Prolongation of Poisson 2-form on Weil bundles

Norbert MAHOUNGOU MOUKALA*, Basile Guy Richard BOSSOTO†

Abstract

In this paper, $M$ denotes a smooth manifold of dimension $n$, $A$ a Weil algebra and $M^A$ the associated Weil bundle. When $(M, \omega_M)$ is a Poisson manifold with 2-form $\omega_M$, we construct the 2-Poisson form $\omega_M^A$, prolongation on $M^A$ of the 2-Poisson form $\omega_M$. We give a necessary and sufficient condition for that $M^A$ be an $A$-Poisson manifold.

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1 Introduction

1.1 Weil algebra and Weil bundle

In what follows, all structures are assumed to be of class $C^\infty$. We denote by $M$ a smooth differential manifold, $C^\infty(M)$ the algebra of differentiable functions on $M$ and by $\mathfrak{X}(M)$, the $C^\infty(M)$-module of vectors field on $M$.

A Weil algebra is a real, unitary, commutative algebra of finite dimension with a unique maximal ideal of codimension 1 on $\mathbb{R}$ [15].

Let $A$ be a Weil algebra and $m$ be its maximal ideal. We have $A = \mathbb{R} \oplus m$ and the first projection

$$A = \mathbb{R} \oplus m \longrightarrow \mathbb{R}$$

is a homomorphism of algebras which is surjective, called augmentation and the unique non zero integer $h \in \mathbb{N}$ such that $m^h \neq (0)$ and $m^{h+1} = (0)$ is the height of $A$ [15].

If $M$ is a smooth manifold, and $A$ a Weil algebra of maximal ideal $m$, an infinitely near point to $x \in M$ of kind $A$ is a homomorphism of algebras

$$\xi : C^\infty(M) \longrightarrow A$$

such that $[\xi(f) - f(x)] \in m$ for any $f \in C^\infty(M)$. i.e the real part of $\xi(f)$ is exactly $f(x)$ [15].

We denote by $M^A_x$ the set of all infinitely near points to $x \in M$ of kind $A$ and $M^A = \bigcup_{x \in M} M^A_x$ the manifold of infinitely near points of kind $A$. We have $\dim M^A = \dim M \times \dim A$ [7].

*nmahomouk@yahoo.fr
†bossotob@yahoo.fr
When both $M$ and $N$ are smooth manifolds and when $h : M \rightarrow N$ is a differentiable application, then the map 

$$h^A : M^A \rightarrow N^A, \xi \mapsto h^A(\xi),$$

such that, for any $g \in C^\infty(N)$,

$$[h^A(\xi)](g) = \xi(g \circ h)$$

is also differentiable. When $h$ is a diffeomorphism, it is the same for $h^A$ [2].

Moreover, if $\varphi : A \rightarrow B$ is a homomorphism of Weil algebras, for any smooth manifold $M$, the map

$$\varphi_M : M^A \rightarrow M^B, \xi \mapsto \varphi \circ \xi$$

is differentiable. In particular, the augmentation

$$A \rightarrow \mathbb{R}$$

defines for any smooth manifold $M$, the projection

$$\pi_M : M^A \rightarrow M,$$

which assigns every infinitely near point to $x \in M$ to its origin $x$. Thus $(M^A, \pi_M, M)$ defines the bundle of infinitely near points or simply Weil bundle [4], [7], [15], [9].

If $(U, \varphi)$ is a local chart of $M$ with coordinate functions $(x_1, x_2, ..., x_n)$, the application

$$U^A \rightarrow A^n, \xi \mapsto (\xi(x_1), \xi(x_2), ..., \xi(x_n)),$$

is a bijection from $U^A$ into an open of $A^n$. The manifold $M^A$ is a smooth manifold modeled over $A^n$, that is to say an $A$-manifold of dimension $n$ [1], [13].

The set, $C^\infty(M^A, A)$ of differentiable functions on $M^A$ with values in $A$ is a commutative, unitary algebra over $A$. When one identifies $\mathbb{R}^A$ with $A$, for $f \in C^\infty(M)$, the map

$$f^A : M^A \rightarrow A, \xi \mapsto \xi(f)$$

is differentiable and the map

$$C^\infty(M) \rightarrow C^\infty(M^A, A), f \mapsto f^A,$$

is an injective homomorphism of algebras and we have:

$$(f + g)^A = f^A + g^A; (\lambda \cdot f)^A = \lambda \cdot f^A; (f \cdot g)^A = f^A \cdot g^A$$

for $\lambda \in \mathbb{R}$, $f, g \in C^\infty(M)$.

We denote $\mathfrak{X}(M^A)$, the set of all vector fields on $M^A$. According to [11], [8] We have the following equivalent assertions:

**Theorem 1.** The following assertions are equivalent:

1. A vector field on $M^A$ is a differentiable section of the tangent bundle $(TM^A, \pi_{M^A}, M^A)$.
2. A vector field on $M^A$ is a derivation of $C^\infty(M^A)$.
3. A vector field on $M^A$ is a derivation of $C^\infty(M^A, A)$ which is $A$-linear.
4. A vector field on $M^A$ is a linear map $X : C^\infty(M) \to C^\infty(M^A, A)$ such that

$$X(f \cdot g) = X(f) \cdot g^A + f^A \cdot X(g), \text{ for any } f, g \in C^\infty(M).$$

Consequently [8].

**Theorem 2.** The map

$$\mathfrak{x}(M^A) \times \mathfrak{x}(M^A) \to \mathfrak{x}(M^A), (X, Y) \mapsto [X, Y] = X \circ Y - Y \circ X$$

is skew-symmetric $A$-bilinear and defines a structure of $A$-Lie algebra over $\mathfrak{x}(M^A)$.

Thus, if $Der_A[C^\infty(M^A, A)]$ denotes the $C^\infty(M^A, A)$-module of derivations of $C^\infty(M^A, A)$ which are $A$-linear, a vector field on $M^A$ is a derivation of $C^\infty(M^A, A)$ which is $A$-linear i.e a $A$-linear map

$$X : C^\infty(M^A, A) \to C^\infty(M^A, A)$$

such that

$$X(\varphi \cdot \psi) = X(\varphi) \cdot \psi + \varphi \cdot X(\psi), \text{ for any } \varphi, \psi \in C^\infty(M^A, A).$$

Thus, we have

$$\mathfrak{x}(M^A) = Der_A[C^\infty(M^A, A)].$$

**Proposition 3.** [1], [8] If $\theta : C^\infty(M) \to C^\infty(M)$ is a vector field on $M$, then there exists one and only one $A$-linear derivation

$$\theta^A : C^\infty(M^A, A) \to C^\infty(M^A, A)$$

such that

$$\theta^A(f^A) = [\theta(f)]^A$$

for any $f \in C^\infty(M)$.

**Proposition 4.** [1], [8] If $\theta, \theta_1$ and $\theta_2$ are vector fields on $M$ and if $f \in C^\infty(M)$, then we have:

1. $(\theta_1 + \theta_2)^A = \theta_1^A + \theta_2^A$,
2. $(f \cdot \theta)^A = f^A \cdot \theta^A$,
3. $[\theta_1, \theta_2]^A = [\theta_1^A, \theta_2^A]$.

