ON LOCALLY COMPACT TOPOLOGICAL GRAPH INVERSE SEMIGROUPS

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Abstract. In this paper we characterise graph inverse semigroups which admit only discrete locally compact semigroup topology. This characterization provides a complete answer on the question of Z. Mesyan, J. D. Mitchell, M. Morayne and Y. H. Péresse posed in [26].

In this paper all topological spaces are assumed to be Hausdorff. We shall follow the terminology of [12, 15, 22, 29]. By \( \mathbb{N} \) we denote a set of all positive integers. A semigroup \( S \) is called an inverse semigroup if for each element \( a \in S \) there exists a unique element \( a^{-1} \in S \) such that \( a a^{-1} a = a \) and \( a^{-1} a a^{-1} = a^{-1} \).

A map which associates to any element of an inverse semigroup its inverse is called the inversion.

A directed graph \( E = (E^0, E^1, r, s) \) consists of sets \( E^0, E^1 \) of vertices and edges, respectively, together with functions \( s, r : E^1 \to E^0 \) which are called source and range, respectively. In this paper we shall refer to directed graph as simply ”graph”. A path \( x = e_1 \ldots e_n \) in graph \( E \) is a finite sequence of edges \( e_1, \ldots, e_n \) such that \( r(e_i) = s(e_{i+1}) \) for each positive integer \( i < n \). We extend functions \( s \) and \( r \) on the set \( \text{Path}(E) \) of all pathes in graph \( E \) by the following way: for each \( x = e_1 \ldots e_n \in \text{Path}(E) \) put \( s(x) = s(e_1) \) and \( r(x) = r(e_n) \). By \( |x| \) we denote the length of a path \( x \). Observe that each vertex is a path which has zero length. Edge \( e \) is called a loop if \( s(e) = r(e) \). A path \( x \) is called a cycle if \( s(x) = r(x) \). Graph \( E \) is called finite if the sets \( E^0 \) and \( E^1 \) are finite and infinite in the other case.

A topological (inverse) semigroup is a Hausdorff topological space together with a continuous semigroup operation (and an inversion, respectively). If \( S \) is a semigroup (an inverse semigroup) and \( \tau \) is a topology on \( S \) such that \( (S, \tau) \) is a topological (inverse) semigroup, then we shall call \( \tau \) a (inverse) semigroup topology on \( S \).

A bicyclic monoid \( \mathcal{C}(p, q) \) is the semigroup with the identity 1 generated by two elements \( p \) and \( q \) subject to the condition \( pq = 1 \). The bicyclic semigroup admits only the discrete semigroup topology [14]. In [11] this result was extended over the case of semitopological semigroups. The closure of a bicyclic semigroup in a locally compact topological inverse semigroup was described in [14]. A locally compact semitopological bicyclic monoid with adjoined zero is either compact or discrete space [16]. The problem of an embedding of the bicyclic monoid into compact-like topological semigroups discussed in [3, 6, 5, 17, 18].

One of the generalizations of the bicyclic semigroup is a \( \lambda \)-polycyclic monoid. For a non-zero cardinal \( \lambda \), \( \lambda \)-polycyclic monoid \( \mathcal{P}_\lambda \) is the semigroup with identity and zero given by the presentation:

\[
\mathcal{P}_\lambda = \left\langle \{p_i\}_{i \in \lambda}, \{p_i^{-1}\}_{i \in \lambda} \mid p_i^{-1} p_i = 1, p_i p_j = 0 \text{ for } i \neq j \right\rangle.
\]

Polycyclic monoid \( \mathcal{P}_k \) over a finite cardinal \( k \) was introduced in [27]. Observe that the bicyclic semigroup with adjoined zero is isomorphic to the polycyclic monoid \( \mathcal{P}_1 \). Algebraic properties of a semigroup \( \mathcal{P}_k \) were investigated in [23] and [24]. Algebraic and topological properties of the \( \lambda \)-polycyclic monoid were investigated in [3] and [10]. In particular, it was proved that for every non-zero cardinal \( \lambda \) the only locally compact semigroup topology on the \( \lambda \)-polycyclic monoid is a discrete topology. In [17] it was showed that a locally compact semitopological \( \lambda \)-polycyclic monoid is either compact or discrete space.

