Galerkin FEM for a time-fractional Oldroyd-B fluid problem

Mariam Al-Maskari · Samir Karaa

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Abstract
We consider the numerical approximation of a generalized fractional Oldroyd-B fluid problem involving two Riemann-Liouville fractional derivatives in time. We establish regularity results for the exact solution which play an important role in the error analysis. A semidiscrete scheme based on the piecewise linear Galerkin finite element method in space is analyzed, and optimal with respect to the data regularity error estimates are established. Further, two fully discrete schemes based on convolution quadrature in time generated by the backward Euler and the second-order backward difference methods are investigated and related error estimates for smooth and nonsmooth data are derived. Numerical experiments are performed with different values of the problem parameters to illustrate the efficiency of the method and confirm the theoretical results.

Keywords Time-fractional Oldroyd-B fluid problem · Finite element method · Convolution quadrature · Error estimate · Nonsmooth data

Mathematics Subject Classification (2010) 65M60 · 65M12 · 65M15

1 Introduction

Let $\Omega$ be a bounded convex polygonal domain in $\mathbb{R}^2$ with a boundary $\partial \Omega$ and let $T > 0$ be a fixed time. We consider the initial boundary-value problem for the following time-fractional Oldroyd-B fluid equation

\[
(1 + a \partial_t^\alpha) u_t(x, t) = \mu (1 + b \partial_t^\beta) \Delta u(x, t) + f(x, t) \quad \text{in} \ \Omega \times (0, T],
\]

\[
(1.1a)
\]
with a homogeneous Dirichlet boundary condition
\[ u(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T], \quad (1.1b) \]
and initial conditions
\[ u(x, 0) = v(x), \quad (I^{1-\alpha} u_t)(x, 0) = 0 \quad \text{in } \Omega, \quad (1.1c) \]
where \( f \) and \( v \) are given functions, the parameters \( \alpha, \beta \in (0, 1) \), and \( \mu, a, b \) are positive constants. In (1.1a), \( u_t \) denotes the partial derivative of \( u \) with respect to time, and \( \partial_t^\alpha \) is the Riemann-Liouville fractional derivative in time defined for \( 0 < \alpha < 1 \) by the following:
\[
\partial_t^\alpha \varphi(t) := \frac{d}{dt} \frac{\Gamma(1-\alpha) \varphi(t)}{\Gamma(1-\alpha)} = \frac{d}{dt} \int_0^t \omega_{1-\alpha}(t-s) \varphi(s) \, ds \quad \text{with} \quad \omega_{\alpha}(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)},
\]
where \( \Gamma(\cdot) \) is the Gamma function. The second initial condition in (1.1c) is an appropriate condition for the model equation (1.1a), ensuring a unique solvability. This condition replaces the standard strong initial condition \( u_t(x, 0) = 0 \). Indeed, one can show for instance that the solution \( u \) satisfies (see, Theorem 2.1)
\[ u(t) = O(t^{\alpha-\beta+1}) \quad \text{and} \quad u_t(t) = O(t^{\alpha-\beta}) \quad \text{as } t \to 0. \]
This indicates that when \( \alpha < \beta \), the time derivative \( u_t \) has a singularity at \( t = 0 \), and as a consequence, the condition \( u_t(x, 0) = 0 \) is not satisfied. However, we see that
\[ (I^{1-\alpha} u_t)(t) = O(t^{1-\beta}) \quad \text{as } t \to 0. \]
Since \( 0 < \beta < 1 \), we have \( (I^{1-\alpha} u_t)(0) = 0 \), showing that our initial condition is valid for all \( \alpha, \beta \in (0, 1) \). We notice that similar arguments are presented in [6].

The time-fractional problem (1.1) has received considerable attention in recent years due to its capacity of modeling a large class of fluids for different values of \( a \) and \( b \). For example, when \( a = 0 \) and \( b > 0 \), (1.1) describes a Rayleigh-Stokes problem for a generalized fractional second-grade fluid [3, 7–9, 12, 19, 25, 32]. This plays an important role in investigating the behavior of some non-Newtonian fluids. The problem with \( b = 0 \) and \( a > 0 \) describes a generalized fractional Maxwell model [1, 5, 15, 29, 30], while the case with \( a = b = 0 \) corresponds to classical Newtonian fluids (classical diffusion).

Analytical studies for different versions of problem (1.1), in one- and two-spatial dimensions, have been presented by several authors (see, for instance [3, 15, 16, 33]). In most of these works, the solution is represented in a form of a series with the help of Fourier and Laplace transforms. For the general case \( (a > 0 \text{ and } b > 0) \), the solution was derived using double Fourier sine series in [13, 17]. In [18], the solution was represented in an integral form in terms of the Mittag-Leffler function. The results are formal as the convergence of the series and the regularity of the solutions have not been considered.

The numerical approximation of the model (1.1) has also attracted the attention of many authors. The case with \( a = 0 \) (generalized second-grade fluid) has, in particular, been studied extensively in the literature (see [7–9, 12, 19, 25, 32]). In [8] and [9], implicit and explicit finite difference schemes have been examined, for one- and two-dimensional problems, and Fourier analysis has been conducted to analyze
the stability and convergence of the methods. In [32] and [12], an implicit numerical approximation method is developed by transforming the problem into an integral equation. In [19], the authors have investigated a numerical scheme derived by the reproducing kernel technique, and in [25], a compact finite difference method and a radial basis function method was proposed. The convergence analysis in most of these studies requires the solution \( u \) of problem (1.1) to be sufficiently regular, including at \( t = 0 \), which is not practically the case. Most recently in [7], Jin et al. have considered the numerical approximation of the solution of the homogeneous problem (1.1) with \( a = 0 \) by piecewise linear finite elements in space and convolution quadrature in time, and have derived optimal error estimates with respect to the solution smoothness, expressed through the initial data \( v \).

Numerical studies for the general case when \( a \neq 0 \) in (1.1) are still limited. In [4], the authors have derived explicit and implicit schemes in time based on the Grünwald-Letnikov approximation of fractional derivatives, and performed numerical experiments to investigate the behavior of the solution. In [6], the problem is reformulated as a Volterra integral equation, where its kernel is represented in terms of Mittag-Leffler functions. A special attention has been given to the behavior of the solution. In [31], finite difference schemes of second- and fourth-order accuracy in space are proposed with a second-order convolution quadrature in time. An ADI algorithm was then developed for the computation of the numerical solution. In these studies, the theoretical analysis of the error is not presented. In [27], the authors have used a standard Galerkin finite element procedure in space and the famous \( L^1 \)-scheme in time to approximate the solution of (1.1a). They also obtained error estimates under high regularity assumptions on the exact solution. It is worth mentioning that a time-fractional differential equation involving two Riemann-Liouville fractional derivatives was investigated in [2, 34]. Similar to (1.1a) with \( a = 0 \), the equation was considered with only one initial condition \( u(x, 0) = v(x) \).

The aim of this paper is to develop a Galerkin FEM for problem (1.1) and derive optimal with respect to data regularity error estimates. Our analysis is based on exploiting Laplace transform tools with semigroup type properties of the FE solution operator. Further, we investigate two fully discrete schemes for the semidiscrete FE problem based on convolution quadrature in time generated by the backward Euler and the second-order backward difference methods. Error estimates with respect to the data regularity are established (see, Theorems 4.1 and 4.3). Compared to [7], where \( a \) and \( f \) in (1.1a) are both zeros, we have derived error estimates in the \( H^1(\Omega) \)-norm as well as in the \( L^\infty(\Omega) \)-norm. We have further analyzed the inhomogeneous problem and obtained new error estimates.

The paper is organized as follows: In Section 2, we present the solution theory of the mathematical model (1.1) and derive properties of the solution operator, which will play an important role in our subsequent error analysis. In Section 3, we introduce the semidiscrete piecewise linear FE scheme and derive optimal error estimates for the homogeneous and inhomogeneous problems. In Section 4, two fully discrete schemes based on convolution quadrature in time are considered. Finally, in Section 5, we conduct numerical experiments to validate the theoretical results.

Throughout the paper, \( C \) and \( c \) denote generic positive constants that may depend on \( \alpha, \beta, \mu, b, \) and \( a \), but are independent of the mesh size \( h \) and the time step \( \tau \).
2 Regularity results

To express the regularity properties of the solution of problem (1.1), we introduce the Hilbert space $H^r(\Omega) \subset L^2(\Omega)$ induced by the norm

$$\| v \|^2_{H^r(\Omega)} = \| A^{r/2} v \|^2 = \sum_{j=1}^{\infty} \lambda_j^r (v, \phi_j)^2,$$

where $(\cdot, \cdot)$ is the inner product on $L^2(\Omega)$, $\| \cdot \|$ is the induced norm, and $\{ \lambda_j \}_{j=1}^{\infty}$ and $\{ \phi_j \}_{j=1}^{\infty}$ are, respectively, the Dirichlet eigenvalues and eigenfunctions of $A := -\Delta$, with $\{ \phi_j \}_{j=1}^{\infty}$ being an orthonormal basis in $L^2(\Omega)$. Then, for $r \geq 0$, form a Hilbert scale of interpolation spaces.

