ON GRADIENT RICCI SOLITONS

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Abstract. In the first part of the paper we derive integral curvature estimates for complete gradient shrinking Ricci solitons. Our results and the recent work in [14] classify complete gradient shrinking Ricci solitons with harmonic Weyl tensor. In the second part of the paper we address the issue of existence of harmonic functions on gradient shrinking Kähler and gradient steady Ricci solitons. Consequences to the structure of shrinking and steady solitons at infinity are also discussed.

1. Introduction and the results

A complete Riemannian metric $g$ on a smooth manifold $M$ is called a gradient Ricci soliton if there is a function $f$ so that $\text{Ric} + \text{Hess}(f) = \rho \cdot g,$ where $\rho \in \mathbb{R}$. After rescaling the metric $g$ we may assume that $\rho \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$. Gradient Ricci solitons arise often as singularity models of the Ricci flow and that is why understanding them is an important question in the field. Depending on the behavior of the Ricci flow on solitons, they are called shrinking if $\rho = \frac{1}{2}$, steady if $\rho = 0$ and expanding if $\rho = -\frac{1}{2}$.

The classification of gradient shrinking Ricci solitons has been a subject of interest for many people. Hamilton ([16]) showed that the only closed gradient shrinking Ricci solitons in two dimensions are Einstein. In three dimensions, Ivey proved that all compact, gradient shrinking Ricci solitons must have constant positive curvature. The recent work of Böhm and Wilking ([1]) implies the compact gradient shrinking Ricci solitons with positive curvature operator in any dimension have to be of constant curvature, generalizing Ivey’s result. In higher dimensions, Koiso, Cao, Feldman, Ilmanen and Knopf constructed examples of gradient shrinking Ricci solitons that are not Einstein. [5, 13].

The Hamilton-Ivey estimate shows that three dimensional complete solitons have nonnegative sectional curvatures. Combining this with the results of Perelman yields that the three dimensional gradient shrinking solitons with bounded sectional curvatures are $\mathbb{S}^3$, $\mathbb{R}^3$, $\mathbb{S}^2 \times \mathbb{R}$ and their quotients.

Recently, Ni and Wallach ([26]) have studied the classification of complete gradient shrinking Ricci solitons with vanishing Weyl curvature tensor, in any dimension, under the assumptions of nonnegative Ricci curvature and at most exponential growth of the norm of curvature operator. They showed that the only shrinkers satisfying these assumptions are $\mathbb{S}^n$, $\mathbb{R}^n$, $\mathbb{S}^{n-1} \times \mathbb{R}$, and their quotients. In [7] the assumption on nonnegative Ricci curvature has been relaxed to having the Ricci...
curvature bounded from below. Using a technique developed in the compact setting by [11], Petersen and Wylie have obtained [28] the same classification of complete locally conformally flat gradient shrinking solitons assuming only an integral bound of the Ricci curvature:

\[ \int_M |Ric|^2 e^{-f} < \infty. \]

In [32] Zhang proved that gradient shrinking Ricci solitons with vanishing Weyl tensor must have nonnegative curvature operator, which by any of the results mentioned above proved the classification of such solitons as finite quotients of \( \mathbb{R}^n \), \( \mathbb{S}^{n-1} \times \mathbb{R} \) or \( \mathbb{S}^n \).

The question whether certain integral curvature estimates such as (1.1) are true for complete gradient shrinking Ricci solitons has been raised for example in [28, 7, 2]. Besides being interesting on their own, such estimates would have as a consequence an alternate, simpler proof of the classification proved in [32] and should be useful in proving more general results. In this paper we prove that (1.1) is true for any gradient shrinker. In fact, we establish the following.

**Theorem 1.1.** For any complete gradient shrinking Ricci soliton \((M, g)\) we have

\[ \int_M |Ric|^2 e^{-\lambda f} < \infty, \text{ for any } \lambda > 0. \]

Another curvature quantity which is of interest for classification of shrinking solitons is \( \int_M |\nabla Ric|^2 e^{-f} \). If this integral is finite, it implies a useful identity (6)

\[ \int_M |\nabla Ric|^2 e^{-f} = \int_M |\text{div} (Rm)|^2 e^{-f} < \infty, \]

which is crucial in the classification result of [7], mentioned above. At this time, we do not know if (1.2) should hold true for any gradient shrinker. Our next result says that the identity is true assuming a weighted \( L^2 \) bound of the Riemann curvature tensor. In view of the right hand side of (1.2), such an assumption is quite natural.

**Theorem 1.2.** Let \((M, g)\) be a gradient shrinking Ricci soliton. If for some \( \lambda < 1 \) we have \( \int_M |Rm|^2 e^{-\lambda f} < \infty \), then the following identity holds:

\[ \int_M |\nabla Ric|^2 e^{-f} = \int_M |\text{div} (Rm)|^2 e^{-f} < \infty. \]

As a consequence, we can prove that (1.2) is true for gradient shrinking Ricci solitons with harmonic Weyl tensor. Furthermore, we have the following classification result for complete gradient shrinking Ricci solitons that have harmonic Weyl tensor. This extends the results from [26, 7, 28].

**Theorem 1.3.** Any \( n \)-dimensional complete gradient shrinking Ricci soliton with harmonic Weyl tensor is a finite quotient of \( \mathbb{R}^n \), \( \mathbb{S}^{n-1} \times \mathbb{R} \) or \( \mathbb{S}^n \).

The fact that shrinking and steady solitons have many properties common to manifolds with non-negative Ricci curvature motivates us to study the issue of existence of harmonic functions on these manifolds. It is known, see [20], that the existence of certain classes of harmonic functions is related to the existence of ends of the manifold. Some other results about the topology of shrinking Ricci solitons have been obtained in [30, 12].
We first recall some known terminology. A manifold is called nonparabolic if it admits a positive symmetric Green’s function. Otherwise it is called parabolic. An end of a manifold is called nonparabolic if it admits a positive symmetric Green’s function that satisfies the Neumann boundary condition on the boundary of the end. Otherwise, it is called parabolic.

We recall that a gradient shrinking Kähler–Ricci soliton satisfies
\[ R_{\alpha\bar{\beta}} + f_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} \]
for a smooth function \( f \) that has the property \( f_{\alpha\beta} = f_{\bar{\alpha}\bar{\beta}} = 0 \).

We will establish the following Liouville-type theorem for gradient shrinking Kähler–Ricci solitons.

**Theorem 1.4.** Let \((M, g)\) be a gradient shrinking Kähler–Ricci soliton. If \( u \) is a harmonic function with \( \int_M |\nabla u|^2 < \infty \) then \( u \) is a constant function.

Let us point out that on a manifold with a weighted volume \( e^{-f} dv \), a naturally defined operator is the \( f \)-Laplacian \( \Delta_f := \Delta - \nabla f \cdot \nabla \), which is self adjoint with respect to the weighted volume. Though this operator will be used in our proofs, let us point out that in Theorem 1.4 and everywhere in the paper the assumption of being harmonic refers to the usual Laplace operator i.e. \( \Delta u = 0 \).