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For a given directed graph $E = (E^0, E^1, r, s)$ a graph inverse semigroup $G(E)$ over a graph $E$ is a semigroup with zero generated by the sets $E^0$, $E^1$ together with a set $E^{-1} = \{ e^{-1} : e \in E^1 \}$ satisfying the following relations for all $a, b \in E^0$ and $e, f \in E^1$:

(i) $a \cdot b = a$ if $a = b$ and $a \cdot b = 0$ if $a \neq b$;
(ii) $s(e) \cdot e = e \cdot r(e) = e$;
(iii) $e^{-1} \cdot s(e) = r(e) \cdot e^{-1} = e^{-1}$;
(iv) $e^{-1} \cdot f = r(e)$ if $e = f$ and $e^{-1} \cdot f = 0$ if $e \neq f$.

Graph inverse semigroups are a generalization of the polycyclic monoids. In particular, for every non-zero cardinal $\lambda$, $\lambda$-polycyclic monoid is isomorphic to the graph inverse semigroup over the graph $E$ which consists of one vertex and $\lambda$ distinct loops.

By [19 Lemma 2.6] each non-zero element of a graph inverse semigroup $G(E)$ is of the form $uv^{-1}$ where $u, v \in \text{Path}(E)$ and $r(u) = r(v)$. A semigroup operation in $G(E)$ is defined by the following way:

$$u_1v_1^{-1} \cdot u_2v_2^{-1} = \begin{cases} u_1uv_2^{-1}, & \text{if } u_2 = v_1w \text{ for some } w \in \text{Path}(E); \\
u_1(v_2w)^{-1}, & \text{if } v_1 = u_2w \text{ for some } w \in \text{Path}(E); \\
0, & \text{otherwise}, \end{cases}$$

and $uv^{-1} \cdot 0 = 0 \cdot uv^{-1} = 0 \cdot 0 = 0$.

Simply verifications show that $G(E)$ is an inverse semigroup, moreover, $(uv^{-1})^{-1} = vu^{-1}$.

Graph inverse semigroups play an important role in the study of rings and $C^*$-algebras (see [11, 14, 13, 21, 28]). Algebraic properties of graph inverse semigroups were studied in [2, 8, 19, 20, 23, 25]. In [26] Z. Mesyan, J. D. Mitchell, M. Morayne and Y. H. Péresse investigated topological properties of graph inverse semigroups. In particular, one of the main results from [26] is the following:

**Theorem 1** ([26 Theorem 10]). If $E$ is a finite graph, then the only locally compact Hausdorff semigroup topology on $G(E)$ is the discrete topology.

Also in [26] the authors asked the following natural question: *Can the Theorem 1 be generalized to all graphs?*

In this paper we give a complete answer to the above question. In particular, we describe all graph inverse semigroups which allow only discrete locally compact semigroup topology and propose a method of constructing a non-discrete locally compact metrizable inverse semigroup topology on graph inverse semigroups.

1. **Main result**

We shall say that a graph inverse semigroup $G(E)$ satisfies the condition ($\ast$) if for each countable subset $A = \{ x_n \}_{n \in \mathbb{N}} \subseteq \text{Path}(E)$ there exists an infinite subset $B = \{ x_{n_k} \}_{k \in \mathbb{N}} \subseteq A$ and an element $\mu \in G(E)$ such that $\mu \cdot x_{n_k} \in \text{Path}(E)$ and $|\mu \cdot x_{n_k}| > |x_{n_k}|$, for each $k \in \mathbb{N}$. Each graph inverse semigroup $G(E)$ over a finite graph $E$ satisfies the condition ($\ast$) (see [20 Lemma 9]).

**Remark 2.** If a graph inverse semigroup $G(E)$ satisfies the condition ($\ast$) then the set $E^0$ of all vertices of graph $E$ is finite. Indeed, if the set $E^0$ is infinite then for an arbitrary countable infinite subset $A \subseteq E^0 \subseteq \text{Path}(E)$ and for each element $\mu \in G(E)$ there exists a vertex $f \in A \subseteq E^0$ such that $\mu \cdot f = 0 \notin \text{Path}(E)$ which implies that semigroup $G(E)$ does not satisfy the condition ($\ast$). However, there exists a semigroup $G(E)$ over an infinite graph $E$ which satisfies the condition ($\ast$). In particular, each $\lambda$-polycyclic monoid satisfies the condition ($\ast$). Moreover, if for each vertex $e$ of a graph $E$ there exists a cycle $u \in \text{Path}(E)$ such that $s(u) = r(u) = e$ and graph $E$ contains a finite amount of vertices then graph inverse semigroup $G(E)$ satisfies the condition ($\ast$).

**Lemma 3.** Let $X$ be a first countable Hausdorff topological space which has only one non-isolated point $y$. Then $X$ a is metrizable space.

