Phases in Leptonic Mass Matrices: Higher Order Invariants and Applications to Models

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Abstract

We discuss complex rephasing invariants of charged lepton and neutrino mass matrices and associated theorems which determine in general (i) the number of physically meaningful phases in these matrices and (ii) which elements of these matrices can be rendered real by rephasings. New results are presented on higher order complex invariants and the application of the methods to several models.

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1 Introduction

Understanding fermion masses and mixing remains one of the most important outstanding problems in particle physics. In particular, the issue of possible neutrino masses and associated lepton mixing is of fundamental interest. Although there is no definite direct evidence for nonzero neutrino masses\(^1\) they are expected on general grounds: given only the known left-handed neutrino fields and the usual Higgs field(s) in the standard model (and supersymmetric extensions thereof which stabilize the hierarchy), nonzero neutrino masses result generically from higher-dimension operators which are expected to occur at a scale near to that of quantum gravity\(^2\), suppressed by associated inverse powers of the (reduced) Planck mass, \(\bar{M}_P = \sqrt{\hbar c / (8\pi G_N)} = 2.4 \times 10^{18} \text{ GeV}\). For example, gauge-invariant dimension-5 operators of this type could produce neutrino masses of order \(m_\nu \sim a v^2 / \bar{M}_P\), where \(v = 250 \text{ GeV}\) is the scale of electroweak symmetry breaking, and \(a\) is a dimensionless constant. (Here \(\bar{M}_P\) is an approximate upper bound on the mass which suppresses such operators; it is possible that new physics occurs at some intermediate mass scale \(v << M_t < \bar{M}_P\) in such a way that dimension 5 operators of this type would give rise to neutrino masses of order \(a' v^2 / M_t\).) Thus one can understand on general grounds why neutrino masses are so small. It is not known whether there exist any electroweak-singlet neutrino fields. If they do exist, then they could lead, via renormalizable, dimension-4 operators, to neutrino masses \(m_\nu \sim v^2 / M_R\), where the scale \(M_R\) of the electroweak-singlet neutrino mass is naturally \(>> v\), again yielding, albeit for a different reason, very small \(m_\nu\)\(^3\).

In turn, a natural concomitant of (nondegenerate) neutrino masses is lepton mixing, which is thus also a generic expectation\(^4\). It is thus of interest to study models for the leptonic mass matrices in the charge \(Q = 0\) and \(Q = -1\) sectors. The diagonalization of

\(^1\)For reviews and current limits, see Refs. \([1]\) and \([2]\). The apparent solar neutrino deficit is the most suggestive indirect evidence at present. The situation with atmospheric neutrinos is unclear.

\(^2\)This expectation is based on the general consensus that pointlike theories of quantum gravity (in particular, supergravity) are non-renormalizable and is borne out by the \(E << (\alpha')^{-1/2}\) limit of string theories.

\(^3\)The lepton mixing angles are functions of ratios of elements of neutrino matrix elements and of charged lepton mass matrix elements, and even though left-handed neutrino masses are small, some of these ratios could, in principle, be \(O(1)\). However, a set of conditions for natural suppression of observable lepton flavor violation were formulated, and it was shown that the standard model (generalized to include nonzero \(m_\nu\)) satisfies these \([3]\).
these matrices determines both the masses and the observable lepton mixing. In addition to
the search for neutrino masses and mixing in solar and atmospheric neutrino experiments,
searches continue at accelerators. In particular, the CHORUS and NOMAD experiments at
CERN are currently looking for $\nu_\mu \rightarrow \nu_\tau$ oscillations.\footnote{The NOMAD experiment uses a method proposed in Ref. \cite{ref5} while the CHORUS experiment combines this with a search for the $\tau$ tracks in an emulsion.}

In analyzing models of fermion masses and mixing, an important step is to determine
the number of real amplitudes and unremovable, and hence physically meaningful, phases
in the mass matrices. Recently, we reported a general solution to this problem \cite{ref7}. Here
we extend our explicit construction of rephasing invariants and give further applications to
various models. We also compare the situation concerning unremovable phases and associ-
ated invariants in the leptonic sector with that in the quark sector, for which we have also
recently presented a general analysis \cite{ref8}.

The organization of this paper is as follows. In section 2 we briefly review our theorem on
the number of unremovable phases. In section 3 we deal with the question of which elements
of the mass matrices can be made real by rephasings. We discuss the connection between
this and the complex rephasing invariants. This section includes explicit expressions for the
sixth order complex invariants. In section 4 we apply our general results to several models
with and without electroweak singlet neutrinos. Concluding comments are given in section
5.

2 Theorem on the Number of Physical Phases

The leptonic mass terms are taken to arise from interactions which are invariant under the
standard model gauge group $G_{SM} = SU(3) \times SU(2) \times U(1)$, via the spontaneous symmetry
breaking of $G_{SM}$. In the standard model and its supersymmetric extensions, the resul-
tant mass terms appear at the electroweak level via (renormalizable, dimension-4) Yukawa
It follows that the mass terms for the charged leptons can be written in terms of the $G_{SM}$ lepton fields as

$$-L_{mass,\ell} = \sum_{j,k=1}^{3} \left[ (\bar{L}_j L)_j L_{jk}^\ell \ell_{kR} \right] + h.c. \quad (2.1)$$

where $j, k = 1, 2, 3$ denote generation indices, and $L_{jl} = \begin{pmatrix} \nu_\ell \\ \ell \end{pmatrix}_{L}$ are the left-handed SU(2) doublets of leptons (where $\ell_1 = e, \ell_2 = \mu, \ell_3 = \tau$). The index $a = 2$ in $(\bar{L}_j L)_j L_{jk}^\ell \ell_{kR}$ is the SU(2) index. Further, $\ell_{k,R}$ are the right-handed SU(2) singlets, and $M^{(\ell)}$ is the charged lepton mass matrix. We use the result here from LEP and SLC that there are three generations of standard model fermion with associated light neutrinos [1].

In contrast to the charged lepton sector, where one at least knows the relevant fields, in the neutral lepton sector, a priori, one does not; in addition to the three known left-handed $I = 1/2, I_3 = 1/2$ Weyl neutrino fields $\nu_{jL}, j = 1 - 3$, there could be some number $n_s$ of electroweak-singlet neutrino fields $\chi_{j,R}, j = 1, 2, \ldots n_s$. The general neutrino mass matrix is given by

$$-L_{mass,N} = \frac{1}{2} (\bar{\nu}_L \hat{X}_L^c) \begin{pmatrix} M^{(L)}_L & M^{(D)}_R \\ (M^{(D)}_R)^T & M^{(R)}_R \end{pmatrix} \begin{pmatrix} \nu^c_R \\ \chi^c_R \end{pmatrix} + h.c. \quad (2.2)$$

where $\nu_L = (\nu_e, \nu_\mu, \nu_\tau)_L$, $\chi_R = (\chi_1, \ldots, \chi_{n_s})_R$, $M^{(L)}$ and $M^{(R)}$ are $3 \times 3$ and $n_s \times n_s$ Majorana mass matrices, and $M^{(D)}$ is a 3-row by $n_s$-column Dirac mass matrix. We denote the $(3 + n_s) \times (3 + n_s)$ neutrino mass matrix in (2.2) as $M^{(N)}$, where $N$ denotes “neutral”, $Q = 0$. (In Ref. [6] this was labelled equivalently as $M^{(0)}$.) Thus,

$$M^{(L)}_{jk} = M^{(N)}_{jk} \quad (2.3)$$

5 We do not consider models in which fermion masses arise via multifermion operators, which are not perturbatively renormalizable. Note that at a nonperturbative level, lattice studies [7] show that a lattice theory with a specific multifermion action and no scalar fields may yield the same continuum limit as a theory with a Yukawa interaction.

