SPLITTING BRAUER CLASSES USING THE UNIVERSAL ALBANESE  
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ABSTRACT. We prove that every Brauer class over a field splits over a torsor under an 
abelian variety. If the index of the class is not congruent to 2 modulo 4, we show that 
the Albanese variety of any smooth curve of positive genus that splits the class also 
splits the class, and there exist many such curves splitting the class. We show that 
this can be false when the index is congruent to 2 modulo 4, but adding a single genus 
1 factor to the Albanese suffices to split the class.

1. INTRODUCTION

Our main goal in this note is to construct torsors under abelian varieties that split 
Brauer classes over fields. For a variety $X$ over a field $K$ and a Brauer class $\alpha \in \text{Br}(K)$, 
let $\alpha_X$ denote the pullback of $\alpha$ to $\text{Br}(X)$. We say that $X$ splits $\alpha$ if $\alpha_X$ is trivial, or 
equivalently, if there is a rational map from $X$ to any Brauer-Severi variety with 
cohomology class $\alpha$.

**Theorem 1.0.1.** Given a field $K$ and a Brauer class $\alpha \in \text{Br}(K)$, there exists a torsor $T$ 
under an abelian variety over $K$ such that $\alpha_T = 0$. Equivalently, the torsor $T$ admits a 
 rational map to any Brauer-Severi variety $V$ associated to $\alpha$.

In fact, we show that there are many such torsors splitting a given Brauer class 
by studying Albanese varieties. For a smooth proper geometrically connected curve $C$ 
over $K$, let $C \to \text{Alb}_C := \text{Pic}^1_{C/K}$ denote the Albanese morphism for $C$, so the Albanese 
variety $\text{Alb}_C$ is a torsor under the Jacobian variety $\text{Jac}_C := \text{Pic}^0_{C/K}$.

**Theorem 1.0.2.** Let $K$ be a field and $\alpha \in \text{Br}(K)$. Write $\text{ind}(\alpha)$ for the index of $\alpha$.

1. If $\text{ind}(\alpha) \not\equiv 2 \pmod{4}$, then for any smooth proper geometrically connected curve $C$ 
over $K$ of positive genus such that $\alpha_C = 0$, we have that $\alpha_{\text{Alb}_C} = 0$.

2. If $\text{ind}(\alpha) \equiv 2 \pmod{4}$, then for any smooth proper geometrically connected curve $C$ 
over $K$ of positive genus such that $\alpha_C = 0$, there exists a genus 1 curve $C'$ over 
$K$ such that $\alpha_{C' \times \text{Alb}_C} = 0$.

Note that there are many curves splitting any given Brauer class $\alpha$ of index $m$. For 
example, for $m \geq 3$, any complete intersections of $m - 2$ sections of the anticanonical sheaf 
of an associated $(m - 1)$-dimensional Brauer-Severi variety will split the class, 
and a general such complete intersection is a smooth proper geometrically connected 
curve by Bertini’s theorem. (This argument needs to be slightly tweaked if $K$ is finite, 
but in that case the theorems above are trivial since $\text{Br}(K) = 0$.) Thus, Theorem 1.0.1 
follows immediately from Theorem 1.0.2.

This result grew out of considering the following well-known question.

**Question 1.0.3.** Given a field $K$ and a Brauer class $\alpha \in \text{Br}(K)$, is there a genus 1 curve 
$C$ over $K$ such that $\alpha_C = 0$?
This question was asked explicitly by Pete L. Clark on his website and in [RV11], and an affirmative answer was given when $\alpha$ has index 3 by Swets [Swe95], index at most 5 in [dJH12], and index 6 under some assumptions on $K$ by Auel [Aue15]. The techniques in [dJH12, Aue15] are unlikely to generalize to higher index, however, and the general question seems quite difficult.

To prove Theorem 1.0.2, we use the basic theory of big monodromy to deform a curve splitting the class $\alpha$ to a curve whose Jacobian has minimal Néron-Severi group. When the Néron-Severi group is minimal, it is relatively easy to compare obstruction classes for sections of the Picard scheme of the curve with sections of the Picard scheme of its Albanese, giving the result for the general curve. Specializing back to the Albanese of the original curve finishes the proof.

Assumption 1.0.4. Since the Brauer group of a finite field is trivial, we assume from now on that $K$ is an infinite field.

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2. The Jacobian of the General Curve

Our overarching goal in this section is to describe a proof of the following folk theorem. Write $\mathcal{M}_g$ for the stack of smooth proper geometrically connected curves of genus $g \geq 2$. This is a smooth algebraic stack over $\text{Spec} \mathbb{Z}$ with irreducible geometric fibers (see, e.g., [DM69]).

Proposition 2.0.1. Suppose $C$ is a smooth proper geometrically connected curve of genus $g$ over an algebraically closed field $k$ such that the induced map $\text{Spec} \ k \to \mathcal{M}_g$ sends the point of $\text{Spec} \ k$ to the generic point of a fiber of $\mathcal{M}_g \to \text{Spec} \mathbb{Z}$. Then we have that $\text{NS}(\text{Jac}_C) = \mathbb{Z}\Theta$, where $\Theta$ is the class of the $\Theta$-divisor on $\text{Jac}_C$.