As a consequence of the previous theorem we proved that any gradient shrinking Kähler–Ricci soliton has at most one nonparabolic end. Furthermore, if in addition to being Kähler we have an upper bound on the scalar curvature of the form \( \sup_M R < \frac{n}{2} - 1 \) on \( M \), then \( M \) is connected at infinity, i.e., it has one end.

Let us now consider the case of gradient steady Ricci solitons \((M, g)\), which by definition satisfy the equation
\[ \text{Ric} = \text{Hess}(f). \]

**Theorem 1.5.** Let \((M, g)\) be a gradient steady Ricci soliton. If \( u \) is harmonic with \( \int_M |\nabla u|^2 < \infty \) then \( u \) is constant on \( M \).

As a consequence, we have that any gradient steady Ricci soliton has at most one nonparabolic end. In the case we assume more on geometry of steady solitons we can prove the following structural result.

**Theorem 1.6.** If \( M \) is a gradient steady Kähler–Ricci soliton with Ricci curvature bounded below and such that for any \( x \in M \), \( \text{Vol}(B_x(1)) \geq C > 0 \) for some constant \( C \) independent of \( x \), then either it is connected at infinity or it splits isometrically as \( M = \mathbb{R} \times N \), for a compact Ricci flat manifold \( N \).

In proving these results, a good knowledge of the volume growth and asymptotics of the potential function is quite important. While for gradient shrinking solitons the results in [4] provide such estimates, not much is known for steady Ricci solitons. In this sense, we have established the following result.

**Theorem 1.7.** Let \((M, g)\) be any gradient steady Ricci soliton. There exist constants \( a, c, r_0 > 0 \) so that for any \( r > r_0 \),
\[ ce^{\alpha r^\gamma} \geq \text{Vol}(B_p(r)) \geq c \cdot r. \]

As a consequence of this Theorem, we can prove estimates for the potential function \( f \), see Corollary 5.2. Related to Theorem 1.7 we note that it is known that shrinking Ricci solitons with bounded Ricci curvature have at least linear volume...
growth, see [9]. This is a consequence of the log Sobolev inequality established by
Carillo and Ni in [8] and Perelman’s argument in [27].

The organization of the paper is as follows. In Section 2 we prove Theorem 1.1
and Theorem 1.2. We show how to use them to give the proof of Theorem 1.3. In
Section 3 we prove Theorem 1.4 about gradient shrinking Kähler-Ricci solitons. In
Section 4 we prove Theorem 1.5 and Theorem 1.6. In Section 5 we prove Theorem
1.7 about volume of any gradient steady Ricci soliton.

2. Integral curvature estimates for shrinking solitons

Let \((M, g)\) be a complete gradient shrinking Ricci soliton, given by equation

\[ R_{ij} + f_{ij} = \frac{1}{2}g_{ij}. \]

It is well known that after normalizing the potential function we have the following
set of identities satisfied by the soliton,

\[ R + \Delta f = \frac{n}{2}, \quad |\nabla f|^2 + R = f \quad \text{and} \quad \nabla_i \left( R_{ij} e^{-f} \right) = 0. \tag{2.1} \]

We have denoted with \(R\) the scalar curvature of \(M\). In [4] Cao and Zhou have
proved that for any fixed point \(p \in M\) there are uniform constants \(C, c > 0\) so that

\[ \text{Vol}(B_p(r)) \leq Cr^n \quad \text{and} \quad \frac{1}{4} (r(x) - c)^2 \leq f(x) \leq \frac{1}{4} (r(x) + c)^2, \tag{2.2} \]

where \(r(x) = \text{dist}(x, p)\). We note that asymptotic estimates for the potential func-
tion \(f\) were previously studied in [27, 25, 26, 12] and interesting volume growth
properties of solitons were also investigated in [8].

In [9] Chen proved that every complete ancient solution to the Ricci flow has
nonnegative scalar curvature for all times of their existence, see also Proposition 5.5
in [3]. In particular, this holds for gradient shrinking Ricci solitons and therefore
using (2.2) and (2.1) we have:

\[ 0 \leq R \leq \frac{(r(x) + c)^2}{4}. \tag{2.3} \]

In this section we will prove Theorem 1.1 and Theorem 1.2. We will say how to
use them to prove Theorem 1.3.

**Theorem 2.1.** For any complete gradient shrinking Ricci soliton \((M, g)\) we have

\[ \int_M |\text{Ric}|^2 e^{-\lambda f} < \infty, \quad \text{for any } \lambda > 0. \]

**Proof of Theorem 2.1.** For a cut-off function \(\phi\) on \(M\) we have, integrating by parts
and using (2.1):

\[ \int_M |\text{Ric}|^2 e^{-\lambda f} \phi^2 = \int_M R_{ij} \left( \frac{1}{2}g_{ij} - f_{ij} \right) e^{-\lambda f} \phi^2 \]

\[ = \frac{1}{2} \int_M R e^{-\lambda f} \phi^2 + \int_M f_i \nabla_j \left( R_{ij} e^{-\lambda f} \phi^2 \right) \]

\[ = \frac{1}{2} \int_M R e^{-\lambda f} \phi^2 + (1 - \lambda) \int_M R_{ij} f_i f_j e^{-\lambda f} \phi^2 + \int_M R_{ij} f_i e^{-\lambda f} (\phi^2)_j. \tag{2.4} \]
By simple algebraic manipulations we have:

\[
(1 - \lambda) \int_M R_{ij} f_i f_j e^{-\lambda f} \phi^2 \leq \frac{1}{4} \int_M |\text{Ric}|^2 e^{-\lambda f} \phi^2 + |1 - \lambda|^2 \int_M |\nabla f|^4 e^{-\lambda f} \phi^2 \\
\int_M R_{ij} f_i f_j e^{-\lambda f} (\phi^2)_j \leq \frac{1}{4} \int_M |\text{Ric}|^2 e^{-\lambda f} \phi^2 + 4 \int_M |\nabla f|^2 e^{-\lambda f} |\nabla \phi|^2.
\]

Notice that from (2.2) we know \( \int_M |\nabla f|^4 e^{-\lambda f} < \infty \) and \( \int_M R e^{-\lambda f} < \infty \). Therefore, from (2.3) it easily follows that \( \int_M |\text{Ric}|^2 e^{-\lambda f} < \infty \). This proves the Theorem.

Let us also point out that in the special case when \( \lambda = 1 \), (2.4) implies in particular that

\[
\int_M |\text{Ric}|^2 e^{-f} = \frac{1}{2} \int_M Re^{-f} < \infty.
\]

Next denote

\[
|\nabla \text{Ric}|^2 = \sum |\nabla_k R_{ij}|^2, \quad \text{div}(Rm)_{jkl} = \nabla_i R_{ijkl}\]

**Theorem 2.2.** Let \((M, g)\) be a gradient shrinking Ricci soliton. If for some \( \lambda < 1 \) we have \( \int_M |Rm|^2 e^{-\lambda f} < \infty \), then the following identity holds

\[
\int_M |\nabla \text{Ric}|^2 e^{-f} = \int_M |\text{div}(Rm)|^2 e^{-f} < \infty.
\]

**Proof of Theorem 2.2.** We use the following formulas, true for any gradient shrinking Ricci soliton:

\[
(2.5) \quad \nabla_k R_{ij} - \nabla_j R_{ik} = R_{kjhi} f_h \\
(2.6) \quad \nabla_i (R_{jkl} e^{-f}) = 0 \\
(2.7) \quad \text{div}(Rm)_{jkl} = R_{ikjp} f_p
\]

It is known that the Ricci curvature satisfies the equation

\[
\Delta_f R_{ij} = R_{ij} - 2 R_{ikjl} R_{kl}
\]

were \( \Delta_f R_{ij} := \Delta R_{ij} - \langle \nabla f, \nabla R_{ij} \rangle \) is the \( f \)-Laplacian of the Ricci tensor.