6 In a number of interesting models, some of these Yukawa couplings are viewed as originating at a higher mass scale, such as that of a hypothetical supersymmetric grand unified theory (SUSY GUT) or some other (supersymmetric) theory resulting from the $E << (\alpha')^{-1/2}$ limit of a string theory. It is possible that interactions which appear to be Yukawa couplings at a given mass scale, could be effective, in the sense that some of the elements of the associated Yukawa matrices could actually arise from higher-dimension operators at a yet higher mass scale. The plausibility of such higher-dimension operators can be inferred either from the nonrenormalizability of supergravity or as a consequence of the $E << (\alpha')^{-1/2}$ limit of a string theory.

7 In the context of supersymmetric extensions of the standard model, we assume unbroken $R$ parity so that the neutrinos do not mix with the neutralinos (higgsinos and neutral color-singlet gauginos).
where \( j, k = 1, 2, 3 \);

\[
M_{jk}^{(R)} = M_{j+3,k+3}^{(N)} \quad (2.4)
\]

with \( j, k = 1, \ldots n_s \); and

\[
M_{jk}^{(D)} = M_{j,k+3}^{(N)} \quad (2.5)
\]

where \( j = 1, 2, 3, k = 1, \ldots, n_s \). In general, all of these matrices are complex. Recall that the anticommutativity of fermion fields in the path integral and the property \( C^T = -C \) (where \( C \) is the Dirac charge conjugation matrix) imply that

\[
M^{(f)} = M^{(f)T} \quad \text{for} \quad f = L, R
\quad (2.6)
\]

so that

\[
M^{(N)} = M^{(N)T} \quad (2.7)
\]

Thus, the situation in the leptonic sector is qualitatively more complicated than that in the quark sector because of the general presence of three types of neutral fermion bilinears, Dirac, left-handed Majorana, and right-handed Majorana, each with its own gauge and rephasing properties. The diagonalization of \( M^{(\ell)} \) yields the three charged lepton mass eigenstates \( e_m, \mu_m, \) and \( \tau_m \), while the diagonalization of \( M^{(N)} \) yields, in general, \( 3 + n_s \) nondegenerate Majorana neutrino mass eigenstates (some could, of course, be degenerate in magnitude, leading to Dirac neutrino states).

A comment is in order concerning possible electroweak-singlet neutrinos. In the supersymmetric generalization of the standard model, which is motivated by its ability to maintain the condition \( v \ll M_P \) beyond tree level without fine-tuning, the component fields \( \chi_{j,L}^c \) would arise as part of \( G_{SM} \)-singlet chiral superfields \( \hat{\chi}_j^c \) (all chiral superfields will be written as left-handed). One may recall that, in general, chiral superfields which are singlets under the standard model gauge group \( G_{SM} \) can destabilize the hierarchy \([10]\). However, in commonly used models, the symmetry (such as matter parity) which prevents excessively rapid proton decay also excludes the types of terms which would destabilize the hierarchy.

To count the number of unremovable, and hence physically meaningful, phases in \( M^{(\ell)} \) and \( M^{(N)} \), one rephases the lepton fields in (2.1) and (2.2) so as to remove all possible phases
in these matrices. We gave a general theorem on this counting problem in Ref. [7]. To make our discussion reasonably self-contained, we briefly review it here. The rephasings are

$$L_{jL} = e^{-i\alpha_j} L'_{jL}$$  

$$\ell_{jR} = e^{i\beta_{j}^{(\ell)}} \ell'_{jR}$$  

for $j = 1, 2, 3$, and, if there exist any $\chi_{jR}$'s, also

$$\chi_{jR} = e^{i\beta_{j}^{(\chi)}} \chi'_{jR}$$

for $j = 1, \ldots, n_s$. In terms of the primed (rephased) lepton fields, the mass matrices have elements

$$M^{(\ell)'}_{jk} = e^{i(\alpha_j + \beta_k^{(\ell)})} M^{(\ell)}_{jk}$$

for the charged leptons, and, for the neutrino sector,

$$M^{(L)'}_{jk} = e^{i(\alpha_j + \alpha_k)} M^{(L)}_{jk}$$

for $j, k = 1, 2, 3$;

$$M^{(R)'}_{jk} = e^{i(\beta_j^{(\chi)} + \beta_k^{(\chi)})} M^{(R)}_{jk}$$

for $j, k = 1, \ldots, n_s$, and

$$M^{(D)'}_{jk} = e^{i(\alpha_j + \beta_k^{(\chi)})} M^{(D)}_{jk}$$

for $j = 1, 2, 3$ and $k = 1, \ldots, n_s$.

Thus, if $M^{(\ell)}$ has $N_{\ell}$ nonzero (and, in general, complex) elements, then the $N_{\ell}$ equations for making these elements real are

$$\alpha_j + \beta_k^{(\ell)} = -\arg(M^{(\ell)}_{jk}) + \eta_{jk}^{(\ell)} \pi$$

where the set $\{jk\}$ runs over each of these nonzero elements, and $\eta_{jk}^{(\ell)} = 0$ or 1.\footnote{The $\eta_{jk}$ term allows for the possibility of making the rephased element real and negative; this will not affect the counting of unremovable phases.}
Similarly, in the neutrino sector, if $M^{(D)}$, $M^{(L)}$, and $M^{(R)}$ have, respectively, $N_D$, $N_L$, and $N_R$ nonzero (and, in general, complex) elements, then the corresponding equations for making these elements real are

\begin{align}
\alpha_j + \beta^\chi_k &= -\arg(M^{(D)}_{jk}) + \eta^{(D)}_{jk} \pi \\
\alpha_j + \alpha_k &= -\arg(M^{(L)}_{jk}) + \eta^{(L)}_{jk} \pi \\
\beta^\chi_j + \beta^\chi_k &= -\arg(M^{(R)}_{jk}) + \eta^{(R)}_{jk} \pi
\end{align}

where the various ranges of indices are obvious from (2.2). Let us define the $(6 + n_s)$-dimensional vector of fermion field phases

$$v = (\{\alpha_i\}, \{\beta_i^\ell\}, \{\beta_i^\chi\})^T$$

where $\{\alpha_i\} \equiv \{\alpha_1, \alpha_2, \alpha_3\}$, $\{\beta_i^\ell\} \equiv \{\beta_1^\ell, \beta_2^\ell, \beta_3^\ell\}$, and $\{\beta_i^\chi\} \equiv \{\beta_1^\chi, ..., \beta_n^\chi\}$. We also define the vector of phases of elements of the various mass matrices

$$w = (\{-\arg(M^{(\ell)}_{jk}) + \eta^{(\ell)}_{jk} \pi\}, \{-\arg(M^{(N)}_{mn}) + \eta^{(N)}_{mn} \pi\})^T$$

The dimension of the vector $w$ is equal to the number of rephasing equations $N_{eq} = N_\ell + N_N$, where $N_N = N_L + N_R + N_D$. The indices $jk$ and $mn$ in (2.20) run over the nonzero elements of $M^{(\ell)}$ and the independent nonzero elements of the symmetric matrix $M^{(N)}$. We can then write (2.15)-(2.18) as

$$Tv = w$$

which defines the matrix $T$. Since $T$ is an $N_{eq}$-row by $(6 + n_s)$-column matrix, clearly

$$\text{rank}(T) \leq \min\{N_{eq}, 6 + n_s\}$$

Recall that in the quark case [8], the analogous inequality was $\text{rank}(T) \leq \min\{N_{eq}, 9\}$, but for realistic models where $N_{eq} > 9$, one had the stronger inequality that $\text{rank}(T) \leq 8$ because one overall rephasing left the Yukawa (or equivalently, mass) matrices invariant.