The proof can be achieved using the theory of big monodromy in the Hodge or $\ell$-adic context. The $\ell$-adic proof applies in all characteristics, while the Hodge-theoretic proof only applies in characteristic 0. Since we assume that the reader may not be intimately familiar with the theory, we briefly sketch both arguments here. The results that prove this Proposition are Corollary 2.3.3 and Corollary 2.4.2 below. There is also a direct proof using results of Zarhin [Zar00, Zar04] (building on earlier work of Mori); see Section 2.5.
2.1. Representation theory and notation. We will use the following notation and results in this section.

1. Given a ring $R$, we write $\text{Sp}(2g, R)$ for the symplectic group associated to the standard symplectic form of dimension $2g$. We write $\text{GL}(n, R)$ for the algebraic general linear group over $R$ (not just the $R$-points). Given a field $L$ and an $L$-vector space $V$, we will write $\text{GL}(V)$ for the algebraic group of automorphisms of $V$ (not just the $L$-points). This is non-canonically isomorphic to $\text{GL}(\text{dim}_L V, L)$.

2. Given an abstract group $\pi$ and a representation $\rho: \pi \to \text{GL}(n, L)$ over a field $L$, we will write $G(\rho) \subset \text{GL}(n, L)$ for the connected component of the identity of the Zariski closure of the image of $\rho$. If $V \to B$ is a local system of $L$-vector spaces on a topological space, we will write $G(V)$ for $G(\rho)$, where $\rho: \pi_1(B, b) \to \text{GL}(V)$ is the monodromy representation attached to $V$ and $V = V_b$.

3. Let $V$ be a vector space over a field $L$ of dimension $2g$ equipped with the standard symplectic form. The pairing defines an $\text{Sp}(2g, L)$-invariant map $\wedge^2 V \to L$. The kernel $V_2 \subset \wedge^2 V$ of the pairing map is an absolutely irreducible representation of $\text{Sp}(2g, L)$; see [FH91, Theorem 17.5].

2.2. The Néron-Severi sheaf. Suppose $X \to \text{Spec} \, k$ is a smooth proper geometrically connected variety over a field $k$. (The theory we describe here generalizes, but we avoid such a digression.) Let $\text{Pic}_{X/k}$ be the Picard scheme of $X$ over $k$ and $\text{Pic}_{X/k}^0$ its connected component. When $X$ is a curve, the Jacobian variety $\text{Jac}_X$ is identified with $\text{Pic}_{X/k}^0$.

**Definition 2.2.1.** The Néron-Severi sheaf of $X$ is the fppf quotient sheaf

$$\text{NS}_{X/k} = \text{Pic}_{X/k} / \text{Pic}_{X/k}^0.$$ 

By construction, the sheaf $\text{NS}_{X/k}$ is representable by an étale group scheme over $k$. (The key point is that the tangent space at the identity section is trivial, which one can see using the fact that $\text{NS}_{X/k}$ is also the sheaf of connected components of $\text{Pic}_{X/k}$.) The classical Néron-Severi group $\text{NS}(X)$ is defined as $\text{Pic}(X) / \text{Pic}^0(X)$. With this notation, we see that $\text{NS}(X) = \text{NS}_{X/k}(k)$ if $k$ is algebraically closed.

**Observation 2.2.2.** Since $\text{NS}_{X/k}$ is étale, we have that for any extension $L \subset L'$ of separably closed extension fields of $k$, the induced map $\text{NS}_{X/k}(L) \to \text{NS}_{X/k}(L')$ is an isomorphism. In particular, given a separable closure $k^s$ contained in an algebraic closure $\bar{k}$ of $k$, we have $\text{NS}_{X/k}(k^s) \cong \text{NS}_{X/k}(\bar{k})$. In particular, any Néron-Severi class on $X_{\bar{k}}$ is defined over some finite separable extension of $k$. Under the additional assumption that $\text{Pic}_{X/k}^0$ is smooth (e.g., for $X$ a curve or an abelian variety), we have that any Néron-Severi class defined over $k^s$ is induced by an invertible sheaf on $X \otimes_k k^s$. (Indeed, in this case the fppf and étale cohomology of $\text{Pic}_{X/k}^0$ agree by Grothendieck’s theorem [Gro68c, Théorème 11.7], so we have that $H^1_{\text{fppf}}(\text{Spec} \, k^s, \text{Pic}_{X/k}^0) = 0$.)

2.3. Big monodromy: $\ell$-adic realization. We first show that the Néron-Severi group of the Jacobian of the geometric generic fiber for a curve is generated by the class of the $\Theta$-divisor if the relevant Galois representation has large image. We then use results of Katz–Sarnak to find families of curves over finite fields with large monodromy.
Proposition 2.3.1. Let $k$ be any field. Fix a prime $\ell$ invertible in $k$ and assume $k$ contains all $\ell$-power roots of unity, and fix an isomorphism $\overline{Q}_\ell \cong Q(1)$ of Galois modules. Let $C$ over $k$ be a smooth proper geometrically connected curve of genus $g$ such that the identity component of the Zariski closure of the image of the Galois representation

$$\rho_0 : \text{Gal}(\overline{F}/k) \to \text{Sp}(2g, \overline{Q}_\ell)$$

induced by the Galois action on $H^1(C, \overline{Q}_\ell)$ is all of $\text{Sp}(2g, \overline{Q}_\ell)$. Then the geometric generic fiber $C := C_\overline{F}$ has the property that $\text{NS}(\text{Jac}_C) = \mathbb{Z}\Theta$.

