Bilateral Generating Functions Involving Generalized Hypergeometric Polynomials of Two Variables

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Abstract

Objectives: The object of the paper is to obtain novel results on bilateral generating relations for the generalized hypergeometric polynomials of two variables (GHP2D) $\sum_{n=0}^{\infty} a_n (\beta; \gamma; a, b)$.

Methods/Statistical Analysis: We obtain the results by employing the Weisner’s group theoretic method which is an efficient method to obtain various types of generating relations.

Findings: A new generating relation derived for the generalized hypergeometric polynomials by using an ascending recurrence relation. Further, we proved a general theorem on bilateral generating relations for generalized hypergeometric polynomials $\sum_{n=0}^{\infty} a_n (\beta; \gamma; a, b)$. Application/Improvements: It is worth noting that the main theorem can be applied to yield numerous results involving known (unknown) generating functions for various hypergeometric polynomials which are natural arises in the study of many problems in different fields of the fundamental performance metrics of wireless communications systems.

Keywords: Generating Functions, Group-Theoretic Method, Hypergeometric Polynomials, Mathematics Subject Classification (2010): 33C45, 33C65, 42C05, Special Functions, Wireless Communications Systems

1. Introduction

Special functions are the solutions of a wide class of relevant functional equations. Generating functions play a key role in the study of special functions. There are number of methods to get generating functions¹². But it has been found that the group-theoretic method of obtaining generating functions is much potent one in comparison to analytical methods³. Group-theoretic method was introduced by⁴ in the year 1955 and he employed this technique to find generating relations for variety of special functions.

Hypergeometric polynomials arise naturally and frequently in different type of problems related to theoretical physics, applied mathematics, engineering science, statistics and operation research. A considerable field of the fundamental performance metrics of wireless communications systems⁵,⁶ i.e. capacity, error probability and outage probability can be frequently obtained in closed form by the common classical special functions (e.g., Bessel’s functions, Hypergeometric functions or Meijer functions), the Gaussian Q-function, the Marcum Q-function and the Nuttal Q-function . In order to obtain new analytical closed-form results for the basic performance metrics, this set of special functions has been recently extended to include either less common classical special function (e.g., Lauricella functions, Fox’s H function). From a computational point of view, certain Lauricella functions are easier to compute than Fox type functions because either they have Euler type single integral representations with elementary integrands or their univariate Laplace transform is elementary being easily inverted numerically. Also in mathematical physics⁷. The theory of generating functions for generalized hypergeometric polynomials plays a vital role.

In¹ derived bilateral generating functions for the hypergeometric polynomial by Weisner method. Recently, bilateral generating functions for modified Jacobi polynomials were discussed by⁹ using group-theoretic method¹⁰ has also done some work on...
generating functions for Laguerre 2D polynomials, by using group-theoretic method. Also, in\(^1\) studied special linear group \(SL(2,\mathbb{C})\) and generating functions for two variable Legendre polynomials.

2. Definition

Recently, Bhagavan V.S. and Rama Kameswari P.L. introduced generalized hypergeometric 2D polynomials \((GH2DP) \ U_n(\beta;\gamma;a,b)\) and discussed the generating functions with the help of the following differential equation and an ascending recurrence relation respectively:

- The differential equation Satisfied by \(U_n(\beta;\gamma;a,b)\) is:
  \[
  b(a-b)D^2U_n(a,b)-[n+\beta-1)a-(\gamma+2n-2)b]DU_n(a,b)-n(\gamma+n-1)U_n(a,b)=0. \tag{2.1}
  \]

- The ascending recurrence relation is deduced as:
  \[
  DU_n(a,b)=\frac{1}{a(a-b)}\{(\gamma+n)U_{n+1}(a,b)+[(n+\beta)a-(\gamma+2n)b]U_n(a,b)\}, \tag{2.2}
  \]

Where \(D=\frac{d}{db}\).

- The generating functions for \(U_n(\beta;\gamma;a,b)\) are:
  \[
  \sum_{k=0}^{\infty} \frac{U_n(a,b)}{k!}t^k=e^{bt}F_{1}^1(\beta;\gamma;-at). \tag{2.3}
  \]

and

\[
\sum_{k=0}^{\infty} \frac{(c)_kU_k(a,b)}{k!}t^k=(1-bt)^{-c}2F_{1}(c,\beta;\gamma;-tb) \tag{2.4}
\]

The proofs of the above results are clear.

3. Applications

1. \(\lim_{\beta\to\infty} \beta^n U_n(\beta;1+\alpha;a,b) = \frac{n!}{(1+\alpha)^n}L_n^{\alpha}(a,b). \tag{2.5}\)

Where \(L_n^{\alpha}(a,b)\) is the two variables Laguerre polynomial.