We shall now derive an integral representation of the solution $u$ of (1.1) and describe the smoothing properties of the solution operator. For a given $\theta \in (\pi/2, \pi)$, we denote by $\Sigma_\theta$ the sector $\{ z \in \mathbb{C}, \; z \neq 0, \; |\arg z| < \theta \}$. Since $A$ is selfadjoint and positive definite, its resolvent $(zI + A)^{-1} : L^2(\Omega) \to L^2(\Omega)$ satisfies

$$\|(zI + A)^{-1}\| \leq M_\theta |z|^{-1} \quad \forall z \in \Sigma_\theta,$$

(2.1)

where $M_\theta = 1/\text{dist}(\Sigma_\theta, -1) = 1/\sin(\pi - \theta)$. Here, and in what follows, we use the same notation $\| \cdot \|$ to denote the operator norm from $L^2(\Omega) \to L^2(\Omega)$. We shall employ the Laplace transform $\hat{u} := \mathcal{L}(u)$ defined by $\hat{u}(x, z) = \int_0^\infty e^{-zt} u(x, t) \, dt$, and its inverse $u(x, t) = \frac{1}{2\pi i} \oint_C e^{zt} \hat{u}(x, z) \, dz$, where the contour of integration $C$ is any line in the right-half plane parallel to the imaginary axis and with $\text{Im}(z)$ increasing. By noting that

$$\partial_t^2 u_t(z) = z^\alpha \hat{u}_t(z) - (I^{1-\alpha}u_t)(0),$$

and using the condition $(I^{1-\alpha}u_t)(0) = 0$, an application of the Laplace transform to (1.1a) yields

$$(1 + az^\alpha)\hat{u}_t(z) = -\mu(1 + bz^\beta) A\hat{u}(z) + \hat{f}(z).$$

Substituting $\hat{u}_t(z)$ by $z\hat{u}(z) - v$, we find that

$$\left( (z + az^{\alpha+1})I + \mu(1 + bz^\beta)A \right) \hat{u}(z) = (1 + az^\alpha) v + \hat{f}(z),$$

or

$$\mu(1 + bz^\beta) \left( \frac{z + az^{\alpha+1}}{\mu(1 + bz^\beta)} I + A \right) \hat{u}(z) = (1 + az^\alpha) v + \hat{f}(z).$$

Then, we formally have

$$\hat{u}(z) = \hat{E}(z) \left( v + \frac{1}{1 + az^\alpha} \hat{f} \right),$$

(2.2)

where

$$\hat{E}(z) := \frac{g(z)}{z} (g(z)I + A)^{-1},$$

(2.3)
and \( g(z) := \mu^{-1}(z + \alpha z^{\alpha+1})/(1 + \beta z^{\beta}) \). Note that we can recast problem (1.1) with \( f = 0 \) to a Volterra integral equation of the form

\[
u(x, t) = v - \int_0^t k(t - \tau) A u(x, \tau) d\tau.
\tag{2.4}
\]

Indeed, by applying the Laplace transform to (2.4), we get

\[
\hat{u}(z) = \frac{1}{z} (1 + \hat{k}(z) A)^{-1} v.
\tag{2.5}
\]

Comparing (2.5) to (2.2), we obtain by the uniqueness of the Laplace transform that equation (1.1) is equivalent to (2.4) with the kernel \( k(t) \) satisfying \( \hat{k}(z) = g(z)^{-1} \).

In the next lemma, we state a basic property of the function \( g(z) \).

**Lemma 2.1** Let \( \theta \in \left(\pi/2, \frac{\pi}{1 + \alpha}\right) \) be fixed and set \( \bar{\theta} = (1 + \alpha) \theta < \pi \). Then, for any \( z \in \Sigma_\theta \), \( g(z) \in \Sigma_{\bar{\theta}} \) and

\[
|g(z)| \leq M_\theta \frac{1}{\mu} (|z| + a |z|^{\alpha+1}) \min \left\{ 1, \frac{1}{b} |z|^{-\beta} \right\}.
\tag{2.6}
\]

**Proof** Let \( z \in \Sigma_\theta \), i.e., \( z = re^{i\phi} \) with \( |\phi| < \theta \) and \( r > 0 \). Then,

\[
g(z) = \frac{re^{i\phi} + ar^{1+\alpha} e^{i(1+\alpha)\phi} + br^{1+\beta} e^{i(1-\beta)\phi} + abr^{1+\alpha+\beta} e^{i(1+\alpha-\beta)\phi}}{\mu \left( (1 + br^\beta \cos(\beta \phi))^2 + b^2 r^{2\beta} \sin^2(\beta \phi) \right)},
\]

which shows that \( g(z) \in \Sigma_{\bar{\theta}} \) since the four terms in the numerator are in \( \Sigma_{\bar{\theta}} \) and their imaginary parts have the same sign. To prove (2.6), we note that

\[
|g(z)| = \left| \frac{z + az^{\alpha+1}}{\mu |1 + \beta z^{\beta}|} \right| \leq \frac{|z + az^{\alpha+1}|}{\mu \text{dist}(-1, b^{-1}z^{-\beta})} \leq \frac{M_\theta}{\mu} \left( |z| + a |z|^{\alpha+1} \right).
\]

We also have

\[
|g(z)| = \left| \frac{z + az^{\alpha+1}}{\mu b |z|^{\beta} \text{dist}(-1, b^{-1}z^{-\beta})} \right| \leq \frac{M_\theta}{b \mu} \left( |z|^{1-\beta} + a |z|^{1+\alpha-\beta} \right).
\]

This completes the proof of the lemma. \( \square \)

Now, using the resolvent estimate (2.1), we have

\[
\|(g(z)I + A)^{-1}\| \leq M_{\bar{\theta}} |g(z)|^{-1} \quad \forall z \in \Sigma_\theta.
\tag{2.7}
\]

Then, from the definition of \( \hat{E}(z) \) in (2.3), we get

\[
\|\hat{E}(z)\| \leq M_{\bar{\theta}} |z|^{-1} \quad \forall z \in \Sigma_\theta.
\tag{2.8}
\]

Hence, by means of the inverse Laplace transform, we deduce that the solution operator is given by

\[
E(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{zt} \hat{E}(z) \, dz,
\tag{2.9}
\]

where, for \( \theta \in (\pi/2, \pi/(1 + \alpha)) \) and \( \delta > 0 \), \( \Gamma_{\theta,\delta} := \{ \rho e^{\pm i\theta} : \rho \geq \delta \} \cup \{ \delta e^{i\psi} : |\psi| \leq \theta \} \) is the contour oriented with an increasing imaginary part. Hence, by (2.8)
and [26, Theorem 2.1 and Corollary 2.4], we conclude that, for the case \( f = 0 \) and \( v \in L^2(\Omega) \), there exists a unique solution \( u \) to (1.1) given by \( u(t) = E(t)v \) and satisfying
\[
 u \in C([0, T]; L^2(\Omega)) \cap C((0, T]; \dot{H}^2(\Omega)).
\]
To get more details on the smoothing properties of the solution operator, we first state some properties for the operator \( \hat{E}(z) \).

**Lemma 2.2** The following estimates hold:
\[
\|A \hat{E}(z) \chi\| \leq C_\theta |z|^{-1} |g(z)|^{(2-p)/2} \| \chi \|_{\dot{H}^p(\Omega)} \quad \forall z \in \Sigma_\theta, \quad 0 \leq p \leq 2, \tag{2.10}
\]
\[
\|A^\nu \hat{E}(z) \chi\| \leq C_\theta |z|^{-1} |g(z)|^\nu \| \chi \| \quad \forall z \in \Sigma_\theta, \quad 0 \leq \nu \leq 1, \tag{2.11}
\]
where \( C_\theta \) depends only on \( \theta \).

**Proof** Since \( A \) commutes with \( \hat{E}(z) \) on \( \chi \in \dot{H}^2(\Omega) \), we have by (2.8),
\[
\|\hat{E}(z) A \chi\| \leq M_\theta |z|^{-1} \| A \chi \| \quad \forall \chi \in \dot{H}^2(\Omega). \tag{2.12}
\]
On the other hand, by noting that
\[
g(z) \hat{E}(z) = \frac{g(z)}{z} I - A \hat{E}(z),
\]
that is
\[
A \hat{E}(z) = \frac{g(z)}{z} I - g(z) \hat{E}(z), \tag{2.13}
\]
we conclude that
\[
\|A \hat{E}(z) \chi\| \leq (1 + M_\theta) |z|^{-1} |g(z)| \| \chi \| \quad \forall \chi \in L^2(\Omega). \tag{2.14}
\]
The estimate (2.10) follows by interpolating (2.12) and (2.14) for \( p \in [0, 2] \). The second estimate (2.11) follows by interpolating (2.8) and (2.14) for \( \nu \in [0, 1] \), which completes the proof.

Based on Lemma 2.2, stability and smoothing properties of the solution operator \( E(t) \) are established in the following Theorem.

**Theorem 2.1** The following estimates hold for \( t \in (0, T] \) and \( v = 0, 1 \):
\[
\|A^\nu E^{(m)}(t)v\| \leq C t^{-m-v(\alpha-\beta+1)} \| v \|, \quad v \in L^2(\Omega), \ m \geq 0, \tag{2.15}
\]
\[
\|A^\nu E^{(m)}(t)v\| \leq C t^{-m+(1-v)(\alpha-\beta+1)} \| Av\|, \quad v \in \dot{H}^2(\Omega), \ v + m \geq 1, \tag{2.16}
\]
where \( C \) is a constant depending on \( \alpha, \beta, \mu, a, b, \) and \( T \).