\footnotetext{In eq. (2.20) we change notation slightly from Ref. [7], where an overall minus sign was absorbed in $T$.}
contrast, in the corresponding present case, where \( N_{eq} > 6 + n_s \), there is no overall rephasing (unless \( M^{(L)} \) and \( M^{(R)} \) both vanish) which leaves \( M^{(\ell)} \) and \( M^{(N)} \) invariant, and hence the inequality may be realized as an equality, i.e., \( \text{rank}(T) \) may be equal to \( 6 + n_s \). This will be illustrated in specific models to be discussed in section 4.

The first main theorem is then [7]: The number of unremovable and hence physically meaningful phases \( N_p \) in \( M^{(\ell)} \) and \( M^{(N)} \) is

\[
N_p = N_{eq} - \text{rank}(T) \tag{2.23}
\]

For the proof, see Ref. [7]. As noted there, this is independent of whether or not the matrices are initially (complex) symmetric. There is a one-to-one correspondence between each such unremovable phase and the phase of a rephasing-invariant product of elements of mass matrices. We discuss these invariants next.

3 Rephasing Invariants and Theorems on Locations of Phases

A fundamental question concerns which elements of \( M^{(\ell)} \) and \( M^{(N)} \) can be made real by rephasings of lepton fields. In Ref. [7] several theorems were given which provide a general answer to these questions. The general method is to construct all independent complex products of elements of the \( M^{(f)} \), \( f = \ell, N \), having the property that these products are invariant under the rephasings (2.8)-(2.10). We refer to these as complex invariant products. Since in general, by construction, these invariant products are complex, each one implies a constraint, which is that the set of elements which comprise it cannot all be made real by the rephasings (2.8)-(2.10) of the lepton fields. We define an irreducible complex invariant to be one which cannot be factorized purely into products of lower-order complex invariants. Since reducible complex invariants do not yield any new phase constraints, it suffices to consider only irreducible complex invariants, and we shall do so. We define a set of independent (irreducible) complex invariants to be a set of complex (irreducible) invariants with the property that no invariant in the set is equal to (i) the complex conjugate of another invariant in the set or (ii) another element in the set with its indices permuted. This does not imply
that the arguments of a set of independent (irreducible) complex invariants are linearly independent. We define \( N_{ia} \) to denote the number of linearly independent arguments among the independent complex invariants. Further, we define \( N_{inv} \) to be the total number of independent (irreducible) complex invariants in a given model. It is useful to define \( N_{inv,2n} \) to denote the number of independent complex invariants of order \( 2n \).

In Ref. [7] we discussed the general construction of complex rephasing invariants, enumerated the various types of these invariants at fourth and sixth order, and gave explicit expressions for the fourth order and a number of sixth order complex invariants. Here we shall give a general set of explicit expressions for complex invariants up to sixth order, inclusive. We recall our classification of rephasing invariants into two general types: (1) \( P \)-type, which involve elements of mass matrices in one charge sector (charged leptons or neutrinos), and (2) \( Q \)-type, which involve elements of mass matrices in both of these charge sectors. We recall that the two quartic \( P \)-type invariants are

\[
P^{(f)}_{4j1k1,j2k2;\tau} \equiv P^{(f)}_{j1k1,j2k2} = M^{(f)}_{j1k1} M^{(f)}_{j2k2} M^{(f)*}_{j2k1} M^{(f)*}_{j1k2} \tag{3.1}
\]

for \( f = \ell, N \). Note the symmetries

\[
P^{(f)}_{j1k1,j2k2} = P^{(f)}_{j2k2,j1k1} \tag{3.2}
\]

\[
P^{(f)*}_{j1k1,j2k2} = P^{(f)*}_{j1k2,j2k1} \tag{3.3}
\]

(whence also \( P^{(f)}_{j1k1,j2k2} = P^{(f)*}_{j2k1,j1k2} \) for \( f = \ell, N \) and, for \( f = N \), the additional symmetry

\[
P^{(N)}_{j1j2,j3j4} = P^{(N)}_{j2j1,j4j3} \tag{3.4}
\]

In a convenient shorthand notation, the complex invariants in \( (3.1) \) for \( f = \ell, N \) may be denoted as \( \ell\ell\ell^*\ell^* \) and \( NNN^*N^* \), respectively. The \( P_4 \) complex invariants of the form \( NNN^*N^* \) may be further classified into six subtypes according to which submatrices in \( M^{(N)} \) they involve, \( M^{(L)}, M^{(R)}, \) and/or \( M^{(D)} \):

\[
NNN^*N^* : \quad \begin{array}{ccc}
LLL^*L^* & LDL^*D^* \\
RRR^*R^* & RDR^*D^* \\
DDD^*D^* & LRD^*D^*
\end{array} \tag{3.5}
\]
(It is easily seen that the rephasing invariants of the subtype \( LRL^*R^* \) are real.) Explicitly, the first three complex invariants are given by (3.1) for \( f = L, R, D \), respectively, and the other three by the products

\[
\Pi_{j_1j_2,j_3k_1}^{(LD)} = M_{j_1j_2}^{(L)} M_{j_3k_1}^{(D)*} M_{j_2k_1}^{(D)*} \tag{3.6}
\]

\[
\Xi_{k_1k_2,j_1k_3}^{(RD)} = M_{k_1k_2}^{(R)} M_{j_1k_3}^{(D)*} M_{k_1k_3}^{(D)*} \tag{3.7}
\]

and

\[
\Omega_{j_1j_2,k_1k_2}^{(LRDD)} = M_{j_1j_2}^{(L)} M_{k_1k_2}^{(R)*} M_{j_1k_1}^{(D)*} M_{j_2k_2}^{(D)*} \tag{3.8}
\]

It is straightforward to express these in terms of the elements of \( P^{(N)} \), using (2.3)-(2.5):

\[
P_{j_1j_2,j_3j_4}^{(L)} = P_{j_1j_2,j_3j_4}^{(N)} \tag{3.9}
\]

\[
P_{k_1k_2,k_3k_4}^{(R)} = P_{k_1+3, k_2+3, k_3+3, k_4+3}^{(N)} \tag{3.10}
\]

\[
P_{j_1j_2,k_2}^{(D)} = P_{j_1,j_2+3, k_2+3}^{(N)} \tag{3.11}
\]

\[
\Pi_{j_1j_2,j_3k_1}^{(LD)} = P_{j_2,j_1, j_3, k_1+3}^{(N)} \tag{3.12}
\]

\[
\Xi_{k_1k_2,j_1k_3}^{(RD)} = P_{k_1+3, k_2+3, j_1, k_3+3}^{(N)} \tag{3.13}
\]

\[
\Omega_{j_1j_2,k_1k_2}^{(LRDD)} = P_{j_2, j_1, k_1+3, k_2+3}^{(N)} \tag{3.14}
\]

At quartic order, we found two \( Q \)-type complex invariants, namely

\[
Q_{j_1k_1,j_2m_1}^{(D\ell)} = M_{j_1k_1}^{(D)} M_{j_2m_1}^{(\ell)} M_{j_2k_1}^{(D)*} M_{j_1m_1}^{(\ell)*} \tag{3.15}
\]

and

\[
\Pi_{j_1j_2,j_3m_1}^{(L\ell)} = M_{j_1j_2}^{(L)} M_{j_3m_1}^{(\ell)} M_{j_1j_3}^{(L)*} M_{j_2m_1}^{(\ell)*} \tag{3.16}
\]
In the short notation, these are denoted $D\ell D\ell^*$ and $L\ell L^\ast \ell^\ast$. Note that

$$\Pi^{(L_f)}_{j_1j_2j_3m_1} = Q^{(L_f)}_{j_2j_1j_3m_1}$$

(3.17)

At sixth order there are first the complex invariants each of which involves only one charge sector, $f f f f f^* f^*$ for $f = \ell, N$. The explicit expressions for these are

$$P^{(f)}_{j_1k_1,j_2k_2,j_3k_3} = M^{(f)}_{j_1k_1} M^{(f)}_{j_2k_2} M^{(f)*}_{j_3k_3} M^{(f)*}_{j_1k_1} M^{(f)*}_{j_3k_2} M^{(f)*}_{j_1k_3}$$

