Functions of several quaternion variables and quaternion manifolds.

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Abstract

Functions of several quaternion variables are investigated and integral representation theorems for them are proved. With the help of them solutions of the $\bar\partial$-equations are studied. Moreover, quaternion Stein manifolds are defined and investigated.

1 Introduction

Superanalysis and its enormous applications in mathematical and theoretical physics are developing fastly in recent times, but mainly for supercommutative superalgebras [1, 3, 4, 11]. For nonsupercommutative superalgebras superanalysis is rather new [3, 25] and less known, even (super)analysis over Clifford algebras and, in particular, quaternions is very little known.

Quaternion manifolds appear to be very useful, since the spin structure for them naturally arise due to the embedding of the unitary group $U(2)$ of all complex $2 \times 2$ unitary matrices into the quaternion skew field $\mathbf{H}$: $U(2) \hookrightarrow \mathbf{H}$. Moreover, due to Theorem 4.9 for each complex manifold $N$ there exists a quaternion manifold $M$ and a complex holomorphic embedding $\theta : N \hookrightarrow M$. In view of the isomorphism of the spin group $Spin(4)$ with the direct product of the special unitary groups $SU(2) \otimes SU(2)$ and the embedding $SU(2) \otimes SU(2) \hookrightarrow \mathbf{H}^2$ each spin manifold $N$ has the embedding into the corresponding quaternion manifold $M$ of the quaternion dimension $\dim_{\mathbf{H}} M = 2n$, where $\dim_{\mathbf{C}} N = 2n \in \mathbb{N}$ (about complex spin manifolds see [16, 23]).
This is possible for manifolds with connecting mappings of charts being in the class of quaternion holomorphic functions considered in [22], but in general this is impossible in the class of quaternion (right-)superlinearly superdifferentiable functions, since the latter leads to the additional restrictions in a complex manifold $N$ associated with the graded $\mathbf{H}$ structure in the tangent space $T_xN$ for each $x \in N$. The latter manifolds are called in the literature quaternion manifolds (see, for example, [24] and references therein), but practically they are complex manifolds with the additional quaternion structure in $TN$, since they are modelled on $\mathbb{C}^n$ and their connecting functions of charts are complex holomorphic, but $\mathbf{H}$ is not the algebra over $\mathbb{C}$, though there is the embedding of $\mathbb{C}$ into $\mathbf{H}$ as the $\mathbb{R}$-linear space.

On the other hand, a quaternion manifold $M$ modelled on $\mathbf{H}^n$ with quaternion holomorphic connecting functions is not a complex manifold. Even if $M$ is foliated by the corresponding complex local coordinates it is like the product of four manifolds $l,m M$, $l,m \in \{1, 2\}$, with the complex holomorphic structure in $1,1 M$ and $1,2 M$ and the complex antiholomorphic structure in $2,1 M$ and $2,2 M$ factorized by the equivalence relation $Y$ such that $z \in 1,1 M$ is $Y$-equivalent to $z' \in 2,2 M$ and $\xi \in 1,2 M$ is $Y$-equivalent to $\xi' \in 2,1 M$, where $1,1 \phi_a(z) = 2,2 \phi_a(z')$ and $1,2 \phi_a(\xi) = -2,1 \phi_a(\xi')$, where $At(M) = \{(U_a, \phi_a) : a \in \Lambda\}$, $At(l,m M) = \{(l,m U_a, l,m \phi_a) : a \in \Lambda\}$, $\phi_a : U_a \to \phi_a(U_a) \subset \mathbf{H}^n$ is a homeomorphism for each $a$, $\phi_a \circ \phi_a^{-1}$ is quaternion holomorphic on $\phi_a(U_a \cap U_b)$, $U_a$ is open in $M$, $\phi_a(U_a)$ is open in $\mathbf{H}^n$, $\bigcup_a U_a = M$. It is known from the theory of complex manifolds, that the product $S^{2n+1} \times S^{2m+1}$ of two odd-dimensional unit real spheres can be supplied with the complex manifold structure [17]. In view of the discussion above and Theorem 4.9 the product $S^{2n+1} \times S^{2m+1} \times S^{2p+1} \times S^{2q+1}$ with $n + m = p + q$ can be supplied with the quaternion manifold structure, where $n, m, p, q$ are nonnegative integers. In some sense quaternion manifolds may be related with hyperbolic manifolds.

Therefore, in the class of quaternion holomorphic functions the quantum mechanics on complex manifolds is embeddable into the quantum mechanics on quaternion manifolds. Moreover, this is natural, since the Dirac operator can be expressed through the differentiation by quaternion variables [16]. Some attempts to spread quantum mechanics from the complex case into the quaternion case were made in [5], but he had not any mathematical tool of functions of quaternion variables and quaternion manifolds.
In [22] the quaternion line integral \( \int_{\gamma} f(z, \tilde{z})dz \) on the space \( C^0_b(U, \mathbb{H}) \) of all bounded continuous functions \( f: U \to \mathbb{H} \) was investigated, where \( U \) is an open subset in \( \mathbb{H} \), \( \gamma \) is a rectifiable path in \( U \), such that it has some properties like the complex Cauchy integral:

\[
\int_{\gamma} f(z, \tilde{z})dz = \int_{\gamma_1} f(z, \tilde{z})dz + \int_{\gamma_2} f(z, \tilde{z})dz
\]

for each \( \gamma = \gamma_1 \cup \gamma_2 \) with \( \gamma_1(1) = \gamma_2(0) \), \( \gamma_1, \gamma_2 : [0, 1] \to U \), \( \int_{\gamma}(af(z, \tilde{z}) + bg(z, \tilde{z})dz = a \int_{\gamma} f(z, \tilde{z})dz + b \int_{\gamma} g(z, \tilde{z})dz \) for each \( a, b \in \mathbb{H}, f, g \in C^0_b(U, \mathbb{H}) \),

\[
(\partial(f_{\gamma \eta = \gamma(1)}) f(z, \tilde{z})dz)/\partial \eta). I = f(\eta, \tilde{\eta}),
\]

though the quaternion line integral is not \( \mathbb{H} \)-linear. Then there quaternion holomorphic functions were introduced and investigated with the specific definition of the superdifferentiability (in general \( \mathbb{H} \)-nonlinear). There were found many quaternionic features in comparison with the complex case and the theory of quaternion holomorphic functions can not be reduced to the theory of complex holomorphic functions.

In this paper functions of several quaternion variables are investigated (see §§2 and 3) and their theory is applied to the definition and the studies of quaternion analogs of complex Stein manifolds (see §4). In this article theorems about integral representations of quaternion functions are proved. Among them there are the quaternion analogs of the Cauchy-Green, Martinelli-Bochner and Leray formulas, but they are quite different from the complex case, since quaternion integrals are noncommutative and quaternion differential forms does not satisfy the same properties as complex differential forms. They are applied to solve the \( \bar{\partial} \)-equations. This is important not only for the theory of quaternion functions, but also for investigations of quaternion manifolds. Then this can be used for developments of sheaves, quantum sheaves and quantum field theory on quaternion manifolds. Naturally free loop spaces and transformation groups of quaternion manifolds such as groups of diffeomorphisms, groups of geometric loops, groups of hoops can be further investigated continuing previous works of the author on these spaces and groups [18, 19, 20, 21], which is interesting for theory of groups and their representations and also for their usages in quantum field theory, quantum gravity, superstrings, etc.
2 Differentiable functions of several quaternion variables

2.1. Theorem. Let $U$ be an open subset in $H$ with a $C^1$-boundary $\partial U$ $U$-homotopic with a product $\gamma_1 \times \gamma_2 \times \gamma_3$, where $\gamma_j(s) = a_j + r_j \exp(2\pi M_j s)$, $M_j \in H_1$, $|M_j| = 1$, $s \in [0, 1]$, $\gamma_j([0, 1]) \subset U$, $0 < r_j < \infty$, $j = 1, 2, 3$, where $M_j$ are linearly independent over $\mathbb{R}$. Let also $f : cl(U) \to H$ be a continuous function on $cl(U)$ such that $(\partial f(z)/\partial \bar{z})$ is defined in the sense of distributions in $U$ is continuous in $U$ and has a continuous extension on $cl(U)$, where $U$ and $\gamma_j$ for each $j$ satisfy conditions of Theorem 3.9 [22]. Then

\begin{equation}
(2\pi)^{-3} \int_{\gamma_3} \int_{\gamma_2} \int_{\gamma_1} f(\zeta_1)(\partial_{\zeta_1} \text{Ln}(\zeta_1-\zeta_2))M_1^{-1}(\partial_{\zeta_2} \text{Ln}(\zeta_2-\zeta_3))M_2^{-1}(\partial_{\zeta_3} \text{Ln}(\zeta_3-z))M_3^{-1}
\end{equation}

\begin{equation}
-(2\pi)^{-3} \int_U \{(\partial \hat{f}(\zeta_1)/\partial \zeta_1), d\hat{\zeta}_1\wedge \partial_{\zeta_1} \text{Ln}(\zeta_1-\zeta_2))M_1^{-1}(\partial_{\zeta_2} \text{Ln}(\zeta_2-\zeta_3))M_2^{-1}(\partial_{\zeta_3} \text{Ln}(\zeta_3-z))M_3^{-1}\}.
\end{equation}

Proof. We have the identities $d_{\zeta}[\hat{f}(\zeta) \partial_{\zeta} \text{Ln}(\zeta-z)] = \{(\partial \hat{f}(\zeta)/\partial \zeta), d\hat{\zeta}\} \wedge \partial_{\zeta} \text{Ln}(\zeta-z) + \{(\partial \hat{f}(\zeta)/\partial \zeta), d\zeta\} \wedge \partial_{\zeta} \text{Ln}(\zeta-z) = 0$ for $\zeta$ varying along a path $\gamma$, where for short $f(z) = f(z, \bar{z})$, since there is the bijection of $z$ with $\bar{z}$ on $H$, there exist quaternion-valued functions $g(\zeta, z)$ and $h(\zeta, z)$ such that $\partial_{\zeta} \text{Ln}(\zeta-z) = g(\zeta, z) d\zeta = (d\zeta) h(\zeta, z)$ (see also §§2.1 and 2.6 [22]). As in [22] $\hat{f}(z, \bar{z}) := \partial g(z, \bar{z})/\partial z$, where $g(z, \bar{z})$ is a quaternion-valued function such that $(\partial g(z, \bar{z})/\partial z).I = f(z, \bar{z})$. Since $\zeta_1$ varies along the path $\gamma_1$, then $d\zeta_1 \wedge d\zeta_1|_{\zeta_1 \in \gamma_1} = 0$. Consider $z \in U$ and $\epsilon > 0$ such that the torus $\text{T}(z, \epsilon, H)$ is contained in $U$, where $\partial \text{T}(z, \epsilon, H) = \psi_3 \times \psi_2 \times \psi_1$, $\psi_j$ are of the same form as $\gamma_j$ but with $z$ instead of $a_j$ and with $r_j = \epsilon$. Applying Stokes formula for regions in $\mathbb{R}^4$ and componentwise to $H$-valued differential forms we get

\begin{equation}
\int_{\partial U} \omega - \int_{\partial \text{T}(z, \epsilon, H)} \omega = \int_{U \setminus \text{T}(z, \epsilon, H)} dw, \text{ where}
\end{equation}

\begin{equation}
w = \hat{f}(\zeta_1), (\partial_{\zeta_1} \text{Ln}(\zeta_1-\zeta_2))M_1^{-1}(\partial_{\zeta_2} \text{Ln}(\zeta_2-\zeta_3))M_2^{-1}(\partial_{\zeta_3} \text{Ln}(\zeta_3-z))M_3^{-1}.
\end{equation}

In view of Theorem 3.9 [22] we have that $\lim_{\epsilon \to 0, \epsilon > 0}(2\pi)^{-3} \int_{\gamma_3} \int_{\gamma_2} \int_{\gamma_1} w = f(z)$ and $\lim_{\epsilon \to 0, \epsilon > 0} \int_{U \setminus \text{T}(z, \epsilon, H)} d\omega = f_U d\omega$. From this formula (2.1) follows.
2.1.1. Remark. Formula (2.1) is the quaternion analog of the (complex) Cauchy-Green formula. Since in the sense of distributions $\partial \hat{f} / \partial \bar{z} = \partial (\partial g / \partial z) / \partial z$, then from $\partial \hat{f} / \partial \bar{z} = 0$ it follows, that $\partial \hat{f} / I / \partial \bar{z} = \partial f / \partial \bar{z} = 0$. If $\partial f / \partial \bar{z} = 0$, then $g$ can be chosen such that $\partial g / \partial \bar{z} = 0$ [22]. Therefore, from Formula (2.1) it follows, that $f$ is quaternion holomorphic in $U$ if and only if $\partial f / \partial \bar{z} = 0$ in $U$.

2.2. Remark. Instead of curves $\gamma$ of Theorem 2.1 above or Theorems 3.9, 3.24 [22] it is possible to consider their natural generalization $\gamma(\theta) + z_0 = z_0 + r(\theta) \exp(2\pi S(\theta))$, where $r(\theta)$ and $S(\theta)$ are continuous functions of finite total variations, $\theta \in [0, 1] \subset \mathbb{R}$, $r(\theta) \geq 0$, $S(\theta) \in \mathbb{H}$. Therefore, $\gamma$ is a rectifiable path. If $S(0) = S(1) \mod(S_i)$ and $r(0) = r(1)$, then $\gamma$ is a closed path (loop): $\gamma(0) = \gamma(1)$, where $S_1 := \{z \in H_i : |z| = 1\}$, $H_i := \{z \in H : z + \bar{z} = 0\}$.

