Structure of sufficient quantum coarse-grainings

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Abstract: Let \(\mathcal{H}\) and \(\mathcal{K}\) be finite dimensional Hilbert spaces, \(T : B(\mathcal{H}) \to B(\mathcal{K})\) be a coarse-graining and \(D_1, D_2\) be density matrices on \(\mathcal{H}\). In this paper the consequences of the existence of a coarse-graining \(\beta : B(\mathcal{K}) \to B(\mathcal{H})\) satisfying \(\beta T(D_s) = D_s\) are given. (This means that \(T\) is sufficient for \(D_1\) and \(D_2\).) It is shown that \(D_s = \sum_{p=1}^{r_s} \lambda_s(p) S_s^\mathcal{H}(p) R^\mathcal{H}(p)\) \((s = 1, 2)\) should hold with pairwise orthogonal summands and with commuting factors and with some probability distributions \(\lambda_s(p)\) for \(1 \leq p \leq r\) \((s = 1, 2)\). This decomposition allows to deduce the exact condition for equality in the strong subadditivity of the von Neumann entropy.

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1 Introduction

Let \(\mathcal{H}\) and \(\mathcal{K}\) be finite dimensional Hilbert spaces and let \(T : B(\mathcal{H}) \to B(\mathcal{K})\) be a trace-preserving completely positive (or at least 2-positive) mapping. The mapping \(T\) sends density matrices acting on \(\mathcal{H}\) into density matrices acting on \(\mathcal{K}\). Such a mapping is called channeling transformation in quantum information theory, if \(\mathcal{H} = \mathcal{K}\), then \(T\) may describe the dynamical change of state. We use the term coarse-graining, because the statistical aspects get emphasis. Let \(D_1\) be a density of a quantum state on \(\mathcal{H}\).

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Then the coarse-grained density $T(D_1)$ contains less information about the original quantum state and provides a partial knowledge of $D_1$. The statistical inference is manifested by a mapping $\beta : B(\mathcal{K}) \to B(\mathcal{H})$ and in the good case $\beta T(D_1) = D_1$ and the original state is recovered.

In this paper, we study the scenario, where two states, density matrices $D_1$ and $D_2$, are given and we want to distinguish between them. If this is not more difficult than distinguishing between $T(D_1)$ and $T(D_2)$, then the coarse-graining is called sufficient for this pair. Formally we say that $T$ is sufficient for $D_1$ and $D_2$ if there exists a trace preserving 2-positive mapping $\beta : B(\mathcal{K}) \to B(\mathcal{H})$ such that

$$\beta T(D_1) = D_1 \quad \text{and} \quad \beta T(D_2) = D_2. \quad (1)$$

This $\beta$ plays the role of recovery and it is not at all unique. Early references concerning sufficiency in this quantum mechanical setting are [11, 12] and our general reference is Chap. 9 of [8]. (In classical mathematical statistics sufficiency is a standard subject included in most books, our terminology is close to [16].)

Algebraically $\beta$ in (1) is the left inverse of $T$ as far as the densities $D_1$ and $D_2$ are concerned. It is easy to give an example where such a $\beta$ exists. If $T$ is implemented by a unitary $U : \mathcal{H} \to \mathcal{K}$, then $\beta$ can be implemented by $U^* : \mathcal{K} \to \mathcal{H}$. This is a trivial situation. It is a bit less trivial that $\beta$ exists also in the case when $D_1 = D_2$.

The aim of this paper is to characterize the situation when the above $\beta$ exists. Actually, this was done a long time ago. It was proved in [11] (see also [14]) that $\beta$ exists if and only if

$$T^*\left(T(D_2)^i T(D_1)^{-i}\right) = D_2^i D_1^{-i} \quad (2)$$

for all real $t$, where $T^*$ is the standard transpose of $T$. Although this is a necessary and sufficient condition, it is not completely satisfactory, since it does not give any hint about the interrelation of $T$, $D_1$ and $D_2$.

The main result of the present paper is to show that (2) implies the decomposition

$$D_s = \sum_{p=1}^{r} \lambda_s(p) S^H_s(p) R^H(p),$$

where $S^H_s(p)$ commutes with $R^H(p)$, there are pairwise orthogonal projections $q_p$ such that $S^H_s(p)$ and $R^H(p)$ are supported in $q_p$ for all $1 \leq p \leq r$ and $\lambda_s(p)$ are some probability distributions ($s = 1, 2$). The point is that the second factor is the same for $s = 1$ and for $s = 2$.