**Proof.** Fix any countable open neighborhood base $\mathcal{B}_y = \{ U_n \}_{n \in \mathbb{N}}$ of the point $y$. For each $x \in X \setminus \{ y \}$ put $V_x = \{ x \}$. Simply verifications show that family $\mathcal{B} = \{ V_x \}_{x \in X \setminus \{ y \}} \cup \mathcal{B}_y$ is a $\sigma$-locally finite base...
of the topological space $X$. Since a family $B$ consists of closed and open subsets of $X$ the space $X$ is regular. Hence by Nagata-Smirnov Metrization Theorem (see [15, Theorem 4.4.7]) the space $X$ is metrizable.

**Theorem 4.** Discrete topology is the only locally compact semigroup topology on a graph inverse semigroup $G(E)$ if and only if $G(E)$ satisfies the condition $(\ast)$.

**Proof.** ($\Leftarrow$) Assuming the contrary, let $G(E)$ be a locally compact non-discrete topological graph inverse semigroup which satisfies the condition $(\ast)$. Observe that by [20, Theorem 3] each non-zero element of $G(E)$ is an isolated point. Hence there exists an open neighborhood base of the point 0 which consists of open compact subsets of $G(E)$. Since $G(E) \setminus \{0\}$ is a discrete subspace of $G(E)$ the local compactness of $G(E)$ implies that for each two open compact neighborhoods $U \subset V$ of the point 0 the set $V \setminus U$ is finite. Hence element 0 is a limit point of each infinite sequence $(x_n)_{n \in \mathbb{N}}$ which is contained in an arbitrary compact open neighborhood $U$ of 0.

Fix an arbitrary compact open neighborhood $U$ of 0. Then one of the following three conditions holds:

1. For each sequence $(x_n)_{n \in \mathbb{N}} \subset \text{Path}(E)$, $\lim_{n \in \mathbb{N}} x_n \neq 0$ and $\lim_{n \in \mathbb{N}} x_n^{-1} \neq 0$.
2. There exists a sequence $(x_n)_{n \in \mathbb{N}} \subset \text{Path}(E)$ such that $\lim_{n \in \mathbb{N}} x_n = 0$.
3. There exists a sequence $(x_n)_{n \in \mathbb{N}} \subset \text{Path}(E)$ such that $\lim_{n \in \mathbb{N}} x_n^{-1} = 0$.

Suppose that condition (1) holds. For each $y \in \text{Path}(E)$ denote

$$X_y = \{x \in \text{Path}(E) : xy^{-1} \in U \setminus \{0\}\}.$$  

We claim that for each element $y \in \text{Path}(E)$ the set $X_y$ is finite. Indeed, suppose that there exists an element $y \in \text{Path}(E)$ and an infinite subset $\{x_n\}_{n \in \mathbb{N}} \subset \text{Path}(E)$ such that $x_n y^{-1} \in U \setminus \{0\}$ for each $n \in \mathbb{N}$. Observe that for any elements $a, b \in G(E) \setminus \{0\}$ the sets

$$\{x \in G(E) : a \cdot x = b\} \quad \text{and} \quad \{x \in G(E) : x \cdot a = b\}$$

are finite (see [20, Lemma 1]). Hence the set $\{x_n y^{-1} : n \in \mathbb{N}\}$ is an infinite subset of $U$ which implies that $\lim_{n \in \mathbb{N}} x_n y^{-1} = 0$. Since $0 \cdot y = 0$ the continuity of the semigroup operation in $G(E)$ implies that

$$0 = \lim_{n \in \mathbb{N}} x_n y^{-1} = \lim_{n \in \mathbb{N}} x_n y^{-1} \cdot y = \lim_{n \in \mathbb{N}} (y^{-1} y) = \lim_{n \in \mathbb{N}} x_n$$

which contradicts the assumption. Analogously it can be proved that for each $x \in \text{Path}(E)$ the set

$$Y_x = \{y \in \text{Path}(E) : xy^{-1} \in U \setminus \{0\}\}$$

is finite.

