Fractional linear maps in general relativity and quantum mechanics

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This paper studies the nature of fractional linear transformations in a general relativity context as well as in a quantum theoretical framework. Two features are found to deserve special attention: the first is the possibility of separating the limit-point condition at infinity into loxodromic, hyperbolic, parabolic and elliptic cases. This is useful in a context in which one wants to look for a correspondence between essentially self-adjoint spherically symmetric Hamiltonians of quantum physics and the theory of Bondi-Metzner-Sachs transformations in general relativity. The analogy therefore arising, suggests that further investigations might be performed for a theory in which the role of fractional linear maps is viewed as a bridge between the quantum theory and general relativity. The second aspect to point out is the possibility of interpreting the limit-point condition at both ends of the positive real line, for a second-order singular differential operator, which occurs frequently in applied quantum mechanics, as the limiting procedure arising from a very particular Kleinian group which is the hyperbolic cyclic group. In this framework, this work finds that a consistent system of equations can be derived and studied. Hence one is led to consider the entire transcendental functions, from which it is possible to construct a fundamental system of solutions of a second-order differential equation with singular behavior at both ends of the positive real line, which in turn satisfy the limit-point conditions. Further developments in this direction might also be obtained by constructing a fundamental system of solutions and then deriving the differential equation whose solutions are the independent system first obtained. This guarantees two important properties at the same time: the essential self-adjointness of a second-order differential operator and the existence of a conserved quantity which is an automorphic function for the cyclic group chosen.

1. Introduction

Projective geometry was developed in the nineteenth century as a form of geometry that describes graphical rather than metric properties [1,2]. Nevertheless, over the years, it has been found to play a role in leading to new directions both in differential geometry [3] and in pseudo-Riemannian geometry. In the latter case, projective transformations play an important role in the asymptotic symmetry group of an
asymptotically flat spacetime [4,5] and hence in many asymptotic properties of classical and quantum field theories [6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22].

In particular, the recent work in Ref. [5] has exploited the analysis of fixed points of fractional linear transformations in order to classify the Bondi-Metzner-Sachs transformations, and has even suggested that a link exists between such transformations and their counterpart in the theory of singular self-adjoint boundary-value problems. It has been therefore our aim to understand whether such a correspondence is truly conceivable, because it might imply that general relativity can be seen as the bridge between classical and quantum physics.

For this purpose, Sect. 2 studies limit-point limit-circle theory and its link with Bondi-Metzner-Sachs transformations. Section 3 is devoted to the limit-point case at both ends of the positive real line, with the associated hyperbolic cyclic groups. Eventually, our concluding remarks are presented in Sect. 4.

2. Limit Point, Limit Circle Theory and Bondi-Metzner-Sachs Transformations

2.1. Spherically Symmetric Hamiltonians

We here introduce the limit-point, limit-circle theory for self-adjointness of Sturm-Liouville-like second-order differential operator on the real axis. The importance of this special class of operators in one particle quantum theory is well known, especially in the context of spherical symmetric Hamiltonians for bound states

\[ \hat{\mathcal{H}} = -\frac{\hbar^2}{2m} \Delta + V(r), \]  

where the Euclidean \( n \)-dimensional Laplacian can be written in spherical coordinates as

\[ -\Delta = -\frac{d^2}{dr^2} - \frac{(n-1)}{r} \frac{d}{dr} - \Delta \mathcal{S}, \]  

and the spherical Laplacian takes the form [23]

\[ -\Delta \mathcal{S} = \frac{\hat{L}^2}{\hbar^2 r^2}, \]

in which \( \hat{L}^2 \) is the squared angular momentum operator of the particle. The Hilbert space to which the eigenfunctions of the operator in Eq. (2.1) should belong, is the closure [24] of the tensor space

\[ \mathcal{L}^2 \left( \mathbb{R}^+, r^{(n-1)} dr \right) \otimes \mathcal{L}^2 \left( S^{(n-1)}, d\Omega \right), \]

in \( \mathcal{L}^2 \left( \mathbb{R}^n, r^{(n-1)} dr d\Omega \right) \), for which \( S^{(n-1)} \) is the \((n-1)\)-sphere embedded in \( \mathbb{R}^n \) and \( d\Omega \) is the surface element of such a sphere. The closure could be obtained by adjoining each limit of any sequence of functions of the space, to the space itself. The
closure is thus a Hilbert space and coincides with $L^2(\mathbb{R}^n, r^{(n-1)} dr d\Omega)$. Therefore, we can consider the eigenvalue problem for the operator (2.1)

$$\hat{H} \varphi = E \varphi,$$

and from previous reasoning, we can factorize the eigenfunctions by the product of a purely angular function and a purely radial one

$$\varphi(r; \theta_1, \ldots, \theta_{n-1}) = \psi(r) \Theta(\theta_1, \ldots, \theta_{n-1}),$$

but from Eq. (2.2), $\varphi$ is an eigenfunction of the operator (2.1) if and only if $\Theta$ is an eigenfunction of $\Delta_S$, that is, if and only if one has \[23\]

$$\frac{\hbar^2}{2} \frac{\partial^2}{\partial r^2} \Theta(\theta_1, \ldots, \theta_{n-1}) = \frac{(l+ n - 2)}{r} \Theta(\theta_1, \ldots, \theta_{n-1}), \quad l = 0, 1, \ldots$$

Thus, the eigenvalue equation reduces to

$$\left[-\left(\frac{d^2}{dr^2} + \frac{(n-1)}{r} \frac{d}{dr}\right) + \frac{\kappa_{n,l}}{r^2} + \frac{2m}{\hbar^2} V(r)\right] \psi(r) = \frac{2m}{\hbar^2} E \psi(r),$$

and one can see that the energy levels are strongly subjected to the angular momentum values and to the dimensionality of the Euclidean space under consideration. Therefore, it could be convenient to consider a family of two-integer-parameter operators by setting

$$\kappa_{n,l} = l (l + n - 2),$$

thus

$$\hat{H}_r (n, l) = -\left[\frac{d^2}{dr^2} + \frac{(n-1)}{r} \frac{d}{dr}\right] + \frac{\kappa_{n,l}}{r^2} + \frac{2m}{\hbar^2} V(r). \quad (2.3)$$

This is a family of operators acting on the $L^2(\mathbb{R}^+, r^{(n-1)} dr)$ space. One can thus consider the following norm-preserving unitary map \[24,5\]

$$\hat{U} : \psi \in L^2(\mathbb{R}^+, r^{(n-1)} dr) \rightarrow \tilde{\psi} = r^{(n-1)/2} \psi \in L^2(\mathbb{R}^+, dr), \quad (2.4)$$

which maps the operator (2.3) into

$$\hat{\tilde{H}}_r (n, l) = \hat{U} \hat{H}_r (n, l) \hat{U}^{-1} = \left[-\frac{d^2}{dr^2} + \frac{(n-1)(n-3) + \kappa_{n,l}}{4r^2} \right] + \frac{2m}{\hbar^2} V(r). \quad (2.5)$$

By setting

$$\lambda_{n,l} = \left(l + \frac{1}{2} (n - 2)\right),$$

and by using the equation for $\kappa_{n,l}$, we can write Eq. (2.5) as

$$\hat{\tilde{H}} (n, l) = -\frac{d^2}{dr^2} + \left(\frac{\lambda_{n,l}^2 - \frac{1}{4}}{r^2} + \frac{2m}{\hbar^2} V(r)\right). \quad (2.6)$$
The unitary operator (2.4) leaves the spectrum of \( \hat{H}_r (l, n) \) unaffected when the transformation (2.5) is applied. Hamiltonians of kind (2.6) have been studied in the literature for example in Refs. \([21,25,26]\) and our attention is mainly focused on them because their form is suitable for the application of self-adjointness criterions first developed by H. Weyl in his early work \([27]\). In the subsequent developments we will review the limit-point, limit-circle theory and we hope it will be clear that the required self-adjointness for quantum mechanical Hamiltonians, does not only satisfy the empirical desire to conduct some sort of reasonable experiment, but also satisfies the curiosity of the theoretician which can investigate the matter of facts by looking with his mathematical lens, behind what is already known.

2.2. Limit-Point, Limit-Circle Theory

Throughout the present subsection we will investigate the spectral properties of the following differential operator on the real axis defined by

\[
L x = - (px')' + qx,
\]

where the function \( x \) is assumed to be a function of some \( r \) variable on the real axis while \( p^{-1}, p' \) and \( q \) are summable functions on any compact subinterval of interest \([28]\), and \( p > 0 \) \([29,30]\). Note that the operator defined in (2.7) is analogous to the operator (2.6). We will call singular points of Eq. (2.7) each point which is a singular point for its coefficients or each point at infinity. For example, the operator (2.6) has two singular points: the point at infinity taken as the limit \( r \to \infty \) and the point \( r = 0 \) which is a singular point for

\[
q (r) = \left( \xi_{n,l}^2 - \frac{1}{4} \right) r^2 + \frac{2m}{\hbar^2} V(r).
\]

We note that in the case we are dealing with s-waves in three dimensions by picking up the operator \( \hat{H}_r (0, 3) \), we have that \( \lambda_{3,0}^2 = \frac{1}{4} \) and no singularity comes from the first term of the right-hand side of Eq. (2.8). Nevertheless, many physical potentials, for example the Coulomb potential, possess singular behaviour at \( r = 0 \). Therefore, we will investigate operators of the type (2.7) by assuming such singular behaviour at both ends of the positive real axis.

The limit-point, limit-circle theory treats singular self-adjoint problems of the second order whose differential equation is established in Eq. (2.7). For what follows, it is essential to consider the Green’s formula which states that if \( [r_1, r_2] \) is any interval in which the operator \( L \) is defined and \( f \) and \( g \) are two functions for which \( Lf \) and \( Lg \) are meaningful, then

\[
\int_{r_1}^{r_2} dr \left( \sqrt{m} L f - f L \sqrt{m} g \right) = [fg](r_2) - [fg](r_1),
\]

where

\[
[fg](r) = p(r) (f(r) \overline{g}'(r) - f'(r) \overline{g}(r)),
\]

\[
\overline{g}'(r) = \frac{d}{dr} \overline{g}(r).
\]
and \( \overline{f} \) is the complex conjugate of the function \( f \).

**Definition 2.1.** Let \( \tilde{r} \) be a singular point for Eq. (2.7) If for a particular complex number \( l_0 \) each solution of the equation

\[
Lx = l_0x,
\]

is square summable in some neighborhood of \( \tilde{r} \), then \( L \) is said to be in the “limit-circle” case at \( \tilde{r} \). If this is not the case, then \( L \) is said to be in the “limit-point” case at \( \tilde{r} \).

The geometrical interpretation of this nomenclature will be clear soon. As already mentioned, we are mainly interested in only two singular points: the point at infinity and the point \( r = 0 \) thus we will adapt each theorem and proof of Ref. [29] to these particular singular points.

**Theorem 2.1.** Suppose that the only singular point in Eq. (2.7) is the point at infinity. If every solution of \( Lx = l_0x \) is of class \( L^2(c, \infty) \) for some \( c > 0 \) and some complex number \( l_0 \), then, for every arbitrary complex number \( l \), every solution of \( Lx = lx \) is of class \( L^2(c, \infty) \).

**Proof.** Suppose \( \varphi \) and \( \psi \) are two linearly independent solutions of \( Lx = l_0x \). Let \( \chi \) be any solution of \( Lx = lx \), or equivalently, of

\[
Lx = l_0x + (l - l_0)x.
\]

Upon multiplying \( \varphi \) by a constant in order to achieve \( [\varphi \psi](r) = 1 \) (note that if \( f \) and \( g \) are two solutions of \( Lx = l_0x \) then \( [\varphi \psi](r) \) is the Wronskian of the differential equation which is a constant for a fundamental system of solutions), the Lagrange variation of parameters formula yields

\[
\chi(r) = c_1 \varphi + c_2 \psi + (l - l_0) \int_\tilde{c}^r dr' \varphi(r) (r') \psi(r') - \varphi(r') \psi(r) \chi(r') \quad (2.9)
\]

where \( c_1, c_2 \) and \( \tilde{c} \geq c \) are three constants. If we set

\[
\|\chi\|_{\tilde{c}}^2 = \int_\tilde{c}^r dr' |\chi|'^2, \quad r \geq \tilde{c},
\]

then there exists a constant \( M \) such that \( \|\varphi\|_{\tilde{c}}, \|\psi\|_{\tilde{c}} \leq M \) for all \( r > \tilde{c} \). The Schwarz inequality then gives

\[
\left| \int_\tilde{c}^r dr' \varphi(r) \psi(r') - \varphi(r') \psi(r) \chi(r') \right| \leq M (|\varphi| + |\psi|) \|\chi\|_{\tilde{c}}.
\]

By using the Minkowski inequality jointly with the previous Schwarz inequality

\[
\left( \int_\tilde{c}^r dr' (f + g)^2 \right)^{\frac{1}{2}} \leq \left( \int_\tilde{c}^r dr' f^2 \right)^{\frac{1}{2}} + \left( \int_\tilde{c}^r dr' g^2 \right)^{\frac{1}{2}},
\]

into Eq. (2.9) we easily get

\[
\|\chi\|_{\tilde{c}} \leq (|c_1| + |c_2|) M + 2M^2 \|l - l_0\| \|\chi\|_{\tilde{c}},
\]
and if \( \tilde{c} \) is chosen large enough so that \( M^2 |l - l_0| < \frac{1}{4} \), then
\[
\| \chi \|_{\tilde{c}} \leq 2 (|c_1| + |c_2|) M,
\]
and since the right-hand side of this inequality is independent of \( r \), then \( \chi \in L^2(\tilde{c}, \infty) \) for all \( \tilde{c} \geq c \) and thus is \( L^2(c, \infty) \). Q.E.D.

Theorem 2.2. Suppose that the only singular point in Eq. (2.7) is \( r = 0 \). If every solution of \( Lx = l_0 x \) is of class \( L^2(0, c) \) for some \( c > 0 \) and some complex number \( l_0 \), then, for every arbitrary complex number \( l \), every solution of \( Lx = lx \) is of class \( L^2(0, c) \).

Proof. The proof is not different from that of the previous theorem. In this case Eq. (2.9) still holds but we are interested in the inequality chain \( 0 < r \leq \tilde{c} \leq c \) and this forces a modification for Eq. (2.10) which should be written as
\[
\| \chi \|_{\tilde{c}}^2 = \int_{r}^{\tilde{c}} dr' |\chi|^2, \quad r \leq \tilde{c}.
\]
Then there must exist a constant \( M \) such that \( \| \varphi \|_{\tilde{c}}, \| \psi \|_{\tilde{c}} \leq M \) from which it follows the Schwarz and Minkowski inequality as stated above. Then, one can always chose a small enough \( \tilde{c} \) such that \( M^2 |l - l_0| < \frac{1}{4} \) and thus \( \chi \in L^2(0, \tilde{c}) \) for every \( \tilde{c} \leq c \). Q.E.D.

The above theorems show that in the limit-point case, at most one linearly independent solution of \( Lx = lx \) is of class \( L^2 \) near the singular point, which we have chosen to be \( r = 0 \) and infinity. Now we will show that in the limit-point case there is indeed one and only one square integrable function near the singular point for each \( l \) such that the imaginary part \( \Im(l) \neq 0 \). This proof will be carried out via a very powerful geometrical interpretation of the limit-point and limit-circle cases.

(a) Geometrical interpretation of the limit-point, limit-circle cases at infinity

Suppose \( Lx = lx \) to be defined in \( [c, \infty] \) with \( c > 0 \) and that the only singular point in this interval is the point at infinity. Let \( \varphi \) and \( \psi \) be two independent solutions satisfying
\[
\varphi(c, l) = \sin \alpha, \quad \psi(c, l) = \cos \alpha,
\]
\[
p(c) \varphi'(c, l) = -\cos \alpha, \quad p(c) \psi'(c, l) = \sin \alpha,
\]
where \( \alpha \in [0, \pi] \). Clearly, \( \varphi \) and \( \psi \) are linearly independent solutions. Note that for each \( \alpha \in [0, \pi] \), conditions (2.11) can be always achieved by setting up a rather general Cauchy problem with initial point \( c > 0 \). From general arguments about the existence of solutions for the equation \( Lx = lx \), one can state that \( \varphi, \varphi', \psi, \psi' \) are entire functions of \( l \) and continuous in the variables \( (r, l) \). Obviously, we have \( \varphi(r) = 1 \) and thus \( \varphi(r) = 1 \) for each \( r \). These solutions are real for real \( l \) and satisfy the following mixed boundary conditions in \( c \):
\[
\cos \alpha \varphi(c, l) + \sin \alpha p(c) \varphi'(c, l) = 0,
\]
Every solution to $Lx = lx$ must be of the form 

$$\chi = \varphi + m \psi,$$

with some constant $m$ which depends upon $l$. Now consider the following boundary conditions at $b$ with $c < b < \infty$:

$$\cos \beta x(b) + \sin \beta p(b)x'(b) = 0, \quad \beta \in [0, \pi[) \quad (2.12)$$

One can see that if $\chi$ must satisfy condition (2.12) then it must be

$$m = -\frac{\cot \beta \varphi(b, l) + p(b)\varphi'(b, l)}{\cot \beta \psi(b, l) + p(b)\psi'(b, l)},$$

which is a function of the triplet $(l, b, \beta)$. Since $\varphi, \varphi', \psi, \psi'$ are entire and continuous functions of $(l, r)$, then it follows that $m$ is meromorphic in $l$ and real for real $l$. By setting $z = \cot \beta$, this function becomes

$$m = \frac{A z + B}{C z + D}, \quad (2.13)$$

where the coefficients $A, B, C, D$ are functions of the pair $(l, b)$ and one can easily see what these correspond to. Equation (2.13) is a fractional linear transformation when we freely let $z$ run on $\hat{\mathbb{C}}$. We already know that such kind of transformations are responsible of a one-to-one mapping between circles of the complex plane. Therefore, the $z$ variable runs over the real line when we let $\beta$ vary on its range $[0, \pi[$ and the map (2.13) transforms such a line into a circle $C_b$ (note that the circle is strictly related to the coefficients appearing in Eq. (2.13) and thus to the upper boundary point $b$) on the $m$ complex plane. Thus, $\chi$ satisfies the condition (2.12) if and only if $m$ lies on the circle $C_b$.

The derivation of the equation for such a circle is not different from that obtained in Ref. [31], and it is

$$(A + \overline{C}m)(B + Dm) - (A + Cm)(B + \overline{D}m) = 0.) \quad (2.14)$$

One can show that the centre and the radius for $C_b$ must respectively be

$$\bar{m}_b = \frac{AD - BC}{CD - \overline{CD}},$$

$$r_b = \frac{|AD - BC|}{|CD - \overline{CD}|}.$$
one can see that Eq. (2.14) can be written as
\[ [\chi \chi](b) = 0, \]  
(2.16)
while
\[ [\varphi \psi](b) = A\overline{D} - B\overline{C}, \]
\[ [\psi \psi](b) = C\overline{D} - \overline{C}D, \]
\[ [\varphi \bar{\psi}](b) = AD - BC = 1, \]
thus
\[ \tilde{m}_b = \frac{[\varphi \psi](b)}{[\psi \psi](b)}, \quad r_b = \frac{1}{[\psi \psi](b)}. \]  
(2.17)
Since the coefficient of \( m \overline{m} \) in Eq. (2.14) is \( [\psi \psi](b) \), it follows that the interior of \( C_b \) is given by
\[ \frac{[\chi \chi](b)}{[\psi \psi](b)} < 0. \]  
(2.18)
Now, by using the following Green’s formula:
\[ \int_c^b dr (\bar{\psi}L\psi - \psi L\bar{\psi}) = (l - \bar{l}) \int_c^b dr \psi \bar{\psi} = [\psi \psi](b) - [\psi \psi](c), \]
and recalling that \( [\psi \psi](c) = p(c)(\psi(c)\bar{\psi}(c) - \psi'(c)\bar{\psi}(c)) \), from Eqs. (2.11) one obtains
\[ [\psi \psi](b) = 2i\Im(l) \int_c^b dr' |\psi|^2, \]  
(2.19)
as well as
\[ [\chi \chi](b) = 2i\Im(l) \int_c^b dr' |\chi|^2 + [\chi \chi](c), \]  
(2.20)
and since \( [\chi \chi](c) = -2i\Im(m) \), Eq. (2.18) becomes
\[ \int_c^b dr' |\chi|^2 < \frac{\Im(m)}{2|l|}, \quad \Im(l) \neq 0. \]  
(2.21)
Hence, all interior points of \( C_b \) are defined by the previous equation while all points on the circle \( C_b \) satisfy the equality sign in place of the inequality sign into Eq. (2.21). The radius is thus
\[ r_b = \left( 2|\Im(l)| \int_c^b dr' |\psi|^2 \right)^{-1}. \]  
(2.22)
If one chooses some other upper end \( \tilde{b} < b \), Eq. (2.14) defines another circle \( C_{\tilde{b}} \) whose radius is larger than that of \( C_b \). One can ask how \( C_b \) and \( C_{\tilde{b}} \) are related one to the other and it follows from Eq. (2.21) that
\[
\int_c^{\tilde{b}} |r'|^2 < \int_c^{b} |r'|^2 \leq \frac{\Im(m)}{\Im(l)},
\]
thus all points of \( C_b \) are contained in the interior of \( C_{\tilde{b}} \). This means that while increasing the upper end \( b \) taking the limit \( b \to \infty \), this process lets the circles converge to a limit point \( m_\infty \) or to a limit circle \( C_\infty \). In the former case, the radius of the circle \( C_b \) must converge to zero, i.e. \( r_b \to 0 \) and thus
\[
\lim_{b \to \infty} \int_c^b |r'|^2 = \infty,
\]
and the function \( \psi \) does not belong to \( L^2(c, \infty) \), i.e. not all solutions of the equation \( Lx = lx \) are square summable in the neighbourhood of infinity and this coincides with the limit-point case previously defined. But from Eq. (2.21) one sees that
\[
\int_c^b |r'|^2 < \frac{\Im(m_\infty)}{\Im(l)},
\]
where \( m_\infty \) is the limit point. Therefore, by letting \( b \) approach infinity in the previous equation, one deduces that \( \chi \in L^2(c, \infty) \), and from the fact that \( \psi \) is not square summable we obtain that there is one and only one independent solution which is square summable near infinity, as we have already mentioned above.