**Corollary 5.** The map

$$\mathfrak{x}(M) \to Der_A[C^\infty(M^A, A)], \theta \mapsto \theta^A$$

is an injective homomorphism of $\mathbb{R}$-Lie algebras. If $\mu : A \to A$, is a $\mathbb{R}$-endomorphism, and $\theta : C^\infty(M) \to C^\infty(M)$ a vector field on $M$, then

$$\theta^A(\mu \circ f^A) = \mu \circ [\theta(f)]^A, \text{ for any } f \in C^\infty(M).$$
1.2 Poisson manifold

We recall that a Poisson structure on a smooth manifold $M$ is due to the existence of a bracket $\{,\}_M$ on $C^\infty(M)$ such that the pair $(C^\infty(M),\{,\}_M)$ is a real Lie algebra such that, for any $f \in C^\infty(M)$ the map

$$ad(f) : C^\infty(M) \to C^\infty(M), g \mapsto \{f,g\}_M$$

is a derivation of commutative algebra i.e

$$\{f,g \cdot h\}_M = \{f,g\}_M \cdot h + g \cdot \{f,h\}_M$$

for $f,g,h \in C^\infty(M)$. In this case we say that $C^\infty(M)$ is a Poisson algebra and $M$ is a Poisson manifold \cite{5,14,10}.

Let $\Omega_{\mathbb{R}}[C^\infty(M)]$ be the $C^\infty(M)$-module of K"ahler differentials of $C^\infty(M)$ and

$$\delta_M : C^\infty(M) \to \Omega_{\mathbb{R}}[C^\infty(M)], f \mapsto f \otimes 1_{C^\infty(M)} - 1_{C^\infty(M)} \otimes f$$

the canonical derivation which the image of $\delta_M$ generates the $C^\infty(M)$-module $\Omega_{\mathbb{R}}[C^\infty(M)]$ i.e for $x \in \Omega_{\mathbb{R}}[C^\infty(M)]$,

$$x = \sum_{i \in I \text{ finite}} f_i \cdot \delta_M(g_i),$$

with $f_i, g_i \in C^\infty(M)$ for any $i \in I$ \cite{3,10,11}.

The manifold $M$ is a Poisson manifold if and only if there exists a skew-symmetric 2-form

$$\omega_M : \Omega_{\mathbb{R}}[C^\infty(M)] \times \Omega_{\mathbb{R}}[C^\infty(M)] \to C^\infty(M)$$

such that for any $f$ and $g$ in $C^\infty(M)$,

$$\{f,g\}_M = -\omega_M[\delta_M(f),\delta_M(g)]$$

defines a structure of Lie algebra over $C^\infty(M)$ \cite{10,11}. In this case, we say that $\omega_M$ is the Poisson 2-form of the Poisson manifold $M$ and we denote $(M,\omega_M)$ the Poisson manifold of Poisson 2-form $\omega_M$.

The main goal of this paper is to study the prolongation of the Poisson 2-form $\omega_M$ of Poisson manifold on Weil bundles.

2 The algebra of K"ahler forms on $C^\infty(M^A, A)$

**Definition 1.** The $C^\infty(M^A)$-module of K"ahler differentials of $C^\infty(M^A)$ is the set

$$\Omega_{\mathbb{R}}[C^\infty(M^A)] = J \frac{\mathcal{J}}{J^2}$$

where $J$ is the $C^\infty(M^A)$-submodule of $C^\infty(M^A) \otimes C^\infty(M^A)$ generated by the elements of the form $F \otimes 1_{C^\infty(M^A)} - 1_{C^\infty(M^A)} \otimes F$ with $F \in C^\infty(M^A)$. Thus, the map

$$d_{M^A} : C^\infty(M^A) \to \Omega_{\mathbb{R}}[C^\infty(M^A)], F \mapsto F \otimes 1_{C^\infty(M^A)} - 1_{C^\infty(M^A)} \otimes F$$

is a derivation and the image of $d_{M^A}$ generates $\Omega_{\mathbb{R}}[C^\infty(M^A)]$. 

The $A$-algebra $C^\infty(M^A, A) \otimes_A C^\infty(M^A, A)$ admits a structure of $C^\infty(M^A, A)$-module defined by the homomorphism of $A$-algebras

$$C^\infty(M^A, A) \to C^\infty(M^A, A) \otimes_A C^\infty(M^A, A), \varphi \mapsto \varphi \otimes 1_{C^\infty(M^A, A)}.$$  

In this case, we say that $C^\infty(M^A, A) \otimes_A C^\infty(M^A, A)$ admits a structure of $C^\infty(M^A, A)$-module defined by the first factor. The second factor is defined by

$$C^\infty(M^A, A) \to C^\infty(M^A, A) \otimes_A C^\infty(M^A, A), \varphi \mapsto 1_{C^\infty(M^A, A)} \otimes \varphi.$$  

The map

$$C^\infty(M^A, A) \times C^\infty(M^A, A) \to C^\infty(M^A, A), (\varphi, \psi) \mapsto \varphi \cdot \psi$$  

being $A$-bilinear, then there exists a unique $A$-linear map

$$m : C^\infty(M^A, A) \otimes C^\infty(M^A, A) \to C^\infty(M^A, A)$$  

such that

$$m(\varphi \otimes \psi) = \varphi \cdot \psi.$$  

The kernel of $m$ is the $C^\infty(M^A, A)$-submodule of $C^\infty(M^A, A) \otimes C^\infty(M^A, A)$ generated by the elements of the form $\varphi \otimes 1_{C^\infty(M^A, A)} - 1_{C^\infty(M^A, A)} \otimes \varphi$ with $\varphi \in C^\infty(M^A, A)$.

We denote $\Omega_A[\omega C^\infty(M^A, A)]$, the $\omega C^\infty(M^A, A)$-module of Kähler differentials of $C^\infty(M^A, A)$ which are $A$-linears. In this case, for $\varphi \in C^\infty(M^A, A)$, we denote $\varphi \otimes 1_{C^\infty(M^A, A)} - 1_{C^\infty(M^A, A)} \otimes \varphi$, the class of $\varphi \otimes 1_{C^\infty(M^A, A)} - 1_{C^\infty(M^A, A)} \otimes \varphi$ in $C^\infty(M^A, A)$.

The map

$$C^\infty(M) \to \Omega_A[\omega C^\infty(M^A, A)], f \mapsto f^A \otimes 1_{C^\infty(M^A, A)} - 1_{C^\infty(M^A, A)} \otimes f^A$$  

is a derivation.

Thus,

**Proposition 6.** There exists a unique $A$-linear derivation

$$\delta^A_M : \omega C^\infty(M^A, A) \to \Omega_A[\omega C^\infty(M^A, A)]$$  

such that

$$\delta^A_M(f^A) = [\delta_M(f)]^A$$  

for any $f \in C^\infty(M)$.