(3.18)

for $f = \ell, N$. For any $f$, the invariants in (3.18) satisfy

$$P^{(f)}_{j_1k_1,j_2k_2,j_3k_3} = P^{(f)}_{j_2k_2,j_3k_3,j_1k_1} = P^{(f)}_{j_3k_3,j_1k_1,j_2k_2}$$

(3.19)

and

$$P^{(f)}_{j_1k_1,j_2k_2,j_3k_3} = P^{(f)*}_{j_3k_2,j_1k_1,j_2k_3}$$

(3.20)

For the cases where $M^{(f)}$ is symmetric, namely $M^{(f)} = M^{(N)}$ or, for submatrices, $M^{(L)}$, $M^{(R)}$, one has the additional symmetry

$$P^{(f)}_{j_1k_1,j_2k_2,j_3k_3} = P^{(f)}_{j_2k_2,j_3k_3,j_1k_1}$$

(3.21)

The $NNNN^*NN^*$ invariants may be divided into 11 subtypes according to which submatrices of $M^{(N)}$ they involve:

$$NNNN^*NN^* : \quad LLLL^* L^* L^* \quad LLDL^* L^* D^* \quad RRDR^* R^* D^* \quad LRD^* D^* R^*$$

(3.22)

Complex conjugates of these invariants yield the same phase constraints and hence are not listed. Further, in the shorthand notation, the ordering of the factors is not important; e.g. $DDLD^* D^* L^*$, $DLDD^* L^* D^*$, and $LDDL^* D^* D^*$ represent the same subtype, and so
for the others. Note that the 6’th order rephasing invariants $\ell RR\ell^* R^*$, $\ell R\ell^* \ell^* R^*$, $LRRL^* R^* R^*$, $LLRL^* L^* R^*$, and $\ell LR\ell^* L^* R^*$ all reduce to real factors times quartic complex invariants and hence are not irreducible 6’th order complex invariants. Given an explicit formula for a particular type of $N_{NNN}^* N^* N^*$ complex invariant in terms of submatrices of $M^{(N)}$, it is straightforward to use eqs. (2.3)-(2.5) to reexpress it in terms of $M^{(N)}$. Thus, for $LLLL^* L^* L^*$, we have the relation $P_{j_1 j_2, j_3 j_4, j_5 j_6}^{(L)} = P_{j_1 j_2, j_3 j_4, j_5 j_6}^{(N)}$; for $D D D D^* D^* D^*$ the relation $P_{j_1 k_1, j_2 k_2, j_3 k_3}^{(D)} = P_{j_1 k_1 + 3, j_2 k_2 + 3, j_3 k_3 + 3}^{(N)}$, and so forth for the others.

Explicit expressions for the other subtypes of $N_{NNN}^* N^* N^*$ complex invariants in (3.22) are given below (in some cases there is more than one kind of structure for a given subtype):

$$DDDL^* D^* L^* : \quad M_{j_1 k_1}^{(D)} M_{j_2 k_2}^{(D)} M_{j_3 k_3}^{(L)} M_{j_4 k_4}^{(D)} M_{j_5 k_5}^{(D)*} M_{j_6 k_6}^{(L)*}$$

(3.23)

$$M_{j_1 k_1}^{(D)} M_{j_2 k_2}^{(D)} M_{j_3 k_3}^{(L)*} M_{j_4 k_4}^{(D)} M_{j_5 k_5}^{(D)*} M_{j_6 k_6}^{(L)*}$$

(3.24)

$$DDRD^* D^* R^* : \quad M_{j_1 k_1}^{(D)} M_{j_2 k_2}^{(D)} M_{j_3 k_3}^{(R)} M_{j_4 k_4}^{(D)} M_{j_5 k_5}^{(D)*} M_{j_6 k_6}^{(R)*}$$

(3.25)

$$M_{j_1 k_1}^{(D)} M_{j_2 k_2}^{(D)} M_{j_3 k_3}^{(R)} M_{j_4 k_4}^{(D)} M_{j_5 k_5}^{(D)*} M_{j_6 k_6}^{(R)*}$$

(3.26)

$$LLDL^* L^* D^* : \quad M_{j_1 j_2}^{(L)} M_{j_3 j_4}^{(L)} M_{j_5 k_5}^{(D)} M_{j_1 j_4}^{(L)*} M_{j_2 j_5}^{(L)*} M_{j_3 k_3}^{(D)*}$$

(3.27)

$$RRDR^* R^* D^* : \quad M_{k_1 k_2}^{(R)} M_{k_3 k_4}^{(R)} M_{j_1 k_5}^{(D)} M_{k_1 k_4}^{(R)*} M_{k_2 k_5}^{(D)} M_{j_1 k_3}^{(R)*}$$

(3.28)

$$LRDL^* R^* D^* : \quad M_{j_1 j_2}^{(L)} M_{k_1 k_2}^{(R)} M_{j_3 k_3}^{(D)} M_{j_1 j_3}^{(L)*} M_{k_1 k_3}^{(R)*} M_{j_2 k_2}^{(D)*}$$

(3.29)

$$LLRL^* D^* D^* : \quad M_{j_1 j_2}^{(L)} M_{j_3 j_4}^{(L)} M_{j_1 j_3}^{(R)} M_{j_2 k_2}^{(D)} M_{j_3 k_3}^{(D)*}$$

(3.30)

$$LRRD^* D^* R^* : \quad M_{j_1 j_2}^{(L)} M_{k_1 k_2}^{(R)} M_{j_3 k_3}^{(D)} M_{j_1 k_2}^{(D)*} M_{j_2 k_4}^{(R)*} M_{k_1 k_3}^{(R)*}$$

(3.31)
and
\[
LRDD^* D^* D^* : \quad M_{j_1 j_2}^{(L)} M_{k_1 k_2}^{(R)} M_{j_3 k_3}^{(D)} M_{j_1 k_3}^{(D^*)} M_{j_2 k_3}^{(D^*)} M_{j_3 k_1}^{(D^*)} \tag{3.32}
\]
\[
M_{j_1 j_2}^{(L)} M_{k_1 k_2}^{(R)} M_{j_3 k_3}^{(D)} M_{j_1 k_2}^{(D^*)} M_{j_2 k_3}^{(D^*)} M_{j_3 k_1}^{(D^*)} \tag{3.33}
\]

Note that the invariants in (3.23) and (3.27) are of the form
\[
Q_{j_1 k_1 j_2 k_2 j_3 m_1}^{(f f f')} = M_{j_1 k_1}^{(f)} M_{j_2 k_2}^{(f)} M_{j_3 m_1}^{(f')} M_{j_1 k_1}^{(f^*)} M_{j_2 k_2}^{(f^*)} M_{j_3 m_1}^{(f^*)} \tag{3.34}
\]
for \((f f f') = (DDL)\) and \((LLD)\), respectively.