Consider $S$ absolutely continuous such that there exists $T \in L^1([0, 1], H)$ for which $S(\theta) = S(0) + \int_0^\theta T(\tau) d\tau$ (see Satz 2 and 3 (Lebesgue) in §6.4 [14]) and let $r(\theta) > 0$ for each $\theta \in [0, 1]$. Evidently, $Mn := S(1) - S(0) = \int_0^1 T(\tau) d\tau$ is invariant relative to reparametrizations $\phi \in Diff_+([0, 1])$ of diffeomorphisms of $[0, 1]$ preserving the orientation, $n$ is a real number, $M \in S_i$. Then $\Delta \text{Arg}(\gamma) := \text{Arg}(\gamma)|_0^1 = 2\pi\int_0^1 T(\tau) d\tau$ (see also Formula (3.7) and §3.8 [22]). In view of Theorem 3.8 [22] for each loop $\gamma : \Delta \text{Arg}(\gamma) \in ZS_i$. For each $\epsilon > 0$ for the total variation there is the equality $V(\gamma \epsilon) = V(\gamma)\epsilon$. Since $\gamma([0, 1])$ is a compact subset in $H$, then there exists $r_m := \sup_{\theta \in [0, 1]} r(\theta) < \infty$. Hence $z_0 + (\gamma \epsilon)([0, 1]) \subset B(H, z_0, r_m\epsilon)$.

Therefore, Theorems 3.9, 3.24, 3.30 and Formulas (3.9, 3.9’) [22] and Theorem 2.1 above are true for such paths $\gamma$ also and Formula (3.9) [22] takes the form

$$(2.2) \quad f(z) = (2\pi n)^{-1}\left(\int_\psi f(\zeta)(\zeta - z)^{-1} d\zeta\right)\hat{M},$$

where $0 \neq n \in \mathbb{Z}$ for a closed path $\gamma$, $M \in S_i$, Formula (2.2) generalizes Formula (3.9), when $|n| > 1$. When $\hat{M}(0, \gamma) = 0$, then $\left(\int_\psi f(\zeta)(\zeta - z)^{-1} d\zeta\right) = 0$ (see also §3.23 [22]).

2.3. Theorem. Let $U$ be a bounded open subset in $H$ and $f : U \rightarrow H$ be a bounded continuous function. Then there exists a continuous function $u(z)$ which is a solution of the equation

$$(1) \quad (\partial u(z) / \partial \bar{z}) = \hat{f}$$

in $U$, in particular, $(\partial u(z) / \partial \bar{z}).I = f(z)$. 

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Proof. Take quaternion one-forms $d\zeta$ expressible through $d\zeta$ as $\sum_{j=1}^{k(l)} P_{j,1,l} d\zeta P_{j,2,l}$ with fixed nonzero quaternions $P_{j,q,l}$, where $l = 1, 2, 3, 4$, $k(l) \in \mathbb{N}$, $\zeta \in U$.

Choose them satisfying conditions $d\zeta_1 \wedge \nu = \xi(v, w, x, y)dv \wedge dw \wedge dx \wedge dy$, $d\zeta_2 \wedge \nu = 0$, where $\nu = (\partial_1 Ln(\zeta_1 - \zeta_2))M_1^{-1}(\partial_3 Ln(\zeta_2 - \zeta_3))M_2^{-1}(\partial_3 Ln(\zeta_3 - z))M_3^{-1}$ as in $\S 2.1$, $z = vI + wJ + xK + yL$, $I, J, K$ and $L$ are Pauli-matrices.

$v, w, x$ and $y \in \mathbb{R}$, $\xi : U \to \mathbf{H}$ is a function nonzero and finite almost everywhere on $U$ relative to the Lebesgue measure. Then there exist $d\zeta_4$ and $\nu$ such that the continuous function

$$ (2) \quad u(z) := -(2\pi)^{-3} \int_U (\hat{f}(\zeta_1).d\zeta_4) \wedge \nu $$

is a solution of equation (1). To demonstrate this take closed curves $\gamma_j$ in $U$ as in $\S 2.1$ and $\S 2.2$, for example, such that $\zeta_j \in \gamma_j$ satisfy conditions:

$$ (\zeta_1 - \zeta_2) = \eta_1, \quad (\zeta_2 - \zeta_3) = \eta_3, \quad \zeta_1 = r\eta_3^T $$

with $0 < r < 1$, where $\eta_1 = \left( t \quad 0 \right)$, $\eta_2 = \tilde{\eta}_3$, $\tilde{\eta}_3 = \left( 0 \quad u \quad 0 \right)$, $\tilde{\eta}_4 = \eta_4$, $\eta_4 = \left( \bar{t} \quad 0 \right)$, where $t$ and $u \in \mathbb{C}$, $\zeta = \left( \bar{t} \quad u \quad \bar{t} \right) \in U$, $a^T$ denotes the transposed matrix of a matrix $a$.

Hence $d\eta_1 \wedge d\eta_2 = 0$, $d\eta_2 \wedge d\eta_3^T = 0$, $d\eta_3 \wedge d\eta_4^T = 0$, $d\eta_4 \wedge d\eta_4 = 0$,

(i) $\eta_1^T d\eta_1 = (d\eta_1)\eta_1^k$ and $\eta_3^T d\eta_3 = [(d\eta_3)\eta_3^k]$.

for $k = 1$ and $k = -1$. These variables are expressible as $\zeta_1 = \sum_{j=1}^{k(l)} P_{j,1,l} \zeta P_{j,2,l}$ (see $\S \S 3.7$ and 3.28 [22]). Therefore, there exists a quaternion variable $\zeta_4$ expressible through $\zeta$ as above such that

(ii) $d\zeta_4 \wedge \nu = \xi(v, w, x, y)dv \wedge dw \wedge dx \wedge dy$, $d\zeta_4 \wedge \nu = 0$. Therefore, there exists a subgroup of the group of all quaternion holomorphic diffeomorphisms of $U$ preserving Conditions (ii) and the construction given above has natural generalizations.

On the other hand, $\zeta$ is expressible through $\zeta_1, ..., \zeta_4$ as $\zeta = \sum_i P_{i,1} \zeta P_{i,2}$, where $P_{i,1}$, ..., $P_{i,2}$ are quaternion constants. Let at first $f$ be continuously differentiable in $U$. Each $\zeta_j$ is expressible in the form $\zeta_j = \sum_{\ell} \tilde{t}_{b_j} S_{\ell}$, where $\tilde{t}_{b_j} \in \mathbb{R}$ are real variables, $S_{\ell} \in \{ I, J, K, L \}$, hence differentials $(\partial f/\partial \zeta_j).d\zeta_j = \sum_{\ell} ((\partial f/\partial z).S_{\ell} d\tilde{t}_{b_j} + (\partial f/\partial \bar{z}).\bar{S}_{\ell} d\tilde{t}_{b_j})$ are defined. Consider a fixed $z_0 \in U$.

We take a $C^\infty$-function $\chi$ on $H$ such that $\chi = 1$ in a neighbourhood $V$ of $z_0$, $V \subset \mathbb{R}$, $\chi = 0$ in a neighbourhood of $H \setminus \mathbb{R}$. Then $u = u_1 + u_2$, where

$$ u_1(z) := -(2\pi)^{-3} \int_U \chi(\zeta_1)(\hat{f}(\zeta_1).d\zeta_4) \wedge \nu, $$

$$ u_2(z) := -(2\pi)^{-3} \int_U (1 - \chi(\zeta_1))(\hat{f}(\zeta_1).d\zeta_4) \wedge \nu. $$
Then 
\[ u(z) := -(2\pi)^{-3} \int_{\mathbb{H}} \chi(\zeta_1 + z)(\hat{f}(\zeta_1 + z).d\bar{\zeta}_4) \wedge \psi, \]
where
\[ \psi := (\partial_{\zeta_1} \ln(\zeta_1 - \zeta_2))M_1^{-1}(\partial_{\zeta_2} \ln(\zeta_2 - \zeta_3))M_2^{-1}(\partial_{\zeta_3} \ln(\zeta_3)M_3^{-1}. \]
Since \( \partial_{\zeta_4} \{[\chi(\zeta_1 + z)\hat{f}(\zeta_1 + z)]\wedge \psi\}.d\bar{\zeta}_4 \]
\[ = \partial_{\zeta_4} \{[\chi(\zeta_1 + z)\hat{f}(\zeta_1 + z)]\wedge \psi\} \]
then due to Equations (i, ii)
\[ (\partial u(z)/\partial \bar{z}) = -(2\pi)^{-3} \int_{\mathbb{H}} \partial_{\zeta_4} \{[\chi(\zeta_1 + z)\hat{f}(\zeta_1 + z)]\wedge \psi\}.\]
In view of Theorem 2.1 applied to \( f,S \) for each \( S \in \{I, J, K, L\} \) we have
\[ (\partial u_1/\partial \bar{z}) = \hat{f} \text{ in } V, \text{ consequently, } (\partial u/\partial \bar{z}) = \hat{f} \text{ in a neighbourhood of } z_0. \]
Taking a sequence \( f^n \) of continuously differentiable functions uniformly converging to \( f \) on \( U \) we get the corresponding \( u^n \) such that in the sense of distributions \( (\partial u^n/\partial \bar{z}) = \lim_{n \to \infty} (\partial u^n/\partial \bar{z}) = \lim_n \hat{f}^n = \hat{f} \).

2.4. **Theorem.** Let \( U \) be an open subset in \( \mathbb{H}^n \). Then for every compact subset \( K \) in \( U \) and every multi-order \( k = (k_1, ..., k_n) \), there exists a constant \( C > 0 \) such that
\[ \max_{z \in K} |\partial^k f(z)| \leq C \int_U |f(z)|d\sigma_{4n} \]
for each quaternion holomorphic function \( f \), where \( d\sigma_{4n} \) is the Lebesgue measure in \( \mathbb{H}^n \).

2.5. **Corollary.** Let \( U \) be an open subset in \( \mathbb{H}^n \) and let \( f_i \) be a sequence of quaternion holomorphic functions in \( U \) which is uniformly bounded on every compact subset of \( U \). Then there is a subsequence \( f_{k_j} \) converging uniformly on every compact subset of \( U \) to a limit in \( C^w_z(U, \mathbb{H}) \).

Proofs of Theorem 2.4 and Corollary 2.5 follow from Theorem 2.1 above and Theorem 3.9 [22] analogously to Theorem 1.1.13 and Corollary 1.1.14 [9].

2.6. **Definitions.** Let \( U \) be an open subset in \( \mathbb{H}^n \) and \( f : U \to \mathbb{H}^m \) be a quaternion holomorphic function, then the matrix: \( J_f(z) := (\partial f_j(z)/\partial z_k) \)
is called the quaternion Jacobi matrix, where \( j = 1, ..., m, k = 1, ..., n \). To this quaternion operator matrix there corresponds a real \((4m) \times (4n)\)-matrix. Denote by \( \text{rank}_R(J_f(z)) \) a rank of a real matrix corresponding to \( J_f(z) \). Then \( f \) is called regular at \( z \in U \), if \( \text{rank}_R(J_f(z)) = 4 \min(n,m) \). If \( U \) and \( V \) are two open subsets in \( \mathbb{H}^n \), then a bijective surjective mapping \( f : U \to V \) is called quaternion biholomorphic if \( f \) and \( f^{-1} : V \to U \) are quaternion holomorphic.

2.7. **Proposition.** Let \( U \) and \( V \) be open subsets in \( \mathbb{H}^n \) and \( \mathbb{H}^m \) respectively. If \( f : U \to \mathbb{H}^m \) and \( g : V \to \mathbb{H}^k \) are quaternion holomorphic functions
such that \( f(U) \subset V \), then \( g \circ f : U \to \mathbb{H}^k \) is quaternion holomorphic and \( J_{g \circ f}(z) = f'(z) \cdot (J_f(z)) \cdot h \) for each \( h \in \mathbb{H}^n \).

**Proof.** In view of Definition 2.1 and Theorem 3.10 [22] \((\partial g_j(f(z))/\partial z_l) \cdot \zeta = \sum_{s=1}^{m_l} \sum_{l=1}^{m_j} (\partial g_j(\xi))/\partial \xi_s) \cdot (\partial f_s(z)/\partial z_l) \cdot h_l \), where \( h = (h_1, ..., h_n) \), \( h_l \in \mathbb{H} \) for each \( l = 1, ..., n \), since \( f(U) \subset V \) and this is evident for quaternion polynomial functions and hence for locally converging series of quaternion holomorphic functions.

2.8. Proposition. Let \( U \) be a neighbourhood of \( z \in \mathbb{H}^n \) and let \( f : U \to \mathbb{H}^n \) be a quaternion holomorphic function. Then \( f \) is quaternion biholomorphic in some neighbourhood of \( z \) if and only if \( \text{rank}_R J_f(z) = 4n \).

**Proof.** From Proposition 2.7 it follows, that the condition \( \text{rank}_R J_f(z) = 4n \) is necessary. In view of Definition 2.1, Theorem 3.10 and Note 3.11 [22] an increment of \( f \) can be written in the form \( f(z + \zeta) = f(z) + J_f(z) \cdot \zeta + O(|\zeta|^2) \) for each \( \zeta \in \mathbb{H}^n \) such that \( z + \zeta \in U \). As in the proof of Theorem 1.1.18 [9] we get, that there exists a quaternion holomorphic function \( h \) on an open neighbourhood \( W \) of \( z \) in \( U \) satisfying the condition \(|g(z + \zeta)| \leq C|\zeta|^2, W \supset B(z, 2\epsilon, \mathbb{H}^n), 0 < \epsilon < (2C)^{-1}, \) where \( C \) is a positive constant and \( h \) is given by the series \( h = \sum_{k=1}^{\infty} g_k \), where \( g_k+1 = g \circ g_k \) for each \( k \in \mathbb{N} \) and \( g_1 := g, g := id - f \), since for each \( \eta \in W \) there exists \( r > 0 \) such that \( B(\eta, r, \mathbb{H}) \subset U \) and the series for \( h \) is convergent on \( B(\eta, r, \mathbb{H}) \) with \( h(B(z, \epsilon, \mathbb{H}^n)) \subset B(z, 2\epsilon, \mathbb{H}^n) \). The operator \( J_f(\eta) \) is continuous by \( \eta \) on \( U \), hence there exists a neighbourhood \( V \) of \( z \) such that \( \text{rank}_R J_f(z) = 4n \) on it, hence \( f(V) \) is open in \( \mathbb{H}^n \). From \((id + h) \circ f = f \circ (id + h) = id \) on \( B(z, \epsilon, \mathbb{H}^n) \) it follows, that \( f \) is quaternion holomorphic on a neighbourhood of \( z \).