Proof. The first Chern class defines a morphism

$$c : \text{NS}(\text{Jac}_C) \to H^2(\text{Jac}_C, \overline{Q}_\ell),$$

which is injective modulo torsion. (Note that we may ignore Tate twists in this proof since we fixed an isomorphism $\overline{Q}_\ell \cong Q(1)$ above.) Because any class of $\text{NS}(\text{Jac}_C)$ is defined over a finite separable extension of $k$ by Observation 2.2.2, the image of $c$ is contained in the union of all the subspaces $H^2(\text{Jac}_C, \overline{Q}_\ell)\Gamma$, where $\Gamma$ ranges over open subgroups of the absolute Galois group $G_k := \text{Gal}(\overline{F}/k)$.

By assumption $G(\rho_0) = \text{Sp}(2g, \overline{Q}_\ell)$, but also for any open subgroup $\Gamma \subset G_k$, the group $G(\rho_0|\Gamma) = \text{Sp}(2g, \overline{Q}_\ell)$, since passage to finite index subgroups does not change the identity component of the Zariski closure.

The cup product pairing defines a Galois-invariant map

$$p : H^2(\text{Jac}_C, \overline{Q}_\ell) = \bigwedge^2 H^1(\text{Jac}_C, \overline{Q}_\ell) = \bigwedge^2 H^1(C, \overline{Q}_\ell) \to \overline{Q}_\ell$$

with kernel $V$. For an open subgroup $\Gamma \subset G_k$, the space $H^2(\text{Jac}_C, \overline{Q}_\ell)^\Gamma \cap V$ is a subspace stable under $\Gamma$, hence under $G(\rho_0|\Gamma) = \text{Sp}(2g, \overline{Q}_\ell)$. Since $V$ is an irreducible representation of $\text{Sp}(2g, \overline{Q}_\ell)$ by Section 2.1.3, the intersection must be 0.

We thus find that the composition map

$$\text{NS}(\text{Jac}_C) \otimes \overline{Q}_\ell \overset{c}{\rightarrow} \bigcup_{\Gamma \subset G_k} H^2(\text{Jac}_C, \overline{Q}_\ell)^\Gamma \hookrightarrow H^2(\text{Jac}_C, \overline{Q}_\ell) \overset{p}{\rightarrow} \overline{Q}_\ell$$

is an isomorphism (since $\text{NS}(\text{Jac}_C)$ is not 0).

As a consequence, we have an injection $\text{NS}(\text{Jac}_C) \hookrightarrow \text{NS}(C)$. It is well known that the pullback of the $\Theta$-divisor class to $C$ has degree $g$ (see, e.g., [Pol03, Theorem 17.4]), and since $\Theta$ is a principal polarization, it is indivisible in $\text{NS}(\text{Jac}_C)$. We therefore have $\text{NS}(\text{Jac}_C) = \mathbb{Z}\Theta$; furthermore, the injection $\text{NS}(\text{Jac}_C) \hookrightarrow \text{NS}(C)$ is identified with $\mathbb{Z} \to g\mathbb{Z} \hookrightarrow \mathbb{Z}$. \qed

In [KS99, Chapter 10], Katz and Sarnak produce families of curves over finite fields with large monodromy groups. Recall that, given a family of curves $f : C \to B$ over a finite field $F_q$ with $\ell$ an invertible prime, the geometric monodromy group $G_{\text{geom}}$ of $f$ is the $\overline{Q}_\ell$-algebraic group $G(\rho)$ associated to the representation

$$\rho : \pi_1(B \otimes_{F_q} \overline{F}_q) \to \text{Sp}(2g, \overline{Q}_\ell)$$

attached to the lisse sheaf $R^1f_*\overline{Q}_\ell$ (as in Section 2.1.2). The following theorem is a summary of [KS99, Theorem 10.1.16 and Theorem 10.2.2].
**Theorem 2.3.2** (Katz–Sarnak). For any genus $g$ and any finite field $\mathbf{F}_q$ with $\ell$ an invertible prime, there is an open subset $U \subset \mathbf{A}_{\mathbf{F}_q}$ and a family $\mathcal{C} \to U$ of smooth proper geometrically connected genus $g$ curves such that $G_{\text{geom}} = \text{Sp}(2g, \mathbf{Q}_\ell)$.

**Corollary 2.3.3.** Let $\mathcal{C} \to U$ be a family as in Theorem 2.3.2. The geometric generic fiber $\overline{\mathcal{C}} := \mathcal{C} \times_U \text{Spec} \mathbf{F}_q(t)$ has the property that $\text{NS}(\text{Jac}_{\overline{\mathcal{C}}}) = \mathbf{Z} \Theta$.

