CHARACTERISTIC VARIETIES OF HYPERSURFACE COMPLEMENTS

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Abstract. We give divisibility results for the (global) characteristic varieties of hypersurface complements expressed in terms of the local characteristic varieties at points along one of the irreducible components of the hypersurface. As an application, we recast old and obtain new finiteness and divisibility results for the classical (infinite cyclic) Alexander modules of complex hypersurface complements. Moreover, using Suciu’s notion of locally straight spaces, we translate our divisibility results for characteristic varieties in terms of the corresponding resonance varieties.

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1. Introduction

The study of the topology of complex algebraic varieties is a classical subject going back to Zariski, Enriques, Deligne, Hirzebruch or Arnol’d. A particularly fruitful approach consists of using local topological information at singular points in order to extract global topological properties of algebraic varieties.

A celebrated result in this circle of ideas is Libgober’s divisibility theorem for the classical (infinite cyclic) Alexander invariants of complements to complex affine hypersurfaces with only isolated singularities, including at infinity (see [Lib82, Lib83, Lib94]). More precisely, Libgober showed that the (only nontrivial) global Alexander polynomial of the complement divides the product of the local Alexander polynomials associated with each singular point. The second-named author ([Max06]) extended these divisibility results to the case of hypersurfaces with arbitrary singularities and in general position at infinity. Furthermore, this global-to-local approach was used in [DM07] to show that such divisibility results also hold for certain multivariable Alexander-type invariants, called the Alexander varieties (or support loci) of the hypersurface complement.

The Alexander varieties of a topological space are closely related to the so-called characteristic varieties (or jumping loci of rank-one local systems), which are topological invariants of the space. A fundamental result of Arapura ([A97]) showed that the characteristic varieties of complex hypersurface complements have a rigid structure. More precisely, the characteristic varieties of a plane curve complement are unions of subtori of the character torus, possibly translated by unitary characters. The result is true more generally, for any smooth quasi-projective variety, by work of Budur-Wang [BW14]. It is worth noting that Libgober extended this rigidity property also in the local hypersurface complement context [Lib09]. The interplay between the characteristic and Alexander varieties of any topological space is described in detail by Papadima and Suciu in [PS10].

Let \( f : \mathbb{C}^{n+1} \to \mathbb{C} (n \geq 2) \) be a reduced homogeneous polynomial of degree \( d > 1 \). Then \( f \) defines a global Milnor fibration \( f : M \to \mathbb{C}^* \) with total space \( M = \mathbb{C}^{n+1} \setminus f^{-1}(0) \) and Milnor fiber \( F = f^{-1}(1) \). Let \( V \) be the hypersurface in \( \mathbb{CP}^n \) defined by \( f \), and set \( M^* = \mathbb{CP}^n \setminus V \). Also, denote by \( H \) the hyperplane at infinity in \( \mathbb{CP}^n \). Assume that \( f = f_1 \cdots f_r \), where \( f_i \) are the irreducible factors of \( f \). Then \( V_i = \{ f_i = 0 \} \) are the irreducible components of \( V \).

Let \( V_i(M) \) and \( V_i(M^*) \) denote the \( i \)-th characteristic varieties of \( M \) and \( M^* \), respectively (see Definition 2.13). The aim of this paper is to obtain general divisibility results for these global characteristic varieties. In more detail, we prove that the characteristic varieties of \( M \) and \( M^* \) (up to degree \( n - 1 \)) are contained in the union of the local characteristic varieties at points along one of the irreducible component of \( V \); see Theorem 5.4 for a precise formulation. These divisibility results are very general, in the sense that no additional assumptions on the
hypersurface are needed. Moreover, our results improve on the divisibility theorems for the Alexander varieties obtained by Dimca and the second-named author in [DM07], see Remarks 4.4 and 5.5.

In the proof of the above-mentioned divisibility results for \( V_i(M) \) and \( V_i(M^*) \), we also study (and obtain divisibility results for) the Alexander and characteristic varieties of \( U := \mathbb{C}P^n \setminus V \cup H \), by employing methods developed by the second-named author in [Max06]. (Here the hyperplane at infinity \( H \) intersects \( V \) transversally.) These are then related to \( V_i(M) \) and \( V_i(M^*) \) via the \( n \)-homotopy equivalence \( U \hookrightarrow M \) and the Hopf map \( p : M \to M^* \).

The paper is organized as follows.

In Section 2, we describe a closed relationship between the homology Alexander varieties and cohomology Alexander varieties. In Section 3, we compute the stalks of the Sabbah specialization complex associated to an affine hypersurface in terms of the corresponding local cohomology Alexander modules. The proofs of the above-mentioned divisibility results are carried out in Sections 4 and 5, respectively. In particular, we give some vanishing results for the homology Alexander polynomials in Section 5.2. In Section 5.3, we obtain sharper divisibility results in the special case of hyperplane arrangements.

Sections 6 and 7 are devoted to applications of these divisibility results.

In section 6, we first recall the definition of the Dwyer-Fried invariants. Dwyer and Fried observed that the homological finiteness properties of free abelian covers are completely determined by the Alexander varieties. Moreover, Papadima andSuciu noted that in the previous sentence one can replace the Alexander varieties by the characteristic varieties. By combining this fact with our divisibility results for characteristic varieties, we recast several finiteness and divisibility results for classical (infinite cyclic) Alexander modules of hypersurface complements, e.g., Libgober’s results for hypersurfaces with only isolated singularities, including at infinity [Lib94], or the second-named author’s results for hypersurfaces in general position at infinity [Max06]. We also obtain a divisibility result for the classical Alexander modules of essential hyperplane arrangements, which (to our knowledge) seems to be new (see Remark 6.8).

In Section 7, we first introduce the resonance varieties and recall the tangent cone inclusion (cf. [Lib02]), which establishes a nice relationship between the characteristic varieties and the resonance varieties. We then recall Suciu’s notion of locally straight spaces (cf. [Su12A, Definition 6.1]), a class which already includes hyperplane arrangement complements. In Proposition 7.6, we give another set of examples of locally straight spaces. Moreover, using the tangent cone inclusion, we translate our divisibility results for the Alexander and characteristic varieties into similar results for the corresponding resonance varieties.

Convention: Unless otherwise specified, all homology and cohomology groups will be assumed to have \( \mathbb{C} \)-coefficients.

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2. Alexander varieties and Characteristic varieties

In this section, we introduce three types of jumping loci associated to a space and obtain a relation between the homology Alexander varieties and cohomology Alexander varieties.

2.1. Algebraic preliminaries. Let \( \Gamma_r = \mathbb{C}[t_1, t_1^{-1}, \cdots, t_r, t_r^{-1}] \) be the Laurent polynomial ring in \( r \) variables. \( \Gamma_r \) is a regular Noetherian domain, and in particular it is factorial. If \( A \) is a finitely generated \( \Gamma_r \)-module, the support locus of \( A \) is defined by the annihilator ideal of \( A \):

\[
\text{Supp}(A) = V(\text{ann}(A)).
\]

By construction, \( \text{Supp}(A) \) is a Zariski closed subset of the \( r \)-dimensional affine torus \( (\mathbb{C}^*)^r \), where \( (\mathbb{C}^*)^r \) is the scheme defined by \( \Gamma_r \). \( \text{Supp}(A) \) is called proper, if it has at least codimension 1 in \( (\mathbb{C}^*)^r \).

For \( \lambda = (\lambda_1, \cdots, \lambda_r) \in (\mathbb{C}^*)^r \), let \( m_\lambda \) be the corresponding maximal ideal in \( \Gamma_r \). For a \( \Gamma_r \)-module \( A \), we denote by \( A_\lambda \) the localization of \( A \) at the maximal ideal \( m_\lambda \). For \( A = \Gamma_r \), we use the simpler notation \( \Gamma_\lambda \). Then, if \( A \) is of finite type, we have that

\[
\text{Supp}(A) = \{ \lambda \in (\mathbb{C}^*)^r \mid A_\lambda \neq 0 \}.
\]

Let \( P \) be a prime ideal in \( \Gamma_r \), of height 1. Since \( \Gamma_r \) is a regular Noetherian domain, the localization of \( \Gamma_r \) at \( P \), denoted by \( \Gamma_P \), is a regular local ring of dimension 1. So \( \Gamma_P \) is a discrete valuation ring, hence a principal ideal domain. A prime ideal \( P \) of height 1 in \( \Gamma_r \) is principal, and we let \( \Delta(P) \) denote the generator of \( P \), which is well-defined up to multiplication by units of \( \Gamma_r \).

Assume that \( \text{Supp}(A) \) is proper in \( (\mathbb{C}^*)^r \), i.e., \( \text{ann}(A) \neq 0 \). Then, if \( V(P) \not\subseteq \text{Supp}(A) \), \( A_P = 0 \). On the other hand, if \( V(P) \subseteq \text{Supp}(A) \), then \( \text{ann}(A) \subseteq P \) and \( \text{ann}(A_P) = \{\text{ann}(A)\}P \neq 0 \), hence \( A_P \) is a torsion \( \Gamma_P \)-module. This shows that \( A_P \) has finite length as a \( \Gamma_P \)-module, which is denoted by \( \text{lg}(A_P) \). The characteristic polynomial of \( A \) is defined by

\[
\Delta(A) = \prod_P \Delta(P)^{\text{lg}(A_P)},
\]

where the product is over all the prime ideals in \( \Gamma_r \), of height 1 such that \( V(P) \subseteq \text{Supp}(A) \). Since \( \text{Supp}A \) is proper, this product is indeed a finite product. The prime factors of \( \Delta(A) \) are in one-to-one correspondence with the codimension one irreducible hypersurfaces of \( (\mathbb{C}^*)^r \) contained in \( \text{Supp}(A) \). Note that if \( \text{codim} \text{Supp}(A) > 1 \) then \( \Delta(A) = 1 \). If \( \text{Supp}(A) = (\mathbb{C}^*)^r \), then \( \Delta(A) = 1 \) by convention.

Remark 2.1. Sabbah used the alternating product of the above characteristic polynomials to define the Zeta function of a \( \Gamma_r \)-module complex, see [Sab90, Definition 2.1.12]. The support of the \( \Gamma_r \)-module \( A \) can also be defined by the first Fitting ideal of \( A \), see [DM07].

Lemma 2.2. If \( A \to B \to C \) is an exact sequence of finitely generated \( \Gamma_r \)-modules, then \( \text{Supp}(B) \subseteq \text{Supp}(A) \cup \text{Supp}(C) \). Moreover, if \( \text{Supp}(A) \) and \( \text{Supp}(C) \) are proper in \( (\mathbb{C}^*)^r \), then so is \( \text{Supp}(B) \), and \( \Delta(B) \) divides \( \Delta(A) \times \Delta(C) \).
Proof. Using the exactness of the localization (see [Wei94, p.76]), we get an exact sequence $A_\lambda \to B_\lambda \to C_\lambda$. If $\lambda \notin \text{Supp}(A) \cup \text{Supp}(C)$, then $A_\lambda = 0 = C_\lambda$. Thus $B_\lambda = 0$, i.e., $\lambda \notin \text{Supp}(B)$.

Let $P$ be a prime ideal in $\Gamma_r$ of height 1. Using the exactness of the localization over $P$, we get an exact sequence $A_P \to B_P \to C_P$. Assume that $\text{Supp}(A)$ and $\text{Supp}(C)$ are proper in $(\mathbb{C}^*)^r$, then $\lg(B_P) \leq \lg(A_P) + \lg(C_P)$. So $\Delta(B)$ divides $\Delta(A) \times \Delta(C)$. \hfill $\square$

Lemma 2.3. [Ser65, Proposition 4(c)] If $A$ and $B$ are finitely generated $\Gamma_r$-modules, then

$$\text{Supp}(A \otimes_{\Gamma_r} B) = \text{Supp}(A) \cap \text{Supp}(B).$$

2.2. Alexander varieties. Assume that $X$ is a finite, connected, $n$-dimensional CW complex with $\pi_1(X) = G$. Let $\nu : G \to \mathbb{Z}^r$ be an epimorphism, and consider the corresponding free abelian cover $X^\nu$ of $X$. The group of covering transformations of $X^\nu$ is isomorphic to $\mathbb{Z}^r$ and acts on the covering space. By choosing fixed lifts of the cells of $X$ to $X^\nu$, we obtain a free basis for the chain complex $C_*(X^\nu, \mathbb{C})$ of $\Gamma_r$-modules.

Remark 2.4. ([DM07, Remark 2.2]) Though the ring $\Gamma_r$ is commutative, it should be regarded as a quotient ring of $\mathbb{C}[\pi_1(X)]$, which is non-commutative in general. Hence, one should be careful to distinguish the right from the left $\Gamma_r$-modules. The ring $\Gamma_r$ has a natural involution denoted by an overbar, sending each $t_i$ to $\overline{t}_i := t_i^{-1}$. To a left $\Gamma_r$-module $A$, we associate the right $\Gamma_r$-module $\overline{A}$, with the same underlying abelian group but with the $\Gamma_r$-action given by $a \cdot c \mapsto \overline{c} \cdot a$, for $a \in A$ and $c \in \Gamma_r$. In this paper, we regard $C_*(X^\nu, \mathbb{C})$ as a complex of right $\Gamma_r$-modules. Conversely, to a right $\Gamma_r$-module $A$, we can associate a corresponding left module $\overline{A}$ by using the involution. Moreover, it can be seen that $A = (\overline{A})$.

Let us now define \( \overline{\text{Supp}}(A) = \{(t_1, \cdots, t_r) \in (\mathbb{C}^*)^r \mid (t_1^{-1}, \cdots, t_r^{-1}) \in \text{Supp}(A)\} \) and \( \Delta(A)(t_1, \cdots, t_r) = \Delta(A)(t_1^{-1}, \cdots, t_r^{-1}) \). Then \( \text{Supp}(A) = \overline{\text{Supp}}(A) \) and \( \Delta(A) = \overline{\Delta}(A) \).

Definition 2.5. The $i$-th homology Alexander module $A_i(X, \nu)$ of $X$ associated to the epimorphism $\nu$ is by definition the $\Gamma_r$-module $H_i(X^\nu, \mathbb{C})$. Similarly, the $i$-th cohomology Alexander module $A^i(X, \nu)$ of $X$ is by definition the $i$-th cohomology module of the dual complex $\text{Hom}_{\Gamma_r}(C_*(X^\nu, \mathbb{C}), \Gamma_r)$, where $\Gamma_r$ is considered here with the induced right $\Gamma_r$-module structure, as in Remark 2.4.

Since $X$ is a finite $n$-dimensional CW complex, it is clear by definition that $A_i(X, \nu) = 0 = A^i(X, \nu)$ for $i > n$.

Definition 2.6. The $i$-th homology Alexander variety of $X$ associated to the epimorphism $\nu$ is the support locus of the annihilator ideal of $A_i(X, \nu)$:

$$W_i(X, \nu) = \text{Supp}(A_i(X, \nu)),$$

and, the corresponding $i$-th homology Alexander polynomial is $\Delta_i(X, \nu) = \Delta(A_i(X, \nu))$. Similarly, the $i$-th cohomology Alexander variety of $X$ associated to the epimorphism $\nu$ is the support locus of the annihilator ideal of $A^i(X, \nu)$:

$$W^i(X, \nu) = \text{Supp}(A^i(X, \nu)),$$

and, the corresponding $i$-th cohomology Alexander polynomial is $\Delta^i(X, \nu) = \Delta(A^i(X, \nu))$. 

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Since $X$ is a finite CW-complex, the chain complex $C_\ast(X^\nu; \mathbb{C})$ is of finite type, so the corresponding (homology or cohomology) Alexander varieties are well-defined. The (homology or cohomology) Alexander varieties and (homology or cohomology) Alexander polynomials are homotopy invariants of $X$.

For the one-variable case, the homology Alexander varieties and the homology Alexander polynomial coincide with the set of roots of Alexander polynomials and, resp., the Alexander polynomials from [Lib94, Max06].

If $r = b_1(X)$, i.e., $\text{Free} H_1(X, \mathbb{Z}) \cong \mathbb{Z}^r$, we simply write $A_i(X) := A_i(X, \nu), W_i(X) := W_i(X, \nu), \Delta_i(X) := \Delta_i(X, \nu)$, and similar notations are used for the cohomology versions. If $H_1(X, \mathbb{Z})$ is free, this corresponding cover is called the universal abelian cover, and the covering map is denoted by $ab$.

Here we only give the definition of these Alexander invariants with $C_\ast$-coefficients, but see [PS10, Section 3.2] for arbitrary field coefficients.