In the latter case, the radius \( r_b \) approaches a limit \( r_\infty > 0 \) and this implies that \( \psi \in L^2(c, \infty) \). If \( \tilde{m}_\infty \) is any point on the limit circle \( C_\infty \), it obviously gives rise to the following equation:
\[
\int_c^b |r'|^2 < \frac{\Im(\tilde{m}_\infty)}{\Im(l)},
\]
and by taking the limit \( b \to \infty \), one deduces that besides \( \psi \), also \( \chi \) is square summable near infinity thus every solution is \( L^2(c, \infty) \) and this coincides with the limit-circle case previously defined. In this case \( m \) lies on \( C_\infty \) if and only if
\[
\Im(l) \int_c^\infty |r'|^2 = \Im(m),
\]
and since \( |\chi\chi| (c) = -2i\Im(m) \), from Eqs. (2.20) and (2.21) we deduce that \( m \) is on \( C_\infty \) if and only if \( |\chi\chi| (\infty) = 0 \). We have thus proved the following theorem:

**Theorem 2.3.** Let \( \Im(l) \neq 0 \) and \( \varphi, \psi \) be linearly independent solutions of \( Lx = lx \), where the equation is defined on \([c, \infty]\) with \( c > 0 \) and have its only singular point at infinity. Suppose that these solutions satisfy Eq. (2.11), then the solution \( \chi = \varphi + m\psi \) satisfies the real boundary condition (2.12) if and only if \( m \) lies on the circle \( C_b \) in the complex plane whose equation is \( |\chi\chi| (b) = 0 \). As \( b \to \infty \) either \( C_b \to C_\infty \), a limit circle, or \( C_b \to m_\infty \), a limit point. All solutions of \( Lx = lx \) are \( L^2(c, \infty) \) in...
the former case, and if \( \Im(l) \neq 0 \), there is exactly one linearly independent solution which is \( L^2(c, \infty) \) in the latter case. Moreover, in the limit-circle case, a point is on the limit circle \( C_\infty(l) \) if and only if \( [\chi(\infty)] = 0 \).

At this stage of the theory it is often convenient to state some criterion \([27,24]\) for the establishment of the limit-point or limit-circle case in such a way that one can always obtain limiting properties by simply looking at the coefficients of the second order differential operator. Our aim is quite distinct here: we do not want to recover the maximal amount of information about limit-circle, limit-point properties for some special kind of operators, but we want to establish the fundamental fact that the requirement of self-adjointness for Eq. (2.7) is always accompanied by a pictorial geometrical interpretation which could suggest some investigation paths. This is the motivation for treating explicitly the geometrical interpretation of limit-point, limit-circle cases at \( r = 0 \), which, as we will see, is carried out with a slight modification of some equations derived for the geometrical interpretation at infinity.

(b) Geometrical interpretation of the limit-point, limit-circle cases at the origin.

Suppose \( Lx = lx \) to be defined in \( ]0, c[ \) with \( c > 0 \) and that the only singular point in this interval is at \( r = 0 \). Let \( \varphi \) and \( \psi \) be two independent solutions of the equation. We generally want that such solutions coincide with that assumed for the case (a) from the fact that operators of type (2.6) are defined on the entire positive real line and thus they often possess singular behaviours at \( r = 0 \) and infinity, as we have already mentioned. Therefore we require that Eqs. (2.11) should hold also for \( \varphi \) and \( \psi \) here introduced. We thus have

\[
\begin{align*}
\cos \alpha \varphi(c,l) + \sin \alpha p(c) \varphi'(c,l) &= 0, \\
\sin \alpha \psi(c,l) - \cos \alpha p(c) \psi'(c,l) &= 0,
\end{align*}
\]

as in the case (a). Every solution of the equation must be of the form \( \chi = \varphi + m' \psi \).

The boundary condition (2.11) must be modified by considering a point \( a \) for which \( 0 < a < c \). It is replaced by

\[
\begin{align*}
\cos \beta' x(a) + \sin \beta' \ p(a) x'(a) &= 0, \quad \beta' \in [0, \pi[,
\end{align*}
\]

and if we require \( \chi \) to satisfy Eq. (2.24), we must have

\[
m' = -\frac{\cot \beta' \varphi(a,l) + p(a) \varphi'(a,l)}{\cot \beta' \psi(a,l) + p(a) \psi'(a,l)},
\]

and it is a function of the triplet \((l, a, \beta')\). Since \( \varphi, \varphi', \psi, \psi' \) are entire and continuous functions of \((l, r)\), then it follows that \( m' \) is also meromorphic in \( l \) and real for real \( l \). By setting \( z' = \cot \beta' \) we obtain the analogous equation of Eq. (2.13) which is

\[
m' = -\frac{A'z' + B'}{C'z' + D'},
\]

(2.25)
and this last equation maps the real line into a circle $C'_a$ on the $m'$ plane as before. The equation for $C'_a$ is analogous to Eq. (2.14) by letting all quantities be primed. The center and the radius are

$$m'_a = \frac{A'D' - B'C'}{C'D' - C'D'},$$

$$r_a = \frac{|A'D' - B'C'|}{|C'D' - C'D'|},$$

where

$$A' = \varphi(a, l), \quad B' = p(a) \varphi'(a, l),$$

$$C' = \psi(a, l), \quad D' = p(a) \psi'(a, l).$$

(2.26)

from which

$$[\varphi \psi](a) = A'D' - B'C',$$

$$[\psi \psi](a) = C'D' - C'D',$$

$$[\varphi \psi](a) = A'D' - B'C' = 1.$$ 

The equation for $C'_a$ is equally given by the concise form $[\chi \chi](a) = 0$ while the centre and the radius can be written as

$$m'_a = \frac{[\varphi \psi](a)}{[\psi \psi](a)}, \quad r_a = \frac{1}{|[\psi \psi](a)|}.$$

The interior of $C'_a$ is obtained by a modification of Eq. (2.18) as

$$[\chi \chi](a) < 0,$$ 

(2.27)

and from the Green’s formula we obtain a modification of Eq. (2.19) which in this case must be written as

$$[\psi \psi](a) = -2i \mathfrak{I}(l) \int_a^c dr' |\psi|^2,$$ 

(2.28)

by making use of the Green’s formula jointly with Eq. (2.11), as below:

$$\int_a^c dr(\bar{\psi} L \psi - \psi L \bar{\psi}) = (l - \bar{l}) \int_c^b dr \psi \bar{\psi} = [\psi \psi](c) - [\psi \psi](a),$$

while Eq. (2.20) is replaced by

$$[\chi \chi](a) = -2i \mathfrak{I}(l) \int_a^c dr' |\chi|^2 + [\chi \chi](c),$$ 

(2.29)

by also making use of the Green’s formula

$$\int_a^c dr(\bar{\chi} L \chi - \chi L \bar{\chi}) = (l - \bar{l}) \int_c^b dr \chi \bar{\chi} = [\chi \chi](c) - [\chi \chi](a).$$
In this way, Eq. (2.27) must be written for all \( l \) such that \( \Im(l) \neq 0 \), as
\[
\int_a^c dr' |\chi|^2 < -\frac{\Im(m')}{\Im(l)}, \quad \Im(l) \neq 0,
\]
and this last equation defines all the interior points of \( C'_a \). The equality sign defines all points that lie on \( C'_a \). One can see that Eq. (2.30) is very different from Eq. (2.21) because here \( \Im(m') \) is required to be opposite in sign to \( \Im(l) \) while in Eq. (2.21) the same sign is required. Thus, if we fix the complex number \( l \), then the functions \( m(l, b, \beta) \) and \( m'(l, a, \beta') \) must lie on opposite complex half-planes and thus have opposite sign for their imaginary part. As before, the radius of \( C'_a \) is
\[
a = \left( 2|\Im(l)| \int_a^c dr' |\psi|^2 \right)^{-1},
\]
which can approach a finite limit or tend to zero in the limiting procedure \( a \to 0 \). However, if \( \tilde{a} \) is such that \( 0 < a < \tilde{a} \), then it defines a circle \( C'_{\tilde{a}} \) and from the fact that
\[
\int_{\tilde{a}}^c dr' |\chi|^2 < \int_a^c dr' |\chi|^2 < -\frac{\Im(m')}{\Im(l)}
\]
we deduce that \( C'_{\tilde{a}} \) contains \( C'_a \). As it happened in the case (a) for the geometrical interpretation at infinity, we distinguish the limit-point case when the radius (2.31) approaches zero as \( a \to 0 \), from the limit-circle case when Eq. (2.31) approaches a finite value. In the limit-point case, \( \psi \) is not square summable near the origin, while it happens that
\[
\int_0^c dr' |\chi|^2 < -\frac{\Im(m'_{\infty})}{\Im(l)}
\]
where \( m'_{\infty} \) is the limit point. Thus there is one and only one solution which is square summable near the origin. In the limit-circle case, \( \psi \) is square summable near the origin and if we choose a point \( \hat{m}'_{\infty} \) lying on the limit circle \( C'_{\infty} \), then
\[
\int_0^c dr' |\chi|^2 < -\frac{\Im(\hat{m}'_{\infty})}{\Im(l)}
\]
and we have that all solutions to the equation \( L_x = lx \) are \( L^2(0, c) \).

We have thus proved the analogous of Theorem 2.3, i.e.

**Theorem 2.4.** Let \( \Im(l) \neq 0 \) and \( \varphi, \psi \) be linearly independent solutions of \( L_x = lx \), where the equation is defined in \( [0, c] \) with \( c > 0 \) and have its only singular point at \( r = 0 \). Suppose that these solutions satisfy Eq. (2.11), then the solution \( \chi = \varphi + m\psi \) satisfies the real boundary condition (2.24) if and only if \( m \) lies on the circle \( C'_a \) in the complex plane whose equation is \( [\chi\chi](a) = 0 \). As \( a \to 0 \) either \( C'_a \to C'_{\infty} \), a limit circle, or \( C'_a \to m'_{\infty} \), a limit point. All solutions of \( L_x = lx \) are \( L^2(0, c) \) in the former case, and if \( \Im(l) \neq 0 \), there is exactly one linearly independent solution which is \( L^2(0, c) \) in the latter case. Moreover, in the limit-circle case, a point is on the limit circle \( C'_{\infty} \) if and only if \( [\chi\chi](\infty) = 0 \).
Now that we have provided the geometrical interpretation of the limit-point, limit-circle theory, we revert to another important question which can be answered by this theory, that is the self-adjointness of the operator (2.7) in the case it is singular at both ends of the real positive line.

2.3. Singular Behavior at Both Ends of the Interval

Here we consider the interval \((0, \infty)\) and suppose that the coefficients of the operator \(L\) have a singular behaviour at \(r = 0\). Thus we are treating singular behaviour at both ends of the positive real line. We suppose that \(p(r) > 0\) on such a semi-infinite interval and that \(p, p', q\) are real and continuous on \(\mathbb{R}^+\) (these conditions can be relaxed somewhat). Let \(c > 0\) and let \(\varphi_1, \varphi_2\) be two solutions to \(Lx = lx\), real for real \(l\), and satisfying the following conditions at \(c\):

\[
\varphi_1(c, l) = 1, \quad \varphi_2(c, l) = 0,
\]

\[
p(c) \varphi'_1(c, l) = 0, \quad p(c) \varphi'_2(c, l) = 1,
\]

then \(\varphi_1\) and \(\varphi_2\) form a fundamental system of solutions for the equation and they are also entire functions of \(l\) for fixed \(r\). Let \(\delta = [a, b] \subset \mathbb{R}^+\) be a finite interval containing \(c\) and consider the following self-adjoint problem:

\[
\begin{align*}
Lx &= lx, \\
\cos \beta x(a) + \sin \beta p(a)x'(a) &= 0, \\
\cos \beta x(b) + \sin \beta p(b)x'(b) &= 0,
\end{align*}
\]

with \(\beta, \beta' \in [0, \pi]\). Then there exists a countable sequence of eigenvalues \(\{\lambda_n(\delta)\}, n = 1, 2, \ldots\), and a complete set of orthonormal eigenfunctions \(\{h_n(\delta)\}\) in \(L^2(\delta)\). If there is some degeneracy for any of the eigenvalues, we indicate all the eigenfunctions belonging to the eigenspace under consideration, by substituting the index \(n\) with \(n_m, m = 1, 2, \ldots\) when it is necessary. When it is not specified, all summations over \(n\) are meant to be summations over the complete set of eigenfunctions regardless of the order which can be established between eigenvectors belonging to the same eigenspace. In this way the Parseval equality can be written down as

\[
\int_\delta dr|f(r)|^2 = \sum_{n=1}^{\infty} \left| \int_\delta dr f(r) \overline{h_n(\delta)} \right|^2,
\]

while the Hilbert product in \(L^2(0, \infty)\) between \(f_1\) and \(f_2\) is given by

\[
\int_\delta dr f_1(r) \overline{f_2(r)} = \sum_{n=1}^{\infty} \int_\delta dr f_1(r) \overline{h_n(\delta)} \int_\delta dr f_2(r) \overline{h_n(\delta)}.
\]

But \(\varphi_1\) and \(\varphi_2\) form a fundamental system, thus

\[
h_n(\delta)(r) = t_{n, 1}^{(\delta)} \varphi_1\left(r, \lambda_n(\delta)\right) + t_{n, 2}^{(\delta)} \varphi_2\left(r, \lambda_n(\delta)\right),
\]
where \( t_{n,j}^{(\delta)} \) for every \( n = 1, 2, \ldots \), and \( j = 1, 2 \), are complex constants. By inserting Eq. (2.35) into Eq. (2.33) we can write this last equation as

\[
\int_{\delta} dr |f(r)|^2 = \sum_{n=1}^{\infty} \sum_{j,k=1}^{2} \int_{\delta} dr f(r) \bar{t}_{n,j}^{(\delta)} \varphi_j \left( r, \lambda_n^{(\delta)} \right) \int_{\delta} dr f(r) t_{n,k}^{(\delta)} \varphi_k \left( r, \lambda_n^{(\delta)} \right),
\]

(where we have used the fact that \( \varphi_j \) are real functions) then we can set

\[
g_j^{(\delta)} (\lambda) = \int_{\delta} dr f(r) \varphi_j (r, \lambda),
\]

\[
\rho_{jk}^{(\delta)} (\lambda) = \begin{cases} 
0 & \text{for } \lambda = 0 \\
\sum_m \bar{t}_{m,j}^{(\delta)} t_{m,k}^{(\delta)} + \rho_{jk}^{(\delta)} (\lambda_{n-1}^{(\delta)}) & \text{for } \lambda \in [\lambda_n^{(\delta)}, \lambda_{n+1}^{(\delta)}]
\end{cases}
\]

where the summation over \( m \) in Eq. (2.37) stands for a summation over all indices such that \( \lambda_n^{(\delta)} = \lambda_{m}^{(\delta)} \) when some degeneracy may occur. In terms of Eqs. (2.36) and (2.37), Eq. (2.33) can be written as

\[
\int_{\delta} dr |f(r)|^2 = \int_{-\infty}^{\infty} d\lambda \rho_{jk}^{(\delta)} (\lambda) \sum_{j,k=1}^{2} \bar{g}_j^{(\delta)} (\lambda) g_k^{(\delta)} (\lambda).
\]

The matrix \( \rho_{jk}^{(\delta)} (\lambda) \) is called spectral matrix associated to the self-adjoint problem (2.32) and it satisfies the following three requirements:

(i) It is Hermitian, i.e. \( \rho_{jk}^{(\delta)} (\lambda) = \bar{\rho}_{kj}^{(\delta)} (\lambda) \).

(ii) \( \rho^{(\delta)} (\Delta) = \rho^{(\delta)} (\lambda) - \rho^{(\delta)} (\mu) \) is positive semidefinite if \( \lambda > \mu \), where \( \Delta = |\mu, \lambda| \).

(iii) The total variation of \( \rho_{jk}^{(\delta)} (\lambda) \) is finite on every finite \( \lambda \) interval.

Any matrix satisfying (ii) is said to be nondecreasing.

By applying the Parseval equality (2.38) to any continuous function on \( \mathbb{R}^+ \) which vanishes outside some interval \( \delta_1 \subset \delta \), one obtains the same Eq. (2.38) but instead of the function \( \bar{g}_j^{(\delta)} \) defined in Eq. (2.36) it is more convenient to use

\[
g_j^{(\delta)} = \int_{-\infty}^{\infty} dr f(r) \varphi_j (r, \lambda).
\]

We can now show that if Eq. (2.7) is in the limit-point case both at \( r = 0 \) and infinity, there exists an unique matrix \( \rho \) satisfying the properties (i), (ii) and (iii) such that \( \rho^{(\delta)} \to \rho \) when the limit \( \delta \to (0, \infty) \) is taken. Then, for every \( f \in L^2(0, \infty) \), Eq. (2.38) holds with \( \rho \) in place of \( \rho^{(\delta)} \) and Eq. (2.39) in place of Eq. (2.36). If one of the ends is in the limit-circle case, then the limiting spectral matrix still exists but the uniqueness is not guaranteed.

The existence of a limiting spectral matrix.
The key for proving the existence of this limiting spectral matrix, resides in the possibility of showing that the integral
\[
\int_{-\infty}^{\infty} \frac{dp_{jk}^{(\delta)}(\lambda)}{|\lambda - l|},
\]
is uniformly convergent when one takes the limit $\delta \to \mathbb{R}^+$. This is sufficient for proving the existence. In doing this, we must construct this particular type of integral and this will be our effort for the next few pages.

Let $\chi_a = \varphi_1 + m'_a \varphi_2$ be a solution of the equation $Lx = lx$ which satisfies
\[
\cos \beta' x(a) + \sin \beta' p(a)x'(a) = 0,
\]
and $\chi_b = \varphi_1 + m_b \varphi_2$ another solution which satisfies
\[
\cos \beta x(b) + \sin \beta p(b)x'(b) = 0,
\]
then $m_b$ and $m'_a$ lie on the circles $C_b$ and $C'_a$, respectively, of equations
\[
[\chi_a \chi_b](b) = 0, \quad [\chi_a \chi_a](a) = 0.
\]
The Green’s function for the problem (2.32) exists and it can be easily calculated, provided $\exists (l) \neq 0$, and is given by
\[
G^{(\delta)}(r, \varrho, l) = \begin{cases} 
\frac{\chi_a(r, l)\chi_b(p, l)}{m'_a(l) - m_b(l)} & 0 < \varrho \\
\frac{\chi_a(p, l)\chi_b(r, l)}{m'_a(l) - m_b(l)} & \varrho > \varrho
\end{cases}
\]
We now want to apply the completeness relation (2.34) to the following functions:
\[
f_1(r) = \frac{\partial^s G^{(\delta)}}{\partial \varrho^s}(r, c, l), \quad f_2(r) = \frac{\partial^p G^{(\delta)}}{\partial \varrho^p}(r, c, l), \quad (s, p = 0, 1).
\]
Take the above definition of the Green’s function and calculate it at $\varrho = c$. We have
\[
G^{(\delta)}(r, c, l) = \begin{cases} 
\frac{\chi_a(r, l)}{m'_a(l) - m_b(l)} & 0 < r \\
\frac{\chi_b(r, l)}{m'_a(l) - m_b(l)} & r > c
\end{cases}
\]
and one can also easily compute the first derivative of the Green function with respect to the $\varrho$ variable and calculate it for $\varrho = c$
\[
\frac{\partial G^{(\delta)}}{\partial \varrho}(r, c, l) = \begin{cases} 
\frac{m_b(l)\chi_a(r, l)}{l c(m'_a(l) - m_b(l))} & 0 < r \\
\frac{m_b(l)\chi_b(r, l)}{l c(m'_a(l) - m_b(l))} & r > c
\end{cases}
\]
In order to derive the above functions, the previously stated conditions $\varphi_1(c, l) = 1$, $p(c) \varphi'_1(c, l) = 0$, $\varphi_2(c, l) = 0$ and $p(c) \varphi'_2(c, l) = 1$ have been implicitly used.

For example, we can evaluate one of the required integrals underlying the completeness relations we are looking for. Take $f_1 = f_2 = G^{(\delta)}(r, c, l)$ in Eq. (2.40) and apply to them the completeness relation (2.34)
\[
2i \mathcal{I}(l) \int_{\delta} dr |G^{(\delta)}(r, c, l)|^2 = 2i \mathcal{I}(l) |m'_a(l) - m_b(l)|^{-2} \left\{ \int_{\delta} dr |\chi_a(r, l)|^2 \right\}.
\]
\begin{align*}
&\quad + \int_c^b dr |\chi_b (r, l)|^2 \bigg) \\
&= |m'_a (l) - m_b (l)|^{-2} \{[\chi_a \chi_a] (c) - [\chi_b \chi_b] (c)\} \\
&= 2i \Im (m'_a (l) - m_b (l)) |m'_a (l) - m_b (l)|^{-2} ,
\end{align*}
where we have used the Green’s formula jointly with the equations $[\chi_a \chi_a] (a) = 0$ and $[\chi_b \chi_b] (b) = 0$ for the circles $C'_a$ and $C_b$, respectively. Thus
\begin{equation}
\int_b \delta |G^{(\delta)} (r, c, l)|^2 = \frac{3 (m'_a (l) - m_b (l))^{-1}}{3 (l)}. \tag{2.41}
\end{equation}
Similarly
\begin{align*}
(m'_a (l) - m_b (l)) \left( 1 - \lambda_n^{(\delta)} \right) \int_b \delta |G^{(\delta)} (r, c, l)|\tilde{h}_n^{(\delta)} (r) \\
= \left[ \chi_b h_n^{(\delta)} \right] (b) - \left[ \chi_a h_n^{(\delta)} \right] (c) + \left[ \chi_a h_n^{(\delta)} \right] (c) - \left[ \chi_a h_n^{(\delta)} \right] (a) \\
= (m'_a (l) - m_b (l)) \left[ \varphi_2 h_n^{(\delta)} \right] (c) \\
= (m'_a (l) - m_b (l)) \tilde{t}_n^{(\delta)} ,
\end{align*}
where the Green’s formula has been used for the passage from the first to the second line, while the relation $\left[ \chi_b h_n^{(\delta)} \right] (b) = \left[ \chi_a h_n^{(\delta)} \right] (a) = 0$ has been used in the passage from the second to the third line which follows from the fact that $\chi_b$ and $h_n^{(\delta)}$ satisfy the same boundary condition at $b$ and the same holds for $\chi_a$ and $h_n^{(\delta)}$ in $a$. The passage from the third to the fourth line follows from Eq. (2.35).