2.9. Corollary. Let \( X \) be a subset in \( \mathbb{H}^n \) and \( k \in \{1, 2, ..., n-1\} \), then the following conditions are equivalent:

(i) for each \( \zeta \in X \) there exists a quaternion biholomorphic map \( f = (f_1, ..., f_n) \) in some neighbourhood \( U \) of \( \zeta \) such that \( \text{rank}_R f = 4n \) on \( U \) and \( X \cap U = \{z \in U : f_{k+1}(z) = 0, ..., f_n(z) = 0\} \);

(ii) for each \( \zeta \in X \) there exists a neighbourhood \( V \) of \( \zeta \) and a regular quaternion holomorphic map \( g : V \to \mathbb{H}^{n-k} \) such that \( X \cap V = \{z \in V : g(z) = 0\} \).

**Proof** is analogous to the proof of Corollary 1.1.19 [9] and with a consideration of the determinant function of the real \( 4n \times 4n \) matrix \( J_{fR}(z) \) corresponding to the quaternion operator \( J_f(z) \), since \( \det J_{fR}(z) \neq 0 \) if and only if \( \text{rank}_R f(z) = 4n \).
2.10. Definitions. Let $U$ be an open subset in $\mathbb{H}^n$. A subset $X$ in $U$ is called a quaternion submanifold of $\mathbb{H}^n$ if the equivalent conditions of Corollary 2.9 are satisfied. If in addition $X$ is a closed subset in $U$, then $X$ is called a closed quaternion submanifold of $U$. This definition is the particular case of the following general definition.

A quaternion holomorphic manifold of quaternion dimension $n$ is a real $4n$-dimensional $C^\infty$-manifold $X$ together with a family \(\{(U_j, \phi_j) : j \in \Lambda\}\) of charts such that

(i) each $U_j$ is an open subset in $X$ and $\bigcup_{j \in \Lambda} U_j = X$, where $\Lambda$ is a set;

(ii) for each $j \in \Lambda$ a mapping $\phi_j : U_j \to V_j$ is a homeomorphism on an open subset $V_j$ in $\mathbb{H}^n$;

(iii) for each $j, l \in \Lambda$ a connection mapping $\phi_j \circ \phi_l^{-1}$ is a quaternion biholomorphic map (see §2.6) from $\phi_l(U_j \cap U_l)$ onto $\phi_j(U_j \cap U_l)$. Such system is called a quaternion holomorphic atlas $\text{At}(X) := \{(U_j, \phi_j) : j \in \Lambda\}$. Each chart $(U_j, \phi_j)$ provides a system of quaternion holomorphic coordinates induced from $\mathbb{H}^n$. For short we shall write quaternion manifold instead of quaternion holomorphic manifold and quaternion atlas instead of quaternion holomorphic atlas if other will not be specified.

For two quaternion manifolds $X$ and $Y$ with atlases $\text{At}(X) := \{(U_j, \phi_j) : j \in \Lambda_X\}$ and $\text{At}(Y) := \{(W_l, \psi_l) : l \in \Lambda_Y\}$ a function $f : X \to Y$ is called quaternion holomorphic if $\psi_l \circ f \circ \phi_j^{-1}$ is quaternion holomorphic on $\phi_j(U_j \cap f^{-1}(W_l))$. If $f : X \to Y$ is a quaternion biholomorphic epimorphism, then $X$ and $Y$ are called quaternion biholomorphically equivalent.

A subset $Z$ of a quaternion manifold $X$ is called a quaternion submanifold, if $\phi_j(U_j \cap Z)$ is a quaternion submanifold in $\mathbb{H}^n$ for each chart $(U_j, \phi_j)$. If additionally $Z$ is closed in $X$, then $Z$ is called a closed quaternion submanifold.

2.11. Theorem. Let $n \geq 2$, $f_1, ..., f_n \in C^1_{0,(z,\bar{z})}(\mathbb{H}^n, \mathbb{H})$ be a family of continuously quaternion $(z, \bar{z})$-differentiable functions satisfying compatibility conditions:

\[ (i) \quad \partial f_j/\partial \bar{z}_k = \partial f_k/\partial \bar{z}_j \text{ for each } j, k = 1, ..., n, \]

where in $C^1_{0,(z,\bar{z})}(\mathbb{H}^n, \mathbb{H})$ is the subspace of $C^1_{(z,\bar{z})}(\mathbb{H}^n, \mathbb{H})$ of functions with compact support. Then there exists $u \in C^1_{0,(z,\bar{z})}(\mathbb{H}^n, \mathbb{H})$ satisfying the following $\bar{\partial}$-equation:

\[ (ii) \quad \partial u/\partial \bar{z}_j = f_j, \quad j = 1, ..., n; \]
in particular, \((\partial u/\partial \bar{z}_j).I = f_j\).

**Proof.** We put

\[(iii)\quad u(z) := -(2\pi)^{-3} \int_{H} \left[ (\hat{f}_1(\zeta_1, z_2, ..., z_n).d\zeta_4) \wedge \eta \right], \text{ where} \]

\[\eta := (\partial_{\zeta_1}Ln(\zeta_1 - \zeta_2))M_1^{-1}(\partial_{\zeta_2}Ln(\zeta_2 - \zeta_3))M_2^{-1}(\partial_{\zeta_3}Ln(\zeta_3 - z)M_3^{-1} \]

(see §2.3). By changing of variables we get

\[u(z) := -(2\pi)^{-3} \int_{H} \left[ (\hat{f}_1(z_1 + \zeta_1, z_2, ..., z_n).d\zeta_4) \wedge \psi \right], \text{ where} \]

\[\psi := (\partial_{\zeta_1}Ln(\zeta_1 - \zeta_2))M_1^{-1}(\partial_{\zeta_2}Ln(\zeta_2 - \zeta_3))M_2^{-1}(\partial_{\zeta_3}Ln(\zeta_3)M_3^{-1}. \]

Therefore, \(u \in C^1_{(\vec{z},\bar{z})}(H^n, H)\). Due to Theorem 2.3 \(\partial u/\partial \bar{z}_1 = \hat{f}_1\) in \(H^n\). In view of Theorem 2.1 and the condition \(\partial f_v/\partial \bar{z}_k = \partial f_v/\partial \bar{z}_1\) the following equality is satisfied

\[\hat{f}_k(z) = -(2\pi)^{-3} \int_{H} \left\{ [\partial f_v(\zeta_1, z_2, ..., z_n)/\partial \zeta_1].d\zeta_4 \right\} \wedge \psi, \]

hence \(\partial u/\partial \bar{z}_k = \hat{f}_k\) for \(k = 2, ..., n\), that is, \(u\) satisfies equations (ii). From this it follows, that \(u\) is quaternion holomorphic in \(H^n \setminus supp(f_1) \cup ... \cup supp(f_n)\). In view of formula (iii) it follows, that there exists \(0 < r < \infty\) such that

\[(iv)\quad u(z) = 0 \text{ for each } z \in H^n \text{ with } |z_2| + ... + |z_n| > r. \text{ From } \partial u/\partial \bar{z}_1 = \hat{f}_1 \text{ it follows, that } \partial u/\partial \bar{z}_1 = \hat{f}_1. \text{ Consequently, there exists } 0 < R < \infty \text{ such that } u \text{ may differ from } 0 \text{ on } H^n \setminus B(H^n, 0, R) \text{ only on a quaternion constant (see Theorem 3.28 and Note 3.11 in [22]). Together with (iv) this gives, that } u(z) = 0 \text{ on } H^n \setminus B(H^n, 0, \max(R, r)). \]

**2.12. Theorem.** Let \(U\) be an open subset in \(H^n\), where \(n \geq 2\). Suppose \(K\) is a compact subset in \(U\) such that \(U \setminus K\) is connected. Then for every quaternion holomorphic function \(h\) on \(U \setminus K\) there exists a function \(H\) quaternion holomorphic in \(U\) such that \(H = h\) in \(U \setminus K\).

**Proof.** Take any infinite \((z, \bar{z})\)-differentiable function \(\chi\) on \(U\) with compact support such that \(\chi|_V = 1\) on some (open) neighbourhood \(V\) of \(K\). Then consider a family of functions \(f_j\) such that \(\hat{f}_j(z, \bar{z}).S = -\{(\partial \chi/\partial \bar{z}).S\}h\) in \(U \setminus K\) and \(\hat{f}_j = 0\) outside \(U \setminus K\) for each \(S \in \{I, J, K, L\}\), where \(j = 1, ..., n\), \(f_j(z) = \hat{f}_j(z).I\). Therefore, conditions of Theorem 2.11 are
satisfied and it gives a function \( u \in C^1_{\alpha(z, z)}(\mathbb{H}^n, \mathbb{H}) \) such that \( \partial u / \partial \bar{z}_j = \hat{f}_j \) for each \( j = 1, ..., n \). A desired function \( H \) can be defined by the formula \( H := (1 - \chi)h - u \) such that \( H \) is quaternion holomorphic in \( U \). Since \( \chi \) has a compact support, then there exists an unbounded connected subset \( W \) in \( \mathbb{H}^n \setminus \text{supp}(\chi) \). Therefore, \( u|_W = 0 \), consequently, \( H|_{U \cap W} = h|_{U \cap W} \). From \((U \setminus K) \cap W \neq \emptyset\) and connectedness of \( U \setminus K \) it follows, that \( H|_{U \setminus K} = h|_{U \setminus K} \).

2.13. Remark. In the particular case of a singleton \( K = \{z\} \) Theorem 2.12 gives nonexistence of isolated singularities, that is, each quaternion holomorphic function in \( U \setminus \{z\} \) for \( U \) open in \( \mathbb{H}^n \) with \( n \geq 2 \) can be quaternion holomorphically extended to \( z \). Theorem 2.12 is the quaternion analog of the Hartog’s theorem for \( \mathbb{C}^n \).

2.14. Corollary. Let \( U \) be an open connected subset in \( \mathbb{H}^n \) and \( n \geq 2 \). Suppose that \( f \) is a right superlinearly superdifferentiable function \( f : U \rightarrow \mathbb{H} \) and \( N(f) := \{z \in U : f(z) = 0\} \), then

(i) \( U \setminus N(f) \) is connected,

(ii) \( N(f) \) is not compact.

Proof. (i). We have \( f = (f_{1,1}, f_{1,2}) \), where \( f_{1,1}(t, u) \) is holomorphic by \( t \) and antiholomorphic by \( u \), \( f_{1,2}(t, u) \) is holomorphic by \( u \) and antiholomorphic by \( t \), where \( jz = (i_u, j_u, i_t, j_t) \), \( jz \in \mathbb{H}, z = (1z, ..., n_z) \in U \). Therefore, \( N(f) = N(f_{1,1}) \cap N(f_{1,2}) \), consequently, \( U \setminus N(f) = (U \setminus N(f_{1,1})) \cup (U \setminus N(f_{1,2})) \). Then from Corollary 1.2.4 [9] for complex holomorphic functions (i) follows.

(ii). Suppose that \( N(f) \) is compact. In view of (i) and Theorem 2.12 the function \( 1/f \) can be quaternion holomorphically extended on \( N(f) \). This is a contradiction, since \( f = 0 \) on \( N(f) \).

2.14.1. Note. Corollary 2.14 is not true for arbitrary quaternion holomorphic functions, for example, \( f(1z, 2z) = f_1(1z) f_2(2z) \) on \( B(\mathbb{H}^2, 0, 2) \), where \( f_1(1z) := 1z - r_1, f_2(2z) := 2z - r_2, 0 < r_1, 0 < r_2, r_1^2 + r_2^2 < 4 \).

2.15. Theorem. Let \( U \) be an open subset in \( \mathbb{H}^n \), \( f_1, ..., f_n \) be infinite differentiable (by real variables) functions on \( U \) and Conditions 2.11.(i) are satisfied in \( U \). Then for each open bounded polytor \( P = P_1 \times ... \times P_n \) such that \( \text{cl}(P) \) is a subset in \( U \), there exists a function \( u \) infinite differentiable (by real variables) on \( P \) and satisfying Conditions 2.11.(ii) on \( P \).

Proof. Suppose that the theorem is true for \( f_{m+1} = ... = f_n = 0 \) on \( U \). The case \( m = 0 \) is trivial. Assume that the theorem is proved for \( m - 1 \). Consider \( U' = U'_1 \times ... \times U'_n \) and \( U'' = U''_1 \times ... \times U''_n \) open polytors in \( \mathbb{H}^n \)
such that $P \subset \text{cl}(P) \subset U'' \subset \text{cl}(U'') \subset U' \subset \text{cl}(U') \subset U$. Take an infinite differentiable (by real variables) function $\chi$ on $U'_{m}$ with compact support such that $\chi|_{U''_{m}} = 1$, $\chi = 0$ in a neighbourhood of $H \setminus U'_{m}$. There exists a function

$$
\eta(z) := -(2\pi)^{-3} \int_{U'_{m}} \left\{ \left[ \chi(\zeta) \hat{f}_{m}(1z, \ldots, m^{-1}z, \zeta_{1}, m^{+1}z, \ldots, n^{z}) \right. \right. \left. \left. d\zeta_{4} \right. \right\} \land \nu,
$$

where a differential form $\nu$ is given in §2.3 with $\zeta_{1}, \zeta_{2}, \zeta_{3} \in U'_{m}$ and $m^{z}$ here for $\nu$ instead of $z$ in §2.3. By changing of variables as in §2.3 we get

$$
\eta(z) := -(2\pi)^{-3} \int_{H} \chi(\zeta_{1}+z)(\hat{f}_{m}(1z, \ldots, m^{-1}z, \zeta_{1}+m^{z}, m^{+1}z, \ldots, n^{z}).d\zeta_{4}) \land \psi,
$$

where

$$
\psi := (\partial_{\zeta_{4}} \ln(\zeta_{1} - \zeta_{2}))M^{-1}_{1}(\partial_{\zeta_{4}} \ln(\zeta_{2} - \zeta_{3}))M^{-1}_{2}(\partial_{\zeta_{3}} \ln(\zeta_{3})M^{-1}_{3}.
$$

Consequently, $\partial \hat{\eta}/\partial \hat{\zeta}_{m} = \hat{f}_{m}$ in $U''$. The final part of the proof is analogous to that of Theorem 1.2.5 [9].

