Abstract. We explain how the distributional index of an operator on a principal G-manifold P that is obtained by lifting a Dirac operator on \( P/G \) can serve as a link between the Duflo isomorphism and Chern-Weil forms.

1. Introduction

The theme of this article arises in the typical context of a closed even-dimensional spin (or spin-c) manifold \( M \). There one naturally has a principal \( G \)-bundle \( P \to M \), where \( G \) is some compact Lie group, and a Dirac operator \( D \) acting on sections of an equivariant vector bundle \( P \times_G E \). By picking a connection on \( P \), one can horizontally lift the Dirac operator on \( M \) to \( \tilde{P} \). The resulting differential operator \( \tilde{D} \), which may operate on \( \mathcal{C}^\infty (P) \otimes E \), is non-elliptic. However, it is transversally elliptic in the sense of Atiyah and Singer [1, Def. 1.3, p. 7], which means in our situation that \( \tilde{D} \) is elliptic in horizontal directions relative to the selected connection on \( P \).

Continuing with the idea of Atiyah and Singer, one has the distributional index \( [\tilde{D}] \) of \( \tilde{D} \), which is a formal \( \mathbb{Z} \)-linear combination of characters of \( G \). It is a theorem of Atiyah and Singer that \( [\tilde{D}] \) is a genuine distribution on \( G \) [1, Thm. 2.2, p. 10].

More is true in our case. Owing to the free \( G \)-action on \( P \), the distribution \( [\tilde{D}] \) is supported at the identity, and it can be identified as an element of the center \( \mathcal{Z}(\mathfrak{g}) \) of the universal enveloping algebra generated by the Lie algebra \( \mathfrak{g} \) of \( G \). Thus \( [\tilde{D}] \) is subject to (the inverse of) the Duflo isomorphism

\[
\text{Duf} : \mathcal{S}^G(\mathfrak{g}) \to \mathcal{Z}(\mathfrak{g}).
\]

Here \( \mathcal{S}^G(\mathfrak{g}) \) denotes the \( G \)-invariant subalgebra of the symmetric algebra generated by \( \mathfrak{g} \); it can be identified as the algebra of \( G \)-invariant distributions on \( \mathfrak{g} \) that are supported at the origin. So it makes sense to pair \( \text{Duf}^{-1}[\tilde{D}] \) with a \( G \)-invariant analytic function \( \varphi \) defined on some neighborhood of the origin in \( \mathfrak{g} \). One can now ask whether the following equation holds:

\[
\langle \text{Duf}^{-1}[\tilde{D}], \varphi \rangle = \langle \hat{D} \varphi, \hat{M} \rangle.
\]

Here \( \hat{D} \) is the index class of \( D \); \( \hat{\varphi} \) is the characteristic class on \( M \) obtained from \( \varphi \) by the Chern-Weil homomorphism; and \( \hat{M} \) is the fundamental homology class of \( M \) determined by the spin structure.

Our aim is to show that Equation (1) holds under a reasonable condition on the connection associated with \( \tilde{D} \) and if we restrict the domain of \( \tilde{D} \) to

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the $G$-invariant subspace $(C^\infty(P) \otimes E)^G$. Denote the restriction of $\hat{D}$ by $D_M$. Then Equation (1) explicitly shows how the distributional index of $D_M$ serves has a link between the Duflo isomorphism and the Chern-Weil forms.

In a sense this is just another manifestation of the Atiyah-Singer index theorem. Indeed, when $\varphi$ is the constant polynomial 1, the pairing $\langle \hat{D}, \hat{M} \rangle$ calculates the usual (graded) index of $D$.

2. Setup

Throughout this article, $M$ denotes a closed oriented Riemannian manifold of dimension $n$. Let $D$ be a geometric Dirac operator acting on the sections of an equivariant vector bundle

$$P \times_G E \to M,$$

where $P$ is the total space of a principal bundle

$$\kappa: P \to M$$

whose fibers are isomorphic to a compact Lie group $G$, and $E$ is a finite-dimensional $G$-vector space over $\mathbb{C}$ associated with a representation

$$\nu: G \to \text{Aut}(E).$$

We denote by

$$\nu_*: g \to \text{End}(E)$$

the Lie algebra representation induced by the differential of $\nu$ at the identity.