**Proof** We differentiate both sides of (2.9) with respect to \( t \) and then apply the operator \( A^\nu \) so that
\[
A^\nu E^{(m)}(t)v = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\beta}} z^m e^{zt} A^\nu \hat{E}(z)v \, dz.
\]
Then, by (2.11),

$$
\| A^\nu E^{(m)}(t) \| \leq C_\theta \int_{\Gamma_{\theta,\delta}} |z|^m e^{\text{Re}(z)t} |z|^{-1} |g(z)|^\nu |dz|.
$$  (2.17)

To evaluate the integral in (2.17), we choose $\delta = 1/t$ and set $z = re^{i\varphi}$. Then, for $\nu = 0$, we deduce that

$$
\| A^\nu E^{(m)}(t) \| \leq C_\theta \int_{\Gamma} |z|^{m-1} e^{\text{Re}(z)t} |dz| \leq C_\theta t^{-m},
$$

where here and throughout the paper $\Gamma := \Gamma_{\theta,1/t}$. For the case $\nu = 1$, we obtain

$$
\| A E^{(m)}(t) \| \leq C_\theta \int_{\Gamma} |z|^{m-1} e^{\text{Re}(z)t} \left( |z| + a |z|^{\alpha+1} \right) \min \left\{ 1, \frac{1}{b} |z|^{-\beta} \right\} |dz|
\leq C_\theta t^{-m} (t^{-1} + at^{-\alpha-1}) \min \left\{ 1, \frac{t^\beta}{b} \right\}
\leq C_\theta t^{-m} (t^{-1} + at^{-\alpha-1}) \frac{t^\beta}{b}
\leq Ct^{-m-\alpha-1+\beta},
$$

where the last inequality holds since $t^{-1} \leq T^\alpha t^{-\alpha-1}$. To prove the estimate (2.16), we note that by (2.13),

$$
E^{(m)}(t) v = \frac{1}{2\pi i} \int_{\Gamma} z^m e^{zt} (z^{-1} - g(z))^{-1} \hat{E}(z) v \, dz.
$$

As $\int_{\Gamma} e^{zt} z^{m-1} \, dz = 0$ for $m \geq 1$, and

$$
\| g(z)^{-1} \hat{E}(z) \| \leq M_\theta \mu \min \left\{ |z|^{-2} + b|z|^{\beta-2}, \frac{1}{a} (|z|^{-2-\alpha} + b|z|^{-2-\alpha+\beta}) \right\},
$$

we conclude that

$$
\| E^{(m)}(t) v \| \leq C \int_{\Gamma} |z|^m e^{\text{Re}(z)t} \mu \min \left\{ |z|^{-2} + b|z|^{\beta-2}, \frac{1}{a} (|z|^{-2-\alpha} + b|z|^{-2-\alpha+\beta}) \right\} \| Av \| |dz|
\leq Ct^{-m} \min \left\{ t + bt^{1-\beta}, \frac{1}{a} (t^{\alpha+1} + bt^{\alpha-\beta+1}) \right\} \| Av \|
\leq Ct^{-m} (t + bt^{1-\beta}) \frac{t^\alpha}{a} \| Av \|
\leq Ct^{-m} (t + bt^{1-\beta}) \frac{t^\alpha}{a} \| Av \|
\leq Ct^{-m+1-\beta+\alpha} \| Av \|,
$$

which shows (2.16) for $\nu = 0$. Finally, (2.16) with $\nu = 1$ can be obtained from (2.15) by setting $\nu = 0$ and replacing $v$ by $Av$.  

\[ \square \]
3 The spatially semidiscrete problem

In this section, we describe the Galerkin FE procedure in space and derive optimal error estimates with respect to the smoothness of the solution expressed through the initial data $v$ and the right-hand side $f$. Let $\mathcal{T}_h$ be a shape regular and quasi-uniform triangulation of the domain $\bar{\Omega}$ into triangles $K$, and let $h = \max_{K \in \mathcal{T}_h} h_K$, where $h_K$ denotes the diameter of $K$. The approximate solution $u_h$ of the Galerkin FEM will be sought in the finite element space $V_h$ of continuous piecewise linear functions over the triangulation $\mathcal{T}_h$:

$$V_h = \{ v_h \in C^0(\bar{\Omega}) : v_h|_K \text{ is linear for all } K \in \mathcal{T}_h \text{ and } v_h|_{\partial \Omega} = 0 \}.$$

The semidiscrete Galerkin FEM for problem (1.1) is to seek $u_h : (0, T] \to V_h$ such that

$$((1 + a\partial_t^a)u_h, \chi) + a \left( \mu (1 + b\partial_t^b) u_h \right) = (f, \chi) \quad \forall \chi \in V_h, \quad t \in (0, T], \quad u_h(0) = v_h, \quad (3.1)$$

with $v_h$ is an appropriate approximation to the initial data $v$ in $V_h$, and $a(v, w) := (\nabla v, \nabla w)$ is the bilinear form associated with the operator $A$. On the space $V_h$, we define the $L^2$-projection $P_h : L^2(\Omega) \to V_h$ and the Ritz projection $R_h : H^1_0(\Omega) \to V_h$, respectively, by

$$(P_h \varphi, \chi) = (\varphi, \chi) \quad \forall \chi \in V_h,$n

$$a(R_h \varphi, \chi) = a(\varphi, \chi) \quad \forall \chi \in V_h.$$

The operators $P_h$ and $R_h$ satisfy the following approximation properties (see, [10] and [28]).

**Lemma 3.1** The operators $P_h$ and $R_h$ satisfy

$$\| P_h \psi - \psi \| + h \| \nabla (P_h \psi - \psi) \| \leq c h^q \| \psi \|_{\dot{H}^q(\Omega)} \quad \forall \psi \in \dot{H}^q(\Omega), \quad q = 1, 2, \quad (3.2)$$

$$\| R_h \psi - \psi \| + h \| \nabla (R_h \psi - \psi) \| \leq c h^q \| \psi \|_{\dot{H}^q(\Omega)} \quad \forall \psi \in \dot{H}^q(\Omega), \quad q = 1, 2. \quad (3.3)$$

In particular, (3.2) indicates that $P_h$ is stable in $\dot{H}^1(\Omega)$.

We next introduce the discrete operator $A_h : V_h \to V_h$ defined by

$$(A_h \psi, \chi) = (\nabla \psi, \nabla \chi) \quad \forall \psi, \chi \in V_h,$$

so that the semidiscrete scheme (3.1) can be rewritten as follows:

$$(1 + a\partial_t^a)u_h + \mu (1 + b\partial_t^b) A_h u_h(t) = P_h f(t), \quad t \in (0, T], \quad (3.4)$$

with $u_h(0) = v_h$ and $(I^{1-a}u_h)(0) = 0$. 

$\square$ Springer
3.1 The homogeneous problem

Following the analysis in Section 2, the solution of the homogeneous semidiscrete problem (3.4) can be represented by the following:

\[ u_h(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{zt} \hat{E}_h(z) v_h \, dz =: E_h(t)v_h , \]

where \( \hat{E}_h(z) = z^{-1}g(z)(g(z)I + A_h)^{-1} \). Since \( A_h \) is self-adjoint and positive definite on \( V_h \), the estimate in Lemma 2.2 are also valid for \( \hat{E}_h \).

**Lemma 3.2** With \( \chi \in V_h \), the following estimates hold:

\[ \| A_h \hat{E}_h(z) \chi \| \leq C_\theta |z|^{-1} |g(z)|^{(2-p)/2} \| A_h^{p/2} \chi \| \quad \forall z \in \Sigma_\theta , \quad 0 \leq p \leq 2 , \]

\[ \| A_h^v \hat{E}_h(z) \chi \| \leq C_\theta |z|^{-1} |g(z)|^v \| \chi \| \quad \forall z \in \Sigma_\theta , \quad 0 \leq v \leq 1 . \]

where \( C_\theta \) is independent of the mesh size \( h \).

By choosing \( v_h = P_h v \), the error of the FE approximation \( e_h(t) := u_h(t) - u(t) \) at time \( t \) can be represented by the following:

\[ e_h(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{zt} (\hat{E}_h(z) P_h - \hat{E}(z)) v \, dz = \frac{1}{2\pi i} \int_{\Gamma} e^{zt} \frac{g(z)}{z} S_h(z) v \, dz , \quad (3.5) \]

where \( S_h(z) = (g(z)I + A_h)^{-1} P_h - (g(z)I + A)^{-1} \). In the next lemma, we state important properties of the operator \( S_h(z) \) which will play a key role in the error analysis.

**Lemma 3.3** The following estimate holds for all \( z \in \Sigma_\theta \),

\[ \| S_h(z) v \| + h \| \nabla S_h(z) v \| \leq ch^2 \| v \| , \quad (3.6) \]

\[ \| S_h(z) v \|_{L^\infty(\Omega)} \leq ch^2 l_h^2 \| v \|_{L^\infty(\Omega)} , \quad (3.7) \]

with \( l_h = | \ln h | \).

The first estimate is given in [7, 14, 23]. The second one can be found in [24]. Now we are in position to prove a nonsmooth data error estimate for the semidiscrete problem.

**Theorem 3.1** Let \( u \) and \( u_h \) be the solutions of problems (1.1) and (3.4) with \( v \in L^2(\Omega) \) and \( v_h = P_h v \), respectively, and \( f = 0 \). Then, for \( t > 0 \),

\[ \| e_h(t) \| + h \| \nabla e_h(t) \| \leq Ch^2 t^{\beta - \alpha - 1} \| v \| . \quad (3.8) \]
Proof The $L^2$-estimate of the error follows from the representation (3.5) and the estimate (3.6). Indeed, we have

$$\|e_h(t)\| \leq C h^2 \|v\| \int_{\Gamma} e^{\text{Re}(z)t} \frac{|g(z)|}{|z|} |dz|$$

$$\leq C \left( t^{-1+\beta} + at^{-1-\alpha+\beta} \right) h^2 \|v\|$$

$$\leq Ct^{\beta-\alpha-1} h^2 \|v\|.$$

The $H^1(\Omega)$-error estimate is established analogously. □

In the next theorem, an error bound is obtained for smooth initial data $v \in \dot{H}^2(\Omega)$.

Theorem 3.2 Let $u$ and $u_h$ be the solutions of problems (1.1) and (3.4) with $v \in \dot{H}^2(\Omega)$ and $v_h = R_h v$, respectively, and $f = 0$. Then, for $t > 0$,

$$\|e_h(t)\| + h \|\nabla e_h(t)\| \leq ch^2 \|v\|_{\dot{H}^2(\Omega)}.$$ (3.9)

Proof By (2.13) and the identity $A_h R_h = P_h A$, we obtain

$$e_h(t) = \frac{1}{2\pi i} \left( - \int_{\Gamma} e^{zt} - S_h(z) A v \, dz + \int_{\Gamma} e^{zt} - (R_h v - v) \, dz \right).$$

Then, using (3.6) and (3.3), we deduce that

$$\|e_h(t)\| \leq ch^2 \|A v\| \int_{\Gamma} e^{\text{Re}(z)t} |z|^{-1} |dz|$$

$$\leq ch^2 \|A v\|.$$

The $H^1(\Omega)$-error estimate is derived by a similar argument. □

Finally, we derive an error estimate in the maximum norm by considering (3.7) and following the argument in the proof of Theorem 3.1.