We also find the following seven independent types of 6'th order complex invariants linking the charge \(Q = -1\) and \(Q = 0\) sectors:
\[
\begin{align*}
\ell \ell \ell^* \ell^* L^* & : \quad \ell \ell \ell^* \ell^* L^* \\
L \ell L^* L^* & : \quad \ell \ell \ell^* \ell^* L^* \\
\ell \ell D^* \ell^* D^* & : \quad \ell DR^* D^* R^* \\
DD D^* D^* \ell^* & : \quad \ell LR^* D^* D^* \\
\end{align*}
\tag{3.35}
\]

The explicit forms for these are
\[
\begin{align*}
\ell \ell \ell^* \ell^* L^* : & \quad M_{j_1 m_1}^{(f)} M_{j_2 m_2}^{(f)} M_{j_3 m_3}^{(L)} M_{j_1 m_2}^{(f^*)} M_{j_3 m_3}^{(L^*)} M_{j_1 m_1}^{(L^*)} \tag{3.36} \\
& \quad M_{j_1 m_1}^{(f)} M_{j_2 m_2}^{(f)} M_{j_3 m_3}^{(L)} M_{j_1 m_2}^{(f^*)} M_{j_3 m_2}^{(f^*)} M_{j_1 m_2}^{(L^*)} \tag{3.37} \\
L \ell L^* L^* & : \quad M_{j_1 j_2}^{(L)} M_{j_3 j_4}^{(L)} M_{j_5 m_1}^{(L)} M_{j_1 j_4}^{(L^*)} M_{j_5 j_2}^{(L^*)} M_{j_3 m_1}^{(L^*)} \tag{3.38} \\
\ell \ell D^* \ell^* D^* & : \quad M_{j_1 m_1}^{(f)} M_{j_2 m_2}^{(f)} M_{j_3 k_1}^{(D)} M_{j_1 m_2}^{(f^*)} M_{j_3 m_1}^{(D^*)} M_{j_2 k_1}^{(D^*)} \tag{3.39} \\
\ell DR^* D^* R^* & : \quad M_{j_1 m_1}^{(f)} M_{j_2 k_1}^{(D)} M_{j_3 k_3}^{(R)} M_{j_1 m_2}^{(f^*)} M_{j_2 k_2}^{(D^*)} M_{j_3 k_3}^{(R^*)} \tag{3.40} \\
\end{align*}
\]
The methods for determining these are similar to those discussed in detail for the quark case in Ref. [7]. For example, the general explicit form of the $\ell D L^* D^* L^*$ invariants is

$$M_{j_1 m_1}^{(L)} M_{j_2 k_1}^{(D)} M_{j_3 j_4}^{(L)^*} M_{j_2 m_1}^{(D)^*} M_{j_4 k_1}^{(L)^*} M_{j_3 j_2}^{(L)^*}$$ (3.42)

$$M_{j_1 m_1}^{(L)} M_{j_2 k_1}^{(D)} M_{j_3 j_4}^{(L)} M_{j_4 m_1}^{(D)^*} M_{j_2 k_1}^{(L)^*} M_{j_3 j_2}^{(L)}$$ (3.43)

$$M_{j_1 m_1}^{(L)} M_{j_2 k_1}^{(D)} M_{j_3 j_4}^{(L)} M_{j_2 m_1}^{(D)^*} M_{j_3 k_1}^{(L)^*} M_{j_4 j_2}^{(L)^*}$$ (3.44)

$$M_{j_1 m_1}^{(L)} M_{j_2 k_1}^{(D)} M_{j_3 j_4}^{(L)} M_{j_3 m_1}^{(D)^*} M_{j_2 k_1}^{(L)^*} M_{j_1 j_2}^{(L)^*}$$ (3.45)

$$M_{j_1 m_1}^{(L)} M_{j_2 k_1}^{(D)} M_{j_3 j_4}^{(L)} M_{j_4 k_1}^{(D)^*} M_{j_3 m_1}^{(L)^*} M_{j_1 j_2}^{(L)^*}$$ (3.46)

$$M_{j_1 m_1}^{(L)} M_{j_2 k_1}^{(D)} M_{j_3 j_4}^{(L)} M_{j_4 m_1}^{(D)^*} M_{j_3 k_1}^{(L)^*} M_{j_1 j_2}^{(L)^*}$$ (3.47)

where $\sigma \in S_4$, the group of permutations of 4 indices. We then determine the full subset of these 4! products which yields independent irreducible sixth order complex invariants, and obtain the results listed in (3.42)-(3.47). Note that (3.36) and (3.38) are again of the form (3.34) for $(fff) = (\ell \ell L)$ and $(LL\ell)$, respectively.

In the case of quark mass matrices (with $N_G = 3$ generations of quarks), we showed [8] that the fourth and sixth order complex invariants sufficed to describe all of the phase constraints on the mass matrices. An analogous result does not hold in the leptonic sector, however; in general, it may be necessary to include complex invariants of higher order than
sixth to obtain all of the phase constraints. The reason for this is just that the neutral part of the lepton sector is, in general, more complicated than either the up or down quark sector. Rather than enumerate in general all of the possible eighth-order complex invariants, which would be quite involved, we will make clear how one uses them in practical calculations via two specific models in section 4.

We next review our theorems on invariants, as applied to lepton mass matrices. For a given model, one constructs the maximal set of $N_{ia}$ independent complex invariants of lowest order(s), whose arguments (phases) are linearly independent. Then (a) each of these invariants implies a constraint that the elements contained within it cannot, in general, all be made simultaneously real; (b) this constitutes the complete set of constraints on which elements of $M^{(L)}$ and $M^{(N)}$ can be made simultaneously real; and hence (c) $N_p = N_{ia}$.

An immediate corollary is: If in a given model there are as at least as many independent quartic complex invariants with independent arguments as there are unremovable phases, then the arguments of all higher order complex invariants are expressible in terms of those of the quartic invariants and hence yield no new phase constraints.

In the quark case, we presented a useful graphical representation for the complex invariants in Ref. [8]. This again works directly for invariants involving only the charged lepton sector. For invariants involving the neutral lepton sector, there is no $1 \to 1$ correspondence between a matrix element $M^{(N)}_{jk}$ occurring in a given complex invariant and a point on an graphical array representing the matrix $M^{(N)}$; instead, as a result of the symmetry (2.7), $M^{(N)T} = M^{(N)}$, there is a obvious 2-fold homomorphism according to which the element $M^{(N)}_{jk} = M^{(N)}_{kj}$ corresponds to the two points ($j$'th row, $k$'th column) and ($k$'th row, $j$'th column) in the array representing $M^{(N)}$. For an invariant involving $2n$ elements of $M^{(N)}$ of which a (possibly null) subset of $r$ are diagonal elements, there are $2^{2n-k}$-ways of representing it graphically. Among these ways, however, one can always find a graphical representation which is analogous to those for the quark sector complex invariants discussed in Ref. [8]. Moreover, as a tool for quickly determining the complex invariants, given the forms of the mass matrices, the graphical method is equally useful in the leptonic case as in the quark case. We shall illustrate it in this context below.
Since the full set of $N_{\text{inv}}$ independent complex invariants will have arguments which are not, in general, linearly independent, it follows that

$$N_{\text{inv}} \geq N_{\text{ia}}$$

(3.49)

It may also happen that, e.g. for order $2n = 4$, the number of independent quartic complex invariants, $N_{\text{inv},4}$ is greater than $N_{\text{ia}}$. Using the methods of Ref. [7] (see also Ref. [8]), one can determine how many of the complex invariants of all orders, or of a particular order, have linearly independent arguments, and can thus construct sets of such invariants with this property.

4 Applications to Specific Models

Although our theorems are quite general, it is useful to see how they apply to various specific models. In Ref. [7] we gave a brief discussion of applications. Here we will consider several more models. In contrast to the quark sector, where one has reasonably accurate measurements of most Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix elements, in the leptonic sector, there is not even any definite direct evidence of leptonic mixing, let alone accurate measurements of various mixing matrix elements. As noted above, probably the strongest indirect evidence comes from the apparent deficiency of the solar neutrino flux. The situation concerning possible evidence for atmospheric neutrino oscillations is unclear at present [1, 2]. We shall tentatively assume that the solar neutrino deficit does indicate neutrino oscillations.\footnote{For an recent phenomenological study which also includes fits to atmospheric neutrino oscillations, see, e.g., Ref. [11].} Moreover, since our purpose is to illustrate the application of our general results on phases, we will not discuss the details of the phenomenology of the models.

