Brane World Gravity in an $AdS$ Black Hole *

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Abstract

We consider a model of brane world gravity in the context of non-conformal non-SUSY matter. In particular we modify the earlier strong coupling solution to the glueball spectrum in an $AdS^7$ Black Hole by introducing a Randall-Sundrum Planck brane as a UV cut-off. The consequence is a new normalizable zero mass tensor state, which gives rise to an effective Einstein-Hilbert theory of gravity, with exponentially small corrections set by the mass gap to the discrete glueball spectrum. However the simplest microscopic theory for the Planck brane is found to have a tachyonic instability in the radion mode.

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1 Introduction

An interesting idea that has emerged in recent years is that the energy scale of gravity might not be significantly higher than the scale of other fundamental interactions [1, 2]. Models to achieve this goal have invoked extra dimensions, which may be either compact circles, or “effectively compact” directions where the spacetime metric decreases exponentially with the proper distance in the internal direction [3, 4]. Gravity lives in all the dimensions, while matter is confined to a hypersurface (the brane).

We study a model that follows the spirit of the latter class of theories, but which has some interesting features [5, 6, 7, 8] not found in earlier models. Our compact direction ends in a smooth way at some value of the internal coordinate [9, 10, 11, 12, 13, 14, 15, 16, 17], instead of terminating in a second brane or continuing forever. Thus the “warped spacetime” lies in the region $r_{\text{min}} < r < r_c$, with the graviton localized near a brane at $r_c$ and $r = r_{\text{min}}$ being a smooth end to the internal radial direction $r$. Such a metric ending smoothly at $r = r_{\text{min}}$ (and becoming approximately anti-de-Sitter at large $r$) arises in the $AdS^7 \times S^4$ Black Hole gravity dual of $d=4$ QCD at strong coupling [9]. We modify the metric emerging from this construction, by placing a “Planck brane” at $r = r_c$ and impose reflection symmetry about $r = r_c$.

The original AdS/CFT dual description for $QCD_4$ at strong coupling suggested by Witten [9] is in fact a finite temperature 5-d Yang-Mills theory at the boundary of a deformed AdS space with radial coordinate $r_{\text{min}} < r < \infty$. Anti-periodic boundary condition on the thermal axis for fermions breaks conformal and SUSY symmetries giving rise to a discrete “glueball” spectrum with mass scale set by $\Lambda_{qcd} = r_{\text{min}}/R_{\text{ads}}^2$ or equivalently by the temperature of the resultant Black Hole. In our present analysis we find the following features. The Planck brane at $r = r_c$ must be asymmetric – it must have one tension along the “space time” directions of the brane, but a different tension along the “thermal” direction $\tau$, the ratio of these two tensions fixing the brane location to $r = r_c$. We construct a model, which yields a brane with effective tensions that are asymmetric in the desired manner.

We then look at the small fluctuations around this background. We find a massless graviton “localized on the brane”. But we also have a “radion” mode that is essentially a fluctuation of the proper distance from the brane at $r_c$ to the horizon of the Black Hole at $r = r_{\text{min}}$. For our microscopic construction of the asymmetric brane, we find that this radion mode is an unstable solution. We then examine a 1-parameter family of “effective potentials” for the brane, which generalize the behavior found in this explicit brane construct and solve analytically for all zero-mass excitations. We find that beyond a certain critical value for this
parameter the radion mode indeed becomes stable. However we have not been able to find an explicit construction of branes which leads to these effective potentials.

An interesting feature of the model is the presence of both a graviton mode localized at the Planck brane and a discrete set of radial “KK states” analogous to the glueball modes found in the dual description of QCD. The large mass hierarchy required for the effective 4-d Planck mass relative to the QCD scale is exponential in the proper distance from $r_{\text{min}}$ to $r_c$. The “glueball states” suffer only an exponentially small correction due to the fact that the coordinate $r$ has an UV cut-off at $r = r_c$ instead of continuing to infinity. It is true that the model we are considering has not strictly been derived from string theory and therefore may not exhibit Gravity/Gauge duality. But we have chosen the metric for $r < r_c$ to be equal to that found in the strong coupling dual to d=4 QCD, so it is interesting to speculate that these “glueball modes” still represent some kind of gauge excitations in the current model as well. The fact that both gravity and glueball modes are excitations of a single string description in AdS space opens up interesting questions on the relations between these two entities, and we comment on these issues at the end of the paper.
2 The Model

We introduce an effective low energy model for brane world gravity interacting with non-conformal matter. Our approach is based on modifying the AdS/CFT example proposed by Witten as a gravity dual for $QCD_4$ in the strong coupling limit. We modify the UV behavior with the insertion of a so called “Planck brane” in the spirit of Randall-Sundrum brane world gravity.

Let us recapitulate Witten’s suggestion of the gravity dual of $QCD_4$. One begins with the AdS/CFT correspondence for the 11-d M-theory background metric $AdS_7 \times S^4$. The 11th coordinate is compactified on a small circle (with radius $R_{11} = g_s l_s$) reducing M-theory to IIA string theory and the boundary (0,2) 6-d CFT to 5-d SUSY Yang-Mills. Then a second circle is introduced with anti-periodic boundary coordinate for all fermionic modes so that conformal and all supersymmetries are broken. The second circle will be designated here by $\tau$ or simply the thermal coordinate. It is conjectured that at weak coupling the corresponding field theory is a confining 4-d Yang-Mills theory. This new metric is an AdS Black Hole solution to the bosonic sector of 11-d supergravity,

$$\begin{align*}
S &= -\frac{1}{2\kappa_{11}} \int d^{11}x \sqrt{-g_{11}} (R_{11} - |F_4|^2) + \frac{1}{12\kappa_{11}} \int A_3 \wedge F_4 \wedge F_4 \text{ fermions} ,
\end{align*}$$

written in term of the metric tensor $g_{MN}$ and the 3-form gauge field $A_{MNL}$ and its field strength. In the black brane solution a constant background for $A_{MNL}$ for $N$ units of magnetic flux gives rise to an effective cosmological constant; ignoring fluctuations in $A$ and nonzero R charges in $S^4$, it is adequate for our present purpose to consider a simpler action in the $AdS^D$ subspace,

$$\begin{align*}
S &= -\frac{1}{2\kappa_D} \int_M d^Dx \sqrt{-g} (R - 2\Lambda) .
\end{align*}$$

$M_D \sim 1/\kappa_D^{1/d}$ is the bulk Planck mass in $AdS^{d+2}$ with $D = d + 2$. The $D = d + 2$ coordinates are designated by $x^M = (r, \tau, x^\mu)$ with $\mu = 1, \cdots d$. We also find it convenient to consider a general $AdS^{d+2}$ instead of restricting our discussion to $AdS^7$ appropriate for the M theory construction.

2.1 Black Hole Background

Substituting in to Einstein’s equations the ansatz

$$\begin{align*}
ds^2 &= \hat{g}_{MN} dx^M dx^N = \frac{1}{f(r)} dr^2 + f(r)d\tau^2 + f_0(r)g_{\mu\nu}(x) dx^\mu dx^\nu ,
\end{align*}$$

with one “radial” coordinate $r$, one angular coordinate $\tau$ periodic in $[0, \beta)$ and $f_0(r) \equiv k^2 r^2$, one finds the general solution, $f(r,c) = c + (kr)^2 - (kr_{min})^{d+1}/(kr)^d - 1$, with $c = 0, -1, +1$ for
\[ ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \] being Minkowski (or Euclidean), de Sitter and anti-de Sitter respectively. These spaces are all asymptotic to \( AdS^{d+2} \) as \( r \to \infty \) with the AdS radius \( R_{ads} = 1/k \) fixed by the cosmological constant, \( \Lambda = -(D-1)(D-2)k^2/2 \). In particular we are interested in the asymptotically flat Minkowski background \( ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \), thus \( r_c = 0 \) and

\[ f(r) \equiv (kr)^2 - \frac{(kr_{min})^{d+1}}{(kr)^{d-1}} \]  

The period for the thermal axis (\( \tau \to \tau + \beta \)), or inverse “Hawking temperature”, is fixed to be \( \beta = (4\pi)/((D-1)k^2r_{min}) \) by the requirement that the horizon is a coordinate singularity at \( r = r_{min} \). For the case \( r_{min} = 0 \), the space becomes pure AdS, \( f(r) \to f_0(r) = k^2r^2 \) and there is no horizon and no condition on the periodicity. Often it is more convenient to work with units where \( R_{ads} \equiv k^{-1} = 1 \). We shall do so for the most part, when necessary restoring \( R_{ads} \) by dimensional analysis.