We assume that $E$ is a graded $\text{Cl}(n)$-module. Regarding the irreducible ones, there is only one (up to isomorphism) if $n$ is odd, and there are two if $n$ is even. If we fix one over the other for the even case, then we may speak of the irreducible graded $\text{Cl}(n)$-module $S$. Under such circumstances, $E$ is of the form

$$E = S \otimes W,$$  \hspace{1cm} (2)

where $\text{Cl}(n)$ acts canonically on $S$ and trivially on $W$.

The existence of a Dirac operator on the sections of $P \times_G E$ implies that the vector bundle $P \times_G E$ admits a $\text{Cl}(M)$-module structure. This means that $E$ is a module over the Clifford algebra $\text{Cl}(n)$ generated by the Euclidean space $\mathbb{R}^n$ and that we have a bundle map

$$
\begin{array}{ccc}
\text{Cl}(M) & \longrightarrow & P \times_G \text{End}(E) \\
& & \downarrow \text{c} \\
& & M
\end{array}
$$

such that, when restricted to each fiber, we have an isomorphism of algebras

$$\text{Cl}(T_x M) \cong \text{Cl}(n) \subset \text{End}(E).$$

Let $\Gamma(P \times_G E)$ denote the space of sections of $P \times_G E$. Our assumption that $D$ is a geometric Dirac operator means that there is a Clifford connection $\nabla$ on $\Gamma(P \times_G E)$, relative to which $D$ is locally of the form

$$D = \sum_{i=1}^{n} c(\xi_i) \nabla_{\xi_i}.$$
for any local orthonormal frame \( \{ \xi_i \}_{i=1}^n \) for the tangent bundle \( TM \) of \( M \).

Now let \( \tilde{\nabla} \) be the covariant derivative for the trivial bundle \( P \times E \to P \) that agrees with \( \nabla \) under the usual identification of \( \Gamma(P \times_G E) \) with \( \Gamma(P \times E)_G = (C^\infty(P) \otimes E)_G \). The covariant derivative \( \tilde{\nabla} \) determines a connection \( \theta \) on \( P \), which in turn determines the horizontal subspace at each tangent space of \( P \). We define the lift \( \tilde{D} \) of \( D \) as the differential operator on \( \Gamma(P \times E) = C^\infty(P) \otimes E \) that is locally of the form

\[
\tilde{D} = \sum_{i=1}^n c(\xi_i) \tilde{\nabla}_{\xi_i},
\]

where \( \tilde{\xi}_i \) denotes the horizontal lift of \( \xi_i \).

The connection \( \theta \) induces a \( G \)-invariant metric on \( P \) in the following way: Let \( \langle , \rangle_M \) be the pullback of the metric of \( M \) along the bundle projection \( \kappa \), and let \( \langle , \rangle_g \) be an inner product on \( g \) that is invariant under the adjoint action of \( G \) on \( g \). Then we define the metric \( \langle , \rangle \) on \( P \) by

\[
\langle X, Y \rangle = \langle X, Y \rangle_M + \langle \theta(X), \theta(Y) \rangle_g.
\]

With the metric on \( P \) at hand, we have the Riemannian connection \( \nabla^P \) for \( \mathfrak{X}(P) := \Gamma(TP) \). For each \( X \in \mathfrak{g} \), denote the fundamental vector field it generates on \( P \) by \( \tilde{\xi} \). Let \( \tilde{\xi} \) be a horizontal vector field. The vector field \( \nabla^P \tilde{\xi} \) is again horizontal; denote it \( \tilde{\xi}' \). Let \( \xi := \kappa_\ast \tilde{\xi} \), where \( \kappa_\ast \) is the pushforward induced by \( \kappa \), and denote its value at \( x \in M \) by \( \xi_x \). Define \( \alpha_x(X) \in \text{End}(T_xM) \) by \( 2\xi'_x = \alpha_x(X)\xi_x \). This gives a Lie algebra representation

\[
\alpha_x : \mathfrak{g} \to \mathfrak{so}(T_xM).
\]