In general, models for fermion masses and mixing can be classified according to whether they assume a theoretical framework of perturbative or nonperturbative electroweak symmetry breaking. We shall concentrate here on the class of models in which the observed electroweak symmetry breaking is perturbative. In this class, an appealing framework is provided by supersymmetric extensions of the standard model, which stabilize the Higgs sector
and hence stabilize the hierarchy between the electroweak scale and the Planck scale. In these models, one typically hypothesizes some simple forms at a high mass scale (e.g., a mass scale characterizing a possible grand unified theory (GUT) or a mass scale near to the string scale, $M_P$) for the Yukawa and higher-dimensional operators which are responsible for fermion mass generation. As part of this, one assumes some symmetries to prevent various Yukawa couplings and higher-dimensional operators, and thereby render various entries in the (effective) Yukawa matrices zero. The purpose of this is, of course, to minimize the number of parameters and hence increase the apparent predictiveness. One then evolves these forms for the Yukawa matrices down to the electroweak mass scale using the appropriate renormalization group equations. For this evolution, we use the renormalization group equations of the minimal supersymmetric standard model (MSSM). Parenthetically, we recall that when one evolves these forms for the Yukawa matrices down to the electroweak level, the zero entries do not, in general, remain zero. Clearly, the entries which are modelled as being exactly zero might well be nonzero, but small quantities. For example, one possibility is that they are suppressed by certain powers of $(M_r/M_P)$, where $M_r$ is the reference scale at which one analyzes the effective Yukawa interaction. Since our results do not depend on whether or not the mass matrices are symmetric at some mass scale, we shall consider the general case of non-symmetric mass matrices.

4.1 Model Without Electroweak-Singlet Neutrinos

We first consider a model with no electroweak-singlet neutrinos, i.e., with $n_s = 0$. This is defined by:

$$M^{(\ell)} = \begin{pmatrix} 0 & E_{12} & 0 \\ E_{21} & E_{22} & E_{23} \\ 0 & E_{32} & E_{33} \end{pmatrix}$$ (4.1.1)

$$M^{(L)} = \begin{pmatrix} 0 & L_{12} & 0 \\ L_{12} & L_{22} & L_{23} \\ 0 & L_{23} & L_{33} \end{pmatrix}$$ (4.1.2)

(where each element is, in general, complex, since no symmetry forces it to be real). We have verified that, for appropriate choices of the parameters, this model is able to fit current
solar neutrino data and other established limits on neutrino masses and mixing. For this model, there are $N_{eq} = 10$ complex elements in $M^{(t)}$ and $M^{(D)}$ and corresponding rephasing equations. We calculate the matrix $T$ and find $\text{rank}(T) = 6$. Our theorem (2.23) then implies that there are $N_p = N_{eq} - \text{rank}(T) = 4$ unremovable phases in $M^{(t)}$ and $M^{(L)}$.

(This model provides an illustration of how $\text{rank}(T)$ may realize the inequality (2.22) as an equality.) We find $N_{\text{inv,4}} = 8$ independent complex (quartic) invariants:

$$P^{(t)}_{22,33} = E_{22}E_{33}E_{32}^*E_{23}^*$$  \hspace{1cm} (4.1.3)

$$P^{(L)}_{22,33} = L_{22}L_{33}L_{23}^2$$  \hspace{1cm} (4.1.4)

$$\Pi^{(L)}_{12,22} = L_{12}E_{22}L_{22}^*E_{12}^*$$  \hspace{1cm} (4.1.5)

$$\Pi^{(L)}_{12,32} = L_{12}E_{32}L_{23}^*E_{12}^*$$  \hspace{1cm} (4.1.6)

$$\Pi^{(L)}_{22,32} = L_{22}E_{32}L_{23}^*E_{22}^*$$  \hspace{1cm} (4.1.7)

$$\Pi^{(L)}_{22,33} = L_{22}E_{33}L_{23}^*E_{23}^*$$  \hspace{1cm} (4.1.8)

$$\Pi^{(L)}_{23,32} = L_{23}E_{32}L_{33}^*E_{22}^*$$  \hspace{1cm} (4.1.9)

$$\Pi^{(L)}_{23,33} = L_{23}E_{33}L_{33}^*E_{23}^*$$  \hspace{1cm} (4.1.10)

From these we calculate the corresponding $Z$ matrix and find that it has rank 4, so that of the eight arguments of the complex invariants, there are $N_{ia} = 4$ linearly independent ones, in accord with the equality $N_{ia} = N_p$ and our result that $N_p = 4$. There are correspondingly four independent phase constraints. In particular, for this model it is not, in general, possible
by any rephasings to make either the charged lepton or neutrino mass matrices real. A set of complex invariants with linearly independent arguments is given by

\[
\{ P_{22,33}^{(\ell)}, P_{22,33}^{(L)}, \Pi_{12,22}^{(LL)}, \Pi_{22,32}^{(LL)} \}
\] (4.1.11)

We denote the arguments of the invariants in eq. (4.1.11 as \(\theta_j\), \(j = 1, \ldots, 4\); i.e., \(\theta_1 = \arg(P_{22,33}^{(\ell)})\), etc. The only element which does not appear in any complex invariant and hence may be rephased freely is \(E_{21}\).

A necessary task which a model builder must carry out is to determine which of the elements of the mass matrices can be rendered real by the rephasing of the lepton fields. We display below an example of an allowed form for the mass matrices after such rephasing is

\[
M^{(\ell)'} = \begin{pmatrix}
0 & |E_{12}| e^{-i\theta_3} & 0 \\
|E_{21}| & |E_{22}| & |E_{23}| \\
0 & |E_{32}| e^{i\theta_4} & |E_{33}| e^{i(\theta_1+\theta_4)}
\end{pmatrix}
\] (4.1.12)

\[
M^{(L)'} = \begin{pmatrix}
0 & |L_{12}| & 0 \\
|L_{12}| & |L_{22}| & |L_{23}| \\
0 & |L_{32}| & |L_{33}| e^{i\theta_2}
\end{pmatrix}
\] (4.1.13)

This model has \(N_{\text{inv,6}} = 4\) sixth order complex invariants:

\[
Q_{22,33,12}^{(LL)} = L_{22} L_{33} E_{12} L_{23}^* L_{12}^* E_{32}^*
\] (4.1.14)

\[
Q_{22,33,12}^{(LL)} = E_{22} E_{33} L_{12} E_{23}^* E_{12}^* L_{23}^*
\] (4.1.15)

\[
Q_{33,12,22}^{(LL)} = L_{33} L_{12} E_{22} L_{23}^2 E_{12}^*
\] (4.1.16)

\[
Q_{33,12,22}^{(LL)} = E_{33} E_{12} L_{22} E_{32}^* E_{23}^* L_{12}^*
\] (4.1.17)

However, as a result of our corollary 1 above, since we have already exhibited a set of quartic complex invariants with independent arguments and comprised of \(N_{ia}\) members equal to the number \(N_p\) of unremovable phases in the mass matrices, it follows that all higher order invariants, and, in particular, the sixth order ones listed above, have arguments which can be expressed as linear combinations of those of the set of \(N_{ia}\) fourth order invariants and hence do not yield any new phase constraints.
4.2 Models with Electroweak-Singlet Neutrinos

4.2.1 $n_s = 3$ Model 1

We will concentrate on models with $n_s = N_G = 3$. Our methods are easily applied to models with other values of $n_s$. We begin with an idealized model in which left-handed Majorana mass terms are assumed to be negligible, i.e., $M^{(L)} = 0$, and the other mass matrices are given by

$$M^{(L)} = \begin{pmatrix} 0 & E_{12} & 0 \\ E_{21} & E_{22} & E_{23} \\ 0 & E_{32} & E_{33} \end{pmatrix}$$

(4.2.1)

$$M^{(R)} = \begin{pmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & R_{23} \\ 0 & R_{23} & R_{33} \end{pmatrix}$$

(4.2.2)

$$M^{(D)} = \begin{pmatrix} 0 & D_{12} & 0 \\ D_{21} & 0 & D_{23} \\ 0 & D_{32} & D_{33} \end{pmatrix}$$

