Global behaviour of bistable solutions for gradient systems in one unbounded spatial dimension

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This paper is concerned with spatially extended gradient systems of the form

\[ u_t = -\nabla V(u) + u_{xx}, \]

where spatial domain is the whole real line, state-parameter \( u \) is multidimensional, and the potential \( V \) is coercive at infinity. For such systems, under generic assumptions on the potential, the asymptotic behaviour of every bistable solution — that is, every solution close at both ends of space to spatially homogeneous stable equilibria — is described. Every such solutions approaches, far to the left in space a stacked family of bistable fronts travelling to the left, far to the right in space a stacked family of bistable fronts travelling to the right, and in between a pattern of stationary solutions homoclinic or heteroclinic to homogeneous stable equilibria. This result pushes one step further the program initiated in the late seventies by Fife and MacLeod about the global asymptotic behaviour of bistable solutions, by extending their results to the case of systems. In the absence of maximum principle, the arguments are purely variational, and call upon previous results obtained in companion papers.

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1 Introduction

This paper deals with the global dynamics of nonlinear parabolic systems of the form

(1) \[ u_t = -\nabla V(u) + u_{xx}, \]

where time variable \( t \) and space variable \( x \) are real, spatial domain is the whole real line, the function \( (x,t) \mapsto u(x,t) \) takes its values in \( \mathbb{R}^n \) with \( n \) a positive integer, and the nonlinearity is the gradient of a scalar potential function \( V : \mathbb{R}^n \to \mathbb{R} \), which is assumed to be regular (of class at least \( C^2 \)) and coercive at infinity (see hypothesis \( (H_{\text{coerc}}) \) in subsection 2.1 on page 4).

A fundamental feature of system (1) is that it can be recast, at least formally, as the gradient flow of an energy functional. If \( (v,w) \) is a pair of vectors of \( \mathbb{R}^n \), let \( v \cdot w \) and \( |v| = \sqrt{v \cdot v} \) denote the usual Euclidean scalar product and the usual Euclidean norm, respectively, and let us write simply \( v^2 \) for \( |v|^2 \). If \( (x,t) \mapsto u(x,t) \) is a solution of system (1), the energy (or Lagrangian or action) functional of the solution reads:

(2) \[ E[u(\cdot,t)] = \mathcal{E}[x \mapsto u(x,t)] = \int_{\mathbb{R}} \left( \frac{u_x(x,t)^2}{2} + V(u(x,t)) \right) dx. \]

Its time derivative reads, at least formally,

(3) \[ \frac{d}{dt} \mathcal{E}[u(\cdot,t)] = -\int_{\mathbb{R}} u_t(x,t)^2 \, dx \leq 0, \]

and system (1) can formally be rewritten as:

\[ u_t(\cdot,t) = -\frac{\delta}{\delta u} \mathcal{E}[u(\cdot,t)]. \]

If system (1) is considered on a bounded spatial domain with boundary conditions that preserve this gradient structure, then the integrals in (2) and (3) converge, thus the system is really — and not only formally — of gradient type. In this case the dynamics is (at least from a qualitative point of view) fairly well understood, up to a fine description of the global attractor that is compact and made of the unstable manifolds of stationary solutions [11, 27]. According to LaSalle’s principle, every solution approaches the set of stationary solutions under rather general additional hypotheses [26].

If space is the whole real line and the solutions under consideration are only assumed to be bounded, then the gradient structure above is only formal and allows a much richer phenomenology (the full attractor is by the way far from being fully understood in this case, see the introduction of [10] and references therein). A salient feature is the occurrence of travelling fronts, that is travelling waves connecting homogeneous equilibria at both ends of space. Those solutions are known to play a major role in the asymptotic behaviour of “many” initial conditions.

This crucial role of travelling fronts can be viewed, abstractly, as a consequence of another fundamental feature of system (1): the fact that a formal gradient structure.
exists not only in the laboratory frame, but also in every frame travelling at a constant speed. Indeed, for every real quantity \( c \), if a function \( (x, t) \mapsto u(x, t) \) is a solution of system (1), then the function \( (y, t) \mapsto v(y, t) \) defined by

\[
v(y, t) = u(ct + y, t)
\]

(the same function viewed in a frame travelling at speed \( c \)) is a solution of

(4)

\[
v_t - cv_y = -\nabla V(v) + v_{yy},
\]

and if we consider the energy

(5)

\[
\mathcal{E}_c[v(\cdot, t)] = \int_{\mathbb{R}} e^{cy} \left( \frac{v_y(y, t)^2}{2} + V(v(y, t)) \right) dy,
\]

then, at least formally,

(6)

\[
\frac{d}{dt} \mathcal{E}_c[v(\cdot, t)] = -\int_{\mathbb{R}} e^{cy} v_t(y, t)^2 dy,
\]

and the system (4) can formally be rewritten as:

(7)

\[
v_t(\cdot, t) = -e^{-cy} \frac{\delta}{\delta v} \mathcal{E}_c[v(\cdot, t)].
\]

This gradient structure has been known for a long time, but it was not until recently that it received a more detailed attention from several authors (among them C. B. Muratov, Th. Gallay, R. Joly, and the author [8, 9, 14, 22]), and that it was shown that this structure is sufficient (in itself, that is without the use of the maximum principle) to prove results of global convergence towards travelling fronts. These ideas have been applied since in different contexts, for instance by G. Chapuisat [3], Muratov and M. Novaga [15–17], N. D. Alikakos and N. I. Katzourakis [1], C. Luo [13].

Among travelling fronts, two main classes can be distinguished: monostable fronts, where an unstable equilibrium is replaced by a stable one, and bistable fronts, where the invaded equilibrium also is stable. A reasonably wide class of solutions, sufficiently large to capture the convergence to travelling fronts while limiting the complexity of the dynamics encountered is made of solutions that are close to homogeneous equilibria at both ends of space, at least for large times. And among such initial conditions the simplest case is that of bistable initial conditions, when those equilibria at both ends of space are stable.

In the late seventies, substantial breakthroughs have been achieved by P. C. Fife and J. MacLeod about the global behaviour of such bistable solutions in the scalar case (\( n \) equals 1). Their results comprise global convergence towards a bistable front [5], global convergence towards a “stacked family of bistable fronts” [6], and finally, in the case of a bistable potential, a rather complete description of the global asymptotic behaviour of all solutions that are sufficiently close, at infinity in space, to the local (non global) minimum point [7]. Note by the way that the gradient structure (7) above (in a travelling
referential) was known to Fife and MacLeod who used it in their initial paper [5] of 1977. Several extensions and generalizations of these results have been achieved since, but mostly in the scalar case $n = 1$ (using maximum principles and order-preserving properties of the solutions), see the recent papers [4, 20] and references therein.

The aim of this paper, completing the companion papers [22, 23], is to make a step further in this program, by extending these results to the case of systems of the form (1), and by providing for such systems a complete description of the asymptotic behaviour of every bistable solution (Theorem 1 below).

2 Assumptions, notation, and statement of the results

This section presents strong similarities with section 2 of [23], where more details and comments can be found.

2.1 Semi-flow in uniformly local Sobolev space and coercivity hypothesis

Let us denote by $X$ the uniformly local Sobolev space $H^1_{ul}(\mathbb{R}, \mathbb{R}^n)$. System (1) defines a local semi-flow in $X$ (see for instance D. B. Henry’s book [12]).

As in [23], let us assume that the potential function $V : \mathbb{R}^n \to \mathbb{R}$ is of class $C^k$ where $k$ is an integer not smaller than 2, and that this potential function is strictly coercive at infinity in the following sense:

$$(H_{\text{coerc}}) \quad \liminf_{R \to +\infty} \inf_{|u| \geq R} \frac{u \cdot \nabla V(u)}{|u|^2} > 0$$

(or in other words there exists a positive quantity $\varepsilon$ such that the quantity $u \cdot \nabla V(u)$ is larger than $\varepsilon |u|^2$ as soon as $|u|$ is sufficiently large).

According to this hypothesis ($H_{\text{coerc}}$), the semi-flow of system (1) is actually global (see Lemma 1 on page 18). Let us denote by $(S_t)_{t \geq 0}$ this semi-flow.

2.2 First generic hypothesis on the potential: critical points are nondegenerate

The results of this paper require several generic hypotheses on the potential $V$. The simplest of those hypotheses is:

$$(H_{\text{non-deg}}) \quad \text{Every critical point of } V \text{ is nondegenerate.}$$

In other words, for all $u$ in $\mathbb{R}^n$, if $\nabla V(u)$ vanishes, then the Hessian $D^2 V(u)$ possesses no vanishing eigenvalue. As a consequence, in view of hypothesis $(H_{\text{coerc}})$, the number of critical points of $V$ is finite. Everywhere in this paper, the term “minimum point” denotes a point where a function — namely the potential $V$ — reaches a local or global minimum.

Notation. Let $\mathcal{M}$ denote the set of (nondegenerate, local or global) minimum points of $V$: $\mathcal{M} = \{u \in \mathbb{R}^n : \nabla V(u) = 0 \text{ and } D^2 V(u) \text{ is positive definite}\}$.
2.3 Bistable solutions: definition and notation

Our targets are bistable solutions, let us recall their definition already stated in [23].

**Definition.** A solution \((x, t) \mapsto u(x, t)\) of system (1) is called a *bistable solution* if there are two (possibly equal) points \(m_−\) and \(m_+\) in \(\mathcal{M}\) such that the quantities:

\[
\limsup_{x \to -\infty} |u(x, t) - m_-| \quad \text{and} \quad \limsup_{x \to +\infty} |u(x, t) - m_+|
\]

both approach 0 when time approaches \(+\infty\). More precisely, such a solution is called a *bistable solution connecting \(m_-\) to \(m_+\)* (see figure [1]). A function \(u_0\) in \(X\) is called a *bistable initial condition (connecting \(m_-\) to \(m_+\))* if the solution of system (1) corresponding to this initial condition is a bistable solution (connecting \(m_-\) to \(m_+\)).

Let \(m_-\) and \(m_+\) denote two (possibly equal) points in \(\mathcal{M}\).

**Notation.** Let

\[X_{\text{bist}}(m_-, m_+)\]

denote the subset of \(X\) made of bistable initial conditions connecting \(m_-\) to \(m_+\).

By construction, this set is positively invariant under the semi-flow of system (1). As proved in [22, 23], this set is in addition nonempty and open in \(X\) (for the usual norm on this function space), and contains all functions sufficiently close to the minimum points \(m_-\) and \(m_+\) at the ends of space.

The aim of this paper is to study the asymptotic behaviour of solutions belonging to the sets \(X_{\text{bist}}(m_-, m_+)\). The description of this asymptotic behaviour involves two kinds of particular solutions: stationary solutions connecting (stable) equilibria and (bistable) fronts travelling at a constant speed.

2.4 Stationary solutions and travelling fronts: definition and notation

Let \(c\) be a real quantity. A function

\[\phi : \mathbb{R} \to \mathbb{R}^n, \quad \xi \mapsto \phi(\xi)\]
is the profile of a wave travelling at speed \( c \) (or is a stationary solution if \( c \) vanishes) for system \( \Box \) if the function \( (x,t) \mapsto \phi(x-ct) \) is a solution of this system, that is if \( \phi \) is a solution of the differential system

\[
\phi'' = -c\phi' + \nabla V(\phi) .
\]

This system can be viewed as a damped oscillator (or a conservative oscillator if \( c \) vanishes) in the potential \(-V\), the speed \( c \) playing the role of the damping coefficient.

**Notation.** If \( u_- \) and \( u_+ \) are critical points of \( V \) (and \( c \) is a real quantity), let \( \Phi_c(u_-, u_+) \) denote the set of nonconstant solutions of system \( \Box \) connecting \( u_- \) to \( u_+ \). With symbols,

\[
\Phi_c(u_-, u_+) = \{ \phi : \mathbb{R} \to \mathbb{R}^n : \phi \text{ is a nonconstant solution of system } (\Box) \text{ and } \phi(\xi) \xrightarrow{\xi \to -\infty} u_- \text{ and } \phi(\xi) \xrightarrow{\xi \to +\infty} u_+ \} .
\]

Let us make some comments about this set and the notation.

- The notation “\( \phi \)” and “\( \Phi \)” refers to the concept of “front”.
- If \( \phi \) is an element of some set \( \Phi_c(u_-, u_+) \) for some nonzero quantity \( c \), then the difference \( V(u_+) - V(u_-) \) is nonzero (and of the sign of \(-c\)) so that \( u_- \) and \( u_+ \) are unequal, thus \( \phi \) is indeed a travelling front.
- If conversely \( \phi \) is an element of some set \( \Phi_0(u_-, u_+) \) (for a null speed), then \( V(u_+) = V(u_-) \) and we may have \( u_- = u_+ \); in such a case, \( \phi \) should better be called a “pulse” than a “front”. Nevertheless, for convenience and homogeneity purposes, we shall keep the notation \( \Phi_0(u_-, u_+) \) and \( \phi \) also in the case of stationary solutions (be they heteroclinic or homoclinic to critical points of \( V \) as those that will be considered).
- If \( c \) is a nonzero real quantity, then every solution of system \( \Box \) that is defined on the whole real line and bounded must approach the set of critical points of \( V \) when \( \xi \to -\infty \) and \( \xi \to +\infty \). Since according to \((\text{H}_{\text{non-deg}})\) the critical points of \( V \) are isolated, such a solution must actually approach (single) critical points at both ends of \( \mathbb{R} \). In other words, this solution must belong to the set \( \Phi_c(u_-, u_+) \) for some critical points \( u_- \) and \( u_+ \). In other words again, every bounded travelling wave is a travelling front.
- By contrast, if \( c \) equals 0, there are in general many bounded global solutions of system \( \Box \) that are neither homoclinic nor heteroclinic to critical points of \( V \), in other words that belong to none of the sets \( \Phi_0(u_-, u_+) \). However, stationary solutions in these sets are (together with homogeneous stationary solutions) the only one that are involved in the asymptotic behaviour of bistable solutions of system \( \Box \), thus only for them do we need a notation.
- Every stationary solution that will be involved in the asymptotic behaviour stated in the results below is actually **bistable**, that is belongs to a set \( \Phi_c(m_-, m_+) \) for some real quantity \( c \) and some points \( m_- \) and \( m_+ \) that are minimum points — and not
only critical points — of $V$. The reason why the notation above encompasses other critical points is that it will be used to state hypothesis ($H_{bist}$) in subsection 2.6 below.

2.5 Breakup of space translation invariance for stationary solutions and travelling fronts

Due to space translation invariance, nonconstant solutions of system (8) go by one-parameter families. For various reasons, it is convenient to pick a single “representative” in each of these families. This is done through the next definitions.

Let $\lambda_{\min} (\lambda_{\max})$ denote the minimum (respectively, maximum) of all eigenvalues of the Hessian matrices of the potential $V$ at (local) minimum points. In other words, if $\sigma(D^2V(u))$ denotes the spectrum of the Hessian matrix of $V$ at a point $u$ in $\mathbb{R}^n$, $\lambda_{\min} = \min_{m \in M} \min_{\sigma(D^2V(m))}$ and $\lambda_{\max} = \max_{m \in M} \max_{\sigma(D^2V(m))}$ (recall that the set $M$ is finite). Obviously, $0 < \lambda_{\min} \leq \lambda_{\max} < +\infty$.

**Notation.** For the remaining of this paper, let us fix a positive quantity $d_{Esc}$, sufficiently small so that, for every (local) minimum point $m$ of $V$ and for all $u$ in $\mathbb{R}^n$ satisfying $|u - m| \leq d_{Esc}$, every eigenvalue $\lambda$ of $D^2V(u)$ satisfies:

\[
\frac{\lambda_{\min}}{2} \leq \lambda \leq 2\lambda_{\max}.
\]

The reason for the subscript “Esc” in this notation is that this distance $d_{Esc}$ will be used to “track” the position in space where a solution “escapes” a neighbourhood of a minimum point of $V$ (this position is called “leading edge” by Cyrill B. Muratov [14, 17, 18]). Inside this neighbourhood, the potential essentially behaves like a positive definite quadratic form; and every nonconstant stationary solution of system (8), connecting two points of $M$, “escapes” at least at distance $d_{Esc}$ from each of these two points (whatever the value of the speed $c$ and even if these two points are equal) at some position of space (see figure 2). In other words, for every real quantity $c$, for every pair $(m_-, m_+)$ of points of $M$, and for every function $\phi$ in $\Phi_c(m_-, m_+)$,

\[
\sup_{\xi \in \mathbb{R}} |\phi(\xi) - m_-| > d_{Esc} \quad \text{and} \quad \sup_{\xi \in \mathbb{R}} |\phi(\xi) - m_+| > d_{Esc}
\]

(see Lemma 43 on page 87 for the case where $c$ is nonzero, and [23] for the case where $c$ equals 0). This provides a way to pick a representative among the family of all translates of $\phi$, by demanding that, say, the translate be exactly at distance $d_{Esc}$ of his right-end limit $m_+$ at $\xi = 0$, and closer for every positive $\xi$ (see figure 3). Here is a more formal definition. For $c$ in $\mathbb{R}$ and $(m_-, m_+)$ in $M^2$, let us consider the set of normalized bistable fronts/stationary solutions connecting $m_-$ to $m_+$:

\[
\Phi_{c,\text{norm}}(m_-, m_+) = \{ \phi \in \Phi_c(m_-, m_+) : |\phi(0) - m_+|_D = d_{Esc} \text{ and } |\phi(\xi) - m_+|_D < d_{Esc} \text{ for all } \xi > 0 \}.
\]
Figure 2: Every function in $\Phi_c(m_-, m_+)$ (that is, stationary in a frame travelling at a zero or nonzero speed and connecting two minimum points and nonconstant) escapes at least at distance $d_{Esc}$ of these minimum points.

Figure 3: Normalized stationary solution.
2.6 Additional generic hypotheses on the potential

The result below requires additional generic hypotheses on the potential $V$, that will now be stated. A formal proof of the genericity of this hypothesis is scheduled (work in progress by Romain Joly and the author).

**(H$_{\text{min}}$)** Every critical point of $V$ that belongs to the same level set as a (local) minimum point is itself a (local) minimum point.

In other words, for every pair $(u_1, u_2)$ in $\mathbb{R}^n$,

$$[\nabla V(u_1) = \nabla V(u_2) = 0 \text{ and } V(u_1) = V(u_2) \text{ and } D^2V(u_1) > 0] \Rightarrow D^2V(u_2) > 0.$$ 

This hypothesis is required in order to apply the relaxation results of [23] (where, by the way, the question of whether this hypothesis is really required for the validity of the results stated is not answered).

**(H$_{\text{bist}}$)** Every front travelling at a nonzero speed and invading a stable equilibrium (a minimum point of $V$) is bistable.

In other words, for every minimum point $m_+$ in $\mathcal{M}$, every critical point $u_-$ of $V$, and every positive quantity $c$, if the set $\Phi_c(u_-, m_+)$ is nonempty, then $u_-$ must belong to $\mathcal{M}$. As a consequence of this hypothesis, only bistable travelling fronts will be involved in the asymptotic behaviour of bistable solutions.

The statement of the two remaining hypotheses requires the following notation.

**Notation.** If $m_+$ is a point in $\mathcal{M}$ and $c$ is a positive quantity, let $\Phi_c(m_+)$ denote the set of fronts travelling at speed $c$ and “invading” the equilibrium $m_+$ (note that according to hypothesis (H$_{\text{bist}}$) all these fronts are bistable), and let us define similarly $\Phi_{c,\text{norm}}(m_+)$. With symbols,

$$\Phi_c(m_+) = \bigcup_{m_- \in \mathcal{M}} \Phi_c(m_-, m_+) \quad \text{and} \quad \Phi_{c,\text{norm}}(m_+) = \bigcup_{m_- \in \mathcal{M}} \Phi_{c,\text{norm}}(m_-, m_+).$$

The two additional generic hypotheses that will be made on $V$ are the following.

**(H$_{\text{disc-c}}$)** For every $m_+$ in $\mathcal{M}$, the set:

$$\{c \in [0, +\infty) : \Phi_c(m_+) \neq \emptyset\}$$

has an empty interior.

**(H$_{\text{disc-\phi}}$)** For every minimum point $m_+$ in $\mathcal{M}$ and every real quantity $c$, the set

$$\{(\phi(0), \phi'(0)) : \phi \in \Phi_{c,\text{norm}}(m_+)\}$$

is totally discontinuous — if not empty — in $\mathbb{R}^{2n}$. That is, its connected components are singletons. Equivalently, the set $\Phi_{c,\text{norm}}(m_+)$ is totally disconnected for the topology of compact convergence (uniform convergence on compact subsets of $\mathbb{R}$).
In these two last definitions, the subscript “disc” refers to the concept of “discontinuity” or “discreteness”.

Finally, let us define the following “group of generic hypotheses”:

\[(G) \quad (H_{\text{non-deg}}) \text{ and } (H_{\text{min}}) \text{ and } (H_{\text{bist}}) \text{ and } (H_{\text{disc-c}}) \text{ and } (H_{\text{disc-\Phi}}).\]

### 2.7 Propagating and standing terraces of bistable solutions

This subsection is devoted to several definitions. Their purpose is to enable a compact formulation of the main result of this paper (Theorem 1 below). Some comments on the terminology and related references are given at the end of this subsection.

**Definition** (propagating terrace of bistable fronts, figure 4). Let \(m_-\) and \(m_+\) be two minimum points of \(V\) (satisfying \(V(m_-) \leq V(m_+)\)). A function

\[\mathcal{T} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n, \quad (x, t) \mapsto \mathcal{T}(x, t)\]

is called a **propagating terrace of bistable fronts travelling to the right**, connecting \(m_-\) to \(m_+\), if there exists a nonnegative integer \(q\) such that:

1. if \(q\) equals 0, then \(m_- = m_+\) and, for every real quantity \(x\) and every nonnegative time \(t\),
   \[\mathcal{T}(x, t) = m_- = m_+;\]

2. if \(q\) equals 1, then there exist:
   - a positive quantity \(c_1\)
   - and a function \(\phi_1\) in \(\Phi_{c_1}(m_-, m_+)\) (that is, the profile of a bistable front travelling at speed \(c_1\) and connecting \(m_-\) to \(m_+)\),
   - and a \(C^1\)-function \(t \mapsto x_1(t)\), defined on \(\mathbb{R}_+\), and satisfying \(x_1'(t) \to c_1\) when \(t\) approaches \(+\infty\),

such that, for every real quantity \(x\) and every nonnegative time \(t\),

\[\mathcal{T}(x, t) = \phi_1(x - x_1(t));\]
3. if \( q \) is not smaller than 2, then there exists \( q-1 \) minimum points \( m_1, \ldots, m_{q-1} \) of \( V \), satisfying (if we denote \( m_+ \) by \( m_0 \) and \( m_- \) by \( m_q \))

\[
V(m_0) > V(m_1) > \cdots > V(m_q),
\]

and there exist \( q \) positive quantities \( c_1, \ldots, c_q \) satisfying:

\[
c_1 \geq \cdots \geq c_q,
\]

and for each integer \( i \) in \( \{1, \ldots, q\} \), there exist:

- a function \( \phi_i \) in \( \Phi_{c_i}(m_i, m_{i-1}) \) (that is, the profile of a bistable front travelling at speed \( c_i \) and connecting \( m_i \) to \( m_{i-1} \))
- and a \( C^1 \)-function \( t \mapsto x_i(t) \), defined on \( \mathbb{R}_+ \), and satisfying \( x_i'(t) \to c_i \) when \( t \) approaches \( +\infty \)

such that, for every integer \( i \) in \( \{1, \ldots, q-1\} \),

\[
x_{i+1}(t) - x_i(t) \to +\infty \text{ when } t \to +\infty,
\]

and such that, for every real quantity \( x \) and every nonnegative time \( t \),

\[
\mathcal{T}(x,t) = m_0 + \sum_{i=1}^{q} \left[ \phi_i(x - x_i(t)) - m_{i-1} \right].
\]

Obviously, item 2 may have been omitted in this definition, since it fits with item 3 with \( q \) equals 1.

A propagating terrace of bistable fronts travelling to the left may be defined similarly.

The next three definitions deal with stationary solutions. They are exactly identical to those of [23].

**Figure 5: Standing terrace (with four items, \( q = 4 \)).**

**Definition** (standing terrace of bistable stationary solutions, figure 5). Let \( h \) be a real quantity and let \( m_- \) and \( m_+ \) be two minimum points of \( V \) such that \( V(m_-) = V(m_+) = h \). A function

\[
\mathcal{T} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n, \quad (x,t) \mapsto \mathcal{T}(x,t)
\]

is called a standing terrace of bistable stationary solutions, connecting \( m_- \) to \( m_+ \), if there exists a nonnegative integer \( q \) such that:

11
1. if \( q \) equals 0, then \( m_\pm = m_+ \) and, for every real quantity \( x \) and every nonnegative time \( t \),
\[
T(x, t) = m_\pm = m_+ ;
\]

2. if \( q = 1 \), then there exist:
   - a bistable stationary solution \( u_1 \) connecting \( m_- \) to \( m_+ \)
   - and a \( C^1 \)-function \( t \mapsto x_1(t) \) defined on \( \mathbb{R}_+ \) and satisfying \( x'_1(t) \to 0 \) when \( t \)
     approaches \( +\infty \)
such that, for every real quantity \( x \) and every nonnegative time \( t \),
\[
T(x, t) = u_1(x - x_1(t)) ;
\]

3. if \( q \) is not smaller than 2, then there exist \( q - 1 \) minimum points \( m_1, \ldots, m_{q-1} \) of \( V \) (not necessarily distinct), all in the level set \( V^{-1}(\{h\}) \), and if we denote \( m_- \) by \( m_0 \) and \( m_+ \) by \( m_q \), then for each integer \( i \) in \( \{1, \ldots, q\} \), there exist:
   - a bistable stationary solution \( u_i \) connecting \( m_{i-1} \) to \( m_i \)
   - and a \( C^1 \)-function \( t \mapsto x_i(t) \) defined on \( \mathbb{R}_+ \) and satisfying \( x'_i(t) \to 0 \) when \( t \)
     approaches \( +\infty \)
such that, for every integer \( i \) in \( \{1, \ldots, q-1\} \),
\[
x_{i+1}(t) - x_i(t) \to +\infty \quad \text{when } t \to +\infty ,
\]
and such that, for every real quantity \( x \) and every nonnegative time \( t \),
\[
T(x, t) = m_0 + \sum_{i=1}^{q} [u_i(x - x_i(t)) - m_{i-1}] .
\]

Obviously, once again item 2 may have been omitted in this definition, since it fits with item 3 with \( q \) equals 1.

**Definition** (energy of a bistable stationary solution). Let \( x \mapsto u(x) \) be a bistable stationary solution connecting two local minima \( m_- \) and \( m_+ \) of \( V \), and let \( h \) denote the quantity \( V(m_+) \) (which is equal to \( V(m_-) \)). The quantity
\[
E[u] = \int_{\mathbb{R}} \left( \frac{|u'(x)|^2}{2} + V(u(x)) - h \right) dx
\]
is called the energy of the (bistable) stationary solution \( u \). Observe that this integral converges, since \( u(x) \) approaches its limits \( m_- \) and \( m_+ \) at both ends of space at an exponential rate.

**Definition** (energy of a standing terrace). Let \( h \) denote a real quantity and let \( \mathcal{T} \) denote a standing terrace of bistable stationary solutions connecting two local minima of \( V \) in the level set \( V^{-1}(\{h\}) \). With the notation of the two definitions above, the quantity \( E[\mathcal{T}] \) defined by:
1. if $q$ equals 0, then $\mathcal{E}[\mathcal{T}] = 0$,

2. if $q$ equals 1, then $\mathcal{E}[\mathcal{T}] = \mathcal{E}[u]$,

3. if $q$ is not smaller than 2, then $\mathcal{E}[\mathcal{T}] = \sum_{i=1}^{q} \mathcal{E}[u_i]$,

is called the *energy of the standing terrace* $\mathcal{T}$.

The terminology “propagating terrace” was introduced by A. Ducrot, T. Giletti, and H. Matano in [4] (and subsequently used by P. Poláčik, [19–21]) to denote a stacked family (a layer) of travelling fronts in a (scalar) reaction-diffusion equation. This led the author to keep the same terminology in the present context, and to introduce the term “standing terrace” for sake of homogeneity. Those terminologies are convenient to denote objects that would otherwise require a long description. They are also used in the companion papers [24, 25].

The author hopes that these advantages balance some drawbacks of this terminological choice. Like the fact that the word “terrace” is probably more relevant in the scalar case $n = 1$ (see the pictures in [4, 20]) than in the more general case of systems considered here. Or the fact that the definitions above and in [23] are different from the original definition of [4] in that they involve not only the profiles of particular (standing or travelling) solutions, but also their positions (denoted above by $x_i(t)$).

To finish, observe that in the present context terraces are only made of bistable solutions, by contrast with the propagating terraces introduced and used by the authors cited above; that (still in the present context) terraces are approached by solutions but are (in general) not solutions themselves; and that a (standing or propagating) terrace may be nothing but a single stable homogeneous equilibrium (when $q$ equals 0) or may involve a single travelling front or a single inhomogeneous stationary solution (when $q$ equals 1).

### 2.8 Main result (asymptotic behaviour)

The following theorem, illustrated by figure 6, is the main result of this paper.

![Figure 6: Asymptotic behaviour of a bistable solution (illustration of Theorem 1).](image)

13
Theorem 1 (asymptotic behaviour). Assume that $V$ satisfies the coercivity hypothesis $(H_{\text{coerc}})$ and the generic hypotheses $(G)$. Then for every bistable solution $(x,t) \mapsto u(x,t)$ of system (1) there exist:

- a propagating terrace $T_{\text{left}}$ of bistable fronts travelling to the left,
- a standing terrace $T_{\text{center}}$ of bistable stationary solutions,
- a propagating terrace $T_{\text{right}}$ of bistable fronts travelling to the right,

such that, for every sufficiently small positive quantity $\varepsilon$,

$$ \sup_{x \in (-\infty, -\varepsilon t]} |u(x, t) - T_{\text{left}}(x, t)| \to 0 $$
$$ \sup_{x \in [-\varepsilon t, \varepsilon t]} |u(x, t) - T_{\text{center}}(x, t)| \to 0 $$
$$ \sup_{x \in [\varepsilon t, +\infty)} |u(x, t) - T_{\text{right}}(x, t)| \to 0 $$

when $t$ approaches $+\infty$.

In addition to this statement, if we denote by $m_{\text{left}}$ and $m_{\text{right}}$ the local minimum points of $V$ connected by the solution $(x,t) \mapsto u(x,t)$, and if we denote by $m_{\text{center-left}}$ and $m_{\text{center-right}}$ the two local minimum points of $V$ connected by the standing terrace $T_{\text{center}}$, and if we denote by $h$ the quantity $V(m_{\text{center-left}})$ (according to the definition of a standing terrace above this quantity must also be equal to $V(m_{\text{center-right}})$), then:

- the propagating terrace $T_{\text{left}}$ connects $m_{\text{left}}$ to $m_{\text{center-left}}$,
- the propagating terrace $T_{\text{right}}$ connects $m_{\text{center-right}}$ to $m_{\text{right}}$,
- for every sufficiently small positive quantity $\varepsilon$, the quantity:

$$ \int_{-\varepsilon t}^{\varepsilon t} \left( \frac{u_x(x,t)^2}{2} + V(u(x,t)) - h \right) dx $$

(that may be called the “residual asymptotic energy” of the solution) approaches the energy of the center terrace when $t$ approaches $+\infty$.