For a cut-off function \( \phi \) on \( M \) we have

\[
\int_M |\nabla \text{Ric}|^2 e^{-f} \phi^2 = -\int_M (\Delta_f R_{ij}) R_{ij} e^{-f} \phi^2 = \int_M (\nabla_k R_{ij}) R_{ij} e^{-f} (\phi^2)_k \\
- \int_M |R_{ij}|^2 e^{-f} \phi^2 + 2 \int_M R_{ikjl} R_{ij} R_{kl} e^{-f} \phi^2 - \int_M (\nabla_k R_{ij}) R_{ij} e^{-f} (\phi^2)_k
\]

The Riemann curvature term can be computed using the soliton equation:

\[
2 \int_M R_{ikjl} R_{ij} R_{kl} e^{-f} \phi^2 = \int_M |R_{ij}|^2 e^{-f} \phi^2 - 2 \int_M R_{ikjl} R_{ij} f_{kl} e^{-f} \phi^2.
\]

This gives

\[
\int_M |\nabla \text{Ric}|^2 e^{-f} \phi^2 = -2 \int_M R_{ikjl} R_{ij} f_{kl} e^{-f} \phi^2 - \int_M (\nabla_k R_{ij}) R_{ij} e^{-f} (\phi^2)_k.
\]
Using (2.6) we now get
\[-2 \int_M R_{ikjl}R_{ij}f_{kl}e^{-f} \phi^2 = 2 \int_M f_k \nabla_l (R_{ikjl}e^{-f} R_{ij} \phi^2)\]
\[= 2 \int_M f_k R_{ikjl}e^{-f} \nabla_l (R_{ij} \phi^2)\]
\[= 2 \int_M R_{ikjl} (\nabla_l R_{ij})f_k e^{-f} \phi^2 + 2 \int_M R_{ikjl} R_{ij} f_k e^{-f} (\phi^2)_{1}.\]

Notice moreover, using (2.5) and (2.7) that
\[2R_{ikjl} (\nabla_l R_{ij}) f_k = -2R_{ljik} (\nabla_l R_{ij}) = 2R_{ljik} f_k (\nabla_l R_{ij})\]
\[= (R_{ljik} f_k) (\nabla_j R_{il} - \nabla_l R_{ij}) = \text{div}Rm^2\]

This proves the Theorem.

Returning to (2.8) we see that all terms involving \(\nabla \phi\) imply that
\[\int_M \nabla R\text{ic}^2 e^{-f} \phi^2 = \int_M |\text{div}Rm|^2 e^{-f} \phi^2\]
\[+ 2 \int_M R_{ikjl} R_{ij} f_k e^{-f} (\phi^2)_{1} - \int_M (\nabla_k R_{ij}) R_{ij} e^{-f} (\phi^2)_{k}.\]

Our hypothesis and soliton identities imply that
\[\int_M |\text{div}Rm|^2 e^{-f} \leq C \int_M |\text{Rm}|^2 |\nabla f|^2 e^{-f} \leq C \int_M |Rm|^2 e^{-\lambda f} < \infty\]
\[\int_M |R_{ikjl} R_{ij} f_k \phi_l| e^{-f} \leq C \int_M |Rm|^2 |\nabla f|^2 e^{-f} \leq C \int_M |Rm|^2 e^{-\lambda f} < \infty,\]
for \(\lambda < 1\) given in the statement of Theorem 2.2. Hence, (2.8) and the arithmetic-mean inequality imply that
\[\int_M \nabla R\text{ic}^2 e^{-f} \phi^2 \leq C + 2 \int_M |\nabla_k R_{ij}| |R_{ij}| e^{-f} \phi |\nabla \phi|\]
\[\leq C + \frac{1}{2} \int_M |\nabla R\text{ic}^2 e^{-f} \phi^2 + 2 \int_M |R\text{ic}|^2 e^{-f} |\nabla \phi|^2.\]

This clearly shows that
\[\int_M |\nabla R\text{ic}^2 e^{-f} < \infty.\]

Returning to (2.8) we see that all terms involving \(\nabla \phi\) must converge to zero as \(r \to \infty\). More precisely, taking \(\phi\) such that \(\phi = 1\) on \(B_p(r), \phi = 0\) on \(M \setminus B_p(2r)\) and \(|\nabla \phi| \leq \frac{1}{4}\), it follows that as \(r \to \infty\)
\[\left| \int_M R_{ikjl} R_{ij} f_k e^{-f} (\phi^2)_{l} \right| \leq C \int_{B_p(2r) \setminus B_p(r)} |Rm|^2 e^{-\lambda f} \to 0,\]
\[\left| \int_M (\nabla_k R_{ij}) R_{ij} e^{-f} (\phi^2)_{k} \right| \leq C \left( \int_M |\nabla R\text{ic}|^2 e^{-f} \right)^{\frac{1}{2}} \left( \int_{B_p(2r) \setminus B_p(r)} |R\text{ic}|^2 e^{-f} \right)^{\frac{1}{2}} \to 0.\]

This proves the Theorem.

\[\square\]

Remark 2.3. In [7] the integral identity in Theorem 2.2 was established under a pointwise assumption on the Riemann tensor, that is,
\[|R_{ijkl}| (x) \leq e^{ar(x)+1}.\]
Then Shi’s derivative estimate will imply that $|\nabla \text{Ric}|$ has a similar growth and therefore $\int_M |\nabla \text{Ric}|^2 e^{-f} < \infty$. Then it follows that the integration by parts argument is valid in the noncompact setting. The advantage of our argument is that it requires only weak integral control of the Riemann tensor.