2.16. Definition. Let $W$ be an open subset in $H^{n}$ and for each open subsets $U$ and $V$ in $H^{n}$ such that

(i) $\emptyset \neq U \subset V \cap W \neq V$

(ii) $V$ is connected

there exists a quaternion holomorphic (right superlinear superdifferentiable, in short RSS, correspondingly) function $f$ in $W$ such that there does not exist any quaternion holomorphic (RSS) function $g$ in $V$ such that $g = f$ in $U$. Then $W$ is called a domain of quaternion (RSS, respectively) holomorphy. Sets of quaternion holomorphic (RSS) functions in $W$ are denoted by $H(W)$ ($H_{RSS}(W)$ respectively).

2.17. Definition. Suppose that $W$ is an open subset in $H^{n}$ and $K$ is a compact subset of $W$, then

(i) $\hat{K}_{W}^{H} := \{ z \in W : |f(z)| \leq \sup_{\zeta \in K} \| \hat{f}(\zeta) \| \text{ for each } f \in H(W) \}$;

(ii) $\hat{K}_{W}^{H_{RSS}} := \{ z \in W : |f(z)| \leq \sup_{\zeta \in K} |f(\zeta)| \text{ for each } f \in H_{RSS}(W) \}$;

these sets are called the $H(W)$-convex hull of $K$ and the $H_{RSS}(W)$-convex hull of $K$ respectively, where $\|\hat{f}(\zeta)\| := \sup_{h \in H^{n}, |h| \leq 1} |\hat{f}(\zeta), h|$. If $K = \hat{K}_{W}^{H}$ or $K = \hat{K}_{W}^{H_{RSS}}$, then $K$ is called $H(W)$-convex or $H_{RSS}(W)$-convex correspondingly.

2.18. Proposition. For each compact set $K$ in $H^{n}$, the $H(H^{n})$-hull and $H_{RSS}(H^{n})$-hull of $K$ are contained in the $R$-convex hull of $K$.

Proof. I. Consider at first the $H(H^{n})$-hull of $K$. Each $z \in H^{n}$ can be written in the form $z = (1z, \ldots, n^{z})$, $j^{z} \in H$, $j^{z} = \sum_{l=1}^{4} x_{l}j_{l}S_{l}$, where
\[ x_{i,j} = x_{i,j}(z) \in \mathbb{R}, \; \mathbf{S}_l \in \{I, J, K, L\}. \] If \( w \in \mathbb{H}^n, \; w \notin \text{co}_R(K) \), then there are \( y_1, \ldots, y_n \in \mathbb{R} \) such that \( \sum_{i,j=1}^n \sum_{l=1}^4 x_{i,j}(w)y_{i,j(l-1)+l} = 0 \), but \( \sum_{i,j=1}^n \sum_{l=1}^4 x_{i,j}(w)y_{i,j(l-1)+l} < 0 \) if \( z \in K \), where
\[
\text{co}_R(K) := \{z \in \mathbb{H}^n : \text{there are } a_1, \ldots, a_s \in \mathbb{R} \text{ and } v_1, \ldots, v_s \in K \text{ such that } z = a_1v_1 + \ldots + a_kv_k \} \text{ denotes a } \mathbb{R}-\text{convex hull of } K \text{ in } \mathbb{H}^n. \]
Put \( \zeta_j = \sum_{l,j} y_{i,j(l-1)+l}S_l \), then \( f(z) := \exp(\sum_{j=1}^n \zeta_j\hat{\zeta}_j) \) is the quaternion holomorphic function in \( \mathbb{H}^n \) such that \( |f(z)| < 1 \) for each \( z \in K \) and \( |f(w)| = 1 \) for the marked point \( w \) above (see Corollary 3.3 [22]), since \( J^2 = K^2 = L^2 = -I \). From \( \|f(\zeta)\| \geq |f(\zeta)| \) the first statement follows.

II. Consider now the \( \mathcal{H}_{RSL}(\mathbb{H}^n) \)-hull of \( K \). Each \( f \in \mathcal{H}_{RSL}(W) \) has the form \( f = \left( \begin{array}{cc} f_{1,1} & f_{1,2} \\ -f_{1,2} & f_{2,2} \end{array} \right) \) such that \( f_{1,1} \) and \( f_{1,2} \) are complex holomorphic by complex variables \( j^2t \) and \( j^4u \) respectively and antiholomorphic by complex variables \( j^4u \) and \( j^2t \) correspondingly (see Proposition 2.2 [22]). The set \( K \) has projection \( K_{1,1} \) and \( K_{1,2} \) on the complex subspaces \( \mathbb{C}^n \) corresponding to variables \( 1^t, \ldots, n^t \) and \( 1^u, \ldots, n^u \) respectively. Therefore, \( (\mathcal{K}^\mathcal{H}_{RSL})_{1,1} \subset \hat{K}^\mathcal{O}_{1,1}\mathbb{C}^n \) for \( l = 1 \) and \( l = 2 \), where \( \hat{K}^\mathcal{O}_{1,1}\mathbb{C}^n \) denotes the complex holomorphic hull of \( K_{1,1} \) in \( \mathbb{C}^n \). In view of Proposition 1.3.3 \( \mathcal{K}^\mathcal{O}_{1,1}\mathbb{C}^n \subset \text{co}_R(K_{1,1}) \), hence \( \mathcal{K}^\mathcal{H}_{RSL} \subset \text{co}_R(K) \).

2.18.1. Note. Due to Proposition 2.18 above Corollary 1.3.4 [9] can be transferred on \( \mathcal{H} \) and \( \mathcal{H}_{RSL} \) for \( \mathbb{H}^n \) instead of \( \mathbb{C}^n \). Quaternion versions of Theorems 1.3.5, 7, 11, Corollaries 1.3.6, 8, 9, 10, 13 and Definition 1.3.12 are true in the \( \mathcal{H}_{RSL} \)-class of functions instead of complex holomorphic functions.

### 3 Integral representations of functions of quaternion variables

3.1. Definitions and Notations. Consider an \( \mathbb{H} \)-valued function on \( \mathbb{H}^n \) such that

(i) \( (\zeta, \zeta) = ae \) with \( a \geq 0 \) and \( (\zeta, \zeta) = 0 \) if and only if \( \zeta = 0 \),

(ii) \( (\zeta, z + \xi) = (\zeta, z) + (\zeta, \xi) \),

(iii) \( (\zeta + \xi, z) = (\zeta, z) + (\xi, z) \),

(iv) \( (a\zeta, z\beta) = a(\zeta, z)\beta \),

(v) \( (\zeta, z)^\ast = (z, \zeta) \) for each \( \zeta, \xi \) and \( z \in \mathbb{H}^n, \alpha \) and \( \beta \in \mathbb{H}, n \in \mathbb{N} \). Then this function is called the scalar product in \( \mathbb{H}^n \). The corresponding norm is:

(vi) \( |\zeta| = \{((\zeta, \zeta))^{1/2} \}. \) In particular, it is possible to take the canonical
scalar product:
(vii) \( <\zeta; z> := (\zeta, z) = \sum_{i=1}^{n} \bar{\zeta}^i l^i z \),
where \( z = (1, \ldots, n) \), \( l^i \in H \).

Consider differential forms on \( H \):

1. \( \omega_1(\zeta) = \bar{\zeta}d\bar{\zeta} \), \( \omega_1(\zeta - z) = (\bar{\zeta} - \bar{z})d(\bar{\zeta} - \bar{z}) \),
2. \( \nu_1(\zeta) = (d\bar{\zeta})(\bar{\zeta}), \nu_1(\zeta - z) = (d(\bar{\zeta} - \bar{z}))(\bar{\zeta} - \bar{z}) \),
3. \( \omega_2(\zeta) = (jd\bar{\zeta}j)(\bar{\zeta}j), \omega_2(\zeta - z) = j(d\bar{\zeta} - d\bar{z})j + j(d\bar{\zeta} - d\bar{z})j, \)
4. \( \nu_2(\zeta) = d\bar{\zeta} \land d\bar{\zeta}, \nu_2(\zeta, z) = (d\bar{\zeta} - d\bar{z}) \land (d\bar{\zeta} - d\bar{z}) \),
5. \( \omega_4(\zeta) = \nu_2(\zeta) \land \nu_2(\zeta), \)
6. \( \omega_4(\zeta, z) = \nu_2(\zeta, z) \land \nu_2(\zeta) \),
7. \( \bar{\omega}_4(\zeta, z) = \nu_2(\zeta, z) \land \nu_2(\zeta, z) \).

With the help of them construct differential forms on \( H^n \):

\[ (8) \quad \theta(\zeta) := \frac{(2n - 1)!(2\pi)^{-2n}|\zeta - z|^{-4n}}{\sum_{s=1}^{n}} \{ \omega_4(\zeta) \land \ldots \land \omega_4(s - 1, z) \land \nu_1(s - 1, z) \land \omega_4(s - 1, z) \land \omega_4(s - 1, z) \land \omega_4(s - 1, z) \land \ldots \land \omega_4(n, z) \}; \]

\[ (9) \quad \theta(\zeta, z) := \frac{(2n - 1)!(2\pi)^{-2n}|\zeta - z|^{-4n}}{\sum_{s=1}^{n}} \{ \omega_4(\zeta, z) \land \ldots \land \omega_4(n, z) \}; \]

\[ (10) \quad \bar{\theta}(\zeta, z) := \frac{(2n - 1)!(2\pi)^{-2n}|\zeta - z|^{-4n}}{\sum_{s=1}^{n}} \{ \bar{\omega}_4(\zeta, z) \land \ldots \land \bar{\omega}_4(n, z) \}; \]

where \( \zeta \) and \( z \in H^n \). If \( U \) is an open subset in \( H^n \) and \( f \) is a bounded quaternion differential form on \( U \), then by the definition:

\[ (11) \quad (B_U f)(z) := \int_{\zeta \in U} f(\zeta) \land \theta(\zeta, z) \]

for each \( z \in H^n \). If in addition \( U \) is with a piecewise \( C^1 \)-boundary (by the corresponding real variables) and \( f \) is a bounded differential form on \( \partial U \), then by the definition:

\[ (12) \quad (B_{\partial U} f)(z) := \int_{z \in \partial U} f(\zeta) \land \theta(\zeta, z) \]
for each \( z \in \mathbb{H}^n \).

**3.2. Theorem.** Let \( U \) be an open subset in \( \mathbb{H}^n \) with piecewise \( C^1 \)-boundary \( \partial U \). Suppose that \( f \) is a continuous function on \( \text{cl}(U) \) and \( \bar{\partial}f \) is continuous on \( U \) in the sense of distributions and has a continuous extension on \( \text{cl}(U) \). Then

\[
(1) \quad f = \mathcal{B}_{\partial U} f - \mathcal{B}_U \bar{\partial}f \quad \text{on} \quad U,
\]

where \( \mathcal{B}_U \) and \( \mathcal{B}_{\partial U} \) are the quaternion integral operators given by Equations 3.1.(11, 12).

**Proof.** The differential form \( \theta(\zeta, z) \) has the decomposition

\[
(2) \quad \theta(\zeta, z) = \sum_{q=0}^{2n-1} \Upsilon_q(\zeta, z),
\]

where \( \Upsilon_q(\zeta, z) \) is the quaternion differential form with all terms of degree \( 4n - q - 1 \) by \( \zeta \) and \( \bar{\zeta} \) and their multiples on quaternion constants and of degree \( q \) by \( z \) and \( \bar{z} \) and their multiples on quaternion constants. The differential form \( f(\zeta) \) has the decomposition

\[
(3) \quad f(\zeta) = \sum_{r=0}^{m} f_r(\zeta),
\]

where \( m = \text{deg}(f) \), \( f_r(\zeta) \) is with all terms of degree \( r \) by \( \zeta \) and \( \bar{\zeta} \) and their multiples on quaternion constants. Then \( f_r \wedge \Upsilon_q = 0 \), when \( r > q + 1 \). By the definition of integration \( \int_{\zeta \in U} f_r(\zeta) \wedge \Upsilon_q(\zeta, z) = 0 \) for \( r < q + 1 \). If \( f \) is a function, then \( \int_{\zeta \in \partial U} f(\zeta) \Upsilon_q(\zeta, z) = 0 \) for each \( q > 0 \), since \( \partial U \) has the dimension \( 4n - 1 \), hence

\[
(4) \quad (\mathcal{B}_{\partial U} f)(z) = \int_{\zeta \in \partial U} f(\zeta) \theta_z(\zeta),
\]

since \( \Upsilon_0(\zeta, z) = \theta_z(\zeta) \). If \( f \) is a 1-form, then \( \int_{\zeta \in U} f(\zeta) \wedge \Upsilon_q(\zeta, z) = 0 \) for each \( q > 0 \), since \( U \) has the dimension \( 4n \), consequently,

\[
(5) \quad (\mathcal{B}_U f)(z) = \int_{\zeta \in U} f(\zeta) \wedge \theta_z(\zeta).
\]

Write \( \xi \in \mathbb{H} \) in the form \( \xi = \alpha e + \beta j \), then \( \bar{\xi} = \alpha - \beta j \), where \( \alpha = \alpha_0 + \alpha_i \) and \( \beta = \beta_0 + \beta_i \) \( \epsilon \mathbb{C} \), \( \alpha_0, \alpha_i, \beta_0 \) and \( \beta_i \) \( \epsilon \mathbb{R} \), \( i := (-1)^{1/2} \). There is the identity
\[ \beta j = j \bar{\beta}, \text{ since } ij = -ji = k, \text{ where } \bar{\beta} = \beta_0 = \beta_j i, \ H = Re \oplus Ri \oplus Rj \oplus Rk, \]
\[ i^2 = j^2 = k^2 = -e, \ i = i e. \]
Then
\[ d\beta j \land d\bar{\beta}j = 0, \ \nu_2(\xi) = -d\alpha \land d\beta j - d\beta j \land d\beta - d\beta \land d\bar{\beta} e, \]
\[ \omega_2(\xi) = d\alpha \land d\alpha e + 2d\alpha \land d\bar{\beta}j - d\beta \land d\bar{\beta} e, \]
\[ d\xi \land d\xi \land \omega_2(\xi) = 0, \text{ since } d\xi \land d\xi = d\alpha \land d\alpha e - 2d\alpha \land d\beta j + d\beta \land d\bar{\beta} e, \]
\[ \nu_2(\xi) \land \omega_2(\xi) = -d\alpha \land d\alpha e \land d\beta \land d\bar{\beta} e = 4d\alpha_0 \land d\alpha \land d\beta_0 \land d\bar{\beta} e. \]
Therefore, \( \nu_2(\xi) \land \omega_2(\xi) \) plays the role of the volume element in \( H \). Hence
\[ d_\zeta(|\zeta - z|^{4n+1}) = [(2n - 1)!(2\pi)^{-2n}](2n)!d^1\alpha_0 \land d^1\alpha_1 \land d^1\beta_0 \land d^1\beta_1 \land .. \land d^n\alpha_0 \land d^n\alpha_1 \land d^n\beta_0 \land d^n\beta_1, \text{ since } d_\zeta = \partial_\zeta + \partial_\bar{\zeta} \text{ (see Formula (2.15) [22]).} \]
In view of Proposition 1.7.1 [9] and Formulas (6 - 9), 3.1.\( vi, vii \) above the differential form \( \theta_\zeta(\zeta) \) is closed in \( U \setminus \{z\} \).