The \( r-\tau \) manifold, \( r \in [r_{min}, \infty) \) and \( \tau \in [0, \beta) \), at fixed \( x^\mu \) can be regarded as the entire 2-d plane with origin at \( r = r_{min} \). At times we will prefer to replace \( r \) by the proper distance from \( r = r_{min} \),

\[ y(r) \equiv \int_{r_{min}}^{r} \frac{dr}{\sqrt{f(r)}} = \frac{2}{(d+1)} \log \left[ (r/r_{min})^{(d+1)/2} + \sqrt{(r/r_{min})^{d+1} - 1} \right] , \]  

or \( r(y) = r_{min} \cosh \frac{(d+1)}{2}y \). In terms of \( y \), the metric becomes

\[ ds^2 = dy^2 + r^2(y) \tanh^2 \left( \frac{d+1}{2}y \right) d\tau^2 + r^2(y)dx^\mu dx_\mu , \]  

with the AdS form recovered in the large \( y \) limit where \( r/r_{min} \simeq \exp[ky] \).
2.2 Low Energy Effective Action

Next we introduce a Planck brane at $r = r_c$ replacing the $r$-$\tau$ plane by a compact disk, with a $Z_2$ reflection $r \rightarrow r_c^2/r$. These two disks are then patched together to form a $S^2/Z_2$ orbifold of the 2-sphere. In order to see how this effects the spectrum, we must build a model of the brane and study the small fluctuations of Einstein’s equations subject to the Israel junction condition. We model the Planck brane as an infinitely thin shell by adding a surface term to the action,

$$S_{\text{eff}} = -\frac{1}{2\kappa_D} \int_M d^d x d\tau d\rho \sqrt{-g} \left( R - 2\Lambda \right) + \frac{1}{2\kappa_D} \int_{\partial M} d^{d-1} x d\tau \sqrt{-q} V_{\text{brane}} ,$$

where $q$ is the determinant of the induced metric. As we will argue below, a particular microscopic construction can lead to a potential of the form

$$V_{\text{brane}} = \lambda_1 + \lambda_2 \exp[-\sigma(x)] ,$$

where $g_{\tau\tau} \equiv \exp[2\sigma(x)]$. Unlike the Randall-Sundrum construction, two independent parameters, $(\lambda_1, \lambda_2)$, are needed because the cusp due to the $Z_2$ orbifold is not the same in $\tau$ direction versus the Euclidean $x^\mu$ directions. We shall demonstrate shortly that in order to have a metrically flat brane on a $Z_2$ orbifold at $r = r_c$ the parameters in Eq. (8), must be adjusted to fit the background metric

$$\lambda \equiv \frac{d}{dy} \log f_0(r_c - \epsilon) = \frac{\lambda_1}{2d} ,$$
$$\lambda_{\tau} \equiv \frac{d}{dy} \log f(r_c - \epsilon) = \frac{\lambda_1}{2d} + \frac{\lambda_2}{2\sqrt{f}} .$$

(9)

The ratio of $\lambda$ and $\lambda_{\tau}$ fixes the position of the Planck brane $r = r_c$.

Let us now see if it is possible to construct such an asymmetric brane from the microphysics that we are allowed to assume. The brane must be characterized as an object defined in an intrinsically covariant fashion, in order that we may place it in the ambient geometry and consistently couple its fluctuations to the fluctuations of the metric. This fact places constraints on how we may obtain the asymmetric brane. Consider a set of $d$-branes with world volume extending over all the directions $(x^\mu, \tau)$; these branes have the usual isotropic tension. Now take a collection of $(d-1)$-branes, and place them perpendicular to the $\tau$ direction. These branes have tension only along $x^\mu, \mu = 1 \ldots d$. We let the density of these branes be uniform along the circle $\tau$. This model leads to a brane world potential give by Eq. (8). The combined tension of this set of branes is clearly asymmetric. The two kinds of branes are not bound to each other, but since we only consider symmetric fluctuations, by symmetry both kinds of branes will stay at the same location $r = r_c$. We will also study fluctuations that are independent of the coordinate $\tau$, so the distribution of the $d$-branes in the $\tau$ direction will automatically remain uniform.
In this construction we see the difficulties with other ways of achieving asymmetric tensions. The basic objects that we have in our theory are (i) branes and (ii) momentum modes. Suppose we put a set of 1-branes wrapping the direction $\tau$, distributed uniformly in the directions $x_i$. We do of course want to have vibrations that are functions of the $x_i$. But after such a wave passes through this gas of 1-branes, the 1-branes will no longer be uniformly distributed in the coordinates $x_i$. Thus their stress tensor will not be mimicked by a single asymmetric brane; rather we will have to introduce a density function $\rho(x_i, t)$ to characterize the evolving distribution of the 1-branes in the directions $x_i$. The same applies to momentum modes carrying momentum in the direction $\tau$, which we could naively have considered as another way to generate asymmetric tensions.

Since we want our physics to be $x^\mu$ dependent (but we can choose $\tau$ independence), the above construction using $d$-branes and $(d - 1)$-branes appears to be the only simple choice available. However our subsequent analysis of this brane world leads to instability signaled by a tachyonic radion mode. To understand this instability, we have generalized our model to an effective brane world potential,

$$V_{brane} = \tilde{\lambda}_1 + \tilde{\lambda}_2 \exp[-\alpha \sigma(x)],$$

where $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are appropriately redefined to give the flat brane solution. The free parameter $\alpha$ allows one to adjust the quadratic term about the background, i.e., varying $V_{brane}'(\sigma_0)$ while holding $V_{brane}(\sigma_0)$ and $V_{brane}'(\sigma_0)$ fixed in the background metric ($\exp[2\sigma_0] = f(r)$), thus shifting the radion to positive mass squared and in turn stabilizing the brane.

### 2.3 Definition of Spectral Problem

To find the spectrum around this background solutions we must find the linearized Einstein’s equations. More precisely the linearized Euler-Lagrange equations of bulk gravity coupled to our low energy effective brane world action. Our approach is to write down the EOM in an axial gauge ($h_{Mr} = 0$ for $M \neq r$) and integrate from the IR horizon at $r = r_{min}$ to the UV boundary at the Planck brane where $r = r_c$. Although the actual Minkowski time is parallel to the brane, we are in a sense treating the radial coordinate as a sort of time variable. This appears to be the best way to approach the spectral problem.