**Corollary 2.4.** Let $(M, g)$ be a gradient shrinking Ricci soliton with harmonic Weyl tensor. Then

$$\int_M |\nabla \text{Ric}|^2 e^{-f} = \int_M |\text{div} \text{Rm}|^2 e^{-f} < \infty.$$  

**Proof of Corollary 2.4.** We start with a formula established in the proof of Theorem 2.2:

$$\int_M |\nabla \text{Ric}|^2 e^{-f} \phi^2 = \int_M |\text{div} \text{Rm}|^2 e^{-f} \phi^2 + 2 \int_M R_{ijkl} R_{ij} e^{-f} (\phi^2)_l - \int_M (\nabla_k R_{ij}) R_{ij} e^{-f} (\phi^2)_k,$$

It is known that if the Weyl tensor is harmonic i.e. $\text{div} W = 0$, we have the following identity for gradient shrinkers, \[2.9\]:

$$R_{ijkl} \nabla_l f = \frac{1}{(n-1)} (R_{il} f^l g_{jk} - R_{jl} f^l g_{ik}).$$

Since on shrinking soliton we have the identity $(\text{div} \text{Rm})_{kji} = R_{ijkl} f^l$, by \[2.9\] we obtain

$$\int_M |\text{div} \text{Rm}|^2 e^{-f} \leq C \int_M |\text{Ric}|^2 |\nabla f|^2 e^{-f} \leq \int_M |\text{Ric}|^2 e^{-\mu f} < \infty,$$

for $\mu < 1$. Moreover, for a cut-off as in Theorem 2.2 we get:

$$\int_M |R_{ijkl} f_k R_{ij} (\phi^2)| e^{-f} \leq \frac{C}{r} \left( \int_M |\text{div} \text{Rm}|^2 e^{-f} + \int_M |\text{Ric}|^2 e^{-f} \right) \leq \frac{C}{r} \to 0 \text{ as } r \to \infty.$$

The rest of the proof is same as the proof of Theorem 2.2. □

Either Theorem 2.2 combined with results in [7] or Theorem 2.1 combined with the results in [28] can be used to show that any locally conformally flat gradient shrinking Ricci soliton is rigid. More generally, in the case of a harmonic Weyl tensor we have the following.

**Theorem 2.5.** Any $n$-dimensional complete shrinking gradient Ricci soliton with harmonic Weyl tensor is a finite quotient of $\mathbb{R}^n$, $\mathbb{S}^{n-1} \times \mathbb{R}$ or $\mathbb{S}^n$.

**Proof of Theorem 2.5.** In [14] it has been proved that once we have Corollary 2.4 then any gradient shrinking Ricci soliton with harmonic Weyl tensor must be a finite quotient as in the statement of the Theorem. This proves the Theorem. □

### 3. Gradient shrinking Ricci solitons and harmonic functions

As mentioned in the introduction, shrinking solitons have many properties in common with manifolds with non-negative Ricci curvature. For manifolds with non-negative Ricci curvature S.-T. Yau ([31]) proved that positive harmonic functions are necessarily trivial. The question is whether this generalizes to gradient shrinking
Ricci solitons. One motivation for studying the existence of harmonic functions comes from its relation to the structure of the manifold at infinity, that is, the number of ends.

Let \((M, g)\) be a gradient shrinking Kähler-Ricci soliton, that is,

\[
R_{\alpha\bar{\beta}} + f_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}},
\]

for a smooth function \(f\) that has the property

\[
(3.1) \quad f_{\alpha\beta} = f_{\bar{\alpha}\bar{\beta}} = 0.
\]

Note that in complex coordinates we have, for any \(u, v \in C^\infty(M)\)

\[
\langle \nabla u, \nabla v \rangle = \frac{1}{2} (u_{\alpha} v_{\bar{\alpha}} + u_{\bar{\alpha}} v_{\alpha})
\]

\[
\Delta u = u_{\alpha\bar{\alpha}}
\]

Our next result says there are no harmonic functions with bounded total energy on complete gradient shrinking Kähler-Ricci solitons.

**Theorem 3.1.** Let \((M, g)\) be a gradient shrinking Kähler-Ricci soliton. If \(u\) is a harmonic function with \(\int_M |\nabla u|^2 < \infty\) then \(u\) is a constant function.

**Proof of Theorem 3.1.** Let \(u\) satisfy \(\Delta u = 0\) on \(M\) and \(\int_M |\nabla u|^2 < \infty\). We first prove that \(\nabla f\) and \(\nabla u\) are orthogonal to each other. Then this implies that \(u\) is in fact \(f\)-harmonic and this fact forces \(u\) to be constant.

Let \(\phi : M \to [0, 1]\) be a cut off function such that \(\phi = 1\) on \(B_p(r)\) (a geodesic ball centered at some fixed point \(p \in M\) of radius \(r\)), \(\phi = 0\) outside \(B_p(2r)\) and \(|\nabla \phi| \leq \frac{C}{r}\).

We recall that, according to a result of P. Li in [18], if \(u\) is harmonic and with finite total energy on a Kähler manifold then it is in fact pluriharmonic, that is \(u_{\alpha\bar{\beta}} = 0\).

Let us define \(F \in C^\infty(M)\) to be

\[
F := \langle \nabla f, \nabla u \rangle = \frac{1}{2} (u_{\alpha} f_{\bar{\alpha}} + u_{\bar{\alpha}} f_{\alpha}).
\]

We show that \(F \equiv 0\). To this end, observe that

\[
(u_{\alpha} f_{\bar{\alpha}})_{\bar{\delta}} = u_{\alpha\bar{\delta}} f_{\bar{\alpha}} + u_{\alpha} f_{\bar{\delta} \bar{\alpha}} = 0,
\]

where we have used (3.1) and \(u_{\alpha\bar{\beta}} = 0\). Similarly, \((u_{\bar{\alpha}} f_{\alpha})_{\bar{\delta}} = 0\). This implies that

\[
\Delta F = 0,
\]

hence we have:

\[
\int_M |\nabla F|^2 \phi^2 = - \int_M (\Delta F) F \phi^2 - 2 \int_M F \phi \langle \nabla F, \nabla \phi \rangle
\]

\[
\leq 2 \int_M |\nabla F| |F| |\nabla \phi| \phi \leq \frac{1}{2} \int_M |\nabla F|^2 \phi^2 + 2 \int_M |F|^2 |\nabla \phi|^2.
\]

From here we get

\[
\int_M |\nabla F|^2 \phi^2 \leq 4 \int_M |F|^2 |\nabla \phi|^2 \leq 4 \int_M |\nabla u|^2 |\nabla f|^2 |\nabla \phi|^2
\]

\[
\leq C \int_{B_p(2r) \setminus B_p(r)} |\nabla u|^2 \to 0 \text{ as } r \to \infty.
\]
In the last line we have used that $|\nabla \phi| \leq \frac{C}{r}$, the fact that $|\nabla f|$ grows linearly on $M$ and the assumption that $u$ has finite total energy. This yields $F = \text{const}$ on $M$. The asymptotic behavior (2.2) of $f$ guarantees that $f$ attains its minimum somewhere on a compact subset of $M$ and therefore its gradient vanishes at the minimum point. In particular, that means $F \equiv 0$ on $M$. Define the $f$-Laplacian of a function to be

$$\Delta_f = \Delta - \langle \nabla f, \nabla \rangle.$$ 

We now prove that we have the following:

(3.2) $\Delta_f |\nabla u| \geq \frac{1}{2} |\nabla u|$, whenever $|\nabla u| \neq 0$.

Since we have proved $\langle \nabla u, \nabla f \rangle = 0$, it follows that $\Delta_f u = \Delta u - \langle \nabla f, \nabla u \rangle = 0$.

The Bochner formula implies

$$\Delta_f |\nabla u|^2 = 2Ric_f (\nabla u, \nabla u) + 2 |\nabla \Delta_f u, \nabla u| + 2 |u_{ij}|^2 \geq |\nabla u|^2 + 2 |\nabla |\nabla u||^2.$$ 

In the last line we have used the Kato inequality. Since on the other hand,

$$\Delta_f |\nabla u|^2 = 2 |\nabla u| \Delta_f |\nabla u| + 2 |\nabla |\nabla u||^2,$$

it is clear that we get (3.2).

