ESSENTIAL MERIDIONAL SURFACES
FOR TUNNEL NUMBER ONE KNOTS

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ABSTRACT. We show that for each pair of positive integers \(g\) and \(n\), there are tunnel number one knots, whose exteriors contain an essential meridional surface of genus \(g\), and with \(2n\) boundary components. We also show that for each positive integer \(n\), there are tunnel number one knots whose exteriors contain \(n\) disjoint, non-parallel, closed incompressible surfaces, each of genus \(n\).

1. Introduction

In this paper we consider essential surfaces, closed or meridional, properly embedded in the exteriors of tunnel number one knots. The exterior of a knot \(k\) is denoted by \(E(k) = S^3 - \text{int } N(k)\). Recall that a knot \(k\) in \(S^3\) has tunnel number one if there exists an arc \(\tau\) embedded in \(S^3\) with \(k \cap \tau = \partial \tau\), such that \(S^3 - \text{int } N(k \cup \tau)\) is a genus 2 handlebody. Such an arc is called an unknotting tunnel for \(k\). Equivalently, a knot \(k\) has tunnel number one if there is an arc \(\tau\) properly embedded in \(E(k)\), such that \(E(k) - \text{int } N(\tau)\) is a genus 2-handlebody; in general, the unknotting tunnels we consider are of this type. Sometimes it is convenient to express a tunnel \(\tau'\) for a knot \(k\) as \(\tau' = \tau_1 \cup \tau_2\), where \(\tau_1\) is a simple closed curve and \(\tau_2\) is an arc connecting \(\tau_1\) and \(\partial N(k)\); by sliding the tunnel we can pass from one expression to the other.

A surface \(S\) properly embedded in a 3-manifold \(M\) is essential if it is incompressible, \(\partial\)-incompressible, and non-boundary parallel. A surface properly embedded in the exterior of a knot \(k\) is meridional if each component of \(\partial S\) is a meridian of \(k\). Let \(M\) be a compact 3-manifold, and let \(S\) be a surface in \(M\), either properly embedded or contained in \(\partial M\). Let \(k\) be a knot in the interior of \(M\), intersecting \(S\) transversely. Let \(\hat{S} = S - \text{int } N(k)\). The surface \(\hat{S}\) is properly embedded in \(M - \text{int } N(k)\), and its boundary on \(\partial N(k)\), if any, consists of meridians of \(k\). We say that \(\hat{S}\) is meridionally compressible in \((M, k)\), if there is an embedded disk \(D\) in \(M\), intersecting \(k\) at most once, with \(\hat{S} \cap D = \partial D\), so that \(\partial D\) is a nontrivial curve on \(\hat{S}\), and is not parallel to a component of \(\partial \hat{S}\) lying on \(\partial N(k)\). Otherwise

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$\hat{S}$ is called meridionally incompressible. In particular if $\hat{S}$ is meridionally incompressible in $(M, k)$, then it is incompressible in $M - k$.

Some results are available on incompressible surfaces in tunnel number one knot exteriors. Regarding meridional surfaces, it is shown in [GR] that the exterior of a tunnel number one knot does not contain any essential meridional planar surface. Another proof of this fact is given in [M]. This says that any tunnel number one knot is indecomposable with respect to tangle sum. Considering closed surfaces, it is shown in [MS] that there are tunnel number one knots whose complements contain an essential torus, and such knots are classified. In [E2] it is proved that for each $g \geq 2$, there exist infinitely many tunnel number one knots whose complements contain a closed incompressible surface of genus $g$; such surfaces are also meridionally incompressible.

In this paper we prove the following,

**Theorem 3.2.** For each pair of integers $g \geq 1$ and $n \geq 1$, there are tunnel number one knots $K$, such that there is an essential meridional surface $S$ in the exterior of $K$, of genus $g$, and with $2n$ boundary components. Furthermore, $S$ is meridionally incompressible.

This gives a positive answer to question 1.8 in [GR]. It follows from [CGLS] that any of the knots of Theorem 3.2 also contains a closed essential surface of genus $\geq 2$. That surface is obtained by somehow tubing the meridional surface. However such a surface will be meridionally compressible.

Combining the construction of [E2,§6] with that of Theorem 3.2, we get the following,

**Theorem 3.3.** For each positive integer $n$, there are tunnel number one knots $K$, such that in the exterior of $K$ there are $n$ disjoint, non-parallel, closed incompressible surfaces, each of genus $n$.

It follows from the construction that one of the surfaces, say $S_1$, is meridionally compressible while the others are meridionally incompressible. It follows also that the surface $S_1$ is the closest to $\partial E(K)$, that is, $S_1$ and $\partial E(K)$ bound a submanifold $M$ which does not contain any of the other surfaces. It follows from [CGLS,2.4.3] that $S_1$ remains incompressible after performing any non-integral Dehn surgery on $K$, and then so does any of the other surfaces. This fact, Theorem 3.3 and the observation that the exterior of a tunnel number one knot is a compact 3-manifold with Heegaard genus 2, imply the following.

**Corollary.** For each positive integer $n$, there are closed, irreducible 3-manifolds $M$, with Heegaard genus 2, such that in $M$ there are $n$ disjoint, non-parallel, closed incompressible surfaces, each of genus $n$.

This corollary improves one of the results of [Q], where it is shown that for each $n$, there exist closed irreducible 3-manifolds with Heegaard genus 2 which contain an incompressible surface of genus $n$.

In Theorem 3.3 the genus of the surfaces grows as much as the number of surfaces. This fact is essential, i.e., it is not just a consequence of the construction method. It follows from the main Theorem of the recent paper [ES], that it is impossible for an irreducible 3-manifold with Heegaard genus $g$, with or without boundary, to contain an arbitrarily large number of disjoint and closed incompressible surfaces of bounded genus.
The idea of the proof of Theorem 3.2 is the following: Start with a tunnel number one knot \( k \), and unknotting tunnel \( \tau \), and a closed incompressible surface in the complement of \( k \) which intersects \( \tau \) in two points. We know by \([MS]\) and \([E2]\) that such knots do exist. Now take an iterate of \( k \) and \( \tau \), i.e., a knot \( k^* \) formed by the union of two arcs \( k^* = k_1 \cup k_2 \), where \( k_1 = \tau \) and \( k_2 \) is an arc lying on \( \partial N(k) \). Thus \( k^* \) intersects \( S \) in two points. It follows that \( k^* \) is a tunnel number one knot (see Lemma 3.1); an unknotting tunnel \( \tau^* \) for \( k^* \) is formed by the union of \( k \) and an arc joining \( k \) to a point in \( k_1 \cap k_2 \). Slide \( \tau^* \) so that it becomes an arc with endpoints on \( k^* \), also denoted by \( \tau^* \). Now take an iterate of \( k^* \) and \( \tau^* \); this is a knot \( k^{**} \) with tunnel number one which intersects \( S \) in as many points as desired. If \( k^* \) and \( k^{**} \) satisfy certain conditions (Theorem 2.1), the surfaces \( S_1 = S - \text{int } N(k^*) \) and \( S_2 = S - \text{int } N(k^{**}) \) are essential meridional surfaces in the exterior of \( k^* \) and \( k^{**} \), respectively.

