FINITE TIME/INFINITE TIME BLOW-UP BEHAVIORS FOR THE INHOMOGENEOUS NONLINEAR SCHRODINGER EQUATION

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Abstract. In this work, we consider the following focusing inhomogeneous nonlinear Schrödinger equation

\[ i\partial_t u + \Delta u + |x|^{-b}|u|^pu = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N \]

with \( 0 < b < \min\{2, N\} \) and \( \frac{4-2b}{N} < p < \frac{4-2b}{2} \). Assume that \( u_0 \in H^1(\mathbb{R}^N) \) and beyond the ground state threshold, then we prove the following two statements,

1. when \( \frac{4-2b}{N} < p < \min\{\frac{4}{N}, \frac{4-2b}{N}\} \), or \( p = \frac{4}{N} \) when \( b \in (0, \frac{2}{N}) \), then the corresponding solution blows up in finite time;

2. when \( \frac{4}{N} < p < \frac{4-2b}{2} \), we prove the finite or infinite time blow-up. Moreover, we can further obtain a precise lower bound of infinite time blow-up rate, that is

\[ \sup_{t \in [0, T]} \|\nabla u(t)\|_{L^2} \gtrsim T^\kappa, \quad \text{for some } \kappa > 0. \]

To our knowledge, the statement (1) establishes the first finite time blow-up result for this equation in the intercritical case when the initial data \( u_0 \) doesn’t have finite variance and is non-radial. The statement (2) gives the first result for the infinite time blow-up rate for this equation.

1. Introduction

In this paper, we consider the following focusing inhomogeneous nonlinear Schrödinger (INLS) equation

\[
\begin{cases}
    i\partial_t u + \Delta u + |x|^{-b}|u|^pu = 0, \\
u(x, 0) = u_0,
\end{cases}
\]

(1.1)

where \( u = u(t, x) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C} \) is a complex-valued function, \( 0 < b < \min\{2, N\} \) and \( \frac{4-2b}{N} < p < 2_b^* \), where \( 2_b^* = \infty \) if \( N = 1, 2 \), and \( 2_b^* = \frac{4-2b}{N-2} \) if \( N \geq 3 \). The equation has various physical contexts such as stable high power propagation can be achieved in plasma by sending a preliminary laser beam that creates a channel with a reduced electron density, and thus reduces the nonlinearity inside the channel (see [23, 27]). In particular, if \( b = 0 \), equation (1.1) is the following classical nonlinear Schrödinger (NLS) equation

\[ i\partial_t u + \Delta u + |u|^pu = 0. \]  

(1.2)

It is well known that if \( u(t, x) \) is a solution of (1.1), so is

\[ u_{\lambda}(t, x) = \lambda^{2-b} u(\lambda^2 t, \lambda x), \quad \lambda > 0. \]
Denote \( s_c = \frac{N}{2} - \frac{2b}{p} \), then the scaling leaves \( \dot{H}^{s_c} \) norm invariant, that is,
\[
\|u\|_{\dot{H}^{s_c}} = \|u_\lambda\|_{\dot{H}^{s_c}}.
\]
Then, \( s_c \) is called the critical regularity. If \( s_c = 0 \) \( (p = \frac{4-2b}{N}) \), the problem is known as mass-critical or \( L^2 \)-critical; if \( s_c < 0 \) \( (0 < p < \frac{4-2b}{N}) \), it is called mass-subcritical or \( L^2 \)-subcritical; if \( 0 < s_c < 1 \), it is known as mass-supercritical and energy-subcritical (or intercritical).

