Torsion of a thin-walled structure of a waveguide with rectangular cross-section

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Abstract. The paper has considered the problem of determining the stress state of thin-walled waveguide structures during torsion. An analytical solution using the Bredt’s formula has been considered and its comparison with a numerical solution using the finite element method has been performed. The numerical solution has been carried out using a created volume model, and the load has been modeled as a set of concentrated forces distributed over the entire cross-section of the waveguide. It has allowed us to reduce the local effects in the form of complex stress state and increase the accuracy of the results. Comparison of the results has shown that the stresses calculated by the finite element method are always higher than those obtained by the Bredt’s formula and it is necessary to take into account their maximum values. The deformation evaluation also has shown that the FEM calculations give larger twist angles than those obtained by the Bredt’s formula for all the considered section sizes.

1. Introduction
Calculation of the action of almost any load for thin-walled beams is a very complicated task [1-15]. Contrary to the theory of beams, a thin-walled cross-section cannot be considered as absolutely rigid and unchangeable; therefore its deformation must be taken into account. This requires the application of a more complex theory of plates and shells that leads to system of differential equations for the two unknown values. Due to the complexity of differential equations, it is not always possible to obtain the accurate analytical solutions.

The development of computers, numerical methods and universal calculation programs based on them made it possible to automate the calculations of thin-walled sections and in many respects pushed analytical calculation methods into the background. Among the numerical methods, the most widespread is the finite element method due to its versatility, ease of use and implementation in many ready-made software products.

However, numerical methods have their own peculiarities and limitations, which require modification of the original design scheme to obtain a correct solution. For example, from the accuracy point of view it is preferable to use volume models and volume finite elements [16-20]. But this approach does not allow us to specify any torques, since volume finite elements do not have rotational degrees of freedom, they have only translational ones. As a result, the calculator needs to convert the torque load into an equivalent distribution of concentrated forces. The accuracy of such a conversion will greatly affect the results obtained, especially in the local area of application of this equivalent load, which makes it difficult to calculate when there is a torque load.
This paper considers the problem of torsion of a thin-walled beam with rectangular cross-section as applied to waveguide structures. The example of analytical and numerical solutions with a comparison of the results is given. In the numerical solution, a rather complicated method is proposed for decomposing the torque into an equivalent set of concentrated forces uniformly distributed over the cross-section of the waveguide.

2. Analytical solution according to R. Bredt’s formula

We consider the problem of pure torsion of an extended beam, which is rigidly fixed on one side and loaded with a torque \( T \) on the other side (figure 1).

![Figure 1. Torsion of the waveguide.](image)

In 1896, R. Bredt proposed a simple solution to this torsion problem for thin-walled beams with almost any cross-section shape [5]. Using the analogy with the theory of beams, to determine the stresses and strains during torsion, it is necessary to know the polar moment of inertia and the polar moment of the cross-section during torsion. With regard to the rectangular thin-walled cross-section of the waveguide, the expressions for the polar moment of inertia and the moment of resistance are:

\[
J_\rho = \frac{4\omega^2}{\int ds}, \quad W_\rho = 2\omega t .
\]

(1)

where \( \omega \) is the cross-sectional area enclosed within the median line of the thin-walled cross-section; \( \int ds \) is the integral over the closed contour of the median line of the cross-section.

In our case:

\[
\omega = b'h'; \quad \int ds = \frac{2}{t}(b' + h') .
\]

(2)

The substitution of the expressions (2) in the formulas (1) gives:

\[
J_\rho = 2t\left(\frac{b'h'}{b' + h'}\right)^2, \quad W_\rho = 2t \cdot b'h'.
\]

(3)

(4)

Then the shear stresses and strains during torsion are determined by equations equivalent to the dependences of beam theory:

\[
\tau_\rho = \frac{T}{W_\rho} .
\]

(5)
\[ \varphi = \frac{T \cdot l}{G \cdot J_\rho}, \]  

(6)

where G is the shear modulus.