2.3. Homology versus cohomology Alexander varieties. We employ here the spectral sequence approach from [DM07, section 2.3] to relate the homology and resp. cohomology Alexander varieties. More precisely, $A^\ast(X)$ can be computed from $A_s(X)$ by using the Universal Coefficient spectral sequence:

$$\text{Ext}_{\Gamma_r}^q(A_s(X), \Gamma_r) \Rightarrow A^{s+q}(X).$$

Using the exactness of the localization, we get the following localized spectral sequence for any $\lambda \in (\mathbb{C}^*)^r$:

$$\text{Ext}_{\Gamma_{\lambda}}^q(A_s(X)_{\lambda}, \Gamma_{\lambda}) \Rightarrow A^{s+q}(X)_{\lambda}.$$  

The next proposition is a direct consequence of [DM07, Proposition 2.3].

**Proposition 2.7.** For $X$ a finite, connected CW-complex and any $k \geq 0$,

$$\bigcup_{i=0}^k W^i(X) \subset \bigcup_{i=0}^k W_i(X)$$

**Proof.** If $\lambda \notin \bigcup_{i=0}^k W_i(X)$, (2.2) yields that $A_i(X)_{\lambda} = 0$ for $0 \leq i \leq k$. Then the spectral sequence (2.5) gives that $A^i(X)_{\lambda} = 0$ for $i \leq k$. Hence $\lambda \notin \bigcup_{i=0}^k W^i(X)$. \hfill \square

Since $C_\ast(X^{ab}, \mathbb{C})$ is a complex of finitely generated free $\Gamma_r$-modules, we have the following $\Gamma_r$-module complex isomorphism:

$$\text{Hom}_{\Gamma_r}(\text{Hom}_{\Gamma_r}(C_\ast(X^{ab}, \mathbb{C}), \Gamma_r), \Gamma_r) \cong C_\ast(X^{ab}, \mathbb{C}).$$

So, similarly, we also have a Universal Coefficient spectral sequence computing $A^\ast(X)$ from $A_s(X)$:

$$\text{Ext}_{\Gamma_s}^q(A^{n-s}(X), \Gamma_r) \Rightarrow A_{n-s-q}(X)$$

and the localized spectral sequence:

$$\text{Ext}_{\Gamma_{\lambda}}^q(A^{n-s}(X)_{\lambda}, \Gamma_{\lambda}) \Rightarrow A_{n-s-q}(X)_{\lambda}.$$
**Proposition 2.8.** For $X$ a finite, connected, CW-complex of dimension $n$, and any $k \leq n$, we have:

\[(2.9)\]

\[\bigcup_{i=k}^{n} W_i(X) \subset \bigcup_{i=k}^{n} W_i(X)\]

Moreover,

\[(2.10)\]

\[\bigcup_{i=0}^{n} W_i(X) = \bigcup_{i=0}^{n} W_i(X)\]

**Proof.** The first claim follows from the spectral sequence (2.8) by an argument similar to that of Proposition 2.7. The second claim follows from (2.6) and (2.9). \[\Box\]

**Remark 2.9.** Since $A_0(X) = \Gamma_r/m_1$, where $1 = (1, \cdots, 1)$, it follows from [DM07, Proposition 2.3] that $A^0(X) = 0$.

**Remark 2.10.** Since $X$ is a finite $n$-dimensional CW-complex, $A_n(X)$ is torsion-free as $\Gamma_r$-module. So, if $A_n(X) \neq 0$, then $W_n(X) = (\mathbb{C}^*)^r$. Moreover, (2.9) yields that $W_n(X) \subset W^n(X)$, hence $W^n(X) = (\mathbb{C}^*)^r$. In particular, $\Delta_n(X) = \Delta^n(X) = 1$.

Next, we investigate the relationship between the Alexander polynomials $\Delta_*(X)$ and $\Delta^*(X)$.

**Proposition 2.11.** Assume that $\bigcup_{i=0}^{k} W_i(X)$ is proper in $(\mathbb{C}^*)^r$. Then, for any $i \leq k$,

\[(2.11)\]

\[\Delta_{i-1}(X) = \Delta^i(X)\]

Therefore, there is an exact correspondence between the codimension 1 irreducible hypersurfaces of $(\mathbb{C}^*)^r$ contained in $W_{i-1}(X)$ and $W^i(X)$, respectively, for $i \leq k$.

**Proof.** Let $P$ be a prime ideal in $\Gamma_r$ of height 1. Using the exactness of the localization at $P$ applied to the spectral sequence (2.4), we get:

\[(2.12)\]

\[E_2^{s,q} = \text{Ext}^q_{\Gamma_p}(A_s(X)_P, \Gamma_P) \Rightarrow A^{s+q}(X)_P.\]

$\Gamma_p$ is a discrete valuation ring, hence a principal ideal domain. So $E_2^{s,q} = 0$ for $q > 1$. Note also that $E_2^{s,0} = \text{Ext}^0_{\Gamma_p}(A_s(X)_P, \Gamma_P)$ is always free, and $E_2^{s,1} = \text{Ext}^1_{\Gamma_p}(A_s(X)_P, \Gamma_P)$ is always torsion as a $\Gamma_p$-module. So the spectral sequence (2.12) degenerates at $E_2$. Since $\bigcup_{i=0}^{k} W_i(X)$ is proper, we have that

\[\text{Ext}^0_{\Gamma_r}(A_s(X), \Gamma_r) = \text{Hom}_{\Gamma_r}(A_s(X), \Gamma_r) = 0 \text{ for any } s \leq k.\]

Then the spectral sequence (2.12) yields that $E_2^{i-1}(X)_P, \Gamma_P) \cong A^i(X)_P$, for $i \leq k$. The claim follows now from the Universal Coefficient Theorem for the principal ideal domain $\Gamma_p$. \[\Box\]

**Remark 2.12.** The results stated in this subsection hold more generally, for the Alexander varieties and Alexander polynomials associated to any epimorphism $\nu : G \twoheadrightarrow \mathbb{Z}^r$. 

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2.4. Characteristic varieties. Let $X$ be, as before, a finite $n$-dimensional connected CW-complex with $\pi_1(X) = G$. Let $\mathbb{C}^*$ be the multiplicative group of non-zero complex numbers. Then the group of $\mathbb{C}$-valued characters, $\text{Hom}(G, \mathbb{C}^*)$, is a commutative, affine algebraic group. Each character $\rho \in \text{Hom}(G, \mathbb{C}^*)$ defines a rank one local system on $X$, denoted by $L_\rho$.

**Definition 2.13.** The $i$-th characteristic variety of $X$ is the Zariski closed subset:

$$V_i(X) = \{ \rho \in \text{Hom}(G, \mathbb{C}^*) | \dim_\mathbb{C} H_i(X, L_\rho) \neq 0 \}.$$  

**Remark 2.14.** $V_i(X)$ is filtered by the closed subsets

$$V_k^i(X) = \{ \rho \in \text{Hom}(G, \mathbb{C}^*) | \dim_\mathbb{C} H_i(X, L_\rho) \geq k \},$$

for all $k > 0$.

**Remark 2.15.** Some references define the characteristic varieties by using cohomology rather than homology:

$$V^i(X) = \{ \rho \in \text{Hom}(G, \mathbb{C}^*) | \dim_\mathbb{C} H^i(X, L_\rho) \neq 0 \}.$$  

Let $\overline{\rho}$ denote the inverse of $\rho$ in the character group $\text{Hom}(G, \mathbb{C}^*)$, then

$$H^i(X, L_{\overline{\rho}}) \cong \text{Hom}(H_i(X, L_\rho), \mathbb{C}).$$

Therefore, $V_i(X) = \overline{V^i(X)}$, where $\overline{V^i(X)} = \{ \rho \in \text{Hom}(G, \mathbb{C}^*) | \overline{\rho} \in V^i(X) \}$.

The characteristic varieties $V_i(X)$ are homotopy invariants of $X$, see [Su14] Lemma 2.4.

Recall now the projection map $\nu : G \to \mathbb{Z}^r$ from section 2.2. The map $\nu$ induces an embedding $\nu^* : (\mathbb{C}^*)^r = \text{Hom}(\mathbb{Z}^r, \mathbb{C}^*) \hookrightarrow \text{Hom}(G, \mathbb{C}^*)$. Papadima and Suciu established the following relationship between the characteristic varieties and the homology Alexander varieties:

**Theorem 2.16.** ([PS10] Theorem 3.6) For any $k \geq 0$ and any epimorphism $\nu : G \to \mathbb{Z}^r$,

$$\bigcup_{i=0}^k W_i(X, \nu) = \text{im}(\nu^*) \cap \left( \bigcup_{i=0}^k V_i(X) \right).$$

Let $\text{Hom}(G, \mathbb{C}^*)^0$ denote the identity component of the algebraic group $\text{Hom}(G, \mathbb{C}^*)$. Then the following result follows at once from Theorem 2.16.

**Proposition 2.17.** Let $X$ be a finite connected CW-complex of dimension $n$. Set $V(X) = \bigcup_{i=0}^n V_i(X)$, and $W(X) = \bigcup_{i=0}^n W_i(X)$. Then

$$W(X) = V(X) \cap \text{Hom}(G, \mathbb{C}^*)^0.$$  

**Remark 2.18.** It is known that for any character $\rho$ of $\pi_1(X)$, the following identity holds: $\chi(X, L_\rho) = \chi(X)$ (see [Di04] Proposition 2.5.4]). Hence, if $\chi(X) \neq 0$, we have that $V(X) = \text{Hom}(G, \mathbb{C}^*)$ and $W(X) = \text{Hom}(G, \mathbb{C}^*)^0$. So, $V(X)$ and $W(X)$ are interesting to study only when $\chi(X) = 0$.  

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3. Sabbah Specialization complex and local Alexander modules

In this section, we compute the stalks of the Sabbah specialization complex associated to an affine hypersurface in terms of the local cohomology Alexander modules. Most results here are based on [DM07, section 2.2] and [Bud12, section 3].

For any complex algebraic variety $X$ and any commutative ring $R$, we denote by $D^b_c(X, R)$ the derived category of bounded cohomologically $R$-constructible complexes of sheaves on $X$. For a quick introduction to derived categories, the reader is advised to consult [Di04].

**Definition 3.1.** For $\mathcal{F} \in D^b_c(X, \Gamma_r)$ and a point $x \in X$, the $i$-th support of $\mathcal{F}$ at $x$ is defined by $\text{Supp}_i^x(\mathcal{F}) := \text{Supp}(\mathcal{H}^i(\mathcal{F})_x) \subset (\mathbb{C}^*)^r$, and the $i$-th characteristic polynomial of $\mathcal{F}$ at $x$ is defined by $\delta^i(\mathcal{F}_x) := \Delta(\mathcal{H}^i(\mathcal{F})_x)$. The support of $\mathcal{F}$ at $x$ is defined by $\text{Supp}_x(\mathcal{F}) := \bigcup_i \text{Supp}_i^x(\mathcal{F})$, and the multi-variable monodromy zeta-function of $\mathcal{F}$ at $x$ is defined by

$$Z(\mathcal{F}_x) := \prod_i \delta^i(\mathcal{F}_x)^{(-1)^i}.$$  

3.1. Alexander modules of hypersurface complements. Let $V$ be a hypersurface in $\mathbb{CP}^n$ defined by a degree $d$ reduced homogeneous polynomial $f$. Assume that $f = f_1 \cdots f_r$, where $f_i$ are the irreducible factors of $f$, so $V_i = \{f_i = 0\}$ are the irreducible components of $V$.

Choose a hyperplane $H \not\subseteq V$ in $\mathbb{CP}^n$ as the hyperplane at infinity. Without loss of generality, assume that $\mathbb{CP}^n$ has coordinates $[x_0 : \cdots : x_n]$ and $H = \{x_0 = 0\}$. Then $\mathbb{C}^n = \mathbb{CP}^n \setminus H$ has coordinates $(x_1, \cdots, x_n)$. Set $g_i = f_i(1, x_1, \cdots, x_n)$ and $g = \prod_{i=1}^r g_i$, then $g = (g_1, \cdots, g_r)$ gives a polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^r$. Set

$$D_i = \{g_i = 0\}, \quad D = \bigcup_{i=1}^r D_i,$$

and

$$\mathcal{U} = \mathbb{C}^n \setminus D = \mathbb{CP}^n \setminus (V \cup H).$$

It is known that $H_1(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}^r$ is torsion free (see [Di92], (4.1.3),(4.1.4)), generated by meridian loops $\gamma_i$ about each irreducible component $V_i$, $1 \leq i \leq r$. If $\gamma_\infty$ denotes the meridian about the hyperplane at infinity, then there is a relation in $H_1(\mathcal{U}, \mathbb{Z}) : \sum_{i=1}^r d_i \gamma_i + \gamma_\infty = 0$, where $d_i = \text{deg} f_i$.

We can associate to the hypersurface complement $\mathcal{U}$ the homology (resp. cohomology) Alexander varieties $\mathcal{W}_i(\mathcal{U})$ (resp. $\mathcal{W}^i(\mathcal{U})$) and homology (resp. cohomology) Alexander polynomials $\Delta_i(\mathcal{U})$ (resp. $\Delta^i(\mathcal{U})$) as defined in Section 2.2. Note that $\mathcal{U}$ has the homotopy type of a finite $n$-dimensional CW complex. So $A_i(\mathcal{U}) = 0 = A^i(\mathcal{U})$ for $i > n$, and $A_n(\mathcal{U})$ is torsion-free as $\Gamma_r$-module.

Define a local system $\mathcal{L}$ on $\mathcal{U}$ with stalk $\Gamma_r$, and representation of the fundamental group determined by the composition:

$$\pi_1(\mathcal{U}) \to H_1(\mathcal{U}, \mathbb{Z}) \to \text{Aut}(\Gamma_r),$$

where the first map is the abelianization, and the last map is defined by $\gamma_i \mapsto t_i$. Here $t_i$ is the action given by the multiplication with $t_i$. Let $\mathcal{L}^\vee$ be the dual local system, whose stalk at a point $x \in \mathcal{U}$ is $\mathcal{L}_x^\vee := \text{Hom}_{\Gamma_r}(\mathcal{L}_x, \Gamma_r)$. 


For $F \in D^b_c(X, \Gamma_r)$, let $\mathcal{D}F$ denote its Verdier dual and $\overline{F}$ denote its conjugation (i.e., $\overline{F}$ is defined by composing all module structures with the involution). Then there is an isomorphism of local systems: $\mathcal{L}^\vee \cong \overline{\mathcal{L}}$, see [DM07, section 2.2].

The Alexander modules of $\mathcal{U}$ are related to the above local system by the following $\Gamma_r$-module isomorphisms:

\begin{align}
H^i(\mathcal{U}, \mathcal{L}) &\cong A^i(\mathcal{U}) \\
H^i(\mathcal{U}, \overline{\mathcal{L}}) &\cong A^i(\mathcal{U})
\end{align}

for all $i$. Note that since $\mathcal{U}$ is smooth of complex dimension $n$, we also have that $\mathcal{D}\mathcal{L} = \overline{\mathcal{L}}[2n]$ and $\mathcal{D}(\mathcal{D}\mathcal{L}) = \mathcal{L}$.

**Remark 3.2.** The following isomorphism follows by an argument similar to the one used in the proof of [Max06, Corollary 3.4]:

\begin{align}
A^i(\mathcal{U}) &\cong H^2n-i_c(\mathcal{U}, \mathcal{L}).
\end{align}

The second named author proved this isomorphism in the one-variable case by using intersection homology, see [Max06, Corollary 3.4]. Since the intersection homology theory is defined for any Noetherian commutative ring of finite cohomological dimension ([B83, p68]), this isomorphism also holds for the multi-variable Laurent polynomial ring $\Gamma_r$.

### 3.2. Sabbah specialization complex.

The Sabbah specialization complex (see [Sab90], and its reformulation in [Bud12]) can be regarded as a generalization of Deligne’s nearby cycle complex.

In our notations, consider the following commutative diagram of spaces and maps:

\[
\begin{array}{ccc}
D & \xrightarrow{i} & \mathbb{C}^n \\
\downarrow{g} & & \downarrow{g} \\
\mathbb{C}^r & \xrightarrow{\tilde{g}} & (\mathbb{C}^r)^r
\end{array}
\]

where $\tilde{g}$ is the universal covering of $(\mathbb{C}^r)^r$, and the right-hand square of the diagram is cartesian.

**Definition 3.3.** ([Bud12, Definition 3.2]) The Sabbah specialization complex functor of $g$ is defined by

$$
\psi_g = i^* Rl_* R\pi_! (l \circ \pi)^*: D^b_c(\mathbb{C}^n, \mathbb{C}) \to D^b_c(D, \Gamma_r),
$$

and we call $\psi_g\mathbb{C}_{\mathbb{C}^n}$ the Sabbah specialization complex.

**Remark 3.4.** ([Bud12, Remark 3.3]) The Sabbah specialization complex can be viewed as a generalization of Deligne’s nearby cycles. In fact, when $r = 1$, $\psi_g\pi$ as defined here equals $\psi_g\pi[1]$ as defined by Deligne (see [Bry86], page 13), where $R\pi_!$ is replaced by $R\pi_*$. For short, in the following we write $\mathbb{C}$ for the constant sheaf $\mathbb{C}_{\mathbb{C}^n}$ on $\mathbb{C}^n$.