Obviously in this theorem the profiles involved in the propagating and standing terrace $T_{\text{left}}$ and $T_{\text{right}}$ and $T_{\text{center}}$ (and the order of these profiles) are uniquely defined by the solution (uniquely if profiles are normalized and uniquely up to space translation if they are not), but not the positions $t \mapsto x_i(t)$ of these profiles.

2.9 Regularity of the correspondence between a solution and its asymptotic pattern

Notation. Let

$$ X_{\text{bist}}(M) = \bigsqcup_{(m_-, m_+) \in M^2} X_{\text{bist}}(m_-, m_+) $$
For \( u_0 \) in \( X_{\text{bist}}(\mathcal{M}) \), if \((x,t) \mapsto u(x,t)\) denotes the corresponding solution, using the notation of Theorem 1 above and the forthcoming remark, let:

- \( q_{\text{right}} \) denote the number of items involved in the right-propagating terrace \( T_{\text{right}} \),
- \( q_{\text{left}} \) denote the number of items involved in the left-propagating terrace \( T_{\text{left}} \),
- \( c_{1,\text{right}} \) denote the real quantity defined by:
  - if \( q_{\text{right}} \) equals 0 then \( c_{1,\text{right}} = 0 \),
  - if \( q_{\text{right}} \) is not smaller than 1 then \( c_{1,\text{right}} \) is the (positive) speed of the “first” travelling front involved in the right-propagating terrace \( T_{\text{right}} \) (the one invading \( m_+ \)),
- \( c_{1,\text{left}} \) denote the real quantity defined by:
  - if \( q_{\text{left}} \) equals 0 then \( c_{1,\text{left}} = 0 \),
  - if \( q_{\text{left}} \) is not smaller than 1 then \( c_{1,\text{left}} \) is the (negative) speed of the “first” travelling front involved in the right-propagating terrace \( T_{\text{left}} \) (the one invading \( m_- \)),
- \( h \) denote the quantity \( V(m_{\text{center-left}}) = V(m_{\text{center-right}}) \),
- \( E \) denote the energy of the center standing terrace \( T_{\text{center}} \).

This defines maps:

\[
q_{\text{right}} : X_{\text{bist}}(\mathcal{M}) \to \mathbb{N}, \\
q_{\text{left}} : X_{\text{bist}}(\mathcal{M}) \to \mathbb{N}, \\
h : X_{\text{bist}}(\mathcal{M}) \to \mathbb{R}, \\
E : X_{\text{bist}}(\mathcal{M}) \to [0, +\infty), \\
c_{1,\text{right}} : X_{\text{bist}}(\mathcal{M}) \to (0, +\infty), \\
c_{1,\text{left}} : X_{\text{bist}}(\mathcal{M}) \to (-\infty, 0).
\]

Finally, let

\[
X_{\text{bist, no-inv}}(\mathcal{M}) = X_{\text{bist}}(\mathcal{M}) \cap q_{\text{right}}^{-1}(\{0\}) \cap q_{\text{left}}^{-1}(\{0\}).
\]

In this notation the subscript "no-inv" refers to the fact that these solutions are those for which none of the two stable equilibria at both ends of space is “invaded” by a travelling front. Note that for every solution in \( X_{\text{bist, no-inv}}(\mathcal{M}) \), the equilibria approached by the solution at both ends of spatial domain must belong to the same level set of \( V \) (this follows from Theorem 1).

The following proposition states some regularity properties (upper or lower semi-continuity) of the “correspondences” between a solution and its asymptotic pattern defined above. The underlying phenomenon is in essence nothing else than the standard upper semi-continuity with respect to initial condition of the asymptotic level set of a (descendent) gradient flow (of say a Morse function on a finite-dimensional manifold). All the assertions stand with respect to the topology induced by \( \|\cdot\|_X \) (for the domain spaces) and the topology induced by the usual distance on \( \mathbb{R} \) (for the arrival spaces).
**Proposition 1** (continuity properties of the asymptotic pattern with respect to initial data). The following assertions hold:

1. the maps $c_{1, \text{right}}$ and $-c_{1, \text{left}}$ are lower semi-continuous;
2. the restriction of the map $\mathcal{E}$ to the set $X_{\text{bist, no-inv}}(\mathcal{M})$ is upper semi-continuous.

*Proof.* The first assertion is proved in [22]. The second assertion is proved in [23].

There are many other natural questions concerning the regularity of those correspondences. For instance it seems likely that the asymptotic potential level (the map $h[\cdot]$) is upper semi-continuous on $X_{\text{bist}}(\mathcal{M})$. And it would be nice to equip the “space of asymptotic patterns” with a topological structure ensuring similar semi-continuity properties for other (all?) features of the asymptotic pattern (speeds of the travelling fronts, values of the potential at minimum points connected by these travelling fronts, energies of the components of the center standing terrace). This question goes beyond the scope of this paper; it is discussed in more details (in a more restricted case) in [23].

### 2.10 Additional questions

Besides the questions concerning the regularity of the correspondence between a solution and its asymptotic pattern mentioned above, here are some other natural questions (either that I have not been able to solve, or that are beyond the scope of this paper).

- Does Theorem 1 hold without hypothesis ($H_{\text{disc-c}}$)? The question is legitimate since this hypothesis is not required to prove convergence towards the “first travelling fronts” (those called $\phi_{1,+}$ and $\phi_{1,-}$ in the statement of Theorem 1); indeed no hypothesis of this kind is made in [22]. However, I have not been able to get rid of that hypothesis to prove convergence towards a travelling front “following” a previous one. For additional comments and details see subsection 4.9.

- Does Theorem 1 hold without hypothesis ($H_{\text{min}}$)? This question is discussed in [23].

- All the convergence results stated in this paper are purely qualitative (there is no quantitative estimate about the rate of convergence of a solution towards its asymptotic pattern). Recently, F. Béthuel, G. Orlandi and D. Smets [2] have obtained such quantitative estimates for the same gradient systems but in a different setting (they do not consider convergence towards travelling fronts but only relaxation towards stationary solutions, for solution connecting global minimum points of $V$). It would be interesting to see if the same approach can yield similar quantitative estimates but for general bistable solutions (and asymptotic behaviour involving travelling fronts).
2.11 Extensions

Results similar to Theorem 1 hold in the following two cases that are considered in the companion papers [24, 25].

- Damped hyperbolic systems of the form
  \[ \alpha u_{tt} + u_t = -\nabla V(u) + u_{xx} \]

obtained by adding an inertial term \( \alpha u_{tt} \) (where \( \alpha \) is a positive non necessarily small quantity) to the parabolic system (1) considered here. Note that this extension was already achieved by Gallay and Joly in the scalar case \( n = 1 \) (for a bistable potential) using the gradient structure in every travelling frame (6), see [8].

- Radially symmetric solutions in higher space dimension \( d \), governed by systems of the form
  \[ u_t = -\nabla V(u) + \frac{d-1}{r} u_r + u_{rr} \]

where the nonnegative quantity \( r \) denotes the radius (distance to the origin) in \( \mathbb{R}^d \).

2.12 Organization of the paper

The next section 3 is devoted to some preliminaries (existence of solutions, preliminary computations on spatially localized functionals, notation).

Proof of Theorem 1 is essentially based on two propositions: Propositions 2 and 7, together with the results of the companion paper [23]. Sections 4 and 5 are devoted to these two propositions.

Proposition 2 is the main step and is proved in section 4 on page 21. It is an extension of the main result of global convergence towards travelling fronts proved in the previous paper [22]. As in [22], the proof is based on estimates about the time derivatives of energy functionals of the form (5), with the exponential weight replaced by an integrable one. By contrast with the situation investigated in [22], the hypotheses of Proposition 2 cope with the case where the solution is not necessarily close to a single local minimum point of the potential in the whole domain ahead of the “next” travelling front, but may behave in this domain as a propagating terrace of bistable travelling fronts. With respect to [22], the main difficulty is then that an additional term appears in the time derivatives of the localized energy functionals. For this reason, the relaxation scheme requires more care.

Proposition 7 is much easier and is proved in section 5 on page 74. It follows readily from Proposition 8 which states that, if a solution is close to stable homogeneous equilibria in large spatial domains on the left and on the right, and if these domains are not invaded (at a nonzero mean speed) from the “center” area in between, then the energy of the solution (in a standing frame) between these two areas remains nonnegative. Again, the proof is based on estimates for localized energy functionals in a travelling frame — this time at a very small speed.
The proof of Theorem 1 is completed in section 6 on page 86 by combining Proposition 2, Proposition 7, and the results of [23]. Finally, elementary properties of the profiles of travelling fronts are recalled in section 7 on page 87.

3 Preliminaries

3.1 Global existence of solutions and attracting ball for the semi-flow

The following lemma follows from general existence results and the coercivity property \((H_{\text{coerc}})\). For the proof see for instance [22, 23]. The two conclusions of this lemma are somehow redundant, since an upper bound for the \(H^1_{ul}\)-norm immediately yields an upper bound for the \(L^\infty\)-norm, and the converse is also true since the flow is regularizing. The reason for this redundant statement is that both quantities \(R_{\text{att},\infty}\) and \(R_{\text{att},X}\) used to express these bounds will be explicitly used at some places.

**Lemma 1** (global existence of solutions and attracting ball). For every function \(u_0\) in \(X\), system (1) has a unique globally defined solution \(t \mapsto S_tu_0\) in \(C^0([0, +\infty), X)\) with initial data \(u_0\). In addition, there exist positive quantities \(R_{\text{att}, X}\) and \(R_{\text{att}, \infty}\) (radius of attracting ball for the \(H^1_{ul}\)-norm and the \(L^\infty\)-norm, respectively), depending only on the potential \(V(\cdot)\), such that, for every sufficiently large time \(t\),

\[
\| (S_tu_0)(\cdot) \|_X \leq R_{\text{att}, X} \quad \text{and} \quad \sup_{x \in \mathbb{R}} | (S_tu_0)(x) | \leq R_{\text{att}, \infty}.
\]

In addition, system (1) has smoothing properties (Henry [12]). Due to these properties, since \(V\) is of class \(C^k\) (with \(k\) not smaller than 2), every solution \(t \mapsto S_tu_0\) in \(C^0([0, +\infty), X)\) actually belongs to

\[
C^0((0, +\infty), C^{k+1}_b(\mathbb{R}, \mathbb{R}^n)) \cap C^1((0, +\infty), C^{k-1}_b(\mathbb{R}, \mathbb{R}^n)),
\]

and, for every positive quantity \(\varepsilon\), the following quantities

\[
\sup_{t \geq \varepsilon} \| S_tu_0 \|_{C^{k+1}_b(\mathbb{R}, \mathbb{R}^n)} \quad \text{and} \quad \sup_{t \geq \varepsilon} \left\| \frac{d(S_tu_0)}{dt} \right\|_{C^{k-1}_b(\mathbb{R}, \mathbb{R}^n)}
\]

are finite.

3.2 Time derivative of (localized) energy and \(L^2\)-norm of a solution in a travelling referential

Let \(u_0\) be a function in \(X\) and, for all \(x\) in \(\mathbb{R}\) and \(t\) in \([0, +\infty)\), let \(u(x, t) = (S_tu_0)(x)\) denote the corresponding solution of system (1). Let \(c\) be a real quantity, and, for all \(y\) in \(\mathbb{R}\) and \(t\) in \([0, +\infty)\), let

\[
v(y, t) = u(ct + y, t)
\]
denote the same solution viewed in a referential travelling at speed $c$. As mentioned in introduction, $(y, t) \mapsto v(y, t)$ is a solution of the system:

$$v_t - cv_y = -\nabla V(v) + v_{yy}$$

that can be formally rewritten as the (descendent) gradient of the following energy (Lagrangian, action) functional:

$$\int_{\mathbb{R}} e^{cy} \left( \frac{v_y(y, t)^2}{2} + V(v(y, t)) \right) dy.$$  

The key ingredients of the proofs rely on appropriate combinations of this functional with the other most natural functional to consider, namely the $L^2$-norm (with the same exponential weight):

$$\int_{\mathbb{R}} e^{cy} \frac{v(y, t)^2}{2} dy.$$ 

Of course it is necessary to localize the integrands to ensure the convergence of the integrals. Let $(y, t) \mapsto \psi(y, t)$ be a function defined on $\mathbb{R} \times [0, +\infty)$ and such that, for all $t$ in $(0, +\infty)$, the function $y \mapsto \psi(y, t)$ belongs to $W^{2,1}(\mathbb{R}, \mathbb{R})$ and its time derivative $y \mapsto \psi_t(y, s)$ is defined and belongs to $L^1(\mathbb{R}, \mathbb{R})$. Then, for every positive time $t$, the time derivatives of the two aforementioned functionals — localized by the “weight” function $\psi(x, t)$ — reads:

$$\frac{d}{dt} \int_{\mathbb{R}} \psi \left( \frac{v_y^2}{2} + V(v) \right) dy = \int_{\mathbb{R}} \left[ -\psi v_t^2 + \psi_t \left( \frac{v_y^2}{2} + V(v) \right) + (c\psi - \psi_y) v_y \cdot v_t \right] dy,$$

and

$$\frac{d}{dt} \int_{\mathbb{R}} \psi \frac{v^2}{2} dy = \int_{\mathbb{R}} \left[ \psi \left( -v \cdot \nabla V(v) - v_y^2 \right) + (\psi_t + \psi_{yy} - c\psi_y) \frac{v^2}{2} \right] dy.$$  

Here are some basic observations about these expressions.

- The time derivative of the (localized) energy is the sum of a (nonpositive) “dissipation” term and two additional “flux” terms.
- The time derivative of the (localized) $L^2$-norm is similarly made of two “main” terms and an additional “flux” term. Among the two main term, the second is nonpositive, and so is the first if $|v|$ is small and if $0_{\mathbb{R}}$ is a local minimum of the potential (we will use the $L^2$-norm functional in that kind of situation).
- The flux terms involving $\psi_t$ are obviously small if $\psi$ varies slowly with respect to time.
- The other flux term in the time derivative of energy is small if the quantity $c\psi - \psi_y$ is small, that is if $\psi(y, t)$ is close to behave $\exp(cy)$ (up to a positive multiplicative constant). Of course, this cannot hold for all $y$ in $\mathbb{R}$ since $y \mapsto \psi(y, t)$ must be in $L^1(\mathbb{R}, \mathbb{R})$. As a consequence, it will not be possible to avoid that this second flux term be weighted by a “non small” weight of the order of $c\psi$, at least somewhere in space.
• Hopefully, the remaining flux term in the time derivative of the $L^2$-norm is “nicer”, in the sense that it is small under any of two conditions instead of a single one:
  – either if $\psi(y, t)$ is close to behave like $\exp(cy)$ (up to a positive multiplicative constant, like the previous flux term),
  – or if $\psi(y, t)$ varies slowly with $y$.

• Finally, the “tricky” flux term in the time derivative of energy can be balanced by the other terms of those expressions if we consider a linear combination of energy and $L^2$-norm with a sufficiently large weighting of the $L^2$-norm with respect to the weighting of energy.

These observations will be put in practice several times along the following pages.

3.3 Miscellanea

3.3.1 Estimates derived from the definition of the “escape distance”

For every minimum point $m$ in $\mathcal{M}$ and every vector $v$ in $\mathbb{R}^n$ satisfying $|v - m|_D \leq d_{Esc}$, it follows from inequalities [9] on page 7 that

$$\begin{align*}
\frac{\lambda_{\min}}{4} (u - m)^2 & \leq V(u) - V(m) \leq \lambda_{\max} (u - m)^2, \\
\frac{\lambda_{\min}}{2} (u - m)^2 & \leq (u - m) \cdot \nabla V(u) \leq 2\lambda_{\max} (u - m)^2.
\end{align*}$$

(15)

3.3.2 Minimum of the convexities of the lower quadratic hulls of the potential at local minimum points

For the computations carried in subsection 4.4 below, it will be convenient to introduce the quantity $q_{low-hull}$ defined as the minimum of the convexities of the negative quadratic hulls of $V$ at the points of $\mathcal{M}$ (see figure 7). With symbols:

$$q_{low-hull} = \min_{m \in \mathcal{M}} \min_{u \in \mathbb{R}^n \setminus \{m\}} \frac{V(u) - V(m)}{(u - m)^2}$$

Figure 7: Lower quadratic hull of the potential at a minimum point (definition of the quantity $q_{low-hull}$).
(a similar quantity was defined in [23]). This definition ensures (as obviously displayed by figure 7) that, for every minimum point \( m \) of \( V \) and for all \( u \) in \( \mathbb{R}^n \),

\[
V(u) - q_{\text{low-hull}}(u - m)^2 \geq 0.
\]

Let us consider the following quantity (it will be used to define the \textit{weighting of the energy} in the firewall functions defined in subsection 4.4 and sub-subsection 4.7.1):

\[
w_{\text{en},0} = \frac{1}{\max(1, -4 q_{\text{low-hull}})}.
\]

It follows from this definition that, for every \( m \) in the set \( M \) and for all \( u \) in \( \mathbb{R}^n \),

\[
(16)\quad w_{\text{en},0} V(u) + \frac{(u - m)^2}{4} \geq 0.
\]

4 Invasion implies convergence

4.1 Definitions and hypotheses

Let us assume that \( V \) satisfies the coercivity hypothesis (H\textsubscript{coerc}) and the generic hypotheses (G) (see subsection 2.6). Let us consider a minimum point \( m \) in \( M \), a function (initial data) \( u_0 \) in \( X \), and, for all quantities \( x \) in \( \mathbb{R} \) and \( t \) in \( [0, +\infty) \), the corresponding solution \( (x, t) \mapsto u(x, t) = (S_t u_0)(x) \).

We are not going to assume that this solution is bistable, but (as stated by the next hypothesis (H\textsubscript{hom-right})) that there exists a growing interval, travelling at a positive speed, where the solution is close to \( m \) (the subscript “hom” in the definitions below refers to this “homogeneous” area), see figure 8.

\[(H_{\text{hom-right}})\quad \text{There exists a positive quantity } c_{\text{hom}} \text{ and a } C^1\text{-function } x_{\text{hom}} : [0, +\infty) \to \mathbb{R}, \text{ satisfying } x'_{\text{hom}}(t) \to c_{\text{hom}} \text{ when } t \to +\infty, \text{ such that, for every positive quantity } L,
\]

\[
\|y \mapsto u(x_{\text{hom}}(t) + y, t) - m\|_{H^1([-L,L])} \to 0 \text{ when } t \to +\infty.
\]
For every $t$ in $[0 + \infty)$, let us denote by $x_{\text{Esc}}(t)$ the supremum of the set:

$$\left\{ x \in (-\infty, x_{\text{hom}}(t)) : |u(x, t) - m| = d_{\text{Esc}} \right\},$$

with the convention that $x_{\text{Esc}}(t)$ equals $-\infty$ if this set is empty. In other words, $x_{\text{Esc}}(t)$ is the first point at the left of $x_{\text{hom}}(t)$ where the solution “Escapes” at the distance $d_{\text{Esc}}$ from the stable homogeneous equilibrium $m$. We will refer to this point as the “Escape point” (there will also be an “escape point”, with a small “e” and a slightly different definition later). Let us consider the upper limit of the mean speeds between 0 and $t$ of this Escape point:

$$c_{\text{Esc}} = \limsup_{t \to +\infty} \frac{x_{\text{Esc}}(t)}{t},$$

and let us make the following hypothesis, stating that the area around $x_{\text{hom}}(t)$ where the solution is close to $m$ is “invaded” from the left at a nonzero (mean) speed.

$$(H_{\text{inv}})$$ The quantity $c_{\text{Esc}}$ is positive.

### 4.2 Statement

The aim of section 4 is to prove the following proposition, which is the main step in the proof of Theorem 1. The proposition is illustrated by figure 9.

**Proposition 2** (invasion implies convergence). *Assume that $V$ satisfies the coercivity hypothesis $(H_{\text{coerc}})$ and the generic hypotheses $(G)$, and, keeping the definitions and notation above, let us assume that for the solution under consideration hypotheses $(H_{\text{hom-right}})$ and $(H_{\text{inv}})$ hold. Then the following conclusions hold.*

- The function $t \mapsto x_{\text{Esc}}(t)$ is of class $C^1$ as soon as $t$ is sufficiently large and
  $$x'_{\text{Esc}}(t) \to c_{\text{Esc}} \text{ when } t \to +\infty.$$

- There exist:
  - a minimum point $m_{\text{next}}$ in $\mathcal{M}$ satisfying $V(m_{\text{next}}) < V(m)$,
a profile of travelling front \( \phi \) in \( \Phi_{\text{Esc,norm}}(m_{\text{next}},m) \),
- a \( C^1 \)-function \([0, +\infty) \to \mathbb{R}, t \mapsto x_{\text{hom-next}}(t)\),
such that, when time approaches \(+\infty\), the following limits hold:

\[
x_{\text{Esc}}(t) - x_{\text{hom-next}}(t) \to +\infty \quad \text{and} \quad x'_{\text{hom-next}}(t) \to c_{\text{Esc}}
\]

and

\[
\sup_{x \in [x_{\text{hom-next}}(t), x_{\text{hom}}(t)]} \left| u(x, t) - \phi(x - x_{\text{Esc}}(t)) \right| \to 0
\]

and, for every positive quantity \( L \),

\[
\left\| y \mapsto u\left(x_{\text{hom-next}}(t) + y, t\right) - m_{\text{next}}\right\|_{H^1([-L,L], \mathbb{R}^n)} \to 0.
\]

In this statement, the very last conclusion is almost entirely redundant with the previous one (indeed according to the regularizing feature of the semi-flow the norm used to express convergence does not matter). The reason why this last conclusion is stated this way is that it emphasizes the fact that a property similar to (H_{\text{hom-right}}) is recovered “behind” the travelling front. As can be expected this will be used to prove Theorem 1 by re-applying Proposition 2 as many times as required (to the left and to the right), as long as “invasion of the equilibria behind the last front” occurs.

### 4.3 Settings of the proof, 1: normalization and choice of origin of times

Let us keep the notation and assumptions of subsection 4.1, and let us assume that the hypotheses (H_{\text{coerc}}) and (G) and (H_{\text{hom-right}}) and (H_{\text{inv}}) of Proposition 2 hold.

Before doing anything else, let us clean up the place.

- For notational convenience, let us assume without loss of generality that \( m = 0_{\mathbb{R}^n} \) and \( V(0_{\mathbb{R}^n}) = 0 \).

- According to Lemma 1, we may assume (without loss of generality, up to changing the origin of time) that, for all \( t \) in \([0, +\infty)\),

\[
\sup_{x \in \mathbb{R}} |u(x, t)| \leq R_{\text{att,}\infty} \quad \text{and} \quad \| x \mapsto u(x, t) \|_X \leq R_{\text{att},X}.
\]

- According to the a priori bounds (11) on page 18, we may assume (without loss of generality, up to changing the origin of time) that

\[
\sup_{t \geq 0} \| x \mapsto u(x, t) \|_{C^{k+1}_{\text{b}}(\mathbb{R}, \mathbb{R}^n)} < +\infty \quad \text{and} \quad \sup_{t \geq 0} \| x \mapsto u_t(x, t) \|_{C^{k-1}_{\text{b}}(\mathbb{R}, \mathbb{R}^n)} < +\infty.
\]

- According to (H_{\text{hom-right}}), we may assume (without loss of generality, up to changing the origin of time) that, for all \( t \) in \([0, +\infty)\),
Unfortunately, the Escape point \( x_{\text{esc}}(t) \) presents a significant drawback: there is no a priori reason why it should display any form of continuity (it may jump back and forth while time increases). This lack of control is problematic with respect to our purpose, which is to write down a dissipation argument precisely around the position in space where the solution escapes from \( 0_{\mathbb{R}^n} \).

The answer to this difficulty will be to define another “escape point” (this one will be denoted by \( x_{\text{esc}}(t) \) — with a small “e” — instead of \( x_{\text{Esc}}(t) \)). This second definition is a bit more involved than that of \( x_{\text{Esc}}(t) \), but the resulting escape point will have the significant advantage of growing at a finite (and even bounded) rate (Lemma 4 below). The material required to define this escape point is introduced in the next subsection.

### 4.4 Firewall functional in the laboratory frame

The content of this subsection and of the next one is almost identical to that of section 4 of [23], where details, proofs, and comments can be found (the sole difference in [23] is the existence of a positive definite “diffusion matrix” — whereas in the present paper this diffusion matrix equals identity). Here we shall just provide the minimum required definitions and statements.

The notation is the same as in section 4 of [23] with an additional subscript “0” to point out that all these objects are defined in the (standing) laboratory frame. Similar objects will be defined in the next subsection 4.7 but this time in a travelling referential (this time the notation will not comprise this subscript “0”).

Let

\[
\kappa_0 = \min\left(\sqrt{\frac{2}{u_{\text{en,0}}}}, \sqrt{\frac{\lambda_{\text{min}}}{2}}\right)
\]

and let us consider the weight function \( \psi_0 \) defined by

\[
\psi_0(x) = \exp(-\kappa_0|x|).
\]

For \( \xi \in \mathbb{R} \), let \( T_\xi \psi_0 \) denote the translate of \( \psi_0 \) by \( \xi \), that is the function defined by:

\[
T_\xi \psi_0(x) = \psi_0(x - \xi)
\]

(see figure 10). For all \( t \) in \([0, +\infty)\) and \( \xi \) in \( \mathbb{R} \), let us consider the “firewall” function

![Figure 10: Graph of the weight function \( x \mapsto T_\xi \psi(x) \) used to define the firewall function \( F(\xi, t) \). The slope is small, according to the definition of \( \kappa \).](image)
\[ F_0(\xi, t) = \int_{\mathbb{R}} T_\xi \psi_0(x) \left( w_{en,0} \left( \frac{u_x(x,t)^2}{2} + V(u(x,t)) \right) + \frac{u(x,t)^2}{2} \right) \, dx, \]

(the quantity \( w_{en,0} \) was defined in sub-subsection [3.3.2]). Let
\[ \nu_{F_0} = \min\left( \frac{1}{w_{en,0}}, \frac{\lambda_{\min}}{4 \, w_{en,0} \, \lambda_{\max}} \right), \]

let
\[ \Sigma_{Esc,0}(t) = \{ x \in \mathbb{R} : |u(x,t)| > d_{Esc} \}, \]

and
\[ K_{F_0} = \max_{|v| \leq R_{att,\infty}} \left( \nu_{F_0} \left( w_{en,0} \, V(v) + \frac{v^2}{2} \right) - v \cdot \nabla V(v) + \frac{\lambda_{\min}}{4} \, v^2 \right). \]

**Lemma 2** (firewall decrease up to pollution term). For all \( \xi \) in \( \mathbb{R} \) and \( t \) in \([0, +\infty)\),
\[ \partial_t F_0(\xi, t) \leq -\nu_{F_0} \, F_0(\xi, t) + K_{F_0} \int_{\Sigma_{Esc,0}(t)} T_\xi \psi_0(x) \, dx. \]

*Proof.* See lemma 2 of [23].

### 4.5 Upper bound on the invasion speed

Let
\[ d_{esc} = d_{Esc} \left[ \min \left( \frac{w_{en,0}}{2}, \frac{1}{2} \right) \right]. \]

As the quantity \( d_{Esc} \) defined in subsection [23] this quantity \( d_{esc} \) will provide a way to measure the vicinity of the solution to the minimum point \( 0_{\mathbb{R}^n} \), this time in terms of the firewall function \( F_0 \). The value chosen for \( d_{esc} \) ensures the validity of the following lemma.

**Lemma 3** (escape/Escape). For all \( \xi \) in \( \mathbb{R} \) and \( t \) in \([0, +\infty)\), the following assertion holds:
\[ F_0(\xi, t) \leq d_{esc}^2 \implies |u(\xi, t)| \leq d_{esc}. \]

*Proof.* See lemma 3 of [23].

Let \( L \) be a positive quantity, sufficiently large so that
\[ 2K_{F_0} \exp(-\kappa_0 L) \leq \nu_{F_0} \, d_{esc}^2, \] namely \( L = \frac{1}{\kappa_0} \log \left( \frac{16 \, K_{F_0}}{\nu_{F_0} \, d_{esc}^2 \, \kappa_0} \right) \),
Figure 11: Graph of the hull function $\eta_{\text{no-esc}}$.

let $\eta_{\text{no-esc}} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ ("no-escape hull") be the function defined by (see figure 11):

$$
\eta_{\text{no-esc}}(x) = \begin{cases} 
+\infty & \text{for } x < 0, \\
\frac{d_{\text{esc}}^2}{2} \left(1 - \frac{x}{2L}\right) & \text{for } 0 \leq x \leq L, \\
\frac{d_{\text{esc}}^2}{4} & \text{for } x \geq L,
\end{cases}
$$

and let $c_{\text{no-esc}}$ ("no-escape speed") be a positive quantity, sufficiently large so that

$$
c_{\text{no-esc}} \frac{d_{\text{esc}}^2}{4L} \geq \frac{2K_{\alpha}}{\kappa_0}, \quad \text{namely } \quad c_{\text{no-esc}} = \frac{8K_{\alpha} L}{\kappa_0 d_{\text{esc}}^2}.
$$

The following lemma, illustrated by figure 12 is a variant of lemma 4 of [23].

Figure 12: Illustration of Lemma 4 if the firewall function is below the maximum of two mirror hulls at a certain time $t_0$ and if these two hulls travel at opposite speeds $\pm c_{\text{no-esc}}$, then the firewall will remain below the maximum of those travelling hulls in the future (note that after they cross this maximum equals $+\infty$ thus the assertion of being "below" is empty).

**Lemma 4** (bound on invasion speed). For every pair $(x_{\text{left}}, x_{\text{right}})$ of points of $\mathbb{R}$ and every $t_0$ in $[0, +\infty)$, if

$$
\mathcal{F}(x,t_0) \leq \max(\eta_{\text{no-esc}}(x - x_{\text{left}}), \eta_{\text{no-esc}}(x_{\text{right}} - x)) \quad \text{for all } x \in \mathbb{R},
$$

then, for every date $t$ not smaller than $t_0$ and all $x$ in $\mathbb{R}$,

$$
\mathcal{F}(x,t) \leq \max\left(\eta_{\text{no-esc}}(x_{\text{left}} - c_{\text{no-esc}}(t - t_0)), \eta_{\text{no-esc}}(x_{\text{right}} + c_{\text{no-esc}}(t - t_0) - x)\right).
$$
Proof. See lemma 4 of [23].

4.6 Settings of the proof, 2: escape point and associated speeds

With the notation and results of the previous subsections 4.4 and 4.5 in pocket, let us pursue the settings for the proof of Proposition 2 “invasion implies convergence”.