Since the set $U$ is infinite we see that the set $C = \{y \in \text{Path}(E) : X_y \neq \emptyset\}$ is infinite as well. Fix an arbitrary countable subset $B = \{y_n\}_{n \in \mathbb{N}} \subset C$. For each element $y_n \in B$ fix an element $x_{y_n} \in X_{y_n}$ such that $|x_{y_n}| \geq |x|$ for an arbitrary element $x \in X_{y_n}$. Since for each element $x \in \text{Path}(E)$ the set $Y_x$ is finite we obtain that the set $\{x_{y_n}\}_{n \in \mathbb{N}}$ is infinite and $\lim_{n \in \mathbb{N}} x_{y_n} y_n^{-1} = 0$. Since a semigroup $G(E)$ satisfies the condition $(\ast)$ we can find an infinite subsequence $\{x_{y_{n_k}}\}_{k \in \mathbb{N}} \subset \{x_{y_n}\}_{n \in \mathbb{N}}$ and an element $\mu \in G(E)$ such that $\mu \cdot x_{y_{n_k}} \in \text{Path}(E)$ and $|\mu \cdot x_{y_{n_k}}| > |x_{y_{n_k}}|$. Since $\mu \cdot 0 = 0$ the continuity of the semigroup operation in $G(E)$ implies that $\lim_{k \in \mathbb{N}} \mu \cdot x_{y_{n_k}} y_{n_k}^{-1} = 0$. Observe that $\mu \cdot x_{y_{n_k}} y_{n_k}^{-1} \neq 0$, because $\mu \cdot x_{y_{n_k}} \in \text{Path}(E)$ and $r(\mu \cdot x_{y_{n_k}}) = r(x_{y_{n_k}}) = r(y_{n_k})$. Hence $\mu \cdot x_{y_{n_k}} \in X_{y_{n_k}}$ and $|\mu \cdot x_{y_{n_k}}| > |x_{y_{n_k}}|$ for some sufficiently large $k$ which contradicts the choice of $x_{y_{n_k}}$. Hence condition (1) does not hold.

Suppose that condition (2) holds. Put $A = \text{Path}(E) \cap U$. Since there exists a sequence of elements from $\text{Path}(E)$ which converges to 0 the set $A$ is infinite. For each $x \in \text{Path}(E)$ put

$$Y_x^A = \{y \in A : x y^{-1} \in U \setminus \{0\}\}.$$  

Clearly that for each $x \in \text{Path}(E)$ the set $Y_x^A$ is finite. Put $D = \{x \in \text{Path}(E) : Y_x^A \neq \emptyset\}$. Then one of the following two cases holds:

1. The set $D$ is finite.
2. The set $D$ is infinite.
Consider case (2.1). Finiteness of the set $D$ implies that the set $T = \{xy^{-1} : x \in D \text{ and } y \in Y^A\}$ is finite as well. Then $V = U \setminus T$ is an open neighborhood of 0 which satisfies the following condition: for each non-zero element $xy^{-1} \in V$, $y \notin A$. Since the set $E^0$ of all vertices of graph $E$ is finite (see Remark 2) and the set $A$ is infinite there exists a vertex $e$ and an infinite subset $\hat{A} \subset A$ such that for each element $y \in \hat{A}$, $r(y) = e$. Fix an arbitrary element $y \in \hat{A}$. Since $0 \cdot y^{-1} = 0$ the continuity of the semigroup operation in $G(E)$ implies that there exists an open compact neighborhood $W$ of 0 such that $W \cdot y^{-1} \subset V$. Since $\hat{A}$ is an infinite subset of $U$ we obtain that the set $W \cap \hat{A}$ is infinite as well. Fix an arbitrary element $z \in W \cap \hat{A}$. Then $z \cdot y^{-1} \in W \cdot y^{-1} \subset V \setminus \{0\}$ which contradicts the choice of the set $V$. Hence case (2.1) does not hold.

Consider case (2.2). Fix an arbitrary infinite countable subset $H = \{x_n\}_{n \in \mathbb{N}}$ of $D$. For each element $x_n \in H$ by $y_{x_n}$ we denote an element of $Y^A$ such that $|y_{x_n}| \geq |y|$ for an arbitrary element $y \in Y^A$. The set $B = \{y_{x_n} : n \in \mathbb{N}\}$ is infinite, because for each element $y \in \text{Path}(E)$ the set $X_y$ is finite. Observe that $\lim_{n \in \mathbb{N}} x_n y_{x_n}^{-1} = 0$. Since semigroup $G(E)$ satisfies the condition ($\star$) there exists an element $\mu \in G(E)$ and an infinite subset $C = \{y_{x_n} : k \in \mathbb{N}\} \subset B$ such that $\mu \cdot y_{x_n} \in \text{Path}(E)$ and $|\mu \cdot y_{x_n}| > |y_{x_n}|$, for each $k \in \mathbb{N}$. We claim that the set $(\mu \cdot C) \cap U$ is infinite. Indeed, since $\mu \cdot 0 = 0$ the continuity of the semigroup operation in $G(E)$ implies that there exists an open compact neighborhood $W$ of 0 such that $\mu \cdot W \subset U$. Since for a fixed non-zero elements $a, b$ of $G(E)$ the set $\{x \in G(E) : a \cdot x = b\}$ is finite and the set $W \cap C$ is infinite we obtain that the set $(\mu \cdot C) \cap U$ is infinite.