**Proof.** Let

$$\delta^A_M : C^\infty(M^A, A) \to C^\infty(M^A, A) \sigma^{-1} \otimes_C C^\infty(M^A) \xrightarrow{id_A \otimes_M id^A} A \otimes \Omega_A[\omega C^\infty(M^A)] \xrightarrow{\varphi} \Omega_A[\omega C^\infty(M^A, A)]$$  

be that map, where

$$\sigma^{-1} : \varphi = \sum_{a=1}^{\dim A} (a^a \circ \varphi) \cdot a_a \mapsto \sum_{a=1}^{\dim A} a_a \otimes (a^a \circ \varphi)$$  

with $(a_a)_{a=1,\ldots,\dim A}$ a basis of $A$ and $(a^a)_{a=1,\ldots,\dim A}$ the dual basis of the basis $(a_a)_{a=1,\ldots,\dim A}$,

$$id_A \otimes_M d_{M^A} : \sum_{a=1}^{\dim A} a_a \otimes (a^a \circ \varphi) \mapsto \sum_{a=1}^{\dim A} a_a \otimes d_{M^A}(a^a \circ \varphi) = \sum_{a=1}^{\dim A} a_a \otimes [(a^a \circ \varphi) \otimes 1_{C^\infty(M^A)} - 1_{C^\infty(M^A)} \otimes (a^a \circ \varphi)].$$
\[
\delta_M : C^\infty(M) \rightarrow \Omega[\mathbb{R}[C^\infty(M)]
\]

is a derivation, then the map

\[
C^\infty(M) \rightarrow \Omega[\mathbb{R}[C^\infty(M,A)], f \mapsto [\delta_M(f)]^A
\]

is a derivation, then the map

\[
\delta_M : C^\infty(M) \rightarrow \Omega[\mathbb{R}[C^\infty(M)]
\]

is a derivation, then the map

\[
C^\infty(M) \rightarrow \Omega[\mathbb{R}[C^\infty(M,A)], f \mapsto [\delta_M(f)]^A
\]

Thus,

\[
\delta^A_M(\varphi) = \sum_{\alpha=1}^{\dim A} (a^\alpha \circ \varphi) a_\alpha \in \delta^A_M(\varphi) \cdot \varphi
\]

- For any \( \varphi, \psi \in C^\infty(M^A, A) \), we have

\[
\delta^A_M(\varphi + \psi) = [\varphi \circ (id_A \otimes d_{M^A}) \circ \sigma^{-1}](\varphi + \psi)
\]

- For any \( \varphi \in C^\infty(M^A, A) \) and \( \alpha \in A \), we have

\[
\delta^A_M(\varphi \cdot \psi) = [\varphi \circ (id_A \otimes d_{M^A}) \circ \sigma^{-1}](\varphi \cdot \psi)
\]

As

\[
\delta_M : C^\infty(M) \rightarrow \Omega[\mathbb{R}[C^\infty(M)]
\]

is a derivation, then the map

\[
C^\infty(M) \rightarrow \Omega[\mathbb{R}[C^\infty(M,A)], f \mapsto [\delta_M(f)]^A
\]
is a derivation. Thus, for any \( f \in C^\infty(M) \)

\[
\delta^A_{\mathcal{M}^A}(f^A) = \sigma \circ (id_A \otimes d_{M^A}) \circ \sigma^{-1}(f^A)
\]

\[
= \sum_{a=1}^{\dim A} (a^* \circ f^A)a_a \otimes 1_{C^\infty(M^A,A)} - 1_{C^\infty(M^A,A)} \otimes (a^* \circ f^A)a_a
\]

\[
= \sum_{a=1}^{\dim A} (a^* \circ f^A)a_a \otimes 1_{C^\infty(M^A,A)} - 1_{C^\infty(M^A,A)} \otimes \sum_{a=1}^{\dim A} (a^* \circ f^A)a_a
\]

\[
= f^A \otimes 1_{C^\infty(M^A,A)} - 1_{C^\infty(M^A,A)} \otimes f^A
\]

\[
= [f \otimes 1_{C^\infty(M)} - 1_{C^\infty(M)} \otimes \bar{f}]^A
\]

i.e

\[
\delta^A_{\mathcal{M}^A}(f^A) = [\delta_M(f)]^A.
\]

\[\square\]

**Proposition 7.** The map

\[\Omega_\mathbb{R}[C^\infty(M)] \to \Omega_A[C^\infty(M^A, A)], x \mapsto x^A\]

is an injective homomorphism of \( \mathbb{R} \)-modules.

**Proof.** Let

\[\Psi : \Omega_\mathbb{R}[C^\infty(M)] \to \Omega_A[C^\infty(M^A, A)], x \mapsto x^A\]

be that map. \[\square\]

For any \( x, y \in \Omega_\mathbb{R}[C^\infty(M)] \),

\[
\Psi(x + y) = (x + y)^A = (x^A + y^A)
\]

\[
= \left( \sum_{i \in \text{finite}} f_i \cdot \delta_M(f'_i) + \sum_{j \in \text{finite}} g_j \cdot \delta_M(g'_j) \right)^A
\]

\[
= \left( \sum_{i \in \text{finite}} f_i \cdot \delta_M(f'_i) \right)^A + \left( \sum_{j \in \text{finite}} g_j \cdot \delta_M(g'_j) \right)^A
\]

\[
= x^A + y^A.
\]

For any \( x \in \Omega_\mathbb{R}[C^\infty(M)] \) and for \( \lambda \in \mathbb{R} \),

\[
\Psi(\lambda \cdot x) = (\lambda \cdot x)^A
\]

\[
= (\lambda \cdot \sum_{i \in \text{finite}} f_i \cdot \delta_M(f'_i))^A
\]

\[
= \lambda \cdot \left( \sum_{i \in \text{finite}} f_i \cdot \delta_M(f'_i) \right)^A
\]

\[
= \lambda \cdot x^A.
\]

The pair \( (\Omega_A[C^\infty(M^A, A)], \delta^A_{\mathcal{M}^A}) \) satisfies the following universal property: for every \( C^\infty(M^A, A) \)-module \( E \) and every \( A \)-derivation

\[\Phi : C^\infty(M^A, A) \to E,\]