Since the complete positivity of $T$ is not assumed, the Stinespring dilation cannot be used. For this reason and also due to the algebraic methods, our approach is different from [5], where the conditions $T(D_1) = D_1$ and $T(D_2) = D_2$ are studied and variety of physical motivations is given.
We apply our structure theorem to deduce a sufficient and necessary condition for the equality case in the strong subadditivity of quantum entropy and obtain the result of [4] as an application.

In the whole paper, an algebraic approach is followed.

2 Preliminaries

Let $\mathcal{H}$ and $\mathcal{K}$ be finite dimensional Hilbert spaces. Recall that 2-positivity of $\tau : B(\mathcal{H}) \to B(\mathcal{K})$ means that

$$\begin{bmatrix} \tau(A) & \tau(B) \\ \tau(C) & \tau(D) \end{bmatrix} \geq 0 \text{ if } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \geq 0.$$  

It is well-known that a 2-positive unit-preserving mapping $\tau$ satisfies the Schwarz inequality $\tau(A^*A) \geq \tau(A)^*\tau(A)$.

The most important 2-positive mappings are of the form $\tau(A) = \sum_i L_i A L_i^*$ with some linear operators $L_i : \mathcal{H} \to \mathcal{K}$. (This is the Kraus representation of the completely positive maps.) We call $L_i$ operator coefficients.

In this paper $T$ always denotes a trace-preserving 2-positive mapping $T : B(\mathcal{H}) \to B(\mathcal{K})$ and we assume that the density matrices $D_1, D_2, T(D_1)$ and $T(D_2)$ are all invertible. If $T$ admits a Kraus representation, then the operator coefficients satisfy $\sum_i L_i^*L_i = I$. Lots of applications of such mappings are given in [7] in the setting of quantum information theory.

The spaces $B(\mathcal{H})$ and $B(\mathcal{K})$ are Hilbert spaces when they are endowed with the standard Hilbert-Schmidt inner product

$$\langle A, B \rangle := \text{Tr} A^*B.$$  

For a trace-preserving 2-positive mapping $T : B(\mathcal{H}) \to B(\mathcal{K})$, its adjoint $T^*$ is a unital 2-positive mapping. It follows that $T^* : B(\mathcal{K}) \to B(\mathcal{H})$ satisfies the Schwarz inequality.

The spaces $B(\mathcal{H})$ and $B(\mathcal{K})$ admit also the inner products

$$\langle A, B \rangle_{D_1} := \text{Tr} A^*D_1^{1/2}BD_1^{1/2} \quad (A, B \in B(\mathcal{H}))$$  

and

$$\langle X, Y \rangle_{T(D_1)} := \text{Tr} X^*T(D_1)^{1/2}YT(D_1)^{1/2} \quad (X, Y \in B(\mathcal{K})).$$

The dual $\alpha$ of $T^*$ with respect to these inner products is 2-positive and unital, and it is characterized by the properties

$$\alpha : B(\mathcal{H}) \to B(\mathcal{K}), \quad \langle X, \alpha(A) \rangle_{T(D_1)} = \langle T^*(X), A \rangle_{D_1} \quad (A \in B(\mathcal{H}), X \in B(\mathcal{K})).$$  

(4)
It is easy to give $\alpha$ concretely:

$$\alpha(A) = T(D_1)^{-1/2} T \left( D_1^{1/2} A D_1^{1/2} \right) T(D_1)^{-1/2}$$  \hspace{1cm} \text{(5)}$$

It is seen from this formula that if $T$ has operator coefficients $L_i$, then the operator coefficients of $\alpha$ are $T(D_1)^{-1/2} L_i D_1^{1/2}$.

Note that if $T^*$ is an embedding, then $\alpha$ is the generalized conditional expectation introduced in [11], see [10] for generalizations and for a systematic study. This kind of dual was called transpose in [8] and makes appearance in several places, for example in connection with the best quantum recovery map [3], or in the theory of Connes-Narnhofer-Thirring dynamical entropy [8].

The standard dual $T^\#: B(K) \to B(H)$ of $\alpha$ is trace preserving. The next few lines follow simply from the definition of $T^\#$ and the concrete form of the above inner products:

$$\text{Tr} \, T^\#(X) A = \langle T^\#(X^*), A \rangle = \langle X^*, \alpha(A) \rangle$$

$$= \text{Tr} \, X \alpha(A) = \langle T(D_1)^{-1/2} X^* T(D_1)^{-1/2}, \alpha(A) \rangle_{T(D_1)}$$

$$= \langle T^*(T(D_1)^{-1/2} X^* T(D_1)^{-1/2}), A \rangle_{D_1}$$

$$= \text{Tr} \, T^*(T(D_1)^{-1/2} X T(D_1)^{-1/2}) D_1^{1/2} A D_1^{1/2}.$$  

Hence

$$T^\#(X) = D_1^{1/2} T^* \left( T(D_1)^{-1/2} X T(D_1)^{-1/2} \right) D_1^{1/2}. \hspace{1cm} \text{(6)}$$

Observe that $T^\#(T(D_1)) = D_1$.