One thus obtains
\begin{equation}
\int_b \delta |G^{(\delta)} (r, c, l)|\tilde{h}_n^{(\delta)} (r) = \frac{\tilde{t}_n^{(\delta)}}{l - \lambda_n^{(\delta)}}. \tag{2.42}
\end{equation}
From Eq. (2.37) one can show by using the standard theory of generalized functions, that
\begin{equation}
d\rho_{jk}^{(\delta)} (\lambda) = \sum_{n=1}^{\infty} d\lambda \ \delta \left( \lambda - \lambda_n^{(\delta)} \right) \sum_m \tilde{t}_{nm,j} \tilde{t}_{nm,k}^{(\delta)}, \tag{2.43}
\end{equation}
(where the sum over the $m$ index takes into account the degeneracy of the eigenvalue) hence, if we divide Eq. (2.43) by $|\lambda - l|^2$ and integrate over the $\lambda$ variable in the range $(-\infty, \infty)$ we get
\begin{equation}
\int_{-\infty}^{\infty} \frac{d\rho_{jk}^{(\delta)} (\lambda)}{|\lambda - l|^2} = \sum_{n=1}^{\infty} \sum_m \frac{\tilde{t}_{nm,j} \tilde{t}_{nm,k}^{(\delta)}}{|\lambda_n^{(\delta)} - l|^2}.
\end{equation}
Now, recalling that
\[
\int_\delta dr |G^{(\delta)}(r, c, l)|^2 = \sum_{n=1}^{\infty} \sum_m \int_\delta dr G^{(\delta)}(r, c, l) \overline{h}_{nm}^{(\delta)}(r) \int_\delta dr G^{(\delta)}(r, c, l) \overline{h}_{nm}^{(\delta)}(r),
\]
and by using Eqs. (2.41) and (2.42) one can deduce that
\[
\int_{-\infty}^{\infty} \frac{d\rho_{11}^{(\delta)}(\lambda)}{|\lambda - t|} = \frac{\mathcal{F}(M_{11}^{(\delta)})}{\mathcal{F}(t)}
\]
with \(M_{11}^{(\delta)} = (m_a'(l) - m_b'(l))^{-1}\). At this stage, if one carries all calculations for completeness by using Eq. (2.34) between the other functions in Eqs. (2.40) by setting \(s \neq p\), one finds that
\[
\int_{-\infty}^{\infty} \frac{d\rho_{jk}^{(\delta)}(\lambda)}{|\lambda - t|^2} = \frac{\mathcal{F}(M_{jk}^{(\delta)})}{\mathcal{F}(t)},
\]
where
\[
M_{11}^{(\delta)} = \frac{1}{m_a'(l) - m_b'(l)}, \quad M_{12}^{(\delta)} = M_{21}^{(\delta)} = \frac{1}{2} \frac{m_a'(l) + m_b'(l)}{m_a'(l) - m_b'(l)}, \quad M_{22}^{(\delta)} = \frac{m_a'(l)m_b(l)}{m_a'(l) - m_b(l)}.
\]

Now we have to show the existence of the limiting spectral matrix by applying some fundamental theorems such as the Helly selection theorem and one integration theorem, to Eq. (2.44). But first, let us recall that \(m_b(l)\) and \(m_a'(l)\) must lie on opposite half-planes from Eqs. (2.21) and (2.30). Suppose \(l = i\). Then, the points \(m_a'(i)\) must lie in \(C_1\) for \(a < 1\), whereas the points \(m_b(i)\) must lie on \(C_1\) which is in \(C_1\) for \(b > 1\). Thus, there exists a positive constant \(u\) such that \(|m_a'(i) - m_b(i)| > u\) for \(a < 1\) and \(b > 1\). But since \(m_b(l)\) and \(m_a'(l)\) are uniformly bounded for \(a < 1\) and \(b > 1\), it follows from Eq. (2.44) that
\[
\int_{-\infty}^{\infty} \frac{d\rho_{jj}^{(\delta)}(\lambda)}{1 + \lambda^2} = \mathcal{F}(M_{jj}^{(\delta)}) < K, \quad j = 1, 2,
\]
for some constant \(K\). By looking at the differential (2.43), one can easily see that the products of the type \(l_{n,m,j}^{(\delta)}l_{n,m,k}^{(\delta)}\) must be absolutely bounded from the law of cosines which yields
\[
2 \left| l_{n,m,j}^{(\delta)}l_{n,m,k}^{(\delta)} \right| \leq \left| l_{n,m,j}^{(\delta)} \right|^2 + \left| l_{n,m,k}^{(\delta)} \right|^2,
\]
and from which it follows:
\[
\int_{-\infty}^{\infty} \frac{|d\rho_{jk}^{(\delta)}(\lambda)|}{\lambda^2 + 1} < K, \quad (2.46)
\]
that holds also for \(j \neq k\). Now we introduce the following theorems which are the Helly selection theorem and a particular integration theorem, respectively [29].
Theorem 2.5. Let \( \{h_n\}, n = 1, 2, \ldots, \) be a sequence of real nondecreasing functions on \( \lambda \in \mathbb{R}, \) and let \( H \) be a continuous nonnegative function on the same interval. If
\[
|h_n(\lambda)| \leq H(\lambda), \; n = 1, 2, \ldots, \lambda \in \mathbb{R},
\]
then there exists a subsequence \( \{h_{n_k}\} \) and a nondecreasing function \( h \) such that
\[
|h(\lambda)| \leq H(\lambda), \; \lambda \in \mathbb{R},
\]
and
\[
\lim_{k \to \infty} h_{n_k}(\lambda) = h(\lambda).
\]

Theorem 2.6. Suppose \( \{h_n\} \) is a real, uniformly bounded, sequence of nondecreasing functions on a finite interval \( \lambda \in [a, c] \), and assume
\[
\lim_{n \to \infty} h_n(\lambda) = h(\lambda), \; \lambda \in [a, c].
\]
If \( f \) is any continuous function on \( \lambda \in [a, c] \), then
\[
\lim_{n \to \infty} \int_a^c dh_n(\lambda)f(\lambda) = \int_a^c dh(\lambda)f(\lambda).
\]

Consider now Eq. (2.46). Therefore, if we take any \( \nu > 0 \) there must be
\[
\int_{-\nu}^{\nu} \left| d\rho^{(\delta)}_{jk}(\lambda) \right| < K (1 + \nu^2),
\]
and this, together with the condition \( \rho^{(\delta)}_{jk}(0) = 0 \) in Eq. (2.37) gives
\[
\left| \rho^{(\delta)}_{jk}(\lambda) \right| \leq K (1 + \lambda^2).
\]
Now, if we apply Theorem 2.5 by choosing a sequence of intervals \( \delta_n = [a_n, b_n] \) such that \( \delta_n \to \mathbb{R}^+ \), there remains defined a sequence of real nondecreasing functions \( \rho^{(\delta_n)}_{jk}(\lambda) \) (the reality and nondecreasing behaviour follows from property (ii) for spectral matrices and Eq. (2.37)) for which there exists a subsequence converging to a limit function \( \rho_{jk}(\lambda) \) which is monotone, nondecreasing and satisfies
\[
|\rho_{jk}(\lambda)| \leq K (1 + \lambda^2),
\]
that is a spectral matrix for which properties (i), (ii) and (iii) hold. It is also possible to show that the Parseval equality (2.38) still holds with \( \rho_{jk}(\lambda) \) in place of \( \rho^{(\delta_n)}_{jk}(\lambda) \) for every \( f \in \mathcal{L}^2 (0, \infty) \) by an application of the integration theorem 2.6.

When the existence of the limiting spectral matrix is proved, then the following theorem also holds [29]:

Theorem 2.7. Let \( \rho \) be any limiting matrix of the set \( \{\rho^{(\delta)}\} \). If \( f \in \mathcal{L}^2 (0, \infty) \), the vector \( g = (g_1, g_2) \), where
\[
g_j(\lambda) = \int_{-\infty}^{\infty} dr f(r) \varphi_j(r, \lambda),
\]
and \( \varphi_j \) with \( j = 1, 2 \) form a fundamental system of solutions for the equation \( Lx = lx \) satisfying the conditions

\[
\varphi_1 (c, l) = 1, \quad \varphi_2 (c, l) = 0,
\]
\[
p (c) \varphi'_1 (c, l) = 0, \quad p (c) \varphi'_2 (c, l),
\]

for some \( c > 0 \), then \( g_j (\lambda) \) converges in \( L^2 (\rho) \), i.e. in the Hilbert space of all square summable functions on the measure space whose measure is given by \( \rho \), that is, there exists a \( g \in L^2 (\rho) \) such that

\[
\| g - g_{cd} \| \rightarrow 0 \text{ for } c \rightarrow 0 \text{ and } d \rightarrow \infty,
\]

where

\[
g_{cd,j} (\lambda) = \int_c^d dr f (r) \varphi_j (r, \lambda), \quad c < d.
\]

In terms of this \( g \), the Parseval equality

\[
\int_{-\infty}^{\infty} dr |f(r)|^2 = \int_{-\infty}^{\infty} d\rho_{jk} (\lambda) \sum_{j,k=1}^{2} \bar{g}_j (\lambda) g_k (\lambda),
\]

and the expansion

\[
f (r) = \int_{-\infty}^{\infty} d\rho_{jk} (\lambda) \sum_{j,k=1}^{2} \varphi_j (r, \lambda) g_k (\lambda),
\]

are valid, the latter integral converges in the \( L^2 (0, \infty) \) norm.

**Uniqueness of the limiting spectral matrix**

The uniqueness for the spectral matrix relies on the existence of the following limit for every pair of continuity points \( \lambda, \mu \in \mathbb{R}^+ \) for \( \rho_{jk} \):

\[
\rho_{jk} (\lambda) - \rho_{jk} (\mu) = \lim_{\delta \rightarrow \mathbb{R}^+} \left( \rho_{jk}^{(\delta \mu)} (\lambda) - \rho_{jk}^{(\delta \mu)} (\mu) \right), \quad j, k = 1, 2,
\]

Now we will show that this limit exists if Eq. (2.7) is in limit-point case at both ends of the real positive line. This is possible if we show that

\[
\rho_{jk} (\lambda) - \rho_{jk} (\mu) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi} \int_{\mu}^{\lambda} d\nu \Im (M_{jk} (\nu + i\epsilon)),
\]

(2.47)

because the limit-point case at both ends guarantees the uniqueness of the limits

\[
M_{jk}^{(\delta)} \rightarrow M_{jk}, \quad \text{for } \delta \rightarrow \mathbb{R}^+, \quad j, k = 1, 2,
\]

with \( M_{jk}^{(\delta)} \) given in Eqs. (2.45), and thus the existence of the limit (2.47).
In proving Eq. (2.47), let us consider Eq. (2.44). For any fixed $l$ with $I(l) \neq 0$, there exists a constant $K$ such that

$$\int_{-\mu}^{\mu} \frac{d\rho_{jk}(\lambda)}{|\lambda - l|^2} \leq K, \quad j, k = 1, 2,$$

for $b > 1$ and $a < 1$. Upon choosing a sequence $\delta_n \to \mathbb{R}^+$ as done before, it follows that the above equation is also true with $\rho_{jk}$ in place of $\rho_{jk}^{(4)}$. Since the above equation holds for every $\mu > 0$, then

$$\int_{-\infty}^{\infty} \frac{d\rho_{jk}(\lambda)}{|\lambda - l|^2} < \infty, \quad j, k = 1, 2.$$

From Eq. (2.46) there exists a constant $\tilde{K}$ such that, for $b > 1$ and $a < 1$, one has

$$\int_{-\infty}^{\infty} \frac{d\rho_{jk}(\lambda)}{\lambda^3} < \frac{\tilde{K}}{\mu}, \quad j, k = 1, 2.$$

This relation similarly holds if the integration is taken over $]-\infty, -\mu[$. If $I(l) \neq 0$ and $I(l_0) \neq 0$ and the equation

$$\int_{-\infty}^{\infty} \frac{d\rho_{jk}(\lambda)}{\lambda^3} \left( \frac{1}{|\lambda - l|^2} \right) \left( \frac{1}{|\lambda - l_0|^2} \right), \quad j, k = 1, 2, \quad (2.48)$$

is considered over the intervals $]-\infty, -\mu[,$ $]-\mu, \mu[,$ and $][\mu, \infty[$, it follows that, if $\delta \to \infty$ through a chosen subsequence and if then $\mu \to \infty$, the integration Theorem 2.6 guarantees that Eq. (2.48) tends to

$$\int_{-\infty}^{\infty} \frac{d\rho_{jk}(\lambda)}{\lambda^3} \left( \frac{1}{|\lambda - l|^2} \right) \left( \frac{1}{|\lambda - l_0|^2} \right), \quad j, k = 1, 2.$$

But if we make use of Eq. (2.44), we can write Eq. (2.48) as

$$\frac{\mathcal{J}(M_{jk}(l))}{\mathcal{J}(l)} - \frac{\mathcal{J}(M_{jk}(l_0))}{\mathcal{J}(l_0)}, \quad j, k = 1, 2,$$

which tends to

$$\frac{\mathcal{J}(M_{jk}(l))}{\mathcal{J}(l)} - \frac{\mathcal{J}(M_{jk}(l_0))}{\mathcal{J}(l_0)}, \quad j, k = 1, 2,$$

where

$$M_{11}(l) = \frac{1}{m'_\infty(l) - m_\infty(l)}, \quad M_{12}(l) = M_{21}(l) = \frac{1}{2} \frac{m'_\infty(l) + m_\infty(l)}{m'_\infty(l) - m_\infty(l)}, \quad M_{22}(l) = \frac{m'_\infty(l) m_\infty(l)}{m'_\infty(l) - m_\infty(l)},$$

and $m'_\infty(l)$, $m_\infty(l)$ are the limit points at the origin and at infinity, respectively. Therefore

$$\frac{\mathcal{J}(M_{jk}(l))}{\mathcal{J}(l)} = \int_{-\infty}^{\infty} \frac{d\rho_{jk}(\lambda)}{|\lambda - l|^2} + c_{jk}, \quad j, k = 1, 2, \quad (2.49)$$
where \( c_{jk} \) are four constants independent of \( l \), provided \( I(l) \neq 0 \). Now, letting \( R(l) = 0 \) and \( I(l) \to \infty \), it readily follows that \( c_{jk} = 0 \). Now, let \( \lambda, \mu \) be points of continuity for \( \rho_{jk} \). Then, from Eq. (2.49) it follows

\[
\lim_{\epsilon \to 0^+} \int_{\lambda}^{\mu} d\nu \mathcal{I} (M_{jk} (\nu + i\epsilon)) = \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} d\nu \frac{c \rho_{jk} (\sigma)}{(\sigma - \nu)^2 + \epsilon^2}
\]

\[
= \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} d\rho_{jk} (\sigma) \left[ \tan^{-1} \left( \frac{\lambda - \sigma}{\epsilon} \right) - \tan^{-1} \left( \frac{\mu - \sigma}{\epsilon} \right) \right]
\]

\[
= \pi (\rho_{jk} (\lambda) - \rho_{jk} (\mu)), \ j, k = 1, 2.
\]

This proves Eq. (2.47) and hence the uniqueness of the limiting spectral matrix in the limit-point case at both ends of the real positive line. Thus, we have proved the following fundamental theorem:

**Theorem 2.8.** Let \( L \) be in the limit point case at \( r = 0 \) and \( r = \infty \). There exists a nondecreasing Hermitian matrix \( \rho = (\rho_{jk}) \) whose elements are of bounded variation on every finite \( \lambda \) interval, and which is essentially unique in the sense that

\[
\rho_{jk} (\lambda) - \rho_{jk} (\mu) = \lim_{\delta_n \to \mathbb{R}^+} \left( \rho_{jk}^{(\delta_n)} (\lambda) - \rho_{jk}^{(\delta_n)} (\mu) \right), \ j, k = 1, 2,
\]

at points of continuity \( \lambda, \mu \) of \( \rho_{jk} \). Furthermore,

\[
\rho_{jk} (\lambda) - \rho_{jk} (\mu) = \frac{1}{\pi} \int_{\mu}^{\lambda} d\nu \mathcal{I} (M_{jk} (\nu + i\epsilon)), \ j, k = 1, 2,
\]

(2.50)

where

\[
M_{11} (l) = \frac{1}{m'(l) - m(l)},
\]

\[
M_{12} (l) = M_{21} (l) = \frac{1}{2} \frac{m'(l) + m(l)}{m'(l) - m(l)},
\]

\[
M_{22} (l) = \frac{m'(l) m(l)}{m'(l) - m(l)}.
\]

(2.51)

The spectrum associated with a problem for which \( \rho \) is uniquely determined, is the set of nonconstancy points of \( \rho \), that is, the set of all nonconstancy points of all elements \( \rho_{jk} \). Since \( \rho \) is Hermitian and nondecreasing, it follows that

\[
|\rho_{jk} (\Delta)|^2 \leq \rho_{jj} (\Delta) \rho_{kk} (\Delta),
\]

where

\[
\rho_{jk} (\Delta) = \rho_{jk} (\lambda) - \rho_{jk} (\mu), \ \Delta = |\mu, \lambda|.
\]

Hence the set of nonconstancy points of all elements of the limiting spectral matrix is the same of all nonconstancy points for its diagonal elements. Clearly the spectrum
is a closed set. The point spectrum is the set of all discontinuity points of $\rho$, and
the continuous spectrum is the set of continuity points of $\rho$. Points in the spectrum
are called eigenvalues and the solutions to the eigenvalue problem for such points
are called eigenfunctions.

It is essential to remark that each physical problem whose Hamiltonian is given
by Eq. (2.6) and for which $r = 0$ and $r = \infty$ are singular points (thus we are
taking aside the case $\lambda_{3,0} = \frac{1}{2}$ into Eq. (2.6) with a nonsingular potential $V$) must
possess one and only one limiting spectral matrix in such a way that every experi-
ment about spherical symmetric quantum particles, would be always reproducible.
This is the core of the predictability of quantum mechanics: the energy levels are
always theoretically established within a margin of error that can be estimated only
through experimental data.

Taking aside the case $\lambda_{3,0} = \frac{1}{2}$, quantum mechanical Hamiltonians of type (2.6)
must be in limit-point case at both $r = 0$ and $r = \infty$ and this leads us to the
possibility to give a geometrical interpretation for such problems. We recall that
this kind of uniqueness for self-adjoint problems we are treating, is often called
in the literature essential self-adjointness. i.e., the closure of the operator is self-adjoint, which implies in turn that the self-adjoint extension exists and is
unique. From the fact that the establishment of an essentially self-adjoint problem,
within the theory we have developed, is carried out by a limiting procedure of
self-adjoint problems

$$\delta = [a, b] \rightarrow \mathbb{R}^+, \quad a \rightarrow 0, \quad b \rightarrow \infty,$$

we can always choose a sequence of intervals $\{\delta_n = [a_n, b_n]\}$ which converges to
the real positive line in the limit $n \rightarrow \infty$ and for which there remain defined two families of circles $\{C'_{a_n}\}$ and $\{C_{b_n}\}$, where $C'_{a_n}$ lies on the $m'_{a_n}(l)$ plane while $C_{b_n}$
lies on the $m_{b_n}(l)$ plane, and which lie on opposite half-planes for each fixed $n$. The
equations for these circles are $[\chi_{a_n} \chi_{a_n}] (a_n) = 0$ for $C'_{a_n}$ and $[\chi_{b_n} \chi_{b_n}] (b_n) = 0$ for
$C_{b_n}$. As we already know, the limit-point cases at both ends (case which must occur
for spherically symmetric quantum mechanical Hamiltonians) are characterized by
the existence of limit points

$$C'_{a_n} \rightarrow m'_{\infty}, \quad C_{b_n} \rightarrow m_{\infty},$$

thus, for each essentially self-adjoint quantum mechanical problem for Hamiltonians
(2.6) with singular behavior at both ends of the positive real line, there remain
defined two families of circles lying on opposite half-planes and such that each
family is formed by circles enclosed one into the other whose radii approach zero.

Singular behavior at infinity only

As the last argument of the limit-point, limit-circle theory we will give some
fundamental results for the case in which Eq. (2.7) has singular behavior only
at infinity. This is the case of three-dimensional $s$-waves, i.e. $\lambda_{3,0} = \frac{1}{2}$ in Eq. (2.6)
with nonsingular potential $V$. The proof for the existence and uniqueness of spectral
Let us consider the following self-adjoint problem:

\[
\begin{align*}
    \cos \alpha x(c) + \sin \alpha p(c)x'(c) &= 0 \\
    \cos \beta x(b) + \sin \beta p(b)x'(b) &= 0
\end{align*}
\]

with \(\alpha, \beta \in [0, \pi]\) and \(0 < c < b\). The problem here is identical to Eq. (2.32) but we will fix throughout the exposition, the value of \(c < b\). Hence, there exists a countable sequence of eigenvalues \(\{\lambda_n^{(b)}\}\), \(n = 1, 2, \ldots\), and a complete set of orthonormal eigenfunctions \(\{h_n^{(b)}\}\) in \(L^2(c, b)\). Let \(\varphi\) and \(\psi\) be two independent solutions to \(Lx = lx\) satisfying conditions (2.11), thus \(\psi\) satisfies the first boundary condition of Eq. (2.52) and no solution independent of \(\psi\) can satisfy this condition. Therefore, each eigenfunction must be of the form

\[
h_n^{(b)} = t_n^{(b)} \psi \left( r, \lambda_n^{(b)} \right), \quad n = 1, 2, \ldots,
\]

with \(t_n^{(b)}\) complex constants independent of \(r\). If \(f\) is any continuous function on \((c, b)\), then the Parseval equality is written as

\[
\int_c^b dr |f(r)|^2 = \sum_{n=1}^{\infty} \left| t_n^{(b)} \right|^2 \left| \int_c^b dr f(r) \psi \left( r, \lambda_n^{(b)} \right) \right|^2.
\]

Let

\[
g(\lambda) = \int_0^\infty dr f(r) \psi(r, \lambda),
\]

and let \(\rho^{(b)}\) be a monotone nondecreasing step function of \(\lambda\) having a jump of \(\left| t_n^{(b)} \right|\) at each eigenvalue \(\lambda_n^{(b)}\)

\[
\rho^{(b)}(\lambda) = \begin{cases} 
0 & \text{for } \lambda = 0 \\
\left| t_n^{(b)} \right|^2 + \rho^{(b)}(\lambda_{n-1}) & \text{for } \lambda \in [\lambda_n^{(b)}, \lambda_{n+1}^{(b)}]
\end{cases}
\]

which is called spectral function for the problem (2.52). Then the Parseval equality should be written as

\[
\int_c^\infty dr |f(r)|^2 = \int_0^\infty d\rho^{(b)}(\lambda) |g(\lambda)|^2.
\]

At this stage, the fundamental idea behind the generalization of Eq. (2.54) to the case of the entire positive real line, is to show the existence of a nondecreasing function \(\rho\) which is the limit \(\rho^{(b)} \to \rho\) when \(b \to \infty\), and such that Eq. (2.54) holds when we replace \(\rho^{(b)}\) with \(\rho\) in it. The following theorem, of which we omit the proof, holds:
Theorem 2.9. Let $L$ be in the limit-point case at infinity. Then there exists a monotone nondecreasing function $\rho(\lambda)$ on $\mathbb{R}$ such that it is unique in the sense of
\[ \rho(\lambda) - \rho(\mu) = \lim_{b \to \infty} \left( \rho^{(b)}(\lambda) - \rho^{(b)}(\mu) \right), \] (2.55)
at the points of continuity $\lambda, \mu$ of $\rho$. Furthermore
\[ \rho(\lambda) - \rho(\mu) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{\mu}^{\lambda} d\nu \, \mathcal{J}(m_{\infty}(\nu + i\epsilon)), \] (2.56)
where $m_{\infty}(l)$ is the limit point at infinity for $l$ fixed.