In terms of \( \alpha_x \), the condition for the covariant derivative \( \tilde{\nabla} \) on \( C^\infty(P) \otimes E \) to be a Clifford connection is that

\[
[\nu_\ast(X), c(\xi)] = c(\alpha_x(X)\xi)
\]

holds for all \( X \in \mathfrak{g} \), \( \xi \in T_xM \), and \( x \in M \). It is appropriate to call this as the Clifford condition for \( \tilde{\nabla} \). If this condition holds, then the family \( \{ \alpha_x \}_{x \in M} \) of Lie algebra representations, together with the bundle map \( (3) \), defines collectively a Lie algebra representation

\[
\alpha : \mathfrak{g} \to \mathfrak{so}(n).
\]

This representation is equivalent to a Lie algebra representation

\[
\gamma : \mathfrak{g} \to \mathfrak{spin}(n) \subset \text{Cl}(n),
\]

where the two representations are related by

\[
\alpha(X)(v) = [\gamma(X), v].
\]

Here the bracket on the right-hand side denotes commutation in \( \text{Cl}(n) \).

In terms of the Lie algebra representation \( \gamma \), the Clifford condition \( (4) \) can be written as

\[
[\nu_\ast(X), c(\xi)] = [\gamma(X), c(\xi)].
\]

This commutation relation implies that the \( g \)-action on \( E \) respects the factorization \( (2) \), so that

\[
\nu_\ast = \gamma \otimes 1 + 1 \otimes \tau
\]
for some Lie algebra representation
\[ \tau: g \to \text{End}(W). \]

**Remark.** The Clifford condition (4) is always satisfied if the Lie algebra representation \( \gamma \) is induced by a Lie group representation \( G \to \text{Spin}(n) \); in that case we have \( TM \cong P \times_G \mathbb{R}^n \), where \( G \) acts on \( \mathbb{R}^n \) through the double covering \( \text{Spin}(n) \to \text{SO}(n) \).

3. **Distributional Index**

The lifted operator \( \tilde{D} \) on \( P \) is by construction a transversally elliptic operator on \( E \)-valued functions on \( P \). Because the connection \( \theta \) is \( G \)-invariant, \( \tilde{D} \) is also \( G \)-invariant, so the kernel of \( \tilde{D} \) is a \( G \)-space. Denote the even and odd parts of the kernel as \( \ker(\tilde{D}^+) \) and \( \ker(\tilde{D}^-) \), respectively. Let \( \hat{G} \) be the unitary dual of \( G \). It is a result of Atiyah and Singer [1, Lem. 2.3, p. 10] that the pairings
\[ \langle \ker \tilde{D}^\pm, V \rangle := \dim \text{Hom}_G(\ker \tilde{D}^\pm, V) \]
are finitely valued for all \( [V] \in \hat{G} \). So
\[ [\ker \tilde{D}^\pm] := \sum_{[V] \in \hat{G}} \langle \ker \tilde{D}^\pm, V \rangle [V] \]
are well-defined elements of the formal representation group
\[ \hat{R}(G) := \prod_{[V] \in \hat{G}} \mathbb{Z} \cdot [V]. \]

Moreover, identifying \( [V] \) with its character, the formal sums \( [\ker \tilde{D}^\pm] \) converge in the distributional sense [1, Thm. 2.2, p. 10]. The **distributional index** of \( \tilde{D} \) is then defined as
\[ [\tilde{D}] := [\ker \tilde{D}^+] - [\ker \tilde{D}^-]. \]

Our real interest, however, lies in the “trivial part” of \([\tilde{D}]\), that is, the distributional index \([D_M] \) where \( D_M \) is the restriction of \( \tilde{D} \) onto the \( G \)-invariant subspace \((C^\infty(P) \otimes E)^G\). The reason we write the restricted operator as \( D_M \) is that its operation on the \( G \)-invariants agree with the operation of \( D \) on the sections of \( P \times_G E \).