In addition, the solution satisfies the conservation of mass and energy, defined respectively by
\[
M(u(t)) := \int_{\mathbb{R}^N} |u|^2 dx = M(u_0),
\]
\[
E(u(t)) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u|^{p+2} dx = E(u_0).
\]

The well-posedness theory of equation (1.1) was first studied by Genoud-Stuart [24]. Using the abstract theory of Cazenave [2], they showed that the equation (1.1) is local well-posed in \( H^1(\mathbb{R}^N) \) if \( 0 < p < 2^*_c \) \( (s_c < 1) \), as well as global well-posedness for any initial data if \( 0 < p < \frac{4-2b}{N} \) \( (s_c < 0) \) and sufficiently small initial data if \( \frac{4-2b}{N} \leq p < 2^*_c \) \( (0 \leq s_c < 1) \) in \( H^1(\mathbb{R}^N) \). Farah [14] proved a Gagliardo-Nirenberg type estimate and used it to establish sufficient conditions for global well-posedness in \( H^1(\mathbb{R}^N) \). Guzmán [25] used the contraction mapping principle based on the known classical Strichartz estimates to establish the local and global well-posedness for the INLS equation (1.1) in Sobolev spaces \( H^s(\mathbb{R}^N) \), \( 0 \leq s \leq 1 \). As for the scattering theory of equation (1.1), Farah and Guzmán [15] proved that when the initial data is radial, \( 0 < b < \frac{N}{2} \), and \( p = 2 \), then the corresponding solution scatters in \( H^1(\mathbb{R}^3) \). Later, Farah and Guzmán [16] extended the range of \( b \) and \( p \), and obtained the energy scattering to all spacial dimensions \( N \geq 2 \). Some other results related to the well-posedness and scattering can be found in [3, 6, 9–12, 35] and the references therein.

Furthermore, when \( p = \frac{4-2b}{N} \), \( u_0 \in H^1(\mathbb{R}^N) \) and \( \|u_0\|_{L^2} < \|Q\|_{L^2} \), Genoud [22] showed that equation (1.1) is globally well-posed in \( H^1(\mathbb{R}^N) \), and proved the existence of critical mass blow-up solutions. Here \( Q \) is the ground state of the elliptic equation
\[
-\Delta Q + |x|^{-b}Q|Q|^p = 0.
\] (1.3)

We give a brief introduction of the existence and uniqueness of ground state solutions. The existence of the ground state was obtained by Genoud-Stuart [19, 24] in dimensions \( N \geq 2 \), and by Genoud [20] in dimension \( N = 1 \). Uniqueness was obtained in dimensions \( N \geq 3 \) by Yanagida [36] (see also Genoud [19]), in dimensions \( N = 2 \) by Genoud [21] and in dimension \( N = 1 \) by Toland [34].

In recent years, the blow-up theory of equation (1.1) has also been extensively studied. Farah [14] proved that if the initial data \( u_0 \in \{f \in H^1(\mathbb{R}^N); |x|f \in L^2(\mathbb{R}^N)\} \) and satisfies
\[
E(u_0)^{s_c}M(u_0)^{1-s_c} < E(Q)^{s_c}M(Q)^{1-s_c},
\] (1.4)
and
\[
\|\nabla u_0\|_{L^2}^2\|u_0\|_{L^2}^{-s_c} > \|\nabla Q\|_{L^2}^{s_c}\|Q\|_{L^2}^{1-s_c}
\] (1.5)
with \( 0 < s_c < 1 \), then the corresponding solution blows up in finite time. In the intercritical case, Dinh [8] showed the finite time blow-up for radial initial data \( u_0 \in H^1(\mathbb{R}^N) \) with negative energy or below ground state in dimensions \( N \geq 2 \). Campos and Cardoso [4] established a scattering and blow-up dichotomy above the mass-energy threshold, which extended the condition (1.4). Later, Cardoso and Farah [5] proved the finite time blow-up if
the initial data $u_0$ is radial and $E(u_0) \leq 0$ in dimensions $N \geq 3$ with $0 < s_c < 1$. Ardila and Cardoso [11] proved that if $u_0 \in H^1(\mathbb{R}^N)$ satisfies (1.3) and (1.5) with $0 < s_c < 1$, then the corresponding solution blows up in finite or infinite time. Recently, Dinh and Keraani [12] obtained a new blow-up criterion to equation (1.1) and when $u(t)$ has finite variance then it blows up in finite time. See also for examples some other blow-up results in [7, 29, 33, 38] and references therein for $b < 0$ and more general nonlinear terms.