The proof is similar to that given for singularities at both ends of the positive real line. The uniqueness of the spectral function is established in Eqs. (2.55) and (2.56).

As it happens for singular behaviors at both ends of the positive real line, when we apply these arguments to that class of Hamiltonians among the family (2.6) which have singular behavior only at infinity, then we expect that these should be in limit-point case. The geometrical interpretation is as follows: the limiting procedure $b \to \infty$ gives rise to a family of circles $\{C_b\}$ which are contained one into the other and which lie in one of the two complex half-planes of opposite imaginary part, and their radii approach zero as $b \to \infty$.

2.4. Connection Between Limit-Point, Limit-Circle Theory and BMS Transformations

In this section we are characterizing the linear transformations (2.13), showing which are the basic requirements to be made for their coefficients in order to establish the limit-point cases at infinity. Of course, we must require $r_b \to 0$ as $b \to 0$ where
\[ r_b = \frac{1}{\left| CD - CD \right|}, \] (2.57)
and the coefficients $C, D$ are given in Eqs. (2.15).

We can write Eq. (2.13) in the form
\[ m_b = \frac{\alpha z + \beta}{\gamma z + \delta}, \]
\[ \left\{ \begin{array}{l} \alpha = -A\tau \\ \beta = -B\tau \\ \gamma = \tau C \\ \delta = \tau D \end{array} \right. \]
\[ \tau = \pm i, \] (2.58)
and in this way we ensure that $AD - BC = 1$, thus we can always deal with the fractional linear transformations in terms of $\alpha, \beta, \gamma, \delta$. The trace of this transformation is $j = \alpha + \beta$ and from the classification given in Ref. [31], we know that if $j$ is real and $|j| < 2$ the transformation is elliptic, if $j$ is real and $|j| = 2$ then it is parabolic while if $j$ is real and $|j| > 2$, it is hyperbolic. In the case $j^2 \notin [0, \infty[$,
or equivalently $j$ is not real, the transformation is loxodromic. From Eq. (2.57), we observe that

$$r_b^{-1} = |CD - CD| = |\gamma \delta - \gamma \delta| = 2 |\gamma \delta|,$$

(2.59)

and in the limit-point case at infinity, it must be $|\gamma \delta| \to \infty$ as $b \to \infty$. This implies two possibilities:

(1) The modulus of the $\delta$ variable must tend to infinity;

(2) The modulus of the $\gamma$ variable must tend to infinity.

In the case (1) we observe that the trace $j = \alpha + \delta$ of Eq. (2.58) must diverge, thus, the transformations (2.58) must reduce to hyperbolic or loxodromic as $b$ increases. For what follows, it is helpful to set up the following nomenclature:

$$\alpha(b) = \alpha_1(b) + i\alpha_2(b), \beta(b) = \beta_1(b) + i\beta_2(b),$$

$$\gamma(b) = \gamma_1(b) + i\gamma_2(b), \delta(b) = \delta_1(b) + i\delta_2(b),$$

(2.60)

although the $b$ dependence will be explicitly omitted hereafter.

(1) The $\delta$ variable approaching infinity

We can make use of the equation $\alpha \delta - \beta \gamma = 1$ in Eq. (2.58), and solve it in terms of the $\delta$ variable

$$\delta = \frac{1 + \beta \gamma}{\alpha},$$

(2.61)

from which

$$\delta_1 + i\delta_2 = \frac{1 + (\beta_1 + i\beta_2)(\gamma_1 + i\gamma_2)}{\alpha_1 + i\alpha_2} = \frac{1 + (\beta_1 + i\beta_2)(\gamma_1 + i\gamma_2)}{|\alpha|^2} \frac{(\alpha - i\alpha_2)}{\alpha_1 - i\alpha_2}$$

$$= \frac{\alpha_1 - i\alpha_2 + [\beta_1 \gamma_1 + i\beta_1 \gamma_2 + i\beta_2 \gamma_1 - \beta_2 \gamma_2](\alpha_1 - i\alpha_2)}{|\alpha|^2}$$

and thus

$$\delta_1 = |\alpha|^{-2} [(1 + \beta_1 \gamma_1 - \beta_2 \gamma_2) \alpha_1 + (\beta_1 \gamma_2 + \beta_2 \gamma_1) \alpha_2],$$

$$\delta_2 = |\alpha|^{-2} [(\beta_1 \gamma_2 + \beta_2 \gamma_1) \alpha_1 + (-1 + \beta_2 \gamma_2 - \beta_1 \gamma_1) \alpha_2],$$

(2.62)

which can be put in the following matrix form:

$$\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \frac{1}{|\alpha|^2} \begin{pmatrix} 1 + \beta_1 \gamma_1 - \beta_2 \gamma_2 & \beta_1 \gamma_2 + \beta_2 \gamma_1 \\ \beta_1 \gamma_2 + \beta_2 \gamma_1 & -1 + \beta_2 \gamma_2 - \beta_1 \gamma_1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$  

(2.63)

There are two mutually exclusive cases arising from the requirement $\delta_2 \to \infty$, which can be treated: the hyperbolic case and the loxodromic case (hereafter we will write $H$ for hyperbolic and $L$ for loxodromic)

Subcase (1.H)
The hyperbolic case is obtained by evaluating
\[ j^2 = (\alpha + \delta)^2 = (\alpha_1 + \delta_1)^2 - (\alpha_2 + \delta_2)^2 + 2i (\alpha_1 + \delta_1) (\alpha_2 + \delta_2), \]
and then requiring
\[ j^2 > 4. \]
Thus we must set
\[ \alpha_2 = -\delta_2, \tag{2.64} \]
to obtain \( j^2 \in \mathbb{R} \) while the condition \( j^2 > 4 \) is automatically ensured in the limit \( b \to \infty \) when \( \delta \to \infty \) as in the case we are treating. In this case, Eq. (2.63) takes the form
\[ \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \frac{1}{\hat{\alpha}_1^2 + \hat{\delta}_2^2} \begin{pmatrix} 1 + \beta_1 \gamma_1 - \beta_2 \gamma_2 & \beta_1 \gamma_2 + \beta_2 \gamma_1 \\ \beta_1 \gamma_2 + \beta_2 \gamma_1 & -1 - \beta_1 \gamma_1 + \beta_2 \gamma_2 \end{pmatrix} \begin{pmatrix} \hat{\alpha}_1 \\ -\hat{\delta}_2 \end{pmatrix}. \tag{2.66} \]
We are mainly interested in the limit value of all variables at infinity, therefore we can set
\[ \lim_{b \to \infty} \alpha_k = \hat{\alpha}_k, \quad \lim_{b \to \infty} \beta_k = \hat{\beta}_k, \]
\[ \lim_{b \to \infty} \gamma_k = \hat{\gamma}_k, \quad \lim_{b \to \infty} \delta_k = \hat{\delta}_k, \tag{2.65} \]
for \( k = 1, 2 \). Of course, in this case we must have \( \hat{\delta}_2 = \infty \), but if some estimates of the order of infinity of \( \hat{\delta}_2 \) are needed, as well as the limits in Eq. (2.65), it is convenient to write Eq. (2.63) in the limit point case as
\[ \begin{pmatrix} \hat{\delta}_1 \\ \hat{\delta}_2 \end{pmatrix} = \frac{1}{\hat{\alpha}_1^2 + \hat{\delta}_2^2} \begin{pmatrix} 1 + \hat{\beta}_1 \hat{\gamma}_1 - \hat{\beta}_2 \hat{\gamma}_2 & \hat{\beta}_1 \hat{\gamma}_2 + \hat{\beta}_2 \hat{\gamma}_1 \\ \hat{\beta}_1 \hat{\gamma}_2 + \hat{\beta}_2 \hat{\gamma}_1 & -1 - \hat{\beta}_1 \hat{\gamma}_1 + \hat{\beta}_2 \hat{\gamma}_2 \end{pmatrix} \begin{pmatrix} \hat{\alpha}_1 \\ -\hat{\delta}_2 \end{pmatrix}. \tag{2.66} \]
Subcase (1.L)
The loxodromic case is obtained by the property
\[ j^2 = (\alpha + \delta)^2 = (\alpha_1 + \delta_1)^2 - (\alpha_2 + \delta_2)^2 + 2i (\alpha_1 + \delta_1) (\alpha_2 + \delta_2) \in \mathbb{C} - \mathbb{R}^+, \]
thus we must require
\[ (\alpha_1 + \delta_1) (\alpha_2 + \delta_2) \neq 0, \tag{2.67} \]
and this automatically ensures \( j^2 \in \mathbb{C} - \mathbb{R}^+ \). In this case Eq. (2.63) does not require any modification. The limiting equation can be written as
\[ \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \frac{1}{|\alpha|^2} \begin{pmatrix} 1 + \hat{\beta}_1 \hat{\gamma}_1 - \hat{\beta}_2 \hat{\gamma}_2 & \hat{\beta}_1 \hat{\gamma}_2 + \hat{\beta}_2 \hat{\gamma}_1 \\ \hat{\beta}_1 \hat{\gamma}_2 + \hat{\beta}_2 \hat{\gamma}_1 & -1 - \hat{\beta}_1 \hat{\gamma}_1 + \hat{\beta}_2 \hat{\gamma}_2 \end{pmatrix} \begin{pmatrix} \hat{\alpha}_1 \\ \hat{\delta}_2 \end{pmatrix}. \tag{2.68} \]
(2) The \( \gamma \) variable approaching infinity.
In this case it is useful to solve the equation \( \alpha \delta - \beta \gamma = 1 \) in terms of the \( \gamma \) variable. Simple calculations, which are similar to those which led us to Eq. (2.62), show that

\[
\begin{align*}
\gamma_1 &= |\beta|^{-2} \left[ (-1 + \alpha_1 \delta_1 - \alpha_2 \delta_2) \beta_1 + (\alpha_1 \delta_2 + \alpha_2 \delta_1) \beta_2 \right], \\
\gamma_2 &= |\beta|^{-2} \left[ (\alpha_1 \delta_2 + \alpha_2 \delta_1) \beta_1 + (1 + \alpha_2 \delta_2 - \alpha_1 \delta_1) \beta_2 \right],
\end{align*}
\]

(2.69)

thus we can write Eq. (2.68) in matrix form as

\[
\begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix} = 1 \\
|\beta|^{-2} \begin{pmatrix}
1 & \alpha_1 \delta_2 + \alpha_2 \delta_1 \\
\alpha_1 \delta_2 - \alpha_2 \delta_1 & 1 - \alpha_1 \delta_1 - \alpha_2 \delta_2
\end{pmatrix} \begin{pmatrix}
\beta_1 \\
\beta_2
\end{pmatrix}.
\]

(2.70)

In this case we can distinguish four subcases: the hyperbolic, the loxodromic, the parabolic and the elliptic cases (hereafter, we write H for hyperbolic, L for loxodromic, P for parabolic and E for elliptic).

**Subcase (2.H)**

The hyperbolic case is obtained by evaluating

\[
\begin{align*}
 j^2 &= (\alpha + \delta)^2 = (\alpha_1 + \delta_1)^2 - (\alpha_2 + \delta_2)^2 + 2i (\alpha_1 + \delta_1) (\alpha_2 + \delta_2),
\end{align*}
\]

and then requiring

\[
 j^2 > 4.
\]

Thus we must set

\[
\alpha_2 = -\delta_2,
\]

jointly with the condition

\[
(\alpha_1 + \delta_1)^2 > 4.
\]

In this case, Eq. (2.70) takes the form

\[
\begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix} = 1 \\
|\beta|^{-2} \begin{pmatrix}
1 & \alpha_1 \delta_1 + \delta_2^2 - \alpha_1 \delta_1 - \delta_2^2\delta_1 - \delta_2 \delta_2 \delta_1 - \delta_2^2 \delta_1 \\
\alpha_1 \delta_1 + \delta_2^2 \delta_1 - \delta_2 \delta_2 \delta_1 - \delta_2^2 \delta_1 & 1 - \alpha_1 \delta_1 - \delta_2 \delta_2 
\end{pmatrix} \begin{pmatrix}
\beta_1 \\
\beta_2
\end{pmatrix},
\]

which in the limit-point case can be written as

\[
\begin{pmatrix}
\dot{\gamma}_1 \\
\dot{\gamma}_2
\end{pmatrix} = 1 \\
|\beta|^{-2} \begin{pmatrix}
1 & \dot{\alpha}_1 \dot{\delta}_1 + \dot{\delta}_2^2 - \dot{\alpha}_1 \dot{\delta}_1 - \dot{\delta}_2 \dot{\delta}_2 \dot{\delta}_1 - \dot{\delta}_2^2 \dot{\delta}_1 \\
\dot{\alpha}_1 \dot{\delta}_1 + \dot{\delta}_2^2 \dot{\delta}_1 - \dot{\delta}_2 \dot{\delta}_2 \dot{\delta}_1 - \dot{\delta}_2^2 \dot{\delta}_1 & 1 - \dot{\alpha}_1 \dot{\delta}_1 - \dot{\delta}_2 \dot{\delta}_2 
\end{pmatrix} \begin{pmatrix}
\dot{\beta}_1 \\
\dot{\beta}_2
\end{pmatrix}.
\]

**Subcase (2.L)**

The loxodromic case is obtained by the property

\[
\begin{align*}
 j^2 &= (\alpha + \delta)^2 = (\alpha_1 + \delta_1)^2 - (\alpha_2 + \delta_2)^2 + 2i (\alpha_1 + \delta_1) (\alpha_2 + \delta_2) \in \mathbb{C} - \mathbb{R}^+,
\end{align*}
\]

thus we must require

\[
(\alpha_1 + \delta_1) (\alpha_2 + \delta_2) \neq 0.
\]
In this case, Eq. (2.70) does not need any modification. In the limit-point case, we must set
\[
\begin{pmatrix}
\hat{\gamma}_1 \\
\hat{\gamma}_2
\end{pmatrix}
= \frac{1}{|\beta|^2} \begin{pmatrix}
-1 + \hat{\alpha}_1 \hat{\delta}_1 - \hat{\alpha}_2 \hat{\delta}_2 & \hat{\alpha}_1 \hat{\delta}_2 + \hat{\alpha}_2 \hat{\delta}_1 \\
\hat{\alpha}_1 \hat{\delta}_2 + \hat{\alpha}_2 \hat{\delta}_1 & 1 - \hat{\alpha}_1 \hat{\delta}_1 + \hat{\alpha}_2 \hat{\delta}_2
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2
\end{pmatrix}.
\]

*Subcase (2.P)*

In the parabolic case, we start with the equation
\[
j^2 = (\alpha + \delta)^2 = (\alpha_1 + \delta_1)^2 - (\alpha_2 + \delta_2)^2 + 2i(\alpha_1 + \delta_1)(\alpha_2 + \delta_2) = 4,
\]
and this can be fulfilled if and only if
\[
\alpha_2 = -\delta_2,
\]
and hence Eq. (2.69) reduces to
\[
\begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix}
= \frac{1}{|\beta|^2} \begin{pmatrix}
-1 \pm 2\delta_1 - \delta_1^2 + \delta_2^2 & \pm 2\delta_2 - 2\delta_2 \delta_1 \\
\pm 2\delta_2 - 2\delta_2 \delta_1 & 1 \mp 2\delta_1 + \delta_1^2 - \delta_2^2
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2
\end{pmatrix},
\]
which in the limit-point case, by using Eq. (2.67), can be written as
\[
\begin{pmatrix}
\hat{\gamma}_1 \\
\hat{\gamma}_2
\end{pmatrix}
= \frac{1}{|\beta|^2} \begin{pmatrix}
-1 \pm 2\hat{\delta}_1 - \hat{\delta}_1^2 + \hat{\delta}_2^2 & \pm 2\hat{\delta}_2 - 2\hat{\delta}_2 \hat{\delta}_1 \\
\pm 2\hat{\delta}_2 - 2\hat{\delta}_2 \hat{\delta}_1 & 1 \mp 2\hat{\delta}_1 + \hat{\delta}_1^2 - \hat{\delta}_2^2
\end{pmatrix}
\begin{pmatrix}
\hat{\beta}_1 \\
\hat{\beta}_2
\end{pmatrix},
\]
in which some analysis of the behaviors of \(\hat{\delta}_k\) and \(\hat{\beta}_k\) for \(k = 1, 2\) would be necessary for further developments.

*Subcase (2.E)*

In the elliptic case, the squared trace
\[
j^2 = (\alpha + \delta)^2 = (\alpha_1 + \delta_1)^2 - (\alpha_2 + \delta_2)^2 + 2i(\alpha_1 + \delta_1)(\alpha_2 + \delta_2),
\]
must lie in \([0, 4]\) and hence
\[
\alpha_2 = -\delta_2,
\]
and Eq. (2.70) reduces to
\[
\begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix}
= \frac{1}{|\beta|^2} \begin{pmatrix}
-1 + \alpha_1 \delta_1 + \delta_2^2 & \alpha_1 \delta_2 - \delta_2 \delta_1 \\
\alpha_1 \delta_2 - \delta_2 \delta_1 & 1 - \alpha_1 \delta_1 - \delta_2^2
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2
\end{pmatrix},
\]
which in the limit-point case can be written as
\[
\begin{pmatrix}
\hat{\gamma}_1 \\
\hat{\gamma}_2
\end{pmatrix}
= \frac{1}{|\beta|^2} \begin{pmatrix}
-1 + \hat{\alpha}_1 \hat{\delta}_1 + \hat{\delta}_2^2 & \hat{\alpha}_1 \hat{\delta}_2 - \hat{\delta}_2 \hat{\delta}_1 \\
\hat{\alpha}_1 \hat{\delta}_2 - \hat{\delta}_2 \hat{\delta}_1 & 1 - \hat{\alpha}_1 \hat{\delta}_1 - \hat{\delta}_2^2
\end{pmatrix}
\begin{pmatrix}
\hat{\beta}_1 \\
\hat{\beta}_2
\end{pmatrix},
\]
and also in this case, the behaviors of $\alpha_k, \beta_k$ and $\delta_k$, for $k = 1, 2$ are necessary for further developments.

The evaluation of the limits (2.66), (2.68), (2.72) and (2.74) might be very involved because the limit-point, limit-circle theory does not impose any a priori restriction on the behavior of $\alpha, \beta, \gamma$ and $\delta$ defined in Eq. (2.58) at infinity. This gives us only some advice on the square integrability near infinity for the functions $\varphi$ and $\psi$ (thus on $\alpha$ and $\gamma$, respectively). The lack of square integrability near infinity for the function $\psi$, results from Eq. (2.22) which, in the limit-point case we are treating, coincides with the requirement $r_b \to 0$ for large $b$. The lack of square integrability near infinity for the function $\varphi$, can be appreciated by using the method developed in Ref. [30], where it is shown, by making use of

$$\int_0^b dr' |\varphi + m\psi|^2 \leq \frac{|m|}{\mathcal{J}(l)}$$

that it must be

$$\frac{1}{2}|m|^2 \int_c^b dr' |\psi|^2 - \int_0^b dr' |\varphi|^2 \leq \int_0^b dr' |\varphi + m\psi|^2 \leq \frac{|m|}{\mathcal{J}(l)},$$

which is a second degree algebraic inequality for the variable $|m|^2$, thus we must also have

$$|m|^2 \int_c^b dr' |\psi|^2 - 2 \frac{|m|}{\mathcal{J}(l)} - 2 \int_0^b dr' |\varphi|^2 \leq 0.$$ 

The admissible roots lie within the interval defined by the associated algebraic equation but we must of course rule out the negative root, finding therefore

$$|m| \leq \frac{1}{\mathcal{J}(l) \int_0^b dr' |\psi|^2} + \left\{ \frac{2 \int_0^b dr' |\varphi|^2}{\int_0^b dr' |\psi|^2} + \frac{1}{\mathcal{J}^2(l) \left( \int_0^b dr' |\psi|^2 \right)^2} \right\}^{\frac{1}{2}}.$$ (2.75)

From the fact that $\mathcal{J}(m) \neq 0$ for every $b > 0$, then $|m|$ is always positive also in the limit $b \to \infty$ and this can be reached if the $L^2$ squared norm near infinity of $\varphi$, has the same order of infinity of that of $\psi$. Therefore, $\varphi$ is not square integrable near infinity, as well as $\psi$.

