Superfield Formulation of the Phase Space Path Integral

I.A. BATALIN
Lebedev Physics Institute
53 Leninisky Prospect
Moscow 117924
Russia

K. BERING
Institute for Fundamental Theory
Department of Physics
University of Florida
Gainesville, FL, 32611, USA

and

P.H. DAMGAARD
The Niels Bohr Institute
Blegdamsvej 17
DK-2100 Copenhagen
Denmark

Abstract

We give a superfield formulation of the path integral on an arbitrary curved phase space, with or without first class constraints. Canonical tranformations and BRST transformations enter in a unified manner. The superpartners of the original phase space variables precisely conspire to produce the correct path integral measure, as Pfaffian ghosts. When extended to the case of second-class constraints, the correct path integral measure is again reproduced after integrating over the superpartners. These results suggest that the superfield formulation is of first-principle nature.
1. Introduction. In a recent paper [1], we have shown that an arbitrary Hamiltonian quantum field theory can be given a superfield formulation. Although the formalism of Ref. [1] and the constructions explained below can be formulated in operator language, we shall here focus on the path integral formalism. The needed superspace is two-dimensional, consisting of time $t$ and a new Grassmann-odd superpartner, which we denote by $\theta$. All original phase space coordinates $z^A_0(t)$ are then treated as zero-components of super phase space coordinates

$$z^A(t, \theta) = z^A_0(t) + \theta z^A_1(t).$$  \hspace{1cm} (1)

In particular, $z^A(t, \theta)$ has the same statistics as $z^A_0(t)$, which we denote by $\epsilon_A$. One essential ingredient of Ref. [1] was the introduction of a superspace derivative

$$D \equiv \frac{d}{d\theta} + \theta \frac{d}{dt},$$  \hspace{1cm} (2)

which acts like a “square root” of the time derivative:

$$D^2 = \frac{d}{dt}.$$  \hspace{1cm} (3)

The superspace extends in an obvious manner to a $(d+1)$-dimensional superspace of coordinates $(x^\mu, \theta)$ when considered in the context of a Lorentz invariant quantum field theory in $d$ dimensions, but we shall here restrict ourselves to the finite-dimensional case of $2N$ phase space variables.

The purpose of the present paper is to demonstrate that the superspace formalism developed in [1] reaches one step deeper than could have been anticipated. By considering here the extension to a phase space with a non-constant symplectic metric, we shall show that the required superspace generalization of the phase space path integral [2,3] leads, after integrating out the fermionic coordinate $\theta$, to precisely the correct path integral measure. This is a quite non-trivial fact, completely independent of whether we consider a system with (first class) constraints or not. Moreover, when considered in the presence of second-class constraints it turns out that our formalism also here directly yields all required factors in the path integral.

2. Symplectic Structure. In addition to eqs. (1) and (2), the few ingredients we need are as follows. Define a graded Poisson bracket by

$$\{F, G\} \equiv F \overset{\leftarrow}{\partial}_A \omega^{AB} \overset{\rightarrow}{\partial}_B G,$$  \hspace{1cm} (4)

for functions $F = F(z(t, \theta))$, $G = G(z(t, \theta))$. Here the (non-degenerate) symplectic metric,

$$\omega^{AB} = \omega^{AB}(z(t, \theta)) = \{z^A(t, \theta), z^B(t, \theta)\},$$  \hspace{1cm} (5)

is allowed to depend on $z^A(t, \theta)$. We will in what follows suppress some of the arguments to make the formulas more readable. For precise details we refer to the original paper [1]. The symmetry properties are as follows:

$$\omega^{BA} = -(-1)^{\epsilon_A \epsilon_B} \omega^{AB}, \hspace{1cm} \epsilon(\omega^{AB}) = \epsilon_A + \epsilon_B,$$  \hspace{1cm} (6)

which ensures

$$\{F, G\} = -(-1)^{\epsilon(F)\epsilon(G)} \{G, F\}.$$  \hspace{1cm} (7)