Criteria for the existence of finite-time blow-up solutions for the INLS equation (1.1) are known only in cases where $|x|u_0 \in L^2(\mathbb{R}^N)$ or $u_0$ is radial. In this work, we aim to study the finite time blow-up theory for the INLS equation (1.1) for the non-radial data without finite variance. We prove that the solution for INLS equation (1.1) blows up in finite time when $\frac{4-2b}{N} < p < \min\{\frac{N}{2}, 2^*_c\}$ or $p = \frac{4}{N}$. The main blow-up mechanism is that for an ODE equation $f^l < f'$, if $l > 1$, then $f$ will blow up in finite time. Besides, we give a lower bound for infinite time blow-up rate when $\frac{4}{N} < p < 2^*_c$. More precisely, we prove the followings.

**Theorem 1.1.** Let $s_c = \frac{N}{2} - \frac{2-b}{p}$ and $0 < b < \min\{2, N\}$. Assume that $u_0 \in H^1(\mathbb{R}^N)$ and satisfies (1.4) and (1.5). Let $u(t)$ be the solution of (1.1) defined in the maximal time interval of existence, say $I$. If one of the following cases holds,

1. $\frac{4-2b}{N} < p < \min\{\frac{N}{2}, 2^*_c\}$, or
2. $p = \frac{4}{N}, 0 < b < \frac{4}{N},$

then $I$ is finite.

**Theorem 1.2.** Let $s_c = \frac{N}{2} - \frac{2-b}{p}$ and $0 < b < \frac{4}{N}$. Assume that $u_0 \in H^1(\mathbb{R}^N)$ and satisfies (1.4) and (1.5). Let $u(t)$ be the solution of (1.1) defined in the maximal time interval of existence, say $I$. If $\frac{4}{N} < p < 2^*_c$, then either $I$ is finite, or $I = \mathbb{R}$, and for any $T > 0$,

$$\sup_{t \in [0, T]} \|\nabla u(t)\|_{L^2} \gtrsim T^\frac{b}{\alpha}, \quad \text{with} \quad \alpha = \frac{Np}{4}. \quad (1.6)$$

To our best knowledge, this is the first finite time blow-up result for the equation (1.1) in the intercritical case when $u_0$ satisfies the conditions (1.4) and (1.5), without the condition that $|x|u_0 \in L^2(\mathbb{R}^N)$ or $u_0$ is radial.

For the classical nonlinear Schrödinger equation (1.2), in the case of general $H^1$ data (not necessarily finite variance or radially symmetric), the first blow-up result was proved by Glangetas and Merle [18]. They proved that negative energy solutions either blows up in finite time or blows up in infinite time in the sense that

$$\sup_{t \in \mathbb{R}} \|\nabla u(t)\|_{L^2} = +\infty. \quad (1.7)$$

In [26], Holmer and Roudenko showed that under the conditions that (1.4) and (1.5), then the corresponding solution to the focusing 3D cubic NLS either blows up in finite time or blows up in infinite time. Later, Du, Wu, and Zhang [13] gave an alternative simple proof for this result and extended it to all dimensions. So, it is important to note that for the classical NLS, it is still unknown that $H^1$ solutions having negative energy or satisfying (1.4) and (1.5) blow up in finite time (except the 1D mass-critical NLS [32]). The results of blow-up in some other situations can be referred to [17, 28, 30, 31, 37] and references therein.

Compared with this, for the inhomogeneous nonlinear Schrödinger equation (1.1), when $0 < s_c < \min\{\frac{bN}{4}, 1\}$ or $s_c = \frac{bN}{4}$, we prove the finite time blow-up, and when $\frac{bN}{4} < s_c < 1$, prove the finite or infinite time blow-up and obtain a lower bound of blow-up rate (1.6),
which is finer than \(|1.7|\). The key to our proof is the use of the localised virial identity. Thanks to the existence of \(|x|^{-b}\), we can obtain the good remainder estimates by using the Gagliardo-Nirenberg inequality when \(\frac{4-2b}{N} < p \leq \frac{4}{N}\). When \(\frac{4}{N} < p < 2b\), the above argument fails, so we choose a time-dependent radius in the localised virial identity. In particular, we take

\[ R = R(T) = C \sup_{t \in [0,T]} \|\nabla u(t)\|_{L^2}^{2a-2}. \]

This allows us to use the radius to control the blow-up time through delicate analysis, and thus gives the lower bound for infinite time blow-up rate.