(4.2.3)

We have again checked that with appropriate choices of the parameters, this model can fit the experimental constraints discussed at the beginning of the paper. The model has $N_{eq} = 15$, and we compute $\text{rank}(T) = 9$, so that there are $N_p = 6$ unremovable phases in the mass matrices. The model has $N_{inv,4} = 6$ quartic complex invariants:

$$P_{22,33}^{(L)} = E_{22}E_{33}E_{23}^*E_{32}^*$$

(4.2.4)

$$Q_{23,33}^{(DL)} = D_{23}E_{33}D_{33}^*E_{23}^*$$

(4.2.5)

$$Q_{23,32}^{(DL)} = D_{23}E_{32}D_{33}^*E_{22}^*$$

(4.2.6)

$$Q_{12,32}^{(DL)} = D_{12}E_{32}D_{32}^*E_{12}^*$$

(4.2.7)

$$\Xi_{23,33}^{(RD)} = R_{23}D_{33}R_{33}^*D_{32}^*$$

(4.2.8)
\[ P_{22,33}^{(R)} = R_{22}R_{33}R_{23}^{*2} \]  

Using our Z matrix method, we find that among these six invariants there are five linearly independent arguments. A set of five quartic complex invariants with linearly independent arguments is provided by two among the subset \( \{ P_{22,33}^{(\ell)}, Q_{23,33}^{(D\ell)}, Q_{23,32}^{(D\ell)} \} \) together with the three invariants \( Q_{12,32}^{(D\ell)}, \Xi_{23,33}^{(RD)}, \) and \( P_{22,33}^{(R)} \). To obtain an invariant encoding the sixth unremovable phase, we consider 6’th order complex products. We find \( N_{inv,6} = 7 \) of these:

\[ I_{6a} = E_{12}E_{23}D_{32}E_{22}^{*}E_{33}^{*}D_{12}^{*} \]  

\[ I_{6b} = D_{23}D_{32}E_{12}D_{12}^{*}D_{33}^{*}E_{22}^{*} \]  

\[ I_{6c} = E_{12}D_{23}R_{22}^{*}E_{22}^{*}D_{12}^{*}R_{23}^{*} \]  

\[ I_{6d} = E_{12}D_{33}R_{22}^{*}E_{22}^{*}D_{12}^{*}R_{23}^{*} \]  

\[ I_{6e} = E_{12}D_{23}R_{23}^{*}E_{22}^{*}D_{12}^{*}R_{33}^{*} \]  

\[ I_{6f} = E_{12}D_{33}R_{23}^{*}E_{32}^{*}D_{12}^{*}R_{33}^{*} \]

and

\[ I_{6g} = D_{23}^{2}R_{11}^{*}D_{21}^{2}R_{33}^{*} \]

Among these seven 6’th order invariants, the arguments of the first six are linearly dependent upon the five arguments from the quartic invariants, but the argument of \( I_{6g} \) is linearly independent of these and hence accounts for the sixth unremovable phase in the model. Therefore a complete set of complex invariants is given by

\[ \{ Q_{23,33}^{(D\ell)}, Q_{23,32}^{(D\ell)}, Q_{12,32}^{(D\ell)}, \Xi_{23,33}^{(RD)}, P_{22,33}^{(R)}, I_{6g} \} \]
We denote the arguments of these complex invariants as $\phi_j$, $j = 1, ..., 6$, i.e., $\phi_1 = \arg(Q^{(D\ell)}_{23,32})$, etc. An example of an allowed form for the mass matrices in this model after rephasing of the lepton fields is

$$M^{(\ell)'} = \begin{pmatrix} 0 & |E_{12}|e^{i(\phi_2 - \phi_3)} & 0 \\ |E_{21}| & |E_{22}| & |E_{23}| \\ 0 & |E_{32}|e^{i\phi_2} & |E_{33}|e^{i\phi_1} \end{pmatrix} \quad (4.2.18)$$

$$M^{(R)'} = \begin{pmatrix} |R_{11}|e^{i\phi_6} & 0 & 0 \\ 0 & |R_{22}|e^{i(\phi_5 + 2\phi_4)} & |R_{23}|e^{i\phi_4} \\ 0 & |R_{23}|e^{i\phi_4} & |R_{33}| \end{pmatrix} \quad (4.2.19)$$

and

$$M^{(D)'} = \begin{pmatrix} 0 & |D_{12}| & 0 \\ |D_{21}| & 0 & |D_{23}| \\ 0 & |D_{32}| & |D_{33}| \end{pmatrix} \quad (4.2.20)$$

By considering a special case of this model, we can illustrate our statement in section 3 that for leptonic mass matrices, in contrast to quark mass matrices, the quartic and sixth order complex invariants do not necessarily suffice to encode all of the unremovable phases. For this purpose, let us, for example, set $R_{33} = 0$. In this case, $N_{eq} = 14$ and the rank of the resultant $T$ is again 9, so that $N_p = 5$. Among the quartic invariants, the first four remain, but the last two, $\Xi^{(RD)}_{23,33}$ and $P^{(R)}_{22,33}$, evidently vanish. From our previous analysis it follows that among the remaining four quartic invariants, there are three linearly independent arguments. Of the original seven sixth-order complex invariants, the last three evidently vanish. Since the arguments of the first four sixth order invariants are all linearly dependent upon those of the quartic invariants, they does not account for either of the two remaining unremovable phases. Consequently, one must proceed to construct the complex 8'th order complex invariants for this special case. We find the following two:

$$I_{8a} = E_{22}D_{23}D_{32}R_{11}E_{32}^*D_{21}^2R_{23}^* \quad (4.2.21)$$

$$I_{8b} = E_{23}D_{23}D_{32}R_{11}E_{33}^*D_{21}^2R_{23}^* \quad (4.2.22)$$

The arguments of these two invariants are linearly independent of the three arguments from the quartic invariants, yielding the full set of $N_{ia} = N_p = 5$ independent arguments for
this special case of the model where $R_{33} = 0$. A set of complex invariants with linearly independent phases is provided by (i) three out of eqs. (4.2.4)-(4.2.9) together with (ii) $I_{8a}$ and $I_{8b}$. (The general model with $R_{33} \neq 0$ has several additional 8’th order complex invariants, but they do not yield any new phase constraints.)

4.2.2 $n_s = 3$ Model 2

A second model with $n_s = 3$ electroweak-singlet neutrinos is defined by the mass matrices

$$M^{(\ell)} = \begin{pmatrix} 0 & E_{12} & 0 \\ E_{21} & 0 & E_{23} \\ 0 & E_{32} & E_{33} \end{pmatrix} \quad (4.2.23)$$

$$M^{(L)} = \begin{pmatrix} 0 & L_{12} & 0 \\ L_{12} & 0 & 0 \\ 0 & 0 & L_{33} \end{pmatrix} \quad (4.2.24)$$

$$M^{(R)} = \begin{pmatrix} 0 & R_{12} & 0 \\ R_{12} & 0 & R_{23} \\ 0 & R_{23} & R_{33} \end{pmatrix} \quad (4.2.25)$$

$$M^{(D)} = \begin{pmatrix} 0 & D_{12} & 0 \\ D_{21} & 0 & D_{23} \\ 0 & D_{32} & D_{33} \end{pmatrix} \quad (4.2.26)$$