There exists \( \epsilon_0 > 0 \) such that for each \( 0 < \epsilon < \epsilon_0 \) the ball \( B(H^n, z, \epsilon) := \{\zeta \in H^n : |\zeta - z| \leq \epsilon\} \) and hence the sphere \( S(H^n, z, \epsilon) := \{\zeta \in H^n : |\zeta - z| = \epsilon\} \) are contained in \( U \). Apply the Stoke’s formula for matrix-valued functions and differential forms componentwise, then
\[ \int_{S(H^n, z, \epsilon)} f(\zeta) \theta_\zeta(\zeta) = \int_{\partial U} f(\zeta) \theta_\zeta(\zeta) \land \theta_\zeta(\zeta), \text{ where } U_\epsilon := U \setminus B(H^n, z, \epsilon), 0 < \epsilon < \epsilon_0. \]

There are identities: \( d\xi \land jd\xi = d\xi j \land d\xi \) and \( (\xi d\xi)^\sim = [d\xi] \bar{\zeta} \) for each \( \xi, \zeta \in H \). Then from Identity (8) it follows, that
\( i \)
\[ d\xi \land jd\xi \land d\xi \land d\xi = 0, \]
\( ii \)
\[ d\xi \land d\xi \land jd\xi \land d\xi = 0, \]
\( iii \)
\[ d\xi \land d\xi \land d\xi \land d\xi = 0, \]
\( iv \)
\[ d\xi \land jd\xi \land d\xi \land d\xi = 0, \]
\( v \)
\[ d\xi \land jd\xi \land jd\xi \land d\xi = 0, \]
\( vi \)
\[ d\xi \land jd\xi \land jd\xi \land jd\xi = 0. \]
Therefore, from (8) and (vi) it follows, that
\[ \mathcal{B}_\xi df = \partial_\xi df + \partial_\zeta df, \text{ where } \partial df(\zeta) = (\partial f(\zeta)/\partial \zeta).d\zeta, \]
\[ \partial df(\zeta) = (\partial f(\zeta)/\partial \bar{\zeta}).d\bar{\zeta}, f(\zeta) = f(\zeta, \bar{\zeta}) \text{ is the abbreviated notation.} \]

In view of Formula (10) and the Stoke’s formula:
\[ \int_{S(H^n, z, \epsilon)} \theta_\zeta(\zeta) = (2n - 1)!(2\pi)^{-2n} [4^n \epsilon^{-4n} 2n] \int_{B(H^n, z, \epsilon)} (dV)e = e, \text{ where } dV \text{ is the standard volume element of the Euclidean space } R^{4n}. \]
From Formula (13) it follows, that
\[ \lim_{\epsilon \to 0} \int_{S(H^n, z, \epsilon)} f(\zeta) \theta_\zeta(\zeta) = f(z), \]
\[ \int_{S(H^n, z, \epsilon)} (f(\zeta) - f(z)) \theta_\zeta(\zeta) = (2n - 1)!(2\pi)^{-2n} \epsilon^{-4n+1} \int_{S(H^n, z, \epsilon)} (f(\zeta) - f(z))[\zeta- \bar{\zeta}] \]
\[ z^{4n-1} \theta_2(\zeta). \] The form \([|\zeta - z^{4n-1} \theta_2(\zeta)|]\) is bounded on \(U\), consequently, \(|f(\zeta) - f(z))\theta_2(\zeta)| \leq C_1 \max\{|f(\zeta) - f(z)| : \zeta \in B(\mathbb{H}^n, z, \epsilon)\}\), where \(C_1\) is a positive constant independent of \(f\) and \(\epsilon\) for each \(0 < \epsilon < \epsilon_0\). Therefore, Formula (1) follows from Formula (11) by taking the limit when \(\epsilon > 0\) tends to zero and using Identity (12).

3.3. Corollary. Let \(U\) be an open set in \(\mathbb{H}^n\) and \(f\) be a continuous function on \(cl(U)\) and quaternion holomorphic on \(U\). Then

\[ f = B_{\partial U} f \text{ on } U, \]

where \(B_U\) and \(B_{\partial U}\) are the integral operators given by Equations 3.1.(11, 12).

Proof. From \(\partial f = 0\), since \(\partial f(\zeta)/\partial \zeta = 0\), and Formula 3.2.(1) it follows Formula 3.3.(1).

3.4. Definitions and Notations. Suppose that \(U\) is a bounded open subset in \(\mathbb{H}^n\) and \(\psi(\zeta, z)\) be a quaternion-valued \(C^1\)-function (by the corresponding real variables) defined on \(V \times U\), where \(V\) is a neighbourhood of \(\partial U\) in \(\mathbb{H}^n\), such that

\[ (1) \quad < \psi(\zeta, z); \zeta - z > \neq 0 \text{ for each } (\zeta, z) \in \partial U \times U. \] Then \(\psi\) is called a quaternion boundary distinguishing map. Consider the function:

\[ (2) \quad \eta^\psi(\zeta, z, \lambda) := \lambda(\zeta - z) < \zeta - z; \zeta - z >^{-1} \]
\[ +(1 - \lambda)\psi(\zeta, z) < \zeta - z; \psi(\zeta, z) >^{-1}, \]

(see Formula 3.1.(viii)) and the differential forms:

\[ (3) \quad \omega_1(\tilde{s} \eta^\psi(\zeta, z, \lambda)) := \tilde{s} \eta^\psi(\zeta, z, \lambda)(\tilde{\partial}_{s\zeta} s_z + d\lambda) \tilde{s} \eta^\psi(\zeta, z, \lambda), \]
\[ \omega_2(\tilde{s} \eta^\psi(\zeta, z, \lambda)) := \tilde{s} \eta^\psi(\zeta, z, \lambda)(\tilde{\partial}_{s\zeta} s_z + d\lambda) \tilde{s} \eta^\psi(\zeta, z, \lambda), \]
\[ (4) \quad \nu_1(\tilde{s} \eta^\psi(\zeta, z, \lambda)) := [\tilde{\partial}_{s\zeta} s_z + d\lambda] \tilde{s} \eta^\psi(\zeta, z, \lambda) \tilde{s} \eta^\psi(\zeta, z, \lambda), \]
\[ \nu_1(\tilde{s} \eta^\psi(\zeta, z, \lambda)) := [\tilde{\partial}_{s\zeta} s_z + d\lambda] \tilde{s} \eta^\psi(\zeta, z, \lambda) \tilde{s} \eta^\psi(\zeta, z, \lambda), \]
\[ (5) \quad \nu_2(\tilde{s} \eta^\psi(\zeta, z, \lambda)) := [\tilde{\partial}_{s\zeta} s_z + d\lambda] \tilde{s} \eta^\psi(\zeta, z, \lambda) \tilde{s} \eta^\psi(\zeta, z, \lambda), \]

analogously to (3 - 5) are defined \(\omega_1(\tilde{s} \psi(\zeta, z)), \nu_1(\tilde{s} \psi(\zeta, z)), \nu_2(\tilde{s} \psi(\zeta, z))\) with \(\tilde{s} \psi(\zeta, z)\) instead of \(\tilde{s} \eta^\psi(\zeta, z)\).

\[ (6) \quad \phi_{\zeta, z} := \phi_{\zeta, z}(\psi(\zeta, z); \zeta) := (2n - 1)! (2\pi)^{-2n} < \psi(\zeta, z); \zeta - z >^{-2n} \]
\[ \sum_{s=1}^{n} \nu_2(\tilde{s} \psi(\zeta, z)) \wedge \omega_2(\tilde{s} \zeta) \wedge ... \wedge \nu_2(\tilde{s}^{-1} \psi(\zeta, z)) \wedge \omega_2(\tilde{s}^{-1} \zeta) \]
\[ \wedge [\omega_1(\tilde{s} \psi(\zeta, z)) \wedge \omega_2(\tilde{s} \zeta) + \nu_1(\tilde{s} \psi(\zeta, z)) \wedge \omega_2(\tilde{s} \zeta)] \wedge \nu_2(\tilde{s}^{-1} \psi(\zeta, z)) \wedge \omega_2(\tilde{s}^{-1} \zeta) \wedge ... \wedge \nu_2(\tilde{n} \psi(\zeta, z)) \wedge \omega_2(\tilde{n} \zeta)); \]
\[ (7) \quad \phi_{\zeta, z, \lambda} := \phi_{\zeta, z, \lambda}(\psi(\zeta, z), \zeta) := (2n - 1)! (2\pi)^{-2n} \]
$$\sum_{s=1}^{n} \{ \nu_2(1\bar{\eta}_{\psi}(\zeta, z, \lambda)) \wedge \omega_2(1\zeta) \wedge ... \wedge \nu_2(s^{-1}\bar{\eta}_{\psi}(\zeta, z, \lambda)) \wedge \omega_2(s^{-1}\zeta)$$

$$\wedge [\omega_1(s\bar{\eta}_{\psi}(\zeta, z, \lambda)) \wedge \omega_2(s\zeta) + \nu_1(s\bar{\eta}_{\psi}(\zeta, z, \lambda)) \wedge \omega_2(s\zeta)] \wedge 

\nu_2(s+1\bar{\eta}_{\psi}(\zeta, z, \lambda)) \wedge \omega_2(s+1\zeta) \wedge ... \wedge \nu_2(n\bar{\eta}_{\psi}(\zeta, z, \lambda)) \wedge \omega_2(n\zeta) \}. $$

If $f$ is a bounded differential form on $U$, then define:

$$\text{(8)} \quad (L_{\partial U}^\psi f)(z) := \int_{\zeta \in \partial U} f(\zeta) \wedge \phi_{\zeta, z}(\psi(\zeta, z); \zeta),$$

$$\text{(9)} \quad (R_{\partial U}^\psi f)(z) := \int_{\zeta \in \partial U, 0 \leq s \leq 1} f(\zeta) \wedge \phi_{\zeta, z}(\psi(\zeta, z); \zeta).$$

3.5. Theorem. Let $U$ be an open subset in $\mathbb{H}^n$ with a piecewise $C^1$ boundary and let $\psi$ be a quaternion boundary distinguishing map for $U$. Suppose that $f$ is a continuous mapping $f : \text{cl}(U) \rightarrow \mathbb{H}$ such that $\bar{\partial} f$ is also continuous on $U$ in the sense of distributions and has a continuous extension on $\text{cl}(U)$. Then

$$\text{(1)} \quad f = (L_{\partial U}^\psi f) - (R_{\partial U}^\psi \bar{\partial} f) - (B_U \bar{\partial} f) \text{ on } U,$$

where the quaternion integral operators $B_U, L_{\partial U}^\psi$ and $R_{\partial U}^\psi$ are given by Equations 3.1.(11), 3.4.(8), (9).