We can provide an interpretation of cases (1.H), (1.L), (2.P) and (2.E) by looking at the type of BMS transformations which we can call “purely hyperbolic”, “purely loxodromic”, “purely parabolic” or “purely elliptic”. Recall that a BMS transformation is given by the pair of transformations [5]

$$\zeta \to \frac{a\zeta + b}{c\zeta + d},$$ (2.76)

$$u \to \frac{(1 + |\zeta|^2) [u + \alpha(\zeta, \zeta)]}{|a\zeta + b|^2 + |c\zeta + d|^2},$$ (2.77)
for the conformal infinity of an asymptotically flat space time. The nontrivial point here, is that such maps can be equally well described if we choose a particular form for the fractional linear transformation in them. It is well known from Ref. [31], that each fractional linear transformation with one fixed point only, i.e. a parabolic transformation, can be always reduced to a pure translation of the form

\[ \zeta' = \zeta + b, \]  

(2.78)

while all the other transformations with two fixed points, can be always written in the form

\[ \zeta' = K\zeta, \]  

(2.79)

with \( K = Ae^{i\theta} \). The “purely elliptic” case corresponds to setting \( A = 1 \), the “pure hyperbolic” case corresponds to setting \( \theta = 2k\pi \) with \( k \in \mathbb{Z} \), while the “pure loxodromic” case corresponds to setting \( A \neq 1 \) and \( \theta \neq 2k\pi \) with \( k \in \mathbb{Z} \). We also recall that the procedure which enables us to write Eqs. (2.78) and (2.79), is a process which admits the possibility to conjugate the fixed points of the transformation, to particular points which are often chosen as the point \( \zeta = \infty \) and \( \zeta = 0 \), when a 2-fixed-point transformation is considered, while in the case of a transformation with one fixed point only, the fixed point is conjugated to the point at infinity. In this way, despite the position of the fixed points for the superrotation in Eq. (2.76), we can always reduce it to one among Eq. (2.78) or (2.79). Once this step is done, we can distinguish four cases as below [5]:

(H) \textit{Hyperbolic BMS}

\[ \zeta \rightarrow A\zeta, \ A \in \mathbb{R}^+ - \{1\}, \]

\[ u \rightarrow F_H \cdot [u + \alpha], \ F_H = \frac{\left(1 + |\zeta|^2\right)}{1 + A^2|\zeta|^2}. \]

(L) \textit{Loxodromic BMS}

\[ \zeta \rightarrow Ae^{i\theta}\zeta, \ A \in \mathbb{R}^+ - \{1\}, \ \theta \neq 2k\pi, \ k \in \mathbb{Z}, \]

\[ u \rightarrow F_L \cdot [u + \alpha], \ F_L = \frac{\left(1 + |\zeta|^2\right)}{1 + A^2|\zeta|^2}. \]

(P) \textit{Parabolic BMS}

\[ \zeta \rightarrow \zeta + \beta, \]

\[ u \rightarrow F_P \cdot [u + \alpha], \ F_P = \frac{\left(1 + |\zeta|^2\right)}{1 + |\zeta + \beta|^2}. \]
(E) **Elliptic BMS**

\[ \zeta \rightarrow e^{i\theta} \zeta, \quad \theta \neq 2k\pi, \quad k \in \mathbb{Z}, \]

\[ u \rightarrow F_E \cdot [u + \alpha], \quad F_E = \frac{\left(1 + |\zeta|^2\right)}{\left(1 + |\zeta|^2\right)} = 1. \]

This characterization of BMS transformations, leads to a correspondence between half of the BMS transformations and singular second order self-adjoint problems in quantum mechanics. If we can solve a limit-point, limit-circle problem for a given Hamiltonian in quantum mechanics, and thus we can calculate the functions \( \varphi \) and \( \psi \) in the limit-point case, then Eq. (2.59) forces the parameters \( \alpha, \beta, \gamma \) and \( \delta \) occurring in Eq. (2.58) to fall back in one of the cases (1.H), (1.L), (2.H), (2.L), (2.P) and (2.E) for \( b \rightarrow \infty \) and thus to give rise to “purely hyperbolic”, “purely loxodromic”, “purely parabolic” or “purely elliptic” BMS transformations, whose functional form is expressed in the transformations (H), (L), (P) and (E) above. We can thus suggest that the limit-point case at infinity admits a profound interpretation in terms of symmetries of the space-time itself and thus that a self-adjoint problem in quantum mechanics is strictly related to a particular class of BMS transformations for an asymptotically flat space-time. Suppose first to have solved a limit-point case at infinity in such a way that the two independent solutions \( \varphi \) and \( \psi \) are known. From our previous analysis, it is clear that the limit-point requirement must force these two independent solutions to fall back in one of the cases (1.H), (1.L), (2.H), (2.L), (2.P) and (2.E). This means that there must exist a lower bound \( M \) such that for each \( b > M \), all the fractional linear transformations in Eq. (2.58) are of one special kind and cannot be of some other kind. This ensures us that all such transformations, for \( b > M \), can give rise to one and only one type of BMS transformations between (H), (L), (P) and (E). Further developments in this direction, can be accomplished only if the particular limit-point theory is solved and thus if a concrete case is chosen as an example of application for the theory here treated.

Other questions which can arise from solving a concrete limit-point, limit-circle problem, can regard the possibility of constructing precise discrete subgroups of \( PSL(2, \mathbb{C}) \) in the limit \( b \rightarrow \infty \) or equivalently, if it is possible to construct a bounded sequence \( \{b_n\} \) on the real positive line, chosen in such a way that Eq. (2.58) falls back in one of the cases discussed above for each \( b_n \) and, at the same time, which forces the BMS transformations such arising, to form a Kleinian group.
3. Limit-Point Case at Both Ends of the Positive Real Line and Hyperbolic Cyclic Groups

3.1. The Arrangement of the Isometric Circles in a Hyperbolic Cyclic Group

In this section we try to obtain a relation between the limit-point, limit-circle theory and the theory of hyperbolic cyclic groups of fractional linear transformations. As we have already established in the previous section, each second-order self-adjoint problem which is singular at both ends of the positive real line, is accompanied by two families of circles which reduce to a pair of points on the complex plane when the interval in which the problem is first studied is stretched out by covering the whole positive real line. We have also mentioned the importance of singular self-adjoint problems in quantum mechanics thus, by solving one of such boundary problems in the framework of the limit-point, limit-circle theory one can also obtain some insight about other aspects of physics, which cannot be immediately viewed but nevertheless can arise in a very elegant way as it happened in the connection between limit-point, limit-circle theory and the BMS transformations discussed in the last subsection of the previous section. In this framework we can expect to interpret our previous results and all of that which will come next, as an example of how some areas of quantum theory can give rise to a non-trivial connection with some areas of the theory of gravitation.

The basic idea that we will follow hereafter, relies on an evident similitude between the arrangement of the isometric circles of a hyperbolic cyclic group (that we denote by $I_n$), and the arrangement of all circles $C_b$ in the limit-point, limit-circle theory. Actually, a hyperbolic cyclic group gives rise to two families of isometric circles, families which we can indicate as $\{I_n\}$ and $\{I_{-n}\}$ referring to $I_n$ as the isometric circle of the hyperbolic transformation $f^n$ while referring to $I_{-n}$ as the isometric circle of the inverse $f^{-n}$ (note that $f^n \circ f^{-n} = 1$), which satisfies the following requirements:

(i) Each member of the family $\{I_n\}$ is exterior to each member of the family $\{I_{-n}\}$.

(ii) For $m > n$, $I_m$ is contained in the interior of $I_n$ and $I_{-m}$ is contained in the interior of $I_{-n}$.

(iii) In the limit $n \to \infty$, both $I_n \to \xi_1 \in \mathbb{C}$ and $I_{-n} \to \xi_2 \in \mathbb{C}$, where $\xi_1$ and $\xi_2$ are the fixed points of the generating hyperbolic transformation $f$.

We also know that each isometric circle of a hyperbolic cyclic group, contains at most one fixed point of the generating transformation. We will show actually, that all members of a disjoint family of circles contain the same fixed point. All properties (i), (ii) and (iii) above, are very similar to that established for the arrangement of the circles $C_b$ and $C'_a$ for a second-order self-adjoint problem on the positive real line which may occur if and only if the second-order differential operator of
the theory is in limit point case at both ends of the interval. In that case, we can establish three properties that should hold in order to get the limit-point case at both ends.

(i') Each member of the family \( \{ C_b \} \) is exterior to each member of the family \( \{ C'_a \} \).

(ii') For \( b > \tilde{b} \) and \( a < \tilde{a} \), \( C_b \) is contained in the interior of \( C_{\tilde{b}} \) and \( C'_a \) is contained in the interior of \( C'_{\tilde{a}} \).

(iii') In the limit \( b \to \infty \) and \( a \to 0 \), both \( C_b \to m_\infty \in \mathbb{C} \) and \( C'_a \to m'_\infty \in \mathbb{C} \), where \( m_\infty \) and \( m'_\infty \) are the limit-points at \( r = \infty \) and \( r = 0 \), respectively.

From previous discussions, we know that all circles belonging to the same disjoint family, contain one and only one of the two limit points. We can thus construct somehow, a monotonic increasing sequence \( \{ b_n \} \) and a monotonic decreasing sequence \( \{ a_n \} \), in such a way that \( \delta_n = [a_n, b_n] \to \mathbb{R}^+ \), which is the goal of the limit-point theory at both ends of the positive real line. In this way, properties (i'), (ii') and (iii') can be reformulated as follows.

(i'') Each member of the family \( \{ C_{b_n} \} \) is exterior to each member of the family \( \{ C'_{a_n} \} \).

(ii'') For \( m > n \), \( C_{b_m} \) is contained in the interior of \( C_{b_n} \) and \( C'_{a_m} \) is contained in the interior of \( C'_{a_n} \).

(iii'') In the limit \( n \to \infty \), both \( C_{b_n} \to m_\infty \in \mathbb{C} \) and \( C_{a_n} \to m'_\infty \in \mathbb{C} \), where \( m_\infty \) and \( m'_\infty \) are the limit-points at \( r = \infty \) and \( r = 0 \), respectively.

Now, one can see from the similar meanings of (i) and (i''), (ii) and (ii''), (iii) and (iii''), that a complete identification can be obtained if we impose the following restrictions:

\[
I_n = C_{b_n}, \quad I_{-n} = C'_{a_n}, \quad \forall n \in \mathbb{N},
\]

\[
\xi_1 = m_\infty, \quad \xi_2 = m'_\infty.
\] (3.1)

Despite the similarity first mentioned, these last conditions are not trivial for the different nature of circles \( C_{b_n} \) and \( I_n \), thus we will see that a consistent set of restrictions must be imposed is such a way that our desired relations (3.1) could be satisfied. The point here is that we are guided by the analogy between relations (i), (ii), (iii) and (i''), (ii''), (iii''), but nothing ensures that each limit-point condition at both ends of the positive real line could return us a hyperbolic cyclic group, hence some restriction may occur for the fundamental system of solutions \( \varphi_1 \) and \( \varphi_2 \) to the equation \( Lx = lx \) as well as for the existence of two monotonic sequences of points \( \{ a_n \} \) and \( \{ b_n \} \) (which will be clear in the following).

In the remainder of this section, we will try to recover some further notions
about hyperbolic cyclic groups. First, we must prove properties (i), (ii) and (iii). We will be concerned with the following type of fractional linear transformation:

$$f(z) = \frac{az + b}{cz + d}, \quad c \neq 0.$$  

(3.2)

As we already know, the requirement $c \neq 0$ corresponds to demanding that the map (3.2) has two finite fixed points. In this case, the isometric circle of the map (3.2) is given by

$$I : \quad |cz + d| = 1, \quad c \neq 0,$$

(3.3)

and it is the locus of points of the complex plane whose arcs are unaltered in length when Eq. (3.2) is applied. Following the arguments contained in Ref. [31], the case $c \neq 0$ is appropriate for a powerful conjugation of fixed points of Eq. (3.2). Call its fixed points $\xi_1$ and $\xi_2$ and use the four-point ratio

$$\frac{(z' - z'_1)}{(z' - z'_2)} = \frac{(z - z_1)}{(z - z_2)} \frac{(z_2 - z_3)}{(z_1 - z_3)},$$

(3.4)

by setting

$$z'_1 = z_1 = \xi_1, \quad z'_2 = z_2 = \xi_2, \quad z'_3 = \frac{a}{c},$$

and solving it with respect to the $z' = f(z)$ variable

$$z' = \frac{\xi_1 - \xi_2 K}{1 - K \left(\frac{z - \xi_1}{z - \xi_2}\right)},$$

(3.5)

where $K$ is the multiplier

$$K = \frac{(\xi_1 - \frac{a}{c})}{(\xi_2 - \frac{a}{c})}.$$  

We can obtain the same result in Eq. (2.5) by using the conjugation process which involves the following variables:

$$Z = g(z) = \frac{(z - \xi_1)}{(z - \xi_2)}, \quad Z' = g(z') = \frac{(z' - \xi_1)}{(z' - \xi_2)},$$

(3.6)

which conjugate the fixed points to $z = 0$ and $z = \infty$. Equation (3.4) reduces to

$$g(z') = Kg(z),$$

(3.7)

and by defining

$$K(z) = Kz,$$

(3.8)

Eq. (3.4) reduces to

$$g(z') = K(g(z)) = K \circ g(z),$$

from which

$$z' = g^{-1} \circ K \circ g(z),$$
but $g^{-1}$ can be written as
\[ z = g^{-1}(Z) = \frac{\xi_2 Z - \xi_1}{(Z - 1)}, \]
thus
\[ z' = g^{-1}(KZ) = g^{-1}\left(K\frac{z - \xi_1}{z - \xi_2}\right) = \frac{\xi_2 K\left(\frac{z - \xi_1}{z - \xi_2}\right) - \xi_1}{K\left(\frac{z - \xi_1}{z - \xi_2}\right) - 1}, \]
from which we recover Eq. (3.5). We recall the fact that a hyperbolic transformation is characterized by a real and positive value of $K$ not equal to one, i.e. $K \in \mathbb{R}^+ - \{1\}$; for $K \in [0, 1]$ we get a contraction about the fixed point $\xi_1$ and a dilation about $\xi_2$, while if $K \in [1, \infty]$ we get a dilation about $\xi_1$ and a contraction about $\xi_2$ thus, in the former case $\xi_1$ is an attractive point and $\xi_2$ is a repulsive point while in the latter the reverse holds. We also know that for a hyperbolic transformation each circle through the fixed points is mapped into another such circle, the interior of a circle through the fixed points is mapped into itself, any circle orthogonal to any circle through the fixed points is mapped into another such circle and that the fixed points are inverse one to the other with respect to each circle orthogonal to any circle through the fixed points. Therefore, if a fractional linear map is a hyperbolic transformation, its trace $j = a + c$ must satisfy
\[ |j| > 2. \]
We are interested in cyclic groups hence, given Eq. (3.2), we can construct the variables (3.6). Now, by defining the following sequence of transformations for the $Z$ plane defined in Eq. (3.6)
\[ Z'' = K^2 Z, \quad Z''' = K^3 Z, \quad \ldots, \quad Z^{(n)} = K^n Z, \quad (3.9) \]
we are interested in finding which are the corresponding transformations, whose general mapping is given in Eq. (3.2), for the $z$ plane. Obviously, from previous reasoning, we have
\[ f = g^{-1} \circ K \circ g, \quad (3.10) \]
where the $K$ transformation is given in Eq. (3.8). One can see that the following maps
\[ f^2 = g^{-1} \circ K \circ g \circ g^{-1} \circ K \circ g = g^{-1} \circ K^2 \circ g, \]
\[ f^3 = g^{-1} \circ K \circ g \circ g^{-1} \circ K \circ g \circ g^{-1} \circ K \circ g = g^{-1} \circ K^3 \circ g, \]
\[ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots, \]
\[ f^n = \underbrace{g^{-1} \circ K \circ g \circ \cdots \circ g^{-1} \circ K \circ g}_{n} = g^{-1} \circ K^n \circ g, \]
are of the form given in Eq. (3.10) and thus suffice for defining transformations of type (3.7), which are explicitly given in Eq. (3.9). We have thus recovered the fact that the \( n \)-th power of some linear transformation \( f \) (with two fixed points), has a multiplier which is the \( n \)th power of the multiplier for the map \( f \). We can thus write the following expression from Eq. (3.5):

\[
 f^n (z) = z^{(n)} = \frac{\xi_1 - \xi_2 K^n \left( \frac{z - \xi_1}{z - \xi_2} \right)}{1 - K^n \left( \frac{z - \xi_1}{z - \xi_2} \right)},
\]

(3.11)

and hence, the net effect of several applications of the same transformation becomes merely a substitution of the multiplier for the original transformation. We also stress the fact that the multiplier is the only object which defines a particular transformation (once his fixed points are chosen) and the way in which it varies, while we are applying several copies of the same map, is independent of the representation used whatever is given in Eqs. (3.9) or in Eq. (3.11).

We can also discuss some generalization of Eq. (3.10) taking into account the possibility of conjugating any triplet \( z_1, z_2 \) and \( z_3 \) to any other triplet \( z_1', z_2' \) and \( z_3' \) despite of the particular choice which leads us to the multiplier \( K \), by using Eq. (3.4). Notice that Eq. (3.4) has always a finite value even if we choose some of the points \( z_1, z_2, z_3 \) or \( z_1', z_2', z_3' \) to be not finite. But if we set

\[
 \sigma = \frac{z_2' - z_3'}{z_1' - z_3'}, \quad \rho = \frac{z_2 - z_3}{z_1 - z_3}, \quad \gamma = \frac{\rho}{\sigma},
\]

then Eq. (3.4) can be written as

\[
 \left( \frac{z' - z_1'}{z' - z_2'} \right) = \gamma \left( \frac{z - z_1}{z - z_2} \right),
\]

and by using the notation

\[
 Z = h(z) = \frac{z - z_1}{z - z_2}, \quad Z' = h'(z') = \frac{z' - z_1'}{z' - z_2'},
\]

jointly with the following definition:

\[
 \gamma (Z) = \gamma Z,
\]

we can write the map \( f (z) = z' \) as

\[
 f = h^{-1} \circ \gamma \circ h.
\]

(3.12)

Of course, by solving the equation \( Z' = \gamma Z \) in the variable \( z' \), the map \( f \) takes the form

\[
 f (z) = z' = \frac{z_1' - z_2' \gamma \left( \frac{z - z_1}{z - z_2} \right)}{1 - \gamma \left( \frac{z - z_1}{z - z_2} \right)},
\]
and the same arguments given above, hold for subsequent applications of the same map (3.12), thus we can state that subsequent applications of the same map \( f \) lead to a substitution of the multiplier \( \gamma \) by powers of it and hence, the maps
\[
Z'' = \gamma^2 Z, \ldots, Z^{(n)} = \gamma^n Z,
\]
correspond to the maps
\[
f^2 = g^{-1} \circ K^2 \circ g, \ldots, f^n = g^{-1} \circ K^n \circ g,
\]
on the \( z \) plane, where
\[
f^n (z) = z^{(n)} = \frac{z'_{1} - z'_{2} \gamma^n \left( \frac{z - z_1}{z - z_2} \right)}{1 - \gamma^n \left( \frac{z - z_1}{z - z_2} \right)}, \tag{3.13}
\]
Now we want to derive the expression for the isometric circle for the transformation (3.11) which will be the isometric circle of the \( n \)-th power of any transformation whose multiplier \( K \) is already known. First, let us note that the determinant of Eq. (3.5) is not equal to one, as well for Eq. (3.11), thus we must divide and multiply this latter by
\[
K^{\frac{t}{2}}(\xi_1 - \xi_2),
\]
hence Eq. (3.11) becomes
\[
z^{(n)} = \frac{\xi_1 - \xi_2 K^n \left( \frac{z - z_1}{z - z_2} \right)}{1 - K^n \left( \frac{z - z_1}{z - z_2} \right)} = \frac{(K^n \xi_2 - \xi_1) z + \xi_1 \xi_2 (1 - K^n)}{(K^n - 1) z + (\xi_2 - K^n \xi_1)}
\]
\[
= \frac{(K^n \xi_2 - \xi_1)}{K^n (\xi_1 - \xi_2)} z + \frac{\xi_1 \xi_2 (1 - K^n)}{K^n (\xi_1 - \xi_2)}, \tag{3.14}
\]
and the last expression has determinant equal to one. The general case in Eq. (3.14) can also be worked out. In this case we can write Eq. (3.14) as
\[
z^{(n)} = \frac{z'_{1} - z'_{2} \gamma^n \left( \frac{z - z_1}{z - z_2} \right)}{1 - \gamma^n \left( \frac{z - z_1}{z - z_2} \right)} = \frac{z'_{1} z - z'_{1} z_2 - z'_{2} \gamma^n z + z'_{2} z_1 \gamma^n}{z - z_2 - \gamma^n z + \gamma^n z_1}
\]
\[
= \frac{(z'_{2} \gamma^n - z'_{1}) z + (z'_{1} z_2 - z'_{2} z_1 \gamma^n)}{(\gamma^n - 1) z + (z_2 - \gamma^n z_1)},
\]
whose determinant is
\[
t = (z'_{2} \gamma^n - z'_{1}) (z_2 - \gamma^n z_1) - (\gamma^n - 1) (z'_{1} z_2 - z'_{2} z_1 \gamma^n),
\]
and we have just to multiply and divide by \( \sqrt{t} \) to get a fractional linear transformation with determinant equal to one
\[
z^{(n)} = \frac{(z'_{2} \gamma^n - z'_{1}) z + (z'_{1} z_2 - z'_{2} z_1 \gamma^n)}{\sqrt{t}} \frac{\sqrt{t}}{(\gamma^n - 1) z + (z_2 - \gamma^n z_1)}, \tag{3.15}
\]
which follows at once from the relation
\[
\frac{a}{\sqrt{t}} \frac{d}{\sqrt{t}} - \frac{b}{\sqrt{t}} \frac{c}{\sqrt{t}} = \frac{t}{t} = 1.
\]
Taking into account Eq. (3.3), the isometric circle of the map (3.14) must be written as
\[
I_n: \left| z + \frac{(\xi_2 - K^n \xi_1)}{(K^n - 1)} \right| = \left| \frac{\xi_1 - \xi_2}{K^n - K^{-n}} \right|,
\]
while the inverse transformation of Eq. (3.14) is obviously written as
\[
f^{-n}(z) = z(-n) = -\left[ \frac{(\xi_2 - K^n \xi_1)}{K^n (\xi_1 - \xi_2)} \right] z + \left[ \frac{\xi_1 \xi_2 (1 - K^n)}{K^n (\xi_1 - \xi_2)} \right],
\]
whose isometric circle is written as
\[
I_{-n}: \left| z + \frac{(\xi_1 - K^n \xi_2)}{(K^n - 1)} \right| = \left| \frac{\xi_1 - \xi_2}{K^n - K^{-n}} \right|,
\]
and $I_n$ and $I_{-n}$ have the same radius but different centre. In the case of a hyperbolic or loxodromic cyclic group with $K = Ae^{i\theta}$, with $A \in \mathbb{R}^+ - \{1\}$ we can observe that the radii of circles $I_n$ and $I_{-n}$ approach zero as $n \to \infty$. From the theory developed in Ref. [31], we know that the limit points of a Kleinian group are all the points in the neighborhoods of which an infinity of arcs of isometric circles fall. Take for example a hyperbolic or loxodromic cyclic group, then we have
\[
A > 1, \quad I_n \to \xi_1, \quad I_{-n} \to \xi_2,
\]
\[
A < 1, \quad I_n \to \xi_2, \quad I_{-n} \to \xi_1,
\]
thus, the isometric circles wrap up at the fixed points showing that these are the limit points of the cyclic group. This proves the relation (iii) we have stated so far for hyperbolic cyclic groups.

