n}} \\ 0 & N^{(\infty)}_\mu & M^{(\infty)}_{f_{23}} & \cdots & M^{(\infty)}_{f_{2n}} \\ 0 & 0 & N^{(\infty)}_\mu & \cdots & M^{(\infty)}_{f_{3n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N^{(\infty)}_\mu \end{pmatrix}_{n \times n},
\]
(3.8)

where $\mu$ is stated as in (1.7). Then the strongly irreducible decomposition of $A^{(\infty)}$ is not unique up to similarity.

**Proof.** We need to construct two bounded maximal abelian sets of idempotents in $\{A^{(\infty)}\}'$ such that they are not similar to each other.

We write $N^{(\infty)}_\mu$ in the form $N_\mu \otimes I_{l^2}$, where $I_{l^2}$ is the identity operator on $l^2$. Denote by $\mathcal{P}$ the set of all the spectral projections of $N_\mu$. This set forms a bounded maximal abelian set of idempotents in $\{N_\mu\}'$. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $l^2$. Denote by $E_k$ the projection such that $\text{ran}E_k = \{\lambda e_k : \lambda \in \mathbb{C}\}$. Define
\[
\mathcal{L}_1 \triangleq \{P \in \mathcal{L}(l^2) : P = P^* = P^2 \in \{E_k : k \in \mathbb{N}\}'\}.
\]
(3.9)

Denote by $\chi_S$ the characteristic function for a Borel subset $S$ in $\sigma(N_\mu)$ and define
\[
\mathcal{L}_2 \triangleq \{M_{\chi_S} \in \mathcal{L}(L^2(\mu)) : S \subseteq \sigma(N_\mu) \text{ is Borel}\}.
\]
(3.10)

There is a unitary operator $U : L^2[0, 1] \to l^2$ such that $UPU^* \in \mathcal{L}(l^2)$ for every $P \in \mathcal{L}_2$. The sets $\mathcal{L}_2 \triangleq U\mathcal{L}_2U^*$ and $\mathcal{L}_1$ are two bounded maximal abelian sets of idempotents in $\mathcal{L}(l^2)$ but they are not unitarily equivalent.

The fact that $W^*(\mathcal{P}) \otimes W^*(\mathcal{L}_1)$ and $W^*(\mathcal{P}) \otimes W^*(\mathcal{L}_2)$ are both maximal abelian von Neumann algebras yields that
\[
\mathcal{F}_1 \triangleq \{P \in W^*(\mathcal{P}) \otimes W^*(\mathcal{L}_1) : P = P^* = P^2\}
\]
(3.11)
and

\[ \mathcal{F}_2 \triangleq \{ P \in W^*(\mathcal{P}) \otimes W^*(\mathcal{Q}_2) : P = P^* = P^2 \} \tag{3.12} \]

are both bounded maximal abelian sets of idempotents in \( \{ N_{12} \otimes I_{i2} \}' = L^\infty(\mu) \otimes \mathscr{L}(l^2) \).

We assert that \( \mathcal{F}_i^{(n)} \) is a bounded maximal abelian set of idempotents in \( \{ A^{(\infty)} \}' \) for \( i = 1, 2 \).

An operator \( X \) in \( \{ A^{(\infty)} \}' \) can be expressed in the form

\[
X = \begin{pmatrix}
X_{11} & X_{12} & X_{13} & \cdots & X_{1n} \\
X_{21} & X_{22} & X_{23} & \cdots & X_{2n} \\
X_{31} & X_{32} & X_{33} & \cdots & X_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
X_{n1} & X_{n2} & X_{n3} & \cdots & X_{nn}
\end{pmatrix}_{n \times n}.
\tag{3.13}
\]

By a similar proof of Lemma 2.1, we obtain that \( X_{ij} \) is in \( \{ N_{12} \otimes I_{i2} \}' \) and the equation \( X_{ij} = 0 \) holds for \( i > j \) and \( X_{ii} = X_{11} \) for \( i = 2, \ldots, n \) in (3.13). Furthermore, if \( X \) as in (3.13) is an idempotent, then so is every main diagonal entry \( X_{ii} \) of \( X \).

We assume that \( X \) is an idempotent in \( \{ A^{(\infty)} \}' \) and commutes with \( \mathcal{F}_1^{(n)} \). Hence \( X_{ii} \) commutes with \( \mathcal{F}_1 \). The fact that \( \mathcal{F}_1 \) is a maximal abelian set of idempotents implies that \( X_{ii} \) belongs to \( \mathcal{F}_1 \). Thus \( X_{ii} \) commutes with \( X_{ij} \). For the 1-diagonal entries, the equation \( 2X_{ii}X_{i,i+1} - X_{i,i+1} = 0 \) yields \( X_{i,i+1} = 0 \), for \( i = 1, \ldots, n - 1 \). By this way, the \( k \)-diagonal entries of \( X \) are all zero, for \( k = 2, \ldots, n \). Therefore \( X \) is in \( \mathcal{F}_1^{(n)} \). Both \( \mathcal{F}_1^{(n)} \) and \( \mathcal{F}_2^{(n)} \) are bounded maximal abelian sets of idempotents in \( \{ A^{(\infty)} \}' \).

Every operator \( X \) in \( \{ A^{(\infty)}_n \}' \) can be expressed in the form

\[
X = \int_{\sigma(N_{12})} X(\lambda)d\mu(\lambda).
\tag{3.14}
\]

Suppose that there is an invertible operator \( X \) in \( \{ A^{(\infty)}_n \}' \) such that

\[ X\mathcal{F}_2^{(n)}X^{-1} = \mathcal{F}_1^{(n)}. \]

For each \( P \) in \( \mathcal{F}_2^{(n)} \), the projection \( P(\lambda) \) is either of rank \( \infty \) or 0, for almost every \( \lambda \) in \( \sigma(N_{12}) \). But there exists an projection \( Q \) in \( \mathcal{F}_1^{(n)} \) such that \( Q(\lambda) \) is of rank \( n \), for almost every \( \lambda \) in \( \sigma(N_{12}) \). This is a contradiction. Therefore \( \mathcal{F}_1^{(n)} \) and \( \mathcal{F}_2^{(n)} \) are not similar in \( \{ A^{(\infty)} \}' \).

As an application of the preceding proposition, we obtain the following corollary.

**Corollary 3.3.** Let \( A \) be an operator assumed as in (1.16). If the multiplicity function \( m_f \) of the main diagonal operator \( M_f \) takes finitely many values and there exists a bounded N-valued simple function \( r_A \) on \( \sigma(A) \) such that

\[
K_0(\{ A \}') \cong \{ \phi(\lambda) \in \mathbb{Z}^{(r_A(\lambda))} : \phi \text{ is bounded Borel on } \sigma(A) \}, \tag{3.15}
\]

then the strongly irreducible decomposition of \( A \) is unique up to similarity.
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