Proof. There is the decomposition:

$$\text{(2)} \quad \phi_{\zeta, z, \lambda} = \sum_{q=0}^{2n-1} Y_q^\psi(\zeta, z, \lambda),$$

where $Y_q^\psi(\zeta, z, \lambda)$ is a differential form with all terms of degree $q$ by $z$ and $\bar{z}$ and their multiples on quaternion constants and of degree $(4n - q - 1)$ by $(\zeta, \lambda)$ (including $\zeta$ and multiples of $\zeta$ and $\bar{\zeta}$ on quaternion constants). A differential form $f$ has Decomposition 3.2.(3). If $\psi(z)$ is a quaternion $z$-superdifferentiable nonzero function on an open set $V$ in $\mathbb{H}^n$, then differentiating the equality $(\psi(z))(d\psi(z))^{-1} = e$ gives $[d_1(\psi(z))^{-1}]. h = -(\psi(z))^{-1}(d_1\psi(z)). (\psi(z))^{-1}$ for each $z \in V$ and each $h \in \mathbb{H}$. Then

$$\int_{\zeta \in \partial U, 0 \leq \lambda \leq 1} f_r(\zeta) \wedge Y_q^\psi(\zeta, z, \lambda) = 0 \text{ for each } r \neq q + 1,$$

since $\dim(\partial U) = 4n - 1$, $d\lambda \wedge d\lambda = 0$ and $d\lambda$ commutes with each $b \in \mathbb{H}$. Therefore,

$$\text{(3)} \quad R_{\partial U}^\psi f_r = \int_{\zeta \in \partial U, 0 \leq \lambda \leq 1} f_r(\zeta) \wedge Y_{r-1}^\psi(\zeta, z, \lambda) \text{ for each } 1 \leq r \leq 2n \text{ and }$$

$R_{\partial U}^\psi f_r = 0$ for $r = 0$ or $r > 2n$. In particular, if $f = f_1$, then

$$\text{(4)} \quad R_{\partial U}^\psi f_1 = \int_{\zeta \in \partial U, 0 \leq \lambda \leq 1} f_1(\zeta) \wedge \phi_{\zeta, \lambda}(\psi(\zeta, z); \zeta),$$

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where \( \bar{\phi}_{\zeta, \lambda}(\psi(\zeta, z); \zeta) \), is obtained from \( \bar{\phi}_{\zeta, z, \lambda}(\psi(\zeta, z); \zeta) \) by substituting all \( \bar{\partial} s_{\zeta} \) in Formulas 3.4.(3 – 5, 7, 9) on \( \bar{\partial} s_{\zeta} \). On the other hand, with the help of Formulas 3.2.(8), (\( vi \)) each quaternion external derivative \( \bar{\partial} s_{\zeta} \) can be replaced on \( d s_{\zeta} \) in \( \phi_{\zeta, z}(\psi(\zeta, z); \zeta) \) in Formula (4). For \( \phi_{\zeta, z} \) there is the decomposition:

\[
\phi_{\zeta, z} = \sum_{q=0}^{2n-1} \Upsilon_q(\zeta, z), \quad \text{where } \Upsilon_q(\zeta, z) \text{ is a differential form with all terms of degree } q \text{ by } z \text{ and } \bar{z} \text{ and their multiples on quaternion constants and of degree } 4n - q - 1 \text{ by } \zeta \text{ and } \bar{z} \text{ and their multiples on quaternion constants. Therefore,}
\]

\[
(6) \quad L^\psi_{\partial U} f_r = \int_{\zeta \in \partial U} f_r \wedge \Upsilon_r(\zeta, z) \quad \text{for each } 0 \leq r \leq 2n - 1
\]

and \( L^\psi_{\partial U} f_r = 0 \) for \( r \geq 2n \). In particular, for \( f = f_0 \):

\[
(7) \quad L^\psi_{\partial U} f_0 = \int_{\zeta \in \partial U} f_0(\zeta) \phi_{\zeta}(\psi(\zeta, z); \zeta)
\]

where \( \phi_{\zeta}(\psi(\zeta, z)) \), is obtained from \( \phi_{\zeta, z}(\psi(\zeta, z); \zeta) \) by substituting all \( \bar{\partial} s_{\zeta} \) in Formulas 3.4.(3 – 5, 6, 8) on \( \bar{\partial} \zeta \).

In view of Formula 3.2.(1) it remains to prove, that \( R^\psi_{\partial U} \bar{\partial} f = L^\psi_{\partial U} f - B_{\partial U} f \) on \( U \). For each \( \zeta \) in a neighbourhood of \( \partial U \) there is the identity:

\[
(8) \quad < \eta^\psi(\zeta, z, \lambda), \zeta - z > = 1 \quad \text{for each } 0 \leq \lambda \leq 1, \text{ hence } d_{\zeta, z, \lambda} < \eta^\psi(\zeta, z, \lambda), \zeta - z > = 0. \quad \text{By Proposition 1.7.1 [9] and Formulas 3.2.(8), (9), (i – vi):}
\]

\[
(9) \quad d_{\zeta, \lambda} \bar{\phi}_{\zeta, z, \lambda} = 0. \quad \text{From Identities 3.2.(8), (\( vi \)) it follows, that}
\]

\[
(10) \quad \bar{\partial} f \wedge \bar{\phi}_{\zeta, \lambda} = 0. \quad \text{Therefore, from (4), (9), (10) it follows, that}
\]

\[
(11) \quad d_{\zeta, \lambda}[f(\zeta) \bar{\phi}_{\zeta, \lambda}] = [\bar{\partial} f(\zeta)] \wedge \bar{\phi}_{\zeta, \lambda}, \text{ since } \bar{\partial} f(\zeta) = \sum_{s=1}^{n} (\partial f(\zeta, \zeta)) / \partial s. \quad \text{In view of Proposition 1.7.3 [9] and Formulas 3.2.(8), (9), (i – vi); 3.4.(1 – 7)}:
\]

\[
(12) \quad \bar{\phi}_{\zeta, \lambda, \lambda} = 0 = \phi_{\zeta}, \quad \bar{\phi}_{\zeta, \lambda, \lambda} = 1 = \theta_{\zeta}(\zeta).
\]

From the Stoke’s formula for matrix-valued differential forms, in particular, for \( \left[ f(\zeta) \bar{\phi}_{\zeta, z, \lambda}(\psi(\zeta, z); \zeta) \right] \) on \( \partial U \times [0, 1] \) and Formulas (4), (7), (11), (12) above it follows the statement of this theorem.

3.6. Corollary. Let conditions of Theorem 3.5 be satisfied and let \( f \) be a quaternion holomorphic function on \( U \), then \( f = L^\psi_{\partial U} f \) on \( U \).

3.7. Remark. For \( n = 1 \) Formula 3.2.(1) produces another analog of the Cauchy-Green formula (see Theorem 2.1 and Remark 2.1.1) without using the quaternion line integrals. This is caused by the fact that the dimension of \( H \) over \( R \) is greater, than 2: \( \text{dim}_R H = 4 \), that produces new integral relations. Theorem 3.2 can be used instead of Theorem 2.1 to prove theorems 2.3 and 2.11 (with differential forms of Theorem 3.2 instead of differential forms of Theorem 2.1). If \( \psi(\zeta, z) = \zeta - z \), then \( L^\psi_{\partial U} = B_{\partial U} \) and \( R^\psi_{\partial U} = 0 \),
hence Formula 3.5(1) reduces to Formula 3.2.(1). For a function \( f \) or a 1-form \( \tilde{f} \) Formulas 3.2.(4), (5) respectively are valid as well for \( \tilde{\theta}(\zeta, z) \) instead of \( \theta(Z, \zeta) \), where \( d_{\zeta,z} \tilde{\theta}(\zeta, z) = 0 \) for each \( \zeta \neq z \). A choice of \( \omega_4 \) (see 3.1.(5)) and the corresponding to it \( \omega_1, \omega_2, \omega_3 \) is not unique, for example, \( d_{\zeta} \Lambda d_{\zeta} \Lambda d_{\zeta} \Lambda d_{\zeta} = 0 \). Formulas 3.2.(1) and 3.5.(1) for functions of quaternion variables are the quaternion analogs of the Martinelli-Bochner and the Leray formulas for functions of complex variables respectively, where \( \psi(\zeta, z) \) is the quaternion analog of the Leray complex map (see §3.4). In the quaternion case the algebra of differential forms bears the additional gradation structure and have another properties, than in the complex case (see also §§2.8 and 3.7 [22]). Lemma 3.9 below shows, that quaternion boundary distinguishing maps exist.

### 3.8. Definitions and Notations

Let a subset \( U \) in \( \mathbb{H}^n \) be given by:

1. \( U := \{z \in \mathbb{H}^n : \rho(z) < 0\} \), where \( \rho \) is a real-valued function such that there exists a constant \( \epsilon_0 > 0 \) for which:
   
   \[ \sum_{l,m=1}^{4n}(\partial^2\rho(z)/\partial x_l \partial x_m)t_l t_m \geq \epsilon_0 |t|^2 \]  
   
   for each \( t \in \mathbb{R}^{4n} \), where \( z = (1, z, \ldots, n, z) \), \( t_z \in \mathbb{H} \), \( t_z = \sum_{l,m=1}^{4n}x_{4(l-1)+m}S_m \), \( S_1 := e \), \( S_2 := i \), \( S_3 := j \), \( S_4 := k \), \( x_t \in \mathbb{R} \). Then \( U \) is called a strictly convex open set (with \( C^2 \)-boundary). Let

2. \( w_\rho(z) := (\partial \rho(z)/\partial 1, z, \ldots, \partial \rho(z)/\partial n, z) \) and \( v_\rho(z) := \sum_{m=1}^{4n}(w_\rho.S_m)S_m \), where as usually \( w_\rho.S_m = (dz(\rho(z))).S_m \) is the differential of \( \rho \).

### 3.9. Lemma

Let the function \( v_\rho \) be as in §3.8. Then \( v_\rho \) is the quaternion boundary distinguishing map for \( U \).

**Proof.** Since \( S_mS_l = (-1)^{\kappa(S_m)+\kappa(S_l)}S_lS_m \), where \( \kappa(S_1) = 0 \), \( \kappa(S_2) = \kappa(S_3) = \kappa(S_4) = 1 \), then

\[ < v_\rho(\zeta) ; \zeta - z > = \rho(\zeta) / \rho(z) = 2 \sum_{l,m=1}^{4n}(\partial \rho(z)/\partial x_l) x_l(\zeta - z) \]

where \( x_l = x_l(\zeta) \) and \( x_l(\zeta - z) \) are real coordinates corresponding to \( \zeta \) and \( \zeta - z \). By the Taylor’s theorem: \( \rho(z)e = \rho(z) - \rho(z) / 2 - < \zeta - z; v_\rho(\zeta) >/2 + \sum_{l,m=1}^{4n}(\partial^2 \rho(\zeta)/\partial x_l \partial x_m) x_l(\zeta - z) x_m(\zeta - z)e/2 + o(\zeta - z)^2)e \)

Therefore, there exists a neighbourhood \( V \) of \( \partial U \) and \( \epsilon_1 > 0 \) such that

1. \( (< v_\rho(\zeta) ; \zeta - z > + < \zeta - z; v_\rho(\zeta) >) / 2 \geq -\rho(z) + \epsilon_0 |\zeta - z|^2 / 4 \) for each \( \zeta \in V \) and \( |\zeta - z| \leq \epsilon_1 \), where \( a = a_ee + a_ii + a_jj + a_kk \) for each \( a \in \mathbb{H} \), \( a_ee, a_ii, a_jj \) and \( a_kk \) are reals. If \( z \in U \), \( \zeta \in \partial U \), \( |z - \zeta| \leq \epsilon_1 \), then by

1. \( (< v_\rho(\zeta) ; \zeta - z > + < \zeta - z; v_\rho(\zeta) >) / 2 \geq -\rho(z) > 0 \) if \( |z - \zeta| > \epsilon_1 \), put \( z_1 := (1 - \epsilon_1|\zeta - z|^{-1})(\zeta + \epsilon_1|\zeta - z|^{-1}z) \), then \( \zeta - z_1 = \epsilon_1|\zeta - z|^{-1}(\zeta - z) \),
consequently, \( <v_\rho(\zeta); \zeta - z > + <\zeta - z; v_\rho(\zeta) >)_e/2 \geq -\rho(z_1) \). Evidently, \( U \) is convex and \( z_1 \in U \).

3.10. Theorem. Let \( U \) be a strictly convex open subset in \( \mathbb{H}^n \) (see 3.8.(1)) and let \( f \) be a continuous function on \( U \) with continuous \( \tilde{\partial}f \) on \( U \) in the sense of distributions having a continuous extension on \( \text{cl}(U) \) such that 2.11.(i) is satisfied. Then there exists a function \( u \) on \( U \) which is a solution of the \( \tilde{\partial} \)-equation 2.11.(ii).

Proof. In proofs of Theorems 2.3 and 2.11 take in Formula 3.5.(1) \( \chi f \) instead of \( f \), which is possible due to Lemma 3.9, choosing \( \psi = v_\rho \) and \( \text{supp}(\chi) \) as a proper subset of \( U \). Then \( L_{\partial U}^s \chi f = 0 \) and \( B_{\partial U}^s \chi f = 0 \), hence \( \chi f = -B_U \tilde{\partial} \chi f \). For each fixed \( z \in U \) a subset \( ^1U_\eta := \{ \xi \in \mathbb{H} : \rho(1z, ..., l-1z, \xi, l+1z, ..., nz) < 0 \} \) is strictly convex in \( \mathbb{H} \) due to 3.8.(1,2), where \( \eta := (1z, ..., l-1z, l+1z, ..., nz) \). Apply 3.5.(1) by a variable \( \xi \) in \( ^1U_\eta \), in particular, for \( l = 1 \), for which \( v_\rho \) by the variable \( \xi \) is the quaternion boundary distinguishing map for \( ^1U_\eta \). Therefore, \( u(z) := -B_{^1U_\eta} \chi f(\xi, \eta), d\xi \) with \( z = (\xi, \eta), \xi \in ^1U_\eta \) solves the problem.

4 Quaternion manifolds

4.1. Definitions and Notations. Suppose that \( M \) is a quaternion manifold and let \( GL(N, \mathbb{H}) \) be the group of all invertible quaternion \( N \times N \) matrices. Then a quaternion holomorphic vector bundle \( Q \) of quaternion dimension \( N \) over \( M \) is a \( C^\infty \)-vector bundle \( Q \) over \( M \) with the characteristic fibre \( \mathbb{H}^n \) together with a quaternion holomorphic atlas of local trivializations: \( g_{a,b} : U_a \cap U_b \rightarrow GL(N, \mathbb{H}) \), where \( U_a \cap U_b \neq \emptyset \), \( \{(U_a, h_a) : a \in \Upsilon\} = At(Q) \), \( \cup_a U_a = M \), \( U_a \) is open in \( M \), \( h_a : Q|_{U_a} \rightarrow U_a \times \mathbb{H}^n \) is the bundle isomorphism, \( (z, g_{a,b}(z)v) = h_a \circ h_b^{-1}(z, v), z \in U_a \cap U_b, v \in \mathbb{H}^n \). Since \( M \) is also the real manifold there exists the tangent bundle \( TM \) such that \( T_x M \) is isomorphic with \( \mathbb{H}^n \) for each \( x \in M \), since \( TU_a = U_a \times \mathbb{H}^n \) for each \( a \), where \( \text{dim}_H M = n \) is the quaternion dimension of \( M \). If \( X \) is a Banach space over \( \mathbb{H} \) (with left and right distributivity laws relative to multiplications of vectors in \( X \) on scalars from \( \mathbb{H} \)), then denote by \( X^*_q \) the space of all additive \( \mathbb{R} \)-homogeneous functionals on \( X \) with values in \( \mathbb{H} \). Clearly \( X^*_q \) is the Banach space over \( \mathbb{H} \). Then \( T^* M \) with fibres \( (\mathbb{H}^n)^* \) denotes the quaternion cotangent bundle of \( M \) and \( \Lambda^r T^* M \) denotes the vector bundle whose sections are quaternion \( r \)-forms on \( M \), where \( S_b dx_b \wedge S_a dx_a = -(-1)^{(k(S_a)))(k(S_b))} S_a dx_a \wedge S_b dx_b \) for each
$S_a, S_b \in \{e, i, j, k\}$, $dz = cdx_e + idx_i + jdx_j + kdx_k$, $z \in \mathbb{H}$, $x_b \in \mathbb{R}$.