there exists a unique \( C^\infty(M^A, A) \)-linear map

\[\bar{\Phi} : \Omega_A[C^\infty(M^A, A)] \to E\]
such that
\[ \widetilde{\Phi} \circ \delta_{M^+}^A = \Phi. \]
In other words, there exists a unique \( \widetilde{\Phi} \) which makes the following diagram commutative
\[
\begin{array}{ccc}
\Omega_A[C^\infty(M^A, A)] & \xrightarrow{\delta_{M^+}^A} & C^\infty(M^A, A) \\
\downarrow{\delta_{M^+}^A} & & \downarrow{\Phi} \\
\end{array}
\]
This fact implies the existence of a natural isomorphism of \( C^\infty(M^A, A) \)-modules
\[
\text{Hom}_{C^\infty(M^A, A)}(\Omega_A[C^\infty(M^A, A)], E) \rightarrow \text{Der}_A[C^\infty(M^A, A)], E), \psi \mapsto \psi \circ \delta_{M^+}^A.
\]
In particular, if \( E = C^\infty(M^A, A) \), we have
\[
\Omega_A[C^\infty(M^A, A)]^* \cong \text{Der}_A[C^\infty(M^A, A)] = \mathfrak{L}(M^A).
\]
For any \( p \in \mathbb{N} \), \( \Lambda^p(\Omega_A[C^\infty(M^A, A)]) = \text{Hom}_{C^\infty(M^A, A)}(\Omega_A[C^\infty(M^A, A)], C^\infty(M^A, A)) \) denotes the \( C^\infty(M^A, A) \)-module of skew-symmetric multilinear forms of degree \( p \) from \( \Omega_A[C^\infty(M^A, A)] \) into \( C^\infty(M^A, A) \) and
\[
\Lambda(\Omega_A[C^\infty(M^A, A)]) = \bigoplus_{p \in \mathbb{N}} \Lambda^p(\Omega_A[C^\infty(M^A, A)])
\]
the exterior \( C^\infty(M^A, A) \)-algebra of \( \Omega_A[C^\infty(M^A, A)] \).
\[
\Lambda^0(\Omega_A[C^\infty(M^A, A)]) = C^\infty(M^A, A),
\]
\[
\Lambda^1(\Omega_A[C^\infty(M^A, A)]) = \Omega_A[C^\infty(M^A, A)]^*.
\]
We denote,
\[
\delta_{M^+}^A = \delta_{M^+} : \Lambda(\Omega_A[C^\infty(M^A, A)]) \rightarrow \Lambda(\Omega_A[C^\infty(M^A, A)])
\]
a unique derivation, of degree +1, which extends the canonical derivation
\[
\delta_{M^+}^0 : C^\infty(M^A, A) \rightarrow \Omega_A[C^\infty(M^A, A)].
\]
For any \( \varphi, \psi, \psi_1, \psi_2, ..., \psi_p \in C^\infty(M^A, A) \) and \( \omega \in \Omega_A[C^\infty(M^A, A)]^* \), we get
1. \( \delta_{M^+}^i(\varphi \cdot \delta_{M^+}^i(\psi_1) \wedge ... \wedge \delta_{M^+}^i(\psi_p)) = \delta_{M^+}^i(\varphi) \wedge \delta_{M^+}^i(\psi_1) \wedge ... \wedge \delta_{M^+}^i(\psi_p). \)
2. \( \delta_{M^+}^i(\psi \cdot \delta_{M^+}^0(\varphi)) = \delta_{M^+}^0(\psi) \wedge \delta_{M^+}^0(\varphi). \)
3. \( \delta_{M^+}^i(\varphi \cdot \omega) = \delta_{M^+}^0(\varphi) \wedge \omega + \varphi \cdot \delta_{M^+}^i(\omega). \)

**Proposition 8.** If \( \eta \in \Lambda^p(\Omega_A[C^\infty(M)]) \), then \( \eta^A \in \Lambda^p(\Omega_A[C^\infty(M^A, A)]) \).
Proof. Indeed, for any \( \eta \in \Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)]) \), \( \eta \) is of the form \( \delta_M(f_1) \wedge \cdots \wedge \delta_M(f_p) \) with \( f_1, f_2, \ldots, f_p \in C^\infty(M) \).

\[
\eta^A = [\delta_M(f_1) \wedge \cdots \wedge \delta_M(f_p)]^A
\]

\[
= [\delta_M(f_1)]^A \wedge \cdots \wedge [\delta_M(f_p)]^A
\]

\[
= \delta_M^0(\varphi_1) \wedge \cdots \wedge \delta_M^0(\varphi_p).
\]

Thus, the \( C^\infty(M^A, A) \)-module \( \Lambda^p(\Omega_A[C^\infty(M^A, A)]) \) is generated by elements of the form

\[
\eta^A = \delta_M^0(\varphi_1) \wedge \cdots \wedge \delta_M^0(\varphi_p)
\]

with \( \varphi_1 = f_1^A, \ldots, \varphi_p = f_p^A \in C^\infty(M^A, A). \)

The algebra

\[
\Lambda(\Omega_A[C^\infty(M^A, A)]) = \bigoplus_{p \in \mathbb{N}} \Lambda^p(\Omega_A[C^\infty(M^A, A)])
\]

is the algebra of Kähler forms on \( C^\infty(M^A, A) \).

The pair \( (\Lambda(\Omega_A[C^\infty(M^A, A)]), \delta_M^A) \) is a differential complex and the map

\[
A \times \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow \Omega_A[C^\infty(M^A, A)], (a, x) \longmapsto a \cdot x^A
\]

induces the morphism of the differential complex \( (A \otimes \Lambda(\Omega_{\mathbb{R}}[C^\infty(M)]), id_A \otimes \delta_M) \) into the differential complex \( (\Lambda(\Omega_A[C^\infty(M^A, A)]), \delta_M^A). \)

### 3 Lie derivative with respect to a derivation on \( M^A \)

Let

\[
\theta : C^\infty(M) \longrightarrow C^\infty(M)
\]

be a derivation and

\[
\sigma_{\theta} : [\Omega_{\mathbb{R}}[C^\infty(M)]]^p \longrightarrow \Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)]),
\]

be the \( C^\infty(M) \)-skew-symmetric multilinear map such that for any \( x_1, x_2, \ldots, x_p \in \Omega_{\mathbb{R}}[C^\infty(M)] \),

\[
\sigma_{\theta}(x_1, x_2, \cdots, x_p) = \sum_{i=1}^{p} (-1)^{i+1} \tilde{\theta}(x_i) \cdot x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_p,
\]

where

\[
\tilde{\theta} : \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow C^\infty(M)
\]

is a unique \( C^\infty(M) \)-linear map such that \( \tilde{\theta} \circ \delta_M = \theta \). Then,

\[
\sigma_{\theta}^A : [\Omega_A[C^\infty(M^A, A)]]^p \longrightarrow \Lambda^p(\Omega_A[C^\infty(M^A, A)])
\]

is a unique \( C^\infty(M^A, A) \)-skew-symmetric multilinear map such that

\[
\sigma_{\theta}^A(x_1^A, x_2^A, \ldots, x_p^A) = [\sigma_{\theta}(x_1, x_2, \ldots, x_p)]^A.
\]

We denote

\[
\tilde{\sigma}_{\theta}^A : \Lambda^p(\Omega_A[C^\infty(M^A, A)]) \longrightarrow \Lambda^{p-1}(\Omega_A[C^\infty(M^A, A)]),
\]

...
the unique $C^\infty(M^A, A)$-skew-symmetric multilinear map such that

$$
\widetilde{\sigma}^A_{\varphi^p}(x^A_1 \wedge x^A_2 \wedge \cdots \wedge x^A_p) = \sigma^A_{\varphi^p}(x^A_1, x^A_2, \ldots, x^A_p)
$$
i.e. $\sigma^A_{\varphi^p}$ induces a derivation

$$
i_p^p = \widetilde{\sigma}^A_{\varphi^p} : \Lambda(\Omega_A[C^\infty(M^A, A)]) \rightarrow \Lambda(\Omega_A[C^\infty(M^A, A)])
$$
of degree $-1$.