In the analysis of condition (11) we first establish that the existence of $\beta$ implies that from the set of all possible $\beta$’s satisfying (11) we can choose one canonically, namely $T^\#$. Remember that the definition of $T^\#$ depends on the density $D_1$, although this dependence is not included in the notation.

Assume now the existence of $\beta$ for (11). According to Theorem 2 in [14] we have (2) for all real $t$. Under our hypothesis $u_t := T(D_1)^{it} T(D_2)^{-it}$ and $w_t := D_1^{it} D_2^{-it}$ are unitaries and condition (2) tells us that $u_t \in \mathcal{A}_T$, for every $t \in \mathbb{R}$, see Lemma 1 below for $\mathcal{A}_T$- and its properties. Consequently, $T^*(u_t Y) = T^*(u_t) T^*(Y)$ and by analytic continuation we have

$$T^* \left( T(D_1)^{-1/2} T(D_2) T(D_1)^{-1/2} \right) = T^* \left( T(D_1)^{-1/2} T(D_2) T(D_1)^{-1/2} \right) T^* \left( T(D_2)^{1/2} T(D_1)^{-1/2} \right)$$

$$= D_1^{-1/2} D_2^{1/2} D_2^{1/2} D_1^{-1/2}$$

which implies

$$D_1^{1/2} T^* \left( T(D_1)^{-1/2} T(D_2) T(D_1)^{-1/2} \right) D_1^{1/2} = D_2. \hspace{1cm} \text{(7)}$$

Therefore the relation $T^\#(T(D_2)) = D_2$ can be concluded and in this way the following has been shown in [11].
**Proposition 1** If there exists a trace preserving 2-positive mapping \( \beta : B(\mathcal{K}) \to B(\mathcal{H}) \) such that \( \beta T(D_1) = D_1 \) and \( \beta T(D_2) = D_2 \), then \( T^* T(D_1) = D_1 \) and \( T^* T(D_2) = D_2 \).

Consider now the 2-positive unital mapping \( (T^* T)^* = T^* \circ \alpha =: \gamma_\mathcal{H} \). If \( L_i \) are the operator coefficients of \( T \), then \( \gamma_\mathcal{H} \) has coefficients 

\[
    L_i^* T(D_1)^{-1/2} L_i D_1^{1/2}.
\]

Let \( \mathcal{D}_\mathcal{H} \) be the set of its fixed points. Since \( \gamma_\mathcal{H} \) leaves the states corresponding to \( D_1 \) and \( D_2 \) invariant, the mean ergodic theorem applies and tells us the existence of conditional expectation \( E \) from \( B(\mathcal{H}) \) to \( \mathcal{D}_\mathcal{H} \) which commutes with \( \gamma_\mathcal{H} \) and has the property \( E^*(D_s) = D_s \) \((s = 1, 2)\). Takesaki’s theorem ([17, 18], cf. Theorem 4.5 in [8]) tells us that

\[
    D^t \mathcal{D}_\mathcal{H} D^{-it} \subset \mathcal{D}_\mathcal{H} \quad (t \in \mathbb{R}).
\]

(In another formulation, \( \mathcal{D}_\mathcal{H} \) is stable under the modular groups, see Chapter 4 of [8] for a concise overview of the modular theory.)

### 3 Structure of certain unitaries

In order to understand condition (8), we analyze the relation

\[
    u^* A u \subset A
\]

for a unitary \( u \) and for a subalgebra \( A \) of \( B(\mathcal{H}) \). The result is formulated in the propositions below. We shall use the emerging structure in the next section but the result is interesting in itself.

Since \( A \) is finite dimensional, it is isomorphic to a direct sum of full matrix algebras, so in an appropriate basis, elements of \( A \) have a blockdiagonal form

\[
    A = \oplus_{(m, d)} \left( \oplus_{i=1}^{K(m, d)} \left( \oplus_{i=1}^{m} A(m, d, i) \right) \right),
\]

where \( m \) denotes the multiplicity and \( d \) the dimension of the block \( A(m, d, i) \).

For example, if there are three different blocks with multiplicity two, two of them with dimension two and one of them with dimension three, and another block with multiplicity four and dimension one, then \( K(2, 2) = 2 \), \( K(2, 3) = 1 \), \( K(4, 1) = 1 \) and every element \( A \in A \) has the form

\[
    A = \text{Diag} (B_1, B_1, B_2, B_2, C, C, d, d, d, d)
    = \text{Diag} (A(2, 2, 1), A(2, 2, 1), A(2, 2, 2), A(2, 2, 2), A(2, 3, 1), A(2, 3, 1),
    A(4, 1, 1), A(4, 1, 1), A(4, 1, 1), A(4, 1, 1))
\]

with \( B_1, B_2 \in \mathcal{M}_2, C \in \mathcal{M}_3 \) and \( d \in \mathcal{M}_1 = \mathbb{C} \).