Theorem 3.3 Let $u$ and $u_h$ be the solutions of problems (1.1) and (3.4) with $v \in L^\infty(\Omega)$ and $v_h = P_h v$, respectively, and $f = 0$. Then, for $t > 0$,

$$\|e_h(t)\|_{L^\infty(\Omega)} \leq ch^2 t^{\frac{2}{2}-\alpha-1} \|v\|_{L^\infty(\Omega)}.$$ (3.10)

Remark 3.1 In Theorem 3.2, the estimate (3.9) is still valid if one chooses the approximation $v_h = P_h v$ (see, the arguments in [7, Remark 3.1]). Then, by interpolation with (3.8), we obtain for $q \in [0, 2]$,

$$\|e_h(t)\| + h \|\nabla e_h(t)\| \leq ch^2 t^{(\beta-\alpha-1)(2-q)/2} \|v\|_{\dot{H}^q(\Omega)}.$$ (3.11)

3.2 The inhomogeneous problem

We now turn back to the inhomogeneous problem (1.1). With a vanishing initial data $v$, Laplace transforms yield $\hat{u}(z) = \mu^{-1} (1 + bz)^{-1} (g(z)I + A)^{-1} \hat{f}(z)$. Then, with $\hat{f}(z)$ being analytic in the sector $\Sigma_\theta$, we can follow the analysis presented for the
previous subsection to obtain, after noting that \(|(1 + b z^\beta)^{-1} \leq M(b^{-1}|z|^{-\beta})|\) the following error estimates:

\[ \| e_h(t) \| + h \| \nabla e_h(t) \| \leq c h^2 t^{\beta - 1} \| \hat{f}(z) \|_{L^2(\Omega), \Gamma} \quad (3.12) \]

and

\[ \| e_h(t) \|_{L^\infty(\Omega)} \leq c h^2 l_h^2 t^{\beta - 1} \| \hat{f}(z) \|_{L^\infty(\Omega), \Gamma}, \quad (3.13) \]

where \( \Gamma = \Gamma_{\theta, 1/\tau} \) and

\[ \| \hat{f}(z) \|_{B, \Gamma} := \sup_{z \in \Gamma} \| \hat{f}(z) \|_B. \]

Note that it is possible to remove the term \( t^{\beta - 1} \) from the right-hand sides of (3.12) and (3.13) if \( \| z^{1-\beta} \hat{f}(z) \|_{L^2(\Omega), \Gamma} \) and \( \| z^{1-\beta} \hat{f}(z) \|_{L^\infty(\Omega), \Gamma} \) are finite. A serious restriction in this approach is that it requires the Laplace transform \( \hat{f}(z) \) to exist, to be analytic in the whole sector \( \Sigma_{\theta} \) and to be such that the norms indicated above are finite. The simple case with \( f(t) = e^t \) shows that the Laplace transform \( \hat{f}(z) = (z - 1)^{-1} \) is not analytic in \( \Sigma_{\theta} \) and so the method is not applicable.

We shall now follow a different approach and prove results showing a classical-type nonsmooth data error estimate that does not use \( \hat{f}(z) \).

**Theorem 3.4**  Let \( u \) and \( u_h \) be the solutions of problems (1.1) and (3.4), respectively. Let \( q \in [0, 2] \) and \( p > 1/\beta \). Then, for \( t > 0 \), the following error estimates hold:

(a) If \( v \in \dot{H}^q(\Omega) \) and \( f \in L^p(0, T; L^2(\Omega)) \), then

\[ \| e_h(t) \| + h \| \nabla e_h(t) \| \leq c h^2 \left( t^{(\beta - \alpha - 1)(2-q)/2} \| v \|_{\dot{H}^q(\Omega)} + t^{\beta - 1/p} \| f \|_{L^p(0,T;L^2(\Omega))} \right). \quad (3.14) \]

(b) If \( v \in L^\infty(\Omega) \) and \( f \in L^p(0, T; L^\infty(\Omega)) \), then

\[ \| e_h(t) \|_{L^\infty(\Omega)} \leq c h^2 l_h^2 \left( t^{\beta - \alpha - 1} \| v \|_{L^\infty} + t^{\beta - 1/p} \| f \|_{L^p(0,T;L^\infty(\Omega))} \right). \quad (3.15) \]

**Proof** Set \( G_h(t) = E_h(t) P_h - E(t) \) and \( \tilde{G}_h(t) = H_h(t) P_h - H(t) \) where \( H(t) = \mathcal{L}^{-1} \left\{ \frac{1}{\mu(1 + b z^\beta)} \left( g(z) I + A \right)^{-1} \right\} \). Then, by applying the Laplace transform to (1.1a) and (3.4), we represent the error by

\[ e_h(t) = G_h(t) v + \int_0^t \tilde{G}_h(t - s) f(s) \, ds := I + II. \]

The first term \( I \) is already bounded in (3.11). For the second term, we apply Lemma 3.3 to get

\[ \| \tilde{G}_h(t) \| = \left\| \frac{1}{2\pi i} \int_{\Gamma} e^{zt} \frac{1}{\mu(1 + b z^\beta)} S_h(z) \, dz \right\| \leq c h^2 \min \left\{ t^{-1}, t^{\beta - 1} \right\} \leq c_T h^2 t^{\beta - 1}. \]
Using this bound and Hölder’s inequality, it follows that
\[
\|II\| \leq ch^2 \int_0^t (t - s)^{\beta - 1} \|f(s)\| \, ds
\]
\[
\leq ch^2 \left( \frac{t(\beta + 1)\hat{p} + 1}{(\beta - 1)\hat{p} + 1} \right)^{1/\hat{p}} \|f\|_{L^p(0,T;L^2(\Omega))},
\]
where \(\hat{p}\) is the conjugate exponent of \(p\). This proves (3.14). For the \(L^\infty(\Omega)\)-estimate, we first notice the bound
\[
\|\tilde{G}_h(t)\chi\|_{L^\infty(\Omega)} \leq ch^2 l^2 ht^{\beta - 1} \|\chi\|_{L^\infty(\Omega)},
\]
and, as a consequence, we obtain
\[
\|II\|_{L^\infty(\Omega)} \leq ch^2 \left( \frac{t(\beta - 1)\hat{p} + 1}{(\beta - 1)\hat{p} + 1} \right)^{1/\hat{p}} \|f\|_{L^p(0,T;L^\infty(\Omega))} \, ds.
\]
This bound with (3.10) yields the desired estimate (3.15).

\section{Time discretization}

Now, we consider the time discretization of the semidiscrete problem (3.4) by convolution quadratures generated by the backward Euler and the second-order backward difference methods. We divide the time interval \([0, T]\) into a uniform grid with a time step size \(\tau = T/N\), \(N\) a positive integer, and set \(t_j = j\tau\). Let \(K(z)\) be the Laplace transform of a distribution \(k(t)\) on the real line, which vanishes for \(t < 0\), has its singular support empty or concentrated at \(t = 0\) and which is analytic for \(t > 0\).

With \(\partial_t\) denoting time differentiation, we then define \(K(\partial_t)\) as the operator of (distributional) convolution with the kernel \(k\): \(K(\partial_t)\varphi = k * \varphi\) for a function \(\varphi(t)\) with suitable smoothness. The convolution quadrature developed in [11, 23] and initiated in [20, 21] refers to an approximation of \(K(\partial_t)\varphi\) by a discrete convolution \(K(\partial_\tau)\varphi\) at \(t = t_n\) as
\[
K(\partial_\tau)\varphi(t_n) = \sum_{j=0}^n \omega_n - j(\tau)\varphi(t_j),
\]
where the quadrature weights \(\{\omega_j(\tau)\}_{j=0}^\infty\) are determined by the generating power series
\[
\sum_{j=0}^\infty \omega_j(\tau)\xi^j = K(\delta(\xi)/\tau)
\]
with \(\delta(\xi)\) being a rational function, chosen as the quotient of the generating polynomials of a stable and consistent linear multistep method. We shall consider the Backward Euler (BE) and the second-order backward difference (SBD) methods, for which \(\delta(\xi) = 1 - \xi\) and \(\delta(\xi) = (1 - \xi) + (1 - \xi)^2/2\), respectively. For the BE
method, the convolution quadrature formula for approximating the fractional integral
\( \partial_t^{-\alpha} \varphi \) is given by
\[ \partial_t^{-\alpha} \varphi(t_n) = \sum_{j=0}^{n} \omega_{n-j} \varphi(t_j), \]
where \( \sum_{j=0}^{\infty} \omega_j \xi^j = [(1 - \xi)/\tau]^{-\alpha} \), \( \omega_j = \tau^\alpha (-1)^j \binom{-\alpha}{j} \),
while for the SBD method, the quadrature weights are provided by the formula [20]:
\[ \omega_j = \tau^\alpha (-1)^j \left( \frac{2}{3} \sum_{l=0}^{j} 3^{-l} \binom{-\alpha}{j-l} \binom{-\alpha}{l} \right). \]

An important property of the convolution quadrature is that it maintains some relations of the continuous convolution. For instance, the associativity of convolution is valid for the convolution quadrature [22], such as
\[ K_1(\partial_\tau) K_2(\partial_\tau) = K_1 K_2(\partial_\tau) \quad \text{and} \quad K_1(\partial_\tau)(k * \varphi) = (K_1(\partial_\tau)k) * \varphi. \quad (4.1) \]

The approximation properties of the convolution quadrature, which represent the key ingredient in the error analysis, are described in the following lemma (see, [21, Theorem 4.1] and [22, Theorem 2.2]).