This model has $N_{eq} = 15$. We calculate the $15 \times 9$ matrix $T$ and find $\text{rank}(T) = 9$, so that, according to our theorem, there are $N_p = 6$ unremovable phases in the leptonic mass matrices. To determine which elements can be made real, we construct a set of complex invariants which yields the full set of phase constraints. We find that there are $N_{\text{inv,4}} = 8$ quartic complex invariants:

$$Q_{12,32}^{(DL)} = D_{12} E_{32} D_{32}^* E_{12}^* \quad (4.2.27)$$

$$Q_{23,33}^{(DL)} = D_{23} E_{33} D_{33}^* E_{23}^* \quad (4.2.28)$$

$$\Xi_{21,23}^{(RD)} = R_{12} D_{23} R_{23}^* D_{21}^* \quad (4.2.29)$$

$$\Xi_{33,32}^{(RD)} = R_{33} D_{32} R_{23}^* D_{33}^* \quad (4.2.30)$$
\[ \Xi_{21,32}^{(RD)} = R_{12}D_{32}R_{23}^*D_{21}^* \]  
(4.2.31)

\[ \Omega_{12,12}^{(LRDD)} = L_{12}R_{12}D_{21}^*D_{12}^* \]  
(4.2.32)

\[ \Omega_{12,23}^{(LRDD)} = L_{12}R_{23}D_{23}^*D_{12}^* \]  
(4.2.33)

\[ \Omega_{33,23}^{(LRDD)} = L_{33}R_{23}D_{32}^*D_{33}^* \]  
(4.2.34)

By our usual Z matrix method, we determine that among these eight invariants there are six linearly independent arguments, so that the quartic complex invariants suffice to yield all phase constraints in this model. We find that a subset of the eight quartic invariants which have independent arguments is given, e.g., by

\[ \{ Q_{12,32}^{(DL)}, Q_{23,33}^{(DL)}, \Xi_{33,32}^{(RD)}, \Omega_{12,12}^{(LRDD)}, \Omega_{12,23}^{(LRDD)}, \Omega_{33,23}^{(LRDD)} \} \]  
(4.2.35)

We denote the arguments of these complex invariants as \( \omega_j, j = 1, \ldots, 6 \), respectively. An example of an allowed set of mass matrices after rephasing is

\[ M^{(L)} = \begin{pmatrix} 0 & |E_{12}| & 0 \\ |E_{21}| & 0 & |E_{23}| \\ 0 & |E_{32}|e^{i\omega_1} & |E_{33}|e^{i\omega_2} \end{pmatrix} \]  
(4.2.36)

\[ M^{(L)} = \begin{pmatrix} 0 & |L_{12}| & 0 \\ |L_{12}| & 0 & 0 \\ 0 & 0 & |L_{33}| \end{pmatrix} \]  
(4.2.37)

\[ M^{(R)} = \begin{pmatrix} 0 & |R_{12}|e^{i\omega_3} & 0 \\ |R_{12}|e^{-i\omega_3} & 0 & |R_{23}| \\ 0 & |R_{23}| & |R_{33}|e^{i\omega_4} \end{pmatrix} \]  
(4.2.38)

\[ M^{(D)} = \begin{pmatrix} 0 & |D_{12}| & 0 \\ |D_{21}|e^{-i\omega_5} & 0 & |D_{23}|e^{-i\omega_6} \\ 0 & |D_{32}| & |D_{33}| \end{pmatrix} \]  
(4.2.39)

By our corollary 1, since we have constructed a set \( \Xi_{21,32}^{(RD)} \) of quartic invariants with the full \( N_{ia} = N_p \) set of linearly independent arguments, all higher order complex invariants have
arguments which can be expressed in terms of those of this set, and hence they do not yield any new phase constraints. For completeness, we exhibit the 6’th order complex invariants in this model. Two of these are of \( LRDD^*D^*D^* \) type,

\[
P^{(N)}_{12,66,35} = L_{12}R_{33}D_{32}D_{12}^*D_{23}^*D_{33}^*
\]

(4.2.40)

\[
P^{(N)}_{26,33,54} = L_{33}R_{12}D_{23}D_{21}^*D_{32}^*D_{33}^*
\]

(4.2.41)

while the third is of \( DDLD^*D^*L^* \) type:

\[
P^{(N)}_{12,63,35} = D_{32}D_{33}L_{12}D_{12}^*D_{23}^*L_{23}^*
\]

(4.2.42)

(Several choices of indices on the \( P^{(N)}_{j_1k_1j_2k_2j_3k_3} \) yield the same invariants; we have listed only one for each case.)

4.2.3 \( n_s = 3 \) Model 3

We next give two toy models to illustrate some theoretical points. The first is an example of a model with no complex quartic invariants, where the first complex invariant occurs at sixth order. It is defined by

\[
M^{(\ell)} = \begin{pmatrix} E_{11} & 0 & 0 \\ 0 & E_{22} & 0 \\ 0 & 0 & E_{33} \end{pmatrix}
\]

(4.2.43)

\[
M^{(L)} = \begin{pmatrix} L_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

(4.2.44)

\[
M^{(R)} = \begin{pmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & 0 \\ 0 & 0 & R_{33} \end{pmatrix}
\]

(4.2.45)

\[
M^{(D)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & D_{23} \\ 0 & D_{32} & D_{33} \end{pmatrix}
\]

(4.2.46)
We calculate that \( \text{rank}(T) = 9 \) for this model, so that there is \( N_p = 1 \) unremovable phase. The single independent complex sixth order invariant is of \( DDRD^*D^*R^* \) type:

\[
D^2_{32}R_{33}D^*_{33}R^*_{22}
\]  
(4.2.47)

The corresponding phase may be placed in any of the mass matrix elements involved in (4.2.47).

4.2.4 \( n_s = 3 \) Model 4

This model illustrates that, in contrast to the case in the quark sector, the quartic and sixth order complex invariants do not necessarily suffice to yield all phase constraints in the leptonic sector. The model is defined by \( M^{(L)} \) and \( M^{(D)} \) as in eqs. (4.2.43) and (4.2.46), respectively, together with

\[
M^{(L)} = \begin{pmatrix}
L_{11} & 0 & 0 \\
0 & L_{22} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]  
(4.2.48)

and

\[
M^{(R)} = \begin{pmatrix}
R_{11} & 0 & 0 \\
0 & R_{22} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]  
(4.2.49)

The \( T \) matrix for this model has rank 9, so that \( N_p = 1 \). There are no (nonzero) complex quartic or sixth order invariants. The model has the single independent 8’th order invariant

\[
L_{22}R_{22}D^2_{33}D^*_{33}D^*_{33}D^*_{33}
\]  
(4.2.50)

As before, the corresponding phase may be placed in any of the mass matrix elements involved in (4.2.47).

4.3 Unified Models of Quark and Lepton Mass Matrices

One of the appeals of grand unified theories has always been the fact that they allow one naturally to relate quark and lepton Yukawa, and hence mass, matrices.\footnote{This appeal is offset somewhat by the fact that even when one considers supersymmetric grand unified theories to stabilize the overall gauge hierarchy, there is a still a serious problem associated with the necessity of splitting the electroweak doublet Higgs fields from color triplet Higgs fields which occur in the same}
note that the methods for the counting and allowed placements of the unremovable phases in
the lepton sector discussed here and in Ref. [7] can be combined with our analogous methods
for the quark sector discussed in Ref. [8] in the context of these unified models of lepton and
quark masses. If various elements of the quark and lepton mass matrices are equal, or differ
only by some real factor (because of the types of Yukawa couplings allowed by the grand
unified gauge invariance and by various other symmetries which are imposed to restrict these
couplings), some of the invariants in the quark and lepton sectors may coincide, leading to
a reduction in the total number of phases. We gave an example of this in Ref. [7].

5 Conclusions

The goal of understanding fermion masses and mixing remains one of the most important
outstanding problems in particle physics. In this paper we have given a detailed discussion of
our methods for determining the number of unremovable phases in a given model of leptonic
mass matrices, including new results on higher order invariants and illustrative applications
to specific models. As ongoing and future experiments searching for evidence of neutrino
masses and lepton mixing yield new information, the construction of more tightly constrained
predictive models of fermion masses will continue. We believe that the general tools discussed
here will be of use in these studies.

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