In the following we will be mainly interested in purely hyperbolic cyclic groups, i.e. cyclic groups generated by a hyperbolic transformation. The reason is that we can easily show property (i) for hyperbolic cyclic groups by assuming $K = A \in \mathbb{R}^+ - \{1\}$. This will be explicitly done later but for the moment we want to establish the fundamental fact that if one of the fixed points, say $\xi_1$, is in the interior of any $I_m$ for fixed $m$, then each member of the family $\{I_n\}$ contains in its interior the same fixed point.

From the standard theory developed Ref. [31], we know that the isometric circle of a hyperbolic transformation contains one and only one fixed point while the isometric circle of its inverse contains the other and that such circles are exterior one to the other. Hence, it is not useless to establish that if $\xi_1$ (or $\xi_2$) is interior to any $I_m$, then it is also contained in the interior of $I_n$ for each $n$ and thus each family $\{I_n\}$ and $\{I_{-n}\}$ contains one and only one fixed point. This will pave the
way for the proof of properties (i) and (ii) stated at the beginning of this section for the hyperbolic cyclic groups.

All the interior points of of the isometric circle $I_m$ satisfy

$$\left| z + \frac{(\xi_2 - K^m \xi_1)}{(K^m - 1)} \right| < \left| \frac{\xi_1 - \xi_2}{K^{m/2} - K^{-m/2}} \right|,$$

and our aim is to obtain all conditions which admit the possibility that $\xi_2$ (or $\xi_1$) lies in the interior of $I_m$ for some $m$, thus

$$\left| \xi_2 + \frac{(\xi_2 - K^m \xi_1)}{(K^m - 1)} \right| < \left| \frac{\xi_1 - \xi_2}{K^{m/2} - K^{-m/2}} \right|$$

should hold and the previous equation can be put in the form

$$|q_m r_m| < |r_m|,$$  \hspace{1cm} (3.19)

where

$$q_m = K^{m/2}, \quad r_m = \frac{(\xi_1 - \xi_2)}{(K^{m/2} - K^{-m/2})}.$$  

Equation (3.19) is fulfilled if and only if $|q_m| < 1$ and this can be accomplished if the modulus of $K$ is less than 1, thus

$$|q_m| < 1 \implies |K| = |A e^{i\theta}| = A < 1,$$

but this case is independent of the index $m$, i.e. if $A < 1$, then $|q_m| < 1$ for every $m \in \mathbb{N}$. We have thus proved that all members of the family $\{I_n\}$ contain the fixed point $\xi_2$ (and hence every member of the family $\{I_{-n}\}$ contains the other fixed point) when $A < 1$. Same reasoning can be applied if we try to fulfill the request that $\xi_1$ lies in the interior of $I_m$ for some $m$. In this case,

$$|r_m| < |q_m r_m|$$  \hspace{1cm} (3.20)

should hold instead of Eq. (3.19) and hence $A > 1$ is also required. The fulfillment of Eq. (3.20) is independent of the $m$ index once any $A > 1$ is chosen. Therefore, each member of the family $\{I_n\}$ contains in its interior the fixed point $\xi_1$ (and hence every member of the family $\{I_{-n}\}$ contains the other fixed point) when $A > 1$.

We will now prove the fundamental fact that $I_m$ is always in the interior of $I_n$ for $m > n$. This statement, follows at once by proving that $I_n$ and $I_{-m}$ are always exterior for each $n$ and $m$ and by applying Theorem 5.18 of Ref. \[31\], which we report here below:

Let $I_h$, $I_g$, $I'_g$, $I'_{gh}$ be the isometric circles of the transformations $h, \ g, \ g^{-1}, \ g \circ h$, respectively. If $I_h$ and $I_{gh}$ are exterior to one another, then $I_{gh}$ is contained in $I_g$.

We are thus concerned with the transformations $h = f^l, \ g = f^n$ and $g \circ h = f^m$ where $m > n$ and $(m - n) = l$. We should set up all conditions which ensure that $I_l$ is exterior to $I_{-m}$ in such a way that we can establish the fact that $I_m$ is in the interior of $I_n$ for $m > n$. We will soon see that this condition is independent of the choices made for $l, \ m, \ n$ and thus $I_m$ is always in the interior of $I_n$ for every $m > n$. 


Take into account Eqs. (3.16) and (3.18) for $I_l$ and $I_{-m}$. The requirement that these two circles be exterior one to the other, follows from requiring that the distance between their centres exceeds the sum of their radii, i.e.,

$$\left|\frac{(\xi_2 - K^l \xi_1)}{(K^l - 1)} - \frac{(\xi_1 - K^m \xi_2)}{(K^m - 1)}\right| > \left|\frac{\xi_1 - \xi_2}{K^l - K^{-l}}\right| + \left|\frac{\xi_1 - \xi_2}{K^m - K^{-m}}\right|.$$  \hspace{1cm} (3.21)

We can deal with such inequality, by setting

$$\alpha_s = K^l - K^{-l}, \quad \gamma_s = \xi_2 K^{-l} - \xi_1 K^l, \quad \gamma_{-s} = \xi_2 K^l - \xi_1 K^{-l},$$  \hspace{1cm} (3.22)

from which Eq. (3.21) reads as

$$\left|\gamma_l \alpha_m + \gamma_{-m} \alpha_l\right| < |\xi_1 - \xi_2| \left(\frac{1}{|\alpha_l|} + \frac{1}{|\alpha_m|}\right),$$  \hspace{1cm} (3.23)

but some algebraic calculations make it possible to simplify the numerator of the left-hand side of Eq. (3.23)

$$\gamma_l \alpha_m + \gamma_{-m} \alpha_l = (\xi_2 - \xi_1) \left(K^{\frac{m+l}{2}} - K^{-\frac{m+l}{2}}\right),$$

and also let us write Eq. (3.23) in the elegant form

$$|\alpha_l| + |\alpha_m| < \left|K^{\frac{m+l}{2}} - K^{-\frac{m+l}{2}}\right| \implies$$

$$\left|K^l - K^{-l}\right| + \left|K^{\frac{m}{2}} - K^{-\frac{m}{2}}\right| < \left|K^{\frac{m+l}{2}} - K^{-\frac{m+l}{2}}\right|.$$  \hspace{1cm} (3.24)

We can take the move from Eq. (3.24) only if we suppose some functional form for the multiplier $K$. We can obtain some interesting identities in the case of a loxodromic generating transformation with $K = \alpha e^{i\theta}$, but this is not the easiest one which we can deal with, and thus we postpone the discussion about the general loxodromic case for the fulfilment of Eq. (3.24) and take into account the easier case of a hyperbolic generating transformation with $K = \alpha \in \mathbb{R}^+ - \{1\}$. Therefore, we must distinguish two cases: $A > 1$ and $A < 1$.

**Case $A > 1$**

We have

$$\left|A^l - A^{-l}\right| + \left|A^{m/2} - A^{-m/2}\right| < \left|A^{\frac{m+l}{2}} - A^{-\frac{m+l}{2}}\right|,$$

but $A^k - A^{-k} > 0$ for $k > 0$, hence

$$A^l - A^{-l} + A^{m/2} - A^{-m/2} < A^{\frac{m+l}{2}} - A^{-\frac{m+l}{2}}.$$  \hspace{1cm} (3.25)

But one has also the majorization

$$A^l + A^{m/2} < A^{\frac{m+l}{2}},$$  \hspace{1cm} (3.26)

from the fact that the product of quantities bigger than 1, exceeds always their sum. We have also that

$$-A^l - A^{m/2} < -A^{\frac{m+l}{2}},$$
which follows from
\[
\left( \frac{1}{A} \right)^{\frac{m}{2}} + \left( \frac{1}{A} \right)^{\frac{l}{2}} > \left( \frac{1}{A} \right)^{\frac{m+l}{2}}.
\] (3.27)

By noticing that the sum of quantities less then 1, exceeds always their product, Eqs. (3.26) and (3.27) completely prove the fulfilment of the inequality (3.25). Note that the fulfilment of (3.25) is independent of the choice of \(l\) and \(m\). This shows that in the present case, the circle \(I_l\) is always exterior to the circle \(I_{-m}\) and therefore, from Ref. [31], \(I_m\) must lie in the interior of \(I_n\) with \(m = l + n > n\). From the arbitrariness of \(l\), we can say that if \(K = A > 1\), then \(I_{n+1}\) is contained in the interior of \(I_n\) for each \(n \in \mathbb{N}\). In this case, all properties (i) and (ii) for the isometric circles of a hyperbolic cyclic group stated at the beginning of the present section, are satisfied (which was our main interest for further developments). Of course, the case \(A < 1\) can equally be treated by establishing the fundamental fact that \(I_l\) is exterior to \(I_{-m}\) for each \(l\) and \(m\), and that \(I_{n+1}\) is contained in the interior of \(I_n\) for each \(n \in \mathbb{N}\).

**Case A < 1.**

Inequality (3.25) is replaced by
\[
A^{\frac{l}{2}} - A^{-\frac{l}{2}} + A^{-\frac{m}{2}} - A^{\frac{m}{2}} > A^{\frac{m+l}{2}} - A^{-\frac{m+l}{2}},
\] (3.28)
while conditions (3.26) and (3.27) should be replaced by
\[
A^{\frac{l}{2}} + A^{\frac{m}{2}} > A^{\frac{m+l}{2}},
\]
\[
\left( \frac{1}{A} \right)^{\frac{m}{2}} + \left( \frac{1}{A} \right)^{\frac{l}{2}} < \left( \frac{1}{A} \right)^{\frac{m+l}{2}},
\]
which lead to the fulfilment of Eq. (3.28). Thus, properties (i) and (ii) are satisfied also in case of a hyperbolic cyclic group with \(A < 1\).

We have just proved properties (i), (ii) and (iii) for a generic hyperbolic cyclic group. This circumstance will make it possible for us to obtain a consistent set of equations which can be derived by imposing the validity of Eqs. (3.1).

### 3.2. Some Remarks on Loxodromic Cyclic Groups

We revert to Eq. (3.24) and discuss the possibility to fulfil such equation in the case of a loxodromic cyclic group for which \(K = Ae^{i\theta}\). When \(\theta = 0\), such an inequality is always satisfied and thus the question arises about which circumstances might lead to its fulfilment when \(\theta \neq 0\). Note that in the limits \(l \to \infty\) or \(m \to \infty\), Eq. (3.24) is trivially satisfied letting \(I_{-m}\) be exterior to \(I_l\). However, nothing ensures that for low values of \(l\) and \(m\) this could occur, but some development might be obtained as well although the complexity which arises from direct calculations suggests a very strong correlation between the values of \(l, m\) and \(\theta\), and this complexity can be faced only via numerical computation whenever needed.
Let us note the presence of the following recurrent function in Eq. (3.24), which we can write as

$$F(\lambda) = |K^\lambda - K^{-\lambda}|,$$  \hspace{1cm} (3.29)
from which Eq. (3.24) reads as

$$F \left( \frac{l}{2} \right) + F \left( \frac{m}{2} \right) < F \left( \frac{l + m}{2} \right).$$  \hspace{1cm} (3.30)

We want to study the function (3.29) and obtain useful relations which might be used for further developments. Let us note that

$$\text{arg} \left( K^\lambda \right) = \lambda \theta.$$

We will be mainly interested in the smallest angle between the directions of $K^\lambda$ and $K^{-\lambda}$ which we will call $\theta_i$.

(1) Suppose that

$$0 < \lambda \theta < \frac{\pi}{2}.$$  \hspace{1cm} (3.31)

The interpretation for the value of the function $F(\lambda)$ is given in Fig. 1 and we are thus interested in the value of the angle $\theta_i = 2\lambda \theta$.

(2) Suppose now that

$$\lambda \theta = \frac{\pi}{2}.$$  \hspace{1cm} (3.32)

Thus the geometrical meaning of $F(\lambda)$ can be given as in Fig. 2, and the angle in

\[
\begin{align*}
\text{Fig. 1. Case 1}
\end{align*}
\]
which we are interested is $\theta_i = \pi$, while the function $F(\lambda)$ reduces to

$$F(\lambda) = A^\lambda + A^{-\lambda}.$$  

(3) Suppose that

$$\frac{\pi}{2} < \lambda \theta < \pi,$$  

hence the geometrical interpretation is the one shown in Fig. 3, where $\theta_i = 2\pi - 2\lambda\theta$.

(4) In the case

$$\lambda \theta = \pi,$$  

(3.34)
this leads to the picture in Fig. 4 where \( \theta_i = 0 \) and
\[
F(\lambda) = \left| A^\lambda - A^{-\lambda} \right|.
\]

(5) The case
\[
\pi < \lambda \theta < \frac{3\pi}{2},
\]
is shown in Fig. 5 and the angle \( \theta_i = 2\left[\pi - (2\pi - \lambda \theta)\right] = 2\lambda \theta - 2\pi \) is obtained.

(6) Suppose next that
\[
\lambda \theta = \frac{3\pi}{2},
\]
One therefore obtains the picture in Fig. 6 Then \( \theta_i = \pi \) and
\[
F(\lambda) = A^\lambda + A^{-\lambda}.
\]
(7) The case

\[ \frac{3\pi}{2} < \lambda \theta < 2\pi, \]

is represented in Fig. 7 and the angle \( \theta_i = 2(2\pi - \lambda \theta) = 4\pi - 2\lambda \theta \), is obtained as

(8) The last case is

\[ \lambda \theta = 2\pi, \]

in FIG. 8 where \( \theta_i = 0 \) and

\[ F(\lambda) = |A^\lambda - A^{-\lambda}|. \]
Fig. 8. Case 8

All the above cases are very useful for evaluating the function $F(\lambda)$. From the theorem of cosines, we can set

$$F(\lambda)^2 = A^2 + A^{-2\lambda} - 2 \cos \theta_i,$$

and one can obtain $F(\lambda)$ once the angle $\theta_i$ is known. From the above discussion it is evident that $\theta_i$ is a function of the $\lambda$ variable and one can recover the fact that such a function is a continuous, periodic and bounded function of $\lambda$ whose functional expression can be easily obtained. Let us sum up all the cases obtained so far:

1. $\lambda \theta \in ]0, \frac{\pi}{2}[$ $\Rightarrow \theta_i = 2\lambda \theta$.
2. $\lambda \theta = \frac{\pi}{2}$ $\Rightarrow \theta_i = \pi$.
3. $\lambda \theta \in ]\frac{\pi}{2}, \pi[$ $\Rightarrow \theta_i = 2\pi - 2\lambda \theta$.
4. $\lambda \theta = \pi$ $\Rightarrow \theta_i = 0$.
5. $\lambda \theta \in ]\pi, \frac{3\pi}{2}[$ $\Rightarrow \theta_i = 2\lambda \theta - 2\pi$.
6. $\lambda \theta = \frac{3\pi}{2}$ $\Rightarrow \theta_i = \pi$.
7. $\lambda \theta \in ]\frac{3\pi}{2}, 2\pi[$ $\Rightarrow \theta_i = 4\pi - 2\lambda \theta$.
8. $\lambda \theta = 2\pi$ $\Rightarrow \theta_i = 0$.

We notice that for $\lambda \theta > 2\pi$, several cases may occur but all of them return us a value of the $\theta_i$ function which falls back into one of the cases from (1) to (8). This enables us to state the periodicity (with period $2\pi$ in the $\lambda \theta$ variable) of the $\theta_i(\lambda \theta)$ function. A remark is also needed for cases (2), (4), (6) and (8): since Eq. (3.39) is continuous in the $\theta_i$, it is also continuous at the point $\theta_i = 0, \pi$.

Thus we can incorporate the case (2) into case (1), the case (4) into case 3), the case (6) into case (5) and the case (8) into case (7) by simply looking at the functional...
form of $\theta_i$ given in cases (1), (3), (5) and (7). Therefore, there are just four cases left:

- $(1') \lambda \theta \in \left[0, \frac{\pi}{2}\right] \implies \theta_i = 2\lambda \theta.$
- $(2') \lambda \theta \in \left[\frac{\pi}{2}, \pi\right] \implies \theta_i = 2\pi - 2\lambda \theta.$
- $(3') \lambda \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \implies \theta_i = 2\lambda \theta - 2\pi.$
- $(4') \lambda \theta \in \left[\frac{3\pi}{2}, 2\pi\right] \implies \theta_i = 4\pi - 2\lambda \theta.$

The functional form of $\theta_i(\lambda \theta)$ is (here $k \in \mathbb{Z}$)

$$\theta_i(\lambda \theta) = \begin{cases} 
2\lambda \theta - 2k\pi, & \lambda \theta \in \left[k\pi, (2k + 1)\frac{\pi}{2}\right] \\
-2\lambda \theta + 2k\pi, & \lambda \theta \in \left[(2k + 1)\frac{\pi}{2}, (k + 1)\pi\right]
\end{cases}$$

and it is represented in Fig. 9

![Plot of the function describing the internal angle](image)

The inequality (3.30), which is

$$F\left(\frac{l}{2}\right) + F\left(\frac{m}{2}\right) < F\left(\frac{l + m}{2}\right),$$

might be proved with the help of Eqs. (3.39) and (3.40). By recalling that such an inequality is satisfied in the limit $l \to \infty$ (and thus $m \to \infty$ from the relation $m = n + l$), one can ask whether there exists a minimum value of $l$ such that for
any fixed \( n \), the inequality is disproved. Notice that there is a strong correlation between the values of \( l, m \) and \( \theta \), thus in the perspective of further developments, it would be convenient to fix a definite value of \( \theta \) and try to obtain a lower bound, for the \( l \) variable and for a fixed \( n \), below which the inequality is not satisfied. Take for example the configuration of isometric circles of Ref. [31]. In that case, the isometric circles intersect each other for low values of \( n \), thus it should be convenient to start with low values of the indices \( n, l, m \).

As far as we can see, further developments are not immediate but nevertheless they might be obtained by taking the move from the results of the present subsection.

### 3.3. Correspondence Between Hyperbolic Cyclic Groups and Limit-Point Case at Both Ends

Our aim is now to establish the identification stated in Eq. (3.1) between the circles arising from the limit-point theory at both ends of the positive real line and the isometric circles of a hyperbolic cyclic group with multiplier \( K = A \in \mathbb{R}^+ - \{1\} \).

Suppose that the operator \( L \)

\[
Lx(r) = -(p(r)x(r))' + q(r)x(r),
\]

defined in \( \mathbb{R}^+ \), is in the limit-point case at both \( r = 0 \) and \( r = \infty \). If \( \varphi_1 \) and \( \varphi_2 \) are two linearly independent solutions of the equation

\[
Lx = lx, \quad l \in \mathbb{C},
\]

satisfying the conditions

\[
\varphi_1 (c, l) = 1, \quad \varphi_2 (c, l) = 0,
\]

\[
p(c) \varphi_1' (c, l) = 0, \quad p(c) \varphi_2' (c, l) = 1,
\]

then we can set up a self-adjoint problem on the interval \( \delta = [a,b] \subset \mathbb{R}^+ \), with \( c \in \delta \), given in Eq. (2.32) and thus there exists a countable set of eigenfunctions and eigenvalues of \( L \) from which the completeness and unitarity relation (also called Parseval equality) follows for every \( f \in \mathcal{L}^2(\delta) \). Following the arguments of Sect. 2, we know that if we choose two solutions in the form

\[
\chi_a (r) = \varphi_1 + m'_a \varphi_2, \quad \chi_b (r) = \varphi_1 + m_b \varphi_2,
\]

of Eq. (3.41), which also satisfy the following conditions:

\[
\cos \beta' \chi_a (a) + \sin \beta' p(a)\chi_a' (a) = 0, \quad \beta' \in [0, \pi],
\]

\[
\cos \beta \chi_b (b) + \sin \beta p(b)\chi_b' (b) = 0, \quad \beta \in [0, \pi],
\]

then, there remain defined two circles \( C'_a \) and \( C_b \), in the complex plane of the \( m'_a \) and \( m_b \) variables, whose equations are

\[
[\chi_a \chi_a] (a) = 0, \quad [\chi_b \chi_b] (b) = 0,
\]
or equivalently, whose centres and radii are
\[ C'_a : \tilde{m}'_a = \left( \frac{[\varphi_1 \varphi_2](a)}{2i \Im (l) \int_a^c dr' |\varphi_2|^2} \right), \quad r'_a = \left( \frac{2i \Im (l) \int_a^c dr' |\varphi_2|^2} {2i \Im (l) \int_a^c dr' |\varphi_2|^2} \right)^{-1}, \]
and
\[ C_b : \tilde{m}_b = -\left( \frac{[\varphi_1 \varphi_2](b)}{2i \Im (l) \int_c^b dr' |\varphi_2|^2} \right), \quad r_b = \left( \frac{2i \Im (l) \int_c^b dr' |\varphi_2|^2} {2i \Im (l) \int_c^b dr' |\varphi_2|^2} \right)^{-1}. \] (3.42)

The limit point case is characterized by the following limiting values for centres and radii:
\[ \lim_{a \to 0} r_a = 0, \quad \lim_{a \to 0} \tilde{m}'_a = m'_\infty \in \mathbb{C}, \]
\[ \lim_{b \to \infty} r_b = 0, \quad \lim_{b \to \infty} \tilde{m}_b = m_\infty \in \mathbb{C}, \] (3.43)
and \(m'_\infty\) and \(m_\infty\) are the values of the limit points at \(r = 0\) and \(r = \infty\), respectively.

We recall that the limit point condition at both ends of the interval guarantees the uniqueness of the spectral matrix of the problem, therefore the essential self-adjointness of the operator \(L\) on the whole positive real line. The existence of a complete set of eigenfunctions and eigenvalues belonging to the point spectrum and the continuum spectrum is ensured, hence every \(f \in L^2(\mathbb{R}^+)\) has a unique spectral decomposition in terms of the eigenfunctions of \(L\) over \(L^2(\mathbb{R}^+)\).