Since in particular, $\Delta_f |\nabla u| \geq 0$ and $\int_M |\nabla u|^2 e^{-f} < \infty$, from a standard argument of Yau, see also [22] and Theorem 4.2 in [28] for the $f$–Laplacian case, it follows that $|\nabla u| = C$ on $M$. Then (3.2) implies that $|\nabla u| = 0$, hence $u$ is constant on $M$. □

**Corollary 3.2.** Let $M$ be a gradient shrinking Kähler-Ricci soliton. It then has at most one nonparabolic end.

**Proof of Corollary 3.2.** From the theory of Li and Tam [20] it is known that if a manifold $M$ has at least two nonparabolic ends, then there exists a bounded harmonic function $u$ on $M$ which has finite total energy $\int_M |\nabla u|^2 < \infty$. This is impossible by Theorem 3.1. □

The next result shows that under some upper bound for the scalar curvature of $M$, all ends are nonparabolic. Here we do not make the assumption of $M$ being Kähler.

**Proposition 3.3.** Let $(M, g)$ be a gradient shrinking Ricci soliton such that for some constant $\alpha$ we have $R \leq \alpha < \frac{n}{2} - 1$. Then all the ends of $M$ are nonparabolic.

**Proof of Proposition 3.3.** For $a = \frac{n}{2} - \alpha - 1 > 0$ we compute

$$\Delta f^{-a} = -af^{-a-1}\Delta f + a(a+1)|\nabla f|^2 f^{-a-2} \leq a \left( \frac{n}{2} - \alpha \right) f^{-a-1} - a(a+1) R f^{-a-2} \leq a \left( \alpha - \frac{n}{2} + a + 1 \right) f^{-a-1} = 0.$$ 

This proves that there exists a positive super-harmonic function which converges to zero at infinity. Then it is known [19] [15] that any end of $M$ (and hence $M$) is nonparabolic. □
Remark 3.4. If \( R = \frac{n}{2} - 1 \) the conclusion in Proposition 3.3 no longer holds. For example, \( S^{n-2} \times \mathbb{R}^2 \), where \( \mathbb{R}^2 \) is the Gaussian soliton has \( R = \frac{n}{2} - 1 \) and it is parabolic.

Combining Proposition 3.3 and Corollary 3.2 we conclude the following.

**Corollary 3.5.** If \((M, g)\) is a gradient shrinking Kähler-Ricci soliton with scalar curvature
\
R \leq \alpha < \frac{n}{2} - 1
\
for some constant \( \alpha \), then it is connected at infinity.

We conclude this section with the observation that under the hypothesis in Proposition 3.3 the manifold satisfies a Poincaré inequality, similar to Hardy’s inequality. Indeed, for any function \( \phi \) with compact support,
\
\int_M (\Delta f) f^{-1} \phi^2 = \int_M f^{-2} |\nabla f|^2 \phi^2 - 2 \int_M \phi f^{-1} \langle \nabla f, \nabla \phi \rangle
\
\leq (1 + \varepsilon) \int_M f^{-2} |\nabla f|^2 \phi^2 + \varepsilon^{-1} \int_M |\nabla \phi|^2.
\
On the other hand,
\
\int_M (\Delta f) f^{-1} \phi^2 \geq (\frac{n}{2} - \alpha) \int_M f^{-1} \phi^2 \quad \text{and} \quad \int_M f^{-2} |\nabla f|^2 \phi^2 \leq \int_M f^{-1} \phi^2.
\
Therefore, we arrive at
\
\varepsilon \left( \frac{n}{2} - \alpha - 1 - \varepsilon \right) \int_M f^{-1} \phi^2 \leq \int_M |\nabla \phi|^2.
\
Choosing \( \varepsilon = \frac{1}{2} \left( \frac{n}{2} - \alpha - 1 \right) > 0 \) we obtain a weighted Poincaré inequality
\
\int_M \rho \phi^2 \leq \int_M |\nabla \phi|^2, \quad \text{for} \quad \rho := \frac{1}{4} \left( \frac{n}{2} - \alpha - 1 \right)^2 f^{-1}.
\
Weighted Poincaré inequalities are known to be equivalent to the manifold being nonparabolic [21]. Moreover, \( M \) satisfies the property \((P\rho)\) in the sense of [21]. For gradient steady solitons a weighted Poincaré inequality was proved in [8].

### 4. Steady gradient Ricci solitons

In this section we will study the existence of harmonic functions on gradient steady solitons, which is, as we have mentioned earlier, tightly related to the structure of a given manifold.

Let \((M, g)\) be a gradient steady Ricci soliton, that is,
\
Ric = Hess(f).
\
In [17] Hamilton showed that
\
R + |\nabla f|^2 = \lambda,
\
for some constant \( \lambda > 0 \). Every steady soliton is in particular an ancient solution to the Ricci flow and therefore \( R \geq 0 \). This implies \( |\nabla f| \leq \sqrt{\lambda} \) and therefore
\( f(x) \leq f(p) + \sqrt{\lambda} r(x) \) (4.1)
\
where \( p \in M \) is a fixed point, \( r(x) = dist(x, p) \) and \( x \in M \) is an arbitrary point on \( M \). We claim the following result.
Theorem 4.1. If \((M, g)\) is a gradient steady Ricci soliton and \(\Delta u = 0\) with \(\int_M |\nabla u|^2 < \infty\), then \(u\) is a constant function.

Proof of Theorem 4.1. For a cut-off \(\phi\) on \(M\) it follows that

\[
\int_M \text{Ric}(\nabla u, \nabla u) \phi^2 = \int_M f_{ij} u_i u_j \phi^2 = -\int_M u_{ij} f_i u_j \phi^2 - \int_M f_i u_i (\phi^2)_j.
\]

Note that, integrating by parts we have:

\[
-\int_M u_{ij} f_i u_j \phi^2 = \frac{1}{2} \int_M (\Delta f) |\nabla u|^2 \phi^2 + \frac{1}{2} \int_M |\nabla u|^2 \langle \nabla f, \nabla \phi \rangle.
\]

Plug this in formula (4.2) and get that

\[
\int_M \text{Ric}(\nabla u, \nabla u) \phi^2 = \frac{1}{2} \int_M R |\nabla u|^2 \phi^2 + \frac{1}{2} \int_M |\nabla u|^2 \langle \nabla f, \nabla \phi \rangle.
\]

A similar integration by parts argument was used in [10]. We now recall the Bochner formula

\[
\Delta |\nabla u|^2 = 2 \text{Ric}(\nabla u, \nabla u) + 2 |u_{ij}|^2 \geq 2 \text{Ric}(\nabla u, \nabla u) + 2 |\nabla |\nabla u||^2.
\]

We multiply this by \(\phi^2\), use (4.3), and integrate by parts:

\[
2 \int_M |\nabla u|^2 \phi^2 + \int_M R |\nabla u|^2 \phi^2 \leq -\int_M \langle \nabla u \nabla^2 u, \phi \rangle
- \int_M |\nabla u|^2 \langle \nabla f, \nabla \phi \rangle + 2 \int_M \langle \nabla f, \nabla u \rangle \cdot \langle \nabla u, \nabla \phi \rangle.
\]

We have thus proved that

\[
\int_M |\nabla u|^2 \phi^2 + \int_M R |\nabla u|^2 \phi^2 \leq 4 \int_M |\nabla u|^2 |\nabla \phi|^2
- \int_M |\nabla u|^2 \langle \nabla f, \nabla \phi \rangle + 2 \int_M \langle \nabla f, \nabla u \rangle \cdot \langle \nabla u, \nabla \phi \rangle
\]

\[
\leq C \int_M |\nabla u|^2 |\nabla \phi|,
\]

where in the last line we have used that \(|\nabla f| \leq C\). Letting \(r \to \infty\) and using that \(u\) has finite total energy it results that \(|\nabla |\nabla u|| = R |\nabla u|^2 = 0\). This implies \(|\nabla u| = C\). But since \(M\) is nonparabolic, we know it has infinite volume. This is in fact true in general, see Theorem 1.7. But \(\int_M |\nabla u|^2 < \infty\), therefore \(|\nabla u| = 0\) and this proves the Theorem.