We begin by parameterizing the metric in the traditional Kaluza-Klein form,

$$ds^2 = \tilde{g}_{mn} dx^m dx^n + e^{2\sigma}(d\tau + A_m dx^m)^2.$$

with $x^m = (r, x^1, \cdots x^d)$. As we noted previously in our glueball analysis [2, 13], it is natural to classify modes with respect to the number of $\tau$ indices ( zero for $g_{mn} = \tilde{g}_{mn} + \exp[2\sigma]A_n A_m$,
one for $g_{\nu \tau} = \exp[2\sigma]A_m$, and two for $g_{\tau \tau} = \exp[2\sigma]$ respectively). Expanding to linear order in the fluctuations in axial gauge ($h_{\tau M} = 0$ for $M \neq r$),

$$ds^2 = f(r)^{-1}(1 + \rho)d\tau^2 + r^2(\eta_{\mu \nu} + h_{\mu \nu})dx^\mu dx^\nu + f(r)(1 + S)d\tau^2 + 2f(r)A_\mu dx^\mu d\tau,$$

and ignoring $\tau$ dependence (or excitation in the thermal direction) the trace reversed Einstein equations in the bulk as follows.\footnote{To be precise about our conventions, the linearized Einstein equations are written in the form, $h^M_{\cdots L} + h_{L \cdots M} - h^M_{\cdots L} - h_{L \cdots M} + 2(D-2)^{-1}h^M_{\cdots N} = \Sigma^M_{\cdots N}$ where the bulk energy momentum $\Sigma^M_{\cdots N} = 2\kappa(T^M_{\cdots N} - (D-2)^{-1}g^M_{\cdots N})$ has been introduced to set the normalization. The $M$-index is raised by the background metric \cite{12}, except when we use Lorentzian signature $\eta_{\mu \nu}$ (mostly positive) for flat Minkowski space.}

**Equations of Motion:** We enumerate these equations as 3 second order wave equations, (tensor, vector and scalar), followed by 3 constraint equations,

$$\Sigma_{\mu \nu} = \nabla^2 h_{\mu \nu} + \frac{1}{r^2}[\partial_\mu \partial_\nu (h + S + \rho) - \partial_\mu \partial_\lambda h_{\lambda \nu} - \partial_\nu \partial_\lambda h_{\mu \lambda}] + (\frac{f}{r^2}(h' + S' - \rho') - 2(d + 1)\rho)\eta_{\mu \nu} = 0,$$

$$\Sigma^\tau = \nabla^2 A_\mu - \frac{1}{r^2}\partial_\mu \partial^3 A_\lambda + r^2(\frac{f}{r^2})' A'_\mu = 0,$$

$$\Sigma^{r} = \nabla^2 S + \frac{f'}{r^2}(h' + S' - \rho') - 2(d + 1)\rho = 0,$$

$$\Sigma''_r = f\partial^3 A'_\lambda = 0,$$

$$\Sigma''_r = f(S'' + h'') + \frac{3f'}{2}S' + (\frac{2f'}{r} + \frac{f''}{2})h' - (\frac{df}{r} + \frac{f'}{2})\rho' + \frac{1}{r^2}\partial^3 \partial_\lambda \rho - 2(d + 1)\rho = 0,$$

where $h' = \partial_\tau h$, etc. We note that the diagonal $rr$-equation is second order in $r$-derivatives; it only becomes a first-order constraint in “trace-unreversed” form, e.g., for $\Sigma''_r - \frac{1}{2}(\Sigma''_r + \Sigma^\lambda_\lambda + \Sigma''^r_r)$. We also note that not all these equations are independent due to the Bianchi identity.

It is interesting to point out that these equations have the expected pure AdS ($f(r) \to f_0(r) = r^2$) limits. It is easy to see that the 3 wave equations, (13)-(15), collapse into a single tensor equation for $h_{ab}, a, b = \tau, 1, \ldots, d$, with the identification $A_\mu \equiv h_{\tau \mu}$ and $S \equiv h_{\tau \tau}$. Similarly, the first two constraints, (13) and (17), after taking into account that $\tau$ derivatives have been dropped, also become a single vector equation. Thus in pure AdS we have the 3 equations,

$$\Sigma_{ab} = \nabla^2 h_{ab} + \frac{1}{r^2}[\partial_a \partial_b (h + \rho) - \partial_a \partial^\sigma h_{cb} - \partial_b \partial^\sigma h_{ac}] + [r(h' - \rho') - 2(d + 1)\rho]\eta_{ab} = 0,$$

$$\Sigma^\tau = r^2 \partial^\sigma (h'_{ca} - \eta_{ca} h') + d r \partial_a \rho = 0,$$

$$\Sigma''_r = r^2 h'' + 3rh' + (d + 1)\rho' + \frac{1}{r^2}\partial^\sigma \partial_\rho \rho - 2(d + 1)\rho = 0.$$
Lastly, we introduce for later convenience a differential operator in \( r \), \( \nabla^2_r \phi \equiv \sqrt{g}^{-1}(\sqrt{g} f \phi')' \), where \( \sqrt{g} = r^d \), so that the covariant Laplacian can be expressed as
\[
\nabla^2 \phi = \frac{1}{\sqrt{g}} \partial_M (\sqrt{g} g^{MN} \partial_N \phi) = \nabla^2_r \phi + \frac{1}{r^2} \partial^\lambda \partial_\lambda \phi = f \phi'' + \left( \frac{d}{r} + f' \right) \phi' + \frac{1}{r^2} \partial^\lambda \partial_\lambda \phi.
\]

**Boundary Conditions:** The tension on the Planck surface must be adjusted (or fine tuned) to preserve the Black Hole metric with a metrically flat \((R^d \times S^1)\) brane world at \( r = r_c \). This is done through the Israel junction conditions, which can also be expressed conveniently in Gaussian normal coordinates (or axial gauge \( g_{Mr} = 0 \) for \( M \neq r \)) with the brane fixed at \( r = r_c \). The extrinsic curvature takes the form,
\[
K_{ab}(r_c \pm \epsilon) = \frac{1}{2\sqrt{g_{rr}}} \partial_r g_{ab}(x, r_c \pm \epsilon) , \tag{19}
\]
and jump condition,
\[
K_{ab}(r_c + \epsilon) - K_{ab}(r_c - \epsilon) = -2\kappa_D \int_{r_c - \epsilon}^{r_c + \epsilon} dr (T_{ab}^{brane} - \frac{1}{D-2} g_{ab} T^{brane}) , \tag{20}
\]
where \( g_{ab} \) is the induced metric (namely \( a, b \) take on all coordinates except \( r \)).

The trace-reversed energy-momentum on for our brane model is given by
\[
2\kappa_D (T_{ab}^{brane} - \frac{1}{D-2} g_{ab} T^{brane}) = \begin{bmatrix} \lambda_T[\sigma] g_{rr} & 0 \\ 0 & \lambda[\sigma] g_{\mu\nu} \end{bmatrix} \delta(r - r_c) , \tag{21}
\]
where we define the “tensions” in an arbitrary background, \( \sigma \), by \( \lambda[\sigma] \equiv (V[\sigma] + V'[\sigma])/2d \) and \( \lambda_T[\sigma] \equiv (V[\sigma] - (d-1)V'[\sigma])/2d \). To zeroth order in the background metric \( \exp[2\sigma_0(r)] = f(r) \), their values are now fixed by (19)-(21): \( \lambda[\sigma_0] = \lambda \) and \( \lambda_T[\sigma_0] = \lambda_T \). These reduce to Eq. (9) for \( \alpha = 1 \), as promised.

To first order, the Israel junction conditions provide boundary conditions for our EOM at \( r = r_c \),
\[
\begin{align*}
\h_{\mu\nu}' &= \left( \frac{\lambda_T}{2} \rho + \left( \frac{\lambda - \lambda_T}{2\sqrt{f}} \right) \frac{1 - \alpha}{d} \right) S_{\mu\nu} , \\
S' &= \frac{\lambda_T}{2} \rho + \left( \frac{\lambda - \lambda_T}{2\sqrt{f}} \right) \frac{1 - \alpha}{d} + \alpha S , \\
A'_\mu &= 0 . \tag{22}
\end{align*}
\]
(We have also imposed \( Z_2 \) symmetry so that the jump condition is defined through the radial derivative to the left of the cusp at \( r_c - \epsilon \).) Note that the simplest model for an asymmetric brane, Eq. (8), corresponds to having \( \alpha = 1 \) in which we must further restrict \( A_\mu(r_c) = 0 \). To maintain the horizon as a coordinate singularity also dictates that
\[
S(r_{min}) = \rho(r_{min}) , \tag{23}
\]
and requires that \( \rho, \h_{\mu\nu}, \) and \( S \) are regular at \( r = r_{min} \) [11, 12, 13].
3 Analysis of Zero Mass Spectrum

Randall and Sundrum have observed that the gravitational field in the bulk spacetime AdS space gives rise to a zero mass graviton. The graviton is a domain wall state confined to the Planck brane with an exponential tail into the bulk when measured in proper distance. We will see below that their mechanism for producing brane world gravity generalizes to our spacetime background as well. However in the first Randall-Sundrum model \cite{3} there is another massless bulk mode, the “radion”, representing the fluctuations in the proper distance separating the positive and negative tensions branes at the $Z_2$ orbifold planes. We wish to see if we have a similar radion mode in our model.