We want to obtain more useful relations by imposing the restriction contained in Eq. (3.1) in such a way that the circles \(C'_a\) and \(C_b\) so arising can be interpreted as the isometric circles of a hyperbolic cyclic group. In doing this, the discreteness of such a Kleinian group should let us reinterpret the limiting procedure of Eq. (3.43): we define a monotonic decreasing sequence of points \(\{a_n\}\) such that \(a_n \to 0\) and a monotonic increasing sequence of points \(\{b_n\}\) such that \(b_n \to \infty\)
\[ a_1 > a_2 > \cdots > a_n > ... > 0, \]
\[ b_1 < b_2 < \cdots < b_n < ... < \infty, \] (3.44)
and thus, in each interval \(\delta_n = [a_n, b_n] \subset \mathbb{R}^+,\) Eq. (2.32) defines a self-adjoint problem for the operator \(L.\) The limit-point condition at both ends guarantees the essential self-adjointness of \(L\) when the limit \(\delta_n \to \mathbb{R}^+\) is taken and hence, by using Eq. (3.42), the following circles remain defined:
\[ C'_{a_n} : z + \frac{[\varphi_1 \varphi_2](a_n)}{[\varphi_2 \varphi_2](a_n)} = \frac{1}{[\varphi_2 \varphi_2](a_n)}, \]
\[ C_{b_n} : z + \frac{[\varphi_1 \varphi_2](b_n)}{[\varphi_2 \varphi_2](b_n)} = \frac{1}{[\varphi_2 \varphi_2](b_n)}, \] (3.45)
in which the concise form has been used for
\[ [\varphi_2 \varphi_2](a) = -2i \Im (l) \int_a^c dr' |\varphi_2|^2, \quad [\varphi_2 \varphi_2](b) = 2i \Im (l) \int_c^b dr' |\varphi_2|^2, \] (3.46)
derived in Sect. 2. The limiting relations in Eq. (3.43) must be replaced by the following:

$$\lim_{n \to \infty} r_{an} = 0, \quad \lim_{n \to \infty} \tilde{m}'_{an} = m'_{\infty} \in \mathbb{C},$$

$$\lim_{n \to \infty} r_{bn} = 0, \quad \lim_{n \to \infty} \tilde{m}_{bn} = m_{\infty} \in \mathbb{C}.$$ 

By making use of Eqs. (3.45), (3.16) and (3.18), conditions (3.1) lead us to the system of equations

$$\begin{align*}
\int_{b_n}^{c_n} d\gamma |\varphi_2|^2 &\doteq \int_{a_n}^{c_n} d\gamma' |\varphi_2|^2 \\
\eta_n &= \frac{(\xi_1 - K^n \xi_2)}{(K^n - 1)} \doteq \frac{|\varphi_1 \varphi_2|}{|\varphi_1 \varphi_2|^2} = -\tilde{m}'_{an} \\
\mu_n &= \frac{\xi_1 - \xi_2}{K^n - K^{-1}} \doteq \frac{1}{25(t)} \int_{b_n}^{c_n} d\gamma' |\varphi_2|^2
\end{align*}$$

(3.47)

The first equation corresponds to requiring that the radii of the isometric circles $I_n$ and $I_n^{-1}$ coincide (and thus $r_{an} = r_{bn}$). Notice that $r_b^{-1}$ is a monotonic increasing function of the $b$ variable and it takes all values between 0 and $\infty$. The same holds for $r_a^{-1}$ which is a monotonic decreasing function of the $a$ variable, and takes all values between $\infty$ and 0. Thus, we are just claiming that $\varphi_2$ is not square integrable near infinity nor near zero (which was proved at the end of Sect. 2). Therefore, for any chosen increasing and divergent sequence $\{b_n\}$ there always exists a corresponding decreasing sequence $\{a_n\}$ such that $a_n \to 0$. The monotonic behavior of the functions $r_a^{-1}$ and $r_b^{-1}$, jointly with their common lower bound, tell us that for every $b_k \in \{b_n\}$ there exists one value $a_k$ such that $a_n \to 0$. The monotonic decreasing sequence $\{a_n\}$ is defined, providing the fulfilment of the first equation in Eqs. (3.47).

The second and the third equation in Eqs. (3.47) impose a strong restriction on the locus of the centres $\tilde{m}'_{an}$ and $\tilde{m}_{bn}$. Notice that for fixed value of $K$, $\xi_1$ and $\xi_2$, the cyclic group is uniquely defined and thus, the locus of points above which $\tilde{m}'_{an}$ and $\tilde{m}_{bn}$ are required to lie (from the definition claimed), must be a straight line. In fact, let us consider

$$\mu_n = \frac{(\xi_2 - K^n \xi_1)}{(K^n - 1)},$$

into Eqs. (3.47) which, for a fixed hyperbolic cyclic group with $K = A > 1$ (and fixed point chosen), describe a set of points of the complex plane obtained by letting the index $n$ vary. All these points have the same phase:

$$\arg(\mu_{n+1} - \mu_n) = \text{const.},$$

thus they have to lay on a straight line as first mentioned. Then, by setting

$$\mu(t) = \frac{\xi_2 - t \xi_1}{t - 1}, \quad t \in [1, \infty[, $$

one can see that this function is a continuous curve on the complex plane which has one end at the point

$$\mu(\infty) = -\xi_1,$$
hence $\mu(t)$ describes a segment on the complex plane and all points $\mu_n$ lie on such a segment. The same arguments hold for the points

$$\eta_n = \frac{\xi_1 - K^n \xi_2}{K^n - 1},$$

which have to lie on another segment of the complex plane. The limit-point condition at $r = 0$ and $r = \infty$, suggests that such points must lie on opposite halfplanes, i.e. $\mu_n$ has positive imaginary part for every $n$ (resp. negative imaginary part for every $n$) and $\eta_n$ has negative imaginary part for every $n$ (resp. positive imaginary part for every $n$). We also observe that, by setting

$$D_n = |\mu_{n+1} - \mu_n| = |\xi_1 - \xi_2| \left[ \frac{K^n(K - 1)}{(K^{n+1} - 1)(K^n - 1)} \right] = |\eta_{n+1} - \eta_n| = D'_n,$$

the sequence of distances $\{D_n = D'_n\}$ is a monotonic decreasing sequence for sufficiently large $K$. This last claim can be easily proved by simply setting $K^n = t$ and by differentiating the function

$$D(t) = \frac{at(K - 1)}{(Kt - 1)(t - 1)}, \quad t > 1, \ a > 0.$$ 

One obtains

$$\frac{dD}{dt}(K^n) < 0 \iff K^n > \frac{1}{\sqrt{K - 1}},$$

which is always satisfied for large $n$; but if we want that such an inequality should hold for every $n \in \mathbb{N}$, then we should search for the real roots of the equation

$$K^3 - K^2 - 1 = 0.$$

Such an equation has only one real root which is

$$\tilde{K} \approx 1.4656,$$

and this is the value which provides a lower bound above which the sequence $\{D_n\}$ is a monotonic decreasing sequence.
The fourth equation in Eqs. (3.47) can be satisfied by any monotonic sequence \( \{b_n\} \) (once the hyperbolic cyclic group is chosen, i.e. \( \xi_1, \xi_2 \) and \( K \) are fixed) provided that a function \( \varphi_2 \) not in \( L^2(c, \infty) \) nor in \( L^2(0, c) \) could be constructed somehow. We notice that it is rather convenient to start with a hyperbolic cyclic group instead of solving some limit-point, limit-circle problem on the positive real line. This is because the restrictions on the functions \( \varphi_1 \) and \( \varphi_2 \) given in the second and the third equations of Eqs. (3.47) are so strict that it is hopeless trying to fulfil them once \( \varphi_1 \) and \( \varphi_2 \) are already known as independent solutions of a limit-point problem. We suggest that the function \( \varphi_2 \) should be obtained by starting from the fourth equation of Eqs. (3.47) once that \( \xi_1, \xi_2 \) and \( K = A \in \mathbb{R}^+ - \{1\} \) have been fixed.

The problem here is to construct a pair of functions \( \varphi_1 \) and \( \varphi_2 \) which are not square integrable near infinity nor near zero, and which satisfy Eqs. (3.47) jointly with the system of conditions at \( c \in \delta_n = [a_n, b_n] \)

\[
\begin{align*}
\varphi_1 (c, l) &= 1, \quad \varphi_2 (c, l) = 0, \\
p(c)\varphi'_1 (c, l) &= 0, \quad p(c)\varphi'_2 (c, l) = 1.
\end{align*}
\]

In this way, \( \varphi_1 \) and \( \varphi_2 \) can be viewed as a system of independent solutions for the equation

\[
Lx = tx,
\]

and the form so chosen for them, should force the coefficients of the operator \( L \) to have a particular functional expression, and this is the main goal which one can hope to accomplish in a more advanced theory. Actually, in the context of the limit-point, limit-circle theory for second-order singular self-adjoint problems, a lot of efforts have been produced in this direction and a fruitful theory has been developed in Refs. [32,33]. Here it is shown how to construct the differential equation of a second-order singular self-adjoint problem by starting from the knowledge of its spectral function (see Sect. 2). In the literature, functions in Eq. (2.51) are also known as Weyl-Titchmarsh functions and they have been widely studied in the context of limit-point, limit-circle theory. We write them here below for problems with singular behaviour at both ends of the domain of definition for the operator \( L \):

\[
\begin{align*}
M_{11} (l) &= \frac{1}{m'_{\infty} (l) - m_{\infty} (l)}, \\
M_{12} (l) &= M_{21} (l) = \frac{1}{2} \frac{m'_{\infty} (l) + m_{\infty} (l)}{m'_{\infty} (l) - m_{\infty} (l)}, \\
M_{22} (l) &= \frac{m'_{\infty} (l) m_{\infty} (l)}{m'_{\infty} (l) - m_{\infty} (l)}.
\end{align*}
\]

One can thus see that these functions only depend upon the limit points \( m_{\infty} (l) \) and \( m'_{\infty} (l) \) on the complex plane. In Ref. [32] it is shown how to construct the function
$q(r)$ occurring in the operator $L$ in Eq. (3.41), by starting only from the knowledge of Weyl-Titchmarsh functions. This makes us hope that further developments might also be achieved in the context in which Eqs. (3.47) are meaningful.

**Existence of monotonic sequences satisfying the second and the third equations**

We now revert to the second and third equations in Eqs. (3.47). Let us refer only to the third equation for simplicity of reasoning

$$
\mu_n = \frac{(\xi_2 - K^n \xi_1)}{(K^n - 1)} = \frac{[\varphi_1 \varphi_2](b_n)}{[\varphi_2 \varphi_2](b_n)} = -\tilde{m}_{b_n}.
$$

We know that this equation defines a countable sequence of points which lie on a segment of the complex plane. Let us consider the case in which $K = A > 1$: one of the ends of such a segment is the point $-\xi_1$ while the distances between successive points are $D_n$ given in Eq. (3.48) where $D_n \to 0$ for $n \to \infty$ (if $K > \tilde{K} \approx 1.4656$, then $\{D_n\}$ is a monotonic decreasing sequence approaching zero). Therefore, we must require that the function

$$
-\tilde{m}_r = \frac{[\varphi_1 \varphi_2](r)}{[\varphi_2 \varphi_2](r)} (3.49)
$$

should intersect such a segment for $r > c \in \delta_n$ in correspondence of the values $r = b_n$ for $n \in \mathbb{N}$. The function in Eq. (3.49) is a continuous parametric curve of the $r$ variable on the complex plane which follows at once from the continuity of the functions $\varphi_1, \varphi_2, \varphi'_1$ and $\varphi'_2$. Hence, we are dealing with a continuous curve of the complex plane which intersects a given segment at most in a countable sequence of points $\{\mu_n\}$, while the distance between consecutive points of intersection decreases as $n \to \infty$. Of course, although this segment lies on the straight line which passes through the point $-\xi_1$ and which form an angle

$$
\theta = \arg (\mu_{n+1} - \mu_n) = \text{const}.
$$

with the positive direction of the real line (we will denote such a straight line with $\ell$), such a curve can always be viewed as the transformed curve of another curve which in turn intersects the real axes precisely at points $\tilde{\mu}_n$ for which

$$
|\tilde{\mu}_{n+1} - \tilde{\mu}_n| = |\mu_{n+1} - \mu_n| = D_n. \quad (3.50)
$$

and it is obtained by an isometry which brings the straight line $\ell$ into the real axes. Thus, it will be convenient to study the curve which intersects the real axis instead of Eq. (3.49).

Examples of curves which intersect the real axis in a countable sequence of points for which distances decrease as the points approach a finite limit, can be easily provided. Take for example the following curve of the complex plane:

$$
\tilde{\gamma}(r) = u(r) + iv(r),
$$
\[
\begin{aligned}
&\left\{ 
\begin{array}{l}
u(r) = 1 - e^{-r} \\
u(r) = \sin \left[ (1 - e^{-r}) \left( \prod_{n=1}^{5} r - 1 - \frac{1}{(\sigma)^{n}} \right) \right], \quad \sigma > 1, \ r \in [0, \infty[:
\end{array}
\right.
\end{aligned}
\]

whose imaginary part is plotted in Figs. 11 and 12 for \( \sigma = 1.1 \) (small and large values of \( r \), respectively). This function has three useful properties: in the limit \( r \to \infty \),

the curve reaches the point \( \tilde{z} = 1 \); the distance between two subsequent points of intersection with the positive real line, decreases when such points approach \( \tilde{z} = 1 \); among all points of intersection with the real line, five of them, i.e.

\[
\hat{\mu}_n = 1 + \frac{1}{(\sigma)^n},
\]
can be arbitrarily chosen by letting vary the number \( \sigma > 1 \).

Despite the properties mentioned earlier, the function (3.51) is not of the type we need because we cannot choose a number \( \sigma \) such that Eq. (3.50) is satisfied for every \( n \), thus we should look for another procedure which makes it possible to
construct the function (3.49) rather than simply guess its functional form. Notice
that the points of intersection with the real line for the curve (3.51), correspond
to the zeros of the function \( v(r) \) defined therein. Hence, when we act on Eq. (3.49)
with an isometry which brings \( \ell \) into the real line, we obtain a curve of the form
\[
\gamma(r) = x(r) + iy(r),
\]
(3.52)
and we should impose that the function \( y \) has an infinite number of zeros among
which it is possible to find a countable sequence \( \{b_n\} \) such that
\[
x(b_n) = \tilde{\mu}_n,
\]
(3.53)
and the sequence \( \{\tilde{\mu}_n\} \) is necessarily written as below:
\[
\tilde{\mu}_n = \tilde{\mu}_1 + \sum_{k=1}^{n-1} D_k,
\]
(3.54)
where \( D_k \) is given in Eq. (3.48). Therefore, once a hyperbolic cyclic group is chosen
and a function \( y(r) \) is so constructed that the set of its zeros contains at least the
sequence \( \{b_n\} \), then the third equation in Eqs. (3.47) can be easily fulfilled. Notice
that we can choose at will the value of \( \tilde{\mu}_1 \in \mathbb{R} \) in Eq. (3.54) but there is also another
constraint which arises form the third equation in Eqs. (3.47). We must have
\[
\lim_{n \to \infty} |\mu_n - \mu_1| = \left| \xi_1 + \frac{\xi_2 - K \xi_1}{K - 1} \right| = \frac{|\xi_1 - \xi_2|}{K - 1}, \quad K = A > 1,
\]
and hence we must also have
\[
\lim_{n \to \infty} \tilde{\mu}_n = \tilde{\mu}_1 + \frac{|\xi_1 - \xi_2|}{K - 1}, \quad (3.55)
\]
which is finite for every chosen cyclic hyperbolic group. However, Eq. (3.53) defines
the sequence \( \{b_n\} \) and it is not difficult to find a function \( x \) such that this sequence
can be made monotonic and divergent. Of course, this sequence is defined by
\[
b_n = x^{-1}(\tilde{\mu}_n).
\]
(3.56)
Thus, the function
\[
x : r \in [c, \infty[ \longrightarrow \left[ \tilde{\mu}_1, \tilde{\mu}_1 + \frac{|\xi_1 - \xi_2|}{K - 1} \right], \quad c \in \delta_n, \ n \in \mathbb{N},
\]
(3.57)
is required to be strictly monotonic and thus invertible in its domain. This is the
only requirement upon the function \( x \). Once \( x \) is so constructed, then there must
exist a divergent monotonic sequence \( \{b_n\} \). As an example of constructive process
for the \( x \) function, consider the \( u \) function into Eq. (3.51) defined on \( r \in [0, \infty[ \)
and its values are taken in the interval \( [0, 1] \).

If we can easily obtain some functional form for Eq. (3.57), then it is a rather
difficult task to guess a suitable form of \( y \) by simply knowing which are its zeros
defined in Eq. (3.56). We can nevertheless achieve its construction with the help
of the theory of canonical products in complex analysis [34]. As we will soon see,
given the divergent series (3.56), there always exists a representation of an entire function with zeros at \( \{ b_n \} \) and no other zeros.

Take for example the following infinite product of complex numbers

\[
P = \lim_{n \to \infty} P_n = \lim_{n \to \infty} \prod_{k=1}^{n} p_k = \prod_{k=1}^{\infty} p_k.
\]

(3.58)

If such a product is convergent, then \( p_k \) must tend to 1. This is clear from

\[
p_n = \frac{P_n}{P_{n-1}}.
\]

In view of this fact, it is convenient to write Eq. (3.58) as

\[
P = \lim_{n \to \infty} P_n = \prod_{k=1}^{\infty} (1 + \omega_k),
\]

(3.59)

where \( \omega_k \to 0 \) is a necessary condition for its convergence. Take the infinite sum

\[
S = \sum_{k=1}^{\infty} \log(1 + \omega_k),
\]

(3.60)

suppose that \( S \) is finite and denote its partial sum by \( S_n \). From the fact that \( \omega_n \) are complex numbers, we must choose the principal branch of the logarithm in each term of Eq. (3.60). We obviously have

\[
P_n = e^{S_n},
\]

and in the case in which \( S_n \to S \), we also have \( P_n \to P = e^S \neq 0 \). Thus, the convergence of the series (3.60) is a sufficient condition for the convergence of the infinite product (3.60). It can be proved that such a condition is also necessary. We can state the following theorem:

**Theorem 3.1.** The infinite product \( \prod_{k=1}^{\infty} (1 + \omega_k) \) with \( (1 + \omega_n) \neq 0 \) converges simultaneously with the series \( \sum_{k=1}^{\infty} \log(1 + \omega_k) \) whose terms represent the values of the principal branch of the logarithm.

This theorem proves that the problem of convergence for an infinite product can be always reduced to the more familiar question concerning the convergence of a series. It can also be noticed that the series (3.60) converges absolutely and simultaneously with the simpler series \( \sum_{k=0}^{\infty} |\omega_k| \). This can be deduced by the limit

\[
\lim_{z \to 0} \frac{\log(1 + z)}{z} = 1,
\]

jointly with the double inequality

\[
(1 - \epsilon)|\omega_n| < |\log(1 + \omega_n)| < (1 + \epsilon)|\omega_n|,
\]

which holds for \( \epsilon > 0 \) and large \( n \).

**Theorem 3.2.** A necessary and sufficient condition for the absolute convergence of the product \( \prod_{k=1}^{\infty} (1 + \omega_k) \) is the convergence of the series \( \sum_{k=1}^{\infty} |\omega_k| \).
Nevertheless, some examples can be found which show that the convergence of the series \( \sum_{k=1}^{\infty} \omega_k \) is neither sufficient nor necessary for the convergence of the infinite product (3.59). We can now revert to the problem of the convergence for infinite products whose factors are functions of a variable. This will be extremely useful for us because we might, in this way, obtain some representation of the function \( y(r) \) in Eq. (3.52) by using a generalization of the fundamental theorem of algebra as we will see.

Take an entire function \( g(z) \) on the complex plane. Then \( e^{g(z)} \neq 0 \) is an entire function as well. Conversely, if \( f(z) \neq 0 \) is an entire function, we can show that it can be always represented as \( e^{g(z)} \). We point out that the logarithmic derivative
\[
\frac{d}{dz} \log f(z) = \frac{f'(z)}{f(z)}
\]
is analytic in the whole plane and thus it is the derivative of an entire function \( g(z) \). From this fact, by direct computation, we can state that \( f(z)e^{-g(z)} \) has everywhere vanishing derivative, thus \( f(z) \) should be a constant multiple of \( e^{g(z)} \). The constant factor can be absorbed in the definition of the function \( g(z) \). This method leads us to a powerful representation of entire functions which may have zeros on the complex plane. Assume that \( f(z) \) has a zero at \( z = 0 \) of multiplicity \( s \) and a finite sequence of zeros \( b_1, b_2, \ldots, b_N \) (multiple zeros being repeated). From the above discussion, we can write such a function \([34,35]\) as
\[
f(z) = z^s e^{g(z)} \prod_{k=1}^{N} \left( 1 - \frac{z}{b_k} \right) \tag{3.61}
\]
If there exist infinitely many zeros, we can try to obtain a generalization of Eq. (3.61) by introducing an infinite product
\[
f(z) = z^s e^{g(z)} \prod_{k=1}^{\infty} \left( 1 - \frac{z}{b_k} \right) \tag{3.62}
\]
This last representation is valid if and only if the infinite product converges uniformly on every compact set of the complex plane. In fact, if this is the case, the infinite product occurring in Eq. (3.62) represents an entire function with zeros at the same points of \( f(z) \) and same multiplicity as \( f(z) \), and their quotient can be expressed as
\[
z^s e^{g(z)} = \frac{f(z)}{\prod_{k=1}^{\infty} \left( 1 - \frac{z}{b_k} \right)}.
\]
The product in Eq. (3.62), converges absolutely if and only if \( \sum_{k=1}^{\infty} \frac{1}{|b_k|} \) is convergent and in this case the convergence is also uniform in every compact disc with \( |z| < R \) for some \( R \). It is only under this special condition that we can obtain a representation of the form (3.62). But a method is available which makes it possible to introduce some convergence-producing factors for treating the general case. One
can prove the existence of polynomials \( p_k(z) \) such that, for any chosen sequence of complex numbers \( \{b_k\} \) and \( b_n \to \infty \), the function

\[
\prod_{k=1}^{\infty} \left( 1 - \frac{z}{b_k} \right) e^{p_k(z)},
\]

(3.63)

converges to an entire function and the product converges together with the series

\[
\sum_{k=1}^{\infty} r_k(z) = \sum_{k=1}^{\infty} \left[ \log \left( 1 - \frac{z}{b_k} \right) + p_k(z) \right],
\]

(3.64)

where the branch of the logarithm shall be chosen so that the imaginary part of the leading term \( r_k(z) \) lies in \([-\pi, \pi]\). For a given \( R \) we can consider the only terms with \( |b_k| > R \). In the region \(|z| < R\), the principal branch of \( \log \left( 1 - \frac{z}{b_k} \right) \) can be expanded in a Taylor series

\[
\log \left( 1 - \frac{z}{b_k} \right) = -\frac{z}{b_k} - \frac{1}{2} \left( \frac{z}{b_k} \right)^2 - \frac{1}{3} \left( \frac{z}{b_k} \right)^3 - \ldots.
\]

We reverse the signs and choose \( p_k(z) \) as partial sums

\[
p_k(z) = -\frac{z}{b_k} - \frac{1}{2} \left( \frac{z}{b_k} \right)^2 - \ldots - \frac{1}{s_k} \left( \frac{z}{b_k} \right)^{s_k}.
\]

In this way the leading term of the series (3.64) has the representation

\[
r_k(z) = -\frac{1}{(s_k + 1)} \left( \frac{z}{b_k} \right)^{s_k+1} - \frac{1}{(s_k + 2)} \left( \frac{z}{b_k} \right)^{s_k+2} - \ldots
\]

and we easily obtain the estimate

\[
|r_k(z)| \leq \frac{1}{(s_k + 1)} \left( \frac{R}{|b_k|} \right)^{s_k+1} \left( 1 - \frac{R}{|b_k|} \right)^{-1}.
\]

(3.65)

From the previous estimate, it follows \( r_k(z) \to 0 \) and by supposing that the series

\[
\sum_{k=1}^{\infty} \frac{1}{(s_k + 1)} \left( \frac{R}{|b_k|} \right)^{s_k+1},
\]

(3.66)

converges, it follows that \( \sum_{k=1}^{\infty} r_k(z) \) is absolutely and uniformly convergent for \(|z| \leq R\), and thus the product (3.63) represents an analytic function is such a disk.