Throughout, 3-manifolds and surfaces are assumed to be compact, connected and orientable. If \( X \) is contained in a 3-manifold \( M \), then \( N(X) \) denotes a regular neighborhood of \( X \) in \( M \); if \( X \) is contained in a surface \( S \), then \( \eta(X) \) denotes a regular neighborhood of \( X \) in \( S \). \( \Delta(\alpha, \beta) \) denotes the minimal intersection number of two essential simple closed curves on a torus \( T \).

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2. Construction of essential meridional surfaces

Let \( k \) be a knot in \( S^3 \), and let \( \tau' = \tau_1 \cup \tau_2 \) be an unknotting tunnel for \( k \), where \( \tau_1 \) is a simple closed curve, and \( \tau_2 \) is an arc with endpoints in \( \partial N(k) \) and \( \tau_1 \). Let \( S \) be a closed surface of genus \( g \) contained in the exterior of \( k \); then \( S \) divides \( S^3 \) into two parts, denoted by \( M_1 \) and \( M_2 \), where, say, \( k \) lies in \( M_2 \). We say that \( S \) is special with respect to \( k \) and \( \tau' \) if it satisfies:

1. \( \tau_1 \) is disjoint from \( S \), and \( \tau_2 \) intersects \( S \) transversely in one point, so \( \tau_1 \) lies in \( M_1 \);
2. \( S \) is essential in \( E(k) \).

This definition is a variation of the one given in \([E2, \S6]\).

Note that by \([MS]\), \([E1]\), there exist knots with these properties when \( g = 1 \); when \( g \geq 2 \), the existence of knots like these follows from \([E2, \S6.1]\). Note that \( M_2 \cap N(\tau_2) \) is a cylinder \( R \cong D^2 \times I \), so that \( R \cap S \) is a disk \( D_1 \cong D^2 \times \{1\} \), and \( R \cap N(k) \) is a disk \( D_0 \cong D^2 \times \{0\} \). Slide \( \tau_1 \) over \( \tau_2 \), to get an arc \( \tau \) with both endpoints on \( D_0 \subset \partial N(k) \), so that \( \tau \cap M_2 \) consists of two straight arcs contained in \( R \), i.e., arcs which intersect each disk \( D^2 \times \{x\} \) transversely in one point. The surface \( S \) and the arc \( \tau \) then intersect in two points. The arc \( \tau \) has a neighborhood \( N(\tau) \cong D^2 \times I \), so that \( N(\tau) \cap M_2 \subset R \).

Let \( P \) be a solid torus, \( D_0 \) a disk contained in \( \partial P \), and \( \rho = \{ \rho_1, \ldots, \rho_n \} \), a collection of arcs properly embedded in \( P \), so that its endpoints lie in \( D_0 \). We say that this forms a toroidal tangle with respect to \( D_0 \), and denote it by \( (P, D_0, \rho) \).

Recall that the wrapping number of a knot in a solid torus is defined as the minimal number of times that the knot intersects any meridional disk of such solid torus. We define the wrapping number of an arc \( \rho_i \) in \( P \) as the wrapping number of the knot obtained by
joining the endpoints of \( \rho_i \) with an arc in \( D_0 \), and then pushing it into the interior of \( P \). This is well defined.

The tangle \((P, D_0, \rho)\) is good if:

1. Each arc \( \rho_i \) has wrapping number \( \geq 1 \) in \( P \), and there is at least one arc \( \rho_i \) whose wrapping number in \( P \) is \( \geq 2 \);
2. Each arc \( \rho_i \) has no local knots, i.e., if a sphere \( S \) intersects \( \rho_i \) in two points, then \( S \) bounds a ball \( B \) such that \( B \cap \rho_i \) is an unknotted spanning arc.

If the tangle \((P, D_0, \rho)\) is good, then \( D_0 - \partial \rho \) is incompressible in \( P - \rho \), i.e., there is no disk \( D \) properly embedded in \( P \), disjoint from \( \rho \), with \( \partial D \subset D_0 \), and such that \( \partial D \) is essential in \( D_0 - \partial \rho \).

Let \( A \) be an annulus in \( \partial P \), essential in \( P \), so that \( D_0 \subset A \). The tangle \((P, D_0, \rho)\) is good with respect to \( A \) if:

1. \((P, D_0, \rho)\) is good.
2. No arc \( \rho_i \) is isotopic relative to \( \partial \rho_i \) to an arc \( \lambda \) contained in \( A \) (ignoring the other arcs).

Let \( \hat{k} \) be a knot contained in the interior of \( N(k) \cup N(\tau) \). We say that \( \hat{k} \) is specially knotted if:

1. \( \hat{k} \) intersects the disk \( D_0 \) transversely in \( 2n \) points, so that \( \hat{k} \cap N(\tau) \) consists of \( n \) straight arcs in \( N(\tau) \) (\( \cong D^2 \times I \));
2. the toroidal tangle \((N(k), D_0, \rho)\) is good, where \( \rho = \hat{k} \cap N(k) \);
3. in the case that \( k \) is parallel to a curve lying on \( S \), assume also the following: let \( \gamma \) be the curve on \( \partial N(k) \) which cobounds an annulus with a curve on \( S \), and so that \( \gamma \) meets \( D_0 \) in one arc. Let \( A = \eta(\gamma \cup D_0) \). Then \((N(k), D_0, \rho)\) is good with respect to \( A \),

As \( N(\tau) \cap M_2 \subset R \), it follows that \( \hat{k} \cap R \) consists of \( 2n \) straight arcs. So \( \hat{k} \) intersects \( S \) in \( 2n \) points. Let \( \hat{S} = S \cap E(\hat{k}) \). This is a surface properly embedded in \( E(\hat{k}) \), whose boundary consists of \( 2n \) meridians of the knot \( \hat{k} \).

**Theorem 2.1.** Let \( k \) be a knot, \( \tau' = \tau_1 \cup \tau_2 \) an unknotted tunnel for \( k \), and \( S \) a surface which is special with respect to \( k \) and \( \tau' \). Let \( \hat{k} \subset N(k) \cup N(\tau) \) be a knot which is specially knotted. Then the surface \( \hat{S} = S \cap E(\hat{k}) \) is an essential meridional surface in the exterior of \( \hat{k} \). Furthermore, if the surface \( S \) is meridionally incompressible in \((S^3, k)\), then \( \hat{S} \) is meridionally incompressible in \((S^3, \hat{k})\). If \( S \) is meridionally compressible, but the wrapping number of some arc \( \rho_i \) in \( N(k) \) is \( \geq 3 \), where \( \rho = N(k) \cap \hat{k} \), then \( \hat{S} \) is meridionally incompressible.

**Proof.** To prove that the surface \( \hat{S} \) is essential in the exterior of \( \hat{k} \), it suffices to show that it is incompressible, because any two-sided, connected, incompressible surface in an irreducible 3-manifold with incompressible torus boundary must be \( \partial \)-incompressible, unless it is a boundary-parallel annulus, which is not the case here.

Let \( S' = (S \cup \partial R - \text{int} \, (D_1)) \), so \( S' \) is isotopic to \( S \), and let \( \hat{S} = S' \cap E(\hat{k}) \); then \( \hat{S} \) is a surface isotopic to \( \hat{S} \). Denote by \( M_1' \) and \( M_2' \) the complementary regions of \( \hat{S} \) in \( E(\hat{k}) \),
where \( \partial N(k) \cap E(\hat{k}) \) lies in \( M_2' \). Let \( T = \partial N(k) - \text{int} \left( D_0 \right) \). This is a once punctured torus, which is properly embedded in \( M_2' \), i.e., \( \tilde{S} \cap T = \partial T = \partial D_0 \).