Let \( P_{m, d} \) be the projection in \( A \) corresponding to multiplicity \( m \) and dimension \( d \), that is

\[
    P_{m, d}(m', d', i) := \delta(m', m) \delta(d', d) I_d \quad (1 \leq i \leq K(m, d)),
\]

where \( \delta \) is the Kronecker delta.
where $I_d$ is the $d \times d$ identity matrix, and let

$$P_m := \oplus_d P_{m,d}$$

be the projection corresponding to the multiplicity $m$. We denote by $\mathcal{H}_m$ the range of $P_m$. Note that $P_{m,d}$ and hence $P_m$ commutes with elements of $A$, so

$$\mathcal{A}_{m,d} := P_{m,d}A \quad \text{and} \quad \mathcal{A}_m := P_mA$$

are unital algebras with unit $P_{m,d}$ and $P_m$, respectively.

We fix an orthonormal basis

$$\{e(m, d, k) : 1 \leq k \leq mdK(m, d)\}$$

in the range of $P_{m,d}$ for every possible $m$ and $d$, such that $|e(m, d, k)\rangle\langle e(m, d, k)| \in \mathcal{A}$.

**Proposition 2** Let $u$ be a unitary in $B(\mathcal{H})$ such that the map $A \mapsto u^*Au$ leaves $\mathcal{A}$ invariant. Then $u$ commutes with $P_m$ for every multiplicity $m$.

**Proof.** The statement is trivial when only one multiplicity exists, and we apply mathematical induction in the number of multiplicities.

Note that the rank of any minimal projection in $P_mA$ is $m$. Let $m_1$ denote the smallest multiplicity, and let $q \in P_{m_1}A$ be a minimal projection, then $q$ is of rank $m_1$. $P_mu^*qu$ is a projection again and its rank is at most $m_1$. Since all non-zero projections in $P_mA$ has rank at least $m$, we can conclude that $P_mu^*qu = 0$ if $m > m_1$. Every element of $P_{m_1}A$ is a linear combination of the above $q$’s, hence we have $P_mu^*au = 0$ if $m > m_1$ and $a \in P_{m_1}A$.

Since $|e(m_1, d, k)\rangle\langle e(m_1, d, k)| \leq P_{m_1}$ for every possible $d, k$, it follows that

$$P_mu^*|e(m_1, d, k)\rangle\langle e(m_1, d, k)|uP_m \leq P_m(u^*P_{m_1}u) = 0$$

if $m > m_1$. So we can conclude that $u^*e(m_1, d, k) \in \mathcal{H}_{m_1}$ which gives that both $\mathcal{H}_{m_1}$ and its orthogonal complement are invariant subspaces for $u$, that is, $P_{m_1}u = uP_{m_1}$.

Now we can restrict the whole problem to the orthogonal complement of $\mathcal{H}_{m_1}$ and use induction hypothesis in the number of multiplicities. \(\square\)

We have obtained that $u$ has a blockdiagonal structure $u = \oplus_m P_muP_m$, and to explore the finer structure of $u$, we can restrict our attention to the case when all the multiplicities are the same number $m$, i.e. the elements of $\mathcal{A}$ have the form

$$A = \oplus_d \left( \oplus_{i=1}^{K(d)} (\oplus_{i=1}^{m} A(d, i)) \right).$$

As before, we can define projections $P_{d,i}$ ($1 \leq i \leq K(d)$) by the formula

$$P_{d,i}(d', i') := \delta(d', d)\delta(i', i)I_d,$$
and the projection corresponding to dimension \( d \) is
\[
P_d := \bigoplus_{i=1}^{K(d)} P_{d,i}.
\]
Again, all these projections commute with all elements of \( \mathcal{A} \).