Since $0 \cdot \mu^{-1} = 0$ the continuity of the semigroup operation in $G(E)$ implies that $\lim_{k \in \mathbb{N}} x_n k y_{x_n}^{-1} \cdot \mu^{-1} = \lim_{k \in \mathbb{N}} x_n k (\mu \cdot y_{x_n})^{-1} = 0$.

Observe that for each $k \in \mathbb{N}$, $x_n k (\mu \cdot y_{x_n})^{-1} \neq 0$, because $\mu \cdot y_{x_n} \in \text{Path}(E)$ and $r(\mu \cdot y_{x_n}) = r(y_{x_n})$. Hence there exists a positive integer $k$ such that $\mu \cdot y_{x_n} \in Y^A_{x_n}$ and $|\mu \cdot y_{x_n}| > |y_{x_n}|$ which contradicts the choice of $y_{x_n}$. The obtained contradiction implies that case (2.2) is not possible. Hence condition (2) does not hold.

Similar arguments imply that condition (3) does not hold as well. The obtained contradiction implies that the only locally compact semigroup topology on a graph inverse semigroup $G(E)$ which satisfies the condition ($\star$) is the discrete topology.

$(\Rightarrow)$ Suppose that a semigroup $G(E)$ does not satisfy the condition ($\star$). Then there are two cases to consider.

1. Graph $E$ contains infinitely many vertices.
2. Graph $E$ contains finitely many vertices.

Consider case (1). Fix an arbitrary countable infinite subset $A = \{e_n : n \in \mathbb{N}\} \subset E^0$. Define a topology $\tau$ on the semigroup $G(E)$ by the following way: each non-zero element of $G(E)$ is an isolated point and a family $B = \{U_n : n \in \mathbb{N}\}$ forms an open neighborhood base of the point 0, where $U_n = \{e_k \in A : k > n\} \cup \{0\}$. By Lemma 3 topological space $(G(E), \tau)$ is metrizable. Since for each $n \in \mathbb{N}$ the set $U_n$ is compact we obtain that $(G(E), \tau)$ is a locally compact space. Observe that $U_n^{-1} = U_n$ and hence inversion is continuous in $(G(E), \tau)$. Let $U_n$ be an arbitrary open neighborhood of 0 and $ab^{-1} \in G(E) \setminus \{0\}$. Then the continuity of the semigroup operation in $(G(E), \tau)$ can be derived from the following equalities:

- $U_n \cdot U_n = U_n$;
- $ab^{-1} \cdot (U_n \setminus \{s(b)\}) = 0 \subset U_n$;
- $(U_n \setminus \{s(a)\}) \cdot ab^{-1} = 0 \subset U_n$.

Hence $(G(E), \tau)$ is a locally compact metrizable topological inverse semigroup.

Consider case (2). Since semigroup $G(E)$ does not satisfy the condition ($\star$) there exists an infinite subset $A = \{x_n\}_{n \in \mathbb{N}} \subset \text{Path}(E)$ such that for each infinite subset $B = \{x_{n_k}\}_{k \in \mathbb{N}} \subset A$ and for each element $\mu \in G(E)$ there exists an element $x_{n_k} \in B$ such that either $\mu \cdot x_{n_k} \notin \text{Path}(E)$ or $|\mu \cdot x_{n_k}| \leq |x_n|$. Observe that with no loss of generality we can assume that for each element $x_n \in A$, $s(x_n) = e_0$ for some fixed vertex $e_0 \in E^0$. We claim the following:


Claim. There exists a pair of vertices \( e_n \) and \( e_{n+1} \) such that the set \( \{ u \in \text{Path}(E) : r(u) = e_n \} \) is finite and the set \( \{ a \in E^1 : s(a) = e_n \text{ and } r(a) = e_{n+1} \} \) is infinite.

Proof. Step 1. Observe that the choice of the set \( A \) implies that there exists no edge which range is a vertex \( e_0 \). Since the set of all vertices of the graph \( E \) is finite and \( A \) is an infinite set there exists a vertex \( e_1 \) and an infinite subset \( B_1 \subset A \) such that each element \( x_n \in B_1 \) is of the form \( x_n = a_{(1,n)}v \) where \( a_{(1,n)} \) is an edge such that \( s(a_{(1,n)}) = e_0, r(a_{(1,n)}) = e_1 \) for each \( n \in \mathbb{N} \) and \( s(v) = e_1 \). If the set \( T_1 = \{ a_{(1,n)} : n \in \mathbb{N} \} \) is infinite then put \( e_n = e_0, e_{n+1} = e_1 \) and proof of the claim ends.