Let \( D \) be a compression disk for \( \tilde{S} \). Suppose first that it lies in \( M_1' \). As \( S' \) is essential in \( E(k) \), it follows that \( \partial D \) is a trivial curve on \( S' \) which bounds a disk \( D' \subset S' \), and \( D \cup D' \) bounds a 3-ball \( B \). As \( \partial D \) is supposed to be essential in \( \tilde{S} \), one arc \( \alpha \) of \( \hat{k} \) contained in \( M_1' \) must in fact be contained in the 3-ball \( B \). We may assume that \( \alpha = \tau \). Note that \( \partial D \) must be isotopic in \( S' - \partial \tau \) to \( \partial D_0 \). Then the tunnel \( \tau \) is contained in a 3-ball, which implies that \( k \) is the trivial knot. This is a contradiction.

Suppose then that \( D \) lies in \( M_2' \). Consider the intersection between \( T \) and \( D \). If they do not intersect, then there are two cases: (1) \( D \) is contained in \( N(k) \). In this case \( \partial D \) must lie on \( D_0 \), which implies that \( \partial D \) is trivial on \( \tilde{S} \), or that \( D_0 - \rho \) is compressible in \( N(k) - \rho \), which contradicts the hypothesis. (2) \( D \) is disjoint from \( N(k) \). One possibility is that \( \partial D \) is isotopic to \( \partial D_0 \), but in this case the tunnel \( \tau \) is, as above, contained in a 3-ball which is impossible. Otherwise, by isotoping \( D \) we may assume that \( \partial D \) is contained in \( S \), and then \( D \) is also a compression disk for \( S \) disjoint from \( N(k) \), which contradicts the hypothesis that \( S \) is incompressible in \( E(k) \).

Assume then that \( D \) and \( T \) have nonempty intersection. This intersection consists of a finite number of arcs and simple closed curves. Assume also that \( D \) has been chosen, among all compression disks, to have a minimal number of intersections with \( T \). This implies that any curve or arc of intersection is essential in \( T \), for if one curve (arc) is trivial, then doing surgery on \( D \) with the disk bounded by an innermost curve (outermost arc) we get a disk with fewer intersections with \( T \).

Let \( \sigma \) be a simple closed curve of intersection which is innermost in \( D \), so it bounds a disk \( D' \) whose interior is disjoint from \( T \). If \( D' \) lies in \( N(k) \), then \( \sigma \) is either a meridian of \( T \), or it is parallel to \( \partial T \), but in both cases it follows that \( D_0 - \rho \) is compressible in \( N(k) - \rho \). If the interior of \( D' \) is disjoint from \( N(k) \), then as \( k \) is a nontrivial knot, \( \sigma \) must be trivial on \( T \), which contradicts the choice of \( D \).

Assume then that the intersections between \( D \) and \( T \) consists only of arcs. Let \( \sigma \) be an outermost arc in \( D \) which bounds a disk \( E \). Suppose first that \( E \subset N(k) \). Then \( \partial E = \sigma \cup \delta \), where \( \delta \subset D_0 \). It follows that \( \partial E \) is nontrivial on \( \partial N(k) \), i.e., it is a meridian of \( N(k) \), and then each of the \( \rho_i \) has wrapping number \( \leq 1 \) in \( N(k) \), which contradicts the hypothesis. So \( E \) cannot be contained in \( N(k) \). Again let \( \partial E = \sigma \cup \delta \), where \( \sigma \) is contained in \( T \) and \( \delta \) in \( \tilde{S} \). As \( \sigma \) is nontrivial in \( T \) then \( \delta \) is also nontrivial in \( \tilde{S} - D_0 \). By isotoping \( D \) we can ensure that \( \delta \cap \partial R \) consists of two arcs; let \( E' \subset \partial R \) be a disk containing these arcs in its boundary. Now \( E \cup E' \) is an annulus with one boundary component on \( S \), and the other on \( \partial N(k) \). Here we apply [CGLS.2.4.3], where \( M, S, T, r_0 \) of that theorem correspond in our notation to \( M_2 - \text{int} N(k), S, \partial N(k) \), and the component of \( \partial (E \cup E') \) lying on \( \partial N(k) \), which we denote also by \( r_0 \). Clearly \( S \) compresses after performing meridional surgery on \( \partial N(k) \). Then part (b) of [CGLS.2.4.3] implies that \( \Delta(\mu, r_0) \leq 1 \), where \( \mu \) is a meridian of \( \partial N(k) \). So either \( \mu = r_0 \), or \( r_0 \) goes around \( \partial N(k) \) once longitudinally. The first possibility implies that \( S \) is meridionally compressible, and the second one implies that \( k \) is parallel to a curve lying on \( S \). So we are done, unless one of these cases happens. Note that each of these possibilities excludes the other, for if \( k \) is parallel to a curve on \( S \),
and $S$ is meridionally compressible, then either $S$ is compressible or $S$ is isotopic to $\partial N(k)$.

Suppose first that $S$ is meridionally compressible. Let $\sigma$ be an outermost arc in $D$, which bounds a disk $E$, so that $\partial E = \sigma \cup \delta$, where $\sigma$ is contained in $T$ and $\delta$ in $\tilde{S}$. As above, there is a disk $E' \subset \partial R$, such that $E \cup E'$ is an annulus with one boundary component on $S$, and the other is a meridian of $\partial N(k)$. Consider all the outermost arcs on $D$; by the argument given above we can assume that any one of them determines a curve on $T$ parallel to $\sigma$. Let $F$ be a region on $D$ adjacent to one of the outermost arcs, so that all of its intersections with $T$, except at most one, are outermost arcs. To find such an $F$, take the collection of arcs in $D$ which are not outermost arcs, and among these choose one which is outermost. $F \subset N(k)$, and then either $\partial F$ is trivial on $\partial N(k)$, or $\partial F$ is a meridian of $N(k)$. $\partial F$ consists of, say, $2m$ consecutive arcs, $\partial F = \sigma_1, \delta_1, \ldots, \sigma_m, \delta_m$, where $\sigma_i \subset T$, and $\delta_i \subset D_0$. Then at least $m-1$ of the arcs are parallel to $\sigma$, say $\sigma_1, \ldots, \sigma_{m-1}$. If $\sigma_m$ is not parallel to $\sigma$, then $\partial F$ would go around $N(k)$ once longitudinally, which is impossible, for $\partial F$ bounds a disk in $N(k)$. We conclude that all the arcs $\sigma_i$ are parallel in $T$, as in Figure 1.

![Figure 1](image_url)

Let $E_1, E_2$ be two meridian disks of $N(k)$ whose boundaries are disjoint from $D_0$ and $\cup \sigma_i$. Then $E_1 \cup E_2$ bounds a ball $B$ in $N(k)$ which contains $D_0$ and $F$, after possibly isotoping $F$.

There are two cases:

1. $F$ is parallel to a disk $D_1 \subset \partial N(k)$. Clearly $D_1 \subset B$. $F$ and $D_1$ cobound a 3-ball $B_1$. Suppose that an arc $\rho_i$ is contained in $B_1$. By joining the endpoints of $\rho_i$ with an arc contained in $D_0$, we get a simple closed curve $\rho'_i$, which is contained in $B$, and then its wrapping number in $N(k)$ is 0, which contradicts the hypothesis.