Note that \( D_M \) on \((C^\infty(P) \otimes E)^G\) is effectively elliptic and Fredholm. Moreover, if we write the (scalar) Laplacian on \( P \) as \( \Delta_P \), then \( D_M^2 + \Delta_P \) is equal to an operator \( F \) of order zero (see [2, Prop. 5.6, p. 172]). So \( L := -\Delta_P + F \) is a generalized Laplacian that agrees with \( D_M^2 \) on \((C^\infty(P) \otimes E)^G\). We shall denote by \( P_t \) the heat kernel associated with \( L \), that is, the integral kernel of the operator \( e^{tL} \).

For \( [V] \in \hat{G} \) and \( g \in G \), let \( [V]_g \) denote the value of the character of \( [V] \) at \( g \). Let
\[ [D_M]_g := \sum_{[V] \in \hat{G}} (\langle \ker D_{M^+}, V \rangle - \langle \ker D_{M^-}, V \rangle) [V]_g. \]
Owing to the equivariant McKean-Singer formula (see \cite{2}, Prop. 5.6, p. 173; Prop. 6.3, p. 185),

\[[DM]_g = \text{Str}(ge^{\nu L}) = \int_M \int_G \text{Str}(P_t(x, x \cdot g) \nu(g)^{-1}) \, dg \, dx.\]

Here \text{Str} denotes the super trace for graded operators. Thus, the pairing of \[[DM]\] with a function \(f \in C^\infty(G)\) can be calculated in the following way:

\[
\langle [DM], f \rangle = \int_M \int_G \text{Str}(P_t(x, x \cdot g)f(g) \nu(g)^{-1}) \, dg \, dx. \tag{6}
\]

Equation (6) shows that \([DM]\) indeed has point support at the identity, owing to the finite-propagation property of the heat kernel (see \cite{5}, Prop. 7.24, p. 107).

4. Chern-Weil Forms

We quickly recall the construction of the Chern-Weil homomorphism. As a preliminary remark, suppose we have a formal power series \(\varphi \in \mathbb{R}[[g^*]]\). Let \(\wedge(N)\) be the exterior algebra generated by some finite-dimensional vector space \(N\) over \(\mathbb{R}\), and let \(\wedge^+(N)\) be its subalgebra comprising all elements of even degree. Then the formal power series \(\varphi\) defines a map \(g \otimes \wedge^+(N) \to \wedge^+(N)\) in the following way. Identify \(g\) with \(g \otimes \{1\} \subset g \otimes \wedge^+(N)\). By duality, the evaluation of \(\chi \in g^*\) at an arbitrary element \(\eta = \sum X_j \otimes \eta_j \in g \otimes \wedge^+(N)\) takes the value \(\chi(\eta) = \sum \chi(X_j) \eta_j\). The evaluation map \(\text{ev}_\eta : g^* \to \wedge^+(N), \chi \mapsto \chi(\eta)\), extends uniquely as an algebra homomorphism to \(\text{ev}_\eta : \mathbb{R}[[g^*]] \to \wedge^+(N)\). Then \(\text{ev}_\eta(\varphi) = \varphi(\eta)\) is the evaluation of \(\varphi\) at \(\eta\). Note that \(\text{ev}_\eta\) factors through \(S(g)\). All of this makes sense even if we replace \(N\) with \(N_C := N \otimes \mathbb{C}\).