**Proposition 3** In the above setting \( u \) commutes with \( P_d \) for every dimension \( d \), and so \( u \) has the blockdiagonal structure
\[
u = \bigoplus_d P_d u P_d.
\]

**Proof.** Since \( \mathcal{A} \) is isomorphic to \( \mathcal{M} := \bigoplus_d \bigoplus_{i=1}^{K(d)} \mathcal{M}_d \), then \( \text{Ad}_u \) induces an automorphism \( \gamma \) of \( \mathcal{M} \). The inclusion matrix of \( \gamma \) is a permutation matrix corresponding to a permutation \( \tau \) of the set of all possible pairs \((d, i)\) such that
\[
\tau(d, i) = (d', i') \quad \text{gives} \quad d = d'.
\]
This implies that
\[
P_d u^* P_d u = 0 \quad \text{for} \quad d \neq d',
\]
which, by the same argument as in Proposition 2, implies the desired statement. \( \square \)

In the view of the above Propositions we can suppose that all the blocks in \( \mathcal{A} \) have the same multiplicity \( m \) and the same dimension \( d \), consequently \( \mathcal{A} \) is isomorphic to \( \bigoplus_{j=1}^{K} \mathcal{M}_d \). In this case \( \text{dim}(\mathcal{H}) = mdK \), and \( \mathcal{B} \) is isomorphic to \( \mathcal{M}_d \otimes \mathcal{M}_m \otimes \mathcal{M}_K \). Elements of \( \mathcal{A} \) have the form
\[
\sum_i A_i \otimes I_m \otimes E_{ii},
\]
where \( A_i \) is an element of \( \mathcal{M}_d \) and \( \{ E_{ij} : 1 \leq i, j \leq K \} \) are the standard matrix units of \( \mathcal{M}_K \). It is easily seen that in this representation \( u \) has the form
\[
u = \sum_{i=1}^{K} u_i \otimes E_{\sigma(i)i},
\]
where \( \sigma \) is a permutation of the set \( \{1, 2, \ldots, K\} \), and the \( u_i \)'s are easily seen to be unitary elements of \( \mathcal{M}_d \otimes \mathcal{M}_m \) that leave the subalgebra \( \mathcal{M}_d \otimes I_m \) invariant.

The final step to describe the structure of a possible unitary \( u \in \mathcal{M}_d \otimes \mathcal{M}_m \) such that \( \text{Ad}_u \) leaves the subalgebra \( \mathcal{M}_d \otimes I_m \) invariant. Since \( \text{Ad}_u \) induces an automorphism of \( \mathcal{M}_d \otimes I_m \), we have a unitary \( v \) such that
\[
\text{Ad}_u(a \otimes I_m) = (v \otimes I_m)^* (a \otimes I_m)(v \otimes I_m).
\]
The automorphism \( \text{Ad}_u \circ \text{Ad}_v^{-1} \) leaves the subalgebra fixed and is induced by a unitary \( W \in \mathcal{M}_j \otimes \mathcal{M}_m \). Hence \( W \) must be in the commutant of the subalgebra, that is, \( W = I_d \otimes w \). From this we conclude that \( u = v \otimes w \).

We arrived at the following:
Proposition 4  In the case when all the multiplicities and all the dimensions are the same, \( u \) must be of the form

\[
  u = \sum_{i=1}^{K} v_i \otimes w_i \otimes E_{\sigma(i)i},
\]

where \( v_i \in \mathcal{M}_d \) and \( w_i \in \mathcal{M}_m \) are unitaries and \( \sigma \) is a permutation of the set \( \{1, 2, \ldots, K\} \).

Note that \( u \mapsto \sigma \) is a homomorphism on the group of allowed \( u \)'s (while \( u \mapsto v_i \otimes w_i \) is not).

The general situation is put together from the above propositions: \( u \) commutes with \( P_{m,d} \) and \( u P_{m,d} \) is described by Proposition 4.

4 Sufficient coarse-grainings

Let \( T : B(\mathcal{H}) \to B(\mathcal{K}) \) be a trace-preserving 2-positive mapping and \( D_t \) be density matrices on \( \mathcal{H} \) (\( t = 1, 2 \)). We assume the existence of a trace preserving 2-positive mapping \( \beta : B(\mathcal{K}) \to B(\mathcal{H}) \) such that \( \beta \circ T(D_1) = D_1 \) and \( \beta \circ T(D_2) = D_2 \). In other words, we suppose that \( T \) is sufficient for \( D_1 \) and \( D_2 \). Our goal is to describe the structure coming from this assumption.

In this section we work with positive unital mappings, so are the adjoint \( T^* : B(\mathcal{K}) \to B(\mathcal{H}) \) and \( \alpha : B(\mathcal{H}) \to B(\mathcal{K}) \) defined by (5). Let the fixed point algebra of \( \gamma_{\mathcal{H}} := T^* \circ \alpha \) be \( \mathcal{D}_{\mathcal{H}} \), that of \( \gamma_{\mathcal{K}} := \alpha \circ T^* \) be \( \mathcal{D}_{\mathcal{K}} \). The mapping \( \gamma_{\mathcal{H}} \) leaves the state corresponding to \( D_1 \) invariant: \( \text{Tr} D_1 \gamma_{\mathcal{H}}(A) = \text{Tr} D_1 A \) follows by easy computation.

\[
  \text{Tr} D_2 \gamma_{\mathcal{H}}(A) = \text{Tr} D_2 A
\]

was shown in the equivalent form \( T^#(T(D_2)) = D_2 \) in (7). Similarly, \( \text{Tr} T(D_1) \gamma_{\mathcal{K}}(X) = \text{Tr} T(D_1)X \) and \( \text{Tr} T(D_2) \gamma_{\mathcal{K}}(X) = \text{Tr} T(D_2)X \).