Quaternion holomorphic Cousin data in $Q$ is a family $\{f_{a,b}: a, b \in \mathbb{Y}\}$ of quaternion holomorphic sections $f_{a,b}: U_a \cap U_b \to Q$ such that $f_{a,b} + f_{b,l} = f_{a,l}$ in $U_a \cap U_b \cap U_l$ for each $a, b, l \in \mathbb{Y}$. A finding of a family $\{f_a: a \in \mathbb{Y}\}$ of quaternion holomorphic sections $f_a: U_a \to Q$ such that $f_{a,b} = f_a - f_b$ in $U_a \cap U_b$ for each $a, b \in \mathbb{Y}$ will be called the quaternion Cousin problem.

4.2. Theorem. Let $M$ be a quaternion manifold and $Q$ be a quaternion holomorphic vector bundle on $M$. Then Conditions (i, ii) are equivalent:

(i) each quaternion holomorphic Cousin problem in $M$ has a solution;

(ii) for each quaternion holomorphic section $f$ of $Q$ such that $\partial f = 0$ on $M$, there exists a $C^\infty$-section $U$ of $Q$ such that $(\partial u/\partial \overline{z}) = \hat{f}$ on $M$.

Proof. (i) \(\to\) (ii). In view of Theorems 2.11 and 3.10 there exists an (open) covering $\{U_a: a\}$ of $M$ and $C^\infty$-sections $u_b: U_b \to Q$ such that $(\partial u_b/\partial \overline{z}) = \hat{f}$ in $U_b$. Then $(u_b - u_l)$ is quaternion holomorphic in $U_b \cap U_l$ and their family forms quaternion holomorphic Cousin data in $Q$. Put $u := u_b - h_b$ on $U_b$, where $u_b - u_l = h_b - h_l$, $h_b: U_b \to Q$ is a quaternion holomorphic section given by (i).

(ii) \(\to\) (i). Take a $C^\infty$-partition of unity $\{\chi_b: b\}$ subordinated to $\{U_b: b\}$ and $c_b := -\sum_a \chi_a f_{a,b}$ on $U_b$, then $f_{l,b} = \sum_a \chi_a(f_{l,a} + f_{a,b}) = c_l - c_b$ in $U_l \cap U_b$, hence $(\partial c_l/\partial \overline{z}) = (\partial c_b/\partial \overline{z})$ in $U_l \cap U_b$. By (ii) there exists a $C^\infty$-section $u: M \to Q$ with $(\partial u/\partial \overline{z}) = (\partial c_b/\partial \overline{z})$ on $U_b$ and $h_b := c_b - u$ on $U_b$ gives the solution.

4.3. Definitions. Suppose $U$ is an open subset in $\mathbb{H}$, then a $C^2$-function $\rho: U \to \mathbb{R}$ is called subharmonic (strictly subharmonic) in $U$ if $\Sigma^4_{m=1} \partial^2 \rho/\partial x^2_m \geq 0$ ($\Sigma^4_{m=1} \partial^2 \rho/\partial x^2_m > 0$ correspondingly) for each $z = x_1e + x_2i + x_3j + x_4k \in U$, where $x_1, ..., x_4 \in \mathbb{R}$. If $U$ is an open subset in $\mathbb{H}^n$, then a $C^2$-function $\rho: U \to \mathbb{R}$ such that the function $\zeta \mapsto \rho(v + \zeta w)$ is subharmonic (strictly subharmonic) on its domain for each $v, w \in \mathbb{H}^n$ is called plurisubharmonic (strictly plurisubharmonic correspondingly) function, where $\zeta \in \mathbb{H}$.

A $C^p$-function $\rho$ on a quaternion manifold $M$ is called a strictly plurisubharmonic exhausting $C^p$-function for $M$, $2 \leq p \in \mathbb{N}$, if $\rho$ is a strictly plurisubharmonic $C^p$-function on $M$ and for each $\alpha \in \mathbb{R}$ the set $\{z \in M: \rho(z) < \alpha\}$ is relatively compact in $M$.

4.4. Theorem. Let $M$ be a quaternion manifold with strictly plurisubharmonic exhausting function $\rho$ such that $\rho e$ is a $C^\infty_{z,\overline{z}}$-function and let $Q$ be a quaternion holomorphic vector bundle on $M$, $U_\alpha := \{z \in M: \rho(z) < \alpha\}$
for $\alpha \in \mathbb{R}$.

(i). Suppose that $d \rho(z) \neq 0$ for each $z \in \partial U_\alpha$ for a marked $\alpha \in \mathbb{R}$. Then every continuous section $f : cl(U_\alpha) \to Q$ quaternion holomorphic on $U_\alpha$ can be approximated uniformly on $cl(U_\alpha)$ by quaternion holomorphic sections of $Q$ on $M$.

(ii). For each continuous mapping $f : M \to Q$ such that $\bar{\partial} f = 0$ on $M$ there exists a continuous mapping $u : M \to Q$ such that $\partial u/\partial \bar{z} = \hat{f}$ on $M$.

**Proof.** For a $C^{\omega}_{z,\bar{z}}$-function $\rho : U \to \mathbb{R}$ (that is, $\rho$ is locally analytic in variables $(z, \bar{z})$, $\mathbb{R} = \mathbb{R} \hookrightarrow \mathbb{H}$) there is the identity:

$$\sum_{t,m,a,b}(\partial^2 \rho/\partial^l x_a \partial^m x_b) t_4(l-1)+a t_4(l-1)+b = \sum_{m,l=1}^n(\partial^2 \rho(z)/\partial^l z \partial^m z).((\partial^l z/\partial^l x_a) t_4(l-1)+a, (\partial^m z/\partial^l x_b) t_4(l-1)+b)$$

$$= \sum_{m,l=1}^n(\partial^2 \rho(z)/\partial^l z \partial^m z).1^{l \xi, m \xi}, \quad \text{since} \quad \partial^l z/\partial^l x_a = S_a, \partial^l z/\partial^l x_a = (-1)^{\kappa(S_a)} S_a$$

where $1^{l \xi} = \sum_{a=1}^a t_4(l-1)+a S_a, S_1 = e, S_2 = i, S_3 = j, S_4 = k$,

(i) $\rho - \rho_\epsilon$ together with its first and second derivatives is not greater than $\epsilon$ on $M$;

(ii) the set Crit($\rho_\epsilon$) := $\{ z \in M : d \rho_\epsilon(z) = 0 \}$ is discrete in $M$;

(iii) $\rho_\epsilon = \rho$ on $A$ (see also Lemma 2.1.2.2 [9] in the complex case).

The space $C^\omega(U, \mathbb{H})$ is dense in $C^0(U, \mathbb{H})$ for each open $U$ in $\mathbb{H}^n$ (see §2.7 and Theorem 3.28 in [22]). Suppose $\beta \in \mathbb{R}$ and $d \rho(z) \neq 0$ for $z \in \partial U_\beta$ and $f : cl(U_\beta) \to Q$ is a continuous section quaternion holomorphic on $U_\beta$. Therefore, for each $\beta \leq \alpha < \infty$ if $d \rho(z) \neq 0$ for each $z \in \partial U_\alpha$, then $f$ can be uniformly approximated on $cl(U_\beta)$ by continuous sections on $cl(U_\alpha)$ that are holomorphic on $U_\alpha$. There exists a sequence $\beta < \alpha_1 < \alpha_2 < ...$ such that $\lim \alpha_l = \infty$ and $d \rho(z) \neq 0$ for each $z \in \partial U_{\alpha_l}$, since Crit($\rho$) is discrete. For each $\epsilon > 0$ there exists a continuous section $f_l : cl(U_{\alpha_l}) \to Q$ such that $f_l$ is quaternion holomorphic on $U_{\alpha_l}$ and $\|f_{l+1} - f_l\|_{C^0(U_{\alpha_l})} < 2^{\epsilon-l-1}$ for each $l \in \mathbb{N}$, where $f_0 := f$. Therefore, the sequence $\{f_l : l \in \mathbb{N}\}$ converges to the quaternion holomorphic section $g : M \to Q$ uniformly on each compact subset $P$ in $M$ and $\|f - g\|_{C^0(U_\beta)} < \epsilon$. 

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The second statement \((ii)\) follows from \((i)\) and Theorems 2.11, 3.10, since \(\text{Crit}(\rho)\) is discrete in \(M\) and there exists a sequence of continuous \(Q\)-valued functions on \(\text{cl}(U_{\alpha})\) such that \(\partial u_l/\partial z = \hat{f}\) on \(U_{\alpha}\), \(\bigcup U_{\alpha} = M\) (see also the complex case in §2.12.3 [9] mentioning, that Lemma 2.12.4 there can be reformulated and proved for a quaternion manifold \(M\) on \(\mathbb{H}^n\) instead of a complex manifold on \(\mathbb{C}^n\)).

4.5. Definitions. Let \(M\) be a quaternion manifold (see §2.10). For a compact subset \(G\) in \(M\) put: \(\hat{G}_M^H := \{z \in M : |f(z)| \leq \sup_{\zeta \in G} |\hat{f}(\zeta)| \forall f \in \mathcal{H}(M)\}\). Such \(\hat{G}_M^H\) is called the \(\mathcal{H}(M)\)-hull of \(G\). If \(G = \hat{G}_M^H\), then \(G\) is called \(\mathcal{H}(M)\)-convex. A quaternion manifold \(M\) is called quaternion holomorphically convex if for each compact subset \(G\) in \(M\) the set \(\hat{G}_M^H\) is compact.

A quaternion manifold \(M\) with a countable atlas \(At(M)\) having dimension \(n\) over \(\mathbb{H}\) and satisfying \((i, ii)\):

\((i)\) \(M\) is quaternion holomorphically convex;

\((ii)\) for each \(z \in M\) there are \(1f, ..., n f \in \mathcal{H}(M)\) and there exists a neighbourhood \(U\) of \(z\) such that the map \(U \ni \zeta \mapsto (1f(\zeta), ..., nf(\zeta))\) is quaternion biholomorphic (see §2.6), then \(M\) is called a quaternion Stein manifold.

4.6. Remark. If \(M_1\) and \(M_2\) are two quaternion Stein manifolds, then \(M_1 \times M_2\) is a quaternion Stein manifold. If \(N\) is a closed quaternion submanifold of a quaternion Stein manifold \(M\), then \(N\) is also a quaternion Stein manifold.

4.7. Theorem. Let \(M\) be a quaternion Stein manifold. Then for each \(\mathcal{H}(M)\)-convex compact subset \(P\) in \(M\), \(P \neq M\) and each neighbourhood \(V_P\) of \(P\) there exists a strictly plurisubharmonic exhausting \(C^\infty_{\zeta, \bar{\zeta}}\)-function \(\rho\) on \(M\) such that \(\rho < 0\) on \(P\) and \(\rho > 0\) on \(M \setminus V_P\).

The proof of this theorem is analogous to that of Theorem 2.3.14 [9] in the complex case taking \(\rho(z) := -1 + \sum_{l=1}^\infty \sum_{k=1}^{N(l)} f_l^k(z)(f_l^k(z))^*\) for each \(z \in M\), where \(f_l^k \in C^\infty_{\zeta, \bar{\zeta}}\), \(M = \bigcup P_l\), \(P_l \subset \text{Int}(P_{l+1})\) for each \(l \in \mathbb{N}\), each \(P_l\) is \(\mathcal{H}(M)\)-convex, \(\sum_{k=1}^{N(l)} |f_l^k(z)|^2 < 2^{-l}\) for each \(z \in P_l\), \(\sum_{k=1}^{N(l)} |f_l^k(z)|^2 > l\) for each \(z \in P_{l+2} \setminus U_l\), \(U_l := \text{Int}(P_{l+1})\), \(\text{rank}[(\partial f_l^k/\partial z_m)]_{m=1, ..., n} = n\) for each \(z \in P_l\).

4.8. Theorem. A quaternion manifold \(M\) is a quaternion Stein manifold if and only if there exists a strictly plurisubharmonic exhausting \(C^\infty_{\zeta, \bar{\zeta}}\)-function \(\rho\) on \(M\), then \(\{z \in M : \rho(z) \leq \alpha\}\) is \(\mathcal{H}(M)\)-convex for each \(\alpha \in \mathbb{R}\).

Proof. The necessity follows from Theorem 4.7. To prove sufficiency
suppose \( \eta = (\eta^1, ..., \eta^n) \) are quaternion holomorphic coordinates in a neighbourhood \( V_\xi \) of \( \xi \in M \). Consider

\[
(1) \quad u(z) := 2 \sum_{l=1}^{n} <\rho, (\eta^l - \eta^l(\xi)) > + \sum_{l=1}^{n} (\partial^2 \rho(\xi)/\partial \eta^l \partial \eta^l) \eta^l (\eta^l(\xi) - \eta_l^l(\eta)) + o(\eta^l(\xi) - \eta_l^l(\eta))^2)
\]

where \( v_{\rho}(\xi) \) is given by 3.8.(3). Then \( u \) is holomorphic in \( V_\xi \) and \( u(\xi) = 0 \).

By Lemma 3.9:

\[
(2) \quad (u(z) + \bar{u}(z))/2 = \rho(z) \rho(\xi) - 2 \sum_{l=1}^{n} (\partial^2 \rho(\xi)/\partial \eta^l \partial \eta^l) \eta^l (\eta^l(z) - \eta_l^l(\eta)) + o(\eta^l(\xi) - \eta_l^l(\eta))^2). \]

From the strict plurisubharmonicity of \( \rho \) it follows, that there exists \( \beta > 0 \) and \( V_\xi \) such that

\[
(3) \quad (u(z) + \bar{u}(z))/2 < \rho(z) - \rho(\xi) - \beta |\eta(z) - \eta(\xi)|^2 \text{ for each } z \in V_\xi. \]

Then \( \exp(u(\xi)) = 1 \) and \( |\exp(u(z))| < 1 \) for each \( \xi \neq z \in cl(U_\alpha) \cap V_\xi \) (see Corollary 3.3 [22]).