Similarly, the super Jacobi identity

$$\{\{F, G\}, H\}(-1)^{\epsilon(F)\epsilon(H)} + \text{cyclic perm.}(F, G, H) = 0,$$  \hspace{1cm} (8)
is satisfied if
\[ \omega^{AD} \partial_D \omega^{BC} (-1)^{\epsilon_A \epsilon_C} + \text{cyclic perm.}(A,B,C) = 0 . \tag{9} \]
As usual, we define an inverse symplectic metric \( \omega_{AB} \) by \( \omega^{AB} \omega_{BC} = \delta^A_C \). Its symmetry properties are quite different:
\[ \omega_{BA} = (-1)^{(\epsilon_A+1)(\epsilon_B+1)} \omega_{AB} , \quad \epsilon (\omega_{AB}) = \epsilon_A + \epsilon_B . \tag{10} \]
Crucial in this context is that the Jacobi identity turns into a closedness relation
\[ \partial_C \omega_{AB} (-1)^{(\epsilon_C+1)\epsilon_B} + \text{cyclic perm.}(A,B,C) = 0 , \tag{11} \]
which implies that locally we can represent \( \omega_{AB} \) in terms of a symplectic potential \( V_A \):
\[ \omega_{AB} = (\partial_A V_B - (-1)^{\epsilon_A \epsilon_B} \partial_B V_A) (-1)^{\epsilon_B} . \tag{12} \]
Our primary aim is not to elaborate on global issues. We shall for simplicity assume that the phase space is simply connected and that there exists a globally defined symplectic potential.

3. **Super Hamiltonian.** Let there now be given a Grassmann-odd BRST generator \( \Omega = \Omega(z(t, \theta)) \) and an Hamiltonian \( H = H(z(t, \theta)) \) with the properties \[ \{ \Omega, \Omega \} = 0 \quad \text{and} \quad \{ H, \Omega \} = 0 . \tag{13} \]
We combine these two fundamental objects into a Grassmann-odd superfield \( Q \):
\[ Q(z(t, \theta), \theta) \equiv \Omega(z(t, \theta)) + \theta H(z(t, \theta)) . \tag{14} \]
It is nilpotent in terms of the Poisson bracket, by virtue of eq. (13):
\[ \{ Q, Q \} = 0 . \tag{15} \]
This nilpotency condition is preserved under super canonical transformations
\[ Q \mapsto Q_\Psi \equiv e^{\text{ad} \Psi} Q , \tag{16} \]
which infinitesimally are generated by the adjoint action
\[ \text{ad} \Psi \equiv \{ \Psi, \cdot \} . \tag{17} \]
Here \( \Psi \) is a superfield,
\[ \Psi (z(t, \theta), \theta) = \Psi_0 (z(t, \theta)) + \theta \Psi_1 (z(t, \theta)) , \tag{18} \]
which plays the rôle of a generalized gauge-fixing fermion. More precisely, \( \Psi \) itself is Grassmann-even, and it is the 1-component \( \Psi_1 \) which directly corresponds to the gauge-fixing fermion. Instead, the bosonic zero-component \( \Psi_0 \) is a generator of ordinary canonical transformations \( \Psi \).

4. **The Action.** The classical equations of motion are taken to be
\[ Dz^A = - \{ Q_\Psi, z^A \} . \tag{19} \]
As was shown in ref. [1], these reduce to the standard equations of motion in the original phase space variables \( z_0^A(t) \). An action which yields these equations of motion is
\[ S[z] = \int_{t_i}^{t_f} dt \, d\theta \left[ z^A \bar{\omega}_{AB} Dz^B (-1)^{\epsilon_B} - Q_\Psi \right] \tag{20} \]
where
\[ \bar{\omega}_{AB} \equiv (z^C \partial_C + 2)^{-1} \omega_{AB} = \int_0^1 \omega_{AB}(\alpha z) \alpha d\alpha . \] (21)

Note that we regain the well-known kinetic term in the case of a constant \( \omega_{AB} \) for which \( \bar{\omega}_{AB} = \frac{1}{2} \omega_{AB} \).