**Lemma 3.5.** ([Bud12, Lemma 3.4]) We have the following distinguished triangle in $D^b_c(\mathbb{C}^n, \Gamma_r)$:

\begin{align}
l_! \mathcal{L} &\to Rl_* \mathcal{L} \to i_! \psi_g \mathbb{C} \xrightarrow{[1]}.
\end{align}

In particular, $\psi_g\mathbb{C} = i^* Rl_* \mathcal{L}$.
Verdier duality yields that (cf. [Sch03 Corollary 4.2.2]):
\[ D\psi_!\mathcal{L} = R\psi_!\mathcal{L}[2n], \text{ and } D\psi_!\mathcal{L} = i\mathcal{L}[2n]. \]
Therefore, by dualizing \((3.3)\), we note that, up to a shift, \(\psi_g^!\mathbb{C}\) is a self-dual (i.e., \(\psi_g^!\mathbb{C} \cong D\psi_g^!\mathcal{C}[-2n + 1]\)), \(\Gamma_r\)-perverse sheaf on \(D\).

3.3. Local Alexander modules. Let \(x\) be a point in \(D\) and \(B_x\) be a small open ball centered at \(x\) in \(\mathbb{C}^n\). Set \(U_x = U \cap B_x\). Denote by \(I_x = \{ i \mid g_i(x) = 0 \}\) and \(r_x = \# |I_x|\). It is known that \(H_1(U_x, Z)\) is torsion free. The map \(H_1(U_x, Z) \to H_1(U, Z)\) induced by the natural inclusion can be viewed as a composition of the following two maps:
\[ H_1(U_x, Z) \to \mathbb{Z}^{r_x} \hookrightarrow \mathbb{Z}^r = H_1(U, Z) \]
where \(\mathbb{Z}^{r_x}\) is the free abelian group generated by the meridian loops \(\{ \gamma_i \}_{i \in I_x}\). The first map in \((3.4)\) is surjective and the second map is injective. (In the special case of hyperplane arrangements, the first map in \((3.4)\) is always an isomorphism.)

Let \(\nu_x\) denote the epimorphism:
\[ \pi_1(U_x) \xrightarrow{\nu_x} \mathbb{Z}^{r_x}. \]
and \(\mathcal{V}(U_x)\) denote the union of the characteristic varieties of \(U_x\). Then Theorem 2.16 yields that \(\mathcal{W}(U_x, \nu_x) = \mathcal{V}(U_x) \cap \text{im}(\nu_x)\). The embedding \(\nu_x^*\) identifies the variables along branches that come from the same global irreducible component. For a concrete situation, see Examples 3.9 and 3.10.

The second map in \((3.4)\) is injective, so this shows that the local cover induced by the map \(H_1(U_x, Z) \to H_1(U, Z) \cong \mathbb{Z}^r\) is disconnected. In particular, it has exactly \([\mathbb{Z}^r : \mathbb{Z}^{r_x}]\) connected components. Let \(\Gamma_{r_x}\) be the sub-ring of \(\Gamma_r\) with variables \(\{ t_i \}_{i \in I_x}\), and \(\Gamma_{r-r_x}\) be the sub-ring of \(\Gamma_r\) in the remaining variables. Then \(\Gamma_r = \Gamma_{r_x} \otimes_{\mathbb{C}} \Gamma_{r-r_x}\).

Definition 3.6. The uniform Alexander varieties of \(U_x\) are defined by
\[ \mathcal{W}_i^{unif}(U_x, \nu_x) = \mathcal{W}_i(U_x, \nu_x) \times (\mathbb{C}^*)^{r-r_x} \text{ and } \mathcal{W}^{unif}(U_x, \nu_x) = \mathcal{W}(U_x, \nu_x) \times (\mathbb{C}^*)^{r-r_x}, \]
where \((\mathbb{C}^*)^{r-r_x}\) is the scheme corresponding to \(\Gamma_{r-r_x}\).

The characteristic varieties of the local complement \(U_x\) have been intensely studied by Libgober in [Lib09]. In particular, he showed that \(\mathcal{V}(U_x)\) is a finite union of torsion-translated subtori (cf. [Lib09 Theorem 1.1]). Note that the Alexander variety \(\mathcal{W}(U_x, \nu_x) = \mathcal{V}(U_x) \cap \text{im}(\nu_x)\) can be defined over \(\mathbb{Q}\), hence the involution does not change \(\mathcal{W}(U_x, \nu_x)\). Moreover, if \(\mathcal{W}(U_x, \nu_x)\) is proper, then the involution keeps \(\Delta^i(U_x, \nu_x) = \Delta_{i-1}(U_x, \nu_x)\) unchanged.

Proposition 3.7. With the above assumptions and notations, for any \(x \in D\), we have:
\[ \text{Supp}_x^i(\psi_g^!\mathbb{C}) = \mathcal{W}_i(U_x, \nu_x) \times (\mathbb{C}^*)^{r-r_x}; \]
\[ \text{Supp}_x(\psi_g^!\mathbb{C}) = \mathcal{W}^{unif}(U_x, \nu_x). \]
Moreover, \(\text{Supp}_x(\psi_g^!\mathbb{C})\) is proper in \((\mathbb{C}^*)^r\), so, for any \(i \geq 0\),
\[ (3.6) \]
\[ \delta^i(\psi_g^!\mathbb{C}) = \Delta^i(U_x, \nu_x) = \Delta_{i-1}(U_x, \nu_x). \]
Proof. By Lemma 3.5 there exist $\Gamma_r$-module isomorphisms: $H^i_\Gamma(C_{x}^{\nu}(U_x),\nu)$, for all $i$. On the other hand, we have:

\[(3.7) \quad C^*(U_x, \overline{\mathcal{L}}|_{U_x}) \cong \operatorname{Hom}_{\Gamma_r}(C_*(U_x^{\nu_r},\Gamma_r) \cong \operatorname{Hom}_{\Gamma_r}(C_*(U_x^{\nu_r},\Gamma_r) \otimes \mathcal{C} \Gamma_r,\nu_r),\]

where the first isomorphism follows from the definition of the cohomology with local coefficients (compare with (3.1)). In fact, since $C_*(U_x^{\nu_r})$ is isomorphic to a direct sum of copies of the ring $\Gamma_r$, we only need to show that the second isomorphism of (3.7) holds for $C_*(U_x^{\nu_r}) = \Gamma_r$. In order to see this, note that

\[
\operatorname{Hom}_{\Gamma_r}(\Gamma_r,\nu) \cong \Gamma_r \cong \Gamma_r \otimes \mathcal{C} \Gamma_r \cong \operatorname{Hom}_{\Gamma_r}(\Gamma_r,\nu_r) \otimes \mathcal{C} \Gamma_r,\nu_r,
\]

where all the isomorphisms are $\Gamma_r$-module isomorphisms. Hence (3.7) follows.

So $H^i_\Gamma(C_{x}^{\nu}(U_x,\nu_x) \otimes \mathcal{C} \Gamma_r$, and $\operatorname{ann}(H^i_\Gamma(C_{x}^{\nu}(U_x,\nu_x))) = \langle \operatorname{ann}(A^i(U_x,\nu_x)) \rangle$, where $\langle \operatorname{ann}(A^i(U_x,\nu_x)) \rangle$ is the ideal in $\Gamma_r$ generated by $\operatorname{ann}(A^i(U_x,\nu_x))$. Therefore, $\operatorname{Supp}_x^{\nu_r}(C_{x}^{\nu}(U_x,\nu_x)) = \mathcal{W}(U_x,\nu_x) \times (C^*)^{r-r_x}$, and $\partial^{i}(C_{x}^{\nu}(U_x,\nu_x)) = \partial^{i}(U_x,\nu_x)$. Altogether, since the involution does not change $\mathcal{W}(U_x,\nu_x)$, we get that $\operatorname{Supp}_x^{\nu_r}(C_{x}^{\nu}(U_x,\nu_x)) = \operatorname{Supp}_x^{\nu_r}(C_{x}^{\nu}(U_x,\nu_x))$.

As we will show later on in Proposition 6.5, $\mathcal{W}(U_x,\nu_x)$ is proper in $(C^*)^{r_x}$. Thus $\operatorname{Supp}_x^{\nu_r}(C_{x}^{\nu}(U_x,\nu_x))$ is proper in $(C^*)^{r_x}$. As the involution keeps $\mathcal{V}(U_x,\nu_x)$ unchanged, the identity $\mathcal{V}(U_x,\nu_x) = \mathcal{V}(U_x,\nu_x)$ follows from Proposition 2.11.

Remark 3.8. Proposition 3.7 is similar to, but more detailed than [Bud12, Lemma 3.18], which in turn is an extension of [Sab90, 2.2.5]. [Bud12] draws two conclusions out of this calculation: that one needs and can introduce $\mathcal{W}^{\text{unif}}$, as here, and $\operatorname{Supp}^{\text{unif}}$. The last conclusion is not correct, as Proposition 3.7 points out, namely $\operatorname{Supp}^{\text{unif}}$ should stay simply $\operatorname{Supp}$ in [Bud12]. N. Budur has communicated to us that, indeed, our Proposition 3.7 above is the correct version, and therefore it simplifies all the statements in [Bud12] concerning $\operatorname{Supp}^{\text{unif}}$, by replacing it with $\operatorname{Supp}$.

Since $\operatorname{Supp}_x^{\nu_r}(C_{x}^{\nu}(U_x,\nu_x))$ is proper in $(C^*)^{r_x}$, we say that the Sabbah specialization complex is a torsion $\Gamma_r$-module complex, i.e., the stalks of its cohomology sheaves are torsion $\Gamma_r$-modules.

Example 3.9. Nodal curve case: Choose $g = x_1^2 - x_2^2(x_2 + 1)$, and $x = (0,0)$. Then $H_1(U_x,\mathbb{Z}) = \mathbb{Z}^2$ and $\mathcal{V}(U_x) = \{(1,1)\}$. The map $\nu_x^*$ gives an embedding: $\mathbb{C}^* \rightarrow (\mathbb{C}^*)^2$, where $\nu_x^*(t) = (t,t)$. So $\mathcal{W}(U_x,\nu_x) = \{1\}$.

Example 3.10. Choose $g_1 = x_1^2 - x_2^2(x_2 + 1)$, and let $g_2$ be a generic linear polynomial. Set $x = (0,0)$. Then $U_x$ is homotopy equivalent to $S^1 \setminus (S^1 \cup S^1)$, and $H_1(U_x,\mathbb{Z}) = \mathbb{Z}^3$. Hence $\mathcal{V}(U_x) = V(s_1s_2s_3 = 1)$. The map $\nu_x^*$ gives the embedding $(\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^3$, where $\nu_x^*(t_1,t_2) = (t_1,t_1,t_2)$. So $\mathcal{W}(U_x,\nu_x) = V(t_1^2t_2 = 1) \subset (\mathbb{C}^*)^2$.

Next, we give some vanishing results for $\Delta_i(U_x,\nu_x)$.

Example 3.11. Assume that $D$ is a normal crossing divisor (NC) at $x$, then $U_x$ is contractible. Thus $\mathcal{V}(U_x) = \{1\}$, and $\mathcal{W}(U_x,\nu_x) = \text{im}(\nu_x^*) \cap \mathcal{V}(U_x) = \{1\}$. So $\operatorname{Supp}_x^{\nu_r}(C_x^{\nu}(U_x,\nu_x)) = \mathcal{W}^{\text{unif}}(U_x,\nu_x) = (\mathbb{C}^*)^{r-r_x}$. If $r_x \geq 2$, then $\Delta_i(U_x,\nu_x) = 1$ for all $i$. 

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Example 3.12. Assume that $D$ is an isolated non-normal crossing divisor (INNC) at $x$, then $U^b_x$ has the homotopy type of a bouquet of $(n-1)$-spheres, see [DL07, Theorem 3.2]. Then $A_i(U_x) = \Gamma_{r_x}/m_1$, $A_i(U_x) = 0$ for $0 < i < n - 1$, and $A_{n-1}(U_x)$ is a finite dimensional $\mathbb{C}$-vector space, hence an Artinian $\Gamma_{r_x}$-module. In particular, $V(U_x)$ is a finite set of points (see [E74, Theorem 2.13]). Hence $W(U_x, \nu_x) = \text{im}(\nu_x^*) \cap V(U_x)$ is also a finite set of points. Then $\Delta_i(U_x, \nu_x) = 1$ for all $i$, if $r_x \geq 2$.

Example 3.13. Choose $x \in \cap_{i=1}^r D_i$, where $r \geq 2$. Assume that $D_1$ is smooth and $D_1$ intersects the other components of $D$ transversally in $B_x$. Set $U' := B_x - \cap_{i=2}^r D_i$. If follows that $U^b_x = (U'_x)^b \times \mathbb{R}$, hence $A_i(U_x) = A_i(U'_x) \otimes \mathbb{C}[t_1, t_1^{-1}]$. Then $W(U_x, \nu_x)$ is proper in the sub-torus of $(\mathbb{C}^*)^r$ defined by $\{t_1 = 1\}$, so it has at least codimension 2 in $(\mathbb{C}^*)^r$, and $\Delta_i(U_x, \nu_x) = 1$ for all $i$.

4. Divisibility results for hypersurfaces transversal at infinity

From now on, we always assume that $V$ is transversal to $H$ in the stratified sense (i.e., $g = \prod_{i=1}^r g_i$ is transversal at infinity). Then the affine hypersurface $D$ is homotopy equivalent to a bouquet of $(n-1)$-spheres, i.e.,

\begin{equation}
D \sim \bigvee_{\mu} S^{n-1},
\end{equation}

where $\mu$ denotes the number of spheres in the above join (cf. [DP03, page 476]). It is shown in loc.cit. that $\mu$ can be determined topologically as the degree of the gradient map associated to $f$.

In this section, we give a general divisibility result (compare with [DM07, Theorem 3.2]) on Alexander varieties of $\mathcal{U}$ by using the residue complex.

4.1. Residue complex and its stalks. First, let us introduce the residue complex.

Definition 4.1. Let $j$ be the inclusion of $\mathcal{U}$ into $\mathbb{CP}^n$. The residue complex $\mathcal{R}^\bullet$ associated to $\mathcal{U}$ is defined by the distinguished triangle:

\begin{equation}
j_*\mathcal{L} \to Rj_*\mathcal{L} \to \mathcal{R}^\bullet \xrightarrow{[1]} \%
\end{equation}

Remark 4.2. The residue complex is a generalization of the peripheral complex used in [Max06]. In fact, when $r = 1$, the residue complex $\mathcal{R}^\bullet$ as defined here equals the peripheral complex $\mathcal{R}^\bullet[-2n]$ as defined by Cappell and Shaneson, see [CS91] or [Max06].

As already pointed out, we have:

$$Dj_*\mathcal{L} = Rj_*\mathcal{L}[2n], \text{ and } DRj_*\mathcal{L} = j_*\mathcal{L}[2n].$$

So, by dualizing (4.2), we see that, up to a shift, $\mathcal{R}^\bullet$ is a self-dual (i.e., $\mathcal{R}^\bullet \cong D\mathcal{R}^\bullet[-2n + 1]$), $\Gamma_\nu$-perverse sheaf.

Next, we compute the stalk cohomology of $\mathcal{R}^\bullet$. It is clear from definition that $\mathcal{R}^\bullet$ has compact support on $V \cup H$. Fix a Whitney $b$-regular stratification $\mathcal{S}$ of $V$. By the transversality assumption, there is a stratification $\mathcal{S}'$ of $V \cup H$ with strata of the form: $H \setminus V \cap H$, $S \cap H$ and $S \setminus S \cap H$, where $S \in \mathcal{S}$. Then $\text{Supp}_{x,\mathcal{R}^\bullet}$ is constant for $x$ in a given stratum of $\mathcal{S}'$. The
to the type of strata in $S$.

(1) If $x \in D = V \setminus H$, then by comparing the two distinguished triangles $(3.3)$ and $(4.2)$, we have that $\mathcal{R}^\bullet_D = \psi_g \mathcal{C}$, and $\mathcal{H}^i(\mathcal{R}^\bullet)_x$ with $x \in D$ is computed in Section 3.3. In particular, $\text{Supp}_x(\mathcal{R}^\bullet) = \mathcal{W}_{\text{unif}}(\mathcal{U}_x, \nu_x)$.

(2) If $x \in H \setminus V$, the link pair of $H \setminus V$ is $(S^1, 0)$, and the corresponding generator $\gamma_\infty$ is mapped to $\prod_{i=1}^r t_i^{-d_i}$ under the representation defining the local system $\mathcal{L}$. So, in this case,

$$\mathcal{H}^i(\mathcal{R}^\bullet)_x \cong \begin{cases} \Gamma_r / \langle \prod_{i=1}^r t_i^{-d_i} = 1 \rangle, & i = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and $\text{Supp}_x(\mathcal{R}^\bullet) = V(\prod_{i=1}^r t_i^{d_i} = 1)$.