According to hypothesis (H_{hom-right}), we may assume, up to changing the origin of time, that, for all $t$ in $[0, +\infty)$ and for all $x$ in $\mathbb{R}$,

\begin{equation}
\mathcal{F}_0(x, t) \leq \max\left(\eta_{\text{no-esc}}(x - (x_{\text{hom}}(t) - 1)), \eta_{\text{no-esc}}(x_{\text{hom}}(t) - x)\right).
\end{equation}

As a consequence, for all $t$ in $[0, +\infty)$, the set

\[ I_{\text{Hom}}(t) = \{x_\ell \leq x_{\text{hom}}(t) : \text{for all } x \text{ in } \mathbb{R}, \mathcal{F}_0(x, t) \leq \max\left(\eta_{\text{no-esc}}(x - x_\ell), \eta_{\text{no-esc}}(x_{\text{hom}}(t) - x)\right)\} \]

is a nonempty interval (containing $[x_{\text{hom}}(t) - 1, x_{\text{hom}}(t)]$) that must be bounded from below (see figure 13). Indeed, if at a certain time it was not bounded from below — in other words if it was equal to $(-\infty, x_{\text{hom}}(t)]$ — then according to Lemma 4 this would remain unchanged in the future, thus according to Lemma 3 the point $x_{\text{Esc}}(t)$ would remain equal to $-\infty$ in the future, a contradiction with hypothesis (H_{inv}).

For all $t$ in $[0, +\infty)$, let

\begin{equation}
x_{\text{esc}}(t) = \inf(I_{\text{Hom}}(t)) \quad (\text{thus } x_{\text{esc}}(t) > -\infty).
\end{equation}

Somehow like $x_{\text{Esc}}(t)$, this point represents the first point at the left of $x_{\text{hom}}(t)$ where the solution “escapes” (in a sense defined by the firewall function $\mathcal{F}_0$ and the no-escape hull $\eta_{\text{no-esc}}$) at a certain distance from $0_{\mathbb{R}^+}$. In the following, this point $x_{\text{esc}}(t)$ will be called the “escape point” (by contrast with the “Escape point” $x_{\text{Esc}}(t)$ defined before). According to the “hull inequality” (24) and Lemma 3 (“escape/Escape”), for all $t$ in $[0, +\infty)$,

\begin{equation}
x_{\text{Esc}}(t) \leq x_{\text{esc}}(t) \leq x_{\text{hom}}(t) - 1 \quad \text{and} \quad \Sigma_{\text{Esc},0}(t) \cap [x_{\text{Esc}}(t), x_{\text{hom}}(t)] = \emptyset,
\end{equation}

Figure 13: Interval $I_{\text{Hom}}(t)$ and definition of $x_{\text{esc}}(t)$. 

27
and, according to hypothesis \(H_{\text{hom-right}}\),

\[(27) \quad x_{\text{hom}}(t) - x_{\text{esc}}(t) \to +\infty \quad \text{when} \quad t \to +\infty.\]

The big advantage of \(x_{\text{esc}}(\cdot)\) with respect to \(x_{\text{Esc}}(\cdot)\) is that, according to Lemma 4, the growth of \(x_{\text{esc}}(\cdot)\) is more under control. More precisely, according to this lemma, for every pair \((t, s)\) of points of \([0, +\infty)\),

\[(28) \quad x_{\text{esc}}(t + s) \leq x_{\text{esc}}(t) + c_{\text{no-esc}} s.\]

For every \(s\) in \([0, +\infty)\), let us consider the “upper and lower bounds of the variations of \(x_{\text{esc}}(\cdot)\) over all time intervals of length \(s\)” (see figure 14):

\[
\begin{align*}
x = x_{\text{esc}}(t_0) + x_{\text{esc}}(t-t_0) & \quad x = x_{\text{esc}}(t_1) + c_{\text{no-esc}} (t-t_1) \\
x = x_{\text{esc}}(t_0) + c_{\text{no-esc}} (t-t_0) & \quad x = x_{\text{esc}}(t_1) + x_{\text{esc}}(t-t_1) \\
x = x_{\text{esc}}(t_0) + x_{\text{esc}}(t-t_0) & \quad x = x_{\text{esc}}(t_1) + x_{\text{esc}}(t-t_1)
\end{align*}
\]

![Figure 14: Illustration of the bounds \((29)\).](image)

\[
\bar{x}_{\text{esc}}(s) = \sup_{t \in [0, +\infty)} x_{\text{esc}}(t + s) - x_{\text{esc}}(t) \quad \text{and} \quad \underline{x}_{\text{esc}}(s) = \inf_{t \in [0, +\infty)} x_{\text{esc}}(t + s) - x_{\text{esc}}(t).
\]

According to these definitions and to inequality \((28)\) above, for all \(t\) and \(s\) in \([0, +\infty)\),

\[(29) \quad -\infty \leq \underline{x}_{\text{esc}}(s) \leq x_{\text{esc}}(t + s) - x_{\text{esc}}(t) \leq \bar{x}_{\text{esc}}(s) \leq c_{\text{no-esc}} s.\]

Let us consider the four limit mean speeds:

\[
\begin{align*}
\bar{c}_{\text{esc-inf}} &= \liminf_{t \to +\infty} \frac{x_{\text{esc}}(t)}{t} & \text{and} & \quad \bar{c}_{\text{esc-sup}} &= \limsup_{t \to +\infty} \frac{x_{\text{esc}}(t)}{t}
\end{align*}
\]

and

\[
\begin{align*}
\bar{c}_{\text{esc-inf}} &= \liminf_{s \to +\infty} \frac{\underline{x}_{\text{esc}}(s)}{s} & \text{and} & \quad \bar{c}_{\text{esc-sup}} &= \limsup_{s \to +\infty} \frac{\bar{x}_{\text{esc}}(s)}{s}.
\end{align*}
\]
The following inequalities follow readily from these definitions and from hypothesis (H\textsubscript{inv}):

\[-\infty \leq \underline{c}_{\text{esc-inf}} \leq c_{\text{esc-inf}} \leq c_{\text{esc-sup}} \leq \bar{c}_{\text{esc-sup}} \leq c_{\text{no-esc}} \quad \text{and} \quad 0 < c_{\text{Esc}} \leq c_{\text{esc-sup}}.\]

We are going to prove that the four limit mean speeds defined just above are equal. The proof is based of the “relaxation scheme” set up in the next subsection.

Remark. In the previous paper [22] where convergence towards a single travelling front was proved, no object similar to the lower bound \(\underline{c}_{\text{esc}}(\cdot)\) or the quantity \(\underline{c}_{\text{esc-inf}}\) was defined. Here those objects will be specifically required to prove convergence towards the travelling fronts “following” the “first” ones (in the statement of Theorem 1, the “first” ones are \(\phi_{i,+}\) and \(\phi_{i,-}\), and the “following” ones are the \(\phi_{i,+}\) and \(\phi_{i,-}\) with \(i\) larger than 1). Indeed, for those “following” travelling fronts, a tighter control over the escape point will be required.

4.7 Relaxation scheme in a travelling frame

The aim of this subsection is to set up an appropriate relaxation scheme in a travelling frame. This means defining an appropriate localized energy and controlling the “flux” terms occurring in the time derivative of this localized energy. The considerations made in subsection 3.2 will be put in practice.

4.7.1 Preliminary definitions

Let us keep the notation and hypotheses introduced above (since the beginning of subsection 4.3), and let us introduce the following real quantities that will play the role of “parameters” for the relaxation scheme below (see figure 15):

- the “initial time” \(t_{\text{init}}\) of the time interval of the relaxation;
- the position \(x_{\text{init}}\) of the origin of the travelling frame at initial time \(t = t_{\text{init}}\) (in practice it will be chosen equal to \(x_{\text{esc}}(t_{\text{init}})\)).

Figure 15: Space coordinate \(y\) and time coordinate \(s\) in the travelling frame, and parameters \(t_{\text{init}}\) and \(x_{\text{init}}\) and \(c\) and \(y_{\text{cut-init}}\).
• the speed $c$ of the travelling frame;

• a quantity $y_{\text{cut-init}}$ that will be the position of the maximum point of the weight function $y \mapsto \chi(y, t_{\text{init}})$ localizing energy at initial time $t = t_{\text{init}}$ (this weight function is defined below); the subscript “cut” refers to the fact that this weight function displays a kind of “cut-off” on the interval between this maximum point and $+\infty$. Thus the maximum point is in some sense the point “where the cut-off begins”.

Let us make on these parameters the following hypotheses:

\begin{equation}
0 \leq t_{\text{init}} \quad \text{and} \quad 0 < c \leq c_{\text{no-esc}} \quad \text{and} \quad 0 \leq y_{\text{cut-init}}.
\end{equation}

The relaxation scheme will be applied several time in the next pages, for various choices of this set of three parameters.

For all $y$ in $\mathbb{R}$ and $s$ in $[0, +\infty)$, let

$$v(y, s) = u(x, t) \quad \text{where} \quad x = x_{\text{init}} + cs + y \quad \text{and} \quad t = t_{\text{init}} + s.$$  

This function $(y, s) \mapsto v(y, s)$ is thus defined from the initial solution $(x, t) \mapsto u(x, t)$ by considering this solution in a frame travelling at speed $c$, with $t_{\text{init}}$ as the origin of times and $x_{\text{init}}$ as the origin of space. It satisfies the differential system already written in (4) on page 3 and (12) on page 19, that is:

$$v_s - cv_y = -\nabla V(v) + v_{yy}.$$  

We are going to define two functions, each with its own weight function:

• a localized energy $s \mapsto \mathcal{E}(s)$;

• a localized “firewall” function $s \mapsto \mathcal{F}(s)$, that will be a combination of the energy and the $L^2$-norm with appropriate weightings.

Here are the constraints that have to be faced in the choice of these definitions. Some of them have already been mentioned in subsection 3.2. Note that these constraints are slightly more involved than in a frame at rest (this latter case is easier and treated in details in [23]).

1. Both weight functions should vary slowly with time.

2. The weight function for the energy should be equal — or at least close — to $\exp(cy)$ (up to a positive multiplicative constant) in a subset of the space real line “as large as possible” since such a subset does not “contribute” to the flux terms in the time derivative of localized energy.

3. The weight function for the firewall functional should either be equal — or at least close — to $\exp(cy)$ (up to a positive multiplicative constant), or vary slowly with respect to $y$, since each of these conditions ensures the smallness of the flux term of the $L^2$-norm.
4. The relative weighting of the energy and the $L^2$-norm in the definition of the firewall functional should face two independent constraints, both in favour of a large weighting of the $L^2$-norm with respect to the energy:

- the firewall should be coercive;
- in the time derivative of the firewall, the “non small” flux terms of the derivative of energy (where the weight function of energy is not close to $\exp(cy)$) should be balanced by the “main” (nonpositive) terms in the derivative of the $L^2$-norm.

5. Finally, if we want the total contribution of the flux terms of the energy to be small (and not only bounded), as will be required to prove convergence towards stationary solutions in a travelling referential, the initial weight function for the energy will have to be chosen equal to $\exp(cy)$ up to far to the right, and it is the parameter $y_{cut-init}$ that will enable this “tuning”.

Let $\kappa$ (rate of decrease of the weight functions), $c_{cut}$ (speed of the cut-off point in the travelling frame), and $w_{en}$ (weighting of energy in the “firewall” function) be three positive quantities, sufficiently small so that

$$\frac{w_{en}c_{cut}(c + \kappa)}{2} \leq \frac{1}{4}$$
and
$$w_{en}(c + 2\kappa)^2 \leq \frac{1}{4}$$

$$w_{en}c_{cut}(c + \kappa) \leq \frac{\lambda_{min}}{8\lambda_{max}}$$
and
$$\frac{(c_{cut} + \kappa)(c + \kappa)}{2} \leq \frac{\lambda_{min}}{8}$$

these conditions will be used to prove inequality (38) on page 35 and

$$w_{en} \leq w_{en,0},$$

(the quantity $w_{en,0}$ was defined in sub-subsection 3.3.2) namely

$$\kappa = \min\left(1, \frac{\lambda_{min}}{8(c_{no-esc} + 1)}\right),$$
$$c_{cut} = \min\left(\frac{\lambda_{min}}{8\lambda_{max}}, \frac{\lambda_{min}}{8(c_{no-esc} + 1)}\right),$$
$$w_{en} = \min\left(w_{en,0}, \frac{1}{(c_{no-esc} + 1)^2}\right).$$

4.7.2 Localized energy

For every real quantity $s$, let us consider the two intervals:

$$I_{main}(s) = (-\infty, y_{cut-init} + c_{cut}s],$$
$$I_{right}(s) = [y_{cut-init} + c_{cut}s, +\infty),$$

and let us consider the function $\chi(y, s)$ (weight function for the localized energy) defined by:

$$\chi(y, s) = \begin{cases} 
\exp(cy) & \text{if } y \in I_{main}(s), \\
\exp((c + \kappa)(y_{cut-init} + c_{cut}s) - \kappa y) & \text{if } y \in I_{right}(s),
\end{cases}$$
(see figure [16]), and, for all $s$ in $[0, +\infty)$, let us define the “energy function” $\mathcal{E}(s)$ by:

$$\mathcal{E}(s) = \int_{\mathbb{R}} \chi(y, s) \left( \frac{v_y(y, s)^2}{2} + V(v(y, s)) \right) dy.$$ 

Note that the choice of the decrease rate of $\chi(\cdot, s)$ on the interval $[y_{\text{cut-init}} + c_{\text{cut}} s, +\infty)$ is not crucial, we may have chosen a rate equal to 1, for instance. The sole (slight) advantage of this choice is that the weight functions $\chi$ and $\psi$ (the latter one defined below) are identical on this interval, and this is rather convenient for the estimates below.

4.7.3 Time derivative of the localized energy

According to expression [13] on page 19 for the derivative of a localized energy, the quantities $\chi_s$ and $c \chi - \chi_y$ will be involved in the derivative of $\mathcal{E}(s)$; it follows from the definition of $\chi$ that:

$$\chi_s(y, s) = \begin{cases} 0 & \text{if } y \in I_{\text{main}}(s), \\ c_{\text{cut}}(c + \kappa) \chi(y, s) & \text{if } y \in I_{\text{right}}(s), \end{cases}$$

and

$$(c \chi - \chi_y)(y, s) = \begin{cases} 0 & \text{if } y \in I_{\text{main}}(s), \\ (c + \kappa) \chi(y, s) & \text{if } y \in I_{\text{right}}(s). \end{cases}$$

Thus, if for all $s$ in $[0, +\infty)$ we define the “dissipation” function by

$$(33) \quad \mathcal{D}(s) = \int_{\mathbb{R}} \chi(y, s) v_s(y, s)^2 dy,$$
then, for all \( s \) in \([0, +\infty)\), it follows from expression (13) on page 19 for the derivative of a localized energy that

\[
E'(s) = -D(s) + \int_{I_{right}(s)} \chi \left( c_{cut}(c + \kappa) \left( \frac{v_y^2}{2} + V(v) \right) + (c + \kappa) v_y \cdot v_s \right) dy
\]

\[
\leq -\frac{1}{2} D(s) + \int_{I_{right}(s)} \chi \left( c_{cut}(c + \kappa) \left( \frac{v_y^2}{2} + V(v) \right) + (c + \kappa)^2 \frac{v_y^2}{2} \right) dy.
\]

(34)

4.7.4 Definition of the “firewall” function and bound on the time derivative of energy

A second function (the “firewall”) will now be defined, to get some control over the second term of the right-hand side of inequality (34). Let us consider the function \( \psi(y, s) \) (weight function for the firewall function) defined by:

\[
\psi(y, s) = \begin{cases} 
\exp((c + \kappa) y - \kappa (y_{cut-init} + c_{cut} s)) & \text{if } y \in I_{main}(s), \\
\chi(y, s) & \text{if } y \in I_{right}(s).
\end{cases}
\]

(see figure 16), and, for all \( s \) in \([0, +\infty)\), let us define the “firewall” function by:

\[
F(s) = \int_{R} \psi(y, s) \left( w_{en} \left( \frac{v_y(y, s)^2}{2} + V(v(y, s)) \right) + \frac{v(y, s)^2}{2} \right) dy.
\]

In view of the property (16) on page 21 concerning \( w_{en,0} \) and since according to inequality (32) the quantity \( w_{en} \) is not larger than \( w_{en,0} \), this function is coercive in the sense that, for all \( s \) in \([0, +\infty)\),

\[
F(s) \geq \min \left( \frac{w_{en}}{2}, \frac{1}{4} \right) \int_{R} \psi(y, s) \left( v_y(y, s)^2 + v(y, s)^2 \right) dy.
\]

(35)

(however, by contrast with the firewall function \( F_0(t) \) in the laboratory frame introduced in subsection 4.4, this coercivity property will not be directly used in the next pages — only the fact that \( F(s) \) is nonnegative will be, for instance to derive (41) from (37)).

Inequality (34) above can be rewritten (without changing the value of the right-hand side) as follows (note the substitution of \( \chi \) by \( \psi \)):

\[
E'(s) \leq -\frac{1}{2} D(s) + \int_{I_{right}(s)} \psi \left( c_{cut}(c + \kappa) \frac{v_y^2}{2} + V(v) \right) + \frac{(c + \kappa)^2}{w_{en}} \frac{v_y^2}{2} dy.
\]

\[
\leq -\frac{1}{2} D(s) + \int_{I_{right}(s)} \psi \left( c_{cut}(c + \kappa) \left( w_{en} \left( \frac{v_y^2}{2} + V(v) \right) + \frac{v^2}{2} \right) + \frac{(c + \kappa)^2}{w_{en}} \frac{v_y^2}{2} \right) dy;
\]

33
thus, again in view of the property (16) on page 21 derived from the definition of $w_{en,0}$ and since $w_{en}$ is not larger than $w_{en,0}$, it follows that

\[ E'(s) \leq -\frac{1}{2} D(s) + \frac{c_{cut} \left(c + \kappa\right) + \left(c + \kappa\right)^2}{w_{en}} \int_{I_{\text{right}}(s)} \psi \left( w_{en} \left( \frac{v_y^2}{2} + V(v) \right) + \frac{v^2}{2} \right) dy \]

\[ \leq -\frac{1}{2} D(s) + \frac{c_{cut} \left(c_{\text{no-esc}} + \kappa\right) + \left(c_{\text{no-esc}} + \kappa\right)^2}{w_{en}} \int_{\mathbb{R}} \psi \left( w_{en} \left( \frac{v_y^2}{2} + V(v) \right) + \frac{v^2}{2} \right) dy . \]

Let us consider the (positive) quantity (depending only on $V$):

\[ K_E = \frac{c_{\text{cut}} \left(c_{\text{no-esc}} + \kappa\right) + \left(c_{\text{no-esc}} + \kappa\right)^2}{w_{en}} . \]

According to the property (16) on page 21 concerning $w_{en,0}$, it follows from the upper bound (34) on $E'(\cdot)$ that, for all $s$ in $[0, +\infty)$,

\[ E'(s) \leq -\frac{1}{2} D(s) + K_E F(s) . \]

Let $s_{\text{fin}}$ be a nonnegative quantity (denoting the length of the time interval on which the relaxation scheme will be applied). It follows from the previous inequality that

\[ \frac{1}{2} \int_0^{s_{\text{fin}}} D(s) \, ds \leq E(0) - E(s_{\text{fin}}) + K_E \int_0^{s_{\text{fin}}} F(s) \, ds . \]

The approximate decrease of of the localized energy (up to a remaining term controlled by the firewall function) above, and more specifically the integrated form (36) are the core of the relaxation scheme we are setting up. Indeed, if some control can be obtained over the right-hand side (upper bound on the firewall function, upper bound on the initial localized energy, lower bound on the final localized energy), then it will follow that the dissipation is bounded, or small (at least its integral), an information obviously related to the vicinity of the solution to travelling fronts. Our next goal (in the next sub-subsection) is to gain some control over the firewall function.

### 4.7.5 Time derivative of the firewall function

For every $s$ in $[0, +\infty)$, let us consider the set (the domain of space where the solution “Escapes” at distance $d_{\text{Esc}}$ from $0_{\mathbb{R}^n}$):

\[ \Sigma_{\text{Esc}}(s) = \{ y \in \mathbb{R} : |v(y, s)| > d_{\text{Esc}} \} . \]

To make the connection with the definition (20) on page 25 of the related set $\Sigma_{\text{Esc},0}(t)$, observe that, for all $s$ in $[0, +\infty)$ and $y$ in $\mathbb{R}$,

\[ y \in \Sigma_{\text{Esc}}(s) \iff x_{\text{init}} + cs + y \in \Sigma_{\text{Esc},0}(t_{\text{init}} + s) . \]

Our next goal is to prove the following lemma (observe the strong similarity with Lemma 2 on page 25).
Lemma 5 (firewall decrease up to pollution term). There exist positive quantities $\nu_F$ and $K_F$, depending only on $V$, such that, for all $s$ in $[0, +\infty)$,

\begin{equation}
F'(s) \leq -\nu_F F(s) + K_F \int_{\Sigma_{\text{Esc}}(s)} \psi(y, s) \, dy.
\end{equation}

Proof. According to expressions (13) and (14) on page 19 for the time derivatives of a localized energy and a localized $L^2$ functional, for all $s$ in $[0, +\infty)$,

\[ F'(s) = \int_{\mathbb{R}} \left[ \psi \left( -w_{en} v_s^2 - v \cdot \nabla V(v) - v_y^2 \right) + \psi_s \left( w_{en} \left( \frac{v_s^2}{2} + V(v) \right) + \frac{v_y^2}{2} \right) \right. \]
\[ + \left. (c\psi_y \psi - \psi_y v_s + (\psi_{yy} - c\psi_y) \frac{v_y^2}{2} \right] dy. \]

According to the definition of $\psi(\cdot, \cdot)$, the following inequalities hold for all values of its arguments:

\[ |\psi_s| \leq c_{\text{cut}} (c + \kappa) \psi \quad \text{and} \quad |c\psi - \psi_y| \leq (c + \kappa) \psi \quad \text{and} \quad \psi_{yy} - c\psi_y \leq \kappa(c + \kappa) \psi \]

(Indeed, $\psi_{yy} - c\psi_y$ equals $\kappa(c + \kappa) \psi$ plus a Dirac mass of negative weight at $y = y_{\text{cut-init}} + c_{\text{cut}} s$). Thus, for all $s$ in $[0, +\infty)$,

\[ F'(s) \leq \int_{\mathbb{R}} \psi \left[ -w_{en} v_s^2 - v \cdot \nabla V(v) - v_y^2 + c_{\text{cut}} (c + \kappa) \left( w_{en} \left( \frac{v_s^2}{2} + |V(v)| \right) + \frac{v_y^2}{2} \right) \right. \]
\[ + \left. w_{en} (c + \kappa)|v_y \cdot v_s| + \kappa(c + \kappa) \frac{v_y^2}{2} \right] dy, \]

thus, polarizing the scalar product $v_y \cdot v_s$,

\[ F'(s) \leq \int_{\mathbb{R}} \psi \left[ \left( w_{en} \left( \frac{c_{\text{cut}} (c + \kappa)}{2} + \frac{(c + \kappa)^2}{4} \right) - 1 \right) v_y^2 - v \cdot \nabla V(v) \right. \]
\[ + \left. w_{en} c_{\text{cut}} (c + \kappa) |V(v)| + \frac{(c_{\text{cut}} + \kappa)(c + \kappa)}{2} v_y^2 \right] dy. \]

According to the conditions (31) on page 31 satisfied by $\kappa$ and $c_{\text{cut}}$ and $w_{en}$, it follows that

\begin{equation}
F'(s) \leq \int_{\mathbb{R}} \psi \left[ -\frac{v_y^2}{2} - v \cdot \nabla V(v) + \frac{\lambda_{\min}}{8\lambda_{\max}} |V(v)| + \frac{\lambda_{\min}}{8} v_y^2 \right] dy.
\end{equation}

If the quantity $v(y, s)$ was close to $0_{\mathbb{R}^n}$ for all $y$ in $\mathbb{R}$, then the right-hand side of this inequality would be bounded from above by $-\varepsilon F(t)$ for some positive quantity $\varepsilon$; indeed, for $|v(\cdot, \cdot)|$ not larger than $d_{\text{Esc}}$, according to the estimates (15) on page 20 for $V(u)$ and $u \cdot \nabla V(u)$ when $|u|$ is not larger than $d_{\text{Esc}}$, the last term is dominated by the term
$-v \cdot \nabla V(v)$, and the quantities $v \cdot \nabla V(v)$ and $v^2$ and $V(v)$ do no differ by more than a bounded factor. What will actually follow (inequality (37) below) is indeed an upper bound of this form plus an additional “remaining” term that comes from the part of space where $v(y, s)$ is not close to $0_{\mathbb{R}^n}$.

Let $\nu F$ be a positive quantity, sufficiently small so that

$$\nu F w_{en} \leq 1 \quad \text{and} \quad \nu F \left( w_{en} \lambda_{max} + \frac{1}{2} \right) \leq \frac{\lambda_{min}}{4}$$

(39)

(these two properties will be used in estimates below), namely

$$\nu F = \min \left( \frac{1}{w_{en}}, \frac{\lambda_{min}}{4(w_{en} \lambda_{max} + 1/2)} \right).$$

Let us add and subtract to the right-hand side of inequality (38) the same quantity (with the purpose of making appear a term proportional to $-F'(s)$), as follows:

$$F'(s) \leq \int_{\mathbb{R}} \psi \left[ -\frac{v^2}{2} - \nu F \left( w_{en} V(v) + \frac{v^2}{2} \right) \right] dy$$

$$+ \int_{\mathbb{R}} \psi \left[ \nu F \left( w_{en} V(v) + \frac{v^2}{2} \right) - v \cdot \nabla V(v) + \frac{\lambda_{min}}{8 \lambda_{max}} |V(v)| + \frac{\lambda_{min}}{8} v^2 \right] dy.$$

(40)

The following observations can be made about the two integrals at the right-hand side of this inequality.

- According to the fist of conditions (39) on $\nu F$, the first integral is bounded from above by $-\nu F F(s)$.

- According to estimates (15) on page 20 for $V(u)$ and $u \cdot \nabla V(u)$ when $|u|$ is not larger than $d_{esc}$, the integrand of the second integral is nonpositive as long as $y$ does not belong to the set $\Sigma_{esc}(s)$; thus the inequality still holds if the domain of integration of that second integral is restricted to $\Sigma_{esc}(s)$.

Thus it follows from (40) that

$$F'(s) \leq -\nu F F(s)$$

$$+ \int_{\Sigma_{esc}(s)} \psi \left[ \nu F \left( w_{en} V(v) + \frac{v^2}{2} \right) - v \cdot \nabla V(v) + \frac{\lambda_{min}}{8 \lambda_{max}} |V(v)| + \frac{\lambda_{min}}{8} v^2 \right] dy.$$

Let us introduce the quantity (depending only on $V$):

$$K_F = \max_{u \in \mathbb{R}^n, \ |u| \leq R_{att, \infty}} \left( \nu F \left( w_{en} V(u) + \frac{u^2}{2} \right) - u \cdot \nabla V(u) + \frac{\lambda_{min}}{8 \lambda_{max}} |V(u)| + \frac{\lambda_{min}}{8} u^2 \right).$$

With this notation, inequality (37) follows from the last inequality above. Lemma 5 is proved.
For all $s$ in $[0, +\infty)$, let
\[
G(s) = \int_{\Sigma_{\text{Esc}}(s)} \psi(y, s) \, dy.
\]

Integrating inequality (37) between 0 and a nonnegative quantity $s_{\text{fin}}$ yields, since $F(s_{\text{fin}})$ is nonnegative,
\[
\int_0^{s_{\text{fin}}} F(s) \, ds \leq \frac{1}{\nu_F} \left( F(0) + K_F \int_0^{s_{\text{fin}}} G(s) \, ds \right),
\]
and the “relaxation scheme” inequality (36) becomes:
\[
(41) \quad \frac{1}{2} \int_0^{s_{\text{fin}}} D(s) \, ds \leq \mathcal{E}(0) - \mathcal{E}(s_{\text{fin}}) + \frac{K_\xi}{\nu_F} \left( F(0) + K_F \int_0^{s_{\text{fin}}} G(s) \, ds \right).
\]

Our next goal is to gain some control over the quantity $G(s)$.

### 4.7.6 Control over the flux term in the time derivative of the firewall function

For every nonnegative quantity $s$, let
\[
y_{\text{hom}}(s) = x_{\text{hom}}(t_{\text{init}} + s) - x_{\text{init}} - cs,
\]
and
\[
y_{\text{esc}}(s) = x_{\text{esc}}(t_{\text{init}} + s) - x_{\text{init}} - cs
\]
(see figures 17 and 18). According to properties (26) on page 27 for the set $\Sigma_{\text{Esc},0}(t)$,

![Diagram showing definitions of escape and homogeneous points](image)

Figure 17: Definitions of the “escape” point $y_{\text{esc}}(s)$ and the point $y_{\text{hom}}(s)$ marking the “homogeneous” area in the travelling frame.

Thus if we consider the quantities
\[
G_{\text{back}}(s) = \int_{-\infty}^{y_{\text{esc}}(s)} \psi(y, s) \, dy \quad \text{and} \quad G_{\text{front}}(s) = \int_{y_{\text{hom}}(s)}^{+\infty} \psi(y, s) \, dy,
\]

\[
\Sigma_{\text{Esc}}(s) \subset (-\infty, y_{\text{esc}}(s)] \cup [y_{\text{hom}}(s), +\infty),
\]
Figure 18: Typical relative positions of the points $y_{esc}(s)$, $y_{cut-init} + c_{cut}s$, $y_{hom}(0) - cs$, and $y_{hom}(s)$ in the travelling referential, and graph of $y \mapsto \psi(y, s)$.

then

$\mathcal{G}(s) \leq \mathcal{G}_{back}(s) + \mathcal{G}_{front}(s)$.

The aim of this sub-subsection is to prove the bounds on $\mathcal{G}_{back}(s)$ and $\mathcal{G}_{front}(s)$ provided by the next lemma. The following additional technical hypothesis will be required for the bound on $\mathcal{G}_{back}(s)$:

\begin{equation}
\bar{c}_{esc-sup} - \frac{\kappa c_{cut}}{4(c_{no-esc} + \kappa)} \leq c.
\end{equation}

This hypothesis is satisfied as soon as the speed is close enough to $\bar{c}_{esc-sup}$. It ensures that the escape point $y_{esc}(s)$ remains “more and more far away to the left” with respect to the position $y_{cut-init} + c_{cut}s$ of the cut-off, when $s$ increases.

**Lemma 6** (upper bounds on flux terms in the derivative of the firewall). There exists a positive quantity $K[u_0]$, depending only on $V$ and on the initial condition $u_0$ under consideration, such that for every nonnegative quantity $s$ the following estimates hold:

\begin{equation}
\mathcal{G}_{back}(s) \leq K[u_0] \exp(-\kappa y_{cut-init}) \exp\left(-\frac{\kappa c_{cut}}{2}s\right),
\end{equation}

\begin{equation}
\mathcal{G}_{front}(s) \leq \frac{1}{\kappa} \exp\left[(c_{no-esc} + \kappa)y_{cut-init} + (c_{no-esc} + \kappa)(c_{cut} + \kappa)s - \kappa y_{hom}(0)\right].
\end{equation}

**Proof.** First here are some considerations about the way the various parameters will be chosen for the relaxation scheme; according to these considerations, figure 18 displays the expected position of the various relevant points in the travelling frame.