To enumerate clearly the physical modes, we choose to completely fix the gauge. It is natural in our argument to study each wave equation, as a function of the radial coordinate, by imposing first the boundary condition at the Black Hole horizon and then integrating out to the Planck brane. A convenient gauge for this analysis begins with our choice of an axial gauge

$$h_{ra} \rightarrow h'_{ra} = h_{ra} + \xi_{a,r} + \xi_{r,a} = 0, \quad \text{for } a = \tau, 1, \cdots d,$$

in a coordinate system with the Planck brane fixed at $r = r_c$ and the horizon at $r = r_{\min}$, independent of the fluctuations around the background. In the above gauge there remains one additional $r$-dependent transformation $r \rightarrow r + \xi^r(x, \tau, r)$ with

$$\xi^\mu(x, \tau, r) = -r^2 \partial^\mu \int^r dr \frac{\xi^r(x, \tau, r)}{r^2 f(r)}, \quad \text{and} \quad \xi^\tau(x, \tau, r) = -f(r) \partial^\tau \int^r dr \frac{\xi^r(x, \tau, r)}{f^2(r)},$$

as well as $d + 1$ $r$-independent gauge transformations, $x^a \rightarrow x^a + \xi_0^a(x, \tau)$. These residual gauge transformations will be used to eliminate additional unphysical states. Since the charged KK modes do not contribute to the zero mass states we may simplify the discussion by dropping all $\tau$-dependence in what follows, leaving us the usual gauge transformation $\xi_0^\tau(x)$ for the KK “photon”, $h_{\tau \mu} = A_{\mu}(x)$, and $d$ $r$-independent gauge transformations, $\xi_0^a(x)$, for our brane world gravity.

3.1 Graviton Solution

We begin by looking for the analog of the Randall-Sundrum solution, thus obtaining the propagating degrees of freedom for the massless graviton on the brane. Our Planck brane spans a $d$-dimensional Lorentz space and a compact $\tau$-axis. We set the gauge field $A_{\mu}$ to zero. By inspection we see from Eq. (13) that the traceless ($\sum_i h_{ri}^{-1} = 0$) transverse plane wave state,
\( h_{ij}^\perp (r) \exp [ip_{\mu} x^\mu] \), satisfy the equation for minimally coupled scalar,

\[
\nabla_r^2 h_{ij}^\perp + \frac{m^2}{r^2} h_{ij}^\perp = 0 .
\] (25)

We note that the constant polarization vectors, \( h_{ij}^\perp (r) = \epsilon_{ij} \), are the solution for a zero-mass graviton in transverse gauge, (with two helicity states, \( \lambda = \pm 2 \), for \( d = 4 \)).

To see that these modes indeed represent only a single graviton in the boundary theory, we must see that the gauge freedom and gauge constraints arising from the bulk description agree exactly with the gauge properties expected from a graviton propagating in the boundary. It is useful to review the standard argument in flat space by dropping the \( r \)-dependence and the coupling to the additional longitudinal modes \( S, \rho \) in Eq. (13). In this case we would have the usual linearized Einstein equation for the tensor field,

\[
h_{\mu\nu,\lambda\lambda} + h_{,\mu\nu} - h_{\mu\lambda,\nu\lambda} - h_{\nu\lambda,\mu\lambda} = 0 .
\] (26)

To be explicit, we work in Minkowski space with conventional lightcone axes, \( e_\mu^\pm = (0, ..., 0, 1, \pm 1) \), so that all vector indices \( \mu \), (e.g., \( V_\mu \)), are replaced by \( i, +, - \), (e.g., \( V_i, V_\pm \equiv e_\mu^\pm V_\mu \)), with \( i = 1, ..., d - 2 \) and metric \( \eta_{++} = \eta_{--} = 2 \). All states are taken to be plane waves with momentum \( p_\mu = (0, ..., p, E) \), \( m^2 = -p^2 = -p_+ p_- \) and \( p_+ = p + E = 2p \) for \( m \rightarrow 0 \) so that graviton solution in momentum space is

\[
h_{\mu\nu} = \epsilon_{\mu\nu} e^{ip_+ x^+} .
\] (27)

In Eq. (26), setting \( \mu\nu = i+, ++ \), and ++, we find

\[
\epsilon_{i-} = 0 ,
\] (28)

\[
\epsilon - \epsilon_{+-} = 0 ,
\] (29)

\[
\epsilon_{--} = 0 ,
\] (30)

respectively where we have defined \( \epsilon = \epsilon_{\mu\nu} \eta^{\mu\nu} \). Thus the polarizations found from the bulk analysis must have the gauge freedom

\[
\epsilon \rightarrow \epsilon_{\mu\nu} + p_\mu \xi_\nu + p_\nu \xi_\mu ,
\] (31)

and be subject to exactly the above three constraints. By the standard argument [18] the physical states in the quotient space are the transverse graviton polarization \( \epsilon_{ij} \) given above.

We now must generalize this argument keeping track of possible \( r \)-dependence and the longitudinal coupling to \( S \) and \( \rho \). For this we need to be more precise about the spectrum of the radial equation for a minimal scalar. This is a Sturm-Liouville eigenvalue problem of the form,

\[
- \nabla_r^2 \phi_n (r) = \frac{m_n^2}{r^2} \phi_n (r) ,
\] (32)
with orthonormal condition, $\int_{r_{\text{min}}}^{r_c} dr \sqrt{-g} r^{-2} \phi_n(r) \phi_m(r) = \delta_{nm}$ and a Neumann boundary conditions, $\phi'(r_c) = 0$ at the Planck brane and regularity at the origin, $r = r_{\text{min}}$. At $r = r_{\text{min}}$, the solution has the form $C_1 + C_2 \log(r - r_{\text{min}})$. Regularity requires that we set $C_2 = 0$. The resulting operator is self-adjoint with a discrete positive semi-definite spectrum, $m_n^2 \geq 0$, and a single null vector, which is a constant in $r$: $\phi_0(r) = C$. Thus the graviton arises as the unique normalizable state which is present only for finite $r_c$. Our earlier tensor glueball calculations $[12]$ with $r_c = \infty$ exhibited a mass gap simply because this state was excluded from the Hilbert space due its logarithmically divergent norm. We postpone to Section 5 a more detailed comment on the effective gravity theory implied by this solution and its relation to the original Randall-Sundrum construction.

3.2 Longitudinal Modes

We proceed by solving the full set of equations, Eqs. 13-18, for massless modes. The KK “photon” $A_\mu(r)$ has a zero mode but this equation decouples from the rest and in our effective theory we have set $A_\mu(r_c) = 0$ eliminating this mode. Viewed from the perspective of the little group for a light-like state, the tensor field $h_{\mu\nu}$, in addition to the graviton, has in general $2(d-2)$ massless vector modes, $(h_{i\pm})$, and 4 scalars, $(h_{-\cdot}, h_{++}, h_{+\cdot}$ and $h = \eta^{\mu\nu} h_{\mu\nu} = \sum_i h_{ii} + h_{++})$. These mix with an additional KK scalar, $S$, and the radion, $\rho$. By finding explicitly all solutions for $m^2 = 0$, we can settle the issue concerning the existence of additional propagating scalars, i.e., the radion. A careful analysis of this sector is a bit involved. Here we give the outline of the argument.