**Proof.** Let $k = \overline{\mathbf{F}}_q(t)$. Theorem 2.3.2 gives that $G_{\text{geom}} = G(\rho) = \text{Sp}(2g, \mathbf{Q}_\ell)$, and the natural surjection $G_k \cong \pi_1(\text{Spec} k) \to \pi_1(U \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q)$, implies that the composition map $\rho_0 : G_k \to \pi_1(U \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q) \to \text{Sp}(2g, \mathbf{Q}_\ell)$ also has the property that $G(\rho_0) = \text{Sp}(2g, \mathbf{Q}_\ell)$. We thus apply Proposition 2.3.1 to obtain the desired result.

**2.4. Big monodromy: Hodge realization.** We give a briefer sketch of the Hodge version of the argument, as it seems to be more widespread in the literature. For example, [BL04, Theorem 17.5.2] and the discussion leading up to it are a valuable source.

Let $V$ be a local system on a connected space $B$ with monodromy representation $\rho : \pi_1(B, b) \to \text{GL}(V)$, where $V = V_b$. Recall that $V$ is said to have big monodromy if $G(V)$ acts irreducibly on $V_C$.

Given a family of principally polarized abelian varieties $g : A \to B$, the polarization defines a quotient sheaf $R^2g_*\mathbf{Q} = \bigwedge^2 R^1g_*\mathbf{Q} \to \mathbf{Q}$. The kernel of this map is a local system $V_2(A)$. Fiberwise, it is invariant under the symplectic group and is itself an irreducible representation (Section 2.1[3]). We will say that $g : A \to B$ has big monodromy for $H^2$ if $V_2(A)$ has big monodromy.

**Lemma 2.4.1.** Let $B$ be a smooth $\mathbf{C}$-scheme. If $f : X \to B$ is a family of smooth proper curves with $G(\text{R}^1f_*\mathbf{Q}) = \text{Sp}(2g, \mathbf{Q})$, then the Jacobian family $g : \text{Jac}_X = \text{Pic}^0_{X/B} \to B$ has big monodromy for both $H^1$ and $H^2$.

**Proof.** Recall that $R^1g_*\mathbf{Q} = R^1f_*\mathbf{Q}$ as local systems, so we can identify $R^2g_*\mathbf{Q}$ with $\bigwedge^2 R^1g_*\mathbf{Q}$. The rest follows from the definitions.

**Corollary 2.4.2.** Suppose $B$ is a smooth $\mathbf{C}$-scheme. If $f : X \to B$ is a smooth proper family of curves with $G(\text{R}^1f_*\mathbf{Q}) = \text{Sp}(2g, \mathbf{Q})$, then for any very general $b \in B(\mathbf{C})$, the Néron-Severi group $\text{NS}((\text{Jac}_{X_b}))$ is isomorphic to $\mathbf{Z}$ and generated by the theta divisor $\Theta$.

**Proof.** As the Néron-Severi group for abelian varieties is torsion-free, it is enough to show the result after tensoring with $\mathbf{Q}$. For any point $b \in B(\mathbf{C})$, the Néron-Severi group $\text{NS}((\text{Jac}_{X_b})) \otimes \mathbf{Q}$ is a trivial $\mathbf{Q}$-Hodge structure contained in the $\mathbf{Q}$-Hodge structure $H^2(\text{Jac}_{X_b}, \mathbf{Q})$. (In fact, by the Lefschetz (1,1) theorem, it is the maximal such structure.) By [PS08, Theorem 10.20], for a very general point $b$, the sub-Hodge structure $(V_2)_b \subset H^2(\text{Jac}_{X_b}, \mathbf{Q})$ has no rational Hodge substructures. By the exact sequence

$$0 \to (V_2)_b \to H^2(\text{Jac}_{X_b}, \mathbf{Q}) \to \mathbf{Q} \to 0,$$

it follows that $\text{NS}((\text{Jac}_{X_b})) \otimes \mathbf{Q}$ is one-dimensional.

**Corollary 2.4.3.** Let $k$ be a field of characteristic 0 and $X \to \text{Spec} k$ be a smooth proper geometrically connected curve such that the image of the induced map $\text{Spec} k \to \mathcal{M}_g$ is the generic point. Then we have that $\text{NS}(\text{Jac}_{X_k}^{-}) = \mathbf{Z}\Theta$. 
Proof. The statement is invariant under extension of $k$. It is well known that for the universal curve over $\mathcal{M}_g$ the geometric monodromy group attached to $H^1$ is $Sp(2g)$; see, e.g., [DM69, §5] or [FM12, Theorem 6.4]. Thus, if $m$ is a very general complex point of $\mathcal{M}_g$ corresponding to a curve $C$, then by Corollary 2.4.2 we have that $\text{NS}(\text{Jac}_C) = \mathbb{Z}\Theta$. Since Néron-Severi groups can only grow under specialization, it follows that the geometric generic point must have the same property, giving the desired result. □

2.5. Appeal to example. Since Néron-Severi rank can only increase under specialization, another proof of Proposition 2.0.1 follows from showing that there exists a single curve of every genus at least 2, in any characteristic, whose Jacobian has Néron-Severi rank exactly 1. Results of Zarhin [Zar00, Zar04] on endomorphism rings of hyperelliptic Jacobians imply that many such curves exist over most characteristics; in fact, Zarhin shows that for any hyperelliptic curve $C : y^2 = f(x)$, where $f(x)$ is an irreducible separable degree $n \geq 5$ polynomial with Galois group either $S_n$ or $A_n$, in any characteristic $p > 3$, the endomorphism ring over the algebraic closure is $\mathbb{Z}$, implying that $\text{NS}(\text{Jac}_C)$ also has rank 1. These results themselves are quite subtle and rely on very different techniques than those sketched in this paper.