So suppose no arc $\rho_i$ is contained in $B_1$. Consider $D_1 \cap \partial D_0$. This is a collection of arcs which divide $D_1$ into regions which are in $D_0$ or in its complement. If there is an outermost arc on $D_1$ which bounds a disk contained in $D_0$, then we can isotope $F$ (and
then $D$) through $D_0$ to get a compression disk with fewer intersections with $T$. If no outermost arc bounds a disk lying in $D_0$, choose any region $D'_0$ of $D_1 \cap D_0$. There is an arc $\alpha \subset D'_0$, whose endpoints lie on $\partial D_1$ (then $\alpha \subset \text{int } D_0$), and there is a disk $E_0 \subset B_1$, so that $\partial E_0 = \alpha \cup \beta$, where $\beta$ is an arc on $F$. Cut $D$ along $E_0$, getting two disks; at least one of them is a compression disk for $\tilde{S}$, but it has fewer intersections with $T$.

(2) $\partial F$ is a meridian of $N(k)$, so $\partial F$ is parallel to $\partial E_1$ (see Figure 1). So $\partial F$ separates the annulus $\partial B - \text{int}(E_1 \cup E_2)$ into two annuli, denoted by $A_1$ and $A_2$, where $\partial A_i = \partial E_i \cup \partial F$. Let $\rho_i$ be an arc of $\rho$, and $\rho'_i$ the simple closed curve obtained by joining the endpoints of $\rho_i$ with an arc in $D_0$. If the endpoints of $\rho_i$ lie in the same annulus $A_j$, then $\rho_i$ is isotopic rel $\partial \rho_i$ (when ignoring the other arcs), to an arc disjoint from $E_1$. This implies that the wrapping number of $\rho'_i$ in $N(k)$ is 0, for $\rho'_i$ is isotopic to a curve disjoint from $E_1$. If the endpoints of $\rho_i$ lie on different annuli, then $\rho_i$ is isotopic rel $\partial \rho_i$ to an arc which intersects $E_1$ in one point. This implies that the wrapping number of $\rho'_i$ in $N(k)$ is 1. This contradicts the hypothesis that at least one of the arcs have wrapping number $\geq 2$. This completes the proof when the surface $S$ is meridionally compressible.

Suppose now that $k$ is parallel to a curve on $S$. As before, let $\sigma$ be an outermost arc in $D$, which bounds a disk $E$, so that $\partial E = \sigma \cup \delta$, where $\sigma$ is contained in $T$ and $\delta$ in $\tilde{S}$. Recall that the union of $\sigma$ and an arc on $D_0$ is a curve $\gamma$ on $\partial N(k)$ which cobounds an annulus $E \cup E'$ with a curve on $S$. Let $A = \eta(\gamma \cup D_0)$. Consider all the outermost arcs on $D$; recall that any one of them determines a curve on $T$ parallel to $\sigma$. Let $F$ be a region on $D$ adjacent to one of the outermost arcs, so that all of its intersections with $T$, except at most one are outermost arcs. $F$ is then a disk properly embedded in $N(k)$, which intersects $D_0$ in $r$ arcs, and all the arcs on $T \cap F$, except at most one are parallel. Let $\partial F = \sigma_1 \cup \delta_1 \cup \ldots \sigma_r \cup \delta_r$, where $\sigma_i \subset T$, $\delta_i \subset D_0$, and $\sigma_1, \ldots, \sigma_{r-1}$ are parallel to $\sigma$. There is an annulus $\Delta$ properly embedded in $N(k)$, $\partial \Delta = \partial A$. We can assume that $D_0, \sigma_1, \ldots, \sigma_{r-1}$ are contained in $A$. If $\sigma_r$ is not parallel to $\sigma$, then it intersects each component of $\partial \Delta$ in one point. It follows that $\partial F$ is trivial in $\partial N(k)$ if and only if each arc $\sigma_i$ is parallel to $\sigma$.

Suppose first that $\partial F$ is trivial in $\partial N(k)$, then $\partial F \subset A$, and $F$ is parallel to a disk $D_1 \subset A$. We can assume that $F$ and $\Delta$ do not intersect. $F$ and $D_1$ cobound a 3-ball $B_1$. Suppose there is an arc $\rho_i \subset B_1$. The arc $\rho_i$ has no local knots, then it is parallel to an arc $\epsilon_i \subset D_1 \subset A$, i.e., the arc $\rho_i$ is isotopic to an arc lying in $A$, contradicting the hypothesis. See Figure 2.

If there is no arc $\rho_i$ in $B_1$, proceed as in the analogous case when $\tilde{S}$ is meridionally compressible, to get a disk $E_0 \subset B_1$, with $\partial E_0 = \alpha \cup \beta$, where $\alpha \subset D_0 \cap D_1$ and $\beta \subset F$, so that by cutting $D$ along $E_0$, we get another compression disk for $\tilde{S}$ with fewer intersections with $T$.

Suppose now that $\partial F$ is a meridian of $N(k)$. Then $\partial F = \alpha \cup \beta$, where $\alpha \subset \partial N(k) - A$, $\beta \subset A$, so $\alpha \subset \sigma_r$. The annulus $\Delta$ can be isotoped so that $\Delta \cap F$ is a single arc. $A$ and $\Delta$ bound a solid torus $\Delta'$, and $F \cap \Delta'$ is a meridian disk for $\Delta'$. If $\rho_i$ is any of the arcs of $\rho$, then $\rho_i$ can be isotoped to be in the 3-ball $\Delta' - \text{int } N(F)$, and so it is parallel to an arc lying on $A$. See Figure 3. This completes the proof.

Now we sketch a proof that $\tilde{S}$ is meridionally incompressible. Suppose there is a disk $D$
embedded in $S^3$, with $\tilde{S} \cap D = \partial D$, which is a nontrivial curve on $\tilde{S}$, and so that $\hat{k}$ intersects $D$ transversely in one point. If the disk $D$ lies in $M'_1$, then as $S'$ is incompressible in $E(k)$, it follows that $\partial D$ is a trivial curve on $S'$, which is the boundary of a disk contained in $S'$ which intersects $\hat{k}$ once, so $D$ is not a disk of meridional compression.

So assume that $D$ lies in $M'_2$. Look at the intersections between $D$ and $T$, and suppose $D$ has been chosen to have minimal intersection with $T$. This implies that any curve or arc of intersection is essential in $T$.

Suppose there is a curve of intersection, innermost in $D$, which bounds a disk $D'$ which meets $\hat{k}$ once. Then $D'$ must lie in $N(k)$. If $\partial D'$ is a meridian of $N(k)$, then each $\rho_i$ has wrapping number $\leq 1$. If $\partial D'$ is not a meridian of $N(k)$, then $\partial D'$ bounds a disk in $\partial N(k)$ which either lies in $T$ or contains $D_0$. In either case it is impossible for $\rho$ to meet $D'$ in
exactly one point. This shows that simple closed curves of intersection cannot bound disks which intersect \( \tilde{k} \), and then these curves can be removed as before. Suppose there is an outermost arc \( \sigma \) in \( D \) which bounds a disk \( E \) disjoint from \( \tilde{k} \). Doing an argument as the one done to prove the incompressibility of \( \tilde{S} \), we have that \( E \) does not lie in \( N(k) \). By the same argument, such a disk can exist only if \( S \) is meridionally compressible, or if \( k \) is parallel to a curve on \( S \). Note that it is always possible to find an outermost arc which bounds a disk disjoint from \( \tilde{k} \). So the proof is complete, except if we have one of the cases just mentioned.