Step 2. If the set \( T_1 = \{ a_{(1,n)} : n \in \mathbb{N} \} \) is finite then similar arguments imply that there exists a vertex \( e_2 \) and an infinite subset \( B_2 \subset B_1 \) such that each element \( x_n \in B_2 \) is of the form \( x_n = a_1a_{(2,n)}v \), where \( a_1 \) is a fixed element from \( T_1, a_{(2,n)} \) is an edge such that \( s(a_{(2,n)}) = e_1, r(a_{(2,n)}) = e_2 \) for each positive integer \( n \) and \( s(v) = e_2 \). We claim that \( |y| \leq 1 \) for each path \( y \in \text{Path}(E) \) such that \( r(y) = e_1 \). Indeed, if there exists an element \( y \in \text{Path}(E) \) such that \( r(y) = e_1 \) and \( |y| > 1 \) then for an infinite subset \( B_2 \subset A \) there exists an element \( \mu = ya_{1}^{-1} \in G(E) \) such that \( \mu \cdot B_2 \subset \text{Path}(E) \) and for each element \( x_k = a_1a_{(2,n)}v \in B_2, \)

\[
|\mu \cdot a_1a_{(2,n)}v| = |ya_{(2,n)}v| > |a_1a_{(2,n)}v|
\]

which contradicts the choice of the set \( A \). If there exists a vertex \( f \) such that the set \( \{ a \in E^1 : s(a) = f \text{ and } r(a) = e_1 \} \) is infinite then put \( e_n = f, e_{n+1} = e_1 \) and proof of the claim ends, because there are no edges which range is a vertex \( f \) (in the other case there exists a path \( y \) such that \( r(y) = e_1 \) and \( |y| > 1 \) which contradicts the above arguments). If for each vertex \( f \) the set \( \{ a \in E^1 : s(a) = f \text{ and } r(a) = e_1 \} \) is finite then the set \( \{ y \in \text{Path}(E) : r(y) = e_1 \} \) is finite as well, because graph \( E \) contains a finite amount of vertices. If the set \( T_2 = \{ a_{(2,n)} : n \in \mathbb{N} \} \) is infinite then put \( e_n = e_1, e_{n+1} = e_2 \) and proof of the claim ends.

Suppose that we have done \( k - 1 \) steps and do not find a desired pair of vertices.

Step \( k \). After the previous steps we obtain two finite sequence of sets \( A \supset B_1 \supset B_2 \supset \ldots \supset B_{k-1}; T_1, T_2, \ldots, T_{k-1} \) and a finite sequence of distinct vertices \( e_1, \ldots e_{k-1} \) where for every \( i < k \) each set \( T_i \) is finite, each set \( B_i \) is infinite and each element \( a_{i,n} \in T_i \) is an edge such that \( s(a_{i,n}) = e_{i-1} \) and \( r(a_{i,n}) = e_i \). Each element \( x_k \in B_{k-1} \) is of the form \( x_k = a_1a_2\ldots a_{k-2}a_{(k-1,n)}v \) where \( a_i \) is a fixed element from \( T_1 \) for each \( i < k - 1 \), \( a_{(k-1,n)} \in T_{k-1} \) for each \( n \in \mathbb{N} \) and \( s(v) = e_{k-1} \). The set \( T_{k-1} = \{ a_{(k-1,n)} : n \in \mathbb{N} \} \) is finite, because in the other case we would find a desired pair of vertices on the previous step. Hence there exists a vertex \( e_k \) and an infinite subset \( B_k \subset B_{k-1} \) such that each element \( x_n \in B_k \) is of the form \( x_n = a_1a_2\ldots a_{k-2}a_{k-1}a_{(k,n)}v \), where \( a_{k-1} \) is a fixed element from \( T_{k-1}, a_{(k,n)} \) is an edge such that \( s(a_{(k,n)}) = e_{k-1}, r(a_{(k,n)}) = e_k \) for each positive integer \( n \) and \( s(v) = e_k \). We claim that \( |y| \leq k - 1 \) for each path \( y \in \text{Path}(E) \) such that \( r(y) = e_{k-1} \). Indeed, if there exists an element \( y \in \text{Path}(E) \) such that \( r(y) = e_{k-1} \) and \( |y| > k - 1 \) then for an infinite subset \( B_k \subset A \) there exists an element \( \mu = y(a_1a_2\ldots a_{k-1})^{-1} \in G(E) \) such that \( \mu \cdot B_k \subset \text{Path}(E) \) and for each element \( x_n = a_1a_2\ldots a_{k-1}a_{(k,n)}v \in B_k, \)

\[
|\mu \cdot a_1a_2\ldots a_{k-1}a_{(k,n)}v| = |ya_{(k,n)}v| > |a_1a_2\ldots a_{k-1}a_{(k,n)}v|
\]

which contradicts the choice of the set \( A \).