**Lemma 4.1** Let \( K(z) \) be analytic in the sector \( \Sigma_\theta \) and satisfies
\[ \| K(z) \| \leq M |z|^{-\mu} \quad \forall z \in \Sigma_\theta, \]
for some real \( \mu \) and \( M \). Assume that the linear multistep method is strongly \( A \)-stable and of order \( p \geq 1 \). Then, for \( \varphi(t) = ct^{\nu-1} \), the convolution quadrature satisfies
\[ \| K(\partial_t)\varphi(t) - K(\partial_\tau)\varphi(t) \| \leq \begin{cases} Ct^{\mu-1+v-p} \tau^p, & \nu \geq p, \\ Ct^{\mu-1} \tau^v, & 0 < \nu \leq p. \end{cases} \]

### 4.1 Error analysis for the BE method

We start by considering the time discretization of (3.4) by the convolution quadrature based on the BE method. To this end, we integrate (3.4) from 0 to \( t \) and use the initial condition \((I^{1-\alpha} u_{\text{ht}})(0) = 0\) to get
\[ (u_h - v_h + a \left( \partial_t^{\alpha-1} u_{\text{ht}} \right)) + \mu \left( \partial_t^{-1} + b \partial_t^{\beta-1} \right) A_h u_h = \partial_t^{-1} f_h, \]
where \( f_h = P_h f \). This can be simplified as follows;
\[ (u_h - v_h + a \left( \partial_t^{\alpha} u_h - \partial_t^{\alpha} v_h \right)) + \mu \left( \partial_t^{-1} + b \partial_t^{\beta-1} \right) A_h u_h = \partial_t^{-1} f_h, \]
and after rearrangements, we get
\[ (1 + a \partial_t^\alpha) u_h + \mu \left( \partial_t^{-1} + b \partial_t^{\beta-1} \right) A_h u_h = (1 + a \partial_t^\alpha) v_h + \partial_t^{-1} f_h. \quad (4.2) \]

The convolution terms in (4.2) can then be approximated by convolution quadratures generated by the BE method. Hence, the fully discrete solution \( U^n_h \) satisfies at \( t_n = n \tau \),
\[ (1 + a \partial_t^\alpha) U^n_h + \mu \left( \partial_t^{-1} + b \partial_t^{\beta-1} \right) A_h U^n_h = (1 + a \partial_t^\alpha) v_h + \partial_t^{-1} f_h. \quad (4.3) \]
If we apply $\partial_\tau$ to (4.3) and use the associativity of convolution in (4.1), we arrive at the following scheme: with $U^0_h = v_h$, find $U^n_h$ for $n = 1, \cdots, N$ such that
\[
\left( \partial_\tau + a \partial_\tau^{1+\alpha} \right) U^n_h + \mu \left( 1 + b \partial_\tau^{\beta} \right) A_h U^n_h = a \partial_\tau^{1+\alpha} v_h + f^n_h,
\]
where $f^n_h = f_h(t_n)$. In view of (4.2) and (4.3), we have for the homogeneous problem,
\[
u_h = G(\partial_t) v_h \quad \text{and} \quad U^n_h = G(\partial_\tau) v_h,
\]
where $G(z) = g(z)(g(z)I + A_h)^{-1}$ and $g(z)$ is the function defined in Lemma 2.1.

Then, we may represent the error $U^n_h - u_h(t_n)$ at $t = t_n$ by
\[
U^n_h - u_h(t_n) = (G(\partial_\tau) - G(\partial_\tau)) v_h.
\]
If we let $G_1(z) = -g^{-1}(z)G(z)$ and noting that
\[
G(z) = I - g^{-1}(z)G(z)A_h,
\]
we deduce that
\[
U^n_h - u_h(t_n) = (G_1(\partial_\tau) - G_1(\partial_\tau)) A_h v_h.
\]
Then, by Lemma 4.1, we obtain the following error estimates for smooth and nonsmooth initial data.

**Lemma 4.2** Let $u_h$ and $U^n_h$ be the solutions of problems (4.2) and (4.3), respectively, with $f = 0$. Then, the following estimates hold:

(a) If $v \in \dot{H}^2(\Omega)$ and $v_h = R_h v$, then
\[
\|U^n_h - u_h(t_n)\| \leq C \tau \left( t_n^{\alpha} + bt_n^{\alpha-\beta} \right) \|v\|_{\dot{H}^2(\Omega)}.
\]

(b) If $v \in L^2(\Omega)$ and $v_h = P_h v$, then
\[
\|U^n_h - u_h(t_n)\| \leq C \tau t_n^{-1} \|v\|.
\]

**Proof** We recall that, by (2.7),
\[
\|G(z)\| \leq M_\theta \quad \forall z \in \Sigma_\theta.
\]
Then, we have
\[
\|G_1(z)\| \leq M_\theta |g(z)|^{-1} \leq M_\theta \mu \left( |z|^{-1} + b|z|^{\beta-1} \right) \min \left\{ 1, \frac{|z|^{-\alpha}}{a} \right\}.
\]
Applying Lemma 4.1 to (4.6) with $v = 1$, $p = 1$ and $\mu = \alpha + 1, 1 + \alpha - \beta$, respectively, we deduce that
\[
\|U^n_h - u_h(t_n)\| \leq C \frac{\mu}{a} \left( t_n^{\alpha} + bt_n^{\alpha-\beta} \right) \|A_h v_h\|.
\]
As $v_h = R_h v$, we use the identity $A_h R_h = P_h A$ so that
\[
\|A_h v_h\| = \|A_h R_h v\| = \|P_h A v\| \leq C \|A v\| = C \|v\|_{\dot{H}^2(\Omega)},
\]
where the last inequality follows from the $L^2(\Omega)$-stability of $P_h$. This shows (4.7).
To derive (4.8), we use (4.9) and apply Lemma 4.1 to (4.4) with $\mu = 0$, $\nu = 1$ and $p = 1$, to obtain
\[ \| U^n_h - u_h(t_n) \| \leq C\tau t_n^{-1} \| v_h \|. \]
The estimate follows then by the $L^2(\Omega)$-stability of $P_h$, which completes the proof.

Now, recalling the estimates derived in Theorems 3.1 and 3.2 for the semidiscrete problem, we summarize our results in the next theorem as follows.

**Theorem 4.1** Let $u$ and $U^n_h$ be the solutions of problems (1.1) and (4.3), respectively, with $f = 0$. Then, the following error estimates hold:

(a) If $v \in \dot{H}^2(\Omega)$ and $v_h = R_h v$, then
\[ \| U^n_h - u(t_n) \| \leq C(h^2 + \tau t_n^\alpha + \tau bt_n^{\alpha-\beta}) \| v \|_{\dot{H}^2(\Omega)}. \]

(b) If $v \in L^2(\Omega)$ and $v_h = P_h v$, then
\[ \| U^n_h - u(t_n) \| \leq C(h^2 t_n^{\beta - \alpha - 1} + \tau t_n^{-1}) \| v \|. \]

**Remark 4.1** Note that, from (4.4),
\[ \nabla (U^n_h - u_h(t_n)) = ((\nabla G)(\partial \tau) - (\nabla G)(\partial t)) v_h. \]
Since $\| \nabla G(z) \| \leq c|g(z)|^{1/2} \leq c|z|^{(1+\alpha-\beta)/2}$, an application of Lemma 4.1 with $\nu = 1$ and $\mu = -(1+\alpha-\beta)/2$ yields
\[ \| \nabla (U^n_h - u_h(t_n)) \| \leq c \tau t_n^{(-1-\alpha+\beta)/2-1} \| v \|. \]
Similarly, we obtain for smooth initial data
\[ \| \nabla (U^n_h - u_h(t_n)) \| \leq c \tau t_n^{(-1+\alpha-\beta)/2} \| v \|_{\dot{H}^2(\Omega)}. \]
Thus, by interpolation, it follows that
\[ \| \nabla (U^n_h - u_h(t_n)) \| \leq c \tau t_n^{-1} \| v \|_{\dot{H}^1(\Omega)}. \]

We now derive error estimates for the inhomogeneous problem with $v = 0$.

**Theorem 4.2** Let $u$ be the solution of the problem (1.1) with $v = 0$ and $f \in L^\infty(0,T; L^2(\Omega))$, and let $U^n_h$ be the solution of (4.3) with $v_h = 0$. Then, the following error estimate holds:
\[ \| U^n_h - u(t_n) \| \leq c \left( h^2 t_n^\beta \| f \|_{L^\infty(0,T; L^2(\Omega))} + \tau t_n^\alpha \| f(0) \| + \tau \int_0^{t_n} (t_n-s)^\alpha \| f'(s) \| ds \right). \] (4.10)

**Proof** Taking into account the estimate derived in Theorem 3.4 for $v = 0$, it suffices to bound $U^n_h - u_h(t_n)$. In view of (4.2) and (4.3), the error is represented by
\[ U^n_h - u_h(t_n) = (F_h(\partial \tau) - F_h(\partial t)) f_h. \]
where $F_h(z) = \frac{1}{\mu(1 + bz^{\beta})}(g(z)I + A_h)^{-1}$. Using the expansion $f_h(t) = f_h(0) + (1 \ast f'_h)(t)$ and the second relation in (4.1), we have

$$U^n_h - u_h(t_n) = (F_h(\partial_\tau) - F_h(\partial_t))f_h(0) + ((F_h(\partial_\tau) - F_h(\partial_t))1 \ast f'_h)(t_n) =: I + II.$$ 

Then, by Lemma 4.1 (with $\mu = 1 + \alpha$ and $\nu = 1$) and the $L^2(\Omega)$-stability of $P_h$, we obtain the following:

$$\|I\| \leq c\tau t^n_n \|f_h(0)\| \leq c\tau t^n_n \|f(0)\|.$$ 

For the second term, Lemma 4.1 yields

$$\|II\| \leq \int_0^{t_n} \|(F_h(\partial_\tau) - F_h(\partial_t))1(t_n - s)f'_h(s)\| \leq c\tau \int_0^{t_n} (t_n - s)^\alpha \|f'_h(s)\| ds.$$ 

Together, these estimates with (3.14) obtained for $\nu = 0$ and $f \in L^\infty(0, T; L^2(\Omega))$ complete the proof of (4.10). 