(3) If $x \in H \cap V$, assume that $x \in S \cap H$ with $\dim \mathbb{C} S > 0$, where $S \in \mathcal{S}$. Choose arbitrary $x_S \in S \setminus S \cap H \subset D$. Then, due to the transversality assumption, the Künneth formula yields that

$$\mathcal{H}^i(\mathcal{R}^\bullet)_x \cong H^i(\mathcal{U}_{x_S}, \mathcal{L}|_{\mathcal{U}_{x_S}}) \otimes_{\Gamma_r} \Gamma_r / \langle \prod_{i=1}^r t_i^{-d_i} = 1 \rangle$$

$$\oplus H^{i-1}(\mathcal{U}_{x_S}, \mathcal{L}|_{\mathcal{U}_{x_S}}) \otimes_{\Gamma_r} \Gamma_r / \langle \prod_{i=1}^r t_i^{-d_i} = 1 \rangle$$

In particular, Lemma 2.3 shows that $\text{Supp}_x(\mathcal{R}^\bullet) = \mathcal{W}_{\text{unif}}(\mathcal{U}_{x_S}, \nu_{x_S}) \cap V(\prod_{i=1}^r t_i^{d_i} = 1)$.

The above stalk calculations, combined with Proposition 6.5 below, show that $\text{Supp}_x(\mathcal{R}^\bullet)$ is always proper, for any $x \in V \cup H$. So $\mathcal{R}^\bullet$ is a torsion $\Gamma_r$-module complex. Moreover, the involution does not change $\text{Supp}_x(\mathcal{R}^\bullet)$.

4.2. Divisibility results for Alexander varieties and characteristic varieties. The following theorem is proved by using a similar approach as that of [Max06], where the second named author obtained general divisibility results for the classical (one-variable) Alexander modules of the hypersurface complement $\mathcal{U}$ by making use the peripheral complex.

**Theorem 4.3.** Assume that the hypersurface $V \subset \mathbb{C}^{\mathbb{P}_n}$ is transversal to the hyperplane at infinity $H$. Fix the stratification $\mathcal{S}'$ of $V \cup H$ as above, and let $S$ denote a stratum in $\mathcal{S}'$. Then, for any $0 \leq k \leq n - 1$,

$$\bigcup_{i=0}^k \mathcal{V}_i(\mathcal{U}) = \bigcup_{i=0}^k \mathcal{W}_i(\mathcal{U}) \subset \bigcup_{S \subset D_1} \mathcal{W}_{\text{unif}}(\mathcal{U}_{x_S}, \nu_{x_S}),$$

where the last union is over the strata $S$ contained in a fixed irreducible component $D_1$ of $D$ with $\dim \mathbb{C} S \geq n - k - 1$ and $x_S \in \mathcal{S}'$ is arbitrary. Moreover,

$$\bigcup_{i=0}^{n-1} \mathcal{V}_i(\mathcal{U}) = \bigcup_{i=0}^{n-1} \mathcal{W}_i(\mathcal{U}) \subset V(\prod_{i=1}^r t_i^{d_i} = 1).$$

In particular, $\mathcal{W}_i(\mathcal{U})$ and $\mathcal{V}_i(\mathcal{U})$ are proper in $(\mathbb{C}^*)^r$ for all $i \leq n - 1$.

**Proof.** The proof is divided into 3 steps.
Step 1: We first interpret the Alexander modules $A_\iota(U)$ ($i \leq n-1$) in terms of the residue complex $\mathcal{R}^\bullet$.

Let $u$ and $v$ be the inclusions of $\mathbb{CP}^n \setminus V_1$ and respectively $V_1$ into $\mathbb{CP}^n$. By applying the compactly supported hypercohomology functor to the distinguished triangle

$$v_*v^!j_!\mathcal{L} \to j_!\mathcal{L} \to u_*u^*j_!\mathcal{L} \overset{[1]}{\to},$$

we have the following long exact sequence:

$$(4.5) \quad \cdots \to H^{2n-i}(V_1, v^!j_!\mathcal{L}) \to H^{2n-i}(\mathbb{CP}^n, j_!\mathcal{L}) \to H^{2n-i}(\mathbb{CP}^n \setminus V_1, u^*j_!\mathcal{L}) \to \cdots$$

Since $U$ is smooth of complex dimension $n$, we have that $\mathcal{L}[n] \in \text{Perv}(U)$. The inclusion $k : U \hookrightarrow \mathbb{CP}^n \setminus V_1$ is a quasi-finite affine morphism, hence $k_!\mathcal{L}[n] \in \text{Perv}(\mathbb{CP}^n \setminus V_1)$ (e.g., see [Di04], Corollary 5.2.17). Since $\mathbb{CP}^n \setminus V_1$ is an $n$-dimensional affine variety, Artin’s vanishing theorem (e.g., see [Sch03], Corollary 6.0.4) yields:

$$H^{2n-i}(\mathbb{CP}^n \setminus V_1, k_!\mathcal{L}) = 0 \text{ for } i < n.$$  

Note that $u^*j_!\mathcal{L} = k_!\mathcal{L}$, and recall that by Remark 3.2 we have that $A_i(U) \cong H^{2n-i}(U, \mathcal{L}) \cong H^{2n-i}(\mathbb{CP}^n, j_!\mathcal{L})$. Therefore, the long exact sequence (4.5), together with the vanishing from (4.6), yields:

$$A_i(U) \cong H^{2n-i}(V_1, v^!j_!\mathcal{L}) \text{ for } i < n - 1,$$

and $A_{n-1}(U)$ is a quotient of the $\Gamma_r$-module $H^{n+1}(V_1, v^!j_!\mathcal{L})$.

Step 2: Next, we prove the divisibility result (4.3) for $k = n - 1$.

Let $c$ be the map from $V_1$ to a point. Then

$$H^{2n-i}(V_1, v^!j_!\mathcal{L}) \cong H^{2n-i}(Rc_!, v^!j_!\mathcal{L})$$

$$\overset{(1)}{=} H^{-i}(Rc_!, \mathcal{D}(\mathcal{L}))$$

$$\cong H^{-i}(Rc_!, \mathcal{D}(\mathcal{L}))$$

$$\cong H^{-i}(\mathcal{D}(\mathcal{L}))$$

where (1) follows from $\mathcal{D}\mathcal{L} \cong \mathcal{L}[2n]$.

Note that $\mathcal{D}(\mathcal{L}) = R\text{Hom}_{\Gamma_r}(\mathcal{L}, \mathcal{L})$. Then we have the following spectral sequence (e.g., see [B83], p.243)

$$(4.7) \quad \text{Ext}_{\Gamma_r}^{q,s}(H^c(V_1, \mathcal{L}^\bullet), \mathcal{L}) \Rightarrow H^{2n-q-s}(V_1, v^!j_!\mathcal{L}).$$

On the other hand, $\mathcal{D}(\mathcal{L}) = \mathcal{L}$, so we have a similar spectral sequence:

$$(4.8) \quad \text{Ext}_{\Gamma_r}^{q,s}(H^{2n-s}(V_1, v^!j_!\mathcal{L}), \mathcal{L}) \Rightarrow H^{s+q}(V_1, \mathcal{L}^\bullet).$$

By using the same arguments as in the proofs of Proposition 2.7 and Proposition 2.8, we obtain

$$\bigcup_{i=0}^{2n} \text{Supp}(H^{2n-i}(V_1, v^!j_!\mathcal{L})) = \bigcup_{i=0}^{2n} \text{Supp}(H^{i}(V_1, \mathcal{L}^\bullet)).$$
Recall that $A_i(\mathcal{U}) \cong H^{2n-i}(V_1, v^j_{ij}L)$ for $i < n - 1$, and $A_n(\mathcal{U})$ is a quotient of the $\Gamma_n$-module $H^{n+1}(V_1, v^j_{ij}L)$. By Lemma 2.2 we have that $W_i(\mathcal{U}) \subset \text{Supp}(H^{2n-i}(V_1, v^j_{ij}L))$ for $i \leq n - 1$. Thus:

\begin{equation}
\bigcup_{i=0}^{n-1} W_i(\mathcal{U}) \subset \bigcup_{i=0}^{n-1} \text{Supp}(H^{2n-i}(V_1, v^j_{ij}L)) \subset \bigcup_{i=0}^{2n} \text{Supp}(H^i_c(V_1, \mathcal{R}^\bullet))
\end{equation}

By using the compactly supported hypercohomology long exact sequence for the inclusion of strata of $V_1$, we get by Lemma 2.2 that

$$\text{Supp}(H^i_c(V_1, \mathcal{R}^\bullet)) \subset \bigcup_{S \subset V_1} \text{Supp}(H^i(S, \mathcal{R}^\bullet|S)),$$

where $S$ runs over all the strata of $V_1$. Note that $H^i(S, \mathcal{R}^\bullet|S)$ is the abutment of a spectral sequence with the $E_2$-term defined by $E_2^{s,q} = H^s(S, \mathcal{R}^q(\mathcal{R}^\bullet))$, and $\text{Supp}_x(\mathcal{R}^\bullet)$ is constant along this stratum. By using the exactness of the localization and the corresponding localized spectral sequence, it follows that if $\lambda \notin \text{Supp}(\mathcal{R}^\bullet|S)$, then $\lambda \notin \text{Supp}(H^i(S, \mathcal{R}^\bullet|S))$. So

$$\text{Supp}(H^i_c(S, \mathcal{R}^\bullet|S)) \subset \text{Supp}(\mathcal{R}^\bullet|S) = \text{Supp}(\mathcal{R}^\bullet|S),$$

where the last identification follows from the fact that the involution does not change $\text{Supp}_x(\mathcal{R}^\bullet)$.

If $S \subset D_1$, then it follows from the discussion of the previous section that $\text{Supp}(\mathcal{R}^\bullet|S) = \mathcal{W}^\text{unif}(\mathcal{U}_S, \nu_S)$, for $x_S \in S$ arbitrary. Similarly, if $S \subset V_1 \cap \mathcal{H}$, there exists a stratum $S' \subset D_1$ such that $\text{Supp}(\mathcal{R}^\bullet|S) = \mathcal{W}^\text{unif}(\mathcal{U}_{S'}, \nu_{S'}) \cap V(\prod_{i=1}^r t_i^{d_i} = 1)$, where $x_{S'} \in S'$. Altogether,

\begin{equation}
\bigcup_{i=0}^{n-1} W_i(\mathcal{U}) \subset \bigcup_{S \subset D_1} \mathcal{W}^\text{unif}(\mathcal{U}_S, \nu_S).
\end{equation}

If we replace $V_1$ by the hyperplane at infinity $H$, then the stalk calculations for $\mathcal{R}^\bullet$ show that

\begin{equation}
\bigcup_{i=0}^{n-1} W_i(\mathcal{U}) \subset V(\prod_{i=1}^r t_i^{d_i} = 1).
\end{equation}

Step 3: We use the Lefschetz hyperplane section theorem to complete the proof.

Fix $1 \leq k < n - 1$. Consider $L = \mathbb{CP}^{k+1}$ a generic $(k + 1)$-dimensional linear subspace of $\mathbb{CP}^n$, such that $L$ is transversal to $V \cup \mathcal{H}$. Then $W = L \cap \mathcal{V}$ is a $k$-dimensional, reduced hypersurface in $L$, which is transversal to the hyperplane at infinity $H \cap L$ of $L$. Moreover, by the transversality assumption, $L \cap \mathcal{D}$ has a Whitney stratification induced from $\mathcal{D}$, with strata of the form $S \cap L$.

By applying the Lefschetz hyperplane section theorem ([Di92, 1.6.5]) to the section of $\mathcal{U}$ by $L$, we see that the inclusion $\mathcal{U} \cap L \hookleftarrow \mathcal{U}$ is a homotopy $(k + 1)$-equivalence. Hence $\mathcal{W}_i(\mathcal{U} \cap L) = \mathcal{W}_i(\mathcal{U})$ for $i \leq k$. Using the fact that the link pair of a stratum $S \cap L$ in $L \cap \mathcal{D}$ is the same as the link pair of $S$ in $\mathcal{D}$, the claim follows by reindexing (replace $s$ by $s - (n - k - 1)$, where $s = \dim_{\mathcal{C}} S$). \qed
Remark 4.4. Similar results are stated in [DM07] for $\mathcal{W}^i(\mathcal{U})$ ($i \leq n - 1$), see Theorem 3.2 and Theorem 3.6 in loc.cit. However, by using (2.6), it can be seen that the statement of Theorem 4.3 is sharper than the above mentioned results from [DM07].

Corollary 4.5. $A_n(\mathcal{U}) = 0$ if and only if $\mu = 0$.

Proof. If $\mu = 0$, then $\mathcal{U}$ admits an infinite cyclic cover which has the homotopy type of a finite CW-complex of dimension $(n - 1)$ (e.g., the generic fibre of $g$, see [LiM14, Proposition 4.6(c)]). It then follows that the universal abelian cover $\mathcal{U}^\text{ab}$ has the homotopy type of an $(n - 1)$-dimensional CW-complex, so $A_n(\mathcal{U}) = 0$.

Conversely, if $A_n(\mathcal{U}) = 0$, then $\mathcal{V}_n(\mathcal{U}) = 0$, so $\bigcup_{i=0}^{n} \mathcal{V}_i(\mathcal{U}) = \bigcup_{i=0}^{n} \mathcal{V}_i(\mathcal{U}) = \mathcal{V}(\mathcal{U})$. Theorem 4.3 shows that $\mathcal{V}(\mathcal{U})$ is proper in $(\mathbb{C}^*)^r$, hence by Remark 2.18 we get that $\chi(\mathcal{U}) = 0$. Thus $\mu = (-1)^n \chi(\mathcal{U}) = 0$. \qed

Remark 4.6. A typical example with $\mu = 0$ is the affine hypersurface complement $M$ defined in the introduction. Moreover, in the isolated singularities case, the condition $\mu = 0$ implies that $\mathcal{U}$ is the affine complement of a homogeneous polynomial, up to a change of coordinates (see [Huh12, Theorem 1] or the reformulation from [Liu14, Theorem 4.3]). The hypersurface complement $\mathcal{U}$ with the transversality assumption and $\mu = 0$ exhibits many similar properties with $M$, e.g., $\mathcal{V}(\mathcal{U}) \subset \mathcal{V}(\prod_{i=1}^{r} t_i^{d_i} = 1)$. For more results in this direction, see [Liu14, Proposition 4.2] and [LiM14, Proposition 4.6].

Example 4.7. Let $\mathcal{U}$ be the complement of the arrangement of $4$ lines in $\mathbb{C}^2$ defined by $g = x(x-y)(x+y)(2x-y+1)$. Then $g$ is transversal at infinity and the line $\{2x-y+1 = 0\}$ intersects the other components transversally. So $\mathcal{V}_1(\mathcal{U}) \subset \mathcal{V}(t_4 = 1) \cap \mathcal{V}(t_1 t_2 t_3 t_4 = 1)$. (In fact, these two sets are equal, see [Su00, Examples 2.5 and 3.7].)

5. A more general setting

Let $V \subset \mathbb{CP}^n$ be, as before, a degree $d$ hypersurface defined by the homogeneous polynomial $f = \prod_{i=1}^{r} f_i$, and set

$$M^* := \mathbb{CP}^n \setminus V.$$  

Then $f$ also defines a hypersurface $f^{-1}(0) \subset \mathbb{CP}^n+1$, the affine cone on $V$. The complement $M := \mathbb{CP}^n+1 \setminus f^{-1}(0)$ is the total space of the global Milnor fibration $f : M \to \mathbb{C}^*$, whose fiber

$$F := f^{-1}(1)$$

is called the Milnor fiber of $f$. Then there exists a $d$-fold covering map $p_d : F \to M^*$.

The Hopf map induces a fibration:

$$p : M \to M^*.$$  

As we will show later on in the proof of Proposition 5.3 below, the transversality assumption implies that $\pi_1(\mathcal{U}) \cong \pi_1(M) = G$, so $\{\gamma_i\}_{1 \leq i \leq r}$ are also generators of $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^r$.

In this section, we pass on the divisibility results from $\mathcal{U}$ to $M$ and $M^*$, respectively. As an application, we give some vanishing results for the homology Alexander polynomial $\Delta_i(M)$, where $i \leq n - 1$. 

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5.1. Divisibility results for the general case. To obtain the relation between the characteristic varieties of $M$ and $M^*$, we need to use the total turn monodromy operator defined in [CD02]. The following description can be found in [Di04, Section 6.4].

Fix a base point $a \in M$ and denote by $\sigma_a$ the loop $t \mapsto \exp(2\pi it)a$ for $t \in [0, 1]$. Choosing a generic line $L$ passing through $a$, transversal to $f^{-1}(0)$, and close to the line $\mathbb{C}a$ (which contains the loop $\sigma_a$), we see that the element $\sigma_a \in \pi_1(M, a)$ is given by the product (in a certain order) of the elementary loops $\sigma_j$ for $j = 1, \cdots, d$, based at $a$ and associated to the intersection points in $L \cap f^{-1}(0)$.