2. \(U_n\left(-c;\gamma;1,\left(1-\rho^{-1}\right)^{-1}\right)=\left(1-\rho^{-1}\right)^{-n}M_n(c;\gamma,\rho), \tag{2.6}\)

Provided \(\gamma>0\) \quad 0<\rho<1 \quad y=0,1,2,...,\

Where \(M_n(c;\gamma,\rho)\) is the Meixner polynomial.\(^4\)

3. \(U_n\left(-z;1,\left(1-e^{\lambda}\right)^{-1}\right)=\left(e^{-\lambda}-1\right)^{-n}\phi_n(c,\lambda) , \tag{2.7}\)

Where \(\phi_n(c,\lambda)\) is the Gottlieb polynomial.\(^6\)

4. Bilateral Generating Functions

4.1 Theorem

If there exist a unilateral generating relation of the form:

\[
G(a,b,t) = \sum_{k=0}^{\infty} \mu_k U_k(a,b)t^k \tag{3.1}
\]

then the following generating relation will exist:

\[
(1-wb)^{\beta-\gamma}\left(1+w(a-b)\right)^{-\beta}G(\frac{a,b+wb(a-b)}{1-wb(1+w(a-b))}) = \sum_{n=0}^{\infty} \sigma_n(t)U_n(a,b)w^n. \tag{3.2}
\]

where

\[
\sigma_n(t) = \sum_{k=0}^{n} \frac{(\gamma+k)n-k}{{n-k}!} \frac{(c)_k}{k!}t^k. \tag{3.3}
\]

4.2 Proof

From the recurrence relation (2.2), we define the following linear partial differential operator \(C\) as follows:

\[
C = \frac{\partial}{\partial b}+\frac{\partial}{\partial c}+C. \tag{3.4}
\]

Consider the operator \(E=R_1\frac{\partial}{\partial b}+R_2\frac{\partial}{\partial c}\).

We now proceed to derive the extended form of operator \(C\):

\[
C = cb(a-b)\frac{\partial}{\partial b}+c^2(2b-a)\frac{\partial}{\partial c}+c(\gamma b-\beta a) \tag{3.4}
\]

We then proceed to derive the extended form of operator \(E\):

\[
E = cb(a-b)\frac{\partial}{\partial b}+c^2(2b-a)\frac{\partial}{\partial c}+c(\gamma b-\beta a) \tag{3.4}
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\[
C = cb(a-b)\frac{\partial}{\partial b}+c^2(2b-a)\frac{\partial}{\partial c}+c(\gamma b-\beta a) \tag{3.4}
\]
Finally, we select new variables $B, C$ then $E$ is transformed into $D = \frac{\partial}{\partial b}$ with this transformation, $\phi^{-1}(a,b,c) f(a,b,c)$ be changed into $F(A, B, C)$. With the help of Taylor’s theorem, we get $e^{wC} f(a,b,c) = \phi(a,b,c) g(a,b,c)$.

Let $F(A, B, C)$ be transformed into $g(a,b,c)$ by inverse substitution. Now, we derive $e^{wC} f(a,b,c)$,

$$
\frac{\partial}{\partial B} \left( c (2b-a) \frac{\partial}{\partial c} + c^2 (2b-a) \right).
$$

Let $\phi(a,b,c)$ be a function such that $R \phi(a,b,c) = 0$ On solving, we get $\phi(a,b,c) = b^{\beta+1} (a-b)^{-\beta+1} c.$

Therefore
$$
E = \phi^{-1}(a,b,c) C \phi(a,b,c) = \frac{\partial}{\partial b} \left( c (2b-a) \frac{\partial}{\partial c} + c^2 (2b-a) \right).
$$

Now, we insert new variables $B, C$ and then $E$ will be transformed into $D = \frac{\partial}{\partial b}$.

We get a set of solutions as follows:

$$
A = a, \quad B = \frac{1}{cb} \quad \text{and} \quad C = \frac{1}{cb(a-b)}.
$$

From which we get

$$
a = A, \quad b = A-BC^{-1}, \quad c = \frac{1}{B(A-BC^{-1})}.
$$

Recalling
$$
C = \phi \ E \phi^{-1} \quad \text{where} \phi(a,b,c) = b^{\beta+1} (a-b)^{-\beta+1} c \quad \text{we get}
$$

$$
e^{wC} f(a,b,c) = \phi(a,b,c) g(a,b,c) \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left. \left.
On replacing \( t \) by \( wzt \) in (4.4) and then operating \( e^{wc} \) on both sides, we get
\[
e^{wc} \ G(a,b,twc) = e^{wc} \sum_{k=0}^{\infty} \mu_k \ U_k(a,b) w^k t^k c^k
\] (4.5)