**Proposition 9.** For any $\vartheta \in \text{Der}_p[\Omega_\mathbb{R}[C^\infty(M)]$ and for any $\eta \in \Lambda^p(\Omega_\mathbb{R}[C^\infty(M)])$, we have

$$
i_\vartheta(\eta^A) = [i_\vartheta(\eta)]^A.
$$

**Proof.** If $\eta \in \Lambda^p(\Omega_\mathbb{R}[C^\infty(M)])$, then there exists $f_1, f_2, \ldots, f_p \in C^\infty(M)$, such that $\eta = \delta_M(f_1) \wedge \cdots \wedge \delta_M(f_p)$.

Thus,

$$
i_\vartheta(\eta^A) = i_\vartheta\left([\delta_M(f_1) \wedge \cdots \wedge \delta_M(f_p)]^A\right)
$$

$$
= i_\vartheta\left([\delta_M(f_1)]^A \wedge \cdots \wedge [\delta_M(f_p)]^A\right)
$$

$$
= \sigma^A_{\varphi^p}([\delta_M(f_1)]^A, \ldots, [\delta_M(f_p)]^A)
$$

$$
= [\sigma^p(\delta_M(f_1), \ldots, \delta_M(f_p))]^A
$$

$$
= [i_\vartheta(\delta_M(f_1)) \wedge \cdots \wedge \delta_M(f_p)]^A
$$

$$
= [i_\vartheta(\eta)]^A.
$$

For $p = 1$, we have

$$
i_\vartheta = \widetilde{\sigma}^A_{\varphi^1} : \Lambda^1(\Omega_A[C^\infty(M^A, A)]) = \Omega_A[C^\infty(M^A, A)]^* \rightarrow \Lambda^0(\Omega_A[C^\infty(M^A, A)]) = C^\infty(M^A, A),
$$

and for any $\gamma \in \Omega_\mathbb{R}[C^\infty(M)]$,

$$
i_\vartheta(\gamma^A) = \widetilde{\theta}^A(\gamma^A).
$$

For $p = 2$, we have

$$
\sigma^A_{\varphi^p} : \Omega_A[C^\infty(M^A, A)] \times \Omega_A[C^\infty(M^A, A)] \rightarrow C^\infty(M^A, A)
$$

and for any $x, y \in \Omega_\mathbb{R}[C^\infty(M)]$,

$$
\sigma^A_{\varphi^p}(x^A, y^A) = \widetilde{\theta}^A(x^A) \cdot y^A - \widetilde{\theta}^A(y^A) \cdot x^A.
$$

Thus, the map

$$
i_\vartheta : \Lambda^2(\Omega_A[C^\infty(M^A, A)]) \rightarrow \Omega_A[C^\infty(M^A, A)]^*
$$
is the unique $C^\infty(M^A, A)$-linear map such that

$$
i_\vartheta(x^A \wedge y^A) = \sigma^A_{\varphi^p}(x^A, y^A) = \widetilde{\theta}(x^A) \cdot y^A - \widetilde{\theta}(y^A) \cdot x^A.
$$

$\square$

**Definition 2.** The Lie derivative with respect to $D \in \text{Der}_A[C^\infty(M^A, A)]$ is the derivation of degree 0

$$
\mathcal{L}_D = i_D \circ \delta^A_M + \delta^A_M \circ i_D : \Lambda(\Omega_A[C^\infty(M^A, A)]) \rightarrow \Lambda(\Omega_A[C^\infty(M^A, A)]).
Proposition 10. For any $\theta \in \mathfrak{X}(M)$, the map

$$\mathcal{L}_{\theta^*} : \Lambda(\Omega_\Lambda[C^\infty(M^A, A)]) \rightarrow \Lambda(\Omega_\Lambda[C^\infty(M^A, A)])$$

is a unique $A$-linear derivation such that

$$\mathcal{L}_{\theta^*}(\eta^A) = [\mathcal{L}_{\theta}(\eta)]^A,$$

for any $\eta \in \Lambda(\Omega_\mathbb{R}[C^\infty(M)])$.

Proof. For any $\eta \in \Lambda(\Omega_\mathbb{R}[C^\infty(M)])$, we have

$$\mathcal{L}_{\theta^*}(\eta^A) = \eta^A + \delta_{M^A}([\eta^A])^A$$

$$= \eta^A + (\delta_{M^A}(\eta))^A$$

$$= \eta^A + (\delta_{M^A}(\eta))^A = (\delta_{M^A}(\eta))^A$$

$$= \mathcal{L}_{\theta}(\eta)^A.$$

$\square$

Proposition 11. For any $\theta \in \mathfrak{X}(M)$, for any $x \in \Omega_\mathbb{R}[C^\infty(M)]$ and for any $f \in C^\infty(M)$, we have

1. $$\mathcal{L}_{f^* \theta^*}(x^A) = [\mathcal{L}_{f \theta}(x)]^A.$$

2. $$\mathcal{L}_{f^* \theta^*}(f^A \cdot x^A) = [\mathcal{L}_{f \theta}(f \cdot x)]^A.$$

3. $$\mathcal{L}_{f^* \theta^*}[\delta_{M^A}(f^A)] = [\mathcal{L}_{f \theta}(\delta_{M^A}(f))]^A.$$

Proof. For any $\theta \in \mathfrak{X}(M)$, for any $x \in \Omega_\mathbb{R}[C^\infty(M)]$ and for any $f \in C^\infty(M)$, we have

1. $$\mathcal{L}_{f^* \theta^*}(x^A) = i_{f^* \theta^*}[\delta_{M^A}^A(x^A)] + \delta_{M^A}^A[i_{f^* \theta^*}(x^A)]$$

$$= f^A \cdot i_{\theta^*}[\delta_{M^A}^A(x^A)] + \delta_{M^A}^A[f^A \cdot i_{\theta^*}(x^A)]$$

$$= f^A \cdot i_{\theta^*}(\delta_{M^A}(x)^A) + \delta_{M^A}^A(f^A \cdot [i_{\theta^*}(x)])^A$$

$$= f^A \cdot i_{\theta^*}(\delta_{M^A}(x)^A) + i_{\theta^*}(x^A) \cdot \delta_{M^A}^A(f^A) + f^A \cdot \delta_{M^A}^A[i_{\theta^*}(x^A)]$$

$$= f^A \cdot (i_{\theta^*}(\delta_{M^A}(x)^A) + \delta_{M^A}^A(f)^A + f^A \cdot \delta_{M^A}^A[i_{\theta^*}(x)])$$

$$= f^A \cdot \delta_{M^A}(x^A) + \delta_{M^A}(f)^A + f \cdot \delta_{M^A}[i_{\theta^*}(x)]^A$$

$$= \delta_{M^A}(f^A) + f \cdot \delta_{M^A}[i_{\theta^*}(x)]^A$$

$$= \delta_{M^A}(f^A) + \delta_{M^A}[i_{\theta^*}(x)]^A$$

Thus,

$$\mathcal{L}_{f^* \theta^*}(x^A) = [\mathcal{L}_{f \theta}(x)]^A.$$
Proposition 12. For any $D \in \text{Der}_A[C^{\infty}(M^A, A)]$, $X \in \Omega_A[C^{\infty}(M^A, A)]$, and $\varphi \in C^{\infty}(M^A, A)$, we have

1. $\mathfrak{L}_{\varphi} D(X) = \varphi \cdot \mathfrak{L}_{D} (X) + \mathfrak{D}(X) \cdot \delta^A_{M^A} (\varphi)$;

2. $\mathfrak{L}_{D} (\varphi \cdot X) = D(\varphi) \cdot X + \varphi \cdot \mathfrak{L}_{D} (X)$;

3. $\mathfrak{L}_{D} \left[ \delta^A_{M^A} (\varphi) \right] = \delta^A_{M^A} [D(\varphi)]$.