If \( g : \mathbb{R} \to H \) is a \( C^\infty \)-function with compact support, then \( g(z\bar{z}) =: \chi(z) \) is a \( C^\infty \)-function on \( H^n \) with compact support such that \( \chi \) is quaternion \((z, \bar{z})\)-superdifferentiable. Therefore, there exists a neighbourhood \( W_\xi \subset V_\xi \) of \( \xi \) and an infinitely \((z, \bar{z})\)-superdifferentiable function \( \chi \) such that \( \chi(W_\xi) = 1 \), \( supp(\chi) \) is a proper subset of \( V_\xi \), consequently, \( \lim_{\beta \to \infty} \| \exp(mu(z))(\partial \chi(z)/\partial \bar{z}) \|_{C^0(U_\alpha)} = 0 \), where \( (\partial \chi(z)/\partial \bar{z}) = (\partial \chi(z)/\partial \bar{z}, ..., \partial \chi(z)/\partial \bar{z}) \). In view of Theorem 3.10 there exist continuous functions \( v_m \) on \( cl(U_\alpha) \) such that \( (\partial v_m/\partial \bar{z}) = \exp(mu(z))(\partial \chi(z)/\partial \bar{z}) \) in \( U_\alpha \) and \( \lim_{\beta \to \infty} \| v_m \|_{C^0(U_\alpha)} = 0 \).

Put \( g_m(z) := \exp(mu(z))(\chi(z) - v_m(z) + v_m(\xi)) \), hence \( g_m \) is continuous on \( cl(U_\alpha) \) and quaternion holomorphic on \( U_\alpha \). Since \( supp(\chi) \) is the proper subset in \( V_\xi \), then \( g_m(\xi) = 1 \) for each \( m \in \mathbb{N} \), \( sup \| g_m \|_{C^0(U_\alpha)} < \infty \) and for each compact subset \( P \) in \( cl(U_\alpha) \setminus \{\xi\} \) there exists \( \lim_{\beta \to \infty} \| g_m \|_{C^0(P)} = 0 \).

In view of Theorem 4.4.(ii) there exists a sequence of functions \( f_m \in \mathcal{H}(M) \) and \( C = const < \infty \) such that \( a \) \( f_m(\xi) = 1 \) for each \( m \in \mathbb{N} \); \( b \) \( \| f_m \|_{C^0(U_\alpha)} \leq C \) for each \( m \in \mathbb{N} \); \( c \) \( \lim_{\beta \to \infty} \| g_m \|_{C^0(P)} = 0 \) for each compact subset \( P \subset cl(U_\alpha) \setminus \{\xi\} \).

Consider a quaternion holomorphic function \( f \) on a neighbourhood of \( \xi \) such that \( f(\xi) = 0 \). Put \( \phi_m := f \exp(mu)\partial \chi/\partial \bar{z} \), then \( supp(\phi_m) \) is the proper subset in \( V_\xi \setminus W_\xi \). In view of Inequality (3) there exists \( \delta > 0 \) such that \( \lim_{\beta \to \infty} \| \phi_m \|_{C^0(U_{\alpha+\delta})} = 0 \). As in §4.4 it is possible to assume, that \( \text{Crit}(\rho) \) is discrete in \( M \). Take \( 0 < \epsilon < \delta \) such that \( \partial \rho \neq 0 \) on \( \partial U_{\alpha+\epsilon} \). In view of Theorem 4.4.(ii) there exists a continuous function \( v_m \) on \( cl(U_{\alpha+\epsilon}) \) such that \( \partial v_m/\partial \bar{z} = \hat{\phi}_m \) on \( U_{\alpha+\epsilon} \) and \( \lim_{\beta \to \infty} \| v_m \|_{C^0(U_{\alpha+\epsilon})} = 0 \). Each \( v_m \) is quaternion holomorphic on \( W_\xi \), since \( \phi_m = 0 \) on \( W_\xi \), hence \( \lim_{\beta \to \infty} \partial v_m(\xi) = 0 \). Since \( f(\xi) = u(\xi) = 0 \) and \( \chi = 1 \) on \( W_\xi \), then \( \partial g_m(\xi)/\partial x = \partial f(\xi)/\partial x \). In view of
Theorem 4.4. 

There exists $f_m \in \mathcal{H}(M)$ such that $\|f_m - g_m\|_{C^0(U_{a+\delta})} < m^{-1}$ and inevitably $\lim_m \|\partial f_m(\xi)/\partial \xi - \partial g_m(\xi)/\partial \xi\| = 0$.

Let $V_\xi$ and $W_\xi$ be as above, then there exists $\delta > 0$ such that $(u(z) + \tilde{u}(z))/2 < -\delta$ for each $z \in U_{a+\delta} \cap (V_\xi \setminus W_\xi)$. Therefore, there exists a branch of the quaternion logarithm $Ln(u) \in \mathcal{H}(U_{a+\delta} \cap (V_\xi \setminus cl(W_\xi)))$ (see §§3.7, 3.8 [22]). From Theorems 4.2, 4.4 it follows that each quaternion holomorphic Cousin problem over $U_{a+\delta}$ has a solution. Hence $Ln(u) = w_1 - w_2$ for suitable $w_1 \in \mathcal{H}(V_\xi \cap U_{a+\delta})$ and $w_2 \in \mathcal{H}(U_{a+\delta} \setminus cl(W_\xi))$. Put $f := u \exp(-w_1)$ in $U_{a+\delta} \cap V_\xi$ and $f := \exp(-w_2)$ in $U_{a+\delta} \setminus cl(W_\xi)$. Then $f \in \mathcal{H}(U_{a+\delta})$ and $f(\xi) = 0$. In view of Inequality (3) $f(z) \neq 0$ for each $\xi \neq z \in cl(U_\alpha)$. Verify now that $cl(U_\alpha)$ is $\mathcal{H}(M)$-convex. Consider $\xi \in M \setminus cl(U_\alpha)$. Due to §4.4 there exists a strictly plurisubharmonic exhausting $C^\omega_{1,1}$-function $\psi$ for $M$ such that $Crit(\psi)$ is discrete and $U_\alpha \subset G(\psi,\xi)$, where $G_\beta := \{z \in M : \psi(z) < \beta\}$ for $\beta \in \mathbb{R}$. Considering shifts $\psi \mapsto \psi + const$ assume $d\psi(z) \neq 0$ for each $z \in \partial G(\psi,\xi)$. From the proof above it follows, that there exists $f \in \mathcal{H}(M)$ such that $f(\xi) = 1$ and $|f(z)| < 1$ for each $z \in cl(U_\alpha)$.

4.8.1. Remark. With the help of Theorem 4.8 it is possible to spread certain modifications of Theorems 3.2 and 3.5 on quaternion Stein manifolds.

4.9. Theorem. Let $N$ be a complex manifold, then there exists a quaternion manifold $M$ and a complex holomorphic embedding $\theta : N \hookrightarrow M$.

Proof. Suppose $At(N) = \{(V_a, \psi_a) : a \in \Lambda\}$ is any holomorphic atlas of $N$, where $V_a$ is open in $N$, $\bigcup_a V_a = N$, $\psi_a : V_a \rightarrow \psi_a(V_a) \subset \mathbb{C}^n$ is a homeomorphic isomorphism for each $a, n = dim_C M \in \mathbb{N}$, $\{V_a : a \in \Lambda\}$ is a locally finite covering of $N$, $\psi_b \circ \psi_a^{-1}$ is a holomorphic function on $\psi_a(V_a \cap V_b)$ for each $a, b \in \Lambda$ such that $V_a \cap V_b \neq \emptyset$. For each complex holomorphic function $f$ on an open subset $V$ in $\mathbb{C}^n$ there exists a quaternion holomorphic function $F$ on an open subset $U$ in $\mathbb{H}^n$ such that $\pi_{1,1}(U) = V$ and $F_{1,1}|_V = f|_V$, where $\pi_{1,1} : \mathbb{R}^n e \oplus \mathbb{R}^n i \oplus \mathbb{R}^n j \oplus \mathbb{R}^n k \rightarrow \mathbb{R}^n e \oplus \mathbb{R}^n i = \mathbb{C}^n$ is the projection, $F_{1,1} := \pi_{1,1} \circ F$ (see Proposition 3.13 [22] and use a locally finite covering of $V$ by balls). Therefore, for each two charts $(V_a, \psi_a)$ and $(V_b, \psi_b)$ with $V_{a,b} := V_a \cap V_b \neq \emptyset$ there exists $U_{a,b}$ open in $\mathbb{H}^n$ and a quaternion holomorphic function $\Psi_{b,a}$ such that $\Psi_{b,a}|_{\psi_a(V_{a,b})} = \psi_{b,a}|_{\psi_a(V_{a,b})}$, where $\psi_{b,a} := \psi_{b} \circ \psi_{a}^{-1}$, $\pi_{1,1}(U_{a,b}) = \psi_a(V_{a,b})$. Consider $Q := \bigoplus_a Q_a$, where $Q_a$ is open in $\mathbb{H}^n$, $\pi_{1,1}(Q_a) = \psi_a(V_a)$ for each $a \in \Lambda$. The equivalence relation $C$ in the topological space $\bigoplus_a \psi_a(V_a)$ generated by functions $\psi_{b,a}$ has an extension to the equivalence relation $\mathcal{H}$ in $Q$. Then $M := Q/\mathcal{H}$ is the desired quaternion manifold with $At(M) = \{(\Psi_a, U_a) : a \in \Lambda\}$ such that $\Psi_a \circ \Psi_{a}^{-1} = \Psi_{b,a}$ for each $U_a \cap U_b \neq \emptyset$, $\Psi_{a}^{-1}|_{\psi_a(V_a)} = \psi_{a}^{-1}|_{\psi_a(V_a)}$ for
each $a, \Psi_a^{-1} : Q_a \to U_a$ is the quaternion homeomorphism. Moreover, each homeomorphism $\psi_a : V_a \to \psi_a(V_a) \subset \mathbb{C}^n$ has the quaternion extension up to the homeomorphism $\Psi_a : U_a \to \Psi_a(U_a) \subset \mathbb{H}^n$. The family of embeddings $\eta_a : \psi_a(V_a) \to Q_a$ such that $\pi_{1,1} \circ \eta_a = id$ together with $At(M)$ induces the complex holomorphic embedding $\theta : N \to M$.

4.10. **Definition.** Let $M$ be a quaternion manifold. Suppose that for each chart $(U_a, \phi_a)$ of $At(M)$ there exists a quaternion superdifferentiable mapping $\Gamma : u \in \phi_a(U_a) \mapsto \Gamma(u) \in L_q(X, X, X^*_q; H) = L_q(X, X; X)$, where $L_q(X^n, (X^*_q)^m; Y)$ denotes the space of all quasi-linear mappings from $X^n \times (X^*_q)^m$ into $Y$ (that is, additive and $\mathbb{R}$-homogeneous by each argument $x$ in $X$ or in $X^*_q$), where $X$ and $Y$ are Banach spaces over $\mathbb{H}, X^*_q$ denotes the space of all quasi-linear functionals on $X$ with values in $\mathbb{H}$ (see §4.1), $X^*_q = L_q(X; H)$.

If $U_a \cap U_b \neq \emptyset$, let

1. $D(\phi_b \circ \phi_a^{-1}).\Gamma(\phi_a) = D^2(\phi_b \circ \phi_a^{-1}) + \Gamma(\phi_b) \circ (D(\phi_b \circ \phi_a^{-1}) \times D(\phi_b \circ \phi_a^{-1}))$.

These $\Gamma(\phi_a)$ are called the Christoffel symbols. Let $\mathcal{B} = \mathcal{B}(M)$ be a family of all quaternion holomorphic vector fields on $M$. For $M$ supplied with $\{\Gamma(\phi_a) : a\}$ define a covariant derivation $(X, Y) \in \mathcal{B}^2 \mapsto \nabla_X Y \in \mathcal{B}$:

2. $\nabla_X Y(u) = DY(u).X(u) + \Gamma(u)(X(u), Y(u))$, where $X(u)$ and $Y(u)$ are the principal parts of $X$ and $Y$ on $(U_a, \phi_a), u = \phi_a(z), z \in U_a$. In this case it is said that $M$ possesses a covariant derivation.

4.11. **Remark.** Certainly for a quaternion manifold there exists a neighbourhood $V$ of $M$ in $TM$ such that $\exp : V \to M$ is quaternion holomorphic (see the real case in [12]).

4.12. **Theorem.** Let $f$ be a quaternion holomorphic function such that $\hat{f}$ is quaternion (right) superlinear on a compact quaternion manifold $M$, then $\hat{f}$ is constant on $M$.

**Proof.** By the supposition of this theorem $(f \circ \phi_b^{-1})$ is quaternion (right) superlinear for each chart $(U_b, \phi_b)$ of $M$. Since $M$ is compact and $|f(z)|$ is continuous, then there exists a point $q \in M$ at which $|f(z)|$ attains its maximum. Let $q \in U_b$, then $f \circ \phi_b^{-1}$ is the quaternion holomorphic function on $V_b := \phi_b(U_b) \subset \mathbb{H}^n$, where $\dim_{\mathbb{H}} M = n$. Consider a polydisk $V$ in $\mathbb{H}$ with the centre $y = \phi_b(q)$ such that $V \subset V_b$. Put $g(w) = f \circ \phi_b^{-1}(y + (z-y)w)$, where $w$ is the quaternion variable. Then for each $z \in V$ there exists $\varepsilon_z > 0$ such that the function $g(w)$ is quaternion holomorphic on the set $W_z := \{w : w \in \mathbb{H}, |w| < 1 + \varepsilon_z\}$ and $|g(w)|$ attains its maximum at $w = 0$. In view of Theorem 3.15 and Remark 3.16 [22] $g$ is constant on $W_z$, hence $f$ is constant on $U_b$. By the quaternion holomorphic continuation $f$ is constant on $M$. 27
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