First, consider first a pair of equations from Eq. (13):

$$\begin{align*}
- \nabla_i^2 h_{i-} &= 0, \\
- \nabla_i^2 h_{i+} &= \frac{2h_{i-}}{r^2}.
\end{align*}$$

(33)

Let us integrate the first equation from $r_{\text{min}}$ outwards. Imposing regularity at $r = r_{\text{min}}$ and satisfying boundary conditions at $r_c$ gives $h_{i-}' = 0$, i.e., a null state $h_{i-} = \text{constant}$. But the second equation leads to a contradiction if this constant is non-zero – we get a logarithmic singularity in $h_{i+}$ at $r = r_{\text{min}}$. Put succinctly, the image of our radial operator in the Hilbert space cannot include the null vector. Thus we conclude that we must have $h_{i-} = 0$, which reproduces (28), and we then find that $h_{i+} = \text{const}$ is also a null vector. This latter constant can be gauged away to zero by a diffeomorphism $\xi^i \sim x^+$, just as would be expected for a graviton on the brane.
Similarly there is a pair of equations for

\[- \nabla^2_r h_{--} = 0 \, , \, \quad - \nabla^2_r [h_{+-} - \frac{2}{d} h] = \frac{2(d - 2)}{dr^2} h_{--} \, . \tag{34}\]

These give the condition \( h_{--} = 0 \), which reproduces (31) above, and the condition that \( h_{+-} - \frac{2}{d} h = constant \); this constant can again be gauged away by a \( r \)-independent diffeomorphism.

Next, we note that there are 3 equations for the remaining 4 scalars, \( S, \rho, h_{++}, \) and \( h \). This would appear to be an undetermined system, but we can perform an \( r \)-dependent diffeomorphims to reduce the radion field to a single constant, \( \rho(r) \rightarrow \rho_0 \), representing the fluctuation for the proper distance separating the Planck brane (at \( r = r_c \)) from the Black Hole horizon (at \( r = r_{min} \)).

Consider the ++ component of the tensor equation for \( h_{\mu\nu} \)

\[- \nabla^2_r h_{++} = - \frac{4}{r^2} (S + \rho_0 + h - h_{++}) \, , \tag{35}\]

and the “trace-unreversed” form of \( \tau\tau \) and the \( rr \) equations:

\[ \nabla^2_r h - \frac{3f'}{2} r^{d-1} h' - d(d + 1)\rho_0 = - \frac{1}{r^2} h_{--} \, , \tag{36} \]
\[ d\frac{f}{r} S' + ((d-1)\frac{f}{r} + \frac{f'}{2})h' - d(d + 1)\rho_0 = - \frac{1}{r^2} h_{--} \, . \tag{37} \]

Since we have shown that the function \( h_{--} \) is zero, the last two equations can be integrated rather trivially from \( r = r_{min} \), yielding

\[ S(r) = \frac{f'\rho_0 y}{2 \sqrt{f}} \, , \quad \text{and} \quad h(r) = \frac{d\sqrt{f}\rho_0 y}{r} + h(r_{min}) \, , \tag{38} \]

where \( y(r) = \int_{r_{min}}^r dr / \sqrt{f(r)} \) is given in Eq. (5). Note that \( S(r_{min}) = \rho_0 \) and both \( h \) and \( S \) are regular at \( r_{min} \). The integration constant \( h(r_{min}) \) remains to be specified. When we substitute this solution into the boundary condition at \( r = r_c \),

\[ S'(r_c) = \frac{\lambda_r}{2} \rho_0 + (\frac{\lambda - \lambda_r}{2\sqrt{f_c}})(\frac{1 - \alpha}{d} + \alpha)S_c \, , \]
\[ h'(r_c) = \frac{d\lambda}{2} \rho_0 + (\frac{\lambda - \lambda_r}{2\sqrt{f_c}})(1 - \alpha)S_c \, . \tag{39} \]

we see that we are forced to set \( \rho_0 = 0 \), (unless we arbitrarily tune \( \alpha \) to a special value as discussed in the next section). It follows from Eq. (38) that we also get \( S = 0 \) and \( h = const. \). Returning to the \( h_{++} \) equation and using arguments similar to those above, we find that we must have \( h - h_{+-} = 0 \), thus reproducing (29). (We have previously shown that \( h_{+-} - \frac{2}{d} h = constant \). It follows that a single \( r \)-independent diffeomorphism can gauge both \( h_{+-} \) and \( h \) to zero.) Lastly, we also have \( h_{++} = constant \), which can again be gauged away to zero.
Thus we see that the radion vanishes in the zero mass sector, unless we tune the parameter $\alpha$ in the above relations to a specific value; we will discuss such tuning further in the next section. One might hope that without such tuning we have eliminated the radion and obtained just a massless graviton on the brane. But as we will see below the radion actually has negative $m^2$ at $\alpha = 1$, which is the value of $\alpha$ for our physical construction of the ‘asymmetric brane’. Thus the brane construction we started with is actually unstable. We will explore the nature of the solutions for arbitrary $\alpha$ in the search for a stable solution; however we do not know if these other values of $\alpha$ can be reproduced by a well-defined microscopic matter Lagrangian.
4 Radion Instability

To address the question of stability, imagine expanding the Euclidean action, Eq. (39), to quadratic order in the fluctuating fields relative to the AdS Black Hole background. The quadratic terms will take the form

$$S_E = \frac{\beta}{4\kappa_D} \int d^d p \int_1^{r_c} dr \sqrt{g} \left[ \phi^\dagger(r) \mathcal{L} \phi(r) + \frac{p^2}{r^2} \phi^\dagger(r) Z \phi(r) + \partial_r X_{\text{surface}} + X_{\text{brane}} \delta(r - r_c) \right], \quad (40)$$

choosing units so that $k = r_{\text{min}} = 1$. $\mathcal{L}$ and $Z$ may in general have Lorentz indices (see examples below). The general concept of stability of a classical solution requires that all fluctuations increase the action so perturbations in the partition function ($\int dg_{MN} \exp[-S_E]$) are well defined. However it is well known that off-shell Euclidean Einstein-Hilbert action is unbounded and does not even satisfy this stability condition to quadratic order in flat space. Consequently we restrict ourselves to a narrower criterion for on-shell stability where all eigenmodes are neither tachyonic (negative $m^2$) nor ghost-like (negative norm, i.e., negative coefficient for the $p^2$ kinetic term). To investigate the question of on-shell stability, one looks at the condition of stationarity. Stationarity of the the surface terms ($X_{\text{surface}}, X_{\text{brane}}$) is equivalent to imposing the Israel matching condition, so once we impose these boundary conditions we can drop the surface terms from the action. Thus we are led back to the problem of finding the spectrum by solving the eigenvalue equations, $\mathcal{L} \phi(r) = (-p^2/r^2)Z \phi(r)$, for each eigenvector $\phi^{(n)}(r)$ with eigenvalue $p^2 = -m_n^2$. The sign of the norm of an eigenvector is defined by sign of the kinetic term,

$$\int_1^{r_c} dr \frac{\sqrt{g}}{r^2} \phi^{(n)}(r) Z \phi^{(n)}(r), \quad (41)$$

in the Euclidean action.

4.1 Radion Mass

To see if the radion is tachyonic, we begin by examining the constraint of the two boundary conditions at the Planck brane more carefully for our potential with a single variable parameter $\alpha$. Since $S$, $S'$ and $h'$ are homogeneous in $\rho_0$, the trivial solution $\rho_0 = 0$ always exists corresponding no massless radion, as indicated earlier. When can we have a solution with $\rho_0 \neq 0$? Substituting the solution for $h$ and $S$, (38), into the junction conditions, we find remarkably that both equations lead to a single condition:

$$\left( \alpha - 1 - \frac{2d f(r_c)}{r_c f'(r_c)} \right) \rho_0 = 0. \quad (42)$$

When $\alpha$ takes on a critical value,

$$\alpha_{\text{critical}} = 1 + \frac{2d f(r_c)}{r_c f'(r_c)} = 1 + d \frac{\lambda}{\lambda_r}, \quad (43)$$

in the Euclidean action.
both boundary conditions for $S'$ and $h'$ at $r_c$ are met, with $\rho_0 \neq 0$. Therefore a massless radion can exist by fine tuning $V_{brane}$.