Suppose first \( S \) is meridionally compressible. In this case we suppose that the wrapping number of some arc \( \rho_i \) in \( N(k) \) is \( \geq 3 \). Take an outermost arc of intersection in \( D \), and suppose it bounds a disk \( D' \) contained in \( N(k) \), which intersects \( \tilde{k} \) in at most one point. \( \partial D' \) is a meridian of \( N(k) \), and then the wrapping number of any arc \( \rho_i \) in \( N(k) \) is \( \leq 2 \), contradicting the hypothesis in this case. So suppose all outermost arcs bound disks which do not lie in \( N(k) \). As in the proof of the incompressibility of \( \tilde{S} \), these arcs in \( T \) are all parallel, and each one of them, together with an arc in \( D_0 \) is a meridional curve on \( \partial N(k) \).

If there is a region \( F \subset D \), such that all the intersections of \( F \) with \( T \), except at most one, are outermost arcs, and so that \( F \) is disjoint from \( \tilde{k} \), proceed as in the proof of the incompressibility of \( \tilde{S} \). If there is no such region \( F \), then \( T \cap D \) consists of \( m \) arcs, all of which are outermost arcs in \( D \), so that the complement of the arcs is a single region \( F' \) contained in \( N(k) \) and intersecting \( \tilde{k} \) once. There are two cases:

1. \( F' \) is parallel to a disk \( D_1 \subset \partial N(k) \). \( F' \) and \( D_1 \) cobound a 3-ball \( B_1 \). If some arc \( \rho_j \) is contained in \( B_1 \) then its wrapping number in \( N(k) \) is 0, contradicting the hypothesis. So there is just one arc \( \rho_i \) which intersects \( B_1 \); one of its endpoints is in \( D_1 \) and the arc intersects \( F' \) in one point. As \( \rho_i \) has no local knots, \( B_1 \cap \rho_i \) is an unknotted spanning arc in \( B_1 \). As in the case of the incompressibility, there is a disk \( E_0 \subset B_1 \), so that \( \partial E_0 = \alpha \cup \beta \), where \( \beta \) is an arc on \( F' \), and \( \alpha \) is an arc in \( D_0 \cap D_1 \). Cut \( D \) along \( E_0 \), getting two disks; at least one of them is a meridional compression disk for \( \tilde{S} \), but it has fewer intersections with \( T \).

2. \( \partial F' \) is a meridian of \( N(k) \). The same proof as in the case of the incompressibility show that if this happens then the wrapping number of any arc \( \rho_i \) in \( N(k) \) is \( \leq 2 \).

Suppose now that \( k \) is parallel to a curve on \( S \). If \( \sigma \) is an outermost arc of intersection in \( D \), bounding a disk \( E \) which does not intersect \( \tilde{k} \), then \( E \) is not contained in \( N(k) \), and \( \partial E = \sigma \cup \delta \), where \( \delta \) is contained in \( \tilde{S} \). The union of \( \sigma \) and an arc on \( D_0 \) is a curve \( \gamma \) on \( \partial N(k) \) which cobounds an annulus with a curve on \( S \); so \( \gamma \) is a curve which goes around \( N(k) \) once longitudinally. If there is an outermost arc of intersection in \( D \) bounding a disk \( D' \) which intersects \( \tilde{k} \) in one point, then \( D' \) is a meridian disk of \( N(k) \). In particular this shows that \( D \cap T \) cannot consist of just one arc. As before, if \( \sigma' \) is another outermost arc in \( D \) which bounds a disk disjoint from \( \tilde{k} \), then \( \sigma' \) is parallel to \( \sigma \).

Now proceed as in the proof of the incompressibility of \( \tilde{S} \) in the case that \( k \) is parallel to a curve on \( S \). The point is to find a region \( F \subset D \), such that all the intersections of \( F \) with \( T \), except at most one, are outermost arcs, and so that \( F \) is disjoint from \( \tilde{k} \). If such region exists we are done. If there is an outermost arc on \( D \) which bounds a disk which intersects \( \tilde{k} \), then such region \( F \) does exists, for otherwise \( D \cap T \) will consist of just one arc. If such
a region \( F \) does not exist, the only possibility left is that \( D \cap T \) consists of \( m \) arcs, all of which are outermost arcs, and \( D \cap N(k) \) is a single disk \( F' \) which meets \( k \) once. Then \( \partial F' \) is completely contained in the annulus \( A = \eta(\gamma \cup D_0) \), which implies that \( \partial F' \) is trivial on \( A \). So \( F' \) bounds a disk \( D_1 \) contained in \( A \); \( F' \) and \( D_1 \) cobound a ball \( B_1 \). If some arc \( \rho_j \) is contained in \( B_1 \) then it is parallel to an arc lying on \( A \), contradicting the hypothesis. So there is just one arc \( \rho_i \) which intersects \( B_1 \); one of its endpoints is in \( D_1 \) and the arc intersects \( F' \) in one point. As \( \rho_i \) has no local knots, \( B_1 \cap \rho_i \) is an unknotted spanning arc in \( B_1 \). As in the proof of the incompressibility of \( \hat{S} \), we can boundary compress \( D \), getting another meridional compression disk for \( \hat{S} \), but with fewer intersections with \( T \). □

**Remark.** The conditions imposed on the tangle \((N(k), D_0, \rho)\) are somehow local, i.e., they consider each arc separately. Giving to the tangle some global property might produce a slightly stronger theorem.

### 3. Tunnel number one knots and meridional surfaces.

Let \( k \) be a tunnel number one knot, and \( \tau \) an unknotting tunnel for \( k \) which is an embedded arc with endpoints lying on \( \partial N(k) \). Assume that a neighborhood \( N(k \cup \tau) \) is decomposed as \( N(k \cup \tau) = N(k) \cup N(\tau) \), where \( N(k) \) is a solid torus, \( N(\tau) \cong D^2 \times I \), \( N(k) \cap N(\tau) \) consists of two disks \( E_0 \) and \( E_1 \), and \( \tau = \{0\} \times I \).

Let \( k^* \) be a knot formed by the union of two arcs, \( k^* = k_1 \cup k_2 \), such that \( k_1 \) is contained in \( \partial N(k) \), and \( k_2 = \tau \). We say that \( k^* \) is an iterate of \( k \) and \( \tau \).

**Lemma 3.1.** Let \( k \) and \( \tau \) be as above, and let \( k^* \) be an iterate of \( k \) and \( \tau \). Then \( k^* \) is a tunnel number one knot. An unknotting tunnel \( \beta' \) for \( k^* \) is given by the union of \( k \) and a straight arc in \( N(k) \) connecting \( k^* \) and \( k \).