Now let \(\Omega(P)\) denote as usual the algebra of differential forms on \(P\). The pullback

\[\kappa^* : \Omega(M) \to \Omega(P)\]

induced by the bundle projection is an injective algebra homomorphism; its image is the algebra \(\Omega_{\text{bas}}(P)\) of basic forms on \(P\). So \(\kappa^*\) has a left-inverse (pushforward), which we denote by

\[\kappa_* : \Omega_{\text{bas}}(P) \to \Omega(M).\]

The curvature \(\Theta\) of our connection \(\theta\) is an element of \(g \otimes \Omega_{\text{bas}}^+(P)\), so it makes sense to evaluate a formal power series \(\varphi \in \mathbb{R}[[g^*]]\) at \(\Theta/2\pi i \in g \otimes \Omega_{\text{bas}}^+(P)\). The resultant \(\varphi(\Theta/2\pi i)\) is a basic form on \(P\), so we can apply the pushforward \(\kappa_*\). This process yields an algebra homomorphism, namely,

\[\text{CW} : \mathbb{R}[[g^*]]^g \to \Omega^+(M)_{\mathbb{C}}, \quad \varphi \mapsto \kappa_* \varphi(\Theta/2\pi i).\]

We refer to \(\text{CW}(\varphi)\) as the Chern-Weil form of \(\varphi\). The high point of Chern-Weil theory is that the de Rham cohomology class of the Chern-Weil form \(\text{CW}(\varphi)\) is a characteristic class.

As demonstrated by Berline and Vergne \cite{3}, a similar “construction” occurs in heat kernel calculations. This is because there is a vector space isomorphism

\[\text{spin}(n) \cong \wedge^2(\mathbb{R}^n)\]
by virtue of the Chevalley map

$$\sigma: \text{Cl}(n) \to \wedge(\mathbb{R}^n),$$

which is defined, in terms of the standard orthonormal basis \(\{e_i\}_{i=1}^n\) of \(\mathbb{R}^n\), by the equation

$$\sigma(e_{i_1} \cdots e_{i_k}) = e_{i_1} \wedge \cdots \wedge e_{i_k}$$

for any subset \(\{e_{i_1}, \ldots, e_{i_k}\}\) of the basis. The Chevalley map is a vector space isomorphism, and the image of \(\text{spin}(n)\) is exactly \(\wedge^2(\mathbb{R}^n)\).

The calculation that mimics the construction of the Chern-Weil map is captured in Lemma 4.2 below. But first, we setup some notations.

**Definition 4.1.** We denote by

$$\lambda: \mathfrak{g} \to \wedge^2(\mathbb{R}^n)$$

the composition of the Lie algebra homomorphism (5) with the Chevalley map. We set

$$\Lambda := \sum_{i=1}^{\dim \mathfrak{g}} X_i \otimes \lambda(X_i) \in \mathfrak{g} \otimes \wedge^+(\mathbb{R}^n),$$

where \(\{X_i\}_{i=1}^{\dim \mathfrak{g}}\) is any orthonormal basis for \(\mathfrak{g}\). (The definition does not depend on the choice of the basis.)

**Lemma 4.2.** Let \(h_t\) be the Gaussian function on \(\mathfrak{g}\). Let \(\varphi\) be an analytic function defined near the origin of \(\mathfrak{g}\). Let \(\psi\) be a \(G\)-invariant bump function supported within the domain of \(\varphi\). Then the \(\wedge(\mathbb{R}^n)\)-valued function

$$t \mapsto \int_{\mathfrak{g}} h_t(X)\psi(X)\varphi(X)e^{-\lambda(X)} dX$$

has an asymptotic expansion \(\sum_{k=0}^\infty \Psi_k t^k\) for \(t \to 0^+\). (The asymptotic expansion is independent of the choice of \(\psi\).) The \(k\)th coefficient \(\Psi_k\) is contained in \(\bigoplus_{q=0}^k \wedge^{2q}(\mathbb{R}^n)\). If \(k \leq n/2\), then the component of \(\Psi_k\) of degree 2\(k\) (the highest degree part) is equal to that of \(\varphi(\Lambda)\in \wedge^+(\mathbb{R}^n)\).