We have the analogous result to Corollary 3.2 in the case of gradient steady Ricci solitons.

Corollary 4.2. Let \(M\) be a gradient steady Ricci soliton. Then it has at most one nonparabolic end.
Proof of Corollary 4.2. As in the proof of Corollary 3.2, we apply the results in [20]. It is known that if a manifold $M$ has at least two nonparabolic ends, then there exists a bounded harmonic function $u$ on $M$ which has finite total energy $\int_M |\nabla u|^2 < \infty$. This is impossible by Theorem 4.1. □

If we assume more on geometry of $(M, g)$ we can say more about its ends. Notice that the previous Corollary does not tell us anything about parabolic ends if any.

We will prove Theorem 1.6 in two steps, depending whether $M$ is nonparabolic or parabolic, in the following two propositions.

Proposition 4.3. Assume $(M, g)$ is a complete, nonparabolic, gradient steady Kähler-Ricci soliton with Ricci curvature bounded below and such that for every $x \in M$, $Vol(B_x(1)) \geq C$, for a uniform constant $C > 0$. Then $M$ is connected at infinity.

Proof of Proposition 4.3. Since the manifold is nonparabolic, it has at least one nonparabolic end. Assume that $M$ has more than one end. By Corollary 4.2 we may assume it has a parabolic end, call it $H$. Then $E := M \setminus H$ is nonparabolic (since otherwise our manifold would be parabolic). By [23] (see also [24]) there exists a positive harmonic function $u$ on $M$ so that

(i) $\int_E |\nabla u|^2 < \infty$, $\inf_E u = 0$ and
(ii) $\lim_{x \to \infty, x \in H} u(x) = \infty$.

We will obtain a contradiction by following a similar argument as in Theorem 3.1. We first show that $u$ is pluriharmonic and use it to deduce that $\langle \nabla u, \nabla f \rangle = 0$.

Let us start with the observation that there exists a uniform constant $C$ so that

$$\sup_H |\nabla u| \leq C.$$  

Indeed, this was proved in [24], Theorem 2.1. Now we prove that there exists a uniform constant $C$ so that

$$\int_{B_p(r)} |\nabla u|^2 \leq Cr.$$  

Since $|\nabla u|$ is bounded on $M$, and $r(x) \leq r$ then $u(x) \leq C \cdot r$. Then, using the co-area formula it follows that

$$\int_{B_p(r)} |\nabla u|^2 = \int_{B_p(r) \cap E} |\nabla u|^2 + \int_{B_p(r) \cap H} |\nabla u|^2 \leq C + \int_{\{u \leq C \cdot r\} \cap H} |\nabla u|^2$$

$$= C + \int_0^{Cr} \int_{\{u = t\} \cap H} |\nabla u| \leq Cr.$$  

Note that since $u$ is harmonic it follows $\int_{u=1} |\nabla u| = \text{const}$. Lemma 3.1 in [18] and (4.2.5) now imply that $u$ is pluriharmonic. As in Theorem 3.1, let us denote

$$F := \langle \nabla f, \nabla u \rangle.$$  

Then, following the argument in Theorem 3.1 we get

$$\int_M |\nabla F|^2 g^2 \leq \frac{C}{r^2} \cdot \int_{B_p(2r) \setminus B_p(r)} |\nabla u|^2 |\nabla f|^2 \leq \frac{C}{r} \to 0 \text{ as } r \to \infty.$$  

We have used that $|\nabla f| \leq C$ in the case of a steady soliton, and (4.2.5). This implies that $F$ is constant on $M$ i.e., $\langle \nabla u, \nabla f \rangle = a \in \mathbb{R}$. Moreover, we can show that
in fact $a = 0$, because otherwise, $|a| = |\langle \nabla u, \nabla f \rangle| \leq |\nabla u| \cdot |\nabla f| \leq C |\nabla u|$, which implies $|\nabla u| \geq \delta > 0$ on $M$. Since $\int_E |\nabla u|^2 < \infty$ and any nonparabolic end $E$ has infinite volume \[19\], we get a contradiction. This proved indeed that

$$\langle \nabla u, \nabla f \rangle \equiv 0.$$  

Recall the inequality \[14\] obtained in the proof of Theorem \[1.1\]

$$\int_M |\nabla |\nabla u||^2 \phi^2 + \int_M R |\nabla u|^2 \phi^2 \leq 4 \int_M |\nabla u|^2 |\nabla \phi|^2$$

$$- \int_M |\nabla u|^2 \langle \nabla f, \nabla \phi^2 \rangle + 2 \int_M \langle \nabla f, \nabla u \rangle \cdot \langle \nabla u, \nabla \phi^2 \rangle.$$  

This holds true here as well, since $u$ is harmonic. Let us choose the cut-off $\phi$ as follows. On the nonparabolic end $E$ we define for $A$ large enough

$$\phi = \begin{cases} 
1 & \text{on } B_p(A) \cap E, \\
A + 1 - r & \text{on } (B_p(A + 1) \setminus B_p(A)) \cap E, \\
0 & \text{on } E \setminus B_p(A + 1). 
\end{cases}$$  

On the parabolic end $H$ we define for $T$ large enough

$$\phi = \begin{cases} 
1 & \text{on } u \leq T, \\
\frac{2T - u}{T} & \text{on } T < u < 2T, \\
0 & \text{on } 2T \leq u.
\end{cases}$$  

Notice that $\phi$ defined in this manner is indeed with compact support, as $u$ is proper on the parabolic end. Observe that since $\langle \nabla u, \nabla f \rangle = 0$ we have

$$\int_M \langle \nabla f, \nabla u \rangle \cdot \langle \nabla u, \nabla \phi^2 \rangle = 0,$$

$$\int_H |\nabla u|^2 \langle \nabla f, \nabla \phi^2 \rangle = 0.$$  

Notice that this is true for any $A$ and $T$. Moreover, we also have

$$\left| \int_E |\nabla u|^2 \langle \nabla f, \nabla \phi^2 \rangle \right| \leq C \int_{(B_{p(A+1)} \setminus B_p(A)) \cap E} |\nabla u|^2 \to 0 \text{ as } A \to \infty.$$  

Here we have used that $u$ has finite Dirichlet integral on the end $E$. It is not difficult to see by the co-area formula and since $|\nabla u|$ is bounded we also have

$$\int_M |\nabla u|^2 |\nabla \phi|^2 \to 0 \text{ as } T, A \to \infty.$$  

Hence, letting $A, T \to \infty$ in \[4.6\] we get that $|\nabla |\nabla u|| = R |\nabla u|^2 = 0$. This implies $|\nabla u| = C$. Since the energy of $u$ on the nonparabolic end is infinite, this implies $u = \text{const.}$ \[\square\]

We now discuss the case when $M$ is parabolic.