Lemma 1 Let

\[
  \mathcal{A}_{T^*} := \{ X \in B(\mathcal{K}) : T^*(XX^*) = T^*(X)T^*(X^*) \text{ and } T^*(X^*X) = T^*(X^*)T^*(X) \}
\]

and

\[
  \mathcal{A}_\alpha := \{ A \in B(\mathcal{H}) : \alpha(AA^*) = \alpha(A)\alpha(A^*) \text{ and } \alpha(A^*A) = \alpha(A^*)\alpha(A) \}. \]

Then \( \mathcal{D}_K \subset \mathcal{A}_{T^*} \) and \( \mathcal{D}_\mathcal{H} \subset \mathcal{A}_\alpha \). Moreover, \( T^* \) restricted to \( \mathcal{D}_K \) is an algebraic isomorphism onto \( \mathcal{D}_\mathcal{H} \) with inverse \( \alpha \) and

\[
  \alpha(AB) = \alpha(A)\alpha(B) \quad (10)
\]

for all \( A \in \mathcal{A}_\alpha \) and \( B \in B(\mathcal{H}) \).
The lemma is stated for reference, concerning the proof see 9.1 in [17].

We have \( T(D_1)T(D_1)^*-i \subset \mathcal{D}_K \) and to the unitaries \( T(D_1)^t \) we can apply the arguments in the previous section. \( A \in \mathcal{D}_K \) has the form of [2] and we have the central projections \( P_{m,d} \) of \( \mathcal{D}_K \) at our disposal. As above elements of \( P_{m,d} \mathcal{D}_K \) have the form

\[
\sum_i X_i \otimes I_m \otimes E_{ii},
\]

where \( X_i \) is an element of \( \mathcal{M}_j \) and \( \{ E_{ij} : 1 \leq i, j \leq k \} \) are the standard matrix units of \( \mathcal{M}_k \). We can imagine \( P_{m,d} \mathcal{B}(K)P_{m,d} \) in the form \( \mathcal{M}_j \otimes \mathcal{M}_m \otimes \mathcal{M}_k \).

According to Proposition 4 every unitary \( P_{m,d}T(D_1)^t \) is of the form

\[
\sum_{i=1}^k v_i \otimes w_i \otimes E_{\sigma(i)i},
\]

where \( v_i \in \mathcal{M}_j \) and \( w_i \in \mathcal{M}_m \) are unitaries and \( \sigma \) is a permutation of the set \( \{1, 2, \ldots , k\} \), all of them depend on the real parameter \( t \). Since this dependence is obviously continuous, the only possibility is \( \sigma = \text{identity} \). It follows that

\[
P_{m,d}T(D_1) = \sum_{i=1}^{k(m,d)} S_{11}(m,d,i) \otimes S_{12}(m,d,i) \otimes E_{ii}(m,d).
\]

Similar argument applies to \( T(D_2) \) and we have

\[
P_{m,d}T(D_2) = \sum_{i=1}^{k(m,d)} S_{21}(m,d,i) \otimes S_{22}(m,d,i) \otimes E_{ii}(m,d).
\]

If we want both factors to be normalized, then positive coefficients should be included in the front.

We refer to Theorem 9.11 from [8], this tells that \( T(D_1)^t T(D_2)^{-it} \) belongs to \( \mathcal{D}_K \). Therefore

\[
S_{12}(m,d,i) = S_{22}(m,d,i) \quad (1 \leq i \leq k(m,d)) \tag{11}
\]

We want to see the densities \( T(D_1) \) and \( T(D_2) \) in the central decomposition of the algebra \( \mathcal{D}_K \). Assume that \( z_1, z_2, \ldots, z_r \) the minimal central projections in \( \mathcal{D}_K \). Then \( z_p \mathcal{D}_K \) is isomorphic to a full matrix algebra \( \mathcal{M}_{n_p} \) and \( \mathcal{D}_K \) is isomorphic to \( \bigoplus_{p=1}^r \mathcal{M}_{n_p} \).