**Remark on the proof.** This lemma is similar in form to Lemma 11.3 in Duistermaat [4, p. 137]. The proof given there can be carried over almost verbatim. The only extra thing that needs to be checked is that \(\sum_{i=1}^{\dim \mathfrak{g}} \lambda(X_i)\lambda(X_i) = 0\) when \(\{X_i\}_{i=1}^{\dim \mathfrak{g}}\) is an orthonormal basis for \(\mathfrak{g}\); this can be verified using the Jacobi identity of the Lie bracket. We omit the details. \(\square\)

The element \(\varphi(\Lambda)\) appearing in Lemma 4.2 is a Chern-Weil form in disguise (provided that \(\varphi\) is \(G\)-invariant), as implied by the next lemma. Before stating the lemma, recall that we have a smooth map \(c: \text{Cl}(M) \to \text{Cl}(n)\) that is an algebra isomorphism when restricted to \(\text{Cl}(T_xM) \subset \text{Cl}(M)\) for any \(x \in M\). This yields, via the Chevalley identification, a smooth map

$$\wedge(TM) \to \wedge(\mathbb{R}^n)$$

that is a vector space isomorphism when restricted to \(\wedge(T_xM)\). Though it is an abuse of notation, we shall denote this map also as

$$c: \wedge(TM) \to \wedge(\mathbb{R}^n).$$
One more notation: consider the map $\sharp: \Omega^1(M) \to \mathfrak{X}(M)$ that maps a 1-form to its dual vector field relative to the metric. This induces an algebra isomorphism

$$\sharp: \Omega(M) \to \Lambda \mathfrak{X}(M),$$

which maps differential forms to polyvector fields (so-called the “raising of indices”).

**Lemma 4.3.** Consider the polyvector field $(\kappa_*\Theta)^\sharp$, namely, the one obtained by taking the pushforward $\kappa_*\Theta$ of the curvature form $\Theta$ along the bundle projection and then raising its indices. Denote the value of this polyvector field at an arbitrary point $x \in M$ by $(\kappa_*\Theta)^\sharp_x$. Then $c((\kappa_*\Theta)^\sharp_x) = \Lambda$.

**Remark.** A straightforward consequence is that, for any $\varphi \in \mathbb{R}[\mathfrak{g}^*]^\theta$ and any $x \in M$, we have

$$A(\varphi) = CW(\varphi)_x,$$

where $A$ is the following composition of algebra homomorphisms:

$$\mathbb{R}[\mathfrak{g}^*]^\theta \to \wedge(\mathbb{R}^n) \to \wedge T_x M \to \wedge T^*_x M,$$

where the first map is the evaluation at $\Lambda/2\pi i$; the second is the inverse of $\wedge(T_x M) \overset{\kappa}{\to} \wedge(\mathbb{R}^n)$; and the last is the lowering of indices.

**Proof.** For $X \in \mathfrak{g}$, let $\langle X, \Lambda \rangle$ denote the inner product of $X$ with the $\mathfrak{g}$-factors of $\Lambda$, so that $\langle X, \Lambda \rangle = \lambda(X)$. We need to check that this is equal to $c(x, \kappa_*\Theta^\sharp_x)$. Since $\lambda(X) \in \wedge^2(\mathbb{R}^n)$, we may write

$$\lambda(X) = \frac{1}{2} \sum_{i,j=1}^n \langle \alpha(X)e_i, e_j \rangle e_i e_j,$$

where $\{e_i\}_{i=1}^n$ is the standard orthonormal basis for $\mathbb{R}^n$. Let $\xi_i$ denote the image of $e_i$ under the inverse of $\text{Cl}(T_x M) \overset{\kappa}{\to} \text{Cl}(n)$, and let $\tilde{\xi}_i$ be the horizontal lift of $\xi_i$ at, say, $p \in \kappa^{-1}[x]$. Then

$$c(x, \kappa_*\Theta^\sharp_x) = \frac{1}{2} \sum_{i,j=1}^n \langle X, \Theta_p(\tilde{\xi}_i, \tilde{\xi}_j) \rangle e_i e_j.$$

The curvature form $\Theta$ and the Riemannian connection $\nabla^P$ for $\mathfrak{X}(P)$ satisfy

$$\langle X, \Theta_p(\tilde{\xi}_i, \tilde{\xi}_j) \rangle = 2\langle \nabla^P_X \tilde{\xi}_i, \tilde{\xi}_j \rangle.$$