2.6. General deformations of smoothable curves. In this section we describe how to put any smoothable curve in a family with the generic curve. This will be useful for studying the splitting of Brauer classes, as we explain in Section 3.0.1. (We will only apply this to smooth curves, but we suspect that the full statement for smoothable curves may be useful in the future, so we record it here.)

Corollary 2.6.1. Suppose $C$ is a smoothable proper geometrically connected curve over a field $K$ such that $\text{Ext}^2(L_{C/K}, \Theta_C) = 0$ (for example, a proper nodal curve). Let $W$ be a complete DVR with residue field $K$. There is a proper flat family $\mathcal{C} \to \text{Spec} W$ such that

1. $\mathcal{C} \otimes W K \cong C$, and
2. if $\eta \to \text{Spec} W$ is a geometric generic point, then we have $\text{NS}(\text{Jac}_{\mathcal{C}_\eta}) = \mathbb{Z}\Theta$, where $\Theta$ is the usual theta-divisor class associated to $\mathcal{C}_\eta$.

Proof. Write $Q$ for the fraction field of $W$. The assumptions on $C$ ensure that the universal formal deformation of $C$ is represented by a proper morphism of schemes $\mathcal{C} \to \text{Spec} W[x_1, \ldots, x_n]$ with a smooth fiber. Since the stack of proper curves is an Artin stack locally of finite presentation, Artin’s algebraization theorem tells us that there is a pair $(X, x)$ with $X$ a smooth $W$-scheme and $x \in X(K)$ and a family $\mathcal{C} \to X$ such that the restriction of $\mathcal{C} \to X$ to $\hat{\mathcal{O}}_{X,x}$ is isomorphic to the universal deformation. Let $X^\circ$ denote the locus over which $\mathcal{C}$ is smooth and let $F$ be the prime field of $Q$. There is an induced map $\mu : X^\circ \to \mathcal{M}_{g,F}$, and by the openness of versality we know that this map is dominant.

By Proposition 2.0.1 it suffices to show that there is a map $\text{Spec} W \to X$ whose closed point lands at $x$ and whose generic point maps to the generic point of $\mathcal{M}_{g,F}$. Since $F$ is countable, $\mathcal{M}_{g,F}$ has only countably many closed substacks. Consider the polycylinder $B^n = \{(a_0, \ldots, a_n) | a_i \in Q, |a_i| < 1\}$ parametrizing all $W$-points of $\hat{\mathcal{O}}_{X,x}$. Since $\mu$ is dominant, no closed subscheme of $\mathcal{M}_{g,F}$ contains all of $B^n$. Since $Q$ is uncountable it follows (e.g., by induction on $n$) that there is a point of $B^n$ not in the pullback of any closed substack of $\mathcal{M}_{g,F}$. This gives a $W$-point of $X$ with the desired properties. □
3. Proof of Theorem 1.0.2

Lemma 3.0.1. Suppose $C$ is a smooth proper geometrically connected curve of genus $g$ over a field $K$ and $C \to X$ is an Albanese morphism. Suppose further that the morphism $\text{NS}(X_K) \to \text{NS}(C_K)$ is injective with image $g\text{NS}(C_K)$. If $\alpha \in \text{Br}(K)$ is a class of order prime to $g$ such that $\alpha_C = 0$, then $\alpha_X = 0$.

Proof. By the Leray spectral sequence, we have $\alpha_C = 0$ if and only if $\alpha$ is the obstruction class for a global section of $\text{Pic}_{C/K}$. Consider the diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & \text{Pic}_X^0 & \longrightarrow & \text{Pic}_X & \longrightarrow & \text{NS}_X & \longrightarrow & 0 \\
\downarrow & & \cong & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Pic}_C^0 & \longrightarrow & \text{Pic}_C & \longrightarrow & \text{NS}_C & \longrightarrow & 0
\end{array}
$$

of sheaves on $\text{Spec} K$. The Snake Lemma applied to the diagram yields an exact sequence of sheaves

$$
0 \to \text{Pic}_X^0 \to \text{Pic}_C^0 \to \mathbb{Z}/g\mathbb{Z} \to 0.
$$

Thus, the map on global sections

$$
\text{Pic}_X(K) \to \text{Pic}_C(K)
$$

is injective with cokernel annihilated by $g$. Since $g$ is prime to the order of $\alpha$ and the obstruction map for sections of the Picard scheme is a group homomorphism, we see that $\alpha_X = 0$ if and only if $g\alpha_X = 0$. If $s \in \text{Pic}_C(K)$ is a section with obstruction $\alpha$, then the preimage of $gs$ in $\text{Pic}_X(K)$ is a section with obstruction $g\alpha$, and the desired result follows.