4.2 Error analysis for the SBD method

Now, we consider the time discretization of the semidiscrete problem (3.4) by convolution quadrature based on the second-order backward difference formula, and study convergence rates for smooth and nonsmooth initial data. Recall that the semidiscrete solution $u_h$ is given by the following:

$$u_h = G(\partial_t) \left( v_h + (1 + a\partial_t^\alpha)^{-1}\partial_t^{-1}f_h \right).$$ 

From the estimates in Lemma 4.1, it is clear that a second-order error bound cannot be achieved, if for instance, $\varphi$ is constant (i.e., $\nu = 1$), even if a high-order multistep is used. To maintain the second-order time accuracy, we modify the scheme following the strategy proposed in [7, 11, 23]. To do so, we use (4.5) and the splitting $f_h = f_h(0) + \tilde{f}_h$, where $\tilde{f}_h = f_h - f_h(0)$, so that $u_h$ can be represented by the following:

$$u_h = v_h + G(\partial_t) \left( -g^{-1}(\partial_\tau)A_h v_h + (1 + a\partial_\tau^\alpha)^{-1}(\partial_\tau^{-1}f_h(0) + \partial_\tau^{-1}\tilde{f}_h) \right).$$ (4.11) 

This leads to the corrected numerical scheme

$$U^n_h = v_h + G(\partial_t) \left( -g^{-1}(\partial_\tau)A_h v_h + (1 + a\partial_\tau^\alpha)^{-1}(\partial_\tau^{-1}f_h(0) + \partial_\tau^{-1}\tilde{f}_h) \right),$$ (4.12) 

where the symbol $\partial_\tau$ refers to time approximation based on the SBD method, and the exact contribution $\partial_\tau^{-1}$ is kept in the formula in order to preserve the second-order time accuracy.

For numerical purposes, it is essential to rewrite (4.12) as a time-stepping algorithm. Letting $1_\tau = (0, 3/2, 1, \cdots)$ so that $1_\tau = \partial_\tau \partial_\tau^{-1}$ at grid point $t_n$, we apply the operator $(I + g(\partial_\tau)^{-1}A_h)$ to both sides of (4.12) and use the associativity of convolution to finally arrive at the equivalent form

$$(I + g(\partial_\tau)^{-1}A_h)(U^n_h - v_h) = -g(\partial_\tau)^{-1}A_h 1_\tau v_h + (1 + a\partial_\tau^\alpha)^{-1}(\partial_\tau^{-1}1_\tau f_h(0) + \partial_\tau^{-1}\tilde{f}_h).$$
Applying the operator $\partial_\tau (1 + a \partial_\tau^{\alpha+1})$, we obtain the following:

$$((\partial_\tau + a \partial_\tau^{\alpha+1}) I + \mu (1 + b \partial_\tau^\beta) A_h)(U^n_h - v_h) = -\mu(1 + b \partial_\tau^\beta) A_h 1_\tau v_h + 1_\tau f_h(0) + \tilde{f}_h.$$  

(4.13)

Since $1 v_h - 1_\tau v_h = (v_h, -1/2 v_h, 0, \cdots)$, we thus define the time stepping scheme as: with $U^n_0 = v_h$, find $U^n_h$ such that

$$\left(\frac{3}{2} \tau^{-1} + a \partial_\tau^{\alpha+1}\right) (U^n_h - v_h) + \mu \left(1 + b \tilde{\partial}_\tau^\beta\right) A_h U^n_1 + \frac{\mu}{2} A_h v_h = f^n_1 + \frac{1}{2} f^n_0,$$

and for $n \geq 2$

$$\partial_\tau U^n_h + a \partial_\tau^{\alpha+1} (U^n_h - v_h) + \mu \left(1 + b \tilde{\partial}_\tau^\beta\right) A_h U^n_1 = f^n,$$

where the modified convolution quadrature $\tilde{\partial}_\tau^\beta$ is given by

$$\tilde{\partial}_\tau^\beta \varphi^n = \left(\sum_{j=1}^{n} \omega^\beta_{n-j} \varphi^j + \frac{1}{2} \omega^\beta_{n-1} \varphi^0\right),$$

with the weights $\{\omega^\beta_{j}\}$ being generated by the SBD method.

Now, using Lemma 4.1, we derive the following error bounds for the homogeneous problem with smooth and nonsmooth initial data $v$.

**Lemma 4.3** Let $u_h$ and $U^n_h$ be the solutions of problems (3.4) and (4.13), respectively, with $f = 0$. Then, the following estimates hold:

(a) If $v \in \dot{H}^2(\Omega)$ and $v_h = R_h v$, then

$$\|U^n_h - u_h(t_n)\| \leq C \left( \tau^{2} t_n^{\alpha-1} + \tau^{2} t_n^{\alpha-\beta-1}\right) \|v\|_{\dot{H}^2(\Omega)}.$$  

(4.14)

(b) If $v \in L^2(\Omega)$ and $v_h = P_h v$, then

$$\|U^n_h - u_h(t_n)\| \leq C \tau^{2} t_n^{-2} \|v\|.$$  

(4.15)

**Proof** Note that, in view of (4.5), we can split the error into

$$U^n_h - u_h(t_n) = (G_1(\tilde{\partial}_\tau) - G_1(\partial_t)) \partial_\tau^{-1} A_h v_h,$$  

(4.16)

where $G_1(z) = -z g^{-1}(z) G(z)$. We easily verify that

$$\|G_1(z)\| \leq M_0 \mu |z| \left( |z|^{-1} + b |z|^{\beta-1}\right) \min \left\{1, \frac{|z|^{-\alpha}}{a}\right\}.$$  

Next, we apply Lemma 4.1 to (4.16) with $\nu = 2$, $p = 2$ and $\mu = \alpha, \alpha - \beta$, respectively, and the desired estimate (4.14) follows then by using the identity $A_h R_h = P_h A$.

For the estimate (4.15), we write the error as follows:

$$U^n_h - u_h(t_n) = (G_2(\tilde{\partial}_\tau) - G_2(\partial_t)) \partial_\tau^{-1} v_h.$$  

(4.17)
where $G_2(z) = z(G(z) - I)$. By noting that $\|G_2(z)\| \leq (1 + M_\theta)|z|$ \(\forall z \in \Sigma_\theta\), a use of (4.17), Lemma 4.1 (with $\mu = -1$, $v = 2$ and $p = 2$) and the $L^2(\Omega)$-stability of $P_h$ yields the estimate (4.15).

Recalling the estimates derived in Theorems 3.1 and 3.2, we summarize our results for the SBD method as follows.

**Theorem 4.3** Let $u$ and $U^n_h$ be the solutions of problems (1.1) and (4.13), respectively, with $f = 0$. Then, the following error estimates hold:

(a) If $v \in \dot{H}^2(\Omega)$ and $v_h = R_h v$, then

$$\|U^n_h - u(t_n)\| \leq C(h^2 + \tau^2 t_n^{\alpha-1} + \tau^2 t_n^{\alpha-\beta-1})\|v\| \dot{H}^2(\Omega).$$

(b) If $v \in L^2(\Omega)$ and $v_h = P_h v$, then

$$\|U^n_h - u(t_n)\| \leq C(h^2 \tau^2 t_n^{\alpha-1} + \tau^2 t_n^{\alpha-2})\|v\|.$$

**Remark 4.2** Following the same analysis as in Remark 4.1, we obtain for the SBD method

$$\|\nabla(U^n_h - u_h(t_n))\| \leq c\tau^2 t_n^{(-3+\alpha+\beta)/2-1}\|v\|,$$

and

$$\|\nabla(U^n_h - u_h(t_n))\| \leq c\tau^2 t_n^{(-3+\alpha-\beta)/2}\|v\| \dot{H}^2(\Omega).$$

Then, by interpolation, it follows that

$$\|\nabla(U^n_h - u_h(t_n))\| \leq c\tau^2 t_n^{-2}\|v\| \dot{H}^1(\Omega).$$

For the inhomogeneous problem with $v = 0$, we have the following error estimates.

**Theorem 4.4** Let $u$ be the solution of the problem (1.1) with $\nu = 0$ and $f \in L^\infty(0, T; L^2(\Omega))$, and let $U^n_h$ be the solution (4.13) with $v_h = 0$. Then, the following error estimate holds:

$$\|U^n_h - u(t_n)\| \leq c\left(h^2 \tau^2 t_n^{\beta} \|f\|_{L^\infty(0, T; L^2(\Omega))} + \tau^2 t_n^{\alpha-1} \|f(0)\| + \tau^2 \int_0^{t_n} (t_n - s)^{\alpha-1} \|f''(s)\| ds\right).$$

**Proof** By Theorem 3.4, it suffices to bound $U^n_h - u_h(t_n)$. Let $\tilde{F} = zF(z)$. By using the expansion $\tilde{f}_h = tf'_h + t^* f''_h$ in (4.11) and (4.12), we rewrite the solutions $u_h(t_n)$ and $U^n_h$ as follows:

$$u_h(t_n) = \tilde{F}(\partial_t) tf_h(0) + F(\partial_t) tf'_h(0) + (F(\partial_t) t) * f''_h,$$

$$U^n_h = \tilde{F}(\partial_t) tf_h(0) + F(\partial_t) tf'_h(0) + (F(\partial_t) t) * f''_h,$$

respectively. Then, we have the following:

$$U^n_h - u_h(t_n) = (\tilde{F}(\partial_t) - \tilde{F}(\partial_t)) tf_h(0) + (F(\partial_t) - F(\partial_t)) tf'_h(0)$$

$$+ ((F(\partial_t) - F(\partial_t)) t) * f''_h =: I + II.$$
For the first term, Lemma 4.1 with $\mu = \alpha$ and $\nu = 2$ and the $L^2(\Omega)$-stability of $P_h$ give $\|I\| \leq c t_n^{\alpha - 1} \tau^2 \|f'(0)\|$. For the second term, Lemma 4.1 with $\mu = \alpha + 1$ and $\nu = 2$ and the $L^2(\Omega)$-stability of $P_h$ yield

$$\|II\| \leq \|(F(\partial \tau) - F(\partial \tau)) tf_h'(0)\| + \int_0^{t_n} \|((F(\partial \tau) - F(\partial \tau)) (t_n - s) f_h''(s))\| \, ds \leq c \tau^2 \left( t_n^\alpha \|f_h'(0)\| + \int_0^{t_n} (t_n - s) s \|f_h''(s)\| \, ds \right),$$

which completes the proof.