The right-hand side is, by definition, $\langle \alpha_x(X)\xi_i, \xi_j \rangle$. This proves that $c(x, \kappa_*\Theta^\sharp_x) = \lambda(X)$ as desired. \qed

## 5. Proof of the Theorem

**Theorem 5.1.** Let $M$ be a closed oriented Riemannian manifold of dimension $n$. Let $P$ be a principal bundle over $M$ whose fibers are isomorphic to a compact Lie group $G$. Let $E$ be a $G$-vector space that is also a graded $\text{Cl}(n)$-module. Let $D$ be a geometric Dirac operator on $\Gamma(P \times_G E)$, and let $D_M$ be the restriction of the lift of $D$ onto the domain $(C^\infty(P) \otimes E)^\Gamma$. Suppose the covariant derivative associated with $D_M$ is a Clifford connection. If $n$ is even, then the distributional index $[D_M]$ of $D_M$ satisfies

$$\langle \text{Duf}^{-1}[D_M], \varphi \rangle = \langle \hat{D} \sim \hat{\varphi}, \hat{M} \rangle$$

(8)
for any \( G \)-invariant analytic function \( \varphi \) defined on some neighborhood of the origin in \( \mathfrak{g} \). The pairing is zero if \( n \) is odd.

**Proof.** We begin by recalling the definition of the Duflo isomorphism in terms of distributions. Let \( \mathcal{E}'(\mathfrak{g})_0^G \) denote the algebra of \( G \)-invariant distributions on \( \mathfrak{g} \) supported at the origin. Likewise, let \( \mathcal{E}'(G)_e^G \) denote the algebra of \( G \)-invariant distributions on \( G \) supported at the identity. Let \( j_\theta \) be the analytic function on \( \mathfrak{g} \) defined by:

\[
j_\theta(X) = \det^{1/2} \left[ \frac{\sinh(\text{ad}(X/2))}{\text{ad}(X/2)} \right].
\]

The Duflo isomorphism is then defined as

\[
\text{Duf} = \exp^* \circ j : \mathcal{E}'(\mathfrak{g})_0^G \to \mathcal{E}'(G)_e^G,
\]

where \( \exp^* \) denotes the pushforward induced by the exponential map, and \( j \) denotes the multiplication by \( j_\theta \).

Let \( \exp^*: C^\infty(G) \to C^\infty(\mathfrak{g}) \) be the pullback along the exponential map. Since the exponential map is a local diffeomorphism on some neighborhood \( U \) of the origin in \( \mathfrak{g} \), there is an isomorphism

\[
\log^*: C^\infty(U) \to C^\infty(\exp[U])
\]

to which \( \exp^* \) serves as a left-inverse. This induces, by duality, a linear map

\[
\log^*: \mathcal{E}'(G)_e^G \to \mathcal{E}'(\mathfrak{g})_0^G,
\]

which is inverse to \( \exp^* \). Then we have \( \text{Duf}^{-1} = j^{-1} \circ \log^* \), so

\[
\langle \text{Duf}^{-1}[D_M], \varphi \rangle = \langle [D_M], \log^*(j^{-1}_\theta \psi \varphi) \rangle.
\]

Here we have included a suitable \( G \)-invariant bump function \( \psi \) so that the pullback \( \log^*(j^{-1}_\theta \psi \varphi) \) makes sense. This is fine since \( [D_M] \) has point support.

Applying Equation (6), we get:

\[
\langle \text{Duf}^{-1}[D_M], \varphi \rangle = \int_M \int_G \text{Str}(P_t(x, x \cdot g) \log^*(j^{-1}_\theta \varphi \psi)(g) \nu(g)^{-1}) \ dg \ dx.
\]

Because the left-hand side is independent of \( t \), it is sufficient to show that the right-hand side is asymptotically equal to \( \langle \hat{D} \varphi, \hat{M} \rangle \) as \( t \to 0^+ \) when \( n \) is even, and that it is of \( O(t^{-1/2}) \) when \( n \) is odd.