In the above decomposition of \( T(D_1) \) we have

\[
S_{11} \otimes S_{12} \otimes E_{ii} = (S_{11} \otimes I \otimes E_{ii})(I \otimes S_{12} \otimes E_{ii}),
\]

where the first factor belongs to a central summand \( z_p \mathcal{D}_K \) and the second one is in \( z_p \mathcal{D}_K' \). Hence we arrived at the following structure.
Theorem 1 Let $T : B(\mathcal{H}) \to B(\mathcal{K})$ be a trace-preserving 2-positive mapping which is sufficient for the invertible density matrices $D_s$ on $\mathcal{H}$ ($s = 1, 2$). Assume that $T(D_1)$ and $T(D_2)$ are invertible as well. Then there exists a subalgebra $\mathcal{D}_s \subset B(\mathcal{K})$ with minimal central projections $z_1, z_2, \ldots, z_r \in \mathcal{D}_s$ such that

(a) $T^*(XY) = T^*(X)T^*(Y)$ for $X \in B(\mathcal{K})$ and $Y \in \mathcal{D}_s$.

(b) $T(D_s) = \sum_{p=1}^r \lambda_s(p)S_s(p)R(p)$ for some density operators $S_s(p) \in z_p\mathcal{D}_s$ and $R(p) \in z_p\mathcal{D}_s^*$ and for probability distributions $\lambda_s(p)$ ($1 \leq p \leq r$) for $s = 1, 2$.

The theorem is formulated in the Hilbert space $\mathcal{K}$ but similar formulation is possible in $\mathcal{H}$ as well. One starts with the observation $T^*(\mathcal{D}_s) = T^*(\mathcal{K}) \cong \bigoplus_{p=1}^r M_{n_p}$. $q_p := T^*(z_p)$ are the minimal central projections in $\mathcal{D}_s$. $S_s(p) = \alpha(S_s^H(p))$ for some $S_s^H(p) \in q_p\mathcal{D}_s^H$ ($1 \leq p \leq r$). Property \[10\] is reformulated for the standard dual $T^\#$ as

$$AT^\#(B) = T^\#(\alpha(A)B)$$

and we have

$$T^\#(S_s(p)R(p)) = T^\#(\alpha(S_s^H(p))R(p)) = S_s^H(p)T^\#(R(p)),$$

therefore

$$D_s = T^\#(T(D_s)) = \sum_{p=1}^r \lambda_s(p)S_s^H(p)T^\#(R(p))$$

where the support of $S_s^H(p)$ is in $q_p$ and $T^\#(R(p))$ commutes with $S_s^H(p)$ for all $p$.

We first note that the structure formulated in the theorem is derived from the sufficiency condition but on the other hand that structure implies sufficiency. Namely, the structure above guarantees condition \[2\] by a simple calculation.

$$T^*\left(T(D_2)^iT(D_1)^{-i}\right) = T^* \left(\sum_{p=1}^r \lambda_2^i(p)\lambda_1^{-i}(p)S_2(p)^iS_1(p)^{-i}\right)$$

$$= \sum_{p=1}^r \lambda_2^i(p)\lambda_1^{-i}(p)T^*\left(S_2(p)^iS_1(p)^{-i}\right)$$

$$= \sum_{p=1}^r \lambda_2^i(p)\lambda_1^{-i}(p)T^*\left(S_2(p)^i\right)T^*\left(S_1(p)^{-i}\right)$$

$$= \sum_{p=1}^r \lambda_2^i(p)\lambda_1^{-i}(p)S_2^H(p)^iS_1^H(p)^{-i}$$

$$= D_2^iD_1^{-i}$$

Our theorem extends the result in \[3\] whose setting corresponds to the case $\mathcal{H} = \mathcal{K}$ and $D_1 = D_2$ in our notation and the decomposition

$$\mathcal{H} = \bigoplus_{p=1}^r \mathcal{H}_p^{left} \otimes \mathcal{H}_p^{right}$$
Our theorem extends obviously to more density matrices. If $T$ is sufficient for $D_1, D_2, \ldots, D_k$, then all density matrices have the above form and in each summand the first factor depends on $1 \leq s \leq k$ while the second does not.