By [Di92 Proposition 4.1.3] we have that

$$H_1(M^*, \mathbb{Z}) = \mathbb{Z}^{r-1} \oplus \left(\mathbb{Z}/\gcd(d_1, \cdots, d_r)\mathbb{Z}\right).$$

Moreover, $H_1(M^*, \mathbb{Z}) \cong H_1(M, \mathbb{Z})/\langle [\sigma_1] + \cdots + [\sigma_d] \rangle$, and note that

$$[\sigma_a] = [\sigma_1] + \cdots + [\sigma_d] = d_1[\gamma_1] + \cdots + d_r[\gamma_r]$$

in $H_1(M, \mathbb{Z})$.

For $\lambda = (\lambda_1, \cdots, \lambda_r) \in (\mathbb{C}^*)^r = \text{Hom}(G, \mathbb{C}^*)$, consider the corresponding rank one local system $\mathcal{L}_\lambda$ on $M$ defined by the representation $\rho_\lambda \in \text{Hom}(\pi_1(M, a), \mathbb{C}^*)$, which sends each generator $\gamma_i$ of $H_1(M, \mathbb{Z})$ to $\lambda_i$. We define the total turn monodromy operator of $\mathcal{L}_\lambda$ to be the invertible operator: $T(\mathcal{L}_\lambda) = \rho_\lambda(\sigma_a) \in \text{Aut}(\mathbb{C}) \cong \mathbb{C}^*$. Since $[\sigma_a] = d_1[\gamma_1] + \cdots + d_r[\gamma_r]$ in $H_1(M, \mathbb{Z})$, we get that

$$T(\mathcal{L}_\lambda) = \prod_{i=1}^r \lambda_i^{d_i} \in \mathbb{C}^*.$$

As we will see below, the complex number $\prod_{i=1}^r \lambda_i^{d_i}$ plays a role in describing the local system $R^q p_* \mathcal{L}_\lambda$. In fact, as shown in [Di04, Section 6.4], for any $x \in M^*$, we have:

$$\mathfrak{H}(R^q p_* \mathcal{L}_\lambda)_x = \begin{cases} E^0 = \ker(\prod_{i=1}^r \lambda_i^{d_i} - 1), & q = 0, \\ E^1 = \text{coker}(\prod_{i=1}^r \lambda_i^{d_i} - 1), & q = 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $\lambda \notin V(\prod_{i=1}^r t_i^{d_i} = 1)$, i.e., $\prod_{i=1}^r \lambda_i^{d_i} \neq 1$, then $R^q p_* \mathcal{L}_\lambda = 0$ for all $q$. Thus, for all $i$, we obtain this case that $H^i(M, \mathcal{L}_\lambda) = H^i(M^*, p^* \mathcal{L}_\lambda) = 0$. Then the duality result (2.13) yields that $H_i(M, \mathcal{L}_\lambda) = 0$, or equivalently $\lambda \notin V(M)$. Thus $V(M) \subseteq V(\prod_{i=1}^r t_i^{d_i} = 1)$, which yields that $\mathcal{V}(\mathcal{L}_\lambda) \subseteq V(\prod_{i=1}^r t_i^{d_i} = 1)$, since the involution $\lambda \mapsto \bar{\lambda} := \lambda^{-1}$ keeps $V(\prod_{i=1}^r t_i^{d_i} = 1)$ unchanged. The latter inclusion was already mentioned in Remark 4.6.

If $\lambda \in V(\prod_{i=1}^r t_i^{d_i} = 1)$, then $E^0 \cong \mathbb{C} \cong E^1$. By the arguments used in [Di04 Proposition 6.4.3], there exists a rank one local system $\mathcal{L}_\lambda^*$ on $M^*$ such that $\mathcal{L}_\lambda = p^* \mathcal{L}_\lambda^*$. In fact, the representation of $\mathcal{L}_\lambda$ on $M$ factors through the representation of $\mathcal{L}_\lambda^*$ on $M^*$:

$$\pi_1(M) \xrightarrow{p_*} \pi_1(M^*) \xrightarrow{\rho_\lambda} \mathbb{C}^*$$
where $p_*$ is the epimorphism induced by the Hopf map $p$. Here $\rho_\lambda$ and $\rho_\lambda^s$ are the representation homomorphisms defining $\mathcal{L}_\lambda$ and $\mathcal{L}_\lambda^s$, respectively. (Note that the relation $\sum_{i=1}^r d_i \gamma_i = 0$ in $H_1(M^*, \mathbb{Z})$ is sent by $\rho_\lambda^s$ to $\prod_{i=1}^r t_i^{d_i} = 1$.) The sheaves $R^q p_* \mathcal{L}_\lambda$ are local systems on $M^*$ for $q = 0, 1$, and trivial otherwise. As shown in [Di04, Proposition 6.4.3], the following construction gives the representations defining $R^q p_* \mathcal{L}_\lambda$ ($q = 0, 1$): for $v \in E^q$, and $\alpha \in \pi_1(M^*)$, we set
\[
\alpha \cdot v = \rho_\lambda(\beta)(v)
\]
where $\beta$ is any lifting of $\alpha$ under the epimorphism $p_*$. According to the commutative diagram (5.1), this shows that the local systems $R^q p_* \mathcal{L}_\lambda$ ($q = 0, 1$) coincide with $\mathcal{L}_\lambda^s$. As a consequence, the $E_2$-page of the Leray spectral sequence induced by the Hopf map $p : M \to M^*$,
\[
E_2^{s,q} = H^s(M^*, R^q p_* \mathcal{L}_\lambda) \Rightarrow H^{s+q}(M, \mathcal{L}_\lambda)
\]
reduces to $E_2^{s,q} = H^s(M^*, \mathcal{L}_\lambda^s)$ for $q = 0, 1$ and trivial otherwise. Hence this spectral sequence degenerates at $E_3$.

**Proposition 5.1.** Under the above assumptions and notations, we have that
\[
\mathcal{V}_i(M) \subset \mathcal{V}_i(M^*) \cup \mathcal{V}_{i-1}(M^*).
\]
These two sets are equal, if, in addition, one of the irreducible components of $V$ is a hyperplane.

**Proof.** By Remark 2.15 it suffices to prove the claim for $\mathcal{V}_i(M)$. Assume $\lambda \notin \mathcal{V}_i(M^*) \cup \mathcal{V}_{i-1}(M^*)$. Then $E_2^{0,0} = H^i(M^*, \mathcal{L}_\lambda^s) = 0$ and $E_2^{-1,1} = H^{i-1}(M^*, \mathcal{L}_\lambda^s) = 0$. The spectral sequence (5.2) yields that $H^i(M, \mathcal{L}_\lambda) = 0$, hence $\lambda \notin \mathcal{V}_i(M)$.

If, in addition, one of the irreducible components of $V$ is a hyperplane, then $M = M^* \times \mathbb{C}^*$ (see [Di04, Proposition 6.4.1]). So the spectral sequence (5.2) degenerates at $E_2$, and $H^i(M, \mathcal{L}_\lambda) \cong H^i(M^*, \mathcal{L}_\lambda^s) \oplus H^{i-1}(M^*, \mathcal{L}_\lambda^s)$. The claim follows. \[\square\]

The next result establishes a close relation between the characteristic varieties of $M$ and $M^*$, respectively.

**Theorem 5.2.** For any $k \geq 0$,
\[
\bigcup_{i=0}^k \mathcal{V}_i(M) = \bigcup_{i=0}^k \mathcal{V}_i(M^*).
\]

**Proof.** ($\supseteq$) It follows from Proposition 5.1.

$(\subseteq)$ The proof is done by induction on $k$. By Remark 2.15 it suffices to prove the claim for $\mathcal{V}_i$. For $k = 0$, $\mathcal{V}^0(M) = \{(0)\} = \mathcal{V}^0(M^*)$. Suppose $\lambda \in \bigcup_{i=0}^{k-1} \mathcal{V}_i(M^*)$. We may assume that $\lambda \notin \bigcup_{i=0}^{k-1} \mathcal{V}_i(M^*)$, and $\lambda \in \mathcal{V}_k(M^*)$, for otherwise the proof is complete, by the induction hypothesis. These assumptions translate into $E_2^{s,q} = 0$ for $s < k$ and $H^k(M^*, \mathcal{L}_\lambda^s) \neq 0$. Thus,
\[
H^k(M, \mathcal{L}_\lambda) = E_3^{k,0} = E_2^{k,0} = H^k(M^*, \mathcal{L}_\lambda^s) \neq 0.
\]
So $\lambda \in \mathcal{V}_k(M)$ and the proof is completed. \[\square\]
Set \( G^* = \pi_1(M^*) \) and \( d'_i = \frac{d_i}{\gcd(d_1, \ldots, d_r)} \). The Hopf map induces an injective map \( \text{Hom}(\pi_1(M^*), \mathbb{C}) \rightarrow \text{Hom}(\pi_1(M), \mathbb{C}) \cong \mathbb{C}^* \). So we can regard \( \text{Hom}(\pi_1(M^*), \mathbb{C}) \) as a subset of \( \mathbb{C}^* \). In fact, we have that

\[
\text{Hom}(G^*, \mathbb{C}^*) = V\left(\prod_{i=1}^r t^{d'_i}_i = 1\right), \quad \text{and} \quad \text{Hom}(G^*, \mathbb{C}^*)^0 = V\left(\prod_{i=1}^r t^{d_i}_i = 1\right).
\]

The following proposition is a direct consequence of Theorem 5.2 and [Liu14, Corollary 6.5]. In particular, it reproves (4.4).

**Proposition 5.3.** Assume that the hyperplane at infinity \( H \) is transversal to the hypersurface \( V \subset \mathbb{C}P^n \). Then, for any \( k \leq n - 1 \), we have:

\[
\bigcup_{i=0}^k \mathcal{W}_i(\mathcal{U}) = \bigcup_{i=0}^k \mathcal{V}_i(M) = \bigcup_{i=0}^k \mathcal{V}_i(M^*) \subset V\left(\prod_{i=1}^r t^{d'_i}_i = 1\right).
\]

**Proof.** Since \( H \) is transversal to \( V \), it follows from [Liu14, Corollary 6.5] that the affine hyperplane \( \{x_0 = 1\} \) in \( \mathbb{C}^{n+1} \) is transversal to \( F_0 = f^{-1}(0) \). Note that \( \mathcal{U} \) is isomorphic to \( \{x_0 = 1\} \cap M \). The Lefschetz hyperplane section theorem shows that the natural inclusion \( \mathcal{U} \cong \{x_0 = 1\} \cap M \hookrightarrow M \) is an \( n \)-homotopy equivalence; in particular, \( \pi_1(\mathcal{U}) = \pi_1(M) = G \).

Then, for any \( i \leq n - 1 \), we have:

\[
\mathcal{W}_i(\mathcal{U}) = \mathcal{W}_i(M), \quad \Delta_i(\mathcal{U}) = \Delta_i(M) \text{ and } \mathcal{V}_i(\mathcal{U}) = \mathcal{V}_i(M).
\]

The claim follows now from Theorem 5.2. \( \square \)

Fix a Whitney b-regular stratification \( S \) of \( V \). Since \( V \) is transversal to \( H \), for any \( \mathcal{S} \subset S \) we can find a point \( x \in \mathcal{S} \cap \mathcal{D} \) such that \( \mathcal{U}_x = M^*_x \) where \( M^*_x = M^* \cap B_x \). Here \( B_x \) is a small ball in \( \mathbb{C}P^n \) centered at \( x \). By using Proposition 5.3, the divisibility results for \( \mathcal{U} \) (cf. Theorem 4.3) can be now translated into similar properties for \( M \) and \( M^* \), respectively.

**Theorem 5.4.** With the above assumption and notations, for any \( k \leq n - 1 \) we have that:

\[
\bigcup_{i=0}^k \mathcal{W}_i(M) = \bigcup_{i=0}^k \mathcal{V}_i(M) = \bigcup_{i=0}^k \mathcal{V}_i(M^*) \subset \bigcup_{S \subset \mathcal{V}_1} \mathcal{W}^{\text{unif}}(M^*_S, \nu_S) \cap V\left(\prod_{i=1}^r t^{d'_i}_i = 1\right),
\]

\[
\bigcup_{i=0}^k \mathcal{W}_i(M^*) \subset \bigcup_{S \subset \mathcal{V}_1} \mathcal{W}^{\text{unif}}(M^*_S, \nu_S) \cap V\left(\prod_{i=1}^r t_{d'_i}^i = 1\right),
\]

where the unions are over the strata of \( \mathcal{V}_1 \) with \( \dim_C S \geq n - k - 1 \) and \( x_S \in S \) is arbitrary.

**Remark 5.5.** Theorem 5.4 is a generalization of Theorem 4.3. Indeed, let us replace \( V \) by \( V \cup H \), where we view \( H \) as \( V_0 \). As shown in Example 4.3, the transversality assumption implies that, for any \( x \in H \), \( \mathcal{W}^{\text{unif}}(\mathcal{U}_x) \subset V(t_0 = 1) \). Note that

\[
V\left(\prod_{i=0}^r t^{d'_i}_i = 1\right) \cap V(t_0 = 1) = V\left(\prod_{i=1}^r t^{d_i}_i = 1\right) 
\times V(t_0 = 1).
\]
Then the divisibility result (5.5) along $V_0$ gives (4.4), while (4.3) can be obtained from the divisibility results (5.5) for $V_i$ by intersecting with $V(t_0 = 1)$.

Since $W(U_x, \nu_x) = \mathcal{V}(U_x) \cap \text{im}(\nu_x^*)$, the divisibility results (5.5) can be restated as saying that the global characteristic varieties (except the top one $V(R(M^*))$) are contained in the union of the local characteristic varieties along one irreducible component.

For the one-variable case, the above divisibility results give the corresponding divisibility results for the classical Alexander modules induced by the infinite cyclic cover. More results in this direction are given in next section.

Since divisibility results of Theorem [5.4] hold for each irreducible component of $V$, we get better estimates by taking the intersection over all the irreducible components.

**Corollary 5.6.** Assume that $V$ is a normal crossing (NC) or isolated non-normal crossing divisor (INNC) divisor at any point of the components $V_1, \ldots, V_m$ of $V$, where $m < r$. Then $\dim V_i(M^*) \leq r - m - 1$ for all $i \leq n - 1$. In particular, if $m = r - 1$, then $\bigcup_{i=0}^{n-1} V_i(M) = \bigcup_{i=0}^{n-1} V_i(M^*)$ are just finite sets of torsion points.

**Proof.** It suffices to prove the case $m = 1$, whereas the case $m \geq 2$ follows by taking intersections. If $V$ is only a NC or INNC divisor along $V_1$, then Examples [3.11] and [3.12] show that the local Alexander varieties along $V_1$ are contained in a finite union of subtori of the form $V(t_1 = \lambda_1)$, i.e., $W^{\text{uni}f}(U_x, \nu_x) \subset \bigcup V(t_1 = \lambda_1)$. Then

$$\bigcup_{i=0}^{n-1} V_i(M^*) \subset V(\prod_{i=1}^{r} t_i^d_i = 1) \cap (\bigcup V(t_1 = \lambda_1)),$$

so its dimension is at most $(r - 2)$.

**Corollary 5.7.** Assume that every irreducible component $\{V_1, \ldots, V_m\}$, with $m < r$, is smooth and intersects transversally all other components of $V$. Then $\dim V_i(M^*) \leq r - m - 1$ for all $i \leq n - 1$. In particular, if $m = r - 1$, then $\bigcup_{i=0}^{n-1} V_i(M) = \bigcup_{i=0}^{n-1} V_i(M^*)$ are just finite sets of torsion points.

**Proof.** It suffices to prove the case $m = 1$, whereas the case $m \geq 2$ follows by taking intersections. As shown in Example [3.13], the transversality assumption yields that the local Alexander varieties along $V_1$ are contained in $V(t_1 = 1)$, i.e., $W^{\text{uni}f}(U_x, \nu_x) \subset V(t_1 = 1)$. Then $\bigcup_{i=0}^{n-1} V_i(M^*) \subset V(\prod_{i=1}^{r} t_i^d_i = 1) \cap V(t_1 = 1)$, so its dimension is at most $(r - 2)$.

**Example 5.8.** Assume that $V$ is at most a NC divisor at any point $x \in V$. Then

$$\bigcup_{i=0}^{n-1} V_i(M) = \bigcup_{i=0}^{n-1} V_i(M^*) = \{(1)\}.$$

**Example 5.9.** Assume that $r = 2$, and say that $V_1$ and $V_2$ have at most isolated singularities, and they intersect transversally. Then, for $x \in V_1 \cap V_2$, $W(M_x^*) = \{(1, 1)\}$, while for $x \in V_i$ $(i = 1, 2)$, but $x \notin V_1 \cap V_2$, $W^{\text{uni}f}(M_x^*, \nu_x) \subset \bigcup V(t_i = \lambda_i)$, where $\lambda_i$ is the eigenvalues of $21$.
factors of the local Alexander polynomials $\Delta$ follows at once from Remark 2.18 and Theorem 5.2.