Proof. For any $D \in \text{Der}_A[C^{\infty}(M^A, A)]$, $X \in \Omega_A[C^{\infty}(M^A, A)]$, and $\varphi \in C^{\infty}(M^A, A)$, we have

1. $\mathfrak{L}_{\varphi} D(X) = i_{\varphi} D[\delta^A_{M^A} (X)] + \delta^A_{M^A} [i_{\varphi} D(X)]$

2. $\mathfrak{L}_{D} (\varphi \cdot X) = \varphi \cdot i_{D} \left[ \delta^A_{M^A} (X) \right] + i_{D} (X) \cdot \delta^A_{M^A} (\varphi)$

3. $\mathfrak{L}_{D} \left[ \delta^A_{M^A} (\varphi) \right] = \delta^A_{M^A} [D(\varphi)]$. 
2. \( \Omega_D(\varphi \cdot X) = i_D[\delta^A_{M^\varphi}(\varphi \cdot X)] + \delta^A_{M^\varphi}[i_D(\varphi \cdot X)] \)

= \( i_D[\delta^A_{M^\varphi}(\varphi)X + \varphi \cdot \delta^A_{M^\varphi}(X)] + \delta^A_{M^\varphi}[\varphi \cdot i_D(X)] \)

= \( \tilde{D}[\delta^A_{M^\varphi}(\varphi)] \cdot \varphi \cdot i_D(X) + \varphi \cdot [\delta^A_{M^\varphi}(X)] \varphi \cdot i_D(X) \)

= \( D(\varphi) \cdot X + \varphi \cdot i_D[\delta^A_{M^\varphi}(X)] + \varphi \cdot \delta^A_{M^\varphi}[i_D(X)] \)

= \( D(\varphi) \cdot X + \varphi \cdot i_D(X) \).

3. \( \Omega_D[\delta^A_{M^\varphi}(\varphi)] \)

= \( i_D[\delta^A_{M^\varphi}(\varphi)] + \delta^A_{M^\varphi}[i_D(\varphi)] \)

= \( 0 + \delta^A_{M^\varphi}[\tilde{D} \circ \delta^A_{M^\varphi}(\varphi)] \)

= \( \delta^A_{M^\varphi}[D(\varphi)]. \)

\( \square \)

4 The Poisson 2-form on Weil bundles

We recall that, when \( M \) is a smooth manifold, \( A \) a Weil algebra and \( M^A \) the associated Weil bundle, the \( A \)-algebra \( C^\infty(M^A, A) \) is a Poisson algebra over \( A \) if there exists a bracket \( \{,\} \) on \( C^\infty(M^A, A) \) such that the pairing \( (C^\infty(M^A, A), \{,\}) \) is a Lie algebra over \( A \) satisfying

\( \{\varphi, \psi_1 \cdot \psi_2\} = \{\varphi, \psi_1\} \cdot \psi_2 + \psi_1 \cdot \{\varphi, \psi_2\} \)

for any \( \varphi, \psi_1, \psi_2 \in C^\infty(M^A, A) \). When \( C^\infty(M^A, A) \) is a Poisson \( A \)-algebra, we will say that the manifold \( M^A \) is a \( A \)-Poisson manifold \([2],[6]\).

When \( (M, \{,\}) \) is a Poisson manifold, the map

\( ad : C^\infty(M) \rightarrow Der_A[C^\infty(M)], f \mapsto ad(f) \)

such that \( [ad(f)](g) = [f, g] \) for any \( g \in C^\infty(M) \), is a derivation. Thus:

**Proposition 13.** There exists a derivation

\( ad^A : C^\infty(M^A, A) \rightarrow Der_A[C^\infty(M^A, A)] \)

such that

\( ad^A(f^A) = [ad(f)]^A. \)

Let’s consider the following diagram commutative:

\[
\begin{array}{ccc}
C^\infty(M^A, A) & \xrightarrow{\Phi} & Der_A[C^\infty(M^A, A)] \\
\uparrow \gamma_M & & \uparrow \Phi \\
C^\infty(M) & \xrightarrow{ad} & Der_R[C^\infty(M)]
\end{array}
\]
\[ \tilde{\tau} \circ \gamma_M = \Phi \circ ad, \]

where
\[ \gamma_M : C^\infty(M) \rightarrow C^\infty(M^A, A), f \mapsto f^A \]

and
\[ \Phi : \text{Der}_R[C^\infty(M)] \rightarrow \text{Der}_A[C^\infty(M^A, A)], \theta \mapsto \theta^A. \]

For any \( f \in C^\infty(M) \), we have
\[ \tilde{\tau} \circ \gamma_M(f) = \tilde{\tau}(f^A) \]

and
\[ \Phi \circ ad(f) = \Phi[ad(f)] = [ad(f)]^A. \]

Thus, there exists \( ad^A = \tilde{\tau} \) such that
\[ ad^A(f^A) = [ad(f)]^A. \]

As
\[ ad^A : C^\infty(M^A, A) \rightarrow \text{Der}_A[C^\infty(M^A, A)] \]

is a derivation, then there exists a unique \( C^\infty(M^A, A) \)-linear map
\[ \bar{ad}^A : \Omega_A[C^\infty(M^A, A)] \rightarrow \text{Der}_A[C^\infty(M^A, A)] \]

such that
\[ \bar{ad}^A \circ \delta^A_{M^A} = ad^A. \]

Let’s consider the canonical isomorphism
\[ \sigma_{M^A} : \Omega_A[C^\infty(M^A, A)]^* \rightarrow \text{Der}_A[C^\infty(M^A, A)], \Psi \mapsto \Psi \circ \delta^A_{M^A} \]

and let
\[ \sigma_{M^A}^{-1} \circ \bar{ad}^A : \Omega_A[C^\infty(M^A, A)] \xrightarrow{\bar{ad}^A} \text{Der}_A[C^\infty(M^A, A)] \xrightarrow{\sigma_{M^A}^{-1}} \Omega_A[C^\infty(M^A, A)]^* \]

be the map.