**Proof.** \( N(k) - k \) is homeomorphic to a product \( T \times [0,1] \). Let \( \delta \) be a straight arc in \( N(k) \) connecting \( k \) and one of the points \( k_1 \cap k_2 \), i.e., it is an arc which intersects each torus \( T \times \{x\} \) in one point. Then \( \beta' = k \cup \delta \) is an unknotting tunnel for \( k^* \). To see that, slide \( k_1 \) over \( \delta \) and then over \( k \) to get a 1-complex which is clearly equivalent to \( k \cup \tau \), so its complement is a genus 2 handlebody. □

Let \( k^* \) be an iterate of \( k \) and \( \tau \). It follows by construction that \( k^* \subset N(k \cup \tau) \). Also if \( \beta' \) is the unknotting tunnel for \( k^* \) given by the lemma, then \( k^* \cup \beta' \subset N(k \cup \tau) \). Now \( \beta' \) can be modified to be an arc \( \beta \) with endpoints in \( k^* \). It follows that if \( k^{**} \) is an iterate of \( k^* \) and \( \beta \), then \( k^{**} \) can be isotoped to lie in \( N(k \cup \tau) \). By isotoping \( k^{**} \), if necessary, we have that \( k^{**} \cap N(\tau) \) consists of a collection of arcs parallel to \( \tau \).

**Theorem 3.2.** For each pair of integers \( g \geq 1 \) and \( n \geq 1 \), there are tunnel number one knots \( K \) such that there is an essential meridional surface \( \hat{S} \) in the exterior of \( K \), of genus \( g \), and with \( 2n \) boundary components. Furthermore, \( \hat{S} \) is meridionally incompressible.

**Proof.** Let \( k \) be a tunnel number one knot. Suppose that \( k \) has an unknotting tunnel \( \tau' = \tau_1 \cup \tau_2 \), where \( \tau_1 \) is a simple closed curve, and \( \tau_2 \) is an arc connecting \( k \) and \( \tau_1 \). Suppose there is a closed surface \( S \) of genus \( g \geq 1 \) embedded in the exterior of \( k \), which is special with respect to \( k \) and \( \tau' \).
$S$ divides $S^3$ into two parts, $M_1$ and $M_2$, where, say, $\tau_1$ is contained in $M_1$. $M_2 \cap N(\tau')$ is a cylinder $R \cong D^2 \times I$, so that $R \cap S$ is a disk $D_1$, and $R \cap N(k)$ is a disk $D_0$. Slide $\tau_1$ over $\tau_2$, to get an arc $\tau$ with both endpoints on $D_0 \subset N(k)$, so that $\tau \cap M_2$ consists of two straight arcs contained in $R$. The surface $S$ and the arc $\tau$ intersect in two points.

Let $k^*$ be an iterate of $k$ and $\tau$; then $k^* = k_1 \cup k_2$, where $k_2$ is an arc parallel to $\tau$, so it intersects $S$ in two points. Now $k_1$ is an arc in $\partial N(k)$ whose endpoints lie on $D_0$. By pushing $k_1$ into the interior of $N(k)$ we get a properly embedded arc in $N(k)$. Clearly $k_1$ can be chosen so that $(N(k), D_0, k_1)$ forms a good tangle, just by taking an arc whose wrapping number in $N(k)$ is $\geq 2$. Note that $k_1$ has no local knots in $N(k)$, for it is parallel to an arc lying in $\partial N(k)$. If $k$ is parallel to a curve on $S$, let $\lambda$ be the curve on $\partial N(k)$ which cobounds an annulus with a curve on $S$, so that $\lambda$ meets $D_0$ in one arc; let $A = \eta(\gamma \cup D_0)$. Clearly $k_1$ can be chosen so that $(N(k), D_0, k_1)$ is good with respect to $A$, say by twisting $k_1$ meridionally as many times as necessary; this can be done because the annulus $A$ goes longitudinally once around $N(k)$, and the wrapping number of $k_1$ in $N(k)$ is $\geq 2$. So $k^*$ can be chosen to be specially knotted in $N(k \cup \tau)$. It follows from Theorem 2.1 that $\hat{S} = S \cap E(k^*)$ is an essential meridional surface in $E(k^*)$, and $\partial \hat{S}$ consists of two meridians of $k^*$.

This implies that $k^*$ is a tunnel number one knot which has an unknotted tunnel $\beta' = k \cup \delta$, where $\delta$ is a straight arc connecting $k^*$ and $k$. Let $\beta$ be the arc obtained after sliding $k$ over $\delta$. $N(k^* \cup \beta)$ can be chosen so that it is contained in $N(k \cup \tau)$. Let $k^{**}$ be an iterate of $k^*$ and $\beta$, so $k^{**} = k_1 \cup k_2$, where $k_1 \subset \partial N(k^*)$, and $k_2$ is the tunnel $\beta$. Note that $k^{**} \subset N(k \cup \tau)$. The arc $k_1$ can be isotoped so that $k_1 \cap N(\tau)$ consist of straight arcs, and it can be chosen so that $k_1 \cap N(\tau)$ consists of $n$ arcs, $n$ being a fixed positive integer. So $k^{**}$ intersects $S$ in $2n$ points. $k^{**} \cap N(k)$ then consists of $n$ arcs, $\rho_0, \ldots, \rho_n$, which are properly embedded on $N(k)$, and whose endpoints lie in $D_0$. Clearly $k^{**}$ can be chosen so that $(N(k), D_0, \rho)$ forms a good tangle, say by choosing them so that each arc $\rho_i$, except one, is parallel to $k_1$, and so that each has wrapping number $\geq 2$. The remaining arc can be chosen to be a band sum of the arcs $k_1$ and the knot $k$, so it can be chosen to have wrapping number $\geq 3$. If $k$ is parallel to $S$, again $k^{**}$ can be chosen so that $(N(k), D_0, \rho)$ is good with respect to $A$. Then by Theorem 2.1, the surface $\hat{S} = S \cap E(k^{**})$ is an essential meridional surface in $E(k^{**})$, and $\partial \hat{S}$ consists of $2n$ meridians of $k^{**}$.

If $S$ is meridionally incompressible, then $\hat{S}$ is meridionally incompressible. If $S$ is meridionally compressible, then $k^*$ and $k^{**}$ can be chosen so that $\hat{S}$ is meridionally incompressible. \hfill \Box

It follows from the proof of Theorem 3.2 that for the knots $k$ constructed in [E2], there are many iterates of $k$, whose exteriors contain an essential meridional surface. This is because for such knots, there is an unknotted tunnel $\tau'$ and a surface $S$ which is special with respect to $k$ and $\tau'$. Note also that some of these knots $k$ are parallel to the surface $S$, while others are not [E2,8.2].

An example which illustrates Theorem 3.2 is shown in Figure 4. Let $k$ be the (2,-11)-cable of the left hand trefoil; there is a torus $S$ and unknotted tunnel $\tau'$ for $k$, so that $S$ is special with respect to $k$ and $\tau'$. Note that $k$ is parallel to a curve on $S$. The knot $k^*$ shown in Figure 4 is an iterate of $k$ and the tunnel $\tau'$. It is not difficult to check that
$k^*$ satisfies the conditions of Theorem 2.1. So it follows that $\hat{S}$ is an essential meridional surface in $E(k^*)$.

Combining the last theorem and the construction given in [E2,§6], we get the following.

**Theorem 3.3.** For each positive integer $n$, there are tunnel number one knots $K$, such that in the exterior of $K$ there are $n$ disjoint, non-parallel, closed incompressible surfaces. Each of the surfaces has genus $n$. One of the surfaces is meridionally compressible; the others are meridionally incompressible.