With $\alpha = \alpha_{critical}$, the remaining fluctuation $h_{++}$ can next be found by solving Eq. (35). We obtain the solution

$$h_{++}'(r) = \frac{4}{r^2 f'} y \rho_0 - \frac{4}{r^3 \sqrt{f_c}} y_c \rho_0,$$

with $h_{+-} = 2h/d$. Regularity at $r = r_{min}$ has fixed the integration constant for $h$ so that

$$h(r_{min}) = -\frac{d(d-1)r_c}{(d-2)\sqrt{f_c}} y_c \rho_0.$$

The last remaining gauge freedom can be used to specify the integration constant, thus providing a complete specification for fluctuations associated with the massless radion.

As one smoothly varies $\alpha$ so that $\Delta \alpha \equiv \alpha - \alpha_{critical} \neq 0$, one expects this mode to survive with the radion acquiring a mass. For $\Delta \alpha$ small, this mass can be calculated perturbatively. Denote $S \simeq S_0 + m^2 S_1$ and $h \simeq h_0 + m^2 h_1$, where $S_0$ and $h_0$ are solution at $\alpha = \alpha_{critical}$. Substituting these into Eqs. (15) and (18), we find, to first order in $m^2$, a set of equations for $S_1$ and $h_1$:

$$r^2 f (S''_1 + h''_1) + (2rf + \frac{r^2 f'}{2}) h'_1 + \frac{3r^2 f'}{2} S'_1 = -\rho_0 \quad \text{and} \quad r^2 f h''_1 + 2rf h'_1 - d rf S'_1 = S_0 - \rho_0. \quad (44)$$

By imposing appropriate boundary conditions at both $r = r_{min}$ and $r = r_c$, these first order perturbations can be obtained. In particular, the radion mass is fixed to this order. For $r_c$ large, one obtains a simple analytic expression

$$m^2 \simeq \frac{(d-2)(d+1)}{2} \frac{r_{min}}{r_c} \Delta \alpha. \quad (45)$$

Consequently the system becomes tachyonic when reducing $\alpha$ below $\alpha_{critical}$ in the direction of our model for the Planck brane at $\alpha = 1$. We have check this result numerically that finding for $\alpha < \alpha_{crit}$ a small negative $m^2 < 0$ of order $0((r_{min}/r_c)^{d-1})$ set by the mass hierarchy. At $\alpha = 1$, it remains negative, giving $m^2 \simeq -35(r_{min}/r_c)^{d-1}$.

### 4.2 Ghost Analysis

To establish our sign conventions consider first the physical transverse traceless tensor field for tensor glueballs or graviton mode, $h_{\mu \nu}^\perp(r)$ with $p^\lambda h_{\lambda \nu}^\perp(r) = h_{\mu \lambda}^\perp(r)p^\lambda = \eta^{\mu \nu} h_{\mu \nu}^\perp(r) = 0$. Their contribution to the action is

$$S_E = \frac{\beta}{4\kappa_D} \int d^d p dr \sqrt{g} \left[ - h_{\mu \nu}^\perp \nabla^2 h_{\mu \nu}^\perp + \frac{p^2}{r^2} h_{\mu \nu}^\perp h_{\mu \nu}^\perp \right]. \quad (46)$$
The signs are such that all on-shell massive modes are neither tachyonic nor ghost-like. More subtle problem of the decoupling of null states for the zero-mass graviton has already been discussed leading to only two physical states to the on-shell graviton to with ±2 helicities for d=4.

In the massive scalar sector, we have a coupled problem for four amplitudes, \( h(r,p) = \eta^{\mu\nu} h_{\mu\nu}(r,p), \) \( h_T(r,p) = (\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}) h_{\mu\nu}(r,p), \) \( S(r,p), \) and \( \rho(r,p), \) with

\[
S_E = \frac{\beta}{4\kappa_D} \int d^4p d^4r \sqrt{g} \left[ - \frac{d-1}{d} h_T^* \nabla^2 h_T + h_T^* \nabla^2 h + h^* \nabla^2 h_T + S^* \nabla^2 h + h^* \nabla^2 S \\
+ \frac{(d+1)f}{2} (h^* S' - S^* h') + \frac{p^2}{r^2} \phi^\dagger Z \phi + \mathcal{U}(\phi^\dagger, \phi) \right] \tag{47}
\]

The “kinetic” term is expressed in a matrix form, with

\[
\phi^\dagger Z \phi = \frac{d-2}{d-1} h_T^* h_T - (h_T^* S + S^* \rho + \rho^* h_T + \text{h.c.}) , \tag{48}
\]

where the matrix \( Z \) defines the norm for these scalar fluctuations. The remaining non-vanishing terms for \( \rho \neq 0 \) are

\[
\mathcal{U}(\phi^\dagger, \phi) = \left( \frac{f'}{2} + \frac{(d+1)f}{r} \right) (\rho^* h' - h^* \rho') + \frac{df}{r} (\rho^* S' - S^* \rho') - \frac{d(d+1)}{2} (\rho^* (h_T + S + \rho) + \text{h.c.}) .
\]

Unlike the traceless-transverse tensor modes, the situation is now more complicated. Since the \( Z \)-matrix has both negative and positive eigenvalues, the possibility of ghosts is present. Consequently one must solve the coupled eigenvalue condition and check the actual norm,

\[
\int_1^{r_c} dr r^{d-2} \phi^\dagger Z \phi , \text{ for each mode.}
\]

Without the Planck brane, positivity of the norm for on shell physical states presumably is required if the Maldacena hypothesis relating a unitary Yang-Mills theory to a ghost free super string is in fact valid. Nonetheless the quantization and proof of the no-ghost theorem in Ramond-Ramond backgrounds is far from trivial. Nonetheless for all the massive glueball states which survive when the Planck brane is removed are expected to be stable — neither tachyonic nor ghost-like. However, in the presence of the Planck brane new states do appear.

For the radion we have calculated its norm near the critical point \( (\alpha \approx \alpha_{\text{critical}}) \) to zeroth order in \( m^2 \). With \( h_T = (\frac{d-1}{d}) h_0 + 0(m^2) \), and, for \( r \approx r_c \) large, \( S_0(r) \approx y(r) \rho_0 \), \( h_0(r) \approx d[y(r) - (\frac{d+1}{d-1}) y_c] \rho_0 \), it follows that

\[
\int_1^{r_c} dr r^{d-2} \phi^\dagger Z \phi \approx \frac{1}{(d-2)} r_c^{d-1} y_c^2 \rho_0^2 > 0 \tag{49}
\]

so radion appears not to be a ghost. We have checked numerically that this result holds for a range of values of \( \alpha \) including our model for the Planck brane at \( \alpha = 1 \). Therefore, the radion is a perfectly normal physical mode with a tachyonic mass for our microscopic construction of the asymmetric brane.
5 Brane World Gravity

Now let us return to discuss the graviton in our model and compare it with the Randall-Sundarm constructions. As noted above the transverse tensor \( h_{\mu\nu}^\perp = \epsilon_{\mu\nu}(p) T(r, p) e^{ipx} \), obeys the same equation as a minimally coupled massless scalar,

\[
\frac{1}{r^d} \left( r^d f T' \right)' - \frac{p^2}{r^2} T = 0,
\]

(50)

with \( T'(r_c) = 0 \). A minimal scalar field, \( \nabla^2 \phi + M^2 \phi = 0 \), of mass \( M \) has two possible asymptotic forms, \( \phi(r) \sim (r)^{-\Delta} \), or \( \phi(r) \sim (r)^{\Delta-(d+1)} \), for conformal dimension \( \Delta = \frac{1}{2}(d+1) + \sqrt{\frac{1}{4}(d+1)^2 + (R_{ads}M^2)^2} \). By a general argument for AdS\(^d+2\)/CFT duality, only the first normalizable mode corresponds to an operator in correlation function for the boundary Yang-Mills theory — in our present example, \( T(r) \sim (r)^{-6} \) with conformal mass \( M = 0 \) corresponding to the energy-momentum operator in the boundary theory. The second non-normalizable mode is present only when the boundary theory is deformed by including the operator in the Yang-Mills Lagrangian. Our new graviton solutions, \( T(r) = \text{const} \) with \( p^2 = 0 \), apparently corresponds to the second possibility where the Planck brane defect provides the deformation in terms of a UV cut-off so that this state can be normalized.