**Proposition 14.** If \((M, \omega_M)\) is a Poisson manifold, then the map,
\[ \omega_{A^M} : \Omega_A[C^\infty(M^A, A)] \times \Omega_A[C^\infty(M^A, A)] \rightarrow C^\infty(M^A, A) \]

such that for any \( X, Y \in \Omega_A[C^\infty(M^A, A)] \)
\[ \omega_{A^M}(X, Y) = -[\sigma_{M^A}^{-1} \circ \bar{ad}^A(X)](Y) \]

is a skew-symmetric 2-form on \( \Omega_A[C^\infty(M^A, A)] \) such that
\[ \omega_{M^A}(x^A, y^A) = [\omega_M(x, y)]^A, \]

for any \( x \) and \( y \) in \( \Omega_R[C^\infty(M)] \).
Proof. For any $X \in \Omega_1[C^\infty(M^A, A)]$, we have $X = \sum_{i \in I: \text{finite}} \varphi_i \cdot \delta_{M^A}^A(\psi_i)$, with $\varphi_i \in C^\infty(M^A, A)$, $\psi_i \in C^\infty(M^A, A)$.

\[
\begin{align*}
\omega_{M^A}(X, X) & = -\left[\sigma_{M^A}^{-1} \circ \widetilde{ad}^A(\psi_1)\right](X) \\
& = -\sum_{j \in I: \text{finite}} \varphi_j \cdot \left[\sigma_{M^A}^{-1} \circ \widetilde{ad}^A(\psi_1)\right] \delta_{M^A}^A(\psi_j) \\
& = -\sum_{j \in I: \text{finite}} \varphi_j \cdot \left[\widetilde{ad}^A(\psi_1)\right](\psi_j) \\
& = -\sum_{j, k \in I: \text{finite}} \varphi_j \cdot \varphi_k \cdot \left[\widetilde{ad}^A(\psi_k)\right](\psi_j) \\
& = -\sum_{j, k \in I: \text{finite}} \varphi_j \cdot \varphi_k \cdot \psi_k, \psi_j \\
& = 0.
\end{align*}
\]

For any $X_1, X_2$ and $Y \in \Omega_1[C^\infty(M^A, A)]$ and for any $\varphi \in C^\infty(M^A, A)$, we have

\[
\begin{align*}
\omega_{M^A}[(\varphi \cdot X_1 + X_2), Y] & = -\left[\sigma_{M^A}^{-1} \circ \widetilde{ad}^A(\varphi \cdot X_1 + X_2)\right](Y) \\
& = -\left[\sigma_{M^A}^{-1} \circ \widetilde{ad}^A(\varphi \cdot X_1) + \widetilde{ad}^A(X_2)\right](Y) \\
& = -\varphi \cdot \left[\sigma_{M^A}^{-1} \circ \widetilde{ad}^A(\varphi \cdot X_1)\right](Y) + (\sigma_{M^A}^{-1} \circ \widetilde{ad}^A(X_2))\right](Y) \\
& = \varphi \cdot \omega_{M^A}(X_1, Y) + \omega_{M^A}(X_2, Y).
\end{align*}
\]

For any $x$ and $y$ in $\Omega_1[C^\infty(M)]$,

\[
x^A = \sum_{i \in I: \text{finite}} f^A_i \cdot \delta_{M^A}^A(f^A_i) \quad \text{and} \quad y^A = \sum_{j \in I: \text{finite}} g^A_j \cdot \delta_{M^A}^A(g^A_j).
\]
Thus,

\[
\omega^A_{M^\times}(x^A, y^A) = -[\sigma^{-1}_{M^\times} \circ \ad^A(\sigma^A_{M^\times})](y^A)
\]

\[
= -[\sigma^{-1}_{M^\times} \circ \ad^A \left( \sum_{i \in I: \text{finite}} f_i^A \cdot \delta^A_{M^\times}(f_i^A) \right) \left( \sum_{j \in J: \text{finite}} g_j^A \cdot \delta^A_{M^\times}(g_j^A) \right)]
\]

\[
= - \sum_{i, j \in I: \text{finite}} f_i^A \cdot g_j^A \cdot [\sigma^{-1}_{M^\times} \circ \ad^A(\sigma^A_{M^\times}(f_i^A))](\delta^A_{M^\times}(g_j^A))
\]

\[
= - \sum_{i, j \in I: \text{finite}} f_i^A \cdot g_j^A \cdot [\sigma^{-1}_{M^\times}(\ad^A(f_i^A))](\delta^A_{M^\times}(g_j^A))
\]

\[
= - \sum_{i, j \in I: \text{finite}} f_i^A \cdot g_j^A \cdot [\sigma^{-1}_{M^\times}(\ad^A(f_i^A))] \cdot [\delta^A_{M^\times}(g_j^A)]^A
\]

\[
= - \sum_{i, j \in I: \text{finite}} f_i \cdot g_j \cdot [\sigma^{-1}_{M^\times}(\ad(f_i))] \cdot [\delta_{M^\times}(g_j)]^A
\]

\[
= - \sum_{i, j \in I: \text{finite}} f_i \cdot g_j \cdot [\sigma^{-1}_{M^\times}(\ad(f_i))] \cdot [\delta_{M^\times}(g_j)]^A
\]

\[
= -[\sigma^{-1}_{M^\times} \circ \ad^A \left( \sum_{i \in I: \text{finite}} f_i \cdot \delta_{M^\times}(f_i) \right) \left( \sum_{j \in J: \text{finite}} g_j \cdot \delta_{M^\times}(g_j) \right)]^A
\]

\[
= [\omega_M(x, y)]^A.
\]

\[\square\]

**Proposition 15.** When \((M, \omega_M)\) is a Poisson manifold of Poisson 2-form \(\omega_M\), then \((M^A, \omega^A_{M^\times})\) is a Poisson manifold.

**Proof.** For any \(f\) and \(g\) in \(C^\infty(M)\),

\[
\omega^A_{M^\times}(\delta^A_{M^\times}(f^A), \delta^A_{M^\times}(g^A)) = \omega^A_{M^\times}(\left[\delta_{M^\times}(f)\right]^A, \left[\delta_{M^\times}(g)\right]^A)
\]

\[
= [\omega_{M^\times}(\delta_{M^\times}(f), \delta_{M^\times}(g))]^A
\]

\[
= -[f, g]^A_{M^\times}
\]

and

\[
\omega^A_{M^\times}(x^A, y^A) = [\omega_{M^\times}(x, y)]^A,
\]

for any \(x, y \in \Omega_2[C^\infty(M)]\). We deduce that \((M^A, \omega^A_{M^\times})\) is a Poisson manifold. \[\square\]