By using (3.4) and (3.5), we obtain
\[
e^{wc} \ G(a,b,twc) = (1 - wc(a-b))^{\beta - \gamma} \left(1 + wc(a-b)\right)^{-\beta}
\] (4.6)
\[
G(a,b(1 + cw(a-b))), \frac{twc}{(1 - wc)(1 + w(a-b))}
\]
so that
\[
e^{wc} \sum_{k=0}^{\infty} \mu_k \ U_k(a,b) w^k t^k c^k
\] (4.7)

Now equating (4.6) and (4.7), we obtain
\[
(1 - wc)^{\beta - \gamma} \left(1 + wc(a-b)\right)^{-\beta}
\] (4.8)
\[
G(a,b(1 + cw(a-b))), \frac{twc}{(1 - wc)(1 + w(a-b))}
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mu_k \frac{(\gamma + k)_{n-k}}{(n-k)!} U_n(a,b)(wc)^n t^k.
\]

Put \( z = 1 \) in (4.8) we get
\[
(1 - wb)^{\beta - \gamma} \left(1 + w(a-b)\right)^{-\beta}
\] (4.9)
\[
G(a,b(1 + w(a-b))), \frac{tw}{(1 - wb)(1 + w(a-b))}
\]
\[
= \sum_{n=0}^{\infty} w^n U_n(a,b) \sigma_n(t),
\]
Where \( \sigma_n(t) = \sum_{k=0}^{n} \mu_k \frac{(\gamma + k)_{n-k}}{(n-k)!} t^k. \)
Hence the theorem.

6. Applications

- In the generating relation (2.3) let us take
\[
\mu_k = \frac{1}{k!} \quad \text{and} \quad G(a,b,t) = e^{bt} \ _1F_1[\beta; \gamma; -at]
\]

By the above theorem we have
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\gamma + k)_{n-k}}{k!(n-k)!} U_n(a,b)(w)^n t^k
\]
\[
= (1 - wb)^{-\gamma} \left(1 + w(a-b)\right)^{-\beta} \exp\left\{\frac{twb}{1 - wb}\right\} _1F_1[\beta; \gamma; -atw]
\]
\[
\left|\frac{1}{(1 - wb)(1 + w(a-b))}\right|
\]
Now replacing \(-t\) by \( c\) and \( w\) by \( t\) we get
\[
\sum_{n=0}^{\infty} \frac{\gamma \ _1F_1[\gamma; \zeta; t] U_n(a,b)t^n}{n!}
\]
\[
= (1 - tb)^{\beta - \gamma} \left(1 + t(a-b)\right)^{-\beta} \exp\left\{-\frac{thc}{1 - tb}\right\} _1F_1[\beta; \gamma; -atc]
\]
\[
\left|\frac{1}{(1 - tb)(1 + t(a-b))}\right|
\]
Which is the bilateral generating relation for \( U_n(\beta; \gamma; a, b)\).

- Consider another generating relation (2.4). Let us suppose that
\[
\mu_k = \left(\frac{d}{k!}\right), \quad \text{and} \quad G(a,b,t) = (1 - bt)^{-d} \ _2F_1[d, \beta; \gamma; \frac{-at}{1 - bt}]
\]

In a similar way, we get
\[
\sum_{n=0}^{\infty} \frac{\left(\frac{\gamma}{x}\right) \ _2F_1[\gamma; \zeta; x] U_n(a,b)t^n}{n!}
\]
\[
= (1 - tb)^{\beta - \gamma} \left(1 + t(a-b)\right)^{-\beta} \left(1 - th(1-c)\right)^{-\beta} \ _2F_1[d, \beta; \gamma; \frac{-atc}{(1 - tb)(1 + tb)}]
\]
Which is the bilateral generating relation for \( U_n(\beta; \gamma; a, b)\).

In similar way, one can also derive generating relation for the generalized hypergeometric polynomials of two variables (GHP2D), by using descending recurrence relation.

7. Remark

The corresponding generating functions for various orthogonal polynomials can be obtained from the generating functions (5.1) and (5.2) by using the conditions of section 2.

8. Conclusion

Generating functions involving GH2DP are derived by Weisner’s group-theoretic method. Certain known or new generating relations to some classical orthogonal
polynomials are discussed as special cases which come out many problems in different fields of the fundamental performance metrics of wireless communications systems by the common classical special functions. The application of this GH2DP is to develop and design high performance communication systems is the further scope of this research.

9. References

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