This paper is organized as follows. In Section 2 we give some basic notation and identity. In Section 3 we give the proofs of Theorem 1.1 and Theorem 1.2.

2. Notation and Local Virial identity

2.1. Notation. We write \(X \lesssim Y\) or \(Y \gtrsim X\) to denote the estimate \(X \leq CY\) for some constant (which may depend on \(p\), \(b\), \(N\) and the mass \(M(u_0)\)). Throughout the whole paper, the letter \(C\) will denote different positive constants which are not important in our analysis and may vary line by line.

Before stating this theorem, we introduce some quantities. Let the quantity

\[ K(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{Np + 2b}{2(p+2)} \int_{\mathbb{R}^N} |x|^{-b}|u|^{p+2} dx, \]

then it is known as the virial identity that for the solution \(u\) of the equation (1.1),

\[ \frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx = 8K(u(t)). \]

2.2. Local Virial identity. Let us start by introducing the local Virial identity

\[ I(t) = \int \phi(x) |u(t,x)|^2 dx, \]

then by direct computations (see for example [1]), we have that

**Lemma 2.1.** For any \(\phi \in C^4(\mathbb{R}^N)\),

\[ I'(t) = 2Im \int \nabla \phi \cdot \nabla u \overline{u} dx, \]

and

\[ I''(t) = 4Re \sum_{j,k=1}^N \int \partial_j \partial_k \phi \cdot \partial_j \partial_k u \overline{u} dx - \int |u|^2 \Delta^2 \delta dx - \frac{2p}{p+2} \int |x|^{-b}|u|^{p+2} \Delta \phi dx + \frac{4}{p+2} \int \nabla(|x|^{-b}) \cdot \nabla \phi |u|^{p+2} dx. \]
From this lemma, if $\phi$ is radial, then one may find that

$$I'(t) = 2\Im \int \frac{\phi' x \cdot \nabla u}{r} \tilde{u} dx,$$  \hspace{1cm} (2.1)$$

and

$$I''(t) = 4 \int \frac{\phi'}{r} |\nabla u|^2 dx + 4 \int \left( \frac{\phi''}{r^2} - \frac{\phi'}{r^3} \right) |x \cdot \nabla u|^2 dx$$

$$- \frac{2p}{p+2} \int  \left[ \phi'' + (N-1 + \frac{2b}{p}) \frac{\phi'}{r} \right] |x|^{-b} |u|^{p+2} dx - \int \Delta^2 \phi |u|^2 dx,$$  \hspace{1cm} (2.2)$$

here and in the sequel, $r$ denotes $|x|$.

We rewrite $I''(t)$ in (2.2) as

$$I''(t) = 8K(u(t)) + R_1 + R_2 + R_3,$$  \hspace{1cm} (2.3)$$

where

$$R_1 = 4 \int \left( \frac{\phi'}{r} - 2 \right) |\nabla u|^2 dx + 4 \int \left( \frac{\phi''}{r^2} - \frac{\phi'}{r^3} \right) |x \cdot \nabla u|^2 dx,$$

$$R_2 = - \frac{2p}{p+2} \int  \left[ \phi'' + (N-1 + \frac{2b}{p}) \frac{\phi'}{r} - 2N - \frac{4b}{p} \right] |x|^{-b} |u|^{p+2} dx,$$

$$R_3 = - \int \Delta^2 \phi |u|^2 dx.$$

We choose $\phi$ such that

$$0 \leq \phi \leq r^2, \quad \phi'' \leq 2, \quad \phi^{(4)} \leq \frac{4}{R^2},$$

and

$$\phi = \begin{cases} r^2, & 0 \leq r \leq R, \\ 0, & r \geq 2R. \end{cases}$$

According to our needs, we choose the appropriate $R$ later.