**Proof.** Recall the construction given in [E2,§6]. Let $k_1$ be a knot, $\tau' = \tau_1 \cup \tau_2$ an unknotted tunnel, and $S_1$ an essential surface of genus $g$ embedded in $E(k_1)$, which intersects $\tau'$ in one point. So $S_1$ is special with respect to $k_1$ and $\tau'$ (in both definitions, the one given here and the one in [E2,§6]; see [E2,6.1], which shows that this is true). Let $T = \partial N(k_1)$.

Let $A$ be an annulus contained in $T$, and let $\alpha$ be the core of this annulus. Suppose that $\alpha$ wraps around $N(k_1)$ at least twice longitudinally. If $k_1$ is parallel to $S_1$, suppose also that $\Delta(\gamma, \alpha) \geq 2$, where $\gamma$ is a curve on $\partial N(k_1)$ which cobounds an annulus with a curve on $S_1$.

$S_1$ divides $S^3$ into two parts, $M_1$ and $M_2$, where, say, $k_1$ is contained in $M_2$. Let $\tau'_2 = M_2 \cap \tau_2$; so $\tau'_2$ is an arc with an endpoint on $S_1$ and the other on $\partial N(k)$, which we assume lies on the curve $\alpha$. The curve $\alpha$ goes around $N(k)$ at least twice longitudinally, then it is a toroidal graph of type 1 in $N(k)$, as defined in [E2,§4]. Let $M = M_2 - \text{int } N(k)$. $M$ is a 3-manifold with incompressible boundary. To show that $\tau'_2 \cup \alpha$ is a cabled graph in $M_2$, as defined in [E2,§6], it suffices to prove that $S_1$ remains incompressible after Dehn filling $M$ along $\partial N(k)$ with slope $\alpha$. If $k$ is not parallel to a curve on $S_1$, then as $\Delta(\alpha, \mu) \geq 2$, this follows from the main Theorem of [Wu]. If $k$ is parallel to a curve
on $S_1$, then by hypothesis, $\Delta(\alpha, \gamma) \geq 2$, and by [CGLS,2.4.3] it follows that $S_1$ remains incompressible.

$N(\tau_2')$ is a cylinder $R \cong D^2 \times I$, so that $R \cap S_1$ is a disk $D_1$, and $R \cap N(k_1)$ is a disk $D_0$. Assume that $D_0 \subset A$. Consider the manifold $W = M_1 \cup R \cup N(A)$, and let $\Sigma = \partial W$. As $\tau_2' \cup \alpha$ is a cabled graph in $M_2$, it follows from [E2,6.3] that $\Sigma$ is incompressible in $S^3 - \text{int } W$.

Let $\tau$ be the arc obtained by sliding $\tau_1$ over $\tau_2$, so that $M_2 \cap \tau \subset R$. Now take an iterate $k_2$ of $k_1$ and $\tau$ of a special form. As before $k_2 = \kappa_1 \cup \kappa_2$, where $\kappa_2 = \tau$, and $\kappa_1$ is an arc in $\partial N(k_1)$. Suppose that $\kappa_1$ is contained in $A$, so that its wrapping number in $N(A)$ is $\geq 2$ (i.e., $\rho = k_2 \cap N(A)$ is a properly embedded arc in $N(A)$ whose endpoints lie on $D'_0 = R \cap \partial N(A)$, and we are requiring that the curve obtained from $\rho$ by joining its endpoints with an arc lying on $D'_0$ has wrapping number $\geq 2$ in $N(A)$). Then $k_2 \subset W$, and it follows from [E2,6.4] that $\Sigma$ is incompressible and meridionally incompressible in $(W, k_2)$. So $\Sigma$ is a meridionally incompressible surface contained in the exterior of $k_2$ of genus $g + 1$. By [E2,8.2], it follows that $k_2$ is not parallel to a curve lying on $\Sigma$.

It is not difficult to see that the knot $k_2$ also satisfies the hypothesis of Theorem 2.1; in particular, note that the arc $\kappa_1$ has wrapping number $\geq 4$ in $N(k_1)$ (for $\kappa_1$ has wrapping number $\geq 2$ in $N(A)$, and $\alpha$ has winding number $\geq 2$ in $N(k_1)$). Therefore the surface $\hat{\Sigma} = S_1 \cap E(k_2)$ is meridionally incompressible in $E(k_2)$, its boundary consists of two meridians of $k_2$. By tubing $\hat{\Sigma}$, we get two closed surfaces in $E(k_2)$, of genus $g + 1$. By an application of the handle addition Lemma [J], one of the surfaces must be incompressible in $E(k_2)$; this has to be the surface lying on $M_2$, for the one lying in $M_1$ bounds a handlebody. Denote by $\hat{\Sigma}$ such an incompressible surface; note that it is meridionally compressible. Then there are two different closed incompressible surfaces in $E(k_2)$, $\Sigma$ and $\hat{\Sigma}$. By isotoping $\hat{\Sigma}$ into $W$, these surfaces become disjoint and are obviously non-parallel.

Note that there is an unknotted tunnel $\beta' = k_1 \cup \delta$ for $k_2$, where $\delta$ is a straight arc in $N(k_1)$ connecting $k_1$ and $k_2$ which intersects both surfaces $\Sigma$ and $\hat{\Sigma}$ in one point. Then $\Sigma$ and $\hat{\Sigma}$ are both special with respect to $k_2$ and $\beta$. Note that $\hat{\Sigma}$ is closer to $k_2$ and $\Sigma$ is closer to $k_1$; that is, the arc $\delta$, when going from $k_2$ to $k_1$, intersects first $\hat{\Sigma}$ and then $\Sigma$.

We have proved that there is a tunnel number one knot $k_2$ which has an unknotted tunnel $\tau' = \tau_1 \cup \tau_2$, and two disjoint, non-parallel closed incompressible surfaces in its exterior, each of genus $g + 1$, denoted by $\Sigma$ and $\hat{\Sigma}$, and which are special with respect to $k_2$ and $\tau'$. $\Sigma$ is meridionally incompressible and $\hat{\Sigma}$ is meridionally compressible, and the arc $\tau_2$, when going from $k_2$ to $\tau_1$ intersects first $\hat{\Sigma}$ and then $\Sigma$. Furthermore, $k_2$ is not parallel to a curve lying on any of the two surfaces.

Suppose by induction that we have a tunnel number one knot $k_n$, which has an unknotted tunnel $\tau' = \tau_1 \cup \tau_2$, and $n$ disjoint, non-parallel closed incompressible surfaces in its exterior, of genus $g + n$, denoted by $S_1, S_2, \ldots, S_n$, which are special with respect to $k_n$ and $\tau'$. $S_2, \ldots, S_n$ are meridionally incompressible and $S_1$ is meridionally compressible, and the arc $\tau_2$, when going from $k_n$ to $\tau_1$ intersects the surfaces in the order $S_1, S_2, \ldots, S_n$. Furthermore, $k_n$ is not parallel to a curve lying on any of the surfaces.