5 Numerical experiments

In this section, we conduct numerical experiments to validate the convergence theory presented in Section 4. We provide two sets of numerical examples with zero and nonzero right-hand side data $f$, defined on the square domain $\Omega = (0, 1)^2$. We examine separately the spatial and temporal convergence rates at a fixed final time $T = 0.5$. In our computation, we fix the parameters $\mu = a = b = 1$. For the homogeneous problem, we consider the following smooth and nonsmooth initial data:

(a) $v(x,y) = xy(1-x)(1-y) \in \dot{H}^2(\Omega)$,
(b) $v(x,y) = \chi_{(0,1/2) \times (0,1)}(x,y)$, where $\chi_S$ is the characteristic function of the set $S$. Here $v \in \dot{H}^\epsilon(\Omega)$ for $0 \leq \epsilon < 1/2$.

The exact solution is difficult to obtain for these examples, so we compute a reference solution on a very refined mesh with $h = 1/512$ and employ a time step size $\tau = 1/500$. To examine the temporal convergence rates of the proposed BE and SBD schemes, we employ a uniform mesh in time with a time step size $\tau = T/N$, and choose a sufficiently small mesh size $h = 1/512$ so that the error incurred by spatial discretization is negligible. We measure the error $e^n := u(t_n) - U^n$ by the normalized $L^2(\Omega)$-norm $\|e^n\|/\|v\|$. The numerical results are presented in Table 1 for cases (a) and (b) with different values of $\alpha$ and $\beta$.

In the table, the rate refers to the empirical convergence rate, when the time step size $\tau$ halves. The numerical results show convergence rates of order $O(\tau)$ and $O(\tau^2)$ for the BE and SBD schemes, respectively. We also observe that both schemes exhibit a steady convergence for both smooth and nonsmooth data, which confirms the theoretical convergence rates.

To study the temporal error more closely, we investigate the prefactors in Theorems 4.1 and 4.3. By neglecting the spatial error (i.e., dropping the $O(h^2)$ term) and taking the number of time steps $N$ as fixed, the error bounds in Theorems 4.1 show as $t_N \to 0$,

$$\|U_h^N - u(t_N)\| \leq c t_N^{(\alpha - \beta + 1)} N^{-1} \|v\|_{\dot{H}^2(\Omega)},$$

and

$$\|U_h^N - u(t_N)\| \leq c N^{-1} \|v\|.$$
Table 1  \(L^2\)-error for cases (a) and (b) with \(h = 1/512\)

| \(\alpha\), \(\beta\) | Case (a) | Case (b) |
|----------------|---------|---------|
| \(\alpha = 0.25\) | \(\beta = 0.75\) | \(\beta = 0.5\) | \(\beta = 0.25\) | \(\beta = 0.5\) |
| \(N\) | BE | Rate | SBD | Rate | BE | Rate | SBD | Rate |
| 20 | 1.43e-3 | 7.69e-5 | 8.93e-4 | 4.83e-5 | 2.76e-4 | 4.37e-5 | 1.58e-4 | 2.51e-5 |
| 40 | 7.10e-4 | 1.01 | 1.85e-5 | 2.06 | 9.58e-5 | 1.53 | 1.07e-5 | 2.03 | 5.50e-5 | 1.53 | 5.18e-6 | 2.16 |
| 80 | 3.54e-4 | 1.01 | 4.46e-6 | 2.05 | 3.91e-5 | 1.29 | 2.50e-6 | 2.10 | 1.01e-5 | 1.15 | 3.21e-7 | 2.16 |
| 160 | 1.77e-4 | 1.00 | 1.02e-6 | 2.13 | 1.76e-5 | 1.15 | 5.58e-7 | 2.16 | 1.01e-5 | 1.15 | 3.21e-7 | 2.16 |
| 320 | 8.82e-5 | 1.00 | 1.67e-7 | 2.61 | 3.31e-5 | 0.96 | 1.66e-5 | 2.62 | 1.26e-3 | 0.93 | 1.26e-5 | 2.63 |

By interpolating these results, we obtain for \(q \in [0, 2]\)

\[
\| U^N_h - u(t_N) \| \leq c (\alpha - \beta + 1)^{q/2} N^{-1} \| v \|_{\dot{H}^q(\Omega)}. \tag{5.1}
\]

Similarly, for the SBD scheme, there holds from Theorem 4.3 as \(t_N \to 0\),

\[
\| U^N_h - u(t_N) \| \leq c (\alpha - \beta + 1)^{q/2} N^{-2} \| v \|_{\dot{H}^q(\Omega)}. \tag{5.2}
\]

For fixed \(N = 10\) and \(h = 1/512\), we present the computed normalized \(L^2\)-norm of the error in Table 2 for cases (a) and (b) as \(t_N \to 0\). The predicted rate with respect to \(t_N\) computed from (5.1) and (5.2) is given between brackets. The temporal error should theoretically behaves like \(O(t_N^{(\alpha-\beta+1)/2})\) for both the BE and SBD schemes.

From the table, we notice that the error decreases like \(O(t_N^{1/2})\) in the smooth case (a),

Table 2  \(L^2\)-error for cases (a) and (b) with \(\alpha = 0.25, \beta = 0.75\) as \(t_N \to 0\), \(N = 10\)

| Case | Meth. | 1e-3 | 1e-4 | 1e-5 | 1e-6 | 1e-7 | Rate |
|------|-------|------|------|------|------|------|------|
| (a)  | BE    | 5.37e-3 | 2.41e-3 | 8.54e-4 | 2.82e-4 | 9.09e-5 | 0.49 (0.50) |
|      | SBD   | 3.99e-4 | 1.47e-4 | 4.85e-5 | 1.57e-5 | 5.01e-6 | 0.49 (0.50) |
| (b)  | BE    | 5.38e-3 | 4.32e-3 | 3.21e-3 | 2.45e-3 | 1.85e-3 | 0.12 (0.125) |
|      | SBD   | 4.99e-4 | 3.63e-4 | 2.78e-4 | 2.11e-4 | 1.59e-4 | 0.12 (0.125) |
whereas it decreases like $O(t^{1/8})$ in the nonsmooth case (b). The results agree well with the convergence theory.

Now, we investigate the spatial discretization error. To do so, we fix the time step size $\tau = 1/500$ and perform the computations using the SBD scheme so that the temporal discretization error is negligible. In Table 3, we list the normalized $L^2(\Omega)$-norm and the $L^\infty(\Omega)$-norm of the error for cases (a) and (b). We observe a convergence rate $O(h^2)$ for the $L^2(\Omega)$-norm of the error for smooth and nonsmooth initial data, which confirm the predicted rates. The results also show the validity of the convergence rates in the $L^\infty(\Omega)$-norm (ignoring a logarithmic factor).

By neglecting the temporal error and fixing $h$, we investigate the spatial prefactors in Theorems 4.1 and 4.3. In Table 4, we report the numerical results obtained as $t \to 0$ with $h$ being constant. The results indicate that the spatial error essentially

### Table 3  Error for cases (a) and (b) with $\tau = 1/500$

| $\alpha$, $\beta$ | $M$ | $L^2$-error | Rate | $L^\infty$-error | Rate | $L^2$-error | Rate | $L^\infty$-error | Rate |
|-------------------|-----|--------------|------|------------------|------|--------------|------|------------------|------|
| $\alpha = 0.25$   |     |              |      |                  |      |              |      |                  |      |
| $\beta = 0.75$    | 8   | 2.50e-3      |       | 1.78e-4          |       | 1.35e-3      |       | 3.92e-3          |       |
|                  | 16  | 6.44e-4      | 1.96  | 4.63e-5          | 1.95  | 3.47e-4      | 1.96  | 1.27e-3          | 1.62  |
|                  | 32  | 1.62e-4      | 1.99  | 1.17e-5          | 1.99  | 8.75e-5      | 1.99  | 3.91e-4          | 1.70  |
|                  | 64  | 4.02e-5      | 2.01  | 2.90e-6          | 2.01  | 2.18e-5      | 2.00  | 1.16e-4          | 1.75  |
|                  | 128 | 9.67e-6      | 2.06  | 7.08e-7          | 2.04  | 5.35e-6      | 2.03  | 3.34e-5          | 1.79  |
| $\alpha = 0.5$   |     |              |      |                  |      |              |      |                  |      |
| $\beta = 0.5$    | 8   | 1.73e-5      |       | 1.16e-6          |       | 9.45e-6      |       | 1.34e-5          |       |
|                  | 16  | 4.99e-6      | 1.80  | 3.40e-7          | 1.77  | 2.70e-6      | 1.81  | 3.90e-6          | 1.78  |
|                  | 32  | 1.29e-6      | 1.95  | 8.85e-8          | 1.94  | 6.95e-7      | 1.96  | 1.01e-6          | 1.95  |
|                  | 64  | 3.22e-7      | 2.00  | 2.21e-8          | 2.00  | 1.73e-7      | 2.00  | 2.53e-7          | 2.00  |
|                  | 128 | 7.69e-8      | 2.07  | 5.31e-9          | 2.06  | 4.13e-8      | 2.07  | 6.07e-8          | 2.06  |
| $\alpha = 0.75$ |     |              |      |                  |      |              |      |                  |      |
| $\beta = 0.25$  | 8   | 1.61e-2      |       | 1.10e-3          |       | 9.59e-3      |       | 1.63e-2          |       |
|                  | 16  | 4.14e-3      | 1.96  | 2.88e-4          | 1.93  | 2.51e-3      | 1.93  | 4.31e-3          | 1.92  |
|                  | 32  | 1.04e-3      | 1.99  | 7.28e-5          | 1.99  | 6.33e-4      | 1.99  | 1.08e-3          | 1.99  |
|                  | 64  | 2.57e-4      | 2.01  | 1.81e-5          | 2.01  | 1.57e-4      | 2.01  | 2.69e-4          | 2.01  |
|                  | 128 | 6.13e-5      | 2.07  | 4.35e-6          | 2.06  | 3.74e-5      | 2.07  | 6.44e-5          | 2.06  |

### Table 4  $L^2$-error for cases (a) and (b) with $\alpha = 0.25$, $\beta = 0.75$: $t \to 0$, $h = 1/64$, $N = 500$

| Method | 1e-3  | 1e-4  | 1e-5  | 1e-6  | 1e-7  | Rate |
|--------|-------|-------|-------|-------|-------|------|
| (a)    |       |       |       |       |       |      |
| BE     | 3.46e-4| 4.41e-4| 4.86e-4| 5.04e-4| 5.12e-4| 0.01 (0) |
| SBD    | 4.37e-4| 4.94e-4| 5.08e-4| 5.12e-4| 5.15e-4| 0.00 (0) |
| (b)    |       |       |       |       |       |      |
| BE     | 3.55e-4| 7.88e-4| 1.77e-3| 4.00e-3| 9.12e-3| 0.36 (− 0.375) |
| SBD    | 4.05e-4| 8.28e-4| 1.80e-3| 4.02e-3| 9.14e-3| 0.35 (− 0.375) |
stays unchanged in the smooth case, whereas it deteriorates as $t \to 0$ in the nonsmooth case. Since the initial data in case (b) belongs to $H^\epsilon(\Omega)$ for $0 \leq \epsilon < 1/2$, it is expected that the error grows like $O((\beta - \alpha - 1)(2 - q)/2)$. Hence, the empirical convergence rate in Table 4 agrees with the theoretical one, i.e., $3(\beta - \alpha - 1)/4 = -0.375$ for $\alpha = 0.25$ and $\beta = 0.75$.