**Theorem 3.3.** There exists an entire function with arbitrarily prescribed zeros \( b_n \) provided that, in the case of infinitely many zeros, \( b_n \to \infty \). Every entire function with these and no other zeros can be written in the form

\[
f(z) = z^s e^{g(z)} \prod_{k=1}^{\infty} \left( 1 - \frac{z}{b_k} \right) e^{\left[ \frac{s_k}{b_k} \left( \frac{z}{b_k} \right)^2 + \ldots + \frac{1}{s_k} \left( \frac{z}{b_k} \right)^{s_k} \right]},
\]

(3.67)
where the product is taken over all $b_n \neq 0$, the $s_k$ are certain integers, and $g(z)$ is an entire function.

This theorem is due to Weierstrass and it answers a problem which in the literature is known as the Weierstrass problem \[35\] for the representation of entire functions starting from the knowledge of their zeros. Functions of type (3.67) are also called entire transcendental functions. The representation (3.67) can be made considerably more interesting if we can choose all $s_k$ equal to each other. In the previous proof it has been shown that the product

$$\prod_{k=1}^{\infty} \left( 1 - \frac{z}{b_k} \right) e^{\left[ \frac{z}{b_k} + \frac{1}{2} (\frac{z}{b_k})^2 + \cdots + \frac{1}{h} (\frac{z}{b_k})^h \right]},$$

which is commonly called canonical product if $h$ is the smallest integer which makes convergent the following series for all $R$:

$$\sum_{k=1}^{\infty} \frac{1}{(h+1) \left\{ \frac{R}{|b_k|} \right\}^{h+1}} < \infty,$$

which may occur if

$$\sum_{k=1}^{\infty} \frac{1}{|b_k|^{h+1}} < \infty.$$

The integer $h$ is called the genus of the sequence $\{b_k\}$. Whenever possible it is rather convenient to use the canonical product into the representation (3.67) (which is uniquely determined). If in this representation $g(z)$ reduces to a polynomial, the function $f(z)$ is said to be of finite genus, and its genus is equal to the degree of this polynomial or equal to the genus of the canonical product, whichever is the larger. For instance, a function of genus zero is of the form

$$C z^m \prod_{k=1}^{\infty} \left( 1 - \frac{z}{b_k} \right),$$

with

$$\sum_{k=1}^{\infty} \frac{1}{|b_k|} < \infty.$$

A function with genus one has two representations. It can be of the form

$$C z^m e^{\alpha z} \prod_{k=1}^{\infty} \left( 1 - \frac{z}{b_k} \right) e^{\frac{z}{b_k}},$$

with

$$\sum_{k=1}^{\infty} \frac{1}{|b_k|} = \infty, \sum_{k=1}^{\infty} \frac{1}{|b_k|^2} < \infty.$$
or of the form
\[ Cz^m e^{\alpha z} \prod_{k=1}^{\infty} \left( 1 - \frac{z}{b_k} \right), \]
with
\[ \sum_{k=1}^{\infty} \frac{1}{|b_k|} < \infty, \alpha \neq 0. \]

As we can see, this Theorem 3.3 fits our expectations: we were interested in constructing the function \( y \) occurring in Eq. (3.52) by starting from the knowledge of its zeros \( b_n \), which are given in Eq. (3.56). We have already mentioned that such a sequence of zeros, can be always chosen as a strictly monotonic divergent sequence on the positive real line by simply requiring the monotonicity of the function \( x \) occurring in Eq. (3.52) and defined in Eq. (3.57). Recall that the strict monotonicity of \( \{b_n\} \) ensures that the circle \( C_{b_{n+1}} \) is contained in the interior of \( C_{b_n} \) for each \( n \) and thus we must require so for the accomplishment of the identification between isometric circles of a cyclic hyperbolic group and that of a limit-point theory. By using Eq. (3.56) jointly with Eq. (3.54), our \( y(r) \) function admits the following representation:
\[
y(r) = e^{g(r)} \prod_{k=1}^{\infty} \left( 1 - \frac{r}{x^{-1}(\tilde{\mu}_k)} \right) e^{\left[ \frac{r}{x^{-1}(\tilde{\mu}_k)} + \frac{1}{2} \left( \frac{r}{x^{-1}(\tilde{\mu}_k)} \right)^2 + \cdots + \frac{r}{x^{-1}(\tilde{\mu}_k)^k} \right]}, \tag{3.68}
\]
in which we have supposed that the value of \( \tilde{\mu}_1 \) into Eq. (3.54) is so chosen that none of the \( \tilde{\mu}_n \) is equal to zero (this can be easily obtained by simply setting \( \tilde{\mu}_1 > 0 \) and invoking the freedom we have upon such a variable).

A few more remarks are now in order. The first concerns the oscillations of the function \( y(r) \) which must tend to zero in the limit \( r \to \infty \). This is evident from Eqs. (3.52), (3.53) and (3.54) by taking the limit
\[
\lim_{n \to \infty} \gamma(b_n) = \tilde{\mu}_1 + \frac{\xi_1 - \xi_2}{K - 1} \in \mathbb{R},
\]
thus
\[
\lim_{r \to \infty} y(r) = 0.
\]
This last condition, for example, can be fulfilled by requiring \( g(r) = -r \) in Eq. (3.68).

The second remark concerns the possibility of using the representation given in Eq. (3.62), which is easier to handle, instead of that given in Eq. (3.68). Equation (3.62) can be used if and only if the series
\[
\sum_{k=1}^{\infty} \frac{1}{|b_k|} = \sum_{k=1}^{\infty} \frac{1}{|x^{-1}(\tilde{\mu}_k)|},
\]
is convergent. For example, if we choose

\[
x(r) = \left( \hat{\mu}_1 + \frac{|\xi_1 - \xi_2|}{K-1} \right) (1 - e^{-r}), \quad \hat{\mu}_1 > 0,
\]

(and this function satisfies Eqs. (3.54), (3.55), (3.57) and it is also monotonic in its domain, thus it provides the divergent series given in Eq. (3.56)) then, by setting \( \nu = \left( \hat{\mu}_1 + \frac{|\xi_1 - \xi_2|}{K-1} \right) \), there remains defined the sequence of values

\[
b_n = \log \left( \frac{\nu}{\nu - \beta_n} \right),
\]

(3.69)

where the \( \hat{\mu}_n \) form a bounded sequence from Eqs. (3.54) and (3.55). In this case, we can adopt the representation (3.62), if and only if the series

\[
\sum_{k=1}^{\infty} \frac{1}{\log \left( \frac{\nu}{\nu - \beta_k} \right)},
\]

(3.70)

converges. But this does not converge from the fact that there always exists some integer \( N \) such that for each \( n > N \) the following inequality holds:

\[
\left| \log \left( \frac{\nu}{\nu - \beta_n} \right) \right|^{-1} > \frac{1}{n}, \quad n > N.
\]

(3.71)

Equation (3.71) follows at once by noticing that \( \hat{\mu}_k \to \nu \), thus the terms in the series (3.70) are bounded from below by the terms of a harmonic series. In this case we cannot use the representation (3.62) but we hope that a suitable choice for the monotonic function \( x \) can be always made in such a way that the resulting series of the type (3.70) can be made convergent. In every case, the representation (3.68) can be always used.

The last remark concerns the possibility of treating in the same way the second equation in Eqs. (3.47). The point here is that we cannot apply directly Theorem 3.3, because in such a circumstance there should remain defined a monotonic decreasing sequence of points \( \{a_n\} \) instead of a monotonic increasing sequence, which is one of the hypotheses of Theorem 3.3. Nevertheless, we may expect that all the constructive procedure adopted in the last few pages for the fulfilment of the third equation in Eqs. (3.47), should also be applicable for the fulfilment of the second equation among Eqs. (3.47). This goal should be achieved by simply repeating all the reasoning here developed, but now for the function

\[
-\hat{m}'_\rho = \frac{[\varphi_1 \varphi_2] \left( \frac{1}{\rho} \right)}{[\varphi_2 \varphi_2] \left( \frac{1}{\rho} \right)}, \quad r = \frac{1}{\rho}, \quad r \in [0, c], \quad c \in \delta_n
\]

(3.72)

where the definition made for the \( r \) variable, enables us to replace the limit \( r \to 0 \) (which is the case of the second equation in Eqs. (3.47)) with the limit \( \rho \to \infty \). Therefore, Eq. (3.7) defines a parametric continuous curve on the complex plane in the \( \rho \) variable, while the second equation in Eqs. (3.47) forces such a curve to
intersect a particular segment of the complex plane which lies on a straight line \( \ell' \), which passes through the point \(-\xi_2\) and forms an angle \( \vartheta' = \arg(\eta_{n+1} - \eta_n) \), with the positive direction of the real axis. The result of the analysis should possibly end with the construction of a curve

\[
\gamma'(\rho) = x'(\rho) + iy'(\rho),
\]
(here the primed functions do not refer to their derivatives) where \( x'(\rho) \) should be taken as a monotonic invertible function to guarantee the existence of a decreasing sequence \( \{a_n\} \) such that \( a_n \to 0 \). The function \( y' \) should also be obtained from Eq. (3.68) (or Eq. (3.62) when the infinite product can be made convergent) by substituting the \( r \) variable with the \( \rho \) variable.

### 3.4. Further Considerations

In the previous subsection we have built a method which might lead to the fulfilment of the system of equations (3.47). The aim of the previous pages, is to set up a technique which can prove the existence of solutions for the variables

\[
\tilde{m}_{a_n}, \quad \tilde{m}_{b_n}, \quad a_n, \quad b_n, \quad \forall n \in \mathbb{N},
\]
(3.73)

where \( a_n \) and \( b_n \) satisfy the strict monotonicity of Eqs. (3.44). We have shown, by a constructive process, that such solutions always exist and \( a_n \) and \( b_n \) can always be chosen to fulfil Eqs. (3.44). This has been achieved by the exclusive use of the third and the second equations in Eqs. (3.47). The first and the fourth equations in Eqs. (3.47) can be satisfied in a very easy way once the solutions \( a_n \) and \( b_n \) are known. These latter equations bring us to another constructive process which, this time, involves the function \( \varphi_2 \) for values \( r > c \) and \( r < c \) (taking care to guarantee its differentiability at the point \( c \in \delta_n \)). Thus, after the functional form of \( \varphi_2 \) is taken, we should try to obtain the functional form of \( \varphi_1 \) for values \( r < c \) and \( r > c \) by retaining its differentiability at the point \( c \in \delta_n \) and by making use of the functions \( \tilde{m}_r \) and \( \tilde{m}_r' \). In doing this, we should require that

\[
\varphi_1(c,l) = 1, \quad \varphi_2(c,l) = 0,
\]
\[
p(c) \varphi'_1(c,l) = 0, \quad p(c) \varphi'_2(c,l) = 1,
\]
(3.74)

where this time \( p(c) \) should be considered as an unknown positive real number which represents the value of the coefficient \( p(r) > 0 \) of the equation

\[
Lx(r) = -(px(r))' + qx(r) = lx(r), \quad l \in \mathbb{C}, \quad r > 0,
\]
(3.75)

at the point \( c \). Equations (3.74) are extremely important because they are necessary conditions to let the functions \( \varphi_1 \) and \( \varphi_2 \) be independent solutions of Eq. (3.75). At this stage, the problem is to find the coefficients \( p(r) > 0 \) and \( q(r) \) in such a way that \( \varphi_1 \) and \( \varphi_2 \), which satisfy conditions (3.74), could be independent solutions of
the second order singular equation given in Eq. (3.75). As one can see, this is the inverse problem of finding solutions of a given differential equation. Such a problem has been studied in Refs. [32,33]. We note that our method depends on the value of the imaginary part of the complex number \( l \) occurring in (3.75). There are good possibilities to make the fixed points \( \xi_1 \) and \( \xi_2 \) dependent on the number \( l \), thus one should be able to obtain the spectral matrix from Eq. (2.50). This makes us hope that the problem of finding the differential equation by starting from the knowledge of an independent set of functions \( \varphi_1 \) and \( \varphi_2 \), in the context of the present section, might be solvable or at least partially solvable in future.

There are two main advantages which arise from the fulfillment of the system (3.47) and which should be merely taken in consideration from a purely physical point of view. The first one is that such a system of equations guarantees the essential self-adjointness of a second-order, singular operator at both ends of the positive real line of type (3.75). This cannot be only a mathematical property of operators, although the retained self-adjointness arising from a Kleinian group deserves by itself a careful consideration. The point here is that the limit-point theory has been extensively used in the context of the quantum theory for operators of type (2.6) (for example in Refs. [24,26] and it would be very surprising if some of them could give rise to a pair of independent solutions \( \varphi_1 \) and \( \varphi_2 \) and a pair of limit points \( m_\infty = \xi_1 \) and \( m'_\infty = \xi_2 \) which fulfil Eqs. (3.47) for some value of the parameter \( K \), because of the strict constraints inherent to the system (3.47). Anyway, we suggest the possibility of retaining some sort of generality in the decomposition of the functional space upon which quantum Hamiltonians should be defined. For example, we might require that

\[
\mathcal{L}^2 (\mathbb{R}^n, d^n x) = \mathcal{L}^2 (\mathbb{R}^+, \zeta (r) dr) \otimes \mathcal{L}^2 (\mathcal{T}, \mathcal{O}),
\]

(3.76)

where \( \mathcal{T} \) is a smooth topological manifold homeomorphic to the \( S^{n-1} \) sphere, \( dO \) is the measure on \( \mathcal{T} \), while \( \zeta (r) \) is a function of \( r \in \mathbb{R}^+ \) such that \( d^n x = \zeta (r) dr dO \).

If this decomposition can be achieved from a functional analytic point of view, then we could define a set of coordinates above the \( \mathcal{T} \) manifold and we could express the Laplacian \( P = -\Delta \) in terms of these coordinates by starting from its expression in orthonormal coordinates

\[
P = -\Delta = -\text{div grad} = -\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}.
\]

This procedure should lead to a decomposition of the \( \Delta \) operator which in turn might be written as

\[
\Delta = \Delta_r + \Delta_T,
\]

(3.77)

in which \( \Delta_r \) depends only on the \( r \) coordinate (and this is the case if the validity of Eq. (3.76) can be ensured) while \( \Delta_T \) is dependent on the other coordinates previously defined on \( \mathcal{T} \). Note that the operator \( \Delta_T \) reduces to the operator

\[
\Delta_S = -\frac{\tilde{l}^2}{\hbar^2 r^2},
\]
when $T$ is chosen to be the $S^{(n-1)}$ sphere ($\hat{L}^2$ is the squared angular momentum of the particle in $n$ dimensions). Therefore, when the decomposition (3.77) is allowed, we can regard the operator $\Delta_T$ to be strictly related to some sort of generalized angular momentum of a quantum particle which is also a conserved quantity, and thus we can study its self-adjointness properties as well as its spectrum. If it is self-adjoint, then the spectrum could also be obtained leading to a second-order differential equation (as it happens for Eq. (2.3)) in the radial variable, which can be treated with the machinery of the limit-point, limit-circle theory.

All the generality retained so far for the Laplacian operator, leads us to the second advantage which arises from the fulfilment of Eq. (3.47). In this case, a hyperbolic cyclic group is defined, and its discreteness is a necessary condition for the existence of non-constant automorphic functions. Let us denote with $T_1, T_2, \ldots$ all the elements of a generic Kleinian group $[36]$. Then we can define an automorphic function $[31]$ as

**Definition 3.1.** A function $f$ of the complex variable $z$, is said to be automorphic with respect to a group of linear transformations $T_1, T_2, \ldots$ provided that

1. $f(z)$ is a single-valued analytic function.
2. If $z$ lies in the domain of existence of $f(z)$, the same holds for $T_n(z)$.
3. $f(T_n(z)) = f(z)$.

It turns out that an automorphic function can be non-constant if and only if the group of transformations is a properly discrete group, i.e. there are no infinitesimal transformations. The existence for the domain of such functions is ensured by the following theorem:

**Theorem 3.4.** The domain of existence of an automorphic function extends into the neighborhood of every limit point of the group.

The hyperbolic cyclic groups considered in this section are Kleinian groups with two limit points, i.e. $\xi_1$ and $\xi_2$, thus, if an automorphic function exists, then its domain of existence must extend into the neighborhoods of the limit points of the group. Other properties of automorphic functions can be stated as well. For example, the limit points of the group are essential singularities for the automorphic functions.

In general, given a Kleinian group, nothing ensures the unicity of its automorphic function, but one can always establish their existence. In this context it can be extremely useful to consider the Poincaré $\theta$ series which is defined as

$$\theta(z) = \sum_{k=1}^{\infty} (c_k + d_k)^{-2m} H(T_k(z)), \quad (3.78)$$

where $c_k$ and $d_k$ are the coefficients of the fractional linear trasformation

$$T_k(z) = \frac{a_k z + b_k}{c_k z + d_k}, \quad a_k d_k - b_k c_k = 1, \quad (3.79)$$
The number of transformations which the Kleinian group contains and \( H(z) \) is any rational function of the \( z \) variable none of whose poles is at a limit point of the group. In this context, the group is viewed as being finite, but one can derive a convergent series (3.78) when the limit \( m \to \infty \) is taken. It can be shown that

\[
\theta(T_j(z)) = (c_jz + d_j)^{2m}\theta(z),
\]

and hence, the quotient between two Poincaré \( \theta \) series \( \theta_1(z) \) and \( \theta_2(z) \) leads to an automorphic function

\[
F(T_j(z)) = \frac{\theta_1(T_j(z))}{\theta_2(T_j(z))} = \frac{(c_jz + d_j)^{2m}\theta_1(z)}{(c_jz + d_j)^{2m}\theta_2(z)} = \frac{\theta_1(z)}{\theta_2(z)} = F(z). \tag{3.80}
\]

We can thus state the following theorem:

**Theorem 3.5.** If \( m \geq 2 \) and if the point at infinity is an ordinary point of the group, then the \( \theta \) series (3.78) defines a function which is analytic except possibly for poles in any connected region not containing limit points of the group in its interior.

From the fact that in a hyperbolic cyclic group the point at infinity is an ordinary point, this last theorem enables us to state that a convergent theta series can be always written down for this type of groups and thus, an automorphic function given in Eq. (3.80), can be always constructed for them.

When we are in the context of the isometries of an asymptotically flat space-time, we expect that a suitable group of isometries should give rise to some conserved quantity, i.e. a quantity which does not change when we transform some suitable system of coordinates by applying the isometries of which such a group is composed. It should be clear that conserved quantities are not merely constants if and only if the group of isometries is a discrete group. This is the second advantage of treating cyclic hyperbolic groups in physical context. If we can interpret discrete Kleinian groups [36] as discrete groups of isometries of an asymptotically flat space-time (and thus we are just treating discrete subgroups of the BMS group) then there remains defined a non-constant function which is unaffected when the system of coordinates is transformed according to such a discrete Kleinian group. We could also call such a “conserved” function a “constant of motion” for a particle whose motion is in accordance with the discreteness inherent to the Kleinian group considered, on the conformal infinity of an asymptotically flat space-time.

4. Concluding Remarks

In our paper we have studied the nature of fractional linear transformations in a general relativity context as well as in a quantum theoretical framework. Two features deserve special attention: the first is the possibility of separating the limit-point condition at infinity into loxodromic, hyperbolic, parabolic and elliptic cases. This is useful in a context in which one wants to look for a correspondence between
essentially self-adjoint spherically symmetric Hamiltonians of quantum physics and the theory of Bondi-Metzner-Sachs transformations in general relativity [4,5]. The analogy therefore arising, suggests that further investigations might be performed for a theory in which the role of fractional linear maps is viewed as a bridge between the quantum theory and general relativity.

The second aspect to point out is the possibility of interpreting the limit-point condition at both ends of the positive real line, for a second-order singular differential operator, which occurs frequently in applied quantum mechanics, as the limiting procedure arising from a very particular Kleinian group which is the hyperbolic cyclic group. In this framework, we have found in Sect. 3 that a consistent system of equations can be derived and studied. Hence we are led to consider the entire transcendental functions, from which it is possible to construct a fundamental system of solutions of a second-order differential equation with singular behavior at both ends of the positive real line, which in turn satisfy the limit-point conditions. Further developments in this direction might also be obtained by constructing a fundamental system of solutions and then deriving the differential equation whose solutions are the independent system first obtained. This guarantees two important facts at the same time: the essential self-adjointness of a second-order differential operator and the existence of a conserved quantity which is an automorphic function for the cyclic group chosen. By accomplishing this process, we hope that some sort of interpretation, in terms of discrete symmetries of space-time, might also be established. Moreover, it remains to be seen whether all basic properties of (global) general relativity have a quantum counterpart, if the correspondence that we have suggested is found to be viable.

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