**Proposition 4.4.** Let $M$ be a parabolic gradient steady Kähler-Ricci soliton with Ricci curvature bounded below and such that for any $x \in M$, $\text{Vol}(B_x(1)) \geq C > 0$, for a uniform $C > 0$. Then either it is connected at infinity or it splits isometrically as $\mathbb{R} \times N$, for a compact Ricci flat manifold $N$.  


Proof of Proposition 4.4. Suppose $M$ has at least two parabolic ends (by assumption all its ends are parabolic). Let $E$ be one end, then $F := M \setminus E$ is another end of $M$.

There exists a harmonic function $u$ on $M$ such that

$$\lim_{x \to \infty, x \in E} u(x) = \infty \quad \text{and} \quad \lim_{x \to \infty, x \in F} u(x) = -\infty.$$ 

Applying Theorem 2.1 in [24] on $E$ and $F$ separately, it follows that $u$ has bounded gradient on each of the two ends, therefore

$$|\nabla u| \leq C \quad \text{on} \quad M.$$

Furthermore, with a similar argument as in the proof of (4.5) in Proposition 4.3 we obtain that

$$\int_{B_p(r)} |\nabla u|^2 \leq Cr,$$

for any $r > 0$ large enough. Applying again Lemma 3.1 in [18] we get that $u$ is pluriharmonic i.e. $u_{\alpha \bar{\beta}} = 0$. Denoting $F := \langle \nabla f, \nabla u \rangle$, by the argument in Theorem 3.1 we get

$$\int_M |\nabla F|^2 \phi^2 \leq \frac{C}{r^2} \cdot \int_{B_p(2r) \setminus B_p(r)} |\nabla u|^2 |\nabla f|^2 \leq \frac{C}{r} \to 0 \quad \text{as} \quad r \to \infty.$$

This implies that $F$ is constant on $M$ i.e.,

$$\langle \nabla u, \nabla f \rangle = a \in \mathbb{R}. \quad (4.7)$$

We use again the inequality (4.3) proved in Theorem 4.1, which also holds true in our setting, because $u$ is harmonic. We now choose the following cut-off $\phi$, defined on the level sets of $u$ (which are compact). For $T$ large enough let

$$\phi(x) = \begin{cases} 
0 & \text{on} \quad u \geq 2T, \\
\frac{2T-u}{T} & \text{on} \quad T < u < 2T, \\
1 & \text{on} \quad -T \leq u \leq T, \\
\frac{u+2T}{T} & \text{on} \quad -2T < u < -T, \\
0 & \text{on} \quad u \leq -2T.
\end{cases}$$

Observe now that for any $T$ we have

$$2 \int_M \langle \nabla f, \nabla u \rangle \cdot \langle \nabla u, \nabla \phi \rangle = 4a \int_M \phi \langle \nabla u, \nabla \phi \rangle$$

$$= -\frac{4a}{T^2} \int_{T < u < 2T} (2T - u) |\nabla u|^2 + \frac{4a}{T^2} \int_{-2T < u < -T} (u + 2T) |\nabla u|^2$$

$$= -\frac{4a}{T^2} \left( \int_T^{2T} (2T - t) \, dt \right) \int_{u=t} |\nabla u| + \frac{4a}{T^2} \left( \int_{-2T}^{-T} (t + 2T) \, dt \right) \int_{u=t} |\nabla u|$$

$$= -2a \int_{u=t} |\nabla u| + 2a \int_{u=t} |\nabla u| = 0.$$
We have used the co-area formula and, since \( u \) is harmonic, \( \int_{u=t} |\nabla u| \) is finite and independent of \( t \) for all \( t \in \mathbb{R} \). Moreover, similarly
\[
\int_M |\nabla u|^2 \langle \nabla f, \nabla \phi \rangle = 2 \int_{T<u<T} |\nabla u|^2 (2T - u) + 2a \int_{-2T<u<-T} |\nabla u|^2 (u + 2T)
\]
\[
= -a \int_{u=t} |\nabla u| + a \int_{u=t} |\nabla u| = 0.
\]
This shows that the last two terms involving \( \phi \) in (4.4) are in fact zero for all \( T \).

Moreover, by co-area formula and since \( |\nabla u| \) is bounded we see that
\[
\int_M |\nabla u|^2 |\nabla \phi|^2 \to 0 \text{ as } T \to \infty
\]
Then letting \( T \to \infty \) in (4.4) we conclude that \( |\nabla |\nabla u|| = R |\nabla u|^2 = 0 \). This implies \( |\nabla u| = C \). If \( C = 0 \) this means \( u = \text{const} \) and we are done. Otherwise, \( R \equiv 0 \), therefore \( \text{Ric} \equiv 0 \). This gives now that \( M \) is isometric to \( \mathbb{R} \times N \), for a compact Ricci flat manifold \( N \). This concludes the proof.

From Proposition 4.3 and Proposition 4.4 we obtain the following.

**Theorem 4.5.** If \( M \) is a steady gradient Kähler-Ricci soliton with Ricci curvature bounded below and such that for any \( x \in M \), \( \text{Vol}(B_x(1)) \geq C > 0 \) for some constant \( C \) independent of \( x \), then either it is connected at infinity or it splits isometrically as \( M = \mathbb{R} \times N \), for a compact Ricci flat manifold \( N \).

5. Volume of steady Ricci solitons

We now study the volume of gradient steady Ricci solitons. Our motivation is the results obtained in [2, 4, 8] for shrinking Ricci solitons. We recall that \((M, g)\) is a complete noncompact gradient steady Ricci soliton, i.e.

\[ \text{Ric} = Hess(f). \]

It is known that there exists a positive constant \( \lambda > 0 \) such that

\[ |\nabla f|^2 + R = \lambda. \]

**Theorem 5.1.** If \((M, g)\) is a gradient steady Ricci soliton there exist uniform constants \( a, c \) and \( r_0 \) so that for any \( r > r_0 \)

\[ ce^{a\sqrt{T}} \geq \text{Vol}(B_p(r)) \geq c^{-1} r. \]

**Proof of Theorem 5.1.** We first establish the volume lower bound. If for every \( r_0 \) we have \( \int_{B_p(r_0)} R = 0 \) then \( R \equiv 0 \) on \( M \). By

\[ \Delta f R = R - 2 |\text{Ric}|^2 \]

this implies \( \text{Ric} \equiv 0 \). In this case it is known that we have (5.1).