Unlike the beam theory, formula (5) determines not the maximum, but the average stresses in the thin-walled section of the waveguide. This follows from the assumptions made by Bredt when deriving these simple analytical relations. The maximum stresses are supposed to act in the inner corners of a rectangular cross-section and depend on the radii of rounding of these corners \( r \) (figure 1, b). The values of these maximum shear stresses can be determined through the stress concentration factor according to the formula [X]:

\[ \tau_{\max} = \tau_{op} \cdot \alpha_K = \tau_{op} \left( 1 + \frac{t}{2r + t} - \frac{b' + h'}{b'h'} \right), \]  

(7)

Analytical dependencies (5-7) are valid for sufficiently long beams, i.e. when the ratio of length to maximum cross-section size is not less than 5...6.

3. Preparation of a numerical solution

The numerical solution of the torsion problem by the method is carried out for a three-dimensional model of a thin-walled waveguide, which will allow us to obtain a more accurate solution. This approach is complicated by the need to decompose the concentrated torque into a set of concentrated forces. This follows from the implementation of the finite element method for solid bodies that do not have rotational degrees of freedom. Therefore, we transform the torque into an equivalent load in the form of distributed forces \( F_i \) (figure 2).

![Figure 2. The system of the forces equivalent to the torque.](image)

The value of the \( i \)-th force is calculated from the condition that the moment of this force is constant and equal to

\[ T_i = F_i \cdot h_i = \frac{T}{N} = \text{const}, \]  

(8)

where \( h_i \) is the moment arm for \( F_i \).

\( N \) is the specified total number of forces into which the moment \( T \) is expanded.

A feature of the cross-sectional geometry of the waveguide is that \( H \leftrightarrow B \), as a result of which the distribution of forces along the walls will be uneven. In the case of a thin-walled cross-section with a small wall thickness the points of application of forces are naturally chosen at points on its middle surface. Then the forces \( F_i \), which simulate the torsion, will be distributed along the entire perimeter of
the middle surface of the waveguide cross-section. Let’s represent the external torque $T$ as consisting of two components:

$$T = T_I + T_{II},$$

where $T_I$ is the component of the total torque from the concentrated forces $F_i$, located at 8 characteristic points of a rectangular cross-section (figure 2, a) and forming the main system of forces; $T_{II}$ is the component of the total torque from concentrated forces $F_i$ located in the intervals between 8 characteristic points of the cross-section and forming an additional system of forces.

### 3.1 Basic system of forces

The $T_I$ component in equation (2) is expressed as

$$T_I = 8 \cdot T_i = \sum F_i h_i = 4 F_i \cdot \frac{2B'}{\cos \alpha} + 2 F_h \cdot H + 2 F_H \cdot B,$$

where $F_i = F_2 = F_4 = F_6 = F_8 = T_i \frac{\cos \alpha}{2B'}$ is the forces at the corners of the cross-section;

$F_h = F_2 = F_4 = \frac{T}{2H}$ is the forces on the lower and upper sides of the cross-section;

$F_H = F_3 = F_7 = \frac{T}{2B}$ is the forces on the lateral sides of the cross-section;

$$\alpha = \arctg \left( \frac{H}{B} \right)$$

is the cross-section diagonal angle.

### 3.2 Additional system of forces

Using only the main system of forces $F_i$ during translation, located only at 8 characteristic points of a rectangular cross-section (figure 2, a) will lead to a highly complicated loading of the cross-section and the incorrect resulting solution. Consequently, it is necessary to create an additional system of forces in the intervals between the forces of the main system, while, due to the rectangular waveguide cross-section, the amount of additional forces on different walls will be different.