Vanishing of Alexander polynomials. 5.2.

$H_{\dim}$ formula ($\text{Sab90, 2.6.2}$).

Theorem 5.11. Assume that the hypersurface $\text{Proposition 5.13.}$

The prime factors of $\Delta_i(M)$ for $i \leq n - 1$.

Theorem 5.11. Assume that the hypersurface $V \subset \mathbb{CP}^n$ is transversal to the hyperplane at infinity $H$. Then, for fixed $i \leq n - 1$, the prime factors of $\Delta_i(U) = \Delta_i(M)$ are among the prime factors of the local Alexander polynomials $\Delta_q(U_{x_S}, \nu_{x_S})$ associated to strata $S \subset \bigcap_{i=1}^r D_i$, with $\dim C = s \geq n - k - 1$ and $0 \leq q \leq n - s - 1$.

Proof. The prime factors of $\Delta_i(U)$ are in one-to-one correspondence to the codimension one irreducible hypersurfaces in $(C^*)^r$ contained in $W_i(U)$. Theorem 4.3 shows that the prime factors of $\Delta_q(U_{x_S}, \nu_{x_S})$ associated to strata $S \subset D$. We only need to show that the local Alexander polynomials $\Delta_q(U_{x_S}, \nu_{x_S})$, with $S \not\in \bigcap_{i=1}^r D_i$, are coprime with $\Delta_i(U)$.

Assume that $x \in D$, but $x \not\in \bigcap_{i=1}^r D_i$ (i.e., $r_x < r$). Then $\Delta_q(U_{x_S}, \nu_{x_S})$ is a polynomial in the variables $\{t_i\}_{i \in I_x}$. Since $r_x < r$, it follows that $\gcd(\Delta_q(U_{x_S}, \nu_{x_S}), \prod_{i=1}^r t_i^{d_i} - 1) = 1$. On the other hand, by (4.4) we get that $\Delta_i(U) (i \leq n - 1)$ divides $1$. Altogether, we get that $\gcd(\Delta_q(U_{x_S}, \nu_{x_S}), \Delta_i(U)) = 1$. \hfill \Box

Corollary 5.12. Assume that $V$ is transversal to $H$. Set $l = \dim(\bigcap_{i=1}^r V_i)$, and $l = -1$, when $\bigcap_{i=1}^r V_i \neq \emptyset$, by convention. Then $\Delta_i(U) = \Delta_i(M) = 1$, for $i \leq n - l - 2$. Moreover, $\dim(\bigcup_{i=0}^{n-l-2} V_i(U)) = \dim(\bigcup_{i=0}^{n-l-2} V_i(M)) = \dim(\bigcup_{i=0}^{n-l-2} V_i(M^*)) \leq r - 2$.

Proof. The case $l = -1$ follows from Theorem 5.11. Indeed, in this case we obtain that since $\Delta_i(M) = 1 (i \leq n - 1)$, hence $\bigcup_{i=0}^{n-l-2} V_i(M)$ has codimension at least two in $(C^*)^r$. The claim for $l \geq 0$ follows by the Lefschetz hyperplane section theorem. \hfill \Box

Set $f = (f_1, \cdots, f_r)$. Then $f$ gives a polynomial map from $\mathbb{C}^{n+1}$ to $\mathbb{C}^r$. The following proposition is proved in [Bud12, Corollary 3.9] by using Sabbah’s generalization of A’Campo’s formula ($\text{Sab90, 2.6.2}$).

Proposition 5.13. $Z(\psi_f^0) = (\prod_{i=1}^r t_i^{d_i} - 1)^{-\chi(M^*)}$, where 0 is the origin of $\mathbb{C}^{n+1}$.
One of the irreducible components of $V$, say $V_1$, is smooth and transversal to $\bigcap_{i=2}^{r} V_i$. Then $\Delta_i(\mathcal{U}) = \Delta_i(M) = 1$ for all $i \leq n-1$, and

$$\dim\left(\bigcup_{i=0}^{n-1} V_i(\mathcal{U})\right) = \dim\left(\bigcup_{i=0}^{n-1} V_i(M)\right) = \dim\left(\bigcup_{i=0}^{n-1} V_i(M^*)\right) \leq r - 2.$$ 

On the other hand, $\Delta_n(M) = (\prod_{i=1}^{r} t_i^{-d_i} - 1)^{-1} \chi(M^*)$. In particular,

$$(5.7) \quad (-1)^n \chi(F) = (-1)^n d \cdot \chi(M^*) \geq 0.$$ 

**Proof.** Since $\mathcal{W}(M) \subset V(\prod_{i=1}^{r} t_i^{d_i} = 1)$, then $\mathcal{W}(M)$ is proper in $(\mathbb{C}^*)^r$. By using Proposition 3.7 we have that $\delta^i(\psi_{r\mathbb{C}}) = \Delta_i(M) = \Delta_{i-1}(M)$ for all $i$. Therefore,

$$(\prod_{i=1}^{r} t_i^{d_i} - 1)^{-\chi(M^*)} = Z(\psi_{r\mathbb{C}}) = \prod_i \delta_i(\psi_{r\mathbb{C}})^{(-1)^i} = \prod_i \Delta_i(M)^{(-1)^i+1}$$

By using Theorem 5.11 and Examples 3.11 3.12 3.13 we see that in all three situations above we get: $\Delta_i(\mathcal{U}) = \Delta_i(M) = 1$ for all $i \leq n-1$. Hence $(\prod_{i=1}^{r} t_i^{d_i} - 1)^{-\chi(M^*)} = \Delta_n(M)^{(1)^n+1}$, so

$$(5.8) \quad \Delta_n(M) = (\prod_{i=1}^{r} t_i^{d_i} - 1)^{(-1)^n \chi(M^*)}.$$ 

Since the degree of $\Delta_n(M)$ is non-negative, we must then have that

$$(5.9) \quad (-1)^n \chi(M^*) \geq 0.$$ 

**Remark 5.15.** The inequality (5.9) generalizes several known results obtained by different techniques:

(1) The first situation generalizes [Di02, Corollary 2.2] for essential hyperplane arrangements, where Dimca proved it by using $M_0$-tame polynomials. In fact, he showed that $D$ is homotopy equivalent to a bouquet of $(n-1)$-spheres. The same result is also proved in [DJL07, Proposition 2.1] by induction.

(2) For a class of examples as described in the second situation, see Remark 6.8(2) below.

(3) A typical example for the third situation is an affine hypersurface complement $\mathcal{U}$ with the transversality assumption at infinity. In fact, in this case we have that $(-1)^n \chi(\mathcal{U}) = \mu \geq 0$.

A different proof of (5.9) will be given in Corollary 6.9 of the next section.
5.3. Hyperplane arrangements. Assume now that $M^*$ is the complement of a hyperplane arrangement, so $r = d$. Set $l = \dim(\bigcap_{i=0}^r V_i)$, and let $l = -1$ if $\bigcap_{i=0}^r V_i = \emptyset$. Then $M^* = M'' \times \mathbb{C}^{l+1}$, where $M''$ is the complement of an essential hyperplane arrangement. The inequality (5.9) then gives that $(-1)^{n-l-1}\chi(M'') \geq 0$, so

\[(5.10) \quad (-1)^{n-l-1}\chi(M^*) \geq 0.\]

Without loss of generality, we may assume that $V$ is an essential hyperplane arrangement. Note that $H_1(M^*_x, \mathbb{Z}) = \mathbb{Z}^r$, so the first map in (3.4) is isomorphism. Moreover, since the local defining polynomial is homogeneous, we also have that:

$$V(M^*_x) = W(M^*_x) \subset V(\prod_{i \in I_x} t_i = 1).$$

Then the following divisibility result for the hyperplane arrangements holds:

**Theorem 5.16.** Let $M^* = \mathbb{C}\mathbb{P}^n \setminus V$ be the complement of an essential hyperplane arrangement. Then, for any $k \leq n - 1$,

\[(5.11) \quad \bigcup_{i=0}^k \mathcal{V}_i(M) = \bigcup_{i=0}^k \mathcal{V}_i(M^*) \subset \bigcup_{S \subseteq V_1} \{\mathcal{V}(M^*_x \times V(\prod_{i \notin I_x} t_i = 1))\},\]

where the union is over the strata of $V_1$ with $\dim_{\mathbb{C}} S \geq n - k - 1$, and $x_S \in S$ is arbitrary.

**Proof.** The claim follows from Theorem 5.4 and the following observation:

$$V(\prod_{i \in I_x} t_i = 1) \cap V(\prod_{i=1}^r t_i = 1) = V(\prod_{i \in I_x} t_i = 1) \times V(\prod_{i \notin I_x} t_i = 1).$$

For $\lambda \in V(\prod_{i=1}^r t_i = 1)$, let $\mathcal{L}_x^\lambda$ be the corresponding local system on $M^*$.

**Definition 5.17.** $\lambda$ is called generic at $x$ if $\prod_{i \in I_x} \lambda_i \neq 1$ or $\prod_{i \notin I_x} \lambda_i \neq 1$.

The next result should be compared with [DJL07, Theorem 5.3].

**Corollary 5.18.** Let $M^*$ be the complement of an essential hyperplane arrangements. Assume that $\lambda$ is generic for any point $x \in V_1$. Then

$$H_i(M^*, \mathcal{L}_x^\lambda) = \begin{cases} \mathbb{C}(-1)^n\chi(M^*), & i = n, \\ 0, & \text{otherwise}. \end{cases}$$

**Proof.** By Theorem 5.16 we see that if $\lambda$ is generic along $V_1$, then $\lambda \notin \bigcup_{i=0}^r \mathcal{V}_i(M^*)$, and the claim follows. \[\square\]
6. Dwyer-Fried covers and classical Alexander modules

In this section, we introduce the Dwyer-Fried sets, and establish a relation with the characteristic varieties. As before, we assume that $X$ is a finite, connected, $n$-dimensional CW complex, with $\pi_1(X) = G$. Let $H_1(X, \mathbb{Z})/\text{torsion} \cong \mathbb{Z}^r$ be the maximal torsion free abelian quotient of $G$, where $r = b_1(X)$. The following brief introduction on the Dwyer-Fried sets can be found in [Su12B].

Fix an integer $1 \leq m \leq r$, and consider the regular covers of $X$, with deck groups $\mathbb{Z}^m$. Each such cover, $X^\nu \to X$, is determined by an epimorphism $\nu : H_1(X, \mathbb{Z}) \to \mathbb{Z}^m$. The induced homomorphism in rational cohomology, $\nu^\mathbb{Q} : \mathbb{Q}^m \to H^1(X, \mathbb{Q})$, defines an $m$-dimensional $\mathbb{Q}$-vector subspace, $P_\nu = \text{im}(\nu^\mathbb{Q})$, in $H^1(X, \mathbb{Q}) \cong \mathbb{Q}^r$. On the other hand, any $m$-dimensional $\mathbb{Q}$-vector subspace can be realized by some epimorphism $H_1(X, \mathbb{Z}) \to \mathbb{Z}^m$.

**Proposition 6.1.** [DF87] The connected, regular covers of $X$, with deck groups $\mathbb{Z}^m$, are parametrized by the Grassmannian of the $m$-planes in $H^1(X, \mathbb{Q})$, via the correspondence:

\[
\{ \text{regular } \mathbb{Z}^m\text{-covers of } X \} \longleftrightarrow \{ \text{m-planes in } H^1(X, \mathbb{Q}) \}
\]

\[X^\nu \to X \longleftrightarrow P_\nu = \text{im}(\nu^\mathbb{Q}).\]

**Definition 6.2.** The Dwyer-Fried invariants of $X$ are the subsets

\[\Omega^k_m(X) = \{ P_\nu \in Gr_m(H^1(X, \mathbb{Q})) \mid b_i(X^\nu) < \infty \text{ for } i \leq k \}\]

The $\Omega$-sets are homotopy-type invariants of $X$, see [Su14, Lemma 3.5]. The next result was first proved by Dwyer and Fried in [DF87, Theorem 1], and it was improved by Papadima and Suciu in [PS10].

**Theorem 6.3.** [PS10, Corollary 6.2] Assume that $X$ is a finite connected CW complex. Let $\exp$ denote the exponential map from $\mathbb{C}^r$ to $(\mathbb{C}^\ast)^r$. Then, for any $k \geq 0$,

\[(6.1) \quad \Omega^k_m(X) = \{ P_\nu \in Gr_m(H^1(X, \mathbb{Q})) \mid \left( \exp(P_\nu \otimes \mathbb{C}) \cap \left( \bigcup_{i=0}^k V_i(X) \right) \right) \text{ is finite} \} \]

**Example 6.4.** In the setup of Example 5.9, we get that $\bigcup_{i=0}^{n-1} W_i(\mathcal{U})$ is finite. So $\Omega^{n-1}_2(\mathcal{U}) \neq \emptyset$.

6.1. Classical Alexander modules. We now analyze in detail one particular regular map for the hypersurface complement $\mathcal{U}$ in $\mathbb{C}^n$. In this subsection and the next one, we will not assume that the hypersurface $V \subset \mathbb{C}P^n$ is transversal to the hyperplane at infinity $H$.

Recall that $H_1(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}^r$. Let $lk$ denote the epimorphism $H_1(\mathcal{U}, \mathbb{Z}) \to \mathbb{Z}$, which maps the generators $\gamma_i$ to $1_\mathbb{Z}$. (Here $P_{lk} = \langle [1, \cdots, 1] \rangle \in Gr_1(H^1(\mathcal{U}, \mathbb{Q}))$). Consider the infinite cyclic cover $\mathcal{U}^c$ of $\mathcal{U}$ defined by $lk$, which is usually referred to as the linking number infinite cyclic cover. The Alexander modules $H_i(\mathcal{U}^c)$ are finitely generated $\Gamma = \mathbb{C}[t, t^{-1}]$-module. If $H_i(\mathcal{U}^c)$ is a torsion $\Gamma$-module, let $\omega_i(t)$ denote the corresponding Alexander polynomial of $H_i(\mathcal{U}^c)$. Many of the classical finiteness and divisibility results about $H_i(\mathcal{U}^c)$ for $i \leq n-1$ can be recovered by applying Theorems 5.4 and 6.3 to this specific regular cover.

Note that if $V$ is irreducible (i.e., $r = 1$), the infinite cyclic cover coincides with the universal abelian cover. In this case, $H_i(\mathcal{U}^c)$ is a torsion $\Gamma$-module if and only if $W_i(\mathcal{U})$ is proper, and $W_i(\mathcal{U})$ consists of the collection of the roots of $\omega_i(t)$.
Proposition 6.5. For $x \in D$, $\mathcal{V}(U_x)$ is proper. Moreover, $\mathcal{W}(U_x, \nu_x)$ is proper in $(C^*)^{r_x}$.

Proof. The local infinite cyclic cover $U_x^{C}$ is homotopy equivalent to the Milnor fibre $F_x$, so Theorem 6.3 yields that $\exp(P_{lk_x} \otimes \mathbb{C}) \cap \mathcal{W}(U_x)$ is finite. Note that $H_1(U_x)$ is torsion free. If $\mathcal{V}(U_x)$ is not proper, then $\mathcal{V}(U_x) = (C^*)^{b_1(U_x)}$ and $\exp(P_{lk_x} \otimes \mathbb{C}) \cap \mathcal{V}(U_x) = \exp(P_{lk_x} \otimes \mathbb{C})$ is infinite, which gives a contradiction.

For the second claim, note that the map $lk_x$ factors through $Z^{r_x}$. In fact, we have a diagram

$$
\begin{array}{ccc}
\pi_1(U_x) & \xrightarrow{\nu_x} & H_1(U_x, \mathbb{Z}) \\
\downarrow{\pi_1(U_x)} & & \downarrow{\nu_x} \\
Z^{r_x} & \xrightarrow{lk_x} & H_1(U, \mathbb{Z}) \\
\end{array}
$$

where the horizontal map is the composed map in \ref{3.4}. It follows that $\exp(P_{lk_x} \otimes \mathbb{C}) \subset \text{im}(\nu_x^*)$. Recall that $\mathcal{W}(U_x, \nu_x) = \text{im}(\nu_x^*) \cap \mathcal{V}(U_x)$. Then

$$
\exp(P_{lk_x} \otimes \mathbb{C}) \cap \mathcal{W}(U_x, \nu_x) = \exp(P_{lk_x} \otimes \mathbb{C}) \cap \mathcal{V}(U_x) \cap \text{im}(\nu_x^*) = \exp(P_{lk_x} \otimes \mathbb{C}) \cap \mathcal{V}(U_x).
$$

So $\exp(P_{lk_x} \otimes \mathbb{C}) \cap \mathcal{W}(U_x, \nu_x)$ is also finite. A similar argument as above then shows that $\mathcal{W}(U_x, \nu_x)$ is proper in $(C^*)^{r_x}$.