- The position $x_{init}$ of the origin of the travelling frame at time $t_{init}$ will always be chosen equal to the position $x_{esc}(t_{init})$ of the escape point at this time.
- The time $t_{init}$ will be chosen very large, so that $y_{hom}(0)$ is very large, and $y_{hom}(s) \geq y_{hom}(0) - cs$ remains large for the values of $s$ under consideration.
- The initial position $y_{cut-init}$ of the cut-off will be chosen either equal to 0, or large positive (thus in both cases nonnegative).
- The speed $c$ will be chosen close in the interval $(0, \bar{c}_{esc-sup})$ and very close to the upper bound of this interval. As a consequence, the point $y_{esc}(s)$ (the “escape
point” viewed in the travelling frame) is expected to remain, for most of values of $s$, and for sure for $s$ large, at the left of the “cut-off” point $y_{\text{cut-init}} + c_{\text{cut}} s$, since this cut-off point travels to the right at a definite nonzero speed $c_{\text{cut}}$ in the travelling frame.

These considerations lead us to bound the integrand $\psi(y, s)$ in the expression of $G_{\text{back}}(s)$ and $G_{\text{front}}(s)$ by:

$$G_{\text{back}}(s) \leq \frac{1}{c + \kappa} \exp\left((c + \kappa)y_{\text{esc}}(s) - \kappa y_{\text{cut-init}} - \kappa c_{\text{cut}} s\right)$$

$$\leq \frac{1}{\kappa} \exp\left((c + \kappa)y_{\text{esc}}(s) - \kappa y_{\text{cut-init}} - \kappa c_{\text{cut}} s\right).$$

According to the definition of $y_{\text{esc}}(\cdot)$ and $\bar{x}_{\text{esc}}(\cdot)$ and provided that $x_{\text{init}} = x_{\text{esc}}(t_{\text{init}})$, for all $s$ in $[0, +\infty),$

$$y_{\text{esc}}(s) \leq \bar{x}_{\text{esc}}(s) - cs.$$ It follows that

$$G_{\text{back}}(s) \leq \frac{1}{\kappa} \exp(-\kappa y_{\text{cut-init}}) \exp\left((c + \kappa)(\bar{x}_{\text{esc}}(s) - cs) - \kappa c_{\text{cut}} s\right)$$

$$(45) \quad \leq \frac{1}{\kappa} \exp(-\kappa y_{\text{cut-init}}) \exp\left((c + \kappa)(\bar{x}_{\text{esc}}(s) - cs) - \frac{\kappa c_{\text{cut}}}{2} s\right) \exp\left(-\frac{\kappa c_{\text{cut}}}{2} s\right).$$

Let us us denote by $\beta(s)$ the argument of the second exponential of the right-hand side of this last inequality:

$$\beta(s) = (c + \kappa)(\bar{x}_{\text{esc}}(s) - cs) - \frac{\kappa c_{\text{cut}}}{2} s$$

$$= (c + \kappa)(\bar{x}_{\text{esc}}(s) - \bar{c}_{\text{esc-sup}} s) + \left((c + \kappa)(\bar{c}_{\text{esc-sup}} - c) - \frac{\kappa c_{\text{cut}}}{2}\right)s.$$ According to hypothesis (42) above, the following inequality holds:

$$(c + \kappa)(\bar{c}_{\text{esc-sup}} - c) \leq \frac{\kappa c_{\text{cut}}}{4},$$

thus, for all $s$ in $[0, +\infty),$

$$\beta(s) \leq (c + \kappa)(\bar{x}_{\text{esc}}(s) - \bar{c}_{\text{esc-sup}} s) - \frac{\kappa c_{\text{cut}}}{4} s,$$

and thus, according to the definition of $\bar{c}_{\text{esc-sup}}$ this quantity $\beta(s)$ approaches $-\infty$ when $s$ approaches $+\infty$. It follows from the definition of $\bar{x}_{\text{esc}}(\cdot)$ that $\beta(0)$ equals $0$, and, for all $s$ in $(0, +\infty)$, it follows from the last inequality that

$$\beta(s) > 0 \implies \bar{x}_{\text{esc}}(s) - \bar{c}_{\text{esc-sup}} s > 0.$$
Thus, the following (nonnegative) quantity:

\[ \bar{\beta}[u_0] = \sup_{s \geq 0} (c_{no-esc} + \kappa) (\bar{x}_{esc}(s) - \bar{c}_{esc-sup} s) - \frac{\kappa c_{cut}}{4} s , \]

is an upper bound for all the values of \( \beta(s) \), for all \( s \) in \([0, +\infty)\). This quantity depends on \( V \) and on the function \( x \mapsto \bar{x}_{esc}(s) \), in other words on the initial condition \( u_0 \), but not on the parameters \( t_{init} \) and \( c \) and \( y_{cut-init} \) of the relaxation scheme. Let

\[ K[u_0] = \frac{1}{\kappa} \exp(\bar{\beta}[u_0]) ; \]

with this notation, the upper bound (43) on \( G_{back}(s) \) follows from inequality (45).

Now let us consider the second quantity \( G_{front}(s) \). The control that will be required on this quantity is simpler, since it only relies on the value of \( y_{hom}(0) \), which can be assumed to be arbitrarily large if \( t_{init} \) is sufficiently large. Since \( x'_h(\cdot) \) is nonnegative (see (19) on page 24), for all \( s \) in \([0, +\infty)\),

\[ y'_h(s) \geq -c \quad \text{thus} \quad y_{hom}(s) \geq y_{hom}(0) - cs . \]

By explicit calculation, it follows from the upper bound on the integrand of \( G_{front}(s) \) that

\[ G_{front}(s) \leq \frac{1}{\kappa} \exp((c + \kappa) y_{cut-init}) \exp\left( ((c + \kappa) c_{cut} + \kappa c) s \right) \exp(-\kappa y_{hom}(0)) , \]

and inequality (44) on \( G_{front}(s) \) follows. Lemma 6 is proved.

4.7.7 Final form of the “relaxation scheme” inequality

Let us consider the quantity

\[ K_{G_{back}[u_0]} = \frac{2 K_F K_{back}[u_0]}{\nu_F \kappa c_{cut}} , \]

and, for every nonnegative quantity \( s \), the quantity

\[ K_{G_{front}(s)} = \frac{K_F K_F}{\nu_F \kappa (c_{no-esc} + \kappa)(c_{cut} + \kappa)} \exp((c_{no-esc} + \kappa)(c_{cut} + \kappa)s) . \]

Then, according to inequalities (43) and (44) of Lemma 6 the “relaxation scheme” inequality (41) on page 24 can be rewritten as follows:

\[\frac{1}{2} \int_0^{s_{fin}} D(s) ds \leq \mathcal{E}(0) - \mathcal{E}(s_{fin}) + \frac{K_F}{\nu_F} \mathcal{F}(0) + K_{G_{back}[u_0]} \exp(-\kappa y_{cut-init}) \]
\[ + K_{G_{front}(s_{fin})} \exp((c_{no-esc} + \kappa) y_{cut-init}) \exp(-\kappa y_{hom}(0)) .\]

In order to derive from this estimate useful information (a nice upper bound on the dissipation integral, stating that this dissipation integral is bounded or even small), the following conditions should be fulfilled:
• the “initial” value $E(0)$ of the localized energy should be bounded from above;
• the “final” value $E(s_{\text{fin}})$ of the localized energy should be bounded from below (or, better, close to $E(0)$);
• the “initial” value $F(0)$ of the firewall function should be small (or at least bounded);
• the “initial” position $y_{\text{cut-init}}$ of the “cut-off point” should be large (recall that it is assumed to be nonnegative, and this already provides some control);
• the “initial” position $y_{\text{hom}}(0)$ of the “homogeneous point” should be large, the condition on its side depending on $y_{\text{cut-init}}$ and $s_{\text{fin}}$.

4.7.8 Time derivative of the dissipation

The following lemma provides a convenient way to turn upper bounds on the integral of $D(s)$ over a time interval into upper bounds over $D(s)$ itself. It will be used in the proof of Lemma 17 on page 58 (and nowhere else). Let us mention however that this step could be avoided by a compactness argument altogether on space and time intervals, as do Gallay and Joly in [8].

**Lemma 7** (upper bound on time derivative of dissipation). There exists a positive quantity $K_D$, depending only on $V$, such that for every positive quantity $s$, the following estimate holds:

\[
D'(s) \leq K_D D(s).
\]

**Proof.** For all $s$ in $(0, +\infty)$,

\[
D'(s) = \int_R \left( \chi_s v_s^2 + 2 \chi v_s \left( c v_{ys} - D^2 V(v) \cdot v_s + v_{ys} \right) \right) dy
\]

\[
\leq \int_R \left( c_{\text{cut}} (c + \kappa) \chi v_s^2 - 2 \chi v_s \cdot D^2 V(v) \cdot v_s + 2 \left( c \chi - \chi y \right) v_s \cdot v_{ys} - 2 \chi v_{ys}^2 \right) dy
\]

\[
\leq \int_R \chi \left( c_{\text{cut}} (c + \kappa) v_s^2 - 2 v_s \cdot D^2 V(v) \cdot v_s + 2 \left( c + \kappa \right) |v_s| |v_{ys}| - 2 v_{ys}^2 \right) dy.
\]

Let us consider the quantity (depending only on $V$):

\[
K_D = c_{\text{cut}} (c_{\text{no-esc}} + \kappa) - 2 \min_{|u| \leq R_{\text{att}, \infty}} \sigma(D^2 V(u)) + \frac{(c_{\text{no-esc}} + \kappa)^2}{2};
\]

With this notation, inequality (47) readily follows from inequality (48). □

4.8 Compactness

The end of the proof of Proposition 2 “invasion implies convergence” will make use several times of the following compactness argument.
Lemma 8 (compactness). Let \((x_p, t_p)_{p \in \mathbb{N}}\) denote a sequence in \(\mathbb{R} \times [0, +\infty)\), and for every integer \(p\) let us consider the functions \(x \mapsto u_p(x)\) and \(x \mapsto \tilde{u}_p(x)\) defined by:

\[
u_p(x) = u(x_p + x, t_p) \quad \text{and} \quad \tilde{u}_p(x) = u_t(x_p + x, t_p).
\]

Then, up to replacing the sequence \((x_p, t_p)_{p \in \mathbb{N}}\) by a subsequence, there exist functions \(u_\infty \) in \(C^k_b(\mathbb{R}, \mathbb{R}^n)\) and \(\tilde{u}_\infty \) in \(C^{k-2}_b(\mathbb{R}, \mathbb{R}^n)\) such that, for every positive quantity \(L\),

\[
\|u_p(\cdot) - u_\infty(\cdot)\|_{C^k([-L, L], \mathbb{R}^n)} \to 0 \quad \text{and} \quad \|\tilde{u}_p(\cdot) - \tilde{u}_\infty(\cdot)\|_{C^{k-2}([-L, L], \mathbb{R}^n)} \to 0
\]

when \(p \to +\infty\), and such that, for all \(x\) in \(\mathbb{R}\),

\[
\tilde{u}_\infty(x) = -\nabla V(u_\infty(x)) + u'_\infty(x).
\]

Proof. According to the a priori bounds (18) on page 23 for the derivatives of the solutions of system (1), by compactness and a diagonal extraction procedure, there exist functions \(u_\infty\) and \(\tilde{u}_\infty\) such that, up to extracting a subsequence,

\[
u_p(\cdot) \to u_\infty(\cdot) \quad \text{and} \quad \tilde{u}_p(\cdot) \to \tilde{u}_\infty \quad \text{when} \quad p \to +\infty,
\]

uniformly on every compact subset of \(\mathbb{R}\). The limits \(u_\infty\) and \(\tilde{u}_\infty\) belong respectively to \(C^k_b(\mathbb{R}, \mathbb{R}^n)\) and \(C^{k-2}_b(\mathbb{R}, \mathbb{R}^n)\) and the convergences hold in \(C^k([-L, L], \mathbb{R}^n)\) and in \(C^{k-2}([-L, L], \mathbb{R}^n)\) respectively, for every positive quantity \(L\).

Passing to the limit in system (1) the last conclusion follows. \(\square\)

4.9 Convergence of the mean invasion speed

The aim of this subsection is to prove the following proposition.

Proposition 3 (mean invasion speed). The following equalities hold:

\[
c_{\text{esc-inf}} = c_{\text{esc-sup}} = \bar{c}_{\text{esc-sup}}.
\]

Proof. Let us proceed by contradiction and assume that

\[
c_{\text{esc-inf}} < \bar{c}_{\text{esc-sup}}.
\]

Then, let us take and fix a positive quantity \(c\) satisfying the following conditions:

\[
c_{\text{esc-inf}} < c < \bar{c}_{\text{esc-sup}} \leq c + \frac{\kappa_{\text{cut}}}{4(c_{\text{no-esc}} + \kappa)} \quad \text{and} \quad \Phi_c(0_{\mathbb{R}^n}) = \emptyset.
\]

The first condition is satisfied as soon as \(c\) is smaller than and sufficiently close to \(\bar{c}_{\text{esc-sup}}\), thus existence of a quantity \(c\) satisfying the two conditions follows from hypothesis \(H_{\text{disc-c}}\).

The contradiction will follow from the relaxation scheme set up in subsection 4.7. The main ingredient is: since the set \(\Phi_c(0_{\mathbb{R}^n})\) is empty, some dissipation must occur permanently around the escape point in a referential travelling at speed \(c\). This is stated by the following lemma.
Lemma 9 (nonzero dissipation in the absence of travelling front). There exist positive quantities $L$ and $\varepsilon_{\text{dissip}}$ such that, for every $t$ in $[0, +\infty)$,

$$
\| y \mapsto u_t(x_{\text{esc}}(t) + y, t) + cu_x(x_{\text{esc}}(t) + y, t) \|_{L^2([-L,L],\mathbb{R}^n)} \geq \varepsilon_{\text{dissip}}.
$$

Proof of Lemma 9. Let us proceed by contradiction and assume that the converse is true. Then, for every nonzero integer $p$, there exists $t_p$ in $[0, +\infty)$ such that

$$
(50) \quad \| y \mapsto u_t(x_{\text{esc}}(t_p) + y, t_p) + cu_x(x_{\text{esc}}(t_p) + y, t_p) \|_{L^2([-p,p],\mathbb{R}^n)} \leq \frac{1}{p}.
$$

By compactness (Lemma 8), up to replacing the sequence $(t_p)_{p\in\mathbb{N}}$ by a subsequence, there exist $u_\infty$ in $C^k_b(\mathbb{R}, \mathbb{R}^n)$ and $\tilde{u}_\infty$ in $C^{k-2}_b(\mathbb{R}, \mathbb{R}^n)$ such that, for every positive quantity $L$,

$$
\| u_p(\cdot) - u_\infty(\cdot) \|_{C^k([-L,L],\mathbb{R}^n)} \to 0 \quad \text{and} \quad \| \tilde{u}_p(\cdot) - \tilde{u}_\infty(\cdot) \|_{C^{k-2}([-L,L],\mathbb{R}^n)} \to 0
$$

when $p \to +\infty$ and such that, for all $x$ in $\mathbb{R}$,

$$
\tilde{u}_\infty(x) = -\nabla V(u_\infty(x)) + u''_\infty(x).
$$

According to hypothesis (50), the function $u_\infty + c\tilde{u}_\infty$ vanishes identically, so that $u_\infty$ is a solution of

$$
u''_\infty + cu'_\infty - \nabla V(u_\infty) = 0.
$$

According to the properties of the escape point (26) and (27) on page 27 and on page 28

$$
\sup_{y\in[0, +\infty)} |u_\infty(y)| \leq d_{\text{Esc}},
$$

thus it follows from Lemma 43 on page 87 that $u_\infty(y)$ approaches $0_{\mathbb{R}^n}$ when $y$ approaches $+\infty$. On the other hand, according to the a priori bounds on the solution, $|u_\infty(\cdot)|$ is bounded (by $R_{\text{att},\infty}$), and since $\Phi_c(0_{\mathbb{R}^n})$ is empty, it follows from Lemma 42 on page 87 that $u_\infty(\cdot)$ vanishes identically, a contradiction with the definition of $x_{\text{esc}}(\cdot)$.

The next step is the choice of the time interval and the travelling frame (at speed $c$) where the relaxation scheme will be applied. Those should display the following features:

1. the escape point should not go far to the left (or else, due to the exponential weight, no lower bound on the dissipation would hold);
2. at the end of the time interval the escape point should not be far to the right, so that the final value of the localized energy be bounded from below;
3. the length of the time interval should be sufficiently large, in order a sufficient dissipation to occur;
4. for a given length of that time interval, its lower bound should be chosen arbitrarily large, in order to control the “front” flux term involving the quantity $K_{\mathcal{G},\text{front}}$ in the relaxation scheme final inequality (46) on page 40.
The following lemma is a first attempt to find such a time interval (see figure 19).

**Lemma 10** (large excursions to the right and returns for escape point in travelling frame). There exist sequences \((t_p)_p \in \mathbb{N}\) and \((s_p)_p \in \mathbb{N}\) and \((\bar{s}_p)_p \in \mathbb{N}\) of real quantities such that the following properties hold.

- For every integer \(p\), the following inequalities hold: \(0 \leq t_p \text{ and } 0 \leq s_p \leq \bar{s}_p\);
- \(x_{\text{esc}}(t_p + s_p) - x_{\text{esc}}(t_p) - c s_p \to +\infty \text{ when } p \to +\infty\);
- For every integer \(p\), the following inequality holds: \(x_{\text{esc}}(t_p + \bar{s}_p) - x_{\text{esc}}(t_p) - c \bar{s}_p \leq 0\).

**Proof of Lemma 10.** According to the definition of \(\bar{c}_{\text{esc-sup}}\), there exists a sequence \((s_p)_p \in \mathbb{N}\) of positive real quantities, satisfying

\[
s_p \to +\infty \text{ and } \frac{x_{\text{esc}}(s_p)}{s_p} \to \bar{c}_{\text{esc-sup}} \text{ when } p \to +\infty.
\]

Then, by definition of \(\bar{x}_{\text{esc}}(\cdot)\), for every integer \(p\) there exists a nonnegative quantity \(t_p\) such that

\[
x_{\text{esc}}(t_p + s_p) - x_{\text{esc}}(t_p) \geq \bar{x}_{\text{esc}}(s_p) - 1.
\]

Then,

\[
\frac{x_{\text{esc}}(t_p + s_p) - x_{\text{esc}}(t_p) - c s_p}{s_p} \geq \frac{\bar{x}_{\text{esc}}(s_p) - 1 - c s_p}{s_p} \xrightarrow[p \to +\infty]{} \bar{c}_{\text{esc-sup}} - c > 0
\]

thus

\[
x_{\text{esc}}(t_p + s_p) - x_{\text{esc}}(t_p) - c s_p \to +\infty \text{ when } p \to +\infty.
\]

On the other hand, for every integer \(p\),

\[
\liminf_{s \to +\infty} \frac{x_{\text{esc}}(t_p + s) - x_{\text{esc}}(t_p) - c s}{s} = c_{\text{esc-inf}} - c < 0,
\]

thus

\[
\liminf_{s \to +\infty} x_{\text{esc}}(t_p + s) - x_{\text{esc}}(t_p) - c s = -\infty,
\]

and thus there exists \(\bar{s}_p\) larger than \(s_p\) such that \(x_{\text{esc}}(t_p + \bar{s}_p) - x_{\text{esc}}(t_p) - c \bar{s}_p \leq 0\). Lemma 10 is proved. \(\square\)
Intervals \([t_p, t_p + s_p]\) defined by the previous lemma are not convenient to apply the relaxation scheme because their length is arbitrary large (thus the “front” flux term involving the quantity \(K_{G, \text{from}}\) in the relaxation scheme final inequality (46) on page 40 cannot be controlled on such time intervals). The following lemma provides another sequence of intervals (derived from the sequences defined in Lemma 10) without this drawback (see figure 20).

Let \(\tau\) denote a (large) positive quantity, to be chosen below. This quantity will determine the length of the time intervals where the relaxation scheme will be applied (more precisely that length will be between \(\tau\) and \(2\tau\)). The value of \(\tau\) will be chosen sufficiently large so that a sufficient amount of dissipation occurs during the relaxation scheme.

Lemma 11 (escape point remains to the right and ends up to the left in travelling frame, controlled duration). There exist sequences \((t'_p)_{p \in \mathbb{N}}\) and \((s'_p)_{p \in \mathbb{N}}\) such that, for every integer \(p\) the following properties hold:

- \(0 \leq t'_p\) and \(\tau \leq s'_p \leq 2\tau\),
- for all \(s\) in \([0, \tau]\), the following inequality holds: \(x_{\text{esc}}(t'_p + s) - x_{\text{esc}}(t'_p) - cs \geq 0\),
- \(x_{\text{esc}}(t'_p + s'_p) - x_{\text{esc}}(t'_p) - cs'_p \leq 1\),

and such that

\[t'_p \rightarrow +\infty \quad \text{when} \quad p \rightarrow +\infty.\]

Proof of Lemma 11. For every integer \(p\) let us consider the set:

\[\{\Delta \in [0, +\infty) : \text{there exists } t \in [t_p, t_p + s_p] \text{ such that, for all } s \in [0, \tau], x_{\text{esc}}(t + s) - x_{\text{esc}}(t) - c(t + s - t_p) \geq \Delta\}\]

and let us denote by \(\Delta_p(\tau)\) the supremum of this set (with the convention that \(\Delta_p(\tau)\) equals \(-\infty\) if this set is empty).
First let us prove that $\Delta_p(\tau)$ approaches $+\infty$ when $p$ approaches $+\infty$. For this purpose, observe that, according to the control on the growth of $x_{\text{esc}}(\cdot)$ on page 28, for every integer $p$ and for all $s$ in $[0, \tau]$,
\[
x_{\text{esc}}(t_p + s_p - \tau + s) \geq x_{\text{esc}}(t_p + s_p) - c_{\text{no-esc}}(\tau - s),
\]
and as a consequence,
\[
x_{\text{esc}}(t_p + s_p - \tau + s) - x_{\text{esc}}(t_p) - c(s_p - \tau + s) \\
\geq x_{\text{esc}}(t_p + s_p) - x_{\text{esc}}(t_p) - cs_p - (c_{\text{no-esc}} - c)(\tau - s) \\
\geq x_{\text{esc}}(t_p + s_p) - x_{\text{esc}}(t_p) - cs_p - (c_{\text{no-esc}} - c)\tau,
\]
and according to Lemma 10 this last quantity approaches $+\infty$ when $p$ approaches $+\infty$. This shows that $\Delta_p(\tau)$ approaches $+\infty$ when $p$ approaches $+\infty$. Up to replacing the sequence $(t_p, s_p, \bar{s}_p)_{p \in \mathbb{N}}$ by a subsequence, let us assume that, for every integer $p$ the quantity $\Delta_p(\tau)$ is not smaller than 1. Then, for every integer $p$, according to the definition of $\Delta_p(\tau)$, the set
\[
\{ t \in [t_p, t_p + \bar{s}_p] : \text{for all } s \in [0, \tau], x_{\text{esc}}(t + s) - x_{\text{esc}}(t_p) - cs(t + s - t_p) \geq \Delta_p(\tau) - \frac{1}{2} \}
\]
is nonempty. Let $t'_p$ denote the infimum of this set. Then $t'_p$ is larger than $t_p$, and, according to the control on the growth of $x_{\text{esc}}(\cdot)$ on page 28,
\[
x_{\text{esc}}(t'_p) - x_{\text{esc}}(t_p) - c(t'_p - t_p) = \Delta_p(\tau) - \frac{1}{2},
\]
and, for all $s$ in $[0, \tau]$,
\[
x_{\text{esc}}(t'_p + s) - x_{\text{esc}}(t_p) - c(t'_p + s - t_p) \geq \Delta_p(\tau) - \frac{1}{2}.
\]
Since $t'_p$ is not larger than $t_p + \bar{s}_p$ and according to the last assertion of Lemma 10 and since $\Delta_p(\tau) - 1/2$ is positive, this shows that:
\[
t'_p + \tau \leq t_p + \bar{s}_p.
\]
As a consequence, according to the definition of $\Delta_p(\tau)$, there exists $s'_p$ in $[\tau, 2\tau]$ such that
\[
x_{\text{esc}}(t'_p + s'_p) - x_{\text{esc}}(t_p) - c(t'_p + s'_p - t_p) \leq \Delta_p(\tau) + \frac{1}{2}.
\]
Finally, it follows from equality (51) and inequality (52) that, for all all $s$ in $[0, \tau]$,
\[
x_{\text{esc}}(t'_p + s) - x_{\text{esc}}(t'_p) - cs \geq 0,
\]
and it follows from inequalities (52) and (53) that
\[
x_{\text{esc}}(t'_p + s'_p) - x_{\text{esc}}(t'_p) - cs'_p \leq 1.
\]
In addition, according to equality (51) and to the control on the growth of $x_{\text{esc}}(\cdot)$ on page 28, since $\Delta_p(\tau)$ approaches $+\infty$ when $p$ approaches $+\infty$,
\[
t'_p - t_p \to +\infty \text{ when } p \to +\infty.
\]
Lemma 11 follows from these three last assertions. \qed
Lemma 11 provides intervals \([t'_{p}, t'_{p} + s'_{p}]\) suitable to apply the relaxation scheme set up in subsection 4.7. For every integer \(p\) we are going to apply this relaxation scheme for the following parameters:

\[
\begin{align*}
t_{\text{init}} &= t'_{p} \\
x_{\text{init}} &= x_{\text{esc}}(t_{\text{init}}) \\
c &= \text{as chosen above, and} \\
y_{\text{cut-init}} &= 0
\end{align*}
\]

(the relaxation scheme thus depends on \(p\)). Observe that, according to hypothesis \((49)\) on page 42 for the speed \(c\), both hypotheses \((30)\) on page 30 and \((42)\) on page 38 (required to apply this relaxation scheme) hold. Let us denote by

\[
v^{(p)}(\cdot, \cdot) \quad \text{and} \quad E^{(p)}(\cdot) \quad \text{and} \quad F^{(p)}(\cdot) \quad \text{and} \quad y_{\text{esc}}^{(p)}(\cdot) \quad \text{and} \quad y_{\text{hom}}^{(p)}(\cdot)
\]

the objects defined in subsection 4.7 (with the same notation except the “\((p)\)” superscript that is here to remind that all these objects depend on the integer \(p\)). By definition the quantity \(y_{\text{esc}}^{(p)}(0)\) equals 0, and according to the conclusions of Lemma 11, \(y_{\text{esc}}^{(p)}(s) \geq 0\) for all \(s\) in \([0, \tau]\) and \(y_{\text{esc}}^{(p)}(s'_{p}) \leq 1\).

The two following lemmas will be shown to be in contradiction with the relaxation scheme final inequality \((46)\) on page 40 (see inequality \((55)\) below), and this will complete the proof of Proposition 3.

**Lemma 12** (bounds on energy and firewall at the ends of relaxation scheme). The quantities \(E^{(p)}(0)\) and \(F^{(p)}(0)\) are bounded from above and the quantity \(E^{(p)}(s'_{p})\) is bounded from below, and these bounds are uniform with respect to \(\tau\) and \(p\).

**Lemma 13** (large dissipation integral). The quantity

\[
\int_{0}^{s'_{p}} D^{(p)}(s) \, ds
\]

approaches \(+\infty\) when \(\tau\) approaches \(+\infty\), uniformly with respect to \(p\).

**Proof of Lemma 12**. The fact that \(E^{(p)}(0)\) and \(F^{(p)}(0)\) are bounded from above follows from the fact that \(y_{\text{cut-init}}\) equals 0 and from the a priori bound \((17)\) on page 23 for the solution. The fact that \(E^{(p)}(s'_{p})\) is bounded from below follows from the fact that \(y_{\text{esc}}^{(p)}(s'_{p}) \leq 1\) and from the fact that, according to hypothesis \((H_{\text{coerc}})\), \(V\) is bounded from below.

**Proof of Lemma 13**. By definition of \(D^{(p)}(\cdot)\), for all \(s\) in \([0, s'_{p}]\),

\[
D^{(p)}(s) \geq \int_{-\infty}^{c_{\text{cut}}} e^{sy} v^{(p)}(y, s)^2 \, dy.
\]

thus, for all \(s\) in \([0, \tau]\), since \(y_{\text{esc}}^{(p)}(s) \geq 0\) (performing the change of variables \(y = y_{\text{esc}}^{(p)}(s) + z\)),

\[
D^{(p)}(s) \geq \int_{-\infty}^{c_{\text{cut}} - y_{\text{esc}}^{(p)}(s)} e^{cz} v^{(p)}(y_{\text{esc}}^{(p)}(s) + z, s)^2 \, dz.
\]

47
For all $s$ in $[0, +\infty)$ and $z$ in $\mathbb{R}$,
\[
v_s^{(p)}(y_{\text{esc}}^{(p)}(s) + z, s) = u_t x_{\text{esc}}(t_p' + s) + w, t_p' + s) + c u_x x_{\text{esc}}(t_p' + s) + w, t_p' + s),
\]
thus, according to Lemma 9 on page 43, there exist positive quantities $L$ and $\varepsilon_{\text{diss}}$ such that, uniformly with respect to $p$ in $\mathbb{N}$ and $s$ in $[0, +\infty)$,
\[
\int_{-L}^{L} v_s^{(p)}(y_{\text{esc}}^{(p)}(s) + z, s)^2 dz \geq \varepsilon_{\text{diss}}.
\]
Observe that the condition (49) on page 42 satisfied by $c$ implies that $\bar{c}_{\text{esc-sup}} - c$ is not larger than $c_{\text{cut}}/4$. Thus, since
\[
y_{\text{esc}}^{(p)}(s) \leq \bar{x}_{\text{esc}}(s) - cs \quad \text{for all } s \in [0, +\infty),
\]
there exists a positive quantity $s_0$, depending only on $L$ and on the function $\bar{x}_{\text{esc}}(\cdot)$ such that, for every $s$ in $[s_0, +\infty)$,
\[
c_{\text{cut}}s - y_{\text{esc}}^{(p)}(s) \geq L.
\]
It follows from inequality (54) that, for all $s$ in $[s_0, \tau]$,
\[
\mathcal{D}^{(p)}(s) \geq e^{-cL\varepsilon_{\text{diss}}},
\]
and finally,
\[
\int_{0}^{s_p'} \mathcal{D}^{(p)}(s) ds \geq (\tau - s_0)e^{-cL\varepsilon_{\text{diss}}},
\]
and this finishes the proof of Lemma 13. \hfill \Box

For every integer $p$ the relaxation scheme final inequality (46) on page 40 yields (since $y_{\text{cut-init}}$ equals 0):
\[
\frac{1}{2} \int_{0}^{s_p'} \mathcal{D}^{(p)}(s) ds \leq c^{(p)}(0) - c^{(p)}(s_p') + \frac{K \varepsilon_{\text{diss}}}{\nu_F} \tau^{(p)}(0) + K G_{\text{back}}[u_0]
\]
\[+ K G_{\text{front}}(2\tau \exp(-\kappa y_{\text{hom}}^{(p)}(0))).
\]
Since $t_p'$ approaches $+\infty$ when $p$ approaches $+\infty$, the quantity $y_{\text{hom}}^{(p)}(0)$ also approaches $+\infty$ when $p$ approaches $+\infty$. Thus, according to Lemma 12, the right-hand side of inequality (55) is bounded, uniformly with respect to $\tau$, provided that $p$ (depending on $\tau$) is sufficiently large. This is contradictory to Lemma 12 and completes the proof of Proposition 3 on page 42. \hfill \Box

According to Proposition 3, the three quantities $c_{\text{esc-inf}}$ and $c_{\text{esc-sup}}$ and $\bar{c}_{\text{esc-sup}}$ are equal; let
\[
c_{\text{esc}}
\]
denote their common value.
4.10 Further control on the escape point

At this stage two cases can be distinguished.