![Figure 1: The effective potential \( V_{eff}(y) \) (short dashed curve) with the horizon at \( y = 0 \) (or \( r = 1 \)) and the delta function Planck brane at \( y_c = 5.08308 \) (or \( r_c = 128 \)). Superimposed profile of the graviton wave function (long dashed curve) compared to the first three glueball wave functions (solid curve).](image)

For a direct comparison with the Randall-Sundrum solution, it is instructive to convert to the proper distance \( \tilde{r} \) from the Black Hole horizon and introduce an integrating factor,
\[ \Psi = e^{-W(y)} T, \] so that the tensor equation has the SUSY form,

\[ \left[ \frac{\partial}{\partial y} - W'(y) \right] \left[ \frac{\partial}{\partial y} + W'(y) \right] \Psi(y) = \frac{p^2}{r^2} \Psi(r). \] (51)

The “prepotential”

\[ W(y) = -\frac{1}{2} \log( r^d \sqrt{f(r)} ) = -\frac{1}{2} \log( \sinh((d + 1)y) ) + \text{const} \]

is defined in the interval \( y \in [0, y_c] \) between the Black Hole horizon \( (y = 0) \) and the Planck brane \( (y = y_c) \). The effective potential \( V_{\text{eff}}(y) = (W')^2 - W'' \) is plotted in Fig 1. In the limit \( r_{\text{min}} \rightarrow 0 \) (or large \( y \approx y_c \)), we recover the Randall-Sundrum solution, \( V(y) \approx (d + 1)k|y - y_c| \), for pure AdS.

Note the “volcano” potential at the Planck brane is a delta-function at the edge of the disk. In addition, our effective potential has an attractive double pole at the origin of the disk \( (y = 0) \) at the Black Hole horizon \( (r = r_{\text{min}}) \), which accounts for the discrete glueball or radial KK spectrum. In this representation, the zero-mass bound-state graviton has wave function, \( \Psi = \Psi_0 \exp(-W(y)) \) localized close to the Planck brane. Conversely the glueballs are localized close to the Black Hole horizon with mass on order of \( m_{\text{GB}} \sim k^2 r_{\text{min}} \). Note that, with \( r_{\text{min}} = 0 \), there would be no confinement, and one reverts back to the standard graviton equation [4] with no mass gap.

Without the Planck brane, the zero mass solution is no longer a normalizable state in the tensor spectrum. With the Planck brane, the graviton appears but the massive states are very slightly shifted by \( \Delta m/m_{\text{GB}} = O(r_{\text{min}}/r_c) \). The mass hierarchy between the low energy glueballs and the Planck scale is therefore a reflection of the fact that glueballs wave functions are confined to a region close to \( r = r_{\text{min}} \) where the graviton is exponentially suppressed as a function of proper distance, \( r_{\text{min}}/r_c \approx \exp[-k\Delta y] \).

Once we have a graviton and general covariance in the d-dimensional Minkowski space parallel to our \( d+1 \) dimensional Planck brane we expect to find a low energy effective theory for classical brane world gravity coupled to matter. To see explicitly how this low energy theory emerges, we should split the metric into heavy and light modes, \( g_{\mu\nu}(r, \tau, x) = \hat{g}_{\mu\nu}(r, x) + h_{\mu\nu}(r, \tau, x) \). Assuming that the graviton is the only zero mode, the new background metric \( \hat{g} \) includes the gravitation fluctuations \( g_{\mu\nu}(x) \) to all orders: \( ds^2 = \frac{1}{f(r)} dr^2 + f(r) d\tau^2 + r^2 g_{\mu\nu}(x) dx^\mu dx^\nu \). Expanding the Euclidean action (7) in the \( h_{\mu\nu}(r, \tau, x) \), we arrive at an effective action. The zeroth order term gives the dimensionally reduced Einstein-Hilbert action,

\[ M_D^2 \int d^d x \int_0^\beta d\tau \int_{r_{\text{min}}}^{r_c} dr \sqrt{-\hat{g}} \left( kr \right)^{d-2} R(x) = M_{\text{Planck}}^{d-2} \int d^d x \sqrt{-\hat{g}} R. \] (52)
The linear term in $h_{MN}$ should vanish and the quadratic terms provide the kinetic energy of the massive glueballs coupled covariantly to gravity. Higher terms correspond to glueball/graviton scattering, etc.

The strength of the brane world gravity is given by the effective Planck mass which is related to the transverse volume by

$$M_{\text{Planck}}^2 = M_0^2 V_{\perp},$$

where

$$V_{\perp} = \beta \frac{(kr_c)^{d-1} - (kr_{\min})^{d-1}}{k(d-1)}. \tag{53}$$

Specifically if we start with 11-d M-theory for the deformed AdS$_7 \times S^4$ background, the above analysis can be summarize by

$$M_{11}^2 = \frac{1}{l_p^3} \times \left( \frac{R}{2} \right)^4 \Vol(S^4) \times 2\pi R_{11} \times \frac{R \beta}{4} \left( \frac{r_c}{R} \right)^4, \tag{54}$$

where the factors reading from left to right respectively come from: the 11-d Planck scale $M_{11}^2 = 1/l_p^3$, the volume of $S^4$, the length of the 11-th axis and the volume of $r$-$\tau$ disk. The result is

$$M_{\text{Planck}}^2 = \frac{32\pi^6 N^2}{9\alpha'_{\text{qcd}}} (r_c/r_{\min})^4. \tag{55}$$

It is also worth noting that our radion instability is an extremely small scale set by the inverse of this hierarchy: $m_{\text{tachyon}}^2/\alpha'_{\text{qcd}} = O(\alpha'_{\text{qcd}}/M_{\text{Planck}}^2)$. Nonetheless such an instability is still probably too large on a cosmological time scale.

If we regard the Black Hole horizon at $r = r_{\min}$ as the location of a “QCD brane”, this simple expression can be seen as the standard Randall-Sundrum mass hierarchy between the Planck mass and the “QCD” scale given by the exponential ($r_{\min}/r_c \simeq \exp[-k\Delta y]$) of the proper distance, $\Delta y$, separating the Planck brane from the “QCD brane”. In addition there is a factor $N^2$, which sets the scale of the 6 remaining compact dimensions in units of the fundamental Planck length of M-theory. In fact we have found a Randall-Sundrum like mechanism without any explicit introduction of a second matter brane. Moreover we have a truly large GeV scale extra dimensions; however there is no obvious contradiction with gravity at short distance since the radial KK modes that give exponential corrections to gravity are simply intermediate glueball states that are the obvious consequent of QCD corrections to the graviton propagator. An alternative way to understand the hierarchy is to observe that the coupling of glueballs to the graviton represents an overlap of the graviton wavefunction with the square of the glueball wave function (see Fig. 1). The small coupling is now a result of the exponentially small overlap.