□

Proof of Theorem 1.0.2. Suppose $C$ is a smooth proper geometrically connected curve over $K$ of genus $g \geq 1$ such that $\alpha_C = 0$. If $g = 1$, the desired conclusion holds because $\text{Alb}_C = C$. Thus, we assume $g \geq 2$. We will now show that to prove the Theorem it suffices to prove that $\text{Alb}_C$ splits $\alpha$ under the additional assumption that $g$ is relatively prime to the order of $\alpha$.

Observe that $\text{ind}(\alpha)$ divides $2g - 2 - 2(g - 1)$, since the canonical divisor of $C$ has degree $2g - 2$. As a result, all odd divisors of $\text{ind}(\alpha)$ visibly cannot divide $g$, and if $4|\text{ind}(\alpha)$, then $g$ is odd. Hence, if $\text{ind}(\alpha) \equiv 2 \pmod{4}$, we find that $g$ is relatively prime to $\text{ind}(\alpha)$. If $\text{ind}(\alpha) \equiv 2 \pmod{4}$, then we can write $\alpha = \alpha_2 + \alpha'$ with $\alpha_2$ of index 2 and $\alpha'$ of odd order. Since $\alpha_2$ is split by a conic (namely, the Brauer-Severi variety of the associated division algebra) and any conic admits a cover by a genus 1 curve (namely, the branched cover over a general divisor of degree 4), we see upon taking products that it suffices to prove that $\text{Alb}_C$ splits $\alpha'$ (which has odd index relatively prime to $g$).

Let $W$ be a complete dvr with residue field $K$. By [Gro68a, Corollary 6.2], since $W$ is a complete dvr, there is a unique Brauer class $\tilde{\alpha} \in \text{Br}(W)$ lifting $\alpha$; the lifted class has the same period and index as $\alpha$. By Corollary 2.6.1, there is a family $\mathcal{C} \to \text{Spec} W$ such that the generic fiber $\mathcal{C}_\eta$ of $\mathcal{C}$ satisfies the conditions of Lemma 3.0.1. Since $\alpha_C = 0$, the deformation theory of invertible twisted sheaves as in [Lie08] (or [Tat68, Theorem 3.2], which is recorded without proof) tells us that $\tilde{\alpha}_{\mathcal{C}_\eta} = 0$. By Lemma 3.0.1,
we have that $\tilde{\alpha}_{\text{Alb}\,\psi_\eta} = 0$. Since $\text{Alb}_\psi/W$ is regular, it follows (for example, [Gro68b, Corollary 1.10]) that $\tilde{\alpha}_{\text{Alb}\,\psi/W} = 0$. Thus, $\alpha_{\text{Alb}_\psi} = 0$ by specialization.\footnote{Instead of using a very general $W$-point of the universal deformation ring of $C$ over $K$, we could just use the universal deformation directly. Our approach avoids enlarging the fraction field of $W$ at the expense of a small amount of extra work.}

**Antieau and Auel’s proof of Theorem 1.0.2.** A different proof of Theorem 1.0.2 due to Benjamin Antieau and Asher Auel [AA18], uses results on the stable birational geometry of symmetric powers of Brauer–Severi varieties to show that an appropriate symmetric power of a curve splitting a Brauer class also splits the class. Here is a sketch of their proof; more details may appear elsewhere in the future.

Suppose $C$ is a smooth proper geometrically connected curve over $K$ of genus $g \geq 1$ such that $\alpha_C = 0$. As in the first proof, we may reduce to showing that $\text{Alb}_C$ splits $\alpha$ under the assumptions that $\alpha$ is non-trivial and that $g$ is relatively prime to $\text{ind}(\alpha)$.

The image of $C$ in a Brauer–Severi variety $V$ associated to $\alpha$ cannot be a point, so the image of the induced map $\text{Sym}^{2g-1} C \to \text{Sym}^{2g-1} V$ intersects the smooth locus of $\text{Sym}^{2g-1} V$. By [Kol18, Theorem 1 (4)] (see also [KS04]), the space $\text{Sym}^{2g-1} V$ is stably birational to $\text{Sym}^m V$, where $m = \gcd(2g - 1, \text{ind}(\alpha))$; here, we have $m = 1$ since $\text{ind}(\alpha)$ divides $2g - 2$. Thus, the smooth locus $U$ of $\text{Sym}^{2g-1} V$ is stably birational to $V$ and splits $\alpha$, which implies that $\text{Sym}^{2g-1} C$ also splits $\alpha$.

By the Riemann-Roch theorem, we have that $\sigma: \text{Sym}^{2g-1} C \to \text{Pic}^{2g-1}_C/K \cong \text{Alb}_C$ is a Brauer–Severi scheme of relative dimension $g - 1$. Since $\sigma^* \alpha_{\text{Alb}_C} = 0$, the class $\alpha_{\text{Alb}_C}$ is $g$-torsion. But $\alpha_{\text{Alb}_C}$ is also killed by $\text{ind}(\alpha)$, so the assumption that $g$ is relatively prime to $\text{ind}(\alpha)$ implies that $\alpha_{\text{Alb}_C} = 0$, as desired.

4. **The Conditions of Theorem 1.0.2 Are Necessary**

In this section, we show that there are many examples of Brauer classes $\alpha$ with index congruent to 2 modulo 4 that split on a curve $C$ but not on $\text{Alb}_C$. These examples are easily constructed over local fields (and hence over many finitely generated fields, by standard approximation techniques), and we suspect one could also make similar examples over number fields.