In Fig. 1, we display the profile of the numerical solution in case (b) when $a = \mu = 1$, $b = 0$, and $\alpha = 0.5$, at different times. We observe that the solution oscillates with a slow decay, which reflects in particular the wave feature of the model (1.1) when $b = 0$. The oscillations in the figure are not numerical artifacts. Indeed, they are inherited from the $L^2$-projection $P_h v$ which is oscillatory as shown in Fig. 1a. Furthermore, as observed from Fig. 2, the solution is diffusive (the oscillations are quickly damped) for small $\alpha$ and the oscillations are more pronounced for larger $\alpha$, with a slower decay, showing again the wave feature of the model. In Fig. 3, we display the profile of the numerical solution when $b = a = \mu = 1$ and $\alpha = \beta = 0.5$. The oscillations are quickly damped and not visible as in the previous case. For further details on the behavior of the solution of (1.1), see [4] for the case with $a = 0$ and [4, 31] when $a \neq 0$.

![Fig. 1 Numerical solution for case (b) with $b = 0$ and $\alpha = 0.5$, computed using the SBD scheme with $h = 1/32$](image)
Lastly, we consider the inhomogeneous problem (1.1) with a zero initial data $v$ and briefly examine the spatial convergence rates established in Section 4. We consider the following cases with smooth and nonsmooth right-hand side data $f$:

(c) $v = 0$ and $f(x, y, t) = \left( 2t + \frac{2at^{1-\alpha}}{\Gamma(2-\alpha)} + 8\pi^2\mu t^2 + \frac{16\pi^2\mu b t^{2-\beta}}{\Gamma(3-\beta)} \right) \sin(2\pi x) \sin(2\pi y)$.

(d) $v = 0$ and $f(x, y, t) = (1 + t^{0.2})\chi_{(0,1/2)\times(0,1)}(x, y)$.

For case (c), the exact solution is given by $u(x, y, t) = t^2 \sin(2\pi x) \sin(2\pi y)$, whereas an explicit form is not available in case (d). A reference solution is then computed on the very fine grid. The results reported in Table 5 are obtained using the standard Galerkin FEM in space and the SBD method in time. To validate the error bounds derived in Section 4, we choose a very small time step $\tau = 1/500$ and examine the spatial accuracy at the final time $T = 0.5$. From the table, a convergence
rate of order $O(h^2)$ is observed, for both smooth and nonsmooth right-hand side data, which clearly confirm our theoretical results.

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**References**

1. Abdullah, M., Butt, A.R., Raza, N., Haque, E.U.: Semi-analytical technique for the solution of fractional Maxwell fluid. Can. J. Phys. 94, 472–478 (2017)

2. Al-Maskari, M., Karaa, S.: The lumped mass FEM for a time-fractional cable equation. Appl. Numer. Math. 132, 73–90 (2018)

3. Bazhlekova, E.: Subordination principle for a class of fractional order differential equations. Mathematics 3, 412–427 (2015)

4. Bazhlekova, E., Bazhlekov, I.: Viscoelastic flows with fractional derivative models: computational approach by convolutional calculus of Dimovski. Fract. Calc. Appl. Anal. 17, 954–976 (2014)

5. Bazhlekova, E., Bazhlekov, I.: Peristaltic transport of viscoelastic bio-fluids with fractional derivative models. Biomath 5, 1605151 (2016)

6. Bazhlekova, E., Bazhlekov, I.: On the Rayleigh-Stokes problem for generalized fractional Oldroyd-B fluids. AIP Conf. Proc. 1684, 080001-1–080001-12 (2015)

7. Bazhlekova, E., Jin, B., Lazarov, R., Zhou, Z.: An analysis of the Rayleigh-Stokes problem for a generalized second-grade fluid. Numer. Math. 131, 1–31 (2016)

8. Chen, C.M., Liu, F., Anh, V.: Numerical analysis of the Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivatives. Appl. Math. Comput. 204, 340–351 (2008)

9. Chen, C.M., Liu, F., Anh, V.: A Fourier method and an extrapolation technique for Stokes’ first problem for a heated generalized second grade fluid with fractional derivative. J. Comput. Appl. Math. 223, 777–789 (2009)

10. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. SIAM, Philadelphia (2002)

11. Cuesta, E., Lubich, C., Palencia, C.: Convolution quadrature time discretization of fractional diffusion-wave equations. Math. Comput. 75, 673–696 (2006)

12. Dehghan, M., Abbaszadeh, M.: A finite element method for the numerical solution of Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivatives. Eng. Comput. 33, 587–605 (2017)

13. Fetecau, C., Jamil, M., Fetecau, C., Vieru, D.: The Rayleigh-Stokes problem for an edge in a generalized Oldroyd-B fluid. Z. Angew. Math. Phys. 60, 921–933 (2009)

14. Fujita, H., Suzuki, T.: Evolution problems. Handbook of Numerical Analysis, vol. II, pp. 789–928, Handb. Numer. Anal., II. North-Holland, Amsterdam (1991)

15. Jamil, M., Rauf, A., Zafar, A.A., Khan, N.A.: New exact analytical solutions for Stokes’ first problem of Maxwell fluid with fractional derivative approach. Comput. Math. Appl. 62, 1013–1023 (2011)
16. Khan, M., Ali, S.H., Hayat, T., Fetecau, C.: MHD flows of a second grade fluid between two side walls perpendicular to a plate through a porous medium. Int. J. Non Linear Mech. 43, 302–319 (2008)
17. Khan, M., Anjum, A., Fetecau, C., Qi, H.: Exact solutions for some oscillating motions of a fractional Burgers’ fluid. Math. Comput. Model. 51, 682–692 (2010)
18. Khan, M., Anjum, A., Qi, H., Fetecau, C.: On exact solutions for some oscillating motions of a generalized Oldroyd-B fluid. Z. Angew. Math. Phys. 61, 133–145 (2010)
19. Lin, Y., Jiang, W.: Numerical method for Stokes’ first problem for a heated generalized second grade fluid with fractional derivative. Numer. Methods Partial Differential Equations 27, 1599–1609 (2011)
20. Lubich, C.: Discretized fractional calculus. SIAM J. Math. Anal. 17, 704–719 (1986)
21. Lubich, C.: Convolution quadrature and discretized operational calculus-I. Numer. Math. 52, 129–145 (1988)
22. Lubich, C.: Convolution quadrature revisited. BIT 44, 503–514 (2004)
23. McLean, W., Thomée, V.: Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term. Math. Comput. 65, 1–17 (1996)
24. McLean, W., Thomée, V.: Maximum-norm error analysis of a numerical solution via Laplace transformation and quadrature of a fractional order evolution equation. IMA J. Numer. Anal. 30, 208–230 (2010)
25. Mohebbi, A., Abbaszadeh, M., Dehghan, M.: Compact finite difference scheme and RBF meshless approach for solving 2D Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivatives. Comput. Methods Appl. Mech. Eng. 264, 163–177 (2013)
26. Prüss, J.: Evolutionary Integral Equations and Applications Monographs in Mathematics, vol. 87. Basel, Birkhäuser Verlag (1993)
27. Rasheed, A., Wahab, A., Shah, S.Q., Nawaz, R.: Finite difference-finite element approach for solving fractional Oldroyd-B equation. Adv. Difference Equ. 2016(236), 21 (2016)
28. Thomée, V.: Galerkin Finite Element Methods for Parabolic Problems. Springer, Berlin (2006)
29. Tripathi, D., Pandey, S.K., Das, S.: Peristaltic flow of viscoelastic fluid with fractional Maxwell model through a channel. Appl. Math Comput. 215, 3645–3654 (2010)
30. Tripathi, D.: Peristaltic transport of fractional Maxwell fluids in uniform tubes: applications in endoscopy. Comput. Math. Appl. 62, 1116–1126 (2011)
31. Vasileva, D., Bazhlekov, I., Bazhlekova, E.: Alternating direction implicit schemes for two-dimensional generalized fractional Oldroyd-B fluids. AIP Conf. Proc. 1684, 080014-1–080014-16 (2015)
32. Wu, C.: Numerical solution for Stokes’ first problem for a heated generalized second grade fluid with fractional derivative. Appl. Numer. Math. 59, 2571–2583 (2009)
33. Zhao, C., Yang, C.: Exact solutions for electro-osmotic flow of viscoelastic fluids in rectangular micro-channels. Appl. Math. Comput. 211, 502–509 (2009)
34. Zhu, P., Xie, S., Wang, X.: Nonsmooth data error estimates for FEM approximations of the time fractional cable equation. Appl. Numer. Math. 121, 170–184 (2017)