The above construction can be repeated with $k_n, \tau' = \tau_1 \cup \tau_2$ and $S_1, S_2, \ldots, S_n$. $S_i$ divides $S^3$ into $M^i_1$ and $M^i_2$, where $k_n$ lies in $M^i_2$. Clearly, if $i < j$ then $M^i_2 \subset M^j_2$. Let $\alpha$ be
a simple closed curve on \(\partial N(k_n)\), which goes at least twice longitudinally around \(N(k_n)\). Suppose that the endpoint of \(\tau_2\) lies on \(\alpha\). Let \(\tau_2^i = M_2^i \cap \tau_2\); then, as above, \(\alpha \cup \tau_2^i\) is a cabled graph in \(M_2^i\), for \(k_n\) is not parallel to a curve lying on \(S_1\).

Let \(R_i\) be a regular neighborhood of \(\tau_2^i\) in \(M_2^i\), so that \(R_i \cap S_i\) is a disk \(D_1^i\), and \(R \cap N(k_n)\) is a disk \(D_0\). Assume that \(D_0^i \subset A\), where \(A = \eta(\alpha)\). Consider the manifold \(W_i = M_1^i \cup R_i \cup N_i(A)\), where \(N_i(A)\) is a neighborhood of \(A\). Let \(\Sigma_i = \partial W_i\). As \(\tau_2^i \cup \alpha\) is a cabled graph in \(M_2^i\), it follows from [E2,6.3] that \(\Sigma_i\) is incompressible in \(S^3 - \text{int} \, W_i\). The neighborhoods \(R_i \cup N_i(A)\) can be chosen to be thinner if \(j > i\), that is, \(M_2^i \cap (R_j \cup N_j(A)) \subset R_i \cup N_i(A)\) if \(i < j\). Then the surfaces \(\Sigma_1, \Sigma_2, \ldots, \Sigma_n\) are disjoint.

Let \(\tau\) be the arc obtained by sliding \(\tau_1\) over \(\tau_2\), so that \(M_2^2 \cap \tau \subset R_i\), for all \(i\). Now take an iterate \(k_{n+1}\) of \(k_n\) of a special form. As before \(k_{n+1} = \kappa_1 \cup \kappa_2\), where \(\kappa_2 = \tau\), and \(\kappa_1\) is an arc in \(\partial N(k_j)\). Suppose that \(\kappa_1\) is contained in \(A\), so that its wrapping number in \(N_n(A)\) is \(\geq 2\). Then \(k_{n+1} \subset W_i\), and it follows from [E2,6.4] that \(\Sigma_i\) is incompressible and meridionally incompressible in \((W_i, k_{n+1})\). So \(\Sigma_i\) is a meridionally incompressible surface in the exterior of \(k_{n+1}\) of genus \(g + 1\). Again by [E2,8.2], it follows that \(k_{n+1}\) is not parallel to a curve lying on \(\Sigma_i\).

The knot \(k_{n+1}\) intersects the surface \(S_n\) in two points, the wrapping number of \(\kappa_2\) in \(N(k_n)\) is \(\geq 4\), and \(k_n\) is not parallel to a curve on \(S_n\). So \(k_{n+1}\) and \(S_n\) satisfy the conditions of Theorem 2.1, and then \(\hat{S}_n = S_n \cap E(k_{n+1})\) in an incompressible, meridionally incompressible surface in \(E(k_{n+1})\) whose boundary consists of two meridians of \(E(k_{n+1})\). Then as above, by tubing \(\hat{S}_n\) on the side of \(M_2^j\) and isotoping into \(W_n\), we get a closed surface \(\Sigma_{n+1}\) which is incompressible but meridionally compressible in the exterior of \(k_{n+1}\). The tube added to the surface can be chosen so that it lies in the interior of \(R_n \cup N_n(A)\); this ensures that \(\Sigma_{n+1}\) is disjoint from \(\Sigma_i\), for \(1 \leq i \leq n\).

(From the surface \(S_i\) we can also get a meridionally compressible surface \(\Sigma'_i\), but it will intersect \(\Sigma_j\), if \(i < j\). But note that \(\Sigma_{n+1} = \Sigma_n, \Sigma_{n-1}, \ldots, \Sigma_2, \Sigma'_1, \Sigma_1\), are disjoint).

There is an unknotting tunnel \(\beta'\) for \(k_{n+1}\) of the form \(\beta' = k_n \cup \delta\), where \(\delta\) is a straight arc in \(N(k_n)\) connecting \(k_n\) and \(k_{n+1}\). Note that \(\delta\) intersects each surface \(\Sigma_i\) in one point; this implies that \(\Sigma_i\) is special w.r.t. \(k_{n+1}\) and \(\beta'\). Note also that the arc \(\delta\), when going from \(k_{n+1}\) to \(k_n\), intersects the surfaces in the order \(\Sigma_{n+1}, \Sigma_n, \ldots, \Sigma_1\). Finally, note that the surfaces cannot be parallel, for if two of them were, then two of the surfaces \(S_i\) would also be parallel.

This shows that \(k_{n+1}, \beta', \Sigma_{n+1}, \Sigma_n, \ldots, \Sigma_1\) satisfy the induction hypothesis. This completes the proof.

By starting with a surface \(S\) of genus 1, and repeating the construction \(n - 1\) times, we get the desired conclusion. \(\square\)

**Remark.** It follows from the proof of the above theorem that by changing the induction hypothesis, we can find a tunnel number one knot \(k\), with \(n\) incompressible surfaces in its exterior, \(S_1, S_2, \ldots, S_n\), so that \(S_n\) is meridionally incompressible, but \(S_i\), for \(1 \leq i \leq n - 1\), is meridionally compressible, and \(S_n\) is the surface which is farthest from the knot. It follows also that there are tunnel number one knots whose exteriors contain two collection of disjoint incompressible surfaces, \(S_1, \ldots, S_n\), and \(\Sigma_1, \ldots, \Sigma_n\), where the \(S_i\) are meridionally incompressible, and the \(\Sigma_i\) are meridionally compressible.
References.

[CGLS] M. Culler, C.McA. Gordon, J.Luecke and P. Shalen, Dehn surgery on knots, Ann. Math 125 (1987), 237-300.

[E1] M. Eudave-Muñoz, On nonsimple 3-manifolds and 2-handle addition, Topology Appl. 55 (1994), 131-152.

[E2] M. Eudave-Muñoz, Incompressible surfaces in tunnel number one knots complements, Topology Appl. 98 (1999), 167-189.

[ES] M. Eudave-Muñoz and J. Shor, A universal bound for surfaces in 3-manifolds with a given Heegaard genus, Algebraic and Geometric Topology 1 (2001), 31-37.

[GR] C.McA. Gordon and A. Reid, Tangle decompositions of tunnel number one knots and links, Journal of Knot Theory and Its Ramifications 4 (1995), 389-409.

[J] W. Jaco, Adding a 2-handle to a 3-manifold: An application to property R, Proc. Amer. Math. Soc. 92 (1984), 288-292.

[M] K. Morimoto, Planar surfaces in a handlebody and a theorem of Gordon-Reid, Proceedings of Knots 96 (S. Suzuki, ed.), World Scientific Publishing Co., 1997, pp. 123-146.

[MS] K. Morimoto and M. Sakuma, On unknotting tunnels for knots, Math. Ann. 289 (1991), 143-167.

[Q] R. Qiu, Incompressible surfaces in handlebodies and closed 3-manifolds of Heegaard genus 2, Proc. Amer. Math. Soc. 128 (2000), 3091-3097.

[Wu] Ying-Qing Wu, Incompressibility of surfaces in surgered 3-manifolds, Topology 31 (1992), 271-279.

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