3. PROOF OF THEOREM 1.1 AND THEOREM 1.2

In this section, we prove Theorem 1.1 in subsection 3.1 and prove Theorem 1.2 in subsection 3.2.

Before this, we give a lemma about $K(u(t))$, see [1] for its proof.

Lemma 3.1. Let $\frac{1-2b}{N} < p < 2b^*_N$, then for all $t \in I$, there exits $\delta_0 > 0$, such that

$$K(u(t)) < -\delta_0 \| \nabla u(t) \|_{L^2}^2.$$

Moreover,

$$\| \nabla u(t) \|_{L^2} \gtrsim 1.$$
3.1. Proof of Theorem 1.1. Below we will introduce a lemma about $I''(t)$ when $\frac{4-2b}{N} < p \leq \frac{4}{N}$, which is similar to the result when $\frac{4}{N} < p < 2^*_b$. Although there is only one difference in the proof process, this is the essential reason for the different indicators in the two cases. These results play a vital role in proving Theorem 1.1 and Theorem 1.2.

**Lemma 3.2.** Let $\frac{4-2b}{N} < p < \min\{\frac{4}{N}, 2^*_b\}$ or $p = \frac{4}{N}$ with $0 < b < \frac{4}{N}$, then there exists a constant $\delta_0 > 0$, such that

$$I''(t) \leq -4\delta_0 \|\nabla u\|_{L^2}^2.$$

**Proof.** From (2.3), we estimate the terms on the right hand side, term by term. Firstly, notice that when $0 < b < \frac{4}{N}$, we have $p = \frac{4}{N} < 2^*_b$. Hence, by Lemma 3.1, we get

$$K(u(t)) < -\delta_0 \|\nabla u(t)\|_{L^2}^2.$$  \hfill (3.1)

Secondly, for the term $R_1$, we claim that

$$R_1 \leq 0.$$  \hfill (3.2)

Indeed, we divide the space $\mathbb{R}^N$ into two parts:

$$\left\{ \frac{\phi''}{r^2} - \frac{\phi'}{r^3} \leq 0 \right\}$$

and

$$\left\{ \frac{\phi''}{r^2} - \frac{\phi'}{r^3} > 0 \right\}.$$

When $\frac{\phi''}{r^2} - \frac{\phi'}{r^3} \leq 0$, since $\phi' \leq 2r$, (3.2) is obvious. When $\frac{\phi''}{r^2} - \frac{\phi'}{r^3} > 0$, then since $\phi'' \leq 2$, we have

$$R_1 \leq 4 \int \left( \frac{\phi'}{r} - 2 \right) |\nabla u|^2 dx + 4 \int \left( \frac{\phi''}{r^2} - \frac{\phi'}{r^3} \right) |x|^2 |\nabla u|^2 dx$$

$$= 4 \int \left( \frac{\phi'}{r} - 2 \right) |\nabla u|^2 dx + 4 \int \left( \phi'' - \frac{\phi'}{r} \right) |x|^2 |\nabla u|^2 dx$$

$$= 4 \int \left( \phi'' - 2 \right) |\nabla u|^2 dx \leq 0.$$

Next, we treat the term $R_2$. Recall that

$$R_2 = -\frac{2p}{p+2} \int \left[ \phi'' + (N - 1 + \frac{2b}{p}) \frac{\phi'}{r} - 2N - \frac{4b}{p} \right] |x|^{-b}|u|^{p+2} dx.$$

If $0 \leq r \leq R$, then

$$\phi' = 2r, \quad \phi'' = 2,$$

and thus,

$$\phi'' + (N - 1 + \frac{2b}{p}) \frac{\phi'}{r} - 2N - \frac{4b}{p} = 0.$$

Hence,

$$\text{supp} \left[ \phi'' + (N - 1 + \frac{2b}{p}) \frac{\phi'}{r} - 2N - \frac{4b}{p} \right] \subset (R, \infty).$$
By the Gagliardo-Nirenberg inequality we have
\[ R_2 \lesssim \int_{|x| > R} |x|^{-\frac{p}{2}}|u|^{p+2} dx \lesssim R^{-\frac{p}{2}} \|\nabla u\|_{L^2}^{2\alpha} \|u\|_{L^2}^{p+2-2\alpha}, \tag{3.3} \]
where \( \alpha = \frac{Np}{4} \in (0, 1] \).