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\[2\text{Explicitly we use } \Vol(S^n) = 2\pi^{(n+1)/2}/\Gamma((n+1)/2), \ R^4 = 8\pi N l_p^3, \ r_{\min} = R^2 \Lambda_{\text{qcd}}, \ \alpha'_{\text{qcd}} = R^3 l_s^3/ r_{\min}^3, \ R_{11} = g_s l_s, \ \beta = 2\pi R^3/(3r_{\min}), \text{ and } l_p^3 = g_s l_s^3. \]
6 Conclusion

It is tempting to speculate that our construct can serve as a simple model of gravity interacting with non-conformal matter in the confined phase where both the matter and the graviton are a consequence of excitations of the same higher dimensional string fields with no need for external sources. Of course we are aware that this scenario is a long way from a realistic model that can be rigorously constructed within the context of non-perturbative string theory. Even in this limited context, we have found that our first attempt at a microscopic theory for the Planck brane, intersecting $d-1$ and $d$ dimensional branes, actually leads to a tachyonic radion mode. Future efforts will explore other approaches to a microscopic theory to see if this instability can be circumvented.

Finally let us remark on the spectrum in our M-theory approach to the strong coupling $QCD_4$ glueballs, imagining for now that some suitable model can be found to stabilize the radion. The glueball spectrum has been studied in details \[2, \[3\]. The lowest mass “glueball” states found on the gravitational metric $g_{MN}$ restricted to the $AdS^7$ subspace, obey the inequalities

$$m(0^{++}) < m(2^{++}) = m(1^{-}) = m(0^{++}) < m(1^{-}) = m(0^{-}) .$$

The degeneracy in the spectrum reflect an accidental $O(4)$ symmetry at strong coupling due to the fact that the compact 11th coordinate acts like an extra spatial axis. In effect we have a 5-d theory with tensors, $g_{r r}$, $g_{\mu \nu}$ and $g_{\mu r}$, where $\mu = 1, 2, 3, 4, 11$. The massive (“glueball” states) are very slightly displaced by the presences of a Planck brane due the exponential hierarchy discussed above. However, after introducing the Planck brane, there are a finite number of new “light” states. The new additional states are: (i) A 5-d zero mass tensor graviton with five physical degrees of freedom which gives rise to Kaluza-Klein gravity on $R^{(3,1)} \times S^1$ or in 4-d language a $2^{++}$ graviton, a $1^{-}$ KK photon and a zero mass $0^{++}$ KK dilaton; (ii) a new radion state, stabilized by some unknown mechanism; and (iii) a new light 5-d vector again assuming a way is found to introduce a mass term for the $A_\mu$ field coupled to the brane. This decomposes in 4-d into a vector $1^{+-}$/scalar $0^{++}$ multiplet. Note there is no Planck domain wall state corresponding to the lowest scalar glueball $0^{++}$ associated with $g_{rr}$. Instead the radion is an entirely new degree of freedom due to fluctuations in the finite proper distances, $\Delta y$, of the Planck brane form the Black Hole horizon. Away from strong coupling, the $m(2^{++}) = m(0^{++})$ glueball degeneracy is expected to be broken since the 11-th axis is distinguished in many ways (being a compact axis on which the membrane is wrapped) and in fact lattice QCD data also exhibit a significant splitting. Consequently a similar fate should lift the mass of the dilaton. Once $A_\mu$ is reintroduced a standard Higgs mechanism can lift its mass. Of course without a viable microscopic model, it is not possible to be precise, but clearly higher dimensional branes world models, such as the 5+1 dimensional one we are considering here, can give very interesting new low mass states.
We have sketched the effective low energy theory on the gravity side of the AdS/CFT duality. Here the “glueballs” and the graviton are seen as excitations of the same supergravity (or weak coupling string) modes. This suggests an intriguing possibility for the relationship between the “QCD string” and the “fundamental string”. In the confined phase of QCD both strings may merge into a single entity. One possibility is that when (and if) a phenomenologically satisfactory vacuum is found for superstring theory, one may have a phase diagram, which continuously connects quantum gravity to the lower energy confined QCD phase coupled weakly to low energy gravitational modes. Of course such a theory would have to have additional scales in between the Planck brane and the QCD scale for all of the rest of the standard model, TeV physics and whatever might “grow” in the desert. This need not contradict the alternative dual description of a weak coupling Yang Mills limit in which gravity is seen as an additional (semi-classical) background left over from the superstrings in the IR. Since the graviton mode is a non-normalizable state in the standard AdS/CFT dictionary, it presumably corresponds to adding the Einstein-Hilbert action (at low energy) to the Yang-Mills Lagrangian. Although the QCD string in this picture appears simply as a trick for reformulating the color singlet sector of the confined phase by String/Gauge duality it is in fact more directly a consequence of superstring in extra dimensions. It would be nice to be able to make this dual view of the QCD string more precise.
Appendix A: Global Radion Analysis in Randall-Sundrum Two-Brane Scenario

Consider a pure $AdS_5$ background, with two fixed branes located at $r = r_{\text{min}}$ and $r = r_c$, ($0 < r_{\text{min}} < r_c$), and assume $Z_2$-orbifolding at both branes, (we shall adopt notations as close to our $AdS/BH$ case as possible.) We are interested in providing a “global” solution for a massless radion and also in the limit $r_{\text{min}} \to 0$ where the proper distance between branes diverges. Introducing analogous “reduced” fluctuations, $h_{\mu\nu}$ and $\rho$, the metric, for $r_{\text{min}} \leq r \leq r_c$, becomes, $ds^2 = (1 + \rho) \frac{dr^2}{r^2} + (\eta_{\mu\nu} + h_{\mu\nu}) r^2 dx^\mu dx^\nu$. There are now two sets of junction conditions, at $r = r_{\text{min}}$ and $r = r_c$, where $h'_{\mu\nu}(r_{\text{min}} + \epsilon) = \rho(r_{\text{min}})$ $r_{\text{min}} \eta_{\mu\nu}$ and $h'_{\mu\nu}(r_c - \epsilon) = \frac{\rho(r_c)}{r_c} \eta_{\mu\nu}$ respectively.

Leaving aside the transverse graviton, other massless modes can again be analyzed as done for the case of $AdS/BH$ using lightcone variables. There are four vector fluctuations, $h_{i\pm}$, $i = 1, 2$, and four scalar fluctuations, $h$, $h_{++}$, $h_{--}$, and $h_{+-}$. One easily finds that, after global gauge fixings, $h_{i\pm} = 0$, $h_{--} = 0$, and $h = 2h_{+-}$, with $h$ and $h_{++}$ remaining to be specified.

We next choose the gauge where $\rho \to \rho_0$. Linearized Einstein equations,

$$r h' - 4\rho_0 = 0, \quad \nabla^2 h_{++} - \frac{(2h + 4\rho_0)}{r^2} = 0,$$

(A.1)

can be solved after enforcing boundary conditions, leading to

$$h(r) = 4\rho_0 \left\{ \log \left( \frac{r}{r_{\text{min}}} \right) - \left( \frac{r^2_c}{r^2_c - r^2_{\text{min}}} \right) \log \left( \frac{r_c}{r_{\text{min}}} \right) \right\},$$

$$h'_{++}(r) = \frac{4\rho_0}{r^2_c} \left\{ \log \left( \frac{r}{r_c} \right) + \left( \frac{r_{\text{min}}}{r_c} \right)^2 \left( \frac{r^2_c - r^2}{r^2_c - r_{\text{min}}^2} \right) \log \left( \frac{r_c}{r_{\text{min}}} \right) \right\}.$$

(A.2)

The integration constant for $h_{++}$ can be fixed by the last $r$-independent gauge transformation. It is straightforward to verify that the norm for this mode is positive. This massless radion is therefore physical.

Finally, let us find out what happens if one pushes $r_{\text{min}}$ to zero. In this limit, the proper distance between branes $\int_0^{r_c} \frac{dr}{r} \to \infty$, thus reducing to a one-brane RS scenario [4]. As $r_{\text{min}} \to 0$, both $h$ and $h'_{++}$ approach well-defined limits: $h(r) \to 4\rho_0 \log(\frac{r_c}{r})$, and $h'_{++}(r) \to \frac{4\rho_0}{r^3_c} \log(\frac{r_c}{r})$. However, for $h$ and $h_{++}$ to be bounded at $r = 0$, one finds that

$$\rho_0 = 0,$$

(A.3)

so that the radion decouples, as it should.
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