5 Strong subadditivity of entropy

The strong subadditivity of entropy is

$$S(D_{ABC}) + S(D_B) \leq S(D_{AB}) + S(D_{BC})$$

for a system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, where $D_B, D_{AB}, D_{BC}$ are the reduced densities of the state $D_{ABS}$ of the composite system and $S$ stands for the von Neumann entropy [6]. We have the equivalent form

$$S(D_{ABC}, \tau_{ABC}) + S(D_B, \tau_B) \geq S(D_{AB}, \tau_{AB}) + S(D_{BC}, \tau_{BC})$$

in terms of relative entropy [8], $\tau$ denotes the density of the tracial state (for example, $\tau_B$ is $I_B / \dim \mathcal{H}_B$). This inequality is equivalent to the inequality

$$S(D_{ABC}, \tau_A \otimes D_{BC}) \geq S(D_{AB}, \tau_A \otimes D_B), \quad (12)$$

which, on the other hand, is the consequence of the monotonicity of relative entropy. Uhlmann’s theorem should be applied to the partial trace

$$T(X \otimes Y \otimes Z) = (X \otimes Y)\text{Tr} \, Z,$$

since

$$T(\tau_A \otimes D_{BC}) = \tau_A \otimes D_B \quad \text{and} \quad T(D_{ABC}) = D_{AB}.$$  

To use our previous notation we set $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C, \mathcal{K} := \mathcal{H}_A \otimes \mathcal{H}_B, D_1 := \tau_A \otimes D_{BC}$ and $D_2 := D_{ABC}$. Our aim is to study the case of equality in (12) which is known to be equivalent of the sufficiency of $T$ with respect to $D_1$ and $D_2$ (see [11] and [14]).

We recall that $\mathcal{D}_H$ is the fixed point algebra of the mapping

$$\gamma_H(X \otimes Y \otimes Z) = \left((\tau_A \otimes D_B)^{-1/2}T((\tau_A \otimes D_{BC})^{1/2}(X \otimes Y \otimes Z)(\tau_A \otimes D_{BC})^{1/2})(\tau_A \otimes D_B)^{-1/2}\right) \otimes I_C.$$

It is clear that $\gamma_H(X \otimes I_B \otimes I_C) = X \otimes I_B \otimes I_C$, therefore

$$B(\mathcal{H}_A) \otimes \mathbb{C}I_B \otimes \mathbb{C}I_C \subset \mathcal{D}_H \subset B(\mathcal{H}_A) \otimes B(\mathcal{H}_B) \otimes \mathbb{C}I_C$$

and $\mathcal{D}_H$ must be of the form $B(\mathcal{H}_A) \otimes \mathcal{A}_B \otimes \mathbb{C}I_C$ with a subalgebra $\mathcal{A}_B$ of $B(\mathcal{H}_B)$. Elements of $\mathcal{A}_B$ have the form [11] and

$$I_B = \sum_{m,d} p_{m,d},$$

where $D_1$ comes out.
where the (central) projection $P_{m,d}$ corresponds to multiplicity $m$ and dimension $d$.

The algebra $P_{m,d}ABP_{m,d}$ is isomorphic to $\oplus_{t=1}^{K(m,d)} M_d$. In this case $\dim P_{m,d} = mdK$ and elements of $P_{m,d}ABP_{m,d}$ have the form

$$\sum_i B_i \otimes E_{ii} \otimes I_m,$$

where $B_i$ is an element of $M_d$, $\{E_{ij} : 1 \leq i, j \leq K(m,d)\}$ are the standard matrix units of $M_{K(m,d)}$ and $I_m \in M_m$ is the identity. We use these facts to pass to the algebra $D_H$. $P'_{m,d} := I_A \otimes P_{m,d} \otimes I_C$ is a central projection and elements of $D_H P'_{m,d}$ are of the form

$$\sum_i A \otimes B_i \otimes E_{ii} \otimes I_m \otimes I_C.$$

Since the unitaries $D_{2t}^{it}$ commute with $P'_{m,d}$ we have

$$D_{2t}^{it} (D_H P'_{m,d}) D_{2t}^{-it} \subset D_H P'_{m,d}$$

and this allows us to establish the structure of $D_2 P'(m,j)$.

$$D_2 P'_{m,d} = \sum_i D_{AB1}(i) \otimes E_{ii} \otimes D_{B2C}(i),$$

where $D_{AB1}(i)$ and $D_{B2C}(i)$ are density matrices in $B(H_A) \otimes M_j$ and $M_m \otimes B(H_C)$, respectively. We can conclude the form of $D_{ABC}$ which allows equality in the strong subadditivity for the entropy:

$$D_{ABC} = \sum_{m,d}^{K(m,d)} \lambda(i,m,d) D_{AB1}(i,m,d) \otimes E_{ii}(m,d) \otimes D_{B2C}(i,m,d),$$

where $I \otimes E_{ii}(m,d) \otimes I$ is pairwise orthogonal family of projections acting on $H_B$. This structure is the same as the one obtained in [4].

It has been known for a while that the equality in several strong subadditivity inequalities for the von Neumann entropy of the local restriction of states of infinite product chains is equivalent to the Markov property initiated by Accardi (see Proposition 11.5 in [8] or [13]). Therefore, from the structure [14], one can deduce the form of quantum Markov states which was done in [2,9] by different methods, see these papers concerning the details.

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