1. Either $c_{\text{esc}} < c_{\text{hom}}$, and in this case the end of the proof of Proposition 2 “invasion implies convergence” is very similar to the end of the proof of the main result of [22]. The sole difference is the presence of the additional “front” flux term coming from $G_{\text{front}}(\cdot)$. But when $c_{\text{esc}} < c_{\text{hom}}$ this term can be easily handled if the parameter $c$ involved in the relaxation scheme and the quantity $c_{\text{cut}}$ are chosen in such a way that

$$c + c_{\text{cut}} < c_{\text{hom}}$$

(this is possible since $c_{\text{esc}} < c_{\text{hom}}$). Then the flux terms due to $G_{\text{front}}(\cdot)$ decrease at an exponential rate, and can be made arbitrarily small provided that $t_{\text{init}}$ is large enough. We skip the details since this approach is anyway not sufficient in the case where $c_{\text{esc}} = c_{\text{hom}}$.

2. The other case $c_{\text{esc}} = c_{\text{hom}}$ is slightly more delicate, and taking this case into account will require the more precise control on the invasion point provided by Proposition 4 below.

The remaining of the proof of Proposition 2 covers both cases above, but is specifically designed to take into account the (more difficult) second case (again, in the first case $c_{\text{esc}} < c_{\text{hom}}$, adapting the proof of [22] as sketched above would lead to a simpler proof).

The aim of this subsection is to prove the following proposition that enforces the control on the behaviour of the “escape” point $x_{\text{esc}}(\cdot)$ (this will be used for the additional relaxation arguments carried on in the next subsections).

**Proposition 4** (mean invasion speed, further control). The following equality holds:

$$\xi_{\text{esc-inf}} = c_{\text{esc}}.$$  

**Proof.** The proof is rather similar to that of Proposition 3. We know that $\xi_{\text{esc-inf}}$ is not larger than $c_{\text{esc}}$. Let us proceed by contradiction and assume that

$$\xi_{\text{esc-inf}} < c_{\text{esc}}.$$  

Then, let us take and fix a positive quantity $c$ satisfying the following conditions:

$$\xi_{\text{esc-inf}} < c < c_{\text{esc}} < c + c_{\text{cut}} \quad \text{and} \quad c \geq c_{\text{esc}} - \frac{\kappa c_{\text{cut}}}{4(\xi_{\text{no-esc}} + \kappa)} \quad \text{and} \quad \Phi_c(0_{\mathbb{R}^n}) = \emptyset.$$  

The two first conditions are satisfied as soon as $c$ is smaller than and sufficiently close to $c_{\text{esc}}$, thus existence of a quantity $c$ satisfying the three conditions follows from hypothesis (H$_{\text{disc-c}}$). The following lemma is identical to Lemma 10 on page 44 (but the proof will be slightly different, see figure 21).

**Lemma 14** (large excursions to the right and returns for escape point in travelling frame). There exist sequences $(t_p)_{p \in \mathbb{N}}$ and $(s_p)_{p \in \mathbb{N}}$ and $(\bar{s}_p)_{p \in \mathbb{N}}$ of real quantities such that the following properties hold.
• For every integer $p$ the following inequalities hold: $0 \leq t_p$ and $0 \leq s_p \leq \bar{s}_p$.

• $x_{esc}(t_p + s_p) - x_{esc}(t_p) - c s_p \to +\infty$ when $p \to +\infty$.

• For every integer $p$ the following inequality holds: $x_{esc}(t_p + \bar{s}_p) - x_{esc}(t_p) - c \bar{s}_p \leq 0$.

**Proof of Lemma 14.** According to the definition of $c_{esc-inf}$, there exists a sequence $(s'_p)_{p \in \mathbb{N}}$ of positive real quantities satisfying

$$s'_p \to +\infty \quad \text{and} \quad \frac{\underline{x}_{esc}(s'_p)}{s'_p} \to c_{esc-inf} \quad \text{when} \quad p \to +\infty.$$

We may assume in addition (up to replacing the sequence $(s'_p)_{p \in \mathbb{N}}$ by a subsequence) that

$$\underline{x}_{esc}(s'_p) > -\infty \quad \text{for all} \quad p \in \mathbb{N}.$$

Then, by definition of $\underline{x}_{esc}(\cdot)$, for every integer $p$ there exists a nonnegative real quantity $t'_p$ such that

$$x_{esc}(t'_p + s'_p) \leq x_{esc}(t'_p) + \underline{x}_{esc}(s'_p) - 1.$$

Observe that $t'_p$ must approach $+\infty$ when $p$ approaches $+\infty$, or else this last inequality would yield

$$\liminf_{p \to +\infty} \frac{x_{esc}(t'_p + s'_p)}{t'_p + s'_p} \leq c_{esc-inf} < c_{esc}$$

and this would be contradictory to the definition of $c_{esc-inf}$ and Proposition 3. For every integer $p$ and all $t$ in $[0, +\infty)$, let

$$y_p(t) = x_{esc}(t) - \left( x_{esc}(t'_p + s'_p) + c(t - (t'_p + s'_p)) \right).$$

Then $y_p(t'_p + s'_p) = 0$ and

$$\frac{y_p(t'_p)}{s'_p} = \frac{x_{esc}(t'_p) - x_{esc}(t'_p + s'_p)}{s'_p} + c \geq \frac{1 - \underline{x}_{esc}(s'_p)}{s'_p} + c.$$
thus
\[
\liminf_{p \to +\infty} \frac{y_p(t'_p)}{s'_p} \geq c - \zeta_{\text{esc-inf}} > 0,
\]
and finally
\[
y_p(t'_p) \to +\infty \quad \text{when} \quad p \to +\infty.
\]
On the other hand,
\[
y_p(0) = \frac{x_{\text{esc}}(0) - y_p(t'_p) - s'_p}{t'_p + s'_p} + c
\]
and this quantity approaches the negative quantity \(c - c_{\text{esc}}\) when \(p\) approaches \(+\infty\). As a consequence,
\[
y_p(0) \to -\infty \quad \text{when} \quad p \to +\infty.
\]
Thus, up to replacing the sequence \((t'_p, s'_p)_{p \in \mathbb{N}}\) by a subsequence, we may assume that \(y_p(0) < 0\) for every integer \(p\). Then, for every integer \(p\) let
\[
t_p = \sup \{ t \in [0, t'_p] : y_p(t) \leq 0 \} \quad \text{and} \quad s_p = t'_p - t_p \quad \text{and} \quad \bar{s}_p = t'_p + s'_p - t_p.
\]
Then, according to the control on the growth of \(x_{\text{esc}}(\cdot)\) on page 28, the quantity \(y(t_p)\) must be equal to 0, and all the conclusions of Lemma 14 are satisfied. \(\square\)

Since the conclusions of Lemma 14 are identical to those of Lemma 10 on page 44, the end of the proof of Proposition 4 (“mean invasion speed, further control”) can be achieved exactly as for Proposition 3 (“mean invasion speed”). \(\square\)

4.11 Dissipation approaches zero at regularly spaced times

The key argument behind Propositions 3 and 4 on page 42 and on page 49 (“mean invasion speed” and “… further control”) is that a dissipation uniformly bounded from below around the escape point cannot occur during large time intervals, since it is forbidden by the relaxation scheme set up in subsection 4.7 (and in particular by the relaxation scheme final inequality (46) on page 40). The aim of this subsection is to state another result (Proposition 5 below) that just formalizes this argument, this time considering the dissipation in a frame travelling precisely at the “sole relevant” speed \(c_{\text{esc}}\) given by Propositions 3 and 4.

For all \(t\) in \([0, +\infty)\), the following set:
\[
\left\{ \varepsilon \in (0, +\infty) : \int_{-1/\varepsilon}^{1/\varepsilon} \left( u_p(x_{\text{esc}}(t) + y, t) + c_{\text{esc}} u x(x_{\text{esc}}(t) + y, t) \right)^2 \, dy \leq \varepsilon \right\}
\]
is (according to the a priori bounds (17) on page 23 for the solution) a nonempty interval (which by the way is unbounded from above). Let
\[
\delta_{\text{dissip}}(t)
\]
denote the infimum of this interval. This quantity measures to what extent the solution is, at time $t$ and around the escape point $x_{\text{esc}}(t)$, close to be stationary in a frame travelling at speed $c_{\text{esc}}$. The aim of the next subsection will be to prove that

$$\delta_{\text{dissip}}(t) \to 0 \quad \text{when} \quad t \to +\infty.$$ 

Proposition 5 below can be viewed as a first step towards this goal.

**Proposition 5** (regular occurrence of small dissipation). For every positive quantity $\varepsilon$, there exists a positive quantity $T(\varepsilon)$ such that, for every $t$ in $[0, +\infty)$,

$$\inf_{t' \in [t, t + T(\varepsilon)]} \delta_{\text{dissip}}(t') \leq \varepsilon.$$ 

**Proof.** Let us proceed by contradiction and assume that the converse holds. Then there exist a positive quantity $\varepsilon_0$ and a sequence $(t'_p)_{p \in \mathbb{N}}$ of nonnegative quantities such that, for every $t$ in $[t'_p, t'_p + p]$,

$$\int_{-1/\varepsilon_0}^{1/\varepsilon_0} \left( u_t(x_{\text{esc}}(t) + y, t) + c_{\text{esc}}u_x(x_{\text{esc}}(t) + y, t) \right)^2 \, dy \geq \varepsilon_0.$$ 

(56)

Up to replacing $t'_p$ by $t'_p + p$, we may assume that $t'_p$ approaches $+\infty$ when $p$ approaches $+\infty$.

As for the proof of Propositions 3 and 4, we would like to apply the relaxation scheme in travelling frame where the escape point $x_{\text{esc}}(\cdot)$:

- remains around the origin or to the right during a significant time interval (to recover enough dissipation);
- finishes around the origin or to the left (so that the final energy be bounded from below).

In order these two conditions to hold simultaneously, we are not going to apply the relaxation scheme on the whole intervals $[t'_p, t'_p + p]$, but rather on smaller convenient intervals that we will introduce. Because of the second of these conditions, we are going to choose a speed slightly larger than $c_{\text{esc}}$ for the travelling frame. Let $\tau$ denote a positive quantity to be chosen later (this quantity will play the same kind of role as in the proof of Proposition 3). We are going to consider a frame travelling at a speed slightly larger than $c_{\text{esc}}$, namely:

$$c_{\text{esc}} + \frac{1}{\tau}.$$ 

According to Proposition 3, there exists a positive quantity $\bar{\tau}$ larger than $\tau$ (thus depending on $\tau$) such that:

$$\frac{x_{\text{esc}}(\bar{\tau})}{\bar{\tau}} \leq c_{\text{esc}} + \frac{1}{\tau}.$$
(this will ensure that the second of the conditions above is satisfied on every interval of length \( \tau \)). Besides, according to hypothesis (56) and to the a priori bounds on the solution, we may assume that \( \tau \) is large enough so that, for every \( t \) in \([t'_p, t'_p + p]\),

\[
\int_{-1/\varepsilon_0}^{1/\varepsilon_0} \left( u_t \left( x_{\text{esc}}(t) + y, t \right) + \left( c_{\text{esc}} + \frac{1}{\tau} \right) u_x \left( x_{\text{esc}}(t) + y, t \right) \right)^2 \, dy \geq \frac{\varepsilon_0}{2}.
\]

The following lemma provides the initial times of the time intervals where the relaxation scheme will be applied, ensuring that the first of the two conditions above is fulfilled despite the fact that the travelling speed is slightly above \( c_{\text{esc}} \).

**Lemma 15** (escape point remains to the right in travelling frame). For every sufficiently large integer \( p \), there exists a time \( t_p \) in the interval \([t'_p, t'_p + p - \tau]\) such that, for every \( s \) in \([0, \tau]\),

\[
x_{\text{esc}}(t_p + s) - x_{\text{esc}}(t_p) - c_{\text{esc}} s \geq -1.
\]

**Proof of Lemma 15.** Let us proceed by contradiction and assume that the converse holds. Then, there exists an arbitrarily large integer \( p \) such that, for every \( t \) in the interval \([t'_p, t'_p + p - \tau]\), there exists \( s \) in \((0, \tau]\) such that

\[
x_{\text{esc}}(t + s) - x_{\text{esc}}(t) - c_{\text{esc}} s < -1,
\]

ensuring that the mean speed of \( x_{\text{esc}}(\cdot) \) on the interval \([t, t + s]\) is smaller than \( c_{\text{esc}} - 1/\tau \). This shows that there exists \( t' \) in the interval \([t'_p + p - \tau, t'_p + p]\) such that the interval \([t'_p, t']\) can be cut into a finite number of subintervals (defined one after another, starting from \( t'_p \)) so that, on each of these subintervals, the mean speed of \( x_{\text{esc}}(\cdot) \) is smaller than \( c_{\text{esc}} - 1/\tau \). As a consequence, the mean speed of \( x_{\text{esc}}(\cdot) \) on the whole interval \([t'_p, t']\) is smaller than \( c_{\text{esc}} - 1/\tau \).

But on the other hand, according to Proposition 4, for every sufficiently large \( s \), the following inequality holds:

\[
x_{\text{esc}}(s) \geq \left( c_{\text{esc}} - \frac{1}{\tau} \right) s,
\]

in other words the mean speed of \( x_{\text{esc}}(\cdot) \) cannot be smaller than \( c_{\text{esc}} - 1/\tau \) on a sufficiently large time interval, a contradiction with the previous assertion when \( p \) is sufficiently large. Lemma 15 is proved.

Thus, according to this lemma, for every integer \( p \) sufficiently large so that \( t_p \) is defined, the following assertion hold. For every \( s \) in \([0, \tau]\),

\[
x_{\text{esc}}(t_p + s) - x_{\text{esc}}(t_p) - \left( c_{\text{esc}} + \frac{1}{\tau} \right) s \geq -2,
\]

and

\[
x_{\text{esc}}(t_p + \tau) - x_{\text{esc}}(t_p) - \left( c_{\text{esc}} + \frac{1}{\tau} \right) s \leq 0.
\]

These two last assertions are of the same nature as those of Lemma 10 and as a consequence the end of the proof of Proposition 5 can be carried out along the same lines as the proof of Proposition 3 (or Proposition 4).

\[53\]
4.12 Relaxation

The aim of this subsection is to prove the following proposition.

**Proposition 6** (relaxation). *The following assertion holds:*

\[ \delta_{\text{dissip}}(t) \to 0 \text{ when } t \to +\infty. \]

*Proof.* Let us proceed by contradiction and assume that the converse assertion holds. Then there exists a positive quantity \( \varepsilon_0 \) and a sequence \((t_p)_{p \in \mathbb{N}}\) of (positive) times such that \( t_p \) approaches \(+\infty\) when \( p \) approaches \(+\infty\) and such that, for every integer \( p \),

\[ \delta_{\text{dissip}}(t_p) \geq \varepsilon_0. \]

In other words, there is a “bump” of dissipation at each time \( t_p \). On the other hand, according to Proposition 5 on page 52, on every sufficiently large time interval, there exist times where the dissipation around the escape point is low. Roughly speaking, our strategy will be to apply the relaxation scheme set up in subsection 4.7 on a time interval containing a dissipation bump (at a certain time \( t_p \)) and bounded by two times where the dissipation is low. At both ends of the intervals, it will follow from the smallness of the dissipation that the solution is close to a front travelling at speed \( c_{\text{esc}} \), and therefore that its energy \( E_{c_{\text{esc}}} \) (properly localized) is close to 0. Provided that the energy fluxes can be sufficiently controlled along the relaxation scheme on this time interval, this will be in contradiction with the dissipation bump occurring at time \( t_p \).

In order to reach this contradiction, a number of conditions need to be fulfilled. Here are three of them.

- The relaxation scheme will actually be applied twice, on each side of the dissipation bump occurring at time \( t_p \). Indeed, as illustrated on figure 23, applying the relaxation scheme only once on the whole interval may lead to a dissipation bump occurring “far to the left” in the appropriate travelling frame, ending up with a negligible influence on the (localized) energy \( E_{c_{\text{esc}}} \) of the solution.

![Figure 22: Time of “dissipation bump” \( t_p \), framed by two times where dissipation is almost zero.](image)

- The lengths of the two time intervals where the relaxation scheme will be applied need to be large, in order the mean speed of the escape point on each of these interval to be close to its asymptotic value \( c_{\text{esc}} \) (indeed, it is for this speed \( c_{\text{esc}} \) that the dissipation will be close to 0 at the ends and that a dissipation bump occurs at time \( t_p \)).
Finally, depending on the length of the time intervals where the relaxation schemes is applied, the integer \( p \) will have to be chosen sufficiently large, in order the “front” flux term (the last term in the right hand side of inequality (46) on page 40) to be controlled.

The third of these conditions leads us to introduce a second integer parameter \( q \) that will be related altogether to the length of these two time intervals and to the smallness of the dissipation at both ends. With such a parameter at hand, we are now in position to introduce more precise notation.

We will use again the notation \( T(\cdot) \) introduced in Proposition 5. Up to replacing the sequence \( (t_p)_{p \in \mathbb{N}} \) by a subsequence, we may assume that this sequence is increasing. Then, since \( t_p \) approaches \(+\infty\) when \( p \) approaches \(+\infty\), for every nonzero integer \( q \) there exists an integer \( p_{\text{min}}(q) \) such that, for every integer \( p \geq p_{\text{min}}(q) \),
\[
\begin{align*}
p &\geq p_{\text{min}}(q) \iff t_p \geq q + T(1/q) .
\end{align*}
\]

Note that \( p_{\text{min}}(q) \) approaches \(+\infty\) when \( q \) approaches \(+\infty\). According to Proposition 5, for every nonzero integer \( q \) and every integer \( p \) not smaller than \( p_{\text{min}}(q) \), there exist (see figure 22):

\[
\begin{align*}
t_{\text{bef}}_{p,q} &\text{ in } [t_p - q - T(1/q), t_p - q] \text{ and } t_{\text{aft}}_{p,q} \text{ in } [t_p + q, t_p + q + T(1/q)]
\end{align*}
\]

such that
\[
\delta_{\text{dissip}}(t_{\text{bef}}_{p,q}) \leq 1/q \quad \text{and} \quad \delta_{\text{dissip}}(t_{\text{aft}}_{p,q}) \leq 1/q .
\]

The mentions “bef” and “aft” are reminders of the fact that these times occur “before” or “after” the “dissipation bump time \( t_p \)”. In those definitions, the significant features are that both quantities
\[
\delta_{\text{dissip}}(t_{\text{bef}}_{p,q}) \quad \text{and} \quad \delta_{\text{dissip}}(t_{\text{aft}}_{p,q})
\]
approach 0 when \( q \) approaches \(+\infty\), and both quantities
\[
t_p - t_{\text{bef}}_{p,q} \quad \text{and} \quad t_{\text{aft}}_{p,q} - t_p
\]
approach \(+\infty\) when \( q \) approaches \(+\infty\), while remaining bounded with respect to \( p \) for every fixed nonzero integer \( q \). According to inequality (28) (controlling the growth of the escape point), both quantities
\[
\begin{align*}
\frac{x_{\text{esc}}(t_p) - x_{\text{esc}}(t_{\text{bef}}_{p,q})}{t_p - t_{\text{bef}}_{p,q}} \quad \text{and} \quad \frac{x_{\text{esc}}(t_{\text{aft}}_{p,q}) - x_{\text{esc}}(t_p)}{t_{\text{aft}}_{p,q} - t_p}
\end{align*}
\]
(that is the mean speeds of the escape point on the two intervals surrounding the dissipation bump time \( t_p \)) are bounded from above by \( c_{\text{no-esc}} \), and according to Propositions 3 and 4, both quantities approach \( c_{\text{esc}} \) when \( q \) approaches \(+\infty\), uniformly with respect to \( p \) not smaller than \( p_{\text{min}}(q) \).
By compactness (subsection 4.8), up to replacing the sequence \((t_p)_{p \in \mathbb{N}}\) by a subsequence, there exists a function \(u_\infty\) in \(C^k_b([0, \infty), \mathbb{R}^n)\) such that, of every positive quantity \(L\),
\[
\| y \mapsto u(x_{\text{esc}}(t_p) + y, t_p) - u_\infty(y) \|_{C^k([-L, L], \mathbb{R}^n)} \to 0 \quad \text{when} \quad p \to +\infty .
\]
In the two next sub-subsections, we are going to apply the relaxation scheme set up in subsection 4.7 to the two intervals \([t_{\text{bef}}, t_p]\) and \([t_p, t_{\text{aft}}]\). We will make an extensive use of the following notation.

**Notation.** For every nonnegative time \(t\) and every real quantity \(y\), let
\[
E(y, t) = \frac{u_x(x_{\text{esc}}(t) + y, t)^2}{2} + V(u(x_{\text{esc}}(t) + y, t)) ,
\]
\[
F(y, t) = \frac{u_x(x_{\text{esc}}(t) + y, t)^2}{2} + V(u(x_{\text{esc}}(t) + y, t)) + \frac{u(x_{\text{esc}}(t) + y, t)^2}{2} ,
\]
and, for every function \(\phi\) in \(C^k([0, \infty), \mathbb{R}^n)\), let
\[
E[\phi](y) = \frac{\phi'(y)^2}{2} + V(\phi(y)) ,
\]
\[
F[\phi](y) = \frac{\phi'(y)^2}{2} + V(\phi(y)) + \frac{\phi(y)^2}{2} .
\]

### 4.12.1 Relaxation scheme to the left of the dissipation bump

The aim of this sub-subsection is to prove the following lemma, stating that the energy at the right-end of the “left-hand” interval \([t_{\text{bef}}, t_p]\) is negative. To reach this goal, we will apply the relaxation scheme set up in subsection 4.7 to this interval.
Lemma 16 (negative energy at right-end of left-hand interval). The following inequality holds (and the integral on the left hand side converges):

\[
\int_{-\infty}^{+\infty} \exp(c_{\text{esc}}y) E_{[u, \infty]}(y) \, dy < 0.
\]

Proof of Lemma 16. Let us still consider two integers \( p \) and \( q \) with \( q \) nonzero and \( p \) not smaller than \( p_{\text{min}}(q) \), and let

\[
s_{p,q} = t_p - t_{p,q}^{\text{bef}} \quad \text{and} \quad c_{p,q} = \frac{x_{\text{esc}}(t_p) - x_{\text{esc}}(t_{p,q}^{\text{bef}})}{s_{p,q}}.
\]

Let us assume that \( q \) is large enough so that

\[
0 < c_{p,q} \quad \text{and} \quad c_{p,q} \geq c_{\text{esc}} - \frac{\kappa c_{\text{cut}}}{4(c_{\text{no-esc}} + \kappa)},
\]

and let \( \ell \) denote a nonnegative quantity to be chosen below. We are going to apply the relaxation scheme set up in subsection 4.7 for the following parameters:

\[
t_{\text{init}} = t_{p,q}^{\text{bef}} \quad \text{and} \quad x_{\text{init}} = x_{\text{esc}}(t_{\text{init}}) \quad \text{and} \quad c = c_{p,q} \quad \text{and} \quad y_{\text{cut-init}} = \ell.
\]

Thus the relaxation scheme will depend on the three parameters \( (p, q, \ell) \). Observe that both hypotheses (30) on page 30 and (42) on page 38 (required to apply the relaxation scheme) hold. Let us denote by

\[
\psi^{(p,q)}(\cdot, \cdot) \quad \text{and} \quad \chi^{(p,q,\ell)}(\cdot, \cdot) \quad \text{and} \quad \mathcal{E}^{(p,q,\ell)}(\cdot) \quad \text{and} \quad \mathcal{D}^{(p,q,\ell)}(\cdot)
\]

and

\[
\psi^{(p,q,\ell)}(\cdot, \cdot) \quad \text{and} \quad \mathcal{F}^{(p,q,\ell)}(\cdot)
\]

the objects defined in subsection 4.7 (with the same notation except the “\((p, q)\)” or “\((p, q, \ell)\)” superscripts to emphasize the dependency with respect to the parameters).

The proof is based on the relaxation scheme final inequality (46) on page 40 on the \( s \)-time interval \([0, s_{p,q}]\), which will provide an upper bound on the quantity \( \mathcal{E}^{(p,q,\ell)}(s_{p,q}) \) (the localized energy at the right end of this time interval). This will require a careful choice of the three parameters \( p, q, \) and \( \ell \) to control the various other quantities in this inequality and the difference between \( \mathcal{E}^{(p,q,\ell)}(s_{p,q}) \) and the integral (58). Here are the quantities we have to control:

1. the dissipation term \( \int_0^{s_{p,q}} \mathcal{D}^{(p,q,\ell)}(s) \, ds \);
2. the initial value \( \mathcal{E}^{(p,q,\ell)}(0) \) of the localized energy;
3. the initial value \( \mathcal{F}^{(p,q,\ell)}(0) \) of the firewall function;
4. the “back flux” term (involving the factor \( K_{\text{G.back}}[u_0] \));
5. the “front flux” term (involving the factor \( K_{\text{G.front}} \));
6. the difference between the final energy \( \mathcal{E}^{(p,q,\ell)}(s_{p,q}) \) and the integral (58).
Those controls will be stated by a series of lemmas (Lemmas 17 to 22). Figure 24 summarizes the requirements on the three parameters $p$, $q$, and $\ell$, and the dependencies between those parameters as well, in order all these controls to hold. As illustrated by this figure, the final choice of those parameters, satisfying all these requirements, will be, in short:

- a quantity $\ell$ large enough,
- an integer $q$ large enough (depending on $\ell$),
- an integer $p$ large enough (depending on $\ell$ and $q$).

The proof of Lemma 16 will be presented as a sequence of lemmas. Here is the first one.

**Lemma 17** (lower bound on dissipation). There exists a positive quantity $\varepsilon_{\text{dissip}}$ and a nonzero integer $q_{\text{min, dissip}}$ such that, for every integer $q$ not smaller than $q_{\text{min, dissip}}$, for every integer $p$ not smaller than $p_{\text{min}}(q)$, and for every nonnegative quantity $\ell$, the following inequality holds:

$$\frac{1}{2} \int_{0}^{s_{p,q}} D^{(p,q,\ell)}(s) \, ds \geq \varepsilon_{\text{dissip}}.$$

**Proof of Lemma 17**. Indeed, if the converse was true, there would exist a sequence $(q_j)_{j \in \mathbb{N}}$ of integers, approaching $+\infty$ when $j$ approaches $+\infty$, and a sequence $(p_j)_{j \in \mathbb{N}}$ of integers with $p_j$ not smaller than $p_{\text{min}}(q_j)$ for every integer $j$ (thus $p_j$ also approaches $+\infty$ when $j$ approaches $+\infty$), and a sequence $(\ell_j)_{j \in \mathbb{N}}$ of nonnegative quantities, such that

$$\int_{0}^{s_{p,q}} D^{(p_j,q_j,\ell_j)}(s) \, ds \rightarrow 0 \quad \text{when} \quad j \rightarrow +\infty.$$

According to inequality (47) on page 41 controlling the growth of the dissipation, this would yield

$$D^{(p_j,q_j,\ell_j)}(s_{p_j,q_j}) \rightarrow 0 \quad \text{when} \quad j \rightarrow +\infty.$$
Then, since
\[ D(p_j, q_j)(s_{p_j, q_j}) \geq \int_{-\infty}^{\ell_j + c_{\text{cut}} s_{p_j, q_j}} \exp(c_{p_j, q_j} y) v_s^{(p_j, q_j)}(y, s_{p_j, q_j})^2 \, dy \]

\[ \geq \int_{-\infty}^{c_{\text{cut}} s_{p_j, q_j}} \exp(c_{p_j, q_j} y) v_s^{(p_j, q_j)}(y, s_{p_j, q_j})^2 \, dy; \]

and since \( s_{p_j, q_j} \) approaches \( +\infty \) when \( j \) approaches \( +\infty \), the functions
\[ y \mapsto v_s^{(p_j, q_j)}(y, s_{p_j, q_j}) \]

would necessarily converge to \((0, \ldots, 0)\) in \( L^2([-L, L]) \), for every positive quantity \( L \), when \( j \) approaches \( +\infty \). Since
\[ v_s^{(p_j, q_j)}(y, s_{p_j, q_j}) = u_t(x_{\text{esc}}(t_{p_j}) + y, t_{p_j}) + c_{p_j, q_j} u_x(x_{\text{esc}}(t_{p_j}) + y, t_{p_j}), \]

and since \( c_{p_j, q_j} \) approaches \( c_{\text{esc}} \) when \( j \) approaches \( +\infty \), this would be contradictory to the fact that \( \delta_{\text{dissip}}(t_{p_j}) \) must be larger than \( \varepsilon_0 \). This finishes the proof of Lemma 17. \( \square \)

The conclusions of Lemma 16 will follow from the next five (independent) lemmas.

**Lemma 18** (upper bound on initial energy). For every nonnegative quantity \( \ell \), there exists a nonzero integer \( q_{\min, \text{init-en}}(\ell) \) such that, for every integer \( q \) not smaller than \( q_{\min, \text{init-en}}(\ell) \) and every integer \( p \) not smaller than \( p_{\min}(q) \),
\[ E^{(p, q, \ell)}(0) \leq \frac{\varepsilon_{\text{dissip}}}{8}. \]

**Lemma 19** (lower bound on final energy). For every nonnegative quantity \( L \), there exists a nonzero integer \( q_{\min, \text{fin-en}}(L) \) such that, for every integer \( q \) not smaller than \( q_{\min, \text{fin-en}}(L) \), and for every nonnegative quantity \( \ell \), there exists an integer \( p_{\min, \text{fin-en}}(L, \ell, q) \) such that, for every integer \( p \) not smaller than \( p_{\min, \text{fin-en}}(L, \ell, q) \) and not smaller than \( p_{\min}(q) \),
\[ E^{(p, q, \ell)}(s_{p, q}) \geq \int_{-\infty}^{L} \exp(c_{\text{esc}} y) E_{[u_{\infty}]}(y) \, dy - \frac{\varepsilon_{\text{dissip}}}{8}. \]

**Lemma 20** (upper bound on initial firewall). There exists a positive quantity \( \ell_{\min, \text{init-fire}} \) such that, for every quantity \( \ell \) not smaller than \( \ell_{\min, \text{init-fire}} \), there exists a nonzero integer \( q_{\min, \text{init-fire}}(\ell) \) such that, for every integer \( q \) not smaller than \( q_{\min, \text{init-fire}}(\ell) \), and for every integer \( p \) not smaller than \( p_{\min}(q) \), the following inequality holds (the constants \( K_E \) and \( \nu_{\mathcal{F}} \) being those of inequality (46) on page 40):
\[ \frac{K_E}{\nu_{\mathcal{F}}} \mathcal{F}^{(p, q, \ell)}(0) \leq \frac{\varepsilon_{\text{dissip}}}{8}. \]

**Lemma 21** (upper bound on back flux for the firewall). There exists a nonnegative quantity \( \ell_{\min, \text{back-fire}} \) such that, for every nonzero integer \( q \) and every integer \( p \) not smaller than \( p_{\min}(q) \), the “back-flux” term in inequality (46) on page 40 is not larger than \( \varepsilon_{\text{dissip}}/8. \)
Lemma 22 (upper bound on front flux for the firewall). For every nonnegative quantity \( \ell \) and for every nonzero integer \( q \), there exists an integer \( p_{\min,\text{front-fire}}(\ell, q) \) such that, for every integer \( p \) not smaller than \( p_{\min,\text{front-fire}}(\ell, q) \) and not smaller than \( p_{\min}(q) \), the “front flux” term in inequality (46) on page 40 is not larger than \( \varepsilon_{\text{dissip}}/8 \).