Given a smooth proper geometrically connected curve $C$ over a field $K$, recall that the index of $C$ is the smallest degree of a divisor on $C$, and the period of $C$ is the smallest degree of a divisor class. (Equivalently, the period of $C$ is the order of the Albanese variety $\text{Alb}_C = \text{Pic}^1_{C/K}$ in $\text{H}^1(\text{Spec} K, \text{Jacc}_C)$.)

**Lemma 4.0.1.** Suppose $C$ is a curve over a field $K$ and $\alpha \in \text{Br}(K)$ is a class of order 2. If $\alpha_{\text{Pic}_C} = 0$, then $\alpha_{\text{Pic}^m_C} = 0$ for any odd number $m$.

**Proof.** The $m$th power map on the Picard stack descends to a morphism

$$\lambda_m: \text{Pic}^1_C \to \text{Pic}^m_C$$

that is an étale form of the multiplication by $m$ on $\text{Jacc}_C$. In particular, $\lambda_m$ is finite flat of degree $m^{2g}$, where $g$ is the genus of $C$. By assumption, the class $\alpha_{\text{Pic}^m_C}$ vanishes upon pullback along the morphism $\lambda_m$ of odd degree. By standard calculations in Galois cohomology (see, for example, [Lie08, Proposition 4.1.1.1]), this implies that $\alpha_{\text{Pic}^m_C} = 0$, as desired. \qed
Proposition 4.0.2. Let $m$ be an odd positive integer and suppose $C$ is a smooth proper geometrically connected curve over a local field $K$ of index $2m$, period $m$, and genus $m + 1$. Then the unique non-zero Brauer class $q \in \text{Br}(K)[2]$ is killed by $C$ but not by $\text{Alb}_C$. Thus, there are Brauer classes $\alpha$ of all even indices dividing $2m$ that are killed by $C$ but not by $\text{Alb}_C$.

Proof. Since $m$ is odd, any class $\alpha$ in $\text{Br}(K)[2m]$ can be written as $q + h$ with $h \in \text{Br}(K)[m]$. The relative Brauer group $\text{Br}(C/K)$ is precisely $\text{Br}(K)[2m]$ by the theorem of Roquette–Lichtenbaum [Lic69, Theorem 3]. In addition, period and index are equal over a local field. Thus, to prove the full statement, it suffices to prove the first part.

From Roquette–Lichtenbaum, we have that $q = 0$. If $q = 0$ also, then $q = 0$ by Lemma 4.0.1. Since $C$ has period $m$, there is a $K$-point of $\text{Pic}^m_C$, and restricting to that point would imply that $q = 0$, which is a contradiction. 

By a result of Sharif [Sha07, Theorem 2], for a local field $K$ of characteristic not 2 and for any odd $m$, there exists a curve $C$ over $K$ of index $2m$, period $m$, and genus $m + 1$. Proposition 4.0.2 then shows that the Albanese of $C$ cannot kill numerous classes in $\text{Br}(C/K)$, implying that the conditions of Theorem 1.0.2 are sharp.

5. SOME OBSERVATIONS

5.1. Products of genus one curves. One might attempt to answer Question 1.0.3 by first splitting the class on a family of Albanese varieties with a member that splits as a product of genus 1 curves, and hoping that this decomposition will have implications for splitting the class over a factor. As we briefly explain, there are two reasons that this is unlikely to work.

First, the results of Section 4 show that one cannot hope to use only Albanese varieties of curves from the beginning, because there are examples where the decisive role is played by a genus 1 factor added after the fact, whose presence is necessary to split a single quaternion factor of the Brauer class.

Second, we make a simple observation about products: suppose $T$ and $T'$ are genus 1 curves over $K$, with $T$ of index 2 and $T'$ of index 3. Any Brauer class $\alpha$ killed by $T$ has order 2 and any Brauer class $\alpha'$ killed by $T'$ has order 3. The natural tensoring map $\text{Pic}_T \times \text{Pic}_{T'} \to \text{Pic}_{T \times T'}$ is additive on obstruction classes (since it is equivariant for the multiplication map $G_m \times G_m \to G_m$ of bands for the Picard stacks). Thus, $\alpha + \alpha'$ is killed by $T \times T'$, which is a torsor under $\text{Jac}_T \times \text{Jac}_{T'}$, but $\alpha + \alpha'$ is not killed by either $T$ or $T'$.

Examples of both types are easily constructed over local fields.

5.2. The universal Albanese doesn’t do anything on its own. In light of the method used here – that is, splitting Brauer classes by splitting them on particular base changes of the Albanese map of the universal curve – one might be tempted to ask the following question.

Question 5.2.1. Given a field $K$ and a positive integer $g > 2$, let $C \to \text{Spec } k(\mathcal{M}_{g,K})$ be the universal curve of genus $g$ over the function field of the stack of all curves of genus $g$. What is the kernel of the map $\text{Br}(K) \to \text{Br}(C)$?