We may rewrite the action as
\[ S[z] = \int_{t_i}^{t_f} dt \, d\theta \left[ V_A Dz^A - Q_{\Psi} - [W(z(t,0))]_{t_i}^{t_f} \right] , \] (22)
where the boundary term is given by
\[ W(z) \equiv z^A \bar{V}_A , \] (23)
and
\[ \bar{V}_A \equiv (z^C \partial_C + 1)^{-1} V_A = \int_0^1 V_A(\alpha z) \alpha d\alpha . \] (24)

From eqs. (12) and (21) it follows that
\[ \bar{\omega}_{AB} = (\partial_A \bar{V}_B - (-1)^{\epsilon_A \epsilon_B} \partial_B \bar{V}_A)(-1)^{\epsilon_B} . \] (25)

5. Partition Function. We therefore take the action eq. (20), or equivalently eq. (22), as the correct candidate to be exponentiated, and integrated over in the superfield path integral:
\[ Z = \int [dz] \exp \left[ \frac{i}{\hbar} S[z] \right] . \] (26)

Note that this path integral contains no additional measure factors. This is not needed because the measure \([dz]\) remarkably transforms as a scalar under general coordinate transformations, due to the balance between bosonic and fermionic degrees of freedom in the superfield formulation. In this case, on a curved phase space manifold, a crucial test of the present formalism is to see if we recover the correct path integral measure after integrating out the fermionic \( \theta \)-coordinate. The calculation is straightforward, and results in \( S = S_0 + S_1 \) with
\[ S_0 = \int_{t_i}^{t_f} dt \left[ \dot{z}_0^A V_A(z_0) - H_{\Psi}(z_0) \right] - [W(z(t,0))]_{t_i}^{t_f} \]
\[ S_1 = \int_{t_i}^{t_f} dt \left[ \frac{1}{2} z_1^A \omega_{AB}(z_0) z_1^B (-1)^B - z_1^A \partial_A \Omega_{\Psi}(z_0) \right] . \] (27)

Here \( H_{\Psi} \) and \( \Omega_{\Psi} \) are defined according to eq. (14) and eq. (16). After a gaussian integration over the superpartner \( z_1^A \), and use of the nilpotency relation \( \{\Omega_{\Psi}, \Omega_{\Psi}\} = 0 \), one arrives at the standard form
\[ Z = \int [dz_0] \text{Pf}(\omega) \exp \left[ \frac{i}{\hbar} S_0[z_0] \right] , \] (28)
where the Pfaffian of an arbitrary even supermatrix is given by \( \text{Pf}(M) = (\text{Ber}(M))^{1/2} \).

6. Second Class Constraints. To test yet again how fundamental the present superfield formulation is, let us now consider the case of \( 2n \) second class constraints \( \Phi_\alpha = \Phi_\alpha(z(t,\theta)) \) of Grassmann parity \( \epsilon_\alpha \).