Postponing the proofs of these five lemmas, let us first conclude with the proof of Lemma 18. For every nonnegative quantity \( L \), let:

\[
\ell = \max(\ell_{\min,\text{init-fire}}, \ell_{\min,\text{back-fire}}), \\
q = \max(q_{\min,\text{dissip}}, q_{\min,\text{init-en}}(\ell), q_{\min,\text{fin-en}}(L), q_{\min,\text{init-fire}}(\ell)), \\
p = \max(p_{\min}(q), p_{\min,\text{fin-en}}(L, \ell, q), p_{\min,\text{front-fire}}(\ell, q)).
\]

Then, according to Lemmas 17 to 22 it follows from inequality (46) on page 40 that:

\[
\int_{-\infty}^{L} \exp(c_{\text{esc}}y) E_{[u_{\infty}]}(y) \, dy \leq -\frac{3}{8} \varepsilon_{\text{dissip}},
\]

and since the nonnegative quantity \( L \) is any, this finishes the proof of Lemma 16 (provided that Lemmas 18 to 22 hold).

Proof of Lemma 18 (upper bound on initial energy). Let us proceed by contradiction and assume that the converse holds. Then there exists a nonnegative quantity \( \ell_0 \), a sequence \( (q_j)_{j \in \mathbb{N}} \) of nonzero integers approaching \( +\infty \) when \( j \) approaches \( +\infty \), and a sequence \( (p_j)_{j \in \mathbb{N}} \) of integers such that \( p_j \) is not smaller than \( p_{\min}(q_j) \) for all integer \( j \) (thus \( p_j \) approaches \( +\infty \) when \( j \) approaches \( +\infty \)) and such that, for every integer \( j \),

\[
\mathcal{E}^{2\left(p_j, q_j, \ell_0\right)}(0) \geq \frac{\varepsilon_{\text{dissip}}}{8}.
\]

By compactness (subsection 4.8), up to replacing the sequence \( (p_j, q_j)_{j \in \mathbb{N}} \) by a subsequence, we may assume that there exists \( \phi \) in \( C^1_b(\mathbb{R}, \mathbb{R}^n) \) and \( \tilde{\phi} \) in \( C^{k-2}_b(\mathbb{R}, \mathbb{R}^n) \) such that, for every positive quantity \( L \),

\[
\left\| y \mapsto u_t(x_{\text{esc}}(t_{p_j, q_j}^\text{bef}) + y, t_{p_j, q_j}^\text{bef}) - \phi(y)\right\|_{C^k([-L, L], \mathbb{R}^n)} \to 0 \quad \text{when} \quad j \to +\infty
\]

and

\[
\left\| y \mapsto u_t(x_{\text{esc}}(t_{p_j, q_j}^\text{bef}) + y, t_{p_j, q_j}^\text{bef}) - \tilde{\phi}(y)\right\|_{C^{k-2}([-L, L], \mathbb{R}^n)} \to 0 \quad \text{when} \quad j \to +\infty.
\]

According to the definition (57) on page 55 of \( t_{p_j, q_j}^\text{bef} \) for every integer \( j \),

\[
\delta_{\text{dissip}}(t_{p_j, q_j}^\text{bef}) \leq \frac{1}{q_j},
\]

therefore the function

\[
y \mapsto \tilde{\phi}(y) + c_{\text{esc}} \phi'(y)
\]

must be identically zero on \( \mathbb{R} \), thus passing to the limit in system (1) on page 2, it follows that, for every real quantity \( y \),

\[
\phi''(y) + c_{\text{esc}} \phi'(y) - \nabla \left( \phi(y) \right) = 0.
\]
Since (as stated in (27) on page 28) the difference \( x_{\text{hom}}(t) - x_{\text{esc}}(t) \) approaches \(+\infty\) when \( t \) approaches \(+\infty\), the following estimate holds:

\[
|\phi(y)| \leq d_{\text{esc}} \quad \text{for all } y \in [0, +\infty),
\]

thus according to Lemma 13 on page 87

\[
\phi(y) \to 0_{\mathbb{R}^n} \quad \text{when } y \to +\infty.
\]

On the other hand, according to the a priori bound (17) on page 23, the function \(|\phi(\cdot)|\) is bounded on \(\mathbb{R}\), and according to the definition (25) on page 27 of \(x_{\text{esc}}(\cdot)\) this function \(|\phi(\cdot)|\) cannot be identically zero. In short, the function \(\phi\) must belong to the set \(\Phi_{c_{\text{esc}}} (0_{\mathbb{R}^n})\) of profiles of fronts travelling at speed \(c_{\text{esc}}\) and “invading” the equilibrium \(0_{\mathbb{R}^n}\). As a consequence, according to Lemma 44 on page 88

\[
\int_{\mathbb{R}} \exp(c_{\text{esc}} y) E[\phi](y) \, dy = 0
\]

(and this integral converges). Recall that

\[
\chi^{(p_j, q_j, \ell_0)}(y, 0) = \begin{cases} 
\exp(c_{p_j, q_j} y) & \text{if } y \leq \ell_0, \\
\exp(c_{p_j, q_j} \ell_0 - \kappa(y - \ell_0)) & \text{if } y \geq \ell_0,
\end{cases}
\]

and let

\[
\chi^{(\infty, \infty, \ell_0)}(y) = \begin{cases} 
\exp(c_{\text{esc}} y) & \text{if } y \leq \ell_0, \\
\exp(c_{\text{esc}} \ell_0 - \kappa(y - \ell_0)) & \text{if } y \geq \ell_0.
\end{cases}
\]

Since the convergence

\[
\chi^{(p_j, q_j, \ell_0)}(y, 0) \to 0 \quad \text{when } y \to \pm\infty
\]
is uniform with respect to \(j\) provided that \(j\) is sufficiently large, it follows that

\[
\chi^{(p_j, q_j, \ell_0)}(\cdot, 0) \to \chi^{(\infty, \infty, \ell_0)}(\cdot) \quad \text{in } L^1(\mathbb{R}) \quad \text{when } j \to +\infty
\]

and (according to the definition (60) of \(\phi\) and the a priori bounds (18) on page 23 for the solution) that

\[
y \mapsto \chi^{(p_j, q_j, \ell_0)}(y, 0) E(y, t_{\text{bef}}^{p_j, q_j}) \quad \text{approaches} \quad y \mapsto \chi^{(\infty, \infty, \ell_0)}(y) E[\phi](y) \quad \text{in } L^1(\mathbb{R})
\]

when \(j\) approaches \(+\infty\). As a consequence,

\[
E^{(p_j, q_j, \ell_0)}(0) \to \int_{\mathbb{R}} \chi^{(\infty, \infty, \ell_0)}(y) E[\phi](y) \, dy \quad \text{when } j \to +\infty.
\]

According to inequality (61) the quantity \(V(\phi(y))\) is nonnegative (actually positive) for all \(y\) in \([0, +\infty)\), therefore according to the identity (62),

\[
\int_{\mathbb{R}} \chi^{(\infty, \infty, \ell_0)}(y) E[\phi](y) \, dy \leq 0.
\]

The contradiction with hypothesis (59) follows from (64) and (65). Lemma 18 is proved.
Proof of Lemma 19 (lower bound on final energy). Recall that
\[ E(p,q,\ell)(s_{p,q}) = \int_{\mathbb{R}} \chi(p,q,\ell)(y, s_{p,q}) E(y, t_p) \, dy \]
and that
\[ \chi(p,q,\ell)(y, s_{p,q}) = \begin{cases} 
\exp(c_{p,q}y) & \text{if } y \leq \ell + c_{\text{cut}} s_{p,q}, \\
\exp(c_{p,q}(\ell + c_{\text{cut}} s_{p,q}) - \kappa(y - (\ell + c_{\text{cut}} s_{p,q}))) & \text{if } y \geq \ell + c_{\text{cut}} s_{p,q}.
\end{cases} \]

It follows from the definition of \( x_{\text{esc}}(\cdot) \) that
\[ V(u(x_{\text{esc}}(t_p) + y, t_p)) \geq 0 \text{ for all } y \text{ in } [0, x_{\text{hom}}(t_p) - x_{\text{esc}}(t_p)]. \]

Let \( L \) be a positive quantity, and let us assume that \( p \) is sufficiently large so that \( L \leq x_{\text{hom}}(t_p) - x_{\text{esc}}(t_p) \) (this is possible according to assertion (27) on page 28). Then,
\[ E(p,q,\ell)(s_{p,q}) \geq \int_{-\infty}^{L} \chi(p,q,\ell)(y, s_{p,q}) E(y, t_p) \, dy + \int_{x_{\text{hom}}(t_p) - x_{\text{esc}}(t_p)}^{+\infty} \chi(p,q,\ell)(y, s_{p,q}) E(y, t_p) \, dy. \]  

(66)

Recall that the choice of \( t_{\text{bef}}^{p,q} \) ensures that \( s_{p,q} \) is not smaller than \( q \), therefore as soon as \( q \) is sufficiently large (namely not smaller than \( L/c_{\text{cut}} \)), the first integral at the right-hand of this last inequality (66) equals
\[ \int_{-\infty}^{L} \exp(c_{p,q}y) E(y, t_p) \, dy. \]

(67)

Since the convergence
\[ \exp(c_{p,q}y) \to 0 \text{ when } y \to -\infty \]
is uniform with respect to \( p \) and \( q \) provided that \( q \) is sufficiently large, and according to the a priori bounds (18) on page 23,
\[ y \mapsto \exp(c_{p,q}y) E(y, t_p) \text{ approaches } y \mapsto \exp(c_{\text{esc}}y) E_{[u,\infty]}(y) \]
in \( L^1((-\infty, L]) \), when \( p \) and \( q \) approach \( +\infty \).

On the other hand, for \( \ell \) and \( q \) fixed and \( p \) sufficiently large (depending on the values of \( \ell \) and \( q \)), the second integral of the right-hand of inequality (66) equals
\[ \exp((c_{p,q} + \kappa)(\ell + c_{\text{cut}} s_{p,q})) \int_{x_{\text{hom}}(t_p) - x_{\text{esc}}(t_p)}^{+\infty} e^{-\kappa y} E(y, t_p) \, dy. \]

(68)

According to the a priori bounds (18) and since according to the choice of \( t_{\text{bef}}^{p,q} \) the quantity \( s_{p,q} \) is not larger than \( q + T(1/q) \) (namely it is bounded uniformly with respect to \( p \)), this last quantity (68) approaches 0 when \( p \) approaches \( +\infty \) and \( \ell \) and \( q \) are fixed.

The desired lower bound of Lemma 19 thus follows from inequality (66). This finishes the proof of Lemma 19. \( \Box \)
Our next task is to prove Lemma 20. Let us first introduce some notation and state an intermediate lemma, before actually proceeding to the proof.

Let

\[
\psi_{(\infty, \infty, \ell)}(y) = \begin{cases} 
\exp((c_{\text{esc}} + \kappa)y - \kappa \ell) & \text{if } y \leq \ell \\
\exp((c_{\text{esc}} + \kappa)\ell - \kappa y) & \text{if } y \geq \ell .
\end{cases}
\]

Observe that, for every \(y \in \mathbb{R}\),

\[
\psi_{(p,q,\ell)}(y,0) \to \psi_{(\infty, \infty, \ell)}(y) \quad \text{when } q \to +\infty,
\]

uniformly with respect to \(p\) not smaller than \(p_{\text{min}}(q)\). Let

\[
\varepsilon_{\text{dissip,fire}} = \frac{\nu F \varepsilon_{\text{dissip}}}{8K_{\xi}}.
\]

The aim of Lemma 20 is to prove, under suitable conditions on the integers \(p\) and \(q\) and the quantity \(\ell\), that \(F_{(p,q,\ell)}(0)\) is not larger than \(\varepsilon_{\text{dissip,fire}}\). The following intermediate lemma provides a step towards this purpose.

**Lemma 23** (smallness of firewall with a weight bulk far to the right for a travelling front). There exists a positive quantity \(\ell_1\) such that, for every quantity \(\ell\) not smaller than \(\ell_1\), and for every function \(\phi\) in the set \(\Phi_{c_{\text{esc}}}(0\mathbb{R}^n)\) of profiles of travelling fronts invading \(0\mathbb{R}^n\), such that \(|\phi(y)|\) is not larger than \(d_{\text{Esc}}\) for every nonnegative quantity \(y\), the following inequality holds:

\[
(70) \quad \int_{\mathbb{R}} \psi_{(\infty, \infty, \ell)}(y) F_{[\phi]}(y) \, dy \leq \frac{\varepsilon_{\text{dissip,fire}}}{2}.
\]

**Proof of Lemma 23**. Observe that, for every real quantity \(y\) in \((-\infty, \ell]\),

\[
\psi_{(\infty, \infty, \ell)}(y) = \exp\left(-\frac{\kappa \ell}{2}\right) \exp\left((c_{\text{esc}} + \kappa)\left(y - \frac{\kappa \ell}{2(c_{\text{esc}} + \kappa)}\right)\right) .
\]

Thus, according to the a priori bounds \(18\), there exists a (large) positive quantity \(\ell_2\) such that, for every \(\ell\) not smaller than \(\ell_2\),

\[
\int_{-\infty}^{\ell/2(c_{\text{esc}}+\kappa)} \psi_{(\infty, \infty, \ell)}(y) F_{[\phi]}(y) \, dy \leq \frac{\varepsilon_{\text{dissip,fire}}}{4}.
\]

It follows that

\[
\int_{\mathbb{R}} \psi_{(\infty, \infty, \ell)}(y) F_{[\phi]}(y) \, dy \leq \frac{\varepsilon_{\text{dissip,fire}}}{4} + \int_{\ell/2(c_{\text{esc}}+\kappa)}^{+\infty} \exp(c_{\text{esc}}y) F_{[\phi]}(y) \, dy .
\]

According to Lemma \(13\) on page \(87\), the second term (the integral) in the right-hand side of this inequality approaches 0 when \(\ell\) approaches \(+\infty\), and this convergence is uniform with respect to the profile \(\phi\) belonging to \(\Phi_{c_{\text{esc}}}(0\mathbb{R}^n)\) and satisfying \(|\phi(y)| \leq d_{\text{Esc}}\) for every nonnegative quantity \(y\). This proves Lemma 23. \(\square\)
Proof of Lemma 20 (upper bound on initial firewall). Let us proceed by contradiction and assume that the converse holds. Then, there exists a quantity $\ell_3$ not smaller than the quantity $\ell_1$ introduced in Lemma 23, a sequence $(q_j)_{j \in \mathbb{N}}$ of positive integers approaching $+\infty$, and a sequence $(p_j)_{j \in \mathbb{N}}$ of integers with $p_j$ not smaller than $p_{\text{min}}(q_j)$ for all $j$ (thus $p_j$ also approaches $+\infty$ when $j$ approaches $+\infty$), such that, for every integer $j$,

$$F^{(p_j,q_j,\ell_3)}(0) \geq \varepsilon_{\text{dissip,fire}}.$$ 

Up to replacing the sequence $((p_j,q_j))_{j \in \mathbb{N}}$ by a subsequence, and proceeding as in the proof of Lemma 18, we may assume that there exists a function $\phi$ in the set $\Phi_{\text{esc}}(0 R^n)$ of profiles of fronts travelling at speed $c_{\text{esc}}$ and “invading” the equilibrium $0 R^n$, such that, for every positive quantity $L$,

$$\left\| y \mapsto \psi^{(p_j,q_j)}(y,0) - \phi(y) \right\|_{C^k([-L,L], R^n)} \to 0 \quad \text{when} \quad j \to +\infty.$$ 

Moreover, according to the definition of $x_{\text{esc}}(\cdot)$ and to Lemma 3 on page 25 (escape / Escape), the quantity $|\phi(y)|$ is not larger than $d_{\text{Esc}}$ for every nonnegative quantity $y$. According to the a priori bounds (18) on page 23, the function

$$(71) \quad y \mapsto \psi^{(p_j,q_j,\ell_3)}(y,0) F(y,t_{\text{bef}}^{p_j,q_j})$$

approaches 0 when $y$ approaches $\pm\infty$, and this convergence is uniform with respect to $j$ (provided that $j$ is large enough). As a consequence, the function $(71)$ above approaches

$$y \mapsto \psi^{(\infty,\infty,\ell_3)}(y) F_{\phi}(y)$$

in $L^1(\mathbb{R})$, when $j$ approaches $+\infty$. It follows that $F^{(p_j,q_j,\ell_3)}(0)$ approaches the quantity

$$\int_{\mathbb{R}} \psi^{(\infty,\infty,\ell_3)}(y) F_{\phi}(y) dy$$

when $j$ approaches $+\infty$, a contradiction with Lemma 23. Lemma 20 is proved.

Proof of Lemma 21 (upper bound on back flux for the firewall). In view of the expression of the back flux term in the relaxation scheme final inequality (46) on page 40, the statement is obvious.

Proof of Lemma 22 (upper bound on front flux for the firewall). In view of the expression of the front flux term in the relaxation scheme final inequality (46) on page 40, the statement is obvious.

The proof of Lemma 16 is complete.
4.12.2 Relaxation scheme to the right of the dissipation bump

The purpose of this sub-subsection is to apply once again the relaxation scheme set up in subsection 4.7 to the “second” sub-interval (between \(t_p\) and \(t_{p,q}^{\text{aft}}\), see figure 23 on page 56), in order to complete the proof of Proposition 6. The arguments are very similar to those of the proof of Lemma 16 in the previous sub-subsection.

Let us consider the positive quantity \(\varepsilon_{\text{energy}}\) defined by:

\[
\int_{\mathbb{R}} \exp(c_{\text{esc}} y) E_{[u_{\infty}]}(y) \, dy = -\varepsilon_{\text{energy}}.
\]

For many objects, we will use the same symbols as in the previous sub-subsection although these objects are now defined with respect to the “second” subinterval \([t_p, t_{p,q}^{\text{aft}}]\) (by contrast with the “first” one \([t_{\text{bef}}, t_p]\)). In other words, for notational simplicity we omit the superscripts “aft” (versus “bef”) to differentiate these objects. This begins with the following notation.

For every nonzero integer \(q\) and every integer \(p\) not smaller than \(p_{\min}(q)\), let

\[
s_{p,q} = t_{p,q}^{\text{aft}} - t_p \quad \text{and} \quad c_{p,q} = \frac{x_{\text{esc}}(t_{p,q}^{\text{aft}}) - x_{\text{esc}}(t_p)}{s_{p,q}}.
\]

Let us assume that \(q\) is large enough so that

\[0 < c_{p,q} \quad \text{and} \quad c_{p,q} \geq c_{\text{esc}} - \frac{\kappa_c}{4(e_{\text{no-esc}} + \kappa)} ,\]

and let \(\ell\) denote a nonnegative quantity to be chosen below. We are going to apply the relaxation scheme set up in subsection 4.7 for the following parameters:

\[t_{\text{init}} = t_p \quad \text{and} \quad x_{\text{init}} = x_{\text{esc}}(t_{\text{init}}) \quad \text{and} \quad c = c_{p,q} \quad \text{and} \quad y_{\text{cut-init}} = \ell .\]

As in the previous sub-subsection, the relaxation scheme thus depends on the three parameters \(p\), \(q\), and \(\ell\). Observe that both hypotheses (30) on page 30 and (42) on page 38 (required to apply the relaxation scheme) hold. Let us denote by

\[v^{(p,q)}(\cdot, \cdot) \quad \text{and} \quad \chi^{(p,q,\ell)}(\cdot, \cdot) \quad \text{and} \quad E^{(p,q,\ell)}(\cdot) \quad \text{and} \quad D^{(p,q,\ell)}(\cdot) \]

and \[w^{(p,q,\ell)}(\cdot, \cdot) \quad \text{and} \quad F^{(p,q,\ell)}(\cdot)\]

the objects defined in subsection 4.7 (with the same notation except the “\((p,q)\)” or “\((p,q,\ell)\)” superscripts to emphasize the dependency with respect to the parameters).

The contradiction completing the proof of Proposition 6 will follow from the next five lemmas.

**Lemma 24** (upper bound on initial energy). For every nonnegative quantity \(\ell\), there exists a nonzero integer \(q_{\min,\text{init-en}}(\ell)\) such that, for every integer \(q\) not smaller than \(q_{\min,\text{init-en}}(\ell)\) and every integer \(p\) not smaller than \(p_{\min}(q)\),

\[E^{(p,q,\ell)}(0) \leq -\frac{7}{8} \varepsilon_{\text{energy}} .\]
Lemma 25 (lower bound on final energy). There exists a nonzero integer \( q_{\text{min,fin-en}} \) such that, for every integer \( q \) not smaller than \( q_{\text{min,fin-en}} \) and for every nonnegative quantity \( \ell \), there exists an integer \( p_{\text{min,fin-en}}(\ell, q) \) such that, for every integer \( p \) not smaller than \( p_{\text{min,fin-en}}(\ell, q) \) and not smaller than \( p_{\text{min}}(q) \),
\[
\mathcal{E}(p,q,\ell)(s_{p,q}) \geq -\frac{\varepsilon_{\text{energy}}}{8}.
\]

Lemma 26 (upper bound on initial firewall). There exists a positive quantity \( \ell_{\text{min,init-fire}} \) such that, for every quantity \( \ell \) not smaller than \( \ell_{\text{min,init-fire}} \), there exists a nonzero integer \( q_{\text{min,init-fire}}(\ell) \) such that, for every integer \( q \) not smaller than \( q_{\text{min,init-fire}}(\ell) \), and for every integer \( p \) not smaller than \( p_{\text{min}}(q) \), the following inequality holds (the constants \( K_{\mathcal{E}} \) and \( \nu_F \) being those of inequality \((46)\) on page 40):
\[
\frac{K_{\mathcal{E}}}{\nu_F} f(p,q,\ell)(0) \leq \frac{\varepsilon_{\text{energy}}}{8}.
\]

Lemma 27 (upper bound on back flux for the firewall). There exists a nonnegative quantity \( \ell_{\text{min,back-fire}} \) such that, for every nonzero integer \( q \) and every integer \( p \) not smaller than \( p_{\text{min}}(q) \), the “back-flux” term in inequality \((46)\) on page 40 is not larger than \( \varepsilon_{\text{energy}}/8 \).

Lemma 28 (upper bound on front flux for the firewall). For every nonnegative quantity \( \ell \) and for every nonzero integer \( q \), there exists an integer \( p_{\text{min,front-fire}}(\ell, q) \) such that, for every integer \( p \) not smaller than \( p_{\text{min,front-fire}}(\ell, q) \) and not smaller than \( p_{\text{min}}(q) \), the “front flux” term in inequality \((46)\) on page 40 is not larger than \( \varepsilon_{\text{energy}}/8 \).

Postponing the proofs of these five lemmas, let us first conclude with the proof of Proposition 6. Let
\[
\ell = \max(\ell_{\text{min,init-fire}}, \ell_{\text{min,back-fire}}),
\]
\[
q = \max(q_{\text{min,init-en}}(\ell), q_{\text{min,fin-en}}, q_{\text{min,init-fire}}(\ell)),
\]
\[
p = \max(p_{\text{min}}(q), p_{\text{min,fin-en}}(\ell, q), p_{\text{min,front-fire}}(\ell, q)).
\]
Then, according to Lemmas 24 to 28, inequality \((46)\) on page 40 leads to an immediate contradiction. This finishes the proof of Proposition 6 (provided that Lemmas 24 to 28 hold).

Proof of Lemma 24 (upper bound on initial energy). Let \( \ell \) denote a nonnegative quantity, and let (as in the definition \((63)\) on page 61)
\[
\chi^{(\infty,\infty,\ell)}(y) = \begin{cases} 
\exp(c_{\text{esc}}y) & \text{if } y \leq \ell, \\
\exp(c_{\text{esc}}\ell - \kappa(y - \ell)) & \text{if } y \geq \ell.
\end{cases}
\]
For every sufficiently large positive integer \( q \), and every integer \( p \) not smaller than \( p_{\text{min}}(q) \), the mean speed is close to \( c_{\text{esc}} \), thus (say) larger than \( c_{\text{esc}}/2 \). As a consequence, the convergence
\[
\chi^{(p,q,\ell)}(y,0) \to \chi^{(\infty,\infty,\ell)}(y) \text{ when } y \to \pm \infty
\]
is uniform with respect to \( p \) and \( q \) (provided that \( q \) is large enough and that \( p \) is not smaller than \( p_{\text{min}}(q) \) and that \( \ell \) is fixed). It follows from the convergence above that
\[
y \mapsto \chi^{(p,q,\ell)}(y,0) E(y, t_p) \quad \text{approaches} \quad y \mapsto \chi^{(\infty,\infty,\ell)}(y) E_{[u_\infty]}(y)
\]
in \( L^1(\mathbb{R}) \) when \( q \) approaches \(+\infty\) (uniformly with respect to \( p \) not smaller than \( p_{\text{min}}(q) \)). Thus
\[
\mathcal{E}^{(p,q,\ell)}(0) \to \int_\mathbb{R} \chi^{(\infty,\infty,\ell)}(y) E_{[u_\infty]}(y) \, dy \quad \text{when} \quad q \to +\infty,
\]
uniformly with respect to \( p \) not smaller than \( p_{\text{min}}(q) \). Since the quantity \( V(u_\infty(y)) \) is nonnegative for every nonnegative quantity \( y \), the following inequality holds:
\[
\int_\mathbb{R} \chi^{(\infty,\infty,\ell)}(y) E_{[u_\infty]}(y) \, dy \leq \int_\mathbb{R} \exp(c_{\text{esc}}y) E_{[u_\infty]}(y) \, dy = -\varepsilon_{\text{energy}},
\]
and Lemma 24 follows.

Proof of Lemma 25 (lower bound on final energy). According to Lemma 44 on page 88, there exists a positive quantity \( L \) such that, for every function \( \phi \) in the set \( \Phi_{c_{\text{esc}}}(0_{\mathbb{R}^n}) \) of profiles of fronts travelling at speed \( c_{\text{esc}} \) and “invading” the equilibrium \( 0_{\mathbb{R}^n} \), satisfying \(|\phi(y)| \leq d_{\text{Esc}} \) for every nonnegative quantity \( y \), the following estimate holds:
\[
\int_{-\infty}^L \exp(c_{\text{esc}}y) E_{[\phi]}(y) \, dy \geq -\frac{\varepsilon_{\text{energy}}}{24}.
\]
As in the proof of Lemma 18, let us assume that \( p \) is sufficiently large so that
\[
L \leq x_{\text{hom}}(t_{\text{aft}}^{p,q}) - x_{\text{esc}}(t_{\text{aft}}^{p,q})
\]
(this is possible according to assertion 27 on page 28). Then,
\[
\mathcal{E}^{(p,q,\ell)}(s_{p,q}) \geq \int_{-\infty}^L \chi^{(p,q,\ell)}(y, s_{p,q}) E(y, t_{\text{aft}}^{p,q}) \, dy + \int_{x_{\text{hom}}(t_{\text{aft}}^{p,q}) - x_{\text{esc}}(t_{\text{aft}}^{p,q})}^{+\infty} \chi^{(p,q,\ell)}(y, s_{p,q}) E(y, t_{\text{aft}}^{p,q}) \, dy.
\]
(72)

Recall that the choice of \( t_{\text{aft}}^{p,q} \) ensures that \( s_{p,q} \) is not smaller than \( q \), therefore as soon as \( q \) is sufficiently large (namely not smaller than \( L/c_{\text{cut}} \)), the quantity \( \ell + c_{\text{cut}}s_{p,q} \) is not smaller than \( L \), and therefore the first integral at the right-hand of this last inequality (72) equals
\[
\int_{-\infty}^L \exp(c_{p,q}y) E(y, t_{\text{aft}}^{p,q}) \, dy.
\]
(73)

The following lemma deals with this integral.

67
Lemma 29 (lower bound on final energy, integral between $-\infty$ and $L$). There exists a nonzero integer $q_{\min,\text{fin-en}}$ such that, for every integer $q$ not smaller than $q_{\min,\text{fin-en}}$, for every integer $p$ not smaller than $p_{\min}(q)$, and for every nonnegative quantity $\ell$,

$$\int_{-\infty}^{L} \exp(c_{p,q}y) E(y, t_{p,q}^{\alpha}) \, dy \geq -\frac{\varepsilon_{\text{energy}}}{12}.$$ 

Proof of Lemma 29

Let us proceed by contradiction and assume that the converse holds. Then there exist a sequence $(q_j)_{j \in \mathbb{N}}$ of nonzero integers approaching $+\infty$ when $j$ approaches $+\infty$, a sequence $(p_j)_{j \in \mathbb{N}}$ such that $p_j$ is not smaller than $p_{\min}(q_j)$ for every integer $j$, and a sequence $(\ell_j)_{j \in \mathbb{N}}$ of nonnegative quantities such that, for every integer $j$,

$$\int_{-\infty}^{L} \exp(c_{p_j,q_j}y) E(y, t_{p_j,q_j}^{\alpha}) \, dy \leq -\frac{\varepsilon_{\text{energy}}}{12}.$$ 

Up to replacing the sequence $((p_j, q_j, \ell_j))_{j \in \mathbb{N}}$ by a subsequence, we may assume (proceeding as in the proof of Lemma 18) that there exists a function $\phi$ in the set $\Phi_{\text{esc}}(0_{\mathbb{R}^n})$ of profiles of fronts travelling at speed $c_{\text{esc}}$ and “invading” the equilibrium $0_{\mathbb{R}^n}$, such that, for every positive quantity $L'$,

$$\left\| y \mapsto u(x_{\text{esc}}(p_j,q_j) + y, t_{p_j,q_j}^{\alpha}) - \phi(y) \right\|_{C^k([-L',L'],\mathbb{R}^n)} \to 0 \quad \text{when} \quad j \to +\infty,$$

and such that $|\phi(y)| \leq d_{\text{esc}}$ for all $y$ in $[0, +\infty)$. Since the convergence

$$\exp(c_{p_j,q_j}y) \to 0 \quad \text{when} \quad y \to -\infty$$

is uniform with respect to $j$ (provided that $j$ is large enough), it follows that

$$y \to \exp(c_{p_j,q_j}y) E(y, t_{p_j,q_j}^{\alpha})$$

approaches $y \to \exp(c_{\text{esc}}y) E[\phi](y)$ in $L^1((-\infty, L])$ when $j$ approaches $+\infty$. According to Lemma 44 on page 88,

$$\int_{\mathbb{R}} \exp(c_{\text{esc}}y) E[\phi](y) \, dy = 0 \quad \text{thus} \quad \int_{-\infty}^{L} \exp(c_{\text{esc}}y) E[\phi](y) \, dy \leq 0,$$

a contradiction with hypothesis (74). Lemma 29 is proved. 