For each positive integer \( i < k \) by \( P_i \) we denote the set of all vertices \( f \) such that there exists a path \( u \in \text{Path}(E) \) such that \( s(u) = f, r(u) = e_{k-1} \) and \( |u| = i \).

If there exist a vertex \( f \in P_{k-1} \) and a vertex \( g \) such that the set

\[
\{ a \in E^1 : s(a) = f \text{ and } r(a) = g \}
\]

is infinite then put \( e_n = f, e_{n+1} = g \) and proof of the claim ends, because there are no pathes which range is vertex \( f \in P_{k-1} \) (in the other case there exists a path \( y \) such that \( r(y) = e_{k-1} \) and \( |y| > k - 1 \) which contradicts the above arguments).

If for each vertex \( g \), for each vertex \( f \in P_{k-1} \) the set \( \{ a \in E^1 : s(a) = f \text{ and } r(a) = g \} \) is finite then the set \( \{ u \in \text{Path}(E) : r(u) \in P_{k-2} \} \) is finite as well. If there exist a vertex \( f \in P_{k-2} \) and a vertex \( g \) such that the set

\[
\{ a \in E^1 : s(a) = f \text{ and } r(a) = g \}
\]
is infinite then put \( e_n = f, e_{n+1} = g \) and proof of the claim ends.

If for each vertex \( g \), for each vertex \( f \in P_{k-1} \) the set \( \{a \in E^1 : s(a) = f \text{ and } r(a) = g \} \) is finite then the set \( \{u \in \text{Path}(E) : r(u) \in P_{k-3} \} \) is finite as well. We repeat our arguments and either find a desired pair of vertices or obtain that the set \( \{u \in \text{Path}(E) : r(u) = e_{k-1} \} \) is finite.

If the set \( \{u \in \text{Path}(E) : r(u) = e_{k-1} \} \) is finite and the set \( T_k = \{a_{(k,n)} : n \in \mathbb{N} \} \) is infinite then put \( e_n = e_{k-1}, e_{n+1} = e_k \) and proof of the claim ends. If the set \( T_k \) is finite we do the next step. Since the set \( E^0 \) of all vertices of graph \( E \) is finite and the set \( A \) is infinite we obtain that after a finite amount of steps we find a pair of vertices \( e_n \) and \( e_{n+1} \) such that the set \( \{u \in \text{Path}(E) : r(u) = e_n \} \) is finite and the set \( \{a \in E^1 : s(a) = e_n \text{ and } r(a) = e_{n+1} \} \) is infinite. \( \square \)

Fix those pair of vertices \( e_n \) and \( e_{n+1} \). Denote \( L = \{a \in E^1 : s(a) = e_n, r(a) = e_{n+1} \} \). Let \( M = \{a_n : n \in \mathbb{N} \} \) be an arbitrary countable infinite subset of \( L \). Put

\[
N = \{ua_n a_n^{-1} v^{-1} : u, v \in \text{Path}(E) \text{ such that } r(u) = r(v) = e_n \text{ and } a_n \in M \}.
\]

Define a topology \( \tau \) on the semigroup \( G(E) \) by the following way: each non-zero element of \( G(E) \) is an isolated point and a family \( B = \{U_n : n \in \mathbb{N} \} \) forms an open neighborhood base of the point 0, where \( U_n = \{ua_ka_k^{-1} v^{-1} \in N : k > n \} \cup \{0\} \). Since for a fixed element \( a_k \in M \) the set \( \{ua_k a_k^{-1} v^{-1} : u, v \in \text{Path}(E) \text{ such that } r(u) = r(v) = e_n \} \) is finite each basic open neighborhood of 0 is compact. Hence \( (G(E), \tau) \) is a locally compact topological space. By Lemma 3, \( (G(E), \tau) \) is a metrizable topological space. The equality \( U_n = U_n^{-1} \) provides the continuity of inversion in the semigroup \( (G(E), \tau) \). To prove the continuity of the semigroup operation in \( (G(E), \tau) \) we need to check it in the following three cases:

\begin{enumerate}
  \item \( bc^{-1} \cdot 0 = 0; \)
  \item \( 0 \cdot bc^{-1} = 0; \)
  \item \( 0 \cdot 0 = 0. \)
\end{enumerate}