We focus our attention to the integral over \( G \),

\[
I(t) := \int_G \text{Str}(P_t(x, x \cdot g) \log^*(j^{-1}_\theta \varphi \psi)(g) \nu(g)^{-1}) \ dg.
\]

Let

\[
j_M(X) := \det^{1/2} \left[ \frac{\sinh(\alpha(X/2))}{\alpha(X/2)} \right].
\]

And let \( \text{vol} \) denote the Riemannian volume form on \( M \) associated with the measure \( dx \) on \( M \). We claim that, if \( n \) is odd, then

\[
I(t) \ \text{vol} = O(t^{1/2})
\]

for \( t \to 0^+ \); if \( n \) is even, then

\[
I(t) \ \text{vol} = \text{CW}(j^{-1}_M e^{-\tau}) \ \text{CW}(\varphi)\text{top} + O(t),
\]
where CW is the Chern-Weil map induced by the connection \( \theta \) on \( P \) (which is determined by the Clifford connection associated with \( D \)), and the decoration \( |^{\text{top}} \) picks out the top degree part of the differential form at hand. Assume for the moment that the claim is true; then, since the de Rham cohomology class of the Chern-Weil form \( \text{CW}(j_M e^{-\tau}) \) is the index class of \( D \), we indeed have Equation (8).

To prove our claim, we change the domain of the integral \( I(t) \) from \( G \) to \( g \) by means of the exponential map. As it is well-known, \( j_g^2(X) \) calculates the Jacobian determinant of the exponential map when the exponential map is diffeomorphic near \( X \). So

\[
I(t) = \int_g \text{Str}(P_t(x, x \cdot \exp(X))j_g(X)\psi(\exp(X))\varphi(X)e^{-\nu_e(X)}) \, dX.
\]

In the limit \( t \to 0^+ \), the function \( t \mapsto P_t(x, x \cdot \exp(X)) \) is of \( O(t^\infty) \) if \( X \) is outside any neighborhood of the identity. Thus, the bump function \( \psi \) may be dropped without affecting the asymptotic behavior. In other words,

\[
I(t) \sim \int_g \text{Str}(P_t(x, x \cdot \exp X)j_g(X)\varphi(X)e^{-\tau(X)}e^{-\gamma(X)}) \, dX
\]
as \( t \to 0^+ \). Note that we have used the relation \( \nu_e = \gamma + \tau \).

The asymptotic expansion of \( P_t \) is well-known. See, for instance, Berline, Getzler, and Vergne [2, Thm. 5.9, p. 176], from which we deduce that

\[
I(t) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{m=0}^{\infty} t^m \text{Str}\left( \int_g h_t(X)\Phi_m(x, X)e^{-\lambda(X)} \, dX \right),
\]

where

\[
\Phi_0(x, X) = j_g(X)j_M^{-1}(X)e^{-\tau(X)}\varphi(X).
\]

The rest of the argument proceeds similarly to that found in Berline and Vergne’s proof [3] of the Atiyah-Singer index theorem (see [2, § 5.4]). In short, by Lemma 4.2 and the representation theory of Clifford algebras, we conclude that

\[
I(t) = O(t^{1/2})
\]

if \( n \) is odd, and

\[
I(t) = \frac{1}{(4\pi t)^{n/2}} \text{Str}\left( \int_g h_t(X)\Phi_0(x, X)e^{-\lambda(X)} \, dX \right) + O(t)
\]

\[
= \frac{1}{(4\pi)^{n/2}} \text{Str}(\Phi_0(x, -\Lambda)) + O(t)
\]

if \( n \) is even. Owing to Equation (7), we have

\[
I(t) \, \text{vol} = \text{CW}(j_M^{-1}e^{-\tau}\varphi)|^{\text{top}} + O(t)
\]

when \( n \) is even. Hence Equation (9) and (10) hold as claimed, and we are done. \( \square \)

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