Assume there is an \( r_0 > 0 \) so that \( C_0 := \int_{B_p(r_0)} R > 0 \). Then, since \( R \geq 0 \), for \( r \geq r_0 \) we have

\[ C_0 \leq \int_{B_p(r)} R = \int_{B_p(r)} \Delta f = \int_{\partial B_p(r)} \frac{\partial f}{\partial n} \leq \int_{\partial B_p(r)} |\nabla f| \leq \sqrt{\lambda} \cdot A(\partial(B_p(r))), \]

where \( A(\partial(B_p(r))) \) is the area of the boundary of the ball of radius \( r \) centered at \( p \).
where we have used $|\nabla f| \leq \sqrt{\lambda}$ on a gradient steady soliton. This implies for $r \geq r_0$

$$\text{Area}(\partial B_p(r)) \geq c > 0,$$

for a uniform constant $c$. If we integrate the previous inequality over $[r_0, r]$ we obtain for $r \geq 2r_0$

$$\text{Vol}(B_p(r)) \geq c(r - r_0) \geq c_0 \cdot r.$$

We now prove the volume upper bound.

We denote by $dV|_{\text{exp}_p(r\xi)} = J(r, \xi) / \sqrt{\lambda}$ the volume form of $M$, where $\xi \in S_pM$.

We will omit the dependence on $\xi$.

It is known that along a normal minimizing geodesic starting from $p$,

$$\left(\frac{J'}{J}\right)'(r) + \frac{1}{n-1} \left(\frac{J'}{J}\right)^2(r) + \text{Ric} \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \leq 0.$$

We integrate this from 1 to $r \geq 1$ and use that $\text{Ric} \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = f''(r)$ to get

$$\frac{J'}{J}(r) + \frac{1}{n-1} \int_1^r \left(\frac{J'}{J}\right)^2(t) \, dt \leq -f'(r) + C_0,$$

for some constant $C_0 > 0$, independent of $r$. Let us denote

$$u(t) := \frac{J'}{J}(r).$$

Since $f$ has bounded gradient we get, for any $r \geq 1$,

$$u(r) + \frac{1}{n-1} \int_1^r u^2(t) \, dt \leq C.$$

Notice that by the Cauchy-Schwarz inequality it follows:

(5.2) \quad $u(r) + \frac{1}{(n-1)r} \left(\int_1^r u(t) \, dt\right)^2 \leq C.$

We claim that for any $r \geq 1$,

(5.3) \quad $\int_1^r u(t) \, dt \leq \sqrt{(n-1)Cr}.$

To prove this, define

$$v(r) := \sqrt{(n-1)Cr} - \int_1^r u(t) \, dt.$$

We prove (5.3) by showing that

$$v(r) \geq 0 \quad \text{for all} \quad r \geq 1.$$

Clearly, $v(1) > 0$. Assume by contradiction that $v$ is not positive for all $r \geq 1$, so let $r_0 > 1$ be the first number for which $v = 0$.

Since $v(r_0) = 0$, it follows that

$$\int_1^{r_0} u(t) \, dt = \sqrt{(n-1)Cr_0}.$$
By (5.2), this implies

\[ u(r_0) \leq C - \frac{1}{(n-1)r_0} \left( \int_1^{r_0} u(t) \, dt \right)^2 = C - \frac{1}{(n-1)r_0} (n-1) Cr_0 = 0. \]

Consequently, we obtain

\[ v'(r_0) = \sqrt{\frac{(n-1)C}{2r_0}} - u(r_0) > 0. \]

This implies the existence of a small enough \( \delta > 0 \) such that

\[ v(r_0 - \delta) < v(r_0) = 0, \]

which contradicts the choice of \( r_0 \).

We have proved that (5.3) is true for any \( r \geq 1 \), which by the definition of \( u \) means that

\[ \log J(r) - \log J(1) \leq \sqrt{(n-1)Cr}. \]

This proves that for any \( r \geq 1 \) we have an area bound of the form

\[ \text{Area}(B_p(r)) \leq Ce^{\sqrt{\lambda}}. \]

The Theorem is proved. \( \square \)

Let us note that the lower bound of the volume is sharp. Indeed, the product of the cigar soliton \( (\mathbb{R}^2, dx^2 + dy^2 + 1/x^2 + y^2) \) with any compact Ricci flat \( n-2 \) dimensional manifold is a nonflat steady Ricci soliton with linear volume growth. However, we do not know any examples of steady Ricci solitons with faster than polynomial volume growth. While the volume upper bound in Theorem 5.1 may not be sharp, it is still a very useful information. In fact, it is crucial in the following estimate for the potential function \( f \).

**Corollary 5.2.** Let \( (M, g) \) be a steady nonflat gradient Ricci soliton. Then there exist \( \lambda > 0 \) and \( c > 0 \) such that for any \( r \geq 1 \)

\[ \sqrt{\lambda} + \frac{c}{r} \geq \frac{1}{r} \sup_{\partial B_p(r)} f(x) \geq \sqrt{\lambda} - c\sqrt{\lambda}. \]

**Proof of Corollary 5.2.** It is known that there exists a constant \( \lambda > 0 \) such that

\[ R + |\nabla f|^2 = \lambda. \]

Since \( R \geq 0 \), we have \( |\nabla f| \leq \sqrt{\lambda} \), which proves the upper bound estimate for \( f \).

We now show the lower bound. We check directly that

\[ \Delta e^f = \left( \Delta f + |\nabla f|^2 \right) e^f = \lambda e^f. \]

Integrating this on \( B_p(r) \), it follows

\[ \lambda \int_{B_p(r)} e^f = \int_{\partial B_p(r)} \Delta e^f = \int_{\partial B_p(r)} \frac{\partial}{\partial r} (e^f) \leq \sqrt{\lambda} \int_{\partial B_p(r)} e^f. \]

In the last inequality, we have used that \( |\frac{\partial}{\partial r}| \leq |\nabla f| \leq \sqrt{\lambda} \). Denoting

\[ w(r) := \int_{B_p(r)} e^f \]

it follows from (5.4) that

\[ \sqrt{\lambda} w(r) \leq w'(r). \]
We integrate this from 1 to $r$ to conclude that $w(r) \geq c e^{\sqrt{\lambda} r}$, for some $c > 0$. By (5.4) this means that
\[
\int_{\partial B_p(r)} e^{f} \geq c e^{\sqrt{\lambda} r}, \text{ for any } r \geq 1.
\]
Combining with our area estimate from Theorem 5.1 we get:
\[
Ce^{\alpha \sqrt{r}} \left( \sup_{B_p(r)} e^{f} \right) \geq \left( \sup_{B_p(r)} e^{f} \right) \text{Area}(\partial B_p(r)) \geq \int_{\partial B_p(r)} e^{f} \geq c e^{\sqrt{\lambda} r}.
\]
This implies the lower bound for $f$ and proves the Corollary. \qed

We want to point out that in contrast to shrinking Ricci solitons, where we have (2.2), for steady solitons it is not possible to obtain such estimates for the potential. This is because if $M$ is a gradient steady Ricci soliton, then so is $\mathbb{R} \times M$, where the potential is now constant on $\mathbb{R}$. Thus, in this case the potential does not grow linearly on $M$, and in the general setting the result in Corollary 5.2 seems to be the best we can say. We should also note that Corollary 5.2 was independently proved by P. Wu in [29].

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