Set $T = \{(\alpha, \cdots, \alpha) \mid \alpha \in C^*\} \subset (C^*)^r$. It is clear that $\exp(P_{lk} \otimes \mathbb{C}) = T$. Similarly, set $T_x = \{(\alpha, \cdots, \alpha) \mid \alpha \in C^*\} \subset (C^*)^{r_x}$.

Proposition 6.6. Assume that $P_{lk} \in \Omega_1^1(U)$. Then, for all $0 \leq k \leq j$, $(\bigcup_{i=0}^k \mathcal{W}_i(U)) \cap T$ equals the collection of the roots of the Alexander polynomials $\{\omega_i(t)\}_{0 \leq i \leq k}$. Similarly, for $x \in D$, $\mathcal{W}(U_x, \nu_x) \cap T_x$ is the collection of all the eigenvalues for the local Milnor fibration of $g$ at $x$.

Proof. If $(\alpha, \cdots, \alpha) \in \bigcup_{i=0}^k \mathcal{W}_i(U) = \bigcup_{i=0}^k \mathcal{V}_i(U)$, then there exists $0 \leq i \leq k$ such that $H_i(U, L_\alpha) \neq 0$. Since $P_{lk} \in \Omega_1^1(U)$, the Milnor long exact sequence shows that $\dim H_i(U, L_\alpha) = N(i, \alpha) + N(i-1, \alpha) > 0$, where $N(i, \alpha)$ is the number of the direct summands in the $(t-\alpha)$ part of $H_i(U)$, see \cite{DN04} Theorem 4.2. So $\alpha$ must be a root of $\omega_i(t)$ or $\omega_{i-1}(t)$, and vice versa.

Note that, for any $x \in D$, $P_{lk_x} \in \Omega_1^1(U_x)$ for $k \geq 0$. As shown in the proof of Proposition 6.5, $T_x \cap (\bigcup_{i=0}^k \mathcal{W}_i(U_x, \nu_x)) = \exp(P_{lk_x} \otimes \mathbb{C}) \cap (\bigcup_{i=0}^k \mathcal{V}_i(U_x))$. So the second claim follows by a similar argument as the first part of the proof. \[\square\]
Note that, by combining Theorems 5.4 and 5.3 we get that, if \( \bigcup \mathcal{W}^{\text{unif}}(\mathcal{U}_x, \nu_x) \cap \mathbb{T} \) is finite along one of the irreducible components of \( V \cup H \), then \( P_{lk} \in \Gamma_1^{n-1}(\mathcal{U}) \).

Let us fix now some notations. Set \( V_0 = H, \, d_0 = 1, \) and let \( t_0 \) be the corresponding coordinate in the torus defined by the fundamental group. We analyze the local \( \alpha \)-sets in the following three cases:

(a) \( x \in D \). Note that \( P_{lk} \in \Gamma_1(\mathcal{U}_x) \) for any \( i \geq 0 \), so \( \mathcal{W}^{\text{unif}}(\mathcal{U}_x, \nu_x) \cap \mathbb{T} = \mathcal{W}(\mathcal{U}_x, \nu_x) \cap \mathbb{T}_x \) is always a finite set.

(b) \( x \in H \setminus V \cap H \). Points of \( V(\prod_{i=0}^{r-1} t_i^d_i = 1) \cap \mathbb{T} \) satisfy \( t_0 = \alpha^{-d} \). Since \( H \setminus V \cap H \) is smooth, then \( \mathcal{W}^{\text{unif}}(\mathcal{U}_x, \nu_x) = V(t_0 = 1) \), hence \( \alpha^d = 1 \). Therefore, we have only finite choices for \( \alpha \).

(c) \( x \in V \cap H \). This case will be treated in more detail below. More precisely, we aim to show that, under some good assumptions for \( V \cap H \), we have that \( P_{lk} \in \Gamma_1^{n-1}(\mathcal{U}) \).

**Proposition 6.7. Assume that \( V \) and \( H \) is in one of the following three cases:**

1. \( V \cup H \) has only NC and INNC singularities along \( V \cap H \);
2. \( V \) intersects \( H \) transversally;
3. \( V \cup H \) is an essential hyperplane arrangement.

Then \( P_{lk} \in \Gamma_1^{n-1}(\mathcal{U}) \), i.e., \( H_t(\mathcal{U}^c) \) is a torsion \( \Gamma \)-module for \( i \leq n - 1 \).

**Proof.** In the first case, the NC and INNC assumption along \( V \cap H \) implies that \( \mathcal{W}^{\text{unif}}(\mathcal{U}_x, \nu_x) \) is contained in a finite union of subtori of the form \( V(t_0 = \lambda_0) \) for \( x \in V \cap H \). Hence there can be only finitely many choices for \( \alpha \).

For the second case, the transversality assumption implies that \( \mathcal{W}^{\text{unif}}(\mathcal{U}_x, \nu_x) \cap V(t_0 = 1) \) for any \( x \in V \cap H \), hence \( \alpha^d = 1 \).

In the third case, \( V \cup H \) is an essential hyperplane arrangement, so without loss of generality we can assume that, for \( x \in V \cap H \), \( x \in V_1 \cap \cdots \cap V_k \cap H \), where \( 1 \leq k < r \). Then \( \mathcal{W}^{\text{unif}}(\mathcal{U}_x, \nu_x) \subset V(\prod_{i=0}^{k} t_i^d_i = 1) \). Note that points of \( V(\prod_{i=0}^{r-1} t_i = 1) \cap V(\prod_{i=0}^{k} t_i = 1) \cap \mathbb{T} \) satisfy \( \alpha^{r-k} = 1 \) (as, in the hyperplane arrangements case, we have \( d = r \)). Here \( \mathbb{T} \subset (\mathbb{C}^*)^r \), where \( (\mathbb{C}^*)^r \) has coordinates \( t = (t_1, \cdots, t_r) \in (\mathbb{C}^*)^r \). Since \( k < r \), we can have only finitely many choices for \( \alpha \).

**Remark 6.8.** Proposition 6.7 shows that in the above three cases the Alexander module \( H_t(\mathcal{U}^c) \) is a torsion \( \Gamma \)-module for all \( i \leq n - 1 \). Note that by Theorem 5.4 we have that

\[
\bigcup_{i=0}^{k} \mathcal{W}(\mathcal{U}_x) \cap \mathbb{T} \subset \{ \bigcup_{S \subset V_1} \mathcal{W}^{\text{unif}}(\mathcal{U}_{xS}, \nu_{xS}) \} \cap V(\prod_{i=0}^{r-1} t_i^d_i = 1) \cap \mathbb{T},
\]

where the union is over the strata of one irreducible component \( V_1 \) of \( V \cup V_0 \) with \( \dim_{\mathbb{C}} S \geq n - k - 1 \) and \( xS \in S \) is arbitrary. By using Proposition 6.6 we obtain the corresponding divisibility result for \( \omega_i(t) \) \( (i \leq n - 1) \). Let us now explain in more detail some consequences of our divisibility results.

(1) Assume that \( V \) has only isolated singularities including at infinity. This can be viewed as a special situation of case (1) in Proposition 6.7. It follows by the Lefschetz hyperplane section theorem, that \( \omega_i(t) = 1 \) for \( 1 \leq i \leq n - 2 \), see [Lib94], so the only interesting Alexander polynomial is \( \omega_{n-1}(t) \). The divisibility result along \( V \) yields that the roots of \( \omega_{n-1}(t) \) (besides 1) are among the roots of \( \omega(x)(t) \), for \( x \in \text{Sing}(V) \cup \text{Sing}_\infty(V) \). Here, for \( x \in D, \omega(x)(t) \) is
Assume that \( H \) intersects \( V \) transversally. The transversality assumption implies that \( \Omega^{\text{unif}}(U_x, v_x) \subset V(t_0 = 1) \) for any \( x \in H \). The divisibility result along \( H \) yields that the roots of \( \omega_i(t) \) \( (i \leq n - 1) \) have order \( d \), a result already obtained by the second named author in [Max06, Theorem 4.1]. Moreover, the divisibility result along \( V_1 \) shows that the roots of \( \omega_i(t) \) \( (i \leq n - 1) \) are among the roots of \( \omega_q(x_S)(t) \), in the range \( \dim C S = s \geq n - 1 - i \) and \( 0 \leq q \leq n - 1 - s \). Here \( \omega_q(x_S)(t) \) is the \( g \)-th characteristic polynomial of the monodromy of the local Milnor fibration of \( g \) at \( x_S \in S \subset D_1 \). So we also recover the second-named author’s divisibility results for Alexander polynomials, see [Max06, Theorem 4.2].

(3) Assume that \( V \cup H \) is an essential hyperplane arrangement. It was proved by Dimca that \( H_i(U^c) \) is a torsion \( \Gamma \)-module for \( i \leq n - 1 \), see [Di02, Theorem 2.2]. To our knowledge, the following results are new. First, the divisibility result along \( H \) yields that the roots of \( \omega_i(t) \) \( (i \leq n - 1) \) have order \( r \). Secondly, the divisibility result along one of the irreducible components, say \( V_1 \), yields that the roots of \( \omega_i(t) \) \( (i \leq n - 1) \) are among \( \{ \alpha \mid \alpha^{(r-1)!} = 1 \} \) and the zeros of local characteristic polynomial \( \omega_q(x_S)(t) \) associated to \( x_S \in S \subset D_1 \), in the range \( \dim C S = s \geq n - 1 - i \) and \( 0 \leq q \leq n - 1 - s \).

**Corollary 6.9.** Assume that \( V \cup H \) is in one of the three cases considered in Proposition 6.7. Then

\[
(6.2) \quad (-1)^n \chi(U) \geq 0.
\]

**Proof.** Since \( U \) is an affine variety, \( U \) is homotopy equivalent to a finite \( n \)-dimensional CW complex. Hence \( H_n(U^c) \) is a free \( \Gamma \)-module. Proposition 6.7 shows that if \( V \cup H \) is in one of these 3 cases, then \( H_i(U^c) \) is a torsion \( \Gamma \)-module for \( i \leq n - 1 \). So

\[
\chi(U) = (-1)^n \text{rank} H_n(U^c),
\]

hence \( (-1)^n \chi(U) = \text{rank} H_n(U^c) \geq 0 \). \( \square \)

6.2. **Monodromy representation of the generic fibre of a polynomial map.** The monodromy representation of the generic fibre of \( g \) is closely related to the infinite cyclic cover of \( U \), see [DN04].

Let \( B_g \) be the set of bifurcation points of \( g \). Set \( B_g^* = B_g \setminus \{ 0 \} \). If \( c \in \mathbb{C} \setminus B_g \), then \( G = g^{-1}(c) \) is called the generic fibre of \( g \). Set \( T_b = g^{-1}(D_b) \), where \( D_b \) is a sufficiently small open disk around \( b \in B_f \). Then \( H_i(U^c) \) is a torsion \( \Gamma \)-module if and only if \( H_i(T_b, G) = 0 \) for all \( b \in B_g^* \), see [DN04, Theorem 2.10]. Assume now that \( V \cup H \) is in one of the 3 cases considered in Proposition 6.7. Then \( H_i(U^c) \) is torsion for \( i \leq n - 1 \), so \( H_i(T_b, G) = 0 \) for all \( b \in B_g^* \) and \( i \leq n - 1 \). Moreover, when \( H_i(G) \) is viewed as an \( \Gamma \)-module by the monodromy action, we have \( \Gamma \)-module isomorphisms: \( H_i(G) \cong H_i(U^c) \) for \( i \leq n - 2 \).

**Proposition 6.10.** Assume that \( V \cup H \) is in one of the 3 cases considered in Proposition 6.7. Then, for any integer \( 0 < i \leq n - 2 \), the restriction of the constructible sheaf \( R^i g_* \mathbb{C} \) to \( \mathbb{C}^* \) is a local system corresponding to the monodromy operator \( \mathcal{M}^i : H^i(G) \to H^i(G) \). Here \( \mathcal{M} \) can
be taken to be either the monodromy at the origin or, equivalently, the monodromy at infinity of \( g \). Moreover, we have \( \Gamma \)-module isomorphisms: \( H_i(G) \cong H_i(U^c) \) for \( i \leq n - 2 \), and the corresponding divisibility results for \( M^i (i \leq n - 2) \) follow from Remark 6.8.

7. Resonance varieties and straightness

In this section, we introduce the resonance varieties, and describe their relation with characteristic varieties. As before, we assume that \( X \) is a finite, connected, \( n \)-dimensional CW complex. Consider the cohomology algebra \( A^\bullet = H^\bullet(X, \mathbb{C}) \), with \( \dim_{\mathbb{C}} A^1 = r \). For each \( a \in A^1 \), we have \( a^2 = 0 \). Then, multiplication by \( a \) defines a cochain complex:

\[
(A^\bullet, \cdot a) : A^0 \overset{a}{\longrightarrow} A^1 \overset{a}{\longrightarrow} A^2 \overset{a}{\longrightarrow} \cdots,
\]

known as the Aomoto complex.

**Definition 7.1.** The resonance varieties of \( X \) are defined by:

\[
\mathcal{R}^i(X) = \{ a \in A^1 \mid \dim_{\mathbb{C}} H^i(A^\bullet, \cdot a) \neq 0 \}
\]

Note that, if \( A^i = 0 \), then \( \mathcal{R}^i(X) = \emptyset \). The sets \( \mathcal{R}^i(X) \) are homogeneous algebraic varieties of the affine space \( A^1 = \mathbb{C}^r \) (see \cite[Lemma 3.2]{Su12A} ), and they are homotopy invariants of \( X \), see \cite[Lemma 3.3]{Su12A}. Set

\[
\mathcal{R}(X) = \bigcup_{i=0}^n \mathcal{R}^i(X).
\]

7.1. Tangent cone inclusion and locally straight space. Assume that \( W \subset (\mathbb{C}^*)^r \) is a Zariski closed set. Let \( J \) be the ideal in the ring of analytic functions \( \mathbb{C}\{t_1, \ldots, t_r\} \) defining the germ of \( W \) at \( 1 \), and let \( \text{in}(J) \) be the ideal in the polynomial ring \( \mathbb{C}[t_1, \ldots, t_r] \) spanned by the initial forms of non-zero elements of \( J \).

**Definition 7.2.** The tangent cone of \( W \) is defined by

\[
TC_1(W) = V(\text{in}(J)).
\]

Libgober \cite{Lib02} established a connection between characteristic varieties and the resonance varieties by the following tangent cone inclusion:

**Theorem 7.3.** Let \( X \) be a finite, connected, \( n \)-dimensional CW complex. Then, for any \( k \leq n \),

\[
TC_1(\bigcup_{i=0}^k W_i(X)) \subset \bigcup_{i=0}^k \mathcal{R}^i(X).
\]

Assume that \( X \) is a finite, connected, \( n \)-dimensional CW complex. For any \( k \leq j \), consider the following conditions:

(a) All components of \( \bigcup_{i=0}^k W_i(X) \) passing through the origin are algebraic subtori.

(b) \( TC_1(\bigcup_{i=0}^k W_i(X)) = \bigcup_{i=0}^k \mathcal{R}^i(X) \).
**Definition 7.4.** [Su12A, Definition 6.1] We say $X$ is locally $j$-straight if conditions $(a)$ and $(b)$ hold for each $k \leq j$. If the conditions $(a)$ and $(b)$ hold for all $j \geq 1$, we say $X$ is a locally straight space.

**Remark 7.5.** The variety $M^* = \mathbb{CP}^n \setminus V$ is affine and smooth, so, by [BW14, Theorem 1.1], the characteristic varieties $V^i(M^*)$ are finite unions of torsion translated subtori. Recall also that by Theorem 2.16, we have that

$$\bigcup_{i=0}^{k} W_i(M^*) = (\bigcup_{i=0}^{k} V^i(M^*)) \cap \text{Hom}(G^*, \mathbb{C}^*)^0.$$  

So condition $(a)$ is always satisfied for $M^*$. Similar considerations apply to the spaces $U = \mathbb{CP}^n \setminus (V \cup H)$ and $M = \mathbb{C}^n \setminus f^{-1}(0)$, where $f$ is the homogeneous polynomial defining $V$.

The following result gives a new class of examples of locally straight spaces.

**Proposition 7.6.** Assume that $V$ intersects $H$ transversally, and let $D = V \setminus V \cap H$. If $D$ is a rational homology manifold, then $U = \mathbb{CP}^n \setminus (V \cup H)$ is a locally straight space.