Let us pursue with the end of the proof of Lemma 25. Let us assume that $q$ is larger than $q_{\min,\text{fin-en}}$. For such fixed integer $q$ and every nonnegative fixed quantity $\ell$, since $s_{p,q}$ is bounded from above uniformly with respect to $p$, the following inequality holds for every large enough integer $p$:

$$\ell + c_{\text{cut}} s_{p,q} \geq x_{\text{hom}}(t_{p,q}^{\alpha}) - x_{\text{esc}}(t_{p,q}^{\alpha}).$$

Thus, the second integral of the right-hand side of inequality (72) reads

$$\exp((c_{p,q} + \kappa)(\ell + c_{\text{cut}} s_{p,q})) \int_{x_{\text{hom}}(t_{p,q}^{\alpha}) - x_{\text{esc}}(t_{p,q}^{\alpha})}^{+\infty} \exp(-\kappa y) E(y, t_{p,q}^{\alpha}) \, dy.$$

According to the a priori bounds (18), this last quantity approaches 0 when $p$ approaches $+\infty$ and $\ell$ and $q$ are fixed. This finishes the proof of Lemma 25.
Our next task is to prove Lemma 26. Let us first state an intermediate result. For every nonnegative quantity \(\ell\), every nonnegative integer \(q\), every integer \(p\) not smaller than \(p_{\min}(q)\), let us define the function \(y \mapsto \psi(\infty,\infty,\ell)(y)\) as in (69). Let

\[
\tilde{\varepsilon}_{\text{energy}} = \frac{\nu_F}{8K_F} \varepsilon_{\text{energy}}.
\]

**Lemma 30** (smallness of firewall with a weight bulk far to the right for \(u_{\infty}\)). There exists a positive quantity \(\ell_4\) such that, for every quantity \(\ell\) not smaller than \(\ell_4\), the following inequality holds:

\[
\int_{\mathbb{R}} \psi(\infty,\infty,\ell)(y) F[u_{\infty}](y) \, dy \leq \frac{\tilde{\varepsilon}_{\text{energy}}}{2}.
\]

**Proof of Lemma 30.** Let us proceed like in the proof of Lemma 23. According to the a priori bounds (18), there exists a (large) positive quantity \(\ell_5\) such that, for every \(\ell\) not smaller than \(\ell_5\),

\[
\int_{-\infty}^{\infty} \psi(\infty,\infty,\ell)(y) F[u_{\infty}](y) \, dy \leq \frac{\tilde{\varepsilon}_{\text{energy}}}{4}.
\]

It follows that

\[
\int_{\mathbb{R}} \psi(\infty,\infty,\ell)(y) F[u_{\infty}](y) \, dy \leq \frac{\tilde{\varepsilon}_{\text{energy}}}{4} + \int_{\kappa\ell/(2(c_{\text{esc}}+\ell))}^{+\infty} \exp(c_{\text{esc}}y) F[u_{\infty}](y) \, dy.
\]

According to Lemma 16 on page 57, the second term (the integral) on the right-hand side of this inequality approaches 0 when \(\ell\) approaches \(+\infty\) (indeed for every nonnegative quantity \(y\) the quantity \(|u_{\infty}(y)|\) is not larger than \(d_{\text{Esc}}\), thus \(u_{\infty}(y)^2\) and \(V(u_{\infty}(y))\) are of the same order of magnitude). This proves Lemma 30.

**Proof of Lemma 26 (upper bound on initial firewall).** Let us assume that \(\ell\) is not smaller than the quantity \(\ell_4\) introduced in Lemma 30 above. According to the a priori bounds (18), the function

\[
y \mapsto \psi(p,q,\ell)(y,0) F(y,t_p)
\]

approaches 0 when \(y\) approaches \(\pm\infty\), and this convergence is uniform with respect to \(p\) and \(q\) (provided that \(q\) is large enough and that \(p\) is not smaller than \(p_{\min}(q)\)). As a consequence, the function (76) above approaches the function

\[
y \mapsto \psi(\infty,\infty,\ell)(y) F[u_{\infty}](y)
\]

in \(L^1(\mathbb{R})\) when \(q\) approaches \(+\infty\), uniformly with respect to \(p\) not smaller than \(p_{\min}(q)\). This proves Lemma 26.

**Proof of Lemma 27 (upper bound on back flux for the firewall).** As for Lemma 21 on page 69, the statement follows obviously from the expression of the back flux term in the relaxation inequality (46) on page 40.
Proof of Lemma 28 (upper bound on front flux for the firewall). As for Lemma 22 on page 60, the statement follows obviously from the expression of the back flux term in the relaxation inequality (46) on page 40.

The proof of Proposition 6 on page 54 ("relaxation") is complete. Note that, at this stage, hypothesis (H_{disc-$\Phi$}) has not been used yet.

4.13 Convergence

The end of the proof of Proposition 2 on page 22 ("invasion implies convergence") is a straightforward consequence of Proposition 6, and is very similar to the end of the proof of the main result of [22] or of [23], thus this subsection is rather similar to the corresponding sections of those papers.

We will make use of the notation $x_{Esc}(t)$ and $x_{esc}(t)$ and $x_{hom}(t)$ introduced in subsections 4.1 and 4.6. Recall that, according to (26) on page 27 and to the hypotheses of Proposition 2, for every nonnegative time $t$,

$$-\infty \leq x_{Esc}(t) \leq x_{esc}(t) \leq x_{hom}(t) < +\infty.$$ 

Lemma 31 (existence of Escape point and transversality). The following inequalities hold:

$$\limsup_{t \to +\infty} x_{Esc}(t) - x_{Esc}(t) < +\infty \quad \text{and} \quad \limsup_{t \to +\infty} u(x_{Esc}(t), t) \cdot u_x(x_{Esc}(t), t) < 0.$$

Proof. To prove the first assertion, let us proceed by contradiction and assume that the converse holds. Then there exists a sequence $(t_p)_{p \in \mathbb{N}}$ of nonnegative quantities approaching $+\infty$ such that $x_{esc}(t_p) - x_{Esc}(t_p)$ approaches $+\infty$ when $p$ approaches $+\infty$. Proceeding as in the proof of Lemma 18 on page 59, we may assume, up to replacing the sequence $(t_p)_{p \in \mathbb{N}}$ by a subsequence, that there exists a function $\phi_1$ in the set $\Phi_{c_{esc}}(0_{\mathbb{R}^n})$ of profiles of fronts travelling at speed $c_{esc}$ and "invading" the local minimum $0_{\mathbb{R}^n}$ such that, for every positive quantity $L$,

$$\| y \mapsto u(x_{esc}(t_p) + y, t_p) - \phi_1(y) \|_{C^0([-L, L], \mathbb{R}^n)} \to 0 \quad \text{when} \quad p \to +\infty.$$

In addition, it follows from the definition of $x_{Esc}(\cdot)$ that

$$|\phi_1(y)| \leq d_{Esc} \quad \text{for all} \quad y \in \mathbb{R},$$

a contradiction with Lemma 43 on page 87.

The proof of the second assertion is similar. Let us proceed by contradiction and assume that the converse holds. Then there exists a sequence $(t'_p)_{p \in \mathbb{N}}$ of nonnegative quantities approaching $+\infty$ such that, for every nonzero integer $p$,

$$u(x_{Esc}(t'_p), t'_p) \cdot u_x(x_{Esc}(t'_p), t'_p) \geq -\frac{1}{p}.$$

Proceeding as in the proof of Lemma 18 on page 59, we may assume, up to replacing the sequence $(t'_p)_{p \in \mathbb{N}}$ by a subsequence, that there exists a function $\phi_2$ in the set $\Phi_{c_{esc}, \text{norm}}(0_{\mathbb{R}^n})$.
of (normalized) profiles of fronts travelling at speed $c_{\text{esc}}$ and “invading” the local minimum $0_{\mathbb{R}^n}$ such that, for every positive quantity $L$,
$$
\| y \mapsto u(x_{\text{Esc}}(t_p) + y, t_p) - \phi_2(y) \|_{C^k([-L,L],\mathbb{R}^n)} \to 0 \quad \text{when} \quad p \to +\infty.
$$
It follows from the last inequality that
$$
\phi_2(0) \cdot \phi_2'(0) \geq 0,.
$$
a contradiction with Lemma 43 on page 87. 

**Lemma 32** (regularity of Escape point). The map $t \mapsto x_{\text{Esc}}(t)$ is of class (at least) $C^1$ on a neighbourhood of $+\infty$ and
$$
x'_{\text{Esc}}(t) \to c_{\text{esc}} \quad \text{when} \quad t \to +\infty.
$$

**Proof.** Let us consider the function
$$
f : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}, \quad (x, t) \mapsto \frac{1}{2} (u(x, t)^2 - d_{\text{esc}}^2).
$$
According to the regularity of the solution (see subsection 3.1), this function is of class at least $C^1$, and, for every sufficiently large time $t$, the quantity $f(x_{\text{Esc}}(t), t)$ is equal to 0, and
$$
\partial_x f(x_{\text{Esc}}(t), t) = u(x_{\text{Esc}}(t), t) \cdot u_x(x_{\text{Esc}}(t), t) < 0.
$$
Thus it follows from the implicit function theorem that the function $x \mapsto x_{\text{Esc}}(t)$ is of class (at least) a neighbourhood of $+\infty$, and that, for every sufficiently large time $t$,
$$
x'_{\text{Esc}}(t) = - \frac{\partial_t f(x_{\text{Esc}}(t), t)}{\partial_x f(x_{\text{Esc}}(t), t)} = - \frac{u(x_{\text{Esc}}(t), t) \cdot u_t(x_{\text{Esc}}(t), t)}{u(x_{\text{Esc}}(t), t) \cdot u_x(x_{\text{Esc}}(t), t)}.
$$
According to Lemma 31 above, the denominator of this expression remains bounded away from 0 when time approaches plus infinity. On the other hand, according to the first assertion of Lemma 31 and to Proposition 6 on page 54 and to the a priori bounds (11) on page 18 for the solution,
$$
u_t(x_{\text{Esc}}(t) + y, t) + c_{\text{esc}} u_x(x_{\text{Esc}}(t) + y, t) \to 0 \quad \text{when} \quad t \to +\infty.
$$
Thus it follows from (77) that $x'_{\text{Esc}}(t)$ approaches $c_{\text{esc}}$ when time approaches $+\infty$. Lemma 32 is proved. 

The next lemma is the only place throughout the proof of Proposition 2 where hypothesis (H$_{\text{disc-}\Phi}$) — which is part of the generic hypotheses (G) — is required.

**Lemma 33** (convergence around Escape point). There exists a function $\phi$ in the set $\Phi_{\text{esc-norm}}(0_{\mathbb{R}^n})$ of (normalized) profiles of fronts travelling at speed $c_{\text{esc}}$ and invading the equilibrium $0_{\mathbb{R}^n}$ such that, for every positive quantity $L$,
$$
\sup_{x \in [x_{\text{Esc}}(t) - L, x_{\text{Esc}}(t) + L]} |u(x, t) - \phi(x - x_{\text{Esc}}(t))| \to 0 \quad \text{when} \quad t \to +\infty.
$$
In particular, the set $\Phi_{\text{esc-norm}}(0_{\mathbb{R}^n})$ is nonempty.

71
Proof. Take a sequence \((t_p)_{p \in \mathbb{N}}\) of positive times approaching \(+\infty\) when \(p\) approaches \(+\infty\). Proceeding as in the proof of Lemma 18 on page 59, we may assume, up to replacing the sequence \((t_p)_{p \in \mathbb{N}}\) by a subsequence, that there exists a function \(\phi\) in the set \(\Phi_{\text{esc}}(0_{\mathbb{R}^n})\) of profiles of fronts travelling at speed \(c_{\text{esc}}\) and “invading” the local minimum \(0_{\mathbb{R}^n}\) such that, for every positive quantity \(L\),

\[
\|y \mapsto u(x_{\text{Esc}}(t_p) + y, t_p) - \phi(y)\|_{C^k([-L, L], \mathbb{R}^n)} \to 0 \quad \text{when} \quad p \to +\infty.
\]

According to the definition of \(x_{\text{Esc}}(\cdot)\),

\[
|\phi(0)| = d_{\text{Esc}} \quad \text{and} \quad |\phi(y)| \leq d_{\text{Esc}} \quad \text{for all} \quad y \in [0, +\infty),
\]

thus according to Lemma 43 on page 87, it follows that \(\phi\) actually belongs to the set \(\Phi_{\text{esc,norm}}(0_{\mathbb{R}^n})\) of “normalized” profiles of fronts.

Let \(L\) denote the set of all possible limits (in the sense of uniform convergence on compact subsets of \(\mathbb{R}\)) of sequences of maps

\[
y \mapsto u(x_{\text{Esc}}(t'_{p}) + y, t'_{p})
\]

for all possible sequences \((t'_{p})_{p \in \mathbb{N}}\) such that \(t'_{p}\) approaches \(+\infty\) when \(p\) approaches \(+\infty\). This set \(L\) is included in the set \(\Phi_{\text{esc,norm}}(0_{\mathbb{R}^n})\), and, because the semi-flow of system 1 is continuous on \(X\), this set \(L\) is a continuum (a compact connected subset) of \(X\).

Since on the other hand — according to hypothesis (H\(_{\text{disc-}\Phi}\)) — the set \(\Phi_{\text{esc,norm}}(0_{\mathbb{R}^n})\) is totally disconnected in \(X\), this set \(L\) must actually be reduced to the singleton \(\{\phi\}\). Lemma 33 is proved. \(\square\)

Lemma 34 (convergence up to \(x_{\text{hom}}(t)\)). For every positive quantity \(L\),

\[
\sup_{x \in [x_{\text{Esc}}(t) - L, x_{\text{hom}}(t)]} |u(x, t) - \phi(x - x_{\text{Esc}}(t))| \to 0 \quad \text{when} \quad t \to +\infty.
\]

Proof. Let us proceed by contradiction and assume that the converse holds. Then, according to Lemma 33 above, there exists a positive quantity \(\varepsilon\), and sequences \((t_p)_{p \in \mathbb{N}}\) and \((y_p)_{p \in \mathbb{N}}\) of real quantities both approaching \(+\infty\) when \(p\) approaches \(+\infty\), such that, for every integer \(p\),

\[
|u(x_{\text{Esc}}(t_p) + y_p, t_p) - \phi(y_p)| \geq \varepsilon,
\]

and thus, for every sufficiently large \(p\),

\[
|u(x_{\text{Esc}}(t_p) + y_p, t_p)| \geq \varepsilon.
\]

Using the notation \(\mathcal{F}_0(\cdot, \cdot)\) of subsection 4.4, this yields the existence of a positive quantity \(\varepsilon'\) such that, for every sufficiently large \(p\),

\[
\mathcal{F}_0(x_{\text{Esc}}(t_p) + y_p, t_p) \geq \varepsilon'.
\]

According to hypothesis (H\(_{\text{hom-right}}\)) (subsection 4.1 on page 21),

\[
x_{\text{hom}}(t_p) - (x_{\text{Esc}}(t_p) + y_p) \to +\infty \quad \text{when} \quad p \to +\infty.
\]

72
According to inequality (21) on page 25 about the time derivative of \( F_0 \) and to the fact that both derivatives \( x'_{\text{Esc}}(t) \) and \( x'_{\text{hom}}(t) \) approach a finite value when \( t \) approaches \(+\infty\), this shows that the function:

\[
[0, t_p] \to \mathbb{R}, \quad s \mapsto F_0(x_{\text{Esc}}(t_p) + y_p, t_p - s)
\]

is increasing at an exponential rate with respect to \( s \), on an arbitrarily large time interval starting from 0, provided that \( p \) is sufficiently large, a contradiction with the a priori bounds (18) on the solution.

\[\square\]

### 4.14 Homogeneous point behind the travelling front

According to Lemma 42 on page 87 and hypothesis (H_{bist}), the limit

\[
\lim_{x \to -\infty} \phi(x)
\]

exists and belongs to \( \mathcal{M} \); let us denote by \( m_{\text{next}} \) this limit. According to the same Lemma 42,

\[
V(m_{\text{next}}) < 0.
\]

The following lemma completes the proof of Proposition 2 ("invasion implies convergence").

**Lemma 35** ("next" homogeneous point behind the front). There exists a \( \mathbb{R} \)-valued function \( x_{\text{hom-next}} \), defined and of class \( C^1 \) on a neighbourhood of \( +\infty \), such that the following limits hold when \( t \) approaches \(+\infty\):

\[
x_{\text{Esc}}(t) - x_{\text{hom-next}}(t) \to +\infty \quad \text{and} \quad x'_{\text{hom-next}}(t) \to c_{\text{esc}}
\]

and

\[
\sup_{x \in [x_{\text{hom-next}}(t), x_{\text{hom}}(t)]} \left| u(x, t) - \phi(x - x_{\text{Esc}}(t)) \right| \to 0,
\]

and, for every positive quantity \( L \),

\[
\| y \mapsto u(x_{\text{hom-next}}(t) + y, t) - m_{\text{next}} \|_{H^1([-L, L], \mathbb{R}^n)} \to 0.
\]

**Proof.** Let us consider the sequence of times \( (t_p)_{p \in \mathbb{N}} \) defined as follows: \( t_0 \) is positive and sufficiently large so that the map \( t \mapsto x_{\text{Esc}}(t) \) is defined and of class \( C^1 \) on \( [t_0, +\infty) \), and, for every nonzero integer \( p \),

\[
t_p = \max\left( t_{p-1} + p, \sup\{ t \in [0, +\infty) : \sup_{y \in [-2p, 0]} |u(x_{\text{esc}}(t) + y, t) - \phi(y)| \geq \frac{1}{p} \} \right)
\]

(the key point being that, according to Lemma 33 above, this quantity \( t_p \) is finite). Let \( \chi \) denote a smooth function \( \mathbb{R} \to \mathbb{R} \) satisfying

\[
\chi \equiv 0 \text{ on } (-\infty, 0] \quad \text{and} \quad \chi \equiv 1 \text{ on } [1, +\infty) \quad \text{and} \quad 0 \leq \chi \leq 1 \text{ and } \chi' \geq 0 \text{ on } [0, 1]
\]

(see figure 25) and let us define the function \( [0, +\infty) \to \mathbb{R}, \quad x \mapsto x_{\text{hom-next}}(t) \) by:
Figure 25: Illustration of the definition of the function $x_{\text{hom-next}}(\cdot)$.

$$x_{\text{hom-next}}(t) = x_{\text{Esc}}(t) - (p - 1) - \chi \left( \frac{t - t_{p-1}}{t_{p} - t_{p-1}} \right) \quad \text{for} \quad t \in [t_{p-1}, t_{p}].$$

This function is of class (at least) $C^1$ on $[0, +\infty)$ (it is as smooth as $x_{\text{Esc}}(\cdot)$) and the other conclusions of Lemma 35 follow readily from the definition of $x_{\text{hom-next}}(\cdot)$ (and, for the last conclusion, from the a priori bounds (11) on page 18 for the solution).

The proof of Proposition 2 is complete.

5 Non invasion implies relaxation

In this section the generic hypotheses (G) are not required, let us just assume that $V$ satisfies hypothesis (H$_{\text{coerc}}$).

5.1 Definitions and hypotheses

Let us consider two minimum points $m_-$ and $m_+$ in $\mathcal{M}$, a function (initial condition) $u_0$ in $X$, and, for all $x$ in $\mathbb{R}$ and $t$ in $[0, +\infty)$, let $u(x,t) = (S_t u_0)(x)$ denote the corresponding solution.

Without assuming that this solution is bistable, let us make the following hypothesis (H$_{\text{hom}}$), which is similar to hypothesis (H$_{\text{hom-right}}$) made in section 4 (“invasion implies convergence”), but this time both to the right and to the left in space (see figure 26).

Figure 26: Illustration of hypothesis (H$_{\text{hom}}$) and of Proposition 7.
**Proposition 7** The aim of section 5 is to prove the following proposition.

5.2 Statement

If the quantity \( c_{\text{hom},+} \) and a negative quantity \( c_{\text{hom},-} \) and \( C^1 \)-functions satisfy

\[
x_{\text{hom},+} : [0, +\infty) \to \mathbb{R} \quad \text{satisfying} \quad x'_{\text{hom},+}(t) \to c_{\text{hom},+} \quad \text{when} \quad t \to +\infty
\]

and \( x_{\text{hom},-} : [0, +\infty) \to \mathbb{R} \) satisfying \( x'_{\text{hom},-}(t) \to c_{\text{hom},-} \) when \( t \to +\infty \)

such that, for every positive quantity \( L \),

\[
\| y \mapsto u(x_{\text{hom},+}(t) + y, t) - m_+ \|_{H^1([-L,L])} \to 0 \quad \text{when} \quad t \to +\infty
\]

and

\[
\| y \mapsto u(x_{\text{hom},-}(t) + y, t) - m_- \|_{H^1([-L,L])} \to 0 \quad \text{when} \quad t \to +\infty.
\]

For every \( t \) in \([0, +\infty)\), let us denote by \( x_{\text{Esc},+}(t) \) the supremum of the set

\[
\{ x \in \mathbb{R} : x_{\text{hom},-}(t) \leq x \leq x_{\text{hom},+}(t) \quad \text{and} \quad |u(x, t) - m_+| = d_{\text{Esc}} \}
\]

(with the convention that \( x_{\text{Esc},+}(t) \) equals \(-\infty\) if this set is empty), and let us denote by \( x_{\text{Esc},-}(t) \) the infimum of the set

\[
\{ x \in \mathbb{R} : x_{\text{hom},-}(t) \leq x \leq x_{\text{hom},+}(t) \quad \text{and} \quad |u(x, t) - m_-| = d_{\text{Esc}} \}
\]

(with the convention that \( x_{\text{Esc},+}(t) \) equals \(+\infty\) if this set is empty). Let

\[
c_{\text{Esc},+} = \limsup_{t \to +\infty} \frac{x_{\text{Esc},+}(t)}{t} \quad \text{and} \quad c_{\text{Esc},-} = \liminf_{t \to +\infty} \frac{x_{\text{Esc},+}(t)}{t}
\]

(see figure 26). Obviously, for all \( t \) in \([0, +\infty)\),

\[
x_{\text{Esc},+}(t) \leq x_{\text{hom},+}(t) \quad \text{and} \quad x_{\text{hom},-}(t) \leq x_{\text{Esc},-}(t)
\]

thus

\[
c_{\text{hom},-} \leq c_{\text{Esc},-} \quad \text{and} \quad c_{\text{Esc},+} \leq c_{\text{hom},+}.
\]

If the quantity \( c_{\text{Esc},+} \) was positive or if the quantity \( c_{\text{Esc},-} \) was negative, this would mean that the corresponding equilibrium is “invaded” at a nonzero mean speed, a situation already studied in section 4. Here we shall make the following converse hypothesis:

**\( H_{\text{no-inv}} \)** The following inequalities hold:

\[
c_{\text{Esc},-} \geq 0 \quad \text{and} \quad c_{\text{Esc},+} \leq 0.
\]

5.2 Statement

The aim of section 5 is to prove the following proposition.

**Proposition 7** (non-invasion implies relaxation). Assume that \( V \) satisfies hypothesis \( H_{\text{cerc}} \) (only) and that the solution \( (x, t) \mapsto u(x, t) \) under consideration satisfies hypotheses \( (H_{\text{hom}}) \) and \( (H_{\text{no-inv}}) \). Then,

\[
V(m_-) = V(m_+) \quad \text{and} \quad \sup_{x \in [x_{\text{hom},-}(t), x_{\text{hom},+}(t)]} |u_t(x, t)| \to 0 \quad \text{when} \quad t \to +\infty.
\]
5.3 Settings of the proof

Let us keep the notation and assumptions of subsection 5.1, and let us assume that hypotheses \((H_{coerc})\) and \((H_{hom})\) and \((H_{no-inv})\) of Proposition 7 hold. Before doing anything else, let us clean up the place.

- For notational convenience, let us assume, without loss of generality, that

\[
\max(V(m_-), V(m_+)) = 0.
\]

- According to Lemma 1 on page 18, we may assume (without loss of generality, up to changing the origin of times) that, for all \(t\) in \([0, +\infty)\),

\[
\sup_{x \in \mathbb{R}} |u(x, t)| \leq R_{att, \infty}.
\]

5.4 Relaxation scheme in a standing or almost standing frame

The aim of this subsection is to set up an appropriate relaxation scheme in a standing or almost standing frame. This means defining an appropriate localized energy and controlling the “flux” terms occurring in the time derivative of that localized energy. The argument will be quite similar to that of subsection 4.7 (the relaxation scheme in the travelling frame), the main difference being that \(c\) is now either equal or close to 0, and as a consequence the weight function for the localized energy will be defined with a cut-off on the right and another on the left, instead of a single one; accordingly firewall functions will be introduced to control the fluxes along each of these cut-offs.

5.4.1 Preliminary definitions

Let us keep the notation and hypotheses introduced above (since the beginning of subsection 5.3 and let \(c\) denote a real quantity. By contrast with subsection 4.7, the other parameters — namely the initial time \(t_{init}\), the initial position of origin of travelling frame \(x_{init}\), and the initial position of the cut-off \(y_{cut-init}\) — are not required here; or in other words they are chosen equal to 0). The relaxation scheme will be applied in the next subsection 5.5 for \(c\) very close or equal to 0.

For all \(y\) in \(\mathbb{R}\) and \(t\) in \([0, +\infty)\), let

\[
v(y, t) = u(ct + y, t).
\]

This function is a solution of the differential system

\[
v_t - cv_y = -\nabla V(v) + v_{yy}.
\]

We are going to define a localized energy and two firewall functions associated with this solution. As in sub-subsection 4.7.1 on page 29, we are going to use the quantities

\(\kappa\) and \(c_{cut}\) and \(w_{en}\),

76
with exactly the same definitions as the ones following conditions (31) and (32) on page 31. Let
\[ c_{\text{cut},0} = \min \left( c_{\text{cut}} , \frac{c_{\text{hom},+}}{2} , \frac{|c_{\text{hom},-}|}{2} \right), \]
and let us make on the parameter \( c \) the following hypotheses (see comments below):

\[ |c| \leq \frac{\kappa}{6} \text{ and } |c| \leq c_{\text{no-esc}} \text{ and } |c| \leq \frac{c_{\text{cut},0}}{6}. \]

On the other hand, according to \((H_{\text{hom}})\) and \((H_{\text{no-inv}})\) and to the choice of \(c_{\text{cut},0}\) above, we may assume, up to changing the origin of time, that, for all \( t \) in \([0, +\infty)\),

\[ x_{\text{hom},-}(t) \leq -\frac{11}{6} c_{\text{cut},0} t \text{ and } -\frac{1}{6} c_{\text{cut},0} t \leq x_{\text{Esc},-}(t) \]
\[ \text{and } x_{\text{Esc},+}(t) \leq \frac{11}{6} c_{\text{cut},0} t \text{ and } \frac{1}{6} c_{\text{cut},0} t \leq x_{\text{hom},+}(t). \]

For every nonnegative time \( t \), let
\[ y_{\text{hom},+}(t) = x_{\text{hom},+}(t) - ct \text{ and } y_{\text{hom},-}(t) = x_{\text{hom},-}(t) - ct \]
\[ \text{and } y_{\text{Esc},+}(t) = x_{\text{Esc},+}(t) - ct \text{ and } y_{\text{Esc},-}(t) = x_{\text{Esc},-}(t) - ct. \]

It follows from hypotheses (81) and from the last hypothesis of (80) that, for every nonnegative time \( t \),

\[ y_{\text{hom},-}(t) \leq -\frac{5}{3} c_{\text{cut},0} t \text{ and } -\frac{1}{3} c_{\text{cut},0} t \leq y_{\text{Esc},-}(t) \]
\[ \text{and } y_{\text{Esc},+}(t) \leq \frac{5}{3} c_{\text{cut},0} t \text{ and } \frac{1}{3} c_{\text{cut},0} t \leq y_{\text{hom},+}(t). \]

(see figure 27). Let us briefly comment these hypotheses (80) and (81).

\[ y_{\text{hom},+}(t) \]
\[ y_{\text{hom},-}(t) \]
\[ y_{\text{Esc},+}(t) \]
\[ y_{\text{Esc},-}(t) \]
\[ y_{\text{Esc},+}(t) \]
\[ y_{\text{hom},-}(t) \]

Figure 27: Illustration of setting assumptions for the proof of Proposition 7.

- The relaxation scheme below will be applied either for \( c \) equals 0, or for \( c \) very close to 0. Thus, in principle, the speed \( c_{\text{no-esc}} \) should not play any role in this relaxation scheme. The reason why it is nevertheless assumed in (80) that \(|c|\) is not larger than \( c_{\text{no-esc}} \) and why the values of the quantities \( \kappa \) and \( w_{\text{es}} \) and \( c_{\text{cut},0} \) are chosen so that the hypotheses of sub-subsection 4.7.1 be satisfied — those hypotheses involve \( c_{\text{no-esc}} \) — is that a number of estimates proved in subsection 4.7.1 will therefore remain exactly identical in the present context (thus it will not be necessary to prove variants of them).
• The choice of the value $1/6$ as upper bound of the quotients $|c|/\kappa$ and $|c|/c_{\text{cut,}0}$ in (80) will turn out to be convenient to prove Lemma 38 on page 81 (that is, more precisely, to control the flux terms involved in the derivative of the firewall functions that will be defined below). The choice of the factors $\pm 1/6$ and $\pm 11/6$ in (81) leading to inequalities (82) follows from exactly the same reason.