Consider case (1). We have the following three subcases:

\begin{enumerate}
  \item \( cd = u \) for some pathes \( d, u \in \text{Path}(E) \) such that \( r(u) = e_n; \)
  \item \( c = ua_n d \) for some pathes \( u, d \in \text{Path}(E) \) such that \( r(u) = e_n \) and \( a_n \in M \);
  \item otherwise.
\end{enumerate}

Consider subcase (1.1). Fix an arbitrary open basic neighborhood \( U_k \) of 0. The continuity of the semigroup operation in \( (G(E), \tau) \) can derived from the inclusion \( bc^{-1} \cdot U_k \subseteq U_k \).

Consider subcase (1.2). Fix an arbitrary open basic neighborhood \( U_k \) of 0. The continuity of the semigroup operation in \( (G(E), \tau) \) can derived from the inclusion \( bc^{-1} \cdot U_{\max\{n_0+1, k\}} = 0 \in U_k \).

If there exists no elements \( d, u \in \text{Path}(E) \) and \( a_n \in M \) such that \( r(u) = e_n \) and \( cd = u \) or \( c = ua_n d \) then \( bc^{-1} \cdot U_1 = 0 \in U_k \). Hence the semigroup operation in \( (G(E), \tau) \) is continuous in case (1).

Continuity of the semigroup operation in \( (G(E), \tau) \) in case (2) immediately follows from the continuity of the semigroup operation in \( (G(E), \tau) \) in case (1) and from the continuity of the inversion in \( (G(E), \tau) \).

Consider case (3). We claim that \( U_k \cdot U_k \subseteq U_k \) for each \( k \in \mathbb{N} \). Indeed, fix two arbitrary elements \( u_1 p_n p_n^{-1} v_1^{-1} \) and \( u_2 p_m p_m^{-1} v_2^{-1} \) from \( U_k \). Observe that \( n > k \) and \( m > k \). Then

\[
u_1 p_n p_n^{-1} v_1^{-1} \cdot u_2 p_m p_m^{-1} v_2^{-1} = 0 \in U_k \quad \text{if either } v_1 \neq v_2 \text{ or } n \neq m\]

and

\[
u_1 p_n p_n^{-1} v_1^{-1} \cdot u_2 p_m p_m^{-1} v_2^{-1} = u_1 p_n p_n^{-1} v_2^{-1} \in U_k \quad \text{if } v_1 = v_2 \text{ and } n = m.
\]

Hence \( (G(E), \tau) \) is a metrizable locally compact topological inverse semigroup. \( \square \)

**Corollary 5.** There exists a non-discrete locally compact semigroup topology on a graph inverse semigroup \( G(E) \) if and only if there exists a non-discrete topology \( \tau \) such that \( (G(E), \tau) \) is a locally compact metrizable topological inverse semigroup.

Theorem 4 can be reformulated in the following way:
Theorem 6. Discrete topology is the only locally compact semigroup topology on the graph inverse semigroup $G(E)$ if and only if graph $E$ contains a finite amount of vertices and there does not exist a pair of vertices $e, f \in E^0$ such that the set $\{u \in \text{Path}(E) : r(u) = e\}$ is finite and the set $\{a \in E^1 : s(a) = e \text{ and } r(a) = f\}$ is infinite.

Proof. If the set $E^0$ of all vertices of graph $E$ is infinite or there exists a pair a vertices $e, f \in E^0$ such that the set $\{u \in \text{Path}(E) : r(u) = e\}$ is finite and the set $\{a \in E^1 : s(a) = e \text{ and } r(a) = f\}$ is infinite then we can introduce a non-discrete locally compact metrizable inverse semigroup topology $\tau$ on the semigroup $G(E)$ by the same way as in proof of the Theorem 4.

If there exists a non-discrete locally compact semigroup topology on the graph inverse semigroup $G(E)$ then by Theorem 4 semigroup $G(E)$ does not satisfy the condition $(\ast)$. Hence by the proof of the Theorem 4 either the set of all vertices of graph $E$ is infinite or graph $E$ contains a pair of vertices $e, f \in E^0$ such that the set $\{u \in \text{Path}(E) : r(u) = e\}$ is finite and the set $\{a \in E^1 : s(a) = e \text{ and } r(a) = f\}$ is infinite. \qed

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