To impose such constraints in the path integral, we introduce an auxiliary superfield
\[ \lambda^\alpha(\theta) = \lambda_0^\alpha + \theta \lambda_1^\alpha \] (29)
of Grassmann parity \( \epsilon_\alpha + 1 \), and consider the partition function
\[ Z = \int [dz][d\lambda] \exp \left[ \frac{i}{\hbar} \left( S[z] + \int_{t_i}^{t_f} dt \, d\theta \, \Phi_\alpha \lambda^\alpha \right) \right] . \] (30)
Note again the absence of any non-trivial measure factors. Let us show that this superfield partition function completely reproduces the standard version of the partition function with second class constraints. The crucial property of second-class constraints is that the matrix
\[ \omega_{\alpha\beta} = \{ \Phi_\alpha, \Phi_\beta \} = -(-1)^{\epsilon_\alpha \epsilon_\beta} \omega^{\beta\alpha} \]  
(31)
is invertible. Let us denote the inverse matrix
\[ \omega^{\alpha\beta} = (-1)^{(\epsilon_\alpha+1)(\epsilon_\beta+1)} \omega^{\beta\alpha} \]  
(32)
According to the standard Dirac procedure, the Poisson bracket should be replaced by the Dirac bracket:
\[ \{ F, G \}_D = \{ F, G \} - \{ F, \Phi_\alpha \} \omega^{\alpha\beta} \{ \Phi_\beta, G \} \]  
(33)
Let us now trace the additional terms in the path integral due to the second-class constraints. We do this as before by integrating over the fermionic coordinate \( \theta \). The result is as follows. First, the zero component part \( S_0 \) of the action picks up a delta function term that precisely enforces the second-class constraints in the original phase space variables:
\[ S_0 \rightarrow S_0 + \int_{t_i}^{t_f} dt \lambda_0^a \Phi_\alpha(z_0) \]  
(34)
In the \( S_1 \)-part of the action, integration over \( \theta \) effectively just corresponds to replacing \( \Omega_\Psi(z_0) \) by \( \Omega_\Psi(z_0) - \Phi_\alpha(z_0) \lambda_0^\alpha \). Therefore the gaussian integration over the superpartner \( z_1^A \), besides yielding the correct Pfaffian \( \text{Pf}(\omega) \) as before, also produces a term
\[ \frac{1}{2} \{ \Omega_\Psi(z_0) - \Phi_\alpha(z_0) \lambda_0^\alpha, \Omega_\Psi(z_0) - \Phi_\beta(z_0) \lambda_0^\beta \} \]  
(35)
in the action by completing the square. If we next perform also the gaussian integration over the zero component \( \lambda_0^a \), we get \( \text{Pf}(\{ \Phi, \Phi \}) \). The rest of the action conspires to yield
\[ \frac{1}{2} \{ \Omega_\Psi, \Omega_\Psi \} - \frac{1}{2} \{ \Omega_\Psi, \Phi_\alpha \} \omega^{\alpha\beta} \{ \Phi_\beta, \Omega_\Psi \} = \frac{1}{2} \{ \Omega_\Psi, \Omega_\Psi \}_D = 0 \]  
(36)
Therefore we quite remarkably arrive at just the standard form of the partition function:
\[ Z = \int [dz_0] \text{Pf}(\omega) \exp \left[ \frac{i}{\hbar} S_0[z_0] \right] \delta(\Phi) \text{Pf}(\{ \Phi, \Phi \}) \]  
(37)

7. Conclusions. The superfield formulation introduced in thus in a very precise and non-trivial manner encodes all the information required for Hamiltonian path integral quantization for systems with or without any combination of first and second class constraints, on an arbitrary curved phase space. In view of this, we propose to consider our superfield formalism as a first principle on which to base quantization. An operatorial formulation of precisely the same superfield formulation also exists, with or without first and second class constraints, and with possibly non-constant symplectic \( \omega_{AB} \).

Acknowledgement: I.A.B. and K.B. would like to thank the Niels Bohr Institute for the warm hospitality extended to them there. The work of I.A.B. and P.H.D. is partially supported by grant INTAS-RFBR 95-0829. I.A.B. also acknowledges the funding by grants INTAS 96-0308, RFBR 96-01-00482 and RFBR 96-02-17314. The work of K.B. is supported by DoE grant DE-FG02-97ER41029 and Nordita.

*Without second-class constraints this term was just \( \frac{1}{2} \{ \Omega_\Psi, \Omega_\Psi \} \), which in that case would vanish on account of the nilpotency condition eq. (13).
References

[1] I.A. Batalin, K. Bering and P.H. Damgaard, Nucl. Phys. B515 (1998) 455.

[2] I.A. Batalin and E.S. Fradkin, Mod. Phys. Lett. A4 (1989) 1001.

[3] I.A. Batalin and E.S. Fradkin, Nucl. Phys. B326 (1989) 701.

[4] E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. B55 (1975) 224.
   I.A. Batalin and G.A. Vilkovisky, Phys. Lett. B69 (1977) 309.
   E.S. Fradkin and E.S. Fradkina, Phys. Lett. B72 (1978) 343.
   I.A. Batalin and E.S. Fradkin, Phys. Lett. B122 (1983) 157.

[5] E.S. Fradkin, in Proc. X Winter School of Theo. Phys., Karpacs (1973) No. 207, pp. 93.
   P. Senjanovic, Ann. Phys. 100 (1976) 227.