**Proof.** We only need to check the condition $(b)$. In [Liu14, Proposition 5.5], the first-named author showed that if $D$ is a rational homology manifold, then $V$ is irreducible, and

$$H^i(U) = \begin{cases} 
\mathbb{C}, & i = 0, 1, \\
\mathbb{C}^\mu, & i = n, \\
0, & \text{otherwise.}
\end{cases}$$

(7.2)

Since $V$ is irreducible, the infinite cyclic cover $U^c$ of $U$ coincides with the universal abelian cover. Note also that the transversality assumption yields that $H_i(U^c)$ is a torsion $\Gamma$-module if $i \leq n - 1$, see [Max06, Theorem 4.1] and Proposition 6.7. Thus $\bigcup_{i=0}^{k} W_i(U)$ (for $0 \leq k < n$) is a finite set of points (containing 1), so for any $0 \leq k < n$ we get that

$$TC_1(\bigcup_{i=0}^{k} W_i(U)) = \{0\}.$$  

(7.3)

For $k = n$, $TC_1(\bigcup_{i=0}^{k} W_i(U))$ depends on the number $\mu$, where $\mu$ is defined by (4.1). In fact, since $H_n(U^c) \cong \Gamma^\mu$ (cf. [Max06, Corollary 3.8]), then, if $\mu = 0$, (7.3) also holds for $k = n$, while if $\mu > 0$, then $TC_1(\bigcup_{i=0}^{n} W_i(U)) = \mathbb{C}$.

On the other hand, (7.2) yields that, for $k \leq n - 1$,

$$\bigcup_{i=0}^{k} \mathcal{R}^i(U) = \{0\}.$$  

(7.4)

For $k = n$, $\bigcup_{i=0}^{n} \mathcal{R}^i(U)$ depends on $\mu$. In fact, if $\mu = 0$, then (7.4) also holds for $k = n$, while if $\mu > 0$ we get that $\bigcup_{i=0}^{n} \mathcal{R}^i(U) = \mathbb{C}$. Therefore condition $(b)$ is fulfilled, and the proof is completed.
7.2. Divisibility results for resonance varieties. In this section, we establish a relation between the resonance varieties $\mathcal{R}^i(M^*)$ and $\mathcal{R}^i(M)$ of $M^*$ and $M$, respectively. Note that $\mathcal{R}^i(M) \subseteq H^1(M) \cong \mathbb{C}^r$, while $\mathcal{R}^i(M^*) \subseteq H^1(M^*) \cong \mathbb{C}^{r-1}$. However, the Hopf map $p : M \to M^*$ induces a monomorphism $p^* : H^1(M^*) \to H^1(M)$, so we can regard $\mathcal{R}^i(M^*)$ also as subsets of $H^1(M)$. More precisely, $V(\prod_{i=1} r_i t_i^{d_i} = 1)$ has $\gcd(d_1, \cdots, d_r)$ irreducible components in $(\mathbb{C}^*)^r$, with $V(\prod_{i=1} t_i^{d_i} = 1)$ being the component passing through $(1, \cdots, 1)$. (Recall here that $d_i = d_i / \gcd(d_1, \cdots, d_r)$.) The linear subspace associated to $V(\prod_{i=1} t_i^{d_i} = 1)$ in $\mathbb{C}^r$ is defined by $\sum_{i=0} d_i z_i = 0$, and we denote by $L$ this $(r-1)$-dimensional $\mathbb{C}$-vector subspace. Then $L \cong p^*(H^1(M^*))$ and, by Propositions 5.10 and 2.17 and the tangent cone inclusion, we have that

$$\mathcal{R}(M^*) \subseteq L.$$ 

If, in addition, $\chi(M^*) \neq 0$, then Proposition 5.10 and the tangent cone inclusion yield that $\mathcal{R}(M^*) = L$.

Proposition 7.7. For any $i \geq 0$,

$$\mathcal{R}^i(M) = \mathcal{R}^i(M^*) \cup \mathcal{R}^{i-1}(M^*).$$

Proof. We have an isomorphism of graded algebras (see [Di04, Proposition 6.4.1]):

$$H^\bullet(M) \cong H^\bullet(M^*) \otimes H^\bullet(\mathbb{C}^*).$$

Then

$$\mathcal{R}^i(M) = \bigcup_{p+q=i} \mathcal{R}^p(M^*) \times \mathcal{R}^q(\mathbb{C}^*) = \mathcal{R}^i(M^*) \cup \mathcal{R}^{i-1}(M^*),$$

where the first equality follows from the (proof of the) product formula in [PS10, Proposition 13.1], and the last equality follows from the fact that $\mathcal{R}^q(\mathbb{C}^*) = \{0\}$ for $q = 0, 1$, and empty, otherwise. \(\square\)

The following corollary is a direct consequence of Proposition 7.7.

Corollary 7.8. For any $k \geq 0$,

$$\bigcup_{i=0}^k \mathcal{R}^i(M) = \bigcup_{i=0}^k \mathcal{R}^i(M^*).$$

Moreover, $M^*$ is a locally $j$-straight space if and only if so is $M$.

Proof. The identification in (7.7) follows at once from (7.5). As already pointed out in Remark 7.5, the locally straightness of $M^*$ and $M$ only depend on condition (b) for each of these spaces. Note that Theorem 5.2 combined with Theorem 2.16 yields that

$$\bigcup_{i=0}^k \mathcal{W}_i(M^*) = (\bigcup_{i=0}^k \mathcal{V}_i(M^*)) \cap \text{Hom}(G^*, \mathbb{C}^*)^0 = (\bigcup_{i=0}^k \mathcal{W}_i(M)) \cap \text{Hom}(G^*, \mathbb{C}^*)^0.$$ 

Then the straightness claim follows from the equality (7.7). \(\square\)
In view of the considerations preceding Proposition 7.7, Corollary 7.8 yields that:

$$R(M) = R(M^*) \subseteq L.$$ 

For the local case, the map $\nu_x$ induces an embedding: $\nu_x^*: Q^x \rightarrow H^1(M^*_x, \mathbb{C})$. Motivated by the equality $\bigcup_{i=0}^k W_i(M^*_x, \nu_x) = im(\nu_x^*) \cap (\bigcup_{i=0}^k V_i(M^*))$, we now define local resonance varieties by

$$R(M^*_x, \nu_x) = im(\nu_x^*) \cap R(M^*_x).$$

The corresponding uniform local resonance variety $R^{unif}(M^*_x, \nu_x)$ is then defined by

$$R^{unif}(M^*_x, \nu_x) = R(M^*_x, \nu_x) \times \mathbb{C}^{r_x}.$$

**Theorem 7.9.** Assume that $M^*$ is a locally $j$-straight space. Then, for any $k \leq \min\{j, n-1\}$, we have

$$\bigcup_{i=0}^k R^i(M) = \bigcup_{i=0}^k R^i(M^*) \subseteq \bigcup_{S \subseteq V_1} \{R(M^*_x, \nu_x) \times V(\sum_{i \notin I_x} z_i = 0)\},$$

where the union is over the strata of $V_1$ with $\dim S \geq n - k - 1$ and $x_S \in S$ is arbitrary.

**Proof.** For $k \leq \min\{j, n-1\}$, we have that

$$\bigcup_{i=0}^k R^i(M^*) \subseteq TC_1(\bigcup_{i=0}^k W_i(M^*))$$

$$(2) \subseteq TC_1(\bigcup_{S \subseteq V_1} \bigcup_{i=0}^k W^i_{unif}(M^*_x, \nu_x)) \cap V(\prod_{i=1}^r t_i = 1)$$

$$(3) \subseteq (\bigcup_{S \subseteq V_1} R^{unif}(M^*_x, \nu_x)) \cap L$$

where (1) follows from the locally $j$-straightness of $M^*$, (2) follows from (5.6) and the fact that, if $W_1 \subseteq W_2$, then $TC_1(W_1) \subseteq TC_1(W_2)$, and (3) is a consequence of the tangent cone inclusion. \[\square\]

Let $V$ be a hyperplane arrangement in $\mathbb{C}P^n$, with complement $M^*$. Then it is known that $M^*$ is locally straight, see [Su12A, Proposition 11.1]. The following result follows from Theorem 5.16 by using an argument similar to that from Theorem 7.9.

**Theorem 7.10.** Let $M^*$ be the complement of an essential hyperplane arrangements. Then, for any $k \leq n - 1$,

$$\bigcup_{i=0}^k R^i(M) = \bigcup_{i=0}^k R^i(M^*) \subseteq \bigcup_{S \subseteq V_1} \{R(M^*_x) \times V(\sum_{i \notin I_x} z_i = 0)\},$$

where the union is over the strata of $V_1$ with $\dim S \geq n - k - 1$ and $x_S \in S$ is arbitrary.

It is well known that the resonance varieties of the complement of the hyperplane arrangements are determined by the intersection lattice. A basic open problem is to find concrete formulas for this dependence. Theorem 7.10 gives some insight into this problem.
7.3. A special case of hyperplane arrangements. We end this paper by applying all the results obtained up to this point to a special case of hyperplane arrangements.

Assume that \( V \) is a line arrangement in \( \mathbb{CP}^2 \) (so \( n = 2 \)) with \( r \) components. Then \( M^* \) is homotopy equivalent to a finite two-dimensional CW-complex. We will thus obtain divisibility results for

\[
\mathcal{V}_0(M^*) \cup \mathcal{V}_1(M^*) = \mathcal{W}_0(M^*) \cup \mathcal{W}_1(M^*).
\]

We will assume that \( r \) is large enough (e.g., \( r \geq 4 \)), so that \( M^* \) has interesting topology. We can also assume that \( V \) is essential, for if not, then \( V \) can be viewed as defined by the homogeneous polynomial \( \{x_0^r + x_1^r = 0\} \) and \( M^* \) is homotopy equivalent to \((r-1)\)-fold wedge of circles \( \vee_{r-1}S^1 \). So all the topological information about \( M^* \) is well-understood in the non-essential case. Finally, if \( V \) has only normal crossing singularities, then \( \pi_1(M^*) \) is abelian, see [Di92, Theorem 4.1.13]. So the universal abelian cover of \( M^* \) coincides with its universal cover, and \( \mathcal{W}_1(M^*) = \emptyset \). So this case does not present any interesting phenomena either.

We are going to analyze in detail the following situation: assume that \( V \) has only normal crossing singularities except along one fixed hyperplane of \( V \), say \( V_r \). Note that if there exists a point \( x \in V_r \) such that the multiplicity at \( x \) is 2, say \( x \in V_1 \cap V_r \), then our assumptions imply that \( V_1 \) intersects all the other components of \( V \) transversally. Then by using Proposition 5.3

in which \( V_1 \) plays the role of the line at infinity, we get

\[
\mathcal{V}_0(M^*) \cup \mathcal{V}_1(M^*) = \mathcal{V}_0(\mathbb{CP}^2 - \bigcup_{i=2}^{r} V_i) \cup \mathcal{V}_1(\mathbb{CP}^2 - \bigcup_{i=2}^{r} V_i).
\]

So, without loss of generality, we can assume that, for any singular point \( x \in V_r \), \( x \) has multiplicity at least 3.

Assume now that \( V_r \) has \( k \) singular points, with corresponding multiplicities \( m_1, \cdots, m_r \), so that each \( m_i \geq 3 \). Then \( \sum_{i=1}^{k} (m_i - 1) = r - 1 \), so there is a partition \( P = (P_1 \mid \cdots \mid P_k) \) of \{1, 2, \cdots, r - 1\} associated to these multiplicities, where \( P_i \) is a set of cardinality \((m_i - 1)\).

Up to reordering the hyperplanes in \( V \), we can assume that \( P_1 = \{1, \cdots, m_1 - 1\} \), \( P_2 = \{m_1, \cdots, m_1 + m_2 - 2\} \), etc.

Now we analyze the local characteristic varieties along each irreducible component of \( V \). The points of \( V_1 \) can only be of one of the following three distinct local types. First, if \( x \in V_1 \) is a smooth point of \( V \), then \( V^{\text{unif}}(M^*_r) = V(t_1 = 1) \). Secondly, if \( V \) has a normal crossing singularity at the point \( x \), say \( x \in V_1 \cap V_2 \), then \( V^{\text{unif}}(M^*_r) = V(t_1 = t_2 = 1) \subset V(t_1 = 1) \). Thirdly, if \( x \in V_1 \cap V_r \) has multiplicity \( m_1 \geq 3 \), then \( V^{\text{unif}}(M^*_r) = V(t_r \cdot \prod_{i \in P_1} t_i = 1) \). So the union of the local characteristic varieties at points along \( V_1 \) is \( V(t_1 = 1) \cup V(t_r \cdot \prod_{i \in P_1} t_i = 1) \). By repeating this argument for the irreducible components \( V_i, i \in P_1 \), and taking the intersections of the resulting unions of local characteristic varieties along each \( V_i \) \((i \in P_1)\), we obtain the set:

\[
\Theta_1 := V(t_r \cdot \prod_{i \in P_1} t_i = 1) \cup \{ t \in (\mathbb{C}^*)^r \mid t_i = 1 \text{ for all } i \in P_1 \}.
\]
Similar sets $\Theta_j$ can be attached to the other $P_j$. Let $\Theta$ denote the intersection of the sets $\Theta_j$, for $1 \leq j \leq k$. Then $t \in \Theta$ if and only if $t$ satisfies the following condition for any $j = 1, \ldots, k$:

$$t_i = 1 \text{ for all } i \in P_j \text{ or } t_r \cdot \prod_{i \in P_j} t_i = 1.$$  

Recall also that by Theorem 5.4 we have that $V_0(M^*) \cup V_1(M^*) \subset V(\prod_{i=1}^r t_i = 1)$. Altogether,

$$(7.10) \quad V_0(M^*) \cup V_1(M^*) \subset \Theta \cap V(\prod_{i=1}^r t_i = 1).$$

Note that the union of the local characteristic varieties along $V_r$ is

$$\bigcup_{j=1}^k (V(t_r \cdot \prod_{i \in P_j} t_i = 1)) \cup V(t_r = 1),$$

but this set already contains $\Theta \cap V(\prod_{i=1}^r t_i = 1)$. So the divisibility result (7.10) is our final claim. For $t \in \Theta \cap V(\prod_{i=1}^r t_i = 1)$, if $t_i = 1$ for all $i \in P_j$ holds for $l$ parts of $P$, where $0 \leq l \leq k$, then it is easy to see that $t_r^{k-l-1} = 1$. If $l \neq k-1$, then there are only finite choices for $t_r$.

The corresponding resonance varieties are much simpler: if $z \in R^0(M^*) \cup R^1(M^*)$, then $z$ satisfies the following condition:

$$\sum_{i=1}^r z_i = 0,$$

and for all $j = 1, \ldots, k$: $z_i = 0$ for all $i \in P_j$ or $z_r + \sum_{i \in P_j} z_i = 0$.

If $l \neq k-1$, then $z_r = 0$.

As a quick application of (7.10), let us now study the eigenvalues of the corresponding monodromy action. Let $F$ be the Milnor fibre associated to the line arrangement in $\mathbb{C}P^2$, with monodromy homeomorphism $h : F \to F$ given by $h(x) = \exp(2\pi i/r) \cdot x$. Consider the induced monodromy action on $H_1(F, \mathbb{C})$. The Milnor fibre $F$ is homotopy equivalent to the linking number infinite cyclic cover of $M$. Moreover, since $h^r = id$, so the monodromy action is diagonalisable, and $H_1(F)$ is a semisimple $G$-module.

The eigenvalues of the monodromy action on $H_1(F)$ belong to $T \cap (V_0(M) \cup V_1(M))$, see Proposition 6.6. Here $T = \{ (\alpha, \cdots, \alpha) | \alpha \in \mathbb{C}^* \} \subset (\mathbb{C}^*)^r$. Recall also that $V_0(M^*) \cup V_1(M^*) = V_0(M) \cup V_1(M)$, see Theorem 5.2. By applying the divisibility result (7.10), i.e., by intersecting $\Theta \cap V(\prod_{i=1}^r t_i = 1)$ with $T$, we get $\alpha = 1$, except at points in the intersection

$$V(t_r = 1) \cap (\bigcap_{j=1}^k V(t_r \cdot \prod_{i \in P_j} t_i = 1)) \cap T.$$

The latter intersection yields $\alpha^r = 1$ and $\alpha^{m_i} = 1$ for all $1 \leq i \leq k$. So the monodromy eigenvalues have order $\text{gcd}(r, m_1, \cdots, m_k)$. Therefore, if $\text{gcd}(r, m_1, \cdots, m_k) = 1$, then the monodromy action on $H_1(F)$ is trivial, i.e., it has only the eigenvalue 1. Moreover, since $H_1(F)$ is semisimple, we get that $H_1(F) = \mathbb{C}^{r-1}$ in this case. This follows from the Wang exact sequence:

$$\to H_1(F) \xrightarrow{h- \text{id}} H_1(F) \to H_1(M) \to H_0(F) \xrightarrow{h- \text{id}} H_0(F) \to H_0(M) \to 0,$$
since $H_1(M) = \mathbb{C}^r$. Finally, recall that we assumed $V$ to be essential. Then (5.7) shows that, for such an essential line arrangement, we have $\chi(F) \geq 0$, i.e.,
\[ 1 - \dim H_1(F) + \dim H_2(F) \geq 0, \]
hence
\[ \dim H_2(F) \geq r - 2. \]

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