5.4.2 Localized energy

For every nonnegative time $t$, let us consider the three intervals:

$$I_{\text{left}}(t) = (-\infty, -c_{\text{cut,}0}t],$$
$$I_{\text{main}}(t) = [-c_{\text{cut,}0}t, c_{\text{cut,}0}t],$$
$$I_{\text{right}}(t) = [c_{\text{cut,}0}t, +\infty),$$

and let us consider the function $\chi(y, t)$ (weight function for the localized energy) defined by:

$$\chi(y, t) = \begin{cases} 
\exp(-c_{\text{cut,}0}t + \kappa(y + c_{\text{cut,}0}t)) & \text{if } y \in I_{\text{left}}(t), \\
\exp(cy) & \text{if } y \in I_{\text{main}}(t), \\
\exp((c + \kappa)c_{\text{cut,}0}t - \kappa y) & \text{if } y \in I_{\text{right}}(t)
\end{cases},$$

(see figure 28), and, for all $t$ in $[0, +\infty)$, let us define the “energy function” by:

$$\mathcal{E}(t) = \int_{\mathbb{R}} \chi(y, t) \left( \frac{v_y(y, t)^2}{2} + V(v(y, t)) \right) dy.$$

Figure 28: Graphs of the weight functions $y \mapsto \chi(y, t)$ and $y \mapsto \psi_+(y, t)$ and $y \mapsto \psi_-(y, t)$. The case shown on the figure is that of a positive quantity $c$. The assumption that $|c|$ is smaller than $\kappa$ ensures that the value of $\chi_y(y, t)$ is smaller slightly to the left of $-c_{\text{cut,}0}t$ than slightly to the right (otherwise it would not be legitimate to call the change of slope at $-c_{\text{cut,}0}t$ a “cut-off”).
5.4.3 Time derivative of the localized energy

It follows from the definition of $\chi$ that:

$$
\chi_t(y,t) = \begin{cases} 
  c_{\text{cut},0}(-c + \kappa)\chi(y,t) & \text{if } y \in I_{\text{left}}(t), \\
  0 & \text{if } y \in I_{\text{main}}(t), \\
  c_{\text{cut},0}(c + \kappa)\chi(y,t) & \text{if } y \in I_{\text{right}}(t),
\end{cases}
$$

and

$$
(c\chi - \chi_y)(y,t) = \begin{cases} 
  (c - \kappa)\chi(y,t) & \text{if } y \in I_{\text{left}}(t), \\
  0 & \text{if } y \in I_{\text{main}}(t), \\
  (c + \kappa)\chi(y,t) & \text{if } y \in I_{\text{right}}(t).
\end{cases}
$$

Thus, if for all $t$ in $[0, +\infty)$ we define the “dissipation” function by

$$
D(t) = \int_{\mathbb{R}} \chi(y,t) v_t(y,t)^2 dy,
$$

then, for all $t$ in $[0, +\infty)$, it follows from expression (13) on page 19 (time derivative of a localized energy) that

$$
E'(t) = -D(t) + \int_{I_{\text{left}}(t)} \chi c_{\text{cut},0}(-c + \kappa)\left(\frac{v_y^2}{2} + V(v)\right) + (c - \kappa)v_y \cdot v_t \, dy \\
+ \int_{I_{\text{right}}(t)} \chi c_{\text{cut},0}(c + \kappa)\left(\frac{v_y^2}{2} + V(v)\right) + (c + \kappa)v_y \cdot v_t \, dy,
$$

thus

$$
E'(t) \leq -\frac{1}{2} D(t) + \int_{I_{\text{left}}(t)} \chi c_{\text{cut},0}(-c + \kappa)\left(\frac{v_y^2}{2} + V(v)\right) + \frac{(c + \kappa)^2}{2}v_y^2 \, dy \\
+ \int_{I_{\text{right}}(t)} \chi c_{\text{cut},0}(c + \kappa)\left(\frac{v_y^2}{2} + V(v)\right) + \frac{(c + \kappa)^2}{2}v_y^2 \, dy.
$$

5.4.4 Definition of the “firewall” functions and bound on the time derivative of energy

We are going to define two firewall functions to control the two last terms of the right-hand side of inequality (84) above. Let us consider the functions $\psi_+(y,t)$ and $\psi_-(y,t)$ (weight functions for those firewall functions) defined by (see figure 28):

$$
\psi_+(y,t) = \begin{cases} 
  \exp(c c_{\text{cut},0}t + \kappa(y - c_{\text{cut},0}t)) & \text{if } y \in I_{\text{left}}(t) \cup I_{\text{main}}(t), \\
  \chi(y,t) & \text{if } y \in I_{\text{right}}(t); \\
\end{cases}
\psi_-(y,t) = \begin{cases} 
  \chi(y,t) & \text{if } y \in I_{\text{left}}(t), \\
  \exp(-c c_{\text{cut},0}t - \kappa(y + c_{\text{cut},0}t)) & \text{if } y \in I_{\text{main}}(t) \cup I_{\text{right}}(t),
\end{cases}
$$
and, for all \( t \) in \([0, +\infty)\), let us define the “firewall” functions \( \mathcal{F}_+(t) \) and \( \mathcal{F}_-(t) \) by:

\[
\mathcal{F}_+(t) = \int_{\mathbb{R}} \psi_+(y, t) \left[ \frac{w_{en}(v_y(y, t))^2}{2} + V(v(y, t) - V(m_+)) + \frac{(v(y, t) - m_+)^2}{2} \right] dy,
\]

\[
\mathcal{F}_-(t) = \int_{\mathbb{R}} \psi_-(y, t) \left[ \frac{w_{en}(v_y(y, t))^2}{2} + V(v(y, t) - V(m_-)) + \frac{(v(y, t) - m_-)^2}{2} \right] dy.
\]

Let us consider the same quantity \( K_{\mathcal{E}} \) as in sub-subsection 4.7.4 on page 33.

**Lemma 36** (energy decrease up to pollution term). *For every nonnegative time \( t \),

\[
(85) \quad \mathcal{E}'(t) \leq -\frac{1}{2} D(t) + K_{\mathcal{E}}(\mathcal{F}_+(t) + \mathcal{F}_-(t)).
\]

*Proof*. According to the definition of \( w_{en,0} \) (see [16] on page 21) and since \( w_{en} \) is not larger than \( w_{en,0} \), these two functions are coercive in the sense that, for all \( t \) in \([0, +\infty)\),

\[
\mathcal{F}_+(t) \geq \min \left( \frac{w_{en}}{2}, \frac{1}{4} \right) \int_{\mathbb{R}} \psi_+(v_y^2 + (v - m_+)^2) \, dy,
\]

and

\[
\mathcal{F}_-(t) \geq \min \left( \frac{w_{en}}{2}, \frac{1}{4} \right) \int_{\mathbb{R}} \psi_-(v_y^2 + (v - m_-)^2) \, dy,
\]

however we will not directly use this property (we will only use directly the positivity of these firewall functions).

Since — according to hypotheses [80] — the quantity \(|c|\) is smaller than \( \kappa \), both factors \(-c + \kappa\) and \(c + \kappa\) in inequality (84) are positive; it thus follows from this inequality that (observe the substitution of \( \chi \) by \( \psi_+ \) and \( \psi_- \) and the added nonnegative terms \(-V(m_+) \) and \(-V(m_-)\))

\[
\mathcal{E}'(t) \leq -\frac{1}{2} D(t) + \int_{\mathcal{I}_{en}(t)} \psi_- \left[ \frac{c_{\text{cut},0}(c + \kappa)}{w_{en}} \left( \frac{w_{en}(v_y^2)}{2} + V(v) - V(m_-) \right) + \frac{(v - m_-)^2}{2} \right] \, dy
\]

\[
+ \int_{\mathcal{I}_{\text{cut},0}(t)} \psi_+ \left[ \frac{c_{\text{cut},0}(-c + \kappa)}{w_{en}} \left( \frac{w_{en}(v_y^2)}{2} + V(v) - V(m_+) \right) + \frac{(v - m_+)^2}{2} \right] \, dy
\]

Thus inequality (85) follows from the last inequality (and from property [16] on page 21 about \( w_{en,0} \) and the fact that \( w_{en} \) is not larger than \( w_{en,0} \)). Lemma 36 is proved. \( \square \)
5.4.5 Time derivative of the firewall function

For all $t$ in $[0, +\infty)$, let

$$\Sigma_{\text{Esc,}+}(t) = \{ y \in \mathbb{R} : |v(y, t) - m_+| > d_{\text{Esc}} \}$$

and

$$\Sigma_{\text{Esc,}-}(t) = \{ y \in \mathbb{R} : |v(y, t) - m_-| > d_{\text{Esc}} \},$$

and let us consider the same quantity $\nu_F$ as in sub-subsection 4.7.5 on page 34.

**Lemma 37** (firewall decrease up to pollution term). There exist a nonnegative quantity $K_F$ such that, for all $t$ in $[0, +\infty)$,

$$F'_\pm(t) \leq -\nu_F F_\pm(t) + K_F \int_{\Sigma_{\text{Esc,}\pm}(t)} \psi_\pm(y, t) \, dy. \tag{86}$$

*Proof.* As for $\psi$ in sub-subsection 4.7.5, according to the definitions of $\psi_+$ and $\psi_-$, the following inequalities hold for all values of their arguments:

$$|\partial_t \psi_\pm| \leq c_{\text{cut},0} (|c| + \kappa) \psi_\pm,$$

$$|c \psi_\pm - \partial_y \psi_\pm| \leq (|c| + \kappa) \psi_\pm,$$

$$\partial_y^2 \psi_\pm - c \partial_y \psi_\pm \leq \kappa (|c| + \kappa) \psi_\pm.$$

Thus, if we consider the quantity $K_F$ defined as follows:

$$K_F = \max_{m \in M, u \in \mathbb{R}^n, |u| \leq R_{\text{att},\infty}} \left( \nu_F \left( w_{\text{en}}(V(u) - V(m)) + \frac{(u - m)^2}{2} \right) + \frac{\lambda_{\min}}{8 \lambda_{\max}} |V(u) - V(m)| - (u - m) \cdot \nabla V(u) + \frac{\lambda_{\min}}{8} (u - m)^2 \right),$$

then inequalities (86) follow from exactly the same computations as in sub-subsection 4.7.5 on page 34; this proves Lemma 37. \qed

5.4.6 Control over the flux term in the time derivative of the firewall function

**Lemma 38** (firewall decrease up to pollution term, continuation). For every nonnegative time $t$,

$$F'_\pm(t) \leq -\nu_F F_\pm(t) + \frac{2 K_F}{\kappa} \exp\left( -\frac{\kappa c_{\text{cut},0}}{2} t \right) \tag{87}.$$

*Proof.* For all $t$ in $[0, +\infty)$, let

$$G_+(t) = \int_{\Sigma_{\text{Esc,}+}(t)} \psi_+(y, t) \, dy \quad \text{and} \quad G_-(t) = \int_{\Sigma_{\text{Esc,}-}(t)} \psi_-(y, t) \, dy.$$

According to the definition of $x_{\text{Esc,}+}(t)$ and $x_{\text{Esc,}-}(t)$, for all $t$ in $[0, +\infty)$,

$$\Sigma_{\text{Esc,}+}(t) \subset (-\infty, y_{\text{Esc,}+}(t)] \cup [y_{\text{hom},+}(t), +\infty)$$

and

$$\Sigma_{\text{Esc,}-}(t) \subset (-\infty, y_{\text{Esc,}-}(t)] \cup [y_{\text{hom},-}(t), +\infty),$$

81
thus, if we consider the quantities

\[ G_{\text{front,}+}(t) = \int_{y_{\text{hom,}+}(t)}^{+\infty} \psi_+(y, t) \, dy \quad \text{and} \quad G_{\text{back,}+}(t) = \int_{-\infty}^{y_{\text{Esc,}+}(t)} \psi_+(y, t) \, dy, \]

\[ G_{\text{front,}-}(t) = \int_{-\infty}^{-\infty} \psi_-(y, t) \, dy \quad \text{and} \quad G_{\text{back,}-}(t) = \int_{y_{\text{Esc,}-}(t)}^{+\infty} \psi_-(y, t) \, dy, \]

then, for all \( t \) in \([0, +\infty)\),

\[ G_{+}(t) \leq G_{\text{front,}+}(t) + G_{\text{back,}+}(t) \quad \text{and} \quad G_{-}(t) \leq G_{\text{front,}-}(t) + G_{\text{back,}-}(t). \]

According to the definition of \( \psi_+ \) and \( \psi_- \) and according to hypotheses (80) and inequalities (82) on page 77 it follows from explicit calculations that (see the comment below):

\[ G_{\text{front,}+}(t) \leq \frac{1}{k} \exp \left( c_{\text{cut,}0}(c + \kappa)t - \kappa y_{\text{hom,}+}(t) \right) \leq \frac{1}{k} \exp \left( -\frac{Kc_{\text{cut,}0}}{2} t \right), \]

\[ G_{\text{back,}+}(t) \leq \frac{1}{k} \exp \left( c_{\text{cut,}0}(c - \kappa)t + \kappa y_{\text{Esc,}+}(t) \right) \leq \frac{1}{k} \exp \left( -\frac{Kc_{\text{cut,}0}}{2} t \right), \]

\[ G_{\text{front,}-}(t) \leq \frac{1}{k} \exp \left( c_{\text{cut,}0}(-c + \kappa)t - \kappa y_{\text{hom,}-}(t) \right) \leq \frac{1}{k} \exp \left( -\frac{Kc_{\text{cut,}0}}{2} t \right), \]

\[ G_{\text{back,}-}(t) \leq \frac{1}{k} \exp \left( c_{\text{cut,}0}(-c - \kappa)t + \kappa y_{\text{Esc,}-}(t) \right) \leq \frac{1}{k} \exp \left( -\frac{Kc_{\text{cut,}0}}{2} t \right). \]

The choice of the value 1/6 as an upper bound for the quotients \(|c|/\kappa\) and \(|c|/c_{\text{cut,}0}\) in hypotheses (80) are indeed convenient to derive these inequalities from inequalities (82). As a consequence inequality (87) follows from from inequality (86). Lemma 38 is proved.

### 5.4.7 Upper bound on time derivative of energy — final form

**Lemma 39** (energy decrease up to pollution term, final form). There exist positive quantities \( K_{E,\text{final}} \) and \( \tilde{\nu}_F \), depending only on \( V \), such that, for every nonnegative time \( t \),

\[ E'(t) \leq -\frac{1}{2} D(t) + K_{E,\text{final}} \exp(-\tilde{\nu}_F t). \]

**Proof.** Let us consider the quantity:

\[ \tilde{\nu}_F = \min\left( \nu_F, \frac{Kc_{\text{cut,}0}}{4} \right). \]

It follows from the inequality (87) that, for all \( t \) in \([0, +\infty)\),

\[ F_\pm(t) \leq \left( F_\pm(0) + \frac{4K_F}{K^2 c_{\text{cut,}0}} \right) \exp(-\tilde{\nu}_F t). \]

According to Lemma 1 on page 18 (existence of an attracting ball for the \( H^1_{ul}(\mathbb{R}, \mathbb{R}^n) \)-norm for the semi-flow), there exists a positive quantity \( K_{F,\text{init}} \), depending only on \( V \), such that, up to changing the origin of times, the following estimates hold:

\[ F_+(0) \leq K_{F,\text{init}} \quad \text{and} \quad F_-(0) \leq K_{F,\text{init}}. \]
Thus, if we consider the following positive quantity:

$$K_{\xi,\text{final}} = 2K_{\xi,\text{init}} + \frac{4K_F}{K_{\text{cut},0}}$$

then inequality (88) follows from inequalities (85) and (89) on page 80 and on page 82. Lemma 39 is proved.

Inequality (88) is the key ingredient that will be applied in the next subsection 5.5.

5.4.8 Time derivative of the dissipation

As in sub-subsection 4.7.8 the following estimates still holds with exactly the same quantity $K_D$ (the computation holds unchanged): for all $t$ in $[0, +\infty)$,

$$\mathcal{D}'(t) \leq K_D\mathcal{D}(t)$$

(again, this step could be avoided by a compactness argument altogether on space and time intervals, as do Gallay and Joly in [8]).

5.5 Lower bound on localized energy

Let us keep the notation and hypotheses of subsections 5.1 and 5.3, together with hypothesis (81) on page 77. For every quantity $c$ sufficiently close to 0 so that hypotheses (80) on page 77 be satisfied, let us denote by

$$v^{(c)}(\cdot, \cdot) \quad \chi^{(c)}(\cdot, \cdot) \quad \mathcal{E}^{(c)}(\cdot) \quad \mathcal{D}^{(c)}(\cdot)$$

the objects that were defined in subsection 5.4 (with the same notation except the “$c$” superscript that is here to remind that these objects depend on the quantity $c$). For every such $c$, let us consider the quantity $\mathcal{E}^{(c)}(+\infty)$ in $\mathbb{R} \cup \{-\infty\}$ defined by:

$$\mathcal{E}^{(c)}(+\infty) = \lim \inf_{t \to +\infty} \mathcal{E}^{(c)}(t),$$

and let us call “asymptotic energy at speed $c$” this quantity. According to estimate (88) above, for every such $c$,

$$\mathcal{E}^{(c)}(t) \to \mathcal{E}^{(c)}(+\infty) \quad \text{when} \quad t \to +\infty.$$

The aim of this subsection is to prove the following proposition.

**Proposition 8** (nonnegative asymptotic energy). The quantity $\mathcal{E}^{(0)}(+\infty)$ (the asymptotic energy at speed zero) is nonnegative.

The proof proceeds through the following lemmas and corollaries, that are rather direct consequences of the relaxation scheme set up in the previous subsection 5.4 and in particular of the estimate (88) on the time derivative of the energy.

Since according to hypothesis (78) on page 76 the maximum of $V(m_+)$ and $V(m_-)$ is assumed to be equal to 0, we may assume (without loss of generality) that:

$$V(m_+) = 0.$$
Lemma 40 (nonnegative asymptotic energy in frames travelling at small nonzero speed). For every quantity $c$ sufficiently close to 0 so that hypotheses [80] on page 77 be satisfied, if in addition $c$ is positive, then

$$E^{(c)}(+\infty) \geq 0.$$ 

Proof. Let $c$ be a positive quantity, sufficiently close to 0 so that hypotheses [80] be satisfied. With the notation of subsection 5.4 (for the relaxation scheme in a frame travelling at speed $c$), for all $t$ in $[0, +\infty)$,

$$E^{(c)}(t) = \int_{\mathbb{R}} \chi^{(c)}(y,t) \left( \frac{v_y^{(c)}(y,t)^2}{2} + V(v^{(c)}(y,t)) \right) dy$$

$$\geq \int_{\mathbb{R}} \chi^{(c)}(y,t)V(v^{(c)}(y,t)) dy$$

$$\geq \int_{\Sigma_{Esc,+}(t)} \chi^{(c)}(y,t)V(v^{(c)}(y,t)) dy.$$ 

Thus, if we consider the global minimum value of $V$:

$$V_{\min} = \min_{u \in \mathbb{R}^n} V(u),$$

then, for all $t$ in $[0, +\infty)$, according to hypotheses [81] on page 77,

$$E^{(c)}(t) \geq V_{\min} \int_{\Sigma_{Esc,+}(t)} \chi^{(c)}(y,t) dy$$

$$\geq V_{\min} \left( \int_{-\infty}^{x_{Esc,+}(t)-ct} \chi^{(c)}(y,t) dy + \int_{x_{hom,+}(t)-ct}^{+\infty} \chi^{(c)}(y,t) dy \right)$$

$$\geq V_{\min} \left( \frac{1}{c} \exp \left( c(x_{Esc,+}(t) - ct) \right) + \frac{1}{\kappa} \exp \left( c_{cut,0}(c + \kappa)t - \kappa(x_{hom,+}(t) - ct) \right) \right)$$

$$\geq V_{\min} \left( \frac{1}{c} \exp \left( c(x_{Esc,+}(t) - ct) \right) + \frac{1}{\kappa} \exp \left( -\frac{\kappa c_{cut,0}}{2} t \right) \right),$$

and the conclusion follows. \qed

Corollary 1 (almost nonnegative energy in a travelling frame). For every quantity $c$ sufficiently close to 0 so that hypotheses [80] on page 77 be satisfied, if in addition $c$ is positive, then, for all $t$ in $[0, +\infty)$,

$$E^{(c)}(t) \geq -\frac{K_{E,final}}{\tilde{\nu}_F} \exp(-\tilde{\nu}_F t).$$

Proof. The proof follows readily from previous Lemma [40] and inequality [88]. \qed

Lemma 41 (continuity of energy with respect to the speed at $c = 0$). For every $t$ in $(0, +\infty)$,

$$E^{(c)}(t) \rightarrow E^{(0)}(t) \quad \text{when} \quad c \rightarrow 0.$$
Proof. For all \( t \) in \((0, +\infty)\),
\[
\mathcal{E}^{(0)}(t) = \int_{\mathbb{R}} \chi^{(0)}(x, t) \left( \frac{u_x(x, t)^2}{2} + V(u(x, t)) \right) \, dx,
\]
and, for every quantity \( c \) sufficiently close to 0 so that hypotheses (80) on page 77 be satisfied (substituting the notation \( y \) used in subsection 5.4 by \( x \)),
\[
\mathcal{E}^{(c)}(t) = \int_{\mathbb{R}} \chi^{(c)}(x, t) \left( \frac{u_x(ct + x, t)^2}{2} + V(u(ct + x, t)) \right) \, dx,
\]
and the result follows from the continuity of \( \chi^{(c)}(\cdot, \cdot) \) with respect to \( c \) and from the bounds \(11\) on the derivatives of \( u(\cdot, \cdot) \).

Corollary 2 (almost nonnegative energy in a standing frame). For all \( t \) in \((0, +\infty)\),
\[
\mathcal{E}^{(0)}(t) \geq - \frac{K_{\varepsilon, \text{final}}}{\tilde{\nu}_F} \exp(-\tilde{\nu}_F t).
\]

Proof. The proof follows readily from Corollary 1 and Lemma 41.

Proposition 8 (“nonnegative asymptotic energy”) follows from Corollary 2.

5.6 End of the proof of Proposition 7

According to the estimate (88) on the time derivative of energy, it follows from Proposition 8 that the map
\[
t \mapsto D^{(0)}(t)
\]
is integrable on \((0, +\infty)\), and according to inequality (90) on the time derivative of the dissipation, it follows that
\[
D^{(0)}(t) \to 0 \quad \text{when} \quad t \to +\infty.
\]
Thus it follows from the bounds \(11\) on page 18 for the derivatives of the solution that
\[
\sup_{x \in [-c_{\text{cut}, 0}, 0]} |u_t(x, t)| \to 0 \quad \text{when} \quad t \to +\infty.
\]
Besides, the fact that the quantity
\[
\sup_{x \in [c_{\text{hom}, -}(t), -c_{\text{cut}, 0}]} |u_t(x, t)|
\]
also approaches 0 when time approaches \(+\infty\) can be derived (for instance) from inequality \(21\) of Lemma 2 on page 23 (“linear stability up to pollution term”). The same argument shows by the way (according to hypothesis \(H_{\text{no-inv}}\)) that, for every positive quantity \( \varepsilon \),
\[
\sup_{x \in [c_{\text{hom}, -}(t), -\varepsilon t]} |u(x, t) - m_-| \to 0 \quad \text{and} \quad \sup_{x \in [\varepsilon t, c_{\text{hom}, +}(t)]} |u(x, t) - m_+| \to 0
\]
}\]

85
when time approaches $+\infty$.

The sole assertion of Proposition 7 that remains to prove is the fact that $V(m_-)$ is nonnegative. But if $V(m_-)$ was negative, then, according to the assertions (91) to (93) above (and according to the bounds (11) on page 18 for the solution), the following estimate would hold:

$$E^{(0)}(t) \sim V(m_-) c_{\text{cut}} t \quad \text{when} \quad t \to +\infty,$$

a contradiction with Proposition 8. The proof of Proposition 7 on page 75 (“non-invasion implies relaxation”) is complete.

6 Proof of Theorem 1

The aim of this section is to combine Propositions 2 and 7 (“invasion implies convergence” and “non-invasion implies relaxation”) to complete the proof of Theorem 1. Not much remains to be said.

Let us assume that the coercivity hypothesis (H$_{\text{coerc}}$) and the generic hypotheses (G) hold for the potential $V$, and let us consider two minimum points $m_-$ and $m_+$ in $\mathcal{M}$ and a bistable initial condition $u_0$ connecting $m_-$ to $m_+$.

For all $t$ in $[0, +\infty)$ and $x \in \mathbb{R}$, let $u(x, t) = (S(t) u_0)(x)$ denote the corresponding solution, and let

$$c_{\text{hom}, +} = c_{\text{no-esc}} + 1 \quad \text{and} \quad x_{\text{hom}, +}(t) = c_{\text{hom}, +} t$$

$$c_{\text{hom}, -} = -(c_{\text{no-esc}} + 1) \quad \text{and} \quad x_{\text{hom}, -}(t) = c_{\text{hom}, -} t.$$

According to the results of subsections 4.4 and 4.5 (namely Lemma 4 on page 26 and inequality (21) on page 25), for every positive quantity $L$,

$$\| y \mapsto u(x_{\text{hom}, +}(t) + y) - m_+ \|_{H^1([-L,L])} \to 0 \quad \text{when} \quad t \to +\infty$$

$$\| y \mapsto u(x_{\text{hom}, -}(t) + y) - m_- \|_{H^1([-L,L])} \to 0 \quad \text{when} \quad t \to +\infty,$$

in other words hypotheses (H$_{\text{hom}}$) of Proposition 7 and (H$_{\text{hom-right}}$) of Proposition 2 (and the symmetric hypothesis on the left) hold for these definitions. Thus we can define, for all $t$ in $[0, +\infty)$, two points $x_{\text{Esc}, +}(t)$ and $x_{\text{Esc}, -}(t)$ and the corresponding asymptotic mean-sup speeds $c_{\text{Esc}, +}$ and $c_{\text{Esc}, -}$ exactly as in subsection 5.1.

At this stage, two cases must be distinguished:

1. “invasion”: $\max(c_{\text{Esc}, +}, |c_{\text{Esc}, -}|)$ is positive;

2. “non-invasion”: $c_{\text{Esc}, +}$ is nonpositive and $c_{\text{Esc}, -}$ is nonnegative.

If the first case “invasion” occurs, then Proposition 2 can be applied (either to the right, or to the left, or on both sides). According to the statements of this proposition, behind the front(s) (to the right, or to the left, or on both sides) approached by the solution, there are “new” points $x_{\text{hom}, +, \text{next}}(t)$ and $x_{\text{hom}, -, \text{next}}(t)$ and new speeds $c_{\text{hom}, +, \text{next}}$ and

86
for which hypothesis \(H_{\text{hom}}\) is again satisfied. Thus the same procedure (definition of new “Escape points” and definition of their asymptotic mean-sup speeds and discussion at above about the signs of these speeds) can be repeated. And it can be repeated again, as long as case 1 (invasion) occurs.

Eventually, case 2 (non-invasion) must occur at some step, since at each “invasion”, the new bistable front approached by the solution replaces a stable homogeneous equilibrium by another one with a lower value of the potential. And according to hypotheses \((H_{\text{coerc}})\) and \((H_{\text{non-deg}})\), the set \(M\) of minimum points of \(V\) is finite.

Then, when case 2 (non-invasion) “finally” occurs, Proposition 7 applies, that is the time derivative \(u_t(x, t)\) of the solution approaches 0 uniformly in the “center” area between the two propagating terraces of travelling fronts, when time approaches \(+\infty\). In this case the results proved in [23] can be applied, and show that the solution approaches in this “center area” a standing terrace of bistable stationary solutions as stated by Theorem 1.

The proof of Theorem 1 is complete.

7 Some properties of the profiles of travelling fronts

Let us assume that \(V\) satisfies the coercivity hypothesis \((H_{\text{coerc}})\) (subsection 2.1 on page 4) and the non-degeneracy hypothesis \((H_{\text{non-deg}})\) (subsection 2.2 on page 4).

7.1 Asymptotic behaviour

Let \(c\) denote a positive quantity, and let us consider the differential system governing the profiles of fronts travelling at speed \(c\):

\[
\phi'' = -c\phi' + \nabla V(\phi).
\]

**Lemma 42** (travelling waves approach critical points). There exists a quantity \(C\), depending only on \(V\), such that, for every bounded global solution \(x \mapsto \phi(x)\) of the differential system (94),

\[
\sup_{x \in \mathbb{R}} |\phi(x)| \leq C,
\]

and there are two critical points \(u_-\) and \(u_+\) of \(V\) such that

\[
\phi(x) \xrightarrow{x \to -\infty} u_- \quad \text{and} \quad \phi(x) \xrightarrow{x \to +\infty} u_+ \quad \text{and} \quad V(u_-) < V(u_+).
\]

**Proof.** The uniform bound follows from the coercivity hypothesis \((H_{\text{coerc}})\), a proof of this bound can be found in [22] (lemma 9). The asymptotics follows readily from La Salle’s principle, and the last assertion is standard.

**Lemma 43** (spatial asymptotics of travelling waves). Let \(m\) be a (local) minimum point of \(V\), and let \(x \mapsto \phi(x)\) be a global solution of the differential system (94) satisfying

\[
|\phi(x) - m| \leq d_{\text{Esc}} \quad \text{for every } x \text{ in } [0, +\infty) \quad \text{and} \quad \phi(\cdot) \not\equiv m.
\]

Then the following conclusions hold.
1. The pair \((\phi(x), \phi'(x))\) approaches \((m, 0)\) when \(x\) approaches \(+\infty\).

2. The supremum \(\sup_{x \in \mathbb{R}} |\phi(x) - m|\) is larger than \(d_{\text{Esc}}\).

3. For all \(x\) in \([0, +\infty)\), the scalar product \(\phi(x) - m \cdot \phi'(x)\) is negative.

4. For all \(x\) in \((0, +\infty)\), the distance \(|\phi(x) - m|\) is smaller than \(d_{\text{Esc}}\).

5. There exists a positive quantity \(C\), depending only on \(V\) and \(c\) (not on \(\phi\)) such that, for all \(x\) in \([0, +\infty)\).

\[
|\phi(x) - m| \leq Ce^{-cx} \quad \text{and} \quad |\phi'(x)| \leq Ce^{-cx}.
\]

Proof. Let us denote by \(D\) the Hessian matrix \(D^2V(m)\). Then the linearization at \((m, 0)\) of system (94) is:

\[
\phi'' = -c\phi' + D\phi \Leftrightarrow \begin{pmatrix} \phi' \\ \psi' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ D & -c \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}.
\]

The set of eigenvalues of the linear system (95) is:

\[
\left\{ \frac{-c \pm \sqrt{c^2 + 4\mu}}{2} : \mu \text{ is an eigenvalue of } D \right\}.
\]

Since every eigenvalue of \(D\) is positive, it follows that every nonpositive eigenvalue of system (95) is smaller than \(-c\). Assertion 5 thus follows from the local stable manifold theorem. A proof of assertions 1 to 4 can be found in [22] (lemma 9). \(\square\)

7.2 Vanishing energy

The following observation was to our knowledge first made by Muratov (proposition 3.3 of [14]). For sake of completeness, a proof is given below.

Lemma 44 (weighted energy of a travelling front vanishes). For every minimum point \(m\) of \(V\), every positive quantity \(c\), and every function \(\phi\) in the set \(\Phi_c(m)\) of profiles of fronts travelling at speed \(c\) and “invading” the stable equilibrium \(m\), the weighted energy

\[
\int_{\mathbb{R}} e^{cx} \left( \frac{\phi'(x)^2}{2} + V(\phi(x)) - V(m) \right) dx,
\]

is a convergent integral and its value is 0.

Proof. Let us take a point \(m\) in the set \(\mathcal{M}\) of minimum points of \(V\), a positive quantity \(c\), and a function \(\phi\) in the set \(\Phi_c(m)\) of profiles of fronts travelling at speed \(c\) and “invading” the stable equilibrium \(m\), and let us consider the function

\[
x \mapsto \mathcal{E}_{c, \phi}(x) = \int_{\mathbb{R}} e^{c(y-x)} \left( \frac{\phi'(y)^2}{2} + V(\phi(y)) - V(m) \right) dy = e^{-cx}\mathcal{E}_{c, \phi}(0).
\]

88
According to Lemma 43 above the integral converges thus this function is defined for all $x$ in $\mathbb{R}$. Let us consider the derivative $E'_{c,\phi}(x)$ of this function with respect to $x$. On the one hand this quantity obviously equals $-ce^{-cx}E_{c,\phi}(0)$, and on the other hand, since

$$ E_{c,\phi}(x) = \int_{\mathbb{R}} e^{c(y)} \left( \frac{\phi'(x+y)^2}{2} + V(\phi(x+y)) - V(m) \right) dy, $$

differentiating under the integral we get

$$ E'_{c,\phi}(x) = \int_{\mathbb{R}} e^{c(y)} \left( \phi'(x+y) \cdot \phi''(x+y) + \nabla V(\phi(x+y)) \cdot \phi'(x+y) \right) dy $$

thus according to the differential system (94) the expression below this integral actually equals

$$ \frac{d}{dy} \left( e^{cy} \phi'(y)^2 \right). $$

As a consequence $E'_{c,\phi}(\cdot)$ vanishes, and since $c$ is nonzero this proves the desired result. □

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