**Theorem 16.** The manifold \(M^A\) is a Poisson manifold if and only if there exists a skew-symmetric 2-form

\[
\omega^A_{M^\times} : \Omega^2(C^\infty(M^A, A) \times \Omega^2(C^\infty(M^A, A) \rightarrow C^\infty(M^A, A)
\]

such that for any \(\varphi\) and \(\psi\) in \(C^\infty(M^A, A)\),

\[
[\varphi, \psi]_{M^\times} = -\omega^A_{M^\times}(\delta^A_{M^\times}(\varphi), \delta^A_{M^\times}(\psi))
\]

defines a structure of \(A\)-Lie algebra over \(C^\infty(M^A, A)\). Moreover, for any \(f\) and \(g\) in \(C^\infty(M)\),

\[
\{f^A, g^A\}_{M^\times} = \{f, g\}_{M^\times}^A
\]
Proof. Indeed, according to the previous proposition, the bracket
\[ \{\varphi, \psi\}_{M^A} = -\omega_{M^A}^A(\delta_{M^A}^A(\varphi), \delta_{M^A}^A(\psi)) \]
defines a structure of $A$-Lie algebra over $C_*^\infty(M^A, A)$. For any $f$ and $g$ in $C_*^\infty(M^A)$,
\[ [f^A, g^A]_{M^A} = -\omega_{M^A}^A(\delta_{M^A}^A(f^A), \delta_{M^A}^A(g^A)) \]
\[ = \{f, g\}^A_{M^A}. \]
In this case, we will say that $\omega_{M^A}^A$ is the Poisson 2-form of the $A$-Poisson manifold $M^A$ and we denote $(M^A, \omega_{M^A}^A)$ the $A$-Poisson manifold of Poisson 2-form $\omega_{M^A}^A$. □

**Proposition 17.** When $(M, \omega_M)$ is a Poisson manifold of Poisson 2-form $\omega_M$, then for any $x, y \in \Omega_2[C_*^\infty(M)]$ and for any $f, g \in C_*^\infty(M)$, we get

1. \[ [\tilde{ad}^A(x^A)](f^A) = ([\tilde{ad}(x)](f))^A. \]
2. \[ [\tilde{ad}^A(x^A)](y^A) = ([\tilde{ad}(x)](y))^A. \]
3. \[ g^A = ([\tilde{ad}(\varphi)(f)^A]) \quad \text{and} \quad [\tilde{ad}^A(\varphi^A)](f^A) = ([\tilde{ad}(\varphi)(f)]^A). \]

Proof. 1. For any $x \in \Omega_2[C_*^\infty(M)]$, $x^A = g^A \cdot \delta_{M^A}^A(h^A)$ with $g$ and $h$ in $C_*^\infty(M)$, and for any $f, g \in C_*^\infty(M)$, we have
\[ [\tilde{ad}^A(x^A)](f^A) = [\tilde{ad}^A(g^A \cdot \delta_{M^A}^A(h^A))](f^A) \]
\[ = [g^A \cdot \tilde{ad}^A(\delta_{M^A}^A(h^A))](f^A) \]
\[ = [g^A \cdot ad(h)](f^A) \]
\[ = (g \cdot [\tilde{ad}(h)](f))^A \]
\[ = (g \cdot ad(\delta_M(h))(f))^A \]
\[ = ([\tilde{ad}(x)](f))^A. \]

2. When $y \in \Omega_2[C_*^\infty(M)]$, $y^A = g^A \cdot \delta_{M^A}^A(h^A)$ with $g$ and $h$ in $C_*^\infty(M)$
\[ [\tilde{ad}^A(x^A)](y^A) = [\tilde{ad}^A(x^A)](g^A \cdot \delta_{M^A}^A(h^A)) \]
\[ = [g^A \cdot ([\tilde{ad}^A(x^A)] \circ \delta_{M^A}^A)(h^A)] \]
\[ = [g^A \cdot [\tilde{ad}^A(x^A)](h^A)] \]
\[ = [g^A \cdot ad(\delta_M(h))(f)](f^A) \]
\[ = (g \cdot [\tilde{ad}(x)](f))^A \]
\[ = (g \cdot [\tilde{ad}(x)](h))^A \]
\[ = ([\tilde{ad}(x)](g \cdot \delta_M(h)))^A \]
\[ = ([\tilde{ad}(x)](y))^A. \]
Proposition 18. When $(M, \omega_M)$ is a Poisson manifold of Poisson 2-form $\omega_M$, then for any $X, Y \in \Omega_A[C^\infty(M^A, A)]$ and for any $\varphi, \psi \in C^\infty(M^A, A)$, we get

1. $[\tilde{\text{ad}}^A(X)](\varphi) = -\omega^A_M(X, \delta^A_M(\varphi));$

2. $[\tilde{\text{ad}}^A(X)](Y) = -\omega^A_M(X, Y);$

3. $\Omega_{\tilde{\text{ad}}^A(X)}(\varphi) \delta^A_M(\psi) = \delta^A_M((\varphi, \psi)_M).$

Proof. When $X$ and $Y \in \Omega_A[C^\infty(M^A, A)]$, $X = \sum_{i \in I, f_m} \varphi_i \cdot \delta^A_M(\varphi'_i)$, $Y = \sum_{j \in J, f_m} \psi_j \cdot \delta^A_M(\psi'_j)$ with $\varphi_i, \varphi'_i, \psi_j, \psi'_j \in C^\infty(M^A, A)$

1. $[\tilde{\text{ad}}^A(X)](\varphi) = [\tilde{\text{ad}}^A(\sum_{i \in I, f_m} \varphi_i \cdot \delta^A_M(\varphi'_i))](\varphi)$
   $\quad = \sum_{i \in I, f_m} \varphi_i \cdot ([\tilde{\text{ad}}^A(\varphi'_i)])(\varphi)$
   $\quad = \sum_{i \in I, f_m} \varphi_i \cdot [\tilde{\text{ad}}^A(\varphi'_i)](\varphi)$
   $\quad = \sum_{i \in I, f_m} \varphi_i \cdot \varphi'_i, \varphi'_{i_M} = -\sum_{i \in I, f_m} \varphi_i \cdot \omega^A_M(\delta^A_M(\varphi'_i), \delta^A_M(\varphi))$
   $\quad = -\omega^A_M(\sum_{i \in I, f_m} \varphi_i \cdot \delta^A_M(\varphi'_i), \delta^A_M(\varphi))$
   $\quad = -\omega^A_M(X, \delta^A_M(\varphi)).$

2. $[\tilde{\text{ad}}^A(X)](Y) = [\tilde{\text{ad}}^A(\sum_{j \in J, f_m} \psi_j \cdot \delta^A_M(\psi'_j)))(\varphi)$
   $\quad = \sum_{j \in J, f_m} \psi_j \cdot ([\tilde{\text{ad}}^A(X)](\delta^A_M(\psi'_j))$
   $\quad = \sum_{j \in J, f_m} \psi_j \cdot ([\text{ad}^A(X)] \circ \delta^A_M)(\psi'_j))$
   $\quad = \sum_{j \in J, f_m} \psi_j \cdot [\tilde{\text{ad}}^A(X)](\psi'_j)$
   $\quad = -\sum_{j \in J, f_m} \psi_j \cdot \omega^A_M(X, \delta^A_M(\psi'_j))$
   $\quad = -\omega^A_M(X, \sum_{j \in J, f_m} \psi_j \cdot \delta^A_M(\psi'_j))$
   $\quad = -\omega^A_M(X, Y);.$
\[ \sum_{\text{ad}^A[\delta M^A(\varphi)]} \delta M^A(\psi) = \sum_{\text{ad}^A(\varphi)} \delta M^A(\psi) = \delta M^A[\text{ad}^A(\varphi)](\psi) = \delta M^A([\varphi, \psi]_{M^A}). \]

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