Finally, for the term \( R_3 \), we have
\[ R_3 \lesssim R^{-2} \|u\|_{L^2}^2 \lesssim R^{-2}. \tag{3.4} \]

Thus, from (2.3), (3.1)-(3.4), we have
\[ I''(t) \lesssim -8\delta_0 \|\nabla u(t)\|_{L^2}^2 + R^{-\frac{p}{2}} \|\nabla u(t)\|_{L^2}^{2\alpha} + R^{-2}. \]

If \( \alpha = 1 \), then the second term can be absorbed by the first one by fixing \( R > 0 \) large enough, and thus we have
\[ I''(t) \lesssim -4\delta_0 \|\nabla u(t)\|_{L^2}^2. \]

If \( 0 < \alpha < 1 \), by using Young’s inequality for the second term, we can further obtain the above estimates. The lemma is now proved. \( \square \)

We are now in a position to prove Theorem 1.1.

**Proof.** Suppose that the maximal existence interval is \( I = (-T_*, T^*) \). We proceed by contradiction and assume that \( T^* = +\infty \).

First, by the Lemma 3.1 there exists \( C_0 > 0 \) such that
\[ \|\nabla u(t)\|_{L^2}^2 \geq C_0, \]
for all \( t \in (-T_*, T^*) \). Get directly from the Lemma 3.2 we have
\[ I''(t) \leq -4\delta_0 C_0. \]

Further, integrate from 0 to \( t \) for the above formula,
\[ I'(t) \leq -4\delta_0 C_0 t + I'(0). \tag{3.5} \]

We now may choose \( T_0 > 0 \) sufficiently large such that \( I'(0) < 2\delta_0 C_0 T_0 \). From this and (3.5),
\[ I'(t) \leq -2\delta_0 C_0 t < 0, \quad t \geq T_0. \tag{3.6} \]

By Lemma 3.2 and (3.6), we obtain
\[ I'(t) = \int_{T_0}^{t} I''(s) ds + I'(T_0) \leq -4\delta_0 \int_{T_0}^{t} \|\nabla u(s)\|_{L^2}^2 ds, \quad t \geq T_0. \tag{3.7} \]

Moreover, from (2.1) and Hölder’s inequality we deduce
\[ |I'(t)| = |2iRm \int \phi \frac{x \cdot \nabla u}{r} \bar{u} dx| \lesssim R \|\nabla u(t)\|_{L^2} \|u(t)\|_{L^2} \lesssim R \|\nabla u(t)\|_{L^2}. \tag{3.8} \]
From (3.6)–(3.8), we can obtain
\[ \int_{T_0}^{t} \left\| \nabla u(s) \right\|^2_{L^2} ds \lesssim |I'(t)| \lesssim \| \nabla u(t) \|_{L^2}. \]

Define \( f(t) := \int_{T_0}^{t} \left\| \nabla u(s) \right\|^2_{L^2} ds \) and thus we have \( f^2(t) \lesssim f'(t) \). Finally, taking \( T' > T_0 \) and integrating on \([T', t)\) we get
\[ t - T' \lesssim \int_{T'}^{t} \frac{f'(s)}{f^2(s)} ds = \frac{1}{f(T')} - \frac{1}{f(t)} \leq \frac{1}{f(T')} . \]
Noting that \( f(T') > 0 \), letting \( t \to \infty \) we arrive to a contradiction. Hence Theorem 1.1 is completed.

3.2. Proof of Theorem 1.2. Next, we give the proof of Theorem 1.2. Before that, we will also give an important lemma. Here we can see the difference between two cases in the treatment of \( \mathcal{R}_2 \).