Figure 2, b shows, for example, additional forces in one quadrant of the cross-section. Obviously, the appearance of one force in one quadrant causes the appearance of a similar force in all other quadrants of the cross-section. Thus, the total number of forces simulating the torque will be equal to:

$$N = 8 + 4 \cdot (n + k)$$  \hspace{1cm} (11)

and the moment of each force $F_i$ will be equal to:

$$T_i = \frac{T}{N} = \frac{T}{4(2 + n + k)}.$$  \hspace{1cm} (12)

To determine the amount of additional forces, it is necessary to determine the moment arms to the points of their location. For the upper wall of the section (figure 1, b), the position of this moment arm is determined by the geometric sizes:

$$b_i = \frac{B \cdot i}{2(n + 1)} \hspace{0.5cm} \alpha_i = \arctg \left( \frac{H \cdot (n + 1)}{B \cdot i} \right) \hspace{0.5cm} h_i = \frac{1}{2} \sqrt{H^2 + \left( \frac{B \cdot i}{n + 1} \right)^2}$$  \hspace{1cm} (13)

Knowing the moment arm and the torque value, the corresponding value of the force $F_i$ we define as:
\begin{align*}
F_{bi} &= \frac{2 \cdot M_i}{\sqrt{H^2 + \left( \frac{B \cdot i}{n+1} \right)^2}}. \quad (14)
\end{align*}

Similarly, the geometric dimensions and the value of the forces for the vertical sides can be found:
\begin{align*}
h_j &= \frac{H \cdot j}{2(k+1)}; \quad \alpha_j = \arctg \left( \frac{H \cdot i}{B \cdot (k+1)} \right); \quad F_{hi} = \frac{2 \cdot M_i}{\sqrt{B^2 + \left( \frac{H \cdot j}{k+1} \right)^2}}. \quad (15)
\end{align*}

Using the obtained dependencies, a program was developed that, for the given loading conditions, creates a macro for the Ansys software. Let us perform calculations on analytical dependences based on the Bredt’s formula, obtain a numerical solution with the expansion of the torque into forces, and then compare the results.

4. Results and discussion
A cantilever waveguide was calculated with the following sizes: \( l = 1 \) m, \( B = 0.0374 \) mm, \( H = 0.0174 \) mm, \( t = 0.0012 \) mm, \( r = 0 \). Material: \( E = 7 \times 10^9 \) Pa, \( G = 2.665 \times 10^9 \), Poisson’s ratio 0.3. Load: \( T = 1 \) N\( \cdot \)m, which was represented by a set of forces (figure 3).

The numerical calculation was carried out by the finite element method in the Ansys software using 208,793 finite elements Solid98. The graphical results are shown in the figure 4, and the numerical data are summarized in the table 1.

**Figure 3.** Results of numerical calculation of the waveguide.

**Figure 4.** Results of the numerical calculation in the cross-section of the waveguide.
Table 1. Calculation results.

|                  | Average shear stress, MPa | Maximum shear stress, MPa | Angle of twist, radian |
|------------------|---------------------------|---------------------------|------------------------|
| Bredt’s solution | 0.71                      | 1.34                      | 0.0024                 |
| FEM solution     | 0.78                      | 1.35                      | 0.0025                 |
| Deviation, %     | 9.14                      | 0.383                     | 5.34                   |

The results of calculating the stresses in the waveguide show that the Saint-Venant principle is fulfilled and the edge effect from the application of the load is persist at a distance of one section width (figure 3, b), despite the model used in the form of a distribution of a large number of concentrated forces.

Comparison of the numerical values of the results in table 1 shows that the volume model predicts a little higher stresses and strains than the Bredt’s formula does. Such a discrepancy can be explained by the influence of the cross-section deformations on the solution. The change in the values of shear stresses along the wall thickness is also noticeable.

The conducted studies of the dependence of the obtained calculation results on the number of points N showed that with an increase in their number from N=18 to N=656, the obtained stress-deformed data changes by only 3-5%. The variation in the number of finite elements from 8,000 to 46,690 showed that the results of calculating also changed insignificantly, by 3-4%. Thus, there is no significant refinement of the results due to a change in the FE mesh size or the number N of forces modeling the torque. The practice of applying the developed technique shows that the number of areas and the corresponding distributed forces in the cross-section must correspond to the number of finite elements lying in the plane of the cross-section.

5. Conclusion

In this work, analytical and numerical calculations of an extended waveguide of rectangular cross-section for the action of a torque were carried out. The calculation results showed a fairly good convergence of the results of analytical and numerical calculations. Thus, the results obtained make it possible to recommend the Bredt’s formula for calculating stresses and strains in waveguides with rectangular cross-section.

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