Proposition 5.2.2. The kernel of the map of Question 5.2.1 is 0.
Proof. Over any field $K$, there is a curve $C_0$ of genus $g$ with a $K$-point. Since the universal curve is regular, any class that is trivialized over $C$ is trivialized over any specialization, such as $C_0$. Further specializing to the $K$-point shows that the class itself is 0. □

It might be more interesting to study the relative Brauer group of the universal curve over its field of definition.

6. SOME QUESTIONS

Some natural questions arise from the results we describe here. As mentioned in the introduction, one way to produce curves splitting Brauer classes is as complete intersections of sections of the anticanonical sheaf in a Brauer-Severi variety. On the other hand, if we restrict to a single anticanonical divisor, we obtain Calabi-Yau varieties that split the class. This observation leads to several directions for further exploration.

Question 6.0.1. Is there a fixed positive integer $n$ such that every Brauer class over a field is split by a torsor under an abelian variety of dimension $n$, independent of the index of the class? (Note that Theorem 1.0.2 applied to complete intersections of anti-canonical divisors gives a torsor of dimension $1 + \frac{1}{2}m^{m-1}(m - 3)$ for classes of index $m$, but this depends on $m$.)

Question 6.0.2. Is there a fixed positive integer $n$ such that every Brauer class over a field is split by a Calabi-Yau variety of dimension $n$?

Question 6.0.3. Is every Brauer class over a field split by a K3 surface?

Question 6.0.4. Is every Brauer class over a field split by a curve sitting in a K3 surface?

REFERENCES

[AA18] Benjamin Antieau and Asher Auel, private communication, 2018.
[Aue15] Asher Auel, Algebras of composite degree split by genus one curves, http://www.birs.ca/events/2015/5-day-workshops/15w5016/videos/watch/201509151021-Auel.html, 2015.
[BL04] Christina Birkenhake and Herbert Lange, Complex abelian varieties, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 302, Springer-Verlag, Berlin, 2004. MR 2062673
[dJH12] Aise Johan de Jong and Wei Ho, Genus one curves and Brauer-Severi varieties, Math. Res. Lett. 19 (2012), no. 6, 1357–1359. MR 3091612
[DM69] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 75–109. MR 0262240
[FH91] William Fulton and Joe Harris, Representation theory, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR 1153249
[FM12] Benson Farb and Dan Margalit, A primer on mapping class groups, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012. MR 2850125
[Gro68a] Alexander Grothendieck, Le groupe de Brauer. I. Algèbres d’Azumaya et interprétations diverses, Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 46–66. MR 244269
[Gro68b] ______, Le groupe de Brauer. II. Théorie cohomologique, Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 67–87. MR 244270
[Gro68c] ______, Le groupe de Brauer. III. Exemples et compléments, Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 88–188. MR 244271

[Kol18] János Kollár, Symmetric powers of Severi-Brauer varieties, [https://arxiv.org/abs/1603.02104](https://arxiv.org/abs/1603.02104), 2018.

[KS99] Nicholas M. Katz and Peter Sarnak, Random matrices, Frobenius eigenvalues, and monodromy, American Mathematical Society Colloquium Publications, vol. 45, American Mathematical Society, Providence, RI, 1999. MR 1659828

[KS04] Daniel Krashen and David J. Saltman, Severi-Brauer varieties and symmetric powers, Algebraic transformation groups and algebraic varieties, Encyclopaedia Math. Sci., vol. 132, Springer, Berlin, 2004, pp. 59–70. MR 2090670

[Lic69] Stephen Lichtenbaum, Duality theorems for curves over p-adic fields, Invent. Math. 7 (1969), 120–136. MR 0242831

[Lie08] Max Lieblich, Twisted sheaves and the period-index problem, Compos. Math. 144 (2008), no. 1, 1–31. MR 2388554

[Pol03] Alexander Polishchuk, Abelian varieties, theta functions and the Fourier transform, Cambridge Tracts in Mathematics, vol. 153, Cambridge University Press, Cambridge, 2003. MR 1987784

[PS08] Chris A. M. Peters and Joseph H. M. Steenbrink, Mixed Hodge structures, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 52, Springer-Verlag, Berlin, 2008. MR 2393625

[RV11] Anthony Ruozzi and Uzi Vishne, Open problem session from the conference “Ramification in algebra and geometry”, [http://www.mathcs.emory.edu/RAGE/RAGE-open-problems.pdf](http://www.mathcs.emory.edu/RAGE/RAGE-open-problems.pdf), 2011.

[Sha07] Shahed Sharif, Curves with prescribed period and index over local fields, J. Algebra 314 (2007), no. 1, 157–167. MR 2331756

[Swe95] Paul Kenneth Swets, Global sections of higher powers of the twisting sheaf on a Brauer-Severi variety, ProQuest LLC, Ann Arbor, MI, 1995, Thesis (Ph.D.)–The University of Texas at Austin. MR 2693834

[Tat68] John Tate, On the conjectures of Birch and Swinnerton-Dyer and a geometric analog [see MR1610977], Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 189–214. MR 3202555

[Zar00] Yuri G. Zarhin, Hyperelliptic Jacobians without complex multiplication, Math. Res. Lett. 7 (2000), no. 1, 123–132. MR 1748293

[Zar04] ______, Non-supersingular hyperelliptic Jacobians, Bull. Soc. Math. France 132 (2004), no. 4, 617–634. MR 2131907