**Lemma 3.3.** Let \( \frac{4}{N} < p < 2b^* \), then there exists \( \alpha > 1 \), such that
\[ I''(t) \lesssim -8\delta_0 \| \nabla u(t) \|^2_{L^2} + R^{-b} \| \nabla u(t) \|^{2\alpha}_{L^2} + R^{-2} . \]

**Proof.** The estimates for \( K(u(t)) \), \( \mathcal{R}_1 \) and \( \mathcal{R}_3 \) are the same as in Lemma 3.1 and Lemma 3.2, it reduces to estimate \( \mathcal{R}_2 \). By the Gagliardo-Nirenberg inequality we have
\[ \mathcal{R}_2 \lesssim \int_{|x| > R} |x|^{-b} |u|^{p+2} dx \lesssim R^{-b} \| \nabla u(t) \|^{2\alpha}_{L^2} \| u \|^{p+2-2\alpha}_{L^2} \lesssim R^{-b} \| \nabla u(t) \|^{2\alpha}_{L^2} , \]
where \( \alpha = \frac{Np}{4} \in (1, \frac{2b^* N}{4}) \). Combing (2.3), (3.2), (3.4), Lemma 3.1 with the last inequality, we get the desired result.

Now, we give the proof of Theorem 1.2.

**Proof.** Fixing \( T > 0 \), we let
\[ R(T) = (3\delta_0)^{-\frac{1}{8}} \sup_{t \in [0, T]} \| \nabla u(t) \|^{\frac{2\alpha - 2}{8}}_{L^2} . \]
Then, we have
\[ R(T)^{-b} \| \nabla u(t) \|^{\frac{2\alpha}{L^2}} \leq 3\delta_0 \| \nabla u(t) \|^{2-2\alpha}_{L^2} \| \nabla u(t) \|^{2\alpha}_{L^2} = 3\delta_0 \| \nabla u(t) \|^{2}_{L^2} \]
and
\[ R(T)^{-2} \lesssim 3\delta_0 \| \nabla u(t) \|^{2}_{L^2} . \]
Combing (3.9), (3.10) and Lemma 3.3 we obtain
\[ I''(t) \lesssim -\delta_0 \| \nabla u(t) \|^{2}_{L^2} . \]
Applying the classical analysis identity
\[ I(T) = I(0) + I'(0) \cdot T + \int_{0}^{T} \int_{0}^{s} I''(t) dt ds. \]
From the definition of \( I(t) \) and the choosing of \( \phi \), we have
\[ I(T) \geq 0 , \]
and
\[ I(0) \lesssim R(T)^2. \]

By (3.8), we get
\[ I'(0) \lesssim R(T)\|\nabla u_0\|_{L^2} \lesssim R(T). \]

Collecting the estimates above we have
\[ \delta_0 \int_0^T \int_0^s \|\nabla u(t)\|^2_{L^2} dt\, ds \lesssim R(T)^2 + R(T) \cdot T. \quad (3.11) \]

Recalling that \( \|\nabla u(t)\|^2_{L^2} \geq C_0 > 0 \), for all \( t \in [0, T) \), we get
\[ \delta_0 \int_0^T \int_0^s \|\nabla u(t)\|^2_{L^2} dt\, ds \gtrsim T^2. \quad (3.12) \]

Further, by (3.11), (3.12) and the elementary inequality we have
\[ T^2 \lesssim R(T)^2 + R(T) \cdot T \leq R(T)^2 + \frac{R(T)^2 + T^2}{2}. \]

From this, we can obtain
\[ R(T)^2 \gtrsim T^2. \]

Noting \( R(T) > 0 \), hence
\[ R(T) \gtrsim T. \]

That is
\[ \sup_{t \in [0, T]} \|\nabla u(t)\|_{L^2} \gtrsim T^\frac{b}{2m-2}. \]

This proves Theorem 1.2.

In conclusion, we conclude the proofs of Theorem 1.1 and Theorem 1.2.

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